THE S.V.D. OF THE POISSON KERNEL

GILES AUCHMUTY

Abstract. This paper describes a singular value decomposition (SVD) for the Poisson kernel associated with the Laplacian on bounded regions \( \Omega \) in \( \mathbb{R}^N \), \( N \geq 2 \). The singular functions and singular values are related to certain Steklov eigenvalues and eigenfunctions of the biharmonic operator on \( \Omega \). This eigenproblem and its properties are studied on an appropriate space. This enables a description of the Bergman harmonic projection and an orthonormal basis of the real harmonic Bergman space \( L^2_H(\Omega) \) is found. A reproducing kernel for \( L^2_H(\Omega) \) is constructed and also an orthonormal basis of the space \( L^2(\partial \Omega, d\sigma) \). The Poisson kernel may be regarded as the harmonic extension operator from \( L^2(\partial \Omega, d\sigma) \) to \( L^2_H(\Omega) \) and has an explicit spectral representation that yields the SVD of the Poisson kernel. The singular values of the Poisson kernel are related to the eigenvalues of the DBS eigenproblem. This enables the description of optimal finite rank approximations of the Poisson kernel with error estimates. Explicit spectral formulae for the normal derivatives of eigenfunctions for the Dirichlet Laplacian on \( \partial \Omega \) are found and used to identify a constant in an inequality of Hassell and Tao.

1. Introduction

This paper describes some new representation results for harmonic functions on a bounded region \( \Omega \) in \( \mathbb{R}^N \); \( N \geq 2 \). In particular an explicit description of the Reproducing Kernel for the harmonic Bergman space \( L^2_H(\Omega) \) and the SVD of the Poisson kernel will be obtained. In a earlier paper \([8]\) the author used harmonic Steklov eigenfunctions to represent reproducing kernels on the family \( \mathcal{H}^s(\Omega) \) of harmonic functions. These spaces are modeled on Sobolev spaces and the Steklov eigenfunctions were not, in general, \( L^2 \) orthogonal - though their gradients are.

Here we shall show how similar methods may be used to construct an orthonormal basis of the harmonic Bergman space \( L^2_H(\Omega) \) for a general class of bounded regions. This basis is obtained by constructions involving the eigenfunctions of the Dirichlet Biharmonic Steklov (DBS) eigenproblem. G. Fichera \([19]\) in 1955 showed that the norm of the Poisson kernel as a map of \( L^2(\partial \Omega, d\sigma) \) to \( L^2(\Omega) \) is a function of the smallest eigenvalue of this problem. Here an expression for this harmonic extension operator \( E_H \) in terms of these

Date: April 19, 2016.

The author gratefully acknowledges research support by NSF award DMS 11008754.

2010 Mathematics Subject classification. Primary 46E22, Secondary 35J40, 46E35, 33E20.

Key words and phrases. Reproducing kernel, harmonic Bergman space, biharmonic Steklov eigenfunctions, Poisson kernel, Laplacian eigenfunctions.
DBS eigenfunctions and eigenvalues is found. This eigenfunction expansion provides an SVD for the Poisson kernel that holds under mild boundary regularity requirements.

The results obtained here are found under weaker hypotheses on the boundary ∂Ω of the region than have previously been used for studies of these problems. This is possible due to the use of special Sobolev spaces related to the Laplacian that are introduced in section 3. A crucial result is theorem 3.2 that proves a continuity result for the normal derivative operator \( D_\nu \).

Results about some Steklov eigenproblems related to the Laplacian are described in sections 4 and 5. A nice summary of recent results on the DBS eigenproblem may be found in chapter 3 of Gazzola-Grunau-Sweers [20]. In particular properties of the spectrum were obtained by Ferrero, Gazzola and Weth [18] and properties of the first eigenvalue were studied in Bucur, Ferrero and Gazzola [13]. Here the DBS eigenproblem is studied in the space \( H_0(\Delta, \Omega) \) which may be different to the space used in previous treatments. The results follow from an algorithm to construct an explicit sequence of Dirichlet Biharmonic Steklov (DBS) eigenfunctions that yield an orthonormal basis \( B_H \) of \( L^2_H(\Omega) \). This construction is described in section 5.

In section 6, the Bergman harmonic projection \( \mathbb{P}_H \) of \( L^2(\Omega) \) onto \( L^2_H(\Omega) \) is first defined. Some differences between the harmonic projection on \( H^1(\Omega) \) used in elliptic PDEs and the Bergmann harmonic projection are described. Then the sequence of DBS eigenfunctions is used to construct an orthonormal basis of \( L^2_H(\Omega) \). A representation result for the Bergman harmonic projection \( \mathbb{P}_H \) of \( L^2(\Omega) \) onto \( L^2_H(\Omega) \) in terms of the DBS eigenfunctions is proved as theorem 6.2.

A formula for the reproducing kernel (RK) for \( L^2_H(\Omega) \) follows as corollary 6.3. This RK may be viewed as a Delta function on the class of harmonic functions as \( (6.10) \) holds. The reproducing kernel described here appears to be a spectral representation of a reproducing kernel constructed by J.L. Lions in [24] using control theory methods. Lions showed that there is a reproducing kernel for \( L^2_H(\Omega) \) that is a perturbation of the fundamental solution of the biharmonic operator; the perturbation depending on the region \( \Omega \). Subsequently, Englis et al in [15] and J.L. Lions in [25], have studied the construction of other Reproducing Kernels for various classes of harmonic and other elliptic operators on bounded regions.

The SVD of the classical Poisson integral operator regarded as a linear transformation from \( L^2(\partial\Omega, d\sigma) \) to the harmonic Bergman space \( L^2_H(\Omega) \) is described in section 7. A spectral representation of the harmonic extension operator is first described and shown to be compact. When this operator is regarded as an integral operator then its kernel is the Poisson kernel and we observe that the representation may be regarded as a SVD. The singular vectors are the orthonormal bases \( B_H \) and \( W \) involving DBS eigenvalues and eigenfunctions and the singular values are related to the DBS eigenvalues. Moreover associated finite rank approximations of the Poisson operator have error estimates depending on appropriate DMS eigenvalues. See theorem 7.2 and the error estimates for finite rank approximations of the Poisson operator in theorem 7.3.
In section 8 an explicit formula for the normal derivative of Dirichlet Laplacian eigenfunctions on $\Omega$ is found. This provides a quantification of a constant described by Hassell and Tao [22] for the 2-norms of such eigenfunctions in the case where the domain is a nice bounded region of $\mathbb{R}^N$.

The results here are stated under a weak regularity condition (B2) on the boundary $\partial \Omega$. This condition has been the subject of recent interest as it is related to phenomena that arise in the study of biharmonic boundary value problems. Some comments about these issues may be found in section 2.7 of the monograph of Gazzola, Grunau and Sweers [20] including a description of some apparent ”paradoxes”.

2. Definitions and Notation.

A region is a non-empty, connected, open subset of $\mathbb{R}^N$. Its closure is denoted $\overline{\Omega}$ and its boundary is $\partial \Omega := \overline{\Omega} \setminus \Omega$. A standard assumption about the region is the following.

**B1:** $\Omega$ is a bounded region in $\mathbb{R}^N$ and its boundary $\partial \Omega$ is the union of a finite number of disjoint closed Lipschitz surfaces; each surface having finite surface area.

When this holds there is an outward unit normal $\nu$ defined at $\sigma$ a.e. point of $\partial \Omega$. The definitions and terminology of Evans and Gariepy [17] will be followed except that $\sigma$, $d\sigma$, respectively, will represent Hausdorff $(N-1)$--dimensional measure and integration with respect to this measure. All functions in this paper will take values in $\mathbb{R} := [-\infty, \infty]$ and derivatives should be taken in a weak sense.

The real Lebesgue spaces $L^p(\Omega)$ and $L^p(\partial \Omega, d\sigma)$, $1 \leq p \leq \infty$ are defined in the standard manner and have the usual norms denoted by $\|u\|_p$ and $\|u\|_{p,\partial \Omega}$. When $p = 2$, these spaces will be Hilbert spaces with inner products

$$\langle u, v \rangle := \int_{\Omega} u(x) v(x) \, dx \quad \text{and} \quad \langle u, v \rangle_{\partial \Omega} := |\partial \Omega|^{-1} \int_{\partial \Omega} u \, v \, d\sigma.$$

Let $H^1(\Omega)$ be the usual real Sobolev space of functions on $\Omega$. It is a real Hilbert space under the standard $H^1$--inner product

$$[u, v]_1 := \int_{\Omega} [u(x) v(x) + \nabla u(x) \cdot \nabla v(x)] \, dx.$$  \hspace{1cm} (2.1)

Here $\nabla u$ is the gradient of the function $u$ and the associated norm is denoted $\|u\|_{1,2}$.

