Accurate fields of a radially polarized Gaussian laser beam

Yousef I Salamin
Physics Department, American University of Sharjah, PO Box 26666, Sharjah, United Arab Emirates
E-mail: ysalamin@aus.edu

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Abstract. Explicit truncated series expressions modelling the fields of a focused radially polarized Gaussian laser beam are derived, accurate to order $\varepsilon^{15}$, where $\varepsilon$ is the associated fundamental Gaussian beam diffraction angle. The new terms make significant corrections to the known paraxial field components, the corrected fields satisfy Maxwell’s equations exactly, and their series converge for $\varepsilon < 1$ and diverge for $\varepsilon \geq 1$.

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1. Introduction

One way to produce radially polarized light is to submit it to an axicon of some sort [1, 2]. The output beam from an axicon optical element is radially polarized: it has an on-axis pencil-like focus, a single (azimuthally polarized) magnetic field component and two electric field components: one axially polarized and the other transversely polarized. An output beam of these characteristics may be useful in a number of applications. Take, for example, laser electron acceleration [3]–[5]. Electrons injected axially into the beam will be accelerated by the axial electric field component and confined by the magnetic field component [6]. The fact that the transverse electric field component vanishes on the beam axis is also a welcome feature for this particular application in that it does not work to diffract the accelerated electrons out of the electron beam.

Some recent experiments [7, 8] have shown that radially polarized light may be focused to a spot size significantly smaller than in the case of linearly polarized light. Focusing, in general, is required for many applications. In the case of a radially polarized beam, it results in a more intense axial electric field component. Theoretically, the paraxial approximation suffices to describe the electromagnetic fields of such a beam, provided the focusing is not too tight. But when focusing is made to a spot size of the order of magnitude of the laser wavelength or less, the paraxial approximation ceases to model the fields accurately enough.

The present author has recently shown [9] that significant corrections (of order up to $\varepsilon^5$, where $\varepsilon$ is the Gaussian beam diffraction angle) ought to be added to the paraxial approximation field expressions if they are to model a tightly focused radially polarized beam reasonably accurately. The aim of this paper is to develop the truncated series method and use it to arrive at field expressions for the radially polarized laser beam, accurate to $O(\varepsilon^{15})$. It will be shown that the derived electromagnetic field modes of a Gaussian beam are radially polarized in the sense described above for the fields of a beam submitted to an axicon, and reduce to the fields of an axicon exactly in the paraxial approximation limit. Convergence of the obtained series on the beam axis, together with the relative uncertainties introduced by employing the paraxial approximation in describing the fields there, will be discussed. Finally, an accurate expression for the output beam power will be derived and the relative uncertainty in the calculated beam power stemming from using the paraxial approximation will be discussed.

The fields will be derived from a truncated series representation of the vector potential, along the lines of the work of Lax et al [10] and Davis [11] advanced many years ago. In this method a judicious choice is made for the fields $\mathbf{E}$ and $\mathbf{B}$ (or, alternatively, the vector and scalar potentials $A$ and $\Phi$, respectively, as will be done below). The appropriate wave equations are then solved in cylindrical coordinates, suitable for the description of a Gaussian beam, using a truncated series in powers of $\varepsilon^2$. This method may, in principle, be used to calculate high-order corrections to several kinds of beams, including the fundamental Gaussian, the Laguerre–Gaussian and the Hermite–Gaussian beams [12]. Fields of a radially polarized beam, to be treated in this paper, will often be referred to as the axicon-Gaussian fields, because they will be modelled in terms of the parameters of a fundamental Gaussian beam, namely, its waist radius at focus $w_0$, its Rayleigh range $z_r \equiv kw_0^2/2$, where $k$ is the wavenumber, and its diffraction angle $\varepsilon \equiv w_0/z_r$. Every field component will be modelled by a truncated series with $\varepsilon^2$ playing the role of expansion parameter of smallness.
2. Derivation of the fields

