Affine reductive spaces of small dimension and left A-loops

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Abstract

In this paper we determine the at least 4-dimensional affine reductive homogeneous manifolds for an at most 9-dimensional simple Lie group or an at most 6-dimensional semi-simple Lie group. Those reductive spaces among them which admit a sharply transitive differentiable section yield local almost differentiable left A-loops. Using this we classify all global almost differentiable left A-loops \( L \) having either a 6-dimensional semi-simple Lie group or the group \( SL_3(\mathbb{R}) \) as the group topologically generated by their left translations. Moreover, we determine all at most 5-dimensional left A-loops \( L \) with \( PSU_3(\mathbb{C}, 1) \) as the group topologically generated by their left translations.

1 Introduction

The affine reductive spaces are essential objects of differential geometry (cf. [8], [19], [12]). They are homogeneous manifolds \( G/H \) such that there exists an \( Ad(H) \)-invariant subspace \( \mathfrak{m} \) of the Lie algebra \( \mathfrak{g} \) of \( G \) that is complementary to the subalgebra \( \mathfrak{h} \) in \( \mathfrak{g} \).

The explicite knowledge of affine reductive spaces plays an important role in many investigations (cf. [21], [4], [13]). This paper is an application to differentiable loops since the affine reductive spaces are the key for the classification of almost differentiable left A-loops \( L \); these are loops in which any mapping \( x \mapsto [(ab)^{-1}(a(bx))], a, b \in L \) is an automorphism of \( L \). The relations between them and reductive homogeneous spaces are explicitly discussed in [10], [11] and [18].

Using the fact that the groups topologically generated by the left translations of almost differentiable left A-loops \( L \) are Lie groups (cf. [17]), we treat \( L \) as images of global differentiable sections \( \sigma : G/H \to G \), where \( G \) is a connected
Lie group, $H$ is a closed subgroup containing no non-trivial normal subgroup of $G$ such that the subset $\sigma(G/H)$ is invariant under the conjugation with the elements of $H$. Since the tangent space $T_1(\sigma(G/H))$ is a complementary reductive subspace to the Lie algebra $h$ of $H$ the affine reductive spaces are crucial for the classification of almost differentiable left $A$-loops.

In contrast to the compact connected Lie groups in which for any connected closed subgroup there is a reductive complement (cf. [12], p. 199), for non-compact Lie groups the situation is complicated already if they have small dimension. This is documented by Section 3 and Proposition 20, where we determine all at least 4-dimensional affine reductive homogeneous spaces $(g, h, m)$, such that $g$ is either an at most 9-dimensional simple Lie algebra or it is isomorphic to $\mathfrak{sl}_2(\mathbb{R}) \oplus g_2$, where $g_2$ is a 3-dimensional simple Lie algebra.

The exponential images $\exp m$ of reductive complements $m$ of the triples $(g, h, m)$ obtained in Section 3 and in Proposition 20 yield local left $A$-loops. In Section 4 and Proposition 21 we discuss which of these left $A$-loops can be extended to global ones. They are precisely those exponential images $\exp m$ which form systems of representatives for the cosets $\{xH \mid x \in G\}$ in $G$ and do not contain any element conjugate to an element of $H$.

Since differentiable Bruck loops have realizations on differentiable affine symmetric spaces $G/H$, where $H$ is the set of fixed elements of an involutory automorphism of $G$ and $\sigma(G/H)$ is the exponential image of the $(-1)$-eigenspace of the corresponding automorphism of the Lie algebra $g$ of $G$, the class of differentiable Bruck loops form a proper subclass of almost differentiable left $A$-loops. An important subclass of Bruck loops are the Bruck loops of hyperbolic type which correspond to Lie groups $G$ and involutions $\tau$ fixing elementwise a maximal compact subgroup of $G$ (cf. [7], 64.9, 64.10). Almost differentiable left $A$-loops $L$ having dimension at most 3 and semi-simple Lie groups as the groups topologically generated by their left translations are classified in [18], Section 27 and in [6]. Hence in the following main result of this paper only at most 4-dimensional almost differentiable left $A$-loops occur.

**Theorem** Let $L$ be a connected almost differentiable left $A$-loop such that $\dim L \geq 4$ and the group topologically generated by the left translations of $L$ is semi-simple.

If $\dim G = 6$ then $G$ is isomorphic to $\text{PSL}_2(\mathbb{R}) \times G_2$, where $G_2$ is either $\text{PSL}_2(\mathbb{R})$ or $\text{SO}_3(\mathbb{R})$ and the loop $L$ is either a Scheerer extension of $G_2$ by the hyperbolic plane loop $\mathbb{H}_2$ (cf. [18], Section 22) or the direct product $\mathbb{H}_2 \times \mathbb{H}_2$.

If the group $G$ is simple and $7 \leq \dim G \leq 9$ then $G$ is isomorphic either to $\text{SL}_3(\mathbb{R})$ or to $\text{PSU}_3(\mathbb{C}, 1)$. In the first case $L$ is the 5-dimensional Bruck loop of hyperbolic type having the group $\text{SO}_3(\mathbb{R})$ as the stabilizer of $e \in L$ (cf. [7]).
In the case $G \cong \text{PSU}_3(\mathbb{C},1)$ every loop $L$ with $\dim L < 6$ is the complex hyperbolic plane loop $L_0$ having the group $\text{Spin}_3 \times \text{SO}_2(\mathbb{R})\langle (-1,-1) \rangle$ as the stabilizer of $e \in L_0$ (cf. [5], p. 9).

2 Some basic notions

A binary system $(L, \cdot)$ is called a loop if there exists an element $e \in L$ such that $x = e \cdot x = x \cdot e$ holds for all $x \in L$ and the equations $a \cdot y = b$ and $x \cdot a = b$ have precisely one solution which we denote by $y = a\backslash b$ and $x = b/a$. Let $(L_1, \cdot)$ and $(L_2, *)$ be two loops. The set $L = L_1 \times L_2 = \{(a, b) \mid a \in L_1, b \in L_2\}$ with the componentwise multiplication is again a loop, which is called the direct product of $L_1$ and $L_2$, and the loops $(L_1, \cdot)$, $(L_2, *)$ are subloops of $L$.

A loop is called a left $A$-loop if each mapping $\lambda_{x,y} = \lambda_x^{-1} \lambda_x \lambda_y : L \to L$ is an automorphism of $L$.

Let $G$ be the group generated by the left translations of $L$ and let $H$ be the stabilizer of $e \in L$ in the group $G$. The left translations of $L$ form a subset of $G$ acting on the cosets $\{xH \mid x \in G\}$ such that for any given cosets $aH$ and $bH$ there exists precisely one left translation $\lambda_z$ with $\lambda_z aH = bH$.

Conversely, let $G$ be a group, $H$ be a subgroup containing no normal non-trivial subgroup of $G$ and $\sigma : G/H \to G$ be a section with $\sigma(H) = 1 \in G$ such that the set $\sigma(G/H)$ of representatives for the left cosets $\{xH \mid x \in G\}$ acts sharply transitively on the space $G/H$ of $\{xH \mid x \in G\}$ (cf. [18], p. 18). Such a section we call a sharply transitive section. Then the multiplication defined by $xH \ast yH = \sigma(xH)yH$ on the factor space $G/H$ or by $x \ast y = \sigma(xyH)$ on $\sigma(G/H)$ yields a loop $L(\sigma)$. The group $G$ is isomorphic to the group generated by the left translations of $L(\sigma)$.

If $G$ is a Lie group and $\sigma$ is a differentiable section satisfying the above conditions then the loop $L(\sigma)$ is almost differentiable. This loop is a left $A$-loop if and only if the subset $\sigma(G/H)$ is invariant under the conjugation with the elements of $H$. Moreover the manifold $L$ is parallelizable since the set of the left translations is sharply transitive.

Let $L_1$ be a loop defined on the factor space $G_1/H_1$ with respect to a section $\sigma_1 : G_1/H_1 \to G_1$ the image of which is the set $M_1 \subset G_1$. Let $G_2$ be a group, let $\varphi : H_1 \to G_2$ be a homomorphism and $(H_1, \varphi(H_1)) = \{(x, \varphi(x)) \mid x \in H_1\}$. A loop $L$ is called a Scheerer extension of $G_2$ by $L_1$ if the loop $L$ is defined on the factor space $(G_1 \times G_2)/(H_1, \varphi(H_1))$ with respect to the section $\sigma : (G_1 \times G_2)/(H_1, \varphi(H_1)) \to G_1 \times G_2$ the image of which is the set $M_1 \times G_2$.

If $L$ is a connected almost differentiable left $A$-loop, then the group $G$ topologically generated by the left translations of $L$ within the group of auto-homeomorphisms is a connected Lie group (cf. [17]; [18], Proposition 5.20.
p. 75), and we may describe \( L \) by a differentiable section.

Let \( L \) be a connected almost differentiable left A-loop. Let \( G \) be the Lie group topologically generated by the left translations of \( L \), and let \((g,[\cdot,\cdot])\) be the Lie algebra of \( G \). Denote by \( h \) the Lie algebra of the stabilizer \( H \) of the identity \( e \in L \) in \( G \) and by \( m = T_1\sigma(G/H) \) the tangent space at \( 1 \in G \) of the image of the section \( \sigma : G/H \to G \) corresponding to \( L \). Then \( m \) generates \( g \) and the homogeneous space \( G/H \) is reductive, i.e. we have \( g = m \oplus h \) and \([h,m] \subseteq m\). (cf. [18], Proposition 5.20. p. 75) If \([m,m] \subseteq h\) then the factor space \( G/H \) is an affine symmetric space ([16]) and the corresponding loop \( L \) is called a Bruck loop.

In our computation we often use the following facts about the Lie algebras \( \mathfrak{sl}_2(\mathbb{R}) \) and \( \mathfrak{so}_3(\mathbb{R}) \).

As a real basis of \( \mathfrak{sl}_2(\mathbb{R}) \) we choose the following

\[(*) \quad e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\]

(cf. [9], pp. 19-20).

With respect to this basis the Lie algebra multiplication is given by:

\[[e_1,e_2] = 2e_3, \quad [e_1,e_3] = 2e_2, \quad [e_3,e_2] = 2e_1.\]

1.1 An element \( X = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 \in \mathfrak{sl}_2(\mathbb{R}) \) is elliptic, parabolic or hyperbolic according whether

\[k(X) = k(X,X) = \lambda_1^2 + \lambda_2^2 - \lambda_3^2 \text{ is smaller, equal, or greater 0.}\]

The basis elements \( e_1, e_2 \) are hyperbolic, \( e_3 \) is elliptic and the elements \( e_2+e_3, \quad e_1+e_3 \) are both parabolic. All elliptic elements, all hyperbolic elements as well as all parabolic elements of \( \mathfrak{sl}_2(\mathbb{R}) \) are conjugate in this order to \( e_3 \), to \( e_1 \) respectively to \( e_2+e_3 \) (cf. [9], p. 23). There are 3 conjugacy classes of the one dimensional subgroups of \( \text{PSL}_2(\mathbb{R}) \). As representatives of these classes we can choose \( \exp e_3, \exp e_1, \exp e_2 + e_3 \). There is precisely one conjugacy class \( C \) of the two dimensional subgroups of \( \text{PSL}_2(\mathbb{R}) \), as a representative of \( C \) we choose

\[\mathcal{L}_2 = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a > 0, b \in \mathbb{R} \right\}.\]

The Lie algebra of \( \mathcal{L}_2 \) is generated by the elements \( e_1, e_2 + e_3 \).

According to [9] for the exponential function \( \exp : \mathfrak{sl}_2(\mathbb{R}) \to \text{SL}_2(\mathbb{R}) \) we have

\[\exp X = C(k(X)) I + S(k(X)) X.\]

Here is

\[C(x) = \begin{cases} \cosh \sqrt{x} & \text{for } 0 \leq x, \\ \cos \sqrt{-x} & \text{for } 0 > x, \end{cases}\]

\[S(x) = \begin{cases} \sinh \sqrt{x} & \text{for } 0 \leq x, \\ \sin \sqrt{-x} & \text{for } 0 > x. \end{cases}\]
As a real basis of the Lie algebra $\mathfrak{so}_3(\mathbb{R}) \cong \mathfrak{su}_2(\mathbb{C})$ we can choose the basis elements $\{i e_1, i e_2, e_3\}$, where $i^2 = -1$. Every element of $\mathfrak{so}_3(\mathbb{R})$ is conjugate to $e_3$.

If $X \in \mathfrak{so}_3(\mathbb{R})$ has the decomposition

$$X = \lambda_1 i e_1 + \lambda_2 i e_2 + \lambda_3 e_3$$

then the normalized real Cartan-Killing form $k : \mathfrak{so}_3(\mathbb{R}) \times \mathfrak{so}_3(\mathbb{R}) \rightarrow \mathbb{R}$; $k(X, Y) = \frac{1}{8} \text{trace}(\text{ad}X \text{ ad}Y)$ satisfies

$$k(X) = k(X, X) = -\lambda_1^2 - \lambda_2^2 - \lambda_3^2.$$

For the exponential function $\exp : \mathfrak{su}_2(\mathbb{C}) \rightarrow SU_2(\mathbb{C})$ one has

$$\exp X = C(k(X)) \ I + S(k(X)) \ X,$$

where $C(x) = \cosh(\sqrt{-x} i)$ and $S(x) = \frac{\sinh(\sqrt{-x} i)}{\sqrt{-x} i}$.

**Proposition 1.** There is no connected almost differentiable left $A$-loop $L$ such that the group $G$ topologically generated by its left translations is a compact quasi-simple Lie group $G$ with $\dim G \leq 9$.

**Proof.** If $G$ is a quasi-simple Lie group then it admits a continuous section if and only if $G$ is locally isomorphic to $SO_8(\mathbb{R})$ (cf. [20], pp. 149-150). □

An important tool to exclude certain stabilizers $H$ is the fundamental group $\pi_1$ of a connected topological space. This shows the following lemma which is proved in [5], p. 6.

**Lemma 2.** Denote by $G$ a connected Lie group and by $H$ a connected subgroup of $G$. Let $\sigma : G/H \rightarrow G$ be a global section. Then $\pi_1(K) \cong \pi_1(\sigma(G/H)) \times \pi_1(K_1)$, where $K$ respectively $K_1$ is a maximal compact subgroup of $G$ respectively $H$.

¿From [6] we use Lemma 2, which reads as follows.

**Lemma 3.** Let $L$ be an almost differentiable loop and denote by $\mathfrak{m}$ the tangent space $T_1 \sigma(G/H)$, where $\sigma : G/H \rightarrow G$ is the section corresponding to $L$. Then $\mathfrak{m}$ does not contain any element of $\text{Ad}_g \mathfrak{h}$ for some $g \in G$. Moreover, every element of $G$ can be uniquely written as a product of an element of $\sigma(G/H)$ with an element of $H$.

### 3 Affine reductive spaces of small dimension

In this section we determine all affine reductive homogeneous spaces $(\mathfrak{g}, \mathfrak{h}, \mathfrak{m})$, where $\mathfrak{g}$ is a simple non-compact Lie algebra of dimension at most 9 and $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ such that $\dim \mathfrak{g} - \dim \mathfrak{h} > 3$. 

5
First we deal with the Lie algebra $g = \mathfrak{sl}_2(\mathbb{C})$. A real basis of $g$ is given by \{e_1, e_2, e_3, ie_1, ie_2, ie_3\}, where \{e_1, e_2, e_3\} is the basis of $\mathfrak{sl}_2(\mathbb{R})$ described by (*).

Using the classification of Lie (see Theorem 15 in [15], p. 129) we obtain that every 2-dimensional Lie algebra $h$ of $g$ has (up to conjugation) one of the following shapes:

$$h_1 = \langle e_1, e_2 + e_3 \rangle, \quad h_2 = \langle i(e_2 + e_3), e_2 + e_3 \rangle, \quad h_3 = \langle e_3, ie_3 \rangle,$$

and every 1-dimensional Lie algebra $h$ of $g$ is one of the following:

$$h_4 = \langle e_1 \rangle, \quad h_5 = \langle e_2 + e_3 \rangle, \quad h_6 = \langle e_3 \rangle.$$

**Proposition 4.** The Lie algebra $g = \mathfrak{sl}_2(\mathbb{C})$ is reductive with respect to the following pairs $(h, m)$, where $h$ is an at most 2-dimensional subalgebra of $g$ and $m$ is a complementary subspace to $h$ generating $g$

1) $h_3 = \langle e_3, ie_3 \rangle$, $m = \langle e_1, e_2, ie_1, ie_2 \rangle$,

2) $h_4 = \langle e_1 \rangle$, $m_a = \langle e_2, e_3, ie_1 + ae_1, ie_2, ie_3 \rangle$, where $a \in \mathbb{R}$,

3) $h_6 = \langle e_3 \rangle$, $m_b = \langle e_1, e_2, ie_1, ie_2, ie_3 + be_3 \rangle$, where $b \in \mathbb{R}$.

