Local well-posedness in the Wasserstein space for a chemotaxis model coupled to Navier-Stokes equations

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Abstract

We consider a coupled system of Keller-Segel type equations and the incompressible Navier-Stokes equations in spatial dimension two and three. In the previous work [17], we established the existence of a weak solution of a Fokker-Plank equation in the Wasserstein space using the optimal transportation technique. Exploiting this result, we constructed solutions of Keller-Segel-Navier-Stokes equations such that the density of biological organism belongs to the absolutely continuous curves in the Wasserstein space. In this work, we refine the result on the existence of a weak solution of a Fokker-Plank equation in the Wasserstein space. As a result, we construct solutions of Keller-Segel-Navier-Stokes equations under weaker assumptions on the initial data.

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1 Introduction

In this paper, we study an aerotaxis model formulating the dynamics of oxygen, swimming bacteria, and viscous incompressible fluids in $Q_T := \mathbb{R}^d \times [0, T)$, $d = 2, 3$, $T > 0$.

\begin{align}
\partial_t \rho + u \cdot \nabla \rho - \Delta \rho &= -\nabla \cdot (\chi(c) \rho \nabla c), \\
\partial_t c + u \cdot \nabla c - \Delta c &= -k(c) \rho, \\
\partial_t u + u \cdot \nabla u - \Delta u + \nabla p &= -\rho \nabla \phi, \quad \text{div } u = 0.
\end{align}

Here $\rho(t, x) : Q_T \to \mathbb{R}^+$, $c(t, x) : Q_T \to \mathbb{R}^+$, $u(t, x) : Q_T \to \mathbb{R}^d$ and $p(t, x) : Q_T \to \mathbb{R}$ denote the biological cell concentration, oxygen concentration, fluid velocity, and scalar pressure, respectively, where $\mathbb{R}^+$ indicates the set of non-negative real numbers. The oxygen consumption rate $k(c)$ and the aerobic sensitivity $\chi(c)$ are nonnegative as functions of $c$, namely $k, \chi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $k(c) = k(c(x, t))$ and $\chi(c) = \chi(c(x, t))$ and the time-independent function $\phi = \phi(x)$ denotes the potential function, e.g., the gravitational force or centrifugal force. Initial data are given by $(\rho_0(x), c_0(x), u_0(x))$ with $\rho_0(x), c_0(x) \geq 0$ and $\nabla \cdot u_0 = 0$. Tuval et al. proposed in [27], describing behaviors of swimming bacteria, Bacillus subtilis (see also [7]).

The above system (1.1)-(1.3) seems to have similarities to the classical Keller-Segel model suggested by Patlak[24] and Keller-Segel[18, 19], which is given as

\begin{align}
\rho_t = \Delta \rho - \nabla \cdot (\chi \rho \nabla c), \\
\partial_t c = \Delta c - \alpha c + \beta \rho,
\end{align}

where $\rho = \rho(t, x)$ is the biological cell density and $c = c(t, x)$ is the concentration of chemical attractant substance. Here, $\chi$ is the chemotatic sensitivity, and $\alpha \geq 0$ and $\beta \geq 0$ are the decay and production rate of the chemical, respectively. The system (1.4) has been comprehensively studied and results are not listed here (see e.g. [13, 16, 22, 23, 29] and the survey papers [14, 15]).

We remark that in the case that the effect of fluids is absent, i.e., $u = 0$ and $\phi = 0$, the system (1.1)-(1.3) becomes a Keller-Segel type model with the negative term $-k(c) \rho$. It is due to the fact that
the oxygen concentration is consumed, while the chemical substance is produced by $\rho$ in the Keller-Segel system (1.4).

Our main objective is to establish the existence of solution $\rho$ for the system (1.1)-(1.3) in the Wasserstein space, which will be described later in detail.

We review some known results related to well-posedness of solutions for the system (1.1)-(1.3). Existence of local-in-time solutions was proven for bounded domains in $\mathbb{R}^3$. It was shown in [10] that smooth solutions exist globally in time, provided that initial data are very near constant steady states and $\chi(\cdot), k(\cdot)$ hold the following conditions:

$$
\chi'(\cdot) \geq 0, \quad k'(\cdot) > 0, \quad \left( \frac{k(\cdot)}{\chi(\cdot)} \right)'' < 0.
$$

Global well-posedness of regular solutions was proved in [30] for large initial data for bounded domains in $\mathbb{R}^2$ with boundary conditions $\partial_\nu \rho = \partial_\nu c = u = 0$ under a similar conditions as (1.5) on $\chi(\cdot)$ and $k(\cdot)$:

$$
\left( \frac{k(\cdot)}{\chi(\cdot)} \right)', \quad (\chi(\cdot)k(\cdot))' \geq 0, \quad \left( \frac{k(\cdot)}{\chi(\cdot)} \right)'' \leq 0.
$$

Different structure conditions are given in [1], where (1.6) is slightly relaxed, in case that $\chi$ is a constant.

In [3, Theorem 1.1] the first author et al. constructed unique regular solutions for general $\chi$ and $k$ in following function classes:

$$
(\rho, c, u) \in C \left( [0, T^*); H^m(\mathbb{R}^3) \right) \times C \left( [0, T^*); H^{m+1}(\mathbb{R}^3) \right) \times C \left( [0, T^*); H^{m+1}(\mathbb{R}^3) \right),
$$

in case that $\phi \in H^m$ and initial data belong to

$$
(\rho_0, c_0, u_0) \in H^m(\mathbb{R}^3) \times H^{m+1}(\mathbb{R}^3) \times H^{m+1}(\mathbb{R}^3).
$$

In addition, if $\chi(\cdot)$ and $k(\cdot)$ satisfy the following conditions, motivated by experimental results in [7] and [27] (compare to (1.5) or (1.6)): There is an $\epsilon > 0$ such that

$$
\chi(c), k(c), \chi'(c), k'(c) \geq 0, \quad \text{sup} \left| \chi(c) - \mu k(c) \right| < \epsilon \quad \text{for some} \quad \mu > 0,
$$

global-in-time regular solutions exist in $\mathbb{R}^2$ with no smallness of the initial data (see [3, Theorem 1.3]).

We remark that was shown in [6] that if $\|\rho_0\|_{L^1(\mathbb{R}^2)}$ is sufficiently small, global well-posedness and temporal decays can be established, in only case that $\chi(c), k(c), \chi'(c), k'(c) \geq 0$. One can also consult [5], [31] and [4] with reference therein for temporal decay and asymptotics, and also refer to e.g. [8], [11], [26] and [9] for the nonlinear diffusion models of a porous medium type.

Our main ingredient is to establish existence of weak solutions for the Fokker-Plank equations in the Wasserstein space. Compared to previous result, in [17], the authors assumed that $\rho_0$ is in $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and constructed cell concentration $\rho$ in the Wasserstein space. More precisely, by understanding the following Fokker-Planck equation as an absolutely continuous curve in the Wasserstein space,

$$
\left\{ \begin{array}{ll}
\partial_t \rho = \nabla \cdot (\nabla \rho - v \rho) \\
\rho(\cdot, 0) = \rho_0
\end{array} \right. \quad \text{in} \quad Q_T := [0, T] \times \mathbb{R}^d,
$$

we solved (1.10) under the assumption (refer Theorem 1.1 in [17])

$$
v \in L^2(0, T; L^\infty(\mathbb{R}^d)) \quad \text{and} \quad \text{div} \ v \in L^1(0, T; L^\infty(\mathbb{R}^d)).
$$

As a result, by exploiting the above result with $v := u - \chi(c) \nabla c$, we constructed the solution of Keller-Segel-Navier-Stokes system (1.1)-(1.3) under the assumption (refer Theorem 1.2 in [17])

$$
\rho_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d), \quad c_0 \in W^{3,m}(\mathbb{R}^d) \cap W^{2,2}(\mathbb{R}^d), \quad u_0 \in W^{2,b}(\mathbb{R}^d),
$$

for any $m > d$ and $b > d + 2$. 

2
In this paper, by exploiting approximation argument to (1.10) and being able to control the uniform speed of approximated absolutely continuous curves in the Wasserstein space, we solve the Fokker-Plank equation (1.10) under the weaker regularity assumption on the velocity field

$$v \in L^\beta(0,T;L^\alpha(\mathbb{R}^d)), \quad \frac{d}{\alpha} + \frac{2}{\beta} \leq 1, \quad \alpha > d.$$ \hspace{1cm} (1.13)

This is certainly weaker than (1.11) and it turns out that initial condition $\rho_0$ can be in a function space larger than $L^1 \cap L^\infty(\mathbb{R}^d)$.

More precise statement of the above result is stated in Theorem 1.

**Theorem 1** Let $\frac{d}{p} + \frac{2}{q} = 1$ and $p > d$. Suppose

$$\rho_0 dx \in \mathcal{P}_2(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} \rho_0 \ln \rho_0 dx < \infty.$$ \hspace{1cm} (1.14)

Assume further that

$$v \in L^q(0,T;L^p(\mathbb{R}^d)).$$ \hspace{1cm} (1.15)

Then, there exists an absolutely continuous curve $\mu \in AC_2(0,T;\mathcal{P}_2(\mathbb{R}^d))$ such that $\mu(t) := \rho(t,x)dx \in \mathcal{P}^{ac}_2(\mathbb{R}^d)$ for all $t \in [0,T]$, and $\mu$ solves (1.10) in the sense of distributions, namely, for any $\varphi \in C^\infty_c([0,T) \times \mathbb{R}^d)$

$$\int_0^T \int_{\mathbb{R}^d} \{\partial_t \varphi(t,x) + \Delta \varphi(t,x) + \nabla \varphi(t,x) \cdot v(t,x)\} \rho(t,x)dxdt = -\int_{\mathbb{R}^d} \varphi(0,x)\rho_0(x)dx,$$ \hspace{1cm} (1.16)

and

$$W_2(\rho(t),\mu(s)) \leq C \sqrt{T-s} \quad \forall \quad 0 \leq s \leq t \leq T,$$ \hspace{1cm} (1.17)

where the constant $C = C\left(T, \int_{\mathbb{R}^d} \rho_0 \ln \rho_0 dx, \int_{\mathbb{R}^d} |x|^2 \rho_0 dx, \int_0^T \|v(t)\|_{L^p}^{2p} dt\right)$.

