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A set–valued framework for birth–and–growth process

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Summary. We propose a set–valued framework for the well–posedness of birth–and–growth process. Our birth–and–growth model is rigorously defined as a suitable combination, involving Minkowski sum and Aumann integral, of two very general set–valued processes representing nucleation and growth respectively. The simplicity of the used geometrical approach leads us to avoid problems arising by an analytical definition of the front growth such as boundary regularities. In this framework, growth is generally anisotropic and, according to a mesoscale point of view, it is not local, i.e. for a fixed time instant, growth is the same at each space point.

Introduction

Nucleation and growth processes arise in several natural and technological applications (cf. [7, 8] and the references therein) such as, for example, solidification and phase–transition of materials, semiconductor crystal growth, biomineralization, and DNA replication (cf., e.g., [15]).

A birth–and–growth process is a RaCS family given by \( \Theta_t = \bigcup_{n:T_n \leq t} \Theta_{T_n}(X_n) \), for \( t \in \mathbb{R}_+ \), where \( \Theta_{T_n}(X_n) \) is the RaCS obtained as the evolution up to time \( t > T_n \) of the germ born at (random) time \( T_n \) in (random) location \( X_n \), according to some growth model.

An analytical approach is often used to model birth–and–growth process, in particular it is assumed that the growth is driven according to a non–negative normal velocity, i.e. for every instant \( t \), a border point \( x \in \partial \Theta_t \) “grows” along the outward normal unit (e.g. [3–6, 11, 13, 22]). Thus, growth is pointwise isotropic; i.e. given a point belonging \( \partial \Theta_t \), the growth rate is independently from outward normal direction. Note that, the existence of the outward normal vector imposes a regularity condition on \( \partial \Theta_t \) and also on the nucleation process (it cannot be a point process).

This paper is an attempt to offer an original alternative approach based on a purely geometric stochastic point of view, in order to avoid regularity assumptions describing birth–and–growth process. In particular, Minkowski sum (already employed in [19] to describe self–similar growth for a single convex germ) and Aumann
integral are used here to derive a mathematical model of such process. This model, that emphasizes the geometric growth without regularity assumptions on \( \partial \Theta_t \), is rigorously defined as a suitable combination of two very general set–valued processes representing nucleation \( \{ B_t \}_{t \in [s_0, T]} \) and growth \( \{ G_t \}_{t \in [s_0, T]} \) respectively

\[
\Theta_t = \left\{ \Theta_{t_0} \oplus \int_{t_0}^t G_s ds \right\} \cup \bigcup_{s \in [t_0, t]} dB_s \\
\frac{d\Theta_t}{dt} = \oplus G_t dt \cup dB_t \quad \text{or} \quad \Theta_{t+dt} = (\Theta_t \oplus G_t dt) \cup dB_t.
\]

Roughly speaking, increment \( d\Theta_t \), during an infinitesimal time interval \( dt \), is an enlargement due to an infinitesimal Minkowski addend \( G_t dt \) followed by the union with the infinitesimal nucleation \( dB_t \).

As a consequence of Minkowski sum definition, for every instant \( t \), each point \( x \in \Theta_t \) (and then each point \( x \in \partial \Theta_t \)) grows up by \( G_t dt \) and no regularity border assumptions are required. Then we deal with not–local growth; i.e. growth is the same Minkowski addend for every \( x \in \Theta_t \). Nevertheless, under mesoscale hypothesis we can only consider constant growth region as described, for example, in [6]. On the other hand, growth is anisotropic whenever \( G_t \) is not a ball.

The aim of this paper is to ensure the well–posedness of such a model and, hence, to show that above “integral” and “differential” notations are meaningful.

In view of well–posedness, in [1], the authors show how the model leads to different and significant statistical results.

The article is organized as follows. Section 1.1 contains some assumptions about (random) closed sets and their basilar properties. Model assumptions are collected in Section 1.2 and integrability properties of growth process are studied in Section 1.3. For the sake of simplicity, we present, in Section 1.4, main results of the paper (that imply well-posedness of the model), whilst correspondent proofs are in Section 1.4.1. At the last, Section 1.5 proposes a discrete time point of view, also justifying integral and differential notations.

1.1 Preliminary results

Let \( \mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{R}_+ \) be the sets of all non–negative integer, integer, real and non–negative real numbers respectively. Let \( X, X^*, B_0^+ \) be a Banach space, its dual space and the unit ball of the dual space centered in the origin respectively. We shall consider

\[
\mathfrak{P}^0(X) = \text{the family of all subsets of } X, \quad \mathfrak{P}(X) = \mathfrak{P}^0(X) \setminus \{ \emptyset \} \\
\mathfrak{F}^0(X) = \text{the family of all closed subsets of } X, \quad \mathfrak{F}(X) = \mathfrak{F}^0(X) \setminus \{ \emptyset \}.
\]

The suffixes \( c \) and \( b \) denote convexity and boundedness properties respectively (e.g. \( \mathfrak{F}^b(X) \) denotes the family of all closed, bounded and convex subsets of \( X \)).

For all \( A, B \in \mathfrak{P}^0(X) \) and \( \alpha \in \mathbb{R}_+ \), let us define

\[
A + B = \{ a + b : a \in A, \ b \in B \} = \bigcup_{b \in B} b + A, \quad \text{(Minkowski Sum)}
\]

\[
\alpha \cdot A = \{ \alpha a : a \in A \}, \quad \text{(Scalar Product)}
\]

By definition, \( \forall A \in \mathfrak{P}^0(X), \ \alpha \in \mathbb{R}_+ \), we have \( \emptyset + A = \emptyset = \alpha \emptyset \). It is well known that \(+\) is a commutative and associative operation with a neutral element but \( (\mathfrak{P}(X), +) \) is not a group (cf. [20]). The following relations are useful in the sequel (see [21]): for all \( \forall A, B, C \in \mathfrak{P}(X) \)

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In the following, we shall work with closed sets. In general, if \( A, B \in \mathbb{F}^0(\mathcal{X}) \) then \( A + B \) does not belong to \( \mathbb{F}^0(\mathcal{X}) \) (e.g., in \( \mathcal{X} = \mathbb{R} \) let \( A = \{ n + 1/n : n > 1 \} \) and \( B = \mathbb{Z} \), then \( \{ 1/n = (n + 1/n) + (-n) \} \subseteq A + B \) and \( 1/n \downarrow 0 \), but \( 0 \notin A + B \)). In view of this fact, we define \( A \oplus B = A + B \) where \( (\cdot) \) denotes the closure in \( \mathcal{X} \).

