DYNAMICS AT INFINITY AND JACOBI STABILITY OF TRAJECTORIES FOR THE YANG-CHEN SYSTEM

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(Communicated by Miguel Sanjuan)

ABSTRACT. The present work is devoted to giving new insights into a chaotic system with two stable node-foci, which is named Yang-Chen system. Firstly, based on the global view of the influence of equilibrium point on the complexity of the system, the dynamic behavior of the system at infinity is analyzed. Secondly, the Jacobi stability of the trajectories for the system is discussed from the viewpoint of Kosambi-Cartan-Chern theory (KCC-theory). The dynamical behavior of the deviation vector near the whole trajectories (including all equilibrium points) is analyzed in detail. The obtained results show that in the sense of Jacobi stability, all equilibrium points of the system, including those of the two linear stable node-foci, are Jacobi unstable. These studies show that one might witness chaotic behavior of the system trajectories before they enter in a neighborhood of equilibrium point or periodic orbit. There exists a sort of stability artifact that cannot be found without using the powerful method of Jacobi stability analysis.

2020 Mathematics Subject Classification. Primary: 34D20, 34D45; Secondary: 34K18.
Key words and phrases. Poincaré compactification, KCC-theory, chaotic system, dynamics at infinity, Jacobi analysis.

The first author is supported by National Natural Science Foundation of China (Grant No. 11961074), Natural Science Foundation of Guangxi Province (Grant Nos. 2018GXNSFDA281028, 2017GXNSFAA198234), the High Level Innovation Team Program from Guangxi Higher Education Institutions of China (Document No. [2018] 35), and the Science Technology Program of Yulin Normal University (Grant No. 2017YJKY28). The second author is supported by the Postgraduate Innovation Program of Guangxi University for Nationalities (Grant No. GXUNCHXZS2018042). The third author is supported by National Natural Science Foundation of China (Grant No. 11772306), Zhejiang Provincial Natural Science Foundation of China under Grant (No.LY20A020001), and the Fundamental Research Funds for the Central Universities, China University of Geosciences (CUGGC05).

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1. Introduction. Chaos refers to the dynamic state that can produce randomness (intrinsic randomness) in a certain nonlinear dynamic system without any random conditions. Chaotic motion is a reciprocating non-periodic motion, which is very sensitive to initial conditions and becomes more and more unpredictable with the increase of time. Therefore, in some cases, ones wish to weaken or eliminate chaos. In a word, it is very important to have an essential understanding of chaos. Since the Lorenz system [37] was proposed in 1963, many chaotic systems have been proposed. It should be noted that almost all such three-dimensional (3D) autonomous chaotic systems have one saddle and two unstable saddle-foci [37, 11, 38, 51]. The other 3D chaotic systems, such as the original Rössler system [43] and diffusionless Lorenz system [45], have two unstable saddle-foci. In 2008, Yang and Chen [52] modeled the following chaotic system, which is the earlier chaotic system with stable equilibrium point.

\[
\begin{align*}
\dot{x} &= a(y - x), \\
\dot{y} &= cx - xz, \\
\dot{z} &= xy - bz,
\end{align*}
\]

(1)

where \(a, b > 0\) and \(c\) is a real number. Later, this system was named the Yang-Chen system [39, 29]. On the one hand, when \((a, b, c) = (35, 3, 35)\), system has one positive Lyapunov Exponent \(\lambda_{L_1} = 1.0742\), which implies that system (1) yields a chaotic attractor, as shown in Fig. 1. And what’s interesting is that the system has two (Lyapounov) stable equilibrium points at the same time. Thus, the originator of the system points that there coexist a chaotic attractor and two stable node-foci in system (1). When parameters \((a, b, c) = (10, 8/3, 16)\), the same “unusual” thing will happen. On the other hand, the Yang-Chen system is closely related but not topologically equivalent to the original Lorenz and other Lorenz-like systems. In fact, Vanecek and Celikovsky [46] introduced the generalized Lorenz system of the following form: \(\dot{x} = Ax + Q(x)\), where \(A\) is a 3 × 3 real matrix \((a_{ij})_{3 \times 3}\), \(x = [x_1, x_2, x_3]\), \(Q\) is quadratic parts of the system. Lorenz system satisfies \(a_{12}a_{21} > 0\), Chen system satisfies \(a_{12}a_{21} < 0\) and Yang-Chen system (1) satisfies \(a_{12}a_{21} = 0\). In this sense, the Chen system is considered as the dual system to the Lorenz system, and Yang-Chen system (1) represents the transition between Lorenz system and Chen system. The Yang-Chen system can generate complex and abundant

![Figure 1](image-url)
dynamics within wide parameter ranges, including chaos, Hopf bifurcation, period-doubling bifurcation, periodic orbit, sink and source, and so on. There have been some detailed bifurcation analyses and global analyses [30, 39] concerning the Yang-Chen system. The literature [53] presents and investigates a 4D new hyperchaotic system that is constructed by a linear controller for the Yang-Chen system. Some other papers are devoted to the mathematical analysis of the Yang-Chen system [31, 27, 28, 7]. All of these show that the Yang-Chen system has abundant nonlinear dynamics and are of great significance in practical application.

On the one hand, it is very significant to study how much effect the infinite equilibrium point has on the chaos complexity for the unusual system with stable equilibrium point. This paper describes furtherly the global dynamics of unusual system (1). More precisely, by using the Poincaré compactification for the global analysis of unusual system, we give a complete description of its dynamical behavior on the sphere at infinity. On this type of global analysis of other chaotic systems, Llibre et al have done good work [35, 40, 36, 41, 32, 33, 34]. For the sake of completeness, the technique for making such an extension is precisely the Poincaré compactification and is described in detail in the Appendix A.

On the other hand, we also want to know how much the stability of the system trajectory (including equilibrium point) affects the chaotic complexity of systems. The Jacobi stability is a natural generalization of the stability of the geodesic flow on a differentiable manifold equipped with Riemannian or Finslerian metrics to the non-metric manifold. The Jacobi analysis, which was attributed to the contribution of the works of Kosambi [25], Cartan [10], and Chern [13] (KCC-theory), studies the robustness of a second-order differential equation. Robustness is a measure of insensitivity and adaptation to change of the system internal parameters and the environment. KCC theory has been applied to the dynamical system in cosmology [9, 22, 15, 8], gravitation [42, 1, 47], biology or ecology [4, 48, 49, 17, 44, 5, 6]. Nowadays, Jacobi stability analysis has become a useful tool in the study of the complexity of typical chaotic systems. This type of geometric analysis concerning many chaotic systems, such as Lorenz system, Chen system, Rössler system, Hamiltonian system, modified Chua circuit system, Rikitake system, Navier-Stokes system, Rabinovich-Fabrikant system, an unusual Lorenz-like chaotic system, can be found in the recent literature [18, 24, 3, 19, 47, 20, 50, 26, 21, 16]. Research has shown that Jacobi instability can be used to interpret the nature of chaos. The literature [23] has quantitatively described the behavior of the deviation curvature tensor by analyzing the signed geometric curvature of the curve, and revealed the underlying chaotic evolution of the Lorenz system. It should be notice that Lyapunov stability theory is a classical method to study the stability of solutions of differential equations. In the process of describing the equilibrium stability, the basic quantity is Lyapunov index, which quantitatively gives the exponential deviation of the trajectory. Boehmer et al. [8] have compared the relation between Lyapunov stability and Jacobi stability for the 2-dimensional dynamical systems case. And they found that Jacobi stability and Lyapunov stability are not always consistent. However, a complete agreement between two methods regarding the motion of a torque-free rigid body was shown in [2]. Chen and Yin [12] discussed a 4D Lorenz-type multistable hyperchaotic system by using KCC-theory. They pointed that under a certain parameter condition the curve of equilibria of hyperchaotic system is Jacobi unstable, and a Lyapunov unstable periodic orbit is always Jacobi
unstable. Therefore, we can found that Jacobi analysis method can not only analyze the stability of the equilibrium points of the system, but also the stability of any point on the trajectory of the system without the equilibrium point.

