An Analysis of the Weak Finite Element Method for Convection-Diffusion Equations

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We study the weak finite element method solving convection-diffusion equations. A weak finite element scheme is presented based on a spacial variational form. We established a weak embedding inequality that is very useful in the weak finite element analysis. The optimal order error estimates are derived in the discrete $H^1$-norm, the $L^2$-norm and the $L^\infty$-norm, respectively. In particular, the $H^1$-superconvergence of order $k + 2$ is given under certain condition. Finally, numerical examples are provided to illustrate our theoretical analysis.

Keywords: weak finite element method; optimal error estimate; superconvergence; convection-diffusion equation.

1 Introduction

Recently, the weak Galerkin finite element method attracts much attention in the field of numerical partial differential equations [1, 2, 3, 4, 5, 6, 7, 8, 9]. This method is presented originally by Wang and Ye for solving general elliptic problems in multi-dimensional domain [1]. Since then, some modified weak Galerkin methods have also been studied, for example, see [10, 11, 12, 13]. In general, a weak finite element method can be considered as an extension of the standard finite element method where classical derivatives are replaced in the variational equation by the weakly defined derivatives on discontinuous functions. The main feature of this method is that it allows the use of totally discontinuous finite element function and the trace of finite element function on element edge may
be independent with its value in the interior of element. This feature make this method possess all advantages of the usual discontinuous Galerkin (DG) finite element method\cite{14,15,16}, and it has higher flexibility than the DG method. The readers are referred to articles\cite{2,3,15} for more detailed explanation of this method and its relation with other finite element methods.

In this paper, we study the weak finite element method for convection-diffusion equations:

$$-\text{div}(A\nabla u) + b \cdot \nabla u + cu = f, \text{ in } \Omega,$$

where $\Omega \subset \mathbb{R}^d$, coefficient matrix $A = (a_{ij})_{d \times d}$ and $b$ is a vector function.

We first introduce the weak gradient and discrete weak gradient following the way in\cite{1}. Then, we consider how to discretize problem (1.1) by using weak finite elements. In order to make the weak finite element equation have a positive property, we present a spacial weak form for problem (1.1). This weak form is different from the conventional one and is very suitable for the weak finite element discretization. We establish a discrete embedding inequality on the weak finite element space which provides a useful tool for the weak finite element analysis. In analogy to the usual finite element research, we derive the optimal order error estimates in the discrete $H^1$-norm, the $L_2$-norm and the $L_\infty$-norm, respectively. In particular, for the pure elliptic problems in divergence form $(b = 0, c = 0)$, we obtain an $O(h^{k+2})$-order superconvergence estimate for the gradient approximation of the weak finite element solution, when the $(k, k+1)$-order finite element polynomial pair (interior and edge of element) is used. Both our theoretical analysis and numerical experiment show that this weak finite element method is a high accuracy and efficiency numerical method.

This paper is organized as follows. In Section 2, we establish the weak finite element method for problem (1.1). In Section 3, some approximation functions are given and the stability of the weak finite element solution is analyzed. Section 4 is devoted to the optimal error estimate and superconvergence estimate in various norms. In Section 5, we discuss how to solve the weak finite element discrete system of equations and then provide some numerical examples to illustrate our theoretical analysis.

Throughout this paper, for a real $s$, we adopt the notations $W^{s,p}(D)$ to indicate the
usual Sobolev spaces on domain $D \subset \Omega$ equipped with the norm $\| \cdot \|_{s,p,D}$ and semi-norm $| \cdot |_{s,p,D}$, and if $p = 2$, we set $W^{s,p}(D) = H^s(D)$, $\| \cdot \|_{s,p,D} = \| \cdot \|_{s,D}$. When $D = \Omega$, we omit the index $D$. The notations $(\cdot, \cdot)$ and $\| \cdot \|$ denote the inner product and norm in the space $L_2(\Omega)$, respectively. We will use letter $C$ to represent a generic positive constant, independent of the mesh size $h$.

2 Problem and its weak finite element approximation

Consider the convection-diffusion equations:

\[
\begin{cases}
-\text{div}(A \nabla u) + b \cdot \nabla u + cu = f, & \text{in } \Omega, \\
u = g, & \text{on } \partial \Omega,
\end{cases}
\]

where $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) is a polygonal or polyhedral domain with boundary $\partial \Omega$, coefficient matrix $A = (a_{ij})_{d \times d}$ is uniformly positive definite in $\Omega$, i.e., there exists a positive constant $a_0$ such that

\[
a_0 \xi^T \xi \leq \xi^T A(x) \xi, \quad \forall \xi \in \mathbb{R}^d, \quad x \in \Omega.
\]

We assume that $a_{ij}(x) \in [W^{1,\infty}(\Omega)]^{d \times d}$, $b \in [W^{1,\infty}(\Omega)]^d$ and $c(x) \in L_\infty(\Omega)$. As usual, we further assume that

\[
c(x) - \frac{1}{2} \text{div} b(x) \geq 0, \quad x \in \Omega.
\]

Let $T_h = \bigcup \{K\}$ be a regular triangulation of domain $\Omega$ so that $\overline{\Omega} = \bigcup_{K \in T_h} \{K\}$, where the mesh size $h = \max h_K$, $h_K$ is the diameter of element $K$.

In order to define the weak finite element approximation to problem (2.1), we first need to introduce the concepts of weak derivative and discrete weak derivative, which are originally presented in [1].

Let element $K \in T_h$, denote the interior and boundary of $K$ by $K^0$ and $\partial K$, respectively. A weak function on element $K$ refers to a function $v = \{v^0, v^b\}$ with $v^0 = v|_{K^0} \in L_2(K^0)$, $v^b = v|_{\partial K} \in L_2(\partial K)$. Note that for a weak function $v = \{v^0, v^b\}$, $v^b$ may not be necessarily the trace of $v^0$ on $\partial K$.

