Laceability on a class of line graph of cartesian product of graphs

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Abstract. Determining Hamiltonian path connectivity relationship among the nodes of the network increases its optimal connectivity. If a and b are nodes in a graph G so that d(a,b) = t and then P is a non-Hamilton path in G. G is referred to as hypo-edge-Hamilton-t-laceable if forms Hamilton path joining a and b, for each t for 1 ≤ t ≤ diam(G). In this research article we investigate the hypo-edge-Hamilton laceability of line graph of Cartesian product of paths and cycles. Also we discuss the results on Cartesian product of path and wheel.

1. Introduction
For standard definitions and terminologies of graphs we allude to Harary [1]. In this article, we focus on undirected, finite, simple graphs. Let G be a graph which is Hamiltonian-t-laceable[2] (H-t-L), in the case that a Hamiltonian exists or Hamilton path between u & v in G, if there exists a non-Hamilton path P in G, then G is called hypo-edge-Hamilton-t-laceable (H-E-H-t-L) if P + re this is Hamilton path from u to v, also in [3,4], they have discussed these properties for flower snarks and modified flower snarks. Murali R et. al. [5] discussed i-Hamilton laceability properties in product graphs. Further in [6,7] authors explored laceability properties of brick product graphs and H-t-L properties of line graph of Sunlet graph, Helm graph and Gear Graph for t =1,2 and 3. Based on this, in [8, 9] authors investigated laceability properties of Interleaver graph of brick product graphs and Edge fault tolerance in brick product graphs. In this paper hypo-edge Hamilton-t-laceability properties of Cartesian product of (i) paths (ii) path and cycle (iii) path and wheel graphs are discussed.

2. Wheel graph
A wheel graph Wn [1] is a graph constructed by connecting a single universal node to all nodes of Cn.
Example:

![Wheel graphs $W_5$](image)

**Figure 1.** Wheel graphs $W_5$

### 3. Main Results

We now discuss main results.

**Theorem 3.1.** Let graph $G = P_n \times P_n$ then $L(G) = L(P_n \times P_n)$ for $n \geq 3$ is $K^+1$ hypo edge $H$-l-$L$ for even $t, 2 \leq t \leq n-2$ and $H$-l-$L$ for odd $t, 1 \leq t \leq n-2$

**Proof:** Let $G = P_n \times P_n$ and $L(G) = L(P_n \times P_n)$.

Let $x_i$ and $x_j$ be any two vertices of $G$ at a distance $t$ i.e $d_{ij}(x_i, x_j) = t$

In the following instances, we have

**Case i)** For $t = 1$

Here, we see that $d(x_i, x_j) = 1$ for $i=1, j=2n$ in $G$ then the path that leads

$$
\begin{align*}
(x_1 \cdot x_{2n-1}) \bigcup_{k=1}^{n-1} (x_{2n-k} \cdot x_{2n}) \cup (x_{2n} \cdot x_{2n-(k+1)}) \bigcup_{l=1}^{3} (x_{n-l} \cdot x_{n-(l+1)})
\end{align*}
$$

This is Hamilton-$1^*$-laseable path between $x_i$ and $x_j$

**Case ii)** For $t = 2$

Here, we note that $d(x_i, x_j) = 2$ for $i=1, j=2n$ in $G$ then the path

$$
\begin{align*}
\bigcup_{k=1}^{n-1} (x_k \cdot x_{k+1}) \bigcup_{k=1}^{n-2} \{ (x_{n+(k-1)} \cdot x_{2n+(n-1)-k}) \cup (x_{2n+(n-1)-k} \cdot x_{n+k}) \} \\
\cup (x_{2n-2} \cdot x_{2n-1}) \cup (x_{2n-1} \cdot x_{2n})
\end{align*}
$$

is a $K^{+1}$ hypo edge-$H$-$2^*$-$L$ path linking the nodes $x_i$ and $x_j$.

The $K^{+1}$ hypo edge-$H$-$2^*$-$L$ path linking the nodes $x_1$ and $x_{10}$ is shown in figure 4.

![Paths $P_2$ and $P_5$](image)

**Figure 2.** Paths $P_2$ and $P_5
Figure 3. Cartesian product of $P_2$ and $P_5$

Figure 4. Hamiltonian-2*-laceable path in $L(P_2 \times P_5)$ from $x_1$ to $x_{10}$

Case iii) If $d(x_i, x_j) = t$ where $i = t, j = 2n$ and $4 \leq t \leq diam(G)$ where $t$ is even in $G$ and the path

$$
\bigcup_{k=t}^{n} \{x_k, x_{k+1}\} \bigcup_{l=1}^{n-2} \{x_{n+(l-1)}, x_{3n-l-1}\} \cup \{x_{3n-l-1}, x_{n+l}\}\{x_{2n-2}, x_1\}
$$

$$
\bigcup_{m=1}^{t-1} \{x_m, x_{m+1}\} \cup \{x_{t-1}, x_{2n-1}\} \cup \{x_{2n-1}, x_{2n}\}
$$

this is $K^{+1}$ hypo edge- $H$-t-L path linking nodes $x_i$ and $x_j$ where even $t$, $4 \leq t \leq diam(G)$

The $K^{+1}$ hypo edge- $H$-t-L path linking the nodes $x_3$ and $x_{10}$ is shown in figure 7.

