THE CONTINUOUS PRIMITIVE INTEGRAL IN THE PLANE

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ABSTRACT. An integral is defined on the plane that includes the Henstock–Kurzweil and Lebesgue integrals (with respect to Lebesgue measure). A space of primitives is taken as the set of continuous real-valued functions $F(x, y)$ defined on the extended real plane $[-\infty, \infty]^2$ that vanish when $x$ or $y$ is $-\infty$. With usual pointwise operations this is a Banach space under the uniform norm. The integrable functions and distributions (generalised functions) are those that are the distributional derivative $\partial^2/(\partial x \partial y)$ of this space of primitives. If $f = \partial^2/(\partial x \partial y)F$ then the integral over interval $[a, b] \times [c, d] \subseteq [-\infty, \infty]^2$ is $\int_a^b \int_c^d f = F(a, c) + F(b, d) - F(a, d) - F(b, c)$ and $\int_{-\infty}^\infty \int_{-\infty}^\infty f = F(\infty, \infty)$. The definition then builds in the fundamental theorem of calculus. The Alexiewicz norm is $\|f\| = \|F\|_{\infty}$ where $F$ is the unique primitive of $f$. The space of integrable distributions is then a separable Banach space isometrically isomorphic to the space of primitives. The space of integrable distributions is the completion of both $L^1$ and the space of Henstock–Kurzweil integrable functions. The Banach lattice and Banach algebra structures of the continuous functions in $\|\cdot\|_{\infty}$ are also inherited by the integrable distributions. It is shown that the dual space are the functions of bounded Hardy–Krause variation. Various tools that make these integrals useful in applications are proved: integration by parts, Hölder inequality, second mean value theorem, Fubini theorem, a convergence theorem, change of variables, convolution. The changes necessary to define the integral in $\mathbb{R}^n$ are sketched out.

1. Introduction

The continuous primitive integral is discussed in $\mathbb{R}^2$ and then briefly in $\mathbb{R}^n$. This is an integral defined by taking primitives (indefinite integrals) as continuous functions. It includes the Lebesgue and Henstock–Kurzweil integrals. The essential idea is to take a Banach space $\mathcal{B}$ of primitives and define the entities that can be integrated as the distributional derivative of each item in $\mathcal{B}$. Here $\mathcal{B}$ is taken as the continuous functions on the extended real plane. Each such function is differentiated with the partial differential operator $\partial_{12} = \partial^2/(\partial y \partial x)$. This automatically makes the distributions integrable in this sense into a Banach space isometrically isomorphic to the continuous functions under the uniform norm.

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The same process can be repeated with other classes of primitives. There is the regulated primitive integral \([51]\). A function on the real line is regulated if it has a left and right limit at each point, or from within each orthant in \(\mathbb{R}^n\). There is the \(L^p\) primitive integral \([54]\). And there are higher order distributional integrals for which each continuous function is differentiated multiple times \([52]\).

The name, continuous primitive integral, was introduced at the end of \([49]\). Some authors refer to the same integral as the distributional Henstock–Kurzweil or distributional Denjoy integral. As there are several integrals defined by their primitives, as above, we prefer the name continuous primitive integral.

First define the primitives. The extended real line is \(\mathbb{R} = [-\infty, \infty]\). A function \(F : \mathbb{R} \to \mathbb{R}\) is continuous on \(\mathbb{R}\) if it equals its limit at each point, \(F(x) = \lim_{t \to x} F(t)\), where the limit is necessarily one-sided if \(x = \infty\) or \(-\infty\). The extended real plane is \(\mathbb{R}^2\) endowed with the product topology. We then take as a space of primitives \(B_c(\mathbb{R}^2)\) which consists of the continuous functions \(F(x, y)\) on \(\mathbb{R}^2\) that vanish when \(x = -\infty\) or \(y = -\infty\). Under the uniform norm \(B_c(\mathbb{R}^2)\) is a Banach space. A distribution (generalised function), \(f\), has a continuous primitive integral if there is a function \(F \in B_c(\mathbb{R}^2)\) such that \(f = \partial_{12} F\), the partial derivative being understood in the distributional sense. Since \(F(x, y) = 0\) if \(x\) or \(y\) is \(-\infty\), the primitive is unique. If \((x, y) \in \mathbb{R}^2\) then the integral is \(\int_{-\infty}^{x} \int_{-\infty}^{y} f = F(x, y)\), with a similar definition on compact intervals. In this way the definition builds in the fundamental theorem of calculus.

The Alexiewicz norm of \(f\) is

\[
\|f\| = \sup_{(x,y) \in \mathbb{R}^2} \left| \int_{-\infty}^{x} \int_{-\infty}^{y} f \right| = \|F\|_{\infty}.
\]

Write the set of integrable distributions as \(A_c(\mathbb{R}^2)\). Then \(A_c(\mathbb{R}^2)\) is a Banach space that is isometrically isomorphic to \(B_c(\mathbb{R}^2)\). Since the Lebesgue and Henstock–Kurzweil integrals have continuous primitives they form dense subspaces of \(A_c(\mathbb{R}^2)\) but neither is complete in this norm. The continuous primitive integral then provides the completion with respect to the Alexiewicz norm of the space of Henstock–Kurzweil integrable functions. The Henstock–Kurzweil integral allows conditional convergence and so does the continuous primitive integral.

The Henstock–Kurzweil integral is a well-established integration process based on Riemann sums that includes the Lebesgue and improper Riemann integrals in \(\mathbb{R}^n\) (with respect to Lebesgue measure). For early results see \([29]\), \([37]\), \([46]\) and \([33]\). It is discussed on the real line and briefly in \(\mathbb{R}^2\) or \(\mathbb{R}^n\) in the monographs \([38]\), \([39]\), \([48]\) and \([34]\). A detailed treatment of the Henstock–Kurzweil integral on compact intervals in \(\mathbb{R}^n\) is given in \([12]\) and \([35]\), where there is also an extensive review of the literature. See also \([32]\). The Denjoy integral is equivalent to the Henstock–Kurzweil integral and is defined via properties of the primitive. See \([13]\).

Under the usual pointwise operations, \(B_c(\mathbb{R}^2)\) is a Banach lattice and Banach algebra; and \(A_c(\mathbb{R}^2)\) inherits these properties.
The simple structure of $B_c(\mathbb{R}^2)$ makes it easy to prove various results in $A_c(\mathbb{R}^2)$. The corresponding space of primitives for the Lebesgue integral are the absolutely continuous functions. There are many different notions of absolute continuity for functions of two variables, due to Tonelli and other authors. If $f \in L^1(\mathbb{R}^2)$ and $F(x,y) = \int_{-\infty}^x \int_{-\infty}^y f$ then $F$ is absolutely continuous in the sense of Carathéodory. See [47] for the definition and references to Carathéodory’s original work. The primitives for the Henstock–Kurzweil integral in $\mathbb{R}^2$ are much more complicated than $B_c(\mathbb{R}^2)$. See [13]. The primitives for Lebesgue and Henstock–Kurzweil integrals are continuous and the pointwise derivative $\partial_{12}$ exists almost everywhere. Being merely continuous, primitives in $B_c(\mathbb{R}^2)$ need not have a pointwise derivative anywhere but the distributional derivative $B_c(\mathbb{R}^2)$ is well-defined. See following Definition 4.1.

There are many different notions of bounded variation for functions of two variables ([14], [1], [2]). If $g$ is of bounded Hardy–Krause variation then the product $fg$ is in $A_c(\mathbb{R}^2)$ for all $f \in A_c(\mathbb{R}^2)$ and we can prove an integration by parts formula and Hölder inequality. Functions of bounded Hardy–Krause variation also form the dual space of $A_c(\mathbb{R}^2)$.

The paper is laid out as follows.

Section 2 gives the necessary background in distributions. Functions on the extended real plane are discussed in Section 3.

In Section 4 the continuous primitive integral is defined on intervals in $\mathbb{R}^2$ and various basic properties, such as linearity and the fundamental theorem of calculus, are proved. It is shown that $A_c(\mathbb{R}^2)$ is a separable Banach space isometrically isomorphic to the space of primitives $B_c(\mathbb{R}^2)$. The test functions, the real analytic functions, $L^1$ and the Henstock–Kurzweil integrable functions are all shown to be dense in $A_c(\mathbb{R}^2)$. It is shown that the integral can be defined as the limit of a sequence of Lebesgue integrals.

Various examples are given in Section 5. We have already noted above that the continuous primitive integral includes the Lebesgue and Henstock–Kurzweil integrals. If $F \in B_c(\mathbb{R}^2)$ and $f = \partial_{12}F$ then an example of note is the case when the primitive $F$ has a pointwise derivative $\partial_{12}F$ nowhere. Then $\int_a^b \int_c^d f$ is well-defined in $A_c(\mathbb{R}^2)$ but the Lebesgue integral of $f$ does not exist. Also, if $\partial_{12}F = 0$ almost everywhere then the Lebesgue integral of $f$ is 0 over every interval but the continuous primitive integral gives the value we would expect from the fundamental theorem of calculus. In this section we also discuss other compactifications of $\mathbb{R}^2$.

Functions of Hardy–Krause bounded variation are defined in Section 6 and some examples are given.

In Section 7 it is shown that the functions of Hardy–Krause bounded variation form the multipliers and allow us to prove an integration by parts formula in terms of Henstock–Stieltjes integrals. This leads to versions of the first and second mean value theorems for integrals. It is shown that $A_c(\mathbb{R}^2)$ is invariant under translations and that translations are continuous in the Alexiewicz norm.
A type of Hölder inequality is proved in Section 8. This gives the inequality $|\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} fg| \leq \|f\|\|g\|_{bv}$ for $f \in \mathcal{A}_c(\mathbb{R}^2)$ and $g$ of Hardy–Krause bounded variation. Some norms equivalent to $\|\cdot\|$ are introduced. It is shown that the dual space of $\mathcal{A}_c(\mathbb{R}^2)$ is the space of functions of Hardy–Krause bounded variation.

A convergence theorem is given in Section 9 for taking the limit under integrals $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} fg_n$ where $f \in \mathcal{A}_c(\mathbb{R}^2)$ and $g_n$ is a sequence of functions of bounded Hardy–Krause variation.

If $f \in \mathcal{A}_c(\mathbb{R}^2)$ then, in general, the only subsets $f$ is integrable on are finite unions of intervals in $\mathbb{R}^2$. Hence, a change of variables theorem can only map intervals to finite unions of intervals. In Section 10 a change of variables theorem is given where each variable $(x, y)$ is transformed to a linear combination of just one variable.

In Section 11 a partial ordering is introduced on $\mathcal{A}_c(\mathbb{R}^2)$ that makes this into a Banach lattice isomorphic to $\mathcal{B}_c(\mathbb{R}^2)$ under the usual pointwise ordering. Both $\mathcal{B}_c(\mathbb{R}^2)$ and $\mathcal{A}_c(\mathbb{R}^2)$ are abstract $M$-spaces.

In Section 12 the pointwise algebra structure on $\mathcal{B}_c(\mathbb{R}^2)$, defined as usual by $(FG)(x, y) = F(x, y)G(x, y)$, is extended to $\mathcal{A}_c(\mathbb{R}^2)$ so that it becomes a Banach algebra, without a unit but with an approximate identity, isomorphic to $\mathcal{B}_c(\mathbb{R}^2)$.

A sufficient condition for changing the order of iterated integrals is given in Section 13. Some examples are given for which iterated integrals are not equal. Examples of this type can be resolved by showing the primitive is not continuous on the closure of the interval of integration, although it may be continuous on the interior of the interval of integration.

Convolutions $f \ast g$ are defined in Section 14 for $f \in \mathcal{A}_c(\mathbb{R}^2)$ and $g$ of Hardy–Krause bounded variation. These behave similarly to convolutions when $f \in L^1$ and $g \in L^\infty$. Convolutions are also defined for $g \in L^1(\mathbb{R}^2)$ and these behave similarly to convolutions when $f, g \in L^1$.

Finally, some of the changes needed to define the integral in $\mathbb{R}^n$ are sketched out in Section 15.

The notion of using continuous functions for primitives appears to have first been considered by K. Ostaszewski in [44]. Then the definition of the integral was sketched out in the setting of compact intervals in $\mathbb{R}^n$ by P. Mikusinski and K. Ostaszewski in [40] and [41]. In the context of the real line it was also mentioned briefly by B. Bongiorno [10]; B. Bongiorno and T.V. Panchapagesan [11]; B. Bäumer, G. Lumer and F. Neubrander [9]. It was studied in more detail on compact intervals in $\mathbb{R}^2$ by D.D. Ang, K. Schmidt and L.K. Vy in [5] (with some results repeated in [6]) and (on the real line) by E. Talvila [49].

The integral was applied to Fourier series [53] and a type of Salem–Zygmund–Rudin–Cohen factorization was proved there. See also [43].

Various other properties were studied in [16], [20], [21], [50].

A number of our results are generalisations of similar results proved for the Henstock–Kurzweil integral in [35].
2. Distributions

Here we briefly describe notation and a few of the major properties of distributions that we will use. All of the results in distributions we use can be found in [18] and [19].

The support of a function \( \phi : \mathbb{R}^2 \to \mathbb{R} \) is the closure of the set on which it does not vanish, denoted \( \text{supp}(\phi) \). The test functions are \( D(\mathbb{R}^2) = C^\infty(\mathbb{R}^2) = \{ \phi : \mathbb{R}^2 \to \mathbb{R} \mid \phi \in C^\infty(\mathbb{R}^2) \text{ with compact support} \} \). Note that \( D(\mathbb{R}^2) \) is a linear space closed under differentiation. If \( \{ \phi_n \} \) is a sequence of functions in \( D(\mathbb{R}^2) \) and \( \phi \in D(\mathbb{R}^2) \) then \( \phi_n \to \phi \) if there is a compact set \( K \subset \mathbb{R}^2 \) such that for each \( n \in \mathbb{N} \) we have \( \text{supp}(\phi_n) \subseteq K \) and for all integers \( k, \ell \geq 0 \) we have \( \| \partial_1^k \partial_2^\ell \phi_n - \partial_1^k \partial_2^\ell \phi \|_\infty \to 0 \) as \( n \to \infty \), i.e., all partial derivatives converge uniformly to \( \phi \). The symbol \( \partial_i \) represents the partial derivative with respect to the \( i \)th Cartesian variable.

The distributions are the continuous linear functionals on \( D(\mathbb{R}^2) \). This is the dual space of \( D(\mathbb{R}^2) \), written \( D'(\mathbb{R}^2) \). For \( T \in D'(\mathbb{R}^2) \) its action on test function \( \phi \) is written as \( \langle T, \phi \rangle \in \mathbb{R} \). Distributions are linear: \( \langle T, a\phi + b\psi \rangle = a\langle T, \phi \rangle + b\langle T, \psi \rangle \) for all \( \phi, \psi \in D(\mathbb{R}^2) \) and all \( a, b \in \mathbb{R} \). Distributions are continuous: if \( \phi_n \to \phi \) in \( D(\mathbb{R}^2) \) then \( \langle T, \phi_n \rangle \to \langle T, \phi \rangle \) in \( \mathbb{R} \). To define distributions on an open set \( \Omega \subset \mathbb{R}^2 \) we use test functions with compact support in \( \Omega \).

All distributions have derivatives of all orders and all such derivatives are distributions. For each \( i = 1, 2 \) the derivative of \( T \in D'(\mathbb{R}^2) \) is \( \langle \partial_i T, \phi \rangle = -\langle T, \partial_i \phi \rangle \) for each \( \phi \in D(\mathbb{R}^2) \). Write \( \partial_{12} = \partial_1 \partial_2 \). Then \( \langle \partial_{12} T, \phi \rangle = \langle T, \partial_{12} \phi \rangle \). All Cartesian derivative operators commute on test functions and distributions.

3. Extended real plane

The extended real line is \( \mathbb{R} = [-\infty, \infty] \). It is a compact topological space with a topological base given by usual open intervals in \( \mathbb{R} \) together with intervals \([a, \infty) \), \((a, \infty) \) for all \( a \in \mathbb{R} \). This is then a two-point compactification of \( \mathbb{R} \). A function \( F : \mathbb{R} \to \mathbb{R} \) is continuous at \( x \in \mathbb{R} \) if \( \lim_{y \to x} F(y) = F(x) \), continuous at \(-\infty \) if \( \lim_{y \to -\infty} F(y) = F(x) \), continuous at \( \infty \) if \( \lim_{y \to \infty} F(y) = F(x) \). The last two limits are necessarily one-sided. For example, the function \( \arctan \) is continuous on \( \mathbb{R} \) if we define \( \arctan(\pm \infty) = \pm \pi / 2 \) and no definition at \( \pm \infty \) can make the functions \( \sin \) or \( \exp \) continuous on \( \mathbb{R} \).

