On the geometry of some algebras related to the Weyl groupoid.

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Abstract

Let $k$ be an algebraically closed field of characteristic zero. Let $g$ be a finite dimensional classical simple Lie superalgebra over $k$ or $gl(m, n)$. In the case that $g$ is a Kac-Moody algebra of finite type with set of roots $\Delta$, Sergeev and Veselov introduced the Weyl groupoid $\mathcal{W} = \mathcal{W}(\Delta)$, which has significant connections with the representation theory of $g$. Let $h$, $W$ and $Z(g)$ be a Cartan subalgebra of $g_0$, the Weyl group of $g_0$ and the center of $U(g)$ respectively. Also let $G$ be a Lie supergroup with Lie $G = g$. There are several important commutative algebras related to $\mathcal{W}$. Namely

- The image $I(h)$ of the injective Harish-Chandra map $Z(g) \rightarrow S(h)^W$.
- The supercharacter $Z$-algebras $J(g)$ and $J(G)$ of finite dimensional representations of $g$ and $G$.

Let $\mathcal{A} = \mathcal{A}(g)$ be denote either $I(h)$ or $J(G) \otimes_k k$. The purpose of this paper is to investigate the algebraic geometry of $\mathcal{A}$. In many cases, the algebra $\mathcal{A}$ satisfies the Nullstellensatz. This gives a bijection between radical ideals in $\mathcal{A}$ and superalgebraic sets (zero loci of such ideals). Any superalgebraic set is uniquely a finite union of irreducible superalgebraic components. In the Kac-Moody case, we describe the smallest superalgebraic set containing a given (Zariski) closed set, and show that the superalgebraic sets are exactly the closed sets that are unions of groupoid orbits.
1 Introduction

1.1 Background.

1.1.1 The classical Lie superalgebras.

We are interested in the following classical Lie superalgebras $\mathfrak{g}$. All have $\mathfrak{g}_0$ reductive and $\mathfrak{g}_0 \neq \mathfrak{g}$.

(a) The KM$^1$ algebras $\mathfrak{gl}(m|n)$, $\mathfrak{osp}(r|2n)$, $\mathfrak{sl}(m|n)$ with $m \neq n$, $D(2|1; \alpha)$, $F(4)$, $G(3)$,

(b) The strange Lie superalgebras $\mathfrak{p}(n)$, $\mathfrak{q}(n)$ and their simple relatives $\mathfrak{sp}(n)$, $\mathfrak{psq}(n)$.

There are some conditions on $r, m, n \in \mathbb{N}$ and $\alpha \in \mathbb{k}$ which we omit. The last three algebras listed in (a) are called exceptional. The algebras listed in (a) with $\mathfrak{gl}(m|n)$ replaced by $\mathfrak{psl}(n|n)$, together with the simple algebras from (b) are precisely the finite dimensional simple Lie superalgebras $\mathfrak{g}$ with $\mathfrak{g}_0$ reductive and $\mathfrak{g}_0 \neq \mathfrak{g}$.

1.1.2 Algebras related to the Weyl groupoid.

The Weyl groupoid $\mathcal{W}(\Delta)$ of Sergeev and Veselov, \cite{SV11} has become a fundamental tool in the representation theory of a KM algebra $\mathfrak{g}$. Here $\Delta$ is the root system of $\mathfrak{g}$. This groupoid was introduced in relation to the algebras $J(\mathfrak{g})$ and $J(G)$ listed in the abstract. The Lie supergroups $G$ we consider are defined in Lemma 3.4. In particular the supergroups $P(n)$ and $Q(n)$ have Lie $P(n) = \mathfrak{p}(n)$ and Lie $Q(n) = \mathfrak{q}(n)$. Weyl groupoids for $P(n)$ and $Q(n)$ were defined in \cite{IRS21} Section 5.4 and \cite{Rei22}, Section 5 respectively.

For a KM algebra $\mathfrak{g}$, an important role is played by a non-degenerate, symmetric bilinear form $(,)$ on $\mathfrak{g}$. This can be constructed in a uniform way, \cite{Mus12} Theorem 5.4.1, Remark 5.4.2. The restriction of this form to $\mathfrak{h}$ remains non-degenerate and this yields an isomorphism $\mathfrak{h} \rightarrow \mathfrak{h}^*$, denoted $h_\alpha \mapsto \alpha$. Thus we obtain a non-degenerate, symmetric bilinear form on $\mathfrak{h}^*$ which we also denote by $(, )$. This latter form is invariant under the Weyl group $W$ of $\mathfrak{g}_0$, and we can normalize the whole construction by requiring that its Gram matrix is the (symmetrized) Cartan matrix $(\alpha_i, \alpha_j)$, where $\{\alpha_i\}$ is the distinguished basis of simple roots for $\mathfrak{g}$, \cite{Mus12} Section 3.2. The issue with $\mathfrak{psl}(n|n)$ is that any set of simple roots $\{\alpha_i\}$ is linearly dependent, and the matrix $A = (\alpha_i, \alpha_j)$ is singular. Thus there is no bilinear form as above. The usual solution to this problem is to take a minimal realization of $A$, leading to the construction of $\mathfrak{gl}(n|n)$, which has $\mathfrak{psl}(n|n)$ as a subquotient. For $\alpha \in \Delta_{iso}$, the set of isotropic roots of $\mathfrak{g}$, set

$$\Pi_\alpha = \{\lambda \in \mathfrak{h}^* | (\lambda, \alpha) = 0\}. \quad (1.1)$$

\footnotetext[1]{Kac-Moody Lie superalgebras of finite type. These algebras are also known as contragredient Lie superalgebras. \cite{Kac77}, \cite{Mus12} Chapter 5.
The other algebra of interest \( I(h) \) is isomorphic to \( Z(g) \) via the Harish-Chandra map. This algebra may be defined as

\[
I(h) = \{ f \in S(h)^W | f(\lambda) = f(\lambda + t\alpha) \text{ for all } \alpha \in \Delta_{iso}, \lambda \in \Pi_\alpha \text{ and } t \in k \}. \tag{1.2}
\]

Several other definitions of \( I(h) \) can be given based on [Mus12], Lemma 12.1.1. By work of Gorelik and Kac [Gor04], [Kac84], the Harish-Chandra map yields an isomorphism \( Z(g) \rightarrow I(h) \), see also [Mus12], Theorem 13.1.1. For a description of \( Z(g) \) when \( g = p(n) \) and \( q(n) \), see [Gor01], [Gor06] respectively.

There is a description of \( J(G) \) parallel to (1.2). Let \( T \) be a maximal torus in \( G_0 \), and \( X(T) = \text{Hom}(T, k^*) \) (resp. \( Y(T) = \text{Hom}(k^*, T) \)) the group of characters (resp. one-parameter subgroups) of \( T \). Let \( D_\alpha \) be the derivation of \( Z[X(T)] \) given by

\[
D_\alpha(e^\beta) = (\alpha, \beta)e^\beta.
\]

Then in the KM case, we have by [SV11] Equation (1),

\[
J(G) = \{ f \in Z[X(T)]^W | D_\alpha f \in (e^\alpha - 1) \text{ for all } \alpha \in \Delta_{iso} \}. \tag{1.3}
\]

Various conditions equivalent to that defining \( J(G) \) are given in [Mus22] Section 3.

1.1.3 Continuous Weyl groupoids.

In the theory of algebraic groups acting on varieties, a key role is played by closed orbits. However the orbits of \( W \) on \( h^* \) or \( T \) are not closed unless they are finite, see Remark 5.4, and for this reason in Section 5 we introduce the continuous Weyl groupoids \( \mathcal{W}^c \) and \( \mathcal{W}^c_* \). The action of a groupoid \( \mathcal{G} \) on an affine variety \( X \) and the invariant ring \( O(X)^{\mathcal{G}} \) are defined in Section 5. In Proposition 5.9 we show there are actions of \( \mathcal{W}^c \) and \( \mathcal{W}^c_* \) on \( h^* \) and \( T \) respectively such that

\[
I(h) = S(h)^{2\mathcal{W}^c} \text{ and } J(G) \otimes_k k = O(T)^{2\mathcal{W}^c_*}.
\]

**Notation 1.1.** We assume one of the following holds

(i) \( X = h^*, \mathcal{G} = \mathcal{W}^c, B = O(X)^W = S(h)^W \) and \( A = I(h) \).

(ii) \( X = T, \mathcal{G} = \mathcal{W}^c_*, B = O(X)^W \) and \( A = J(G) \otimes_k k \).

**Example 1.2.** If \( g = gl(m|n) \) the algebras \( I(h) \) and \( J(G) \otimes_k k \) are the algebras of supersymmetric polynomials and Laurent supersymmetric polynomials over \( k \) respectively. These algebras have very beautiful combinatorics, see [Mus12] Chapter 12, [PT92] and [Ser19]. Surprisingly though, the algebras of (Laurent) supersymmetric polynomials have received little attention from a geometric point of view, perhaps because they are not Noetherian.

In the simplest case where \( m = n = 1 \), \( \mathcal{W} \) is the smallest groupoid that is not
a disjoint union of groups. Thus $\mathfrak{M}$ has two objects, and two non-identity morphisms. To describe the action of $\mathfrak{M}$ on the plane $\mathfrak{h}^* = \mathfrak{k}^{11}$, take the objects to be $\pm a$ where $a \neq 0$ is a point in $\mathfrak{h}^*$. The functor sends $\pm a$ to the line $L$ they span, and the non-identity morphisms add or subtract $a$ to a point on $L$. The invariant ring $S(\mathfrak{h})^{\mathfrak{M}}$ consists of all polynomial functions that are constant on $L$. So $S(\mathfrak{h})^{\mathfrak{M}} = \mathbb{k} + T S(\mathfrak{h})$ where $S(\mathfrak{h}) = \mathbb{k}[S, T]$ and $L$ is the line defined by $T$. The algebra $S(\mathfrak{h})^{\mathfrak{M}}$ is a well-known example of a non-Noetherian domain of Krull dimension two. This answers a question of S. Paul Smith.

1.1.4 The Duflo-Serganova functor.

We recall the definition of the Duflo-Serganova functor $DS_x$, in the case that $x$ is a root vector such that $[x, x] = 0$. For $M$ as above, set $M_x = \text{Ker } M x/x M$. Then $M_x$ is a module for $\mathfrak{g}_x$ where $\mathfrak{g}_x = \text{Ker } x/\text{Im } x$, which is a Lie superalgebra of smaller rank than $\mathfrak{g}$. Although $\mathfrak{g}_x$ is a quotient of a subalgebra of $\mathfrak{g}$, it was shown in [DS05] Lemma 6.3, that it embeds in $\mathfrak{g}$ at least in the KM case. We follow the description of [HR18] section 2C in the case that $x$ is a root vector corresponding to the isotropic root $\beta$.

Lemma 1.3. Let $x$ be a root vector corresponding to the isotropic root $\beta$. We can identify $\mathfrak{g}_x$ with a subalgebra of $\mathfrak{g}$ such that the centralizer $\mathfrak{g}^x$ of $x$ in $\mathfrak{g}$ is a semidirect sum $\mathfrak{g}^x = \mathfrak{g}_x \ltimes [x, \mathfrak{g}]$. The subalgebra $\mathfrak{g}_x$ is given by

$$\mathfrak{g}_x = \mathfrak{h}(\beta) \oplus \bigoplus_{\alpha \in \Delta(\beta)} \mathfrak{g}^\alpha, \quad (1.4)$$

where $\Delta(\beta) = \{ \alpha \in \Delta | (\alpha, \beta) = 0, \alpha \neq \pm \beta \}$, and $\mathfrak{h}(\beta) = \text{span } \{ h_\alpha | \alpha \in \Delta(\beta) \}$. We have $\mathfrak{h}(\beta) = \mathfrak{g}_x \cap \mathfrak{h}$.