The research of the geometrization of the Yang-Chen system has rarely been studied in the literature. In this paper, we analyze the stability of the Yang-Chen system (1) from the viewpoint of KCC-theory. We find that all equilibrium points of the system, including those of the two stable node-foci, are Jacobi unstable. The obtained results show that Jacobi instability may be an important source of chaos. For reasons of simplicity, the definition of Jacobi stability is presented in Appendix B.

The rest of this paper is organized as follows. In Section 2, dynamical behavior of the system (1) at infinity is analyzed. In Section 3, the Jacobi stability at equilibrium points and a periodic orbit of the system (1) is studied. We find that all equilibrium points, including those of the two linear stable node-foci, are all Jacobi unstable. In Section 4, the behavior of the deviation vector near the equilibrium points is investigated. The last section presents conclusions. For the sake of completeness, in the Appendixes A and B, the Poincaré compactification for a polynomial vector field in \( \mathbb{R}^3 \), the basic concepts of the KCC-theory, including the definition of Jacobi stability are presented, respectively.

2. Dynamical behavior at infinity. In this section, we will analyze the dynamic behavior of the system at infinity. In order to do so in the following four subsection one will analyze the Poincaré compactification of the system in the locals \( U_1 \) and \( V_i \), \( i = 1, 2, 3 \) (see Appendix A for details).

2.1. In the local chart \( U_1 \) and \( V_1 \). From the process of Appendix A, the expression of the Poincaré compactification \( p(X) \) of the Yang-Chen system in the local chart \( U_1 \) is provided by

\[
\begin{align*}
\dot{z}_1 &= -z_2 + cz_3 + a z_1 z_3 - az_1^2 z_3 \\
\dot{z}_2 &= z_1 + (a - b) z_2 z_3 - az_1 z_2 z_3 \\
\dot{z}_3 &= az_2^3 - az_1 z_2^2 
\end{align*}
\]

(2)

If \( z_3 = 0 \), which corresponds to the points of the sphere \( S^2 \) at infinity, and the unique singular point of the system (2) is \((0, 0, 0)\) and the eigenvalues of the linear part of the system at this point are \( i \), \(-i\) and 0, and the zero eigenvalue exists eigenvector \((0, 0, 1)\).

Generally speaking, the dynamics near a nonhyperbolic singular point of this case can be fairly complex. Luckily for a property of the compactification procedure, the plane \( z_1 - z_2 \) is invariant under the flow of the system (2), which makes the analysis simpler. Taking \( z_3 = 0 \) the equations of the system (2) receives

\[
\begin{align*}
\dot{z}_1 &= -z_2, \\
\dot{z}_2 &= z_1, \\
\end{align*}
\]

(3)

which has \((0, 0)\) as a unique singular point. The system (3) has the first integral \( H = z_1^2 + z_2^2 \). So the origin of the chart \( U_1 \) confined to the infinite sphere is a center.

When reverse the time the flow in the local chart \( V_1 \) is the same as in the local chart \( U_1 \), because of the consistency between the compactified vector field \( p(X) \) in \( V_1 \) and the vector field \( p(X) \) in \( U_1 \) multiplied by \(-1\). Hence the Yang-Chen system also has a center on the infinite sphere at the negative endpoint of the \( x \)-axis.

In order to study the dynamics of the system (1) in a neighborhood of the infinite sphere on the local charts \( U_1 \) and \( V_1 \), aiming to understand how the solutions
come from and go to infinity, One will consider firstly $z_3 > 0$ small. The unique equilibrium point of the system (2) is $p_1 = (0, 0, 0)$ and the eigenvalues of the linearized system at this point are $i$, $-i$ and 0, the zero eigenvalue having eigenvector $(0, 0, 1)$. The pure imaginary eigenvalues correspond to the center at infinity. It remains to analyze the dynamics of the system for $z_3 > 0$ small in the 1-dimensional center manifold connected with the zero eigenvalue. This is done in the following proposition.

**Proposition 1.** When $a \neq 0$ the singular point $(0, 0, 0)$ of the system (2) has a center manifold, which is unstable (resp. stable) if $a > 0$ (resp. $a < 0$).

*Proof.* From the Center Manifold Theorem (see [54]) it follows that the system (2) has 1-dimensional center manifold at the singular point $(0, 0, 0)$, the graph of a function $h : \mathbb{R} \to \mathbb{R}$ given by $(z_1, z_2) = h(z_3) = (h_1(z_3), h_2(z_3))$ satisfying the conditions

$$h(0) = (0, 0), \quad Dh(0) = (0, 0), \quad (4)$$

and

$$\dot{z}_1 - Dh_1(z_3)\dot{z}_3 = 0, \quad \dot{z}_2 - Dh_2(z_3)\dot{z}_3 = 0. \quad (5)$$

Moreover, the flow on the center manifold is governed by the one dimensional equation

$$\dot{z}_3 = a z_3^2 - ah_1(z_3)z_3^2. \quad (6)$$

Assume that

$$z_1 = h_1(z_3) = a_1 z_3^2 + a_2 z_3^3 + O(z_3^4), \quad z_2 = h_2(z_3) = b_1 z_3^2 + b_2 z_3^3 + O(z_3^4). \quad (7)$$

Now from (4) and considering the expressions for $\dot{z}_1$ and $\dot{z}_2$ given by the system (2) we have

$$-h_2 + cz_3 + ah_1z_3^2 - ah_1^2 z_3 - Dh_1(a z_3^2 - ah_1 z_3^2) = 0, \quad h_1 + (a - b)h_2 z_3 - ah_1 z_3^2 - Dh_2(a z_3^2 - ah_1 z_3^2) = 0, \quad (8)$$

where $h_1 = h_1(z_3)$ and $h_2 = h_2(z_3)$ are received by (6). Equating the coefficients of the powers of $z_3$ in equations (7) shows that $a_i = b_i = 0$ for all $i \geq 1$. Hence $h_1 \equiv h_2 \equiv 0$ and the local center manifold at $(0, 0, 0)$ is the graph of the function $(z_1, z_2) = (h(z_3) = (h_1(z_3), h_2(z_3))) = (0, 0)$, which the coordinates $(z_1, z_2, z_3)$ is given by

$$W_{loc}(0, 0, 0) = \{(z_1, z_2, z_3) = (0, 0, z_3) | z_3 \in (-\varepsilon, \varepsilon), \varepsilon > 0\}.$$

Hence from (5) it follows that the equation $\dot{z}_3 = a z_3^2$ commands the flow on this manifold, which implies that for $z_3 > 0$ the singular point $(0, 0, 0)$ is locally asymptotically stable if $a < 0$ (resp. unstable if $a > 0$) along its center manifold.

