Dynamics of Spherically Symmetric Gravitational Collapse in $f(R)$ Theory

M. Sharif† and H. Rizwana Kausar ‡
Department of Mathematics, University of the Punjab,
Quaid-e-Azam Campus, Lahore-54590, Pakistan
E-mail: † msharif.math@pu.edu.pk
‡ rizwa_math@yahoo.com

Abstract. We study dynamics of spherically symmetric gravitational collapse in modified $f(R)$ gravity. The matter filled inside the collapsing body is of two types: a) the usual matter that consists of isotropic non-viscous fluid satisfying the energy conditions; b) the matter for which the effective stress-energy tensor contains higher order curvature derivatives violating the null energy conditions. Thus the higher order curvature terms are interpreted as the gravitational fluid or $f(R)$ dark energy. We discuss the effects of such type of dark energy on the coupling of dynamical and transport equations.

Keywords: $f(R)$ dark energy; Dynamical equations; Transport equation.

PACS: 04.50.Kd

1. Introduction
What is the origin of the perceived dark energy (DE) component which drives the acceleration? This is a well-posed and interesting question which has gained much attraction recently. Several radically different candidates have been proposed to explain it ranging from the introduction of a cosmological constant [1] in General Relativity (GR) to the development of modified theories of gravity [2]. In the latter context, one may assume that at large scales, GR breaks down and a more general function describes the gravitational field. The task of these modifications is to fit the astrophysical data without adding exotic ingredients.

Recently, many generalizations/modifications have been proposed. In particular, there is a considerable interest in generalizing the Ricci scalar curvature $R$ with an arbitrary function of Ricci scalar, $f(R)$, in the Einstein-Hilbert action. This task was initially developed by Buchdahl [3] and further extended by some other people [4]. Such a modification successfully describes the DE epoch and passes the solar system tests [5]-[9]. It is remarkable that even the form of $f(R)$ gravity may be reconstructed from the known universe [10]. Hence, this approach may suggest gravitational alternative for DE.

The action for the modified $f(R)$ theory of gravity is given by

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(R).$$

(1)

Here the presence of $f(R)$ function may be understood as the introduction of an effective fluid which is not restricted to hold the usual energy conditions [11]. In this way, DE can be thought
of as having the geometrical origin rather than some additional scalar fields which are added by hand to the matter part. It is also possible to show that \( f(R) \) theory can play a major role at astrophysical scales due to modification of gravitational potential in the low energy limit. Such type of modified potential offers the possibility to fit galaxy rotation and galaxy cluster potential without need of huge amount of dark matter [12].

In this work, we are concerned with spherically symmetric stars to investigate how \( f(R) \) dark energy affects the process of gravitational collapse. To carry out this work, we have taken care of the literature [13, 14]. The scheme of the paper is as follows. In next section 2, we present some basic material useful for this work. Section 3 is devoted to formulate the dynamical equations in \( f(R) \) gravity. In section 4, the transport equation and the coupling with the dynamical equations is given. The last section 5 provides a summary and discussion of the results.

2. \( f(R) \) Field Equations

We consider the spherically symmetric line element as follows

\[
\begin{align*}
    ds^2 &= A^2(t, r)dt^2 - B^2(t, r)dr^2 - C^2(t, r)(d\theta^2 + \sin^2 \theta d\phi^2).
\end{align*}
\]  

(2)

An isotropic distribution of matter is assumed inside the collapsing sphere which satisfies the energy conditions and undergoes dissipation in the form of heat flux. It is given by

\[
T^m_{\alpha\beta} = (\rho + p)u_\alpha u_\beta - pg_{\alpha\beta} + u_\alpha q_{\beta} + q_\alpha u_\beta, \quad (\alpha, \beta = 0, 1, 2, 3).
\]  

(3)

Here, we have taken \( \rho \) as the energy density, \( p \) the pressure, \( q_\alpha \) the heat flux and \( u_\alpha \) the four-velocity of the fluid. These quantities satisfy the relations

\[
u^\alpha u_\alpha = 1, \quad u^\alpha q_\alpha = 0.
\]  

