Conjugacy Classes of Involutive Automorphisms of the $C^{(1)}_\ell$ Affine Kac-Moody Algebras

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Abstract

The conjugacy classes of the involutive automorphisms of the affine Kac-Moody algebras $C^{(1)}_\ell$ for $\ell \geq 2$ are determined using the matrix formulation of automorphisms of an affine Kac-Moody algebra. PACS: 2.20.+b, 03.30.+p, 11.17.+y, 12.40Aa.

1 Introduction

The significant role played by the automorphism groups of semi-simple Lie algebras is well known. In particular, the study of the involutive automorphisms of complex semi-simple Lie algebras by Gantmacher [1] allowed Gantmacher [2] to obtain a very elegant systematic determination of all the simple real Lie algebras. As one can expect a similar situation for the affine Kac-Moody algebras, it is clearly important to determine the conjugacy classes of the group of automorphisms of these algebras as well. Moreover, there are also significant implications for the Virasoro algebra and conformal field theory through the Sugawara construction. The part played by Kac-Moody automorphisms in this context has been discussed by Bernard [3], Walton [4], Bouwknegt [5], and Font [6].

The structure of the affine Kac-Moody algebras and their Weyl groups is now very well established. (For reviews see Kac [7], Goddard and Olive [8, 9], and Cornwell [10]). Unless otherwise stated all the notations and conventions that will be employed in the present paper are those of [10]. In particular, quantities belonging to the simple complex Lie algebra $\hat{L}^0$ associated with an untwisted affine Kac-Moody algebra $\hat{L}$ are distinguished from the corresponding quantities belonging to $\hat{L}$ by a superscript 0, so that, for example, $\alpha$ is the linear functional on the Cartan subalgebra $H$ of $\hat{L}$ that is the extension of the linear functional $\alpha^0$ on the Cartan subalgebra $H^0$ of $\hat{L}^0$. 
An automorphism of an affine Kac-Moody algebra \( \tilde{\mathcal{L}} \) that maps every element of its Cartan subalgebra \( \mathcal{H} \) into an element of \( \mathcal{H} \) is called a “Cartan preserving automorphism”. The study of this type of automorphism was initiated by Gorman et al. [11] and extended further by Cornwell [12]. Although such automorphisms are very important, in that every conjugacy class of the automorphism group contains at least one Cartan preserving automorphism, it is necessary to go beyond such automorphisms. The main reason for this is that it is possible for two Cartan preserving automorphisms to be conjugate members of the group of all automorphisms of an affine Kac-Moody algebra, even though they are not conjugate within the subgroup of Cartan preserving automorphisms. That is, conjugacy of Cartan preserving automorphisms within the group of all automorphisms of an affine Kac-Moody algebra is often achieved via “non-Cartan preserving automorphisms”.

To enable this problem to be tackled systematically, a comprehensive method of dealing with all the automorphisms of an untwisted affine Kac-Moody algebra based on a matrix formulation of the untwisted affine Kac-Moody algebras was developed by Cornwell [12] (extending some previous work on the corresponding “derived subalgebra” \( \tilde{\mathcal{L}}' \) by Levstein [13]. It was shown in [12] that in general there are four types of automorphism within this matrix formulation, which were called “type 1a”, “type 1b”, “type 2a”, and “type 2b”. The explicit derivation of all of these types of automorphism, as well as the investigation of identical automorphisms and of the identity automorphism, the formulae for the products of automorphisms, the conditions for an automorphism to be involutive, the formulae for the inverses of automorphisms, and the conjugacy conditions for automorphisms may all be found [12]. The account given below just describes the parts that are essential for the present analysis. Previous papers (Cornwell [14, 15, 16], Clarke and Cornwell [18]) have discussed in detail the conjugacy classes of the involutive automorphisms of the algebras \( A^{(1)}_{\ell} \) and \( B^{(1)}_{\ell} \) (for \( \ell \geq 1 \)).

The present paper is devoted to a derivation of the conjugacy classes of the involutive automorphisms within the group of all automorphisms of \( C^{(1)}_{\ell} \), the analysis of which presents a number of new features. Its organization is as follows. In Section 2 the essential facts concerning the structure of \( C^{(1)}_{\ell} \) and the relevant formulae of the matrix formulation are very briefly summarized, and a concise notation for certain matrices is presented. The transformations of the Weyl group of \( C^{(1)}_{\ell} \) are given in Section 3 and are used in the first subsection of Section 4 to deduce the matrices that generate the required involutive automorphisms. Some general results on the conjugacy problem for \( C^{(1)}_{\ell} \) are obtained in the second subsection of Section 4, and these are developed further for the three specific types of automorphism that occur for \( C^{(1)}_{\ell} \) in Section 5, Section 6, and Section 7. The results are summarized in Section 8.

2 Preliminaries

2.1 The structure of \( \tilde{\mathcal{L}} \)

It is well known ([5, 6, 7, 8]) that every complex untwisted affine Kac-Moody algebra \( \tilde{\mathcal{L}} \) can be constructed from its corresponding simple complex Lie algebra \( \tilde{\mathcal{L}}^0 \) in the following way. \( \tilde{\mathcal{L}} \) may be taken to have a general element of the form

\[
\sum_{j=-\infty}^{\infty} \sum_{p=1}^{n^0} \mu_{jp} l^j \otimes a_p^0 + \mu_c c + \mu_d d, \tag{1}
\]

where \( \mu_{jp}, \mu_c, \) and \( \mu_d \) are arbitrary complex numbers, with only a finite number of the \( \mu_{jp} \) being non-zero. Here \( j \) takes any integer value, \( a_p^0 \) are the basis elements of \( \mathcal{L}^0 \) (where \( p = 1, 2, \ldots, n^0 \),
$n^0$ being the order of $\tilde{L}^0$, and where $t$ is a complex number). The commutators of $\tilde{L}$ are given by

$$[t^j \otimes a^0, t^k \otimes b^0] = t^{j+k} \otimes [a^0, b^0] + j\delta^{j+k,0}B^0(a^0, b^0)c$$

(2)

(for all integers $j$ and $k$ and all $a^0, b^0 \in \tilde{L}^0$, where the commutators and Killing form $B^0(, )$ of the right-hand side are those of $\tilde{L}^0$),

$$[t^j \otimes a^0, c] = 0$$

(3)

(for all integers $j$ and all $a^0 \in \tilde{L}^0$, and where again the commutator on the right-hand side of (2) is that of $\tilde{L}^0$),

$$[d, t^j \otimes a^0] = j t^j \otimes a^0,$$

(4)

(for all integers $j$ and all $a^0 \in \tilde{L}^0$), and

$$[d, c] = 0.$$  

(5)

If $h^0_{\alpha_k}$ (for $k = 1, 2, \ldots, \ell$) are the Weyl basis elements of the Cartan subalgebra $H^0$ of $\tilde{L}^0$ corresponding to the simple roots $\alpha^0_k$ (for $k = 1, 2, \ldots, \ell$) of $\tilde{L}^0$, then the $\ell + 2$ basis elements $c$, $d$, and $t^0 \otimes h^0_{\alpha_k}$ (for $k = 1, 2, \ldots, \ell$) constitute a basis for a Cartan subalgebra $H$ of $\tilde{L}$. The basis of the "derived subalgebra" $\tilde{L}$ of $\tilde{L}$ consists of all the basis elements of $\tilde{L}$ except $d$. Every linear functional, and, in particular, every root, $\alpha^0$ that is defined on $H^0$ can be "extended" to give a linear functional $\alpha$ on $H$ by the definitions

$$\alpha(t^0 \otimes h^0_{\alpha_k}) = \alpha^0(h^0_{\alpha_k}) \text{ (for } k = 1, 2, \ldots, \ell), \alpha(c) = 0, \alpha(d) = 0.$$  

(6)

Moreover, if $\delta$ is the linear functional on $H$ defined by

$$\delta(t^0 \otimes h^0_{\alpha_k}) = 0 \text{ (for } k = 1, 2, \ldots, \ell), \delta(c) = 0, \delta(d) = 1,$$

(7)

then, for every integer value of $j$, $t^j \otimes c^0_{\alpha_k}$ corresponds to a root $j\delta + \alpha$ of $\tilde{L}$ and $t^j \otimes h^0_{\alpha_k}$ corresponds to a root $j\delta$ of $\tilde{L}$ (for $k = 1, 2, \ldots, \ell$). Moreover, for $k = 1, 2, \ldots, \ell$ the simple roots $\alpha_k$ of $\tilde{L}$ are just the extensions of the simple roots $\alpha^0_k$ of $\tilde{L}^0$, and the remaining simple root $\alpha_0$ of $\tilde{L}$ is given by $\alpha_0 = \delta - \alpha_H$, where $\alpha_H$ is the extension of the highest root $\alpha_H^0$ of $\tilde{L}^0$.

### 2.2 Matrix formulation of the automorphisms of $\tilde{L}$

Let $\Gamma$ be a faithful irreducible representation of some dimension $d_\Gamma$ of $\tilde{L}^0$, and let $\gamma$ be the Dynkin index of $\Gamma$. Then the first term $\sum_{j=-\infty}^{\infty} \sum_{p=1}^{n^0} \mu_{jp} t^j \otimes a^0_p$ of the general element (2) of the affine untwisted Kac-Moody algebra $\tilde{L}$ is represented by the $d_\Gamma \times d_\Gamma$ matrix $\sum_{j=-\infty}^{\infty} \sum_{p=1}^{n^0} \mu_{jp} t^j \Gamma(a^0_p)$. A typical matrix of this form will be denoted by $a(t)$, i.e.

$$a(t) = \sum_{j=-\infty}^{\infty} \sum_{p=1}^{n^0} \mu_{jp} t^j \Gamma(a^0_p).$$

(8)

Clearly all the entries of $a(t)$ are Laurent polynomials in the complex variable $t$. A typical element of $\tilde{L}$ can then be written as

$$a(t) + \mu_c c + \mu_d d.$$

(9)

(Of course in no sense do the + signs in (9) represent ordinary matrix addition).