**Proof.** The basis elements of an arbitrary complement $m_1$ to $h_1$ in $g$ are

$$X_1 = e_2 + a_1e_1 + b_1(e_2 + e_3), \quad X_2 = ie_1 + a_2e_1 + b_2(e_2 + e_3),$$

$$X_3 = ie_2 + a_3e_1 + b_3(e_2 + e_3), \quad X_4 = ie_3 + a_4e_1 + b_4(e_2 + e_3),$$

where $a_j, b_j, j = 1, 2, 3, 4$ are real parameters.

An arbitrary complement $m_2$ to $h_2$ in $g$ has as generators

$$Y_1 = e_1 + a_1(e_2 + e_3) + b_1i(e_2 + e_3), \quad Y_2 = e_2 + a_2(e_2 + e_3) + b_2i(e_2 + e_3),$$

$$Y_3 = ie_1 + a_3(e_2 + e_3) + b_3i(e_2 + e_3), \quad Y_4 = ie_2 + a_4(e_2 + e_3) + b_4i(e_2 + e_3),$$

where $a_j, b_j \in \mathbb{R}, j = 1, 2, 3, 4$.

We can choose as basis elements of an arbitrary complement $m_3$ to $h_3$ the following:

$$Z_1 = e_1 + a_1e_3 + b_1ie_3, \quad Z_2 = e_2 + a_2e_3 + b_2ie_3,$$

$$Z_3 = ie_1 + a_3e_3 + b_3ie_3, \quad Z_4 = ie_2 + a_4e_3 + b_4ie_3,$$

where $a_j, b_j \in \mathbb{R}, j = 1, 2, 3, 4$ are real numbers.

An arbitrary complement $m_4$ to $h_4$ in $g$ has as basis elements

$$W_1 = e_2 + a_1e_1, \quad W_2 = e_3 + a_2e_1, \quad W_3 = ie_1 + a_3e_1,$$

$$W_4 = ie_2 + a_4e_1, \quad W_5 = ie_3 + a_5e_1$$

with the real parameters $a_j, j = 1, 2, 3, 4, 5$.

The generators of an arbitrary complement $m_5$ to $h_5$ in $g$ are

$$V_1 = e_1 + a_1(e_2 + e_3), \quad V_2 = e_2 + a_2(e_2 + e_3), \quad V_3 = ie_1 + a_3(e_2 + e_3),$$

$$V_4 = ie_2 + a_4e_3, \quad V_5 = ie_3 + a_5e_1.$$
\[ V_4 = ie_2 + a_4(e_2 + e_3), \quad V_5 = ie_3 + a_5(e_2 + e_3), \]

where \( a_j, j = 1, 2, 3, 4, 5 \) are real parameters.

An arbitrary complement \( m_6 \) to \( h_6 \) in \( g \) has as generators

\[
U_1 = e_1 + a_1e_3, \quad U_2 = e_2 + a_2e_3, \quad U_3 = ie_1 + a_3e_3,
\]

\[
U_4 = ie_2 + a_4e_3, \quad U_5 = ie_3 + a_5e_3
\]

with \( a_1, a_2, a_3, a_4, a_5 \in \mathbb{R} \).

Using the relation \([h_i, m_i] \subseteq m_i, i = 1, \ldots, 6,\) we obtain the contradictions that \([e_2 + e_3, X_1] = 2e_1 - 2a_1(e_2 + e_3) \in h_1 \) and \([e_2 + e_3, Y_1] = [e_2 + e_3, V_1] = -2(e_2 + e_3) \in h_2 \cap h_5 \) and the assertion follows. \( \Box \)

Now we consider the Lie algebra \( g = sl_3(\mathbb{R}) \). It is isomorphic to the Lie algebra of matrices

\[
(\lambda_1e_1 + \lambda_2e_2 + \lambda_3e_3 + \lambda_4e_4 + \lambda_5e_5 + \lambda_6e_6 + \lambda_7e_7 + \lambda_8e_8) \mapsto
\[
\begin{pmatrix}
-\lambda_5 - \lambda_8 & \lambda_1 & \lambda_2 \\
\lambda_3 & \lambda_5 & \lambda_6 \\
\lambda_4 & \lambda_7 & \lambda_8
\end{pmatrix}; \lambda_i \in \mathbb{R}, i = 1, \ldots, 8.
\]

In this representation the Lie multiplication of \( g \) is given by

\[
[e_1, e_2] = [e_1, e_7] = [e_2, e_6] = [e_3, e_4] = [e_3, e_6] = [e_4, e_7] = [e_5, e_8] = 0,
\]

\[
[e_1, e_6] = [e_2, e_5] = \frac{1}{2}[e_2, e_8] = e_2, \quad [e_1, e_8] = [e_2, e_7] = \frac{1}{2}[e_1, e_5] = e_1,
\]

\[
[e_4, e_6] = [e_3, e_8] = \frac{1}{2}[e_3, e_5] = -e_3, \quad [e_3, e_7] = [e_4, e_5] = \frac{1}{2}[e_4, e_8] = -e_4,
\]

\[
[e_6, e_8] = [e_5, e_6] = [e_3, e_2] = [e_2, e_4] = [e_5, e_7] = [e_7, e_8] = -e_7,
\]

\[
[e_1, e_3] = -e_5, \quad [e_2, e_4] = -e_8, \quad [e_6, e_7] = e_5 - e_8.
\]

Now using the classification of Lie, who has determined all subalgebras of \( sl_3(\mathbb{R}) \) (cf. [13], pp. 288-289 and [14], p. 384) we obtain that every 4-dimensional Lie algebra \( h \) of \( g \) has (up to conjugation) one of the following forms:

\[
h_1 = \langle e_1, e_2, e_6, e_5 + ce_8 \rangle, \quad h_2 = \langle e_3, e_5, e_6, e_8 \rangle, \quad h_3 = \langle e_1, e_2, e_6, e_8 \rangle,
\]

\[
h_4 = \langle e_2, e_5, e_6, e_8 \rangle, \quad h_5 \cong gl_2(\mathbb{R}) = \langle e_5, e_6, e_7, e_8 \rangle, \text{ where } c \in \mathbb{R}.
\]

The 3-dimensional subalgebras \( h \) of \( g \) (up to conjugation) are the following:

\[
h_6 \cong so_3(\mathbb{R}) = \langle e_1 - e_3, e_2 - e_4, e_7 - e_6 \rangle, \quad h_7 \cong sl_2(\mathbb{R}) = \langle e_1 + e_3, e_2 + e_4, e_6 - e_7 \rangle,
\]

\[
h_8 \cong sl_2(\mathbb{R}) = \langle e_5 - e_8, e_6, e_7 \rangle, \quad h_9 = \langle a(e_5 + e_8) + e_6 - e_7, e_1, e_2 \rangle, \quad a \geq 0,
\]

\[
h_{10} = \langle e_5 - e_8, e_2 + e_3, e_6 \rangle, \quad h_{11} = \langle e_3, e_6, e_8 + e_2 \rangle, \quad h_{12} = \langle e_2, e_6, e_5 + e_8 - e_3 \rangle,
\]

\[
h_{13} = \langle e_1, e_2, e_6 \rangle, \quad h_{14} = \langle e_5, e_8, e_6 \rangle, \quad h_{15} = \langle e_2, e_5 + e_8, e_6 \rangle, \quad h_{16} = \langle e_3, e_6, e_8 \rangle,
\]

\[
h_{17} = \langle e_2, e_6, (b - 1)e_5 + be_8 \rangle, \quad b \in \mathbb{R}, \quad h_{18} = \langle e_3, e_6, e_5 + ce_8 \rangle, \quad c \in \mathbb{R}.
\]
The 2-dimensional subalgebras $h$ of $g$ are given (up to conjugation) by

$$h_{19} = \langle e_6, e_2 + e_3 \rangle, \quad h_{20} = \langle e_6, e_2 + e_8 \rangle, \quad h_{21} = \langle e_3, e_6 + e_5 \rangle,$$

$$h_{22} = \langle e_3, e_5 + a e_8 \rangle, \quad a \in \mathbb{R}\{0, 1\}, \quad h_{23} = \langle e_5, e_6 \rangle, \quad h_{24} = \langle e_2, e_6 \rangle,$$

$$h_{25} = \langle e_6, e_3 \rangle, \quad h_{26} = \langle e_5, e_8 \rangle, \quad h_{27} = \langle e_6, e_5 + e_8 \rangle, \quad h_{28} = \langle e_6, e_8 \rangle,$$

$$h_{29} = \langle e_5 - e_8, e_2 + e_3 \rangle, \quad h_{30} = \langle e_5 + e_8, e_6 - e_7 \rangle.$$  

Moreover, every 1-dimensional subalgebra $h$ of $g$ has one of the following shapes:

$$h_{31} = \langle e_5 + a e_8 \rangle, \quad a \in \mathbb{R}\{0\}, \quad h_{32} = \langle e_2 + e_8 \rangle, \quad h_{33} = \langle e_2 + e_3 \rangle,$$

$$h_{34} = \langle e_6 \rangle, \quad h_{35} = \langle e_6 - e_7 + b(e_5 + e_8) \rangle, \quad b \geq 0.$$  

**Proposition 5.** The Lie algebra $g = \mathfrak{sl}_3(\mathbb{R})$ is reductive with respect to a 4-dimensional subalgebra $h$ of $g$ and a complementary subspace $m$ generating $g$ only in the case $h_5 \cong \mathfrak{sl}_2(\mathbb{R})$ and $m_5 = \langle e_1, e_2, e_3, e_4 \rangle$.

**Proof.** The basis elements of an arbitrary complement $m_i$ to the subalgebra $h_i$ are:

For $i = 1$

$$e_3 + a_1 e_1 + a_2 e_2 + a_3(e_5 + c e_8) + a_4 e_6, \quad e_4 + b_1 e_1 + b_2 e_2 + b_3(e_5 + c e_8) + b_4 e_6,$$

$$e_7 + c_1 e_1 + c_2 e_2 + c_3(e_5 + c e_8) + c_4 e_6, \quad e_8 + d_1 e_1 + d_2 e_2 + d_3(e_5 + c e_8) + d_4 e_6,$$

for $i = 2$

$$e_1 + a_1 e_3 + a_2 e_5 + a_3 e_6 + a_4 e_8, \quad e_2 + b_1 e_3 + b_2 e_5 + b_3 e_6 + b_4 e_8,$$

$$e_4 + c_1 e_3 + c_2 e_5 + c_3 e_6 + c_4 e_8, \quad e_7 + d_1 e_3 + d_2 e_5 + d_3 e_6 + d_4 e_8,$$

for $i = 3$

$$e_3 + a_1 e_1 + a_2 e_2 + a_3 e_6 + a_4 e_8, \quad e_4 + b_1 e_1 + b_2 e_2 + b_3 e_6 + b_4 e_8,$$

$$e_5 + c_1 e_1 + c_2 e_2 + c_3 e_6 + c_4 e_8, \quad e_7 + d_1 e_1 + d_2 e_2 + d_3 e_6 + d_4 e_8,$$

for $i = 4$

$$e_1 + a_1 e_2 + a_2 e_5 + a_3 e_6 + a_4 e_8, \quad e_3 + b_1 e_2 + b_2 e_5 + b_3 e_6 + b_4 e_8,$$

$$e_4 + c_1 e_2 + c_2 e_5 + c_3 e_6 + c_4 e_8, \quad e_7 + d_1 e_2 + d_2 e_5 + d_3 e_6 + d_4 e_8,$$

for $i = 5$

$$e_1 + a_1 e_5 + a_2 e_6 + a_3 e_7 + a_4 e_8, \quad e_2 + b_1 e_5 + b_2 e_6 + b_3 e_7 + b_4 e_8,$$

$$e_3 + c_1 e_5 + c_2 e_6 + c_3 e_7 + c_4 e_8, \quad e_4 + d_1 e_5 + d_2 e_6 + d_3 e_7 + d_4 e_8,$$

where $a_j, b_j, c_j, d_j$ are real numbers $j = 1, 2, 3, 4$. The assertion follows now from the relation $[h, m] \subseteq m$.  

8
Proposition 6. The Lie algebra \( g = \mathfrak{sl}_3(\mathbb{R}) \) is reductive with a 3-dimensional subalgebra \( h \) and a 5-dimensional complementary subspace \( m \) generating \( g \) in precisely one of the following cases:

1) \( h_6 \cong \mathfrak{so}_3(\mathbb{R}) \), \( m_6 = \langle e_5, e_8, e_1 + e_3, e_2 + e_4, e_7 + e_6 \rangle \),

2) \( h_7 = \langle e_1 + e_3, e_2 + e_4, e_6 - e_7 \rangle \), \( m_7 = \langle e_5, e_8, e_1 - e_3, e_2 - e_4, e_7 + e_6 \rangle \),

3) \( h_8 = \langle e_5 - e_8, e_6, e_7 \rangle \), \( m_8 = \langle e_1, e_2, e_3, e_4, e_5 + e_8 \rangle \).

Both Lie algebras \( h_i \) and \( h_i \) are isomorphic to \( \mathfrak{sl}_i(\mathbb{R}) \).

Proof. The generators of an arbitrary complement \( m_i \) to \( h_i \) in \( g \) are:

For \( i = 6 \)

\[
\begin{align*}
& e_3 + a_1(e_1 - e_3) + a_2(e_2 - e_4) + a_3(e_7 - e_6), \\
& e_4 + b_1(e_1 - e_3) + b_2(e_2 - e_4) + b_3(e_7 - e_6), \\
& e_5 + c_1(e_1 - e_3) + c_2(e_2 - e_4) + c_3(e_7 - e_6), \\
& e_6 + d_1(e_1 - e_3) + d_2(e_2 - e_4) + d_3(e_7 - e_6), \\
& e_8 + f_1(e_1 - e_3) + f_2(e_2 - e_4) + f_3(e_7 - e_6),
\end{align*}
\]

for \( i = 7 \)

\[
\begin{align*}
& e_3 + a_1(e_1 + e_3) + a_2(e_2 + e_4) + a_3(e_6 - e_7), \\
& e_4 + b_1(e_1 + e_3) + b_2(e_2 + e_4) + b_3(e_6 - e_7), \\
& e_5 + c_1(e_1 + e_3) + c_2(e_2 + e_4) + c_3(e_6 - e_7), \\
& e_6 + d_1(e_1 + e_3) + d_2(e_2 + e_4) + d_3(e_6 - e_7), \\
& e_8 + f_1(e_1 + e_3) + f_2(e_2 + e_4) + f_3(e_6 - e_7),
\end{align*}
\]

for \( i = 8 \)

\[
\begin{align*}
& e_1 + a_1(e_5 - e_6) + a_2e_6 + a_3e_7, \\
& e_2 + b_1(e_5 - e_8) + b_2e_6 + b_3e_7, \\
& e_3 + c_1(e_5 - e_8) + c_2e_6 + c_3e_7, \\
& e_4 + d_1(e_5 - e_8) + d_2e_6 + d_3e_7, \\
& e_5 + f_1(e_5 - e_8) + f_2e_6 + f_3e_7,
\end{align*}
\]

for \( i = 9 \)

\[
\begin{align*}
& e_3 + a_1e_1 + a_2e_2 + a_3(e_6 - e_7 + a(e_5 + e_8)), \\
& e_4 + b_1e_1 + b_2e_2 + b_3(e_6 - e_7 + a(e_5 + e_8)), \\
& e_5 + c_1e_1 + c_2e_2 + c_3(e_6 - e_7 + a(e_5 + e_8)), \\
& e_6 + d_1e_1 + d_2e_2 + d_3(e_6 - e_7 + a(e_5 + e_8)), \\
& e_8 + f_1e_1 + f_2e_2 + f_3(e_6 - e_7 + a(e_5 + e_8)),
\end{align*}
\]

