Closing the Gap between Weighted and Unweighted Matching in the Sliding Window Model

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Abstract

We consider the Maximum-weight Matching (MWM) problem in the streaming sliding window model of computation. In this model, the input consists of a sequence of weighted edges on a given vertex set \( V \) of size \( n \). The objective is to maintain an approximation of a maximum-weight matching in the graph spanned by the \( L \) most recent edges, for some integer \( L \), using space \( \tilde{O}(n) \).

Crouch et al. [ESA’13] gave a \((3 + \varepsilon)\)-approximation for (unweighted) Maximum Matching (MM), and, very recently, Biabani et al. [ISAAC’21] gave a \((3.5 + \varepsilon)\)-approximation for MWM. In this paper, we give a \((3 + \varepsilon)\)-approximation for MWM, thereby closing the gap between MWM and MM.

Biabani et al.’s work makes use of the smooth histogram technique introduced by Braverman and Ostrovsky [FOCS’07]. Rather than designing sliding window algorithms directly, this technique reduces the problem to designing so-called lookahead algorithms that have certain smoothness properties. Biabani et al. showed that the one-pass MWM streaming algorithm by Paz and Schwartzman [SODA’17] constitutes a lookahead algorithm with approximation factor \( 3.5 + \varepsilon \), which yields their result.

We first give a hard instance, showing that Paz and Schwartzman’s algorithm is indeed no better than a \( 3.5 \)-approximation lookahead algorithm, which implies that Biabani et al.’s analysis is tight. To obtain our improvement, we give an alternative and more complex definition of lookahead algorithms that still maintains the connection to the sliding window model. Our new definition, however, reflects further smoothness properties of Paz and Schwartzman’s algorithm, which we exploit in order to improve upon Biabani et al.’s analysis, thereby establishing our result.

1 Introduction

The data streaming model is a well-established computational model that provides a framework for studying massive data set algorithms. The defining features of the model are restricted access to the input data and sublinear space. A data streaming algorithm processes its input sequentially in a single pass while maintaining only a small summary of the input data in memory.

In this paper, we study the Maximum-weight Matching (MWM) problem in the (streaming) sliding window model. In this variant of the streaming model, the input consists of a potentially infinite
sequence $e_1, e_2, \ldots$ of weighted edges on an underlying vertex set $V$ of size $n$. The objective is to maintain a matching of large weight in the graph spanned by the $L$ most recent edges, for some integer $L$, using only semi-streaming space $O(n \log n) = \tilde{O}(n)$. In more detail, after having processed the current edge $e_i$, for every $i$, the objective is to report an approximation of a maximum-weight matching in the graph spanned by the current sliding window $E_i := \{e_j : \max\{i - L + 1, 1\} \leq j \leq i\}$.

In the following, all sliding window algorithms discussed use semi-streaming space, and, unless stated otherwise, we will omit to mention their space requirements explicitly.

While sliding window algorithms have been studied for two decades [DGIM02], sliding window algorithms for graph problems were first considered by Crouch et al. [CMS13] in 2013. Amongst other results, they showed that there is a $(3 + \varepsilon)$-approximation sliding window algorithm for unweighted Maximum Matching (MM) and a $9.027$-approximation sliding window algorithm for MWM.

While no improved results are known for MM, Crouch and Stubbs [CS14] subsequently improved upon the result for MWM and gave a $(6 + \varepsilon)$-approximation, and, very recently, Biabani et al. [BdBM21] gave a $(3.5 + \varepsilon)$-approximation for MWM. The state-of-the-art results for MM and MWM in the sliding window model therefore do not yet line up.

1.1 Our Results

In this paper, we close the gap between MM and MWM in the sliding window model. To this end, we give a sliding window algorithm for MWM that matches the approximation guarantee of the best known sliding window algorithm for MM.

**Theorem 1** (simplified version). There is a deterministic streaming sliding window algorithm for Maximum-weight Matching with an approximation factor $3 + \varepsilon$ that uses space $O\left(\frac{\log(1/\varepsilon)}{\varepsilon^2} \cdot n \log^2 n\right)$, for any $0 < \varepsilon \leq 0.1$.

Similar to previous results, our result holds if edge weights can be stored using $O(\log n)$ bits and the maximum weight $w_{\text{max}}$ and minimum weight $w_{\text{min}}$ of an edge are such that $\log(w_{\text{max}}/w_{\text{min}}) = O(\log n)$.

Table 1 summarizes all results known for MM and MWM in the sliding window model.

| Algorithm | Approximation Factor | Reference |
|-----------|----------------------|-----------|
| MM        | $3 + \varepsilon$    | Crouch et al. [CMS13] |
| MWM       | $9.027$              | Crouch et al. [CMS13] |
|           | $6 + \varepsilon$    | Crouch and Stubbs [CS14] |
|           | $3.5 + \varepsilon$  | Biabani et al. [BdBM21] |
|           | $3 + \varepsilon$    | This paper |

Table 1: Known results on semi-streaming sliding window algorithms for MM and MWM.

1.2 Techniques

We will first explain the techniques behind Biabani et al.’s $(3.5 + \varepsilon)$-approximation sliding window algorithm for MWM and then discuss our new ideas which yield the improved approximation guarantee.

Bibani et al.’s algorithm combines the smooth histogram technique for sliding window algorithms by Braverman and Ostrovsky [BO07] and the one-pass $(2 + \varepsilon)$-approximation streaming algorithm for MWM by Paz and Schwartzman [PS17]. We will first expand on the smooth histogram technique and discuss how it is applied in this context, and then discuss Paz and Schwartzman’s algorithm.
Smooth Histogram Technique. Braverman and Ostrovsky [BO07] showed that if a function $f$ fulfills certain smoothness criteria\(^1\) then a sliding window algorithm for approximating $f$ can be obtained from a traditional (non-sliding window) streaming algorithm for $f$ at the expense of only a logarithmic increase in the space requirements (as long as the approximation factor of the streaming algorithm is constant), and a slight increase in the approximation factor. In the context of MWM, the smoothness criteria are captured by Biabani et al. [BdBM21] via the notion of a lookahead algorithm.

