Solution of Klein Gordon equation for hyperbolic cotangent potential in the presence of a minimal length using Hypergeometric method

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Abstract. This research was to investigate the solution of Klein Gordon equation within minimal length formalism. The Hypergeometric method was used to solve Klein Gordon equation to obtain the relativistic energy and wave functions within minimal length formalism. The relativistic energy was calculated numerically using Matlab software and un-normalized wave function was expressed in hypergeometric terms. The results showed that the relativistic energy within minimal length formalism was greater than the original relativistic energy and the relativistic energy increase by the increasing of the minimal length parameter. The minimal length parameter caused the shape change in wave functions.

1. Introduction

The study of behavior microscopic spin-zero particles in relativistic quantum mechanics was described using Klein Gordon equation [1,2]. An exact solution of Klein Gordon equation with a potential vector \( V \) and scalar potential \( S \) was obtainable in the case of exact symmetric spin condition when \( S=V \) and \( S=-V \) for exact symmetric pseudospin condition. In these two cases, Klein Gordon equation reduces to Schrodinger like equation thus a state solution was obtained using methods in nonrelativistic quantum mechanics [3]. Klein Gordon equation can be solved by Supersymmetric Quantum Mechanics (SUSY) [4], Algebraic solution [5], Nikiforov Uvarov (NU) method [6], and Asymptotic Iteration Method (AIM) [7-9,20].

The minimal length idea in quantum mechanics was to modify Generalized Uncertainty Principle (GUP) [10]. This modified version of quantum mechanics was based on the following deformed commutation relations between position and momentum operators [10]. The minimal length may be viewed as an intrinsic scale characterizing the system [11]. The minimal length to Klein Gordon equation had done by Bouaziz [12], Janna and Roy [11], and Merad et al. [13]. The solution of minimal length had been done by Alimmohamadi and Hassanabadi in 2017 for Bohr Mottelson equation [14].

In this paper, we solved symmetric spin condition of Klein Gordon equation with equal scalar and vector potentials in the presence of a minimal length. The radial part solution of Klein-Gordon equation in the presence of a minimal length for hyperbolic cotangent function potential has been studied. The hyperbolic cotangent potential was a part hyperbolic Manning Rosen potential. The hyperbolic Manning Rosen potential was used to study quark-gluon dynamic [15-17]. The hypergeometric method has been applied to obtain the relativistic energy and radial wave functions in
the presence of a minimal length. The work was organized as follows. In Section 2, Klein Gordon equation with minimal length formalism was introduced. In Section 3, we described briefly Hypergeometric method which was used to solve Klein Gordon equation. In Section 4, the relativistic energy and wave function of Klein Gordon equation was presented. The conclusion of Klein Gordon equation with minimal length was presented in Section 5.

2. Klein Gordon Equation with Minimal Length Formalism

The uncertainty of particle position was explained using commutation relations between position and momentum operators as expressed in Heisenberg’s uncertainty principle. The general equation is given as [18,19],

\[
\left[ X, P \right] \geq \hbar
\]  

(1)

For the case of minimal length occurring in position uncertainty and momentum uncertainty was influenced by a field then equation (1) can be written as [18,19]

\[
\left[ X, P \right] \geq \hbar \left( 1 + \alpha_{ML} \right) \left( \Delta P \right)^2
\]  

(2)

where \( X \) was position and \( P \) was the corresponding momentum. By equation (2), the position and momentum operators [18-20] were defined as,

\[
\hat{X}_i = \hat{x}_i
\]  

(3)

\[
\hat{P}_i = \left( 1 + \alpha_{ML} \right) \hat{p}_i
\]  

(4)

where \( \alpha_{ML} \) was a minimal length parameter that has value \( 0 \leq \alpha_{ML} \leq 1 \). Klein Gordon equation was given by [9,11]

\[
\left( E - V\left( r, \theta, \phi \right) \right)^2 \psi\left( r, \theta, \phi \right) = \left[ \hat{p}^2 c^2 + \left( \hat{M}_c^2 c^2 + S\left( r, \theta, \phi \right) \right)^2 \right] \psi\left( r, \theta, \phi \right)
\]  

(5)

where \( V\left( r, \theta, \phi \right) \) and \( S\left( r, \theta, \phi \right) \) were vector and scalar potentials, respectively. \( E \) was relativistic energy and \( M_c \) was rest mass. By setting \( S\left( r, \theta, \phi \right) = V\left( r, \theta, \phi \right) \) in equation (5) and substituting equation (4) into equation (5) with \( \hat{p} = -i\hbar \nabla \) and \( c = \hbar = 1 \) (natural unit), we had

\[
\left( E^2 - M_c^2 - \left( E + M_c \right) V\left( r, \theta, \phi \right) \right) \psi\left( r, \theta, \phi \right) = -\left( \Delta - 2\alpha_{ML} \Delta r \right) \psi\left( r, \theta, \phi \right)
\]  

(6)

where we must set \( 2V\left( r, \theta, \phi \right) \) from equation (5) becomes \( V\left( r, \theta, \phi \right) \), so we obtained equation (6).

