What do quantum “weak” measurements actually measure?

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Abstract

A precise definition of “weak [quantum] measurements” and “weak value” (of a quantum observable) is offered, and simple finite-dimensional examples are given showing that weak values are not unique and therefore probably do not correspond to any physical attribute of the system being “weakly” measured, contrary to impressions given by most of the literature on weak measurements.

A possible mathematical error in the seminal paper introducing “weak values” is explicitly identified. A mathematically rigorous argument obtains results similar to, and more general than, the main result of that paper and concludes that even in the infinite-dimensional context of that paper, weak values are not unique. This implies that the “usual” formula for weak values is not universal, but can apply only to specific physical situations.

The paper is written in a more pedagogical and informal style than is usual in the research literature in the hope that it might serve as an introduction to weak values.

1 Introduction

I have just spent several months pondering the implications of a 2 cm stack of recent papers concerning “weak measurements” in quantum mechanics. They include papers originally introducing this concept, various rederivations and extensions, experimental “weak” measurements, and their use to supposedly resolve “Hardy’s paradox”.

I have come to the conclusion that the concept of “weak measurement”, as presented in this literature, cannot withstand careful scrutiny. That does not mean that there can be no truth or value in it, but that generally accepted interpretations of its meaning are mistaken. This note explains why.

It will be written from a semi-historical viewpoint. “Semi” because it seems clearest to first introduce the basic ideas in a simple, finite-dimensional context. Then we shall describe the more complicated infinite-dimensional situation in which they were first introduced, resulting in the widely quoted formulas \( \langle \psi_f, A \psi_{in} \rangle / \langle \psi_f, \psi_{in} \rangle \) or \( \Re(\langle \psi_f, A \psi_{in} \rangle / \langle \psi_f, \psi_{in} \rangle) \) for the “weak” value of the observable \( A \), given that the pre-measurement state was \( \psi_i \), and that after measurement the state was “post-selected” to be \( \psi_f \). (All of these terms will be precisely defined later; they are included here only to orient readers who may

\footnote{For contact information, go to \url{http://www.math.umb.edu/~sp}}
have some prior acquaintance with weak values.) Then we shall observe that though the concept of “weak measurement” may have some valid uses, we see no good reason to think that the results of most “weak” measurements will be given by one of these formulas.

2 Reader’s guide to this paper

The aim of this paper is to share what I have learned in my study of “weak values” over the past few months. It is written primarily for physicists. (I am a mathematician.) It is hoped that it could serve as an introduction to “weak values”. In order to avoid possible misunderstandings, it begins at a level more elementary than would typically be assumed by a journal article. However, the reader is assumed to be thoroughly familiar with quantum mechanics, particularly in finite dimensions at the level of the book *Quantum Computation and Quantum Information* by Nielsen and Chuang [2]. Readers who are already familiar with “weak values” can skim or skip the introductory Sections 3 and 4.

The main ideas of the paper are presented in Section 5, “A simple finite-dimensional model”. The main new result is the observation that weak values are not unique. I suspect that for many readers, this will be all they care to learn about weak values.

The presentation of that section may seem quite different from most presentations in the literature. It would be natural for a reader unfamiliar with weak values to wonder if it is oversimplified, particularly since the model is finite dimensional, whereas most presentations in the literature are set in a more complicated infinite dimensional context. Section 6 discussing the traditional approach is written for such readers. It obtains the usual formula for a weak value in the traditional infinite dimensional context in what we hope is a mathematically rigorous way.

So-called “postselection” is fundamental to the traditional approach. We have described it in a mathematically precise way but have given no examples. That is because despite the mathematical simplicity of the concept, we do not know of any physically realistic examples which could be presented using a reasonable space and without elaborate diagrams (which we lack the facilities to prepare efficiently). The most enlightening examples we have seen are related to “Hardy’s paradox”. Readable descriptions of that can be found in [13] and [5].

Section 9 presents another simple proof that “weak values” are not unique. Since it is closer to the methods common in the literature (such as the seminal paper [3] of Aharonov, Albert, and Vaidman) than the previous exposition, it may be more congenial to those already familiar with weak values. As described there, it was discovered after the rest of the paper had already been written and typeset. Were I writing it again, I might try to find a way to put this new proof at center stage, but at this stage, it doesn’t seem worth the effort. The exposition as originally written still seems more suitable for beginners.

Appendices 1 and 2 discuss mathematical subtleties which I suspect will be
of limited or no interest to most physicist readers. Since I had to work them out to be confident that the mathematics was rigorous, I thought I might as well include them.

3 Notation and point of view

We attempt to stay as close as possible to traditional physics notation, reverting to notation more common in mathematics only when it seems less ambiguous or complicated. The inner product of vectors \( v, w \) in a complex Hilbert space \( H \) will be denoted \( \langle v, w \rangle \), with the physics convention that this be linear in the second variable \( w \), and conjugate-linear in the first variable \( v \). The norm of a vector \( v \) will usually be denoted as \( |v| := \langle v, v \rangle^{1/2} \), but (with due notice to the reader) this may occasionally be changed to \( ||v|| \) when dealing with Hilbert spaces of functions, to distinguish the \( L^2 \) norm of a function from its absolute value.

The mathematics of quantum mechanics describes a (pure) state of a physical system as a ray in a complex Hilbert space, a “ray” being defined as the set of all multiples \( \alpha v \), \( \alpha \) complex, of a nonzero vector \( v \). Thus a ray is uniquely described by any nonzero vector in it, which may be normalized to norm 1 if desired. In a context of detailed calculations, normalization is often helpful, but otherwise it may merely introduce nuisance numerical factors. We shall normalize only where it seems helpful. Also, we follow convention by speaking, regarding a vector \( v \), of the (pure) “state” \( v \) instead of the more pedantic “ray determined by \( v \”).

The projector to a subspace \( E \) will typically be denoted \( P_E \), in place of the common but unnecessarily complicated physics notation \( \sum_i |e_i \rangle \langle e_i| \), where \( \{e_i\} \) is an orthonormal basis for \( E \). When \( E \) is the entire Hilbert space of states, \( P_E \) is called the identity operator and denoted \( I := P_E \). When \( E \) is the one-dimensional subspace spanned by a vector \( w \), we may write \( P_w \) for \( P_E \). We shall make constant use of the formula for \( P_w \) when \( |w| = 1 \): \( P_w v = \langle w, v \rangle w \). Note, however, that under our convention, \( P_w = P_{w/|w|} \), so this formula for \( P_w \) only applies for \( |w| = 1 \).

Our mathematical formulation of quantum mechanics follows that of Chapter 2 of the book of Nielsen and Chuang [2]. We shall consider only projective measurements (as opposed to the more general “positive operator valued” measurements emphasized there).

Given a system in a pure state and a subspace \( E \), the projective measurement associated with \( E \) answers the question: “After the measurement, is the state of the system in \( E \) or the orthogonal complement \( E^\perp \) of \( E \)?” (According to usual interpretations of quantum mechanics, those are the only possibilities for such a projective measurement.)

If before that projective measurement the system was in pure state \( v \), then after the measurement it will be either in (unnormalized) state \( P_E v \) or \((I - P_E)v \), with respective probabilities \(|P_E v|^2/|v|^2\) and \(|(I - P_E)v|^2/|v|^2\) ([2] p. 87 ff.). The measurement is sometimes said to “project” the pre-measurement state \( v \)
Orthogonal subspaces with \( \sum v \).

Suppose we decide to choose randomly one of the pure states. The state is \( w \) by a vector \( w \), which for normalized states \( v \) and \( w \) will occur with probability \( |P_w v|^2 = |\langle v, w \rangle|^2 \). Note also that when \( E \) is the entire pure state space, so that \( P_E = I \), the measurement has no effect on the system.

More generally let \( E_1, E_2, \ldots \) be a finite or infinite collection of pairwise orthogonal subspaces with \( \sum_i P_{E_i} = I \). The collection of projectors \( \{ P_{E_i} \} \) is called a resolution of the identity. With any such collection is associated a projective measurement whose result is to “project” an initial (pre-measurement) pure state \( v \) onto one of the \( P_{E_i} v \) with probability \( |P_{E_i} v|^2 / |v|^2 \).

The above refers to pure states. We shall deal mainly with pure states, but it is sometimes difficult to discuss measurements of pure states without introducing the slightly subtle concept of “mixed” state. For simplicity of exposition, we shall review this concept in a context less general than necessary, but as general as we shall need it.

Let the unit vectors \( v_1, v_2, \ldots \) represent a (finite or countably infinite) collection of pure states and \( p_1, p_2, \ldots \) corresponding probabilities which sum to 1. Suppose we decide to choose randomly one of the pure states \( v_i \) with probability \( p_i \) and then perform some measurements on the chosen state. Just before the random choice, we say that the system is in a mixed state which we’ll denote by the traditional symbol \( \rho \). Mathematically, a mixed state \( \rho \) is described by a positive operator of trace 1, traditionally called a density matrix: for the above example, \( \rho = \sum_i p_i P_{v_i} \). If after the random choice we measure to see if the (pure) state of the system lies in a subspace \( E \), the probability that it does lie in \( E \) is \( \text{tr} (\rho P_E) \). When all but one of the probabilities \( p_i \) vanish, say the nonvanishing probability is \( p_k = 1 \), then \( \rho = P_{v_k} \) and \( \text{tr} (\rho P_E) = \text{tr} (P_{v_k} P_E) = |P_{E v_k}|^2 / |v_k|^2 \), which is the same as the corresponding formula for a system in pure state \( v_k \).

For example, suppose the Hilbert space of pure states is two-dimensional with orthonormal basis \( e_1, e_2 \), and consider a measurement to determine if a particle with initial normalized state \( v \) is (after the measurement) in state \( e_1 \) or \( e_2 \). (This corresponds to a projective measurement with respect to the resolution of the identity \( P_{e_1}, P_{e_2} \).) If we know that the measurement has been made but have not been told the result, then from our point of view the system is in a mixed state: it is in pure state \( e_1 \) with probability \( |P_{e_1} v|^2 = |\langle e_1, v \rangle|^2 \) and in pure state \( e_2 \) with probability \( |\langle e_2, v \rangle|^2 \). But if we know the result of the measurement, then the system is in a pure state, either \( e_1 \) or \( e_2 \), according to the result.

A subtle aspect of the concept of “mixed state” is that the state of the system may depend on the observer: an observer who knows the result of a measurement will “see” a pure state, but the state will be mixed from the perspective of an observer who knows that a measurement has been made (or

\[ \text{This terminology is suggestive but slightly misleading because to “project” } v \text{ in this sense is not the same as applying a single projector to } v. \]

\[ \text{Since our formulation of “mixed state” does not exclude the case of exactly one nonvanishing } p_i, \text{ a pure state is technically a mixed state. Some authors use “mixed state” to denote what we would have to call a “non-pure mixed state”.} \]
will be made) but does not know the result.

4 The problem which “weak” measurements address

Consider the following trial of an experiment which is to be repeated many times. A quantum system $S$ is prepared in a given initial state $s_{\text{in}}$. We have an observable (Hermitian operator) $A : S \rightarrow S$. For simplicity, assume that $A$ has a complete set of eigenvectors. We would like to measure the expectation of $A$ in the state $s_{\text{in}}$, namely $\langle s_{\text{in}}, A s_{\text{in}} \rangle$, without changing the state of $S$ (from $s_{\text{in}}$ to something else).

What was just proposed may seem impossible because measuring $A$ constitutes a projective measurement which will project $s_{\text{in}}$ onto one of the eigenvectors of $A$ (thus changing it, unless $s_{\text{in}}$ happened to be an eigenvector to start). However, we did not say we needed to measure $A$, but only its expectation in the state $s_{\text{in}}$, which is not quite the same thing. And we are willing to weaken the requirement of not changing $s_{\text{in}}$ to only changing $s_{\text{in}}$ negligibly.

Why would we want to do this? Well, we might want to perform a subsequent experiment on $s_{\text{in}}$ and speak of “the expectation of $A$ in the state $s_{\text{in}}$ given that” the subsequent experiment yielded a particular result. Classically, this would be called a conditional expectation of $A$ given the particular result of the subsequent experiment.

In practice, the “subsequent experiment” is usually a so-called “postselection” to a given final state $s_{\text{f}}$. We measure $\langle s_{\text{in}}, A s_{\text{in}} \rangle$ while changing $s_{\text{in}}$ negligibly and then perform a projective measurement to see if the final state is $s_{\text{f}}$ (as described in a preceding section). If the final state is $s_{\text{f}}$, we record the measurement of $\langle s_{\text{in}}, A s_{\text{in}} \rangle$; otherwise we discard it. The average of all the recorded measurements $\langle s_{\text{in}}, A s_{\text{in}} \rangle$ is taken as (a good approximation to) the average value of $A$ in all states $s_{\text{in}}$ which project to $s_{\text{f}}$ in the final step. Classically, this would be called the conditional expectation of $A$ given that the final measurement yielded $s_{\text{f}}$.

The reader who feels a sense of unease at this proposal should be congratulated on his perspicuity. Indeed, serious objections can be raised to the proposal, but we are presenting it as it seems typically (though usually not so explicitly) presented in the research literature. Objections will be discussed later. Right now we are trying to present a rapid overview.

Part of the above proposal is both sound and experimentally feasible. A seminal paper of Aharonov, Albert and Vaidman [3] cleverly expands a theory of measurement of von Neumann to accomplish measurement of $\langle s_{\text{in}}, A s_{\text{in}} \rangle$ while negligibly changing the state of the system.

\footnote{Or her, of course. We follow the long-standing and sensible grammatical convention that either “his” or “her” means “his or her” or “her or his” in contexts like this. We find it distracting to alternate “his” with “her”, and make no attempt to equalize the number of each.}
They couple the original system $S$ to an auxiliary “meter” system $M$ in such a way that a particular “meter” observable in the meter system has the same mathematical expectation as $A$, and in addition the coupling between the meter and the original system $S$ is so weak that measuring the meter observable negligibly affects the state of $S$.

This involves a tradeoff between weakness of the coupling and accuracy of individual meter measurements. A weaker coupling generally entails wider dispersion of the meter measurements. The average of a large number of meter measurements will approximate $\langle s_{in}, A s_{in}\rangle$, but a very large number may be required when the coupling is very weak.

5 A simple finite-dimensional model

5.1 Introduction of the model

We present a simple model which illustrates how weak measurements can be performed. Let $S$ denote the Hilbert space of the system which is the primary object of study, and $M$ the Hilbert space of an auxiliary “meter” system. We will take both $S$ and $M$ to be two-dimensional with orthonormal bases $s_0, s_1$ for $S$ and $m_0, m_1$ for $M$. No confusion should result if we use the same symbol $S$ for the physical system itself and for its Hilbert space, and similarly for $M$.

The Hilbert space of the composite system comprising both $S$ and $M$ is the tensor product $S \otimes M$, which has the orthonormal basis $\{s_i \otimes m_j | 0 \leq i, j \leq 1\}$. If the system $S$ is in state $s$ and system $M$ in state $m$, then the composite system is in state $s \otimes m$, and conversely. Not every state of $S \otimes M$ can be written in the form $s \otimes m$. Those which can be written in this way are are called product states, and those which cannot are called entangled. An example of a state which can be proved to be entangled is $s_0 \otimes m_0 + s_1 \otimes m_1$.

5.2 The states of $S$ and of $M$ derived from the state of $S \otimes M$

When the composite system $S \otimes M$ is in an entangled pure state, each individual system $S$ or $M$ is in mixed state, which is never pure. This follows from the
general rule ([2], p. 107) for passing from a mixed state \( \rho \) (which could be pure) of the composite system \( S \otimes M \) to states of its factors \( S \) and \( M \). The rule is that the state of \( S \) is obtained by taking the partial trace with respect to \( M \) of \( \rho \), denoted \( \text{tr}_M \rho \). A consequence of this rule is that when \( S \otimes M \) is in a pure state, a (projective) measurement in \( M \) can affect the state of \( S \) if and only if the state of \( S \otimes M \) is entangled.\(^6\)

Appendix I reminds the reader of the definition and properties of the partial trace. One very useful calculation done there encapsulates most of the properties of the partial trace which will be used in our discussion of weak values. It goes as follows. Let \( \{m_i\} \) be an orthonormal basis (finite or infinite) for \( M \) and \( r \) a pure state in \( S \otimes M \) with \( |r| = 1 \) It is easy to show that \( r \) may be uniquely written as

\[
    r = \sum_i s_i \otimes m_i \quad \text{with} \quad s_i \in S \quad \text{and} \quad \sum_i |s_i|^2 = 1. \tag{1}
\]

(The \( s_i \) need not be orthogonal.) Considered as a mixed state, \( r \) is represented by the density matrix \( P_r \), and the corresponding mixed state of \( S \) is \( \text{tr}_M P_r \). The calculation yields

\[
    \text{tr}_M P_r = \sum_i |s_i|^2 P_{s_i}, \tag{2}
\]

which explicitly exhibits the mixed state of \( S \) corresponding to the pure state \( r \) of \( S \otimes M \) as a convex linear combination of pure states of \( S \). This rule for passing from a pure state of \( S \otimes M \) to a mixed state of \( S \) will be illustrated by an example which will also illustrate other subtleties.