Definition 2 (($f, \alpha, \beta$)-lookahead algorithm [BdBM21]). Let $\beta \in (0, 1)$ and $\alpha > 0$ be real numbers. Let $X$ be a ground set, $S$ a stream of items of $X$, and let $f : 2^X \to \mathbb{R}^+$ be a non-decreasing function. We say that a streaming algorithm $\mathcal{ALG}$ is a $(f, \alpha, \beta)$-lookahead algorithm if, for any partitioning of $S$ into three substreams $A, B, C$ with $\mathcal{ALG}(B) \geq (1 - \beta) \cdot \mathcal{ALG}(AB)$, the following holds: $f(ABC) \leq \alpha \cdot \mathcal{ALG}(BC)$.

We observe that the previous definition holds for real-valued non-decreasing functions. In the context of MWM, the weight of a maximum-weight matching rather than the matching itself fulfills these conditions. We will therefore consider the problem of determining the weight of a maximum-weight matching instead, and, in order to be able to output an actual matching as required in these conditions. We will therefore consider the problem of determining the weight of a maximum-weight matching in stream $S$.

Biabani et al. [BdBM21] showed that if there is a $(\text{MWM}, \alpha, \beta)$-lookahead algorithm that uses space $s$ then there exists a sliding-window algorithm with approximation ratio $\alpha$ and space $O\left(\frac{1}{\varepsilon} \cdot s \log \sigma\right)$, where $\sigma = \frac{n}{2} \cdot \frac{w_{\text{max}}}{w_{\text{min}}}$ and $w_{\text{max}}$ and $w_{\text{min}}$ are the maximum and minimum weights of an edge in the input stream, respectively. Observe that, under the usual assumption that $w_{\text{max}}/w_{\text{min}}$ is polynomial in $n$, we have $\log \sigma = O(\log n)$.

The main part of their analysis is to show that a monotonic version of the one-pass streaming algorithm for MWM by Paz and Schwartzman constitutes a $(\text{MWM}, (3.5 + \varepsilon), \beta)$-lookahead algorithm, for small values of $\varepsilon$ and $\beta \leq \varepsilon/9$. Combined, this yields a $(3.5 + \varepsilon)$-approximation sliding window algorithm with space $O\left(\frac{1}{\varepsilon} \cdot s \log \sigma\right)$, where $s$ is the space required by (the monotonic version of) Paz and Schwartzman’s algorithm.

Paz and Schwartzman’s MWM Algorithm. Paz and Schwartzman’s original algorithm [PS17] uses space $O\left(\frac{1}{\varepsilon} \cdot n \log^2 n\right)$ and is based on the local ratio technique – see the survey by Bar-Yehuda et al. [BBFR04] for further details on this technique. Ghaffari and Wajc [GW19] provided a simplified version of their algorithm and improved the space complexity to the (optimal in $n$) bound $O\left(\frac{\log(1/\varepsilon)}{\varepsilon} \cdot n \log n\right)$.

The Paz and Schwartzman algorithm (with Ghaffari and Wajc’s improvement) works as follows. It maintains a potential $\varphi(v)$ that is initialized with 0, for every vertex $v \in V$, and uses a stack data structure Stack. When an edge $e = \{u, v\}$ arrives in the stream, $e$ is pushed onto Stack if its weight $w(e)$ exceeds the sum of the potentials of its incident vertices by a factor of at least $(1 + \varepsilon)$, i.e., $w(e) \geq (1 + \varepsilon) \cdot (\varphi(u) + \varphi(v))$. The discrepancy between $w(e)$ and $\varphi(u) + \varphi(v)$ is denoted the reduced weight of $e$ and is abbreviated by $w'_e := w(e) - (\varphi(u) + \varphi(v))$. Then, the potentials $\varphi(u)$ and $\varphi(v)$ are updated as $\varphi(u) = \varphi(u) + w'_e$ and $\varphi(v) = \varphi(v) + w'_e$. Last, if either $u$ or $v$ is adjacent to at least $\frac{3 \log(1/\varepsilon)}{\varepsilon} + 1$ edges in Stack then the oldest (and thus lightest) edge incident to the vertex is removed from Stack, thereby limiting the number of edges on Stack. After having processed all the edges in the stream, the output matching $\hat{M}$ is computed in a post-processing step. The edges in Stack

\(^1\)Informally, a function $f : 2^X \to \mathbb{R}$ is consider to be smooth if it satisfies the following: If $f(A)$ is close to $f(B)$ for $A, B \subseteq X$, for a suitable notion of closeness, then the values $f(A \cup C)$ and $f(B \cup C)$ are close for all $C \subseteq X$. 

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are popped one by one and greedily inserted into \( \hat{M} \) if possible, i.e., as long as \( \hat{M} \) remains a matching. We denote the Paz and Schwartzman algorithm by \( \mathcal{ALG}_{PS} \). See Section 2 for a formal description of the algorithm.

Biabani et al. employed a monotonic version of \( \mathcal{ALG}_{PS} \). The smooth histogram technique requires the underlying algorithm \( \mathcal{ALG} \) to be such that \( \mathcal{ALG}(A) \leq \mathcal{ALG}(AB) \), for substreams \( A \) and \( B \), which is not necessarily fulfilled by \( \mathcal{ALG}_{PS} \) since the algorithm as described above only reports a Greedy (maximal) matching that is computed in reverse order in how the edges were added to the stack rather than, for example, a heaviest matching that can be formed using these edges. Biabani et al. addressed this by computing Greedy matchings in reverse order after every edge insertion into \( \text{Stack} \) and maintain and report the heaviest matching found thus far. We call this version of the Paz and Schwartzman algorithm the monotonic version of the algorithm and abbreviate it by \( \mathcal{ALG}_{mon} \).

Our Contributions. We first show that the analysis of Biabani et al. is best possible in that the Paz and Schwartzman algorithm and its monotonic version are no better than \((MWM, 3.5, \beta)\)-lookahead algorithms. This is achieved by providing a concrete stream of edges \( S \) that can be partitioned into substreams \( A, B, C \) such that \( \mathcal{ALG}_{mon}(B) \geq (1 - \beta) \cdot \mathcal{ALG}_{mon}(AB) \) and \( MWM(ABC) \) is roughly \( 3.5 \cdot \mathcal{ALG}_{mon}(BC) \). The smooth histogram technique applied to lookahead algorithms as defined in Definition 2 thus cannot give an improved approximation guarantee when Paz and Schwartzman’s algorithm is used as the underlying algorithm.

