SECOND-ORDER MASS ESTIMATES FOR STATIC VACUUM METRICS
WITH SMALL BARTNIK DATA

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Abstract. Given on the 2-sphere Bartnik data (prescribed metric and mean curvature) that is a small perturbation of the corresponding data for the standard unit sphere in Euclidean space, we estimate to second order, in the size of the perturbation, the mass of the asymptotically flat static vacuum extension (unique up to diffeomorphism) which is a small perturbation of the flat metric on the exterior of the unit ball in Euclidean space and induces the prescribed data on the boundary sphere. As an application we obtain a new upper bound on the Bartnik mass of small metric spheres to fifth order in the radius.

1. Motivation and statement of the results

Bartnik’s definition of quasilocal mass ([11–13]) has inspired a number of interesting questions and results in mathematical relativity; see for example the surveys [6, 17]. The present article is motivated by a desire to refine existing estimates of the Bartnik mass of 2-spheres with small data, meaning metric and mean curvature close to those of the standard sphere. Over time Bartnik’s original definition has given rise in the literature to several variants, whose relation has been analyzed in [38, 40], but here we adopt a somewhat restrictive version, defined in (1.5), after establishing some notation and preliminary definitions. The statement of our results begins on page 4.

Basic definitions and notation. Set \( M := \{ \vec{x} \in \mathbb{R}^3 : |\vec{x}| \geq 1 \} \) and let \( \iota : \partial M \to M \) be the inclusion map for its boundary \( \mathbb{S}^2 := \partial M \). Given a Riemannian metric \( g = g_{ab} \) on \( M \), we write \( R_{abcd}[g] = R_{abcd} \), \( R_{ab}[g] = R_{ab} \), and \( R[g] = R \) for the corresponding Riemann, Ricci, and scalar curvature of \( g \); our conventions for \( R_{abcd} \) are declared in Appendix A. We also write \( H[\iota, g] \) for the corresponding mean curvature function induced on \( \partial M \): by our convention the \( g \)-divergence of the outward (so pointing into the unit ball) \( g \)-unit normal on \( \partial M \). The volume and area densities induced by \( g \) and \( \iota^*g \) respectively will be denoted by \( \sqrt{|g|} \) and \( \sqrt{|\iota^*g|} \). We reserve \( \delta = \delta_{ab} \) for the Euclidean metric on \( \mathbb{R}^3 \) (and its restriction to \( M \)) and caution that by our convention \( H[\iota, \delta] = -2 \). In integrals of functions on \( M \) and \( \mathbb{S}^2 \) the densities \( \sqrt{|\delta|} \) and \( \sqrt{|\iota^*\delta|} \) respectively should be understood, unless a different density is explicitly specified.

In tensor expressions we will use Roman indices for components in \( M \) and Greek indices for components in \( \partial M \). Mostly we employ abstract index notation (that is coordinate-free tensor notation), but we make two types of occasional exceptions. First, we will sometimes work in the standard Cartesian coordinates on \( \mathbb{R}^3 \), for which purpose we reserve the indices \( i, j, k \); second, the label \( r \) will always refer to the standard radial coordinate on \( \mathbb{R}^3 \). Roman indices will be raised and lowered via \( \delta \) and Greek via \( \iota^*\delta \). We shall use a semicolon (in the case of fields over \( M \)) or colon (in the case of fields over \( \partial M \)) to indicate differentiation with respect to the Levi-Civita connection induced by \( \delta \) or \( \iota^*\delta \) respectively; in the case of covariant differentiation of functions (which does not depend on the choice of metric) and in the case of coordinate differentiation (with respect to a Cartesian or radial coordinate) we prefer to use a comma instead.

Given \( k \in \mathbb{Z} \cap [0, \infty), \alpha \in [0, 1), \beta \in \mathbb{R}, \) and a \( C^k_{\text{loc}} \) tensor field \( F \) over a manifold \( S \), possibly with boundary, smoothly embedded in \( \mathbb{R}^3 \) (for example \( S = M \) or \( S = \partial M \)), we define the standard...
Hölder norm

\[ \|F\|_{k,\alpha} := \sum_{i=0}^{k} \sup_{x \in S} |D^i_{\delta} F(\bar{x})|_{\delta} + \sup_{x \neq \bar{y} \in S} \frac{|D^i_{\delta} F(\bar{x}) - D^i_{\delta} F(\bar{y})|}{|\bar{x} - \bar{y}|^\alpha} \]

and weighted Hölder norm

\[ \|F\|_{k,\alpha,\beta} := \sum_{i=0}^{k} \sup_{x \in S} (1 + |\bar{x}|)^{\beta+i} |D^i_{\delta} F(\bar{x})|_{\delta} \]