Lemma 1.4. The restriction of $(\ , \ )$ to $\text{span } \Delta(\beta)$ is non-degenerate.

Proof. Let $\{ \alpha_1, \ldots, \alpha_r \}$ be a basis for $\text{span } \Delta(\beta)$ and suppose $\sigma \in \mathfrak{h}^*$ with $(\sigma, \beta) = 1$. For $i \in [r]$, let $\alpha'_i = \alpha_i - (\sigma, \alpha_i)$. The Gram matrix of $(\ , \ )$ using the basis $\{ \alpha_1, \ldots, \alpha_r, \beta, \sigma \}$ for $\mathfrak{h}^*$ has the same determinant as the Gram matrix using the basis where each $\alpha_i$ is replaced by $\alpha'_i$. Since the latter matrix is block diagonal and $(\alpha_i, \alpha_j) = (\alpha'_i, \alpha'_j)$, the result follows. □

Under the isomorphism $\mathfrak{h}^* \longrightarrow \mathfrak{h}^*$, span $\Delta(\beta)$ maps to span $\{ h_\alpha | \alpha \in \Delta(\beta) \}$ which we identify with $\mathfrak{h}(\beta)$. We have span $\Delta(\beta) = \mathfrak{h}(\beta)^*$. If $(\sigma(\beta), \beta) = 1$, we have an orthogonal direct sum

$$\mathfrak{h}^* = \mathfrak{h}(\beta)^* \oplus \text{span } \{ \beta, \sigma(\beta) \}. \quad (1.5)$$

In the table below, taken from [GHSS22] Section 4.5, the first column lists the Lie superalgebras we consider, $x$ is a root vector, and the second column describes $\mathfrak{g}_x$ up to isomorphism. If $\mathfrak{g} = \mathfrak{p}(n)$, we assume that $x \in \mathfrak{g}^{e_1 + e_2}$ in the notation of [Mus12],
Since we use induction on the rank as a proof technique, we make some brief remarks about the base cases. These occur when $g_x$ has no isotropic roots. Then the Weyl groupoid for $g_x$ is just the Weyl group. The desired results hold trivially in these cases. By convention $\operatorname{gl}(0|k) = \operatorname{gl}(k|0) = \operatorname{gl}(k)$, and other low dimensional entries in the column for $g_x$ should be interpreted similarly.

The main use we make of the functor $DS_x$ is to construct certain algebra maps $ds_x$ in Subsections 3.1 and 3.2. These algebra maps can be realized in two equivalent ways, and both realizations can be found in the literature, at least implicitly. Both methods can be easily adapted to obtain maps on suitable overrings as in the diagram preceding Lemma 2.7. In the situation of Lemma 1.3, $h^*$ has been identified with a subspace of $h^*$, so we have a restriction map $\text{res} : S(h) \rightarrow S(h_x), f \mapsto f|_{h^*}$. This map descends to $S(h)^W$ and $I(h)$ and we have $\text{res} = ds_x : I(h) \rightarrow I(h_x)$. On the other hand the algebra $I(h)$ can be defined in terms of polynomials satisfying certain partial evaluation conditions, and then it is natural to view $ds_x$ as evaluation $\text{ev} : I(h) \rightarrow I(h_x)$). Similar remarks apply to $ds_x : J(G) \rightarrow J(G_x)$. 

1.1.5 Harish-Chandra pairs and representations of Lie supergroups.

An (algebraic) Lie supergroup $G$ can be defined using a Hopf superalgebra $O(G)$ and the functor of points. Denote the category of supercommutative $k$-algebras $\text{Alg}$. Then for any supercommutative algebra $A$, the $A$-points of $G$ are given by

$$G(A) = \text{Mor}_{\text{Alg}}(O(G), A).$$

The Hopf superalgebra structure on $O(G)$ induces a natural group structure on $G(A)$. Another useful approach (due to Kostant [Kos77] and Koszul [Kos83]) to $G$ uses Harish-Chandra pairs. Following [CCFT11] Definition 7.1, a Harish-Chandra pair (HCP) is a pair $(G_0, g)$ where consisting of an algebraic group $G_0$ and a Lie superalgebra $g$ such that

(a) $g_0 = \text{Lie } G_0$,

(b) There is a representation $\rho$ of $G_0$ on $g$ such that $\rho(G_0)|_{g_0} = \text{Ad}$ and the differential of $\rho$ acts on $g$ as the adjoint representation: for $X \in g_0, Y \in g$, $d\rho(X)Y = [X, Y]$. 

\[ \begin{array}{|c|c|} 
\hline
\mathfrak{g} & \mathfrak{g}_x \\
\hline
\mathfrak{gl}(m|n) & \mathfrak{gl}(m-1|n-1) \\
\mathfrak{sl}(m|n), m \neq n & \mathfrak{sl}(m-1|n-1) \\
\mathfrak{osp}(m|2n) & \mathfrak{osp}(m-2|2n-2) \\
D(2|1; \alpha) & k \\
F(4) & \mathfrak{sl}(3) \\
G(3) & \mathfrak{sl}(2) \\
\mathfrak{q}(n) & \mathfrak{q}(n-2) \\
\mathfrak{p}(n) & \mathfrak{p}(n-2) \\
\hline
\end{array} \]
We write \((G_0, \mathfrak{g}, \rho)\) if we want to stress the role of \(\rho\). Harish-Chandra pairs form a category, for morphisms see [CCF11] Definition 7.4.2. Given a Lie supergroup \(G\), we obtain a HCP \((G_0, \text{Lie} \, G, \text{Ad})\). Conversely given a HCP, we can put a Hopf superalgebra structure on 
\[ \mathcal{O}(G) := \text{Hom}_{U(\mathfrak{g}_0)}(U(\mathfrak{g}), \mathcal{O}(G_0)), \]  
(1.6)
and thus obtain a Lie supergroup \(G\). In (1.6), \(U(\mathfrak{g})\) is naturally a left \(U(\mathfrak{g}_0)\)-module, and the action of \(x \in \mathfrak{g}_0\) on \(\mathcal{O}(G_0)\) is given by a left-invariant differential operator, [CCF11] page 125. This is the basis for the following result, see [CCF11] Theorem 7.4.5.

**Theorem 1.5.** There is an equivalence of categories between the category of Lie supergroups and the category of Harish-Chandra pairs.

The connection between representations of Harish-Chandra pairs and the corresponding Lie supergroup was explained in [CC11]. A representation of a HCP \((G_0, \mathfrak{g})\) consists of the following data

(a) A representation \(\Sigma : G_0 \rightarrow GL(V)\),

(b) A representation \(\sigma : \mathfrak{g} \rightarrow \mathfrak{gl}(V)\), such that \(d\Sigma = \sigma|_{\mathfrak{g}_0}\) and a further compatibility condition given in [CC11] Definition 6 (2) holds.

**Theorem 1.6.** There is a bijection between representations of the HCP \((G_0, \mathfrak{g})\) and representations of \(G\) on \(V\).

**Proof.** See [CC11] Proposition 6. 

Let \(\mathcal{F}_\mathfrak{g}\) (resp. \(\mathcal{F}_G\)) be the category of finite dimensional \(\mathbb{Z}_2\)-graded \(\mathfrak{g}\)-modules (resp. \(G\)-modules), and \(\Pi\) the parity change functor on \(\mathcal{F}_\mathfrak{g}\). The extra condition mentioned above essentially says that the representations of \(G\) that arise correspond to representations of \(\mathfrak{g}_0\) that are integrable as \(G_0\)-modules. For example, suppose that \(\mathfrak{g}_0\) has a non-trivial center. If \(\mathfrak{a}\) is an abelian Lie algebra and \(T\) is a torus with \(\text{Lie} \, T = \mathfrak{a}\), then integrable representations of \(\mathfrak{a}\) correspond to elements of the character group \(X(T)\). Hence \(\mathfrak{g}_0\) has many more finite dimensional modules than \(G_0\), and these modules can be induced to give graded modules for \(\mathfrak{g}\). Thus the category \(\mathcal{F}_G\) is more manageable than \(\mathcal{F}_\mathfrak{g}\). In addition, the results we quote from [HR18], [IRS21] and [Rei22] work best for supergroups. Let \(\mathcal{F} = \mathcal{F}_G\) or \(\mathcal{F}_\mathfrak{g}\). The Grothendieck group \(K(\mathcal{F})\) of \(\mathcal{F}\) is the free abelian group on the symbols \([M]\), where \(M\) is an object in \(\mathcal{F}\) modulo the relations that \([M] = [L] + [N]\) whenever there is an exact sequence \(0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0\) in \(\mathcal{F}\). Since \(\mathcal{F}\) is closed under tensor product, \(K(\mathcal{F})\) has a ring structure determined by \([M][N] = [M \otimes N]\). The supercharacter ring (resp. character ring) is the factor ring of \(K(\mathcal{F}_\mathfrak{g})\) by the ideal generated by all \([\Pi(M)] + [M]\) (resp. \([\Pi(M)] - [M]\)) where \(M \in \mathcal{F}_\mathfrak{g}\). For the classical Lie superalgebras \(\mathfrak{g}\) listed in Subsection 1.1.1 we define \(J(\mathfrak{g})\) to be the character ring of \(\mathfrak{g}\) if \(\mathfrak{g}\) has type \(q\) and the supercharacter ring in all other cases. The \(\mathbb{Z}\)-algebra \(J(G)\) is the factor ring of \(K(\mathcal{F}_G)\) defined in a similar way. The Lie superalgebras \(\mathfrak{g}\) listed in 1.1.1 such that
$g_0$ has a non-trivial center are $g = \mathfrak{gl}(m|n), \mathfrak{sl}(m|n)$ and $g = \mathfrak{osp}(2|2n)$. In these cases $J(g)$ is graded by an abelian group, and $J(G)$ is the identity component of this grading, see [SV11] Propositions 7.1, 7.2, 7.3 and 7.5.

We mention an immediate consequence of the weak Nullstellensatz for $Z(g)$, Theorem 4.2. Let $g$ be any classical Lie superalgebra. A central character of $g$ is an algebra map $Z(g) \rightarrow k$, [GHSS22] Section 3. Any central character $\chi$ determines, and is determined by the maximal ideal $\text{Ker} \chi$. An element $z \in Z(g)$ acts on any Verma module $M(\lambda)$ as a scalar $\chi_\lambda(z)$ and $\chi_\lambda$ is called the central character afforded by $M(\lambda)$, [Mus12] 8.2.4. The next result proves and extends Conjecture 13.5.1 from [Mus12].

**Corollary 1.7.** Any central character of $g$ is afforded by a Verma module.

This paper replaces the unpublished preprint [Mus19] which dealt only with the cases where $g = \mathfrak{gl}(m|n)$ or $\mathfrak{osp}(r|2n)$. I am grateful to Maria Gorelik, Hanspeter Kraft and Shifra Reif for helpful correspondence.

## 2 General Results.

Throughout $[k]$ denotes the set of the first $k$ positive integers and iff means if and only if. We start with some general results on commutative rings.

### 2.1 Certain Ring Extensions.

In this Subsection we assume the following hypothesis on the ring extension $A \subseteq B$.

**Hypothesis 2.1.** For some $T \in A$, a non-zero divisor in $B$, we have $TB \subseteq A$.

If $I$ is an ideal of $A$, let

$$\text{rad} (I) = \{ f \in A | f^n \in I, \text{ for some } n > 0 \}.$$

If $I = \text{rad} (I)$ we say $I$ is a radical ideal. Suppose $R$ is a commutative ring and $T \in R$, is a non-zero divisor. Denote the localization by $R_T$.

**Lemma 2.2.** Assume Hypothesis 2.1

(a) We have an equality of localizations $A_T = B_T$.

(b) If $P$ is a radical ideal of $A$ with $T \in P$, then $TB \subseteq P$.