Furthermore, if $\rho_0 \in L^\alpha(\mathbb{R}^d)$ for $\alpha > 1$ then we have

$$\|\rho(t)\|_{L^\infty(\mathbb{R}^d)} \leq \|\rho_0\|_{L^\alpha} \exp\left(C\|v\|_{L^\infty(0,T;L^p(\mathbb{R}^d))}^q\right) \quad \forall \quad t \in [0,T].$$ \hspace{1cm} (1.18)

**Remark 1** The limiting case, $(p,q) = (d,\infty)$, in (1.15) can be included, and it requires, however, an extra smallness of its norm, i.e. there is an $\epsilon > 0$ such that $\|v\|_{L^\infty(0,T;L^p(\mathbb{R}^d))} < \epsilon$. Thus, we do not consider such case in Theorem 1.

With the aid of Theorem 1, we construct weak solutions for the aerotaxis-fluid model (1.1)-(1.3) in the Wasserstein space. For convenience, we introduce some function classes, which are defined as

$$X_a((0,t) \times \mathbb{R}^d) := L^\infty(0,t;L^a(\mathbb{R}^d)), \hspace{1cm} (1.19)$$

$$Y_a((0,t) \times \mathbb{R}^d) := L^2(0,t;W^{2,a}(\mathbb{R}^d)) \cap W^{1,2}(0,t;L^a(\mathbb{R}^d)), \hspace{1cm} (1.20)$$

where the function spaces $X_a$ and $Y_a$ are equipped with the following norms:

$$\|f\|_{X_a} := \|f\|_{L^\infty((0,T);L^a(\mathbb{R}^d))}, \hspace{1cm} (1.21)$$

$$\|f\|_{Y_a} := \|f\|_{L^2((0,T);W^{2,a}(\mathbb{R}^d))} + \|f\|_{W^{1,2}((0,T);L^a(\mathbb{R}^d))}. \hspace{1cm} (1.22)$$

Next, we establish the existence of solutions in the classes $(\rho,c,u) \in X_a \times Y_a \times Y_a$ for the system (1.1)-(1.3). Our result reads as follows:
**Theorem 2** Let $d = 2, 3$, and $d/2 < a < \infty$. Suppose that $\chi, k, \chi', k'$ are all non-negative and $\chi$, $k \in C^j(\mathbb{R}^+)$ and $k(0) = 0$, $\|\nabla \phi\|_{L^1 \cap L^\infty} < \infty$ for $1 < j < \infty$. Let the initial data $(\rho_0, c_0, u_0)$ be given as

$$\rho_0 \in (L^1 \cap L^a)(\mathbb{R}^d), \quad c_0, u_0 \in W^{2,a}(\mathbb{R}^d).$$

Then there exists $T > 0$ and a unique weak solution $(\rho, c, u)$ of (1.1)-(1.3) such that

$$\rho \in X_a(Q_T), \quad c, u \in Y_a(Q_T),$$

where $Q_T = [0, T] \times \mathbb{R}^d$. Furthermore, we have

$$W_2\left(\frac{\rho(t)}{\|\rho_0\|_{L^1(\mathbb{R}^d)}}, \frac{\rho(s)}{\|\rho_0\|_{L^1(\mathbb{R}^d)}}\right) \leq C\sqrt{t-s} \quad \forall \ 0 \leq s \leq t \leq T.$$  \hspace{1cm} (1.25)

Here the constant $C$ in (1.25) depends on

$$\|(|u| + |\nabla c|)||_{L^{\alpha}(0,T;L^{\infty}(\mathbb{R}^d))}, \quad \int_{\mathbb{R}^d}|x|^2\rho_0 dx, \quad \|\rho_0\|_{L^\alpha(\mathbb{R}^d)},$$

where $\alpha > d$ and $\beta \geq 2$ are numbers that come from Theorem 1 when $p = a$.

We make several comments for Theorem 2.

(i) It was shown in [20] that if the initial data are sufficiently small in invariant classes, it is known that mild solutions globally exist in time. More precisely, in case $\kappa(c) = c$ and under the assumption that $\max \left\{\|\rho_0\|_{L^\frac{d}{2}}, \|c_0\|_{L^{\infty}(\mathbb{R}^d)}, \|u_0\|_{L^\infty(\mathbb{R}^d)}, \|\nabla \phi\|_{L^\infty(\mathbb{R}^d)}\right\}$ is small (if $d \geq 3$, $\|\rho_0\|_{L^\frac{d}{2}}$ is relaxed by $\|\rho_0\|_{L^\alpha(\mathbb{R}^d)}$, mild solutions exist globally in time, e.g. $\mathcal{L}^{\frac{\beta}{2} - \frac{\alpha}{2}} \rho \in BC_{w}([0, \infty); L^\alpha(\mathbb{R}^d))$ (see [20, Theorem 1]). The initial data in Theorem 2 are assumed to be subcritical, and thus stronger than those in [20]. Nevertheless, the weak solutions constructed in Theorem 2 is for the case of large data, not small data and furthermore, the estimate (1.25) is valid continuously up to initial time (compare to [20]).

(ii) It is not clear if local well-posedness can be established in case that $\rho_0 \in L^{\frac{\beta}{2}}(\mathbb{R}^d)$ with no smallness in $L^{\frac{\beta}{2}}(\mathbb{R}^d)$-norm or not. We also do not know that local well-posedness can be extened to global well-posedness, when $\rho_0 \in L^p(\mathbb{R}^d)$ for $p > \frac{d}{2}$, and thus we leave these as open questions.

(iii) The results of Theorem 2 also hold for $d > 3$. We do not, however, include the case for the system (1.1)-(1.3), since a dimension higer than three doesn’t seem to be empirically relevant.

(iv) We note that, due to regularized effect of diffusion, solutions in Theorem 2 become regular in $[\delta, T] \times \mathbb{R}^d$ for all $\delta > 0$, assuming additionally that $\chi$ are $\kappa$ belong to $C^\infty(\mathbb{R}^+)$.

(vi) The initial data $c_0$ and $u_0$ can be relaxed compared to assumptions in (1.23). To be more precise, $W^{2,a}(\mathbb{R}^d)$ in (1.23) can be replaced by $D_a^{1 - \frac{\beta}{2}, a}(\mathbb{R}^d)$ defined by

$$D_a^{1 - \frac{\beta}{2}, a}(\mathbb{R}^d) := \{h \in L^a(\mathbb{R}^d) : \|h\|_{D_a^{1 - \frac{\beta}{2}, a}} = \|h\|_{L^a(\mathbb{R}^d)} + \left(\int_0^\infty \|t^{\frac{\beta}{2}} A_a e^{-tA_a} h\|_{L^a(\mathbb{R}^d)}^2 dt\right)^{\frac{\beta}{2}} < \infty\},$$

where $A_a$ is the Heat or Stokes operator (see e.g. [12, Theorem 2.3]).

This paper is organized as follows. In Section 2, preliminary works are introduced. Section 3 and Section 4 are devoted to proving Theorem 1 and Theorem 2, respectively.
2 Preliminaries

2.1 Wasserstein space

In this subsection, we introduce the Wasserstein space and remind some properties of it. For more detail, readers may refer to e.g. [2] and [28].

Definition 3 Let $\mu$ be a probability measure on $\mathbb{R}^d$. Suppose there is a measurable map $T: \mathbb{R}^d \mapsto \mathbb{R}^d$. Then, the map $T$ induces a probability measure $\nu$ on $\mathbb{R}^d$ which is defined as

$$
\int_{\mathbb{R}^d} \varphi(y) d\nu(y) = \int_{\mathbb{R}^d} \varphi(T(x)) d\mu(x) \quad \forall \varphi \in C(\mathbb{R}^d).
$$

We denote, for convenience, $\nu := T_#\mu$ and say that $\nu$ is the push-forward of $\mu$ by $T$.

Let us denote by $\mathcal{P}_2(\mathbb{R}^d)$ the set of all Borel probability measures on $\mathbb{R}^d$ with a finite second moment. For $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, we consider

$$
W_2(\mu, \nu) := \left( \inf_{\gamma \in \Gamma(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 d\gamma(x, y) \right)^{\frac{1}{2}},
$$

where $\Gamma(\mu, \nu)$ denotes the set of all Borel probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ which has $\mu$ and $\nu$ as marginals, i.e.

$$\gamma(A \times \mathbb{R}^d) = \mu(A) \quad \gamma(\mathbb{R}^d \times A) = \nu(A)$$

for every Borel set $A \subset \mathbb{R}^d$.

Equation (2.1) defines a distance on $\mathcal{P}_2(\mathbb{R}^d)$ which is called the Wasserstein distance. Equipped with the Wasserstein distance, $\mathcal{P}_2(\mathbb{R}^d)$ is called the Wasserstein space. It is known that the infimum in the right hand side of Equation (2.1) always achieved. We will denote by $\Gamma_o(\mu, \nu)$ the set of all $\gamma$ which minimize the expression.

If $\mu$ is absolutely continuous with respect to the Lebesgue measure then there exists a convex function $\phi$ such that $\gamma := (Id \times \nabla \phi)_#\mu$ is the unique element of $\Gamma_o(\mu, \nu)$, that is, $\Gamma_o(\mu, \nu) = \{(Id \times \nabla \phi)_#\mu\}$.

