Convergence of binomial tree methods to Black Scholes model on determining stock option prices

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Abstract. There are many alternative investment tools that can be used to be the choice of investors. One of them is a derivative product. Derivative products that are more widely known and traded on financial markets are options. Option is a contract or agreement between two parties to buy or sell an instrument. To minimize risk, it is necessary to determine the option price by determining the fair price of the option. This option pricing can be done by using the Binomial Tree method and the Black Scholes method. Some factors that influence options are stock prices, strike prices, maturity, volatility, and interest rates. This paper discusses the European option pricing call on the shares of Bank Central Asia (BCA) with the Binomial Tree method and the Black Scholes Method. From the results of the research, it is found that the Binomial Tree method will converge to the Black Scholes method if the time partition increases.

1. Introduction
The development of the investment in the world is now increasingly rapid and growing. This matter is indicated by the increasing number of alternative investment tools used to be the choice of investors. One of them is derivative products. Derivative products are financial instruments whose value depends on the value of underlying assets can also be used to manage and minimize losses caused by fluctuations in the prices of underlying assets [1]. Derivative products are widely known and traded in financial markets, namely options. Options as one of the financial products that are widely assessed mathematically, both deterministically and stochastically.

Option is a contract or agreement between two parties, whereby the contract holder has the right not the obligation to buy or sell an underlying asset from the seller to a certain asset at a certain price and time [2]. Based on the type of option rights are divided into two, namely the purchase option (call option) and the selling option (put option). Buy options give the buyer the right to buy a particular stock at a predetermined price at any time during the contract, while the selling option gives the buyer the right to sell a certain price at a predetermined price at any time during the contract [3]. Options are divided into two types based on the type of time period of implementation, namely the type option.

America and European type options. The American type option is an option that is done at or before maturity, while the European type option is an option that is done at maturity [4].

In buying shares that are more expensive than the market or selling stocks that are cheaper than the market, it is necessary to determine the option price to minimize risk, namely by determining the fair price of the option. This option pricing can be done in several ways, including the Binomial Tree Method and the Black and Scholes Method [5] (Boddie et al. 2014)
The Binomial Tree Method was first introduced by Cox, Ross, and Rubinstein in 1979. This method reveals that stock price movements in the market have two possibilities, namely up and down [6]. Meanwhile, the Black Scholes method has been published since 1973 by Fisher Black and Myron Scholes in "The Pricing of Option and Corporate Liabilities". This method can only be done on the European type option pricing which can only be done at maturity, and has the assumption that no dividends will be paid during the option period, there is not transaction fees, risk free and constant interest rates over time, and the stock price follows geometric Brown motion [7].

2. Methodology
Option contracts are an agreement for option holders to buy an underlying asset on a certain date (maturity date) and at a certain price level (strike price). An option holder can make a decision to buy or sell an option in a particular market situation. In the European type call option, the option holder can ignore the option if the stock price at maturity is lower than the purchase price because it does not provide benefits for option holders, this condition can be called a strike price. Conversely, the option holder will sell shares at a strike price that is higher than the market price, the option holder will benefit by buying a low price share then selling it at a high price to the writer (seller). Writer must buy shares from the option holder who has bought the selling option as a transaction risk.

Option contracts based on the rights granted are as follows:

On the buy option (call option):
1. In-the-money: strike price is less than the stock price at the time of transaction (\(K < S_t\)).
2. At-the-money: strike price equals the stock price at the time of the transaction (\(K = S_t\)).
3. Out-of-the-money: strike price is greater than the stock price at the time of transaction (\(K > S_t\)).

On the sell option (put option):
1. In-the-money: strike price is greater than the stock price at the time of transaction (\(K > S_t\)).
2. At-the-money: strike price equals the stock price at the time of the transaction (\(K = S_t\)).
3. Out-of-the-money: strike price is less than the stock price at the time of transaction (\(K < S_t\)).

