ABSTRACT. Let $X$ be a complete simplicial toric variety over a finite field. For any matrix $Q$ with integer entries, we are interested in evaluation codes on the subset $Y_Q$ of $X$ parameterized by the columns of $Q$. We give an algorithmic method relying on elimination theory for finding generators for the vanishing ideal $I(Y_Q)$ which can be used to compute basic parameters of these codes. Proving that $I(Y_Q)$ is a lattice ideal $I_L$, we also give algorithms for obtaining a basis for the unique lattice $L$ and share procedures implementing them in Macaulay2. Finally, we give a direct algorithmic method to compute lengths of these codes.

1. INTRODUCTION

Let $K = \mathbb{F}_q$ be a fixed finite field and $\Sigma \subset \mathbb{R}^n$ be a complete simplicial fan with rays $\rho_1, \ldots, \rho_r$ generated by the primitive lattice vectors $v_1, \ldots, v_r \in \mathbb{Z}^n$, respectively. We consider the corresponding toric variety $X$ with a split torus $T_X \cong (K^*)^n$. We assume that the class group $\text{Cl}(X)$ have no torsion. Given an element $a = (a_1, \ldots, a_s) \in \mathbb{Z}^s$ we use $t^a$ to denote $t^a_1 \cdots t^a_s$. Recall the construction of $T_X$ as a geometric quotient (see [4] and [5]) via the following two key dual exact sequences:

$$\Psi : 0 \longrightarrow \mathbb{Z}^n \overset{\phi}{\longrightarrow} \mathbb{Z}^r \overset{\beta}{\longrightarrow} A \longrightarrow 0,$$

where $\phi$ denotes the matrix $[v_1 \cdots v_r]^T$, and

$$\Psi^* : 1 \longrightarrow G \overset{i}{\longrightarrow} (K^*)^r \overset{\pi}{\longrightarrow} T_X \longrightarrow 1,$$

where $\pi : (t_1, \ldots, t_r) \mapsto [t^{u_1} : \cdots : t^{u_n}]$, with $u_1, \ldots, u_n$ being the columns of $\phi$ and $G = \ker(\pi)$. The exact sequence $\Psi^*$ gives $T_X$ a quotient representation $T_X \cong (K^*)^r / G$, meaning that every element in the torus $T_X$ can be represented as $[p_1 : \cdots : p_r] := G \cdot (p_1, \ldots, p_r)$ for some $(p_1, \ldots, p_r) \in (K^*)^r$.

Denote by $S = \mathbb{K}[x_1, \ldots, x_r]$ the homogeneous coordinate ring of $X$. As $A \cong \text{Cl}(X)$ is torsion-free, we have $A = \mathbb{Z}^d$ for $d = r - n$ and $S$ is $\mathbb{Z}^d$-graded via $\deg_A(x_j) := \beta_j := \beta(e_j)$ from the exact sequence $\Psi$. Thus, $S = \bigoplus_{\alpha \in A} S_\alpha$, where $S_\alpha$ is the vector space spanned by the monomials having degree $\alpha$.

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For any matrix \( Q = [q_1, q_2, \ldots, q_r] \in M_{s \times r}(\mathbb{Z}) \), the subset \( Y_Q = \{ [t^{q_1} : \cdots : t^{q_r}] | t \in (\mathbb{K}^*)^s \} \subset T_X \) is called the toric set parameterized by \( Q \). We are interested in evaluation codes defined on these subsets of the torus \( T_X \). Since different representatives and orderings give equivalent codes, fix an ordering \( \{ P_1, \ldots, P_N \} \) of some representatives of all the elements \( \{ [P_1], \ldots, [P_N] \} \) in \( Y_Q \) and consider the following evaluation map \( \text{ev}_{Y_Q} : S_\alpha \to \mathbb{K}^N, \quad F \mapsto (F(P_1), \ldots, F(P_N)) \). The map \( \text{ev}_{Y_Q} \) is \( \mathbb{K} \)-linear and \( \text{ev}_{Y_Q}(S_\alpha) \subset \mathbb{P}_q^N \) is an evaluation code on \( Y_Q \) of length \( N = |Y_Q| \). It is called the parameterized toric code associated to \( Q \) and is denoted by \( C_{\alpha,Y_Q} \). The dimension of \( C_{\alpha,Y_Q} \), denoted \( k = \dim_{\mathbb{K}}(C_{\alpha,Y_Q}) \), is the dimension as a subspace \( \text{ev}_{Y_Q}(S_\alpha) \subset \mathbb{P}_q^N \). The number of non-zero entries in any \( c \in C_{\alpha,Y_Q} \) is called its weight and minimum distance of \( C_{\alpha,Y_Q} \) is the smallest weight among all non-zero \( c \in C_{\alpha,Y_Q} \setminus \{0\} \).

Parameterized toric codes are among evaluation codes on a toric variety showcasing champion examples, see [2, 3, 14]. The vanishing ideal \( I(Y_Q) \) is the graded ideal generated by homogeneous polynomials in \( S \) vanishing at every point of \( Y_Q \). Since the kernel of the linear map \( \text{ev}_{Y_Q} \) equals the homogeneous piece \( I(Y_Q)_\alpha \) of degree \( \alpha \), we have an isomorphism of \( \mathbb{K} \)-vector spaces \( S_\alpha/I(Y_Q)_\alpha \cong C_{\alpha,Y_Q} \). The dimension and the length of \( C_{\alpha,Y_Q} \) can be computed using the multigraded Hilbert function of \( I(Y_Q) \) as shown by Şahin and Soprunov in [23]. This motivates to develop methods and algorithms for computing a generating set of the vanishing ideal \( I(Y_Q) \). When \( I(Y_Q) \) is complete intersection, Soprunov gave bounds for the minimum distance of \( C_{\alpha,Y_Q} \), see [25].

Parameterized toric codes include toric codes, as a special case where \( Q = I_r \), which were constructed by Hansen in [11]. The length in this special case is \( |T_X| = (q - 1)^n \). Assuming that the evaluation map is injective, the dimension is the number of monomials of degree \( \alpha \). The minimum distance has been studied by a number of mathematicians since then, see [12, 13, 15, 16, 21, 26].

