Research Article

New Operators of Cubic Picture Fuzzy Information with Applications

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Received 13 March 2021; Revised 12 April 2021; Accepted 20 April 2021; Published 20 May 2021

Academic Editor: Naeem Jan

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The researcher has been facing problems while handling imprecise and vague information, i.e., the problems of networking, decision-making, etc. For encountering such complicated data, the notion of fuzzy sets (FS) has been considered an influential tool. The notion was extended to its generalizations by a number of researchers in different ways which helps to understand and assess even more complex issues. This article characterizes imprecision with four kinds of values of membership. In this work, we aim to define and examine cubic picture fuzzy sets and give an application on averaging aggregation operators. We first introduce the notion of a cubic picture fuzzy set, which is a pair of interval-valued picture fuzzy set and a picture fuzzy set by giving examples. Then, we define two kinds of ordering on these sets and also discuss some set-theoretical properties. Moreover, we introduce three kinds of averaging aggregation operators based on cubic picture fuzzy sets and, at the end, we illustrate the results with a decision-making problem by using one of the provided aggregation operators.

1. Introduction

In 1965, Zadeh generalized the classical set and perceived the idea of fuzzy sets [1] to deal with uncertainty. This idea allows creating some new dimensions in the field of research and has been applied in many fields such as decision-making, medical diagnosis, and pattern recognition [1–6]. But in fuzzy set only the membership degree is considered. The limitation of fuzzy sets is that the nonmembership degree cannot be defined independently. To overcome this limitation, several extensions have been made by many researchers such as interval-valued fuzzy sets [7], intuitionistic fuzzy sets (IFSs) [8], cubic sets [9], and neutrosophic sets [10]. Among these various extensions of fuzzy sets, cubic set is one of the most prominent extensions. Jun [9] presented the idea of cubic sets in terms of interval-valued fuzzy set and fuzzy set in 2012. The very basic properties of cubic sets were studied, and some useful operations were defined successfully in his paper. Khir et al. [11] presented the idea of fuzzy sets and fuzzy logic and their application. Later on, the idea of cubic sets was applied to various fields by many authors (see [12–17]).

In recent years, the notion of fuzzy sets was further generalized by Couung et al. and they proposed the concept of picture fuzzy sets [18, 19], and this idea gained more and more attention from the researchers. Several similarity measures, correlation coefficients, and entropy measures for picture fuzzy sets were defined by many authors and they applied these sets in various fields (see [20–28]). Recently, Couung et al. [29] have extended the picture fuzzy sets to the interval-valued picture fuzzy sets. For some works on picture fuzzy sets and several types of aggregation operators, we refer the reader to [24, 25, 30–35].

Inspiring from the above study, we propose the concept of cubic picture fuzzy sets, which is an extension of cubic sets, picture fuzzy sets, and interval-valued picture fuzzy sets.

The rest of the paper is organized as follows. In Section 2, some basic definitions and results which are necessary for the main sections are discussed. In Section 3, the concept of cubic picture fuzzy sets which is a mixture of an interval-valued
picture fuzzy set and a picture fuzzy set is introduced, and some basic operations on these sets were defined by giving several examples. Then the related theorems are studied. In Section 4, three types of aggregation operators in the environment of cubic picture fuzzy sets are discussed and, finally, one of them is applied in decision-making problem in the last section.

2. Preliminaries

Definition 1 (see [1]). Let $\tilde{S}$ be a nonempty set. Then $U = \{\tilde{s}, \mathcal{U}_U(\tilde{s}) | \tilde{s} \in S\}$ is called a fuzzy set, where $\mathcal{U}_U$ is a membership function that maps each element of $S$ in $[0, 1]$. Here we say that $U$ is a fuzzy subset of $S$.

Definition 2 (see [8]). Consider a closed interval $[\mathcal{U}_U, \mathcal{U}_U^*]$ of $I$ where $I = [0, 1]$ is called an interval number, where $0 \leq \mathcal{U}_U \leq \mathcal{U}_U^*$. The set of all interval numbers is denoted by $[I]$. A function $\beta : S \rightarrow [I]$ is said to be an interval-valued fuzzy (IVF) set of $S$. The set of all IVF sets of $S$ is denoted by $[I]_S$. For each $\mathcal{U}_U \in [I]_S$ and $\tilde{s} \in S$, $\mathcal{U}_U(\tilde{s}) = [\mathcal{U}_U(\tilde{s}), \mathcal{U}_U^*(\tilde{s})]$ is called the degree of membership of an element $\tilde{s}$ to $\mathcal{U}_U$ in this case $U^*_S : S \rightarrow I$ and $U^*_S : S \rightarrow I$ are fuzzy subsets of $S$; these sets are known as lower fuzzy set and upper fuzzy subset of $S$, respectively.

Definition 3 (see [9]). The cubic set of a nonempty set $\tilde{S}$ is defined as follows: $\tilde{I} = \{\tilde{s}, \tilde{A}(\tilde{s}), \tilde{B}(\tilde{s}) | \tilde{s} \in \tilde{S}\}$, where $\tilde{A}$ is an interval-valued fuzzy (IVF) set of $\tilde{S}$ and $\tilde{A}$ is a fuzzy subset of $\tilde{S}$. A cubic set is simply denoted by $\tilde{I} = \langle \tilde{A}, \tilde{B} \rangle$.

Definition 4 (see [9]). A cubic set $\tilde{I} = \langle \tilde{A}, \tilde{B} \rangle$ is known to be

1. an internal cubic set (briefly, ICS), if $\tilde{A}(\tilde{s}) \leq \tilde{B}(\tilde{s}) \leq \tilde{A}(\tilde{s})$, $\forall \tilde{s} \in \tilde{S}$

2. an external cubic set, (briefly, ECS), if $\tilde{A}(\tilde{s}) \notin (\tilde{A}(\tilde{s}), \tilde{A}^+(\tilde{s}))$, $\forall \tilde{s} \in \tilde{S}$

Example 1. If $\tilde{A}$ is an IVF set of $\tilde{S}$, then $U = \{\tilde{s}, \tilde{A}(\tilde{s}), 1(\tilde{s}) | \tilde{s} \in \tilde{S}\}$, $V = \{\tilde{s}, \tilde{A}(\tilde{s}), 0(\tilde{s}) | \tilde{s} \in \tilde{S}\}$ and $C = \{\tilde{s}, \tilde{A}(\tilde{s}), \tilde{B}(\tilde{s}) \} | \tilde{s} \in \tilde{S}\}$, where $\tilde{A}^+(\tilde{s}) = (\tilde{A}(\tilde{s}) + \tilde{A}^+(\tilde{s})) / 2$ are cubic sets of $\tilde{S}$.

Example 2. Let $\tilde{U} = \{\tilde{s}, \tilde{A}(\tilde{s}), 1(\tilde{s}) | \tilde{s} \in \tilde{S}\}$ be a cubic set of $\tilde{S}$ and $\tilde{A}(\tilde{s}) = [0.3, 0.7]$ and $\tilde{B}(\tilde{s}) = 0.4$, for each $\tilde{s} \in \tilde{S}$. Then $\tilde{U}$ is an ICS. If $\tilde{A}(\tilde{s}) = [0.3, 0.7]$ and $\tilde{B}(\tilde{s}) = 0.8$, for each $\tilde{s} \in \tilde{S}$, then $\tilde{U}$ is an ECS. If $\tilde{A}(\tilde{s}) = [0.3, 0.7]$ and $\tilde{B}(\tilde{s}) = \tilde{s}$, for each $\tilde{s} \in \tilde{S}$, then $\tilde{U}$ is neither an ICS nor an ECS.

