Kondo physics in the Josephon junction with DIII-class topological and s-wave superconductors

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Abstract – We discuss the Kondo effect in the Josephson junction formed by the indirect coupling between a one-dimensional DIII-class topological and s-wave superconductors via a quantum dot. By performing the Schrieffer-Wolff transformation, we find that the single-electron occupation in the quantum dot induces various correlation modes, such as the Kondo and singlet-triplet correlations between the quantum dot and the s-wave superconductor and the spin-exchange correlation between the dot and Majorana doublet. Also, it modifies the Josephson effect by introducing the high-order anisotropic Kondo-like correlation and extra spin-exchange correlations. However, the Kondo temperature is still governed by the antiferromagnetic correlation between the dot and s-wave superconductor. We believe that this work shows the fundamental property of the DIII-class topological superconductor.

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Introduction. – The observation of the Kondo effect in semiconducting quantum dots (QDs) has opened a new area of research about the quantum transport through nanoscale and mesoscale systems [1]. Since QDs have the advantages to couple and then form QD molecules, the Kondo effect in multi-QD systems exhibits various forms. For instance, the spin and orbital Kondo effects, two-stage Kondo effect, and Kondo-Fano effect have been observed, which induce abundant transport behaviors [2]. In addition, the Kondo effect plays a special role in affecting the Andreev reflection in the systems with superconducting leads. In one Josephson junction with an embedded QD, the interplay between the Kondo and Josephson effects efficiently drives the Josephson phase transition [3,4]. Namely, if the Kondo temperature \( T_K \) is much greater than the superconducting gap \( \Delta_s \), the Kondo screening will dominate the system and the Josephson current will be at its 0 phase. Instead, if \( T_K \ll \Delta_s \), the ground state is a BCS-like singlet state, so the \( \pi \)-junction behavior takes place [5].

In recent years, the topological superconductor (TS) has become one main concern in the field of mesoscopic physics because Majorana modes appear at the ends of the one-dimensional TS which are of potential application in topological quantum computation [6–8]. Owning to the presence of Majorana zero mode, fractional Josephson effect comes into being, manifested as the \( 4\pi \)-periodic oscillation and parity-related direction of the supercurrent [9–14]. Besides, in the complicated junction with Majorana mode and s-wave superconductor, the supercurrent blockade takes place [15]. What is more important is that in the DIII-class TS, which is time-reversal-invariant, the Majorana mode appears in pairs due to Kramers’s theorem [16–20]. Hence at each end of the DIII-class TS nanowire there will exist one Majorana doublet [21–25]. Since the Majorana doublet is protected by the time-reversal symmetry, the period of its driving Josephson current becomes related to the fermion parity of the DIII-class TS junction [26]. What is more interesting is that in the junction formed by the coupling between the DIII-class TS and the s-wave superconductor, the Josephson effect presents an apparent time-reversal anomaly phenomenon [27].

The special property of the DIII-class TS motivates us to think that in its presented Josephson junction, the Kondo effect will be certain to arouse new physical results. Basing on such an idea, in this letter we aim to investigate the new feature of the Kondo effect in a hybrid Josephson junction, i.e., the junction formed by the indirect coupling...
between a DIII-class TS and a s-wave superconductor via one QD. Our investigation shows that the single-electron occupation in the QD induces multiple correlation modes, including the Kondo and singlet-triplet correlations between the QD and the s-wave superconductor, and the spin correlation between the QD and Majorana doublet. Besides, the Kondo QD modifies the Andreev reflection between the superconductors, leading to the occurrence of anisotropic high-order Kondo correlation and extra spin-exchange correlations which contribute to the Josephson effect.

Theoretical model. – The considered Josephson junction is illustrated in fig. 1(a), where a one-dimensional DIII-class TS couples to one s-wave superconductor via one QD. The system Hamiltonian can be written as

\[ H = H_p + H_s + H_D + H_T. \]

The first two terms denote the Hamiltonians of the DIII-class TS and the s-wave superconductor, which can be expressed as [27]

\[
\begin{align*}
H_p &= -\mu_p \sum_{j\gamma} c_{j\uparrow}^\dagger c_{j\uparrow} - \sum_{j\gamma} \left( c_{j+1\gamma}^\dagger c_{j\gamma} + \text{h.c.} \right) \\
H_s &= \sum_{k\sigma} \xi_k f_{k\sigma}^\dagger f_{k\sigma} + \sum_{k\sigma} \left( \Delta_{s,k} f_{k\uparrow}^\dagger f_{k\downarrow}^\dagger + \text{h.c.} \right).
\end{align*}
\]