The extended real plane is \( \mathbb{R}^2 \) and has the product topology. It is then a compact Hausdorff space. The continuous functions on \( \mathbb{R}^2 \) are denoted \( C(\mathbb{R}^2) \). Note that they are real-valued. We define

**Definition 3.1.**

\[
B_0(\mathbb{R}) = \{ F : \mathbb{R} \to \mathbb{R} \mid F \text{ is continuous on } \mathbb{R}, F(-\infty) = 0 \}
\]

\[
B_c(\mathbb{R}^2) = \{ F : \mathbb{R}^2 \to \mathbb{R} \mid F \text{ is continuous on } \mathbb{R}^2, F(-\infty, s) = F(s, -\infty) = 0 \text{ for all } s \in \mathbb{R} \}.
\]

Hence, a function \( F \in B_c(\mathbb{R}^2) \) is continuous at \((x, y) \in \mathbb{R}^2 \) if for each \( \epsilon > 0 \) there is \( \delta > 0 \) such that if \((x - \xi)^2 + (y - \eta)^2 < \delta^2 \) then \(|F(x, y) - F(\xi, \eta)| < \epsilon \). If
if $G \in \mathbb{R}^2$ implies uniform continuity in $\mathbb{R}^2$ but uniform continuity in $\mathbb{R}^2$ does not imply continuity or boundedness in $\mathbb{R}^2$. With the uniform norm, $\| \cdot \|_\infty$, $\mathcal{B}_c(\mathbb{R}^2)$ is a Banach space. Note that if $F \in C(\mathbb{R}^2)$ then

$$\|F\|_\infty = \sup_{(x,y) \in \mathbb{R}^2} |F(x,y)| = \sup_{(x,y) \in \mathbb{R}^2} |F(x,y)| = \max_{(x,y) \in \mathbb{R}^2} |F(x,y)|.$$  

4. The continuous primitive integral

We can now define the integrable distributions as the derivatives of functions in $\mathcal{B}_c(\mathbb{R}^2)$. Parts of Propositions 4.2, 4.5, 4.6 were proved for compact intervals in [5].

**Definition 4.1.**

$$\mathcal{A}_c(\mathbb{R}^2) = \{ f \in \mathcal{D}'(\mathbb{R}^2) \mid f = \partial_{12}F \text{ for some } F \in \mathcal{B}_c(\mathbb{R}^2) \}.$$  

In this definition the function $F$ is called the primitive of $f$. If $f \in \mathcal{A}_c(\mathbb{R}^2)$ has primitive $F \in \mathcal{B}_c(\mathbb{R}^2)$ then the action of $f$ on test function $\phi$ is $\langle f, \phi \rangle = \langle F, \partial_{12} \phi \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(x,y) \partial_{12} \phi(x,y) \, dy \, dx$. Since $\phi$ is smooth with compact support this last integral exists in the Riemann sense.

If $F$ is a function in $\mathcal{B}_c(\mathbb{R}^2)$ then $F(x,y)$ vanishes when $x$ or $y$ is $-\infty$. Primitives are then unique. The derivative operator $\partial_{12}$ is a linear isomorphism between $\mathcal{A}_c(\mathbb{R}^2)$ and $\mathcal{B}_c(\mathbb{R}^2)$. We define its inverse to be the integral and then $\mathcal{A}_c(\mathbb{R}^2)$ inherits the Banach space structure of $\mathcal{B}_c(\mathbb{R}^2)$.

**Proposition 4.2.** (a) If $f \in \mathcal{A}_c(\mathbb{R}^2)$ then it has a unique primitive in $\mathcal{B}_c(\mathbb{R}^2)$. (b) If $f \in \mathcal{A}_c(\mathbb{R}^2)$ with primitive $F \in \mathcal{B}_c(\mathbb{R}^2)$ then define the Alexiewicz norm of $f$ by $\| f \| = \| F \|_\infty$. Then $\mathcal{A}_c(\mathbb{R}^2)$ is a Banach space. The derivative $\partial_{12}$ provides a linear isometry and isomorphism between $\mathcal{B}_c(\mathbb{R}^2)$ and $\mathcal{A}_c(\mathbb{R}^2)$. (c) If $G$ is continuous on $\mathbb{R}^2$ then $\partial_{12}G \in \mathcal{A}_c(\mathbb{R}^2)$. (d) For all $f, g \in \mathcal{A}_c(\mathbb{R}^2)$; $c_1, c_2 \in \mathbb{R}$; $\phi \in \mathcal{D}(\mathbb{R}^2)$ we have $\langle c_1 f + c_2 g, \phi \rangle = c_1 \langle f, \phi \rangle + c_2 \langle g, \phi \rangle$.

The Alexiewicz norm first appears in [3].

**Proof.** (a) Suppose $f \in \mathcal{A}_c(\mathbb{R}^2)$ and $f = \partial_{12}F = \partial_{12}G$ for $F, G \in \mathcal{B}_c(\mathbb{R}^2)$. Let $\Phi = F - G$. Then $\Phi \in \mathcal{B}_c(\mathbb{R}^2)$ and $\partial_{12} \Phi = 0$. But then $\Phi(x,y) = \Theta(x) + \Psi(y)$ for some functions $\Theta, \Psi \in C(\mathbb{R})$. Fixing $x \in \mathbb{R}$ and letting $y \to -\infty$ and then fixing $y \in \mathbb{R}$ and letting $x \to -\infty$ shows $\Theta$ and $\Psi$ are constant functions with sum 0. (b) The derivative operator $\partial_{12}$ is linear. By (a) it is one-to-one on $\mathcal{B}_c(\mathbb{R}^2)$. By definition it is onto $\mathcal{A}_c(\mathbb{R}^2)$. The definition of $\| \cdot \|$ makes it into an isometry. (c) If $G \in C(\mathbb{R}^2)$ define $\Theta, \Psi \in C(\mathbb{R})$ by $\Theta(x) = G(x, -\infty)$ and $\Psi(y) = G(-\infty, y)$. Define $F \in \mathcal{B}_c(\mathbb{R}^2)$ by $F(x,y) = G(x,y) + G(-\infty, -\infty) - \Theta(x) - \Psi(y)$. Then $\partial_{12}G = \partial_{12}F$. (d) The derivative $\partial_{12}$ is linear.

We can now define the integral of a distribution in $\mathcal{A}_c(\mathbb{R}^2)$.
Definition 4.3. Let \( f \in \mathcal{A}_c(\mathbb{R}^2) \) with primitive \( F \in \mathcal{B}_c(\mathbb{R}^2) \). We define its continuous primitive integral on interval \( I = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \) by \( \int_I f = \int_a^b \int_c^d f = F(a, c) + F(b, d) - F(a, d) - F(b, c) \).

If \( a = b \) or \( c = d \) then the integral of \( f \) over \( I \) is zero. This shows the integral over any line parallel to the \( x \) or \( y \) axis is zero. Hence, the integral over the boundary of an interval always vanishes and there is no distinction between integrating over open or closed intervals. We also have the usual convention that \( \int_a^d f = -\int_d^a f \). Note that \( \int_{\mathbb{R}^2} f = F(\infty, \infty) \) and \( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f = F(x, y) \) for all \( (x, y) \in \mathbb{R}^2 \). As well, \( \int_I (c_1 f + c_2 g) = c_1 \int_I f + c_2 \int_I g \).

The definition builds in the fundamental theorem of calculus.

Proposition 4.4 (Fundamental theorem of calculus). (a) Let \( f \in \mathcal{A}_c(\mathbb{R}^2) \) and define \( \Phi(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f \). Then \( \Phi \in \mathcal{B}_c(\mathbb{R}^2) \) and \( \partial_{12} \Phi = f \). (b) Let \( G \in C(\mathbb{R}^2) \). Then \( \int_{-\infty}^{x} \int_{-\infty}^{y} \partial_{12} G = G(-\infty, -\infty) + G(x, y) - G(-\infty, y) - G(x, -\infty) \).

Proof. (a) See Proposition 4.2 (a). (b) See Proposition 4.2 (c).

The space \( \mathcal{B}_c(\mathbb{R}^2) \) is separable and hence \( \mathcal{A}_c(\mathbb{R}^2) \) is as well.

Proposition 4.5. (a) Step functions are dense in \( \mathcal{B}_c(\mathbb{R}^2) \). (b) Both \( \mathcal{B}_c(\mathbb{R}^2) \) and \( \mathcal{A}_c(\mathbb{R}^2) \) are separable. (c) The real analytic functions are dense in \( \mathcal{B}_c(\mathbb{R}^2) \) and \( \mathcal{A}_c(\mathbb{R}^2) \). (d) If \( f: \mathbb{R}^2 \to \mathbb{R} \) is a function in \( L^1(\mathbb{R}^2) \) (with respect to Lebesgue measure), or integrable in the sense of Henstock–Kurzweil or as a Denjoy integral then \( f \in \mathcal{A}_c(\mathbb{R}^2) \) and the integrals agree on intervals in \( \mathbb{R}^2 \).

Proof. (a) For each \( n \in \mathbb{N} \) we can make a partition of \( \mathbb{R} \) by \(-\infty = p_0 < p_1 < \ldots < p_n = \infty \) and hence of \( \mathbb{R}^2 \) using \((p_i, p_j)\). Let \( P_{ij} = (p_{i-1}, p_i) \times (p_{j-1}, p_j) \). Let \( \sigma_{ij} \in \mathbb{R} \). A step function is

\[
\sigma(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} \chi_{P_{ij}}(x, y)
\]

with \( \sigma(-\infty, y) = \sigma(x, -\infty) = 0 \) for \( x, y \in \mathbb{R} \). If \( p_i \) and \( \sigma_{ij} \) are taken in \( \mathbb{Q} \), then the collection of all such step functions is countable. Since \( \mathbb{R}^2 \) is compact, given \( \epsilon > 0 \) and \( F \in \mathcal{B}_c(\mathbb{R}^2) \) there is a step function \( \sigma \) with \( \sigma_{11} = \sigma_{i1} = 0 \) and \( \|F - \sigma\|_\infty < \epsilon \).

(b) The half-space Poisson kernel is \( \Phi_z(x, y) = z(x^2 + y^2 + z^2)^{-3/2}/(2\pi) \) where \( z > 0 \). For example, see [8]. Note that \( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi_z(x, y) \, dy \, dx = 1 \). For a step function \( \sigma \) as above, define

\[
(4.1) \quad u_z(x, y) = \sigma * \Phi_z(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma(x - \xi, y - \eta) \Phi_z(\xi, \eta) \, d\eta \, d\xi
\]

\[
(4.2) \quad = \Phi_z * \sigma(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma(\xi, \eta) \Phi_z(x - \xi, y - \eta) \, d\eta \, d\xi.
\]
In (4.1) we can have \((x, y) \in \mathbb{R}^2\); since \(\sigma(x, y)\) has limits with one of \(x\) and \(y\) fixed in \(\mathbb{R}\) and the other going to \(\infty\) or \(-\infty\) we define

\[
    u_z(\infty, y) = \sigma \ast \Phi_z(\infty, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma(\infty, y - \eta) \Phi_z(\xi, \eta) \, d\eta \, d\xi.
\]

Similarly at other points of the boundary of \(\mathbb{R}^2\). With this convention, note that

\[
    u_z(\infty, \infty) = \sigma \ast \Phi_z(\infty, \infty) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma(\infty, \infty) \Phi_z(\xi, \eta) \, d\eta \, d\xi = \sigma(\infty, \infty)
\]

since the Poisson kernel integrates to 1. Similarly at other corners of \(\mathbb{R}^2\). We use (4.2) only for \((x, y) \in \mathbb{R}^2\).

If \((x, y)\) is in a compact set \(K \subset \mathbb{R}^2\) then there is a constant \(k\) (depending on \(K\) and \(z\)) such that \(|(x - \xi)^2 + (y - \eta)^2 + z^2|^{-3/2} \leq k[\xi^2 + \eta^2 + z^2]^{-3/2}\) for all \((\xi, \eta) \in \mathbb{R}^2\). By dominated convergence we can differentiate under the integral in (4.2) at each \((x, y, z) \in \mathbb{R}^3\). Hence, \(u_z \in C^\infty(\mathbb{R}^3)\). (Harmonic functions are in fact real analytic.)

Since \(\sigma\) is bounded, dominated convergence allows us to take limits under the integral and this shows \(u_z\) is continuous on the boundary of \(\mathbb{R}^n\) at points of continuity of \(\sigma\). To show continuity at other points on the boundary note that \(\sigma_z\) is the sum of a finite number of terms of type, say,

\[
    \sigma_{m_j} \int_{p_{m-1}}^{p_1} \Phi_z(\xi - x, \eta - y) \, d\eta \, d\xi = \sigma_{m_j} \int_{p_{m-1}}^{p_1} \int_{p_{j-1}}^{p_j} \Phi_z(\xi, \eta) \, d\eta \, d\xi
\]

which is clearly continuous on \(\mathbb{R}^2\). Hence, \(u_z \in B_c(\mathbb{R}^2)\).

Convolution with the Poisson kernel is known to approximate a continuous function uniformly on compact sets in \(\mathbb{R}^2\) as \(z\) decreases to 0. Our function \(\sigma\) need not be continuous but is still approximated in this sense. If \((0, 0) \in P_{ij}\) let \(Q_0 = \cup \{P_{\alpha\beta} \mid \alpha = i - 1, i, i + 1, \beta = j - 1, j, j + 1\}\). If \(i\) or \(j\) is 1 or \(n\) then this union might contain fewer than nine rectangles. Since \(F\) is continuous and \(\|F - \sigma\|_\infty < \epsilon\) we have \(|\sigma_{\alpha\beta} - \sigma_{\gamma\delta}| \leq 2\epsilon\) if \(|\alpha - \gamma| \leq 1\) and \(|\beta - \delta| \leq 1\). For \((x, y) \in \mathbb{R}^2\) we have

\[
    u_z(x, y) - \sigma(x, y) = \sigma \ast \Phi_z(x, y) - \sigma(x, y) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi_z(\xi, \eta) \, d\eta \, d\xi
\]

so that \(|u_z(x, y) - \sigma(x, y)| \leq I_1 + I_2\) where

\[
    I_1 = \int \int_{Q_0} |\sigma(x - \xi, y - \eta) - \sigma(x, y)| \Phi_z(\xi, \eta) \, d\eta \, d\xi \leq 2\epsilon \int \int_{Q_0} \Phi_z(\xi, \eta) \, d\eta \, d\xi \leq 2\epsilon,
\]

\[
    I_2 = \int \int_{\mathbb{R}^2 \setminus Q_0} |\sigma(x - \xi, y - \eta) - \sigma(x, y)| \Phi_z(\xi, \eta) \, d\eta \, d\xi \leq 2(\|F\|_\infty + \epsilon) \int \int_{\mathbb{R}^2 \setminus Q_0} \Phi_z(\xi, \eta) \, d\eta \, d\xi \to 0 \text{ as } z \downarrow 0.
\]
The last line above follows with dominated convergence. If we let $z$ decrease to 0 through rational values then we see $B_c(\mathbb{R}^2)$ is separable. This also shows $\mathcal{A}_c(\mathbb{R}^2)$ is separable.

(c) The proof of (b) shows the real analytic functions are dense in $B_c(\mathbb{R}^2)$ and hence dense in $\mathcal{A}_c(\mathbb{R}^2)$. (d) Primitives of Lebesgue integrable, Henstock–Kurzweil integrable and Denjoy integrable functions are continuous. When an absolutely continuous function is differentiated pointwise the derivative agrees with the distributional derivative. The same applies to primitives of the other two integrals, which are discussed in [13]. Hence, the continuous primitive integral includes the Lebesgue, Henstock–Kurzweil and Denjoy integrals in the sense that the integrals agree on intervals.

It is known that $C(X)$ is separable exactly when $X$ is a compact metric space. For example, [30, p. 221]. This then shows that $B_c(\mathbb{R}^2)$ is separable. However, our construction in the above proof lets us conclude real analytic functions are dense in $B_c(\mathbb{R}^2)$.

If $f$ is a function in $L^1(\mathbb{R}^2)$ then $f \in \mathcal{A}_c(\mathbb{R}^2)$ and $\|f\| \leq \|f\|_1$ with equality if $f \geq 0$ almost everywhere. The norms are not equivalent on $L^1(\mathbb{R}^2)$. For example, for $n \in \mathbb{N}$ let $f_n(x,y) = \sin(nx)\chi_{(0,2\pi)}(x)\chi_{(0,1)}(y)$. Then $\|f_n\|_1 = \int_0^{2\pi} |\sin(nx)| \, dx = 4$. Hence, there can be no inequality $c_1\|f\| \leq \|f\|_1 \leq c_2\|f\|$ for some constants $c_1$, $c_2$ and all $f \in L^1(\mathbb{R}^2)$.

