BOUNDED CONTRACTIONS FOR AFFINE BUILDINGS

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Abstract. We consider affine buildings with refined chamber structure. For each vertex \( x \) we construct a contraction, based at \( x \), that is used to prove exactness of Schneider-Stuhler resolutions of arbitrary depth.

1. Introduction

Let \( k \) be a local non-archimedean field and \( G \) the group of \( k \)-points of an algebraic reductive group defined over \( k \). Let \( X \) be the Bruhat-Tits building attached to \( G \), which we view as a \( CAT(0) \) metric space. In [MP2], to every \( x \in X \) and non-negative real number \( r \), Moy and Prasad attach a subgroup \( G_{x,r} \) such that \( G_x := G_{x,0} \) is the parahoric subgroup of \( G \) attached to \( x \), \( G_{x,s} \subseteq G_{x,r} \) whenever \( r \leq s \), and \( G_{x,r} \) are normal subgroups of \( G_x \). For every \( r \geq 0 \) we let \( G_{x,r,+} = \bigcup_{s > r} G_{x,s} \). The building \( X \) has a (standard) structure of a chamber complex such that, for every facet \( \sigma \) and \( r \) an integer, the function \( x \mapsto G_{x,r,+} \) is constant for \( x \) in the interior of \( \sigma \). Let \( r \) be a rational number. Then the chamber structure can be refined so that \( x \mapsto G_{x,r,+} \) is constant on the interior points of each facet [BKV]. For example, if \( G = SL_2(k) \), the building \( X \) is a tree and if we divide each edge into two, of equal lengths, then \( x \mapsto G_{x,r,+} \) is constant on the interior points of each edge if \( r \) is half-integral. Fix \( r \) and a refined chamber structure on \( X \). Let \( C_i(X) \) be the free abelian group with the basis consisting of the \( i \)-dimensional facets of \( X \). Let \( C_{-1}(X) = \mathbb{Z} \). Let \( x \in X \) be a vertex. Recall that a contraction \( c \) of \( C_*(X) \) based at \( x \) is a sequence of homomorphisms \( c_i : C_i(X) \to C_{i+1}(X) \), \( i = -1, 0, 1, \ldots \), such that \( c_{-1}(1) = x \) and \( c_{-1} \partial + \partial c_i = 1 \). Our main technical result is the construction of a contraction of \( \sigma \mapsto c(\sigma) = \sum c(\sigma, \tau) \tau \) with the following properties:

1. the contraction \( c \) is \( G_x \)-equivariant.
2. if \( c(\sigma, \tau) \neq 0 \) then \( \tau \) is contained in the smallest subcomplex of \( X \) containing the cone with the vertex \( x \) and the base \( \sigma \).
3. the coefficients \( c(\sigma, \tau) \) are uniformly bounded.

A more detailed meaning of (2) is the following. The cone is the union of all geodesic segments connecting \( x \) to a point in \( \sigma \). If \( c(\sigma, \tau) \neq 0 \) then there exists a point \( y \in \sigma \) and a point \( z \) on the geodesic \([x, y]\) such that \( z \) is either an interior point of \( \tau \) or \( \tau \) is in the boundary of a facet containing \( z \) as an interior point.

Let \( V \) be a smooth representation of \( G \) of depth \( r \). Following Schneider-Stuhler [SS] we define a projective resolution of \( V \) using the chain complex \( C_*(X) \). Since we use a refined

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chamber structure on $X$ the projective modules in the resolution have the same depth $r$. The exactness of resolution is a simple consequence of the existence of the contraction $c$ and niceness of Moy-Prasad groups $G_{x,r}$ if $r > 0$. An open compact subgroup $K$ of $G$ is nice if, for any smooth representation $V$ of $G$ generated by its $K$-fixed vectors, any subquotient of $V$ is generated by its $K$-fixed vectors. It is recorded and used in the literature that $G_{x,r}$ is nice if $x$ is a special point in the building. Niceness of all positive depth Moy-Prasad groups is possibly known to experts, however, we have not found a precise reference in the literature and hence have included a proof. We thank C. Bushnell and G. Henniart for a discussion on this matter. The property (3) is not used in this paper, however, it is critical to prove that the resolution stays exact after passing to a Schwartz completion, see [OS] for details.