**Definition 2.1.** The weak derivative $\partial_{x_i}^w v$ of a weak function $v$ with respect to variable $x_i$ is defined as a linear functional in the dual space $H^{-1}(K)$ whose action on each
\( q \in H^1(K) \) is given by
\[
\int_K \partial_{x_i}^w v q dx = - \int_K v^0 \partial_{x_i} q K dx + \int_{\partial K} v^b q \cos \theta_i ds, \forall q \in H^1(K), 1 \leq i \leq d. \tag{2.4}
\]

where \( n = (\cos \theta_1, \ldots, \cos \theta_d)^T \) is the outward unit normal vector on \( \partial K \).

Obviously, as a bounded linear functional on \( H^1(K) \), \( \partial_{x_i}^w v \) is well defined. Moreover, for \( v \in H^1(K) \), if we consider \( v \) as a weak function with components \( v^0 = v|_{K^0}, v^b = v|_{\partial K} \), then by the Green formula, we have for \( q \in H^1(K) \) that
\[
\int_K \partial_{x_i} v q dx = - \int_K v \partial_{x_i} q dx + \int_{\partial K} v^0 \partial_{x_i} q ds = - \int_K v^0 \partial_{x_i} q dx + \int_{\partial K} v^b q \cos \theta_i ds, \tag{2.5}
\]

which implies that \( \partial_{x_i}^w v = \partial_{x_i} v \) is the usual derivative of function \( v \in H^1(K) \).

According to Definition 2.1, the weak gradient \( \nabla_w v \) of a weak function \( v \) should be such that \( \nabla_w v = (\partial_{x_1}^w v, \ldots, \partial_{x_d}^w v)^T \in [H^{-1}(K)]^d \) satisfies
\[
\int_K \nabla_w v \cdot q dx = - \int_K v^0 \text{div} q dx + \int_{\partial K} v^b q \cdot nds, \forall q \in [H^1(K)]^d. \tag{2.6}
\]

Below we introduce the discrete weak gradient which is actually used in our analysis.

For any non-negative integer \( l \geq 0 \), let \( P_l(D) \) be the space composed of all polynomials on set \( D \) with degree no more than \( l \). Introduce the discrete weak function space on \( K \)
\[
W(K; k, r) = \{ v = (v^0, v^b) : v^0 \in P_k(K^0), v^b \in P_r(\partial K) \}. \tag{2.7}
\]

**Definition 2.2.** For \( v \in W(K, k, r) \), the discrete weak derivative \( \partial_{x_i}^w v \in P_r(K) \) is defined as the unique solution of equation:
\[
\int_K \partial_{x_i}^w v q dx = - \int_K v^0 q dx + \int_{\partial K} v^b q \cos \theta_i ds, \forall q \in P_r(K), 1 \leq i \leq d. \tag{2.8}
\]

According to Definition 2.2, for weak function \( v \in W(K, k, r) \), its discrete weak gradient \( \nabla_w v = (\partial_{x_1}^w v, \ldots, \partial_{x_d}^w v)^T \in [P_r(K)]^d \) is the unique solution of equation:
\[
\int_K \nabla_w v \cdot q dx = - \int_K v^0 \text{div} q dx + \int_{\partial K} v^b q \cdot nds, \forall q \in [P_r(K)]^d. \tag{2.9}
\]

**Remark 2.1.** We here first define the (discrete) weak derivative, and then the (discrete) weak gradient follows naturally. This method of defining (discrete) weak gradient is slightly different from that in [1], in which the (discrete) weak gradient is defined solely.

From (2.6) and (2.9), we have
\[
\int_K (\nabla_w v - \nabla_w v) \cdot q dx = 0, \forall q \in [P_r(K)]^d.
\]
This shows that $\nabla_{w,r} v$ is a discrete approximation of $\nabla v$ in $[P_r(K)]^d$. In particular, if $v \in H^1(K)$, we have from (2.5) and (2.9) that
\[
\int_K (\nabla v - \nabla_{w,r} v) \cdot q \, dx = 0, \quad \forall q \in [P_r(K)]^d.
\]
That is, $\nabla_{w,r} v$ is the $L_2$ projection of $\nabla v$ in $[P_r(K)]^d$ if $v \in H^1(K)$.

We have introduced the weak derivative (gradient) and discrete weak derivative (gradient), but only the discrete weak gradient given in (2.9) will be used throughout this paper. The others also should be useful in the study of numerical partial differential equations.

A important property of $\nabla_{w,r} v$ can be stated as follows, see [1, Lemma 5.1].

**Lemma 2.1.** Let $v = \{v^0, v^b\} \in W(K, k, r)$ be a weak function and $r > k$. Then, $\nabla_{w,r} v = 0$ on $K$ if and only if $v = \text{constant}$, that is, $v^0 = v^b = \text{constant}$ on $K$.

Now, we construct the weak finite element space. Denote by $\mathcal{E}_h^0 = \bigcup \{ e \in \partial K \setminus \partial \Omega : K \in T_h \}$ the union of all boundary faces or edges ($d = 2$) of elements in $T_h$ that are not contained in $\partial \Omega$. Let $K_1$ and $K_2$ be two adjacent elements with the common face $e = \partial K_1 \cap \partial K_2$, and $n_1$ and $n_2$ are the outward unit normal vectors on $\partial K_1$ and $\partial K_2$, respectively. For weak function defined on $T_h$, set $v|_{K_1} = \{v^0_1, v^b_1\}$, $v|_{K_2} = \{v^0_2, v^b_2\}$. We define the jump of weak function $v$ on $e$ by
\[
[v]_e = v^b_1 n_1 + v^b_2 n_2 = (v^b_1 - v^b_2) n_1, \quad e \in \mathcal{E}_h^0.
\]
Then, weak function $v$ is single value on $e$ if and only if $[v]_e = 0$. The weak finite element space is now defined by
\[
S_h = \{ v : v|_K \in W(K, k, r), K \in T_h, [v]_e = 0, e \in \mathcal{E}_h^0 \},
\]
\[
S_h^0 = \{ v : v \in S_h, v|_{\partial \Omega} = 0 \}.
\]