(4)

Using the metric approach, i.e., varying the action in Eq.(1) with respect to\( g_{\alpha\beta} \) provides the following field equations

\[
F(R)R_{\alpha\beta} - \frac{1}{2}f(R)g_{\alpha\beta} - \nabla_\alpha \nabla_\beta F(R) + g_{\alpha\beta}F(R) = \kappa T_{\alpha\beta},
\]  

\[
F(R) = df(R)/dR.
\]  

(5)

By taking all the higher order corrections to the curvature on the right hand side of the field equations and defining it as a "dark source" term or "effective fluid" [15], this equation can be written as

\[
G_{\alpha\beta} = \frac{\kappa}{F}(T^{(m)}_{\alpha\beta} + T^{(D)}_{\alpha\beta}),
\]  

(6)

where

\[
T^{D}_{\alpha\beta} = \frac{1}{\kappa} \left[ \frac{f(R) + RF(R)}{2} g_{\alpha\beta} + \nabla_\alpha \nabla_\beta F(R) - g_{\alpha\beta}F(R) \right].
\]  

(7)

This makes it easy to compare \( f(R) \) theory with GR. For the metric (2), the field equations (5) take the following form

\[
\left( \frac{2B}{B} + \frac{\dot{C}}{C} \right) \frac{\dot{C}}{C} - \left( \frac{A}{B} \right)^2 \left[ \frac{2C''}{C} + \left( \frac{C'}{C} \right)^2 - \frac{2B'C'}{BC} - \left( \frac{B}{C} \right)^2 \right] = \frac{\kappa}{F} \rho A^2,
\]  

\[
+ \frac{A^2}{\kappa} \left\{ \frac{f - RF}{2} + \frac{E''}{B^2} + \left( \frac{2C'}{C} - \frac{\dot{B}}{B} \right) \frac{\dot{F}}{A^2} + \left( \frac{2C'}{C} - \frac{\dot{B}}{B} \right) \frac{F''}{B^2} \right\},
\]  

\[
-2 \left( \frac{C'}{C} - \frac{CA'}{CA} - \frac{B'C'}{BC} \right) = \frac{\kappa}{F} \left[ -qAB + \frac{1}{\kappa} \left( \dot{F}' - \frac{A'}{A} \dot{F} - \frac{\dot{B}}{B} F' \right) \right],
\]  

(8)

(9)
which dot and prime represent derivatives with respect to \( t \) and \( r \) respectively. The mass function introduced by Misner and Sharp is defined as follows [16]

\[
m(t, r) = \frac{C}{2} (1 + g^{\mu\nu} C_{,\mu} C_{,\nu}) = \frac{C}{2} \left( 1 + \frac{\dot{C}^2}{A^2} - \frac{C'^2}{B^2} \right)
\]

which provides total energy inside a spherical body of radius "\( C \)".

3. Dynamical Equations

The dynamical equations can be obtained from the contracted Bianchi identities [14]. Consider the following two equations

\[
\left( T^{(m)\alpha\beta} + T^{(D)\alpha\beta} \right)_{;\beta} u_{\alpha} = 0, \quad \left( T^{(m)\alpha\beta} + T^{(D)\alpha\beta} \right)_{;\beta} \chi_{\alpha} = 0
\]

which yield respectively

\[
\frac{\dot{\rho}}{A} + \frac{1}{A} (\rho + p) \left( \frac{\dot{B}}{B} + \frac{2\dot{C}}{C} \right) + \frac{q}{B} \left( \frac{A'}{A} + \frac{2C'}{C} \right) + \frac{q'}{B} + D_1,
\]

\[
\frac{\dot{q}}{A} + \frac{2q}{A} \left( \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) + \frac{A'}{AB} (\rho + p) + \frac{q'}{B} + D_2,
\]

where \( D_1 \) and \( D_2 \) are given in Appendix. The proper time and radial derivatives are given by

\[
D_T = \frac{1}{A} \frac{\partial}{\partial t}, \quad D_C = \frac{1}{C} \frac{\partial}{\partial r}.
\]

