Mapping the surgery exact sequence for topological manifolds to analysis

Vito Felice Zenobi

Abstract

In this paper we prove the existence of a natural mapping from the surgery exact sequence for topological manifolds to the analytic surgery exact sequence of N. Higson and J. Roe. This generalizes the fundamental result of Higson and Roe, but in the treatment given by Piazza and Schick, from smooth manifolds to topological manifolds. Crucial to our treatment is the Lipschitz signature operator of Teleman.

We also give a generalization to the equivariant setting of the product defined by Siegel in his Ph.D. thesis. Geometric applications are given to stability results for rho classes. We also obtain a proof of the APS delocalised index theorem on odd dimensional manifolds, both for the spin Dirac operator and the signature operator, thus extending to odd dimensions the results of Piazza and Schick. Consequently, we are able to discuss the mapping of the surgery sequence in all dimensions.

Contents

1 Introduction 1
2 Signature operator on Lipschitz manifolds 2
3 $\varrho$ classes 3
   3.1 Perturbed signature operator 4
   3.2 Perturbed signature operator on manifolds with cylindrical ends 7
      3.2.1 Bunke’s relative index theorem for Lipschitz manifolds 8
4 Mapping surgery to analysis: the odd dimensional case 12
5 Products 13
   5.1 The analytic structure set and Siegel’s product 13
   5.2 Stability of $\varrho$ classes 16
      5.2.1 The signature operator 16
      5.2.2 Dirac operators and positive scalar curvature 18
   5.3 The delocalized APS index Theorem in the odd-dimensional case 19
6 Mapping surgery to analysis: the even dimensional case 21
References 21

1 Introduction

Let $M$ be a $n$-dimensional topological manifold, with $\Gamma = \pi_1(M)$ and let $\tilde{M} \to M$ be its universal covering. We assume $n$ greater than 5 and, initially, odd.

In [20] Sullivan proves that there always exists a Lipschitz manifold structure on $M$ and that it is unique up to bi-Lipschitz homeomorphism isotopic to the identity. In [21][22] Teleman studies index theory in the Lipschitz context and in [7] Hilsum develops it in the framework...
of unbounded Kasparov theory. In particular there is a signature operator arising from the Lipschitz structure and this operator determines a well defined class in the K-homology of $M$.

Thanks to these results it is possible to extend the work by Piazza and Schick [14] (that follows the one by Higson and Roe [4, 5, 6]) from the smooth to the topological category. Let us recall that in [14] Piazza and Schick built a natural transformation

$$
L_{n+1}(\mathbb{Z} \Gamma) \xrightarrow{\text{Ind}_\Gamma} S(M) \xrightarrow{\varphi} N(M) \xrightarrow{\beta} L_n(\mathbb{Z} \Gamma)
$$

from the surgery exact sequence for smooth manifolds to the analytic exact sequence of Higson and Roe, using tools and methods in coarse index theory.

In this paper we verify that this mapping also exists for the surgery sequence for topological manifolds. To this aim we will use as key tool the Lipschitz structure given by Sullivan theorem [20]. In particular we prove that the key results by Wahl, Piazza and Schick, have a true abstract and K-theoretical meaning, that does not depend on the smooth structure and the pseudodifferential calculus.

One significant difference between the smooth SES and the topological SES is that the second one is an exact sequence of groups, whereas the first one is not. In this paper we deal with the mapping at the set level: to prove that diagram is commutative as a diagram of groups, the main difficulty is that the group structure of the topological structure set is rather hard to handle. The following question is wide open:

• is the map $\varphi: S(M) \to K_{n+1}(D^*(\tilde{M}^\Gamma))$ a homomorphism of groups?

A positive answer to this question would have direct consequences to Conjecture 3.8 in [28], using the methods in [29].

In the second part of the paper we generalize to the equivariant setting a product formula proved by Siegel in his Ph.D. thesis [18]. This product allows us to give stability results for $\varphi$-invariants and to prove the delocalized APS index theorem of Piazza and Schick ([15]) in the odd dimensional case. This last result leads to define a natural mapping from the SES to the analytic SES of Higson and Roe when $\dim(M) = n$ is even, in both the smooth and the topological setting.

The same method apply to extend the construction in [15], about the Stolz exact sequence, to the case of even dimensional manifolds, but this was already proven in [30].

We refer the reader to [14] for a more detailed overview of the problem in the smooth setting.

Acknowledgements.

I am very thankful to my advisors Paolo Piazza and Georges Skandalis for their support and teachings.

2 Signature operator on Lipschitz manifolds

We start recalling fundamental results on Lipschitz manifolds. For further details we refer to [21, 22, 7, 20, 25].

Definition 2.1. A Lipschitz atlas on a topological manifold $M$ is an atlas such that the map $\varphi \circ \psi^{-1}$ is a Lipschitz homeomorphism for any two charts $\varphi: U \to \mathbb{R}^n$ and $\psi: V \to \mathbb{R}^n$. By definition a Lipschitz manifold structure on $M$ is a maximal Lipschitz atlas.

Theorem 2.1 ([20]). Any topological manifold of dimension $n \neq 4$ has a Lipschitz atlas of coordinates. For any two such structures $L_1$ and $L_2$, there exists a Lipschitz homeomorphism $h: L_1 \to L_2$ isotopic to the identity.
Theorem 2.2 ([21, 7]). Let \( M \) be a closed oriented Lipschitz manifold of even dimension. Then from the complex of \( L^2 \)-differential forms on \( M \) (with respect to some choice of a Lipschitz Riemannian metric \( g \)) one obtains a signature operator \( D_g \) which is closed and self-adjoint. Therefore \( D_g \) determines a class \([D]\) in \( K_0(M) \simeq KK(C(M), \mathbb{C}) \) which is independent of the choice of the metric \( g \). The image of \([D]\) in \( K_0(pt) \simeq KK(\mathbb{C}, \mathbb{C}) \) (i.e., the index of \( D_g \)) is the usual signature of the manifold.

In [8] Hilsum proves that the signature operator gives a KK-class as above in the case of non compact manifolds too, provided the manifold \( M \) is endowed with a metric \( g \) such that it is metrically complete with respect to the induced structure of metric space. Moreover he showed a result on the finite propagation speed for solutions of the wave equation.

Theorem 2.3 (Hilsum). Let \( M \) be an oriented Lipschitz manifold with a Riemannian structure, such that the manifold is complete as metric space. Let \( d \) be the associated distance function and let \( D \) be the associated signature operator. For all \( t \in \mathbb{R} \), we have that:

\[
\text{supp}(e^{itD}) \subset \{(x, y) \in M \times M \mid d(x, y) \leq t\}.
\]

For \( f \in \mathcal{S}(\mathbb{R}) \) such that \( \text{supp}(\hat{f}) \subset [-a, a] \), with \( a > 0 \), we have that:

\[
\text{supp}(f(D)) \subset \{(x, y) \in M \times M \mid d(x, y) \leq a\}.
\]

This theorem will be key in the coarse geometrical setting.

### 3 \( \varrho \) classes

We refer the reader to [6, sect.1] and [15, sect.1] for notations about coarse geometry and coarse algebras.

Let \( N \) be an oriented topological manifold of dimension \( n \geq 5 \); an element of the topological topological structure set \( N \) is given by an orientation preserving homotopy equivalence \( f : M \rightarrow N \). Two different homotopy equivalences \( f : M \rightarrow N \) and \( f' : M' \rightarrow N \), are equivalent if there is a \( h \)-cobordism \( W \) between them and a homotopy equivalence \( F : W \rightarrow N \times [0, 1] \), such that \( F_{|M} = f \) and \( F_{|M'} = f' \).

**Definition 3.1.** We define the topological structure set \( S^{TOP}(N) \) of \( N \) as the set of the \( h \)-cobordism classes of oriented homotopy equivalences.

Given a class \([f : M \rightarrow N]\), we set \( Z = M \cup N \). Let \( \Gamma \) be the fundamental group of \( N \). The universal covering \( \tilde{N} \rightarrow N \) is induced by a map \( u : N \rightarrow B\Gamma \): \( \tilde{N} = u^*E\Gamma \), where \( B\Gamma \) is the classifying space of \( \Gamma \) and \( E\Gamma \) its universal covering. Let \( \tilde{M} \) be the \( \Gamma \)-Galois covering induced by \( u_M := u \circ f \), then we get a \( \Gamma \)-Galois covering \( \tilde{Z} = \tilde{M} \cup -\tilde{N} \) on \( Z \). Let \( F = \tilde{Z} \times_{\Gamma} C^*(\Gamma) \) be the associated Mischenko bundle.

Now, starting from a Lipschitz structure on \( Z \) given by Theorem 2.1 consider the \( L^2 \)-forms complex \( L^2(Z, \Lambda_3(Z)) \), see [2, Section 2].

We get a differential \( d_Z \) and an involution \( \tau_Z \); \( \tau_Z \) is the operator \( \omega \mapsto i^{p(p-1)+\frac{3}{2}} * \omega \) on forms of degree \( p \). Like in the classical Hodge theory we can define the Lipschitz signature operator (with coefficients) as

\[
D_Z = (d_Z - \tau_Z d_Z \tau_Z)
\]

if \( n \) is even and

\[
D_Z = (\tau_Z d_Z + d_Z \tau_Z)
\]

if \( n \) is odd.

Like in [7], we have that \((L^2(Z, \Lambda_3(Z)), \mu, D_Z)\) defines an unbounded class \([D_Z] \in KK(C(Z), \mathbb{C})\), where \( \mu \) is the representation that associates the multiplication operator \( \mu_f \) to a function \( f \).
3.1 Perturbed signature operator

We want to associate a class in the K-group $K_*(D^*(M)^\Gamma)$ to a homotopy equivalence $f: M \to N$ and show that this mapping is well defined on the $h$-cobordism classes.

The key result for what follows is the homotopy invariance of the index class of the signature operator for compact oriented smooth manifolds, proved by M. Hilsum and G. Skandalis in [10]. Remember that, in the equivariant setting, this class is given by

$$\text{Ind}_r(D_Z) = [\mathcal{F}] \otimes_{C(X) \otimes C^*_r(\Gamma)} [D_Z] \in KK(C, C^*_r(\Gamma)),$$

where $[\mathcal{F}]$ is the class of Mishchenko bundle in $KK(C, C(X) \otimes C^*_r(\Gamma))$.

**Theorem 3.1** (Hilsum-Skandalis). Let $f: M \to N$ be a homotopy equivalence. Then the class $\text{Ind}_r(D_Z) \in KK(C, C^*_r(\Gamma))$ vanishes.

In remark [10, p.95] the authors observe that all arguments can be applied to the Lipschitz case: we can easily check that the smoothness of the objects is not necessary.

**Remark 3.1.** Of particular interest to us is a byproduct of the proof of Theorem 3.1, namely the construction of a homotopy $D_0$ between the signature operator $D_Z = D_0$ and an invertible operator $D_1$. This is the reason for the vanishing of the index class $\text{Ind}_r(D_Z)$. Here $D_Z$ is the signature operator twisted by the Mishchenko bundle. Moreover we point out that the perturbation creates a gap in its spectrum near 0.

**Proposition 3.1.** The difference $D_0 - D_1$ is a $C^*$-module compact operator on $L^2(Z, \Lambda_C(Z) \otimes \mathcal{F})$ both in the smooth and in the Lipschitz case.

**Proof.**

The proof of [10, Theorem 3.3] is based on the construction of an operator $T_{\alpha,\beta}$, that plays the role of the pull-back of forms.

Let us take the following data:

- a submersion $p: M \times B^k \to N$, where $B^k$ is the unit open disk of $\mathbb{R}^k$;
- a smooth $k$-form $v$ with compact support on $B^k$, such that $\int_{B^k} v = 1$. Put then $\omega = p_{B^k}^*(v)$.