2. Lipschitz simplicial approximation

We fix a cell structure $A$ on $\mathbb{R}^n$ so that each (closed) cell $\sigma$ is a convex polyhedron, and when $\tau \subset \sigma$ are cells in $A$ then $\tau$ is a polyhedral face of $\sigma$. For simplicity, we will assume that $A$ is invariant under a discrete group of translations of rank $n$, although the arguments below apply when $A$ has bounded combinatorics and contains only finitely many isometry types of cells. We also fix an orientation of each cell. By $A^i$ we denote the $i$-skeleton of $A$. Consider the augmented cellular chain complex $C_\ast(A)$ of $A$ where $C_i(A)$ is the free abelian group with the basis consisting of the $i$-cells of $A$, and $C_{-1}(A) = \mathbb{Z}$. The boundary morphisms are defined by

$$\partial \sigma = \sum_{\tau} \epsilon(\sigma, \tau) \tau$$

where the sum runs over codimension 1 faces $\tau$ of $\sigma$ and $\epsilon(\sigma, \tau) = 1$ if the orientation of $\sigma$ is obtained from the orientation of $\tau$ by appending an outward normal to $\tau$, and otherwise $\epsilon(\sigma, \tau) = -1$.

We fix a vertex $v_0$ of $A^0$. Without loss of generality we can assume that $v_0 = 0$. Recall that a contraction $c$ of $C_\ast(A)$ based at $v_0$ is a sequence of homomorphisms $c_i : C_i(A) \rightarrow C_{i+1}(A)$, $i = -1, 0, 1, \cdots$, where $c_{-1}(1) = v_0$, such that $c_{i-1} \partial + \partial c_i = 1$. The goal of this section is to construct $\sigma \mapsto c(\sigma) = \sum c(\sigma, \tau) \tau$ with the coefficients $c(\sigma, \tau)$ uniformly bounded and $c(\sigma, \tau) \neq 0$ only if $\tau$ is contained in every convex subcomplex of $A$ containing $v_0$ and $\sigma$.

The standard method of deforming a map $f : \mathbb{R}^k \rightarrow \mathbb{R}^n$ so it misses a given cell $\sigma$ of dimension $> k$ is to radially project to $\partial \sigma$ from a point $x_\sigma$ in the interior of $\sigma$ and in the complement of the image of $f$. For the purposes of controlling the coefficients $c(\sigma, \tau)$ in the contraction we need to control the Lipschitz constant, which will dramatically increase if $x_\sigma$ is close to the image of $f$.

Proposition 2.1. For every $N, L > 0$ there is $L' > 0$ with the following properties. Let $D \subset \mathbb{R}^k$ be a polyhedral cell and $f : D \rightarrow \mathbb{R}^n$ a map such that

- $f$ is $L$-Lipschitz,
- $f(\partial D) \subset A^{k-1}$,
- $f(D) \cap \sigma$ is contained in the union of $N$ $k$-planes for every cell $\sigma$ of $A$. 

Then there is a map $g : D \to \mathbb{R}^n$ such that

(i) $g = f$ on $\partial D$, and on $f^{-1}(A^{k-1})$,
(ii) $g$ is $L'$-Lipschitz,
(iii) for every $x \in D$, $g(x)$ belongs to the closure of the open cell of $A$ that contains $f(x)$,
(iv) $g(D) \subset A^k$.

Proof. We inductively construct maps $f = g_n, g_{n-1}, \ldots, g_k = g : D \to \mathbb{R}^n$ satisfying (i), (iv), (modified) (ii)-(iii): $g_i(D) \subset A^i$, $g_i$ is $L_i$-Lipschitz, and in addition $g_i(D) \cap \sigma$ is contained in the union of $N_i$ $i$-planes. The construction from $g_i$ to $g_{i-1}$ is as follows.