Parameterized codes were defined and studied for the first time by Villarreal, Simis and Renteria in [20] when \( X \) is a projective space. Among other interesting results, they gave a method for computing a generating set of \( I(Y_Q) \) and showed that \( I(Y_Q) \) is a lattice ideal of dimension 1. Later, the lattice of the vanishing ideal is determined more explicitly, when \( Q \) is a diagonal matrix in [17]. When \( Y_Q \) is the torus \( T_X \) lying in the projective space \( X = \mathbb{P}^n \), that is \( Q = I_r \), the main parameters are determined in [24]. Dias and Neves generalized parametrized codes to weighted projective spaces and showed that the vanishing ideal of the torus \( T_X \) is a lattice ideal of dimension 1 in [7].

We generalize some of these results to parameterized toric codes obtained from a general toric variety. In section 2, we give an expression for \( I(Y_Q) \), see Theorem 2.3 leading to a method, see Algorithm 1 via Elimination Theory, for computing a generating set. Thus, one can find the length and the dimension of the parameterised code on \( Y_Q \) by computing the hilbert function of \( I(Y_Q) \). We share a Macaulay2 code which implements Algorithm 1 in Procedure 2.4. In section 3, firstly we prove that \( I(Y_Q) \) is a lattice ideal, see
Our second main result, Theorem 3.4, gives a practical description of the unique lattice $L$ for which $I(Y_Q) = I_L$, see Algorithm 2. We imply the algorithm in Macaulay2, see Procedure 3.5. In Section 4, we present more conceptual expressions of the lattice $L$ in Theorem 4.1 and Theorem 4.10, generalizing previous results in the case of the projective space. Section 5 gives a direct method for computing the length of $C_{\alpha,Y_Q}$. We provide examples computed in Macaulay2.

## 2. Vanishing Ideal via Elimination Theory

In this section, we give a method yielding an algorithm for computing the generators of vanishing ideal of $Y_Q$. The following basic theorems will be used to obtain our first main result giving this method.

**Lemma 2.1.** [1] Let $K$ any field and $f$ be polynomial in $K[x_1, \ldots, x_s]$ such that $\deg x_i f \leq k_i$ for all $i$. Let $K_i \subset K$ be finite set with $|K_i| \geq k_i + 1$ for $1 \leq i \leq s$. If $f$ vanishes on $K_1 \times K_2 \times \cdots \times K_s$, then $f = 0$.

**Theorem 2.2.** [6] Let $I \subset K[x_1, \ldots, x_k]$ be an ideal and let $G$ be a Gröbner basis of $I$ with respect to lex order where $x_1 > x_2 > \cdots > x_k$. Then, for every $0 \leq l \leq k$ the set $G_l = G \cap K[x_{l+1}, \ldots, x_k]$ is a Gröbner basis of the $l$-th elimination ideal $I_l = I \cap K[x_{l+1}, \ldots, x_k]$.

When $X = \mathbb{P}^n$, $S$ is graded by $A = \mathbb{Z}$ and variables have standard degree $\beta_i = 1$ for all $i$. In this case, there is a method, see [20, Theorem 2.1], to compute the vanishing ideal $I(Y_Q)$. In [7], Dias and Neves extended the same result to weighted projective spaces if $Q$ is the identity matrix, that is, $Y_Q = T_X$. Now, we generalize these works to more general toric varieties. Recall that $m = m^+ - m^-$, where $m^+, m^- \in \mathbb{N}^r$, and $x^m$ denotes the monomial $x_1^{m_1} \cdots x_r^{m_r}$ for any $m = (m_1, \ldots, m_r) \in \mathbb{Z}^r$.

**Theorem 2.3.** Let $R = K[x_1, \ldots, x_r, y_1, \ldots, y_s, z_1, \ldots, z_d, w]$ be a polynomial ring which is an extension of $S$. Then $I(Y_Q) = J \cap S$, where

$$J = \langle \{x_i y^q_i z^{-\beta_i} - y^q_i z^{\beta_i} \}_{i=1}^r \cup \{y_i^q - 1 \}_{i=1}^s, w y^q_i z^{\beta_i} - \cdots y^q_i z^{\beta_r} - 1 \rangle.$$

**Proof.** First, we show the inclusion $I(Y_Q) \subset J \cap S$. Since $I(Y_Q)$ is a homogeneous ideal, it is generated by homogeneous polynomials. Pick any generator $f = \sum_{i=1}^k c_i x^{m_i}$ of degree $\alpha = \sum_{j=1}^r \beta_j m_{ij}$. We use
binomial theorem to write any monomial $x^{m_i}$ as

$$x^{m_i} = x_1^{m_{i1}} \cdots x_r^{m_{ir}}$$

$$= \left( x_1 - \frac{y q_i^+ z_{i1}^+ + y q_i^+ z_{i1}^-}{y q_i^- z_{i1}^-} \right)^{m_{i1}} \cdots \left( x_r - \frac{y q_r^+ z_{r1}^+ + y q_r^+ z_{r1}^-}{y q_r^- z_{r1}^-} \right)^{m_{ir}}$$

$$= \sum_{j=1}^{r} g_{ij} \left( \frac{x_j y q_j^+ z_{j1}^+ - y q_j^+ z_{j1}^-}{y q_j^- z_{j1}^-} \right)^{m_{j1}} \cdots \left( \frac{x_j y q_j^+ z_{j1}^+}{y q_j^- z_{j1}^-} \right)^{m_{jr}}$$

where $g_{i1}, \ldots, g_{ir} \in R$ for all $i$. Substituting all the $x^{m_i}$ in $f$, we get

$$f = \sum_{j=1}^{r} g_{ij} \frac{x_j y q_j^+ z_{j1}^+ - y q_j^+ z_{j1}^-}{y q_j^- z_{j1}^-} \left( \frac{y q_j^+ z_{j1}^+}{y q_j^- z_{j1}^-} \right)^{m_{j1}} + z_{j}^{\alpha} f(y_{q1}, \ldots, y_{qr}),$$

where $g_j = \sum_{i=1}^{k} c_i g_{ij}$ and $m_j = \sum_{i=1}^{k} m_{ij}$. Then set $m = \sum_{j=1}^{r} \sum_{i=1}^{r} m_{ij}$. We multiply $f$ by $h^m$ to clear all the denominators which yields that

$$f h^m = \sum_{j=1}^{r} G_j \left( x_j y q_j^+ z_{j1}^+ - y q_j^+ z_{j1}^- \right) + z_{j}^{\alpha'} \left( y q_j^- \cdots y_{qr}^+ \right)^{m} f(y_{q1}, \ldots, y_{qr}),$$

where $h = y q_{1}^- z_{11}^- \cdots y q_{r}^- z_{r1}^-$ and

$$\alpha' = \sum_{j=1}^{r} \left( \beta_j^- (m - m_{ij}) + \beta_j^+ m_{ij} \right) \in \mathbb{N}^r, \quad G_j = g_j \prod_{j=1}^{r} \left( y q_j^- z_{j1}^- \right)^{m - m_{ij}} \in R.$$