Definition 4 Let $\phi : \mathcal{P}_2(\mathbb{R}^d) \mapsto (-\infty, \infty]$. We say that $\phi$ is convex in $\mathcal{P}_2(\mathbb{R}^d)$ if for every couple $\mu_1, \mu_2 \in \mathcal{P}_2(\mathbb{R}^d)$ there exists an optimal plan $\gamma \in \Gamma_o(\mu_1, \mu_2)$ such that

$$\phi(\mu^{1 \to 2}_t) \leq (1 - t)\phi(\mu_1) + t\phi(\mu_2) \quad \forall t \in [0, 1],$$

where $\mu^{1 \to 2}_t$ is a constant speed geodesic between $\mu_1$ and $\mu_2$ defined as

$$\mu^{1 \to 2}_t := ((1 - t)\pi_1 + t\pi_2)_#\gamma.$$

Here, $\pi_1 : \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}^d$ and $\pi_2 : \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}^d$ are the first and second projections of $\mathbb{R}^d \times \mathbb{R}^d$ onto $\mathbb{R}^d$ defined by

$$\pi_1(x, y) = x, \quad \pi_2(x, y) = y \quad \forall x, y \in \mathbb{R}^d.$$

As we have seen in Theorem 1 and Theorem 2, we will find a solution of $\rho$ equation (1.1) in the class of absolutely continuous curves in the Wasserstein space. Now, we introduce the definition of absolutely continuous curve and its relation with the continuity equation.

Definition 5 Let $\sigma : [a, b] \mapsto \mathcal{P}_2(\mathbb{R}^d)$ be a curve. We say that $\sigma$ is absolutely continuous and denote it by $\sigma \in AC_2(a, b; \mathcal{P}_2(\mathbb{R}^d))$, if there exists $m \in L^2([a, b])$ such that

$$W_2(\sigma(s), \sigma(t)) \leq \int_s^t m(r) dr \quad \forall a \leq s \leq t \leq b.$$ (2.2)
If $\sigma \in AC_2(a, b; P_2(\mathbb{R}^d))$, then the limit

$$|\sigma'|(t) := \lim_{s \to t} \frac{W_2(\sigma(s), \sigma(t))}{|s - t|},$$

exists for $L^1$-a.e $t \in [a, b]$. Moreover, the function $|\sigma'|$ belongs to $L^2(a, b)$ and satisfies

$$|\sigma'|(t) \leq m(t) \quad \text{for } L^1 \text{-a.e. } t \in [a, b],$$

for any $m$ satisfying (2.2). We call $|\sigma'|$ by the metric derivative of $\sigma$.

**Lemma 6** ([2], Theorem 8.3.1) If $\sigma \in AC_2(a, b; P_2(\mathbb{R}^d))$ then there exists a Borel vector field $v : \mathbb{R}^d \times (a, b) \rightarrow \mathbb{R}^d$ such that

$$v_t \in L^2(\sigma_t) \quad \text{for } L^1 \text{-a.e } t \in [a, b],$$

and the continuity equation

$$\partial_t \sigma_t + \nabla \cdot (v_t \sigma_t) = 0,$$

holds in the sense of distribution sense.

Conversely, if a weak* continuous curve $\sigma : [a, b] \rightarrow P_2(\mathbb{R}^d)$ satisfies the continuity equation (2.4) for some Borel vector field $v_t$ with $\|v_t\|_{L^2(\sigma_t)} \in L^2(a, b)$, then $\sigma : [a, b] \rightarrow P_2(\mathbb{R}^d)$ is absolutely continuous and $|\sigma'| \leq \|v_t\|_{L^2(\sigma_t)}$ for $L^1$-a.e $t \in [a, b]$.

**Notation**: In Lemma 6, we use notation $v_t := v(\cdot, t)$ and $\sigma_t := \sigma(t)$. Throughout this paper, we keep this convention, unless any confusion is to be expected, and a usual notation $\partial_t$ is adopted for temporal derivative, i.e. $f_t := f(\cdot, t)$ and $\partial_t f := \frac{df}{dt}$.

**Lemma 7** Let $\sigma_n \in AC_2(a, b; P_2(\mathbb{R}^d))$ be a sequence and suppose there exists $m \in L^2([a, b])$ such that

$$W_2(\sigma_n(s), \sigma_n(t)) \leq \int_s^t m(r)dr \quad \forall \ a \leq s \leq t \leq b,$$

for all $n \in \mathbb{N}$. Then there exists a subsequence $\sigma_{n_k}$ such that

$$\sigma_{n_k}(t) \text{ weak* converges to } \sigma(t), \quad \text{for all } t \in [a, b],$$

for some $\sigma \in AC_2(a, b; P_2(\mathbb{R}^d))$ satisfying

$$W_2(\sigma(s), \sigma(t)) \leq \int_s^t m(r)dr \quad \forall \ a \leq s \leq t \leq b.$$

**Proof**: Refer proposition 3.3.1 in [2] with the fact that $P_2(\mathbb{R}^d)$ is weak* compact.

### 2.2 Estimates of heat equation and Stokes system

We first recall some estimates of the heat equation, which are useful for our purpose. For convenience, we denote $Q_t = \mathbb{R}^d \times [0, t]$ for $t > 0$.

Let $w$ be the solution of the following heat equation:

$$\partial_t w - \Delta w = \nabla \cdot g + h \quad \text{in } Q_t, \quad w(x, 0) = w_0,$$

where $g$ is a $d$-dimensional vector field and $h$ is a scalar function.

Let $(\alpha, \beta)$ with $d/\alpha + 2/\beta \leq 1$ and $d < \alpha < \infty$. Suppose that $\nabla w_0 \in L^\alpha_{\gamma}Q_t$, $g \in L^\beta_{x}L^\alpha_{\gamma}$ and $h \in L^\beta_{x}L^\alpha_{\gamma}$, where $1 < \gamma \leq \alpha$ with $1/r < 1/\alpha + 1/d$. Then, it follows that

$$\|\nabla w\|_{L^\beta_{x}L^\alpha_{\gamma}Q_t} \leq C \left( \|g\|_{L^\beta_{x}L^\alpha_{\gamma}} + t^{\frac{\beta(\frac{\alpha}{\beta} - 1)\gamma}{\alpha}} \|h\|_{L^\beta_{x}L^\alpha_{\gamma}} + t^{\frac{\beta}{d}} \|\nabla w_0\|_{L^\alpha_{\gamma}Q_t} \right)$$

(2.9)
The estimate (2.9) is well-known, but, for clarity, we give a sketch of its proof.

Indeed, we decompose $w = w_1 + w_2 + w_3$ such that $w_i, i = 1, 2, 3$ satisfies

$$\partial_t w_1 - \Delta w_1 = \nabla \cdot g \quad \text{in} \ Q_i, \quad w_1(x, 0) = 0$$
$$\partial_t w_2 - \Delta w_2 = h \quad \text{in} \ Q_i, \quad w_2(x, 0) = 0$$
$$\partial_t w_3 - \Delta w_3 = 0 \quad \text{in} \ Q_i, \quad w_3(x, 0) = w_0$$

Via representation formula of the heat equation for example $w_1$,

$$w_1(x, t) = -\int_0^t \int_{\mathbb{R}^d} \nabla \Gamma(x - y, t - s)g(y, s)dyds,$$

where $\Gamma$ is the heat kernel. Due to maximal regularity, we observe that

$$\|\nabla w_1\|_{L^p_t L^q_x} \leq C \|g\|_{L^p_t L^q_x}.$$

On the other hand, it is easy to see that

$$\|\nabla w_3\|_{L^p_t L^q_x} \leq C t^{\frac{d}{p} - 1} \|\nabla w_0\|_{L^p_t L^q_x}.$$ (2.10)

Indeed, via representation formula of the heat equation, we have

$$w_2(x, t) = \int_0^t \int_{\mathbb{R}^d} \Gamma(x - y, t - s)h(y, s)dyds.$$

Using potential estimate, we have

$$\|\nabla w_2(t)\|_{L^p_t L^q_x} \leq C \int_0^t (t - s)^{-\frac{d}{2} + \frac{d}{p} - \frac{d}{q} - \frac{1}{2}} \|h(s)\|_{L^p_t L^q_x} ds.$$

Integrating in time, we obtain

$$\|\nabla w_2\|_{L^p_t L^q_x} \leq C t^{-\frac{d}{p} + \frac{d}{2} + \frac{1}{2}} \|h\|_{L^p_t L^q_x}.$$ (2.11)

We deduce the estimate (2.9).

**Remark 2** In case that $g = 0$, for any $(\alpha, \beta)$ with $d/\alpha + 2/\beta \leq 1$, $d < \alpha \leq \infty$ and $\gamma < \infty$

$$\|\nabla w\|_{L^p_t L^q_x} \leq C \left(t^{\frac{d}{p} + 1 - \frac{d}{2}} \|h\|_{L^p_t L^q_x} + t^{\frac{d}{p}} \|\nabla w_0\|_{L^q_x}\right).$$ (2.11)

**Compared to (2.9), the case that $(\alpha, \beta) = (\infty, 2)$ is also included in (2.11).**

We also remind the maximal regularity of heat equation and Stoke system (see e.g. [21] and [25]). Let $w$ be a solution of

$$w_t - \Delta w = f \quad \text{in} \ Q_T := \mathbb{R}^d \times [0, T]$$
$$w(x, 0) = w_0(x) \quad \text{in} \ \mathbb{R}^d.$$

Then, the following a priori estimate holds for any $1 < p, q < \infty$

$$\|w_t\|_{L^p_t L^q_x(Q_T)} + \|\nabla^2 w\|_{L^p_t L^q_x(Q_T)} \leq C \|f\|_{L^p_t L^q_x(Q_T)} + CT^{\frac{d}{p}} \|\nabla^2 w_0\|_{L^q_x(\mathbb{R}^d)}.$$ (2.12)

In case of the following Stokes system

$$v_t - \Delta v + \nabla \pi = f, \quad \text{div} \ v = 0 \quad \text{in} \ Q_T := \mathbb{R}^d \times [0, T]$$
\[ v(x, 0) = v_0(x) \quad \text{in} \quad \mathbb{R}^d. \]

Similarly, it is known that the following a priori estimate holds:
\[ \|v_t\|_{L^p_t(L^q_x(Q_T))} + \|\nabla^2 v\|_{L^p_t(L^q_x(Q_T))} + \|\nabla \pi\|_{L^p_t(L^q_x(Q_T))} \leq C \|f\|_{L^p_t(L^q_x(Q_T))} + CT^\frac{\beta}{\alpha} \|\nabla^2 v_0\|_{L^p_x}, \quad (2.13) \]

where \(1 < p, q < \infty\).

