Regular colored graphs of positive degree

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Abstract. Regular colored graphs are dual representations of pure colored D-dimensional complexes. These graphs can be classified with respect to an integer, their degree, much like maps are characterized by the genus. We analyze the structure of regular colored graphs of fixed positive degree and perform their exact and asymptotic enumeration. In particular we show that the generating function of the family of graphs of fixed degree is an algebraic series with a positive radius of convergence, independent of the degree. We describe the singular behavior of this series near its dominant singularity, and use the results to establish the double scaling limit of colored tensor models.

1 Introduction

Context. In this article a colored graph is a connected bipartite graph such that each edge has a color in \(\{0, 1, \ldots, D\}\) and each vertex is incident to exactly one edge of each color. Colored graphs appear naturally in the crystallization theory of manifolds [12] and in colored tensor models [7] (or colored group field theory). They are dual to colored triangulations of piecewise linear orientable \((D+1)\)-dimensional pseudo-manifolds [5,8]. Although not all \((D+1)\)-triangulations can be properly colored, colored graphs are fundamental because any orientable topological manifold in any dimension admits a colored triangulation [13] and any triangulation in any dimension can be transformed into a colored triangulation by a barycentric subdivision.

To each colored graph is associated a natural invariant, its degree [9], which for \(D = 2\) corresponds to the genus of the dual (2-dimensional) triangulation. Unlike the genus however, the degree is not a topological invariant of the associated pseudo-manifold for \(D \geq 3\). Be that as it may, classifying graphs in terms of the degree offers a first rough classification of triangulations of pseudo-manifolds in any dimension. It also plays a distinctive role in tensor models, where this classification allows access to subsequent orders in their \(1/N\) expansion, as this expansion is indexed by the degree (exactly like the \(1/N\) expansion of matrix models is indexed by the genus).

Our results. Our main result is a structural analysis of rooted colored graphs of fixed degree, which yield on the one hand an exact and an asymptotic enumerations of these graphs, and on the other hand to the construction of the double scaling limit of colored tensor models.

The structural analysis we perform relies on the reduction of colored graphs via a precise algorithm to some terminal forms of the same degree, which we call schemes, such that the number of schemes of a given degree is finite and the number of graphs sharing a scheme is exponentially bounded. More precisely we show

**Theorem 1.** For any fixed dimension \(D \geq 3\) and degree \(\delta \geq 1\), there exist a finite set \(S_\delta\) of schemes of degree \(\delta\), families of polynomials \((P_s(t))_{s \in S_\delta}\) and simple parameters \(a, b : S_\delta \to \mathbb{N}\) of the schemes such that the generating function of rooted colored graphs of degree \(\delta\) with respect to the number of black vertices is

\[
F_\delta(z) = T(z) \sum_{s \in S_\delta} \frac{P_s(U(z))}{(1 - U(z)^2)^a(s)(1 - DU(z))^b(s)}
\]

where \(U(z) = zT(z)^{D+1}\) with \(T\) the unique power series solution of the equation

\[
T(z) = 1 + zT(z)^{D+1}.
\]

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Previous classifications in terms of the degree exist \cite{9}, but while the number of terminal forms identified in \cite{9} is finite at fixed degree, there is no control over the number of graphs associated to a terminal form. Our approach is instead reminiscent of the classification of maps of fixed genus performed in \cite{3}, or that of simplicial decompositions of surfaces with boundaries in \cite{11}, and more generally of Wright’s approach to the enumeration of labeled graph with fixed excess \cite{15,16}.

From our main theorem we are able to extract the leading terms in the singular expansion of the generating functions of colored graphs of degree $\delta$.

**Theorem 2.** For any fixed $D \geq 3$ and $\delta \geq 1$, the generating function of schemes of degree $\delta$ has a dominant singularity at $z_0 = D^D/(D+1)^{D+1}$ and a singular expansion in a slit domain around $z_0$ of the form

$$ F_\delta(z) = c_\delta(1 - z/z_0)^\frac{D}{2}(-\beta+1)(1+o(1)) $$

where $\beta$ is the maximum of a simple integer linear program:

$$ \beta = \max(2x + 3y - 1 \mid (D-2)x + Dy = \delta, x \in \mathbb{N}, y \in \mathbb{N}) $$

In particular $\beta$ roughly grows linearly with $\delta$ and for fixed $D$ we determine the largest linearity factor $\max(\beta/\delta)$ and for which $\delta$ it is obtained:

| $\max(\beta/\delta)$ | $3 \leq D \leq 5$ | $D = 6$ | $D \geq 7$ |
|----------------------|-------------------|---------|----------|
| $\frac{1}{D-2}$     | $\frac{1}{D}$    | $\frac{1}{D}$ | $\frac{1}{D}$ |

Moreover the constant $c_\delta$ have combinatorial interpretations, which for $3 \leq D \leq 5$ involve Catalan numbers.

**Discussion.** From a probabilistic point of view the above result implies that we can give a description of large random colored graphs of fixed degree. It was indeed shown in \cite{10} that upon scaling edge length by a factor $k^{-1/2}$ and letting $k$ go to infinity, the degree 0 colored graphs with $2k$ vertices converge in the sense of Hausdorff-Gromov to the Continuum Random Tree. As we shall discuss in a sequel of this paper, our results allow to identify more generally $k^{-1/2}$ as the proper scaling for which uniform random rooted colored graphs of fixed degree $\delta \geq 1$ can have a non-trivial continuum limit when the number of vertices go to infinity, and to describe this limit.

Another major outcome of our results is the so-called double scaling limit of colored tensor models. Although the number of colored graphs with $2n$ vertices grows super-exponentially with $n$, we can give a meaning to the resummation of the generating series of graphs of fixed degree. Balancing the singular behavior of these generating series around some critical point $z_0$ with the scaling in $N$ we can take the double limit $N \to \infty$, $z \to z_0$ in a correlated way and exhibit a regime in which graphs of arbitrary large degree contribute.

Together with the parallel result obtained in \cite{4} by different methods, for a simpler model of quartically perturbed uncolored tensor model, these are the first results of this kind in the realm of tensor models.

A number of very difficult questions remain open. Prominent among them is the following. A given topology (say spherical) can be represented by graphs of arbitrary degree. It is a difficult open question whether the number of triangulations of a fixed topological manifold is exponentially bounded or not (the so called Gromov question \cite{14} in the case of the spherical topology). In view of our results the question can now translate in finding an exponential bound (in the degree) on the number of schemes to which graphs representing a given topology can reduce.

**Organization of the paper** After stating some definitions and elementary properties of colored graphs in Section 2 we embark in Section 3 in the description of our classification, and in particular of the set $S_\delta$ of schemes indexing the sum in Theorem 3 above. The next two Sections 4 and 5 are devoted to the proof that the set $S_\delta$ is finite for each $\delta$. At this point the exact enumeration is easily obtained in Section 6. Section 7 is then dedicated to the description of the schemes giving the asymptotically dominant contribution to the enumeration, and a discussion of implications on the double scaling of colored tensor models.
The resulting combinatorial map arrangement of edges around black vertices and the counterclockwise arrangement of edges around each vertex is incident to exactly one edge of each color. Multiple edges are allowed, but, due to the color constraints, self-loops are not. A rooted colored graph is a colored graph with a distinguished edge, called the root edge.

2 Notation and generalities on colored graphs

From now on in this article, an integer $D \geq 3$ is fixed and a colored graph is a connected bipartite $(D + 1)$-regular graph with black and white vertices and colored edges, such that the colors of edges are taken in the set $\{0, \ldots, D\}$, and each vertex is incident to exactly one edge of each color. Let $G$ be a colored graph. In view of the bipartiteness and regularity constraints, $G$ has an equal number $k$ of black and white vertices, and by construction it also has $(D + 1)k$ edges. Let us define the faces of $G$ as its bicolored connected components: more precisely, given $0 \leq c < c' \leq D$, the faces of color $(c, c')$ are the connected components of the subgraph consisting of edges that have color $c$ or $c'$, and they form a set of cycles (since every vertex in the subgraph has degree 2).

Let $F_{p}^{c,c'}$ denote the number of faces of color $(c, c')$ of length $2p$, $F_{c}^{c'} = \sum_{p \geq 1} F_{p}^{c,c'}$ the total number of faces of color $(c, c')$ and $F = \sum_{0 \leq c < c' \leq D} F_{c}^{c'}$ the total number of faces. Finally, let the reduced degree $\delta$ be the integer defined by the relation:

$$\delta = \frac{1}{2}D(D-1)k + D - F$$

**Proposition 1 (\cite{7}).** The reduced degree $\delta$ of a colored graph $G$ is a non negative integer.

**Proof.** Each of the $D!$ cyclic permutations $\pi$ of the colors $\{0, \ldots, D\}$ induces a unique embedding of the graph $G$ in a compact oriented surface upon requiring that $\pi$ describes the clockwise arrangement of edges around black vertices and the counterclockwise arrangement of edges around white ones. The resulting combinatorial map $G_{\pi}$, called in \cite{7} a jacket of $G$, has $(D + 1)k$ edges, $2k$ vertices and $\sum_{c} F_{c}^{c,\pi(c)}$ faces since its faces are precisely the faces of color $(c, \pi(c))$ of $G$. Euler’s formula yields the relation $2k + \sum_{c} F_{c}^{c,\pi(c)} = (D + 1)k + 2 - 2g_{\pi}$, where $g_{\pi}$ denotes the genus of $G_{\pi}$, and upon summing over all $\pi$ we obtain that $\delta = \frac{1}{(D-1)!} \sum_{c} g_{\pi}$. The positivity of $\delta$ thus follows from that of the genera of all jackets. \hfill \Box

The following corollary summarizes the relations we shall need on colored graphs:
Corollary 1. Let \( G \) be a colored graph with \( 2k \) vertices, \( (D + 1)k \) edges, \( F \) faces, \( F_p \) of which have length \( 2p \), and degree \( \delta \). Then:

\[
\begin{align*}
D(D + 1)k &= 2 \sum_{p \geq 1} p F_p, \quad (1) \\
D(D - 1)k &= 2F + 2\delta - 2D, \quad (2) \\
(D + 1)\delta + 2F_1 &= D(D + 1) + \sum_{p \geq 2} ((D - 1)p - D - 1)F_p. \quad (3)
\end{align*}
\]

Proof. The first equation follows from double counting of the edges at vertex incidences and at face incidences. The second is the definition of the degree. The third one is a simple linear combination of the first two.