Take both \( S \) and \( M \) to be two-dimensional with orthonormal bases \( s_0, s_1 \) and \( m_0, m_1 \), respectively. Suppose we perform the following experiment a large number of times. Start with the composite system \( S \otimes M \) in the normalized entangled state

\[
    e := (s_0 \otimes m_0 + s_1 \otimes m_1)/\sqrt{2}.
\]

Then perform a projective measurement in the meter system to determine if it is in state \( m_0 \) or \( m_1 \). (This corresponds to a projective measurement in the composite system with respect to the resolution of the identity \( I \otimes P_{m_0}, I \otimes P_{m_1} \).) The result of the projective measurement will be \( m_0 \) with probability \( |(I \otimes P_{m_0})e|^2 = 1/2 \), and in this case the post-measurement state will be the product state \( (I \otimes P_{m_0})e = s_0 \otimes m_0/\sqrt{2} \). (The observable \( I \otimes P_{m_0} \) corresponds to measuring \( P_{m_0} \) in the meter system and nothing in the \( S \) system.)

\(^6\)It is immediate that if a pure state of \( S \otimes M \) is not entangled (i.e., is a product state \( s \otimes m \)), then a measurement in \( M \) cannot affect the state \( s \) of \( S \). To see that if a pure state is entangled, then a measurement in \( M \) will change the state of \( S \), let \( m_0, m_1, \ldots \) be an orthonormal basis for \( M \). Then a general normalized pure state \( r \) of \( S \otimes M \) can be written uniquely as \( r = \sum s_i \otimes m_i \), with \( s_i \in S \) satisfying \( \sum_i |s_i|^2 = 1 \).

A projective measurement in \( M \) with respect to the basis \( \{m_i\} \) corresponds to a projective measurement in \( S \otimes M \) with respect to the resolution of the identity \( \{I \otimes P_{m_i}\} \). This results in outcome \( m_k \) occurring with probability \( |s_k|^2 \), and when \( m_k \) does occur, the resulting state is the pure state \( (I \otimes P_{m_k})r = s_k \otimes m_k \). That is, the measurement changes \( r \) into \( s_k \otimes m_k \) with probability \( |s_k|^2 \). In order for this not to be an actual change, \( r \) must be a multiple of \( s_k \otimes m_k \), which says that \( r \) is a product state.
corresponding state of \( S \) expressed as a density matrix is \( \text{tr}_M P_{s_0 \otimes m_0} = P_{s_0} \), which expressed as a vector state is \( s_0 \). Similarly, it will be found to be \( m_1 \) with probability \( 1/2 \), in which case the post-measurement state will be \( s_1 \otimes m_1 \), with \( S \) in state \( s_1 \).

This seems to imply that a measurement in the meter system can affect the state of \( S \), and indeed it can, according to what seems the generally accepted physical interpretation of the mathematics. (This seems the interpretation of [2] and is also our interpretation.) But the way in which it affects the state of \( S \) is subtle and worth further examination.

In principle, the two systems \( S \) and \( M \) could be spacelike separated, so that, according the principle of relativistic locality, a measurement in \( M \) at a particular time (in some Lorentz frame) could not affect a measurement in \( S \) at the same time. That implies that if the \( S \) system can be said to have a definite state, that state cannot be affected by a particular \( M \) measurement. This seems to contradict the interpretation which we adopted in the preceding paragraph.

The contradiction is one manifestation of a pervasive tension between “realism” and “locality”, two philosophical concepts which are typically only vaguely defined in the literature. It would seem that if we accept that system \( S \) must have a definite state (one aspect of “realism”) and that a measurement in a spacelike separated \( M \) at a particular time can affect that state at that time, then the principle of locality cannot hold.

The most common resolution of this contradiction seems to be to abandon “locality” as a general principle (i.e., reject the proposal that measurements in the meter system at a particular time cannot influence the \( S \) system at the same time) and replace it by the weaker statement that measurements in the meter system cannot be used to send superluminal messages (i.e., messages which travel faster than light) to the \( S \) system.

That removes the contradiction, but leaves one a little uneasy that although there seems no obvious way that the mathematics of quantum mechanics could facilitate superluminal communication, nevertheless some clever person might someday find a way to do it.

We will accept this resolution because without something like it, we could not use the accepted physical interpretations of the mathematics of quantum mechanics. The resolution applies as follows to the experiment introduced above of measuring for \( m_0 \) or \( m_1 \) in the meter system.

Perform the experiment a large number of times, and divide the outcomes into two classes, those which resulted in \( m_0 \) (Class 0) and those resulting in \( m_1 \) (Class 1). Then any measurements made in \( S \) for the Class 0 states will have identical statistics to measurements made in a single system \( S \) (i.e., forgetting entirely about \( M \)) in state \( s_0 \), and similarly the Class 1 states will have statistics identical to those of system \( S \) in state \( s_1 \). The combined classes will have statistics identical to those of a mixed state which is in state \( s_0 \) with probability \( 1/2 \) and \( s_1 \) with the same probability.

\(^7\)Including this literature! We are not trying to prove a theorem here, but to communicate a point of view. An attempt at careful definitions (which would not be easy) would only be a distraction.
Although the $S$ observer may not realize it, half the time he is measuring in state $s_0$, and half the time in $s_1$. Considered as a density matrix (trace one positive operator) on $S$, his state is $\frac{1}{2}P_{s_0} + \frac{1}{2}P_{s_1} = I/2$ until he is informed of the result of the meter measurement. If the measurement in the meter system results, say, in outcome $m_0$, and if the $S$ observer is informed of this fact, then he will condition his future calculations on this fact. When so conditioned, his state becomes $P_{s_0}$ (considered as a density matrix) or $s_0$ (considered as a vector).

Only in this weak statistical sense, can the existence or nonexistence of a measurement in $M$ (and its result) influence the state of $S$. But for $S$ to perceive this influence seems to require “classical” (i.e., not faster than light) communication between $M$ and $S$.

### 5.3 Introduction to “weak” measurements

We have noted that when $S \otimes M$ is in a pure state $s \otimes m$, a projective measurement in $M$ cannot affect the state of $S$. More generally, the effect of a measurement in $M$ on the state of $S$ is expected to be arbitrarily small if the pre-measurement pure state of $S \otimes M$ is close enough to a product state. To make this precise, we would have to commit ourselves to definitions of “effect of a measurement in $M$” and “close enough to a product state”. When the Hilbert spaces $S$ and $M$ are finite dimensional, so are the linear spaces in which the mixed states reside, and since all norms on a finite dimensional normed linear space are equivalent, closeness can be measured in any convenient norm.

For example, we could measure the distance between two mixed states $\rho, \sigma$ using the trace norm: $||\rho - \sigma||_{tr} := \sqrt{\text{tr}((\rho - \sigma)^2)}$. The reader who wants a definite definition may use the one just given, though other norms may be easier to calculate in specific contexts.

For pure states in $S \otimes M$, the reader will probably find it easier to think in terms of the Hilbert space norm instead of the trace norm. (We didn’t formulate our definition in terms of the Hilbert space norm because we needed it to apply also to mixed states, in order to talk about the “effect of a measurement in $M$ on the (generally mixed) state of $S$”.)

In infinite dimensions, more care would be necessary. These issues seem not to have been considered in the literature on weak measurements. We do not pursue them here because our aim is to explain the concept of weak measurements in the context in which they historically arose, in which technical mathematics would only be a distraction.

For the rest of this section, we assume that all Hilbert spaces occurring are finite dimensional. This implies that all Hermitian operators occurring can be written as a linear combination (with coefficients the eigenvalues of the operator) of orthogonal projectors.

With these preliminary observations out of the way, we can describe the idea of weak measurement in a more precise way. Suppose we are given a reproducible, normalized pure state $s$ of $S$. By “reproducible” we mean that we have
an apparatus that can produce any number of replicas of this state. Suppose we are also given an observable (Hermitian operator) $A$ on $S$. Our goal is to determine the expectation $\langle s, As \rangle$ of $A$ to arbitrary accuracy by making measurements only in $M$ which have arbitrarily small effect on the pre-measurement state $s$ of $S$. We shall call these “weak” measurements.

We now describe a simple way to do this. Let $m_0, m_1$ be an orthonormal basis for $M$. Let $\epsilon > 0$ be a real parameter which will measure the strength of the effect of a “meter” measurement in the $m_0, m_1$ basis on the state of $S$.

First we consider the special case in which $A$ is both Hermitian and unitary (as are the Pauli matrices, for example). Let $s$ be a normalized pure state of $S$, and $m_0, m_1$ an orthonormal basis for $M$. Start with the pure state of $S \otimes M$

$$r := r(\epsilon) := s \otimes m_0 \sqrt{1 - \epsilon^2} + As \otimes m_1 \epsilon .$$

The assumptions that $A$ is unitary and $s$ is normalized imply that $r$ is normalized: $|r| = 1$.

From the point of view of $S$, this state is very close to the given state $s$ when $\epsilon$ is small. More precisely, the partial trace with respect to $M$ of $P_r$, $\text{tr}_M P_r$, is arbitrarily close to $P_s$ for $\epsilon$ sufficiently close to 0.

Let

$$B := \begin{bmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{bmatrix}$$

be the matrix with respect to the $m_0, m_1$ basis of a Hermitian operator $B$ on $M$. Our main conclusion will require the assumption that $B_{00} = 0$, but we initially include an arbitrary $B_{00}$ for purposes of later discussion.

A short calculation reveals that

$$\langle r(\epsilon), (I \otimes B) r(\epsilon) \rangle = B_{00}(1 - \epsilon^2) + 2\epsilon \sqrt{1 - \epsilon^2} \langle s, As \rangle \Re(B_{10}) + \epsilon^2 \langle As, As \rangle B_{11} .$$

Under the assumption $B_{00} = 0$,

$$\lim_{\epsilon \to 0} \frac{\langle r(\epsilon), (I \otimes B) r(\epsilon) \rangle}{\epsilon} = 2\langle s, As \rangle \Re(B_{10}) .$$

This says that when $\Re B_{10} \neq 0$, measuring the average value of $B$ in $M$ and normalizing appropriately (by dividing by $\epsilon$) will approximate the average value of $A$ in state $s$ of $S$ up to an inessential constant factor. Moreover, we shall

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8 This does not contradict the supposed impossibility of cloning quantum states (under certain restricted hypotheses a “no-cloning” theorem can be proved) because we do not require to be able to copy any unknown state given to us, only that we have a device which can reliably produce some particular state in which we are interested.

9 In the literature, the term “weak measurement” may sometimes refer to experiments which involve not only weak measurements in our sense, but also “postselection” to be described below.
show in a moment that this approximation is obtained with negligible change (for small $\epsilon$) in the state $s$ of $S$.

Although we assumed that $A$ was unitary in order to make the algebra simple, similar weak measurement protocols can easily be obtained for arbitrary Hermitian operators $A$ (on a finite dimensional $S$). The following are two possible approaches.

1. If $P$ is a projector, then $U := 2P - I$ is unitary (and Hermitian), and $P = (U + 1)/2$. Since we showed above how to obtain $\langle s, U s \rangle$ to arbitrary accuracy by measurements in $M$, the simple algebraic transformation $\langle s, Ps \rangle = \langle s, U s \rangle/2 + 1/2$ gives us $\langle s, Ps \rangle$ with the same (negligible for small $\epsilon$) disturbance of the state $S$. The case of general Hermitian $A$ is then obtained by writing $A$ as a linear combination of projectors.

This approach has the advantage of conceptual simplicity, but may be inconvenient in specific applications.

2. The case of general Hermitian $A$ can also be obtained by normalizing \( \langle s, Ps \rangle \), i.e., replacing $r(\epsilon)$ in (3) by the normalized state

$$\tilde{r}(\epsilon) := \frac{r(\epsilon)}{|r(\epsilon)|} s \otimes m_0 \sqrt{1 - \epsilon^2} + As \otimes m_1 \epsilon \over \sqrt{1 - \epsilon^2 + \epsilon^2 |As|^2}.$$  \quad (6)

Though the algebra is slightly messier, \( \tilde{r}(\epsilon) \) with $r(\epsilon)$ in place of $r(\epsilon)$ is still obtained:

$$\lim_{\epsilon \rightarrow 0} \frac{\langle \tilde{r}(\epsilon), (I \otimes B)\tilde{r}(\epsilon) \rangle}{\epsilon} = 2 \langle s, As \rangle \Re(B_{10}). \quad (7)$$

Taking $B_{10} := 1/2 =: B_{01}$, we see that measuring the average value of $B$ relative to $\epsilon$ will yield the average value of $A$ in state $s$ to arbitrary accuracy. If the measurement of $B$ has arbitrarily small effect on the state of $S$ for sufficiently small $\epsilon$, then $B$ “weakly” measures $\langle s, As \rangle$. (This is essentially our definition of “weakly measures”, which will be formalized in Definition 1 below.)

It is not hard to see that for small $\epsilon$, measuring $B$ does not appreciably affect the state of $S$. For small $\epsilon$, $\tilde{r}(\epsilon)$ is arbitrarily close to $s \otimes m$, and the corresponding mixed state of $S$, namely $tr_M P_{\tilde{r}}$, is arbitrarily close to $tr_M P_{s \otimes m} = P_s$ (which is the density matrix equivalent of vector state $s$). Here we have used the facts that in finite dimensions, $r \mapsto P_r (|r| = 1)$ and $\rho \mapsto tr_M \rho$ are continuous relative to the topologies induced by any norms.

Let $b_0, b_1$ be orthonormal eigenvectors of $B$. The measurement of $B$ will project $\tilde{r}(\epsilon)$ onto either $(I \otimes P_{b_0})\tilde{r}(\epsilon)$ or $(I \otimes P_{b_1})\tilde{r}(\epsilon)$. For small $\epsilon$, this projection will be arbitrarily close to either $(I \otimes P_{b_0})(s \otimes m) = s \otimes P_{b_0} m$ or $s \otimes P_{b_1} m$, respectively. Taking the first alternative for convenience, we may conclude that the post-measurement state of $S$, namely the density matrix $tr_M P_{(I \otimes P_{b_0})\tilde{r}}$, is arbitrarily close to $tr_M P_{(I \otimes P_{b_0})(s \otimes m)} = tr_M P_{s \otimes P_{b_0} m} = P_s$. This shows that for small $\epsilon$, measuring $B$ has arbitrarily small effect on the state of $S$.

Measuring the average value of $A$ without significantly changing the state of $S$ may seem too good to be true, but close scrutiny reveals that a tradeoff
is involved. Measuring the average value of $B$ requires measuring a very small quantity (on the order of $\epsilon$). To obtain a reliable average value for $B$ we will need to average a large number of individual observations. The smaller $\epsilon$, the larger the number of observations required. (From general statistical theory, we expect that the required number of observations should scale like $1/\epsilon^2$.)

In an ideal situation in which measuring $B$ produces only random errors, the random errors will average to zero, and the procedure described will be feasible. But systematic errors need not average out and can make the weak measurement infeasible when $\epsilon$ is too small. This drawback, or tradeoff, is implicit in all “weak measurement” schemes which we have seen.

For example, in a Stern-Gerlach experiment to measure the spin of particle, the slightest misalignment of the magnets can produce a systematic error which will cause the limit $\epsilon \to 0$ in (6) to fail to exist. Mathematically, this shows up in the necessity of the assumption $B_{00} = 0$. In a physical experiment, there is generally no way to assure that $B_{00}$ will be exactly zero. But if $B_{00}$ is small, we can hope to obtain meaningful experimental results consistent with (6) when $\epsilon$ is not too small.

Since we shall often refer to “weak measurements”, we offer the following physical definition of such measurements.

**Definition 1** Let $S$ be a quantum system and $A$ an observable on $S$. Let $s$ be a state of $S$ of which we can make an arbitrary number of copies. A weak measurement protocol is a procedure which can determine the expectation $\langle s, As \rangle$ of $A$ in the state $s$ to arbitrary accuracy while perturbing each copy of $s$ used in the procedure by an arbitrarily small amount.

Weak measurements are of particular interest when we want to use the (copies of the) state $s$ used in the weak measurement for future experiments. If we had measured observable $A$ in state $s$ in the normal way, it would project $s$ onto one of the eigenvectors of $A$, thus changing $s$ to that eigenvector.

A kind of “future experiment” which has received great attention in the literature is “postselection”. Indeed, interest in weak measurements stems from the seminal paper of Aharonov, Albert, and Vaidman [3] which considers weak measurements in the context of postselection and coins the term “weak value of a quantum variable”.

Let $f$ be a desired “final state” in $S$. A “postselection” experiment follows a weak measurement protocol by a projective measurement relative to the resolution of the identity $P_f \otimes I_M, (I_S - P_f) \otimes I_M$, where the subscripts on the

10 Previously, the initial state was called $s_{i\alpha}$ and the final state $s_f$ to correspond more closely to the notation $\psi_{i\alpha}$ and $\psi_f$ commonly used in the literature. But both notations are unnecessarily complicated. The only additional information contained in $s_f$ as opposed to $f$ is an implicit suggestion that it probably refers to a state in $S$. But since the final state $f$ always occurs in expressions like $f \otimes m \in S \otimes M$ which explicitly imply the suggestion, this additional information is redundant and complicates the typesetting.

