Transaction costs: a new point of view

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Abstract

We consider a new approach to portfolio selection in presence of transaction costs which allows to map the problem into one without costs. The proposed approach connects all the quantities of interest to exit times and probabilities to reach barriers. This leads to analytic results in the Wiener case and to directly measurable quantities on a historical dataset in real markets.

1 Introduction

Optimal allocation of wealth among portfolios in presence of transaction costs is a well known problem in literature [1, 2, 3, 4, 5]. The investment we consider in this paper is on a risk-less bank account paying a fixed interest rate and on a risky asset with the logarithmic price increments modeled by a Wiener process.

The general approach to face this problem is to follow the asset price during all the time and sometimes “control” the portfolio buying or selling stocks. This approach leads to problems of singular control Brownian motion (see e.g. [6, 7] for a review) which seem the most adequate in this context. In an instantaneous control technique one has access (in principle) to the entire information; i.e. the process is continuously observed, even between two changes of trader’s portfolio. However, for all practical purposes, the necessary information is limited only to the moments the investor acts: therefore, we shall focus our attention only on these moments when something happens relevant from his point of view.

A trading rule is given if, each time the investor changes his portfolio, one specifies:

- which fraction of his capital has to be invested in the risky asset,
- when the investor should modify again his position;
then among all possible strategies the investor is interested in the optimal one. Following Kelly [8] we consider an investor who desires to maximize the growth rate of his capital. It seems to be the most natural approach if the trader is interested only in the long run behavior of his capital: in the following this kind of investor will be called speculator.

In financial markets assets have a bid and ask price. In this case the asset value is no longer defined in a unique way: it can take on any value in the bid-ask spread. At every time one can buy assets at the ask price or sell them obtaining the bid price, which is always lower then the previous one. In this paper we shall consider this spread in asset price as the only source of trading costs.

The most common way to treat the portfolio optimization in presence of such costs [1, 3] is to define an absolute asset value depending on the bid and ask price. However in a financial strategy it is natural to introduce a value relative to speculator’s behavior, i.e. an asset value which depends on the kind of operation (ask or bid) the market trader performs. The value is a matter of conventions: there are no financial reasons to prefer one choice or another. Of course the optimal growth rate cannot depend on the convention used. We shall show that this new point of view allows a mapping of the portfolio optimization problem to a similar one without costs. Therefore it connects a singular control problem to information theory, with the main advantage that information à la Shannon [9] of an asset price can be measured on financial data and also computed in an elementary way for processes as the Wiener one considered here.

The paper is organized as follows: in section 2 we state the portfolio selection problem. We summarize the case of absence of costs in a version appropriate for our purposes in section 3. In section 4 the relative value approach is treated: both the exact solution and an approximation of it are considered in detail and we discuss the relations and differences between the relative and absolute approach. Finally section 5 is devoted to summarize the results. In appendix A we deduce the probabilities and average time to exit from a barrier in a Wiener process and in appendix B we discuss the absolute value approach in an approximate case.

2 The model

We consider a speculator who diversifies his portfolio in a risk-less bank account with a deterministic rate of growth $R$ and a risky asset, for example a stock in a capital market. The ask and bid prices of the stock’s shares at calendar (continuous) time $t$ are related by the following equation

$$S^a_t = e^{\gamma} S^b_t,$$  \hspace{1cm} (1)

where we consider the transaction cost $\gamma$ time independent.

The return at calendar time $t$ is defined

$$r_t \equiv \ln \frac{S^b_{t+1}}{S^b_t}.$$  \hspace{1cm} (2)

The speculator is interested to choose a strategy which maximizes the exponential growth

2
rate of his capital $W_t$

$$\lambda \equiv \lim_{\Omega \to \infty} \frac{1}{\Omega} \ln \frac{W_{\Omega}}{W_0}. \tag{3}$$

In particular we shall consider the case where the trader modifies his position in the market only when a relevant change of the asset price appears. This point of view is natural in realistic situations where the trader waits until the return’s variations are significant for him and then rebalances his portfolio. We call, as in [10], $\Delta$-trading time the rank of such an investment.

At $\Delta$-trading time $k$ the speculator keeps a fraction $l_k$ of his capital in assets, while the remaining part is left in the bank account. We shall consider only self-financing strategies and we desire to repeat the game at every $\Delta$-trading time: therefore we shall limit our analysis to the fraction $l_k$ such that the capital is always strictly greater than zero. The speculator waits until

$$r_{t,t_k} = \ln \frac{S_t^b}{S_{t_k}^b} \tag{4}$$

raises up to $\Delta^+_k$ or decreases to $-\Delta^-_k$ (with $\Delta^+_k, \Delta^-_k \geq 0$), where both barriers are fixed by the speculator at $\Delta$-trading time $k$. When the return (4) “hits” one of the two barriers the speculator can buy or sell assets. We call exit time

$$T_k \equiv t_{k+1} - t_k$$

the lag between two portfolio changes.