In equation (6), \( \alpha_{ML} \) was ignored because \( \alpha_{ML} \) had very small value. In the case for Klein Gordon without minimal length formalism with \( \alpha_{ML} = 0 \) [13], then equation (6) was reduced into

\[
\left( E_0^2 - M_c^2 - \left( E_0 + M_c \right) V\left( r, \theta, \phi \right) \right) \psi\left( r, \theta, \phi \right) = -\left( \Delta \right) \psi\left( r, \theta, \phi \right) = -\left( \Delta_0 \right) \psi\left( r, \theta, \phi \right)
\]  

(7)

\( E_0 \) was relativistic energy at any level without the presence of a minimal length, by rewriting equation (7) as

\[
\Delta_0^2 = \left( E_0^2 - M_c^2 - \left( E_0 + M_c \right) V\left( r, \theta, \phi \right) \right)^2
\]  

(8)

In equation (8) the expression of \( \Delta_0^2 \) depends only on the potential \( V\left( r, \theta, \phi \right) \) and it was not considered to be spherical Laplacian, thus by inserting equation (8) into equation (6) we got

\[
\left( E^2 - M_c^2 - \left( E + M_c \right) V\left( r, \theta, \phi \right) \right) \psi\left( r, \theta, \phi \right) = -\left( \Delta - 2\alpha_{ML} \left( E_0^2 - M_c^2 - \left( E_0 + M_c \right) V\left( r, \theta, \phi \right) \right)^2 \right) \psi\left( r, \theta, \phi \right)
\]  

(9)

By applying spherical Laplacian operator was given as,

\[
\Delta = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2}
\]  

(10)
By setting $\psi (r, \theta, \varphi) = R (r) \Theta (\theta) \Phi (\varphi)$, so we had radial part,

$$
\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial R (r)}{\partial r} + \left[ \left( E^2 - M_o^2 - (E + M_o) V (r) \right) \right. \\
\left. \left( \frac{E_o^2 - M_o^2}{2} \right)^2 - 2 \left( E_o^2 - M_o^2 \right) (E_o + M_o) V (r) \right. \\
\left. + \left( E_o + M_o \right)^2 V^2 (r) \right] R (r) = \frac{\lambda}{r^2} R (r) \tag{11}
$$

By setting $R (r) = \chi (r)/r$, so equation (11) becomes,

$$
\frac{d^2 \chi (r)}{dr^2} - \frac{L (L+1)}{r^2} \chi (r) + \left[ \left( E^2 - M_o^2 - (E + M_o) V (r) \right) \right. \\
\left. \left( \frac{E_o^2 - M_o^2}{2} \right)^2 - 2 \left( E_o^2 - M_o^2 \right) (E_o + M_o) V (r) \right. \\
\left. + \left( E_o + M_o \right)^2 V^2 (r) \right] \chi (r) = 0 \tag{12}
$$

Equation (12) was Klein Gordon equation in the presence of a minimal length for the case symmetric spin condition.

3. Hypergeometric Method
The principles of Hypergeometric method is substituted with new variable and parameter to obtain a second-order differential equation of hypergeometric function which is expressed [22,23],

$$
z (1-z) \frac{d^2 \phi}{dz^2} + \left( c-(a+b+1)z \right) \frac{d\phi}{dz} - ab \phi = 0 \tag{13}
$$

with the solution of a wave function is given as [22]

$$
\phi (z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n \tag{14}
$$

The energy eigenvalue is obtained from the condition in equation (13), [20]

$$
a = -n \text{ or } b = -n \tag{15}
$$

where $n=0,1,2,\ldots$ Equation(14) can be finite series of polynomials of rank $n$ by equation(15).

4. Klein Gordon Equation with Minimal Length Formalism
By applying the hyperbolic cotangent potential [24] was given as,

$$
V (r) = V_o \coth (\omega r) + V_1 \tag{16}
$$

where $V_o$ and $V_1$ were constant of potential $\omega$ was a range of potential into equation (12), then equation (12) becomes

$$
\frac{d^2 \chi (r)}{dr^2} - \frac{L (L+1)}{r^2} \chi (r) - 2 \alpha mL \gamma V_o^2 \frac{1}{\cosh^2 (\omega r)} \chi (r) \\
+ \left( \left( 4 \alpha mL \mu - (E + M_o) \right) V_1 \right) + \left( E_o^2 - M_o^2 \right) \coth (\omega r) \chi (r) \\
+ \left( \left( E^2 - M_o^2 - 2 \alpha mL \gamma^2 \right) V_1 \right) + \left( -2 \alpha mL \gamma V_o^2 \right) \chi (r) = 0 \tag{17}
$$
where,
\[ \gamma = (E_0 + M_o), \quad \mu = (E_0^2 - M_o^2)(E_0 + M_o), \quad \delta = (E_0^2 - M_o^2) \]  
(18)