The velocity of the collapsing fluid is defined by the proper time derivative of \( C \), i.e.,

\[
U = D_T C = \frac{\dot{C}}{A}
\]

which is always negative in the case of collapse. Using this definition, we can write Eq.(12) as follows

\[
E = \frac{C'}{B} \left[ 1 + U^2 + \frac{2m}{C} \right]^{1/2}.
\]

The variation of total energy of the collapsing sphere of radius \( C \) with respect to proper time can be obtained by applying Eq.(16) to (12) along with Eqs.(9), (10) and (18) as follows

\[
D_T m = \frac{-C'^2}{2} \left[ \left( \frac{\kappa}{Fp} - \frac{1}{4C^2} - D_3 \right) U + \left( \frac{\kappa q}{F} + D_4 \right) E \right]
\]
where $D_3$ and $D_4$ are written in Appendix. In Eq. (19), terms inside the first round brackets increase the energy density through the rate of work being done by the effective pressure $p$ and the DE in $D_3$ whose negative effect is balanced with collapsing velocity $U$. The terms in the second round brackets have negative sign indicating that energy is leaving the system due to heat flux and repulsive effect of curvature terms. Similarly, the variation of energy between adjacent spherical surfaces inside the considered matter distribution can be obtained by using Eqs. (8) and (9) as follows

$$D_{Cm} = \frac{C^2}{2} \left[ \left( \frac{\kappa q}{F} + D_4 \right) \frac{U}{E} + \frac{\kappa \rho}{F} + D_5 \right],$$

(20)

where $D_5$ being the dark source term given in Appendix. Equation (20) shows the contribution of energy density, heat flux with the coefficients and $D_4$, $D_5$ containing higher order curvature corrections (DE) which affect the adjacent layers of collapsing sphere.

The acceleration $D_T U$ of the collapsing matter inside the spherical body is found by using Eqs. (10), (12), (17) and (18) as follows

$$D_T U = \frac{A'}{AB} E - \frac{\kappa}{2F} \left[ p - \frac{1}{\kappa} \left\{ \frac{f(R) - RF}{2} - \dot{F} \right\} \frac{\dot{F}}{A^2} \right. $$

$$\left. + \frac{\dot{F}}{A^2} \left( \frac{\dot{A}}{A} + \frac{2\dot{C}}{C} \right) + \frac{F'}{B^2} \left( \frac{A'}{A} + \frac{2C'}{C} \right) \right].$$

(21)

This equation implies that

$$\frac{A'}{AB}(\rho + p) = \frac{(\rho + p) E}{2} D_T U + \frac{(\rho + p) \kappa}{2FE} \left[ p - \frac{1}{\kappa} \left\{ \frac{f(R) - RF}{2} - \dot{F} \right\} \right.$$  

$$\left. + \frac{\dot{F}}{A^2} \left( \frac{\dot{A}}{A} + \frac{2\dot{C}}{C} \right) + \frac{F'}{B^2} \left( \frac{A'}{A} + \frac{2C'}{C} \right) \right].$$

(22)

Replacing this value with the third term of Eq. (15), we get

$$(\rho + p) D_T U = - (\rho + p) D_6 - E^2 D_C p$$  

$$- E \left[ 2 q \left( \frac{D_T B}{B} - \frac{D_T C}{C} \right) + D_T q \right] + D_7,$$