While on the subject, we also warn the reader that earlier versions of this paper used $q$ instead of $f$ or $s_f$ for the final state. Should the reader encounter a $q$ in the context of a final state, it is probably a typo.
various identity operators denote the spaces on which they act. (In the future, we shall generally omit the subscripts, writing, for example, $P_f \otimes I, (I - P_f) \otimes I$ for the above.)

Suppose the state of $S \otimes M$ after the weak measurement protocol is a pure state (an atypical situation, but the easiest to discuss). Then informally, the postselection asks (and answers) the question: “Is the state of $S \otimes M$ after postselection in $[f] \otimes M$ or $[f]_M$, where $[f]$ denotes the subspace spanned by $f$, and $[f]_M$ its orthogonal complement. If the postselected state of $S \otimes M$ is in $[f] \otimes M$, then it is of the form $f \otimes r$ with $r \in M$, and the corresponding state of $S$ is $f$. Otherwise, the state of $S$ is a mixture of pure states which are orthogonal to $f$.

“Postselection” discards cases in which the postselected state of $S$ is not the pure state $f$. This amounts to replacing probabilities by probabilities conditional on successful (i.e., the postselected state of $S$ is $f$) postselection.

In the more typical case in which the state of $S \otimes M$ after the weak measurement protocol is a mixed state $\rho$ (a density matrix), then after the postselection it will be either in state $(P_f \otimes I)\rho(P_f \otimes I)/\text{tr}((P_f \otimes I)\rho(P_f \otimes I))$ or $((I - P_f) \otimes I)\rho((I - P_f) \otimes I)/\text{tr}(((I - P_f) \otimes I)\rho((I - P_f) \otimes I))$, so the postselection may be regarded as resolving the question: Does the postselected state $\sigma$ satisfy $(P_f \otimes I)\sigma = 0$ (successful postselection) or $(P_f \otimes I)\sigma = 0$?

The cumbersome language of mixed states can usually be finessed in the context of weak measurements in finite dimensions by noting that a sufficiently accurate weak measurement protocol starting with initial state $s \in S$ leaves $S$ in a state close to $s$. It is routine to show that this implies that the state of $S \otimes M$ is close to (the density matrix analog $P_s \otimes r$ of) a pure product state $s \otimes r$. Then one can apply the simpler discussion previously given to the pure state $s \otimes r$.

In the literature, the result of measuring the average value of an observable in a state $s$ by a procedure which negligibly disturbs the state (such as [5] above) followed by conditioning on successful postselection to a final state $f$ is said to result in a “weak value” of the observable. If in the above example we choose $B \rangle := 1/2 =: B_0$, we obtain the so-called “weak value”

$$
\lim_{\epsilon \to 0} \frac{\langle \tilde{r}(\epsilon), (P_f \otimes B)\tilde{r}(\epsilon) \rangle}{\epsilon \langle \tilde{r}(\epsilon), (P_f \otimes I)\tilde{r}(\epsilon) \rangle} = \frac{\langle As, P_f s \rangle / 2 + \langle s, P_f As \rangle / 2}{\langle s, P_f s \rangle} = \frac{(f, s)\langle As, f \rangle / 2 + \langle s, f \rangle\langle As, f \rangle / 2}{\langle f, s \rangle\langle s, f \rangle} = \frac{\text{Re} \langle f, As \rangle}{\langle f, s \rangle}. \tag{8}
$$

The mathematical procedure may be understood as follows. Asking the postselection question “Is $S$ in state $f$ or in a mixture of states orthogonal to $f$?” is the same as measuring $P_f \otimes I$, and the expectation of $P_f \otimes I$ in state $\tilde{r}(\epsilon)$ is the proportion of “yes” answers (in the limit of a large number of measurements). Measuring $P_f \otimes B = (P_f \otimes I)(I \otimes B)$ corresponds to simultaneously measuring $B$ and postselecting to $f$; its average is the average of the observable
whose value is the value of \( B \) for successful postselections and zero otherwise.
The conditional expectation of \( B \) given successful postselection is the average of the just mentioned observable \( P_f \otimes B \) divided by the probability of successful postselection. This conditional expectation is \( O(\epsilon) \), so we normalize by dividing by \( \epsilon \).11

The term “weak value” is not used consistently in the literature. The seminal paper [3] of Aharonov, Albert, and Vaidman introduced the concept of “weak value of a quantum variable” in a way conceptually similar to the above, but using a very different “meter system” along with a very different measurement procedure (which will be discussed in the next section). They identified \( \frac{\langle f, As \rangle}{\langle f, s \rangle} \) (a quantity which in general need not be real) as their “weak value”. Other authors (e.g., [10]) obtain the real part of this as their “weak value” (as we did above, using different methods). Some authors seem to define the “weak value” of an observable \( A \) to be one of these expressions. Most of the literature gives the impression that any “weak measurement” (in the sense of Definition 1) followed by postselection will result in a “weak value” \( \Re(\frac{\langle f, As \rangle}{\langle f, s \rangle}) \), no matter what the measurement procedures. One of the purposes of this note is to dispel this belief; a counterexample will be given below.

At this point we have to warn the reader that in order to continue with the description of the weak measurement process as typically presented in the literature (e.g. as in [3]), we need to temporarily use language which we think questionable, as will be discussed in detail later. Suppose we start with a state \( s \in S \), construct \( \tilde{r}(\epsilon) \in S \otimes M \) for suitably small \( \epsilon \), measure \( B \) in \( M \) (i.e., measure \( I \otimes B \) in \( S \otimes M \)), and then postselect to \( f \in S \). We do this as many times as necessary to obtain a reliable average of the results (i.e., measurements of \( B \) followed by successful postselection). That average is taken as an estimate of the expectation of \( B \) conditioned on postselection to \( f \), which (after normalization by division by \( \epsilon \)) in turn is used as an estimate of the expectation of \( A \) in the state \( s \), conditioned on the postselection of \( f \). The emphasized phrase is problematic, but to the best of my understanding, it accurately reflects the meaning of less precise language typical in the literature.

Why is the emphasized phrase problematic? After all, conditional expectations are uncontroversial in classical probability theory. Insight can be obtained by digressing to review classical conditional expectations on finite probability spaces.

Let \( \Sigma \) be a finite probability space, \( \Delta \) a subset of \( \Sigma \), and \( X \) a (real-valued) random variable on \( \Sigma \) taking on values \( x_1, x_2, \ldots, x_m \). Then the expectation

\[
\mathbb{E}[X | \Delta] = \frac{\sum_{x \in \Delta} x \cdot \mathbb{P}(X = x | \Delta)}{\mathbb{P}(\Delta)}
\]

is the conditional expectation of \( X \) given \( \Delta \).

---

11In the narrative, we have been speaking of measuring \( I \otimes B \) then postselecting to \( f \). We are modeling this as measuring \( I \otimes B \) and simultaneously postselecting (which is possible because \( P_f \otimes I \) and \( I \otimes B \) commute), which is the same as measuring \( P_f \otimes B \). The result of this simultaneous measurement is the result of the \( I \otimes B \) measurement if the postselection was successful, and zero otherwise. The narrative could be revised to speak of simultaneous measurement of \( B \) and postselection, but it seems easier to think about first measuring \( B \), then postselecting immediately after.
Exp(X) of X, is defined by

\[ \text{Exp}(X) := \sum_{i=1}^{m} x_i p(X = x_i) \]

where \( p \) denotes probability. Note that \( \text{Exp}(X) \) is necessarily a convex linear combination of the values \( x_i \) of X. Conditioning on \( \Delta \) means passing to a new probability space whose underlying set is \( \Delta \), with new probabilities \( p_{\Delta}(\cdot) \) obtained by dividing original probabilities by \( p(\Delta) \): for any subset \( K \) of \( \Delta \), the new probability \( p_{\Delta}(K) := p(K)/p(\Delta) \). Relative to this new probability space, the new expectation of \( X \), denoted \( \text{Exp}(X | \Delta) \), is

\[ \text{Exp}(X | \Delta) := \sum_{i=1}^{m} x_i p_{\Delta}(X = x_i) \]

which is again a convex linear combination of the values of \( X \).

The point is that in ordinary probability theory, conditional expectations of an observable (i.e., random variable) are always convex linear combinations of the values of the observable. In quantum mechanics, the possible measured values of an observable on a finite-dimensional Hilbert space are its eigenvalues. But (8) need not be a convex linear combination of the eigenvalues of \( A \). So it seems questionable to think of (8) as a conditional expectation (as seems the usual interpretation in the literature).

This is emphasized by the provocative title of Ahaharonov, Albert, and Vaidman’s seminal paper [3]: “How the Result of a Measurement of a Component of the Spin of a Spin-1/2 Particle Can Turn Out to be 100”. The measurement they seem to be talking about is not a single measurement (which certainly could be far from expected if experimental error is large), but a measurement of the average value of the spin, conditioned by a postselection. The authors seem to believe that they have explained how the average postselected value of a spin-1/2 measurement can be 100. We are not convinced that the measurements they describe are measurements of the spin, conditioned on successful postselection, as will be elaborated below.

Although the widely accepted (8) surely describes some quantum measurement, we see no reason that it should correspond to conditioning a measurement of the average value of \( A \) on postselection to \( f \). What it does correspond to for our toy model is given precisely by the left side of equation (8), namely conditioning the meter measurement on postselection to \( f \). The sentence

1. “The average value of the normalized (i.e., by division by \( \epsilon \)) meter measurement equals the average value of \( A \)”

is true (in the limit \( \epsilon \to 0 \)), but the sentence

2. “The average value of the normalized meter measurement conditioned on postselection to \( f \) equals the average value of \( A \) conditioned on postselection to \( f \)”
is either false, meaningless, or tautological, depending on how it is interpreted.

In order to speak meaningfully of “the average value of $A$ conditioned on postselection to $f$”, we need to say how this quantity is measured. If we measure $A$ in $S$, successfully postselect to $f$, and average the results, we do not necessarily obtain $\mathbb{R}(\langle f, As \rangle / \langle f, s \rangle)$.

Physically, this is because measuring $A$ can significantly disturb the original state $s$ of the system. To explicitly calculate what happens, suppose that $A$ has two eigenvalues $\alpha_1, \alpha_2$ with corresponding normalized eigenvectors $a_1, a_2$, and take $|s| = 1$. After $A$ is measured, $S$ is in state $a_i$ with probability $|\langle a_i, s \rangle|^2$, $i = 1, 2$. Subsequently, the postselection succeeds with (conditional) probability $|\langle f, a_i \rangle|^2$. The total probability that the postselection succeeds is

$$\sum_{i=1}^{2} |\langle a_i, s \rangle|^2 |\langle f, a_i \rangle|^2 .$$

Hence the conditional expectation of $A$ given that the postselection succeeds is

$$\frac{\alpha_1 |\langle a_1, s \rangle|^2 |\langle f, a_1 \rangle|^2 + \alpha_2 |\langle a_2, s \rangle|^2 |\langle f, a_2 \rangle|^2}{|\langle a_1, s \rangle|^2 |\langle f, a_1 \rangle|^2 + |\langle a_2, s \rangle|^2 |\langle f, a_2 \rangle|^2} .$$

(9)

It is easy to see that this need not equal (8), $\mathbb{R}(\langle f, As \rangle / \langle f, s \rangle)$. For example, (9) is a convex linear combination of $\alpha_1, \alpha_2$, and therefore cannot be arbitrarily large, whereas $\mathbb{R}(\langle f, As \rangle / \langle f, s \rangle)$ can be arbitrarily large when $f$ is nearly orthogonal to $s$ and the numerator $\langle f, As \rangle$ is not close to 0. (Such examples are easy to construct).

Therefore, if sentence 2 is to be true, the “average value of $A$ conditioned on postselection to $f$” cannot refer to normal measurement in $S$. To what could it refer? If it refers to measurement in $M$ of the normalized average value of $B$ postselected to $f$, then sentence 2 becomes a tautology, true by definition and containing no useful information.

5.4 “Weak values” are not unique

We think that the strongest argument that the “weak value” $\mathbb{R}(\langle f, As \rangle / \langle f, s \rangle)$ does not correspond to any simple physical attribute of the system $S$ which is being weakly measured is that one can obtain different expressions corresponding to weak measurements in $M$ followed by postselection to $f$. These different expressions will be obtained by reasoning conceptually identical to the reasoning which led to (8).

Consider the setup described above of a system $S$ coupled to a two-dimensional meter system $M$ with orthonormal basis $m_0, m_1$. Given a Hermitian operator $A$ on $S$ and an initial state $s$ of $S$, we want to approximate $\langle s, As \rangle$ to arbitrary accuracy by measuring the average value of an observable $B$ on $M$. The notation will be the same as in the previous discussion leading to the “weak value” (8).
Let $V$ be any unitary operator on $S$ satisfying $Vs = s$. In place of the state $	ilde{r}(\epsilon)$ of $S \otimes M$ in (10), define
\[
\tilde{r}(\epsilon) := (V \otimes I)\tilde{r}(\epsilon) = \frac{s \otimes m_0 \sqrt{1 - \epsilon^2 + A_s \otimes m_1 \epsilon}}{\sqrt{1 - \epsilon^2 + |A_s|^2}} = \frac{Vs \otimes m_0 \sqrt{1 - \epsilon^2 + V As \otimes m_1 \epsilon}}{\sqrt{1 - \epsilon^2 + |A_s|^2}} = \frac{s \otimes m_0 \sqrt{1 - \epsilon^2 + V As \otimes m_1 \epsilon}}{\sqrt{1 - \epsilon^2 + |A_s|^2}}.
\]

The condition $Vs = s$ guarantees that the corresponding state of $S$ will approximate $s$ for small $\epsilon$.

Define the operator $B$ on $M$ as in (11) with $B_{00} = 0$ and $B_{01} = 1/2 = B_{10}$. Since $V \otimes I$ and $I \otimes B$ commute,
\[
\langle \tilde{r}, (I \otimes B)\tilde{r} \rangle = \langle (V \otimes I)\tilde{r}, (I \otimes B)(V \otimes I)\tilde{r} \rangle = \langle \tilde{r}, (I \otimes B)\tilde{r} \rangle \approx \epsilon \langle s, As \rangle,
\]
and
\[
\lim_{\epsilon \to 0} \frac{\langle \tilde{r}(\epsilon), (I \otimes B)\tilde{r}(\epsilon) \rangle}{\epsilon} = \langle s, As \rangle,
\]
so averaging measured values of $I \otimes B$ in state $\tilde{r}$ and normalizing by dividing by $\epsilon$ is a weak measurement protocol in the sense of Definition 4.

As in the discussion leading to (3), suppose that after measuring $I \otimes B$ with $S \otimes M$ in state $\tilde{r}(\epsilon)$, we postselect the state of $S$ to $f \in S$ (which as previously explained means a successful measurement of $P_f \otimes I$ in $S \otimes M$). As before, the conditional expectation $E_\epsilon(B|f)$ of $B$ given success of the postselection is
\[
E_\epsilon(B|f) := \frac{\langle \tilde{r}(\epsilon), (P_f \otimes B)\tilde{r}(\epsilon) \rangle}{\langle \tilde{r}, (P_f \otimes I)\tilde{r} \rangle} \approx \frac{(1 - \epsilon^2)B_{00}\langle s, P_f s \rangle + 2\epsilon \sqrt{1 - \epsilon^2} \Re(B_{10}\langle s, P_f V As \rangle) + \epsilon^2 B_{11}\langle V As, P_f V As \rangle}{(1 - \epsilon^2)\langle s, P_f s \rangle + \epsilon^2 \langle V As, P_f V As \rangle}.
\]

Although we are assuming that $B_{00} = 0$ and $B_{10} = 1/2 = B_{01}$, we included these above so that $\langle \tilde{r}, (P_f \otimes I)\tilde{r} \rangle$ in the denominator could be read off by setting $B := I$ in the numerator. Also note that the normalization factor making $|\tilde{r}| = 1$ is the same in the numerator and denominator and hence cancels.

Under our assumption $B_{00} = 0$, the expression (12) for $E_\epsilon$ is of order $\epsilon$. As in the previous protocol leading to the traditional “weak value” (8), we normalize by dividing by $\epsilon$ and take the limit as $\epsilon \to 0$, obtaining as the result of our new protocol the new “weak value” for $\langle s, As \rangle$ conditional on postselection to $q$:
\[
\lim_{\epsilon \to 0} \frac{E_\epsilon(B|f)}{\epsilon} = \frac{\Re\langle s, P_f V As \rangle}{\langle s, P_f s \rangle} = \frac{\Re\langle f, V As \rangle}{\langle f, s \rangle},
\]
where the last inequality was obtained by manipulations similar to those leading to (8), to which this this reduces when $V = I$. 
To see that (8) and (13) need not be equal, take $S$ to be two-dimensional with orthonormal basis $s, s^\perp$. Define $A$ and $V$ by the following matrices with respect to this basis,

$$A := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad V := \begin{bmatrix} 1 & 0 \\ 0 & \eta \end{bmatrix}$$

with $|\eta| = 1$, so that $Vs = s$, $Vs^\perp = \eta s^\perp$, $As = s^\perp$ and $As^\perp = s$. Let $f := (s + s^\perp)/\sqrt{2}$. Then the numerator of (13) becomes

$$\Re\langle s, P_f A s \rangle = \Re(\langle s, P_f s^\perp \rangle \eta) = \Re(\langle s, (f, s^\perp) f \rangle \eta) = \Re\eta/2.$$

Equation (13) for this special case becomes

$$\lim_{\epsilon \to 0} \frac{E_\epsilon(B|f)}{\epsilon} = \Re\eta . \quad (14)$$

Obviously, this is not independent of $\eta$, as it would have to be if “weak values” were unique. Any real number between $-1$ and $1$ can be obtained as a “weak value” for the specified $s, f$ and $A$.

