K3-fibered Calabi-Yau threefolds II, singular fibers

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Introduction

In part I of this paper we have introduced a twist map from a product of weighted hypersurfaces onto another weighted hypersurface. This map realized a quotient formation. The image was shown to have the structure of fibration. As it is known how to resolve singularities of these hypersurfaces with the methods of toric geometry, this gives a method for explicitly determining the singular fibers in that fibration. The twist map is defined for a pair of weighted hypersurfaces $V_1 \subset \mathbb{P}(w_0, \ldots, w_n)$ and $V_2 \subset \mathbb{P}(v_0, \ldots, v_m)$, and maps the product onto a hypersurface in the weighted projective space $\mathbb{P}(v_0 w_1, \ldots, v_n w_0, w_0 v_1, \ldots, w_0 v_m)$. We are interested in the case that the image is Calabi-Yau, i.e., has vanishing first Chern class, in which case this Calabi-Yau has a constant-modulus fibration, either elliptic or K3.

For an introduction to this see part I. We will first recall the twist map and some further notations, then begin by describing the Kodaira fibers of types I*0, II, III, IV, II*, III* and IV*, from the same point of view: as resolutions of images of the twist map. Here the reader will see that this method is convenient and effective. Then we proceed to consider the degenerate fibers in families of K3 surfaces. Recall that there is a vast literature on stable degenerations of K3 surfaces. Looking at elliptic curves, the stable degenerate ones are exactly the fibers of type I_n, corresponding to a resolution of a surface $A_n$-singularity. The singular fibers of types I*0, II, III, IV, II*, III* and IV* in Kodaira’s classification, are all non-stable, and it is an analog of these which we consider in this paper. We remark that in all cases we consider, the Calabi-Yau threefold has both a K3- as well as an elliptic fibration. The elliptic fibrations are more thoroughly studied, and we could have indeed considered these. However we prefer the picture of K3-fibration for the following reason: all birational transformations (i.e., non-uniqueness of models) take place in the fibers, whereas for elliptic fibrations, to get certain models one must modify the base of the fibration. Furthermore, this research may be considered a first detailed look at non-stable K3-degenerations.

1 Weighted projective spaces

We will be working with weighted projective spaces, which are certain (singular) quotients of usual projective space. Alternatively, they may be described as quotients of $\mathbb{C}^{n+1}$ by a $\mathbb{C}^*$-action. We assume the weights $(w_0, \ldots, w_n)$ are given, let $\mu_{w_i}$ denote the group of $w_i$th roots of unity, and
consider the action of \( \mu := \mu_{w_1} \times \cdots \times \mu_{w_n} \) on \( \mathbb{P}^n \) as follows. Let \( g = (g_0, \ldots, g_n) \in \mu \), and consider for \((z_0: \ldots: z_n)\) homogenous coordinates on \( \mathbb{P}^n \) the action

\[
(g, (z_0: \ldots: z_n)) \mapsto (g_0 z_0: \ldots: g_n z_n).
\]

Alternatively, consider the action of \( \mathbb{C}^* \) on \( \mathbb{C}^{n+1} \) given by

\[
(t, (z_0, \ldots, z_n)) \mapsto (t^{w_0} z_0, \ldots, t^{w_n} z_n).
\]

In both cases, the resulting quotient is the weighted projective space, which we will denote by \( \mathbb{P}(w_0, \ldots, w_n) \). General references for weighted projective spaces are [3] and [2]. A weighted hypersurface is the zero locus of a weighted homogenous polynomial \( p \). We will assume the weights are normalized in the sense that no \( n \) of the \( n+1 \) weights have a common divisor \( > 1 \). Both for the weighted projective spaces as well as for the weighted hypersurfaces this assumption is no restriction (cf. [2] 1.3.1 and [3], pp. 185-186). We will write such isomorphisms in the sequel without further comment, for example \( \mathbb{P}(2,3,6) \cong \mathbb{P}(2,1,2) \cong \mathbb{P}(1,1,1) = \mathbb{P}^2 \), where the first equality is because the last two weights are divisible by 3. The second while the first and last are divisible by 2.

We will use the notation \( \mathbb{P}(w_0, \ldots, w_n)[d] \) to denote either a certain weighted hypersurface of degree \( d \), or to denote the whole family of such (the context will make the usage clear). In the particular case that the weighted polynomial \( p \) is of Fermat type, then there is a useful fact, corresponding to the above normalizations. For example, in \( \mathbb{P}(2,3,6) \) consider the weighted hypersurface \( x_0^2 + x_1^4 + x_2^3 = 0 \). Then the isomorphism \( \mathbb{P}(2,3,6) \cong \mathbb{P}(2,1,2) \) above is given by the introduction of a new variable \((x'_0) = x_0^2\), which is in spite of appearances a one to one coordinate transformation (becuase of admissible rescalings), and the Fermat polynomial becomes \((x'_0)^2 + x_1^4 + x_2^3 = 0\). Again, the isomorphism \( \mathbb{P}(2,1,2) \cong \mathbb{P}(1,1,1) \) is given by setting \((x'_1) = x_1^2\), and the Fermat polynomial becomes \((x'_0)^2 + (x'_1)^2 + x_2^3 = 0\), which is a quadric in the projective plane. We denote this process by the symbolic expressions

\[
\mathbb{P}(2,3,6)[12] \cong \mathbb{P}(2,1,2)[4] \cong \mathbb{P}(1,1,1)[2].
\]

It is well-known how to resolve the weighted projective space \( \mathbb{P}(w_0, \ldots, w_n) \). For this, one takes the following vectors in \( \mathbb{R}^n \),

\[
v_0 = \frac{1}{w_0} \begin{pmatrix} -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix}, \quad v_1 = \frac{1}{w_1} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad v_n = \frac{1}{w_n} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},
\]

and considers the lattice \( \mathcal{L} = \mathbb{Z} v_0 + \mathbb{Z} v_1 + \cdots + \mathbb{Z} v_n \) in \( \mathbb{R}^n \). The \( n+1 \) vectors \( v_0, \ldots, v_n \) give a cone decomposition of \( \mathbb{R}^n \), and this decomposition is refined until the resulting decomposition satisfies: each cone has volume 1, where the volume is normalized in such a way that the standard simplex in \( \mathbb{R}^n \) has volume \( \prod w_i \). This is equivalent to: any set of \( n \) vectors spanning one of the cones of the decomposition form a \( \mathbb{Z} \)-basis of the lattice \( \mathcal{L} \).

Now suppose we are given a weighted hypersurface of degree \( d \) in \( \mathbb{P}(w_0, \ldots, w_n) \), such that \( d = \sum w_i \). Then, as is well-known, this is a sufficient condition for the variety to be Calabi-Yau, i.e., the dualizing sheaf is trivial. Supposing moreover that the hypersurface is quasi-smooth, then in dimensions 2 and 3, by work of Roan and Yau ([4], section 3), there is a resolution of singularities such that the smooth variety is still Calabi-Yau. In this case, the resolution (described in [3]) is easier than of the ambient projective spaces themselves. The reason is that it effectively reduces
to a question of cones in one dimension less. In particular, in the case of Calabi-Yau threefolds, the resolution is described in terms of a simplicial decomposition of a triangle (which is the face of one of the cones mentioned above). Let $X \subset P^w$ denote the singular weighted hypersurface. Then under the assumption that $X$ is quasi-smooth, the singularities are all quotient singularities by abelian groups. Locally they can be written as quotients of $\mathbb{C}^3$ by the following transformations

$$\psi: \mathbb{C}^3 \rightarrow \mathbb{C}^3$$

$$(z_1, z_2, z_3) \rightarrow \left( \exp\left(\frac{a}{d}z_1, \exp\left(\frac{b}{d}z_2, \exp\left(\frac{c}{d}z_3\right) \right) \right),$$

which describes an action of the group of $d$th roots of unity $\mu_d$ on $\mathbb{C}^3$. Let $e_i, i = 1, 2, 3$ denote the standard unit vectors in $\mathbb{R}^3$, and let

$$v = \begin{pmatrix} a/d \\ b/d \\ c/d \end{pmatrix}.$$

Let $\mathcal{L}$ denote the lattice in $\mathbb{R}^3$ spanned by the $e_i$ and $v$. Finally, let $\sigma$ denote the cone

$$\sigma = \left\{ \sum_{i=1}^{3} x_i e_i \in \mathbb{R}^3 \mid x_i \geq 0, i = 1, 2, 3 \right\}.$$

Then this affine cone determines a toric variety, which is a neighborhood of the singularity in question. Next one uses the fact that, since $X$ is Calabi-Yau, $a + b + c$ is divisible by $d$, and this in turn implies that the integral vectors we require to decompose the cone to get a smooth cone decomposition all lie in a hypersurface. This is given by

$$\mathcal{F} = \sigma \cap \left\{ \sum x_i = 1 \right\}.$$

This is a triangle, and we need to determine the number of vertices, edges and two-simplices of the simplicial decomposition of $\mathcal{F}$ to determine the number of resolution divisors, the number of intersection curves and intersection points. First of all, since the area of $\mathcal{F}$ is equal to $d$, there must be a total of $s = d$ simplices in the decomposition.