In order to define the weak finite element approximation to problem (2.1), we need to derive a spatial weak form for problem (2.1). From the differential formula
\[
b \cdot \nabla u = \frac{1}{2} b \cdot \nabla u + \frac{1}{2} \text{div}(b u) - \frac{1}{2} \text{div}b \, u,
\]
and the Green formula, we see that a weak form for problem (2.1) is to find $u \in H^1(\Omega)$ $u|_{\partial \Omega} = g$ such that
\[
(A \nabla u, \nabla v) + \frac{1}{2} (b \cdot \nabla u, v) - \frac{1}{2} (u, b \cdot \nabla v) + (c b u, v) = (f, v), \quad \forall v \in H^1_0(\Omega),
\]
(2.11)
where \( c_b = c - \frac{1}{2} \text{div} b \geq 0 \). Denote the discrete \( L_2 \) inner product and norm by
\[
(u, v)_h = \sum_{K \in T_h} (u, v)_K = \sum_{K \in T_h} \int_K u v \, dx, \quad \|u\|_h^2 = (u, u)_h.
\]
Motivated by weak form (2.11), we define the weak finite element approximation of problem (2.1) by finding \( u_h \in S_h, u_h|_{\partial \Omega} = g_h \) such that
\[
a_h(u_h, v) = (f, v^0), \quad \forall v \in S_h^0, \tag{2.12}
\]
where \( g_h \) is a proper approximation of function \( g \) and the bilinear form
\[
a_h(u, v) = (A \nabla_{w,r} u, \nabla_{w,r} v)_h + \frac{1}{2}(b \cdot \nabla_{w,r} u, v^0)_h - \frac{1}{2}(u^0, b \cdot \nabla_{w,r} v)_h + (c_b u^0, v^0). \tag{2.13}
\]
Bilinear form \( a_h(u, v) \) is not based on the conventional one:
\[
a(u, v) = (\nabla u, \nabla v) + (b \cdot \nabla u, v) + (c u, v).
\]
The advantage of our bilinear form is that it always is positive definite on the weak function space \( S_h \times S_h \), and the conventional one is not, since the integration by parts does not hold on weak function space \( S_h \) or \( S_h^0 \).

**Theorem 2.1.** Let \( r > k \). Then, the solution of weak finite element equation (2.12) uniquely exists.

**Proof.** Since equation (2.12) is essentially a linear system of equations, we only need to prove the uniqueness. Let \( f = g_h = 0 \), we need to prove \( u_h = 0 \). Taking \( v = u_h \in S_h^0 \) in (2.12), we obtain
\[
a_0 \|\nabla_{w,r} u_h\|_h^2 \leq a_h(u_h, u_h) = 0.
\]
This implies that \( \nabla_{w,r} u_h = 0 \) on \( T_h \). Thus, from Lemma 2.1, we know that \( u_h \) is a piecewise constant on \( T_h \). Since \([u_h]_e = 0 \) and \( u_h|_{\partial \Omega} = 0 \), so we have \( u_h = 0 \).

## 3 Projection and approximation

In this section, we give some projections and approximation properties which will be used in next section.

In order to balance the approximation accuracy between spaces \( S_h \) and \( P_r(K) \) used to compute \( \nabla_{w,r} u_h \), from now on, we always set the index \( r = k + 1 \) in \((2.7) - (2.8)\). The other choice of weak finite element space can be found in \([1, 3]\).
For $l \geq 0$, let $P^l_h$ is the local $L_2$ projection operator, restricted on each element $K$, $P^l_h : u \in L_2(K) \to P^l_h u \in P_l(K)$ such that

$$
(u - P^l_h u, q)_K = 0, \ \forall q \in P_l(K), \ K \in T_h. \tag{3.1}
$$

By the Bramble-Hilbert lemma, it is easy to prove that (see [16])

$$
\|u - P^l_h u\|_{0,K} \leq Ch^s_K \|u\|_{s,K}, \ 0 \leq s \leq l + 1. \tag{3.2}
$$

We now define a projection operator $Q_h : u \in H^1(\Omega) \to Q_h u \in S_h$ such that

$$
Q_h u|_K = \{Q^0_h u, Q^b_h u\} = \{P^k_h u, P^{k+1}_h u^b\}, \ K \in T_h, \tag{3.3}
$$

where $P^{k+1}_{\partial K}$ is the $L_2$ projection operator in space $P_{k+1}(\partial K)$.

**Lemma 3.1.** Let $u \in H^{1+s}(\Omega), \ s \geq 0$. Then, $Q_h u$ has the following approximation properties

$$
\|u - Q^0_h u\|_{0,K} \leq Ch^s_K \|u\|_{s,K}, \ 0 \leq s \leq k + 1, \ K \in T_h, \tag{3.4}
$$

$$
\|\nabla_{w,r} Q_h u - \nabla u\|_{0,K} \leq Ch^s_K \|u\|_{1+s,K}, \ 0 \leq s \leq k + 2, \ K \in T_h. \tag{3.5}
$$

**Proof.** Since $Q^0_h u = P^k_h u$, then estimate (3.4) follows from (3.2). Furthermore, from (2.9) and the definition of $Q_h u$, we have

$$
\int_K \nabla_{w,r} Q_h u \cdot q dx = - \int_K Q^0_h u \div q dx + \int_{\partial K} Q^b_h u q \cdot nds
$$

$$
= - \int_K \div q dx + \int_{\partial K} u q \cdot nds = \int_K \nabla q dx, \ \forall q \in [P_r(K)]^2.
$$

This implies $\nabla_{w,r} Q_h u = P^r_h \nabla u$ and estimate (3.5) holds, noting that $r = k + 1$. ■

For the error analysis, we still need to introduce a special projection function [17]. For simplifying, we only consider the case of two-dimensional domain ($d = 2$).

Let $e_i$ and $\lambda_i$ ($1 \leq i \leq 3$) are the edge and barycenter coordinate of $K$, respectively.

For function $\phi$, $\text{curl} \phi = (\partial_{x_2} \phi, -\partial_{x_1} \phi)^T$. Let Space $P^0_{k+2}(K) = \{ p \in P_{k+2}(K) : p|_{\partial K} = 0 \} = \lambda_1 \lambda_2 \lambda_3 P_{k-1}(K)$ and $H(\text{div}; \Omega) = \{ u \in [L_2(\Omega)]^2 : \text{div} u \in L_2(\Omega) \}$.