Derivation of the (monochromatic) axicon-Gaussian fields will follow the procedure outlined by McDonald [13], apart from a slight change of notation, the use of SI units throughout, and the assumption that the fields have a time-dependence of the form \( e^{i\omega t} \), where \( \omega \) is the frequency. Propagation along the \( z \) axis and a stationary focus at the origin of coordinates will be assumed. According to McDonald [13], the axicon modes of a Gaussian laser beam (wave number \( k = \frac{\omega}{c} \), with \( c \) the speed of light in vacuum) may be derived from a longitudinally-polarized vector potential \( A \) and the corresponding scalar potential \( \Phi \). The vector potential satisfies the wave equation

\[
\nabla^2 A = \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2},
\]

and is related to the scalar potential \( \Phi \) by the Lorenz condition

\[
\nabla \cdot A + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} = 0.
\]

The electric and magnetic fields then follow from the potentials by differentiation

\[
E = -\nabla \Phi - \frac{\partial A}{\partial t}; \quad B = \nabla \times A.
\]

Employing cylindrical coordinates \( (r, \theta, z) \), where the \( z \)-axis is aligned with the laser beam propagation direction, the desired fields may be obtained from

\[
A(r, \theta, z, t) = \hat{z} A_0 \Psi(r, z) e^{i\eta},
\]

with \( A_0 \) a constant amplitude and \( \eta = \omega t - k z \). Direct substitution of this expression for \( A \) into the wave equation yields

\[
\nabla^2 \Psi - 2i k \frac{\partial \Psi}{\partial z} = 0.
\]

A change to the scaled coordinates

\[
\xi = \frac{x}{w_0}, \quad \nu = \frac{y}{w_0}, \quad \zeta = \frac{z}{z_r}, \quad \rho^2 = \xi^2 + \nu^2,
\]

then transforms equation (5) into

\[
\nabla^2_\perp \Psi - 4i \frac{\partial \Psi}{\partial \xi} + \epsilon^2 \frac{\partial^2 \Psi}{\partial \xi^2} = 0, \quad \nabla^2_\perp \equiv \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right).
\]

For small diffraction angles, \( \epsilon < 1 \), it is tempting to use \( \epsilon^2 \) as an expansion parameter for \( \Psi \). So, let us write

\[
\Psi = \sum_{n=0}^{\infty} \epsilon^{2n} \Psi_{2n}.
\]
This expansion, often referred to as the Lax series in treatments of various types of Gaussian beams [12], has been the subject of a number of papers recently. Linear operators, introduced by Wünsche [14] to generate exact solutions of the wave equation from paraxial solutions of the same equation, have recently been shown to lead to field expressions that may be summed under certain conditions [15, 16]. To the best of our knowledge, the axicon modes of a Gaussian beam have not been discussed in any of these publications.

When the series expansion (8) is inserted into equation (7) and after the vanishing of the coefficient of each power of \( \epsilon^2 \) is demanded, there results

\[
\nabla_\perp^2 \Psi_0 - 4i \frac{\partial \Psi_0}{\partial \zeta} = 0, \quad (n = 0),
\]

\[
\nabla_\perp^2 \Psi_{2n} - 4i \frac{\partial \Psi_{2n}}{\partial \zeta} + \frac{\partial^2 \Psi_{2n-2}}{\partial \zeta^2} = 0, \quad (n \geq 1).
\]

Equation (9) gives the paraxial approximation fields. The well known solutions of the equations corresponding to \( n = 0, 1 \) and 2, giving the paraxial and the first- and second-order corrections, are \([11, 13], [17]–[19]\).

\[
\Psi_0 = fe^{-f\rho^2},
\]

\[
\Psi_2 = \left( \frac{f}{2} - \frac{\rho^4 f^3}{4} \right) \Psi_0,
\]

\[
\Psi_4 = \left( \frac{3 f^2}{8} - \frac{3 \rho^4 f^4}{16} - \frac{\rho^6 f^5}{8} + \frac{\rho^8 f^6}{32} \right) \Psi_0,
\]

where

\[
f = \frac{i}{\xi + i} = \frac{e^{i\psi_G}}{\sqrt{1 + \xi^2}}, \quad \psi_G = \tan^{-1} \xi.
\]

In equation (14) \( \psi_G \) is the Guoy phase associated with the fundamental Gaussian beam. Based on equations (11)–(13) expressions for the fields, accurate to \( O(\epsilon^5) \), have recently been obtained [9]. In what follows we explore the need for higher-order corrections.