Since $F = \left( y q_{1}^- \cdots y_{qr}^+ \right)^{m} f(y_{q1}, \ldots, y_{qr})$ is a polynomial in $K[y_{1}, \ldots, y_{s}]$, we can apply division algorithm and divide $F$ by $\{y_{k} q_{i1}^- - 1\}^{s}_{i=1}$ which leads to

$$f h^m = \sum_{j=1}^{r} G_j \left( x_j y q_j^+ z_{j1}^+ - y q_j^+ z_{j1}^- \right) + z_{j}^{\alpha'} \left( \sum_{i=1}^{s} H_i (y q_i - 1) + E(y_{1}, \ldots, y_{s}) \right).$$

We claim that $E(t) = 0$ for all $t = (t_1, \ldots, t_s) \in (\mathbb{K}^*)^s$. Since $f \in S$, we have $f(x_1, \ldots, x_r) = f(x_1, \ldots, x_r, y_1, \ldots, y_{s}, z_1, \ldots, z_d, w)$. Thus, using $f \in I(Y_Q)$ and substituting $x_i = t a_i, y_j = t_j$, and $z_k = 1$ for all $i, j, k$, in equation (1) yields

$$0 = 0 h^m = \sum_{j=1}^{r} G_j 0 + \sum_{i=1}^{s} H_i 0 + E(t_1, \ldots, t_s).$$

So, $E$ vanishes on all of $(\mathbb{K}^*)^s$ as claimed. Since the usual degree of $E$ in the variable $y_i$ satisfies $deg y_i(E) < q - 1$, it must be the zero polynomial by Lemma [2.1].
Multiplying now the equation (1) by $w^m$, we have
\[ f(hw)^m = \sum_{j=1}^{r} w^m G_j \left( x_j y^{q_j} z^{\beta_j} - y^{q_j} z^{\beta_j} \right) + w^m z^j \sum_{i=1}^{s} H_i(y_i q_i - 1). \]

As $f(hw)^m = f(hw - 1 + 1)^m = f(H(hw - 1) + 1) = fH(hw - 1) + f$ for some $H \in R$ by binomial theorem, it follows that
\[ f = \sum_{j=1}^{r} w^m G_j \left( x_j y^{q_j} z^{\beta_j} - y^{q_j} z^{\beta_j} \right) + w^m z^j \sum_{i=1}^{s} H_i(y_i q_i - 1) - fH(hw - 1) \]
which establishes the inclusion $I(Y_Q) \subset J \cap S$.

In order to get the reverse inclusion notice that $J$ is generated by binomials. Thus, any Gröbner bases of $J$ consists of binomials by Buchberger algorithm \cite[Theorem 2, p.87]{6}. It follows from Theorem \ref{thm:2.3} that $J \cap S$ is also generated by binomials. To see the opposite inclusion, it suffices to take $f = x^a - x^b \in J \cap S$. So, we can write
\[ f = \sum_{j=1}^{r} G_j \left( x_j y^{q_j} z^{\beta_j} - y^{q_j} z^{\beta_j} \right) + \sum_{i=1}^{s} H_i(y_i q_i - 1) + H(hw - 1), \]
for some polynomials $G_1, \ldots, G_r, H_1, \ldots, H_s, H$ in $R$. As the last equality is valid also in the ring $R[z_1^{-1}, \ldots, z_d^{-1}]$ setting $y_i = 1, x_i = z^{\beta_j}, w = 1/z^{\beta_1} \cdots z^{\beta_r}$ gives
\[ z^{a_1 \beta_1 + \cdots + a_r \beta_r} - z^{b_1 \beta_1 + \cdots + b_r \beta_r} = 0. \]
This means that $a_1 \beta_1 + \cdots + a_r \beta_r = b_1 \beta_1 + \cdots + b_r \beta_r$. Hence $f = x^a - x^b$ is homogeneous. Let us see that $f$ vanishes on any element $[t^{q_1}: \cdots : t^{q_r}] \in Y_Q$. Since $f(t^{q_1}, \ldots, t^{q_r})$ can be computed by setting $x_i = t^{q_j}, y_j = t_j, z_k = 1$ for all $i, j, k$ and $w = 1/t^{q_1} \cdots t^{q_r}$ on the right hand side of equation (2), we obtain $f(t^{q_1}, \ldots, t^{q_r}) = 0$ and so $f \in I(Y_Q)$. Consequently, $J \cap S \subset I(Y_Q)$.

Theorem \ref{thm:2.3} gives rise to the following algorithm for computing the binomial generators of $I(Y_Q)$.

**Algorithm 1** Computing the generators of vanishing ideal $I(Y_Q)$.

**Input** The matrices $Q \in M_{s \times r}(\mathbb{Z}), \beta \in M_{d \times r}(\mathbb{Z})$ and a prime power $q$.

**Output** The generators of $I(Y_Q)$.

1. Write the ideal $J$ of $R$ using Theorem \ref{thm:2.3}
2. Find the Gröbner basis $G$ of $J$ wrt. lex order $w > z_1 > \cdots > z_d > y_1 > \cdots > y_s > x_1 > \cdots > x_r$.
3. Find $G \cap S$ so that $I(Y_Q) = \langle G \cap S \rangle$.

Using the function `toBinomial` creating a binomial from a list of integers (see \cite{9}), we write a Macaulay2 code which implements this algorithm.

**Procedure 2.4.** Given a particular input $q, Q, \beta$, the following procedure find the generators of $I(Y_Q)$. 
Example 2.5. Let \( X = \mathcal{H}_2 \) be the Hirzebruch surface corresponding to a fan in \( \mathbb{R}^2 \) generated by \( v_1 = (1, 0) \), \( v_2 = (0, 1) \), \( v_3 = (-1, 2) \), and \( v_4 = (0, -1) \). Thus we have the short exact sequence

\[
0 \rightarrow \mathbb{Z}^2 \xrightarrow{\phi} \mathbb{Z}^4 \xrightarrow{\beta} \mathbb{Z}^2 \rightarrow 0,
\]

where

\[
\phi = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & -1 \end{bmatrix}^T \quad \text{and} \quad \beta = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.
\]

This shows that the class group is \( A = \mathbb{Z}^2 \) and the total coordinate ring is \( S = \mathbb{K}[x_1, x_2, x_3, x_4] \) where \( \deg_A(x_1) = \deg_A(x_3) = (1, 0), \deg_A(x_2) = (-2, 1), \deg_A(x_4) = (0, 1) \). Consider the toric set parameterized by \( Q = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \), that is, \( Y_Q = \{(t : t^2 : t^3 : t^4) | t \in \mathbb{K}^* \} \). In order to compute the generators of \( I(Y_Q) \) in Macaulay2 it suffices to enter \( q \) and the matrices \( Q \) and \( \beta \):

\[
i1 : q=11;\text{Beta}=\text{matrix} \{ \{1,-2,1,0\}, \{0,1,0,1\} \}; \text{Q}=\text{matrix} \{ \{1,2,3,4\} \};
\]

and we obtain \( I(Y_Q) = \langle x_1^5 - x_3^5, x_2^2 x_2 - x_4 \rangle \) by using Procedure 2.4.