Then $p^*: L^2(N, \Lambda_C(N) \otimes \mathcal{F}_N) \to L^2(M \times B^k, \Lambda_C(M \times B^k) \otimes p^*\mathcal{F}_N)$ is a bounded operator and $T_{\alpha,\beta}$ is defined as the operator $\xi \mapsto q_*(\omega \wedge p^*(\xi))$. Consider the following commutative diagram

$$\begin{array}{ccc}
M \times B^k & \xrightarrow{p} & N \\
\downarrow{q} & \swarrow{t} & \\
M \times N & \xrightarrow{p_N} & N
\end{array}$$

where $t = id_M \times p$. We get that, for $\xi \in L^2(N, \Lambda_C(N) \otimes \mathcal{F}_N)$

$$T_{\alpha,\beta}(\xi) = q_*(\omega \wedge p^*(\xi)) = (p_{B^k})_*(t_*(\omega \wedge (t)^*p_x^*(\xi))) = (p_M)_*(t_*(\omega \wedge p_{B^k}^*(\xi))).$$

Notice that $(p_M)_*$ is nothing else than the integration over $N$. Assume that $k$ and $p$ are chosen so that $t$ is a submersion. If we denote the form $t_*(\omega)$ on $M \times N$ with $k(y, x)$, it turns out that $T_{\alpha,\beta}(\xi) = \int_N k(x, y)\xi(x)$ is an integral operator with smooth kernel and consequently a smoothing operator.

The operator $Y$ in [10, Lemma 2.1(c)], such that $1 + T_{\alpha,\beta} \circ T_{\alpha,\beta} = d_N \circ Y + Y \circ d_N$, is bounded of order $-1$ (see the proof of [27, Lemma 2.2] for an explicit expression of $Y$).
Now we can follow word by word the proof of \cite{[13]} Lemma 9.14, using the conventions in \cite{[27]} Section 3. For simplicity let us consider the odd case. The perturbed signature operator is then given by

\[ \mathcal{D}_t = -iU_t(d_t \circ S_t + S_t \circ d_t) \circ U_t^{-1} \]

where

\[ d_t = \begin{pmatrix} d_M & tT_{p,v}^* \cr 0 & -d_N \end{pmatrix}, \quad S_t = \text{sign} (\tau_Z \circ L_t) \text{ and } U_t = |\tau_Z \circ L_t|^{1/2}, \]

with

\[ L_t = \begin{pmatrix} 1 + T_{p,v}^* \circ T_{p,v} & (1 - i t \gamma \circ Y) \circ T_{p,v}^* \\
T_{p,v} \circ (1 + i t \gamma \circ Y) & 1 \end{pmatrix} \]

One can easily see that \( L_t = 1 + H_t \), with \( H_t \) smoothing. Moreover one has that \( |\tau_Z \circ L_t| = \sqrt{L_t^* L_t} = \sqrt{1 + R_t} \), with \( R_t \) smoothing. Observe that \( 0 < L_t^* L_t = 1 + R_t \) implies that \( R_t > 1 \). It follows that \( \sqrt{L_t^* L_t} - 1 = f(R_t) \), where \( f(x) = \sqrt{1 + x} - 1 \) is holomorphic on the spectrum of \( R_t \) (-1 is a branch point for \( f \)). Since \( f(0) = 0 \), we have that \( f(z) = az + zh(z)z \), where \( h \) is a holomorphic function.

Let us point out that if \( S_0 \) and \( S_1 \) are smoothing operators and \( T \) is a bounded operator, then \( S_0 \circ T \circ S_1 \) is smoothing. Then we immediately get that \( F_t := |\tau_Z \circ L_t| - 1 = f(R_t) \) is

smoothing.

With the same argument one can prove that \( U_t = 1 + H_t \) with \( H_t \) smoothing.

By \cite{[13]} Lemma A.12, \( |\tau_Z \circ L_t|^{-1} = 1 + F_t^* \) and \( U_t = 1 + H_t^* \) with \( F_t^* \) and \( H_t^* \) smoothing. Then one obtains that, \( S_t = \tau_Z + G_t \) and \( d_t = d_Z + E_t \), where \( G_t \) and \( E_t \) are smoothing operators.

Consequently one has that

\[ \mathcal{D}_t = -i(1 + H_t) \left( (d + E_t) \circ (\tau_Z + G_t) + (\tau_Z + G_t) \circ (d + E_t) \right) \circ (1 + H_t^*) \]

is equal to \( \mathcal{D} + C_f \) with \( C_f \) a compact operator.

Now we have to prove that the Lipschitz Hilsum-Skandalis perturbation is bounded. In the smooth case we tackled the problem geometrically, here we try with a more analytical approach. An operator of order \(-n\) is a bounded operator between \( H^s(Z, E) \) and \( H^{s+n}(Z, E) \), the Sobolev sections of \( E \) of order \( s \) and \( s + n \), for any \( s \). An operator is regularizing if it is of order \(-\infty\). Equivalently one can say that an operator \( T \) is regularizing (of order \(-\infty\)) if \( D^m \circ T \circ D^n \) is a bounded operator on \( L^2\)-section for any \( m, n \in \mathbb{Z} \).

By \cite{[7]} Proposition 5.6 we know that the signature operator has compact resolvent, therefore its spectrum is a countable and discrete subset \( \{ \lambda_n \}_{n \in \mathbb{N}} \) of \( \mathbb{R} \) such that \( \lim_{n \to \infty} \lambda_n = +\infty \).

Now let \( \psi \in C_0^\infty(\mathbb{R}) \) be a rapidly decreasing even function such that \( \psi(1) = 1 \). Since \( \psi \) is even, it turns out that \( \psi(d_N + d_N^*) \) preserves the degree of forms and it is a Hilbert-Schmidt operator: the proof of the first statement of \cite{[16]} Prop. 5.31 works putting 'Hilbert-Schmidt' instead of 'smoothing'. Let us denote its kernel by \( k(x, y) \in L^2(N \times N) \).

Define the compact operator \( T_f : L^2(N, \Lambda_C(N) \otimes F_N) \to L^2(M, \Lambda_C(M) \otimes f^* F_N) \) as the integral operator with kernel \( (f \times \text{id}_N)^* k \in L^2(M \times N) \).

This operator verifies the hypothesis of \cite{[10]} Lemma 2.1. Indeed, because of our choice of \( \psi \), we have that \( 1 - \psi(x) = x \cdot \varphi(x) \), where \( \varphi \) is a rapidly decreasing odd function. Moreover \( d_N^* \circ \varphi(d_N + d_N^*) = \varphi(d_N + d_N^*) \circ d_N \), since \( \varphi \) is odd. Then we get the following formula

\[ 1 - \psi(d_N + d_N^*) = d_N \circ \varphi(d_N + d_N^*) + \varphi(d_N + d_N^*) \circ d_N \]

and by construction \( T_f \mathcal{D} = \psi(d_N + d_N^*) \psi(d_N + d_N^*) \). Now it’s easy to verify that there exists an operator \( Y \in \mathcal{B}(L^2(N, \Lambda_C(N) \otimes F_N)) \) such that \( Y(\text{dom}(d_N)) \subset \text{dom}(d_N) \) and that
\[ 1 - T^*_f \circ T_f = d_N \circ Y + Y \circ d_N : \]
\[ 1 - T^*_f \circ T_f = 1 - \psi(d_N + d^*_N) \circ \psi(d_N + d^*_N) = \]
\[ = 1 - (1 - d_N \circ \varphi(d_N + d^*_N) + \varphi(d_N + d^*_N) \circ d_N) \cdot \varphi(d_n + d^*_N) \circ d_N \]
\[ = -\varphi(d_N + d^*_N) \circ d_N \varphi(d_N + d^*_N) + \varphi(d_N + d^*_N) \circ d_N + \varphi(d_N + d^*_N) \circ \varphi(d_N + d^*_N) \circ d_N + \]
\[ + d_N \circ \varphi(d_N + d^*_N) \circ d_N \circ \varphi(d_N + d^*_N) + \varphi(d_N + d^*_N) \circ \varphi(d_N + d^*_N) \circ d_N + \]
\[ + d_N \circ \varphi(d_N + d^*_N) \circ d_N \circ \varphi(d_N + d^*_N) + d_N \circ \varphi(d_N + d^*_N) \circ \varphi(d_N + d^*_N) \circ d_N = \]
\[ = d_N \circ Y + d_N \circ Y, \]

with \( Y = \varphi(d_N + d^*_N) - \varphi(d_N + d^*_N) \circ d_N \circ \varphi(d_N + d^*_N). \)

It is easy to verify that the operator \( T_f \) is a regularizing operator (and hence a compact operator), therefore the image of \( T_f \) is in the domain of the Lipschitz signature operator.

Then the boundedness of the Lipschitz Hilsum-Skandalis perturbation follows as in the smooth case.

The only thing we have to care about is the dependence of this construction on the choice of the metric on \( N \). In particular we have to check that \( \psi(d_N + d^*_N) \) is Hilbert-Schmidt no matter which metric we use to take the adjoint.

If we have two different metrics \( g_0 \) and \( g_1 \) on \( N \), then by [7, Lemma 5.1] we can complete the \( Lip(N) \)-module \( Lip(N, \mathcal{A}(\mathcal{C}(N)) \circ \mathcal{F}_N) \) with respect to the two metrics and we obtain two isomorphic \( C(N) \)-Hilbert modules with compatible metrics:

\[ K^{-1} \| \cdot \|_1 \leq \| \cdot \|_0 \leq K \| \cdot \|_1 \exists K \in \mathbb{R}^+ \setminus \{0\}. \]

Then by the Minmax Theorem \( |\lambda_n| \leq K^2|\lambda_n| \), where for any \( n \in \mathbb{N} \), \( \lambda_n \) is the \( n \)-th eigenvalue of \( d + d^*_n \).

So it is easy to check that if \( \psi \) is a rapidly decreasing function on the spectrum of \( d + d^*_0 \), it is rapidly decreasing on the spectrum of \( d + d^*_1 \) too. Therefore \( \psi(d + d^*) \) is Hilbert-Schmidt independently of the metric we choose.

\[ \square \]

**Definition 3.2.** Let \( f : M \to N \) be a homotopy equivalence and \( Z = M \cup -N \). Denote by \( C_f \) the perturbation of \( D_Z \) arising in the previous remark and call it a trivializing perturbation. Note that it depends on the homotopy equivalence \( f \).

We recall from [14] that there is an isomorphism of \( \mathbb{C}^* \)-algebras

\[ \mathbb{K}(L^2(Z, \mathcal{A}(\mathcal{C}(Z) \circ \mathcal{F})) \simeq C^*(\tilde{Z})^\Gamma. \]

By [12] Proposition 2.1, the above isomorphism gives an isomorphism at level of multiplier algebras

\[ \mathbb{B}(L^2(Z, \mathcal{A}(\mathcal{C}(Z) \circ \mathcal{F})) \simeq \mathcal{M}(C^*(\tilde{Z})^\Gamma). \]

This isomorphism is given by the map \( L_\pi \) defined in [14] Section 2.2.1. Hence we can go from the Mishchenko bundle setting to the covering one. From now on \( C_f \) will be the element in \( \mathcal{M}(C^*(\tilde{Z})^\Gamma) \) associated to \( C_f \in \mathbb{B}(L^2(Z, \mathcal{A}(\mathcal{C}(Z) \circ \mathcal{F})) \) through the map \( L_\pi \). Moreover \( D_Z \) will indicate the operator on the covering induced by the signature without coefficients in the Mishchenko bundle.

**Remark 3.2.** Consider a chopping function \( \psi \in C_b(\mathbb{R}) \) with compactly supported Fourier transform. Thanks to Theorem 2.3 we can prove that the functional calculus through \( \psi \) of the operator \( D_Z \) is an operator of finite propagation. The pseudolocality of \( D_Z \) comes from [7, 6.1]. Hence \( \psi(\tilde{D}_Z) \in D^*(\tilde{Z})^\Gamma \).