To obtain a lighter notation we define the (final) state of the speculator $\eta$, depending on the kind of operation (ask/bid) he will perform at $\Delta$-trading time $k + 1$:

$$\eta = \begin{cases} a & (\text{ask}) \quad \text{when the trader buys} \\ b & (\text{bid}) \quad \text{when the trader sells} \end{cases}$$

The fraction of the capital in assets is exp($z_k \Delta^+_k$)$l_k W_k$ just before the $k + 1$-th trade and $l_{k+1} W_{k+1}$ immediately after, where the random variable $z_k$ can assume the values

$$z_k \equiv \begin{cases} -1 & \text{if } r_{t_{k+1},t_k} < 0 \\ +1 & \text{if } r_{t_{k+1},t_k} > 0 \end{cases}.$$

In particular we shall consider the most natural investment

$$\eta = a \quad \text{when } z_k = -1$$
$$\eta = b \quad \text{when } z_k = +1 \tag{5},$$

i.e. buy at the lower price and sell at the higher, or equivalently the following condition must be satisfied

$$\exp(-\Delta^-_k) l_k W_k \leq l_{k+1} W_{k+1} \quad \text{if } \eta = a$$
$$\exp(\Delta^+_k) l_k W_k \geq l_{k+1} W_{k+1} \quad \text{if } \eta = b \tag{6}.$$

In the proposed trading rule there are two classes of parameters, connected to the two main characteristics of a financial strategy, we have mentioned in the introduction:

- $\{l\}$ which specifies the fraction of the capital invested on assets,
• \{\Delta\} which is connected to the time the speculator has to keep his position.

We have split the optimal trading rule in the two main questions for a speculator; we shall show that it will allow us to map the problem with transaction costs into one where no costs are present.

If we limit our attention to the trading times we can rewrite the growth rate of the capital

\[ \lambda = \frac{h}{T}, \]  

(7)

where we call

\[ h \equiv \lim_{K \to \infty} \frac{1}{K} \sum_{k=0}^{K} \ln \frac{W_{k+1}}{W_k}, \]  

(8)

the mean lyapunov exponent of the capital, \( K \) is the number of investments (or equivalently exits of the process) up to the time \( t \) and

\[ T \equiv \lim_{K \to \infty} \frac{1}{K} \sum_{k=1}^{K} T_k \]  

(9)

is the mean exit time.

In the case ergodic processes are considered, one can substitute the \( \Delta \)-trading time average in equations (8) and (9) with the expectation value of the associate process.

We notice that this analysis, where the returns are considered only when they hit the barriers, resembles Poincaré maps in dynamical systems. Going from continuous to discrete time setting, the quantities of dynamical interest are simply rescaled by the mean exit time \( T \) [11].

The discrete time framework is clearly relevant in finance because it allows to show the connection of the capital growth rate, obtained via an optimal control policy, with quantities strictly related to Shannon entropy [9]. The main advantage is that these quantities can be directly measured on a historical dataset of financial assets [12, 15] or computed via elementary probability theory in simple cases. Furthermore, analyzing transaction costs from an unusual point of view this new framework will allow us to map the problem into one without costs.

This approach is directly connected to the way a financial decision is taken in practice. The speculator is interested in a growth rate which is optimal with respect to what has happened up to \( m \) \( \Delta \)-time steps before: it is an intuitive fact that events far in the past are irrelevant in the present portfolio selection. In other words, portfolio depends only on the last \( m \) investments, i.e on a finite set of parameters \( \{l\} \) and \( \{\Delta\} \), where the Markovian order \( m \) is chosen according to the “memory” of the asset process [12].

In this paper we shall consider the no-memory case modeling the returns with

\[ dr_t = \mu dt + \sigma dw_t, \]  

(10)

where \( \mu \) is the drift and \( w_t \) a Wiener process with unitary variance.

We can safely assume a null risk-free interest rate \( R \). The case \( R > 0 \) can always be recovered by simply replacing \( \mu \) with \( \mu - R \) and adding \( R \) to the growth rate of the capital (3).
3 Absence of costs

Using the notation introduced above we summarize in this section the well known case of portfolio selection in absence of transaction costs. The capital at $\Delta$-trading time $k + 1$ is given by

$$W_{k+1} = [1 - l_k + l_k \exp(z_k \Delta^z_k)] W_k .$$

(11)

We are considering a “scale invariant” trader: maximizing the growth rate his behavior does not depend on the capital he has when he starts an investment. In the case under consideration the return process has no memory and no transaction costs are involved. The speculator repeats then exactly the same game at every $\Delta$-trading time. This implies that the optimal choice of both the fractions $\{l\}$ and of barriers $\{\Delta\}$ do not depend on the considered time $k$.

The mean lyapunov exponent of the capital is

$$h(l; \Delta^+, \Delta^-) = p \ln [1 + l(\exp(-\Delta^-) - 1)] + (1 - p) \ln [1 + l(\exp(\Delta^+) - 1)] .$$

(12)

where $p = \pi(\Delta^+, \Delta^-)$ is the probability to exit from the lower barrier. The capital growth rate as a function of the chosen strategy, is obtained dividing the mean lyapunov exponent (12) by the average exit time $T = \tau(\Delta^+, \Delta^-)$. We have derived in the appendix the values of $\pi$ (equation (A.1)) and $\tau$ (equation (A.2)) for the Wiener process (10).

