Mass and temperature limits for blackbody radiation

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A spherically symmetric distribution of classical blackbody radiation is considered, at conditions in which gravitational self-interaction effects become not negligible. Static solutions to Einstein field equations are searched for, for each choice of the assumed central energy density. Spherical cavities at thermodynamic equilibrium, i.e. filled with blackbody radiation, are then studied, in particular for what concerns the relation among the mass $M$ of the ball of radiation contained in them and their temperature at center and at the boundary. For these cavities it is shown, in particular, that: i) there is no absolute limit to $M$ as well to their central and boundary temperatures; ii) when radius $R$ is fixed, however, limits exist both for mass and for boundary energy density $\rho_B$: $M \leq KS(R)$ and $\rho_B \leq Q/R^2$, with $K = 0.493$ and $Q = 0.02718$, dimensionless, and $S(R)$ the Schwarzschild mass for that radius. Some implications of the existence and the magnitude of these limits are considered. Finally the radial profiles for entropy for these systems are studied, in their dependence on the mass (or central temperature) of the ball of radiation.

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I. INTRODUCTION

The original motivation for this work is to explore the relation between entropy and gravity in a classical context (i.e. not quantum). The goal is to see if, under strong gravitational interaction circumstances, some properties peculiar to black hole entropy (such as for example its area-scaling law), apparent in a semiclassical approach, can also be retrieved for conventional entropy in a purely classical scenario, in which the fields too, in addition to the background, are classical. Having this in mind, systems consisting entirely of blackbody radiation are chosen for investigation, with the hope that for them, both entropy can be readily calculated and, also, strong gravitational interaction circumstances can be reached. During the analysis of such systems we have found for them the existence of peculiar mass and temperature limits for each assigned radius $R$.

The idea that under strong gravitational interaction conditions conventional entropy can acquire properties similar to black hole entropy is not new. In particular in [1] it has been shown that the entropy of a spherically symmetric self-gravitating system in equilibrium becomes area scaling as the radius approaches the Schwarzschild value; in this limit the total entropy of the body is accounted for only by the entropy of the more external spherical shells. In [2] the entropy of a spherically symmetric system in equilibrium consisting of a perfect fluid with constant density has been explicitly calculated. The entropy has been found to loose its extensivity property and to reside more and more in the external spherical shells as the gravitational interaction goes stronger. The limit $R = 2M$ ($M$ is the mass, or mass-energy, as spelled out in [2], known also as ADM mass) cannot be reached in this case, as equilibrium is possible only if $R \geq (9/4)M$ (see, for instance, [2]). One would like to explicitly calculate entropy for equilibrium systems with other equations of state, possibly going to explore also the limit $R \simeq 2M$. But it turns out that for all viable examples the radius of the system becomes infinite. This is a problem because in this case it becomes unclear what is intended with the concepts of “more external shells” and “mass” of the system. The recourse to different equations of state for different radii can give realistic models with finite size, but can hide the relation between entropy and gravity behind the circumstantial peculiarities of the constructed model.

As well known (see for example [3]) and we will recall later, a gravitationally self-interacting static distribution of blackbody radiation is in general precisely one of these infinite size systems. From a thermodynamical point of view this system is particularly appealing due to its simplicity, as its state is locally determined by only one thermodynamical observable, for example its temperature $T$. It is then tempting to use such a system to investigate the connection between entropy and gravity and we can imagine to circumvent the problem of its being of infinite size by forcing it to be adiabatically confined inside a finite size cavity. By applying a sufficiently high compression one could then expect the system to go towards its Schwarzschild limit. In principle it becomes then possible to study the behaviour of a pure classical system, this cavity filled with blackbody radiation (in our study all what is needed of blackbody radiation are purely classical properties, as we will see), in a strong gravitational regime. At these conditions, by the way, some quantum behaviour is expected to come on for this...
system, as, of course, pair production or other effects. We decide however to study the system at a purely classical level, and to postpone any quantum consideration (pair production) at the end, once the characteristics classically expected are known. All this is the content of the following section.