\[ + \sup_{x \neq \bar{y} \in S} \left[ 1 + \min \{|\bar{x}|, |\bar{y}|\} \right]^{\alpha+\beta+k} \frac{|D^i_{\delta} F(\bar{x}) - D^i_{\delta} F(\bar{y})|}{|\bar{x} - \bar{y}|^\alpha}, \]

where the derivatives \(D_{\delta}\) and differences are taken componentwise relative to the standard Cartesian coordinates \(\{x^1, x^2, x^3\}\) on \(\mathbb{R}^3\). For each tensor bundle \(E\) over \(S\) we define the Banach spaces \(C^{k,\alpha,\beta}(E)\) and \(C^{k,\alpha}(E)\) (written simply \(C^{k,\alpha,\beta}(S)\) and \(C^{k,\alpha}(S)\) as usual when \(E\) is the trivial bundle \(S \times \mathbb{R}\) of sections of \(E\) with finite \(\|\cdot\|_{k,\alpha,\beta}\) and \(\|\cdot\|_{k,\alpha}\) norms respectively. (Frequently the letters \(\alpha\) and \(\beta\) will also appear as indices for tensors over \(\partial M\), but this double duty should never cause confusion.)

We will make routine use of the standard asymptotic “big O” notation. Specifically, suppose \(D\) (for data) is a set, \(\mathcal{X}\) is a vector space equipped with norm \(\|\cdot\|\), and \(x, y : D \rightarrow \mathcal{X}\) and \(c : D \rightarrow [0, \infty)\) are functions; we write

\[ x = y + O(c) \]

if there exist constants \(C, \epsilon_0 > 0\) such that

\[ \|x(d) - y(d)\| \leqCc(d) \text{ for all } d \in D \text{ with } c(d) < \epsilon_0. \]

In instances of this notation the set \(D\), vector space \(\mathcal{X}\), and norm \(\|\cdot\|\) should always be clear from context. Frequently \(\mathcal{X}\) will simply be \(\mathbb{R}\) with \(\|\cdot\|\) the absolute value, and typically \(D\) will consist either of Bartnik data (prescribed metric and mean curvature on a given surface) close to that of \(\partial M\) or \(\mathcal{X}\) will simply be \(\mathcal{X}\) with \(\|\cdot\|\) as in \([14]\). Note that, by the connectedness (as in \([45]\) or \([53]\)) of the Bartnik mass \([9, 10, 23, 47]\), for \(\partial M\) outer-minimizing in \((M, g)\) if its area (measured by \(g\)) is no greater than that of any surface in \(M\) enclosing it. Note that \(\partial M\) is outer-minimizing in \((M, g)\) for every \(g\) in a sufficiently small \(C^1\) neighborhood of \(\delta\).

Now to a given \(C^2\) Riemannian metric \(\gamma = \gamma_{\alpha,\beta}\) and \(C^1\) function \(H\) on \(S^2 = \partial M\) we associate the Bartnik mass

\[ m_B[\gamma, H] := \inf \{m_{\text{ADM}}[g] : g \in \text{Met}^{2,0,1}(M), R[g] \geq 0, \iota^* g = \gamma, \mathcal{H}[\iota, g] \geq H, \text{ and } \partial M \text{ is outer-minimizing in } (M, g) \} \]

(provided this infimum exists), where

\[ m_{\text{ADM}}[g] := \frac{1}{16\pi} \lim_{r \rightarrow \infty} \int_{|x| = r} \left( g_{ik} k^k - g^{k}_{ik} \right) \frac{x^i}{|x|} \]

is the ADM mass ([9, 10, 23, 47]) of \((M, g)\), commas indicating coordinate differentiation as usual. The inequality on mean curvature was adopted from \([43]\) and enforces nonnegative scalar curvature across the boundary in a distributional sense; we remind the reader that by our conventions the standard unit sphere in Euclidean space has mean curvature \(\mathcal{H}[\iota, \delta] = -2\). The outer-minimizing condition was suggested in \([14]\). Note that, by the connectedness (as in \([45]\) or \([53]\)) of the
space of orientation-preserving diffeomorphisms of \( S^2 \), definition (1.5) is diffeomorphism-invariant: 
\[
m_B[\gamma,H] = m_B[\phi^*\gamma,\phi^*H]
\]
for any smooth diffeomorphism \( \phi : S^2 \to S^2 \).

