Lie algebras

A remark on boundary level admissible representations

*Une remarque sur les représentations admissibles de niveau limite*

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**Abstract**

We point out that it is immediate by our character formula that in the case of a boundary level the characters of admissible representations of affine Kac–Moody algebras and the corresponding $W$-algebras decompose in products in terms of the Jacobi form $\vartheta_{11}(\tau, z)$. © 2017 Published by Elsevier Masson SAS on behalf of Académie des sciences.

**Résumé**

Nous remarquons la conséquence suivante de notre formule de caractères. Pour un niveau limite, les caractères d’une représentation admissible d’une algèbre de Kac–Moody affine ainsi que de la $W$-algèbre correspondante s’expriment comme des produits de formes de Jacobi $\vartheta_{11}(\tau, z)$. © 2017 Published by Elsevier Masson SAS on behalf of Académie des sciences.

Recently a remarkable map between 4-dimensional superconformal field theories and vertex algebras has been constructed [1]. This has led to new insights in the theory of characters of vertex algebras. In particular it was observed that in some cases these characters decompose in nice products [10,8].

The purpose of this note is to explain the latter phenomena. Namely, we point out that it is immediate by our character formula [5,6] that in the case of a boundary level the characters of admissible representations of affine Kac–Moody algebras and the corresponding $W$-algebras decompose in products in terms of the Jacobi form $\vartheta_{11}(\tau, z)$.

We would like to thank Wenbin Yan for drawing our attention to this question.

Let $g$ be a simple finite-dimensional Lie algebra over $\mathbb{C}$, let $\mathfrak{h}$ be a Cartan subalgebra of $g$, and let $\Delta \subset \mathfrak{h}^*$ be the set of roots. Let $Q = Z\Delta$ be the root lattice and let $Q^* = \{h \in \mathfrak{h}^* \mid \alpha(h) \in \mathbb{Z} \text{ for all } \alpha \in \Delta\}$ be the dual lattice. Let $\Delta_+ \subset \Delta$ be a subset of positive roots, let $\{\alpha_1, \ldots, \alpha_t\}$ be the set of simple roots and let $\rho$ be half of the sum of positive roots. Let $W$ be the Weyl group. Let $(\cdot | \cdot)$ be the invariant symmetric bilinear form on $g$, normalized by the condition $(\alpha | \alpha) = 2$ for a long root $\alpha$, and let $h^{\vee}$ be the dual Coxeter number ($= \frac{1}{2}$ eigenvalue of the Casimir operator on $g$). We shall identify $\mathfrak{h}$ with $\mathfrak{h}^*$ using the form $(\cdot | \cdot)$.

Let $\widehat{\mathfrak{g}} = g[t, t^{-1}] + \mathbb{C}K + \mathbb{C}d$ be the associated with $g$ affine Kac–Moody algebra (see [3] for details), let $\widehat{\mathfrak{h}} = \mathfrak{h} + \mathbb{C}K + \mathbb{C}d$ be its Cartan subalgebra. We extend the symmetric bilinear form $(\cdot | \cdot)$ from $\mathfrak{h}$ to $\widehat{\mathfrak{h}}$ by letting $(\mathfrak{h}|\mathbb{C}K + \mathbb{C}d) = 0$, $(K|K) = 0$, $(d|d) = 1$. We extend the Jacobi form $\vartheta_{11}(\tau, z)$ to this larger domain.
Let \( |d| = 1 \), and we identify \( \hat{\mathfrak{h}}^* \) with \( \mathfrak{h} \) using this form. Then \( d \) is identified with the 0th fundamental weight \( \Lambda_0 \in \hat{\mathfrak{h}}^* \), such that \( \Lambda_0|_{\mathfrak{g}^{\mathfrak{h}} \cap \mathfrak{t} + \mathfrak{c}} = 0 \), \( \Lambda_0(K) = 1 \), and \( K \) is identified with the imaginary root \( \delta \in \hat{\mathfrak{h}}^* \). Then the set of real roots of \( \hat{\mathfrak{g}} \) is \( \hat{\Delta}^r = \{ \alpha + n\delta \mid \alpha \in \Delta, n \in \mathbb{Z} \} \) and the subset of positive real roots is \( \hat{\Delta}^r_+ = \Delta_+ \cup \{ \alpha + n\delta \mid \alpha \in \Delta, n \in \mathbb{Z}_{>0} \} \). Let

\[
\hat{\Pi}_u = \{ u\delta - \theta, \alpha_1, \ldots, \alpha_\ell \},
\]

where \( \theta \in \Delta_+ \) is the highest root, so that \( \hat{\Pi}_1 \) is the set of simple roots of \( \hat{\mathfrak{g}} \). For \( \alpha \in \hat{\Delta}^r_+ \) one lets \( \alpha^\vee = 2\alpha/(\langle \alpha, \alpha \rangle) \). Finally, for \( \beta \in Q^* \) define the translation \( t_\beta \in \text{End} \hat{\mathfrak{h}}^* \) by

\[
t_\beta(\lambda) = \lambda + \lambda(K)\beta - (\langle \lambda, \beta \rangle) + \frac{1}{2}\lambda(K)|\beta|^2 \delta.
\]