For any \( A, B \in \mathbb{F}(\mathcal{X}) \) the Hausdorff distance (or metric) is defined by

\[
\delta_H(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} \|a - b\|_\mathcal{X}, \sup_{b \in B} \inf_{a \in A} \|a - b\|_\mathcal{X} \right\}.
\]

For all \( (x^*, A) \in B_1^* \times \mathbb{F}(\mathcal{X}) \), the support function is defined by \( s(x^*, A) = \sup_{a \in A} x^*(a) \). It can be proved (cf. \([2, 14]\)) that for each \( A, B \in \mathbb{F}_{bc}(\mathcal{X}) \),

\[
\delta_H(A, B) = \sup \{|s(x^*, A) - s(x^*, B)| : x^* \in B_1^*\}.
\] (1.1)

Let \((\Omega, \mathfrak{F})\) be a measurable space with \( \mathfrak{F} \) complete with respect to some \( \sigma \)-finite measure, let \( X : \Omega \to \mathbb{F}^0(\mathcal{X}) \) be a set–valued map, and

\[
D(X) = \{ \omega \in \Omega : X(\omega) \neq \emptyset \}
\]

be the domain of \( X \)

\[
X^{-1}(A) = \{ \omega \in \Omega : X(\omega) \cap A \neq \emptyset \}, \quad A \subseteq \mathcal{X},
\]

be the inverse image of \( X \).

Roughly speaking, \( X^{-1}(A) \) is the set of all \( \omega \) such that \( X(\omega) \) hits set \( A \).

Different definitions of measurability for set–valued functions are developed over the years by several authors (cf. \([2, 10, 16, 17]\) and reference therein). Here, \( X \) is measurable if, for each \( \Omega \), open subset of \( \mathcal{X} \), \( X^{-1}(\Omega) \in \mathfrak{F} \).

**Proposition 1.1.1** (See \([17]\)) \( X : \Omega \to \mathbb{F}^0(\mathcal{X}) \) is a measurable set–valued map if and only if \( D(X) \in \mathfrak{F} \), and \( \omega \mapsto d(x, X(\omega)) \) is a measurable function of \( \omega \in D(X) \) for each \( x \in \mathcal{X} \).

From now on, \( \mathcal{U}(\Omega, \mathfrak{F}; \mathbb{F}(\mathcal{X})) \) (= \( \mathcal{U}(\Omega; \mathbb{F}(\mathcal{X})) \)) if the measure \( \mu \) is clear) denotes the family of \( \mathbb{F}(\mathcal{X}) \)-valued measurable maps (analogous notation holds whenever \( \mathbb{F}(\mathcal{X}) \) is replaced by another family of subsets of \( \mathcal{X} \)).

Let \((\Omega, \mathfrak{F}, \mu)\) be a complete probability space and let \( X \in \mathcal{U}(\Omega, \mathfrak{F}; \mathbb{F}(\mathcal{X})) \), then \( X \) is a RaCS.

It can be proved (see \([18]\)) that, if \( X, X_1, X_2 \) are RaCS and if \( \xi \) is a measurable real–valued function, then \( X_1 \oplus X_2, X_1 \cap X_2, \xi X \) and \( (\text{Int } X)^\mathcal{F} \) are RaCS. Moreover, if \( \{X_n\}_{n \in \mathbb{N}} \) is a sequence of RaCS then \( X = \bigcup_{n \in \mathbb{N}} X_n \) is so.

Let \((\Omega, \mathfrak{F}, \mu)\) be a finite measure space (although most of the results are valid for \( \sigma \)-finite measures space). The Aumann integral of \( X \in \mathcal{U}(\Omega, \mathfrak{F}; \mathbb{F}(\mathcal{X})) \) is defined by

\[
\int_{\Omega} X d\mu = \left\{ \int_{\Omega} x d\mu : x \in S_X \right\},
\]

where \( S_X = \{ x \in L^1[\Omega, \mathcal{X}] : x \in X \mu - a.e. \} \) and \( \int_{\Omega} x d\mu \) is the usual Bochner integral in \( L^1[\Omega, \mathcal{X}] \). Moreover, \( \int_{\Omega} X d\mu = \left\{ \int_{\Omega} x d\mu : x \in S_X \right\} \) for \( A \in \mathfrak{F} \). If \( \mu \) is a probability measure, we denote the Aumann integral by \( EX = \int_{\Omega} X d\mu \).

Let \( X \in \mathcal{U}(\Omega, \mathfrak{F}; \mathbb{F}(\mathcal{X})) \), it is integrably bounded, and we shall write \( X \in L^1[\Omega, \mathfrak{F}; \mathbb{F}(\mathcal{X})] = L^1[\Omega; \mathbb{F}(\mathcal{X})] \), if \( \|X\|_h \in L^1[\Omega, \mathfrak{F}; \mathbb{R}] \).

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1.2 Model assumptions

Let us consider

\[
\Theta_t = \left( \Theta_0 \oplus \int_{t_0}^t G_s ds \right) \cup \bigcup_{s \in [t_0, t]} dB_s,
\]

\[
d\Theta_t = \oplus G_t dt \cup dB_t \quad \text{or} \quad \Theta_{t+dt} = (\Theta_t \oplus G_t dt) \cup dB_t.
\]

In fact, above equation is not a definition since, for example, problems arise handling non–countable union of (random) closed sets. The well–posedness of \(\text{(1.2)}\) and hence the existence of such a process are the main purpose of this paper.

From now on, let us consider the following assumptions.

(A-0) - \((X, \| \cdot \|_X)\) is a reflexive Banach space with separable dual space \((X^*, \| \cdot \|_{X^*})\).

then, \(X\) is separable too, see [12, Lemma II.3.16 p. 65]).

- \([t_0, T] \subset \mathbb{R}\) is the time observation interval (or time interval).

- \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [t_0, T]}, P)\) is a filtered probability space, where the filtration \(\{\mathcal{F}_t\}_{t \in [t_0, T]}\) is assumed to have the usual properties.

