Long Time Behavior of Solutions
to Nernst–Planck and Debye–Hückel
Drift–Diffusion Systems

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Abstract

We study the convergence rates of solutions to drift-diffusion systems (arising from plasma, semiconductors and electrolytes theories) to their self-similar or steady states. This analysis involves entropy-type Lyapunov functionals and logarithmic Sobolev inequalities.

Key words and phrases: drift-diffusion systems, asymptotic behavior of solutions, logarithmic Sobolev inequalities

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1 Introduction

We consider the long time asymptotics of solutions to drift-diffusion systems

\[ u_t = \nabla \cdot (\nabla u + u \nabla \phi) , \quad (1.1) \]

\[ v_t = \nabla \cdot (\nabla v - v \nabla \phi) , \quad (1.2) \]

\[ \Delta \phi = v - u , \quad (1.3) \]

where \( u, v \) denote densities of negatively, respectively positively, charged particles. The Poisson equation (1.3) defines the electric potential \( \phi \) coupling the equations (1.1)-(1.2) for the temporal evolution of charge distributions. The system (1.1)-(1.3) was formulated by W. Nernst and M. Planck at the end of the nineteenth century as a basic model for electrodiffusion of ions in electrolytes filling the whole space \( \mathbb{R}^3 \). Note that the case of multicharged particles is also covered by (1.1)-(1.3) since \( u \) and \( v \) denote the charge densities.

Supplemented with the no-flux boundary conditions

\[ \frac{\partial u}{\partial \nu} + u \frac{\partial \phi}{\partial \nu} = 0 , \quad (1.4) \]

\[ \frac{\partial v}{\partial \nu} - v \frac{\partial \phi}{\partial \nu} = 0 \quad (1.5) \]

on the boundary of a bounded domain \( \Omega \subset \mathbb{R}^d, d \leq 3 \), and either

\[ \phi = 0 \quad \text{on } \partial \Omega , \quad (1.6) \]

or

\[ \phi = E_d \ast (v - u) , \quad (1.7) \]

where \( E_d \) is the fundamental solution of the Laplacian in \( \mathbb{R}^d \), the system (1.1)-(1.3) was also studied by P. Debye and E. Hückel in the 1920’s. (1.6) signifies a conducting boundary of the container, while in the case of a bounded domain the “free” boundary condition (1.7) corresponds to a container immersed in a medium with the same dielectric constant as the solute.

These equations, together with their generalizations including e.g. an exterior potential, known as drift-diffusion Poisson systems, also appear in plasma physics and (supplemented with some mixed linear boundary conditions instead of (1.4)-(1.5)) in semiconductor device modelling.
To determine completely the evolution, the initial conditions

\[ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) \tag{1.8} \]

are added. Obviously, positivity of \( u_0 \geq 0, v_0 \geq 0 \) is conserved: \( u(x,t) \geq 0, v(x,t) \geq 0 \), as well as the total charges

\[ M_u = \int u_0(x) \, dx = \int u(x,t) \, dx, \quad M_v = \int v_0(x) \, dx = \int v(x,t) \, dx \tag{1.9} \]

Here \( M_u, M_v \) are not necessarily the same, i.e. the electroneutrality condition

\[ M_u = M_v \tag{1.10} \]

is not, in general, required. Condition (1.10) must be satisfied in the case of the homogeneous Neumann boundary conditions \( \frac{\partial \phi}{\partial n} = 0 \) (i.e. an isolated wall of the container) leading together with (1.4)-(1.5) to

\[ \frac{\partial u}{\partial v} = \frac{\partial v}{\partial u} = \frac{\partial \phi}{\partial n} = 0. \]

Our results (Theorem 1.2 below) are valid in that case, with even a simpler proof.

The asymptotic properties of solutions to (1.1)-(1.3), (1.7) have been studied recently in [1]. The authors proved that (for \( d \geq 3, M_u = M_v = 1 \) and \( u_0 \) and \( v_0 \) regular enough) \( u, v \) tend to their self-similar asymptotic states at an algebraic rate. We improve these results by relaxing assumptions on the initial data and showing a stronger (still algebraic) decay rate, which we expect to be optimal (see Theorem 1.1 below).