**Proposition 3.2.** The difference between \( \psi(\tilde{D}_Z) \) and \( \psi(\tilde{D}_Z + C_f) \) belongs to \( C^*(\tilde{Z})^\Gamma \).
Proof. Moving to the Mishchenko bundle setting through \( [3.1] \), we should prove that the difference \( \psi(D_Z) - \psi(D_Z + C_f) \) belongs to \( \mathbb{K}(L^2(Z, \Lambda_\mathcal{C}(Z) \otimes F)) \). If \( \psi(t) = t(1 + t^2)^{-\frac{1}{2}} \), by Proposition 2.2 we have that \([\psi_1(D_Z), a] \) belongs to the algebra of compact \( C^* \)-module operators. So if we consider the matrices \( \begin{bmatrix} D_Z & 0 \\ 0 & D_Z + C_f \end{bmatrix} \) and \([1 0] \), their bracket consists in \( \begin{bmatrix} 0 & -C_f' \\ C_f' & 0 \end{bmatrix} \), that is known to be bounded. Then, after applying the functional calculus through \( \psi_1 \), we deduce that the matrix components in the bracket

\[ \pm(\psi_1(D_Z) - \psi_1(D_Z + C_f)) \]

are compact.

Now notice that two different chopping functions differ by a function in \( C_0(\mathbb{R}) \). Taking into account the correspondence stated in \( [3.1] \), we have that the resolvent of \( D_Z \), given by \((i + D_Z)^{-1} \), is compact (see \([7] \) Proposition 5.6) and since \( \phi(t) = (i + t)^{-1} \) generates \( C_0(\mathbb{R}) \), the functional calculus of \( D_Z \) through a function in \( C_0(\mathbb{R}) \) gives a compact operator. Then if \( \psi' \) is any chopping function, it turns out that

\[ \psi(D_Z) - \psi(D_Z + C_f) = \psi_1(D_Z) - \psi_1(D_Z + C_f) \]

is compact operators and the right-hand side term is compact. \( \square \)

Corollary 3.1. The operator \( \chi(\tilde{D}_Z + C_f) \), with \( \chi(x) = \frac{x}{|x|} \), is a bounded involution in \( D^*(\tilde{Z})^\Gamma \).

Thanks to Corollary 3.1 we can define a class by setting

\[ \varrho(\tilde{D}_Z + C_f) = \begin{bmatrix} 1 \quad 0 \\ 0 \quad 1 \end{bmatrix} (1 + \chi(\tilde{D}_Z + C_f)) \in K_0(D^*(\tilde{Z})^\Gamma). \]

Now consider the map \( \varphi : Z \to M \) such that \( \varphi|_N = f \) and \( \varphi|_{-M} = \text{Id}_M \); we can clearly see that \( \varphi \) is covered by a \( \Gamma \)-equivariant map \( \bar{\varphi} : \tilde{Z} \to M \).

Definition 3.3. Let \( f : M \to N \) be a homotopy equivalence between two compact oriented Lipschitz manifolds. Consider \( Z = M \cup -N \) and its covering \( \tilde{Z} \) associated, as above, to a classifying map \( u : Z \to B\Gamma \). Let \( \tilde{D}_Z \) be the Lipschitz signature operator and let \( C_f \) be the trivializing perturbation associated to \( f \). We define

\[ \varrho(f : M \to N) := \bar{\varphi} \ast \begin{bmatrix} 1 \quad 0 \\ 0 \quad 1 \end{bmatrix} (1 + \chi(\tilde{D}_Z + C_f)) \in K_0(D^*(\tilde{M})^\Gamma) \]

and

\[ \varrho_f(f : M \to N) = u \ast \varrho(f : M \to N) \in K_0(D^*_\Gamma). \]

Proposition 3.3. The \( \varrho \)-class does not depend on the choice of the Lipschitz structure.

Proof. The second part of Theorem 2.1 can be formulated as follows: let \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) be two different Lipschitz structures on \( Z \), then there exists a bi-Lipschitz homeomorphism \( \phi : Z \to Z \), isotopic to the identity through a path \( \phi^t \) and such that \( \phi^t(\mathcal{L}_2) = \mathcal{L}_1 \), where \( \phi^t : C(Z) \to C(Z) \) is the induced \( * \)-homomorphism. Because of the functoriality of the Teleman’s construction we know that \( \phi_* (\varrho_1) = \varrho_2 \), where \( \varrho_1 \) and \( \varrho_2 \) are the invariants associated to two different Lipschitz structures. The isotopy \( \phi^t \) induces a paths of \( * \)-isomorphisms \( \phi^*_t : D^*(\tilde{Z})^\Gamma \to D^*(\tilde{Z})^\Gamma \). Then \( \phi^*_t(\varrho_1) \) gives a homotopy between \( \varrho_2 \) and \( \varrho_1 \). \( \square \)

3.2 Perturbed signature operator on manifolds with cylindrical ends

In this section we verify that the construction we made of \( \varrho \) and \( \varrho_f \) are well defined on the structure set \( S^{TOP}(N) \).

For this purpose we will use results in \([27, 15, 14] \), where the authors have developed the theory in the smooth setting. Their methods are rather abstract and they also hold in the Lipschitz context.
In order to develop the theory for manifolds with cylindrical ends, we are going to use the same notations as [14, 2.19].

The geometrical setting is the following: let $f: M \to N$ and $f': M' \to N$ be two topological structures for $N$; let $W$ be a cobordism between $\partial_0 W = M$ and $\partial_1 W = M'$ and let $W_\infty$ be the manifold with the infinite semi-cylinder $\partial W \times [0, \infty)$ attached to the boundary; let $V = N \times [0, 1]$ and $V_\infty$ be the complete cylinders with base $\partial V = N$; there is a homotopy equivalence $F: W_\infty \to V_\infty$ which has the product form $F_{\partial_0} \times \text{id}_{[0, 1]}$ on the cylindrical ends, where $F_{\partial_0} = f: M \to N$ and $F_{\partial_1} = f': M' \to N$, both of them being homotopy equivalences.

Thanks to the results in [8] we have a well defined Lipschitz signature complex on $X = W_\infty \cup -V_\infty$. Notice that $\partial_0 X = Z$ and $\partial_1 X = Z'$.

First of all we need a generalization of Theorem 3.1 for manifolds with cylindrical ends. This result is given by [27, Proposition 8.1], where a perturbation is associated to the signature operator to the homotopy equivalence $F$. Such a perturbation makes the operator invertible, like in the usual case.

**Remark 3.3.** Like the case presented in Theorem 3.1, the generalization developed in [27, Proposition 8.1] is still valid in the Lipschitz setting.

The goal of this section is to verify that the $\varrho$-class is well defined on the h-cobordism classes: as pointed out in [14, Proposition 4.7], this is obtained from the combination of [27, Theorem 8.4] and [14, Corollary 3.3].

In [27, Theorem 8.4] all constructions work in the Lipschitz framework, where we do not consider the parameter $\varepsilon$. Wahl builds a perturbation of the signature operator $C^0_F$, supported on the cylindrical ends, from the perturbations on $Z$ and $Z'$; hence she constructs a homotopy of operators between $D_X + C^0_F$ and an other operator that, thanks to the Bunke’s relative index theorem, has vanishing index.

For the proof of the equality we just mentioned, the only point that is not obvious in the Lipschitz case is the one concerning the use of the relative index theorem proved in [2], since what remains of the proof uses abstract theory of unbounded operators and spectral flow methods.

It is worth formulating Bunke’s Theorem in the Lipschitz case and giving a sketch of its proof.

### 3.2.1 Bunke’s relative index theorem for Lipschitz manifolds

The idea of the theorem is the following: let $X$ be a manifold, let $E \to X$ be a bundle and $D$ a Fredholm operator on the sections of this bundle; if there exists a hypersurface $Y$ in $X$ such that the operator is invertible near $Y$, we can cut the manifold (and the bundle) along $Y$ and we can paste a semicylinder to the boundary of both parts obtained after cutting, extending the bundle and the operator along the semicylinder. Then we obtain an operator whose index equals the index of the original operator.

More precisely we are considering the following data: the Lipschitz manifold $X$ we have defined in the previous subsection, the Hilbert module $L^2(X, \Lambda_C(X) \otimes \mathcal{F})$ of $L^2$-forms on $X$ twisted by the Mishchenko bundle, that we are going to denote by $\mathcal{H}^0$; a regular operator $G$ that is the twisted Lipschitz signature operator, possibly perturbed by a bounded operator; we suppose that there is a Lipschitz function with compact support $f \geq 0$ and $(G^2 + f)^{-1} \in \mathcal{B}(\mathcal{H}^0, \mathcal{H}^2)$ (here $\mathcal{H}^2$ is the maximal domain of the square of the signature operator).

**Definition 3.4.** Let $\text{Lip}_K(X)$ be the set of bounded Lipschitz functions $h$ such that for all $\varepsilon > 0$ there exists a compact $C \subset X$, with $||dh|_{X \setminus C}||_{L^\infty} < \varepsilon$. Let us call $C_K(X)$ the closure of $\text{Lip}_K(X)$ in the sup-norm.

For the comfort of the reader, we recall the theorem stated in the Lipschitz setting. Let $E_i \to X_i$, $i = 1, 2$, be the two $C^\infty(\Gamma)$-bun $\Lambda_C(X_i) \otimes \mathcal{F}_i$, with operator $G_i$, associated to them as above. Let $W_i \cup_\gamma V_i$ be partitions of $X_i$ where $Y_i$ are compact hypersurfaces. Assume
Lemma 3.3. For any where the $U(Y_i)$ are commutable neighbourhoods of $Y_i$, $i = 1, 2$. We cut $X_i$ at $Y_i$ glue the pieces together interchanging the boundary components and obtain $X_3 := W_2 \cup Y V_2$ and $X_4 := W_2 \cup Y V_1$. Moreover, we glue the bundles using $\Psi$, which yields $A$-$C^*$ bundles $E_3 \to X_3$ and $E_4 \to X_4$ and we assume that $G_i$, $i = 3, 4$ are again invertible at infinity. We define $[X_i]$ as the class $[\mathcal{H}^i_0, \frac{G_i}{G_i + 1}] \in KK(C_K(X_i), C^*_r(\Gamma))$. The algebra $C_K(X)$ is unital. Hence, there is an embedding $i: \mathbb{C} \to C_K(X)$ and an induced map

$$i^*: KK(C_K(X), C^*_r(\Gamma)) \to KK(\mathbb{C}, C^*_r(\Gamma)).$$

Set $\{X_i\} := i^*[X_i] \in KK(\mathbb{C}, C^*_r(\Gamma))$ for $i = 1, \ldots, 4$.

Theorem 3.2 [2].

$$\{X_1\} + \{X_2\} - \{X_3\} - \{X_4\} = 0.$$

Here are two facts:

- thanks to [21] Theorem 7.1 we have the following Rellich-type result: the inclusion $\mathcal{H}^2 \hookrightarrow \mathcal{H}^0$ is compact;

- for any $f$ Lipschitz function compactly supported on $X$, the multiplication operator $f: \mathcal{H}^2 \to \mathcal{H}^0$ is compact. And this also holds for the Clifford multiplication by $\text{grad}(f)$, the gradient of $f$.

Let $R(\lambda)$ be the bounded operator $(G^2 + f + \lambda)^{-1} \in \mathbb{B}(\mathcal{H}^0, \mathcal{H}^2)$, for $\lambda \geq 0$; because of the Rellich-type result, we know that $R(\lambda)$ defines a compact operator in $\mathbb{B}(\mathcal{H}^0)$ and that there is a positive constant $C$ such that $||R(\lambda)|| \leq (C + \lambda)^{-1}$.

Lemma 3.1. The integral

$$F = \frac{G}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} R(\lambda) d\lambda$$

is convergent and defines an operator in $\mathbb{K}(\mathcal{H}^0)$.

Lemma 3.2. The operator $[D, R(\lambda)]$ extends to a bounded operator that coincides with $-R(\lambda)\text{grad}(f)R(\lambda)$.

Moreover such an operator is compact.

Proof. See [2] Lemma 1.7 and Lemma 1.8. \hfill \Box

Lemma 3.3. For any $h \in C_K(X)$, $h(F^2 - I) \in \mathbb{K}(\mathcal{H}^0)$.