**Lemma 1.1** Define integers $d_i$ as follows.

$$d_1 = \gcd(a, d), \quad d_2 = \gcd(b, d), \quad d_3 = \gcd(c, d).$$

Then the number $v$ of vertices and $e$ of edges in a smooth decomposition is

$$v = \frac{d + 2 + (d_1 + d_2 + d_3)}{2}, \quad e = \frac{3d + (d_1 + d_2 + d_3)}{2}.$$

Here, $d$ is the order of the group acting, and is arbitrary (not necessarily odd as in [3]).

**Remark:** This formula is different than that given in [3]. In that paper, the authors only consider cases in which $d_i = 1$ all $i$, which is the same thing as only having isolated singular points, and in these cases, our formula does agree with theirs.

**Proof:** Note that if the singular point is not isolated, then there are singular curves meeting at the point; in such a case, if the singularity along the curve is $\mathbb{Z}/e\mathbb{Z}$, then $e - 1$ divisors are introduced.
to resolve the curve. This corresponds to \( e - 1 \) vertices of the decomposition, which lie on one of the edges. Let \( y \) denote the number of vertices which lie on the boundary of \( F \), i.e., on one of the edges. This number is determined exactly as in \([6]\), and is \( y + 3 = d_1 + d_2 + d_3 \), where the 3 are the original vertices of the triangle. Next, it is easy to see that we may assume that all \( y \) of these lie on one edge, as in the following picture; the number of vertices, edges and simplices remains constant:

Now it is easy to count the number of simplices (which is \( d \)) in terms of \( x = \) the number of inner vertices, and \( y \). As in the following picture, one gets the equation

\[
x + 1 + x + y = d.
\]

Furthermore, the total number of vertices is \( v = 3 + x + y \), and from these two equations we get the formula for \( v \). Since the triangle has Euler number \( 1 = v - e + d \), the number of edges \( e \) follows from this. \( \square \)

2 The twist map

Let \( V_1, V_2 \) be weighted hypersurfaces defined as follows.

\[
V_1 = \{x^\ell + p(x_1, \ldots, x_n) = 0\} \subset \mathbb{P}(w_0, w_1, \ldots, w_n),
\]

\[
V_2 = \{y^\ell + q(y_1, \ldots, y_m) = 0\} \subset \mathbb{P}(v_0, v_1, \ldots, v_m),
\]

where we assume both \( p \) and \( q \) are quasi-smooth. The degrees of these hypersurfaces are

\[
\nu = \deg(V_1) = \ell \cdot w_0, \quad \mu = \deg(V_2) = \ell \cdot v_0.
\]

We then consider the hypersurface

\[
X := \{p(z_1, \ldots, z_n) - q(t_1, \ldots, t_m) = 0\} \subset \mathbb{P}(v_0w_0, \ldots, v_0w_n, w_0v_1, \ldots, w_0v_m).
\]

Note that the degree of \( X \) is \( v_0 \cdot \deg(p) = w_0 \cdot \deg(q) = v_0w_0\ell \). The following was shown in part I of this paper. The rational map

\[
\Phi : \mathbb{P}(w_0, w_1, \ldots, w_n) \times \mathbb{P}(v_0, v_1, \ldots, v_m) \rightarrow \mathbb{P}(v_0w_0, \ldots, v_0w_n, w_0v_1, \ldots, w_0v_m)
\]

\[
((x_0, \ldots, x_n), (y_0, \ldots, y_m)) \mapsto (y_0/w_0 \cdot x_1, \ldots, y_0/w_0 \cdot x_n, x_0/v_0 \cdot y_1, \ldots, x_0/v_0 \cdot y_m)
\]
Monodromy singular fibers

\( C, C \)

8

6 \times IV

A

8 \times III

B

12 \times II

C, C^2 = A

6 \times \Pi, 1 \times I_{0}^*

B, B^{-2} = -1

9 \times \Pi, 1 \times I_{0}^*

C, C^{-3} = -1

4 \times IV, 1 \times IV^*

A, A^{-1}

8 \times \Pi, 1 \times IV^*

C, C^{-2} = A^{-1}

5 \times III, 1 \times III^*

B, B^{-1}

7 \times \Pi, 1 \times II^*

C, C^{-1}

5 \times \Pi, 1 \times IV^*, 1 \times I_{0}^*

C, C^{-2}, C^{-3} = -1

2 \times \Pi, 2 \times II^*

C, C^{-1}

Table 1: K3 surfaces with constant modulus elliptic fibrations

restricts to \( V_1 \times V_2 \) to give a rational generically finite map onto \( X \). Under the assumption that \( w_0, v_0 \) and \( \ell \) have no non-trivial common divisor, this map is generically \( \ell \) to one and \( V_1 \times V_2 \longrightarrow X \) is the projection onto the quotient of \( V_1 \times V_2 \) by \( \mu_\ell \), which acts on the product \( V_1 \times V_2 \). Moreover, assuming that \( X \) is Calabi-Yau, \( V_2 \) is Calabi-Yau and \( w_0 > 1 \), there is a resolution of singularities \( \tilde{X} \) of \( X \) which possesses a fibration onto a resolution \( Y \) of \( V_1/\mu_\ell \) (\cite{[5]}, Lemma 3.4).

3 Kodaira’s singular elliptic fibers with torsion Monodromy

Before turning to degenerations of K3 surfaces we show how to rederive Kodaira’s classification of singular fibers of elliptic surfaces as an application of the twist map. So pretend we had no idea about this classification. We will characterize singular fibers in terms of the relations which their monodromy matrices must fulfill. In aftermath, using the fact that the monodromy matrices are elements of \( SL(2, \mathbb{Z}) \), we could, using the known properties of \( SL(2, \mathbb{Z}) \), derive the classification given by Kodaira.

These singular fibers were classified by Kodaira; his method was to construct these fibers as quotients of smooth families of elliptic curves of the form \( D \times E \), where \( D \) is a disc and \( E \) is an elliptic curve with an automorphism, i.e., of modulus either \( i \) or \( \varrho \). His construction was hence in terms of a local group action. We will show how this can be easily derived upon application of our twist map, which displays things in terms of global quotients. We consider the following K3 surfaces which are images under the twist map of products of a curve \( C \) and an elliptic curve \( E \).

In Table 1, the K3 surfaces of Fermat type with the named weights are described as images under the twist map of products \( C \times E \). There are three elliptic curves which occur, namely:

\[
E_1 = \{ y_0^3 + y_1^3 + y_2^3 = 0 \} \subset \mathbb{P}_{(1,1,1)} = \mathbb{P}^2.
\]
\[
E_2 = \{ y_0^4 + y_1^4 + y_2^4 = 0 \} \subset \mathbb{P}_{(1,1,2)}.
\]
\[
E_3 = \{ y_0^6 + y_1^6 + y_2^6 = 0 \} \subset \mathbb{P}_{(1,2,3)}.
\]

and the curves \( C \) are those curves of degree \( w_0 \ell \) in \( \mathbb{P}_{(w_0,w_1,w_2)} \), which we take to be of Fermat type (except for case 11). In the last columns we list the Monodromy matrices, without using what they
look like. For example, in the fourth case, we have six singular fibers of type III (this is seen by finding the number of zeros of the polynomial $x_1^2 + x_2^6 = 0 \subset P_{(1,2)}$, which is six), and each has monodromy matrix $B$. Recall that the monodromy gives a representation of $\pi_1(B - \Delta)$, where $B$ is the base of the fibration and $\Delta$ is the ramification locus. Since in our case $B = P^1$, it is known what this fundamental group is: $\pi_1(P^1 - \{n \text{ points}\}) = \langle \alpha_1, \ldots, \alpha_n | \prod_i \alpha_i = 1 \rangle$. In case 4, it follows that since we have six singular fibers of type III, the remaining monodromy matrix (it is easily verified that there is just one) is given by a matrix $M$ satisfying the relation $B^6 \cdot M = 1$, and hence $M^2 = 1$. Similar considerations apply in all other cases.

