A CAMERON–STORVICK TYPE THEOREM ON $C_{a,b}^2[0,T]$ WITH APPLICATIONS

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ABSTRACT. The purpose of this paper is to establish a very general Cameron–Storvick theorem involving the generalized analytic Feynman integral of functionals on the product function space $C_{a,b}^2[0,T]$. The function space $C_{a,b}^2[0,T]$ can be induced by the generalized Brownian motion process associated with continuous functions $a$ and $b$. To do this we first introduce the class $F_{A_1,A_2}$ of functionals on $C_{a,b}^2[0,T]$ which is a generalization of the Kallianpur and Bromley Fresnel class $F_{A_1,A_2}$. We then proceed to establish a Cameron–Storvick type theorem on the product function space $C_{a,b}^2[0,T]$. Finally we use our Cameron–Storvick type theorem to obtain several meaningful results and examples.

1. INTRODUCTION

Let $C_0[0,T]$ denote one-parameter Wiener space, that is, the space of all real-valued continuous functions $x$ on $[0,T]$ with $x(0) = 0$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0,T]$ and let $m_w$ denote Wiener measure. Then, as is well-known, $(C_0[0,T], \mathcal{M}, m_w)$ is a complete measure space.

In [1] Cameron established an integration by parts formula for the Wiener measure $m_w$. More precisely, in [1], Cameron introduced the first variation (a kind of Gâteaux derivative) of functionals on the classical Wiener space $C_0[0,T]$ and established a formula involving the Wiener integral of the first variation. This was the first infinite dimensional integration by parts formula. In [15] Donsker also established this formula using a different method, and applied it to study Fréchet–Volterra differential equations. In [22, 23] Kuo and Lee established an integration by parts formula for abstract Wiener space which they then used to evaluate various functional integrals.

In [2], Cameron and Storvick also established an integration by parts formula involving the analytic Feynman integral of functionals on $C_0[0,T]$. They also applied their celebrated parts formula to establish the existence of the analytic Feynman integral of unbounded functionals on $C_0[0,T]$. The integration by parts formula on $C_0[0,T]$ suggested in [2] was improved in [24] to study the parts formulas involving the analytic Feynman integral and the analytic Fourier–Feynman transform. Since then the parts formula for the analytic Feynman integral is called the Cameron–Storvick theorem by many mathematicians.

The Cameron–Storvick type theorem and related topics were also developed for functionals on the very general function space $C_{a,b}^2[0,T]$ in [6, 11]. The function

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space $C_{a,b}[0,T]$, induced by generalized Brownian motion process (GBMP), was introduced by J. Yeh [27, 28].

A GBMP on a probability space $(\Omega, \Sigma, P)$ and a time interval $[0,T]$ is a Gaussian process $Y = \{Y_t\}_{t \in [0,T]}$ such that $Y_0 = c$ almost surely for some constant $c \in \mathbb{R}$, and for any set of time moments $0 = t_0 < t_1 < \cdots < t_n \leq T$ and any Borel set $B \subset \mathbb{R}^n$, the measure $P(I_{t_1, \ldots, t_n, B})$ of the cylinder set $I_{t_1, \ldots, t_n, B}$ of the form

$$I_{t_1, \ldots, t_n, B} = \{\omega \in \Omega : (Y(t_1, \omega), \ldots, Y(t_n, \omega)) \in B\}$$

is equal to

$$\left((2\pi)^n \prod_{j=1}^{n} (b(t_j) - b(t_{j-1}))\right)^{-1/2} \times \int_B \exp \left\{ -\frac{1}{2} \sum_{j=1}^{n} \frac{(\eta_j - a(t_j)) - (\eta_{j-1} - a(t_{j-1}))^2}{b(t_j) - b(t_{j-1})}\right\} d\eta_1 \cdots d\eta_n,$$

where $\eta_0 = c$, $a(t)$ is a continuous real-valued function on $[0,T]$, and $b(t)$ is an increasing continuous real-valued function on $[0,T]$. Thus, the GBMP $Y$ is determined by the continuous functions $a(\cdot)$ and $b(\cdot)$. For more details, see [27, 28]. Note that when $c = 0$, $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, the GBMP is the standard Brownian motion (Wiener process). In this paper we set $c = a(0) = b(0) = 0$. Then the function space $C_{a,b}[0,T]$ induced by the GBMP $Y$ determined by the $a(\cdot)$ and $b(\cdot)$ can be considered as the space of continuous sample paths of $Y$.

In this paper, we establish a general Cameron–Storvick theorem involving the generalized analytic Feynman integral of functionals on the product function space $C_{a,b}^2[0,T]$. We also present a meaningful example to which our Cameron–Storvick type theorem can be applied. To do this, we introduce the class $F_{A_1, A_2}$ of functionals on $C_{a,b}^2[0,T]$ which is a generalization of the Kallianpur and Bromley Fresnel class $F_{A_1, A_2}$, see [21].

2. Preliminaries

In this section we first give a brief background of some ideas and results which are needed to establish our new results in Sections 3, 4, and 5 below.

Let $a(t)$ be an absolutely continuous real-valued function on $[0,T]$ with $a(0) = 0$ and $a'(t) \in L^2[0,T]$, and let $b(t)$ be a strictly increasing, continuously differentiable real-valued function with $b(0) = 0$ and $b'(t) > 0$ for each $t \in [0,T]$. The generalized Brownian motion process $Y$ determined by $a(t)$ and $b(t)$ is a Gaussian process with mean function $a(t)$ and covariance function $r(s,t) = \min\{b(s), b(t)\}$.

We consider the function space $(C_{a,b}[0,T], W(C_{a,b}[0,T]), \mu)$ induced by the generalized Brownian motion process $Y$, where $C_{a,b}[0,T]$ denotes the set of continuous sample paths of the generalized Brownian motion process $Y$ and $W(C_{a,b}[0,T])$ is the $\sigma$-field of all $\mu$-Carathéodory measurable subsets of $C_{a,b}[0,T]$. For the precise procedure used to construct this function space, we refer to the references [4, 5, 6, 10, 11, 27, 28].

We note that the coordinate process defined by $e_t(x) = x(t)$ on $C_{a,b}[0,T]$ is also the generalized Brownian motion process determined by $a(t)$ and $b(t)$, i.e., for each $t \in [0,T]$, $e_t(x) \sim N(a(t), b(t))$, and the process $\{e_t : 0 \leq t \leq T\}$ has nonstationary and independent increments.
Recall that the process \( \{ e_t : 0 \leq t \leq T \} \) on \( C_{a,b}[0,T] \) is a continuous process. Thus the function space \( C_{a,b}[0,T] \) reduces to the classical Wiener space \( C_0[0,T] \), considered in papers [1, 2, 18, 20, 24] if and only if \( a(t) \equiv 0 \) and \( b(t) = t \) for all \( t \in [0,T] \).

In [6, 9, 11], the generalized analytic Feynman integral and the generalized analytic Fourier–Feynman transform of functionals on \( C_{a,b}[0,T] \) were investigated. The functionals considered in [6, 9, 11] are associated with the separable Hilbert space

\[
L_{a,b}^2[0,T] = \left\{ v : \int_0^T v^2(s) \, db(s) < +\infty \text{ and } \int_0^T v^2(s) \, d|a|(s) < +\infty \right\}
\]

where \( |a|(\cdot) \) denotes the total variation function of \( a(\cdot) \). The inner product on \( L_{a,b}^2[0,T] \) is given by the formula

\[
(u,v)_{a,b} = \int_0^T u(s)v(s) \, dm_{|a|,b}(s) \equiv \int_0^T u(s)v(s) \, db(s) + |a|(s)
\]

where \( m_{|a|,b} \) is the Lebesgue–Stieltjes measure induced by the increasing function \( |a| + b(\cdot) \) on \([0,T]\). We note that \( \|u\|_{a,b} = (u,u)_{a,b} = 0 \) if and only if \( u(\cdot) = 0 \) a.e. on \([0,T]\).