Proof. We have

$$\left(\frac{G}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} R(\lambda) d\lambda\right) \left(\frac{G}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} R(\lambda) d\lambda\right) = \frac{G^2}{\pi^2} \left(\int_0^\infty \lambda^{-\frac{1}{2}} R(\lambda) d\lambda\right)^2 + \frac{G}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} R(\lambda) d\lambda \int_0^\infty \lambda^{-\frac{1}{2}} R(\lambda) d\lambda =$$

$$\frac{G^2}{\pi^2} (G^2 + f)^{-1} - \frac{G}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} R(\lambda) \text{grad}(f)R(\lambda) d\lambda \int_0^\infty \lambda^{-\frac{1}{2}} R(\lambda) d\lambda \sim$$

$$\frac{G^2}{\pi^2} (G^2 + f)^{-1},$$
where in the third step we have used Lemma 3.2. Here $\sim$ means “equal modulo compacts”. Hence
\[
h(F^2 - I) \sim h \frac{f}{G^2 + f}
\]
that is compact, since multiplication by $f$ is.

**Lemma 3.4.** For any $h \in C_K(X)$, $[F, h] \in \mathbb{K}(H_0)$.

**Proof.** Since we chose as $G$ as a perturbation of the signature operator $D$ and since the perturbation becomes compact under bounded transform, we have that
\[
[F, h] \sim \left[ \frac{D}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} R(\lambda) d\lambda, h \right] =
\frac{D}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} R(\lambda) d\lambda h + \left[ \frac{D}{\pi}, h \right] \int_0^\infty \lambda^{-\frac{1}{2}} R(\lambda) d\lambda \sim
\frac{D}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} [R(\lambda), h] d\lambda + \text{grad}(h) \int_0^\infty \lambda^{-\frac{1}{2}} R(\lambda) d\lambda.
\]
The term in the last line is compact as in the proof of [2, Lemma 1.12].

**Lemma 3.5.** Let $f, f_1$ be two positive and compactly supported Lipschitz functions such that $(G^2 + f)^{-1}, (G^2 + f_1)^{-1} \in B(H_0, H^2)$. Then the two associated operators $F, F_1$ differ each other by a compact operator.

**Proof.** See [2, Lemma 1.10].

The lemmas we presented yield to the following

**Proposition 3.4.** The pair $(H_0, F)$ defines a Kasparov $(C_K(X), C^*_r(\Gamma))$-module and its class in $KK(C_K(X), C^*_r(\Gamma))$ does not depend on the choice of $f$.

After checking this technical part, the proof of Theorem 3.2 is completely abstract and it follows in the Lipschitz case as in the smooth one.

Now we treat another fundamental result proved by Piazza ans Schick: the delocalized Atiyah-Patodi-Singer index theorem. As noticed in [14, Section 5.2], the proof of the delocalized APS index theorem is based on abstract functional analysis for unbounded operators on Hilbert spaces. The reader can verify that it works almost completely in the same way in the Lipschitz context and we will not give all details again.

The only proof to be modified is [14, Prop 5.33]. Being the context and the notation understood, we state the following Proposition.

**Proposition 3.5.** Given a Dirac type operator $D$, the operator $(1 + D^2)^{-1}; L^2 \to H^2$ is a norm limit of finite propagation operators $G_n; L^2 \to H^2$ with the property that $[\varphi, G_n]; L^2 \to H^2$ is compact for any compactly supported continuous function on $M$.

**Proof.** It is an easy computation to show that
\[
\frac{1}{1 + x^2} = \int_{-\infty}^{+\infty} \frac{e^{-|t|}}{2} e^{-txt} dt.
\]
Let $f: \mathbb{R} \to \mathbb{R}$ be a $C^\infty$ function such that
- $0 \leq f \leq 1$,
- $f = 1$ on a neighbourhood of $0$, 

• $f$ has compact support.

Define $G_n = \int_{-\infty}^{+\infty} f \left( \frac{t}{n} \right) \frac{e^{-|t|}}{2} e^{-itD} dt$.

Finite propagation: since $f \left( \frac{t}{n} \right) \frac{e^{-|t|}}{2}$ has compact support, $G_n$ has finite propagation speed.

Pseudolocality: thanks to the above formula, $(1+D^2)^{-1} - G_n = \int_{-\infty}^{+\infty} (1-f \left( \frac{t}{n} \right)) \frac{e^{-|t|}}{2} e^{-itD} dt$.

Notice that $(1-f \left( \frac{t}{n} \right)) \frac{e^{-|t|}}{2}$ is $C^\infty$ and moreover it is a rapidly decreasing function on the spectrum of $D$. By [16, Prop 5.31], $(1+D^2)^{-1} - G_n$ is a bounded operator from $H^m$ to $H^k$ for any $m, k \in \mathbb{N}$, hence $G_n$ is pseudolocal because so is $(1+D^2)^{-1}$. Indeed, using Jacobi identities for commutators and the fact that $[\phi, D] = c(d\phi)$, $[\phi, (1+D^2)^{-1}] = (1+D^2)^{-1} c(d\phi) D (1+D^2)^{-1} + (1+D^2)^{-1} D c(d\phi) (1+D^2)^{-1}$ is compact, because the Clifford multiplication $c(d\phi)$ is compact too.

In fact we need less: it is sufficient to show that $(1+D^2)^{-1} - G_n$ is a bounded operator from $L^2$ to $H^3$ and then, by Rellich Theorem, the commutator $[\phi, (1+D^2)^{-1}] - G_n$ turns out to be a compact operator from $L^2$ to $H^2$. To prove this, we only need that the third derivative of $(1-f \left( \frac{t}{n} \right)) \frac{e^{-|t|}}{2}$ has a bounded supremum norm (less than being rapidly decreasing).

In fact, under these hypotheses and by the properties of the Fourier transform, we get that

$$||(1+D^2)^{-1} - G_n||_{L^2 \to H^3} = \left\| \left(1-f \left( \frac{t}{n} \right) \right) \frac{e^{-|t|}}{2} \right\|_\infty$$

is bounded. Moreover $\left(1-f \left( \frac{t}{n} \right) \right) \frac{e^{-|t|}}{2}$ is equal to

$$- \frac{1}{n^3} t'' \left( \frac{t}{n} \right) e^{-|t|} + \frac{3}{n^2} t' \left( \frac{t}{n} \right) t e^{-|t|} - \frac{3}{n} t' \left( \frac{t}{n} \right) e^{-|t|} - (1-f \left( \frac{t}{n} \right)) |t|^3 e^{-|t|}$$

that clearly goes to zero as $n$ goes to infinity. This also holds in the Lipschitz case.

Now we can state the delocalized Atiyah-Patodi-Singer index theorem, that holds in the Lipschitz context too.

**Theorem 3.3** ([14]). If $i: C^*(\widetilde{X})^\Gamma \to D^*(\widetilde{X})^\Gamma$ is the inclusion and $j_* : D^*(\partial \widetilde{X})^\Gamma \to D^*(\widetilde{X})^\Gamma$ is the map induced by the inclusion $j : \partial \widetilde{X} \to \widetilde{X}$, we have

$$i_* (\text{Ind}_F(D_X + C_F^\text{cy})) = j_* (\varrho(D_{\partial X} + C_{F_0})) \in K_0(D^*(\widetilde{X})^\Gamma).$$

Using the functoriality of the classifying map $u \circ F \cup u : \widetilde{X} \to E \Gamma$ and the map $\Phi := \pi_1 \circ (F \cup -i d \gamma_{x,0})$ we obtain

$$i_* \Phi_* (\text{Ind}_F(D_X + C_F^\text{cy})) = \varrho(F_\partial) \in K_0(D^*(\widetilde{V})^\Gamma)$$

$$i_* u_* \Phi_* (\text{Ind}_F(D_X + C_F^\text{cy})) = \varrho(F_\partial) \in K_0(D_F^\Gamma).$$

Observe that $\varrho$ is additive on disjoint unions like $\partial X = Z \cup -Z'$ and in particular that

$$\varrho(F_\partial) = \varrho(f) - \varrho(f').$$

Combining this with [27, Theorem 8.4], we finally have that

$$\varrho(f) = \varrho(f'),$$

and similarly for $g$, hence they are well defined on $\mathcal{S}^{TOP}(N)$. 

4 Mapping surgery to analysis: the odd dimensional case

Here we refer the reader to [14] Section 4 for definitions. We state the main theorem.

**Theorem 4.1.** Let \( N \) be an \( n \)-dimensional closed oriented topological manifold with fundamental group \( \Gamma \). Assume that \( n \geq 5 \) is odd. Then there is a commutative diagram with exact rows

\[
\begin{array}{cccc}
L_{n+1}(Z\Gamma) & \rightarrow & \mathcal{S}^{TOP}(N) & \rightarrow & \mathcal{N}^{TOP}(N) & \rightarrow & L_n(Z\Gamma) \\
\text{Ind}_r \downarrow & & \phi \downarrow & & \beta \downarrow & & \text{Ind}_r \\
K_{n+1}(C^*_r(\Gamma)) & \rightarrow & K_{n+1}(D^*(\tilde{N}^\Gamma)) & \rightarrow & K_n(N) & \rightarrow & K_n(C^*_r(\Gamma))
\end{array}
\]

and through the classifying map \( u: N \rightarrow B\Gamma \) of the universal cover \( \tilde{N} \) of \( N \), we have the analogous commutative diagram that involves the universal Higson-Roe exact sequence

\[
\begin{array}{cccc}
L_{n+1}(Z\Gamma) & \rightarrow & \mathcal{S}^{TOP}(N) & \rightarrow & \mathcal{N}^{TOP}(N) & \rightarrow & L_n(Z\Gamma) \\
\text{Ind}_r \downarrow & & \phi \downarrow & & \beta \downarrow & & \text{Ind}_r \\
K_{n+1}(C^*_r(\Gamma)) & \rightarrow & K_{n+1}(D^*_r) & \rightarrow & K_n(B\Gamma) & \rightarrow & K_n(C^*_r(\Gamma))
\end{array}
\]

**Remark 4.1.** Let us recall the fact that, despite \( \mathcal{S}^{TOP}(N) \) has a group structure, we do not deal with it and the top row is considered just as a sequence of sets, as in the smooth case.

Thanks to the results in the previous section we can check that the results in [14] Sections 4.2 and 4.3 hold in terms of the category of topological manifolds instead of the one of smooth manifolds: all proofs are still valid in the Lipschitz context. Thanks to the work by C.Whal [27, Theorem 9.1], that can be combined with Theorem 2.3, the first vertical arrow is well defined in the Lipschitz setting. The second one is also well defined for the previous section. Concerning the third one there are no significant problems.

The same method in Proposition 3.3 applies to the class of the signature and its index class, then all vertical arrows do not depend on the chosen Lipschitz structure.

One has to check the commutativity of the three squares.

- The third square is obviously commutative: let \( (f: M \rightarrow N) \) be a normal map in \( \mathcal{N}^{TOP}(N) \), it is sent horizontally to the same map forgetting the fact that it is normal and then through \( \text{Ind}_r \) to the difference \( \text{Ind}_r(D_M) - \text{Ind}_r(D_N) \); on the other hand \( \beta(f: M \rightarrow N) = f_*[D_M] - [D_N] \), that gives, through the analytic assembly map, the index class just founded.

- Let us study the second square: let \( (f: M \rightarrow N) \) be a structure in \( \mathcal{S}^{TOP}(N) \), it goes to the same map forgetting the fact that \( f \) is a homotopy equivalence; the \( \varphi \)-class \( \phi(f) \), as in Definition 3.3, is the push-forward through \( \tilde{\varphi} \) of the class \( \frac{1}{2}(1 + \chi(\hat{D}_Z + C_f)) \in K_0(D^*(\tilde{Z}^\Gamma)) \); this goes horizontally to the class in \( K_0(D^*(\tilde{Z}^\Gamma))/\mathcal{C}^*(\tilde{Z}^\Gamma) \) that represents, by Pashcke duality, the K-homology class of the signature operator of \( Z \); then by functoriality of \( \tilde{\varphi}_* \) and the fact that \( \beta(f: M \rightarrow N) = f_*[D_M] - [D_N] \), we obtain the commutativity of the second square.