The region $\Omega$ is said to satisfy Rellich’s theorem provided the embedding of $H^1(\Omega)$ into $L^p(\Omega)$ is compact for $1 \leq p < p_S$ where $p_S := 2N/(N - 2)$; $N \geq 3$, or $p_S = \infty$ when $N = 2$. There are a number of different criteria on $\Omega$ and $\partial \Omega$ that imply this result. When (B1) holds it is theorem 1 in section 4.6 of [17]; see also Amick [2]. DiBenedetto [14], in theorem 14.1 of chapter 9 shows that the result holds when $\Omega$ is bounded and satisfies a ”cone property”. Adams and Fournier give a thorough treatment of conditions for this result in chapter 6 of [3] and show that it also holds for some classes of unbounded regions.
When (B1) holds, then the trace of a Lipschitz continuous function on $\overline{\Omega}$ to $\partial \Omega$ is continuous and there is a continuous extension of this map to $W^{1,1}(\Omega)$. This linear map $\gamma$ is called the trace on $\partial \Omega$ and each $\gamma(u)$ is Lebesgue integrable with respect to $\sigma$; see [17], section 4.2 for details. In particular, when $\Omega$ satisfies (B1), then the Gauss-Green theorem holds in the form

$$\int_{\Omega} u(x) D_j v(x) \, dx = \int_{\partial \Omega} u v \nu_j \, d\sigma - \int_{\Omega} v(x) D_j u(x) \, dx$$

for $1 \leq j \leq N$. (2.2)

and all $u, v$ in $H^1(\Omega)$. Often, as here, $\gamma$ is omitted in boundary integration.

The region $\Omega$ is said to satisfy a compact trace theorem provided the trace mapping $\gamma : H^1(\Omega) \to L^2(\partial \Omega, d\sigma)$ is compact. Evans and Gariepy [17], section 4.3 show that $\gamma$ is continuous when $\partial \Omega$ satisfies (B1). Theorem 1.5.1.10 of Grisvard [21] proves an inequality that implies the compact trace theorem when $\partial \Omega$ satisfies (B1). This inequality is also proved in [14], chapter 9, section 18 under stronger regularity conditions on the boundary.

We will generally use the following equivalent inner product on $H^1(\Omega)$

$$[u, v]_{\partial} := \int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\partial \Omega} u v \, d\tilde{\sigma}. \quad (2.3)$$

The related norm is denoted $\|u\|_{\partial}$. $\tilde{\sigma}$ is the normalized surface area measure defined by $\tilde{\sigma}(E) := |\partial \Omega|^{-1} \sigma(E)$ where $|\text{bdy}| := \sigma(\partial \Omega)$ is the surface measure of the boundary. The proof that this norm is equivalent to the usual $(1,2)$--norm on $H^1(\Omega)$ when (B1) holds is Corollary 6.2 of [6] and also is part of theorem 21A of [30].

When $F \in L^2(\Omega; \mathbb{R}^N)$ and there is a function $\varphi \in L^2(\Omega)$ satisfying

$$\int_{\Omega} u \varphi \, dx = \int_{\Omega} \nabla u \cdot F \, dx \quad \text{for all } u \in C^1_c(\Omega)$$

then we say that $\text{div} F := \varphi$ is the divergence of $F$. The class of all $L^2$--vector fields on $\Omega$ whose divergence is in $L^2(\Omega)$ is denoted $H(\text{div}, \Omega)$ and is a real Hilbert space with the inner product

$$[F, G]_{\text{div}} := \int_{\Omega} [F \cdot G + \text{div} F \, \text{div} G] \, dx. \quad (2.5)$$

The results described here depend on techniques and results of variational calculus. Relevant notations and definitions are those of Attouch, Buttazzo and Michaille [4].

3. The spaces $H(\Delta, \Omega)$ and $H_0(\Delta, \Omega)$.

Henceforth the region $\Omega$ is assumed to satisfy (B1). Define $H(\Delta, \Omega)$ to be the subspace of all functions $u \in H^1(\Omega)$ with $\nabla u \in H(\text{div}, \Omega)$. Write $\Delta u := \text{div}(\nabla u)$ so $\Delta$ is the usual Laplacian. $H(\Delta, \Omega)$ is a real Hilbert space with the inner product

$$[u, v]_{\partial, \Delta} := [u, v]_{\partial} + \int_{\Omega} \Delta u \Delta v \, dx. \quad (3.1)$$
A function $u \in L^2(\Omega)$ is said to be *harmonic* on $\Omega$ provided

$$\int_{\Omega} u \Delta v \, dx = 0 \quad \text{for all} \quad v \in C^2_c(\Omega). \quad (3.2)$$

Thus a function $u \in H^1(\Omega)$ will be harmonic on $\Omega$ provided

$$\int_{\Omega} \nabla u \cdot \nabla v \, dx = 0 \quad \text{for all} \quad v \in H^1_0(\Omega). \quad (3.3)$$

Let $\mathcal{H}(\Omega)$ be the class of all $H^1$–harmonic functions on $\Omega$. The following result has been used in a variety of ways in some preceding papers, [6] and [7], that study other issues. Here a different statement and a direct proof is provided for completeness.

**Lemma 3.1.** Suppose that $\Omega$ satisfies (B1). Then there are closed subspaces $H^1_0(\Omega)$, $\mathcal{H}(\Omega)$ of $H^1(\Omega)$ and projections $P_0, P_H$ onto these spaces such that

$$u = P_0 u + P_H u \quad \text{for all} \quad u \in H^1(\Omega). \quad (3.4)$$

Moreover $\gamma(u) = \gamma(P_H u)$ and $[P_0 u, P_H u]_\partial = 0$ for all $u \in H^1(\Omega)$.

**Proof.** Given $u \in H^1(\Omega)$, consider the variational problem of minimizing

$$\mathcal{F}(v) := \|v - u\|^2_\partial \quad \text{over} \quad v \in H^1_0(\Omega).$$

This problem has a unique minimizer $u_0 \in H^1_0(\Omega)$ as $\mathcal{F}$ is convex, coercive and continuous on $H^1_0(\Omega)$. Evaluation of the G-derivative of $\mathcal{F}$ implies that the minimizer satisfies

$$D\mathcal{F}(u_0)(v) = 2 \int_{\Omega} \nabla (u_0 - u) \cdot \nabla v \, dx = 0 \quad \text{for all} \quad v \in H^1_0(\Omega).$$

That is $u_h := u - u_0$ is harmonic on $\Omega$. Define $P_0 u = u_0$ and $P_H u = u_h$, these are continuous maps into $H^1_0(\Omega), \mathcal{H}(\Omega)$ respectively. These are projections with closed range from corollary 3.3 of Auchmuty [5]. Since $\gamma(u_0) = 0$ one has $\gamma(u) = \gamma(u_h)$ and the orthogonality follows from the extremality condition above. \[\square\]

This lemma provides a $\partial$–orthogonal decomposition of $H^1(\Omega)$ and the operator $P_H$ defined here is the standard harmonic projection of $H^1$ functions.

Define $H_0(\Delta, \Omega)$ to be the range of $P_0$ when restricted to $H(\Delta, \Omega)$. It is a closed subspace of $H(\Delta, \Omega)$ and the orthogonal decomposition

$$H(\Delta, \Omega) = H_0(\Delta, \Omega) \oplus_{\partial} \mathcal{H}(\Omega) \quad (3.5)$$

holds with respect to the inner product (3.1).

The following theorem shows that when $u \in H_0(\Delta, \Omega)$, the boundary flux $D_\nu u$ has further regularity.

**Theorem 3.2.** Suppose that (B1) holds and $u \in H_0(\Delta, \Omega)$. Then $D_\nu u$ is in $L^2(\partial\Omega, d\sigma)$ and there is a $C_\Omega$ such that $\|D_\nu u\|_2 \leq C_\Omega \|\Delta u\|_2$ for all $u \in H_0(\Delta, \Omega)$. 