We shall look for the maximum capital growth rate finding first the optimal fraction $l$ at fixed barriers $\{\Delta\}$ and then searching the optimal $\Delta$s.

The optimal $l$ satisfies

$$l(\Delta^+, \Delta^-) = \frac{1 - p}{1 - \exp(-\Delta^-)} - \frac{p}{\exp(\Delta^+ - 1)} .$$

(13)

Substituting the optimal value of $l$ (13) in equation (12) we can write the mean lyapunov exponent

$$h = p \ln \frac{p}{q} + (1 - p) \ln \frac{1 - p}{1 - q}$$

(14)

as the Kullback entropy [14] between the probabilities $p$ and $q$, where $q = q(\Delta^+, \Delta^-)$ is the martingale probability (A.3). As pointed out by Kelly [8] in a particular case, relation (14) plays a central role in log-optimal portfolio selection: the growth rate obtained modifying the portfolio according to an optimal control strategy [6, 7] is linked to the entropy of the underlying process in information theory [9]. Speculator’s capital growth rate in absence and, as we shall show, in presence of transaction costs depends on the information present in the asset price. Such an information can be measured in financial series [12, 15] using a technique inspired by the Kolmogorov $\epsilon$-entropy [16].

In Figure 1 we have plotted the capital growth rate $\lambda(\Delta^+, \Delta^-)$ for the optimal choice of $l$. We notice that it is a non increasing function of its arguments. The maximum is

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1The same property is true for all the utilities in the HARA class (see e.g. [13] for a list of the general characteristics of these functions).
Figure 1: Capital growth rate in absence of costs for the optimal choice of $l$ as a function of $\Delta^+$ and $\Delta^-$. The parameters chosen are $\mu = 0.1$ and $\sigma = 1$.

reached for $(\Delta^+, \Delta^- = 0)$, obtaining the well known result that the optimal policy is continuous. It is an intuitive result due to the fact that a change in the portfolio does not cost anything and then the best solution is to use this free opportunity to rebalance continuously the investment.

We also observe that the maximum $(\Delta^+, \Delta^- = 0)$, is the only point where the gradient of the growth rate is zero. Therefore, if the speculator prefers to change his position at finite $\Delta$s, the error in the growth rate is small, i.e. of the second order in $\Delta$ for small $\Delta$.

Performing the limit $\Delta^+, - \rightarrow 0$ in the equations (13,14) one obtains the optimal capital fraction

$$l^* = \frac{\mu}{\sigma^2} + \frac{1}{2}$$

and the optimal growth rate

$$\lambda^* = \frac{\sigma^2}{2} l^{*2}.$$  (16)

We shall not allow the trader to borrow money from a bank or short selling of stock. An optimal portfolio suggests to keep the same fraction for ever ($l^*$ does not depend on $k$) and, because we are thinking to model a portfolio including shares as risky assets, a never-ending borrowing or short selling position does not appear realistic. The considered cases correspond to

$$-1 \leq \frac{2\mu}{\sigma^2} \leq 1.$$  (17)
If this ratio is 1 the best solution is to transfer all the money to the stock, instead the opposite limit corresponds to have no money in the risky asset. In the following we shall only consider transaction costs in the case with drift $\mu$ and variance $\sigma$ which satisfy condition (17).

## 4 The relative value approach

In the portfolio selection in absence of costs there is a unique asset price. It is natural to associate the asset value to this price. The value is a way to quantify the amount of money the trader has in his risky assets: he diversifies his portfolio according to it, investing a fraction of his capital $l_k$ which depends on the asset value at $\Delta$-trading time $k$. If transaction costs are present the asset value is instead a convention. One obtains the bid price selling the asset and buys it at the ask price. The value can be any price between these two.

To define the asset value, two points of view seem natural:

- An absolute value $S_t$ which is considered the “true price”, different from the ask and bid prices the trader finds in the market

$$
S_t^a = e^{\gamma_a} S_t \\
S_t^b = e^{-\gamma_b} S_t
$$

where $\gamma = \gamma_a + \gamma_b$ is the transaction cost defined in (1).

- A value relative to the speculator’s last trade: it is the bid price when he sells and the ask price when he buys.

The point of view considered in the literature (see e.g. [1, 3] and [4] and the references there in) is the absolute value approach\(^2\). In this approach an asset is bought (or sold) at a price different from the value of an asset of the same kind already included in trader’s portfolio. The absolute difference between the value and the price gives the cost. In the appendix B we summarize this approach in a approximate situation.

We stress here that the absolute value is a pure convention. A financial asset is well defined only when a trader buys (sells) it; its price is the ask (bid) price. It seems natural to associate the same “wealth” not only to the assets just traded but also to all the others present in his portfolio even before this transaction. In this way at every trading time it exists only one value for the speculator. This sounds reasonable because there is no difference between the assets owned and the ones just bought (sold). We are considering a value relative to the state of the speculator, as it as been defined in section 2. A consequence of a relative value is that one has four possibilities depending on the kind of investment (ask/bid) the speculator is performing at $\Delta$-trading time $k$ (initial state $\xi$) and he will perform at $k+1$ (final state $\eta$) depending on which barrier the return (4) will hit. To underline this fact we shall denote the barriers not only with the $\Delta$-trading time of the operation but also with the initial and final state.