To determine the structure of the degenerate fibers, note that for the first three examples, we have (let us use $(z_1, z_2)$ as weighted homogenous coordinates on the image weighted projective three-space) that $z_1$ and $z_2$ are both non-vanishing, while for the sum of $d$th powers we have $z_1^d + z_2^d = 0$. It is known that for weighted projective spaces for weights of the form $(1, k_2, k_3, k_4)$, the affine open subset $z_1 \neq 0$ is really just a $\mathbb{C}^3$. It follows that we may view, for each pair $(z_1, z_2)$ such that $z_1^d + z_2^d = 0$, the corresponding fiber of the K3 as the curve given by the affine equation in $\mathbb{C}^2$ which results. In the three cases of interest, these affine equations are

| affine equation | picture | Monodromy matrix |
|-----------------|---------|------------------|
| $x^3 + y^2 = 0$ | ![image](https://via.placeholder.com/150) | $C, C^6 = 1$ |
| $x^4 + y^2 = 0$ | ![image](https://via.placeholder.com/150) | $B, B^4 = 1$ |
| $x^3 + y^3 = 0$ | ![image](https://via.placeholder.com/150) | $A, A^3 = 1$ |

To describe the other singular fibers which occur, one must consider now the resolution of the singular weighted projective space. Then additional singular fibers occur if: one of the “special” sections $C_0 := \{z_2 = 0\} \cap X$ or $C_{\infty} := \{z_1 = 0\} \cap X$ is not a smooth plane cubic, an elliptic curve. We will describe this for cases giving rise to the singular fibers of types I* and IV*, respectively. These are the cases 4 (or 5), 6, 8 and 9 above. The monodromy matrices of these fibers are $-1$, $C^{-1}$, $B^{-1}$ and $A^{-1}$, respectively, as we have explained above.

**Case 4:** $P_{(1,2,3,6)}$. To see whether the curves $C_0$ and $C_{\infty}$ are indeed rational, we use the following compact method explained above:

$$C_0 = P_{(1,3,6)[12]} \cong P_{(1,1,2)[4]},$$

which is the original elliptic curve we started with.

$$C_{\infty} = P_{(2,3,6)[12]} \cong P_{(2,1,2)[4]} \cong P_{(1,1,1)[2]},$$

which means this curve is isomorphic to a quadric in the usual projective plane, hence rational.

Next we need the singular locus of the ambient space. This is $\Sigma = \{z_1 = z_2 = 0\} \cup \{z_1 = z_3 = 0\} \cap \Sigma_2 \cup \Sigma_1$, and the equation of the hypersurface is $X = \{z_1^2 + z_2^6 + z_3^3 + z_4^3 = 0\}$, so for the intersections we have $\Sigma_1 \cap X = P_{(2,6)[12]} \cong P_{(1,3)[6]} \cong P_{(1,1)[2]} = 2$ points, while for $\Sigma_2 \cap X = P_{(3,6)[12]} \cong P_{(1,2)[4]} \cong P_{(1,1)[2]} = 2$ points. Note that all four points lie on the curve

\[1\text{the notation } \{\ldots\}z_k \text{ indicates that } \{\ldots\} \text{ is fixed under a } \mathbb{Z}/k\mathbb{Z} \text{ stabilizer} \]
$C_\infty$, while two of them lie on the curve $C_0$. In particular, the two curves meet in two points. Now resolve the singularities of the ambient space; since we have two $\mathbb{Z}_2$ points and two $\mathbb{Z}_3$ points, we get a total of 6 exceptional curves, and as already mentioned, there are two “fibers”, one rational curve and one elliptic curve. The elliptic curve $C_0$ is clearly a smooth fiber; it intersects two of the singular points, which has only one explanation: there are two sections meeting it, which are components of the exceptional locus. A picture will make this clearer:

We see easily the smooth fiber $C_0$ and the fiber of type $I_0^*$, $F_\infty$, consisting of the proper transform of $C_\infty$ and four exceptional $\mathbb{P}^1$'s introduced in the resolution. Since we already deduced above that the monodromy matrix fulfills $M^2 = 1$, it follows that we have derived the structure of the singular fiber of type $I_0^*$.

Case 6: $\mathbb{P}(1,3,4,4)$ The Fermat hypersurface is $X = \{ z_1^4 + z_2^4 + z_3^4 + z_4^4 = 0 \}$ and the singular locus consists of a single component $\Sigma = \{ z_1 = z_2 = 0 \} \mathbb{Z}_4$. The intersection with $X$ is $\mathbb{P}(4,4)[12] \cong \mathbb{P}(1,1)[3] = 3$ points. Note that at these three points the two curves $C_\infty$ and $C_0$ intersect. They are $C_\infty = \mathbb{P}(3,4,4)[12] \cong \mathbb{P}(3,1,1)[3]$ which is rational and $C_0 = \mathbb{P}(1,4,4)[12] \cong \mathbb{P}(1,1,1)[3]$, which is the smooth elliptic curve. There are on the intersection three $\mathbb{Z}_4$ points, resolving them gives, in
addition to the three sections, one smooth fiber and one fiber of type IV*. The picture is as follows

Case 8: \( P_{(1,4,5,10)}[20] \) The singular locus here is \( \Sigma_1 = \{z_1 = z_3 = 0\} \mathbb{Z} \) and \( \Sigma_2 = \{z_1 = z_2 = 0\} \mathbb{Z} \). The curve \( C_0 \) is \( P_{(1,5,10)}[20] \cong P_{(1,2)}[4] \), which is elliptic, and \( C_\infty = P_{(4,5,10)}[20] \cong P_{(4,1,2)}[4] \cong P_{(2,1,1)}[2] \), which is rational. The two curves \( C_0 \) and \( C_\infty \) meet at \( X \cap \Sigma_2 = P_{(5,10)}[20] \cong P_{(1,2)}[4] \cong P_{(1,1)}[2] = 2 \) points. Hence there are two sections in the exceptional locus. There are \( X \cap \Sigma_1 = P_{(4,10)}[20] \cong P_{(2,5)}[10] \cong P_{(1,1)}[1] = 1 \) point more singularities. Resolving singularities we easily find a smooth fiber, a fiber of type III* and two sections of the fibration.

Finally, the most interesting case is the one giving rise to the II* type fiber.

Case 9: \( P_{(1,6,14,21)} \) The Fermat hypersurface is given by

\[ X = \{z_1^4 + z_2^7 + t_1^2 + t_2^2 = 0\} \subset P_{(1,6,14,21)}. \]

The two special curves are given by setting \( z_1 \) and \( z_2 \) to be zero:

\[ C_\infty = \{z_1 = 0\} \cap X = \{z_2^7 + t_1^3 + t_2^2 = 0\} \subset P_{(6,14,21)} = \{(z_2')^1 + (t_1')^1 + (t_2')^1 = 0\} \subset P_{(1,1,1)} \]

which is clearly just a linear \( P^1 \), and

\[ C_0 = \{z_1^4 + t_1^3 + t_2^2 = 0\} \subset P_{(1,14,21)} \cong \{(z_2')^6 + t_1^3 + t_2^2 = 0\} \subset P_{(1,2,3)}, \]

which is clearly just our elliptic curve. The singular locus of the ambient space is \( \Sigma_1 = \{z_1 = z_2 = 0\} \mathbb{Z} \) and \( \Sigma_2 = \{z_1 = z_3 = 0\} \mathbb{Z} \) and \( \Sigma_3 = \{z_1 = z_4 = 0\} \mathbb{Z} \). Furthermore, \( X \cap \Sigma_1 = P_{(14,21)}[42] \cong P_{(2,3)}[6] \cong P_{(1,1)}[1] = 1 \) point, \( X \cap \Sigma_2 = P_{(6,21)}[42] \cong P_{(2,7)}[14] \cong P_{(1,1)}[1] = 1 \) point,
\( X \cap \Sigma_3 = P_{(6,14)}[42] \cong P_{(3,7)}[21] \cong P_{(1,1)}[1] = 1 \text{ point. All these points are on the curve } C_\infty, \text{ one of them is the intersection with } C_0. \text{ This is described in the following picture}

For completeness we discuss briefly the last two cases in the table.