The title of this subsection, “weak values are not unique”, summarizes its conclusion. The reasoning leading this conclusion may be summarized as follows.

Given an observable $A$ on $S$ and a reproducible pure state $s$ of $S$, we defined a one-parameter family, indexed by $|\eta| = 1$, of weak measurement protocols (in the sense of Definition 1) to approximate $\langle s, As \rangle$ while negligibly changing the state $s$ of $S$. Applying one of these weak measurement protocols conditional on successful postselection of the (negligibly changed) state of $S$ to $f$ gives a conditional expectation called a “weak value” of $A$. This “weak value” is not independent of the protocol, i.e., not independent of $\eta$.

Since all these different “weak values” were obtained by conceptually identical reasoning, we see no reason to identify any one of them with some intrinsic, measurement-independent property of system $S$. The “standard” weak value (8) is one of these and seems generally identified in the literature (often implicitly) with the expectation of $A$ conditional on successful postselection to $f$. We think such an identification fallacious.

6 Historical summary and simplification of traditional approach

6.1 Overview

This section describes an approach similar, but not identical, to that pioneered by Aharonov, Albert, and Vaidman [3], which builds on a classical theory of measurement due to von Neumann [6]. The cited paper of Aharonov, Albert, and Vaidman will be called “AAV” below. It culminates in a formula for weak values identical to (13) except that the real part symbol $\Re$ is omitted.
The omission is rather curious. Their footnote 4 seems to recognize that the real part should be taken, but the uncontrolled (and questionable) approximations in the main text yield (13) without the real part. Subsequent literature mostly takes the real part, so when we refer to the “usual” weak value formula, we mean (expressed in the notation of [3], which is commonly used in the literature)

“usual” weak value of observable $A$
with initial state $\psi_{in}$ and post-selected to state $\psi_f$

$$= \Re \langle \psi_f, A\psi_{in} \rangle \langle \psi_f, \psi_{in} \rangle.$$ (15)

In notation used above and to be used below, this would read

“usual” weak value of observable $A$
with initial state $s$ and post-selected to state $f$

$$= \Re \langle f, As \rangle \langle f, s \rangle.$$ (16)

All literature on weak values known to me gives the impression that (15) (or (15) without the real part) is a universal formula which is to be expected in all experimental situations. We have seen in the last section that this is not so.

Most “derivations” (which really should be called “motivations”) of (15) in the literature are mathematically imprecise and overly complicated. We hope that a cleaner motivation may help clarify the domain of applicability of (15).

Our conclusion will be that (15) may well hold in some experimental situations, but that claims to its universality should be critically examined.

6.2 Preparation of the initial state for a weak measurement

The analysis of the previous section started with initial states called $\tilde{r}(\epsilon)$ in (6) or $\hat{r}(\epsilon)$ in (10), and it was implicitly assumed that such states are physically realizable. By contrast, the literature such as AAV generally starts with an initial product state $s \otimes m \in S \otimes M$ (product states are generally considered physically realizable) and obtains the desired entangled state to measure in the meter system by applying a unitary operator $U$ to obtain $U(s \otimes m)$ as the analog of our slightly entangled starting state $\tilde{r}(\epsilon)$. The unitary operator $U$ is usually considered as a time evolution operator $U = e^{-iHt}$ with $H$ the Hamiltonian, and its only function is to assure that the initial state $U(s \otimes m)$ is physically realizable. In our version of the AAV approach, $t$ will be considered as a small parameter, which subsequently will be called $\epsilon$.

Both von Neumann and AAV use a “meter” Hilbert space $M$ defined as the space of all square integrable complex-valued functions on the real line $\mathbb{R}$. This space is known to physicists as the Hilbert space of a single spinless particle in
one dimension and to mathematicians as $L^2(\mathbb{R})$, the space of all complex-valued functions $g = g(q)$ of a real variable $q$ which are square-integrable:

$$\int_{-\infty}^{\infty} |g(q)|^2 dq < \infty.$$ 

The inner product on $M = L^2(\mathbb{R})$ is defined as usual for $g, h \in M$ by

$$\langle g, h \rangle := \int_{-\infty}^{\infty} g(q)^* h(q) dq.$$ 

Two important Hermitian operators on $L^2(\mathbb{R})$ are the position operator $Q$ defined for $g \in L^2(\mathbb{R})$ satisfying certain technical conditions (which we do not list here because they will not be important to us) by

$$Qg(q) := qg(q) \text{ for all } q \in \mathbb{R}, \quad (17)$$

and the momentum operator $P$ defined by

$$(Pg)(q) := -i \frac{dq}{dq} \text{ for all } q \in \mathbb{R}. \quad (18)$$

For any real $\alpha$, if we expand $e^{-i\alpha P}$ in a formal power series and apply that to a smooth function $g \in L^2(\mathbb{R})$, using Taylor’s theorem we obtain

$$(e^{-i\alpha P} g)(q) = g(q - \alpha). \quad (19)$$

In other words, $e^{-i\alpha P}$ translates the graph of $g$ a distance of $\alpha$ units to the right. We shall denote this translate of $g$ by $g_\alpha$:

$$g_\alpha(q) := g(q - \alpha). \quad (20)$$

This calculation was purely formal, but it is well known how to formulate definitions under which it can be rigorously proved.\footnote{In careful treatments of the foundations of quantum mechanics such as \cite{8}, the momentum operator $P$ is defined as the infinitesimal generator of translations, so that $(e^{-i\beta P} f)(q) = f(q - \beta)$ is true by definition. Then it is later verified that $P = -i\frac{dq}{dq}$ on an appropriate domain.}

In the finite dimensional context considered in the last section, physical realizability of the entangled state to be meter measured ($\hat{r}$ or $\tilde{r}$ of the previous sections) is not an issue because any unitary operator on a finite dimensional Hilbert space can be physically realized to an arbitrary approximation by physically constructible quantum gates (\cite{2}, Chapter 4), and any state can obviously be obtained by applying a unitary operator to a product state.

The issue in infinite-dimensional contexts is sidestepped by both von Neumann and AAV via the assumption that the required Hamiltonian will be physically realizable. They both essentially use a Hamiltonian $H$ of the form

$$H = A \otimes P, \quad (21)$$

where $A$ is the Hermitian operator on $S$ whose expectation $\langle s, As \rangle$ is to be approximated by a meter measurement.\footnote{AAV uses a slightly more complicated expression, but in the context of their argument it is essentially the same as \cite{21}.}
7 AAV’s extension of Von Neumann’s theory of measurement

7.1 Our adaptation of Von Neumann’s general framework

Von Neumann [6] did not consider “weak” measurements *per se*. For the purpose of dealing with them, we shall modify (21) by inserting a small positive parameter $\epsilon$ to measure the strength of the interaction:

$$H(\epsilon) := \epsilon A \otimes P.$$  \hspace{1cm} (22)

(Alternatively, $\epsilon$ could be considered as a small time; the mathematics is insensitive to this sort of variation of the physical picture.) Eventually, we shall take a limit $\epsilon \to 0$ after appropriate normalization.

The setup of AAV is superficially different in that they interchange $Q$ and $P$, so that their “meter” measures the momentum of the “pointer” rather than its position. Since Fourier transformation implements a unitary equivalence which takes $P$ to $-Q$ and $Q$ to $P$, there is no essential mathematical difference between the two formulations.

The significance of the Hamiltonian (22) is most easily seen in the case in which $A$ has only one eigenvalue $\alpha$, so that $A = \alpha I$, where $I$ is the identity operator on an $n$-dimensional Hilbert space. Let $a_1, a_2, \ldots, a_n$ be an orthonormal basis for $S$. Then $S \otimes M$ identifies naturally with a direct sum of $n$ copies $[a_i] \otimes M \cong M$ of $M$, where $[a_i]$ denotes the one-dimensional subspace of $S$ spanned by $a_i$, and $\cong$ denotes isomorphism. Moreover, each of the copies $[a_i] \otimes M$ is invariant under $H(\epsilon)$ and may be naturally identified with $M = L^2(\mathbb{R})$. When so identified, $e^{-iH(\epsilon)}$ acts as translation by $\epsilon \alpha$: $(e^{-iH(\epsilon)}g)(q) = g(q - \epsilon \alpha)$.

Thus when $A$ has just one eigenvalue $\alpha$, $e^{-iH(\epsilon)}$ simply translates the initial probability distribution of pointer readings by $\epsilon \alpha$. If the starting meter state $m$ yields an average reading $\gamma = \langle m, Qm \rangle$, then the average reading in a state $e^{-iH(\epsilon)}(s \otimes m)$ will be $\gamma + \epsilon \alpha$. For simplicity of language, we choose the origin of $\mathbb{R}$ so that $\gamma = 0$, i.e., so that the average meter reading in state $m$ is 0.

We “read the meter” by starting with the reproducible product state $s \otimes m$, applying $e^{-iH(\epsilon)}$, and measuring $I \otimes Q$. Repeating this many times obtains an average value for $I \otimes Q$ of $\epsilon \alpha$, which for fixed $\epsilon$ can be determined to arbitrary accuracy by measuring sufficiently many times. Dividing by $\epsilon$ gives $\alpha$. It seems reasonable to hope that this measurement might be “weak” in the sense of Definition[1] because for small $\epsilon$, $e^{-iH(\epsilon)}(s \otimes m) = s \otimes e^{-i \epsilon \alpha P} m = s \otimes m_{\epsilon \alpha}$ is arbitrarily close in norm to $s_0 \otimes m_0$ (because translations are strongly continuous).\footnote{That is a hand-waving plausibility argument of the type often accepted in the physics literature, not a proof. It is actually wrong in the sense that it cannot easily be made into a proof; one sticking point is that the trace and operator norms are not equivalent when $M$ is infinite-dimensional. However, the weakness of the measurement can be justified by other methods.}

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[1] Reference or note to be included here.
The situation is similar if $A$ has several eigenvalues $\alpha_1, \alpha_2, \ldots, \alpha_n$ (not necessarily distinct) with a corresponding orthonormal basis $a_1, a_2, \ldots, a_n$ of eigenvectors: $A_i a_i = \alpha_i a_i, i = 1, 2, \ldots, n$. In that case, $S \otimes M$ decomposes as a direct sum $\oplus_i ([a_i] \otimes M)$, where each $[a_i] \otimes M$ is invariant under $H(\epsilon)$ and has a natural identification with $M = L^2(\mathbb{R})$. The action of $e^{-iH(\epsilon)}$ on $[a_i] \otimes M$ is as described above when $A$ had only one eigenvalue.

Denote the decomposition of the initial normalized state $s \in S$ as a direct sum of orthogonal states in $[a_i]$ as

$$s = \oplus_i s_i$$

with $s_i \in [a_i]$ (so that each $s_i$ is a multiple of $a_i$), and $\sum_i |s_i|^2 = 1$. Then

$$\langle s \otimes m, (A \otimes I)(s \otimes m) \rangle = \sum_i \alpha_i |s_i|^2$$

is the expectation $\langle s, As \rangle$ of $A$ in the state $s$, which we shall show is also equal to

$$\frac{\langle e^{-iH(\epsilon)}(s \otimes m), Q e^{-iH(\epsilon)}(s \otimes m) \rangle}{\epsilon}.$$  

(24)

Denoting (as always) by $g(\beta)$ the translate of a function $g$ by $\beta$, $g(\beta)(q) := g(q - \beta)$, we have

$$\langle e^{-iH(\epsilon)}(s \otimes m), (I \otimes Q)e^{-iH(\epsilon)}(s \otimes m) \rangle = \langle \oplus_j s_j \otimes (m)_{e\alpha_j}, \oplus_k s_k \otimes Q((m)_{e\alpha_k}) \rangle$$

$$= \sum_j |s_j|^2 \langle m_{e\alpha_j}, Q m_{e\alpha_j} \rangle$$

$$= \sum_j |s_j|^2 \int_{-\infty}^{\infty} dq \ m^* (q - e\alpha_j) \ q \ m(q - e\alpha_j)$$

$$= \sum_j |s_j|^2 \int_{-\infty}^{\infty} dq \ m^* (q + e\alpha_j) \ m(q)$$

$$= \sum_j |s_j|^2 e \alpha_j$$

$$= \epsilon \langle s, As \rangle.$$  

(25)

In the second line we used the orthogonality of the $s_j$ to convert a $\sum_{j,k}$ into a $\sum_j$, and the passage to the next to last line uses the assumption that the expectation $\langle m, Qm \rangle$ of $Q$ in the state $m$ is 0.

The expectation $\langle s, As \rangle$ of $A$ can be determined to arbitrary accuracy by averaging a sufficiently large number of measurements of pointer position $Q$ for small, fixed $\epsilon$, and finally dividing by $\epsilon$:

$$\langle s, As \rangle = \frac{\langle e^{-iH(\epsilon)s}(s \otimes m), (I \otimes Q)e^{-iH(\epsilon)s}(s \otimes m) \rangle}{\epsilon}.$$  

(26)

means under additional hypotheses. Appendix 2 examines the surprisingly delicate issue of “weakness” in more detail.
The normalized (by dividing by $\epsilon$, which is our definition of “normalized” in this context) conditional expectation of $(I \otimes Q)$ given postselection to $f \in S$ is
\begin{equation}
\frac{1}{\epsilon} \frac{\langle e^{-iH(\epsilon)}(s \otimes m), (P_f \otimes Q)e^{-iH(\epsilon)}(s \otimes m) \rangle}{\langle e^{-iH(\epsilon)}(s \otimes m), (P_f \otimes I)e^{-iH(\epsilon)}(s \otimes m) \rangle}.
\end{equation}
(27)

The numerator is the expectation in the state $e^{-iH(\epsilon)}(s \otimes m)$ of the product of the commuting observables $I \otimes Q$ and $P_f \otimes I$. This product has the value of $Q$ if the postselection succeeds and 0 otherwise. The denominator is the probability that the postselection succeeds.

Assuming weakness of the measurement, i.e., that after applying $e^{-iH(\epsilon)}$ to $s \otimes m$ and then measuring $Q$, the state of $S$ will be close to the original state $s$, we have just described a weak measurement protocol (in the sense of Definition 1) for approximating $\langle s, As \rangle$ to arbitrary accuracy while making an arbitrarily small change in the original state $s$ of $S$. The technical problem of proving weakness will be examined in Appendix 2. For the moment, we assume it.

The end of Section 4 pointed out the logical fallacy of identifying (27) with the expectation of $A$ in the initial state $s$ conditional on postselection to $f$, as equation (26) tempts. If nevertheless we make this identification (as AAV do), then we shall show that the limit as $\epsilon \to 0$ of (27) produces a “weak value” \begin{equation}
\Re \left( \frac{\langle f, As \rangle}{\langle f, s \rangle} \right)
\end{equation}
(28)
for $A$. This coincides with the “weak value” obtained by Lundeen and Steinberg [10] by different methods. By contrast, the “weak value” obtained by AAV was
\begin{equation}
\frac{\langle f, As \rangle}{\langle f, s \rangle}.
\end{equation}
(29)

Before calculating (27), we continue the historical exposition by indicating how AAV obtained (29). Instead of assuring weakness of the interaction by making the Hamiltonian $H(\epsilon)$ close to 0, as we did above, they use the above Hamiltonian $H(\epsilon)$ with $\epsilon := 1$. To obtain a “weak” interaction which affects the state of $S$ only slightly, they vary the initial meter state $m$, which they assume real with a square (which is a probability distribution on $\mathbb{R}$) of Gaussian form:
\begin{equation}
m^2(q) = \frac{\exp(-q^2/2\sigma^2)}{(2\pi)^{1/2}\sigma}.
\end{equation}
(30)

They attempt to obtain weakness of the measurement by taking $\sigma$ large, meaning that the Gaussian is very spread out. One hopes that this might assure weakness because then $(m)_{\alpha_i}$ will be close in norm to $m$ for all of the eigenvalues $\alpha_i$ of $A$, so that tracing the state $e^{-iH(s \otimes m)}$ over $M$ to obtain the state of $S$ might be hoped to yield\begin{equation}
e^{-iH(s \otimes m)} = \sum_i s_i \otimes m_{\alpha_i} \approx \sum_i s_i \otimes m \quad \text{trac}_M \sum_i s_i = s.
\end{equation}
(31)
(The above implicitly identifies vector states $v \in S \otimes M$ with the corresponding density matrices $P_v$, in order to apply $\text{tr}_M$.)