\(^2\)Generally $\gamma_a = \gamma_b$ or one of the two parameters is kept equal to zero.
We remind that the bid and ask price are linked by relation (1) and that the trader buys at the lower price and sells at the higher (see relations (5)). The evolution of the capital invested in assets between time $k$ and $k+1$ is then:

$$\begin{align*}
\frac{S^a_{k+1}}{S^a_k} l_k W_k &= \exp(-\Delta_{aa;k}) l_k W_k \quad \text{if } \xi = a, \eta = a \\
\frac{S^b_{k+1}}{S^b_k} l_k W_k &= \exp(\Delta_{ab;k} - \gamma) l_k W_k \quad \text{if } \xi = a, \eta = b \\
\frac{S^a_{k+1}}{S^a_k} l_k W_k &= \exp(-\Delta_{ba;k} + \gamma) l_k W_k \quad \text{if } \xi = b, \eta = a \\
\frac{S^b_{k+1}}{S^b_k} l_k W_k &= \exp(\Delta_{bb;k}) l_k W_k \quad \text{if } \xi = b, \eta = b
\end{align*}$$

(18)

where $\Delta_{ab;k}, \Delta_{ba;k} \geq \gamma$, i.e. one should at least wait a time long enough to pay the costs!

We are considering the simple case of a no-memory process. The only piece of information the speculator has to remember of his past is whether at the previous $\Delta$-time step he has bought or sold assets, i.e. his initial state $\xi$. So, the barriers $\Delta_{\xi\eta;k}$ can only depend on the initial and final states and the fraction $l_k$ only on the initial state, i.e when the $k^{th}$ investment is decided.

To obtain a lighter notation we introduce

$$\Delta^- \equiv \Delta_{aa}, \quad \Delta^+ \equiv \Delta_{ab} - \gamma, \quad \Delta^- \equiv \Delta_{ba} - \gamma, \quad \Delta^+ \equiv \Delta_{bb}.$$  

The capital at $\Delta$-trading time $k + 1$ is

$$W_{k+1} = \left[1 + l_\xi (\exp(z_k \Delta^-) - 1)\right] W_k \equiv \omega_{\xi\eta} W_k ,$$  

(19)

where $z_k$ has been defined in section 2 as the sign of the return between $k$ and $k + 1$.

The problem is then completely defined by a Markovian transition matrix between the initial and final state

$$V = \begin{pmatrix} 1 - \alpha & \alpha \\ \beta & 1 - \beta \end{pmatrix} ,$$

(20)

where

$$\begin{align*}
\alpha &= 1 - \pi(\Delta^-_a, \Delta^+_a + \gamma) \\
\beta &= \pi(\Delta^-_b + \gamma, \Delta^+_b).
\end{align*}$$

The probabilities of the states $a$ and $b$ satisfy the following relation

$$\frac{p_a}{p_b} = \frac{V_{ba}}{V_{ab}} = \frac{\beta}{\alpha}.$$  

(21)

We have mapped the original portfolio selection into one where no costs are present but a Markovian memory must be considered. The capital growth rate for a speculator who follows this strategy is:

$$\lambda = \frac{\sum_{\xi,\eta} p_\xi V_{\xi\eta} \ln \omega_{\xi\eta}}{\sum_{\xi} p_\xi T_\xi} ,$$
where we have defined an average exit time which depends on the initial state $\xi$ of the trader:

$$
T_a = \tau(\Delta_a^+ + \gamma, \Delta_a^-),
$$

$$
T_b = \tau(\Delta_b^+, \Delta_b^- + \gamma).
$$

Let us comment equation (21), which is the core of the paper. The selection of the optimal trading rule from the prospective of the relative value has led us to reduce our task to an optimization problem with a stochastic return driven by a finite-state Markovian chain. The problem can be then faced with elementary probability theory in discrete spaces! In the following we shall find the solution for $\{l\}$ and $\{\Delta\}$ which optimize the growth rate (21). We consider then an ansatz, with $l$ and $\{\Delta^+, \Delta^-\}$ independent from the initial state, which appears quite natural in the case of absence of memory for the underlying process, we are dealing with in this paper. We discuss in detail this approximation underlining why it can be relevant for practical purposes and compare the results with the absolute value approach.

### 4.1 Exact solution

We have shown how the portfolio selection in presence of transaction costs has been mapped into a Markovian problem in absence of costs. We find here the optimal solution following the same route of previous section maximizing first $\{l\}$ and then finding the optimal barriers.

The two optimal values of $l$ are

$$
l_a(\Delta_a^+, \Delta_a^-) = \frac{\alpha}{1 - \exp(-\Delta_a^-)} - \frac{1 - \alpha}{\exp(\Delta_a^+) - 1},
$$

$$
l_b(\Delta_b^+, \Delta_b^-) = \frac{\beta}{1 - \exp(-\Delta_b^-)} - \frac{1 - \beta}{\exp(\Delta_b^+) - 1}.
$$

Substituting $l_a$ and $l_b$ in (21) one obtains

$$
\lambda = \frac{\sum_{\xi, \eta} p_\xi V_{\xi \eta} \ln \frac{V_{\xi \eta}}{Q_{\xi \eta}}}{\sum_{\xi} p_\xi T_\xi},
$$

where

$$
Q = \begin{pmatrix}
1 - q_a & q_a \\
q_b & 1 - q_b
\end{pmatrix},
$$

with

$$
\begin{cases}
q_a = 1 - q(\Delta_a^+, \Delta_a^-) \\
q_b = q(\Delta_b^+, \Delta_b^-)
\end{cases}.
$$

We notice that the numerator of equation (23) is for a Markov chain the quantity equivalent to the Kullback entropy of equation (14), obtained in the no memory case: this is the available information introduced in a simpler case by [10].