Each of the four types (1a, 1b, 2a, and 2b) of automorphism of $\tilde{L}$ depends on the following three quantities (although the dependence is different for the different types):
1. A \( d_\mathcal{T} \times d_\mathcal{T} \) matrix \( U(t) \), which is assumed to be invertible and for which all the entries of \( U(t) \) and \( U(t)^{-1} \) are assumed to be Laurent polynomials in the complex variable \( t \);

2. A non-zero complex parameter \( u \);

3. An arbitrary complex parameter \( \xi \).

It is convenient to exhibit these together as a triple in the form \( \{ U(t), u, \xi \} \). As noted below, for \( C^{(1)}_t \) the type 1b automorphisms and the type 2b automorphisms coincide with the type 1a and the type 2a automorphisms respectively, so only the actions of type 1a and type 2a automorphisms \( \phi \) need be explicitly displayed:

1. Actions on \( a(t) \):
   
   (a) For type 1a automorphisms:
   
   \[
   \phi(a(t)) = U(t)a(ut)U(t)^{-1} + \frac{1}{\gamma} \text{Res}\{\text{tr}(U(t)^{-1} \frac{dU(t)}{dt} a(ut))\}c. \tag{10}
   \]

   (b) For type 2a automorphisms:
   
   \[
   \phi(a(t)) = U(t)a(ut^{-1})U(t)^{-1} + \frac{1}{\gamma} \text{Res}\{\text{tr}(U(t)^{-1} \frac{dU(t)}{dt} a(ut^{-1}))\}c. \tag{11}
   \]

2. Actions on \( c \) and \( d \):
   
   \[
   \phi(c) = \mu c, \tag{12}
   \]

   and

   \[
   \phi(d) = \mu \Phi(U(t)) + \xi c + \mu d, \tag{13}
   \]

   where \( \Phi(U(t)) \) is the \( d_\mathcal{T} \times d_\mathcal{T} \) matrix that depends on \( U(t) \) according to the definition

   \[
   \Phi(U(t)) = \{-t \frac{dU(t)}{dt} U(t)^{-1} + \frac{1}{d_\mathcal{T}} \text{tr}(t \frac{dU(t)}{dt} U(t)^{-1}) 1\}, \tag{14}
   \]

   and

   \[
   \mu = \begin{cases} 
   1 & \text{for type 1a}, \\
   -1 & \text{for type 2a}. 
   \end{cases} \tag{15}
   \]

   If the triples \( \{ U(t), u, \xi \} \) and \( \{ U'(t), u', \xi' \} \) specify two automorphisms of the same type, then these automorphisms are identical if and only if

   \[
   u' = u, \xi' = \xi, \tag{16}
   \]

   and there exists a non-zero complex number \( \eta \) and an integer \( k \) such that

   \[
   U'(t) = \eta t^k U(t). \tag{17}
   \]

   A type 1a automorphism corresponding to the triple \( \{ U(t), u, \xi \} \) is involutive if and only if the following three conditions are all satisfied:

   \[
   U(t) U(ut) = \eta t^k 1 \tag{18}
   \]
for some complex number $\eta$ and some integer $k$, 
\[ u^2 = 1, \]  
and 
\[ \xi = -\frac{1}{2\gamma} \text{Res}\{\text{tr}(U(t)^{-1} \frac{dU(t)}{dt} \Phi(U(ut)))\}. \]  
Similarly a type 2a automorphism corresponding to the triple $\{U(t), u, \xi\}$ is involutive if and only if the following two conditions are all satisfied:
\[ U(t) U(ut^{-1}) = \eta^k 1 \]  
for some complex number $\eta$ and some integer $k$, and 
\[ \xi = -\frac{1}{2\gamma} \text{Res}\{\text{tr}(U(t)^{-1} \frac{dU(t)}{dt} \Phi(U(ut^{-1})))\}. \]  

The necessary and sufficient conditions for the conjugacy of a pair of automorphisms $\phi_1$ and $\phi_2$ of $\tilde{\mathcal{L}}$ corresponding to the triples $\{U_1(t), u_1, \xi_1\}$ and $\{U_2(t), u_2, \xi_2\}$ respectively via an automorphism $\phi$ of $\tilde{\mathcal{L}}$ corresponding to the triple $\{S(t), s, \xi\}$ will now be stated. More precisely, these are the necessary and sufficient conditions for the automorphism equality $\phi_1 = \phi \circ \phi_2 \circ \phi^{-1}$ to hold. In each case there three such conditions, but only two will be exhibited explicitly. These are the conditions relating the matrix $U_1(t)$ to the matrix $U_2(t)$ and the parameter $u_1$ to the parameter $u_2$. It is possible to also relate the parameter $\xi_1$ to the parameter $\xi_2$, but the resulting expressions will be omitted as they are very complicated and will not be needed in the subsequent analysis. If $\phi_1$ and $\phi_2$ are conjugate then they must be of the same type.

1. If $\phi_1$ and $\phi_2$ are two type 1a automorphisms and $\phi$ is a type 1a automorphism, the conditions are:
\[ \eta^k U_1(t) = S(t) U_2(st) S(u_2 t)^{-1}, \]  
where $\eta$ is any non-zero complex number and $k$ is any integer, and 
\[ u_1 = u_2. \]

2. If $\phi_1$ and $\phi_2$ are two type 1a automorphisms and $\phi$ is a type 2a automorphism, the conditions are:
\[ \eta^k U_1(t) = S(t) U_2(st^{-1}) S(u_2^{-1} t)^{-1}, \]  
where $\eta$ is any non-zero complex number and $k$ is any integer, and 
\[ u_1 = u_2^{-1}. \]

3. If $\phi_1$ and $\phi_2$ are two type 2a automorphisms and $\phi$ is a type 1a automorphism, the conditions are:
\[ \eta^k U_1(t) = S(t) U_2(st) S(s^{-2}u_2 t^{-1})^{-1}, \]  
where $\eta$ is any non-zero complex number and $k$ is any integer, and 
\[ u_1 = s^{-2}u_2. \]
4. If \( \phi_1 \) and \( \phi_2 \) are two type 2a automorphisms and \( \phi \) is a type 2a automorphism, the conditions are:

\[
\eta t^k U_1(t) = S(t) U_2(st^{-1}) S(s^2 u_2^{-1} t^{-1})^{-1},
\]

where \( \eta \) is any non-zero complex number and \( k \) is any integer, and

\[
u_1 = s^2 u_2^{-1}.
\]

(29)

(30)

Taken together, these results imply that in searching for the conjugacy classes of involutive automorphisms of \( C^{(1)}_\ell \) only three categories of class representatives need be considered, namely:

1. type 1a involutive automorphisms with \( u = 1 \),
2. type 1a involutive automorphisms with \( u = -1 \),
3. type 2a involutive automorphisms with \( u = 1 \).

2.3 The affine Kac-Moody algebra \( C^{(1)}_\ell \)

The algebra \( C^{(1)}_\ell \) has \((\ell + 1)\) simple roots \( \alpha_0, \ldots, \alpha_\ell \) and generalized Cartan matrix \( A \), where

\[
A = \begin{pmatrix}
2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-2 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & \cdots & -1 & 2 & -2 \\
0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2
\end{pmatrix}
\]

The quantities \( <\alpha_j, \alpha_k> \), are given by

\[
<\alpha_j, \alpha_k> = B_{jk},
\]

where the matrix \( B \) is

\[
B = \frac{1}{4(\ell + 1)} \begin{pmatrix}
4 & -2 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-2 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & \cdots & -1 & 2 & -2 \\
0 & 0 & 0 & 0 & \cdots & 0 & -2 & 4
\end{pmatrix}
\]

(For the matrices \( A \) and \( B \) the index set is taken to be \( \{0, 1, \ldots, \ell\} \). As the associated simple complex Lie algebra for \( C^{(1)}_\ell \) is the algebra \( C_\ell \), it follows that \( \alpha_H = 2(\alpha_1 + \cdots + \alpha_{\ell - 1}) + \alpha_\ell \), and hence that

\[
\delta = \alpha_0 + 2(\alpha_1 + \cdots + \alpha_{\ell - 1}) + \alpha_\ell
\]

\[
c = h_0 + 2(h_{\alpha_1} + \cdots + h_{\alpha_{\ell - 1}}) + h_{\alpha_\ell}.
\]
As \( C_\ell \) is the complexification of the real Lie algebra \( sp(\ell) \), \( C_\ell \) may be taken to be the set of \( (2\ell \times 2\ell) \) complex matrices \( a \) that satisfy

\[
\tilde{a}J + Ja = 0, \quad \text{where } J = \begin{pmatrix} 0 & 1_\ell \\ -1_\ell & 0 \end{pmatrix}.
\] (31)

If a matrix \( U(t) \) is to correspond to an automorphism of \( C_\ell^{(1)} \), then the mapping

\[
a(t) \mapsto U(t)a(ut)U(t)^{-1}
\]

must stabilize the ‘matrix part’ of the algebra. That is, the image of \( a(t) \) must satisfy (31). Schur’s lemma implies that this condition is satisfied if

\[
\tilde{U}(t)JU(t) = f(t)J,
\]

where \( f(t) \) is some function of \( t \). With the assumption that \( U(t) \) and \( U(t)^{-1} \) have entries that are Laurent polynomials, then \( \det(U(t)) = \alpha t^\beta \), where \( \alpha \) is some non-zero complex number and \( \beta \) is some integer.

For \( C_\ell \) the representation \( \Gamma \) of (8) may be defined as follows. Let

\[
\alpha^j_\ell = \begin{cases} 
\varepsilon_j - \varepsilon_{j+1}, & j = 1, 2, \ldots, \ell - 1 \\
2\varepsilon_\ell & j = \ell,
\end{cases}
\]

so that the roots of \( C_\ell \) may be expressed in the form

\[
(\varepsilon_j - \varepsilon_k) \quad \text{for } 1 \leq j < k \leq \ell \\
(\varepsilon_j + \varepsilon_k) \quad \text{for } 1 \leq j \leq k \leq \ell.
\]

Then the matrices representing the basis elements of \( C_\ell \) may be taken to be

\[
\Gamma(h_{\alpha_j}) = \begin{cases} 
\{4(\ell + 1)\}^{-1}(X_{jj} - X_{j+1,j+1}) & \text{for } j = 1, 2, \ldots, \ell - 1 \\
\{2(\ell + 1)\}^{-1}X_{\ell\ell} & \text{for } j = \ell
\end{cases}
\]

\[
\Gamma(e_{\varepsilon_j - \varepsilon_k}) = \{4(\ell + 1)\}^{-\frac{1}{2}}X_{jk} \quad \text{for } j < k; j, k = 1, 2, \ldots, \ell
\]

\[
\Gamma(e_{\varepsilon_j + \varepsilon_k}) = \{4(\ell + 1)(1 + \delta_{jk})\}^{-\frac{1}{2}}Y_{jk} \quad \text{for } j \leq k; j, k = 1, 2, \ldots, \ell,
\]

where the matrices \( X \) and \( Y \) are defined below in (33). This representation \( \Gamma \) is equivalent to its contragredient representation, since

\[
\tilde{\Gamma} = -J\Gamma J^{-1},
\]

which implies that the type 1b automorphisms and the type 2b automorphisms coincide with the type 1a and the type 2a automorphisms respectively. The Dynkin index of this representation is given by

\[
\gamma = \frac{1}{2(\ell + 1)}.
\] (32)

2.4 Concise notation for matrices

It is necessary to introduce some notation here to make subsequent analysis more concise. The expressions ‘dsum’ and ‘offsum’ are analogous to the commonly-used ‘diag’.
1. The first of these indicates a direct sum. For example

\[
dsum\{a, b, \ldots, y, z\} = \begin{pmatrix}
a & 0 & \cdots & 0 & 0 \\
0 & b & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & y & 0 \\
0 & 0 & \cdots & 0 & z
\end{pmatrix},
\]

where \( a, b, \ldots, y, z \) are square matrices.

2. The expression ‘offsum’ is similar to the previous expression. Thus

\[
offsum\{a, b, \ldots, y, z\} = \begin{pmatrix}
0 & 0 & \cdots & 0 & a \\
0 & 0 & \cdots & b & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & y & \cdots & 0 & 0 \\
z & 0 & \cdots & 0 & 0
\end{pmatrix},
\]

where \( a, b, \ldots, y, z \) are all square matrices.