Figure 5: Paths $P_2$ and $P_6$

Figure 6. Cartesian product of $P_2$ and $P_6$
Case iv) If \(d(x_i, x_j) = t\) where \(i=t, j=2n\) and \(3 \leq t \leq n-2\) where \(t\) is odd then the path

\[
\bigcup_{k=t}^{n} (x_{k}, x_{k+1}) \bigcup_{l=1}^{n-t+1} (x_{n+(l-1)}, x_{3n-l-1}) \cup (x_{3n-l-1}, x_{n+t}) \cup (x_{2n-t+1}, x_{t-1})
\]

\[
\bigcup_{m=1}^{t-1} \{(x_{t-(2m-1)}, x_{t-2m}) \cup (x_{t-2m}, x_{2n-t+2m}) \cup (x_{2n-t+2m}, x_{2n+t-(2m+1)}) \}
\]

\[
\cup (x_{2n+t-(2m+1)}, x_{2n+t-(2m+2)}) \cup (x_{2n+t-(2m+2)}, x_{2n-t+(2m+1)}) \cup (x_{2n-2}, x_{2})
\]

\[
\cup (x_{2}, x_{1}) \cup (x_{1}, x_{2n-1}) \cup (x_{2n-1}, x_{2n})
\]

(4)

this is \(K^{1+}\) hypo edge- \(H-t\)-laceable path that connects nodes \(x_i\) and \(x_j\) where \(3 \leq t \leq n-2\).

Therefore from the above equations (1),(2),(3) and (4), we conclude that \(G\) is accordingly hypo edge \(H-t\)-laceable.

**Remark 3.1.1.** If \(t = n-1\) in \(L(P_2 \times P_n)\), we get the Hamilton path from \(x_{n-1}\) to \(x_{2n}\)

Then the path

\[
\begin{align*}
x_{n-1} &\rightarrow x_n \\
x_n &\rightarrow x_{3n-2} \\
x_{3n-2} &\rightarrow x_{n+1} \\
x_{n+1} &\rightarrow x_{2n-3} \\
x_{2n-3} &\rightarrow x_{n+2} \\
x_{n+2} &\rightarrow x_{n-2} \\
x_{n-2} &\rightarrow x_{n-3} \\
x_{n-3} &\rightarrow x_{n+3} \\
\end{align*}
\]

\[
\cdots \\
\rightarrow x_{2n-1} \rightarrow x_{2n}
\]

(5)

**Example 3.1.2.** In \(L(P_2 \times P_6)\)

Hamiltonian path from \(x_5\) to \(x_{12}\) is

\[
\begin{align*}
x_5 &\rightarrow x_n \\
x_n &\rightarrow x_6 \\
x_6 &\rightarrow x_{16} \\
x_{16} &\rightarrow x_4 \\
x_4 &\rightarrow x_8 \\
x_8 &\rightarrow x_4 \\
x_4 &\rightarrow x_3 \\
x_3 &\rightarrow x_9 \\
x_9 &\rightarrow x_{14} \\
x_{14} &\rightarrow x_{13} \\
x_{13} &\rightarrow x_{10} \\
x_2 &\rightarrow x_1 \\
x_1 &\rightarrow x_{11} \\
x_{11} &\rightarrow x_{12}
\end{align*}
\]

(6)

**Theorem 3.2.** Let \(G = P_2 \times C_n\) be a graph then \(L(G) = L(P_2 \times C_n)\) is Hamiltonian- \(t^*\)-laceable such that

\[
\begin{align*}
t = 1 &\quad \text{for } n \geq 3 \\
t = 2,3 &\quad \text{for } n \geq 4
\end{align*}
\]

**Proof:** Let \(G = P_2 \times C_n\) and \(L(G) = L(P_2 \times C_n)\).

Let \(x_i\) and \(x_j\) be any two vertices of \(G\) at a distance \(t\) i.e. \(d(x_i, x_j) = t\)

We have the following instances.
Case i) For $t = 1$
In this case, for $n \geq 3$ in $G$, we observe that if $d(x_i, x_j) = 1$ for $i=1, j=2$ then the path
\[
(x_1 \cdot x_3n-1) \cup (x_3n-1 \cdot x_2n-1) \cup (x_2n-1 \cdot x_2n) \bigcup_{k=1}^{n-2} \{(x_2n+(k-1)) \cdot x_2n-(k+1)\}
\cup (x_2n-(k+1) \cdot x_2n+k) \} \cup (x_3n-2 \cdot x_3n) \cup (x_3n \cdot x_n) \bigcup_{l=1}^{n-2} (x_n \cdot x_{n-l})
\]
(7)
is a Hamilton-$1^*$-laceable path joining $x_i$ and $x_j$.

