EXISTENCE OF SOLUTION FOR SCHRÖDINGER EQUATION WITH DISCONTINUOUS NONLINEARITY AND CRITICAL GROWTH

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Abstract. This paper concerns with the existence of nontrivial solution for the following problem
\[
\begin{aligned}
-\Delta u + V(x)u &= \gamma H_e(|u| - a)|u|^{q-2}u + |u|^{2^*-2}u \quad \text{in } \mathbb{R}^N, \\
u &\in H^1(\mathbb{R}^N),
\end{aligned}
\]
where, \( N \geq 3, \gamma \geq 0, H_e : \mathbb{R} \to \mathbb{R} \) denotes the Heaviside function, \( a \geq 0, 2 < q < 2^* \) and \( V : \mathbb{R}^N \to \mathbb{R} \) is \( \mathbb{Z}^N \)-periodic with \( \beta = 0 \) does not belong to the spectrum of \( -\Delta + V \).

1. Introduction

In this paper we study the existence of nontrivial solution for the following class of elliptic problems
\[
(P_{\gamma}^*) \quad \begin{cases}
-\Delta u + V(x)u = f_\gamma(x, u) \quad \text{in } \mathbb{R}^N \\
u &\in H^1(\mathbb{R}^N),
\end{cases}
\]
where, \( f_\gamma(x, s) = \gamma H_e(|s| - a)|s|^{q-2}s + |s|^{2^*-2}s \) and \( H_e : \mathbb{R} \to \mathbb{R} \) denotes the Heaviside function, that is,
\[
H_e(t) = \begin{cases} 
0, & \text{if } t \leq 0 \\
1, & \text{if } t > 0.
\end{cases}
\]
In addition, \( \gamma \geq 0, N \geq 3 \) and \( V \) is continuous function, periodic with respect to \( x \)-variable,
\[(V_1) \quad 0 \notin \sigma(-\Delta + V), \text{ the spectrum of } -\Delta + V; \]
and
\[(V_2) \quad \sigma(-\Delta + V) \cap (-\infty, 0) \neq \emptyset. \]

Problems of type \((P_{\gamma}^*)\), when nonlinearity is considered continuous, has been studied by Chabrowski and Szulkin [16], M. Schechter and W. Zou [41], Alves and Germano [3]. In [16] Chabrowski and Szulkin studied the following class of problems
\[
(P) \quad \begin{cases}
-\Delta u + V(x)u = K(x)|u|^{2^*-2}u + g(x, u) \quad \text{in } \mathbb{R}^N \\
u &\in H^1(\mathbb{R}^N),
\end{cases}
\]
where \( N \geq 4, V, K, g \) are periodic in \( x_j \) for \( 1 \leq j \leq N, K > 0, g \) is of subcritical growth and 0 is in a gap of the spectrum of \( -\Delta + V \). They proved the existence of a nontrivial solution using the linking theorem.

Knowing that the equation
\[
-\Delta u + \beta u = |u|^{2^*-2}u, \quad \text{in } \mathbb{R}^N,
\]

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when $\beta \neq 0$, has only the trivial solution in $H^1(\mathbb{R}^N)$ (cf [12]). Thereby, when $\gamma = 0$ in $(P^*_\gamma)$ we are left with an equation similar to (1.1) and so the existence of nontrivial solution of $(P^*_\gamma)$ is an interesting problem. One of the pioneering results involving critical problems was obtained in article of Brézis and Nirenberg [11] and has helped and motivated a great amount of research on this class of problem involving critical exponent.

In several articles, the authors are focus on equations with subcritical growth, since critical growth bring many difficulties, because, yet in bounded domain $\Omega \subset \mathbb{R}^N$, the Sobolev embedding $H^1_0(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is not compact. In recent years, the study of equations with critical growth has made great progress and has attracted the attention of many authors. One way to regain compactness is constrain the functional value in a suitable interval, see for example [16]. When the functional is strongly defined, then the functional has mountain pass geometry, so it is easy to estimate the functional level. For example, under some weaker conditions, Lins and Silva [26] considered the existence of nontrivial solutions

$$-\Delta u + V(x)u = f(x, u)$$

when $f$ is asymptotically periodic in $x$. However, for the strongly indefinite case, the problem becomes very difficult e.g. see [3, 16, 41].

We will study a class of semilinear problems with discontinuous linearity in $\mathbb{R}^N$, strongly indefinite and critical growth $(P^*_\gamma)$. The interest in the study of nonlinear partial differential equations with discontinuous nonlinearities has increased because many free boundary problems arising in mathematical physics may be stated in this from. Among these problems, we have the seepage surface problem and the Elenbaas equation, see for example [19–21].

Variational methods for problems with discontinuous nonlinearity and critical exponent have been applied to several problems. Alves and Bertone [8], got two nonnegative solutions for the following quasilinear problem

$$-\Delta_p u = H_e(u - a)u^{p^*-1} + \lambda h(x), \text{ in } \mathbb{R}^N$$

where $H_e$ is the Heaviside function. Alves, Bertone and Goncalves in [6] studied the equation

$$-\Delta u = u^{2^*-1} + \lambda h(x)H(u - a)u^q, \text{ in } \mathbb{R}^N$$

obtained the existence of positive solutions for $0 \leq q \leq 1$ and $1 < q < 2^* - 1$.

A rich literature is available for problems with discontinuous nonlinearities, and we refer the reader to Chang [19], Ambrosetti and Badiale [10], Alves and Patricio [2], Cerami [14], Alves et al. [6], Alves et al. [7], Alves and Bertone [8], Alves and Nascimento [9], Cerami [14, 15], Badiale [13], Dinu [22], Gasínski and Papageorgiou [25], Kourogenis and Papageorgiou [27], Mironescu and Rădulescu [30], Rădulescu [33–38] and their references. Several techniques have been developed or applied in their study, such as variational methods for nondifferentiable functionals, lower and upper solutions, global branching, and the theory of multivalued mappings.

In this paper a study is made of a strongly indefinite problem with discontinuous nonlinearity that involves two extreme conditions that result in loss of compactness, namely, when we consider the space $\mathbb{R}^N$ and the critical growth. More precisely, we will find nontrivial solution to problem $(P^*_\gamma)$ and for this we will use a link theorem for a class of functional locally Lipschitz due to Alves and Patricio [1], in which they propose a generalization of the link theorem due to Kryszewski and Szulkin [28] in which they consider class $C^1$ functional. One of the main difficulty in the present paper was to make sure that the weak limit of sequences $(PS)$ is nontrivial for a given level.
The present paper is relevant, because it is the first result that establishes the existence of solutions for Schrödinger equation strongly indefinite with discontinuous nonlinearity and critical growth.

The main result is the following

**Theorem 1.1.** Suppose that conditions \((V_1) - (V_2)\) are satisfied. If \(N \geq 4\), the problem \((P_\gamma^*)\) has a nontrivial solution. If \(N = 3\), there are \(\gamma > 0\) and \(a_0 = a_0(\gamma) > 0\) such that, \(0 \leq a \leq a_0\), the problem \((P_\gamma^*)\) has a nontrivial solution.

**Definition 1.2.** For a solution to the problem \((P_\gamma^*)\) we understand it to be a function \(u \in H^1(\mathbb{R}^N) \cap W^2_{\text{loc}}(\mathbb{R}^N)\), for some \(p > 1\), such that

\[
-\Delta u(x) + V(x)u(x) \in \partial_t F_\gamma(x, u(x)) \text{ a.e in } \mathbb{R}^N,
\]

where \(F_\gamma(x, t) = \int_0^t f_\gamma(x, s)ds\) and \(\partial_t F_\gamma\) denotes the generalized gradient of \(F_\gamma\) with respect to variable \(t\). When

\[
-\Delta u(x) + V(x)u(x) = f_\gamma(x, u(x)), \text{ a.e in } \mathbb{R}^N,
\]

then we say that \(u\) is strong solution.