(Nucleation Process). \(B = \{B(\omega, t) = B_t : \omega \in \Omega, t \in [t_0, T]\}\) is a process with non–empty closed values, i.e. \(B : \Omega \times [t_0, T] \rightarrow \mathcal{F}(X)\) such that

(A-1) \(B(\cdot, t) \in \mathcal{U}[\Omega, \mathcal{G}_t, P; \mathcal{F}(X)]\), for every \(t \in [t_0, T]\), i.e. \(B_t\) is an adapted (to \(\{\mathcal{G}_t\}_{t \in [t_0, T]}\)) process.

(A-2) \(B_t\) is increasing: for every \(t, s \in [t_0, T]\) with \(s < t\), \(B_s \subseteq B_t\).

(A-3) for every \(\omega \in \Omega\) and \(t \in [t_0, T]\), \(0 \in G(\omega, t)\).

(A-4) for every \(\omega \in \Omega\) and \(t \in [t_0, T]\), \(G(\omega, t)\) is convex, i.e. \(G : \Omega \times [t_0, T] \rightarrow \mathcal{F}_c(X)\).

(A-5) there exists \(K \in \mathbb{F}_b(X)\) such that \(G(\omega, t) \subseteq K\) for every \(t \in [t_0, T]\) and \(\omega \in \Omega\).

As a consequence, \(G(\omega, t) \in \mathbb{F}_b(X)\) and \(\|G(\omega, t)\|_h \leq \|K\|_h, \forall (\omega, t) \in \Omega \times [t_0, T]\).

In order to establish the well–posedness of integral \(\int_{t_0}^t G_s ds\) in \(\mathbb{F}_b(X)\), let us consider a suitable hypothesis of measurability for \(G\) (analogously to what is).

\(\mathcal{F}(X)\)–valued process \(G = \{G_t\}_{t \in [t_0, T]}\) has left continuous trajectories on \([t_0, T]\) if, for every \(\mathbf{T} \in [t_0, T]\) with \(t < T\),

\[
\lim_{t \to T} \delta_{\mathbf{T}} (G(\omega, t), G(\omega, T)) = 0, \quad \text{a.s.}
\]

The \(\sigma\)-algebra on \(\Omega \times [t_0, T]\) generated by the processes \(\{G_t\}_{t \in [t_0, T]}\) with left continuous trajectories on \([t_0, T]\), is called the previsible (or predictable) \(\sigma\)-algebra and it is denoted by \(\mathcal{P}\).

**Proposition 1.2.1** The previsible \(\sigma\)-algebra is also generated by the collection of random sets \(A \times t_0\) where \(A \in \mathcal{G}_{t_0}\) and \(A \times (s, t]\) where \(A \in \mathcal{G}_s\) and \((s, t] \subset [t_0, T]\).
Proof. Let the $\sigma$-algebra generated by the above collection of sets be denoted by $\mathcal{P}'$. We shall show $\mathcal{P} = \mathcal{P}'$. Let $G$ be a left continuous process and let $\alpha = (T - t_0)$, consider for $n \in \mathbb{N}$

$$G_n(\omega, t) = \begin{cases} G(\omega, t_0), & t = t_0 \\ G\left(\omega, t_0 + \frac{k_0}{2^n}\right), & (t_0 + \frac{k_0}{2^n}) < t \leq \left(t_0 + \frac{(k_0+1)\alpha}{2^n}\right) \\ k \in \{0, \ldots , (2^n - 1)\} \end{cases}$$

It is clear that $G_n$ is $\mathcal{P}'$-measurable, since $G$ is adapted. As $G$ is left continuous, the above sequence of left-continuous processes converges pointwise (with respect to $\delta_H$) to $G$ when $n$ tends to infinity, so $G$ is $\mathcal{P}'$-measurable, thus $\mathcal{P} \subseteq \mathcal{P}'$.

Conversely consider $A \times (s, t) \in \mathcal{P}'$ with $(s, t) \subset [t_0, T]$ and $A \in \mathcal{F}_s$. Let $b \in X \setminus \{0\}$ and $G$ be the process

$$G(\omega, v) = \begin{cases} b, & v \in (s, t], \omega \in A \\ 0, & \text{otherwise} \end{cases}$$

this function is adapted and left continuous, hence $\mathcal{P}' \subseteq \mathcal{P}$. 

Then let us consider the following assumption.

(A-6) $G$ is $\mathcal{P}$-measurable.

1.3 Growth process properties

Theorem 1.3.2 is the main result in this section. It shows that $\omega \mapsto \int_a^b G(\omega, \tau)d\tau$ is a RaCS with non-empty bounded convex values. This is the first step in order to obtain well-posedness of (1.2).

Proposition 1.3.1 Suppose (A-3), . . . , (A-6) and let $\mu_\lambda$ be the Lebesgue measure on $[t_0, T]$, then

- $G(\omega, \cdot) \in \mathcal{U}\left([t_0, T], \mathcal{B}[t_0, T], \mu_\lambda; \mathcal{F}_{bc}(X)\right)$ for every $\omega \in \Omega$.
- $G(\cdot, t) \in \mathcal{U}(\Omega, \mathcal{F}_{t-}, \mathcal{P}; \mathcal{F}_{bc}(X))$ for each $t \in [t_0, T]$, where $\mathcal{F}_{t-}$ is the so called history $\sigma$-algebra i.e. $\mathcal{F}_{t-} = \sigma(\mathcal{F}_s : 0 \leq s < t) \subseteq \mathcal{F}$.
- $G \in L^1([t_0, T], \mathcal{B}[t_0, T], \mu_\lambda; \mathcal{F}_{bc}(X)) \cap L^1(\Omega, \mathcal{F}, \mathcal{P}; \mathcal{F}_{bc}(X))$

Proof. Assumptions (A-3) and (A-4) imply that $G$ is non-empty and convex. Measurability and integrability properties are consequence of (A-6) and (A-5) respectively.

Theorem 1.3.2 Suppose (A-3) . . . (A-6) For every $a, b \in [t_0, T]$, the integral $\int_a^b G(\omega, \tau)d\tau$ is non-empty and the set-valued map

$$G_{a,b} : \Omega \to \mathcal{P}(X) \quad \omega \mapsto \int_a^b G(\omega, \tau)d\tau$$

is measurable. Moreover, $G_{a,b}$ is a non-empty, bounded convex RaCS.
In order to prove Theorem 1.3.2 consider following properties for real processes. A real-valued process \( X = \{ X_t \}_{t \in [0,T]} \) is predictable with respect to filtration \( \mathcal{F}_t \), if it is measurable with respect to the predictable \( \sigma \)-algebra \( P_\mathcal{F} \), i.e. the \( \sigma \)-algebra generated by the collection of random sets \( A \times \{ 0 \} \) where \( A \in \mathcal{F}_0 \) and \( A \times (s,t] \) where \( A \in \mathcal{F}_s \).