To find expressions for \( \Psi_{2n} \) corresponding to \( n \geq 3 \), we will be guided by hindsight and follow the work of Davis [11]. Consider a spherical wave emanating from the origin of a Cartesian coordinate system. Its fields depend on the spatial coordinates through the exponential factor \( \exp \left[ -ik\sqrt{x^2 + y^2 + z^2} \right] \). For points near the propagation axis, where \( \rho \ll z \), this factor has the following series expansion

\[
\exp \left[ -ik\sqrt{1 + \left( \frac{w_0 \rho}{z} \right)^2} \right] = e^{-kz - i\rho^2 / \zeta} \left\{ 1 + \epsilon^2 \left( \frac{i\rho^4}{4\zeta^3} \right) + \epsilon^4 \left( \frac{-i\rho^6}{8\zeta^5} - \frac{\rho^8}{32\zeta^6} \right) + \cdots \right\}.
\]

Note that the quantity between curly brackets has been expressed in terms of powers of \( \epsilon^2 \). For a tightly focused Gaussian beam, and for points satisfying the same condition, \( \rho \ll z \), we are
going to assume that dependence of the $\Psi_{2n}$ upon the coordinates $\rho$ and $\zeta$ will be of the same general structure as in equation (15), for all values of $n$. Specifically, we will write

$$\Psi_2 = (a_0 + a_4 \rho^4)\Psi_0, \quad \Psi_{2n} = \left(b_0 + \sum_{j=2}^{2n} b_{2j} \rho^{2j}\right)\Psi_0,$$

(16)

where the quantities $a_0$, $a_4$, $b_0$ and $b_{2j}$ depend only upon $\zeta$, and the second of equations (16) holds for $n \geq 2$. The ansatz (16) has been arrived at by hindsight stemming from some rigor. The exponential term in the expansion (15) suggests the place of $\Psi_0$ and its dependence upon $\rho$, which is also supported already by the known explicit forms of $\Psi_2$ and $\Psi_4$ in equations (12) and (13). Similar support from (12) and (13), as well as from the expansion (15), is lent to our choice of limits on the sum in the second of equations (16). The fact that only even powers of $\rho$ appear in (12), (13) and (15) has forced the $\rho^{2j}$ dependence in (16) upon us.

Analytic expressions for the $\Psi_{2n}$ may then be obtained from substituting equation (16) of a particular value of $n$ in the corresponding differential equation (see equation (10)) and demanding that the coefficients of all powers of $\rho^{2j}$ in the resulting expression vanish identically. The analytic expressions of the $\Psi_{2n}$ are collected in appendix A.

An ansatz for the scalar potential of the same general structure as that of the vector potential, namely, $\Phi = \phi(r, \theta, z) e^{i\omega t}$ will now be employed. This gives $\partial\Phi/\partial t = i\omega\Phi$. Using this result in the Lorenz condition one gets

$$\Phi = \frac{ic}{k} \nabla \cdot A.$$

(17)

The electric and magnetic fields then follow from equations (3)

$$E = -i\omega A - \frac{ic}{k} \nabla(\nabla \cdot A), \quad B = \nabla \times A.$$

(18)

Although we have stopped at terms in $\Psi$ of order $\varepsilon^{14}$ in the vector potential expansion, or equivalently in equation (16), the space derivatives in equations (18) will ultimately give rise to terms of order $\varepsilon^{15}$, $\varepsilon^{16}$ and so on. Similarly, when one goes for the term of order $\varepsilon^{16}$ in equation (16) one gets terms in the fields of order $\varepsilon^{16}$, $\varepsilon^{17}$... etc. Thus, terms in the fields of order $\varepsilon^{16}$ can not be accurate, if one stops at terms of order $\varepsilon^{14}$ in equation (16). Hence, we will only retain terms in the fields that are of a maximum order of $\varepsilon^{15}$ in the present work, for the accuracy of such terms is guaranteed.

According to equations (18) the components $E_\theta$, $B_r$ and $B_z$ vanish identically. This feature is characteristic of light output of an axicon. With $E_0 \equiv \omega A_0$, the remaining complex field components may be written as

$$E_r = E_0 e^{-f_0^2 + in} \sum_{\ell=0}^{7} \varepsilon^{2\ell+1} E_{2\ell+1},$$

(19)

$$E_\zeta = -i E_0 e^{-f_0^2 + in} \sum_{\ell=0}^{6} \varepsilon^{2\ell+2} E_{2\ell+2},$$

(20)

$$c B_0 = E_0 e^{-f_0^2 + in} \sum_{\ell=0}^{7} \varepsilon^{2\ell+1} B_{2\ell+1}.$$

(21)
In these equations, the coefficients in the sums are electric and magnetic field terms. These terms are collected in appendix B.