**Proposition 4.6.** (a) $L^1(\mathbb{R}^2)$ is dense in $\mathcal{A}_c(\mathbb{R}^2)$. (b) The test functions are dense in $\mathcal{A}_c(\mathbb{R}^2)$.

**Proof.** (a) The construction in Proposition 4.5 shows $L^1(\mathbb{R}^2)$ is dense in $\mathcal{A}_c(\mathbb{R}^2)$ since we can differentiate $u_\epsilon$ under the integral sign. Integration then shows $\|\partial_{12}u_\epsilon\|_1 \leq 4 \sum_{i,j} |\sigma_{ij}|$. (b) If $f \in \mathcal{A}_c(\mathbb{R}^2)$ and $\epsilon > 0$ there is $g \in L^1(\mathbb{R}^2)$ such that $\|f - g\|_1 < \epsilon$. If $\phi$ is a test function then $\|f - \phi\| \leq \|f - g\| + \|g - \phi\|_1$ and test functions are known to be dense in $L^1(\mathbb{R}^2)$. For example, [18, Proposition 8.17].

This then gives an alternative way to define the integral. We have $\mathcal{A}_c(\mathbb{R}^2)$ is the completion of $L^1(\mathbb{R}^2)$ in the Alexiewicz norm. If $\{f_n\} \subset L^1(\mathbb{R}^2)$ is a Cauchy sequence in the Alexiewicz norm then it converges to a distribution $f \in \mathcal{A}_c(\mathbb{R}^2)$. Let $F_n, F \in B_c(\mathbb{R}^2)$ be the respective primitives of $f_n$ and $f$. Then $\partial_{12}(F_n - F) = \partial_{12}F_n - \partial_{12}F = f_n - f$ so $\|F_n - F\|_\infty \rightarrow 0$. This gives

$$\int_a^b \int_c^d f = F(a,c) + F(b,d) - F(a,d) - F(b,c)$$

$$= \lim_{n \rightarrow \infty} [F_n(a,c) + F_n(b,d) - F_n(a,d) - F_n(b,c)]$$

$$= \lim_{n \rightarrow \infty} \int_a^b \int_c^d f_n$$

and defines the integral of $f \in \mathcal{A}_c(\mathbb{R}^2)$ using only Lebesgue integrals of functions in $L^1(\mathbb{R}^2)$. 

**CONTINUOUS PRIMITIVE INTEGRAL 9**
5. Examples

If $f$ and $\tilde{f}$ are functions in $\mathcal{A}_c(\mathbb{R}^2)$ such that $f$ and $\tilde{f}$ agree except on a set of Lebesgue measure zero then they have the same primitive in $\mathcal{B}_c(\mathbb{R}^2)$ and hence the same integral on all subintervals on $\mathbb{R}^2$. Of course, this pointwise comparison is not possible for distributions in $\mathcal{A}_c(\mathbb{R}^2)$ that do not happen to be functions.

Functions that have a conditionally convergent Henstock–Kurzweil or improper Riemann integral are in $\mathcal{A}_c(\mathbb{R}^2) \setminus L^1(\mathbb{R}^2)$. For example, we can define $f(x, y) = \sin(x) \sin(y)/(xy)$ with $f(x, y) = 0$ if $x = 0$ or $y = 0$. For another example take $G(x, y) = x^2 y^2 \sin(x^{-4}) \sin(y^{-4})$ with $G(x, y) = 0$ if $x = 0$ or $y = 0$. Then $G \in \mathcal{B}_c(\mathbb{R}^2)$ so $\partial_{12} G \in \mathcal{A}_c(\mathbb{R}^2) \setminus L^1_{loc}(\mathbb{R}^2)$.

The above examples are products of a function of $x$ and a function of $y$. More generally, note that $F, G \in \mathcal{B}_c(\mathbb{R}^2)$ and $\partial_{12} F \in \mathcal{A}_c(\mathbb{R}^2)$.

Similarly, the function $(x, y) \mapsto F(x, y) G(x) \in \mathcal{B}_c(\mathbb{R}^2)$ if now $G \in C(\mathbb{R})$ or if $G \in C((-\infty, \infty))$ and is bounded. In general we cannot apply a differentiation product rule except when $G$ is of bounded variation. See Section 7.

If $f, g \in \mathcal{A}_c(\mathbb{R})$ then define $h \in \mathcal{A}_c(\mathbb{R}^2)$ by $h(x, y) = f(x) g(y)$. This is in fact a tensor product but we will not employ any special notation. We can take $F, G \in \mathcal{B}_c(\mathbb{R})$ to be functions of Weierstrass type that are continuous but pointwise differentiable nowhere. Then the distributional derivative is $\partial_{12} (FG) = f'g'$.

Neither the Lebesgue nor Henstock–Kurzweil integral of $f'g'$ exists on any interval but the continuous primitive integral is $\int_a^b \int_c^d f'g' = [F(b) - F(a)][G(d) - G(c)]$ for all $[a, b] \times [c, d] \subseteq \mathbb{R}$. If we take $F, G \in \mathcal{B}_c(\mathbb{R})$ to be singular, i.e., continuous, not constant, with pointwise derivative equal to 0 almost everywhere, then the Lebesgue integral exists and gives $\int_a^b \int_c^d f'g' = 0$ while the continuous primitive integral is again $\int_a^b \int_c^d f'g' = [F(b) - F(a)][G(d) - G(c)]$.

An example of $F \in \mathcal{B}_c(\mathbb{R}^2)$ that is not a product of functions in $\mathcal{B}_c(\mathbb{R})$ is $F(x, y) = \exp(-\sqrt{x^2 + y^2})$. Proposition 5.1 also gives a procedure for constructing such primitives.

In Section 10 we discuss change of variables. It is worth noting here that $C(\mathbb{R}^2), \mathcal{B}_c(\mathbb{R}^2), \mathcal{A}_c(\mathbb{R}^2)$ and $\partial_{12}$ are not invariant under rotations. For example, if $F(x, y) = x/(1 + |x|)$ then $F \in C(\mathbb{R}^2)$ and $\partial_{12} F = 0$. Rotate to get $G(x, y) = (x + y)/(1 + |x + y|)$. For $(x, y) \in \mathbb{R}^2$ we have $G(\infty, y) = 1, G(-\infty, y) = -1, G(x, \infty) = 1, G(x, -\infty) = -1$. Hence, $G \notin C(\mathbb{R}^2)$. And, $\partial_{12} G \neq 0$.

The topology of $\mathbb{R}^2$ depends on the Cartesian coordinate system. In $\mathbb{R}^2$ we employ a four-point compactification of $\mathbb{R}^2$. Stereographic projection uses a one-point compactification so that a function continuous in the polar coordinates extended plane must have $\lim_{r \to \infty} F(r, \theta)$ equal to a constant, independent of angle $\theta$. If $F$ is continuous in this sense then $F \in C(\mathbb{R}^2)$. The converse is not true; for example, $F(x, y) = \arctan(x) \arctan(y)$.

Also, in polar coordinates we could use a compactification of $\mathbb{R}^2$ with a continuum of points at infinity. In this sense, a function $(r, \theta) \mapsto F(r, \theta)$ is continuous
Proposition 5.1. Now consider $\Theta_2$ with these boundary values. Then there is a function $F(x, y) = \lambda_{1}(\Theta_1(y), \Theta_2(x)), F(x, -\infty) = \Theta_4(x)$. If $g: \mathbb{R} \to \mathbb{R}$ is a Banach space under the norm $\|g\|_{BV} = \|g\|_{\infty} + Vg$. If $f \in \mathcal{A}_c(\mathbb{R}^2)$ with primitive $F \in \mathcal{B}_c(\mathbb{R}^2)$ and $g \in \mathcal{BV}(\mathbb{R})$ then there is the integration by parts formula $\int_{-\infty}^{\infty} f dg = F(\infty) g(\infty) - \int_{-\infty}^{\infty} F dg$. The last integral is a Henstock–Stieltjes integral. See [39]. If we take $\lambda_1, \lambda_2 \in [0, 1]$ such that $\lambda_1 + \lambda_2 = 1$ (a convex combination) and require $g \in \mathcal{BV}$ to satisfy $g(x) = \lambda_1 g(x^-) + \lambda_2 g(x^+)$ for each $x \in \mathbb{R}$ and $g(\infty) = \lim_{x \to \infty} g(x)$, $g(-\infty) = \lim_{x \to -\infty} g(x)$ then $g$ is said to be of normalised bounded variation. For example, taking $\lambda_1 = 0$ and $\lambda_2 = 1$ makes $g$ right continuous on $\mathbb{R}$. Sometimes different conditions are imposed at $\pm \infty$. A function of bounded variation need only be changed on a countable collection of points to make it of normalised bounded variation. Note that the $\mathcal{BV}$ norm of a function is the same for any
normalisation. A normalisation can then be fixed and the resulting space labeled \( N BV \).

Two intervals in \( \mathbb{R}^2 \) are **nonoverlapping** if their intersection is of Lebesgue (planar) measure zero. A **division** of \( \mathbb{R}^2 \) is a finite collection of nonoverlapping intervals whose union is \( \mathbb{R}^2 \). If \( g : \mathbb{R}^2 \to \mathbb{R} \) then its Vitali variation is \( V g = \sup_D \sum_i |g(a_i, c_i) + g(b_i, d_i) - g(a_i, d_i) - g(b_i, c_i)| \) where the supremum is taken over all divisions \( D \) of \( \mathbb{R}^2 \) and interval \( I_i = [a_i, b_i] \times [c_i, d_i] \) is an interval in \( D \). If we fix one variable and find the one-variable variation as a function of the remaining variable we write \( V g(\cdot, y_0) \) or \( V g(x_0, \cdot) \) according as the second variable has been fixed as \( y_0 \) or the first variable fixed as \( x_0 \). The space of Hardy–Krause bounded variation is defined as follows.

**Definition 6.1** (Hardy–Krause variation). Let \( g : \mathbb{R}^2 \to \mathbb{R} \). Suppose \( V g = \sup_D \sum_i |g(a_i, c_i) + g(b_i, d_i) - g(a_i, d_i) - g(b_i, c_i)| \) where the supremum is taken over all divisions \( D \) of \( \mathbb{R}^2 \) and interval \( I_i = [a_i, b_i] \times [c_i, d_i] \) is an interval in \( D \). We write \( \| g \|_{bv} = \| g \|_{\infty} + \| V_1 g \|_{\infty} + \| V_2 g \|_{\infty} + \| V_{12} g \|_{\infty} \).

Basic results about functions of finite Hardy–Krause variation are proved in [14] and [7]. Our definition is slightly different but the same results hold. In particular, functions in \( HKBV(\mathbb{R}^2) \) are bounded and if \( V_1 g(\cdot, y_0) \) and \( V_2 g(x_0, \cdot) \) are finite for some \( x_0 \) and \( y_0 \) then \( \| V_1 g \|_{\infty} \) and \( \| V_2 g \|_{\infty} \) are finite and \( HKBV(\mathbb{R}^2) \) is a Banach space.

There are many types of variation for functions of two or more variables but Hardy–Krause variation is the most appropriate for nonabsolute integration. See [14], [1], [2] and [28].

**Example 6.2.** For \( (x, y) \in \mathbb{R}^2 \) let \( g = \chi_{(-\infty, x) \times (-\infty, y)} \). Then \( \| g \|_{\infty} = 1 \). And,

\[
V_1 g(\cdot, t) = \begin{cases} 
0, & t \geq y \\
1, & t < y,
\end{cases}
\]

\[
V_2 g(s, \cdot) = \begin{cases} 
1, & s < x \\
0, & s \geq x,
\end{cases}
\]

so that \( \| V_1 g \|_{\infty} = \| V_2 g \|_{\infty} = 1 \). Note that \( V_{12} g = 1 \) since there is exactly one interval in each division of \( \mathbb{R}^2 \) with exactly one corner in \( (-\infty, x) \times (-\infty, y) \). Therefore \( \| g \|_{bv} = 4 \).

Similarly, if

\[
g(x, y) = \begin{cases} 
1, & x \geq 0 \\
0, & x < 0
\end{cases}
\]

then \( \| g \|_{\infty} = 1, \| V_1 g \|_{\infty} = 1, \| V_2 g \|_{\infty} = 0, V_{12} g = 0 \) so that \( \| g \|_{bv} = 2 \).

If \( I \) is a finite interval in \( \mathbb{R}^2 \) and \( g = \chi_I \) then we have \( \| g \|_{\infty} = 1, \| V_1 g \|_{\infty} = \| V_2 g \|_{\infty} = 2, V_{12} g = 4 \) so that \( \| g \|_{bv} = 9 \).

**Example 6.3.** The function

\[
g(x, y) = \begin{cases} 
1, & y > x \\
0, & y \leq x.
\end{cases}
\]
is not of bounded Hardy–Krause variation. Only intervals with one corner on the line \( y = x \) contribute to \( V_{12} \). But there can be a countable number of these. For a similar example see [45].

It can be shown that \( \chi_I \) is of bounded Hardy–Krause variation if and only if \( I \) is a finite union of intervals (in the fixed Cartesian coordinate system).

## 7. Integration by parts

Note the classical formula, valid for \( F, g \in C^2(\mathbb{R}^2) \) and all \( a, b, c, d \in \mathbb{R} \),

\[
\int_a^b \int_c^d \partial_{12} (Fg)(x, y) \, dy \, dx
= F(a, c)g(a, c) + F(b, d)g(b, d) - F(a, d)g(a, d) - F(b, c)g(b, c)
= \int_a^b \int_c^d F_{12}(x, y)g(x, y) \, dy \, dx + \int_a^b [F(x, d)g_1(x, d) - F(x, c)g_1(x, c)] \, dx
+ \int_c^d [F(b, y)g_2(b, y) - F(a, y)g_2(a, y)] \, dy - \int_a^b \int_c^d F(x, y)g_{12}(x, y) \, dy \, dx.
\]

This gives us the form the integration by parts formula should have in \( \mathcal{A}_c(\mathbb{R}^2) \). It is essentially the same as the formula for Henstock–Kurzweil integrals [35, Example 6.5.11]. See also [56].

**Definition 7.1** (Integration by parts). Let \( f \in \mathcal{A}_c(\mathbb{R}^2) \) with primitive \( F \in \mathcal{B}_c(\mathbb{R}^2) \). Let \( g \in HKBV(\mathbb{R}^2) \). Let \([a, b] \times [c, d] \subseteq \mathbb{R}^2 \). Define

\[
\int_a^b \int_c^d fg = F(a, c)g(a, c) + F(b, d)g(b, d) - F(a, d)g(a, d) - F(b, c)g(b, c)
- \int_a^b F(x, d)g_1(x, d) + \int_a^b F(x, c)g_1(x, c)
- \int_c^d F(b, y)g_2(b, y) + \int_c^d F(a, y)g_2(a, y)
+ \int_a^b \int_c^d F(x, y)g_{12}(x, y).
\]

Subscripts indicate a Henstock–Stieltjes integral with respect to the relevant variable. This is defined as follows [12]. A **tagged division** of \( \mathbb{R}^2 \) is a division for which each interval in the division has an associated **tag**, which is a point in the interval. A **gauge** is a mapping \( \gamma \) from \( \mathbb{R}^2 \) to the open sets in \( \mathbb{R}^2 \) such that \( \gamma(x, y) \) is an open set containing point \((x, y)\). An interval-point pair in a tagged division is \( \gamma \)-fine if \( I \subset \gamma(x, y) \) where \((x, y)\) is the tag associated with interval \( I \). It is possible to choose \( \gamma \) such that if \((x, y) \in \mathbb{R}^2 \) then \( \gamma(x, y) \subset \mathbb{R}^2 \). This means that if an interval in a \( \gamma \)-fine tagged division intersects the boundary of \( \mathbb{R}^2 \) then its tag must be on the boundary of \( \mathbb{R}^2 \). (We always assume this of \( \gamma \).) Existence of \( \gamma \)-fine tagged divisions is proven in [35], [39] and [48], and is a consequence of the compactness of \( \mathbb{R}^2 \). If \( F, g : \mathbb{R}^2 \to \mathbb{R} \) then the Henstock–Stieltjes integral

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(x, y) \, d_{12}g(x, y)
\]
exists with value \( A \in \mathbb{R} \) if for every \( \epsilon > 0 \) there is a
gauge \( \gamma \) such that for each \( \gamma \)-fine tagged division \( \{[a_i, b_i] \times [c_i, d_i], (x_i, y_i)\}_{i=1}^N \) we have
\[
\left| \sum_{i=1}^N F(x_i, y_i)[g(a_i, c_i) + g(b_i, d_i) - g(a_i, d_i) - g(b_i, c_i)] - A \right| < \epsilon.
\]

There are various other Stieltjes type integrals, including Riemann-Stieltjes, but they are all equivalent when \( F \) is continuous. See [28], [39] and [22].