Because the flow in the local chart $U_1$ reversing the time, under the same arguments stated above, one can infer that the flow on the center manifold of the origin in the local chart $V_1$ is controlled by the equation

$$\dot{z}_3 = -a z_3^2 - ah_1(z_3)z_3^2.$$
sphere in the local charts $U_1$ and $V_1$, it’s not completely described. In fact, it could occur that a solution enters the infinity and tends to one of the periodic orbits of the centers at infinity. Then, for the sake of completing the analysis, one has used numerical simulations for several values of the parameters and different initial conditions, which indicate that the periodic orbits at infinity are normally unstable, that is any solution starting in the interior of the Poincaré ball and near any periodic orbit of the centers at infinity go far from them (see Figs. 2 and 3), following the behavior on the center manifolds described above.

2.2. In the local chart $U_2$ and $V_2$. Again by means of the results stated in Appendix A, it shows that the expression of the Poincaré compactification $p(X)$ of the Yang-Chen system in the local chart $U_2$ is obtained by

\begin{align*}
\dot{z}_1 &= az_3 - az_1z_3 + z_1^2z_2 - cz_2^2z_3 \\
\dot{z}_2 &= z_1 - bz_2z_3 - cz_1z_2z_3 + z_1z_2^2 \\
\dot{z}_3 &= -cz_3^2 + z_1z_2z_3
\end{align*}

(9)

If $z_3 = 0$, the system (9) has a line of equilibria given by the $z_2$-axis and the linear part of the system at these equilibria has three null eigenvalues. Let us try to study the flow near these equilibria. From the compactification procedure described in Appendix A one knows that the $z_1$-$z_2$ plane is invariant under the flow of the system (9) restricted to the $z_1$-$z_2$ plane is given by

\begin{align*}
\dot{z}_1 &= z_2^2z_2, \quad \dot{z}_2 = z_1 + z_1z_2^2,
\end{align*}

(10)

which has the first integral $H = \frac{1}{2}z_1 - t$.

It is an integrable system, since if $z_1 \neq 0$ which has the first integral $H(z_1, z_2) = \ln |z_1| - \ln(1 + z_2^2)/2$. Using this first integral and observing that the system (10) has the $z_2$-axis as a line of equilibria, we have that the global phase portrait in the local chart $U_2$ on the infinite sphere is shown in Fig. 4. If reverse the time the
Figure 3. Dynamics of the Yang-Chen system near the sphere at infinity in the local charts $V_1$ (blue) $(a, b, c) = (0.5, 1.01, 1)$ with initial conditions $(z_1(0), z_2(0), z_3(0)) = (0.03, 0.03, 0.03)$, (red) $(a, b, c) = (1, 1.01, 1)$ with initial conditions $(z_1(0), z_2(0), z_3(0)) = (0.03, 0.03, 0.03)$, (black) $(a, b, c) = (0.1, 1.01, 1)$ with initial conditions $(z_1(0), z_2(0), z_3(0)) = (0.03, 0.03, 0.01)$, respectively.

Figure 4. Phase portrait of the system (10), which corresponds to the phase portrait of the Yang-Chen system at infinity in the local charts $U_2$. 

Flow in the local chart $V_2$ is the same as the flow in the local chart $U_2$, because the vector field $p(X)$ in $U_2$ multiplied by $-1$ is the same as the compactified vector field $p(X)$ in $V_2$. Therefore the phase portrait on the chart $V_2$ is the same as Fig. 4 shown in next, reversing approximatively the direction of time.
2.3. In the local chart $U_3$ and $V_3$. The expression of the Poincaré compactification $p(X)$ of the system (1) in the local chart $U_3$ is given by

\[
\begin{align*}
\dot{z}_1 &= az_2z_3 + (b - a)z_1z_3 - z_1^2z_2 \\
\dot{z}_2 &= -z_1z_3^2 + cz_1z_3 + bz_2z_3 - z_1 \\
\dot{z}_3 &= z_3(bz_3 - z_1z_2).
\end{align*}
\]

(11)

For $z_3 = 0$, the above system is the same as the equations (10) multiplied by $-1$ and then the analysis on the sphere at infinity is alike the one analyzed in the previous subsection. Now we will study dynamic behavior of the system (11) in a neighborhood of the infinite sphere on the chart $U_3$, by considering small $z_3 > 0$, since we are interested in the behavior of the solutions which tend to infinity on the $z$-axis. The $z_3$-axis is invariant by the flow of the system (11), since for $z_1 = z_2 = 0$ the system gained $z_1 = 0$, $z_2 = 0$, $z_3 = bz_3^2$, hence the origin is asymptotically stable (resp. unstable) if $b < 0$ (resp. $b > 0$). Furthermore, if $b = 0$, the system has a line of equilibria in the $z_3$-axis and the following proposition holds.

**Proposition 2.** For $b = 0$, the equilibrium points $(0, 0, z_3)$ ($z_3 > 0$) of the system (11) are stable focus which is normally hyperbolic to the $z_3$-axis, that is the linear part of the system at each equilibria $(0, 0, z_3)$ have a pair complex conjugate eigenvalues with negative real part and the corresponding two-dimensional stable manifolds normally to the $z_3$-axis under the following condition

\[
\frac{1}{z_3} > \frac{(a + 4ac)}{4a}.
\]

**Proof.** For $b = 0$ the Jacobian matrix of the system (11) at the unique equilibrium point $(0, 0, z_3)$ is given by

\[
\begin{pmatrix}
-az_3 & az_3 & 0 \\
0 & c_3 - 1 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

which has the eigenvalues

\[
\lambda_{1,2} = \frac{-az_3 \pm \sqrt{z_3[(a + 4ac)z_3 - 4a]}}{2}, \lambda_3 = 0,
\]

with corresponding eigenvectors

\[
v_{1,2} = \left(\frac{-az_3 \pm \sqrt{z_3[(a + 4ac)z_3 - 4a]}}{2(cz_3 - 1)}, 1, 0\right), v_3 = (0, 0, 1),
\]

from which the proposition follows, since we consider $a > 0$ and $z_3 > 0$. \hfill \square

When reverse the time the flow in the local chart $V_3$ is the same as the flow in the local chart $U_3$. Hence the same type of analysis as made above, thinking about near the infinity in the local chart $V_3$ we have $z_3 < 0$, allows us to prove the following preposition. We can prove the following proposition by doing the same thing above.

**Proposition 3.** For $b = 0$, the equilibrium points $(0, 0, z_3)$ ($z_3 < 0$) of the system (11) with reversed time (that is in the local chart $V_3$) are saddles normally hyperbolic to $z_3$-axis, that is the linear part of the system at each equilibrium $(0, 0, z_3)$ has two real eigenvalues with contrary sign signs and the corresponding one-dimensional stable and one-dimensional unstable manifolds normal to the $z_3$-axis.