In next lemma, we obtain some estimates of functions in \(Y_a\), which we will use later.

**Lemma 8** Let \(d/2 < a \leq d\) and \(d/\alpha + 2/\beta = 1\). Assume that \(\alpha, \beta, p\) and \(q\) are numbers satisfying
\[ \frac{d}{\alpha} + \frac{2}{\beta} = 1, \quad d < \alpha < \infty, \quad \frac{d}{p} + \frac{2}{q} = \frac{d}{a}. \]

Suppose that \(f \in Y_a(Q_T)\) and \(f(x, 0) = f_0 \in (L^a \cap L^\infty)(\mathbb{R}^d)\). Then,
\[
\|f\|_{L^2_tL^\infty_x(Q_T)} \leq CT^{1 - \frac{\beta}{\alpha}} \|f\|_{Y_a(Q_T)} + CT^\frac{\beta}{\alpha} \|f_0\|_{L^\infty_x}, \quad (2.14)
\]
\[
\|f\|_{L^2_tL^2_x(Q_T)} \leq CT^{1 - \frac{\beta}{\alpha}} \|f\|_{Y_a(Q_T)} + CT^\frac{\beta}{\alpha} \|f_0\|_{L^a_x}, \quad (2.15)
\]

**Proof.** Indeed, we consider
\[ \partial_t f - \Delta f = g \quad \text{in} \quad Q_t, \quad f(x, 0) = f_0(x) \quad \text{in} \quad \mathbb{R}^d. \]

Via representation formula of the heat equation, we have
\[ f(x, t) = \int_{\mathbb{R}^d} \Gamma(x - y, t)f_0(y)dy + \int_0^t \int_{\mathbb{R}^d} \Gamma(x - y, t - s)g(y, s)dyds, \]

where \(\Gamma\) is the heat kernel. Since \(g \in L^2_tL^2_x(Q_T)\), direct computations show that
\[
\|f\|_{L^2_tL^2_x(Q_T)} \leq C \left\| \int_0^T (t - s)^{-\frac{\beta}{\alpha}} \|g(s)\|_{L^2_x} ds \right\|_{L^1_t} + CT^\frac{\beta}{\alpha} \|f_0\|_{L^2_x}
\]
\[
\leq CT^{\frac{\beta}{\alpha}(1 - \frac{\beta}{\alpha})} \|g\|_{L^2_tL^2_x} + CT^\frac{\beta}{\alpha} \|f_0\|_{L^2_x}
\]
\[
\leq CT^{\frac{\beta}{\alpha}(1 - \frac{\beta}{\alpha})} \|f\|_{Y_a(Q_T)} + CT^\frac{\beta}{\alpha} \|f_0\|_{L^2_x}. \]

Since the other estimates can be similarly verified, we skip its details. \(\square\)

### 3 Proof of Theorem 1

In this section, we provide the proof of Theorem 1. We start with some a priori estimates.

**Lemma 9** Let \(\frac{2}{p} + \frac{2}{q} = 1\) and \(p > d\). Suppose that a non-negative function \(\rho_0 : \mathbb{R}^d \to \mathbb{R}\) and a vector field \(v : \mathbb{R}^d \to \mathbb{R}^d\) satisfy
\[ \rho_0 \in L^a(\mathbb{R}^d) \cap C^\infty_c(\mathbb{R}^d) \quad \text{and} \quad v \in L^q(0, T; L^p(\mathbb{R}^d)) \cap C^\infty_c([0, T] \times \mathbb{R}^d). \]

Let \(\alpha > 1\) and \(\rho\) be a solution of
\[ \partial_t \rho = \nabla \cdot (\nabla \rho - v \rho), \quad \rho(0, \cdot) = \rho_0. \quad (3.1) \]

Then we have
\[ \sup_{t \in [0, T]} \|\rho(t)\|_{L^a} \leq \|\rho_0\|_{L^a} e^{CT^\frac{\beta}{\alpha} \|v(t)\|_{L^p}}. \quad (3.2) \]
Proof. We multiply (3.1) with $\rho^{α-1}$ and integrate w.r.t spatial variable to get
\[
\frac{1}{α} \frac{d}{dt} \int_{\mathbb{R}^d} ρ^α \, dx + \frac{4(α-1)}{α^2} \int_{\mathbb{R}^d} |∇ ρ_ρ|^2 \, dx = -\int_{\mathbb{R}^d} ∇ \cdot (vρ) \, ρ^{α-1} \, dx.
\]
Due to the Gagliardo-Nirenberg inequality, namely
\[
C \leq \|∇ g\|_2^{1-θ'} \|g\|_2^{θ'} \quad \text{for} \quad \frac{1}{β} = \left(\frac{1}{2} - \frac{1}{d}\right)θ' + \frac{1}{2}, \quad 2 < β < 2^* = \frac{2d}{d-2}, \quad (3.3)
\]
the righthand side is estimated as follows:
\[
-\int_{\mathbb{R}^d} ∇ \cdot (vρ) \, ρ^{α-1} \, dx = (α-1) \int_{\mathbb{R}^d} ρ^{α-1} v \cdot ∇ ρ \, dx = \frac{2(α-1)}{α} \int_{\mathbb{R}^d} ρ^α v \cdot ∇ ρ \, dx
\leq \|v\|_{L^p} \|∇ ρ_ρ\|_{L^2} \|ρ\|_{L^2} \leq \frac{2(α-1)}{α^2} \|∇ ρ_ρ\|_{L^2}^2 + C \|v\|_{L^p}^{2p} \|ρ\|_{L^α}^p. \quad (3.4)
\]
Therefore, we obtain
\[
\frac{1}{α} \frac{d}{dt} \|ρ\|_{L^α}^α + \frac{2(α-1)}{α^2} \|∇ (ρ_ρ)\|_{L^2}^2 \leq C \|v\|_{L^p}^{2p} \|ρ\|_{L^α}, \quad (3.5)
\]
which yields
\[
\|ρ(t)\|_{L^α}^α \leq \|ρ_0\|_{L^α}^α e^{C \int_0^t \|v(τ)\|_{L^p}^{2p} \, dτ}. \quad (3.6)
\]
This completes the proof. □

Next, we estimate the Wasserstein distance, which turns out to be Hölder continuous.

Lemma 10 Let $\frac{d}{p} + \frac{2}{d} = 1$ and $p > d$. Suppose that a non-negative function $ρ_0 : \mathbb{R}^d \to \mathbb{R}$ and a vector field $v : \mathbb{R}^d \to \mathbb{R}^d$ satisfy
\[
ρ_0 \in \mathcal{P}_2(\mathbb{R}^d) \cap C^∞(\mathbb{R}^d) \quad \text{and} \quad v \in L^q(0, T; L^p(\mathbb{R}^d)) \cap C^∞([0, T] × \mathbb{R}^d).
\]
Let $ρ \in AC_2(0, T; L^∞(\mathbb{R}^d))$ be a solution of
\[
∂_t ρ = ∇ \cdot (∇ ρ - vρ), \quad ρ(0, \cdot) = ρ_0. \quad (3.7)
\]
Then we have
\[
W_2(ρ_s, ρ_t) \leq C\sqrt{T-s}, \quad \text{for all} \quad 0 ≤ s < t ≤ T \quad (3.8)
\]
for some positive constant $C = C(T, ∫_{\mathbb{R}^d} ρ_0 \ln ρ_0 \, dx, ∫_{\mathbb{R}^d} |x|^2 ρ_0 \, dx, ∫_{\mathbb{R}^d} |v|^p \, dx, ∫_{\mathbb{R}^d} |v|^q \, dx)$.

Proof. First, we estimate the second moment of $ρ$. We multiply $|x|^2$ to (3.7) and integrate
\[
\frac{d}{dt} \int_{\mathbb{R}^d} ρ_t |x|^2 \, dx = \int_{\mathbb{R}^d} |x|^2 ∇ \cdot (∇ ρ_t - v_t ρ_t) \, dx = 2d \int_{\mathbb{R}^d} ρ_t \, dx - ∫_{\mathbb{R}^d} x v_t ρ_t \, dx \quad (3.9)
\]
\[
≤ C + εμ \|∇ ρ_t\|_{L^2}^2 + C \|x ρ_t^{\frac{1}{2}}\|_{L^2}^2 + C \|v_t\|_{L^p}^{2p},
\]
where we used
\[
∫_{\mathbb{R}^d} |xv_t| ρ_t \, dx ≤ ∫_{\mathbb{R}^d} |x^2 ρ_t^{\frac{1}{2}}\|_{L^2} \|ρ_t^{\frac{1}{2}}\|_{L^∞} \|v_t\|_{L^p} \leq C ∫_{\mathbb{R}^d} |x ρ_t^{\frac{1}{2}}\|_{L^2}^2 ∫_{\mathbb{R}^d} |v_t|_{L^p}^{2p} \quad (3.10)
\]
Here $μ ∈ (0, 1)$ which will be specified later.
Next, we prove $\rho \in AC_2(0, T; P_2(\mathbb{R}^d))$. To do this, we rewrite (3.7) as follows

$$\partial_t \rho + \nabla \cdot (w\rho) = 0, \quad \text{where} \quad w := -\frac{\nabla \rho}{\rho} + v.$$  