Let us stress that, for fixed barriers $\{\Delta\}$, the optimal growth rate of speculator’s capital is linked to the Shannon entropy of the underlying process and then in general to a measurable quantity even if transaction costs are present.
The speculator tries then to select the barriers $\Delta$ that maximize the information per unit of time (23) associated to the process. We observe in equations (18) that the trader pays costs only when he changes his state: every modification of his portfolio still remaining in the same state costs nothing. Of course he will use this free opportunity to rebalance his portfolio as often as he can, i.e

$$\Delta^{-}, \Delta^{+} \to 0.$$  \hfill (24)

In this case the growth rate of the capital becomes:

$$\lambda(\Delta^{+}, \Delta^{-}) = \mu \frac{\tilde{\beta} \left( \tilde{\alpha} \ln \tilde{\alpha} + \tilde{q}_a - \tilde{\alpha} \right) + \tilde{\alpha} \left( \tilde{\beta} \ln \tilde{\beta} + \tilde{q}_b - \tilde{\beta} \right)}{\tilde{\beta} (-1 + (\Delta^{+} + \gamma)\tilde{\alpha}) + \tilde{\alpha} \left( 1 - (\Delta^{-} + \gamma)\tilde{\beta} \right)},$$ \hfill (25)

where

$$\begin{align*}
\tilde{\alpha} &= \frac{\partial \alpha}{\partial \Delta_a} |_{\Delta_a=0} \\
\tilde{q}_a &= \frac{\partial q_a}{\partial \Delta_a} |_{\Delta_a=0} \\
\tilde{\beta} &= \frac{\partial \beta}{\partial \Delta_b} |_{\Delta_b=0} \\
\tilde{q}_b &= \frac{\partial q_b}{\partial \Delta_b} |_{\Delta_b=0}.
\end{align*}$$

We notice that even in this case the growth rate (25) is a non increasing function of its arguments.

The optimal fractions (22) become

$$\begin{align*}
l_a(\Delta^{+}) &= \tilde{\alpha} - \tilde{q}_a \\
l_b(\Delta^{-}) &= \tilde{q}_b - \tilde{\beta}.
\end{align*}$$ \hfill (26)

![Figure 2: The values of $\Delta$s allowed by conditions (28) for $\mu/\sigma^2 = 0.1$ and $\gamma = 0.01$. The full line represents $\nu^{-1}_a[l_b(\Delta^{-})]$ and the dashed $\nu^{-1}_b[l_a(\Delta^{+})]$.](image)
We remind that we are looking for a solution where the trader sells at the higher price
and buys at the lower, and then we have to check that conditions (6) are satisfied, which
imply:

\[
\begin{cases}
\nu_a(\Delta_a^+) &\equiv \frac{e^{\Delta_a^+} l_a(\Delta_a^+)}{1 + (e^{\Delta_a^+} - 1) l_a(\Delta_a^+)} \\ 
\nu_b(\Delta_b^-) &\equiv \frac{e^{-\Delta_b^-} l_b(\Delta_b^-)}{1 + (e^{-\Delta_b^-} - 1) l_b(\Delta_b^-)}
\end{cases}
\geq \begin{cases}
l_b(\Delta_b^-) \\
l_a(\Delta_a^+)
\end{cases}. \tag{27}
\]

It is easy to verify that \(\nu\) are monotone and in particular \(\nu_a\) is a strictly increasing
function of its argument and \(\nu_b\) strictly decreasing. Inverting (27) one obtains:

\[
\begin{cases}
\Delta_a^+ \geq \nu_a^{-1} [l_b(\Delta_b^-)] \\
\Delta_b^- \geq \nu_b^{-1} [l_a(\Delta_a^+)]
\end{cases} \tag{28}
\]

In Figure 2 we have shown the region of \(\Delta\)s allowed by condition (28). We observe
in particular that \(\nu_a^{-1} [l_b(\Delta_b^-)]\) and \(\nu_b^{-1} [l_a(\Delta_a^+)]\) cross at the point \(\Delta_a^+ = \Delta_b^- = \Delta\) where \(\Delta\) is given by equation

\[
\frac{2\mu}{\sigma^2} \sinh \left(\frac{\Delta}{2}\right) = \sinh \frac{\mu}{\sigma^2} (\Delta + \gamma). \tag{29}
\]

We notice that it is possible to show after some algebra that the tangents of the two
curves are parallel to the axes in the crossing point \((\Delta_a^+, \Delta_b^- = \Delta)\).

Equation (29) gives the optimal choice for the barriers of the portfolio selection problem
in presence of transaction costs, because the growth rate (25) is a non increasing function
of the two barriers.

