Approximate Simulation for Template-Based Whole-Body Control

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Abstract—Reduced-order template models are widely used to control high degree-of-freedom legged robots, but existing methods for template-based whole-body control rely heavily on heuristics and often suffer from robustness issues. In this letter, we propose a template-based whole-body control method grounded in the formal framework of approximate simulation. Our central contribution is to demonstrate how the Hamiltonian structure of rigid-body dynamics can be exploited to establish approximate simulation for a high-dimensional nonlinear system. The resulting controller is passive, more robust to push disturbances, uneven terrain, and modeling errors than standard QP-based methods, and naturally enables high center of mass walking. Our theoretical results are supported by simulation experiments with a 30 degree-of-freedom Valkyrie humanoid model.

I. INTRODUCTION

Reduced-order “template” models are foundational to controlling high degree-of-freedom legged robots [1], [2]. Such models, like the Linear Inverted Pendulum (LIP), abstract away complicating details of a robot’s dynamics, enabling focus on key aspects of balancing and locomotion, such as regulating the center of mass (CoM) and center of pressure (CoP) [3]. Template models allow for longer planning horizons and reduce computational costs, and in many ways have enabled the expanded use of legged robots in recent years.

Despite the widespread usage of template models, however, surprisingly little is known about formal connections between templates and full-order rigid-body robot models. This means that whole-body controllers used to track template models include many heuristic elements in their design. While such methods can achieve good results in practice, they require extensive parameter tuning and are not always robust.

Addressing the theoretical gap between template and whole-body models, as well as the associated performance issues this gives rise to in practice, is the focus of a growing body of literature. A strict approach was taken in [4], where the dynamics of a Spring-Loaded Inverted Pendulum (SLIP) were embedded as the the Hybrid Zero Dynamics [5] of an asymmetric hopper. Reachability analysis was used in [6] to constrain the template to enforce constraints in the full-order model. Our proposed approach is related most closely to [7], which used the framework of approximate simulation to establish a hierarchical relation between the centroidal dynamics of a multi-link balancer and the LIP model.

Approximate simulation, an extension of simulation/bisimulation from formal methods to continuous systems, ensures that the outputs of two systems can remain $\epsilon$-close [8]. In [7], it was shown that establishing such a relation enables the projection of contact friction constraints to the template model, and the resulting controller enabled better recovery from large push disturbances than a standard approach.

These prior results have several important limitations, however. First, no formal connection was made between the CoM dynamics and the full rigid-body dynamics. Instead, it was assumed that desired CoM accelerations could be enforced via task-space feedback linearization: this enabled the use of existing results on approximate simulation for linear systems. Second, since full rigid-body dynamics were not considered explicitly, the approach of [7] does not account for subtasks essential to locomotion such as controlling swing foot trajectories. Finally, external disturbances arising from contact events and modeling errors were not considered.

In this letter, we address each of these limitations. We establish approximate simulation between a simple linear CoM model and the (nonlinear) whole-body dynamics by drawing on the Hamiltonian structure of rigid-body dynamics. This enables a hierarchical approximate simulation relationship between the LIP model, CoM model, and whole-body model. We propose an optimization-based whole-body controller that enforces approximate simulation, manages complex whole-body constraints, and enables lower priority tasks when possible (e.g., swing foot tracking). We explicitly account for disturbances from impact events and modeling errors using the recently proposed framework of robust approximate simulation [9]. In addition to enforcing approx-
imate simulation, our proposed controller ensures passivity, suggesting better safety and robustness properties [10].

These theoretical results are validated in simulation experiments with a 30 degree-of-freedom (DoF) Valkyrie humanoid. We find that our proposed whole-body controller is more robust to push disturbances, modeling errors, and uneven terrain than a standard whole-body controller. Furthermore, our approach continues to function effectively even in singular configurations, such as with straight legs, enabling more efficient walking with a high CoM.

The remainder of this letter is organized as follows: background on approximate simulation is introduced in Section II. Our main theoretical results are presented in Section III, and simulation experiments are discussed in Section IV. We conclude with Section V.