Their argument involves detailed calculations with Gaussians employing uncontrolled approximations. After calculating (27), we shall note that the same argument (assuming weakness, as do AA V) is easily adapted to produce a mathematically rigorous version of the calculation attempted by AA V, but concluding with a different result.

### 7.2 Calculation of the AA V-type “weak value”

Before beginning, we summarize the notation to be used. Since we shall be working with a “meter” Hilbert space $M := L^2(\mathbb{R})$, which is a function space, we change notation for the norm of an element $g$ of $M$ to

$$||g|| := \left[ \int_{\mathbb{R}} |g(q)|^2 dq \right]^{1/2},$$

to distinguish it from its absolute value $|g(q)|$. The Hilbert space $S$ of the system of primary interest will be assumed finite dimensional, and we continue to denote the norm of elements of $S$ by $|s|$. Also, we denote the norm of $u \in S \otimes M$ by $|u|_1$.

We assume given a reproducible product state $s \otimes m \in S \otimes M = S \otimes L^2(\mathbb{R})$ with $|s| = 1 = ||m||$ to which we shall apply the unitary operator $e^{-iH(\epsilon)} = e^{-iA \otimes P}$ to obtain a state on which to perform a meter measurement, which means measuring $(I \otimes Q)$. After performing the meter measurement, we postselect to a given final state $f \in S$. Postselecting to $f$ means measuring $P_f \otimes I$ and discarding results in which this last measurement gives 0. The results which are not discarded are then averaged and normalized (by dividing by $\epsilon$) to produce a final “weak value” which is often interpreted in the literature (incorrectly, in our view) as an approximation (becoming exact as $\epsilon \rightarrow 0$) to the average value of $A$ in the state $s$ conditioned on successful postselection to $f$.

We assume that $S$ is finite dimensional. Let $A : S \rightarrow S$ be a given Hermitian operator, for which we desire that the average meter measurement will

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15. This argument is highly suspect because the “≈” is only obvious for the Hilbert space norm, and $\text{tr}_M$ is not continuous from the Hilbert space norm on $S \otimes M$ to any norm on $S$. However, Appendix 2 shows how it can be fixed. We include it primarily to provide a motivation for the AA V approach which would probably be accepted in the physics literature and secondarily to underscore the need for care when making approximations.

16. A. Peres [9] has characterized equation (3) of AA V as a “faulty approximation”. It seems surprising that questions about basic mathematical procedures leading to new and striking conclusions have remained unresolved in the literature for over 20 years.

17. More precisely, we model the procedure of measuring $Q$ followed by postselection to $f$ as a measurement of $P_f \otimes Q$, the value of which is the value of $Q$ if $f$ is obtained, and 0 otherwise. Discarding the results for which $f$ is not obtained then corresponds to division of the average value of this measurement by the average value of a measurement of $P_f \otimes I$. But it is easier to think and speak of first measuring $Q$, then $P_f$. The subtle difficulty in the latter way of thinking is that the state of the system after a measurement of $Q$ (but before the postselection) is not precisely defined by standard quantum mechanics, as discussed in Appendix 2.
approximate \langle s, As \rangle. Our goal will be to calculate
\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \frac{\langle e^{-iH(\epsilon)} (s \otimes m), (P_f \otimes Q) e^{-iH(\epsilon)} (s \otimes m) \rangle}{\langle e^{-iH(\epsilon)} (s \otimes m), (P_f \otimes I) e^{-iH(\epsilon)} (s \otimes m) \rangle},
\]
which corresponds to the procedure just described. The numerator of the second fraction is the expectation of the measurement of the product of the commuting operators \( I \otimes Q \) and \( P_f \otimes I \), which is the expectation of a measurement of \( Q \) with measurements in which the postselection fails counted as zeros. The denominator is the probability that the postselection succeeds, and the entire quotient the expectation of \( Q \) conditional on success of the postselection.

The result of the calculation will be the “weak value”
\[
\Re \frac{\langle f, As \rangle}{\langle f, s \rangle},
\]
assuming that \( \langle f, s \rangle \neq 0 \). When \( \langle f, s \rangle = 0 \), the expression is undefined.

However to obtain this result, we shall need the following additional assumptions on the initial meter state \( m \in L^2(\mathbb{R}) \). All but one are mild regularity and growth conditions. All are satisfied by the Gaussian meter states used by AAV. These assumptions are:

1. We assume that \( m \) is a real-valued function. A previous version of this paper stated that this was not an essential assumption, but that was a mistake. The proof as given in the rest of the section does require this assumption. A later section added after the mistake was discovered completes the proof to be given below without this assumption, obtaining still more non-traditional “weak values”.

2. We continue to assume, as in the preceding discussion, that \( \langle m, Qm \rangle = 0 \), i.e., that the average meter reading is initially “0”.

3. We assume that \( m \) satisfies the growth condition
\[
\lim_{q \to \pm \infty} q m^2(q) = 0
\]

4. We assume that \( m \) has a continuous second derivative \( m'' = m''(q) \) satisfying the growth condition
\[
\lim_{q \to \pm \infty} q^3 |m''(q)| = 0
\]
These assumptions can be weakened in various ways which seem not very interesting. Most are the kind of regularity and growth conditions typically assumed without mention in physics calculations. The statements given were chosen for their simplicity.

Actually, we shall perform a more general calculation which will yield a “weak value” which is in general nothing like the traditional value, in order
to illustrate that even in the AAV context, “weak values” are not unique. The ideas are the same as in the finite dimensional example of nonunique weak values in Section 5.4.

We assume given a normalized “initial state” \( s \) of \( S \), and a normalized “meter state” \( m \) of \( M \) satisfying the above conditions. As before, \( A \) is a given Hermitian operator on \( S \) whose expectation in the state \( s, \langle s, As \rangle \) is to be measured. Let \( a_1, a_2, \ldots, a_n \) an orthonormal basis of eigenvectors for \( A \), and write
\[
    s = \sum_{i=1}^{n} s_i, \quad \text{with } s_i \text{ a multiple of } a_i \text{ and } \sum_{i} |s_i|^2 = 1. \tag{34}
\]

For the more general calculation, we also assume given a unitary operator \( V : S \to S \) which satisfies \( Vs = s \). In place of the unitary operator \( e^{-iH(\epsilon)} \) which prepared the state for the meter measurement (by applying it to \( s \otimes m \)), we shall use
\[
    (V \otimes I)e^{-iH(\epsilon)} = (V \otimes I)e^{-iAs \otimes P}, \tag{35}
\]
resulting in an initial state
\[
    (V \otimes I)e^{-iH(\epsilon)}(s \otimes m) = \sum_j Vs_j \otimes m_{\epsilon a_j}, \tag{36}
\]
just before the meter measurement, which will use
\[
    \frac{\langle (V \otimes I)e^{-iH(\epsilon)}s \otimes m, (I \otimes Q)(V \otimes I)e^{-iH(\epsilon)}s \otimes m \rangle}{\epsilon} \tag{37}
\]
to approximate \( \langle s, As \rangle \). The “weak value” to be calculated is \(e^{-32}\) with \( e^{-iH(\epsilon)} \) replaced by \((V \otimes I)e^{-iH(\epsilon)}\):
\[
    \lim_{\epsilon \to 0} \frac{1}{\epsilon} \frac{\langle (V \otimes I)e^{-iH(\epsilon)}(s \otimes m), (P_f \otimes Q)(V \otimes I)e^{-iH(\epsilon)}(s \otimes m) \rangle}{\langle (V \otimes I)e^{-iH(\epsilon)}(s \otimes m), (P_f \otimes I)(V \otimes I)e^{-iH(\epsilon)}(s \otimes m) \rangle}. \tag{38}
\]
Of course, taking \( V := I \) recovers the AAV-type situation previously discussed and \(e^{-32}\). The only reason for the assumption \( Vs = s \) is to assure the weakness of the measurement, as will be shown in Appendix 2. Since this assumption will play no role in the following calculation, we refrain from replacing \( Vs \) by \( s \).

First we calculate the limit as \( \epsilon \to 0 \) of the denominator of \(e^{-32}\):
\[
    \lim_{\epsilon \to 0} \langle (V \otimes I)e^{-iH(\epsilon)}(s \otimes m), (P_f \otimes I)(V \otimes I)e^{-iH(\epsilon)}(s \otimes m) \rangle
    = \lim_{\epsilon \to 0} \langle (V \otimes I) \sum_{i} s_i \otimes m_{\epsilon a_i}, (P_f \otimes I)(V \otimes I) \sum_{j} s_j \otimes m_{\epsilon a_j} \rangle
\]
\footnote{That section employed a unitary operator \( V \) on \( S \) with \( Vs = s \). The choice \( V := I \) yielded the traditional weak value \( \Re \langle f, As \rangle / \langle f, s \rangle \), and other choices yielded other weak values. By using a non-real meter function \( m \in L^2(\mathbb{R}) \), one can obtain non-traditional weak values even with \( V := I \), as explained in a later section. Thus the reader should feel free to set \( V := I \) on first reading.}
\[ \sum_{i,j} \langle Vs_i, Pf Vs_j \rangle \lim_{\epsilon \to 0} \langle m_{\epsilon \alpha_i}, m_{\epsilon \alpha_j} \rangle \]
\[ = \sum_{i,j} \langle Vs_i, Pf Vs_j \rangle \]
\[ = \langle f, Vs \rangle \langle Vs, f \rangle, \quad (39) \]

where we have used the fact that translations are continuous in the Hilbert space norm on \( L_2(\mathbb{R}) \) to eliminate the factor involving the inner product of the \( m \)'s, and obtained the last line by recalling from (34) that \( \sum_i s_i = s \).

Next we calculate
\[ \lim_{\epsilon \to 0} \langle (V \otimes I)e^{-iH(\epsilon)}s \otimes m, (T \otimes Q)(V \otimes I)e^{-iH(\epsilon)}s \otimes m \rangle, \quad (40) \]

where \( T : S \to S \) is an arbitrary Hermitian operator. For \( T := I \), this will validate (37) as an approximation to \( \langle s, As \rangle \), and for \( T := P_f \), it will give the numerator of (38). We have
\[
\langle (V \otimes I)e^{-iH(\epsilon)}s \otimes m, (T \otimes Q)(V \otimes I)e^{-iH(\epsilon)}s \otimes m \rangle
= \langle (V \otimes I) \sum_i s_i \otimes m_{\epsilon \alpha_i}, (TV \otimes Q) \sum_j s_j \otimes m_{\epsilon \alpha_j} \rangle
= \sum_{i,j} \langle Vs_i, TV s_j \rangle \langle m_{\epsilon \alpha_i}, (Qm)_{\epsilon \alpha_j} \rangle. \quad (41)
\]

The inner product involving \( m \) is
\[
\langle m_{\epsilon \alpha_i}, Qm_{\epsilon \alpha_j} \rangle = \int_{q = -\infty}^{\infty} m^*(q - \epsilon \alpha_i) q m(q - \epsilon \alpha_j) \, dq
= \int m(q) (q + \epsilon \alpha_i) m(q - \epsilon (\alpha_j - \alpha_i)) \, dq
= \int m(q) \epsilon \alpha_i m(q - \epsilon (\alpha_j - \alpha_i)) \, dq
+ \int m(q) q [m(q) - m'(q) \epsilon (\alpha_j - \alpha_i)] \, dq + O(\epsilon^2). \quad (42)
\]

The second line was obtained by a linear change of variable in the integral. For the last line, we performed a power series expansion about \( q \) to order 1 with remainder and used the growth conditions to estimate the remainder term as \( O(\epsilon^2) \). In detail, for any \( q \) and \( \beta \),
\[
m(q + \epsilon \beta) = m(q) + m'(q) \epsilon \beta + m''(\zeta_q)(\epsilon \beta)^2 / 2,
\]
where \( \zeta_q \) is between \( q \) and \( q + \epsilon \beta \), and the growth conditions on \( m \) and \( m'' \) ensure that the integral of the terms involving \( m'' \) is \( O(\epsilon^2) \).

The first integral in the last line of (42) is \( \epsilon(\alpha_i + o(1)) \) (because for any \( \beta \), \( ||m_{\epsilon \beta} - m|| = o(1) \), where as usual in this context, \( o(1) \) represents a term which goes to 0 as \( \epsilon \to 0 \). Integrating the second integral by parts yields
\[
\langle m_{\epsilon \alpha_i}, Qm_{\epsilon \alpha_j} \rangle = \epsilon [\alpha_i + \frac{\alpha_j - \alpha_i}{2}] + O(\epsilon^2) + o(1). \quad (43)
\]
Hence
\[
\lim_{\epsilon \to 0} \frac{(m_{\alpha_i}, Qm_{\alpha_j})}{\epsilon} = \frac{\alpha_i + \alpha_j}{2}.
\] (44)

Combining this with (41) gives (40) as
\[
\lim_{\epsilon \to 0} \frac{\langle (V \otimes I)e^{-iH(\epsilon)}s \otimes m, (T \otimes Q)Ve^{-iH(\epsilon)A}s \otimes m \rangle}{\epsilon} = \sum_i \sum_j \langle Vs_i, TVAs \rangle \frac{\alpha_i + \alpha_j}{2} = \langle V, TVAs \rangle + \langle VAs, TVs \rangle \frac{2}{2} = \Re \langle V, TVAs \rangle.
\] (45)

To understand these manipulations, recall that the \(s_i\) were defined in (34) by
\[
s = \sum_i s_i \quad \text{with} \quad As_i = \alpha_i s_i.
\]
Specializing (45) to \(T := I\) gives
\[
\lim_{\epsilon \to 0} \frac{\langle (V \otimes I)e^{-iH(\epsilon)}s \otimes m, (I \otimes Q)(V \otimes I)e^{-iH(\epsilon)A}s \otimes m \rangle}{\epsilon} = \Re \langle V, VAs \rangle = \langle s, As \rangle,
\] (46)
justifying (37) as an approximation to \(\langle s, As \rangle\).

Specializing (45) to \(T := P_f\) gives
\[
\lim_{\epsilon \to 0} \frac{\langle (V \otimes I)e^{-iH(\epsilon)}s \otimes m, (P_f \otimes Q)(V \otimes I)e^{-iH(\epsilon)A}s \otimes m \rangle}{\epsilon} = \Re \langle (f, VAs)(V, s) \rangle.
\] (47)

Combining this with equations (39) and (45) yields the “weak value”
\[
\lim_{\epsilon \to 0} \frac{\langle (V \otimes I)e^{-iH(\epsilon)}(s \otimes m), (P_f \otimes Q)(V \otimes I)e^{-iH(\epsilon)A}(s \otimes m) \rangle}{\epsilon} \quad = \quad \Re \langle (f, VAs)(V, s) \rangle.
\] (48)

Specializing to \(V = I\) yields the AAV-type “weak value”:
\[
\lim_{\epsilon \to 0} \frac{\langle e^{-iH(\epsilon)}(s \otimes m), (P_f \otimes Q)e^{-iH(\epsilon)A}(s \otimes m) \rangle}{\epsilon} \quad = \quad \Re \langle (f, As)(f, s) \rangle.
\] (49)

7.3 “Weak values” are not unique, even in an AAV-type context

We have seen that applying \((V \otimes I)e^{-iAP}\) to a starting product state \(s \otimes m\) to obtain a state to be measured and postselected results in a “weak value” given
by (47),
\[ \Re \frac{\langle f, VAs \rangle}{\langle f, Vs \rangle} = \Re \frac{\langle f, VAs \rangle}{\langle f, s \rangle}, \tag{50} \]
where we have finally used the assumption \( Vs = s \), which will be shown in Appendix 2 to guarantee “weakness” of the measurement. This does not look like the AAV-type “weak value” of equation (49), namely
\[ \Re \frac{\langle f, As \rangle}{\langle f, s \rangle}, \tag{51} \]
but for logical completeness, we should check that they can be numerically different, as well as different in appearance. The calculational argument given at the end of Section 5 is easily adapted to show this, but for the reader’s convenience and for variety, we give here a more general argument which avoids calculation.

Suppose the two “weak values” given by equations (50) and (51) are always equal. Then
\[ \Re \frac{\langle f, VAs \rangle}{\langle f, s \rangle} = \Re \frac{\langle f, As \rangle}{\langle f, s \rangle}, \tag{52} \]
for all states \( s, f \in S \) with \( \langle f, s \rangle \neq 0 \), all Hermitian \( A : S \to S \), and all unitary \( V : S \to S \) satisfying \( Vs = s \).

Take \( A \) to be any Hermitian operator with \( As \) not a scalar multiple of \( s \), i.e., \( s \) is not an eigenvector of \( A \). Let \( u := As - \langle s, As \rangle s \) denote the component of \( As \) orthogonal to \( s \). Then for any \( w \in S \) with \( |w| = |u| \) and \( \langle w, s \rangle = 0 \), there exists a unitary \( V \) on \( S \) with \( Vs = s \) and \( Vu = w \), i.e., \( VAs = w + \langle s, As \rangle s \), and
\[ \Re \frac{\langle f, VAs \rangle}{\langle f, s \rangle} = \Re \frac{\langle f, Ws \rangle}{\langle f, s \rangle} + \langle s, As \rangle. \]
By varying \( w \), we can change \( \Re(\langle f, VAs \rangle/\langle f, s \rangle) \) except in the special case in which \( f \) is a multiple of \( s \).