Note that according to Definition 7.1
\[
\int_{-\infty}^x \int_{-\infty}^y fg = F(x, y)g(x, y) - \int_{-\infty}^x F(s, \infty) d_1 g(s, \infty)
\]
\[
- \int_{-\infty}^x F(\infty, t) d_2 g(\infty, t) + \int_{-\infty}^x \int_{-\infty}^y F(s, t) d_{12} g(s, t).
\]

**Remark 7.2.** Every distribution, \( T \), can be multiplied by every \( C^\infty \) function, \( \psi \), using \( \langle T \psi, \phi \rangle = \langle T, \phi \psi \rangle \) for test function \( \phi \). The pointwise product \( \phi \psi \) is again a test function. And, Definition 7.1 now defines the product \( fg \). Since \( fg \) is integrable for each \( f \in \mathcal{A}_c(\mathbb{R}^2) \) we say \( g \in \mathcal{HKBV}(\mathbb{R}^2) \) is a *multiplier* for the continuous primitive integral.

The integration by parts formula then induces the multiplication \( \mathcal{A}_c(\mathbb{R}^2) \times \mathcal{HKBV} \to \mathcal{A}_c(\mathbb{R}^2) \) and \( \mathcal{A}_c(\mathbb{R}^2) \) is then a Banach \( \mathcal{HKBV} \)-module. See [15] for the definition. Theorems similar to those in [51, §7] can then be proved.

We can justify the above definition with the following observation.

**Proposition 7.3.** Suppose \( F \in C(\mathbb{R}^2) \), \( f = \partial_{12} F \), \( g \in \mathcal{HKBV}(\mathbb{R}^2) \). For \((x, y) \in \mathbb{R}^2 \) define
\[
\Phi(x, y) = F(x, y)g(x, y) - \int_{-\infty}^x F(s, y) d_1 g(s, y) - \int_{-\infty}^y F(x, t) d_2 g(x, t)
\]
\[
+ \int_{-\infty}^x \int_{-\infty}^y F(s, t) d_{12} g(s, t).
\]
Then \( \Phi \in C(\mathbb{R}^2) \). If \( F \in \mathcal{B}_c(\mathbb{R}^2) \) then \( \Phi \in \mathcal{B}_c(\mathbb{R}^2) \) and \( \partial_{12} \Phi \in \mathcal{A}_c(\mathbb{R}^2) \).

**Proof.** To prove continuity at \((x, y) \in \mathbb{R}^2 \) let \((\xi, \eta) \in \mathbb{R}^2 \). It suffices to consider \( \xi \leq x \) and \( \eta \leq y \). Then
\[
\Phi(x, y) - \Phi(\xi, \eta) = F(x, y)g(x, y) - F(\xi, \eta)g(\xi, \eta)
\]
\[
- \int_{-\infty}^x F(s, y) d_1 g(s, y) + \int_{-\infty}^\xi F(s, \eta) d_1 g(s, \eta)
\]
\[
- \int_{-\infty}^y F(x, t) d_2 g(x, t) + \int_{-\infty}^\eta F(\xi, t) d_2 g(\xi, t)
\]
\[
+ \int_{-\infty}^x \int_{-\infty}^y F(s, t) d_{12} g(s, t) - \int_{-\infty}^\xi \int_{-\infty}^\eta F(s, t) d_{12} g(s, t).
\]
The right side of (7.1) is written
\[
[F(x, y) - F(\xi, \eta)]g(\xi, \eta) + F(x, y)[g(x, y) - g(\xi, \eta)].
\]
Line (7.2) is written
\[-\int_{-\infty}^{\xi} [F(s, y) - F(s, \eta)] \, d_1 g(s, \eta) + \int_{-\infty}^{\xi} F(s, y) \, d_1 g(s, \eta) - \int_{-\infty}^{x} F(s, y) \, d_1 g(s, y).\]

Line (7.3) is written
\[-\int_{-\infty}^{\eta} [F(x, t) - F(\xi, t)] \, d_2 g(\xi, t) + \int_{-\infty}^{\eta} F(x, t) \, d_2 g(\xi, t) - \int_{-\infty}^{y} F(x, t) \, d_2 g(x, t).\]

Line (7.4) is written
\[\int_{-\infty}^{\xi} \int_{-\infty}^{\eta} [F(s, t) - F(s, y)] \, d_{12} g(s, t) + \int_{-\infty}^{\xi} F(s, y) \, d_{12} g(s, y) + \int_{-\infty}^{\xi} \int_{-\infty}^{\eta} [F(s, t) - F(\xi, t)] \, d_{12} g(s, t) + \int_{-\infty}^{\xi} F(\xi, t) \, d_{12} g(\xi, t) + \int_{-\infty}^{\xi} \int_{-\infty}^{\eta} [F(s, t) - F(y)] \, d_{12} g(s, t) + \int_{-\infty}^{\xi} F(y) \, d_{12} g(s, y).\]

Combining the above four terms gives
\[|\Phi(x, y) - \Phi(\xi, \eta)| \leq |F(x, y) - F(\xi, \eta)||g(\xi, \eta)| + \left|\int_{-\infty}^{\xi} [F(s, y) - F(s, \eta)] \, d_1 g(s, \eta)\right| + \left|\int_{-\infty}^{x} F(s, y) \, d_1 g(s, y)\right| + \left|\int_{-\infty}^{-\infty} \int_{-\infty}^{\eta} [F(s, t) - F(\xi, t)] \, d_{12} g(s, t)\right| + \left|\int_{-\infty}^{\xi} \int_{-\infty}^{\eta} [F(s, t) - F(x, t)] \, d_{12} g(s, t)\right| + \left|\int_{-\infty}^{\xi} \int_{-\infty}^{\eta} [F(s, t) - F(x, y)] \, d_{12} g(s, t)\right| + \left|\int_{-\infty}^{x} \int_{-\infty}^{\eta} [F(s, t) - F(\xi, t)] \, d_{12} g(s, t)\right| + \left|\int_{-\infty}^{\xi} \int_{-\infty}^{\eta} [F(s, t) - F(y)] \, d_{12} g(s, t)\right| + \left|\int_{-\infty}^{x} \int_{-\infty}^{\eta} [F(s, t) - F(\xi, t)] \, d_{12} g(s, t)\right| + \left|\int_{-\infty}^{\xi} \int_{-\infty}^{\eta} [F(s, t) - F(x, y)] \, d_{12} g(s, t)\right|.

The integrals in the above line with respect to $d_1 g$ are bounded by $2\|F\|_\infty\|V_1 g\|_\infty$; those with respect to $d_2 g$ are bounded by $2\|F\|_\infty\|V_2 g\|_\infty$; those with respect to $d_{12} g$ are bounded by $2\|F\|_\infty\|V_{12} g\|_\infty$. By dominated convergence and the continuity of $F$ it then follows that they all tend to 0 as $(\xi, \eta) \to (x, y)$. This shows $\Phi$ is continuous on $\mathbb{R}^2$. Minor changes show continuity on $\mathbb{R}_c^2$. It follows from the definition of $\Phi$ that if $F \in B_c(\mathbb{R}^2)$ then $\Phi \in B_c(\mathbb{R}^2)$. □

**Remark 7.4.** Note that Definitions 4.3 and 7.1 agree in the case when $g$ is the characteristic function of an interval. Suppose $g = \chi_I$ where $I = [a, b] \times [c, d]$ is a compact interval in $\mathbb{R}^2$. Since $F(x, y)$ vanishes when $x = -\infty$ or $y = -\infty$,
Definition 7.1 gives
\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} fg = F(\infty, \infty)g(\infty, \infty) - \int_{-\infty}^{\infty} F(x, \infty) d_1 g(x, \infty) \]
\[ - \int_{-\infty}^{\infty} F(\infty, y) d_2 g(\infty, y) + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(x, y) d_{12} g(x, y) \]
\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(x, y) d_{12} g(x, y), \]
\]last line following since \( g \) is constant in a neighbourhood of \( \partial \mathbb{R}^2 \).

Now use Definition 4.3. Suppose \([s, t] \times [u, v] \) is an interval in a tagged division of \( \mathbb{R}^2 \). Let \( \Delta g = g(s, u) + g(t, v) - g(s, v) - g(t, u) \). A Riemann sum consists of terms \( F(z_1, z_2) \Delta g \) for some tag \((z_1, z_2) \in [s, t] \times [u, v] \). Consider the point \((a, c)\). For any gauge \( \gamma \), a \( \gamma \)-fine tagged division can be chosen so that \((a, c)\) is in the interior of exactly one interval and \((a, c)\) is the tag for this interval. For this interval \( \Delta g = 1 \). Similarly with the points \((b, d), (a, d), (b, c)\). And \( \Delta g \) vanishes for all but four intervals in the Riemann sum. This shows
\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(x, y) d_{12} g(x, y) = F(a, c) + F(b, d) - F(a, d) - F(b, c). \]

Similarly, if \( I \subseteq \mathbb{R}^2 \).

By Proposition 4.6 every distribution in \( \mathcal{A}_c(\mathbb{R}^2) \) is the limit in the Alexiewicz norm of a sequence of functions in \( L^1(\mathbb{R}^2) \). We can show how this also holds for \( \partial_{12} \Phi \) from Proposition 7.3.

**Proposition 7.5.** Let \( f \in \mathcal{A}_c(\mathbb{R}^2) \). Let \( \{f_n\} \subset L^1(\mathbb{R}^2) \) such that \( \| f - f_n \| \to 0 \). Let \( F \) and \( F_n \) be the respective primitives in \( \mathcal{B}_c(\mathbb{R}^2) \). With \( \Phi \) as in Proposition 7.3 and \( \Phi_n \) similarly for \( F_n \) we have \( \| \partial_{12} \Phi_n - \partial_{12} \Phi \| \to 0 \) as \( n \to \infty \).

**Proof.** We have the estimate
\[ |\Phi_n(x, y) - \Phi(x, y)| \leq \| F_n - F \|_\infty (\| g \|_\infty + \| V_1 g \|_\infty + \| V_2 g \|_\infty + V_{12} g) \]
from which the result follows. \( \square \)

If \( \phi \in \mathcal{D}(\mathbb{R}^2) \) then \( \phi \in \mathcal{HKBV}(\mathbb{R}^2) \) so integration by parts gives another interpretation of the action of \( f \in \mathcal{A}_c(\mathbb{R}^2) \) as a distribution. Let \( F \in \mathcal{B}_c(\mathbb{R}^2) \) be the primitive of \( f \). We have
\[ \langle f, \phi \rangle = \langle \partial_{12} F, \phi \rangle = \langle F, \partial_{12} \phi \rangle = F(\infty, \infty) \phi(\infty, \infty) - \int_{-\infty}^{\infty} F(s, \infty) d_1 \phi(s, \infty) \]
\[ - \int_{-\infty}^{\infty} F(\infty, t) d_2(\infty, t) + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f \phi \]
\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f \phi. \]
If $F$ is a continuous function on $\mathbb{R}$ and $g$ is of bounded variation the one-dimensional formula is well-known:

$$\int_{-\infty}^{\infty} g \, dF = F(\infty)g(\infty) - F(-\infty)g(-\infty) - \int_{-\infty}^{\infty} F \, dg.$$ 

For example, see [39]. There is an analogue in $\mathcal{A}_c(\mathbb{R}^2)$.

**Proposition 7.6.** Let $F \in \mathcal{B}_c(\mathbb{R}^2)$ and $f = \partial_{12} F$. Let $g \in \mathcal{HKBV}(\mathbb{R}^2)$. Then for $[a, b] \times [c, d] \subseteq \mathbb{R}^2$

$$\int_a^b \int_c^d [F(a, c) + F(x, y) - F(a, y) - F(x, c)] \, d_{12} g(x, y)$$

$$= \int_a^b \int_c^d [g(x, y) + g(b, d) - g(x, d) - g(b, y)] \, d_{12} F(x, y).$$

In particular,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F \, d_{12} g = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g \, d_{12} F + g(\infty, \infty) F(\infty, \infty) - \int_{-\infty}^{\infty} g(x, \infty) \, d_F(x, \infty)$$

$$- \int_{-\infty}^{\infty} g(\infty, y) \, d_F(\infty, y)$$

(7.6)

so that if $g(x, y)$ vanishes when $x$ or $y$ is $\infty$ then

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f \, g = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F \, d_{12} g = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g \, d_{12} F.$$

If $g(x, y)$ vanishes when $x$ or $y$ is $-\infty$ then

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f \, g = F(\infty, \infty) g(\infty, \infty) - \int_{-\infty}^{\infty} F(x, \infty) \, d_{12} g(x, \infty)$$

$$- \int_{-\infty}^{\infty} F(\infty, y) \, d_{12} g(\infty, y) + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F \, d_{12} g$$

(7.7)

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g \, d_{12} F.$$  

(7.8)

Proven. The first line is from Theorem 8.8, page 127 in [28], which is easily extended from a compact interval to $\mathbb{R}^2$. The author considers various Stieltjes integrals but these are all equal under the hypotheses of our theorem. This first line can be written

$$F(a, c) [g(a, c) + g(b, d) - g(a, d) - g(b, c)] + \int_a^b \int_c^d F \, d_{12} g$$

$$- \int_c^d F(a, y) [d_{2} g(b, y) - d_{2} g(a, y)] - \int_a^b F(x, c) [d_{1} g(x, d) - d_{1} g(x, c)]$$

$$= \int_a^b \int_c^d g \, d_{12} F + g(b, d) [F(a, c) + F(b, d) - F(a, d) - F(b, c)]$$

$$- \int_a^b g(x, d) [d_{1} F(x, d) - d_{1} F(x, c)] - \int_c^d g(b, y) [d_{2} F(b, y) - d_{2} F(a, y)].$$
Taking limits gives (7.6) and (7.7). Interchanging \( F \) and \( g \) in the first line and repeating these steps gives the final equations. \( \square \)

**Proposition 7.7** (First and second mean value theorem for integrals). Suppose \( g \in \mathcal{HBV}(\mathbb{R}^2) \) such that \( g(a,c) + g(b,d) - g(a,d) - g(b,c) \geq 0 \) for all \( [a,b] \times [c,d] \subseteq \mathbb{R}^2 \). (a) Suppose \( F \in C(\mathbb{R}^2) \). Then there exists \( (\xi,\eta) \in \mathbb{R}^2 \) such that

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F \, d1_{12}g = F(\xi,\eta) \left[ g(-\infty,-\infty) + g(\infty,\infty) - g(-\infty,\infty) - g(\infty,-\infty) \right].
\]

(b) Let \( f \in \mathcal{A}_c(\mathbb{R}^2) \) and let \( F \in \mathcal{B}_c(\mathbb{R}^2) \) be its primitive. Suppose \( g(x,y) \) vanishes when \( x \) or \( y \) is infinity. Then there is \( (\xi,\eta) \in \mathbb{R}^2 \) such that

\[
\int_{-\infty}^{\infty} fg = F(\xi,\eta) \left[ g(-\infty,-\infty) + g(\infty,\infty) - g(-\infty,\infty) - g(\infty,-\infty) \right].
\]

**Proof.** (a) Let \( \Delta = g(-\infty,-\infty) + g(\infty,\infty) - g(-\infty,\infty) - g(\infty,-\infty) \). The function \( \Psi(x,y) = F(x,y) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d1_{12}g \) is continuous. Both \( \Psi \) and \( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F \, d1_{12}g \) have \( \min_{\mathbb{R}^2} F)\Delta \) and \( \max_{\mathbb{R}^2} F)\Delta \) as respective lower and upper bounds. Use of the intermediate value theorem completes the proof. (b) Use part (a) and (7.7). \( \square \)

 Versions of the first mean value theorem, part (a), are known for Henstock–Stieltjes and Lebesgue integrals. See page 209 in [39] and Problem 6, page 190, in [17]. For the second mean value theorem for one-dimensional Henstock–Kurzweil integrals see §1.10 in [13] and page 211 in [39]. See Theorems 6.4.2 and 6.5.13 in [35] for \( n \)-dimensional Henstock–Kurzweil integrals. See also [56].