II. FIELD EQUATIONS AND THEIR SOLUTIONS

We consider a perfect fluid consisting of a photon gas at thermodynamic equilibrium (somehow along the lines of [5], in which this same system is approached, even if with different context and motivations) or, without quantum terminology, consisting of blackbody radiation. Its equation of state is

\[ p = \rho/3, \]  

where \( \rho \) and \( p \) are respectively the energy density and pressure in the rest frame of the fluid element. Its stress-energy tensor is

\[ T_{\alpha\beta} = (\rho + p)u_\alpha u_\beta + pg_{\alpha\beta}, \]  

where \( u_\alpha \) is the fluid element 4-velocity and \( g_{\alpha\beta} \) is the metric tensor. In the case of static equilibrium configurations (necessarily with spherical symmetry), from (1) and (2) Einstein field equations in Schwarzschild coordinates read

\[ \frac{h'}{r^2} + \frac{h - 1}{hr^2} = 8\pi \rho (tt) \]

\[ \frac{f'}{rfh} - \frac{h - 1}{hr^2} = \frac{8\pi}{3} \rho (rr) \]

\[ \frac{f'}{rfh} - \frac{h'}{r^2} + \frac{1}{\sqrt{fh}} \left( \frac{f'}{\sqrt{fh}} \right)' = \frac{16\pi}{3} \rho (\theta\theta) = (\phi\phi), \]

where the derivative symbol means derivative with respect to radial coordinate (area radius) \( r \) and where the metric is

\[ ds^2 = -f(r) \, dt^2 + h(r) \, dr^2 + r^2 \, d\Omega^2, \]

with \( f \) and \( h \) two non-negative functions of \( r \), to be determined.

For static gravitational fields, Tolman relation [6] prescribes that \( T\sqrt{-g_{00}} \) be constant throughout the system, where \( T \) is the temperature in the rest frame of the fluid element. This means for us that

\[ T(r)\sqrt{f(r)} = \text{const.} \]

For blackbody radiation, Stefan-Boltzmann classical result reads (in conventional units)

\[ \rho = \frac{4}{c} \sigma T^4, \]

with \( c \) the speed of light in vacuum and \( \sigma \) the Stefan-Boltzmann constant. Applying this equation locally throughout our system (equivalence principle), from equation [6] we have thus

\[ \rho(r)f^2(r) = \text{const} = \frac{C}{8\pi}, \]

with \( C \) some constant with the dimensions of energy density (length\(^{-2} \) in geometrized units). Note that while the value of \( C \) depends on the chosen unit for length, the quantities \( \rho/C \) and \( r\sqrt{C} \) are dimensionless.

Making use of [7], the first and the second of equations [8] become
\[
\frac{C}{f^2} = \frac{h'}{rh^2} + \frac{h - 1}{hr^2} \tag{8}
\]
\[
\frac{C}{3f^2} = \frac{f'}{rf} - \frac{h - 1}{hr^2} \tag{9}
\]
while the third, as can be verified (and as noticed in [5]), is always identically satisfied if equations (8-9) are satisfied, as a confirmation of Tolman result.

Before we start to solve these coupled differential equations, let us study some general characteristics which the solutions \( f(r) \) and \( h(r) \) should satisfy. \( h(r) \) can be put in the form
\[
h(r) = \frac{1}{1 - 2m(r)/r}, \tag{10}
\]
(see for example [3]) with \( m(r) \) given by
\[
m(r) = \int_0^r 4\pi r^2 \rho(\tilde{r}) d\tilde{r} = \frac{C}{2} \int_0^r \tilde{r}^2 \frac{f^2(\tilde{r})}{f^2(\tilde{r})} d\tilde{r}, \tag{11}
\]
that is, \( m(r) \) is the ADM mass inside \( r \).

For each allowed equilibrium configuration, in the limit \( r \to 0 \)
\[
m(r) \simeq \frac{4}{3} \pi r^3 \rho(0), \tag{12}
\]
so that \( m(r)/r \to 0 \). This implies \( h(0) = 1 \) for each solution with \( \rho(0) \) finite. As regards \( f \), from (7) we have
\[
\rho(0)f^2(0) = \frac{C}{8\pi}, \tag{13}
\]
so that finite \( \rho(0) \) implies a non zero \( f(0) \). Taking the derivative of (10) and using (12) and (13), we have that in the limit \( r \to 0 \), \( h'(r) \simeq \frac{16}{3}\pi \rho(0)/r \) and \( f'(r) \simeq r f(0) \frac{16}{3}\pi \rho(0) \), so that \( h'(0) = 0 \) and \( f'(0) = 0 \).