We then show that

$$\int_0^T \|u_t\|_{L^2(\mu_t)}^2 dt < \infty. \quad (3.11)$$

Once we obtain (3.11), from Lemma 6, we are done with proving $\rho \in AC_2(0, T; P_2(\mathbb{R}^d))$. Thus, it suffices to prove (3.11). We multiply $\ln \rho$ to (3.7) and then integrate

$$\frac{d}{dt} \int_{\mathbb{R}^d} \rho_t \ln \rho_t dx + \int_{\mathbb{R}^d} \frac{\nabla \rho_t}{\rho_t} \cdot \nabla \rho_t dx = \int_{\mathbb{R}^d} v_t \cdot \nabla \rho_t dx$$

$$\leq \|v_t\|_{L^p} \|\nabla \rho_t^\frac{1}{2} \|_{L^2} \frac{1}{\rho_t^\frac{1}{2}} \leq \epsilon \|v_t\|_{L^p} \|\nabla \rho_t^\frac{1}{2} \|_{L^2}^{1+\frac{\epsilon}{2}} \leq \epsilon \|\nabla \rho_t^\frac{1}{2} \|_{L^2}^{2} + C \|v_t\|_{L^p}^{2n}, \quad (3.12)$$

where we used that

$$\|\rho_t^\frac{1}{2} \|_{L^\infty} \leq \|\rho_t^\frac{1}{2} \|_{L^2} \|\nabla \rho_t^\frac{1}{2} \|_{L^2}. \quad (3.13)$$

We note there exist constants $a, b > 0$ (independent of $\rho$) such that

$$-\int_{\rho(x) < 1} \rho \ln \rho \, dx \leq \int_{\mathbb{R}^d} |x|^2 \rho(x) \, dx + a. \quad (3.14)$$

Combining (3.9) and (3.12), we have

$$\frac{d}{dt} \left( \mu \int_{\mathbb{R}^d} \rho_t \ln \rho_t dx + \int_{\mathbb{R}^d} |x|^2 \rho_t dx \right) + \mu (1 - 2\epsilon) \int_{\mathbb{R}^d} \frac{\nabla \rho_t^2}{\rho_t}$$

$$\leq C + C \int_{\mathbb{R}^d} |x|^2 \rho_t dx + C \|v_t\|_{L^p}^{2n}. \quad (3.15)$$

On the other hand,

$$\int_{\mathbb{R}^d} |v_t|^2 \rho_t dx \leq \|v_t\|_{L^p}^2 \|\nabla \rho_t^\frac{1}{2} \|_{L^2}^{2n} \leq \|v_t\|_{L^p}^2 \|\nabla \rho_t^\frac{1}{2} \|_{L^2}^{2n} \leq C \|v_t\|_{L^p}^{2n} + \mu \|\nabla \rho_t^\frac{1}{2} \|_{L^2}^2. \quad (3.16)$$

From (3.15) and (3.16), we get

$$\frac{d}{dt} \left( \mu \int_{\mathbb{R}^d} \rho_t \ln \rho_t dx + \int_{\mathbb{R}^d} |x|^2 \rho_t dx \right) + \mu (1 - 3\epsilon) \int_{\mathbb{R}^d} \frac{\nabla \rho_t^2}{\rho_t} + \int_{\mathbb{R}^d} |v_t|^2 \rho dx$$

$$\leq C + C \int_{\mathbb{R}^d} |x|^2 \rho_t dx + C \|v_t\|_{L^p}^{2n}. \quad (3.17)$$

Integrating in time, we have

$$\left( \mu \int_{\mathbb{R}^d} \rho_t \ln \rho_t dx + \int_{\mathbb{R}^d} |x|^2 \rho_t dx \right) + \mu (1 - 3\epsilon) \int_0^t \int_{\mathbb{R}^d} \frac{\nabla \rho_t^2}{\rho_t} \, dx \, dt + \int_0^t \int_{\mathbb{R}^d} |v_t|^2 \rho_t \, dx \, dt$$

$$\leq \left( \mu \int_{\mathbb{R}^d} \rho_0 \ln \rho_0 dx + \int_{\mathbb{R}^d} |x|^2 \rho_0 dx \right) + C t + C \int_0^t \int_{\mathbb{R}^d} |x|^2 \rho_t \, dx \, dt + C \int_0^t \|v_t\|_{L^p}^{2n} \, dt.$$

We denote

$$f(t) = \mu \int_{\mathbb{R}^d} \rho_t (\ln \rho_t) dx + \int_{\mathbb{R}^d} |x|^2 \rho_t dx,$$

$$g(t) = \mu (1 - 3\epsilon) \int_0^t \int_{\mathbb{R}^d} \frac{\nabla \rho_t^2}{\rho_t} \, dx \, dt + \int_0^t \int_{\mathbb{R}^d} |v_t|^2 \rho_t \, dx \, dt.$$
Then

\[ f(t) + g(t) \leq f(0) + Ct + C \int_0^t \int_{\mathbb{R}^d} |x|^2 \rho_t \, dx \, d\tau + C \int_0^t \|v_t\|_{L^p} \, d\tau + \mu \int_{\mathbb{R}^d} \rho_t (\ln \rho_t)^- dV_t \]

\[ \leq f(0) + Ct + C \int_0^t f(\tau) \, d\tau + C \int_0^t \|v\|_{L^p} \, d\tau + \mu \int_{\mathbb{R}^d} \rho_t (\ln \rho_t)^- \, dx \]

\[ \leq f(0) + C(t + 1) + C \int_0^t f(\tau) \, d\tau + C \int_0^t \|v\|_{L^p} \, d\tau + \mu \int_{\mathbb{R}^d} |x|^2 \rho_t \, dx \]

\[ \leq f(0) + C(t + 1) + C \int_0^t f(\tau) \, d\tau + C \int_0^t \|v\|_{L^p} \, d\tau + \mu f(t). \]  

(3.18)

Taking \( \mu \) sufficiently small, it follows that

\[ f(t) + g(t) \leq Cf(0) + C(t + 1) + C \int_0^t f(\tau) \, d\tau + C \int_0^t \|v\|_{L^p} \, d\tau. \]  

(3.19)

Gronwall’s inequality implies that

\[ f(t) + g(t) \leq \left(1 + te^{Ct}\right) \left(f(0) + C(t + 1) + C \int_0^t \|v\|_{L^p} \, d\tau\right). \]  

(3.20)

Therefore, we obtain

\[ \int_{\mathbb{R}^d} \rho_t (\ln \rho_t)_+ \, dx + \int_{\mathbb{R}^d} |x|^2 \rho_t \, dx + \int_0^t \int_{\mathbb{R}^d} \frac{|\nabla \rho_t|^2}{\rho_t} \, dx \, d\tau + \int_0^t \int_{\mathbb{R}^d} |v_t|^2 \rho_t \, dx \, d\tau \leq C = C(t), \]  

(3.21)

where \( C(t) \) depends on

\[ t, \int_{\mathbb{R}^d} \rho_0 \ln \rho_0 \, dx, \int_{\mathbb{R}^d} |x|^2 \rho_0 \, dx, \int_0^t \int_{\mathbb{R}^d} \|v\|_{L^p} \, d\tau. \]

It follows that

\[ \int_{\mathbb{R}^d} \rho_t (\ln \rho_t)^- \leq C = C(t), \]  

(3.22)

which leads to

\[ \int_{\mathbb{R}^d} \rho_t |\ln \rho_t| \, dx + \int_{\mathbb{R}^d} |x|^2 \rho_t \, dx + \int_0^t \int_{\mathbb{R}^d} \left(\frac{|\nabla \rho_t|^2}{\rho_t} + |v_t|^2 \rho_t\right) \, dx \, d\tau \leq C = C(t). \]  

(3.23)

Finally, for any \( t \in [0, T] \), we have

\[ \int_0^t \|w_t\|_{L^2(\rho_t)}^2 \, d\tau \leq 2 \int_0^t \int_{\mathbb{R}^d} \frac{|\nabla \rho_t|^2}{\rho_t} \, dx \, d\tau + 2 \int_0^t \int_{\mathbb{R}^d} |v_t|^2 \rho_t \, dx \, d\tau \]

\[ \leq C(T). \]  

(3.24)

Due to (3.23), the constant \( C(T) \) in (3.24) only depends on

\[ T, \int_{\mathbb{R}^d} \rho_0 \ln \rho_0 \, dx, \int_{\mathbb{R}^d} |x|^2 \rho_0 \, dx, \int_0^T \int_{\mathbb{R}^d} \|v_t\|_{L^p} \, dt. \]

Hence, we have

\[ W_2(\rho(s), \rho(t)) \leq \int_s^t \|w_t\|_{L^2(\rho_t)} \, d\tau \leq C(T) \sqrt{t - s}, \]  

(3.25)

which concludes (3.8).
Now we present the proof of Theorem 1.