Definition 5 (see [9]). Let $\tilde{S}$ be a nonempty set and let $\tilde{U} = \langle \tilde{A}, \tilde{B} \rangle$ and $\tilde{V} = \langle I, K \rangle$ be two cubic sets of $\tilde{S}$. Then the orderings are defined in the following way:

1. (Equality) $\tilde{U} = \tilde{V}$ if $\tilde{A} = \tilde{I}$ and $\tilde{B} = \tilde{K}$

2. (P-Order) $\tilde{U} \subseteq \tilde{V} \Rightarrow \tilde{A} \subseteq I$ and $\tilde{B} \leq K$

3. (R-Order) $\tilde{U} \subseteq \tilde{V} \Rightarrow \tilde{A} \subseteq I$ and $\tilde{B} \geq K$

Definition 6 (see [9]). For arbitrary indexed family of cubic sets $\tilde{U}_i = \{\tilde{s}, \tilde{A}_i(\tilde{s}), \tilde{B}_i(\tilde{s}) \} | \tilde{s} \in S\}$, where $i \in \lambda$, we define the $P$-union, $P$-intersection, R-union, and R-intersection as follows:

1. (P-Union) $\cap_{i \in \lambda} \tilde{U}_i = \{\tilde{s}, \cap_{i \in \lambda} \tilde{A}_i(\tilde{s}), \cap_{i \in \lambda} \tilde{B}_i(\tilde{s}) \} | \tilde{s} \in S\}$

2. (P-Intersection) $\cap_{i \in \lambda} \tilde{U}_i = \{\tilde{s}, \cap_{i \in \lambda} \tilde{A}_i(\tilde{s}), \cap_{i \in \lambda} \tilde{B}_i(\tilde{s}) \} | \tilde{s} \in S\}$

3. (R-Union) $\cap_{i \in \lambda} \tilde{U}_i = \{\tilde{s}, \cap_{i \in \lambda} \tilde{A}_i(\tilde{s}), \cap_{i \in \lambda} \tilde{B}_i(\tilde{s}) \} | \tilde{s} \in S\}$

4. (R-Intersection) $\cap_{i \in \lambda} \tilde{U}_i = \{\tilde{s}, \cap_{i \in \lambda} \tilde{A}_i(\tilde{s}), \cap_{i \in \lambda} \tilde{B}_i(\tilde{s}) \} | \tilde{s} \in S\}$

Definition 7 (see [18, 19]). A picture fuzzy set (briefly, PFS) $\tilde{S}$ of a universe $\tilde{S}$ is an object in the form of $\{\tilde{s}, \tilde{U}_U(\tilde{s}), \tilde{A}_U(\tilde{s}), \tilde{B}_U(\tilde{s}) | \tilde{s} \in \tilde{S}\}$, where $\tilde{U}_U, \tilde{A}_U, \tilde{B}_U : \tilde{S} \rightarrow [0, 1]$ are fuzzy sets that satisfy $0 \leq \tilde{U}_U(\tilde{s}) + \tilde{A}_U(\tilde{s}) \leq 1$ for each $\tilde{s} \in \tilde{S}$. Then the values $\tilde{U}_U(\tilde{s}), \tilde{A}_U(\tilde{s}), \tilde{B}_U(\tilde{s})$ are called the degree of positive membership of $\tilde{s}$ in $\tilde{U}$, the degree of neutral membership of $\tilde{s}$ in $\tilde{U}$, and the degree of negative membership of $\tilde{s}$ in $\tilde{U}$, respectively. Now $(1 - \tilde{U}_U(\tilde{s}) + \tilde{A}_U(\tilde{s}) + \tilde{B}_U(\tilde{s}))$ could be called the degree of refusal membership of $\tilde{s}$ in $\tilde{S}$. Let $\tilde{P}(\tilde{S})$ represent the set of all picture fuzzy sets of a universe $\tilde{S}$.

Definition 8 (see [18, 19]). Let $\tilde{U}$ and $\tilde{V}$ be the PFSs. Then the set of operations are defined as follows:

1. $\bigcup_{\tilde{U}} \subseteq \tilde{V}$ if $f(\tilde{s}) \in \tilde{S}, \tilde{U}_U(\tilde{s}) \leq \tilde{V}_U(\tilde{s})$ and $\tilde{A}_U(\tilde{s}) \leq \tilde{A}_V(\tilde{s})$ and $\tilde{B}_U(\tilde{s}) \leq \tilde{B}_V(\tilde{s})$

2. $\bigcup_{\tilde{U}} \subseteq \tilde{V}$ if $f(\tilde{s}) \in \tilde{S}$

3. $\bigcup_{\tilde{U}} \subseteq \tilde{V}$ if $f(\tilde{s}) \in \tilde{S}$

4. $\bigcup_{\tilde{U}} \subseteq \tilde{V}$ if $f(\tilde{s}) \in \tilde{S}$
Now, a generalization of interval-valued fuzzy set \( U \) is proposed. Here \( \text{int}[0,1] \) stands for the set of all closed subintervals of \([0,1] \).

\[
M_U: \tilde{S} \longrightarrow \text{int}([0,1]), \quad M_U(\tilde{s}) = \left( \left( M_{UL}(\tilde{s}) \right), \left( M_{UR}(\tilde{s}) \right) \right) \in \text{int}([0,1]), \\
L_U: \tilde{S} \longrightarrow \text{int}([0,1]), \quad L_U(\tilde{s}) = \left( \left( L_{UL}(\tilde{s}) \right), \left( L_{UR}(\tilde{s}) \right) \right) \in \text{int}([0,1]), \\
N_U: \tilde{S} \longrightarrow \text{int}([0,1]), \quad N_U(\tilde{s}) = \left( \left( N_{UL}(\tilde{s}) \right), \left( N_{UR}(\tilde{s}) \right) \right) \in \text{int}([0,1]).
\]

The following condition is satisfied: \( \sup M_U(\tilde{s}) + \sup L_U(\tilde{s}) + \sup N_U(\tilde{s}) \leq 1 \), \((\forall \tilde{s} \in \tilde{S})\). The IVPFS(\( \tilde{S} \)) denotes the set of all interval-valued picture fuzzy sets of \( \tilde{S} \).

### 3. Cubic Picture Fuzzy Sets

In this section, we propose the notion of a cubic picture fuzzy set and investigate its set-theoretical operations and some basic properties by giving illustrative examples.

**Definition 9** (see [29]). An interval-valued picture fuzzy set (briefly, IVPS) \( U \) of a universe \( S \) is an object in the following form: \( U = \left\{ (M_U(\tilde{s}), L_U(\tilde{s}), N_U(\tilde{s})) | \tilde{s} \in \tilde{S} \right\} \), where

\[
\left\{ \begin{array}{l}
\tilde{s} \in \text{int}([0,1]), \\
\left( M_U(\tilde{s}), L_U(\tilde{s}), N_U(\tilde{s}) \right) \in \text{int}([0,1])
\end{array} \right\}
\]

is an internal cubic picture fuzzy set of \( \tilde{S} \).

**Example 4.** Let \( \tilde{S} = \left\{ \tilde{s}_1, \tilde{s}_2, \tilde{s}_3 \right\} \) be given; then the CPFS,

\[
U = \left\{ \begin{array}{l}
\tilde{s}_1, [0.1, 0.3], [0.3, 0.4], [0.1, 0.3], 0.2, 0.3, 0.2, \\
\tilde{s}_2, [0.1, 0.3], [0.3, 0.5], [0.0, 0.2], 0.2, 0.4, 0.1, \\
\tilde{s}_3, [0.1, 0.3], [0.1, 0.4], [0.0, 0.2], 0.2, 0.3, 0.1
\end{array} \right\}
\]

is an external cubic picture fuzzy set of \( \tilde{S} \).

**Theorem 1.** If \( C_P = \langle \tilde{A}, \tilde{B} \rangle \) is a cubic picture fuzzy set, which is not an ECPFS, then there exists \( \tilde{s} \in S \) such that \( \tilde{B}_1(\tilde{s}) \in (\tilde{B}_1(\tilde{s}), \tilde{B}_1(\tilde{s})) \), \( \tilde{B}_2(\tilde{s}) \in (\tilde{B}_2(\tilde{s}), \tilde{B}_2(\tilde{s})) \), and \( \tilde{B}_3(\tilde{s}) \in (\tilde{B}_3(\tilde{s}), \tilde{B}_3(\tilde{s})) \).