By volume considerations, there is $\epsilon > 0$ that depends only on $N_i$ and $A$, so that for every $i$-cell $\sigma$ there is a point $x_\sigma \in \bar{\sigma}$ at distance $> \epsilon$ from $g_i(D) \cap \sigma$ and also distance $> \epsilon$ from $\partial \sigma$. Compose $g_i$ with the map which is the radial projection from $x_\sigma$ to $\partial \sigma$ on $\sigma$ for each $i$-simplex $\sigma$. This composition is $g_{i-1}$. The Lipschitz constant $L_{i-1}$ of $g_{i-1}$ is uniformly bounded, more precisely,

$$L_{i-1} \leq \left(\frac{d}{\epsilon}\right)^2 L_i$$

where $d$ is a bound on the diameter of all simplices, and $N_{i-1} \leq N_i M_i P_i$, where $M_i$ is the maximal number of codimension 1 faces of a cell in $A$ and $P_i$ is the maximal number of cells intersecting a given cell.

**Proposition 2.2.** Let $v_0 = 0$ be a vertex of $A$. For every cell $\sigma$ of $A$ there is a homotopy $H_\sigma : \sigma \times [0, D_\sigma] \to \mathbb{R}^n$, with $D_\sigma = \text{diam}(\sigma \cup \{v_0\})$, so that the following holds.

1. $H_\sigma(x, 0) = v_0$ for every $x \in \sigma$,
2. $H_\sigma(., 1)$ is the inclusion,
3. the image of $H_\sigma$ is contained in $A^{\text{dim}(\sigma)+1}$, and it is also contained in every convex subcomplex of $A$ that contains $v_0$ and $\sigma$,
4. the restriction of $H_\sigma$ to $\tau \times [0, D_\sigma]$ for a face $\tau < \sigma$ is the linear reparametrization of $H_\tau$,
5. $H_\sigma$ is uniformly Lipschitz,
6. $H_\sigma$ is uniformly bounded distance away from the straight line homotopy $(x, t) \mapsto tx/D_\sigma$.

In the proof we will need the following lemma.

**Lemma 2.3.** For every $L_0, \epsilon, D, \alpha_0 > 0$ there exists $L' > 0$ such that the following holds.

Let $\sigma \subset \mathbb{R}^k$ be a convex polyhedral cell, $v \in \mathbb{R}^k$, $f : \sigma \to \mathbb{R}^n$, $w \in \mathbb{R}^n$ such that:

(a) $f$ is $L$-Lipschitz,
(b) $|v - x| > \epsilon$ for every $x \in \sigma$,
(c) $|w - y| \leq D$ for every $y \in f(\sigma)$,
(d) $\angle_y(x, v) > \alpha_0$ for every $x, y \in \sigma$, $x \neq y$.

Then the map $F : v * \sigma \to \mathbb{R}^n$ defined on the cone by

$$F((1-t)v + tx) = (1-t)w + tf(x)$$

is $L'$-Lipschitz.
Proof. We may assume $v = w = 0$. The extension $F$ is clearly $L$-Lipschitz when restricted to each slice $t \sigma$ by (a), and by (b) and (c) $F$ is also uniformly Lipschitz when restricted to each radial line \{tx | t \in [0, 1]\} for any $x \in \sigma$. Any two points in the cone $v \ast \sigma$ can be connected by a segment in a slice followed by a segment in a radial line. By (d) the angle between these segments is bounded below, so the total length is bounded by a fixed multiple of the distance between the two points, implying the result. \hfill \Box