Proof. When (B1) holds let $v = v_0 + v_h$ with $v_0 \in H^1_0(\Omega)$ and $v_h \in H(\Omega)$ be a decomposition of $v \in H(\Delta, \Omega)$ as in lemma 3.1. When $u \in H_0(\Delta, \Omega)$, Green’s formula for Sobolev functions on $\Omega$ becomes

$$
\int_{\Omega} [v_h \Delta u - u \Delta v_h] \, dx = \int_{\partial \Omega} \gamma(v_h) D_v u \, d\sigma.
$$

(3.6)

Since $v_h$ is harmonic on $\Omega$ and $\gamma(v_h) = \gamma(v)$, this becomes

$$
|\partial \Omega| |\langle \gamma(v), D_v u \rangle_{\partial \Omega}| \leq \|v_h\|_{2,\Omega} \|\Delta u\|_2 \quad \text{for all} \quad v \in H^1(\Omega).
$$

From theorem 6.3 of [8], $H^{1/2}(\Omega)$ and $L^2(\partial \Omega, d\sigma)$ are isometrically isomorphic, so

$$
\|D_v u\|_{2,\partial \Omega} = \sup_{\|v\|_{2,\partial \Omega} \leq 1} |\langle v, D_v u \rangle_{\partial \Omega}| \leq \sup_{\|v_h\|_{1/2,\Omega} \leq 1} \|v_h\|_{2,\Omega} \|\Delta u\|_2
$$

$$
\leq C_\Omega \|\Delta u\|_2 \quad \text{with} \quad C_\Omega := \sup_{\|v_h\|_{1/2,\Omega} \leq 1} \|v_h\|_{2,\Omega}.
$$

This $C_\Omega$ is finite and attained as the imbedding of $H^{1/2}(\Omega)$ into $L^2(\Omega)$ is compact. \qed

Now consider the inner product on $H_0(\Delta, \Omega)$ defined by

$$
[u, v]_{\Delta} := \int_{\Omega} \Delta u \Delta v \, dx.
$$

(3.7)

The following inequality shows that this generates an equivalent norm to that of $(\partial, \Delta)$.

Lemma 3.3. Suppose that (B1) holds, $u \in H_0(\Delta, \Omega)$ and $\lambda_1$ is the first eigenvalue of the Dirichlet Laplacian on $\Omega$. Then

$$
\|u\|_{\Delta}^2 \leq \|u\|_{\partial,\Delta}^2 \leq (1 + \frac{1}{\lambda_1}) \|u\|_{\Delta}^2 \quad \text{for all} \quad u \in H_0(\Delta, \Omega).
$$

(3.8)

Proof. The first inequality is trivial, while the second follows from the fact that

$$
\int_{\Omega} |\Delta u|^2 \, dx \geq \lambda_1 \int_{\Omega} |\nabla u|^2 \, dx \quad \text{for all} \quad u \in H_0(\Delta, \Omega).
$$

\qed

When $\partial \Omega$ satisfies further regularity conditions, it is well known that $H_0(\Delta, \Omega) = H^2(\Omega) \cap H^1_0(\Omega)$. This is proved in Evans [16] chapter 6, section 6.3.2 when $\partial \Omega$ is $C^2$. Adolfsson [1] has shown that this holds when $\partial \Omega$ is bounded, Lipschitz and satisfies a uniform outer ball condition.

For this paper a slightly stronger assumption than (B1) about the region $\Omega$ is needed namely;

(B2): $\Omega$ is a bounded region with a boundary $\partial \Omega$ for which (B1) holds and $D_v$ is a compact mapping of $H_0(\Delta, \Omega)$ into $L^2(\partial \Omega, d\sigma)$.

This condition has been verified under various regularity conditions on the boundary $\partial \Omega$. Necas [26] chapter 2, theorem 6.2 has shown that (B2) holds when $\Omega$ is Lipschitz and satisfies a uniform outer ball condition. Grisvard [21] chapter 1.5 has a further discussion.
of this. (B2) also holds when each component of the boundary $\partial \Omega$ is a $C^2$-manifold. More literature about this is described in section 2.7 of [20] where some related "paradoxes" are presented.

4. Harmonic Steklov Representations and Boundary Traces.

The methods used here depend on results about boundary traces described in some earlier papers of the author. In particular, the spectral characterization of trace spaces described in Auchmuty [7] and results about spaces of harmonic functions proved in [8] will be used. For convenience some of these results are summarized below. Henceforth $\Omega$ is a bounded region in $\mathbb{R}^N$ satisfying (B2).

A function $s \in H^1(\Omega)$ is said to be a harmonic Steklov eigenfunction provided it is a non-zero solution of

$$\int_{\Omega} \nabla s \cdot \nabla v \, dx = \delta \int_{\partial \Omega} sv \, d\tilde{\sigma} \quad \text{for all} \quad v \in H^1(\Omega). \quad (4.1)$$

When this holds then $\delta$ is the associated Steklov eigenvalue.

Let $S := \{s_j : j \geq 0\}$ be a maximal orthogonal sequence of harmonic Steklov eigenfunctions as described in [7]. Assume that they are normalized so that their boundary traces are $L^2$-orthonormal; $\langle s_j, s_k \rangle_{\partial \Omega} = \delta_{jk}$ for all $j, k$. A function $f \in L^2(\partial \Omega, d\tilde{\sigma})$ has the usual representation

$$f(x) = \sum_{j=0}^{\infty} \hat{f}_j s_j(x) \quad \text{on} \quad \partial \Omega \quad \text{with} \quad \hat{f}_j := \langle f, s_j \rangle_{\partial \Omega} \quad (4.2)$$

with respect to this basis. Here $\hat{f}_j$ is called the j-th Steklov coefficient of $f$ and (3.1) is called the Steklov representation of $f$.

When $f \in L^2(\partial \Omega, d\tilde{\sigma})$ then the function $Ef \in L^2(\Omega)$ defined by

$$Ef(x) := \sum_{j=0}^{\infty} \hat{f}_j s_j(x) \quad (4.3)$$

is a harmonic function on $\Omega$.

This $f$ is defined to be in the trace space $H^s(\partial \Omega)$ with $s \geq 0$ provided its Steklov coefficients satisfy

$$\|f\|_{s,\partial \Omega}^2 := \sum_{j=0}^{\infty} (1 + \delta_j)^{2s} |\hat{f}_j|^2 < \infty. \quad (4.4)$$

Thus $s = 0$ is the usual Lebesgue space $L^2(\partial \Omega, d\sigma)$ from Parseval’s identity. When $f \in H^s(\partial \Omega)$, then $Ef$ is in a space $H^{s+1/2}(\Omega)$ and moreover $E$ is an isometric isomorphism of these spaces. See theorems 6.2 and 6.3 in [5] for full statements and proofs of this.
A linear functional $G$ is in the dual space $H^{-s}(\partial \Omega)$ provided $G$ has Steklov coefficients $\hat{G}_j$ and there is a constant $C$ such that

$$G(f) := \langle G, f \rangle_{\partial \Omega} := \sum_{j=0}^{\infty} \hat{G}_j \hat{f}_j \leq C \|f\|_{s, \partial \Omega}$$

for all $f \in H^s(\partial \Omega).$ (4.5)

This pairing of $H^s(\partial \Omega)$ and $H^{-s}(\partial \Omega)$ extends the usual $L^2-$inner product. That is, functionals in $H^{-s}(\partial \Omega)$ have well-defined Steklov representations so they may be regarded as generalized functions on $\partial \Omega$.

When $u \in H^1(\Omega)$, then its boundary trace $\gamma(u)$ will be in $H^{1/2}(\partial \Omega)$ and its normal derivative $D_\nu u$ is a generalized function in $H^{-1/2}(\partial \Omega)$. Suppose

$$\gamma(u)(x) = \sum_{j=0}^{\infty} \hat{u}_j s_j(x), \quad \text{then} \quad D_\nu u = \sum_{j=1}^{\infty} \delta_j \hat{u}_j s_j.$$ (4.6)

Note that the inner product on $H^{1/2}(\partial \Omega)$ associated with (4.4) is

$$[u, v]_{1/2, \partial \Omega} := \hat{u}_0 \hat{v}_0 + \langle D_\nu u, v \rangle_{\partial \Omega}$$

and this expression is symmetric in $u, v$. When $u \in H_0(\Delta, \Omega)$, then theorem 3.2 implies the last term here is a standard boundary integral as $D_\nu u \in L^2(\partial \Omega, d\sigma)$.