**Proof.** The proof is straightforward and therefore is omitted. \( \square \)

**Theorem 2.** If \( C_P = \langle \tilde{A}, \tilde{B} \rangle \) is both ICPFS and ECPFS, then the following is satisfied for each \( \tilde{s} \in S \):

\[
\tilde{B}_1(\tilde{s}) \in (\tilde{B}_1(\tilde{s}), \tilde{B}_1(\tilde{s})), \quad \tilde{B}_2(\tilde{s}) \in (\tilde{B}_2(\tilde{s}), \tilde{B}_2(\tilde{s})), \quad \text{and} \quad \tilde{B}_3(\tilde{s}) \in (\tilde{B}_3(\tilde{s}), \tilde{B}_3(\tilde{s})).
\]

**Proof.** Assume that \( C_P \) is both ICPFS and ECPFS. Then, by using the definitions of ICPFS and ECPFS, we have\( \tilde{B}_1(\tilde{s}) \not\in \tilde{B}_1(\tilde{s}) \), \( \tilde{B}_1(\tilde{s}) \not\in \tilde{B}_1(\tilde{s}) \), and \( \tilde{B}_1(\tilde{s}) \not\in \tilde{B}_1(\tilde{s}) \). Thus \( \tilde{B}_1(\tilde{s}) = \tilde{B}_1(\tilde{s}) = \tilde{B}_1(\tilde{s}) \), implying that \( \tilde{B}_1(\tilde{s}) \not\in \tilde{B}_1(\tilde{s}) \), \( \tilde{B}_1(\tilde{s}) \not\in \tilde{B}_1(\tilde{s}) \), and \( \tilde{B}_1(\tilde{s}) \not\in \tilde{B}_1(\tilde{s}) \).
Proposition 1. Let $\mathcal{U} = C_p = \langle \tilde{A}, \mathcal{R} \rangle$ and $\mathcal{V} = C_p = (J, K)$ be the cubical picture fuzzy sets, then equality, $P$-order, and $R$-order are defined as follows:

1. (Equality) $U = \mathcal{V} \iff \tilde{A} = I$ and $\mathcal{R} = K$
2. (P-order) $U \subseteq_p \mathcal{V} \iff \tilde{A} \subseteq I$ and $\mathcal{R} \subseteq K$
3. (R-order) $U \subseteq_r \mathcal{V} \iff \tilde{A} \subseteq I$ and $\mathcal{R} \supseteq K$

Definition 13. For any indexed family of CPFSs $\mathcal{U} = \{\tilde{A}_i, \mathcal{R}_i\}$, $\mathcal{V} = (J, K)$, $C = (L, M)$, and $D = (O, T)$, we have the following:

1. $\sqcup_i \mathcal{U}_i = \{\tilde{s}, (\sqcup_i \tilde{A}_i)(\tilde{s}), (\sqcup_i \mathcal{R}_i)(\tilde{s}) | \tilde{s} \in \tilde{S}\}$ (P-union)
2. $\sqcap_i \mathcal{U}_i = \{\tilde{s}, (\sqcap_i \tilde{A}_i)(\tilde{s}), (\sqcap_i \mathcal{R}_i)(\tilde{s}) | \tilde{s} \in \tilde{S}\}$ (P-intersection)
3. $\sqcup_i \mathcal{U}_i = \{\tilde{s}, (\sqcup_i \tilde{A}_i)(\tilde{s}), (\sqcup_i \mathcal{R}_i)(\tilde{s}) | \tilde{s} \in \tilde{S}\}$ (P-intersection)
4. $\sqcap_i \mathcal{U}_i = \{\tilde{s}, (\sqcap_i \tilde{A}_i)(\tilde{s}), (\sqcap_i \mathcal{R}_i)(\tilde{s}) | \tilde{s} \in \tilde{S}\}$ (R-intersection)

The complement of $U = \langle \tilde{A}, \mathcal{R} \rangle$ is also a cubical picture fuzzy set which is defined by $U^c = \{\tilde{s}, \tilde{A}^c(\tilde{s}), \mathcal{R}^c(\tilde{s}) | \tilde{s} \in \tilde{S}\}$.

Proposition 1. For any CPFS $\mathcal{U} = \langle \tilde{A}, \mathcal{R} \rangle$, $\mathcal{V} = (J, K)$, $C = (L, M)$, and $D = (O, T)$, the following:

1. If $U \subseteq_p \mathcal{V}$ and $V \subseteq_p C$ then $U \subseteq_p C$
2. If $U \subseteq_p \mathcal{V}$ and $U \subseteq_p C$ then $U \subseteq_p \mathcal{V} \cap_p C$
3. If $U \subseteq_p \mathcal{V}$ and $U \subseteq_p C$ then $U \cup_p C \subseteq_p \mathcal{V}$
4. If $U \subseteq_p \mathcal{V}$ and $C \subseteq_p D$ then $U \cup_p C \subseteq_p \mathcal{V} \cup_p D$
5. If $U \subseteq_r \mathcal{V}$ and $V \subseteq_r C$ then $U \subseteq_r C$
6. If $U \subseteq_r \mathcal{V}$ and $U \subseteq_r C$ then $U \cap_p C \subseteq_r \mathcal{V}$
7. If $U \subseteq_r \mathcal{V}$ and $C \subseteq_r D$ then $U \cup_p C \subseteq_r \mathcal{V} \cup_r D$
8. If $U \subseteq_r \mathcal{V}$ and $C \subseteq_r D$ then $U \cap_p C \subseteq_r \mathcal{V} \cap_r D$

Proof. The proof is straightforward and therefore is omitted.

Remark 1. The following are noted:

1. If $U \subseteq_p \mathcal{V}$, then $\mathcal{V}^c \nsubseteq_p U^c$
2. If $U \subseteq_r \mathcal{V}$, then $\mathcal{V}^c \nsubseteq_r U^c$

Example 5. Let

$\mathcal{U} = \{[0.0, 0.025], [0.0, 0.025], [0.0, 0.25], (0.1, 0.1, 0.2), (0.2, 0.2, 0.1)\}$

and then $U \subseteq_p \mathcal{V}$.

Since

$U^c = \{[0.2, 0.04], [0.2, 0.25], [0.1, 0.2], (0.2, 0.2, 0.1)\}$

we obtain $\mathcal{V}^c \nsubseteq_p U^c$.

Example 6. Let

$\mathcal{U} = \{[0.1, 0.2], [0.2, 0.25], [0.2, 0.4], (0.1, 0.2, 0.3)\}$

and then $U \subseteq_r \mathcal{V}$.

Since

$\mathcal{V}^c = \{[0.2, 0.05], [0.2, 0.3], [0.1, 0.2], (0.3, 0.2, 0.1)\}$

we have $\mathcal{V}^c \nsubseteq_r U^c$.

Theorem 3. Let $\mathcal{U} = \langle \tilde{A}, \mathcal{R} \rangle$ be a cubic picture fuzzy set. If $\mathcal{U}$ is an ICPFS (resp., ECPFS), then $\mathcal{U}$ is an ICPFS (resp., ECPFS).

Proof. The proof is straightforward and therefore is omitted.

Theorem 4. P-union and P-intersection of arbitrary indexed family of ICPFS $\{\mathcal{U}_i = \langle \tilde{A}_i, \mathcal{R}_i \rangle | i \in \Lambda\}$ are ICPFSs.

Proof. As $\mathcal{U}_i$ is an ICPFS,

$\mathcal{R}_i^c(\tilde{s}) \subseteq \mathcal{R}_i^c(\tilde{s}) \subseteq \mathcal{R}_i(\tilde{s}) \subseteq \mathcal{R}_i(\tilde{s}) \subseteq \mathcal{R}_i^c(\tilde{s})$

and, likewise,

$\mathcal{R}_i^c(\tilde{s}) \subseteq \mathcal{R}_i^c(\tilde{s}) \subseteq \mathcal{R}_i(\tilde{s}) \subseteq \mathcal{R}_i(\tilde{s}) \subseteq \mathcal{R}_i^c(\tilde{s})$

Hence, P-union and P-intersection of $\mathcal{U}_i$ are CPFSs.

Remark 2. P-union and P-intersection of ECPFSs need not be an ECPFS.

Example 7. Let $\mathcal{U} = \langle \tilde{A}, \mathcal{R} \rangle$ and $\mathcal{V} = (J, K)$ be the ECPFSs of $I = [0, 1]$, where
\[ \tilde{A}(\tilde{s}) = \{ [0.2, 0.3], [0.0, 0.2], [0.0, 0.2] \}, \quad \tilde{B}(\tilde{s}) = \{ 0.1, 0.3, 0.3 \} \]
\[ J(\tilde{s}) = \{ [0.0, 0.2], [0.1, 0.3], [0.2, 0.3] \}, \quad K(\tilde{s}) = \{ 0.3, 0.0, 0.1 \} \]

for all \( \tilde{s} \in I \).