8 Relation of the AAV approach to ours

As mentioned earlier, our method of obtaining weakness of the measurement differs from that of AAV. This section notes that the difference is only cosmetic, and it also notes what we suspect may be an essential error in the mathematics of AAV.

We obtain weakness of the meter measurement by replacing the state preparation Hamiltonian \( A \otimes P \) with \( H(\epsilon) := \epsilon A \otimes P \). This makes the normalized (i.e., divided by \( \epsilon \)) expectation of the meter measurement conditional on postselection to \( f \) (cf. (32)) equal to
\[ \frac{1}{\epsilon} \frac{\langle e^{-iH(\epsilon)}(s \otimes m), (Pf \otimes Q)e^{-iH(\epsilon)}(s \otimes m) \rangle}{\langle e^{-iH(\epsilon)}(s \otimes m), (Pf \otimes I)e^{-iH(\epsilon)}(s \otimes m) \rangle}. \tag{53} \]
By contrast, AAV attempts to obtain weakness by using a fixed preparation Hamiltonian $A \otimes P$ (multiplied by an inessential constant) and instead replacing our fixed meter state $m$ by an $\epsilon$-dependent meter state $m_{AAV}[\epsilon](\cdot)$ defined for $\epsilon > 0$ by

$$m_{AAV}[\epsilon](q) := \left[ \exp(-\epsilon^2 q^2 / 2) \right]^{1/2}_{\sqrt{2\pi}/\epsilon},$$

which makes $m_{AAV}(\epsilon)^2$ a Gaussian centered at 0 with variance $1/\epsilon^2$. Here we are translating AAV into our notation. A reader consulting AAV should remember that they interchange $P$ and $Q$ relative to our convention; i.e., their “meter” is $P$, and their preparation Hamiltonian is $-A \otimes Q$ instead of our $A \otimes P$.

To see the relation of the two approaches, let $m$ be a fixed meter state (not necessarily a Gaussian as in AAV) satisfying the conditions $1 - 4$ of Subsection 7.2, and for $\epsilon > 0$ define a new meter state $m[\epsilon](\cdot) \in L^2(\mathbb{R})$ by

$$m[\epsilon](q) := m(\epsilon q) \sqrt{\epsilon}.$$  

The graph of $m[\epsilon](\cdot)$ is the graph of $m$ expanded by a factor of $1/\epsilon$ and then normalized to make its $L^2$ norm $\|m[\epsilon]\|$ equal to 1. For $m := m_{AAV}[1]$ as given by (54) with $\epsilon := 1$, $m[\epsilon]$ coincides with $m_{AAV}(\epsilon)$ for arbitrary $\epsilon > 0$.

We shall show that (55) can be rewritten as

$$\frac{1}{\epsilon} \langle e^{-iH(\epsilon)}(s \otimes m), (P_f \otimes Q)e^{-iH(\epsilon)}(s \otimes m) \rangle / \langle e^{-iH(\epsilon)}(s \otimes m), (P_f \otimes I)e^{-iH(\epsilon)}(s \otimes m) \rangle = \frac{\langle e^{-iH(1)}(s \otimes m[\epsilon]), (P_f \otimes Q)e^{-iH(1)}(s \otimes m[\epsilon]) \rangle}{\langle e^{-iH(1)}(s \otimes m[\epsilon]), (P_f \otimes I)e^{-iH(1)}(s \otimes m[\epsilon]) \rangle}.$$  

In other words, the result of using the fixed Hamiltonian $H(1)$ with the $\epsilon$-dependent meter state $m[\epsilon]$ is the same as using our $\epsilon$-dependent Hamiltonian $H(\epsilon)$ with our fixed meter state $m$, and then normalizing by dividing by $\epsilon$.

We shall show that the numerator of the left side of (56) equals the numerator of the right side, leaving the similar calculation of the denominators to the reader. The notation will be that of the previous calculation of the AAV-type weak value; in particular, $s = \sum_i s_i$ is the decomposition of $s$ as a sum of orthogonal eigenvectors $s_i$ of $A$ with respective eigenvalues $\alpha_i$, $As_i = \alpha_i s_i$, and for $g \in L^2(\mathbb{R})$, $g_\beta(q) := g(q - \beta)$. Recalling that

$$e^{-iH(1)}(s \otimes g) = \sum_i s_i \otimes g_{\alpha_i},$$

we have, setting $q' := \epsilon q$,

$$\langle e^{-iH(1)} s \otimes m[\epsilon], (P_f \otimes Q)(s \otimes m[\epsilon]) \rangle = \sum_{i,j} \langle s_i, P_f s_j \rangle \int_{-\infty}^{\infty} m[\epsilon]_{\alpha_i}(q) q m[\epsilon]_{\alpha_j}(q) dq.$$  

\footnote{We say “attempts” not to cast doubt on the “weakness” of their procedure, but because they do not prove weakness, nor even discuss it.}
\[\begin{align*}
&= \sum_{i,j} (s_i, P_j s_j) \int_{-\infty}^{\infty} m(\epsilon(q - \alpha_i)) m(\epsilon(q - \alpha_j)(\sqrt{\epsilon})^2 dq \\
&= \sum_{i,j} (s_i, P_j s_j) \int_{-\infty}^{\infty} m(q' - \epsilon \alpha_i)) (q'/\epsilon) m(q' - \epsilon \alpha_j) \frac{dq'}{\epsilon} \\
&= \frac{1}{\epsilon} \sum_{i,j} (s_i, P_j s_j) \langle m_{\epsilon \alpha_i}, Qm_{\epsilon \alpha_j} \rangle, \quad (57)
\end{align*}\]

which is identical to the the numerator of (53) as calculated in (41), (42), and the following equations.

Since the AAV-setup seems essentially mathematically identical to ours, one may wonder how the two approaches end with different weak values, namely \(\langle f, A s \rangle / \langle f, s \rangle\) for AAV and the real part of that for us. Few readers will want to slog through the mostly routine but tedious mathematics of both to look for errors, so it may be helpful to point out what we think might be an essential error in AAV.

It is well known and routine to calculate that if \(g = g(q)\) is a function in \(L^2(\mathbb{R})\) and \(\tilde{g} = \tilde{g}(p)\) denotes its Fourier transform, then multiplying \(g(q)\) by \(e^{i\alpha q}\), with \(\alpha\) real, translates its Fourier transform by \(\alpha\). That is, if \(h(q) := e^{i\alpha q} g(q)\), then \(\tilde{h}(p) = \tilde{g}(p - \alpha)\). But this holds only for real \(\alpha\); for nonreal \(\alpha\), \(e^{i\alpha q}\) grows exponentially for large \(|q|\) and \(h\) cannot be expected to even have a Fourier transform, since it cannot be expected to be in \(L^2\). But in passing from their fundamental equation (2) (via their uncontrolled approximation (3)) to their (5), AAV seems to assume the just-mentioned fact about Fourier transforms for complex \(\alpha\). We suspect that this may be the origin of the difference between their “weak value” and ours.

9 Another way to obtain non-standard “weak values”

We present another way to obtain “weak values” which differ from the usual formulas \(\Re(\langle f, A s \rangle / \langle f, s \rangle)\) or \(\langle f, A s \rangle / \langle f, s \rangle\). It is motivated by a comment on the Internet newsgroup sci.physics.research which noted that the unitary operator \(V\) appearing in the previous examples depends on the initial state \(s\). But what if the initial state is not known? The commenter (who posts under the pseudonym “student” and whom I thank) thought that it would be a rather peculiar type of measurement that would not work for all states.

We agree that this does complicate the situation, but not fatally. First of all, it is frequently the case that the initial state \(s\) is given by the physical situation; for example this is the case for the investigations of “Hardy’s paradox” in [13] and [5]. The nonuniqueness of weak values for such situations is a counterexample to claims that weak values must be given by the above “usual” formulas.

Second, in all formulations of weak measurements known to us, the initial state \(s\) has to be assumed to be “reproducible”, i.e., the experimenter must have
an apparatus which will produce any number of copies of \( s \). This is because for very weak coupling between the system of interest \( S \) and the meter system, a large number of meter measurements may be required to obtain a reliable average. And in finite dimensions, a reproducible state is effectively known because its components with respect to any given basis can be determined to arbitrary accuracy by quantum tomography ([2] pp. 389 ff). If we want to make a weak measurement involving a starting state \( s \in S \), we can first calculate its components and then construct the desired operator \( V \) (to arbitrary accuracy) using quantum gates ([2], Chapter 4).

However, the comment did cause us to look for other ways which did not depend explicitly on the initial state to obtain non-traditional weak values. One such way was already known from our study of the Yokota, et al. paper [5] (as described in the next section), but this seemed undesirably complicated for readers without detailed knowledge of that paper.

The result of the search seemed surprising. We shall present below a way to obtain weak values such that the preparation Hamiltonian (the analog of \( H(\epsilon) := \epsilon A \otimes P \) of (22)) does not depend on the initial state \( s \in S \). The preparation Hamiltonian used is very similar to the (22) used by von Neumann and AAV except that it requires only a finite dimensional meter space (which can be as small as dimension 2), instead of the infinite dimensional meter space \( L^2(\mathbb{R}) \) used by them. Since we continue to assume that the system \( S \) of interest is finite dimensional, the finite dimensionality of the meter space means that the calculations will be purely algebraic, and rigorous. The algebra will be identical for an infinite dimensional meter space, though further considerations would be necessary to make the calculation rigorous.

The general method will be identical to that used in Subsection 7.2 to calculate the AAV-type “weak value”. The difference is that instead of the preparation Hamiltonian \( H(\epsilon) := \epsilon A \otimes Q \) used there, we shall use a preparation Hamiltonian of the form

\[
H(\epsilon) := \epsilon A \otimes G
\]  

(58)

with \( G : M \to M \) Hermitian. The Hermitian operator on \( M \) whose value gives the meter reading will be called \( B : M \to M \) instead of the \( Q \) used in Subsection 7.2. Thus \( G \) plays the role of \( P \) and \( B \) the role of \( Q \) in Subsection 7.2.

A starting state \( s \otimes m \) is assumed given, with \(|s| = 1 = |m|\) and

\[
\langle m, Bm \rangle = 0
\]  

(59)

as before. The state to be meter-measured will be obtained as before by applying \( e^{-iH(\epsilon)} \) to \( s \otimes m \):

\[
e^{-i(\epsilon(A \otimes G))}(s \otimes m) = s \otimes m - i\epsilon As \otimes Gm + O(\epsilon^2)
\]  

(60)

Then the normalized (by division by \( \epsilon \)) average value of the meter measurement is

\[
\frac{\langle e^{-i(\epsilon(A \otimes G))}(s \otimes m), (I \otimes B)e^{-i(\epsilon(A \otimes G))}(s \otimes m) \rangle}{\epsilon}
\]
\[ \langle s, s \rangle \langle m, Bm \rangle - ie \langle s, As \rangle \langle m, BGm \rangle - \langle As, s \rangle \langle Gm, Bm \rangle + O(c^2) \]
\[
\varepsilon
\]
\[ = -i \langle s, As \rangle \langle m, BG - GBm \rangle + O(e) \quad , \tag{61} \]

and
\[
\lim_{\epsilon \to 0} \frac{\langle e^{-i(A \otimes G)}(s \otimes m), (I \otimes B) \langle e^{-i(A \otimes G)}(s \otimes m) \rangle \rangle}{\epsilon} \]
\[ = -i \langle m, (BG - GB)m \rangle \langle s, As \rangle . \tag{62} \]

Thus if we choose \( m, B, \) and \( G \) such that
\[
1 = -i \langle m, (BG - GB)m \rangle = 23 \langle m, Bm \rangle , \tag{63} \]

measuring \( B \) in the meter space and normalizing by dividing by \( \epsilon \) (for small \( \epsilon \)) constitutes a weak measurement protocol in the sense of Definition 1. (We omit the proof that the measurement negligibly affects the state of \( S \) for small \( \epsilon \), which is identical to that given in Subsection 5.3.)

The average value of the normalized meter reading conditional on successful postselection to \( f \in S \) is
\[
\frac{1}{\epsilon} \frac{\langle e^{-i(A \otimes G)}(s \otimes m), (P_f \otimes B) \langle e^{-i(A \otimes G)}(s \otimes m) \rangle \rangle}{\langle f, s \rangle \langle s, f \rangle} \]
\[ = -i \langle f, As \rangle \langle m, BGm \rangle - \langle As, f \rangle \langle s, m, GBm \rangle + O(e) \]
\[ = \frac{\langle f, As \rangle \langle m, BGm \rangle - \langle As, f \rangle \langle s, GBm \rangle}{\langle f, s \rangle} + O(e) \]
\[ = 23 \langle f, As \rangle \langle m, BGm \rangle - \frac{\langle f, s \rangle \langle s, f \rangle}{f, s} + O(e) \quad . \tag{64} \]

when \( \langle f, s \rangle \neq 0 \) and undefined if \( \langle f, s \rangle = 0 \). The following calculations assume \( \langle f, s \rangle \neq 0 \). Write
\[
\langle m, BGm \rangle = \rho + i \kappa \quad \text{with } \rho, \kappa \text{ real} . \tag{65} \]

From (63) \( \kappa = 1/2 \). Then the limit as \( \epsilon \to 0 \) of (64) becomes
\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \frac{\langle e^{-i(A \otimes G)}(s \otimes m), (P_f \otimes B) \langle e^{-i(A \otimes G)}(s \otimes m) \rangle \rangle}{\langle f, s \rangle \langle s, f \rangle} \]
\[ = 2\kappa \Re\langle f, As \rangle \langle f, s \rangle + 2\rho \Im\langle f, As \rangle \langle f, s \rangle \]
\[ = 2\Re\langle f, As \rangle \langle f, s \rangle + 2\rho \Im\langle f, As \rangle \langle f, s \rangle \quad . \tag{66} \]
The first term in (66) is the “usual” weak value $\Re ((f, As)/(f, s))$, but when $\Im ((f, As)/(f, s)) \neq 0$, we shall show that the second term can be chosen arbitrarily by adjusting the value of $\rho = \Re (m, BGm)$.

Recall that $m, B$, and $G$ are arbitrary subject to $\Im (m, BGm) = 1/2$ and $\langle m, Bm \rangle = 0$. To see that any number can be obtained for $\Re (m, BGm)$ under these conditions, take the meter space $M$ to be two-dimensional with orthonormal basis $m, m^\perp$, and define $G$ and $B$ by the following matrices with respect to this basis:

$$ G := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B := \begin{bmatrix} 0 & \rho + i/2 \\ \rho - i/2 & 0 \end{bmatrix} \quad \text{with} \quad \rho \text{ real.} \quad (67) $$

Then $BG$ has the following form, where entries denoted “*” have not been calculated because they are irrelevant to calculation of $\langle m, BGm \rangle$ (which is the upper left entry of BG):

$$ BG = \begin{bmatrix} \rho + i/2 & * \\ * & * \end{bmatrix} = \begin{bmatrix} \langle m, BGm \rangle & * \\ * & * \end{bmatrix}. \quad (68) $$

When $\Im ((f, As)/(f, s)) \neq 0$, by varying $\rho$, one can obtain any number whatever as a “weak value” for $A$.

The above calculations are rigorous for finite-dimensional $S$ and $M$, and still algebraically correct in infinite dimensions. In the physics literature, such algebraic calculations are typically accepted as “proofs”. If we relax mathematical rigor to this extent, we obtain from the above a very simple “proof” of the “usual” weak value $\Re ((f, As)/(f, s))$ by taking

$$ M := L^2(\mathbb{R}), \quad G := P, \quad B := Q, \quad m(q) := \frac{1}{\sqrt{2\pi}} e^{-q^2/2}, \quad (69) $$

where $P$ and $Q$ are the usual momentum and position operators, respectively, defined in Subsection 6.2.

This “proof” is deeply flawed because the starting equation (66),

$$ e^{-i\epsilon (A \otimes G)(s \otimes m)} = s \otimes m - i\epsilon As \otimes Gm + O(\epsilon^2), $$

would probably be difficult to justify for our unbounded $G := P$. Some “proofs” of the “usual” weak value formula in the literature rely on uncontrolled approximations like this. In honesty, they should be called something like “algebraic motivations” instead of proofs.

If we are willing to accept uncritically such uncontrolled approximations, we can obtain arbitrary weak values in an AAV-type framework by taking $B$ and $m$ as in equation (69) and

$$ G : P + \delta Q \quad \text{with} \quad \delta \neq 0 \text{ real.} \quad (70) $$

This results in

$$ \langle m, BGm \rangle = \delta + \frac{i}{2}, \quad (71) $$
so that when $\Im(\langle f, As \rangle/\langle f, s \rangle) \neq 0$, any “weak value” whatever can be obtained using the preparation Hamiltonian $H(\epsilon) := \epsilon A \otimes (P + \delta Q)$.

Since $\delta$ can be arbitrarily small, anyone who claims that the “usual” weak value formula gives the only experimentally possible result should be obligated to explain how the von Neumann/AAV Hamiltonian $H(\epsilon) := \epsilon A \otimes P$ can be guaranteed in any experimental situation, and assuming that, how the Hamiltonian $A \otimes P$ can be experimentally distinguished from $A \otimes (P + \delta Q)$ for arbitrarily small $\delta$. On the level of uncontrolled approximations, this will probably be impossible.

