Compressible structures in incompressible hydrodynamics and their role in turbulence onset

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Abstract. The formation of the coherent vortical structures in the form of thin pancakes is studied for three-dimensional flows at the high Reynolds regime when, in the leading order, the development of such structures can be described within the 3D Euler equations for ideal incompressible fluids. Numerically and analytically on the base of the vortex line representation we show that compression of such structures and respectively increase of their amplitudes are possible due to the compressibility of the continuously distributed vortex lines. It is demonstrated that this growth can be considered as analog of breaking for the divergence-free vorticity field. At high amplitudes this process has a self-similar behavior connected the maximal vorticity and the pancake width by the Kolmogorov type relation $\omega_{\max} \propto l^{-2/3}$. The role of such structures in the Kolmogorov spectrum formation is also discussed.

1. Introduction

According to the Kolmogorov-Obukhov theory of developed hydrodynamic turbulence \cite{1, 2}, the velocity fluctuations at intermediate spatial scales $l$ obey the power-law $\langle |\delta v| \rangle \propto \varepsilon^{1/3} l^{1/3}$, where $\varepsilon$ is the mean energy flux from large to small scales (equivalent to the Kolmogorov spectrum $E_k \propto \varepsilon^{2/3} k^{-5/3}$, see e.g. \cite{3, 4}). Consequently, fluctuations of the vorticity $\omega = \text{rot} \mathbf{v}$ diverge at small scales as $\langle |\delta \omega| \rangle \propto \varepsilon^{1/3} l^{-2/3}$ and therefore the Kolmogorov spectrum is linked with the small-scale structures of intense vorticity.

The Kolmogorov-Obukhov arguments are based on the isotropy of the flow and locality of nonlinear interaction at intermediate scales. Then, the dynamics in the inertial interval can be described by the Euler equations and the emergence of the Kolmogorov spectrum can be expected before the viscous scales get excited. Thus, the Kolmogorov spectrum formation must have a close link to the mechanism of vorticity growth with a possible finite-time blowup in the incompressible 3D Euler equations. This problem was intensively studied over the last decades because of its relation to subsequent transition to turbulence. Several analytical blowup and no-blowup criteria were established; see the reviews in \cite{5} and \cite{6}. The central result is the Beale–Kato–Majda theorem \cite{7}, which states that at a singular point (if it exists) the time integral of maximum vorticity must explode.

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In parallel, a large effort was made with numerical analysis. However, numerical evidence of blow-up in the 3D Euler remains very controversial question because a large increase of vorticity makes a flow asymptotically quasi two-dimensional. As well known for 2D flows a blow-up is forbidden (see, e.g. [8, 9, 10, 11]). Thus, further numerical studies were mainly focused on specific initial conditions providing enhanced vorticity growth, e.g., antiparallel or orthogonal vortices; we refer to [6] and [12] for a brief review for examples of recent numerical works. It is worth noting that interaction of 3D flow with a rigid boundary may be a reason of the finite time singularity appearance observed in the numerical experiment [13].

2. Vortical pancake structures: direct integration

In this paper we discuss our recent findings [12, 14, 15, 16] concerning the general properties of the high-vorticity regions developing from generic large-scale initial conditions in 3D Euler equations in the periodic box \( r = (x, y, z) \in [-\pi, \pi]^3 \). The initial flows are chosen as a superposition of the shear flow \( \omega_x = \sin z, \omega_y = \cos z, \omega_z = 0 \) and a random (not necessarily small) perturbation. Here we consider only two initial conditions designated as \( I_1 \) and \( I_2 \) in [12].

As was first found by Brachet et.al [17], the regions of high vorticity in the generic case represent exponentially compressing pancake-like structures. Contrary to the previous studies, we show [12] by means of the direct integration of the 3D Euler equations that evolution of the pancakes is governed by two different exponents for thickness \( l(t) \propto e^{-\beta_1 t} \) and maximal vorticity \( \omega_{\text{max}}(t) \propto e^{\beta_2 t} \) describing compression of the pancake with the universal ratio \( \beta_2/\beta_1 \approx 2/3 \), leading to the Kolmogorov-type scaling law

$$\omega_{\text{max}}(t) \propto l(t)^{-2/3},$$

see Fig. 1 and Fig 2(a).