In the case of a bounded domain, the convergence (with no specific speed) in the \( L^1 \)-norm of \( u \) and \( v \) solving (1.1)-(1.5) to their corresponding steady states has been proved in [5] (as well as the \( L^\infty \)-convergence for more regular \( u_0, v_0 \)). Here we prove the exponential convergence towards the steady states with a decay rate depending on \( d \geq 2, M_u, M_v \), and the initial value of the entropy functional only (see Theorem 1.2 below).

Notation. The \( L^p \)-norm in \( \mathbb{R}^d \) or \( \Omega \subseteq \mathbb{R}^d \) is denoted by \( | \cdot |_p \), and inessential constants (which may vary from line to line) are denoted generically by \( C \).
Define the asymptotic states in $\mathbb{R}^d$ by

$$u_{as}(x,t) = \frac{M_u}{(2\pi(2t+1))^{d/2}} \exp\left(-\frac{|x|^2}{2(2t+1)}\right),$$  \hspace{1cm} (1.11)

$$v_{as}(x,t) = \frac{M_v}{(2\pi(2t+1))^{d/2}} \exp\left(-\frac{|x|^2}{2(2t+1)}\right),$$  \hspace{1cm} (1.12)

where the charges of the solution $\langle u, v \rangle$ of (1.1)-(1.2) are given by (1.9), and the entropy functional by

$$L(t) = \int u(x,t) \log\left(\frac{u(x,t)}{u_{as}(x,t)}\right) dx + \int v(x,t) \log\left(\frac{v(x,t)}{v_{as}(x,t)}\right) dx + \frac{1}{2} |\nabla \phi(t)|^2.$$  \hspace{1cm} (1.13)

**Theorem 1.1** There exists a constant $C = C(d, M_u, M_v, L_0)$ such that for each solution $\langle u, v \rangle$ of (1.1)-(1.3), (1.7)-(1.8) in $\mathbb{R}^d$, $d \geq 3$, if $L(0) = L_0$, then for all $t \geq 0$,

$$L(t) \leq CH(t)$$  \hspace{1cm} (1.14)

and

$$|u(t) - u_{as}(t)|_1^2 + |v(t) - v_{as}(t)|_1^2 + |\nabla \phi(t)|_2^2 \leq C H(t),$$  \hspace{1cm} (1.15)

where

$$H(t) = \begin{cases} (2t+1)^{-1/2}, & d = 3, \\ (2t+1)^{-1} \left( \log(2t+1) + 1 \right), & d = 4, \\ (2t+1)^{-1}, & d > 4. \end{cases}$$

Moreover if $M_u = M_v$, then $H(t) = (2t+1)^{-1}$ for any $d \geq 3$.

In the case of a bounded domain, define the entropy functional

$$W(t) = \int u(x,t) \log\left(\frac{u(x,t)}{U(x)}\right) dx + \int v(x,t) \log\left(\frac{v(x,t)}{V(x)}\right) dx$$

$$+ \frac{1}{2} \int (\phi - \Phi)(u - U - v + V) dx,$$  \hspace{1cm} (1.16)

for the solution $\langle u, v, \phi \rangle$ of (1.1)-(1.5), (1.6) or (1.7), (1.8) and the unique steady state $\langle U, V, \Phi \rangle$ of the Debye-Hückel system with

$$M_u = \int U(x) dx, \hspace{0.5cm} M_v = \int V(x) dx.$$  \hspace{1cm} (1.17)
Note that for the condition (1.6) the third term in \( W(t) \) takes the form \( \frac{1}{2} |\nabla (\phi - \Phi)|_2^2 \).

**Theorem 1.2** If \( d = 2 \), then there exist two constants \( \lambda = \lambda(d, M_u, M_v, W_0) \) and \( C = C(M_u, M_v, W_0) \) such that for each solution \( \langle u, v, \phi \rangle \) of (1.1)-(1.6), (1.8) in a bounded domain \( \Omega \), if \( W(0) = W_0 \), then for all \( t \geq 0 \),

\[
W(t) \leq W(0) e^{-\lambda t},
\]

and

\[
|u(t) - U|_1^2 + |v(t) - V|_1^2 + |\nabla (\phi - \Phi)|_2^2 \leq C e^{-\lambda t}.
\]

(1.19)

If \( d \geq 3 \), the same results hold except that \( \lambda \) and \( C \) may also depend on \( \sigma = \sup_{t \geq 0} |\phi(t)|_\infty \), which is supposed to be finite.