The space \( \mathcal{A}_c(\mathbb{R}^2) \) is invariant under translations, as is the Alexiewicz norm. We also have continuity of translations. If \( f \in \mathcal{D}'(\mathbb{R}^2) \) and \( (s,t) \in \mathbb{R}^2 \) then the translation is defined by \( \langle \tau_{(s,t)}f, \phi \rangle = \langle f, \tau_{(-s,-t)}\phi \rangle \) where \( \tau_{(s,t)}\phi(x,y) = \phi(x-s, y-t) \) for \( \phi \in \mathcal{D}(\mathbb{R}^2) \).

**Proposition 7.8.** (a) If \( f \in \mathcal{D}'(\mathbb{R}^2) \) then \( f \in \mathcal{A}_c(\mathbb{R}^2) \) if and only if \( \tau_{(s,t)}f \in \mathcal{A}_c(\mathbb{R}^2) \) for all \( (s,t) \in \mathbb{R}^2 \). (b) Let \( f \in \mathcal{A}_c(\mathbb{R}^2) \). Then \( \|f\| = \|\tau_{(s,t)}f\| \) for all \( (s,t) \in \mathbb{R}^2 \). (c) Let \( f \in \mathcal{A}_c(\mathbb{R}^2) \). Then \( \lim_{(s,t) \to (0,0)} \|f - \tau_{(s,t)}f\| = 0 \).

The proofs are based on the corresponding properties in \( \mathcal{B}_c(\mathbb{R}^2) \). See also [49, Theorem 28].

8. Hölder inequality and dual space

The integration by parts formula, Definition 7.1, leads to a version of the Hölder inequality. It is known that the dual space of the Henstock–Kurzweil integrable functions is \( \mathcal{HBV}(\mathbb{R}^2) \) ([35, §6.6]). The Hölder inequality and density of the Henstock–Kurzweil integrable functions in \( \mathcal{A}_c(\mathbb{R}^2) \) then show that the dual space of \( \mathcal{A}_c(\mathbb{R}^2) \) is also \( \mathcal{HBV}(\mathbb{R}^2) \).
Proposition 8.1 (Hölder inequality). Let \( f \in \mathcal{A}_c(\mathbb{R}^2) \) and \( g \in \mathcal{HKBV}(\mathbb{R}^2) \). Then for all \([a,b] \times [c,d] \subseteq \mathbb{R}^2\) and all \((x,y) \in \mathbb{R}^2\),
\[
\left| \int_a^b \int_c^d fg \right| \leq \|f\| \left( 4\|g\|_\infty + 2\|V_1g\|_\infty + 2\|V_2g\|_\infty + V_{12}g \right)
\]
\[
\left| \int_{-\infty}^x \int_{-\infty}^y fg \right| \leq \|f\| \left( \|g\|_\infty + \|V_1g\|_\infty + \|V_2g\|_\infty + V_{12}g \right) = \|f\|\|g\|_{bv}.
\]

We now give two equivalent norms.

Proposition 8.2 (Equivalent norms). For \( f \in \mathcal{A}_c(\mathbb{R}^2) \) define \( \|f\|' = \sup_{I} |\int_I f| \) where the supremum is taken over all intervals \( I \subseteq \mathbb{R}^2 \); \( \|f\|'' = \sup_{g} |\int_{-\infty}^\infty \int_{-\infty}^\infty fg| \) where the supremum is taken over all \( g \in \mathcal{HKBV}(\mathbb{R}^2) \) such that \( \|g\|_{bv} \leq 1 \).

Proof. Since the characteristic function of an interval is of bounded variation integration by parts establishes existence of \( \|\cdot\|' \). Clearly, \( \|f\| \leq \|f\|' \). And, there is the decomposition into nonoverlapping intervals,
\[
\int_a^b \int_c^d f = \int_{-\infty}^a \int_{-\infty}^c f + \int_{-\infty}^b \int_{-\infty}^d f - \int_{-\infty}^a \int_{-\infty}^d f - \int_{-\infty}^b \int_{-\infty}^c f,
\]
so that \( \|f\|' \leq 4\|f\| \).

If \( g \in \mathcal{HKBV}(\mathbb{R}^2) \) with \( \|g\|_{bv} \leq 1 \) then the Hölder inequality (Proposition 8.1) establishes \( \|f\|'' \leq \|f\| \). For a reverse inequality let \( \epsilon > 0 \). There exists \((x,y) \in \mathbb{R}^2 \) such that \( |\int_{-\infty}^x \int_{-\infty}^y f| \geq \|f\| - \epsilon \). Let \( g = (1/4)\chi_{(-\infty,x) \times (-\infty,y)} \). Then \( \|g\|_{bv} = 1 \) (Example 6.2). And,
\[
\|f\|'' \geq \left| \int_{-\infty}^\infty fg \right| = \frac{1}{4} \left| \int_{-\infty}^x \int_{-\infty}^y f \right| \geq \frac{\|f\| - \epsilon}{4}.
\]
Hence, \( \|f\|/4 \leq \|f\|'' \). \( \Box \)

The integral \( \int_{-\infty}^\infty \int_{-\infty}^\infty fg \) is not changed when \( g \) is changed on a coordinate line.

Proposition 8.3. Let \( f \in \mathcal{A}_c(\mathbb{R}^2) \) and \( g \in \mathcal{HKBV}(\mathbb{R}^2) \). If \( g \) is changed on a coordinate line the integral \( \int_{-\infty}^\infty \int_{-\infty}^\infty fg \) is not changed.

Note that this includes the result that if \((s,t) \in \mathbb{R}^2 \) and \( g = \chi_{(s,t)} \) then \( \int_{-\infty}^\infty \int_{-\infty}^\infty fg = 0 \) for all \( f \in \mathcal{A}_c(\mathbb{R}^2) \).

Proof. Let \( f \in \mathcal{A}_c(\mathbb{R}^2) \) with primitive \( F \in \mathcal{B}_c(\mathbb{R}^2) \).

First show that if \( g \) is the characteristic function of a point then \( \int_{-\infty}^\infty \int_{-\infty}^\infty fg = 0 \). Let \( g = \chi_{(s,t)} \). Then \( \int_{-\infty}^\infty \int_{-\infty}^\infty fg = \int_{-\infty}^\infty \int_{-\infty}^\infty gd_{12}F \) by (7.7) or (7.9). A gauge \( \gamma \) can always be chosen so that if interval \( I = [a,b] \times [c,d] \) is in a \( \gamma \)-fine tagged division and \((s,t) \in I \) then its tag is \((s,t)\). The only terms in a Riemann sum that do not necessarily vanish are \( g(s,t)(F(a,c) + F(b,d) - F(a,d) - F(b,c)) \) but the gauge can force this term to be arbitrarily small due to the uniform continuity of \( F \). There can be at most four such terms. Hence, \( \int_{-\infty}^\infty \int_{-\infty}^\infty fg = 0 \).
Now consider 
\[ g(x, y) = \begin{cases} 
\psi(x); & x \in \mathbb{R}, y = \infty \\
0; & \text{else} 
\end{cases} \]
where \( \psi : \mathbb{R} \to \mathbb{R} \) is of bounded variation and \( \psi(\pm \infty) = 0 \). We again have 
\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_{12} F. \]
Given a gauge \( \gamma \) we can always choose a \( \gamma \)-fine tagged division so that there is a point \( y \in \mathbb{R} \) such that if \( I \) is an interval in the tagged division then \( I = [x_{i-1}, x_i] \times [y, \infty] \) with associated tag \( (z_i, \infty) \) for which \( x_{i-1} \leq z_i \leq x_i \). And, there is \( N \in \mathbb{N} \) so that \( -\infty = x_0 < x_1 < \ldots < x_N = \infty \).

We then have a partition of the line \( \{ (s, \infty) \in \mathbb{R}^2 \mid s \in \mathbb{R} \} \). We can assume \( z_1 = -\infty \) and \( z_N = \infty \). The terms that do not necessarily vanish in a Riemann sum for such a division are
\[
\sum_{i=1}^{N} \psi(z_i) [F(x_{i-1}, y) + F(x_i, \infty) - F(x_{i-1}, \infty) - F(x_i, y)]
\]
\[
= \sum_{i=1}^{N-1} \psi(z_i) [F(x_i, \infty) - F(x_i, y)] + \sum_{i=1}^{N-1} \psi(z_{i+1}) [F(x_i, y) - F(x_i, \infty)]
\]
\[
= \sum_{i=1}^{N-1} [\psi(z_i) - \psi(z_{i+1})] [F(x_i, \infty) - F(x_i, y)].
\]
The Riemann sum is then bounded by \( V\psi \sup_{x,y \in \mathbb{R}} [F(x, \infty) - F(x, y)] \). Since \( F \) is uniformly continuous this can be made arbitrarily small by choosing \( \gamma \) to force \( y \) close enough to \( \infty \). Hence, \( \int_{-\infty}^{\infty} f g = 0 \).

Changing \( g \) on other lines is handled similarly. \( \Box \)

To discuss the dual space of \( A_c(\mathbb{R}^2) \) we need to define normalisations for functions of bounded variation. Fix \( \alpha_-, \alpha_+, \alpha_{++}, \alpha_{+-}, \alpha_{-+} \in [0, 1] \) such that \( \alpha_- + \alpha_{++} + \alpha_{-+} + \alpha_{+-} = 1 \). If \( g \in \mathcal{HKBV}(\mathbb{R}^2) \) then define it’s normalisation \( \tilde{g} \) as follows. For \( (x, y) \in \mathbb{R}^2 \) put \( \tilde{g}(x, y) = \alpha_- \lim_{(s, t)\to(x, y-)} g(s, t) + \alpha_+ \lim_{(s, t)\to(x, y+)} g(s, t) + \alpha_{++} \lim_{(s, t)\to(x-, y)} g(s, t) + \alpha_{+-} \lim_{(s, t)\to(x-, y+)} g(s, t) + \alpha_{-+} \lim_{(s, t)\to(x, y-)} g(s, t) + \alpha_{+-} \lim_{(s, t)\to(x, y+)} g(s, t) \). This involves changing \( g \) on a set that is at most countable. There is a similar procedure on the boundary of \( \mathbb{R}^2 \). For example, fix \( \beta_- , \beta_+ \in [0, 1] \) such that \( \beta_- + \beta_+ = 1 \). For \( y \in \mathbb{R} \) we define \( \tilde{g}(\infty, y) = \beta_- \lim_{(s, t)\to(\infty, y-)} g(s, t) + \beta_+ \lim_{(s, t)\to(\infty, y+)} g(s, t) \). A single limit is required at each of the four corner points of \( \mathbb{R}^2 \).

Finally, we have a characterisation of the dual space of \( A_c(\mathbb{R}^2) \). It is clear from Proposition 8.3 that if two elements of the dual space differ only on a coordinate line then they represent the same dual space element. This is dealt with by fixing a normalisation on \( \mathcal{HKBV}(\mathbb{R}^2) \).

Proposition 8.4 (Dual space). Fix any normalisation on \( \mathcal{HKBV}(\mathbb{R}^2) \). The dual space of \( A_c(\mathbb{R}^2) \) is \( A_c(\mathbb{R}^2)' = \mathcal{HKBV}(\mathbb{R}^2) \).

Proof. The Hölder inequality shows that every function of bounded variation generates a bounded linear functional on \( A_c(\mathbb{R}^2) \) via \( f \mapsto \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f g \) (f \in \mathcal{HKBV}(\mathbb{R}^2))
\( \mathcal{A}_c(\mathbb{R}^2) \), \( g \in \mathcal{HKBV}(\mathbb{R}^2) \)). And, in [35], Section 6.6, it is shown that each element of the dual space of the Henstock–Kurzweil integrable functions is given by integration against a function of bounded variation that vanishes on the boundary. (This is done on a compact interval but the proof extends immediately to \( \mathbb{R}^2 \).) Since the space of Henstock–Kurzweil integrable functions is dense in \( \mathcal{A}_c(\mathbb{R}^2) \) (Proposition 4.5) this shows that the dual space of \( \mathcal{A}_c(\mathbb{R}^2) \) is also \( \mathcal{HKBV}(\mathbb{R}^2) \). By Proposition 8.3 we get the same result for our given normalisation. □

Note that each normalisation on \( \mathcal{HKBV}(\mathbb{R}^2) \) gives an isometrically isomorphic representation of the dual space. Equivalently, we can say the dual space is the set of functions of essential bounded variation. This is the set of equivalence classes of functions agreeing almost everywhere with a function of bounded variation. Choosing a normalisation merely selects one element of each equivalence class. It is often misstated in the literature that the dual space of the continuous functions on the real line is \( \text{BV} \) (including in [49]) but the dual space is more properly given as normalised bounded variation or essential bounded variation. The formulas in Definition 7.1 and Proposition 7.6 are not defined if \( g \) is of essential bounded variation but the integral \( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} fg \) can computed using sequences of \( L^1 \) functions as in Proposition 7.5.

9. Convergence theorems

A number of convergence theorems are given in [5] and [49] that can be extended to \( \mathcal{A}_c(\mathbb{R}^2) \), including a necessary and sufficient condition for interchanging limits and integrals. The required changes are minor so we do not present them here. Instead, we give the convergence theorem that seems to be the most useful in practice. It refers to limits of \( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} fg_n \) where \( \{g_n\} \) is a sequence of functions of bounded variation. This can occur, for example, in a convolution product. See [55] for an application on the real line.

**Proposition 9.1.** Let \( f \in \mathcal{A}_c(\mathbb{R}^2) \). Let \( \{g_n\} \subset \mathcal{HKBV}(\mathbb{R}^2) \) such that \( \|g_n\|_{bv} \) is bounded and \( \lim_{n \to \infty} g_n = g \) pointwise on \( \mathbb{R}^2 \) for a function \( g : \mathbb{R}^2 \to \mathbb{R} \). Then \( g \in \mathcal{HKBV}(\mathbb{R}^2) \) and \( \lim_{n \to \infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} fg_n = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} fg \).

**Proof.** We can write \( \|g_n\|_{bv} \leq M \).

To prove \( g \) is bounded note that

\[
|g(x, y)| \leq |g(x, y) - g_n(x, y)| + |g_n(x, y)| \leq |g(x, y) - g_n(x, y)| + M.
\]

Now let \( n \to \infty \).
Fix a finite collection of nonoverlapping intervals \( \{[a_i, b_i] \times [c_i, d_i]\}_{i=1}^N \). We have the inequality

\[
\sum_{i=1}^N |g(a_i, c_i) + g(b_i, d_i) - g(a_i, d_i) - g(b_i, c_i)|
\]

(9.1) \[
\leq \sum_{i=1}^N |g(a_i, c_i) - g_n(a_i, c_i)| + \sum_{i=1}^N |g(b_i, d_i) - g_n(b_i, d_i)|
\]

(9.2) \[
+ \sum_{i=1}^N |g(a_i, d_i) - g_n(a_i, d_i)| + \sum_{i=1}^N |g(b_i, c_i) - g_n(b_i, c_i)|
\]

(9.3) \[
+ \sum_{i=1}^N |g_n(a_i, c_i) + g_n(b_i, d_i) - g_n(a_i, d_i) - g_n(b_i, c_i)|.
\]

For these fixed finite sums we can take \( n \) large enough so that the sums in (9.1) and (9.2) contribute less than any prescribed \( \epsilon > 0 \). The sum in (9.3) is always less than \( M \). Hence, \( V_{12}g < \infty \).

To show \( V_1 g(\cdot, y_0) \) is finite for some \( y_0 \in \mathbb{R} \) write

\[
\sum_{i=1}^N |g(x_i, y_0) - g(x_{i-1}, y_0)|
\]

\[
\leq \sum_{i=1}^N |g(x_i, y_0) - g_n(x_i, y_0)| + \sum_{i=1}^N |g(x_{i-1}, y_0) - g_n(x_{i-1}, y_0)| + V_1 g_n(\cdot, y_0).
\]

As above, we can take \( n \) large enough to make these last two sums small. Similarly with \( V_2 g(x_0, \cdot) \). Hence, \( g \in \mathcal{HKBV}(\mathbb{R}^2) \).

