BROWNIAN SUPER-EXPONENTS

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Abstract. We introduce a transform on the class of stochastic exponentials for d-dimensional Brownian motions. Each stochastic exponential generates another stochastic exponential under the transform. The new exponential process is often merely a supermartingale even in cases where the original process is a martingale. We determine a necessary and sufficient condition for the transform to be a martingale process. The condition links expected values of the transformed stochastic exponential to the distribution function of certain time-integrals.

1. Introduction

If $X(t)$ is a d-dimensional progressively measurable process and $W$ is a Brownian motion under a measure $P$, the stochastic exponential determined by $X$ is the process

$$Z_X(t) = \exp\left\{ \int_0^t X(u) \cdot dW(u) - \frac{1}{2} \int_0^t |X(u)|^2 \, du \right\}.$$ 

The problem of checking whether $Z_X(t)$ is a true martingale is important for the use of Girsanov’s theorem. Two well-known sufficient conditions are due to Novikov and to Kazamaki; see for example Revuz and Yor [7]. Examples where the process $Z_X(t)$ is strictly a supermartingale appear in Goodman and Kim [3], Levental and Skorohod [6], and Wong and Heyde [9].

In their recent paper, Wong and Heyde [9] present a necessary and sufficient condition for any stochastic exponential to form a martingale process. Their condition is formulated in terms of an explosion time. We consider a class of stochastic exponentials for which their condition becomes more explicit. We begin with any stochastic exponential and we describe a modification, or transform, of it which generates another stochastic exponential.

The transform involves a time-integral of the form

$$\int_0^t |X(u)|^2 Z_X(u) \, du.$$ 

We derive a necessary and sufficient condition for the transform to be a martingale. Our condition is formulated in terms of the distribution of time integrals, and we use the relation to obtain bounds on the tail behavior of these distributions.

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Definition 1.1. Suppose that \( X(t) \) is a progressively measurable process such that for some \( T > 0 \),
\[
P \left\{ \int_0^T ||X(u)||^2 du < \infty \right\} = 1 \tag{1.1}\]
If \( Z_X(t) \) is the stochastic exponential generated by \( X(t) \) and \( y > 0 \), the associated super-exponent process \( Y_X(t) \), defined for \( t \leq T \), is
\[
Y_{X,y}(t) = \frac{Z_X(t)}{y^{-1} + \frac{1}{2} \int_0^t ||X(u)||^2 Z_X(u)du} \tag{1.2}
\]
Notice from Equation (1.2) that \( Y_{X,y}(0) = y \). In addition, \( Y_{X,y}(t) \) is positive so that the random variable
\[
\exp(Y_{X,y}(t))
\]
is greater than one. We show that this random variable has a finite expected value which is less than or equal to \( e^y \). This result is surprising since \( Y_{X,y}(t) \) is used as an exponent here. According to Definition 1.1, \( Y_{X,y}(t) \) itself contains an exponential factor \( Z_X(t) \). For this reason, we say that the process \( Y_{X,y}(t) \) is a Brownian super-exponent.

2. Transform Properties

Proposition 2.1. Suppose that a progressively measurable process \( X \) satisfies condition (1.1). Let \( Y_{X,y}(t) \) denote the super-exponent process in Definition 1.1. Then for each \( t \leq T \),
\[
Y_{X,y}(t) = y + \int_0^t Y_{X,y}(u)X(u) \cdot dW(u) - \frac{1}{2} \int_0^t ||X(u)||^2 Y_{X,y}^2(u)du \tag{2.1}
\]
Moreover, the process
\[
\exp(Y_{X,y}(t)) \tag{2.2}
\]
is a positive supermartingale on the interval \( 0 \leq t \leq T \). In addition, the process
\[
\tilde{Z}(t) = \exp(Y_{X,y}(t) - y) \tag{2.3}
\]
is a stochastic exponential for \( W \). This stochastic exponential is generated by the \( d \)-dimensional process
\[
Y_{X,y}(t)X(t) \tag{2.4}
\]
Proof. It follows from the definition of $Z_X(t)$ that