We want now to solve equations (8-9). We can write equation (9) (as in [5]) as
\[
h = \frac{3f(f + rf')}{Cr^2 + 3f^2} \tag{14}
\]
and, substituting this expression of \( h \) in equation (8), we obtain
\[
f'' = \frac{6Crf^2 + 6Cr^2 f f' - 6f^3 f' + 2Cr^3 f'^2}{Cr^3 f^3} \tag{15}
\]
so that equations (14-15) are equivalent to the starting equations (8-9). Using, in addition, the previous results about the behaviour of \( f \) near the origin, the problem is thus reduced to find the solutions \( f \) of the second order equation (15) with initial conditions
\[
f(0) = \sqrt{\frac{C}{8\pi \rho(0)}}, \quad f'(0) = 0. \tag{16}
\]

For each choice of the pair \( C, \rho(0) \) we have a solution \( f(r) \). Note however from equation (7) that physical configurations \( \rho = \rho(r) \) depend only on the ratio \( C/f^2 \) and not on \( C \) and \( f \) separately. This implies that for any chosen \( \rho(0) \), the (infinite number of) solutions \( f \) that we find when \( C \) is varied, correspond always to the same physical configuration \( \rho = \rho(r) \). To fix \( \rho(0) \) corresponds to uniquely fix the physical configuration, irrespective of the separate values of \( C \) and \( f \). Without loss of generality we can then study the solutions to equation (16) for a single arbitrary choice of the value of \( C (\neq 0) \); let us put for simplicity \( C = 1 \). All possible static configurations of our system are then in 1-1 correspondence with all the solutions \( f \) of equation (15).
\[ f'' = \frac{6r f'^2 + 6r^2 f f' - 6f^3 f' + 2r^3 f'^2}{r^3 f + 3r f^3} \] (17)

with initial conditions

\[ f(0) = \sqrt{\frac{1}{8\pi \rho(0)}}, \quad f'(0) = 0, \] (18)

where \( \rho(0) \) is any positive number and with \( h \) given by (14). Due to (6), each specification of \( \rho(0) \) is equivalent to a specification of the central temperature \( T_C \).

Some symmetry considerations can now help to understand how the set of solutions \( f \) (and \( h \)) is arranged. Consider the transformation

\[ r \to \tilde{r} = \lambda r \]
\[ f \to \tilde{f} = \lambda f. \] (19) (20)

We have

\[ \tilde{f}(\tilde{r}) = \lambda f(\tilde{r}/\lambda) \]

or

\[ \tilde{f}(r) = \lambda f(r/\lambda) \]

and then

\[ \tilde{f}'(r) = f'(r/\lambda) \]

and

\[ \tilde{f}''(r) = \frac{1}{\lambda} f''(r/\lambda) \]

so that, if \( f(r) \) is a given solution, \( \tilde{f}(r) \) is also a solution, as can be easily verified going through equation (17). \( \tilde{f}(r) \) is such that \( \tilde{f}(0) = \lambda f(0) \). On the other hand we know that, starting from a given solution \( f_1 \), we can obtain all solutions if we change the initial conditions, \( f_1(0) \to f_\lambda(0) = \lambda f_1(0) \), with \( \lambda \) spanning all positive real values. Thus, in terms of an assigned solution \( f_1(r) \), a function \( f_\lambda \) is a solution if and only if

\[ f_\lambda(r) = \lambda f_1(r/\lambda) \] (21)

with \( \lambda > 0 \). For the solutions \( h \) we obtain

\[ h_\lambda(r) = h_1(r/\lambda), \] (22)

where \( h_\lambda(r) \) and \( h_1(r) \) corresponds to \( f_\lambda(r) \) and \( f_1(r) \) through (14).

Summing up, once we know the behaviour of solutions \( f \) and \( h \) for a given particular initial value \( f(0) = a_1 \) of \( f \), arbitrarily fixed (> 0), we can readily get the corresponding behaviours of the solutions for every other assigned initial value \( a_2 \). Let us study for simplicity the case \( f(0) = 1 \), that is the solutions to (17) with initial conditions

\[ f(0) = 1; \quad f'(0) = 0. \] (23)

From numerical integration of equation (17) and from equation (14) we obtain the solutions \( f \) and \( h \) reported in Figures 1 and 2. The structure of the corresponding equilibrium configuration can be seen in Figures 3 and...
that respectively report \( \rho \) and \( 2m/r \), as a function of \( r \). Our choice \( C = 1 \) implies that \( \rho \) and \( r \) are given here respectively in units of \( C \) and \( C^{-1/2} \). From (14) and (21) the properties of \( \rho \) for every other solution can readily be inferred. Note that \( \rho(r) \) approaches asymptotically 0 as \( r \to \infty \). This implies that for this configuration and, by scaling property (21), for every other solution, the radius of the system is infinite.