In turn we also define the Bartnik mass for compact Riemannian 3-balls: if \((\Omega, h)\) is the image of a smooth diffeomorphism \( \phi \) from origin-centered closed unit ball in \( \mathbb{R}^3 \) and if \( h \) induces on \( \partial \Omega \) metric \( \gamma \) and mean curvature \( H \) (the \( h \) divergence of the inward unit normal), then to \((\Omega, h)\) we associate the Bartnik mass
\[
m_B[\Omega, h] := m_B[\phi^*\gamma, \phi^*H].
\]

Note that the definition does not depend on the choice of diffeomorphism. By the positive mass theorem with corners of [41] (as well as the extendibility of diffeomorphisms between balls established in [48]) \( m_B[\Omega, h] \) exists and is nonnegative whenever \((\Omega, h)\) admits at least one embedding into an asymptotically flat \( \mathbb{R}^3 \) having nonnegative scalar curvature and in which \( \partial \Omega \) is outer-minimizing. Moreover it follows from [36] that in this case \( m_B[\Omega, h] = 0 \) only when \( h \) is flat.

**Quasilocally defined**. When \( \Omega \) lies in a time-symmetric slice whose source fields contribute nonnegative energy density at each point, definition (1.7) is intended as a measure of the (quasilocally defined) mass of \( \Omega \). In this article we specialize to the case where \( \Omega \) is a metric ball \( B_\tau \) of small radius \( \tau \) in some such slice \((N, h)\). Fixing \((N, h)\) and the center of \( B_\tau \) while taking \( \tau \) small, the leading contribution to the mass of \( B_\tau \), according to any physically reasonable definition, must be given by its volume times the source energy density at its center. Indeed, as reviewed in a moment, the Bartnik mass does not fail to meet this natural expectation, and the present article is motivated by a desire to identify the next most significant contributions. See for example [15, 20, 21, 28, 33, 54, 55, 59] for estimates of other quasilocal masses of small spheres in both Riemannian and spacetime settings.

For context and further motivation we now recall analogous estimates for two quasilocal masses which provide well-known lower and upper bounds for the Bartnik mass (at least under the assumptions of interest here). Namely, for a surface \( \Sigma \) having induced metric \( \gamma \) and mean curvature \( H \) in some time-symmetric slice we have the Hawking mass ([31]) \( m_H[\Sigma] \) and the Brown-York mass ([16]) \( m_{BY}[\Sigma] \) of \( \Sigma \):
\[
m_H[\Sigma] := \sqrt{\frac{\text{Area}(\Sigma)}{16\pi}} \left( 1 - \frac{1}{16\pi} \int \sqrt{\gamma} \right) \quad \text{and}
\]
\[
m_{BY}[\Sigma] := \frac{1}{8\pi} \int (H - H_0) \sqrt{\gamma},
\]
where \( \text{Area}(\Sigma) \) is the area of \( \Sigma \) and for \( m_{BY}[\Sigma] \) we assume that \( \Sigma \) has positive Gaussian curvature and that \( H_0 \) is the mean curvature of an isometric embedding of \( \Sigma \) in Euclidean \( \mathbb{R}^3 \) (whose existence and uniqueness up to rigid motions are guaranteed by [32, 46, 49, 50]); we also remind the reader that \( \sqrt{\gamma} \) denotes the area density induced by \( \gamma \) and that by our conventions the standard unit sphere has mean curvature \(-2\).

We now fix a smooth Riemannian 3-manifold \((N, h)\) having nonnegative scalar curvature, we also fix a point \( p \in N \), and for small variable \( \tau > 0 \) we consider the closed metric ball \( B_\tau \) of center \( p \) and radius \( \tau \). It is straightforward to compute the expansion
\[
m_H[B_\tau] := m_H[\partial B_\tau] = \frac{\tau^3}{12} R + \frac{\tau^5}{720} (6\Delta R - 5R^2) + O(\tau^6),
\]
where \( R \) is the scalar curvature, \( \Delta \) the Laplacian, and all terms are evaluated at \( p \); see for example [28]. Much less straightforward but also accomplished in [28] is the corresponding calculation
\[
m_{BY}[B_\tau] := m_{BY}[\partial B_\tau] = \frac{\tau^3}{12} R + \frac{\tau^5}{1440} (24|\text{Ric}|^2 - 13R^2 + 12\Delta R) + O(\tau^6),
\]
where $|\text{Ric}|^2$ is the squared norm of the Ricci curvature (and again all terms are evaluated at $p$).