Fig.1 (a) shows evolution of the global vorticity maximum \( \omega_{\text{max}}(t) = \max_r |\omega(r,t)| \), demonstrating the vorticity increase from \( \omega_{\text{max}}(0) = 1.5 \) up to 18.4 at \( t = 7.75 \) and the asymptotically exponential vorticity growth at late times. Panel (b) of the same figure shows the three-dimensional regions containing the vorticity \( \omega = |\omega| \geq 0.85 \omega_{\text{max}}(t) \) at different times, and one can clearly see the formation of a thin pancake structure. It is convenient to introduce the pancake mid-plane as a surface, where the vorticity attains a maximum within the pancake thickness. The color in Fig.1 (b) describes the mid-plane vorticity, from blue for 0.85 \( \omega_{\text{max}}(t) \) to yellow for \( \omega_{\text{max}}(t) \). At \( t = 3 \) and 4, the pancake spans the whole periodic domain in \( x \)-direction; for larger times its lateral sizes decrease, but eventually stabilize and remain almost constant at \( t \geq 6 \). On the contrary, the thickness keeps decreasing rapidly. Thus, at late times, vorticity variations become large (small) in transversal (tangential) directions to the pancake. Our findings are based on direct numerical integration of the Euler equations (in the vorticity formulation)

$$\partial_t \omega = \text{rot} (v \times \omega), \quad v = \text{rot}^{-1} \omega,$$

for more than 30 initial conditions, and on the recent simulations [16] of the equations in the vortex line representation (VLR). In all our simulations we used the pseudo-spectral Runge-Kutta fourth-order method in adaptive anisotropic rectangular grid, which is uniform in each direction and adapted independently along each of the three coordinates, with up to 2048\(^3\) total number of nodes and observed in details evolution of high-vorticity regions. These regions represent pancake-like structures and the flow near the pancake is described locally by a novel exact self-similar solution of the Euler equations combining a shear flow with an asymmetric straining flow [14]. It is worth noting also that for more than 30 initial conditions used in simulations we observed formation of pancake structures with the 2/3 relation (1) that allows us to state that this law can be considered as universal.
Figure 1.  (a) Global vorticity maximum as a function of time (logarithmic vertical scale). The red dashed line indicates the slope $\propto e^{\beta_2 t}$ with $\beta_2 = 0.5$. The thin vertical line marks the final simulation time $t = 6.89$ in [12]. (b) Regions of the largest vorticity, $\omega \geq 0.85 \omega_{\max}(t)$, at different times. Color represents vorticity in the mid-plane of the pancake: from blue for $0.85 \omega_{\max}(t)$ to yellow for $\omega_{\max}(t)$. The structures are shifted vertically for better visualization. (c) Characteristic spatial scales $\ell_1$ (black), $\ell_2$ (blue) and $\ell_3$ (red) of the pancake structure. The red dashed line indicates the slope $\propto e^{-\beta_1 t}$ with $\beta_1 = 0.74$.

3. Vortex line representation
As a transformation, the VLR is based on a simple observation. According to (2) the vorticity can change only due to the velocity component $v_n$ normal to the vector $\omega$ (because of the vector product). Therefore, in the general case, $\text{div } v_n \neq 0$. Just this is the origin of the vorticity field compressibility, in spite of its divergence-free. From another side, as well known, Eq. (2) describes dynamics of the frozen-in-fluid vector field $\omega$. This property means that any fluid particle moves together with its own vortex line and therefore $v_n$ indeed represents a velocity of the vortex line.