for \( i = 10 \)

\[
\begin{align*}
& e_1 + a_1(e_2 + e_3) + a_2(e_5 - e_8) + a_3e_6, \\
& e_2 + b_1(e_2 + e_3) + b_2(e_5 - e_8) + b_3e_6.
\end{align*}
\]
\[ e_4 + c_1(e_2 + e_3) + c_2(e_5 - e_8) + c_3e_6, \quad e_5 + d_1(e_2 + e_3) + d_2(e_5 - e_8) + d_3e_6, \]
\[ e_7 + f_1(e_2 + e_3) + f_2(e_5 - e_8) + f_3e_6, \]

for \( i = 11 \)
\[ e_1 + a_1(e_2 + e_8) + a_2e_3 + a_3e_6, \quad e_2 + b_1(e_2 + e_8) + b_2e_3 + b_3e_6, \]
\[ e_4 + c_1(e_2 + e_8) + c_2e_3 + c_3e_6, \quad e_5 + d_1(e_2 + e_8) + d_2e_3 + d_3e_6, \]
\[ e_7 + f_1(e_2 + e_8) + f_2e_3 + f_3e_6, \]

for \( i = 12 \)
\[ e_1 + a_1e_2 + a_2e_6 + a_3(e_5 + e_8 - e_3), \quad e_3 + b_1e_2 + b_2e_6 + b_3_e_5 + e_8 - e_3, \]
\[ e_4 + c_1e_2 + c_2e_6 + c_3(e_5 + e_8 - e_3), \quad e_7 + d_1e_2 + d_2e_6 + d_3(e_5 + e_8 - e_3), \]
\[ e_8 + f_1e_2 + f_2e_6 + f_3(e_5 + e_8 - e_3), \]

for \( i = 13 \)
\[ e_3 + a_1e_1 + a_2e_2 + a_3e_6, \quad e_4 + b_1e_1 + b_2e_2 + b_3e_6, \quad e_5 + c_1e_1 + c_2e_2 + c_3e_6, \]
\[ e_7 + d_1e_1 + d_2e_2 + d_3e_6, \quad e_8 + f_1e_1 + f_2e_2 + f_3e_6, \]

for \( i = 14 \)
\[ e_1 + a_1e_5 + a_2e_6 + a_3e_8, \quad e_2 + b_1e_5 + b_2e_6 + b_3e_8, \quad e_3 + c_1e_5 + c_2e_6 + c_3e_8, \]
\[ e_4 + d_1e_5 + d_2e_6 + d_3e_8, \quad e_7 + f_1e_5 + f_2e_6 + f_3e_8, \]

for \( i = 15 \)
\[ e_1 + a_1e_2 + a_2(e_5 + e_8) + a_3e_6, \quad e_3 + b_1e_2 + b_2(e_5 + e_8) + b_3e_6, \]
\[ e_4 + c_1e_2 + c_2(e_5 + e_8) + c_3e_6, \quad e_5 + d_1e_2 + d_2(e_5 + e_8) + d_3e_6, \]
\[ e_7 + f_1e_2 + f_2(e_5 + e_8) + f_3e_6, \]

for \( i = 16 \)
\[ e_1 + a_1e_3 + a_2e_6 + a_3e_8, \quad e_2 + b_1e_3 + b_2e_6 + b_3e_8, \quad e_4 + c_1e_3 + c_2e_6 + c_3e_8, \]
\[ e_5 + d_1e_3 + d_2e_6 + d_3e_8, \quad e_7 + f_1e_3 + f_2e_6 + f_3e_8, \]

for \( i = 17 \) and \( b \neq 0 \)
\[ e_1 + a_1e_2 + a_2e_6 + a_3((b - 1)e_5 + be_8), \quad e_3 + b_1e_2 + b_2e_6 + b_3((b - 1)e_5 + be_8), \]
\[ e_4 + c_1e_2 + c_2e_6 + c_3((b - 1)e_5 + be_8), \quad e_5 + d_1e_2 + d_2e_6 + d_3((b - 1)e_5 + be_8), \]
\[ e_7 + f_1e_2 + f_2e_6 + f_3((b - 1)e_5 + be_8), \]

for \( i = 17 \) and \( b = 0 \)
\[ e_1 + a_1e_2 + a_2e_6 - a_3e_5, \quad e_3 + b_1e_2 + b_2e_6 - b_3e_5, \quad e_4 + c_1e_2 + c_2e_6 - c_3e_5, \]
\[ e_7 + d_1e_2 + d_2e_6 - d_3e_5, \quad e_8 + f_1e_2 + f_2e_6 - f_3e_5, \]

for \( i = 18 \)
\text{in the case}
\begin{align*}
e_1 + a_1 e_3 &+ a_2 (e_5 + ce_8) + a_3 e_6, \quad e_2 + b_1 e_3 + b_2 (e_5 + ce_8) + b_3 e_6, \\
e_4 + c_1 e_3 &+ c_2 (e_5 + ce_8) + c_3 e_6, \quad e_7 + d_1 e_3 + d_2 (e_5 + ce_8) + d_3 e_6, \\
e_8 + f_1 e_3 &+ f_2 (e_5 + ce_8) + f_3 e_6,
\end{align*}
where $a_j, b_j, c_j, d_j, f_j \in \mathbb{R}, \ j = 1, 2, 3$. Using the relation $[h_i, m_i] \subseteq m_i, i = 6, \cdots, 18$, we obtain the assertion. \hfill \Box

\textbf{Proposition 7.} The Lie algebra $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{R})$ is reductive with respect to a pair $(\mathfrak{h}, \mathfrak{m})$, where $\mathfrak{h}$ is a 2-dimensional subalgebra of $\mathfrak{g}$ and $\mathfrak{m}$ is a complementary subspace to $\mathfrak{h}$ generating $\mathfrak{g}$ in exactly one of the following cases:

1) $\mathfrak{h}_{26} = \langle e_5, e_8 \rangle$ and $\mathfrak{m}_{26} = \langle e_1, e_2, e_3, e_4, e_6, e_7 \rangle$.

2) $\mathfrak{h}_{30} = \langle e_5 + e_8, e_6 - e_7 \rangle$ and $\mathfrak{m}_{30} = \langle e_1, e_2, e_3, e_4, e_5 - e_8, e_6 + e_7 \rangle$.

\textbf{Proof.} An arbitrary complement $\mathfrak{m}_i$ to the subalgebra $\mathfrak{h}_i, i = 19, \cdots, 30$, in $\mathfrak{g}$ has as generators in the case $i = 19$
\begin{align*}
e_1 + b_1 e_6 &+ c_1 (e_2 + e_3), \quad e_2 + b_2 e_6 + c_2 (e_2 + e_3), \quad e_4 + b_3 e_6 + c_3 (e_2 + e_3) \\
e_5 + b_4 e_6 &+ c_4 (e_2 + e_3), \quad e_7 + b_5 e_6 + c_5 (e_2 + e_3), \quad e_8 + b_6 e_6 + c_6 (e_2 + e_3),
\end{align*}
in the case $i = 20$
\begin{align*}
e_1 + b_1 e_6 &+ c_1 (e_2 + e_8), \quad e_2 + b_2 e_6 + c_2 (e_2 + e_8), \quad e_3 + b_3 e_6 + c_3 (e_2 + e_8), \\
e_4 + b_4 e_6 &+ c_4 (e_2 + e_8), \quad e_5 + b_5 e_6 + c_5 (e_2 + e_8), \quad e_7 + b_6 e_6 + c_6 (e_2 + e_8),
\end{align*}
in the case $i = 21$
\begin{align*}
e_1 + b_1 e_3 &+ c_1 (e_6 + e_5), \quad e_2 + b_2 e_3 + c_2 (e_6 + e_5), \quad e_4 + b_3 e_3 + c_3 (e_6 + e_5), \\
e_5 + b_4 e_3 &+ c_4 (e_6 + e_5), \quad e_7 + b_5 e_3 + c_5 (e_6 + e_5), \quad e_8 + b_6 e_3 + c_6 (e_6 + e_5),
\end{align*}
in the case $i = 22$
\begin{align*}
e_1 + b_1 e_3 &+ c_1 (e_5 + ae_8), \quad e_2 + b_2 e_3 + c_2 (e_5 + ae_8), \quad e_4 + b_3 e_3 + c_3 (e_5 + ae_8), \\
e_6 + b_4 e_3 &+ c_4 (e_5 + ae_8), \quad e_7 + b_5 e_3 + c_5 (e_5 + ae_8), \quad e_8 + b_6 e_3 + c_6 (e_5 + ae_8),
\end{align*}
in the case $i = 23$
\begin{align*}
e_1 + b_1 e_5 &+ c_1 e_6, \quad e_2 + b_2 e_5 + c_2 e_6, \quad e_3 + b_3 e_5 + c_3 e_6, \\
e_4 + b_4 e_5 &+ c_4 e_6, \quad e_7 + b_5 e_5 + c_5 e_6, \quad e_8 + b_6 e_5 + c_6 e_6,
\end{align*}
in the case $i = 24$
\begin{align*}
e_1 + b_1 e_2 &+ c_1 e_6, \quad e_3 + b_2 e_2 + c_2 e_6, \quad e_4 + b_3 e_2 + c_3 e_6, \\
e_5 + b_4 e_2 &+ c_4 e_6, \quad e_7 + b_5 e_2 + c_5 e_6, \quad e_8 + b_6 e_2 + c_6 e_6,
\end{align*}
in the case $i = 25$
\begin{align*}
e_1 + b_1 e_3 &+ c_1 e_6, \quad e_2 + b_2 e_3 + c_2 e_6, \quad e_4 + b_3 e_3 + c_3 e_6, \\
e_5 + b_4 e_3 &+ c_4 e_6, \quad e_7 + b_5 e_3 + c_5 e_6, \quad e_8 + b_6 e_3 + c_6 e_6.
in the case $i = 26$
\[
e_1 + b_1e_5 + c_1e_8, \ e_2 + b_2e_5 + c_2e_8, \ e_3 + b_3e_5 + c_3e_8, \ e_4 + b_4e_5 + c_4e_8, \ e_6 + b_5e_5 + c_5e_8, \ e_7 + b_6e_5 + c_6e_8,
\]
in the case $i = 27$
\[
e_1 + b_1e_6 + c_1(e_5 + e_8), \ e_2 + b_2e_6 + c_2(e_5 + e_8) \ e_3 + b_3e_6 + c_3(e_5 + e_8), \ e_4 + b_4e_6 + c_4(e_5 + e_8), \ e_5 + b_5e_6 + c_5(e_5 + e_8), \ e_7 + b_6e_6 + c_6(e_5 + e_8),
\]
in the case $i = 28$
\[
e_1 + b_1e_6 + c_1e_8, \ e_2 + b_2e_6 + c_2e_8 \ e_3 + b_3e_6 + c_3e_8, \ e_4 + b_4e_6 + c_4e_8, \ e_5 + b_5e_6 + c_5e_8, \ e_7 + b_6e_6 + c_6e_8,
\]
in the case $i = 29$
\[
e_1 + b_1(e_2 + e_3) + c_1(e_5 - e_8), \ e_2 + b_2(e_2 + e_3) + c_2(e_5 - e_8), \ e_4 + b_3(e_2 + e_3) + c_3(e_5 - e_8), \ e_5 + b_4(e_2 + e_3) + c_4(e_5 - e_8), \ e_6 + b_5(e_2 + e_3) + c_5(e_5 - e_8), \ e_7 + b_6(e_2 + e_3) + c_6(e_5 - e_8),
\]
in the case $i = 30$
\[
e_1 + b_1(e_5 + e_8) + c_1(e_6 - e_7), \ e_2 + b_2(e_5 + e_8) + c_2(e_6 - e_7), \ e_3 + b_3(e_5 + e_8) + c_3(e_6 - e_7), \ e_4 + b_4(e_5 + e_8) + c_4(e_6 - e_7), \ e_5 + b_5(e_5 + e_8) + c_5(e_6 - e_7), \ e_6 + b_6(e_5 + e_8) + c_6(e_6 - e_7),
\]
where $b_j, c_j \in \mathbb{R}, \ j = 1, \ldots, 6$. The relation $[h_i, m_i] \subseteq m_i, \ i = 19, \ldots, 30,$ yields the assertion. \hfill \Box

**Proposition 8.** The Lie algebra $g = sl_3(\mathbb{R})$ is reductive with a 1-dimensional subalgebra $h$ and a 7-dimensional complementary subspace $m$ generating $g$ in precisely one of the following cases:

1) $h_{31.1} = (e_5 + ae_8), \ a \in \mathbb{R} \setminus \{0, 1, -\frac{1}{2}, -2\}$ and

$\mathfrak{m}_b = \langle e_1, e_2, e_3, e_4, e_6, e_7, e_8 + b(e_5 + ae_8) \rangle, \ b \in \mathbb{R},$

2) $h_{31.2} = (e_5 - 2e_8)$ and

$\mathfrak{m}_{b,c,d} = \langle e_6, e_7, e_1 + b(e_5 - 2e_8), e_3 + c(e_5 - 2e_8), e_2, e_4, e_8 + d(e_5 - 2e_8) \rangle, \ b, c, d \in \mathbb{R},$

3) $h_{31.3} = (e_5 - \frac{1}{2}e_8)$ and

$\mathfrak{m}_{b,c,d} = \langle e_6, e_7, e_1 + b(e_5 - \frac{1}{2}e_8), e_3 + c(e_5 - \frac{1}{2}e_8), e_2, e_4, e_8 + d(e_5 - \frac{1}{2}e_8) \rangle, \ b, c, d \in \mathbb{R},$

4) $h_{31.4} = (e_5 + e_8)$ and

$\mathfrak{m}_{b,c,d} = \langle e_1, e_2, e_3, e_4, e_6 + b(e_5 + e_8), e_7 + c(e_5 + e_8), e_8 + d(e_5 + e_8) \rangle, \ b, c, d \in \mathbb{R},$

5) $h_{32} = (e_2 + e_8)$ and $\mathfrak{m}_d = \langle e_1, e_2, e_3, -e_8 + 2e_4, e_6, e_7, e_5 + de_8 \rangle, \ d \in \mathbb{R},$

6) $h_{35} = (e_6 - e_7 + b(e_5 + e_8)), \ b \geq 0$ and

$\mathfrak{m}_c = \langle e_1, e_2, e_3, e_4, e_6 + e_7, e_5 - e_8, e_8 - 2ce_7 + 2cbe_8, \ c \in \mathbb{R}.$
Proof. An arbitrary complement \( \mathfrak{m}_i \) to the subalgebra \( \mathfrak{h}_i \), \( i = 31, \cdots, 35 \), in \( \mathfrak{g} \) has as generators in the case \( i = 31 \)
\[
e_1 + a_1(e_5 + ae_8), \ e_2 + a_2(e_5 + ae_8), \ e_3 + a_3(e_5 + ae_8), \ e_4 + a_4(e_5 + ae_8), \ e_6 + a_5(e_5 + ae_8), \ e_7 + a_6(e_5 + ae_8), \ e_8 + a_7(e_5 + ae_8),
\]
in the case \( i = 32 \)
\[
e_1 + a_1(e_2 + e_8), \ e_3 + a_2(e_2 + e_8), \ e_4 + a_3(e_2 + e_8), \ e_5 + a_4(e_2 + e_8), \ e_6 + a_5(e_2 + e_8), \ e_7 + a_6(e_2 + e_8), \ e_8 + a_7(e_2 + e_8),
\]
in the case \( i = 33 \)
\[
e_1 + a_1(e_2 + e_3), \ e_3 + a_2(e_2 + e_3), \ e_4 + a_3(e_2 + e_3), \ e_5 + a_4(e_2 + e_3), \ e_6 + a_5(e_2 + e_3), \ e_7 + a_6(e_2 + e_3), \ e_8 + a_7(e_2 + e_3),
\]
in the case \( i = 34 \)
\[
e_1 + a_1e_6, \ e_2 + a_2e_6, \ e_3 + a_3e_6, \ e_4 + a_4e_6, \ e_5 + a_5e_6, \ e_7 + a_6e_6, \ e_8 + a_7e_6,
\]
in the case \( i = 35 \)
\[
e_1 + a_1(e_6 - e_7 + b(e_5 + e_8)), \ e_2 + a_2(e_6 - e_7 + b(e_5 + e_8)), \ e_3 + a_3(e_6 - e_7 + b(e_5 + e_8)), \ e_4 + a_4(e_6 - e_7 + b(e_5 + e_8)), \ e_5 + a_5(e_6 - e_7 + b(e_5 + e_8)), \ e_7 + a_6(e_6 - e_7 + b(e_5 + e_8)), \ e_8 + a_7(e_6 - e_7 + b(e_5 + e_8)),
\]
where \( a_j \in \mathbb{R}, \ j = 1, \cdots, 7 \). Using the relation \([\mathfrak{h}_i, \mathfrak{m}_j] \subseteq \mathfrak{m}_i, \ i = 31, \cdots, 35 \), we obtain the assertion. \( \square \)