With (1.5) as our definition of Bartnik mass (and also under more general conditions) we have the well-known inequalities

\begin{equation}
(1.11) \quad m_B[B_\tau] \leq m_B[B_{\tau,0}] \leq m_{BY}[B_\tau],
\end{equation}

where the lower bound follows from [36] and the upper bound from [52] (at least for $\tau$ small enough that $\partial B_\tau$ has positive Gaussian curvature and inward pointing mean curvature). Since the upper and lower bounds evidently agree to fourth order, we immediately obtain the estimate

\begin{equation}
(1.12) \quad m_B[B_\tau] = \frac{1}{12} R \tau^3 + O(\tau^5),
\end{equation}

confirming (via the energy constraint) the expectation mentioned above that as $\tau$ shrinks to 0 the ratio of the ball’s Bartnik mass to its volume tends to the source energy density at its center. On the other hand the fifth-order terms for the Hawking and Brown-York mass differ in general (even for spherically symmetric $B_\tau$).

**Results.** Bartnik conjectured ([11–13]) that his quasilocal mass should be realized by a unique asymptotically flat static vacuum extension inducing the given boundary metric and mean curvature. The necessity of the boundary and static vacuum conditions for a minimizer have been established (see [7], [24–26] and [34], as well as [2, 35], for the spacetime version), but the existence of a minimizer is known only (at least to the author) for (i) apparent horizons, by virtue of [36] (or [51, 56]), and (ii) for data which can be realized as an outer-minimizing embedded sphere (required for our definition, (1.5)) enclosing the horizon in a time-symmetric Schwarzschild slice (or any outer-minimizing embedded sphere in Euclidean space), by virtue of [36] (or [51, 56]). Although the most aggressive formulations of the conjecture are now known to be false (see [7, 39, 44] and for counterexamples in the spacetime setting in higher dimensions also [35]), it does suggest a strategy, pursued in this article, for seeking a tighter upper bound than the Brown-York mass affords in (1.11). Our results are contained in the following theorem and corollary.

**Theorem 1.13** (Mass estimate; cf. [1, 3, 5, 42] regarding existence and uniqueness). Let $M := \{\bar{x} \in \mathbb{R}^3 : |\bar{x}| \geq 1\}$, let $\iota : \partial M \to M$ be the inclusion map of $\partial M = S^2$ in $M$, and let $\delta$ be the standard Euclidean metric on $M$, so that $\iota^* \delta$ is the round metric of area $4\pi$ and $\partial M$ has mean curvature (by our conventions) $-2$ in $(M, \delta)$. There exists $\epsilon_0 > 0$ such that for any Riemannian metric $\gamma = \iota^* \delta + 1$ and function $H = -2 + \frac{1}{\delta}$ on $S^2$ with $\epsilon := \|\gamma\|_{3, \alpha} + \|H\|_{2, \alpha} < \epsilon_0$ there is an asymptotically flat static vacuum metric $g$ on $M$, unique up to diffeomorphism (for $\epsilon_0$ sufficiently small), which induces metric $\gamma$ and mean curvature $H$ on $\partial M$ and which satisfies $\|g - \delta\|_{3, \alpha, 1} = O(\epsilon)$. Moreover $g$ has mass $m_{ADM}[g] = \bar{m} + \bar{m} + O(\epsilon^3)$, where

\begin{equation}
(1.14) \quad \bar{m} = \frac{1}{16\pi} \int_{\partial M} \left( 2\frac{(1)}{(1)} \gamma - \frac{(1)}{(1)} \sigma \sigma \right),
\end{equation}

and

\begin{equation}
(1.15) \quad \frac{(1)}{(1)} \sigma := \frac{(1)}{(1)} \gamma - \frac{1}{2} \left( \gamma \gamma \sigma \sigma \sigma \right).
\end{equation}
the \((\iota^*\delta)\) trace-free part of \(\gamma, v : M \to \mathbb{R}\) the unique \((\delta)\) harmonic function on \(M\) vanishing at infinity and satisfying

\[
(1.16) \quad \iota^* v_{,rr} = 2H + \gamma^{(1)}_{\alpha\beta} : \beta^\alpha - \frac{1}{2}(\Delta + 2)\gamma^{(1)}_{\sigma},
\]

and \(f : S^2 \to \mathbb{R}\) any solution to

\[
(1.17) \quad (\Delta + 2)f = H - \iota^*(v - v_r).
\]

The preceding equation has a solution for any data, and the choice of solution does not affect the value of \(m\). In the above, \(v_r\) and \(v_{,rr}\) are the first and second derivatives of \(v\) in the radial direction (into \(M\)), and the integrals, the raising of indices, the norm operation, the Laplacian \(\Delta\), and the covariant differentiation indicated by the colon in front of indices are all defined via the round metric \(\iota^*\delta\).