II. BACKGROUND

Approximate simulation is a relationship between two dynamical systems, $\Sigma_1$ and $\Sigma_2$:

$$\Sigma_1 : \begin{cases} \dot{x}_1 = f_1(x_1, u_1, d) \\ y_1 = g_1(x_1) \end{cases}, \quad \Sigma_2 : \begin{cases} \dot{x}_2 = f_2(x_2, u_2) \\ y_2 = g_2(x_2) \end{cases},$$

where $x_i \in \mathcal{X}_i \subseteq \mathbb{R}^{m_i}$ are the system states, $u_i \in \mathcal{U}_i \subseteq \mathbb{R}^{p_i}$ are the control inputs, and $d \in \mathbb{R}^d$ is a disturbance signal representing external disturbances or modeling errors, and $y_i \in \mathbb{R}^{m_i}$ are the system outputs. Note that the states may be of different sizes but the outputs must be of the same size. We typically consider $\Sigma_1$ to be a more complex system model (e.g., full rigid-body dynamics) and $\Sigma_2$ to be a simpler model (e.g., template dynamics). $\Sigma_1$ approximately simulates $\Sigma_2$ if $\Sigma_1$ can always track the output of $\Sigma_2$ with $\epsilon$ precision:

**Definition 1 (Approximate Simulation):** Let $u_i(t)$ be input signals to $\Sigma_i$ with corresponding state and output trajectories $x_i(t)$ and $y_i(t)$. We say that $\Sigma_1$ approximately simulates $\Sigma_2$ if for every $u_2(t)$ and every $(x_1(0), x_2(0))$ there exists an $\epsilon > 0$ such that if $\|y_1(0) - y_2(0)\|^2 \leq \epsilon$, there exists $u_1(t)$ such that $\|y_1(t) - y_2(t)\|^2 \leq \epsilon$ for $t \geq 0$.

We can certify approximate simulation by finding a Lyapunov-like simulation function $\mathcal{V}$ and an interface $u_V$ [11]. The notion of robust approximate simulation [9] extends these results to account for external disturbances $d$.

**Definition 2 (Robust Simulation Function [9]):** Let $\mathcal{V} : \mathcal{X}_1 \times \mathcal{X}_2 \rightarrow \mathbb{R}^+$ be a smooth function and $u_V : \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{U}_2 \rightarrow \mathcal{U}_1$ be a continuous function. $\mathcal{V}$ is a robust simulation function of $\Sigma_2$ by $\Sigma_1$ and $u_V$ is an associated interface if there exist class-$\mathcal{K}$ functions $\gamma_1, \gamma_2$ such that for all $(x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2$,

$$\mathcal{V}(x_1, x_2) \geq \|g_1(x_1) - g_2(x_2)\|^2$$

and for all $(u_2, d) \in \mathcal{U}_2 \times \mathbb{R}^d$ satisfying $\gamma_1(\|d\|) + \gamma_2(\|u_2\|) < \mathcal{V}(x_1, x_2),$

$$\frac{\partial \mathcal{V}}{\partial x_2} f_2(x_2, u_2) + \frac{\partial \mathcal{V}}{\partial x_1} f_1(x_1, u_V(x_1, x_2, u_2), d) < 0.$$  \hspace{1cm} (3)

\footnote{A function $\gamma : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a class-$\mathcal{K}$ function if it is continuous, strictly increasing, and $\gamma(0) = 0$.}

\textbf{Fig. 2.} Model hierarchy considered in this approach. Our primary contribution is in establishing a robust approximate simulation relation between the intermediate CoM model and the whole-body model.

The conditions (2) and (3) essentially state that the simulation function bounds the output error, and as long as $u_2$ and $d$ are not too large, the simulation function decreases. If $\Sigma_1$ approximately simulates $\Sigma_2$, we can bound the output error $\|y_1 - y_2\|^2$ [8], [9]. Typically, a controller is designed for the simplified model $\Sigma_2$: the concrete model $\Sigma_1$ can also be guaranteed to complete the given task via the interface $u_V$.

The main challenge is finding a Lyapunov-like (robust) simulation function $\mathcal{V}(x_1, x_2)$ and an associated interface $u_V(x_1, x_2, u_2)$. For linear systems, there are well-established conditions for the existence of quadratic simulation functions [9], [11]. Systematic design procedures for nonlinear systems remain an area of ongoing research, though state-of-the-art techniques based on Sum-of-Squares programming [12] do not currently scale to high dimensional systems like legged robots. To address this challenge, we draw inspiration from energy shaping control [10] to find a simulation function.