Case ii) For $t = 2$
In this case, for $n \geq 4$ in $G$, we observe that if $d(x_i, x_j) = 2$ for $i=1, j=3$ then the path
\[
(x_i \cdot x_3n-1) \cup (x_3n-1 \cdot x_2n-1) \cup (x_2n-1 \cdot x_2n) \cup (x_2n-2 \cdot x_2n-2) \cup (x_2n-2 \cdot x_2n-1)
\cup (x_2n+1 \cdot x_2n+2) \bigcup_{k=1}^{n-4} \{(x_2n+(k-1)) \cdot x_2n-(k+3) \cup x_2n-(k+3) \cdot x_2n+(k-2)\}
\cup (x_3n-2 \cdot x_3n) \cup (x_3n \cdot x_n) \bigcup_{l=1}^{n-3} (x_n \cdot x_{n-l})
\]
(8)
is a H-$2^*$.L path linking the nodes $x_i$ and $x_j$.

The Hamilton-$2^*$-laceable path joining the vertices $x_1$ and $x_3$ is shown in figure 8.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure8.png}
\caption{Path $P_2$ and cycle $C_6$}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{figure9.png}
\caption{Cartesian product of $P_2$ and $C_6$}
\end{figure}
Case iii) For $t=3$

In this case, for $n \geq 4$ in $G$, we observe that if $d(x_i, x_j) = 3$ for $i=1, j=4$ then the path
\[
(x_i, x_{3n-1}) \cup (x_{3n-1}, x_{2n-1}) \cup (x_{2n-1}, x_{2n}) \cup (x_{2n}, x_{2n-2}) \cup (x_{2n-2}, x_2) \\
\cup (x_2, x_3) \cup (x_3, x_{2n-3}) \cup (x_{2n-3}, x_{2n+1}) \cup (x_{2n+1}, x_{2n+2}) \\
\bigcup_{k=1}^{n-4} \{ (x_{2n+(k+1)}, x_{2n-(k+3)}), x_{2n-(k+3)}, x_{2n+(k+2)} \} \cup (x_{3n-2}, x_{3n})
\]

is a $H-3^* - L$ path joining the vertices $x_i$ and $x_j$ for $n \geq 4$ in $G$.

Therefore from the above equations (7), (8) and (9), we conclude that $G$ is $H-t^* - L$.

The $H-3^* - L$ path joining the vertices $x_1$ and $x_4$ is shown in figure 11.
Theorem 3.3. Let $G = P_2 \times W_n$ then for $n \geq 4$, $G$ is Hamiltonian- $t^*$-laceable such that it is $1 \leq t \leq \text{diam}(G)$

Proof: Let $G = P_2 \times W_n$ with $2n$ vertices and $\text{Diam}(G)=3$ for $n \geq 5$ and is $\text{Diam}(G)=3$ for $n = 4$.

Let $x_i$ and $x_j$ be any two vertices of $G$ at a distance $i$ e $d(x_i, x_j) = t$.

For $n \geq 4$ and $1 \leq t \leq \text{diam}(G)$, we have the following instances

Case i) If $t = 1$

Here, we note that $d(x_i, x_j) = 1$ for $i=1, j= n+1$ in $G$ then the path that leads

$$\bigcup_{k=1}^{n-1} (x_k, x_{k+1}) \cup (x_n, x_{2n}) \cup \bigcup_{l=1}^{n-1} (x_{2n}, x_{2n-l})$$

is a Hamiltonian- $t^*$-laceable path joining $x_i$ and $x_j$.

Case ii) If $t = 2$

Here, we note that $d(x_i, x_j) = 2$ for $i=1, j= n+2$ in $G$ and the path

$$\bigcup_{k=1}^{n-1} \{ (x_k, x_{k+1}) \} \cup (x_n, x_{2n}) \cup (x_{2n}, x_{2n+3}) \cup \bigcup_{l=1}^{n-1} \{ (x_{n+l}, x_{n+l+2}) \}$$

is a Hamilton- $t^*$-laceable path joining the vertices $x_i$ and $x_j$.

The Hamilton- $t^*$-laceable path joining vertices $x_1$ and $x_3$ is shown in figure 12.

Figure 12. Hamilton- $t^*$-laceable path in $P_2 \times W_n$ from $x_1$ to $x_8$

Case iii) If $t = 3$

In this case, we note that $d(x_i, x_j) = 3$ for $i=1, j= n+3$ in $G$ and the path

$$\bigcup_{k=1}^{n-1} \{ (x_k, x_{k+1}) \} \cup (x_n, x_{2n}) \cup (x_{2n}, x_{n+4}) \cup \bigcup_{l=1}^{n-5} \{ (x_n+l, x_{n+l+4}) \}$$

is a Hamilton- $t^*$-laceable path between the vertices $x_i$ and $x_j$.

Therefore from the above equations (10), (11) and (12), we conclude that $G$ is Hamilton- $t^*$-laceable.
4. Conclusion
In this research article we discussed Hypo-edge Hamilton lacealibility of Line graph of Cartesian product of path (two vertices) with cycle and path and also Laceability property in Cartesian product of path and wheel graph. This concept can be extended to line graph of other graph structures.

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