ON THE THREE-ANYON HARMONICS

Giovanni AMELINO-CAMELIA\(^{(a)}\) and Chaiho RIM\(^{(b)}\)

\(^{(a)}\) Theoretical Physics, University of Oxford, 1 Keble Rd., Oxford OX1 3NP, UK
\(^{(b)}\) Center for Theoretical Physics, Laboratory for Nuclear Science, and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA

Abstract

The 3-anyon problem is studied using a set of variables recently proposed in an anyon gauge analysis by Mashkevich, Myrheim, Olaussen, and Rietman (MMOR). Boundary conditions to be satisfied by the wave functions in order to render the Hamiltonian self-adjoint are derived, and it is found that the boundary conditions adopted by MMOR are one of the ways to satisfy these general self-adjointness requirements. The possibility of scale-dependent boundary conditions is also investigated, in analogy with the corresponding analyses of the 2-anyon case. The structure of the known solutions of the 3-anyon in harmonic potential problem is discussed in terms of the MMOR variables. Within a series expansion in a boson gauge framework the problem of finding any anyon wavefunction is reduced to a (possibly infinite) set of algebraic equations, whose numerical analysis is proposed as an efficient way to study anyon physics.

OUTP-95-49-P/MIT-CTP-2503

\(^{*}\)Work supported in part by funds provided by the U.S. Department of Energy (D.O.E.) under cooperative agreement #DE-FC02-94ER40818, the European Community under contract #ER-BCHBGCT940685, the Yonam Foundation, and the Korea Science and Engineering Foundation.

\(^{†}\)Permanent address: Chonbuk National University, Department of Physics, Chonju, 560-756, Korea.
I. INTRODUCTION

The realization\(^1\) that particles with anomalous exchange statistics—anyons—could be consistently introduced in 2+1 dimensions has had a great impact on theoretical physics, and particularly on our description of certain effectively 2+1 dimensional condensed matter phenomena, such as the fractional quantum Hall effect. However, a complete understanding of anomalous exchange statistics has not yet been achieved; most notably, even very simple \(N\)-anyon quantum mechanical systems (free anyons, anyons in harmonic potential, ...) have proven too hard to be solved, with the exception of the rather trivial case \(N = 2\) in which the (usually problematic) 3-anyon interactions\(^2\) are obviously irrelevant.

A large effort, of which Refs.[2-6] are just a small (but significative) sample, has been devoted to the investigation of 3-anyon problems, but these studies have had only partial success, identifying only an incomplete set of eigenfunctions. One of the reasons of interest in investigations of 3-anyon problems is that we can expect that it would be easy to solve a given \(N\)-anyon problem, if the corresponding 3-anyon eigenfunctions were known; in fact, since there are only 2- and 3-anyon interactions, by considering \(N > 3\) one should encounter no more complications than those present in the \(N = 3\) case. This renders the 3-anyon problem of fundamental importance for the understanding of anomalous exchange statistics.

In this paper we study certain aspects of the 3-anyon problem using the set of variables recently proposed\(^6\) by Mashkevich, Myrheim, Olaussen, and Rietman (MMOR). Sec.II and III are devoted to a review of this formalism, and the way in which it leads to the introduction of the “anyon harmonics”. In Sec.IV we derive some self-adjointness restrictions necessary for a physical set up of the 3-anyon problem, a point which was not discussed in Ref.\(^6\). Also using a relation between the 3-anyon Hamiltonian and the 2-anyon Hamiltonian, and observing that one of the relevant operators is positive semidefinite, we find general self-adjointness restrictions. In section V, we derive the explicit form of those anyon harmonics which correspond to the known solutions of the 3-anyon in harmonic potential problem. In section VI, we analyze anyon harmonics within a series expansion in boson gauge, and compare the results with the results of the corresponding anyon gauge analysis given in Ref.\(^6\). Our closing remarks are given in Sec.VII.

II. SETTING UP THE 3-ANYON PROBLEM

A. Variables

As shown in Ref.\(^3\), in the analysis of the 3-anyon problem it is useful to introduce variables \(Z, u, v\) defined in terms of the complex particle coordinates \(z_j (z_j \equiv x_j + iy_j)\) by the relations

\[^1\]In using the expressions “2-anyon interactions” and “3-anyon interactions” we adopt a, possibly confusing, terminology that has been established in the literature; in fact, we are referring to the description of noninteracting anyons as bosons with certain 2- and 3-body interactions.
\[ Z = \frac{1}{\sqrt{3}} (z_1 + z_2 + z_3) , \]
\[ u = \frac{1}{\sqrt{3}} (z_1 + \eta z_2 + \eta^2 z_3) , \]
\[ v = \frac{1}{\sqrt{3}} (z_1 + \eta^2 z_2 + \eta z_3) , \]  

(2.1)

where \( \eta = e^{i \frac{2 \pi}{3}} \). \( Z \) is the center of mass coordinate, whereas \( u \) and \( v \) are relative motion variables. Notice that from (2.1) using \( 1 + \eta + \eta^2 = 0 \) it follows that

\[ z_2 - z_1 = \frac{\sqrt{3}}{\eta - 1} (u - \eta v) , \]
\[ z_3 - z_1 = \frac{\sqrt{3}}{\eta - 1} (v - \eta u) , \]
\[ z_2 - z_3 = \frac{\sqrt{3}}{\eta - 1} (1 + \eta)(u - v) . \]  

(2.2)

Once the center of mass motion is separated out (see later), it is often convenient to work with the four real variables \( r,q,\theta,\phi \) that are related to \( u,v \) by

\[ u = r \frac{q}{\sqrt{1 + q^2}} e^{i\theta + i\phi} \]
\[ v = r \frac{1}{\sqrt{1 + q^2}} e^{i\theta - i\phi} \]  

(2.3)

\( r \) and \( q \) are non-negative, whereas the angular variables \( \theta \) and \( \phi \) can be taken to run from 0 to \( 2\pi \).

For later convenience, we record here the measure for the integration over \( Z,r,q,\theta,\phi \) that is induced by the (flat) measure of integration over the coordinate variables \( z_j \):

\[ d^2z_1 d^2z_2 d^2z_3 = d^2Z dr dq d\theta d\phi \frac{4r^3q}{(1 + q^2)^2} . \]  

(2.4)

B. Statistical boundary conditions

Anyons can be described as bosons interacting through the mediation of an abelian Chern-Simons gauge field; in this “boson gauge” description the (free) anyons quantum mechanics is governed by the Hamiltonian (N.B. we set \( \hbar \) and the anyon mass to 1)

\[ H_b = \frac{1}{2} \sum_n \left( p_n + \nu a_n \right)^2 , \]  

(2.5)

\[ a_n^k \equiv \epsilon^{kij} \sum_{m(\neq n)} \frac{r_j^m - r_j^m}{|r_n - r_m|^2} \]  

(2.6)

and the wave functions are required to have trivial (symmetric) behavior under interchange of particle positions.
In alternative one can describe anyons in the “anyon gauge”, which is related to the boson gauge description by the following transformation

\[ \Psi_b \rightarrow \Psi_a = U \Psi_b, \quad H_b \rightarrow H_a = U H_b U^{-1} = \sum_n (p_n)^2, \quad (2.7) \]

where

\[ U \equiv \exp \left[ i \nu \sum_{m \neq n} \theta_{nm} \right], \quad (2.8) \]

and \( \theta_{nm} \) are the azimuthal angles of the relative vectors \( r_n - r_m \). The price payed for the simple form of the Hamiltonian \( H_a \) is that, since \( \Psi_b \) is single-valued and \( U \) is not, \( \Psi_a \) is multivalued, and this multivaluedness is related to the anomalous quantum statistics of anyons. In the study of the relative motion of two anyons the multivaluedness can be effectively described by the introduction of (anyonic) polar coordinates \( r_{12} \) and \( \theta_{12} \), in which the relative angle \( \theta_{12} \) runs from \(-\infty\) to \(\infty\), without identifying angles which differ by multiples of \(2\pi\), so that it keeps track of the number of windings \([1]\). In such coordinates the quantum mechanical wave functions describing the relative motion of two anyons satisfy the following condition

\[ \Psi_a(r_{12}, \theta_{12} + \pi) = e^{i\nu\pi} \Psi_a(r_{12}, \theta_{12}). \quad (2.9) \]

\( \nu \), called “statistical parameter”, characterizes the type of anyons, i.e. their statistics; in particular, from Eq.\((2.9)\) one realizes that anyons with even (odd) integer \( \nu \) verify bosonic (fermionic) statistics whereas the noninteger values of \( \nu \) correspond to particles with statistics interpolating between the bosonic and the fermionic case. Without any loss of generality \([1]\), one can restrict the values of \( \nu \) to be in the interval \([0,1]\).