3. The \((\ell \times \ell)\) matrices \(e_{pq}\) (where \(1 \leq p, q \leq \ell\)) are defined by

\[
(e_{pq})_{jk} = \delta_{jp}\delta_{kq}.
\]

Then \(X_{pq}\) and \(Y_{pq}\) (where \(1 \leq p, q \leq \ell\)) may be defined by

\[
X_{pq} = dsum\{e_{pq}, -e_{qp}\}, \quad Y_{pq} = offsum\{(e_{pq} + e_{qp}), 0\}.
\]

(33)

4. Some ‘general’ matrices will now be defined. These are matrices which are used to typify whole collections of specific matrices. The ‘general’ matrix \(D_{j,k}\) is defined to be the general \((k - j + 1) \times (k - j + 1)\) matrix that satisfies

\[
D_{j,k} = \text{diag}\{1, \lambda_{j+1}t^{\mu_{j+1}}, \ldots, \lambda_{k}t^{\mu_{k}}\} \quad \lambda_q^2 = 1 \quad \text{(for} \quad j + 1 \leq q \leq k \text{)},
\]

while \(D_{j,k}^o\) is of the general form \(D_{j,k}\) but has the additional constraint that \(\mu_{j+1} = \cdots = \mu_k = 0\). (For all of the general matrices it is assumed that \(\lambda_q\) is a non-zero complex number for \(1 \leq q \leq \ell\).

5. The general matrix \(G_{j,k}\) is defined by

\[
G_{j,k} = \text{diag}\{\lambda_{j}t^{\mu_{j}}, \ldots, \lambda_{k}t^{\mu_{k}}\}.
\]

\(E_{j,k}, F_{j,k}\) and \(H_{j,k}\) are defined to be variants of \(G_{j,k}\). In the case of \(E_{j,k}\) it is required that \(\mu_{j}, \ldots, \mu_{k}\) be even. In the case of \(F_{j,k}\) the requirement is that they be odd and in the case of \(H_{j,k}\) that they be zero. Furthermore \(\tilde{E}_{j,k}, \tilde{G}_{j,k}\) and \(\tilde{H}_{j,k}\) are defined to be the variants obtained by setting \(\lambda_j t^{\mu_j} = 1\) in \(E_{j,k}, G_{j,k}\) and \(H_{j,k}\) respectively.

6. The general matrices \(L_{j,k}, M_{j,k}\), and \(N_{j,k}\) are defined such that

\[
\{(L_{j,k})_{ab}, (M_{j,k})_{ab}, (N_{j,k})_{ab}\} = \\
\begin{cases}
\lambda_a t^{\mu_a}, \lambda_a t^{\mu_a}, \lambda_a t^{\mu_a} & \text{for} \quad a = b - 1; (a - j) \quad \text{even} \\
\lambda_{a-1} t^{-\mu_{a-1}}, (-1)^{\mu_{a-1}} \lambda_{a-1} t^{-\mu_{a-1}}, \lambda_{a-1} t^{\mu_{a-1}} & \text{for} \quad a = b + 1; (a - j) \quad \text{even} \\
\{0, 0, 0\} & \text{otherwise},
\end{cases}
\]

where \(a, b\) have the index set \(\{j, \ldots, k\}\).
7. The general matrices $L'_{j,k}$, $M'_{j,k}$, and $N'_{j,k}$ are defined by

\[
\{(L'_{j,k})_{ab}, (M'_{j,k})_{ab}, (N'_{j,k})_{ab}\} = \\
\{1,1,1\} \text{ for } a = j; b = j + 1 \\
\{\lambda_a t^{\mu_a}, \lambda_a t^{\mu_a}, \lambda_a t^{\mu_a}\} \text{ for } a = b = 1; a \neq j; (a - j) \text{ even} \\
\{\lambda_2 t^{\mu_2}, \lambda_2 t^{\mu_2}, \lambda_2\} \text{ for } a = 2; b = 1 \\
\{\lambda_2 \lambda_a^{-1} t^{\mu_2-\mu_2}, (-1)^{\mu_2-\mu_2} \lambda_2 \lambda_a^{-1} t^{\mu_2-\mu_2}, \lambda_2 \lambda_a^{-1} t^{\mu_2-\mu_2}\} \ a = b + 1; a \neq j; (a - j) \text{ even} \\
\{0,0,0\} \text{ otherwise},
\]

where in $M'_{j,k}$ the additional constraint is imposed that $\mu_2$ be even. The matrices $L''_{j,k}$ are defined to be of the form $L'_{j,k}$ but such that $\mu_j = \mu_{j+2} = \cdots = \mu_{k-2} = \mu_k = 0; \mu_2 = 1$.

8. Let $W_{j,k}$ be the $(k - j + 1) \times (k - j + 1)$ matrix defined by

\[
W_{j,k} = \text{diag}\{1,-1,\ldots,1,-1\},
\]

where $k$ and $j$ are such that $(k - j)$ is even.

9. For the quantities $\mu_j$ (where $1 \leq j \leq \ell$) that are encountered later on, quantities $\rho_j$ are defined to be functions of the $\mu_j$. Let

\[
\rho_j = \begin{cases} 
1 & \text{if } \mu_j \text{ is odd} \\
0 & \text{if } \mu_j \text{ is even.}
\end{cases}
\]

Furthermore, $\sigma_j$ is defined in terms of the above by letting $\sigma_j = \frac{1}{2}(\rho_j - \mu_j)$.

In general, throughout this paper, otherwise undefined quantities such as $\alpha, \lambda$ are taken to be non-zero complex numbers. In expressions like $\alpha t^{\beta}$ therefore, $\alpha$ is taken to be a non-zero complex number, with $\beta$ being implicitly assumed to be an integer.

3 The Weyl group of $C_\ell$

The following list of the representatives of each conjugacy class of involutions of $W^{(1)}$, where $W^{(1)}$ is the Weyl group of $C_\ell$, was obtained using the results of Richardson([17]). As in the case of $B_\ell$ the representatives lend themselves to listing by “families”, although such a listing is slightly more complicated for the present analysis. In this section the quantities $q$ and $r$ take integer values between 0 and $\ell$. For some families $q$ and $r$ will have other restrictions placed upon them. Thus, by taking all possible values of $q$ and $r$ (subject to restrictions), the conjugacy class representatives are obtained. In the following list the numbers $1,2,\ldots,9$ refer to the “families”. $\tau^o$ is the most general form of representative in each family. These results used in the next section.

1. This family contains just one representative given by

\[
\tau^o(\alpha_j^o) = \alpha_j^o \, \, j = 1,2,\ldots,\ell.
\]

2. This family contains only the representative given by

\[
\tau^o(\alpha_j^o) = -\alpha_j^o \, \, j = 1,2,\ldots,\ell.
\]
3. Let \( q \) range over the integers \( 2,3,\ldots,\ell - 1 \) and let \( \tau^o \) be given by
\[
\begin{align*}
\tau^o(\alpha^o_j) &= \alpha^o_j & j < q - 1 \\
\tau^o(\alpha^o_{q-1}) &= \alpha^o_{q-1} + 2(\alpha^o_1 + \cdots + \alpha^o_{\ell-1}) + \alpha^o_\ell \\
\tau^o(\alpha^o_k) &= -\alpha^o_k & k \geq q.
\end{align*}
\]
In the case where \( q = 2 \), (3) simplifies to
\[
\begin{align*}
\tau^o(\alpha^o_1) &= \alpha^o_1 + 2(\alpha^o_2 + \cdots + \alpha^o_{\ell-1}) + \alpha^o_\ell \\
\tau^o(\alpha^o_k) &= -\alpha^o_k & k \geq q.
\end{align*}
\]
4. Let \( q \) and \( r \) be such that \( q \) is odd, \( r \neq \ell \) and \( r - q > 2 \). \( \tau^o \) is given by
\[
\begin{align*}
\tau^o(\alpha^o_j) &= -\alpha^o_j & j \text{ odd and } 1 \leq j \leq q \\
\tau^o(\alpha^o_k) &= \alpha^o_{k-1} + \alpha^o_k + \alpha^o_{k+1} & k \text{ is even and } 2 \leq k \leq q - 1 \\
\tau^o(\alpha^o_{q+1}) &= \alpha^o_q + \alpha^o_{q+1} \\
\tau^o(\alpha^o_m) &= \alpha^o_m & q + 2 \leq m \leq r - 2 \\
\tau^o(\alpha^o_{r-1}) &= \alpha^o_{r-1} + 2(\alpha^o_r + \cdots + \alpha^o_{\ell-1}) + \alpha^o_\ell \\
\tau^o(\alpha^o_n) &= -\alpha^o_n & r \leq n \leq \ell.
\end{align*}
\]
5. Let \( q \) be odd and such that \( \ell - 2 > q \). Then
\[
\begin{align*}
\tau^o(\alpha^o_j) &= -\alpha^o_j & j \text{ odd and } 1 \leq j \leq q \\
\tau^o(\alpha^o_k) &= \alpha^o_{k-1} + \alpha^o_k + \alpha^o_{k+1} & k \text{ is even and } 1 < k < q \\
\tau^o(\alpha^o_{q+1}) &= \alpha^o_q + \alpha^o_{q+1} + 2(\alpha^o_{q+2} + \cdots + \alpha^o_{\ell-1}) + \alpha^o_\ell \\
\tau^o(\alpha^o_m) &= -\alpha^o_m & q + 2 \leq m \leq \ell.
\end{align*}
\]
6. Let \( q \) be odd and such that \( q \neq \ell \) if \( \ell \) is odd and also such that \( q \neq \ell - 1 \) if \( \ell \) is even. Then
\[
\begin{align*}
\tau^o(\alpha^o_j) &= -\alpha^o_j & j \text{ odd and } 1 \leq j \leq q \\
\tau^o(\alpha^o_k) &= \alpha^o_{k-1} + \alpha^o_k + \alpha^o_{k+1} & k \text{ is even and } 1 < k < q \\
\tau^o(\alpha^o_{q+1}) &= \alpha^o_q + \alpha^o_{q+1} \\
\tau^o(\alpha^o_m) &= \alpha^o_m & q + 2 \leq m \leq \ell.
\end{align*}
\]
7. In this case \( \ell - q \) is even, \( q \neq 1 \) and \( q \neq 2 \).
\[
\begin{align*}
\tau^o(\alpha^o_j) &= \alpha^o_j & 1 \leq j \leq q - 2 \\
\tau^o(\alpha^o_{q-1}) &= \alpha^o_{q-1} + Aq \\
\tau^o(\alpha^o_k) &= -\alpha^o_k & q \leq l \leq \ell \text{ and } \ell - k \text{ is even} \\
\tau^o(\alpha^o_m) &= \alpha^o_{m-1} + \alpha^o_m + \alpha^o_{m+1} & q < m < \ell \text{ and } \ell - m \text{ is odd}.
\end{align*}
\]
8. This root transformation is associated only with odd values of \( \ell \).
\[
\begin{align*}
\tau^o(\alpha^o_j) &= -\alpha^o_j & 1 \leq j \leq \ell \text{ and } j \text{ is odd} \\
\tau^o(\alpha^o_k) &= \alpha^o_{k-1} + \alpha^o_k + \alpha^o_{k+1} & 1 < k \leq \ell \text{ and } k \text{ is even}.
\end{align*}
\]
9. This root transformation is associated only with even values of \( \ell \).
\[
\begin{align*}
\tau^o(\alpha^o_j) &= -\alpha^o_j & 1 \leq j \leq \ell \text{ and } j \text{ is odd} \\
\tau^o(\alpha^o_k) &= \alpha^o_{k-1} + \alpha^o_k + \alpha^o_{k+1} & 1 < k \leq \ell \text{ and } k \text{ is even} \\
\tau^o(\alpha^o_\ell) &= 2\alpha^o_{\ell-1} + \alpha^o_\ell.
\end{align*}
\]
4 Listing of involutive automorphisms of $C_{\ell}^{(1)}$