To illustrate our improvement, we first provide insight into the structure of Biabani et al.’s analysis. In order to prove that \( \mathcal{ALG}_{mon} \) is a \((MWM, 3.5 + \varepsilon, \beta)\)-lookahead algorithm, Biabani et al. relate \( MWM(ABC) \) to the output of \( \mathcal{ALG}_{mon} \) on various substreams of \( ABC \):

\[
MWM(ABC) \leq 2(1 + \varepsilon) \cdot (\mathcal{ALG}_{mon}(AB) + \mathcal{ALG}_{mon}(BC)) - \frac{1}{2(1 + \varepsilon)} \cdot \mathcal{ALG}_{mon}(B) .
\] (1)

They subsequently use the smoothness assumption from Definition 2 and a monotonicity property of \( \mathcal{ALG}_{mon} \) to relate \( \mathcal{ALG}_{mon}(AB) \) and \( \mathcal{ALG}_{mon}(B) \) to \( \mathcal{ALG}_{mon}(BC) \). This ultimately yields the desired bound \( MWM(ABC) \leq (3.5 + \varepsilon) \cdot \mathcal{ALG}_{mon}(BC) \).

To obtain our improvement, we observe that a similar inequality to Inequality 1 can be obtained by considering sums of reduced weights of the respective runs of \( \mathcal{ALG}_{mon} \) instead of the weights of the output matchings of \( \mathcal{ALG}_{mon} \) on the different substreams. This idea is motivated by the fact that the sum of reduced weights is a lower bound on the weight of the matching produced by the algorithm, which can therefore give a more precise analysis. However, when departing from such an inequality involving sums of reduced weights, we unfortunately cannot immediately complete our analysis since, unlike when considering the outputs of \( \mathcal{ALG}_{mon} \) directly, we do not have a sufficient smoothness property regarding sums of reduced weights at our disposal that would allow us to relate these quantities to each other. Our key idea is as follows. To establish the necessary smoothness properties, we give an alternative definition of lookahead algorithms, denoted refined-lookahead algorithms (see Definition 6 for details), which enables us to incorporate the required smoothness property of sums of reduced weights into the definition. This is achieved by requiring refined-lookahead algorithms to produce two outputs, with one corresponding to the sum of reduced weights and the other one to the matching (size) itself. We then prove that, similar to lookahead algorithms, refined-lookahead algorithms can still be turned into sliding window algorithms with a similar small increase in the space complexity. Last, we finish our argument by proving that \( \mathcal{ALG}_{PS} \) is a refined-lookahead algorithm with an approximation factor of \( 3 + \varepsilon \), which establishes our result.
Finally, we would like to point out that our technique allows us to work with $\mathcal{ALG}_{PS}$ directly instead of using its monotonic version $\mathcal{ALG}_{mon}$. This is possible since sums of reduced weights are naturally monotonic, and they take on the role of the output matching size in the smooth histogram technique. See Section 4 for details.

1.3 Further Related Work

The sliding window model can be regarded as a streaming insertion-deletion model with highly structured deletions since, for each incoming edge, the oldest edge in the current window is deleted. Interestingly, the complexities of MM and MWM in the sliding window model are much closer to those in the insertion-only model, where no deletions are allowed, as opposed to the insertion-deletion model, where arbitrary deletions are allowed. In the insertion-only model, the currently best one-pass algorithm known for MM is the Greedy matching algorithm, which produces a 2-approximation and uses semi-streaming space $\tilde{O}(n)$. It is known that computing a $(1 + \ln 2)$-approximation requires strictly more space than $\tilde{O}(n)$ [Kap21], see also the previous lower bounds [GKK12, Kap13]. It remains a key open problem to close this gap. Regarding MWM, a series of works [FKM+05, McG05, Zel12, ELMS11, CS14, PS17, GW19] culminated in the Paz and Schwartzman algorithm, which closes the gap between MWM and MM from an algorithmic perspective in the insertion-only model. In the insertion-deletion model, where arbitrary previously inserted edges can be deleted again, it is known that space $\Theta(n^2/\alpha^3)$ is necessary and sufficient for computing an $\alpha$-approximation to MM, see [AS22] for the algorithm and [DK20] for a matching lower bound (see also the previous works [Kon15, AKLY16]). MWM reduces easily to MM in the insertion-deletion model, by, for example, grouping edges of similar weights into groups and running the MM algorithm a logarithmic number of times in parallel at the expense of only a marginal increase in the approximation factor.

The sliding window model is inspired by the problem of inferring statistics of data occurring within a certain time frame over a continuous stream of data (e.g., maintaining the number of distinct users who have accessed a social media page in the last 24 hours). The model was introduced by Datar et al. [DGIM02], and Crouch et al. [CMS13] were the first to consider graph problems in the sliding window model. Among others, they showed that testing Connectivity and Bipartiteness, and constructing $(1+\varepsilon)$-sparsifiers can be done in the sliding window model using semi-streaming space. Furthermore, as previously mentioned, they also gave the first sliding window algorithms for MM and MWM.

The smooth histogram technique used in our work was introduced by Braverman and Ostrovsky [BO07] and can be regarded as an improvement of the exponential histogram technique [DGIM02] for smooth functions. This technique has successfully been applied to a wide range of problems, such as the computation of coresets for geometric problems [WLT19] and for clustering problems [BLLM16].

1.4 Outline

We first give notation and a discussion of Paz and Schwartzman’s algorithm (with Ghaffari and Wajc’s improvements) including its properties in Section 2. Next in Section 3, we show that the Paz and Schwartzman’s algorithm is no better than a 3.5-approximation lookahead algorithm. Our definition of refined-lookahead algorithms is then given in Section 4. We also establish the connection between refined-lookahead algorithms and sliding window algorithms in this section. Then, we show that the Paz and Schwartzman’s algorithm is a refined lookahead algorithm with approximation factor $3 + \varepsilon$ in Section 5, which establishes our main result. Finally, we conclude with open questions in Section 6.
2 Preliminaries

In this section, we start with some important notation and a formal description of the improved version of Paz and Schwartzman’s algorithm by Ghaffari and Wajc (see Algorithm 1). This is followed by some key insights about the algorithm.

Let $S$ be an input stream representing an edge-weighted graph $G = (V, E, w)$ with a weight function $w : E \to \mathbb{R}^+$. For any subset of edges $F \subseteq E$, let $w(F) = \sum_{e \in F} w(e)$ be the sum of their weights. Then, for any maximum-weight matching in $G$, denoted by $M^*(S)$, we have that $\text{MWM}(S) = w(M^*(S))$.

