Ternary codes, biplanes, and the nonexistence of some quasisymmetric and quasi-3 designs

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Abstract

The dual codes of the ternary linear codes of the residual designs of biplanes on 56 points are used to prove the nonexistence of quasisymmetric 2-(56, 12, 9) and 2-(57, 12, 11) designs with intersection numbers 0 and 3, and the nonexistence of a 2-(267, 57, 12) quasi-3 design. The nonexistence of a 2-(149, 37, 9) quasi-3 design is also proved.

KEYWORDS
biplane, linear code, quasi-3 design, quasisymmetric design, residual design, symmetric design

1 | INTRODUCTION

We assume familiarity with basic facts and notions from combinatorial design theory and coding theory [1,5,22].

A combinatorial design (or an incidence structure) is a pair $\mathcal{D} = (X, \mathcal{B})$ of a finite set $X = \{x_i\}_{i=1}^v$ of points, and a collection $\mathcal{B} = \{B_j\}_{j=1}^b$ of subsets $B_j \subseteq X$, called blocks. The (points by blocks) incidence matrix $A = (a_{ij})$ of a design $\mathcal{D}$ with $v$ points and $b$ blocks is a $(0, 1)$-matrix with $v$ rows indexed by the points and $b$ columns indexed by the blocks, where $a_{ij} = 1$ if the $i$th point belongs to the $j$th block, and $a_{ij} = 0$ otherwise. The transposed matrix $A^T$ is called the blocks by points incidence matrix of $\mathcal{D}$. If $p$ is a prime number, the $p$-rank of $A$ (or $\text{rank}_p A$), is defined as the rank of $A$ over a finite field of characteristic $p$.

Given a design $\mathcal{D}$ with $v$ points and $b$ blocks, and a finite field $F = GF(q)$, one can define two linear codes over $F$ associated with $\mathcal{D}$: the code of length $b$ spanned by the rows of the $v$ by $b$ incidence matrix $A$ is called the code of $\mathcal{D}$ spanned by the points, while the code of length $v$ spanned by the columns of $A$ is called the code of $\mathcal{D}$ spanned by the blocks.

Let $\mathcal{D} = (X, \mathcal{B})$ be a design, and let $B \in \mathcal{B}$ be a block of $\mathcal{D}$. The incidence structure $\mathcal{D}^B = (X', \mathcal{B}')$, where
is called the \textit{derived} design of $\Delta$ with respect to block $B$.

The incidence structure $\mathcal{D}_B = (X'', B'')$, where

$$X'' = X \setminus B, \quad B'' = \{B_j \cap B \mid B_j \in B, B_j \neq B\}$$

is called the \textit{residual} design of $\Delta$ with respect to block $B$.

**Definition 1.** Let $A$ be the $v \times b$ incidence matrix of a design $\Delta$ with $v$ points and $b$ blocks, and let $A''$ be the incidence matrix of a residual design $\mathcal{D}_B$ with respect to a block $B$. The residual design $\mathcal{D}_B$ is said to be \textit{linearly embeddable} \cite{20} over $F = GF(p)$ ($p$ prime), if

$$\text{rank}_p A = \text{rank}_p A'' + 1. \quad (1)$$

**Note 2.** A sufficient condition for a residual design $\mathcal{D}_B$ to be linearly embeddable is that the minimum distance of the linear code over $F$ spanned by the blocks of $\Delta$ is equal to $|B|$ \cite[Theorem 2.2]{20}.

Given a design $\Delta = (X, B)$, its \textit{dual} design $\Delta^*$ is the incidence structure having as points the blocks of $\Delta$, and having as blocks the points of $\Delta$, where a point and a block of $\Delta^*$ are incident if and only if the corresponding block and point of $\Delta$ are incident. If $A$ is the incidence matrix of $\Delta$, then $A^T$ is the incidence matrix of the dual design $\Delta^*$.