Equation (3) already give reasons for the classical case \( D = 2 \), which we do not consider here, to be much different from \( D \geq 4 \). On the one hand when \( D = 2 \), the coefficient of \( F_2 \) in the right hand side is negative, so that the number of large faces (or the degree of the largest face) can be arbitrarily large even if \( \delta \) and \( F_1 \) are fixed. On the other hand when \( D \geq 4 \), the right hand sum has only positive coefficients and the number of large faces (or the degree of the largest face) is bounded by \( (D + 1)\delta + 2F_1 \). The case \( D = 3 \) is a priori intermediate in the sense that faces of degree 4 could proliferate at fixed \( \delta \) and \( F_1 \) (since the coefficient of \( F_2 \) is zero in this case), but we shall see later that this does not happen, and the dichotomy is really between \( D = 2 \) and \( D \geq 3 \).
3 Structural analysis

3.1 The core of a rooted colored graph

First attempts to define the core. Let a melon $m$ in a graph $G$ be an open subgraph of $G$ that consists of $D$ parallel edges joining two vertices, and two half-edges of the same color (one on each vertex of $m$). The melon removal of $m$ in $G$ consists in deleting $m$ from $G$ and gluing the two resulting half-edges to recreate an edge.

We would like to define the core of a rooted colored graph as “the” graph obtained by a maximal sequence of melon removals. However in this “definition”, it is not clear that the core is uniquely defined. We will therefore follow an alternative approach, which is reminiscent of an old result of Tutte [17] about 2-connected graphs, that is, graphs that cannot be disconnected by removing a single vertex. In the modern phrasing of [2], Tutte’s result states that any 2-connected graph $G = (X, E)$ can be decomposed into an arborescent structure called its RMT-tree by a careful analysis of all its 2-cuts, that is, its pairs $\{x,y\}$ of vertices such that $G|_{X \setminus \{x,y\}}$ is not connected. Our interest in this result arises from the following lemma.

**Lemma 1.** Colored graphs are 2-connected.

**Proof.** Consider a colored graph $G$ containing a white vertex $x$ such that $G' = G|_{X \setminus \{x\}}$ is not connected and let $C$ be a connected component of $G'$. Let $\ell$ denote the number of edges between $x$ and vertices of $C$. By hypothesis there is at least one other connected component in $G'$ so that $1 \leq \ell \leq D$. Let us consider the subgraph of $G$ induced by $x$ and the vertices of $C$. In this subgraph all black vertices and all white vertices except $x$ have degree $D + 1$, while $x$ has degree $1 \leq \ell \leq D$: double counting of edges yields a contradiction.

In view of this lemma one could directly apply Tutte’s result to colored graphs and obtain a decomposition by describing the resulting RMT-trees, and this was our approach in an earlier version of this article. However it turns out to be easier to derive the decomposition we need from a direct analysis of 2-edge-cuts.

2-edge-cuts in colored graphs. A 2-edge-cut of a 2-connected graph $G = (X, E)$ is a pair of edges $\{e,e'\}$ such that the graph $G - \{e,e'\} = (X, E \setminus \{e,e'\})$ is not connected (see Fig. 2 left hand side). A simple cycle of a graph $G$ is a cycle visiting each vertex of $G$ at most once.

**Lemma 2.** Let $G$ be a 2-connected graph. Then $\{e,e'\}$ is a 2-edge-cut in $G$ if and only if any simple cycle visiting $e$ also visits $e'$.

**Proof.** Let $e = \{x,y\}$ and let $C$ be a simple cycle visiting $e$. Then $G - \{e,e'\}$ has 2 connected components, one containing $x$ and the other containing $y$. But $C - e$ is a path from $x$ to $y$ and $e'$ is the only edge connecting the two components in $G - \{e\}$.

Conversely if $\{e,e'\}$ is not a 2-edge-cut then there exists a path in $G - \{e,e'\}$ between any two vertices, and in particular between the endpoints of $e$. □

**Lemma 3.** Let $\{e,e'\}$ and $\{e,e''\}$ be two 2-edge-cuts in a 2-connected graph $G$. Then $\{e',e''\}$ is also a 2-edge-cut of $G$. Moreover if two oriented cycles visit $e$ in the same direction, then they both visit $e'$ and $e''$ in the same order after $e$.

**Proof.** Any cycle visiting $e'$ visits also $e$ (since $\{e,e'\}$ is a 2-edge-cut) and thus also $e''$ (since $\{e,e''\}$ is a 2-edge-cut), and we conclude by Lemma 2. Next consider two oriented cycles $c_1 = (e,p_1,e',p_1',e'',p_2)$ and $c_2 = (e,p_2,e',p_2',e'',p_2)$ that visit $e$ in the same direction. But then the path $p_2$ would connect the two connected components of $G - \{e,e'\}$ without visiting $e$ or $e'$. □

A proper cut-set of a 2-connected graph $G$ is a maximal set $C$ of edges such that any 2 edges of $C$ form a 2-edge-cut. In view of the previous lemma, an edge can belong to at most one proper cut-set, so that we define the cut-set of an edge as the unique proper cut-set it belongs if it exists, or the edge itself otherwise.
Lemma 4. Let $G = (X, E)$ be a colored graph and let $C$ a cut-set of $G$. Then there exists a unique way to cyclically arrange the edges of $C$ as $(e_0, \ldots, e_\ell)$ and a unique partition $X_0, \ldots, X_\ell$ of $X$ such that $E = E_{X_0} \cup \ldots \cup E_{X_\ell}$ and for all $i = 0, \ldots, \ell, e_i$ connects a black vertex of $X_i$ to a white vertex of $X_{i+1}$ (with indices taken modulo $\ell + 1$).

Proof. Let $e_0$ be an edge of $C$. Since $G$ is 2-connected, there exists a simple cycle $c$ visiting $e_0$, and this cycle also visits the other edges in $C$. Orient this cycle so that $e_0$ is visited from its black to its white endpoint and let $(e_0, e_1, \ldots, e_\ell)$ describe the cyclic arrangement of the edges of $C$ along this oriented cycle. Then $\{e_i, e_{i+1}\}$ forms a 2-edge-cut and one of the components of $G - \{e_i, e_{i+1}\}$ contains the part of $c$ visiting $e_{i+2}, \ldots, e_{i+1}$. Let $X_i$ denote the vertex set of other component. Then the $X_i$ are disjoint and form a partition of $X$ and the other required properties are immediate. The uniqueness follows from the fact that any other cycle $c'$ has to visit the edges of $C$ in the same order (Lemma 5).

The set $\mathcal{O}_C = \{G_0, \ldots, G_\ell\}$ of open components of a cut-set $C$ is the set of open connected graphs obtained by cutting each edge of the cut-set into two half-edges: with the notation of the lemma, the vertex set of $G_i$ is $X_i$ and the two half-edges of $G_i$ arise from $e_i$ and $e_{i+1}$ respectively. The set $\mathcal{C}_C$ of closed components of $C$ is then the set of rooted colored graphs obtained from the open components by reconnecting in each of them the unique available pair of half-edges to form a root edge: $C_C = \{\text{cl}(G_i), i = 0, \ldots, \ell\}$. These definitions are illustrated by Fig. 2.

Finally the following immediate lemma will be useful.

Lemma 5. Let $C'$ and $C''$ be two distinct cut-sets in a colored graph $G$. Then there is an open component $G'$ of $C'$ containing $C''$, and an open component $G''$ of $C''$ containing $C'$.

Proof. Assume $C''$ is not contained in any of the open components of $C'$, and let $e''_p, e''_q$ be two edges of $C''$ appearing in two different components $G''_p$ and $G''_q$ of $C'$. Then any cycle visiting $e''_p$ also visits $e''_q$, and thus $e''_p$ and $e''_{p+1}$ and so that $C'$ and $C''$ are not disjoint.

Melons and melonic graphs. Let $\Delta$ denote the only rooted colored graph with 2 vertices: it consists of a $(D + 1)$-uple of parallel edges. In agreement with the initial discussion of this section, let us call melon $\text{op}(\Delta)$, the open colored graph associated to $\Delta$: it consists of 2 vertices connected by $D$ parallel edges, each carrying a half-edge.
In the previous literature (see e.g. [7,8]) the graphs that can be obtained from $\Delta$ by cutting edges and inserting melons are called melonic graphs. We give an alternative inductive definition of melonic graphs and prime melonic graphs as follows (see Fig. 3):

- A rooted colored graph is a melonic graph if all the closed components of the cut-set of its root edge are prime melonic graphs. An open colored graph $G$ is melonic if $\text{cl}(G)$ is.
- A rooted colored graph with root edge $r = \{x, y\}$ is a prime melonic graph if all the non-trivial open components obtained by cutting the edges incident to $x$ and $y$ into half-edges are melonic graphs.

Although we do not need it, we now recall the (known) consistency of this definition with the previous one. Let us define a proper sub-melon $m$ of a rooted colored graph $G$ as an open subgraph of $G$ which is a melon and such that the root of $G$ is not one of the parallel edges of $m$. Let $G \setminus m$ denote the rooted colored graph obtained by the removal of a proper sub-melon $m$ in the rooted colored graph $G$, i.e. by removing $m$ and gluing together the two resulting unmatched half-edges to reform an edge. Conversely let $G[e \leftarrow m]$ denote the rooted colored graph obtained by the insertion of a melon $m$ at a non-root edge $e$ of the rooted colored graph $G$, i.e. by cutting $e$ into two half-edges and gluing these half-edges to those of $m$ in the unique way that makes the result bipartite. The insertion at the root edge can be performed in two different ways depending on which half of the root of $G$ is taken as root after the insertion.

**Proposition 2.** A graph is melonic if and only if it can be obtained from $\Delta$ by a sequence of melon insertions.

**Proof.** By induction. 

Our interest in melonic graphs arises from the following two known results (e.g. [7,8]):

**Proposition 3.** The degree of $G$ and $G \setminus m$ are the same.

**Proof.** By construction $G \setminus m$ has two fewer vertices and $\binom{D}{2}$ fewer faces than $G$, hence by Equation (2) they have the same degree.

