ON THE HEREDITARY PROXIMITY TO $\ell_1$

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Abstract. In the first part of the paper we present and discuss concepts of local and asymptotic hereditary proximity to $\ell_1$. The second part is devoted to a complete separation of the hereditary local proximity to $\ell_1$ from the asymptotic one. More precisely for every countable ordinal $\xi$ we construct a separable reflexive space $X_\xi$ such that every infinite dimensional subspace of it has Bourgain $\ell_1$-index greater than $\omega^\xi$ and the space itself has no $\ell_1$-spreading model. We also present a reflexive HI space admitting no $\ell_p$ as a spreading model.

1. Introduction

Concepts of proximity to a classical $\ell_p$ (or $c_0$) space play a significant role to the understanding of the structure of a Banach space. They are categorized as follows:

The first one is the global proximity to $\ell_p$ which simply means that $\ell_p$ is isomorphic to a subspace $Y$ of $X$. The local proximity which occurs more frequently, due to J.L. Krivine’s theorem [19], is measured through the Bourgain $\ell_p$-index [10]. The last concept is the asymptotic proximity that varies from A. Brunel-L. Sucheston $\ell_p$-spreading models, [11], to the asymptotic $\ell_p$ spaces. The latter class of Banach spaces appeared after B.S. Tsirelson space [28] that answered in negative the famous Banach’s problem by showing that global proximity to some $\ell_p$ is not always possible.

It is easy to see that the global proximity to $\ell_p$ is the strongest one followed by the asymptotic one. The local proximity is the weakest among them. It is also known that the three classes are separated for each $\ell_p$. Namely there are spaces with arbitrarily large local proximity to $\ell_p$ but no asymptotic one and similarly for the asymptotic and the global ones. The present paper is mainly devoted to the separation of the local and asymptotic proximity to $\ell_1$ when the first one is hereditarily large. In particular our work is motivated by a result of the third named author stated as follows.

Theorem. ([25]) Let $X$ be a separable Banach space and $\xi$ be a countable ordinal. If $X$ is boundedly distortable and has hereditary Bourgain $\ell_1$-index greater than $\omega^\xi$ then it is saturated by asymptotic $\ell^\xi_1$ spaces.

Let’s recall that the hereditary Bourgain $\ell_p$-index of a Banach space $X$ is the minimum of Bourgain $\ell_p$-index of its subspaces. In the sequel by the $\ell_p$-index we will mean the Bourgain $\ell_p$-index.

In view of the above theorem it is natural to ask how critical is the bounded distortion of $X$ for the final conclusion. It is also worth adding that heredity assumptions for the local proximity to $\ell_1$ could yield large asymptotic one. In this direction we prove the following.

Proposition. Let $(e_n)$ be a Schauder basis of a Banach space $X$ such that the Bourgain $\ell_1$-tree supported by any subsequence of $(e_n)_{n \in \mathbb{N}}$ has order greater than $\omega^\xi$. Then there exists a subsequence generating an $\ell^\xi_1$-spreading model.
Our aim is to show that large hereditary \( \ell_1 \)-structure in a Banach space \( X \) does not imply in general any asymptotic one. More precisely the main goal at the present paper is to prove the next

**Theorem.** For every countable ordinal \( \xi \) there exists a separable reflexive space \( X_\xi \) with the hereditary \( \ell_1 \)-index greater than \( \omega^\xi \) such that \( \mathfrak{X}_\xi \) does not admit an \( \ell_1 \)-spreading model. Moreover the dual \( X_\xi^* \) has hereditary \( c_0 \)-index greater than \( \omega^\xi \) and does not admit \( c_0 \) as a spreading model.

Our approach in constructing the space \( X_\xi \) is based on mixed Tsirelson extensions of a ground set \( G_\xi \) using the method of attractors. The latter appeared in [8] and is extensively used in [3]. The method of attractors has two separated steps that we are about to describe.

In the first step an auxiliary space is constructed that partially solves the required problem. In our case for a given countable ordinal \( \xi \) we construct a ground set \( G_\xi \) such that the resulting space \( X_{G_\xi} \) is reflexive, has a Schauder basis \( (e_n)_{n \in \mathbb{N}} \) and satisfies the following properties.

i) The space \( X_{G_\xi} \) does not have an \( \ell_1 \)-spreading model.

ii) For every \( L \in [\mathbb{N}] \) the \( \ell_1 \)-index of the subspace \( \langle (e_n)_{n \in L} \rangle \) is greater than \( \omega^\xi \).

The above stated proposition together with property i) of the space \( X_{G_\xi} \) indicate that property ii) requires special attention. Namely for every subsequence \( (e_n)_{n \in I} \) of the basis the \( \ell_1 \)-index of the corresponding subspace should be greater than \( \omega^\xi \) but the \( \ell_1 \)-tree supported by that subsequence should be of small height. To achieve those requirements we include in the set \( G_\xi \) a \( c_0 \)-tree generated by functionals of the form \( m^{-2n_2-j} \sum_{k \in B} e_k \), \( \#B \leq n_{2j-1} \) and the tree order induced by an appropriate coding function.

The proof that \( X_{G_\xi} \) does not have an \( \ell_1 \)-spreading model heavily relies on combinatorics in particular on Ramsey theory.

The next step is to define the space \( X_\xi \). For this purpose we proceed to a mixed Tsirelson extension \( K_\xi \) of the set \( G_\xi \) including also the attracting functionals. The space \( X_\xi \) is the completion of \( (c_0(\mathbb{N}), \| \cdot \|_{K_\xi}) \) where \( \| \cdot \|_{K_\xi} \) is the norm induced by the set \( K_\xi \).

For our approach the attracting functionals are of significant importance. They are the tool for transferring to every block subspace of \( X_\xi^* \) a large \( c_0 \)-subtree from the set \( G_\xi \). Thus we are able to show that the hereditary \( c_0 \)-index of \( X_\xi^* \) is greater than \( \omega^\xi \). This does not yield that the corresponding \( \ell_1 \)-index of \( X_\xi \) is also greater than \( \omega^\xi \). Therefore we need one more step, namely the desired space \( X_\xi \) is a quotient \( X_\xi^*/X_\xi^L \) where \( X_\xi^L = \langle (e_n)_{n \in L} \rangle \) with \( L \) a suitable subset of \( \mathbb{N} \). Both spaces \( X_\xi, X_\xi^* \) do not have an \( \ell_1 \) spreading model.

We proceed now to describe how the paper is organized.

Section 2 is devoted to preliminary notions and results.

In Section 3 we discuss different concepts of local and asymptotic proximity to \( \ell_1 \). More precisely we introduce the hereditary strategic \( \ell_1 \)-index of a Banach space \( X \) denoted as \( I_{hs}(X) \). This index is formulated in terms of Gowers game and Schreier families \( S_\xi, \xi < \omega_1 \). A non-hereditary version of the aforementioned index, related to the asymptotic structures defined in [21], is also presented. The \( I_{hs}(X) \) is essentially equivalent to the hereditary block \( \ell_1 \)-index, denoted as \( I_{hb}(X) \), in the following manner: First we show that for every countable ordinal \( \xi \) if \( I_{hs}(X) > \xi \) then \( I_{hb}(X) > \omega^\xi \). In the opposite direction Gowers dichotomy [15] yields that if \( I_{hb}(X) > \omega^\xi \) then there exists a closed subspace \( Y \) of \( X \) with \( I_{hs}(Y) > \xi \). Furthermore we examine the relation between the concepts of non-hereditary \( \ell_1 \)-proximity. We finish Section 3 by showing strong correlation between the above notions in a subsequence setting, i.e. with block sequences replaced by subsequences of a fixed basis. The main result in this part is Proposition 3.2 which has been mentioned before.

With Section 4 we start dealing with our final goal namely the space \( X_\xi \). Thus in this section for a given countable ordinal \( \xi \) we define a ground set \( G_\xi \) which serves as
a norming set for the aforementioned auxiliary space $X_{G_{\xi}}$. The set $G_{\xi}$ includes a rich $c_0$-tree of height greater than $\omega^k$. This tree is defined as follows. First $G_{\xi}$ contains all functionals of the form $m_{2j-1}^{2} \sum_{k \in B} e_k^*$ with $\#B \leq m_{2j-1}$ where $(n_j)_{j \in \mathbb{N}}$, $(m_j)_{j \in \mathbb{N}}$ are appropriate increasing sequences of natural numbers. Next using a coding function $\sigma$, in a similar manner as in the classical work of B. Maurey and H. Rosenthal [22], we define a well-founded tree of $\sigma$-special sequences $(f_1, \ldots, f_d)$ ordered by the initial segment inclusion. Each $f_i$ is of the form $m_{2j-1}^{2} \sum_{k \in B_i} e_k^*$, $\#B_i \leq n_{2j-1}$. Then we include into $G_{\xi}$ all $\sum_{i=1}^{d} f_i$ where $(f_i)_{i=1}^{d}$ is a $\sigma$-special sequence.

A second ingredient of $G_{\xi}$ is coming from James tree-like spaces [17] (see also [1], Chapter 13 or [3]). Namely we include all rational $\ell_2$-convex combinations of functionals $\sum_{i=1}^{d} f_i$ with disjoint weights. We denote by $X_{G_{\xi}} = (c_{00}(\mathbb{N}), \|n\|_{G_{\xi}})$ the space with the set $G_{\xi}$ as the norming set.

In Section 6 we present the basic properties of the space $X_{G_{\xi}}$. Namely it is reflexive, has a Schauder basis $(e_n)_{n \in \mathbb{N}}$, is $\ell_2$-saturated and for every $L \subseteq [\mathbb{N}]$ the $\ell_1$-index of the subspace $(\langle e_n \rangle_{n \in L})$ is greater than $\omega^k$. Also we show a dual result. Namely for every $(e_n^*)_{n \in L}$ in the dual $X_{G_{\xi}}^*$ the $c_0$-index of $(\langle e_n^* \rangle_{n \in L})$ is similarly large.

In Section 6 it is shown that the space $X_{G_{\xi}}$ does not have an $\ell_1$-spreading model. This is the most involved part of the study of $X_{G_{\xi}}$. This result is critical as with some small additional effort yields that the ultimate space $X_{\xi}$ shares the same property. The difference of the space $X_{G_{\xi}}$ from the earlier examples of spaces with no $\ell_p$-spreading model, (i.e. [24]) is that $X_{G_{\xi}}$ has a rich local $\ell_1$ structure. Therefore the proof requires new tools which are of combinatorial nature. The main part of the proof is given by Proposition 6.3.

In Section 7 we define the set $K_{\xi}$ which is the norming set of the intermediate space $X_{\xi}$. The ingredients of the set $K_{\xi}$ are the following:

i) The set $G_{\xi}$ is included into $K_{\xi}$. In particular $K_{\xi}$ is a mixed Tsirelson extension of $G_{\xi}$.

ii) For the aforementioned sequences $(m_j)_{j \in \mathbb{N}}$, $(n_j)_{j \in \mathbb{N}}$ the set $K_{\xi}$ is closed under the even operations. Namely it contains all $f = m_{2j}^{-1} \sum_{i=1}^{n_{2j}} f_i$ with $f_1 < \cdots < f_{n_{2j}}$ in $K_{\xi}$.

iii) For the odd operations $(A_{n_{2j-1}}, m_{2j-1})$ $K_{\xi}$ includes the attracting functionals. Those are functionals of the form $f = m_{2j-1}^{-1} \sum_{i=2}^{n_{2j-1}} (f_{2i-1} + e_{i,2}^*)$ with $f_1 < e_{i,2}^* < f_3 < e_{i,4}^* < \cdots$ and the whole sequence is selected with the aid of a coding function.

iv) The set $K_{\xi}$ contains all rational $\ell_2$-convex combinations of its weighted functionals with disjoint weights and also it is a rationally convex set.

The set $K_{\xi}$ induces a norm on $c_{00}(\mathbb{N})$ and the space $X_{\xi}$ is its completion. The attracting functionals are responsible for carrying structure from the set $G_{\xi}$ to block subspaces of $X_{\xi}^*$. To make more transparent the role of the attracting functionals let us first notice, that each attracting functional $f$ consists of two parts. Namely $f = g_1 + g_2$ where $g_1 = m_{2j-1}^{-1} (f_1 + f_3 + \cdots)$ and $g_2 = m_{2j-1}^{-1} (e_{i,2}^* + e_{i,4}^* + \cdots)$. The key result for applying the method of attractors is the following

**Lemma.** For every block subspace $Y$ of $X_{\xi}^*$, for every $j_0 \in \mathbb{N}$ and every $\epsilon > 0$ there exists an attracting functional $f = m_{2j_0}^{-1} \sum_{i=1}^{n_{2j_0}} (f_{2i-1} + e_{i,2}^*)$ such that writing $f = g_1 + g_2$ we have $\text{dist}(g_1, Y) < \epsilon$ and

\begin{equation}
\|g_1\| \geq cm_{2j_0}\tag{1}
\end{equation}

where $c \in (0, 1)$ is a universal constant.

Granting (1) we proceed as follows. We observe that $\|f\| \leq 1$ as $f \in K_{\xi}$ and also (1) yields $\|m_{2j_0}^{-1} g_1\| \geq c$. Therefore the functional $m_{2j_0}^{-1} g_1$ has norm bounded from below by $c$ and also

\begin{equation}
\|m_{2j_0}^{-1} g_1 - (m_{2j_0}^{-1} g_2)\| \leq m_{2j_0}^{-1}\tag{2}
\end{equation}
We recall that \((-m^{-1}_{2^{j_0} - 92}) \in G_\xi\) and in fact is a component of the \(c_0\)-tree structure included in \(G_\xi\). Therefore inequality (2) permit us to transfer into an arbitrary block subspace of \(X_\xi^*\) a \(c_0\)-tree structure of height greater than \(\omega^\xi\). It is worth pointing out that the method of attractors offers a considerable reduction to the complexity of the proofs for properties of the ultimate space. For example in the case of the space \(X_\xi\) the proof that the space does not have an \(\ell_1\)-spreading model is essentially given for the auxiliary space \(X_{G_\xi}\) where the norming set \(G_\xi\) is simpler than the set \(K_\xi\).

Sections 8-9-10 are rather technical and include the necessary estimations for proving the aforementioned inequality 11. This part is closely related to the well known estimations in mixed Tsirelson and Hereditarily Indecomposable spaces. The additional complexity of this part, compared to the previous similar results, arises from property iv of the norming set \(K_\xi\) and the local \(c_0\)-structure of the set \(G_\xi\). A consequence of the new estimations is the following theorem.

**Theorem.** There exists a reflexive HI space \(X\) admitting no \(\ell_p\), \(1 \leq p < \infty\), or \(c_0\) as a spreading model.

This result is presented in Section 11 which also includes a general result concerning spaces with no \(\ell_p\) (or \(c_0\)) as a spreading model (Theorem 11.3). E. Odell and Th. Schlumprecht, [24], have presented the first example of a Banach space with no \(\ell_p\) (or \(c_0\)) as a spreading model. The aforementioned result provides an alternative proof of the latter property of E. Odell and Th. Schlumprecht example and also yields that the space \(X_\xi\) does not have an \(\ell_1\)-spreading model.

In Section 12 we show that the space \(X_\xi^*\) has hereditary \(c_0\)-index greater than \(\omega^\xi\). This result does not allow us to conclude that the hereditary \(\ell_1\)-index of \(X_\xi\) is also greater than \(\omega^\xi\). It is worth pointing out that in general the existence of local or asymptotic \(c_0\)-structure in \(X^*\) yields that the corresponding \(\ell_1\) will occur on \(X\). For example it is easy to see that if \(X^*\) has a \(c_0\)-spreading model then \(X\) will have a corresponding \(\ell_1\). This fact seems not to remain valid for the hereditary local or asymptotic structure as \(X_\xi\) indicates.

In Section 13 we make the final step in defining the space \(X_\xi = X_\xi^*/X_\xi^k\) where \(X_\xi^k = \langle \{e_n\}_{n \in L} \rangle\) with a suitable subset \(L \subset \mathbb{N}\). The space \(X_\xi\) has a Schauder basis and \(I_{hb}(X_\xi) > \omega^\xi\). The space \(X_\xi\) does not have an \(\ell_1\)-spreading model since it is quotient of \(X^*\) which satisfies the same property. Actually both spaces \(X_\xi\) and \(X_\xi^*\) do not have any \(\ell_p\) as a spreading model.

2. Preliminaries

We start by recalling some basic definitions and standard notation. Let \(X\) be a Banach space with a basis \((e_i)\). Given any basic sequence \((x_n)\) by \(\langle (x_n) \rangle\) we denote the closed vector subspace spanned by \((x_n)\). The *support* of a vector \(x = \sum_i x_i e_i\) is the set \(\text{supp} x = \{i \in \mathbb{N} : x_i \neq 0\}\), we define the range \(x \in X\) as the smallest interval in \(\mathbb{N}\) containing support of \(x\). Given any \(x = \sum_i x_i e_i\) and finite \(E \subset \mathbb{N}\) put \(Ex = \sum_{i \in E} x_i e_i\). We write \(x < y\) for vectors \(x, y \in X\), if \(\maxsupp(x) < \minsupp(y)\). A *block sequence* is any sequence \((x_i) \subset X\) satisfying \(x_1 < x_2 < \ldots\), a *block subspace* of \(X\) - any closed subspace spanned by an infinite block sequence. A *tail subspace* of \(X\) is any subspace of the form \(\langle (e_n)_{n \geq n_0} \rangle\) for some \(n_0 \in \mathbb{N}\).

Let \((x_n)_{n \in \mathbb{N}}\) be a sequence and \((e_n)_{n \in \mathbb{N}}\) be a sequence of positive numbers. We say that \((y_n)_{n \in \mathbb{N}}\) is \((e_n)\)-close to \((x_n)_{n \in \mathbb{N}}\) if \(\|x_n - y_n\| < e_n\) for every \(n \in \mathbb{N}\).

Given infinite \(M \subset \mathbb{N}\) by \([M]\) we denote the family of all infinite subsets of \(M\), by \([M]^{< \infty}\) - the family of all finite subsets of \(M\). By \([M]^n, n \in \mathbb{N}\), we denote all finite subsets of \(M\) of cardinality \(n\). A family \(\mathcal{F}\) of finite subsets of \(\mathbb{N}\) is regular, if it is *hereditary*, i.e. for any \(G \subset F, F \in \mathcal{F}\) also \(G \in \mathcal{F}\), spreading, i.e. for any integers...
n_1 < \cdots < n_k and m_1 < \cdots < m_k with n_i \leq m_i, i = 1, \ldots, k, if (n_1, \ldots, n_k) \in \mathcal{F} then also (m_1, \ldots, m_k) \in \mathcal{F}, and compact in the product topology of 2^\mathbb{N}.

The families $\mathcal{A}_n, n \in \mathbb{N}$, are defined by the following formula:

$$\mathcal{A}_n = \{ F \subseteq \mathbb{N} : \#F \leq n \}, \quad n \in \mathbb{N}.$$ 

Define the generalized Schreier families $(\mathcal{S}_\xi)_{\xi < \omega_1}$ of finite subsets of $\mathbb{N}$ by the transfinite induction [2]:

$$\mathcal{S}_0 = \{ \{n\} : n \in \mathbb{N} \} \cup \{\emptyset\}$$

$$\mathcal{S}_{\xi+1} = \mathcal{S}_1(\mathcal{S}_\xi) = \{ F_1 \cup \cdots \cup F_m : m \in \mathbb{N}, F_1, \ldots, F_m \in \mathcal{S}_\xi, m \leq F_1 < \cdots < F_m \}$$

for any $\xi < \omega_1$. If $\xi$ is a limit ordinal, choose $\xi_n \not\in \xi$ and set

$$\mathcal{S}_\xi = \{ F : F \in \mathcal{S}_{\xi_n} \text{ and } n \leq F \text{ for some } n \in \mathbb{N} \}.$$ 

It is well known that the families $\mathcal{A}_n, n \in \mathbb{N}, S_\xi, \xi < \omega_1$, are regular families of finite subsets of $\mathbb{N}$.

Let us recall that a set $F \in \mathcal{S}_\xi$ is called a maximal set if there is no $G \in \mathcal{S}_\xi$ such that $F \subseteq G$. In [14] it is proved that $F$ is $\mathcal{S}_\xi$-maximal if and only if there is no $k \in \mathbb{N}$ with $F < k$ and $F \cup \{k\} \in \mathcal{S}_\xi$.

**Definition 2.1.** Let $\mathcal{F}$ be one of the families $\mathcal{A}_n, n \in \mathbb{N}, S_\xi, \xi < \omega_1$, and $\theta \in (0,1)$.

1) A finite sequence $(f_1, \ldots, f_k)$ in $c_00(\mathbb{N})$ is said to be $\mathcal{F}$-admissible if

$$\text{supp}(f_1) < \cdots < \text{supp}(f_k) \text{ and } \{\min(f_1), \ldots, \min(f_k)\} \in \mathcal{F}$$

2) The $(\mathcal{F}, \theta)$-operation on $c_00(\mathbb{N})$ is the operation which assigns to each $\mathcal{F}$-admissible sequence $f_1 < \cdots < f_d$ the vector $\theta(f_1 + \cdots + f_d)$.

Throughout this paper by a tree on a set $X$ we mean a subset $\mathcal{T}$ of $\bigcup_{n=1}^{\infty} X^n$ such that $(x_1, \ldots, x_k) \in \mathcal{T}$ whenever $(x_1, \ldots, x_k, x_{k+1}) \in \mathcal{T}$, $k \in \mathbb{N}$, ordered by the initial segment inclusion. A tree $\mathcal{T}$ is well-founded, if there is no infinite sequence $(x_i) \subset X$ with $(x_1, \ldots, x_k) \in \mathcal{T}$ for any $k \in \mathbb{N}$. Given a tree $\mathcal{T}$ on $X$ put

$$D(\mathcal{T}) = \{(x_1, \ldots, x_k) : (x_1, \ldots, x_k, x) \in \mathcal{T} \text{ for some } x \in X\}.$$ 

Inductively define trees $D^\alpha(\mathcal{T})$: $D^0(\mathcal{T}) = \mathcal{T}$, $D^{\alpha+1}(\mathcal{T}) = D(D^\alpha(\mathcal{T}))$ for $\alpha$ ordinal and $D^\alpha(\mathcal{T}) = \bigcap_{\beta<\alpha} D^\beta(\mathcal{T})$ for $\alpha$ limit ordinal. The order of a well-founded tree $\mathcal{T}$ is given by $o(\mathcal{T}) = \inf\{\alpha : D^\alpha(\mathcal{T}) = \emptyset\}$.

Let $\mathcal{F}$ be a countable family of finite subset of $\mathbb{N}$ endowed with the topology of the pointwise topology. For $\alpha < \omega_1$, we set $\mathcal{F}^{\alpha+1} = \{ F \in \mathcal{F} : F$ a limit point of $\mathcal{F}^\alpha \}$ and for $\alpha$ limit ordinal $\mathcal{F}^\alpha = \bigcap_{\beta<\alpha} \mathcal{F}^\beta$. The Cantor-Bendixson index of $\mathcal{F}$, denoted by $CB(\mathcal{F})$, is defined as the least $\alpha$ for which $\mathcal{F}^\alpha = \emptyset$.

Let $\mathcal{T}$ be a countable tree on $\mathbb{N}$. Then $\mathcal{T}$ defines the family

$$\mathcal{F}_\mathcal{T} = \{ F \in [\mathbb{N}]^{<\omega} : \text{there exists } t \in \mathcal{T} \text{ with } F \text{ a subset of the range of } t \}.$$ 

It follows that the family $\mathcal{F}_\mathcal{T}$ is hereditary. Conversely given $\mathcal{F}$ a countable family of finite subsets of $\mathbb{N}$ with each $F \in \mathcal{F}$ we associate the finite strictly increasing sequence $t_F$ of integers with range equal to the set $F$. We set

$$\mathcal{T}_\mathcal{F} = \{ t \in \bigcup_n [\mathbb{N}]^{\infty} : \text{there exists } F \in \mathcal{F} \text{ such that } t \text{ is initial segment of } t_F \}.$$ 

From the above definitions it follows that $F \in \mathcal{F}_1$ if and only if the corresponding node in the tree has infinitely many immediate successors. It follows $CB(\mathcal{F}) \leq o(\mathcal{T}_\mathcal{F})$. In the case of the Schreier families (c.f [2]) it follows that

$$CB(\mathcal{S}_\xi) = o(T_{\mathcal{S}_\xi}) = \omega^\xi + 1.$$ 

For unexplained notions and notations we refer the reader to [20].
3. Concepts of proximity to $\ell_1$

In this section we introduce the hereditary strategic $\ell_1$-index (Def. 3.3) and show that it is essentially equivalent to the notion of the hereditary Bourgain $\ell_1$-index (Prop. 3.4).