**Proof.** We leave (a) as an exercise. Then (b) holds since $(TB)^2 \subseteq AP = P$. 

If $L$ is an ideal of $R$ define the extension of $L$ to $R_T$ to be $L^e = L_T$, the localization of $L$. If $M$ is an ideal of $R_T$, set $M^c = M \cap R$, the contraction of $M$ to $R$. Then extension and contraction provide order preserving bijections

$$\{ P \in \text{Spec } R | T \notin P \} \leftrightarrow \text{Spec } R_T. \tag{2.1}$$
This applies to $R = A$ or $B$ in Lemma 2.2 so we have a bijection

$$\{P \in \text{Spec } B | T \notin P\} \leftrightarrow \{P \in \text{Spec } A | T \notin P\},$$

(2.2)
given by $P \mapsto P \cap A$. From now on we identify the two sets on either side of (2.1).

**Theorem 2.3.** Let $\phi : A \rightarrow A/\mathfrak{T} = C$ be the natural map. Then we have a disjoint union

$$\text{Spec } A = \text{Spec } B_T \cup \phi^{-1}(\text{Spec } C).$$

**Proof.** As noted above we have $A_T = B_T$. For a prime ideal $Q$ of $A$ there are two possibilities. If $T \notin Q$, then $Q_T$ is a prime ideal of $A_T = B_T$ such that $Q_T \cap A = Q$. If $T \in Q$, then $Q$ is the inverse image of the prime ideal of $Q/B_T$ under $\phi$. □

**Corollary 2.4.** If $m$ is a maximal ideal of $A$ and $T \notin m$, the maximal ideal $M$ of $B$ given by $M = m_T \cap B$ satisfies $m = M \cap A$. Also $mB \neq B$.

**Proof.** This first statement follows from Equation (2.2), and the second is an immediate consequence. □

In Corollary 2.4 if $B = \mathcal{O}(X)$ for an affine variety $X$, with $B$ a finitely generated $k$-algebra, then $m$ and $M$ have the same set of zeroes in $X$, so by the usual Nullstellensatz $M = \text{Rad } mB$. However $M$ can strictly contain $mB$, see [Mus22] Example 2.2.

**Lemma 2.5.** In the situation of Theorem 2.3, suppose that the rings $B_T$ and $C$ satisfy the ascending chain condition on radical ideals. Then so does $A$.

**Proof.** Let $R_1^i \subseteq R_2^i \subseteq \ldots$ be an ascending chain of radical ideals in $A$. Then $R_1^i \subseteq R_2^i \ldots$ is an ascending chain of radical ideals in $A_T = B_T$, so by assumption, there is an $m$ such that $R_T^m = R_T^i$ for all $i \geq m$. Write $R_T^m = p_1 \cap p_2 \cap \ldots \cap p_r$ for some prime ideals of $B_T$. We can assume that this intersection is irredundant, and all the $p_k$ are minimal over $R_T^m$. Then if $P_k = p_k \cap A$ and $i \geq m$, the $P_k$ are exactly the minimal primes over $R^i$ that do not contain $T$. By [Eis95] Corollary 2.12 every radical ideal in a commutative ring is the intersection of (a possibly infinite number of) prime ideals. So for $i \geq m$, write $R^i$ as an intersection of prime ideals in $A$. Say

$$R^i = P_1 \cap \ldots \cap P_r \cap \bigcap_{j \in \Lambda_i} Q_j,$$

where the $Q_j$ are prime ideals that contain $TB$. Thus $D_i = \bigcap_{j \in \Lambda_i} Q_j$ is a radical ideal in $A$ containing $TB$, and if $D_k = D_i/\mathfrak{T}B$, then $D_1 \subseteq D_2 \subseteq \ldots$ is an ascending chain of radical ideals of $C$. So there is an $n \geq m$, such that $D_i = D_n$ for all $i > n$ and then $R^i = R^n$. □

**Corollary 2.6.** With the same assumptions as the Lemma, every radical ideal $I$ of $A$ is a finite intersection of prime ideals. These prime ideals can be taken to be the prime ideals minimal over $I$.

**Proof.** See [Kap74] Theorem 87. □
2.2 Towards the Weak Nullstellensatz.

Consider a ring extension $A \subseteq B$, and $X \subseteq \text{Max } A$. We say the Relative Weak Nullstellensatz (RWN) for $X$ and the pair $(A,B)$, if for every $m \in \text{Max } A$ we have $mB \neq B$. If the RWN holds in the case $X = \text{Max } A$, we say simply that RWN holds for the pair $(A,B)$. If $A, B$ are $k$-algebras, with $B$ finitely generated and $M \in \text{Max } B$, then $B/M \cong k$, so if $m = M \cap A$, then
\[
k \hookrightarrow A/m \cong (A + M)/M \subseteq B/M \cong k.\tag{2.3}
\]

Thus $m \in \text{Max } A$ and we have a map
\[
\psi : \text{Max } B \rightarrow \text{Max } A, \quad M \mapsto M \cap A.\tag{2.4}
\]

If RWN holds for the pair $(A,B)$, and $m \in \text{Max } A$, then by Zorn’s Lemma there is a $M \in \text{Max } B$ containing $mB$, we have $M \cap A = m$, so the map in (2.4) is surjective.

Consider the following diagram of ring homomorphisms where the vertical maps are inclusions, and $\phi$ is the restriction of $\Phi$ to $A$.

\[
\begin{array}{ccc}
A & \xrightarrow{\phi} & A' \\
\downarrow & & \downarrow \\
B & \xrightarrow{\Phi} & B'
\end{array}
\]

**Lemma 2.7.** Suppose $\Phi$ and $\phi$ are onto and RWN holds for the pair $(A',B')$. Then the RWN holds for the pair $(A,B)$ and the set $X = \phi^{-1}(\text{Max } A)$.

**Proof.** Suppose $m \in X$, that is $m$ is a maximal ideal of $A$ containing $\text{Ker } \phi$. Then $\phi(m)$ is a maximal ideal of $A'$, so by assumption $J = \phi(m)B'$ is a proper ideal of $B'$. Since $\Phi(mB) = J$, it follows that $mB$ is a proper ideal of $B$. \qed

**Lemma 2.8.** Suppose the finite group $G$ acts on $S$ and assume $|G|$ is a unit in $S$. If RWN holds for the pair $(R,S)$, then the RWN holds for the pair $(R^G, S^G)$.

**Proof.** In this situation we have a Reynolds operator $R \rightarrow R^G$. Regarded as a $G$-module $R^G$ is the isotypic component of $R$ corresponding to the trivial module. Let $R^+$ be the sum of the remaining isotypic components. Thus $R = R^G \oplus R^+$ is a decomposition into $R^G$-modules. If $m \in \text{Max } R^G$, then $mR = m \oplus mM^+$ is a proper ideal of $R$, so by assumption $mS$ is a proper ideal of $S$. The result follows since $mS^G = S^G$ implies $mS = S$. \qed

RWN does not hold always when $S = k[x,y]$ is a polynomial algebra in 2 variables: if $R = k[x] + (xy - 1)S$, then $m = xk[x] + (xy - 1)S$ is a maximal ideal of $R$ such that $mS = S$. 

9
3 Applications.

In this Section if $A = I(h)$ or $J(G) \otimes_{\mathbb{k}} \mathbb{k}$ we consider a ring extension $A \subset B$ and an element $T \in A$ such that $TB \subset A$. Thus Hypothesis \ref{hypothesis} holds. Because we will need to change the Lie superalgebra, we sometimes write $A = A(g)$. The algebra $B$ is either $S(h)W$ or $O(T)W$, so is in particular a finitely generated $\mathbb{k}$-algebra. To prove the weak Nullstellensatz we consider a map with kernel $T$, $A(g) -\rightarrow A(gx)$, where $gx$ is as in the table from Subsection \ref{table}, and use induction on the rank of $g$.

3.1 Application to $I(h)$.

By \cite{GHSS22} Section 3, the functor $DS_x$ induces an algebra map $Z(g) -\rightarrow Z(gx)$. Using the Harish-Chandra map, this gives rise to a map $ds_x : I(h) -\rightarrow I(hx)$. Because of (1.4) and the above remarks, this map can be thought of as restriction of functions $\text{res}(f) = f|_{h^*_x}$.

3.1.1 The Kac-Moody case.

In the KM case, if $g_1$ is a simple $g_0$-module, let $\Omega = \Delta_{\text{iso}}$. Otherwise $g_1 = g_1^+ \oplus g_1^-$ is a direct sum of two simple $g_0$-modules, and we let $\Omega = \Delta^+_1$ be the set of roots of $g_1^+$. Then $W$ acts transitively on $\Omega$. Define $T = \prod_{\beta \in \Omega} h_\beta$. Then $T$ is $W$-invariant. We describe an automorphism $\sigma$ on $g_x$ which we are going to use below.

(a) For $g = \mathfrak{osp}(2m|2n)$, then $\sigma$ is the diagram automorphism of $g_x$ described in \cite{Mus12} Lemma 5.5.12. This automorphism preserves $h$.

(b) For $g = F(4)$, then $\sigma$ is induced by an involution of the Dynkin diagram for $g_x = \mathfrak{sl}(3)$.

(c) for $g = D(2|1;\alpha)$ we have $g_x = \mathbb{k}$ and $\sigma = -\text{Id}$.

(d) In all other cases $\sigma = \text{Id}$.

Theorem 3.1. We have

(a) $TS(h)^W \subset I(h)$ and the kernel of $\text{res} : I(h) \rightarrow I(h_x)$ equals $TS(h)^W$.

(b) We have $\text{res}(I(h)) = I(h_x)^{\sigma_x}$.

Proof. By construction $T$ is $W$-invariant and $T$ vanishes on all hyperplanes $\Pi_\alpha$. Thus by (1.2), $TS(h)^W \subset \text{Ker res}$. Suppose $x = e_\beta$ is a root vector and $f \in \text{Ker res}$. Then $h_\beta$ divides $f$ in $S(h)$, so by $W$-invariance, so does $T$. Write $f = gT$ with $g \in S(h)$. Since $f,T$ are $W$-invariant, so is $g$ proving (a). For (b) see \cite{Gor20} Theorem 6.4.

\[ \square \]
3.1.2 Algebras of type $q$

There are four related Lie superalgebras of type $q$. We refer to [Rei22] 2.1, [CW12] for full details, but mention that $q(n)$ is constructed as a subalgebra of $\mathfrak{gl}(n|n)$, and has derived subalgebra $\mathfrak{sq}(n)$. We have $q(n)_0 \cong \mathfrak{gl}(n)$ and $\mathfrak{sq}(n)_0 \cong \mathfrak{sl}(n)$. The algebras $q(n), \mathfrak{sq}(n)$ have factor algebras $pq(n)$ and $psq(n)$ respectively. The algebra $psq(n)$ is simple for $n \geq 3$. By [Gor06] Theorem 13.1.

$$Z(q(n)) \cong \{ f \in S(\mathfrak{h})^W | f|_{x_n = -x_{n-1} = t} \text{ is independent of } t \} := I_n. \quad (3.1)$$

Now (3.1) shows that if $T = \prod_{i<j}(x_i + x_j)$, then $T \in A = I_n$. Set $B = S(\mathfrak{h})^W$.

Define $f \text{ ev} : I_n \rightarrow I_{n-2}$ by $ev(f) = f|_{x_n = -x_{n-1} = 0}$.

**Theorem 3.2.** We have

(a) $TB \subset I_n$ and the kernel of $ev : I_n \rightarrow I_{n-2}$ equals $TB$.