$$dZ_X = Z_XX \cdot dW$$

and

$$d \int_0^t ||X||^2 Z_X du = ||X||^2 Z_X dt$$

Direct calculation shows that

$$dY_{X,y} = \frac{dZ_X}{y^{-1} + \frac{1}{2} \int_0^t ||X||^2 Z_X du} + Z_X \frac{d(y^{-1} + \frac{1}{2} \int_0^t ||X||^2 Z_X du)^{-1}}{y^{-1} + \frac{1}{2} \int_0^t ||X||^2 Z_X du}$$

$$= Y_{X,y} X \cdot W - \frac{1}{2} ||X||^2 Y_{X,y}^2 dt$$

From this equation we see that $Y_{X,y} - y$ is the sum of the Itô integral of $Y_{X,y} X$ and the elementary integral of $-\frac{1}{2} ||X||^2 Z_X$. This establishes Equation (2.1). It follows immediately from Equation (2.1) that $Y_{X,y} - y$ is the exponent of a stochastic exponential. Therefore, the process

$$\exp(Y_{X,y}(t) - y)$$

is a positive local martingale. It is well known that a positive local martingale is a supermartingale; see, for instance, Karatzas and Shreve [4]. In addition, Equation (2.3) is a direct consequence of Equation (2.1) and the definition of stochastic exponential processes.

$\square$

Theorem 2.2. Suppose that $X(t)$ is a deterministic function such that for some $T > 0$

$$\int_0^T ||X(u)||^2 du < \infty.$$ 

Let $Z_X(t)$ and $Y_{X,y}$ denote the stochastic exponential and super-exponent process generated by $X(t)$. Then for each non-negative measurable function $G(u), u > 0$, and $t < T$,

$$E[G(Y_{X,y}(t)) \exp(Y(t) - y)]$$

$$= E \left[ G \left( \frac{Z_X(t)}{y^{-1} + \frac{1}{2} \int_0^t ||X(u)||^2 Z_X(u) du} ; \int_0^t ||X(u)||^2 Z_X(u) du < \frac{2}{y} \right) \right]$$

(2.6)

Proof. For $N = 1, 2, \ldots$ let $\tau_N$ be the stopping time defined by

$$\tau_N = \inf\{t \leq T : Y_{X,y}(t) \geq N\}$$

It follows from Equation (2.1) that

$$Y_{X,y}(t \land \tau_N) - y = \int_0^{t \land \tau_N} Y_{X,y}(u)X(u) \cdot dW(u) - \frac{1}{2} \int_0^{t \land \tau_N} ||X(u)||^2 Y_{X,y}^2(u) du$$

From this equation we see that $\exp(Y_{X,y}(t \land \tau_N) - y)$ is another stochastic exponential which is generated by

$$Y_{X,y}(u)1_{\{u < \tau_N\}}X(u)$$
Since this process is uniformly bounded in $L^2[0,T]$, it satisfies Novikov’s condition. It is well known (see Karatzas and Shreve [4]) that the associated stochastic exponential is a martingale. We apply Girsanov’s Theorem to change measure using the Radon-Nykodym derivative

$$\Lambda(T) = \exp(Y_{X,y}(T \land \tau_N) - y)$$

The probability measure $Q_N$ is given by

$$\frac{dQ_N}{dP} = \Lambda(T)$$

Then with respect to $Q_N$ the process

$$\tilde{W}(t) = W(t) - \int_0^{t \land \tau_N} Y_{X,y}(u)X(u)du$$

is a Brownian motion for $t \leq T$. Since $Y_{X,y}(t)$ is a strong solution to equation (2.1), we may consider its SDE with respect to the Brownian motion $\tilde{W}$:

For $t < \tau_N$

$$dY_{X,y} = Y_{X,y}X \cdot d\tilde{W} - \frac{1}{2}||X||^2Y_{X,y}^2dt$$

$$= Y_{X,y}X \cdot \{ d\tilde{W} + Y_{X,y}Xdt \} - \frac{1}{2}||X||^2Y_{X,y}^2dt$$

$$= Y_{X,y}X \cdot d\tilde{W} + \frac{1}{2}||X||^2Y_{X,y}^2dt$$

(2.7)