Given integers $v \geq k \geq t \geq 0$, $\lambda \geq 0$, a $t$-$(v, k, \lambda)$ design (or briefly, a $t$-design) $\Delta$ is an incidence structure with $v$ points and blocks of size $k$ such that every $t$-subset of points is contained in exactly $\lambda$ blocks. A $t$-$(v, k, \lambda)$ design is also an $s$-$(v, k, \lambda_s)$ design for every integer $s$ in the range $0 \leq s \leq t$, with $\lambda_s = \binom{v-s}{t-s}/\binom{k-s}{t-s}$.

Let $\Delta = (X, B)$ be a $t$-$(v, k, \lambda)$ design, and let $x \in X$ be a point. The derived design $\mathcal{D}_x$ with respect to $x$ is the $(t-1)$-$(v-1, k-1, \lambda)$ design with point set $X \setminus \{x\}$, and block set $\{B \setminus \{x\} \mid B \in B, x \notin B\}$.

The residual design $\mathcal{D}_x$ with respect to $x$ is the $(t-1)$-$(v-1, k, \lambda_{t-1} - \lambda)$ design with point set $X \setminus \{x\}$, and block set $\{B \mid B \in B, x \notin B\}$.

If $\Delta$ is a 2-$(v, k, \lambda)$ design with $v > k > 0$, the number of blocks $b = v(v-1)\lambda/(k(k-1))$ satisfies the Fisher inequality:

$$b \geq v, \quad (2)$$

and the equality $b = v$ holds if and only if every two blocks of $\Delta$ share exactly $\lambda$ points.

A 2-$(v, k, \lambda)$ design $\Delta$ with $b = v$ is called \textit{symmetric}. The dual design $\Delta^*$ of a symmetric 2-$(v, k, \lambda)$ design $\Delta$ is a symmetric design having the same parameters as $\Delta$. A symmetric design is \textit{self-dual} if it is isomorphic to its dual design. A \textit{biplane} is a symmetric design with $\lambda = 2$.

**Note 3.** Assume that $\Delta$ is a 2-$(v, k, \lambda)$ design. If $x$ is a point of $\Delta$, the derived design $\mathcal{D}_x$ is a 1-$(v-1, k-1, \lambda)$ design, while the residual design $\mathcal{D}_x$ is a 1-$(v-1, k, r-\lambda)$ design, where $r = \lambda_i = \lambda(v-1)/(k-1)$ is the number of blocks of $\Delta$ that contains $x$. 


If $\mathcal{D}$ is a symmetric $2(v, k, \lambda)$ design, then $r = k$ and $\mathcal{D}_x$ is a $1(v - 1, k, k - \lambda)$ design. In addition, if $B$ is a block, the derived design $\mathcal{D}^B$ is a $2(k, \lambda, \lambda - 1)$ design, while the residual design $\mathcal{D}_B$ is a $2(v - k, k - \lambda, \lambda)$ design.

A $2(v, k, \lambda)$ design is quasisymmetric with intersection numbers $x, y$ ($x < y$), if every two blocks intersect in either $x$ or $y$ points. A brief survey on quasisymmetric designs is given in [18]. Links between quasisymmetric designs and error-correcting codes are discussed in [21].

Examples of quasisymmetric designs are: (a) unions of identical copies of symmetric $2$-designs; (b) nonsymmetric $2(v, k, 1)$ designs; (c) strongly resolvable designs; and (d) residual designs of biplanes. A quasisymmetric $2(v, k, \lambda)$ design with $2k < v$ which does not belong to any of these four classes is referred to as exceptional.

The classification of exceptional quasisymmetric designs is a difficult open problem. A table of admissible parameters for exceptional quasisymmetric designs with number of points $v \leq 70$ is given in [18, Table 48.25]. This table is an updated version of Neumaier's table [19] published in 1982.

Two admissible parameters sets for exceptional quasisymmetric designs whose existence has been unknown since 1982 are $2(56, 12, 9)$, $(x = 0, y = 3)$, and $2(57, 12, 11)$, $(x = 0, y = 3)$. It is the goal of this paper to show that quasisymmetric designs with these parameters do not exist. Note that the nonexistence of the latter is a direct consequence of that of former. We thank the anonymous reviewer who pointed out that the nonexistence of the latter is essentially known, if one notices that the latter is a $3(57, 12, 2)$ design. See Remark 8.