In Figure 3 we plot the optimal values of \(\Delta\) as a function of \(2\mu/\sigma^2\) obtained from
equation (29) for a particular choice of \(\gamma\) observing that \(\Delta\) goes to infinity when \(2\mu/\sigma^2\)
approaches 1. We have considered only the positive values because \(\Delta\) is an even function
of \(\mu\).

One can also show that \((\Delta_a^-, \Delta_b^+ = 0; \Delta_a^+, \Delta_b^- = \Delta)\) is the only point where the gradient of
the growth rate (23) is the null vector. This fact is particularly relevant for a speculator,
because the optimal solution, as a consequence of (24), requires on average to rebalance
the portfolio after a time equal to zero. A null gradient of the capital growth rate on the
optimal solution implies, as in the absence of costs case, that a suboptimal choice of the
barriers causes a small error in the capital growth rate.

In Figure 4 we have shown the values of \(l_x\) (26) on the optimal solution for \(\Delta\)s (29) for
two different values of \(\gamma\) and the fraction \(l^*\) of the capital in the no cost case (13). After
some algebra one can obtain that

\[
l_a(\tilde{\Delta}) = 2l^* - l_b(\tilde{\Delta}), \tag{30}
\]

i.e. \(l^*\) is the mean value of \(l_a(\tilde{\Delta})\) and \(l_b(\tilde{\Delta})\) for all \(\tilde{\Delta}\) and in particular on the optimal
solution (29).

The optimal growth rate of the portfolio in presence of transaction costs is

\[
\lambda_O = \frac{\sigma^2}{2} l_a(\Delta)l_b(\Delta) \leq \lambda^* \tag{31}
\]

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Figure 3: Optimal values of $\Delta$ vs $2\mu/\sigma^2$ for $\gamma = 0.01$.

Figure 4: Optimal values of $l_a$ and $l_b$ vs $2\mu/\sigma^2$. The fraction of the capital $l_a$ is greater than the value $l^*$ obtained in absence of costs (dot dashed line) while $l_b$ is lower. The dashed line corresponds to the value of $\gamma = 0.1$ and the full line to $\gamma = 0.01$. 

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where $\lambda^*$ is the one obtained in the no-costs case (16). We notice that the optimal growth rate of the capital is proportional to the square of the geometric average of $l_a$ and $l_b$ and $\lambda^*$ to the square of their arithmetic average. The geometric average is always lower or equal to the arithmetic average $l^*$ and equal only when $l_a = l_b$, i.e. for $2\mu/\sigma^2 = -1, 1$. This fact has a simple interpretation: these are the cases where the trader maintains his position for ever; in these situations no costs are payed, except at least once when the investment begins.

### 4.2 An approximate solution

In this subsection we consider the simplest ansatz for speculator’s strategy: it will allow us to compare the two approaches of the relative and absolute value and to suggest a feasible approximation of the optimal trading rule. We choose both the barriers and the fraction independent on the initial state $\xi$. This should be a reasonable approximation, as explained in the no-costs case, because we are dealing with a no memory process and a growth rate maximizer. We notice that conditions (6) are automatically satisfied once we have chosen a time independent value of $l$.

The capital growth rate (21) becomes

$$
\lambda = \frac{p \ln [1 + l(\exp(-\Delta^-) - 1)] + (1 - p) \ln [1 + l(\exp(\Delta^+) - 1)]}{pT_a + (1 - p)T_b},
$$

Figure 5: Capital growth rate for the optimal choice of $l$ as a function of $\Delta^+$ and $\Delta^-$. The parameters are $\mu = 0.1, \sigma = 1, \gamma = 0.01$. The capital growth rate (21) becomes

\begin{equation}
\lambda = \frac{p \ln [1 + l(\exp(-\Delta^-) - 1)] + (1 - p) \ln [1 + l(\exp(\Delta^+) - 1)]}{pT_a + (1 - p)T_b},
\end{equation}

13
Figure 6: Optimal values of $\Delta^-$ (full line) and $\Delta^+$ (dashed line) vs $2\mu/\sigma^2$ for $\gamma = 0.01$.

Figure 7: Optimal values of $l$ vs $2\mu/\sigma^2$. The dashed line corresponds to the value of $\gamma = 0.1$ and the full line to $\gamma = 0.01$. We have plotted also the value of $l$ in absence of costs (dot dashed line).
where $p = \beta/(\alpha + \beta)$.

We have mapped the problem into one in absence of costs considered in section 3 with probability $p$ to exit from the lower barrier and average exit time $T = pT_a + (1 - p)T_b$. The optimal value of $l$ is given by equation (13). Substituting this value in equation (32) one obtains again that the growth rate can be written as the ratio between the Kullback entropy (14) and the average exit time $T$.

In Figure 5 we have plotted the capital growth rate (7) as a function of the barriers $\Delta^+$ and $\Delta^-$ calculated for the optimal $l$ (13). We observe that the maximum is reached for finite values of the barriers. Thus the speculator (on average) changes his portfolio after a finite time, i.e. he follows a discrete time trading rule.