Hence the VLR follows if one introduces the Lagrangian trajectories of the vortex lines as solution of the ODEs

$$\frac{dx}{dt} = v_n(x, t) \text{ with } x|_{t=0} = a,$$

which defines the compressible mapping $x = x(a, t)$. Compressibility of the mapping is a direct sequence of the Liouville formula applied to this equation,

$$\frac{dJ}{dt} = \text{div } v_n \cdot J,$$

where $J$ is the mapping Jacobian. In the general situation, $\text{div } v_n \neq 0$ and it is why there are no restrictions imposed on the $J$ value. In particular, the Jacobian can take arbitrary values including zero.
Figure 2. (a) The vorticity maximum $\omega_{\text{max}}$ vs. the pancake thickness $l_1$ in logarithmic scales during the pancake evolution. The circle marks the vorticity maximum at the final time $t = 7.5$ and the dashed line indicates the power-law $\omega_{\text{max}} \propto l_1^{-2/3}$. (b) Isosurfaces of $|\omega| = 0.8 \omega_{\text{max}}$ (red) and Jacobian $J = 1.25 J_{\text{min}}$ (blue) at $t = 7.5$.

In terms of this mapping, Eq. (2) admits explicit integration in the form [18, 19]

$$\omega(x, t) = \frac{\mathbf{J} \omega_0(a)}{J}, \quad \mathbf{J}(a, t) = [J_{ij}(a, t)] = \left[ \frac{\partial x_i}{\partial a_j} \right], \quad J = \det \mathbf{J},$$

(5)

where $\omega_0(a)$ is the initial vorticity which has a meaning of the Cauchy invariants [20] and $\mathbf{J}$ is the Jacobi matrix of the mapping $x = x(a, t)$. Eqs. (3), (5) together with $\text{div} \, \mathbf{v} = 0$ form complete system for the VLR. In a generic case, a sustained growth of vorticity should be related to simultaneous decrease of the Jacobian in the denominator of Eq. (5), what may be seen as formation of high density of vortex lines. For a blowup, it is reasonable to expect that the Jacobian vanishes in a finite time, see for instance [21].

Let us assume that the points of the vorticity maximum and the Jacobian minimum coincide. The vorticity maximum satisfies the so-called vortex-stretching equation,

$$\frac{d\omega_{\text{max}}}{dt} = \omega_{\text{max}} \tau_i \tau_j \frac{\partial v_i}{\partial x_j} \approx \omega_{\text{max}} \frac{\partial v_\tau}{\partial x_\tau} = \omega_{\text{max}} \text{div}(\mathbf{v}_\tau) = -\omega_{\text{max}} \text{div}(\mathbf{v}_n),$$

(6)

where all spatial derivatives are taken at the maximum point, $\tau$ is the unit vector along the direction tangent to the vorticity and $v_\tau$ is the velocity component parallel to the vorticity; summation is implied with respect to repeated indexes. In Eq. (6) we additionally assumed that the vorticity direction does not change significantly near the vorticity maximum, what corresponds to our numerical simulations. According to Eq. (4), the Jacobian minimum satisfies

$$\frac{dJ_{\text{min}}}{dt} = J_{\text{min}} \text{div}(\mathbf{v}_n),$$

(7)

and we conclude that the vorticity maximum should evolve inverse-proportionally to the Jacobian minimum,

$$\omega_{\text{max}}(t) J_{\text{min}}(t) \approx \text{const},$$

(8)

that is in agreement with numerical simulation (see Fig.3). In the next Sections, we show that
compressibility of the vorticity field provides formation of the coherent structures of the pancake form. This process is familiar to the formation of caustics in acoustics and optics where these are the pancake type structures spreading in the plane transverse to the breaking direction.

4. Exact pancake model and the VLR

Our numerical experiment demonstrated also that the formation of the pancake type structures has a self-similar behavior (see [12]). The vorticity component transverse to the pancake turns out to be much smaller the component along the pancake. We established also that asymptotically the pancake structure can be approximated with a high accuracy by a novel exact solution of the Euler equations (suggested in [12]), which combines a shear flow aligned with an asymmetric straining flow, and is characterized by an arbitrary transversal vorticity profile. This solution in the Cartesian coordinates $x = x_1 n_1 + x_2 n_2 + x_3 n_3$ is written as

$$\mathbf{v}(x,t) = -\omega_{\text{max}}(t) l_1(t) \mathbf{f} \begin{pmatrix} x_1 \\ l_1(t) \end{pmatrix} n_3 + \begin{pmatrix} -\beta_1 x_1 \\ \beta_2 x_2 \\ \beta_3 x_3 \end{pmatrix},$$