(b) We have $ev(I_n) = I_{n-2}$

**Proof.** (a) follows as in the proof of Theorem 3.1 (a). For (b) see [Gor20] Proposition 5.8.3.

To unify notation, we set $I(\mathfrak{h}) = I_n$ if $\mathfrak{g} = q(n)$. In this case, by [Mus12] Lemma 12.1.1.

$$I(\mathfrak{h}) = \{ f \in S(\mathfrak{h})^W | f(\lambda) = f(\lambda + t\alpha) \text{ for all } \alpha \in \Delta^+, \lambda \in \Pi_\alpha \text{ and } t \in k \} \quad (3.2)$$

3.1.3 Algebras of type $p$

We are interested in two Lie superalgebras of type $p$. We refer to [CW12] for details, but recall that $p(n)$ is constructed as a subalgebra of $\mathfrak{gl}(n|n)$, and has derived subalgebra $\mathfrak{sp}(n)$. We have $p(n)_0 \cong \mathfrak{gl}(n)$ and $\mathfrak{sp}(n)_0 \cong \mathfrak{sl}(n)$. The algebra $psq(n)$ is simple for $n \geq 3$. If $\mathfrak{g} = p(n)$, then by [Gor01] Theorem 4.1, $Z(\mathfrak{g})$ contains an element $t$ such that $t^2 = 0$ and there is a linear isomorphism $Z(\mathfrak{g}) \cong k \oplus tS(\mathfrak{h})^W$. This means that $Z(\mathfrak{g})$ has a unique prime ideal and the methods of Subsection 2.1 do not apply to the study of $Z(\mathfrak{g})$. For this reason $p(n)$ is excluded in some results. Corollary L.7 holds when $\mathfrak{g} = p(n)$, since there is only one central character. The ring $J(G)$ when $G$ is the Lie supergroup $P(n)$ remains of interest.

3.2 Application to $J(G) \otimes_k \mathbb{Z}$

3.2.1 Lie supergroups

We make a basic hypothesis, compare [GHSS22] Section 8.

**Hypothesis 3.3.** Assume one of the following holds

(i) If $\mathfrak{g} = \mathfrak{gl}(m|n)$, set $G_0 = GL(m) \times GL(n)$.
(ii) If \( g = \mathfrak{osp}(r|2n) \), set \( G_0 = SO(r) \times SP(2n) \), where \( r = 2m + 1 \) or \( 2m \).

In cases (i) and (ii) set

\[
P = \sum_{i=1}^{m} \mathbb{Z} \epsilon_i + \sum_{i=1}^{n} \mathbb{Z} \delta_i.
\]

(iii) If \( g = \mathfrak{sl}(m|n) \) \( m \neq n \), set \( G_0 = \{(A, B) \in GL(m) \times GL(n)| \det A = \det B \} \) and let \( P \) be the root lattice of \( g \).

(iv) If \( g = \mathfrak{p}(n) \), set \( G_0 = \{(A, A^{-t}) \in GL(n) \times GL(n)\} \).

(v) If \( g = \mathfrak{q}(n) \), set \( G_0 = \{(A, A) \in GL(n) \times GL(n)\} \).

In (iv) \( A^{-t} \) is the transpose of \( A^{-1} \). Suppose (iv) or (v) holds. The simple roots are \( \alpha_i = \epsilon_i - \epsilon_{i+1}, i \in [n] \). We use the bilinear form on \( \mathfrak{h}^* \) given by \( (\epsilon_i, \epsilon_j) = \delta_{i,j} \) and set

\[
P = \sum_{i=1}^{n} \mathbb{Z} \epsilon_i.
\]

Recall that \( \mathbb{T} \) is a maximal torus in \( G_0 \).

**Lemma 3.4.**

(a) There is a Harish-Chandra pair \((G_0, g)\) and the Lie supergroup determined by (1.6) is \( G = GL(m|n), OSP(r|2n), SL(m|n), P(n) \) or \( Q(n) \) in cases (i)-(v) respectively.

(b) There is an isomorphism \( P \rightarrow X(\mathbb{T}) \), which we write as \( \alpha \rightarrow e^\alpha \). Here \( e \) is a formal symbol used to convert to multiplicative notation.

**Proof.** The possible weights of \( G_0 \)-modules (and hence also \( G \)-modules) are determined by \( P \). Thus the result follows from the explicit description of the characters or supercharacters of these modules given in [SV11], [IRS21] and [Rei22].

If \( g = \mathfrak{p}(n) \) or \( \mathfrak{q}(n) \), and \( \alpha = \epsilon_i - \epsilon_j \in \Delta_0^+ \), the set of positive even roots, set \( \overline{\alpha} = \epsilon_i + \epsilon_j \) and define in place of (1.1),

\[
\Pi_\alpha = \{\lambda \in \mathfrak{h}^* | (\lambda, \overline{\alpha}) = 0\}.
\]

In the KM case, an iso-set is a linearly independent set of mutually orthogonal isotropic roots. If \( g = \mathfrak{p}(n) \) or \( \mathfrak{q}(n) \), an iso-set is a linearly independent subset \( A \) of \( \Delta_0^+ \) such that \( (\alpha, \overline{\beta}) = 0 \) for all \( \alpha, \beta \in A \). The non-standard convention that \( \Delta_{iso} = \Delta_0^+ \) in types \( p \) and \( q \), allows us to give unified statements of certain results. Let \( P \subset \mathfrak{h}^* \) be as in Hypothesis (3.3).

For all \( \alpha \in P \), define the subtorus \( T_\alpha \) of \( T \) by

\[
T_\alpha = \text{Ker } e^\alpha \text{ in cases (i)-(iii), or } T_\alpha = \text{Ker } e^{\overline{\alpha}} \text{ in cases (iv)-(v)}.
\]

Composition gives a non-degenerate pairing \( X(\mathbb{T}) \times Y(\mathbb{T}) \rightarrow \text{Hom}(\mathfrak{k}^*, \mathfrak{k}^*) \cong \mathbb{Z} \). The next result relates this pairing to the bilinear form on \( P \).

**Lemma 3.5.** For each root \( \beta \) there is a unique \( c_\beta \in Y(\mathbb{T}) \) such that

\[
e^\alpha c_\beta(t) = t^{(\alpha, \beta)} \text{ for all roots } \alpha \text{ and } t \in \mathfrak{k}^*.
\]
Proof. First suppose $G = GL(m|n)$ or $OSP(r|2n)$, with $r = 2m + 1$ or $2m$ and let $p = m + n$. We show (3.5) holds for all $\beta \in P$. It is convenient to relabel the ordered basis of $P$ as follows
\[(\alpha_1, \ldots, \alpha_p) := (\epsilon_1, \ldots, \epsilon_m, \delta_1, \ldots, \delta_n).\]
Then the dual basis is
\[(\varpi_1, \ldots, \varpi_p) := (\epsilon_1, \ldots, \epsilon_m, -\delta_1, \ldots, -\delta_n).\]
There are unique $c_1, \ldots, c_p \in Y(T)$ such that $e^{\alpha_i} c_j(t) = t^{\delta_i,j}$. If $\beta \in P$, write $\beta = \sum a_i \varpi_i$ with $a_i \in \mathbb{Z}$. Then set $c_\beta = \prod a_j^{c_j}$. It is enough to show (3.5) for $\alpha = \alpha_i$, but in this case we have, for $t \in k^*$ we have
\[e^{\alpha_i} c_\beta(t) = t^{\alpha_i} = t^{(\alpha_i, \beta)}.\]
If $G = SL(m|n)$ the distinguished set of simple roots is
\[\alpha_i = \epsilon_i - \epsilon_{i+1}, \quad i \in [m-1]\]
\[\alpha_m = \epsilon_m - \delta_1,\]
\[\alpha_{m+i} = \delta_i - \delta_{i+1}, \quad i \in [m-1]\]
Set $x_i = e^{\epsilon_i}, y_i = e^{\delta_i}$. Then for a simple root $\alpha$, $e^{\alpha} = x_i x_{i+1}^{-1}, x_m y_1^{-1}$ and $y_i y_{i+1}^{-1}$ in the three cases listed above. For a root $\beta$ define $c_\beta \in Y(T)$, by
\[c_\beta(t) = (t^{(\beta, \epsilon_1)}, \ldots, t^{(\beta, \epsilon_m)}, t^{(\beta, \delta_1)}, \ldots, t^{(\beta, \delta_n)}).\]
It suffices to show (3.5) for $\alpha, \beta$ for simple roots and this is done with a short computation. \qed

**Lemma 3.6.** If $G = P(n)$ or $Q(n)$, then for all $\beta \in \Delta_0^+$ there is a unique $c_\beta \in Y(T)$ such that
\[e^{\varpi} c_\beta(t) = t^{(\varpi, \beta)} \text{ for all } \alpha \in \Delta_0^+ \text{ and } t \in k^*.\] (3.6)

**Proof.** Define $c_\beta \in Y(T)$ by $c_\beta(t) = (1, \ldots, t, \ldots, t^{-1}, \ldots, 1)$, where $t, t^{-1}$ are in positions $i$ and $j$. The result holds by a short computation. \qed

In the KM case, it was shown by Hoyt and Reif, [HR18] Proposition 8, that the functor $DS_x$ induces a ring homomorphism $ds_x : J(G) \rightarrow J(G_x)$. Our next aim is to describe the image and kernel of the map $ds_x$, and deduce that the inclusion $\mathcal{A} = J(G) \otimes \mathbb{k} \subseteq \mathcal{B} = \mathbb{Z}[X(T)]^W \otimes \mathbb{k}$ satisfy Hypothesis 2.1. Though not strictly necessary for all the results we cite, we extend scalars to $k$. This brings out the analogy with $I(\mathfrak{h})$. Note that $\mathbb{Z}[X(T)] \otimes \mathbb{k} = \mathcal{O}(T)$. 

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3.2.2 Properties of $ds_x$.

In the KM case, set $\rho_{\text{iso}} = \frac{1}{2} \sum_{\alpha \in \Delta^+_{\text{iso}}} \alpha$.

**Theorem 3.7.** (a) There is an element $T \in B$ such that $\text{Ker } ds_x = TB \subset A$. In particular Hypothesis [2.1] holds.

(b) Let $G$ be one of the Lie supergroups $SL(m|n), m \neq n, GL(m|n), \text{ or } OSP(r|2n)$. Then $ds_x : J(G) \rightarrow J(G_x)$ is surjective.

**Proof.** (a) Define $T$ by

$$T = e^{\rho_{\text{iso}}} \prod_{\alpha \in \Delta^+_{\text{iso}}} (1 - e^{-\alpha}).$$

In the notation of [HR18] Lemma 16 we have $T = k(\rho_{\text{iso}})$. So the result follows from the cited Lemma and [HR18] Theorem 17. Also (b) is [HR18] Theorem 20. \qed

3.2.3 The case of $Q(n)$.

The case $G = Q(n)$ is studied in [Rei22]. Here are some relevant results. In this and the following Subsection, let $T_n$ be a maximal torus in $Q(n) = GL(n)$. Then [Rei22] Proposition 3, gives an explicit description of a ring $J_n$ isomorphic to $J(Q(n))$. For our purposes it is convenient to extend scalars to $\mathbb{Z}[\frac{1}{2}]$. Thus we define

$$J_n = \{ f \in \mathbb{Z}[\frac{1}{2}]|X(T_n)|^W |f|_{x_n = -x_{n-1} = t} \text{ is independent of } t \}. \quad (3.7)$$

With this definition $J_n \cong J(Q(n)) \otimes \mathbb{Z}[\frac{1}{2}]$. Using this isomorphism, $ds_x$ identifies with an evaluation map $A = J_n \rightarrow J_{n-2} = A'$, given by $ev(f) = f|_{x_n = -x_{n-1} = 1}$. Note that $A$ is contained in $B = \mathbb{Z}[\frac{1}{2}]|X(T_n)|^W$. So we have a diagram as in Subsection 2.2, where the map $\phi = ev$, and the map $\Phi : B \rightarrow B' = \mathbb{Z}[\frac{1}{2}]|X(T_{n-2})|^W$ is defined by

$$\Phi(f) = f|_{x_n = -x_{n-1} = 1}.$$ 

**Theorem 3.8.** (a) The map $ev : J_n \rightarrow J_{n-2}$ is surjective.

(b) Define $T \in B = \mathbb{Z}[\frac{1}{2}]|X(T_n)|^W$, by $T = \prod_{i<j} (x_i + x_j)$. Then $TB \subset A$. That is Hypothesis [2.7] holds. Furthermore $\text{Ker } ev = TB$.