I have carefully checked and found that the fields satisfy Maxwell’s equations \( \nabla \cdot \mathbf{B} = 0 \) and \( \nabla \cdot \mathbf{E} = 0 \), exactly. A second check will be to find out whether the fields have the correct paraxial approximation limits. More on this follows in the next section.

3. Discussion

3.1. The paraxial approximation

The lowest-order terms, corresponding to \( \ell = 0 \) in equations (19)–(21), give the following (real) field components

\[
E_r = E_0 \rho C_2, \quad E_z = E_0^2 \left[ S_2 - \rho^2 S_3 \right], \quad B_\theta = E_r / c, \quad \text{\(22\)}
\]

where

\[
E = E_0 e^{-r^2/w^2}, \quad w = w_0 \sqrt{1 + \zeta^2}, \quad \text{\(23\)}
\]

\[
C_n = \left( \frac{w_0}{w} \right)^n \cos(\psi + n \psi_0), \quad n = 2, 3, \ldots, \quad \text{\(24\)}
\]

\[
S_n = \left( \frac{w_0}{w} \right)^n \sin(\psi + n \psi_0), \quad \text{\(25\)}
\]

\[
\psi = \psi_0 + \omega t - k z - \frac{k r^2}{2 R}, \quad R = z + \frac{z^2}{z}, \quad \text{\(26\)}
\]

and \( \psi_0 \) is a constant initial phase. With a minimum of effort equations (22) may be shown to give the well known paraxial axicon fields exactly [5], [20]–[22].

3.2. On the transverse plane through the focus

On the transverse plane through the focus, \( \zeta = 0 \), one has: \( f = 1, \) \( w = w_0 \), and \( \psi = \psi_0 + \omega t \). The resulting field amplitudes, scaled by \( E_0 \), will be denoted by \( e_r, e_z \) and \( b_\theta \). Surface plots of the amplitudes of \( e_r, e_z \) and \( c b_\theta \) are shown in figure 1. Note that the amplitudes of \( e_r \) and \( c b_\theta \) exhibit clear features of a hollow (radially polarized) beam: they vanish identically at \( \rho = 0 \), a distinctive characteristic of light submitted to an axicon. On the other hand, the amplitude of \( e_z \) peaks at \( \rho = 0 \), as is also expected for an axicon output beam [7, 8]. These features, as well as some more details, may be read from figure 2 which exhibits intersections of the surface plots of figures 1(a) and (b) and similar ones for the corresponding paraxial fields with the plane \( y = 0 \). The proliferation of peaks in the field amplitudes as a result of the added corrections is quite evident.
Figure 1. Surface plots of the amplitudes $e_r$, $e_z$, and $cB_\theta$ of $E_r/E_0$, $E_z/E_0$ and $cB_\theta/E_0$, respectively, in the transverse plane through the beam focus. For all plots, the diffraction angle is $\epsilon = 0.75$, which corresponds to $w_0/\lambda = 4/3\pi$.

3.3. On the beam axis

On the beam axis, $\rho = 0$, the (real) components $E_r$ and $B_\theta$ vanish identically, while $E_z$ becomes

$$E_z(0, \zeta, t) = E_0 \left\{ \epsilon^2 S_2 + \epsilon^4 \left( \frac{S_3}{2} \right) + \epsilon^6 \left( \frac{3S_4}{8} \right) + \epsilon^8 \left( \frac{3S_5}{8} \right) + \epsilon^{10} \left( \frac{15S_6}{32} \right) 
+ \epsilon^{12} \left( \frac{45S_7}{64} \right) + \epsilon^{14} \left( \frac{315S_8}{256} \right) \right\},$$

(27)

with $\psi \to \psi_0 + \omega t - kz$. It has been demonstrated in recent experiments [8] that close to three-quarters of the total beam power may be focused in this axial field component. For this reason, this component plays a central role in particle acceleration [3]–[5], [21, 22].