(1)

c_{j\sigma}^\dagger and \( f_{k\sigma}^\dagger (c_{j\sigma} \text{ and } f_{k\sigma}) \) are the electron creation (annihilation) operators in the DIII-class TS and s-wave superconductor, respectively, with spin index \( \sigma \) \( \sigma_l (l = 1, 2, 3) \) is the Pauli matrix. \( \mu_p \) and \( t \) denote the chemical potential and intersite coupling in the DIII-class TS, and \( \xi_k \) is the electron energy at state \( |k\sigma \rangle \) in the s-wave superconductor. \( \Delta_p \) and \( \Delta_s \) are the Cooper-pair hopping terms in the two kinds of superconductors. \( H_D \) describes the Hamiltonian of the QD. We here consider one single-level QD to be embedded in the junction and then \( H_D = \varepsilon_d \sum_{\sigma} n_\sigma + U n_{\uparrow} n_{\downarrow} \) with \( n_\sigma = d_{\sigma}^\dagger d_{\sigma}, d_{\uparrow}^\dagger d_{\downarrow}^\dagger \) and \( d_{\sigma} \) are the annihilation operators in the QD, \( \varepsilon_0 \) is the QD level, and \( U \) denotes the intradot Coulomb repulsion. \( H_T \) describes the couplings between the QD and superconductors. Its expression can be given by

\[
H_T = -\delta t \sum_{k\sigma} \left( e^{i\theta/2}c_{k\sigma}^\dagger d_{\sigma} + \sum_{k\delta} V_{k\delta} e^{i\phi_{\delta}/2} f_{k\sigma}^\dagger f_{k\delta}^\dagger d_{\sigma} + \text{h.c.} \right),
\]

where \( \delta t \) and \( V_{k\delta} \) are the QD-superconductor coupling coefficients, respectively.

For the case of one infinitely long DIII-class TS, one Majorana doublet will form at its end. In the situation of a larger \( \Delta_p \) (e.g., \( \Delta_p > \Delta_s \)), we can project \( H_p \) onto the zero-energy subspace of \( H_p \). As a result, \( H \) can be simplified as

\[
H = \sum_{\sigma} \varepsilon_d d_{\sigma}^\dagger d_{\sigma} + U n_{\uparrow} n_{\downarrow} + \sum_{k\sigma} \xi_k f_{k\sigma}^\dagger f_{k\sigma} \]

\[
+ \left( \sum_{k} \Delta_{s,k} f_{k\uparrow}^\dagger f_{k\downarrow}^\dagger + \sum_{k\sigma} V_k f_{k\sigma}^\dagger d_{\sigma} - \sum_{\sigma} \delta t e^{i\frac{\theta}{2}} \gamma_{0\sigma} d_{\sigma} + \text{h.c.} \right).
\]

(2)

\( \gamma_{0\sigma} \) is the Majorana operator, which obeys the anti-commutation relationship \( \{ \gamma_{0\sigma}, \gamma_{0\sigma}' \} = 2\delta_{\sigma\sigma'} \). Since the existence of a s-wave superconductor, it is necessary to introduce the Bogoliubov unitary transformation to diagonalize its Hamiltonian by defining

\[
\begin{pmatrix}
\frac{f_{k\uparrow}}{\xi_{k\uparrow}} \\
\frac{f_{k\downarrow}}{\xi_{k\downarrow}}
\end{pmatrix} =
\begin{pmatrix}
\frac{u_k}{\xi_k} & \frac{-\gamma_k}{\xi_k} \\
\frac{\gamma_k}{\xi_k} & \frac{u_k}{\xi_k}
\end{pmatrix}
\begin{pmatrix}
f_{k\uparrow} \\
f_{k\downarrow}
\end{pmatrix}.
\]

And then, the system Hamiltonian can be written as \( H = H_0 + H_T \), where

\[
\begin{align*}
H_0 &= \sum_{\sigma} \varepsilon_d d_{\sigma}^\dagger d_{\sigma} + U n_{\uparrow} n_{\downarrow} + \sum_{k\sigma} E_k \alpha_{k\sigma}^\dagger \alpha_{k\sigma}, \\
H_T &= \sum_{k\sigma} V_k \left( u_k d_{\sigma}^\dagger \alpha_{k\sigma} + \text{h.c.} \right) \\
&- \sum_{k} \left( \sum_{\sigma} \left( \varepsilon_k d_{\sigma}^\dagger \alpha_{k\sigma}^\dagger + v_k \alpha_{k\sigma} \alpha_{k\sigma} + \text{h.c.} \right) \right) \\
&- \delta t \sum_{\sigma} \left( e^{i\frac{\theta}{2}} \gamma_{0\sigma} d_{\sigma} + \text{h.c.} \right).
\end{align*}
\]

(3)

with \( E_k = \sqrt{\xi_k^2 + \Delta^2}. \) It is readily found that \( H_0 \) is the diagonalized Hamiltonian including the QD and s-wave superconductor, whereas \( H_T \) describes the hopping between the QD and superconductors. Based on such a form of \( H \), the distribution of relevant quantum states in this system can be clearly illustrated in fig. 1(b). There is no doubt that the presence of a Majorana doublet will modify the Kondo physics in a nontrivial way.