$$\omega(x,t) = \omega_{\text{max}}(t) f'(\frac{x_1}{l_1(t)}) n_2.$$

Here the vorticity has self-similar dependence on $x_1/l_1(t)$ only, and the only one component parallel to $n_2$, $\beta_1$, $\beta_2$ and $\beta_3$ are arbitrary constants with one constraint $-\beta_1 + \beta_2 + \beta_3 = 0$. Here $\omega_{\text{max}}(t) = w_0 e^{\beta_2 t}$, $l_1(t) = h_0 e^{-\beta_1 t}$ are temporal dependences for the maximal vorticity and the pancake thickness, $w_0$ and $h_0$ are positive (initial) values, $f(\xi)$ is arbitrary smooth function with $|\max f'(\xi)| = 1$.

This solution, being the first one capable to describe both the exponential pancake development, can be extended for the Navier-Stokes equations as well [14]. Numerical simulation gives a good fitting with this exact model (up to 10 pancake width), however the model can not explain the $2/3$ law. This law can appear due to account of three-dimensionality. The latter, in particular, follows from the VLR for the exact model (9, 10):

$$x_1 = a_1 e^{-\beta_1 t}, \quad x_2 = a_2, \quad x_3 = a_3 e^{\beta_3 t} - w_0 h_0 f \left( \frac{a_1}{h_0} \right) \frac{\sinh(\beta_3 t)}{\beta_3}.$$

![Figure 3](image-url)
The corresponding Jacobi matrix and its determinant are written as follows,

$$\hat{J}(a, t) = \left[ \frac{\partial r_i}{\partial a_j} \right] = \begin{pmatrix} e^{-\beta_1 t} & 0 & 0 \\ -w_0 f' \left( \frac{a_1}{h_0} \right) \sinh(\beta_3 t) & 1 & 0 \\ 0 & 0 & e^{\beta_3 t} \end{pmatrix}, \quad J = \det \hat{J} = e^{(\beta_1 - \beta_3) t} = e^{-\beta_2 t},$$

and one can verify that expressions (10), (11) and (12) satisfy the equation (5). Note that in terms of the VLR the model solution is degenerate: the initial vorticity is aligned with the eigenvector of the Jacobi matrix corresponding to unit eigenvalue, $\hat{J} \omega_0 = \omega_0$, so that all time-dependency for the vorticity comes from the denominator in (5), i.e., the Jacobian which, in its turn, does not depend on coordinates. The pancake model solution allows for an arbitrary scaling between the vorticity maximum and the pancake thickness, and our observation of the $2/3$-scaling remained unexplained.

The Jacobi matrix for the exact solution contains one off-diagonal element $J_{13}$ growing exponentially with time $\sim e^{\beta_3 t}$. By this reason, an eigen value of the Jacobi matrix does not coincide with a relative stretching. To introduce correctly the relative stretching one needs to consider the so-called singular value problem for matrix $\hat{J} = [\partial r_i / \partial a_j]$ at $J_{\text{min}}$. This problem reduces to seeking for two rotation matrices $\mathbf{U}$ and $\mathbf{V}$ and the diagonal matrix $\Sigma = \text{diag}\{\sigma_1, \sigma_2, \sigma_3\}$, with $0 < \sigma_1 < \sigma_2 < \sigma_3$, called singular eigen values. Then the Jacobi matrix can be represented as follows $\hat{J} = \mathbf{U} \Sigma \mathbf{V}^T$, where index $T$ means transposition. Both rotation matrices $\mathbf{U}$ and $\mathbf{V}$ are found by the standard way (for details see [16]).

In the given case we have

$$\sigma_1^2 = g - \sqrt{g^2 - e^{-2\beta_2 t}}, \quad \sigma_2^2 = 1, \quad \sigma_3^2 = g + \sqrt{g^2 - e^{-2\beta_2 t}} ,$$

where

$$g = \frac{1}{2} \left( e^{-2\beta_2 t} + e^{2\beta_2 t} + \left[ w_0 f' \left( \frac{a_1}{h_0} \right) \frac{\sinh(\beta_3 t)}{\beta_3} \right]^2 \right).$$

