Violation of Bell inequalities by stochastic simulations of Gaussian States based on their positive phase-space Wigner representation

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Abstract

Because of the difference between symmetrically and normally ordered operators, some trajectories in stochastic simulations with a probability density equal to a positive Wigner function can imply negative intensities, despite a positive mean. Hence, Bell inequalities do not apply. We retrieve for a weakly squeezed Gaussian state the maximum violation allowed by quantum mechanics, for the Clauser-Horn-Shimony-Holt (CHSH) as well as for the Clauser-Horn Bell inequalities. With the latter, the influence of the quantum efficiency of the detectors is evidenced.

1) Introduction

As reported by Drummond et al. [1], Feynman answered negatively in 1982 [2] to the question “Can quantum systems be probabilistically simulated by a classical computer?” Following [1] and others, we propose in this paper a more positive answer. However, we first would like to remark that there are simple systems that justify the Feynman’s assertion. Take for example two entangled images formed by spontaneous parametric down conversion (SPDC) on two cameras able to count photons [3]. Let the number of photons on each pixel of the cameras be smaller than, say, 5, and the number of relevant pixels on each camera be $512 \times 512$. In the Schrödinger point of view, the Hilbert space that describes such a system has $512 \times 512$ dimensions, each corresponding to a different couple of images with its own probability. It is evidently impossible to simulate, by using a generator of random numbers corresponding to these probabilities, the successive couples of images obtained by repeating the experiment. On the other hand, all statistical features of the images, like means, variances, pixel correlations and so on, can easily be obtained in the Heisenberg point of view, since quantum quadratures propagate like classical optical fields. To calculate the statistical features of the spatial repartition of SPDC, two methods were proposed in [4], using either the Green functions characterizing the pixel to pixel input-output relations or stochastic simulations based on the Wigner phase-space function.

Bell inequalities involve correlations between remote systems and the simulation of their violation does not necessitate reproducing a succession of individual, realistic, experiments. Nevertheless, Bell argued [5] that a quantum state with a positive-definite Wigner function would not violate a Bell inequality since this Wigner function can be viewed as a probability density function of a local hidden variable. This argument has been proved to be false [6] by showing a relation between the Wigner function and the parity operator, leading to an experimental demonstration of violation of Bell inequalities in spatial parity space using SPDC [7]. Note that SPDC, or squeezed vacuum, does possess a positive-definite Wigner function. Actually, before these works in the spatial domain, SPDC was proved by Kwiat et al to allow a violation of the Bell inequalities in the original scheme using polarizers [8]. The experiment used polarization entanglement between the pairs of photons coming from the two intersections of the two cones corresponding to type-II SPDC, each cone with a
polarization orthogonal to the other: see Figure 2. At detection, the polarizing beam-splitters can be viewed as parity operators. This experiment was analyzed in the Wigner representation by Casado et al. [9], who remarked that the detected intensity is not positive-definite for each realization of the underlying random process, even if the mean intensity, corresponding in simulations to an average of a great number of realizations, is positive. This remark leaded them to propose a modification of the quantum formalism that is compatible with experiment only if the detectors admit some basic dark rate, or in other words if they are directly sensitive to vacuum fluctuations. No experimental evidence has justified this proposition, beyond the evident problems of energy conservation. Brambilla et al. used also stochastic simulations based on the Wigner formalism to describe the spatial properties of SPDC [10].