Case 10: \( P_{(2,3,10,15)}. \) First we have the fixed points given by \( x_0 = 0. \) We are looking for solutions of

\[
\{x_1^{15} + x_2^{10} = 0 \} \subset P_{(2,3)},
\]

which is the same as \( \{(x_1')^5 + (x_2')^5 = 0 \} \subset P_{(1,1)}, \) of which there are obviously only five solutions. So we have five singular fibers of type II. The singular locus of the ambient space is \( \Sigma_1 = \{z_1 = z_2 = 0\}Z_5, \Sigma_2 = \{z_1 = z_3 = 0\}Z_3, \) and \( \Sigma_3 = \{z_2 = z_4 = 0\}Z_2. \) We have for the intersections \( X \cap \Sigma_1 = P_{(10,15)}[30] \cong P_{(2,3)}[6] \cong P_{(1,1)}[1] = 1 \text{ point, } X \cap \Sigma_2 = P_{(3,15)}[30] \cong P_{(1,5)}[10] \cong P_{(1,1)}[2] = 2 \text{ points, and } \Sigma_3 = P_{(2,10)}[30] \cong P_{(1,5)}[15] \cong P_{(1,1)}[3] = 3 \text{ points.} \) The two curves \( C_0 \) and \( C_\infty \) meet in a single point (the \( Z_5 \) point), hence there is a single section in the exceptional locus. There are three \( Z_2 \) points on \( C_0, \) while there are two \( Z_3 \) points on the \( C_\infty, \) in addition to the common \( Z_5 \) point. For the curves \( C_0 \) and \( C_\infty \) we have \( C_0 = P_{(2,10,15)}[30] \cong P_{(1,5,15)}[15] \cong P_{(1,1,3)}[3], \) which is a rational curve, and \( C_\infty = P_{(3,10,15)}[30] \cong P_{(1,10,5)}[10] \cong P_{(1,2,1)}[2], \) which is also a rational curve. We have the picture:
Case 11: $\mathbb{P}_{(5,6,22,33)}$ For the weights in this case, a Fermat hypersurface is not possible. We consider instead the following polynomial:

$$\{z_0^{12}z_1 + z_1^{11} + z_2^2 + z_3^3 = 0\} \subset \mathbb{P}_{(5,6,22,33)}.$$ 

We see without difficulty that this is the image under the twist map $\mathbb{P}_{(1,5,6)} \times \mathbb{P}_{(1,2,3)} \rightarrow \mathbb{P}_{(5,6,22,33)}$

$$(x_0 : x_1 : x_2), (y_0 : y_1 : y_2)) \mapsto (y_0^{5/11} x_1 : y_0^{6/11} x_2 : x_0^2 y_1 : x_0^3 y_2)$$

of the product $\{x_0^6 + x_1^{12} x_2 + x_2^{11} = 0\} \times \{y_0^6 + y_1^3 + y_2^2 = 0\}$. As was explained in part I, this is a K3 surface, and we now describe the singular fibers. The singular locus has the following components:

$\Sigma_1 = \{z_1 = z_2 = 0\}z_{11}, \Sigma_2 = \{z_1 = z_3 = 0\}z_3, \Sigma_3 = \{z_1 = z_4 = 0\}z_2, \Sigma_4 = (\Sigma_2 \cap \Sigma_3)z_6$ and $\Sigma_5 = \{(1,0,0,0)\}z_5$. The intersections with $X$ are as follows. $\Sigma_1 \cap X = \mathbb{P}_{(22,33)}[66] = 1$ point, $\Sigma_2 \cap X = \mathbb{P}_{(6,33)}[66] \cong \mathbb{P}_{(6,3)}[6] \cong \mathbb{P}_{(2,1)}[2] = 1$ point, $\Sigma_3 \cap X = \mathbb{P}_{(6,22)}[66] \cong \mathbb{P}_{(6,2)}[6] \cong \mathbb{P}_{(3,1)}[3] = 1$ point. At the same time the fibers $C_0$ and $C_\infty$ are as follows: $C_0 = \{z_2 = 0\} = \{z_2^2 + z_4^2 = 0\}$, which is a cusp, and $C_\infty = \{z_1 = 0\} = \mathbb{P}_{(6,22,33)}[66] \cong \mathbb{P}_{(6,2,3)}[6] \cong \mathbb{P}_{(1,1,1)}[1]$, which is a rational curve. Note that the $\mathbb{Z}_5$ point is at the cusp of $C_0$. We have the following picture:

The $\mathbb{Z}/5\mathbb{Z}$ point at the cusp of $C_0$ is resolved in the usual manner, being replaced by a chain of length four. The proper transform of $C_0$ is the “center” curve of the resulting configuration, which we have drawn in the picture separately for clarity. Thus we get the two fibers of type $II^*$. 

4 Singular K3 fibers with torsion monodromy

In this section we wish to do the same as above, but now for a set of K3-fibrations.
4.1 Fibers analogous to Kodaira’s type II, III and IV

First we have the analogy to the simple Kodaira fibers. Once again, we have some unknown monodromy matrix, of which we know only the order. In these cases, just as above, we have an affine surface as the singular fiber. These are listed in Table 2.

4.2 Fibers analogous to Kodaira’s type I*6, II*, III* and IV*

Here we repeat the analysis above, this time applied to weighted hypersurfaces which are K3-fibrations. We consider the cases listed in Table 3.

Just as in the case of the elliptic fibrations we have the surfaces $C_0 := \{z_2 = 0\} \cap X$ and $C_\infty := \{z_1 = 0\} \cap X$. In the first three cases it is easy to see that both of these surfaces are just smooth fibers, hence the only singular fibers which occur are those for which the affine surface listed in Table 2 of types IV1, IX1 and XII3, respectively, describe the singular fibers. Those of interest to us here occur in the remaining cases. We begin by discussing the singular fibers denoted IV*, IX*1 and XII*5 in Table 3.

4.3 Fibers analogous to I*6

In this section we consider the cases 4-6 in the table above, in which the monodromy matrix of the singular fiber fulfills $M^2 = 1$. In this respect, each of these is an analog of Kodaira’s I*6 type fiber.

Case IV*1

We describe this example in more detail as a description of the general procedure to be used in the sequel. The bad fiber is $C_\infty \cong \mathbb{P}(2,3,6,6)[18] \cong \mathbb{P}(2,1,2,2)[6] \cong \mathbb{P}(1,1,1,1)[3]$, a cubic surface. (This is a del-Pezzo surface of degree 3, which is $\mathbb{P}^2$ blown up in six points). The singular locus of the ambient space is $\Sigma_1 = \{z_1 = z_2 = 0\} \mathbb{Z}_6$, $\Sigma_2 = \{z_1 = z_3 = 0\} \mathbb{Z}_6$ and their intersection is $(\Sigma_1 \cap \Sigma_2) \mathbb{Z}_6$. Note that $\Sigma_1 \cap X = \mathbb{P}(3,6,6)[18] \cong \mathbb{P}(1,2,2)[6] \cong \mathbb{P}(1,1,1)[3]$, which is a cubic curve, which is elliptic. The other intersection is $\Sigma_2 \cap X = \mathbb{P}(2,6,6)[18] \cong \mathbb{P}(1,3,3)[9] \cong \mathbb{P}(1,1,1)[3]$, which is again an elliptic curve. They meet in the three $\mathbb{Z}_6$-points on $X$ ($\mathbb{P}(6,6)[18] \cong \mathbb{P}(1,1)[3] = \text{three points}$). We have the following picture:

![Diagram](image)

Each of the dots represents a $\mathbb{Z}_6$-point, $\Sigma_1 \cap X$ is the intersection of the two surfaces $C_0$ and $C_\infty$.

The curve $\Sigma_1$ is the base locus of the K3-fibration, i.e., every fiber $X_s$ passes through $\Sigma_1$; we think of the base $\mathbb{P}^1$ of the fibration as the exceptional $\mathbb{P}^1$ of directions through $\Sigma_1$. Note the general fiber is $\mathbb{P}(1,3,6,6)[18] \cong \mathbb{P}(1,1,2,2)[6]$, which has three $\mathbb{Z}_6$ points ($\{z_1 = z_2 = 0\} \mathbb{Z}_6 \cap X = \mathbb{P}(2,2)[6] \cong \mathbb{P}(1,1)[3]$), whose resolutions are in fact sections of the elliptic fibration of the fiber. This exceptional $\mathbb{P}^1$ in each fiber is the intersection of the fiber with the exceptional divisor ($\cong \mathbb{P}(1,2,3)$ described below) at each $\mathbb{Z}_6$-point.