When (B1) holds and $u, v \in H(\Delta, \Omega)$, then a classical Green’s identity becomes

$$\int_\Omega [u \Delta v - v \Delta u] \, dx = |\partial \Omega| [\langle D_\nu v, u \rangle_{\partial \Omega} - \langle D_\nu u, v \rangle_{\partial \Omega}]$$

where the terms on the right hand side are defined by pairings of the form

$$\langle D_\nu u, v \rangle_{\partial \Omega} := \sum_{j=1}^{\infty} \delta_j \hat{u}_j \hat{v}_j$$

that extend standard boundary integrals. As a consequence one sees that

$$\int_\Omega u \Delta v \, dx = \int_\Omega v \Delta u \, dx \quad \text{for all} \quad u, v \in H_0(\Delta, \Omega).$$ (4.10)

5. **Dirichlet Biharmonic Steklov Eigenproblems on $\Omega$.**

In this section we shall describe some properties of solutions of the Dirichlet Biharmonic Steklov (DBS) eigenproblem on a region $\Omega \subset \mathbb{R}^N$ that satisfies (B2). This problem is described in section 5.1 of Kuttler and Sigillito [23] and more recently in section 3.3 of Gazzola, Grunau and Sweers [20]. Note that (B2) is weaker than the requirements on the domain in the analyses of [20] and others.

Our particular aim is to construct an orthonormal basis of $H_0(\Delta, \Omega)$ using the framework of Auchmuty [9]. This involves the solution of a sequence of constrained variational
principles - which are not your standard principles involving Rayleigh quotients and equality constraints.

A function \( b \in H(\Delta, \Omega) \) is said to be (weakly) biharmonic provided
\[
\int_{\Omega} \Delta b \Delta v \, dx = 0 \quad \text{for all} \quad v \in C^2(\Omega). \tag{5.1}
\]
The DBS eigenproblem is to find nontrivial solutions \((q, b) \in \mathbb{R} \times H_0(\Delta, \Omega)\) of the system
\[
\int_{\Omega} \Delta b \Delta v \, dx = q \int_{\partial \Omega} D_\nu b D_\nu v \, d\sigma \quad \text{for all} \quad v \in H_0(\Delta, \Omega). \tag{5.2}
\]
This is a weak version of the problem of finding biharmonic functions \( b \in H_0(\Delta, \Omega) \) that satisfy the boundary conditions
\[
b = \Delta b - q D_\nu b = 0 \quad \text{on} \quad \partial \Omega. \tag{5.3}
\]
Here \( q \) is the DBS eigenvalue and this is a Steklov eigenproblem as the eigenvalue appears only in the boundary condition.

Take \( V = H_0(\Delta, \Omega) \), then (5.2) has the form of the problem studied in Auchmuty \[9\] with the notation,
\[
a(u, v) := \int_{\Omega} \Delta u \Delta v \, dx, \quad m(u, v) := \int_{\partial \Omega} D_\nu u D_\nu v \, d\sigma \quad \text{and} \quad \lambda := q. \tag{5.4}
\]
Functions \( u, v \) in \( H_0(\Delta, \Omega) \) are said to be \( \Delta \)-orthogonal (resp. \( m \)-orthogonal) provided \( a(u, v) = 0 \), \( m(u, v) = 0 \). When \( b_1, b_2 \) are two DBS eigenfunctions satisfying (5.2), then they will be \( \Delta \)-orthogonal if and only if they are \( m \)-orthogonal. First note that if (5.2) has a nontrivial solution \((q, b)\) then by letting \( v = b \) it follows that \( q \geq 0 \). If \( q = 0 \) then \( \Delta b \equiv 0 \) on \( \Omega \) so by uniqueness \( b \equiv 0 \). Thus all DBS eigenvalues must be strictly positive.

To find the smallest DBS eigenvalue, let \( C_1 \) be the closed unit ball in \( H_0(\Delta, \Omega) \) and consider the problem of maximizing
\[
\mathcal{M}(u) := \int_{\partial \Omega} |D_\nu u|^2 \, d\sigma \quad \text{subject to} \quad \|u\|^2_\Delta \leq 1. \tag{5.5}
\]
Define \( \beta_1 := \sup_{u \in C_1} \mathcal{M}(u) \), then the following result generalizes the existence results of theorem 3.17 of [20] and [13]. It requires weaker assumptions on the boundary \( \partial \Omega \) and the solutions are in a different space.

**Theorem 5.1.** Assume that (B2) holds, then there are functions \( \pm b_1 \in C_1 \) that maximize \( \mathcal{M} \) on \( C_1 \). These functions are non-trivial solutions of (5.2) associated with the smallest positive eigenvalue \( q_1 = 1/\beta_1 \) of the DBS eigenproblem and
\[
\int_{\Omega} |\Delta u|^2 \, dx \geq q_1 \int_{\partial \Omega} |D_\nu u|^2 \, d\sigma \quad \text{for all} \quad u \in H_0(\Delta, \Omega). \tag{5.6}
\]
Proof. $C_1$ is weakly compact in, and $\mathcal{M}$ is weakly continuous on $H_0(\Delta, \Omega)$ so $\mathcal{M}$ attains its supremum on $C_1$. If $b_1$ is such a maximizer so also is $-b_1$ as $\mathcal{M}$ is even. Let $I_1(u)$ be the indicator functional of $C_1$, then from part (ii) of Theorem 9.5.5 of Attouch, Buttazzo and Michaille [4] the maximizers are solutions of the inclusion $0 \in D\mathcal{M}(b) + \partial I_1(b)$. Here $D$ is a G-derivative and $\partial$ denotes the subdifferential. In proposition 9.6.1, it is shown that $\partial I_1(u) = \{0\}$ if $\|u\|_\Delta < 1$. Thus if the maximizer occurs at an interior point of $C_1$ then $m(b, v) = 0$ for all $v \in H_0(\Delta, \Omega)$ and the maximum value is 0 which is not true. Thus the maximizer occurs at a $b$ with $\|b\|_\Delta = 1$ and then

$$\partial I_1(b) = \{\mu Da(b, .) : \mu \leq 0\}$$

as in the proof of proposition 9.6.1. of [4]. Thus the maximizers $b$ satisfy

$$\mu a(b, v) = m(b, v) \quad \text{for all} \quad v \in H_0(\Delta, \Omega) \text{ and some} \quad \mu \geq 0. \quad (5.7)$$

Put $v = b$ to see that $\mu$ will be this maximum value $\beta_1$ and then (5.7) shows that a maximizing $b_1$ satisfies (5.2) with $q_1 = \beta_1^{-1}$. Moreover $q_1$ will be the smallest eigenvalue of (5.2) and the inequality (5.6) holds by scaling the constraint. □

Given this first DBS eigenvalue and eigenfunction, a family of successive eigenvalues and eigenfunctions is now constructed sequentially. Suppose that the set $\{q_1, \ldots, q_{k-1}\}$ of (k-1) smallest eigenvalues of (5.2) and a corresponding sequence of $\Delta$-orthonormal eigenfunctions $\{b_1, \ldots, b_{k-1}\}$ has been found. Let $V_k$ be the subspace spanned by this finite set of eigenfunctions.

Define $W_k := \{u \in H_0(\Delta, \Omega) : a(u, b_j) = 0 \text{ for } 1 \leq j \leq k - 1\}, C_k := C_1 \cap W_k$. Consider the problem of maximizing $\mathcal{M}(u)$ on $C_k$ and evaluating $\beta_k := \sup_{u \in C_k} \mathcal{M}(u)$. This problem has maximizers that provide the next smallest eigenvalue and associated normalized eigenfunctions as described next.

**Theorem 5.2.** Assume (B2) holds and the smallest (k-1) DBS eigenvalues $q_j$ are known with an associated family of $\Delta$-orthonormal eigenfunctions $b_j$. When $C_k$ as above, there are functions $\pm b_k \in C_k$ that maximize $\mathcal{M}$ on $C_k$. $b_k$ is a non-trivial solution of (5.2) associated with the next smallest positive eigenvalue $q_k = 1/\beta_k$ of the DBS eigenproblem. Also $a(b_k, b_j) = m(b_k, b_j) = 0$ for all $1 \leq j \leq k - 1$ and $a(b_k, b_k) = 1$.

Proof. For each finite $k$, $C_k$ is non-empty, closed, convex and bounded in $H_0(\Delta, \Omega)$. Hence the weakly continuous functional $\mathcal{M}$ attains a finite maximum $\beta_k$ on $C_k$. $\beta_k > 0$ as there are infinitely many independent functions in $H_0(\Delta, \Omega)$ with $D_v u \neq 0$ on $\partial \Omega$.