**Remark 1.** In case \(N \geq 4\), we will not have restrictions for constants \(a \geq 0\) and \(\gamma \geq 0\) to obtain a nontrivial solution to the problem \((P_\gamma^*)\). Thereby, in case \(N \geq 4\), we will have a strong solution for \((P_\gamma^*)\) choosing \(a > 0\) so that

\[
(1.2) \quad \sup_{x \in \mathbb{R}^N} V(x) < a^{2^*-2}.
\]

First note that \((1.2)\) is possible because we are assuming that \(V : \mathbb{R}^N \to \mathbb{R}\) is \(\mathbb{Z}^N\)-periodic and therefore bounded, i.e, there exists \(K_0 > 0\) such that \(|V(x)| \leq K_0\) for all \(x \in \mathbb{R}^N\). So, just choose \(a > K_0\).

Using the definition of generalized gradient, by a simple calculation, we get:

\[
\partial_t F_\gamma(x, u(x)) = \begin{cases} 
  u(x)^{2^*-1}, & \text{if } 0 \leq u(x) < a \\
  [a^{2^*-1}, a^{2^*-1} + \gamma a^{q-1}], & \text{if } u(x) = a \\
  u(x)^{2^*-1} + \gamma u(x)^{q-1}, & \text{if } u(x) > a.
\end{cases}
\]

or

\[
\partial_t F_\gamma(x, u(x)) = \begin{cases} 
  (-u(x))^{2^*-2}u(x), & \text{if } -a < u(x) \leq 0 \\
  [-a^{2^*-1} + \gamma (-a)^{q-1}, (-a)^{2^*-1}], & \text{if } u(x) = -a \\
  (-u(x))^{2^*-2}u(x) + \gamma (-u(x))^{q-2}u(x), & \text{if } u(x) < -a.
\end{cases}
\]

Suppose that \(u\) is a solution of \((P_\gamma^*)\), then

\[
\begin{cases} 
  -\Delta u(x) + V(x)u(x) = u(x)^{2^*-1}, & \text{if } 0 \leq u(x) < a \\
  -\Delta u(x) + V(x)u(x) \in [a^{2^*-1}, a^{2^*-1} + \gamma a^{q-1}], & \text{if } u(x) = a \\
  -\Delta u(x) + V(x)u(x) = u(x)^{2^*-1} + \gamma u(x)^{q-1}, & \text{if } u(x) > a.
\end{cases}
\]

or

\[
\begin{cases} 
  -\Delta u(x) + V(x)u(x) = (-u(x))^{2^*-2}u(x), & \text{if } -a < u(x) \leq 0 \\
  -\Delta u(x) + V(x)u(x) \in [(-a)^{2^*-1} + \gamma (-a)^{q-1}, (-a)^{2^*-1}], & \text{if } u(x) = -a \\
  -\Delta u(x) + V(x)u(x) = (-u(x))^{2^*-2}u(x) + \gamma (-u(x))^{q-2}u(x), & \text{if } u(x) < -a.
\end{cases}
\]

Set

\[
A_+ = \{x \in \mathbb{R}^N : u(x) = a\} \text{ and } A_- = \{x \in \mathbb{R}^N : u(x) = -a\}.
\]
Under (1.2) we obtain that the Lebesgue measure of $A_+$ (or $A_-$) is zero. Otherwise if measure of $A_+$ (or $A_-$) is positive, then by applying Stampacchia theorem [40],

$$ -\Delta u(x) = 0 \text{ in } A_+ \text{ (or } A_-), $$

that is,

$$ V(x)a \in [a^{2^*-1}, a^{2^*-1} + \gamma a^{9-1}], \text{ for } x \in A_+, $$

or

$$ -V(x)a \in [(-a)^{2^*-1} + \gamma(-a)^{9-1}, (-a)^{2^*-1}], \text{ for } x \in A_. $$

In any of the above cases, once $(-a)^{2^*-2} = a^{2^*-2}$, we have

$$ a^{2^*-2} \leq V(x), \text{ for all } x \in A_+ \text{ (or } A_-), $$

a contradiction with (1.2).

By remark 1, we can enunciate the following:

**Corollary 1.3.** If $N \geq 4$, assuming $(V_1)$ and $(V_2)$, there exist $a > 0$ such that the problem $(P'_\gamma)$ has a nontrivial strong solution.

**Notation:** From now on, otherwise mentioned, we use the following notations:

- $B_r(u)$ is an open ball centered at $u$ with radius $r > 0$, $B_r = B_r(0)$.
- $X^*$ denotes the dual topological space of $X$ and $\| \cdot \|_*$ denotes the norm in $X^*$.
- $C$ denotes any positive constant, whose value is not relevant.
- $\| \cdot \|_p$ denotes the usual norm of the Lebesgue space $L^p(\mathbb{R}^N)$, for $p \in [1, +\infty]$.
- $\| \cdot \|_H$ denotes the usual norm of the Orlicz space $L^H(\mathbb{R}^N)$ associated the $N$-function $H$.
- If $u : \mathbb{R}^N \to \mathbb{R}$ is measurable function, the integral $\int_{\mathbb{R}^N} u \, dx$ will be denoted by $\int_{\mathbb{R}^N} u$.
- $o_n(1)$ denotes a real sequence with $o_n(1) \to 0$ as $n \to +\infty$.
- If $k > 0$, $O(\epsilon^k)$ denotes a function that $\frac{O(\epsilon^k)}{\epsilon^k}$ is bounded as $\epsilon \to 0^+$.

## 2. Basic results from nonsmooth analysis

In this section, for the reader’s convenience, we recall some definitions and basic results on the critical point theory of locally Lipschitz functionals as developed by Chang [19], Clarke [17,18] and Grossinho and Tersian [39].

Let $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$ be a real Banach spaces. A functional $I : X \to \mathbb{R}$ is locally Lipschitz, $I \in \text{Lip}_{loc}(X, \mathbb{R})$ for short, if given $u \in X$ there is an open neighborhood $V := V_u \subset X$ of $u$, and a constant $K = K_u > 0$ such that

$$ |I(v_2) - I(v_1)| \leq K\|v_1 - v_2\|_X, \quad v_i \in V, \quad i = 1, 2. $$

The generalized directional derivative of $I$ at $u$ in the direction of $v \in X$ is defined by

$$ I_\circ(u; v) = \limsup_{h \to 0, \delta \to 0} \frac{1}{\delta} (I(u + h + \delta v) - I(u + h)). $$

The generalized gradient of $I$ at $u$ is the set

$$ \partial I(u) = \{ \xi \in X^* \mid I_\circ(u; v) \geq \langle \xi, v \rangle \; \forall \; v \in X \}. $$

**Lemma 2.1.** If $I$ is continuously differentiable to Fréchet in an open neighborhood of $u \in X$, we have $\partial I(u) = \{ I'(u) \}$.

**Lemma 2.2.** If $Q \in C^1(X, \mathbb{R})$ and $\Psi \in \text{Lip}_{loc}(X, \mathbb{R})$, then for each $u \in X$

$$ \partial(Q + \Psi)(u) = Q'(u) + \partial \Psi(u). $$
Moreover, we denote by \( \lambda_I(u) \) the following real number

\[
\lambda_I(u) := \min\{||\xi||, \xi \in \partial I(u)\}.
\]

We recall that \( u \in X \) is a critical point of \( I \) if \( 0 \in \partial I(u) \), or equivalently, when \( \lambda_I(u) = 0 \).

### 3. On the energy functional of problem \((P_\gamma^*)\)

It follows that the functional

\[
I_\gamma(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) - \int_{\mathbb{R}^N} F_\gamma(x,u), \ u \in H^1(\mathbb{R}^N),
\]

where \( F_\gamma(x,t) = \int_0^t f_\gamma(x,s)ds \), is well defined.

