VARIOUS ASPECTS OF DIFFERENTIAL EQUATIONS
HAVING A COMPLETE SET OF INDEPENDENT FIRST
INTEGRALS

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Abstract. In this paper we study the differential equations in $D \subseteq \mathbb{R}^{2N}$ having a complete set of independent first integrals. In particular we study the case when the first integrals are

$$f_\nu = (Ax_\nu + By_\nu)^2 + \sum_{j=1}^{N} \frac{(x_\nu y_j - x_j y_\nu)^2}{a_\nu - a_j},$$

for $\nu = 1, \ldots, N$, where $A, B$ and $a_1 < a_2 \ldots < a_N$ are constants.

MSC: (2000) 34A34.
Keywords: Hamiltonian systems, first integrals.

1. Introduction.

Let $D$ be an open subset of $\mathbb{R}^{2N}$. By definition an autonomous differential system is a system of the form

$$\dot{x} = X(x), \quad x \in D,$$

where the dependent variables $x = (x_1, \ldots, x_{2N})$ are real, the independent variable (the time $t$) is real and the $C^1$ functions $X(x) = (X_1(x), \ldots, X_K(x))$ are defined in the open set $D$.

For simplicity we shall assume the underlying functions to be of class $C^\infty$, although most results remain valid under weaker hypotheses.

Below we use the following notations

$$|S| = \{h_1, \ldots, h_{2N}\}^* = \begin{vmatrix} dh_1(\partial_1) & \ldots & dh_1(\partial_{2N}) \\ \vdots & \ddots & \vdots \\ dh_{2N}(\partial_1) & \ldots & dh_{2N}(\partial_{2N}) \end{vmatrix},$$

$$x_{N+j} = y_j,$$

for $j = 1, \ldots, N$. Our main result is the following

**Theorem 1.** Let $f_j = f_j(x_1, \ldots, x_N, y_1, \ldots, y_N)$ for $j = 1, 2, \ldots, N$ be a given set of independent functions defined in an open set $D \subset \mathbb{R}^{2N}$ and such that

i) If

$$|S|_0 = \{f_1, \ldots, f_N, x_1, \ldots, x_N\}^* \neq 0.$$
Then the differential systems in \( D \) which admit the set of first integrals \( f_j \) for \( j = 1, 2, \ldots, N \) are
\[
\dot{x}_k = \frac{\partial H}{\partial y_k},
\]
\[
\dot{y}_k = -\frac{\partial H}{\partial x_k} + \frac{1}{|S|_0} \sum_{j=1}^{N} \{ H, f_j \} \{ f_1, \ldots, f_{j-1}, y_k, f_{j+1}, \ldots, f_N, x_1, \ldots, x_N \}^*,
\]
for \( k = 1, \ldots, N \) where \( H = H(x_1, \ldots, x_N, y_1, \ldots, y_N) \), is an arbitrary function and
\[
\sum_{n=1}^{N} \left( \frac{\partial H}{\partial y_j} \frac{\partial f_0}{\partial x_j} - \frac{\partial H}{\partial y_j} \frac{\partial f_0}{\partial x_j} \right) = \{ H, f_0 \}.
\]
ii) If
\[
|S|_0 = 0, \quad |S|_N = \{ f_1, \ldots, f_N, x_1, \ldots, x_{N-1}, y_1 \}^* \neq 0.
\]
Then the differential systems in \( D \) which admit the set of first integrals \( f_j \) for \( j = 1, 2, \ldots, N \) are
\[
\dot{x}_k = \frac{\partial H}{\partial y_k},
\]
\[
\dot{y}_k = -\frac{\partial H}{\partial x_k} + \frac{1}{|S|_N} \sum_{j=1}^{N} \{ H, f_j \} \{ f_1, \ldots, f_{j-1}, y_k, f_{j+1}, \ldots, f_N, x_1, \ldots, x_N \}^* + \lambda \{ f_1, \ldots, f_N, x_1, \ldots, x_{k-1}, y_1, x_{k+1}, \ldots, x_{N-1} \}^*,
\]
for \( k = 1, \ldots, N \), where \( H = H(x_1, \ldots, x_N, y_1, \ldots, y_N) \), and \( \lambda = \lambda(x_1, \ldots, x_N, y_1, \ldots, y_N) \) are arbitrary functions.

2. Proof of Theorems

Proof of Theorem 2 In view of the relations
\[
\sum_{j,k=1}^{N} \{ H, f_j \} \{ f_1, \ldots, f_{j-1}, y_k, f_{j+1}, \ldots, f_N \}^* \frac{\partial f_0}{\partial y_k} = \{ H, f_0 \},
\]
we deduce that \( \dot{f}_0 = \{ H, f_0 \} - \{ H, f_0 \} = 0 \), for \( \alpha = 1, \ldots, N \). Thus the given functions are constant along the solutions of the given differential system. This is the proof of part 1) of the theorem.

Now we prove the part ii). After computation we obtain
\[
\dot{f}_\alpha = \lambda \{ f_1, \ldots, f_N \}^* \frac{\partial f_\alpha}{\partial x_N},
\]
for $\alpha = 1, \ldots, N$, hence in view of the assumptions we obtain the proof of part ii). In short Theorem 1 is proved.