Analysis about Kondo-physics change. – Schrieffer-Wolff transformation. It is known that the Kondo effect arises from the antiferromagnetic correlation between the localized and conduction electrons, and it can be well analyzed by transforming it into one s-d exchange model by performing the Schrieffer-Wolff transformation [28]. Next we would like to discuss the Kondo effect using such a method.

To begin with, we need to project the Hilbert space of this junction into two sectors, namely, a low-energy subspace and a high-energy subspace, with projection operators \( P_L = |L\rangle \langle L| \) and \( P_H = |H\rangle \langle H| = 1 - P_L \). Following the definition of projection operators, the total Hamiltonian can therefore be divided into the diagonal and off-diagonal parts, i.e., \( H_0 = \hat{P}_L H \hat{P}_L + \hat{P}_H H \hat{P}_H = \left[ \begin{array}{cc} H^L & 0 \\ 0 & H^H \end{array} \right] \) and \( H_T = \hat{P}_L H \hat{P}_H + \hat{P}_H H \hat{P}_L = \left[ \begin{array}{cc} 0 & \alpha \\ \alpha & 0 \end{array} \right] \). Next, via a canonical transformation, \( H \) can be rotated into a block-diagonal form, i.e.,

\[
e^{-S} \left[ \begin{array}{cc} \hat{H}_L & 0 \\ 0 & \hat{H}_H \end{array} \right] e^{-S} = \hat{H}.
\]

According to the Baker-Campbell-Hausdorff formula, the transformed Hamiltonian \( \hat{H} \) can be expanded into series \( \hat{H} = H + [S, H] + \frac{1}{2!}[S, [S, H]] + \frac{1}{3!}[S, [S, [S, H]]] + \cdots \). We choose the generator \( S \) of the canonical transformation such
that \([S, H_0] = -H_T\), hence the off-diagonal part \(H_T\) is eliminated in the lowest-order term. And then, \(\hat{H} = H_0 + \frac{1}{2}[S, H_T] + \sum_{n=2}^{\infty} \frac{n}{n+1} [S, H_T]^n\). Within the low-order approximation, the effective Hamiltonian shows the following form: \(\hat{H} = H_0 + \Delta \hat{H}\) with \(\Delta = \frac{1}{2}[S, H_T]\).