Let us consider in some detail the functions \( h \) and \( 2m/r \) (Figures 2 and 4). \( h \) starts from 1 at \( r = 0 \), and \( 2m/r \) from 0; they reach their maxima, after which they go towards their own limiting values [5],

\[
h(\infty) = \frac{7}{4}
\]

(24)

and

\[
\left(\frac{2m}{r}\right)_{\infty} = \frac{3}{7},
\]

(25)

shown in the Figures, with damped oscillations around them, in which both the amplitudes of the deviations from the limiting values as well their derivatives at any order reduce asymptotically to 0. Analytically both limiting values can easily be verified. In fact consider for example that from (10) we have

\[
\frac{2m}{r} = \frac{h - 1}{h}.
\]

(26)

In equation (8) we see then that, at extremal points, that is when \( r \) has values \( r^* \) such that \( h'(r^*) = 0 \) (that is \( (2m(r)/r)_{r=r^*} = 0 \)), we have
As extremal values tend to \((2m/r)_{\infty}\) when \(r \to \infty\), from equation (9) considered in this limit, using (26) we have

\[
\frac{1}{3} \left( \frac{2m}{r} \right)_{\infty} = 1 - \left( \frac{2m}{r} \right)_{\infty} - \left( \frac{2m}{r} \right)_{\infty} = 1 - 2 \left( \frac{2m}{r} \right)_{\infty}
\]

and thus equation (26), as well the result (21). Here we made use of the fact that in the limit \(r \to \infty\), \(r^2/f^2) = 0\) implies \(f' = f/r\). Note that equations (26) and (22) determine \(2m/r\) for every solution; we see that \(2m/r\) scales as \(h\). From scaling property (22) and from equations (18) and (20), the asymptotic behaviour of \(h\) and \(2m/r\) when \(r \to \infty\) implies that in the limit of very large central density \(\rho(0)\) or very large \(T_C\), \(h(r) \simeq \text{const} = 7/4\) and \(2m(r)/r \simeq \text{const} = 3/7\), for every \(r\), except the region of \(r\) very near to 0, in which \(h\) goes from 1 to 7/4 and \(2m/r\) goes from 0 to 3/7, with damped oscillations.

As regards \(f\), from (9) we see that \(f'(r) \geq 0, \forall r\) and from (27) we have that in the limit \(r \to \infty\), \(f(r) \simeq \sqrt{7/3} r\) (see (8)). Due to equation (24) this implies

\[
\lim_{\lambda \to 0} f_\lambda(r) = \lim_{\rho(0) \to \infty} f(r) = \lim_{T_C \to \infty} f_\lambda(r) = \lim_{\lambda \to 0} \lambda f(r/\lambda) = \lim_{\lambda \to 0} \lambda \frac{7}{3} \frac{r}{\lambda} = \sqrt{7/3} r, \quad (r_{fixed})
\]

and then

\[
\lim_{\rho(0) \to \infty} \rho(r) = \lim_{T_C \to \infty} \rho(r) = \lim_{T_C \to \infty} \frac{1}{8\pi f_\lambda^2(r)} = \frac{3}{56\pi} \frac{1}{r^2}, \quad (r_{fixed})
\]

that is when the central density (central temperature) is allowed to increase without limit, the density (temperature) at each fixed radius \(r\) does not increase unlimitedly but approaches a limiting value proportional to \(r^{-2}\) (\(r^{-1/2}\)).

Finally, from scaling properties (24), from initial conditions (15) and from the behaviour of \(f(r), h(r)\) and \(\rho(r)\) near \(r = 0\), shown in Figures 1, 2 and 3, we have that in the limit of very low central density, \(f(r), h(r)\) and \(\rho(r)\) are constant in an ever increasing range of \(r\), so that the flat-space limit for constant low densities is recovered.