In the three-anyon problem the description of the anyon multivaluedness requires the introduction of three angles running from \(-\infty\) to \(\infty\) without identifications. A consistent choice of these angles is given by \( \theta, \phi, \) and \( \xi \equiv 2 \arctan q \), where \( \theta, \phi, \) and \( q \) are the variables defined in the preceding subsection.

In describing all possible exchanges of the positions of the three anyons one can exploit the fact that an arbitrary permutation of \((1,2,3)\) can be obtained as a composition of cyclic permutations \(P: (1,2,3) \rightarrow (2,3,1)\) and exchanges \(E: (1,2,3) \rightarrow (1,3,2)\). The final outcome (after imposing, like in Ref.\([3]\), that the wave function acquires an overall phase \(e^{i\nu\pi}\), when going along a continuous curve in the 3-anyon configuration space, starting and ending with the same configuration, and such that two anyons are interchanged in the counterclockwise direction without encircling the other anyon) is the following set of conditions to be satisfied by the 3-anyon (relative motion) wave functions

\[ \Psi(r, \pi - \xi, -\phi, \theta) = e^{i\nu\pi} \Psi(r, \xi, \phi, \theta), \]
\[ \Psi(r, \xi, \phi + \frac{2\pi}{3}, \theta + \pi) = e^{2i\nu\pi} \Psi(r, \xi, \phi, \theta), \quad (2.10) \]
\[ \Psi(r, \xi, \phi, \theta + 2\pi) = e^{i6\nu\pi} \Psi(r, \xi, \phi, \theta). \]

Of course, since we are dealing with indistinguishable particles we can restrict the analysis to a fundamental domain, such as

\[ 0 \leq r, \quad 0 \leq \xi \leq \frac{\pi}{2}, \quad -\frac{\pi}{3} \leq \phi \leq \frac{\pi}{3}, \quad 0 \leq \theta \leq 2\pi, \quad (2.11) \]
which provides a single covering of the physical configuration space. Then the conditions (2.10) induce the following “statistical boundary conditions”

\[\Psi(r, q = 1, -\phi, \theta) = e^{i\nu\pi} \Psi(r, q = 1, \phi, \theta),\]
\[\Psi(r, q, \phi = \frac{\pi}{3}, \theta + \pi) = e^{i2\nu\pi} \Psi(r, q, \phi = -\frac{\pi}{3}, \theta),\]
\[\Psi(r, q, \phi, \theta = 2\pi) = e^{i6\nu\pi} \Psi(r, q, \phi, \theta = 0).\]  

Note that, in order to make closer contact with the analysis of Ref. [6], we have written these boundary conditions in terms of \(q\) rather than \(\xi\), exploiting the fact that within the fundamental domain (2.11) the map between \(q\) and \(\xi\) is invertible. (In particular \(0 \leq q \leq 1\) in the fundamental domain.)

Once the problem is solved within the fundamental domain one can extend the solutions to the entire plane; however, for this type of procedure one should use the variable \(\xi\), since the variable \(q\) does not keep track of the phases arising depending on the “history of windings” of the given evolution in configuration space.

We close this section by recording, for later convenience, the formulas describing the action of the operations \(P\) and \(E\) on the coordinates of our three anyons

\[P:\begin{cases}
(z_1, z_2, z_3) \rightarrow (z_2, z_3, z_1), \\
(Z, u, v) \rightarrow (Z, \eta^2 u, \eta v), \\
(Z, r, q, \phi, \theta) \rightarrow (Z, r, q, \phi + \frac{2\pi}{3}, \theta + \pi).
\end{cases}\]  

\[E:\begin{cases}
(z_1, z_2, z_3) \rightarrow (z_1, z_3, z_2), \\
(Z, u, v) \rightarrow (Z, v, u), \\
(r, q, \phi, \theta) \rightarrow (r, q^{-1}, -\phi, \theta).
\end{cases}\]  

III. SEPARATION OF VARIABLES AND ANYON HARMONICS

In the anyon gauge, the Hamiltonian describing three anyons in a “central” and nonsingular potential \(V(\sqrt{uu + vv})\) can be written as \(H_{tot} = H_{com} + h\), where

\[H_{com} = -2 \frac{\partial}{\partial Z} \frac{\partial}{\partial \bar{Z}}\]  

\(^5\)The boundary conditions (2.12) follow from continuity of the wave function as it crosses the boundary of the fundamental domain. We do not insist instead on the continuity of the derivatives of the wave function across the boundary of the fundamental domain; we shall only constrain (see Sec.IV) their behavior with the requirement that the Hamiltonian be self-adjoint.
describes the center of mass motion, which is trivial and will be ignored in the following, and
\[ h = -2 \left[ \frac{\partial}{\partial u} \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \frac{\partial}{\partial v} + V(\sqrt{u^2 + v^2}) \right] , \]  
(3.2)
describes the relative motion, which contains all the information on the statistics.

In terms of the \((r,q,\theta,\phi)\) variables the relative motion hamiltonian can be written as
\[ h = -\frac{1}{4r^3} \frac{\partial}{\partial r} (r^3 \frac{\partial}{\partial r}) + \frac{1}{4r^2} M + V(r) \]  
(3.3)
where
\[ M = (1 + q^2) \left\{ -\frac{1}{q} \frac{\partial q}{\partial q} q \frac{\partial q}{\partial q} + \frac{1}{q^2} \left( \frac{1}{2i} \frac{\partial}{\partial \theta} + \frac{1}{i} \frac{\partial}{\partial \phi} \right)^2 + \left( \frac{1}{2i} \frac{\partial}{\partial \theta} - \frac{1}{i} \frac{\partial}{\partial \phi} \right)^2 \right\} . \]  
(3.4)
The \((r\text{-independent})\) operator \( M \), and the relative angular momentum
\[ L = u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v} - \bar{u} \frac{\partial}{\partial \bar{u}} - \bar{v} \frac{\partial}{\partial \bar{v}} = \frac{1}{i} \frac{\partial}{\partial \theta}, \]  
(3.5)
commute among themselves and with the relative motion Hamiltonian
\[ [M, L] = [M, h] = [L, h] = 0 . \]  
(3.6)
The eigenfunctions can be characterized by quantum numbers, \( E, \mu, \) and \( l \) respectively for \( h, M, \) and \( L \):
\[ h \Psi_{E,l,\mu} = E \Psi_{E,l,\mu}, \quad M \Psi_{E,l,\mu} = \mu(\mu + 2) \Psi_{E,l,\mu}, \quad L \Psi_{E,l,\mu} = l \Psi_{E,l,\mu} . \]  
(3.7)
Notice that, for later convenience, we have chosen \( \mu \), rather than the eigenvalue of \( M \), as the quantum number associated with \( M \), and, since \( \mu(\mu + 2) \) is symmetric under \( \mu \rightarrow -\mu - 2 \), \( \mu \) can be taken \(*\) to satisfy \( \mu \geq -1 \).

The eigenfunctions can be put in the factorized form
\[ \Psi_{E,l,\mu} = \Theta_l(\theta) R_{E,\mu}(r) \Phi_{l,\mu}(q, \phi) , \]  
(3.8)
where \( \Theta_l, R_{E,\mu}, \) and \( \Phi_{l,\mu} \) are such that
\[ L \Theta_l(\theta) = l \Theta_l(\theta) \]  
(3.9)
\[ M_l \Phi_{l,\mu} \equiv (1 + q^2) \left\{ -\frac{1}{q} \frac{\partial q}{\partial q} q \frac{\partial q}{\partial q} + \frac{1}{q^2} \left( \frac{1}{2i} \frac{\partial}{\partial \theta} + \frac{1}{i} \frac{\partial}{\partial \phi} \right)^2 + \left( \frac{1}{2i} \frac{\partial}{\partial \theta} - \frac{1}{i} \frac{\partial}{\partial \phi} \right)^2 \right\} \Phi_{l,\mu} = \mu(\mu + 2) \Phi_{l,\mu} \]  
(3.10)
\[ h_{\mu} R_{E,\mu} \equiv \left( -\frac{1}{4r^3} \frac{d}{dr} r^3 \frac{d}{dr} + \frac{\mu(\mu + 2)}{4r^2} + V(r) \right) R_{E,\mu} = ER_{E,\mu} \]  
(3.11)

**One might wonder whether complex values of \( \mu \), still leading to real eigenvalues of \( M \), could be physically relevant for the 3-anyon problem. As we shall see later, this possibility is excluded by the fact that \( M \) turns out to be a positive semidefinite operator, which will allow us to further restrict the range of values of \( \mu \) to \( \mu \geq 0 \).**
The statistical boundary conditions and Eq. (3.3) determine the $\Theta_l(\theta)$:

$$\Theta_l(\theta) = e^{il\theta},$$  \hspace{1cm} (3.12)

where

$$l = 3\nu + m \quad \text{with integer } m.$$  \hspace{1cm} (3.13)