Now we have an explicit solution to the SDE in equation (2.7):

$$Y_{X,y}(t) = \frac{\tilde{Z}_X(t)}{y^{-1} - \frac{1}{2} \int_0^t ||X(u)||^2\tilde{Z}_X(u)du}$$

(2.8)

In this equation, $\tilde{Z}_X(t)$ denotes the stochastic exponential (generated by $X$) with respect to the Brownian motion $\tilde{W}$. Now we consider

$$E[G(Y_{X,y}(t)) \exp(Y_{X,y}(t) - y)1_{\{t < \tau_N\}}]$$

$$= E[G(Y_{X,y}(t))\Lambda(T)1_{\{t < \tau_N\}}]$$

$$= E_{Q_N}[G(Y_{X,y}(t))1_{\{t < \tau_N\}}]$$

$$= E_{Q_N}[G(\frac{\tilde{Z}_X(t)}{y^{-1} - \frac{1}{2} \int_0^t ||X(u)||^2\tilde{Z}_X(u)du})1_{\{t < \tau_N\}}]$$

(2.9)

Here we used the identity for $Y_{X,y}$ in Equation (2.8).

Moreover, from Equation (2.8) we also have

$$t < \tau_N \quad \text{iff.} \quad \max_{s \leq t} \frac{\tilde{Z}_X(s)}{y^{-1} - \frac{1}{2} \int_0^s ||X(u)||^2\tilde{Z}_X(u)du} < N$$

This allows us to write the last expected value in Equation (2.9) as

$$E[G(\frac{Z_X(t)}{y^{-1} - \frac{1}{2} \int_0^t ||X(u)||^2Z_X(u)du}) \cdot \max_{s \leq t} \frac{Z_X(s)}{y^{-1} - \frac{1}{2} \int_0^s ||X(u)||^2Z_X(u)du} < N]$$

since the integrand involves only the distribution of a Brownian motion for each choice of $N$. The limit of this expected value as $N \to \infty$ is
Since the limit of the first expected value in Equation (2.9) is
\[ E[G(Y_{X,Y}(t)) \exp(Y_{X,Y}(t) - y)] \]
the theorem is proved. \( \square \)

3. Examples Using the Transform

**Proposition 3.1.** Suppose that \( X(t) \) is a deterministic function such that
\[ \int_0^t ||X(u)||^2 du \]
is strictly increasing and finite for \( t \leq T < \infty \). Let \( Z_{X}(t) \) and \( Y_{X,Y} \) denote the stochastic exponential and super-exponent process generated by \( X(t) \). Then the process
\[ \exp(Y_{X,Y}(t)) \]
is a strict supermartingale for \( t \leq T \). Moreover,
\[ E[\exp(Y_{X,Y}(t))] = e^y \Pr \left\{ \int_0^t ||X(u)||^2 Z_{X}(u) du < \frac{2}{y} \right\} \quad (3.1) \]

**Proof.** We apply Theorem 2.2 using the choice \( G(u) \equiv 1 \). Equation (2.6) becomes
\[ E[\exp(Y_{X,Y}(t)) - y] = \Pr \left\{ \int_0^t ||X(u)||^2 Z_{X}(u) du < \frac{2}{y} \right\} , \]
and Equation (3.1) follows. Now since each \( Z_{X}(u) \) is a log normal random variable, the process
\[ \int_0^t ||X(u)||^2 Z_{X}(u) du \quad (3.2) \]
has strictly increasing sample paths. It follows that the right hand expression in Equation (3.1) is strictly decreasing. Therefore, \( \exp(Y_{X,Y}(t)) \) is a strict supermartingale. \( \square \)

