Study of Spiral Pattern Formation in Rayleigh-Bénard Convection

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We present a numerical study of a generalized two-dimensional Swift-Hohenberg model of spiral pattern formation in Rayleigh-Bénard convection in a non-Boussinesq fluid. We demonstrate for the first time that a model for convective motion is able to predict in considerable dynamical detail the spontaneous formation of a rotating spiral state from an ordered hexagon state. Our results are in good agreement with recent experimental studies of $CO_2$ gas. The mean flow and non-Boussinesq effects are shown to be crucial in forming rotating spirals.
One of the most striking examples of spatio-temporal self-organized phenomena in non-equilibrium systems is the rotating spiral seen in chemical and biological systems \[1\] \[2\]. The Belousov-Zhabotinsky (BZ) reaction \[3\] has received considerable attention as an example of chemical wave propagation. The formation of spiral patterns in the BZ system resulting from the coupling of a reaction process with a transport process such as diffusion has been extensively studied, both theoretically and experimentally, during the past decade. Remarkably, similar rotating spirals were observed recently in Rayleigh-Bénard (RB) convection in non-Oberbeck-Boussinesq fluids \[4\]. Much of the earlier experimental work has been restricted to Oberbeck-Boussinesq type fluids, in which one observes various configurations of roll patterns. However, in a non-Oberbeck-Boussinesq system with, for example, temperature dependent transport coefficients, both roll and hexagonal patterns can exist. Very recently Bodenschatz et al \[4\] have performed experiments on convection in CO\(_2\) gas and studied the existence of and transitions between convective patterns exhibiting different symmetries. They have observed the competition between a uniform conducting state, a convective state with hexagonal symmetry, and a convective state consisting of rolls. Their most surprising discovery is that the hexagon-roll transition has a tendency to form rotating spirals. In this paper we present the first numerical evidence for the spontaneous formation of a rotating spiral pattern during the hexagon-roll transition in a large aspect ratio cylindrical cell near onset. We also report the numerical results of spiral formation during the conduction-roll transition. We will qualitatively compare our numerical results with the experiments of Bodenschatz et al.

We now describe our two-dimensional (2D) study of spiral patterns formed in a thin layer of a non-Oberbeck-Boussinesq fluid. We model such a fluid by a two dimensional generalized Swift-Hohenberg equation \[5\] \[6\], given by equations (1) and (2) below, which we solve by numerical integration. The Swift-Hohenberg equation and various generalizations of it have proven to be quite successful in explaining many features of convective flow \[7\], particularly near onset. As we show in this paper, the
same holds true for the non-Oberbeck-Boussinesq fluid. Our model is defined by

\[
\frac{\partial \psi(\vec{r}, t)}{\partial t} + g_m \vec{U} \cdot \nabla \psi = \left[ \epsilon - \left( \nabla^2 + 1 \right)^2 \right] \psi - g_2 \psi^2 - \psi^3 + f(\vec{r}),
\]

where \( \vec{U} = (\partial_y \xi) \hat{e}_x - (\partial_x \xi) \hat{e}_y \)

(3)

with boundary conditions,

\[ \psi|_B = \hat{n} \cdot \nabla \psi|_B = 0, \]

(4)

where \( \hat{n} \) is the unit normal to the boundary of the domain of integration, \( B \). This equation with \( g_2 = g_m = 0 \) reduces to the Swift-Hohenberg (SH) equation and has been extensively used to model convection in thin cells and near onset \[7, 8, 9, 10\]. The scalar order parameter \( \psi(\vec{r}, t) \) is related to the fluid temperature in the mid-plane of the convective cell. \( \xi(\vec{r}, t) \) is the vertical vorticity potential. This mean flow field coupling \[6, 11\] with the SH equation has been shown to play a key role in the onset of turbulence in Oberbeck-Boussinesq systems \[12\]. The quantity \( \epsilon \) is the reduced Rayleigh number,

\[ \epsilon = \frac{R}{R^\infty_c} - 1, \]