In Figure 6 we plot the values of the barriers for which the optimal growth rate is reached. The fraction $l$ of the capital computed on the optimal barriers has been plotted in Figure 7. We notice that

$$
\begin{align*}
\Delta^-(-\mu) &= \Delta^+(\mu) \\
\Delta^+(-\mu) &= \Delta^-(\mu) \\
l(-\mu) &= 1 - l(\mu)
\end{align*}
$$

as a consequence of the symmetry

$$
\lambda_{-\mu}(l; \Delta^+, \Delta^-) = \lambda_{\mu}(1 - l; \Delta^-, \Delta^+) - \mu
$$

of the capital growth rate (32).

In Figure 8 we compare the optimal capital growth rate with the approximate ones obtained by the two approaches (absolute and relative). We notice that the differences
between the two are negligible even for a so large (and unrealistic) transaction cost. Furthermore the error committed considering these approximate solutions instead of the exact one is small even for large $\gamma$. As we would expect, it is then reasonable, except for very small values of $\mu$, to limit a trading rule to a time independent $l$, if no memory is present in the return process.

5 Conclusions

We have considered in this paper a new point of view to treat the transaction costs problem. This approach focuses only on the times the trader modifies his position. In the diversification of his portfolio the asset value (of all assets!) depends on the trade (bid or ask) he is performing and then it is relative to the market operator. This approach presents several advantages compared with the “traditional” convention of the absolute value.

The portfolio selection has been transformed to a Markovian problem in absence of costs. We have connected the quantities of interest (growth rate and optimal portfolio strategy) to the average exit times and probabilities to reach a barrier. If the returns can be modeled by a Wiener process, as assumed in this paper, both quantities can be computed using elementary probability theory, allowing to have analytic formulas for both the strategy ($l$ and $\Delta s$) and capital growth rate of the optimal solution. More generally, however, the same technique can be easily extended to all the utilities in the HARA class and this approach allows a straightforward generalization to the case with memory; it is then particularly relevant in the case of a real market, where exit times and probabilities to reach a barrier can be measured directly from a historical dataset [15].

We have shown that the exact solution of the problem in presence of transaction costs breaks the time invariance of the investment. However such a strategy is not feasible in practice because the speculator should modify his portfolio after a time which is on average zero. In subsection 4.2 we have considered a suboptimal strategy feasible for a speculator who wants to use it. This strategy, where the broken symmetry is restored, shows small differences in the capital growth rate with respect to the optimal one and involves only a finite number of transactions in finite time.

We want to stress here that in real markets the variation of the returns are discrete and of the same order of magnitude as the transaction cost $\gamma$. To extend this approach to this more realistic case is enough to consider the probability and the average time to exit from the barriers in the discrete case shown in appendix A.

Even in the case the costs to transact are included, the selection of a log-optimal portfolio depends on the information à la Shannon of the asset price: in the general case this quantity can be measured on real data and it can be computed analytically if an elementary model for the returns is assumed. The Wiener process can be considered a toy model for asset returns, nevertheless for a speculator who waits until relevant changes in the returns appear, this model catches the essential features of the portfolio selection, leading to a deeper understanding of the role of approximate but feasible strategies.
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Appendix A

In this appendix we compute the probabilities and the average exit times of a Wiener process as limit of a random walk on a one dimensional lattice; in this case both quantities can be obtained with elementary probability theory (see for example Cap. 14 of [17]).

The walker goes after a time step $\epsilon$ to the right of a lattice step $\sigma\sqrt{\epsilon}$ with probability

$$p_\epsilon = \frac{1}{2} \left( 1 + \frac{\mu}{\sigma} \sqrt{\epsilon} \right)$$

and to the left with probability $1 - p_\epsilon$, where $\mu$ and $\sigma$ are the parameters of the Wiener process (10).

Starting from zero the probability to reach a barrier situated at $-\Delta^-$ before hitting a barrier at $\Delta^+$ is

$$\pi_\epsilon(\Delta^+, \Delta^-) = \begin{cases} \frac{\rho_\epsilon^{\Delta^+} - 1}{\Delta^+} & \text{if } \mu \neq 0 \\ \frac{\rho_\epsilon^{\Delta^+ + \Delta^-} - 1}{\Delta^+ + \Delta^-} & \text{if } \mu = 0 \end{cases}$$

where

$$\rho_\epsilon = \left( \frac{p_\epsilon}{1 - p_\epsilon} \right)^{1/\epsilon}.$$ 

The average time to exit from one of the two barriers is

$$\tau_\epsilon(\Delta^+, \Delta^-) = \begin{cases} \frac{1}{\mu} \left[ \Delta^+ - (\Delta^+ + \Delta^-)\pi_\epsilon \right] & \text{if } \mu \neq 0 \\ \frac{1}{\sigma^2} \Delta^+ \Delta^- & \text{if } \mu = 0 \end{cases}.$$ 