Define the projection operator $\pi_h : H(\text{div}; \Omega) \to H(\text{div}; \Omega)$, restricted on $K \in T_h$, $\pi_h u \in [P_{k+1}(\Omega)]^2$ satisfies

$$
(u - \pi_h u, \nabla q)_K = 0, \ \forall q \in P_k(K), \tag{3.6}
$$

$$
\int_{e_i} (u - \pi_h u) \cdot nq ds = 0, \ \forall q \in P_{k+1}(e_i), \ i = 1, 2, 3, \tag{3.7}
$$

$$
(u - \pi_h u, \text{curl} q)_K = 0, \ \forall q \in P^0_{k+2}(K). \tag{3.8}
$$
Some properties of projection $\pi_h u$ had been discussed in [17], we here give a more detailed analysis for our argument requirement.

**Lemma 3.2.** For $u \in H(\text{div}; \Omega)$, the projection $\pi_h u$ uniquely exists and satisfies

$$
\langle \text{div}(u - \pi_h u), q \rangle_K = 0, \quad \forall q \in P_k(K), \ K \in T_h. \tag{3.9}
$$

Furthermore, if $u \in H^{1+s}(\Omega), \ s \geq 0$, then

$$
\|\pi_h u\|_{0,K} \leq C\|u\|_{1,K}, \ K \in T_h, \tag{3.10}
$$

$$
\|u - \pi_h u\|_{0,K} \leq Ch^s\|u\|_{s,K}, \ 1 \leq s \leq k + 2, \ K \in T_h. \tag{3.11}
$$

**Proof.** We first prove the unique existence of $\pi_h u$. Since the number of dimensions (noting that (3.6) is trivial for $q = \text{constant}$):

$$
\dim(P_k(K)) - 1 + 3 \dim(P_{k+1}(e_i)) + \dim(P^0_{k+2}(K)) = 2 \dim(P_{k+1}(K)),
$$

so the linear system of equations (3.6)~(3.8) is consistent. Thus, we only need to prove the uniqueness. Assume that $u = 0$ in (3.6)~(3.8), we need to prove $\pi_h u = 0$. From (3.6)~(3.7), we have $\pi_h u \cdot n = 0$ on $\partial K$ and

$$
\langle \text{div}\pi_h u, q \rangle_K = -(\pi_h u, \nabla q)_K + \int_{\partial K} \pi_h u \cdot n q ds = 0, \quad \forall q \in P_k(K).
$$

This implies $\text{div}\pi_h u = 0$ on $K$. So there exists a function $\phi \in P_{k+2}(K)$ so that $\text{curl}\phi = \pi_h u$ (see [18]). Since the tangential derivative $\partial_t \phi = \pi_h u \cdot n = 0$ on $\partial K$, so $\phi = \phi_0 = \text{constant}$ on $\partial K$. Let $p = \phi - \phi_0$. Then, $p \in P^0_{k+2}(K)$ and $\text{curl} p = \pi_h u$. Taking $q = p$ in (3.8), we obtain $\|\pi_h u\|_{0,K} = 0$ so that $\pi_h u = 0$. Next, we prove (3.9)~(3.11). Equation (3.9) comes directly from the Green formula and (3.6)~(3.7). From the solution representation of linear system of equations (3.6)~(3.8), it is easy to see that on the reference element $\hat{K}$,

$$
\|\hat{\pi}_h \hat{u}\|_{0,\hat{K}} \leq \hat{C}(\|\hat{u}\|_{0,\hat{K}} + \|\hat{u}\|_{0,\partial \hat{K}}) \leq \hat{C}(\|\hat{u}\|_{0,\hat{K}} + \|\nabla \hat{u}\|_{0,\hat{K}}), \tag{3.12}
$$

where we have used the trace inequality. Then, (3.10) follows from (3.12) and a scale argument between $\hat{K}$ and $K$. From (3.12), we also obtain

$$
\|\pi_h u\|_{0,K} \leq \hat{C}\|\hat{u}\|_{s,\hat{K}}, 1 \leq s \leq k + 2.
$$
Hence, estimate (3.11) can be derived by using the Bramble-Hilbert lemma.

The following discrete embedding inequality is an analogy of the Poincaré inequality in $H^1_0(\Omega)$.

**Lemma 3.3.** Let $\Omega$ be a polygonal or polyhedral domain. Then, for $v \in S^0_h$, there is a positive constant $C_0$ independent of $h$ such that

$$\|v^0\| \leq C_0 \|\nabla_w r u\|_h, \forall v \in S^0_h.$$ (3.13)

**Proof.** For $v \in S^0_h$, we first make a smooth domain $\Omega' \supset \Omega$ (if $\Omega$ is convex, we may set $\Omega' = \Omega$) and extend $v^0$ to domain $\Omega'$ by setting $v^0|_{\Omega' \setminus \Omega} = 0$. Then, there exists a function $w \in H^1_0(\Omega') \cap H^2(\Omega')$ such that

$$-\Delta w = v^0, \text{ in } \Omega', \|w\|_{2,\Omega'} \leq C\|v^0\|.$$ (3.14)

Now we set $w = -\nabla w$, then $w \in [H^1(\Omega)]^d$ satisfies

$$\text{div } w = v^0, \text{ in } \Omega, \|w\|_1 \leq \|w\|_{2,\Omega'} \leq C\|v^0\|.$$ (3.15)

Hence, we have from (3.9), (3.10) and (2.9) that

$$\|v^0\|^2 = (\text{div } w, v^0) = (\text{div } \pi_h w, v^0) = \sum_{K \in T_h} (-\int_K \nabla_{w,r} v \cdot \pi_h w dx + \int_{\partial K} v^b \pi_h w \cdot nds)$$

$$= \sum_{K \in T_h} -\int_K \nabla_{w,r} v \cdot \pi_h w dx \leq \|\nabla_{w,r} v\|_h \|\pi_h w\|_h$$

$$\leq C \|\nabla_{w,r} v\|_h \|w\|_1 \leq C \|\nabla_{w,r} v\|_h \|v^0\|,$$

where we have used the fact that $[v]_e = 0$ and

$$\sum_{K \in T_h} \int_{\partial K} v^b \pi_h w \cdot nds = \sum_{K \in T_h} \int_{\partial K} v^b w \cdot nds = \sum_{e \in \partial_h} \int_{e} [v]_e w \cdot nds = 0.$$ (3.14)

The proof is completed.