**Proof of Theorem 1** Suppose \( v \in L^q(0, T; L^p(\mathbb{R}^d)) \) and \( \rho_0 \in \mathcal{P}_2(\mathbb{R}^d) \) with \( \int_{\mathbb{R}^d} \rho_0 \ln \rho_0 \, dx < \infty \). By exploiting truncation and mollification, we may choose a sequence of vector fields \( v^n \in C_c^\infty ([0, T] \times \mathbb{R}^d) \) such that

\[
\lim_{n \to \infty} ||v^n - v||_{L^q(0, T; L^p(\mathbb{R}^d))} = 0. \tag{3.26}
\]

Using truncation, mollification and normalization in a similar way, we choose a sequence of functions \( \rho^n_0 \in C_c^\infty (\mathbb{R}^d) \cap \mathcal{P}_2(\mathbb{R}^d) \) satisfying

\[
\lim_{n \to \infty} W_2(\rho^n_0, \rho_0) = 0, \quad \lim_{n \to \infty} \int_{\mathbb{R}^d} \rho^n_0 \ln \rho^n_0 \, dx = \int_{\mathbb{R}^d} \rho_0 \ln \rho_0 \, dx. \tag{3.27}
\]

From Theorem of [17], we have \( \rho^n \in AC_2(0, T; \mathcal{P}_2(\mathbb{R}^d)) \) which is a solution of

\[
\begin{cases}
\partial_t \rho^n = \nabla \cdot (\nabla \rho^n - v^n \rho^n) \\
\rho^n(0, \cdot) = \rho^n_0,
\end{cases}
\]

that is,

\[
\int_0^T \int_{\mathbb{R}^d} \{ \partial_t \varphi(t, x) + \Delta \varphi(t, x) + \nabla \varphi(t, x) \cdot v^n(t, x) \} \rho^n(t, x) = - \int_{\mathbb{R}^d} \varphi(0, x) \rho^n_0(x), \tag{3.29}
\]

for every \( \varphi \in C_c^\infty ([0, T) \times \mathbb{R}^d) \).

**Step 1.** We claim that curves \( \rho^n : [0, T] \mapsto \mathcal{P}_2(\mathbb{R}^d) \) are equi-continuous. First, we exploit Lemma 10 and get

\[
W_2(\rho^n_s, \rho^n_t) \leq C_n \sqrt{t-s} \tag{3.30}
\]

where \( C_n \) only depending on \( T, \int_{\mathbb{R}^d} \rho^n_0 \ln \rho^n_0 \, dx, \int_{\mathbb{R}^d} |x|^2 \rho^n_0 \, dx, \int_0^T ||v^n||_{L^p}^\frac{2p}{p-2} \, dt \) \tag{3.31}

Due to (3.26) and (3.27), we conclude

\[
W_2(\rho^n_s, \rho^n_t) \leq C \sqrt{t-s} \quad \text{for all sufficiently large } n, \tag{3.32}
\]

where \( C \) only depending on \( \int_{\mathbb{R}^d} \rho_0 \ln \rho_0 \, dx, \int_{\mathbb{R}^d} |x|^2 \rho_0 \, dx, \int_0^T ||v||_{L^p}^\frac{2p}{p-2} \, dt \) \tag{3.33}

**Step 2.** Estimation (3.32) says that curves \( \rho^n : [0, T] \mapsto \mathcal{P}_2(\mathbb{R}^d) \) are equi-continuous and hence there exists a curve \( \rho : [0, T] \mapsto \mathcal{P}_2(\mathbb{R}^d) \) such that , as \( n \to \infty \) (up to subsequence)

\[
\rho^n(t) \text{ weak* converges to } \rho(t) \quad \text{and} \quad W_2(\rho(s), \rho(t)) \leq C \sqrt{t-s}, \quad \forall \ 0 \leq s \leq t \leq T. \tag{3.34}
\]

Due to (3.23), we note that the uniform entropy bound on \( \rho^n \) implies \( \rho(t) \in \mathcal{P}_2^{ac}(\mathbb{R}^d) \) for all \( t \in [0, T] \). Furthermore, the convergence in (3.26) gives us

\[
\int_0^T \int_{\mathbb{R}^d} (\nabla \varphi \cdot v^n) \rho^n \, dx \, dt \to \int_0^T \int_{\mathbb{R}^d} (\nabla \varphi \cdot v) \rho \, dx \, dt, \quad \text{as } n \to \infty, \tag{3.35}
\]

for any \( \varphi \in C_c^\infty ([0, T) \times \mathbb{R}^d) \). Indeed, we have

\[
\begin{align*}
\int_0^T \int_{\mathbb{R}^d} [\nabla \varphi \cdot v^n \rho^n - \nabla \varphi \cdot v \rho] \, dx \, dt \\
&= \int_0^T \int_{\mathbb{R}^d} \nabla \varphi \cdot (v^n - v) \rho^n \, dx \, dt + \int_0^T \int_{\mathbb{R}^d} \nabla \varphi \cdot (v \rho^n - v \rho) \, dx \, dt \\
&= I + II,
\end{align*}
\]

12
where

\[ |I| \leq \|\nabla \varphi\|_{L^\infty([0,T] \times \mathbb{R}^d)} \int_0^T \int_{\mathbb{R}^d} |v^n - v|^2 \rho^n \ dx dt \]
\[ \leq T \|\nabla \varphi\|_{L^\infty([0,T] \times \mathbb{R}^d)} \int_0^T \int_{\mathbb{R}^d} |v^n - v|^2 \rho^n \ dx dt. \]  

(3.37)

Due to (3.16), we note that

\[ \int_0^T \int_{\mathbb{R}^d} |v^n - v|^2 \rho^n \ dx dt \leq \int_0^T \|v^n - v\|^2_{L^p} \|\nabla (\rho^n)^{\frac{1}{2}}\|_{L^q}^2 \ dx dt \]
\[ \leq \left( \int_0^T \|v^n - v\|^2_{L^p} \right)^{\frac{2}{2+a}} \left( \int_0^T \|\nabla (\rho^n)^{\frac{1}{2}}\|_{L^q}^2 \right)^{\frac{1}{2}} \]  

(3.38)

where the last inequality follows from (3.23). Combining (3.37) and (3.38), we have

\[ |I| \leq C \|v^n - v\|^2_{L^2(0,T;L^p(\mathbb{R}^d))} \rightarrow 0 \]  

(3.39)

as \( n \rightarrow \infty \) due to (3.26). Also, weak* convergence in (3.34) implies that \( II \) converges to 0 as \( n \rightarrow 0 \). We plug (3.34) and (3.35) into (3.29), and get

\[ \int_0^T \int_{\mathbb{R}^d} \left\{ \partial_t \varphi(t,x) + \Delta \varphi(t,x) + \nabla \varphi(t,x) \cdot v(t,x) \right\} \rho(t,x) \ dx \ dx dt = - \int_{\mathbb{R}^d} \varphi(0,x) \rho_0(x) \ dx, \]  

(3.40)

for each \( \varphi \in C_c(0,T) \times \mathbb{R}^d \). This implies that \( \rho \) solves

\[ \left\{ \begin{array}{l} \partial_t \rho = \nabla \cdot (\nabla \rho - v \rho) \\
\rho(0) = \rho_0. \end{array} \right. \]  

(3.41)

Step 3. If \( \rho_0 \in L^\alpha(\mathbb{R}^d) \) for \( \alpha > 1 \) then, in addition to (3.27), we may choose \( \rho^n_0 \) satisfying

\[ \lim_{n \rightarrow \infty} \|\rho^n_0 - \rho_0\|_{L^\alpha(\mathbb{R}^d)} = 0. \]  

(3.42)

From (3.2), we have

\[ \sup_{t \in [0,T]} \|\rho^n(t)\|_{L^\alpha} \leq \|\rho_0^n\|_{L^\alpha} e^{C \int_0^T \|v^n(t)\|_{L^\alpha}}. \]  

(3.43)

Due to the lower semicontinuity of \( L^\alpha \)-norm with respect to the weak* convergence, we take \( n \rightarrow \infty \) in (3.43) and get

\[ \sup_{t \in [0,T]} \|\rho(t)\|_{L^\alpha} \leq \|\rho_0\|_{L^\alpha} e^{C \int_0^T \|v(t)\|_{L^\alpha}}. \]  

(3.44)

which completes the proof.

\[ \square \]

4 Proof of Theorem 2

Reminding function spaces \( X_a \) and \( Y_a \) defined in (1.19)-(1.20), we denote \( W := X_a \times Y_a \times Y_a \). Let \( M > 0 \) be a positive number and we introduce \( W^M := X_a^M \times Y_a^M \times Y_a^M \), where

\[ X_a^M := \{ f \in X_a \| f \|_{X_a} \leq M \}, \quad Y_a^M := \{ f \in Y_a \| f \|_{Y_a} \leq M \}. \]

For convenience, for \( a > d/2 \) we denote

\[ \eta := 1 - \frac{d}{2a} = \frac{2a - d}{2a}. \]  

(4.1)
To construct solutions, we will use the method of iterations. Setting $\rho_1(x, t) := \rho_0(x)$, $c_1(x, t) := c_0(x)$ and $u_1(x, t) := u_0(x)$, we consider the following: $k = 1, 2, \cdots$

\[ \partial_t \rho_{k+1} = \nabla \cdot (\nabla \rho_{k+1} - u_k \rho_{k+1} - \chi(c_k) \nabla c_{k+1} \rho_{k+1}), \quad \rho_{k+1}(\cdot, 0) = \rho_0, \quad (4.2) \]

\[ \partial_t c_{k+1} - \Delta c_{k+1} = -u_k \nabla c_{k+1} - \kappa(c_k) \rho_k, \quad c_{k+1}(\cdot, 0) = c_0, \quad (4.3) \]

\[ \partial_t u_{k+1} - \Delta u_{k+1} + \nabla \rho_{k+1} = -u_k \nabla u_k - \rho_k \nabla \phi, \quad \text{div } u_{k+1} = 0 \quad u_{k+1}(\cdot, 0) = u_0. \quad (4.4) \]

Here for given $M > 0$ we assume that $\rho_0$ and $c_0$ are non-negative and

\[ \|\rho_0\|_{L^1 \cap L^\infty([0, T])} < \frac{M}{6}, \quad \|c_0\|_{W^{2, \alpha}([0, T])} < \frac{M}{6}, \quad \|u_0\|_{W^{2, \alpha}([0, T])} < \frac{M}{6}. \quad (4.5) \]

We note, due the maximum principle, that $c_k$ is uniformly bounded, i.e., $\|c_k(\cdot, t)\|_{L^\infty([0, T])} \leq \|c_0\|_{L^\infty([0, T])}$. Now we are ready to present the proof of Theorem 2.