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The first term in equation (27) gives the paraxial representation of the axial electric field component and the remaining terms are higher-order corrections. The amplitude of $E_z(0, z, t)$, normalized by $E_0$, is shown in figure 3 as a function of the axial distance from the focus (scaled by $z_r$, the Rayleigh length). Note that the added corrections are quite significant close to the focus and die down quickly beyond $z \sim 2z_r$, for the chosen diffraction angle.

To further elucidate the importance of the added corrections we will now introduce a measure of the relative uncertainty in calculating the axial field amplitude. A plausible measure of the relative error may be exhibited by the function [16]

$$\epsilon(\zeta) \equiv \frac{E_z(0, \zeta) - E_{z, \text{paraxial}}(0, \zeta)}{E_z(0, \zeta)},$$

where $E_z(0, \zeta)$ is the field given by equation (27) and $E_{z, \text{paraxial}}(0, \zeta)$ is its paraxial approximation. This function is plotted in figure 4 for four values of the diffraction angle. The figure shows clearly that the relative error depends strongly upon the diffraction angle, approaching 100% for $\epsilon \geq 1$. The cases corresponding to $\epsilon \geq 1$ in figure 4 should not be taken seriously for two reasons. On the one hand, they describe cases far beyond the diffraction limit. On the other hand, the series giving their fields actually diverge, as will be discussed below. For all diffraction angles, however, the relative error drops quickly with increasing distance from the beam’s central focus.

**Figure 2.** Intersections of surfaces similar to those of figures 1(a) and (b), respectively, with the plane $y = 0$. The legends given in (b) apply to (a) as well.
Figure 3. Amplitude of $E_z$ (scaled by $E_0$) on axis as a function of the distance from the focus (scaled by $z_r$).

Figure 4. Relative error in the amplitude of $E_z$ on axis as a function of the distance from the focus, calculated using equation (28). The given diffraction angles correspond (top to bottom) to $w_0/\lambda = 2/\pi, 5/4\pi, 1/\pi$ and $10/11\pi$, respectively.

Questions regarding convergence of the series (27) may not be adequately answered by looking at figure 4. Better conclusions may be drawn from figure 5 in which the on-axis $E_z$ amplitude is plotted as a function of the distance from the focus for four values of the diffraction angle. In each part, seven curves are shown: one for the paraxial amplitude (of order $\varepsilon^2$), a second for the paraxial amplitude plus the correction of order $\varepsilon^4$, and so on, up to the full non-paraxial amplitude containing all corrections (up to order $\varepsilon^{14}$). Almost independently of the value of $\varepsilon$ all curves coincide with the paraxial term for large $\zeta$. Thus it may be stated in general that the
paraxial approximation describes adequately the on-axis field component at points far away from the central focus. Closer to the focus, however, each correction adds significantly to the paraxial term for large values of $\varepsilon$ and less so for small values. In fact, $\varepsilon = 1$ seems from the figure to separate two regimes. For $\varepsilon < 1$, the added corrections contribute less and less with increasing order. This leads to the conclusion in support of the series expansion converging quickly for values of $\varepsilon \ll 1$ (figure 5(a)) and less so for $\varepsilon$ slightly less than unity (figure 5(b)). The opposite appears to be true for $\varepsilon \gtrsim 1$ as is made quite evident in figures 5(c) and (d). The series expansion clearly diverges for $\varepsilon \gtrsim 1$.

4. The beam power

Laser systems are often characterized by their output beam power. An expression for the beam power may be obtained by integrating the axial component of the Poynting vector over the whole transverse plane through the focus and then averaging the result over the time. In so doing, using the fields above, one arrives at an expression containing terms of $O(\varepsilon^{30})$. However, terms of $O(\varepsilon^{18})$ and beyond cannot be accurate. Consider, for example, the term in the power of $O(\varepsilon^{18})$. This term is expected to contain a contribution from a product of terms in the fields of $O(\varepsilon)$ and $O(\varepsilon^{17})$, with the latter not available in our truncated series. Thus terms to $O(\varepsilon^{16})$ only, in the output power expression given below, will be retained for their accuracy is guaranteed.
A detailed analytic calculation yields ($s = \varepsilon/2$)

$$P = P_0 \left[ 1 + 3s^2 + 9s^4 + 30s^6 + \frac{225}{2}s^8 + \frac{12203}{32}s^{10} - \frac{3445}{32}s^{12} - \frac{928671}{128}s^{14} \right].$$

(29)

$$P_0 = \frac{\pi w_0^2 E_0^2}{2c\mu_0} s^2,$$

(30)

where $P_0$ is the beam power calculated from the fields in the paraxial approximation.