The nonexistence of a quasisymmetric $2(57, 12, 11)$ design implies also the nonexistence of a quasi-3 design with parameters $2(267, 57, 12)$. Similarly, using the nonexistence of quasisymmetric $2(37, 9, 8)$ designs with intersection numbers 1 and 3 [10], we show that a quasi-3 design with parameters $2(149, 37, 9)$ does not exist. The existence of quasi-3 designs with these parameters was a long-standing open question [15].

## 2 Residual 2-(45, 9, 2) Designs and Their Ternary Codes

Our proof of the nonexistence of a quasisymmetric $2(56, 12, 9)$ design is based on the following lemmas.

**Lemma 4.** Suppose that $\mathcal{D} = (X, B)$ is a quasisymmetric $2(56, 12, 9)$ design with intersection numbers 0 and 3, and let $z \in X$ be a point of $\mathcal{D}$.

(i) The derived design $\mathcal{D}^z$ is a $1(55, 11, 9)$ design with 45 blocks whose dual design $(\mathcal{D}^z)^*$ is a $2(45, 9, 2)$ design.
(ii) The residual design $\mathcal{D}_z$ is a $1(55, 12, 36)$ design with 165 blocks. The columns of the $55 \times 165$ incidence matrix of $\mathcal{D}_z$ belong to the dual code $C^\perp$ of the linear code $C$ over GF(3) spanned by the columns of the $55 \times 45$ incidence matrix of $\mathcal{D}^z$.

**Proof.**

(i) Since every two nondisjoint blocks of $\mathcal{D}$ share exactly three points, every two blocks of $\mathcal{D}^z$ share exactly two points. This implies that $(\mathcal{D}^z)^*$ is a $2(45, 9, 2)$ design.
(ii) The inner product of the incidence vector of every block $B$ of $D_z$ with the incidence vector of every block of $D^z$ is either 0 or 3.

By a theorem of Hall and Connor [8], every 2-(45, 9, 2) design is a residual design with respect to a block of a biplane with parameters 2-(56, 11, 2). There are five nonisomorphic biplanes $B_i$ ($1 \leq i \leq 5$) with these parameters [7, 15.8], all five being self-dual. The first biplane, $B_1$, was found by Hall, Lane, and Wales [9], $B_2$ was found by Salwach and Mezzaroba [17], $B_3$ and $B_4$ were found by Denniston [6], and $B_5$ was found by Janko and Trung [12]. The residual 2-(45, 9, 2) designs of the five biplanes fall into 16 isomorphism classes (see [14, Table 2]).

It was shown by an exhaustive computer search (Kaski and Östergård [13]), that up to isomorphism, there are exactly five biplanes with 56 points, and consequently, exactly 16 nonisomorphic 2-(45, 9, 2) designs.

Using Lemma 4, the existence question for a 2-(56, 12, 9) design can be resolved by computing the sets of all $(0, 1)$-vectors of weight 12 in the dual codes of the ternary linear codes spanned by the rows of the $45 \times 55$ incidence matrices of the 16 nonisomorphic 2-(45, 9, 2) designs, and checking if any of these sets contains a subset of 165 vectors that form the incidence matrix of a 1-(55, 12, 36) design with 165 blocks of size 12, such that every two blocks are either disjoint or share exactly three points.

**Lemma 5.** Let $D_B$ be a 2-(45, 9, 2) design with point set $\{1, 2, \ldots, 45\}$, being the residual design of a 2-(56, 11, 2) biplane $D$ with respect to a block $B = \{46, \ldots, 56\}$. Let $A$ be an incidence matrix of $D$ given by (3), where $A''$ is the $45 \times 55$ incidence matrix of $D_B$, and $A'$ is the $11 \times 55$ incidence matrix of the derived 2-(11, 2, 1) design $D^B$.

\[
A = \begin{pmatrix}
0 \\
A'' \\
0 \\
1 \\
A' \\
1
\end{pmatrix}
\]

If $c = (c_1, \ldots, c_{55})$ is a $(0, 1)$-codeword of weight 12 in the dual code $(L'')^\perp$ of the ternary linear code $L''$ spanned by the rows of $A''$, then $c^* = (c_1, \ldots, c_{55}, 0)$ belongs to the dual code $L'$ of the ternary linear code $L$ spanned by the rows of $A$.