The “radial” part of the problem is also simply solved; for example, for $V = 0$ (and $E \geq 0$) one finds that the general solution of Eq. (3.11) has the form

$$R_{E,\mu}(r) \sim A_1 r J_{1+\mu}(\sqrt{E}r) + B_1 r Y_{1+\mu}(\sqrt{E}r),$$  \hspace{1cm} (3.14)

where $A$ and $B$ are parameters, and $J_\nu (Y_\nu)$ is the first (second) kind Bessel function. Notice that Eq. (3.14) is very similar to certain solutions of the two-anyon problem. This is a consequence of the general property of $h_{\mu}$ of being simply related to $H_{2}^{(s)}(\nu = 1 + \mu; V(r))$, the relative motion s-wave Hamiltonian for two anyons of statistical parameter $1 + \mu$ in a potential $V(r)$ (obviously, within $H_{2}^{(s)}$ the variable $r$ is the distance between the two anyons):

$$rh_{\mu}r^{-1} = \frac{1}{4} \left[ -\frac{1}{r} \partial_r (r \partial_r) + \frac{(1 + \mu)^2}{r^2} + V(r) \right] = \frac{1}{4} H_{2}^{(s)}(\nu = 1 + \mu; V(r)).$$  \hspace{1cm} (3.15)

This observation allows one to use the known properties [7,8] of $H_{2}^{(s)}$ in the study of $h_{\mu}$.

The most difficult part of the 3-anyon problem is the identification of the functions $\Phi_{l,\mu}$, also called “anyon harmonics” [6] because, together with $\Theta_l(\theta)$, they give a generalization† of the hyper-spherical harmonics on $S^3$, with Euler angles $(\xi, \rho, \tau)$ related to the variables $(q, \phi, \theta)$ by the relations $q = \tan \frac{\xi}{2}$, $\phi = \rho$, and $\tau = -2\theta + 2\pi$. The statistical boundary conditions are such that the anyon harmonics with $\nu \neq 0, 1$ cannot be obtained as a combination of a finite number of ordinary harmonics on $S^3$ [9].

Some progress in the investigation of the anyon harmonics can be achieved by employing the ladder operators [9]:

$$K_+ = e^{i2\theta} \left\{ i \frac{\partial}{\partial \xi} - \frac{1}{\sin \xi} \frac{\partial}{\partial \phi} - \cot \xi \frac{\partial}{\partial \theta} \right\},$$  \hspace{1cm} (3.16)

$$K_- = e^{-i2\theta} \left\{ i \frac{\partial}{\partial \xi} + \frac{1}{\sin \xi} \frac{\partial}{\partial \phi} + \cot \xi \frac{\partial}{\partial \theta} \right\},$$

$$K_3 = \frac{1}{2i} \frac{\partial}{\partial \theta} = \frac{1}{2} L,$$

† Notice that $M$ in Eq. (3.4) is the laplacian on $S^3$. Unlike in the ordinary case (where $0 \leq \xi < \pi$, $-\pi \leq \rho < \pi$, $-2\pi \leq \tau < 2\pi$), the ranges of the Euler angles are $0 \leq \xi < \frac{\pi}{2}$, $-\frac{\pi}{3} \leq \rho < \frac{\pi}{3}$ and $-2\pi \leq \tau < 2\pi$ (i.e. the harmonics are defined on $S^3/Z_2 \times Z_3$); moreover, the $\nu$-dependent statistical boundary conditions (2.12) are different from the ones for the ordinary hyper-spherical harmonics.
which satisfy the $SU(2)$ Lie-algebra

$$[K_3, K_{\pm}] = \pm K_{\pm}, \quad [K_+, K_-] = 2K_3,$$  \hspace{1cm} (3.17)

and are invariant under $P$ but change sign under $E$. Observing that $M = 2(K_+K_- + K_-K_+) + 4K_3^2$, one finds that $\mu_2$ (see Eq. (3.7)) is the Casimir number of the harmonics on $S^3$. In addition, since $K_+ = K_+^\pm$ with respect to the measure $\sin \xi d\xi d\phi d\tau$, $M$ is a positive semi-definite operator, so that $\mu \geq 0$; this plays an important role in the analysis of the next section.

In Sec.V, we derive the explicit form of those anyon harmonics that correspond to the known solutions of the 3-anyon in harmonic potential problem. We shall refer to these anyon harmonics as “type-I and type-II”, in analogy with the terminology used in the literature for the corresponding wave functions of the 3-anyon in harmonic potential problem. Other types (“type-III and type-IV”) of anyon harmonics correspond to the eigensolutions of the of the 3-anyon in harmonic potential problem that are still unknown‡‡ and we shall refer to them in the following as the “missing anyon harmonics”.

IV. SELF-ADJOINTNESS RESTRICTIONS

In Ref. [6], in setting up the 3-anyon problem, besides the statistical boundary conditions, certain additional boundary conditions were imposed. In order to have square integrable wave functions it was demanded that their radial part satisfy

$$[R(r)]_{r \to 0} \sim r^\tau \text{ with } \tau > -2,$$  \hspace{1cm} (4.1)

and, based on some considerations on the physical significance of the configurations with $q = 0$ or $q = 1$ it was demanded that§§

$$\left[ \frac{\partial}{\partial \phi} \Psi_{E,l,\mu}(r, q, \phi, \theta) \right]_{q=1, \phi = \pm \frac{\pi}{3}} = e^{-i(l-2\nu)\pi} \left[ \frac{\partial}{\partial \phi} \Psi_{E,l,\mu}(r, q, \phi, \theta) \right]_{q=1, \phi = -\frac{\pi}{3}},$$  \hspace{1cm} (4.2)

$$\Psi_{E,l,\mu}(r, q = 0, \phi, \theta) = \text{finite}$$  \hspace{1cm} (4.3)

$$\left[ \frac{\partial}{\partial q} \ln \Psi_{E,l,\mu}(r, q, \phi, \theta) \right]_{q=1} = - \left[ \frac{\partial}{\partial q} \ln \Psi_{E,l,\mu}(r, q, -\phi, \theta) \right]_{q=1}. $$  \hspace{1cm} (4.4)

In this section, we look for boundary conditions that lead to a “physically meaningful” 3-anyon problem, i.e. square integrable wave functions, singular at no more than a finite

‡‡These eigensolutions have been investigated using perturbation theory [7,10] and certain numerical methods [11,12], uncovering some of their general properties, most notably the existence of eigenenergies depending nonlinearly on $\nu$, unlike the type-I and type-II eigenenergies.

§§The conditions (4.3) and (4.4) were imposed explicitly in Ref. [6], whereas (4.2) was contained implicitly in a $2\pi/3 \phi$-periodicity condition for the three-fold covering of the configuration space there considered.
number of points, and self-adjoint Hamiltonian \[ H \]. In particular, we check whether Eqs. (4.1)-(4.4) correspond to one of the consistent choices.

Similar analyses \[ 8 \] in the context of the 2-anyon problem have led to interesting results. Most notably, it was found that there is a one-parameter family of self-adjoint extensions of the 2-anyon Hamiltonian, corresponding to scale-dependent boundary conditions, and it has been argued \[ 8,14 \] that this scale-dependence could be important in the description of some condensed matter systems, like the fractional quantum Hall effect, in which anyonic collective modes are believed to be present.

We start by noticing that, in order for the Hamiltonian \( h \) to be self-adjoint, it is sufficient to impose that \( L, M_l, \) and \( h_\mu \) be self-adjoint. \( L \) is obviously self-adjoint on the space spanned by the functions \( \Theta_i \) defined in Eq. (3.12):

\[
\int_0^{2\pi} d\theta \Theta_i(\theta)^* L \Theta_i(\theta) - \int_0^{2\pi} d\theta (L \Theta_i(\theta))^* \Theta_i(\theta) = \frac{1}{i} \Theta_i(\theta)^* \Theta_i(\theta)) \frac{2\pi}{0} = 0 . \quad (4.5)
\]

The analysis of the self-adjointness of \( h_\mu \) can be simplified by observing that, using Eq. (3.13), the matrix elements of \( h_\mu \) (with the appropriate measure \( r^3 \)) between wavefunctions \( R_{E,\mu} \) and \( R'_{E,\mu} \) can be rewritten in terms of matrix elements of \( H_2^{(s)} \) (with measure \( r \)) between wavefunctions \( r R_{E,\mu} \) and \( r R'_{E,\mu} \) specifically

\[
\int dr r^3 R_{E,\mu} h_\mu R'_{E,\mu} = \int dr (r R_{E,\mu}) (r h_\mu r^{-1}) (r R'_{E,\mu}) = \frac{1}{4} \int dr (r R_{E,\mu}) H_2^{(s)} (r R'_{E,\mu}) \quad (4.6)
\]