The following linear subspace of \( C_{a,b}[0,T] \) plays an important role throughout this paper.

Let

\[
C'_{a,b}[0,T] = \left\{ w \in C_{a,b}[0,T] : w(t) = \int_0^t z(s) \, db(s) \text{ for some } z \in L_{a,b}^2[0,T] \right\}.
\]

For \( w \in C'_{a,b}[0,T] \), with \( w(t) = \int_0^t z(s) \, db(s) \) for \( t \in [0,T] \), let \( D : C'_{a,b}[0,T] \to L_{a,b}^2[0,T] \) be defined by the formula

\[
(Dw)(t) = z(t) = \frac{w'(t)}{b'(t)}.
\]

Then \( C'_{a,b} \equiv C'_{a,b}[0,T] \) with inner product

\[
(w_1,w_2)_{C'_{a,b}} = \int_0^T D_1(s)D_2(s) \, db(s) = \int_0^T z_1(s)z_2(s) \, db(s)
\]

is a separable Hilbert space.

Note that the two separable Hilbert spaces \( L_{a,b}^2[0,T] \) and \( C'_{a,b}[0,T] \) are (topologically) homeomorphic under the linear operator given by equation (2.1). The inverse operator of \( D \) is given by

\[
(D^{-1}z)(t) = \int_0^t z(s) \, db(s), \quad t \in [0,T].
\]

For a more detailed study of the inverse operator \( D^{-1} \) of \( D \), see [8].

Recall that above, as well as in papers [6, 9, 11], we require that \( a : [0,T] \to \mathbb{R} \) is an absolutely continuous function with \( a(0) = 0 \) and with \( \int_0^T |a'(t)|^2 \, dt < \infty \). Our conditions on \( b : [0,T] \to \mathbb{R} \) imply that \( 0 < \delta < b'(t) < M \) for some positive real numbers \( \delta \) and \( M \), and all \( t \in [0,T] \). In this paper, in addition to the conditions put on \( a(t) \) above, we now add the condition

\[
\int_0^T |a'(t)|^2 \, d|a|(t) < +\infty.
\]
One can see that the function $a : [0, T] \to \mathbb{R}$ satisfies the condition \(2.3\) if and only if $a(t)$ is an element of $C'_{a,b}[0, T]$. Under the condition \(2.3\), we also observe that for each $w \in C'_{a,b}[0, T]$ with $Dw = z$,

$$(w, a)_{C'_{a,b}} = \int_0^T Dw(t)Da(t)db(t) = \int_0^T z(t)a'(t)dt = \int_0^T z(t)da(t).$$

For each $w \in C'_{a,b}[0, T]$, the Paley–Wiener–Zygmund (PWZ) stochastic integral $(w, x)\sim$ is given by the formula

\begin{equation}
(w, x)\sim = \lim_{n \to \infty} \int_0^T \sum_{j=1}^n (w, g_j)_{C'_{a,b}}Dg_j(t)dx(t)
\end{equation}

for $\mu$-a.e. $x \in C_{a,b}[0, T]$ where $\{g_j\}_{j=1}^\infty$ is a complete orthonormal set of functions in $C'_{a,b}[0, T]$ such that for each $j \in \mathbb{N}$, $Dg_j$ is of bounded variation on $[0, T]$. For a more detailed study of the space $C'_{a,b}[0, T]$ and the PWZ stochastic integral given by \(2.3\), see \cite{5 10}.

In \cite{25}, Pierce and Skoug used the the inner product $(\cdot, \cdot)_{a,b}$ on $L^2_{a,b}[0, T]$ rather than the inner product $(\cdot, \cdot)_{C'_{a,b}}$ on $C'_{a,b}[0, T]$ to study the PWZ stochastic integral and the related integration formula on the function space $C_{a,b}[0, T]$.

The generalized analytic Feynman integral on the function space $C_{a,b}[0, T]$ was first defined and studied in \cite{[10]} and the study of this integral has continued in \cite{[4 5 9 12]}. We assume familiarity with \cite{[5 6 9 11 12]} and adopt the concepts and the definitions of the generalized analytic Feynman integral on $C_{a,b}[0, T]$.

Based on the references \cite{19, 20, 21}, we present several concepts which involve the scale-invariant measurability to define a generalized analytic Feynman integral of functionals on the product space $C^2_{a,b}[0, T]$.

Let $(C^2_{a,b}[0, T], W(C^2_{a,b}[0, T]), \mu^2)$ be the product function space, where $C^2_{a,b}[0, T] = C_{a,b}[0, T] \times C_{a,b}[0, T]$, $W(C^2_{a,b}[0, T]) \equiv W(C_{a,b}[0, T]) \otimes W(C_{a,b}[0, T])$ denotes the $\sigma$-field generated by measurable rectangles $A \times B$ with $A, B \in W(C_{a,b}[0, T])$, and $\mu^2 \equiv \mu \times \mu$. A subset $B$ of $C^2_{a,b}[0, T]$ is said to be scale-invariant measurable if $\{(\rho_1 x_1, \rho_2 x_2) : (x_1, x_2) \in B\}$ is $W(C^2_{a,b}[0, T])$-measurable for every $\rho_1 > 0$ and $\rho_2 > 0$, and a scale-invariant measurable subset $N$ of $C^2_{a,b}[0, T]$ is said to be scale-invariant null set with $\mu(\mu \times \mu)(\{(\rho_1 x_1, \rho_2 x_2) : (x_1, x_2) \in N\}) = 0$ for every $\rho_1 > 0$ and $\rho_2 > 0$. A property that holds except on a scale-invariant null set is said to hold s-a.e. on $C^2_{a,b}[0, T]$. A functional $F$ on $C^2_{a,b}[0, T]$ is said to be scale-invariant measurable provided $F$ is defined on a scale-invariant measurable set and $F(\rho_1 \cdot, \rho_2 \cdot)$ is $W(C^2_{a,b}[0, T])$-measurable for every $\rho_1 > 0$ and $\rho_2 > 0$. If two functionals $F$ and $G$ defined on $C^2_{a,b}[0, T]$ are equal s-a.e., then we write $F \equiv G$.

We denote the product function space integral of a $W(C^2_{a,b}[0, T])$-measurable functional $F$ by

$$E[F] \equiv E_{[0, T]}[F(x_1, x_2)] = \int_{C^2_{a,b}[0, T]} F(x_1, x_2)d(\mu \times \mu)(x_1, x_2)$$

whenever the integral exists.

Throughout this paper, let $\mathbb{C}$, $\mathbb{C}_+$ and $\mathbb{C}_+$ denote the complex numbers, the complex numbers with positive real part and the nonzero complex numbers with nonnegative real part, respectively. Furthermore, for each $\lambda \in \mathbb{C}_+$, $\lambda^{1/2}$ denotes the principal square root of $\lambda$; i.e., $\lambda^{1/2}$ is always chosen to have positive real part,
so that \( \lambda^{-1/2} = (\lambda^{-1})^{1/2} \) is in \( \mathbb{C}_+ \). We also assume that every functional \( F \) on \( C^2_{a,b}[0,T] \) we consider is s.a.e. defined and is scale-invariant measurable.

The following definition is due to Choi, Skoug and Chang [10, 13].