(23)

where $D_6$ and $D_7$ are provided in Appendix. Equation (23) represents the role of different forces including DE on the collapsing process. It can be interpreted in the form of Newton’s second law of motion, i.e., Force = inertial mass $\times$ acceleration. The term $\rho + p$ on the LHS indicates inertial mass density multiplying with acceleration $D_T U$. The first term on the RHS is the same and can be interpreted as an active gravitational mass by the equivalence principle. We see that how this mass is affected by DE. The terms in the second square brackets appear with negative sign enhancing the collapsing rate which show the gradient of the pressure. Therefore this term would increase the rate of collapse. The terms in the last square brackets play the role of heat flux and a strong contribution of DE in $D_7$ creating repulsive effect on the surroundings, hence reducing the rate of collapse. This leads to a point that spherical body may not collapse fully in any case due to immense of DE terms in Eq. (23), whatsoever gravity pulls towards the center.
4. Transport Equation

Here the transport equation for heat flux is derived using the Müller-Israel-Stewart theory of dissipative fluids [17]. It helps to study the transfer of mass, heat and momentum during the matter collapse. This is given by

\[ \tau h^{\alpha\beta}u^\gamma q_\gamma + q^\alpha = -\eta h^{\alpha\beta}(T_\beta + a_\beta T) - \frac{1}{2}\eta T^2 \left( \frac{\tau u^\beta}{\eta T^2} \right)_\beta q^\alpha, \]  

(24)

where \( h^{\alpha\beta} = g^{\alpha\beta} - u^\alpha u^\beta \) is the projection tensor, \( \eta \) denotes the thermal conductivity, \( T \) is the temperature, \( \tau \) stands for relaxation time which is the time taken by a perturbed system to return into an equilibrium state and \( a_\beta T \) is the Tolman inertial term with \( a_\alpha = u_\alpha u^\beta \) being the acceleration. The independent component of this equation is given by

\[ \tau \dot{q} = -qA - \frac{1}{2} \frac{\eta q T^2}{\tau} \left( \frac{\tau}{\eta T^2} \right)_\beta \frac{B}{B} + 2 \frac{\dot{C}}{C} + \frac{\eta A^2}{B} \left( \frac{T}{A} \right)' \].

(25)

Substituting the value of \( \dot{A} \) from Eq.(22) and using Eqs.(17) and (18), it follows that

\[ \begin{align*}
D_T q &= -\frac{\eta T^2 q}{2\tau} D_T \left( \frac{\tau}{\eta T^2} \right) - \frac{q}{2} \left( \frac{D_T B}{B} + 2 \frac{D_T C}{C} + \frac{1}{\tau} \right) + \frac{\eta E}{\tau} D_C T \\
&- \frac{\eta T}{\tau E} D_T U - \frac{\eta T}{\tau E} D_6.
\end{align*} \]

(26)

Now we couple this transport equation with dynamical Eq.(23) to see the effects of heat flux on collapsing process. Using Eq.(26) in (23), we obtain

\[ (\rho + p)(1 - \alpha)D_T U = -(1 - \alpha)(\rho + p)D_6 - E \left[ -\frac{\eta T^2 q}{2\tau} D_T \left( \frac{\tau}{\eta T^2} \right) - E^2 D_C p - \frac{q}{2} \left( \frac{D_T B}{B} + 2 \frac{D_T C}{C} + \frac{1}{\tau} \right) + \frac{\eta E}{\tau} D_C T \right] + D_7, \]

(27)

where \( \alpha \) is given by

\[ \alpha = \frac{\eta T}{\tau (\rho + p)}. \]

(28)

We see that inertial mass density on the LHS of dynamical equation (27) is affected by the factor \( 1 - \alpha \) arising from Tolman term. The same factor also appears with the gravitational force term on the RHS which shows the consistency of equivalence principle. However, we note that this factor is independent of the curvature fluid terms. It is mentioned here that the evolution process of collapsing sphere now depends upon the values of \( \alpha \) and terms inside the first square brackets in Eq.(27). The terms inside the second square brackets and expressions, namely, \( D_6 \) representing gravitational force in Eq.(34) and \( D_7 \), the DE contributions in Eq.(35), are the same as in dynamical equation (23).