**Algorithm 1** $\mathcal{ALG}_{PS}^\epsilon$ (Paz and Schwartzman’s algorithm with Ghaffari and Wajc’s improvements)

**Input:** A stream $S$ of weighted edges

**Initialization:**
1. Stack $\leftarrow$ an empty stack
2. for every vertex $v \in V$ do $\varphi(v) \leftarrow 0$

**Streaming:**
3. while a new edge $e = \{u, v\}$ of the stream $S$ is revealed do
4. if $w(e) < (1 + \epsilon) \cdot (\varphi(u) + \varphi(v))$ then $w'(e) \leftarrow 0$
5. else
6. $w'(e) \leftarrow w(e) - (\varphi(u) + \varphi(v))$ \hspace{1cm} $\triangleright w'(e)$ is the reduced weight of $e$
7. $\varphi(u) \leftarrow \varphi(u) + w'(e)$; $\varphi(v) \leftarrow \varphi(v) + w'(e)$ \hspace{1cm} $\triangleright$ update potentials
8. Stack.Push($e$)
9. for $x \in \{u, v\}$ do $\triangleright$ optimizing space
10. if $x$ is adjacent to $> \frac{3\log(1/\epsilon)}{\epsilon}$ + 1 edges in Stack then
11. Remove the oldest edge adjacent to $x$ from Stack

**Postprocessing:**
12. Let $\hat{M}$ be an empty matching
13. while Stack is not empty do
14. $e \leftarrow$ Stack.Pop()
15. if $\hat{M} \cup \{e\}$ is a matching then $\hat{M} \leftarrow \hat{M} \cup \{e\}$
16. return $\hat{M}$ \hspace{1cm} $\triangleright$ a Greedy matching of the edges in Stack

$\mathcal{ALG}_{PS}^\epsilon$ (Algorithm 1) uses the notions of reduced weights and vertex potentials. These are respectively represented by the functions $w'_S : E \to \mathbb{R}^+$ and $\varphi_S : V \to \mathbb{R}^+$ when the algorithm is executed on a stream $S$. The sum of all reduced weights is denoted by $W'_S = \sum_{e \in E} w'_S(e)$. For any edge in the stream, its reduced weight is non-negative and is unchanged by the processing of any subsequent edges. In particular, for a stream $AB$ and any edge $e \in A$, we have $w'_A(e) = w'_{AB}(e) \geq 0$. Hence, the sum of the reduced weights is a non-decreasing function, i.e., $W'_A \leq W'_AB$. The output matching of $\mathcal{ALG}_{PS}^\epsilon$ on a stream $S$ is denoted by $\hat{M}(S)$.

Ghaffari and Wajc’s analysis of the algorithm reveals the following key observations and results which we later use in our proofs.

**Observation 3** (Ghaffari and Wajc [GW19]). At any moment there are at most $O\left(\frac{\log(1/\epsilon)}{\epsilon} \cdot n\right)$ edges stored in Stack during the execution of $\mathcal{ALG}_{PS}^\epsilon$.

**Proposition 4** (Ghaffari and Wajc [GW19]). For any edge $e = \{u, v\}$ in a stream $S$, after the execution of $\mathcal{ALG}_{PS}^\epsilon$, its weight is bounded as $w(e) \leq (1 + \epsilon) \cdot (\varphi_S(u) + \varphi_S(v))$. 

Proposition 5 (Ghaffari and Wajc [GW19]). Let $\varepsilon > 0$ and $S$ be a stream of edges. Then, the following inequalities hold:

\[
\begin{align*}
    w(M^*(S)) & \geq W'_S, \\
    w(M(S)) & \geq \frac{1}{1+4\varepsilon} \cdot W'_S = \frac{1}{2(1+4\varepsilon)} \sum_{v \in V} \phi_S(v) \geq \frac{1}{2(1+4\varepsilon)(1+\varepsilon)} \cdot w(M^*(S)).
\end{align*}
\]

Note that Proposition 5 uses the important fact that $W'_S = \frac{1}{2} \sum_{v \in V} \phi_S(v)$ as the potential of a vertex $v$ is actually the sum of reduced weights of edges incident to $v$. Furthermore, its last inequality is due to Proposition 4 since each vertex in a matching is incident to at most one edge.

Indeed, Proposition 5 shows that $\mathcal{ALG}_{PS}$ is a $(2+\varepsilon)$-approximation streaming algorithm for MWM, and since Observation 3 indicates that the algorithm only ever stores $O(n)$ edges on the stack (for constant $\varepsilon$), it thus uses optimal in $n$ space (up to constant factors) since a maximum-weight matching may consist of up to $n/2$ edges.

3 Hard Instance for Paz and Schwartzman’s Algorithm

In this section, we show that the Paz and Schwartzman’s algorithm and its monotonic version are no better than $(\text{MWM}, 3.5, \beta)$-lookahead algorithms. The definition of a lookahead algorithm given by Biabani et al. (Definition 2) together with the Paz and Schwartzman’s algorithm thus cannot be used to improve upon the approximation factor of 3.5.

Recall that a lookahead algorithm relies on the smoothness of the algorithm’s output. More formally, an $(f, \alpha, \beta)$-lookahead algorithm $\mathcal{ALG}$ satisfies the condition that for any stream $ABC$, if $\mathcal{ALG}(B) \geq (1-\beta) \cdot \mathcal{ALG}(AB)$ then $f(ABC) \leq \alpha \cdot \mathcal{ALG}(BC)$ (see Definition 2). In other words, if the algorithm $\mathcal{ALG}$ outputs similar results on the streams $B$ and $AB$ then the algorithm’s output on $BC$ is required to be an $\alpha$-approximation of the objective value $f(ABC)$ of the whole stream $ABC$.

We will present a graph $G$ whose edges are divided into three substreams $A, B$ and $C$ such that $\mathcal{ALG}_{PS}$ outputs matchings of the same weight on substreams $AB$ and $B$, while the outputted matching on substream $BC$ is roughly a 3.5-approximation of a maximum-weight matching of the entire stream $ABC$. The graph $G$ is such that even if we modified $\mathcal{ALG}_{PS}$ to return maximum-weight matchings among the edges stored in Stack then the same properties still hold. Thus, the hard instance is also hard for the monotonic version of the algorithm. The graph $G$ is depicted in Fig. 1.

Figure 1: The edges of the graph $G$ are divided into substreams $A, B$ and $C$. The order of the edges within the substreams is indicated by subscripts (the order of the edges with the same subscript is not important). The thin edges have unit weight and the thick edges have the indicated larger weights.
Matchings computed on $AB$ and $B$. First, we analyze $\mathcal{ALG}_{PS}^\varepsilon$ separately on the substreams $A$ and $B$. See Fig. 2 for the values of the reduced weights and potentials computed by the algorithm.