Given \( \Lambda \in \hat{\mathfrak{h}}^* \) let \( \hat{\Delta}^\Lambda = \{ \alpha \in \hat{\Delta}^r_+ \mid \langle \alpha|\alpha \rangle \in \mathbb{Z} \} \). Then \( \Lambda \) is called an \textit{admissible} weight if the following two properties hold:

(i) \( \langle \Lambda + \hat{\rho}|\alpha^\vee \rangle \notin \mathbb{Z}_{\leq 0} \) for all \( \alpha \in \hat{\Delta}_+ \).

(ii) \( \mathfrak{q}\hat{\Delta}^\Lambda = \mathfrak{q}\hat{\Delta}^\text{re} \).

If instead of (ii) a stronger condition holds:

(ii)' \( \varphi(\hat{\Delta}^\Lambda) = \hat{\Delta}^\text{re} \) for a linear isomorphism \( \varphi: \hat{\mathfrak{h}}^* \to \hat{\mathfrak{h}}^* \).

then \( \Lambda \) is called a \textit{principal} admissible weight. In [6] the classification and character formulas for admissible weights are reduced to that for principal admissible weights. The latter are described by the following proposition.

\textbf{Proposition 1.} [6] Let \( \Lambda \) be a principal admissible weight and let \( k = \Lambda(K) \) be its level. Then

(a) \( k \) is a rational number with denominator \( u \in \mathbb{Z}_{\geq 1} \), such that

\[
k + h^\vee \geq \frac{h^\vee}{u} \quad \text{and} \quad \text{gcd}(u, h^\vee) = \text{gcd}(u, r^\vee) = 1,
\]

where \( r^\vee = 1 \) for \( g \) of type A-D-E, \( r^\vee = 2 \) for \( g \) of type B, C, F, and \( r^\vee = 3 \) for \( g = G_2 \).

(b) All principal admissible weights are of the form

\[
\Lambda = (t_\beta y)(\Lambda^0 - (u - 1)(k + h^\vee)\Lambda_0),
\]

where \( \beta \in Q^* \), \( y \in \mathfrak{w} \) are such that \( (t_\beta y)\hat{\Pi}_u \subset \hat{\Delta}_+ \). \( \Lambda^0 \) is an integrable weight of level \( u(k + h^\vee) - h^\vee \), and \( \dot{\rho} \) denotes the shifted action: \( w.\Lambda = w(\Lambda + \hat{\rho}) - \hat{\rho} \).

(c) For \( g = sl_n \) all admissible weights are principal admissible.

Recall that the normalized character of an irreducible highest weight \( \hat{\mathfrak{g}} \)-module \( L(\Lambda) \) of level \( k \neq -h^\vee \) is defined by

\[
\text{ch}_\Lambda(\tau, z, t) = q^m_A \text{tr} L(\Lambda)e^{2\pi \tau h}
\]

where

\[
h = -\tau d + z + tK, \quad z \in \mathfrak{h}, \quad \tau, t \in \mathbb{C}, \quad \text{Im} \tau > 0, \quad q = e^{2\pi \tau},
\]

and \( m_A = \frac{|\Lambda + \hat{\rho}|^2}{2\delta(\delta + K)} - \frac{3g}{2} \) (the normalization factor \( q^{m_A} \) “improves” the modular invariance of the character).

In [6], the characters of the \( \hat{\mathfrak{g}} \)-modules \( L(\Lambda) \) for arbitrary admissible \( \Lambda \) were computed, see Theorem 3.1, or formula (3.3) there for another version in case of a principal admissible \( \Lambda \). In order to write down the latter formula, recall the normalized affine denominator for \( \hat{\mathfrak{g}} \):

\[
\hat{R}(h) = q^{\frac{1}{2}\dim \mathfrak{g}} e^{\hat{\rho}(h)} \prod_{n=1}^{\infty} (1 - q^n)^{\delta} \prod_{\alpha \in \hat{\Delta}_+} (1 - e^{\alpha(z)q^n})(1 - e^{-\alpha(z)q^n}).
\]