2.1 Lemma Itô
Lemma Itô is a stochastic analogy of the chain rules in ordinary differential equations to solve a stochastic integral. Lemma Itô was discovered by a mathematician, namely K. Itô in 1951.

Suppose the variable \(X\) follows the Itô process

\[
\frac{dX}{dt} = a(X, t) \, dt + b(X, t) \, dW_t
\]

Where \(dW_t\) is the Wiener process and \(a, b\) is a function of \(X, t\). The variable \(X\) has a drift level \(a\) and a variance level of \(b^2\). Lemma Itô shows the functions, \(G\) from \(X\) and \(t\) follows the process:

\[
dG = \left( \frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 \right) \, dt + \frac{\partial G}{\partial x} b \, dW_t
\]

\(dW_t\) is a Wiener in the equation (2). \(G\) follows the Itô process, using the drift rate

\[
\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2
\]

Standard deviation of changes in a short period of time \(\Delta t\) in stock prices and Parameter \(\mu\) is the level of expectation of stock returns, \(\sigma\) is the volatility of stock prices.

\[
dS = \mu S dt + \sigma S dW
\]
\( \mu \) and \( \sigma \) are constant, from lemma Ito\( \text{'} \) follows the process of Ito\( \text{'} \) then the function \( G \) and \( S \) is

\[
dG = \left( \frac{\partial G}{\partial S} \mu S + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S} \sigma S dW_t
\]  

(4)

The solution of equation (4) is

\[
S_t = S_0 \exp \left( \left( \mu + \frac{\sigma^2}{2} \right) t + \sigma W_t \right)
\]  

(5)

2.2 Volatility

Stock price volatility is the movement of the rise and fall of stock prices on the stock exchange in the annual period used to measure the level of risk of a stock. Volatility values are at positive intervals between 0 and infinity \((0 \leq \sigma \leq \infty)\). The higher the level of volatility, the higher the level of rise and fall of stock returns obtained. Volatility can be calculated using the formula:

\[
\sigma = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (R_i - \bar{R})^2}
\]  

(6)

or with the following formula:

\[
\sigma = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} R_i^2 - \frac{1}{n(n-1)} \left( \sum_{i=1}^{n} R_i \right)^2}
\]  

(7)

2.3 Binomial Tree Method

In the Binomial method, the option price is calculated by looking for the present value of the expected profit from using the option (payoff) when the option is used at the loan deadline. Stock price and execution price are needed in calculating payoff. For a simple stock price binomial method, the time interval is \([0, T]\) divided into \(N\) sub intervals, which are the same length as the points for \(0 = t_0 < t_1 < t_2 < \cdots < t_n = T\), where \(t_i = i \Delta t\) for \((i = 0, 1, 2, ..., N)\), \(\Delta t = T/N\) and \(S_t = S_{t_i}\) is stock price at time \(t_i\).

Assumptions:

a. In time interval \(\Delta t\) the stock price can go up or down to \(S_{t_i} \rightarrow S_u\) or \(S_{t_i} \rightarrow S_d\) with \(0 < d < 1 < u\)

b. The opportunity for stock prices to rise \(P\) (up) = \(p\) and \(P\) (down) = \(1 - p\)

c. There is no dividend payment for a certain period of time.

d. Expectation of stock price return equal to risk free interest \(r\). So that the stock price \(S_t\) moves randomly \(S_{t_i}\) and \(t_i\) becomes \(S_{t_{i+1}}\) and at the time \(t_i\) this means:

\[
E(S_{t_{i+1}}) = S_t e^{\mu \Delta t}
\]  

(8)

In the Binomial method, when \(t_i = i \Delta t\) there will be \(i + 1\) the stock price that might occur:

\[
S_{i,j} = S_0 u^i d^{j-i}, i = 0, 1, 2, 3, ..., N \text{ dan } j = 0, 1, 2, 3, ..., i
\]  

(9)

The equation above is not recursive, so it requires an equation, so the calculation does not take long.