It is almost immediate that

$$e^{-iQ^2\delta/2}P e^{iQ^2\delta/2} = P + \delta Q$$

because for any $g \in L^2(\mathbb{R})$, $(e^{-iQ^2\delta/2}g)(q) = e^{-i\delta^2/2}g(q)$ and $P := -id/dq$. (We say “almost” because a rigorous verification would require careful specification of the domain of $P$, which we have not discussed.) In other words, the new Hamiltonian $A \otimes (P + \delta Q)$ is formally unitarily equivalent to the AAV Hamiltonian $A \otimes P$.

Thus the above setup with the new Hamiltonian can be unitarily transformed into one with the AAV Hamiltonian. The transformation will take the AAV meter state $m$ of equation (69) into the new meter state

$$q \mapsto e^{-i\delta^2/2}m(q),$$

whose absolute square defines the same Gaussian probability distribution on position space as $m$. This suggests that it might be possible to rework the rigorous calculation of the traditional weak value $\Re(\langle f, As \rangle/\langle f, s \rangle)$ in Subsection 7.2 into a rigorous calculation which gives arbitrary weak values in a slightly different setup using the AAV Hamiltonian $A \otimes P$, but the slightly different meter state (73). The next section will carry this out.

10 Another way to rigorously obtain non-standard “weak values” in a framework similar to AAV

The rigorous calculation of the traditional weak value $\Re(\langle f, As \rangle/\langle f, s \rangle)$ of Subsection 7.2 assumed that the meter state $m \in L^2(\mathbb{R})$ is a real-valued function. After that proof was typeset, we have noticed that without that assumption, the proof of that section is easily adapted to rigorously obtain non-traditional (in fact, arbitrary) weak values in a setting very similar to AAV. More precisely, it assumes the AAV preparation Hamiltonian with a non-real meter state which is the AAV meter state (the square root of a Gaussian) multiplied by a complex function of modulus 1, so the new meter state defines the same Gaussian probability distribution in position space as the AAV meter state. This section will outline the necessary modifications to the argument of subsection 7.2. We continue to assume that the meter state $m$ satisfies the normalization and growth
conditions 2, 3, and 4 of Section 7.2, but we no longer assume condition 1 that it be real.

The new proof is identical to the old through equation (11), written here with a new number, and with $V := I$:

$$
\langle e^{-iH(\epsilon)}(s \otimes m), (T \otimes Q)e^{-iH(\epsilon)}(s \otimes m) \rangle
= \langle \sum_i s_i \otimes m_{\epsilon \alpha_i}, (T \otimes Q) \sum_j s_j \otimes m_{\epsilon \alpha_j} \rangle
= \sum_{i,j} \langle s_i, Ts_j \rangle \langle m_{\epsilon \alpha_i}, (Qm_{\epsilon \alpha_j}) \rangle .
$$

(74)

The inner product involving $m$ is

$$
\langle m_{\epsilon \alpha_i}, Q m_{\epsilon \alpha_j} \rangle
= \int_{q=-\infty}^{\infty} m^*(q - \epsilon \alpha_i) q m(q - \epsilon \alpha_j) dq
= \int m^*(q)(q + \epsilon \alpha_i)m(q - \epsilon(\alpha_j - \alpha_i)) dq
= \int m^*(q)\epsilon \alpha_i m(q - \epsilon(\alpha_j - \alpha_i)) dq
+ \int m^*(q) q [m(q) - m'(q)\epsilon(\alpha_j - \alpha_i)] dq + O(\epsilon^2) .
$$

(75)

The first integral in (75) is the same as before, namely $\epsilon(\alpha_i + o(1))$, but integration by parts only determines the real part of the second integral, which is the same as before, namely $\epsilon(\alpha_i - \alpha_j)/2$.

We shall not evaluate the imaginary part of the second integral in general, but only for a specific meter function $m$ which we assume to be of the form

$$
m(q) = e^{-i\delta^2/2}m_0(q) ,
$$

(76)

where $\delta$ is a real constant and $m_0$ a real meter function satisfying the same conditions as assumed in Subsection 7.2 together with the additional normalization condition

$$
\langle m_0, Q^2 m_0 \rangle = 1 .
$$

(77)

For example, we could take $m_0$ as the AAV meter function $m_{AAV}[1]$, defined in equation (54) as the square root of a Gaussian with mean 0 and variance 1.

Then, since

$$
m'(q) = e^{-i\delta^2/2}[m_0'(q) - i\delta m_0(q)] ,
$$

$$
\langle m_{\epsilon \alpha_i}, Q m_{\epsilon \alpha_j} \rangle
= \int_{q=-\infty}^{\infty} m^*(q - \epsilon \alpha_i) q m(q - \epsilon \alpha_j) dq
= \epsilon(\alpha_i + o(1)) + \epsilon(\alpha_j - \alpha_i)/2 + O(\epsilon^2)
$$

(78)

Those are that $m_0$ be normalized, i.e., $||m_0|| = 1$, and satisfy the previous condition 2 that $\langle m_0, Qm_0 \rangle$, and the growth conditions 3 and 4.
This is the same result as for a real meter function plus the imaginary part \((\alpha_j - \alpha_i)\delta_i\). (We assume that the reader will take in stride the use of the same symbol \(i\) for the imaginary unit and an index as in \(\alpha_i\). Since the previous version only used real quantities, no ambiguity arose there, and changing notation for the index at this point probably risks more confusion than retaining it.)

The main step to complete the calculation is to evaluate the normalization of \((74)\) in the limit \(\epsilon \to 0\). There are two cases to consider, \(T := I\) to demonstrate that the normalized meter expectation does equal \(\langle s, As \rangle\) and \(T := P_f\) to calculate the weak value. Here we take \(T := P_f\), leaving the case \(T := I\) to the reader. We have

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \langle e^{-iH(\epsilon)}(s \otimes m), (P_f \otimes Q)e^{-iH(\epsilon)}(s \otimes m) \rangle
\]

\[
= \sum_{i,j} \langle s_i, P_f s_j \rangle \langle m_{\epsilon\alpha_j}, (Qm)_{\epsilon\alpha_i} \rangle
\]

\[
= \sum_{i,j} \langle s_i, P_f s_j \rangle \left[ \frac{\alpha_i + \alpha_j}{2} + (\alpha_j - \alpha_i)\delta_i \right]
\]

\[
= \Re \langle As, P_f s \rangle - 2\delta \Im \langle As, P_f s \rangle . \tag{79}
\]

To understand the passage to the last line, first recall that the \(s_i\) are eigenvectors of \(A\) with \(As_i = \alpha_i s_i\), and \(s = \sum_i s_i\). We shall concentrate on evaluating

\[
\sum_{i,j} \langle s_i, P_f s_j \rangle (\alpha_j - \alpha_i) ,
\]

leaving the other term, which is similar and was effectively done in the original proof, to the reader. We have

\[
\sum_{i,j} \langle s_i, P_f s_j \rangle (\alpha_j - \alpha_i) = \sum_i \sum_j \langle s_i, P_f s_j \rangle (\alpha_j - \alpha_i)
\]

\[
= \sum_i [(\langle s_i, P_f As \rangle - \alpha_i \langle s_i, P_f s \rangle)]
\]

\[
= \langle s, P_f As \rangle - \langle As, P_f s \rangle
\]

\[
= \langle P_f s, As \rangle - \langle As, P_f s \rangle
\]

\[
= 2i \Im \langle P_f s, As \rangle .
\]

The desired weak value is

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \langle e^{-iH(\epsilon)}(s \otimes m), (P_f \otimes Q)e^{-iH(\epsilon)}(s \otimes m) \rangle
\]

\[
= \langle e^{-iH(\epsilon)}(s \otimes m), (P_f \otimes I)e^{-iH(\epsilon)}(s \otimes m) \rangle \tag{80}
\]
The limit as \( \epsilon \to 0 \) of the denominator of the second fraction was effectively computed in (39) (which did not require the reality of the meter function) as \( \langle f, s \rangle \langle s, f \rangle \). Combining this with (79) gives the weak value as

\[
\lim_{\epsilon \to 0} \frac{\langle e^{-iH(\epsilon)}(s \otimes m), (P_f \otimes Q)e^{-iH(\epsilon)}(s \otimes m) \rangle}{\langle f, s \rangle \langle s, f \rangle} = \Re \langle A, f \rangle \langle f, s \rangle + 2\delta \Im \langle A, f \rangle \langle f, s \rangle. \quad (81)
\]

This shows rigorously that when \( \Im \langle f, A \rangle / \langle f, s \rangle \neq 0 \), any weak value whatever may be obtained by some postselected weak measurement protocol using the AAV preparation Hamiltonian and a meter state different from that used by AAV, but which defines the same Gaussian probability distribution on position space.

Therefore, any argument that the “usual” weak value \( \Re \langle f, A \rangle / \langle f, s \rangle \) is experimentally inevitable will probably have to argue for the experimental universality of the precise AAV setup (meter observable, meter state, and preparation Hamiltonian). Since there seems no argument in the literature that this precise setup can be realized in any experimental situation, much less all, such an argument would probably have to break new ground.

11 Rambling afterword with conclusions

After spending months learning about weak values in the hope that it might lead to some fundamental new understanding of quantum mechanics, I was disappointed that this hope did not materialize. This work was written mainly to save others in a similar situation from similarly wasting their time.

I do not recall the exact route I followed through the “weak values” literature, but it took me through all of it that I could find. I think the first paper I read might have been a very interesting short paper of Yokota, et al., which describes an experimental measurement of weak values to confirm a quantum mechanical prediction concerning “Hardy’s Paradox”. Then I probably looked at the mostly clear and interesting analysis of “Hardy’s Paradox” in [13], followed by carefully reading AAV [3], but it could have been in the reverse order. I have made no attempt to reference all the papers in my thick folder on weak values.

\[21\] Since there have been experiments which do obtain the “usual” weak value, that might be considered suggestive that the AAV framework can be experimentally realized. However, we have seen that other frameworks can produce the same weak value. We have never seen an argument in the experimental literature that the experimental situation actually does implement the AAV setup.

\[22\] “Hardy’s Paradox” is what it is commonly called in the literature, but Hardy [12] did not present it as a paradox, and I think it questionable to call it that.
I was struck by the fact that all the literature which I saw gave the impression that one of the two versions of the “standard” formulas for weak values, equation (28), \( \Re (\langle f, A_s \rangle / \langle f, s \rangle) \), or (29), \( \langle f, A_s \rangle / \langle f, s \rangle \) was a universal formula to be expected in all experimental situations. But AAV’s motivation of (29) did not seem fundamental because it relied on assumption of the particular Hamiltonian (22), without any discussion of how to guarantee that Hamiltonian in any experimental situation, much less in all experimental situations.

Yokota, et al. [5] obtained the standard weak values formula (28) using a Hamiltonian completely different from that of AAV or other motivations for (28) that I have seen. For example, AAV and Lundeen/Steinberg’s [10] both use (22), which employs an infinite dimensional meter space, while Yokota, et al. use a four-dimensional meter space. It seemed remarkable that these disparate approaches yielded the same formula.

At first, it seemed inconceivable that they could unless they were both reflecting aspects of some fundamental, as yet undiscovered physical reality or mathematical fact. I set out to find an explanation for the fact that AAV and Yokota, et al., had obtained similar “weak values” with such utterly different methods and physical setups.

The effort led to disappointment. I did not discover any new physical or mathematical principles.

On careful reading of Yokota, et al.’s [5] analysis, I noticed some arbitrary elements. Their method for obtaining weak values was not unreasonable, but it seemed idiosyncratic, and I would have done it differently. I worked out the consequences of my method and did indeed obtain a different “weak value”. My method had no claim to being superior to theirs, but the exercise convinced me that weak values are not unique.

I did not present my method for obtaining weak values in the setup of Yokota, et al. [5] in this paper because it is more complicated than the nonuniqueness examples presented and would require extensive analysis of [5]. The examples presented in this paper may seem artificial, but they have the merit of being simple.

The fact that “weak values” different from those of Yokota, et al., [5] can reasonably be obtained for their setup does not imply that there is anything wrong with their paper. Their weak values and mine are both obtained using quantum mechanics so standard that it is unlikely to be questioned. So long as their experimental procedures accurately reflect the mathematical methods used to obtain their weak values, their experiment can be viewed as a confirmation of standard quantum mechanics. Their modestly written paper does not claim anything further. In particular, they (wisely) do not claim to have resolved “Hardy’s Paradox”.

The “Reader’s Guide” section remarked that I suspect that the introduction to weak values and the finite-dimensional examples of Section 5 may turn out to be all that most readers care to learn about weak values. That is because the ideas seem so unremarkable when presented in that context and because I think that the literature which makes them seem remarkable rests on an implicit logical fallacy exposed in that section. But I have no illusions that all readers
will share these views.

Readers who do think it worth their while to read further will find what I hope is a mathematically rigorous derivation of the standard weak value formula. To a non-mathematician it may look complicated, but that is mainly because everything is precisely defined and much more fully written out than is usual in the physics literature. The fact that it applies to almost any real “meter” state, not just a Gaussian meter state as in the existing literature, may be viewed as an advance by those who take seriously the universality of the Hamiltonian (22) which AAV and most of the existing literature implicitly assume.

12 Closing remarks on notation

The notation chosen may irritate some readers. It is intended to stay as close as possible to notation common in the physics literature without sacrificing mathematical accuracy (as physics notation often does). It necessarily involves some compromises.

Physicists who have never seen a proper introduction to tensor products may find notation like $s \otimes m$ strange and excessive. Its merit is that it is precise, unlike physics-type notation like $|s\rangle m$ or $|s\rangle_S m_M$. (Either would probably suggest to many readers that $s$ is a label for an eigenvalue of some operator, which it is not. And if it is already understood, that $s$ is merely a label for a vector state in $S$, what additional information would be given by the brackets and subscript of $|s\rangle_S$?)

The use of “projector” in place of the “projection” nearly universal in the mathematics literature may grate on some mathematicians. Who does this dude think he is to give him the right to substitute his own notation for what is standard?

I chose “projector” mainly because that is the term most common in the physics literature. It also seems to me the more grammatically appropriate alternative. If a “bettor” is one who bets, shouldn’t a “projector” be one who projects? If a “bettor” produces a bet, it seems reasonable that a “projector” should produce a “projection”. The projector $P_u$ produces the projection $[u]$ (the subspace spanned by $u$). Mathematicians who find this unconvincing and are still offended by “projector” may be mollified by the assurance that I intend to revert to “projection” instead of “projector” in my next purely mathematical paper!

Finally, we have bowed to the tradition of the physics literature by describing the time evolution of a quantum system initially in state $\psi$ by $t \mapsto e^{-iHt}\psi$, where $H$ is the Hamiltonian, instead of the simpler and more natural $t \mapsto e^{iHt}\psi$.

\footnote{A referee for my book on relativistic electrodynamics made a similar impassioned objection to my choice of “⊥” for the Hodge dual operation in place of the “∗” which most mathematical literature uses, and is usually called the “Hodge * (or star) operation”. I had chosen “⊥” because it seemed more common in the physics literature after it was popularized by its use in the monumental treatise ‘Gravitation’ of Misner, Thorne, and Wheeler (MTW) and because it more accurately reflects the geometric content of the operation. I imagine that the referee was a pure mathematician who might never have seen MTW.}
The complication of the minus sign, though slight in any given instance, is a constant annoyance to mathematicians. In the complex number system, \(-1\) has two square roots, and “\(i\)” is an arbitrary label for one of them. The two square roots have identical algebraic properties, so either of them could be called “\(i\)”.

The convention \(t \mapsto e^{-iHt}\psi\) suggests that “\(-i\)” is in some fundamental way different from “\(i\)”, and that it would be wrong to write the time evolution as \(t \mapsto e^{iHt}\psi\).

13 Appendix 1: The partial trace

The definition of trace class operators on infinite dimensional Hilbert spaces is surprisingly subtle, as is the definition of partial trace. If one does not feel the need to worry about convergence and orders of summation, the definitions can be lifted verbatim from the corresponding definitions for finite matrices. However, since we are trying to perform rigorously calculations which are done non-rigorously in the physics literature and have led to serious errors, that approach would not fulfill our needs.

On the other hand, we do not want to devote pages to carefully developing the properties of traces. We choose a middle approach stating the properties of traces which we shall use, either giving references or indicating how they can be proved. Good references for partial traces seem particularly hard to find. I would be grateful for any that readers might furnish.

Besides summarizing properties of these traces which we shall need, this appendix performs some calculations which will be used in the main text. Our primary reference for the definition of trace class operator is A. Knapp’s book, *Advanced Real Analysis* [11]. It efficiently develops the properties of trace class operators on infinite dimensional Hilbert spaces, but does not discuss the partial trace.