In what follows, we aim to solve the analytical form of \(\Delta \hat{H}\) by deriving the generator \(S\). Substitute this form into equation \([S, H_0] + H_T = 0\), we can get the result \(S = \sum_{i,j} [H_i] P_{i,j} \frac{1}{\Delta E_{ij}} P_{j,i} - \hbar c\). For a single-level QD, it possesses four electron states, i.e., \([0], [n_\sigma],\) and \([n_\sigma, n_{\bar{\sigma}}]\). It is suitable to introduce a set of projection operators in terms of occupation configuration on the QD, which are \(P_0 = (1 - n_\sigma)(1 - n_{\bar{\sigma}})\), \(P_1 = n_\sigma(1 - n_{\bar{\sigma}}) + n_{\bar{\sigma}}(1 - n_\sigma)\), and \(P_2 = n_\sigma n_{\bar{\sigma}}\). In the case of \(\varepsilon_d < 0\) and \(\varepsilon_d + U > 0\), the Kondo effect have an opportunity to come into being. In such a case, the ground state of the Anderson-type system is a local moment \([1] = [n_\sigma]\) configuration, whereas the high-energy intermediate states correspond to \([0]\) and \([n_\sigma, n_{\bar{\sigma}}]\). As a result, the projector into the low(high)-energy subspace is \(P_L = P_1 (P_H = P_0 + P_2)\). In view of the expression of \(H_T\), one can find that three kinds of coupling manners occur between the QD and superconductors, i.e., \(H_{1L}^T = \sum_{k} V_k \delta \sigma \langle \epsilon_d \rangle \), \(H_{2L}^T = -\delta \sum_{j} (\epsilon_d - \varepsilon_j) |d_j\rangle\langle d_j| - h.c.\), and \(H_{3L}^{T\prime} = \delta \sum_{j} (\epsilon_d - \varepsilon_j) |d_j\rangle\langle d_j| - h.c.\). \(H_{1L}\) is the traditional mixing term between the conduction quasi-electrons and the localized electrons. It shows that when a conduction electron or hole is excited into the localized state in the QD to create these excited-state configurations, the corresponding excitation state is either a state with one conduction quasi-electron with its energy \(E_k\), or a state with two localized electrons and one conduction quasi-hole. In these two processes, the excitation energies are \(\Delta E_{10} = E_k - \varepsilon_d\) and \(\Delta E_{12} = \varepsilon_d + U - E_k\), respectively. Accordingly, \(S^I\) is given by \(S^I = \sum_{k} V_k \delta \sigma \langle \epsilon_d \rangle \), \(H^I = \sum_{k} V_k \delta \sigma \langle \epsilon_d \rangle - h.c.\). \(S^I\) describes the scattering process between the localized electrons and conduction quasi-holes and vice versa. In these processes, the virtual excitation energies are \(\Delta E_{01} = -E_k - \varepsilon_d\) and \(\Delta E_{12} = \varepsilon_d + U - E_k\), and then \(S^{III}\) is written as \(S^{III} = \sum_{k} \delta \sigma \langle \epsilon_d \rangle \), \(H^{III} = \sum_{k} \delta \sigma \langle \epsilon_d \rangle - h.c.\). Next, \(H^{III}\) results from the mixing between the Majorana zero mode and the QD. It contributes to the virtual excitation energies \(\Delta E_{01} = -\varepsilon_d\) and \(\Delta E_{12} = \varepsilon_d + U\). \(S^{III}\) thus reads \(S^{III} = \sum_{k} \delta \sigma \langle \epsilon_d \rangle \), \(H^{III} = \sum_{k} \delta \sigma \langle \epsilon_d \rangle - h.c.\). Up to now, the generator of the Schrieffer-Wolff transformation in such a hybrid structure has been solved, i.e., \(S = S^I + S^{III} + S^{III}\). It is readily verified that \(S\) obeys the necessary conditions \(S^I = -S\) and \([S, H_0 + P_L] = 0\).

Following the expression of \(S\), the analytical form of \(\Delta H\) can be derived using the relation of \(\Delta H = \frac{1}{2}[S, H_T]\). Note that since the Kondo effect occurs under the condition of \(\varepsilon_d < 0\) and \(\varepsilon_d + U > 0\), it is reasonable to only calculate \(\Delta H_{LL}\), which is given by \(\Delta H_{LL} = \frac{1}{2} \sum_{P_L}[S, H_T] P_L\). After a simple derivation, one can find that \(\Delta H_{LL} = \sum_{P_L} P_L H_{TLL} P_L = \sum_{j=0,1} H_{jLL}^j \frac{1}{\Delta E_{ij}} H_{jLL}^j\), due to \(P_L = P_1\) and \(P_H = P_0 + P_2\). According to the result of \(S\), we can readily obtain the detailed form of each quantity in this equation. To be specific, \(H_{01} = \sum_{k} V_k u_{(n_{\bar{\sigma}})} \delta \sigma \langle \epsilon_d \rangle - h.c.\), \(H_{01}^T = -\delta \sum_{j} (\epsilon_d - \varepsilon_j) |d_j\rangle\langle d_j| - h.c.\), \(H_{1L}^{T\prime} = \delta \sum_{j} (\epsilon_d - \varepsilon_j) |d_j\rangle\langle d_j| - h.c.\), \(H_{1L}^{T\prime} = \delta \sum_{j} (\epsilon_d - \varepsilon_j) |d_j\rangle\langle d_j| - h.c.\). Meanwhile, \(H_{12}^{T\prime} = -\delta \sum_{j} (\epsilon_d - \varepsilon_j) |d_j\rangle\langle d_j| - h.c.\), \(H_{12}^{T\prime} = -\delta \sum_{j} (\epsilon_d - \varepsilon_j) |d_j\rangle\langle d_j| - h.c.\). Substituting these quantities in eq. (4), we can readily find that each term of \(\Delta H_{LL}\) consists of four parts. To be concrete, the four parts of its first term are

I): \[\sum_{kk',\sigma\sigma'} V_k V_{k'} \left( \frac{u_k^2}{E_k + \varepsilon_d} + \frac{u_{k'}^2}{E_{k'} + \varepsilon_d} \right) \alpha_{k\sigma} \alpha_{k'\sigma'}^\dagger A_{\sigma\sigma'}^{i};\]

II): \[-\sum_{kk',\sigma\sigma'} V_k V_{k'} \left( \frac{u_k u_{k'}}{E_k + \varepsilon_d} + \frac{w_{k'}^2}{E_{k'} + \varepsilon_d} \right) \alpha_{k\sigma} \alpha_{k'\sigma'}^\dagger A_{\sigma\sigma'}^{i};\]