We discuss also non-hereditary concepts of proximity to $\ell_1$ and show strong relation between these notions in the sequence setting (Prop. 3.7).

**Definition 3.1.** Let $X$ be a Banach space with a basis.
A tree $T$ on $X$ is an $\ell_1$-tree on $X$ with constant $C \geq 1$, if any $(x_1, \ldots, x_k) \in T$ is a normalized sequence $C$-equivalent to the unit vector basis of $\ell_1^k$. Let

$$I(X,C) = \sup \{ o(T) : T - \ell_1$-tree on $X$ with constant $C \}, \quad C \geq 1$$

The (Bourgain) $\ell_1$-index of $X$ is defined by $I(X) = \sup \{ I(X,C) : C \geq 1 \}$.

The block (Bourgain) $\ell_1$-index $I_b(X)$, is defined analogously, using $\ell_1$-trees consisting of block sequences.

It follows by [10] that for a separable Banach space $X$ the $\ell_1$-index is a countable ordinal if and only if $X$ does not contain $\ell_1$. In this case the $\ell_1$-index is of the form $\omega^\xi$ for some $\xi < \omega_1$ and also it is greater than $\ell_1$. The same holds for the block $\ell_1$-index [13]. Recall that if $I(X) \geq \omega^\omega$, then $I(X) = I_b(X)$, if $I(X) = \omega^{n+1}$, then $I_b(X) = \omega^n$ or $I_b(X) = \omega^{n+1}$ [13]. For more information on the block $\ell_1$-index see [18].

The hereditary (Bourgain) $\ell_1$-index is defined as $I_h(X) = \min \{ I(Y) : Y$ is subspace of $X \}$. The hereditary block (Bourgain) $\ell_1$-index, denoted as $I_{hb}(X)$, is defined similarly taking block Bourgain $\ell_1$-indices of block subspaces.

Next we introduce the following notion.

**Definition 3.2.** Let $X$ be a Banach space with basis, $\xi < \omega_1$, $C \geq 1$ and consider an $S_\xi$-game between the players $S$ and $V$ defined as follows:

- in the $i$-th move player $S$ chooses a block subspace $X_i$ of $X$ and player $V$ picks a normalized block vector $x_i \in X_i$.

We say that player $V$ wins, if the resulting sequence $(x_i)_i^{=k}$ is $C$-equivalent to the standard basis of $\ell_1^k$ and maximal $S_\xi$-admissible (i.e. \{\text{minsupp } x_i : i = 1, \ldots, k\} is $S_\xi$-maximal). We say that $V$ has a winning strategy, if $V$ wins the game for any possible choice of the player $S$.

**Definition 3.3.** Let $X$ be a Banach space with a basis. We define the hereditary strategic $\ell_1$-index of $X$ by the formula

$$I_h(X) = \sup \{ \xi < \omega_1 : \text{ for any } \xi < \xi \text{ there is } C \geq 1 \text{ such that } V \text{ has a winning strategy in } S_\xi - \text{game with constant } C \}.$$

If we consider the game above where $S$ chooses only tail subspaces instead of arbitrary block subspaces, then we define the strategic $\ell_1$-index of $X$ denoted by $I_b(X)$.

In non-hereditary case for $\xi = 1$ we obtain Definition 2.1. [21].

The next result describes the relations between the above introduced notions.

**Proposition 3.4.** Let $X$ be a Banach space with a basis, $\xi < \omega_1$.

If $I_h(X) > \xi$ then $I_{hb}(X) > \omega^\xi$.

If $I_h(X) > \omega^\xi$ then $I_h(Y_0) > \xi$ for some block subspace $Y_0$ of $X$.

**Proof.** We can assume that $X$ does not contain a copy of $\ell_1$. Notice that for any $\xi < \omega_1$ we have $I_h(X) > \xi$ if and only if player $V$ has a winning strategy in $S_\xi$-game with some constant $C \geq 1$.

Assume first that player $V$ has a winning strategy in the $S_\xi$-game (with some constant $C$). Fix a block subspace $Y$. Then player $V$ in particular has a winning strategy in producing maximal $S_\xi$-admissible sequences $C$-equivalent to the unit vector basis of $\ell_1$.
in the $S^Y_\ell$-game, in which player $S$ chooses at each step some $m_k \in \mathbb{N}$ and $V$ chooses $x_k \in Y$ with $x_k > m_k$ (i.e. player $S$ chooses only tail subspaces of $Y$).

Take the tree $T$ of all block sequences produced by player $V$ in all $S^Y_\ell$-games for all block subspaces $Y$, with all possible moves of player $S$, i.e. all block sequences $(x_1, \ldots, x_k)$ produced by player $V$ at some point in all $S^Y_\ell$-games according to his winning strategy. It follows that $T$ is an $\ell_1$-tree with a constant $C$. We show now that $o(T) \geq o(S^Y_\ell) = \omega^k + 1$ which implies that $I_h(Y) > \omega^k$.

For any $(m_1, \ldots, m_k) \in S^Y_\ell$ by $T_{m_1, \ldots, m_k}$ denote the set of all block sequences of length $k$ produced by $V$ according to his winning strategy in all $S^Y_\ell$-games, where $S$ has chosen in his first $k$ moves $m_1, \ldots, m_k$.

By induction we show that for any $\alpha < \omega_1$ we have

$$D^\alpha(T) \supset \cup\{T_{m_1, \ldots, m_k} : (m_1, \ldots, m_k) \in D^\alpha(S^Y_\ell)\}.$$ 

Indeed, if $\alpha = \beta + 1$ (in particular if $\beta = 0$), then by the inductive assumption (or by the definition of $T$ in case of $\beta = 0$) we have

$$D(D^\beta T) \supset \cup\{D(T_{m_1, \ldots, m_k}) : (m_1, \ldots, m_k) \in D^\beta(S^Y_\ell)\} = \cup\{T_{m_1, \ldots, m_k-1} : (m_1, \ldots, m_k) \in D^\beta(S^Y_\ell)\} = \cup\{T_{m_1, \ldots, m_1} : (m_1, \ldots, m_1) \in D^\alpha(S^Y_\ell)\}$$

For any limit $\alpha < \omega_1$ by the inductive assumption we have

$$\cap_{\beta < \alpha} D^\beta T \supset \cap_{\beta < \alpha} \cup\{T_{m_1, \ldots, m_k} : (m_1, \ldots, m_k) \in D^\beta(S^Y_\ell)\}$$

$$\supset \cup\{T_{m_1, \ldots, m_k} : (m_1, \ldots, m_k) \in D^\alpha(S^Y_\ell)\}.$$ 

It follows that $o(T) \geq o(S^Y_\ell)$, which ends the proof of the first implication.

Assume now that for any block subspace $Y$ of $X$ we have $I_h(Y) > \omega^k$. We will show now that in any block subspace $Y$ there is a block subspace $Y_0$ and $C \geq 1$ such that in any subspace $W \subset Y_0$ there is a maximal $S^Y_\ell$-admissible block sequence $(x_1, \ldots, x_k)$ $C$-equivalent to the unit vector basis of $\ell_1^k$. Then by Gowers dichotomy for games for families of finite block sequences [13], in some block subspace of $Y$ player $V$ has a winning strategy for producing maximal $S^Y_\ell$-admissible block sequences $2C$-equivalent to unit vector basis of suitable finite dimensional spaces $\ell_1$, which will prove that $I_h(X) > \ell_1$.

First notice that there is some block subspace $Y_0$ and universal constant $C$ such that for any block subspace $S$ of $Y_0$ there is an $\ell_1$-tree $T_S$ on $Z$ with constant $C$ and $o(T_S) \geq \omega^k + 1$. Since we deal with $\ell_1$-trees, we can assume that they are hereditary trees, i.e. for any $\ell_1$-tree $R$, any $(x_i)_{i \in F} \in R$ and any $G \subset F$ we have also $(x_i)_{i \in G} \in R$.

Indeed, otherwise we can produce a decreasing sequence of block subspaces $(Y_n)$ such that in each $Y_n$ there is no $\ell_1$-tree with constant $n$ and order greater than $\omega^k + 1$. Then the diagonal subspace does not contain any $\ell_1$-tree of order greater than $\omega^k + 1$, a contradiction with the assumption, since the block $\ell_1$-index of a Banach space is a limit ordinal.

Take $Y_0$ and the constant $C$ as above, for any block subspace $Z$ of $Y_0$ pick an $\ell_1$-tree $T_Z$ with constant $C$ and $o(T_Z) \geq \omega^k + 1$. Let $T = \bigcup\{T_Z : Z \subset Y_0\}$. Then $T$ is also an hereditary $\ell_1$-tree (with constant $C$).

Pick any block subspace $W = ((w_n)_n) \subset Y_0$. Let $L = \{\minsupp w_n : n \in \mathbb{N}\}$ and

$$F = \{\minsupp x_1, \ldots, \minsupp x_k \subset L : (x_1, \ldots, x_k) \in T \cap W^k, \ k \in \mathbb{N}\}$$

Notice that the tree $F$ is hereditary and well-founded. Indeed, assume that there is $(n_k)_{k \in \mathbb{N}}$ such that $(n_1, \ldots, n_k) \in F$ for any $k \in \mathbb{N}$. Thus for any $k \in \mathbb{N}$ there is $(x_1^k, \ldots, x_k^k) \in T$ with $n_1 \leq x_1^k < n_2 \leq x_2^k < \cdots < n_k \leq x_k^k$. By compactness argument we can assume that for some block sequence $(x_i)$ we have $x_i^k \rightarrow x_i$ in $X$ as $k \rightarrow \infty$. Since each sequence $(x_i^1, \ldots, x_i^k)$ is $C$-equivalent to the unit vector basis of $\ell_1^k$, thus $(x_i)_{i \in \mathbb{N}}$ is equivalent to the unit vector basis of $\ell_1$, a contradiction with the assumption from the beginning of the proof.
By I. Gasparis’ dichotomy [14] there is some infinite \( M \subset L \) such that either \( S_\xi \cap [M]^{-\infty} \subset F \) or \( F \cap [M]^{-\infty} \subset S_\xi \).

We show that the first case holds by the following

**Claim.** Given any tree \( R \) on \( X \) let
\[
F_R = \{(\text{minsupp} x_1, \ldots, \text{minsupp} x_k) : (x_1, \ldots, x_k) \in R\}
\]
Then if trees \( R \) and \( F_R \) are well-founded it follows that \( o(R) = o(F_R) \).

To prove the claim it is enough to show by induction that \( F_{D^\alpha(R)} \subset D^\alpha(F_R) \) for any \( \alpha < \omega_1 \).

Let now \( Z = \langle (w_n)_{\text{minsupp} w_n \in M} \rangle \). By definition of \( T \) and \( F \) we have that \( F \cap [M]^{-\infty} \subset F_T \), therefore by the above Claim
\[
o(F \cap [M]^{-\infty}) \geq o(T_Z) > \omega^\xi + 1 = o(S_\xi)
\]
hence \( S_\xi \cap [M]^{-\infty} \subset F \) must hold.

Take now any \((n_1, \ldots, n_k)\) maximal in \( S_\xi \cap [M]^{-\infty} \subset F \) and the corresponding \((x_1, \ldots, x_k) \in T \cap W^k \) with \( \text{minsupp} x_i = n_i, i = 1, \ldots, k \). It is clear that \((x_1, \ldots, x_k)\) is maximal \( S_\xi \)-admissible. Therefore we picked in \( W \) a maximal \( S_\xi \)-admissible block sequence \( C \)-equivalent to the unit vector basis of some finite dimensional \( \ell_1 \), which by previous remarks ends the proof. \( \square \)

**Remark 3.5.** The non-hereditary case the above proposition takes the following form for a Banach space \( X \) with a basis:

If \( I_s(X) > \xi \), then \( I_b(X) > \omega^\xi \).

If \( I_b(\langle e_n \rangle_{n \in M}) > \omega^\xi \) for any \( M \in [N] \), then \( I_s(X) > \xi \).

The proof goes along the same scheme as above, with passing to subspaces spanned by subsequences of the basis instead of block subspaces.

A stronger representation of \( \ell_1 \) in a Banach space is described by the following notions:

**Definition 3.6.** Let \( X \) be a Banach space with a basis and \( \xi < \omega_1 \) be a countable ordinal.

A normalized basic sequence \((x_i)_{i \in N} \subset X\) generates an \( \ell_1^\xi \)-spreading model, \( \xi < \omega_1 \), with constant \( C \geq 1 \), if for any \( F \in S_\xi \) the sequence \((x_i)_{i \in F} \) is \( C \)-equivalent to the unit vector basis of \( \ell_1^\xi \).

We say that \((x_i)\) generates an \( \ell_1 \)-spreading model if the above property holds for \( \xi = 1 \).

The space \( X \) is \( \ell_1^\xi \)-asymptotic, if any \( S_\xi \)-admissible sequence \((x_i)_{i=1}^n \) is \( C \)-equivalent to the unit vector basis of \( \ell_1^\xi \).

The space \( X \) is said to be \( \ell_1 \)-asymptotic if the above property holds for \( \xi = 1 \).

The asymptotic \( \ell_1 \) spaces were introduced in [23], \( \ell_1 \)-spreading models of higher order were studied in [6, 4].

Replacing \( \ell_1 \) by \( c_0 \) in the above definitions we obtain the (Bourgain) \( c_0 \)-index, a \( c_0^\xi \)-spreading model and an asymptotic \( c_0 \) space.

The structures described in the above definitions give us an hierarchy of the representation of \( \ell_1 \) in a Banach space \( X \) with a basis, not containing \( \ell_1 \). For a countable ordinal \( \xi < \omega_1 \) we consider the following four “local” structures:

A) The space \( X \) is \( \ell_1^\xi \) asymptotic.
B) The space \( X \) contains a sequence generating an \( \ell_1^\xi \)-spreading model.
C) The strategic \( \ell_1 \)-index of \( X \) is greater than \( \xi \).
D) The block \( \ell_1 \)-index of \( X \) is greater than \( \omega^\xi \).

Let us observe that the following implications hold for the above structures
\[
A) \Rightarrow B) \Rightarrow C) \Rightarrow D)
\]
The reverse implications are not true.
First observe that for every $\xi < \omega_1$ the Tsirelson spaces $T[S_\xi, \theta]$, $\theta \in (0, 1)$, are examples of reflexive Banach space which are $\ell_1^{\xi}$-asymptotic however they do not contain $\ell_1$.

The Schreier space $X_\xi$, $\xi < \omega_1$, is an example of Banach space with basis for which every subsequence of the basis generates an $\ell_1$-spreading model and it does not contain an asymptotic $\ell_1^\xi$ subspace for any $\zeta \leq \xi$ since, as it is well known, $X_\xi$ is $c_0$-saturated.

In the next section of the paper for every $\xi < \omega_1$ we provide an example of Banach space $X_G\xi$ with strategic $\ell_1$-index greater than $\xi$ yet $X_G\xi$ does not contain sequence generating an $\ell_1$-spreading model. More precisely the space $X_G\xi$ has basis such that in the game, where $S$ chooses subspaces spanned by subsequences of the basis, player $V$ has a winning strategy. Therefore if a space $X$ has strategic $\ell_1$-index greater than $\xi$, then it does not follow that $X$ contains a sequences generating an $\ell_1$-spreading model.

A richer asymptotic $\ell_1^\xi$ structure than the one described in property C) would be provided by a winning strategy of player $V$ in the following modification of the $S_\xi$-game: player $S$ chooses subsequences of the basis instead of block subspaces and player $V$ picks vectors from subsequences chosen by player $S$. In such a case every subsequence of the basis would admit $\ell_1$-tree (i.e. formed by elements of this subsequence) of order $\omega^\xi$. Let us observe the following

**Proposition 3.7.** Let $(e_n)$ be a basis of $X$ such that any subsequence $(e_n)_{n \in L}$ admit an $\ell_1$-tree $T_L \subset \{(e_n)_{n \in F} : F \in \{L\}^{<\infty}\}$ of order $\omega^\xi$, $\xi < \omega_1$. Then some subsequence of $(e_n)_{n \in \mathbb{N}}$ generates an $\ell_1^\xi$-spreading model.

**Proof.** Assume first that any subsequence $(e_n)_{n \in L}$ of the basis admits an $\ell_1$-tree of order greater than $\omega^\xi + 1$.

Repeating the reasoning and notation from the proof of Prop. 5.4 we can assume that there is some subsequence $(e_n)_{n \in L_0}$ and an hereditary $\ell_1$-tree $T$ formed by elements of this subsequence such that for any $M \subset L_0$ the order of $F_T \cap [M]^{<\infty}$ is greater than $\omega^\xi + 1$.

As before by I.Gasparis' dichotomy [13] pick an infinite $M \subset \mathbb{N}$ such that either $S_\xi \cap [M]^{<\infty} \subset F_T$ or $F_T \cap [M]^{<\infty} \subset S_\xi$. By the above remark the first case holds. Notice that if $M = (m_i)_i$ then for any $F \in S_\xi$ we have $(m_i)_{i \in F} \in S_\xi \cap [M]^{<\infty} \subset F_T$, which implies that $(e_n)_{n \in M}$ generates an $\ell_1^\xi$-spreading model.

Now notice that if $(e_n)$ admits an $\ell_1$-tree of order $\omega^\xi$, then it also admit an $\ell_1$-tree of order greater than $\omega^\xi + 1$ (maybe with worse constant).

It follows by repeating the reasoning in Lemma 5.7 [13] in case of the block $\ell_1$-index defined not by using all block $\ell_1$-trees but only $\ell_1$-trees consisting of finite subsequences of $(e_n)$. Therefore we can reduce the case where all subsequences admit $\ell_1$-trees of order $\omega^\xi$ to the case of order greater than $\omega^\xi + 1$, which was treated above and hence we finish the proof of the observation. \hfill $\Box$

We end this section with the following observation regarding the non-hereditary indices. Assuming only $I_b(X) > \omega^\xi$ does not imply $I_s(X) > 1$, as it shown by the following example.

Let $X$ be an $\ell_2$-direct sum of $\ell_1^n$'s, i.e. $X = (\bigoplus_{n=1}^{\infty} \ell_1^n)_{\ell_2}$.

It is clear that $I_b(X) > \omega$, since $\sup\{a(T) : T \text{\ is } \ell_1\text{-tree with constant 1}\} = \omega$.

On the other hand $I_s(X) = 1$. Indeed, consider $S_1$-game with player $S$ choosing tail subspaces and let in the first move player $S$ pick a tail subspace $(\bigoplus_{n=1}^{\infty} \ell_1^n)_{\ell_2}$. In the $(i+1)$-th move player $S$ chooses tail subspaces after the support of vector $x_i$ picked by $V$ in the $i$-th move, i.e. if in the $i$-th move player $V$ picks a vector $x_i \in (\bigoplus_{n=1}^{\infty} \ell_1^n)_{\ell_2}$, then in the $(i+1)$-th move player $S$ chooses tail subspace $(\bigoplus_{n=n_i+1}^{\infty} \ell_1^n)_{\ell_2}$.

Therefore player $S$ forces the sequence $(x_1, \ldots, x_k)$ produced by player $V$ to be 1-equivalent to the unit vector basis of $\ell_2$, hence there is no universal constant $C$ for which
player $V$ has a winning strategy for producing $S_T$-admissible block sequences $C$-equivalent to the unit vector basis of suitable finite dimensional $\ell_1$.

4. Definition of the ground set $G_\xi$

In this section we define for any $\xi < \omega_1$ ground sets $G_\xi'$ and $G_\xi$ for the auxiliary spaces $Y_{G_{\xi'}}$ and $X_{G_{\xi'}}$ respectively, in particular we introduce the tree of $G_{\xi'}$-special functionals, defined with the use of special coding function $\sigma_1$, and related notions. We show that the space $Y_{G_{\xi'}}$ is $c_0$-saturated (Prop. 11).

We recall that a subset $G$ of $c_0(\mathbb{N})$ is said to be a ground set if:

1. $G$ is symmetric and $\{e_n^*: n \in \mathbb{N}\}$ is contained in $G$.
2. $\|\phi\|_\infty \leq 1$ and $\phi(n) \in \mathbb{Q}$ for $\phi \in G$.
3. $G$ is closed under the restriction of its elements to intervals of $\mathbb{N}$.

A ground set $G$ induces a norm $\|\cdot\|_G$ in $c_0(\mathbb{N})$ defined by

$$\|x\|_G = \sup\{\phi(x) : \phi \in G\} \quad \text{and we set } X_G = (c_0(\mathbb{N}), \|\cdot\|_G).$$

It is easy to see that the natural basis of $c_0(\mathbb{N})$ is a Schauder basis of the space $X_G$. In the opposite for every Banach space with a Schauder basis $(x_n)_{n \in \mathbb{N}}$ there exists a ground set $G$ such that the natural correspondence $e_n \rightarrow x_n$ is extended to an isomorphism between $X_G$ and $X$.

We pass now to define the ground set $G_\xi$. Fix $\xi < \omega_1$. We choose two strictly increasing sequences $(n_j)_j$, $(m_j)_j$ of positive integers, such that

1. $m_1 = 2^5$ and $m_{j+1} = m_j^3$
2. $n_1 = 2^6$ and $n_{j+1} = (2n_j)^{s_j}$ where $2^{s_j} = m_{j+1}^3$.

Let us observe, for later use, that $260m_{2j}^4 \leq n_{2j-1}$ for $j \geq 2$. Set $G_0 = \{\pm e_n^* : n \in \mathbb{N}\}$

$$G_1 = \left\{ \frac{1}{m_{2j-1}^2} \sum_{i \in E} f_i : f_i \in G_0, (f_i)_{i \in E}, \# E \leq n_{2j-1} \right\}, j \in \mathbb{N}.$$

Finally we set $G_1 = \cup_{j \in \mathbb{N}} G_1^j$.

**Notation 4.1.** For every $f \in G_1^j$ we define the weight of $f$ as $\omega(f) = m_{2j-1}^2$ and the index as $\text{ind}(f) = j$. The elements of $G_1$ are called functionals of type I.

We consider the following set

$$\mathcal{U} = \{(f_1, \ldots, f_d) : f_1 < \cdots < f_d, f_i \in G_1^j, j_i < j_{i+1} \text{ for all } i < d \in \mathbb{N}\}.$$ 

Let $\mathbb{N} = M_1 \cup M_2$ where $M_1, M_2$ are infinite disjoint sets. Let also $\sigma_1 : \mathcal{U} \rightarrow M_2$ be a 1-1 function (i.e. a coding function) such that $\sigma_1(f_1, \ldots, f_{i+1}) > \sigma_1(f_1, \ldots, f_i)$ for every $i \in \mathbb{N}$.

**Definition 4.2.** A special sequence is an element $(f_1, \ldots, f_d)$ of $\mathcal{U}$ satisfying the following:

1. $(\text{minsupp } f_i)_{i=1}^d \in S_T$.
2. $f_i \in G_1^j, j_i \in M_1, f_{i+1} \in G_1^j(f_1, \ldots, f_i)$ for every $i = 1, \ldots, d-1$.

We define

$$G_{sp} = \left\{ E \sum_{i=1}^d \epsilon_i f_i : (f_1, \ldots, f_d) \text{ is a special sequence, } \epsilon_i \in \{-1, 1\}, E \text{ interval of } \mathbb{N} \right\}$$

For any element $\phi = E \sum_{i=1}^d \epsilon_i f_i$ of $G_{sp}$ we set $\text{ind}(\phi) = \{\text{ind}(f_i) : Ef_i \neq \emptyset, i \leq d\}$. We define also

$$G_{\ell_2} = \left\{ \sum_{i=1}^d a_i \phi_i : d \in \mathbb{N}, \sum_{i=1}^d a_i^2 \leq 1, (\phi_i)_{i=1}^d \subset G_{sp} \cup G_1, (\text{ind } \phi_i)_{i=1}^d \text{ pairwise disjoint} \right\}.$$


Definition 4.3. The ground set $G_\xi$ is defined to be the set

$$G_\xi = G_0 \cup G_1 \cup G_{sp} \cup G_{\ell_2}.$$ 

Remarks 4.4. 1) The set $G_\xi$ is symmetric and closed under the restriction of its elements on intervals of $\mathbb{N}$.