Most of Theorem 1.13 is old news: existence was established first in [42] assuming reflectional symmetry through each of the coordinate planes and later in generality, along with local uniqueness and in arbitrary dimension, in [5], applying results from [4, 8]. In fact a proof of existence and uniqueness for the more general stationary vacuum extension problem (with small data) has recently been achieved in [1] and, beyond the ball, for small perturbations of a broad class of Euclidean domains in the static case in [3]. The first-order mass estimate \(m\) appears in [57] and actually holds for any sufficiently small vacuum extension by virtue of identity (C.4) in Appendix C. The novel contribution of the present article is the computation of the quadratic term \(m_2\) in the expansion of the extension’s mass, which proceeds by an elaboration of the approach in [57].

Clearly, for \(\epsilon_0\) sufficiently small, \(\partial M\) is outer-minimizing in the extension featured in Theorem 1.13. Consequently (and independently of the validity of Bartnik’s static vacuum extension conjecture in this setting) the ADM mass of this extension is an upper bound for the Bartnik mass \(m_\beta[\gamma, H]\) (as defined in (1.5)) of its boundary. (For an estimate of the Bartnik mass of data which is exactly (rather than approximately, as here) constant-mean-curvature but not necessarily almost round see [18],) In particular we get the following upper bound for the Bartnik mass of small metric spheres.

**Corollary 1.18** (Upper bound on the quintic term of the Bartnik mass of small spheres). Let \(p\) be a point in a Riemannian 3-fold with nonnegative scalar curvature and for each \(\tau > 0\) let \(B_\tau\) be the closed metric ball of radius \(\tau\). Then (recalling (1.7) and (1.5)) for \(\tau\) sufficiently small we have the estimate

\[
(1.19) \quad m_\beta[B_\tau] \leq \frac{\tau^3}{12}R + \frac{\tau^5}{2160} \left(30|Ric|^2 - 25R^2 + 18\Delta R\right) + O(\tau^6),
\]

where \(R\) is the scalar curvature \(|Ric|^2\) the squared norm of the Ricci curvature, \(\Delta\) the Laplacian, and all terms are evaluated at \(p\).

**Remark 1.20** (Conjectural equality). We have equality in Corollary 1.18 in the event that Bartnik’s static vacuum extension conjecture holds in this restricted, weak-field regime.

**Remark 1.21** (Comparison with Hawking and Brown-York masses). Writing \(m_S[B_\tau]\) for the right-hand side of (1.19) (representing the ADM mass of the static vacuum extension from Theorem 1.13 for \(\partial B_\tau\)) and recalling the expansions (1.9) and (1.10), we have

\[
(1.22) \quad m_S[B_\tau] - m_\beta[B_\tau] = \frac{\tau^5}{216} \left(3|Ric|^2 - R^2\right) + O(\tau^6) \quad \text{and}
\]

\[
 (m_{BY}[B_\tau] - m_S[B_\tau]) = \frac{\tau^5}{4320} \left(12|Ric|^2 + 11R^2\right) + O(\tau^6),
\]
so that $m_p[B_\tau] \leq m_s[B_\tau] + O(\tau^6)$, with equality only when Ric is spherically symmetric at $p$, while $m_s[B_\tau] \leq m_{BY}[B_\tau] + O(\tau^6)$, with equality only when $B_\tau$ is flat at $p$.

An alternative computation of $m_s[B_\tau]$, that avoids reference to the details of the construction of the extension, has been given in [30] (after the appearance of the present article in preprint form).

**Outline.** The mass estimate in Theorem 1.13 is proved by carrying out the construction of a static vacuum extension in [57] and keeping track of the mass to second order (in the size of the perturbation from the Bartnik data of the standard sphere). In Section 2 we recall the static vacuum system and review the analysis of the linearized problem in [57], with some refinements. In Section 3 we achieve both the static vacuum conditions and the boundary conditions to first order. In Section 4 we achieve the static vacuum conditions to second order (without sacrificing the satisfaction of the boundary conditions to first order). In Section 5 we complete the construction to second order, enforcing also the boundary conditions. In Section 6 we compute the mass of the approximate solution constructed in the previous steps, so estimating the mass of an exact static vacuum extension to second order and completing the proof of Theorem 1.13. In Section 7 we at last apply the mass estimate of Theorem 1.13 to prove Corollary 1.18.

There are five appendices. The first three present standard identities and computations for ease of reference, while the fourth consists of a straightforward but somewhat lengthy calculation we could not find elsewhere, and the fifth presents an alternative proof and generalization of a step performed in [57]. Appendix A declares our conventions for the Riemann curvature tensor and includes some useful identities, particularly in dimension 3. Appendix B contains the response of Ricci curvature to conformal change of metric. Appendix C is used to relate the variation of the boundary data under change of ambient metric to the corresponding variation of the mass. Appendix D computes the second variation of the induced boundary metric and mean curvature with respect to simultaneous ambient diffeomorphism and conformal change. Finally, Appendix E concerns the linearization of the prescribed Ricci equation on Euclidean domains without boundary conditions and related material on symmetric tensors of prescribed support and (compatibly supported) divergence.