4.1 Outline of method

There are essentially two stages in determining the conjugacy classes of an untwisted affine Kac-Moody algebra $\hat{\mathcal{L}}$ in the matrix formulation. The first is to find, for each type of automorphism, all the $U(t)$ matrices corresponding to Cartan preserving involutive automorphisms. This involves working through all the root preserving transformations of $\hat{\mathcal{L}}^0$ in the manner described in detail in [14, 15, 16, 18]. For $C_{\ell}$ these root preserving transformations are precisely those listed in the Section 3. The resulting $U(t)$ matrices are listed in the following subsection. The second stage, which is much more difficult, is to determine which of the corresponding automorphisms of $\hat{\mathcal{L}}$ are actually conjugate to each other. Some general considerations on this matter for $C_{\ell}^{(1)}$ are described in the next subsection, and these are developed further for type 1a involutive automorphisms with $u = 1$ in Section 5, for type 1a involutive automorphisms with $u = -1$ in Section 6, and for type 2a involutive automorphisms (with $u = 1$) in Section 7.

4.2 Initial listing

(i) For the type 1a involutive automorphisms with $u = 1$ the $U(t)$ matrices are:

$$\text{ds} \sum \{ D_{i,\ell}^0, D_{i,\ell}^1 \}$$

$$\text{ds} \sum \{ D_{i,\ell}^0, D_{i,\ell}^1 \}$$

$$\text{off} \sum \{ \hat{G}_{1,\ell}, \lambda_{\ell+1} t^{\mu_{\ell+1}} \hat{G}_{1,\ell}^{-1} \}$$

$$\begin{pmatrix}
D_{i,q-1}^0 & 0 & 0 & 0 \\
0 & 0 & 0 & G_{q,\ell} \\
0 & 0 & -D_{1,q-1}^0 & 0 \\
0 & G_{q,\ell}^{-1} & 0 & 0
\end{pmatrix}$$

$$\begin{pmatrix}
L_{1,q} & 0 & 0 & 0 & 0 & 0 \\
0 & D_{q+2,\ell-1}^0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & G_{r,\ell} \\
0 & 0 & 0 & -L_{1,q} & 0 & 0 \\
0 & 0 & 0 & 0 & -D_{q+2,\ell-1}^0 & 0 \\
0 & 0 & G_{r,\ell}^{-1} & 0 & 0 & 0
\end{pmatrix}$$

$$\begin{pmatrix}
L_{1,q}' & 0 & 0 & 0 \\
0 & 0 & 0 & G_{q+2,\ell} \\
0 & 0 & -L_{1,q}' & 0 \\
0 & \lambda_2 t^{\mu_2} G_{q+2,\ell}^{-1} & 0 & 0
\end{pmatrix}$$

$$\text{ds} \sum \{ L_{1,q}, D_{q+2,\ell}, L_{1,q}, D_{q+2,\ell}^0 \}$$

$$\text{ds} \sum \{ L_{1,q}, D_{q+2,\ell}, -L_{1,q}, -D_{q+2,\ell}^0 \}$$

$$\begin{pmatrix}
D_{i,q-1}^0 & 0 & 0 & 0 & 0 & 0 \\
0 & L_{q,\ell-2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & G_{\ell,\ell} \\
0 & 0 & 0 & -D_{i,q-1}^0 & 0 & 0 \\
0 & 0 & 0 & 0 & -L_{q,\ell-2} & 0 \\
0 & 0 & G_{\ell,\ell}^{-1} & 0 & 0 & 0
\end{pmatrix}$$
\[
\begin{pmatrix}
L_{1,\ell-2} & 0 & 0 & 0 \\
0 & 0 & 0 & G_{\ell,\ell} \\
0 & 0 & -\tilde{L}_{1,\ell-2} & 0 \\
0 & \lambda_2 t^{\mu_2} G_{\ell,\ell}^{-1} & 0 & 0
\end{pmatrix}
\]  
(43)

\[
dsum\{L_{1,\ell-1}, \tilde{L}_{1,\ell-1}\}
\]
(44)

\[
dsum\{\tilde{L}_{1,\ell-1}, \tilde{L}_{1,\ell-1}\}. 
\]
(45)

(ii) For the type 1a involutive automorphisms with \( u = -1 \) the \( U(t) \) matrices are:

\[
dsum\{D_{1,\ell}, D_{1,\ell}\}
\]
(46)

\[
dsum\{D_{1,\ell}, -D_{1,\ell}\}
\]
(47)

\[
\begin{pmatrix}
0 & \hat{E}_{1,\ell} \\
\lambda_{\ell+1} t^{\mu_{\ell+1}} \hat{E}_{1,\ell}^{-1} & 0
\end{pmatrix}
\]
(48)

\[
\begin{pmatrix}
D_1^{q-1} & 0 & 0 & 0 \\
0 & 0 & 0 & E_{q,\ell} \\
0 & 0 & -D_1^{q-1} & 0 \\
0 & E_{q,\ell}^{-1} & 0 & 0
\end{pmatrix}
\]
(49)

\[
\begin{pmatrix}
D_1^{q-1} & 0 & 0 & 0 \\
0 & 0 & 0 & F_{q,\ell} \\
0 & 0 & -D_1^{q-1} & 0 \\
0 & F_{q,\ell}^{-1} & 0 & 0
\end{pmatrix}
\]
(50)

\[
\begin{pmatrix}
M_{1,q} & 0 & 0 & 0 & 0 & 0 \\
0 & D_{q+2,r-1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & E_{r,\ell} & 0 \\
0 & 0 & 0 & -M_{1,q}^{-1} & 0 & 0 \\
0 & 0 & 0 & 0 & -D_{q+2,r-1} & 0 \\
0 & E_{r,\ell}^{-1} & 0 & 0 & 0 & 0
\end{pmatrix}
\]
(51)

\[
\begin{pmatrix}
M_{1,q} & 0 & 0 & 0 & 0 & 0 \\
0 & D_{q+2,r-1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & F_{r,\ell} & 0 \\
0 & 0 & 0 & -M_{1,q}^{-1} & 0 & 0 \\
0 & 0 & 0 & 0 & -D_{q+2,r-1} & 0 \\
0 & F_{r,\ell}^{-1} & 0 & 0 & 0 & 0
\end{pmatrix}
\]
(52)

\[
\begin{pmatrix}
M'_{1,q} & 0 & 0 & 0 \\
0 & 0 & 0 & F_{q+2,\ell} \\
0 & \lambda_2 t^{\mu_2} F_{q+2,\ell}^{-1} & 0 & 0
\end{pmatrix}
\]
(53)

\[
\begin{pmatrix}
M'_{1,q} & 0 & 0 & 0 \\
0 & 0 & 0 & E_{q+2,\ell} \\
0 & \lambda_2 t^{\mu_2} E_{q+2,\ell}^{-1} & 0 & 0
\end{pmatrix}
\]
(54)
(iii) For the type 2a involutive automorphisms (with $u = 1$) the $U(t)$ matrices are:

\[
\begin{align*}
\text{dsum}\{M_{1,q}, D^o_{q+2,\ell}, \tilde{M}^{-1}_{1,q}, D^o_{q+2,\ell}\} & \quad (55) \\
\text{dsum}\{M_{1,q}, D^o_{q+2,\ell}, -M^{-1}_{1,q}, -D^o_{q+2,\ell}\} & \quad (56) \\
\begin{pmatrix}
D^o_{1,q-1} & 0 & 0 & 0 & 0 & 0 \\
0 & M_{q,\ell-2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -F^{-1}_{\ell,\ell} & 0 & 0 & 0 \\
\end{pmatrix} & \quad (57) \\
\begin{pmatrix}
D^o_{1,q-1} & 0 & 0 & 0 & 0 & 0 \\
0 & M_{q,\ell-2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -E^{-1}_{\ell,\ell} & 0 & 0 & 0 \\
\end{pmatrix} & \quad (58) \\
\begin{pmatrix}
M'_{1,\ell-2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_2 t^{\mu_2} \tilde{M}'_{1,\ell-2} & 0 & 0 \\
0 & -\lambda_2 t^{\mu_2} F^{-1}_{\ell,\ell} & 0 & 0 & 0 \\
\end{pmatrix} & \quad (59) \\
\begin{pmatrix}
M'_{1,\ell-2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_2 t^{\mu_2} \tilde{M}'_{1,\ell-2} & 0 & 0 \\
0 & \lambda_2 t^{\mu_2} E^{-1}_{\ell,\ell} & 0 & 0 & 0 \\
\end{pmatrix} & \quad (60) \\
dsum\{L'_{1,\ell-1}, \lambda_2 t^{\mu_2} \tilde{L}'_{1,\ell-1}\} & \quad (61) \\
dsum\{L'_{1,\ell-1}, -\lambda_2 t^{\mu_2} \tilde{L}'_{1,\ell-1}\} & \quad (62)
\end{align*}
\]
\begin{equation}
\begin{pmatrix}
N'_{1,q} & 0 & 0 & 0 \\
0 & 0 & 0 & t^{\mu_1}H_{q+2,\ell} \\
0 & \lambda_2 t^{\mu_2}H_{q+2,\ell}^{-1} & 0 & 0 \\
0 & 0 & -\lambda_2 t^{\mu_2}H_{q+2,\ell}^{-1} & 0 \\
\end{pmatrix}
\end{equation}
\tag{68}
\] 

\[
dsum\{N_{1,q}, D_{q+2,\ell}, t^{\mu_{\ell+1}}N_{1,q}^{-1}, D_{q+2,\ell}, \ell\}
\]
\tag{69}

\[
\begin{pmatrix}
D_{1,q-1} & 0 & 0 & 0 & 0 \\
0 & N_{q,\ell-2} & 0 & 0 & 0 \\
0 & 0 & 0 & -t^{2\mu_2}D_{1,q-1}^{-1} & 0 \\
0 & 0 & 0 & 0 & -t^{2\mu_2}N_{q,\ell-2}^{-1} \\
0 & 0 & t^{2\mu_2}G_{\ell,\ell}^{-1} & 0 & 0 \\
\end{pmatrix}
\]
\tag{70}

\[
\begin{pmatrix}
N'_{1,\ell-2} & 0 & 0 & 0 \\
0 & 0 & 0 & G_{\ell,\ell} \\
0 & 0 & -\lambda_2 t^{\mu_2}N'_{1,\ell-2}^{-1} & 0 \\
0 & \lambda_2 t^{\mu_2}G_{\ell,\ell}^{-1} & 0 & 0 \\
\end{pmatrix}
\]
\tag{71}

\[
dsum\{N'_{1,\ell-1}, N'_{1,\ell-1}\}
\]
\tag{72}

\[
dsum\{N'_{1,\ell-1}, -N'_{1,\ell-1}\}
\]
\tag{73}

4.3 Simplification of listing

In this subsection, we will investigate the conjugacy of some of the automorphisms specified in the previous subsection.