Just as in the previous proof, the maximizers must obey $a(b_k, b_k) = 1$ by homogeneity. From lemma 4.1 in [9] and the analysis of section 9.6 of [4], a maximizer of $\mathcal{M}$ on $C_k$ satisfies the equation

$$m(b, v) = a(w, v) + \mu a(b, v) \text{ for some} \ w \in V_k, \mu \geq 0 \text{ and all} \ v \in H_0(\Delta, \Omega). \quad (5.8)$$

Put $v = b_j$ here, then $a(w, b_j)$ as $b_k \in W_k$. Thus $w = 0$ and $b_k$ is a solution of $a(b, v) = q m(b, v)$ for all $v \in H_0(\Delta, \Omega)$. Thus $\pm b_k$ is a solution (5.2) associated with
the next smallest positive eigenvalue \( q_k = 1/\beta_k \) of the DBS eigenproblem. By construction \( a(b_k, b_j) = m(b_k, b_j) = 0 \) for all \( 1 \leq j \leq k - 1 \), so the theorem holds.

This theorem says that there one can find a countable sequence of \( \Delta \)-orthonormal DBS eigenfunctions \( \mathcal{B} := \{ b_k : k \geq 1 \} \) with each eigenfunction maximizing \( M \) on a set \( C_k \) as above. Let \( \mathcal{BH}(\Omega) \) be the subspace of all biharmonic functions in \( H_0(\Delta, \Omega) \). It is closed in view of the definition via (5.1). The following result is an analog of parts of theorem 3.18 in [20]. See also Ferrero, Gazzola and Weth [18].

**Theorem 5.3.** Assume that \( \Omega \) satisfies (B2) and \( \mathcal{B} \) is a sequence of DBS eigenfunctions constructed by the above algorithm. The corresponding DBS eigenvalues \( q_j \) each have finite multiplicity and increase to \( \infty \). \( \mathcal{B} \) is a \( \Delta \)-orthonormal basis of the subspace \( \mathcal{BH}(\Omega) \) of \( H_0(\Delta, \Omega) \).

**Proof.** When \( \mathcal{B} \) is constructed as above, it converges weakly to zero in \( H_0(\Delta, \Omega) \) as it is \( \Delta \)-orthonormal. Thus \( M(b_k) = \beta_k \) converges to zero as it is weakly continuous - or \( q_k \) increases to \( \infty \). Hence each eigenvalue has finite multiplicity.

Let \( V \) be the closed subspace spanned by \( \mathcal{B} \). It will be a subspace of \( \mathcal{BH}(\Omega) \) since each \( b_k \in \mathcal{BH}(\Omega) \). If \( v \in \mathcal{BH}(\Omega) \) is \( \Delta \)-orthogonal to \( V \) and \( M(v) > 0 \), then \( \tilde{v} := v/\sqrt{a(v, v)} \) has \( a(\tilde{v}, \tilde{v}) = 1 \) and \( M(\tilde{v}) > \beta_K \) for some large \( K \). This contradicts the definition of \( \beta_K \), so we must have \( M(v) = 0 \) for all that are \( \Delta \)-orthogonal to \( V \). The uniqueness of solutions of the Dirichlet biharmonic problem on regions obeying (B2) then yields that such a \( v \) must be zero, so \( \mathcal{B} \) is a maximal \( \Delta \)-orthonormal set in \( \mathcal{BH}(\Omega) \) as claimed.

This theorem implies that a biharmonic function \( b \) has the spectral, or eigenfunction, representation

\[
b(x) = \sum_{j=1}^{\infty} \langle b, b_j \rangle \Delta b_j(x) \quad \text{on} \quad \Omega \tag{5.9}\]

that converges in the \( \Delta \)-norm to \( b \) from the basic representation theorem for vectors in a Hilbert space.

Define \( H_{00}(\Delta, \Omega) \) to be the class of all functions in \( H_0(\Delta, \Omega) \) that also have \( D_\nu u = 0 \) on \( \partial \Omega \). Since \( D_\nu \) is a continuous linear map, \( H_{00}(\Delta, \Omega) \) is a closed subspace of \( H_0(\Delta, \Omega) \). The following results lead to an orthogonal decomposition that is analogous to that described in theorem 3.19 of [20] as well as to the decomposition (3.5) given above. Namely

\[
H_0(\Delta, \Omega) = H_{00}(\Delta, \Omega) \oplus_\Delta \mathcal{BH}(\Omega) \tag{5.10}\]

where \( \oplus_\Delta \) indicates the orthogonal complement with respect to the \( \Delta \)-inner product.

To see this, assume \( u \in H_0(\Delta, \Omega) \) and \( D_\nu u = \eta \) on \( \partial \Omega \). Define \( K_\eta \) to be the affine subspace of \( H_0(\Delta, \Omega) \) of functions with \( D_\nu u = \eta \) on \( \partial \Omega \).
Lemma 5.4. When \((B2)\) holds, \(\eta\) as above, there is a unique function \(\tilde{b} \in \mathcal{B}\mathcal{H}(\Omega)\) that minimizes \(\|u\|_\Delta\) on \(K_\eta\). It is a solution of
\[
\int_\Omega \Delta b \Delta v \, dx = 0 \quad \text{for all} \quad v \in H_{00}(\Delta, \Omega). \tag{5.11}
\]

Proof. Let \(A(u) := a(u, u)\) as in (5.4) and consider the problem of minimizing \(A\) on \(K_\eta\). \(A\) is strictly convex, coercive and weakly l.s.c. on \(H_{00}(\Delta, \Omega)\) as it is a norm and thus there is a unique minimizer \(\tilde{b}\) of \(A\) on \(K_\eta\). The extremality condition satisfied by \(\tilde{b}\) is that \(a(\tilde{b}, v) = 0\) for all \(v \in H_{00}(\Delta, \Omega)\) so (5.11) holds. \(\square\)

Suppose that \(P_B : H_0(\Delta, \Omega) \to \mathcal{B}\mathcal{H}(\Omega)\) is the linear map defined by \(P_B u = \tilde{b}\) when \(u \in H_0(\Delta, \Omega)\) has \(D_\nu u = \eta\). Define \(P_{00} u := u - P_B u\), then \(P_{00} u \in H_{00}(\Delta, \Omega)\). These are complementary projections of \(H_0(\Delta, \Omega)\) to itself and (5.10) is an orthogonal decomposition since (5.11) holds.

Section 3.3 of [20] describes the DBS spectrum and eigenfunctions for the unit ball in \(\mathbb{R}^N\) explicitly - and the formula for arbitrary balls may then be found using scaling arguments. It would be of great interest to have further information about these eigenfunctions and eigenvalues for simple two and three dimensional regions. There has been some computation of such eigenvalues starting with the work of Sigillito and Kuttler described in [23].

6. Orthonormal Bases and Reproducing Kernels for \(L^2_H(\Omega)\).

The Bergman space \(L^2_H(\Omega)\) is the space of weakly harmonic functions in \(L^2(\Omega)\) - that is, those that satisfy (3.2). Chapter 8 of Axler, Bourdon and Ramey [10] provides an introduction to Bergman spaces and early results regarding these spaces are described in Bergman [11] and Bergman and Schiffer [12]. In this section the orthogonal projection of \(L^2(\Omega)\) onto \(L^2_H(\Omega)\) and an explicit \(L^2\)-orthonormal basis of \(L^2_H(\Omega)\) will be described. This leads to an explicit formula for the Reproducing Kernel of \(L^2_H(\Omega)\) in terms of the sequence of DBS eigenfunctions generated as in the preceding section.