This observation allows us to use the results obtained in the literature on the self-adjointness of \( H_2^{(s)} \) for our analysis of the self-adjointness of \( h_\mu \); from the results of Ref. \[ 8 \] it follows straightforwardly that \( h_\mu \) is a symmetric operator if \( R(r) \) satisfies

\[
\left[ \cos(\sigma) r^{2+\mu} R(r) + \sin(\sigma) (L_0)^{2+2\mu} \frac{d (r^{2+\mu} R(r))}{d(r^{2+2\mu})} \right]_{r=0} = 0 , \quad (4.7)
\]

where \( \sigma \) is a parameter characterizing the boundary conditions and \( L_0 \) is a reference scale, which breaks scale invariance \[ 8 \] when \( \sigma \neq \frac{2}{2} \cdot \text{integer} \). Note, however, that square integrability requires that Eq. (4.1) be satisfied, and therefore, since Eq. (4.7) implies \[ 8 \] that for \( r \approx 0 \) the wavefunctions behave like

\[
\cos(\sigma) r^\mu - \sin(\sigma) (L_0)^{2+2\mu} r^{-2-\mu} , \quad (4.8)
\]

we find that for \( \mu \geq 0 \), which as shown in the preceding section is the case relevant to the 3-anyon problem, only the scale invariant choice \( \sigma = 0 \) is physically acceptable. In particular, this leads to the following boundary condition to be satisfied by \( R(r) \)

\[
[R(r)]_{r=0} \sim r^\tau \quad \text{with} \quad \tau > 0 , \quad (4.9)
\]

which is more restrictive than the corresponding boundary condition (1.1) adopted in Ref. \[ 8 \].

In relation to the results of Ref. \[ 8 \], it is interesting to observe that, as indicated by Eqs. (2.3), configurations with \( r = 0 \) correspond to “3-anyon collisions”, i.e. maximal overlap of three anyons. As a consequence, our analysis of \( h_\mu \) shows that, unlike in the 2-anyon collisions considered in Ref. \[ 8 \], 3-anyon collisions are not associated with the possibility of a family of self-adjoint extensions of the Hamiltonian. However, based on the results for two anyons obtained in Ref. \[ 8 \] for the 2-anyon problem, in our 3-anyon problem one can
expect such a possibility to arise at least for configurations in which two of the anyons collide leaving the third one as a spectator.

Finally, concerning the self-adjointness of \( M_l \) one has to require

\[
0 = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} d\phi \int_0^1 dq \frac{q}{(1 + q^2)^2} \left[ \Phi_{l,\mu}(q, \phi)^* M_l \Phi_{l,\mu'}(q, \phi) - (M_l \Phi_{l,\mu'}(q, \phi))^* \Phi_{l,\mu}(q, \phi) \right],
\]

which can be easily shown to lead to the requirement

\[
C\left(\frac{\pi}{3}\right) - C(0^+) + C(0^-) - C\left(-\frac{\pi}{3}\right) + B^+(1) + B^-(1) - B^+(0) - B^-(0) = 0,
\]

where

\[
B^\pm(q) = \pm \int_{\phi^\pm}^{\pi/3} d\phi \left[ -\Phi_{l,\mu}(q, \phi)^* \left( q \frac{\partial}{\partial q} \Phi_{l,\mu}(q, \phi) \right) + \left( q \frac{\partial}{\partial q} \Phi_{l,\mu}(q, \phi) \right)^* \Phi_{l,\mu}(q, \phi) \right],
\]

\[
C(\phi) = \int_0^1 dq \frac{1}{q} \left[ -\Phi_{l,\mu}(q, \phi)^* \left( \frac{\partial}{\partial \phi} \Phi_{l,\mu}(q, \phi) \right) + \left( \frac{\partial}{\partial \phi} \Phi_{l,\mu}(q, \phi) \right)^* \Phi_{l,\mu}(q, \phi) \right].
\]

Notice that we have divided the \( \phi \) integration in positive-\( \phi \) and negative-\( \phi \) pieces in order to allow singular contributions at the point \( q = 1, \phi = 0 \), corresponding to 2-anyon collisions.

Without a better understanding of the general structure of the anyon harmonics it is not possible to express the conditions for the self-adjointness of \( M_l \) more explicitly than in Eq. (4.11). We observe, however, that Eq. (4.11) is consistent with the following requirements for the anyon harmonics

\[
\Phi_{l,\mu}(q = 1, \phi) = e^{-i\nu\pi} \Phi_{l,\mu}(q = 1, -\phi) \quad \text{for } 0 < \phi \leq \frac{\pi}{3},
\]

\[
\Phi_{l,\mu}(q, \phi = \frac{\pi}{3}) = e^{-i(l-2\nu)\pi} \Phi(q, \phi = -\frac{\pi}{3}),
\]

\[
\left[ \frac{\partial}{\partial \phi} \ln \Phi_{l,\mu}(q = 1, \phi) \right]_{\phi = \frac{\pi}{3}} = \left[ \frac{\partial}{\partial \phi} \ln \Phi_{l,\mu}(q = 1, \phi) \right]_{\phi = -\frac{\pi}{3}},
\]

\[
\Phi_{l,\mu}(q = 0, \phi) = \text{finite},
\]

\[
\left[ \frac{\partial}{\partial q} \ln \Phi_{l,\mu}(q, \phi) \right]_{q = 1} = - \left[ \frac{\partial}{\partial q} \ln \Phi_{l,\mu}(q, -\phi) \right]_{q = 1} \quad \text{for } 0 < \phi \leq \frac{\pi}{3},
\]

where the conditions (4.13) and (4.14) follow from the statistical boundary conditions (2.12), whereas the conditions (4.13), (4.16), and (4.17) follow from the conditions (4.2)-(4.4) imposed in Ref. [6]. We therefore find that the set of boundary conditions imposed on the anyon harmonics in Ref. [6] is consistent with the self-adjointness of \( M_l \).

The form of Eq. (4.11) appears to leave room for other consistent choices of boundary conditions, as expected based on the experience with the 2-anyon problem. However, unlike the 2-anyon case, it seems difficult that a scale could play a role in such boundary conditions,

***Note, however, that whereas in Ref. [6] the conditions for the wave functions involved a multiple covering of the configuration space, we have given here conditions within a fundamental domain (single covering of the configuration space).
since the $M$-eigenproblem involves only the dimensionless variables $q, \phi$. [N.B.: The scale-dependence of the boundary conditions considered in the 2-anyon problem arises as a result of the fact that one of the relevant variables, the distance between the two anyons, is dimensional.] Perhaps, scale anomalies would only arise if one considered even more complicated boundary conditions, renouncing to the a priori requirement that the boundary conditions be compatible with $[M, L] = [M, \hat{h}] = [L, \hat{h}] = 0$

V. TYPE-I AND TYPE-II ANYON HARMONICS

As anticipated, in this section, in order to give the reader some intuition concerning the structure of the anyon harmonics, we derive the explicit form of the type-I and type-II anyon harmonics, the ones that correspond to the known solutions of the 3-anyon in harmonic potential problem, which can be obtained using energy ladder operators.

We consider the harmonic potential $V = r^2 = u\dddot{u} + v\dddot{v}$. The corresponding 3-anyon (relative motion) Hamiltonian can be written in terms of creation and annihilation operators:

$$h = \sum_{k=1}^{2} (a_k^+ a_k + b_k^+ b_k) + 2,$$

(5.1)

where

$$a_1 = \frac{1}{\sqrt{2}}(\dddot{u} + \frac{\partial}{\partial u}), \quad b_1 = \frac{1}{\sqrt{2}}(u + \frac{\partial}{\partial \dddot{u}}), \quad a_1^+ = \frac{1}{\sqrt{2}}(u - \frac{\partial}{\partial \dddot{u}}), \quad b_1^+ = \frac{1}{\sqrt{2}}(\dddot{u} - \frac{\partial}{\partial u}),$$

$$a_2 = \frac{1}{\sqrt{2}}(\dddot{v} + \frac{\partial}{\partial v}), \quad b_2 = \frac{1}{\sqrt{2}}(v + \frac{\partial}{\partial \dddot{v}}), \quad a_2^+ = \frac{1}{\sqrt{2}}(v - \frac{\partial}{\partial \dddot{v}}), \quad b_2^+ = \frac{1}{\sqrt{2}}(\dddot{v} - \frac{\partial}{\partial v}).$$

(5.2)

The following commutation relations hold:

$$[a_k, a_{k'}^+] = \delta_{kk'}, \quad [b_k, b_{k'}^+] = \delta_{kk'},$$

$$[a_k, b_{k'}] = [b_k, a_{k'}] = [a_k^+, a_{k'}^+] = [b_k^+, b_{k'}^+] = 0.$$  

(5.3)

The energy ladder operators are given by

$$P_{ij} = \sum_{k=1}^{2} (a_k^+)^i (b_k^+)^j, \quad Q_{ij} = \sum_{k=1}^{2} (a_k)^i (b_k)^j.$$  

(5.4)

and in particular $P_{11}$ and $Q_{11}$ close with $h$ on the $SO(2, 1)$ algebra:

$$[h, P_{11}] = 2P_{11}, \quad [h, Q_{11}] = -2Q_{11}, \quad [P_{11}, Q_{11}] = -2h.$$  

(5.5)

Clearly, the energy eigenstates can be organized in representations of this $SO(2, 1)$ algebra, and therefore, it is sufficient to find the bottom state of each representation, $\Psi_0$, which satisfies

$$Q_{11}\Psi_0 = 0, \quad h\Psi_0 = E_0\Psi_0.$$  

(5.6)

The other energy eigenstates can be obtained from the bottom states via the $P_{11}$ ladder operator.