**Theorem 3.** [7,8] A rooted colored graph is melonic if and only if it has degree 0.

**Proof.** Propositions 2 and 3 immediately implies that melonic graphs have degree 0. The proof of the other implication is based on the observation that any graph of degree 0 contains a non-root melon [7,8].
The core decomposition. A proper open subgraph of an open colored graph $G$ is a connected open subgraph of $G$. By extension a proper open subgraph of a rooted colored graph $G$ is a proper open subgraph of $\text{op}(G)$. In particular $\text{op}(G)$ is the largest proper open subgraph of $G$. Two proper open subgraphs of an open colored graph $G$ are totally disjoint if their edge sets are disjoint and their half-edges belong to different edges of $G$. The following lemma is illustrated by Fig. 4.

**Lemma 6.** If two proper open melonic subgraphs of a rooted colored graph $G$ are not totally disjoint then their union is an open melonic subgraph of $G$.

**Proof.** First observe that if $M$ is a proper open melonic subgraph of $G$ with half-edges $h \in e$ and $h' \in e'$, then the edges $e$ and $e'$ of $G$ form a 2-edge-cut of $G$, unless $e = e' = r$ the root of $G$, in which case $M$ is $\text{op}(G)$. Indeed since each vertex of $M$ has the same degree in $M$ and in $G$, all the edges of $G$ incident to a vertex of $M$ are in $M$, except for $e$ and $e'$.

Now if $M_1$ and $M_2$ are two open melonic subgraphs of $G$ with half-edges $h_1 \in e_1$ and $h'_1 \in e'_1$, and $h_2 \in e_2$ and $h'_2 \in e'_2$ respectively, then the two cuts $\{e_1, e'_1\}$ and $\{e_2, e'_2\}$ either

- belong to two different cut-sets: in this case, in view of Lemma 5, either $M_1 \subset M_2$ or $M_2 \subset M_1$ (or $M_1$ and $M_2$ are totally disjoint but this contradict the hypothesis).
- or belong to the same cut-set: in this case, since $M_1$ and $M_2$ are not totally disjoint, there are two edges $e''_1 \in \{e_1, e'_1\}$ and $e''_2 \in \{e_2, e'_2\}$ such that $M_1 \cup M_2$ is the component of $G - \{e''_1, e''_2\}$ not containing the root. In particular the closed components of the cut-set of the root of $\text{cl}(M_1 \cup M_2)$ are the union of the closed components of the cut-sets of the roots of $\text{cl}(M_1)$ and $\text{cl}(M_2)$ and they are all prime melonic graphs, so that $M_1 \cup M_2$ is melonic.

□

A maximal melonic subgraph of an open colored graph $G$ is a proper open melonic subgraph which is maximal for inclusion.

**Lemma 7.** The maximal melonic subgraphs of $G$ are totally disjoint.

**Proof.** This is an immediate consequence of Lemma 6. □

The core of an open colored graph $G$ is the graph $\hat{G}$ obtained from $G$ by deleting each of its maximal open melonic subgraphs and gluing the resulting pairs of half-edges to reform edges. Observe that if a half-edge of $G$ belongs to a melonic subgraph then there is no gluing to perform for the deletion of this melonic subgraph, and if both half-edge of $G$ belong to a same melonic subgraph then $G$ is melonic and the core is empty.
A rooted colored graph is *melon-free* if it does not contains a proper sub-melon. By convention the empty graph is considered as a melon-free graph: in view of Proposition 2 it is in fact the only melon-free graph of degree 0. By construction the core of an open colored graph \( G \) is melon-free.

The following characterization (which is in fact not used in the rest of the paper) more generally relates our construction to the initial discussion of this section.

**Proposition 4.** The core \( \hat{G} \) of a rooted colored graph \( G \) is the unique melon-free graph that can be obtained from \( G \) by a sequence of melon removals.

By definition of the core \( \hat{G} \) of an open colored graph \( G \), for each non-root edge \( e = \{x, y\} \) with color \( i \) in \( \hat{G} \),

- either there is a maximal melonic subgraph \( G_e \) in \( G \) whose half-edges have color \( i \) and respectively point to \( x \) and \( y \),
- or there is an edge \( \{x, y\} \) with color \( i \) in \( G \) and this edge is not involved in any melonic subgraph of \( G \), and in this case \( G_e = \emptyset \) by convention.

Similarly if \( x \) is the extremity of the white half-edge \( h_w \) (resp. black half-edge \( h_b \)) of \( \hat{G} \), then

- either there is a maximal melonic subgraph \( G_w \) (resp. \( G_b \)) whose white (resp. black) half-edge is the white (resp. black) half-edge of \( G \),
- or the extremity of the white (resp. black) half-edge of \( G \) is \( x \) and this half-edge is not involved in any melonic subgraph of \( G \), and in this case \( G_w = \emptyset \) (resp. \( G_b = \emptyset \)).

The **core decomposition** of a rooted colored graph \( G \) is the tuple \( (\hat{G}; (G_0, G_1, \ldots, G_{(D+1)p})) \),
where \( e_1, \ldots, e_{D-1} \) is a canonical list of the edges of \( \hat{G} \).

The following theorem summarizes, and is an immediately consequence of, the above discussion.

**Theorem 4 (Core decomposition).** The core decomposition is one-to-one between

- rooted colored graph with \( 2k \) nodes and degree \( \delta \),
- pairs \( (G'; (G_0, G_1, \ldots, G_{(D+1)p})) \) where
  - \( G' \) is melon-free rooted colored graph with \( 2p \) vertices and degree \( \delta \), and
  - for all \( i = 0, 1, \ldots, (D+1)p \), \( G_i \) is a possibly open melonic graph with \( 2q_i \) vertices, such that \( 2p + \sum_{i=0}^{(D+1)p} 2q_i = 2k \).

**Proof.** The core decomposition is clearly injective since all the components of \( G \) have been kept in the decomposition as well as the correspondence between edges of the core and subgraphs. Conversely any such pair yields a rooted colored graph \( G \) by substitution of the \( G_i \) in \( \hat{G} \) and all substituted melonic subgraphs becomes totally disjoint maximal melonic subgraphs in \( G \), so that \( G' \) is the core of \( G \) and the core decomposition gives back the \( G_i \).

The relation between \( k, p \) and the \( q_i \) follows from the remark that the core decomposition is a partition of the vertex set of \( G \), while the fact that a rooted colored graph and its core have the same degree follows from Proposition 3.

### 3.2 Chains

The set of cores of degree \( \delta \) is not finite for \( \delta \geq 1 \), so that we need to refine further the decomposition. In order to do that we will identify and remove maximal chains of some well chosen \( (D-1) \)-dipoles.

A \( (D-q) \)-dipole is a couple of vertices connected by exactly \( D - q \) parallel edges, and thus attached to the rest of the graph by \( q \) pairs of half-edges of the same color. A \( (D-1) \)-dipole of color \((i, j)\) is a \( (D-1) \)-uple of parallel edges attached to the rest of the graph by two half-edges of color \( i \) and two half-edges of color \( j \). (This definition is slightly different from the one found in [12][5].)

As illustrated by Fig. [6] let a pre-chain in a rooted colored graph \( G \) be a configuration made of
Fig. 6. Two chains, with odd and even length respectively (with $D = 4$, edge colors omitted), and a pair of non-disjoint, non-proper chains (of length 1, with $D = 3$)

- two left external half-edges $\ell_\circ$ and $\ell_\bullet$,
- two right external half-edges $r_\bullet$ and $r_\circ$,
- $2p$ internal vertices, $p \geq 1$, forming a sequence $d_1, \ldots, d_p$ of $(D-1)$-dipoles

such that

- the white (resp. black) vertex of $d_1$ is incident to $\ell_\circ$ (resp. $\ell_\bullet$)
- the white (resp. black) vertex of $d_p$ is incident to $r_\circ$ (resp. $r_\bullet$)
- the dipole $d_i$ and $d_{i+1}$ share two edges for each $i = 1, \ldots, p - 1$
- none of the root or the four external half-edges $\ell_\circ, \ell_\bullet, r_\circ, r_\bullet$ are matched together (in other terms two external half-edges cannot be part of a same edge).

The vertices of the dipoles $d_i$ are referred to as internal vertices of the chain. A pre-chain is proper if it contains at least 4 internal vertices (or equivalently at least two $(D-1)$-dipoles). A proper pre-chain is maximal if it cannot be extended into a larger proper chain.

Finally a pre-chain is a chain if its two left external half-edges $\ell_\circ$ and $\ell_\bullet$ have the same color (and then so do the right external half-edges $r_\circ$ and $r_\bullet$), it is a twister if $\ell_\circ$ and $\ell_\bullet$ have different colors $\{i, j\}$ (and in this case, due to color constraints on each dipole $r_\circ$ and $r_\bullet$ also have the same different colors $\{i, j\}$).

**Lemma 8.** If $D \geq 4$, two distinct maximal pre-chains in a melon-free rooted colored graph cannot share an internal vertex. If $D = 3$, two distinct maximal proper pre-chains in a melon-free rooted colored graph cannot share an internal vertex.

**Proof.** Assume first that the rooted colored graph $G$ contains two chains that share a $(D-1)$-dipole. But this dipole has exactly the same neighbors in both chains so that the two chains cannot be both maximal and differ.

Now if two chains share a vertex but no $(D-1)$-dipoles, then this vertex must belong to two distinct $(D-1)$-dipoles. Parallel edge count shows that this is not possible if $D \geq 4$ (unless the vertex belongs to a melon). As illustrated by the right hand side of Fig. 6 for $D = 3$ a vertex $u$ can belong to two 2-dipoles $u - v$ and $u - w$. But if $u - v$ belongs to a proper chain, then $w$ has to belong to the same chain (since the chain has at least 4 internal vertices), and the graph reduces to a double cycle of length 4. But the root edge is then necessarily internal.
3.3 Classification of chains

There are two main types of chains, depending on the way the external edges are connected by faces of $G$:

**Broken chains** A chain with external color $(i, j)$ is broken if for all $k \neq i$, $\ell_\circ$ and $\ell_\bullet$ are in the same face of color $(i, k)$ (and in this case $r_\circ$ and $r_\bullet$ are in the same face of color $(k, j)$ for all $k \neq j$). This situation is schematized by Case (a) in Figure 7.