3. VANISHING IDEALS VIA SATURATION OF LATTICE BASIS IDEALS

In this section, we show that the vanishing ideal \( I(Y_Q) \) is a lattice ideal. For any parametrized toric set, we will determine the unique lattice defining \( I(Y_Q) \).

A binomial ideal is an ideal generated by binomials \( x^a - x^b \), where \( a, b \in \mathbb{N}^r \), see [3] for foundational properties they have. A subgroup \( L \subseteq \mathbb{Z}^r \) is called a lattice, and the following binomial ideal is called the associated lattice ideal:

\[
I_L = \langle x^a - x^b | a - b \in L \rangle = \langle x^{m^+} - x^{m^-} | m \in L \rangle.
\]

For any matrix \( Q \), we denote by \( L_Q \) the lattice \( \ker \mathbb{Z} Q \) of integer vectors in \( \ker Q \).

Lemma 3.1. A binomial \( f = x^a - x^b \) in \( S \) is homogeneous iff \( a - b \in L_\beta \).
Proof. By definition, \( \deg_A(x^a) = a_1 \deg_A(x_1) + \cdots + a_r \deg_A(x_r) = \beta_1 a_1 + \cdots + \beta_r a_r = \beta(a) \). So, \( f \) is homogeneous, that is, \( \deg_A(x^a) = \deg_A(x^b) \) iff \( \beta(a) = \beta(b) \). The latter is equivalent to \( \beta(a - b) = 0 \), which holds true iff \( a - b \in L_\beta \).

It is now time to prove the following key result proving that the vanishing ideal is a lattice ideal. The fact that it is a lattice ideal has recently been observed in [22] without describing the corresponding lattice.

**Lemma 3.2.** The ideal \( I(Y_Q) = I_{L_1} \), for the lattice \( L_1 = \{ m \in L_\beta : Qm \equiv 0 \mod (q - 1) \} \).

**Proof.** Before we go further, let us note that \( x^a(t^{q_1}, \ldots, t^{q_r}) = (t^{q_1})^{a_1} \cdots (t^{q_r})^{a_r} = t^{Qa} \), for \( t \in ( \mathbb{K}^* )^s \). It follows that a binomial \( f = x^a - x^b \) vanishes at a point \( (t^{q_1}, \ldots, t^{q_r}) \) if and only if \( t^{Qa} = t^{Qb} \). As \( t \in ( \mathbb{K}^* )^s \), this is equivalent to \( t^{Q(a-b)} = 1 \).

To prove \( I(Y_Q) \subseteq I_{L_1} \), take a generator \( f = x^a - x^b \) of \( I(Y_Q) \). As \( f \) vanishes on \( Y_Q \), we have that \( t^{Q(a-b)} = 1 \) for all \( t \in ( \mathbb{K}^* )^s \). Then, by substituting \( t = (\eta, 1, \ldots, 1) \) in this equality, we observe that \( q - 1 \) divides the first entry of the row matrix \( Q(a - b) \), where \( \eta \) is a generator of the cyclic group \( \mathbb{K}^* \) of order \( q - 1 \). Similarly, \( q - 1 \) divides the other entries, and so \( Q(a - b) \equiv 0 \mod (q - 1) \). Since \( a - b \in L_\beta \) from Lemma 3.1, \( f \) being homogeneous, we have \( a - b \in L_1 \).

Conversely, let \( f = x^a - x^b \in I_{L_1} \). Then \( a - b \in L_\beta \) and \( Q(a - b) \equiv 0 \mod (q - 1) \). This implies that \( f \) is homogeneous by Lemma 3.1 and that \( t^{Q(a-b)} = 1 \) for all \( t \in ( \mathbb{K}^* )^s \). Hence, \( f(t^{q_1}, \ldots, t^{q_r}) = 0 \) for any \( t \in ( \mathbb{K}^* )^s \), by the first part. Thus, \( f \in I(Y_Q) \) and \( I_{L_1} \subseteq I(Y_Q) \). \( \square \)

For any lattice \( L \), the lattice basis ideal of \( L \) is the ideal of \( S \) generated by the binomials \( x^{m^+} - x^{m^-} \) corresponding to the set of \( m \) which constitutes a \( \mathbb{Z} \)-basis of \( L \).

Let \( I \) and \( J \) be ideals in \( S \). Then the ideal

\[ I : J^\infty = \{ F \in S : F \cdot J^k \subseteq I \text{ for some integer } k \geq 0 \} \]

is called the saturation of \( I \) with respect to \( J \).

**Lemma 3.3.** [18] (Lemma 7.6) Let \( L \) be a lattice. The saturation of the lattice basis ideal of \( L \) with respect to the ideal \( \langle x_1 \cdots x_r \rangle \) is equal to the lattice ideal \( I_L \).

Thus, we can obtain generators of \( I(Y_Q) = I_{L_1} \) from a \( \mathbb{Z} \)-basis of \( L_1 \). Although the lattice \( L_1 \) in Lemma 3.2 is inevitable conceptually, it is not that easy to find its basis. The following result gives another description of \( L_1 \) leading to an algorithm computing its basis.

**Theorem 3.4.** Let \( \pi_s : \mathbb{Z}^{n+s} \to \mathbb{Z}^n \) be the projection map sending \( (c_1, \ldots, c_n, c_{n+1}, \ldots, c_{n+s}) \) to \( (c_1, \ldots, c_n) \).