Proof of Proposition 2.4. The construction is by induction on $\dim(\sigma)$. When $\sigma$ is a vertex apply Proposition 2.1 to the geodesic $f$ joining $v_0$ to $\sigma$. This gives $g = h_{\sigma} : [0, D_{\sigma}] \to \mathbb{R}^n$. Assume now that $\dim(\sigma) = k$ and $H_{\tau}$ has been constructed for all faces $\tau < \sigma$. Define $f : \partial(\sigma \times [0, D_{\sigma}]) \to \mathbb{R}^n$ to be constant $v_0 = 0$ on $\sigma \times \{0\}$, inclusion on $\sigma \times \{D_{\sigma}\}$, and linearly reparametrized $H_{\tau}$ on $\tau \times [0, D_{\sigma}]$, for every face $\tau < \sigma$. We now wish to extend $f$ to all of $\sigma \times [0, D_{\sigma}]$ and apply Proposition 2.1 to this extension. If we extend by coning off from a point in the interior of $\sigma \times [0, D_{\sigma}]$, the Lipschitz constant might blow up since the angles as in Lemma 2.3(d) will be small when $D_{\sigma}$ is large. Instead, we first subdivide $[0, D_{\sigma}]$ into $0 = t_0 < t_1 < \cdots < t_{\sigma} = D_{\sigma}$ so that the length $t_i - t_{i-1}$ of each segment belongs to a fixed interval $[B_1, B_2]$ with $B_1 > 0$. Then we use Lemma 2.3 to extend $f$ to $\sigma \times \{t_i\}$ for each $i$. For this we use $v_i = (x_{\sigma}, t_i)$ for a point $x_{\sigma}$ in the interior of $\sigma$, and we use the same point for this isometry type. The image $f(\partial(\sigma) \times \{t_i\})$ has uniformly bounded diameter, and we can set $w_i \in \mathbb{R}^n$ to be any point in this image. Because of (5) and (6), the assumptions of Lemma 2.3 are satisfied and we get uniformly Lipschitz extensions of $f$. It remains to extend $f$ to each $\sigma \times [t_{i-1}, t_i]$ this is done in exactly the same way, by coning from $(x_{\sigma}, t_i)$ with respect to a point in the image of $\partial(\sigma \times [t_{i-1}, t_i])$. Thus the extension $\sigma \times [0, D_{\sigma}] \to \mathbb{R}^n$ is uniformly Lipschitz. The third bullet in Proposition 2.1 holds since only a bounded number of images of $\sigma \times [t_{i-1}, t_i]$ intersect a given cell. Thus we can apply Proposition 2.1 to this extension to get the desired map $H_{\sigma}$. \hfill \Box

When $\sigma$ is a $k$-cell and $\tau$ a $(k+1)$-cell (both oriented) we denote by $c(\sigma, \tau)$ the degree of the map

$$
(\sigma \times [0, D_{\sigma}], \partial(\sigma \times [0, D_{\sigma}])) \xrightarrow{H_{\sigma}} (A^{k+1}, A^{k+1} \setminus \partial \tau)
$$

(the latter set is by excision equivalent to $(\tau, \partial \tau)$).

Proposition 2.4. The numbers $c(\sigma, \tau) \in \mathbb{Z}$ are uniformly bounded.

Proof. A warmup is the fact that any map $S^n \to S^n$ of large degree must have a large Lipschitz constant. This follows from Arzela-Ascoli. The set of maps $S^n \to S^n$ with Lipschitz constants bounded above is compact and hence represents finitely many homotopy classes.

The proof in our relative case is similar. Suppose there is a sequence of pairs $(\sigma_i, \tau_i)$ with $|c(\sigma_i, \tau_i)| \to \infty$. We may assume all $\sigma_i$ are isometric and have dimension $k$, and all $\tau_i$ are isometric. By (5) the preimage of $\tau_i$ under $H_{\sigma_i}$ is contained in $\sigma_i \times [u_i, u_i + C]$ for some fixed $C$ independent of $i$. The image of $\sigma_i \times [u_i, u_i + C]$ is contained in a subcomplex $Y_i \subset A^{k+1}$ of $\tau_i$ and up to isomorphism there are only finitely many possibilities.

After translating the interval and identifying all $\sigma_i$ and all $\tau_i$ and the $Y_i$ as above we have a sequence of maps $\sigma \times [0, C] \to Y$, and these all have uniformly bounded Lipschitz
constants. By Arzela-Ascoli after a subsequence they will be close to each other and will determine the same degree. Contradiction.

Now \( \sigma \to c(\sigma) = -\sum c(\sigma, \tau) \tau \) defines a contraction.

Assume that \( X \) is a building corresponding to a reductive group \( G \) over a \( p \)-adic field \( k \), with a refined chamber structure i.e. we have divided chambers into smaller ones in a \( G \)-equivariant fashion. Let \( x \) be a vertex in \( G \) and \( G_x \) the (largest) parahoric group fixing \( x \). For every facet \( \sigma \subset X \) the cone with the vertex \( x \) and the base \( \sigma \) is contained in any apartment \( A \) containing \( x \) and \( \sigma \). Thus the contraction \( c(\sigma) \) can be defined working in any such apartment. If \( g \in G_x \) stabilizes \( \sigma \) then \( g \) fixes \( \sigma \) point-wise, hence also the cone and \( c(\sigma) \). Hence \( c \) can be extended to whole \( G_x \)-orbit of \( \sigma \) in an \( G_x \)-equivariant fashion, and we have proved:

**Proposition 2.5.** For every vertex \( x \) in \( X \) there exists a \( G_x \)-invariant contraction \( c \) satisfying the conditions (1)-(3) in Introduction.