**Proof.** For (a) see [Rei22] Proposition 7. Since clearly $T$ is fixed by $W$ and $ev(T) = 0$, we have $TB \subset A$ and $TB \subset \text{Ker } ev$. The other inclusion in (b) follows from the proof of [Rei22] Proposition 6. Without extending scalars, we would not have $TB \subset A$. \qed

3.2.4 The case of $P(n)$.

The case of $G = P(n)$ is studied in [IRS21]. We define

$$J_n = \{ f \in \mathbb{Z}[X(T_n)]^W |f|_{x_n = x_{n-1} = t} \text{ is independent of } t \}. \quad (3.8)$$
By Theorem 1.0.1 $J_n \cong J(P(n))$, the super character ring of $P(n)$. As before this allows us to identify $ds_x$ with an evaluation map $A = J_n \rightarrow J_{n-2} = A'$, given by $ev(f) = f|_{x_n = x_{n-1} = 1}$. Note that $A$ is contained in $B = \mathbb{Z}[X(T_n)]^W$. So we have a diagram as in Subsection 2.2, where the map $\phi = ev$, and the map $\Phi : B \rightarrow B' = \mathbb{Z}[X(T_{n-2})]^W$ is defined by $\Phi(f) = f|_{x_n = x_{n-1} = 1}$.

**Theorem 3.9.** (a) The map $ev : J_n \rightarrow J_{n-2}$ is surjective.

(b) Define $T \in B = \mathbb{Z}[X(T_n)]^W$, by $T = \prod_{i<j} (1 - x_i x_j)$. Then such that $TB \subset A$. That is Hypothesis 2.1 holds. Furthermore $\text{Ker } ev = TB$.

**Proof.** For (a) see Theorem 4.0.1. Statement (b) follows from the proof of Proposition 3.2.1. 

4 The Nullstellensatz.

In this Section we assume one of the following holds, compare Notation 1.1

(i) $g \neq p(n)$, $A = I(h)$, $X = h^*$ and $B = \mathcal{O}(X)^W = S(h)^W$.

(ii) $A = J(G) \otimes_{\mathbb{Z}} k$, $X = T$, $\mathcal{B} = \mathcal{O}(X)^W$ and $G$ is not exceptional KM.

4.1 Maximal Ideals.

First we prove the analog of the weak Nullstellensatz.

**Theorem 4.1.** If $m$ is a maximal ideal of $A$, then there is a maximal ideal $M$ of $\mathcal{B}$ such that $m = M \cap A$.

**Proof.** We give the details in case (i). If $T \notin m$ this follows from Corollary 2.4. Assume is KM and let $W_x$ be the Weyl group of $g_x$. By induction the conclusion holds with $(A,B)$ replaced by the pair $(I(h_x), S(h_x)^W)$. Thus by Lemma 2.8 it holds also for $(I(h_x)^\sigma, S(h_x)^{W_x,\sigma})$ where $\sigma = \sigma_x$ is as in Theorem 3.1. Thus if $T \in m$, the result holds by Lemma 2.7 taking $M$ to be any maximal ideal containing $m\mathcal{B}$. The same argument works for $g = q(n)$ using Theorem 3.2. The proof in case (ii) is similar.

**Theorem 4.2.** If $\lambda \in X$, set

$$m_\lambda = \{ f \in A | f(\lambda) = 0 \}.$$ 

If $m$ is a maximal ideal of $A$, then $m = m_\lambda$ for some $\lambda \in X$.

**Proof.** This follows from Theorem 4.1. Note that maximal ideals in $\mathcal{B}$ correspond to $W$-orbits in $X$.

For $\lambda \in X$, let $M_\lambda \in \text{Max } \mathcal{O}(X)$ be the ideal of functions vanishing at $\lambda$.

**Corollary 4.3.** We have
(a) There are surjective maps
\[ \text{Max } \mathcal{O}(X) \rightarrow \text{Max } \mathcal{B} \rightarrow \text{Max } \mathcal{A}. \]

(b) Denote the composite of the maps in (a) by \( \sigma \) and suppose that \( \sim \) is the smallest equivalence relation on \( X \) such that

(i) \( w \lambda \sim \lambda \) for all \( w \in W \).

(ii) If \( X = h^\ast \) and \( (\lambda, \alpha) = 0 \) for \( \alpha \in \Delta_{iso} \), then \( \lambda \sim \lambda + t\alpha \) for all \( t \in k \).

(iii) If \( X = T \) and \( \lambda \in T_\alpha \) for \( \alpha \in \Delta_{iso} \), then \( \lambda \sim c_\alpha(t)\lambda \) for all \( t \in k \).

Then \( M_\mu \in \sigma^{-1}(m_\lambda) \iff \lambda \sim \mu \).

Proof. In (a) the map \( \text{Max } \mathcal{B} \rightarrow \text{Max } \mathcal{A} \) is given by \( M \rightarrow M \cap A \), and the other map is defined similarly. The first map is surjective since \( \mathcal{B} \) is the fixed ring of \( \mathcal{O}(X) \) under the action of the finite group \( W \). The second map is surjective by Theorem 4.1. If \( X = h^\ast \), part (b) is a reformulation of [Mus12] Theorem 13.5.4, and the proof when \( X = T \) is similar.

4.2 The Strong Nullstellensatz.

We deduce the strong Nullstellensatz from Theorem 4.1. There is a small problem because we need the analogous result when the pair \((\mathcal{A}, \mathcal{B})\) is replaced by \((\mathcal{A} \otimes_k k[z], \mathcal{B} \otimes_k k[z])\). This is easily taken care of by noting that the description of \( \mathcal{A} \) given by (1.2) and (1.3) is independent of the algebraically closed field, and hence holds also for any field extension \( K \) of \( k \).

Corollary 4.4. If \( m \in \text{Max } \mathcal{A} \otimes_k k[z] \), then \( mB \otimes_k k[z] \) is a proper ideal of \( \mathcal{B} \otimes_k k[z] \).

Proof. Let \( S = k[z]\setminus\{0\} \). If \( m \cap S = \emptyset \), then \( m_\mathcal{S} \in \text{Max } \mathcal{A} \otimes_k K \), where \( K = k(z) \), and the result follows by the above remarks. Otherwise, we have \( z - \lambda \in m \) for some \( \lambda \in k \). Now if \( C = \mathcal{A} \) or \( \mathcal{B} \), set \( C_\lambda = C \otimes_k k[z]/(z - \lambda) \). Then \( m : m/(z - \lambda) \in \text{Max } \mathcal{A}_\lambda \).

Now
\[ \mathcal{A} \cong A_\lambda \subseteq B_\lambda \cong \mathcal{B}, \]
so by Theorem 4.1 \( m_\mathcal{B}_\lambda \) is a proper ideal of \( B_\lambda \).

If \( I \) is a subset of \( \mathcal{A} \), let \( \mathcal{V}(I) = \{ x \in X | f(x) = 0 \text{ for all } f \in I \} \). Such a set is called an superalgebraic set. If instead \( I \) is a subset of \( \mathcal{B} \), we say that \( \mathcal{V}(I) \) is an algebraic set (or closed) in \( X \). Thus any superalgebraic set is algebraic. In addition if \( I \) is a subset of \( X \), set
\[ I_\mathcal{A}(V) = \{ f \in \mathcal{A} | f(x) = 0 \text{ for all } x \in V \}. \]

We will also need
\[ I(V) = \{ f \in \mathcal{O}(X) | f(x) = 0 \text{ for all } x \in V \}. \]
Theorem 4.5. The maps $I_A$ and $V$ are inverse bijections between the set of superalgebraic sets in $X$, and the set of radical ideals in $A$. Both maps are order reversing.

Proof. The key point is that $I_A(V(I)) \subseteq \text{rad}(I)$. To show $I$ is not assumed finitely generated, we repeat the well-known “Rabinowitsch trick”, [Eis95] Theorem 4.19. [Ful89] Chapter 1. Equation (4.1) below uses only finitely many elements from $I$. Suppose $G \in I_A(V(I))$, and let $J$ be the ideal of $A \otimes k[z]$ generated by $I$ and $zG - 1$. Then $V(J)$ is empty, since $G$ vanishes wherever all polynomials in $I$ vanish. Therefore by Corollary 4.4, $1 \in J$, and we can write

$$1 = \sum_{i=1}^{r} A_i F_i + B(zG - 1),$$

where the $F_i$ are in $I$ and $B, A_i \in A \otimes k[z]$. Now set $Y = 1/z$ and multiply by a large power of $Y$ to obtain

$$Y^N = \sum_{i=1}^{r} C_i F_i + D(G - Y)$$

where $D, C_i \in A \otimes k[Y]$. The result follows by setting $Y = G$. \qed

Proposition 4.6. The maps $V$, and $I_A$ satisfy the following properties. Suppose that $E_\lambda, V_\lambda$ are subsets of $A$, and $X$ respectively and that $a, b$ are ideals of $A$.

(a) $V(\bigcup_{\lambda \in \Lambda} E_\lambda) = \bigcap_{\lambda \in \Lambda} V(E_\lambda)$.
(b) $V(a \cap b) = V(ab) = V(a) \cup V(b)$.
(c) $I_A(\bigcup_{\lambda \in \Lambda} V_\lambda) = \bigcap_{\lambda \in \Lambda} I_A(V_\lambda)$.

Proof. Left to the reader. \qed

4.3 Prime ideals and irreducible components.

We say that a superalgebraic set is irreducible if it cannot be written as the union of two proper superalgebraic sets.

Proposition 4.7. If $I$ is a radical ideal of $A$ and $V = V(I)$ is the corresponding superalgebraic set, then $I$ is prime iff $V$ is irreducible as a superalgebraic set.

Proof. The proof is completely analogous to the classical case [Ful89] Chapter 1.5, Proposition 1, page 15. \qed

In general if $I$ is a radical ideal of $A$, then using Corollary 2.6 we can write $I$ uniquely in the form $I = P_1 \cap \ldots \cap P_r$, where the $P_i$ are the prime ideals of $A$ which are minimal over $I$. Then $V(I) = \bigcup_{i=1}^{r} V(P_i)$ by Proposition 4.6. We call the superalgebraic sets $V(P_i)$ the irreducible superalgebraic components of $V(I)$. They are irreducible.
Corollary 4.8. Every superalgebraic set is uniquely a finite union of irreducible superalgebraic components.

Proof. This follows from the Nullstellensatz and Proposition 4.6 (b). \hfill \Box

Remark 4.9. Our approach to the Nullstellensatz was to relate ideals of \( A \) and \( B = O(X)^W = O(X/W) \). Therefore it might seem more natural to relate radical ideals in \( A \) to their zero loci in \( X/W \). However the connection between closed sets in \( X \) and \( X/W \) is well understood, when \( O(X) \) is finitely generated, [DK02] Section 2.3.1. If \( U \) is a closed \( W \)-invariant subset of \( X \), then \( I(U) \) is a \( W \)-stable ideal of \( O(X) \). Let \( \pi : X \rightarrow X/W \) be the natural map, corresponding to the inclusion of rings \( O(X)^W \subseteq O(X) \). If \( V \) is closed in \( X/W \), and \( U \) is an irreducible component of \( \pi^{-1}(V) \), then so is \( wU \) for \( w \in W \), and \( \pi(U) = \pi(wU) \). In Section 6 it will be more convenient to work with closed sets in \( X \), rather than \( X/W \). Several closed subsets of \( X \) we consider are identified under \( \pi \), see Lemma 6.10.