Using the coordinates representation of $Q_{11}$,

$$Q_{11} = -\frac{h}{2} + 1 + \frac{r}{2} \frac{\partial}{\partial r} + r^2,$$  

(5.7)
and the fact that $Q_{11}$ commutes with $L$ and $M_I$

$$[Q_{11}, L] = [Q_{11}, M_I] = 0 ,$$  

one can show that the bottom states have the form

$$\Psi_0 = r^\mu e^{-r^2} e^{-i\theta} \Phi_{\mu,l}(q, \phi) ,$$  

with the energy of the bottom state given by $E_0 = \mu + 2$, corresponding to the Casimir number of the representation.

The analytically known bottom states are of two types. The type-I bottom states can be obtained by applying (compositions of) the operators $P_{20}$, $P_{30}$ and $P_{21}$ to the bottom state

$$\Psi_0^I = (v^3 - u^3)^\nu e^{-u\bar u - v\bar v} = r^{3\nu} e^{-r^2} e^{i3\nu \theta} \frac{1}{(q + q^{-1})^{3\nu/2}} [(qe^{i\phi})^{-\frac{3}{2}} - (qe^{i\phi})^{\frac{3}{2}}]^\nu ,$$  

which has $\mu = l = 3\nu$, and, like all bottom states, $E = 2 + \mu$.

Similarly, the type-II bottom states can be obtained by applying (compositions of) $P_{02}, P_{03}$ and $P_{12}$ to the bottom state

$$\Psi_0^{II} = (u^3 \bar u^3 - v^3 \bar v^3)^{2-\nu} e^{-u\bar u - v\bar v} = r^{6-3\nu} e^{-r^2} e^{i(3\nu - 6)\theta} \frac{1}{(q + q^{-1})^{(6-3\nu)/2}} [(qe^{i\phi})^{-\frac{3}{2}} - (qe^{i\phi})^{\frac{3}{2}}]^{2-\nu} ,$$  

which has $\mu = 6 - 3\nu$, and $l = 6 - \mu$.

We can limit our analysis to the type-I states, since any type-II state can be obtained as the complex conjugate of a corresponding type-I state, in which $\nu$ is substituted by $2 - \nu$ (see, e.g., the relation between $\Psi_0^I$ and $\Psi_0^{II}$).

Let us start by noticing that $\Psi_0^I$ corresponds to the anyon harmonic

$$\Phi_0^I(q, \phi) = \frac{1}{(q + q^{-1})^{3\nu/2}} [(qe^{i\phi})^{-\frac{3}{2}} - (qe^{i\phi})^{\frac{3}{2}}]^\nu .$$  

Next, we apply compositions of the ladder operators $P_{20}$ and $P_{30}$ to $\Psi_0^I$ and obtain the bottom states (up to an unimportant constant)

$$\Psi_{0,N_{20},N_{30}}^I = P_{20}^{N_{20}} P_{30}^{N_{30}} \Psi_0^I = e^{i(3\nu + 2N_{20} + 3N_{30})\theta} r^{3\nu + 2N_{20} + 3N_{30}} e^{-r^2} \Phi_{0,N_{20},N_{30}}^I(q, \phi) ,$$  

where $\Phi_{0,N_{20},N_{30}}^I$ is the anyon harmonic

$$\Phi_{0,N_{20},N_{30}}^I = \frac{1}{(q^{-1} + q)^{3\nu + 2N_{20} + 3N_{30}/2}} \frac{1}{(q + q^{-1})^{3\nu/2}} [(qe^{i\phi})^{-\frac{3}{2}} + (qe^{i\phi})^{\frac{3}{2}}]^{N_{30}} [(qe^{i\phi})^{-\frac{3}{2}} - (qe^{i\phi})^{\frac{3}{2}}]^\nu ,$$  

with $\mu = 3\nu + 2N_{20} + 3N_{30}$ and $l = \mu$.

The construction of the most general type-I bottom state also requires the use of the operator $P_{21}$, which involves a complication associated to the fact that the energy eigenfunctions obtained from $\Psi_{0,N_{20},N_{30}}^I$ via application of $P_{21}$ are not bottom states, they are instead mixtures of eigenstates in different representations. For example, (again up to an unimportant constant)
\[ \Psi^* = P_{21} P_{20}^{N_{20}} P_{30}^{N_{30}} \Psi_0^I = \left\{ (u^2 v + v^2 \bar{u})(u \bar{v})^{N_{20}} (u^3 + \bar{v}^3)^{N_{30}} \right\} - \frac{N_{20}}{2} \frac{u^2 v + v^2 \bar{u}}{u \bar{v}} (u \bar{v})^{N_{20}} (u^3 + \bar{v}^3)^{N_{30}} - \frac{3N_{30}}{2} \frac{u^2 v + v^2 \bar{u}}{u \bar{v}} (u \bar{v})^{N_{20}+2} (u^3 + \bar{v}^3)^{N_{30}+1} \Psi_0^I , \] (5.15)

which is not of the form given in Eq. (5.14) and is not annihilated by \( Q_{11} \). This wavefunction is not homogeneous in the scale \( r \), and contains harmonics with \( \mu = 3r + 1 + 2N_{20} + 3N_{30} \) as well as \( \mu = 3r + 3 + 2N_{20} + 3N_{30} \). To extract a bottom state we first rewrite \( \Psi^* \) as

\[ \Psi^* = e^{i \theta (3r+1+2N_{20}+3N_{30})} e^{-r^2} r^{3r+1+2N_{20}+3N_{30}} (r^2 \Phi^{(1)} - \frac{N_{20}}{2} \Phi^{(2)} - \frac{3N_{30}}{2} \Phi^{(3)}) , \] (5.16)

where

\[
\Phi^{(1)} = \frac{(qe^{i3\phi})^{-\frac{1}{2}} + (qe^{i3\phi})^{\frac{1}{2}}}{(q^{-1} + q)^{\frac{1}{2} \frac{3}{2}}} ((qe^{i\phi})^{-\frac{1}{2}} + (qe^{i\phi})^{\frac{1}{2}})^{N_{30}} ((qe^{i\phi})^{-\frac{1}{2}} - (qe^{i\phi})^{\frac{1}{2}})^{N_{20}} , \\
\Phi^{(2)} = \frac{1}{(q^{-1} + q)^{\frac{1}{2} \frac{3}{2}}} (((qe^{i\phi})^{-\frac{1}{2}} + (qe^{i\phi})^{\frac{1}{2}})^{N_{30}+1} ((qe^{i\phi})^{-\frac{1}{2}} - (qe^{i\phi})^{\frac{1}{2}})^{N_{20}} , \\
\Phi^{(3)} = \frac{1}{(q^{-1} + q)^{\frac{1}{2} \frac{3}{2}}} (((qe^{i\phi})^{-\frac{1}{2}} + (qe^{i\phi})^{\frac{1}{2}})^{N_{30}+1} ((qe^{i\phi})^{-\frac{1}{2}} - (qe^{i\phi})^{\frac{1}{2}})^{N_{20}} . \] (5.17)

\( \Phi^{(2)} \) and \( \Phi^{(3)} \) are harmonics with \( \mu = l = 3r + 1 + 2N_{20} + 3N_{30} \) but \( \Phi^{(1)} \) is not. In fact \( \Psi^* \) is degenerate with the state \( \tilde{\Psi}^* \equiv P_{11} P_{20}^{N_{20}-1} P_{30}^{N_{30}+1} \Psi_0^I \); in order to obtain a bottom state, one has to take an appropriate linear combination of \( \Psi^* \) and \( \tilde{\Psi}^* \). In our example the final result is

\[ \Phi^{I, \mu, N_{20}, N_{30}} = \Phi^{(1)} - \frac{N_{20}}{\mu} \Phi^{(2)} - \frac{6N_{30}}{\mu} \Phi^{(3)} , \] (5.18)

with \( \mu = 3r + 2N_{20} + 3N_{30} + 3 \) and \( l = 3r + 1 + 2N_{20} + 3N_{30} \).