Then \( I(Y_Q) = I_L \), for the lattice \( L = \{ \phi c : c \in \pi_s(\ker \mathbb{Z}[Q\phi((q - 1)I_s)]) \} \).
Proof. We have that $I(Y_Q) = I_{L_1}$ where $L_1 = \{ m \in L_β : Qm \equiv 0 \mod (q - 1) \}$ by Lemma 3.2. Therefore it is enough to prove that $L = L_1$. Since $\text{Im}φ = L_β$ by the exact sequence $\mathfrak{P}$, it follows that $m \in L_β$ iff $m = φe$ for some $e \in \mathbb{Z}^n$. This means that

$$L_1 = \{ φe : Qφe \equiv 0 \mod (q - 1) \text{ and } e \in \mathbb{Z}^n \}. $$

Take $φe \in L$ so that $e = (c_1, \ldots, c_n) \in π_s (\ker_{\mathbb{Z}}[Qφ((q - 1)I_s)])$. Then there are $c_{n+1}, \ldots, c_{n+s} \in \mathbb{Z}$ such that $[Qφ((q - 1)I_s)](c_1, \ldots, c_n, c_{n+1}, \ldots, c_{n+s}) = 0$. This is equivalent to the following

$$Qφ(c_1, \ldots, c_n) + (q - 1)I_s(c_{n+1}, \ldots, c_{n+s}) = 0 \quad Qφe = -(q - 1)(c_{n+1}, \ldots, c_{n+s}) .$$

This proves that $Qφe \equiv 0 \mod (q - 1)$. Thus $φe \in L_1$.

For the converse, take $φe \in L_1$. Then $Qφ(φe) \equiv 0 \mod (q - 1)$. It follows that

$$Qφ = (q - 1)(c_{n+1}, \ldots, c_{n+s})$$

for some $c_{n+1}, \ldots, c_{n+s} \in \mathbb{Z}$. Thus, we have $[Qφ((q - 1)I_s)](c_1, \ldots, c_n, -c_{n+1}, \ldots, -c_{n+s}) = 0$. Hence, we have $e = π_s(c_1, \ldots, c_n, -c_{n+1}, \ldots, -c_{n+s}) \in π_s (\ker_{\mathbb{Z}}[Qφ((q - 1)I_s)]).$ □

Theorem 3.4 leads to the following algorithm for computing a $\mathbb{Z}$-basis of the lattice $L = L_1$.

**Algorithm 2** Computing the lattice $L$ such that $I_L = I(Y_Q)$.

**Input** The matrices $Q \in M_{s \times r}(\mathbb{Z}), φ \in M_{r \times n}(\mathbb{Z})$ and a prime power $q$.

**Output** A basis of $L$.

1. Find the generators of the lattice $\ker_{\mathbb{Z}}[Qφ((q - 1)I_s)]$.
2. Find the matrix $M$ whose columns are the first $s$ coordinates of the generators of $\ker_{\mathbb{Z}}[Qφ((q - 1)I_s)]$.
3. Compute the matrix $φM$ whose columns are a $\mathbb{Z}$-basis of the lattice $L$.

The algorithm can be implemented in Macaulay2 as follows.

**Procedure 3.5.** The command $\text{ML}$ gives the matrix whose columns are generators of the lattice $L$.

```
 i2: s=numRows Q; n=numColumns Phi;
 i3: ML=Phi*(id_{ZZ^n})*random(ZZ^n,ZZ^s))*0)*syz (Q*Phi*(q-1)*(id_{ZZ^s}))
```

**Example 3.6.** Let $X = H_2$ over $F_{11}$ and $Q = [1 2 3 4]$. So, we have the following input:

```
i1: q=11; Phi=matrix([1,0],[0,1],[1,2],[0,1]); Q=matrix ([i,j,k,l]);
```

We find $L$ such that $I(Y_Q) = I_L$ using Procedure 3.5.

```
i2: s=numRows Q; n=numColumns Phi;
i3: ML=Phi*(id_{ZZ^n})*random(ZZ^n,ZZ^s))*0)*syz (Q*Phi*(q-1)*id_{ZZ^s}))
```
gives the matrix whose columns consistute a basis of $L$ is

$$ML = \begin{bmatrix} 2 & 1 & 0 & -1 \\ -5 & 0 & 5 & 0 \end{bmatrix}^T.$$  

Finally, we determine $I(Y_Q) = I_L$ as follows

i4 : r=numRows Phi; (D,P,K) = smithNormalForm Phi; Beta=P^{n..r-1};
i5 : S=ZZ/q[x_1..x_r,Degrees=>transpose entries Beta];
i6: toBinomial = (b,S) -> (top := 1_S; bottom := 1_S;
   scan(#b, i -> if b_i > 0 then top = top * S_i^{(b_i)}
   else if b_i < 0 then bottom = bottom * S_i^{(-b_i)}); top - bottom);
i7: IdealYQ=(ML,S)->(J = ideal apply(entries transpose ML, b -> toBinomial(b,S));
   scan(gens S, f-> J=saturate(J,f));J)
i8: IYQ=IdealYQ(ML,S)

Therefore, we get $I_L = \langle x_1^2x_2 - x_4, x_5 - x_3^5 \rangle$.

**Remark 3.7.** Another advantage of finding the matrix $ML$ giving a basis for the lattice is that one can confirm if the lattice ideal is a complete intersection immediately, by checking if $ML$ is mixed dominating.

**Definition 3.8.** Let $A$ be matrix whose entries are all integers. $A$ is called mixed if there is a positive and a negative entry in every columns. If every square submatrix of $A$ is not mixed, it is called dominating.

**Theorem 3.9.** [19] Let $L \subset \mathbb{Z}^r$ be a lattice with basis $m_1, \ldots, m_k$. If $L \cap \mathbb{N}^r = 0$, then $I_L$ is complete intersection iff the matrix $[m_1 \cdots m_r]$ is mixed dominating.

Using Theorem 3.9 one can confirm when $I(Y_Q) = I_L$ is a complete intesection by looking at a basis of the lattice $L$.

**Example 3.10.** Let $X = \mathcal{H}_2$ be the Hirzebruch surface over $\mathbb{F}_{11}$ and $Q = [1 \ 2 \ 3 \ 4]$. In Example 3.6 we have seen that the matrix whose columns consitute a basis of $L$ is

$$ML = \begin{bmatrix} 2 & 1 & 0 & -1 \\ -5 & 0 & 5 & 0 \end{bmatrix}^T.$$  

Since $ML$ is mixed dominating, $I(Y_Q) = I_L$ is complete intersection.