**Proof of Theorem 2** In case that $a > d$ is rather easy. From now on, we consider only the case that $d/2 < a \leq d$. For simplicity, we assume that $\int_{\mathbb{R}^d} \rho_0(x) \, dx = 1$ (if not, we replace $\rho_0$ by $\rho_0/\int_{\mathbb{R}^d} \rho_0 \, dx$, which yield any crucial change for the local existence of solutions). Under the hypothesis that $(\rho_k, c_k, u_k) \in W^M$, we first show that $(\rho_{k+1}, c_{k+1}, u_{k+1}) \in W^M$ for sufficiently small $T$, which will be specified later.

First, recalling the equation (4.2) and using the estimate (1.18), in particular the case that $(\alpha, \beta) = (2a, \tilde{a})$, i.e., $\tilde{a} = 4a/(2a - d)$ we obtain for all $t \in [0, T]$

\[ \|\rho_{k+1}(t)\|_{L^\infty([0, T])} \leq \|\rho_0\|_{L^\infty([0, T])} \exp \left( C(\|u_k + \chi(c_k) \nabla c_{k+1}(t)\|_{L^p}^2)^{\frac{4a}{d - a}} \right) \]

\[ \leq \|\rho_0\|_{L^\infty([0, T])} \exp \left( C(\|u_k\|_{L^p}^2 + \|\nabla c_{k+1}\|_{L^p}^2)^{\frac{4a}{d - a}} \right). \]

We note, due to (2.9), that

\[ \|\nabla c_{k+1}\|_{L^p L^2} \leq C \|u_k c_{k+1}\|_{L^p L^2} + C t^{\frac{1}{2}} (1 - \frac{d}{a}) \|\rho_k\|_{L^p L^2} + C t^{\frac{4a}{d - a}} \|\nabla c_0\|_{L^2} \]

\[ \leq C \|u_k\|_{L^p L^2} + C t^{\frac{1}{2}} (1 - \frac{d}{a}) \|\rho_k\|_{L^p L^2} + C t^{\frac{4a}{d - a}} \|\nabla c_0\|_{L^2} \]

\[ \leq C t^{\frac{1}{2}} \|u_k\|_{Y_\alpha(Q_t)} + C t^{\frac{1}{2}} (1 - \frac{d}{a}) \|\rho_k\|_{L^p \cap L^2} + C t^{\frac{4a}{d - a}} \left( \|u_0\|_{L^2} + \|\nabla c_0\|_{L^2} \right), \quad (4.6) \]

where we used (2.15) and $\|c_{k+1}(t)\|_{L^\infty} \leq \|c_0\|_{L^\infty}$. Thus, summing up the estimates, we have

\[ \|\rho_{k+1}(t)\|_{L^\infty([0, T])} \leq \|\rho_0\|_{L^\infty([0, T])} \times \]

\[ \times \exp \left( C t^0 \|u_k\|_{Y_\alpha(Q_t)} + C t^{\frac{4a}{d - a}} \|\rho_k\|_{X_{\alpha}(Q_t)} + C T^{\frac{4a}{d - a}} \left( \|u_0\|_{L^2} + \|\nabla c_0\|_{L^2} \right) \right). \]

Since $\|u_k\|_{Y_\alpha(Q_t)} \leq M$, $\|\rho_k\|_{X_{\alpha}(Q_t)} \leq M$ and $\|\rho_0\|_{L^\infty([0, T])} < M/2$, by taking a sufficiently small $T > 0$ we obtain

\[ \|\rho_{k+1}\|_{X_{\alpha}(Q_t)} \leq M. \quad (4.7) \]

Next, we consider the equation (4.4). Let $p$ with $a < p < \tilde{a} = ad/(d - a)$ with $a > d/2$ and $q$ with $d/p + 2/q = d/a$. We then define numbers $\tilde{p}$ and $\tilde{q}$ by $1/\tilde{p} + 1/p = 1/a$ and $1/\tilde{q} + 1/q = 1/2$, respectively. We note that $d/\tilde{p} + 2/\tilde{q} = 1$ and $\tilde{p} > d$. We then compute that

\[ \|u_k \nabla u_k\|_{L^2 L^p(Q_t)} \leq \|u_k\|_{L^p} \|\nabla u_k\|_{L^p} \leq \|u_k\|_{L^p L^\infty} \|\nabla u_k\|_{L^p L^p} \]

\[ \leq C \left( T^0 \|u_k\|_{Y_\alpha(Q_t)} + T^{\frac{4a}{d - a}} \|u_0\|_{L^p(\mathbb{R}^d)} \left( \|u_k\|_{Y_\alpha(Q_t)} + T^{\frac{4a}{d - a}} \|\nabla u_0\|_{L^p(\mathbb{R}^d)} \right) \right) \]
\[
\leq C \left( T^n \| u_k \|_{Y_\alpha(Q_\alpha)} + (T^{n+\frac{1}{2}} + T^\frac{1}{2}) \| u_0 \|_{W^{2,\alpha}(\mathbb{R}^d)} \right) \| u_k \|_{Y_\alpha(Q_\alpha)} + CT^\frac{1}{2} \| u_0 \|_{W^{2,\alpha}(\mathbb{R}^d)}. \tag{4.8}
\]

Via maximal regularity (2.13) of the Stokes system and (4.8), we have for all \( k \geq 1 \)

\[
\| \partial_t u_{k+1} \|_{L_T^2 L_x^6(Q_T)} + \| \nabla^2 u_{k+1} \|_{L_T^2 L_x^6(Q_T)} + \| \nabla \rho_{k+1} \|_{L_T^2 L_x^6(Q_T)}
\leq C \left( T^n \| u_k \|_{Y_\alpha(Q_\alpha)} + (T^{n+\frac{1}{2}} + T^\frac{1}{2}) \| u_0 \|_{W^{2,\alpha}(\mathbb{R}^d)} \right) \| u_k \|_{Y_\alpha(Q_\alpha)} + C(T^\frac{1}{2} \| u_0 \|_{W^{2,\alpha}(\mathbb{R}^d)} + T^\frac{1}{2}) \| u_0 \|_{W^{2,\alpha}(\mathbb{R}^d)}, \tag{4.9}
\]

where we used that \( \| \nabla \phi \|_{L_\infty(\mathbb{R}^d)} < C \).

For the control of lower order derivative of \( u_{k+1} \), via potential estimate, we can also have

\[
\| u_{k+1} \|_{L_T^1 L_x^6(Q_T)} \leq C \left( T^\frac{3}{2} + T \right) \| u_k \|_{Y_\alpha(Q_\alpha)}^2 + T^\frac{1}{2} \| u_0 \|_{W_x^6(Q_\alpha)}^2 \]
\[
+ CT^\frac{3}{2} \| \rho_{k+1} \|_{X_\alpha(Q_\alpha)} + CT^\frac{1}{2} \| u_0 \|_{L_\infty(\mathbb{R}^d)}. \tag{4.10}
\]

Thus, combining estimates (4.7), (4.9) and (4.10), and taking \( T \) sufficiently small, we note that

\[
\| u_{k+1} \|_{Y_\alpha(Q_T)} \leq M. \tag{4.11}
\]

On the other hand, similarly as in (4.8), we note that

\[
\| u_k \nabla c_{k+1} \|_{L_T^2 L_x^5(Q_T)} \leq \| u_k \|_{L_T^\infty L_x^6} \| \nabla c_{k+1} \|_{L_T^2 L_x^5} \leq \| u_k \|_{L_T^\infty L_x^6} \| \nabla c_{k+1} \|_{L_T^2 L_x^5}
\leq C \left( T^n \| u_k \|_{Y_\alpha(Q_\alpha)} + T^\frac{1}{2} \| u_0 \|_{L_\infty(\mathbb{R}^d)} \right) \left( \| c_{k+1} \|_{Y_\alpha(Q_\alpha)} + T^\frac{1}{2} \| \nabla c_0 \|_{L_\infty(\mathbb{R}^d)} \right)
\leq C \left( T^n \| u_k \|_{Y_\alpha(Q_\alpha)} + T^\frac{1}{2} \| u_0 \|_{L_\infty(\mathbb{R}^d)} \right) \| c_{k+1} \|_{Y_\alpha(Q_\alpha)}
\leq C T^n \| u_k \|_{Y_\alpha(Q_\alpha)} + \| u_0 \|_{W^{2,\alpha}(\mathbb{R}^d)} \| c_{k+1} \|_{Y_\alpha(Q_\alpha)} + C T^\frac{1}{2} \| c_0 \|_{W^{2,\alpha}(\mathbb{R}^d)} \| u_k \|_{Y_\alpha(Q_\alpha)}
\leq C T^\frac{1}{2} \| \rho_k \|_{X_\alpha} + C(T^\frac{1}{2} \| u_0 \|_{W^{2,\alpha}(\mathbb{R}^d)} + T^\frac{1}{2}) \| c_0 \|_{W^{2,\alpha}(\mathbb{R}^d)}, \tag{4.12}
\]

Using maximal regularity (2.12) for the equation (4.3) and the estimate (4.12), we have

\[
\| \partial_t c_{k+1} \|_{L_T^2 L_x^5(Q_T)} + \| \nabla^2 c_{k+1} \|_{L_T^2 L_x^5(Q_T)}
\leq C \left( T^n \| u_k \|_{Y_\alpha(Q_\alpha)} + (T^{n+\frac{1}{2}} + T^\frac{1}{2}) \| c_0 \|_{W^{2,\alpha}(\mathbb{R}^d)} \right) \| c_{k+1} \|_{Y_\alpha(Q_\alpha)} + C T^{n+\frac{1}{2}} \| c_0 \|_{W^{2,\alpha}(\mathbb{R}^d)} \| u_k \|_{Y_\alpha(Q_\alpha)}
\leq C T^\frac{1}{2} \left( T^n \| u_k \|_{Y_\alpha(Q_\alpha)} + \| u_0 \|_{W^{2,\alpha}(\mathbb{R}^d)} \right) \| c_{k+1} \|_{Y_\alpha(Q_\alpha)} + CT^\frac{1}{2} \| c_0 \|_{W^{2,\alpha}(\mathbb{R}^d)} \| u_k \|_{Y_\alpha(Q_\alpha)}
\leq C \left( T^\frac{1}{2} \| \rho_k \|_{X_\alpha} + C \| u_0 \|_{W^{2,\alpha}(\mathbb{R}^d)} \right) \| c_{k+1} \|_{Y_\alpha(Q_\alpha)} + C T^\frac{1}{2} \| c_0 \|_{W^{2,\alpha}(\mathbb{R}^d)} \| u_k \|_{Y_\alpha(Q_\alpha)}, \tag{4.13}
\]

where we used that \( \| \kappa(c_k) \|_{L_\infty(\mathbb{R}^d)} < C = C(\| c_0 \|_{L_\infty}) \) and \( \eta \) is the number defined in (4.1).