Sufficient conditions for \( \sigma \) to be finite in the case of linear boundary conditions and explicit estimates of \( \lambda \) and \( C \) will be provided in the proof of this result.

## 2 Proof of Theorem 1.1

We begin with a rescaling of the system (1.1)-(1.3) which will lead to a system with a quadratic confinement potential, and therefore (eliminating the dispersion) to the expected exponential convergence to the steady states. This idea was applied in [8] and [7], as well as in [1], to a variety of problems ranging from kinetic equations to porous media equations.

Let \( \bar{x} \in \mathbb{R}^d \), \( \tau > 0 \), be the new variables defined by

\[
\bar{x} = \frac{x}{R(t)}, \quad \tau = \log R(t), \quad R(t) = (2t + 1)^{1/2},
\]

(2.1)

and consider the rescaled functions \( \bar{u}, \bar{v}, \bar{\phi} \) such that

\[
u(x,t) = \frac{1}{R^d(t)} \bar{v}(\bar{x},\tau),
\]

\[
\phi(x,t) = \bar{\phi}(\bar{x},\tau).
\]

(2.2)
This whole section will deal with the rescaled system, so omitting the bars over $x$, $u$, $v$, $\phi$ will not lead to confusions with the original system, which now takes, after rescaling, the form

$$u_\tau = \nabla \cdot (\nabla u + u x + u \nabla \phi), \quad (2.3)$$

$$v_\tau = \nabla \cdot (\nabla v + v x - v \nabla \phi), \quad (2.4)$$

$$\Delta \phi = e^{-\tau(d-2)}(v - u). \quad (2.5)$$

The scaling (2.2) preserves the $L^1$-norms, so the rescaled initial data $u_0$, $v_0$ still satisfy

$$M_u = \int u_0(x) \, dx = \int u(x, \tau) \, dx, \quad M_v = \int v_0(x) \, dx = \int v(x, \tau) \, dx. \quad (2.6)$$

Denote by $\langle u_\infty, v_\infty \rangle$ the steady state of (2.3)-(2.4), that is

$$u_\infty(x) = \frac{M_u}{(2\pi)^{d/2}} \exp\left(-\frac{|x|^2}{2}\right), \quad (2.7)$$

$$v_\infty(x) = \frac{M_v}{(2\pi)^{d/2}} \exp\left(-\frac{|x|^2}{2}\right). \quad (2.8)$$

Of course, going back to the original variables $x$, $t$, $\langle u_\infty, v_\infty \rangle$ corresponds to the asymptotic state $\langle u_{as}, v_{as} \rangle$ defined by (1.11)-(1.12). Writing $\phi = \beta \psi$ with $\beta = \beta(\tau) = e^{-\tau(d-2)} \to 0$ as $\tau \to +\infty$, we introduce the relative entropy

$$W(\tau) = \int u \log \left( \frac{u}{u_\infty} \right) \, dx + \int v \log \left( \frac{v}{v_\infty} \right) \, dx + \frac{\beta}{2} |\nabla \psi|^2 \quad (2.9)$$

corresponding to the original entropy functional $L$ in (1.13). The evolution of $W$ is given by

$$\frac{dW}{d\tau} = -\int u \left| \nabla \left( \log \frac{u}{U} \right) \right|^2 \, dx - \int v \left| \nabla \left( \log \frac{v}{V} \right) \right|^2 \, dx - \left( \frac{d}{2} - 1 \right) \beta |\nabla \psi|^2, \quad (2.10)$$

with $U$, $V$ denoting the local Maxwellians

$$U(x, \tau) = M_u \frac{\exp\left(-\frac{1}{2} |x|^2 - \phi(x, \tau)\right)}{\int \exp\left(-\frac{1}{2} |y|^2 - \phi(y, \tau)\right) \, dy}, \quad (2.11)$$
\[ V(x, \tau) = M_u \frac{\exp \left(-\frac{1}{2} |x|^2 + \phi(x, \tau) \right)}{\int \exp \left(-\frac{1}{2} |y|^2 + \phi(y, \tau) \right) dy}, \]  
(2.12)

so that \( \nabla U/U = -(x + \nabla \phi), \) \( \nabla V/V = -(x - \nabla \phi). \) Using the notation

\[ J = \frac{1}{2} \int u \left| \frac{\nabla u}{u} + x \right|^2 dx + \frac{1}{2} \int v \left| \frac{\nabla v}{v} + x \right|^2 dx, \]  
(2.13)