A positive phase-space Wigner function offers an evident scheme of stochastic simulations that should, for polarization entangled SPDC, violate Bell inequalities since, in the words of Cahill and Glauber [11], the Wigner function may be used to write the ensemble averages of all bounded operators as convergent integrals. We will see in this paper that this is indeed the case. Curiously, such violation seems have not been demonstrated before, though Werner et al. presented [12] a thorough comparison of the advantages and drawbacks of the positive-P and Wigner representation. To our present purpose, we retain that the symmetrically ordered operators corresponding to the electric field in the Wigner representation, i.e. the quantum quadratures, propagate classically and the results are exact inasmuch as the pump field is intense and undepleted. These features were exploited in [13] to simulate a spatially multimode Hong-Ou-Mandel experiment and in [14] to characterize SPDC issued from crystals with complex structures. As also stated in [12], the Wigner representation requires only half the number of variables as does the positive-P method, and requires independent Gaussian noise sources only at the input. Nevertheless, this is the positive-P method that was chosen to show that quantum simulations can be used to demonstrate violation of Bell inequalities [15]. As regards the Wigner formalism, it implies for each trajectory four complex numbers, corresponding to the two orthogonal polarizations of the field at the two remote locations. It has been demonstrated in [16-17] that four complex numbers that look similar do obey the Bell inequalities in the Glauber-Sundarshan representation. A misconception would consist in believing that this demonstration extends to the Wigner representation. Indeed, it is well-known [11, 16] that squeezed vacuum does not possess a regular Glauber-Sundarshan representation, though its Wigner function is positive-definite. On the other hand, the demonstration of Bell inequalities involves positive intensities. Hence Bell inequalities can be violated for squeezed vacuum either because its Glauber-Sundarshan representation is not regular, or because the real number giving the intensity associated to a trajectory in the Wigner representation may be negative, as we will develop in this paper. Of course, it means that a trajectory of the stochastic simulation does not correspond to anything in the real world (looking for such a correspondence leads to doubtful physics [9]). Only averages on a great number of trajectories correspond to physical quantities.

The remaining of the paper is organized as follows. In section 2, we give the theoretical framework. Section 3 deals with numerical results and we conclude in section 4.

2) Theoretical framework

It has been shown by Cahill and Glauber, [11] Eq. 4.23, that the expectation value of a symmetrically ordered product of creation and annihilation operators $a^\dagger$ and $a$, can be always
expressed as an integral in the entire complex plane $\mathbb{C}$ over a c-number $a$, weighted by the Wigner function $W(\alpha)$:

$$< (\alpha^*)^n \alpha^m >_s = \frac{1}{\pi} \int \pi \cdot W(\alpha) (\alpha^*)^n \alpha^m d^2 \alpha$$  \hspace{1cm} (1)

The subscript $s$, or symmetrically ordered, means that all orders are present with an equal weight in the expectation. For example, we have, for $n=m=2$:

$$< (\alpha^*)^2 \alpha^2 >_s = \frac{1}{6} \left( a^{*\dagger} a a^{\dagger} a + a^{\dagger} a a^{*\dagger} a + a a^{*\dagger} a^{\dagger} a + a a^{\dagger} a^{*\dagger} a + a^{*\dagger} a a^{\dagger} a + a^{*\dagger} a a^{*\dagger} a + a^{*\dagger} a^{*\dagger} a + a^{*\dagger} a^{*\dagger} a + a^{*\dagger} a a^{\dagger} a + a^{*\dagger} a a^{*\dagger} a + a a^{\dagger} a^{*\dagger} a + a a^{\dagger} a^{*\dagger} a + a^{*\dagger} a^{*\dagger} a + a^{*\dagger} a^{*\dagger} a ight)$$  \hspace{1cm} (2)

Some useful relations hold between the operator number of photons $N = a^\dagger a$, and the symmetrically ordered operators:

$$N = (a^\dagger a)_s = \frac{1}{2}, \quad N^2 = ((a^\dagger a)_s)^2 - N = \frac{1}{2}$$  \hspace{1cm} (3)

Eqs. 3 are derived by using the commutation relation of the annihilation operator $[a, a^\dagger] = 1$.

We deduce the mean and the variance:

$$< N > = < (a^\dagger a)_s > - \frac{1}{2} \quad V(N) = < N^2 > - < N >^2 = < (a^\dagger a)_s >^2 - < (a^\dagger a)_s >^2 - \frac{1}{2}$$  \hspace{1cm} (4)

Since these relations are based only of the commutation properties of the annihilation operator in a mode, they are general, whatever the wave function involved in the means. If two different modes are implied, the corresponding annihilation operators commute and we obtain for the covariance of the numbers of photons in two modes 1 and 2:

$$\text{Cov}(N_1, N_2) = < N_1 N_2 > - < N_1 > < N_2 > = < a^\dagger_1 a_1 a^\dagger_2 a_2 >_s - < (a^\dagger_1 a_1)_s > < (a^\dagger_2 a_2)_s >$$  \hspace{1cm} (5)