For $z_3 = 0$ the system (11) has a line of equilibria given by the $z_2$-axis and the linear part of the system at these equilibria has three null eigenvalues. Let us try
to understand the flow near these equilibria. From the compactification procedure described in Appendix A we know that the $z_1$-$z_2$ plane is invariant under the flow of system (11) restricted to the $z_1$-$z_2$ plane is given by

\[
\dot{z}_1 = -z_1^2 z_2, \quad \dot{z}_2 = -z_1 z_2^2 - z_1,
\]

which has the first integral $H = \frac{z_1^2}{z_1^2 + 1}$. We have the global phase portrait in the local chart $U_3$ on the infinite sphere as shown in Fig. 5.

![Figure 5. Phase portrait of the system (12), which corresponds to the phase portrait of the Yang-Chen system at infinity in the local charts $U_3$.](image)

2.4. Dynamics on the Poincaré sphere at infinity. Considering the analysis made in the previous subsections we have a global picture of the dynamical behavior of system (1) on the sphere at infinity: there are two centers at the endpoints of the $x$-axis and saddle points at the endpoints of the $y$- and $z$-axes. (see Fig. 6). This proves the following Theorem 2.1. It is important to note that the dynamics at infinity does not depend on the values of the parameters $a$, $b$ and $c$. We observe that the complete description of the phase portrait of system (1) on the Poincaré sphere at infinity was possible because of the invariance of this set under the flow of the compactified system in each local chart.

**Theorem 2.1.** For all $a$, $b$ and $c \in \mathbb{R}$ the phase portrait of unusual system (1) on the Poincaré sphere at infinity is as shown in Fig. 6: there are two centers at the endpoints of the $x$-axis and saddle points at the endpoints of the $y$- and $z$-axes.

3. Jacobi analysis. In this section, the Jacobi stability of the trajectories of the Yang-Chen system (1) at any point is studied from KCC-theory. We consider the special trajectory (i.e. the equilibrium point) of the system firstly, and then analyze Jacobi stability of trajectory at any point. In order for this, in the next three subsections and Sec.4, we shall apply the basic concepts of the KCC-theory, including five KCC-invariants, and the definition of Jacobi stability (see Appendix B for details).
According to the literature [52], when \( a > 0, \ b > 0 \) and \( c \leq 0 \), the system (1) is not chaotic. Therefore, in the following study, the parameters \( a, b, \) and \( c \) are supposed to be positive. Now transform the system (1) to a second order differential equation (SODE). According to the first equation of system (1), variable \( y \) can be expressed as

\[
y = x + \frac{\dot{x}}{a}.
\]

By substituting \( y \) into the second equation of system (1), we obtain

\[
\ddot{x} + a\dot{x} - acx + axz = 0.
\]

Taking the derivative of the third equation of system (1) with respect to \( t \), one finds

\[
\ddot{z} = x(cx - xz) + (\dot{x} - bx)\left(x + \frac{1}{a}\dot{x}\right) + b^2z,
\]

i.e.,

\[
\ddot{z} - b^2z + \left(\frac{b}{a} - 1\right)x\dot{x} - \frac{1}{a}\dot{x}^2 + (b - c)x^2 + x^2z = 0.
\]

Changing the notation as

\[
x = x_1, \dot{x} = y_1, z = x_2, \dot{z} = y_2,
\]

system (1) translates to the following form

\[
\begin{aligned}
\dot{x}_1 + ay_1 - acx_1 + ax_1x_2 &= 0, \\
\dot{x}_2 + (b - c)x_2^2 + x_1^2x_2 - b^2x_2 + \left(\frac{b}{a} - 1\right)x_1y_1 - \frac{1}{a}y_1^2 &= 0.
\end{aligned}
\]

System (1) and the above second-order equations (13) are equivalence.
3.1. **Five KCC invariants.** According to the system (13), we obtain the form of two-order differential equations
\[ \ddot{x} + 2G_i(x, y, t) = 0, \ i \in \{1, 2\}, \]
where
\[ G_1(x, y, t) = \frac{1}{2} (ay_1 - acx_1 + ax_2) \]
and
\[ G_2(x, y, t) = \frac{1}{2} \left[ (b - c)x_1^2 + x_1^2x_2 - b^2x_2 + \left( \frac{b}{a} - 1 \right)x_1x_2 - \frac{1}{a}y_1^2 \right]. \]
According to Appendix B, the components of the nonlinear connection and Berwald connection are obtained as follows,
\[ N_1 = \frac{a}{2}, \ N_2 = 0, \ N_2' = \frac{1}{2} \left( \frac{b}{a} - 1 \right)x_1 - \frac{1}{a}y_1, \ N_2'' = 0, \]
\[ G_{11} = G_{12} = G_{21} = G_{22} = 0, \ G_{11}' = -\frac{1}{a}. \]
The components of the first KCC-invariant and the deviation curvature tensor are given by
\[ \epsilon_1 = \frac{1}{2} ay_1 - acx_1 + ax_2, \epsilon_2 = x_1^2x_2 - b^2x_2 + \frac{1}{2} \left( \frac{b}{a} - 1 \right)x_1y_1 + (b - c)x_1^2, \]
and
\[ P_1^1 = ac - ax_2 + \frac{1}{4} a^2, \ P_2^1 = -ax_1, \]
\[ P_1^2 = \left( 1 - \frac{b}{2a} \right)y_1 - x_1x_2 + \frac{4c - 7b - a}{4}x_1, \ P_2^2 = b^2 - x_1^2, \]
respectively.

The third, fourth, and fifth KCC-invariants (be defined by Eq. (28) of Appendix B) are identically equal to zero for the system (13).

3.2. **Jacobi stability of the equilibrium points.** When \( a > 0, b > 0 \) and \( c > 0 \), system (1) has three real equilibrium points. They are \( E_1(0, 0, 0), \ E_2(\sqrt{bc}, \sqrt{bc}, c) \), and \( E_3(-\sqrt{bc}, -\sqrt{bc}, c) \). Hence, the system (13) also has three the equilibrium points \( E_1(0, 0, 0), \ E_2(\sqrt{bc}, c), \ E_3(-\sqrt{bc}, c) \) under the same parameter conditions.

Owing to Routh-Hurwitz criteria, a result about the equilibrium points can be obtained as follows:

**Theorem 3.1.** The equilibrium points \( E_1(0, 0, 0), \ E_2(\sqrt{bc}, c), \) and \( E_3(-\sqrt{bc}, c) \) of Yang-Chen system (13) are Jacobi unstable.

**Proof.** At the equilibrium point \( E_1 \), calculating the components of the deviation curvature tensor
\[ P_1^1(E_1) = ac + \frac{1}{4} a^2, \ P_2^1(E_1) = 0, \ P_2^2(E_1) = 0, \ P_2^2(S_0) = b^2. \]
For the equilibrium point \( E_2 \) and \( E_3 \),
\[ P_1^1(E_2) = \frac{1}{4} a^2, \ P_2^1(E_2) = -a\sqrt{bc}, \]
\[ P_1^2(E_2) = -\frac{\sqrt{bc}}{4}(7b + a), \ P_2^2(E_2) = b(b - c), \]
and

\[ P_1^1(E_3) = \frac{1}{4} a^2, \quad P_1^2(E_3) = a\sqrt{bc}, \]
\[ P_2^2(E_3) = \frac{\sqrt{bc}}{4}(7b + a), \quad P_2^2(E_3) = b(b - c), \]
respectively.