III): \[\sum_{k,\sigma,\sigma'} V_k \left( \frac{u_k \delta \epsilon \theta^\dagger}{E_k + \varepsilon_d} + \frac{w_{k'} \delta \epsilon \theta^\dagger}{E_{k'} + \varepsilon_d} \right) \alpha_{k\sigma} \gamma_{\sigma\sigma'} A_{\sigma\sigma'}^{i};\]

IV): \[\sum_{\sigma,\sigma'} \delta_{\sigma,\sigma'} (\gamma_{\sigma\sigma'} A_{\sigma\sigma'} + 1) B_{\sigma\sigma'}^{i};\]

with \(A_{\sigma\sigma'} = \delta_{\sigma,\sigma'} d^\dagger d + \delta_{\sigma,\sigma'} \frac{1}{2} + \text{sgn}(\sigma) S^2\). As a counterpart, those in the second term of \(\Delta H_{LL}\) are given by

I): \[\sum_{kk',\sigma\sigma'} V_k V_{k'} \left( \frac{u_k^2}{\varepsilon_d + U - E_k} \right) \alpha_{k\sigma} \alpha_{k'\sigma'}^\dagger + \left( \frac{u_{k'}^2}{E_{k'} + \varepsilon_d + U} \right) \alpha_{k\sigma} \alpha_{k'\sigma'}^\dagger B_{\sigma\sigma'}^{i};\]

II): \[-\sum_{kk',\sigma\sigma'} V_k V_{k'} \left( \frac{u_k u_{k'}}{\varepsilon_d + U - E_k} \right) \alpha_{k\sigma} \alpha_{k'\sigma'}^\dagger + \left( \frac{u_{k'}^2}{E_{k'} + \varepsilon_d + U} \right) \alpha_{k\sigma} \alpha_{k'\sigma'}^\dagger B_{\sigma\sigma'}^{i};\]

57002-p3


where $\Delta S_{\downarrow} = J_{v} V_{k} \left( \frac{u_{k}^{2}}{E_{k} - U} + \frac{u_{k}^{* 2}}{E_{k} + U} - \frac{u_{k}^{2} u_{k}^{* 2}}{E_{k}^{2} - U^{2}} \right)$ and

$$J_{k} = V_{k} V_{k} \left( \frac{u_{k}}{E_{k} - U} + \frac{u_{k}^{*}}{E_{k} + U} - \frac{u_{k} u_{k}^{*}}{E_{k}^{2} - U^{2}} \right) \left( E_{k}^{2} - U^{2} \right)$$

It is clearly shown that $\Delta H_{L, L, L}$ reflects the normal spin correlation between the localized state in the QD and the continuum states in the s-wave superconductor. In the case of positive spin-correlation coefficient, the two quantum states will correlate with each other in the antiferromagnetic manner, as a result, the Kondo effect will come into being.

The second parts in eqs. (5), (6) describe the singlet-triplet correlations. Their contributions to $\Delta H_{L, L, L}$ are

$$\Delta H_{L, L, L} = \sum_{k, \sigma, \sigma'} \left| J_{k} \right|^{2} S_{\sigma} S_{\sigma'}^{+} + \frac{1}{2} \sum_{k, \sigma, \sigma'} \tilde{\Delta}_{k}^{\uparrow} \alpha_{k}^{\uparrow} \alpha_{k'}^{\downarrow} S_{\sigma}^{+} + \frac{1}{2} \sum_{k, \sigma, \sigma'} \Delta_{k} \alpha_{k}^{\uparrow} \alpha_{k'}^{\downarrow} S_{\sigma}^{+}$$

in which $\tilde{\Delta}_{k}^{\downarrow} = V_{k} V_{k} \left( \frac{u_{k}^{2}}{E_{k} - U} + \frac{u_{k}^{* 2}}{E_{k} + U} - \frac{u_{k} u_{k}^{*}}{E_{k}^{2} - U^{2}} \right)$ and $\Delta_{k}^{\downarrow} = V_{k} V_{k} \left( \frac{u_{k}^{2}}{E_{k} - U} + \frac{u_{k}^{* 2}}{E_{k} + U} - \frac{u_{k} u_{k}^{*}}{E_{k}^{2} - U^{2}} \right)$. The first term in this equation shows the correlation between the localized state in the QD and the Cooper pair in the s-wave superconductor. In the second term, it suggests that the correlation between the localized state in the QD and the Cooper pairs in the s-wave superconductor has an opportunity to break up the Cooper pairs and reconstruct the correlation between the localized state and new spin-conserved Cooper pairs. Both $\Delta H_{L, L, L}$ and $\Delta H_{L, L, L, L}$ originate from the coupling between the QD and s-wave superconductor, and they have been observed by Saloman [29].