(i) Let \( U(t) \) given by (68) and let \( U'(t) \) be obtained from \( U(t) \) by replacing \( L_{1,q} \) with \( W_{1,q} \). We define

\[
S(t) = dsum\{V_1(t)(t), \ldots, V_{\frac{1}{2}(q+1)}(t), 1_{\ell-q-1}, V'_1(t), \ldots, V'_{\frac{1}{2}(q+1)}(t), 1_{\ell-q-1}\},
\]

where

\[
V_j(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} \lambda_{2j-1}^{t^{\mu_2j-1}} & 1 \\ 1 & \lambda_{2j-1}^{t^{\mu_2j-1}} \end{pmatrix}, \quad V'_j(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} \lambda_{2j-1}^{-1} t^{-\mu_2j-1} & \lambda_{2j-1}^{-1} t^{-\mu_2j-1} \\ 1 & -1 \end{pmatrix}.
\]

Thus

\[
S(t)U'(t)S(t)^{-1} = U(t), \quad \tilde{S}(t)JS(t) = J.
\]

Consequently in (68) we may assume without loss of generality that \( L_{1,q} \) may be replaced by \( W_{1,q} \). Similarly in (69), (70) and (72) we may assume that the submatrices \( L_{j,k} \) are replaced by the submatrices \( W_{j,k} \). Furthermore, the above analysis may be modified slightly for type 1a automorphisms with \( u = -1 \) and also for type 2a automorphisms with \( u = 1 \). Thus, if \( U(t) \) is given by any one of (51), (52), (53), (54), (57), (58) or (70), then the submatrices \( M_{j,k} \) (or \( N_{j,k} \)) may be replaced by \( W_{j,k} \).

(ii) Let \( U(t) \) be given by (66) and let \( U'(t) \) be obtained from \( U(t) \) by setting \( \lambda_j = 1 \) (for \( 2 \leq j \leq \ell + 1 \)) and also by setting \( \mu_j = \rho_j \) (for \( 2 \leq j \leq \ell + 1 \)). If we define

\[
t = \lambda_{\ell+1}^{-\frac{1}{4}} diag\{1, \lambda_2 t^{\mu_2}, \ldots, \lambda_\ell t^{\mu_\ell}\}, \quad S(t) = dsum\{t^{\sigma+1}, t, t^{-1}\},
\]
\[ \tilde{S}(t)JS(t) = t^{\sigma_{\ell+1}}J, \quad S(t)U'(t)S(t)^{-1} = \lambda^{-\frac{1}{\ell+1}} t^{\sigma_{\ell+1}} U(t). \]

Thus in (66) we may assume that \( \lambda_j t^{\mu_j} = t^\rho \) (for \( 2 \leq j \leq \ell + 1 \)). Furthermore the above argument may be modified slightly for the matrices given by (68) or (69) but with the same result. In fact this result may be extended even further. If \( U(t) \) is given by any one of (57), (58), (66), (67), (68), (70) or (71), then in the submatrices \( E_{j,k} \) (or \( F_{j,k}, G_{j,k}, H_{j,k} \) etc.) we may assume that \( \lambda_m t^{\mu_m} = t^{\rho_m} \) for \( m \leq k \).

(iii) Let \( U(t) \) be given by (72) and let \( U'(t) \) be obtained from \( U(t) \) by putting \( \lambda_j t^{\mu_j} = 1 \) (for \( 3 \leq j \leq q, (q-j) \) is even) and by putting \( \lambda_j t^{\mu_j} = t^\rho \) (for \( j = 2, q + 2 \leq j \leq \ell \)). Then let

\[
\begin{align*}
t_1 &= \text{diag}\{\lambda_2^\frac{1}{2} t^{-\sigma_2}, \ldots, \lambda_2^\frac{1}{2} \lambda_q t^{\mu_q - \sigma_2}, 1\}, \\
&t_2 = \lambda_2^\frac{1}{2} \text{diag}\{\lambda_{q+2}^\frac{1}{2} t^{\sigma_{q+2}}, \ldots, \lambda_\ell^\frac{1}{2} t^{\sigma_\ell}\}, \\
&t_3 = \text{diag}\{\lambda_2^\frac{1}{2} t^{\sigma_2}, \lambda_2^\frac{1}{2} \lambda_3 t^{-\mu_3}, t^{\sigma_3}, \ldots, \lambda_2^\frac{1}{2} \lambda_q t^{\mu_q - \sigma_2}\},
\end{align*}
\]

and

\[ S(t) = \text{dsum}\{t_1, t_2, t_3, t_2^{-1}\}. \]

Then \( \tilde{S}(t)JS(t) = t^{\sigma_2}J \), and \( S(t)U'(t)S(t)^{-1} = t^{\sigma_2} U(t) \). Hence, in (66), we may assume that the matrices \( U(t) \) are all of the form \( U'(t) \), where \( U'(t) \) is as defined above. Similar assumptions may be made about the matrices given by (63), (69), (70), (71), (72) and (73). The proof is very similar to that just given.

(iv) Let \( U(t) \) be any matrix of the form \( \text{dsum}\{a, b\} \), where \( a \) and \( b \) are \( (\ell \times \ell) \) matrices such that

\[ a = \text{diag}\{a_1, \ldots, a_\ell\}, \quad b = \text{diag}\{b_1, \ldots, b_\ell\}, \]

and let \( a' \) and \( b' \) be obtained from \( a \) and \( b \) by exchanging \( j \) and \( k \) (where \( 1 \leq j, k \leq \ell \)) in their respective index sets. Let \( U'(t) \) be given by \( \text{offsum}\{a', b'\} \) and define \( S \) by

\[
S_{ab} = \begin{cases} 
1 & \text{if } a = b, a \neq j, k, j + \ell, k + \ell \\
0 & \text{if } a = b, a = j, k, j + \ell, k + \ell \\
1 & \text{if } (a, b) = (j, k), (k, j), (j + \ell, k + \ell), (k + \ell, j + \ell) \\
0 & \text{otherwise}. 
\end{cases}
\]

Then \( \tilde{S}JS = J \) and \( SU(t)S^{-1} = U'(t) \). Similarly if

\[
U(t) = \text{offsum}\{a, b\} \\
U'(t) = \text{offsum}\{a', b'\},
\]

then we still find that

\[ \tilde{S}JS = J, \quad SU(t)S^{-1} = U'(t). \]

This may be extended further to some of the other matrices under consideration. In fact, \( U(t) \) will almost always have submatrices that are diagonal. The above analysis will extend to these cases, allowing us to alter the index sets of the diagonal parts arbitrarily.

(v) Let \( U(t) = \text{dsum}\{a, b\} \), where \( a \) and \( b \) are as defined in (iv), and let \( U'(t) \) be obtained from \( U(t) \) by exchanging the elements \( a_j \) and \( b_j \) for \( 1 \leq j \leq \ell \). Define \( S \) by

\[
S_{ab} = \begin{cases} 
1 & \text{if } a = b, a \neq j, j + \ell \\
0 & \text{if } a = b = j, j + \ell \\
i & \text{if } (a, b) = (j, j + \ell), (j + \ell, j) \\
0 & \text{otherwise}. 
\end{cases}
\]

Then \( \tilde{S}JS = J \) and \( SU(t)S^{-1} = U'(t) \).
Then $\tilde{S}JS = J$ and $SU(t)S^{-1} = U'(t)$. Thus the order of diagonal elements may be regarded as being arbitrary, up to a certain point. Furthermore, if $U(t)$ is given by

$$U(t) = \text{offsum}\{a, b\},$$

and if $U'(t)$ is obtained from $U(t)$ by exchanging the elements $a_j$ and $b_j$, then the above analysis still holds. That is, $a_j$ and $b_j$ may be exchanged without loss of generality. As in (iv), we may extend this to those matrices $U(t)$ with diagonal submatrices. The conclusion still holds. That is, in the index set $\{1, \ldots, 2\ell\}$ of $U(t)$ we may exchange the $j$th and the $(j + \ell)$th members, without loss of generality. (vi) Let $U(t)$ be of the general form (66) and let $U'(t)$ be obtained from $U(t)$ by setting $\mu_q = 0$. If we let $S(t) = \text{dsum}\{1, t^{-\mu_q}, 1\}$, then

$$S(t)U(t)S(t^{-1}) = U'(t), \quad \tilde{S}JS = t^{-\mu_q}.$$ 

Thus we may assume without loss of generality that $\mu_q = 0$. Similarly, if $U(t)$ is given by (67), (68), (70) or (71) then without loss of generality the quantities $\mu_r, \mu_\ell$, and $\mu_\ell$ may be assumed to be zero.

(vii) Let $U(t)$ be of the form

$$U(t) = \begin{pmatrix} H_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & H_2 & 0 \\ 0 & a^{-1} & 0 & 0 \end{pmatrix},$$

where $H_1$ and $H_2$ are arbitrary $(\ell - q) \times (\ell - q)$ matrices and

$$a = \text{diag}\{t^{\mu_{\ell - q + 1}}, \ldots, t^{\mu_\ell}\}.$$ 

Let $U'(t)$ be given by $\text{dsum}\{H_1, 1, H_2, -1\}$. If we define $S(t)$ by

$$S(t) = \begin{pmatrix} 1_{\ell - q} & 0 & 0 & 0 \\ 0 & (2)^{-\frac{1}{2}}1_{\ell - q} & 0 & -(2)^{-\frac{1}{2}}a \\ 0 & 0 & 1_{\ell - q} & 0 \\ 0 & (2)^{-\frac{1}{2}}a^{-1} & 0 & (2)^{-\frac{1}{2}}1_{\ell - q} \end{pmatrix},$$

then we have that $\tilde{S}(t)JS(t) = J$ and also that $S(t)U'(t)S(t)^{-1} = U(t)$. This allows for a great amount of simplification later on. Many automorphisms can be shown to be conjugate by using this matrix transformation. Moreover, it is also valid for the type 1a involutive automorphisms (with $u = -1$) and the type 2a involutive automorphisms (with $u = 1$). This is because in the former case $S(-t) = S(t)$ and in the latter $S(t^{-1}) = S(t)$.

5 Study of the type 1a involutive automorphisms with $u = 1$

5.1 Conjugacy analysis

Using the results of Section 4, we find that each involutive automorphism of type 1a with $u = 1$ is conjugate to at least one automorphism corresponding to one of the following matrices:

$$U(t) = \text{dsum}\{D_{1, \ell}^0, D_{1, \ell}^0\}$$

(74)
The automorphism corresponding to $U(t)$ in particular. Upon putting (A) containing the automorphism $U$ and we may assume that $\ell - m \leq \lfloor \frac{t}{2} \rfloor$. Thus we are investigating precisely $(1 + \lfloor \frac{t}{2} \rfloor)$ distinct automorphisms, each of which corresponds to a different value of $(\ell - m)$.