**Unbroken chains** Chains that are not broken are unbroken. Let us consider separately chains of external color $(i \neq j)$ and $(i, i)$:

- External color $(i \neq j)$: $\ell_\circ$ has color $i$ and belongs to a face of color $(i, k)$ which does not contain $\ell_\bullet$ (since the chain is not broken): this face has to leave the chain through $r_\circ$ or $r_\bullet$ so that $k = j$. Moreover this face cannot visit two vertices of a same dipole of the chain so it has to travel horizontally. Since both edge colors and vertex colors alternate along the face, it leaves the chain through $r_\bullet$ and the chain contains an odd number of $(D - 1)$-dipoles. This situation is schematized by Case (b) in Figure 7.

- External color $(i, i)$: again there is a face containing $\ell_\circ$ and not $\ell_\bullet$ (since the chain is not broken), which has to travel horizontally and leave the chain by $r_\bullet$ or $r_\circ$, in fact $r_\bullet$ due to color alternation as above. There is only one such face traveling horizontally between $\ell_\circ$ and $r_\bullet$, with color $(i, j)$ (and a parallel face of color $(i, j)$ goes through $\ell_\bullet$ and $r_\circ$). The color $j$ is referred to as the secondary color of the unbroken chain. This situation is schematized by Case (c) in Figure 7.

For completeness let us also to describe twisters:

**Twister with color** $(i, j)$ They are similar to unbroken chains except that the bottom colors are reversed.

In the rest of the paper twisters will not play any role: they will just be considered as peculiar configurations of $D - 1$ dipoles. In particular a twister is not a chain.
Fig. 8. An example of scheme.

### 3.4 Schemes

A *scheme* is a graph with colored edges consisting of two types of vertices:

- Regular black and white vertices of degree $D + 1$, each incident to one edge of each color.
- Chain-vertices of one of the following 3 types:
  - Broken chain-vertices of color $(i, j)$. (Case (a) of Figure 7)
  - Unbroken chain-vertices of color $(i, j)$, $i \neq j$. (Case (b) of Figure 7)
  - Unbroken chain-vertices of color $(i, i)$, with secondary color $j$. (Case (c) of Figure 7)

A scheme is *reduced* if it does not contain any proper chain.

The scheme $\tilde{G}$ of a core $\hat{G}$ is obtained by replacing each maximal proper chains of $\hat{G}$ by the corresponding chain-vertex. Since maximal proper chains are vertex disjoint this can be done independently for each chain. Observe that in the case $D = 3$, this operation would not have been well defined if single $(D - 1)$-dipoles were considered as chains. This is why we restrict our attention to maximal proper chains.

By construction, the scheme of a colored graph is reduced. Observe moreover that we do not associate any chain-vertex to a twister: even if a core contains twister made of many $(D - 1)$-dipoles, these dipoles remains as such in the scheme. In particular a reduced scheme can contain twisters.

**Proposition 5.** There is a bijection between the set of melon-free rooted colored graphs with $2k$ vertices and the set of pairs $(\tilde{G}; (c_1, \ldots, c_q))$ where $\tilde{G}$ is a reduced scheme with $q$ chain-vertices $x_1, \ldots, x_q$, and $c_i$ is a chain of the same type as $x_i$, such that the total number of vertices in $\tilde{G}$ and the $c_1, \ldots, c_q$ is $2k$.

The three types of chain-vertices that we have introduced allow to keep track in $\tilde{G}$ of the cycles of $\hat{G}$ that are not entirely included in a chain.

**Lemma 9.** Let $\hat{G}$ and $\hat{G}'$ be two melon-free rooted colored graphs with the same scheme $\tilde{G}$. Then $\hat{G}$ and $\hat{G}'$ have the same degree.

**Proof.** We postpone the proof of this lemma to Section 4.2 where we analyze the effect on the degree of the removal of a $(D - 1)$-dipole or a chain, and we show it depends only on the type of chain-vertex that represents it.

The previous lemma allows to define the *degree of a scheme*, as the common degree of all cores that have it as scheme.
3.5 Finiteness of the set of schemes of degree $\delta$

The validity of our classification relies on the following main result:

**Theorem 5.** The number of schemes of degree $\delta$ is finite.

The proof of Theorem 5 goes in two parts. First, in Section 4, we analyze the iterative elimination of chain-vertices, $(D - 1)$-dipoles, and in the cases $D \geq 4$, $(D - 2)$-dipoles, to prove the following result.

**Proposition 6.** The number of chain-vertices, $(D - 1)$, and for $D \geq 4$, $(D - 2)$-dipoles in a scheme of degree $\delta$ is bounded by $5\delta$.

Once this result is granted, we observe that the minimal realization of any chain-vertex consists of at most three $(D - 1)$-dipoles, so that there is an injective map of schemes of degree $\delta$ into graphs of degree $\delta$ with at most $15\delta (D - 1)$ and $(D - 2)$-dipoles. In order to prove Theorem 5 it thus only remains to prove that there exists only a finite number of graphs of degree $\delta$ with at most $15\delta (D - 1)$ or $(D - 2)$-dipoles. Section 5 is dedicated to the proof of the following more general result.

**Proposition 7.** For $D = 3$, the number of colored graphs with a fixed number of 2-dipoles is finite. For $D \geq 4$, the number of colored graphs with fixed numbers of $(D - 1)$-dipoles and $(D - 2)$-dipoles is finite.
4 Bounds on the number of chain-vertices and dipoles in a scheme.

The aim of this section is to prove Proposition\[\textit{5}\]

As a preliminary we analyze the effect of the deletion of a single \((D - q)\)-dipole. Next we extend the analysis to the deletion of a chain-vertex. Finally we prove Proposition\[\textit{6}\] through the announced analysis of iterative deletion of all chain-vertices, \((D - 1)\)-dipoles, and, for \(D \geq 4\), \((D - 2)\)-dipoles.

4.1 Analysis of a \((D - q)\)-dipole removal.

Let us define more precisely the removal of a \((D - q)\)-dipole (with \(1 \leq q \leq D - 2\)) of a graph \(G\): assuming the non-parallel edges of the dipole have colors \(i_0, i_1, \ldots, i_q\), we delete the two vertices of the dipole and their incident half-edges and form one new edge of each color \(i_0, i_1, \ldots, i_q\) with the remaining half-edges. By construction the number of vertices decreases by 2 and the number of edges decrease by \((D + 1)\). In order to understand the variation of the degree we need to consider more precisely the variation of the number of faces: we shall see that this depends on the number of connected components the graph separates into upon removal of the \((D - q)\)-dipole and on whether couples of new edges \((i_a, i_b)\) belong to a same cycle or not.

Connected components and faces after a dipole removal. We denote \(G_1, G_2, \ldots, G_C\) the \(C\) connected components obtained after the removal of the \((D - q)\)-dipole, \(1 \leq C \leq q + 1\). As the removal of the dipole deletes two vertices we have

\[
k(G) = k(G_1) + k(G_2) + \cdots + k(G_C) + 1.
\]

We denote \(d_1\) the number of new edges belonging to \(G_1\), \(d_2\) the number of new edges belonging to \(G_2\), and so on. We have

\[
d_1 + d_2 + \cdots + d_C = q + 1.
\]

Without loss of generality we can assume that the colors of the \(d_1\) new edges belonging to \(G_1\) are \(i_0, \ldots, i_{d_1 - 1}\), the colors of the \(d_2\) new edges belonging to \(G_2\) are \(i_{d_1}, \ldots, i_{d_1 + d_2 - 1}\) and so on. The graphs \(G_1, G_2, \ldots, G_C\) are colored graphs.

The faces affected by the \((D - q)\)-dipole removal are the ones containing at least one of its vertices. They fall in three categories:

- Faces with colors \((c, c')\) such that \(\{c, c'\} \cap \{i_0, i_1, \ldots, i_q\} = \emptyset\). For each of the \(\binom{D - q}{2}\) choices of such colors, exactly one face of degree two (made of two parallel edges of the dipole) is deleted by the removal of the dipole:

\[
F^{c c'}(G) = F^{c c'}(G_1) + F^{c c'}(G_2) + \cdots + F^{c c'}(G_C) + 1,
\]

- Faces with colors \((i, c)\), with \(i \in \{i_0, i_1, \ldots, i_q\}\) and \(c \not\in \{i_0, i_1, \ldots, i_q\}\). For each of the \((D - q - 1)(q + 1)\) choices of such colors, exactly one face is incident to the dipole: this face has length at least four in \(G\) and the dipole removal reduces its length by 2, so that the number of faces of this color is unchanged:

\[
F^{i c}(G) = F^{i c}(G_1) + F^{i c}(G_2) + \cdots + F^{i c}(G_C),
\]

- Faces with colors \((i, j)\) with \(\{i, j\} \subset \{i_0, i_1, \ldots, i_q\}\). For each of the \(\binom{q + 1}{2}\) choices of such colors, exactly one face is incident to the dipole. In this case we distinguish between two possibilities:

  - **Type a.** The four edges of color \(i\) and \(j\) belong to the same cycle \((i, j)\) in \(G\). Upon removal of the \((D - q)\)-dipole the cycle \((i, j)\) splits into two disjoint cycles:

    \[
    F^{ij}(G) = F^{ij}(G_1) + F^{ij}(G_2) + \cdots + F^{ij}(G_C) - 1.
    \]

  - **Type b.** The four edges of color \(i\) and \(j\) belong to two distinct cycles \((i, j)\) in \(G\). Upon removal of the \((D - q)\)-dipole the two cycles \((i, j)\) are merged into a unique cycle:

    \[
    F^{ij}(G) = F^{ij}(G_1) + F^{ij}(G_2) + \cdots + F^{ij}(G_C) + 1.
    \]

We are now in position to describe the global effect of the removal of a dipole on the degree.
Case I. Completely separating \((D - q)\)-dipoles. We first consider the case in which the removal of \((D - q)\)-dipoles splits the graph into \(q + 1\) connected components each containing exactly one new edge \((C = q + 1\) with the notation above): we refer to such a dipole as completely separating. We illustrated in Figure 9 the case of a completely separating \((D - 1)\)-dipole.