The derivation of bottom states corresponding to repeated application of \( P_{21} \) involves more complicated, but conceptually analogous, problems related to degeneracy. Rather than describing more examples, we simply give the final result in the form of the general formulas for the quantum numbers \( \mu \) and \( \ell \) of the type-I type-II harmonics

\[ \mu = 3r + 2N_{20} + 3N_{30} + 3N_{21} , \quad l = 3r + 2N_{20} + 3N_{30} + N_{21} \quad \text{for the type I}, \]
\[ \mu = 6 - 3r + 2N_{02} + 3N_{03} + 3N_{12} , \quad l = 3r - 2N_{02} - 3N_{03} - N_{12} \quad \text{for the type II} . \] (5.19)

We remark that the quantum numbers \( \mu \) and \( l \) are not sufficient to specify a representation, i.e. there are different bottom states which have same quantum number \( \mu \) and \( l \). We do not know yet what symmetry, if any, causes this degeneracy.

**VI. SERIES EXPANSIONS IN BOSON GAUGE**

In this section, we investigate the anyon harmonics using the technique of series expansions. An important issue in this analysis is the choice between the anyon gauge and the boson gauge. As we discussed in Sec.II, these two formulations of anyons are equivalent; however, depending on the type of calculation to be performed, one of them may require a simpler analysis. For example, in perturbation theory [1] and numerical investigations [2] and [12], the two formulations suggest very different approaches, and greater progress has been made in boson gauge. It is not surprising that the anyon gauge be more troublesome.
when an expansion is involved, because (in exchange for a simpler Hamiltonian) it involves complicated boundary conditions (multivaluedness), which are not easily kept under control in an expansion (it may even happen that finite orders of the expansion do not satisfy the same boundary conditions as the resummed series). We therefore choose to work out our expansion (it may even happen that finite orders of the expansion do not satisfy the complicated boundary conditions (multivaluedness), which are not easily kept under control when an expansion is involved, because (in exchange for a simpler Hamiltonian) it involves)

\[ \Phi_{l,\mu} = \left( \frac{q}{1 + q^2} \right)^{\frac{l}{4}} \left( \bar{z}^{-3/2} - \bar{z}^{3/2} \right)^{1-\nu} F_{l,\mu}(z, \bar{z}), \quad (6.1) \]

where we introduced the notation \( z = q \exp i\phi \) for future convenience.

The **ansatz** (6.1) corresponds to a boson gauge description because the prefactor \( (\bar{z}^{-3/2} - \bar{z}^{3/2})^{-\nu} \) takes care of the \( \nu \)-dependent phase factors in the statistical boundary conditions, so that the statistical boundary conditions simply require that \( (\text{we remind the reader that } l - 3\nu \text{ is integer}) \]

\[ F_{l,\mu}(\bar{z}^{-1}, \bar{z}^{-1}) = -F_{l,\mu}(z, \bar{z}), \quad (6.2) \]
\[ F_{l,\mu}(\eta z, \eta \bar{z}) = \exp[i\pi(l-3\nu)] F_{l,\mu}(z, \bar{z}). \quad (6.3) \]

[Note that here we have extended the fermionic\[†††\] statistical conditions satisfied by \( F_{l,\mu} \) to the entire plane, which amounts to a six-fold covering of the true configuration space. This can be useful at intermediate stages of the calculation \[†††\], but at the end we will only be concerned with our fundamental domain \( (2.11) \).]

The equation of motion of \( F_{l,\mu} \) (i.e. the equation to be satisfied by \( F_{l,\mu} \) in order to have \( M \Phi_{l,\mu} = \mu(\mu + 2)\Phi_{l,\mu} \)) can be simply written in terms of the differential operator \( \mathcal{L} = \mathcal{L}_+ + \mathcal{L}_- \), where

\[ \mathcal{L}_+(z, \bar{z}) = z^{-\frac{1}{2}} \left( z \frac{\partial}{\partial z} + \frac{\mu + 1}{4} \right) \left\{ \bar{z}^{-2} \left( \bar{z} \frac{\partial}{\partial \bar{z}} + \frac{\mu - l - 6(1 - \nu)}{4} \right) - \bar{z} \left( \bar{z} \frac{\partial}{\partial \bar{z}} + \frac{\mu - l + 6(1 - \nu)}{4} \right) \right\} \quad (6.4) \]

and \( \mathcal{L}_- = -\mathcal{L}_+(z^{-1}, \bar{z}^{-1}) \). In fact, the equation of motion of \( F_{l,\mu} \) is

\[ k_{l,\mu}(z, \bar{z}) \equiv \mathcal{L}(z, \bar{z})F_{l,\mu}(z, \bar{z}) = 0. \quad (6.5) \]

From the properties \( (6.2) \) and \( (6.3) \) of \( F_{l,\mu} \), given above, \( \mathcal{L}(\eta z, \eta \bar{z}) = -\mathcal{L}(z, \bar{z}) \), and \( \mathcal{L}(z^{-1}, \bar{z}^{-1}) = -\mathcal{L}(z, \bar{z}) \), one finds that

\[ k_{l,\mu}(\eta z, \eta \bar{z}) = \exp[i\pi(l-3\nu)] k_{l,\mu}(z, \bar{z}), \quad (6.6) \]
\[ k_{l,\mu}(z^{-1}, \bar{z}^{-1}) = k_{l,\mu}(z, \bar{z}). \quad (6.7) \]

\[†††\] We are calling boson gauge any gauge which involves ordinary (bosonic or fermionic) wave functions; however, in the literature sometimes the terminology “fermion gauge” has been used for cases leading to fermionic statistical conditions, like \( (6.2)-(6.3) \). It would be easy to modify the prefactor \( (\bar{z}^{-3/2} - \bar{z}^{3/2})^{-\nu} \) so that \( (6.2)-(6.3) \) be replaced by bosonic statistical conditions; however, we found that starting from fermionic statistical conditions is useful in the study of the fermionic end ground state, on which we shall ultimately concentrate.
Before using these relations in the study of series expansions of the missing anyon harmonics in subsections VI.C and VI.D, we want to illustrate in subsections VI.A and VI.B, how they are satisfied by the type-II anyon harmonics. (As usual, similar results can be analogously obtained for the type-I harmonics.)

### A. Harmonics with $F_{l,\mu} = f(\bar{z})$

If one tries the ansatz $F_{l,\mu} = f(\bar{z})$, then the equation of motion takes the form

\[ 0 = k_{l,\mu}(z, \bar{z}) = f(\bar{z})z^{-\frac{1}{2}}(\frac{\mu + l}{4}) \times \{ \bar{z}^{-2}(\bar{z} \frac{\partial}{\partial \bar{z}} + \frac{\mu - l - 6(1 - \nu)}{4}) - \bar{z}^{-1}(\bar{z} \frac{\partial f(\bar{z})}{\partial \bar{z}} + \frac{\mu - l + 6(1 - \nu)}{4}) \}, \tag{6.8} \]

which is satisfied if

\[ \mu = -l. \tag{6.9} \]

The functional dependence of $f$ on $\bar{z}$ is therefore only constrained by the statistical boundary conditions and the self-adjointness restrictions, and it is easy to verify that

\[ f(\bar{z}) = (\bar{z}^{-\frac{3}{2}} - \bar{z}^\frac{3}{2})(\bar{z}^{-\frac{3}{2}} + \bar{z}^\frac{3}{2})^N_1 \tag{6.10} \]

where $N_1$ is a non-negative integer, is consistent with Eqs.(4.13)-(4.17) if $\mu \geq 3(2 - \nu) + 3N_1$ and $l = 3\nu - 3N_1 - 2N_2 - 6$, where $N_2$ is a non-negative integer. Combining this observation with Eq.(6.9), we realize that these correspond to the type-II harmonics with

\[ \mu = 6 - 3\nu + 3N_1 + 2N_2, \quad l = 3\nu - 3N_1 - 2N_2 - 6. \tag{6.11} \]

### B. Harmonics with $F_{l,\mu} = (z^{-\frac{1}{2}} + z^\frac{1}{2})(z^{-\frac{1}{2}} \bar{z} + z^\frac{1}{2} \bar{z}^{-1})f(\bar{z})$

Let us start by considering

\[ F_{l,\mu} = (z^{-\frac{1}{2}} + z^\frac{1}{2})(z^{-\frac{1}{2}} \bar{z} + z^\frac{1}{2} \bar{z}^{-1})f(\bar{z}). \tag{6.12} \]