3. Moy-Prasad groups

Let \( k \) be a non-archimedean local field and \( G \) the group of \( k \) points of a reductive algebraic group defined over \( k \). Let \( S \) be a maximal \( k \)-split torus in \( G \) and \( \Phi = \Phi(G, S) \) the corresponding restricted root system. Let \( T \) be the centralizer of \( S \) in \( G \). A decomposition \( \Phi = \Phi^+ \cup \Phi^- \) of the root system into positive and negative roots defines a pair of maximal unipotent subgroups \( U \) and \( \bar{U} \) of \( G \). Let \( A(S) \) be the apartment in the building \( X \) of \( G \) stabilized by \( S \). Let \( x \in A(S) \) and \( r > 0 \). The Moy-Prasad group \( G_{x,r} \) has an Iwahori decomposition ([MP2] Theorem 4.2)

\[
G_{x,r} = \bar{U}_{x,r} T_r U_{x,r}
\]

where the three factors are the intersection of \( G_{x,r} \) with \( \bar{U} \), \( T \) and \( U \), respectively. The factor \( T_r \) is independent of \( x \) as indicated.

**Lemma 3.1.** Let \( x, y \in X \) and \( z \) be a point on the geodesic connecting \( x \) and \( y \). If \( r > 0 \) then

\[
G_{z,r} \subseteq G_{x,r} G_{y,r}.
\]

**Proof.** Without loss of generality we can assume that \( x, y \in A(S) \). The apartment \( A(S) \) is an affine space so \( v = x - y \) is a well defined vector in the space of translations of \( A(S) \). Elements of \( \Phi \) (roots) are functionals on the space of translations of \( A(S) \). Thus \( \alpha(v) \) is a well defined real number for every \( \alpha \in \Phi \). Pick a decomposition \( \Phi = \Phi^+ \cup \Phi^- \) such that \( \alpha(v) \geq 0 \) for all \( \alpha \in \Phi^+ \). Then

\[
\bar{U}_{x,r} \supseteq \bar{U}_{z,r} \quad \text{and} \quad U_{y,r} \supseteq U_{z,r}.
\]

This inclusions are a direct consequence of the definition of \( G_{x,r} \) [MP1] if \( G \) is quasi-split. The general case is then deduced by checking it over a finite unramified extension of \( k \), over which \( G \) is quasi-split, and then taking fixed points for the Galois action. Lemma follows from the Iwahori decomposition of \( G_{z,r} \). □
The Iwahori decomposition and Lemma 3.1 hold for the Groups $G_{x,r,+}$ (for $r \geq 0$) since $G_{x,r,+} = G_{x,s}$ for all $s > r$ and sufficiently close to $r$. Now fix a non-negative rational number $r$. Refine the chamber decomposition of $X$ so that the function $x \mapsto G_{x,r,+}$ is constant on interiors of facets. Thus, for any facet $\sigma$ we define $K_\sigma = G_{x,r,+}$ where $x$ is any interior point of $\sigma$. If $\tau$ is a facet in the boundary of $\sigma$ then $K_\tau \subseteq K_\sigma$. The refinement of $X$ and a check of these inclusions is done explicitly for quasi-split groups. We shall give details in the split case below. The general case follows by unramified Galois descent.

**Lemma 3.2.** Let $x$ be a vertex of $X$ and $\sigma$ a facet for a refined decomposition of $X$. Assume that a facet $\tau$ is contained in the smallest subcomplex of $X$ containing the cone with the vertex $x$ and the base $\sigma$. Then

$$K_\tau \subseteq K_x K_\sigma$$

**Proof.** By the assumption, there exists a point $y \in \sigma$ and a point $z$ on the geodesic $[x, y]$ such that the facet containing $z$, as an interior point, contains $\tau$. Then $K_\tau \subseteq G_{x,r,+}$ and $G_{y,r,+} \subseteq K_\sigma$. Lemma follows from Lemma 3.1. □