![Figure 9](image)

Fig. 9. The decomposition at a completely separating \((D - 1)\)-dipole.

In this case all the faces of colors \((i, j)\) with \(\{i, j\} \subset \{i_0, i_1, \ldots i_q\}\) are of Type \(a\), hence

\[
F(G) = F(G_1) + F(G_2) + \cdots + F(G_{q+1}) + \binom{D - q}{2} - \binom{q + 1}{2} \\
= F(G_1) + F(G_2) + \cdots + F(G_{q+1}) + \frac{1}{2}D(D - 2q - 1).
\]

(5)

Using Equation [2] we get

\[
\delta(G) = \frac{1}{2}D(D - 1)k(G) + D - F(G) \\
= \frac{1}{2}D(D - 1)\left(k(G_1) + k(G_2) + \cdots + k(G_{q+1}) + 1\right) + D \\
- \left(F(G_1) + F(G_2) + \cdots + F(G_{q+1}) + \frac{1}{2}D(D - 2q - 1)\right) \\
= \delta(G_1) + \delta(G_2) + \cdots + \delta(G_{q+1}).
\]

In other terms the degree is distributed between the \(q + 1\) connected components.

Case II. Non completely separating \((D - q)\)-dipoles. We now consider the remaining cases \((1 \leq C \leq q)\): we refer to such a dipole as non completely separating.

All the faces \((i, j)\) with \(\{i, j\} \subset \{i_0, i_1, \ldots i_q\}\) and furthermore \(i \in \{i_0, \ldots i_{a_1 - 1}\}\) and \(j \notin \{i_0, \ldots i_{a_1 - 1}\}\) are again of Type \(a\). On the contrary the faces \((i, j)\) with \(\{i, j\} \subset \{i_0, \ldots i_{a_1 - 1}\}\) can be either of Type \(a\) or of Type \(b\). We represented in Figure 10a \((D - 1)\)-dipole with a face (in \(G\)) \((i, j)\) of Type \(a\), while in figure Figure 11a \((D - 1)\)-dipole of with two faces (in \(G\)) of Type \(b\).

![Figure 10](image)

Fig. 10. The decomposition at a non-separating \((D - 1)\)-dipole resulting into two \((i, j)\)-cycles.
Denote \( r_1 \) the number of such faces of Type \( b \), \( 0 \leq r_1 \leq \binom{d_1}{2} \) in \( G_1 \), \( r_2 \) the number of faces of type \( b \) in \( G_2 \) and so on. We have

\[
F(G) = F(G_1) + F(G_2) + \cdots + F(G_C) + \left( \frac{D-q}{2} \right) - \left( \frac{q+1}{2} \right) + 2r_1 + 2r_2 + \cdots + 2r_C \\
= F(G_1) + F(G_2) + \cdots + F(G_C) + \frac{1}{2}(D-2q-1) + 2r_1 + 2r_2 + \cdots + 2r_C
\]

Using again Equation (2) we get

\[
\delta(G) = \frac{1}{2}D(D-1)k(G) + D - F(G) \\
= \frac{1}{2}D(D-1)\left( k(G_1) + k(G_2) + \cdots + k(G_C) + 1 \right) + D \\
- \left( F(G_1) + F(G_2) + \cdots + F(G_C) + \frac{1}{2}(D-2q-1) + 2r_1 + 2r_2 + \cdots + 2r_C \right) \\
= \delta(G_1) + \delta(G_2) + \cdots + \delta(G_C) + D(q+1-C) - 2r_1 - 2r_2 - \cdots - 2r_C.
\]

In other terms the variation of the degree through the removal depends on the structure of the incident cycles. We now make the various possibilities more explicit in the cases of \((D-1)\)- and \((D-2)\)-dipoles.

**Case II \( (q = 1) \). Non completely separating \((D - 1)\)-dipoles.** In the case \( q = 1 \), we have necessarily \( C = 1 \) and \( d_1 = 2 \). Depending on the value of \( r_1 \) we thus distinguish two cases:

- Case II.A: the face \((i_0, i_1)\) is of Type \( a \), hence \( r_1 = 0 \) (represented in Figure 10), \( \delta(G) = \delta(G_1) + D \).
- Case II.B: the faces \((i_0, i_1)\) are of Type \( b \), \( r_1 = 1 \), (represented in Figure 11) \( \delta(G) = \delta(G_1) + \delta(G_2) + D - 2 \).

Observe that for all \( D \geq 3 \) the degree strictly decreases under the removal of a non completely separating \((D - 1)\)-dipole.

**Case II \( (q = 2) \). Non completely separating \((D - 2)\)-dipoles.** In the case \( q = 2 \) there are two possible values for \( C \), and a total of 6 possible cases:

- \( C = 1, d_1 = 3 \) and
  - \( r_1 = 0, \delta(G) = \delta(G_1) + 2D \).
  - \( r_1 = 1, \delta(G) = \delta(G_1) + 2D - 2 \).
  - \( r_1 = 2, \delta(G) = \delta(G_1) + 2D - 4 \).
  - \( r_1 = 3, \delta(G) = \delta(G_1) + 2D - 6 \).
- \( C = 2, d_1 = 1 \) (hence \( r_1 = 0 \), \( d_2 = 2 \) and
  - \( r_2 = 0, \delta(G) = \delta(G_1) + \delta(G_2) + D \).
  - \( r_2 = 1, \delta(G) = \delta(G_1) + \delta(G_2) + D - 2 \).

Observe that, for all \( D \geq 4 \), the degree strictly decreases under the removal of a non completely separating \((D - 2)\)-dipole.
4.2 Chain-vertex removal vs \((D - 1)\)-dipole removal

The removal of a chain-vertex consists in deleting this vertex and the incident half-edges and creating two new edges by joining two by two the remaining half-edges that arise from the same extremity of the chain-vertex. The removal of a chain-vertex in a scheme can alternatively be performed in the following equivalent way:

- Replace the chain-vertex by its minimal length chain representation: this yields a scheme \(G^+\) with same degree.
- Remove one of the \((D - 1)\)-dipole of the inserted chain: one of the three cases above applies.
- Eliminate the melons that might have been created: these operations do not affect the degree.

This last procedure, although slightly more complex \textit{a priori}, allows to built on the case analysis already done for \((D - 1)\)-dipole removal.

**Case I. Separating chain-vertex.** The removal of a chain-vertex separates the graph \(G\) into two components \(G_1\) and \(G_2\) if and only if the deletion of any \((D - 1)\)-dipole of the equivalent chain separates the graph \(G^+\) into two components \(G_1^+\) and \(G_2^+\). In such a case,

\[
\delta(G) = \delta(G^+) = \delta(G_1^+) + \delta(G_2^+) = \delta(G_1) + \delta(G_2) .
\]

Observe that in this case, the chain-vertex can represent indifferently an unbroken or a broken chain.

**Case II. Non-separating chain-vertex, two resulting cycles** This case is similar to Case II.A of the previous section: the removal of the chain-vertex does not separate the graph \(G\) and in the resulting graph \(G'\) the two new edges belong to two different \((i, j)\)-cycles. Then the removal of the chain-vertex is equivalent to a Case II.A removal of \((D - 1)\)-dipole in the graph \(G^+\), followed by some melon deletions:

\[
\delta(G) = \delta(G^+) = \delta(G_1^+) = \delta(G_2^+) = \delta(G') + D
\]

whether the chain-vertex represent an unbroken or a broken chain.

**Case III.a. Non-separating chain-vertex, unbroken chain, single resulting cycle.** This case is similar to Case II.B of the previous section: the removal of the chain-vertex does not separate the graph \(G\) but in the resulting graph \(G'\) the two new edges belong to a same \((i, j)\)-cycle. If the chain-vertex represents an unbroken chain then the removal of the chain-vertex is equivalent to a Case II.B removal of \((D - 1)\)-dipole in the graph \(G^+\), followed by some melon deletions:

\[
\delta(G) = \delta(G^+) = \delta(G_1^+) = \delta(G_2^+) = \delta(G') + D - 2
\]

**Case III.b. Non-separating chain-vertex, broken chain, single resulting cycle.** The removal of the chain-vertex does not separates the graph \(G\) and in the resulting graph \(G'\) the two new edges belong to a same \((i, j)\)-cycle, but the chain-vertex represent a broken chain: in this case the removal is equivalent to a Case II.A removal of \((D - 1)\)-dipole in the graph \(G^+\), followed by some melon deletions:

\[
\delta(G) = \delta(G^+) = \delta(G_1^+) = \delta(G_2^+) = \delta(G') + D
\]
4.3 Iterative deletion of chain-vertices, \((D - 1)\)-dipoles and \((D - 2)\)-dipoles

Let \(\tilde{G}\) be a scheme of degree \(\delta\), and let \(u_1, \ldots, u_r, v_1, \ldots, v_s\) and \(w_1, \ldots, w_t\) denote respectively the chain-vertices, \((D - 1)\)-dipoles and \((D - 2)\)-dipoles of \(\tilde{G}\), in arbitrary order.

Starting from \(\tilde{G}\), perform the following operations:

- For \(i = 1, \ldots, r\) successively, remove \(u_i\), and mark the two resulting new edges.
- For \(i = 1, \ldots, s\) successively, remove \(v_i\), and mark the two resulting new edges.
- If \(D \geq 4\), for \(i = 1, \ldots, t\) successively:
  - If \(w_i\) is totally separating, delete \(w_i\), form three new coherent colored edges with the remaining half-edges, mark these three new edges, and add a copy of \(\Delta\) with three marked edges to the set of connected components.
  - If \(w_i\) is partially separating, break the pair of separating edges incident to \(w_i\) and form a colored edge with the resulting two half-edges in each of the two resulting connected components, and mark these two new edges.
  - If \(w_i\) is non-separating, deleted \(w_i\), form three new coherent colored edges with the remaining half-edges, and mark these three new edges.

Let \(\bar{G}\) be the resulting graph, and denote by \(\mathcal{C}^+\) the set of its connected components with positive degree plus the root component, and by \(\mathcal{C}_0\) the set of its connected components with degree 0.