By linearity of the integral we can assume \( g_n \to 0 \). Integration by parts then gives

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f g_n = F(\infty, \infty) g_n(\infty, \infty) - \int_{-\infty}^{\infty} F(s, \infty) d_1 g_n(s, \infty)
\]

(9.4) \[
- \int_{-\infty}^{\infty} F(\infty, t) d_2 g_n(\infty, t) + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(s, t) d_{12} g_n(s, t).
\]

The term \( F(\infty, \infty) g_n(\infty, \infty) \to 0 \). To show the last integral above tends to 0 we use the method in [42, p. 126]. Let \( \epsilon > 0 \). Since \( F \) is uniformly continuous, we can take a gauge \( \gamma \) so that for each interval \( I_i \) in a \( \gamma \)-fine tagged division, if \( (x, y) \) and \( (s, t) \) are points in \( I_i \) then \( |F(x, y) - F(s, t)| < \epsilon \). Now suppose \( \{[a_i, b_i] \times [c_i, d_i], (x_i, y_i)\}_{i=1}^N \) is a \( \gamma \)-fine tagged division of \( \mathbb{R}^2 \). Let \( \Delta_i g_n = g_n(a_i, c_i) + \)
CONTINUOUS PRIMITIVE INTEGRAL

\( g_n(b_i, d_i) - g_n(a_i, d_i) - g_n(b_i, c_i) \) and \( I_i = [a_i, b_i] \times [c_i, d_i] \). Then

\[
\left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(x, y) \, d_{12} g_n(x, y) - \sum_{i=1}^{N} F(x_i, y_i) \Delta_i g_n \right|
\]

\[
= \sum_{i=1}^{N} \left| \int_{I_i} F(x, y) \, d_{12} g_n(x, y) - F(x_i, y_i) \int_{I_i} d_{12} g_n(x, y) \right|
\]

\[
= \sum_{i=1}^{N} \int_{I_i} |F(x, y) - F(x_i, y_i)| \, d_{12} g_n(x, y)
\]

\[
\leq \epsilon V_{12} g_n \leq \epsilon M.
\]

Therefore, for a fixed tagged division the Riemann sums approximate the integral uniformly in \( n \). But the Riemann sums tend to \( 0 \) as \( n \to \infty \) since \( g_n \to 0 \). Similarly, for the other two integrals in (9.4).

Note that we also get convergence on every subinterval of \( \mathbb{R}^2 \).

10. CHANGE OF VARIABLES

If \( V \) and \( W \) are open sets in \( \mathbb{R}^n \) a typical change of variables theorem for \( L^1 \) functions is that

\[
\int_W f \, d\lambda = \int_V (f \circ T) |\det J_T| \, d\lambda,
\]

where \( T : V \to W \) is a diffeomorphism, \( J_T \) is the Jacobian and \( f \in L^1(W) \). For a proof see [18].

For the continuous primitive integral on the real line the following theorem appears in [49]:

**Theorem 10.1.** Suppose \( f \in \mathcal{A}_c(\mathbb{R}) \) and \( F' = f \) where \( F \in C(\mathbb{R}) \). Let \( -\infty \leq a < b \leq \infty \). If \( G \in C([a, b]) \) then

\[
\int_{G(b)}^{G(a)} f = \int_a^b (f \circ G') G' = (F \circ G)(b) - (F \circ G)(a).
\]

If \( G \in C((a, b)) \) and \( \lim_{t \to a^+} G(t) = -\infty \) and \( \lim_{t \to b^-} G(t) = \infty \) then

\[
\int_{-\infty}^b f = \int_a^b (f \circ G') G' = F(\infty) - F(-\infty).
\]

The function \( F \circ G \) is continuous so its continuous primitive integral exists. The quantity \( (f \circ G) G' \) is written in place of \( (F \circ G)' \) and it is shown in [49] that if two sequences of differentiable functions converge to \( F \) and \( G \), respectively, then the usual pointwise formula for differentiation of a composite function converges in the Alexiewicz norm to \( (F \circ G)' \). However, this does not imply separate existence of \( f \circ G \) and \( G' \) and the multiplication is purely formal. Indeed, suppose \( F(x) = x^2 \) and \( G \) is continuous but pointwise differentiable nowhere. Then \( F' \circ G = 2G \). An arbitrary distribution can be multiplied by a \( C^\infty \) function and a distribution in \( \mathcal{A}_c(\mathbb{R}) \) can be multiplied by a function of bounded variation but \( G \) is not of bounded variation so the product \( 2GG' \) has no meaning other than shorthand for \( (G^2)' \).

Here we choose to present a restricted change of variables formula that has immediate application to convolutions (Section 14).
There is a well-established method of composing a distribution with a linear bijection. Suppose $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear bijection. For a distribution $T \in \mathcal{D}'(\mathbb{R}^2)$ the composition $T \circ \Psi \in \mathcal{D}'(\mathbb{R}^2)$ is defined for $\phi \in \mathcal{D}(\mathbb{R}^2)$ by $\langle T \circ \Psi, \phi \rangle = (\det \Psi)^{-1} \langle T, \phi \circ \Psi^{-1} \rangle$. For example, [18, p. 285].

If $f \in \mathcal{A}_c(\mathbb{R}^2)$ then integration of $f \circ \Psi$ requires $\Psi$ to map intervals onto finite unions of intervals (in the same Cartesian coordinate system) since these are the only regions in $\mathbb{R}^2$ for which the integral is defined. This can be accomplished by having each component of $\Psi$ depend linearly on only one variable.

**Theorem 10.2.** Let $\alpha, \beta, \gamma_1, \gamma_2 \in \mathbb{R}$ such that $\alpha \beta \neq 0$. Let $[a, b] \times [c, d] \subseteq \mathbb{R}^2$ and let $f \in \mathcal{A}_c(\mathbb{R}^2)$. (a) If $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $\Psi(u, v) = (\alpha u + \gamma_1, \beta v + \gamma_2)$ then

$$
\int_a^b \int_c^d f(x, y) \, dy \, dx = \alpha \beta \int_{(a-\gamma_1)/\alpha}^{(b-\gamma_1)/\alpha} \int_{(c-\gamma_2)/\beta}^{(d-\gamma_2)/\beta} (\alpha u + \gamma_1, \beta v + \gamma_2) \, dv \, du.
$$

(b) If $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $\Psi(u, v) = (\beta v + \gamma_2, \alpha u + \gamma_1)$ then

$$
\int_a^b \int_c^d f(x, y) \, dy \, dx = \alpha \beta \int_{(a-\gamma_2)/\beta}^{(b-\gamma_2)/\beta} \int_{(c-\gamma_1)/\alpha}^{(d-\gamma_1)/\alpha} (\beta v + \gamma_2, \alpha u + \gamma_1) \, dv \, du.
$$

The proof follows from the above definition for composition with a linear bijection. Note that $\Psi$ maps intervals to intervals, as indicated by the limits of integration in the theorem.

The convention following Definition 4.3 on ordering of upper and lower limits of integration has been used. If any of $a, b, c, d$ is in $\{-\infty, \infty\}$ then the usual arithmetic of infinities can be used to determine the limits of integration. For example, if $a = -\infty$ then replace $(a - \gamma_1) / \alpha$ with $\text{sgn}(\alpha) \infty$.

See [38] for similar change of variables for the Henstock–Kurzweil integral.

11. **Banach lattice**

The usual pointwise ordering on $\mathcal{B}_c(\mathbb{R}^2)$ makes it into a Banach lattice and $\mathcal{A}_c(\mathbb{R}^2)$ inherits this structure. This creates a distributional ordering such that all distributions in $\mathcal{A}_c(\mathbb{R}^2)$ have absolutely convergent integrals. For functions in $\mathcal{A}_c(\mathbb{R}^2)$ the usual pointwise ordering leads to conditionally convergent integrals. See the second paragraph of Section 5.

This distributional ordering has been used to solve problems in ordinary and partial differential equations. See S. Heikkilä [23], [24], [25], [26]; S. Heikkilä and E. Talvila [27]; Liu Wei and Ye Guoju with numerous co-authors, for example, [36].

A reference for Banach lattices is [4]. The definitions in this section are largely repeated from [52]. Corresponding lattice results were obtained for distributional integrals on the real line with continuous primitives [49], with regulated primitives [51], and of higher order [52]. We omit proofs in this section since they are so similar to results in these papers.

If $\preceq$ is a binary operation on set $S$ then it is a partial order if for all $x, y, z \in S$ it is reflexive ($x \preceq x$), antisymmetric ($x \preceq y$ and $y \preceq x$ imply $x = y$) and
transitive ($x \leq y$ and $y \leq z$ imply $x \leq z$). If $S$ is a Banach space with norm $\| \cdot \|_S$ and $\leq$ is a partial order on $S$ then $S$ is a Banach lattice if for all $x, y \in S$

1. $x \lor y$ and $x \land y$ are in $S$. The join is $x \lor y = \sup\{x, y\} = w$ such that $x \leq w$, $y \leq w$ and if $x \leq \bar{w}$ and $y \leq \bar{w}$ then $w \leq \bar{w}$. The meet is $x \land y = \inf\{x, y\} = w$ such that $w \leq x$, $w \leq y$ and if $\bar{w} \leq x$ and $\bar{w} \leq y$ then $\bar{w} \leq w$.

2. $x \leq y$ implies $x + z \leq y + z$ for all $z \in S$.

3. $x \leq y$ implies $kx \leq ky$ for all $k \in \mathbb{R}$ with $k \geq 0$.

4. $|x| \leq |y|$ implies $\|x\|_S \leq \|y\|_S$.

If $x \leq y$ we write $y \geq x$. We also define $|x| = x \lor (-x)$, $x^+ = x \lor 0$ and $x^- = (-x) \lor 0$. Then $x = x^+ - x^-$ and $|x| = x^+ + x^-$. The usual pointwise ordering, $F_1 \leq F_2$ if and only if $F_1(x, y) \leq F_2(x, y)$ for all $(x, y) \in \mathbb{R}^2$, is a partial order on $B_c(\mathbb{R})$. Since $B_c(\mathbb{R})$ is closed under the operations $(F_1 \lor F_2)(x, y) = \sup(F_1(x, y), F_2(x, y))$ and $(F_1 \land F_2)(x, y) = \inf(F_1(x, y), F_2(x, y))$, it is then a vector lattice (or Riesz space). Since $\|F_1\| \leq \|F_2\|$ implies $\|F_1\|_\infty \leq \|F_2\|_\infty$ we have that $B_c(\mathbb{R})$ is a Banach lattice.

A partial ordering in $A_c(\mathbb{R}^2)$ is inherited from $B_c(\mathbb{R})$. If $f_1, f_2 \in A_c(\mathbb{R}^2)$ with respective primitives $F_1, F_2 \in B_c(\mathbb{R})$ then $f_1 \leq f_2$ if and only if $F_1 \leq F_2$ in $B_c(\mathbb{R})$. The isomorphism between $A_c(\mathbb{R}^2)$ and $B_c(\mathbb{R})$ now shows $A_c(\mathbb{R}^2)$ is also a Banach lattice.

It is not a linear ordering. For example, if $F(x, y) = \exp(-x^2 - y^2)$ and $G(x, y) = \exp(-(x - 1)^2 - (y - 1)^2)$ then we have neither $F \leq G$ nor $G \leq F$ and similarly in $A_c(\mathbb{R}^2)$.

An element $e \geq 0$ such that for each $x \in S$ there is $\lambda > 0$ such that $|x| \leq \lambda e$ is an order unit for lattice $S$. The order unit for $B_c(\mathbb{R})$ would have to vanish on $\{-\infty\} \times \mathbb{R}$ and on $\mathbb{R} \times \{-\infty\}$ and decay to 0 more slowly than any continuous function. (See [52, Theorem 5.1]). Hence $A_c(\mathbb{R}^2)$ does not have an order unit.

We have absolute integrability: if $f \in A_c(\mathbb{R}^2)$ so is $|f|$. The partial derivative operator $\partial_{12}$ commutes with $\lor$ and $\land$ and hence with $|\cdot|$.

**Theorem 11.1** (Banach lattice). (a) $B_c(\mathbb{R})$ is a Banach lattice. (b) For $f_1, f_2 \in A_c(\mathbb{R}^2)$ with respective primitives $F_1, F_2 \in B_c(\mathbb{R})$, define $f_1 \prec f_2$ if $F_1 \leq F_2$ in $B_c(\mathbb{R})$. Then $A_c(\mathbb{R}^2)$ is a Banach lattice isomorphic to $B_c(\mathbb{R})$. (c) $B_c(\mathbb{R})$ and $A_c(\mathbb{R}^2)$ do not have an order unit. (d) Let $F_1, F_2 \in B_c(\mathbb{R})$. Then $\partial_{12}(F_1 \lor F_2) = (\partial_{12} F_1) \lor (\partial_{12} F_2)$, $\partial_{12}(F_1 \land F_2) = (\partial_{12} F_1) \land (\partial_{12} F_2)$, $|\partial_{12} F| = |\partial_{12} F|$, $\partial_{12}(F^+)$ = $(\partial_{12} F^+)$, and $\partial_{12}(F^-)$ = $(\partial_{12} F^-)$. (e) If $f \in A_c(\mathbb{R})$ with primitive $F \in B_c(\mathbb{R})$ then $|f| \in A_c(\mathbb{R})$ with primitive $|F| \in B_c(\mathbb{R})$ and $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f| = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f|$. (f) If $f \in A_c(\mathbb{R}^2)$ then $f^\pm \in A_c(\mathbb{R}^2)$ with respective primitives $F^\pm \in B_c(\mathbb{R})$. Jordan decomposition: $f = f^+ - f^-$. And, $\int_{-\infty}^{\infty} fg = \int_{-\infty}^{\infty} f^+ g - \int_{-\infty}^{\infty} f^- g$ for every $g \in HKBV(\mathbb{R})$. (g) $A_c(\mathbb{R})$ is distributive: $f_1 \land (f_2 \lor f_3) = (f_1 \land f_2) \lor (f_1 \land f_3)$ and $f_1 \lor (f_2 \land f_3) = (f_1 \lor f_2) \land (f_1 \lor f_3)$.
\[(f_1 \vee f_2) \wedge (f_1 \vee f_3) \text{ for all } f_1, f_2, f_3 \in \mathcal{A}_c(\mathbb{R}^2).\] (h) \(\mathcal{A}_c(\mathbb{R}^2)\) is modular: For all \(f_1, f_2, f_3 \in \mathcal{A}_c(\mathbb{R}^2)\), if \(f_1 \leq f_2\) then \(f_1 \vee (f_2 \wedge f_3) = f_2 \wedge (f_1 \vee f_3)\) for all \(f_3 \in \mathcal{A}_c(\mathbb{R}^2)\).

(i) Let \(F_1\) and \(F_2\) be continuous functions on \(\mathbb{R}^2\). Then

\[
\partial_{12} F_1 \leq \partial_{12} F_2 \iff F_1(-\infty,-\infty) + F_1(x,y) - F_1(-\infty,y) - F_2(x,-\infty) \\
\leq F_2(-\infty,-\infty) + F_2(x,y) - F_2(-\infty,y) - F_2(x,-\infty)
\]

for all \((x,y) \in \mathbb{R}^2\).

Let \(f_1, f_2 \in \mathcal{A}_c(\mathbb{R}^2)\) with respective primitives \(F_1, F_2 \in \mathcal{B}_c(\mathbb{R}^2)\). Note that if \(F_1 \leq F_2\) in \(\mathcal{B}_c(\mathbb{R}^2)\) then we can differentiate both sides of this inequality with \(\partial_{12}\) to get \(f_1 \leq f_2\) in \(\mathcal{A}_c(\mathbb{R}^2)\). And, if \(f_1 \leq f_2\) in \(\mathcal{A}_c(\mathbb{R}^2)\) we can integrate both sides against \(\chi_{(-\infty,x) \times (-\infty,y)}\) to get \(F_1 \leq F_2\) in \(\mathcal{B}_c(\mathbb{R}^2)\). See Theorem 4.4. This also shows the derivative \(\partial_{12}\) is a positive operator on \(\mathcal{B}_c(\mathbb{R}^2)\) and its inverse is a positive operator on \(\mathcal{A}_c(\mathbb{R}^2)\).

In general, \(\int_a^b \int_c^d |f|\) and \(\int_a^b \int_c^d f\) are not comparable. However, if \(a = -\infty\) or \(c = -\infty\) then \(\int_a^b \int_c^d f \leq |\int_a^b \int_c^d f|\).

The usual pointwise ordering makes \(L^1\) into a Banach lattice. But the space of Henstock–Kurzweil integrable functions is not a vector lattice. It is not closed under supremum and infimum since there are functions integrable in this sense for which \(\int_{-\infty}^\infty \int_{-\infty}^\infty f(x,y)\, dy\, dx\) converges but \(\int_{-\infty}^\infty |f(x,y)|\, dy\, dx\) diverges. For example, the function \(f(x,y) = \sin(x)\sin(y)/(xy)\) from Section 5. Thus, even for functions, when we allow conditional convergence we must look elsewhere to find a lattice structure.