Therefore, the characteristic polynomial of the deviation curvature tensor could be written as following:

\[ \lambda^2 - \left(\frac{1}{4} a^2 + b^2 - bc\right) \lambda + \frac{1}{4} b \left( a^2b - 2a^2c - 7abc \right) = 0. \] (14)

It is easy to obtain the eigenvalues of the deviation curvature tensor at the equilibrium points by using Eq. (30) of Appendix B,

\[ \lambda_+(E_1) = ac + \frac{1}{4} a^2, \quad \lambda_-(E_1) = b^2, \]

\[ \lambda_+(E_2) = \lambda_+(E_3) = \frac{1}{2} \left\{ \frac{1}{4} a^2 + b^2 - bc + \sqrt{\frac{1}{4} a^2 + bc - b^2} \right\}, \]

\[ \lambda_-(E_2) = \lambda_-(E_3) = \frac{1}{2} \left\{ \frac{1}{4} a^2 + b^2 - bc - \sqrt{\frac{1}{4} a^2 + bc - b^2} \right\}. \]

As the eigenvalues of the deviation curvature tensor at \( E_1 \) are given by (15), \( \lambda_-(E_1) = b^2 \), which means that this eigenvalue is always positive. According to the definition in Appendix B, the point \( E_1(0, 0) \) of the Yang-Chen system is Jacobi unstable.

Owing to Routh-Hurwitz criteria, the eigenvalues of characteristic polynomial equation (14) are negative or have a negative real part if

\[ \frac{1}{4} a^2 + b(b - c) < 0 \]

and

\[ b \left( a^2b - 2a^2c - 7abc \right) > 0. \] (17)

However, it can be found that there are no positive real numbers \( a, b, \) and \( c \) that make both of these constraints hold.

Let \( F(b) = \frac{1}{4} a^2 + b(b - c), \) \( G(b) = b \left( a^2b - 2a^2c - 7abc \right) \). In fact, if \( a^2 \geq c^2 \), \( F(b) < 0 \) is false. Therefore, the points \( E_2 \) and \( E_3 \) are Jacobi unstable.

Now we discuss what happens when \( a^2 < c^2 \).

Let \( F(b) = 0 \). One can obtain \( b_1 = \frac{-c - \sqrt{c^2 - a^2}}{2}, \) \( b_2 = \frac{c + \sqrt{c^2 - a^2}}{2} \). In this time, if \( b_1 < b < b_2 \), then \( F(b) < 0 \) is true. Under this conditions, (i) when \( 7c - a > 0, \) \( G(b) < 0 \) is false; (ii) when \( 7c - a = 0, \) \( G(b) < 0 \) is false; (iii) for \( 7c - a < 0 \), consider following two cases:

**Case 1)** \( 2ac \geq \frac{1}{4} a^2 \), i.e. \( 8c \geq a \). However, there are no positive real numbers \( a, b \) and \( c \) that satisfy \( a^2 < c^2, 7c - a < 0 \) and \( 8c \geq a \) simultaneously.

**Case 2)** \( 8c < a \). there are also no positive real numbers \( a, b \) and \( c \) that satisfy \( a^2 < c^2, 7c - a < 0 \) and \( 8c < a \) simultaneously.

From what has been discussed above, the equilibrium points \( E_2 \) and \( E_3 \) are Jacobi unstable. \( \square \)
To summarize, all of the three equilibrium points of the Yang-Chen system (13) are Jacobi unstable when \( a > 0, b > 0, c > 0 \). Although the equilibrium points \( E_2 \) and \( E_3 \) are linear stable node-foci, they are unstable in the sense of Jacobi stability, which is different from Lorenz system [23] and Chen system [24]. According to the existing literature, we know that the non-trivial equilibrium points of Lorenz and Chen systems are unstable, but under some parameter conditions these equilibrium points can be Jacobi stable. However, in the Yang-Chen system (13) the linear stable non-trivial equilibrium points of are always Jacobi unstable. There exists a sort of stability artifact that cannot be found without using the powerful method of Jacobi stability analysis.

4. Dynamics of deviation vector \( \xi(t) \). In this section, in order to investigate the occurrence of chaos, we begin to observe the behavior of the deviation vector \( \xi(t) \) and its components \( \xi_i(t), i = 1, 2 \). For the Yang-Chen system (13), according to Eqs. (26) in Appendix B, one can obtain

\[
\frac{d^2 \xi_1}{dt^2} + a \frac{d\xi_1}{dt} + (ax_2 - ac)\xi_1 + ax_1\xi_2 = 0
\]  

and

\[
\frac{d^2 \xi_2}{dt^2} + \left((\frac{b}{a} - 1)x_1 - \frac{2}{a} y_1\right) \frac{d\xi_1}{dt} + \left[2x_1x_2 + (\frac{b}{a} - 1)y_1 + 2(b - c)x_1\right] \xi_1 + (x_1^2 - b^2) \xi_2 = 0.
\]  

The deviation vector is expressed as

\[
\xi(t) = \sqrt{[\xi_1(t)]^2 + [\xi_2(t)]^2}.
\]

Similar to [23], one defines the instability exponents of systems (18) and (19), which are analogues of the Lyapounov exponent:

\[
\delta_i(E) = \lim_{t \to \infty} \frac{1}{t} \ln \left[ \frac{\xi_i(t)}{\xi_i(0)} \right], i = 1, 2,
\]  

and

\[
\delta(E) = \lim_{t \to \infty} \frac{1}{t} \ln \left[ \frac{\xi(t)}{\xi(0)} \right].
\]  

Next, the dynamics of deviation vector and its components are analyzed in detail near the equilibrium points.

4.1. Dynamics of deviation vector near \( E_1(0,0) \). Near the equilibrium point \( E_1(0,0) \), the deviation equations (18) and (19) can be written in the forms

\[
\frac{d^2 \xi_1}{dt^2} + a \frac{d\xi_1}{dt} - ac\xi_1 = 0,
\]  

and

\[
\frac{d^2 \xi_2}{dt^2} - b^2 \xi_2 = 0,
\]

respectively. Under the initial conditions \( \xi_1(0) = \xi_2(0) = 0, \dot{\xi}_1(0) = \xi_{10}, \) and \( \dot{\xi}_2(0) = \xi_{20} \), the general solutions of the above equations are given by

\[
\frac{\xi_1(t)}{\xi_{10}} = \begin{cases} 
2T \sinh \left( \frac{4M_1 t}{M_1} \right), & a + 4c > 0, \\
T, & a + 4c = 0, \\
2T \sin \left( \frac{4M_2 t}{M_2} \right), & a + 4c < 0,
\end{cases}
\]
and
\[ \frac{\xi_2(t)}{\xi_20} = \frac{\sinh(bt)}{b}, \]

where
\[ M_1 = \sqrt{a^2 + 4ac}, \quad \Upsilon = \exp\left(-\frac{at}{2}\right), \quad M_2 = \sqrt{-(a^2 + 4ac)}. \]

Hence, the deviation vector is obtained as following.
\[
\xi(t) = \begin{cases} 
\sqrt{4\Upsilon^2 \frac{\sinh^2(\frac{1}{2}M_1 t)}{M_1^2}} + \frac{\xi_{20} \sinh^2(bt)}{\xi_{20}^2} t, & a + 4c > 0, \\
\sqrt{\Upsilon^2 \frac{\sinh^2(bt)}{\xi_{20}^2}}, & a + 4c = 0, \\
\sqrt{4\Upsilon^2 \frac{\sinh^2(\frac{1}{2}M_2 t)}{M_2^2}} + \frac{\xi_{20} \sinh^2(bt)}{\xi_{20}^2}, & a + 4c < 0.
\end{cases}
\]

Fig. 7 shows the time variation of the deviation vectors and its component near the point \( E_1(0,0) \) for different parameter values, respectively.