**Remark 3.2.** Equation (3.1) provides a useful tool for investigating the distribution of a time integral given by Equation (3.2). Since each super-exponent
\[ Y_{X,Y}(t) = \frac{Z_{X}(t)}{y^{-1} + \frac{1}{2} \int_0^t ||X(u)||^2 Z_{X}(u) du} \]
is point-wise increasing as a function of $y$, it follows from the identity

$$\Pr \left\{ \int_0^t ||X(u)||^2 Z_X(u) \, du < a \right\} = \exp\left(-\frac{2}{a}\right) E[\exp(Y_{X,2/a}(t))]$$

that the distribution function is the product of a decreasing function of $a$ and the explicit factor $\exp(-2/a)$. It is not known whether $\exp(Y_{X,\infty}(t))$ has finite expectation. A finite expected value would produce sharp estimates for the lower tail probability of (3.2). We conjecture that

$$E[ \frac{2Z_X(t)}{\int_0^t ||X(u)||^2 Z_X(u) \, du} ] = \infty.$$  

**Example 3.3.** In the case of $d = 1$ the choice $X(t) \equiv \sigma$ specializes the time integral in (3.2) to a time integral of *geometric Brownian motion*:

$$\int_0^t \exp(\sigma W(u) - \sigma^2 u/2) \, du \tag{3.3}$$

Expected values involving related time integrals appear in computational problems of financial mathematics. Consequently, distribution properties of these time integrals have been studied by many authors; see Dufresne [1], Geman and Yor [2], Rogers and Shi [8], and Goodman and Kim [3]. Although most works have used analytic techniques to express the distribution in various integral forms, in Goodman and Kim [3] martingales techniques are used exclusively. A special case of Equation (3.1) appears in [3], Theorem 4.1:

$$\Pr \left\{ \int_0^t \exp(W(u) - u/2) \, du \leq a \right\}$$

$$= \exp\left(-\frac{2}{a}\right) E[\exp(\frac{2 \exp(W(t) - t/2)}{a + \int_0^t \exp(W(u) - u/2) \, du})]$$

The right hand expression for the distribution can be differentiated with respect to $a$. Consequently, it is shown in [3] that the density function multiplied by $a^2/2$ equals the difference between two distribution functions of time integrals of slightly different geometric Brownian motions.

**Example 3.4.**

In contrast to deterministic choices for $X(t)$, where the stochastic exponential

$$\exp(Y_{X,y}(t))$$

is never a martingale, stochastic choices for $X$ may produce martingales. Of course, the introduction of a stopping time, as we have seen in the proof of Theorem 2.2, may produce a martingale. In other cases, stopping times are not required.

Consider the example of $X(t) = \cos(W(t))$, again in the case $d = 1$. Then

$$Z_X(t) = \exp(\int_0^t \cos(W(u)) \, dW(u) - \frac{1}{2} \int_0^t \cos^2(W(u)) \, du)$$
SUPER-EXPONENTS

\[ = \exp(\sin(W(t)) + \frac{1}{2} \int_0^t \sin(W(u) - \cos^2(W(u)))du) \]

is a bounded random variable. Therefore, its super-exponent, \( Y_{\cos(W),y}(t) \) is also bounded. Then since the local martingale
\[ \exp(Y_{\cos(W),y}(t)) \]
is also bounded, it is a martingale. It is of interest then to know when a super-exponent generates a martingale process.

4. THE MARTINGALE CONDITION

Theorem 1 of Wong and Heyde [9] identifies a necessary and sufficient condition for a progressively measurable process \( \tilde{X} \) to generate a martingale stochastic exponential process. For completeness, we state their result here.

**Proposition 4.1.** ([9], Proposition 1) Consider a d-dimensional progressively measurable process \( \tilde{X}(t) = \xi(W(\cdot), t) \). Then there will also exist a d-dimensional progressively measurable process
\[ \tilde{R}(t) = \xi(W(\cdot) + \int_0^t \tilde{R}(u)du, t) \]
defined possibly up to an explosion time \( \tau^{MR} \) where
\[ \tau^{MR} = \inf \left\{ t \in \mathbb{R}^+ : M_{\tilde{R}}(t) = \int_0^t ||\tilde{R}(u)||^2 du = \infty \right\} \]

**Theorem 4.2.** ([9], Theorem 1) Consider \( \tilde{X}(t) \) and \( \tilde{R}(t) \) as defined in Proposition 4.1. The stochastic exponential \( Z_{\tilde{X}}(T) \) satisfies
\[ P(\tau^{MR} > T) = E_P[Z_{\tilde{X}}(T)] \]
and hence is a martingale if and only if \( P(\tau^{MR} > T) = 1 \).