Next, the third parts in eqs. (5), (6) reflect the contribution of the Kondo QD to the Josephson effect. Their composition can be expressed as $\Delta H_{L, L, L, L, L} = H_{K}^{f} + H_{cross} + H_{r}$, more complicated than the terms above. The first term reads

$$H_{K}^{f} = \sum_{k, \sigma, \sigma'} \left( J_{k} \left( \frac{u_{k}^{2}}{E_{k} - U} + \frac{u_{k}^{* 2}}{E_{k} + U} - \frac{u_{k} u_{k}^{*}}{E_{k}^{2} - U^{2}} \right) \right) \left( \alpha_{k}^{\uparrow} \alpha_{k'}^{\downarrow} + h.c. \right)$$

in which $\delta_{\sigma, \sigma'} = \delta_{\sigma, \sigma'} \delta_{\sigma, \sigma'}^{\downarrow} d_{\sigma} + \delta_{\sigma, \sigma'}^{\downarrow} \delta_{\sigma, \sigma'}^{\downarrow} d_{\sigma}^{\dagger} + \operatorname{sgn} (\sigma) S_{\sigma}^{\dagger}$. The second term in eq. (5), (6) reflects the contribution of the Kondo QD to the Josephson effect. Their composition can be expressed as $\Delta H_{L, L, L, L, L} = H_{K}^{f} + H_{cross} + H_{r}$, more complicated than the terms above. The first term reads

$$H_{K}^{f} = \sum_{k, \sigma, \sigma'} \left( J_{k} \left( \frac{u_{k}^{2}}{E_{k} - U} + \frac{u_{k}^{* 2}}{E_{k} + U} - \frac{u_{k} u_{k}^{*}}{E_{k}^{2} - U^{2}} \right) \right) \left( \alpha_{k}^{\uparrow} \alpha_{k'}^{\downarrow} + h.c. \right)$$

with $K_{k}^{f} = \left( \frac{u_{k}}{E_{k}} \right)^{2} = \frac{1}{2} \left( \frac{u_{k}}{E_{k}} \right)^{2} + \frac{1}{2} \left( \frac{u_{k}}{E_{k}} \right)^{2} \exp \left( -i \theta / 2 \right)$ and $M_{k} = \frac{1}{2} \left( \frac{u_{k}}{E_{k}} \right)^{2} + \frac{1}{2} \left( \frac{u_{k}}{E_{k}} \right)^{2} \exp \left( -i \theta / 2 \right)$ sin $\theta$. Although the terms in eq. (10) seem to be chaotic, they contribute little to the Josephson effect because they are not the real correlation due to only being related to $\sigma S_{\sigma}^{\dagger}$ and $\sigma S_{\sigma}^{\dagger}$ ($n = x, y$). As for $H_{r}$, it is given by $H_{r} = \frac{1}{2} \left( \frac{u_{k}}{E_{k}} \right)^{2} + \frac{1}{2} \left( \frac{u_{k}}{E_{k}} \right)^{2} \exp \left( -i \theta / 2 \right)$ and $M_{k} = 2 \left( \frac{u_{k}}{E_{k}} \right)^{2} + \frac{1}{2} \left( \frac{u_{k}}{E_{k}} \right)^{2} \exp \left( -i \theta / 2 \right)$ sin $\theta$. This term reflects the direct coupling between the Majorana doublet and the s-wave superconductor. It can be understood to be the Kondo screening caused by the Josephson effect.

The last terms in eqs. (5), (6) represent the correlation between the QD and the Majorana doublet. Via a straightforward derivation, we can simplify them to be $\Delta H_{L, L, L, L} = J_{M} \sum_{\sigma, \sigma'} S_{\sigma}^{\dagger} \gamma_{\sigma}^{\sigma} \gamma_{\sigma'}^{\sigma'}$, in which $J_{M} = \delta_{\sigma}^{\dagger} \left( \frac{1}{E_{k}^{2}} - \frac{1}{E_{k}^{2}} \right)$. It seems that $\Delta H_{L, L, L, L}$ describes the Kondo-typed correlation between the QD and Majorana doublet. However, note that $\gamma_{\sigma}^{\sigma} = 2$ and $\gamma_{\sigma}^{\sigma'} = (-1)^{N_{R}}$, which labels the two-fold degenerate ground states by the time-reversal-symmetric sector $Z_{2}$ index. Therefore, $\Delta H_{L, L, L, L}$ does not induce any correlation mechanism. Up to now, we have clarified the complicated correlation properties in the case of one Kondo dot embedded in a hybrid junction with the DIII-class TS and s-wave superconductor.
Fig. 2: (Colour online) Scaling flow to describe the Kondo physics from $H_K$. Relevant parameters are taken to be $\rho = \Delta = 1$ and $D = 100$.

