A NOTE ON LIST-COLORING POWERS OF GRAPHS

NICHOLAS KOSAR, SARKA PETRICKOVA, BENJAMIN REINIGER, AND ELYSE YEAGER

Abstract. Recently, Kim and Park have found an infinite family of graphs whose squares are not chromatic-choosable. Xuding Zhu asked whether there is some \( k \) such that all \( k \)th power graphs are chromatic-choosable. We answer this question in the negative.

1. INTRODUCTION

The list-chromatic number of a graph \( G \), denoted \( \chi^\ell(G) \), is the least \( k \) such that for any assignment of lists of size \( k \) to the vertices of \( G \), there is a proper coloring of \( V(G) \) where the color at each vertex is in that vertex’s list. A graph is said to be chromatic-choosable if \( \chi^\ell(G) = \chi(G) \). The \( k \)th power of a graph \( G \), denoted \( G^k \), is the graph on the same vertex set as \( G \) with an edge \( uv \) if and only if the distance from \( u \) to \( v \) in \( G \) is at most \( k \).

The List-Total-Coloring Conjecture (LTCC) asserts that \( \chi^\ell(T(G)) = \chi(T(G)) \) for every graph \( G \), where \( T(G) \) is the total graph of \( G \). The Square List Coloring Conjecture (SLCC) was introduced in [5], as it would imply the LTCC. The SLCC asserts that squares of graphs are chromatic-choosable. However, the SLCC was recently disproved by Kim and Park [4]. Xuding Zhu asked whether there is any \( k \) such that all \( k \)th powers are chromatic-choosable [7]. In this note we give a negative answer to Zhu’s question.

Theorem 3.4. There is a constant \( c \) such that for every \( k \in \mathbb{N} \), there is an infinite family of graphs \( G \) such that

\[
\chi^\ell(G^k) \geq c \chi(G^k) \log(\chi(G^k)).
\]

While preparing this note, it has come to our attention that Kim, Kwon, and Park have arrived at a similar result [3]. We briefly compare results at the end of this note.

2. CONSTRUCTION

The example of Kim and Park [4] for \( k = 2 \) is based on complete sets of mutually orthogonal latin squares. We will use this structure to find examples for all \( k \), but we find the language of affine planes to be more convenient.

Take an affine plane \((\mathcal{P}, \mathcal{L})\) on \( n^2 \) points. Let \( \{L_0, L_1, \ldots, L_n\} \) be the decomposition of \( \mathcal{L} \) into parallel classes. Recall that we call the elements of \( \mathcal{P} \) the points and the elements of \( \mathcal{L} \) the lines of the plane, and that we have the following properties (see for instance [2]):

[1]

Mathematics Dept., University of Illinois, Urbana-Champaign
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Each line is a set of $n$ points.

- For each pair of points, there is a unique line containing them.
- Two lines in the same parallel class do not intersect.
- Two lines in different parallel classes intersect in exactly one point.
- Such a plane exists whenever $n$ is a (positive) power of a prime.

Form the bipartite graph $H$ with parts $\mathcal{P}$ and $B = \mathcal{L} - L_0$, with $p\ell \in E(H)$ if and only if $p \in \ell$. Let $a_1, \ldots, a_n$ denote the lines of $L_0$. Consider the refinement $\mathcal{V}'$ of the bipartition of $H$ obtained by partitioning $\mathcal{P}$ into $a_1, \ldots, a_n$ and $B$ into $L_1, \ldots, L_n$. Note that $H[a_i, L_j]$ is a matching for each $i$ and $j$. In Figure 2, the graph $H$ is shown with $n = 3$. Edges are drawn differently according to which parallel class their line-endpoint belongs to, and the parts of $\mathcal{V}'$ are indicated.

Let $k \geq 2$. Subdivide the edges of $H$ into paths of different lengths: edges incident to $L_1$ are subdivided into paths of length $k$, while edges not incident to $L_1$ are subdivided into paths of length $k + 1$. For an edge $p\ell \in E(H)$, denote the vertices along the subdivision path as $p = (p\ell)_0, (p\ell)_1, (p\ell)_2, \ldots$. If $\ell \in L_1$, then $(p\ell)_k = \ell$, and if $\ell \notin L_1$, then $(p\ell)_{k+1} = \ell$. For a vertex $(p\ell)_i$, say its level is $i$, its point is $p$, and its line is $\ell$ (levels are well-defined, and points and lines of vertices of degree 2 are well-defined). Form the graph $G$ by, for each $\ell \in \bigcup_{2 \leq i \leq n} L_i$, adding edges to make the neighborhood of $\ell$ a clique and then deleting $\ell$. For each $i, j \in [n]$ and $m \in \{0, \ldots, k\}$, let $V_{i,j,m} = \{(p\ell)_m : p\ell \in E(H), p \in a_i, \ell \in L_j\}$; then $\{V_{i,j,m} : i,j \in [n], m \in \{0, \ldots, k\}\}$ is a partition of $V(G)$ into sets of size $n$, which we call $\mathcal{V}$. In Figure 2, the graph $G$ is shown. Again we use $n = 3$, and here the parts of $\mathcal{V}$ are indicated.