Now we deal with the Lie algebra \( \mathfrak{su}_3(\mathbb{C}, 1) \). It can be treated as the Lie algebra of matrices
\[
(\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4 + \lambda_5 e_5 + \lambda_6 e_6 + \lambda_7 e_7 + \lambda_8 e_8) \mapsto
\]
\[
\begin{pmatrix}
-\lambda_1 i & -\lambda_2 - \lambda_3 i & \lambda_4 + \lambda_5 i \\
\lambda_2 - \lambda_3 i & \lambda_1 i + \lambda_6 i & \lambda_7 + \lambda_8 i \\
\lambda_4 - \lambda_5 i & \lambda_7 - \lambda_8 i & -\lambda_6 i
\end{pmatrix}; \lambda_i \in \mathbb{R}, i = 1, \cdots, 8.
\]
Then the multiplication of \( \mathfrak{g} \) is given by the following:
\[
[e_1, e_6] = 0, \ [e_3, e_2] = 2e_1, \ [e_4, e_5] = 2(e_1 - e_6), \ [e_8, e_7] = 2e_6,
\]
\[
[e_6, e_3] = [e_7, e_4] = [e_8, e_5] = \frac{1}{2} [e_1, e_3] = e_2,
\]
\[
[e_2, e_6] = [e_4, e_8] = [e_7, e_5] = \frac{1}{2} [e_2, e_1] = e_3,
\]

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The Lie algebras up to conjugacy, one of the following:

$[e_7, e_2] = [e_3, e_8] = [e_5, e_6] = [e_1, e_3] = e_4,$

$[e_8, e_2] = [e_7, e_3] = [e_6, e_4] = [e_4, e_1] = e_5,$

$[e_2, e_4] = [e_3, e_5] = [e_8, e_1] = \frac{1}{2}[e_8, e_6] = e_7,$

$[e_2, e_5] = [e_4, e_3] = [e_1, e_7] = \frac{1}{2}[e_6, e_7] = e_8.$

The normalized Cartan-Killing form $k : \mathfrak{su}_3(\mathbb{C}, 1) \times \mathfrak{su}_3(\mathbb{C}, 1) \to \mathbb{R}$ is the map $(X, Y) \mapsto \frac{1}{2}\text{trace}(\text{ad}X \text{ad}Y) = \frac{1}{2}\text{trace}(XY)$. An element $X = \lambda_ie_i \in \mathfrak{su}_3(\mathbb{C}, 1), \lambda_i \in \mathbb{R}, i = 1, \cdots, 8,$ is elliptic, parabolic or loxodromic according whether

$$k(X) = k(X, X) = -\lambda_1^2 - \lambda_2^2 - \lambda_3^2 - \lambda_4^2 + \lambda_5^2 + \lambda_6^2 + 2 \lambda_1 \lambda_6$$

is smaller, equal or greater 0.

Let $H$ be a connected closed subgroup of the group $PSU_3(\mathbb{C}, 1)$. Then according to [1], Satz 1, p. 251 and [2], Section 5, p. 276, the group $H$ is, up to conjugacy, one of the following:

(1) $H$ is a subgroup of $Spin_3 \times SO_2(\mathbb{R})/\langle(-1, -1)\rangle$,

(2) $H$ is a subgroup of the 5-dimensional solvable group $NG_{1,1}$ in [1], p. 253,

(3) $H$ is the group $SL_2(\mathbb{R}) \times SO_2(\mathbb{R})/\langle(-1, -1)\rangle$,

(4) $H$ is the group $SL_2(\mathbb{R}) \times \{1\}/\langle(-1, 1)\rangle \cong PSL_2(\mathbb{R})$,

(5) $H$ is the connected component of the group $SO_3(\mathbb{R}, 1) \cong PSL_2(\mathbb{R}).$

The Lie algebras $\mathfrak{h}_i$, $i = 1, \cdots, 5$, of $H$ in the cases (1) till (5) are given in this order by

$$\overline{\mathfrak{h}}_1 = \langle e_1, e_2, e_3, e_6 \rangle, \quad \overline{\mathfrak{h}}_2 = \langle e_1 - \frac{1}{2}e_6, e_8, e_4 - e_3, e_5 + e_2, e_6 + e_7 \rangle,$$

$$\overline{\mathfrak{h}}_3 = \langle e_1, e_6, e_7, e_8 \rangle, \quad \overline{\mathfrak{h}}_4 = \langle e_6, e_7, e_8 \rangle, \quad \overline{\mathfrak{h}}_5 = \langle e_2, e_4, e_7 \rangle.$$

After a straightforward calculation in $\overline{\mathfrak{h}}_2$ we obtain that the conjugacy classes of the 4-dimensional subalgebras of $\mathfrak{su}_3(\mathbb{C}, 1)$ are the following:

$$\mathfrak{h}_1 = \langle e_1, e_2, e_3, e_6 \rangle, \quad \mathfrak{h}_2 = \langle e_4 - e_3, e_2 + e_5, e_6 + e_7, e_8 \rangle,$$

$$\mathfrak{h}_3 = \langle e_1 - \frac{1}{2}e_6 + ae_8, e_4 - e_3, e_2 + e_5, e_6 + e_7 \rangle, \quad \mathfrak{h}_4 = \langle e_1, e_6, e_7, e_8 \rangle,$$

where $a \in \mathbb{R}.$

Computations in $\overline{\mathfrak{h}}_1$ and $\overline{\mathfrak{h}}_2$ yield that the 3-dimensional subalgebras of $\mathfrak{su}_3(\mathbb{C}, 1)$ have one of the following shapes:

$$\mathfrak{h}_5 = \langle e_1, e_2, e_3 \rangle, \quad \mathfrak{h}_6 = \langle e_2, e_4, e_7 \rangle, \quad \mathfrak{h}_7 = \langle e_6, e_7, e_8 \rangle,$$

$$\mathfrak{h}_8 = \langle e_5 + e_2, e_6 + e_7, e_8 \rangle, \quad \mathfrak{h}_9 = \langle e_4 - e_3 + be_8, e_5 + e_2, e_6 + e_7 \rangle,$$

$$\mathfrak{h}_{10} = \langle e_4 - e_3 + b(e_5 + e_2), e_6 + e_7, e_8 + c(e_5 + e_2) \rangle,$$

$$\mathfrak{h}_{11} = \langle e_1 - \frac{1}{2}e_6 + \frac{1}{2}c(e_4 - e_3) - \frac{1}{2}b(e_5 + e_2), e_8 + b(e_4 - e_3) + c(e_5 + e_2), e_6 + e_7 \rangle,$$

where $b, c \in \mathbb{R}.$
Similarly we obtain that every 2-dimensional subalgebra of $\mathfrak{su}_3(\mathbb{C}, 1)$ has one of the following forms:

\[ h_{12} = \langle e_1, e_6 \rangle, \quad h_{13} = \langle e_4 - e_3, e_6 + e_7 \rangle, \]
\[ h_{14} = \langle e_5 + e_2 + b(e_4 - e_3), e_6 + e_7 \rangle, \quad h_{15} = \langle e_4 - e_3, e_8 + b(e_6 + e_7) \rangle, \]
\[ h_{16} = \langle e_5 + e_2 + b(e_4 - e_3), e_8 + c(e_6 + e_7) \rangle, \]
\[ h_{17} = \langle e_6 + e_7, e_8 + b(e_4 - e_3) + c(e_5 + e_2) \rangle, \]
\[ h_{18} = \langle e_6 + e_7 + a(e_4 - e_3) + b(e_5 + e_2), e_8 + c(e_4 - e_3) + d(e_5 + e_2) \rangle, \]
\[ h_{19} = \langle e_1 - \frac{1}{2}e_6 + ae_8 + b(e_4 - e_3) + c(e_5 + e_2), e_6 + e_7 \rangle, \]
\[ h_{20} = \langle e_1 - \frac{1}{2}e_6 + \frac{3}{2}a(e_4 - e_3) + \frac{3}{2}b(e_5 + e_2) - \frac{3(a^2 + b^2)}{2}(e_6 + e_7), e_8 + b(e_4 - e_3) + a(e_5 + e_2) + c(e_6 + e_7) \rangle, \]

where $a, b, c, d \in \mathbb{R}$ and in the Lie algebra $h_{18}$ one has $bc - ad = \frac{1}{2}$.

Moreover, every 1-dimensional subalgebra $h$ of $g$ is given by

\[ h_{21} = \langle e_1 + ae_6 \rangle, \quad h_{22} = \langle e_6 \rangle, \quad h_{23} = \langle e_8 \rangle, \]
\[ h_{24} = \langle e_6 + e_7 + ce_8 \rangle, \quad h_{25} = \langle e_5 + e_2 + b(e_6 + e_7) + ce_8 \rangle, \]
\[ h_{26} = \langle e_4 - e_3 + a(e_5 + e_2) + b(e_6 + e_7) + ce_8 \rangle, \]
\[ h_{27} = \langle e_1 - \frac{1}{2}e_6 + d(e_4 - e_3) + a(e_5 + e_2) + b(e_6 + e_7) + ce_8 \rangle, \]

where $a, b, c, d$ are real numbers.

**Proposition 9.** The Lie algebra $\mathfrak{su}_3(\mathbb{C}, 1)$ is reductive with a 4-dimensional subalgebra $h$ and a complementary subspace $m$ generating $g$ if and only if the following holds:

1) $h_1 \cong \mathfrak{so}_3(\mathbb{R}) \oplus \mathfrak{so}_2(\mathbb{R}) = \langle e_1, e_2, e_3, e_6 \rangle$ and $m_1 = \langle e_4, e_5, e_7, e_8 \rangle$.
2) $h_4 \cong \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{so}_2(\mathbb{R}) = \langle e_1, e_6, e_7, e_8 \rangle$ and $m_4 = \langle e_2, e_3, e_4, e_5 \rangle$.

**Proof.** For the basis elements of an arbitrary complement $m$ to $h_1$ in $g$ we have

\[ e_4 + a_1 e_1 + b_1 e_2 + c_1 e_3 + d_1 e_6, \quad e_5 + a_2 e_1 + b_2 e_2 + c_2 e_3 + d_2 e_6, \]
\[ e_7 + a_3 e_1 + b_3 e_2 + c_3 e_3 + d_3 e_6, \quad e_8 + a_4 e_1 + b_4 e_2 + c_4 e_3 + d_4 e_6 \]

with the real numbers $a_i, b_i, c_i, d_i, i = 1, 2, 3, 4$.

An arbitrary complement $m$ to $h_2$ in $g$ has as generators

\[ e_1 + a_1(e_4 - e_3) + b_1(e_5 + e_2) + c_1(e_6 + e_7) + d_1 e_8, \]
\[ e_2 + a_2(e_4 - e_3) + b_2(e_5 + e_2) + c_2(e_6 + e_7) + d_2 e_8, \]
\[ e_3 + a_3(e_4 - e_3) + b_3(e_5 + e_2) + c_3(e_6 + e_7) + d_3 e_8, \]
\[ e_6 + a_4(e_4 - e_3) + b_4(e_5 + e_2) + c_4(e_6 + e_7) + d_4 e_8, \]
where \( a_i, b_i, c_i, d_i, i = 1, 2, 3, 4 \) are real parameters.

The basis elements of an arbitrary complement \( \mathfrak{m} \) to \( \mathfrak{h}_3 \) in \( \mathfrak{g} \) are

\[
e_3 + a_1(e_1 - \frac{1}{2}e_6 + ae_8) + b_1(e_4 - e_3) + c_1(e_2 + e_5) + d_1(e_6 + e_7),
\]

\[
e_5 + a_2(e_1 - \frac{1}{2}e_6 + ae_8) + b_2(e_4 - e_3) + c_2(e_2 + e_5) + d_2(e_6 + e_7),
\]

\[
e_7 + a_3(e_1 - \frac{1}{2}e_6 + ae_8) + b_3(e_4 - e_3) + c_3(e_2 + e_5) + d_3(e_6 + e_7),
\]

\[
e_8 + a_4(e_1 - \frac{1}{2}e_6 + ae_8) + b_4(e_4 - e_3) + c_4(e_2 + e_5) + d_4(e_6 + e_7),
\]

where \( a_i, b_i, c_i, d_i \in \mathbb{R}, i = 1, 2, 3, 4 \).

As the generators of an arbitrary complement \( \mathfrak{m} \) to \( \mathfrak{h}_4 \) in \( \mathfrak{g} \) we can choose the following:

\[
e_2 + a_1e_1 + b_1e_6 + c_1e_7 + d_1e_8, \quad e_3 + a_2e_1 + b_2e_6 + c_2e_7 + d_2e_8,
\]

\[
e_4 + a_3e_1 + b_3e_6 + c_3e_7 + d_3e_8, \quad e_5 + a_4e_1 + b_4e_6 + c_4e_7 + d_4e_8,
\]

where \( a_i, b_i, c_i, d_i, i = 1, 2, 3, 4 \) are real numbers.

Now the assertion follows from the relation \([\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m} \). \( \square \)

**Proposition 10.** The Lie algebra \( \mathfrak{g} = \mathfrak{su}_3(\mathbb{C}, 1) \) is reductive with respect to precisely one of the following pairs \((\mathfrak{h}, \mathfrak{m})\), where \( \mathfrak{h} \) is a 3-dimensional subalgebra of \( \mathfrak{g} \) and \( \mathfrak{m} \) is a complementary subspace to \( \mathfrak{h} \) generating \( \mathfrak{g} \):

1) \( \mathfrak{h}_6 \cong \mathfrak{sl}_2(\mathbb{R}) = \langle e_2, e_4, e_7 \rangle \) and \( \mathfrak{m}_6 = \langle e_1, e_3, e_5, e_6, e_8 \rangle \),

2) \( \mathfrak{h}_7 \cong \mathfrak{sl}_2(\mathbb{R}) = \langle e_6, e_7, e_8 \rangle \) and \( \mathfrak{m}_7 = \langle e_1 - \frac{1}{2}e_6, e_2, e_3, e_4, e_5 \rangle \).