5 Weyl Groupoids and their Actions.

5.1 Actions of groupoids on varieties.

A groupoid \( \mathcal{G} \) can be defined as a small category with all morphisms invertible. We denote the set of objects by \( \mathcal{B} \) which we call the base. As in [SV11] we use the same notation \( \mathcal{G} \) for the set of morphisms as for the groupoid itself. For a variety \( X \) consider the groupoid \( \mathfrak{A}(X) \) with base all subvarieties of \( X \), and morphisms all isomorphisms between subvarieties. We say the groupoid \( \mathcal{G} \) acts on \( X \) if there is a functor

\[ F : \mathcal{G} \rightarrow \mathfrak{A}(X). \]

If \( g : b \rightarrow b' \) is a morphism in \( \mathcal{G} \), set \( s(g) = b \). Then for \( \lambda \in X \), set \( \mathcal{B}_\lambda = \{ b \in \mathcal{B} | \lambda \in F(b) \} \) and \( \mathcal{G}_\lambda = \{ g \in \mathcal{G} | s(g) \in \mathcal{B}_\lambda \} \). If \( g \in \mathcal{G}_\lambda \), we say \( g \) is defined at \( \lambda \) and often write \( g\lambda = F(g)\lambda \). Set \( \mathcal{G}_0\lambda = \{ \lambda \} \) and for \( i \geq 0 \), define \( \mathcal{G}_{(i+1)}\lambda = \{ g\mu | \mu \in \mathcal{G}_{(i)}\lambda, g \in \mathcal{G}_{(i)}\mu \} \). The \( \mathcal{G} \)-orbit of \( \lambda \) is \( \mathcal{G}\lambda = \bigcup_{i \geq 0} \mathcal{G}_{(i)}\lambda \). Informally, \( \mathcal{G}_{(i)}\lambda \) is the set of points in \( X \) we can reach from \( \lambda \) using \( i \) morphisms from \( \mathcal{G} \). If \( X \) is affine, the invariant ring for the action of \( \mathcal{G} \) on \( X \) is

\[ O(X)^\mathcal{G} = \{ f \in O(X) | f \text{ is constant on } \mathcal{G}-\text{orbits} \} \]

Unlike the case of a group action on \( X \), the groupoid \( \mathcal{G} \) does not in general act on functions.

Lemma 5.1. The comorphism \( \pi : X \rightarrow \text{Spec } O(X)^\mathcal{G} = Y \) is constant on \( \mathcal{G} \)-orbits.

Proof. This follows from the definition of \( O(X)^\mathcal{G} \). \hfill \Box

For continuous Weyl groupoids the map \( \pi \) has some pleasant properties, see Corollary 5.8.
5.2 Continuous Weyl groupoids and their actions.

In [SV11] Sergeev and Veselov associated a certain groupoid $\mathcal{W} = \mathcal{W}(\Delta)$, which they call Weyl groupoid, to the root system $\Delta$ of a KM algebra. They also defined a functor $\mathcal{W} \rightarrow \mathcal{A}(\mathfrak{h}^*)$. We introduce two closely related groupoids $\mathcal{W}^c = \mathcal{W}^c(\Delta)$ and $\mathcal{W}_c = \mathcal{W}_c^c(\Delta)$.

We need a preliminary construction, namely the semi-direct product groupoid $\Gamma \ltimes \mathcal{G}$. Let $\mathcal{G}$ be a groupoid and $\Gamma$ a group acting on $\mathcal{G}$ by automorphisms of the corresponding category. In particular, $\Gamma$ acts on the base $\mathcal{B}$ of $\mathcal{G}$. Then the semi-direct product groupoid $\Gamma \ltimes \mathcal{G}$ has the same base $\mathcal{B}$, and is defined in [SV11]. To define the continuous Weyl groupoids, we first introduce some auxiliary groupoids $\mathcal{G}^c$ and $\mathcal{G}_c^c$.

5.2.1 An action on $\mathfrak{h}^*$.

In the KM case consider the following groupoid $\mathcal{T}^c_{iso}$ with base $\Delta^c_{iso} = \Delta_{iso} \times \mathbb{R}$. The non-identity morphisms are $\tau_{\alpha,t} : (\alpha, t) \rightarrow (-\alpha, t)$. The group $W$ acts on $\mathcal{T}^c_{iso}$ in a natural way: $\alpha \rightarrow w(\alpha)$, $\tau_{\alpha,t} \rightarrow \tau_{w(\alpha),t}$. Let $\mathcal{G}^c = W \ltimes \mathcal{T}^c_{iso}$ with base $\Delta^c_{iso}$. The continuous Weyl groupoid $\mathcal{W}^c$ is defined by

$$\mathcal{W}^c = W \bigsqcup G^c,$$

(5.1)

the disjoint union of the group $W$ considered as a groupoid with a single point base $[W]$ and $\mathcal{G}^c$. There is an action of $\mathcal{W}^c$ on $\mathfrak{h}^*$, given by a functor from $\mathcal{W}^c$ to $\mathcal{A}(\mathfrak{h}^*)$. Here we adapt [SV11], Section 9. The base point $[W]$ maps to the whole space $\mathfrak{h}^*$. The base element $(\alpha, t)$ maps to the hyperplane $\Pi_\alpha = \Pi_{-\alpha}$. The morphism $\tau_{\alpha,t}$ acts via

$$\tau_{\alpha,t}(x) = x + t\alpha, \quad x \in \Pi_\alpha.$$  

(5.2)

If $\mathfrak{g} = \mathfrak{p}(n)$ or $\mathfrak{q}(n)$, define the groupoid $\mathcal{T}^c$ with base $\mathcal{B}^c = \{ (\pm \alpha, t) \mid \alpha \in \Delta_0^+, t \in \mathbb{R} \}$ and non-identity morphisms $\tau_{\alpha,t} : (\alpha, t) \rightarrow (-\alpha, t)$ and $\tau_{-\alpha,t} : (-\alpha, t) \rightarrow (\alpha, t)$ for $\alpha \in \Delta_0^+$. Next form the semi-direct product $\mathcal{G}^c = W \ltimes \mathcal{T}^c$ with base $\mathcal{B}^c$, and define $\mathcal{W}^c$ as in (5.1). There is an action of $\mathcal{W}^c$ on $\mathfrak{h}^*$ defined as before, except that in (5.2) for $\alpha \in \Delta_0^+$, $\Pi_\alpha$ is given by (5.3). The orbits of $\mathcal{W}^c$ on $\mathfrak{h}^*$ are described in all cases in Theorem 5.7. However, we make no use of this action when $\mathfrak{g} = \mathfrak{p}(n)$, see Subsection 3.1.3.

5.2.2 An action on $\mathbb{T}$.

We need a variant $\mathcal{W}_c^c$ of $\mathcal{W}^c$. In the KM case define $\mathcal{G}_c^c$ in exactly the same way as $\mathcal{G}^c$, except that the base of $\mathcal{G}_c^c$ is $\Delta_{iso} \times \mathbb{R}$. Then set $\mathcal{W}_c^c = W \bigsqcup \mathcal{G}_c^c$. There is an action of $\mathcal{W}_c^c$ on $\mathbb{T}$, corresponding to a functor from $\mathcal{W}_c^c$ to $\mathcal{A}(\mathbb{T})$. As before, the base point $[W]$ maps to the whole space $\mathbb{T}$. The base element $(\alpha, t)$ maps to the subtorus $T_\alpha$ defined in (3.4). The morphism $\tau_{\alpha,t} : T_\alpha \rightarrow T_\alpha$ acts via

$$\tau_{\alpha,t}(x) = c_\alpha(t)x, \quad x \in T_\alpha.$$  

(5.3)
If \( G = P(n) \) or \( Q(n) \), the groupoids \( \mathcal{G}_\lambda^c \) and \( \mathfrak{W}_\lambda^c \) are defined by analogy with the groupoids for \( p(n) \) or \( q(n) \) except that the base of \( \mathcal{G}_\lambda^c \) is \( \{ (\pm \alpha, t) \mid \alpha \in \Delta^+, t \in k^\times \} \). There is a functor \( \mathfrak{W}_\lambda^c \rightarrow \mathfrak{A}(T) \), such that \( W = NG_0(T)/T \) acts in the obvious way. The base elements \( (\pm \alpha, t) \) map to \( T_\alpha = \text{Ker} \text{e}^\alpha \), and for \( \alpha \in \Delta_0^+ \), \( \tau_{\alpha,t} \) acts via \( (5.3) \).

**Lemma 5.2.** If \( (\alpha, \beta) = 0 \), then \( (\beta, \overline{\alpha}) = 0 \).

**Proof.** Suppose \( \alpha = \epsilon_i - \epsilon_j \), and \( \beta = \epsilon_k - \epsilon_\ell \) where \( i < j, k < \ell \). If \( (\alpha, \beta) = 0 \), then \( 0 = \delta_{i,k} + \delta_{i,\ell} - \delta_{j,k} - \delta_{j,\ell} \). If \( i = k \), then \( j = \ell \) and hence \( (\beta, \overline{\alpha}) = \delta_{i,k} - \delta_{i,\ell} + \delta_{j,k} - \delta_{j,\ell} = 0 \). Similar arguments for the cases \( i = \ell \) and \( i \neq k, \ell \) complete the proof. \( \square \)

Equation (5.3) is a multiplicative analog of (5.2). If \( \alpha \in \Delta_{is} \) is isotropic, \( T_\alpha \) is a codimension one subtorus in \( T_\alpha \), and by (3.5) \( \text{Im} c_\alpha \subseteq T_\alpha \). Then \( \tau_{\alpha,t} \) translates \( x \in T_\alpha \) using the one-parameter subgroup \( c_\alpha \). In types P and Q, this holds since \( (\alpha, \overline{\alpha}) = 0 \).

### 5.2.3 Orbits.

If \( \mathfrak{G} \) is the continuous Weyl groupoid as in Notation 1.1, we describe the orbit \( \mathfrak{G}\lambda \) of \( \lambda \in X \). Define the *degree of atypicality* of \( \lambda \) to be

\[
\text{atyp} \lambda = \text{max}\{s| (\lambda, A) = 0 \text{ for some iso-set } A \text{ with } |A| = s \} \quad \text{if } X = \mathfrak{h}^*,
\]

or

\[
\text{atyp} \lambda = \text{max}\{s| \lambda \in \bigcap_{\alpha \in A} T_\alpha \text{ for some iso-set } A \text{ with } |A| = s \} \quad \text{if } X = T.
\]

We say \( \lambda \) is *typical* if \( \text{atyp} \lambda = 0 \). Next set

\[
E(\lambda) = \{ \alpha \in \Delta_{iso} | \lambda \in \Pi_\alpha \}, \quad E_\lambda = \bigcup_{w \in W} \bigcup_{\alpha \in E(\lambda)} (\lambda + k\alpha)
\]

or

\[
E(\lambda) = \{ \alpha \in \Delta_{iso} | \lambda \in T_\alpha \}, \quad E_\lambda = \bigcup_{w \in W} \bigcup_{\alpha \in E(\lambda)} (\text{Im} c_\alpha \lambda)
\]

if \( X = \mathfrak{h}^* \) or \( X = T \) respectively. We need a preliminary result.