We define $(A)^{(j)}$ (for $0 \leq j \leq \lfloor \frac{t}{2} \rfloor$) to be the conjugacy class containing the type 1a (with $u = 1$) automorphism corresponding to $U_j$, where

$$U_j = \text{dsum}\{1_{\ell-j}, -1_{\ell-j}, 1_{\ell-j}, -1_{\ell-j}\}. \quad (80)$$

We will demonstrate that $(A)^{(j)}$ is disjoint from $(A)^{(k)}$ when $j \neq k$. If the classes $(A)^{(j)}$ and $(A)^{(k)}$ did coincide for different values of $j$ and $k$, then it would be necessary for a matrix $S(t)$ to exist such that

$$S(t) \text{ dsum } U_j S(t)^{-1} = \lambda t^\mu U_k,$$  

where $U_j$ and $U_k$ are defined as in (80). This would have to hold for all values of $t$, and for $t = 1$ in particular. Upon putting $t = 1$ in the equation (81) we find that there cannot be any Laurent matrix $S(t)$ that satisfies (81). Thus the automorphisms that correspond to (74) fall into precisely $(1 + \lfloor \frac{t}{2} \rfloor)$ disjoint conjugacy classes which we have called $(A)^{(0)}, \ldots, (A)^{(1 + \lfloor \frac{t}{2} \rfloor)}$. It is clear that $(A)^{(0)}$ contains only the identity automorphism.

We move onto the automorphism corresponding to (75) and define (B) to be the conjugacy class containing the automorphism $U(t) 1, 0\}$, where $U(t)$ is given by (75). We will show that (B) is disjoint from the classes $(A)^{(j)}$ that we have discussed previously. If we suppose that (B) and $(A)^{(j)}$ are the same for some suitable $j$, then the following must hold

$$S(t) U S(t)^{-1} = \lambda t^\mu U_j,$$  

where $U_j$ is as defined in (80). If we substitute $t = 1$ in this equation, then it is easily seen that $\ell$ must be even. Furthermore when $\ell$ is even, the matrix $S(1)$ cannot be chosen such that $S(1) J S(1) = \lambda J$. Thus (B) is a class that is disjoint from all of the others identified so far.

Next we consider the automorphism corresponding to (76). Consideration of the determinant of $U(t)$, where $U(t)$ is given by (76), allows us to infer that the automorphism corresponding to it

$$U(t) = \text{dsum}\{1_{\ell}, -1_{\ell}\} \quad (75)$$

$$U(t) = \text{offsum}\{1_{\ell}, t1_{\ell}\} \quad (76)$$

$$U(t) = \begin{pmatrix} L_{1,q}^0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{\ell-q-1} \\ 0 & 0 & -L_{1,q}^0 & 0 \\ 0 & t1_{\ell-q-1} & 0 & 0 \end{pmatrix} \quad (77)$$

$$U(t) = \text{dsum}\{L_{1,\ell-1}^0, L_{1,\ell-1}^0\} \quad (78)$$

$$U(t) = \text{dsum}\{L_{1,\ell-1}, -L_{1,\ell-1}\}. \quad (79)$$
does not belong to any of the conjugacy classes that we have identified so far. Otherwise we would have
\[ S(t)US(t)^{-1} = \lambda t^k U(t), \]
where \( U \) is a \( t \)-independent matrix. Taking determinants of both sides of (82) implies that \( k = -\frac{1}{2} \), which is obviously a contradiction. Hence the automorphisms corresponding to (36), and for which \( \mu_{\ell+1} \) is odd, belong to some previously unidentified conjugacy class which we call (C).

Let \( U(t) \) be given by (77) and recall that the type 1a involutive automorphism corresponding to \( \text{offsum}\{1_\ell, t1_\ell\} \) belongs to the conjugacy class (C). Then we define a matrix \( S(t) \) by
\[
S(t) = \begin{pmatrix}
Q_1 & 0 & Q_2 & 0 \\
0 & 1_{\ell-q-1} & 0 & 0 \\
Q_3 & 0 & Q_4 & 0 \\
0 & 0 & 0 & 1_{\ell-q-1}
\end{pmatrix},
\]
where the \((ab)\)th elements of the \((q+1)\times(q+1)\) submatrices are given by
\[
\{(Q_1), (Q_2), (Q_3), (Q_4)\}_{ab} = \begin{cases}
\{1, (2)^{-\frac{1}{2}}, (2)^{-\frac{1}{2}}, 1\} & \text{for } a = b; a \text{ odd,} \\
\{-(2)^{-\frac{1}{2}}i, i, -(2)^{-\frac{1}{2}}it, -1\} & \text{for } a = b; a \text{ even,} \\
\{i, -(2)^{-\frac{1}{2}}i, (2)^{-\frac{1}{2}}it, -i\} & \text{for } a = b - 1; a \text{ odd,} \\
\{(2)^{-\frac{1}{2}}t, 1, -1, -(2)^{-\frac{1}{2}}t^{-1}\} & \text{for } a = b - 1; a \text{ even,} \\
\{0, 0, 0, 0\} & \text{otherwise.}
\end{cases}
\]
Then \( S(t)JS(t) = J \) and \( S(t) \) offsum\{1_\ell, t1_\ell\} \( S(t)^{-1} = U(t) \). Hence the automorphisms (77) all belong to the conjugacy class (C).

The only remaining type 1a involutive automorphisms with \( u = 1 \) are those given by (78) and (79). Let us first examine those given by (78). Consideration of the determinant of \( U(t) \) of (78) shows that the automorphism corresponding to it cannot belong to any of the conjugacy classes \( (A)^{(0)}, \ldots, (A)^{(L)} \) or (B). Either it belongs to (C), or to some as yet unidentified conjugacy class. Let us suppose that it belongs to the class (C), so that for some matrix \( S(t) \)
\[
S(t) \text{ dsum}\{L^0_{1_\ell, t-1}, L^0_{1_\ell, \ell-1}\} S(t)^{-1} = \text{offsum}\{1_\ell, t1_\ell\}. \tag{83}
\]
We are assuming that (83) holds for every non-zero value of \( t \), and in particular for \( t = 1 \). Let \( T \) be defined to be obtained from \( L^0_{1_\ell, t-1} \) by setting \( t = 1 \). We know that there exists a \( t \)-independent matrix \( R \) such that
\[
R \text{ dsum}\{T, \tilde{T}\} R^{-1} = \text{dsum}\{W_{1, \ell-1}, W_{1, t-1}\},
\]
but we know that there exists no \( t \)-independent matrix \( Q \) such that both
\[
Q \text{ dsum}\{W_{1, \ell-1}, W_{1, t-1}\} Q^{-1} = \beta \text{offsum}\{1_\ell, 1_\ell\}
\]
and
\[
\tilde{Q}JQ = \gamma J.
\]
Thus our assumption that it belongs to the class (C) is false, and so the automorphism (78) does belong to some new conjugacy class (D).

Finally, we examine the automorphism (79) and we demonstrate that this automorphism belongs to the conjugacy class (C). Let \( U(t) \) be given by (73) and let \( U'(t) \) be given by (74). Then
\[
S(t)U'(t)S(t)^{-1} = U(t), \ S(t)JS(t) = J,
\]
where
\[ S(t) = \begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix} \]

The submatrices \( Q_1, Q_2, Q_3 \) and \( Q_4 \) are of dimension \( \ell \times \ell \) and their \((ab)\)th elements are given by

\[
\{ (Q_1)_{ab}, (Q_2)_{ab}, (Q_3)_{ab}, (Q_4)_{ab} \} = \begin{cases} 
\{1, (2)^{-1/2}, (2)^{-1/2}, 1\} & \text{for } a = b; a \text{ odd,} \\
\{- (2)^{-1/2}it, i, i, - (2)^{-1/2}it^{-1}\} & \text{for } a = b; a \text{ even,} \\
\{i, - (2)^{-1/2}i, (2)^{-1/2}i, -i\} & \text{for } a = b - 1; a \text{ odd,} \\
\{(2)^{-1/2}t, 1, -1, (2)^{-1/2}t^{-1}\} & \text{for } a = b + 1; a \text{ even,} \\
\{0, 0, 0, 0\} & \text{otherwise.}
\end{cases}
\]

5.2 Explicit forms for the automorphisms

With the analysis of the previous subsection we have identified all the conjugacy classes of the type 1A involutive automorphisms within the group of all automorphisms of \( \text{C}_1^{(1)} \). Curiously, there are more conjugacy classes for the case \( \ell = 2m \) than there are for the case \( \ell = 2m + 1 \). This is due to the nature of the conjugacy classes of the Weyl group for the algebra \( \text{C}_\ell \). We recall that for even \( \ell \) there was one extra family of conjugacy classes of involutive Weyl group elements. Some of the automorphisms associated with these Weyl group elements give rise to an ‘extra’ conjugacy class of involutive automorphisms of \( \text{C}_1^{(1)} \).

1. The conjugacy class \( (A)^{(0)} \) is clearly the conjugacy class which contains only the identity automorphism.

2. The conjugacy class \( (A)^{(b)} \) for \( b \neq 0 \) has a representative \( \psi \), where \( \psi \) has the following effect upon the basis elements:

\[
\begin{align*}
\psi(h_{\alpha_k}) &= h_{\alpha_k} \quad \text{for } k = 1, 2, \ldots, \ell \\
\psi(e^{\pm \alpha_H}) &= e^{\pm \alpha_H} \\
\psi(e^{\pm \alpha_k}) &= \begin{cases} 
  e^{\pm \alpha_k} & \text{for } k \neq \ell - b; 1 \leq k \leq \ell \\
  -e^{\pm \alpha_k} & \text{for } k = \ell - b
\end{cases} \\
\psi(c) &= c \\
\psi(d) &= d.
\end{align*}
\]

Its \( U(t) \) matrix is given by the \( U_j \) of (80).