Proof. The ternary codes of the five biplanes with parameters 2-(56, 11, 2) were computed in [14], and all five codes have minimum distance 11. Thus, by Definition 1 and Note 2, every residual 2-(45, 9, 2) design is linearly embeddable over $GF(3)$, and

\[
\text{rank}_3 A = \text{rank}_3 A'' + 1.
\]

This implies

\[
\text{dim } L = \begin{pmatrix}
A'' \\
0 \\
I_{55}
\end{pmatrix}
\]

(4)
Since the sum of all rows of $A$ over $GF(3)$ is the constant vector with all entries equal to 2, the all-one vector $\bar{1}_{56} = (\bar{1}_{55}, 1)$ of length 56 belongs to $L$. Thus, $L$ is spanned by the row vectors of the matrix in the right-hand side of (4), to which $c^*$ is orthogonal. Therefore, $c^* \in L^\perp$. □

Lemmas 4 and 5 imply the following (see Figure 1 for an illustration).

**Lemma 6.** Let $\mathcal{D}$ be a 2-(56, 11, 2) biplane, and let $S$ be the set of all $(0, 1)$-vectors of weight 12 in the dual code of the ternary linear code spanned by the points of $\mathcal{D}$. Let $\mathcal{D}_B$ be a residual 2-(45, 9, 2) design of $\mathcal{D}$ with respect to a block $B$.

A necessary condition for the existence of a quasisymmetric 2-(56, 12, 9) design $\hat{\mathcal{D}}$ having $\mathcal{D}_B^*$ as a derived design is that the subset $S_B \subseteq S$ consisting of all vectors having 0 in the position labeled by $B$, contains a set of 165 vectors that are the incidence vectors of the blocks of a 1-(55, 12, 36) design $\hat{\mathcal{D}}_z$, such that every two blocks of $\hat{\mathcal{D}}_z$ are either disjoint or share exactly three points.

3 | NONEXISTENCE OF QUASISYMMETRIC 2-(56, 12, 9) AND 2-(57, 12, 11) DESIGNS

**Theorem 7.**

(i) A quasisymmetric 2-(56, 12, 9) design with block intersection numbers 0, 3 does not exist.

(ii) A quasisymmetric 2-(57, 12, 11) design with block intersection numbers 0, 3 does not exist.

**Proof:**

(i) By a theorem of Hall and Connor [8], every 2-(45, 9, 2) design is a residual design with respect to a block of a biplane with parameters 2-(56, 11, 2). Thus, by Lemma 6, it is sufficient to inspect the sets of all $(0, 1)$-codewords of weight 12 in the dual codes of the ternary codes of the five biplanes with 56 points. The number of such codewords can be found by computing with Magma [4] the complete weight enumerator of the dual codes. This was done in [14] for three of the biplanes, $B_1$, $B_2$, and $B_4$, while upper bounds 91 and 22 were found for the codes of the biplanes $B_3$ and $B_5$. 

![Figure 1](https://wileyonlinelibrary.com)
[14, Table 1], and these results were used to prove that none of the five biplanes can be extended to a 3-(57, 12, 2) design.

Using Magma, we were able to compute the exact numbers of such codewords for the dual codes of B3 and B5 (84 and 20, respectively). We reproduce some of the properties of these codes in Table 1, where the first column lists the corresponding biplane, column two gives the 3-rank of its incidence matrix, or the code dimension, the third column gives the order of the automorphism group of the biplane, the fourth column gives the minimum distance of the code, and the last column gives the total number of (0, 1)-codewords of weight 12 in the dual code.

Since the dual codes of the biplanes B3, B4, and B5 each contains less than 165 (0, 1)-codewords of weight 12, it follows from Lemma 6 that none of the dual designs of their residual 2-(45, 9, 2) designs can be a derived 1-(55, 11, 9) design of a quasisymmetric 2-(56, 12, 9) design with intersection numbers 0, 3.