We shall work out the details in the case $G$ is simple and split. (A similar treatment for quasi split groups can be easily given based on computations in [PR].) Since $G$ is split, $S$ is a maximal torus, hence $T = S$. Let $O$ be the ring of integers in $k$ and $\pi$ and a uniformizing element. The group $k^x$ has a natural filtration $k^x \supset \mathcal{O}^x \supset 1 + \pi \mathcal{O} \supset 1 + \pi^2 \mathcal{O} \supset \ldots$ which gives rise to a filtration

$$T \supset T_0 \supset T_1 \supset \ldots$$

where $T_0$ is the maximal compact subgroup, which is the same as the set of all $t \in T$ such that $\chi(t) \in O^x$ for all algebraic characters $\chi$ of $T$, and $T_r$ is the set of all $t \in T$ such that $\chi(t) \in 1 + \pi^r \mathcal{O}$ for all algebraic characters $\chi$. The apartment $A = A(S)$ can be identified with $\text{Hom}(\mathbb{G}_m, S) \otimes \mathbb{R}$, so 0 is a special vertex of $A$. The apartment is a Coxeter complex defined by an affine root system $\Psi = \cup_{n \in \mathbb{Z}} \Phi + n$.

For every root $\alpha \in \Phi$ we have a subgroup $U_\alpha$ of $G$ isomorphic to $k$, and for every affine root $a$ we have a subgroup isomorphic to $O$. More precisely, If $a$ is an affine root whose gradient is $\alpha$, then $U_\alpha$ is defined as the subgroup of $U_a$ consisting of elements fixing the half-plane $a \geq 0$. The filtration

$$\ldots \supset U_{a-1} \supset U_a \supset U_{a+1} \supset \ldots$$

of $U_\alpha$ corresponds to the filtration $\ldots \supset \pi^{-1} O \supset \pi O \supset \ldots$ of $k$. For example, if $G = \text{SL}_2(k)$ and $T$ the torus of diagonal matrices, then one can identify $A$ with $\mathbb{R}$ and the set of affine roots with the set of affine functions $x \mapsto \pm x + n$, where $n \in \mathbb{Z}$, so that for $a(x) = x + n$

$$U_a = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \mid x \in \pi^n O \right\} \text{ and } U_{-a} = \left\{ \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \mid x \in \pi^{-n} O \right\}$$

It is convenient to introduce a notion of imaginary roots. The imaginary roots are constant, integer-valued functions on $A$. Let

$$\bar{\Psi} = \Psi \cup \mathbb{Z}.$$
If $a \in \tilde{\Psi}$ is an imaginary non-negative root, we define $U_a = T_a \subseteq T$. Let $x \in A$ and $r \geq 0$, a real number. Then $G_{x,r}$ and $G_{x,r,+}$ are generated by $U_a$ for all $a \in \tilde{\Psi}$ such that $a(x) \geq r$ and $a(x) > r$, respectively. If we fix a decomposition $\Phi = \Phi^+ \cup \Phi^-$ then the groups $U_{x,r}$ and $\tilde{U}_{x,r}$, appearing in the Iwahori decomposition, are generated by $U_a$ such that $a(x) \geq r$ and the gradient of $a$ is in $\Phi^+$ and $\Phi^-$, respectively. Now the inclusions in the proof of Lemma 3.1 are clear.

If $r$ is an integer and $a(x) > r$ for one interior point of a facet $\sigma$ then the same is true for any point in the interior of $\sigma$. Thus the groups $G_{x,r,+}$ are constant on interiors of facets. Moreover, if $\tau$ is contained in the boundary of $\sigma$, and $a > r$ on the interior of $\tau$ then it is so on the interior of $\sigma$. Hence $K_{r, \tau} \subseteq K_{\tau}$. Finally, assume $r = \frac{n}{m}$ is a non-negative rational number. We can refine the Coxeter complex by replacing $\Psi$ with $\Psi_m = \bigcup_{n \in \mathbb{Z}} \Phi + \frac{n}{m}$. Now the groups $G_{x,r,+}$ are constant on interiors of facets.