5. Summary

The aim of this paper is to study the problem of gravitational collapse in \( f(R) \) theory of gravity which is strongly motivated by the observational discovery of DE. Here the higher order curvature terms are imposed to violate the null energy conditions and thought to be the origin of DE. We have considered spherically symmetric collapsing stars and discussed their dynamics under the
influence of \( f(R) \) DE. Effects of outgoing heat flux and anisotropic pressure have also been taken into account. Firstly, we have formulated dynamical equations by using contracted Bianchi identities. Secondly, we have developed transport equation and its coupling with dynamical equation.

The dynamical equations help to investigate the evolution of gravitational collapse with time. These equations yield the variation of total energy inside a collapsing body with respect to time and adjacent surfaces, as found in Eqs.\( (19) \) and \( (20) \) respectively. Here curvature terms appear to affect the passive gravitational mass and rate of collapse. Thus \( f(R) \) DE slows down the rate of the collapse due its repulsive effect.

The transport equation governs the dissipative fluxes and their associated quantities like temperature, relaxation time and thermal conductivity. It helps to construct physically viable models of radiating stars. After coupling the dynamical and transport equations, the above quantities affect the dynamics of collapsing process other than \( f(R) \) DE. The appearance of additional factor \( \alpha \) has a particular effect on the inertial mass density and gravitational force term.

**Acknowledgment**

We would like to thank the Higher Education Commission, Islamabad, Pakistan for its financial support through the *Indigenous Ph.D. 5000 Fellowship Program Batch-III*.

**Appendix**

Here, we explicitly give the expressions for the coefficients \( D_1, D_2, D_3, D_4, D_5, D_6 \) and \( D_7 \) representing the high order terms in the curvature.

\[
D_1 = \frac{A}{\kappa} \left\{ \frac{1}{A^2 B^2} \left( \frac{\dot{F}}{A} - \frac{A'}{A} \frac{\dot{F}}{B} - \frac{\dot{B}}{B} F' \right) \right\},
\]

\[
+ \left\{ \frac{f - RF}{2A^2} + \frac{F''}{A^2 B^2} - \frac{\dot{F}}{A^2} \left( \frac{\dot{B}}{B} - \frac{2C'}{C} \right) - \frac{F'}{B^2} \left( \frac{B'}{B} - \frac{2C'}{C} \right) \right\},
\]

\[
+ \frac{A}{A^2} \left\{ \frac{f - RF}{2A^2} + \frac{F''}{A^2 B^2} - \frac{\dot{F}}{A^2} \left( \frac{\dot{B}}{B} - \frac{2C'}{C} \right) - \frac{F'}{B^2} \left( \frac{B'}{B} - \frac{2C'}{C} \right) \right\},
\]

\[
+ \frac{B}{B A^2} \left\{ \frac{F''}{B^2} + \frac{\dot{F}}{A^2} \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) - \frac{F'}{B^2} \left( \frac{A'}{A} + \frac{B'}{B} \right) \right\},
\]

\[
+ \frac{2C'}{C A^2} \left\{ \frac{\dot{F}}{A^2} - \frac{\dot{F}}{A} \left( \frac{3C'}{A} - \frac{\dot{A}}{A} \right) - \frac{F'}{B^2} \left( \frac{A'}{A} + \frac{C'}{C} \right) \right\},
\]

\[
+ \frac{1}{A^2 B^2} \left( \frac{\dot{F}}{B} - \frac{A'}{A} \frac{\dot{B}}{B} F' \right) \left( \frac{2A'}{A} + \frac{B'}{B} + \frac{C'}{C} \right) \right\},
\]

\[
D_2 = -\frac{B}{\kappa} \left\{ \frac{1}{A^2 B^2} \left( \frac{\dot{F}}{A} - \frac{A'}{A} \frac{\dot{F}}{B} - \frac{\dot{B}}{B} F' \right) \right\},
\]

\[
+ \left\{ \frac{f - RF}{2B^2} - \frac{\dot{F}}{A^2} \left( \frac{\dot{A}}{A} + \frac{2C'}{C} \right) + \frac{F'}{B^2} \left( \frac{A'}{A} + \frac{2C'}{C} \right) \right\},
\]

\[
+ \frac{A'}{A B^2} \left\{ \frac{\dot{F}}{A^2} - \frac{\dot{F}}{A} \left( \frac{A'}{A} + \frac{\dot{B}}{B} \right) - \frac{F'}{B^2} \left( \frac{A'}{A} - \frac{B'}{B} \right) \right\},
\]