**Proposition 1.3.3** (See [9, Propositions 2.30, 2.32 and 2.41]) Let \( X = \{ X_t \}_{t \in [0,T]} \) be a predictable real–valued process, then \( X \) is \( \mathcal{P} \otimes \mathcal{B}_{[0,T]} \)-measurable. Further, for every \( \omega \in \Omega \), the trajectory \( X(\omega, \cdot) : [t_0, T] \to \mathbb{R} \) is \( (\mathcal{B}_{[t_0,T]}, \mathcal{F}_0) \)-measurable.

**Lemma 1.3.4** Let \( x^* \) be an element of the unit ball in the dual space \( B_1^* \), then \( G \mapsto s(x^*, G) \) is a measurable map.

**Proof.** By definition \( s(x^*, G) = \sup \{ x^*(g) : g \in G \} \). Since \( \mathcal{X} \) is separable \([A-0]\) there exists \( \{ g_n \}_{n \in \mathbb{N}} \subset G \) such that \( G = \{ g_n \} \). Then, for every \( x^* \in B_1^* \) we have

\[
s(x^*, G) = \sup_{g \in G} x^*(g) = \sup_{n \in \mathbb{N}} x^*(g_n).
\]

Since \( x^* \) is a continuous map then, \( s(x^*, \cdot) \) is measurable. \( \square \)

**Proof of Theorem 1.3.2** At first, we prove that \( G_{a,b} \) is a measurable map. From Proposition 1.3.1 integral \( G_{a,b} = \int_a^b G(\omega, \tau) d\tau \) is well defined for all \( \omega \in \Omega \). Assumption \([A-3]\) implies \( 0 \in G_{a,b}(\omega) \neq \emptyset \) for every \( \omega \in \Omega \). Hence, the domain of \( G_{a,b} \) is the whole \( \Omega \) for all \( a, b \in [t_0, T] \)

\[
D (G_{a,b}) = \{ \omega \in \Omega : G_{a,b} \neq \emptyset \} = \Omega \in \mathcal{F}.
\]

Thus, by Proposition 1.1.1 and for a fixed couple \( a, b \in [t_0, T] \), \( G_{a,b} \) is (weakly) measurable if and only if, for every \( x \in \mathcal{X} \), the map

\[
\omega \mapsto d \left( x, \int_a^b G(\omega, \tau) d\tau \right) = \delta_H \left( x, \int_a^b G(\omega, \tau) d\tau \right)
\]

is measurable. Equation (1.1) guarantees that (1.3) is measurable if and only if, for every \( x \in \mathcal{X} \), the map

\[
\omega \mapsto \sup_{x^* \in B_1^*} s(x^*, x) - s \left( x^*, \int_a^b G(\omega, \tau) d\tau \right)
\]

is measurable. The above expression can be computed on a countable family dense in \( B_1^* \) (note that such family exists since \( \mathcal{X}^* \) is assumed separable \([A-0]\)):

\[
\omega \mapsto \sup_{n \in \mathbb{N}} \left| s(x^*_n, x) - s \left( x^*_n, \int_a^b G(\omega, \tau) d\tau \right) \right|.
\]

It can be proved ( [18, Theorem 2.1.12 p. 46]) that

\[
s \left( x^*, \int_a^b G(\omega, \tau) d\tau \right) = \int_a^b s \left( x^*, G(\omega, \tau) \right) d\tau, \quad \forall x^* \in B_1^*
\]

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and therefore, since \( s(x^*, x) \) is a constant, \( G_{a,b} \) is measurable if, for every \( x^* \in \{x_i^*\}_{i \in \mathbb{N}} \), the following map
\[
(\Omega, \mathcal{F}) \rightarrow (\mathbb{R}, \mathcal{B}_\mathbb{R}) \quad \omega \mapsto \int_a^b s(x^*, G(\omega, \tau)) \, d\tau
\]
is measurable. Note that \( s(x^*, G(\cdot, \cdot)) \), as a map from \( \Omega \times [t_0, T] \) to \( \mathbb{R} \), is predictable since it is the composition of a predictable map \([A-6]\) with a measurable one (see Lemma \([A-3.4]\):
\[
s(x^*, G(\cdot, \cdot)) : (\Omega \times [t_0, T], \mathcal{P}) \rightarrow (\mathcal{F}(\mathcal{X}), \sigma_f) \rightarrow (\mathbb{R}, \mathcal{B}_\mathbb{R}) \quad (\omega, t) \mapsto G(\omega, t) \mapsto s(x^*, G(\omega, t))
\]
thus, by Proposition \([A-3.3]\) it is a \( \mathcal{P} \)-measurable map and hence \((1.4)\) is a measurable map.

In view of the first part, it remains to prove that \( G_{a,b} \) is a bounded convex set for a.e. \( \omega \in \mathcal{H} \). Since \( X \) is reflexive \([A-0]\) by Proposition \([A-3.1]\) we have that \( G_{a,b} \) is closed (\([18, \text{Theorem 2.2.3}]\)). Further, \( G_{a,b} \) is also convex (see \([18, \text{Theorem 2.1.5 and Corollary 2.1.6}]\)).

To conclude the proof, it is sufficient to show that \( G_{a,b} \) is included in a bounded set:
\[
\int_a^b G(\omega, \tau) \, d\tau = \left\{ \int_a^b g(\omega, \tau) \, d\tau : g(\omega, \cdot) \in G(\omega, \cdot) \subseteq K \right\} \\
\subseteq \left\{ \int_a^b k \, d\tau : k \in K \right\} = \{(b-a)k : k \in K\} = (b-a)K.
\]

\[\blacksquare\]

1.4 Geometric Random Process

For the sake of simplicity, let us present the main results which proofs will be given in Section \([1.4.1]\).