III. MAIN RESULTS

A. Model Hierarchy

In our proposed control approach, we use the model hierarchy shown in Figure 2. The whole-body model $\Sigma_1$ approximately simulates an intermediate CoM model $\Sigma_2$, which in turn approximately simulates the LIP model $\Sigma_3$.

At the highest level of abstraction, the LIP model $\Sigma_3$ is used to generate a constant-height CoM trajectory consistent with a set of pre-planned footsteps. The LIP dynamics are given by [3]

$$\dot{x}_3 = A_{lip} x_3 + B_{lip} u_3, \quad y_3 = C_{lip} x_3,$$  \hspace{1cm} (4)

where $x_3 = [(p_{com}^{lip})^T (v_{com}^{lip})^T]^T \in \mathbb{R}^6$ is the position and velocity of the CoM, $u_3 \in \mathbb{R}^2$ is the CoP, and $y_3 = p_{com}^{lip}$. At the intermediate level of abstraction, we consider a simple single-integrator model of the CoM,

$$\dot{x}_2 = u_2, \quad y_2 = x_2,$$  \hspace{1cm} (5)

where $x_2 \in \mathbb{R}^3$ represents the CoM position. This model adds flexibility over the LIP in the sense that the CoM is not constrained to a constant height. Noting that (4) and (5) are linear systems, the interface

$$u_2 = v_{com}^{lip} + K(x_2 - p_{com}^{lip})$$  \hspace{1cm} (6)

enforces a robust approximate simulation relation between systems (4) and (5), where $K$ is any Hurwitz matrix [9].
At the lowest level of abstraction, we consider the full multi-contact rigid-body dynamics of an \( n \)-link robot

\[
M(q)\ddot{q} + C(q, \dot{q})\dot{q} + \tau_g = S^T \tau + \sum J_{cj}^T f_{cj} + \tau_{ext},
\]

where \( q \in \mathbb{R}^{n+6} \) are generalized coordinates and \( \tau \) are controlled joint torques. \( M(q) \) is the positive definite mass matrix, \( C(q, \dot{q}) \dot{q} \) collects Coriolis and centripetal terms, \( \tau_g \) is the torque due to gravity, \( f_{cj} \in \mathbb{R}^6 \) are forces at contact points \( c_j \), \( J_{cj} \) are the associated contact Jacobians, and \( \tau_{ext} \) are external (disturbance) torques.

These whole-body dynamics form a nonlinear system with

\[
x_1 = \begin{bmatrix} q & \dot{q} \end{bmatrix}^T, \quad y_1 = x_{com}(q),
\]

where \( x_{com} \) is the position of the CoM, and \( J \) is the CoM Jacobian (\( x_{com} = J\dot{q} \)).

### B. Energy Shaping for Approximate Simulation

As discussed in Section \[\text{II}\] establishing approximate simulation for nonlinear systems is a difficult and open problem. In this section, we show that the Hamiltonian structure of \( \Sigma_1 \) can be used to certify that \( \Sigma_1 \) approximately simulates the intermediate CoM model \( \Sigma_2 \).

We first introduce the control transformation

\[
u_1 = S^T \tau + \sum J_{cj}^T f_{cj}.
\]

The resolution of particular joint torques \( \tau \) and contact forces \( f_{cj} \) will be discussed in Section \[\text{III-C}\]. With this control transformation, energy shaping principles can be used to design a simulation function and interface, as follows:

**Theorem 1:** Assume that \( x_1 \) and \( x_2 \) are constrained to compact sets \( X_1, X_2 \). Assume that \( \|u_2\| \leq \delta > 0 \). Let \( \kappa > 0 \) be a scalar constant, and \( K_D > \max_{x_1 \in X_1, x_2 \in X_2} \left\{ \frac{1}{\kappa}M(q) \right\} \) be a symmetric positive definite matrix. Then the interface

\[
u_1 = \tau_g - 2\kappa S^T (y_1 - y_2) - K_D(q - J^T u_2)
\]

certifies that \( \Sigma_1 \) robustly approximately simulates \( \Sigma_2 \) with the associated simulation function

\[
\mathcal{V}(x_1, x_2) = \frac{1}{2\kappa} q^T M(q) \dot{q} + \|y_1 - y_2\|^2,
\]

being the rescaled closed-loop Hamiltonian under (10).