**Definition.** Let \( \mathbb{C}^2_+ = \{ \lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2 : \text{Re}(\lambda_j) > 0 \text{ for } j = 1, 2 \} \) and let \( \bar{\mathbb{C}}^2_+ = \{ \lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2 : \lambda_j \neq 0 \text{ and } \text{Re}(\lambda_j) \geq 0 \text{ for } j = 1, 2 \} \). Let \( F : C^2_{a,b}[0,T] \to \mathbb{C} \) be a scale-invariant measurable functional such that the function space integral

\[
J(\lambda_1, \lambda_2) = \int_{C^2_{a,b}[0,T]} F(\lambda_1^{-1/2} x_1, \lambda_2^{-1/2} x_2) d(\mu \times \mu)(x_1, x_2)
\]

exists and is finite for each \( \lambda_1 > 0 \) and \( \lambda_2 > 0 \). If there exists a function \( J^*(\lambda_1, \lambda_2) \) analytic on \( \mathbb{C}^2_+ \) such that \( J^*(\lambda_1, \lambda_2) = J(\lambda_1, \lambda_2) \) for all \( \lambda_1 > 0 \) and \( \lambda_2 > 0 \), then \( J^*(\lambda_1, \lambda_2) \) is defined to be the analytic function space integral of \( F \) over \( C^2_{a,b}[0,T] \) with parameter \( \tilde{\lambda} = (\lambda_1, \lambda_2) \), and for \( \tilde{\lambda} \in \mathbb{C}^2_+ \) we write

\[
E^{an}_{\tilde{\lambda}}[F] \equiv E_{x_1}^{an}[F(x_1, x_2)] \equiv E_{x_1, x_2}^{an(\lambda_1, \lambda_2)}[F(x_1, x_2)] = J^*(\lambda_1, \lambda_2).
\]

Let \( g_1 \) and \( g_2 \) be nonzero real numbers. Let \( F \) be a functional such that \( E^{an}_{\tilde{\lambda}}[F] \) exists for all \( \tilde{\lambda} \in \mathbb{C}^2_+ \). If the following limit exists, we call it the generalized analytic Feynman integral of \( F \) with parameter \( \tilde{\lambda} = (g_1, g_2) \) and we write

\[
E^{an}_{\tilde{\lambda}}[F] \equiv E_{x_1}^{an, g}[F(x_1, x_2)] \equiv E_{x_1, x_2}^{an(\lambda_1, \lambda_2), g}[F(x_1, x_2)] = \lim_{\tilde{\lambda} \to \tilde{\lambda}} E^{an}_{\tilde{\lambda}}[F].
\]

3. A CAMERON–STORVICK TYPE THEOREM ON \( C^2_{a,b}[0,T] \)

In [1], Cameron (also see [2, Theorem A]) expressed the Wiener integral of the first variation of a functional \( F \) on the Wiener space \( C_0[0,T] \) in terms of the Wiener integral of the product of \( F \) by a linear functional, and in [2, Theorem 1], Cameron and Storvick obtained a similar result for the analytic Feynman integral on \( C_0[0,T] \). In [3, Theorem 2.4], Chang, Song and Yoo also obtained a Cameron–Storvick theorem on abstract Wiener spaces. In [11], Chang and Skoug developed these results for functionals on the function space \( C_{a,b}[0,T] \).

In this section, we establish a Cameron–Storvick theorem for the generalized analytic Feynman integral of functionals on the product function space \( C^2_{a,b}[0,T] \). To do this we first give the definition of the first variation of a functional \( F \) on \( C^2_{a,b}[0,T] \).

**Definition.** Let \( F \) be a functional on \( C^2_{a,b}[0,T] \) and let \( g_1 \) and \( g_2 \) be functions in \( C_{a,b}[0,T] \). Then

\[
\delta F(x_1, x_2; g_1, g_2) = \frac{\partial}{\partial h} \left( F(x_1 + hg_1, x_2) + F(x_1, x_2 + hg_2) \right) \bigg|_{h=0} = \frac{\partial}{\partial h} F(x_1 + hg_1, x_2) \bigg|_{h=0} + \frac{\partial}{\partial h} F(x_1, x_2 + hg_2) \bigg|_{h=0}
\]

(3.1)

(if it exists) is called the first variation of \( F \) in the direction of \( (g_1, g_2) \).

Throughout this section, when working with \( \delta F(x_1, x_2; g_1, g_2) \), we will always require \( g_1 \) and \( g_2 \) to be functions in \( C'_{a,b}[0,T] \).

We first quote the translation theorem [11] for the function space integral using our notations.
Lemma 3.1 (Translation Theorem). Let $F \in L^1(C_{a,b}[0,T])$ and let $w_0 \in C_{a,b}'[0,T]$. Then

$$
\int_{C_{a,b}[0,T]} F(x + w_0) d\mu(x)
$$

(3.2)

$$
= \exp \left\{ - \frac{1}{2} \| w_0 \|_{C_{a,b}'}^2 - (w_0, a)_{C_{a,b}'} \right\} \int_{C_{a,b}[0,T]} F(x) \exp \{(w_0, x)^-\} d\mu(x).
$$

Theorem 3.2 (An integration by parts formula). Let $g_1$ and $g_2$ be nonzero functions in $C_{a,b}'[0,T]$. Let $F(x_1, x_2)$ be $\mu \times \mu$-integrable over $C_{a,b}[0,T]$. Assume that $F$ has a first variation $\delta F(x_1, x_2|g_1, g_2)$ for all $(x_1, x_2) \in C_{a,b}^2[0,T]$ such that for some $\gamma > 0$,

$$
\sup_{|h| \leq \gamma} |\delta F(x_1 + hg_1, x_2 + hg_2|g_1, g_2)|
$$

is $\mu \times \mu$-integrable over $C_{a,b}^2[0,T]$ as a functional of $(x_1, x_2) \in C_{a,b}^2[0,T]$. Then

$$
E_{\mathcal{F}}[\delta F(x_1, x_2|g_1, g_2)]
$$

(3.3)

$$
= E_{\mathcal{F}}[F(x_1, x_2)\{ (g_1, x_1)^- + (g_2, x_2)^- \}]
$$

$$
- \{ (g_1, a)_{C_{a,b}'} + (g_2, a)_{C_{a,b}'} \} E_{\mathcal{F}}[F(x_1, x_2)].
$$

Proof. First note that

$$
\delta F(x_1 + hg_1, x_2 + hg_2|g_1, g_2)
$$

$$
= \frac{\partial}{\partial \lambda} F(x_1 + hg_1 + \lambda g_1, x_2 + hg_2) \bigg|_{\lambda=0} + \frac{\partial}{\partial \mu} F(x_1 + hg_1, x_2 + hg_2 + \lambda g_2) \bigg|_{\lambda=0}
$$

$$
= \frac{\partial}{\partial \lambda} F(x_1 + (h + \lambda)g_1, x_2 + hg_2) \bigg|_{\lambda=0} + \frac{\partial}{\partial \mu} F(x_1 + hg_1, x_2 + (h + \lambda)g_2) \bigg|_{\lambda=0}
$$

$$
= \frac{\partial}{\partial \mu} F(x_1 + \mu g_1, x_2 + hg_2) \bigg|_{\mu=h} + \frac{\partial}{\partial \mu} F(x_1 + hg_1, x_2 + \mu g_2) \bigg|_{\mu=h}
$$

$$
= 2 \frac{\partial}{\partial h} F(x_1 + hg_1, x_2 + hg_2).
$$

But since

$$
\sup_{|h| \leq \gamma} \left| \frac{\partial}{\partial h} F(x_1 + hg_1, x_2 + hg_2) \right|
$$

is $\mu \times \mu$-integrable,

$$
\frac{\partial}{\partial h} F(x_1 + hg_1, x_2 + hg_2)
$$
is $\mu \times \mu$-integrable for sufficiently small values of $h$. Hence by the Fubini theorem and equation (3.2), it follows that