Eq. (1, 4, 5) suggest a scheme of numerical simulation for states whose the Wigner function is definite positive, and remains as such under propagation. To calculate the integral on the right side of Eq. 1, the simplest solution is to randomly sampling the complex plane by using a probability density proportional to the Wigner function. The real part of the obtained c-number corresponds to the position quadrature, quadrature $X_1 = \frac{a + a^\dagger}{\sqrt{2}}$ in optics, and the imaginary part to the momentum quadrature, quadrature $X_2 = \frac{i a - a^\dagger}{\sqrt{2}}$ in optics. It can be easily verified that $X_1^2 + X_2^2 = (a^\dagger a)_s$. Eq. 1 ensures that the quantum mean of $X_1^2 + X_2^2$ is equal to the average of the squared moduli of the numerous random drawn complex numbers. Clearly, the equality does not hold for an individual drawing: if acting on the vacuum, $X_1$ and $X_2$ have a negative covariance, since their commutator is equal to $\frac{i}{\sqrt{2}}$, while the real and imaginary parts of the c-numbers are independent. Indeed, the Wigner function of the vacuum is Gaussian and depends of the squared modulus, [see (11) Eq. 4.38]:

$$W_0(a) = 2 \exp(-2|a|^2)$$  \hspace{1cm} (6)

We have now all the elements to introduce a complete numerical scheme to model the SPDC.
- Divide the input plane of the crystal in sufficiently small pixels to ensure that the sampling theorem is fulfilled at the crystal output: indeed, phase-matching acts in the spatial domain as a low-pass amplifier, ensuring a spatial cut-frequency that defines the conditions of sampling.

- Draw at random for each pixel two c-numbers, whose the real and imaginary part are independently picked from a Gaussian distribution of zero mean and variance \(\frac{\lambda}{4}\), in accordance with Eq. 6. Each c-number correspond to a polarized field along one of the two neutral axes of the crystal, horizontal (H) or vertical (V). We can prove easily from Eqs. (1, 4, 5), that such a drawing ensures \(\langle N \rangle = 0\), \(\langle V(N) \rangle = 0\), \(\text{cov}(N_1,N_2)=0\) (two pixels 1 and 2), as expected for the input vacuum.

- Propagate the field in the crystal using an usual split-step algorithm, where the classical coupled equations of parametric amplification are solved in the direct domain, and diffraction taken into account by propagating the plane wave spectrum in the spatial Fourier domain [4]. It can be proved that quantum quadratures propagate like classical waves in the undepleted pump approximation [12].

- Repeat the entire procedure a great number of times. Each iteration is called a trajectory.

- Calculate at the output all the statistical features of interest on the detected photon-numbers by applying first Eq. 1, to pass from averages of squared moduli of c-numbers to means of symmetrically operators, then Eqs 3 to 5 to apply “quantum corrections” in order to retrieve photon numbers from symmetrically ordered operators. Note that these quantum corrections can be applied either to each trajectory (Eq.3) or to the means (Eqs 4 and 5).

![Fig.1: experimental set-up. The photons 1 and 2 propagate in different but non opposite directions.](image-url)
Fig 2: non corrected mean intensity in the Fourier plane (average of 20,000 trajectories).

The envisioned experimental set-up is similar to that of [8]: see Fig 1. An U.V. pump beam is incident on a BBO crystal in conditions of type II phase-matching. A horizontally polarized signal and a vertically polarized idler beam are created by SPDC and four detectors record the photons coming from the cone intersections, in chosen polarization directions. We see on Fig. 2 the mean intensity obtained in the far-field, or Fourier domain, by averaging 30,000 trajectories. We keep for the following only two pixels, numbered 1 and 2, corresponding respectively to the left and right best intersection of the cones, exactly symmetrical with respect to the direction of the pump beam. The Bell biphoton state corresponding to these two pixels can be written as:

$$|\psi^+\rangle = (|H_1, V_2\rangle + |V_1, H_2\rangle) / \sqrt{2}$$

(7)

The other Bell states can be obtained by using wave-plates [8] but, for sake of conciseness, we consider only $|\psi^+\rangle$ in this paper.