To prove Theorem 1 we first need the following result regarding the structure of the rigid-body dynamics (7):

**Lemma 1:** There exists a matrix \( C(q, \dot{q}) \) such that (7) holds and \( M(q, \dot{q}) - 2C(q, \dot{q}) \) is skew-symmetric.

**Proof:** [Theorem 1] First, define the desired potential

\[
U^{des}(x_1, x_2) = \|y_1 - y_2\|^2.
\]

Since the kinetic energy term \( \frac{1}{2\kappa} q^T M(q) \dot{q} \) is non-negative, \( \mathcal{V} \) always bounds the output error \( U^{des} \), and thus \( \mathcal{V} \) satisfies the first condition for robust approximate simulation (2).

Next, we will show that \( \mathcal{V}(x_1, x_2) \) decreases along system trajectories as long as \( u_2 \) and \( \tau_{ext} \) are small:

\[
\dot{\mathcal{V}} = \frac{1}{2\kappa} q^T M(q) \dot{q} + \frac{1}{\kappa} q^T M(q) \ddot{q} + \frac{\partial U^{des}}{\partial \dot{q}} \dot{q} + \frac{\partial U^{des}}{\partial \dot{x}_2} \ddot{x}_2
\]

\[
= \frac{1}{2\kappa} q^T M(q) \dot{q} + \frac{1}{\kappa} q^T (u_1 + \tau_{ext} - C(q, \dot{q}) \dot{q} - \tau_g)
\]

\[
+ \frac{\partial U^{des}}{\partial \dot{q}} \dot{q} + \frac{\partial U^{des}}{\partial \dot{x}_2} u_2
\]

\[
= \frac{1}{\kappa} q^T (u_1 + \tau_{ext} - \tau_g) + \frac{\partial U^{des}}{\partial \dot{q}} \dot{q} + \frac{\partial U^{des}}{\partial \dot{x}_2} u_2
\]

by Lemma 1. Applying the interface (10) and noting that \( \frac{\partial U^{des}}{\partial \dot{q}} = 0 \), we have that

\[
\dot{\mathcal{V}} = \frac{1}{\kappa} (JK_D \dot{q})^T u_2 + \frac{1}{\kappa} q^T \tau_{ext} + \frac{\partial U^{des}}{\partial \dot{x}_2} u_2 - \frac{1}{\kappa} q^T K_D \dot{q}.
\]

Let \( \gamma_1 \) and \( \gamma_2 \) be class-\( \mathcal{K} \) functions defined by

\[
\gamma_1(s) = \max_{x_1 \in X_1} \left\{ \frac{1}{\kappa}\|s\| \right\} s
\]

\[
\gamma_2(s) = \max_{x_1 \in X_1, x_2 \in X_2} \left\{ \frac{1}{\kappa}\|JK_D \dot{q}\| + 2\|y_1 - y_2\| + \frac{1}{\delta} U^{des} \right\} s.
\]

If \( \gamma_1(\|\tau_{ext}\|) + \gamma_2(\|u_2\|) \leq \mathcal{V}(x_1, x_2) \), then we have that for any \( x_1 \in X_1, x_2 \in X_2 \),

\[
\frac{1}{\kappa} \|\dot{q}\| \|\tau_{ext}\| + \frac{1}{\delta} U^{des} \|u_2\| + 2\|y_1 - y_2\| \|u_2\|
\]

\[
+ \frac{1}{\kappa} \|JK_D \dot{q}\| \|u_2\| \leq \frac{1}{2\kappa} q^T M\dot{q} + U^{des}
\]

\[
\frac{1}{\kappa} q^T \tau_{ext} + 2(y_2 - y_1)^T u_2 + \frac{1}{\kappa} |JK_D \dot{q}|^T u_2 \leq \frac{1}{2\kappa} q^T M\dot{q},
\]

\[
\frac{1}{\kappa} q^T \tau_{ext} + \frac{\partial U^{des}}{\partial \dot{x}_2} u_2 + \frac{1}{\kappa} |JK_D \dot{q}|^T u_2 \leq \frac{1}{\kappa} q^T K_D \dot{q},
\]

which by (13) implies that \( \dot{\mathcal{V}} \leq 0 \). This satisfies the second condition (3) for robust approximate simulation, and thus the Theorem holds.