(2.10) can be rewritten as

\[ \frac{dW}{d\tau} = -2J - 2 \int (\nabla u - \nabla v) \cdot \nabla \phi \, dx - 2 \int (u - v) x \cdot \nabla \phi \, dx \]

\[ - \int (u + v)|\nabla \phi|^2 \, dx - \left( \frac{d}{2} - 1 \right) \beta |\nabla \psi|_2^2 \]  
(2.14)

The quantity \( J \) in (2.13) can be estimated from below using the Gross logarithmic Sobolev inequality

\[ \int f \log \left( \frac{f}{|f|_1} \right) \, dx + d \left( 1 + \frac{1}{2} \log(2\pi a) \right) |f|_1 \leq \frac{a}{2} \int \frac{|
abla f|^2}{f} \, dx \]  
(2.15)

valid for each \( a > 0, \) see e.g. [11] or a thorough discussion of different versions of logarithmic Sobolev inequalities in [2]. (2.15) becomes an equality if and only if \( f(x) = C \exp(-|x|^2/(4a)) \) (up to a translation).

Taking \( a = 1 \) in (2.15), the relation (2.14) leads to

\[ - \left( \frac{dW}{d\tau} + 2W \right) \geq 2\beta |u - v|_2^2 - \beta \frac{d}{2} |\nabla \psi|_2^2 \geq -C\beta (M_u + M_v)^2 \]  
(2.16)

with a constant \( C = C(d) = \frac{2}{d} \left( \frac{d-2}{4} \right)^{(d-2)/2} \Sigma^d/2, \) because by the Hardy-Littlewood-Sobolev inequality and an interpolation

\[ |\nabla \psi|_2^2 \leq \Sigma|u - v|^2_{2d/(d+2)} \leq \Sigma|u - v|_1^d |u - v|_{2d-4/d}^2 \leq \frac{4}{d} |u - v|_{2d-4/d}^2 + C|u - v|_1^2. \]

Clearly, (2.16) implies

\[ \frac{d}{d\tau} \left( e^{2\tau} W(\tau) \right) \leq C(M_u + M_v)^2 e^{\tau(4-d)} \]
and, after one integration, we obtain

\[ W(\tau) \leq \left( W(0) e^{-\tau} + C(M_u + M_v)^2 \right) e^{-\tau} \]  

(2.17)
in the case \( d = 3 \),

\[ W(\tau) \leq \left( W(0) + C(M_u + M_v)^2 \tau \right) e^{-2\tau} \]  

(2.18)
if \( d = 4 \), and finally for all \( d > 4 \)

\[ W(\tau) \leq \left( W(0) + C(M_u + M_v)^2 \right) e^{-2\tau}. \]  

(2.19)

Since from the Csiszár-Kullback inequality (cf. (1.9) in [2], App. D in [7], [6] or [10]) \( W(\tau) \) controls the \( L^1 \)-norm of \( u - u_\infty \) and \( v - v_\infty \), we get the same decay rates as in (2.17)-(2.19) for

\[ |u(\tau) - u_\infty|^2_1 + |v(\tau) - v_\infty|^2_1 + \beta|\nabla \psi(\tau)|^2_2 \leq 2 \left( \max(M_u, M_v) + 1 \right) W(\tau). \]  

(2.20)

Returning to the original variables \( x, t \), this implies, of course, the estimates (1.14)-(1.15) of Theorem 1.1 in the general case.

In the electroneutrality case (1.10): \( M_u = M_v \), since \( u_\infty = v_\infty \), so for \( d = 3 \),

\[ |u - v|^2_1 = |u - u_\infty + v_\infty - v|^2_1 \leq C e^{-\tau}. \]

Next, a modification of (2.16) reads

\[ \frac{d}{d\tau} \left( e^{2\tau} W(\tau) \right) \leq C e^{2\tau} \beta |u - v|^2_1 \leq C, \]

and this leads to \( W(\tau) \leq C(1 + \tau) e^{-2\tau} \). Inserting this into (2.20) and (2.16) once again implies

\[ \frac{d}{d\tau} \left( e^{2\tau} W(\tau) \right) \leq C(1 + \tau) e^{-\tau}, \]

so that \( W(\tau) \leq C e^{-2\tau} \). If \( d = 4 \), the same reasoning once again applies providing also the same improved decay rate. \( \square \)