Consider the example

\[
F(x,y) = \begin{cases} 
\int_0^x \int_0^y \frac{\sin(s)\sin(t)}{st} \, dt \, ds, & x,y \geq 0 \\
0, & \text{otherwise}.
\end{cases}
\]

Then \(F(x,y) \geq 0\) for all \((x,y) \in \mathbb{R}^2\) so \(\partial_{12} F \geq 0\) in \(\mathcal{A}_c(\mathbb{R}^2)\). The integrand is not positive in a pointwise sense so \(\preceq\) is not compatible with the usual pointwise ordering on \(\mathcal{A}_c(\mathbb{R}^2)\). The order \(\preceq\) is also not compatible with the usual order on distributions: if \(T, U \in \mathcal{D}'(\mathbb{R})\) then \(T \geq U\) if and only if \((T - U, \phi) \geq 0\) for all \(\phi \in \mathcal{D}(\mathbb{R})\) such that \(\phi \geq 0\). If \(T \geq 0\) then it is known that \(T\) is a Borel measure. The usual ordering on distributions does not give a vector lattice on \(\mathcal{A}_c(\mathbb{R}^2)\). With the distributional ordering, \(\text{sup}(\partial_{12} F,0)\) is the function equal to \(\sin(x)\sin(y)/(xy)\) when \(x \in [2m\pi,(2m+1)\pi]\) and \(y \in [2n\pi,(2n+1)\pi]\), or, \(x \in [(2m+1)\pi,(2m+2)\pi]\) and \(y \in [(2n+1)\pi,(2n+2)\pi]\) for some integers \(m,n \geq 0\), and is equal to zero otherwise. This function is not in \(\mathcal{A}_c(\mathbb{R}^2)\) since the integral defining \(F\) converges conditionally. The derivative \(\partial_{12} F\) is not positive in the pointwise or distributional sense. Note that in \(\mathcal{A}_c(\mathbb{R}^2)\) we have \((\partial_{12} F)^+ = |\partial_{12} F| = \partial_{12} F\) and \((\partial_{12} F)^- = 0\).

A vector lattice is order complete (or Dedekind complete) if every nonempty subset that is bounded above has a supremum. But \(\mathcal{B}_c(\mathbb{R}^2)\) is not complete. Let \(F_n(x,y) = |\sin(\pi/x)\sin(\pi/y)|\) for \(x,y \geq 1/n\) with \(F_n(x,y) = 0\) if \(x \leq 1/n\) or \(y \leq 1/n\). Let \(S = \{F_n \mid n \in \mathbb{N}\}\) then \(S \subseteq \mathcal{B}_c(\mathbb{R}^2)\). An upper bound for \(S\) is the
function

\[ F(x, y) = \begin{cases} 
1, & x, y > 0 \\
\frac{1}{|x|+1}, & x < 0, y > 0 \\
\frac{1}{|y|+1}, & x > 0, y < 0 \\
\frac{1}{(|x|+1)(|y|+1)}, & x, y < 0.
\end{cases} \]

But \( \sup(S)(x, y) = \chi_{(0,\infty) \times (0,\infty)}(x, y)|\sin(\pi/x)\sin(\pi/y)| \), which is not continuous. Hence, \( A_c(\mathbb{R}^2) \) is also not complete.

A vector lattice is Archimedean if whenever \( 0 \leq x \leq ny \) for all \( n \in \mathbb{N} \) and some \( y \geq 0 \) then \( x = 0 \). Applying the Archimedean property at each point of the domain \( \mathbb{R}^2 \) shows \( B_c(\mathbb{R}^2) \) and hence \( A_c(\mathbb{R}^2) \) are Archimedean. All lattice inequalities that hold in \( \mathbb{R} \) also hold in all Archimedean spaces and all lattice equalities that hold in \( \mathbb{R} \) also hold in all vector lattices. See [4]. This expands the list of identities and inequalities proved in Theorem 11.1.

A Banach lattice is an abstract L-space if \( \|x+y\| = \|x\| + \|y\| \) for all \( x, y \geq 0 \). A Banach lattice is an abstract M-space if \( \|x \vee y\| = \max(\|x\|, \|y\|) \) for all \( x, y \geq 0 \). See, for example, [4]. We next show that \( B_c(\mathbb{R}^2) \) and \( A_c(\mathbb{R}^2) \) are abstract M-spaces but neither is an abstract L-space.

**Theorem 11.2.** Both of \( B_c(\mathbb{R}^2) \) and \( A_c(\mathbb{R}^2) \) are abstract M-spaces. Neither is an abstract L-space.

For a proof see [52, Theorem 5.2].

For every measure \( \mu \) it is known that \( L^1(\mu) \) is an abstract L-space and that a Banach lattice is an abstract L-space if and only if it is lattice isometric to \( L^1(\nu) \) for some measure \( \nu \). Notice that \( L^\infty(\mu) \) is an abstract M-space. A Banach lattice is an abstract M-space with unit if and only if it is lattice isometric to \( C(K) \) for some compact Hausdorff space \( K \). These results are due to S. Kakutani, M. Krein and others. For references see [4]. In our case, \( B_c(\mathbb{R}^2) \) and \( A_c(\mathbb{R}^2) \) are isomorphic to the set of continuous functions that vanish on \( \{-\infty\} \times \mathbb{R} \) and on \( \mathbb{R} \times \{-\infty\} \). It is not clear what the space \( K \) is here. The fact that \( A_c(\mathbb{R}^2) \) is an abstract M-space but not an abstract L-space indicates that what we have termed an integral here is fundamentally different from the Lebesgue integral.

12. Banach algebra

A commutative algebra is a vector space \( V \) over scalar field \( \mathbb{R} \) with a multiplication \( V \times V \to V \) such that for all \( u, v, w \in V \) and all \( a \in \mathbb{R} \), \( u(vw) = (uv)w \) (associative), \( uv = vu \) (commutative), \( u(v+w) = uv + uw \) and \( (u+v)w = uw + vw \) (distributive), \( a(uv) = (au)v \). If \( (V, \| \cdot \|_V) \) is a Banach space and \( \|uv\|_V \leq \|u\|_V \|v\|_V \) then it is a Banach algebra. For any compact Hausdorff space, \( K \), the set of continuous real-valued functions \( C(K) \) is a commutative Banach algebra under pointwise multiplication and the uniform norm. Since \( \mathbb{R}^2 \) is compact and \( B_c(\mathbb{R}^2) \) is closed under pointwise multiplication, \( B_c(\mathbb{R}^2) \) is a subalgebra of \( C(\mathbb{R}^2) \). The usual pointwise multiplication, \( (FG)(x, y) = [F(x, y)][G(x, y)] \) for all \( (x, y) \in \mathbb{R}^2 \), then makes \( B_c(\mathbb{R}^2) \) into a commutative algebra. The inequality
\[ \|F_1 F_2\|_\infty \leq \|F_1\|_\infty \|F_2\|_\infty \] for all \( F_1, F_2 \in \mathcal{B}_c(\mathbb{R}^2) \) shows \( \mathcal{B}_c(\mathbb{R}^2) \) is a commutative Banach algebra.

There is no unit. For suppose \( F(x, y) > 0 \) for all \((x, y) \in \mathbb{R}^2\). If \( eF = F \) then \( e(x, y) = 1 \) for all \((x, y) \in \mathbb{R}^2\) so \( e \notin \mathcal{B}_c(\mathbb{R}^2) \).

Consider the sequence \((u_n) \in \mathcal{B}_c(\mathbb{R}^2)\) defined by \( u_n(x) = 0 \) for \( x \leq -n \), \( u_n(x) = x + n \) for \(-n \leq x \leq 1 - n\) and \( u_n(x) = 1 \) for \( x \geq 1 - n \). Define \( U_n \in \mathcal{B}_c(\mathbb{R}^2) \) by \( U_n(x, y) = u_n(x) u_n(y) \). For each \( F \in \mathcal{B}_c(\mathbb{R}^2) \) we have \( \|F - U_n F\|_\infty \to 0 \). Given \( \varepsilon > 0 \) there is \( M \in \mathbb{R} \) such that \( |F(x, y)| < \varepsilon \) for all \((x, y) \) such that \( x \leq M \) or \( y \leq M \). We then have \( |F(x, y) - U_n(x, y) F(x, y)| = |F(x, y)||1 - U_n(x, y)| < \varepsilon \) if \( x \leq M \) or \( y \leq M \). If \( x \geq M \) and \( y \geq M \) take \( n \geq 1 - M \). Then \( U_n(x, y) = 1 \).

Hence, \( \|F - U_n F\|_\infty \to 0 \). \( \mathcal{B}_c(\mathbb{R}^2) \) is then said to have an approximate identity.

**Theorem 12.1.** If \( f_1, f_2 \in \mathcal{A}_c(\mathbb{R}^2) \) with respective primitives \( F_1, F_2 \in \mathcal{B}_c(\mathbb{R}^2) \) define their product by \( f_1 f_2 = \partial_{12}(F_1 F_2) \). Then \( \mathcal{A}_c(\mathbb{R}^2) \) is a commutative Banach algebra without unit, with approximate identity, isomorphic to \( \mathcal{B}_c(\mathbb{R}^2) \).

There is no difficulty in allowing functions in \( \mathcal{B}_c(\mathbb{R}^2) \) to be complex-valued and using \( \mathbb{C} \) as the field of scalars. Complex conjugation is then an involution on \( \mathcal{B}_c(\mathbb{R}^2) \). Then \( \mathcal{B}_c(\mathbb{R}^2) \) is a \( {C^*} \)-algebra since for each \( F \in \mathcal{B}_c(\mathbb{R}^2) \) we have \( \|F\|_\infty = \|F\|_\infty \) and \( \|F F\|_\infty = \|F\|_\infty^2 \). Thus, \( \mathcal{A}_c(\mathbb{R}^2) \) is also a \( {C^*} \)-algebra.

Suppose \( f_1, f_2 \in \mathcal{A}_c(\mathbb{R}^2) \) have respective primitives \( F_1, F_2 \in \mathcal{B}_c(\mathbb{R}^2) \). Let \( g \in \mathcal{HKBV}(\mathbb{R}^2) \). Then according to Definition 7.1

\[
\int_{-\infty}^{x} \int_{-\infty}^{y} (f_1 f_2) g = F_1(x, y) F_2(x, y) g(x, y) - \int_{-\infty}^{x} F_1(s, \infty) F_2(s, \infty) d_1 g(s, \infty) - \int_{-\infty}^{x} F_1(\infty, t) F_2(\infty, t) d_2 g(\infty, t) + \int_{-\infty}^{x} \int_{-\infty}^{y} F_1(s, t) F_2(s, t) d_{12} g(s, t).
\]

There are zero divisors. Let \( F_1, F_2 \in \mathcal{B}_c(\mathbb{R}^2) \) with disjoint supports. Then \( F_1 F_2 = 0 \) in \( \mathcal{B}_c(\mathbb{R}^2) \) so \( \partial_{12}(F_1 F_2) = 0 \) in \( \mathcal{A}_c(\mathbb{R}^2) \), yet neither \( \partial_{12} F_1 \) nor \( \partial_{12} F_2 \) need be zero. This example also shows the multiplication introduced in \( \mathcal{A}_c(\mathbb{R}^2) \) is not compatible with pointwise multiplication in the case when elements of \( \mathcal{A}_c(\mathbb{R}^2) \) are functions.

The product of a function in \( \mathcal{B}_c(\mathbb{R}^2) \) and a function in \( C(\mathbb{R}^2) \) is in \( \mathcal{B}_c(\mathbb{R}^2) \). Therefore, \( \mathcal{B}_c(\mathbb{R}^2) \) is an ideal of \( C(\mathbb{R}^2) \). The maximal ideals of \( C(\mathbb{R}^2) \) consist of functions vanishing at a single point. See, for example, [30] and [31] for this and other results that hold for continuous functions on a compact Hausdorff space (and hence for \( \mathcal{A}_c(\mathbb{R}^2) \)).

### 13. Iterated integrals

It was shown in [5, Theorem 4] that if \( f \in \mathcal{A}_c(\mathbb{R}^2) \) then a type of Fubini theorem holds in that

\[
\int_a^b \int_c^d f = \int_a^b \left( \int_c^d f(x, y) \, dy \right) \, dx = \int_c^d \left( \int_a^b f(x, y) \, dx \right) \, dy
\]
and the integral over an interval in $\mathbb{R}^2$ is equal to the two iterated integrals.

A sufficient condition for existence of the iterated integrals, that can sometimes take the place of Tonelli’s theorem in $\mathcal{A}_c(\mathbb{R}^2)$, is the following.

**Proposition 13.1.** Let $f \in \mathcal{A}_c(\mathbb{R})$. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be measurable. Assume (i) for each $x \in \mathbb{R}$ the function $y \mapsto g(x, y)$ is in $\mathcal{BV}(\mathbb{R})$; (ii) the function $x \mapsto V_2g(x, \cdot)$ is in $L^1(\mathbb{R})$; (iii) there is $M \in L^1(\mathbb{R})$ such that for each $y \in \mathbb{R}$ we have $|g(x, y)| \leq M(x)$. Then the iterated integrals exist and are equal, $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y)g(x, y) \, dy \, dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y)g(x, y) \, dx \, dy$.

For a proof see [50, Proposition A.3]. The proposition was first proved for the wide Denjoy integral on compact intervals on page 58 in [13].

Calculus and integration texts often contain examples of functions of two variables for which the iterated integrals are not equal. These conundrums can usually be resolved by showing the primitive is not continuous.

**Example 13.2.** Let $\Omega \subset \mathbb{R}^2$ be the interval $\Omega = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < 1, 0 < y < \infty\}$. Let $F : \Omega \to \mathbb{R}$ be given by $F(x, y) = x^y$. Then $F$ is continuous on $\Omega$ but there is no way to extend the domain of $F$ to $\Omega$ so that $F$ is continuous. For, we have the limiting values, $F(0, y) = 0$ for $0 < y < \infty$, $F(1, y) = 1$ for $0 < y < \infty$, $F(x, 0) = 1$ for $0 < x < 1$, $F(x, \infty) = 0$ for $0 < x < 1$. Hence, $F$ cannot be made continuous on $\Omega$. Now we let $f(x, y) = \partial_1 F(x, y) = x^{y-1} + x^{y-1}y \log(x)$ for $(x, y) \in \Omega$. Since $F$ is not continuous on $\Omega$ the integral $\int_{\Omega} f$ does not exist, yet the two iterated integrals are equal. A calculation shows that for each $0 < x < 1$ we have $\int_0^{\infty} f(x, y) \, dy = 0$ so $\int_0^1 \left( \int_0^{\infty} f(x, y) \, dy \right) \, dx = 0$. For each $0 < y < \infty$ we have $\int_0^1 f(x, y) \, dx = 0$ so $\int_0^{\infty} \left( \int_0^1 f(x, y) \, dx \right) \, dy = 0$. Suppose $0 < a < b < 1$, $0 < c < d < \infty$. Taking iterated limits

$$
\lim_{b \to 0^-} \lim_{b \to 1^-} \lim_{a \to 0^+} \lim_{c \to 0^+} \int_a^b \int_c^d f = \lim_{b \to 0^-} \lim_{b \to 1^-} \lim_{a \to 0^+} \lim_{c \to 0^+} \left[ a^c + b^d - a^d - b^c \right] = \lim_{b \to 0^-} \lim_{b \to 1^-} \lim_{a \to 0^+} \lim_{c \to 0^+} [1 + b^d - 0 - 1] = 0
$$

and

$$
\lim_{b \to 0^-} \lim_{b \to 1^-} \lim_{c \to 0^+} \lim_{a \to 0^+} \int_a^b \int_c^d f = -1.
$$

Hence, $\int_{\Omega} f$ does not exist.

**Example 13.3.** Let $F(x, y) = \arctan(xy)$. Then

$$
F_1(x, y) = \frac{y}{x^2y^2 + 1}, \quad F_2(x, y) = \frac{x}{x^2y^2 + 1}, \quad F_{12}(x, y) = \frac{1 - x^2y^2}{(x^2y^2 + 1)^2} = F_{21}(x, y).
$$
We have the iterated improper Riemann integrals
\[
\int_{-\infty}^{\infty} \int_{0}^{1} F_{12}(x, y) \, dy \, dx = \int_{-\infty}^{\infty} \left[ F_{1}(x, 1) - F_{1}(x, 0) \right] \, dx
\]
\[
= \int_{-\infty}^{\infty} \frac{dx}{x^2 + 1} = \pi
\]
\[
\int_{0}^{1} \int_{-\infty}^{\infty} F_{21}(x, y) \, dx \, dy = \int_{0}^{1} \left[ \lim_{x \to \infty} F_{2}(x, y) - \lim_{x \to -\infty} F_{2}(x, y) \right] \, dy
\]
\[
= \int_{0}^{1} 0 \, dy = 0.
\]

Although $F$ is bounded and continuous on $\mathbb{R}^2$, it is not continuous on $\mathbb{R}^2$. This can be seen by examining the behaviour of $F(x, y)$ in a neighbourhood of the point $(0, \infty)$. Hence, $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \partial_{12} F$ does not exist.