The basic idea behind this Theorem is to formulate a desired potential (12) which has a minimum when \( y_1 = y_2 \). The energy-shaping-inspired interface (10) is analogous to a PD controller with gravity compensation. We can then use the resulting (scaled) closed-loop Hamiltonian as a simulation function (11). This is because the closed-loop Hamiltonian bounds the output error, has a minimum when \( y_1 = y_2 \), and decreases for small \( u_2 \) and \( \tau_{ext} \).

**Remark 1:** While the error bound resulting from [11, Theorem 1] and the \( \gamma_1, \gamma_2 \) presented above is not tight, the simulation function \( \mathcal{V} \) does provide a tight error bound. Techniques for finding a smaller \( \epsilon = \max_{x_2} \mathcal{V} \) are outside the scope of this letter, and will be a focus of future work.

**Remark 2:** While a \( u_1 \) that tracks a nominal CoM trajectory could also be chosen using task-space feedback linearization, our approach does not require a left inverse of the CoM Jacobian \( J \). This means that our interface (10) is effective even when the robot is in a singular configuration (i.e., when \( J \) is not full rank, such as when the legs are
fully extended). This enables walking with a higher CoM, as shown in Section [4].

Beyond the fact that the interface [10] and simulation function [11] certify a robust approximate simulation relation, they also enforce passivity:

**Corollary 1:** The interface [10] ensures passivity of the closed-loop system [5] with respect to inputs to the CoM system \( u_2 \) and external disturbances \( \tau_{ext} \).

**Proof:** Consider the closed-loop Hamiltonian

\[
H = \frac{1}{2} \dot{q}^T M(q) \dot{q} + \kappa \| y_1 - y_2 \|^2,
\]

which is a scaled version of the simulation function [11]. From (13), we have that

\[
\dot{H} \leq (2\kappa \| y_1 - y_2 \|^2 + [JK_D \dot{q}]^T) u_2 + \dot{q}^T \tau_{ext}. \tag{14}
\]

For \( u_2 = 0 \), this certifies passivity of the mapping from external disturbances \( \tau_{ext} \) to joint velocities \( \dot{q} \). Similarly, in the absence of disturbances \( \tau_{ext} = 0 \), (14) certifies a passive relationship between the input to the reduced-order model \( u_2 \) and a unusual output \((2\kappa \| y_1 - y_2 \| + JK_D \dot{q})\), composed of a scaled output error \((2\kappa \| y_1 - y_2 \|)\) and a scaled CoM velocity \((JK_D \dot{q})\).

These passivity relationships essentially regulate how additional energy enters the system. This property has important implications for safety and robustness. Specifically, the feedback interconnection of passive systems is always passive, a property that does not always hold for general notions of stability [10]. Furthermore, passivity-based controllers tend to be remarkably robust to modeling errors and external disturbances [14]–[18], a critical property for operation in uncertain real-world settings.

Combining Theorem [4] with existing results for linear systems [9], [11], we can obtain the following main result:

**Theorem 2:** The whole-body model \( \Sigma_1 \) robustly approximately simulates the LIP model \( \Sigma_3 \).

**Proof:** Following [9], we have that the interface [6] ensures that \( \Sigma_2 \) robustly approximately simulates \( \Sigma_3 \). Let \( \epsilon_1 \) denote the associated error bound. By Theorem [1] \( \Sigma_1 \) robustly approximately simulates \( \Sigma_2 \). Let \( \epsilon_2 \) denote the associated error bound. Then for any \( u_3(t) \),

\[
\| y_1 - y_3 \|^2 \leq \| y_1 - y_2 \|^2 + \| y_2 - y_3 \|^2 \leq \epsilon_1 + \epsilon_2 = \epsilon, \tag{15}
\]

and thus the Theorem holds.