**Lemma 5.3.** We have \( \mathfrak{G}\lambda = E_\lambda \).

**Proof.** Clearly \( E_\lambda \subseteq \mathfrak{G}\lambda \). We show that for \( \mu \in \mathfrak{G}_{(i+1)} \lambda \), we have \( E_\mu = E_\lambda \). Write \( \mu = g\nu \) where \( \nu \in \mathfrak{G}_{(i)} \lambda \) and \( g \in \mathfrak{G}_{(\nu)} \). By induction \( E_\nu = E_\lambda \). Thus we can assume that \( \nu = \lambda \). Since \( E_\lambda \) is \( W \)-invariant, we may further assume \( g = \tau_{\alpha,t} \). The action of \( g \) is given by (5.2) or (5.3), so since \( g \) is defined at \( \lambda, \lambda \in \Pi_\alpha \) or \( \lambda \in T_\alpha \). Hence \( \alpha \in E(\lambda) \) and this gives the result. \( \square \)
Remark 5.4. If $\lambda \in h^*$, the proof shows that $W\lambda = \bigcup_{w \in W} w[\bigcup_{\alpha \in E(\lambda)} (\lambda + Z\alpha)]$, which has closure $W^*\lambda$. This explains why our primary interest is in orbits of continuous Weyl groupoids. To give a definitive description of these orbits we need two more Lemmas.

Lemma 5.5. Let $F(\lambda) \subseteq E(\lambda)$ be an iso-set with $|F(\lambda)| = \text{atyp} \lambda$, and set $G(\lambda) = F(\lambda) \cup -F(\lambda)$. If $\beta \in E(\lambda) \setminus F(\lambda)$, then for some reflection $u \in W$ we have $u\lambda = \lambda$ and $u\beta = \pm \alpha \in G(\lambda)$.

Proof. See [Gor20] Equation (14). For convenience we give a short proof. In the KM case, we can argue as follows. There is some $\alpha \in F(\lambda)$ such that $(\alpha, \beta) \neq 0$. The Jacobi identity applied to the three root vectors $e_{\pm\alpha} \in g_{\pm\alpha}$ and $e_\beta \in g_\beta$ implies that one of $[e_\beta, e_{\pm\alpha}]$ is non-zero. Replacing $\alpha$ by $-\alpha$ if necessary, we can assume $\gamma = \beta - \alpha$ is an (even) root. Then from the rank two case, [Mus12] Table 3.4.1 we see that $\gamma$-string through $-\beta$ consists of $-\alpha$ and $-\beta$. Thus if $u = s_\gamma$ is the reflection corresponding to $\gamma$, we have $u\beta = \alpha$. Since $(\lambda, \gamma) = 0$, $u$ fixes $\lambda$. For types $P$ and $Q$ we adapt [Gor20]. We have $(\alpha, \beta) \neq 0$ for some $\alpha \in F(\lambda)$. We can assume $\alpha = e_i - e_j, \beta = \pm(e_i - e_k) \in \Delta^+_0$, where $i < j \neq i, k$. Then

$$(\lambda, \overline{\beta}) = 0 = (\lambda, \overline{\alpha}),$$

and the result holds with $\gamma = e_j - e_k$.

Next set

$$F_\lambda = \bigcup_{w \in W} w(\lambda + \sum_{\alpha \in F(\lambda)} k\alpha) \text{ or } F_\lambda = \bigcup_{w \in W} w\left(\prod_{\alpha \in F(\lambda)} \text{Im } e_\alpha\right)\lambda$$

if $X = h^*$ or $X = T$ respectively. The definitions do not change if $F(\lambda)$ is replaced by $G(\lambda)$.

Lemma 5.6. We have $F_\lambda \subseteq \mathfrak{G}\lambda$.

Proof. By (5.3) and Lemma 5.2 if $\alpha, \beta \in F(\lambda)$, $s, t \in k$ and $\lambda \in T_\alpha \cap T_\beta$ then $c_\alpha(t)\lambda \in T_\beta$. Similar remarks apply to (5.2) if $\lambda \in \Pi_\alpha \cap \Pi_\beta$. This implies the statement.

Theorem 5.7. We have

(a) $\mathfrak{G}\lambda = F_\lambda$,

(b) $\dim \mathfrak{G}\lambda = \text{atyp} \lambda$,

(c) Every $\mathfrak{G}$-orbit is closed.

(d) Let $\sim$ be the relation from Corollary 4.3 (b). Then $\mu \sim \lambda$ iff $\mathfrak{G}\mu = \mathfrak{G}\lambda$.

(e) If $m_\lambda = M_\lambda \cap A$ as in Corollary 4.5, then $\mathcal{V}(m_\lambda) = \mathfrak{G}\lambda$.

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Proof. By Lemmas 5.3 and 5.6 $F_{\lambda} \subseteq E_{\lambda} = \mathcal{G}_{\lambda}$. We show $E_{\lambda} \subseteq F_{\lambda}$. If $\beta \in E(\lambda) \setminus F(\lambda)$, let $u$ be as in Lemma 5.5. Then $u(\lambda + k\beta) = (\lambda + k\alpha)$ and $u(c_{\beta}(t)\lambda) = c_{\alpha}(t)\lambda$ in the two cases under consideration. This shows (a) and (b), (c) are immediate consequences. By definition $\sim$ is the smallest equivalence relation satisfying the conditions in Corollary 4.3 (b). Hence if $\mu \sim \lambda$ we have $G_{\mu} = G_{\lambda}$. Conversely if $G_{\mu} = G_{\lambda}$, then $\mu \sim \lambda$ by Lemma 5.3 proving (d). From Corollary 4.3 and (d) we have

$$m_{\lambda} = m_{\mu} \iff \mu \sim \lambda \iff G_{\mu} = G_{\lambda}. \quad (5.4)$$

Since $\lambda \in V(m_{\lambda})$ this implies $V(m_{\lambda}) \supseteq \mathcal{G}_{\lambda}$. For the other inclusion, if $\mu \in V(m_{\lambda})$, then $m_{\mu} = \mathcal{I}_{A}(\mu) \supseteq m_{\lambda}$, and equality must hold. Thus $\mu \in \mathcal{G}_{\mu} = \mathcal{G}_{\lambda}$, proving (e).

We interpret some of our results geometrically. The map $\pi$ from Lemma 5.1 restricts to the map

$$\psi: \text{Max } O(X) \rightarrow \text{Max } A. \quad (5.5)$$

as in (2.4).

**Corollary 5.8.** (a) The map in (5.5) is surjective.

(b) Identifying $X$ with $\text{Max } O(X)$, the fiber over $m_{\lambda}$ is $\mathcal{G}_{\lambda}$.

**Proof.** (a) is a restatement of the Weak Nullstellensatz, Corollary 4.3 (a) and (b) follows from (5.4).

**Proposition 5.9.** Assume Hypothesis 3.3 holds. Then

(a) $O(T)^{\text{MF}} = J(G) \otimes_{\mathbb{Z}} k$.

(b) $S(h)^{\text{MF}} = S(h)^{\text{MF}} = I(h)$ if $\mathfrak{g} \neq \mathfrak{p}(n)$.

**Proof.** By (1.3), (3.7), (3.8), and [Mus22] Lemma 3.1, if $f \in O(T)$, then $f \in J(G) \otimes_{\mathbb{Z}} k$ iff $f(\lambda) = f(c_{\alpha}(t)\lambda)$ for all relevant roots $\alpha$, $t \in k$ and all $\lambda \in T_{A}$. Equivalently $f \in O(T)^{\text{MF}}$. This shows (a) and the proof of (b) is similar using (1.2). The first equality in (b) holds because any polynomial that is constant on infinitely many points on a line is constant on the whole line.

6 The Geometry of Superalgebraic Sets.

6.1 Outline

Let $\mathfrak{g}$ be a KM Lie superalgebra and let $G$ be the corresponding supergroup as in Lemma 3.4. We use Notation 1.1. We define the $S$-topology on $\text{Spec } O(X)$ by declaring that the $S$-closed sets are the $\mathcal{G}$-stable (Zariski) closed sets. To give an idea of the geometry of superalgebraic sets, we show that they are precisely the $S$-closed sets. We also compute the $S$-closure $V^{S}$ of an arbitrary closed set $V = V_{0} = V(I)$ with $I$ a $W$-invariant radical ideal in $O(X)$. From $V$ we obtain a superalgebraic set which we call the heart and another closed set $V_{1} = V(I_{1})$ with
$I_1$ a $W$-invariant radical ideal of $O(X)$. The process can be repeated giving new closed sets $V_i$ and new hearts. Then $V^S$ is the union of all the hearts and also the union of all the $V_i$. The hearts are determined by the minimal degrees of atypicality in the Zariski irreducible components of the $V_i$. A somewhat simplified version of this procedure and some examples are given in [Mus22]. If the defining equations for $V$ are known, it can be done algorithmically using Groebner bases, see [Mus22] Example 2.11. The heart is defined in the next subsection, and then $V^S$ is analyzed using projections onto intersections of hyperplanes corresponding to iso-sets. Let

$$
\text{atyp } V = \max \{ s \mid \text{for some } \lambda \in V, \text{atyp } \lambda = s \}.
$$

Now suppose $r = \text{atyp } V$ and choose an iso-set $S = \{ \beta = \beta_1, \beta_2, \ldots, \beta_r \}$ with $(\lambda, S) = 0$ for some $\lambda \in V$. Since we can replace $\alpha \in S$ by $-\alpha$ if necessary, we can assume $S \subseteq \Omega$.

### 6.2 The heart

Suppose $I = \bigcap_{i=1}^k P_i$, an irredundant intersection of prime ideals of $B$, where $T \notin P_i$ iff $i \in [k]$. Equivalently $i \in [k]$ iff $\mathcal{V}(P_i)$ contains typical points. For $i \in [k]$, set $Q_i = P_i \cap A$. Then set

$$
\text{heart } V = \text{heart}_0 V = \mathcal{V} \left( \bigcap_{i=1}^k Q_i \right), \quad V_{\text{reg}} = \mathcal{V} \left( \bigcap_{i=1}^k P_i \right).
$$

If $k = 0$, set heart $V = \emptyset$. Note that heart $V$ is superalgebraic, since $\bigcap_{i=1}^k Q_i$ is an ideal of $A$. Set $V^c = \mathcal{V}(I, T)$.

**Lemma 6.1.** For $i \in [k]$,

(a) $\mathcal{V}(P_i) \subseteq \mathcal{V}(Q_i)$ and $V_{\text{reg}} \subseteq \text{heart } V$.

(b) If $\lambda \in \mathcal{V}(Q_i)$, and $T(\lambda) \neq 0$, then $\lambda \in \mathcal{V}(P_i)$.

(c) If $\lambda \in \text{heart } V$, and $T(\lambda) \neq 0$, then $\lambda \in V_{\text{reg}}$.

(d) heart $V \setminus V_{\text{reg}} \subseteq V^c$.

(e) If $i > k$, then $\mathcal{V}(P_i) \subseteq V^c$.

**Proof.** (a) is clear since the $\mathcal{V}(P_i)$ and $\mathcal{V}(Q_i)$ are the irreducible components of $V_{\text{reg}}$ and heart $V$ respectively. Next if $f \in P_i$ we have $T^b f \in Q_i$ for some $b \geq 0$. So if $\lambda \in \mathcal{V}(Q_i)$, and $T(\lambda) \neq 0$ we obtain $f(\lambda) = 0$, proving (b) and (c) is proved similarly. By (c), if $\lambda \in \text{heart } V \setminus V_{\text{reg}}$, then $T(\lambda) = 0$ and (d) follows from this. Finally (e) follows since $T \in P_i$ for $i > k$.

Note that

$$
V = V_{\text{reg}} \cup \bigcup_{i > k} \mathcal{V}(P_i).
$$

(6.1)
Corollary 6.2.  
\[ V = \text{heart } V \cup V^c. \]

Proof. The inclusion \( \subseteq \) follows from (a), (e) in the Lemma and (6.1). The opposite inclusion results from \( V^c \subseteq V \), (d) in the Lemma and (6.1). \( \square \)

6.3 Analysis using projections

If \( q : X \rightarrow Y \) is a morphism of affine varieties, define the comorphism \( q^* : \mathcal{O}(Y) \rightarrow \mathcal{O}(X) \) by \( (q^* f)(x) = (fq)(x) \) for all \( f \in \mathcal{O}(Y) \) and \( x \in X \). It is easy to check the following.