 Depending on the ordering of the \(u_i\), \(v_i\) and \(w_i\), each removal that is performed can be separating or not. Let us introduce notations for the number of removals of each type during the process:

- \(p\) for all separating removals.
- \(q_1\) for removals of non-separating chain-vertices representing an unbroken chain that joins two cycles (Case IIIa, degree decreases by \(D - 2\)).
- \(q_2\) for removals of other non-separating chain-vertices (Cases II and IIIb, degree down by \(D\))
- \(q'\) for removals of non-separating \((D - 1)\)-dipoles (degree decreases by at least \(D - 2\))
- \(q''\) for removals of non-separating \((D - 2)\)-dipoles (degree decreases by at least \(2D - 6\)).

By definition \(p + q_1 + q_2 + q' + q'' = r + s + t\). The number of marked edges created by non-separating removal is at most \(2q_1 + 2q_2 + 2q' + 3q''\). Observe that the number can be much smaller, if the various dipoles and chain vertices share external edges, like in figure 12.

Similarly, the number of marked edges created by separating removal is at most \(2(|\mathcal{C}^+| + |\mathcal{C}_0| - 1)\) (since a separating removal creates at most two new marks and simultaneously increases the number of connected components by one). Letting \(m(c)\) denote the number of marked edges in a component \(c\) of \(\bar{G}\), we get

\[
\sum_{c \in \mathcal{C}^+ \cup \mathcal{C}_0} m(c) \leq 2(q_1 + q_2 + q') + 3q'' + 2(|\mathcal{C}^+| + |\mathcal{C}_0| - 1).
\]

**Lemma 10.** Each non-root component \(c \in \mathcal{C}^+_+\) has at least one marked edge. Each non-root component \(c \in \mathcal{C}_0\) has at least three marked edges.
Proof. Each component has at least one marked edge by construction. A component with degree zero and only two marked edges is a melonic graph with at most two melons: it is therefore a \((D-1)\)-dipole or a chain that was already present in \(G\) and that should have been removed.

In view of the previous lemma

\[
\sum_{c \in C_+ \cup C_0} m(c) \geq |C_+| + 3|C_0|,
\]

so that

\[
2(q_1 + q_2 + q') + 3q'' + |C_+| - 2 \geq |C_0|. \tag{6}
\]

Now the number of positive degree component is bounded by the initial degree \(\delta\), i.e. \(|C_+| \leq \delta + 1\), and the number of marked edges produced by non-separating removals, which is at most \(2(q_1 + q_2 + q') + 3q'' \leq 2\delta\) (each removal contributing to \(q_1 + q'\) decreases the degree by at least \(D - 2 \geq 1\), each removal contributing to \(q''\) decreases the degree by at least \(2D - 6 \geq 2\) and each removal contributing to \(q_2\) decreases the degree by \(D \geq 3\)). In view of Equation (6) above the number of components of degree 0 is in turn bounded by \(3\delta\), and the number of separating removals \(p = |C_+| + |C_0| - 1\) is bounded by \(4\delta\). Finally the total number of removals in the above procedure, which is also the total number \(r + s + t\) of chain-vertices, \((D-1)\)- and \((D-2)\)-dipoles is bounded by \(5\delta\).

This completes the proof of Proposition 6.
5 Graphs with bounded numbers of \((D - 1)\) and \((D - 2)\)-dipoles

In this section we prove Proposition 7.

The case \(D = 3\). For \(D = 3\), we are interested in graphs with fixed number of 2-dipoles, or equivalently, with a fixed number of faces of degree 2. In view of Equation 3, the number of faces of degree 6 or more in such a graph with degree \(\delta\) satisfies

\[
\sum_{p \geq 3} F_p \leq \sum_{p \geq 2} 2(p - 2)F_p \leq (D + 1)\delta + 2F_1
\]

i.e. this number is finite. However there could a priori be an arbitrary number of faces of degree 4 since the coefficient of \(F_2\) is zero for \(D = 3\). Let us rule this out thanks to the following remark: there exists an integer \(N(k, g)\) such that any map of genus \(g\) with more than \(N(k, g)\) vertices and less than \(k\) faces of length not equal to 4 contains a vertex \(v\) such that all faces at distance less than 3 of \(v\) have degree 4. Now upon considering an arbitrary jacket of \(G\), the previous remark implies that the neighborhood of the vertex \(v\) is necessarily a quotient of a four by four toroidal grid, in contradiction with the fact that \(G\) is arbitrarily large.

We conclude that there are only finitely many colored graphs with fixed number of 2-dipoles, which is the statement of Proposition 7 in the case \(D = 3\).

The case \(D \geq 4\). Consider a colored graph \(G\) of degree \(\delta\) with \(2k\) vertices, having \(t_1\) \((D - 1)\)-dipoles and \(t_2\) \((D - 2)\)-dipoles. We will show that the number of such graphs is finite. The bound we established below on the number of such graph is not tight and can be improved with minimal effort.

Let us count faces of degree 2 according to whether they belong to a \((D - 1)\)-dipole, a \((D - 2)\)-dipole or none of these two:

\[
F_1 \leq t_1 \left(\binom{D - 1}{2}\right) + t_2 \left(\binom{D - 2}{2}\right) + \alpha(D)k ,
\]

where

- \(\alpha(4) = 0\) as, for \(D = 4\), all the faces with two vertices must belong to a \((D - 1)\)- or a \((D - 2)\)-dipole (i.e. a 3- or a 2-dipole).
- \(\alpha(5) = 3\) as, for \(D = 5\), a vertex not belonging to a \((D - 1)\)- or \((D - 2)\)-dipole can belong to at most three 2-dipoles.
- \(\alpha(6) = 6\) as, for \(D = 6\), a vertex not belonging to a \((D - 1)\)- or \((D - 2)\)-dipole belongs to the largest number of faces of degree two when it belongs to two 3-dipoles i.e. to six faces of degree two.
- \(\alpha(D) = \frac{(D - 3)(D - 4)}{2} + 6\), for all \(D \geq 7\) as, in this case, a vertex not belonging to a \((D - 1)\)- or \((D - 2)\)-dipole belongs to the largest number of faces of degree two when it belongs to a \((D - 3)\)-dipole and a 4-dipole.

One the one hand, the bound (7) together with Equation 3 gives

\[
\sum_{p \geq 2} ((D - 1)p - D - 1)F_p \leq (D + 1)\delta - D(D + 1) + 2t_1 \left(\binom{D - 1}{2}\right) + 2t_2 \left(\binom{D - 2}{2}\right) + 2\alpha(D)k .
\]

On the other hand, together with Equation 2 it yields

\[
\left(\frac{D(D + 1)}{2} - \alpha(D)\right)k \leq \sum_{p \geq 2} pF_p \leq t_1 \left(\binom{D - 1}{2}\right) + t_2 \left(\binom{D - 2}{2}\right)
\]

Eliminating \(k\) between these two equations and reordering we get:
\[
\sum_{p \geq 2} \left[ \left( D - 1 - \frac{4\alpha(D)}{D(D + 1) - 2\alpha(D)} \right) p - D - 1 \right] F_p \\
\leq (D + 1)\delta - D(D + 1) + \left( 2 + \frac{4\alpha(D)}{D(D + 1) - 2\alpha(D)} \right) t_1 \left( \begin{array}{c} D - 1 \\ 2 \end{array} \right) \\
+ \left( 2 + \frac{4\alpha(D)}{D(D + 1) - 2\alpha(D)} \right) t_2 \left( \begin{array}{c} D - 2 \\ 2 \end{array} \right).
\]

The coefficient of \( F_p \) on the left hand side is
- for \( D = 4 \): \( 3p - 5 \).
- for \( D = 5 \): \( \frac{7}{2}p - 6 \).
- for \( D = 6 \): \( \frac{21}{4}p - 7 \).
- for \( D \geq 7 \): \( \frac{3(D-4)(D+1)}{4(D-3)} (p - 2) + \frac{(D-6)(D+1)}{2(D-3)} \).

In particular this coefficient is strictly positive for \( p \geq 2 \) so that we get for the previous equation an upper bound for each \( F_p, p \geq 2 \) depending only on \( D \) and \( \delta \), and there is a maximal value of \( p \), depending again only on \( D \) and \( \delta \) for which \( F_p \) can be non zero.
On the other hand,

\[
\frac{D(D + 1)}{2} - \alpha(D) = \begin{cases} 
10, & D = 4 \\
12, & D = 5 \\
15, & D = 6 \\
4(D - 3), & D \geq 7 
\end{cases}
\]

is always positive, so that from Equation \( \text{[8]} \) we finally get an upper bound on \( k \) depending only on \( D, \delta, t_1 \) and \( t_2 \).

This completes the proof of Proposition \( \text{[7]} \).
6 Exact enumeration

6.1 Melonic graphs and cores

In view of Proposition 4 we will need, in order to enumerate colored graphs, the generating function of possibly empty melonic graphs.

Proposition 8 (See e.g. [7]). The generating function $T(z)$ of possibly empty rooted melonic graphs having root color 0 with respect to the number of black vertices is the unique power series solution of the equation

$$T(z) = 1 + zT(z)^{D+1}.$$ 

Proof. Let $M(z)$ be the generating function of rooted prime melonic graphs. Then the inductive definition of rooted melonic graphs and prime melonic graphs immediately translate into the equations:

$$T(z) = 1 + \sum_{i \geq 1} M(z)^i = \frac{1}{1 - M(z)}$$

and

$$M(z) = zT(z)^D.$$ 

Proof. The first expansion follows immediately from Proposition 8 using Lagrange inversion formula. The singular expansion is e.g. a direct instance of the standard theory of singularity analysis of simple trees generating functions [6][Chap. VII.4].

Proposition 9. Let $\hat{G}$ be a rooted melon-free graph with $2p$ vertices, and thus $(D + 1)p$ edges. The generating function $F_{\hat{G}}(z)$ of rooted colored graphs with core $\hat{G}$ with respect to the number of black vertices is

$$F_{\hat{G}}(z) = z^p T(z)^{(D+1)p+1}.$$ 

Proof. This immediately follows from the bijection of Theorem 3. The case $p = 0$ corresponds to the empty core graph associated to the melons.
6.2 Chains of \((D - 1)\)-dipoles and schemes

In view of Proposition 5 in order to enumerate cores in terms of schemes, we will need four chain generating series, depending on whether the chain is broken or not, and whether its external edges have identical color or not. Recall that a proper chain has at least 4 internal vertices.