A direct application of Lemma 3.3 is the stability estimate of weak finite element solution $u_h$.

**Lemma 3.4.** Let $u_h \in S_h$ be the solution of problem (2.12), $g_h = Q^h b g$ and $u \in H^1(\Omega)$ the solution of problem (2.1) with $f \in L^2(\Omega)$ and $g \in H^\frac{1}{2}(\partial \Omega)$. Then we have

$$\|u^0_h\| + \|\nabla_{w,r} u_h\|_h \leq C(\|f\| + \|g\|_{H^\frac{1}{2}(\partial \Omega)}).$$
Proof. Let \( e_h = u_h - Q_hu \). Then, from (2.12), we see that \( e_h \in S_h^0 \) satisfies

\[
a_h(e_h, v) = (f, v^0) - a_h(Q_hu, v), \quad v \in S_h^0.
\]

Taking \( v = e_h \) and noting that \( Q_h^0 u = P_h^k u \) and \( \nabla_{w, r} Q_h u = P_h^r \nabla u \), we have

\[
\| \nabla_{w, r} e_h \|_h^2 \leq \| f \| \| e_h \|_h + (|A|_\infty + |b|_\infty + |c_b|_\infty)(\| \nabla u \| + \| u \|)(\| \nabla_{w, r} e_h \|_h + \| e_h \|_h).
\]

Using embedding inequality (3.13) and the a priori estimate for elliptic problem (2.1), the proof is completed.

4 Error analysis

In this section, we do the error analysis for the weak finite element method (2.12). We will see that the weak finite element method possesses the same (or better) convergence order as that of the conventional finite element method.

In following error analysis, we always assume that the data \( A, b \) and \( c \) in problem (2.1) is smooth enough for our argument.

Lemma 4.1. Let \( u \in H^2(\Omega) \) be the solution of problem (2.1). Then we have

\[
(\pi_h(A \nabla u), \nabla_{w, r} v)_h + \frac{1}{2}(b \cdot \nabla u, v^0)_h - \frac{1}{2}(\pi_h(b u), \nabla_{w, r} v)_h + (c_b u, v^0) = (f, v^0), \quad v \in S_h^0.
\]

Proof. Let \( w \in [H^1(\Omega)]^d \). From Lemma 3.2 and (3.14), we have

\[
-(\text{div} w, v^0)_h = -(\text{div} \pi_h w, v^0)_h = (\pi_h w, \nabla_{w, r} v)_h, \quad v \in S_h^0.
\]  

(4.1)

Next, from equations (2.1) and (2.10), we obtain

\[
-(\text{div}(A \nabla u), v^0)_h + \frac{1}{2}(b \cdot \nabla u, v^0)_h - \frac{1}{2}((\text{div}(bu), v^0)_h + (c_b u, v^0) = (f, v^0), \quad v \in S_h^0.
\]

Together with (4.1) in which setting \( w = A \nabla u \) and \( w = bu \), respectively, we arrive at the conclusion.

We first give an abstract error estimate for \( u_h - Q_hu \) in the discrete \( H^1 \)-norm.

Theorem 4.1. Let \( u \) and \( u_h \) be the solutions of problems (2.1) and (2.12), respectively, \( u \in H^2(\Omega) \) and \( g_h = Q_h^g g \). Then, we have

\[
a_0 \| \nabla_{w, r} u_h - \nabla_{w, r} Q_h u \|_h \leq \| A \nabla_{w, r} Q_h u - \pi_h(A \nabla u) \|_h + C_0 \| b \cdot (\nabla_{w, r} Q_h u - \nabla u) \|_h + \| b Q_h^0 u - \pi_h(b u) \|_h + C_0 \| c_b(Q_h^0 u - u) \|_h.
\]  

(4.2)
Proof. From Lemma 4.1, we see that $Q_h u$ satisfies the equation

$$a_h(Q_h u, v) = (f, v^0) + (A \nabla_{w,r} Q_h u - \pi_h(A \nabla u), \nabla_{w,r} v)_h$$
$$+ \frac{1}{2}(b \cdot (\nabla_{w,r} Q_h u - \nabla u), v^0)_h - \frac{1}{2}(b Q_h^0 u - \pi_h(b u), \nabla_{w,r} v)_h$$
$$+ (c_b(Q_h^0 u - u), v^0)_h, \; \forall v \in S_h^0. \tag{4.3}$$

Combining this with equation (2.12), we obtain the error equation

$$a_h(Q_h u - u_h, v) = (A \nabla_{w,r} Q_h u - \pi_h(A \nabla u), \nabla_{w,r} v)_h$$
$$+ \frac{1}{2}(b \cdot (\nabla_{w,r} Q_h u - \nabla u), v^0)_h - \frac{1}{2}(b Q_h^0 u - \pi_h(b u), \nabla_{w,r} v)_h$$
$$+ (c_b(Q_h^0 u - u), v^0)_h, \; \forall v \in S_h^0. \tag{4.4}$$

Taking $v = Q_h u - u_h$ in (4.4) and using embedding inequality (3.13), we arrive at the conclusion of Theorem 4.1.

By means of Theorem 4.1, we can derive the following error estimates.