Observe that the substream $A$ consists of two disjoint paths of length three. While only one of them is shown in Fig. 2, the algorithm computes the same reduced weights and potentials for both paths.

We now analyze the execution of the algorithm on substream $AB$. To this end, consider the moment when the substream $A$ has been fully processed and substream $B$ begins. Observe that each edge of $B$ is now incident to a single vertex with potential $1 + \varepsilon$. Thus, by construction of the algorithm, none of the edges of $B$ are pushed onto Stack. These edges therefore have have reduced weights zero and cannot be outputted by the algorithm. Furthermore, when run on $AB$, the algorithm outputs the two edges in $A_1$, i.e., $w(\hat{M}(AB)) = 2 + 2\varepsilon$, which are the only two edges pushed onto Stack.

As established in Figure 2, when the algorithm runs only on the substream $B$, it outputs the two edges in $B_2$, i.e., $w(\hat{M}(B)) = 2 + 2\varepsilon$. Hence, we have that $w(\hat{M}(B)) = w(\hat{M}(AB))$. It follows that the stream $ABC$ satisfies the condition $w(\hat{M}(B)) \geq (1 - \beta) \cdot w(\hat{M}(AB))$, for any value of $\beta \geq 0$, as required by the definition of a lookahead algorithm.

Matchings computed on $BC$. Now, we analyze the execution of the algorithm on the substream $BC$. At the time when the substream $C$ begins, the reduced weights of edges in $B$ and the current potentials of the incident vertices are the same as when the algorithm is run only on the substream $B$ – see Fig. 2 for these values. See Fig. 3, for the reduced weights of the edges in $C$ when we run the algorithm on the substream $BC$.

By the end of the execution, only the two edges in $C_1$ and $C_3$ are pushed onto Stack since the edges in $C_2$ have reduced weights zero. The algorithm $\mathcal{ALG}_{PS}^\varepsilon$ outputs a greedy matching of the edges pushed onto Stack (in the reverse order they arrived). In particular, it outputs the edges in $C_1$ and $C_3$ and they block all edges in $B$. Hence, $w(\hat{M}(BC)) = 2 + 4\varepsilon$. Observe further that these edges constitute a maximum-weight matching among the edges pushed onto Stack.

Maximum-weight Matching and Approximation Factor. First, observe that the unique maximum-weight matching in $G$ consists of all the edges that have an endpoint of degree 1 (the edges in $A_2, C_2$, and $C_3$) and is thus of weight $7 + 3\varepsilon$. Since $w(\hat{M}(BC)) = 2 + 4\varepsilon$, we conclude that it is not possible for $\mathcal{ALG}_{PS}^\varepsilon$ to yield a $(\text{MWM}, 3.5 - \delta, \beta)$-lookahead algorithm, for any constant $\delta > 0$ and
suitable parameter $\beta$.

4 Refined Lookahead Algorithm

In this section, we give our definition of a refined lookahead algorithm and establish the connection between refined lookahead algorithms and the sliding window model.

**Definition 6** ($(f, \alpha_1, \alpha_2, \beta)$-refined lookahead algorithm). Let $\beta \in (0, 1)$, $\alpha_1, \alpha_2 \geq 1$ and, for a ground set $X$, let $f : 2^X \rightarrow \mathbb{R}^+$ be a non-decreasing function. We say a streaming algorithm $\mathcal{ALG}$ with two outputs $\mathcal{ALG}_1, \mathcal{ALG}_2$ is a $(f, \alpha_1, \alpha_2, \beta)$-refined lookahead algorithm if the following holds for any stream $S$ of items of the set $X$:

1. $\mathcal{ALG}_1(S) \leq f(S) \leq \alpha_1 \cdot \mathcal{ALG}_1(S)$, i.e., the first output is an $\alpha_1$-approximation of $f$.

2. For any partitioning of $S$ into three disjoint sub-streams $A, B, C$ with $\mathcal{ALG}_1(B) \geq (1 - \beta) \cdot \mathcal{ALG}_2(AB)$, we have $\mathcal{ALG}_2(BC) \leq f(ABC) \leq \alpha_2 \cdot \mathcal{ALG}_2(BC)$, i.e., if the first output on the substream $AB$ is similar to the first output on the substream $B$ then the second output on the substream $BC$ is an $\alpha_2$-approximation of $f$ on the whole stream $S = ABC$.

First, observe that if $\mathcal{ALG}_1 = \mathcal{ALG}_2$ and $\alpha_1 = \alpha_2 = \alpha$ then we obtain the standard definition of a $(f, \alpha, \beta)$-lookahead algorithm as given by Biabani et al. (see Definition 2). In the following, we will apply Definition 6 to algorithm $\mathcal{ALG}_{PS}$. To this end, the first output $\mathcal{ALG}_1$ constitutes the sum of reduced weights $W'_S$, the second output constitutes the weight of the outputted matching $w(\hat{M}(S))$, and function $f$ is the weight of a maximum-weight matching $\text{MWM}(S)$. In Section 5, we will prove that this indeed yields a $(\text{MWM}, (2+2\varepsilon), (3+20\varepsilon), \beta)$-refined lookahead algorithm, for a suitable value $\beta > 0$.

Next, in Algorithm 2, we give a slightly modified version of the sliding window algorithm by Biabani et al. [BdBM21] adapted to refined lookahead algorithms $\mathcal{ALG}$. Let $e$ be the current item of the stream being processed by Algorithm 2 and let $E$ be the current sliding window consisting of the $L$ most recently processed items (including $e$). While processing $e$, a new bucket $B_{k+1}$ is created alongside an instance of $\mathcal{ALG}$. Then, $e$ is fed into each of the buckets, $B_1, \ldots, B_{k+1}$, and their corresponding instances of $\mathcal{ALG}$ are updated. Next, starting from the oldest bucket, only its newest similar bucket, determined by Item 2 of Definition 6, is kept and every other bucket in between is deleted. Whether a newest similar bucket exists or not, the process is then repeated with the next oldest remaining bucket.
until reaching the newest bucket. Note that the oldest and newest buckets, \( B_1 \) and \( B_{k+1} \) respectively, are never deleted by this process. However, if the number of items fed into the second oldest remaining bucket \( B_{r+1} \) is at least \( L \), i.e., the current sliding window \( E \) is fully contained in \( B_{r+1} \), then \( B_1 \) and its instance of \( \mathcal{ALG} \) are subsequently deleted. The buckets are then renumbered from the oldest one to the newest, 1 to \( k \) respectively, such that \( k \) is the number of remaining buckets. At this stage, the sliding window \( E \) is sandwiched between buckets \( B_1 \) and \( B_2 \). Finally, after processing the item, if the current sliding window is exactly the items fed into bucket \( B_1 \), then the algorithm reports the second output of the instance of \( \mathcal{ALG} \) on bucket \( B_1 \), \( \mathcal{ALG}_2(B_1) \), as the solution, otherwise it reports \( \mathcal{ALG}_2(B_2) \).