By standard argument, \( Q \in C^1(H^1(\mathbb{R}^N), \mathbb{R}) \) where

\[
Q(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)
\]

and

\[
Q'(u)v = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv), \ \forall \ u, v \in H^1(\mathbb{R}^N).
\]

In addition, we can write

\[
I_\gamma = Q - \Psi_\gamma,
\]

where \( \Psi_\gamma : H^1(\mathbb{R}^N) \to \mathbb{R} \) given by

\[
\Psi_\gamma(u) = \int_{\mathbb{R}^N} F_\gamma(x,u).
\]

By \((V_1)\), it is well known that \( H^1(\mathbb{R}^N) = X^+ \oplus X^- \) is an orthogonal decomposition and there is an equivalent norm \( ||\cdot|| \) to \( ||\cdot||_{H^1(\mathbb{R}^N)} \) (see \([1,2]\)) such that

\[
I_\gamma(u) = \frac{1}{2}||u^+||^2 - \frac{1}{2}||u^-||^2 - \Psi_\gamma(u), \ \forall \ u = u^+ + u^- \in X^+ \oplus X^-.
\]

By definition

\[
\partial_t F_\gamma(x,t) = \{\mu \in \mathbb{R} : F_\gamma^\mu(x,t;r) \geq \mu r, \ r \in \mathbb{R}\},
\]

where \( F_\gamma^\mu(x,t;r) \) denotes the generalized directional derivative of \( t \to F_\gamma(x,t) \) in the direction of \( r \), i.e.,

\[
F_\gamma^\mu(x,t;r) = \limsup_{h \to 0, h \neq 0} \frac{F_\gamma(x,h + \lambda r) - F_\gamma(x,h)}{\lambda}.
\]

Consider

\[
F_\gamma : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}
\]

\[
(x,t) \mapsto F_\gamma(x,t) = \int_0^t f_\gamma(x,s)ds
\]

where \( f_\gamma(x,s) = \gamma H(||s| - a||s|^{q-2}s + |s|^{2^* - 2}s, \ \gamma \in \mathbb{R} \), that is,

\[
F_\gamma(x,t) = \begin{cases} 
\frac{1}{2^*} |t|^{2^*}, & \text{if } |t| \leq a \\
\frac{1}{2^*} |t|^{2^*} + \frac{\gamma}{q} |t|^q - \frac{\gamma}{q} a^q, & \text{if } |t| > a
\end{cases}
\]

In fact it is easy to check that functional \( \Psi_\gamma : H^1(\mathbb{R}^N) \to \mathbb{R} \) given by

\[
\Psi_\gamma(u) = \int_{\mathbb{R}^N} F_\gamma(x,u),
\]
Lemma 3.1. Assume \( \Phi \) satisfies \( \Delta_2 \)-condition, we can guarantee that given \( J \in (L^\Phi(\mathbb{R}^N))^* \), then

\[
J(u) = \int_{\mathbb{R}^N} vu, \quad \forall \, u \in L^\Phi(\mathbb{R}^N),
\]

for some \( v \in L^{\tilde{\Phi}}(\mathbb{R}^N) \), where \( \tilde{\Phi} \) is the conjugate function of \( \Phi \). In general, we need to prove that the inclusion below holds

\[
\partial \Psi_\gamma(u) \subset \partial_t F_\gamma(x,u) = \left[ f_\gamma(x,u(x)), \overline{f}_\gamma(x,u(x)) \right] \text{ a.e in } \mathbb{R}^N,
\]

where

\[
f_\gamma(x,t) = \lim_{r \downarrow 0} \text{ess inf}\left\{ f_\gamma(x,s); |s-t| < r \right\}
\]

and

\[
\overline{f}_\gamma(x,t) = \lim_{r \downarrow 0} \text{ess sup}\left\{ f_\gamma(x,s); |s-t| < r \right\}.
\]

We have that the condition below is satisfied:

(F*) There exist \( C_0, C_1 > 0 \) such that

\[
|\xi| \leq C_0 |u|^{q-1} + |u|^{2^* - 1} \leq C_1 \Phi'(|u|), \quad \forall \xi \in \partial_t F_\gamma(x,u), \quad \forall \, x \in \mathbb{R}^N,
\]

for some \( C_1 > 0 \).

The next three results establish important properties of the functional \( \Psi_\gamma \) given in (3.3).

Lemma 3.1. [See [4] or [3]] Assume (F*). Then, the functional \( \Psi_\gamma : L^\Phi(\mathbb{R}^N) \to \mathbb{R} \) given by

\[
\Psi_\gamma(u) = \int_{\mathbb{R}^N} F_\gamma(x,u), \quad u \in L^\Phi(\mathbb{R}^N),
\]

is well defined and \( \Psi_\gamma \in \text{Lip}_{loc}(L^\Phi(\mathbb{R}^N), \mathbb{R}) \).

Theorem 3.2 (See [19], Theorem 2.1 or [5], Theorem 4.1). Assume (F*), then for each \( u \in L^\Phi(\mathbb{R}^N) \),

(3.4) \[
\partial \Psi_\gamma(u) \subset \partial_t F_\gamma(x,u) = \left[ f_\gamma(x,u(x)), \overline{f}_\gamma(x,u(x)) \right] \text{ a.e in } \mathbb{R}^N.
\]

The inclusion above means that given \( \xi \in \partial \Psi_\gamma(u) \subset (L^\Phi(\mathbb{R}^N))^* \approx L^{\tilde{\Phi}}(\mathbb{R}^N) \), there is \( \tilde{\xi} \in L^{\tilde{\Phi}}(\mathbb{R}^N) \) such that

\[
\begin{align*}
\langle \xi, v \rangle &= \int_{\mathbb{R}^N} \tilde{\xi} v, \quad \forall \, v \in L^{\tilde{\Phi}}(\mathbb{R}^N), \\
\langle \tilde{\xi}(x) \rangle &\in \partial_t F_\gamma(x,u(x)) = \left[ f_\gamma(x,u(x)), \overline{f}_\gamma(x,u(x)) \right] \text{ a.e in } \mathbb{R}^N.
\end{align*}
\]

The following proposition is very important to establish the existence of a critical point for the functional \( I_\gamma \).

Proposition 3.3. (See [1]). If \( (u_n) \subset H^1(\mathbb{R}^N) \) is such that \( u_n \rightharpoonup u_0 \) in \( H^1(\mathbb{R}^N) \) and \( \rho_n \in \partial \Psi_\gamma(u_n) \) satisfies \( \rho_n \rightharpoonup \rho_0 \) in \( (H^1(\mathbb{R}^N))^* \), then \( \rho_0 \in \partial \Psi_\gamma(u_0) \).
4. Generalized linking theorem

From now on, $X$ is a Hilbert space with $X = Y \oplus Z$, where $Y$ is a separable closed subspace of $X$ and $Z = Y^\perp$. If $u \in X$, $u^+$ and $u^-$ denote the orthogonal projections from $X$ in $Z$ and in $Y$, respectively. In $X$ let us define the norm

$$||| \cdot ||| : X \rightarrow \mathbb{R}$$

$$u \mapsto |||u||| = \max \left\{ \|u^+\|, \sum_{k=1}^{\infty} \frac{1}{2^k} |(u^-, e_k)| \right\},$$

where $(e_k)$ is a total orthonormal sequence in $Y$. The topology on $X$ generated by $||| \cdot |||$ will be denoted by $\tau$ and all topological notions related to it will include this symbol.

Let $I : X \rightarrow \mathbb{R}$ a functional locally Lipschitz, $I \in Lip_{loc}(X, \mathbb{R})$. We will say a functional $I : X \rightarrow \mathbb{R}$ verifies the condition $(H)$ when:

$$\begin{align*}
(H) & \quad \begin{cases}
\text{If } (u_n) \subset I^{-1}([\alpha, \beta]) \text{ is such that } u_n \rightharpoonup u_0 \text{ in } X, \text{ then there exists } M > 0 \\
\quad \text{such that } \partial I(u_n) \subset B_M(0) \subset X^*, \quad \forall \ n \in \mathbb{N}. \text{ In addition, if } \xi_n \in \partial I(u_n) \\
\quad \text{with } \xi_n \rightharpoonup \xi_0 \text{ in } X^*, \text{ we have } \xi_0 \in \partial I(u_0).
\end{cases}
\end{align*}$$