$$E_{\vec{x}}[\delta F(x_1, x_2|g_1, g_2)]$$

$$= E_{\vec{x}} \left[ \frac{\partial}{\partial h} \left( F(x_1 + hg_1, x_2) + F(x_1, x_2 + hg_2) \right) \right]_{h=0}$$

$$= \frac{\partial}{\partial h} E_{\vec{x}} \left[ F(x_1 + hg_1, x_2) + F(x_1, x_2 + hg_2) \right]_{h=0}$$

$$= \frac{\partial}{\partial h} E_{\vec{x}} \left[ F(x_1 + hg_1, x_2) \right]_{h=0} + \frac{\partial}{\partial h} E_{\vec{x}} \left[ F(x_1, x_2 + hg_2) \right]_{h=0}$$

$$= \frac{\partial}{\partial h} E_{x_2} \left[ F(x_1 + hg_1, x_2) \right]_{h=0} + \frac{\partial}{\partial h} E_{x_2} \left[ F(x_1, x_2 + hg_2) \right]_{h=0}$$

$$= \frac{\partial}{\partial h} \left( \exp \left\{ - \frac{h^2}{2} \| g_1 \|^2_{C_{a,b}^*} - h(g_1, a) C_{a,b} \right\} \right. \times E_{x_2} \left[ F(x_1, x_2) \exp \{ h(g_1, x_1)^{-} \} \right]_{h=0}$$

$$+ \frac{\partial}{\partial h} \left( \exp \left\{ - \frac{h^2}{2} \| g_2 \|^2_{C_{a,b}^*} - h(g_2, a) C_{a,b} \right\} \right. \times E_{x_1} \left[ F(x_1, x_2) \exp \{ h(g_2, x_2)^{-} \} \right]_{h=0}$$

$$= E_{\vec{x}} \left[ F(x_1, x_2) \left\{ (g_1, x_1)^{-} + (g_2, x_2)^{-} \right\} \right]$$

$$- \left\{ (g_1, a) C_{a,b}^* + (g_2, a) C_{a,b} \right\} E_{\vec{x}} \left[ F(x_1, x_2) \right]$$

as desired. \qed

**Lemma 3.3.** Let $g_1$, $g_2$, and $F$ be as in Theorem 3.2. For each $\rho_1 > 0$ and $\rho_2 > 0$, assume that $F(\rho_1 x_1, \rho_2 x_2)$ is $\mu \times \mu$-integrable. Furthermore assume that $F(\rho_1 x_1, \rho_2 x_2)$ has a first variation $\delta F(\rho_1 x_1, \rho_2 x_2|\rho_1 g_1, \rho_2 g_2)$ for all $(x_1, x_2) \in C^2_{a,b}[0, T]$ such that for some positive function $\gamma(\rho_1, \rho_2)$,

$$\sup_{|h| \leq \gamma(\rho_1, \rho_2)} |\delta F(\rho_1 x_1 + \rho_1 h g_1, \rho_2 x_2 + \rho_2 h g_2|\rho_1 g_1, \rho_2 g_2)|$$

is $\mu \times \mu$-integrable over $C^2_{a,b}[0, T]$ as a functional of $(x_1, x_2) \in C^2_{a,b}[0, T]$. Then

$$E_{\vec{x}}[\delta F(\rho_1 x_1, \rho_2 x_2|\rho_1 g_1, \rho_2 g_2)]$$

(3.4)

$$= E_{\vec{x}} \left[ F(\rho_1 x_1, \rho_2 x_2) \left\{ (g_1, x_1)^{-} + (g_2, x_2)^{-} \right\} \right]$$

$$- \left\{ (g_1, a) C_{a,b}^* + (g_2, a) C_{a,b} \right\} E_{\vec{x}} \left[ F(\rho_1 x_1, \rho_2 x_2) \right].$$

**Proof.** Given a pair $(\rho_1, \rho_2)$ with $\rho_1 > 0$ and $\rho_2 > 0$, let $R_{(\rho_1, \rho_2)}(x_1, x_2) = F(\rho_1 x_1, \rho_2 x_2)$. Then we have that

$$R_{(\rho_1, \rho_2)}(x_1 + \rho_1 h g_1, x_2) = F(\rho_1 x_1 + h g_1, \rho_2 x_2)$$

and

$$R_{(\rho_1, \rho_2)}(x_1, x_2 + \rho_2 h g_2) = F(\rho_1 x_1, \rho_2 x_2 + h g_2)$$

and that

$$\frac{\partial}{\partial h} R_{(\rho_1, \rho_2)}(x_1 + \rho_1 h g_1, x_2)_{h=0} = \frac{\partial}{\partial h} F(\rho_1 x_1 + \rho_1 h g_1, \rho_2 x_2)_{h=0}$$
and
\[
\frac{\partial}{\partial h} R_{(\rho_1, \rho_2)}(x_1, x_2 + hg_2) \bigg|_{h=0} = \frac{\partial}{\partial h} F(\rho_1 x_1, \rho_2 x_2 + \rho_2 g_2) \bigg|_{h=0}.
\]
Thus we have
\[
\delta F(\rho_1 x_1, \rho_2 x_2; \rho_1 g_1, \rho_2 g_2)
= \frac{\partial}{\partial h} F(\rho_1 x_1 + \rho_1 h g_1, \rho_2 x_2) \bigg|_{h=0} + \frac{\partial}{\partial h} F(\rho_1 x_1, \rho_2 x_2 + \rho_2 h g_2) \bigg|_{h=0}
\]
\[
= \frac{\partial}{\partial h} R_{(\rho_1, \rho_2)}(x_1 + hw_1, x_2) + \frac{\partial}{\partial h} R_{(\rho_1, \rho_2)}(x_1, x_2 + hg_2) \bigg|_{h=0}
\]
\[
= \delta R_{(\rho_1, \rho_2)}(x_1, x_2|g_1, g_2).
\]
Hence by equation (3.6) with \(F\) replaced with \(R_{(\rho_1, \rho_2)}\), we have
\[
E_\mathbb{F}[\delta F(\rho_1 x_1, \rho_2 x_2; \rho_1 g_1, \rho_2 g_2)]
= E_\mathbb{F}[\delta R(x_1, x_2|g_1, g_2)]
= E_\mathbb{F}[R(x_1, x_2)((g_1, x_1)^- + (g_2, x_2)^-)]
- \{(g_1, a)C_{a,b}' + (g_2, a)C_{a,b}'\} E_\mathbb{F}[R(x_1, x_2)]
= E_\mathbb{F}[F(\rho_1 x_1, \rho_2 x_2)((g_1, x_1)^- + (g_2, x_2)^-)]
- \{(g_1, a)C_{a,b}' + (g_2, a)C_{a,b}'\} E_\mathbb{F}[F(\rho_1 x_1, \rho_2 x_2)]
\]
which establishes equation (3.4).