3. PROOF

**Claim 3.1.** $G^{4k}$ is multipartite on $\mathcal{V}$.

**Proof.** Let $p$ and $q$ be two points in some $a_i$. Any path from $p$ to $q$ must start by increasing levels, arriving at $(p\ell)_k$. If $\ell \notin L_1$, then the path must move from $(p\ell)_k$ to $(p'\ell)_k$ for some $p'$ not on $a_i$. Continuing along the path to level 0, we arrive at $p'$. Since $p'$ is not on $a_i$, $p'$ and $q$ are on a common line $\ell' \in \bigcup_{i=1}^n L_i$. If $\ell' \in L_1$, the shortest path from $p'$ to $q$ is to increase levels to $\ell'$ and decrease levels to $q$. If $\ell' \in \bigcup_{i=2}^n L_i$, the shortest path from $p'$ to $q$ is to increase levels to $(p'\ell')_k$, move over to $(q\ell')_k$, and then decrease levels to $q$. Notice, if $p$ and $p'$ are on a common
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Figure 2. The graph $G$ when $n = 3$.  

Let $\ell_1, \ell_2 \in L_1$. Any path would have to have both ends decrease to level 0. If both $\ell_1$ and $\ell_2$ connect to points in some $a_i$, then since these vertices are a distance at least $4k+1$ apart, the path between $\ell_1$ and $\ell_2$ would have length at least $4k+1$. Otherwise, the paths from $\ell_1$ and $\ell_2$ arrive at points on different lines in $L_0$, say $p$ and $q$, respectively. These two points are on a common line not in $L_0$ or $L_1$, say $\ell$. The shortest path between $p$ and $q$ is to go from $p$ to $(p\ell)_k$, over to $(q\ell)_k$, and finally to $q$. However, this results in a path between $\ell_1$ and $\ell_2$ of length at least $4k+1$.

Let $(p\ell_1)_k, (q\ell_2)_k$ be two vertices in the same part other than $L_1$; that is, $p, q$ are both on some $a_i$ and $\ell_1, \ell_2$ are two lines in the same parallel line class. If a path joining them starts by decreasing levels from both ends to level 0, that is connects $(p\ell_1)_k$ to $p$ and $(q\ell_2)_k$ to $q$, then since $p$ and $q$ are a distance at least $4m+1$ apart, the path between $(p\ell_1)_k$ and $(q\ell_2)_k$ would have length at least $4m+1$. Otherwise, at least one of $(p\ell_1)_k$ or $(q\ell_2)_k$ must first go to $(p'\ell_1)_k$ or $(q'\ell_2)_k$. Without loss of generality connect $(p\ell_1)_k$ to $(p'\ell_1)_k$. Now, any path must connect $(p'\ell_1)_k$ to $p'$ and $(q\ell_2)_k$ to $q$. These are on a common line not in $L_0$, however, increasing levels from each of $p'$ and $q$ to level $k$ results in a total of at least $4k+1$ steps.

Now consider two degree-two vertices in the same part. Any path joining them has ends that either increase or decrease levels from the endpoint. If the path increases levels from both ends or decreases levels from both ends, then we arrive at different vertices in the same level 0 or level $k$ part. Since the rest of the path must have length at least $4m+1$, the total path must have length at least $4k+1$. Otherwise, one end increases levels and the other decreases levels. The resulting point, $p$, is not on the resulting line, $\ell$. The path must next increase levels from $p$
to a line. If this line is in the same parallel line class as $\ell$, then the resultant path has length over $4k + 1$. Otherwise, since this line is not in the same class as $\ell$, these two lines share a common point. The shortest completion of the path is through this point. However, since at least one of these lines is not in $L_1$, the path must contain at least 3 vertices in level $k$. Thus, the path has length at least $4k + 1$. \qed

**Claim 3.2.** The subgraph of $G^{4k}$ induced by the vertices in levels 0 through $k - 1$ is complete multipartite on the parts of $\mathcal{V}$ restricted to those levels.

**Proof.** Consider two points $p, q$ on different lines in $L_0$. They are on a common line $\ell \in \bigcup_{i=1}^n L_i$. If $\ell \in L_1$, connect $p$ to $\ell$ then $\ell$ to $q$. If $\ell \notin L_1$, connect $p$ to $(p\ell)_k$ to $(q\ell)_k$ to $q$. In each case the path has length at most $2k - 1 < 4k$.