- For the first square commutativity we refer the reader to [14] 4.10. Let \( a \in L_{n+1}(Z\Gamma) \) and let \( (f: M \rightarrow N) \) be a structure in \( \mathcal{S}^{TOP}(N) \). The commutativity of the first square means that the following equation holds:

\[
i_*([\text{Ind}_r(a)]) = \phi(a[f: M \rightarrow N]) - \phi([f: M \rightarrow N]) \in K_0(D^*(\tilde{Z}^\Gamma));
\]

this is proved identifying the right hand side with the class predicted by the APS delocalized index theorem, that, as we know, holds in the Lipschitz case too. The proof is based on an addition formula, as in [27, 7.1], and algebraic identifications of \( \varphi \)-classes, that the reader can verify still holding, word-for-word, in the Lipschitz case.
5 Products

Let $M$ and $N$ be two Cartesian products with a common factor, namely $M = M_1 \times M_2$ and $N = N_1 \times M_2$, and let $f_1 : M_1 \to N_1$ be a homotopy equivalence. Therefore $f = f_1 \times \text{id} : M \to N$ is a homotopy equivalence.

Observe that the signature operator on $Z = M \cup (-N)$ has this form: $D_Z = D_1 \hat{\otimes} 1 + 1 \hat{\otimes} D_2$, i.e. the graded tensor product of the signature operator $D_1$ on $M_1 \cup (-N_1)$ and the signature operator $D_2$ on $M_2$.

As before we construct from $f$ a bounded operator $C_f$ that produces an invertible perturbation $D_Z + C_f$. Notice that, from the construction in [10] and as it has been pointed out in [27] (6.1), the operator $C_f$ has the form $C_f \hat{\otimes} 1$, where all grading operators are understood in the graded tensor product. We have

$$D_Z + C_f = (D_1 + C_{f_1}) \hat{\otimes} 1 + 1 \hat{\otimes} D_2$$

so we can associate an invertible perturbation of $D_Z$ to an invertible perturbation of $D_1$.

We would like to state a product formula involving the $\sigma$-class invariant of the first factor and the K-homology class of the second one. For this purpose we recall some abstract results.

5.1 The analytic structure set and Siegel’s product

We are going to extend Siegel’s construction of Section 6.2.2 [18] to the $\Gamma$-equivariant case. Another approach to product formulas has been worked out recently in [31] and it has been applied to the study of secondary invariants associated to metrics with positive scalar curvature.

In general, if we have an exact sequence of $C^*$-algebras

$$0 \to J \to A \xrightarrow{\alpha} A/J \to 0,$$

we can define a natural isomorphism between $K_*(A)$ and the K-group of the mapping cone $K_*(C_\tau)$, where, if $M(J)$ is the multiplier algebra of $J$, $\tau : A/J \to M(J)/J$ is the Busby invariant of the extension.

More explicitly, since $J$ is an ideal of $A$, we have the following commutative diagram

$$\begin{array}{ccc}
0 & \to & J & \xrightarrow{\alpha} & A & \xrightarrow{\beta} & A/J & \to & 0 \\
\downarrow{=} & & \downarrow{\sigma} & & \downarrow{\tau} & & \downarrow{\pi} & & \downarrow{=} \\
0 & \to & J & \xrightarrow{\iota} & M(J) & \xrightarrow{\pi} & M(J)/J & \to & 0
\end{array}$$

where $\sigma$ is natural map that identifies an element of $A$ with a bounded operator on $J$, seen as Hilbert module over $J$. In the even case, if $p$ is an element of $A$, we set $q = \sigma(p) \in M(J)$; the K-theory of any multiplier algebra vanishes, hence there is a path $q_0$ in $M(J)$ such that $q_0 = q$ and $q_1 = 1 \oplus 0$. Finally we define a map between $A$ and $C_\tau$ that associates the element $(\beta(p), \pi(q_i)) \in C_\tau$ to $p$. In the odd case we construct this map similarly, using unitary elements instead of projections.

**Proposition 5.1** (Prop 6.2.7 [18]). The map defined above induces a natural isomorphism between $K_*(A)$ and $K_*(C_\tau)$.

In particular we take $A = D^*(\tilde{X})^\Gamma$ and $J = C^*(\tilde{X})^\Gamma$ and, considering elements of $D^*(\tilde{X})^\Gamma$ as concrete elements in $M(C^*(\tilde{X})^\Gamma)$, we denote a class in $K_0(C_\tau)$ by $[\beta(p), \pi(q_i)]$.

We prefer to have a realization of this class as in Kasparov’s theory. That is why we would like to give an explicit construction of the cycles of $K_*(D^*(\tilde{X})^\Gamma)$ through Kasparov bimodules. Let $H$ be the Hilbert space used to define the algebra $D^*(\tilde{X})^\Gamma$ and denote by $\xi \mapsto \xi^\gamma$ the action of $\Gamma$ on $H$, where $\xi \in H$ and $\gamma \in \Gamma$. 

The only remark to be made is that, by Paschke duality, we have that
\[
\langle \xi, \eta \rangle_{C^*_\Gamma(\Gamma)} := \sum_{\gamma \in \Gamma} \langle \xi^\gamma, \eta \rangle_{\gamma},
\]
where \( (\xi, \eta) \) is the usual scalar product of \( H \).

The \( C^*_\Gamma(\Gamma) \)-module is isomorphic to the one associated to the Mishchenko-Fomenko bundle. Let us recall that
\[
\lim_{\gamma} \left[ \bigotimes_{\gamma' \in \Gamma} \Gamma(\gamma') \right] = \bigotimes_{\gamma \in \Gamma} \Gamma(\gamma).
\]

Let \( X \) be a compact manifold and let \( \Gamma \) be a discrete group acting by isometries on \( X \). We define \( \tilde{H} \) as the \( C^*_\Gamma(\Gamma) \)-module obtained by completing \( C_c(\tilde{X})H \) with respect to the norm induced by the \( \Gamma \)-valued inner product
\[
\langle \xi, \eta \rangle := \sum_{\gamma \in \Gamma} \langle \xi^\gamma, \eta \rangle_{\gamma},
\]
where \( \langle \xi, \eta \rangle \) is the usual scalar product of \( H \).

The \( C^*_\Gamma(\Gamma) \)-module is isomorphic to the one associated to the Mishchenko-Fomenko bundle. Let us recall that
\[
\lim_{\gamma} \left[ \bigotimes_{\gamma' \in \Gamma} \Gamma(\gamma') \right] = \bigotimes_{\gamma \in \Gamma} \Gamma(\gamma).
\]

Let \( X \) be a compact manifold and let \( \Gamma \) be a discrete group acting by isometries on \( X \). We define \( \tilde{H} \) as the \( C^*_\Gamma(\Gamma) \)-module obtained by completing \( C_c(\tilde{X})H \) with respect to the norm induced by the \( \Gamma \)-valued inner product
\[
\langle \xi, \eta \rangle := \sum_{\gamma \in \Gamma} \langle \xi^\gamma, \eta \rangle_{\gamma},
\]
where \( \langle \xi, \eta \rangle \) is the usual scalar product of \( H \).

Remark 5.1. Let \( X \) be as above. A \( j \)-multigraded \( \Gamma \)-equivariant analytic structure cycle on \( X \) consists of the following data:

- a selfadjoint \( j \)-multigraded Kasparov \( C_0(\tilde{X}) - \mathbb{C} \)-module \( (H, \phi, F) \);
- a norm continuous path of operators \( \tilde{F}(t) \) in \( B(\tilde{H}) \) such that \( \tilde{F}(0) = F \), \( \tilde{F}(1)^2 \geq \varepsilon > 0 \).

Here the Kasparov \( \mathbb{C} \)-module \( C_0(\tilde{X}) \)-module \( B(\tilde{H}) \) is constructed as above.

Definition 5.2. Let \( X \) be as above. A \( j \)-multigraded \( \Gamma \)-equivariant analytic structure cycle on \( X \) consists of the following data:

- a selfadjoint \( j \)-multigraded Kasparov \( C_0(\tilde{X}) - \mathbb{C} \)-module \( (H, \phi, F) \);
- a norm continuous path of operators \( \tilde{F}(t) \) in \( B(\tilde{H}) \) such that \( \tilde{F}(0) = F \), \( \tilde{F}(1)^2 \geq \varepsilon > 0 \).

Here the Kasparov \( \mathbb{C} \)-module \( C_0(\tilde{X}) \)-module \( B(\tilde{H}) \) is constructed as above.

Definition 5.3. Let \( (H_i, \phi_i, F_i, \tilde{F}_i(t)) \), \( i = 0, 1 \), be two \( j \)-multigraded \( \Gamma \)-equivariant analytic structure cycles. We will say that they are homotopic if there exists a path \( (H, \phi, F, \tilde{F}(t)) \) of \( \Gamma \)-equivariant analytic structure cycles that joins them. Then we denote by \( S^J_f(X) \) the Grothendieck group generated by all homotopy classes of \( j \)-multigraded \( \Gamma \)-equivariant analytic structure cycles on \( X \).

Proposition 5.2. Let \( \Phi: K_j(C_\gamma) \to S^J_{j-1}(X) \) be the map \( [p, \pi(q_1)] \mapsto [H, \phi, F_p, \tilde{F}_p(t)] \), with \( F \) is equal to \( 2p - 1 \in D^*(\tilde{X})^\Gamma/C^*(\tilde{X})^\Gamma \) and \( \tilde{F}_p(t) \) is the image of \( 2p - 1 \in M(C^*(\tilde{X})^\Gamma/C^*(\tilde{X})^\Gamma) \) through the isomorphism in \( \ref{5.1} \). This application establishes a natural isomorphism.

Proof. The proof is analogous to the non equivariant case, proved in \[13 \] Proposition 6.2.10. The only remark to be made is that, by Paschke duality, we have that \( K_j(D^*(\tilde{X})^\Gamma/C^*(\tilde{X})^\Gamma) \cong K_j(D^*(\tilde{X})^\Gamma/C^*(\tilde{X})^\Gamma) \).

Remark 5.1. From the above discussion, we get the following commutative diagram
\[
\begin{array}{ccc}
\cdots & \rightarrow & K_j(C^*(\tilde{X})^\Gamma) \rightarrow K_j(D^*(\tilde{X})^\Gamma) \rightarrow K_j(D^*(\tilde{X})^\Gamma/C^*(\tilde{X})^\Gamma) \rightarrow \cdots \\
\downarrow \cong & & \downarrow \cong \\
\cdots & \rightarrow & KK^{j-1}(\mathbb{C}, C^*(\tilde{X})^\Gamma \otimes C_0(0,1)) \rightarrow S^J_{j-1}(\tilde{X}) \rightarrow KK^{j-1}(C(X), \mathbb{C}) \rightarrow \cdots
\end{array}
\]

Lemma 5.1. Let \( Y, X \) be two manifolds and assume that a group \( \Gamma \) acts on \( Y \) and \( X \) freely, isometrically and such that \( Y/\Gamma \) is compact. Let \( f: Y \rightarrow X \) be a \( \Gamma \)-equivariant continuous coarse map and let \( \gamma: H_Y \rightarrow H_X \) be an isometry that covers \( f \) in the \( D^* \)-sense (\[13 \] Definition 1.7)). Then we have the following commutative diagram
\[
\begin{array}{ccc}
K_j(D^*(Y)^\Gamma) & \xrightarrow{f_*} & K_j(D^*(X)^\Gamma) \\
\downarrow \Phi & & \downarrow \Phi \\
S^J_{j-1}(Y) & \xrightarrow{f_*} & S^J_{j-1}(X)
\end{array}
\]

where \( f_* \) sends the class \([H, \varphi, F, \tilde{F}(t)]\) to \([H, \varphi \circ f^*, F, \tilde{F}(t)]\).
Proof. We can assume that $H_Y = L^2(Y)$, $H_X = L^2(X)$ and that representation of $C_0(Y)$ and $C_0(X)$ are given by the function multiplication. Let $p \in D^*(Y)^\ast$ be a projection. Then we get two element of $S^1_{j-1}(X)$:

- the first one is $\Phi([\text{Adv}(p)]) = [L^2(X), \varphi_X, F_{\text{Adv}(p)}, \tilde{F}_{\text{Adv}(p)}(t)]$, where $F_{\text{Adv}(p)} = 2\beta(\text{Adv}(p)) - 1$ and $\tilde{F}_{\text{Adv}(p)}(t) = 2\pi(\text{Adv}(q_t)) - 1$;

- the second one is $f_p(\Phi(p)) = [L^2(Y), \varphi_Y \circ f^\ast, F_p, \tilde{F}_p(t)]$ where $F_p$ and $\tilde{F}_p(t)$ are defined as above and $f^\ast : C_0(X) \to C_0(Y)$ is the pull-back.