Since estimates of lower order derivative of \( c_{k+1} \) can be obtained as in (4.10), we skip its details (from now on the estimate of lower order derivatives are omitted, unless it is necessary to be specified). Taking \( T \) small enough such that

\[
T^\frac{1}{2} \left( T^n \| u_k \|_{Y_\alpha(Q_\alpha)} + \| u_0 \|_{W^{2,\alpha}(\mathbb{R}^d)} \right) \leq C T^\frac{1}{2} \left( T^n M + \| u_0 \|_{W^{2,\alpha}(\mathbb{R}^d)} \right) \leq \frac{1}{2},
\]

the estimate (4.13) becomes

\[
\| c_{k+1} \|_{Y_\alpha} \leq C \left( T^{\frac{1}{2}+\eta} \| c_0 \|_{W^{2,\alpha}(\mathbb{R}^d)} \right) \| u_k \|_{Y_\alpha(Q_\alpha)} + T^\frac{1}{2} \| \rho_k \|_{X_\alpha} + \left( T^\frac{1}{2} \| u_0 \|_{W^{2,\alpha}(\mathbb{R}^d)} + T^\frac{1}{2} \right) \| c_0 \|_{W^{2,\alpha}(\mathbb{R}^d)},
\]
which yields again for a sufficiently small $T$

$$\|c_{k+1}\|_{Y_n} \leq M.$$ (4.14)

Next, we show that this iteration gives a fixed point via contraction, which turns out to be unique solution of the system under considerations. Let $\delta_k \rho := \rho_{k+1} - \rho_k$, $\delta_k c := c_{k+1} - c_k$ and $\delta_k u := u_{k+1} - u_k$. We then see that $(\delta_k \rho, \delta_k c, \delta_k u)$ solves

$$\begin{align*}
\partial_t \delta_k \rho - \Delta \delta_k \rho &= -\nabla \cdot \left( u_k \delta_k \rho + \delta_{k-1} u \rho_k + \chi(c_k) \nabla c_{k+1} \delta_k \rho \right) \\
&\quad + \chi(c_k) \nabla \delta_k c \rho_k + \delta_{k-1} \chi(\cdot) \nabla c_{k+1} \delta_k \rho,
\end{align*}$$

$$\begin{align*}
\partial_t \delta_k c - \Delta \delta_k c &= -\delta_{k-1} u \nabla c_{k+1} - u_k \nabla \delta_k c - \delta_{k-1} \kappa(\cdot) \rho_k - \kappa(c_k) \delta_{k-1} \rho,
\end{align*}$$

$$\begin{align*}
\partial_t \delta_k u - \Delta \delta_k u + \nabla \delta_k p &= -\delta_{k-1} u \nabla u_k - u_k \nabla \delta_{k-1} u - \delta_k \rho \nabla \phi,
\end{align*}$$

where $\delta_{k-1} \chi(\cdot) := \chi(c_k) - \chi(c_{k-1})$ and $\delta_{k-1} \kappa(\cdot) := \kappa(c_k) - \kappa(c_{k-1})$. Here zero initial data are given, namely $\delta_k \rho(\cdot, 0) = 0$, $\delta_k c(\cdot, 0) = 0$ and $\delta_k u(\cdot, 0) = 0$.

For convenience, we denote

$$h_1 := u_k \delta_k \rho, \quad h_2 := \delta_{k-1} u \rho_k, \quad h_3 := \chi(c_k) \nabla c_{k+1} \delta_k \rho,$$

$$h_4 := \chi(c_k) \nabla \delta_k c \rho_k, \quad h_5 := \delta_{k-1} \chi(\cdot) \nabla c_{k+1} \delta_k \rho.$$
The term $H_5$ can be similarly estimated as $H_3$, namely
\[
H_5 \leq C \left\| \int_0^t (t-s)^{-\frac{d}{2}+\frac{1}{2}} \| \delta_{k-1} \cdot \cdot \cdot \|_{L^\infty} \right\|_{L^2_t} \| \rho_k \|_{L^2_t} ds \right\|_{L^\infty_t} \\
\leq C \| \delta_{k-1} \cdot \cdot \cdot \|_{L^\infty} \| \nabla c_k \|_{L^{4,2}} \| \rho_k \|_{L^{4,2}} L^2_t \\
\leq C \left( t^{1-\frac{d}{42}} \| u_k \|_{Y_n(Q_t)} + t^{\frac{d}{4}+1} \| \rho_k \|_{L^{4,2}} \| \cdot \cdot \cdot \|_{L^{4,2}} \right) \| \rho_k \|_{L^{4,2}} \delta_{k-1} \cdot \cdot \cdot \|_{L^\infty} \\
\leq C t^\frac{\alpha}{\alpha+2} (t^{\frac{d}{4}+1} + t^{\frac{d}{4}}) \| \delta_{k-1} \cdot \cdot \cdot \|_{Y_n}.
\]
Before we estimate $H_4$, we let \( \tilde{a} \) with $1/\tilde{a} = 1/a - 1/d$. Choosing $\alpha$ and $\beta$ with $\tilde{a} < \alpha < \infty$ and $d/\alpha + 2/\beta = 1$, we have
\[
\| \nabla \delta c \|_{L^2_t L^2_x} \leq C \left( \| \delta_{k-1} \cdot u_c \cdot \cdot \cdot \|_{L^2_t L^2_x} + \| u_k \|_{L^2_t L^2_x} \right) \| \delta_{k-1} \cdot \cdot \cdot \|_{L^2_t L^2_x} \\
\leq C \left( \| \delta_{k-1} \cdot \cdot \cdot \|_{L^2_t L^2_x} + \| u_k \|_{L^2_t L^2_x} \right) \| \delta_{k-1} \cdot \cdot \cdot \|_{L^\infty} \| \cdot \cdot \cdot \|_{L^\infty} + \| \cdot \cdot \cdot \|_{L^\infty} \| \delta_{k-1} \cdot \cdot \cdot \|_{L^\infty} \\
\leq C t^\frac{\alpha}{\alpha+2} (\| \delta_{k-1} \cdot \cdot \cdot \|_{Y_n} + \| \delta_{k-1} \cdot \cdot \cdot \|_{Y_n} + \| \delta_{k-1} \cdot \cdot \cdot \|_{Y_n} + C t^\frac{\alpha}{\alpha+2} \| \delta_{k-1} \cdot \cdot \cdot \|_{Y_n} + t^\frac{\alpha}{\alpha+2} \| \delta_{k-1} \cdot \cdot \cdot \|_{Y_n} + C t^\frac{\alpha}{\alpha+2} \| \delta_{k-1} \cdot \cdot \cdot \|_{Y_n}.
\]
Using the estimate (4.18), we obtain
\[
H_4 \leq C \int_0^t (t-s)^{-\frac{d}{4}+\frac{1}{4}} \| \nabla \delta c \|_{L^2_t L^2_x} \| \cdot \cdot \cdot \|_{L^\infty_t} \| \cdot \cdot \cdot \|_{L^\infty_t} \\
\leq C \| \nabla \delta c \|_{L^2_t L^2_x} \| \cdot \cdot \cdot \|_{L^\infty_t} \| \cdot \cdot \cdot \|_{L^\infty_t} \\
\leq C t^\frac{\alpha}{\alpha+2} (\| \delta_{k-1} \cdot \cdot \cdot \|_{Y_n} + \| \delta_{k-1} \cdot \cdot \cdot \|_{Y_n} + \| \delta_{k-1} \cdot \cdot \cdot \|_{Y_n} + C t^\frac{\alpha}{\alpha+2} \| \delta_{k-1} \cdot \cdot \cdot \|_{Y_n} + t^\frac{\alpha}{\alpha+2} \| \delta_{k-1} \cdot \cdot \cdot \|_{Y_n} + C t^\frac{\alpha}{\alpha+2} \| \delta_{k-1} \cdot \cdot \cdot \|_{Y_n}.
\]
Therefore, we obtain
\[
\| \delta_{k-1} \cdot \cdot \cdot \|_{Y_n} \leq \max(T^n, T^\frac{\alpha}{\alpha+2}) \| \delta_{k-1} \cdot \cdot \cdot \|_{Y_n} + T^n \| \delta_{k-1} \cdot \cdot \cdot \|_{Y_n}
\]
Next, via maximal regularity of the Stokes system, we estimate $\delta u$ as follows:
\[
\| \delta_{k-1} \cdot \cdot \cdot \|_{L^2_t L^2_x} \leq \max(T^n, T^\frac{\alpha}{\alpha+2}) \| \delta_{k-1} \cdot \cdot \cdot \|_{Y_n} + T^n \| \delta_{k-1} \cdot \cdot \cdot \|_{Y_n}.
\]
This yields fixed point via the theory of the contraction mapping. Once we prove (4.21), we complete the proof. Note that this can be done exactly the same as we did in the proof of Theorem 1.

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