Consider two vertices in different parts at level $i$, $1 \leq i < k - 1$. Either their points are on different lines in $L_0$ or their lines are from different parallel classes. If their points are from different lines in $L_0$, go to these points. These points share a common line not in $L_0$. Connect via the path between this line. This takes at most $2i + 2k + 1 \leq 4k - 1$ steps. If their lines are from different parallel classes, increase levels to level $k$. These two lines share a common point. By, if necessary, first changing vertices at level $k$, connecting through this point, we get a path of length at most $2(k - i) + 2 + 2k = 4k - 2i + 2 \leq 4k$.

Finally, consider two vertices in levels $i$ and $j$, $0 \leq i < j < k$. Start a path joining them by decreasing levels from the lower-level vertex, and increasing levels from the larger-level vertex. Let the point we arrive at from decreasing the lower-level vertex be $p$. If the increasing from the larger-level vertex takes us to a line in $L_1$, we can connect from this line to a point on a different line of $L_0$ than $p$, say $q$. Now $p$ and $q$ are on a common line not in $L_0$. Connecting through this gives us a path of length at most $k - 1 + k + 2k + 1 = 4k$. If instead the increasing from the larger-level vertex takes us to a vertex of the form $(q\ell)_k$, $\ell \notin L_1$, then let $\ell'$ be the line through $p$ in $L_1$. $\ell$ and $\ell'$ intersect at a point, say $q'$. We can complete the path by going from $(q\ell)_k$ to $(q'\ell)_k$ to $q'$ to $\ell'$ to $p$. This takes a total of at most $k - 1 + 1 + 3k = 4k$ steps. \qed

We will use the following result of Alon.

**Lemma 3.3.** Let $K_{r,s}$ denotes the complete $r$-partite graph with each part of size $s$. There are two constants, $d_1$ and $d_2$, such that

$$d_1 r \log s \leq \chi(\mathcal{K}_{r,s}) \leq d_2 r \log s.$$  

Everything is now in place to complete the proof.

**Theorem 3.4.** There is a constant $c$ such that for every $k \in \mathbb{N}$, there is an infinite family of graphs $G^*$ such that

$$\chi(\mathcal{G}^k) \geq c \chi(\mathcal{G}^k) \log(\chi(\mathcal{G}^k)).$$

**Proof.** Since $G^{4k}$ is multipartite on $kn^2 + 1$ parts, $\chi(G^{4k}) \leq kn^2 + 1$, and so $n \geq \sqrt{\chi(G^{4k}) - 1}/k$. 

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Since $G^{4k}$ contains a complete multipartite subgraph with $(k - 1)n^2$ parts of size $n$, we have from Lemma 3.3 that
\[
\chi_\ell(G^{4k}) \geq d_1 \left( \frac{k - 1}{k} \right) (\chi(G^{4k}) - 1) \log \left( \frac{\chi(G^{4k}) - 1}{k} \right)
\]
\[
= d_1 \left( \frac{k - 1}{k} \right) (\chi(G^{4k}) - 1) \left( (\chi(G^{4k}) - 1) - \log k \right)
\]
\[
\geq d_1 \left( \frac{k - 1}{k} \right) (\chi(G^{4k}) - 1) \left( (\chi(G^{4k}) - 1) - \log k \right).
\]
Taking $n$ large enough makes $\chi(G^{4k})$ as large as we like, and so by taking a constant $c$ just smaller than $d_1/4$ and taking $n$ sufficiently large, we obtain
\[
\chi_\ell(G^{4k}) \geq c \chi(G^{4k}) \log \chi(G^{4k}).
\]
The family $\{G_n^4\}$ is an infinite family of graphs whose $k$th powers satisfy the desired inequality. □

4. Remarks

Using similar constructions, we have found infinite families of graphs $G$ whose $k$th powers are complete multipartite on roughly $kn^2/4$ parts each of size $n$, but only when $k \not\equiv 0 \mod 4$. The construction presented here is messier and does not yield complete multipartite powers, but it proves the theorem for all values of $k$ simultaneously.

While preparing this note, it has come to our attention that another team has arrived at a similar result. Kim, Kwon, and Park [3] have found, for each $k$, an infinite family of graphs $G$ whose $k$th powers satisfy $\chi_\ell(G^{k}) \geq \frac{10}{9} \chi(G^{k}) - 1$.

Noel [6] asked whether there is an $f(x) = o(x^2)$ such that $\chi_\ell(G^{k}) \leq f(\chi(G^{k}))$ for all $G$. The example of Kim and Park shows that such an $f$ must satisfy $f(x) = \Omega(x \log x)$. We may ask the same question for larger $k$:

**Question 4.1.** Is there an $f_k(x) = o(x^2)$ such that $\chi_\ell(G^{k}) \leq f(\chi(G^{k}))$ for all $G$?

The present examples show that such an $f_k$ must satisfy $f_k(x) = \Omega(x \log x)$.

**Question 4.2.** Fix $k \geq 2$. Is there a constant $c_k$ such that $\chi_\ell(G^{k}) \leq c_k \chi(G^{k}) \log(\chi(G^{k}))$ for every graph $G$? If so, can such $c_k$ be found independent of $k$?

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