The $\mathbb{Z}_6$-singular points are of the type $\frac{1}{6}(1,2,3)$, which is the usual shorthand for the quotient of $\mathbb{C}_3$ by the action $(z_1, z_2, z_3) \mapsto (e^{2\pi i/6}z_1, e^{2\pi i/6}z_2, e^{\pi i}z_3)$. To describe the resolution of the
| Fiber | Affine Equation | $\mu$ | Euler # | Fiber | Monodromy | Relation |
|-------|----------------|------|--------|-------|-----------|----------|
| IV$_1$ | $z^6 + x^3 + y^3 = 0$ | 20 | 4 | | M(IV$_1$) | $M(IV_1)^6 = 1$ |
| III$_1$ | $z^8 + x^4 + y^2 = 0$ | 21 | 3 | | M(III$_1$) | $M(III_1)^8 = 1$ |
| II$_1$ | $z^{12} + x^3 + y^2 = 0$ | 22 | 2 | | M(II$_1$) | $M(II_1)^{12} = 1$ |
| IX$_1$ | $z^6 + x^4 + y^2 = 0$ | 15 | 9 | | M(IX$_1$) | $M(IX_1)^{12} = 1$ |
| VIII$_1$ | $z^9 + x^3 + y^2 = 0$ | 16 | 8 | | M(VIII$_1$) | $M(VIII_1)^{18} = 1$ |
| XII$_1$ | $z^4 + x^3 + y^3 = 0$ | 12 | 12 | | M(XII$_1$) | $M(XII_1)^{12} = 1$ |
| X$_1$ | $z^8 + x^3 + y^2 = 0$ | 14 | 10 | | M(X$_1$) | $M(X_1)^{24} = 1$ |
| XII$_2$ | $z^5 + x^4 + y^2 = 0$ | 12 | 12 | | M(XII$_2$) | $M(XII_2)^{20} = 1$ |
| XII$_3$ | $z^7 + x^3 + y^2 = 0$ | 12 | 12 | | M(XII$_3$) | $M(XII_3)^{42} = 1$ |
| VI$_1$ | $z^{10} + x^3 + y^2 = 0$ | 18 | 6 | | M(VI$_1$) | $M(VI_1)^{15} = 1$ |

Table 2: List of singular K3-fibers with nilpotent Monodromy
### Table 3: K3-fibered Calabi-Yau weighted hypersurfaces which are also elliptic fibered, have constant modulus and are of Fermat type

| $(w_0, w_1, w_2)$ | $(v_0, v_1, v_2, v_3)$ | $\ell$ | $(k_1, k_2, k_3, k_4, k_5)$ | $d$ | Euler# | singular fibers |
|-------------------|------------------------|--------|-----------------------------|-----|--------|-----------------|
| $(2,1,1)$         | $(1,1,2,2)$            | 6      | $(1,1,2,4,4)$               | 12  | −192   | $12 \times \text{IV}_1$ |
|                   | $(1,2,3,6)$            | 12     | $(1,1,4,6,12)$              | 24  | −312   | $24 \times \text{IX}_1$ |
|                   | $(1,6,14,21)$          | 42     | $(1,1,12,28,42)$            | 84  | −960   | $84 \times \text{XII}_2$ |
| $(3,1,2)$         | $(1,1,2,2)$            | 6      | $(1,2,3,6,6)$               | 18  | −144   | $9 \times \text{IV}_1, 1 \times \text{IV}^*_1$ |
|                   | $(1,2,3,6)$            | 12     | $(1,2,6,9,18)$              | 36  | −228   | $18 \times \text{IX}_1, 1 \times \text{IX}^*_1$ |
|                   | $(1,6,14,21)$          | 42     | $(1,2,18,42,63)$            | 126 | −720   | $63 \times \text{XII}_3, 1 \times \text{XII}^*_3$ |
| $(4,1,3)$         | $(1,1,2,2)$            | 6      | $(1,3,4,8,8)$               | 24  | −120   | $8 \times \text{IV}_1, 1 \times \text{IV}^{**}_1$ |
|                   | $(1,2,3,6)$            | 12     | $(1,3,8,12,24)$             | 48  | −192   | $16 \times \text{IX}_1, 1 \times \text{IX}^{**}_1$ |
|                   | $(1,6,14,21)$          | 42     | $(1,3,24,56,84)$            | 168 | −624   | $56 \times \text{XII}_3, 1 \times \text{XII}^{**}_3$ |
| $(5,1,4)$         | $(1,2,3,6)$            | 12     | $(1,4,10,15,30)$            | 60  | −168   | $15 \times \text{IX}_1, 1 \times \text{IX}^*_1$ |
| $(7,1,6)$         | $(1,2,3,6)$            | 12     | $(1,6,14,21,42)$            | 84  | −132   | $14 \times \text{IX}_1, 1 \times \text{IX}^{**}_1$ |
|                   | $(1,6,14,21)$          | 42     | $(1,6,42,98,147)$           | 294 | −480   | $49 \times \text{XII}_3, 1 \times \text{XII}^{**}_3$ |
| $(5,2,3)$         | $(1,1,2,2)$            | 6      | $(2,3,5,10,10)$             | 30  | −72    | $5 \times \text{IV}_1, 1 \times \text{IV}^{**}_1, 1 \times \text{IV}^*_1$ |
|                   | $(1,2,3,6)$            | 12     | $(2,3,10,15,30)$            | 60  | −108   | $10 \times \text{IX}_1, 1 \times \text{IX}^*_1, 1 \times \text{IX}^{**}_1$ |
|                   | $(1,6,14,21)$          | 42     | $(2,3,30,70,105)$           | 210 | −384   | $35 \times \text{XII}_3, 1 \times \text{XII}^*_3, 1 \times \text{XII}^{**}_3$ |

In which the two (respectively one) vertices on the edge $\kappa_1, \kappa_2$ (respectively $\kappa_1 \kappa_2 \kappa_3$) correspond to the two (respectively one) exceptional divisors over $\Sigma_i$. One of the two divisors over $\Sigma_1$ is a curve-section of the fibration: every fiber passes through $\Sigma_1$, hence the intersection of one of the divisors with each fiber is a curve, in this case isomorphic to the curve $\Sigma_1$, which is elliptic. Thus, only the other two components over the singular loci $\Sigma_1$ and $\Sigma_2$ belong to the singular fiber. The vertex in the middle of the triangle corresponds to an additional exceptional divisor, which one easily sees is just a copy of $\mathbb{P}_{(1,2,3)}$, which is then resolved when the singular curves $\Sigma_i$ are. There are six cones (triangles) decomposing the big one, corresponding to the fact that the lattice is of index $\mathbb{Z}_6$.\footnote{This is a different, but equivalent, description of what we described above.} \footnote{In this and following diagrams, vertices which are circled belong to divisors $E$ which are sections of the fibration (i.e., $E \cap X_s$ is a curve for all $s \in \mathbb{P}^1$), hence do not belong to the singular fiber.}
Three exceptional $\mathbb{P}^1$'s on the blow-up of the cubic surface

Figure 1: This is the singular fiber of type VI$_1^*$. It consists of three components, the image of the K3 surface itself, here denoted $\Theta$, which is the proper transform of a copy of $\mathbb{P}^{(1,1,1,1)}[3]$, a cubic surface. The exceptional divisor over $\Sigma_i$ is denoted $\Theta_i$, and is an elliptic ruled surface, while $\Theta$ is a rational elliptic surface (it is $\mathbb{P}^2$ blown up at nine points), and each of the intersections $\Theta \cap \Theta_i$ is an elliptic curve.

See also [3] for details on these matters. Altogether there are at each $\mathbb{Z}_6$-point a total of four exceptional divisors; three of these are the exceptional divisors over the $\Sigma_i$, the additional one at each point is the $\mathbb{P}^{(1,2,3)}$ just mentioned. Note that these latter exceptional surfaces are also $\mathbb{P}^1$-sections of the fibration, as they lie on $\Sigma_1 \cap X$, hence meet all fibers.

After resolution of singularities, we have the following divisors which were introduced:

1. Over $\Sigma_1$, two elliptic ruled surfaces $\Theta_{1,1}$ and $\Theta_{1,2}$, which intersect each other in a section of the ruling.

2. Over $\Sigma_2$, an elliptic ruled surface, $\Theta_{2,1}$.

3. Over each $\mathbb{Z}_6$-point, a (resolution of a) copy of $\mathbb{P}^{(1,2,3)}$; this intersects the fiber $F_\infty$ in the union of four rational curves, and intersects the other fibers in an exceptional $\mathbb{P}^1$. Let $\Theta_3, \Theta_4, \Theta_5$ denote the three exceptional divisors introduced over the three points.

4. The proper transform $\Theta$ of $C_\infty$.

The proper transform $[C_\infty]$ is the cubic surface $\mathbb{P}^{(2,3,6,6)}[18] \cong \mathbb{P}^{(1,1,1,1)}[3]$ blown up at three disjoint points. Let $\Theta$ denote this surface; it is a rational elliptic surface with $e(\Theta) = 12$, $K_\Theta^2 = 0$. The singular fiber $F_\infty$ is

$$F_\infty = \Theta \cup \Theta_{1,1} \cup \Theta_{2,1}.$$  

The following divisors are sections of the fibration (hence do not belong to the fiber $F_\infty$): $\Theta_{1,2}, \Theta_3, \Theta_4, \Theta_5$. The fiber $F_\infty$ is depicted in Figure 1.