4. **CONCEPTUAL DESCRIPTIONS OF THE LATTICE OF A VANISHING IDEAL**

The following result gives a handier description of the lattice of the ideal $I(Y_Q)$, in terms of $Q$ and $\beta$, under a condition on the lattice $L = QL_{\beta} = \{Qm|m \in L_{\beta}\}$. Before stating it, let us remind that $L : (q - 1) = \{m \in \mathbb{Z}^r|(q - 1)m \in L\}$ and the colon module $L : (q - 1)\mathbb{Z}^s$ are the same.
Theorem 4.1. Let $L = (L_Q \cap L_\beta) + (q-1)L_\beta$. Then $I_L \subseteq I(Y_Q)$. The equality holds if and only if $L = L : (q-1)$.

Proof. We start with the proof of the inclusion $I_L \subseteq I(Y_Q)$. By the virtue of Lemma 3.1, it suffices to prove that $L \subseteq L_1$, as $I(Y_Q) = I_{L_1}$. Take $m \in L$. Since $L = (L_Q \cap L_\beta) + (q-1)L_\beta \subseteq L_\beta$, we have $m \in L_\beta$. On the other hand, we can write $m = m' + (q-1)m''$ for some $m' \in L_Q \cap L_\beta$ and $m'' \in L_\beta$. Since $Qm = (q-1)Qm''$, it follows that $m \in L_1$, completing the proof of the inclusion.

Now, in order to show that $I(Y_Q) \subseteq I_L$ iff $L : (q-1) \subseteq L$, it is enough to prove that $L_1 \subseteq L$ iff $L : (q-1) \subseteq L$. Assume first that $L_1 \subseteq L$ and take $z \in L : (q-1)$. This means that there exist $m \in L_\beta$ such that $(q-1)z = Qm$. So, $m \in L_1 \subseteq L$ and we have $m = m' + (q-1)m''$ for some $m' \in L_Q \cap L_\beta$ and $m'' \in L_\beta$. Thus, $(q-1)z = Qm = (q-1)Qm''$, and we have $z = Qm'' \in L$. Therefore, $L : (q-1) \subseteq L$.

Suppose now that $L : (q-1) \subseteq L$ and let $m \in L_1$. Then $m \in L_\beta$ and $Qm = (q-1)z$ for some $z \in \mathbb{Z}^s$. So, $z \in L : (q-1) \subseteq L$ yielding $z = Qm'$ for some $m' \in L_\beta$. Hence $Q(m - (q-1)m') = 0$ and so, $m - (q-1)m' \in L_Q \cap L_\beta$. This implies that $m = (m - (q-1)m') + (q-1)m' \in L$. Hence, $L_1 \subseteq L$.

We can check if the condition above is satisfied and in the affirmative case we can compute the generators of the lattice using the following code in Macaulay2.

Procedure 4.2. The lattice $L : (q-1)$ is obtained using the command $LL: (q-1) \ast (\mathbb{Z}^s)$ below.

```plaintext
i2: s=numRows Q;
i3: LL=image (Q*Phi);
i4: if LL:(q-1)*(ZZ^s)==LL then print yes else print no;
i5: ML=mings ([(q-1)*image Phi)+intersect(ker Q,image Phi));
```

Example 4.3. Consider the Hirzebruch surface $X = H_2$ over the field $\mathbb{F}_2$ and take $Q = [1, 2, 3, 4]$. The input is as follows:

```plaintext
i1 : q=2;Phi=matrix{{1,0},{0,1},{-1,2},{0,-1}};Q=matrix {{1,2,3,4}};
```

and then Procedure 4.2 gives $L = ((-1, 0, 1, 0), (-2, -1, 0, 1))$ and thus $I(Y_Q) = \langle x_1^2, x_2 + x_4, x_1 + x_3 \rangle$ via saturation. Note that when $q = 11$, the condition $L : (q-1)$ does not hold, in which case the ideal is already found in Example 3.6.

Definition 4.4. $Q$ is called homogeneous, if there is a matrix $A \in M_{d \times s}(\mathbb{Q})$ such that $AQ = \beta$.

Lemma 4.5. Let $Q = [q_1, \ldots, q_r] \in M_{s \times r}(\mathbb{Z})$. $Q$ is homogeneous iff $L_Q \subseteq L_\beta$.

Proof. Suppose that $Q$ is homogeneous. So, $AQ = \beta$ for some matrix $A \in M_{d \times s}(\mathbb{Q})$. Take an element $m$ of $L_Q$. Since $Qm = 0$, we have $\beta m = AQm = 0$. Hence, $m \in L_\beta$ and thus $L_Q \subseteq L_\beta$. Conversely, assume that $L_Q \subseteq L_\beta$. Denote by $Q'$ the $(s + d) \times r$ matrix $[Q \ \beta]^T$. Then $L_Q' = L_Q \cap L_\beta = L_Q$ which implies
that the rows of \( \beta \) belong to the row space of \( Q \). Let the following be the \( i \)-th row of \( \beta \):

\[
\begin{align*}
    a_{i1}[q_{11} \cdots q_{1r}] + \cdots + a_{is}[q_{s1} \cdots q_{sr}] &= \left[ a_{i1}q_{11} + \cdots + a_{is}q_{s1} \cdots a_{i1}q_{1r} + \cdots + a_{is}q_{sr} \right] \\
    &= [a_{i1} \cdots a_{is}] \mathbf{q}_1 + \cdots + [a_{i1} \cdots a_{is}] \mathbf{q}_r.
\end{align*}
\]

So, if \( A = (a_{ij}) \) then \( AQ = \beta \). \( \square \)

The next corollary is a generalization of Theorem 2.5 in [20] studying the case \( X = \mathbb{P}^n \).

**Corollary 4.6.** Let \( Q = [\mathbf{q}_1 \cdots \mathbf{q}_r] \in M_{s \times r}(\mathbb{Z}) \) be a homogeneous matrix and \( L = L_Q + (q-1)L_\beta \). Then \( I_L \subset I(Y_Q) \). The equality holds if and only if \( L = L : (q-1) \).

**Proof.** Since \( Q \) is homogeneous, \( L_Q \subset L_\beta \) which show that \( L = L_Q \cap L_\beta + (q-1)L_\beta = L_Q + (q-1)L_\beta \). Therefore, \( I(Y_Q) = I_L \) from Theorem 4.1. \( \square \)

We give other proofs of the following facts proven for the first time in [22], using Theorem 4.1.

**Corollary 4.7.** \( I(T_X) = I_{(q-1)L_\beta} \)

**Proof.** \( T_X \) is the toric set parametrized by the identity matrix \( Q = I_r \). It is clear that \( L_Q = \text{ker}_Z I_r = \{0\} \).