We have to prove that these two classes are the same. Consider the projection $Q = VV^*$, then we can decompose $\Phi([\text{Adv}(p)])$ in two direct summand:

$$\Phi([\text{Adv}(p)]) = [QL^2(X), \varphi_X, F_1, \tilde{F}_1(t)] \oplus [(1 - Q)L^2(X), \varphi_X, F_2, \tilde{F}_2(t)],$$

where $F_1 = Q F_{\text{Adv}(p)} Q$, $F_2 = (1 - Q) F_{\text{Adv}(p)} (1 - Q)$ and $\tilde{F}_1(t)$ and $\tilde{F}_2(t)$ are defined similarly.

The second summand is clearly degenerate; the first one, thanks to the following commutative diagram

$$
\begin{array}{ccc}
C_0(X) & \xrightarrow{f^\ast} & C_0(Y) \\
\downarrow{\varphi_X} & & \downarrow{\text{Adv}_Y} \\
B(L^2(Y)) & \approx & B(QL^2(X))
\end{array}
$$

is equal to $f_p(\Phi(p))$.

Let $\xi$ be a class in $S^1_j(\tilde{X}_1)$ represented by a cycle $(H_1, \phi_1, F_1, \tilde{F}_1(t))$ and let $\lambda$ be a class in $K_1(\tilde{X}_2/T_2)$ represented by an equivariant Kasparov module $(H_2, \phi_2, F_2)$, where $\tilde{X}_1$ and $\tilde{X}_2$ are two Lipschitz manifolds. Let $(H_1 \hat{\otimes} H_2, \phi_1 \hat{\otimes} \phi_2, F)$ be an exterior Kasparov product of $(H_1, \phi_1, F_1)$ and $(H_2, \phi_2, F_2)$ and $F' = \frac{1}{\sqrt{2}} (F_1 \hat{\otimes} 1 + 1 \hat{\otimes} F_2)$.

**Definition 5.4.** We define a product

$$S^1_j(\tilde{X}_1) \times K_1(\tilde{X}_2/T_2) \to S^1_{j+i}(\tilde{X}_1 \times \tilde{X}_2)$$

that associates to $(\xi, \lambda)$ the class of $(H_1 \hat{\otimes} H_2, \phi_1 \hat{\otimes} \phi_2, F, \tilde{F}(t))$, where $\tilde{F}(t)$ is the concatenation of the paths given by

$$t \mapsto \cos \left( \frac{\pi}{2} t \right) \tilde{F} + \sin \left( \frac{\pi}{2} t \right) \tilde{F}', \quad t \in [0, 1]$$

and

$$t \mapsto \frac{1}{\sqrt{2}} (\tilde{F}_1(t) \hat{\otimes} 1 + 1 \hat{\otimes} \tilde{F}_2), \quad t \in [0, 1].$$

The product is compatible with homotopies in both factors and so it is well defined.

**Remark 5.2.** A similar product is defined in an obvious way on $KK^{j-1}((\mathbb{C}, C^*_r(\Gamma) \otimes C_0(0, 1))$ and $KK^j(C(X), \mathbb{C})$. It is natural in the sense that the following diagram

$$
\begin{array}{ccccccc}
\cdots & \to & KK^j((\mathbb{C}, A) \times K_1(X_2)) & \to & S^1_j(\tilde{X}_1) \times K_1(X_2) & \to & KK^j(C(X_1), C) \times K_1(X_2) & \to & \cdots \\
\downarrow & & & & \downarrow & & \downarrow & \\
\cdots & \to & KK^{j+i}((\mathbb{C}, B) & \to & S^1_{j+i}(\tilde{X}_1 \times \tilde{X}_2) & \to & KK^{j+i}(C(X_1 \times X_2), \mathbb{C}) & \to & \cdots
\end{array}
$$

is commutative. Here $A = C^*_r(\tilde{X}_1) \otimes C_0(0, 1)$ and $B = C^*_r(\tilde{X}_1 \times \tilde{X}_2)^{\Gamma_1 \times \Gamma_2} \otimes C_0(0, 1)$. 
Lemma 5.2. Let $Y, X, Z$ be to manifolds and assume that a group $\Gamma_1$ acts on $Y$ and $X$ and $\Gamma_2$ acts on $Z$, freely and isometrically. Let $f : Y \to X$ be a $\Gamma$-equivariant continuous coarse map. Then the following diagram

\[
\begin{array}{c}
S^\Gamma_1(Y) \times K^\Gamma_2(Z) \\
\downarrow \\
S^\Gamma_1 \times \Gamma_2(Y \times Z) \quad (f \times id_Z)^* \downarrow \quad S^\Gamma_1 \times \Gamma_2(X \times Z)
\end{array}
\]

where the vertical arrows are given by $\tilde{g}_\mathcal{A}$ is commutative.

Proof. This is straightforward since $(\phi_1 \otimes \phi_2) \circ (f \times id_Z)^* = (\phi_1 \circ f^*) \otimes \phi_2$. \hfill \qed

5.2 Stability of $\varrho$ classes

5.2.1 The signature operator

Let $f : M \to N$ be a structure in $\mathcal{S}^{TOP}(N)$ and $\varrho(f)$ be the associated $\varrho$-class in $K_*(D^*(\tilde{Z})^\Gamma)$. Let us see the different realisations of this class with respect to the different models of the analytical structure set.

- In $K_0(D^*(\tilde{Z})^\Gamma)$ we have the element $\left[ \frac{1}{2}(1 + \chi(\tilde{D}Z + C_f)) \right]$.
- In $K_0(C_\Gamma)$ the $\varrho$-class is represented by a pair $[\pi(p), \pi(p(t))]$. Let $p$ be the image of $\frac{1}{2} \left( 1 + \psi(\tilde{D}Z) \right)$ in $D^*(\tilde{Z})^\Gamma/C^*(\tilde{Z})^\Gamma$, where $\psi$ is any chopping function; let $p(t)$ be a path in $M(C^*(\tilde{Z})^\Gamma)$ such that $p(t) = \frac{1}{2} \left( 1 + \psi(D_{2t}) \right)$ for $t \in [0, \frac{1}{2}]$, where $D_\alpha$ is the Hilsum-Skandalis homotopy, and $p(t) = \frac{1}{2} \left( 1 + \psi(t(\tilde{D}Z + C_f)) \right)$ for $t \in [\frac{1}{2}, 1]$, where $\psi(t) = \psi(t) + (t - \frac{1}{2})\pi)$) (notice that $l_{f(t)} \psi_\Gamma = \chi$).
- In $S^\Gamma_1(\tilde{Z})$ this element turns into the quaduple $\left[ H, \phi, F, \tilde{F}(t) \right]$, where $F = \left( \psi(\tilde{D}Z) \right)$ and $\tilde{F}(t) = 2(p(t) - 1)$, with $p(t)$ as above. Here $(H, \phi)$ is the $\Gamma$-equivariant $C_0(\tilde{Z})$-module used to construct $D^*(\tilde{Z})^\Gamma$.

Proposition 5.3. Let $M_1$ and $N_1$ be two $n$-dimensional Lipschitz manifolds with $n$ odd and let $M_2$ be an $m$-dimensional Lipschitz manifold with $m$ even. Let $M$ be $M_1 \times M_2$, let $N$ be $N_1 \times M_2$ and let $f_1 : N_1 \to M_1$ be a homotopy equivalence. Let $\Gamma_i$ be the fundamental groups of $M_i$, with $i = 1, 2$. We get that

$\varrho(f_1 \times id_M) = \varrho(f_1) \cdot [D_2] \in S^\Gamma_1 \times \Gamma_2(M_1 \times M_2)$

and the same holds for $\varrho_\Gamma$.

Proof. Remember the geometrical context described at the beginning of the section. We obtain $F_1 = \psi(\tilde{D}_1)$, where $D_1$ is the signature operator on $Z_1$, $\tilde{F}_1(t)$ as at the begin of the section and $F_2 = \psi(\tilde{D}_2)$.

Let $G = \psi(\tilde{D}_1 \otimes 1 + 1 \otimes \tilde{D}_2)$. Consequently we obtain the following cycles

- $[L^2(\tilde{M}_1, \Lambda^*(\tilde{M}_1)), \phi_1, F_1, \tilde{F}_1(t)];$
- $[D_2] = [L^2(\tilde{M}_2, \Lambda^*(\tilde{M}_2)), \phi_2, F_2];$
- $[L^2(\tilde{M}, \Lambda^*(\tilde{M})), \phi_1 \otimes \phi_2, G, \tilde{G}(t)];$
where \( \tilde{G}(t) \) is the path given by \( \psi(D_{1,2t} \hat{\otimes} 1 + 1 \hat{\otimes} D_2) \) for \( t \in [0, \frac{1}{2}] \) and \( \psi_{t}((D_{1}+C_{f_{t}}) \hat{\otimes} 1 + 1 \hat{\otimes} D_2) \) for \( t \in [\frac{1}{2}, 1] \), with \( \psi_t \) as at the begin of the section. Doing the product between the first two classes, using \( F \) like in Definition 5.4 we obtain the class

\[ [L^2(\tilde{M}, \Lambda^{*}(\tilde{M})), \phi_1 \hat{\otimes} \phi_2, F, \tilde{F}(t)]. \]

Now we would like a homotopy between the class represented by the path \( \tilde{F}(t) \) and the one represented by the path \( \tilde{G}(t) \).

Let \( h \) be the map such that \( (s,t) \mapsto 1 - s(1-t) \) for \( t > 1 - \frac{s}{2} \) and \( (s,t) \mapsto \frac{1-s}{2} \) otherwise. We have that the homotopy \( (t,s) \mapsto \tilde{F}(h(s,t)) \) for \( s \in [0,1] \): this homotopy sends any point of the first part of the path given in Definition 5.4 to the starting point of the second part. For this purpose we should verify that the hypotheses of positivity required in \([18, \text{Lemma 6.2.12}]\) hold: we need that at each \( s \) the square of the operator in \( t = 1 \) is strictly positive. But this follows from \([3, \text{Lemma 10.7.4}]\).

Because of the diagram in \([11, \text{Theorem 3.2}]\), \((L^2(\tilde{M}, \Lambda^{*}(\tilde{M})), \phi_1 \hat{\otimes} \phi_2, G)\) represents a Kasparov product for \((L^2(M_1, \Lambda^*(M_1)), \phi_1, F_1)\) and \((L^2(M_2, \Lambda^*(M_2)), \phi_2, F_2)\), namely there is an operatorial homotopy connecting them. This is true also for \( F(h(t,1)) \) and \( G(t) \) continuously in \( t \), for any \( t \).

Hence we have proved the equality. By functoriality this is true not only on \( Z_4 \) and \( M_2 \times Z_4 \) but even on \( M_4 \) and \( M_2 \times M_4 \). Trivially this holds for \( G_t \) too.

We would like that, after fixing a class \([D_2] \neq 0\), the product we have described behaves injectively with \( \varrho \)-classes. We can achieve this under certain hypotheses on \( \Gamma_2 \): we require the group to have a \( \gamma \) element. This means that there exists a \( C^* \)-algebra \( A \) on which \( \Gamma \) acts properly and elements

\[ \eta \in KK_{\Gamma_2}(\mathbb{C}, A) \quad \text{and} \quad d \in KK_{\Gamma_2}(A, \mathbb{C}), \]

such that \( \gamma = \eta \otimes A d \in KK_{\Gamma_2}(\mathbb{C}, \mathbb{C}) \) satisfies \( p^* \gamma = 1 \in KK_{\mathbb{E}T_2 \times T_2}(C_0(\mathbb{F}T_2), C_0(\mathbb{F}T_2)) \), where \( \mathbb{E}T_2 \) is the classifying space for proper actions of \( \Gamma_2 \) and \( p: \mathbb{E}T_2 \times \Gamma_2 \rightarrow \Gamma_2 \) is the homomorphism defined by \( p(z, g) = g \). We refer the reader to \([23, 24]\).