Let us assume conditions from \([A-0]\) to \([A-6]\). For every \( t \in [t_0, T] \subset \mathbb{R}, n \in \mathbb{N} \) and \( \Pi = (t_i)_{i=0}^n \) partition of \([t_0, t]\), let us define
\[
s_{\Pi}(t) = \left( B_{t_0} \oplus \int_{t_0}^t G(\tau) \, d\tau \right) \cup \bigcup_{i=1}^n \left( \Delta B_{t_i} \oplus \int_{t_i}^t G(\tau) \, d\tau \right) \quad (1.5)
\]
\[
S_{\Pi}(t) = \left( B_{t_0} \oplus \int_{t_0}^t G(\tau) \, d\tau \right) \cup \bigcup_{i=1}^n \left( \Delta B_{t_i} \oplus \int_{t_{i-1}}^t G(\tau) \, d\tau \right) \quad (1.6)
\]
where \( \Delta B_{t_i} = B_{t_i} \setminus B_{t_{i-1}}^\circ \) (\( B_{t_{i-1}}^\circ \) denotes the interior set of \( B_{t_{i-1}} \)) and where the integral is in the Aumann sense with respect to the Lebesgue measure \( d\tau = d\mu_\lambda \).

We write \( s_{\Pi} \) and \( S_{\Pi} \) instead of \( s_{\Pi}(t) \) and \( S_{\Pi}(t) \) when the dependence on \( t \) is clear.

Proposition \([1.4.1]\) guarantees that both \( s_{\Pi} \) and \( S_{\Pi} \) are well defined RaCS, further, Proposition \([1.4.2]\) shows \( s_{\Pi} \subseteq S_{\Pi} \) as a consequence of different time intervals integration: if the time interval integration of \( G \) increases then the integral of \( G \) does not decrease with respect to set-inclusion (Lemma \([1.4.2]\)). Proposition \([1.4.3]\) means that \( \{s_{\Pi}\} \subseteq \{S_{\Pi}\} \) increases (decreases) whenever a refinement of \( \Pi \) is considered.
At the same time, Proposition 1.4.5 implies that $s_H$ and $S_H$ become closer each other (in the Hausdorff distance sense) when partition $\Pi$ becomes finer. The “limit” is independent on the choice of the refinement as consequence of Proposition 1.4.6. Corollary 1.4.7 means that, given any $\{\Pi_j\}_{j \in \mathbb{N}}$ refinement sequence of $[t_0, t]$, the random closed sets $s_{H_j}$ and $S_{H_j}$ play the same role that lower sums and upper sums have in classical analysis when we define the Riemann integral. In fact, if $\Theta_t$ denotes their limit value (see (1.7)), $s_{H_j}$ and $S_{H_j}$ are a lower and an upper approximation of $\Theta_t$, respectively. Note that, as a consequence of monotonicity of $s_{H_j}$ and $S_{H_j}$, we avoid problems that may arise considering uncountable unions in integral expression in (1.2).

**Proposition 1.4.1** Let $\Pi$ be a partition of $[t_0, t]$. Both $s_H$ and $S_H$, defined in (1.5) and (1.6), are RaCS.

**Lemma 1.4.2** Let $X \in L^1([I, \mathbb{R}; \mu_\lambda; \mathbb{F}(X)])$, where $I$ is a bounded interval of $\mathbb{R}$, such that $0 \in X$ $\mu_\lambda$-almost everywhere on $I$ and let $I_1, I_2$ be two other intervals of $\mathbb{R}$ with $I_1 \subset I_2 \subset I$. Then

$$\int_{I_1} X(\tau) d\tau \subseteq \int_{I_2} X(\tau) d\tau.$$ 

**Proposition 1.4.3** Let $\Pi$ be a partition of $[t_0, t]$. Then $s_H \subseteq S_H$ almost surely.

**Proposition 1.4.4** Let $\Pi$ and $\Pi'$ be two partitions of $[t_0, t]$ such that $\Pi'$ is a refinement of $\Pi$. Then, almost surely, $s_H \subseteq s_{H'}$ and $S_H \subseteq S_{H'}$.

**Proposition 1.4.5** Let $\{\Pi_j\}_{j \in \mathbb{N}}$ be a refinement sequence of $[t_0, t]$ (i.e. $|\Pi_j| \to 0$ if $j \to \infty$). Then, almost surely, $\lim_{j \to \infty} \delta_H (s_{H_j}, S_{H_j}) = 0$.

**Proposition 1.4.6** Let $\{\Pi_j\}_{j \in \mathbb{N}}$ and $\{\Pi'_l\}_{l \in \mathbb{N}}$ be two distinct refinement sequences of $[t_0, t]$, then, almost surely,

$$\lim_{j \to \infty} \delta_H (s_{H_j}, s_{H'_l}) = 0 \quad \text{and} \quad \lim_{l \to \infty} \delta_H (S_{H_j}, S_{H'_l}) = 0.$$ 

**Corollary 1.4.7** For every $\{\Pi_j\}_{j \in \mathbb{N}}$ refinement sequence of $[t_0, t]$, the following limits exist

$$\left( \bigcup_{j \in \mathbb{N}} s_{H_j} \right), \quad \left( \lim_{j \to \infty} s_{H_j} \right), \quad \lim_{j \to \infty} S_{H_j}, \quad \bigcap_{j \in \mathbb{N}} S_{H_j},$$  

and they are equals almost surely. The convergences is taken with respect to the Hausdorff distance.

We are now ready to define the continuous time stochastic process.

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Definition 1.4.8 Assume \([A-0] \ldots [A-6]\). For every \(t \in [t_0, T]\), let \(\{\Pi_i\}_{i \in N}\) be a refinement sequence of the time interval \([t_0, t]\) and let \(\Theta_t\) be the RaCS defined by

\[
\bigcup_{j \in N} s_{\Pi_j}(t) = \left( \lim_{j \to \infty} s_{\Pi_j}(t) \right) = \Theta_t = \lim_{j \to \infty} S_{\Pi_j}(t) = \bigcap_{j \in N} S_{\Pi_j}(t),
\]

then, the family \(\Theta = \{\Theta_t : t \in [t_0, T]\}\) is called geometric random process G-RaP (on \([t_0, T]\)).

Theorem 1.4.9 Let \(\Theta\) be a G-RaP on \([t_0, T]\), then \(\Theta\) is a non-decreasing process with respect to the set inclusion, i.e. 

\[P(\Theta_s \subseteq \Theta_t, \forall t_0 \leq s < t \leq T) = 1.\]

Moreover, \(\Theta\) is adapted with respect to filtration \(\{\mathcal{F}_t\}_{t \in [t_0, T]}\).