Two polarizing beam-splitters separate the beams 1 and 2, with their first neutral axes forming respectively an angle $\theta_1$ and $\theta_2$ with the horizontal direction. The four output field amplitudes can be written as:

$$\begin{pmatrix} A_{i+} \\ A_{i-} \end{pmatrix} = \begin{pmatrix} \cos(\theta_i) & \sin(\theta_i) \\ -\sin(\theta_i) & \cos(\theta_i) \end{pmatrix} \begin{pmatrix} A_{i1}^{\prime} \\ A_{i2}^{\prime} \end{pmatrix}$$

(8)

where + and – designate the two output ports of the polarizing beam-splitter and i=1 or 2 refers to the left or right pixel.

After calculating the intensities $I_i^j(\theta_i) = |A_{i}^{\prime}|^2$ expressed in number of photons, we apply the quantum corrections of Eq.3 to obtain the normalized correlation:


The top bars mean: average on a great number of trajectories. Clearly, the quantum corrections vanish at the numerator of Eq. (9). On the other hand, these quantum corrections do hold at the denominator. If we assume that the corrected intensities, i.e. the photon numbers, are positive for each trajectory, we can derive [16] the CHSH form of Bell inequalities [18]: whatever the angles $\theta_1, \theta'_1, \theta_2, \theta'_2$, we have

$$|B| \leq 2, \text{ where } B = E(\theta_1, \theta_2) - E(\theta'_1, \theta'_2) + E(\theta_1, \theta'_2) + E(\theta'_1, \theta_2)$$

(10)

However, for some trajectories, the photon numbers are negative because of the quantum corrections, though the average is, and must be, positive. Hence, the CHSH Bell inequality can be violated only for a high detector quantum efficiency, in fact greater than 83% [20] for a maximally entangled state. Once more, we will see that the negative “corrected intensities” allow this inequality to be numerically violated.

4) Results

All results will be given for $\theta_1 = -\frac{\pi}{8}$, $\theta'_1 = \frac{\pi}{8}$, $\theta_2 = \frac{\pi}{2}$, $\theta'_2 = -\frac{\pi}{4}$, ensuring for the Bell state $|\psi^+\rangle$ the quantum theoretical values $B = 2\sqrt{2} = 2.83$, $C=1.21$, i.e. the maximum violation of the Bell inequalities allowed by quantum mechanics. See for example [16] for the quantum calculation.

Fig. 3 shows, for a mean output intensity of 0.02 photon/pixel (sum of signal and idler), the evolution of B and C for a number of iterations n between 1000 and $1400^2=1.96 \times 10^6$. A simple estimation of the confidence interval on the involved intensities is as follows. The probability density of the non-corrected intensity for a mode, i.e. after the polarizing beam-splitter, is given by a decreasing exponential, with, for a trajectory, a standard deviation equal to the mean (the statistics is that of a speckle of unity contrast [21]). The standard deviation of the total mean intensity $\sigma_j$ is inversely proportional to the square root of the number of trajectories, giving for $n=2.56 \times 10^6$ and a mean non-corrected intensity of the signal or the idler $I_{\text{sum}} = 0.51$, $\sigma_j = \frac{0.51 \times \sqrt{n}}{1400} = 5.2 \times 10^{-4}$ . We use here the fact that the signal and the idler
intensities have independent statistics when adding on either the pixel 1 or 2 and obey each a thermal statistics (standard deviation equal to the mean, see below). On the other hand, the true intensity is the corrected one, giving a relative standard deviation of \( \frac{2 \times 10^{-4}}{2 \times 10^{-4}} = 2.6\% \).

Though the exact computation of the uncertainty range on B is difficult, we can admit that this value is also close to the relative standard deviation of B. We see here the principal drawback of the method: the useful information lies in the corrected values, while the fluctuations scale with the non-corrected ones, leading to the necessity of a great number of trajectories for the small gain that allows a weak squeezed state to reproduce at best the quantum behavior of a biphoton state.

![Graph](image)

**Fig 3:** An example of the evolution of B and C versus the number of trajectories. Colored areas: uncertainty ranges centered on the theoretical biphoton values.