(5)

where \( R \) is the Rayleigh number and \( R^\infty_c \) is the critical Rayleigh number for an infinite system. A phenomenological forcing field \( f \) has been included in Eq.(1) to simulate the lateral sidewall forcing produced by horizontal temperature gradients present in the experiment. As in earlier studies \[13, 14\], we have varied the strength and spatial extent of \( f \) in order to best fit the experimental observations. We have derived a three mode amplitude equation from the generalized SH equation in order to both estimate the threshold values of \( \epsilon \) that separate regions in which roll and hexagonal configurations are stable, as well as the values of the parameters that enter the generalized SH equation in terms of experimentally measurable quantities. From
the experiment [4], we find that \( g_2 \approx 0.35 \), which is the value that we have used in Eq.(1). The value of \( \epsilon \) used in the numerical simulation is related to the real experimental value \( \epsilon^{exp} \) in ref. [4] by \( \epsilon^{exp} = 0.3594 \epsilon \). The nonlinear coupling constant \( g_m \) has been chosen to be 50, which is consistent with earlier studies [11, 6], and \( \epsilon^2 = 10 \) throughout our calculations. We note that Bestehorn et al [15] have reported a study of this model limited to the special case in which the initial configuration is a spiral. This, however, avoids the fundamental question addressed here of how the spiral spontaneously forms in the hexagon-roll transition. In the following we report the results of our calculations.

i) Formation of hexagonal pattern with sidewall forcing field. We have considered a circular cell of radius \( R = 32\pi \), which corresponds to an aspect ratio of \( \Gamma = 2R/\pi = 64 \). A square grid with \( N^2 \) nodes has been used with spacing \( \Delta x = \Delta y = 64\pi/N \), and \( N = 256 \). We approximate the boundary conditions on \( \psi \) by taking \( \psi(\vec{r}, t) = 0 \) for \( \|\vec{r}\| \geq R \), where \( \vec{r} \) is the location of a node with respect to the center of the domain of integration. The initial condition \( \psi(\vec{r}, t = 0) \) is a random variable, gaussianly distributed with zero mean and a variance of \( 10^{-1} \). In this case \( \epsilon = 0.1 \), and \( f = 0.0 \) everywhere except on the nodes adjacent to the boundary with \( f = 0.1 \). Figure 1 presents a typical configuration for a hexagonal pattern which forms in the presence of a strong static sidewall forcing.

ii) Early stage of transition between hexagons and rolls. We use Fig. 1 as the initial configuration with exactly the same forcing field \( f \) as before, but now we increase \( \epsilon \) very slowly up to 0.3. We take \( \epsilon = 0.1 + 1.67 \times 10^{-4}t \) for \( 0 < t < 1200 \) and \( \epsilon = 0.3 \) for \( t > 1200 \). Figure 2 shows two configurations during the early transient regime during the hexagon to roll transition. How the sidewall forcing and defects mediate the transition can be clearly seen in Fig. 2. The rolls are formed near the sidewall with a favorable orientation relative to the symmetry of the sidewall. In the meantime the defects glide toward each other and invade nearby regions of hexagonal order to create a region of rolls that spreads across the cell as the transition
proceeds. The spiral formation is already noticeable in Fig. 2(b). This resembles the experimental observation that there is a tendency to form spirals during the hexagon to roll transition.

iii) Formation of rotating spirals. Fig. 3 show the spatial and temporal formation of the spiral pattern at later times than in Fig. 2. In Fig. 3(a), we see that rolls bend or curve rapidly forming a roughly uniform patch of rolls with a locally disordered texture near the left corner. Further evolution consists primarily of dislocations gliding toward each other and eventually annihilating themselves, ending in a three-armed spiral. The final state of the rotating spiral (Figs. 3(b) and 3(c)) is remarkably similar to one observed in the experiment and occurs at $t \approx 49000 \approx 12$ horizontal-diffusion times. The corresponding experimental times are in the range of 10 to 20 horizontal-diffusion times. Our numerical investigation indicates that the non-Boussinesq effect plays a crucial role in forming spontaneous spirals. In the absence of $g_2$ (with or without the mean flow field), if we start with a random initial condition, there is no occurrence of a spiral pattern. This strongly suggests that the formation of spiral patterns is an intrinsic property of non-Boussinesq systems. We have studied the transition from a conduction state to a rotating spiral state, as well as the transition from a rotating spiral state to a hexagon state. We have also studied the effect of decreasing $\epsilon$ in the hexagon to rotating spiral transition. We find that a stable $n$-armed spiral tends toward one with fewer arms when $\epsilon$ is decreased, in agreement with experimental observation. The details of all the above will be discussed elsewhere.