1. We know that \( U \cup \rho V = \{ \tilde{s}, J(\tilde{s}), B(\tilde{s})|\tilde{s} \in S \} \) and
   \( B_1(\tilde{s}) \in (B_1^-(\tilde{s}), B_1^+(\tilde{s})) \), \( B_2(\tilde{s}) \in (B_2^-(\tilde{s}), B_2^+(\tilde{s})) \), \( B_3(\tilde{s}) \in (B_3^-(\tilde{s}), B_3^+(\tilde{s})) \) for all \( \tilde{s} \in S \). Hence, \( \bigcup \rho V \) is not an ECPFS.

2. We know that \( \bigcap \rho V = \{ \tilde{s}, A(\tilde{s}), K(\tilde{s})|\tilde{s} \in S \} \) and
   \( K_1(\tilde{s}) \in (K_1^-(\tilde{s}), K_1^+(\tilde{s})) \), \( K_2(\tilde{s}) \in (K_2^-(\tilde{s}), K_2^+(\tilde{s})) \) for all \( \tilde{s} \in S \). Hence \( \bigcap \rho V \) is not an ECPFS.

The following example shows that the R-union and R-intersection of CPFSs need not be an CPFS.

**Example 8.** Let \( U = \langle \tilde{A}, \tilde{B} \rangle \) and \( V = (J, K) \) be CPFSs in \( I = [0, 1] \), where
\( A(\tilde{s}) = [0.0, 0.2], [0.1, 0.3], [0.2, 0.3] \),
\( B(\tilde{s}) = [0.1, 0.3, 0.3] \) and \( J(\tilde{s}) = [0.2, 0.3], [0.0, 0.2], [0.0, 0.2] \), \( K(\tilde{s}) = 0.3, 0.0, 0.1 \) for all \( \tilde{s} \in S \).

1. We know that \( U \cup \rho V = \{ \tilde{s}, J(\tilde{s}), Z(\tilde{s})|\tilde{s} \in S \} \) and
   \( B_1(\tilde{s}) \notin (B_1^-(\tilde{s}), B_1^+(\tilde{s})) \), \( B_2(\tilde{s}) \notin (B_2^-(\tilde{s}), B_2^+(\tilde{s})) \), \( B_3(\tilde{s}) \notin (B_3^-(\tilde{s}), B_3^+(\tilde{s})) \) for all \( \tilde{s} \in S \).
   Hence, \( U \cup \rho V \) is not a CPFS.

2. We know that \( \bigcap \rho V = \{ \tilde{s}, A(\tilde{s}), K(\tilde{s})|\tilde{s} \in S \} \) and
   \( K_1(\tilde{s}) \notin (K_1^-(\tilde{s}), K_1^+(\tilde{s})) \), \( K_2(\tilde{s}) \notin (K_2^-(\tilde{s}), K_2^+(\tilde{s})) \), \( K_3(\tilde{s}) \notin (K_3^-(\tilde{s}), K_3^+(\tilde{s})) \) for all \( \tilde{s} \in I \).
   Hence, \( \bigcap \rho V \) is not a CPFS.

The following example shows that “R-union” and “R-intersection” of ECPFS need not be an ECPFS.

**Example 9.** Let \( U = \langle \tilde{A}, \tilde{B} \rangle \) and \( V = (J, K) \) be ECPFSs of \( I = [0, 1] \) in which \( A(\tilde{s}) = [0.0, 0.2], [0.1, 0.4], [0.2, 0.3] \) and \( B(\tilde{s}) = [0.3, 0.0, 0.1], [0.0, 0.2], [0.1, 0.2] \), and \( K(\tilde{s}) = [0.4, 0.3, 0.0] \) for all \( \tilde{s} \in I \).

1. We know that \( U \cup \rho V = \{ \tilde{s}, J(\tilde{s}), B(\tilde{s})|\tilde{s} \in S \} \) and
   \( B_1(\tilde{s}) \notin (B_1^-(\tilde{s}), B_1^+(\tilde{s})) \). Hence, \( U \cup \rho V \) is not a ECPFS in \( I \).

2. We know that \( \bigcap \rho V = \{ \tilde{s}, A(\tilde{s}), K(\tilde{s})|\tilde{s} \in S \} \) and
   \( K_1(\tilde{s}) \notin (K_1^-(\tilde{s}), K_1^+(\tilde{s})) \). Hence, \( \bigcap \rho V \) is not a ECPFS.

**Theorem 5.** Let \( U = \langle \tilde{A}, \tilde{B} \rangle \) and \( V = (J, K) \) be CPFSs, such that
\[
\begin{align*}
\text{Max} \left( (B_1^-)(\tilde{s}), K_1^-(\tilde{s}) \right) & \leq (B_1^+(\tilde{s}), K_1^+(\tilde{s})) \\
\text{Max} \left( (B_2^-)(\tilde{s}), K_2^-(\tilde{s}) \right) & \leq (B_2^+(\tilde{s}), K_2^+(\tilde{s})) \\
\text{Max} \left( (B_3^-)(\tilde{s}), K_3^-(\tilde{s}) \right) & \leq (B_3^+(\tilde{s}), K_3^+(\tilde{s}))
\end{align*}
\]
for each \( \tilde{s} \in \tilde{S} \). Then the “R-union” of \( U \) and \( V \) is a CPFS.

**Proof.** Let \( U = \langle \tilde{A}, \tilde{B} \rangle \) and \( V = (J, K) \) be two CPFSs, which satisfy the conditions given in Theorem 5; then we have
\( B_1^- \leq B_1^+ \leq B_1 \), \( B_2^- \leq B_2 \leq B_2^+ \), \( B_3^- \leq B_3 \leq B_3^+ \), and
\( K_1^- \leq K_1 \leq K_1^+ \), \( K_2^- \leq K_2 \leq K_2^+ \), \( K_3^- \leq K_3 \leq K_3^+ \).

These imply that \( (B_1^+)(\tilde{s})AK_1^- \leq (B_1^+)(\tilde{s})AK_1 \), \( (B_1^+)(\tilde{s})AK_1 \leq (B_1(\tilde{s})AK_1^+) \), \( (B_3(\tilde{s})AK_3^-) \leq (B_3(\tilde{s})AK_3^+) \).

It follows from the assumption that
\[
\begin{align*}
\text{Max} \left( (B_1^-)(\tilde{s}), K_1^-(\tilde{s}) \right) & \leq (B_1^+(\tilde{s}), K_1^+(\tilde{s})) \\
\text{Max} \left( (B_2^-)(\tilde{s}), K_2^-(\tilde{s}) \right) & \leq (B_2^+(\tilde{s}), K_2^+(\tilde{s})) \\
\text{Max} \left( (B_3^-)(\tilde{s}), K_3^-(\tilde{s}) \right) & \leq (B_3^+(\tilde{s}), K_3^+(\tilde{s}))
\end{align*}
\]

**Example 10.** Let \( U = \langle \tilde{A}, \tilde{B} \rangle \) and \( V = (J, K) \) be ECPFSs of \( I = [0, 1] \), in which
\( \tilde{A} (\tilde{s}) = \{0.1, 0.2\}, [0.1, 0.3], [0.2, 0.4]\), \( \tilde{B} (\tilde{s}) = \{0.3, 0.0, 0.5\}, J (\tilde{s}) = \{0.0, 0.2\}, [0.0, 0.3], [0.1, 0.3]\), \( K (\tilde{s}) = \{0.3, 0.4, 0.0\}\),

for all \( \tilde{s} \in I \).

It is seen that \( U^* = (\tilde{A}, K) \) and \( V^* = (J, \tilde{B}) \) are not CPFSs, because in \( U^* \), \( \tilde{B}_1 (\tilde{s}) \notin [0.1, 0.2], \tilde{B}_3 (\tilde{s}) \notin [0.1, 0.3] \), \( \tilde{B}_1 (\tilde{s}) \notin [0.2, 0.4] \) and in \( V^* \), \( K_1 (\tilde{s}) \notin [0.0, 0.2] \), \( K_3 (\tilde{s}) \notin [0.1, 0.3] \).

The following example shows that the "P-union" of two ECPFSs need not be a CPFS.

**Example 11.** Let \( U = (\tilde{A}, \tilde{B}) \) and \( V = (J, K) \) be ECPFSs of \( I = [0, 1] \) in which \( \tilde{A} (\tilde{s}) = \{0.2, 0.4\}, [0.1, 0.2], [0.0, 0.3]\), \( \tilde{B} (\tilde{s}) = [0.1, 0.3, 0.4] \), \( J (\tilde{s}) = [0.0, 0.2], [0.1, 0.4], [0.2, 0.3] \), and \( K (\tilde{s}) = [0.3, 0.0, 0.1] \) for all \( \tilde{s} \in I \). \( \bigcup \bigcup \bigcup V = \left\{ \tilde{s}, J (\tilde{s}), K (\tilde{s}) \right\} \in I \); clearly, \( 0.4 \notin [0.2, 0.3] \). Hence \( \bigcup \bigcup \bigcup \bigcup \) is not a CPFS of \( I \).