**Theorem 4.1.** (See [1]) Let $Y$ be a separable closed subspace of a Hilbert space $X$ and $Z = Y^\perp$. If $u \in X$, as in the previous section, $u^+$ and $u^-$ denote the orthogonal projections in $Z$ and $Y$, respectively. Given $\rho > r > 0$ and $z \in Z$ with $||z|| = 1$, we set

$$M = \{u = y + tz ; ||u|| \leq \rho, t \geq 0 \text{ and } y \in Y\}$$

$$M_0 = \{u = y + tz ; y \in Y, ||u|| = \rho \text{ and } t \geq 0 \text{ or } ||u|| \leq \rho \text{ and } t = 0\}$$

$$S = \{u \in Z ; ||u|| = r\}.$$

Assume $I \in Lip_{loc}(X, \mathbb{R})$ such that

$I$ is $\tau$ - upper semicontinuous

and

$$b = \inf \left\{ \sup_{M_0} I, \sup_{M} I < \infty \right\}$$

If $I$ verifies the condition $(H)$, there is $c \in [b, d]$ and a sequence $(u_n) \subset X$ such that

$$I(u_n) \rightarrow c \text{ and } \lambda_I(u_n) \rightarrow 0.$$
Thus, applying [32, Lemma 2.1], \(u_n \to 0\) in \(L^{2^*}(\mathbb{R}^N)\) and by interpolation on the Lebesgue spaces, \(u_n \to 0\) in \(L^s(\mathbb{R}^N)\) for \(2 < s < 2^*\). On the other hand,

(5.1) \[0 < c = I_\gamma(u_n) - \frac{1}{2} \langle w_n, u_n \rangle + o_n(1) = \left(\frac{1}{2} - \frac{1}{2^*}\right) \int_{\mathbb{R}^N} \rho_n u_n + o_n(1),\]

where \(w_n = Q'(u_n) - \rho_n\) with \(\lambda_{I_\gamma}(u_n) = ||w_n||_*\) and \(\rho_n \in \partial \Psi_\gamma(u_n)\).

Since \[\int_{\mathbb{R}^N} \rho_n u_n \leq \gamma ||u_n||^2_0 + ||u_n||^2_{2^*} \to 0,\]

contrary to (5.1).

From this, going to a subsequence if necessary, there exists \(n_0 \in \mathbb{N}\) such that

\[\sup_{y \in \mathbb{R}^N} \int_{B(y,1)} |u_n|^{2^*} \geq \frac{\delta}{2}, \ n \geq n_0.\]

By definition of supreme, there exists \((y_n) \subset \mathbb{R}^N\) such that

\[\int_{B(y_n,1)} |u_n|^{2^*} \geq \frac{\delta}{4}, \ n \geq n_0.\]

Then, there exists \((z_n) \subset \mathbb{Z}^N\) such that

\[\int_{B(z_n,1+\sqrt{N})} |u_n|^{2^*} \geq \frac{\delta}{4}, \ n \geq n_0.\]

Setting \(v_n(x) = u_n(x+z_n)\), we compute

(5.2) \[\int_{B(0,1+\sqrt{N})} |v_n(x)|^{2^*} = \int_{B(z_n,1+\sqrt{N})} |u_n(x)|^{2^*} \geq \frac{\delta}{4}, \ n \geq n_0.\]

Similarly to what was done in [1, Claim 6.12], we have that \((v_n) \subset H^1(\mathbb{R}^N)\) is also a \((PS)_{c_\gamma}\) sequence for \(I_\gamma\). Going to a subsequence, if necessary, let \(v \in H^1(\mathbb{R}^N)\) the weak limit of the sequence \((v_n) \subset H^1(\mathbb{R}^N)\).

Claim 5.2. If \(c_\gamma < \frac{S}{N}\), then \(v \neq 0\).

Suppose by contradiction \(v = 0\) and assume that

\[|\nabla v_n|^2 \rightharpoonup \mu \quad \text{and} \quad |v_n|^{2^*} \rightharpoonup \nu \quad \text{in} \quad \mathcal{M}(\mathbb{R}^N).\]

By Concentration-Compactness Principle II due to Lions [29], there exist a countable set \(J\), \(\nu_j \in \mathbb{R}_+\) and \((x_j) \subset \mathbb{R}^N\) such that

\[\nu = \sum_{j \in J} \nu_j \delta_{x_j}\]

where \(\delta_x\) denotes the mass of Dirac concentrated in \(x \in \mathbb{R}^N\). In addition,

\[\mu \geq S \sum_{j \in J} \nu_j^{\frac{2^*}{2^*}} \delta_{x_j},\]

where

\[S = \inf \{|\nabla u|^2_2; u \in D^{1,2}(\mathbb{R}^N), |u|_{2^*} = 1\}\]

is the best Sobolev constant for immersion of \(D^{1,2}(\mathbb{R}^N)\) in \(L^{2^*}(\mathbb{R}^N)\). We prove that \(\nu_j = 0\) for all \(j \in J\). Indeed, otherwise, suppose there is \(j_0 \in J\) such that

(5.3) \[\nu_{j_0} > 0.\]

For \(\delta > 0\) consider the function \(\varphi_\delta \in C_0^\infty(\mathbb{R}^N)\)
\[ \varphi_\delta(x) = \begin{cases} 1, & |x - x_{j0}| \leq \frac{\delta}{2} \\ 0, & |x - x_{j0}| > \delta. \end{cases} \]

By definition of convergence in the sense of measure theory, we get

- \[ \int_{\mathbb{R}^N} \varphi_\delta |v_n|^2 \to \int_{\mathbb{R}^N} \varphi_\delta d\nu, \text{ as } n \to +\infty. \]
- \[ \int_{\mathbb{R}^N} \varphi_\delta |\nabla v_n|^2 \to \int_{\mathbb{R}^N} \varphi_\delta d\mu, \text{ as } n \to +\infty. \]

Since \( (v_n)_n \) is bounded in \( L^{2^*}(\mathbb{R}^N) \), then \( |v_n|^q \) is bounded in \( L^{\frac{2^*}{q}}(\mathbb{R}^N) \) where \( q \in (2, 2^*) \) and \( v_n(x) \to 0 \text{ a.e in } \mathbb{R}^N \). So,

\[ |v_n|^q \to 0 \text{ in } L^{\frac{2^*}{q}}(\mathbb{R}^N), \]

that is,

\[ (5.4) \quad \int_{\mathbb{R}^N} |v_n|^q \varphi_\delta \to 0. \]

By the inequality of Hölder and the limitation of \( (v_n)_n \) in \( H^1(\mathbb{R}^N) \), we get

\[ \left| \int_{\mathbb{R}^N} v_n \nabla v_n \nabla \varphi_\delta \right| \leq \left( \int_{\mathbb{R}^N} |v_n|^2 |
abla \varphi_\delta|^2 \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} |
abla v_n|^2 \right)^{\frac{1}{2}} \leq C \left( \int_{B_\delta(x_{j0})} |v_n|^2 |
abla \varphi_\delta|^2 \right)^{\frac{1}{2}}. \]

Once \( v_n \to 0 \) in \( L^2_{loc}(\mathbb{R}^N) \), we obtain

\[ \left( \int_{B_\delta(x_{j0})} |v_n|^2 |
abla \varphi_\delta|^2 \right)^{\frac{1}{2}} \to 0, \text{ as } n \to \infty, \]

showing that

\[ (5.5) \quad \int_{\mathbb{R}^N} v_n \nabla v_n \nabla \varphi_\delta \to 0, \text{ as } n \to +\infty. \]

In addition,

\[ (5.6) \quad \int_{\mathbb{R}^N} V(x) \varphi_\delta |v_n|^2 \to 0, \text{ as } n \to +\infty. \]

We still have

\[ (5.7) \quad \int_{\mathbb{R}^N} \rho_n v_n \varphi_\delta \leq \gamma \int_{\mathbb{R}^N} \varphi_\delta |v_n|^q + \int_{\mathbb{R}^N} \varphi_\delta |v_n|^{2^*}. \]