**Remark 2.1** Note that the constant \( C \) in (1.15) depends on \( d, M_u, M_v \) and \( L(0) \) only, and is independent of e.g. \( |u_0|_r, |v_0|_r \) with some \( r > d/2 \) — as it was in fact in [1]. Conditions like \( |u_0|_r, |v_0|_r < \infty \) are sufficient for (local in
time) existence of solutions to the considered systems (cf. Theorem 2 in [5]), but they can be relaxed — as it was done for a related parabolic-elliptic system describing the gravitational interaction of particles in [4]. Thus, compared to [1], Theorem 1.1 gives not only an improvement of the exponents but also gets rid of the unnecessary dependence on quantities other than $L(0)$, $M_u$, $M_v$. We do not know if the exponents in Theorem 1.1 are optimal, but such a conjecture is supported by the calculations in the proof of the following

**Proposition 2.2** There exists a constant $\lambda > 0$ depending only on $d$ with

$$\lambda \geq \lambda(d) = (d-2)\left(\sqrt{(d-1)^2 + 3} - (d-1)\right),$$

such that

$$W(\tau) \leq W(0)e^{-\lambda\tau}$$

(2.21)

and hence

$$L(t) \leq L(0)(2t+1)^{-\lambda/2}$$

for each solution $(u,v)$ to the Nernst-Planck system.

**Remark 2.3** The interest of this proposition is that the constants controlling the convergence of $W(t)$, $L(t)$, and hence $|u-u_x|_1$, $|v-v_x|_1$ in (1.15), depend on the initial values of $W(0)$, $L(0)$ only (and not on $|u|_1 = M_u$, $|v|_1 = M_v$, which are quantities not comparable with, say, $\int u\log u\,dx$, $\int v\log v\,dx$ in the whole $\mathbb{R}^d$ space case). However, the exponent $\lambda$ — which is evaluated explicitly — is not as good as the one in Theorem 1.1.

**Proof of Proposition 2.2.** Using (2.9), (2.13), (2.14), we may write for any positive $\lambda$

$$-\left(\frac{dW}{d\tau} + \lambda W\right) = \lambda\left(J - \int u\log\left(\frac{u}{u_\infty}\right) - \int v\log\left(\frac{v}{v_\infty}\right)\right) + (2-\lambda)J + B + 2E - \mu F,$$

(2.22)

where

$$B = \beta^2\int (u+v)|\nabla \psi|^2\,dx,$$

$$E = \beta |u-v|^2_2,$$

$$F = \left(\frac{d}{2} - 1\right)\beta |\nabla \psi|^2,$$

$$\mu = 1 + \frac{\lambda}{d-2}.$$
Observe that if we define
\[ G_1 = \int u \left( \frac{\nabla u}{u} + x \right) \cdot \nabla \phi \, dx, \quad G_2 = \int v \left( \frac{\nabla v}{v} + x \right) \cdot \nabla \phi \, dx, \]
then
\[ G_1 - G_2 = \int \nabla (u-v) \cdot \nabla \phi \, dx + \int (u-v) (x \cdot \nabla \phi) \, dx = E - F. \]

Define now
\[ f_1 = \sqrt{2 - \lambda} \cdot \sqrt{u} \left( \frac{\nabla u}{u} + x \right), \quad g_1 = \sqrt{u} \nabla \phi, \]
\[ f_2 = \sqrt{2 - \lambda} \cdot \sqrt{v} \left( \frac{\nabla v}{v} + x \right), \quad g_2 = \sqrt{v} \nabla \phi, \]
\[ a_1 = |f_1|_2, \quad b_1 = |g_1|_2, \quad a_2 = |f_2|_2, \quad b_2 = |g_2|_2. \]

By the Cauchy-Schwarz inequality we have
\[ (2 - \lambda)^{1/2} |E - F| = (2 - \lambda)^{1/2} |G_1 - G_2| = \left| \int (f_1 g_1 - f_2 g_2) \, dx \right| \leq a_1 b_1 + a_2 b_2. \]