The existence of the \( \gamma \) element implies that the Baum-Connes assembly map (with coefficients) is split injective: this gives the existence of a non trivial element \( \zeta \in KK(C_\Gamma^*(\Gamma_2), \mathbb{C}) \).

**Lemma 5.3.** Let \( \Gamma_2 \) have a \( \gamma \) element. If \( x \in K_1(M_2) \) is a non zero element, \( \varrho \in K_0(D^*(\tilde{M}_1)^{\Gamma_1}) \) is not zero, then \( \varrho \cdot x \) is not zero in \( K_0(D^*(M_1 \times M_2)^{\Gamma_1 \times \Gamma_2}) \).

**Proof.** Let \( \varrho = [H_1, \phi_1, F_1, \tilde{F}_1(t)] \) and \( x = [H_2, \phi_2, F_2] \). By Proposition 5.3

\[ \varrho \cdot x = [H_1 \hat{\otimes} H_2, \phi_1 \hat{\otimes} \phi_2, G, \tilde{G}(t)], \]

as in Definition 5.4.

Let \( \iota^* \in KK(\mathbb{C}, C(M_2)) \) be the class induced by the inclusion \( \iota: \mathbb{C} \rightarrow C(M_2) \) and let \( \zeta \in KK(C_\Gamma^*(\Gamma_2), \mathbb{C}) \) be non trivial (it exists by hypotheses). It turns out that

\[ \iota^* \otimes C(M_2) [H_1 \hat{\otimes} H_2, \phi_1 \hat{\otimes} \phi_2, G] \in K_1(M_2) \]

is equal to the class \([H_1, \phi_1, F_1]\).

Hence if \( \varrho \cdot x \) is zero, then the \( K \)-homology cycle \((H_1, \phi_1, F_1)\) is homotopic to the trivial one.

Moreover \((\tilde{H}, \tilde{\phi}, \tilde{G}(t)) \otimes C(\mathbb{F}T_2) \otimes C([0,1])(\zeta \otimes \text{id}) \), where \( H = H_1 \otimes H_2 \), is equal to \((\tilde{H}_1, \tilde{\phi}_1, \tilde{F}_1(t))\). Consequently \( \tilde{F}_1(t) \) is homotopic to the path constantly equal to a trivial operator. Then \( \varrho = [H_1, \phi_1, F_1, \tilde{F}_1] \) is zero, that is absurd. \( \square \)
Remark 5.3. The same argument fit to prove that if we fix a non zero element \( x \in K_1(M_2) \), then the vertical arrows of the following diagram

\[
\cdots \rightarrow KK^j(C, A) \rightarrow S^{j+1}_\Gamma(M_1) \rightarrow KK^j(C(M_1), C) \rightarrow \cdots
\]

\[
\downarrow x \downarrow \quad \downarrow x
\]

\[
\cdots \rightarrow KK^{j+i}(C, B) \rightarrow S^{j+1}_{\Gamma_1 \times \Gamma_2}(M_1 \times \widetilde{M}_2) \rightarrow KK^{j+i}(C(M_1 \times X_2), C) \rightarrow \cdots
\]

are injective. Here \( A = C^*(\widetilde{M}_1)^{\Gamma_1} \otimes C_0(0, 1) \) and \( B = C^*(\widetilde{M}_1 \times \widetilde{M}_2)^{\Gamma_1 \times \Gamma_2} \otimes C_0(0, 1) \).

Corollary 5.1. Let \( M_2 \) be an even dimensional Lipschitz manifold with fundamental group \( \Gamma_2 \) such that it has a \( \gamma \) element and \( [D_2] \neq 0 \) in the \( K \)-homology of \( M_2 \). If \( f_1: N_1 \rightarrow M_1 \) and \( f_1': N_1' \rightarrow M_1 \) are homotopy equivalences between odd dimensional Lipschitz manifolds, with different \( g \)-class invariants, then

\[
[f_1 \times \text{id}_{M_2}] \neq [f_1' \times \text{id}_{M_2}] \in S^{\text{TOP}}(M_1 \times M_2).
\]

5.2.2 Dirac operators and positive scalar curvature

We would like to apply the methods of the previous sections to get similar results about the secondary invariants described in [15].

Let us recall [15, Definition 1.6]: let \((M, g)\) be a Riemannian spin manifold of dimension \( n > 0 \), with fundamental group \( \Gamma \). If \( g \) has uniformly positive scalar curvature then the Dirac operator \( D_M \) is invertible and \( \chi(\widetilde{D}_M) \), the bounded transform of the lift of \( D_M \) to the universal covering of \( M \), defines a class \( \varrho_g \in D^*(\widetilde{M})^\Gamma \).

Thanks to that and the APS-delocalized Theorem, for \( n \) odd, one obtains the following commutative diagram

\[
\begin{array}{cccccc}
\Omega_{n+1}^{\text{spin}}(M) & \longrightarrow & \Omega_{n+1}^{\text{spin}}(M) & \longrightarrow & \Omega_n^{\text{spin}}(M) \\
\beta \downarrow & & \text{Indr} \downarrow & & \beta \\
K_{n+1}(M) & \longrightarrow & K_{n+1}(C_r^*(\Gamma)) & \longrightarrow & K_n(D^*(\widetilde{M})^\Gamma) & \longrightarrow & K_n(M)
\end{array}
\]

where \( M \) is a compact space with fundamental group \( \Gamma \) and universal covering \( \widetilde{M} \). The first row in the diagram is the Stolz exact sequence, see for instance [15, Definition 1.39].

In the \( S^1(M) \) picture of the analytic structure set, the class \( \varrho_g \) is given by the quadruple

\[
[L^2(M, \mathcal{S}), C(M), \varnothing_M, \chi(\varnothing_M)]
\]

Here the last term is the constant path \( \chi(\varnothing_M) \) because the operator is invertible and there is no need to perturb it.

Remark 5.4. If \((M, g)\) has positive scalar curvature and \((N, h)\) is another Riemannian manifold, then for \( \varepsilon > 0 \) small enough, \((M \times N, g \times \varepsilon h)\) has positive scalar curvature. Hence if \( M \) admits a metric with positive scalar curvature, so does \( M \times N \).

Proposition 5.4. Let \( M \) be a spin manifold of dimension \( n \) and let \( g \) be a Riemannian metric with positive scalar curvature on \( M \). Let \( N \) be a spin manifold of dimension \( m \) and \( h \) a Riemannian metric such that \((M \times N, g \times h)\) has positive scalar curvature. Then

\[
\varrho_g \cdot [\varnothing_h] = \varrho_{g \times h} \in S^1_{n+m}(M \times N),
\]

where \( \Gamma_1 \) and \( \Gamma_2 \) are the fundamental groups of \( M \) and \( N \) respectively and \([\varnothing_h]\) is the class of the Dirac operator on \( N \) in \( K_m(N) \).
Proof. We can prove the result as we did. Moreover since the class $g_i$ is represented by a quadraple whose last term is the constant path $\chi(\tilde{\theta}_m)$, it turns out that we can prove it in an easier way (see for instance [13, Proposition 6.2.13]).

Proposition 5.5. Let $M$ be a spin manifold of odd dimension $n$ with fundamental group $\Gamma_1$ and let $g_1$ and $g_2$ be two Riemannian metrics with positive scalar curvature on $M$ such that $\varrho_{g_1} \neq \varrho_{g_2} \in S^{n}_{\Gamma_1}(M)$. Let $(N,h)$ be a Riemannian spin manifold of even dimension $m$ with fundamental group $\Gamma_2$, such that $[\varrho_N] \in K_0(N)$ does not vanish, $\Gamma_2$ has a $\gamma$ element and $g_i \times h$ has positive scalar curvature on $M_i \times N$.

Then

$$\varrho_{g_1 \times h} \neq \varrho_{g_2 \times h} \in S^{F_1 \times \Gamma_2}(M \times N).$$

Proof. We can use the arguments we used for Lemma 5.3 to obtain immediately the result.

5.3 The delocalized APS index Theorem in the odd-dimensional case

Another application of the product formula is the proof of the delocalized APS index theorem for odd dimensional cobordisms.

We will do it for the perturbed signature operator, the theorem for the Dirac operator on a spin manifold with positive scalar curvature is completely analogous.

Because of motivations well explained in [13, Remark 4.6], we will do prove the theorem at the cost of inverting 2. We recall that here and in [13] the signature operator on an odd dimensional manifold is not the odd signature operator of Atiyah, Patodi and Singer, but the direct sum of two (unitarily equivalent) versions of this operator.

Since in the statement of the delocalized APS index theorem in the odd dimensional case we will compare the $g$ invariant of the boundary with the index of the APS odd signature operator on the cobordism, it is worth to specify the notation we shall follow: on an odd dimensional manifold we denote by $D^{APS}$ the odd signature operator of Atiyah, Patodi and Singer and we denote by $D$ the odd signature operator that we used so far.

The strategy of the proof is to reduce the odd dimensional case to the even dimensional one through the product by the $K$-homology class of the signature operator on the circle. Then it is useful to review the behaving of the signature operator with respect to cartesian products of manifolds. For a detailed treatment we refer the reader to sections 5 and 6 of [26].

Let $W$ be an $n$-dimensional manifold with boundary $\partial W$ endowed with a cocompact free $\Gamma$-action. We assume that $n$ odd and that the boundary of $W$ is composed by a pairs of homotopy equivalent manifolds. Let $j: \partial W \hookrightarrow W$ and $j': \partial W \times \mathbb{R} \hookrightarrow W \times \mathbb{R}$ be the obvious inclusions. Let us recall some useful facts:

- the even signature operator $D_{W \times S^1}$ is equivalent to the direct sum of two copies of the exterior product $D^W_{\text{ext}} \otimes 1 + 1 \otimes D^S_{\text{ext}}$, see [26, Section 6.3]. Since $D_{S^1}$ is the sum of two equivalent versions of $D^\text{ext}_{S^1}$, one has that $D_{W \times S^1}$ is equivalent to $D^W_{\text{ext}} \otimes 1 + 1 \otimes D^S_{\text{ext}}$. Consequently the higher index of $(D^W_{\text{ext}} + C^F_{\text{ext}})$ is equal to the class given by the product $\frac{1}{2}(\text{Ind}_1(D^W_{\text{ext}} + C^F_{\text{ext}}})): [D_{S^1}],$ where here $: K_1(C^*_r(\Gamma)) \times K_j(S^1) \to K_{i+j}(C^*_r(\Gamma \times \mathbb{Z}));$

- the operator $D_{\partial W \times S^1}$ is equivalent to the exterior product of the even dimensional signature operator $D_{\partial W}$ and the odd dimensional signature operator $D_{S^1}$. Consequently this means that $\varrho(D_{\partial W} + C_{F_0}) : [D_{S^1}]$ is equal to $\varrho(D_{\partial W \times S^1} + C_{F_0 \times \mathbb{Z}})$, where here $: S^F_{\Gamma}(\partial W) \times K_j(S^1) \to S^{F \times \mathbb{Z}}_{\Gamma \times \mathbb{Z}}(W \times \mathbb{R})$.

Theorem 5.1. If $i: C^*(\hat{W})^\mathbb{F} \hookrightarrow D^*(\hat{W})^\mathbb{F}$ is the inclusion and $j_*: D^*(\partial \hat{W})^\mathbb{F} \to D^*(\hat{W})^\mathbb{F}$ is the map induced by the inclusion $j: \partial \hat{W} \hookrightarrow \hat{W}$, we have

$$i_* \left( \frac{1}{2} \text{Ind}_1(D^W_{\text{ext}} + C^F_{\text{ext}}) \right) = j_* (\varrho(D_{\partial W} + C_{F_0})) \in K_0(D^*(\hat{W})^\mathbb{F}) \otimes \mathbb{Z} \left[\frac{1}{2}\right].$$
where $\frac{1}{2}\text{Ind}_\Gamma(D_{W}^{APS} + C_{F}^{cyl}) \in K_0(C^*(\tilde{W})^\Gamma) \otimes \mathbb{Z}[\frac{1}{2}]$.