### C. Optimization-Based Whole-Body Control

In the previous section, we showed how to select \( u_1 \) to track the intermediate CoM model while ensuring approximate simulation and passivity. But choosing joint torques consistent with this \( u_1 \) while meeting torque limits and contact friction constraints is non-trivial. To do so, we propose the following QP:

\[
\min_{u_2 \in \mathbb{R}^{n_{\tau_{res}}}, \dot{q}, \tau, f_j} \quad w_1 \| u_2 - u_2^{des} \|^2 + w_2 \| J_{foot} \dot{q} + J_{foot} \dot{q} - \dot{x}_{foot}^{des} \|^2 + \ldots \tag{16}
\]

\[
\text{s.t. } M\dot{q} + C\dot{q} + \tau(q) = S^T \tau + \sum J_{cj}^T f_j \tag{18}
\]

\[
u_y(x_1, x_2, u_2) + N^T \tau_0 = S^T \tau + \sum J_{cj}^T f_j \tag{19}
\]

\[
J_{y}\dot{q} + J_{\dot{q}} \dot{q} = 0 \tag{20}
\]

\[
f_j \in \text{friction cones} \tag{21}
\]

\[
\tau \leq \tau \leq \tilde{\tau} \tag{22}
\]

The primary cost function [19] attempts to ensure that \( u_2 \) matches a nominal input \( u_2^{des} \), given by the interface with the LIP model [6]. Additional weighted costs like [17] can be added to regulate lower-priority tasks like tracking a desired swing foot trajectory, minimizing angular momentum, and regulating torso orientation.

The constraint (18) enforces the robot’s dynamics, and (for fixed \( \dot{q}, \ddot{q} \)) is linear in the decision variables. [19] ensures that the robot’s CoM motion is determined by the interface [10], which is linear in \( u_2 \). We use the dynamically consistent null-space projector \( \mathbb{N} \) [19] to add some flexibility—all joint torques are not strictly determined by [10]—while ensuring that the CoM motion matches that requested by the interface. This enables lower-priority tasks like swing foot control. Contact points are fixed in place by (20), while (21) ensures that contact forces remain within their respective (Coulomb) friction cones. This friction constraint is linear if pyramidal inner approximations of the friction cones are taken. Finally, (22) enforces torque limits.

This proposed optimization scheme is closely related to the QPs widely used for whole-body control of high-DoF legged robots [1]. In such schemes, a desired CoM acceleration

\[
\dot{x}_{com}^{des} = \dot{v}_{com}^{lip} + K_{\dot{p}}^{lip} (\dot{v}_{com}^{lip} - \dot{x}_{com}^{des}) + K_D^{lip} (p_{com}^{lip} - x_{com}^{des}) \tag{23}
\]

which tracks the LIP is encoded as a cost, subject to similar dynamics, friction, contact, and torque constraints.

As we will shown in Section [4], our approach is more robust to large push disturbances, uneven terrain, and modeling errors than these standard QPs. This is due to the fact that the interface constraint [19] enforces passivity properties (see Corollary [1]), regulating the energy that can be injected into the system. At the same time, [19] is minimally restrictive. Intuitively, \( u_2 \) regulates the CoM, while \( \tau_0 \) regulates all other (non-CoM) degrees of freedom, allowing the left hand side of [19] to take arbitrary values if needed.

### TABLE I

**PUSH RECOVERY SIMULATION RESULTS (\( x = \text{FAIL} \), \( \checkmark = \text{SUCCESS} \)).**

| Trial | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-------|---|---|---|---|---|---|---|---|---|---|----|
| Ours  | x | x | x | x | x | x | x | x | x | x | x  |
| Standard | x | x | x | x | x | x | x | x | x | x | x  |

### TABLE II

**UNEVEN TERRAIN SIMULATION RESULTS (\( x = \text{FAIL} \), \( \checkmark = \text{SUCCESS} \)).**

| Trial | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-------|---|---|---|---|---|---|---|---|---|---|----|
| Ours  | x | x | x | x | x | x | x | x | x | x | x  |
| Standard | x | x | x | x | x | x | x | x | x | x | x  |
nominal input to the CoM model, $u$, initial CoM position. The interface (6) was used to select a QP (16) was solved to select joint torques. We used position. The CoM model $x$ and $5000$ LIP model was generated, holding constant at the desired position. The CoM model $x_2$ was initialized to match the true initial CoM position. The interface (6) was used to select a nominal input to the CoM model, $u_2$. Then the whole-body QP (16) was solved to select joint torques. We used $k = 5000$ and $K_D = 1000I$. The results are shown in Figure 3. The robot’s CoM moves smoothly to the desired position, while the simulation function (11) bounds the output error and decreases except when $u_2$ is large.