**Theorem 3.4.** Let \(g_1, g_2\), and \(F\) be as in Lemma 3.3. Then if any two of the three generalized analytic Feynman integrals in the following equation exist, then the third one also exists, and equality holds:
\[
E_\mathbb{F}^\text{anf}(g_1, g_2)[\delta F(x_1, x_2|g_1, g_2)]
= -iE_\mathbb{F}^\text{anf}(g_1, g_2)[F(x_1, x_2)((g_1, x_1)^- + g_2(x_2)^-)]
- \{(i q_1)^{1/2}(1, a)C_{a,b}' + (g_2, a)C_{a,b}'\} E_\mathbb{F}^\text{anf}(g_1, g_2)[F(x_1, x_2)].
\]

**Proof.** Let \(\rho_1 > 0\) and \(\rho_2 > 0\) be given. Let \(y_1 = \rho_1^{-1} g_1\) and \(y_2 = \rho_2^{-1} g_2\). By equation (3.4),
\[
E_\mathbb{F}[\delta F(\rho_1 x_1, \rho_2 x_2|g_1, g_2)]
= E_\mathbb{F}[\delta F(\rho_1 x_1, \rho_2 x_2|\rho_1 y_1, \rho_2 y_2)]
= E_\mathbb{F}[F(\rho_1 x_1, \rho_2 x_2)((g_1, x_1)^- + (g_2, x_2)^-)]
- \{(g_1, a)C_{a,b}' + (g_2, a)C_{a,b}'\} E_\mathbb{F}[F(\rho_1 x_1, \rho_2 x_2)]
= E_\mathbb{F}[F(\rho_1 x_1, \rho_2 x_2)((g_1, x_1)^- + \rho_2^{-2}(g_2, \rho_2 x_2)^-)]
- \{\rho_1^{-1}(g_1, a)C_{a,b}' + \rho_2^{-1}(g_2, a)C_{a,b}'\} E_\mathbb{F}[F(\rho_1 x_1, \rho_2 x_2)].
\]
Now let \(\rho_1 = \lambda_1^{-1/2}\) and \(\rho_2 = \lambda_2^{-1/2}\). Then equation (3.6) becomes
\[
E_\mathbb{F}[\delta F(\lambda_1^{-1/2} x_1, \lambda_2^{-1/2} x_2|g_1, g_2)]
= E_\mathbb{F}[F(\lambda_1^{-1/2} x_1, \lambda_2^{-1/2} x_2)((\lambda_1, x_1)^- + \lambda_2(x_2)^-)]
- \{\lambda_1^{1/2}(g_1, a)C_{a,b}' + \lambda_2^{1/2}(g_2, a)C_{a,b}'\} E_\mathbb{F}[F(\lambda_1^{-1/2} x_1, \lambda_2^{-1/2} x_2)].
\]
Since $\rho_1 > 0$ and $\rho_2 > 0$ were arbitrary, we have that equation \((3.7)\) holds for all $\lambda_1 > 0$ and $\lambda_2 > 0$. We now use Definition \(2\) to obtain our desired conclusion. \(\square\)

**Corollary 3.5.** Let $H$ be a $\mu$-integrable functional on $C_{a,b}[0,T]$. Let $g$ be a function in $C'_{a,b}[0,T]$. Then if any two of the three generalized analytic Feynman integrals on $C_{a,b}[0,T]$ in the following equation exist, then the third one also exists, and equality holds:

\[
E_x^{\text{anf}}[\delta H(x)] = -iqE_x^{\text{anf}}[H(x)] - (-iq)^{1/2}(g,a)C_a E_x^{\text{anf}}[H(x)],
\]

where $E_x^{\text{anf}}[H(x)] = \int_{C_{a,b}[0,T]}^\text{anf} H(x)d\mu(x)$ means the generalized analytic Feynman integral of functionals $H$ on $C_{a,b}[0,T]$, see [13, 14, 15, 16, 17, 18, 19].

**Proof.** Simply choose $F(x_1, x_2) = H(x_1)$. \(\square\)

4. **FUNCTIONALS IN THE GENERALIZED FRENSIL TYPE CLASS $F_{A_1,A_2}$**

In this section we introduce the generalized Fresnel type class $F_{A_1,A_2}^{a,b}$ to which we apply our Cameron–Storvick type theorem.

Let $\mathcal{M}(C_{a,b}[0,T])$ be the space of complex-valued, countably additive (and hence finite) Borel measures on $C_{a,b}[0,T]$. The space $\mathcal{M}(C_{a,b}[0,T])$ is a Banach algebra under the total variation norm and with convolution as multiplication, see [13, 20].

**Definition.** Let $A_1$ and $A_2$ be bounded, nonnegative self-adjoint operators on $C_{a,b}[0,T]$. The generalized Fresnel type class $F_{A_1,A_2}^{a,b}$ of functionals on $C_{a,b}^2[0,T]$ is defined as the space of all functionals $F$ on $C_{a,b}^2[0,T]$ of the form

\[
F(x_1, x_2) = \int_{C_{a,b}[0,T]} \exp \left\{ \sum_{j=1}^{2} i(A_1^{1/2}w, x_j)^{\sim} \right\} df(w)
\]

for s-a.e. $(x_1, x_2) \in C_{a,b}^2[0,T]$, where $f$ is in $\mathcal{M}(C_{a,b}[0,T])$. More precisely, since we identify functionals which coincide s-a.e. on $C_{a,b}^2[0,T]$, $F_{A_1,A_2}^{a,b}$ can be regarded as the space of all s-equivalence classes of functionals of the form \((4.1)\). For more details, see [10, 13].

**Remark 4.1.** (1) Note that in case $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, the function space $C_{a,b}[0,T]$ reduces to the classical Wiener space $C_0[0,T]$. In this case, the generalized Fresnel type class $F_{A_1,A_2}^{a,b}$ reduces to the Kallianpur and Bromley Fresnel class $F_{A_1,A_2}$, see [21].

(2) In addition, if we choose $A_1 = I$ (identity operator) and $A_2 = 0$ (zero operator), then the class $F_{A_1,A_2}^{a,b}$ reduces to the Fresnel class $F(C_0[0,T])$. It is known, see [18], that $F(C_0[0,T])$ forms a Banach algebra over the complex field.

(3) The map $f \mapsto F$ defined by \((4.1)\) sets up an algebra isomorphism between $\mathcal{M}(C_{a,b}[0,T])$ and $F_{A_1,A_2}^{a,b}$ if $\text{Ran}(A_1 + A_2)$ is dense in $C_{a,b}[0,T]$ where $\text{Ran}$ indicates the range of an operator. In this case, $F_{A_1,A_2}^{a,b}$ becomes a Banach algebra under the norm $\|F\| = \|f\|$. For more details, see [21].

As discussed in [13, Remark 7], for a functional $F$ in $F_{A_1,A_2}^{a,b}$ and a vector $\vec{q} = (q_1, q_2)$ with $q_1 \neq 0$ and $q_2 \neq 0$, the generalized analytic Feynman integral $E^{\text{anf}}[F]$ might not exist. By a simple modification of the example illustrated in [3], we
can construct an example for the functional whose generalized analytic Feynman integral $E^{\text{antf}}[F]$ does not exist. Thus we need to impose additional restrictions on the functionals $F$ in $\mathcal{F}_{A_1, A_2}^{a,b}$.

Given a positive real number $q_0 > 0$, and bounded, nonnegative self-adjoint operators $A_1$ and $A_2$ on $C_{a,b}[0,T]$, let

$$k(q_0; \vec{A}; w) \equiv k(q_0; A_1, A_2; w)$$

(4.2)

$$= \exp \left\{ \sum_{j=1}^{2} (2q_0)^{-1/2} A_j^{1/2} \|w\| \|A\| C_{a,b} \right\}$$

where $\|A_j^{1/2}\|_o$ means the operator norm of $A_j^{1/2}$ for $j \in \{1, 2\}$. For the existence of the generalized analytic Feynman integral of $F$, we define a subclass $\mathcal{F}_{A_1, A_2}^{q_0}$ of $\mathcal{F}_{A_1, A_2}^{a,b}$ by $F \in \mathcal{F}_{A_1, A_2}^{q_0}$ if and only if

$$\int_{C_{a,b}[0,T]} k(q_0; \vec{A}; w) df(w) < +\infty$$

where $f$ and $F$ are related by equation (4.1) and $k$ is given by equation (4.2).