Notice also that \( L = QL_\beta = \{I_r \mathbf{m} | \mathbf{m} \in L_\beta \} = L_\beta \). Since \( L_\beta \) is torsion free, the condition \( L = L : (q-1) \) is satisfied. As \( L = (q-1)L_\beta \), we have that \( I(T_X) = I_{(q-1)L_\beta} \) by Theorem 4.1. \( \square \)

**Corollary 4.8.** Let \( I([1]) \) be the vanishing ideal of \([1 : \cdots : 1]\). Then \( I([1]) = I_{L_\beta} \)

**Proof.** Take \( Q = \beta \). Then \( Y_Q = \{[1 : \cdots : 1]\} \) and \( L_Q = L_\beta \). So, \( L = QL_\beta = \{Q \mathbf{m} | \mathbf{m} \in L_Q \} = 0 \). Thus, \( L = L_\beta + (q-1)L_\beta = L_\beta \). This gives \( I([1 : \cdots : 1]) = I_{L_\beta} \) by Theorem 4.1. \( \square \)

We close this section discussing another special case where \( Q \) is diagonal.

**Definition 4.9.** The toric set parameterized by a diagonal matrix is called a degenerate torus.

Let \( \eta \) be a generator of the cyclic group \( \mathbb{K}^* \), then for all \( t_i \in \mathbb{K}^* \) we can write \( t_i = \eta^{s_i} \) for some \( 0 \leq s_i \leq q - 2 \). The following is the generalized version of the corresponding result in [17] valid for \( X = \mathbb{P}^n \). The first proof is given in [22] we give another here using Lemma 3.2.

**Theorem 4.10.** Let \( Q = \text{diag}(q_1, \ldots, q_r) \in M_{r \times 1}(\mathbb{Z}) \) and \( D = \text{diag}(d_1 \ldots, d_r) \) where \( d_i = |\eta^{s_i}| \). Then \( I(Y_Q) = I_L \) for \( L = D(L_\beta D) \).

**Proof.** By Lemma 3.2 it is enough to show that \( L = L_1 \) where \( L_1 = \{ \mathbf{m} \in L_\beta : Q \mathbf{m} \equiv 0 \mod (q-1) \} \).

Let \( \mathbf{m} \) be any element in \( L \). Then \( \mathbf{m} = D \mathbf{z} \) for some \( \mathbf{z} \in \mathbb{Z}^r \) and \( \mathbf{m} \in L_\beta \). Since \( d_i = (q-1)/\gcd(q-1, q_i) \), \( q_i d_i \equiv 0 \mod (q-1) \) for all \( d_i \). Therefore, \( Q \mathbf{m} = Q D \mathbf{z} \equiv 0 \mod (q-1) \) and so, \( \mathbf{m} \in L_1 \).
Take \( \mathbf{m} \in L_1 \). Then \( Q\mathbf{m} \equiv 0 \mod (q - 1) \), that is, for every \( 1 \leq i \leq r \) there exist \( z_i \in \mathbb{Z} \) such that \( q_i m_i = (q - 1) z_i \). Hence,

\[
\frac{q_i}{\gcd(q - 1, q_i)} m_i = \frac{q - 1}{\gcd(q - 1, q_i)} z_i = d_i z_i.
\]

Since \( q_i/\gcd(q - 1, q_i) \) and \( q - 1/\gcd(q - 1, q_i) \) are coprime, it follows that \( d_i \) divides \( m_i \). Therefore, \( m_i = d_i z'_i \) for some \( z'_i \in \mathbb{Z} \) and so, \( \mathbf{m} = Dz' \) for \( z' = (z'_1, \ldots, z'_r) \). Hence, \( \mathbf{m} \in L \).

\[\square\]

5. The length of \( C_{\alpha,Y_Q} \)

The length of the parameterized toric code \( C_{\alpha,Y_Q} \) can be computed using the vanishing ideal of \( Y_Q \). In this section, we give an algorithm computing the length directly using the parameterization of \( Y_Q \). It is clear that \( T_X \) and \( Y_Q \) are groups under the componentwise multiplication

\[
[p_1 : \cdots : p_r][p'_1 : \cdots : p'_r] = [p_1 p'_1 : \cdots : p_r p'_r]
\]

and the map

\[
\varphi_Q : (\mathbb{K}^*)^s \to Y_Q, \quad t \mapsto [t^{q_1} : \cdots : t^{q_r}]
\]

is a group epimorphism. It follows that \( Y_Q \cong (\mathbb{K}^*)^s/\ker(\varphi_Q) \) and so,

\[
|Y_Q| = |(\mathbb{K}^*)^s/\ker(\varphi_Q)| = (q - 1)^s/|\ker(\varphi_Q)|.
\]

Hence, the length of the code \( C_{\alpha,Y_Q} \) depends on \( |\ker(\varphi_Q)| \).

**Proposition 5.1.** Let \( H = \{1, \ldots, q - 1\} \times \cdots \times \{1, \ldots, q - 1\} \subset \mathbb{Z}^s \) and \( \eta \) be a generator of \( \mathbb{K}^* \). If \( P = \{ \mathbf{h} \in H | hQ \phi \equiv 0 \mod q - 1 \} \), then \( \ker(\varphi_Q) = \{ (\eta^{h_1}, \ldots, \eta^{h_s}) | h = (h_1, \ldots, h_s) \in P \} \). Therefore, \( |\ker(\varphi_Q)| = |P| \).

**Proof.** Let \( \mathbf{t} \in \ker(\varphi_Q) \subset (\mathbb{K}^*)^s \). Then \( [t^{q_1} : \cdots : t^{q_r}] = [1 : \cdots : 1] \), that is, \( (t^{q_1}, \ldots, t^{q_r}) \) is element of the orbit \( G(1, \ldots, 1) = G = \{ \mathbf{x} \in (\mathbb{K}^*)^r | \mathbf{x}^\mathbf{m} = 1 \} \) for all \( \mathbf{m} \in L_\beta \). Since \( L_\beta = \text{im} \phi \), we have \( \mathbf{m} \in L_\beta \) if and only if \( \mathbf{m} = \phi \mathbf{e} \) for some \( \mathbf{e} \in \mathbb{Z}^n \). Therefore, we have

\[
\mathbf{x}^\mathbf{m}(t^{q_1}, \ldots, t^{q_r}) = t^{Q\mathbf{m}} = t^{Q\phi \mathbf{e}} = 1.
\]