The orthogonal projection of \(L^2(\Omega)\) onto \(L^2_H(\Omega)\) may be found by looking at a variational principle for the orthogonal complement. Given \(f \in L^2(\Omega)\), consider the minimum norm variational problem of minimizing the functional \(D : H_{00}(\Delta, \Omega) \to \mathbb{R}\) defined by
\[
D(\psi) := \int_\Omega |\Delta \psi - f|^2 \, dx. \tag{6.1}
\]

Lemma 6.1. Suppose that \((B2)\) holds and \(f \in L^2(\Omega)\), then there is a unique minimizer \(\tilde{\psi}\) of \(D\) on \(H_{00}(\Delta, \Omega)\). \(\tilde{\psi}\) satisfies
\[
\int_\Omega (\Delta \tilde{\psi} - f) \Delta \chi \, dx = 0 \quad \text{for all} \quad \chi \in H_{00}(\Delta, \Omega). \tag{6.2}
\]
Proof. As in the proof of lemma 5.4, \( D \) is continuous, strictly convex and coercive on \( H_{00}(\Delta, \Omega) \). Hence there is a unique minimizer of \( D \) on \( H_{00}(\Delta, \Omega) \). \( D \) is G-differentiable and the standard extremality condition implies (6.2). □

The system (6.2) is the weak version of the boundary value problem

\[
\Delta^2 \psi = \Delta f \quad \text{on} \quad \Omega \quad \text{with} \quad \psi = D_n \psi = 0 \quad \text{on} \quad \partial \Omega.
\]

The solution \( \tilde{\psi} \) will be called the \textit{biharmonic potential} of \( f \). Define \( P_W : L^2(\Omega) \to L^2(\Omega) \) by \( P_W f := \Delta \tilde{\psi} \). It is straightforward to verify that \( P_W \) is a projection onto a subspace \( W := \Delta(H_{00}(\Delta, \Omega)) \subset L^2(\Omega) \). The range of this \( P_W \) is closed from corollary 3.3 of [5].

Define \( P_H := I - P_W \) then \( P_H \) is also be a projection on \( L^2(\Omega) \) with closed range that will be called the \textit{Bergman harmonic projection}. The range of \( P_H \) is the class of all functions \( v \in L^2(\Omega) \) that satisfy

\[
\int_{\Omega} v \Delta \chi \, dx = 0 \quad \text{for all} \quad \chi \in H_{00}(\Delta, \Omega).
\]

Thus \( v \) is harmonic on \( \Omega \) as (3.2) holds we have an \( L^2 \)-orthogonal decomposition

\[
L^2(\Omega) = L^2_H(\Omega) \oplus W.
\]

This decomposition is a version of a result attributed to Khavin described in lemma 4.2 of Shapiro [28].

The Bergman harmonic projection \( P_H \) is not the same as the harmonic projection of lemma 3.1 \( P_H f \) the closest harmonic function to \( f \) in the \( \partial \)-norm on \( H^1(\Omega) \) while \( P_H f \) is the closest harmonic function in the \( L^2 \)-norm on \( \Omega \). In particular the standard projection has \( P_H f = 0 \) for all \( f \in H_0(\Delta, \Omega) \) while \( P_H f \) may be non-zero for functions in \( H_0(\Delta, \Omega) \).

Let \( B = \{ b_j : j \geq 1 \} \) be a maximal \( \Delta \)-orthonormal sequence of DBS eigenfunctions constructed using the algorithm of section 5. Define \( h_j := \Delta b_j \) for each \( j \geq 1 \) and \( B_H := \{ h_j : j \geq 1 \} \subset L^2(\Omega) \). The \( h_j \) are harmonic from (5.2) as

\[
\int_{\Omega} h_j \Delta v \, dx = \int_{\Omega} \Delta b_j \Delta v \, dx = 0 \quad \text{for all} \quad v \in C_c^2(\Omega)
\]

so (3.2) holds as \( C_c^2(\Omega) \subset H_{00}(\Delta, \Omega) \). Since these \( b_j \) are \( \Delta \)-orthonormal, the \( h_j \) are \( L^2 \)-orthonormal.

For \( M \geq 1 \), consider the functions \( H_M : \Omega \times \Omega \to \mathbb{R} \) defined by

\[
H_M(x, y) := \sum_{j=1}^{M} h_j(x) h_j(y) = \sum_{j=1}^{M} \Delta b_j(x) \Delta b_j(y). \tag{6.5}
\]

Since each \( h_j \) is \( C^\infty \) on \( \Omega \) from Weyl’s lemma for harmonic functions, \( H_M \) is also \( C^\infty \) and the integral operator \( \mathcal{H}_M : L^2(\Omega) \to L^2(\Omega) \) defined by

\[
\mathcal{H}_M f(x) := \int_{\Omega} H_M(x, y) f(y) \, dy \tag{6.6}
\]
AUCHMUTY

is a finite rank projection of \( L^2(\Omega) \) into \( L^2_H(\Omega) \).

**Theorem 6.2.** Assume (B2) holds and \( B_H \) is constructed as above, then \( B_H \) is a maximal orthonormal set in \( L^2_H(\Omega) \). The orthogonal projection \( \mathbb{P}_H \) of \( L^2(\Omega) \) onto \( L^2_H(\Omega) \) has the representation

\[
\mathbb{P}_H f(x) = \lim_{M \to \infty} \mathcal{H}_M f(x) = \sum_{j=1}^{\infty} \langle f, h_j \rangle h_j(x) \quad \text{for all } f \in L^2(\Omega). \tag{6.7}
\]

**Proof.** The above comments show that \( B_H \) is an orthonormal subset of \( L^2_H(\Omega) \). It remains to show that it is maximal. Suppose not, then there is a \( k \in L^2_H(\Omega) \) with \( \langle k, h_j \rangle = 0 \) for all \( j \geq 1 \). Let \( \tilde{u} \) be the unique solution in \( H^1(\Omega) \) of the equation

\[
\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega k v \, dx \quad \text{for all } v \in H^1_0(\Omega).
\]

This \( \tilde{u} \) exists, is unique and \(-\Delta \tilde{u} = k \in L^2(\Omega)\). Thus \( \tilde{u} \in H_0(\Delta, \Omega) \) and \( \int_\Omega \Delta \tilde{u} \Delta b_j \, dx = 0 \) for all \( j \geq 1 \). So \( k = \Delta \tilde{u} \) is \( \Delta \)-orthogonal to \( L^2_H(\Omega) \) from theorem 5.3. This implies \( k = 0 \) so \( B_H \) is an orthonormal basis of \( L^2_H(\Omega) \). Given that \( B_H \) is an orthonormal basis of \( L^2_H(\Omega) \), the last sentence follows from the Riesz-Fisher theorem. \( \square \)

**Corollary 6.3.** Suppose that (B2) holds, then \( L^2_H(\Omega) \) is a Reproducing Kernel Hilbert (RKH) space with reproducing kernel

\[
R_\Omega(x, y) := \sum_{j=1}^{\infty} \Delta b_j(x) \Delta b_j(y) \quad \text{for } (x, y) \in \Omega \times \Omega. \tag{6.8}
\]

**Proof.** The fact that \( L^2_H(\Omega) \) is a RKH space is classical; see theorem 8.4 of \([10] \). From theorem 6.2 when \( k \in L^2_H(\Omega) \) then

\[
k(x) = \sum_{j=1}^{\infty} \hat{k}_j h_j(x) = \sum_{j=1}^{\infty} \langle k, \Delta b_j \rangle \Delta b_j(x) \quad \text{on } \Omega \tag{6.9}
\]

This series converges in the \( L^2 \)-norm so \( k(x) = \langle R_\Omega(x, .), k \rangle \) when \( x \in \Omega \) and \( R_\Omega \) is defined by (6.8).

\( \square \)

It appears that this expression is an eigenfunction expansion of the very different reproducing kernel found by J.L. Lions in \([24] \). Note in particular that the reproducing kernel \( R_\Omega \) acts as a *Delta function* on the subspace \( L^2_H(\Omega) \) of \( L^2(\Omega) \) in that

\[
k(x) = \int_\Omega R_\Omega(x, y) k(y) \, dy \quad \text{for all } x \in \Omega \quad \text{and} \quad k \in L^2_H(\Omega). \tag{6.10}
\]

When \( b_j \) is a DBS eigenfunction, define \( w_j := \sqrt{q_j |\partial \Omega|} D_\nu b_j \). From theorem 3.2 \( w_j \in L^2(\partial \Omega, d\sigma) \). The boundary conditions on the DBS eigenfunctions imply that

\[
\gamma(h_j) = q_j D_\nu b_j = \sqrt{\frac{q_j}{|\partial \Omega|}} w_j \quad \text{for each } j \geq 1. \tag{6.11}
\]
Let $\mathcal{W} := \{w_j : j \geq 1\}$, then the following result says $\mathcal{W}$ is an orthonormal basis of $L^2(\partial\Omega, d\tilde{\sigma})$.

**Theorem 6.4.** Assume (B2) holds, then $\mathcal{W}$ is a maximal orthonormal set in $L^2(\partial\Omega, d\tilde{\sigma})$.