**Unbroken chains** Let us fix two colors \(i \neq j\). There is exactly one \((i, j)\)-unbroken chain with \(2k\) vertices, for \(k \geq 1\), so their generating function \(C(u)\) with respect to the number of \((D - 1)\)-dipoles, and the generating function \(C^+(u)\) of the proper ones are

\[
C(u) = \frac{u}{1-u}, \quad C^+(u) = \frac{u^2}{1-u}.
\]

External edges have different colors if the number of dipoles is odd, equal colors if it is even: the generating functions for \((i, j)\)-unbroken proper chains with left extremity of color \(i\) and right extremity of color \(j\), or \(i\), are thus respectively:

\[
C^+_i(u) = \frac{u^3}{1-u^2}, \quad C^+_j(u) = \frac{u^2}{1-u^2}.
\]

**Arbitrary chains** Let us fix one color \(i\). The generating function of arbitrary non-empty, and arbitrary proper chains of dipoles with left external color \(i\) are

\[
A(u) = \frac{Du}{1-Du}, \quad A^+(u) = \frac{(Du)^2}{1-Du}.
\]

A non-empty chain with external colors \((i, i)\) consists of a non-empty chain not reusing color \(i\) followed by a \((D - 1)\)-dipole with right external color \(i\) and a possibly empty chain with external colors \((i, i)\):

\[
A_=(u) = \frac{Du}{1-(D-1)u} : u : (1 + A_=(u)) = \frac{Du^2}{1-(D-1)u} : \frac{1}{1-(Du^2/(D-1)u)} = \frac{Du^2}{(1+u)(1-Du)}
\]

Now fix moreover a second color \(j \neq i\). A non-empty chain with external colors \((i, j)\), \(i \neq j\) is an arbitrary non-empty chain which has not equal external colors:

\[
A_\neq(u) = \frac{1}{D} (A(u) - A=(u)) = \frac{u}{(1+u)(1-Du)}.
\]

**Broken chains** Let us fix two colors \(i\) and \(j\) (maybe equal). A proper broken chain with external color \((i, i)\) is an arbitrary proper chain which is not unbroken. If \(i = j\), all the \(D\) possible second colors for the unbroken chain have to be considered:

\[
B^+_i(u) = A=(u) - DC^+_i(u) = \frac{Du^2}{1+(1-Du)} - \frac{Du^2}{1-u^2} = \frac{D(D-1)u^2}{(1-u^2)(1-Du)} \quad (11)
\]

\[
B^+_\neq(u) = A_\neq(u) - u - C^+_\neq(u) = \frac{u}{(1+u)(1-Du)} - \frac{u}{1-u^2} = \frac{D-Du^2}{(1-u^2)(1-Du)} \quad (12)
\]

**Proposition 10.** Let \(S\) be a reduced scheme with \(2p\) black and white vertices, \(c_w\) (resp. \(b_\neq\)) unbroken (resp. broken) chains with equal color extremities and \(c_\neq\) (resp. \(b_w\)) unbroken (resp. broken) chains with different color extremities. The generating function \(G_S(u)\) of rooted melon-free graphs with scheme \(S\) is

\[
G_S(u) = \frac{u^p B^+_w(u)^b C^+_w(u)^c}{P_S(u)} = \frac{u^p B^+_\neq(u)^b C^+_\neq(u)^c}{P_S(u)}
\]

where \(b = b_w + b_\neq\) and \(c = c_w + c_\neq\) and \(P_S\) is the monomial

\[
D^{b+c} (D-1)^{b+w+c} u^{2b+2c}.
\]

**Proof.** Again this follows immediately from the bijection in Proposition 5 and the explicit formulae for \(B^+_w, C^+_w, B^+_\neq, C^+_\neq\).
6.3 The enumeration of rooted colored graph with degree $\delta$

To conclude the proof of Theorem 1, we now have only to put together Theorem 5 with Propositions 4 and 5:

**Theorem 6.** Let $\delta \geq 1$. The generating function of rooted colored graphs with root edge of color 0 and degree $\delta$ with respect to the number of black vertices is

$$F^0_\delta(z) = T(z) \sum_{s \in S^0_\delta} G_s(zT(z)^{D+1})$$

where the sum is over the finite set $S^0_\delta$ of reduced scheme with degree $\delta$ and root edge of color 0.

The sum over all rooted graphs is recovered by multiplying by an overall factor $(D+1)$ for the color of the root. The first values can be computed explicitly. Observe that to do these computations by hand it is useful to group schemes that differ by “irrelevant” differences (like broken vs unbroken separating chains). We do it for $\delta = D-2$ to recover the known result: Degree $D-2$ ($\delta = D-2$). Up to some choices of colors, there is just one “non-degenerate” scheme, plus a few degenerate ones:

- The lollipop with a separating chain-vertex as arm and an unbroken chain-vertex as loop. It consists more precisely of a root melon with root of color 0 attached by some color $j \neq 0$ to an arbitrary chain with left external color $j$, attached in turn on the other side to a $(D-2)$-dipole, carrying in turn an unbroken chain with distinct colors at extremities:

$$F_{lolly}(z) = T(z) \cdot u \cdot D \cdot A(u) \cdot u \cdot \frac{D}{2} \cdot \frac{u}{1-u^2} \bigg|_{u=zT(z)^{D+1}}$$

Observe that $A(u)$ includes the case when the arm consist of a unique $D-1$ dipole, and the last term includes the case when the loop (unbroken chain) is reduced to a $(D-1)$-dipole.

- The “degenerate” ones occur when

  - the arm is absent. The scheme consists of a rooted $D-1$ dipole carrying an unbroken chain with distinct end colors.
  - the arm is present but it is empty. The scheme is formed by a rooted $D-1$ dipole connected by two parallel edges to a $D-2$ dipole, and the $D-2$ dipole carries an unbroken chain with distinct end colors.

The various generating functions are obtained upon replacing the relevant terms in $F_{lolly}$ by constants terms.

In this case it is more efficient to consider directly the various part of the scheme and reconstruct the generating function: the arm is either absent, or a (possibly empty) chain, with the root on one side and the (possibly reduced to a $(D-1)$-dipole) loop attached on the other side:

$$F_{D-2}(z) = T(z) \cdot u \cdot \left(1 + D \cdot u + D \cdot \frac{D}{2} \cdot \frac{u}{1-u^2} \bigg|_{u=zT(z)^{D+1}}\right)$$

This result is consistent with the one found in [11] and gives, as discussed below, the first order correction in the expansion of colored tensor model.

More generally, given a scheme $s$ whose root edge is not on a (root) melon, the schemes formed of a root melon hooked to the root of $s$, or attached to a $(D-1)$-dipole or a chain-vertex, in turn attached to the root of $s$ are valid schemes with the same degree as $s$, that can be considered to derive from $s$.

It turns out to be more efficient to list all schemes of $\cup_{\delta \geq 1} S^0_\delta$ according to their size, rather than going degree by degree: however computing all terms by hand is quite tedious. The first terms are listed below.
**without chain-vertex, 2 black vertices.** Two $D - q$ dipoles and two $q + 1$ dipoles for $q \geq 1$, disposed in a square (recall that chain-vertices represent only proper chains). Degree $(D - q - 1)q$.

**with 1 chain-vertex, 1 black vertex.** A $D - 1$ dipole connected to the chain vertex. Degree $D - 2$ if the chain vertex is unbroken or $D$ if it is broken.

**with 2 chain-vertices, 1 black vertex.** A $D - 3$ dipole (hence exists only for $D \geq 4$) attached to two chain vertices. Degree $2(D - 2)$, $2D - 2$ or $2D$ depending whether the chain vertices are both unbroken, one broken and one unbroken or both unbroken.

**with 1 chain-vertex, 2 black vertices.** The chain-vertex is connected to a vertical sequence of a $D - 1$ dipole, a 2 dipole and another $D - 1$ dipole. Degree is at least $2(D - 2)$.

**without chain-vertex, 3 black vertices.** There are numerous possibilities. For example every black vertex can be connected to each white vertex by a $p$ dipole a $q$ dipole and a $D + 1 - p - q$ dipole (the dipoles form a hexagon and its diagonals).
Dominant schemes

Comparing Proposition 10 and Theorem 6 to the singular expansion (2), we immediately see that $F_δ(z)$ has radius of convergence $z_0$ and admits a singular expansion near $z = z_0$ of the form

$$F_δ(z) = \frac{D + 1}{D} \sum_{S \in S_δ} \frac{D^{-p} P_S(1/D)}{(1 - 1/D^2)^{S+c}(2(D + 1)/D^2)^{b/2}(1 - z/z_0)^{b/2}(1 + O(\sqrt{1 - z/z_0}))}$$

which is dominated by schemes that maximize $b$. In other terms, with probability tending to 1 when $k$ goes to infinity, a uniform random colored graph with degree $\delta$ will have a scheme that maximizes $b$.

In order to identify these schemes we improve in this section the analysis of the number of broken chain-vertices in a scheme.

7.1 A bound on the number of broken chain-vertices

Starting from a scheme $\tilde{G}$ we iteratively remove all the broken chain-vertices to get a graph $\bar{G}'$. Let us denote by $q$ the number of broken chains that are not separating during this process, by $C_0'$ the set of non root connected components of degree 0 in $\bar{G}'$, and by $C_+'$ the set of connected components of positive degree plus the root component. Then the variation of degree between $\tilde{G}$ and $\bar{G}'$ is

$$\delta(\bar{G}') = \sum_{c \in C_+'} \delta(c) + Dq.$$

In view of the analysis of schemes of low degree in the previous section, we see that the minimal degree of a positive degree component is $\alpha D = (D - 2)$. Applying this to $C_+'$, the previous equation rewrites into a bound in terms of the number $c_+ = |C_+'| - 1$ of positive degree components:

$$(D - 2)c_+ + Dq \leq \delta(\bar{G}).$$

Moreover, our analysis of marked edges in $\tilde{G}$ of Section 4.3 can be adapted to $\bar{G}'$. However since we do not delete unbroken chains and $(D - 1)$-dipoles, we need a different argument to rule out degree zero components having only two marks: such a component is a $(D - 1)$-dipole, or a chain of $(D - 1)$-dipoles with one mark on each extremity, and it got its mark from a removed adjacent chain. But then that chain was not maximal. Therefore all the components in $C_0'$ have at least three marks and we get

$$2q + 2(|C_0'| + c_+) \geq \sum_{c \in C_+' \cup C_0'} m(c) \geq c_+ + 1 + 3|C_0'|$$

which implies

$$2q + c_+ - 1 \geq |C_0'|.$$

Moreover, this inequality is saturated only if all the zero degree components in $C_0'$ have exactly three marks and all the components in $C_+'$ have exactly one mark.