By applying the centrifugal approximation [25] in equation (19) was given as,
\[ \frac{1}{r^2} = \sinh^2(\omega r) \]
(19)

Then equation (17) change into
\[ \left[ \frac{d^2 \chi(r)}{dr^2} + \left( \omega L(L+1) + 2\alpha_M \gamma V_o^2 \right) \right] \sinh^2(\omega r) \chi(r) + \left( 4\alpha_M \mu - (E + M_o) \right) V_o - 2\alpha_M \gamma V_o \coth(\omega r) \chi(r) = 0 \]
(20)

By setting,
\[ \nu = \omega L(L+1) + 2\alpha_M \gamma V_o^2 \]
(21)
\[ 2q = \left( 4\alpha_M \mu - (E + M_o) \right) V_o - 2\alpha_M \gamma V_o \]
(22)
\[ -k^2 = \left( 4\alpha_M \mu - (E + M_o) \right) V_i + \left( E^2 - M_o^2 - 2\alpha_M \lambda^2 \right) - 2\alpha_M \gamma V_i^2 - 2\alpha_M \gamma V_o^2 \]
(23)

Then equation (20) becomes,
\[ \frac{d^2 \chi(r)}{dr^2} + \left( \nu - 2q \coth(\omega r) + k^2 \right) \chi(r) = 0 \]
(24)

Equation (24) was a differential equation which had to be reduced to hypergeometric differential equation type. By using the suitable variable change \( \coth(\omega r) = 1 - 2z \) so we got,
\[ z(1-z) \frac{d^2 \chi(r)}{dz^2} + (1-2z) \frac{d \chi(r)}{dz} + \left[ \nu - 2q - 4\alpha^2 - 4\beta^2 \frac{4}{4(1-z)} \right] \chi(r) = 0 \]
(25)

with
\[ -2q + k^2 = 4\alpha^2 \]
(26)
\[ 2q + k^2 = 4\beta^2 \]
(27)
\[ \nu - 2q = \frac{\nu - 1}{\omega^2} \]
(28)

Equation (25) was an intermediate of the hypergeometric differential equation. We reduced equation (25) by using new wave function,
\[ \chi(r) = z^\alpha (1-z)^\beta f(z) \]
(29)

and by inserting equations (26-29) into equation (25), we had
\[ z(1-z) \frac{d^2 f(z)}{dz^2} + \left[ (2\alpha+1)-(2\alpha+2\beta+2)z \right] \frac{d f(z)}{dz} + \left[ \nu - 1 -(\alpha + \beta)(\alpha + \beta + 1) \right] f(z) = 0 \]
(30)

Equation (30) shows the hypergeometric differential equation therefore with the solution in the form of \( f(z) =_2 F_1(a,b,c,z) \), with
\[ a = \alpha + \beta - (\nu - 1) \]
(31)
\[ b = \alpha + \beta + \nu \]
(32)
The hypergeometric parameter with increasing of potentials and minimal length parameters. We obtain the relativistic energy equation, was given as,

\[
E^2 - M_o^2 = -\omega^2 \left( \delta_e - \frac{1}{2} n \right)^2 + \frac{(4\alpha_M - (E + M_o))V_o - 2\alpha_M \gamma^2 V_o^2}{2\alpha^2} \left( \delta_e - \frac{1}{2} n \right)^2 - (4\alpha_M - (E + M_o))V_o + \eta_{rel}
\]

(34)

with

\[
\delta_e = \sqrt{\omega L (L+1) + 2\alpha_M \gamma^2 V_o^2 + \frac{1}{4}}
\]

(35)

\[
\eta_{rel} = 2\alpha_M \xi^2 + 2\alpha_M \gamma^2 V^2 + 2\alpha_M \gamma^2 V^2
\]

(36)

where \( L \) was angular momentum and \( n \) was a quantum number. Equation (34) was the relativistic energy of Klein Gordon equation in the presence of a minimal length for hyperbolic cotangent potential. From Equation (34), we calculated either the relativistic energy \( E_o \) for \( \alpha_{ML} = 0 \) and \( E \) for \( \alpha_{ML} \neq 0 \). The relativistic energy value was calculated numerically by using Matlab software which was shown in Table 1.