We shall illustrate these results in the following example.

**Example 1.** For the case when

\[ f_1 = H = \frac{y_1^2}{2m_1} + \frac{y_2^2}{2m_2} + \frac{y_3^2}{2m_3} + \frac{a}{(x-y)^2} + \frac{b}{(z-x)^2(z-y)^2}, \]

\[ f_2 = x_1y_1 + x_2y_2 + x_3y_3, \]

\[ f_3 = y_1 + y_2 + y_3, \]

where $a, b, c$ are constants, we obtain that

\[ |S|_0 = \frac{(x_2-x_1)y_3}{m_3} + \frac{(x_1-x_3)y_2}{m_2} + \frac{(x_3-x_2)y_1}{m_1}. \]

By considering that

\[ \{f_1, f_2\} = 2f_1, \quad \{f_3, f_2\} = -f_3, \quad \{f_1, f_3\} = 0, \]

we deduce that differential system (2) takes the form

\[ \dot{x}_1 = \frac{\partial H}{\partial y_1}, \quad \dot{x}_2 = \frac{\partial H}{\partial y_2}, \quad \dot{x}_3 = \frac{\partial H}{\partial y_3}, \]

\[ \dot{y}_1 = -\frac{\partial H}{\partial x_1} + 2H \left( \frac{y_3}{m_3} - \frac{y_2}{m_2} \right), \]

\[ \dot{y}_2 = -\frac{\partial H}{\partial x_2} + 2H \left( \frac{y_1}{m_1} - \frac{y_3}{m_3} \right), \]

\[ \dot{y}_3 = -\frac{\partial H}{\partial x_3} + 2H \left( \frac{y_2}{m_2} - \frac{y_1}{m_1} \right). \]

We observe that these first integrals appear when we study the movement of three particles with masses $m_1, m_2, m_3$, which interact with each other with a force inversely proportional to the cube of the distance between them.

**Example 2.** It is well known that in the Kepler problem there are six first integrals

\[ \mathbf{M} = (M_1, M_2, M_3) = \mathbf{x} \times \mathbf{y} = c_1, \quad \mathbf{W} = (W_1, W_2, W_3) = \mathbf{y} \times \mathbf{M} + \frac{\mu \mathbf{x}}{r} = c_2, \]

where $r = ||\mathbf{x}|| = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

We determine differential system with three first integrals $(M_1, M_2, M_3)$ and $(W_1, W_2, W_3)$. For the first case we obtain that

\[ f_1 = M_1, \quad f_2 = M_2, \quad f_3 = M_3, \]

\[ |S|_1 = x_1M_3, \quad |S|_0 = 0, \]

\[ \{f_1, f_2\} = f_3, \quad \{f_2, f_3\} = f_1, \quad \{f_3, f_1\} = f_2. \]
Choosing $H = \frac{1}{2} (y_1^2 + y_2^2 + y_3^2)$ then system (4) takes the form
\[ \dot{x} = y, \quad \dot{y} = \lambda x. \]
These equations can be interpreted as geodesic flow of a particle which is constrained to move on the unit sphere $x_1^2 + x_2^2 + x_3^2 = 1$.

For the second case we obtain that
\[ f_1 = W_1, \quad f_2 = W_2, \quad f_3 = W_3, \]
where $\langle x, y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3$, $||M||^2 = M_1^2 + M_2^2 + M_3^2$, Taking $H = \frac{1}{2} (y_1^2 + y_2^2 + y_3^2)$ then from (2) we obtain the differential system
\[ \dot{x} = y, \quad \dot{y} = -\frac{x}{||x||^3}. \]
It is easy to observe that the first integrals $W_1, W_2, W_3$ admit the representation
\[ W_j = \frac{\partial F}{\partial x_j} \]
for $j = 1, 2, 3$ where
\[ F = 1/2 \left( ||x||^2 ||y||^2 - \langle x, y \rangle + \mu ||x|| \right). \]