For \( a = 35 \) and \( b = 3 \), with the increase of the parameter \( c \), one can see that in Fig. 7, the component \( \xi_1 \) increases rapidly after \( t = 0.6 \). The component \( \xi_2 \) whose behavior is only dependent on the values of \( b \), also increases quickly in time, and the vector \( \xi \) exponentially increases in time. The time evolution of the deviation vector and its components near equilibrium point \( E_1 \) of the Yang-Chen system is similar to that in literature [23].

From (20) and (21), one can obtain instability exponent at point \( E_1 \) as follows,
\[
\delta_1(E_1) = \begin{cases} 
\frac{1}{2}(M_1 - a), & a + 4c > 0, \\
-\frac{a}{2}, & a + 4c \leq 0,
\end{cases}
\]
and \( \delta \) can be estimated as
\[
\delta(E_1) = \begin{cases} 
\frac{1}{2} \ln \left[ \sqrt{2 \cosh(2bt)} - 2 + \exp(-(M_1 + a)t)(\exp(M_1 t) - 1)^2 \right], & a + 4c > 0, \\
b, & a + 4c \leq 0.
\end{cases}
\]

Fig. 8 shows the time variation of the instability exponent \( \delta(E_1) \) for different parameter values.

4.2. Dynamics of deviation vector near \( E_2(\sqrt{bc}, c) \) and \( E_3(-\sqrt{bc}, c) \). For both critical points \( E_2(\sqrt{bc}, c) \) and \( E_3(-\sqrt{bc}, c) \), the deviation equations take the forms
\[
\frac{d^2\xi_1}{dt^2} + \frac{d\xi_1}{dt} \pm a\sqrt{bc} \xi_2 = 0
\]
and
\[
\frac{d^2\xi_2}{dt^2} \pm \sqrt{bc} \left( \frac{b}{a} - 1 \right) \frac{d\xi_1}{dt} \pm 2b\sqrt{bc} \xi_1 + (bc - b^2) \xi_2 = 0,
\]
respectively.

The time evolution of deviation vector and its components near equilibrium point \( E_{2,3}(\sqrt{bc}, c) \) are showed in Fig. 9.
4.3. Curvature of deviation vector. In literature [23], authors found that there is some relation between the curvature of the deviation vector and the chaotic behavior of the system, and numerical result display a correspondence between the two. In this section, we also consider the behavior of curvature $\kappa_0$ of deviation vector in the Yang-Chen system. As in the literature [23], one also defines $\kappa_0$ as

$$\kappa_0(E) = \frac{\dot{\xi}_1(t)\ddot{\xi}_2(t) - \ddot{\xi}_1(t)\dot{\xi}_2(t)}{{\left[\dot{\xi}_1(t)\right]}^2 + {\left[\dot{\xi}_2(t)\right]}^2}^{3/2}.$$
where
\[
\Gamma(t)=\frac{1}{2}Y(M_1-a)\sin\left(\frac{1}{2}M_2t\right),
\]
\[
\Gamma_2=\frac{1}{2}Y(a^2-M_2^2)\sin\left(\frac{1}{2}M_2t\right)\cos\left(\frac{1}{2}M_2t\right).
\]

As shown in Fig. 10, for the chosen range of parameters, the deviation vector is positive for small values of time. Then it reaches the value zero at a certain moment \(t_0\), and enters the region of the negative values. Lastly, in the limit of large times, it tends to zero. The time variation of curvature \(\kappa_0\) of deviation vector near equilibrium points \(E_1\) of the Yang-Chen system is similar to that in literature [23]. Therefore, it can see that the correspondence between the curvature of the deviation vector and the chaotic behavior of the system is still valid in the Yang-Chen system through the test in Fig. 10.

Let \(a = 10\) and \(b = 8/3\). According to Taylor expansion, near \(t = 0\), the curvature of the deviation vector is
\[
\kappa_0(t, c) \approx \frac{53.3333\sqrt{100 + 40c}}{a^2} + \frac{80q_1(2.98402 + 1.26315c - 0.07454c^2)}{(3.7642 + c)^2q_2}t
\]
\[
- \frac{40q_1}{(3.7642 + c)^5q_2} \times (211.136 - 295.455c - 225.284c^2
\]
\[
+ 32.6195c^3 + 37.6138c^4 + 5.21749c^5)t^2 + \cdots
\]

where \(q_1 = \sqrt{5 + 2c}\), \(q_2 = \sqrt{150.568 + 40c}\).

The time interval \(t_0\) for which \(\kappa_0(t, c) = 0\) in the first approximation is given by
\[
t_0 \approx \frac{0.6667(3.7642 + c)^2\sqrt{100 + 40c}}{q_1q_2(0.07454c^2 - 1.26315c - 2.98402)}
\]
5. Conclusions. In this paper, we present new insights into the unusual system with two stable node-foci. Firstly, we give a complete description of the phase portrait of the system at infinity. This is identified with the sphere $S^2$ in $\mathbb{R}^3$ after compactification. The purpose of this is to analyze the influence of equilibrium point at infinity on the chaotic complexity of the system. We find that although the Yang-Chen system has two stable equilibrium points $E_2$ and $E_3$, all of its infinite equilibrium points are unstable: there are two centers at the endpoints of the $x$-axis and saddle points at the endpoints of the $y$- and $z$-axes. Secondly, we discuss that

The time variation of curvature $\kappa_0$ of deviation vector near equilibrium points $E_{2,3}$ as showed in Fig. 11.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure9.png}
\caption{Time variation of the deviation vector and its curvature near $E_{2,3}$ with $a = 35, b = 3$. Initial conditions used to integrate deviation equations are $\xi_1(0) = \xi_2(0) = 0, \dot{\xi}_1(0) = \dot{\xi}_2(0) = 10^{-6}$.}
\end{figure}
the Jacobi stability of the trajectories for the Yang-Chen system from the viewpoint of KCC-theory. The dynamical behavior of the deviation vector near the whole trajectories is analyzed in detail. The obtaining results show that, although the
equilibrium points $E_2$ and $E_3$ of the Yang-Chen system are linear stable node-foci, they are Jacobi unstable for $a > 0$, $b > 0$, $c > 0$. Thirdly, the dynamical behavior of the trajectories near the equilibrium points is analyzed in detail by introducing the instability exponent $\delta$ and the curvature $\kappa_0$ of the deviation vector.

Our study show that one might witness chaotic behavior of the system trajectories before they enter in a neighborhood of equilibrium point. We have here a sort of stability artifact that cannot be found without using the powerful method of Jacobi stability analysis. It is hoped that the investigation presented in this paper can contribute to revealing the stability of dynamical systems, be quite beneficial for further studies of the dynamically rich Yang-Chen system (1), and will shed some light on the true geometrical structure of the interesting chaotic attractor of the chaotic system with stable equilibrium points.

Acknowledgments. We are grateful to the editor and the referees for their valuable comments and suggestions.

Appendix A.