To obtain a system with finite size we can imagine to enclose our radiation inside a perfectly insulating spherical cavity with area radius \(R\), with its center coinciding with the center of the radiation distribution and to assume that thermodynamical equilibrium is reached, that is the temperature of the walls is equal to the temperature of radiation at \(r = R\). This will correspond to a certain central energy density \(\rho_C\) for the cavity. Due to the local uniqueness of solutions to equation (17) with initial conditions (15), this is equivalent to take a certain (physical) solution to equation (17) (that with \(\rho(0) = \rho_C\)), to cancel all radiation outside...
$r = R$ and to put an insulating spherical wall (of negligible thickness and mass) at $r = R$ exactly at the temperature the radiation has there. As regards energy density radial profile $\rho(r)$, this means that there is a 1-1 correspondence between all admitted cavities filled with blackbody radiation (that is every admitted central temperatures or boundary temperatures as well radii) and all possible $\rho(r)$ distributions in finite ranges $[0, R]$, where $\rho(r) = 1/(8\pi f^2(r))$ and $f$ is solution to (17) with initial conditions (18).

Let us assume now, as allowed by Birkhoff theorem [3, 7], that the Schwarzschild coordinates be chosen in such a way that outside the cavity exactly the Schwarzschild form of the metric obtains

$$ds^2 = -(1 - 2M/r) \, dt^2 + \frac{1}{1 - 2M/r} \, dr^2 + r^2 d\Omega^2.$$  

The parameter $M$ here is precisely the mass-energy or ADM mass as given by (14) with $r = R$, that is

$$M = \int_0^R 4\pi r^2 \rho(r) \, dr = \frac{C}{2} \int_0^R \frac{r^2}{f^2(r)} \, dr, \tag{30}$$

This mass parameter $M$ is the mass probed by Kepler-like experiments with gravitating bodies far away from the system. In our context it thus represents meaningfully the mass of the ball of radiation contained in the cavity. As the metric outside has been explicitly chosen, the values of the metric tensor components $g_{\alpha\beta}$ at the boundary are now fixed. Going at this boundary from inside, continuity of metric tensor components at the boundary of the cavity requires that

$$f(R) = 1 - 2M/R \quad \text{and} \quad h(R) = 1/(1 - 2M/R). \tag{32}$$

Equations (10), (11) and (30) show that $h$ fulfills condition (32). Making use of the freedom we have to choice $C$ and $f$ separately, provided the ratio $C/f^2$ be held fixed, we can always find a value of $C$ such that the corresponding $f$ fulfills the boundary condition (11). In other words, once a single coordinate system is explicitly chosen, the arbitrariness in the choice of the constant $C$ in equations (5)–(9) is lost. In the case of a cavity filled with radiation, with central density $\rho_C$, condition (13) requires $C$ to be

$$C = \frac{f^2(R)}{f_1^2(R)} = \frac{(1 - 2M/R)^2}{f_1^2(R)}, \tag{33}$$

where with $f_1$ we intend here the solution $f$ for the case $C = 1$ and $\rho(0) = \rho_C$. Each choice of central density $\rho(0)$ fixes the mass $M$ within a given radius $R$ and then separately $C$ and $f$.

From the results obtained so far some consequences can be drawn.

A first consequence regards mass. We can infer it simply from the behaviour of the functions $2m(r)/r$ for equilibrium configurations, as detailed above. The non-vanishing asymptotic limit (24) implies that there is no limit on the mass $M$; we have however that for each fixed $R$
with \( K \) dimensionless not depending on \( R \). The value of \( K \) can be read directly in Fig. 4 as the value of the maximum of the function \( 2m/r \) there reported. This means that, whichever is the energy density or the temperature of the blackbody radiation at equilibrium inside a cavity of radius \( R \), its mass can never exceed a fraction \( K \) of the Schwarzschild mass \( M_S = R/2 \) for that radius. Let consider a cavity at its mass limit \( M_{\text{limit}} \), for concreteness the configuration described in Fig. 4 with area radius \( R \) equal to the extremal point. Suppose to supply it, in some unspecified manner, with some positive amount of energy \( \Delta E \), while maintaining fixed its radius. What we should obtain at equilibrium, if reachable, is simply a cavity with new conditions of mass \( M \) and central density \( \rho \), fixed its radius. The plot is done in terms of \( z = \rho(0)R^2 \) on \( M/R \). The expression for \( 2m/r \) indicates how to rescale the abscissa to obtain, from the given plot, the actual value of \( 2m/r \). The explicit form of the plotted function, expressed in terms of \( R \) as a function of \( z = \rho(0)R^2 \), is