\[
E(S_N) = S_0 e^{N\mu \Delta t}
\]  

(10)

To calculate the initial stock price of the matured stock price using the discount \(e - \mu \Delta t\) from the above equation with the formula:
So that the value of the European call option is at the time \( t_i \) that is the average of the current option value \( t_i + 1 \) that is \( \{S_iN\} \). Thus, the payoff value of a call option is

\[
C_{iN} = max(S_{iN} - K, 0)
\]  

(13)

The backward binomial tree method for calculating call options at \( t = 0 \), is:

\[
C_{ij} = e^{-\mu\Delta t} E(C_{iN}e^{\mu\Delta t})
\]

(14)

\[
C_{ij} = e^{-\mu\Delta t}(pC_{i+1,j+1} + (1-p)C_{i+1,j})
\]

(15)

2.4 Black Scholes method

The Black-Scholes method is a method used to determine option values. In the financial world, this model was developed by Fischer Black and Myron Scholes in 1973. This method is limited because only can be used to determine the European type option value that is executed at maturity, while this model does not apply to American type options, because the option can be executed at any time until the due date. There are several assumptions for formulating option values developed by Fisher Black and Myron Scholes:

1. The distribution of stock prices is lognormal and the range of returns on stocks is constant.
2. There are no transaction fees and taxes in buying or selling shares or options.
3. The type of option used is the type of European option, where the option can only be executed when the maturity date.
4. There is no possibility of the act of buying low-priced securities in a market and at the same time selling it at a higher price in a different market, so as to gain profits without risk.
5. Short-term risk-free interest rates are known and their value is constant.
6. Trading of underlying assets is continuous.
7. There is no prohibition on Short selling.

The European type Call option prices determined by the Black Scholes formula are as follows:

\[
C = S_0N(d_1) - Ke^{-\mu T}N(d_2)
\]

(16)

Where

\[
d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)T}{\sigma \sqrt{T}}
\]

(17)

\[
d_2 = d_1 - \sigma \sqrt{T}
\]

(18)

where \( N(d_1) \) is the cumulative density function of the normal distribution of \( d_1 \), and \( N(d_2) \) is the normal distributive cumulative density function of \( d_2 \).

3. Results and Discussion

The data used in this study is secondary data, namely the historical data of stock prices from PT. Bank Central Asia Tbk. (BCA) with a period of January 5, 2017 - January 5 2018. The stock price used is the daily closing price (close price) and the option price calculated is only the European type call option. With a risk-free interest rate of 0.065 or 6.5% with an initial stock price of 15,675.00 IDR, a strike price of 500.00 IDR, a maturity of 3 months and the stock price is assumed to follow the
lognormal distribution and no dividends are taken into account. By using the data return of the stock price of PT. Bank Central Asia Tbk. (BBCA) is calculated to find the volatility value as follows:

| Date             | Close Price (IDR) |
|------------------|-------------------|
| January 5, 2017  | 15675             |
| January 6, 2017  | 15600             |
| January 9, 2017  | 15350             |
| ⋮                | ⋮                 |
| January 5, 2018  | 22250             |

By using the equation (7), the stock closing return price is calculated. The expected return of the stock closing price is 0.0014. The calculation results of $R_t - \bar{R}$ can be seen in Table 2.

| t     | Stock Closing Price (IDR) | $R_t$ | $R_t - \bar{R}$ | $\left( R_t - \bar{R} \right)^2$ |
|-------|---------------------------|-------|-----------------|-------------------------------|
| 1     | 15600                     | -0.0048 | -0.0062 | 0.00004 |
| 2     | 15350                     | -0.0162 | -0.0176 | 0.00031 |
| 3     | 15400                     | 0.0033  | 0.0019 | 0.0000 |
| 4     | 15300                     | -0.0065 | -0.0079 | 0.00006 |
| ⋮    | ⋮                         | ⋮      | ⋮       | ⋮       |
| 255   | 22250                     | 0.0011  | -0.0003 | 0.00000 |

And the volatility is $\sigma = 0.1846$. So, obtained the value of $\sigma$ used in this study is equal to 0.1846 or 18.46% means that the stock price movement of PT. Bank Central Asia Tbk. (BCA) of 18.46% during the period of January 5, 2017 - January 5, 2018.