By the fact that \( (v_n) \) is \( (PS)_{c_r} \), there is \( w_n \in \partial I_\gamma(v_n) \) and \( \rho_n \in \partial \Psi(v_n) \) such that

\[ ||w_n|| = \lambda_{I_\gamma}(v_n) = o_n(1) \text{ and } \langle w_n, \phi \rangle = \langle Q'(v_n), \phi \rangle - \langle \rho_n, \phi \rangle, \forall \phi \in H^1(\mathbb{R}^N). \]
By (5.4), (5.5), (5.6) and (5.7)
\[
o_n(1) = \langle w_n, \varphi_\delta v_n \rangle = \int_{\mathbb{R}^N} \nabla v_n \nabla (\varphi_\delta v_n) + \int_{\mathbb{R}^N} V(x)|v_n|^2 \varphi_\delta - \int_{\mathbb{R}^N} \rho_n v_n \varphi_\delta
\]
\[
= \int_{\mathbb{R}^N} |\nabla v_n|^2 \varphi_\delta + \int_{\mathbb{R}^N} v_n \nabla \varphi_\delta \nabla v_n + \int_{\mathbb{R}^N} V(x)|v_n|^2 \varphi_\delta - \int_{\mathbb{R}^N} \rho_n v_n \varphi_\delta
\]
\[
\geq \int_{\mathbb{R}^N} |\nabla v_n|^2 \varphi_\delta + \int_{\mathbb{R}^N} v_n \nabla \varphi_\delta \nabla v_n + \int_{\mathbb{R}^N} V(x)|v_n|^2 \varphi_\delta + \gamma \int_{\mathbb{R}^N} \varphi_\delta |v_n|^q - \int_{\mathbb{R}^N} \varphi_\delta |v_n|^{2^*},
\]
that is,
\[
0 \geq \int_{\mathbb{R}^N} \varphi_\delta d\mu - \int_{\mathbb{R}^N} \varphi_\delta d\nu, \quad \forall \delta > 0.
\]
Crossing the limit when $\delta \to 0$, by the dominated convergence theorem of Lebesgue, we get
\[
\mu(x_{j_0}) \leq \nu(x_{j_0}),
\]
this is,
\[
S_{\nu_{j_0}} \leq \nu_{j_0}.
\]
By (5.3)
\[
(5.8)
\]
Knowing $\rho_n(x) \in \partial_t F_\gamma(x, v_n(x))$
\[
o_n(1) + c_\gamma = I_\gamma(v_n) - \frac{1}{2} \langle w_n, v_n \rangle
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^N} \rho_n v_n - \int_{\mathbb{R}^N} F_\gamma(x, v_n)
\]
\[
= \left( \frac{1}{2} - \frac{1}{2^*} \right) \int_{\mathbb{R}^N} \rho_n v_n \geq \frac{1}{N} \int_{\mathbb{R}^N} |v_n|^{2^*},
\]
this is,
\[
\frac{1}{N} \liminf_n \left( \int_{\mathbb{R}^N} |v_n|^{2^*} \right) \leq c_\gamma.
\]
Since $|v_n|^{2^*} \to \nu$ in $\mathcal{M}^+(\mathbb{R}^N)$ and (5.8), we get
\[
c_\gamma \geq \frac{1}{N} \liminf_n \left( \int_{\mathbb{R}^N} |v_n|^{2^*} \right)
\]
\[
\geq \frac{1}{N} \nu(\mathbb{R}^N)
\]
\[
\geq \frac{1}{N} \nu(\{x_{j_0}\}) = \frac{1}{N} \nu_{j_0} \geq \frac{1}{N} S_{\nu_{j_0}}^N > c_\gamma,
\]
what is absurd.
Therefore, $\nu = 0$ implying in $|v_n|^{2^*} \to 0$ in $\mathcal{M}^+(\mathbb{R}^N)$ and consequently $v_n \to 0$ in $L_{2^*}^{2^*}(\mathbb{R}^N)$ contradicting (5.2).

Now, we are ready to show the estimate from above involving the number $c_\gamma > 0$.

• Case $N \geq 4$.

**Remark 2.** We may assume without loss of generality $V(0) < 0$. So, by continuity of $V : \mathbb{R}^N \to \mathbb{R}$, we can choose $r > 0$ such that $V(x) \leq -\beta < 0$ for $x \in B_r$ and some $\beta > 0$.  

Consider the function
\[ \varphi_\varepsilon(x) := \frac{c_N \psi(x) \varepsilon^{\frac{N-2}{2}}}{(\varepsilon^2 + |x|^2)^{\frac{N-4}{2}}} \]
where \( c_N = (N(N - 2))^{\frac{N-2}{4}} \), \( \varepsilon > 0 \) and \( \psi \in C_0^\infty(\mathbb{R}^N) \) is such that
\[ \psi(x) = 1 \text{ for } |x| \leq \frac{r}{2} \text{ and } \psi(x) = 0 \text{ for } |x| \geq r. \]

We shall need the following asymptotic estimates as \( \varepsilon \to 0^+ \) (see [42]).
\[ ||\nabla \varphi_\varepsilon||^2_2 = S^N_2 + O(\varepsilon^{n-2}), \quad ||\nabla \varphi_\varepsilon||_1 = O(\varepsilon^{\frac{N-2}{2}}), \quad ||\varphi_\varepsilon||^{2^*_N}_2 = S^N_2 + O(\varepsilon^N) \tag{5.9} \]
\[ ||\varphi_\varepsilon||^2_{2^*_N-1} = O(\varepsilon^{\frac{N-2}{2}}), \quad ||\varphi_\varepsilon||_1 = O(\varepsilon^{\frac{N-2}{2}}) \]
and
\[ ||\varphi_\varepsilon||^2_2 = \begin{cases} b \varepsilon^2 |\log(\varepsilon)| + O(\varepsilon^2), & \text{if } N = 4 \\ be^2 + O(\varepsilon^{N-2}), & \text{if } N \geq 5. \end{cases} \tag{5.10} \]
where \( b > 0 \).

**Proposition 5.3.** Suppose \( N \geq 4 \), there is \( \varepsilon_0 > 0 \) such that for all \( \varepsilon \in (0, \varepsilon_0) \) and for all \( O(\varepsilon^{N-2}) \)
\[ \int_{\mathbb{R}^N} \frac{(|\nabla \varphi_\varepsilon|^2 + V(x)\varphi_\varepsilon^2)}{|\varphi_\varepsilon|^{2^*_N}} + O(\varepsilon^{N-2}) < S. \]

**Proof.** In fact, given \( O(\varepsilon^{N-2}) \)
\[ \int_{\mathbb{R}^N} \frac{(|\nabla \varphi_\varepsilon|^2 + V(x)\varphi_\varepsilon^2)}{|\varphi_\varepsilon|^{2^*_N}} + O(\varepsilon^{N-2}) = \frac{S^N_2 + O(\varepsilon^{N-2}) + \int_{\mathbb{R}^N} V(x)\varphi_\varepsilon^2}{[S^N_2 + O(\varepsilon^N)]^{\frac{N-2}{2}}} + O(\varepsilon^{N-2}) \]
\[ = S \left[ 1 + \frac{O(\varepsilon^{N-2})}{S^N_2} + \frac{1}{S^N_2} \int_{\mathbb{R}^N} V(x)\varphi_\varepsilon^2 \right] \]
\[ = S \left[ 1 + O(\varepsilon^{N-2}) + \frac{1}{S^N_2} \int_{\mathbb{R}^N} V(x)\varphi_\varepsilon^2 \right] + O(\varepsilon^{N-2}). \]

On the other hand, by (5.10) and remark 2
\[ \int_{\mathbb{R}^N} V(x)\varphi_\varepsilon^2 \leq -\beta ||\varphi_\varepsilon||^2_2 = \begin{cases} -\beta b \varepsilon^2 |\log(\varepsilon)| + O(\varepsilon^2) & \text{if } N = 4 \\ -\beta b \varepsilon^2 + O(\varepsilon^{N-2}) & \text{if } N \geq 5. \end{cases} \tag{5.11} \]
If $N = 4$, by (5.11), we obtain
\[
\int_{\mathbb{R}^N} \left( |\nabla \phi_\varepsilon|^2 + V(x)\phi_\varepsilon^2 \right) \frac{1}{|\phi_\varepsilon|^2} + O(\varepsilon^2) = S \left[ 1 + O(\varepsilon^2) + \frac{\frac{1}{S^2}}{[1 + O(\varepsilon^4)]^{\frac{1}{2}}} \int_{\mathbb{R}^N} V(x)\phi_\varepsilon^2 \right] + O(\varepsilon^2)
\]
\[
\leq S \left[ 1 + O(\varepsilon^2) - \frac{\frac{1}{S^2}}{[1 + O(\varepsilon^4)]^{\frac{1}{2}}} \beta b \varepsilon^2 |\log(\varepsilon)| + O(\varepsilon^2) \right] + O(\varepsilon^2)
\]
\[
= S \left( \frac{1}{[1 + O(\varepsilon^4)]^{\frac{1}{2}}} \left( \frac{O(\varepsilon^2)}{[1 + O(\varepsilon^4)]^{\frac{1}{2}}} \right) \right) + S \left( \frac{\beta b \varepsilon^2 |\log(\varepsilon)|}{S^2 [1 + O(\varepsilon^4)]^{\frac{1}{2}}} \right) + O(\varepsilon^2).
\]

Note that

1. \( \frac{1}{[1 + O(\varepsilon^4)]^{\frac{1}{2}}} \to 1 \) as \( \varepsilon \to 0^+ \);  
2. \( \frac{O(\varepsilon^2)}{[1 + O(\varepsilon^4)]^{\frac{1}{2}}} = O(\varepsilon^2) \).