Algorithm 2 Sliding Window Algorithm

**Input:** A stream \( S \) with a sliding window of length \( L \)
\( \mathcal{ALG} \): a \( (f, \alpha_1, \alpha_2, \beta) \)-refined lookahead algorithm

**Initialization:**
1: Let \( k \leftarrow 0 \) be the number of buckets

**Streaming:**
2: while a new item \( e \) of the stream \( S \) is revealed do
3: Create an empty bucket \( B_{k+1} \)
4: Let \( \mathcal{ALG}(B_{k+1}) \) be a new instance of \( \mathcal{ALG} \) for the bucket \( B_{k+1} \)
5: Feed \( e \) into each of the \( k+1 \) buckets and update their respective instances of \( \mathcal{ALG} \)
6: \( i \leftarrow 1 \)
7: while \( i < k \) do
8: Let \( j > i \) be the largest index for which \( \mathcal{ALG}_1(B_j) \geq (1 - \beta) \cdot \mathcal{ALG}_1(B_i) \)
9: if no such \( j \) exists then \( j \leftarrow i + 1 \)
10: Delete buckets \( B_r \) for each \( i < r < j \) and their corresponding instances of \( \mathcal{ALG} \)
11: \( i \leftarrow j \)
12: Let \( B_{>1} \) be the next remaining bucket after \( B_1 \)
13: if no such bucket exists then continue to line 16
14: if \( |B_{>1}| \geq L \) then \( \triangleright |B_{>1}| \) is the number of items fed into \( B_{>1} \)
15: Delete \( B_1 \) and its instance of \( \mathcal{ALG} \)
16: Renumber the buckets and let \( k \) be the number of remaining ones
17: if \( |B_1| = L \) then report \( \mathcal{ALG}_2(B_1) \)
18: else report \( \mathcal{ALG}_2(B_2) \)

In essence, the buckets and corresponding instances of \( \mathcal{ALG} \) of Algorithm 2 simulate runs of a traditional streaming algorithm \( \mathcal{ALG} \) on suffixes of the current sliding window. Note that the oldest run always contains all items of the sliding window and potentially some additional ones. The idea is to maintain runs on suffixes such that the \( \mathcal{ALG}_1 \) outputs of any two consecutive runs are not too different, while the \( \mathcal{ALG}_1 \) outputs of any non-consecutive runs are sufficiently different so as to ensure that at most a logarithmic number of parallel runs are executed at any given instance.

This idea is exactly captured when \( \mathcal{ALG} \), with two outputs \( \mathcal{ALG}_1 \) and \( \mathcal{ALG}_2 \), is a \( (f, \alpha_1, \alpha_2, \beta) \)-refined lookahead algorithm (which applies the smooth histogram technique by Braverman and Ostrovsky [BO07]). The first output \( \mathcal{ALG}_1 \) is used to determine how often a run on a suffix should be maintained, which depends on the smoothness criteria given by Item 2 of Definition 6. The second output is a solution which, given the smoothness assumptions of the runs, is always guaranteed to be an \( \alpha_2 \)-approximation of the next oldest run. We highlight that the smoothness assumptions are only guaranteed to hold for consecutive runs whose suffixes differ by more than one item. Then, for a stream \( S \) of items from a set \( X \) and a non-decreasing function \( f : 2^X \rightarrow \mathbb{R}^+ \), the number of runs is at
most logarithmic in \( n \) as long as \( \sigma_f(S) = f(S)/\min(\{f(e) : e \in S\}) \), is polynomial in \( n \). We prove this formally in Theorem 7.

**Theorem 7.** Let \( 0 < \beta < 1 \) and \( \alpha_1, \alpha_2 \geq 1 \), \( S \) be a stream of items from a set \( X \), and \( f : 2^X \to \mathbb{R}^+ \) be a non-decreasing function. Suppose there exists a \((f, \alpha_1, \alpha_2, \beta)\)-refined lookahead algorithm that uses space \( s \). Then, there exists a sliding window algorithm that maintains an \( \alpha_2 \)-approximation of \( f \) using space \( O(\frac{1}{\beta} \cdot s \log(\alpha_1 \sigma)) \) for \( \sigma = \sigma_f(S) \).

**Proof.** We prove that Algorithm 2 satisfies the assertion of the theorem. Let \( \mathcal{ALG}_1 \) be the used \((f, \alpha_1, \alpha_2, \beta)\)-refined lookahead algorithm with the outputs \( \mathcal{ALG}_1 \) and \( \mathcal{ALG}_2 \).

**Approximation.** Let \( E \) be the sliding window at any instance of the algorithm, i.e., the set of the \( L \) most recently processed items. The algorithm ensures that \( E \) is sandwiched between buckets \( B_1 \) and \( B_2 \), in particular, \( B_2 \subseteq E \subseteq B_1 \). We are now in one of two cases, either the items of \( B_1 \) and \( B_2 \) differ by exactly one item or more than one item.

In the former case, the algorithm asserts that \( |B_2| < L \), otherwise \( B_1 \) would have been deleted, and therefore the items fed into \( B_1 \) are exactly those of the sliding window \( E \), i.e., \( |B_1| = L \). The reported solution is then always \( \mathcal{ALG}_2(B_1) = \mathcal{ALG}_2(E) \) which by **Item 2 of Definition 6** (consider the case when \( E = ABC = BC \)) is trivially an \( \alpha_2 \)-approximation of \( f(E) \).