A suitable measure of the correction to the beam power, introduced by the terms in the field expressions beyond the paraxial approximation, may be given by the relative error in the calculated power, defined by

$$\delta = \frac{P - P_0}{P}.$$

(31)

The percentage error ($\delta\%$) is shown in figure 6 for output powers calculated on the basis of equation (29). Note that terms in the fields beyond $O(\varepsilon^5)$ begin to make a difference in the calculated power roughly for $\varepsilon > 0.5$, which corresponds to $w_0/\lambda < 2/\pi$. As $\varepsilon \to 1$, however, the uncertainty exceeds 65%. It should be borne in mind, though, that validity of the truncated series expansion becomes in doubt as $\varepsilon \to 1$.

5. Summary and conclusions

The main goal of this paper has been to develop the truncated series method in detail and to improve the accuracy of the representation, as truncated series in the Gaussian beam diffraction
angle $\varepsilon$, of the fields of a radially polarized laser beam, the kind one may produce by submitting a fundamental Gaussian beam to an axicon of some variety. The explicit expressions obtained employ the well known parameters of a fundamental Gaussian beam, correct to order of the fifteenth power in the diffraction angle and going far beyond the fields of order $\varepsilon^5$ obtained earlier [9]. Three important issues were then tackled which are of importance for such fields to be valid and useful, namely: (a) the derived field expressions have been shown to reduce to the well known paraxial expressions in the appropriate limits, (b) the derived fields have been checked and shown to satisfy Maxwell’s equations, and (c) the series representations have been shown to converge on the beam axis quickly for $\varepsilon < 1$ and to diverge for $\varepsilon \geq 1$.

The author is aware of the existence of series expressions that can be summed up in closed analytic forms for the fields of other types of Gaussian beams [12], [14]–[16]. Unfortunately, similar attempts pertaining to a radially polarized Gaussian beam per se do not exist, to the best of our knowledge.

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**Appendix A. The expansion coefficients**

In this appendix, explicit expressions for $\Psi_{2n}$ are given for $n = 3, 4, \ldots, 7$. The procedure outlined in section 2 leads to

$$
\Psi_6 = \left(\frac{3 f^3}{8} - \frac{3 \rho^4 f^5}{16} - \frac{\rho^6 f^6}{8} - \frac{3 \rho^8 f^7}{64} + \frac{\rho^{10} f^8}{32} - \frac{\rho^{12} f^9}{384}\right) \Psi_0,
$$

(A.1)

$$
\Psi_8 = \left(\frac{15 f^4}{32} - \frac{15 \rho^4 f^6}{64} - \frac{5 \rho^6 f^7}{32} - \frac{15 \rho^8 f^8}{256} - \frac{\rho^{10} f^9}{64} + \frac{37 \rho^{12} f^{10}}{1536} - \frac{\rho^{14} f^{11}}{256} + \frac{\rho^{16} f^{12}}{6144}\right) \Psi_0,
$$

(A.2)

$$
\Psi_{10} = \left(\frac{45 f^5}{64} - \frac{45 \rho^4 f^7}{128} - \frac{15 \rho^6 f^8}{64} - \frac{45 \rho^8 f^9}{512} - \frac{3 \rho^{10} f^{10}}{128} - \frac{5 \rho^{12} f^{11}}{1024} + \frac{9 \rho^{14} f^{12}}{512}
- \frac{17 \rho^{16} f^{13}}{4096} + \frac{\rho^{18} f^{14}}{3072} - \frac{\rho^{20} f^{15}}{122880}\right) \Psi_0,
$$

(A.3)

$$
\Psi_{12} = \left(\frac{315 f^6}{256} - \frac{315 \rho^4 f^8}{512} - \frac{105 \rho^6 f^9}{256} - \frac{315 \rho^8 f^{10}}{2048} - \frac{21 f^{11} \rho^{10}}{512} - \frac{35 f^{12} \rho^{12}}{4096} - \frac{3 f^{13} \rho^{14}}{2048}
+ \frac{211 f^{14} \rho^{16}}{16384} + \frac{f^{15} \rho^{18}}{256} + \frac{7 f^{16} \rho^{20}}{163840} - \frac{f^{17} \rho^{22}}{49152} + \frac{f^{18} \rho^{24}}{2949120}\right) \Psi_0,
$$