Below the term “operator” will always mean “bounded operator on a Hilbert space. A bounded operator \(T\) is said to be of trace class if for all orthonormal sequences \(\{u_i\}_{i=1}^\infty\) and \(\{v_i\}_{i=1}^\infty\),

\[
\sum_{i=1}^\infty |\langle u_i, T v_i \rangle| < \infty.
\]

(82)

Its trace, denoted \(\text{tr } T\), is defined as

\[
\text{tr } T := \sum_{i=1}^\infty |\langle u_i, T u_i \rangle| < \infty.
\]

(83)

This sum can be shown to be independent of the orthonormal basis \(\{u_i\}\). In finite dimensions, this is a routine algebraic calculation, but an efficient proof for infinite dimensions requires careful organization.

To define partial traces, it will be helpful to identify operators with sesquilinear forms: an operator \(T : H \to H\) is identified with the sesquilinear form
$Q_T(\cdot, \cdot)$ defined for all $u,v \in H$ by

$$Q_T(u,v) := \langle u, Tv \rangle .$$

(84)

When naming this form would only be a distraction, we shall refer to it as the form

$$u, v \mapsto \langle u, Tv \rangle .$$

(85)

It is well known (and easy to prove) that this provides a one-to-one correspondence between operators and bounded sesquilinear forms. This identification of operators with forms will allow us to read off properties of the partial trace from corresponding properties of the ordinary trace.

Let $S$ and $M$ be Hilbert spaces, $S \otimes M$ their tensor product, and $L : S \otimes M \rightarrow S \otimes M$ a trace class operator on this tensor product. For given vectors $s,s' \in S$, consider the form on $M$ defined for all $m,m' \in M$ by

$$m, m' \mapsto \langle s \otimes m, L(s' \otimes m') \rangle .$$

(86)

For fixed $s,s'$, this form corresponds to an operator on $M$ which is easily seen to be trace class (because $L$ is trace class on $S \otimes M$). Its trace is the definition of the partial trace with respect to $M$ of $L$, denoted $\text{tr}_M L$, as a sesquilinear form. More concretely, for any orthonormal basis $\{f_i\}$ for $M$,

$$\langle s, (\text{tr}_M L)s' \rangle = \sum_\alpha \langle s \otimes f_\alpha, L(s' \otimes f_\alpha) \rangle .$$

(87)

(We sometimes distinguish dummy indices of summation by Greek.) From the assumption that $L$ is trace class, it follows easily that the sum in the definition converges absolutely, and also that $\text{tr}_M L$ is trace class and $\text{tr} (\text{tr}_M L) = \text{tr} L$.

The definition can also be viewed more concretely as defining $\text{tr}_M L$ as a matrix. If $\{e_k\}$ is an orthonormal basis for $S$, and if the matrix of $L$ with respect to the orthonormal basis $\{e_j \otimes f_k\}$ for $S \otimes M$ is $(L_{ij,\alpha})$, then the matrix for $\text{tr}_M L$ is $(\sum_\alpha L_{i\alpha, j\alpha})$. An advantage of giving the definition in terms of forms instead of matrices is that it avoids the nuisance of checking that it is basis-independent.

Let $\{f_i\}$ be an orthonormal basis for $M$. Then any unit vector $u \in S \otimes M$ can be written uniquely as

$$u = \sum_\alpha s_\alpha \otimes f_\alpha \quad \text{with} \quad \sum_\alpha |s_\alpha|^2 = 1 .$$

(88)

Recall that $P_u$ (the projector onto $u$) represents the mixed state (often called a density matrix on $S \otimes M$) corresponding to the vector $u$. We shall derive a revealing formula for the mixed state $\text{tr}_M P_u$.

24 To see this, note that for normalized $s$ and $s'$, the sequences $\{s \otimes f_i\}$ and $\{s' \otimes f_j\}$ are orthonormal, so the definition of “$L$ is trace class on $S \otimes M$” applies directly to assure the absolute convergence of the sum in (87). The other facts follow from similar observations.
Let \( \{e_i\} \) be an orthonormal basis for \( S \). First note that for any unit vector \( v \in S \), the matrix of the projector \( P_v \) on \( v \) is obtained from
\[
(e_j, P_v e_k) = \langle e_j, \langle v, e_k \rangle v \rangle = \langle v, e_k \rangle \langle e_j, v \rangle.
\] (89)

From the definition (87), for any fixed \( j, k \)
\[
\langle e_j, (\text{tr}_M P_u) e_k \rangle = \sum_{\alpha} \langle e_j \otimes f_\alpha, P_u (e_k \otimes f_\alpha) \rangle
= \sum_{\alpha} \langle u, e_k \otimes f_\alpha \rangle \langle e_j \otimes f_\alpha, u \rangle
= \sum_{\alpha} \langle s_\alpha, e_k \rangle \langle e_j, s_\alpha \rangle
= \sum_{\alpha} |s_\alpha|^2 \langle e_j, P_{s_\alpha} e_k \rangle
= \sum_{\alpha} |s_\alpha|^2 \langle e_j, P_{s_\alpha} e_k \rangle
\] (90)

In summary,
\[
\text{tr}_M P_u = \text{tr}_M P \sum_{\alpha} s_\alpha \otimes f_\alpha = \sum_{\alpha} |s_\alpha|^2 P_{s_\alpha}.
\] (91)

(Recall that we are using the convention that \( P_u = P_{|u|} \).) This exhibits \( \text{tr}_M P_u \) as a convex linear combination of pure states \( P_{s_\alpha} \). (This makes it almost obvious that \( \text{tr}_M P_u \) is pure if and only if \( u \) is a product state, a fact mentioned in the main text.)

It is also of interest to calculate \( \text{tr}_S P_u \) for the \( u = \sum_\alpha s_\alpha \otimes f_\alpha \) given by (88), with \( |u| = 1 \). Let \( \{e_i\} \) and \( \{f_j\} \) be orthonormal bases for \( S \) and \( M \), respectively. Then the matrix of \( \text{tr}_S P_u \) with respect to \( \{f_j\} \) is given by
\[
\langle f_k, (\text{tr}_S P_u) f_j \rangle = \sum_{\alpha} \langle e_\alpha \otimes f_k, P_u (e_\alpha \otimes f_j) \rangle
= \sum_{\alpha} \langle e_\alpha \otimes f_k, \sum_\beta s_\beta \otimes f_\beta, e_\alpha \otimes f_j \rangle \sum_\gamma s_\gamma \otimes f_\gamma
= \sum_{\alpha} \langle s_j, e_\alpha \rangle \langle e_\alpha, s_k \rangle
= \langle s_j, s_k \rangle,
\] (92)

where the last equality is Parseval’s equality.

The statement and proof of equation (92) assumed for simplicity that \( |u| = 1 \), but of course the case of arbitrary nonzero \( u \) can be immediately obtained by normalization. Since this result will be needed in Appendix 2, we state it explicitly for the reader’s convenience. Let \( \{f_i\} \) be an orthonormal basis for \( M \). Then for any nonzero vector \( u = \sum_\alpha s_\alpha \otimes f_\alpha \in S \otimes M \), the matrix \((\langle f_k, (\text{tr}_S P_u) f_j \rangle)\) of \( \text{tr}_S P_u \) is given by
\[
\langle f_k, (\text{tr}_S P_u) f_j \rangle = \frac{\langle s_j, s_k \rangle}{|u|^2}.
\] (93)
14 Appendix 2: Issues in proving weakness of the AAV-type “weak measurement”

Subsection 7.1 outlined a weak measurement protocol and gave a hand-waving motivation (equation (31)) why it might be hoped to be weak in the sense of Definition 11. When examined in detail, the “weakness” issue turns out to be unexpectedly subtle.

So far as I know, this issue has never been recognized in the physics literature. This subsection discusses it in detail.

We shall be discussing the situation discussed in the main text in which the state space $S$ of primary interest is finite dimensional, but the meter space $M = L^2(\mathbb{R})$ is infinite dimensional. We switch to the notation $||m|| := \left[ \int \! |m(q)|^2 \, dq \right]^{1/2}$ for the $L^2$ Hilbert space norm of $m \in L^2(\mathbb{R})$, in order to reserve $|m(q)|$ for the absolute value of the function $m(\cdot) \in L^2(\mathbb{R})$. We continue to use $|s| := (s, s)^{1/2}$ and $|u|$ for the norms of vectors $s \in S$ and $u \in S \otimes M$.

The notation will be as in the main text, summarized here for the reader’s convenience. Let $s \otimes m$ be a given product state in $S \otimes M$ with $|s| = 1 = ||m||$, which will be fixed throughout the discussion. Let $H(\epsilon) := \epsilon A \otimes P$, where $A$ is a given Hermitian operator on $S$, $P$ the momentum operator on $M = L^2(\mathbb{R})$, and $V$ a unitary operator on $S$ with $Vs = s$.

Recall that we apply $(V \otimes I)e^{-iH(\epsilon)}$ to $s \otimes m$ to obtain a state $(V \otimes I)e^{-iH(\epsilon)}(s \otimes m)$ in which the expectation

$$
\langle (V \otimes I)e^{-iH(\epsilon)}(s \otimes m), (I \otimes Q)(V \otimes I)e^{-iH(\epsilon)}(s \otimes m) \rangle
$$

is to be measured, where $Q$ operates on $g \in L^2(\mathbb{R})$ by $(Qg)(q) := qg(q)$. The hope is that this expectation divided by $\epsilon$ will approximate $\langle s, As \rangle$ for small $\epsilon$, and that the measurement will negligibly change the starting state $s$ of $S$.

At first glance, it seems plausible that for small $\epsilon$, the state $(V \otimes I)e^{-iH(\epsilon)}(s \otimes m)$ would be close to $s \otimes m$, and that the corresponding states of $S$, namely $\text{tr}_M(V \otimes I)(e^{-iH(\epsilon)}(s \otimes m))$ and $\text{tr}_M(s \otimes m) = s$ would also be close. But the reader who attempts to rigorously justify this expectation will find that careful thought is necessary. One complication is that the operation $\text{tr}_M$ operates on an infinite dimensional space $S \otimes M$, so that the sense in which it is continuous has to be carefully considered.25 But one welcome simplification is that the definition of “closeness” in $S$ (or in density matrices on $S$) is not in issue: all norms on a finite dimensional space are equivalent, so we may define “closeness” in $S$ by any convenient norm, and reflect this in language which would otherwise be sloppy.

A priori, it is not enough to show that $\text{tr}_M P_{(V \otimes I)e^{-iH(\epsilon)}(s \otimes m)} \to P_s$ as $\epsilon \to 0$ because of the possibility that the measurement of $I \otimes Q$ may affect the (mixed) state $\text{tr}_M P_{(V \otimes I)e^{-iH(\epsilon)}(s \otimes m)}$ of $S$.

25 For example, for infinite dimensional $M$, the mapping $u \otimes v \to \text{tr}_M P_u \otimes v$ is not continuous with respect to the norm topologies, so $e^{-iH(\epsilon)}(s \otimes m) \to s \otimes m$ as $\epsilon \to 0$ does not immediately imply that $\text{tr}_M P_{e^{-iH(\epsilon)}(s \otimes m)} \to \text{tr}_M P_s \otimes m = P_s$. 
Since $I \otimes Q$ has continuous spectrum, the precise manner in which the measurement of $Q$ affects the state is not even defined by usual formulations of quantum mechanics\textsuperscript{26}

It would be defined if $Q$ had pure point spectrum, so to carry out a rigorous analysis we shall need to approximate $Q$ by an operator with pure point spectrum. To approximate $Q$ in norm within $\lambda > 0$ by such an operator $B$, define $B$ for $g \in L^2(\mathbb{R})$ by

$$ (Bg)(q) := \lambda[q/\lambda]g(q) $$

where the bracket denotes the greatest integer function: for $x \in \mathbb{R}$, $[x]$ denotes the greatest integer less than or equal to $x$. The function $Bg$ need not be in $L^2(\mathbb{R})$, but restricting to the set of all $g$ for which $Bg$ is in $L^2(\mathbb{R})$ makes $B$ Hermitian (we omit the details).

Then $B$ has the set $\{k\lambda\}_{k=-\infty}^{\infty}$ of integer multiples of $\lambda$ as pure point spectrum. For each integer $k$, let $P_k$ denote the projector on the eigenspace of $B$ with eigenvalue $k\lambda$. This eigenspace consists of all functions with support in $[k\lambda, (k + 1)\lambda)$.

A measurement of $B$ is the same as a projective measurement with respect to the resolution of the identity $\{I \otimes P_k\}_{k=-\infty}^{\infty}$. If the composite system is in normalized pure state $r \in S \otimes M$ before a measurement of $B$, then after the measurement, it will be in (unnormalized) pure state $(I \otimes P_k)r$ with probability $|(I \otimes P_k)r|^2$, i.e., in mixed state $\sum_k |(I \otimes P_k)r|^2 P(I \otimes P_k)r$. The state of $S$ will then be

$$ \sum_k |(I \otimes P_k)r|^2 \text{tr}_M P(I \otimes P_k)r. $$

Let $\{a_j\}_{j=1}^n$ be an orthonormal basis for $S$ of eigenvectors $a_j$ of $A$: $Aa_j = \alpha_j a_j$. For our given $s \in S$, let $s = \sum_j \sigma_j a_j$ be its expansion as a linear combination of these eigenvectors. We are interested in the case

$$ r := (V \otimes I)e^{-iH(c)}(s \otimes m) = \sum_j \sigma_j Va_j \otimes m_{\epsilon a_j} = \sum_j Va_j \otimes \sigma_j m_{\epsilon a_j}, $$

where $m_{\beta}$ denotes the translate of $m$ by $\beta$: $m_{\beta}(q) := m(q - \beta)$ for all $q \in \mathbb{R}$. By formula 43\textsuperscript{26} of Appendix 1 (with the roles of $S$ and $M$ reversed), the $i,j$ matrix element of $\text{tr}_M P(I \otimes P_k)r$ with respect to the orthonormal basis $\{Va_i\}$ is:

$$ \langle Va_i, \text{tr}_M P(I \otimes P_k)r \rangle \langle Va_j \rangle = \frac{\langle P_k m_{\epsilon a_j}, P_k m_{\epsilon a_j} \rangle \sigma_j^* \sigma_i}{|(I \otimes P_k)r|^2}. $$

Substituting in (96) gives the state of $S$ after the projective measurement, expressed as a density matrix with respect to the orthonormal basis $\{Va_j\}$, as

$$ \sum_k \langle P_k m_{\epsilon a_j}, P_k m_{\epsilon a_j} \rangle \sigma_j^* \sigma_i. $$

\textsuperscript{26}If one tries to substitute “Dirac delta functions” for eigenvectors of $Q$ one is led outside the Hilbert space $L^2(\mathbb{R})$. For example, if one measures $Q$ and obtains the value $3.2$, one cannot say that the subsequent state of the system is $q \mapsto \delta(q - 3.2)$. So far as the author knows, this problem has never been resolved in a complete and rigorous way.
Our goal is to show that the limit as $\epsilon \to 0$ of expression (99) is $P_s$ (which is $s$ expressed as a density matrix). This is immediate if it is legitimate to interchange this limit with the summation in (99) because

$$\sum_k \lim_{\epsilon \to 0} \langle P_k m, P_k m \rangle \sigma_j^* \sigma_i = \sum_k |P_k m|^2 \sigma_j^* \sigma_i = \sigma_j^* \sigma_i,$$

(100)
since $\{P_k\}$ is a resolution of the identity and $|m| = 1$. The operator on $S$ whose matrix with respect to the basis $\{V a_j\}$ is given by (100) as $\langle \sigma_j^* \sigma_i \rangle$ is the projector onto the vector

$$\sum_j \sigma_j V a_j = V \sum_j \sigma_j a_j = V s = s.$$

Thus we have established weakness of the measurement of $B$ if the meter state $m$ satisfies

$$\lim_{\epsilon \to 0} \sum_k \langle P_k m_{e\alpha_j}, P_k m_{e\alpha_i} \rangle = \sum_k \lim_{\epsilon \to 0} \langle P_k m_{e\alpha_j}, P_k m_{e\alpha_i} \rangle .$$

(101)

It is legitimate to interchange limits in an expression like (101) if all of the indicated limits exist and if at least one of the inner limits is uniform. A simple hypothesis which assures this is that $m$ have compact support. In that case, the sum on the left side of (101) is actually finite with a maximum number of nonzero terms independent of $\epsilon$ for small $\epsilon$, so the situation is trivial. Since physical “meters” have a bounded range of possible readings, this hypothesis is physically reasonable.

Of course, many other conditions will also suffice. For those who like to work with Gaussians, if $m$ is a Gaussian (or more generally, if $m(q)$ decays sufficiently rapidly as $q \to \pm \infty$), uniform convergence of the sum follows from brute force upper bounds on $|m(q)|$ for large $q$.

As for existence of the limits, the limits in the right-hand expression in (101) obviously exist. If the left-hand limits are in question, one could pass to a sequence $\{\epsilon_l\}_{l=1}^\infty$ with $\epsilon_l \to 0$ for which $\lim_{l \to \infty} \sum_k \langle P_k m_{e\alpha_j}, P_k m_{e\alpha_i} \rangle$ does exist. (Routine compactness arguments assure the existence of such a subsequence.) Then the uniformity shows that all the double limits are equal (and that the left-hand limit in (101) actually does exist).

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27In case the reader is puzzled that there only appears to be one limit in (101), note that the infinite sum implicitly includes a limit.
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