Poincaré compactification in $\mathbb{R}^3$. A polynomial vector field $X$ in $\mathbb{R}^n$ can be extended to a unique analytic vector field on the sphere $\mathbb{S}^n$. The technique for making such an extension is called the Poincaré compactification and allows us to study a polynomial vector field in a neighborhood of infinity, which corresponds to the Poincaré compactification and allows us to extended to a unique analytic vector field on the sphere $\mathbb{S}^n$. Poincaré introduced this compactification for polynomial vector fields in $\mathbb{R}^2$. Its extension to $\mathbb{R}^n$ for $n > 2$ can be found in [14]. In this section, one describes the Poincaré compactification for polynomial vector fields in $\mathbb{R}^3$ following closely what is made in [14].

In $\mathbb{R}^3$ one considers the polynomial differential system

$$
\dot{x} = P^1(x, y, z) \quad \dot{y} = P^2(x, y, z) \quad \dot{z} = P^3(x, y, z),
$$

or equivalently its associated polynomial vector field $X = (P^1, P^2, P^3)$. The degree $n$ of $X$ is defined as $n = \max\{ \deg(P^i) : i = 1, 2, 3 \}$. Let $S^3 = \{ y = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4 : ||y|| = 1 \}$ be the unit sphere in $\mathbb{R}^4$, and $S_+ = \{ y \in S^3 : y_4 > 0 \}$ and $S_- = \{ y \in S^3 : y_4 < 0 \}$ be the northern and southern hemispheres of $S^3$, respectively. The tangent space to $S^3$ at the point $y$ is denoted by $T_y S^3$. Then the tangent plane $T_{(0,0,0,1)} S^3 = \{(x_1, x_2, x_3, 1) \in \mathbb{R}^4 : (x_1, x_2, x_3) \in \mathbb{R}^3 \}$ is identified with $\mathbb{R}^3$.

One considers the central projections $f_+: \mathbb{R}^3 = T_{(0,0,0,1)} S^3 \rightarrow S_+$ and $f_- : \mathbb{R}^3 = T_{(0,0,0,1)} S^3 \rightarrow S_-$ defined by $f_\pm (x) = \pm (x_1, x_2, x_3, 1)/\Delta x$, where $\Delta x = (1 + \sum_{i=1}^3 x_i^2)^{1/2}$. Through these central projections $\mathbb{R}^3$ is identified with the northern and southern hemispheres of $S^3$. The equator of $S^3$ is $S^2 = \{ y \in S^3 : y_4 = 0 \}$. Clearly $S^2$ can be identified with the infinity of $\mathbb{R}^3$.

The maps $f_+$ and $f_-$ define two copies of $X$ on $S^3$, one $DF_+ \circ X$ in the northern hemisphere and the other $DF_- \circ X$ in the southern one. Denoted by $X$ the vector field on $S^3 \setminus S^2 = S_+ \cup S_-$ which, restricted to $S_+$ coincides with $DF_+ \circ X$ and restricted to $S_-$ coincides with $DF_- \circ X$.

The expression for $X(y)$ on $S_+ \cup S_-$ is

$$
X(y) = y_4 \begin{pmatrix}
1 - y_1^2 & -y_2 y_1 & -y_3 y_1 \\
-y_1 y_2 & 1 - y_2^2 & -y_3 y_2 \\
-y_1 y_3 & -y_2 y_3 & 1 - y_3^2 \\
-y_1 y_4 & -y_2 y_4 & -y_3 y_4
\end{pmatrix}
\begin{pmatrix}
p^1 \\
p^2 \\
p^3
\end{pmatrix}
$$
where \( P_i = P_i(y_1/|y_1|, y_2/|y_2|, y_3/|y_3|) \). Written in this way \( \overline{X}(y) \) is a vector field in \( \mathbb{R}^4 \) tangent to the sphere \( S^3 \).

Now one can extend analytically the vector field \( \overline{X}(y) \) to the whole sphere \( S^3 \) by \( p(X)(y) = y_4^{-1} \overline{X}(y) \). This extended vector field \( p(X) \) is called the Poincaré compactification of \( X \) on \( S^3 \).

As \( S^3 \) is a differentiable manifold in order to compute the expression for \( p(X) \) one can consider the eight local charts \( (U_i, F_i) \), \( (V_i, G_i) \), where \( U_i = \{ y \in S^3 : y_i > 0 \} \) and \( V_i = \{ y \in S^3 : y_i < 0 \} \) for \( i = 1, 2, 3, 4 \); the diffeomorphisms \( F_i : U_i \to \mathbb{R}^3 \) and \( G_i : V_i \to \mathbb{R}^3 \) for \( i = 1, 2, 3, 4 \) are the inverses of the central projections from the origin to the tangent planes at the points \( (\pm 1, 0, 0, 0), (0, \pm 1, 0, 0), (0, 0, \pm 1, 0) \) and \( (0, 0, 0, \pm 1) \), respectively. Now one does the computations on \( U_1 \). Suppose that the origin \( (0, 0, 0, 0) \), the point \( (y_1, y_2, y_3, y_4) \) \( S^3 \) and the point \( (1, z_1, z_2, z_3) \) in the tangent plane to \( S^3 \) at \( (1, 0, 0, 0) \) are collinear. Then one has \( 1/y_1 = z_1/y_2 = z_2/y_3 = z_3/y_4 \), and consequently \( F_1(y) = (y_2/y_1, y_3/y_1, y_4/y_1) = (z_1, z_2, z_3) \) defines the coordinates on \( U_1 \). As

\[
DF_1(y) = DF_1(y) = \begin{pmatrix}
-y_2/y_1^2 & 1/y_1 & 0 & 0 \\
-y_3/y_1^2 & 0 & i/y_1 & 0 \\
-y_4/y_1^2 & 0 & 0 & 1/y_1
\end{pmatrix}
\]

and \( y_4^{n-1} = (z_3/\Delta z)^{n-1} \), the analytical vector field \( p(X) \) becomes \( z_3^n/\Delta z^{n-1} \) \((-z_1P^1 + P^2, -z_2P^1 + P^3, -z_3P^1)\), where \( P_i = P^i(1/z_3, z_1/z_3, z_2/z_3) \). In a similar way one can deduce the expressions of \( p(X) \) in \( U_2 \) and \( U_3 \). These are \( z_3^n/\Delta z^{n-1} \) \((-z_1P^2 + P^1, -z_2P^2 + P^3, -z_3P^2)\), where \( P_i = P^i(z_1/z_3, 1/z_3, z_2/z_3) \) in \( U_2 \), and \( z_3^n/\Delta z^{n-1} \) \((-z_1P^3 + P^1, -z_2P^3 + P^2, -z_3P^3)\), where \( P_i = P^i(z_1/z_3, z_2/z_3, 1/z_3) \) in \( U_3 \).

The expression for \( p(X) \) in \( U_4 \) is \( z_3^{n+1} \) \((-z_1P^1, P^2, P^3)\), now denoting \( P_i = P^i(z_1, z_2, z_3) \). The expression for \( p(X) \) in the local chart \( V_i \) is the same as in \( U_i \) multiplied by \((-1)^{n-1}\).