We can also determine the properties of the monodromy matrix which determines this bad fiber: since the fibration has 9 singular fibers of type IV$_1$, and $M(IV_1)^6 = 1$, it follows that the monodromy matrix $M$ here must fulfill $M(IV_1)^9 \cdot M = 1$, i.e., $M = M(IV_1)^{-3}$ and hence $M^2 = 1$. So this fiber is in a sense an analogue of the Kodaira type $I_6^*$. Moreover, an easy calculation gives
the Euler number of this fiber. Indeed, since the fibered threefold has Euler number $-144$, and there are nine fibers of type $IV_1$ and a single fiber of type $IV_1^*$, we get from the formula

$$e(X) = 24 \cdot (2 - (9 + 1)) + 9 \cdot e(IV_1) + 1 \cdot e(IV_1^*) = -144,$$

that the Euler number of the bad fiber is $12$. We can check this, by calculating the Euler number of our bad fiber: it is

$$e(IV_1^*) = e(\Theta) + e(\Theta_1) + e(\Theta_2) - e(\Theta \cap \Theta_1) - e(\Theta \cap \Theta_2) = 12 + 0 + 0 - 0 - 0,$$

since elliptic ruled surfaces, as well as elliptic curves, have Euler number $0$.

In the sequel we will not go into such detail.

**Case $IX_1^*$**

The bad fiber is again $C_\infty \cong P_{(2,6,9,18)}[36] \cong P_{(2,2,3,6)}[12] \cong P_{(1,1,3,3)}[6]$. The singular locus of the ambient space is $\Sigma_1 = \{z_1 = z_2 = 0\} \cong Z_6$, $\Sigma_2 = \{z_1 = z_4 = 0\} \cong Z_6$ and the lower-dimensional parts are given by $\Sigma_3 = \{z_1 = z_2 = z_3 = 0\} \cong Z_9$ and $\Sigma_4 = \{z_1 = z_2 = x_4 = 0\} \cong Z_9 = \Sigma_1 \cap \Sigma_2$. The intersections with $X$ give $\Sigma_1 \cap X = \Sigma_{(6,9,18)}[36] \cong P_{(2,3,6)}[12] \cong P_{(2,1,2)}[4] \cong P_{(1,1,1)}[2]$, which is a rational curve, and $\Sigma_2 \cap X = \Sigma_{(2,6,18)}[36] \cong P_{(1,9,9)}[18] \cong P_{(1,1,3)}[6]$ which is a curve of genus $2$ (it is a double cover of $P^1$ branched along a sextic, or alternatively, a degree six curve on the Hirzebruch surface $P_{(1,1,3)}$). These two intersect in $\Sigma_1 \cap \Sigma_2 \cap X = \Sigma_{(6,18)}[36] \cong P_{(1,3,3)}[6] \cong P_{(1,1,1)}[2]$ which is two points. This is the locus $\Sigma_4 \cap X$ and consists of two points. The $Z_9$-locus yields $P_{(9,18)}[36] \cong P_{(1,2)}[4] \cong P_{(1,1)}[2]$ which is also two points. The configuration then looks like

![Diagram](image)

in which the $Z_6$ points are the filled circles, the $Z_9$-points the filled squares. The resolution of the curves $\Sigma_i$, $i = 1, 2$ is the same as above, hence the resolution of the $Z_6$ points is precisely as above. To describe the $Z_9$ points, we note that since they lie on a $Z_3$ curve, one of the fractions is $3/9$, hence we have $\frac{1}{9}(1,3,5)$. The resolution of this follows the same pattern as above. We now find the following cone decomposition:
Using Lemma [1.1], we see that in our case of \( \frac{1}{13}(1,3,5) \) we have \( d_1 = d_3 = 1, \ d_2 = 3, \ d_1 + d_2 + d_3 = 5 \) and hence the number of vertices (including the three corners of the original triangle) is 8, the number of edges is 16, and the cone decomposition is as given above. The two vertices on the edge of the triangle of course correspond to the two exceptional divisors over the curve \( \Sigma_1 \), on which the \( \mathbb{Z}_9 \)-points lie. There are at each \( \mathbb{Z}_9 \)-point three additional exceptional divisors. Note that again, one of these three is a \( \mathbb{P}^1 \)-section of the fibration, while the other two are components of the bad fiber. We have circled the vertices corresponding to the exceptional divisors which are \( \mathbb{P}^1 \)-sections of the fibration. After the resolution of singularities, our fiber will look as in Figure 2.

Once again the monodromy is easily seen to satisfy \( M^2 = 1 \), and the Euler number of this singular fiber can be calculated as above; it is 18.

**Case XII**

The ambient space is \( \mathbb{P}(1,2,18,42,63) \), and the singular locus in this space is

- \( \Sigma_1 = \{ z_1 = z_2 = 0 \} \mathbb{Z}_3 \cong \mathbb{P}(18,42,63) \cong \mathbb{P}(1,1,3) \).
- \( \Sigma_2 = \{ z_1 = z_5 = 0 \} \mathbb{Z}_2 \cong \mathbb{P}(2,18,42) \cong \mathbb{P}(1,3,7) \).
- \( \Sigma_3 = \Sigma_1 \cap \Sigma_2 = \{ z_1 = z_2 = z_5 = 0 \} \mathbb{Z}_6 \cong \mathbb{P}^1 \).
- \( \Sigma_4 = \{ z_1 = z_2 = z_3 = 0 \} \mathbb{Z}_21 \cong \mathbb{P}^1 \).

The bad fiber is \( C_\infty \cong \mathbb{P}(2,18,42,63)[126] \cong \mathbb{P}(1,3,7,21)[21] \); it contains the two curves \( X \cap \Sigma_1 \) and \( X \cap \Sigma_2 \), and these two curves intersect in just one point. There is a further singular point \( \Sigma_4 \cap X \), which also lies on \( \Sigma_1 \) but not on \( \Sigma_2 \). Hence we have the picture.
The fiber is the sum \( \Theta + \Theta_1 + \ldots + \Theta_8 + \Theta_{1,1} + \Theta_{2,1} \)
where \( \Theta \) is the quotient fiber
\( \cong \mathbb{P}_{(1,3,7,21)}[21], \Theta_i, \ i = 1, \ldots, 8 \)
are the resolution divisors of the \( \mathbb{Z}_{21} \) point, \( \Theta_9 \) is the one which is a section, \( \Theta_{1,1} \) is rational ruled and \( \Theta_{2,1} \) is \( g = 6 \) ruled.

Figure 3: — The singular fiber of type XII* —

The \( \mathbb{Z}_6 \)-point is resolved just as above, with a single exceptional divisor which is a \( \mathbb{P}^1 \)-section of the fibration. For the \( \mathbb{Z}_{21} \) point, we deduce from the fact that it lies on a \( \mathbb{Z}_3 \) curve, that it is of type \( \frac{1}{21}(1,2,18) \). This yields \( d_1 = d_2 = 1, \ d_3 = 3 \) and hence, by the formula above, \( v = \frac{23+5}{2} = 14 \), of which five are on the boundary of the triangle. As we have already mentioned, any decomposition of the cone with this many vertices yields a smooth resolution of the singular point. As described above, since the singular point is on \( \Sigma_1 \), it meets every fiber, so one of the components is a curve-section of the fibration. We choose the following decomposition:

We have nine exceptional divisors, of which one is a section of the fibration, the other eight belong to the singular fiber. After resolution of singularities, our singular fiber thus looks as in Figure 3.

Note the surface \( \Theta_9 \), which is a section of the fibration, not a component of the singular fiber. This component of the resolution corresponds to the circled vertex above, and meets all the other eight components, while each of the other eight meets only two others, as drawn.

4.4 Fibers analogous to Kodaira’s type IV*

In this section we consider the cases 7-9 in our table. In all these cases, the monodromy matrix is an element of order 3, so in a sense an analog of Kodaira’s type IV* type fiber.

Case IV**:
The fiber is the sum
\[ \Theta + \Theta_1 + \Theta_2 + \Theta_3 + \Theta_{11} + \Theta_{12} \]
where \( \Theta \) is the quotient fiber
\[ \cong P_{(1,2,2,3)}[6], \Theta_i, i = 1, \ldots, 3 \]
are the resolution divisors of the three \( \mathbb{Z}_8 \) points, \( \Theta_{1,i}, i = 1, 2 \)
are the resolution divisors over the singular curve.

Figure 4: — The singular fiber of type IV_{1**} —

The ambient space is \( P_{(1,3,4,8,8)} \), which has the following singularities:

- \( \Sigma_1 = \{ z_1 = z_4 = 0 \} \mathbb{Z}_4 \),
- \( \Sigma_2 = \{ z_1 = z_2 = z_3 = 0 \} \mathbb{Z}_8 \).