The number $p$ of separating broken chain-vertices satisfies

$$p = |C_0'| + c_+ \leq 2q + 2c_+ - 1$$

and the total number $b$ of broken chain-vertices in $\tilde{G}$ satisfies

$$b = p + q \leq 2c_+ + 3q - 1.$$
Proposition 11. The number $b$ of broken chain-vertices in a scheme of degree $\delta$ is at most
\[ b_{\text{max}} = 2c_+ + 3q - 1 , \]
and it saturates this bound only if all the components in $C'_+$ have exactly one mark and all the components in $C'_0$ have exactly three marks.

Furthermore, the parameters $c_+$ and $q$ satisfy
\[ (D-2)c_+ + Dq \leq \delta . \]
and the constraint is saturated only if all the components of positive degree in $C'_+$ have degree exactly $D-2$.

7.2 Realizability and dominant schemes

Given integers $c_+$ and $q$ that satisfy the constraint $(D-2)c_+ + Dq \leq \delta$, it is always possible to construct a scheme with these parameters that has $2c_+ + 3q - 1$ broken chain-vertices: form $c_+$ loops each using one unbroken chain-vertex, and put these loops and a root melon at the $c_+ + 1$ leaves of a binary tree whose $2c_+ - 1$ edges are broken chain-vertices. Finally add $q$ broken chain-vertices and attach their two extremities to existing edges: each attachment creates a $(D-2)$-dipole and add a further broken chain-vertex, so that $3q$ broken chain-vertices are added, to reach the expected total of $2c_+ + 3q - 1$.

The maximal value $b_{\text{max}}$ is therefore always of the form $2c_+ + 3q - 1$ for some $c_+$ and $q$ such that $(D-2)c_+ + Dq = \delta$.

Proposition 12. For any $\delta \geq 1$, the dominant schemes are schemes with $b_{\text{max}}$ broken chains where $b_{\text{max}}$ is the maximum of the following linear program:
\[ 2x + 3y - 1 , \]
subject to the constraint $(D-2)x + Dy = \delta$.

and such that
- All components in $C'_0$ have exactly 3 marks: they are $(D-2)$-dipoles.
- The root component has one mark and degree 0: it is a root melon.
- All positive components in $C'_+$ have one mark and degree $D-2$: each of them is an unbroken chain forming a loop at a $(D-2)$-dipole.
- All the other elements of the schemes are $b_{\text{max}}$ broken chains.

It follows that in all cases the dominant schemes can be seen as graphs whose edges represent the broken chains and whose vertices can either be three valent (representing the $(D-2)$-dipoles in $C'_0$) or univalent (representing the loops in $C'_+)$.

In order to completely characterize these schemes we need to determine $b_{\text{max}}$. Optimizing the above linear program yields the parameters of the candidate schemes with a maximal number of broken chain-vertices: the “pure” solutions are
\[ c_+^a = \delta/(D-2) , \quad q^a = 0 \Rightarrow b^a = 2\delta/(D-2) - 1 \]
\[ c_+^b = 0 , \quad q^b = \delta/D \Rightarrow b^b = 3\delta/D - 1 \]

Pure solutions are not realizable for value of $\delta$ that are not divisible by $D$ or $D-2$, so that for $D \geq 5$ mixed solutions should be considered also.

Let us describe the schemes in the pure cases:
- Case a, with $\delta = n \cdot (D-2)$: since $q^a = 0$, all the $2n - 1$ broken chains are separating: the scheme is a binary tree with $n + 1$ leaves, one carrying the root melon and the other carrying unbroken loops, with $n - 1$ internal nodes each carrying a $(D-2)$-dipole, and with the $2n - 1$ edges carrying the broken chains.
The contribution of such a scheme $S$ is (up to lower order terms)

$$u^{2n} A(u)^{2n-1} = \frac{D^{2n-1} u^{4n}}{(1 - Du)^{2n-1}}, \quad (\delta = n \cdot (D - 2))$$

so that the total contribution of these scheme, taking into account the choices of colors is

$$\frac{1}{n!} \binom{2n-2}{n-1} T(z) \cdot u \cdot D \cdot \left( \frac{Du}{1 - Du} \right)^{2n-1} u^{2n-1} \left( \frac{D}{2} \right)^{n-1} \left( \frac{1}{2} \right)^n \left( \frac{u}{1 - u^2} \right)^n.$$

The sum over the family of graphs corresponding to a scheme of degree $\delta$ and with $b$ broken chain-vertices will be

$$\frac{D^{3n-1} u^{5n-1}}{(1 - Du)^{3n-1}}, \quad (\delta = n \cdot D).$$

As the number of trivalent graphs grows super exponentially these series is not summable in $n$.

- For $D = 3$, we get $b^a = 2\delta$ and $b^b = \delta$ so $c^a = \delta / (D - 2)$, $q^a = 0$ gives the dominant contribution: for all $\delta > 0$, the dominant schemes are the binary trees of Case a above.
- For $D = 4$, we get $b^a = \delta$ and $b^b = 3\delta / 4$, so again $c^a = \delta / (D - 2)$ (hopefully here $\delta$ is always even) $q^a = 0$ gives the dominant contributions and Case a schemes dominate again, for all (even) $\delta > 0$.
- For $D = 5$, we get $b^a = 2\delta / 3$ and $b^b = 3\delta / 5$, and binary trees again, but only for values of $\delta$ that are multiples of 3. For the other values of $\delta$, the dominant graphs have a lower ratio $b/\delta$ (so that they don’t appear in the scaling limit of next section).
- For $D = 6$, we get $b^a = \delta / 2$ and $b^b = \delta / 2$ so its a draw: all combinations are possible. The dominant graphs are both binary trees with loop leaves and 3-regular graphs, as well as mixed graphs.
- For $D \geq 7$, we get $b^a = 2\delta / (D - 2) < b^b = 3\delta / D$ so $q^b = \delta / D$ wins, and the dominant graphs are the rooted trivalent graphs.

### 7.3 Double scaling

The Feynman amplitude of graphs (maps) in matrix models is $N^{2-2g}$, where $g$ is the genus of the map. In colored tensor models, the edge $(D + 1)$-colored graphs come equipped with a scaling in $N$,

$$N^{D-\delta}.$$

The sum over the family of graphs corresponding to a scheme of degree $\delta$ and with $b$ broken chain-vertices will be

$$N^D \frac{1}{N^{d(z_0 - z)^2}}. \quad (18)$$
which we have identified above.

In particular in the case $3 \leq D \leq 5$ we can actually form a convergent double scaling series:

$$T(z) + T(z) D \sum_{n \geq 1} \frac{1}{N^{D-2}} \frac{1}{n!} \left( \frac{2n-2}{n-1} \right) \left( \frac{D u}{1 - D u} \right)^{2n-1} u^{2n} \left( \frac{D}{2} \right)^{2n-1} \left( \frac{u}{1 - u^2} \right)^n$$

$$= T(z) + T(z) D \frac{1}{N^{D-2}} \frac{D u}{1 - D u} \frac{u^2}{2} \left( \frac{D}{2} \right) \frac{u}{1 - u^2} \sum_{n \geq 0} \frac{1}{n+1} \left( 2n \right) \left[ \frac{1}{N^{D-2}} \frac{D u}{1 - D u} u^2 \left( \frac{D}{2} \right)^2 \frac{u}{1 - u^2} \right]^n.$$  \hspace{1cm} (19)

Upon letting $N \to \infty$ and $z \to z_0$, keeping $N^{D-2} \left( 1 - z/z_0 \right) = x^{-1}$ fixed and large enough the above sum over $n$ converges. For each such choice of $x$ we can thus define a non trivial limit distribution on the set of all rooted colored graphs with dominant schemes such that large schemes are favored in this distribution.

The double scaling series computes further to

$$T(z) \left\{ 1 + \frac{1 + 1}{D (D-1) u} - \frac{1 - D u}{D (D-1) u} \sqrt{1 - 4 \frac{1}{N^{D-2}} \left( \frac{D u}{1 - D u} \right)^2 u^2 \left( \frac{D}{2} \right)^2 \frac{u}{1 - u^2}} \right\}$$

and taking into account that in the critical regime $z \to z_0$ we have

$$T(z) \sim \frac{1 + D}{D} - \sqrt{2 \frac{D + 1}{D^3}} \sqrt{1 - \frac{z}{z_0}}, \quad u \sim \frac{1}{D} - \sqrt{2 \frac{D + 1}{D^3}} \sqrt{1 - \frac{z}{z_0}}$$

$$1 - D u \sim \sqrt{2 D + 1} \sqrt{1 - \frac{z}{z_0}},$$

we obtain the leading critical behavior

$$\frac{D + 1}{D} + \frac{2}{D - 1} \sqrt{2 \frac{D + 1}{D^3}} \sqrt{1 - \frac{z}{z_0}} - \frac{D + 1}{D - 1} \sqrt{2 \frac{D + 1}{D^3}} \sqrt{1 - \frac{z}{z_0}} = \frac{D^2}{N^{D-2} (D+1)^3}.$$  \hspace{1cm} (21)

Note that, as expected, we recover the singular behavior of $T(z)$ in the $N \to \infty$ limit. At finite $N$ the singular behavior is distributed between two terms, one which diverges at $z_0$ and the second one which diverges for the shifted critical value $z_1 = z_0 \left( 1 - \frac{D^2}{N^{D-2} (D+1)^3} \right) < z_0$. The latter governs the critical behavior of the double scaling series.

The double scaling regime identified in this paper must be studied further. The next step is to study the geometry of the resulting continuum random space starting with its Hausdorff and spectral dimensions. In the case $3 \leq D \leq 5$, as the double scaling is summable, one can attempt to iterate this procedure and construct a multiple scaling limit in which a genuinely new random space is obtained.

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