**Proof.** Let $W$ be as above. Because of Proposition 5.2 and Lemma 5.1 we will prove the theorem in the $S_0^1(\cdot)$ setting.

Let $\Pi_D: S_0^1(\tilde{W}) \to S_0^1(\tilde{W} \times \mathbb{R})$ and $\Pi_C: K_1(C^*(\tilde{W})^\Gamma) \to K_1(C^*(\tilde{W} \times \mathbb{R})^{\mathbb{Z} \times \Gamma})$ be the morphism induced by the product with the of class of the signature operator $D_{S^1}$ in $K_1(S^1)$. By Lemma 5.3, we have that

$$i_*\left(\frac{1}{2}\text{Ind}_\Gamma(D_{\tilde{W}}^{APS} + C_{\tilde{F}}^{cyl})\right) = j_*(\varrho(D_\partial W + C_{F_0})) \quad \text{(5.2)}$$

holds if and only if

$$\Pi_D\left(i_*\left(\frac{1}{2}\text{Ind}_\Gamma(D_{\tilde{W}}^{APS} + C_{\tilde{F}}^{cyl})\right)\right) = \Pi_D(j_*(\varrho(D_\partial W + C_{F_0}))$$

holds,

But by Remark 5.2 it turns out that

$$\Pi_D\left(i_*\left(\frac{1}{2}\text{Ind}_\Gamma(D_{\tilde{W}}^{APS} + C_{\tilde{F}}^{cyl})\right)\right) = i_*\left(\Pi_C\left(\frac{1}{2}\text{Ind}_\Gamma(D_{\tilde{W}}^{APS} + C_{\tilde{F}}^{cyl})\right)\right)$$

and, by Proposition 5.4, that

$$\Pi_C\left(\frac{1}{2}\text{Ind}_\Gamma(D_{\tilde{W}}^{APS} + C_{\tilde{F}}^{cyl})\right) = \text{Ind}_\Gamma(D_{W \times S^1} + C_{F_0 \times \text{id}})$$

Moreover by Lemma 5.2 it follows that

$$\Pi_D(j_*(\varrho(D_\partial W + C_{F_0}))) = j'_S(\Pi_D(\varrho(D_\partial W + C_{F_0}))$$

and, by Proposition 5.4 that

$$\Pi_D(\varrho(D_\partial W + C_{F_0})) = \varrho(D_\partial W \times S^1 + C_{F_0 \times \text{id}})$$

Thus we have that (5.2) holds if and only if

$$i_*\left(\text{Ind}_\Gamma(D_{W \times S^1}^{APS} + C_{F_0 \times \text{id}}^{cyl})\right) = j'_S(\varrho(D_\partial W \times S^1 + C_{F_0 \times \text{id}}))$$

holds. But, since $W \times S^1$ is even dimensional, the equality on the right-hand side holds by 3.3 and the Theorem is proved.

If $W$ is a Spin Riemannian manifold with boundary, such that the metric on the boundary has positive scalar curvature, then we can state the analogous theorem for the $\varrho$ invariants associated to Dirac operators.

**Theorem 5.2.** If $i: C^*(\tilde{W})^\Gamma \hookrightarrow D^*(\tilde{W})^\Gamma$ is the inclusion and $j_*: D^*(\partial \tilde{W})^\Gamma \to D^*(\tilde{W})^\Gamma$ is the map induced by the inclusion $j: \partial \tilde{W} \hookrightarrow \tilde{W}$, we have

$$i_*(\text{Ind}_\Gamma(D_{\tilde{W}})) = j_*(\varrho(D_\partial W)) \in K_0(D^*(\tilde{W})^\Gamma)$$

Notice that in this case it is not necessary to invert 2. Moreover the proof of the theorem is very similar to the case of the signature operator, but easier because we have not to perturb the Dirac operator to obtain an invertible operator.
6 Mapping surgery to analysis: the even dimensional case

The extension to the odd dimensional case of the delocalized APS index theorem allows us to state the following result (with proof almost identical to the the one given in the odd dimensional case).

**Theorem 6.1.** Let $N$ be an $n$-dimensional closed oriented topological manifold with fundamental group $\Gamma$. Assume that $n \geq 5$ is even. Then there is a commutative diagram with exact rows

$$
\begin{array}{cccccc}
L_{n+1}(\mathbb{Z}\Gamma) & \rightarrow & S^{\text{TOP}}(N) & \rightarrow & N^{\text{TOP}}(N) & \rightarrow & L_n(\mathbb{Z}\Gamma) \\
\downarrow \text{Ind}_r & & \downarrow \varrho & & \uparrow \beta & & \downarrow \text{Ind}_r \\
K_{n+1}(C^*_r(\Gamma)) \otimes \mathbb{Z} \left[ \frac{1}{2} \right] & \rightarrow & K_{n+1}(D^*(\tilde{N})^\Gamma) \otimes \mathbb{Z} \left[ \frac{1}{2} \right] & \rightarrow & K_n(\mathbb{Z}\Gamma) \otimes \mathbb{Z} \left[ \frac{1}{2} \right] & \rightarrow & K_n(C^*_r(\Gamma)) \otimes \mathbb{Z} \left[ \frac{1}{2} \right]
\end{array}
$$

and through the classifying map $u: N \rightarrow B\Gamma$ of the universal cover $\tilde{N}$ of $N$, we have the analogous commutative diagram that involves the universal Higson-Roe exact sequence

$$
\begin{array}{cccccc}
\Omega^{\text{spin}}_{n+1}(M) & \rightarrow & R^{\text{spin}}_{n+1}(M) & \rightarrow & \text{Pos}^{\text{spin}}_n(M) & \rightarrow & \Omega^{\text{spin}}_n(M) \\
\downarrow \beta & & \downarrow \text{Ind}_r & & \uparrow \varrho & & \downarrow \beta \\
K_{n+1}(M) & \rightarrow & K_{n+1}(C^*_r(\Gamma)) & \rightarrow & K_{n+1}(D^*(\tilde{M})^\Gamma) & \rightarrow & K_n(M)
\end{array}
$$

with $n \geq 5$ even.

**Remark 6.1.** Thanks to Theorem 5.2, we can enunciate the analogous statement for the Stolz sequence. With the same notations as in subsection 5.2.2, we obtain the following commutative diagram

$$
\begin{array}{cccccc}
\Omega^{\text{spin}}_{n+1}(M) & \rightarrow & R^{\text{spin}}_{n+1}(M) & \rightarrow & \text{Pos}^{\text{spin}}_n(M) & \rightarrow & \Omega^{\text{spin}}_n(M) \\
\downarrow \beta & & \downarrow \text{Ind}_r & & \uparrow \varrho & & \downarrow \beta \\
K_{n+1}(M) & \rightarrow & K_{n+1}(C^*_r(\Gamma)) & \rightarrow & K_{n+1}(D^*(\tilde{M})^\Gamma) & \rightarrow & K_n(M)
\end{array}
$$

with $n \geq 5$ even.

**References**

[1] Saad Baaj and Pierre Julg. Théorie bivariante de Kasparov et opérateurs non bornés dans les $C^*$-modules hilbertiens. *C. R. Acad. Sci. Paris Sér. I Math.*, 296(21):875–878, 1983.

[2] Ulrich Bunke. A $K$-theoretic relative index theorem and Callias-type Dirac operators. *Math. Ann.*, 303(2):241–279, 1995.

[3] Nigel Higson and John Roe. *Analytic $K$-homology*. Oxford Mathematical Monographs. Oxford University Press, Oxford, 2000. Oxford Science Publications.

[4] Nigel Higson and John Roe. Mapping surgery to analysis. I. Analytic signatures. *K-Theory*, 33(4):277–299, 2005.

[5] Nigel Higson and John Roe. Mapping surgery to analysis. II. Geometric signatures. *K-Theory*, 33(4):301–324, 2005.

[6] Nigel Higson and John Roe. Mapping surgery to analysis. III. Exact sequences. *K-Theory*, 33(4):325–346, 2005.
[7] Michel Hilsum. Signature operator on Lipschitz manifolds and unbounded Kasparov bimodules. In *Operator algebras and their connections with topology and ergodic theory (Buşteni, 1983)*, volume 1132 of *Lecture Notes in Math.*, pages 254–288. Springer, Berlin, 1985.

[8] Michel Hilsum. Fonctorialité en $K$-théorie bivariante pour les variétés lipschitiennes. *K-Theory*, 3(5):401–440, 1989.

[9] Michel Hilsum. L’invariant $\eta$ pour les variétés lipschitiennes. *J. Differential Geom.*, 55(1):1–41, 2000.

[10] Michel Hilsum and Georges Skandalis. Invariance par homotopie de la signature à coefficients dans un fibré presque plat. *J. Reine Angew. Math.*, 423:73–99, 1992.

[11] Kjeld Knudsen Jensen and Klaus Thomsen. *Elements of $KK$-theory*. Mathematics: Theory & Applications. Birkhäuser Boston Inc., Boston, MA, 1991.

[12] E. C. Lance. *Hilbert $C^*$-modules*, volume 210 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1995. A toolkit for operator algebraists.

[13] Paolo Piazza and Thomas Schick. Bordism, rho-invariants and the Baum-Connes conjecture. *J. Noncommut. Geom.*, 1(1):27–111, 2007.

[14] Paolo Piazza and Thomas Schick. The surgery exact sequence, $k$-theory and the signature operator. *arXiv:1309.4370v1*, 2013.

[15] Paolo Piazza and Thomas Schick. Rho-classes, index theory and Stolz’ positive scalar curvature sequence. *J. Topol.*, 7(4):965–1004, 2014.

[16] John Roe. *Elliptic operators, topology and asymptotic methods*, volume 179 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow, copublished in the United States with John Wiley & Sons, Inc., New York, 1988.

[17] John Roe. Comparing analytic assembly maps. *Q. J. Math.*, 53(2):241–248, 2002.

[18] Paul Siegel. Homological calculations with analytic structure groups. *PhD thesis*, 2012.

[19] D. Sullivan and N. Teleman. An analytic proof of Novikov’s theorem on rational Pontrjagin classes. *Inst. Hautes Études Sci. Publ. Math.*, (58):79–81 (1984), 1983.

[20] Dennis Sullivan. Hyperbolic geometry and homeomorphisms. In *Geometric topology (Proc. Georgia Topology Conf., Athens, Ga., 1977)*, pages 543–555. Academic Press, New York, 1979.

[21] Nicolae Teleman. The index of signature operators on Lipschitz manifolds. *Inst. Hautes Études Sci. Publ. Math.*, (58):39–78 (1984), 1983.

[22] Nicolae Teleman. The index theorem for topological manifolds. *Acta Math.*, 153(1-2):117–152, 1984.

[23] Jean Louis Tu. La conjecture de Novikov pour les feuilletages hyperboliques. *K-Theory*, 16(2):129–184, 1999.

[24] Jean-Louis Tu. The gamma element for groups which admit a uniform embedding into Hilbert space. In *Recent advances in operator theory, operator algebras, and their applications*, volume 153 of *Oper. Theory Adv. Appl.*, pages 271–286. Birkhäuser, Basel, 2005.

[25] P. Tukia and J. Väisälä. Lipschitz and quasiconformal approximation and extension. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 6(2):303–342 (1982), 1981.

[26] Charlotte Wahl. Product formula for Atiyah-Patodi-Singer index classes and higher signatures. *J. K-Theory*, 6(2):285–337, 2010.
[27] Charlotte Wahl. Higher $\rho$-invariants and the surgery structure set. *J. Topol.*, 6(1):154–192, 2013.

[28] S. Weinberger and G. Yu. Finite part of operator $\kappa$-theory for groups finitely embeddable into hilbert space and the degree of non-rigidity of manifolds. *arXiv:1308.4744*.

[29] Zhizhang Xie and Guoliang Yu. Higher rho invariants and the moduli space of positive scalar curvature metrics. *arXiv:1310.1136*.

[30] Zhizhang Xie and Guoliang Yu. Positive scalar curvature, higher rho invariants and localization algebras. *Adv. Math.*, 262:823–866, 2014.

[31] Rudolf Zeidler. Positive scalar curvature and product formulas for secondary index invariants. *arXiv:1412.0685*. 