For the intersections we have \( \Sigma_1 \cap X = P_{(4,8,8)}[24] \cong P_{(1,2,2)}[6] \cong P_{(1,1,1)}[3] \), which is an elliptic curve. \( \Sigma_2 \cap X = P_{(8,8)}[24] \cong P_{(1,1)}[3] \), which consists of three points. So the singular locus of \( X \) is a curve and three additional points on that curve. The bad fiber is \( C_\infty = P_{(3,4,8,8)} \cong P_{(1,2,3,2)}[6] \), which is a double cover of the space \( P_{(1,2,3)} \) branched over a sextic curve. Since the \( \mathbb{Z}_8 \) points lie on a \( \mathbb{Z}_4 \) curve, their type is \( \frac{1}{8}(1, 3, 4) \), and we have \( d_1 = d_2 = 1, d_3 = 4 \) so by our formula above we get \( v = \frac{10 + 6}{2} = 8 \), of which 3 lie on the edge (corresponding to the exceptional divisors of the \( \mathbb{Z}_4 \)-curve), and we take the following cone decomposition:

This means that at each of the three \( \mathbb{Z}_8 \) points, we have two exceptional divisors, one of which is a section of the fibration. In the picture of the cone decomposition we have circled the two vertices which correspond to exceptional divisors which are sections of the fibration. The others then belong to the singular fiber, so we have a total of five components in addition to the image of the K3 itself. After resolution of singularities, we get the fiber in Figure 4.

Case IX_{1**}:
The ambient space is \( P_{(1,3,8,12,24)} \), which has as singular locus:

- \( \Sigma_1 = \{ z_1 = z_2 = 0 \} \mathbb{Z}_4 \),
- \( \Sigma_2 = \{ z_1 = z_3 = 0 \} \mathbb{Z}_8 \),
- \( \Sigma_3 = \Sigma_1 \cap \Sigma_2 = \{ z_1 = z_2 = z_3 = 0 \} \mathbb{Z}_{12} \),
\[ \Sigma_4 = \{z_1 = z_2 = z_4 = 0\}\mathbb{Z}_8. \]

The intersections with \(X\) are \(\Sigma_1 \cap X = \mathbb{P}_{(8,12,24)}[48] \cong \mathbb{P}_{(1,1,1)}[2]\), a rational curve, \(\Sigma_2 \cap X = \mathbb{P}_{(3,12,24)}[48] \cong \mathbb{P}_{(1,1,2)}[4]\), an elliptic curve, \(\Sigma_3 \cap X = \mathbb{P}_{(12,24)}[48] \cong \mathbb{P}_{(1,1)}[2]\), consisting of two points, and \(\Sigma_4 \cap X = \mathbb{P}_{(8,24)}[48] \cong \mathbb{P}_{(1,1)}[2]\), again two points. Hence, before resolution, our fiber \(C_\infty = \mathbb{P}_{(3,8,12,24)}[48] \cong \mathbb{P}_{(1,1,2,2)}[3]\) looks as follows:

![Diagram](image)

The situation at each of the \(\mathbb{Z}_8\) points is just as in the previous example, as each lies again on a \(\mathbb{Z}_4\) curve. Hence the resolution of these points introduces two exceptional divisors each, one of which is a component of the singular fiber. Looking at the \(\mathbb{Z}_{12}\) points, we see that as they are again on \(\mathbb{Z}_3\) curves, they are of type \(\frac{1}{12}(1,2,9)\). Hence we have \(d_1 = 1, d_2 = 2, d_3 = 3\) and the formula for the number of vertices of a cone decomposition yields \(v = \frac{14+6}{2} = 10\) vertices, of which there are three on the boundary corresponding to the \(\mathbb{Z}_4\) curve, and two others on another boundary, corresponding to the \(\mathbb{Z}_3\) curve. Hence there are two vertices in the interior, and we have the cone decomposition:

![Diagram](image)

In total, in addition to the proper transform \(\Theta\) of the singular fiber \(C_\infty\), we have two divisors over \(\Sigma_1\) (the third is a section of the fibration and not a component of the fiber), two over \(\Sigma_2\), and over each of the two \(\mathbb{Z}_8\) and two \(\mathbb{Z}_{12}\) points we also have two exceptional divisors, one of which is not a section, hence a component of the fiber. After resolution of singularities, we have the picture of Figure 3.

**Case XII**

The ambient space is \(\mathbb{P}_{(1,3,24,56,84)}\), which has the following singular locus:

- \(\Sigma_1 = \{z_1 = z_2 = 0\}\mathbb{Z}_4\),
- \(\Sigma_2 = \{z_1 = z_4 = 0\}\mathbb{Z}_4\),
- \(\Sigma_3 = \{z_1 = z_2 = z_5 = 0\}\mathbb{Z}_8\),
- \(\Sigma_4 = \{z_1 = z_2 = z_3 = 0\}\mathbb{Z}_{28}\),
- \(\Sigma_5 = \Sigma_1 \cap \Sigma_2 = \{z_1 = z_2 = z_4 = 0\}\mathbb{Z}_{12}\).
The singular fiber is the sum
\[ \Theta + \Theta_1 + \Theta_2 + \Theta_3 + \Theta_4 + \Theta_{1,1} + \Theta_{1,2} + \Theta_{2,1} + \Theta_{2,2} \]
where \( \Theta \) is the proper transform of the surface \( C_\infty \) and the \( \Theta_i \)
\( i = 1, \ldots, 4 \) are the resolution surfaces of the two \( \mathbb{Z}_8 \) points and the
two \( \mathbb{Z}_{12} \) points; the \( \Theta_{1,i}, i = 1, 2 \)
are rational ruled, the \( \Theta_{2,i}, i = 1, 2 \)
are elliptic ruled.

Figure 5: — The singular fiber of type IX\(_1^*\) —

For the intersections with \( X \) we get \( \Sigma_1 \cap X \cong \mathbb{P}(1,1,1)[1] \) is rational, \( \Sigma_2 \cap X \cong \mathbb{P}(1,2,7)[14] \), a
g = 3 curve, \( \Sigma_3 \cap X \), \( \Sigma_4 \cap X \) and \( \Sigma_5 \cap X \) all consist of just a single point. The singular fiber is
\( C_\infty \cong \mathbb{P}(1,2,7,14)[14] \), a rational surface. This looks as follows:

The \( \mathbb{Z}_{12} \) point is just the same as above, yielding upon resolution one additional component
to the fiber. Similarly, the \( \mathbb{Z}_8 \) point is the same as above, yielding also one additional component
to the fiber. It remains to resolve the \( \mathbb{Z}_{28} \) point. Note that as it lies on a \( \mathbb{Z}_4 \) curve, it is of type
\( \frac{1}{28}(1,3,24) \). We have \( d_1 = d_2 = 1, d_3 = 4 \) and for the number of vertices we get \( v = \frac{30+6}{2} = 18 \), of
which only six are on the boundary. Hence we must insert 12 vertices in the interior. We choose
the following cone decomposition:
The singular fiber is the sum
\[ \Theta + \Theta_1 + \cdots + \Theta_{13} + \Theta_{1,1} + \Theta_{1,2} + \Theta_{2,1} + \Theta_{2,2} \]
where \( \Theta \) is the proper transform of \( C_\infty \)
and \( \Theta_i, i = 1, \ldots, 11 \) are the divisors of the resolution of the \( \mathbb{Z}_{28} \) point
\( \Theta_{12} \) is from the \( \mathbb{Z}_{12} \) point
\( \Theta_{13} \) is from the \( \mathbb{Z}_8 \) point
\( \Theta_{1,i}, i = 1, 2 \) are rational ruled
\( \Theta_{2,i}, i = 1, 2 \) are \( g = 3 \) ruled.

Figure 6: — The singular fiber of type XII$_3^{**}$ —

Once again, the two components of the resolution which give rise to sections of the fibration instead of components of the singular fiber are circled. We again see in the middle the eight components giving rise to an \( A_7 \) configuration, but this time the component which meets all eight is a component of the singular fiber instead of a section. In addition, we have two more components, each of which meets the special component and four of the components of the \( A_7 \) chain. After resolution, the fiber looks as in Figure 6.

4.5 An analog of Kodaira’s type III$^*$ fiber

We consider now the tenth case in the table. The monodromy matrix has order four, so in this sense this is an analog of Kodaira’s type III$^*$ fiber.

Case IX$^{***}$:

The ambient space is \( \mathbb{P}_{(1,4,10,15,30)} \) with singular locus

- \( \Sigma_1 = \{ z_1 = z_2 = 0 \} \mathbb{Z}_5 \),
- \( \Sigma_2 = \{ z_1 = z_4 = 0 \} \mathbb{Z}_2 \),
- \( \Sigma_3 = \Sigma_1 \cap \Sigma_2 = \{ z_1 = z_2 = z_4 = 0 \} \mathbb{Z}_{10} \),
- \( \Sigma_4 = \{ z_1 = z_2 = z_3 = 0 \} \mathbb{Z}_{15} \).