**Proof.** An arbitrary complement \( \mathfrak{m}_i \) to the subalgebra \( \mathfrak{h}_i, i = 5, \cdots, 11 \), in \( \mathfrak{g} \) has as generators in the case \( i = 5 \)

\[
e_4 + a_1e_1 + b_1e_2 + c_1e_3, \quad e_5 + a_2e_1 + b_2e_2 + c_2e_3, \quad e_6 + a_3e_1 + b_3e_2 + c_3e_3,
\]

\[
e_7 + a_4e_1 + b_4e_2 + c_4e_3, \quad e_8 + a_5e_1 + b_5e_2 + c_5e_3,
\]

in the case \( i = 6 \)

\[
e_1 + a_1e_2 + b_1e_4 + c_1e_7, \quad e_3 + a_2e_2 + b_2e_4 + c_2e_7, \quad e_5 + a_3e_2 + b_3e_4 + c_3e_7,
\]

\[
e_6 + a_4e_2 + b_4e_4 + c_4e_7, \quad e_8 + a_5e_2 + b_5e_4 + c_5e_7,
\]

in the case \( i = 7 \)

\[
e_1 + a_1e_6 + b_1e_7 + c_1e_8, \quad e_2 + a_2e_6 + b_2e_7 + c_2e_8, \quad e_3 + a_3e_6 + b_3e_7 + c_3e_8,
\]

\[
e_4 + a_4e_6 + b_4e_7 + c_4e_8, \quad e_5 + a_5e_6 + b_5e_7 + c_5e_8,
\]

in the case \( i = 8 \)

\[
e_1 + a_1(e_2 + e_5) + b_1(e_6 + e_7) + c_1e_8, \quad e_2 + a_2(e_2 + e_5) + b_2(e_6 + e_7) + c_2e_8,
\]

\[
e_3 + a_3(e_2 + e_5) + b_3(e_6 + e_7) + c_3e_8, \quad e_4 + a_4(e_2 + e_5) + b_4(e_6 + e_7) + c_4e_8,
\]

\[
e_6 + a_5(e_2 + e_5) + b_5(e_6 + e_7) + c_5e_8,
\]

in the case \( i = 9 \)

\[
e_1 + a_1(e_2 + e_5) + b_1(e_6 + e_7) + c_1e_8, \quad e_2 + a_2(e_2 + e_5) + b_2(e_6 + e_7) + c_2e_8,
\]

\[
e_3 + a_3(e_2 + e_5) + b_3(e_6 + e_7) + c_3e_8, \quad e_4 + a_4(e_2 + e_5) + b_4(e_6 + e_7) + c_4e_8,
\]

\[
e_6 + a_5(e_2 + e_5) + b_5(e_6 + e_7) + c_5e_8,
\]

in the case \( i = 9 \).
\[ e_1 + a_1(e_2 + e_5) + b_1(e_6 + e_7) + c_1(e_4 - e_3 + be_8), \]
\[ e_2 + a_2(e_2 + e_5) + b_2(e_6 + e_7) + c_2(e_4 - e_3 + be_8), \]
\[ e_3 + a_3(e_2 + e_5) + b_3(e_6 + e_7) + c_3(e_4 - e_3 + be_8), \]
\[ e_4 + a_4(e_2 + e_5) + b_4(e_6 + e_7) + c_4(e_4 - e_3 + be_8), \]
\[ e_5 + a_5(e_2 + e_5) + b_5(e_6 + e_7) + c_5(e_4 - e_3 + be_8), \]
in the case \( i = 10 \)
\[ e_1 + a_1(e_4 - e_3 + b(e_2 + e_5)) + b_1(e_6 + e_7) + c_1(e_8 + c(e_2 + e_5)), \]
\[ e_2 + a_2(e_4 - e_3 + b(e_2 + e_5)) + b_2(e_6 + e_7) + c_2(e_8 + c(e_2 + e_5)), \]
\[ e_3 + a_3(e_4 - e_3 + b(e_2 + e_5)) + b_3(e_6 + e_7) + c_3(e_8 + c(e_2 + e_5)), \]
\[ e_5 + a_4(e_4 - e_3 + b(e_2 + e_5)) + b_4(e_6 + e_7) + c_4(e_8 + c(e_2 + e_5)), \]
\[ e_6 + a_5(e_4 - e_3 + b(e_2 + e_5)) + b_5(e_6 + e_7) + c_5(e_8 + c(e_2 + e_5)), \]
and in the case \( i = 11 \)
\[ e_2 + a_1(e_1 - \frac{1}{2}e_6 + \frac{3}{2}c(e_4 - e_3) - \frac{3}{2}b(e_5 + e_2)) + b_1(e_8 + b(e_4 - e_3) + c(e_5 + e_2)) + c_1(e_6 + e_7), \]
\[ e_3 + a_2(e_1 - \frac{1}{2}e_6 + \frac{3}{2}c(e_4 - e_3) - \frac{3}{2}b(e_5 + e_2)) + b_2(e_8 + b(e_4 - e_3) + c(e_5 + e_2)) + c_2(e_6 + e_7), \]
\[ e_4 + a_3(e_1 - \frac{1}{2}e_6 + \frac{3}{2}c(e_4 - e_3) - \frac{3}{2}b(e_5 + e_2)) + b_3(e_8 + b(e_4 - e_3) + c(e_5 + e_2)) + c_3(e_6 + e_7), \]
\[ e_5 + a_4(e_1 - \frac{1}{2}e_6 + \frac{3}{2}c(e_4 - e_3) - \frac{3}{2}b(e_5 + e_2)) + b_4(e_8 + b(e_4 - e_3) + c(e_5 + e_2)) + c_4(e_6 + e_7), \]
\[ e_7 + a_5(e_1 - \frac{1}{2}e_6 + \frac{3}{2}c(e_4 - e_3) - \frac{3}{2}b(e_5 + e_2)) + b_5(e_8 + b(e_4 - e_3) + c(e_5 + e_2)) + c_5(e_6 + e_7), \]
where \( a_j, b_j, c_j \in \mathbb{R}, j = 1, \cdots, 5 \). The relation \( [h_i, m_i] \subseteq m_i, i = 5, \cdots, 11, \) yields the assertion. \( \square \)

**Proposition 11.** The Lie algebra \( g = \text{su}_3(\mathbb{C}, 1) \) is reductive with respect to the following pairs \((h, m)\), where \( h \) is a 2-dimensional subalgebra of \( g \) and \( m \) is a complementary subspace to \( h \) generating \( g \), if and only if one of the following holds:

1) \( h_{12} = \langle e_1, e_6 \rangle \) and \( m_{12} = \langle e_2, e_3, e_4, e_5, e_7, e_8 \rangle \),

2) \( h_{20} = \langle e_1 - \frac{1}{2}e_6 + \frac{3}{2}a(e_4 - e_3) - \frac{3}{2}b(e_5 + e_2) - \frac{3(a^2 + b^2)}{2}(e_6 + e_7), e_8 + b(e_4 - e_3) + a(e_5 + e_2) + c(e_6 + e_7) \rangle \)

and

\( m_{20} = \langle e_6 + e_7, e_2 + e_5, e_4 - e_3, e_1 - be_8 + 2ae_1 - ae_6, e_2 + ae_8 + 2be_1 - be_6, e_6 + ce_8 + be_5 - ae_4 \rangle, a, b, c \in \mathbb{R} \).
Proof. An arbitrary complement $\mathfrak{m}_i$ to the subalgebra $\mathfrak{h}_i$, $i = 12, \cdots, 20$, in $\mathfrak{g}$ has as generators in the case $i = 12$

\[
e_2 + a_1 e_1 + b_1 e_6, \quad e_3 + a_2 e_1 + b_2 e_6, \quad e_4 + a_3 e_1 + b_3 e_6, \quad e_5 + a_4 e_1 + b_4 e_6, \quad e_7 + a_5 e_1 + b_5 e_6, \quad e_8 + a_6 e_1 + b_6 e_6.
\]

in the case $i = 13$

\[
e_1 + a_1(e_4 - e_3) + b_1(e_6 + e_7), \quad e_2 + a_2(e_4 - e_3) + b_2(e_6 + e_7), \quad e_3 + a_3(e_4 - e_3) + b_3(e_6 + e_7), \quad e_5 + a_4(e_4 - e_3) + b_4(e_6 + e_7), \quad e_6 + a_5(e_4 - e_3) + b_5(e_6 + e_7), \quad e_8 + a_6(e_4 - e_3) + b_6(e_6 + e_7),
\]

in the case $i = 14$

\[
e_1 + a_1(e_2 + e_5 + b(e_4 - e_3)) + b_1(e_6 + e_7), \quad e_2 + a_2(e_2 + e_5 + b(e_4 - e_3)) + b_2(e_6 + e_7), \quad e_3 + a_3(e_2 + e_5 + b(e_4 - e_3)) + b_3(e_6 + e_7), \quad e_4 + a_4(e_2 + e_5 + b(e_4 - e_3)) + b_4(e_6 + e_7), \quad e_5 + a_5(e_2 + e_5 + b(e_4 - e_3)) + b_5(e_6 + e_7), \quad e_6 + a_6(e_2 + e_5 + b(e_4 - e_3)) + b_6(e_6 + e_7),
\]

in the case $i = 15$

\[
e_1 + a_1(e_4 - e_3) + b_1(e_8 + b(e_6 + e_7)), \quad e_2 + a_2(e_4 - e_3) + b_2(e_8 + b(e_6 + e_7)), \quad e_3 + a_3(e_4 - e_3) + b_3(e_8 + b(e_6 + e_7)), \quad e_5 + a_4(e_4 - e_3) + b_4(e_8 + b(e_6 + e_7)), \quad e_6 + a_5(e_4 - e_3) + b_5(e_8 + b(e_6 + e_7)), \quad e_7 + a_6(e_4 - e_3) + b_6(e_8 + b(e_6 + e_7)),
\]

in the case $i = 16$

\[
e_1 + a_1(e_5 + e_2 + b(e_4 - e_3)) + b_1(e_8 + c(e_6 + e_7)), \quad e_2 + a_2(e_5 + e_2 + b(e_4 - e_3)) + b_2(e_8 + c(e_6 + e_7)), \quad e_3 + a_3(e_5 + e_2 + b(e_4 - e_3)) + b_3(e_8 + c(e_6 + e_7)), \quad e_4 + a_4(e_5 + e_2 + b(e_4 - e_3)) + b_4(e_8 + c(e_6 + e_7)), \quad e_5 + a_5(e_5 + e_2 + b(e_4 - e_3)) + b_5(e_8 + b(e_6 + e_7)), \quad e_6 + a_6(e_5 + e_2 + b(e_4 - e_3)) + b_6(e_8 + b(e_6 + e_7)),
\]

in the case $i = 17$

\[
e_1 + a_1(e_6 + e_7) + b_1(e_8 + b(e_4 - e_3) + c(e_5 + e_2)),
\]

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\[ e_2 + a_2(e_6 + e_7) + b_2(e_8 + b(e_4 - e_3) + c(e_5 + e_2)), \]
\[ e_3 + a_3(e_6 + e_7) + b_3(e_8 + b(e_4 - e_3) + c(e_5 + e_2)), \]
\[ e_4 + a_4(e_6 + e_7) + b_4(e_8 + b(e_4 - e_3) + c(e_5 + e_2)), \]
\[ e_5 + a_5(e_6 + e_7) + b_5(e_8 + b(e_4 - e_3) + c(e_5 + e_2)), \]
\[ e_6 + a_6(e_6 + e_7) + b_6(e_8 + b(e_4 - e_3) + c(e_5 + e_2)), \]
in the case \( i = 18 \)
\[ e_1 + a_1(e_6 + e_7 + a(e_4 - e_3) + b(e_5 + e_2)) + b_1(e_8 + c(e_4 - e_3) + d(e_5 + e_2)), \]
\[ e_2 + a_2(e_6 + e_7 + a(e_4 - e_3) + b(e_5 + e_2)) + b_2(e_8 + c(e_4 - e_3) + d(e_5 + e_2)), \]
\[ e_3 + a_3(e_6 + e_7 + a(e_4 - e_3) + b(e_5 + e_2)) + b_3(e_8 + c(e_4 - e_3) + d(e_5 + e_2)), \]
\[ e_4 + a_4(e_6 + e_7 + a(e_4 - e_3) + b(e_5 + e_2)) + b_4(e_8 + c(e_4 - e_3) + d(e_5 + e_2)), \]
\[ e_5 + a_5(e_6 + e_7 + a(e_4 - e_3) + b(e_5 + e_2)) + b_5(e_8 + c(e_4 - e_3) + d(e_5 + e_2)), \]
\[ e_6 + a_6(e_6 + e_7 + a(e_4 - e_3) + b(e_5 + e_2)) + b_6(e_8 + c(e_4 - e_3) + d(e_5 + e_2)), \]
in the case \( i = 19 \)
\[ e_2 + a_1(e_1 - \frac{1}{2}e_6 + ae_8 + b(e_4 - e_3) + c(e_5 + e_2)) + b_1(e_6 + e_7), \]
\[ e_3 + a_2(e_1 - \frac{1}{2}e_6 + ae_8 + b(e_4 - e_3) + c(e_5 + e_2)) + b_2(e_6 + e_7), \]
\[ e_4 + a_3(e_1 - \frac{1}{2}e_6 + ae_8 + b(e_4 - e_3) + c(e_5 + e_2)) + b_3(e_6 + e_7), \]
\[ e_5 + a_4(e_1 - \frac{1}{2}e_6 + ae_8 + b(e_4 - e_3) + c(e_5 + e_2)) + b_4(e_6 + e_7), \]
\[ e_7 + a_5(e_1 - \frac{1}{2}e_6 + ae_8 + b(e_4 - e_3) + c(e_5 + e_2)) + b_5(e_6 + e_7), \]
\[ e_8 + a_6(e_1 - \frac{1}{2}e_6 + ae_8 + b(e_4 - e_3) + c(e_5 + e_2)) + b_6(e_6 + e_7), \]
and in the case \( i = 20 \)
\[ e_2 + a_1(e_1 - \frac{1}{2}e_6 + \frac{3}{2}a(e_4 - e_3) - \frac{3}{2}b(e_5 + e_2) - \frac{3(a^2 + b^2)}{2})(e_6 + e_7) + b_1(e_8 + b(e_4 - e_3) + a(e_5 + e_2) + c(e_6 + e_7)), \]
\[ e_3 + a_2(e_1 - \frac{1}{2}e_6 + \frac{3}{2}a(e_4 - e_3) - \frac{3}{2}b(e_5 + e_2) - \frac{3(a^2 + b^2)}{2})(e_6 + e_7) + b_2(e_8 + b(e_4 - e_3) + a(e_5 + e_2) + c(e_6 + e_7)), \]
\[ e_4 + a_3(e_1 - \frac{1}{2}e_6 + \frac{3}{2}a(e_4 - e_3) - \frac{3}{2}b(e_5 + e_2) - \frac{3(a^2 + b^2)}{2})(e_6 + e_7) + b_3(e_8 + b(e_4 - e_3) + a(e_5 + e_2) + c(e_6 + e_7)), \]
\[ e_5 + a_4(e_1 - \frac{1}{2}e_6 + \frac{3}{2}a(e_4 - e_3) - \frac{3}{2}b(e_5 + e_2) - \frac{3(a^2 + b^2)}{2})(e_6 + e_7) + b_4(e_8 + b(e_4 - e_3) + a(e_5 + e_2) + c(e_6 + e_7)), \]
\[ e_6 + a_5(e_1 - \frac{1}{2}e_6 + \frac{3}{2}a(e_4 - e_3) - \frac{3}{2}b(e_5 + e_2) - \frac{3(a^2 + b^2)}{2})(e_6 + e_7) + b_5(e_8 + b(e_4 - e_3) + a(e_5 + e_2) + c(e_6 + e_7)), \]
\[ e_7 + a_6(e_1 - \frac{1}{2}e_6 + \frac{3}{2}a(e_4 - e_3) - \frac{3}{2}b(e_5 + e_2) - \frac{3(a^2 + b^2)}{2})(e_6 + e_7) + b_6(e_8 + b(e_4 - e_3) + a(e_5 + e_2) + c(e_6 + e_7)), \]
where \( a_j, b_j \in \mathbb{R}, j = 1, \cdots 6 \). Using the relation \( [h_i, m_i] \subseteq m_i, i = 12, \cdots 20 \), we obtain the assertion. \( \square \)
Proposition 12. The Lie algebra $\mathfrak{g} = \mathfrak{su}_3(\mathbb{C}, 1)$ is reductive with a 1-dimensional subalgebra $\mathfrak{h}$ and a 7-dimensional complementary subspace $\mathfrak{m}$ generating $\mathfrak{g}$ in precisely one of the following cases:

1) $\mathfrak{h} = \langle e_1 - 2e_6 \rangle$, $\mathfrak{m}_{b,c,d} = \langle e_2 + b(e_1 - 2e_6), e_3 + c(e_1 - 2e_6), e_6 + d(e_1 - 2e_6), e_4, e_5, e_7, e_8 \rangle$, when $b, c, d \in \mathbb{R}$.

2) $\mathfrak{h} = \langle e_1 + e_6 \rangle$ and $\mathfrak{m}_{b,c,d} = \langle e_2, e_3, e_7, e_8, e_4 + d(e_1 + e_6), e_5 + b(e_1 + e_6), e_6 + c(e_1 + e_6) \rangle$ with $b, c, d \in \mathbb{R}$.

3) $\mathfrak{h} = \langle e_1 - \frac{1}{2}e_6 \rangle$, $\mathfrak{m}_{b,c,d} = \langle e_2, e_3, e_4, e_5, e_6 + b(e_1 - \frac{1}{2}e_6), e_7 + c(e_1 - \frac{1}{2}e_6), e_8 + d(e_1 - \frac{1}{2}e_6) \rangle$ and $b, c, d \in \mathbb{R}$.