4. Schneider-Stuhler complex

We fix a non-negative rational number $r$ throughout this section. We refine the chamber decomposition of $X$ so that the groups $G_{x,r,+}$ are constant on the interior of facets and we set $K_{\sigma} := G_{x,r,+}$ where $x$ is any interior point of a facet $\sigma$. Let $V$ be a smooth representation of $G$, this means that any vector $v \in V$ is fixed by an open compact subgroup of $G$, depending on $v$. For every facet $\sigma$ let $V_{\sigma}$ be the subspace of all vectors in $V$ fixed by $K_{\sigma}$. The complex $C_* (X) \otimes_\mathbb{Z} V$ admits a natural representation of $G$ defined by $g(\sigma \otimes v) = g(\sigma) \otimes g(v)$, for all $g \in G$. Let $C_* (X, V)$ be the subcomplex spanned by $\tau \otimes v$ where $v \in V_{\tau}$. The boundary $\partial$ preserves $C_* (X, V)$ because $V_{\sigma} \subseteq V_{\rho}$ any time $\rho$ is in the boundary of $\sigma$. The action of $G$ on $C_* (X) \otimes V$ preserves the subcomplex $C_* (X, V)$.

Theorem 4.1. Let $x$ be a vertex in $X$ and $c$ an $x$-based contraction of $C_* (X)$ satisfying the conditions (1) and (2) in the Introduction. Then the complex $C_* (X, V)^{K_x}$ is exact.

Proof. The contraction $c$ defines a contraction on $C_* (X) \otimes V$ by $c(\sigma \otimes v) = c(\sigma) \otimes v$. To prove the theorem, it suffices to show that the contraction preserves the subcomplex $C_* (X, V)^{K_x}$. Let $e_x : V \to V_x$ be the projection given by averaging the action of $K_x$ on $V$ with respect to a Haar measure on $K_x$ of volume one. The subcomplex $C_* (X, V)^{K_x}$ is spanned by elements $e_x (\sigma \otimes v)$ where $v \in V_{\sigma}$. Since $c$ is $K_x$-invariant,

$$c(e_x (\sigma \otimes v)) = e_x (c(\sigma) \otimes v) = \sum_\tau c(\sigma, \tau) e_x (\tau \otimes v).$$

By the property (2) and Lemma 3.2 for every $\tau$ appearing in this sum, there exist $g_1, \ldots, g_n \in K_x \cap K_{\tau}$ such that

$$K_{\tau} = \bigcup_{i=1}^n g_i (K_{\sigma} \cap K_{\tau})$$

(a disjoint sum). Since $g_i \in K_x$, it follows that $e_x \cdot g_i = e_x$, as operators on any representation of $G$. Hence

$$e_x (\tau \otimes v) = \frac{1}{n} \sum_{i=1}^n e_x g_i (\tau \otimes v).$$
Since $g_i \in K_\tau$, these elements fix $\tau$. Hence

$$e_x(\tau \otimes v) = e_x(\tau \otimes \frac{1}{n} \sum_{i=1}^{n} g_i v).$$

Since $v$ is fixed by $K_\sigma \cap K_\tau$ and $g_i$ are representatives of all $K_\sigma \cap K_\tau$-cosets in $K_\tau$, it follows that

$$\frac{1}{n} \sum_{i=1}^{n} g_i v = e_\tau(v) \in V_\tau$$

where $e_\tau : V \to V_\tau$ is the projection given by averaging the action of $K_\tau$ on $V$ with respect to a Haar measure on $K_\tau$ of volume one. Hence $e_\tau(\tau \otimes v) \in C_*(X, V)^{K_\tau}$ as desired. \qed

5. Nice open compact subgroups

Let $S$ be a maximal split torus in $G$. Let $P$ be a parabolic subgroup of $G$. Without loss of generality we shall assume that $P$ contains $S$. In particular, we have a “standard” choice of the Levi $L$ and the opposite $U$ of the radical $U$ of $P$, both normalized by $S$. Let $K$ be a an open compact subgroup of $G$ and $K^G$ the set of all $G$-conjugates of $K$. The parabolic group $P$ acts on $K^G$ with finitely many orbits. We say that $K$ is nice with respect to $P$ if in any $P$-conjugacy class in $K^G$ there is $K'$ such that the Iwahori decomposition holds:

$$K' = (K' \cap U)(K' \cap L)(K' \cap \bar{U}).$$

If that is the case, then $K'_L = K' \cap L$ is isomorphic to $K' \cap P/K' \cap U$. This observation implies the first of the following properties, for the second see Proposition 3.5.2. in \cite{BD}.