For the intersections with \( X \) we have \( \Sigma_1 \cap X \cong \mathbb{P}_{(1,1,1)}[2] \), a rational curve, \( \Sigma_2 \cap X \cong \mathbb{P}_{(2,3,1)}[6] \), an elliptic curve, and the two intersections \( X \cap \Sigma_3 \) and \( X \cap \Sigma_4 \) both consist of two points. Hence the picture is just as in the case IX$^{**}$ above, with the \( \mathbb{Z}_{12} \) points now replaced by \( \mathbb{Z}_{10} \) points, and the \( \mathbb{Z}_8 \) points there replaced by \( \mathbb{Z}_{15} \) points here. The resolution of the \( \mathbb{Z}_2 \) curve is an elliptic ruled surface, a component of the singular fiber, and the resolution of the \( \mathbb{Z}_5 \) curve is a union of four rational ruled surfaces, of which one is a section of the fibration, while the other three give components of the singular fiber. So we just have to resolve the \( \mathbb{Z}_{10} \) and \( \mathbb{Z}_{15} \) points. Note that since both lie on a \( \mathbb{Z}_5 \) curve, they are of types \( \frac{18}{18}(1,4,5) \) and \( \frac{18}{18}(1,4,10) \) (or \( (2,3,1) \), it doesn’t matter). Hence we have \( d_1 = 1, d_2 = 2, d_3 = 5 \) in the first and \( d_1 = d_2 = 1, d_3 = 5 \) in the second case. Hence we have \( v = \frac{12+18}{2} = 10 \) vertices in the first case and \( v = \frac{12+18}{2} = 12 \) vertices in the second. We take the two following cone decompositions:
The singular fiber is the union of 16 components
\[ \Theta + \Theta_{1,1} + \cdots + \Theta_{1,4} + \Theta_{2,1} + \Theta_1 + \Theta_2 + \Theta_{3,1} + \cdots + \Theta_{3,4} + \Theta_{4,1} + \cdots + \Theta_{4,4} \]
where \( \Theta \) is the proper transform of \( C_\infty \), and \( \Theta_{1,i}, i = 1, \ldots, 4 \) are the components resolving the \( \mathbb{Z}_5 \) curve, \( \Theta_{2,1} \) resolving the \( \mathbb{Z}_2 \) curve, \( \Theta_{i,j}, i = 3, 4, j = 1, \ldots, 4 \) are the divisors resolving the \( \mathbb{Z}_{15} \) points, \( \Theta_i, i = 1, 2 \) resolving the \( \mathbb{Z}_{10} \) points.

![Diagram](image)

Figure 7: — The singular fiber of type IX_1** —

From the singular curves we get \( 1 + 4 = 5 \) components, from each \( \mathbb{Z}_{10} \) point an additional one and at each \( \mathbb{Z}_{15} \) point, we get four further components. Hence the singular fiber has a total of 15 components, in addition to \( \Theta \), the proper transform of \( C_\infty \).

After resolution of singularities, the singular fiber looks as in Figure 7.

4.6 Analogs of Kodaira’s type II∗ fiber

Case IX_1**: The ambient space is \( \mathbb{P}_{(1,6,14,21,42)} \), which has the following singular locus:

- \( \Sigma_1 = \{ z_1 = z_2 = 0 \} \mathbb{Z}_7 \).
The singular fiber is a sum of 26 components. Θ is the proper transform of $C_\infty$, and for the exceptional divisors we have chosen the notation so that the divisors $\Theta_{i,j}$ resolve the singular locus $\Sigma_j$. There are then $5 + 2 + 1 + 10 + 4 + 2 = 25$ components of the various loci.

- $\Sigma_2 = \{z_1 = z_3 = 0\} \mathbb{Z}_3$,
- $\Sigma_3 = \{z_1 = z_4 = 0\} \mathbb{Z}_2$,
- $\Sigma_4 = (\Sigma_1 \cap \Sigma_2) \mathbb{Z}_{21}$,
- $\Sigma_5 = (\Sigma_1 \cap \Sigma_3) \mathbb{Z}_{14}$,
- $\Sigma_6 = (\Sigma_2 \cap \Sigma_3) \mathbb{Z}_6$.

The intersections $\Sigma_i \cap X$ are all rational curves, which meet two at a time. We have the following configuration in $C_\infty$:

The $Z_6$ points are resolved precisely as in the cases above; there is one exceptional divisor, which this time is a component of the singular fiber, as it is not contained in all fibers, but only in $C_\infty$. It remains to resolve the $Z_{14}$ and $Z_{21}$ points. They are of types $\frac{1}{11}(1, 6, 7)$ and $\frac{1}{21}(1, 6, 14)$, and we have $d_1 = 1$, $d_2 = 2$, $d_3 = 7$ in the first case and $d_1 = 1$, $d_2 = 3$, $d_3 = 7$ in the second case. By our formula above, this means we have $v = 13$ and $v = 17$, respectively, leading to 3 and 6 inner vertices, respectively. We choose the following cone decompositions:
Again the vertices corresponding to the sections of the fibration are circled. Just as above, from this we can without difficulty derive the singular fiber. It will look as in Figure 8.

**Case XII\(^{**}_3\):**

The ambient projective space is \(\mathbb{P}(1,6,42,98,147)\) with singular locus

- \(\Sigma_1 = \{z_1 = z_2 = 0\} \mathbb{Z}_7\),
- \(\Sigma_2 = \{z_1 = z_4 = 0\} \mathbb{Z}_3\),
- \(\Sigma_3 = (\Sigma_1 \cap \Sigma_2) \mathbb{Z}_{21}\),
- \(\Sigma_4 = \{z_1 = z_2 = z_3 = 0\} \mathbb{Z}_{49}\).

The intersections \(\Sigma_i \cap X, \ i = 1, 2\) are both rational curves, which meet in a single point. Furthermore, \(\Sigma_4 \cap X\) consists also of a single point. One sees easily that the \(\mathbb{Z}_{21}\) point is resolved exactly as above in the case IX\(^{**}_3\). It remains to resolve the \(\mathbb{Z}_{49}\) point. Since this lies on a \(\mathbb{Z}_7\) curve, the singularity is of the type \(\frac{1}{49}(1,6,42)\), and we have \(d_1 = d_2 = 1, \ d_3 = 7\). The number of vertices is then \(\frac{31+9}{2} = 30\), of which \(3 + 6\) are on the boundary. It follows that we have to include 21 inside vertices. We take the cone decomposition of Figure 9.

The singular fiber will look quite a bit like that of type XII\(^{**}_3\), but will have nine additional components. This will look as displayed in Figure 10.
The singular fiber is the union of 33 components. $\Theta$ is the proper transform of $C_{\infty}$, and again the components $\Theta_{i,j}$ are the exceptional divisors resolving $\Sigma_i$. Of the 20 components resolving the $\mathbb{Z}_{49}$ point, we have not drawn two of them, which correspond to the vertices labeled with a square in the above cone decomposition, as they would have cluttered up the picture too much.
4.7 The other cases

We now consider the last three cases in Table 3. The difference between these and the above cases is that there are now two bad fibers of type. These are the fibers which are the total transforms of the surfaces $C_\infty$ and $C_0$. The fiber $C_0$ is now also singular because of the fact that the first weight is no longer unity. A detailed analysis is not necessary. Consider case 13, i.e., the projective space $\mathbb{P}_{(2,3,5,10,10)}$. The surface $C_\infty$ is $\mathbb{P}_{(2,5,10,10)}[30] \cong \mathbb{P}_{(1,1,1,1)}[3]$, a cubic surface, and the two singular curves on it are elliptic and meet in three points. Without difficulty we recognize the singular fiber of type IV$^*_1$. The surface $C_0$ is $\mathbb{P}_{(3,5,10,10)}[30] \cong \mathbb{P}_{(1,2,2,3)}[6]$, and we recognize the singular fiber of type IV$^{**}_1$. Note that the singular curve $\Sigma_1 - \{z_1 = z_2 = 0\}$ is $\mathbb{Z}_5$, and yields upon resolution four components, one of which is a section of the fibration. The other three split; one of the components belongs to the fiber at $\infty$, while the other belongs to the fiber of type IV$^{**}_1$, and indeed, the first has one, the second has two such components. Similarly, one can check that the exceptional divisors which lie over the singular points split, one component is a section, the others belong to one or the other fiber. The same methods apply to the remaining cases, and the results are listed in Table 3.

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