In $\mathbb{R}^2$ the iterated integrals theorem takes the following form.

**Proposition 13.4.** Let $f \in \mathcal{A}_c(\mathbb{R}^2)$. Let $g : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be measurable on $\mathbb{R}^2 \times \mathbb{R}^2$. Assume (i) for each $(x, y) \in \mathbb{R}^2$ the function $(s, t) \mapsto g(x, y; s, t)$ is in $\mathcal{HKBV}(\mathbb{R}^2)$; (ii) for each $t \in \mathbb{R}$ the function $(x, y) \mapsto V_1 g(x, y; \cdot, t)$ is in $L^1(\mathbb{R}^2)$, for each $s \in \mathbb{R}$ the function $(x, y) \mapsto V_2 g(x, y; s, \cdot)$ is in $L^1(\mathbb{R}^2)$, the function $(x, y) \mapsto V_{12} g(x, y; \cdot, \cdot)$ is in $L^1(\mathbb{R}^2)$; (iii) there is $M \in L^1(\mathbb{R}^2)$ such that for each $(s, t) \in \mathbb{R}^2$ we have $|g(x, y; s, t)| \leq M(x, y)$. Then the iterated integrals exist and are equal, $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(s, t) g(s, t; x, y) \, dt \, ds \, dy \, dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(s, t) g(s, t; x, y) \, dy \, dx \, dt \, ds$.

Note that the variation in (ii) is computed with respect to the second pair of variables in $g$, while the integration in (ii) and (iii) is computed with respect to the first pair of variables. The proof is similar to that of Proposition 13.1. The final step uses the density of step functions in $\mathcal{B}_c(\mathbb{R}^2)$ (Theorem 4.5).

### 14. Convolution

In this section the convolution $f * g$ is defined for $f \in \mathcal{A}_c(\mathbb{R}^2)$ and $g \in \mathcal{HKBV}(\mathbb{R}^2)$ and then for $g \in L^1(\mathbb{R}^2)$.

In Theorem 14.1 it is shown that when $g \in \mathcal{HKBV}(\mathbb{R}^2)$ the convolution has similar properties to the case when $f \in L^1$ and $g \in L^\infty$. Since $L^\infty$ is the dual space of $L^1$ this mirrors the fact that $\mathcal{HKBV}(\mathbb{R}^2)$ is the dual space of $\mathcal{A}_c(\mathbb{R}^2)$. In Theorem 14.3 the density of $L^1(\mathbb{R}^2)$ in $\mathcal{A}_c(\mathbb{R}^2)$ is used to define the convolution for $f \in \mathcal{A}_c(\mathbb{R}^2)$ and $g \in L^1(\mathbb{R}^2)$. This type of convolution has properties analogous to convolutions on $L^1 \times L^1$.

Convolutions in $\mathcal{A}_c(\mathbb{R})$ were introduced in [50]. Here we extend the two most important theorems from $\mathbb{R}$ to $\mathbb{R}^2$. Many other results, such as differentiation and integration of convolutions, can also be carried over to $\mathbb{R}^2$.

First we show the convolution is well-defined. Fix $(x, y) \in \mathbb{R}^2$. If $f \in \mathcal{A}_c(\mathbb{R}^2)$ has primitive $F \in \mathcal{B}_c(\mathbb{R}^2)$ define $\Phi(s, t) = F(x - s, y - t)$. Then $\Psi \in \mathcal{B}_c(\mathbb{R}^2)$ and $\partial_{12} \Psi(s, t) = \partial_{12} F(x - s, y - t)$. We can then define $\psi(s, t) = f(x - s, y - t) = \phi(s) \ast \psi(t)$.

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \ast g(x, y) \, dx \, dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(x, y) \ast G(x, y) \, dx \, dy
\]
\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \cdot \partial_{12} g(x, y) \, dx \, dy + \partial_{12} f(x, y) \cdot \int_{-\infty}^{\infty} g(x, y) \, dx \, dy
\]
\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \cdot \partial_{12} g(x, y) \, dx \, dy + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \partial_{12} f(x, y) \cdot g(x, y) \, dx \, dy
\]
\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \cdot \partial_{12} g(x, y) \, dx \, dy + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \partial_{12} f(x, y) \cdot g(x, y) \, dx \, dy
\]
\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \cdot \partial_{12} g(x, y) \, dx \, dy + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \partial_{12} f(x, y) \cdot g(x, y) \, dx \, dy
\]
\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \cdot \partial_{12} g(x, y) \, dx \, dy + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \partial_{12} f(x, y) \cdot g(x, y) \, dx \, dy
\]
\[ \partial_{12} \Psi(s, t). \] Then \( f \ast g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-s, y-t)g(s, t) \, dt \, ds \) is well-defined for each \( g \in HKBV(\mathbb{R}^2). \) See Theorem 10.2.

**Theorem 14.1.** Let \( f \in A_0(\mathbb{R}^2), \) let \( F \in B_c(\mathbb{R}^2) \) be its primitive and let \( g \in HKBV(\mathbb{R}^2). \) Then (a) \( f \ast g \) exists on \( \mathbb{R}^2 \) (b) \( f \ast g = g \ast f \) (c) \( \|f \ast g\|_{\infty} \leq \|f\|_{\infty} \|g\|_{bv} \) (d) \( f \ast g \in C(\mathbb{R}^2). \) Let \( \epsilon_1, \epsilon_2 \in \{+, -\}. \) Then \( \lim_{y \to \pm \infty} f \ast g(x) = g(\epsilon_1 \infty, \epsilon_2 \infty)F(\infty, \infty). \) (e) If \( h \in L^1(\mathbb{R}^2) \) then \( f \ast (g \ast h) = (f \ast g) \ast h \in C(\mathbb{R}^2). \)

**Proof.** (a) The above definition and integration by parts show \( f \ast g \) exists on \( \mathbb{R}^2. \) (b) If \( g \in HKBV(\mathbb{R}^2) \) then the function \( (s, t) \mapsto g(x-s, y-t) \) is also in \( HKBV(\mathbb{R}^2). \) Hence, \( g \ast f \) exists in \( \mathbb{R}^2. \) We can change variables as in Theorem 10.2. (c) This follows from Proposition 7.8 and the Hölder inequality (Proposition 8.1). (d) To show continuity at \( (x, y) \in \mathbb{R}^2, \) let \( (\xi, \eta) \in \mathbb{R}^2. \) Then

\[
|f \ast g(x, y) - f \ast g(\xi, \eta)| \leq \|f(x-\cdot, y-\cdot) - f(\xi-\cdot, \eta-\cdot)\|_{bv}.
\]

This last expression tends to 0 as \( (\xi, \eta) \to (x, y) \) by continuity in the Alexiewicz norm (Proposition 7.8). It is clear from the proof of Proposition 9.1 that the convergence theorem applies for limits of two continuous variables. We can then take limits as \( x \) and \( y \) tend to \( \infty \) or \( -\infty \) under the integral signs of \( g \ast f. \) Note that

\[
g \ast f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(s, t)g(x-s, y-t) \, dt \, ds.
\]

And,

\[
\lim_{y \to \pm \infty} g(x-s, y-t) = \begin{cases} g(\infty, \infty), & (s, t) \neq (\infty, \infty) \\ g(-\infty, -\infty), & (s, t) = (\infty, \infty). \end{cases}
\]

As per Proposition 8.3 we can ignore the value of the integrand in \( g \ast f(x, y) \) on two coordinate lines. Hence, the limit of \( f \ast g(x, y) \) as \( x, y \to \infty \) gives \( F(\infty, \infty)g(\infty, \infty). \) Similarly, for the other cases. This also shows \( f \ast g \in C(\mathbb{R}^2). \)

Part (d) can also be proved with integration by parts. (e) To show \( g \ast h \in HKBV(\mathbb{R}^2) \) let \( (a_i, b_i) \times (c_i, d_i) \) be disjoint intervals in \( \mathbb{R}^2. \) By dominated convergence and the Fubini–Tonelli theorem we have

\[
\sum |g \ast h(a_i, c_i) + g \ast h(b_i, d_i) - g \ast h(a_i, d_i) - g \ast h(b_i, c_i)| \leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |g(a_i - x, c_i - y) + g(b_i - x, d_i - y) - g(a_i - x, d_i - y) - g(b_i - x, c_i - y)||h(x, y)| \, dy \, dx.
\]

From this it follows that \( V_{12}g \ast h \leq V_{12}g \|h\|_1. \) Similarly, \( \|V_1g \ast h\|_\infty \leq \|V_1g\|_\infty \|h\|_1 \) and \( \|V_2g \ast h\|_\infty \leq \|V_2g\|_\infty \|h\|_1. \) Also, \( \|g \ast h\|_\infty \leq \|g\|_\infty \|h\|_1. \) Hence, \( g \ast h \in HKBV(\mathbb{R}^2). \) Part (d) now shows \( f \ast (g \ast h) \in C(\mathbb{R}^2). \) It is known that convolution with an \( L^1 \) function and a bounded function produces a function in \( C(\mathbb{R}^2). \) For example, [18, Proposition 8.8]. Hence, \( (f \ast g) \ast h \in C(\mathbb{R}^2). \) To show \( f \ast (g \ast h) = (f \ast g) \ast h \) requires a change in order of integration and this is justified by Proposition 13.4. \( \square \)
Hence, as
Proof. Let \( f \in \mathcal{A}_c(\mathbb{R}^2) \). Then, by (a),
\[
(f * g)(s, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(r, \xi) g(s - r, t - \eta) dr d\eta.
\]
It follows that \( \|f * g\| \leq \|f\| \|g\|_1 \). Hence, \( \{f_n \} \) is a Cauchy sequence in \( \mathcal{A}_c(\mathbb{R}^2) \) and therefore converges to a unique element of \( \mathcal{A}_c(\mathbb{R}^2) \). This also shows that \( f * g \) is independent of the defining sequence \( \{f_n\} \).

**Theorem 14.3.** Let \( f \in \mathcal{A}_c(\mathbb{R}^2) \) and \( g \in L^1(\mathbb{R}^2) \). Define \( f * g \) as in Definition 14.2. Then (a) \( f * g \in \mathcal{A}_c(\mathbb{R}^2) \) and \( \|f * g\| \leq \|f\| \|g\|_1 \). (b) Let \( h \in L^1(\mathbb{R}^2) \). Then \( (f * g) * h = f * (g * h) \in \mathcal{A}_c(\mathbb{R}^2) \). (c) Define \( g_r(x, y) = r^{-2} g(r^{-1} x, r^{-1} y) \) for \( r > 0 \). Let \( A = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_r(x, y) dy dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g \). Then \( \|f * g - Af\| \to 0 \) as \( r \to 0^+ \).

**Proof.** Let \( \{f_n\} \subset L^1(\mathbb{R}^2) \) such that \( \|f_n - f\| \to 0 \). (a) We have \( \|f_n\| \to \|f\| \) and the inequality
\[
\|f_n * g\| - \|f * g - f_n * g\| \leq \|f * g\| \leq \|f_n * g\| + \|f * g - f_n * g\|.
\]
Hence,
\[
\|f * g\| = \lim_{n \to \infty} \|f_n * g\| \leq \lim_{n \to \infty} \|f_n\| \|g\|_1 = \|f\| \|g\|_1.
\]
(b) From the \( L^1 \) theory of convolutions it is known that \( g * h \in L^1(\mathbb{R}^2) \). For example, [18]. Then, by (a), \( f * (g * h) \in \mathcal{A}_c(\mathbb{R}^2) \). And, \( f * g \in \mathcal{A}_c(\mathbb{R}^2) \) so by (a), \( (f * g) * h \in \mathcal{A}_c(\mathbb{R}^2) \). Hence, both \( f * (g * h) \) and \( (f * g) * h \) exist in \( \mathcal{A}_c(\mathbb{R}^2) \). To show they are equal note that convolutions are associative in \( L^1(\mathbb{R}^2) \). Therefore,
\[
0 = \lim_{n \to \infty} \|f_n * (g * h) - f * (g * h)\| = \lim_{n \to \infty} \|(f_n * g) * h - f * (g * h)\|.
\]
And, \( f_n * g \in \mathcal{A}_c(\mathbb{R}^2) \) such that \( \|f_n * g - f * g\| \to 0 \). Therefore, \( \|(f_n * g) * h - (f * g) * h\| \to 0 \). It now follows that \( f * (g * h) = (f * g) * h \). (c) If suffices to prove that \( \|f_n * g - Af_n\| \to 0 \). Accordingly,
\[
f_n * g_r(s, t) - Af_n(s, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [f_n(s - r\xi, t - r\eta) - f_n(s, t)] g(\xi, \eta) d\eta d\xi.
\]
We can change variables by Theorem 10.2. To find the Alexiewicz norm, the above expression is integrated from \( s = -\infty \) to \( x \) and from \( t = -\infty \) to \( y \), for some \((x, y) \in \mathbb{R}^2\). In the integral with \( f_n(s - r\xi, t - r\eta) \) the order of integration can be changed due to the Fubini–Tonelli theorem. In the integral with \( f_n(s, t) \)
the order of integration can be changed since the \((s,t)\) variables separate from the \((\xi,\eta)\) variables. This then gives
\[
\left| \int_{-\infty}^{x} \int_{-\infty}^{y} [f_n * g_r(s,t) - Af_n(s,t)] \, dt \, ds \right|
\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{x} \int_{-\infty}^{y} [f_n(s-r\xi,t-r\eta) - f_n(s,t)] \, dt \, ds \, |g(\xi,\eta)| \, d\eta \, d\xi
\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \|f_n(\cdot - r\xi, \cdot - r\eta) - f_n(\cdot, \cdot)\| |g(\xi,\eta)| \, d\eta \, d\xi.
\]

Dominated convergence and continuity in the Alexiewicz norm (Proposition 7.8) allows us to take the limit \(n \to \infty\) under the integral sign. \(\square\)

**Example 14.4.** Part (c) of this theorem is useful for showing the solution of a differential equation takes on initial or boundary values in the Alexiewicz norm. For example, if \(\Phi_z(x,y) = z(x^2 + y^2 + z^2)^{-3/2}/(2\pi)\) is the half-space Poisson kernel from Proposition 4.5, then \(\lim_{z \to 0^+} \|f * \Phi_z - f\| = 0\). Then the convolution \(u(x,y,z) = f * \Phi_z(x,y)\) satisfies the boundary condition \(u = f\) in the Alexiewicz norm when \(z \to 0^+\). The partial derivatives of \(\Phi_z\) are of bounded variation. Proposition 9.1 can then be used to show we can differentiate under the integrals and this shows \(u\) is harmonic in the half-space \((x,y,z) \in \mathbb{R}^2 \times (0, \infty)\).

15. **The integral in \(\mathbb{R}^n\)**

Here we will briefly sketch out the differences between the integral in \(\mathbb{R}^2\) and in \(\mathbb{R}^n\).

We now let \(D_n = \partial_1 \partial_2 \ldots \partial_n\) and define
\[
\mathcal{B}_c(\mathbb{R}^n) = \{ F \in C(\mathbb{R}^n) \mid F(x) = 0 \text{ if } x_i = -\infty \text{ for some } 1 \leq i \leq n \}.
\]
\[
\mathcal{A}_c(\mathbb{R}^n) = \{ f \in \mathcal{D}'(\mathbb{R}^n) \mid f = D_nF \text{ for some } F \in \mathcal{B}_c(\mathbb{R}^n) \}.
\]

As before, primitives are unique. It is convenient to use matrix notation to define the integral over interval \(I = [a_{21}, a_{11}] \times [a_{22}, a_{12}] \times \ldots \times [a_{2n}, a_{1n}]\) by
\[
\int_I f = \int_{a_{21}}^{a_{11}} \int_{a_{22}}^{a_{12}} \cdots \int_{a_{2n}}^{a_{1n}} f = (-1)^n \sum_{i_1, \ldots, i_n \in \{1,2\}} (-1)^{i_1+\ldots+i_n} F(a_{i_11}, a_{i_22}, \ldots, a_{i_n n}).
\]

There are \(2^n\) summands. This formula can be proved by induction by writing iterated integrals.

In the proof of Proposition 4.5 the half-space Poisson kernel in \(\mathbb{R}^n\) is given in [8, p. 145].

Hardy–Krause variation in \(\mathbb{R}^n\) is defined in Definition 6.5.2 in [35]. The integration by parts formula, due to J. Kurzweil, is given in [35], Theorem 6.5.9. See also [56]. Various forms of the second mean value theorem are given in [35] and [56].

J. Mawhin has listed the coordinate transformations that map intervals to intervals and this will give a change of variables theorem as in Theorem 10.2. See [38].
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