In coordinates (3) this becomes:

\[
\hat{R}(\tau, z, t) = (-1)^{\dim \mathfrak{g}} e^{2\pi h^\vee \tau} \eta(\tau)^{\frac{1}{2}(3\dim \mathfrak{g})} \prod_{\alpha \in \hat{\Delta}_+} \theta_{11}(\tau, \alpha(z)),
\]

where
\[ \vartheta_{11}(\tau, z) = -i q \frac{1}{\pi} e^{-\pi i z} \eta(\tau) \prod_{n=1}^{\infty} (1 - e^{-2\pi i q^n})(1 - e^{2\pi i q^n-1}) \]

is one of the standard Jacobi forms \( \vartheta_{ab} \), \( a, b = 0 \) or \( 1 \) (see, e.g., Appendix to [7]), and \( \eta(\tau) \) is the Dedekind eta function.

For a principal admissible \( \Lambda \), given by (2), formula (3.3) from [6] becomes in coordinates (3):

\[
(\hat{\mathcal{R}}\mathcal{H}_{\Lambda})(\tau, z, t) = (\hat{\mathcal{R}}\mathcal{H}_{\Lambda^0}) \left( u \tau, y^{-1}(z + \tau \beta), \frac{1}{u} (t + (z|\beta) + \frac{\tau|\beta|^2}{2}) \right).
\]

(5)

It follows from (5) that if \( \Lambda^0 = 0 \) in (2) (so that \( \text{ch}_{\Lambda^0} = 1 \)), which is equivalent to

\[ k + h^\vee = \frac{h^\vee}{u} \text{ and } \gcd(u, h^\vee) = \gcd(u, r^\vee) = 1, \]

(6)

the (normalized) character \( \text{ch}_{\Lambda} \) turns into a product. The level \( k \), defined by (6), is naturally called the boundary principal admissible level in [4], see formula (3.5) there. We obtain from Proposition 1, (4) and (5)

**Proposition 2.**

(a) All boundary principal admissible weights are of level \( k \), given by (6), and are of the form

\[ \Lambda = (t_{p,1}(k\Lambda^0),) \]

(7)

where \( \beta \in \mathbb{Q}^*, \ y \in \mathcal{W} \) are such that \( (t_{p,1}(k\Lambda^0),) \subset \mathcal{A} \). In particular, \( k \Lambda^0 \) is a principal admissible weight of level (6).

(b) If \( \Lambda \) is of the form (7), then

\[ \text{ch}_{\Lambda}(\tau, z, t) = e^{2\pi i k t} \frac{\eta(u\tau)}{\eta(\tau)} \prod_{\alpha \in \Delta_+} \vartheta_{11}(u\tau, \alpha(z)) \]

\[ \times \prod_{\alpha \in \Delta_+} \vartheta_{11}(u\tau, \alpha(z))^{\frac{1}{2}(3\epsilon - \dim g)}. \]

**Remark 1.** For the vacuum module \( L(k\Lambda^0) \) of the boundary principal admissible level \( k \) the character formula from Proposition 2(b) becomes

\[ \text{ch}_{k\Lambda^0}(\tau, z, t) = e^{2\pi i k t} \frac{\eta(u\tau)}{\eta(\tau)} \prod_{\alpha \in \Delta_+} \vartheta_{11}(u\tau, \alpha(z)) \]

\[ \times \prod_{\alpha \in \Delta_+} \vartheta_{11}(u\tau, \alpha(z))^{\frac{1}{2}(3\epsilon - \dim g)}. \]

**Example 1.** Let \( g = sl_2 \), so that \( h^\vee = 2 \). Then the boundary levels are \( k = \frac{2}{u} - 2 \), where \( u \) is a positive odd integer, and all admissible weights are

\[ \Lambda_{k,j} := t_{-\frac{j}{u}}(k\Lambda^0) = (k + \frac{2j}{u})\Lambda^0 - \frac{2j}{u} \Lambda_1, \ j = 0, 1, \ldots, u - 1, \]

and the character formula from Proposition 2(b) becomes:

\[ \text{ch}_{\Lambda_{k,j}} = e^{2\pi i (k - \frac{j}{u})} q^{\frac{2}{u}} \prod_{\alpha \in \Delta_+} \vartheta_{11}(u\tau, z - j\tau) \]

For \( u = 3 \) and 5 some of these formulas is conjectured in [8].