4) $\mathfrak{h}_a = \langle e_1 + ae_6 \rangle$ and $\mathfrak{m}_b = \langle e_2, e_3, e_4, e_5, e_6 + b(e_1 + ae_6), e_7, e_8 \rangle$, where $a \in \mathbb{R} \setminus \{-\frac{1}{2}, -2, 1\}$, $b, c, d \in \mathbb{R}$.

5) $\mathfrak{h} = \langle e_6 \rangle$ and $\mathfrak{m}_a = \langle e_1 + ae_6, e_2, e_3, e_4, e_5, e_7, e_8 \rangle$, $a \in \mathbb{R}$.

6) $\mathfrak{h} = \langle e_8 \rangle$ and $\mathfrak{m}_a = \langle e_1 + ae_8, e_2, e_3, e_4, e_5, e_6, e_7 \rangle$, $a \in \mathbb{R}$.

7) $\mathfrak{h} = \langle e_6 + e_7 + ce_8 \rangle$ and $\mathfrak{m}_b = \langle e_1 + bce_8, e_2, e_3, e_4, e_5, e_6 + e_7, e_7 - \frac{1}{2}e_8 \rangle$ with $c \in \mathbb{R} \setminus \{0\}$, $b, \in \mathbb{R}$.

8) $\mathfrak{h}_{b,c} = \langle e_5 + e_2 + b(e_6 + e_7) + ce_8 \rangle$ and $\mathfrak{m}_{d} = \langle e_1 - \frac{c^3d-cd-bc}{2c}e_8, e_2 + \frac{1}{c}e_8, e_3 + cdce_8, e_7 - \frac{b+c+d}{c}e_8, e_4 - e_3, e_2 + e_5, e_6 + e_7 \rangle$, when $b, c, d \in \mathbb{R}$, $c \neq 0$.

9) $\mathfrak{h}_{a,b,c} = \langle e_1 - e_3 + a(e_5 + e_2) + b(e_6 + e_7) + ce_8 \rangle$ and $\mathfrak{m}_{d} = \langle e_2 - dce_8, e_3 - \frac{1+b+c+a}{2}e_8, e_6 - \frac{a^3+1+bc+bc^2+bc^2}{2c}e_8, e_5 + e_2, e_6 + e_7, e_4 - e_3, e_1 + \frac{a+b+c}{c}e_8, e_7 - \frac{b+c}{c}e_8, e_3 + \frac{a}{c}e_4 + \frac{d}{c}e_2, e_8 - \frac{8ac-4ftc^2+24fd^2-9t+12d}{2(8ac-3a+4ac^2)}(e_1 - \frac{1}{7}e_6 + ce_8) \rangle$, $a, b, c, d, f \in \mathbb{R}$, $c \neq 0$, $8dc - 3a + 4ac^2 \neq 0$.

10) $\mathfrak{h}_{a,b,c,d} = \langle e_1 - \frac{1}{2}e_6 + d(e_4 - e_3) + a(e_5 + e_2) + b(e_6 + e_7) + ce_8 \rangle$ and $\mathfrak{m}_{f} = \langle e_6 + e_7, e_4 - e_3, e_5 + e_2, e_3 + f(e_1 - \frac{1}{7}e_6 + ce_8), e_2 - \frac{2c}{3}e_4 - \frac{4a}{3}e_1 - \frac{2b}{3}e_7 + \frac{2d}{3}e_8, e_7 - \frac{b}{3}e_8 + \frac{a}{3}e_4 + \frac{d}{3}e_2, e_8 - \frac{8ac-4ftc^2+24fd^2-9t+12d}{2(8ac-3a+4ac^2)}(e_1 - \frac{1}{7}e_6 + ce_8) \rangle$, $a, b, c, d, f \in \mathbb{R}$, $c \neq 0$, $8dc - 3a + 4ac^2 \neq 0$.

Proof. An arbitrary complement $\mathfrak{m}_i$ to the subalgebra $\mathfrak{h}_i$, $i = 21, \ldots, 27$, in $\mathfrak{g}$ has as generators in the case $i = 21$

$e_2 + a_1(e_1 + ae_6)$, $e_3 + a_2(e_1 + ae_6)$, $e_4 + a_3(e_1 + ae_6)$, $e_5 + a_4(e_1 + ae_6)$, $e_6 + a_5(e_1 + ae_6)$, $e_7 + a_6(e_1 + ae_6)$, $e_8 + a_7(e_1 + ae_6)$,

in the case $i = 22$

$e_1 + a_1e_6$, $e_2 + a_2e_6$, $e_3 + a_3e_6$, $e_4 + a_4e_6$, $e_5 + a_5e_6$, $e_7 + a_6e_6$, $e_8 + a_7e_6$, $e_9 + a_8e_6$,

in the case $i = 23$

$e_1 + a_1e_8$, $e_2 + a_2e_8$, $e_3 + a_3e_8$, $e_4 + a_4e_8$, $e_5 + a_5e_8$, $e_6 + a_6e_8$, $e_7 + a_7e_8$, $e_8 + a_8e_8$. 

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in the case $i = 24$

$$e_1 + a_1(e_6 + e_7 + ce_8), \quad e_2 + a_2(e_6 + e_7 + ce_8),$$
$$e_3 + a_3(e_6 + e_7 + ce_8), \quad e_4 + a_4(e_6 + e_7 + ce_8),$$
$$e_5 + a_5(e_6 + e_7 + ce_8), \quad e_7 + a_6(e_6 + e_7 + ce_8),$$
$$e_8 + a_7(e_6 + e_7 + ce_8),$$

in the case $i = 25$

$$e_1 + a_1(e_5 + e_2 + b(e_6 + e_7) + ce_8), \quad e_2 + a_2(e_5 + e_2 + b(e_6 + e_7) + ce_8),$$
$$e_3 + a_3(e_5 + e_2 + b(e_6 + e_7) + ce_8), \quad e_4 + a_4(e_5 + e_2 + b(e_6 + e_7) + ce_8),$$
$$e_5 + a_5(e_5 + e_2 + b(e_6 + e_7) + ce_8), \quad e_7 + a_6(e_5 + e_2 + b(e_6 + e_7) + ce_8),$$
$$e_8 + a_7(e_5 + e_2 + b(e_6 + e_7) + ce_8),$$

in the case $i = 26$

$$e_1 + a_1(e_4 - e_3 + a(e_5 + e_2) + b(e_6 + e_7) + ce_8),$$
$$e_2 + a_2(e_4 - e_3 + a(e_5 + e_2) + b(e_6 + e_7) + ce_8),$$
$$e_3 + a_3(e_4 - e_3 + a(e_5 + e_2) + b(e_6 + e_7) + ce_8),$$
$$e_5 + a_4(e_4 - e_3 + a(e_5 + e_2) + b(e_6 + e_7) + ce_8),$$
$$e_6 + a_5(e_4 - e_3 + a(e_5 + e_2) + b(e_6 + e_7) + ce_8),$$
$$e_7 + a_6(e_4 - e_3 + a(e_5 + e_2) + b(e_6 + e_7) + ce_8),$$
$$e_8 + a_6(e_4 - e_3 + a(e_5 + e_2) + b(e_6 + e_7) + ce_8),$$

in the case $i = 27$

$$e_2 + a_1(e_1 - \frac{1}{2}e_6 + d(e_4 - e_3) + a(e_5 + e_2) + b(e_6 + e_7) + ce_8),$$
$$e_3 + a_2(e_1 - \frac{1}{2}e_6 + d(e_4 - e_3) + a(e_5 + e_2) + b(e_6 + e_7) + ce_8),$$
$$e_4 + a_3(e_1 - \frac{1}{2}e_6 + d(e_4 - e_3) + a(e_5 + e_2) + b(e_6 + e_7) + ce_8),$$
$$e_5 + a_4(e_1 - \frac{1}{2}e_6 + d(e_4 - e_3) + a(e_5 + e_2) + b(e_6 + e_7) + ce_8),$$
$$e_6 + a_5(e_1 - \frac{1}{2}e_6 + d(e_4 - e_3) + a(e_5 + e_2) + b(e_6 + e_7) + ce_8),$$
$$e_7 + a_6(e_1 - \frac{1}{2}e_6 + d(e_4 - e_3) + a(e_5 + e_2) + b(e_6 + e_7) + ce_8),$$
$$e_8 + a_7(e_1 - \frac{1}{2}e_6 + d(e_4 - e_3) + a(e_5 + e_2) + b(e_6 + e_7) + ce_8),$$

where $a_j, j = 1, \cdots, 7$, are real parameters. The relation $[h_i, m_i] \subseteq m_i$, $i = 21, \cdots, 27$, yields the assertion. 

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4 Left A-loops as sections in simple Lie groups

The connected almost differentiable left A-loops $L$ with $\dim L \leq 2$ are classified in [18], Section 27 and Theorem 18.14. Furthermore, all 3-dimensional left A-loops which are differentiable sections in a non-solvable Lie group are determined in [6]. In this section we deal with the at least 4-dimensional almost differentiable left A-loops having an at most 9-dimensional simple Lie group $G$ as the group topologically generated by their left translations. According to Lemma 1 the group $G$ is not compact.

**Proposition 13.** There exists no at least 4-dimensional differentiable left A-loop having a group locally isomorphic to $\text{PSL}_2(\mathbb{C})$ as the group topologically generated by its left translations.

**Proof.** Since the tangent space $T_e L$ for an almost differentiable left A-loop $L$ is reductive only the pairs $(h, m)$ in Proposition 4 can occur as the tangent objects $(T_1 H, T_e L)$, where $H$ is the stabilizer of the identity $e$ of $L$. A maximal compact subalgebra of the Lie algebra $h_3$ as well as of $h_6$ is isomorphic to $\text{so}_2(\mathbb{R})$. Hence the Lie group corresponding to $h_3$ as well as to $h_6$ cannot be the stabilizer of $e \in L$ (cf. Lemma 2). Moreover, the hyperbolic elements $e_1 \in h_4$ and $e_2 \in m_a$ are conjugate (see 1.1). This contradiction to Lemma 3 yields the assertion.

**Proposition 14.** Let $G$ be locally isomorphic to $\text{SL}_3(\mathbb{R})$. Every connected almost differentiable left A-loop having $G$ as the group topologically generated by its left translations is isomorphic to the 5-dimensional Bruck loop $L_0$ of hyperbolic type having the group $\text{SO}_3(\mathbb{R})$ as the stabilizer of $e \in L_0$.

**Proof.** Since the tangent space $T_e L$ for an almost differentiable left A-loop $L$ is reductive we have to investigate the pairs $(h, m)$ listed in Propositions 5, 6, 7 and 8. According to Lemma 2 the Lie groups belonging to the Lie algebras $h_5, h_7, h_8, h_30$ and $h_35$ for $b = 0$ cannot be stabilizers of $e \in L$. The element $-e_5 + e_8 \in h_{26}$ is conjugate to $\frac{1}{2} e_1 + 2 e_3 \in m_{26}$ under $g = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -\frac{1}{2} & 0 \\ 1 & \frac{1}{2} & 0 \end{pmatrix}$.

The element $e_2 + e_8 \in h_{32}$ is conjugate to $e_1 + 2 e_7 - e_8 + 2 e_4 \in m_4$ under $g = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -\frac{1}{2} & 0 \\ 2 & 2 & 0 \end{pmatrix}$ and $e_6 - e_7 + b(e_5 + e_8) \in h_{35}$, $b > 0$, is conjugate to $(b^2 + 1)e_1 - e_3 + 2b(e_5 - e_8) \in m_b$ under $g = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -b & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Moreover, the element $e_8 + \frac{1}{a} e_5 \in h_{31,1}$ is conjugate to $\frac{a^2 + a + 1}{a^2} e_1 + e_2 + e_3 + e_4 - e_6 - e_7 \in m_b$ under $g = \begin{pmatrix} 1 & -\frac{1}{a} & -1 \\ 1 & \frac{a+1}{a} & -1 \\ 0 & \frac{a}{2+a} & \frac{a}{2+a} \end{pmatrix}$. 

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In the case 2) of Proposition 8 we choose \( k \in \mathbb{R}\setminus\{0\} \) in such a way that 
\( l := k^2c + k + b \neq 0 \). Then the element \( l(e_5 - 2e_8) \in h_{31,2} \) is conjugate to 
\( e_1 + b(e_5 - 2e_8) + 3(e_2 - ke_6) + k(e_3 + c(e_5 - 2e_8)) + \frac{2k^2c + k + 3b}{3}(ke_4 - e_7) \in m_{b,c,d} \) under \( g \) such that 
\[
\begin{pmatrix}
0 & -\frac{2k^2c + k + 3b}{3m} & 1 \\
k & 1 & 0 \\
-\frac{k}{3m} & \frac{1}{3m} & 1
\end{pmatrix}.
\]
In the case 3) of Proposition 8 we take \( k \in \mathbb{R} \) such that \( n := k^2b - 2k + c \neq 0 \). Then the element \( n(e_5 - \frac{1}{2}e_8) \in h_{31,3} \) is conjugate to 
\( -ke_1 + k^2(e_2 + b(e_5 - \frac{1}{2}e_8)) + \frac{3k^2b - 2k + 3c}{2}(e_3 - ke_6) + e_4 + c(e_5 - \frac{1}{2}e_8) + e_7 \in m_{b,c,d} \) under \( g \) such that 
\[
\begin{pmatrix}
0 & \frac{2}{3m} & -\frac{3k^2b + 2b - 3c}{3m} \\
1 & 1 & -k \\
1 & 0 & k
\end{pmatrix}.
\]
In the case 4) of Proposition 8 we take \( k \in \mathbb{R} \) such that \( m := k^2b + k + c \neq 0 \). Then the element \( m(e_5 + e_8) \in h_{31,4} \) is conjugate to 
\( (3c + 3k^2b + k)(ke_2 - e_1) + e_4 - ke_3 + e_7 + c(e_5 + e_8) + k^2(e_6 + b(e_5 + e_8)) \in m_{b,c,d} \) under \( g \) such that 
\[
\begin{pmatrix}
1 & 1 & -k \\
-\frac{1}{3m} & 0 & -\frac{3c - k - 3k^2b}{3m} \\
0 & 1 & \frac{k}{3m}
\end{pmatrix}. \]
These facts contradict Lemma 8.

In the remaining case one has \([m_6, m_6] = h_6 \) and the loop \( L \) with \( T_eL = m_6 \) is a Bruck loop. The assertion follows now from the proof of the Theorem 13 in [5], p. 12.

Since the exponential image of the Lie algebra \( g = su_3(\mathbb{C}, 1) \) is much more complicated than the exponential image of \( g = sl_3(\mathbb{R}) \) we treat the almost differentiable left \( A \)-loops having \( PSU_3(\mathbb{C}, 1) \) as the group topologically generated by the left translations under the assumption that their dimension is at most 5.

**Proposition 15.** Let \( G \) be locally isomorphic to \( PSU_3(\mathbb{C}, 1) \). Every at most 5-dimensional connected almost differentiable left \( A \)-loop having \( G \) as the group topologically generated by the left translations is isomorphic to the complex hyperbolic plane loop \( L_0 \) having the group \( Spin_3 \times SO_2(\mathbb{R})/\langle(-1, -1) \rangle \) as the stabilizer of \( e \in L_0 \).

**Proof.** Since the tangent space \( T_eL \) for an almost differentiable left \( A \)-loop \( L \) is reductive we have to deal only with the pairs \( (h, m) \) described in the Propositions 9, 10. The complex hyperbolic plane loop \( L_0 \) is realized on the exponential image of the subspace \( m_1 \) (cf. [5], p. 8). The Lie group corresponding to \( h_4 \) cannot be the stabilizer of a 4-dimensional topological loop \( L \) (see Lemma 8). According to 1.2 the element \( e_2 \in h_6 \) is conjugate to \( e_1 \in m_6 \), which is a contradiction to Lemma 8. Two loxodromic elements of \( su_3(\mathbb{C}, 1) \) are conjugate in \( SU_3(\mathbb{C}, 1) \) if and only if they have the same...
eigenvalues (cf. Prop. 3.2.3 (d) in [3], p. 65) and therefore they are conjugate in $SL_3(\mathbb{C})$. Since the elements $e_7 \in h_7$ and $e_4 \in m_7$ are loxodromic and $Ad_g(e_7) = e_4$ with $g = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \sqrt{2} \\ \sqrt{2} & 0 & 1 \end{pmatrix} \in SL_3(\mathbb{C})$ we have also a contradiction to Lemma 3.