**Scaling.** The analysis above shows that two kinds of Kondo mechanisms coexist in this junction. We next present a detailed discussion about them by using the perturbation method. We start by rewriting $\Delta H_{LL1} = H_K$ and $H_K = \frac{1}{n} \sum_{kk',ss} \sum_{j,j'} J_j (\sigma_{s's'} \sigma_{ss'}) \alpha_k \alpha_{k'} d_{j}^d d_{j'}^\dagger + \text{h.c.}$ where $J_1 = J_2 = J_{k'k}$ and $J_3 = J_{k'k}$. Then, the Green function that describes the scattering processes can be defined, i.e., $G_{s'\eta\eta'} (k',k',\tau') = \sum_n G_{s\eta\eta'}^{(n)} (k',k',\tau') = - \sum_{n=0}^{\infty} \left( \frac{\omega}{\pi} \right)^n \int_0^{\infty} \cdots \int_0^{\infty} d\tau_1 \cdots d\tau_n \left[ \bar{T}_2 [H_K (\tau_1) \cdots H_K (\tau_n)] \alpha_{k'} (\tau) \alpha_k (\tau') \right]$. With the help of the Wick theorem, the scattering process can be simplified as the multiplication among free propagators, which can certainly be illustrated by the Feynman diagrams. Next, the total particle-particle interaction vertex in the Feynman diagram can be written out, according to the Dyson series in the random-phase-approximation (RPA) theory:

$$\Gamma_{s_1 v_1 s_2 v_2} (k_1, \omega n; k_2, \omega m) = \frac{1}{2} J_j (\sigma_{s_1 s_2} \sigma_{v_1 v_2})^j + \frac{1}{2} \sum_{p,\omega s_1 s_2} \sum_{p,\omega s_1 s_2} J_j (\sigma_{s_1 s_2} \sigma_{v_1 v_2})^j \times G_{s_2 \eta}^{(0)} (p,\omega s_1) G_{d_\eta \eta}^{(0)} (p,\omega s_2; k_2; \omega m),$$

where $G_{s}^{(0)}$ and $G_{d}^{(0)}$ are the free propagators of the s-wave superconductor and QD, respectively. Suppose $\Gamma_{s_1 v_1 s_2 v_2} (k_1, \omega n; k_2, \omega m) = \Gamma_0^{1j} (\sigma_{s_1 s_2} \sigma_{v_1 v_2})^j (\Gamma_1^1 = \Gamma_2^1 = \Gamma_3^1 = \Gamma_4^1)$ and perform spin summation, we arrive at:

$$\Gamma_{s_1 v_1 s_2 v_2} (k_1, \omega n; k_2, \omega m; \omega m') = \frac{\Pi^SP}{2} \sum_{j=1}^3 J_1 \delta_{s_1 s_2} (1 + \Pi^SP \Gamma_0)^j J_j - \Pi^SP (J_1 \Gamma_1^j + \Gamma_1^j J_j) \times (\sigma_{s_1 s_2} \sigma_{v_1 v_2})^j.$$  \hspace{1cm} (11)

We can thus find out the divergent condition for the vertex: \(2J_1^2 - J_j^2 \approx 3\left| (1 + \Pi^SP \Gamma_0)^j J_j - \Pi^SP (J_1 \Gamma_1^j + \Gamma_1^j J_j) \right| \approx 3\). On the other hand, the Kondo-pair bubble diagram $\Pi^SP$ near the Fermi level (zero frequency) can be calculated, i.e., $\Pi^SP \approx \rho \ln (\frac{\Delta}{\rho}) - 2 \rho \lambda_0 (\frac{\Delta}{\rho})$ where $\lambda_0$ is the zero-order Bessel function. Based on this result, the scaling RG-flow can be sketched, as shown in fig. 2. Surely, the formation of Kondo physics is tightly dependent on the interplay between $J^\perp$ and $J^\parallel$. And it also shows the competition between the superconductor-order parameter (i.e., gap $\Delta$, resembling the Cooper-pair combination energy) and the Kondo screening cloud. The superconductor gap suppresses the Kondo resonance, leading to the disappearance of some parts of the scaling lines in the flow diagram in comparison with the normal Kondo effect.

Fig. 3: (Colour online) Mimic of conductance influenced by the change of structural parameters.