In the latter case, the algorithm would have, at some point, deleted buckets which caused \( B_1 \) and \( B_2 \) to become adjacent. Consider the moment when they first became adjacent and let \( B_1^* \) and \( B_2^* \) be the items that had been fed into their respective buckets. At this moment, the algorithm asserts that \( \mathcal{ALG}_1(B_2^*) \geq (1-\beta) \cdot \mathcal{ALG}_1(B_1^*) \). Let \( C \) be the remaining items fed into the buckets such that \( B_1 = B_1^* C \) and \( B_2 = B_2^* C \). Then, by **Item 2 of Definition 6** and \( f \) being non-decreasing,

\[
\mathcal{ALG}_2(B_2) \leq f(B_2) \leq f(E) \leq f(B_1) \leq \alpha_2 \cdot \mathcal{ALG}_2(B_2).
\]

Hence, we have that, \( \mathcal{ALG}_2(B_2) \), is an \( \alpha_2 \)-approximation of \( f(E) \). Now, if \( |B_1| \neq L \) the solution reported is \( \mathcal{ALG}_2(B_2) \), otherwise \( |B_1| = L \) and the solution reported is \( \mathcal{ALG}_2(B_1) = \mathcal{ALG}_2(E) \). We conclude that in either case an \( \alpha_2 \)-approximation of \( f(E) \) is reported.

**Space.** Let \( k \) be the maximum number of buckets stored by the algorithm after processing an item. During the process of deleting buckets, the algorithm ensures that for any buckets \( B_i \) and \( B_{i+2} \), \( \mathcal{ALG}_1(B_{i+2}) < (1-\beta) \cdot \mathcal{ALG}_1(B_i) \). Thus for the largest odd number \( k' \) not exceeding \( k \),

\[
(1 + \beta)^{\frac{k-1}{2}} \mathcal{ALG}_1(B_{k'}) < \mathcal{ALG}_1(B_1).
\]

Recall that \( \frac{f(B_1)}{f(B_{k'})} \leq \sigma \). Then, by **Item 1 of Definition 6**, we have that \( \frac{\mathcal{ALG}_1(B_1)}{\mathcal{ALG}_1(B_{k'})} \leq \alpha_1 \sigma \). It follows that

\[
\frac{k' - 1}{2} < \log_{1+\beta}(\alpha_1 \sigma) \quad \text{and} \quad k' \in O\left(\frac{1}{\beta} \cdot \log(\alpha_1 \sigma)\right).
\]

This implies the result since there are only ever \( k + 1 \leq k' + 2 \) instances of \( \mathcal{ALG}_1 \), each of which uses space \( s \).
5 Sliding Window Algorithm For MWM

In this section, we will prove that $\text{ALG}_\varepsilon$ is a $(\text{MWM}, (2+2\varepsilon), (3+20\varepsilon), \beta)$-refined lookahead algorithm, which, by Theorem 7, implies the existence of a $(3 + \varepsilon)$-approximation sliding window algorithm for MWM.

In the proof, we consider the sum of reduced weights $W'$ as the first output $\text{ALG}_1$ and the weight of the computed matching $\hat{w}(\hat{M})$ as the second output $\text{ALG}_2$. For a motivating example of this choice, consider the graph given in Section 3 (see Fig. 1). We have that $w(\hat{M}(AB)) = w(\hat{M}(B)) = 2 + 2\varepsilon$, $W'_{AB} = 2 + 2\varepsilon$ and $W'_{B} = 1 + 2\varepsilon$. We showed in Section 3 that this is indeed a hard instance for (standard) lookahead algorithms when the weight of the matching computed is used as the smoothness constraint (recall that $(1 - \beta) \cdot w(\hat{M}(AB)) \leq w(\hat{M}(B))$ is then required in a hard instance, which is the case here). On the other hand, refined lookahead algorithms allow us to use the sum of reduced weights as the smoothness constraint. Since $(1 - \beta) \cdot W'_{AB} \not\leq W'_{B}$, for small enough $\beta$, the instance therefore is not hard for refined lookahead algorithms.

We obtain the following main result.

Theorem 8. Let $0 < \varepsilon \leq \frac{1}{10}$ and $0 < \beta \leq \frac{\varepsilon}{9}$. The algorithm $\text{ALG}_\varepsilon$ is a $(\text{MWM}, (2+2\varepsilon), (3+20\varepsilon), \beta)$-refined lookahead algorithm.

To prove Theorem 8, we follow the idea of Biabani et al. [BdBM21]. Let an input stream $S$ be partitioned into three substreams $ABC$. They split the maximum matching of the stream $M^*(ABC)$ into two parts $M^*_{AB}$ and $M^*_C$ where $M^*_{AB} := M^* \cap AB$ is the restriction of $M^*$ to the edges in $AB$. Note that these are not necessarily the maximum matchings in substreams $AB$ and $C$, respectively. Biabani et al. then bound the weights of these two parts separately. To this end, they use the following notion of critical subgraph.

Definition 9 (Critical Subgraph [BdBM21]). Consider a graph $G$ specified by a stream $S$ of edges. Let $A,B,C$ be disjoint substreams of $S$ such that $S = ABC$. Then, the critical subgraph of $G$ with respect to the maximum matching $M^*(ABC)$ and the substreams $A,B,C$ is the subgraph $H = (V_H, E_H)$ such that

- $E_H := \{e \in B \mid e$ is adjacent to two edges in $M^*_C \}$.
- $V_H := V(E_H)$, i.e., $V_H$ is the set of endpoints of the edges in $E_H$.

Next, we present three auxiliary lemmas for bounding $w(M^*_{AB})$ and $w(M^*_C)$. The first one is already proved by Biabani et al. [BdBM21] in the exact formulation as we need, and we highlight that their proof holds for any run of $\text{ALG}_\varepsilon$ on an arbitrary stream.

Lemma 10 (Biabani et al. [BdBM21], Lemma 15). For any stream $AB$,

$$(1 + \varepsilon) \cdot \sum_{v \in V_H} \varphi_{AB}(v) \geq \sum_{e \in E_H} w'_{B}(v) .$$

The following two lemmas are analogous to lemmas proved by Biabani et al. [BdBM21] (Lemmas 13 and 14). They bound the weight of $M^*_{AB}$ and $M^*_C$ in terms of the output of the algorithm. However, we bound the weights by sums of reduced weights computed by the algorithm. The presented proofs resemble the proofs of the original lemmas.
Lemma 11 (Analogue of Lemma 13, [BdBM21]). For any stream $ABC$,

$$w(M_{AB}^*) \leq 2(1 + \varepsilon) \cdot W_{AB}' - \sum_{e \in E_H} w_B'(e).$$

Proof. By definition, we have $w(M_{AB}^*) = \sum_{e \in M_{AB}^*} w(e)$. Let $e = \{u, v\} \in M_{AB}^*$. Note that the vertices $u$ and $v$ are not in $V_H$. Thus, we can bound the sum as follows.