Just see that
\[
\frac{O(\varepsilon^2)}{[1 + O(\varepsilon^4)]^{\frac{1}{2}}} = \frac{O(\varepsilon^2)}{[1 + O(\varepsilon^4)]^{\frac{1}{2}}} \cdot \frac{1}{[1 + O(\varepsilon^4)]^{\frac{1}{2}}}
\]
is bounded for \( \varepsilon \approx 0^+ \).

Follow from (1), \( \varepsilon \approx 0^+ \), that
\[
\frac{1}{[1 + O(\varepsilon^4)]^{\frac{1}{2}}} \geq \frac{1}{2} \iff \frac{-\beta d \varepsilon^2}{[1 + O(\varepsilon^4)]^{\frac{1}{2}}} \leq \frac{-\beta d \varepsilon^2}{2},
\]
and with that
\[
\int_{\mathbb{R}^N} \left( |\nabla \phi_\varepsilon|^2 + V(x)\phi_\varepsilon^2 \right) \frac{1}{|\phi_\varepsilon|^2} + O(\varepsilon^2) = S \left( \frac{1}{[1 + O(\varepsilon^4)]^{\frac{1}{2}}} \right) + S \left( \frac{O(\varepsilon^2)}{[1 + O(\varepsilon^4)]^{\frac{1}{2}}} \right) +
\]
\[
- S \left( \frac{\beta b \varepsilon^2 |\log(\varepsilon)|}{S^2 [1 + O(\varepsilon^4)]^{\frac{1}{2}}} \right) + O(\varepsilon^2)
\]
\[
\leq S \left( \frac{1}{[1 + O(\varepsilon^4)]^{\frac{1}{2}}} \right) + O(\varepsilon^2) - S \left( \frac{\beta d \varepsilon^2}{2S^2 |\log(\varepsilon)|} \right) + O(\varepsilon^2)
\]
\[
= S \left( \frac{1}{[1 + O(\varepsilon^4)]^{\frac{1}{2}}} \right) - S \left( \frac{\beta d \varepsilon^2}{2S^2 |\log(\varepsilon)|} \right) + O(\varepsilon^2)
\]

Consider the application
\[
g : [0, O(\varepsilon^4)] \rightarrow \mathbb{R}
\]
\[
t \mapsto g(t) = \frac{1}{[1 + t]^{\frac{1}{2}}}.
\]
By the mean value theorem there is $\theta \in (0, O(\varepsilon^4))$ such that
\[
\frac{1}{[1 + O(\varepsilon^4)]^{\frac{1}{2}}} - 1 = -\frac{1}{2} [1 + \theta]^{-\frac{3}{2}} O(\varepsilon^4),
\]
that is,
\[
\frac{1}{[1 + O(\varepsilon^4)]^{\frac{1}{2}}} = 1 - \frac{1}{2} [1 + \theta]^{-\frac{3}{2}} O(\varepsilon^4) = 1 - O(\varepsilon^4).
\]
Therefore,
\[
\int_{\mathbb{R}^N} \frac{(|\nabla \varphi_\varepsilon|^2 + V(x)\varphi_\varepsilon^2)}{\varphi_\varepsilon^{\frac{2}{2^*}}} + O(\varepsilon^2) \leq S \left( \frac{1}{[1 + O(\varepsilon^4)]^{\frac{1}{2}}} \right) - S \left( \frac{\beta d \varepsilon^2}{2S^2} |\log(\varepsilon)| \right) + O(\varepsilon^2)
\]
\[
= S - O(\varepsilon^4) - \frac{\beta d \varepsilon^2}{2S^2} |\log(\varepsilon)| + O(\varepsilon^2).
\]

**Claim 5.4.** There is $\varepsilon_0 > 0$ such that
\[
O(\varepsilon^4) + O(\varepsilon^2) - \frac{\beta d \varepsilon^2}{2S^2} |\log(\varepsilon)| < 0, \quad \forall \varepsilon \in (0, \varepsilon_0).
\]

Follows from the fact that
\[
O(\varepsilon^4) + O(\varepsilon^2) - \frac{\beta d \varepsilon^2}{2S^2} |\log(\varepsilon)| = \varepsilon^2 \left[ \varepsilon^2 O(\varepsilon^4) + O(\varepsilon^2) - \frac{\beta d}{2S^2} |\log(\varepsilon)| \right]
\]
with the fact that:
\[
\varepsilon^2 \frac{O(\varepsilon^4)}{\varepsilon^4} + \frac{O(\varepsilon^2)}{\varepsilon^2} \text{ is bounded for } \varepsilon \approx 0^+ \quad \text{and} \quad \lim_{\varepsilon \to 0^+} \frac{\beta d}{2S^2} |\log(\varepsilon)| = -\infty.
\]

Therefore, there is $\varepsilon_0 > 0$ such that
\[
\int_{\mathbb{R}^N} \frac{(|\nabla \varphi_\varepsilon|^2 + V(x)\varphi_\varepsilon^2)}{\varphi_\varepsilon^{\frac{2}{2^*}}} + O(\varepsilon^2) < S, \quad \forall \varepsilon \in (0, \varepsilon_0) \quad \text{and for all } O(\varepsilon^2).
\]

The case $N \geq 5$ is analogous. $\square$

**Remark 3.** (1) Since
\[
I_\gamma u = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) - \int_{\mathbb{R}^N} F_\gamma(x, u) \leq \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) - \frac{1}{2^*} \int_{\mathbb{R}^N} |u|^{2^*},
\]
defining
\[
J(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) - \frac{1}{2^*} \int_{\mathbb{R}^N} |u|^{2^*},
\]
we get $I_\gamma u \leq J(u)$ for all $u \in H^1(\mathbb{R}^N)$.

**Proposition 5.5.** For $u \in H^1(\mathbb{R}^N)$, we have:

(i) If $\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) > 0$, then
\[
\max_{t \geq 0} J(tu) = \frac{1}{N} \left( \frac{\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)}{||u||_{2^*}^2} \right)^{\frac{N}{2}}.
\]
(ii) If
\[
\int_{\mathbb{R}^N} [||\nabla u|^2 + V(x)u^2] \leq 0,
\]
then
\[
\max_{t \geq 0} J(tu) = 0.
\]

**Proof.** Given \( u \in H^1(\mathbb{R}^N) \), set the function
\[
h : [0, +\infty) \rightarrow \mathbb{R}
\]
\[
t \mapsto h(t) = J(tu).
\]

**Case (i).**
By the fact
\[
h(t) \to -\infty \text{ as } t \to +\infty \text{ and } h(t) > 0 \text{ for } t \approx 0,
\]
there is \( t_0 \in (0, +\infty) \) such that
\[
\max_{t \geq 0} h(t) = h(t_0).
\]
More precisely
\[
t_0 = \left( \frac{\int_{\mathbb{R}^N} [||\nabla u|^2 + V(x)u^2]}{||u||^2_{2^*}} \right)^{\frac{1}{N-2}}.
\]
Therefore,
\[
h(t_0) = \frac{1}{N} \left( \frac{\int_{\mathbb{R}^N} [||\nabla u|^2 + V(x)u^2]}{||u||^2_{2^*}} \right)^{\frac{N}{N-2}}.
\]

**Case (ii)**
Just notice that
\[
h(t) \leq 0 \text{ for all } t \geq 0 \text{ and } h(0) = 0.
\]

Before continuing we will make some considerations. First we will need the following proposition, whose proof is in [16, Proposition 2.2].