Remark 1.4.10 We want to point out that, assumptions we considered on Minkowski addition properties, in fact process with “null growth”).

1.4.1 Proofs of Propositions in Section 1.4

Proof of Proposition 1.4.1 For every \(i \in \{0, \ldots, n\}\), \(\int_{I_{i-1}}^t G(\tau)d\tau\) is a RaCS (Theorem 1.3.3). Thus, measurability Assumption [A-1] on \(B\) guarantees that, for every \(t_i \in \Pi, B_{t_i}, \Delta B_{t_i}, \left(\Delta B_{t_i} \oplus \int_{t_i}^t G(\tau)d\tau\right)\), and hence \(s_H\) and \(S_H\) are RaCS.

Proof of Lemma 1.4.2 Let \(y \in \left(\int_{I_i} X(\tau)d\tau\right)\), then there exists \(x \in S_X\), for which \(y = \left(\int_{I_i} x(\tau)d\tau\right)\). Let us define on \(J_2(\supset I_1)\)

\[x'(\tau) = \begin{cases} x(\tau), & \tau \in I_1 \\ 0, & \tau \in J_2 \setminus I_1 \end{cases}\]

then \(x' \in S_X\) and \(y = \left(\int_{I_i} x'(\tau)d\tau\right) \in \left(\int_{I_2} X(\tau)d\tau\right)\).

Proof of Proposition 1.4.3 Thesis is a consequence of Lemma 1.4.2 and Minkowski addition properties, in fact \(\left(\int_{I_{i-1}}^t G(\tau)d\tau\right) \subseteq \left(\int_{I_i}^t G(\tau)d\tau\right)\) implies \(s_H \subseteq S_H\).

Proof of Proposition 1.4.4 Let \(\Pi'\) be a refinement of partition \(\Pi\) of \([t_0, t]\), i.e. \(\Pi \subset \Pi'\). We prove that \(s_H \subseteq s_{\Pi'}\) (\(S_H \subseteq S_{\Pi'}\) is analogous). It is sufficient to show the thesis only for \(\Pi' = \Pi \cup \{t\}\) where \(\Pi = \{t_0, \ldots, t_n\}\) with \(t_0 < \ldots < t_n = t\) and \(t \in (t_0, t)\). Let \(i \in \{0, \ldots, (n - 1)\}\) be such that \(t_i \leq t_i < t_{i+1}\) then

\[
s_H = \left(\int_{t_0}^t G(\tau)d\tau\right) \cup \bigcup_{j = 1}^n \left(\Delta B_{t_j} \oplus \int_{t_j}^t G(\tau)d\tau\right) \cup \left(B_{t_{i+1}} \setminus B_{t_i}\right) \oplus \int_{t_{i+1}}^t G(\tau)d\tau.
\]
such that \( \lim_{m \to \infty} s_H = m \in \mathbb{R} \). By definition of \( \oplus \), if \( y \leq 0 \), for every \( \omega \in \Omega \), we have to prove that, whenever \( d \in \Pi_j \), the \( j \)-partition of the refinement sequence \( \{\Pi_j\}_{j \in \mathbb{N}} \), then

\[
\delta_H (s_{\Pi_j}, s_{\Pi_j}) = \max \left\{ \sup_{x \in s_{\Pi_j}} d(x, s_{\Pi_j}), \sup_{y \in s_{\Pi_j}} d(y, s_{\Pi_j}) \right\}
\]

where \( d(x, s_{\Pi_j}) = \inf_{y \in s_{\Pi_j}} \| x - y \|_X \). By Proposition 1.4.3, \( s_{\Pi_j} \subseteq S_{\Pi_j} \) then

\[
\sup_{x \in s_{\Pi_j}} d(x, S_{\Pi_j}) = 0
\]

and hence we have to prove that, whenever \( j \to \infty \) (i.e. \( |\Pi_j| \to 0 \)),

\[
\delta_H (s_{\Pi_j}, S_{\Pi_j}) = \sup_{y \in S_{\Pi_j}} d(y, s_{\Pi_j}) = \sup_{y \in S_{\Pi_j}} \inf_{x \in s_{\Pi_j}} \| x - y \|_X \to 0.
\]

For every \( \omega \in \Omega \), let \( y \) be any element of \( S_{\Pi_j} (\omega) \), then we distinguish two cases:

1. if \( y \in \left( B_{t_0} (\omega) \oplus \int_{t_0}^t G(\omega, \tau) d\tau \right) \), then it is also an element of \( s_{\Pi_j} (\omega) \), and hence \( d (s_{\Pi_j} (\omega), y) = 0 \).

2. if \( y \notin \left( B_{t_0} (\omega) \oplus \int_{t_0}^t G(\omega, \tau) d\tau \right) \), then there exist \( j \in \{1, \ldots, n\} \) such that

\[
y \in \left( \Delta B_{t_j} (\omega) \oplus \int_{t_{j-1}}^t G(\omega, \tau) d\tau \right).
\]

By definition of \( \oplus \), for every \( \omega \in \Omega \), there exist

\[
\{y_m\}_{m \in \mathbb{N}} \subseteq \left( \Delta B_{t_j} (\omega) + \int_{t_{j-1}}^t G(\omega, \tau) d\tau \right),
\]

such that \( \lim_{m \to \infty} y_m = y \). Then, for every \( \omega \in \Omega \), there exist \( h_m \in \Delta B_{t_j} (\omega) \) and \( g_m \in \left( \int_{t_{j-1}}^t G(\omega, \tau) d\tau \right) \) such that \( y_m = (h_m + g_m) \) and hence

\[
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\[ y = \lim_{m \to \infty} (h_m + g_m) = \lim_{m \to \infty} y_m \]

where the convergence is in the Banach norm, then let \( m \in \mathbb{N} \) be such that \( \|y - y_m\|_X < |II_j| \), for every \( m > m \).