2) The injectivity of the coding function $\sigma_1$ yields that the set of the special sequences has a tree structure i.e. if $(f_1, \ldots, f_d)$, $(g_1, \ldots, g_n)$ are two special sequences then either $f_i \neq g_j$ for all $i,j$ or there exists $i_0 \leq \min\{d,n\}$ such that $f_i = g_i$ for all $i < i_0$ and $f_i \neq g_j$ for all $i,j \geq i_0$, in particular $w(f_i) \neq w(g_j)$ for all $i,j > i_0$.

We shall call every special sequence also a segment of the tree of the special sequences. The elements of $G_{sp}$ are called $G_\xi$–special functionals.

Notation 4.5. For every segment $s = (f_1, \ldots, f_d)$ of the tree of the special sequences we set $F(s) = \{\sum_{i=1}^d \epsilon_i f_i : \epsilon_i \in \{-1,1\}\}$ and for $f \in F(s)$ we set $\text{ind}(f) = \text{ind}(s)$.

If $V = \{s_1, \ldots, s_d\}$ is a set of segments we set

$$\nabla = \left\{ \sum_{i=1}^d \lambda_i \phi_i : \sum_{i=1}^d \lambda_i^2 \leq 1, \phi_i \in F(s_i) \text{ and } \text{ind}(\phi_i) \cap \text{ind}(\phi_j) = \emptyset \text{ for all } 1 \leq i \neq j \leq d \right\}$$

It readily follows that $\nabla$ is symmetric.

We set $G'_{\xi} = G_0 \cup G_1 \cup G_{sp} \cup \{0\}$.

Lemma 4.6. The set $G'_{\xi}$ is a closed subset of $[0,1]^{<\omega}$ in the pointwise topology.

Proof. Let $(\phi_n)_{n \in \mathbb{N}}$ be a sequence of elements of $G'_{\xi}$ such that $\lim_n \phi_n = \phi$ pointwise. It is clear that if $\phi_n \in G_0$ for infinitely many $n$’s then $\phi \in G_0 \cup \{0\}$.

Also if $\phi_n \in G_1$ for all but finitely many $n$’s, then if $\{\phi_n : n \in \mathbb{N}\} \cap G'_1 \neq \emptyset$ for infinitely many $j$’s then $\phi = 0$. Otherwise there exists $j_0$ such that $\phi_n \in G_{\ell_2}^{j_0}$ for all but finitely many $n$’s. Since $A_{n_{j_0}-1}$ is regular we get $\phi \in G_{\ell_2}^{j_0}$.

Finally assume that for all but finitely many $n$’s there exists a segment $s_n = (f_{n1}^1, \ldots, f_{nd}^n)$ of the tree of the special sequences such that $\phi_n = E_n \sum_{i=1}^{d_n} \epsilon_i' f_i^n \in F(s_n)$.

If $\sup_n E_n = +\infty$, then $\phi = 0$. Otherwise we can assume that for all but finitely many $n$’s we have $E_n = [k,K)$, for some $k \in \mathbb{N}$ and $K \in \mathbb{N} \cup \{+\infty\}$.

Since the family $S_\xi$ is regular we get $\{m_i\}_{i=1}^d \in S_\xi$ such that

$$m_1 \leq f^n_1 \leq m_2 \leq \cdots \leq m_d \leq f^n_d \quad \text{(and minsupp} f^n_{d+1} \rightarrow \infty)$$

Let $i_0 = \min\{i \leq d : \text{there is no } n \in \mathbb{N} \text{ such that } f^n_i = f_{i_0}^n \text{ for all } m \geq n\}$. By the reasoning above in case $(\phi_n) \subseteq G_1$ we can assume that for any $i = 1, \ldots, d$ we have $f^n_i \rightarrow f_i$ for some $f_i \in G_1 \cup \{0\}$. Also we may assume that $\epsilon^n_i = \epsilon_i$ for all $i \leq d$ and $n \in \mathbb{N}$.

If $\limsup_n w(f^n_{i_0}) = +\infty$ then $f_{i_0} = 0$. Also since $\sigma_1$ is $1-1$ we get $\limsup_n w(f^n_{i_0+j}) = +\infty$ and therefore $f_{i_0+j} = 0$ for all $j = 1, \ldots, d - i_0$.

If $w(f^n_{i_0}) = m^2_{2j_0-1}$ for all but finitely many $n$’s, since the set $G_{\ell_2}^{j_0}$ is compact we get that $f^n_{i_0} \rightarrow f_{i_0} \in G_{\ell_2}^{j_0}$. Again since $\sigma_1$ is $1-1$ we get $\limsup_n w(f^n_{i_0+j}) = +\infty$ and therefore $f_{i_0+j} = 0$ for all $j = 1, \ldots, d - i_0$.

It follows that $(f_1, \ldots, f_d)$ is a special sequence and therefore $\sum_{i=1}^d \epsilon_i f_i \rightarrow \sum_{i=1}^d \epsilon_i f_i$ is a $G_\xi$–special functional. \qed

We define $X_{G_\xi} = (c_0(\mathbb{N}), \|\cdot\|_{G_\xi})$ and $Y_{G_\xi} = (c_0(\mathbb{N}), \|\cdot\|_{G'_{\xi}})$.

Let us observe that since the sets $G_\xi, G'_{\xi}$ are symmetric and closed under projections on intervals it follows that $(e_n)_{n \in \mathbb{N}}$ is a bimonotone basis for the spaces $X_{G_\xi}$ and $Y_{G_\xi}$.

Proposition 4.7. $Y_{G_\xi}$ is $c_0$-saturated.
Proof. The set \( G_\xi \) is countable by the construction and compact by Lemma 1.10. Since \( G_\xi \) is norming set of \( Y_{G_\xi} \) we conclude that \( Y_{G_\xi} \) is isometric to a subspace of \( C(G_\xi') \) and hence is \( c_0 \)-saturated, see [26] (see also [5] and [1], Theorem 4.5).

We end this section with the following observations regarding the norming set \( G_\xi \).

From the definition of the norming set \( G_\xi \) it follows that for any finite sequence \( (\phi_i) \subset G_1 \cup G_{sp} \) with pairwise disjoint index sets and any \( (a_i) \in \ell_2 \) we have \( \| \sum_i a_i \phi_i \|^*_{G_\xi} \leq (\sum_i a_i^2)^{1/2} \). Therefore for any infinite sequence \( (\phi_i)_i \subset G_1 \cup G_{sp} \) with pairwise disjoint index sets and any \( (a_i)_i \in \ell_2 \) the series \( \sum_i a_i \phi_i \) is convergent in norm and

\[
\| \sum_{i=1}^\infty a_i \phi_i \|^*_{G_\xi} \leq (\sum_{i=1}^\infty a_i^2)^{1/2}.
\]

Notice also that since the set \( G_\xi \) is a norming set for \( X_{G_\xi} \) we obtain that \( B_{X_{G_\xi}} = \overline{\text{conv}(G_\xi)^{w^*}} \). Also it is not hard to show that \( G_\xi^{w^*} = G_\xi \cup F \) where

\[
F = \{ \sum_{i=1}^\infty a_i \phi_i : (a_i)_{i \in \mathbb{N}} \in B_{\ell_2}, \phi_i \in G_1 \cup G_{sp}, \text{ ind}(\phi_i) \cap \text{ind}(\phi_j) = \emptyset \text{ for all } i \neq j \}.
\]

We omit the proof since we shall not make use of this result.

5. Basic Properties of the Space \( X_{G_\xi} \)

In this section we shall prove that the space \( X_{G_\xi} \) has the following properties

1) It is reflexive.

2) For every subsequence \( (e_n)_{n \in M} \) of the basis the subspace \( \langle (e_n)_{n \in M} \rangle \) has \( \ell_1 \)-index greater than \( \omega^k \).

3) Every subspace of \( X_{G_\xi} \) contains \( \ell_2 \).

In order to prove the theorem we shall need the following definition

Definition 5.1. Let \( (x_n)_n \) be a bounded block sequence in \( X_{G_\xi} \) and \( \epsilon > 0 \). We say that \( (x_n)_n \) is \( \epsilon \)-separated if for every \( \phi \in G_1 = \bigcup_{j \in \mathbb{N}} G_j' \)

\[
\# \{ n : \| \phi(x_n) \| \geq \epsilon \} \leq 1.
\]

In addition, we say that \( (x_n)_n \) is separated if for every \( L \in [\mathbb{N}] \) and \( \epsilon > 0 \) there exists an \( M \in [L] \) such that \( (x_n)_{n \in M} \) is \( \epsilon \)-separated.

Concerning the separated sequences the following holds:

Proposition 5.2. Let \( (x_n)_{n \in \mathbb{N}} \) be a separated sequence in \( X_{G_\xi} \) with \( \| x_n \|_{G_\xi} \leq 1 \). Then for all \( m \in \mathbb{N} \) there is \( L \in [\mathbb{N}] \) such that for all \( g \in \ell_2 \)

\[
\# \{ n \in L : \| g(x_n) \| \geq m^{-1} \} \leq 65m^2.
\]

The proof of the above results follows the arguments of Lemma 3.12 and Proposition 3.14 in [3] where we refer for the proofs.

Lemma 5.3. Let \( (x_n)_{n \in \mathbb{N}} \) be a bounded block sequence. Setting \( y_n = \frac{1}{n} \sum_{i \in F_n} x_i \) where \( \# F_n = n \) and \( F_n < F_{n+1} \) we get that \( (y_n)_{n \in \mathbb{N}} \) is separated.

Proof. Let \( \epsilon > 0 \), \( L \in [\mathbb{N}] \) and assume that \( \| y_n \|_{G_\xi} \leq C \) for any \( n \). Pick inductively sequences \( (j_i), (l_i) \subset L \) such that

1) \( cm^2_j \geq 1 \) and 2) \( \epsilon l_i > C m_{j-1} \).

Take now any \( \phi \in G_j' \) and \( l_i \). If \( j \geq j_i \), then by (1) \( |\phi(y_{l_i})| < \epsilon \). If \( j < j_i-1 \), then by (2) \( |\phi(y_{l_i})| < \epsilon \). Therefore \( \# \{ i : |\phi(y_{l_i})| \geq \epsilon \} \leq 1 \) and hence setting \( M = (l_i)_i \), we end the proof that \( (y_l)_{l \in M} \) is \( \epsilon \)-separated.

\( \square \)
Combining the above lemma with Proposition 5.4 and the choice of \((m_j), (n_j)\) we obtain the following.

**Proposition 5.4.** Let \((x_n)_{n \in \mathbb{N}}\) be a bounded block sequence. Assume that \(y_n = \frac{1}{n} \sum_{i \in F_n} x_i\), where \(\# F_n = n\) and \(F_n < F_{n+1}\), satisfy \(\|y_n\| \leq 1\). Then for any \(j \geq 2\) there is an infinite \(L \subseteq \mathbb{N}\) such that for any \(g \in G_\xi\)

\[
\# \{n \in L : |g(y_n)| \geq 2m_{2j}^{-2}\} \leq n_{2j-1}
\]

As corollary of the above proposition we obtain that the basis is shrinking.

**Corollary 5.5.** Every bounded block sequence in \(X_{G_\xi}\) is weakly null.

**Proof.** Assume that there exist a normalized block sequence \((x_n)_{n \in \mathbb{N}}, x^* \in X_{G_\xi}\) of norm one and \(\epsilon > 0\) such that \(x^*(x_n) > \epsilon\) for all \(n\).

Let \(j \in \mathbb{N}\), \(j \geq 2\) such that \(1/2m_{2j}^{-2} < \epsilon/4\). By Proposition 5.4 setting \(y_n = \frac{1}{\# F_n} \sum_{i \in F_n} x_i\), where \(\# F_n = n\) and \(F_n < F_{n+1}\) for all \(n \in \mathbb{N}\), we may assume that for all \(g \in G_\xi\) it holds that

\[
\# \{n : |g(y_n)| \geq 2m_{2j}^{-2}\} \leq n_{2j-1} \Rightarrow |g(\frac{1}{n_{2j}} \sum_{i=1}^{n_{2j}} y_n)| \leq \frac{n_{2j-1}}{n_{2j}} + \frac{1}{2m_{2j}^{-2}} \leq \frac{1}{m_{2j}^{-2}} < \frac{\epsilon}{2}
\]

This yields a contradiction since \(G_\xi\) is a norming set for \(X_{G_\xi}\). \(\square\)

We prove now the reflexivity of \(X_{G_\xi}\).

**Theorem 5.6.** The space \(X_{G_\xi}\) is reflexive.

**Proof.** We show that the basis is shrinking and boundedly complete. Corollary 5.5 yields that the basis \((e_n)_{n \in \mathbb{N}}\) is shrinking. We prove that \((e_n)_{n \in \mathbb{N}}\) is also boundedly complete.

On the contrary assume that \(\sup_{n \in \mathbb{N}} \|\sum_{i=1}^{n} a_i e_i\|_{G_\xi} \leq 1\) and there exist \(\epsilon_0 > 0\), successive intervals \(F_1 < F_2 < \ldots\) of \(N\) such that \(\|\sum_{i \in F_n} a_i e_i\|_{G_\xi} > \epsilon_0\).

For every \(n\) choose \(g_n \in G_\xi\) such that \(g_n(\sum_{i \in F_n} a_i e_i) \geq \epsilon_0\) and range \(g_n \subset F_n\).

We distinguish the following cases

**Case 1.** \(g_n \in G_0\) for infinitely many \(n\).

Then for \(j \in \mathbb{N}\) and \(M > \max F_{n_{2j-1}}\) and we get

\[
\|\sum_{i=1}^{M} a_i e_i\|_{G_\xi} \geq \frac{1}{m_{2j-1}^{-2}} \sum_{n=1}^{n_{2j-1}} \frac{M}{n_{2j}} g_n(\sum_{i=1}^{M} a_i e_i) \geq \frac{\epsilon_0 n_{2j-1}}{m_{2j-1}^{-2}},
\]

a contradiction for large \(j\).

We state the next three cases.

**Case 2.** \(g_n \in G_1\) for all but finitely many \(n\).

**Case 3.** \(g_n \in G_{sp}\) for all but finitely many \(n\).

**Case 4.** \(g_n \in G_{\xi}\) for all but finitely many \(n\).

The proofs of these three cases follow the same argument hence we shall give only the proof of **Case 4.**

If we have that infinitely many \(g_n\)’s have pairwise disjoint index sets then for suitable \(n \in \mathbb{N}\) and \(A \subseteq A_n\) the functional \(g = \frac{1}{n} \sum_{i \in A} g_i \in G_{\xi}\) will give us a contradiction.

Assume that only finitely many \(g_n\)’s have pairwise disjoint index sets.

Let \(g_n = \sum_{i \in D_{n,1}} c_i \phi_i\) where \(\{\phi_i\}_{i \in D_{n,1}}\) have pairwise disjoint index sets. For every \(j \in \mathbb{N}\) we set \(\phi_{i,j} = \phi_{i,j}^1 + \phi_{i,j}^2\) where \(\text{ind}(\phi_{i,j}^1) \subseteq \{1, \ldots, j\}\) and \(\text{ind}(\phi_{i,j}^2) \subseteq \{j + 1, j + 2, \ldots\}\).

Then \(g_n = g_{n,1}^j + g_{n,2}^j = \sum_{i \in D_{n,1}} c_i \phi_{i,j}^1 + \sum_{i \in D_{n,2}} c_i \phi_{i,j}^2\). Let us observe that \(\# D_{n,1} \leq j\).

We distinguish the following two subcases.

**Subcase 4a.** There exists \(j_0 \in \mathbb{N}\) such that for all but finitely many \(n\),

\[
|g_{n,j_0}^1(\sum_{i \in F_n} a_i e_i)| \geq \frac{\epsilon_0}{2}.
\]
Since \( \text{ind}(\phi^1_{i,j}) \subset \{1, \ldots, j_0\} \) it follows that \( \# \supp \phi^1_{i,j_0} \leq \sum_{i \leq j_0} n_{2i-1} = n_0 \) and
\[
|\phi^1_{i,j_0}(\sum_{i \in F_n} a_i e_i)| \leq n_0 \|\phi^1_{i,j_0}\|_\infty \max_{i \in F_n} |a_i| \leq n_0 m^1_{i,j_0} \max_{i \in F_n} |a_i|.
\]
It follows
\[
e_0/2 \leq \sum_{i \in D_{n,1}} c_i |\phi^1_{i,j_0}(\sum_{i \in F_n} a_i e_i)| \leq \sum_{i \in D_{n,1}} |c_i| n_0 m^1_{i,j_0} \max_{i \in F_n} |a_i| \leq j_0 n_0 m^1_{i,j_0} \max_{i \in F_n} |a_i|.
\]
From the above relation as in Case 1 we derive a contradiction.

Subcase 4b. For every \( j, m \in \mathbb{N} \) there exists \( n \in \mathbb{N}, n > m \) with
\[
|g^1_{n,j}(\sum_{i \in F_n} a_i e_i)| \leq e_0/2.
\]
Then we choose inductively an increasing sequence \((n_i)_{i \in \mathbb{N}}\) such that
\[
|g^1_{n_i,j_{i-1}}(\sum_{i \in F_{n_i}} a_i e_i)| \leq e_0/2 \text{ where } j_{i-1} = \max\{\text{ind}(g_{n_k}) : k \leq i - 1\}.
\]
It follows that \( |g^2_{n_i,j_{i-1}}(\sum_{i \in F_{n_i}} a_i e_i)| \geq e_0/2 \) and the functionals \( g^2_{n_i,j_{i-1}} \) have pairwise disjoint index sets.

Setting \( g = \frac{1}{\sqrt{n}} \sum_{i \in A} g^2_{n_i,j_{i-1}} \) for suitable \( n \) and \( A \in A_n \) we derive a contradiction. \( \square \)

From Theorem 5.6 and Proposition 4.7 we obtain the following

**Corollary 5.7.** The identity map \( I_d : X_{G_\xi} \to Y_{G_\xi} \) is strictly singular.

We prove now that every subsequence of the basis generates a subspace with \( \ell_1 \)-index greater that \( \omega^\xi \).

**Proposition 5.8.** For every \( M \in [\mathbb{N}] \) the subspace \( \langle (e_n)_{n \in M} \rangle \) has \( \ell_1 \)-index greater than \( \omega^\xi \).

**Proof.** It is not hard to see that for every \( j \in \mathbb{N} \) the following holds
\[
\frac{1}{m^2_{j-1}} \leq \frac{1}{n_{2j-1}} \sum_{i=1}^{n_{2j-1}} e_k \|G_\xi\| \leq \frac{2}{m^2_{j-1}}.
\]
Let \( (f_1, \ldots, f_d) \) be a special sequence where \( f_i = \frac{1}{m^2_{j_i-1}} \sum_{k \in F_i} e_k^\star \) with \( \#F = n_{2j_i-1} \) and \( F \subset M \).

For every \( i \leq d \) take the vector \( x_i = \frac{m^2_{j_i-1}}{n_{2j_i-1}} \sum_{k \in F_i} e_k \). It follows that \( 1 \leq \|x_i\|_{G_\xi} \leq 2 \) and
\[
\|\sum_{i=1}^d a_i x_i\|_{G_\xi} \geq \sum_{i=1}^d c_i f_i(\sum_{i=1}^d a_i x_i) = \sum_{i=1}^d |a_i|.
\]
Since for every \( F \in S_\xi \) we have a special sequence \((f_i)_{i \in F} \) it follows that the subspace \( \langle (e_n)_{n \in M} \rangle \) contains an \( \ell_1 \)-tree with constant 1 of order \( \omega^\xi \). It follows that the \( \ell_1 \)-index of the subspace \( \langle (e_n)_{n \in M} \rangle \) is greater than \( \omega^\xi \). \( \square \)

From (4) it follows that for every \( M \in [\mathbb{N}] \) the \( c_0 \)-index of the subspace generated by the subsequence \( (e_n^\star)_{n \in M} \) of the basis of \( X_{G_\xi}^* \) is greater than \( \omega^\xi \). Indeed, take an \( G_\xi \)-special sequence \((f_i^\star)_{i=1}^d \) supported in the set \( M \) such that each \( f_i = m^2_{j_i-1} \sum_{k \in B} e_k^\star \) with \( \#B = n_{2j_i-1} \). From (4) we get \( \|f_i\| \geq 1/2 \) for all \( i \leq d \) and hence
\[
1/2 \leq \|\sum_{i=1}^d \pm f_i\| \leq 1.
\]
From the above inequality easily follows that the \( c_0 \)-index of \( \langle (e_n^\star)_{n \in M} \rangle \) is greater than \( \omega^\xi \).

We show now the the space \( X_{G_\xi} \) is \( \ell_2 \)-saturated.
Theorem 5.9. For any subspace $Y$ of $X_{C_1}$ and any $\epsilon > 0$ there is a subspace of $Y$ which is $(1+\epsilon)$ isomorphic to $\ell_2$.

The proof of the theorem follows the arguments of Lemma B.13 and Theorem B.14 from [3] and we omit it.

6. The space $X_{C_1}$ does not have $\ell_1$ as a spreading model.

In this section we prove that $X_{C_1}$ does not contain a sequence generating an $\ell_1$-spreading model. The basic tool for the proof is Proposition 6.3 which is a combinatorial result. We begin with the following lemmas.

Lemma 6.1. For any $x \in c_00[N]$ and $\epsilon > 0$ there is $j_0 = j_0(x, \epsilon) \in N$ such that for any $g \in G_{\ell_2}$ with $\text{ind}(g) \cap \{1, \ldots, j_0\} = \emptyset$ we have $|g(x)| < \epsilon$.

Proof. Let $D = ||x||_{\ell_1}$ and take $j_0$ so that $\sum_{j=j_0+1}^{\infty} m_{2j-1}^{-4} < (\epsilon/D)^2$. Take now any $g \in G_{\ell_2}$, $g = \sum_i a_i \phi_i$ with $\text{ind}(g) \cap \{1, \ldots, j_0\} = \emptyset$ and compute

$$|g(x)| \leq (\sum_i a_i^2)^{1/2} (\sum_i |\phi_i(x)|^2)^{1/2} \leq (\sum_i |\phi_i|^2_\infty^2 \|x\|^2_1)^{1/2}$$

$$\leq D (\sum_{j=j_0+1}^{\infty} m_{2j-1}^{-4})^{1/2} < \epsilon.$$

Lemma 6.2. Let $(x_n)_{n \in N}$ be a normalized block sequence and $\epsilon > 0$. There exist $j_1 \in N$ and $L \in [N]$ such that for every $f$ of type I with $\text{ind}(f) \geq j_1$ it holds that $|f(x_n)| > \epsilon$ for at most one $n \in L$.

Proof. We set $D_1 = \{(n, m) : \text{exists } f \text{ of type I with } |f(x_n)| \geq \epsilon \text{ and } |f(x_m)| \geq \epsilon\}$.

If there exists $L \in [N]$ with $[L]^2 \cap D = \emptyset$ the proof is complete. Otherwise by Ramsey theorem there exists $L_1$ such that $[L_1]^2 \subset D$. Let $l_1 = \min L_1$ and let $j_1$ be such that $|f(x_{l_1})| < \epsilon$ for all $f$ of type I with $\text{ind}(f) \geq j_1$.

We set

$$D_2 = \{(n, m) \in [L_1]^2 : \text{exists } f \text{ of type I with } \text{ind}(f) \geq j_1, |f(x_n)| \geq \epsilon \text{ and } |f(x_m)| \geq \epsilon\}.$$ 

If there exists $L_2 \in [L_1]$ with $[L_2]^2 \cap D_2 = \emptyset$ the proof is complete. Otherwise by Ramsey theorem there exists $L_2$ such that $[L_2]^2 \subset D_2$. Let $l_2 = \min L_2 > l_1$ and let $j_2$ be such that $|f(x_{l_2})| < \epsilon$ for all $f$ of type I with $\text{ind}(f) \geq j_2$.

If the conclusion does not hold, choosing $k_0$ such the $\epsilon \sqrt{k_0} > 1$, following the above arguments we get $L_1 \supset L_2 \supset \cdots \supset L_{k_0+1}$ such that $[L_k]^2 \subset D_k$, where

$$D_1 = \{(n, m) \in [L_{k_0+1}]^2 : \text{exists } f \text{ of type I with } \text{ind}(f) \geq j_{k_0+1}, |f(x_n)| \geq \epsilon \text{ and } |f(x_m)| \geq \epsilon\}$$

and $|f(x_{\min L_k})| < \epsilon$ for all $f$ of type I with $\text{ind}(f) \geq j_k$.

Setting $l_i = \min L_i$ we get $(l_i, l_{k_0+1}) \in D_i$ for every $i = 1, \ldots, k_0$ and therefore there exists $f_i$ of type I with $\text{ind}(f_i) \in [j_{i-1}, j_i)$ such that $f_i(x_{l_{k_0+1}}) \geq \epsilon$.