The following theorem is due to Choi, Skoug and Chang [13].

**Theorem 4.2.** Let $q_0$ be a positive real number and let $F$ be an element of $\mathcal{F}_{A_1, A_2}^{q_0}$. Then for all real numbers $q_1$ and $q_2$ with $q_1 > q_2$, $j \in \{1, 2\}$, the generalized analytic Feynman integral $E^{\text{antf}}[F]$ of $F$ exists and is given by the formula

$$E^{\text{antf}}[F] = \int_{C_{a,b}[0,T]} \psi(-i\vec{q}; \vec{A}; w) df(w),$$

(4.3)

where $\psi(-i\vec{q}; \vec{A}; w)$ is given by

$$\psi(-i\vec{q}; \vec{A}; w) = \exp \left\{ \sum_{j=1}^{2} \left[ - \frac{i(A_j w, w) C_{a,b}}{2q_j} + i(-i\vec{q})^{-1/2}(A_j^{1/2} w, a) C_{a,b} \right] \right\}.$$

(4.4)

For $j \in \{1, 2\}$, let $g_j \in C_{a,b}[0,T]$ and let $F$ be an element of $\mathcal{F}_{A_1, A_2}^{a,b}$ whose associated measure $f$, see equation (4.1), satisfies the inequality

$$\int_{C_{a,b}[0,T]} \|w\| C_{a,b} df(w) < +\infty.$$

Then using equation (4.1), we obtain that

$$\delta F(x_1, x_2|g_1, g_2)$$

$$= \sum_{k=1}^{2} \left[ \frac{\partial}{\partial h} \left( \int_{C_{a,b}[0,T]} \exp \left\{ \sum_{j=1}^{2} i(A_j^{1/2} w, x_j) \right\} \right) \right]_{h=0}$$

$$= \int_{C_{a,b}[0,T]} \left[ \sum_{k=1}^{2} i(A_k^{1/2} w, g_k) C_{a,b} \right] \exp \left\{ \sum_{j=1}^{2} i(A_j^{1/2} w, x_j) \right\} df(w)$$

$$= \int_{C_{a,b}[0,T]} \exp \left\{ \sum_{j=1}^{2} i(A_j^{1/2} w, x_j) \right\} d\sigma(\vec{A}; \vec{w})$$

(4.6)
where the complex measure $\sigma^{A,q}$ is defined by

$$\sigma^{A,q}(B) = \int_B \left[ \sum_{k=1}^{2} i(A_{j_{k}}^1/2 w, g_{j}) c_{a,b}^j \right] df(w), \quad B \in \mathcal{B}(C_{a,b}^1[0,T]).$$

The second equality of (4.6) follows from (4.5) and Theorem 2.27 in [17]. Also, $\delta F(x_1, x_2 | g_1, g_2)$ is an element of $\mathcal{F}_{A_{1},A_{2}}$ as a functional of $(x_1, x_2)$, since by the Cauchy–Schwartz inequality and (4.5),

$$\|\sigma^{A,q}\| \leq \int_{C_{a,b}^1[0,T]} \left[ \sum_{j=1}^{2} |i(A_{j_{j_{1}}}^1/2 w, g_{j}) c_{a,b}^j| \right] df(w)$$

$$\leq \int_{C_{a,b}^1[0,T]} \left[ \sum_{j=1}^{2} \|A_{j_{j_{1}}}^1/2\| \|w\| c_{a,b}^j \|g_{j}\| c_{a,b}^j \right] df(w)$$

$$\leq \left( \sum_{j=1}^{2} \|A_{j_{j_{1}}}^1/2\| \|g_{j}\| c_{a,b}^j \|w\| c_{a,b}^j \right) \int_{C_{a,b}^1[0,T]} \|w\| c_{a,b}^j df(w) < +\infty,$$

where $\|A_{j_{j_{1}}}^1/2\|$ is the operator norm of $A_{j_{j_{1}}}^1/2$.

For the existence of the generalized analytic Feynman integral of the first variation $\delta F$ of a functional $F$ in $\mathcal{F}_{A_{1},A_{2}}$, we also define a subclass of $\mathcal{F}_{A_{1},A_{2}}$ as follows: given a positive real number $q_0$, we define a subclass $G_{A_{1},A_{2}}^{q_0}$ of $\mathcal{F}_{A_{1},A_{2}}$ by $F \in G_{A_{1},A_{2}}^{q_0}$ if and only if

$$\int_{C_{a,b}^1[0,T]} \|w\| c_{a,b}^j k(q_0; A; w) df(w) < +\infty$$

where $f$ is the associated measure of $F$ by equation (4.1) and $k(q_0; A; w)$ is given by equation (4.2).

Our next theorem follows quite readily from the techniques developed in the proof of Theorem 9 of [13].

**Theorem 4.3.** Let $q_0$ be a positive real number and let $g_1$ and $g_2$ be functions in $C_{a,b}^1[0,T]$. Let $F$ be an element of $G_{A_{1},A_{2}}^{q_0}$. Then for all real numbers $q_1$ and $q_2$ with $|q_j| > q_0$, $j \in \{1, 2\}$, the generalized analytic Feynman integral of $\delta F(\cdot, \cdot | g_1, g_2)$ exists and is given by the formula

$$E_{x_{1}}^{q_{1},q_{2}}[\delta F(x_1, x_2 | g_1, g_2)] = \int_{C_{a,b}^1[0,T]} \left[ \sum_{j=1}^{2} i(A_{j_{j_{1}}}^1/2 w, g_{j}) c_{a,b}^j \right] df_{q_{1},q_{2}}^{x_{1}}(w),$$

where $f_{q_{1},q_{2}}^{x_{1}}$ is a complex measure on $B(C_{a,b}^1[0,T])$, the Borel $\sigma$-algebra of $C_{a,b}^1[0,T]$, given by

$$f_{q_{1},q_{2}}^{x_{1}}(B) = \int_{B} \psi(-iq_{1}; A; w) df(w), \quad B \in B(C_{a,b}^1[0,T])$$

and where $\psi(-iq_{1}; A; w)$ is given by (4.3).

5. **Applications of the Cameron–Storvick type theorem**

Let $A$ be a bounded self-adjoint operator on $C_{a,b}^1[0,T]$. Then we can write

$$A = A_+ - A_-$$
where $A_+$ and $A_-$ are each bounded, nonnegative and self-adjoint. Take $A_1 = A_+$ and $A_2 = A_-$ in the definition of $\mathcal{F}_{A_1, A_2}^{a,b}$ above. In this section we consider functionals in the generalized Fresnel type class $\mathcal{F}_{A}^{a,b} = \mathcal{F}_{A_1, A_2}^{a,b}$ where $A$, $A_+$ and $A_-$ are related by equation (5.1) above.

Let $C_{a,b}^{*}[0, T]$ be the set of functions $k$ in $C_{a,b}^{*}[0, T]$ such that $Dk$ is continuous except for a finite number of finite jump discontinuities and is of bounded variation on $[0, T]$. For any $w \in C_{a,b}^{*}[0, T]$ and $k \in C_{a,b}^{*}[0, T]$, let the operation $\odot$ between $C_{a,b}^{*}[0, T]$ and $C_{a,b}^{*}[0, T]$ be defined by

$$w \odot k = D^{-1}(DwDk)$$

so that $D(w \odot k) = DwDk$, where $DwDk$ denotes the pointwise multiplication of the functions $Dw$ and $Dk$. In equation (5.2), the operator $D^{-1}$ is given by (2.2) above. Then $(C_{a,b}^{*}[0, T], \odot)$ is a commutative algebra with the identity $b$. Also we can observe that for any $w, w_1, w_2 \in C_{a,b}^{*}[0, T]$ and $k \in C_{a,b}^{*}[0, T], (5.3)$

$$w \odot k = k \odot w$$

and

$$\text{(5.4)} \quad (w_1, w_2 \odot k)_{C_{a,b}^{*}} = (w_1 \odot k, w_2)_{C_{a,b}^{*}}.$$  

For a more detailed study of the class $C_{a,b}^{*}[0, T]$, see [5].