Theorem 4.2. Let $u$ and $u_h$ be the solutions of problems (2.1) and (2.12), respectively, $u \in H^{2+s}(\Omega), s \geq 0$, and $g_h = Q_h^0 g$. Then we have the optimal order error estimates

$$\|u_h^0 - u\| + \|\nabla_{w,r} u_h - \nabla u\|_h \leq C h^{1+s} \|u\|_{2+s}, \; 0 \leq s \leq k. \tag{4.5}$$

In particular, for the pure elliptic problem in divergence form $(b = 0, c = 0)$ and $u \in H^{3+s}(\Omega), s \geq 0$, we have the superconvergence estimate

$$\|\nabla_{w,r} u_h - \nabla u\|_h \leq C h^{2+s} \|u\|_{3+s}, \; 0 \leq s \leq k. \tag{4.6}$$

Proof. Using the approximation properties (3.4)–(3.5) and (3.11), we obtain

$$\|A \nabla_{w,r} Q_h u - \pi_h(A \nabla u)\|_h$$
$$\leq |A|_{\infty}\|\nabla_{w,r} Q_h u - \nabla u\|_h + \|A \nabla u - \pi_h(A \nabla u)\|_h \leq C h^{1+s} \|u\|_{2+s}, \; 0 \leq s \leq k + 1,$$

$$|b \cdot (\nabla_{w,r} Q_h u - \nabla u)\|_h \leq C h^{1+s} \|u\|_{2+s}, \; 0 \leq s \leq k + 1,$$

$$|b Q_h^0 u - \pi_h(b u)|_h + \|c_b(Q_h^0 u - u)\|_h \leq C h^{1+s} \|u\|_{1+s}, \; 0 \leq s \leq k.$$

Substituting these estimates into (4.4), we obtain

$$\|\nabla_{w,r} u_h - \nabla_{w,r} Q_h u\|_h \leq C h^{1+s} \|u\|_{2+s}, \; 0 \leq s \leq k. \tag{4.7}$$
Hence, using the triangle inequality
\[ \| \nabla_{w,r} u_h - \nabla u \|_h \leq \| \nabla_{w,r} u_h - \nabla_{w,r} Q_h u \| + \| \nabla_{w,r} Q_h u - \nabla u \|_h \]
and approximation property (3.5), estimate (4.5) is derived for the discrete \( H^1 \)-norm.

Since
\[ \| u_h^0 - u \| \leq \| u_h^0 - Q_h^0 u \| + \| Q_h^0 u - u \|, \]
then the \( L_2 \)-error estimate follows from the discrete embedding inequality (3.13), estimates (4.7) and (3.4).

Furthermore, if \( b = 0, c = 0 \), from Theorem 4.1, we have
\[ a_0 \| \nabla_{w,r} u_h - \nabla_{w,r} Q_h u \|_h \leq \| A \nabla_{w,r} Q_h u - \pi_h (A \nabla u) \|_h. \]
Then, the superconvergence estimate (4.6) can be derived by using approximation properties (3.5) and (3.11) and the triangle inequality.

Theorem 4.2 shows that the weak finite element method is a high accuracy numerical method, in particular, in the gradient approximation.

From Theory 4.2, we see that, for \( k \)-order finite element, weak finite element method usually has higher accuracy than other finite element methods in the gradient approximation. The reason is that the discrete weak gradient is computed by using \((k+1)\)-order polynomial. This action will add some computation expense, but all additional computations are implemented locally, in the element level, see (2.9) and Section 5.

Below we give a superclose estimate for error \( Q_h^0 u - u_h^0 \). To this end, we assume problem (2.1) has the \( H^2 \)-regularity and consider the auxiliary problem: \( w \in H^1_0(\Omega) \cap H^2(\Omega) \) satisfies
\[ - \text{div}(A \nabla w) - b \cdot \nabla w + c w = Q_h^0 u - u_h, \quad \text{in} \quad \Omega, \quad \| w \|_2 \leq C \| Q_h^0 u - u_h \|. \tag{4.8} \]
From the argument of Lemma 4.2, we know that \( w \) satisfies equation:
\[ (\pi_h (A \nabla w), \nabla_{w,r} v)_h - \frac{1}{2} (b \cdot \nabla w, v^0)_h + \frac{1}{2} (\pi_h (bw), \nabla_{w,r} v)_h + (c_h w, v^0) = (Q_h^0 u - u_h^0, v^0), \quad \forall v \in S_h. \tag{4.9} \]

**Theorem 4.3.** Let \( u \) and \( u_h \) be the solutions of problems (2.1) and (2.12), respectively, \( u \in H^{2+s}(\Omega), s \geq 0, g_h = Q_h^b g \). Then we have the following superclose estimate
\[ \| Q_h^0 u - u_h^0 \| \leq Ch^{2+s} \| u \|_{2+s}, \quad 0 \leq s \leq k, \tag{4.10} \]
and the optimal $L_2$-error estimate

$$
\|u_h^0 - u\| \leq C h^{k+1} \|u\|_{k+1}, \ k \geq 1.
$$

**Proof.** Taking $v = Q_h u - u_h$ in (4.9), we have

$$
\|Q_h^0 u - u_h^0\|^2 = (\nabla_{w,r}(Q_h u - u_h), \pi_h(A \nabla w))_h
- \frac{1}{2}(b \cdot \nabla w, Q_h^0 u - u_h^0)_h + \frac{1}{2}(\pi_h(b w), \nabla_{w,r}(Q_h u - u_h))_h + (c_b(Q_h^0 u - u_h^0), w)
= (\nabla_{w,r}(Q_h u - u_h), \pi_h(A \nabla w) - A \nabla_{w,r} Q_h w)_h
- \frac{1}{2}(b \cdot (\nabla w - \nabla_{w,r} Q_h w), Q_h^0 u - u_h^0)_h
+ \frac{1}{2}(\pi_h(b w) - b Q_h^0 w, \nabla_{w,r}(Q_h u - u_h))_h + (c_b(w - Q_h^0 w), Q_h^0 u - u_h^0)
+ a_h(Q_h u - u_h, Q_h w)
\leq C h\|w\|_2(\|\nabla_{w,r}(Q_h u - u_h)\|_h + \|Q_h^0 u - u_h^0\|) + a_h(Q_h u - u_h, Q_h w)
\leq C h^{2+s}\|u\|_{2+s}\|w\|_2 + a_h(Q_h u - u_h, Q_h w), 0 \leq s \leq k,
$$

where we have used embedding inequality (4.13), estimate (4.7) and the approximation properties of $\pi_h w, \nabla_{w,r} Q_h w$ and $Q_h^0 w$. Below we only need to estimate $a_h(Q_h u - u_h, Q_h w)$. Using error equation (4.4), we have

$$
a_h(Q_h u - u_h, Q_h w) = (A \nabla_{w,r} Q_h u - \pi_h(A \nabla u), \nabla_{w,r} Q_h w)_h
+ \frac{1}{2}(b \cdot (\nabla_{w,r} Q_h u - \nabla u), Q_h^0 w)_h - \frac{1}{2}(b \pi_h Q_h^0 u - \pi_h(b u), \nabla_{w,r} Q_h w)_h
+ (c_b(Q_h^0 u - u), Q_h^0 w)_h, \ \forall v \in S_h^0
= E_1 + E_2 + E_3 + E_4.
$$