**Theorem 6.** Let \( U = (\tilde{A}, \tilde{B}) \) and \( V = (J, K) \) be the CPFSs in \( \mathcal{W} \) satisfying the following inequalities:

\[
\min \left( \tilde{B}_1 (\tilde{s}), K_1 (\tilde{s}) \right) \geq \left( \tilde{B}_1 \vee K_1 \right) (\tilde{s}),
\]

\[
\min \left( \tilde{B}_2 (\tilde{s}), K_2 (\tilde{s}) \right) \geq \left( \tilde{B}_2 \vee K_2 \right) (\tilde{s}),
\]

\[
\min \left( \tilde{B}_3 (\tilde{s}), K_3 (\tilde{s}) \right) \geq \left( \tilde{B}_3 \vee K_3 \right) (\tilde{s}),
\]

for all \( \tilde{s} \in \tilde{S} \). Then the "R-intersection" of \( U \) and \( V \) is a CPFS.

**Proof.** The proof is straightforward and therefore is omitted.

**Theorem 7.** Let \( U = (\tilde{A}, \tilde{B}) \) and \( V = (J, K) \) be the CPFSs. If \( U^* = (\tilde{A}, K) \) and \( V^* = (J, \tilde{B}) \) are CPFSs, then P-union \( \left( \bigcup \bigcup \bigcup \bigcup V \right) \) of \( U \) and \( V \) is a CPFS.

**Proof.** The proof is straightforward and therefore is omitted.

**Theorem 8.** Let \( U = (\tilde{A}, \tilde{B}) \) and \( V = (J, K) \) be the ECPFSs. If \( U^* = (\tilde{A}, K) \) and \( V^* = (J, \tilde{B}) \) are CPFSs, then P-union (\( \bigcup \bigcup \bigcup \bigcup V \)) of \( U = (\tilde{A}, \tilde{B}) \) and \( V = (J, K) \) is a CPFS.

**Proof.** The proof is straightforward and therefore is omitted.

**Remark 3.** For two ECPFSs \( U \) and \( V \) of \( \mathcal{W} \), the derived CPFSs \( U^* \) and \( V^* \) need not be ECPFSs.

**Example 12.** Let \( U = (\tilde{A}, \tilde{B}) \) and \( V = (J, K) \) be ECPFSs of \( I = [0, 1] \) in which

\( \tilde{A} (\tilde{s}) = \{0.0, 0.3\}, [0.1, 0.2], [0.2, 0.4]\), \( \tilde{B} (\tilde{s}) = \{0.4, 0.0, 0.1\}, J (\tilde{s}) = \{0.1, 0.2\}, [0.0, 0.3], [0.2, 0.3]\), \( K (\tilde{s}) = \{0.3, 0.4, 0.1\}\),

for all \( \tilde{s} \in I \). Now, from the above, we observe that \( U^* = (\tilde{A}, K) \) and \( V^* = (J, \tilde{B}) \) are not CPFSs, because, in \( U^* \), \( K_1 (\tilde{s}) \in [0.0, 0.3] \), and in \( V^* \), \( \tilde{B}_2 (\tilde{s}) \notin [0.0, 0.3] \).

**Theorem 9.** Let \( U = (\tilde{A}, \tilde{B}) \) and \( V = (J, K) \) be two ECPFSs. If \( U^* = (\tilde{A}, K) \) and \( V^* = (J, \tilde{B}) \) are CPFSs, then P-union \( \left( \bigcup \bigcup \bigcup \bigcup V \right) \) of \( U = (\tilde{A}, \tilde{B}) \) and \( V = (J, K) \) is an ECPFS.

**Proof.** Let \( U = (\tilde{A}, \tilde{B}) \) and \( V = (J, K) \) be ECPFSs, such that \( U^* = (\tilde{A}, K) \) and \( V^* = (J, \tilde{B}) \) are CPFSs. Then we obtain that \( \tilde{B}_1 (\tilde{s}) \notin (\tilde{B}_1 (\tilde{s}), \tilde{B}_1 (\tilde{s})), \tilde{B}_2 (\tilde{s}) \notin (\tilde{B}_2 (\tilde{s}), \tilde{B}_2 (\tilde{s})), \) and \( \tilde{B}_3 (\tilde{s}) \notin (\tilde{B}_3 (\tilde{s}), \tilde{B}_3 (\tilde{s})), \) and \( K_1 (\tilde{s}) \notin \{K_1 (\tilde{s}), K_1 (\tilde{s})\}, K_2 (\tilde{s}) \notin \{K_2 (\tilde{s}), K_2 (\tilde{s})\}, \) and \( K_3 (\tilde{s}) \notin \{K_3 (\tilde{s}), K_3 (\tilde{s})\}, \)

Hence,

\[
\left( \tilde{B}_1 \vee K_1 \right) (\tilde{s}) \notin \left\{ \max \left( \tilde{B}_1 (\tilde{s}), K_1 (\tilde{s}) \right) \right\}, \max \left( \tilde{B}_1 (\tilde{s}), K_1 (\tilde{s}) \right), \]

\[
\left( \tilde{B}_2 \vee K_2 \right) (\tilde{s}) \notin \left\{ \max \left( \tilde{B}_2 (\tilde{s}), K_2 (\tilde{s}) \right) \right\}, \max \left( \tilde{B}_2 (\tilde{s}), K_2 (\tilde{s}) \right), \]

\[
\left( \tilde{B}_3 \vee K_3 \right) (\tilde{s}) \notin \left\{ \max \left( \tilde{B}_3 (\tilde{s}), K_3 (\tilde{s}) \right) \right\}, \max \left( \tilde{B}_3 (\tilde{s}), K_3 (\tilde{s}) \right).
\]
This implies that

\[
(\mathcal{B}_1 \cup K_1)(u) \notin \left( (\mathcal{B}_1 \cup K_1)^-(\tilde{s}), (\mathcal{B}_1 \cup K_1)^+(\tilde{s}) \right),
\]

\[
(\mathcal{B}_2 \cup K_2)(u) \notin \left( (\mathcal{B}_2 \cup K_2)^-(\tilde{s}), (\mathcal{B}_2 \cup K_2)^+(\tilde{s}) \right),
\]

\[
(\mathcal{B}_3 \cup K_3)(u) \notin \left( (\mathcal{B}_3 \cup K_3)^-(\tilde{s}), (\mathcal{B}_3 \cup K_3)^+(\tilde{s}) \right).
\]

(18)

\[
\begin{align*}
\min \left\{ \max \left( \mathcal{B}_1^-(\tilde{s}), K_1^-(\tilde{s}) \right), \max \left( \mathcal{B}_1^+(\tilde{s}), K_1^+(\tilde{s}) \right) \right\} & \\
\geq (\mathcal{B}_1 \cup K_1)(\tilde{s}) & > \max \left\{ \min \left( \mathcal{B}_1^-(\tilde{s}), K_1^-(\tilde{s}) \right), \min \left( \mathcal{B}_1^+(\tilde{s}), K_1^+(\tilde{s}) \right) \right\}.
\end{align*}
\]

(19)

for all \( \tilde{s} \in \tilde{S} \).

Then, the P-intersection of \( U \) and \( V \) is an ECPFS.

\[ \square \]

Theorem 10. Let \( \tilde{U} = (\tilde{A}, \tilde{B}) \) and \( \tilde{V} = (J, K) \) be the ECPFSs of \( \tilde{S} \) such that

\[ \square \]

Theorem 11. Let \( \tilde{U} = (\tilde{A}, \tilde{B}) \) and \( \tilde{V} = (J, K) \) be the CPFSs, such that the following implications are valid:

\[ \square \]

Example 13. Let \( \tilde{U} = (\tilde{A}, \tilde{B}) \) and \( \tilde{V} = (J, K) \) be two ECPFSs of \( I = [0, 1] \) defined as follows:

\[ \tilde{A}(\tilde{s}) = [0.1, 0.3], [0.2, 0.3], [0.0, 0.2], \]

\[ \tilde{B}(\tilde{s}) = [0.0, 0.1, 0.3], \]

\[ J(\tilde{s}) = [0.1, 0.2], [0.2, 0.4], [0.1, 0.3], \]

\[ K(\tilde{s}) = [0.3, 0.1, 0.4], \]

for all \( \tilde{s} \in I \).
Theorem 12. Let $U = (A, \mathcal{R})$ and $V = (J, K)$ be two ECPFSSs, such that the following are satisfied:

$$\min \left\{ \min \left( \mathcal{R}_1^+ (\tilde{s}), K_1^- (\tilde{s}) \right), \min \left( \mathcal{R}_1^- (\tilde{s}), K_1^+ (\tilde{s}) \right) \right\} > (\mathcal{R}_1 \lor K_1)(\tilde{s})$$

for all $\tilde{s} \in \tilde{S}$. Then, $P$-union of $U$ and $V$ is an ECPFSS.