When one works with the expression of the compactified vector field \( p(X) \) in the local charts one usually omits the factor \( 1/(\Delta z)^{n-1} \). One can do that through a rescaling of the time variable. In what follows, One will work with the orthogonal projection of \( p(X) \) from the closed northern hemisphere to \( y_4 = 0 \), one continues denoting this projected vector field by \( p(X) \). Note that the projection of the closed northern hemisphere is a closed ball \( B \) of radius one, whose interior is diffeomorphic to \( R^3 \) and whose boundary \( S^2 \) corresponds to the infinity of \( R^3 \). Of course \( p(X) \) is defined in the whole closed ball \( B \) in such a way that the flow on the boundary is invariant. This new vector field on \( B \) will be called the Poincaré compactification of \( X \), and \( B \) will be called the Poincaré ball.

All the points on the invariant sphere \( S^2 \) at infinity in the coordinates of any local chart \( U_i \) and \( V_i \) have \( z_3 = 0 \). Also, the points in the interior of the Poincaré ball, which is diffeomorphic to \( R^3 \), are given in the local charts \( U_1, U_2 \) and \( U_3 \) by \( z_3 > 0 \) and in the local charts \( V_1, V_2 \) and \( V_3 \) by \( z_3 < 0 \).

**Appendix B.**

**KCC theory and Jacobi analysis.** Let \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \) and \( y = (y_1, y_2, \ldots, y_n) = (dx_1/dt, dx_2/dt, \ldots, dx_n/dt) \in \mathbb{R}^n \). Consider a second-order differential equation (SODE) of the form

\[
\dot{x}_i + 2G_i(x, y, t) = 0, \quad i \in \{1, 2, \ldots, n\},
\]  

(22)
where \((x,y,t) \in \Omega \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n\), \(\Omega\) is an open connected set, and \(G_i(x,y,t)\) is \(C^\infty\) in a vicinity of initial conditions \(((x)_0, (y)_0, t_0)\) in \(\Omega\).

Under the non-singular coordinate transformations
\[
\tilde{x}_i = f_i(x_1, x_2, \ldots, x_n), \quad \tilde{t}_i = t, \quad i \in \{1, 2, \ldots, n\},
\]
the KCC-covariant differential of a vector field \(\xi_i(x)\) on \(\Omega\) is defined by
\[
\frac{D\xi_i}{dt} = \frac{d\xi_i}{dt} + N^j_i \xi_j,
\]
where \(N^j_i = \partial G_i / \partial y_j\) are the coefficients of the nonlinear connection. In this paper, we always use the Einstein summation convention. According to Eq.(24), Eq.(22) is rewritten as
\[
\frac{Dy_i}{dt} = N^j_i y_j - 2G_i = -\epsilon_i.
\]
In general, we call the contravariant vector field \(\epsilon_i\) as the first KCC invariant of Eq. (22) on \(\Omega\). Let’s think about the trajectories \(\tilde{x}_i(t)\) of the system (22) around \(x_i(t)\) according to
\[
\tilde{x}_i(t) - x_i(t) = \eta \xi_i(t),
\]
where \(\xi_i(t)\) are the components of a contravariant vector field defined along \(x_i(t)\) and \(|\eta|\) is a small value. Substitute Eq. (25) into Eq. (22) firstly, after then apply the mean value theorem and take the limit \(\eta \to 0\). Finally, the deviation equations are obtained as
\[
\frac{d^2 \xi_i}{dt^2} + 2N^j_i \frac{d\xi_j}{dt} + 2\xi_j \frac{\partial G_i}{\partial x_j} = 0.
\]
Base on the KCC-covariant differential (24), the above equation (26) can be written as
\[
\frac{D^2 \xi_i}{dt^2} = P^j_i \xi_j,
\]
where
\[
P^j_i = -2 \frac{\partial G_j}{\partial x_i} - 2G_j G^i_j + y_l \frac{\partial N^j_i}{\partial x_l} + N^j_i N^l_i + \frac{\partial N^j_l}{\partial t}.
\]
Here, \(G^i_j = \partial N^j_i / \partial y_l\) are the Berwald connection coefficients. Eq.(27) is called the Jacobi equation and \(P^j_i\) is called the second KCC-invariant (i.e. the deviation curvature tensor). The third, fourth and fifth KCC-invariant that are called respectively the torsion tensor, the Riemann-Christoffel curvature tensor, and the Douglas tensor, are defined as
\[
P^j_{ik} = \frac{1}{3} \frac{\partial P^j_i}{\partial y_k} - \frac{1}{3} \frac{\partial P^j_k}{\partial y_i}, \quad P^j_{ikl} = \frac{\partial P^j_i}{\partial y_l}, \quad D^j_{ikl} = \frac{\partial G^j_{ik}}{\partial y_l}.
\]
These tensors always exist in a Berwald space.

Let us consider the trajectories \(x_i = x_i(t)\) of (22) as curves in the Euclidean space \((\mathbb{R}^n, \langle \cdot, \cdot \rangle)\), where \(\langle \cdot, \cdot \rangle\) is the canonical inner product of \(\mathbb{R}^n\), and suppose the deviation vector \(\xi\) satisfies \(\xi(0) = O \in \mathbb{R}^n\), \(\dot{x}(0) = W \neq O\), here \(O\) is the null vector. Considering an adapted inner product
\[
\langle(X,Y)\rangle := \frac{1}{\langle W, W \rangle} \cdot \langle X, Y \rangle
\]
for any vectors \(X, Y \in \mathbb{R}^n\). Obviously, \(\|Y\|^2 := \langle(Y,Y)\rangle = 1\).
The focusing tendency of trajectories around \( t = 0 \) are
- bunching together if \( \|\xi(t)\| < t^2 \); namely, iff the real part of the eigenvalues of \( P^i_j(0) \) are strictly negative.
- disperse if \( \|\xi(t)\| > t^2 \); namely, iff the real part of the eigenvalues of \( P^i_j(0) \) are strictly positive.

The concept of the Jacobi stability for a dynamical system can be defined as follows\([8, 23]\):

**Definition 5.1.** If the SODE (22) satisfies the initial conditions \( \|x_i(t_0) - \tilde{x}_i(t_0)\| = 0, \|\dot{x}_i(t_0) - \dot{\tilde{x}}_i(t_0)\| \neq 0 \), with respect to the norm \( \|\cdot\| \) induced by a positive definite inner product, then the trajectories of (22) are Jacobi stable if and only if the real part of the eigenvalues of the deviation curvature tensor \( P^i_j \) are strictly negative everywhere. Otherwise, the trajectories are Jacobi unstable.

In two-dimensional space, deviation curvature tensor can be written as matrix \( P \)

\[
P = \begin{bmatrix}
P^1_1 & P^1_2 \\
P^2_1 & P^2_2 
\end{bmatrix}.
\]

Let \( S = \det P = P^1_1 + P^2_2, Q = \det P = P^1_1 P^2_2 - P^2_1 P^1_2 \). The characteristic polynomial above matrix is

\[
\lambda^2 - S\lambda + Q = 0,
\]

and its eigenvalues are

\[
\lambda_\pm = \frac{1}{2} \left[ S \pm \sqrt{S^2 - 4Q} \right].
\]

Owing to Routh-Hurwitz criteria, all the roots of polynomial \( P(\lambda) \) are negative or have negative real parts if and only if

\( S < 0, Q > 0 \).

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Received January 2020; 1st revision April 2020; 2nd revision May 2020.

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