Similar to the above discussion manner, we next focus on $H_{k'}$ to analyze its related physics, by defining the Green function $G_{s'\eta\eta'} (k',k',\tau')$. It can be found that the first-order scattering process is zero because the Wick contraction contains $(\alpha_k^\dagger \gamma_{q\omega}^\eta \gamma_{p\eta}^\eta)$ and $(\gamma_{q\omega}^\eta \alpha_k^\dagger \gamma_{p\eta}^\eta)$. Thus, $G_{s'\eta\eta'}^{(2)} (k',k',\tau')$ makes the leading contribution to the Kondo physics. The resulting RPA vertex equation can be given by:

$$\Gamma_{s_1 v_1 s_2 v_2} (k_1, \omega n; 0, \omega_1; \omega m) = \Psi_{s_1 v_1 s_2 v_2} (p, \omega_1) \times G_{d_s}^{(0)} (\omega m - \omega_1) \Psi_{s_1 v_1 s_2 v_2} (p, \omega_1; 0, \omega_1; \omega m),$$

where $\Psi_{s_1 v_1 s_2 v_2} = K_j (\sigma_{s_1 s_2} \sigma_{v_1 v_2})^j$ and $G_M^{(0)}$ are the free propagators of Majorana doublet $(K_1 = K_2 = K_3 = K_4)$. After defining $\Gamma_{s_1 v_1 s_2 v_2} = \Upsilon_0 \delta_{s_1 s_2} \delta_{v_1 v_2} + \Upsilon_1 (\sigma_{s_1 s_2} \sigma_{v_1 v_2})^j (\Gamma_1^1 = \Gamma_2^1 = \Gamma_3^1 = \Gamma_4^1)$ and perform spin summation, we arrive at:

$$\Gamma_{s_1 v_1 s_2 v_2} (k_1, \omega n; k_2, \omega m; \omega m') = \frac{\Pi^SP}{2} \sum_{j=1}^3 J_j \delta_{s_1 s_2} (1 + \Pi^SP \Gamma_0)^j J_j - \Pi^SP (J_1 \Gamma_1^j + \Gamma_1^j J_j) \times (\sigma_{s_1 s_2} \sigma_{v_1 v_2})^j.$$  \hspace{1cm} (11)

We can thus find out the divergent condition for the vertex: $(2J_1^2 - J_j^2) (1 + \Pi^SP \Gamma_0)^j J_j = (2J_1^2 - J_j^2)$ On the other hand, the Kondo-pair bubble diagram $\Pi^SP$ near the Fermi level (zero frequency) can be calculated, i.e., $\Pi^SP \approx \rho \ln (\frac{\Delta}{\rho}) - 2 \rho \lambda_0 (\frac{\Delta}{\rho})$ where $\lambda_0$ is the zero-order Bessel function. Based on this result, the scaling RG-flow can be sketched, as shown in fig. 2. Surely, the formation of Kondo physics is tightly dependent on the interplay between $J^\perp$ and $J^\parallel$. And it also shows the competition between the superconductor-order parameter (i.e., gap $\Delta$, resembling the Cooper-pair combination energy) and the Kondo screening cloud. The superconductor gap suppresses the Kondo resonance, leading to the disappearance of some parts of the scaling lines in the flow diagram in comparison with the normal Kondo effect.
the Kondo singlet, the conductance feature induced by $H_K$ can be estimated by calculating $-2\text{Im}\Sigma(0)$, where
\begin{equation}
\Sigma(\omega_n) = \frac{1}{\omega_n} \prod_{\nu=m}^{\nu=n} \text{cotan} \left( \frac{\omega_n}{2} \right) + \text{cotan} \left( \frac{\omega_n}{2} \right)
\end{equation}
with \( \Gamma = \sum_{J} \hbar J. \) As shown in fig. 3, we plot the dependence of the conductance $G_F = -2\text{Im}\Sigma_F$ on the structural parameters. It can be clearly seen that the conductance is a cosine function of the Josephson phase difference, and it increases with the enhancement of the magnitude of $K_1$ and $\Pi^{SP}$. This means that the Kondo QD plays a nontrivial role in modifying the particle motion in this structure.

Summary. – In summary, we have presented the discussion about the spin-correlation effect in the QD-embedded Josephson junction formed by the indirect coupling between one DIII-class TS and s-wave superconductors. By carrying out the Schrieffer-Wolff transformation, we have found various correlation mechanisms in this system, such as the Kondo and singlet-triplet correlations between the QD and the s-wave superconductor as well as the spin-exchange correlation between the QD and Majorana doublet. Besides, the Kondo QD modifies the Josephson effect by introducing the high-order anisotropic Kondo-like correlation and extra spin-exchange correlations. However, the Kondo temperature is still governed by the correlation between the Kondo QD and the s-wave superconductor. This work helps to understand the Kondo effect in the system with DIII-class TS.

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