We apply Theorem 1 of [9] using \( \tilde{X}(t) = Y_{X,Y}(t)X(t) \). That is, our generating process is the one in Proposition 2.1 where the stochastic exponential process is
\[ \exp(Y_{X,Y}(t) - y) \].

We first show that each generating process \( X \) implicitly defines another process \( X' \). This allows us to identify the process \( R(t) \).
Proposition 4.3. Suppose that a d-dimensional progressively measurable process \( X(t) \) satisfies
\[
Pr \left( \int_0^T ||X(u)||^2 du < \infty \right) = 1
\]
for some \( T > 0 \). Then there exists another progressively measurable process \( X'(t) \), so that if \( \tilde{X}(t) := Y_{X,y}(t)X(t)1_{\{t \leq T\}} \) in Proposition 4.1, then the process \( \tilde{R}(t) \) of the proposition satisfies
\[
\tilde{R}(t) = \frac{Z_{X'}(t)}{y^{-1} - \frac{1}{2} \int_0^t ||X'(u)||^2 Z_{X'}(u) du} X'(t)
\]
for all \( t < \tau_{MR} \).

Moreover,
\[
\tau_{MR} = \inf \left\{ t \in \mathbb{R}^+ : \int_0^{t \wedge T} ||X'(u)||^2 Z_{X'}(u) du = 2/y \right\}
\]

Proof. We follow the proof of Proposition 4.1. Let
\[
\tilde{X}(t) := Y_{X,y}(t)X(t)1_{\{t \leq T\}}.
\]
For each \( N = 1, 2, \ldots \) we define a sequence of stopping times by
\[
\tau_N = \inf \left\{ t \in \mathbb{R}^+ : \int_0^t Y_{X,y}^2(u)||X(u)||^2 1_{\{u \leq T\}} du \geq N \right\}
\]

It follows from Equation (2.1) that
\[
Z_{\tilde{X}}(t \wedge \tau_N) = \exp(Y_{X,y}(t \wedge \tau_N) - y)
\]
forms a martingale. As in the proof of Theorem 2.2, we apply Girsanov’s theorem using the Radon-Nikodym derivative
\[
\Lambda(T) = \exp(Y_{X,y}(T \wedge \tau_N) - y)
\]
to obtain the probability measure \( Q_N \) where
\[
dQ_N = \Lambda(T)dP.
\]

With respect to the measure \( Q_N \), the process
\[
\tilde{W}(t) = W(t) - \int_0^{t \wedge \tau_N} Y_{X,y}(u)X(u) du
\]
is a Brownian motion. Hence, on the set \( \{ t \leq \tau_N \wedge T \} \) we have
\[
\tilde{X}(t) = \xi(\tilde{W}(\cdot) + \int_0^{\tau_N} Y_{X,y}(u)X(u) du, t)
\]
That is,
\[
Y_{X,y}(t)X(t) = \xi(\tilde{W}(\cdot) + \int_0^{\tau_N} Y_{X,y}(u)X(u) du, t)
\] (4.1)
Now the process $Y_{X,y}(t)$ can also be described in terms of the Brownian motion $\tilde{W}$. The calculations in Equation (2.7) also apply to the stochastic case. Equation (2.8) gives an explicit formula for $Y_{X,y}$:

$$Y_{X,y}(t) = \frac{\tilde{Z}_X(t)}{y^{-1} - \frac{1}{2} \int_0^t \frac{||X(u)||^2\tilde{Z}_X(u)du}{||X(u)||^2}} \quad (4.2)$$

We see that each term of Equation (4.1) is a functional of $X$ and the Brownian motion $\tilde{W}$. This demonstrates the existence of a process $X$ so that (4.1) and (4.2) hold up to a time $\tau_N$ defined by the integral of $Y_{X,y}(u)X(u)$, using the Brownian motion $\tilde{W}$.