In this case, Eq.(6.3) contains a term of order $z^{-1}$ that must vanish for consistency with (6.6)-(6.7); this leads to the requirement

\[ \mu + l - 2 = 0. \tag{6.13} \]

After imposing the vanishing of the $O(z^{-1})$ term, Eq.(6.3) reduces to

\[ 0 = k_{l,\mu}(z, \bar{z}) = \frac{(z^{-\frac{3}{2}} - z^\frac{3}{2})^2}{2}[(z^{-\frac{3}{2}} - z^\frac{3}{2})\bar{z} \frac{\partial f(\bar{z})}{\partial \bar{z}} + (z^{-\frac{3}{2}} + z^\frac{3}{2})\mu - l + 6\nu f(\bar{z})]. \tag{6.14} \]

This equation has a solution
\[ f(\bar{z}) = 1, \quad \mu = l + 16 - 6\nu, \tag{6.15} \]

which, also using (6.13), can be seen to correspond to the type-II harmonic with

\[ \mu = 6 - 3\nu + 3, \quad l = 3\nu - 7. \tag{6.16} \]

Notice that \( f(\bar{z}) = (\bar{z}^{-\frac{3}{2}} - \bar{z}^{\frac{3}{2}})(\bar{z}^{-\frac{3}{2}} + \bar{z}^{\frac{3}{2}})^{N_1} \), which would be suggested by the structure of the statistical boundary conditions and the self-adjointness restrictions, is not a solution of Eq. (6.14). Instead, we have to consider the richer structure

\[
F_{l,\mu} = (\bar{z}^{-\frac{3}{2}} - \bar{z}^{\frac{3}{2}})\{(z^{-\frac{3}{2}}z + z^{\frac{3}{2}})z^{-1})(\bar{z}^{-\frac{3}{2}} + \bar{z}^{\frac{3}{2}})^{N_1} \\
+ (z\bar{z})^{-\frac{3}{2}} + (z\bar{z})^{\frac{3}{2}})(\alpha(z^{-\frac{3}{2}} + z^{\frac{3}{2}})z^{-1} + \beta(\bar{z}^{-\frac{3}{2}} + \bar{z}^{\frac{3}{2}})^{N_1+1}\}.
\tag{6.17}
\]

For this ansatz we find again that the vanishing of the \( O(z^{-1}) \) term in Eq. (6.5) requires (6.13). The rest of Eq. (6.5) is satisfied if \( \alpha = -12N_1/\mu, \beta = -2N_2/\mu, \) and \( \mu = l + 16 - 6\nu + 6N_0 + 4N_02; \) this, when combined with Eq. (6.13), shows that the ansatz (6.17) corresponds to the type-II harmonics with

\[ \mu = 9 - 3\nu + 3N_1 + 2N_2, \quad l = 3\nu - 7 - 3N_1 - 2N_2. \tag{6.18} \]

### C. Series expansion of missing harmonics around \( z = 0 \)

For the type-II harmonics (and the same holds for the type-I harmonics) we have found that they can be associated to a terminating series in powers of \( z \) and \( \bar{z} \); however, this is never the case for the missing harmonics. We now want to investigate this and other features of the expansion of the missing harmonics in powers of \( z \) and \( \bar{z} \). For definiteness we concentrate on the most important missing harmonic, the one that in the fermionic end corresponds to the ground state \([10]\), which we denote with \( F_{3\nu - 3,\mu_0} \), since it has \( l = 3\nu - 3 \). We can write an expansion for \( F_{3\nu - 3,\mu_0} \) as follows

\[
F_{3\nu - 3,\mu_0}(z, \bar{z}) = (z\bar{z})^{-\frac{3}{2} + \Delta} F(z, \bar{z}),
\tag{6.19}
\]

where \( \Delta \) depends only on \( \nu \) and satisfies \( \Delta(\nu = 0) = \Delta(\nu = 1) = 0 \), and \( F(z, \bar{z}) \) involves integer powers of \( z \) and \( \bar{z} \):

\[
F(z, \bar{z}) = \sum_{i,j=-\infty}^{\infty} A_{ij} z^i \bar{z}^j.
\tag{6.20}
\]

In agreement with the fact that, as it follows from Eqs. (6.2)-(6.3),

\[
F_{3\nu - 3,\mu_0}(\eta z, \bar{\eta} \bar{z}) = F_{3\nu - 3,\mu_0}(z, \bar{z}),
\tag{6.21}
\]

\[
F_{3\nu - 3,\mu_0}(z^{-1}, \bar{z}^{-1}) = -F_{3\nu - 3,\mu_0}(z, \bar{z}),
\tag{6.22}
\]

we demand

\[
A_{ij} = 0 \quad \text{when} \quad i - j \neq 0 \mod 3.
\tag{6.23}
\]

The term of lowest order in \( z \) in the equation of motion (6.3) determines the relation between \( \mu_0 \) and \( \Delta \):
\[ \mu_0 = 3 - \nu - 4\Delta . \] (6.24)

Once \( \mu_0 \) is expressed in terms of \( \Delta \) as in (6.24), Eq.(6.5) takes the form
\[ \tilde{\mathcal{L}} F = 0 , \] (6.25)

where \( \tilde{\mathcal{L}} \) is the operator
\[
\tilde{\mathcal{L}}(z, \bar{z}) = (z \bar{z})^{\frac{\nu}{2} - \Delta} \mathcal{L}(z, \bar{z})(z \bar{z})^{-\frac{\nu}{2} + \Delta} \\
= z^{-\frac{1}{2}} \bar{z}^{-2} z \frac{\partial}{\partial z} \{ \bar{z} \frac{\partial}{\partial \bar{z}} - \bar{z}^3 (\bar{z} \frac{\partial}{\partial \bar{z}} - l) \} \\
+ z^{\frac{1}{2}} \bar{z}^{-1} (z \frac{\partial}{\partial z} - \nu + 2\Delta) \{(\bar{z} \frac{\partial}{\partial \bar{z}} + l - \nu + 2\Delta) - \bar{z}^3 (\bar{z} \frac{\partial}{\partial \bar{z}} - \nu + 2\Delta) \} ,
\] (6.26)

and it is easy to verify that this implies
\[ A_{ij} = 0 \quad \text{when} \quad i < 0 \quad \text{and/or} \quad j < 0 , \] (6.27)

whereas the value of \( A_{00} \) is only related to the overall normalization, so we can fix
\[ A_{00} = 1 . \] (6.28)

In fact, Eq.(6.25) implies that all \( A_{ij} \)'s can be determined once the \( A_{3n,0} \)'s, the \( A_{0,3n} \)'s (\( n \) denotes a positive integer), and \( \Delta \) are given; for example, for \( i, j \leq 4 \) the \( A_{ij} \)'s not already fixed by Eqs.(6.23),(6.24),(6.27),(6.28) are given by
\[
A_{11} = (\nu - 2\Delta)(-3 + 2\nu + 2\Delta) \\
A_{22} = -\frac{A_{11}}{4}(1 - \nu + 2\Delta)(-2 + 2\nu + 2\Delta) \\
A_{14} = \frac{A_{03}}{4} (\nu - 2\Delta)(2\nu + 2\Delta) + \frac{1}{4}(\nu - 2\Delta)^2 + \frac{A_{11}}{4}(4 - 3\nu) \\
A_{41} = \frac{A_{30}}{4} (3 - \nu + 2\Delta)(3 - 2\nu - 2\Delta) \\
A_{33} = \frac{A_{22}}{9} (2 - \nu + 2\Delta)(1 - 2\nu - 2\Delta) + A_{30}(1 - \nu) \\
A_{44} = \frac{A_{33}}{16} (3 - \nu + 2\Delta)(2\nu - 2\Delta) + \frac{A_{30}}{16} (3 - \nu + 2\Delta)^2 + \frac{A_{41}}{4}(4 - 3\nu) .
\] (6.29)

The \( A_{3n,0} \)'s, the \( A_{0,3n} \)'s, and \( \Delta(\nu) \) are not determined by Eq.(6.25); they should be fixed so that the statistical boundary conditions and the self-adjointness requirements are satisfied. Since some of these conditions are assigned at \(|z| \equiv q = 1\), they involve all orders in the expansion in powers of \( z \), and therefore cannot be handled analytically. The task is however well suited for numerical analysis.

At this point it is appropriate to compare the results of our boson gauge analysis of the expansion in powers of \( z \) with the ones of the corresponding anyon gauge analysis performed by MMOR in Ref. [6]. Essentially, as appropriate for the anyon gauge, MMOR set up an expansion of the type
\[
\sum_{m=-\infty}^{\infty} C_m e^{-i(2m+\nu+1)} \sum_{s=0}^{\infty} D_{s,m} q^s ,
\] (6.30)
(N.B. To make closer contact with our analysis above, we indicated the formula valid for the fermionic end ground state.)