Proof. The proof is straightforward and therefore is omitted. \qed

Theorem 13. Let $U = (A, \mathcal{R})$ and $V = (J, K)$ be two ECPFSSs, which satisfy the following conditions:

$$\min \left\{ \min \left( \mathcal{R}_2^+ (\tilde{s}), K_2^- (\tilde{s}) \right), \min \left( \mathcal{R}_2^- (\tilde{s}), K_2^+ (\tilde{s}) \right) \right\} > (\mathcal{R}_2 \lor K_2)(\tilde{s})$$

for all $\tilde{s} \in \tilde{S}$. Then $R$-union of $U$ and $V$ is an ECPFSS.

Proof. The proof is straightforward. \qed
Theorem 14. Let $U = (\tilde{A}, \mathcal{B})$ and $V = (J, K)$ be the ECPFSs, such that the following are satisfied:

\begin{align}
\min\left\{ \max\left( \mathcal{B}_1^+ (\tilde{s}), K_1^- (\tilde{s}) \right), \max\left( \mathcal{B}_1^- (\tilde{s}), K_1^+ (\tilde{s}) \right) \right\} &> (\mathcal{B}_1 \lor K_1) (\tilde{s}) \\
> \max\left\{ \min\left( \mathcal{B}_1^+ (\tilde{s}), K_1^- (\tilde{s}) \right), \min\left( \mathcal{B}_1^- (\tilde{s}), K_1^+ (\tilde{s}) \right) \right\}.
\end{align}

for all $\tilde{s} \in \mathcal{S}$. Then R-intersection of $U$ and $V$ is an ECPFS.

Proof. The proof is straightforward. \qed

\begin{align}
\min\left\{ \max\left( \mathcal{B}_1^+ (\tilde{s}), K_1^- (\tilde{s}) \right), \max\left( \mathcal{B}_1^- (\tilde{s}), K_1^+ (\tilde{s}) \right) \right\} &> (\mathcal{B}_1 \lor K_1) (\tilde{s}) \\
= &\max\left\{ \min\left( \mathcal{B}_1^+ (\tilde{s}), K_1^- (\tilde{s}) \right), \min\left( \mathcal{B}_1^- (\tilde{s}), K_1^+ (\tilde{s}) \right) \right\}.
\end{align}

Remark 4. Let $U = (\tilde{A}, \mathcal{B})$ and $V = (J, K)$ be two ECPFSs, such that the following are satisfied:

\begin{align}
\min\left\{ \max\left( \mathcal{B}_2^+ (\tilde{s}), K_2^- (\tilde{s}) \right), \max\left( \mathcal{B}_2^- (\tilde{s}), K_2^+ (\tilde{s}) \right) \right\} &> (\mathcal{B}_2 \lor K_2) (\tilde{s}) \\
= &\max\left\{ \min\left( \mathcal{B}_2^+ (\tilde{s}), K_2^- (\tilde{s}) \right), \min\left( \mathcal{B}_2^- (\tilde{s}), K_2^+ (\tilde{s}) \right) \right\}.
\end{align}

for all $\tilde{s} \in \mathcal{S}$. Then R-intersection of $U$ and $V$ may not be an ECPFS.

Theorem 15. Let $U = (\tilde{A}, \mathcal{B})$ and $V = (J, K)$ be two ECPFSs, such that the following are satisfied:

\begin{align}
\min\left\{ \max\left( \mathcal{B}_2^+ (\tilde{s}), K_2^- (\tilde{s}) \right), \max\left( \mathcal{B}_2^- (\tilde{s}), K_2^+ (\tilde{s}) \right) \right\} &> (\mathcal{B}_2 \lor K_2) (\tilde{s}) \\
= &\max\left\{ \min\left( \mathcal{B}_2^+ (\tilde{s}), K_2^- (\tilde{s}) \right), \min\left( \mathcal{B}_2^- (\tilde{s}), K_2^+ (\tilde{s}) \right) \right\}.
\end{align}
for all $s \in S$. Then R-intersection of $U$ and $V$ is both an ECPFS and a CPFS.

Proof. The proof is straightforward.

Theorem 16. Let $U = (\tilde{A}, \tilde{R})$ and $V = (J, K)$ be two CPFSs. If the implications are satisfied, for all $s \in S$,

\[
\begin{align*}
(B_1 \land K_1)(\tilde{s}) &\leq \max(B_1(\tilde{s}), K_1(\tilde{s})), \\
(B_2 \land K_2)(\tilde{s}) &\leq \max(B_2(\tilde{s}), K_2(\tilde{s})), \\
(B_3 \land K_3)(\tilde{s}) &\leq \max(B_3(\tilde{s}), K_3(\tilde{s})),
\end{align*}
\]

(27)

then the R-union of $U$ and $V$ is an EPCFS.

Definition 15 (see [11]). A function $T: [0, 1] \times [0, 1] \rightarrow [0, 1]$ is said to be a t-norm which satisfies the following:

1. Boundary: $T(0, 0) = 0; T(x, 1) = T(1, x) = x$ for all $x \in [0, 1]$
2. Monotonicity: If $x_1 \leq y_1$ and $x_2 \leq y_2$, then $T(x_1, x_2) \leq T(y_1, y_2)$
3. Commutativity: $T(x_1, x_2) = T(x_2, x_1)$
4. Associativity: $T(x_1, T(x_2, x_3)) = T(T(x_1, x_2), x_3)$

A function $S$ defined by $S(x, y) = 1 - T(1 - x, 1 - y)$ is called t-co-norm. A decreasing function $g$ generates a t-norm as $T(x, y) = g^{-1}(g(x) + g(y))$ such that $g(1) = 0$ and function $h$ generates the t-co-norm as $S(x, y) = h^{-1}(h(x) + h(y))$, where $h(t) = g(1 - t)$. Based on these norms’ generators, $g$ and $h$ will be used in the next theorems.

Proof. The proof is straightforward.

Theorem 17. Let $U = (\tilde{A}, \tilde{R})$ and $V = (J, K)$ be two CPFSs. If the following implications are satisfied for all $s \in S, (\tilde{B}_1 \lor K_1)(\tilde{s}) \geq \min\{\tilde{B}_1(\tilde{s}), K_1^+(\tilde{s})\}, (\tilde{B}_2 \lor K_2)(u) \geq \min\{\tilde{B}_2^+(\tilde{s}), K_2^+(\tilde{s})\}, (\tilde{B}_3 \lor K_3)(w) \geq \min\{\tilde{B}_3^+(\tilde{s}), K_3^+(\tilde{s})\}$, then R-intersection of $U$ and $V$ is an ECPFS.

Proof. The proof is straightforward.