### 3.1 Simulation of Stock Option Price Determination Using the Binomial Tree Method

In determining the stock option price using the Binomial Tree method, the first step is to find the value of $E(S_0)$, as follows:

$$E(S_N) = S_0 e^{N\mu \Delta t}$$

| t     | Stock Closing Price (IDR) | $K$ | $C_{IN}$ | $C_{ij}$ | $E(S_N)$ |
|-------|---------------------------|-----|----------|----------|----------|
| 1     | 15600                     | 500 | 15179    | 15175.13 | 15679    |
| 2     | 15350                     | 550 | 15132.99 | 15125.28 | 15682.99 |
| 3     | 15400                     | 600 | 15086.99 | 15075.46 | 15686.99 |
| 4     | 15300                     | 650 | 15040.99 | 15025.66 | 15690.99 |
| ⋮    | ⋮                         | ⋮  | ⋮        | ⋮        | ⋮        |
| 255   | 22.250,00                 | 13200| 3527,718 | 3305,709 | 16727.72 |

So, obtained the European type call option value using the Binomial Tree Method with $K = 500$ and the 3 month maturity is 15,175.13 IDR.
3.2 Simulation of Stock Option Pricing Using the Black Scholes Method

In determining the stock option price using the Black Scholes method, the first step to look for is to calculate the value of $d_1$ and $d_2$. Using equations (18) and (19), it is obtained as follows:

$$d_i = \frac{\ln \left( \frac{S_t}{F} \right) + \left( r - \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}}$$

| $T$ | $S_t$ | $d_1$ | $d_2$ | $N(d_1)$ | $N(d_2)$ |
|-----|-------|-------|-------|----------|----------|
| 1   | 15675 | 37.54847 | 37.45617 | 1.00000000 | 1.00000000 |
| 2   | 15675 | 36.51585 | 36.42356 | 1.00000000 | 1.00000000 |
| 3   | 15675 | 35.57315 | 35.48086 | 1.00000000 | 1.00000000 |
| 4   | 15675 | 34.70595 | 34.61366 | 1.00000000 | 1.00000000 |
| 255 | 15675 | 2.08407 | 1.991773 | 0.98142322 | 0.976901994 |

Furthermore, the option to buy the Black Scholes method is determined as follows:

$$C = S_0 N(d_1) - Ke^{-\mu T} N(d_2)$$

$$C = 15675 \left( 1 - 500. e^{-0.065 \times 0.25} \right) \left( 1 \right)$$

$$C = 15675 - 491.9406595 \left( 1 \right)$$

$$C = 15675 - 491.9406595$$

$$C = 15183.0593$$

So, we get the European type call option value, using the Black Scholes Method, with $K = 500$ and 3 months maturity amounting to 15,183.06 IDR.

4. Conclusion

From the results of the study, it was found that the stock price at the end of the contract is greater than the strike price, this means $St > K$, so that some conclusions are obtained, namely: In the Binomial Tree method, it is possible for investors to exercise their rights and will get a profit equal to the difference between the share price and the strike price when selling the call option of 15,175.13 IDR. In the Black Scholes model, it is possible for investors to use their rights and will benefit as much as the difference between the share price and the strike price, if they sell the call option of 15,183.06 IDR. From the results of the analysis in the discussion of this study it can be concluded that the calculation of the call option price with the Binomial tree method converges close to the calculation of the Black-Scholes model. Based on the value of the Call option price on the Binomial tree method of 15175.13 IDR and the Black Scholes method call option price of 15,183.06 IDR. The greater time partition to $n$ in the binomial tree method, the option value will converge to the option value of the Black-Scholes method.

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