(29)
\[ D_3 = \frac{f(R) - RF}{2F} + \frac{\dot{F} A}{A^2} + \frac{\dot{A}}{A^2} - \frac{\dot{F}}{A^2} \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) - \frac{F'}{B^2} \left( \frac{A'}{A} + \frac{B'}{B} \right) \]
\[ + \frac{2C}{CA^2} \left( \frac{\dot{F}}{A^2} + \frac{\dot{A}}{A^2} \left( \frac{C'}{C} - \frac{\dot{A}}{C} \right) - \frac{\dot{F}'}{B^2} \left( \frac{A'}{A} - \frac{C'}{C} \right) \right) \]
\[ + \frac{1}{A^2 B^2} \left( \frac{\dot{F}'}{A^2} - \frac{A'}{A} \frac{\dot{F}'}{A} - \frac{\dot{B}}{B} F' \right) \left( \frac{\dot{A}}{A} + \frac{3B}{B} + \frac{2C'}{C} \right) , \] (30)
\[ D_4 = \frac{1}{AB} \left( \frac{A' \dot{F} - \dot{F}' + B F'}{F} \right) , \] (31)
\[ D_5 = \frac{F''}{FB^2} + \frac{f(R) - RF}{2F} - \frac{\dot{F}}{FA^2} \left( \frac{\dot{B}}{B} - \frac{2C'}{C} \right) - \frac{F'}{B^2} \left( \frac{B'}{B} - \frac{2C'}{C} \right) , \] (32)
\[ D_6 = \left[ \frac{\kappa p_r}{2F} - \frac{1}{2F} \left( \frac{f(R) - RF}{2} \right) \right] - DT \left( \frac{\dot{F}}{A} \right) + \frac{2D_T F D_T C}{C} + \frac{E}{B} \left( \frac{A'}{A} + \frac{2C'}{C} \right) D_C F , \] (33)
\[ D_T = \frac{E}{\kappa B} \left[ \left( \frac{D_T A}{A} - \frac{D_T B}{B} - 2D_T C \right) \left( D_T F' - \frac{A'}{A} D_T F - \frac{F'}{B} D_T B \right) \right] + \left\{ \frac{2}{C} D_T F D_T C - D_T \left( \frac{\dot{F}}{A} \right) + \frac{E}{B} \left( \frac{A'}{A} + \frac{2C'}{C} \right) D_C F \right\} , \] (34)
\[ + \frac{A'}{A} \left\{ \frac{F''}{B^2} + D_T \left( \frac{\dot{F}}{A} \right) - \frac{D_T F D_T B}{B} - \frac{F'}{B^2} \left( \frac{A'}{A} + \frac{B'}{B} \right) \right\} + \frac{2C'}{C} \left\{ \frac{F''}{B^2} - D_T F \left( \frac{D_T B}{B} + \frac{D_T C}{C} \right) - \frac{F'}{B^2} \left( \frac{B'}{B} + \frac{C'}{C} \right) \right\} + \frac{1}{A^2} \left( \dot{F}' - \frac{A'}{A} \dot{F} - \dot{B} F' \right) , \] (35)