The finally estimated \( B = 2.68 \) is 5.3\% below the quantum theoretical value for a biphoton state, \( B = 2.83 \), i.e. outside the \( \pm 5.2\% \) uncertainty range at 95\% of confidence. Actually, even for a low mean intensity of 0.02, the probability of a double pair in a single experiment cannot be entirely neglected. It leads to a modification of the coincidence rate that lowers the measured value of B. The theoretical value of B that takes into account this effect can be determined as follows. Let be \( G = \sinh^2 (g L) \) the total gain, in photons per mode, for a crystal of length \( L \), \( g \) is the gain per unit length, depending on the pump intensity and the nonlinear crystal coefficient, at perfect phase matching. The statistics of the signal (idler) beam is thermal, ensuring for its mean and variance [21]:

\[
\langle N_s \rangle = \langle N_i \rangle = G, \quad V(N_s) = V(N_i) = G + G^2
\]

Only pairs are emitted, resulting in a signal-idler covariance equal to the variance:

\[
\langle N_s N_i \rangle = G + G^2
\]
\( \text{Cov}(N_S, N_I) = G + G^2 \) \hspace{1cm} (13)

At the intersection of the cones, the signal and idler intensities are added and not correlated, ensuring:
\[
\langle N_1 \rangle = \langle N_2 \rangle = 2G, \quad V(N_1) = V(N_2) = 2 \left( G + G^2 \right) \]
\hspace{1cm} (14)

There is a perfect correlation between the signal (idler) in 1 and the idler (signal) in 2, which allows us to write the covariance between the two pixels as:
\[
\text{Cov}(N_1, N_2) = 2\text{Cov}(N_S, N_I) = 2 \left( G + G^2 \right) \]
\hspace{1cm} (15)

We find now easily all values necessary to compute the theoretical value of \( E(\theta_1, \theta_2) \):
\[
\langle N_1 N_2 \rangle = 2G + 6G^2 \\
\left( \langle N_1^2 \rangle - \langle N_1 \rangle \right) \left( \langle N_2^2 \rangle - \langle N_2 \rangle \right) = 2 \left( G + G^2 \right) \left( \sin^2(\theta_1 + \theta_2) - \cos^2(\theta_1 + \theta_2) \right) 
\]
\hspace{1cm} (16)

Giving, for the angles corresponding to a maximum violation:
\[
B = 2\sqrt{ \frac{1 + \sqrt{5}}{1 + \sqrt{15}} } \]
\hspace{1cm} (17)

For \( G = 0.01 \) used in Fig. 2, the theoretical value of \( B \) is 2.77, i.e. well inside the \( \pm 5.2\% \) uncertainty range around the “experimental” value 2.68.

Figure 4: Numerical and analytical values of \( B \) versus the intensity in a mode. Colored area: uncertainty range.
Figure 4 shows a comparison between the values of B issued from the numerical simulation (10^5 trajectories for all points but the first, 1.96 10^6 trajectories for this point) and the values calculated with Eq. 16, with a good agreement. It is also interesting to note that the relative number of negative values of $N_1, N_2$ goes from 47% for $G=0.01$ to 22% for $G=0.46$. The quantum limit $B=2$ is attained for 31% of negative values.

It should be noted that $C$ increases with $G$. We see immediately from Eq.16 that, for angles corresponding to a maximum violation and unity quantum efficiency, we have:

$$C = 1.2071 (1 + G)$$

Figure 5: $B$ and $C$ versus the quantum efficiency, for an intensity per mode of 0.01. The colored uncertainty range is centered on the maximum value (1.207) multiplied by the quantum efficiency.

Finally we see on Fig.5 the influence of the quantum efficiency, equivalent to a beam splitter before each detector, with quantum vacuum noise entering the free input port. As foreseen, $B$ is independent of the quantum efficiency, while $C$ surpasses the quantum limit only for a high quantum efficiency, since $C$ is simply multiplied by the quantum efficiency.

5) Conclusion

We have shown in this paper that stochastic simulations based on the positive Wigner function of Gaussian states can be used to demonstrate violation of Bell inequalities. The method is simpler than positive P representation [12]. The minimum of trajectories to attain a good precision becomes very important if the mean number of photons per mode is very low, i.e. in the regime where the probability of a second pair in the mode is weak, meaning that the
simulation corresponds to a genuine biphoton. Nevertheless, strong violation (B=2.6) can be obtained with a mean number of photons per mode of 0.05 and 10^5 trajectories, i.e. 20 minutes on a professional PC, and an uncertainty of about ±4%.

Of course, Bell inequalities are well known for polarization entanglement and, once validated on this physics, our method will prove its true potential in less explored situations. We envision to extend our method to high-dimensional systems [22, 23], where the reduced number of variables, compared to the positive-P, could be very interesting.

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