It follows

$$\frac{1}{\sqrt{k_0}} \sum_{i=1}^{k_0} \epsilon_i f_i(x_{l_{k_0+1}}) \geq \epsilon \sqrt{k_0} > 1,$$ 

a contradiction.

Proposition 6.3. Let $0 < \epsilon < 10^{-10}$ and $(x_n)_{n \in N}$ be a normalized block basis such that for all $n_1 < n_2 < n_3$ there exists $\phi \in G_{\ell_2}$ such that

$$|\phi(x_{n_1})| \geq 1 - \epsilon \text{ for all } i=1,2,3.$$ 

Then there exists $j_1 \in N$ and $L \in [N]$ such that for all $n \in L$ there exists $\phi_n = \sum_{i \in B_n} \lambda_i n \phi_{n,i} \text{ with } \text{ind}(\phi_{n,i}) \in \{1, \ldots, j_1\}$ for all $i \in B_n$ and $\sum_{i \in B_n} \lambda_i^2 \leq 1$ such that

$$\phi_n(x_n) \geq 0.75.$$
Proof. From Lemma 6.1 we get \( j_0 \in \mathbb{N} \) such that
\[
(6) \quad \forall \phi = \sum_{i=1}^{d} \lambda_i \phi_i \text{ with ind}(\phi_i) \cap \{1, \ldots, j_0\} = \emptyset \text{ and } \sum_{i \leq d} \lambda_i^2 \leq 1 \Rightarrow |\phi(x_1)| < \epsilon
\]
Let \( \delta > 0 \) such that \( \delta j_0 < \epsilon \). From Lemma 6.2 we get \( M \in [\mathbb{N}] \) and \( j_1 \in \mathbb{N} \) such that
\[
(7) \quad \text{for every } f \text{ of type I with } \text{ind}(f) > j_1, \ |f(x_n)| > \delta \text{ for at most one } n \in M.
\]
Let \( M_1 = \{1\} \cup M \). After reordering we may assume that \( M_1 = \mathbb{N} \).

Let \( n \in \mathbb{N} \) and for every \( 1 < k < n \) consider the triple \((1, k, n) \in [\mathbb{N}]^3\). Let \( \phi_{k,n} = \sum_{s \in S_{k,n}} c_s \phi_s \in G_{k,2} \), where \( \phi_s \in F(s) \), be the functional we obtain from (8) for \( x_1, x_k \) and \( x_n \). We set
\[
J_{k,n} = \{ s \in S_{k,n} : \text{ind}(\phi_s) > j_0 \} \quad \text{and} \quad G_{k,n} \text{ the complement of } J_k.
\]
From (8) we get \( |\sum_{s \in J_{k,n}} c_s \phi_s(x_1)| < 10^{-10} \) and therefore
\[
(8) \quad \left( \sum_{s \in G_{k,n}} c_s^2 \right)^{1/2} \geq |\sum_{s \in G_{k,n}} c_s \phi_s(x_1)| > 1 - 2/10^5.
\]
The disjointness of the index sets of the segments \( s \in S_{k,n} \) yields that \#\( G_{k,n} \leq j_0 \) for every \( k \).

By (8) we get \( \sum_{s \in G_{k,n}} c_s^2 \geq (1 - 2/10^5)^2 \) and therefore
\[
(9) \quad \sum_{s \in J_{k,n}} c_s^2 \leq 1 - (1 - 2/10^5)^2 < 4/10^5 \quad \Rightarrow \quad \| \sum_{s \in J_{k,n}} c_s \phi_s \|_{G_{k,2}}^* \leq 2/10^5.
\]
From (10) and (11) it follows that for \( r = k, n \)
\[
|\sum_{s \in J_{k,n}} c_s \phi_s(x_r)| \geq 1 - 10^{-10} - 2/10^5 > 1 - 3/10^5.
\]
For every \( s \in G_{k,n} \), if \( \phi_s = \sum_{i \in E_s} \epsilon_i f_{s,i} \) let \( f_{s,k,n} \) be its first node with the property \( \text{range}(f_{s,i}) \cap \text{range}(x_n) \neq \emptyset \). Set
\[
G_{k,n}^0 = \{ s \in G_{k,n} : \text{ind}(f_{s,k,n}) \leq j_1 \},
\]
\[
G_{k,n}^1 = \{ s \in G_{k,n} : \text{ind}(f_{s,k,n}) > j_1 \text{ and } |f_{s,k,n}(x_n)| \leq \delta \},
\]
\[
G_{k,n}^2 = \{ s \in G_{k,n} \setminus G_{k,n}^1 : \text{ind}(f_{s,k,n}) > j_1 \text{ and } |f_{s,k,n}(x_n)| \leq \delta \}.
\]
From (7) it follows that for every \( s \not\in (G_{k,n}^0 \cup G_{k,n}^1) \) it holds that \( |f_{s,k,n}(x_k)| \leq \delta \) and hence the sets \( G_{k,n}^i, i = 0, 1, 2 \) give a partition of \( G_{k,n} \).

For every \( 1 < k < n \) we set
\[
(11) \quad H_{k,n} = \begin{cases} 
G_{k,n}^0 & \text{if } |\sum_{s \in G_{k,n}^0} c_s \phi_s(x_k)| \geq 0.75 \ (**) . \\
G_{k,n}^1 & \text{if } (***) \text{ fails and } \sum_{s \in G_{k,n}^1} c_s^2 \geq \sum_{s \in G_{k,n}^2} c_s^2 . \\
G_{k,n}^2 & \text{otherwise}.
\end{cases}
\]
By Ramsey theorem passing to an infinite subset \( N \) of \( \mathbb{N} \) we may assume that there exists \( i \in \{0, 1, 2\} \) such that for all \( 1 < k < n \in N \), \( H_{k,n} = G_{k,n}^i \). Without loss of generality we may assume that \( N = \mathbb{N} \).

If \( H_{k,n} = G_{k,n}^0 \) for all \( k < n \) then the proof is complete. Next we show that that the cases \( H_{k,n} = G_{k,n}^1 \) or \( H_{k,n} = G_{k,n}^2 \) are not possible.

Claim. Assume that either \( H_{k,n} = G_{k,n}^1 \) for all \( k < n \) or \( H_{k,n} = G_{k,n}^2 \) for all \( k < n \).

Then there exists \( N \in \mathbb{N} \) such that for all \( n \in \mathbb{N} \) there exists a family \( U_n \) of segments such that:
1. \( \#U_n \leq N \).
2. For every \( 1 < k < n \), \( \sup \{ \phi(x_k) : \phi \in U_n \} \geq \delta \).
Let’s see first how the conclusion of the claim makes impossible \( H_{k,n} = G_{k,n}^1 \) or \( H_{k,n} = G_{k,n}^2 \).

Let \( U_n = \{ s_1, \ldots, s_{d_n, n} \} \) with \( d_n \leq N \) for every \( n > 1 \). From 2) we obtain that for every \( 1 < k < n \) there exist \( \phi_k \in U_n \) such that
\[
\phi_k(x_k) \geq \delta.
\]

By the compactness of the set \( G_{k,n}^l \) passing to an infinite subset of \( \mathbb{N} \) we get that there exist finite segments \( s_1, \ldots, s_N \) such that \( w^* - \lim_{i \to \infty} s_i, n = s_i \) for every \( i \leq N \).

Let \( k \in \mathbb{N} \) such that \( \text{supp}(x_k) > \max \{ \text{maxsupp}(s_i) : i \leq N \} \). Then since \( w^* - \lim_{i \to \infty} s_i, n = s_i \) we obtain that there exists \( n_0 > k \) such that \( \text{supp}(s, n_0) \cap \text{supp}(x_k) = \emptyset \) for all \( i \leq N \), a contradiction.

We proceed now to the proof of the Claim.

Assume that \( H_{k,n} \neq G_{k,n}^0 \). It follows that \( |\sum_{s \in G_{k,n}^0} c_s \phi_s(x_k)| < 0.75 \) and from \((10)\) we get
\[
| \sum_{s \in G_{k,n}^1 \setminus G_{k,n}^0} c_s \phi_s(x_k) | \geq 1 - 3/10^5 - 0.75 = 0.25 - 3/10^5.
\]

As in \((8) - (9)\) we get that
\[
2 \sum_{s \in H_{k,n}} c_s^2 \geq \sum_{s \in G_{k,n}^1 \setminus G_{k,n}^0} c_s^2 \geq (0.25 - 3/10^5)^2 \Rightarrow \sum_{s \in H_{k,n}} c_s^2 > 0.031.
\]

From \((13)\) we obtain \( \sum_{s \in G_{k,n}^1 \setminus H_{k,n}} c_s^2 \leq 0.97 \Rightarrow \| \sum_{s \in G_{k,n}^1 \setminus H_{k,n}} c_s \phi_s \|_{G_{k,n}^l} < 0.985 \) and therefore for \( l = k, n \) the following holds:
\[
| \sum_{s \in H_{k,n}} c_s \phi_s(x_i) | \geq | \sum_{s \in G_{k,n}^1 \setminus H_{k,n}} c_s \phi_s(x_i) | - | \sum_{s \in G_{k,n}^1 \setminus H_{k,n}} c_s \phi_s(x_i) |
\]
\[
\geq 1 - 3/10^5 - 0.985 = 0.0149.
\]

Assume now that \( H_{k,n} = G_{k,n}^1 \) for all \( k < n \).

For every \( s \in G_{k,n}^1 \) we set \( \tilde{s} = s_{\lfloor 1, \text{maxsupp}(f_{s,k,n}) \rfloor} \).

Using finite induction we define sets \( U_2, n \subset U_{3, n} \subset \cdots \subset U_{n-1, n} \) as follows:

Set \( U_{2, n} = \{ \tilde{s} : s \in G_{1, n}^1 \} \). Let \( k = 3, \ldots, n - 1 \) and assume that the set \( U_{k-1, n} \) has been defined.

If there exists \( \phi = \sum_i c_i \phi_i \in U_{k-1, n} \), see Notation \( 4.5 \) with \( |\phi(x_k)| \geq \delta \) we set \( U_{k, n} = U_{k-1, n} \).

Assume that
\[
| \sum_{s \in H_{k,n}} c_s \phi_s(x_k) | \geq \sum_{s \in E_{k,n}} c_s \phi_s(x_k) \text{ it holds that } |\phi(x_k)| < \delta.
\]

For the functional \( \phi_{k,n} = \sum_{s \in G_{k,n}^1} c_s \phi_s \) we set
\[
E_{k,n} = \{ s \in G_{k,n}^1 : \tilde{s} \in U_{k-1, n} \}.
\]

Notice that the segments \( \tilde{s}, s \) in \( G_{k,n}^1 \setminus E_{k,n} \), are different from the segments in \( U_{k-1, n} \) and therefore the functionals \( \phi_{s}^1 = \phi_{s} |_{\text{maxsupp}(f_{s,k,n})}, +\infty) \), \( s \in G_{k,n}^1 \setminus E_{k,n} \), have disjoint index sets from the functionals in \( U_{k-1, n} \).

From \((13)\) we get
\[
| \sum_{s \in E_{k,n}} c_s \phi_s(x_k) | = | \sum_{s \in E_{k,n}} c_s \phi_s(x_k) | < \delta.
\]

Combining the above inequality with \((10)\) we get
\[
( \sum_{s \in G_{k,n}^1 \setminus E_{k,n}} c_s^2 )^{1/2} \geq | \sum_{s \in G_{k,n}^1 \setminus E_{k,n}} c_s \phi_s(x_k) | > 1 - 3/10^5 - \delta > 1 - 4/10^5.
\]
while from (14)
\[
| \sum_{s \in G_{k,n}^1 \setminus E_{k,n}} c_s \phi_s(x_k) | > 0.0149 - \delta > 0.0148.
\]

From (17) following the arguments we used to get (19) it follows
\[
(\sum_{s \in E_{k,n}} c_s^2)^{1/2} \leq (1 - (1 - 4/10^5)^2)^{1/2} \leq 0.009.
\]

From the definition of the set $H_{k,n}$, using that $|f_{x,k,n}(x_n)| < \delta$, and (19) we obtain
\[
| \sum_{s \in G_{k,n}^1 \setminus E_{k,n}} c_s \phi_s(x_n) | \geq \sum_{s \in G_{k,n}^1 \setminus E_{k,n}} c_s \phi_s(x_n) - \sum_{s \in G_{k,n}^1 \setminus E_{k,n}} |c_s f_{x,k,n}(x_n)| - \sum_{s \in E_{k,n}} c_s \phi_s(x_n) \geq 0.0149 - j_0 \delta - 0.009 > 0.005.
\]

Set $U_{k,n} = U_{k-1,n} \cup \{ \tilde{s} : s \in G_{k,n}^1 \setminus E_{k,n} \}$.

Note that in this case $\#(U_{k,n} \setminus U_{k-1,n}) \leq j_0$ and from (18) we obtain that for the functional $\phi = \sum_{s \in G_{k,n}^1 \setminus E_{k,n}} c_s \phi_s$ in $U_{k,n}$ it holds that $\phi(x_k) \geq 0.0148$.

Let $U_{n-1,n}$ be the set of the segments we get after we complete the above procedure for $k = 2, \ldots, n - 1$.

We show now that there is no $n$ with $\#U_{n-1,n} > j_0 (2 + 1/\delta_0)$, $\delta_0 = 0.005$.

Indeed assume that there exists $n$ such that $\#U_{n-1,n} > j_0 (2 + 1/\delta_0)$. Since $\#U_{2,n} \leq j_0$ and $\#(U_{k,n} \setminus U_{k-1,n}) \leq j_0$ it follows that $U_{k-1,n} \subseteq U_{k,n}$ holds for at least $t = 1 + 1/\delta_0$ different $k$. For every such $k$ we get a functional $\phi^1_k = \sum_{s \in G_{k,n}^2 \setminus E_{k,n}} c_s \phi^1_s$ such that $\phi^1_k(x_n) > \delta_0$ and the functionals $\phi^1_k$ have pairwise disjoint index sets. It follows that
\[
\frac{1}{\sqrt{t}} \sum_k \phi^1_k(x_n) \geq \sqrt{t} \delta_0 > 1,
\]
a contradiction.

**Case 2.** $H_{k,n} = G_{k,n}^2$ for every $1 < k < n$.

For every $s \in G_{k,n}^2$ we set $\tilde{s} = s_{1, \minsupp f_{x,k,n}}$.

Using finite induction we define sets $U_{2,n} \subset U_{3,n} \subset \cdots \subset U_{n-1,n}$ as follows:

Set $U_{2,n} = \{ \tilde{s} : s \in G_{2,n}^2 \}$. Let $k = 3, \ldots, n - 1$ and assume that $U_{k-1,n}$ has been defined.

If there exists $\phi \in U_{k-1,n}$ such that $|\phi(x_k)| > \delta$ we set $U_{k,n} = U_{k-1,n}$.

Assume that
\[
\text{for all } \phi = \sum_{i} c_i \phi_i \text{ in } U_{k-1,n} \text{ it holds that } |\phi(x_k)| < \delta.
\]

Set
\[
E_{k,n} = \{ s \in G_{k,n}^2 : \tilde{s} \in U_{k-1,n} \}.
\]

We get that the segments $\tilde{s}, s \in G_{k,n}^2 \setminus E_{k,n}$ are different from the segments in $U_{k-1,n}$ and therefore the functionals $\phi^2_s = \phi_s_{|\minsupp f_{x,k,n}, \delta} \in U_{k-1,n}$ have disjoint index set from the functionals in $U_{k-1,n}$.

As in the previous case, see (17), (19), we obtain that $\sum_{s \in E_{k,n}} c_s^2 \leq 0.009$.

From (14) and (20) we obtain
\[
| \sum_{s \in G_{k,n}^1 \setminus E_{k,n}} c_s \phi_s(x_k) | \geq | \sum_{s \in G_{k,n}^1} c_s \phi_s(x_k) | - | \sum_{s \in G_{k,n}^1} c_s f_{x,k,n}(x_k) | - | \sum_{s \in E_{k,n}} c_s \phi_s(x_k) | = 0.0149 - j_0 \delta - \delta > 0.0148.
\]
Also we get
\[ | \sum_{s \in G^2 \setminus E_k} c_s \phi_s^2(x_n) | \geq | \sum_{s \in G^2} c_s \phi_s(x_n) | - | \sum_{s \in E_k} c_s \phi_s(x_n) | \]
\[ \geq 0.0149 - 0.009 > 0.005 \]

We set \( U_{k,n} = U_{k-1,n} \cup \{ \hat{s} : s \in G^2 \setminus E_k \} \). Note that in this case \#(U_{k,n}\setminus U_{k-1,n}) \leq j_0 \) and from (21) we get that for the functional \( \phi_k = \sum_{s \in G^2 \setminus E_k} c_s \phi_s \in \overline{U}_{k,n} \) it holds that \( \phi_k(x_k) \geq 0.0148 \). Let \( U_{n-1,n} \) be the set of the segments we obtain after we complete the above procedure. Assuming that there exists \( n \) such that \#\( U_{n-1,n} \) \( > j_0(2 + 1/\delta^2) \) as in the case where \( H_{k,n} = G^1_{k,n} \) we get a contradiction.

**Theorem 6.4.** The space \( X_{G_2} \) does not contain a normalized sequence generating an \( \ell_1 \)-spreading model.

In the proof of the theorem we shall use the following well known fact.

**Fact 6.5.** Let \( X \) be a Banach space and \( (x_n) \) be a normalized sequence generating an \( \ell_1 \)-spreading model with constant \( c \). Then for every \( \epsilon > 0 \) there exists a normalized sequence \( (y_n) \) generating an \( \ell_1 \)-spreading model with constant \( (1 - \epsilon)^{-1} \). In particular \( (y_n) \) can be chosen such that each \( y_n \) is a normalized convex combination of \( (x_n) \).

**Proof of Theorem 6.4.** Let \( (x_n) \) be a sequence which generates an \( \ell_1 \)-spreading model with constant \( c \). Since the space \( X_{G_2} \) does not contain \( \ell_1 \) by standard arguments passing to a subsequence and taking the differences we may assume that \( (x_n) \) is a block sequence and generates a spreading model with constant \( (1 - \epsilon)^{-1}, \epsilon < 10^{-10} \).

By proposition 6.3 we get that there exist \( j_1 \in \mathbb{N} \) and \( L \leq \mathbb{N} \) such that for all \( n \in L \) there exists \( \phi_n \in G_{\ell_2} \) with index set contained in \( \{ 1, \ldots, j_1 \} \) such that \( \phi_n(x_n) \geq 0.75 \).

It follows that for every \( n \in L \) and every \( \phi \in G_{\ell_2} \) with \( \text{ind}(\phi) \subset \{ j_1 + 1, \ldots \} \) it holds that \( |\phi(x_n)| < 0.75 \).

Indeed if there exists such \( \phi \) with \( |\phi(x_n)| \geq 0.75 \) then the functionals \( \phi \) and \( \phi_n \) have disjoint index sets and hence \( \psi = \frac{\phi + \phi_n}{\sqrt{2}} \in K \). It follows that \( \psi(x_n) \geq \frac{0.75 + 0.75}{\sqrt{2}} > 1 \), a contradiction.

Let \( n_0 \in \mathbb{N} \) with \( n_0 > j_1 \). Let \( x = \frac{1}{n_0} \sum_{i=1}^{n_0} x_i \), with \( n_0 \leq l_1 < \cdots < l_n \in L \).

Since \( (x_n) \) is assumed to generate an \( \ell_1 \)-spreading model with constant \( (1 - \epsilon)^{-1} \) it follows that \( \| x \|_{G_2} \geq 1 - \epsilon \). Let \( \phi \in K \) be a functional which norms \( x \). It readily follows that \( \phi = \sum_{i=1}^d \lambda_i \phi_i \in G_{\ell_2} \).

For every \( i \) let \( \phi_i^1 \) be the part of \( \phi_i \) with the index set contained in \( \{ 1, \ldots, j_1 \} \) and \( \phi_i^2 \) be the part with index set contained in \( \{ j_1 + 1, j_1 + 2, \ldots \} \). Let \( \phi_1 = \sum_{i=1}^d \lambda_i \phi_i^1 \) and \( \phi_2 = \sum_{i=1}^d \lambda_i \phi_i^2 \).

By the above note we get that \( \phi_2(x_i) \leq 0.75 \) for all \( i \) and hence \( \phi_2(x) \leq 0.75 \).

Also for every \( f \) of type I with \( \text{ind}(f) \leq j_1 \) it follows that \( \| f(x) \| \leq \frac{\# \text{supp}(f)}{n_0} \leq \mu(f)^{-1} \).

Hence
\[ |\phi^1(x)| \leq \sum_{i=1}^d \sum_{k \in \text{ind}(\phi^i)} m_k^{-1} \leq \sum_{k=1}^\infty m_k^{-1} < 0.1. \]

Therefore \( |\phi(x)| \leq |\phi_1(x)| + |\phi_2(x)| < 0.85 \), a contradiction. \( \square \)

7. The space \( X_\xi \) as an extension of the space \( X_{G_2} \)

In this section we define the space \( X_\xi \) by its norming set \( K_\xi \). We introduce the key ingredient for the definition of \( K_\xi \) - attractor sequences (Def. 7.1), and the basic tool in estimating the norm in \( X_\xi \) - a tree-analysis of a functional from \( K_\xi \) (Def. 7.3).
Let \((\Lambda_i)_{i \in \mathbb{N}}\) be a partition of \(\mathbb{N}\) into pairwise disjoint infinite sets. Let \(Q_s\) denote the set of all finite sequences \((f_1, \ldots, f_d)\) such that for all \(i, f_i \in c_00(\mathbb{N}), f_i \neq 0, f_i(n) \in \mathbb{Q}\) for all \(n \in \mathbb{N}\) and \(f_1 < f_2 < \cdots < f_d\).

We fix a partition \(N_1, N_2\) of \(\mathbb{N}\). Let \(\sigma: Q_s \to N_2\) be an injective function such that

\[
m_{2\sigma(f_1, \ldots, f_d)}^{1/2} > \max \{|f_i(e_i)|^{-1} : l \in \text{supp } f_i, i \leq d\} \text{ maxsupp } f_d.
\]

Such an injective function exists since the set \(Q_s\) is countable.

Let \(K_\xi\) be the minimal subset of \(c_00(\mathbb{N})\) satisfying the following conditions

1. \(K_\xi\) is symmetric i.e. if \(f \in K_\xi\) then \(-f \in K_\xi, K_\xi\) is closed under the restriction of its elements to intervals of \(N\) and \(G_\xi \subset K_\xi\).
2. \(K_\xi\) is closed under \((A_{n_j}, m_{j_1}^{-1})\)-operations.
3. \(K_\xi\) is closed under \((A_{n_j}, m_{j_1}^{-1})\)-operations on attractor sequences.
4. \(K_\xi\) is closed under the operation \(\sum_{i \in A} \lambda_i f_i\) whenever
   a. \(f_i\) is the result of an \((A_{n_j}, m_{j_1}^{-1})\)-operation and \(n_{ji} \neq n_{jk}\) for every \(i \neq k \in A\)
   b. \((\lambda_i)_{i \in A} \in B_{(0)} \cap [\mathbb{Q}]^{<\infty}\).
5. \(K_\xi\) is rationally convex.

In order to complete the definition of the set \(K_\xi\) we have to define the attractor sequences.

**Definition 7.1.** A finite sequence \((f_1, \ldots, f_d)\) is said to be a \(n_{2j-1}\)-attractor sequence provided that

1. \((f_1, \ldots, f_d) \in Q_s\) and \(f_i \in K_\xi\) for all \(i = 1, \ldots, d \leq n_{2j-1} - 1\).
2. The functional \(f_i\) is the result of an \((A_{n_{2j_1}}, m_{j_1}^{-1})\)-operation on a family of functionals of \(K_\xi\) for some \(j_1 \in N_1\) with \(m_{2j_1} > n_{2j_1} - 1\). Also for every \(i = 2, \ldots, n_{2j-1} / 2\), \(f_{2i-1}\) is the result of an \((A_{n_{2j}}, f_{1}, \ldots, f_{2i-2}), m_{2j}^{-1})\)-operation on a family of functionals of \(K_\xi\).
3. \(f_{2i} = e_i^j\) for some \(\lambda^j_i \in \Lambda^j_{n_{1}, \ldots, n_{2j-1}}\).