1. If $W$ is an $L$-module and $(\text{Ind}_P^G W)^K \neq 0$ then $W^{K'_L} \neq 0$ for some $K'$ in $K^G$ with the Iwahori decomposition.

2. For any $G$-module $V$, and $K'$ in $K^G$ with the Iwahori decomposition, the map $V^{K'} \to V^{K'_L}$ is surjective.

We say that $K$ is nice if it is nice with respect to any parabolic $P$ containing $S$.

Proposition 5.1. Assume that $K$ is nice and $V$ is a $G$-module generated by $V^K$. If $V'$ is a non-trivial subquotient of $V$ then $(V')^K \neq 0$.

Proof. It suffices to prove the statement for irreducible subquotients. Without loss of generality we can assume that $V$ is contained in a single Bernstein component corresponding to a pair $(L, W)$ where $L$ is a Levi group containing $S$ and $W$ a quasi-cuspidal representation of $L$. Let $V'$ be an irreducible subquotient of $V$. Then there exists an unramified twist $W'$ of $W$ such that $V'$ is a submodule of $\text{Ind}_P^G W'$ where $P$ is a parabolic subgroup containing $L$. Since $V^K \neq 0$ there exists at least one irreducible subquotient $V_0$ such that $(V_0^K) \neq 0$. Let $W_0$ be an unramified twist of $W$ such that $V_0$ is contained in $\text{Ind}_P^G W_0$. It follows that $(\text{Ind}_P^G W_0)^K \neq 0$. By (1) above, there exists $K' \in K^G$ with the Iwahori decomposition such that $(W_0)^{K_L} \neq 0$. Hence $(W')^{K'_L} \neq 0$ for any unramified twist $W'$ of $W$. Now let $V'$ be any irreducible quotient of $V$. From the Frobenius-reciprocity

$$\text{Hom}_G(V', \text{Ind}_P^G(W')) = \text{Hom}_L(V'_U, W')$$
it follows that $W'$ is a quotient of $V_l'$. Hence $(V_l')^{K_l} \neq 0$ and $(V')^{K'} \neq 0$ by (2) above. Hence $(V')^K \neq 0$ since $K'$ is conjugate to $K$.

**Proposition 5.2.** For every $x \in X$ and $r > 0$ the Moy-Prasad group $G_{x,r}$ is nice.

*Proof.* Without loss of generality we can assume that $x \in A(S)$. Let $P$ be a parabolic subgroup containing $S$. Let $\sigma$ be a chamber in $A(S)$ containing $x$. Then the point-wise stabilizer $G_\sigma$ of $\sigma$ is an Iwahori subgroup of $G$. Let $N$ be the normalizer of $S$ in $G$. It acts naturally on $A(S)$. We have an Iwasawa decomposition ([BT1], Proposition 7.3.1)

$$G = PNG_\sigma.$$ 

Since $G_\sigma \subset G_x$, the Iwahori group $G_\sigma$ normalizes $G_{x,r}$, hence representatives of $P$-orbits in the $G$-conjugacy class of $K = G_{x,r}$ can be taken to be $K' = G_{x',r}$ where $x' = n(x) \in A(S)$ for some $n \in N$. And these groups have the Iwahori decomposition with respect to $P$, since $P$ contains $S$ by Theorem 4.2 in [MP2].

**Corollary 5.3.** Assume that, for every vertex $x \in X$ there exists a contraction of $C_*(X)$ satisfying the properties (1) and (2). Then the Schneider-Stuhler complex is exact for every smooth representation $V$.

*Proof.* It suffices to prove that the complex is exact in every Bernstein component. The complex $C_*(X, V)$ is a direct sum of $G$-modules isomorphic to $\text{ind}_{S_{x'}}^G V_\tau$ where $\tau$ is a facet in the refined chamber complex and $S_\tau$ is the stabilizer of $\tau$. This module is generated by $V^{K_\tau}_\tau = V_\tau$. Let $x$ be a vertex of $\tau$. Since $K_x \subseteq K_\tau$, it follows that $\text{ind}_{G_\tau}^G V_\tau$ is generated $K_x$-fixed vectors. Thus any Bernstein summand of $C_*(X, V)$ is generated by $K_x$-fixed vectors, for some vertex $x$ and exactness can be checked by passing to $K_x$-fixed vectors. 

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