**Example 2.** Let \( g = sl_N \), so that \( h^\vee = N \), let \( N > 1 \) be odd, and let \( u = 2 \). Then the boundary admissible level is \( k = \frac{N}{2} \), and the boundary admissible weights of the form \( t_{p,1}(k\Lambda^0) \) are:

\[ \Lambda_{N,p} = \frac{N}{2} \Lambda_p, \ p = 0, 1, \ldots, N - 1, \]

where \( \Lambda_p \) are the fundamental weights of \( \widehat{g} \). Letting \( z = \sum_{i=1}^{N-1} z_i \beta_i \), where \( \beta_i \) are the fundamental weights of \( g \), the character formula from Proposition 2 (b) becomes:

\[ \text{ch}_{\Lambda_{N,p}}(\tau, z, t) = i^{p(N-p)} e^{-\pi i N t} \left( \frac{\eta(2\tau)}{\eta(\tau)} \right)^{-\frac{1}{2}(N-1)(N-2)} \]

\[ \prod_{1 \leq i < j \leq p \text{ or } p < i \leq j < N} \vartheta_{01}(2\tau, z_i + \ldots + z_j) \prod_{1 \leq i \leq p < j < N} \vartheta_{11}(2\tau, z_i + \ldots + z_j) \]

\[ \times \prod_{1 \leq i < j < N} \vartheta_{11}(\tau, z_i + \ldots + z_j), \]
where
\[
\vartheta_{01}(\tau, z) = \prod_{n=1}^{\infty} (1 - q^n) (1 - e^{2\pi i z} q^{n - \frac{1}{2}})(1 - e^{-2\pi i z} q^{n - \frac{1}{2}}).
\]

This follows from Proposition 2(b) by applying to \(\vartheta_{11}\) an elliptic transformation (see, e.g., [7], Appendix). In particular,
\[
\text{ch}_{-\frac{k}{2} \Lambda_0} = e^{-\pi i N t} \left( \frac{\eta(2\tau)}{\eta(\tau)} \right)^{(N-1)(N-2)} \prod_{1 \leq i \leq N} \vartheta_{11}(2\tau, z_i + \ldots + z_j).
\]

The latter formula was conjectured in [10].

Remark 2. For principal admissible weights \(\Lambda = (t_\rho, y)(k\Lambda_0)\) and \(\Lambda' = (t_\rho', y')(k\Lambda_0)\) of boundary level \(k = \frac{k'}{2} - \frac{k''}{2}\) the \(S\)-transformation matrix \((a(\Lambda, \Lambda'))\), given by [6], Theorem 3.6, simplifies to
\[
a(\Lambda, \Lambda') = |Q/uh| Q^{-1} |e(yy')| \prod_{\alpha \in \Delta_+} 2 \sin \left( \frac{\pi i u(\rho + h)}{h''} + \frac{2\pi i (\rho + h') + \frac{k'}{2} \rho + \frac{k''}{2}}{u} \right).
\]

Remark 3. If \(g = st_2\) and \(k\) is as in Example 1, then
\[
a(\Lambda_{k,j}, \Lambda_{k,j}) = (-1)^{j+j'} e^{-2\pi i j'} \frac{1}{\sqrt{u}} \sin \left( \frac{u \pi}{2} \right).
\]

One can compute fusion coefficients [9] by Verlinde’s formula:
\[
N_{\Lambda_{k,j_1}, \Lambda_{k,j_2}, \Lambda_{k,j_3}} = (-1)^{j_1+j_2+j_3} \text{ if } j_1 + j_2 + j_3 \in u\mathbb{Z}, \text{ and } = 0 \text{ otherwise}.
\]

Example 3. Let \(g = sl_3\), so that \(h'' = 3\), and let \(u\) be a positive integer, coprime to 3. Then all (principal) admissible weights have level \(k = \frac{2}{3}\) and are of the form (7), where
\[
\beta = -(-1)^p (k_1 \tilde{\Lambda}_1 + k_2 \tilde{\Lambda}_2), \quad y = r_{y}\rho, \quad p = 0 \text{ or } 1, \quad k_i \in \mathbb{Z}, \quad k_i = \delta_{p,1}, \quad k_1 + k_2 \leq u - \delta_{p,0}.
\]

Denote this weight by \(\Lambda_{u,k_1,k_2}^{(p)} = (t_{r_{y}} \rho)(k\Lambda_0)\). Using Remark 2, one computes the fusion coefficients by Verlinde’s formula:
\[
N_{\Lambda_{u,k_1,k_2}^{(p)}, \Lambda_{u,k_1,k_2}^{(p')}, \Lambda_{u,k_1,k_2}^{(p'')}} = (-1)^{p+p'+p''} \text{ if } (-1)^p k_1 + (-1)^{p'} k_1' + (-1)^{p''} k_1'' \in u\mathbb{Z} \text{ for } i = 1, 2,
\]

and = 0 otherwise.