Since the automorphism group of B1 acts transitively on the set of blocks, all residual 2-(45, 9, 2) designs of B1 are isomorphic. We define a graph \( \Gamma_1 \) having as vertices the 2100 (0, 1)-codewords of weight 12 in the dual code of the ternary code spanned by the points of B1, where two codewords are adjacent in \( \Gamma_1 \) if their supports are either disjoint or share exactly three points. It follows from Lemma 6 that if a quasisymmetric 2-(56, 12, 9) design exists and has a derived design with respect to a point which is the dual design of a residual 2-(45, 9, 2) design of B1, then \( \Gamma_1 \) will contain a clique of size 165. A quick computation with Cliquer [16] shows that the maximum clique size of \( \Gamma_1 \) is 22, thus none of the residual designs of B1 is embeddable in a quasisymmetric 2-(56, 12, 9) design.

The graph \( \Gamma_2 \) having as vertices the 516 (0, 1)-codewords of weight 12 in the dual code of the ternary code spanned by the points of B2, where two codewords are adjacent in \( \Gamma_2 \) if their supports are either disjoint or share exactly three points, has maximum clique size 18. It follows by Lemma 6 that a quasisymmetric 2-(56, 12, 9) design having a derived design, which is the dual design of some residual 2-(45, 9, 2) design of B2, does not exist. This completes the proof of part (i).

(ii) Suppose there exists a quasisymmetric 2-(57, 12, 11) design \( \Delta \) with block intersection numbers 0, 3. Then the residual design of \( \Delta \) with respect to a point \( p \) is a quasisymmetric 2-(56, 12, 9) design with block intersection numbers 0, 3. Such a design does not exist by (i) (Table 1).

Remark 8. We thank the anonymous reviewer who pointed out that Theorem 7(ii) is essentially known. Indeed, a quasisymmetric 2-(57, 12, 11) design with block intersection

| \( \text{dim } C \) | \( | \text{Aut } Bi | \) | Min | \( \# \text{ wt } 12 \text{ in } C^\perp \) |
|---|---|---|---|
| B1 | 20 | 80 640 | 11 | 2100 |
| B2 | 22 | 288 | 11 | 516 |
| B3 | 26 | 144 | 11 | 84* |
| B4 | 24 | 64 | 11 | 148 |
| B5 | 26 | 24 | 11 | 20* |

*Note: Entries with * were not found exactly, but were only estimated in [14].
numbers 0 and 3 is necessarily a 3-(57, 12, 2) design by using [19, Proposition 12] (see also [3]). The nonexistence of a 3-(57, 12, 2) design has already been established in [13, Corollary 2].

4 | NONEXISTENCE OF SOME QUASI-3 DESIGNS

Definition 9. A symmetric 2-(v, k, λ) design D is a quasi-3 design [15] with triple intersection numbers x and y (x < y) if every three blocks of D intersect in either x or y points.

Clearly, D is a quasi-3 design if and only if every of its derived (with respect to a block) 2-(k, λ, λ − 1) designs is a quasisymmetric design with block intersection numbers x and y.

According to [15, Table 47.14], there are 12 parameter sets of quasi-3 designs with number of points v ≤ 400, for which the existence of a quasi-3 design is unknown. Two of these 12 open cases are the parameters 2-(149, 37, 9), (x = 1, y = 3), and 2-(267, 57, 12), (x = 0, y = 3).

Theorem 10. A quasi-3 design with parameters 2-(267, 57, 12), (x = 0, y = 3), does not exist.

Proof. Any derived design with respect to a block of a quasi-3 2-(267, 57, 12) design with triple intersection numbers x = 0, y = 3 is a quasisymmetric 2-(57, 12, 11) design with block intersection numbers x = 0, y = 3. By Theorem 7(ii), a quasisymmetric design with the latter parameters does not exist.

Theorem 11. A quasi-3 design with parameters 2-(149, 37, 9), (x = 1, y = 3), does not exist.

Proof. Any derived design with respect to a block of a quasi-3 2-(149, 37, 9) design with triple intersection numbers x = 1, y = 3 is a quasisymmetric 2-(37, 9, 8) design with block intersection numbers x = 1, y = 3. However, it was proved in [10] that quasisymmetric designs with the latter parameters do not exist.

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