At the end of this section we show that several reductive spaces $(g, h, m)$, where $g = su_3(\mathbb{C}, 1)$ and dim $h \leq 2$ cannot correspond to an almost differentiable left A-loop.

**Proposition 16.** There is no almost differentiable left A-loop corresponding to one of the following triples: $(g, h_{12}, m_{12})$ in Proposition 11 and $(g, h, m_a)$ in the case 6) as well as $(g, h, m_b)$ in the case 7) of Proposition 12.

**Proof.** Since the elements $e_1 \in h_{12}$ and $e_2 \in m_{12}$ are elliptic in a subalgebra isomorphic to $so_3(\mathbb{R})$ of $g$ (see 1.2) they are conjugate under $Ad P SU_3(\mathbb{C}, 1)$. Since he element $e_8 \in h$ in the case 6 as well as $e_6 + e_7 + ce_8 \in h, c \neq 0$, in the case 7 of Proposition 12 is hyperbolic in a subalgebra isomorphic to $sl_2(\mathbb{R})$ of $g$ (see 1.1), we have that $e_8$ and $e_7 \in m_a$ respectively $e_6 + e_7 + ce_8$ and $-\frac{1}{\sqrt{2} + 2c}(e_7 - \frac{1}{c}e_8) \in m_b$ are conjugate under $Ad P SU_3(\mathbb{C}, 1)$. This contradicts Lemma 3.

5 Reductive loops corresponding to semi-simple Lie groups of dimension 6

Let $G = G_1 \times G_2$ be the group topologically generated by the left translations of a connected almost differentiable left A-loop $L$, such that $G_i, i = 1, 2$, is a 3-dimensional quasi-simple Lie group. In contrast to the non-existence of 3-dimensional almost differentiable left A-loops belonging to $G$ (cf. Propositions 5 and 8 in [6]) we will show that there are such loops $L$ with $G = G_1 \times G_2$ as the group topologically generated by the left translations if dim $L \geq 4$.

The following fact is well known from linear algebra:

**Lemma 17.** Let $g = g_1 \oplus g_2$, where $g_i, i = 1, 2$ are simple Lie algebras of dimension 3. For any subspace $m$ with dimension 4 respectively 5 the intersections $m \cap g_1$ and $m \cap g_2$ have dimension at least 1 respectively at least 2.

The fact that the coset space $G/H$ is parallelizable is reflected in the following lemma.
Lemma 18. Let $G$ be isomorphic to the Lie group $G_1 \times G_2$, such that $G_2 \cong SO_3(\mathbb{R})$ and for the subgroup $H$ of $G$ one has $H = H_1 \times H_2$ with $1 \neq H_2 \leq G_2$. Then $G$ cannot be the group topologically generated by the left translations of a topological loop.

For the proof see Lemma 2 in [5], p. 5.

First let $G$ be locally isomorphic to $SO_3(\mathbb{R}) \times SO_3(\mathbb{R})$. Since the at most 2-dimensional connected subgroups of $G$ are tori and dim $L \geq 4$ Lemma 2 gives

Proposition 19. There is no left $A$-loop as differentiable section in a group locally isomorphic to $SO_3(\mathbb{R}) \times SO_3(\mathbb{R})$.

Now let $G$ be locally isomorphic to $PSL_2(\mathbb{R}) \times G_2$, where $G_2$ is either the group $SO_3(\mathbb{R})$ or $PSL_2(\mathbb{R})$. Using the real basis of $\mathfrak{so}_3(\mathbb{R})$ respectively of $\mathfrak{so}_3(\mathbb{R})$ introduced in 1.1 respectively in 1.2 we can choose $(e_1, 0)$, $(e_2, 0)$, $(e_3, 0)$, $(0, e_1)$, $(0, e_2)$, $(0, e_3)$ as a real basis of the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{g}_2$, where $\varepsilon = i$ with $i^2 = -1$ for $\mathfrak{g}_2 = \mathfrak{so}_3(\mathbb{R})$ and $\varepsilon = 1$ for $\mathfrak{g}_2 = \mathfrak{sl}_2(\mathbb{R})$.

Denote by $H$ a subgroup of $G$. First we assume that $H$ is decomposable into a direct product. If $H$ has dimension 2 then with Lemma 18 we obtain that $H$ is (up to interchanging the components) either $\mathcal{L}_2 \times \{1\}$ or $K_1 \times K_2$, where $K_i$, $i = 1, 2$ are 1-dimensional subgroups of $PSL_2(\mathbb{R})$. Now according to 1.1 the Lie algebra $\mathfrak{h}$ of $H$ has one of the following forms:

$$
\mathfrak{h}_1 = \langle (e_3, 0), (0, e_3) \rangle, \quad \mathfrak{h}_2 = \langle (e_3, 0), (0, e_2 + e_3) \rangle, \quad \mathfrak{h}_3 = \langle (e_3, 0), (0, e_1) \rangle,
$$

$$
\mathfrak{h}_4 = \langle (e_1, 0), (0, e_1) \rangle, \quad \mathfrak{h}_5 = \langle (e_1, 0), (0, e_2 + e_3) \rangle,
$$

$$
\mathfrak{h}_6 = \langle (e_2 + e_3, 0), (0, e_2 + e_3) \rangle, \quad \mathfrak{h}_7 = \langle (e_1, 0), (e_2 + e_3, 0) \rangle.
$$

The Lie algebras $\mathfrak{h}_1$ till $\mathfrak{h}_7$ are subalgebras of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{sl}_2(\mathbb{R})$ but $\mathfrak{h}_7$ is also a subalgebra of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{so}_3(\mathbb{R})$.

If dim $H = 1$ then $H$ has the shape $K_1 \times \{1\}$ with a 1-dimensional subgroup $K_1$ of $PSL_2(\mathbb{R})$. Then according to 1.1 the Lie algebra $\mathfrak{h}$ of $H$ has (up to interchanging the components) one of the following forms:

$$
\mathfrak{h}_8 = \langle (e_3, 0) \rangle, \quad \mathfrak{h}_9 = \langle (e_1, 0) \rangle, \quad \mathfrak{h}_{10} = \langle (e_2 + e_3, 0) \rangle.
$$

These algebras are subalgebras of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{sl}_2(\mathbb{R})$ as well as $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{so}_3(\mathbb{R})$.

Now we suppose that $H$ is not a direct product of two subgroups. In the case dim $H = 2$ one has $H = \{ (x, \varphi(x)) | x \in \mathcal{L}_2 \}$, where $\varphi \neq 1$ is a homomorphism of $\mathcal{L}_2$ into $PSL_2(\mathbb{R})$. If $\varphi$ is injective then the Lie algebra of $H$ is a subalgebra of $\mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{sl}_2(\mathbb{R})$ and has the shape

$$
\mathfrak{h}_{11} = \langle (e_1, e_1), (e_2 + e_3, e_2 + e_3) \rangle.
$$

If $\varphi$ has 1-dimensional kernel then the Lie algebra of $H$ is given by

$$
\mathfrak{h}_{12} = \langle (e_1, k), (e_2 + e_3, 0) \rangle,
$$

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where \( k \) denotes either the element \( e_1 \) or \( e_2 + e_3 \) of \( \mathfrak{sl}_2(\mathbb{R}) \) or \( e_3 \) of \( \mathfrak{sl}_2(\mathbb{R}) \cap \mathfrak{so}_3(\mathbb{R}) \) (see 1.1 and 1.2).

In the case \( \dim H = 1 \) one has \( H = \{(k_1, \varphi(k_1)) \mid k_1 \in K_1\} \), where \( K_1 \) is a 1-dimensional subgroup of \( \text{PSL}_2(\mathbb{R}) \) and \( \varphi \neq 1 \) is a homomorphism of \( K_1 \) into \( \text{PSL}_2(\mathbb{R}) \) or \( \text{SO}_3(\mathbb{R}) \). Then the Lie algebra \( \mathfrak{h} \) of \( H \) has (up to interchanging the components) one of the following forms:

\[
\begin{align*}
\mathfrak{h}_{13} &= \langle (e_1, e_1) \rangle, \quad \mathfrak{h}_{14} = \langle (e_1, e_2 + e_3) \rangle, \quad \mathfrak{h}_{15} = \langle (e_2 + e_3, e_2 + e_3) \rangle, \\
\mathfrak{h}_{16} &= \langle (e_1, e_3) \rangle, \quad \mathfrak{h}_{17} = \langle (e_2 + e_3, e_3) \rangle, \quad \mathfrak{h}_{18} = \langle (e_3, e_3) \rangle.
\end{align*}
\]

The Lie algebra \( \mathfrak{h}_{13} \) till \( \mathfrak{h}_{18} \) are subalgebras of \( \mathfrak{g} = \mathfrak{sl}_4(\mathbb{R}) \oplus \mathfrak{sl}_4(\mathbb{R}) \) but \( \mathfrak{h}_{16}, \mathfrak{h}_{17}, \mathfrak{h}_{18} \) are also subalgebras of \( \mathfrak{g} = \mathfrak{sl}_4(\mathbb{R}) \oplus \mathfrak{so}_3(\mathbb{R}) \).

**Proposition 20.** The Lie algebra \( \mathfrak{g} = \mathfrak{sl}_4(\mathbb{R}) \oplus \mathfrak{g}_2 \), where \( \mathfrak{g}_2 \) is a 3-dimensional simple Lie algebra, is reductive with an at most 2-dimensional subalgebra \( \mathfrak{h} \) and a complementary subspace \( \mathfrak{m} \) generating \( \mathfrak{g} \) in exactly one of the following cases:

1) \( \mathfrak{h}_8 = \langle (e_3, 0) \rangle, \quad \mathfrak{m}_9 = \langle (e_1, 0), (e_2, 0), (0, e_1), (0, e_2), (ae_3, e_3) \rangle, \)
2) \( \mathfrak{h}_8 = \langle (e_3, 0) \rangle, \quad \mathfrak{m}_9 = \langle (e_1, 0), (e_2, 0), (0, e_1), (be_3, e_2), (0, e_3) \rangle, \)
3) \( \mathfrak{h}_8 = \langle (e_3, 0) \rangle, \quad \mathfrak{m}_9 = \langle (e_1, 0), (e_2, 0), (ce_3, e_1), (0, e_2), (0, e_3) \rangle, \)
4) \( \mathfrak{h}_9 = \langle (e_1, 0) \rangle, \quad \mathfrak{m}_9 = \langle (e_2, 0), (e_3, 0), (0, e_1), (0, e_2), (de_1, e_3) \rangle, \)
5) \( \mathfrak{h}_9 = \langle (e_1, 0) \rangle, \quad \mathfrak{m}_9 = \langle (e_2, 0), (e_3, 0), (0, e_1), (fe_1, e_2), (0, e_3) \rangle, \)
6) \( \mathfrak{h}_9 = \langle (e_1, 0) \rangle, \quad \mathfrak{m}_9 = \langle (e_2, 0), (e_3, 0), (ge_1, e_1), (0, e_2), (0, e_3) \rangle, \)
7) \( \mathfrak{h}_{16} = \langle (e_1, e_3) \rangle, \quad \mathfrak{m}_9 = \langle (e_2, 0), (e_3, 0), (0, e_1), (0, e_2), (he_1, (1 + h)e_3) \rangle, \)
8) \( \mathfrak{h}_{17} = \langle (e_2 + e_3, e_3) \rangle, \quad \mathfrak{m}_9 = \langle (e_3, ke_3), (e_1, 0), (0, e_1), (0, e_2), (e_2 + e_3, 0) \rangle, \)
9) \( \mathfrak{h}_{18} = \langle (e_3, e_3) \rangle, \quad \mathfrak{m}_9 = \langle (le_3, (1 + l)e_3), (e_1, 0), (e_2, 0), (0, e_1), (0, e_2) \rangle, \)
10) \( \mathfrak{h}_1 = \langle (e_3, 0) \rangle, \quad \mathfrak{m}_9 = \langle (e_1, 0), (e_2, 0), (0, e_1), (0, e_2) \rangle, \)
11) \( \mathfrak{h}_3 = \langle (e_3, 0), (0, e_1) \rangle, \quad \mathfrak{m}_4 = \langle (e_1, 0), (e_2, 0), (0, e_2), (0, e_3) \rangle, \)
12) \( \mathfrak{h}_4 = \langle (e_1, 0), (0, e_1) \rangle, \quad \mathfrak{m}_4 = \langle (e_2, 0), (e_3, 0), (0, e_2), (0, e_3) \rangle, \)
13) \( \mathfrak{h}_{13} = \langle (e_1, e_1) \rangle, \quad \mathfrak{m}_9 = \langle (e_2, 0), (e_3, 0), (0, e_3), (0, e_2), (me_1, (1 + m)e_1) \rangle, \)
14) \( \mathfrak{h}_{14} = \langle (e_1, e_2 + e_3) \rangle, \quad \mathfrak{m}_9 = \langle (e_2, 0), (e_3, 0), (0, e_1), (0, e_2 + e_3), (ne_1, e_2) \rangle, \)

where \( a, b, c, d, f, g, h, k, l, m, n \in \mathbb{R} \) and \( \varepsilon = i \) for \( \mathfrak{g}_2 = \mathfrak{so}_3(\mathbb{R}) \) whereas \( \varepsilon = 1 \) for \( \mathfrak{g}_2 = \mathfrak{sl}_4(\mathbb{R}) \). The cases 1) till 10) occur for both simple 3-dimensional Lie algebras whereas the cases 10) till 14) occur only for \( \mathfrak{g}_2 = \mathfrak{sl}_4(\mathbb{R}) \).

**Proof.** The basis elements of an arbitrary complement \( \mathfrak{m}_i \) to \( \mathfrak{h}_i \), \( i = 1, \ldots, 18 \), in \( \mathfrak{g} = \mathfrak{sl}_4(\mathbb{R}) \oplus \mathfrak{g}_2 \), where \( \mathfrak{g}_2 \) is either \( \mathfrak{sl}_4(\mathbb{R}) \) or \( \mathfrak{so}_3(\mathbb{R}) \), are:

In the case \( i = 1 \)
\[
\begin{align*}
& (e_1 + a_1e_3, a_2e_3), \quad (e_2 + b_1e_3, b_2e_3), \quad (c_1e_3, e_1 + c_2e_3), \quad (d_1e_3, e_2 + d_2e_3), \\
& \text{in the case } i = 2 \\
& (e_1 + a_1e_3, a_2(e_2 + e_3)), \quad (e_2 + b_1e_3, b_2(e_2 + e_3)), \\
& (c_1e_3, e_1 + c_2(e_2 + e_3)), \quad (d_1e_3, e_3 + d_2(e_2 + e_3)), \\
& \text{in the case } i = 3 \\
& (e_1 + a_1e_3, a_2e_1), \quad (e_2 + b_1e_3, b_2e_1), \quad (c_1e_3, e_2 + c_2e_1), \quad (d_1e_3, e_3 + d_2e_1), \\
& \text{in the case } i = 4 \\
& (e_2 + a_1e_1, a_2e_1), \quad (e_3 + b_1e_1, b_2e_1), \quad (c_1e_1, e_2 + c_2e_1), \quad (d_1e_1, e_3 + d_2e_1), \\
& \text{in the case } i = 5 \\
& (e_2 + a_1e_1, a_2(e_2 + e_3)), \quad (e_3 + b_1e_1, b_2(e_2 + e_3)), \\
& (c_1e_1, e_1 + c_2(e_2 + e_3)), \quad (d_1e_1, e_3 + d_2(e_2 + e_3)), \\
& \text{in the case } i = 6 \\
& (e_1 + a_1(e_2 + e_3), a_2(e_2 + e_3)), \quad (e_3 + b_1(e_2 + e_3), b_2(e_2 + e_3)), \\
& (c_1(e_2 + e_3), e_1 + c_2(e_2 + e_3)), \quad (d_1(e_2 + e_3), e_3 + d_2(e_2 + e_3)), \\
& \text{in the case } i = 7 \\
& (e_3 + a_1e_1 + a_2(e_2 + e_3), 0), \quad (b_1e_1 + b_2(e_2 + e_3), \varepsilon e_1), \\
& (c_1e_1 + c_2(e_2 + e_3), \varepsilon e_2), \quad (d_1e_1 + d_2(e_2 + e_3), e_3), \\
& \text{in the case } i = 8 \\
& (e_1 + a_1e_3, 0), \quad (e_2 + a_2e_3, 0), \quad (a_3e_3, \varepsilon e_1), \quad (a_4e_3, \varepsilon e_2), \quad (a_5e_3, e_3), \\
& \text{in the case } i = 9 \\
& (e_2 + a_1e_1, 0), \quad (e_3 + a_2e_1, 0), \quad (a_3e_1, \varepsilon e_1), \quad (a_4e_1, \varepsilon e_2), \quad (a_5e_1, e_3), \\
& \text{in the case } i = 10 \\
& (e_2 + a_1(e_2 + e_3), 0), \quad (e_1 + a_2(e_2 + e_3), 0), \quad (a_3(e_2 + e_3), \varepsilon e_1), \\
& (a_4(e_2 + e_3), \varepsilon e_2), \quad (a_5(e_2 + e_3), e_3), \\
& \text{in the case } i = 11 \\
& (e_3 + a_1e_1 + a_2(e_2 + e_3), a_1e_1 + a_2(e_2 + e_3)), \\
& (b_1e_1 + b_2(e_2 + e_3), e_1 + b_1e_1 + b_2(e_2 + e_3)), \\
& (c_1e_1 + c_2(e_2 + e_3), e_2 + c_1e_1 + c_2(e_2 + e_3)), \\
& (d_1e_1 + d_2(e_2 + e_3), e_3 + d_1e_1 + d_2(e_2 + e_3)), \\
& \text{in the case } i = 12 \\
& (e_3 + a_1e_1 + a_2(e_2 + e_3), a_4k), \quad (b_1e_1 + b_2(e_2 + e_3), \varepsilon e_1 + b_1k),
\end{align*}
\]
\[(c_1e_1 + c_2(e_2 + e_3), \varepsilon e_2 + c_1k), \quad (d_1e_1 + d_2(e_2 + e_3), e_3 + d_1k),\]
in the case \(i = 13\)
\[(e_2 + a_1e_1, a_1e_1), \quad (e_3 + a_2e_1, a_2e_1), \quad (a_3e_1, e_1 + a_3e_1),\]
\[(a_4e_1, e_2 + a_4e_1), \quad (a_5e_1, e_3 + a_5e_1),\]
in the case \(i = 14\)
\[(e_2 + a_1e_1, a_1(e_2 + e_3)), \quad (e_3 + a_2e_1, a_2(e_2 + e_3)), \quad (a_3e_1, e_1 + a_3(e_2 + e_3)),\]
\[(a_4e_1, e_2 + a_4(e_2 + e_3)), \quad (a_5e_1, e_3 + a_5(e_2 + e_3)),\]
in the case \(i = 15\)
\[(e_2 + a_1(e_2 + e_3), a_1(e_2 + e_3)), \quad (e_1 + a_2(e_2 + e_3), a_2(e_2 + e_3)),\]
\[(a_3(e_2 + e_3), e_1 + a_3(e_2 + e_3)), \quad (a_4(e_2 + e_3), e_2 + a_4(e_2 + e_3)),\]
\[(a_5(e_2 + e_3), e_3 + a_5(e_2 + e_3)),\]
in the case \(i = 16\)
\[(e_2 + a_1e_1, a_1e_3), \quad (e_3 + a_2e_1, a_2e_3), \quad (a_3e_1, \varepsilon e_1 + a_3e_3),\]
\[(a_4e_1, \varepsilon e_2 + a_4e_3), \quad (a_5e_1, e_3 + a_5e_3),\]
in the case \(i = 17\)
\[(e_2 + a_1(e_2 + e_3), a_1e_3), \quad (e_1 + a_2(e_2 + e_3), a_2e_3), \quad (a_3(e_2 + e_3), \varepsilon e_1 + a_3e_3),\]
\[(a_4(e_2 + e_3), \varepsilon e_2 + a_4e_3), \quad (a_5(e_2 + e_3), e_3 + a_5e_3),\]
in the case \(i = 18\)
\[(e_1 + a_1e_3, a_1e_3), \quad (e_2 + a_2e_3, a_2e_3), \quad (a_3e_3, \varepsilon e_1 + a_3e_3),\]
\[(a_4e_3, \varepsilon e_2 + a_4e_3), \quad (a_5e_3, e_3 + a_5e_3),\]

where \(a_i, \ i = 1, 2, \cdots, 5, \ b_j, \ j = 1, 2, \ c_k, \ k = 1, 2, \ d_l, \ l = 1, 2,\) are real parameters, \(\varepsilon = i\) for \(g_2 = \mathfrak{so}_3(\mathbb{R})\) and \(\varepsilon = 1\) for \(g_2 = \mathfrak{sl}_2(\mathbb{R})\).

Using the relation \([h_i, m_i] \subseteq m_i, \ i = 1, \cdots, 18,\) and Lemma \[17\] we obtain the assertion. \(\square\)

**Proposition 21.** Let \(G\) be locally isomorphic to \(PSL_2(\mathbb{R}) \times G_2\), where \(G_2\) is either \(PSL_2(\mathbb{R})\) or \(SO_3(\mathbb{R})\). If \(G\) is the group topologically generated by the left translations of a connected almost differentiable left A-loop \(L\) then \(L\) is either a Scheerer extension of \(G_2\) by \(\mathbb{H}_2\) or the direct product \(\mathbb{H}_2 \times \mathbb{H}_2\), where \(\mathbb{H}_2\) denotes the hyperbolic plane loop. In the second case \(G\) is isomorphic to \(PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})\).

**Proof.** Since we assume that \(\text{dim} \ L \geq 4\) we have to consider only the pairs \((h, m)\) in Proposition \[20\]. Now using \[1.1\] and \[1.2\] we obtain that the element \((0, e_1) \in h_3 \cap h_4\), the element \((e_1, 0) \in h_0\), the element \((e_1, e_1) \in h_{13}\) respectively the element \((e_1, e_2 + e_3) \in h_{14}\) is conjugate in this order to

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Proposition 20. First we assume that

Now we consider the reductive complements $m_a$, $m_b$, $m_c$ in 1) till 3) of Proposition 20. First we assume that $a \neq 0, b \neq 0, c \neq 0$. The vectors $v_{j,l} = (ke_3, \frac{k}{2} ee_3)$, $w_{j,l} = (\sqrt{k^2 - 4\pi^2} e_2 + ke_3, \frac{k}{2} ee_3)$, where $k > 2\pi$ is an integer, are contained in the subspace $m_a$ for $j = 3, l = a$ and $\varepsilon = 1$, in the subspace $m_b$ for $j = 2, l = b$, respectively in $m_c$ for $j = 1, l = c$, where $\varepsilon = 1$ for $g_2 = \mathfrak{sl}_2(\mathbb{R})$ and $\varepsilon = i$ for $g_2 = \mathfrak{so}_3(\mathbb{R})$. According to 1.1 and 1.2 the images of $v_{j,l}, w_{j,l}, j = 1, 2, 3$, under the exponential map have the following representatives in $PSL_2(\mathbb{R}) \times G_2$:

$$m_1 = \exp v_{3,a} = \left( A, \begin{pmatrix} \cos \frac{k}{a} & \frac{k}{a} \sin \frac{k}{a} \\ -\sin \frac{k}{a} & \cos \frac{k}{a} \end{pmatrix} \right),$$

$$m_2 = \exp w_{3,a} = \left( I, \begin{pmatrix} \cos \frac{k}{a} & \frac{k}{a} \sin \frac{k}{a} \\ -\sin \frac{k}{a} & \cos \frac{k}{a} \end{pmatrix} \right),$$

$$m_3 = \exp v_{2,b} = \left( A, \begin{pmatrix} \cosh (\frac{k}{b} \varepsilon) & \sinh (\frac{k}{b} \varepsilon) \\ -\sinh (\frac{k}{b} \varepsilon) & \cosh (\frac{k}{b} \varepsilon) \end{pmatrix} \right),$$

$$m_4 = \exp w_{2,b} = \left( \pm I, \begin{pmatrix} \cosh (\frac{k}{b} \varepsilon) & \sinh (\frac{k}{b} \varepsilon) \\ -\sinh (\frac{k}{b} \varepsilon) & \cosh (\frac{k}{b} \varepsilon) \end{pmatrix} \right),$$

$$m_5 = \exp v_{1,c} = \left( A, \begin{pmatrix} \cosh (\frac{k}{c} \varepsilon) + \sinh (\frac{k}{c} \varepsilon) & 0 \\ 0 & \cosh (\frac{k}{c} \varepsilon) - \sinh (\frac{k}{c} \varepsilon) \end{pmatrix} \right),$$

$$m_6 = \exp w_{1,c} = \left( I, \begin{pmatrix} \cosh (\frac{k}{c} \varepsilon) + \sinh (\frac{k}{c} \varepsilon) & 0 \\ 0 & \cosh (\frac{k}{c} \varepsilon) - \sinh (\frac{k}{c} \varepsilon) \end{pmatrix} \right),$$

where $A = \begin{pmatrix} \cos k & \sin k \\ -\sin k & \cos k \end{pmatrix}$, $\varepsilon = i$ for $g_2 = \mathfrak{so}_3(\mathbb{R})$, whereas $\varepsilon = 1$ for $g_2 = \mathfrak{sl}_2(\mathbb{R})$. For the representatives

$$g_1 = \left( I, \begin{pmatrix} \cos \frac{k}{a} & \frac{k}{a} \sin \frac{k}{a} \\ -\sin \frac{k}{a} & \cos \frac{k}{a} \end{pmatrix} \right),$$

$$g_2 = \left( I, \begin{pmatrix} \cosh (\frac{k}{b} \varepsilon) & \sinh (\frac{k}{b} \varepsilon) \\ -\sinh (\frac{k}{b} \varepsilon) & \cosh (\frac{k}{b} \varepsilon) \end{pmatrix} \right),$$

$$g_3 = \left( I, \begin{pmatrix} \cosh (\frac{k}{c} \varepsilon) + \sinh (\frac{k}{c} \varepsilon) & 0 \\ 0 & \cosh (\frac{k}{c} \varepsilon) - \sinh (\frac{k}{c} \varepsilon) \end{pmatrix} \right)$$

we have $g_1 = m_1 \cdot h_1 = m_2$, $g_2 = m_3 \cdot h_1 = m_4$, $g_3 = m_5 \cdot h_1 = m_6$ such that $h_1 = (A^{-1}, I)$. These facts again contradict Lemma 3.

For $a = 0, b = 0, c = 0$ the complements $m_a, m_b, m_c$ in 1) till 3) of Proposition 20 reduce to $m_0 = \langle (e_1, 0), (e_2, 0), (0, e_1), (0, e_2), (0, e_3) \rangle$. The
exponential image \( \exp m_0 \) is the direct product \( M \times G_2 \), such that \( M \) is the image of the section corresponding to the hyperbolic plane loop \( \mathbb{H}_2 \) (cf. [18], pp. 283-284) and \( G_2 \) is the group \( PSL_2(\mathbb{R}) \) respectively \( SO_3(\mathbb{R}) \) according whether \( \varepsilon = 1 \) or \( \varepsilon = i \). Since \( H \) has the shape \( H_1 \times \{1\} \), where \( H_1 \cong SO_2(\mathbb{R}) \leq PSL_2(\mathbb{R}) \) the global loop \( L_0 \) realized on \( \exp m_0 \) is the direct product of \( \mathbb{H}_2 \) and \( G_2 \).

Now we treat the complements \( m_h, m_k, m_l, h, k, l \in \mathbb{R} \) of the cases 7) till 9) in Proposition 20. The reductive complement \( m_a, a \in \mathbb{R}, m_b, b \in \mathbb{R}, \) respectively \( m_c, c \in \mathbb{R} \) of Lemma 12 in [6], p. 404, is in this order a subspace of \( m_h, m_k, \) respectively \( m_l \). Moreover, the subalgebra \( h_{16} \) in the case 7) coincides with the subalgebra \( h \) in case 1) of Lemma 12 in [6], the subalgebra \( h_{17} \) in the case 8) is equal with the subalgebra \( h \) in case 2) of Lemma 12 in [6], and the subalgebra \( h_{18} \) in the case 9) coincides with the subalgebra \( h \) in case 3) of Lemma 12 in [6], p. 404. Hence the same computations as in the proof of Proposition 13 in [6], pp. 404-406, show that for \( h \neq -1 \) the complement \( m_h \), for \( k \neq 0 \) the complement \( m_k \) and for \( l \notin \{0, -1\} \) the complement \( m_l \) cannot be the tangent space of a global almost differentiable left \( A \)-loop.

It remains to consider the complements \( m_{h=-1}, m_{k=0}, m_{l=0} \) and \( m_{l=-1} \). First let \( \varepsilon = i \). Then the element \( (e_1, e_3) \in h_{16} \) is conjugate to \( (e_2, ie_1) \in m_{h=-1} \), the element \( (e_2 + e_3, e_1) \in h_{17} \) is conjugate to \( (e_2 + e_3, i e_1) \in m_{k=0} \) and the element \( (e_3, e_3) \in h_{18} \) is conjugate to \( (e_3, i e_1) \in m_{l=-1} \) (see 1.2), which are contradictions to Lemma 3. Since the exponential image of the Lie algebra \( h_{18} \) has the shape \( H_n = \{(x, x^n) | x \in SO_2(\mathbb{R}), n \in \mathbb{N}\setminus\{0\}\} \) the exponential image \( M \times SO_3(\mathbb{R}) \) of the complement \( m_{l=0} \), where \( M \) is the image of the section belonging to the hyperbolic plane loop \( \mathbb{H}_2 \) (cf. [18], pp. 283-284), yields Scheerer extensions of \( SO_3(\mathbb{R}) \) by \( \mathbb{H}_2 \) (cf. [18], Section 2).

Finally let \( \varepsilon = 1 \). The complements \( m_{h=-1}, m_{k=0}, m_{l=-1} \) and \( m_{l=0} \) are (up to interchanging the components) equal to the vector space

\[
\mathbf{m}' = \{(e_1, 0), (e_2, 0), (e_3, 0), (0, e_1), (0, e_2)\}
\]

and its exponential image \( \exp \mathbf{m}' \) is the direct product \( PSL_2(\mathbb{R}) \times M \), where \( M \) is the image of the section corresponding to \( \mathbb{H}_2 \). The group

\[
H = \{(\varphi(x), x) | x \in SO_2(\mathbb{R})\}
\]

coincides with the group \( H_{16} \) belonging to \( h_{16} \) respectively with \( H_{17} \) of \( h_{17} \) if \( \varphi \) is a homomorphism from \( SO_2(\mathbb{R}) \) onto a hyperbolic respectively a parabolic 1-parameter subgroup of \( PSL_2(\mathbb{R}) \). The subgroup \( H_{18} \) of \( h_{18} \) has the form:

\[
H_n' = \{(x^n, x) | x \in SO_2(\mathbb{R}), n \in \mathbb{N}\setminus\{0\}\}.
\]

According to [18], Section 2, any loop \( L \) realized on the factor space \( G/H_n \), \( n = 16, 17, 18 \), and having \( \exp \mathbf{m}' \) as the image of its section is a Scheerer extension of the Lie group \( PSL_2(\mathbb{R}) \) by \( \mathbb{H}_2 \).

All Scheerer extensions having \( PSL_2(\mathbb{R}) \times SO_3(\mathbb{R}) \) or \( PSL_2(\mathbb{R}) \times PSL_3(\mathbb{R}) \) as the group topologically generated by their left translations satisfy the Bol identity because of \( [\mathbf{m}, \mathbf{m}], \mathbf{m}] \subset \mathbf{m} \) but they are not Bruck loops since
there is no involutory automorphism $\sigma : g \to g$ such that $\sigma (m) = -m$ and $\sigma (h) = h$.

In the remaining case 10) in Proposition 20 the subgroup $H_1$ of $h_1$ is the direct product $SO_2(\mathbb{R}) \times SO_2(\mathbb{R})$ and the exponential image $M_1$ of $m_1$ is the direct product $M \times M$, where $M$ is the image of the section belonging to $\mathbb{H}_2$. According to Proposition 1.19 in [18], p. 28, the loop $L$ is the direct product $\mathbb{H}_2 \times \mathbb{H}_2$.

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