Since $Q_h^0 = P_h^k$ and $\nabla_{w,r} Q_h = P_h^r$, then by using Green’s formula and Lemma 3.2, we have

$$
E_1 = (A \nabla_{w,r} Q_h u - \pi_h(A \nabla u), \nabla_{w,r} Q_h w - \nabla w)_h + (A \nabla_{w,r} Q_h u - \pi_h(A \nabla u), \nabla w)_h
\leq C h^{2+s}\|u\|_{2+s}\|w\|_2 + (A \nabla_{w,r} Q_h u - A \nabla u, \nabla w)_h + (A \nabla u - \pi_h(A \nabla u), \nabla w)_h
= C h^{2+s}\|u\|_{2+s}\|w\|_2 + (\nabla_{w,r} Q_h u - \nabla u, A^T \nabla w - P_h^k(A^T \nabla w))_h
- (\text{div}(A \nabla u - \pi_h(A \nabla u)), w - P_h^k w)_h \leq C h^{2+s}\|u\|_{2+s}\|w\|_2, 0 \leq s \leq k.
$$
Substituting estimates $E_1 \sim E_4$ into (4.13) and combining (4.12), we arrive at estimate (4.10), noting that $\|w\|_2 \leq C\|Q_h^0 u - u_h\|$. Estimate (4.11) follows from (4.10) and the triangle inequality.

The difference between estimates (4.5) and (4.11) is that for getting the $O(h^{k+1})$-order error estimate in the $L_2$-norm, the regularity requirement in (4.11) is optimal and lower than that in (4.5).

In order to derive the $L_\infty$-error estimate, we need to impose the quasi-uniform condition on partition $T_h$ so that the finite element inverse inequality holds in $S_h$.

**Theorem 4.4.** Assume that partition $T_h$ is quasi-uniform, and $u$ and $u_h$ are the solution of problems (2.1) and (2.12), respectively, $u \in W^{1+s,\infty}(\Omega) \cap H^{2+s}(\Omega), s \geq 0$, $g_h = Q_h^b g$. Then, we have

$$
\|u - u_h\|_{0,\infty} \leq Ch^{2+s-\frac{d}{2}}(\|u\|_{1+s,\infty} + \|u\|_{2+s}), 0 \leq s \leq k. 
$$

(4.14)

**Proof.** From Theorem 4.3 and the finite element inverse inequality, we have that

$$
\|Q_h^0 u - u_h\|_{0,\infty} \leq Ch^{-\frac{d}{2}}\|Q_h^0 u - u_h\| \leq Ch^{2+s-\frac{d}{2}}\|u\|_{2+s}.
$$

Hence, by using the approximation property of $Q_h^0 u = P_h^k u$, we obtain

$$
\|u - u_h\|_{0,\infty} \leq \|u - Q_h^0 u\|_{0,\infty} + \|Q_h^0 u - u_h\|_{0,\infty}
\leq Ch^{1+s}\|u\|_{1+s,\infty} + Ch^{2+s-\frac{d}{2}}\|u\|_{2+s}.
$$
The proof is completed.

For two-dimensional problem \((d = 2)\), Theorem 4.4 gives the optimal order error estimate in the \(L_\infty\)-norm.

5  Numerical experiment

In this section, we discuss how to solve the weak finite element equation (2.12) and give some numerical examples to illustrate our theoretical analysis.

5.1 Weak finite element linear system of equations

In order to form the discrete linear system of equations from weak finite element equation (2.12), we first introduce the basis functions of space \(S_0^h\). Let \(K\) is a element and \(e\) is an edge of \(K\). Further let \(\{\varphi_{j,K}(x), j = 1, \ldots, N_k\}\) be the basis functions of space \(P_k(K)\), \(\{\varphi_{j,e}(x), j = 1, \ldots, N_e\}\) the basis functions of space \(P_{k+1}(e)\) and \(\{\varphi_{j,b}(x), j = 1, \ldots, M_b\}\) be the basis functions of polynomial set \(\{p \in P_{k+1}(\partial K), K \in T_h : p|_e = \varphi_{j,e}(x), e \in \mathcal{E}_h^0\}\). Set the weak basis functions \(\psi_{j,K} = \{\varphi_{j,K}, 0\}\) and \(\psi_{j,b} = \{0, \varphi_{j,b}\}\). Then, we have \(S_0^h = \text{span}\{\psi_{1,K}, \ldots, \psi_{N_k,K}, \psi_{1,b}, \ldots, \psi_{M_b,b}, K \in T_h\}\). By definition (2.9) of discrete weak gradient, we see that the support set of \(\nabla_{w,r} \psi_{j,K}(x)\) is in \(K\) and the support set of \(\nabla_{w,r} \psi_{j,b}(x)\) is in \(K^b = \{K : \partial K \cap \text{supp}(\psi_{j,b}) \neq \emptyset\}\). Thus, weak finite element equation (2.12) is equivalent to the following linear system of equations: \(u_h \in S_h, u_h|_{\partial \Omega} = g_h\), such that

\[
a_K(u_h, v) = (f, v^0)_K, \quad v \in \{\psi_{j,K}\}, \quad K \in T_h, \quad (5.1)
\]

\[
a_K^h(u_h, v) = 0, \quad v \in \{\psi_{j,b}\}, \quad j = 1, \ldots, M_b, \quad (5.2)
\]

where \(a_D(u, v)\) is the restriction of \(a_h(u, v)\) on set \(D\), i.e., all integrals in \(a_h(u, v)\) are restricted on \(D\). Equations (5.1)–(5.2) form a linear system composed of \(N \times N_k + M_b\) equations with \(N \times N_k + M_b\) unknowns, where \(N\) is the total number of elements in \(T_h\).