**Proof.** From the definition (5.2), the DBS eigenfunctions satisfy
$$\langle \Delta b_j, \Delta b_k \rangle = q_j |\partial\Omega| \langle D_\nu b_j, D_\nu b_k \rangle_{\partial\Omega}$$
for all $j,k$. Since the $\Delta b_j$ are orthonormal in $L^2_H(\Omega)$, the $w_j$ are orthonormal in $L^2(\partial\Omega, d\sigma)$.

Suppose these functions are not maximal. Then there is a non-zero $\eta \in L^2(\partial\Omega, d\tilde{\sigma})$ such that $\langle \eta, w_j \rangle_{\partial\Omega} = 0$ for all $j \geq 1$. Let $\tilde{h} = E\eta \in H^{1/2}(\Omega)$ be the harmonic extension of this boundary data defined as in section 4. This function is in $L^2_H(\Omega)$ so $\tilde{h} \neq 0$ and it has a representation of the form (6.7). In this case at least one $\langle \tilde{h}, h_j \rangle \neq 0$. In view of (6.11) this contradicts our assumption, so $\mathcal{W}$ is an orthonormal basis of $L^2(\partial\Omega, d\tilde{\sigma})$. \(\square\)

In view of this result and the fact that $\mathcal{B}_H$ is an orthonormal basis of $L^2_H(\Omega)$, we shall regard the **harmonic trace operator** $\gamma_H$ to be the linear transformation on $L^2_H(\Omega)$ defined by
$$\gamma_H(k) = |\partial\Omega|^{-1/2} \sum_{j=1}^{\infty} \sqrt{q_j} \hat{k}_j w_j \quad \text{when} \quad k = \sum_{j=1}^{\infty} \hat{k}_j h_j \in L^2_H(\Omega). \quad (6.12)$$

From (6.11) this is an unbounded map into $L^2(\partial\Omega, d\sigma)$. In Auchmuty [8], it was shown that the harmonic trace operator defines an isometric isomorphism between $L^2(\Omega)$ and a space that was denoted $H^{-1/2}(\partial\Omega)$ and characterized as the dual space of $H^{1/2}(\partial\Omega)$ with respect to the usual inner product on $L^2(\partial\Omega, d\tilde{\sigma})$.

### 7. SVD Representation of the Poisson Kernel

The preceding analysis of the Dirichlet Biharmonic Steklov eigenfunctions showed that they generate orthonormal bases of the spaces $\mathcal{B}_H(\Omega), L^2_H(\Omega)$ and $L^2(\partial\Omega, d\tilde{\sigma})$. A closer investigation shows that they also provide a **singular value decomposition** (SVD) of the usual Poisson integral operator for solving the Dirichlet problem for harmonic functions.

Given a function $g \in L^2(\partial\Omega, d\sigma)$, consider the harmonic extension problem of a finding $\tilde{u} := E_H g \in L^2_H(\Omega)$ satisfying $\gamma_H(\tilde{u}) = g$. Then $E_H : L^2(\partial\Omega, d\sigma) \rightarrow L^2_H(\Omega)$ is a right inverse of $\gamma_H$ and (6.11) shows that, when $g = \sum_{j=1}^{\infty} \hat{g}_j w_j \in L^2(\partial\Omega, d\tilde{\sigma})$, then
$$E_H g(x) = \sqrt{|\partial\Omega|} \sum_{j=1}^{\infty} \frac{\hat{g}_j}{\sqrt{q_j}} h_j(x). \quad (7.1)$$

**Theorem 7.1.** Assume that (B2) holds and $E_H : L^2(\partial\Omega, d\sigma) \rightarrow L^2_H(\Omega)$ is defined by (7.1). Then $E_H$ is an injective compact linear transformation with $\|E_H\| = \frac{1}{\sqrt{\eta}}$. 

Proof. From (7.1) and orthonormality, Parseval’s equality yields
\[ \| E_H g \|^2 = |\partial \Omega| \sum_{j=1}^{\infty} \frac{1}{q_j} q_j^2 \leq |\partial \Omega| \| g \|^2_{\partial \Omega}. \]
so \( E_H \) is continuous with norm as in the theorem. \( E_H \) is obviously injective. It is compact as the \( q_j \) increase to \( \infty \). \( \square \)

This formula for the norm of \( E_H \) is well-known when the boundary obeys stronger boundary regularity conditions. It was described in Fichera [19] and is equation 3.1 in [29]. Note that (7.1) is essentially a SVD of this harmonic extension operator as it maps one orthonormal basis to another. The singular values of the operator are simple functions of the eigenvalues of the Dirichlet Biharmonic Steklov problem.

Classically this operator has usually been described in terms of the Poisson kernel - see section 2.2 of Evans [16] or most other PDE texts. The solution of this boundary value problem is described in terms of a function \( P : \Omega \times \partial \Omega \rightarrow [0, \infty] \) such that
\[ E_H g(x) := \int_{\partial \Omega} P(x, z) g(z) \, d\sigma. \] 
(7.2)

A comparison of (7.2) and (7.1) leads to the following singular value decomposition of the Poisson kernel as a function on \( \Omega \times \partial \Omega \).

**Theorem 7.2.** Assume that (B2) holds, then the Poisson kernel \( P(x, z) \) has the singular value representation
\[ P(x, z) = |\partial \Omega|^{-1/2} \sum_{j=1}^{\infty} \frac{1}{\sqrt{q_j}} h_j(x) w_j(z) = \sum_{j=1}^{\infty} \frac{1}{q_j} \Delta b_j(x) \Delta b_j(z) \] 
(7.3)
for \((x, z) \in \Omega \times \partial \Omega \).

**Proof.** The formulae in (7.3) hold by comparing (7.2) and (7.1) and using the definitions and properties of the various functions. \( \square \)

The singular value decomposition of theorem 7.1 leads to explicit formulae for the best rank \( M \) approximations of the Poisson operator. For finite \( M \), define \( P_M \) and \( E_M \) by
\[ P_M(x, z) := \sum_{j=1}^{M} \frac{h_j(x)}{\sqrt{|\partial \Omega| q_j}} w_j(z) \quad \text{and} \quad E_M g(x) := \int_{\partial \Omega} P_M(x, z) g(z) \, d\sigma. \] 
(7.4)

Then for each \( z \in \partial \Omega \), \( P_M(\cdot, z) \) is a harmonic function on \( \Omega \) and for each \( x \in \Omega \), \( P_M(x, \cdot) \) is an \( L^2 \)-function on \( \partial \Omega \) with
\[ \| P_M(\cdot, \cdot) \|^2_{\partial \Omega} = \sum_{j=1}^{M} \frac{|h_j(x)|^2}{|\partial \Omega| q_j} \quad \text{and} \quad \int_{\Omega} \int_{\partial \Omega} |P_M(x, z)|^2 \, d\sigma \, dx = \sum_{j=1}^{M} \frac{1}{q_j}. \]

These formulae lead to the following approximation result for the Poisson operator.
Theorem 7.3. Assume that (B2) holds, $E_H$ is the harmonic extension operator and $P_M, E_M$ are defined by (7.2), then

$$\| E_H g - E_M g \| \leq \sqrt{\frac{|\partial \Omega|}{q_{M+1}}} \| g - g_M \|_{\partial \Omega} \quad \text{for all } g \in L^2(\partial \Omega, d\sigma) \quad (7.5)$$

Here $q_{M+1}$ is the $(M+1)$-st DBS eigenvalue and $g_M := \sum_{j=1}^{M} \hat{g}_j w_j$.

Proof. From (7.2) and (7.4), one sees that

$$E_H g(x) - E_M g(x) = |\partial \Omega|^{1/2} \sum_{j=M+1}^{\infty} \frac{\hat{g}_j}{\sqrt{q_j}} h_j(x), \quad \text{so}$$

$$\| E_H g - E_M g \|^2 = |\partial \Omega| \sum_{j=M+1}^{\infty} \frac{1}{q_j} \hat{g}_j^2 \leq \frac{|\partial \Omega|}{q_{M+1}} \| g - g_M \|^2_{\partial \Omega}$$

so (7.5) holds as claimed. \qed

8. Normal Derivatives of Laplacian Eigenfunctions

Theorem 3.2 provided an estimate of the normal derivative of functions in $H_0(\Delta, \Omega)$. Here a result about the normal derivatives of eigenfunctions of the Dirichlet Laplacian will be proved. This quantifies part of theorem 1.1 of Hassell and Tao [22] that answered a question of Ozawa [27].