\[
\frac{2M(R)}{R} \leq K = 0.493,
\]

with \( K \) dimensionless not depending on \( R \). The value of \( K \) can be read directly in Fig. 4 as the value of the maximum of the function \( 2m/r \) there reported. This means that, whichever is the energy density or the temperature of the blackbody radiation at equilibrium inside a cavity of radius \( R \), its mass can never exceed a fraction \( K \) of the Schwarzschild mass \( M_S = R/2 \) for that radius. Let consider a cavity at its mass limit \( M_{\text{limit}} \), for concreteness the configuration described in Fig. 4 with area radius \( R \) equal to the extremal point. Suppose to supply it, in some unspecified manner, with some positive amount of energy \( \Delta E \), while maintaining fixed its radius. What we should obtain at equilibrium, if reachable, is simply a cavity with new conditions of mass \( M \) and central density \( \rho \), fixed its radius. The plot is done in terms of \( z = \rho(0)R^2 \) on \( M/R \). The expression for \( 2m/r \) indicates how to rescale the abscissa to obtain, from the given plot, the actual value of \( 2m/r \). The explicit form of the plotted function, expressed in terms of \( R \) as a function of \( z = \rho(0)R^2 \), is

\[
\frac{2M(z)}{R} = \frac{h_f(0=1)(\sqrt{8\pi z}) - 1}{h_f(0=1)(\sqrt{8\pi z})}
\]

as can be easily derived from equation 24 and from the scaling property 22. In the Figure the maximum is, obviously, \((2M/R)_{\text{max}} = 0.493\) and the extremal point \( \hat{z} \), is \( \hat{z} = 0.439\); this values can be used to calculate various quantities for the extremal equilibrium configuration. If, for example, the radius is the solar radius, \( R = R_\odot \), we can calculate the extremal values of mass \( M_{\text{max}} \) and central density \( \rho(0)_{\text{extr}} \) as follows: \( M_{\text{max}} = (2M/R)_{\text{max}} R_\odot/2 = 1.16 \cdot 10^5 \ M_\odot \), with \( M_\odot \) the solar mass, and \( \rho(0)_{\text{extr}} = \hat{z}/R_\odot^2 = 0.91 \cdot 10^{-22} \) cm\(^{-2}\), to which corresponds, through 11, a temperature \( T_{\text{extr}} = 1.95 \cdot 10^{10} \) K. Note that this temperature gives \( kT_{\text{extr}} = 1.68 \) MeV, with \( k \) the Boltzmann constant. This implies that in the core of this system many photons are well above threshold for pair production and our assumption of a fluid consisting of blackbody radiation alone is no longer viable. Our results reside on two characteristics of blackbody radiation systems, namely the equation of state 11 (and many systems besides blackbody radiation obey it, for example ideal Fermi gases in the extreme relativistic limit) and the peculiar energy-temperature relation 11. Only systems for which both these properties hold true, are describable in the way we have shown and the eventual existence of conditions could be investigated, for which this is the case for a mixture of blackbody photons and electrons and positrons at equilibrium through the reaction \( \gamma \gamma \equiv e^+ e^- \). On the other hand, as \( T_{\text{extr}} \propto R^{-1/2} \), when the radius is of the order of \( \sim 100 \) \( R_\odot \) or...
Fig. 6: $\rho(R)R^2$ versus $z \equiv \rho(0)R^2$ ($R$ is fixed).

larger the contribution of pair production starts to be highly depressed and thus in any case our assumptions and results in principle permit to describe faithfully the system, not only at low densities but even up to its mass limit. As the mass limit is very high, systems of this size can store in this manner mass energies of orders of $10^7 M_\odot$ while remaining at equilibrium. Quite interestingly we can have thus physically allowed and easily (classically) describable systems, which are extremely compact, with $10^7 M_\odot$ masses within a size of few hundreds of light travel seconds (i.e. a fraction of AU).