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2. **The static vacuum system and its linearization**

With assumptions and notation as in Theorem 1.13 we seek an asymptotically flat Riemannian metric $g$ on $M$ inducing on $\partial M$ metric $\gamma$ and mean curvature (by our convention the divergence of the origin-directed unit normal) $H$ together with a real-valued function $\Phi$ on $M$ making $(M, g)$ static vacuum with static potential $\Phi$. In other words $(g, \Phi)$ must satisfy (see for example [22, 24]) the static vacuum system

\begin{equation}
D^2_{ab}[g]\Phi - \Phi R_{ab}[g] = 0 \quad \text{and} \quad \Delta[g]\Phi = 0
\end{equation}

($D^2_{ab}[g]$, $R_{ab}[g]$, and $\Delta[g]$ denoting respectively the Hessian operator, Ricci curvature, and Laplacian associated to $g$) subject to the boundary conditions

\begin{equation}
\iota^*g_{\alpha\beta} = \gamma_{\alpha\beta} \quad \text{and} \quad H[\iota, g] = H,
\end{equation}
where $\mathcal{H}[\iota, g]$ is the $g$ divergence of the outward (meaning pointing away from $M$) $g$ unit normal for $\partial M$, and we also impose the decay conditions

\begin{align}
 g_{ab} - \delta_{ab} &\in C^{3,\alpha,1}(T^* M^\odot 2) \quad \text{and} \quad \Phi - 1 \in C^{3,\alpha,1}(M).
\end{align}

We follow the procedure of [57], with minor modifications and more detailed estimates.

We define the operators

\begin{align}
 \mathcal{S} : \text{Met}^{k,\alpha,\beta}(M) \times (1 + C^{k,\alpha,\beta}(M)) &\to C^{k-2,\alpha,\beta+2}(T^* M^\odot 2) \times C^{k-2,\alpha,\beta+2}(M) \quad \text{and} \\
 \mathcal{B} : \text{Met}^{k,\alpha,\beta}(M) &\to C^{k,\alpha}(T^* \partial M^\odot 2) \times C^{k-1,\alpha}(\partial M) \quad \text{by}
\end{align}

\begin{align}
 \mathcal{S}[g, \Phi] := (D^2_{ab}[g] \Phi - \Phi R_{ab}[g], \Delta [g] \Phi) \quad \text{and} \quad \mathcal{B}[g] := (\iota^* g, \mathcal{H}[\iota, g]),
\end{align}

and for a given metric $k$ and function $\Psi$ on $M$ we denote the linearization of $\mathcal{S}$ at $(k, \Psi)$ and the linearization of $\mathcal{B}$ at $k$ by $\hat{\mathcal{S}}[k, \Psi]$ and $\hat{\mathcal{B}}[k]$ respectively. In particular we have

\begin{align}
 \hat{\mathcal{S}}[\delta, 1](h, u) = \left( u_{;ab} - \hat{R}_{ab}[\delta](h), \Delta [\delta] u \right),
\end{align}

where $u_{;ab}$ and $\Delta [\delta] u$ are the Hessian and Laplacian of $u$ with respect to the Euclidean metric $\delta$ and the linearization $\hat{R}_{ab}[\delta]$ at $\delta$ of the Ricci curvature can be expressed in the well known form

\begin{align}
 \hat{R}_{ab}[\delta](h) = \frac{1}{2} \left( h^c_{;ab;c} + h^c_{;cb} - h^c_{c;ab} - h^c_{ab;c} \right),
\end{align}

the semicolons indicating differentiation via $\delta$.

To solve the boundary-value problem $\mathcal{S}[g, \Phi] = (0, 0)$ with $\mathcal{B}[g] = (\gamma, H)$ (and the desired decay) we will first analyze the linearized problem and construct an exact solution to the nonlinear problem by iteration (or more formally by invoking the contraction mapping lemma, as in [57]). We will split the linearized problem into two parts: first the problem with homogeneous data for $\hat{\mathcal{S}}[\delta, 1]$ and prescribed inhomogeneous boundary data for $\hat{\mathcal{B}}[\delta]$, namely

\begin{align}
 \hat{\mathcal{S}}[\delta, 1](h_{ab}, u) = (0, 0) \quad \text{and} \quad \hat{\mathcal{B}}[\delta](h_{ab}) = \left( \tilde{\gamma}_{\alpha\beta}, \tilde{H} \right),
\end{align}

and second the problem with no boundary constraint and for $\hat{\mathcal{S}}[\delta, 1]$ prescribed inhomogeneous data $(S_{ab}, \sigma)$ modulo the Lie derivative of the Euclidean metric $\delta$ along some vector field $\chi$, another unknown of the system, namely

\begin{align}
 \hat{\mathcal{S}}[\delta, 1](h, u) = (S_{ab} + \chi_{a;b} + \chi_{b;a}, \sigma),
\end{align}

the semicolon indicating differentiation via $\delta$.