Since every \( \mathbf{t} = (\eta^{h_1}, \ldots, \eta^{h_s}) \) in \( (\mathbb{K}^*)^s \) for some \( \mathbf{h} = (h_1, \ldots, h_s) \in H \), the equality \( (3) \) implies that \( \eta^{hQ \phi \mathbf{e}} = 1 \), for all \( \mathbf{e} \in \mathbb{Z}^n \). Thus, \( \mathbf{h}Q \phi \mathbf{e} \equiv 0 \mod q - 1 \) for all \( \mathbf{e} \in \mathbb{Z}^n \). By choosing \( \mathbf{e} \) vectors in the standard basis of \( \mathbb{Z}^n \), we observe that \( \mathbf{h}Q \phi \equiv 0 \mod q - 1 \). This implies that \( \ker(\varphi_Q) \subseteq \{ (\eta^{h_1}, \ldots, \eta^{h_s}) | \mathbf{h} \in P \} \). The other inclusion is straightforward, completing the first part of the proof. Since the order of \( \eta \) is \( q - 1 \) and \( h_i \) lies in \( H \), it is clear that the correspondence between \( \ker(\varphi_Q) \) and \( P \) is one to one. \( \square \)

**Procedure 5.2.** The following code in Macaulay2 computes \( k = |\ker(\varphi_Q)| \) and the length of \( C_{\alpha,Y_Q} \).
Example 5.3. Let us calculate the length of the code corresponding to the Example 3.6 using the Hilbert function of the vanishing ideal found there. Notice that $\beta = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ is different from the $\beta$ of Example 2.5. Thus, degrees of the generators are $\alpha_1 = (0, 1)$ and $\alpha_2 = (-5, 0)$. By [23, Theorem 3.1], the length can be computed as 5 with the following command:

```
hilbertFunction({-5,1},IYQ)
```

The same length can be computed directly using the Procedure 5.2 with the following input:

```
i1 : q=11; Phi=matrix{{1,0},{0,1},{-1,2},{0,-1}}; Q=matrix {{1,2,3,4}};
```

REFERENCES

1. Alon, N., 1999. Combinatorial Nullstellensatz, Combin. Probab. Comput., 8 no. 1-2, 7–29.
2. P. Beelen, D. Ruano, The order bound for toric codes, M. Bras-Amoros, T. Høholdt (Eds.), AAECC 2009, Springer Lecture Notes in Computer Science, vol. 5527 (2009), pp. 1–10.
3. G. Brown, A. Kasprzyk, Seven new champion linear codes, LMS J. Comput. Math., 16 (2013), pp. 109–117.
4. D. A. Cox, The homogeneous coordinate ring of a toric variety, J. Algebr. Geom. 4 (1995), 17–50.
5. Cox, D. A., Little, J., Schenck, H., 2011, Toric Varieties, Graduate Studies in Mathematics, 124, AMS, Providence, RI.
6. Cox, D., Little, J., O’Shea, D., 1992, Ideals, Varieties, and Algorithms, Springer Verlag.
7. E. Dias and J. Neves, Codes over a weighted torus, Finite Fields and Their Appl., 2015, 33, 66-79.
8. Eisenbud, D., Sturmfels, B., 1997. Binomial ideals, Duke Math. J., 84, 1–45.
9. Eisenbud, D., Grayson, D.R., Stillman, M., Sturmfels, B., 2013, Computations in Algebraic Geometry with Macaulay 2, Springer Verlag.
10. D. Grayson, M. Stillman, Macaulay2–A System for Computation in Algebraic Geometry and Commutative Algebra. math.uiuc.edu/Macaulay2.
11. J. Hansen, Toric surfaces and error–correcting codes in: Buchmann, J. et al. (Eds.), Coding Theory, Cryptography, and Related Areas, Springer, Berlin (2000), pp. 132–142.
12. J. Hansen, Toric varieties Hirzebruch surfaces and error-correcting codes Appl. Algebra Eng. Commun. Comput., 13 (2002), pp. 289–300.
13. D. Joyner, Toric codes over finite fields, Appl. Algebra Engrg. Comm. Comput., 15 (2004), 63–79.
14. J. Little, Remarks on generalized toric codes, Finite Fields Appl., 24 (2013), pp. 1–14.
15. J. Little, H. Schenck, Toric Surface Codes and Minkowski sums, SIAM J. Discrete Math. 20, no. 4, 999–1014.
16. J. Little, R. Schwarz, On toric codes and multivariate Vandermonde matrices, Appl. Algebra Engrg. Comm. Comput. 18 (2007), 349–367.
17. H. H. Lopez, R. H. Villarreal and L. Zarate, Complete Intersection Vanishing Ideals on Degenerate Tori over Finite Fields, Arab J. Math, 2013, 189-197.

18. Miller, E. and Sturmfels, B., Combinatorial commutative algebra. Graduate Texts in Mathematics, 227. Springer-Verlag, New York, (2005).

19. M. Morales and A. Thoma, Complete Intersection Lattice Ideals, J. Algebra, 2005, 284, 755-770.

20. Renteria, C., Simis, A., Villarreal, R. H., 2011. Algebraic methods for parameterized codes and invariants of vanishing over finite fields, Finite Fields Appl.,17 no. 1, 81–104.

21. D. Ruano, On the parameters of $r$-dimensional toric codes, Finite Fields Appl., 13 (2007), 962–976.

22. M. Şahin, Toric codes and lattice ideals, https://arxiv.org/abs/1712.00747

23. M. Şahin and I. Soprunov, Multigraded Hilbert functions and toric complete intersection codes, J. Algebra, 459 (2016), 446–467.

24. E. Sarmiento, M. Vaz Pinto, R. H. Villarreal, The minimum distance of parameterised codes on projective tori, Appl. Algebra Engrg. Comm. Comput. 22, 4(2011), 249-264.

25. I. Soprunov, Toric complete intersection codes, J. Symbolic Comput. 50 (2013), 374–385.

26. I. Soprunov, E. Soprunova, Toric surface codes and Minkowski length of polygons, SIAM J. Discrete Math. 23 (2009), pp. 384-400.

(Esma Baran) DEPARTMENT OF MATHEMATICS, ÇANKIRI KARATEKİN UNIVERSITY, ÇANKIRI, TURKEY

E-mail address: esmabaran@karatekin.edu.tr

(Mesut Şahin) DEPARTMENT OF MATHEMATICS, HACETTEPE UNIVERSITY, ANKARA, TURKEY

E-mail address: mesut.sahin@hacettepe.edu.tr