In Fig. 5 we see that once the extremal central energy density or temperature are overcome, equilibrium configurations are still possible, at lower values of $M$. We have thus larger central temperatures, which give rise however to a lower total mass for the system. This is by no way a problem for energy conservation, obviously. The total energy of the system, given by $M$, contains in fact both the component of proper energy $\rho$ and the (negative) component of gravitational potential energy. Evidently, when the extremal temperature is overcome, the system settles down into a new equilibrium configuration with a stronger gravitational energy component.

A second consequence regards energy density or temperature. Note that, on the basis of the correspondence mentioned above between cavities and solutions $\rho(r)$ to equations, we immediately have the following. While there is no limit on the central density $\rho_C$ (or central temperature) of a given cavity at equilibrium (irrespective of being its size fixed or not), from on the contrary when the size is fixed the temperature at its boundary (that is the temperature of the blackbody radiation coming out through a small hole in the walls of the cavity) is limited, whichever the central temperature is. Fig. reports energy density at the boundary of the fixed-radius cavity $\rho(R)$ times $R^2$ as a function of the quantity $z$ defined above. Here again we have an universal plot, not depending on $R$. Its analytic expression in terms of the solution $f_{f(0)=1}$ corresponding to initial conditions is

$$\rho(R) R^2 = \frac{z}{f_{f(0)=1}(\sqrt{8\pi z})}$$

as can be deduced from equation and from the scaling property. This dimensionless function has absolute maximum $Q = 0.02718$ (the extremal point is $z = 0.1021$) and goes to the asymptotic value $3/(56\pi)$ (with damped oscillations around it) when $r \to \infty$ as can be seen from. For a cavity of size $R$ we have thus the following limiting density on the boundary

$$\rho(R) \leq \frac{Q}{R^2}$$

The limit depends on the size $R$ of the cavity. For temperatures at the boundary of the cavity higher than those corresponding to this limit, thermodynamical equilibrium is no longer possible. If we perform a small hole in the cavity, the radiation emerging from it cannot have an energy density (or corresponding temperature or pressure) larger than this limit. Note that $\rho(R)$ never vanishes, whichever is the (\neq 0) value of central density. Due to this implies that for our cavities, self-gravitating equilibrium configurations cannot exist.

We study, finally, how the radial profiles for entropy density in the cavities depend on central density (or temperature) and then on the mass of the radiation inside it. As already said, this study is performed to see if the results on the variation of entropy radial profiles with mass for spherically symmetric bodies with
constant density can be extended/confirmed for other spherically symmetric equilibrium configurations. Instead to refer to entropy density in the proper orthonormal frame \( \sigma(r) \), following [2] we present our results in terms of, call it, entropy radial density \( s(r) \) defined by

\[
S = \int_0^R 4\pi r^2 s(r) \, dr,
\]

where \( S \) is the total entropy of the cavity and \( R \), as before, its radius. We see that \( s(r) \) is the specific contribution to \( S \) per unit area per radial slice \( (r, r + dr) \). Evidently the relation between \( s \) and \( \sigma \) is

\[
s(r) dr = \sigma(\ell) d\ell,
\]

(38)

From first law of thermodynamics and assuming extensivity to hold locally (from equivalence principle) for the pertinent thermodynamical quantities, the Gibbs-Duhem relation [9]

\[
\rho = T \sigma - p + \mu n
\]

(40)

obtains, where \( \mu \) is the chemical potential and \( n \) the particle (photon, in this case) number density in the rest frame of the fluid. Taking into account that for photons \( \mu = 0 \), from (1) and (6) we obtain

\[
s = \frac{4}{3} \sqrt{\hbar} \rho^{3/4} \propto \rho^{3/4} \sqrt{\hbar}.
\]

(41)

From this equation and from scaling properties (21-22), we have that \( s \) scales as

\[
s_\lambda(r) = \frac{1}{\lambda^{3/2}} s_1(r/\lambda),
\]

(42)

where \( s_1 \) is the entropy radial density for the reference configuration, corresponding to the functions \( f_1 \) and \( h_1 \) in equations (21-22). Putting \( x \equiv r/R \), with \( R \) assigned, we have

\[
s_\lambda(Rx) = \frac{1}{\lambda^{3/2}} s_1(Rx/\lambda).
\]

(43)

This law express the scaling (42) in terms of the normalized radius \( x \).