3. The conjugacy class \( (B) \) has the following representative, which corresponds to the matrix \( U(t) = \text{dsum}\{1_\ell, -1_\ell\} \):

\[
\begin{align*}
\psi(h_{\alpha_k}) &= h_{\alpha_k} \quad \text{for } k = 1, 2, \ldots, \ell \\
\psi(e^{\pm \alpha_H}) &= -e^{\pm \alpha_H} \\
\psi(e^{\pm \alpha_k}) &= \begin{cases} 
  e^{\pm \alpha_k} & \text{for } k = 1, 2, \ldots, \ell - 1 \\
  -e^{\pm \alpha_k} & \text{for } k = \ell
\end{cases} \\
\psi(c) &= c \\
\psi(d) &= d.
\end{align*}
\]

\( (84) \)
4. The conjugacy class (C) has the following representative, which corresponds to the matrix
\[ U(t) = \text{offsum}\{1_\ell, t1_\ell\} : \]
\[
\begin{align*}
\psi(h_{\alpha_k}) &= \begin{cases} h_{\alpha_k} & \text{for } k = 1, 2, \ldots, \ell - 1 \\
 h_{\alpha_\ell} + c & \text{for } k = \ell \end{cases} \\
\psi(e_j d \pm \alpha_H) &= e_{(j+1)\delta \mp \alpha_H} \\
\psi(e_j d \pm \alpha_k) &= \begin{cases} e_{j \pm \alpha_k} & \text{for } k = 1, 2, \ldots, \ell - 1 \\
 -e_{(j+1)\delta \mp \alpha_\ell} & \text{for } k = \ell \end{cases} \\
\psi(c) &= c \\
\psi(d) &= d + 4(\ell + 1)\left(\sum_{p=1}^{\ell-1} p h_{\alpha_p}\right) + \frac{\ell(\ell+1)}{2}c.
\end{align*}
\]

5. The conjugacy class (D) which occurs only for even values of \( \ell \), has the following representative, which is the automorphism corresponding to the \( U(t) \) matrix of (86):
\[
\begin{align*}
\psi(h_{\alpha_k}) &= h_{\alpha_k} + (-1)^{k+1}c \quad \text{for } k = 1, 2, \ldots, \ell \\
\psi(e_j d \pm \alpha_H) &= e_{(j+1)\delta \mp \alpha_H + 2a_1} \\
\psi(e_j d \pm \alpha_k) &= \begin{cases} e_{(j+1)\delta \mp \alpha_k} & \text{for } k = 1, \ldots, \ell - 1; k \text{ is odd} \\
 e_{(j+1)\delta \pm \alpha_k} & \text{for } k = 1, \ldots, \ell - 2; k \text{ is even} \\
 e_{(j+1)\delta \mp \alpha_\ell - 1} & \text{for } k = 1 \end{cases} \\
\psi(c) &= c \\
\psi(d) &= d + 2(\ell + 1)\left(\sum_{p=1, p \text{ odd}}^{\ell-1} h_{\alpha_p}\right) - \frac{\ell(\ell+1)}{2}c.
\end{align*}
\]

6 Study of the type 1a involutive automorphisms with \( u = -1 \)

6.1 Conjugacy analysis

We shall begin this section by examining the automorphisms (16). Define \( s_j \) (for \( 2 \leq j \leq \ell \)) by letting \( s_j = 1 \) if \( \lambda_j = -1 \), and by letting \( s_j = 0 \) of \( \lambda_j = 1 \). Then, with
\[ S(t) = \text{diag}\{1, t^{s_2}, \ldots, t^{s_\ell}, 1, t^{-s_2}, \ldots, t^{-s_\ell}\}, \]

it follows that
\[ \tilde{S}(t)JS(t) = J, \quad S(t)1_{2\ell}S(-t)^{-1} = U(t), \]
so all of the automorphisms (16) belong to the same conjugacy class, which we will call (E).

The automorphisms (17) all belong to the class (E). If \( U(t) \) is given by (17) then we may define \( S(t) \) by
\[ S(t) = \text{diag}\{1, t^{\rho_2}, \ldots, t^{\rho_\ell}, t, t^{1-\rho_2}, \ldots, t^{1-\rho_\ell}\}, \]
so that
\[ \tilde{S}(t)JS(t) = J, \quad S(t)1_{2\ell}S(-t)^{-1} = U(t). \]

The analysis of Section 4 implies that that the automorphisms (18) are all conjugate and, furthermore, they all belong to the conjugacy class (E).

Consider the type 1a automorphism with \( s = 1 \) that corresponds to the matrix \( S(t) \) where
\[ S(t) = \text{dsum}\{1_\ell, t1_\ell\}. \quad (88) \]

This automorphism is such that conjugation of an automorphism given by (49) gives rise to an automorphism given by (50). Hence every automorphism given by (50) is conjugate to one given by (48). We infer from the results of the Section 4 that the automorphisms (48) belong to the
conjugacy class (E). Similar results follow for the remaining type 1a involutive automorphisms (with $u = -1$). That is, for those given by (51) through to (62). The results of Section 3 together with conjugation by the automorphism corresponding to (88) imply that all of these automorphisms belong to the conjugacy class (E).

6.2 Explicit form of automorphism

A representative automorphism $\psi$ of the class (E) is given by

$$
\begin{align*}
\psi(h_{\alpha_k}) &= h_{\alpha_k} \text{ for } k = 1, \ldots, \ell \\
\psi(e_{j \dot{\delta} \pm \alpha_H}) &= (-1)^j e_{j \dot{\delta} \pm \alpha_H} \\
\psi(e_{j \dot{\delta} \pm \alpha_k}) &= (-1)^j e_{j \dot{\delta} \pm \alpha_k} \text{ for } k = 1, \ldots, \ell \\
\psi(c) &= c \\
\psi(d) &= d.
\end{align*}
$$

This automorphism corresponds to the matrix $U(t) = 1_{2\ell}$.

7 Study of the type 2a involutive automorphisms with $u = 1$

7.1 Conjugacy analysis

Using the results of Section 3, we find that each type 2a involutive automorphism (with $u = 1$) is conjugate to some type 2a involutive automorphism corresponding to one of:

$$
\begin{align*}
U(t) &= \text{dsum}\{D_{1, \ell}, D_{1, \ell}^{-1}\} \\
U(t) &= \text{dsum}\{D_{1, \ell}, tD_{1, \ell}^{-1}\} \\
U(t) &= \text{dsum}\{D_{1, \ell}, -D_{1, \ell}^{-1}\} \\
U(t) &= \text{dsum}\{D_{1, \ell}, -tD_{1, \ell}^{-1}\}
\end{align*}
$$

The conjugacy conditions (27) and (29) for type 2a automorphisms may be further refined in the involutive case as follows. Let two involutive automorphisms of type 2a and with $u = 1$ have associated matrices $U(t)$ and $U'(t)$. Then these are conjugate via the automorphism $\{S(t), s, \xi\}$ (where $s^2 = 1$) if and only if

$$
\begin{align*}
S(t)U(st)S(t^{-1})^{-1} &= \lambda t^\mu U'(t) \quad \text{(where $S(t)$ is of type 1a)} \\
S(t)U(st^{-1})S(t^{-1})^{-1} &= \lambda t^\mu \quad \text{(where $S(t)$ is of type 2a)}.
\end{align*}
$$

Thus, to prove that any two automorphisms are not conjugate, we may prove that it is not possible to satisfy (94). In addition, it may be possible to satisfy (94) but only if certain forbidden or contradictory conditions are fulfilled. In this subsection we prove that automorphisms are not conjugate by substituting the values $s = 1, -1$ and $t = 1, -1$ into an equation of the form (94). We then find a number of contradictions which complete the proof. In some cases it may be that there is no matrix $S(1)$ or $S(-1)$ that satisfies (94) (with substitutions) and also satisfies

$$
S(\pm 1)JS(\pm 1) = \lambda J.
$$

We shall begin by examining the automorphisms corresponding to (90). It follows from Section 3 that $U(t)$ may be assumed to be of the form

$$
U(t) = \text{dsum}\{1_a, -1_b, t1_c, -t1_d, 1_a, -1_b, t^{-1}1_c, -t^{-1}1_d\},
$$

(95)
where \(a + b + c + d = \ell\). For any matrix of the form \(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\), we define a function \(\theta\), whose argument is the matrix \(U(t)\) and whose resultant value is an ordered quadruplet. Let \(\theta\) be specified by

\[
\theta(U(t)) = (a, b, c, d).
\]

We define \((F)^{(j,k)}\) to be the set containing the automorphism corresponding to \(U(t)\) where \(U(t)\) is such that

\[
\theta(U(t)) = (u_1, u_2, u_3, u_4)
\]

with \((u_2 + u_4) = j\) and \((u_3 - u_4) = k\). One assumption which may be made after examining Section 4 is that \(0 \leq u_2 - u_4 \leq \left\lfloor \frac{k}{2} \right\rfloor\). We will now show that the automorphisms within each set \((F)^{(j,k)}\) are mutually conjugate. To do this, let \(U(t)\) and \(U'(t)\) be given by \(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\) such that \(\theta(U(t)) = (n_+, n_m, n^t_+, n^{t'}_+)\) and \(\theta(U'(t)) = (n_+-1, n_m-1, n^t_+ + 1, n^{t'}_+ + 1)\). We will now show that the automorphisms \(\{U(t), 1, \xi\}\) and \(\{U'(t), 1, \xi'\}\) are conjugate. Let the matrix \(S(t)\) be given by

\[
\begin{pmatrix}
1_{n_+ - 1} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2}(1+t)1_1 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2}(1+t)1_1 & 0 & 0 \\
0 & 0 & 0 & 1_{\ell-2} & 0 \\
0 & 0 & \frac{1}{2}(1-t^{-1})1_1 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2}(1+t^{-1})1_1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1_{\ell-n_+ - 1}
\end{pmatrix}.
\]

This satisfies

\[
\tilde{S}(t)JS(t) = J, \quad S(t)U(t)S(t^{-1})^{-1} = U''(t),
\]

where \(U''(t)\) is some matrix which, according to Section 4, is conjugate to the automorphism corresponding to \(U'(t)\). Hence the automorphisms corresponding to \(U(t)\) and \(U'(t)\) are conjugate, as we set out to show.

We know already that we may restrict the choice of the parameter \(j\) in the expression \((F)^{(j,k)}\) to the values such that \(0 \leq j \leq \left\lfloor \frac{k}{2} \right\rfloor\). In fact it is possible to restrict the choice of \(k\) so that

\[
0 \leq k \leq \left(\left\lfloor \frac{k}{2} \right\rfloor - j\right).
\]

This means that the members of the set \((F)^{(j',k')}\) (where \(k'\) does not satisfy \(0 \leq k' \leq \left(\left\lfloor \frac{k}{2} \right\rfloor - j\right)\)) are all conjugate to the members of \((F)^{(j,k)}\) where \(j, k\) do satisfy \(\left(\left\lfloor \frac{k}{2} \right\rfloor - j\right)\). We prove this by supposing that \(j, k\) are such that they do not satisfy \(\left(\left\lfloor \frac{k}{2} \right\rfloor - j\right)\). We know, however, that the matrix corresponding to \(U(t)\) is conjugate to the matrix corresponding to \(U(-t)\). A brief inspection of \(U(-t)\) coupled with results of Section 4 indicate that the members of \((F)^{(j,k)}\) are indeed conjugate to the members of \((F)^{(m,n)}\) where \(m, n\) do satisfy \(\left(\left\lfloor \frac{k}{2} \right\rfloor - j\right)\).