Lemma 6.3. Let \( i : Y \rightarrow X \) be an embedding of affine varieties and suppose \( q : X \rightarrow Y \) is a morphism such that \( qi = id_Y \). If \( I = \ker i^* \), then
\[ I \cap q^* \mathcal{O}(Y) = 0 \text{ and } I + q^* \mathcal{O}(Y) = \mathcal{O}(X). \]
Furthermore, if \( V \) is the closed subset of \( Y \) defined by the radical ideal \( K \) of \( \mathcal{O}(Y) \), then as a closed subset of \( X \) we have \( V = V(q^* K) \).

Example 6.4. Let \( X = h^* \) and \( X(\beta) = h(\beta)^* \). Define \( q_\beta : h^* \rightarrow h(\beta)^* \) by restriction \( q_\beta(\lambda) = \lambda|_{h(\beta)} \), and let \( i_\beta : h(\beta)^* \rightarrow h^* \) be the map induced by the inclusion \( \Delta(\beta) \subset \Delta \). The map \( q_\beta \) can also be described as follows. IAN pink notebook. Let \( \{\epsilon_i\}_{i=1}^{p} \) and \( \{h_i\}_{i=1}^{p} \) be dual bases for \( h(\beta)^*, h(\beta) \). Extend the \( h_i \) to functionals on \( h^* \) that vanish on the orthogonal complement of \( h(\beta)^* \) in \( h^* \), see (6.5). Then for all \( \lambda \in h^* \) we have \( q_\beta(\lambda) = \sum_{i=1}^{p} h_i(\lambda)\epsilon_i \).

Example 6.5. If \( X = \mathbb{T} \), let \( X(\beta) = \mathbb{T}(\beta) \) be the subtorus of \( X \) generated by \( Im \, c_\alpha \) for \( \alpha \in \Delta(\beta) \). Let \( \{\chi_i\}_{i=1}^{p} \) and \( \{c_i\}_{i=1}^{p} \) be bases for \( X(\mathbb{T}(\beta)), Y(\mathbb{T}(\beta)) \) respectively such that \( \chi_i \beta_j(t) = t^{h_i(\beta)} \). Extend the \( c_i \) to \( Y(\mathbb{T}) \) by analogy with the previous example. Then define \( q_\beta : \mathbb{T} \rightarrow \mathbb{T}(\beta) \) by \( q_\beta(t) = \prod_{i=1}^{p} c_i(\chi_i(t)) \).

Now for \( X = h^* \) or \( \mathbb{T} \), and \( \beta \) an isotropic root, define \( X(\beta) \) as in the Examples. Define \( p_\beta = i_\beta \circ q_\beta \). We have \( \Pi_\beta = h(\beta)^* + k\beta \) and \( T_\beta = \mathbb{T}(\beta) \times \text{Im } c_\beta \). \( \text{(6.2)} \)

For the torus case, \( \mathbb{T}(\beta) \subset T_\beta \) by Lemma 6.5 and equality holds because \( T_\beta \) is the subtorus generated by \( \text{Im } c_\alpha \) for \( \alpha \in \Delta \) with \( (\alpha, \beta) = 0 \).

Lemma 6.6. (a) If \( \lambda \in \Pi_\beta \), then \( \lambda - p_\beta(\lambda) \in k\beta \).

(b) If \( t \in T_\beta \), then \( tp_\beta(t)^{-1} \in \text{Im } c_\beta \).

Proof. We prove (a). The proof of (b) is similar. By (6.2) we can write \( \lambda = \lambda_0 + c\beta \) with \( \lambda_0 \in h(\beta)^*, c \in k \). Since \( i_\beta \circ q_\beta \) is the identity on \( h(\beta)^* \) and \( q_\beta(\beta) = 0 \), we obtain the result. \( \square \)
The action of $W$ on $\mathcal{O}(X)$ is given by $(wf)(x) = f(w^{-1}x)$ for $w \in W, f \in \mathcal{O}(X)$ and $x \in X$. Then $w\Delta(\beta) = \Delta(w\beta)$, and we have $wX(\beta) = X(w\beta)$. Since $W$ acts transitively on $\Omega$ we may assume $\sigma(w\beta) = w\sigma(\beta)$ in (1.3). Then

$$w \circ q_\beta(\lambda) = q_{w\beta}(w\lambda).$$

(6.3)

We have an embedding $q_\beta^* : \mathcal{O}(X(\beta)) \rightarrow \mathcal{O}(X)$ as in Lemma 6.3 with $Y = X(\beta)$ and $q = q_\beta$. In what follows we suppress $q_\beta^*$.

Set $X_\beta = \Pi_\beta$ or $\Pi_\beta = T_\beta$ in case (i) or (ii) of 1.1 holds respectively. For $\beta \in \Delta_{iso}$, set $V_\beta = V \cap X_\beta$. Since $T(\lambda) = 0$ for $\lambda \in V^c$, we have

$$V^c = \bigcup_{\beta \in \Omega} V_\beta.$$ 

Next we study the sets $V_\beta$ in more detail. If $\beta = \beta_1$, set $U_\beta = q_\beta(V_\beta) \subseteq X(\beta)$ and

$$K_\beta = \mathcal{I}(V_\beta) \cap \mathcal{O}(X(\beta)).$$

We denote the closure of $U \subseteq X$ by $\overline{U}$.

**Lemma 6.7.** For all $\beta \in \Omega$, in case (i) $V^S$ contains the sets

$$Y_\beta := \overline{U_\beta + k_\beta} = \overline{U_\beta + k_\beta}.$$  

(6.4)

**Proof.** If $\lambda \in V_\beta$, then by Lemma 6.6, $\lambda - p_\beta(\lambda) = a_\beta$ for some $a \in k$. By $W$-invariance, $V^S$ contains $\lambda + (t - a)\beta = p_\beta(\lambda) + t\beta$ for all $t \in k$. For the second equality in (6.4), note that $\overline{U_\beta \cap k_\beta} = 0$. 

Similarly in case (ii) $V^S$ contains the sets

$$Y_\beta := \overline{U_\beta \times \text{Im } c_\beta} = \overline{U_\beta \times \text{Im } c_\beta}.$$ 

Next if $X = h^*$ set $z_\beta = h_{\sigma(\beta)}$, and if $X = T$ set $z_\beta = e_{\sigma(\beta)} - 1$. Then let $L_\beta$ be the ideal of $\mathcal{O}(X)$ generated by $K_\beta$ and $z_\beta$.

**Lemma 6.8.** We have $\overline{U_\beta} = \mathcal{V}(K_\beta)$ and $Y_\beta = \mathcal{V}(L_\beta)$.

**Proof.** The first statement follows by [Clo97], Theorem 3.2.3, page 122 and the second is an immediate consequence. 

The ideal $K_\beta$ is called an elimination ideal. Next set $V_1 = \bigcup_{\beta \in \Omega} Y_\beta$. Then $V_1 = \mathcal{V}(I_1)$ where $I_1 = \bigcap_{\beta \in \Omega} L_\beta$. Because $I = \mathcal{I}(V)$ is a radical ideal, so too are $K_\beta, L_\beta$ and $I_1$. By Lemma 6.7 we have

**Proposition 6.9.** The set $V^S$ contains heart $V \cup V_1$. So, $V^S = \text{heart } V \cup V_1^S = V \cup V_1^S$.

The next result shows that $V_1$ and $I_1$ are $W$-invariant.


Lemma 6.10. If \( V \) is \( W \)-invariant, then for \( w \in W \), we have \( w(V_\beta) = V_{w \beta} \)

(a) \( w(V_\beta) = V_{w \beta} \), \( w(U_\beta) = U_{w \beta} \) and \( w(Y_\beta) = Y_{w \beta} \)

(b) \( w(K_\beta \mathcal{O}(X)) = K_{w \beta} \mathcal{O}(X) \) and \( w(L_\beta) = L_{w \beta} \).

Proof. The first statement in (a) follows from \( W \)-invariance of the bilinear form \( (\ , \ ) \), the second from \([6.3]\) and the third is an immediate consequence. By Lemma 6.3, the closed sets defined by \( w(K_\beta \mathcal{O}(X)) \) and \( K_{w \beta} \mathcal{O}(X) \) are \( w(U_\beta) \) and \( U_{w \beta} \). Since these are equal, the defining ideals are equal. This gives the first part in (b) and the second also follows.

Now we repeat this process. Set \( A = A(q) = \{ \beta_1, \ldots, \beta_q \} \) and \( V_A = V \cap \bigcap_{\beta \in A} \Pi_\beta \) in case (i) or \( V_A = V \cap \bigcap_{\beta \in A} \Pi_\beta \) in case (ii). Define \( p_A : X \to X \) by \( p_A = p_{\beta_1} \circ \cdots \circ p_{\beta_q} \).

Next, if \( X_A = \bigcap_{\beta \in A} X(\beta) \), then \( \mathcal{O}(X_A) = \bigcap_{\beta \in A} \mathcal{O}(X(\beta)) \). Consider the elimination ideal

\[
K_A = I(V_A) \cap \mathcal{O}(X_A).
\]

Let \( L_A \) be the ideal of \( \mathcal{O}(X) \) generated by \( K_A \) and \( \alpha_{i_1}, \ldots, \alpha_{i_q} \). If \( U_A = p_A(V_A) \), then \( \overline{U_A} = V(K_A) \).

Lemma 6.11. For all \( q \leq r \), \( V^S \) contains the sets

\[
\{ p_A(q)(\lambda) + \sum_{\beta \in A(q)} k_\beta \lambda \in V_A(q) \}.
\]

or \( \{ p_A(q)(\lambda) \prod_{\beta \in A(q)} \text{Im} c_\alpha | \lambda \in V_A(q) \} \) in case (ii).

Hence \( V^S \) also contains the closures \( Y_A(q) = V(L_A(q)) \) of these sets.

Proof. This follows as in the proof of Lemmas 6.7 and 6.8 by \( G \)-invariance.

We define \( V_q = \bigcup_{w \in W} wY_A(q) \) and \( I_q = \bigcap_{w \in W} wL_A(q) \). Then \( V_q = V(I_q) \). By construction if \( \lambda \in V_q \) then atyp \( \lambda \geq q \). Suppose \( I_q = \bigcap_{i=1}^k P_i \), an irredundant intersection of prime ideals, and suppose \( i \in [k] \) iff \( V(P_i) \) contain a point \( \lambda \) with atyp \( \lambda = q \). For \( i \in [k] \), set \( Q_i = P_i \cap A \). Then set \( \text{heart}_q V = V(\bigcap_{i=1}^k Q_i) \). If \( k = 0 \), set \( \text{heart}_q V = \emptyset \).

Theorem 6.12. The \( S \)-closure of \( V \) is \( V^S = \bigcup_{i=0}^r \text{heart}_i V = \bigcup_{i=0}^r V_i \). The set \( V^S \) is superalgebraic.

Proof. As in Proposition 6.9 if \( q < r \), we have \( V_q^S = \text{heart}_q V \cup V_{q+1}^S = V_q \cup V_{q+1}^S \) and \( V_r^S = \text{heart}_r V \). This gives the equalities. Each \( \text{heart}_i V \) is superalgebraic, so the result follows.

Corollary 6.13. The superalgebraic sets are exactly the closed sets that are invariant under the Weyl groupoid \( G \).

Proof. If \( V \) is closed and \( G \)-invariant, then \( V = V^S \) is superalgebraic. Conversely a superalgebraic set has the form \( V = V(I) \), with \( I \) an ideal of \( \mathcal{A} = \mathcal{O}(X) \). Since elements of \( \mathcal{A} \) are constant on orbits, \( V \) is a union of \( G \)-orbits.

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