(A.4)
\[ \Psi_{14} = \left( \frac{315 f^7}{128} - \frac{315 f^9 \rho^4}{256} - \frac{105 f^{10} \rho^6}{128} - \frac{315 f^{11} \rho^8}{1024} - \frac{21 f^{12} \rho^{10}}{256} - \frac{35 f^{13} \rho^{12}}{2048} \right. \]
\[ \left. - \frac{3 f^{14} \rho^{14}}{1024} - \frac{7 f^{15} \rho^{16}}{16384} - \frac{79 f^{16} \rho^{18}}{8192} - \frac{573 f^{17} \rho^{20}}{163840} + \frac{f^{18} \rho^{22}}{2048} \right) \Psi_0. \] (A.5)

**Appendix B. The fields**

The various terms that go into the electric field components of equations (19) and (20) are

\[ E_1 = f^2 \rho, \] (B.1)

\[ E_2 = f^2 - f^3 \rho^2. \] (B.2)

\[ E_3 = -\frac{f^3 \rho}{2} + f^4 \rho^3 - \frac{f^5 \rho^5}{4}. \] (B.3)

\[ E_4 = \frac{f^3}{2} + f^4 \rho^2 - \frac{5 f^5 \rho^4}{4} + f^6 \rho^6. \] (B.4)

\[ E_5 = -\frac{3 f^4 \rho}{8} - \frac{3 f^5 \rho^3}{8} + \frac{17 f^6 \rho^5}{16} - \frac{3 f^7 \rho^7}{8} + \frac{f^8 \rho^9}{32}. \] (B.5)

\[ E_6 = \frac{3 f^4}{8} + \frac{3 f^5 \rho^2}{8} + \frac{3 f^6 \rho^4}{16} - \frac{19 f^7 \rho^6}{16} + \frac{13 f^8 \rho^8}{32} - \frac{f^9 \rho^{10}}{32}. \] (B.6)

\[ E_7 = -\frac{3 f^5 \rho}{8} - \frac{3 f^6 \rho^3}{8} - \frac{3 f^7 \rho^5}{16} + \frac{33 f^8 \rho^7}{32} - \frac{29 f^9 \rho^9}{64} + \frac{f^{10} \rho^{11}}{16} - \frac{f^{11} \rho^{13}}{384}. \] (B.7)

\[ E_8 = \frac{3 f^5}{8} + \frac{3 f^6 \rho^2}{8} + \frac{3 f^7 \rho^4}{16} + \frac{f^8 \rho^6}{16} - \frac{69 f^9 \rho^8}{64} + \frac{31 f^{10} \rho^{10}}{384} - \frac{25 f^{11} \rho^{12}}{384} + \frac{f^{12} \rho^{14}}{384}. \] (B.8)

\[ E_9 = -\frac{15 f^6 \rho}{32} - \frac{15 f^7 \rho^3}{32} - \frac{15 f^8 \rho^5}{64} - \frac{5 f^9 \rho^7}{64} + \frac{247 f^{10} \rho^9}{256} + \frac{127 f^{11} \rho^{11}}{256} \]
\[ + \frac{137 f^{12} \rho^{13}}{1536} - \frac{5 f^{13} \rho^{15}}{768} + \frac{f^{14} \rho^{17}}{6144}. \] (B.9)
Similarly, the magnetic field terms of equation (21) are

\[ B_1 = f^2 \rho, \]  
\[ B_3 = \frac{f^3 \rho}{2} + \frac{f^4 \rho^3}{2} - \frac{f^5 \rho^5}{4}, \]
\[ B_5 = \frac{3 f^4 \rho}{8} + \frac{3 f^5 \rho^3}{8} + \frac{3 f^6 \rho^5}{16} - \frac{f^7 \rho^7}{4} + \frac{f^8 \rho^9}{32}, \quad \text{(B.18)} \]

\[ B_7 = \frac{3 f^5 \rho}{8} + \frac{3 f^6 \rho^3}{8} + \frac{3 f^7 \rho^5}{16} + \frac{f^8 \rho^7}{16} - \frac{13 f^9 \rho^9}{64} + \frac{3 f^{10} \rho^{11}}{64} - \frac{f^{11} \rho^{13}}{384}, \quad \text{(B.19)} \]