But
\[ 0 \leq (a_1 b_2 - a_2 b_1)^2 = (a_1^2 + a_2^2)(b_1^2 + b_2^2) - (a_1 b_1 + a_2 b_2)^2, \]
\[ a_1 b_1 + a_2 b_2 \leq \sqrt{2} \sqrt{(a_1^2 + a_2^2)/2} \sqrt{b_1^2 + b_2^2} \]
\[ \leq \frac{1}{\sqrt{2}} \left( \frac{1}{2} (a_1^2 + a_2^2) + (b_1^2 + b_2^2) \right) \]
\[ = \frac{1}{\sqrt{2}} \left( (2 - \lambda)J + B \right), \]
and thus
\[ (2 - \lambda)^{1/2} |E - F| \leq \frac{1}{\sqrt{2}} \left( (2 - \lambda)J + B \right). \]

Using (2.22) we get
\[ - \left( \frac{dW}{d\tau} + \lambda W \right) \geq \sqrt{2(2 - \lambda)} |E - F| + 2E - \mu F \]
\[ = F \cdot \left( \sqrt{2(2 - \lambda)} |X - 1| + 2X - \mu \right) \quad (2.23) \]
with \( X = E/F \geq 0 \). For either \( d \geq 4 \) and \( \lambda \leq 2 \), or \( d = 3 \) and \( \lambda \leq 1 \), we have \( \mu \leq 2 \). The right hand side of (2.23) (positive for \( X \geq \mu/2 \)) equals (for \( X \leq \mu/2 \leq 1 \))

\[
\sqrt{2(2-\lambda)}(1-X) + 2X - \mu = \left( 2 - \sqrt{2(2-\lambda)} \right) X + \sqrt{2(2-\lambda)} - \mu,
\]

so that

\[
\sqrt{2(2-\lambda)} \geq \mu \tag{2.24}
\]
guarantees \( \frac{dW}{dx} + \lambda W \leq 0 \), which implies (2.21). The condition (2.24) is equivalent to \( \lambda \leq \lambda(d) \). In particular, \( \lambda(d) \) is an increasing function of \( d \), \( \lambda(3) = \sqrt{7} - 2 < 1 \), \( \lambda(4) = 4\sqrt{3} - 6 \) and \( \lim_{d \to +\infty} \lambda(d) = \frac{3}{2} \).

**Remark 2.4** In the case of one species of particles, i.e. \( v \equiv 0 \) as was in [3] and [4], the result of Proposition 2.2 still holds.

Finally, we remark that there is, in general, no hope to have \( \lambda > 2 \) in non-trivial cases. This can be inferred from the formula (2.22), where for each \( \chi > 1 \), \( J - \chi \left( \int u \log \left( \frac{u}{u_e} \right) \, dx + \int v \log \left( \frac{v}{v_e} \right) \, dx \right) \) could be negative and dominate the other terms (for instance, in the limit \( M_u, M_v \to 0^+ \)).

## 3 Proof of Theorem 1.2

First, we recall that steady states \( (U, V, \Phi) \) of (1.1)-(1.3) satisfy the relations

\[
\nabla \cdot (e^{-\Phi} \nabla (e^\Phi U)) = 0, \quad \nabla \cdot (e^\Phi \nabla (e^{-\Phi} V)) = 0,
\]

hence

\[
U = M_u \frac{e^{-\Phi}}{\int e^{-\Phi} \, dx}, \quad V = M_v \frac{e^\Phi}{\int e^\Phi \, dx}. \tag{3.1}
\]
Together with (1.3) this leads to the Poisson-Boltzmann equation

\[
\Delta \Phi = M_u \frac{e^\Phi}{\int e^\Phi \, dx} - M_u \frac{e^{-\Phi}}{\int e^{-\Phi} \, dx}. \tag{3.2}
\]

This equation, supplemented with the Dirichlet boundary condition (1.6) or the free condition (1.7), for every \( M_u, M_v \geq 0 \), has a unique (weak) solution
Φ, see [9] or Proposition 2 in [5] (and this solution is classical whenever ∂Ω is of class \( C^{1+\epsilon} \) for some \( \epsilon > 0 \)).

The evolution of the Lyapunov functional defined by (1.16) in the case of the Dirichlet boundary condition (1.6) or in the case (1.7) is given by

\[
\frac{dW}{dt} = -\int u|\nabla(\log u + \phi)|^2 \, dx - \int v|\nabla(\log v - \phi)|^2 \, dx,
\]

(3.3)

cf. (35) in [5], where the above relation is obtained for weak solutions to the Debye-Hückel system.