Let us choose as radius of the cavity the radius \( R_0 \) for which \( 2m/r \) in Fig. 4 is maximum. In order to compare entropy radial profiles when the central density is changed, Fig. 5 plots radial profiles of the function \( f(0)^{3/2} \rho^{3/4} \sqrt{\hbar} \) for various choices of \( f(0) \). The radial coordinate is expressed as \( x = r/R_0 \) and \( \rho \) is intended again in units of \( C \). The curves are with increasing \( f(0) \) upwards, respectively \( f(0) = 2/3, 1, 10, 100 \). This means that the second curve from below corresponds to the critical configuration, (for which the central density \( \rho(0)_{\text{extr}} \) is such that \( M \) is maximum) and the curves above and below it are for central densities respectively lower and higher than \( \rho(0)_{\text{extr}} \). The Figure shows that at increasing central density the entropy distribution is more and more weighted around the center; further inspection of equation (43) gives that in the limit \( \rho(0) \to \infty \) the plotted function \( f(0)^{3/2} \rho^{3/4} \sqrt{\hbar} \) can be approximated as \( \sim \text{const} \cdot [\lambda/(xR_0 + \lambda)]^{3/2} \), where \( \lambda = f(0) \). Let us put together this Figure with Fig. 5. If we start from low densities (or temperatures) we see that at increasing \( M \) the contribution to total entropy is more and more dominated by the most internal layers in \( r \), at variance with what obtained, for another system, in [2] where at increasing \( M \) the dominating role is gradually played by external layers. We see also that when however we are at densities a bit larger than \( \rho(0)_{\text{extr}} \) the weight of external layers tends to increase with \( M \), confirming in this case what found for the system studied in [2]. This will be reversed again after the next extremum for \( 2M/R_0 \) (a minimum) will be encountered and so on. We see therefore that in our circumstances the dependence on mass of radial entropy profile is different and more complex than in the case examined in [2].
FIG. 7: $f(0)^{3/2} \rho^{3/4} \sqrt{\pi} \rho_0$ versus $x \equiv r/R_0$ for various choices of central temperature, see text. The curves go upwards at decreasing central temperature. The second curve from bottom corresponds to the limit mass configuration.

III. CONCLUSIONS

Our results can be summarized as follows. Considering insulated spherical cavities thermodynamically at equilibrium, i.e. filled with blackbody radiation, at energy densities sufficiently high to make not negligible the gravitational self-interaction of radiation, we determine their mechanical equilibrium configurations. We find that equilibrium is possible without any limit on mass $M$ (ADM mass) of the contained radiation, provided the area radius $R$ of the cavity is arbitrarily large. At fixed $R$, on the contrary, a (very large) mass limit is present, which, if expressed in terms of the Schwarzschild mass $M_S$ for that radius, is $M(R) \leq K M_S(R)$ with $K = 0.493$ not depending on $R$. This implies that no more than this amount of mass energy can be stored as blackbody radiation in a cavity with assigned area radius $R$. Furthermore we find that equilibrium is possible for any value of central energy density (or temperature), irrespective of being the size of the cavity fixed or not. The energy density $\rho_B = \rho(R)$ at the boundary of the cavity can take any non-zero value if the size $R$ of the cavity is arbitrary; it is however bounded if $R$ is fixed. We find $\rho(R) \leq Q/R^2$ with $Q = 0.02718$ not depending on $R$. This can be interpreted as a limit on the density of the radiation emerging from a small hole in a cavity of assigned radius $R$. The fact that $\rho_B$ is never 0 (except in the trivial case for which $\rho(0) = 0$) implies that self-gravitating equilibrium configurations are never allowed. In the limit of very high central density (or temperature) we have an asymptotic situation in which both the quantities $2M/R$ and $\rho(R) R^2$ no longer depends on $R$, being the first fixed to the value $3/7$ and the second to $3/(56\pi)$.

These results are obtained considering the radiation contained in the cavities from a purely classical point of view, that is neglecting any quantum process such as, for example, pair production by photons. This has been purposely done to study the radial profiles of conventional entropy density for classical fields under strong gravitational interaction conditions. For central temperatures not very high this description can be in any case safely adequate. It turns out furthermore that when the considered radii are roughly $\sim 100 R_\odot$ or larger, these systems can reach even their mass limits without any significant appearance of pair production processes. Cavities with $100 R_\odot$ size (a fraction of AU) can be at equilibrium (i.e. not collapsing) even with masses of orders of $10^7 M_\odot$.

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