We say that \(f \in K_\xi\) is of type I if it is a result of some \((A_{n_{j}}, m_{j-1}^{-1})\) operation. In this case we set \(w(f) = m_j\) and \(\text{ind}(f) = j\).

We say that \(f \in K_\xi\) is of type II if \(f = \sum_{i=1}^n \lambda_i f_i\) for some \((f_i)_{i=1}^n \subset K_\xi\) with \(w(f_i) \neq w(f_j)\) for all \(i \neq j\) and \(\sum_{i=1}^n \lambda_i^2 \leq 1\).

We say that \(f\) is of type III if it is a rational combination of elements of \(K_\xi\).

**Remarks 7.2.**

a. Using the partition of \(\mathbb{N} = \bigcup_{i=1}^\infty \Lambda_i\), it follows that the set of the attractor sequences has a tree structure i.e. if \((f_i)_{i=1}^{n_{2i-1} - 1}, (g_i)_{i=1}^{n_{2i-1} - 1}\) are two attractor sequences either \(g_i \neq f_i\) for every \(i, r\) or there exists \(i_0 \leq \min\{n_{2j-1}, n_{2k-1}\}\) such that \(f_i = g_i\) for all \(i \leq i_0 - 1\) and \(f_i \neq g_i\) for all \(i, r \geq i_0\). In particular \(w(f_{2i-1}) \neq w(g_{2r-1})\) for all \(2i - 1, 2r - 1 > i_0\) and \(f_{2i} \neq g_{2r}\) for all \(2i, 2r > i_0\).

b. Note that in the definition of the norming set when we take functionals \(\sum_{i=1}^n \lambda_i f_i\) with \(\sum_{i=1}^n \lambda_i^2 \leq 1\) we require \(w(f_i) \neq w(f_j)\) but do not require \(\text{supp}(f_i) \cap \text{supp}(f_j) = \emptyset\).

This is will be essential in order to get that no sequence generates an \(\ell_1\)-spreading model.

**Definition 7.3.** The tree \(T_f\) of a functional \(f \in K_\xi\) is a tree of a finite family \(T_f = (f_\alpha)_{\alpha \in A}\) indexed by a finite tree \(A\) with a unique root \(0 \in A\) such that the following holds

1. \(f_0 = f\) and \(f_\alpha \in K_\xi\) for all \(\alpha \in A\).
2. An \(\alpha \in A\) is maximal if and only if \(f_\alpha \in K_\xi\).
3. For every \(\alpha \in A\) not maximal, denoting by \(S_\alpha\) the set of the immediate successors of \(\alpha\) one of the following holds
   a. There exists \(j \in \mathbb{N}\) such that \(\#S_\alpha \leq n_j\) and \(f_\alpha\) is the result of the \((A_{n_j}, m_j^{-1})\)-operation in the set \(\{f_\beta : \beta \in S_\alpha\}\). In this case we say that the weight of \(f\) is \(w(f_\alpha) = m_j\).
(b) \( f_\alpha = \sum_{\beta \in S_\alpha} \lambda_\beta f_\beta \) where \( \sum_{\beta \in S_\alpha} \lambda_\beta \leq 1 \) the functionals \( f_\beta \) are of type I and have different weights.

(c) \( f_\alpha \) is a rational convex combination of the elements \( f_\beta, \beta \in S_\alpha \). Moreover for every \( \beta \in S_\alpha \) range \( f_\beta \subset \) range \( f_\alpha \).

The order \( o(f_\alpha) \) for each \( \alpha \in A \) is also defined by backward induction as follows.

If \( f_\alpha \in G_\xi \) then \( o(f_\alpha) = 1 \), otherwise \( o(f_\alpha) = 1 + \max \{ o(f_\beta) : \beta \in S_\alpha \} \).

The order \( o(T_f) \) of the aforementioned tree is defined to be equal to \( o(f_0) \).

**Definition 7.4.** The order \( o(f) \) of an \( f \in K_\xi \), is defined as

\[
o(f) = \min \{ o(T_f) : T_f \text{ is a tree of } f \}.
\]

Every functional \( f \in K_\xi \) admits a tree analysis not necessarily unique.

**Definition 7.5.** We define \( X_\xi = (e_{00}(N), \|\cdot\|_{K_\xi}) \) and we denote the norm \( \|\cdot\|_{K_\xi} \) by \( \|\cdot\| \).

**Definition 7.6.** Let \( k \in \mathbb{N} \). A vector \( x \in e_{00}(N) \) is said to be a \( C - \ell_1^k \) average if there exists \( x_1 < \ldots < x_k, \|x_i\| \leq C\|x\| \) and \( x = \frac{1}{k} \sum_{i=1}^{k} x_i \). Moreover, if \( \|x\| = 1 \) then \( x \) is called a normalized \( C - \ell_1^k \) average.

**Lemma 7.7** ([27], (or [16], Lemma 4)). Let \( j \geq 1 \), \( x \) be a \( C - \ell_1^m \) average. Then for every \( n \leq n_j - 1 \) and every \( E_1 < \ldots < E_n \), we have that

\[
\sum_{i=1}^{n} \|E_i x\| \leq C(1 + \frac{2n}{n_j}) < \frac{3}{2} C.
\]

**Proposition 7.8.** For every block sequence \((y_\ell) \subset X_\xi \), \( \epsilon > 0 \) and every \( k \geq m_2 \) there exists \( x \in ((y_\ell)) \) which is a normalized \( 2 - \ell_1^k \) average.

In particular for every \( k \in \mathbb{N} \) there exists \( n \in \mathbb{N} \) such that for every finite block subsequence \((y_\ell)_{\ell=1}^{n} \) of \((y_\ell) \) there exists \((z_\ell)_{\ell=1}^{k} \), a normalized block sequence of \((y_\ell)_{\ell=1}^{n} \) such that \( \|z_1 + \cdots + z_k\| \geq k/2 \).

Lemma II.22 [7] give us the existence of such average.

**Proposition 7.9.** \( X_\xi \) is a strictly singular extension of \( X_{G_\xi} \), i.e. the identity map \( X_{G_\xi} \to X_\xi \) is strictly singular.

**Proof.** Let \( Y \) be a block subspace of \( X_\xi \) and \( j \in \mathbb{N} \). By the Proposition 7.8 we obtain a sequence \((y_n)_{n \in \mathbb{N}} \subset Y \) of \( \ell_1^k \) average and by Proposition 5.4 for any fixed \( j > 1 \) we get \( L \in [\mathbb{N}] \) such that for every \( g \in G_\xi \)

\[
\{ n \in L : |g(y_n)| \geq m_2^{-2j} \} \leq n_{2j} - 1.
\]

It follows that

\[
\frac{1}{n_{2j}} \sum_{i=1}^{n_{2j}} y_i \|G_\xi \leq \frac{n_{2j} - 1}{n_{2j}} + \frac{1}{m_{2j}} \leq \frac{2}{m_{2j}}
\]

while

\[
\frac{1}{n_{2j}} \sum_{i=1}^{n_{2j}} y_i \| \geq \frac{1}{m_{2j}}. \quad \text{Since } j \text{ was arbitrarily chosen we have a strictly singular extension}. \]

8. The auxiliary space

Now we define for any \( j_0 > 1 \) the auxiliary space \( X_{W_{j_0}} \) and discuss construction and relation between its norming sets: \( W_{j_0} \) and \( W_{j_0}' \). Next in Subsection 8.1 we estimate the supremum norm of any norming functional from \( W_{j_0} \) (Lemma 8.2) and its action on a suitable average of \((e_n)_{n} \) of length \( n_{j_0} \) (Lemma 8.10).

Let \( j_0 > 1 \) be fixed. We set \( C_{j_0} = \{ \sum_{i \in F} \epsilon_i : F \subset A_{n_{j_0} - 1} \} \). Let \( W_{j_0} \) be the minimal subset of \( e_{00}(N) \) satisfying the following conditions

1) \( W_{j_0} \) contains the set \( C_{j_0} \).
2) $W_{j_0}$ is closed under $(A_{2n_j}, m_j^{-1})$-operations for every $j \in \mathbb{N}$.

3) $W_{j_0}$ is closed under the operation $\sum_{i \in A} \lambda_i f_i + \sum_{i \in B} \lambda_i e_i$, whenever $A \cap B = \emptyset$, $t_i \neq t_j$ for all $i \neq j \in B$, $f_i$ is the result of an $(A_{2n_j}, m_j^{-1})$-operation, $m_j \neq m_{j_0}$, for all $i \neq k \in A$ and $\sum_{i \in A \cup B} \lambda_i^2 \leq 1$.

4) $W_{j_0}$ is rationally convex.

The auxiliary space is the space $X_{W_{j_0}} = (c_0(\mathbb{N}), \|\cdot\|_{W_{j_0}})$.

For every functional $f \in W_{j_0}$ which is the result of an $(A_{2n_j}, m_j^{-1})$-operation we set $w(f) = m_j$ and we say that $f$ is of type I. Every functional of the form $\sum_{i \in A} \lambda_i f_i + \sum_{i \in B} \lambda_i e_i \in W_{j_0}$ is called of type II. Every functional which is a convex combination of elements of $W_{j_0}$ is called of type III.

Consider the minimal set $W'_{j_0}$ which satisfies Properties 1)-3) in the definition of $W_{j_0}$. Then following the arguments of Lemma 3.15 [9], every $f \in W_{j_0}$ of type I with $w(f) = m_j$ can be written as a convex combination $\sum_i \lambda_i f_i$ of functionals from the set $W'_{j_0}$ with $w(f_i) = m_j$ for every $i$. Hence in order to get an upper estimate for functionals of type I from the set $W_{j_0}$, it is enough to get an upper estimate for the functionals from the set $W'_{j_0}$.

An alternative definition of the norming set $W'_{j_0}$ is the following:

Set $W_0 = C_{j_0}$ and assume that $W_{n-1}$ has been defined.

We set

$$W_n^1 = \bigcup_j \left\{ \sum_{i=1}^d f_i : (f_i)_i = A_{2n_j} \text{-admissible subset of } W_{n-1} \right\}$$

and

$$W_n^2 = \left\{ \sum_{i \in A} \lambda_i f_i + \sum_{i \in B} \lambda_i e_i : \sum_{i \in A \cup B} \lambda_i^2 \leq 1, t_i \neq t_j \text{ for all } i \neq j \in B, A \cap B = \emptyset, f_i \in W_{n-1} \text{ is of type I for all } i \in A \text{ with } w(f_i) \neq w(f_j) \text{ for all } i \neq j \in A \right\}.$$

We set $W_n = W_n^1 \cup W_n^2$. Then $W'_{j_0} = \bigcup_n W_n$.

Every $f \in W'_{j_0}$ admits a tree analysis $(f_\alpha)_{\alpha \in T}$ satisfying properties similar to Definition [8].

**Remarks 8.1.** Let $f \in W'_{j_0}$ and $(f_\alpha)_{\alpha \in T}$ be a tree analysis of $f$.

a) If $\alpha$ is a terminal node then $f_\alpha \in C_{j_0}$. In particular if $\alpha \in S_\beta$ and $f_\beta$ is of type II then $f_\alpha = \pm e_k^\alpha$ for some $k$.

b) If $\alpha, \beta$ are different maximal nodes it is not necessary true that $\text{supp}(f_\alpha) \cap \text{supp}(f_\beta) = \emptyset$.

c) A consequence of b) is that it does not hold that $\|f\|_\infty \leq 1$ for every $f \in W'_{j_0}$.

We show in Lemma [8.8] that there is a constant $c_1 > 1$ such that $\|f\|_\infty \leq c_1$ for every $f \in W'_{j_0}$.

### 8.1. Estimates on the auxiliary space.

Set $c_0 = \sum_j m_j^{-2}$ and $c_1 = (\sum_{n=0}^\infty c_0^n)^{1/2}$.

**Lemma 8.2.** For every $f \in W'_{j_0}$ it holds that $\|f(e_k)\| \leq \begin{cases} c_1 \|f\| & \text{if } f \text{ is of type I} \\ c_1^2 \|f\| & \text{if } f \text{ is of type II}. \end{cases}$

**Proof.** We prove by induction that for every $f \in W_n$ the following holds:

$$|f(e_k)| \leq u_f(1 + c_0 + \cdots + c_0^n)^{1/2} \tag{22}$$

where $u_f = w(f)^{-1}$ if $f$ is of type I and $u_f = 1$ otherwise.

Assume that for every $f \in W_{n-1}$ it holds that $|f(e_k)| \leq u_f(1 + c_0 + \cdots + c_0^{n-1})^{1/2}$ which is certainly true for $n = 1$.
If \( f = w(f)^{-1} \sum_{i=1}^{d} f_i \in W_n^1 \) then there exists unique \( i \leq d \) such that \( t \in \text{supp}(f_i) \). It follows from the inductive hypothesis that
\[
|f(e_t)| \leq \frac{1}{w(f)} |f_i(e_t)| \leq \frac{1}{w(f)} (1 + c_0 + \cdots + c_0^{n-1})^{1/2}.
\]
If \( f = \sum_{i \in A} \lambda_i f_i + \sum_{i \in B} \lambda_i e_i^* \in W_n^2 \) then either \( t \notin B \) or there exists unique \( i_t \in B \) such that \( e_t = e_{i_t} \). Assume that the latter holds. Setting \( A_t = \{ i \in A : t \in \text{supp}(f_i) \} \) from the inductive hypothesis and H"older inequality we get
\[
|f(e_t)| \leq \sum_{i \in A_t} |\lambda_i f_i(e_t)| + |\lambda_i e_i| \leq (1 + \sum_{i \in A_t} w(f_i)^{-2}(1 + c_0 + \cdots + c_0^{n-1}))^{1/2}
\]
\[
\leq (1 + (1 + c_0 + \cdots + c_0^{n-1}) \sum_{j} m_j^{-2})^{1/2} = (1 + c_0 + \cdots + c_0^{n-1})^{1/2}.
\]
This proves (22) and completes the proof of the lemma.

Recall that in the definition of the space we have chosen \( n_j = (2n_{j-1})^{s_{j-1}} \) where \( 2^{s_{j-1}} = m_j^3 \).

**Lemma 8.3.** Let \( f \in W_{j_0}^1 \) such that \( |f(e_t)| \geq \frac{2}{m_{j_0}} \) for every \( t \in \text{supp}(f) \). Then
\[
\# \text{supp}(f) \leq (2n_{j_0-1})^{s_{j_0-1}-2}.
\]

This lemma is the key ingredient of our evaluations in the auxiliary space. Its proof is given in four steps. As usually we shall consider the tree analysis \((f_\alpha)_{\alpha \in \mathcal{T}}\) of a functional \( f \) in \( W_{j_0}^1 \). In the first two steps we shall show that we may assume that the height of the tree is less or equal to \( s_{j_0-1} - 3 \), and for all functionals \( f_\alpha \) of type I in a tree analysis of \( f \) it holds that \( w(f_\alpha) < m_{j_0} \). This will enable us to use an argument similar to Lemma A4 in [3] and get the result.

**Proof.** Let \((f_\alpha)_{\alpha \in \mathcal{T}}\) be a tree analysis of the functional \( f \). For every \( t \in \text{supp}(f) \) we set
\[
D_1 = \{ \alpha \in \mathcal{T} : \alpha \text{ is a minimal node with } f_\alpha \text{ of type I and } w(f_\alpha) \geq m_{j_0} \}.
\]
Let us observe that the set \( D_1 \) consists of incomparable nodes. We set
\[
M_1 = \{ \gamma \in \mathcal{T} : \gamma \text{ is a maximal node and there exists } \alpha \in D_1 \text{ with } \alpha \prec \gamma \}
\]
and also
\[
M_2 = \{ \gamma \in \mathcal{T} : \gamma \text{ is a maximal node and } \gamma \notin M_1 \}.
\]
The definition of \( M_2 \) yields that for every \( \gamma \in M_2 \) and every \( \alpha \prec \gamma \) with \( f_\alpha \) of type I, \( w(f_\alpha) < m_{j_0} \).

We denote by \( f_1 \) the functional that we get following backwards the tree analysis of \( f \) with maximal nodes the nodes of \( M_1 \) and \( f_2 \) the corresponding one with maximal nodes those of \( M_2 \). Observe that \( f = f_1 + f_2 \).

We also set \( T_1 = \{ \beta : \text{there exists } \alpha \in D_1 \text{ with } \beta \prec \alpha \} \) which is a complete subtree of the tree \( \mathcal{T} \) and \((g_\alpha)_{\alpha \in T_1}\) the tree analysis of the functional \( f_1 \). Let us observe that every maximal node \( \alpha \) of \( T_1 \) belongs to the set \( D_1 \) and \( g_\alpha = f_\alpha \).

**Sublemma 8.4.** For the functional \( f_1 \), its tree analysis \((g_\alpha)_{\alpha \in T_1}\) and \( t \in \mathbb{N} \) the following hold:

1. for every \( \alpha \in D_1 \) we have \( |g_\alpha(e_t)| \leq \frac{m_{j_0}}{9c_1} \).
2. for every non-maximal node \( \alpha \in T_1 \) for which \( g_\alpha \) is of type I we have
\[
|g_\alpha(e_t)| \leq \frac{9c_1}{8w(g_\alpha)m_{j_0}}.
\]
3. For all \( \alpha \in T_1 \) non-maximal with \( g_\alpha = \sum_{i \in S_\alpha} \lambda_\beta y_\beta \) of type II we have
\[
|g_\alpha(e_t)| \leq \frac{9c_1}{8m_{j_0}}.
\]
Proof. (1) follows from Lemma 8.2 and the definition of the set $M_1$.

(2) and (3) are proved inductively. Let $\alpha \in T_1$ and assume the result for all $\alpha < \beta$.

If $g_\alpha = w(g_\alpha)^{-1} \sum_{\beta \in S_\alpha} g_{\beta}$ it is clear that (2) holds.

We pass now to prove (3). Assume that $g_\alpha = \sum_{\beta \in S_\alpha} \lambda_{\beta}g_{\beta}$. Then $|g_\alpha(e_\gamma)| = |\sum_{\beta \in S_\alpha} \lambda_{\beta}g_{\beta}(e_\gamma)|$.

If $\beta \in S_\alpha$ is not maximal then $g_\beta$ is of type I and the inductive assumptions yields

$$\tag{23} |g_\beta(e_\gamma)| \leq \frac{9c_1}{8w(g_\beta)m_{j_0}}.$$  

Let $\beta \in S_\alpha$ be a maximal node, namely $\beta \in D_1$. From the definition of the set $M_1$ it follows that $g_\beta$ is of type I and $w(g_\beta) = m_{j_0+k}$ for some $k \geq 0$. Lemma 8.2 yields

$$\tag{24} |g_\beta(e_\gamma)| \leq \frac{c_1}{w(g_\beta)} \left(\frac{1}{m_{j_0}m_{j_0}^k}\right) \text{ if } k > 0.$$  

We also observe that there exists at most one $\beta \in S_\alpha$ with $w(g_\beta) = m_{j_0}$. Without loss of generality we assume that there does exist $\beta_0 \in S_\alpha$ with $w(g_{\beta_0}) = m_{j_0}$.

Set $S_\alpha^1 = \{ \beta \in S_\alpha : g_\beta \text{ of type I with } w(g_\beta) < m_{j_0} \}$,

$S_\alpha^2 = \{ \beta \in S_\alpha : g_\beta \text{ of type I with } w(g_\beta) > m_{j_0} \} \cup \{ \beta \in S_\alpha : \beta \text{ is maximal} \}$.

It follows from (23), (24) and the fact that $w(g_\beta) \neq w(g_\gamma)$ for every $\beta \neq \gamma \in S_\alpha$, that

$$|g_\alpha(e_\gamma)| \leq \sum_{\beta \in S_\alpha^1} |\lambda_{\beta}g_\beta(e_\gamma)| + |\lambda_{\beta_0}g_{\beta_0}(e_\gamma)| + \sum_{\beta \in S_\alpha^2} |\lambda_{\beta}g_{\beta}(e_\gamma)|$$

$$\leq |\lambda_{\beta_0}g_{\beta_0}(e_\gamma)| + \sum_{\beta \in S_\alpha^1} \frac{2c_1}{m_{j_0}w(g_\beta)}|\lambda_{\beta}| + \sum_{\beta \in S_\alpha^2} \frac{c_1}{w(g_\beta)}|\lambda_{\beta}|$$

$$\leq |g_{\beta_0}(e_\gamma)| + \sum_{i < j_0} \frac{2c_1}{m_{j_0}m_{i}} + \sum_{k \geq 1} \frac{1}{m_{j_0}^k} \leq \frac{c_1}{m_{j_0}} + \frac{4c_1}{m_{j_0}m_{1}} \leq \frac{9c_1}{8m_{j_0}}.$$  

Remark 8.5. Let $(f_\alpha)_{\alpha \in T}$ be a tree analysis of $f_1$ such that for all $f_\alpha$ of type I it holds that $w(f_\alpha) \neq m_{j_0}$. The proof of the above sublemma yields that in this case $|f_1(e_\gamma)| \leq \frac{9c_1}{8m_{j_0}}$.

Let us observe that, since $f = f_1 + f_2$, the previous sublemma yields that supp($f_2$) = supp($f$). So we may assume that $f = f_2$. Let $(f_\alpha)_{\alpha \in T}$ be the tree analysis of $f$ with all maximal nodes belonging to $M_2$. For $\beta \in T$ we denote by $|\beta| = \# \{ \alpha \in T : \alpha < \beta \}$

the order of $\beta$.

We set $d_{j_0} = s_{j_0-1} - 3$ and we observe that $c_12^{-d_{j_0}} < m_{j_0}^{-2}$. We also set

$$D = \{ \alpha : \alpha \text{ is a maximal node of } T \text{ with } \alpha \leq d_{j_0} \text{ and if } \alpha \in S_\beta, \text{ then } f_\beta \text{ is of type II then } |\lambda_\alpha| \geq m_{j_0}^{-1} \}. $$

With the next two sublemmas we show that supp($f$) = $\cup_{\alpha \in D}$ supp($f_\alpha$).

Sublemma 8.6. Let $t \in$ supp($f$) be such that there is no maximal node $\beta$ with $|\beta| \leq d_{j_0}$ and $t \in$ supp($f_\beta$). Then it holds $|f(e_\gamma)| \leq c_12^{-d_{j_0}}$.

Proof. Let $t \in$ supp($f$). By induction we shall show the following: for every $\gamma \in T$ such that $|\gamma| = d_{j_0} - j$, $j = 0, 1, \ldots , d_{j_0}$ we have that

$$\tag{25} |f_\gamma(e_\gamma)| \leq c_12^{-j}u_{\gamma},$$

where $u_{\gamma} = m_{r-1}$ if $f_\gamma$ is of type I and $w(f_\gamma) = m_{r}$, (we make the convention that $m_0 = 2^{-2}$), $u_{\gamma} = 1$ if $f_\gamma$ is of type II.

First we observe that Lemma 8.2 yields $|f_\beta(e_\gamma)| \leq c_1$ for every maximal node $\beta \in T$. 

$$\square$$
Assume that the result holds for all \( \delta \sim \gamma \) with \( |\delta| = d_{j_0} - (j - 1) \) and \( |\gamma| = d_{j_0} - j \).

If \( f_\gamma = m_{r-1}^{-1} \sum_{\delta \in \mathcal{S}_\gamma} f_\delta \) is of type I then since there exists unique \( \delta \in \mathcal{S}_\gamma \) with \( t \in \text{supp} f_\delta \), from the inductive hypothesis we obtain

\[
|f_\gamma(e_t)| \leq m_{r-1}^{-1} \frac{c_1}{2^{j-1}} \leq \frac{c_1}{m_{r-1}2^j}.
\]

If \( f_\gamma = \sum_{\delta \in \mathcal{S}_\gamma} \lambda_\delta f_\delta \) is of type II then each \( f_\delta \) is of type I. This follows from our assumption that every \( \delta \) with \( |\delta| \leq d_{j_0} \) is not a maximal node of the tree \( T \).