To solve equations (5.1)–(5.2), we need to design a solver to compute the discrete weak gradient \(\nabla_{w,r}v\) or \(\nabla_{w,r}u_h\). According to (2.9), for given \(v = \{v^0, v^b\} \in W(K, k, r)\) \((r = k + 1)\), \(\nabla_{w,r}v \in [P_{k+1}(K)]^d\) can be computed by the following formula

\[
M_K V_{w,r} = A_K V^0 + \sum_{e \in \partial K} B_e V^e, \quad (5.3)
\]
where $V_{w,r}$, $V^0$ and $V^e$ are the vectors associated with functions $\nabla_{w,r} v \in [P_{k+1}(K)]^d$, $v^0 \in P_k(K)$ and $v^e = v^b|_e \in P_{k+1}(e)$, respectively. The matrices in (5.3) are as follows

$$M_K = (m_{ij})_{dN_r \times dN_r}, \quad A_K = (a_{ij})_{dN_r \times N_k}, \quad B_e = (b_{ij})_{dN_r \times N_e},$$

$$m_{ij} = (\varphi_j,K, \varphi_i,K)_K, \quad a_{ij} = -\left(\varphi_j,K, \text{div} \varphi_i,K\right)_K, \quad b_{ij} = (\varphi_j,b, \varphi_i,K \cdot n)_e.$$

Now, linear system of equations (5.1)–(5.2) can be solved in the following way. We first use formula (5.3) to derive the linear relation

$$\nabla_{w,r} u_h(K) = L(u_h^0(K), u_h^b(\partial K)).$$

Then, by substituting $\nabla_{w,r} u_h(K)$ into equations (5.1)–(5.2), we can obtain a linear system of equations that only concerns unknowns \{u_h^0(K), u_h^b(\partial K)\}, $K \in T_h$. This linear system can be solved by using a proper linear solver, in which $\nabla_{w,r} v(K)$ is computed by formula (5.3).

### 5.2 Numerical example

Let us consider problem (2.1) with the following data:

$$A = (1 + x_1 x_2)I, \quad b = (1, 2)^T, \quad c = \sin(x_1 x_2), \quad u = \sin(x_1 \pi) \sin(x_2 \pi),$$

and the source term $f = -\text{div}(A \nabla u) + b \cdot \nabla u + cu$, domain $\Omega = (0, 1)^2$.

In the numerical experiments, we first partition $\Omega$ into a regular triangle mesh $T_h$ with mesh size $h = 1/N$. Then, the refined mesh $T_{h/2}$ is obtained by connecting the midpoint of each edge of elements in $T_h$ by straight line. Thus, we obtain a mesh series $T_{h/2^j}, j = 0, 1, \ldots$. We use the polynomial pair $(P_0(K), P_1(\partial K))$ for space $W(K,k,r)$ and set $P_r(K) = P_1(K)$ in the weak finite element discretization (2.12). We examine the computation error in the discrete $H^1$-norm, the $L_2$-norm and the $L_\infty$-norm, respectively. The numerical convergence rate is computed by using the formula $r = \ln(e_h/e_{h_2})/\ln 2$, where $e_h$ is the computation error. Table I gives the numerical results. We see that the convergence rates are consistent with or better than the theoretical prediction. Then, we further examine the superconvergence of the weak finite element solution (see Theorem 4.2). We take the data as in (5.4) with $b = 0$ and $c = 0$. The desired $H^1$-superconvergence is observed from the numerical results, see Table II. In particular, from Table I-II, we see that the numerical convergence rates in $L_2$-norm and $L_\infty$-norm are also superconvergent, although we have no such conclusion in theory.
TABLE I  History of convergence

| mesh $h$ | $\| \nabla_w \nabla r u_h - \nabla u \|_h$ | $\| u - u_0^0 \|_h$ | $\| u_0^0 - u \|_\infty$ |
|---------|-----------------|-----------------|-----------------|
|         | error           | rate            | error           | rate            | error           | rate            |
| $1/4$   | 0.9329          | -               | 0.870e-1        | -               | 1.990e-1        | -               |
| $1/8$   | 4.660e-1        | 1.0013          | 2.427e-2        | 1.8417          | 5.525e-2        | 1.8489          |
| $1/16$  | 2.325e-1        | 1.0033          | 6.246e-3        | 1.9579          | 1.413e-2        | 1.9677          |
| $1/32$  | 1.162e-1        | 1.0011          | 1.573e-3        | 1.9893          | 3.555e-3        | 1.9905          |
| $1/64$  | 5.806e-2        | 1.0003          | 3.940e-4        | 1.9973          | 8.906e-4        | 1.9969          |
| $1/128$ | 2.903e-2        | 1.0001          | 9.868e-4        | 1.9993          | 2.228e-4        | 1.9993          |

TABLE II  History of convergence with $b = 0$ and $c = 0$

| mesh $h$ | $\| \nabla_w \nabla r u_h - \nabla u \|_h$ | $\| u - u_0^0 \|_h$ | $\| u_0^0 - u \|_\infty$ |
|---------|-----------------|-----------------|-----------------|
|         | error           | rate            | error           | rate            | error           | rate            |
| $1/4$   | 1.875e-1        | -               | 3.129e-2        | -               | 0.928e-1        | -               |
| $1/8$   | 4.896e-2        | 1.9370          | 8.538e-3        | 1.8735          | 2.553e-2        | 1.8625          |
| $1/16$  | 1.239e-2        | 1.9821          | 2.184e-3        | 1.9673          | 6.533e-3        | 1.9664          |
| $1/32$  | 3.109e-3        | 1.9948          | 5.490e-4        | 1.9917          | 1.642e-3        | 1.9919          |
| $1/64$  | 7.782e-4        | 1.9984          | 1.375e-4        | 1.9979          | 4.112e-4        | 1.9979          |
| $1/128$ | 1.946e-4        | 1.9995          | 3.437e-5        | 1.9995          | 1.028e-4        | 1.9993          |

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