References

[1] Weinberg, S.: Rev. Mod. Phys. 61(1989)1; Peebles, P.J.E. and Ratra, B.: Rev. Mod. Phys. 75(2003)559.
[2] Brans, C.H. and Dicke, R.H.: Phys. Rev. D124(1961)925; Gasperini, M. and Veneziano, G.: Phys. Lett. B277(1992)256; Sotiriou, T.P. and Barausse, E.: Phys. Rev. D75(2007)084007.
[3] Buchdahl, H.A.: Mon. Not. Roy. Astron. Soc. 150(1979)1.
[4] Kerner, R: Gen. Relativ. Gravit. 14(1982)453; Dorusseau, J.P., Kerner, R. and Eysseric, P. Gen. Relativ. Gravit. 15(1989)797.
[5] Capozziello, S.: Int. J. Mod. Phys. D11(2002)483; Capozziello, S., Carloni, S. and Troisi, A.: Astron. Astrophys. 625(2003)1.
[6] Nojiri, S. and Odinstov, S.D.: Phys. Rev. D68(2003)123512; Phys. Lett. B576(2003)5; Carrol, V., Duvvuri, M. and Turner, M.: Phys. Rev. D70(2004)043528.
[7] Meng, X. and Wang, P.: Class. Quantum Grav. 21(2004)951; Nojiri, S. and Odinstov, S.D.: Gen. Relativ. Gravit. 36(2004)1765.
[8] Allemandi, G. Borowiec and Francaviglia, M.: Phys. Rev. D70(2004)103503; Wu, X. and Zhu, Z.: Phys. Lett. B660(2008)293.
[9] Amendola, L. and Tsujikawa, S.: Phys. Lett. B660 (2008)125; Cognola, G., Elizalde, E., Zerbini, S., Nojiri, S. and Odintsov, S.D.: JCAP 0502 (2005)10.
[10] Nojiri, S. and Odintsov, S.D.: J. Phys. Conf. Ser. 66 (2007)012005.
[11] de la Cruz-Dombrize, A., Dobado A. and Maroto, A.L.: Phys. Rev. D80 (2009)124011.
[12] Kobayashi, T. and Maeda K.I.: Phys. Rev. D78 (2008)064091; ibid. D79 (2009)024009; Tsujikawa, S., Tamaki, T. and Tavakol, R.: JCAP 0905 (2009)020.
[13] Oppenheimer, J.R. and Snyder, H.: Phys. Rev. 56 (1939)455; Misner, C.W. and Sharp, D.: Phys. Rev. 136 (1964)B571.
[14] Sharif, M. and Rehmat, Z.: Gen. Relativ. Gravit. 42 (2010)1795; Sharif, M. and Fatima, S.: Gen. Relativ. Gravit. 43 (2011)127; Sharif, M. and Abbas, G.: J. Phys. Soc. Jpn. 80 (2011)104002; Sharif, M. and Kausar, H.R.: Int. J. Mod. Phys. D (to appear, 2011) Sharif, M. and Kausar, H.R.: Mod. Phys. Lett. A25 (2010)3299; Sharif, M. and Kausar, H.R.: Astrophys. Space Sci. 331 (2011)281; ibid. 332 (2011)463; Sharif, M. and Kausar, H.R.: J. Phys. Soc. Jpn. 80 (2011)044004; Sharif, M. and Kausar, H.R.: Phys. Lett. B697 (2011)01; Sharif, M. and Kausar, H.R.: JCAP 07 (2011)022.
[15] Akbar, M. and Cai, R-G: Phys. Lett. B648 (2007)243; Capozziello, S., Stabile, A. and Troisi, A.: Class. Quantum Grav. 25 (2008)085004; Akbar, M. and Cai, R-G: Phys. Lett. B635 (2006)07; Lobo, F.S.N. and Oliveira, M.A.: Wormhole Geometries in f(R) Modified Theories of Gravity, arXiv:0909.5539v1.
[16] Misner, C.W. and Sharp, D.: Phys. Rev. 136 (1964)B571.
[17] Maartens, R.: Causal Thermodynamics in Relativity, arXiv:0609119; Herrera, L.: Int. J. Mod. Phys. D15 (2006)2197.
[18] W Arnett, W.: Astrophysics J. 218 (1977)815.
[19] Martínez, J.: Phys. Rev. D53 (1996)6921.