As $t \to \infty$

$$\sigma_1 \propto e^{-\beta_1 t}, \quad \sigma_2 = 1, \quad \sigma_3 \propto e^{\beta_3 t},$$

in correspondence with the numerical simulation. Thus, near the Jacobian minimum along the first axis we have strong compression: $\sigma_1 \propto e^{-\beta_1 t} \propto l_1$; along the third axis we have strong expansion $\sigma_3 \propto e^{\beta_3 t} \propto \omega_{\text{max}}^{-1} l_1^{-1}$; in the intermediate direction $\sigma_2$ is close to unity and does not change significantly. In the limit $t \to \infty$, for the exact solution

$$\mathbf{U} \simeq 1, \quad \mathbf{V} \simeq \begin{pmatrix} 1 \sqrt{1+q^2} & 0 & \frac{q}{\sqrt{1+q^2}} \\ 0 & 1 & 0 \\ -q \sqrt{1+q^2} & 0 & \frac{1}{\sqrt{1+q^2}} \end{pmatrix},$$

where

$$q = - \frac{\Omega_0}{2\beta_3} f' \left( \frac{a_1}{h_0} \right).$$

For long time asymptotics, as the simulations show, matrix $\mathbf{U}$ is close to unity, and $\mathbf{V}$ -to anti-diagonal with $V_{13} \approx V_{22} \approx -V_{31} \approx 1$. By this reason, $q$ in (16) can be considered as a large quantity.
5. Origin of the 2/3 scaling

In all our papers [12, 14, 15, 16] the pancake thickness was defined from expansion of $|\omega|$ near the maximal point $\mathbf{x} = \mathbf{x}_{\max}$

$$|\omega| = \omega_{\max} - \frac{1}{2} \Gamma_{ij}^{(\omega)} \tilde{x}_i \tilde{x}_j,$$  \hspace{1cm} (17)

where $\tilde{x} = \mathbf{x} - \mathbf{x}_{\max}$. The corresponding sizes of the structure in three orthogonal directions $l_n = \sqrt{2\omega_{\max}/|\lambda_n|}$ are defined through eigenvalues $\lambda_n$ of matrix $-\partial_i \partial_j |\omega|$, calculated in the maximal point, so that expansion (17) can be written as

$$|\omega| = \omega_{\max} \left(1 - \sum_{n=1,2,3} \frac{\tilde{x}_n^2}{\lambda_n^2}\right).$$

The maximal $|\lambda_i|$ gives the pancake thickness and its eigen vector defines the pancake orientation.

Because of relation (8) (see also Fig. 2(b)) the area of maximal vorticity is familiar to that for the minimal Jacobian. The latter means that both areas in the $x$-space have the comparable sizes, especially at the long-time asymptotics. In order to establish the 2/3 law one needs to perform in Eq. (5) the transition from the auxiliary $a$-variables to the physical $x$-space. In the maximal vorticity areas the enumerator in (5), which defines the vorticity direction, practically does not change, in a full agreement with the exact solution. Thus, spatial-temporal behavior of the vorticity in its maximal area comes from the Jacobian dependence.

As mentioned already, the numerical pancake solution is well approximated by the exact solution (9,10). Therefore, to find the 2/3 scaling the exact solution will be considered as the zero approximation.

Consider the expansion of $J$ near its minimum

$$J = J_{\min} + \frac{1}{2} \Gamma_{ij}^{(x)} a_i a_j,$$  \hspace{1cm} (18)

which, in the main axises of matrix $\Gamma_{ij}^{(x)}$, is written as

$$J = J_{\min} \left(1 + \sum_{n=1,2,3} \frac{x_n^2}{\ell_n^2}\right).$$

Here we omitted sign tilde assuming that $x = 0$ corresponds to the minimal point; eigen value $\lambda_n^{(x)} \approx 2J_{\min}/\ell_n^2$.

Numerical experiment shows that the pancake thickness $\ell_1$ reduces exponentially and two other scales do not practically change:

$$\ell_1 \propto e^{-\beta_1 t}, \ \ell_2 \propto 1, \ \ell_3 \propto 1.$$  \hspace{1cm} (19)

In this case ratio $\beta_2/\beta_1 \approx 0.64$, namely, the vortex structure evolve in accordance with the 2/3 law (1).