Therefore, using the identical distribution of $W$ and the (original) measure $P$, we see that there exists a progressively measurable process $X'(t)$ so that

$$Z_{X'}(t) = \frac{\tilde{Z}_X(t)}{y^{-1} - \frac{1}{2} \int_0^t ||X'(u)||^2\tilde{Z}_{X'}(u)du} \quad X'(t) = \xi(W(\cdot) + \int_0 \xi_{X,y}(u)X'(u)du, t)$$

Here, we have abbreviated the complete expression on the right hand side using (4.2) to provide the notation. That is, $Y_{X',y}$ denotes the expression in Equation (4.2) but in the original Brownian motion and $X$ is replaced by the process $X'$.

As $N \to \infty$ the stopping time $\tau_N$ increases to the stopping time

$$\tau = \inf\{t \leq T : \int_0^t Y_{X,y}^2(u)||X'(u)||^2du = \infty\}$$

By construction, the new process $X'$ satisfies

$$\int_0^T ||X'(u)||^2du < \infty \quad \text{a. s. and} \quad X'(u) = 0 \quad \text{for} \quad u > T$$

Therefore, the process $Y_{X',y}$ (again, defined as in (4.2)) is bounded along each sample path up to the time where its denominator first hits zero. This defines the stopping time $\tau_{M_R}$ of the Proposition.

\[\Box\]

**Theorem 4.4.** Suppose that $X(t)$ and $X'(t)$ are $d$-dimensional processes as defined in Proposition 4.3. Then the super-exponent process $Y_{X,y}(t)$ satisfies

$$E[\exp(Y_{X,y}(t) - y)] = Pr \left\{ \int_0^t ||X'(u)||^2\tilde{Z}_X(u)du < 2/y \right\} \quad (4.3)$$

for $t \leq T$.

**Proof.** From Theorem 4.2 and Proposition 4.3 we have

$$E[\exp(Y_{X,y}(t) - y)] = Pr \left\{ \tau_{M_R} > t \right\}$$

$$= Pr \left\{ \int_0^t Y_{X,y}^2(u)||X'(u)||^2du < \infty \right\}$$
\begin{align*}
= \Pr \left\{ \int_0^t \frac{Z_X'(u)}{y^{-1} - \frac{1}{2} \int_0^u \|X'(r)\|^2 \frac{y}{2} \|X'(r)\|^2 dr} \|X'(u)\|^2 du < \infty \right\} \\
= \Pr \left\{ \int_0^t \|X'(u)\|^2 Z_X(u) du < \frac{2}{y} \right\}
\end{align*}
\hspace{1cm} \Box

\textbf{References}

1. Dufresne, D.: The integral of geometric Brownian motion; \textit{Adv. in Appl. Probab.} \textbf{33} (2001) 223-241
2. Geman, H, and Yor, M.: Asian Options, Bessel Processes and Perpetuities; \textit{Math. Finance} \textbf{2} (1993) 349-375
3. Goodman, V. and Kim, K.: Exponential martingales and time integrals of Brownian motion \textit{preprint}
4. Karatzas, I., and Shreve, S.: \textit{Brownian Motion and Stochastic Calculus}. Springer-Verlag, New York, 1991
5. Kim, K.: Moment Generating function of the inverse of integral of geometric Brownian Motion; \textit{Proc. Amer. Math. Soc.} \textbf{132} (2004) 2753-2759
6. Levental, S. and Skorohod, A. V.; A necessary and sufficient condition for absence of arbitrage with tame portfolios; \textit{Ann. Appl. Prob.} \textbf{5} (1995) 906-925
7. Revuz, D. and Yor, M.; \textit{Continuous Martingales and Brownian Motion}, \textit{3rd edn}. Springer-Verlag, New York, 1999
8. Rogers, L.C.G., and Shi, Z.: The value of an Asian option \textit{J. Appl Appl. Probab.} \textbf{32} (1995) 1077-1088
9. Wong, B. and Heyde, C.C.; On the martingale property of stochastic exponentials; \textit{J. Appl Probab.} \textbf{41} (2004) 654-664
10. Yor, M.: On some exponential functionals of Brownian motion; \textit{Adv. in Appl. Probab.} \textbf{24} (1992) 509-531

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