Theorem 18. Let $U = (\tilde{A}, \tilde{R})$ and $V = (J, K)$ be two ECPFSs, such that the following implications hold:

Definition 16. Let $C = ((M, L_1, N_1), (U, \rho, \vartheta), C_1 = ((M_1, L_1, N_1), (U_1, \rho_1, \vartheta_1)), and C_2 = ((M_2, L_2, N_2), (U_2, \rho_2, \vartheta_2))$ be three CPFNs. Then the operations $\oplus, \otimes, \lambda C$, and $C^\lambda$ are defined as follows:

1. $C_1 \oplus C_2 = \left( \frac{h^{-1}(h(M_1) + h(M_2))}{g^{-1}(g(L_1) + g(L_2))}, \frac{g^{-1}(g(N_1) + g(N_2))}{h^{-1}(h(U_1) + h(U_2))}, \frac{h^{-1}(h(\rho_1) + h(\rho_2))}{g^{-1}(g(\vartheta_1) + g(\vartheta_2))} \right)$
2. $C_1 \otimes C_2 = \left( \frac{g^{-1}(g(M_1) + g(M_2))}{h^{-1}(h(L_1) + h(L_2))}, \frac{h^{-1}(h(N_1) + h(N_2))}{g^{-1}(g(U_1) + g(U_2))}, \frac{h^{-1}(h(\rho_1) + h(\rho_2))}{g^{-1}(g(\vartheta_1) + g(\vartheta_2))} \right)$
3. $\lambda C = \left( \frac{h^{-1}(h(M))}{g^{-1}(g(L))}, \frac{g^{-1}(g(N))}{h^{-1}(h(U))}, \frac{h^{-1}(h(\rho))}{g^{-1}(g(\vartheta))} \right)$
4. $C^\lambda = \left( \frac{g^{-1}(g(M))}{h^{-1}(h(L))}, \frac{h^{-1}(h(N))}{g^{-1}(g(U))}, \frac{g^{-1}(g(\rho))}{h^{-1}(h(\vartheta))} \right)$

Theorem 19. Let $C_1, C_2$, and $C_3$ be three CPFNs and $\lambda, \lambda_1$, and $\lambda_2 > 0$. Then, we have the following:

1. $C_1 \oplus C_2 = C_2 \oplus C_1$
2. $C_1 \otimes C_2 = C_2 \otimes C_1$
3. $\lambda (C_1 \oplus C_2)$
4. $C_1 \otimes C_2^\lambda = C_2^\lambda \otimes C_1^\lambda$
5. $\lambda_1 (C_1 \oplus C_2)$
6. $C_1 \otimes C_2^\lambda = C_2^\lambda \otimes C_1^\lambda$

Proof. It is easily obtained by the above definition.
4.1. Cubic Picture Fuzzy Weighted Averaging (CPFWA) Operators

Definition 17. Let \( \{U_i\}_{i \in \Lambda} \) be a collection of CPFSs. Then the CPFWA \( \bigoplus^n_{i=1} U_i \) is defined as follows:

\[
\text{CPFWA}(U_1, U_2, \ldots, U_n) = \bigoplus^n_{i=1} \tilde{w}_i U_i
\]

(29)

where \( \tilde{w} = [\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_n]^T \) is the weighted vector of \( U_i \), s.t. \( \tilde{w}_i > 0 \) and \( \sum^n_{i=1} \tilde{w}_i = 1 \).

Theorem 20. If \( \tilde{U}_i = (\tilde{A}_i, \tilde{\mathcal{R}}_i, \tilde{s}) \), \( \tilde{s} \in S \) is the collection of CPFSs, then the averaging value by using CPFWA operator is still CPFS and is given by

\[
\text{CPFWA}(U_1, U_2, \ldots, U_n) = \left( h^{-1}\left( \sum^n_{i=1} \tilde{w}_i h(M_i) \right), g^{-1}\left( \sum^n_{i=1} \tilde{w}_i g(L_i) \right), g^{-1}\left( \sum^n_{i=1} \tilde{w}_i g(N_i) \right) \right).
\]

(30)

Proof. We shall prove the result by using the principle of mathematical induction on "n."

**Step 1.** For \( n = 2 \), we have \( U_1 = (\tilde{A}_1, \tilde{\mathcal{R}}_1), U_2 = (\tilde{A}_2, \tilde{\mathcal{R}}_2) \); thus, by the operations of CPFSs, we get

\[
\begin{align*}
\tilde{w}_1 U_1 &= (h^{-1}(\tilde{w}_1 h(M_1)), g^{-1}(\tilde{w}_1 g(L_1)), g^{-1}(\tilde{w}_1 g(N_1))), h^{-1}(\tilde{w}_1 h(U_1)), g^{-1}(\tilde{w}_1 g(\tilde{\mathcal{R}}_1))) \\
\tilde{w}_2 U_2 &= (h^{-1}(\tilde{w}_2 h(M_2)), g^{-1}(\tilde{w}_2 g(L_2)), g^{-1}(\tilde{w}_2 g(N_2))), h^{-1}(\tilde{w}_2 h(U_2)), g^{-1}(\tilde{w}_2 g(\tilde{\mathcal{R}}_2))).
\end{align*}
\]

(31)

Hence, by additive properties of CPFSs, we get

\[
\text{CPFWA}(U_1, U_2) = \tilde{w}_1 U_1 \oplus \tilde{w}_2 U_2
\]

\[
= \left( h^{-1}\left( \tilde{w}_1 h(M_1) + \tilde{w}_2 h(M_2) \right), g^{-1}\left( \tilde{w}_1 g(L_1) + \tilde{w}_2 g(L_2) \right), g^{-1}\left( \tilde{w}_1 g(N_1) + \tilde{w}_2 g(N_2) \right) \right).
\]

(32)

Then, the results hold for \( n = 2 \).
Step 2. If equation (30) holds for \( n = k \), then, for \( n = k + 1 \), we have

\[
\text{CPFWA}(U_1, U_2, \ldots, U_{K+1}) = \bigoplus_{i=1}^{K+1} \omega_i U_i = \omega_1 U_1 \oplus \omega_2 U_2 \oplus \cdots \oplus \omega_{K+1} U_{K+1}
\]

\[
= \bigoplus_{i=1}^{K+1} \omega_i U_i \oplus \omega_{K+1} U_{K+1}
\]

\[
= \left( h^{-1} \left\{ \sum_{i=1}^{K} (\bar{w}_i h(M_i)) \right\} + h^{-1}(\bar{w}_{K+1} h(M_{K+1})) \right) \bigoplus \left( \sum_{i=1}^{K} (\bar{w}_i g(L_i)) + g^{-1}(\bar{w}_{K+1} g(L_{K+1})) \right)
\]

\[
\cdot \left( h^{-1} \left\{ \sum_{i=1}^{K} (\bar{w}_i h(\mathcal{U}_i)) \right\} + h^{-1}(\bar{w}_{K+1} h(\mathcal{U}_{K+1})) \right)
\]

\[
+ \varphi^{-1}(\bar{w}_{K+1} g(\varphi_{K+1}))
\]

Since the results hold for \( n = k + 1 \), hence, by the principle of mathematical induction, the result given in equation (30) holds for all positive integers \( n \). \qed
Remark 5. If \( g(t) \) is taken to be \( g(t) = -\log(t) \), then by equation (30) we have that

\[
\text{CPFWA}(U_1, U_2, \ldots, U_n) = \left( 1 - \prod_{i=1}^{n} (1 - (M_i))^{\frac{w_i}{\bar{w}_i}}, \prod_{i=1}^{n} (L_i)^{\frac{w_i}{\bar{w}_i}}, \prod_{i=1}^{n} (N_i)^{\frac{w_i}{\bar{w}_i}} \right),
\]

which is called cubic picture fuzzy Archimedean weighted averaging operator.

4.2. Cubic Picture Fuzzy Ordered Weighted Averaging (CPFOWA) Operator. In this section, we intend to take the idea of OWA into CPFWA operator and propose a new operator which is defined as follows.

**Definition 18.** Let \( \{U_i\}_{i \in \Lambda} \) be a collection of CPFSs. Then the CPFOWA \( \mathcal{CPF} \rightarrow \mathcal{CPF} \) is defined in the following way:

\[
\text{CPFOWA}(U_1, U_2, \ldots, U_n) = \left( h^{-1} \left( \sum_{i=1}^{n} \bar{w}_i h(M_{o(i)}) \right), g^{-1} \left( \sum_{i=1}^{n} \bar{w}_i g(L_{o(i)}) \right), g^{-1} \left( \sum_{i=1}^{n} \bar{w}_i g(N_{o(i)}) \right) \right),
\]

In particular, if \( L_{o(i)} = \rho_{o(i)} = 0 \) for all \( i \), then equation (36) reduces to

\[
\text{CPFOWA}(U_1, U_2, \ldots, U_n) = \left( h^{-1} \left( \sum_{i=1}^{n} \bar{w}_i h(M_{o(i)}) \right), g^{-1} \left( \sum_{i=1}^{n} \bar{w}_i g(N_{o(i)}) \right) \right),
\]

which becomes cubic intuitionistic OWA operator.

**Proof.** The proof follows from Theorem 19. \qed

4.3. Cubic Picture Fuzzy Hybrid Averaging (CPFHA) Operator. CPFWA operator weighs the CPFSs only, while CPFOWA weighs the ordered positions of it. However, in order to combine these two aspects in one, we introduce CPFHA operator.