| Table 1. The relativistic energy for \( M_o = 1 \), \( V_o = 0.01 \), \( V = 0.1 \), \( V = 0.001 \), and \( L = 0 \). |
|---------------------------------------------------------------|
| \( n \) | \( \omega = 0.1 \) | \( \omega = 0.2 \) |
|--------|----------------|----------------|
| \( \alpha_{ML} = 0 \) | \( \alpha_{ML} = 0.03 \) | \( \alpha_{ML} = 0.09 \) | \( \alpha_{ML} = 0 \) | \( \alpha_{ML} = 0.03 \) | \( \alpha_{ML} = 0.09 \) |
| 1      | 1.0900         | 1.0901         | 1.0904         | 1.0795         | 1.0796         | 1.0798         |
| 2      | 1.0795         | 1.0796         | 1.0799         | 1.0205         | 1.0214         | 1.0231         |
| 3      | 1.0556         | 1.0560         | 1.0567         | 0.9115         | 0.9161         | 0.9252         |
| 4      | 1.0205         | 1.0214         | 1.0232         | 0.7300         | 0.7494         | 0.7825         |
| 5      | 0.9731         | 0.9753         | 0.9796         | 0.3701         | 0.4532         | 0.5818         |

Table 1 shows that the relativistic energy was influenced by quantum numbers, the range of potentials and minimal length parameters. The value of relativistic energy decreased by the increasing of quantum number \( n \) and range of potential \( \omega \). The increasing of relativistic energy value causes the increasing of the minimal length parameter.

Then, for the wave function was obtained from equations (26-29), was given as

\[
\chi(r) = \frac{1}{2} \left( \frac{1 + \coth (\omega r)}{1 - \coth (\omega r)} \right)^\alpha \frac{2}{2} \frac{2}{2} \ 2F_1 (a, b, c, z)
\]

(37)

with

\[
2F_1 (a, b, c, z) = 1 + \frac{ab}{c} z + \frac{a(a+1)b(b+1)z^2}{c(c+1)} + \frac{a(a+1)(a+2)b(b+1)(b+2)z^3}{c(c+1)(c+2)} + ...
\]

(38)

The hypergeometric parameters from equations (31-33) were substituted into equation (13) and equation (14), so we got,

\[
a = -n
\]

(39)

\[
b = 2\gamma - n
\]

(40)

\[
c = \tau - \sigma
\]

(41)
with,

\[
\mathcal{g} = \sqrt{\frac{\omega L (L+1) + 2\alpha_{\text{ML}} \gamma^2 V_o}{\omega^2} + \frac{1}{4}} \quad (42)
\]

\[
\tau = \sqrt{\frac{\omega L (L+1) + 2\alpha_{\text{ML}} \gamma^2 V_o}{\omega^2} + \frac{1}{4} - \frac{1}{2} - n} \quad (43)
\]

\[
\sigma = \frac{\left(4\alpha_{\text{ML}} \mu - (E + M_\gamma) V_o\right) - 2\alpha_{\text{ML}} \gamma^2 V_2}{2 \omega^2 \sqrt{\frac{\omega L (L+1) + 2\alpha_{\text{ML}} \gamma^2 V_o}{\omega^2} + \frac{1}{4}} - \frac{1}{2} - n} \quad (44)
\]

By inserting equations (39-41) into equation (37), we had the wave functions of quantum number \(n=0,1,2\) that were shown in Table 2.

### Table 2. The wave function of Klein Gordon equation in the presence of minimal length

| \(n\) | Wave Function |
|------|--------------|
| 0    | \(X_0 (r) = (-1)^{\frac{\pi}{2} \sigma} \left( \frac{\cosh (\omega r)}{2} \right) \left( \frac{\sinh (\omega r) - \cosh (\omega r)}{2} \right)\) |
| 1    | \(X_1 (r) = (-1)^{\frac{\pi}{2} \sigma} \left( \frac{\cosh (\omega r)}{2} \right) \left( \frac{\sinh (\omega r) - \cosh (\omega r)}{2} \right) \left[ 1 + (-n)(2\gamma - n) \left( \frac{1 - \coth (\omega r)}{2} \right) \right] \) |
| 2    | \(X_2 (r) = (-1)^{\frac{\pi}{2} \sigma} \left( \frac{\cosh (\omega r)}{2} \right) \left( \frac{\sinh (\omega r) - \cosh (\omega r)}{2} \right) \left[ 1 + (-n)(2\gamma - n) \left( \frac{1 - \coth (\omega r)}{2} \right) \right] \left[ 1 + \gamma \right] \) |

### 5. Conclusion

We investigated Klein Gordon equation in the presence of a minimal length for hyperbolic cotangent potential. The hypergeometric method was used to obtain the relativistic energy and radial wave functions for Klein-Gordon equation in the presence of minimal length. The minimal length parameter gives the influence in the relativistic energy value which by increasing the minimal length parameter causes the greater value of energy. The value of relativistic energy decreases respect to various quantum number and range of potential.

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