**Proposition 5.6.** Suppose \( V \in L^\infty(\mathbb{R}^N) \) and \((V_1) - (V_2)\), there is \( c_0 > 0 \) such that
\[
||u^-||_{W^{1,\infty}(\mathbb{R}^N)} \leq c_0||u^-||_2, \forall u^- \in E^-.
\]

By the convexity of the application \( t \mapsto ||t||^{2^*} \), Hölder inequality and Proposition 5.6, there is \( c_1 > 0 \) such that
\[
||u||_2^{2^*} \geq ||s\varphi_\varepsilon||_2^{2^*} + 2^* \int_{\mathbb{R}^N} (s\varphi_\varepsilon)^{2^*-1}u^-
\]
\[
\geq ||s\varphi_\varepsilon||_2^{2^*} - c_1||\varphi_\varepsilon||_2^{2^*-1}||u^-||_2
\]
\[
(5.12)
\]
and
\[
\int_{\mathbb{R}^N} (\nabla \varphi_\varepsilon \nabla u^- + V(x)\varphi_\varepsilon u^-) \leq O(\varepsilon^{\frac{N-2}{2}})||u^-||_2.
\]

**Proposition 5.7.** There is \( \varepsilon_0 > 0 \) such that
\[
\sup_{u \in Z_\varepsilon, ||u||_2^* = 1} \int_{\mathbb{R}^N} (||\nabla u|^2 + V(x)u^2) < S, \forall \varepsilon \in (0, \varepsilon_0),
\]
where \( Z_\varepsilon = E^- \oplus \mathbb{R}\varphi_\varepsilon \equiv E^- \oplus \mathbb{R}\varphi_\varepsilon^+ \).
Proof. Let \( u = u^- + s \varphi _\epsilon \) such that \( ||u^- + s \varphi _\epsilon||_{2^*} = 1 \). By (5.9), (5.12), (5.13) and continuous Sobolev embeddings

\[
\int _{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) = \int _{\mathbb{R}^N} (|\nabla (s \varphi _\epsilon)|^2 + V(x)(s \varphi _\epsilon)^2) + 2s \int _{\mathbb{R}^N} (\nabla \varphi _\epsilon \nabla u^- + V(x)\varphi _\epsilon u^-) + ||u^-||^2
\]

that is,

\[
\sup _{u \in Z_\epsilon, ||u||_{2^*} = 1} \int _{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) \leq \frac{\int _{\mathbb{R}^N} (|\nabla \varphi _\epsilon|^2 + V(x)\varphi _\epsilon^2)}{||\varphi _\epsilon||_{2^*}^2} + O(\varepsilon ^{N-2})||u^-||_2 - \bar{c}||u^-||_2^2.
\]

Knowing that

\[
\alpha \cdot \beta \leq \frac{\alpha^2}{2} + \frac{\beta^2}{2}, \forall \alpha, \beta \geq 0,
\]

we get

\[
O(\varepsilon ^{\frac{N-2}{2}})||u^-||_2 = O(\varepsilon ^{\frac{N-2}{2}})(\bar{c})\frac{1}{2}||u^-||_2^2
\]

\[
\leq \frac{O(\varepsilon ^{\frac{N-2}{2}})^2}{2} + \frac{\bar{c}||u^-||_2^2}{2}
\]

\[
= \frac{O(\varepsilon ^{N-2})}{2} + \frac{\bar{c}||u^-||_2^2}{2}.
\]

Therefore, by the Proposition 5.3, there is \( \varepsilon _0 > 0 \) such that

\[
\sup _{u \in Z_\epsilon, ||u||_{2^*} = 1} \int _{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) < S, \forall \varepsilon \in (0, \varepsilon _0).
\]

\( \square \)
For $\varepsilon \in (0, \varepsilon_0)$, by Proposition 5.5 and 5.7, for $u \in Z_\varepsilon$, we conclude
\[
J(u) \leq J(tu) \leq \frac{1}{N} \left( \int_{\mathbb{R}^N} \left[ |\nabla u|^2 + V(x)u^2 \right] \right)^{\frac{N}{2}}
\]
\[
\leq \frac{1}{N} \left( \sup_{w \in Z_\varepsilon, ||w||_{2^*} = 1} \int_{\mathbb{R}^N} \left[ |\nabla w|^2 + V(x)w^2 \right] \right)^{\frac{N}{2}},
\]
this is,
\[
\sup_{u \in Z_\varepsilon} J(u) \leq \frac{1}{N} \left( \sup_{w \in Z_\varepsilon, ||w||_{2^*} = 1} \int_{\mathbb{R}^N} \left[ |\nabla w|^2 + V(x)w^2 \right] \right)^{\frac{N}{2}} < \frac{1}{N} S^{\frac{N}{2}}.
\]
Since $c_\gamma \in [b_\gamma, d_\gamma]$ and
\[
d_\gamma = \sup_{\mathcal{M}} I_\gamma
\]
where
\[
\mathcal{M} = \{ u = u^- + tu^+ ; ||u|| \leq \rho, t \geq 0 \text{ and } u^- \in E^- \}
\]
for some $u^+ \in E^+ \setminus \{0\}$. We get $\mathcal{M} \subset Z_\varepsilon$ for $u^+ = \varphi^+_\varepsilon$ and consequently
\[
d_\gamma \leq \sup_{u \in Z_\varepsilon} J(u) \leq \frac{1}{N} S^{\frac{N}{2}}.
\]

- Case $N = 3$.

**Remark 4.** Consider
\[
F_1(x, t) = \begin{cases} 
0, & \text{if } |t| \leq a \\
\frac{1}{q}|t|^q - \frac{1}{q}a^q, & \text{if } |t| > a.
\end{cases}
\]
Fixed $R > 0$, for $x \in B_R$, we obtain:
If $|u(x)| \leq a$, $F_1(x, u) = 0$ and
\[
\frac{1}{q} \int_{B_R} |u(x)|^q \leq \frac{1}{q}a^q |B_R|.
\]
In case $|u(x)| > a$,
\[
\int_{B_R} F_1(x, u) = \frac{1}{q} \int_{B_R} |u(x)|^q - \frac{1}{q}a^q |B_R|.
\]
So,
\[
F_1(x, u) \geq \frac{1}{q} \int_{B_R} |u(x)|^q - \frac{1}{q}a^q |B_R|.
\]

**Lemma 5.8.** Given $z_0 \in E^+ \setminus \{0\}$ and $s_0 > 0$. Let $\rho > 0$ given in the Lemma ??, there are $K > 0$ and $R > 0$ such that
\[
K ||sz_0||_{L^p(B_R)} \leq ||u^- + sz_0||_{L^p(B_R)},
\]
u $\in E^-$, $s \geq s_0$, $p \in (2, 2^*)$ and $||u^- + sz_0|| \leq \rho$.

**Proof.** Suppose that there are $s_n \geq s_0$, $u^-_n \in Y$ and $R_n \to +\infty$ such that
\[
\left\| \frac{u^-_n + z_0}{s_n} \right\|_{L^p(B_{R_n})} = \frac{||u^-_n + s_n z_0||_{L^p(B_{R_n})}}{s_n} \leq \frac{||z_0||_{L^p(B_{R_n})}}{n}, \forall n \in \mathbb{N}.
\]
So,

\[ u_n \to -z_0 \text{ in } L^p(\mathbb{R}^N). \]

On the other hand,

\[ \left\| \frac{u_n}{s_n} \right\|^2 \leq \frac{\left\| u_n + s_n z_0 \right\|^2}{s_0^2} \leq \frac{\rho^2}{s_0^2}. \]

There is \( w \in E^- \) such that, going to a subsequence if necessary,

\[ \frac{u_n}{s_n} \to w \text{ in } H^1(\mathbb{R}^N), \]

consequently

\[ \left(5.15\right) \frac{u_n}{s_n} \to w \text{ in } L^p_{\text{loc}}(\mathbb{R}^N). \]

By \(5.14\) and \(5.15\) we obtain

\[ w = -z_0 \text{ a.e in } \mathbb{R}^N, \]

that is, \( w = -z_0 \in E^+ \setminus \{0\} \) which contradicts the fact \( w \in E^- \).

\[ \square \]

**Lemma 5.9.** Given \( z_0 \in E^+ \setminus \{0\} \), there is \( s_0 > 0 \) such that

\[ d_\gamma = \sup_\mathcal{M} I_\gamma = \sup_A I_\gamma, \]

where

\[ A = \{ u^- + sz_0 ; \left\| u^- + sz_0 \right\| \leq \rho, u^- \in E^- \text{ and } s \geq s_0 \}, \]

and \( \rho > 0 \) is given in the Lemma 5.8.