\[ B_9 = \frac{15 f^6 \rho}{32} + \frac{15 f^7 \rho^3}{32} + \frac{15 f^8 \rho^5}{64} + \frac{5 f^9 \rho^7}{64} + \frac{5 f^{10} \rho^9}{256} - \frac{f^{11} \rho^{11}}{256} + \frac{79 f^{12} \rho^{13}}{1536}, \quad \text{(B.20)} \]

\[ B_{11} = \frac{45 f^7 \rho}{64} + \frac{45 f^8 \rho^3}{64} + \frac{45 f^9 \rho^5}{128} + \frac{15 f^{10} \rho^7}{128} + \frac{15 f^{11} \rho^9}{512} + \frac{3 f^{12} \rho^{11}}{512} - \frac{131 f^{13} \rho^{13}}{1024} - \frac{f^{14} \rho^{15}}{192} + \frac{f^{15} \rho^{17}}{6144}. \quad \text{(B.21)} \]

\[ B_{13} = \frac{315 f^8 \rho}{256} + \frac{315 f^9 \rho^3}{256} + \frac{315 f^{10} \rho^5}{512} + \frac{105 f^{11} \rho^7}{512} + \frac{105 f^{12} \rho^9}{2048} + \frac{21 f^{13} \rho^{11}}{2048} - \frac{7 f^{14} \rho^{13}}{4096} + \frac{107 f^{15} \rho^{15}}{1024} - \frac{787 f^{16} \rho^{17}}{16384} - \frac{135 f^{17} \rho^{19}}{16384} + \frac{323 f^{18} \rho^{21}}{491520} - \frac{f^{19} \rho^{23}}{40960} + \frac{f^{20} \rho^{25}}{2949120}. \quad \text{(B.22)} \]

\[ B_{15} = \frac{315 f^9 \rho}{128} + \frac{315 f^{10} \rho^3}{128} + \frac{315 f^{11} \rho^5}{256} + \frac{105 f^{12} \rho^7}{256} + \frac{105 f^{13} \rho^9}{1024} + \frac{21 f^{14} \rho^{11}}{1024} + \frac{7 f^{15} \rho^{13}}{2048} + \frac{f^{16} \rho^{15}}{2048} - \frac{1429 f^{17} \rho^{17}}{16384} + \frac{731 f^{18} \rho^{19}}{16384} - \frac{1453 f^{19} \rho^{21}}{163840} + \frac{431 f^{20} \rho^{23}}{491520} - \frac{269 f^{21} \rho^{25}}{5898240} + \frac{7 f^{22} \rho^{27}}{5898240} - \frac{f^{23} \rho^{29}}{82575360}. \quad \text{(B.23)} \]

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In a recent publication [1] coefficients of the last three terms in equation (29) have been erroneously reported. The correct expression for the laser output power, when terms in the truncated series of the radially polarized fields to $O(\varepsilon^{15})$ are retained, is

$$P = P_0 \left[ 1 + 3s^2 + 9s^4 + 30s^6 + \frac{225}{2} s^8 + \frac{945}{2} s^{10} + 2205s^{12} + 11340s^{14} \right].$$

(1)

The only place in [1] where equation (29) has been used is to produce figure 6 (curve corresponding to field terms to order $\varepsilon^{15}$). In figure 1 this curve is reproduced along with a similar curve based on equation (1). It is obvious that, for $\varepsilon < 0.8$ the deviation is minor, due to the smallness of the last three terms. The sentence before the very last in [1] may be made to read as follows: ‘As $\varepsilon \to 1$, for which validity of the truncated series is in doubt anyway, the uncertainty reaches 80%’. All our other results and conclusions in [1] remain the same.

![Figure 1](image_url)

**Figure 1.** Percentage error in the calculated power when the paraxial approximation is used to describe the laser fields, assuming the fields to $O(\varepsilon^{15})$ are the most accurately known, using equation (1) above (solid line) and using equation (29) of [1] (dotted line).
Equation (29) of [1] has also been used in another publication [2]. Deviations from the results presented there when the above (correct) equation is used are also minor, as long as $\varepsilon < 0.8$, and do not affect any of our conclusions therein [3].

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