In summary, we have investigated the question of pattern formation in a model of convection in a non-Boussinesq fluid that allows patterns of various symmetries. We start with a random initial condition and show that this leads to the ordered hexagonal state observed in the experiment. We then show that upon increasing $\epsilon$ we see a dynamical evolution to a new roll state which contains a rotating spiral pattern. These results are in very good agreement with the experimental studies in $CO_2$ gas.
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References

[1] G. Nicolis and I. Prigogine, *Self-Organization in Non-equilibrium Systems*, (Wiley, New York, 1977).

[2] H.L.Swinney and V.I.Krinsky, *Waves and Patterns in Chemical and Biological Media*, Physica D 49, pp.1 1991.

[3] A.T.Winfree, Science 175, 634 (1972); S.C.Muller, T.Plessser and B.Hess, Science 230, 661 (1985); W.Y.Tam, W.Horsthemke, Z.Noszticzius and H.Swinney, J.Chem.Phys. 88, 3395 (1988); G.S.Skinner and H.Swinney, Physica D 48, 1 (1991).

[4] E. Bodenschatz, J.R. de Bruyn, G. Ahlers and D.S. Cannell, Phys. Rev. Lett. 67, 3078 (1991).

[5] J. Swift and P.C. Hohenberg, Phys. Rev. A 15, 319 (1977).

[6] P. Manneville, J. Physique 44, 759 (1983).
[7] H.S. Greenside and W.M. Coughran Jr., Phys. Fluids, 49, 726 (1982); H.S. Greenside and W.M. Coughran Jr., Phys. Rev. A 30, 398 (1984); H.S. Greenside and M.C. Cross, Phys. Rev. A 31, 2492 (1985).

[8] P. Collet and J.-P. Eckman, Instabilities and Fronts in Extended Systems, (Princeton University Press, New Jersey, 1990).

[9] P. Manneville, Dissipative Structures and Weak Turbulence, (Academic, New York, 1990).

[10] M.C. Cross, Phys. Fluids 23, 1727 (1980); Phys. Rev. A 25, 1065 (1982); Phys. Rev. A 27, 490 (1983).

[11] E.D. Siggia and A. Zippelius, Phys. Rev. Lett. 47, 835 (1981); A. Zippelius and E.D. Siggia, Phys. Rev. A 26, 178 (1982).

[12] H.S. Greenside, M.C. Cross and W.M. Coughran Jr., Phys. Rev. Lett. 60, 2269 (1988).

[13] H.W. Xi, J. Viñals and J.D. Gunton, Physica A 177, 356 (1991).

[14] J. Viñals, H.W. Xi and J.D. Gunton, Phys. Rev. A, in press.

[15] M. Bestehorn, M. Fantz, R. Friedrich, H. Haken and C. Pérez-García, unpublished.

[16] We note that the vertical-diffusion time is $t_b/\Gamma^2$. 
Figure captions

Figure 1. Hexagonal pattern starting from random initial condition obtained in a cylindrical cell with aspect ratio $\Gamma = 64$. The values of the parameters used are $g_2 = 0.35$, $g_m = 50$ and $\epsilon = 0.1$. A non-zero forcing field localized at the boundary with $f = 0.1$ has been used.

Figure 2. We observe an early stage of hexagon-roll transition obtained by changing $\epsilon$ from $\epsilon = 0.1$ to $\epsilon = 0.3$, in a cylindrical cell with an aspect ratio $\Gamma = 64$. The initial condition is the uniform hexagonal pattern shown in Fig. 2. Two different times, $t = 720$ (b) and $t = 960$ (b) are shown. The rolls appearing near the defects and sidewall boundaries spreads through the cell as the transition proceeds.

Figure 3. Pattern formation of a rotating three-armed spiral obtained in a cylindrical cell with aspect ratio $\Gamma = 64$, with $g_2 = 0.35$, $g_m = 50$, $\epsilon = 0.3$ and $f=0.1$. Time series of the pattern evolution at (a) $t= 7440$, (b) $t=48240$ and (c) $t=64080$. The final rotating spiral pattern is shown in (b) and (c).