Next let $\vartheta$ be a function in $C_{a,b}^{*}[0, T]$ with $D\vartheta = \theta$. Define an operator $A : C_{a,b}^{*}[0, T] \to C_{a,b}^{*}[0, T]$ by

$$Aw(t) = (\vartheta \odot w)(t) = \int_0^t D\vartheta(s)Dw(s)db(s)$$

$$= \int_0^t \theta(s)\frac{w'(s)}{b(s)}db(s) = \int_0^t \theta(s)dw(s).$$

It is easily shown that $A$ is a self-adjoint operator. We also see that $A = A_+ - A_-$ where

$$A_+ w(t) = \int_0^t \theta^+(s)Dw(s)db(s) = \int_0^t \theta^+(s)dw(s)$$

and

$$A_- w(t) = \int_0^t \theta^-(s)Dw(s)db(s) = \int_0^t \theta^-(s)dw(s)$$

and where $\theta^+$ and $\theta^-$ are the positive part and the negative part of $\theta$, respectively. Also, $A_{1/2}^+$ and $A_{1/2}^-$ are given by

$$A_{1/2}^+ w(t) = \int_0^t \sqrt{\theta^+}(s)dw(s) \quad \text{and} \quad A_{1/2}^- w(t) = \int_0^t \sqrt{\theta^-}(s)dw(s),$$

respectively. For a more detailed study of this decomposition, see [19] pp.187–189. For notational convenience, let $\vartheta_{1/2}^+ = D^{-1}\sqrt{\theta^+}$ and let $\vartheta_{1/2}^- = D^{-1}\sqrt{\theta^-}$. Then it follows that

$$A_{1/2}^+ w(t) = (\vartheta_{1/2}^+ \odot w)(t) \quad \text{and} \quad A_{1/2}^- w(t) = (\vartheta_{1/2}^- \odot w)(t),$$

respectively.
For fixed \( g \in C'_{a,b}[0,T] \), let \( g_1 = A_1^{1/2}g \) and \( g_2 = A_1^{1/2}(-g) \). Then we see that for any functions \( w \) in \( C'_{a,b}[0,T] \),

\[
(A_+^{1/2}w, g_1)_{C'_{a,b}} + (A_-^{1/2}w, g_2)_{C'_{a,b}} = (A_+^{1/2}w, A_+^{1/2}g)_{C'_{a,b}} - (A_-^{1/2}w, A_-^{1/2}g)_{C'_{a,b}}
\]

(5.7)

Using equations (5.6), (5.3) and (5.4), we also see that

\[
(A_+^{1/2}w, a)_{C'_{a,b}} = (\vartheta_+^{1/2} \odot w, a)_{C'_{a,b}} = (w \odot \vartheta_+^{1/2}, a)_{C'_{a,b}} = (w, \vartheta_+^{1/2} \odot a)_{C'_{a,b}}
\]

and

\[
(A_-^{1/2}w, a)_{C'_{a,b}} = (w, \vartheta_-^{1/2} \odot a)_{C'_{a,b}}.
\]

Assume that \( F \) is an element of \( \mathcal{F}_q^{\delta} \cap \mathcal{G}_q^{\delta} \) for some \( q_0 \in (0,1) \) where \( \mathcal{F}_q^{\delta} \equiv \mathcal{F}_{A_{t+}, A_{t-}}^{q} \) and \( \mathcal{G}_q^{\delta} \equiv \mathcal{G}_{A_{t+}, A_{t-}}^{q} \) (the classes \( \mathcal{F}_q^{\delta} \) and \( \mathcal{G}_q^{\delta} \) are defined in Section 3 respectively).

Applying (4.7) with \((g_1, g_2) = (1, -1)\) and \((g_1, g_2) = (A_1^{1/2}, -A_1^{-1/2})\), respectively, (5.8), and (5.4) with \((A_1, A_2) = (A_{t+}, A_{t-})\), and using (5.7), we obtain that

\[
E^\text{aff}_{x} E^\text{aff}_{x} [\delta F(x_1, x_2|A_1^{1/2}, -A_1^{-1/2})]
\]

(5.8)

\[
E^\text{aff}_{x} E^\text{aff}_{x} [\delta F(x_1, x_2|g_1, g_2)]
\]

(5.9)

\[
\int_{C'_{a,b}[0,T]} \int_{C'_{a,b}[0,T]}
\]

\[
\int_{C'_{a,b}[0,T]} \int_{C'_{a,b}[0,T]} df(w)
\]

(5.10)

We also see that for all \( \rho_1 > 0, \rho_2 > 0 \) and \( h \in \mathbb{R} \)

\[
|\delta F(\rho_1 x_1 + \rho_1 g_1, \rho_2 x_2 + \rho_2 g_2|\rho_1 g_1, \rho_2 g_2)|
\]

\[
\leq \int_{C'_{a,b}[0,T]} \left|(A_+^{1/2}w, A_+^{1/2}(\rho_1 g))_{C'_{a,b}} \right| + \left|(A_-^{1/2}w, A_-^{1/2}(\rho_1 g))_{C'_{a,b}} \right| df(w)
\]

\[
\leq \rho_1 \int_{C'_{a,b}[0,T]} \left|(A_+ w, g)_{C'_{a,b}} \right| df(w) + \rho_2 \int_{C'_{a,b}[0,T]} \left|(A_- w, g)_{C'_{a,b}} \right| df(w)
\]

\[
\leq \rho_1 ||A_+||_o + \rho_2 ||A_-||_o ||g||_{C'_{a,b}} \int_{C'_{a,b}[0,T]} ||w||_{C'_{a,b}} df(w).
\]

But the last expression above is bounded and is independent of \((x_1, x_2) \in C^2_{a,b}[0,T]\).

Hence \(\delta F(\rho_1 x_1 + \rho_1 g_1, \rho_2 x_2 + \rho_2 g_2|\rho_1 g_1, \rho_2 g_2)\) is \(\mu \times \mu\)-integrable in \((x_1, x_2) \in \)
$C^2_{a,b}[0,T]$ for every $\rho_1 > 0$ and $\rho_2 > 0$. Also by Theorems 4.3 and 4.2, the
generalized analytic Feynman integrals $E_{x}^{\text{anf}(1,-)}[\delta F(x_1, x_2[A_1^{1/2} g, -A_1^{1/2} g])]$ and $E_{x}^{\text{anf}(1,-)}[F(x_1, x_2)]$
exist. Thus using (5.10) together with (5.8) and (5.9), (3.5), (5.8) and (5.9) with
$w$ replaced with $g$, it follows that

$$\int_{C_{a,b}[0,T]} i(Aw, g)C'_{a,b} \exp \left\{ -\frac{i}{2}(Aw, w)C'_{a,b} \right\} \times \exp \left\{ i \left[ (-i)^{-1/2}(w, \vartheta_+^{1/2} \circ a)C'_{a,b} + (i)^{-1/2}(w, \vartheta_-^{1/2} \circ a)C_{a,b} \right] \right\} df(w)$$

$$= E_{x}^{\text{anf}(1,-)}[\delta F(x_1, x_2)][A_1^{1/2} g, -A_1^{1/2} g]]$$

$$= -iE_{x}^{\text{anf}(1,-)}[F(x_1, x_2)] \left\{ (A_1^{1/2} g, x_1)^\sim + (A_1^{1/2} g, x_2)^\sim \right\}$$

$$- \left\{ (-i)^{-1/2}(A_1^{1/2} g)C'_{a,b} - (i)^{-1/2}(A_1^{1/2} g)C_{a,b} \right\} E_{x}^{\text{anf}(1,-)}[F(x_1, x_2)]$$

$$= -iE_{x}^{\text{anf}(1,-)}[F(x_1, x_2)] \left\{ (A_1^{1/2} g, x_1)^\sim + (A_1^{1/2} g, x_2)^\sim \right\}$$