Hence setting \( A_\delta = \{ \delta : t \in \text{supp} f_\delta \} \) we get

\[
|f_\gamma(e_t)| \leq \sum_{\delta \in A_\delta} |\lambda_\delta f_\delta(e_t)| \leq \sum_{\delta} \frac{1}{m_{r-1}} \frac{c_1}{2^j} \leq \frac{c_1}{2^j}.
\]

\( \Box \)

By the above sublemma we may assume

\[
\text{supp}(f) = \cup \{ \text{supp}(f_\alpha) : \alpha \text{ maximal node with } |\alpha| \leq d_{j_0} \}.
\]

**Sublemma 8.7.** Let \( t \in \text{supp}(f) \) be such that for every maximal node \( \alpha \in \mathcal{T} \) with \( t \in \text{supp}(f_\alpha) \), we have \( |\alpha| \leq d_{j_0} \) and \( \alpha \in \mathcal{S}_\beta \) with \( f_\beta \) is of type II and \( |\lambda_\alpha| \leq m_{j_0}^{-1} \). Then

\[
|f(e_t)| < \frac{3}{2m_{j_0}}
\]

**Proof.** We prove by induction on \( j = 0, \ldots, d_{j_0} \) that if \( |\gamma| = d_{j_0} - j \) then

\[
|f_\gamma(e_t)| < \begin{cases} 
3/2m_{j_0}, & \text{if } f_\gamma \text{ is of type II} \\
1/2^j m_{j_0} m_{r-1}, & \text{if } f_\gamma \text{ is of type I and } w(f_\gamma) = m_r.
\end{cases}
\]

For the predecessors of maximal nodes we have it by assumption. If \( f_\gamma \) is of type I, then only one of its successors have \( e_t \) in its support and by the inductive hypothesis

\[
|f_\gamma(e_t)| < \frac{3}{2m_{j_0}m_r} \leq \frac{1}{2^2 m_{j_0} m_{r-1}}.
\]

If \( f_\gamma \) is of type II and \( |f_\gamma| = d_{j_0} - j \), let \( f_\gamma = \sum_{\beta \in \mathcal{S}_\gamma} \lambda_\beta f_\beta \). Set \( A_\beta = \{ \beta \in \mathcal{S}_\gamma : t \in \text{supp}(f_\beta) \} \). There exists at most one \( f_\beta_0 = e_t^* \) and all other \( f_\beta \) are of type I with different weights. The inductive hypothesis and Hölder inequality yield

\[
|f_\gamma(e_t)| \leq \sum_{\beta \in A_\beta, \beta \neq \beta_0} |\lambda_\beta| \frac{1}{2^{\beta \beta_0} m_{\beta_0} m_{\beta-1}} + |\lambda_{\beta_0}| e_t^*(e_t) < \frac{1}{2m_{j_0}} + \frac{1}{m_{j_0}} \leq \frac{3}{2m_{j_0}}
\]

and the induction is finished. \( \Box \)

The above sublemmas for the tree analysis \( (f_\alpha)_{\alpha \in \mathcal{T}} \) of \( f \) with all maximal nodes belonging to \( M_2 \) yields that

\[
\text{supp}(f) = \cup \{ \text{supp}(f_\alpha) : \alpha \text{ is a maximal node of } \mathcal{T} \text{ with } |\alpha| \leq d_{j_0} \text{ and }
\]

\[
(26) \quad \text{if } \alpha \in \mathcal{S}_\beta, f_\beta \text{ is of type II then } |\lambda_\alpha| \geq m_{j_0}^{-1}\}
\]

**Sublemma 8.8.** Let \( f \in W^+_{j_0} \) with a tree analysis \( (f_\alpha)_{\alpha \in \mathcal{T}} \) such that

1. the height of \( \mathcal{T} \) is less or equal to \( d_{j_0} \),
2. if \( f_\alpha \) is of type I then \( w(f_\alpha) < m_{j_0} \) holds,
3. if \( \alpha \) is a maximal node, \( \alpha \in \mathcal{S}_\beta, f_\beta \) is of type II then \( |\lambda_\alpha| \geq m_{j_0}^{-1} \).

Then

\[
\# \text{supp}(f) \leq (2n_{j_0-1})^d_{j_0+1}.
\]
Proof. Inductively we show that for every $\alpha$ with $|\alpha| = d_{j_0} - j$, $j = 0, 1, \ldots, d_{j_0}$ holds the following
\begin{equation}
\# \text{supp}(f_\alpha) \leq (2n_{j_0-1})^{j+1}.
\end{equation}
If $f_\alpha$ is a terminal node it is clear that (27) holds.
Assume that the result holds for all $\alpha$ of order $d_{j_0}, \ldots, d_{j_0} - (j-1)$ and let $|\alpha| = d_{j_0} - j$.
If $f_\alpha = w(f_\alpha)^{-1} \sum_{\beta \in S_\alpha} f_\beta$ is of type I, then we get $w(f_\alpha) \leq m_{j_0-1}$ and hence $\# S_\alpha \leq 2n_{j_0-1}$. Using the inductive hypothesis we obtain
\begin{equation}
\# \text{supp}(f_\alpha) \leq 2n_{j_0-1} \max \{ \# \text{supp}(f_\beta) : \beta \in S_\alpha \} \leq (2n_{j_0-1})^{j+1}.
\end{equation}
If $f_\alpha = \sum_{\beta \in S_\alpha} \lambda_\beta f_\beta$, setting $S^1_\alpha = \{ \beta \in S_\alpha : \beta \text{ maximal} \}$ from assumption 3) we get $|\lambda_\beta| \geq m^{-1}_{j_0}$ for all $\beta \in S^1_\alpha$ and therefore $\# S^1_\alpha \leq m^2_{j_0}$.
Note also that for every $\beta \in S_\alpha \setminus S^1_\alpha$ it holds that $w(f_\beta) < m_{j_0}$ and hence $\# (S_\alpha \setminus S^1_\alpha) \leq j_0 - 1$. Hence by the inductive hypothesis we get
\begin{equation}
\# \text{supp}(f_\alpha) \leq (j_0 - 1) \max \{ \# \text{supp}(f_\beta) : \beta \in S_\alpha \setminus S^1_\alpha \} + m^2_{j_0}
\leq (j_0 - 1)(2n_{j_0-1})^j + m^2_{j_0} \leq (2n_{j_0-1})^{j+1}.
\end{equation}

\begin{proof}[Proof of Lemma 8.8 completed] The desired inequality follows from (26) and Sublemma 8.8.\end{proof}

Remark 8.9. Take $f \in W'_{j_0}$ such that for every $t \in \text{supp}(f)$, $|f(e_t)| > 3m^{-2}_{j_0}$ and every $f_\alpha$ in the tree analysis $(f_\alpha)_{\alpha \in T}$ of $f$ satisfies $w(f_\alpha) \neq m_{j_0}$. Then Remark 8.8 Sublemma 8.6 and 8.8 yields that $\# \text{supp}(f) \leq (2n_{j_0-1})^{s_{j_0-1} - 2}$.

Lemma 8.10. Let $f \in W'_{j_0}$ be of type I. Then
\begin{equation}
|f(\frac{1}{n_{j_0}} \sum_{i=1}^{n_{j_0}} e_{k_i})| \leq \begin{cases} \frac{4}{w(f)m_{j_0}} & \text{if } w(f) < m_{j_0} \\ \frac{4}{w(f)} & \text{if } w(f) \geq m_{j_0} \end{cases}
\end{equation}
If moreover we assume that there exists a tree analysis $(f_\alpha)_{\alpha \in T}$ of $f$, such that $w(f_\alpha) \neq m_{j_0}$ for every $\alpha \in T$, we have that
\begin{equation}
|f(\frac{1}{n_{j_0}} \sum_{i=1}^{n_{j_0}} e_{k_i})| \leq \begin{cases} \frac{4}{m_1m_{j_0}} & \text{if } w(f) = m_1 < m_{j_0} \\ \frac{4}{m_1} & \text{if } w(f) = m_1 > m_{j_0} \end{cases}
\end{equation}
\begin{proof}[Proof] If $w(f) \geq m_{j_0}$, Lemma 8.2 yields the result.
Let $f = w(f)^{-1} \sum_{i=1}^{d} f_i$ with $w(f) < m_{j_0}$. We set $A_1 = \{ t : t \in \text{supp}(f_i) \text{ for some } i \leq d \text{ and } f_i(e_t) \leq 3m^{-1}_{j_0} \}$.
It readily follows that
\begin{equation}
|f(e_t)| \leq \frac{3}{w(f)m_{j_0}} \text{ for every } t \in A_1.
\end{equation}
We set $B_i = \text{supp}(f_i) \setminus A_1$. Lemma 8.3 yields that
\begin{equation}
\# \text{supp}(B_i) \leq (2n_{j_0-1})^{s_{j_0-1} - 2}.
\end{equation}
It follows that $\# \{ t : t \not\in A_1 \} \leq 2n_{j_0-1}(2n_{j_0-1})^{s_{j_0-1} - 2} = (2n_{j_0-1})^{s_{j_0-1} - 1}$ and therefore
\begin{equation}
|f(\sum_{t \not\in A_1} e_t)| \leq \frac{(2n_{j_0-1})^{s_{j_0-1} - 1}}{n_{j_0}} < m^{-2}_{j_0}.
\end{equation}

From (30) and (31) we get the result.
For the proof of (29) we consider the set $A_1 = \{ t : t \in \text{supp}(f_i) \text{ for some } i \leq d \text{ and } f_i(t) \leq 3m^{-2}_{j_0} \}$.\end{proof}
and using Remark 8.9 we obtain again the relation (31) for the new set $A_1$. \hfill \Box

9. The Basic Inequality and its consequences

In this section we introduce the notion of rapidly increasing sequences (RIS) and prove the Basic Inequality (Prop. 9.3), which by results from the previous section provides estimation on the norm of suitable averages of RIS (Prop. 9.4). In particular we obtain reflexivity of $X_\xi$.

**Definition 9.1.** Let $\epsilon > 0$. A block sequence $(x_k)$ in $X_\xi$ is said to be a $(C, \epsilon)$-rapidly increasing sequence (RIS), if there exists a strictly increasing sequence $(j_k)$ of positive integers such that

a) $\|x_k\| \leq 1$ for all $k$.

b) $m_2^{-1/2} \epsilon < \#(\text{range}(x_k))m_2^{-1/2} < \epsilon$ for all $k \geq 1$.

c) For every $k = 1, 2, \ldots$ and every $f \in K_\xi$ of type $I$ with $w(f) < m_2j_k$ we have that $|f(x_k)| \leq \frac{\epsilon}{w(f)}$.

**Remark 9.2.** From Proposition 7.8 and Lemma 7.7 we get that for every $\epsilon > 0$ any block subspace contains a $(3, \epsilon)$-RIS $(x_k)_k$ where $x_k$ is a normalized $2 - \epsilon_{1+k}^2$ average with $(2j_k)_k$ satisfying condition b) of the above definition.

**Proposition 9.3** (Basic Inequality). Let $\epsilon > 0$, $1 < j_0 \in \mathbb{N}$, $(x_k)_{k \in \mathbb{N}}$ be a $(C, \epsilon)$-RIS in $X_\xi$ with the associated sequence $(j_k)_k$. Assume that for every $g \in G_\xi$ the set $I_g = \{k : |g(x_k)| \geq \epsilon\}$ has cardinality at most $n_{j_0-1}$.

Let $(c_k)_k$ be a sequence of scalars. Then for every $f \in K_\xi$ and every interval $I$ there exists a functional $g \in W_{j_0}$ such that

$$|f(\sum_{k \in I} c_k x_k)| \leq C(g(\sum_{k \in I} |c_k|^{\epsilon_f}) + \epsilon_f \sum_{k \in I} |c_k|), \quad \epsilon_f \leq \epsilon.$$  

(32)

Moreover, if $f$ is the result of an $(A_{n_j}, m_j^{-1})$-operation then $g = e^*_j$ or $0$ or $g$ is the result of an $(A_{2n_j}, m_j^{-1})$ operation and $\epsilon_f \leq cw(f)^{-1/2}$.

If we additionally assume that for every $f \in K_\xi$ with $w(f) = m_{j_0}$, for every interval $J$ it holds that

$$|f(\sum_{k \in J} c_k x_k)| \leq C(\max_{k \in J} |c_k f(x_k)| + \epsilon m_{j_0}^{-1/2} \sum_{k \in J} |c_k|),$$

(33)

then we may select the functional $g$ to have a tree analysis $(g_\alpha)_\alpha$ with $w(g_\alpha) \neq m_{j_0}$ for all $\alpha \in A$.

**Proof.** We shall treat the case that there exists $j_0$ satisfying (33). We also assume that $\text{range}(x_k) \cap \text{range}(f) \neq \emptyset$ for every $k \in I$. We proceed by induction of the order $o(f)$ of the functional $f$.

Let $o(f) = 1$. Then we have that $f \in G_\xi$ and from the assumptions the set $R_f = \{k \in I : |f(x_k)| \geq \epsilon\}$ has cardinality at most $n_{j_0-1}$. We set $g_f = \sum_{k \in R_f} e^*_k$ and $\epsilon_f = \epsilon$. It readily follows

$$|f(\sum_{k \in I} c_k x_k)| \leq g_f(\sum_{k \in I} |c_k|^e_k) + \epsilon_f \sum_{k \in I} |c_k|.$$ 

Suppose now that the result holds for every functional in $K_\xi$ with order less than $q$ and consider $f \in K_\xi$ with $o(f) = q$.

We consider the following three cases.
Case 1. \( f \) is of type I and \( w(f) = m_{j_0} \). We choose \( k_0 \in I \) with \( |c_{k_0}f(x_{k_0})| = \max_{k \in I} |c_kf(x_k)| \) and we set \( g_f = |f(x_{k_0})|e_{k_0}^{*} \). Then from our assumption (33) it follows

\[
|f(\sum_{k \in I} c_kx_k)| \leq C \left( \max_{k \in I} |c_kf(x_k)| + \epsilon m_{j_0}^{-1/2} \sum_{k \in I} |c_k| \right)
\]

\[
\leq C \left( g_f(\sum |c_k|e_k) + \epsilon m_{j_0}^{-1/2} \sum_{k \in I} |c_k| \right).
\]

Case 2. \( f \) is of type I and \( w(f) \neq m_{j_0} \).

Then \( f = m_{j_0}^{-1} \sum_{i=1}^{d} f_i \) with \( j \neq j_0 \) and \( d \leq n_j \). We consider the following three subcases.

Subcase 2a. \( w(f) < m_{2j_0} \) for all \( k \in I \).

For every \( i \leq d \) we set

\( I_i = \{ k \in I : \text{range}(x_k) \cap \text{range}(f_i) \neq \emptyset \) and \( \text{range}(x_k) \cap \text{range}(f_{i'}) = \emptyset \) for all \( i' \neq i \}. \)

We also set \( I_0 = \{ k \in I : \text{range}(x_k) \cap \text{range}(f_i) \neq \emptyset \) for at least two \( i \in \{1, \ldots, d\} \}. \) We observe that \( \#I_0 \leq d \). Condition c) in the definition of the RIS yields

\[
|f(x_k)| \leq \frac{C}{w(f)} \quad \text{for every } k \in I_0.
\]

For every \( i \leq d \) we have that \( I_i \) is a subinterval of \( I \), hence our inductive assumption yields that there exists \( g_{f_i} \in W_{j_0} \) with \( \text{supp}(g_{f_i}) \subset I_i \) such that

\[
|f_i(\sum_{k \in I_i} c_kx_k)| \leq C(g_{f_i}(\sum_{k \in I_i} |c_k|e_k) + \epsilon_f \sum_{k \in I_i} |c_k|).
\]

The family \( \{I_1, \ldots, I_d\} \cup \{\{k\} : k \in I_0\} \) consists of pairwise disjoint intervals and has cardinality less than or equal to \( 2d \leq 2n_j \). We set

\[
g_f = \frac{1}{w(f)}(\sum_{i=1}^{d} g_{f_i} + \sum_{k \in I_0} e_k).
\]

Then \( g_f \in W_{j_0} \), \( \text{supp} g_f \subset I \), while from (33), (34) we obtain

\[
|f(\sum_{k \in I} c_kx_k)| \leq \sum_{k \in I_0} |c_k||f(x_k)| + \frac{1}{w(f)} \sum_{i=1}^{d} C\left(g_{f_i}(\sum_{k \in I_i} |c_k|e_k) + \epsilon \sum_{k \in I_i} |c_k|\right)
\]

\[
\leq C\left(g_f(\sum |c_k|e_k) + \epsilon_f \sum_{k \in I} |c_k|\right), \quad \text{where } \epsilon_f = \epsilon w(f)^{-1}.
\]

Subcase 2b. \( m_{2j_0} \leq w(f) < m_{2j_0+1} \) for some \( k_0 \in I \).

From condition b) in the definition of RIS we get

\[
|f(x_k)| \leq \epsilon w(f)^{-1/2} = \epsilon_f \quad \text{for all } k \in I \text{ with } k < k_0.
\]

Using that \( w(f) \geq m_{2j_0} \) and condition c) we get

\[
|f(x_k)| \leq Cw(f)^{-1} \leq C\epsilon w(f)^{-1/2} = C\epsilon_f \quad \text{for every } k_0 < k \in I.
\]

Thus setting \( g_f = |f(x_{k_0})|e_{k_0}^{*} \) from (37), (38) we get

\[
|f(\sum_{k \in I} c_kx_k)| \leq |c_{k_0}f(x_{k_0})| + \sum_{k \in I \setminus \{k_0\}} |c_kf(x_k)|
\]

\[
\leq |c_{k_0}f(x_{k_0})| + C\epsilon_f \sum_{k \in I \setminus \{k_0\}} |c_k| \leq C\left(g_f(\sum_{k \in I} |c_k|e_k) + \epsilon_f \sum_{k \in I} |c_k|\right).
\]
Subcase 2c. \( m_{2jk+1} \leq w(f) \) for all \( k \in I \).

In this case as in (32), \( |f(x_k)| \leq \epsilon w(f)^{-1/2} = \epsilon f \) for all \( k \in I \), and we set \( g_f = 0 \). It follows easily that (32) holds.

Case 3. \( f \) is of type II, i.e., \( f = \sum_{i=1}^{d} \lambda_i f_i \).

For the given interval \( I \) the inductive assumption associates to each \( f_i \) a functional \( g_{f_i} \) satisfying (32).

We note that there exists at most one \( i \in \{1, \ldots, d\} \) with \( w(f_i) = m_{j_0} \). Without loss of generality we assume that there does exist such an \( i \), denoted by \( i_0 \). From Case 1 we have \( g_{f_{i_0}} = |f(x_{k_0})|e_{k_0}^* \) for some \( k_0 \in I \).

For every \( 1 \leq i \leq d \) we set \( C_{f_i} = \{k \in I : \text{range}(f_i) \cap \text{range}(x_k) \neq \emptyset\} \).

We partition the set \( M = \{1, \ldots, d\} \setminus \{i_0\} \) as follows:
\[
L_0 = \{i \in M : w(f_i) < m_{2jk} \text{ for all } k \in C_{f_i}\}
\]
and for \( k = 1, \ldots, \), we set
\[
L_k = \{i \in M : k \in C_{f_i} \text{ and } m_{2jk} \leq w(f_i) < m_{2jk+1}\}.
\]

We enlarge \( L_{k_0} \) by adding \( i_0 \) to its elements. Note that if \( i \notin L_0 \cup \cup_k L_k \) then \( g_{f_i} = 0 \). We set
\[
g_f = \sum_{i=1}^{d} |\lambda_i|g_{f_i} = \sum_{i \in L_0} |\lambda_i|g_{f_i} + \sum_{k \in L_k} (\sum_{i \in L_k} |\lambda_i f_i(x_k)|) e_k^*.
\]

We show that \( g_f \in W_{j_0} \). Since for every \( i \in L_0 \) it holds that \( w(g_{f_i}) = w(f_i) \) we have that the functionals \( g_{f_i}, i \in L_0 \), have different weights. Also since the sets \( L_k \) are pairwise disjoint it follows
\[
\sum_{i \in L_0} \lambda_i^2 + \sum_k (\sum_{i \in L_k} |\lambda_i f_i(x_k)|)^2 \leq \sum_{i \in L_0} \lambda_i^2 + \sum_k (\sum_{i \in L_k} \lambda_i^2) ||x_k|| \leq \sum_{i=1}^{d} \lambda_i^2 \leq 1,
\]
hence \( g_f \in W_{j_0} \).

We show that (32) holds. First we observe that since for all \( i, \epsilon_{f_i} \leq \epsilon w(f_i)^{-1/2} \) it follows that \( \sum_{i=1}^{d} |\lambda_i|\epsilon_{f_i} \leq \epsilon \). Also,
\[
|f(\sum_{k \in I} c_k x_k)| \leq \sum_{i=1}^{d} |\lambda_i f_i(\sum_{k \in C_{f_i}} c_k x_k)| \leq \sum_{i=1}^{d} C|\lambda_i|g_{f_i}(\sum_{k \in C_{f_i}} |c_k|e_k) + \epsilon_{f_i} \sum_{k \in C_{f_i}} |c_k|) \leq C \left( \sum_{i \in L_0} |\lambda_i|g_{f_i} + \sum_k (\sum_{i \in L_k} |\lambda_i f_i(x_k)|) e_k^* \right) \left( \sum_{k \in I} |c_k|e_k \right) + C \epsilon \sum_{k \in I} |c_k|.
\]

Case 4. \( f = \sum_{i=1}^{d} r_i f_i \), where \( (r_i)_{i=1}^{d} \subset \mathbb{Q} \), is a rational convex combination.

As in the previous case for every \( i = 1, \ldots, d \) we set
\[
I_i = \{k \in I : \text{range}(f_i) \cap \text{range}(x_k) \neq \emptyset\}.
\]

Take suitable \( (g_{f_i}) \) by the inductive hypothesis. Setting \( g_f = \sum_{i=1}^{d} r_i g_{f_i} \), we get
\[
|f(\sum_{k \in I} c_k x_k)| \leq C(g_f(\sum_{k \in I} |c_k|e_k) + \epsilon \sum_{k \in I} |c_k|)
\]
\[\square\]

**Proposition 9.4.** Let \( \epsilon > 0, 1 < j \in \mathbb{N}, (x_k)_{k \in \mathbb{N}} \) be a \((C, \epsilon)\)-RIS in \( X_\xi \) with the associated sequence \((g_k)_{k} \) and \( j < j_1 \). Assume that for every \( g \in G_\xi \) the set \( I_g = \{k : |g(x_k)| \geq 2m_j^{-2}\} \) has cardinality at most \( n_{j-1} \). Then
a) If $\varepsilon \leq \frac{2}{m_j}$ then for every $f \in K_\varepsilon$ of type I

$$
|f\left(\frac{1}{n_j} \sum_{k=1}^{n_j} x_k\right)| \leq \begin{cases} 
\frac{5C}{m_j w(f)}, & \text{if } w(f) < m_j \\
\frac{C}{m_j w(f)} + \frac{2C}{m_j}, & \text{if } w(f) \geq m_j.
\end{cases}
$$

In particular $\left\|\frac{1}{n_j} \sum_{k=1}^{n_j} x_k\right\| \leq \frac{3C}{m_j}.

b) If moreover for $j_0 = j$ the additional assumption of the Basic Inequality is fulfilled (Proposition 9.3, 33), then for a linear combination $\frac{1}{n_j} \sum_{i=1}^{n_j} b_i x_i$, where $|b_i| \leq 1$, we have

$$
\left\|\frac{1}{n_j} \sum_{i=1}^{n_j} b_i x_i\right\| \leq \frac{3C}{m_j}.
$$

Proof. The proof is an application of the Basic Inequality and Lemma 8.10.

Proposition 9.5. The space $X_\varepsilon$ is reflexive.

The reflexivity of $X_\varepsilon$ is consequence of Proposition 9.4 following standard arguments, see [3, 8].

Remark 9.6. The reflexivity of the space $X_\varepsilon$ yields that $K_\varepsilon$ is norm-dense subset of $B_{X_\varepsilon^*}$. Indeed since $K_\varepsilon$ is norming set it follows that $\text{conv}(K_\varepsilon)$ is $w^*$-dense. The reflexivity of $X_\varepsilon$ implies that $\text{conv}(K_\varepsilon)$ is $w$-dense and hence $\|\cdot\|$-dense. Since $K_\varepsilon$ is rationally convex we get that $K_\varepsilon$ is in fact norm-dense subset of $B_{X_\varepsilon^*}$.

10. Exact pairs and attracting sequences

Now we define exact pairs and show saturation of $X_\varepsilon$ by them. Next we introduce the crucial notion of attracting sequences and estimate the norms of averages of elements forming these sequences: vectors (Corollary 10.5) and functionals (Corollary 10.6).

Definition 10.1. A pair $(x, f)$ with $x \in X_\varepsilon$ and $f \in K_\varepsilon$ is said to be a $(C, 2j)$-exact pair, $C \geq 1$, $j \in \mathbb{N}$ if the following conditions are satisfied

1) $f(x) = 1$ and range $f = \text{range } x$.
2) $f$ is of type I and $w(f) = m_{2j}$.
3) $1 \leq \|x\| \leq 3C, \|x\|_\infty \leq m_{2j}^{-2}$ and for every $g$ of type I with $w(g) < m_{2j}$ it holds that $|g(x)| \leq \frac{5C}{w(f)}$ while for $g$ of type I with $w(g) > m_{2j}$, $|g(x)| \leq 5C m_{2j}^{-1}$.