Since we already control the boundary data in (2.7), we may ignore it when solving (2.8). The Lie derivative term in (2.8) is necessary because of an obstruction arising from the linearization of the twice contracted Bianchi identity; it is acceptable because in every such problem we face in practice this identity will be approximately satisfied by $S$ itself, so that the error introduced by $\chi$ will be higher-order and can be safely absorbed by the next iteration of the scheme. The details clarifying and justifying these assertions will be reviewed below when needed.

We solve (2.7) by deforming the Euclidean metric $\delta$ by linearized diffeomorphism and conformal change and accordingly altering the trivial potential $1$. In doing so we encounter the linear system featured in the following lemma and relevant to the statement of Theorem 1.13.

**Lemma 2.9 (Analysis of the boundary system).** Suppose $\alpha \in (0, 1)$, $\tilde{\gamma} \in C^{3,\alpha}(\partial M)$, $\tilde{H} \in C^{2,\alpha}(\partial M)$. 7
(i) There exist functions $v \in C^{3,\alpha,1}(M)$, $f \in C^{4,\alpha}(\partial M)$, and a vector field $X^\sigma \in C^{4,\alpha}(\partial M)$ satisfying the system

\begin{equation}
\begin{aligned}
(a) \quad & \Delta_\delta v = 0 \\
(b) \quad & \iota^* v + 2f = \frac{1}{2} \gamma^\sigma \sigma - X^\sigma : \sigma \\
(c) \quad & \iota^* (v - v_r) + (\Delta_\ast \delta + 2) f = \tilde{H} \\
(d) \quad & \tilde{\gamma}_{\mu\nu} = \left( \frac{1}{2} \gamma^\sigma \sigma - X^\sigma : \sigma \right) \iota^* \delta_{\mu\nu} + X_{\mu\nu} + X_{\nu\mu},
\end{aligned}
\end{equation}

(2.10)

where $\Delta_\delta = \Delta[\delta]$ is the flat Laplacian on $M \subset \mathbb{R}^3$, $\Delta_\ast = \Delta[\iota^* \delta]$ is the round Laplacian on $S^2 = \partial M$, and the raising and lowering of indices as well as the differentiation indicated by the colon are performed via the round metric $\iota^* \delta$.

(ii) The system (2.10) and data $\tilde{\gamma}$ and $\tilde{H}$ uniquely determine $v$: it is the unique harmonic function on $M$ vanishing at infinity and satisfying

\begin{equation}
\iota^* v_{,rr} = 2\tilde{H} + \tilde{\gamma}_{,\rho\sigma} : : \rho\sigma = \frac{1}{2} (\Delta_\ast \delta + 2) \tilde{\gamma}^\sigma \sigma,
\end{equation}

(2.11)

where $\tilde{\gamma} := \gamma - \frac{1}{2} \gamma^\sigma \sigma \iota^* \delta$ and $\iota^* v_{,rr}$ is the restriction to $\partial M = S^2$ of the second radial derivative (directed into $M$) of $v$. In particular

\begin{equation}
\|v\|_{3,\alpha,1} \leq C (\|\tilde{\gamma}\|_{3,\alpha} + \|\tilde{H}\|_{2,\alpha})
\end{equation}

for some $C > 0$ independent of $\tilde{\gamma}$ and $\tilde{H}$.

(iii) Any two solutions to equation (d) in (2.10) differ by a conformal Killing field on $S^2 = \partial M$; any solution $X$ to (d), together with $v$ satisfying (a) and (2.11), uniquely determines a function $f$ so that (2.10) is satisfied.

Proof. Item (i) was established in [57], but we provide a more streamlined proof here and fill in the remaining items.