$$w(M_{AB}^*) \leq (1 + \varepsilon) \sum_{v \in V \setminus V_H} \varphi_{AB}(v)$$

by Proposition 4

$$= (1 + \varepsilon) \left( \sum_{v \in V} \varphi_{AB}(v) - \sum_{v \in V_H} \varphi_{AB}(v) \right)$$

By Proposition 5 and by Lemma 10, we have

$$\sum_{v \in V} \varphi_{AB}(v) = 2W_{AB}' \quad \text{and} \quad (1 + \varepsilon) \sum_{v \in V_H} \varphi_{AB}(v) \geq \sum_{e \in E_H} w_B'(e).$$

Thus, we can conclude that

$$w(M_{AB}^*) \leq 2(1 + \varepsilon) \cdot W_{AB}' - \sum_{e \in E_H} w_B'(e).$$

For the proof of the next lemma, we need the following notion. Let $S$ be a stream of edges. For an edge $e \in S$, we define the set $P_S(e)$ as the set of edges incident to $e$ (including $e$) arriving no later than $e$, i.e., $P_S(e) = \{e' \in S \mid e' \cap e \neq \emptyset, t_e' \leq t_e\}$, where, for any edge $f$, $t_f$ is the arrival time of edge $f$. Biabani et al. [BdBM21] showed that the weight of any edge $e$ can be bounded by the sum of the reduced weights of the edges in $P_S(e)$ (up to a $(1 + \varepsilon)$ factor).

Lemma 12 (Biabani et al. [BdBM21], Lemma 5). For each edge $e \in S$,

$$w(e) \leq (1 + \varepsilon) \sum_{e' \in P_S(e)} w_S'(e').$$

Lemma 13 (Analogue of Lemma 14, [BdBM21]). For any stream $ABC$,

$$w(M_C^*) \leq 2(1 + \varepsilon) \cdot W_{BC}' - (1 + \varepsilon) \sum_{e \in B \setminus E_H} w_B'(e).$$

Proof. First, when considering a run of the algorithm on $BC$, by Lemma 12, we obtain

$$w(M_C^*) = \sum_{e \in M_C^*} w(e) \leq (1 + \varepsilon) \sum_{e \in M_C^*} \sum_{e' \in P(e)} w_{BC}'(e').$$

Observe that any edge $e \in BC$ is incident to at most two edges of $M_C^*$, and the edges of $B \setminus E_H$ are incident to at most one edge of $M_C^*$. Hence, we can rewrite the previous double sum as follows:
\[
\sum_{e \in M^*} \sum_{e' \in P(e)} w'_{BC}(e') \leq 2 \cdot \sum_{e \in BC} w'_{BC}(e) - \sum_{e \in B \setminus E_H} w'_{BC}(e),
\]
which implies the result.

Now, we are ready to prove Theorem 8, i.e., \(\mathcal{ALG}_PS^\varepsilon\) is a \((\text{MWM}, (2 + 2\varepsilon), (3 + 20\varepsilon), \beta)\)-refined lookahead algorithm for suitable parameters \(\varepsilon\) and \(\beta\).

**Proof of Theorem 8.** We recall that we consider a version of \(\mathcal{ALG}_PS^\varepsilon\) such that the first output is the sum of reduced weight \(W'\) and the second output is the weight of the computed matching \(w(\hat{M})\). First by Proposition 5, we get that for any stream \(S\) it holds that \(W' \leq w(M^*(S)) \leq 2(1 + \varepsilon) \cdot W'\). Thus, it remains to prove that for any stream \(ABC\), if we suppose that \(W'_B \geq (1 - \beta) \cdot W'_{AB}\) then for the maximum matching \(M^* = M^*(ABC)\) holds that \(w(M^*) \leq (3 + 20\varepsilon) \cdot w(\hat{M}(BC))\).

\[
\begin{align*}
w(M^*) & \leq 2(1 + \varepsilon) \cdot W'_{AB} + 2(1 + \varepsilon) \cdot W'_{BC} - W'_B & \text{by Lemmas 11 and 13} \\
& \leq \frac{2(1 + \varepsilon)}{1 - \beta} \cdot W'_B + 2(1 + \varepsilon) \cdot W'_{BC} - W'_B & \text{by } W'_B \geq (1 - \beta) \cdot W'_{AB} \\
& \leq (1 + 3\varepsilon) \cdot W'_B + 2(1 + \varepsilon) \cdot W'_{BC} & \text{since } \beta \leq \frac{\varepsilon}{9} \\
& \leq (3 + 5\varepsilon) \cdot W'_{BC} & \text{by } W' \text{ being non-decreasing} \\
& \leq (3 + 5\varepsilon)(1 + 4\varepsilon) \cdot w(\hat{M}(BC)) & \text{by Proposition 5} \\
& \leq (3 + 20\varepsilon) \cdot w(\hat{M}(BC)) & \text{since } \varepsilon \leq \frac{1}{10}.
\end{align*}
\]

Theorems 7 and 8 together then imply our main result.

**Theorem 1.** There is a deterministic streaming sliding window algorithm for Maximum-weight Matching with an approximation factor \(3 + \varepsilon\) that uses space \(O\left(\frac{\log(1/\varepsilon)}{\varepsilon^2} \cdot n \log^2 \sigma\right)\), for any \(0 < \varepsilon \leq 0.1\) and \(\sigma = \frac{n}{2} \cdot w_{max}/w_{min}\).

### 6 Conclusion

In this paper, we gave a deterministic \((3 + \varepsilon)\)-approximation algorithm for MWM in the sliding window model that uses semi-streaming space. The approximation factor of our algorithm matches the approximation factor of the best sliding window algorithm known for (unweighted) MM [CMS13].

Since further improvements in the approximation factor of our result would also imply improvements for the unweighted version of the problem, the most natural direction for future research is to make further progress on the unweighted version of the problem first. From the perspective of upper bounds, can we obtain a 2.99-approximation sliding window algorithm for unweighted MM using semi-streaming space?

While the known lower bounds for MM for one-pass streaming algorithms in the insertion-only model also apply to the sliding window model, no stronger lower bounds for the sliding window model are known. Can we prove a lower bound on the approximation factor of sliding window algorithms for
MM that use semi-streaming space and are stronger than what is currently known for the insertion-only model, i.e., stronger than $1 + \ln(2)$ [Kap21]?

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