Proposition 10.2. Let $j \in \mathbb{N}$ and $Y$ be a block subspace of $X_\varepsilon$. There exists a $(3, 2j)$-exact pair with $x \in Y$.

Proof. Let $Y$ be a block subspace of $X_\varepsilon$ and $j \in \mathbb{N}$. By Remark 9.2 Proposition 9.4 we can choose for $\varepsilon \leq 2m_{2j}^{-2}$ a $(3, \varepsilon)$-RIS $(x_k)_{k=1}^{n_{2j}}$ satisfying the assumptions of Proposition 9.4. It follows that

$$
\left\|\frac{m_{2j}}{n_{2j}} \sum_{i=1}^{n_{2j}} x_i\right\| \leq 9.
$$

Choosing $f_i \in K_\varepsilon$ such that $f_i(x_i) = 1$ and range($f_i$) $\subset$ range($x_i$) we have that $f = m_{2j}^{-1} \sum_{i=1}^{n_{2j}} f_i \in K_\varepsilon$ and $f \left(\frac{m_{2j}}{n_{2j}} \sum_{i=1}^{n_{2j}} x_i\right) = 1$. Setting $x = E \left(\frac{m_{2j}}{n_{2j}} \sum_{i=1}^{n_{2j}} x_i\right)$, where $E = \text{range}(f)$, Proposition 9.4 yields that $(x, f)$ is a $(3, 2j)$-exact pair.

Definition 10.3. A double sequence $(x_k, f_k)_{k=1}^{n_{2j}^{-1}}$ is called a $(C, 2j - 1)$ attracting sequence, if there is a sequence $(j_k)_{k=1}^{n_{2j}^{-1}}$ such that

1) $(f_k)_{k=1}^{n_{2j}^{-1}}$ is a $(2j - 1)$-attractor sequence with $w(f_{2k-1}) = m_{2j-1}$ and $f_{2k} = e_{j_{2k}}^*$, where $l_{2k} \in \Lambda_{2j}$ for all $k \leq n_{2j-1}/2$.
2) $x_{2k} = e_{l_{2k}}$ for all $k \leq n_{2j-1}/2$. 

(3) \((x_{2k-1}, f_{2k-1})\) is a \((C, j_{2k-1})\) exact pair.
(4) \(j_k = 2r (f_1, \ldots, f_{k-1})\) for any \(k \leq n_{2j-1}\)

**Remark 10.4.** If \((x_k, f_k)_{k=1}^{n_{2j-1}}\) is a \((C, 2j-1)\)-attracting sequence, then \((x_k/(3C))_{k=1}^{n_{2j-1}}\) is a \((5/3, n_{2j-1})\) RIS. Indeed,

\[
\#(\text{range } x_k)^{1/2} \leq \#(\text{range } f_k)^{1/2} < \min\{\|f_i\|_\infty, i \leq k\} \leq m_{j_1}^{-1/2} \leq n_{2j-1}^{-1}
\]

by the condition on \(\sigma\). Condition (c) in definition of RIS is satisfied thanks to \(x_{2k} = e_{l_{2k}}\) and the fact that \((x_{2k-1}, f_{2k-1})\) is a \((C, j_{2k-1})\) exact pair.

**Corollary 10.5.** Let \((x_k, f_k)_{k=1}^{n_{2j-1}}\) be a \((C, 2j-1)\) attracting sequence with associated sequence \((j_k)_k\), with \(\|x_{2k-1}\|_G \leq m_{2j-1}\) for every \(k \leq n_{2j-1}^{-1}\). Then

\[
\frac{1}{2m_{2j-1}} \leq \left| \frac{1}{n_{2j-1}} \sum_{k=1}^{n_{2j-1}} (-1)^{k+1} x_k \right| \leq \frac{15C}{m_{2j-1}^2}
\]

**Proof.** To see the lower estimate in (40) note that \(\tilde{f} = m_{2j-1}^{-2} \sum_{k=1}^{n_{2j-1}^{-1/2}} f_{2k} \in G_\xi \subset K_\xi\). Hence

\[
\left| \frac{1}{n_{2j-1}^{1/2}} \sum_{k=1}^{n_{2j-1}^{-1}} (-1)^{k+1} x_k \right| \geq \left| \tilde{f} \left( \frac{1}{n_{2j-1}^{1/2}} \sum_{k=1}^{n_{2j-1}^{-1}} (-1)^{k+1} x_k \right) \right| \geq m_{2j-1}^{-2}/2.
\]

The upper estimation in (40) follows from Proposition 9.4 b) for \(b_k = (-1)^k\) and \(j_0 = 2j-1\) after we show that

K1) for every \(g \in G_\xi\) it holds that \(\#\{k : |g(x_k)| \geq 2m_{2j-1}^{-2}\} \leq n_{2j-1}\). Note \(G_\xi = \{m_{2j-1}^{-2} \sum_{i \in F} \pm e_i : F \in A_{n_{2j-1}}\}\) if \(r \geq j\) then \(|g(e_{l_{2r}})| \leq m_{2j-1}\) while if \(r < j\) it holds that \(\#\{l_{2r} : |g(e_{l_{2r}})| \geq m_{2j-1}^{-2}\} \leq n_{2r-1}\). From the definition of the \(G_\xi\)-special functionals \(g = E \sum_{r=1}^{d} g_r\) we obtain

\[
|g(e_{l_{2k}})| \geq 2m_{2j-1}^{-2} \Rightarrow l_{2k} \in \supp g_r \forall g_r \in \cup_{r \leq j} G_\xi^r
\]

and therefore

\[
\#\{l_{2k} : |g(e_{l_{2k}})| \geq 2m_{2j-1}^{-2}\} \leq \sum_{r < j} n_{2r-1} \leq n_{2j-2}.
\]

Finally let \(y^* = \sum_{k=1}^{d} a_k y^*_k \in G_{l_2}\). For every \(k = 1, \ldots, d\) let \(y^*_k = y^*_{k,1} + y^*_{k,2}\) with \(\text{ind}(y^*_k) \subset \{1, \ldots, j\}\) and \(\text{ind}(y^*_k) \subset \{j+1, j+2, \ldots\}\). So we may write \(y^* = \sum_{k=1}^{d} a_k y^*_{k,1} + \sum_{k=1}^{d} a_k y^*_{k,2}\). Since \(\sum_{i \geq 2j} m_{2i-1}^{-2} < \frac{3}{2m_{2j-1}^{-2}}\) and the sets \(\text{ind}(y^*_k)\) are pairwise disjoint in order \(|y^*(e_{l_{2i}})| \geq 2/m_{2j-1}^{-2}\) it must hold that \(l_{2i} \in \supp y^*_{k,1}\) for some \(k \leq d\). As in the previous case we get

\[
\#\{l_{2i} : |y^*(e_{l_{2i}})| \geq 2m_{2j-1}^{-2}\} \leq \sum_{r < j} n_{2r-1} \leq n_{2j-2}.
\]

To see K2) we have to show that for any \((2j - 1)\)-attractor functional \(g\) and for any interval \(J \subset \{1, \ldots, n_{2j-1}\}\) we have

\[
|g(\sum_{k \in J} (-1)^k x_k)| \leq 5C(\max |g(x_k)| + m_{2j-1}^{-4} \# J).
\]
Let \( g = m_{2j-1}^{-1} \sum_{i=1}^{d} g_i, \) \( d \leq n_{2j-1} \). If \( g = f = m_{2j-1}^{-1} \sum_{i=1}^{n_{2j-1}} f_i \), then \( |g(\sum_{k=1}^{d}(-1)^kx_k)| = 0 \). Otherwise let \( i_0 = \min\{i \leq d : f_i \neq g_i\} \). Then, by definition of the attracting sequence and since \( J \) is an interval, we have

\[
| \sum_{i=1}^{i_0-1} g_i(\sum_{k \in J}(-1)^kx_k)| \leq 3 \max_{i \leq i_0-1}|g_i(x_i)|.
\]

By Remark 7.2a) for any \( 2i-1 > i_0, 2k-1 > i_0 \) we have \( w(g_{2i-1}) \neq w(f_{2k-1}) \) and for any \( 2i > i_0, 2k > i_0 \) we have \( g_{2i} \neq f_{2k} \). Notice \( g_{2i}(x_{2k}) = 0 \) for any \( 2i > i_0 \) and any \( 2k \in J \). By the definition of the attractor sequence we get for any \( 2k > i_0 \)

\[
| \sum_{i \geq i_0} g_i(x_{2k})| \leq \max_{i \geq i_0} \|g_{2i-1}\|_\infty \leq m_{j_i}^{-1} \leq m_{2j-1}^{-4}.
\]

Now by the definition of the exact sequence for any \( 2i-1 \geq i_0 \) and any \( 2k-1 > i_0 \) we have

\[
|g_{2i-1}(x_{2k-1})| \leq 5C \max\{w(g_{2i-1})^{-1}, m_{j_{2k-1}}^{-1}\} \leq 5Cn_{2j-1}^{-2}.
\]

Hence using that \( \|x_{2k-1}\|_\infty \leq m_{j_{2k-1}}^{-2} \) we obtain

\[
| \sum_{i \geq i_0} g_i(x_{2k})| \leq 5Cn_{2j-1}^{-2}(n_{2j-1} \|x_{2k-1}\|_\infty) \leq 5Cm_{2j-1}^{-4}.
\]

From (40), (42), (43) we get

\[
|g(\sum_{k \in J}(-1)^kx_k)| = \left| \frac{1}{m_{2j-1}} \left( \sum_{i=1}^{i_0-1} g_i + \sum_{i \geq i_0} g_i \right) \left( \sum_{k \in J}(-1)^kx_k \right) \right|
\]

\[
\leq 3 \max_{i \leq i_0}|g(x_i)| + |g(x_{i_0})| + 5C \sum_{k \in J} m_{2j-1}^{-4} \leq 5C \left( \max_{k \in J} |g(x_k)| + \frac{\#J}{m_{2j-1}^{-4}} \right)
\]

which ends the proof of K2) and thus the whole proof.

\[
\square
\]

**Corollary 10.6.** Let \((x_i, f_i)_{i=1}^{n_{2j-1}}\) be a \((C, 2j-1)\)-attracting sequence of length \(n_{2j-1}\) satisfying the assumption of Corollary 10.5. Set

\[
\phi = m_{2j-1}^{-2} \sum_{i=1}^{n_{2j-1}/2} f_{2i-1}, \quad \psi = m_{2j-1}^{-2} \sum_{i=1}^{n_{2j-1}/2} f_{2i}
\]

Then

\[
\frac{1}{30C} \leq \|\psi\| \leq 1, \quad \|\phi + \psi\| \leq m_{2j-1}^{-1}.
\]

*Proof.* Notice that \( m_{2j-1}(\phi + \psi) \in K_\xi \) hence the second inequality holds. To prove the first, from Corollary 10.5 we have \( \| \frac{1}{n_{2j-1}} \sum_{k=1}^{n_{2j-1}} (-1)^{k+1}x_k \| \leq \frac{15C}{m_{2j-1}^{-1}} \) and therefore

\[
\|\psi\| \geq \psi \left( \frac{m_{2j-1}^2}{15Cn_{2j-1}} \sum_{i=1}^{n_{2j-1}/2} (-1)^{i+1}x_i \right) = \frac{m_{2j-1}^2}{15Cn_{2j-1} m_{2j-1}^{-1}} \sum_{i=1}^{n_{2j-1}/2} f_{2i}(x_{2i}) = \frac{1}{30C}.
\]

\[
\square
\]

11. Spaces with no \( \ell_p \) as a spreading model

In this section we show that the space \( X_\xi \) does not admit \( c_0 \) or \( \ell_p, 1 \leq p < \infty \), as a spreading model. Actually we show that this holds for a wider class of Banach spaces which describe now.

Let \( G \) be a ground set. Let \( W_G \) denote the smallest subset of \( c_0(\mathbb{N}) \) which

(1) is symmetric, closed under the projections of its elements on intervals of \( \mathbb{N} \) and \( G \subset W_G \).
(2) for every \( j \in \mathbb{N} \) is closed under the \((A_{n_j}, m_j^{-1})\) operation.

(3) whenever \((f_i)_{i=1}^d\) is the result of an \((A_{n_j}, m_j^{-1})\)-operation with \( n_{jk} \neq n_{jm} \) for \( k \neq m \), then \( \sum_{i=1}^d \lambda_i f_i \in \mathcal{W}_G \) for all \((\lambda_i)_{i=1}^d \in B_{l_2} \cap [Q]^{<\infty} \).

(4) is rationally convex.

**Definition 11.1.** A subset \( D_G \) of \( \mathcal{W}_G \) is said to be an extension of \( G \) if:

(i) The set \( D_G \) is symmetric, closed under the projections of its elements on intervals of \( \mathbb{N} \) and \( G \subseteq D_G \).

(ii) For any \( j \in \mathbb{N} \) we have that \( D_G \) is closed under the \((A_{n_j}, m_j^{-1})\)-operation.

(iii) Whenever \((f_i)_{i=1}^d\) is the result of an \((A_{n_j}, m_j^{-1})\)-operation with \( n_{jk} \neq n_{jm} \) for \( k \neq m \), then \( \sum_{i=1}^d \lambda_i f_i \in D_G \) for all \((\lambda_i)_{i=1}^d \in B_{l_2} \cap [Q]^{<\infty} \).

iv) It is rationally convex.

**Definition 11.2.** Let \( D_G \) be an extension subset of \( \mathcal{W}_G \). We define \( \mathcal{Y}_{D_G} = (c_0(\mathbb{N}), \|\cdot\|_{D_G}) \).

We prove now the following theorem

**Theorem 11.3.** Let \( G \) be a ground set such that the corresponding space \( X_G \) does not admit \( \ell_1 \) as a spreading model. Then for every extension \( D_G \) the space \( \mathcal{Y}_{D_G} \) does not admit any \( \ell_p \) or \( c_0 \) as a spreading model.

**Remark 11.4.** The first example of a Banach space \( X \) with no \( \ell_p \) as a spreading model was given by E. Odell and Th. Schlumprecht [24]. The spaces we consider in Theorem 11.3 are extensions of their example. In particular the space \( \mathcal{Y}_{D_G} \) is similar to their example when \( G = \{ \pm e_n^*: n \in \mathbb{N} \} \) and \( D_G = W_G \). Our proof provides also an alternative proof of their result.

**Proof of the theorem.** First we note that Proposition 11.3 holds for the space \( \mathcal{Y}_{D_G} \). It follows that only \( \ell_1 \) is finitely block representable in \( \mathcal{Y}_{D_G} \) and hence \( \mathcal{Y}_{D_G} \) does not admit \( c_0 \) or \( \ell_p \) for \( p > 1 \), as a spreading model. We prove now that the space \( \mathcal{Y}_{D_G} \) does not contain a normalized sequence generating an \( \ell_1 \)-spreading model.

Since the space \( X_G \) does not admit \( \ell_1 \) as an aspreading model Erdos-Magidor theorem [12], yields that for every bounded sequence \((x_n)_{n \in \mathbb{N}}\) and every \( \epsilon > 0 \) we can choose a block sequence \((y_n)_{n \in \mathbb{N}}\) of \((x_n)_{n \in \mathbb{N}}\) where each \( y_n = \sum_{k \in \mathbb{N}} x_k/n_0, \#F_n = n_0 \), such that \( \|y_n\|_G < \epsilon \).

Also by the Fact 6.3 if a sequence \((x_n)_{n \in \mathbb{N}}\) generates an \( \ell_1 \)-spreading model with constant \( c \) then for every \( \epsilon > 0 \) there exists a block sequence \((y_n)_{n \in \mathbb{N}}\) of \((x_n)_{n \in \mathbb{N}}\) generating an \( \ell_1 \)-spreading model with constant \((1 - \epsilon)^{-1}\). So assuming that a normalized block sequence \((y_n)_{n \in \mathbb{N}}\) generates an \( \ell_1 \)-spreading model with constant \( C \), passing to suitable block sequence \((z_n)_{n \in \mathbb{N}}\) of \((y_n)_{n \in \mathbb{N}}\) we may assume that

A) \((z_n)_{n \in \mathbb{N}}\) generates an \( \ell_1 \)-spreading model with constant \((1 - \epsilon)^{-1}\).

B) \( \|z_n\|_G < \epsilon \) for every \( n \in \mathbb{N} \).

We shall need also the following lemmas

**Lemma 11.5.** Let \( x \in \mathcal{Y}_{D_G} \). Then for every \( \epsilon > 0 \) there exists \( j_0 \in \mathbb{N} \) such that for every \( \phi \in D_G \) of type II with \( \text{ind}(\phi) > j_0 \) it holds that \( |\phi(x)| < \epsilon \).

The proof is similar to the proof of Lemma 11.3 and we omit it.

**Lemma 11.6.** Assume that \((z_n)_{n \in \mathbb{N}}\) satisfies A) and B) for \( \epsilon = 10^{-3} \). Then there exists \( j_0 \in \mathbb{N} \) such that for all \( n \in \mathbb{N} \) there exists \( \phi \) of type II with \( \text{ind}(\phi) \leq j_0 \) such that \( |\phi(z_n)| \geq 0.9 \).

**Proof.** Let \( \phi \in D_G \) be such that \( \phi(z_2 + z_n) \geq 1.998 \). It follows that \( |\phi(z_2)| \geq 0.998 \) and \( |\phi(z_n)| \geq 0.998 \).

By B) we get that \( \phi = \sum_{i=1}^d \lambda_i f_i \), where each \( f_i \) is a weighted functional i.e. is a result of an \((A_{n_i}, m_i^{-1})\)-operation and weights of \((f_i)_{i=1}^d \) are different. Let \( j_0 \in \mathbb{N} \) be the
number we obtain from Lemma 11.5 for $z_2$. Setting $A = \{i : w(f_i) \leq m_{j_0}\}$ and $B$ its complement we get $|\sum_{i \in B} \lambda_i f_i(z_2)| < 0.001$. Therefore

$$\left(\sum_{i \in A} \lambda_i f_i(z_2)\right)^{1/2} \geq |\sum_{i \in A} \lambda_i f_i(z_2)| \geq 0.997.$$ 

It follows that

$$\left(\sum_{i \in A} \lambda_i f_i(z_n)\right)^{1/2} \leq (1 - 0.997^2)^{1/2} \Rightarrow \sum_{i \in B} \lambda_i f_i(z_n) \leq 0.09$$

Hence $|\sum_{i \in A} \lambda_i f_i(z_n)| \geq 0.9$. \hfill \Box

By the above lemma we get that (44) for every $\phi$ of type II with ind($\phi$) $> j_0$ it holds that $|\phi(z_n)| < 0.6$ for all $n \in \mathbb{N}$.

Indeed, assume that there exists $\phi_2$ of type II with ind($\phi_2$) $> j_0$ and $|\phi_2(z_n)| \geq 0.6$. Then by the previous lemma we get $\phi_1$ of type II with ind($\phi_1$) $\leq j_0$ such that $\phi_1(z_n) \geq 0.9$. It follows that $\phi = \frac{1}{\sqrt{2}}(\phi_1 + \phi_2) \in D_G$ and hence $\|z_n\| \geq \phi(z_n) \geq 0.99 \sqrt{2} > 1$, a contradiction.

Consider now the vector $u = \frac{1}{\sqrt{n_{j_0} + 1}} \sum_{i=1}^{n_{j_0} + 1} z_{n_{j_0} + 1 + i}$. Since we have that $(z_n)_{n \in \mathbb{N}}$ generates an $\ell_1$-spreading model with constant $0.999^{-1}$, there exists $\phi \in D_G$ such that $\phi(u) \geq 0.999$. Since $\|z_n\|_G \leq 10^{-3}$ for all $n \in \mathbb{N}$ we get that $\phi = \sum_{i=1}^d \lambda_i f_i$ where each $f_i$ is a weighted functional and their weights are different. Set

$$R_1 = \{i : w(f_i) \leq m_{j_0}\} \text{ and } R_2 = \{i : w(f_i) > m_{j_0}\}.$$ 

By (44) for every $n$ we obtain $|\sum_{i \in R_2} \lambda_i f_i(z_n)| \leq 0.6$ and hence

$$\sum_{i \in R_2} \lambda_i f_i(u) \leq 0.6.$$ 

On the other hand if $i \in R_2$ by Lemma 11.7 we get $|f_i(u)| \leq \frac{2}{w(f_i)}$.

Combining the above inequality with (44) we get

$$0.999 \leq |\phi(u)| \leq \sum_{i \in R_1} \frac{2|\lambda_i|}{w(f_i)} + 0.6 < 0.8,$$ 

a contradiction. \hfill \Box

**Corollary 11.7.** The space $X_\ell$ does not admit any $\ell_p$ (or $c_0$) as a spreading model.

The abstraction of the properties of the set $D_G$ enable us to derive also the following

**Theorem 11.8.** There exists a reflexive Hereditarily Indecomposable Banach space $X_{HI}$ with no $\ell_p$, $1 \leq p < \infty$, or $c_0$ as a spreading model.

**Sketch of the proof.** First we shall define the norming set $K_{HI}$ of the space $X_{HI}$. The set $K_{HI}$ is the smallest subset of $c_0([\mathbb{N}])$ satisfying the following conditions

1. $K_{HI}$ is symmetric i.e. if $f \in K_{HI}$ then $-f \in K_{HI}$, it is closed under the restriction of its elements to intervals of $\mathbb{N}$ and $\{e_n^*: n \in \mathbb{N}\} \subset K_{HI}$.
2. $K_{HI}$ is closed under $(A_{n_{j_2}}, m_{2j_1}^{-1})$-operations.
3. $K_{HI}$ is closed under $(A_{n_{j_1} - 1}, m_{2j_2}^{-1})$-operations on special sequences.
4. $K_{HI}$ is closed under the operation $\sum_{i \in A} \lambda_i f_i$ whenever
   a) $f_i$ is the result of an $(A_{n_{ji}}, m_{2j_i}^{-1})$-operation and $n_{ji} \neq n_{jk}$ for every $i \neq k \in A$
   b) $(\lambda_i)_{i \in A} \in B_{l_1} \cap [\mathbb{Q}]^{<\infty}$.
5. $K_{HI}$ is rationally convex.
To complete the definition of $K_{HI}$ we must define the special sequences. An $n_{2j-1}$-special sequence $(f_i)_{i=1}^{n_{2j-1}}$ is defined as the $n_{2j-1}$-attractor sequence (Def. 7.1) with the exception that the functionals $f_2$ are the results of an $(A_{n_{2j-1}}, n_{2j-1})$-operation of functionals of $K_{HI}$ instead of $(e_{k_{2j}}, i, t)$ we use in the attractor sequence. Observe that as a ground set we take the set $\{c_n^* : n \in \mathbb{N}\}$. The space $X_{HI}$ is the completion of $(c_0(\mathbb{N}), \| \cdot \|_{K_{HI}})$. In order to show that $X_{HI}$ is HI space we follow the method initiated in [10] and extended in [8, 3, 6]. Namely first we observe that Proposition 7.8 holds for the space $X_{HI}$ hence there exist seminormalized $\ell_1$ averages in every block subspace. Next we consider rapidly increasing sequence (RIS) of $\ell_1$ averages, Def. 9.1 and we observe that Basic Inequality, Prop 9.3 holds also for the space $X_{HI}$. It follows that the estimations of Prop. 9.4 holds for RIS in the space $X_{HI}$. This enable us to consider the exact pairs, Def. 10.1 and also the dependent sequences. A $(C, n_{2j-1})$-dependent sequence $(x_k, f_k)_{k=1}^{n_{2j-1}}, j \in \mathbb{N}$, is defined as the attractor sequence, Def. 10.3 with the exception that for all $k \leq n_{2j-1}$, $(x_k, f_k)$ is an exact pair. From the Basic Inequality and the estimations on RIS we obtain that estimations of Corollary 10.5 also holds for the dependent sequences. The estimations in Corollary 10.5 easily yields that $X_{HI}$ is indeed HI space.