In Lagrangian variables $a$ expansion of $J$ has the similar form

$$J = J_{\min} + \frac{1}{2} \Gamma_{ij}^{(a)} a_i a_j,$$  \hspace{1cm} (20)

where $\Gamma_{ij}^{(a)} = \partial_i J/\partial a_i \partial a_j$ is positively definite matrix. Note, that the given expansion is valid in a small vicinity around the minimal point $a = 0$, therefore the transition from the variables $x$ to the variables $a$ in (18) can be approximated in the form:

$$x = U \Sigma V^T a.$$
Substitution of this expression into (18) gives how the matrix $\Gamma^{(a)}$ is expressed through $\Gamma^{(a)}$:

$$\Gamma^{(a)} = J^T \Gamma^{(x)} J = V \gamma V^T,$$

where we introduce the matrix

$$\gamma = \Sigma U^T \Gamma^{(x)} U \Sigma.$$

This matrix asymptotically for large $t$ is close to diagonal, since the matrix $\Gamma^{(x)}$ is diagonal with elements $\lambda_i^{(j)} (i, j = 1, 2, 3)$, and the matrix $U$ tends to unity in this limit. The latter is confirmed by numerical simulation. Hence for diagonal elements of $\gamma$ we approximately have:

$$\gamma_{ii} \approx \sigma_i^2 \lambda_i^{(j)} = 2 J_{\min} \sigma_i^2 / \ell_i^2.$$

Remembering, that $\sigma_1 \propto \ell_1 \propto e^{-\beta_1 t}$, $\sigma_2 \propto \ell_2 \propto 1$, $\ell_3 \propto 1$, for $\gamma_{ii}$ we get the following estimate:

$$\gamma_{11} \propto J_{\min}, \quad \gamma_{22} \propto J_{\min}, \quad \gamma_{33} \propto J_{\min} \sigma_3^2.$$

Thus, two first diagonal elements tend exponentially to zero and the only third one can be of the order of unity that is confirmed by numerical experiment. Moreover, off-diagonal elements of $\gamma$ also are small. Only the $\gamma_{33}$ component survives, which is expressed only in $\omega_{\max}$ and the pancake thickness $l_1$ (recall that $\sigma_3 \propto \omega_{\max}^{-1} l_1^{-1}$):

$$\gamma_{33} \approx 2 J_{\min} \sigma_3^2 / \ell_3^2 \propto J_{\min} \omega_{\max}^{-2} l_1^{-2} \propto \omega_{\max}^{-3} l_1^{-2}.$$

Numerical experiment shows that the $\gamma_{33}$ component is of the order of unity. This means that between $\omega_{\max}$ and the pancake thickness, there is the 2/3 scaling law (1).

In conclusion of this Section, we should say that if the rotation $V$ is applied to the matrix $\gamma$, then according to (21), the largest element for $\Gamma^{(a)}$ becomes $\Gamma_{11}^{(a)}$, which coincides with $\gamma_{33}$. Thus, the Jacobian and, accordingly, the vorticity mainly depend on the coordinate $x_1$, the influence of other coordinates is exponentially weak. This once again emphasizes that this structure is quasi-one-dimensional, but the appearance of scaling (1) is a purely three-dimensional phenomenon, which is due to the compressibility of continuously distributed vortex lines.

6. Pancake structures and the Kolmogorov spectrum

Our numerical simulations [12] show that the pancakes appear in increasing number with different scales and generate strongly anisotropic “jets” in the Fourier space. Each jet is a Fourier image of the corresponding pancake structure. Its width in the $k$-space is small in comparison with its length which, in its turn, grows inverse proportionally to a pancake thickness, $\sim \ell(t)^{-1}$ and, thus, the energy spectrum density in such a case will be very irregular relative to both angles and $|k|$. The numerical experiment demonstrates that these jets dominate in the energy spectrum, where, for some initial flows, we observe clearly (after averaging over angles) the gradual formation of the Kolmogorov spectrum $E_k \propto k^{-5/3}$, in fully inviscid system [12].