**Example 3.** We determine the differential system (11) in the case
\[ f_1 = \Upsilon_1 x_1 + \Upsilon_2 x_2 + \Upsilon_3 x_3, \quad f_2 = \Upsilon_1 y_1 + \Upsilon_2 y_2 + \Upsilon_3 y_3, \quad f_3 = \frac{1}{2} \left( \Upsilon_1 (x_1^2 + y_1^2) + \Upsilon_2 (x_2^2 + y_2^2) + \Upsilon_3 (x_3^2 + y_3^2) \right), \]
where $\Upsilon_j = \text{constants}$ for $j = 1, 2, 3$. In this case we have
\[ |S|_0 = 0, \quad |S|_1 = a_2 a_3 (y_3 - y_2), \]
\[ \{f_2, f_1\} = \Upsilon_1 + \Upsilon_2 + \Upsilon_3, \quad \{f_3, f_1\} = f_2, \quad \{f_3, f_2\} = -f_1 \]
Clearly that $|S|_1 \neq 0$ if $y_2 - y_3 \neq 0$.
Taking
\[ \lambda_4 = \frac{1}{\Upsilon_1} \frac{\partial H}{\partial y_1}, \quad \lambda_5 = \frac{1}{\Upsilon_2} \frac{\partial H}{\partial y_2}, \]
we obtain that (11) takes the form
\[ \Upsilon_1 \dot{x}_1 = \frac{\partial H}{\partial y_1}, \quad \Upsilon_1 \dot{x}_2 = \frac{\partial H}{\partial y_2}, \quad \Upsilon_1 \dot{x}_3 = \frac{\partial H}{\partial y_3}, \]
\[ \Upsilon_1 \dot{y}_1 = -\frac{\partial H}{\partial x_1} + \lambda (y_2 - y_3), \]
\[ \Upsilon_2 \dot{y}_2 = -\frac{\partial H}{\partial x_2} + \lambda (y_3 - y_1), \]
\[ \Upsilon_3 \dot{y}_3 = -\frac{\partial H}{\partial x_3} + \lambda (y_1 - y_2), \]
where \( \lambda \) is an arbitrary function and \( H \) is the Hamiltonian function such that

\[
\sum_{j=1}^{3} \frac{1}{\Upsilon_j} \left( \frac{\partial H \partial f_\alpha}{\partial y_j \partial x_j} - \frac{\partial H \partial f_\alpha}{\partial x_j \partial y_j} \right) = 0,
\]

for \( \alpha = 1, 2, 3 \).

In particular if

\[
H = \sum_{m,k=1 \atop m \neq k}^{3} \Upsilon_m \Upsilon_k \log \sqrt{(x_k - x_m)^2 + (y_k - y_m)^2},
\]

and taking \( \lambda = 0 \) we obtain Hamiltonian differential equations of motion of three vortices with intensities \( \Upsilon_j \) for \( j = 1, 2, 3 \), (see for instance [1]).

**Example 4.** Now we study the following case

\[
f_\nu = (Ax_\nu + By_\nu)^2 + \sum_{j \neq \nu}^{N} \frac{(x_\nu y_j - x_j y_\nu)^2}{a_\nu - a_j},
\]

for \( \nu = 1, \ldots, N \), where \( A \) and \( B \) are constants.

It is easy to show that in all the cases the first integrals are in involution. Thus if we determine \( H = H(f_1, \ldots, f_N) \) then we obtain the completely integrable Hamiltonian system.

After some computation we obtain that \( |S|_0 \neq 0 \) if \( B \neq 0 \).

We apply Theorem [1] for these cases when \( N = 3 \).

For the case when \( B = 0 \) we obtain, after some calculations that

\[
|S|_1 = \frac{K}{\Delta} x_3 x_1, \quad |S|_2 = \frac{K}{\Delta} x_3 x_2, \quad |S|_3 = \frac{K}{\Delta} x_3 x_3,
\]

where \( \Delta = (a_1 - a_2)(a_2 - a_3)(a_1 - a_3) \), and

\[
K = a_1(x_2 y_3 - x_3 y_2)(x_2(x_1 y_2 - x_2 y_1) - x_1(x_3 y_1 - x_1 y_3)) + a_2(x_3 y_1 - x_1 y_3)(x_3(x_2 y_3 - x_3 y_2) - x_1(x_1 y_2 - x_2 y_1)) + a_3(x_1 y_2 - x_2 y_3)(x_1(x_3 y_1 - x_1 y_3) - x_2(x_2 y_3 - x_3 y_2))
\]

thus the differential system (4) takes the form

\[
\dot{x}_k = \frac{\partial H}{\partial y_k},
\]

\[
\dot{y}_k = -\frac{\partial H}{\partial x_k} + \lambda x_k.
\]

These differential equations described the behavior of the particle with Hamiltonian \( H \) and constrained to move on the sphere \( x_1^2 + x_2^2 + x_3^2 = 1 \).

In particular if we take

\[
H = \frac{1}{2} (a_1 f_1 + a_2 f_2 + a_3 f_3) = 1/2 (||x||^2 ||y||^2 - \langle x, y \rangle + a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2)
\]

\[
\lambda = \Psi(x_1^2 + x_1^2 + x_1^2),
\]

where \( \Psi \) is an arbitrary function.
then from the above equations we deduce the equation of motion of a particle on an 3-dimensional sphere, with an anisotropic harmonic potential. This system is one of the best understood integrable systems of classical mechanics (for more details see [2]).

For the case when $B \neq 0$ we obtain that $f_1, f_2, f_3 \neq 0$.

Acknowledgements

The first author was partly supported by the Spanish Ministry of Education through projects TSI2007-65406-C03-01 ”E-AEGIS” and Consolider CSD2007-00004 ”ARES”.

References

[1] V.V. Kozlov., Dynamical system X, General theory of vortices, Springer, (2003).
[2] J. Moser, Various aspects of integrable Hamiltonian systems. In S. Helagason J. Coates, editor, Dynamical Systems, C.I.M.E. Lectures, Bressanone 1978, pages 233(290), Birkhäuser, Boston, 2 edition, 1983.

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