Since the ground set of the space $X_{HI}$ is the set $G = \{\pm c_n^* : n \in \mathbb{N}\}$, Theorem 11.3 yields that the space $X_{HI}$ does not admit any $\ell_p$ or $c_0$ as a spreading model. □

12. The $c_0$-index of the dual of $X_\xi$

We show now that every subspace of the dual space $X_\xi^*$ has $c_0$-index greater than $\omega^\xi$ and does not contain a sequence generating a $c_0$-spreading model. As we have observe in Remark 9.6 the set $K_\xi$ is a norm-dense subset of $B_{X_\xi^*}$. So proving that every block subspace generated by a block sequence of elements of $K_\xi$ has $c_0$-index greater than $\omega^\xi$ we get that the same holds for all block subspaces of $X_\xi^*$. Hence in the sequel we will assume that the block subspaces are generated by block sequences of $K_\xi$.

We shall need the dual result to Proposition 7.8.

**Proposition 12.1.** For every $j \in \mathbb{N}$ every block subspace $Y$ of $X_\xi^*$ contains a $2 - c_0^{n_{2j}}$ average i.e. there exists a block sequence $(x_i^*)_{i=1}^{n_{2j}}$ in $Y$ such that $\|x_i^*\| \geq 2^{-1}$ for every $i \leq n_{2j}$ and $2^{-1} \leq \|\sum_{i=1}^{n_{2j}} x_i^*\| \leq 1$.

For the proof see Lemma 5.4 in [8].

**Proposition 12.2.** a) The $c_0$-index of every subspace $Y$ of $X_\xi^*$ is greater than $\omega^\xi$.

b) The dual space $X_\xi^*$ of $X_\xi$ does not contain a normalized basic sequence generating a $c_0$-spreading model.

**Proof.** a) It is enough to prove the result for the block subspaces. Let $Y$ be a block subspace of $X_\xi^*$. By Proposition 12.1 for every $j \in \mathbb{N}$ we choose $x_i^*$, $i \in \mathbb{N}$ such that $x_i^*$ is a $2 - c_0^{n_{2j}}$ average. Let $x_i^* = \sum_{i=1}^{n_{2j}} x_{i,t}^*$ and $x_i = \sum_{i=1}^{n_{2j}} x_{i,t}^*$ where $x_{i,t}^*(x_i,t) \geq 2^{-1}$ and $\|x_i\| = 1$.

Let $1 < j \in \mathbb{N}$. Passing to a subsequence we may assume that $(x_i)_{i=1}^{n_{2j}}$ is a $(3, 1/n_{2j})$-RIS and by Proposition 6.4 for every $y \in G_\xi$ holds

$$\#\{n \in \mathbb{N} : \|g(x_n)\| \geq \frac{2}{m_{2j}}\} \leq 20m_{2j}^4 \leq n_{2j-1}$$

Set $\tilde{y} = m_{2j}^{n_{2j}} \sum_{i=1}^{n_{2j}} x_i$ and $y^* = m_{2j}^{-1} \sum_{i=1}^{n_{2j}} x_i^*$.

Then setting $y := \lambda \tilde{y}$ for some $\lambda \in [1, 2]$, Proposition 9.4 yields that $(y, y^*)$ is a $(6, 2j)$-exact pair.
Note that for every \( g \in G_\xi \) setting \( A = \{ i : |g(x_i)| \geq \frac{4}{m_{2j}} \} \) we get
\[
|g(y)| \leq \frac{2m_{2j}}{n_{2j}} \left( \sum_{i \in A} |g(x_i)| + \sum_{i \notin A} |g(x_i)| \right) \leq \frac{2m_{2j}}{n_{2j}}(n_{2j} - 1 + \frac{2m_{2j}}{m_{2j}^2}) \leq \frac{6}{m_{2j}}.
\]
It follows that for every \( j \in \mathbb{N} \) and every block subspace we have a \((6, 2j)\)-exact pair \((y, y^* )\) with \(y^* \in Y\) and \(||y^* ||_{G_\xi} \leq m_{2j}/6\).

Therefore we are able to construct for every \( j \in \mathbb{N} \) a \(n_{2j-1}\)-attractor sequence \( (y_i, y_i')_{i=1}^{n_{2j-1}} \) with \( (y_2i-1, y_2i-1) \) - a \((6, 2j-1)\)-exact pair, \(y_2i-1 \in Y^*\) and \(||y_2i-1||_{G_\xi} < m_{2j-1}/4\) for every \( i \). From Corollary 11.3 we get
\[
\frac{1}{2m_{2j-1}^2} \leq \frac{1}{n_{2j-1}} \sum_{i=1}^{n_{2j-1}} (-1)^iy_i \leq \frac{90}{m_{2j-1}^2}.
\]
Setting \( z_j^* = m_{2j-2}^{-1} \sum_{i=1}^{n_{2j-1}/2} y_{2i-1}^* \) and \( w_j^* = m_{2j-2}^{-1} \sum_{i=1}^{n_{2j-1}/2} y_{2i}^* \) we get that
\[
||z_j^*||, ||w_j^*|| \geq \frac{1}{180}, \text{ and } ||z_j^* - w_j^*|| \leq \frac{1}{m_{2j-1}}.
\]
Let \( F \in S_\xi \) and \((w_j^*)_{j \in F}\) be a \(G_\xi\)-special sequence such that \(w_j^*\) satisfies (47) for an appropriate \( z_j^* \) and \( \text{minsupp} w_j^* > j \) for every \( j \in F \).

From the definition of the ground set \( G_\xi \) we have \( \sum_{j \in F} \epsilon_j w_j^* \in G_\xi \subset Bx_\xi^* \) and hence
\[
|| \sum_{j \in F} \epsilon_j z_j^* || \leq || \sum_{j \in F} \epsilon_j w_j^* || + || \sum_{j \in F} \epsilon_j z_j^* - \epsilon_j w_j^* || \leq 2.
\]
Notice that the tree consisting of sequences \((z_j^*)_{j \in F}\), with \( F \in S_\xi \), obtained in the way described above has order greater or equal \( o(S_\xi) \).

b) Notice that if there is a normalized basic sequence in \( X_\xi^* \) generating a \(c_0\)-spreading model, then we get also a block sequence \((f_n)\) generating a \(c_0\)-spreading model, and then a normalized block sequence \((x_n) \subset X_\xi \) with \( f_i(x_j) = \delta_{i,j} \) generates an \(\ell_1\)-spreading model in \( X_\xi \), a contradiction with Corollary 14.7.

\[
13. \text{ The space } X_\xi
\]

In this section we define our final space \( X_\xi \) as a suitable quotient \( X_\xi/X_L \) and show the desired properties of \( X_\xi \) (Corollaries 13.3 and 13.6). In order to this we show for any subspace \( Y \subset X_\xi \) with \( \overline{Y/X_L} \) infinite dimensional the existence of an \( \ell_1 \) average and a norming functional with controlled behavior with respect to \( Y/X_L \) and \( X_\xi^* \) (Lemma 13.2) and consequently the existence of a suitable attracting sequence (Corollary 13.4).

For an infinite subset \( L \subset \mathbb{N} \) we set \( X_L = \{(e_n)_{n \in L}\} \), where \((e_n)_{n \in \mathbb{N}} \) is the basis of \( X_\xi \). We shall prove that that for every \( L \in [\mathbb{N}] \) with \( \mathbb{N} \setminus L \) also infinite and satisfying \( |L \cap A_i| = \infty \) for every \( i \in \mathbb{N} \), the quotient space \( X_L = X_\xi/X_L \), does not contain a (normalized) sequence generating an \(\ell_1\)-spreading model but every of its subspaces has \(\ell_1\)-index greater than \( \omega^\xi \). Let \( Q : X_\xi \to X_\xi \) denotes the quotient map.

Let \( \mathbb{N} = L \cup M \), where \( M \cap L = \emptyset \). First we prove the following

Proposition 13.1. \textit{The sequence} \((Q(e_n))_{n \in M}\) \textit{is a basis for the quotient} \( X_\xi \).

\textit{Proof.} For every \( x = \sum_{n \in N} a_n e_n \in X_\xi \) we get \( Q(x) = \sum_{n \in M} a_n Q(e_n) \). Let \( M = (m_i) \) and \( j < n \). By the bimonotonicity of \((e_n)_{n \in \mathbb{N}} \) we have that \( ||\sum_{i=1}^{m_i} a_i e_i|| \leq ||\sum_{i=m_i+1}^{m_n} a_i e_i|| \) for some \( a_i, i \in [m_1, m_n] \). Using again that \((e_n)_{n \in \mathbb{N}} \) is bimonotone, for every \( j < m \) we obtain
\[
||\sum_{i=m_1}^{m_j} a_i e_i|| \leq ||\sum_{i=m_1}^{m_n} a_i e_i|| = ||\sum_{i=1}^{m_n} a_m Q(e_m)||
\]
and therefore \( ||\sum_{i=m_1}^{m_j} a_m Q(e_m)|| \leq ||\sum_{i=m_1}^{m_j} a_i e_i|| \leq ||\sum_{i=1}^{m_n} a_m Q(e_m)||. \) \(\square\)
We shall need the following result which is analogous to Lemma 11 [15].

**Lemma 13.2.** Let $N, m \in \mathbb{N}$, $\epsilon \in (0, 1/4)$ and $Y$ subspace of $X_\xi$ such that the quotient $Y/X_L$ is infinite dimensional. Then there exist a $2 - \epsilon^N$ average $x \in \{e_i : i \geq m\}$ and $f \in K_\xi$ such that

$$f(x) \geq 1/2, \quad \minsupp f \geq \minsupp x, \quad \text{dist}(Q(x), Y/X_L) < \epsilon \quad \text{and} \quad \text{dist}(f, X_L^\perp) < \epsilon.$$ 

**Proof.** Let $(\epsilon_n)_{n \in \mathbb{N}}$ be positive numbers with $\sum_n \epsilon_n < \epsilon$. Let $(\hat{y}_n)_{n \in \mathbb{N}}$ be a normalized $\varpi-$null sequence in $Y/X_L$. Passing to a subsequence we may assume that $(\hat{y}_n)_{n \in \mathbb{N}}$ is $(\epsilon_n)$-close to a normalized block sequence $(\hat{x}_n)_{n \in \mathbb{N}} \in X_\xi$ and there exists $x_n \in X_\xi$ such that

$$Q(x_n) = \hat{x}_n, \quad ||x_n|| = ||\hat{x}_n|| \quad \text{and range}(x_n) = [\minsupp(\hat{x}_n), \maxsupp(\hat{x}_n)].$$

Let also $y_n^* \in X_\xi^*$ such that $y_n^*(x_n) = 1 = ||\hat{x}_n||$, $||y_n^*|| = 1$ for every $n \in \mathbb{N}$. Since $(\epsilon_n)_{n \in \mathbb{N}}$ is bimonotone basis of $X_\xi^*$, setting $x_n^* = E_n y_n^*$, where $E_n = \text{range}(x_n)$, we get that $(x_n^*)_{n \in \mathbb{N}}$ is a block sequence

$$||x_n^*|| \leq ||y_n^*||, \quad x_n^*(x_n) = y_n^*(x_n) \quad \text{and} \quad x_n^* \in X_L^\perp.$$ 

Let $k, j \in \mathbb{N}$ be such that $2^k > m_2^3$ and $N_k \leq n_{2j}$. Such $k, j$ exist, since by definition $n_{2j} = (2n_{2j-1})^{s_{2j-1}}$ and $2^{s_{2j-1}} = m_2^3$. Hence if $N \leq m_{2j-1}$ setting $k = s_{2j-1}$ we get $N^k \leq n_{2j}$ and $2^{-k} < m_2^3$. We set

$$A_1 = \left\{L \in [\mathbb{N}] : L = (l_i)_{i \in \mathbb{N}} : \left|\frac{1}{N} \sum_{i=1}^{N} \hat{x}_{l_i} \right| > 1/2 \right\}.$$ 

By Ramsey theorem we may find an $L \in [\mathbb{N}]$ such that either $[L] \subset A_1$ or $[L] \cap A_1 = \emptyset$.

Assume first that $[L] \subset A_1$. We may assume that $x_{\min L} \geq m$. Since $\hat{x}_n = Q(x_n)$ for every $n \in \mathbb{N}$ it follows that

$$\left|\frac{1}{N} \sum_{i=1}^{N} x_{l_i} \right| \geq \left|\frac{1}{N} \sum_{i=1}^{N} \hat{x}_{l_i} \right| > 1/2.$$ 

Let $x = \frac{1}{N} \sum_{i=1}^{N} x_{l_i}$. Take $g \in X_L^*$ such that $g(\frac{1}{N} \sum_{i=1}^{N} x_{l_i}) > 1/2$. As before we may assume that minsupp $x = \minsupp g$.

Setting $\hat{g} = Q(\frac{1}{N} \sum_{i=1}^{N} x_{l_i})$ we get

$$\text{dist}(\hat{g}, Y/X_L) \leq \left|\frac{1}{N} \sum_{i=1}^{N} (\hat{x}_{l_i} - \hat{g}_{l_i}) \right| \leq \left|\frac{1}{N} \sum_{i=1}^{N} \hat{x}_{l_i} - \hat{g}_{l_i} \right| < \sum_{i} \epsilon_{l_i} \leq \epsilon.$$ 

Since $B X_\xi^* = K_\xi^{w^*}$ we choose $f \in K_\xi$ with

$$f(x) \geq 1/2, \quad \minsupp f \geq \minsupp x \quad \text{and} \quad ||f - g|| < \epsilon.$$ 

This is possible by taking an interval $B$ such that $||g - Bg|| < \epsilon/2$ and using that minsupp $x = \minsupp g$ and $B X_\xi^* = K_\xi^{w^*}$.

Assume now that $[L] \cap A_1 = \emptyset$. Set $\hat{y}_n^{(1)} = \frac{1}{N} \sum_{i \in F_n} \hat{x}_{l_i}$, where $\#F_i = N$, $F_n < F_{n+1}$ for every $n$ and $\cup_i F_i = L$.

Passing to a subsequence we may assume that there exists $a_1 \geq 2$ such that $\hat{x}_n^{(2)} = a_1 \frac{1}{N} \sum_{i \in F_n} \hat{x}_{l_i}$ is a normalized sequence in $Y/Z$. We may again apply Ramsey theorem defining $A_2$ as before. If we get some $L$ with $[L] \subset A_2$ the proof finishes as before.

Assume that in none of the first $k$ steps we get $L \in [\mathbb{N}]$ with $[L] \subset A_k$. Then there exist $a_1, a_2, \ldots, a_k \geq 2$ and $l_1 < l_2 < \cdots < l_{N_k}$ in $\mathbb{N}$ such that the vector

$$\hat{y} = a_1 a_2 \cdots a_k \frac{1}{N_k} \sum_{i=1}^{N_k} \hat{x}_{l_i}.$$
satisfies $\|\vec{y}\|_{X_L} = 1$.

The functional $y^* = \frac{1}{m_{z_j}} \sum_{i=1}^{N} x^*_i \in X^*_L$ satisfies $\|y^*\| \leq 1$. Therefore

$$1 = \|\vec{y}\|_{X_L} \geq y^*(\vec{y}) = \frac{2^k \mathcal{N}^k}{m_{z_j} \mathcal{N}^k} = \frac{2^k}{m_{z_j}}$$

which contradicts our choice of $k$ and $j$. \hfill \Box

**Corollary 13.3.** Let $Y/X_L$ be an infinitely dimensional subspace of the quotient $X_\xi/X_L$.

For every $j \in \mathbb{N}$, $\epsilon > 0$ there exists a $(6, 2j)$-exact pair $(y, f)$ such that

1. $\text{dist}(Q(y), Y/X_L) < \epsilon$ and $\text{dist}(f, X_L^\perp) < \epsilon$,

2. $\|y\|_{G_\xi} < 6m_{2j}^{-1}$.

**Proof.** Let $j \in \mathbb{N}$ and $(\epsilon_i)_i \subset (0, 1)$ such that $\sum_i \epsilon_i < \epsilon \leq n_{2j}^{-1}$. Using Lemma R32 we choose inductively we choose $(z, f, i)_i \subset (\mathcal{L}_0(\mathbb{N}) \times K_\xi)$ such that

a) $z_i$ is a $2 - \ell_{2j}^{2j_i}$ average and $(z_i)_{i=1}^{n_{2j}}$ is a $(3, \epsilon)$-RIS.

b) $\text{dist}(Q(z_i), Y/X_L) < \epsilon_i$ and $\text{dist}(f, z_i^\perp) < \epsilon_i$ for every $i$.

c) $f(z_i) \geq 1/2$.

d) range$(z_i) \cup \text{range}(f_i) < \text{range}(z_{i+1}) \cup \text{range}(f_{i+1})$ for every $i$.

By Proposition 5.2 we can assume that for every $g \in G_\xi$ the set $\{i \in \mathbb{N} : \|g(z_i)\| \geq 2m_{2j}^{-2}\}$ has cardinality at most $n_{j-1}$. Setting $z = \frac{m_{z_j}}{m_{2j}} \sum_{i=1}^{n_{2j}} z_i$ and $f = m_{2j}^{-1} \sum_{i=1}^{n_{2j}} f_i$ we have $f(z) \geq 1/2$. Taking $y := \lambda z$ for some $\lambda \in [1, 2]$ we get from Proposition 5.2 that $(y, f)$ is a $(6, 2j)$-exact pair. It is easy to see that the exact pair $(y, f)$ satisfies the requirements (1). The argument of Lemma R19 yields that (2) also holds. \hfill \Box

**Corollary 13.4.** For every $j \in \mathbb{N}$, any $Y/X_L$ infinitely dimensional subspace of the quotient $X_\xi/X_L$ and any $\epsilon > 0$ there exists a $(6, 2j - 1)$-attracting sequence $(y_i, f_i)_{i=1}^{n_{2j-1}}$ with the associated sequence $(j_i)$, such that

A) $\text{supp}(f_{2j}) = \text{supp}(y_{2j}) \subset L$,

B) For every $i \leq n_{2j-1}$, $(y_{2j-1}, f_{2j-1})$ is a $(6, 2j-1)$-exact pair and $\|y_{2j-1}\|_{G_\xi} < m_{2j-1}^{-2}$,

C) $\text{dist}(Q(\sum_{i=1}^{n_{2j-1}} y_{2j-1}), Y/X_L) < \epsilon$,

D) There exists $g \in X_L^\perp$ with

$$\|g - \frac{1}{m_{2j-1}} \sum_{i=1}^{n_{2j-1}/2} f_{2j-1}\| \leq m_{2j-1}^{-2} \quad \text{and} \quad \|g - \frac{1}{m_{2j-1}} \sum_{i=1}^{n_{2j-1}/2} f_{2j-1}\| \leq 2m_{2j-1}^{-1}.$$ 

**Proof.** Using Corollary 13.3 and assumption on the set $L$ we can choose an attracting sequence $(y_i, f_i)_{i=1}^{n_{2j-1}}$ such that A) holds and for every $i \leq n_{2j-1}/2$ the couple $(y_{2j-1}, f_{2j-1})$ is a $(6, 2j-1)$-exact pair satisfying the conclusion of Corollary 13.3 for $\epsilon_i = \epsilon 2^{-i}$, i.e.

1. $\text{dist}(Q(y_{2j-1}), Y/X_L) < \epsilon$ and $\text{dist}(f_{2j-1}, X_L^\perp) < \epsilon$.

2. $\|y_{2j-1}\|_{G_\xi} < 6m_{2j-1}^{-1} < m_{2j-1}^{-2}$.

For every $i \leq n_{2j-1}/2$ choose $g_{2j-1} \in X_L^\perp$ with $\|f_{2j-1} - g_{2j-1}\| < \epsilon 2^{-i}$ and let $g = m_{2j-1}^{-2} \sum_{i=1}^{n_{2j-1}/2} g_{2j-1}$. By Corollaries 10.4, 10.6 the chosen vectors and functionals satisfy the desired conditions. \hfill \Box

**Corollary 13.5.** Every subspace of the quotient $X_\xi$ has $\ell_1$-index greater than $\omega^\xi$.

**Proof.** By Corollary 13.4 we can choose for every subspace $Y/X_L$ of the quotient $X_\xi$ and any $j$ an attracting sequence $(y_i, f_i)_{i=1}^{n_{2j-1}}$ and $g_j \in X_L^\perp$ such that for all $l$, $\|y_i\|_l < m_{2j-1}^{-2}$.
and for
\[ z_j^* = m_{2j-1}^{n_{2j-1}/2} \sum_{l=1}^{n_{2j-1}/2} f_{2l}^j \in G_1, \quad f_j = m_{2j-1} \sum_{l=1}^{n_{2j-1}/2} f_{2l-1}^j, \quad u_j = m_{2j-1}^{n_{2j-1}/2} \sum_{l=1}^{n_{2j-1}/2} y_{2l-1}^j \]
we have
\[ \|g_j - z_j^*\| < 2m_{2j-1}^{-1} \text{ and } \|g_j - f_j\| \leq m_{2j-1}^{-2} \text{ and } \text{dist}(Q(u_j), Y/X_L) < 16^{-j}. \]
In particular we get \( \|g_j\| \leq 2. \) Setting \( \hat{u}_j = Q(u_j) = Q(\frac{m_{2j-1}^{n_{2j-1} - 1}}{n_{2j-1}} \sum_{l=1}^{n_{2j-1} - 1} (-1)^{j+1} y_{l}^j) \) we get by Corollary 10.5
\[
\frac{1}{8} \leq \frac{1}{4} - m_{2j-1}^{-2}/2 \leq f_j(u_j)/2 - \|f_j - g_j\|/2 \leq g_j(u_j)/2 \leq \|\hat{u}_j\|_{X_\xi} \leq \|\frac{m_{2j-1}^{n_{2j-1} - 1}}{n_{2j-1}} \sum_{l=1}^{n_{2j-1} - 1} (-1)^{j+1} y_{l}^j\| \leq 90.
\]
For any \( F \in \mathcal{S}_\xi \) we pick \( (z_j^*)_{j \in F}, (g_j)_{j \in F}, (u_j)_{j \in F} \) so that

1. \( z_j^*, g_j, u_j \) are picked as above for any \( j \in F \),
2. \( \text{minsupp}(z_j) > j \) for any \( j \in F \),
3. \( (z_j^*)_{j \in F} \) is a \( G_\xi \)-special sequence.

Then we have that \( \|\sum_{j \in F} \epsilon_j z_j^*\| \leq 1 \) for every \( \epsilon_j \in \{-1, 1\} \). Since \( \|g_j - z_j^*\| \leq 2^{-j} \) for every \( j \in F \) it follows that \( \|\sum_{j \in F} \epsilon_j g_j\| \leq 2 \) for every \( \epsilon_j = \pm 1 \). Therefore
\[
\|\sum_{j \in F} a_j \hat{u}_j\|_{X_\xi} \geq \left\{ \sum_{j \in F} \text{sgn}(a_j) g_j(\sum_{j \in F} a_j u_j) \right\} \geq \frac{1}{2} \sum_{j} \left| a_j \right| \geq \frac{1}{16} \sum_{j} \left| a_j \right|.
\]
Since \( \text{dist}(\hat{u}_j, Y/X_L) < 16^{-j} \) for any \( j \), by the above procedure we can obtain an \( \ell_1 \)-tree in \( Y/X_L \) with order greater or equal \( o(\mathcal{S}_\xi) \), hence the \( \ell_1 \)-index of \( Y/X_L \) is greater than \( \omega^\xi \).

**Corollary 13.6.** The space \( X_\xi = X_\xi/X_L \) does not contain a normalized basic sequence generating an \( \ell_1 \)-spreading model.

**Proof.** If a normalized basic sequence \( (\hat{x}_n)_{n \in \mathbb{N}} \subset X_\xi \) generates an \( \ell_1 \)-spreading model in \( X_\xi \), we may assume that it is a block sequence. Then we get a block sequence in \( X_\xi \) which generates an \( \ell_1 \)-spreading model, a contradiction by Corollary 11.7.

Gathering the results of this and previous sections we get the following

**Theorem 13.7.** For every countable ordinal \( \xi \) there exists a separable reflexive Banach space \( X_\xi \) with the hereditary Bourgain \( \ell_1 \)-index greater than \( \omega^\xi \) such that \( X_\xi \) does not admit an \( \ell_1 \)-spreading model. Moreover the dual \( X_\xi^* \) has hereditary \( c_0 \)-index greater than \( \omega^\xi \) and does not admit a \( c_0 \)-spreading model.

**Remark 13.8.** The results presented above concerning indices of the dual space \( X_\xi^* \) and the quotient space \( X_\xi \) suggest the following problem:

Assume that \( X \) is a reflexive Banach space such that every subspace of the dual has \( c_0 \)-index greater than \( \omega^\xi \). Does there exist a subspace \( Y \) such that every subspace of the quotient has \( X/Y \) has Bourgain \( \ell_1 \)-index greater than \( \omega^\xi \)？

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