Remark 4. If \(\Lambda\) is an arbitrary admissible weight, then \(\hat{\Lambda}^\Lambda\) decomposes in a disjoint union of several affine root systems. Then \(\Lambda\) has boundary level if restrictions of it to each of them has boundary level, and formula (3.4) from [6] shows that \(ch_{\Lambda}\) decomposes in a product of the corresponding boundary level characters. Note also that all the above holds also for twisted affine Kac–Moody algebras [6].

Remark 5. The product character formula for boundary level affine Kac–Moody superalgebras holds as well, see [2], formula (2).

Recall that with any \(st_2\)-triple \([f, x, e]\) in \(g\), where \([x, f] = -f, [x, e] = e\), one associates a \(W\)-algebra \(W^k(g, f)\), obtained from the vacuum \(\hat{g}\)-module of level \(k\) by quantum Hamiltonian reduction, so that any \(\hat{g}\)-module \(L(\Lambda)\) of level \(k\) produces either an irreducible \(W^k(g, f)\)-module \(H(\Lambda)\) or zero. The characters of \(L(\Lambda)\) and \(H(\Lambda)\) are related by the following simple formula ([4] or [7]):
\[
\left( W \hat{R} ch_{H(\Lambda)} \right)(\tau, z) = \left( \hat{R} ch_{\Lambda} \right)(\tau, -\tau x + z, \frac{\tau}{2}(x|x)).
\]

Here \(z \in \mathfrak{h}'\), the centralizer of \(f\) in \(\mathfrak{h}\), and
\[
\left( W \hat{R} (\tau, z) = \eta(\tau)^{2l-\frac{1}{2}} \dim(g_0+g_1/2) \prod_{\alpha \in \Delta_+} \vartheta_{01}(\tau, \alpha(z)) \left( \prod_{\alpha \in \Delta_1/2} \vartheta_{01}(\tau, \alpha(z)) \right)^{1/2},
\]

(10)
where $g = \oplus j g_j$ is the eigenspace decomposition for $\text{ad} x$, $\Delta_j \subset \Delta$ is the set of roots of root spaces in $g_j$ and $\Delta_0 = \Delta_+ \cap \Delta_0$ (we assume that $\Delta_j \subset \Delta_+$ for $j > 0$). If $k$ is a boundary level (6), we obtain from Proposition 2(b) and formulas (9), (10) the following character formula for $H(\Lambda)$ if $\Lambda$ is a principal admissible weight (7) ($z \in \mathfrak{h}^\vee$):

$$\begin{align}
\text{ch}_{H(\Lambda)}(\tau, z) &= (-1)^{|\Lambda| + 1} q^{\frac{h^+}{2}(-\tau - x)} e^{2\pi i \frac{\Lambda}{\mathfrak{h}}} 
\times \frac{\eta(\tau)^\frac{1}{2} \tau^{\frac{1}{2} - \frac{1}{2} \dim g}}{\eta(\tau)^\frac{1}{2} \tau^{\frac{1}{2} \dim (\mathfrak{so} + \mathfrak{g}_1)/2}} \prod_{\alpha \in \Delta_+} \vartheta_{11}(u \tau, y(\alpha)(z + \tau \beta - \tau x)) 
\times \prod_{\alpha \in \Delta_0^+} \vartheta_{11}(\tau, \alpha(z)) \left( \prod_{\alpha \in \Delta_0/2} \vartheta_{01}(\tau, \alpha(z)) \right)^{1/2}.
\end{align}$$

Remark 6. A formula, similar to Proposition 2(b) and to formula (11), holds if $g$ is a basic Lie superalgebra; one has to replace the character by the supercharacter, dim by $\text{sdim}$, and the factor $\vartheta_{ab}$, corresponding to a root $\alpha$, by its inverse if this root is odd. Also, the character is obtained from the supercharacter by replacing $\vartheta_{ab}$ by $\vartheta_{a,b+1 \mod 2}$ if the root $\alpha$ is odd.

Remark 7. An example of (11) is the minimal series representations of the Virasoro algebra with central charge $c = 1 - \frac{3(u - 2)^2}{u}$, obtained by the quantum Hamiltonian reduction from the boundary admissible $\widehat{sl}_2$-modules from Example 1. For $j = u - 1$ one gets 0, for $u = 3$ and $j = 0$, 1 one gets the trivial representation, but for all other $j$ and $u \geq 5$ the characters are the product sides of the Gordon generalizations of the Rogers–Ramanujan identities (the latter correspond to $u = 5$). Another example is the minimal series representations of the $N = 2$ superconformal algebras, see [4], Section 7.

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