A non-zero function $e \in H_0(\Delta, \Omega)$ is said to be a Dirichlet eigenfunction of the Laplacian on $\Omega$ corresponding to an eigenvalue $\lambda$ provided

$$\int_{\Omega} [ \nabla e \cdot \nabla v - \lambda e v ] \, dx = 0 \quad \text{for all } v \in H_0^1(\Omega). \quad (8.1)$$

The eigenfunction is normalized if $\| e \| = 1$. Note that theorem 3.2 already provides a generic upper bound for the constant in the inequality 1.1 of [22]. Here an explicit representation for the normal derivatives of Dirichlet eigenfunctions will be derived that shows this constant may be bounded in terms of the first DBS eigenvalue $q_1$.

Theorem 8.1. Suppose (B2) holds and $e \in H_0(\Delta, \Omega)$ is a normalized Dirichlet eigenfunction of the Laplacian on $\Omega$ with eigenvalue $\lambda$. Then

$$\int_{\partial \Omega} |D_e e|^2 \, d\sigma \leq \frac{\| P_H e \|^2}{q_1} \lambda^2 \leq \frac{1}{q_1} \lambda^2 \quad (8.2)$$

with $P_H$ the Bergman harmonic projection.
Proof. When \( \psi \) is the biharmonic potential of \( e \) then, from (6.4), one has \( e = \Delta \psi + e_H \) with \( \psi \in H_{00}(\Delta, \Omega) \) and \( e_H = \mathbb{P}_He \in L^2_H(\Omega) \). From theorem 6.2 \( e_H = \sum_{j=1}^{\infty} \hat{e}_j \Delta b_j \), so the eigenvalue equation yields that

\[
\Delta \left( \lambda^{-1} e + \psi + \sum_{j=1}^{\infty} \hat{e}_j b_j \right) = 0 \quad \text{on} \quad \Omega.
\]

The function here is also zero on \( \partial \Omega \), so

\[
e(x) \equiv -\lambda \left[ \psi + \sum_{j=1}^{\infty} \hat{e}_j b_j \right] \quad \text{on} \quad \Omega. \tag{8.3}
\]

Thus \( D_\nu e = -\lambda \sum_{j=1}^{\infty} \hat{e}_j D_\nu b_j \) on \( \partial \Omega \) as \( \psi \in H_{00}(\Delta, \Omega) \). The definition of \( w_j \) implies that

\[
D_\nu e(x) = -\lambda \sum_{j=1}^{\infty} \frac{\hat{e}_j}{\sqrt{|\partial \Omega|} q_j} w_j(x) \quad \text{on} \quad \partial \Omega. \tag{8.4}
\]

Since \( q_1 \) is the least DBS eigenvalue, (8.2) now follows from Parseval’s equality as \( \| \mathbb{P}_He \|^2 = \sum_{j=1}^{\infty} \hat{e}_j^2 \) and \( \| \mathbb{P}_H \| = 1 \). \( \square \)

In particular this result shows that the constant \( C \) in the Hassell-Tao inequality for nice bounded regions in \( \mathbb{R}^N \) has \( C \leq 1/\sqrt{q_1} \).

These eigenfunctions illustrate a difference between the Steklov harmonic projection which has \( P_H e = 0 \) for any Dirichlet eigenfunction, and the Bergman harmonic projection \( \mathbb{P}_H \) which must have \( \mathbb{P}_H e \neq 0 \).

References

[1] V. Adolfsson, "\( L^2 \)--integrability of second-order derivatives for Poisson’s equations in non-smooth domains", Math Scand, 70 (1992), 146-160.
[2] C. J. Amick, "Some remarks on Rellich’s theorem and the Poincare inequality", J. London Math Soc (2) 18 (1973), 81-93.
[3] R.A. Adams and J.J.F. Fournier, Sobolev spaces, 2nd ed., Academic Press, 2003.
[4] H. Attouch, G. Buttazzo and G. Michaille, Variational Analysis in Sobolev and BV Spaces SIAM, Philadelphia, (2006).
[5] G. Auchmuty, "Orthogonal Decompositions and Bases for 3-d Vector Fields", Numerical Functional Analysis and Optimization, 15 (1994), 455-488.
[6] G. Auchmuty, "Steklov Eigenproblems and the Representation of Solutions of Elliptic Boundary Value Problems", Numerical Functional Analysis and Optimization, 25 (2004) 321-348.
[7] G. Auchmuty, "Spectral Characterization of the Trace Spaces \( H^s(\partial \Omega) \)”, SIAM J of Mathematical Analysis, 38 (2006), 894-907.
[8] G. Auchmuty, "Reproducing Kernels for Hilbert Spaces of Real Harmonic Functions", SIAM J Math Anal, 41, (2009), 1994-2001.
[9] G. Auchmuty, "Bases and Comparison Results for Linear Elliptic Eigenproblems ", J. Math Anal Appns, 390 (2012), 394-406.
[10] S. Axler, P. Bourdon and W. Ramey, Harmonic Function Theory, 2nd ed., Springer-Verlag, New York, 2001.
[11] S. Bergman, The Kernel Function and Conformal Mapping, AMS Math Surveys, 5, (1970).
[12] S. Bergman and M. Schiffer, Kernel Functions and Elliptic Differential Equations in Mathematical Physics, Academic Press, New York (1953).
[13] D. Bucur, A. Ferrero and F. Gazzola, "On the first eigenvalue of a fourth order Steklov eigenproblem", Cal. Vars and Partial Diff. Eqns, 35, (2009), 103-131.
[14] E. DiBenedetto, Real Analysis, Birkhauser, Boston, (2001).
[15] M. Englis, D. Lukkassen, J. Peetre and L-E Persson, "The last Formula of Jacques-Louis Lions: Reproducing kernels for Harmonic and other functions", J. fur Reine und Angewandte Mathematik,
[16] L.C. Evans, Partial Differential Equations, American Math. Society, Providence, (2000).
[17] L. C. Evans and R. F. Gariepy, Measure Theory and Fine Properties of Functions, CRC Press, Boca Raton (1992).
[18] A. Ferrero, F. Gazzola and T. Weth, "On a Fourth Order Steklov eigenvalue problem", Analysis (Munich) 25, (2005), 315-332.
[19] G. Fichera, "Su un principio di dualita per talune formole di maggiorazione relaive alle equazioni differenziali", Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. 19 (1955), 411-418.
[20] F. Gazzola, H-C Grunau and G. Sweers, Polyharmonic Boundary Value Problems Springer-Verlag, Berlin, (2010).
[21] P. Grisvard, Elliptic problems in non-smooth Domains, Pitman, Boston, (1985).
[22] A. Hassell and T. Tao, "Upper and Lower Bounds for Normal Derivatives of Dirichlet Eigenfunctions" Mathematical Research Letters, 9, (2002) 289-305.
[23] J.R. Kuttler and V.G. Sigillito, Estimating eigenvalues with a posteriori/a priori inequalities, Pitman Research Notes in Mathematics, Vol 135, Pitman, Boston, (1985).
[24] J.L. Lions, "Noyau Reproduisant et Systeme d’Optimalite”, in Aspects of Mathematics and Applications, ed J.A. Barroso, Elsevier (1986), 573-582.
[25] J.L. Lions, "Remarks on Reproducing Kernels of some Function Spaces”, in Function Spaces, Interpolation theory and Related Topics, ed Kufner, Cwikel, Englis Persson and Spar, de Gruyter (2002), 49-59.
[26] J. Necas, Les Methodes directes en Theorie des equations elliptiques, Masson, Paris (1967).
[27] S. Ozawa, Asymptotic Property of eigenfunction of the Laplacian at the boundary, Osaka J. Math., 30, (1993) 303-314.
[28] H.S. Shapiro, The Schwarz Function and its Generalization to Higher Dimensions Wiley-Interscience, New York (1992).
[29] V.G. Sigillito, Explicit a priori inequalities with applications to boundary value problems, Pitman Research Notes in Mathematics, Vol 13, Pitman, London, (1977).
[30] E. Zeidler, Nonlinear Functional Analysis and its Applications, II/A: Linear Monotone Methods. Springer Verlag, New York (1990).

Department of Mathematics, University of Houston, Houston, TX 77204-3008 USA
E-mail address: auchmuty@uh.edu