To evaluate our proposed whole-body controller, we compare the performance of our approach with a standard whole-body QP. For the standard QP, we generally followed the approach of [22], using a LIP reference rather than a SLIP reference. We used a QP version of the controller presented in [23] with the same weighted tasks as our approach. This method of tracking a template model with a QP-based whole-body controller is used broadly for humanoid control [1].

Both methods used the same tuning parameters wherever applicable. For the standard approach, the parameters of [23] were tuned to match the step response of our approach. For each scenario, we specified a sequence of footsteps and swing foot trajectories a-priori. We then generated a corresponding LIP model trajectory [3], and tracked it using either our approach or the standard whole-body QP.

We considered four application scenarios as shown in Figure 1 recovering from push disturbances, walking over uneven terrain, walking despite modeling errors, and walking with a high CoM.

To test push recovery, we applied randomly generated pushes while the robot was walking. The time of each push was sampled uniformly from $[400, 600]$N and applied for $0.05s$ in the $+y$ or $-y$ direction, each with $50\%$ probability. A trial was considered successfully if the robot could walk for $5s$ total without falling. The results are shown in Table [1] The same push was applied for both approaches in each trial. Our approach was successful in $7$ of the $11$ trials, while the standard approach was only successful in $2$.

Next, we tested robustness to uneven terrain by randomly placing $15$ bricks in the robot’s path. These bricks were fixed in place, but were smaller than the robot’s foot, leading to frequent shifting of the stance foot while walking. A trial was considered successful if the robot could cross the bricks without falling. The results are shown in Table [1] with the same brick placement used for both approaches in each trial. Our approach was successful in all but $2$ trials, while the standard approach was not successful in any of the trials.

Empirically, the primary failure mode for our approach was when the swing foot got caught behind a brick. In contrast, the primary failure mode for the standard approach was when the stance foot shifted significantly, an event that occurred many times in each trial. The passivity of our approach helps explain our method’s success in this regard: a shift in stance foot essentially injects energy into the system. Our controller’s passivity (see Section III-B) ensures that the total system energy does not grow unbounded as a result.

Our third test scenario dealt with modeling error. In this scenario, the controller assumed the same robot model as in the trials above, but the true robot model was perturbed randomly. Specifically, the mass of each link, $m_i$, was modified according to

$$m_i^{\text{new}} = \max \{ \epsilon, m_i + \epsilon r \}, \quad r \sim \mathcal{N}(0, 1).$$

A trial was considered successful if the robot could walk for $5s$ without falling. The same randomized model was used for both approaches in each trial. The results are shown in Table [1] Our approach was successfully in every case that the standard approach was successful, as well as four cases in which the standard approach failed.

In the final test scenario, we demonstrated that our approach can be used for walking with a higher CoM. Walking with a high CoM reduces torques at the knees, enabling more efficient locomotion [24]. Standard whole-body control approaches do not perform well in configurations like that shown in Figure 1d, where the CoM is high enough that the back leg is fully extended. In such near-singular

![Simulation Function](https://github.com/vincekurtz/valkyrie_drake)

Fig. 3. Using our approach to regulate the CoM while standing.

**IV. SIMULATION RESULTS**

We implemented the controller described above in simulation using a 30 DoF model of the NASA Valkyrie humanoid. Our implementation[1] is in Python, using the Drake [20] simulation and dynamics platform. The whole-body QP was solved at $200Hz$ with Gurobi [21].

First, we consider a scenario in which the robot moves its CoM to a desired position ($[-0.01 0.05 0.85]m$) while balancing in double support. First, a nominal trajectory of the LIP model $x_3$ was generated, holding constant at the desired position. The CoM model $x_2$ was initialized to match the true initial CoM position. The interface (6) was used to select joint torques. We used $\kappa = 5000$ and $K_D = 1000I$. The results are shown in Figure 3. The robot’s CoM moves smoothly to the desired position, while the simulation function (11) bounds the output error and decreases except when $u_2$ is large.

To evaluate our proposed whole-body controller, we compare the performance of our approach with a standard whole-body QP. For the standard QP, we generally followed the approach of [22], using a LIP reference rather than a SLIP reference. We used a QP version of the controller presented in [23] with the same weighted tasks as our approach. This method of tracking a template model with a QP-based whole-body controller is used broadly for humanoid control [1].