This result together with (5.11) yields the formula

$$(5.11)$$

$$E_{x}^{\text{anf}(1,-)}[F(x_1, x_2)] \left\{ (A_1^{1/2} g, x_1)^\sim + (A_1^{1/2} g, x_2)^\sim \right\}$$

$$= i \left\{ (-i)^{-1/2}(g, \vartheta_+^{1/2} \circ a)C'_{a,b} - (i)^{-1/2}(g, \vartheta_-^{1/2} \circ a)C_{a,b} \right\}$$

$$\times \int_{C_{a,b}[0,T]} \exp \left\{ -\frac{i}{2}(Aw, w)C_{a,b} \right\} \times \exp \left\{ i \left[ (-i)^{-1/2}(w, \vartheta_+^{1/2} \circ a)C_{a,b} + (i)^{-1/2}(w, \vartheta_-^{1/2} \circ a)C_{a,b} \right] \right\} df(w)$$

$$- \int_{C_{a,b}[0,T]} (Aw, g)C_{a,b} \exp \left\{ -\frac{i}{2}(Aw, w)C_{a,b} \right\} \times \exp \left\{ i \left[ (-i)^{-1/2}(w, \vartheta_+^{1/2} \circ a)C_{a,b} + (i)^{-1/2}(w, \vartheta_-^{1/2} \circ a)C_{a,b} \right] \right\} df(w).$$

From this, we obtain more explicit formulas as follows.

**Step 1.** Under the special setting of $\vartheta$ in equation (5.3) above, we first observe the
following table:

| $\vartheta$ | $\theta^+$ | $\theta^-$ | $\vartheta^+$ | $\vartheta^-$ | $\vartheta_+^{1/2}$ | $\vartheta_-^{1/2}$ | $\theta A_+ g$ | $\theta A_- g$ | $\vartheta_+^{1/2} A_+$ | $\vartheta_-^{1/2} A_-$ | $\vartheta_+^{1/2} A_+^{1/2}$ | $\vartheta_-^{1/2} A_+^{1/2}$ |
|-----------|-----------|-----------|------------|------------|------------------|------------------|-------------|-------------|-----------------|-----------------|----------------------|----------------------|
| $b$       | 1         | 1         | 0          | 0          | $b$              | 0                | $I$          | $I$          | $I$             | $I$             | $I$                   | $I$                   |
| $-b$      | -1        | 0         | 1          | 0          | $b$              | $-I$             | $I$          | $I$          | $I$             | $I$             | $I$                   | $I$                   |

**Table 1.** The results from the setting of $\vartheta$ in equation (5.11)
Step 2. Using equation (5.11) above together with Table 1 above we obtain the following two explicit formulas:

\[ E_{g}^{\text{aff}(1,-)} [ (g, x_1) \sim \int_{C_{a,b}[0,T]} \exp \{ i(w, x_1) \} df(w) ] \]

\[ = i(-i)^{1/2}(g, a) C_{a,b} \int_{C_{a,b}[0,T]} \exp \left\{ -\frac{i}{2} \|w\|^2_{C_{a,b}} + i(-i)^{-1/2}(w, a) C_{a,b} \right\} df(w) \]

\[ - \int_{C_{a,b}[0,T]} (w, g) C_{a,b} \exp \left\{ -\frac{i}{2} \|w\|^2_{C_{a,b}} + i(-i)^{-1/2}(w, a) C_{a,b} \right\} df(w) \]

and

\[ E_{g}^{\text{aff}(1,-)} [ (g, x_2) \sim \int_{C_{a,b}[0,T]} \exp \{ -i(w, x_2) \} df(w) ] \]

\[ = -i(i)^{1/2}(g, a) C_{a,b} \int_{C_{a,b}[0,T]} \exp \left\{ i\frac{1}{2} \|w\|^2_{C_{a,b}} + i(i)^{-1/2}(w, a) C_{a,b} \right\} df(w) \]

\[ + \int_{C_{a,b}[0,T]} (w, g) C_{a,b} \exp \left\{ i\frac{1}{2} \|w\|^2_{C_{a,b}} - i(i)^{-1/2}(w, a) C_{a,b} \right\} df(w). \]

Other setting. Letting \( \vartheta = \sin \left( \frac{\pi b(t)}{b(T)} \right) \) in equation (5.11) above, it also follows that

\[ \theta(t) = D\vartheta(t) = \frac{\pi}{b(T)} \cos \left( \frac{\pi b(t)}{b(T)} \right), \]

\[ \sqrt{\vartheta^+}(t) = \frac{\pi}{b(T)} \cos^+ \left( \frac{\pi b(t)}{b(T)} \right) = \frac{\pi}{b(T)} \cos \left( \frac{\pi b(t)}{b(T)} \right) \chi_{[0,b^{-1}(\frac{\pi}{2})]}(t), \]

\[ \sqrt{\vartheta^-}(t) = \frac{\pi}{b(T)} \cos^- \left( \frac{\pi b(t)}{b(T)} \right) = -\frac{\pi}{b(T)} \cos \left( \frac{\pi b(t)}{b(T)} \right) \chi_{[b^{-1}(\frac{\pi}{2}), T]}(t), \]

\[ Aw(t) = \int_{0}^{t} \frac{\pi}{b(T)} \cos \left( \frac{\pi b(t)}{b(T)} \right) dw(t), \]

\[ A_{+}^{1/2}w(t) = \int_{0}^{t} \frac{\pi}{b(T)} \cos^+ \left( \frac{\pi b(t)}{b(T)} \right) dw(t) \]

\[ = \int_{0}^{t} \frac{\pi}{b(T)} \cos \left( \frac{\pi b(t)}{b(T)} \right) \chi_{[0,b^{-1}(\frac{\pi}{2})]}(t)dw(t), \]

and

\[ A_{-}^{1/2}w(t) = \int_{0}^{t} \frac{\pi}{b(T)} \cos^- \left( \frac{\pi b(t)}{b(T)} \right) dw(t) \]

\[ = -\int_{0}^{t} \frac{\pi}{b(T)} \cos \left( \frac{\pi b(t)}{b(T)} \right) \chi_{[b^{-1}(\frac{\pi}{2}), T]}(t)dw(t). \]

Using these we can replace equation (5.11) with the following 8 formulas

\[ (A_{+}^{1/2} g, x_1) = \frac{\pi}{b(T)} \int_{0}^{b^{-1}(\frac{\pi}{2})} \cos \left( \frac{\pi b(t)}{b(T)} \right) Dg(t) dx_1(t), \]
In this example, we chose \( \vartheta \) the functions given by \( \vartheta = \sin \left( \frac{2b(t)}{b(T)} \right) \) for functionals of generalized Brownian motion paths. One can show that the sequence \( \{ \vartheta \} \) can be applicable to evaluate Feynman integral using Fourier series approximation. For more details, see [7]. Thus, under these constructions, our results in this paper can be applicable to evaluate Feynman integral using Fourier series approximation (see [16]) for functionals of generalized Brownian motion paths.
A CAMERON–STORVICK TYPE THEOREM ON $C^2_{a,b}[0,T]$ WITH APPLICATIONS

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