**Theorem 21.** Let \( \{U_i\}_{i \in \Lambda} \) be a collection of CPFSs. Then, based on the CPFOWA operator, the aggregated CPFSs can be expressed as follows:

\[
\text{CPFHA}(U_1, U_2, \ldots, U_n) = \left( \sum_{i=1}^{n} \bar{w}_i U_{o(i)} \right), \left( \sum_{i=1}^{n} \bar{w}_i \right) = \left( \sum_{i=1}^{n} \bar{w}_i U_{o(i)} \right),
\]

where \( \bar{w} = [\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_n]^T \) is the standard weight vector of \( U_i \), such that \( \bar{w}_i > 0 \) and \( \sum_{i=1}^{n} \bar{w}_i = 1 \), \( U_{o(i)} \) is the ith largest of CPFSs \( \{U_i\}_{i \in \Lambda} \).

**Definition 19.** Let \( \{U_i\}_{i \in \Lambda} \) be a collection of CPFSs. Then the CPFHA \( \mathcal{CPF}^n \rightarrow \mathcal{CPF}^n \) is defined as follows:

\[
\text{CPFHA}(U_1, U_2, \ldots, U_n) = \left( \sum_{i=1}^{n} \bar{w}_i U_{o(i)} \right), \left( \sum_{i=1}^{n} \bar{w}_i \right) = \left( \sum_{i=1}^{n} \bar{w}_i U_{o(i)} \right),
\]

where \( \bar{w} = [\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_n]^T \) is the standard weight vector of \( U_i \), such that \( \bar{w}_i > 0 \) and \( \sum_{i=1}^{n} \bar{w}_i = 1 \), \( U_{o(i)} \) is the ith largest of
the weighted CPFSs $U_i(U) = n\bar{w}_iU_i$, $i = 1, 2, \ldots, n$, where $n$ is the number of CPFSs. Then CPFHA is called cubic picture fuzzy hybrid averaging operator.

$$\text{CPFHA}(U_1, U_2, \ldots, U_n) = \left(\begin{array}{c}
\left(\begin{array}{c}h^{-1}\left(\sum_{i=1}^{n} \bar{w}_i h(M_{o(0)})\right)\right), g^{-1}\left(\sum_{i=1}^{n} \bar{w}_i g(L_{o(0)})\right), g^{-1}\left(\sum_{i=1}^{n} \bar{w}_i g(N_{o(0)})\right)
\end{array}\right)\\
\left(\begin{array}{c}h^{-1}\left(\sum_{i=1}^{n} \bar{w}_i h(M'_{o(0)})\right)\\ g^{-1}\left(\sum_{i=1}^{n} \bar{w}_i g(L'_{o(0)})\right)\\ g^{-1}\left(\sum_{i=1}^{n} \bar{w}_i g(N'_{o(0)})\right)
\end{array}\right)
\end{array}\right).$$

**Proof.** The proof is similar to Theorem 20, so it is omitted here. \quad \square

5. MCDM Based on the Proposed Operation

In this section, we need the previous aggregation operators in a decision-making for CPFSs with illustrative example for evaluating the approach.

Let a set of $m'$ alternatives denoted by $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{m'})$ be found by the decision-maker under the set of the unlikely criteria $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n)$ whose weight vector is $\bar{w} = (\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_n)$ such that $\bar{w}_i > 0$ and $\sum_{i=1}^{n} \bar{w}_i = 1$.

Suppose that the ranking of an alternative $x_j, (j = 1, 2, \ldots, m')$ on the criteria $\mathcal{G}_i, (i = 1, 2, \ldots, m')$ is assessed by the decision-maker in the form of CPFSs $U_{ij} = (\bar{A}_{ij}, \mathcal{B}_{ij}), i, j \in (1, 2, \ldots, n)$, where $\bar{A}_{ij}$ is the degree of VPFS and $\mathcal{B}_{ij}$ is the degree of PFS that the alternative $\mathcal{A}_i$ does not satisfy the attribute $\mathcal{G}_i$. So we develop an approach for evaluating the best alternative based on the proposed operators for MCGDM problem whose steps are as follows:

**Step 1.** Construct the decision matrix of CPFSs.

$U_{ij} = (\bar{A}_{ij}, \mathcal{B}_{ij})$, where $\bar{A}_{ij} = (M_{ij}, L_{ij}, N_{ij})$ are VPFS and $\mathcal{B}_{ij} = (\bar{H}_{ij}, \bar{P}_{ij}, \bar{S}_{ij})$ are the PFSs towards the alternative $\mathcal{A}_i$ and hence construct a cubic picture fuzzy decision matrix $\mathcal{D} = (P_{ij})_{m'\times n}$.

**Step 2.** Normalized decision matrix, namely, cost ($C$) and benefits ($B$), so we normalize

$$r_{ij} = \begin{cases} 
\bar{P}_{ij}, & k \in B, \\
\bar{P}_{ij}, & k \in C,
\end{cases}$$

(40)

where $\bar{P}_{ij}$ is the complement of $\bar{P}_{ij}$.

**Step 3.** Aggregated assessment of alternative, based on the decision matrix, as taken from step 2, all the aggregated values of the alternatives $\mathcal{A}_i, (i = 1, 2, \ldots, m')$ under the different criteria $\mathcal{G}_i$ are obtained by using either CPFWA or CPFOWA or CPFHA operator and we collect the value of $r_i$ for each alternative $\mathcal{A}_i, (i = 1, 2, \ldots, m')$.

**Step 4.** We compute the score values of $r_i (i = 1, 2, \ldots, n)$.

**Step 5.** At last, we find that the rank of the alternatives $\mathcal{A}_i, (i = 1, 2, \ldots, m')$ according to the descending value of the score value are most valuable.

6. Illustrative Example

In this section, we illustrate with the mathematical example for the decision-making studied as follows.

Suppose few companies design their financial strategy for the next fiscal year, and according to their plan of strategy, they are picking three alternatives defined as follows: $\mathcal{A}_1$: to invest in the "Chinese markets"; $\mathcal{A}_2$: to invest in the "Indian markets"; and $\mathcal{A}_3$: to invest in "USA markets." These proceed for finding the aspect as follows: $G_1$: "the increases analysis," $G_2$: "the decreases analysis," and $G_3$: "the neutral analysis," whose weight vector $w = (0.5, 0.2, 0.3)^T$.

6.1. Example by the CPFWA Operator. The example is applied in CPFWA operator to calculate the best one.

**Step 1.** Three these alternatives $\mathcal{A}_1, (i = 1, 2, 3)$ are to be solved by an expert under the three aspects $G_j (j = 1, 2, 3)$ by using cubic picture fuzzy decision matrix $\mathcal{D} = (P_{ij})_{3\times 3} = ([A]_{ij}, [B]_{ij})$ for $(i, j = 1, 2, 3)$.

**Step 2.** Since the criteria $G_1$ and $G_2$ are the porches criteria while $G_3$ are losses criteria $\bar{R} = (r_{ij})_{3\times 3}$ equation (40) is used as follows:
The effectiveness of the proposed methodology can be resolved more accurately. We compared a numerical example showed that the proposed operators developed multicriteria decision-making (MCDM) to prove the problem and the results are discussed. Furthermore, we are discussed. Several aggregation operators are defined for aggregation operators are subjected to a decision-making problem in pattern recognition. The proposed operators are subjected to a decision-making problem and the results are discussed. Furthermore, we developed multicriteria decision-making (MCDM) to prove the effectiveness and validity of the proposed methodology. A numerical example showed that the proposed operators can resolve decision-making more accurately. We compared these with predefined operators to show the validity and effectiveness of the proposed methodology.

In the future, some similarity measures for CPFS can be developed and can be applied in pattern recognition problems. We will define other methods with CPFS such as Dombi aggregation operators and introduce the idea of cubic picture fuzzy Dombi weighted average (CPFDOWA), cubic picture fuzzy Dombi weighted geometric (CPFDWG), cubic picture fuzzy Dombi ordered weighted geometric (CPFDOWG), and generalized operators in multicriteria decision-making.

**Data Availability**

No data were used to support the study.

**Conflicts of Interest**

The authors declare that there are no conflicts of interest.

**Acknowledgments**

The authors are grateful to the Deanship of Scientific Research, King Saud University, for funding through Vice Deanship of Scientific Research Chairs.

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