Note that, for every \( \omega \in \Omega \) and \( m \in \mathbb{N} \), by Aumann integral definition, there exists a selection \( \hat{g}_m(\cdot) \) of \( G(\omega, \cdot) \) (i.e. \( \hat{g}_m(t) \in G(\omega, t) \) μ\(X\)-a.e.) such that
\[
g_m = \int_{t_j}^{t_j'} \hat{g}_m(\tau)d\tau \quad \text{and} \quad y_m = h_m + \int_{t_j}^{t_j'} \hat{g}_m(\tau)d\tau.
\]

For every \( \omega \in \Omega \), let us consider
\[
x_m = h_m + \int_{t_j}^{t_j'} \hat{g}_m(\tau)d\tau
\]
then \( x_m \in s_{II_j}(\omega) \) for all \( m \in \mathbb{N} \). Moreover, the following chain of inequalities hold, for all \( m > m \) and \( \omega \in \Omega \),
\[
\begin{align*}
\inf_{x' \in s_{II_j}} \|x' - y\|_X & \leq \|x_m - y\|_X \leq \|x_m - y_m\|_X + \|y_m - y\|_X \\
& \leq \left\| \int_{t_j}^{t_j'} \hat{g}_m(\tau)d\tau \right\|_X + |II_j| \leq \int_{t_j}^{t_j'} \|\hat{g}_m(\tau)\|_X d\tau + |II_j| \\
& \leq \int_{t_j}^{t_j'} \|G(\tau)\|_h d\tau + |II_j| \leq |t_j - t_j'| \|K\|_h + |II_j| \\
& \leq |II_j| (\|K\|_h + 1) \frac{1}{n} \to 0
\end{align*}
\]
since \( \|K\|_h \) is a positive constant. By the arbitrariness of \( y \in S_{II_j}(\omega) \) we obtain the thesis.

**Proof of Proposition 1.4.6** Let \( II_j \) and \( II_j' \) be two partitions of the two distinct refinement sequences \( \{II_j\}_{j \in \mathbb{N}} \) and \( \{II_j'\}_{j \in \mathbb{N}} \) of \([t_0, t]\). Let \( II'' = II_j \cup II_j' \) be the refinement of both \( II_j \) and \( II_j' \). Then Proposition 1.4.4 and Proposition 1.4.3 imply that \( s_{II_j} \subseteq s_{II''} \subseteq s_{II_j'} \). Therefore \( s_{II_j} \subseteq s_{II_j'} \) for every \( j \in \mathbb{N} \). Then
\[
\bigcup_{j \in \mathbb{N}} s_{II_j} \subseteq \bigcap_{j \in \mathbb{N}} S_{II_j'}.
\]

Analogously
\[
\bigcup_{j \in \mathbb{N}} s_{II_j'} \subseteq \bigcap_{j \in \mathbb{N}} S_{II_j}.
\]

Proposition 1.4.6 concludes the proof.

In order to prove Theorem 1.4.9 let us consider the following Lemma.

**Lemma 1.4.11** Let \( s, t \in [t_0, T] \) with \( t_0 < s < t \) and let \( II^s \) and \( II^t \) be two partition of \([t_0, s]\) and \([t_0, t]\) respectively, such that \( II^s \subseteq II^t \). Then
\[
s_{II^t}(s) \subseteq s_{II^s}(t) \quad \text{and} \quad S_{II^t}(s) \subseteq S_{II^s}(t).
\]

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Proof. The proofs of the two inclusions are similar. Let us prove that \( s_{H^*}(s) \subseteq s_{H}(t) \).

Since \( H^* \subset H^t \), then \( H^* = (t_i)_{i=0}^{n} \) and \( H^t = (t_i)_{i=n+1}^{m} \) with \( m \in \mathbb{N} \). By Lemma 1.4.2, we have that

\[
    s_{H^*}(s) = \left( B_{t_0} \oplus \int_{t_0}^{s} G(\tau) d\tau \right) \cup \bigcup_{i=1}^{n} \left( \Delta B_{t_i} \oplus \int_{t_i}^{t} G(\tau) d\tau \right)
\]

\[
    \subseteq \left( B_{t_0} \oplus \int_{t_0}^{t} G(\tau) d\tau \right) \cup \bigcup_{i=1}^{n} \left( \Delta B_{t_i} \oplus \int_{t_i}^{t} G(\tau) d\tau \right)
\]

\[
    \subseteq \left( B_{t_0} \oplus \int_{t_0}^{t} G(\tau) d\tau \right) \cup \bigcup_{i=1}^{n} \left( \Delta B_{t_i} \oplus \int_{t_i}^{t} G(\tau) d\tau \right)
\]

\[
    \cup \bigcup_{i=n+1}^{m} \left( \Delta B_{t_i} \oplus \int_{t_i}^{t} G(\tau) d\tau \right)
\]

i.e. \( s_{H^*}(s) \subseteq s_{H}(t) \).

Proof of Theorem 1.4.9

For every \( s, t \in [t_0, T] \) with \( s < t \), let \( \{ H^*_i \}_{i \in \mathbb{N}} \) and \( \{ H^t_i \}_{i \in \mathbb{N}} \) be two refinement sequences of \([t_0, t] \) respectively, such that \( H^*_i \subset H^t_i \) for every \( i \in \mathbb{N} \). Then, by Lemma 1.4.11 \( s_{H^*_i} \subseteq s_{H^t_i} \). Now, as \( i \) tends to infinity, we obtain

\[
    \Theta_s = \bigcap_{i=\infty} s_{H^*_i} \subseteq \bigcap_{i=\infty} s_{H^t_i} = \Theta_t.
\]

For the second part, note that Theorem 1.4.2 still holds replacing \( \tilde{\mathcal{F}} \) instead of \( \tilde{\mathcal{F}} \), so that for every \( s \in [t_0, T] \), the family \( \left\{ \int_{t}^{T} G(\omega, \tau) d\tau \right\}_{\tau \in [s, T]} \) is an adapted process to the filtration \( \{ \tilde{\mathcal{F}}_t \}_{t \in [t_0, T]} \). This fact together with Assumption \([A-1]\) guarantees that \( \{ s_{H^*_i} \}_{i \in \mathbb{N}} \) is adapted for every partition \( H \) of \([s, T]\) and hence \( \Theta \) is adapted too.