**Proof.** By the definition of supreme \((s_n) \subset \left[0, +\infty\right)\) and \((u_n) \subset E^-\) such that \(\left\| u_n + s_n z_0 \right\| \leq \rho\) and

\[ d_\gamma - \frac{1}{n} \leq I_\gamma(u_n + s_n z_0) < \sup_{B_\rho \cap E} I_\gamma = d_\gamma > 0, \quad \forall \ n \in \mathbb{N}. \]

**Claim 5.10.** There is \( s_0 > 0 \) such that \( s_n \geq s_0 \) for all \( n \in \mathbb{N} \).

In fact, suppose that there is \( (s_{n_j}) \subset (s_n) \) such that \( s_{n_j} \to 0 \), then

\[ I_\gamma(u_{n_j}^- + s_{n_j} z_0) \leq \frac{s_{n_j}^2}{2} \left\| z_0 \right\|^2 \to 0. \]

Choosing \( n_{j_0} \in \mathbb{N} \) such that \( \frac{s_{n_j}^2}{2} \left\| z_0 \right\|^2 < \frac{d_\gamma}{2} \), for \( n_j \geq n_{j_0} \), we obtain

\[ d_\gamma - \frac{1}{n_j} < \frac{d_\gamma}{2} \]

what contradicts \(5.16\).

So \( u_n + s_n z_0 \in A \) and

\[ d_\gamma \geq \sup_A I_\gamma \geq I_\gamma(u_n^- + s_n z_0) = \sup_{B_\rho \cap E} I_\gamma + o_n(1) = d_\gamma + o_n(1). \]

\[ \square \]
Lemma 5.11. There is $\gamma > 0$ such that
\[
\sup_{u \in A} I_\gamma(u) < \frac{S^{N}}{N}.
\]
where $A$ is given in the Lemma 5.9. In addition, for this is $\gamma > 0$,
\[
c_\gamma \leq d_\gamma < \frac{S^{N}}{N}.
\]
Proof. Since $I_\gamma(u) \leq J_\gamma(u)$ for all $u \in H^1(\mathbb{R}^N)$, where
\[
J_\gamma(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) - \gamma \int_{\mathbb{R}^N} F_1(x, u),
\]
so just prove the estimate to $J_\gamma$.

Let $u \in A$, by Remark 4 and Lemma 5.8, there are $K > 0$ and $R > 0$ such that
\[
J_\gamma(u) \leq \frac{s^2}{2} ||z_0||^2 - \frac{1}{2} ||u^-||^2 - \gamma \int_{B_R} |u^- + sz_0|^q + \frac{\gamma}{q} a^q |B_R|,
\]
that is,
\[
\sup_{u \in A} I_\gamma(u) \leq \sup_{s \geq s_0} \left( \frac{s^2}{2} ||z_0||^2 - \frac{K \gamma s^q}{q} \int_{B_R} |z_0|^q + \frac{\gamma}{q} a^q |B_R| \right).
\]
Set the function
\[
h : [0, +\infty) \longrightarrow \mathbb{R}
\]
\[
s \mapsto h(s) = \frac{s^2}{2} ||z_0||^2 - \frac{K \gamma s^q}{q} ||z_0||_{L^q(B_R)}^q + \frac{\gamma}{q} a^q |B_R|.
\]
Note that

• $h(s) \rightarrow -\infty$ as $s \rightarrow +\infty$;

• $h(s) > 0$ for $s \approx 0^+$.

Therefore, there is $t_0 \in (0, +\infty)$ such that $h'(t_0) = 0$, this is,
\[
t_0 = \left( \frac{||z_0||^2}{K \gamma ||z_0||_{L^q(B_R)}} \right)^{\frac{1}{q-2}}.
\]
So,
\[
\max_{s \geq 0} h(s) = h(t_0) = \left( \frac{1}{2} - \frac{1}{q} \right) \left( \frac{||z_0||^2}{||z_0||_{L^q(B_R)}^q} \right)^{\frac{q}{q-2}} \left( \frac{1}{K \gamma} \right)^{\frac{2}{q-2}} + \frac{\gamma}{q} a^q |B_R|.
\]
Fixed
\[
\gamma > \left( \frac{1}{2} - \frac{1}{q} \right)^{\frac{q}{2}} \left( \frac{||z_0||^2}{||z_0||_{L^q(B_R)}^q} \right)^{\frac{q}{2}} \left( \frac{1}{K} \right)^{\frac{2}{q-2}} \left( \frac{2N}{S^N} \right)^{\frac{2}{q-2}},
\]
we obtain
\[
\max_{s \geq 0} h(s) = h(t_0) < \frac{S^{N}}{2N} + \frac{\gamma}{q} a^q |B_R|.
\]
Lastly, choosing $a \geq 0$ such that
\[
0 \leq a \leq \left( \frac{qS^{N}}{2\gamma N|B_R|} \right)^{\frac{1}{q-2}}.
\]
we conclude that
\[
\max_{s \geq 0} h(s) = h(t_0) < \frac{S^N_\gamma}{2N} + \frac{S^N_\gamma}{2N} = \frac{S^N_\gamma}{N}.
\]

Now our goal is to prove that
\[
-\Delta v(x) + V(x)v(x) \in \partial F_\gamma(x, v(x)) \ a.e \ in \ \mathbb{R}^N,
\]
where \(v\) is the weak limit of \((v_n)\) in \(H^1(\mathbb{R}^N)\).

From the study above, there exists \((\tilde{w}_n) \subset \partial I_\gamma(v_n)\) such that \(\tilde{w}_n = Q'(v_n) - \tilde{\rho}_n\) and 
\[
\|\tilde{w}_n\|_* = o_n(1) \text{ where } (\tilde{\rho}_n) \subset \partial \Psi_\gamma(v_n).
\]

For \(\phi \in H^1(\mathbb{R}^N)\), we obtain
\[
\langle \tilde{\rho}_n, \phi \rangle = \langle Q'(v_n), \phi \rangle - \langle \tilde{w}_n, \phi \rangle \to \langle Q'(v), \phi \rangle, \text{ as } n \to +\infty,
\]
that is, \(\tilde{\rho}_n \rightharpoonup^* Q'(v) \text{ in } (H^1(\mathbb{R}^N))^*\). Then, by Proposition 3.3, \(Q'(v) \in \partial \Psi_\gamma(v)\). Thereby, 
\[
Q'(v) = \rho \in \partial \Psi_\gamma(v),
\]
and so,
\[
\int_{\mathbb{R}^N} (\nabla v \nabla \phi + Vv\phi) = \int_{\mathbb{R}^N} \rho \phi \quad \text{for all } \phi \in H^1(\mathbb{R}^N),
\]
where \(\rho(x) \in \partial F_\gamma(x, v(x)) \ a.e \ in \ \mathbb{R}^N\). Hence
\[
\begin{aligned}
-\Delta v + V(x)v &= \rho(x) \quad \text{in } \mathbb{R}^N, \\
v &\in H^1(\mathbb{R}^N).
\end{aligned}
\]

Since \(\rho \in L^{2N+2}_{loc}(\mathbb{R}^N)\), the elliptic regularity theory gives that \(v \in W^{2,2N+2}_{loc}(\mathbb{R}^N)\)
\[
-\Delta v + V(x)v = \rho(x) \quad a.e \ in \ \mathbb{R}^N,
\]
that is,
\[
-\Delta v(x) + V(x)v(x) \in \partial F_\gamma(x, v(x)) \ a.e \ in \ \mathbb{R}^N,
\]
finishing the proof of Theorem 1.1.

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