Fig. 4 (a) shows that, at sufficiently small wavenumbers, we clearly observe the gradual formation of the Kolmogorov interval $E_k \propto k^{-5/3}$. This interval grows with time and extends to a decade of wavenumbers, $2 \lesssim k \lesssim 20$, at the end of the simulation. The Kolmogorov interval corresponds to the “frozen” part of the energy spectrum: $E_k(t)$ changes slightly with time in the Kolmogorov region in contrast to the vast changes at larger wavenumbers. Taking into account the times and logarithmic scale in Fig 4 (a), one can guess that the size of the Kolmogorov interval increases exponentially in time.

Fig. 4 (b) shows the isosurface $|\tilde{\omega}(k)| = 0.2$, where the very thin interior part corresponds to larger vorticity, $|\tilde{\omega}(k)| > 0.2$. (Here we introduce the function $\tilde{\omega}(k) = \omega(k)/\max_{|p| = k} (\omega(p))$)
Figure 4. (a) Energy spectrum $E_k(t)$ at different times. Straight line above the curves indicates the slope of the Kolmogorov power-law, $E_k \propto k^{-5/3}$. (b) Isosurface $|\tilde{\omega}(k)| = 0.2$ of the normalized vorticity field in Fourier space at time $t = 6.89$. The jet is aligned with the eigenvector $w_1$ (dashed black line), which is the normal direction of the pancake structure at the global vorticity maximum in physical space. (c) The closer view with the eigenvector $w_1$ (solid black arrow) for the global vorticity maximum and the respective eigenvector (dash-dot blue arrow) for the third largest local maximum.

representing the Fourier transformed vorticity $\omega(k)$ scaled to the maximal norm within each spherical shell. The reason for such a normalization is to compensate the strong decay of vorticity with $k$. As expected, this isosurface is aligned with the eigenvector $w_1$ computed at the global maximum, which should bring the dominant contribution to Fourier components of vorticity at large $k$.

Fig. 4 (c) demonstrates that the isosurface geometry is different at smaller wavenumbers, $k \lessgtr 20$, corresponding to the Kolmogorov interval in Fig. 3. Here, different jets contribute and some of these jets can be clearly related to the pancakes of other local vorticity maximums (the figure shows directions for the first and third largest local maximums, while the second local maximum yields the direction very close to that for the first one).

7. Conclusion

The main conclusion of this paper is that in the three-dimensional hydrodynamics for incompressible fluids in the stage of turbulence onset, at large Reynolds numbers, the main role is played by coherent vortical structures with increasing amplitudes due to the compressibility of the vorticity field, despite its divergence free. For 3D Euler hydrodynamics these are structures in the form of pancakes with scaling of the Kolmogorov type - the ratio between the maximum vorticity and the pancake thickness $\omega_{\text{max}} \sim l^{-2/3}$. Compression of these structures has an exponential character and can be interpreted as a breaking (overturning) process analogous to the formation of shock waves in gas dynamics. We showed using a combined analytical-numerical approach based on the vortex line representation, that the Kolmogorov type scaling for the pancake vortical structures arises from the three-dimensionality of these structures.

The total number of pancake structures, estimated by the number of local vorticity maxima, increases with time. We demonstrate that at late times most of the pancakes are distributed densely across the corresponding interval of wavenumbers.

We clearly observe the formation of the Kolmogorov energy spectrum $E_k \propto k^{5/3}$ in the inviscid system, together with the exponential (i.e., no finite-time blowup) vorticity growth. The interval with Kolmogorov scaling grows with time and extends to a decade of wavenumbers at the end of the simulations. Thin pancake structures in physical space generate strongly anisotropic vorticity field in Fourier space in the form of jets, which are extended in the directions perpendicular to the pancakes. Within these jets, the Fourier components of the flow are large in comparison with the remaining background. We demonstrate that these jets occupy a small fraction of the
entire spectral band, but provide the leading contribution to the energy spectrum of the system. This means that the energy transfer to small scales is performed through the evolution of the pancake structures, and that the Kolmogorov energy spectrum may be attributed to collective behavior of the pancakes.

Thus, we observe and prove by means of combined analytical-numerical approach that the $2/3$-scaling (1) holds universally, while initial conditions composed of the shear flow and a small perturbation develop the spectrum close to $E_k \propto k^{-5/3}$.

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