1.5 Discrete time case and infinitesimal notations

Let us consider \( \Theta_s \) and \( \Theta_t \) with \( s < t \). Let \( \{ H^*_i \}_{i \in \mathbb{N}} \) and \( \{ H^t_j \}_{j \in \mathbb{N}} \) be two refinement sequences of \([t_0, t] \) respectively, such that \( H^*_j \subset H^t_j \) for every \( j \in \mathbb{N} \) (i.e. \( H^*_j = (t_i)_{i=0}^{n} \) and \( H^t_j = (t_i)_{i=n+1}^{m} \) with \( n, m \in \mathbb{N} \)). It is easy to compute

\[
    s_{H^*_j} = \left( s_{H^*_j} \oplus \int_{s}^{t} G(\tau) d\tau \right) \cup \bigcup_{i=n+1}^{m} \left( \Delta B_{t_i} \oplus \int_{t_i}^{t} G(\tau) d\tau \right).
\]

Then, by Definition 1.4.8 whenever \( |H^*_j| \to 0 \), we obtain

\[
    \Theta_s = \left( \Theta_s \oplus \int_{s}^{t} G(\tau) d\tau \right) \cup \lim_{|H^*_j| \to 0} \bigcup_{i=n+1}^{m} \left( \Delta B_{t_i} \oplus \int_{t_i}^{t} G(\tau) d\tau \right). \tag{1.8}
\]

The following notations

\[
    G_h = \int_{s}^{t} G(\tau) d\tau \quad \text{and} \quad B_h = \lim_{|H^*_j| \to 0} \bigcup_{i=n+1}^{m} \left( \Delta B_{t_i} \oplus \int_{t_i}^{t} G(\tau) d\tau \right)
\]

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lead us to the set-valued discrete time stochastic process

\[ \Theta_k = \begin{cases} 
    (\Theta_{k-1} \oplus G_k) \cup B_k, & k \geq 1, \\
    B_0, & k = 0.
\end{cases} \]

In view of this, we are able to justify infinitesimal notations introduced in (1.2). In particular, from Equation (1.8), whenever \( \bar{\alpha} \to 0 \), we obtain

\[ \Theta_t = \left( B_{t_0} \oplus \int_{t_0}^t G(\tau) d\tau \right) \cup \left\{ dB_s \oplus \int_s^t G(\tau) d\tau \right\}, \quad t \in [t_0, T]. \]

Moreover, with a little abuse of this infinitesimal notation, we get two differential formulations

\[ d\Theta_t = \oplus G_t dt \cup dB_t \quad \text{and} \quad \Theta_{t+dt} = (\Theta_t \oplus G_t dt) \cup dB_t. \]

References

1. G. Aletti, E. G. Bongiorno, and V. Capasso. Statistical aspects of set–valued continuous time stochastic processes. (submitted).
2. J. Aubin and H. Frankowska. Set–valued Analysis, volume 2 of Systems & Control: Foundations & Applications. Birkhäuser Boston Inc., 1990.
3. G. Barles, H. M. Soner, and P. E. Souganidis. Front propagation and phase field theory. SIAM J. Control Optim., 31(2):439–469, 1993.
4. M. Burger. Growth of multiple crystals in polymer melts. European J. Appl. Math., 15(3):347–363, 2004.
5. M. Burger, V. Capasso, and A. Micheletti. An extension of the Kolmogorov–Avrami formula to inhomogeneous birth–and–growth processes. In Math Everywhere (G. Aletti et al., Eds). Springer, Berlin, 63–76, 2007.
6. M. Burger, V. Capasso, and L. Pizzocchero. Mesoscale averaging of nucleation and growth models. Multiscale Model. Simul., 5(2):564–592 (electronic), 2006.
7. V. Capasso, editor. Mathematical Modelling for Polymer Processing. Polymerization, Crystallization, Manufacturing. Mathematics in Industry, Vol. 2, Springer–Verlag, Berlin, 2003.
8. V. Capasso. On the stochastic geometry of growth. In Morphogenesis and Pattern Formation in Biological Systems (Sekimura, T. et al. Eds). Springer, Tokyo, 45–58, 2003.
9. V. Capasso and D. Bakstein. An Introduction to Continuous–Time Stochastic Processes. Modeling and Simulation in Science, Engineering and Technology. Birkhäuser Boston Inc., 2005.
10. C. Castaing and M. Valadier. Convex Analysis and Measurable Multifunctions. Lecture Notes in Mathematics, Vol. 580, Springer–Verlag, Berlin, 1977.
11. S. N. Chiu. Johnson–Mehl tessellations: asymptotics and inferences. In Probability, finance and insurance, pages 136–149. World Sci. Publ., River Edge, NJ, 2004.
12. N. Dunford and J. T. Schwartz. Linear Operators. Part I. Wiley Classics Library. John Wiley & Sons Inc., New York, 1988.
13. H. J. Frost and C. V. Thompson. The effect of nucleation conditions on the
topology and geometry of two-dimensional grain structures. *Acta Metallurgica*,
35:529–540, 1987.

14. E. Giné, M. G. Hahn, and J. Zinn. Limit theorems for random sets: an applica-
tion of probability in Banach space results. In *Probability in Banach Spaces, IV*
(Oberwolfach, 1982). Lecture Notes in Mathematics, Vol. 990, 112–135, Springer,
Berlin, 1983.

15. J. Herrick, S. Jun, J. Bechhoefer, and A. Bensimon. Kinetic model of DNA
replication in eukaryotic organisms. *J. Mol. Biol.*, 320:741–750, 2002.

16. F. Hiai and H. Umegaki. Integrals, conditional expectations, and martingales
of multivalued functions. *J. Multivariate Anal.*, 7(1):149–182, 1977.

17. C. J. Himmelberg. Measurable relations. *Fund. Math.*, 87:53–72, 1975.

18. S. Li, Y. Ogura, and V. Kreinovich. *Limit Theorems and Applications of Set-
Valued and Fuzzy Set-Valued Random Variables*. Vol. 43 of *Theory and Deci-
sion Library. Series B: Mathematical and Statistical Methods*. Kluwer Academic
Publishers Group, Dordrecht, 2002.

19. A. Micheletti, S. Patti, and E. Villa. Crystal growth simulations: a new mathemat-
cal model based on the Minkowski sum of sets. In *Industry Days 2003-2004*
(D. Aquilano et al. Eds), volume 2 of *The MIRIAM Project*, pages 130–140. Es-
culapio, Bologna, 2005.

20. H. Rådström. An embedding theorem for spaces of convex sets. *Proc. Amer.
Math. Soc.*, 3:165–169, 1952.

21. J. Serra. *Image Analysis and Mathematical Morphology*. Academic Press Inc.
[Harcourt Brace Jovanovich Publishers], London, 1984.

22. Bo Su and Martin Burger. Global weak solutions of non-isothermal front prop-
gagation problem. *Electron. Res. Announc. Amer. Math. Soc.*, 13:46–52 (elec-
tronic), 2007.

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