Concerning the global in time existence of solutions to the Debye-Hückel system with nonlinear boundary conditions (1.4)-(1.5), we note that this was done for \( d = 2 \) only in Theorem 3 of [5]. Thus, in higher dimensions \( d \geq 3 \), we assume that \( \langle u(t), v(t) \rangle \) exists for all \( t \geq 0 \) and

\[
\sigma = \sup_{t \geq 0} |\phi(t)|_{\infty} < \infty
\]

(3.4)

holds. If equations (1.1)-(1.3) are supplemented with linear type boundary conditions (as it is the case in semiconductor modelling), the assumption \( u_0, v_0 \in L^r(\Omega) \) with an exponent \( r > d/2 \) (cf. Theorem 2 (ii) in [5] and [1] for the case of the whole space \( \mathbb{R}^d \)) guarantees the existence of \( \langle u(t), v(t) \rangle \) for all \( t \geq 0 \).

For \( d = 2 \) the uniform bound (3.4) for the potential \( \phi \) follows from \( W(0) < \infty \). Indeed, for either of conditions (1.6) or (1.7)

\[
\phi(x,t) = \int_{\Omega} K(x,y) (v(y,t) - u(y,t)) \, dy,
\]

where the kernel \( K \) is either the Green function of the domain \( \Omega \) or the fundamental solution \( E_2 \) of the Laplacian in the plane. Since \( |G(x,y) - E_2(x,y)| \leq C = C(\Omega) \), we have, using the inequality

\[
AB \leq A \log A + e^{B-1}
\]

for all \( A \geq 0, B \in \mathbb{R} \), with \( A = u, v \) and \( B = \frac{1}{2\pi} |\log |x - y|| \),

\[
|\phi(x,t)| \leq \int_{\Omega} (u + v) \left( \frac{1}{2\pi} |\log |x - y|| + C \right) \, dy
\]

\[
\leq C \left( \int_{\Omega} u \log u \, dy + \int_{\Omega} v \log v \, dy + 1 \right),
\]

12
independently of $x$. Now, (3.3) gives

$$|\phi(t)|_\infty \leq C\left(W(0) + 1\right) = C\left(W(0), M_u, M_v, \Omega\right) \equiv \sigma < \infty$$

for all $t \geq 0$.

The calculations below are reminiscent of the proof of Theorem 2.1 in [1], but we do not use the Holley-Stroock perturbation lemma for logarithmic Sobolev inequalities as was in [1]. First, apply the inequality (2.15) with $a = 2e^{-\sigma}/\lambda$ and a small $\epsilon > 0$ (to be determined later) to the functions $f = ue^\phi$ and $f = ve^{-\phi}$. Taking into account (3.3), (1.16) and (3.2) we arrive at

$$-\left(\frac{dW}{dt} + \lambda W\right) \geq \lambda \left(\int ue^\phi \log u \, dx + \int ve^{-\phi} \log v \, dx\right) + \lambda \int \left(ue^\phi - ve^{-\phi}\right)\phi \, dx$$

$$-\lambda |ue^\phi|_1 \log |ue^\phi|_1 - \lambda |ve^{-\phi}|_1 \log |ve^{-\phi}|_1$$

$$+ \lambda d \left(1 + \frac{1}{2} \log \left(\frac{4\pi}{\lambda e^\sigma}\right)\right) \left(|ue^\phi|_1 + |ve^{-\phi}|_1\right)$$

$$-\lambda \int (u \log u + v \log v) \, dx + \lambda (M_u \log M_u + M_v \log M_v)$$

$$-\lambda \int (u - v) \Phi \, dx$$

$$-\lambda M_u \log \left(\int e^{-\Phi} \, dx\right) - \lambda M_v \log \left(\int e^{\Phi} \, dx\right)$$

$$-\frac{\lambda}{2} \int (\Phi - \Phi)(u - U - v + V) \, dx.$$
By the Csiszár-Kullback inequality (as was in Section 2), $W(t)$ controls the $L^1$-convergence to the unique steady state, so the conclusion (1.19) of Theorem 1.2 follows from (3.5). This improves (34) in Theorem 6 of [5] in two ways. First, there is an exponential decay rate. Second, (34) is proved under the assumption (3.4), which is weaker than that in Theorem 6 of [5]. Evidently, this result is also valid for one species case ($M_u$ or $M_v$ equal to 0), so Theorem 2 in [3] is also improved.

\[ \square \]

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