Both methods used the same tuning parameters wherever applicable. For the standard approach, the parameters of [23] were tuned to match the step response of our approach. For each scenario, we specified a sequence of footsteps and swing foot trajectories a-priori. We then generated a corresponding LIP model trajectory [3], and tracked it using either our approach or the standard whole-body QP.

We considered four application scenarios as shown in Figure 1 recovering from push disturbances, walking over uneven terrain, walking despite modeling errors, and walking with a high CoM.

To test push recovery, we applied randomly generated pushes while the robot was walking. The time of each push was sampled uniformly from $[1.0, 3.5]s$. The push’s magnitude was sampled uniformly from $[400, 600]$N and applied for $0.05s$ in the $+y$ or $-y$ direction, each with $50\%$ probability. A trial was considered successfully if the robot could walk for $5s$ total without falling. The results are shown in Table I The same push was applied for both approaches in each trial. Our approach was successful in $7$ of the $11$ trials, while the standard approach was only successful in $2$.

Next, we tested robustness to uneven terrain by randomly placing $15$ bricks in the robot’s path. These bricks were fixed in place, but were smaller than the robot’s foot, leading to frequent shifting of the stance foot while walking. A trial was considered successful if the robot could cross the bricks without falling. The results are shown in Table I with the same brick placement used for both approaches in each trial. Our approach was successful in all but $2$ trials, while the standard approach was not successful in any of the trials.

Empirically, the primary failure mode for our approach was when the swing foot got caught behind a brick. In contrast, the primary failure mode for the standard approach was when the stance foot shifted significantly, an event that occurred many times in each trial. The passivity of our approach helps explain our method’s success in this regard: a shift in stance foot essentially injects energy into the system. Our controller’s passivity (see Section III-B) ensures that the total system energy does not grow unbounded as a result.

Our third test scenario dealt with modeling error. In this scenario, the controller assumed the same robot model as in the trials above, but the true robot model was perturbed randomly. Specifically, the mass of each link, $m_i$, was modified according to

$$m_i^{\text{new}} = \max \{ \epsilon, m_i + \epsilon r \}, \quad r \sim \mathcal{N}(0, 1).$$

A trial was considered successful if the robot could walk for $5s$ without falling. The same randomized model was used for both approaches in each trial. The results are shown in Table I Our approach was successfully in every case that the standard approach was successful, as well as four cases in which the standard approach failed.

In the final test scenario, we demonstrated that our approach can be used for walking with a higher CoM. Walking with a high CoM reduces torques at the knees, enabling more efficient locomotion [24]. Standard whole-body control approaches do not perform well in configurations like that shown in Figure 1d, where the CoM is high enough that the back leg is fully extended. In such near-singular
This flexibility reduces the burden on template trajectories model (desired LIP CoM height is too high, and so the intermediate in this letter will continue to enable approximate simulation extensions to nonlinear template models. In this last regard, focus on implementing these methods on a real robot and on errors, and allows for high CoM walking. Future work will methods to push disturbances, uneven terrain, and modeling performance, our approach is passive, more robust than standard our approach was successful with a higher CoM, enabling lower energy walking.

High CoM walking is facilitated by the fact that approximate simulation allows for some deviation between the LIP model CoM position ($y_3$), the nominal CoM position ($y_2$), and the true CoM position ($y_1$). These three quantities are plotted in Figure 4. When the CoM is above the stance foot in single support, the desired LIP CoM height can be achieved ($t \approx \{3.5, 5.5, 7.5, 9.5\}$s). In double support, however, the desired LIP CoM height is too high, and so the intermediate model ($x_2$) height is adjusted downward ($t \approx \{3, 5, 7, 9\}$s). This flexibility reduces the burden on template trajectories of being exactly replicated by the whole-body model, and is a natural byproduct of approximate simulation relationships.

V. CONCLUSION

We introduced a template-based whole-body control strategy that enforces approximate simulation between the LIP model, a linear CoM model, and the whole-body dynamics. In addition to enabling formal guarantees of tracking performance, our approach is passive, more robust than standard methods to push disturbances, uneven terrain, and modeling errors, and allows for high CoM walking. Future work will focus on implementing these methods on a real robot and on extensions to nonlinear template models. In this last regard, we anticipate that the energy-based approaches put forward in this letter will continue to enable approximate simulation strategies to scale to more complex systems.

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