It follows that every automorphism \((F)^{(j,k)}\) belongs to at least one set \((F)^{(j,k)}\) for suitable values of \(j, k\). Let us then redefine \((F)^{(j,k)}\) slightly to be the conjugacy class containing the automorphism corresponding to \(U(t)\) where \(U(t)\) is such that

\[
\theta(U(t)) = (u_1, u_2, u_3, u_4)
\]

with \((u_2 + u_4) = j\) and \((u_3 - u_4) = k\). We have seen that the values

\[
j = 0, \ldots, \left\lfloor \frac{k}{2} \right\rfloor; \quad k = 0, \ldots, \left(\left\lfloor \frac{k}{2} \right\rfloor - j\right)
\]

(97)
are sufficient, in that every automorphism \((\mathcal{A})\) belongs to one of the classes \((\mathcal{F})^{(j,k)}\). We will now show that, for the values specified in \((\mathcal{B})\), the corresponding classes \((\mathcal{F})^{(j,k)}\) are all disjoint. Suppose that \((\mathcal{F})^{(a,b)}\) coincides with \((\mathcal{F})^{(c,d)}\) for \((a,b) \neq (c,d)\). Then define \(U(t)\) and \(U'(t)\) by

\[
U(t) = \text{dsum}\{1_{n_a-b}, t_1, -1_a, 1_{n_b-b}, t_{-1} 1_b, -1_a\}
\]

\[
U'(t) = \text{dsum}\{1_{n_a-d}, t_1, -1_c, 1_{n_b-d}, t_{-1} 1_d, -1_c\}.
\]

Our supposition implies that for all non-zero \(t\)

\[
S(t) \ U(st^{\pm 1}) \ S(t^{-1})^{-1} = \lambda t^\mu U'(t).
\]  

(98)

We know that \(s\) may take the values 1 or -1. If \(s = 1\) and we put \(t = 1\) in \((98)\) then we have the necessary condition that \(a + c = \ell\), which implies that \(a = c\). In that case, we put \(t = -1\) which implies that \(b = d\). This contradicts our original supposition and so we must have \(s = -1\). However, putting \(t = 1, -1\) in \((98)\) implies again that \(a = c\) and \(b = d\). Thus the conjugacy classes \((\mathcal{F})^{(j,k)}\) where \(0 \leq j \leq \left[\frac{\ell}{2}\right]\) and \(0 \leq k \leq \left(\left[\frac{\ell}{2}\right] - j\right)\) are disjoint for different choices of \((j,k)\). Moreover, every automorphism given by \((\mathcal{B})\) belongs to one of these classes.

Consider next the automorphisms \((\mathcal{B})\) for which we may assume that

\[
S(t) \ U(t) \ S(t^{-1})^{-1} = \lambda t^\mu U'(t).
\]  

(99)

As we have mentioned already, it can be shown that \(b \leq \left[\frac{\ell}{2}\right]\). Thus, since \(a\) can only take values 0, \ldots, \left[\frac{\ell}{2}\right], we are concerned with precisely \((1 + \left[\frac{\ell}{2}\right])\) distinct automorphisms. Each distinct automorphism corresponds to a different value of \(a\). The automorphisms \((\mathcal{B})\) fall therefore into at most \((1 + \left[\frac{\ell}{2}\right])\) disjoint conjugacy classes. Let \((G)^{(j)}\) be the conjugacy class containing the automorphism corresponding to \(U(t)\) where \(U(t)\) is given by \((\mathcal{B})\) but with \(a = j\). We note that \(j\) is restricted to the values \(0, \ldots, \left[\frac{\ell}{2}\right]\). We now show that \((G)^{(j)}\) is disjoint from \((G)^{(k)}\) when \(j \neq k\). Let \(U_j(t)\) and \(U_k(t)\) be given by \((99)\) but with \(a = j, k\) respectively. If \((G)^{(j)}\) and \((G)^{(k)}\) are the same, then

\[
S(t) \ U_j(st^{\pm 1}) S(t^{-1})^{-1} = \lambda t^\mu U_k(t)
\]  

(100)

with \(s = 1, -1\). If we suppose that \(s = 1\), then putting \(t = 1\) yields a contradiction, since \((100)\) then implies that \(j = k\), which is not the case. Thus we suppose that \(s = -1\). If we put \(t = -1\) in \((100)\) then we again find that this is not possible. Hence the conjugacy classes \((G)^{(j)}\) are disjoint for distinct values of \(j\). Similarly, the conjugacy classes \((\mathcal{F})^{(m,n)}\) are disjoint from the classes \((G)^{(k)}\).

This may be proven by substituting the values \(s = \pm 1, t = \pm 1\) into a necessary matrix condition. It will be seen that in both of the cases \(s = 1, -1\) this proves to be impossible. Clearly, the matrix equations will have solutions for \(S(\pm 1)\), but these will not satisfy

\[
\tilde{S}(\pm 1) \ J \ S(\pm 1) = \lambda J.
\]

(Much of this has already been done in Section \(\mathcal{B}\) and is applicable here when dealing with \(t\)-independent matrices).

Consider the automorphisms \((\mathcal{P})\) for which we may take

\[
U(t) = \text{dsum}\{1_{\ell-a}, t 1_a, -1_{\ell-a}, -t 1_a\}.
\]  

(101)

We will show that all of these automorphisms are mutually conjugate. Let \(U(t)\) and \(U'(t)\) be given by \((101)\) but with \(a = j, j + 1\) respectively. Then we define \(S(t)\) by

\[
S(t) = \begin{pmatrix}
1_{\ell-j-1} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2}(1 + t) 1_1 & 0 & \frac{1}{2}(1 - t) 1_1 & 0 \\
0 & 0 & 1_{\ell-1} & 0 & 0 \\
0 & \frac{1}{2}(1 - t^{-1}) 1_1 & 0 & \frac{1}{2}(1 + t^{-1}) 1_1 & 0 \\
0 & 0 & 0 & 0 & 1_{j+2}
\end{pmatrix},
\]
where \( \tilde{S}(t)JS(t) = J \) and \( S(t)U(t)S(t^{-1})^{-1} = U'(t) \). Hence all of the automorphisms \( \mathcal{G} \) are mutually conjugate. Moreover, this class is disjoint from all of the other classes identified previously. The method outlined previously demonstrates that this is so. We call this class \( (H) \).

Consider the automorphisms \( \mathcal{G} \). It follows from Section 4 that \( U(t) \) may be suitably reordered so that

\[
U(t) = \text{dsum}\{1_{t-a}, -1_{a}, -t1_{t-a}, t1_a\}.
\]

We have remarked that the automorphisms corresponding to \( U(t) \) and \( U(-t) \) are conjugate. If we then consider \( U(-t) \), where \( U(t) \) is given by \( \mathcal{G} \), then it is clear that the automorphisms \( \mathcal{G} \) belong to the conjugacy classes \( (G)^{(j)} \) which we defined earlier.

### 7.2 Explicit forms for the automorphisms

Now we give representatives for the classes which we identified in the previous subsection.

1. The class \( (F)^{(a,b)} \) has the following representative:

\[
\begin{align*}
\psi(h_{\alpha_k}) &= \begin{cases} h_{\alpha_k} - c & \text{for } k = 1; b \neq 0 \\
h_{\alpha_k} + c & \text{for } k = b + 1; b \neq 0 \\
h_{\alpha_k} & \text{otherwise.} \end{cases} \\
\psi(e_{j\delta \pm \alpha_H}) &= e_{-(j-1)\delta \pm \alpha_1} \text{ for } k = 1; b \neq 0 \\
&= e^{-j\delta \pm \alpha_k} \text{ for } 1 < k < b + 1; b \neq 0 \\
&= e_{(1-j)\delta \pm \alpha_k} \text{ for } k = b + 1; b \neq 0 \\
&= e^{-j\delta \pm \alpha_k} \text{ for } k = \ell - a; a \neq 0 \\
&= e^{-j\delta \pm \alpha_k} \text{ otherwise} \\
\psi(c) &= -c \\
\psi(d) &= d - 4(\ell + 1)(\sum_{p=1}^{b} ph_{\alpha_p+1} + b \sum_{p=b+2}^{\ell-1} h_{\alpha_p} + (b/2)h_{\alpha_\ell}) - 2b(\ell + 1)c.
\end{align*}
\]

This corresponds to the matrix

\[
\text{dsum}\{1_1, t1_b, 1_{t-\delta-b-1}, -1_a, 1_1, t^{-1}1_b, 1_{t-\delta-a-b-1}, -1_a\}.
\]

2. The class \( (G)^{(b)} \) has the following representative:

\[
\begin{align*}
\psi(h_{\alpha_k}) &= \begin{cases} h_{\alpha_k} & \text{for } k = 1, \ldots, \ell - 1 \\
h_{\alpha_k} - c & \text{for } k = \ell \end{cases} \\
\psi(e_{j\delta \pm \alpha_H}) &= e_{-(j+1)\delta \pm \alpha_1} \text{ for } k = \ell - b; b \neq 0 \\
&= -e^{-j\delta \pm \alpha_k} \text{ for } k = \ell - b; b \neq 0 \\
&= e_{-(j+1)\delta \pm \alpha_\ell} \text{ for } k = \ell \\
&= e^{-j\delta \pm \alpha_\ell} \text{ otherwise} \\
\psi(c) &= c \\
\psi(d) &= d + 2(\ell + 1)(\sum_{p=1}^{\ell-1} ph_{\alpha_p} + (\ell/2h_{\alpha_\ell}) + \{(\ell(\ell + 1))/2\})c.
\end{align*}
\]

This corresponds to the matrix \( \text{dsum}\{1_{\ell-b}, -1_b, t1_{\ell-b}, -t1_b\} \).

3. Finally, the class \( (H) \) has the following representative:

\[
\begin{align*}
\psi(h_{\alpha_k}) &= h_{\alpha_k} \\
\psi(e_{j\delta \pm \alpha_H}) &= -e^{-j\delta \pm \alpha_H} \\
\psi(e_{j\delta \pm \alpha_k}) &= \begin{cases} e^{-j\delta \pm \alpha_k} & \text{for } k = 1, \ldots, \ell - 1 \\
&= -e^{-j\delta \pm \alpha_k} & \text{for } k = \ell \end{cases} \\
\psi(c) &= -c \\
\psi(d) &= -d.
\end{align*}
\]
This corresponds to the matrix $\text{dsum}\{1_\ell, -1_\ell\}$.

8 Summary of conclusions

For the complex untwisted affine Kac-Moody algebra $C^{(1)}_\ell$ the representatives of the conjugacy classes of involutive automorphisms are as follows:

1. Involutive automorphisms of type 1a with $u = 1$:
   
   (a) For the conjugacy classes $(A)^{(j)}$, where $0 \leq j \leq \lfloor \frac{\ell}{2} \rfloor$, the representative for $(A)^{(0)}$ is the identity automorphism, and the representatives for $(A)^{(0)}$ (for $0 < j \leq \lfloor \frac{\ell}{2} \rfloor$) are given in (84).
   
   (b) For the conjugacy class (B) the representative is given in (85).
   
   (c) For the conjugacy class (C) the representative is given in (86).
   
   (d) For the conjugacy class (D), which exists only for $\ell$ even, the representative is given in (87).

2. Involutive automorphisms of type 1a with $u = -1$: There is only one conjugacy class, (E), for which the representative is given in (89).

3. Involutive automorphisms of type 2a (with $u = 1$):
   
   (a) For the conjugacy classes $(F)^{(j,k)}$, where $0 \leq j \leq \lfloor \frac{\ell}{2} \rfloor$ and $0 \leq k \leq (\lfloor \frac{\ell}{2} \rfloor - j)$, the representatives are given in (103).

   (b) For the conjugacy classes $(G)^{(j)}$, where $0 \leq j \leq \lfloor \frac{\ell}{2} \rfloor$, the representatives are given in (105).

   (c) For the conjugacy class (H) the representative is given in (106).

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