They were able to prove that the (anyon gauge) equation of motion implies that the coefficients $D_{s,m}$ be such that

$$
\sum_{s=0}^{\infty} D_{s,m} q^s = g_m(q),
$$

(6.31)

where the $g_m$'s are known functions, simply related to hypergeometric functions. $\mu_0$ and the coefficients $C_m$ are instead left undetermined by the equation of motion, and it was found that they are to be fixed by the boundary conditions, using a numerical analysis.

Clearly our boson gauge coefficients $A_{3n,0}A_{0,3n}$ ($0 < n < \infty$) are analogous to the anyon gauge coefficients $C_m$ ($-\infty < m < \infty$), whereas our relations (6.29) are the analogue of (6.31).

The equation of motion is best handled in anyon gauge, as indicated by the comparison of the elegant general formula for $g_m(q)$ given in Ref. [6] with our anyon gauge formulas (6.29). On the other hand, the statistics is simplified in boson gauge; in fact, whereas the $C_m$ are to be determined by the complicated anyon gauge boundary conditions, our $A_{3n,0}A_{0,3n}$ are to be determined by the simpler boson gauge boundary conditions.

Note that from the point of view of numerical analyses the difference between the compact formula for $g_m(q)$ and the formulas (6.29) is not very significant, since they both involve a well defined and finite number of operations. Instead, the difference between the equations to be satisfied by the coefficients $C_m$ and the ones to be satisfied by the coefficients $A_{3n,0}A_{0,3n}$ can be a very important one, since the numerical handling of multivalued functions is a very nontrivial task. Indeed, MMOR pointed out [6] that this multivaluedness can lead to complications associated with the emergence of a singular operator at intermediate stages of the numerical analysis. Such problems are obviously absent in our boson gauge formulation.

**D. On the series expansions around $z = 1$**

Additional information on the structure of the missing eigensolutions might be gained by considering series expansions around $z = 1$. Obviously such expansions would give information complementary to the one in the expansions around $z = 0$.

The natural variable to be used in such studies is $w \equiv \ln z$: an expansion around $z = 1$, is an expansion around $w = 0$, and the statistical conditions on $F_{l,\mu}$ have a very simple form in the $w, \bar{w}$ variables

$$
F_{l,\mu}(-w, -\bar{w}) = -F_{l,\mu}(w, \bar{w}), \quad F_{l,\mu}(w + i2\pi, \bar{w} - i2\pi) = -e^{i\pi(l-3\nu)} F_{l,\mu}(w, \bar{w}).
$$

(6.32)

The operator $L$, when expanded around $w = 0$, takes a rather simple form; for example, to $O(|w|^3)$

$$
L = -6 \frac{\partial}{\partial w} (1 - \nu + \bar{w} \frac{\partial}{\partial \bar{w}}) - 6 \left( -\frac{\mu + l}{2} w \frac{\partial}{\partial w} + \frac{\mu + l}{4} \bar{w} \left( \frac{\mu - l - 2(1 - \nu)}{4} - \frac{1}{2} \bar{w} \frac{\partial}{\partial \bar{w}} \right) + O(|w|^3). \right.
$$

(6.33)

However, it is not easy to keep the statistical conditions under control in the context of an expansion around $w = 0$ ($z = 1$); in fact, as shown by Eq. (6.32), the properties under $P$-transformations (defined in Sec.II) are not preserved by this type of expansion; specifically, $P$-transformations generate terms of any order $w^m$, with $m \leq n$, from a given term of order $w^n$. 18
We shall not pursue this type of expansion further in the present paper; however, we report that, whereas the type-I and type-II harmonics can be simply described in this expansion, we have tried several *ansatzae* for the structure of the missing harmonics in the expansion in powers of $w$ (and $\bar{w}$) without finding any natural candidate to satisfy all necessary statistical conditions and equation of motion.

**VII. SUMMARY AND OUTLOOK**

In our analysis of the self-adjointness, we made considerable progress by exploiting the properties of the MMOR variables $r, \theta, q, \phi$. In the $r$ and $\theta$ sectors we were able to carry out the analysis completely; most notably, in the $r$ sector we found that the self-adjointness requirement, in combination with the fact that $M_l$ is positive semidefinite, lead to even stronger restrictions than the ones imposed by MMOR. The $q, \phi$ (anyon harmonics) sector is extremely complicated and a complete analysis is still beyond reach, but we derived general conditions which allow some preliminary conclusions, and we verified that the boundary conditions imposed by MMOR on the anyon harmonics are consistent with these conditions.

On the important issue of scale anomalies in anyon quantum mechanics, we found that, as long as one insists on boundary conditions compatible with the separation of variables advocated by MMOR, there appears to be no room for scale-dependent boundary conditions; this conclusion was proven rigorously for the $r$ and $\theta$ sectors, and is based on a simple dimensional argument in the $q, \phi$ sector.

We also derived the explicit form of the type-I and type-II anyon harmonics from the corresponding solutions of the harmonic potential problem, and pointed out some structural differences between these and the other types of anyon harmonics within a power expansion in the boson gauge framework.

We proposed to investigate the properties of the missing anyon harmonics by determining numerically the coefficients $A_{3n,0}$, $A_{0,3n}$ that characterize the boson gauge power expansion. As we emphasized in Sec.VI, the information most easily obtainable with these boson gauge numerical techniques should complement the corresponding information obtainable with the anyon gauge numerical techniques considered by MMOR.

We expect that a better understanding, taking into account the statistical boundary conditions, of the properties of the ladder operators $K_\pm$ (introduced in Sec.III) would be very useful in order to to make further progress along the lines followed in the present paper.
REFERENCES

[1] J. M. Leinaas and J. Myrheim, Nuovo Cimento B37 (1977) 1; G. A. Goldin, R. Menikoff, and D. H. Sharp, J. Math. Phys. 21 (1980) 650; 22 (1981) 1664; F. Wilczek, Phys. Rev. Lett. 48 (1982) 1144; 49 (1982) 957. Also see: F. Wilczek, Fractional Statistics and Anyon Superconductivity, (World Scientific, 1990).

[2] Y. S. Wu, Phys. Rev. Lett. 53 (1984) 111; G. V. Dunne, A. Lerda, and C. A. Trugenberger, Mod. Phys. Lett. A6 (1991) 2819; Chihong Chou, Phys. Lett. A155 (1991) 245; A. Polychronakos, Phys.Lett. B264 (1991) 362; G. V. Dunne, A. Lerda, S. Sciuto, and C. A. Trugenberger, Nucl. Phys. B370 (1992) 601; K. H. Cho and C. Rim, Ann. Phys. 213 (1992) 295; S. Mashkevich, Int. J. Mod. Phys. A7 (1992) 7931.

[3] K.H. Cho, Chaiho Rim, and D.S. Soh, Phys. Lett. A164 (1992) 65.

[4] K.H. Cho and Chaiho Rim, Ann. Phys. (N.Y.) 213 (1992) 295.

[5] S.A. Chin and C.-R. Hu, Phys. Rev. Lett. 69 (1992) 229.

[6] S. Mashkevich, J. Myrheim, K. Olaussen, and R. Rietman, Phys. Lett. B348 (1995) 473.

[7] G. Amelino-Camelia, Phys. Lett. B326 (1994) 282; C. Manuel and R. Tarrach, Phys. Lett. B328 (1994) 113.

[8] C. Manuel and R. Tarrach, Phys. Lett. B268 (1991) 222; M. Bourdeau and R.D. Sorkin, Phys. Rev. D45 (1992) 687; G. Amelino-Camelia and D. Bak, Phys. Lett. B343 (1995) 231.

[9] N. Ja. Vilenkin, Special Functions and the Theory of Group Representations, (Am. Math. Soc., Providence, RI, 1968).

[10] A. Khare and J. McCabe, Phys. Lett. B269 (1991) 330; C. Chou, Phys. Rev. D44 (1991) 2533; Erratum-ibid. D45 (1992) 1433; C. Chou, L. Hua, and G. Amelino-Camelia, Phys. Lett. B286 (1992) 329; G. Amelino-Camelia, Phys. Lett. B299 (1992) 83.

[11] M. Sporre, J. J. M. Verbaarschot, and I. Zahed, Phys. Rev. Lett. 67 (1991) 1813; Phys. Rev. B46 (1992) 5738.

[12] M. V. N. Murthy, J. Law, M. Brack, and R. K. Bhaduri, Phys. Rev. Lett. 67 (1991) 1817.

[13] R. Jackiw, in M.A.B. Bèg memorial volume, A. Ali and P. Hoodbhoy eds. (World Scientific, Singapore, 1991).

[14] S. Mashkevich, HEPTh-9511004 (1995).