Performing the limit $\epsilon \to 0$ one recovers the Wiener process defined in (10) and defining

$$\rho \equiv \lim_{\epsilon \to 0} \rho_\epsilon = \exp \left[ \frac{2\mu}{\sigma^2} \right].$$

we obtain the quantities of interest. The probability to hit the lower barrier is

$$\pi(\Delta^+, \Delta^-) = \begin{cases} \frac{\rho^{\Delta^+} - 1}{\rho^{\Delta^+ + \Delta^-} - 1} & \text{if } \mu \neq 0 \\ \frac{\Delta^+}{\Delta^+ + \Delta^-} & \text{if } \mu = 0 \end{cases}. \quad (A.1)$$
and the average exit time is
\[
\tau(\Delta^+, \Delta^-) = \begin{cases} 
\frac{1}{\mu} [\Delta^+ - (\Delta^+ + \Delta^-)\pi] & \text{if } \mu \neq 0 \\
\frac{1}{\sigma^2 \Delta^+ \Delta^-} & \text{if } \mu = 0 .
\end{cases}
\tag{A.2}
\]

We observe that both the probability (A.1) and the exit time (A.2) are continuous in \( \mu \).

We define martingale probability
\[
q(\Delta^+, \Delta^-) = \frac{1 - e^{-\Delta^+}}{1 - e^{-\Delta^+ - \Delta^-}} ,
\tag{A.3}
\]
the one with respect to which the exponentiated return is a martingale process, i.e.
\[
qe^{-\Delta^-} + (1 - q)e^{\Delta^+} = 1 .
\]

**Appendix B**

In this appendix we consider the case in which the costs are due to the difference between the absolute asset value \( S_t \) and the price the trader finds in the market when he sells \( (S^b_t) \) or buys \( (S^a_t) \) assets. The main advantage of this approach is that one can consider the time evolution of a unique “true price” \( S_t \), and the trader diversifies his portfolio according to it taking into account the costs every time he modifies his position. The costs faced for each asset traded are
\[
(e^{\gamma_a} - 1)S_{k+1} \quad \text{if } \eta = a \\
(1 - e^{-\gamma_b})S_{k+1} \quad \text{if } \eta = b .
\]

After having paid the costs the capital at the next \( \Delta \)-trading time will be worth:
\[
W_{k+1} = \begin{cases} 
1 + k(e^{-\Delta^+} - 1)W_k - (e^{\gamma_a} - 1)W_k & \text{if } \eta = a \\
1 + k(e^{\Delta^-} - 1)W_k - (1 - e^{-\gamma_b})W_k & \text{if } \eta = b
\end{cases}
\tag{B.1}
\]
i.e. the capital evolves as in absence of costs (11), but now the trader pays the costs on the assets he has bought or sold.

We can rewrite equation (B.1) in a multiplicative form as in (11):
\[
W_{k+1} = \begin{cases} 
1 - l_k + l_k e^{\gamma_a - \Delta^+} W_k & \text{if } \eta = a \\
1 - l_{k+1} + l_{k+1} e^{\gamma_a} W_k & \text{if } \eta = a \\
1 - l_k + l_k e^{\gamma_b - \Delta^-} W_k & \text{if } \eta = b \\
1 - l_{k+1} + l_{k+1} e^{-\gamma_b} W_k & \text{if } \eta = b
\end{cases}
\tag{B.2}
\]

To show this well known approach in a simple case, we consider the ansatz of \( l \) and \( \{\Delta^-, \Delta^+\} \) to be time independent, as in the approximate solution of the relative value approach. The mean lyapunov exponent of the capital is
\[
h(l; \Delta^+, \Delta^-) = p \ln \frac{1 - l + le^{\gamma_a - \Delta^+}}{1 - l + le^{\gamma_a}} + (1 - p) \ln \frac{1 - l + le^{-\gamma_b + \Delta^-}}{1 - l + le^{-\gamma_b}}
\tag{B.3}
\]
where \( p = \pi(\Delta^+, \Delta^-) \) is the probability (A.1) to exit from the lower barrier. The optimal value of \( l \) satisfies the equation:

\[
0 = pe^{\gamma a}(e^{\Delta^-} - 1) \left[ 1 + l(e^{\Delta^+ - \gamma b} - 1) \right] \\
(1 - p)e^{-\gamma b}(e^{\Delta^+} - 1) \left[ 1 + l(e^{\Delta^- + \gamma a} - 1) \right] \equiv dl^2 + fl + g .
\]

Solving this second order equation one obtains

\[
l(\Delta^+, \Delta^-) = \frac{-f + \sqrt{f^2 - 4dg}}{2d} ,
\]

which corresponds to the absence of costs solution in the limit \( \gamma \to 0 \). The other solution for \( l \) lies outside the interval of values allowed by a self-financing strategy.

In Figure 9 we show the optimal values for the barriers obtained maximizing numerically the capital growth rate. We notice that \((\Delta^-(-\mu), \Delta^+(\mu)) = (\Delta^+(-\mu), \Delta^-(-\mu)) \) and \( l(-\mu) = 1 - l(\mu) \), because, for \( \gamma_a = \gamma_b = \gamma/2 \), the growth rate (B.3) has the same symmetry in \( \mu \) (see equation (33)) of the relative approach.

![Figure 9: Optimal values of \( \Delta^- \) (full line) and \( \Delta^+ \) (dashed line) vs \( 2\mu/\sigma^2 \) for \( \gamma = 0.01 \) and \( \gamma_a = \gamma_b = \gamma/2 \).](image)

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