Equation (d) in (2.10) is thoroughly understood in the literature ([58]); in this paragraph we briefly summarize its standard analysis. We can rewrite (d) as $LX = \tilde{\gamma}$, with $L$ the conformal Killing operator in two dimensions. Taking the divergence of both sides yields $(L^* LX)_\alpha = \tilde{\gamma}^{\alpha}_{\beta\gamma} : \beta$, with $L^*$ the formal adjoint of $L$. Then $L^* L$ is second-order elliptic with the same kernel as $L$, namely the space of conformal Killing fields, to which the divergence of the trace-free symmetric tensor $\tilde{\gamma}$ is orthogonal. It follows that $L^* LX = \text{div} \tilde{\gamma}$ has a solution $X^\sigma \in C^{4,\alpha}(\partial M)$, so that $\tilde{\gamma} - LX$ has vanishing trace and divergence, but on 2-sphere the only such symmetric 2-tensor is the trivial one (as evident from (A.8), since the sphere has strictly positive curvature). Thus $X$ is a solution of equation (d) in (2.10). It is clear from equation (b) that $f$ is then uniquely determined by $v$ and the choice of $X$, completing the proof of (iii).

Solving (b) for $f$, substituting the result into (c), and rearranging yields $\Delta_\ast \delta \iota^* v + 2v_r = -2\tilde{H} + (\Delta_\ast \delta + 2)(\frac{1}{2} \text{tr} \gamma - \text{div} X)$, but by taking the double divergence of $LX = \tilde{\gamma}$ we get $(\Delta_\ast \delta + 2) \text{div} X = \text{div} \text{div} \tilde{\gamma}$. By applying the expression for the Laplacian in spherical coordinates to (a) we obtain (2.11). Conversely, if (a) and (d) hold, and we define $f$ by (b), then (c) holds too. This completes the proof of (ii) and (i), except for the existence of a harmonic $v \in C^{3,\alpha,1}(M)$ satisfying (2.11) and (2.12) (since the asserted regularity of $f$ then follows from equation (c) of (2.10)).
The Laplacian on $(S^2, \epsilon^* \delta)$ has eigenvalues $-\ell(\ell + 1)$ for $\ell \in \mathbb{Z} \cap [0, \infty)$, and an eigenfunction $w$ with eigenvalue $-\ell(\ell + 1)$ has the unique extension $wr^{-\ell-1}$ to a harmonic function on $M$ vanishing at infinity. Consequently, since $(\ell + 1)(\ell + 2) > 0$ for all $\ell \geq 0$, we are guaranteed that there is a unique harmonic function $v$ on $M$ satisfying (2.11) and vanishing at infinity and that its restriction to $S^2$ lies at least in the Sobolev space $H^2(\partial M)$, so certainly in $C^0(\partial M)$. Moreover, the preceding paragraph shows that this restriction satisfies

$$\label{2.13} (\Delta_v \delta + 2N)\epsilon^* v = -2\tilde{H} - \tilde{\gamma}_\sigma^\rho \sigma^\rho + \frac{1}{2}(\Delta_v \delta + 2)\gamma^\sigma_\sigma,$$

where $N$ is the Dirichlet-to-Neumann map for the harmonic extension problem on $(M, \delta)$. Since for any $k \geq 0$ $N$ is a bounded linear map from $C^{k+2,\alpha}(\partial M)$ to $C^{k+1,\alpha}(\partial M)$, it follows by interpolation and elliptic regularity that $\|w\|_{k+2,\alpha} \lesssim \|(\Delta + 2N)w\|_{k,\alpha} + \|w\|_0$ whenever $w \in C^{k+2,\alpha}(\partial M)$. A standard argument using mollification and compactness of $C^{k+2,\alpha}$ in $C^{k+2}$ completes the proof. \qed

In a moment Lemma 2.9 will be applied to the linearized static vacuum boundary-value problem (2.7). But it also ensures that the functions $f$ and $v$, as well as the mass contribution $\tilde{m}$ itself, appearing in the statement of Theorem 1.13 are well defined.

**Corollary 2.14** (Well-definedness of the expression for $\tilde{m}$). Make the assumptions of Theorem 1.13. (i) Equation (1.16) has a unique harmonic solution in $M$ vanishing at infinity. (ii) Equation (1.17) has a solution, and the value of $\tilde{m}$ in (1.14) does not depend on the choice of solution.

**Proof.** Take $\gamma$ and $\tilde{H}$ as in Theorem 1.13 and apply Lemma 2.9 with $\tilde{\gamma} = \gamma$ and $\tilde{H} = H$ and equations (1.16) and (1.17) supplemented by equations (a), (b), and (d) in (2.10). Item (i) and the existence claim of item (ii) follow directly. For the remaining claim in (ii) suppose that $k$ lies in the kernel of $\Delta_v \delta + 2$. Then by equation (c) of (2.10)

$$\label{2.15} -Hk + \frac{1}{2}(v - v_r)2k = -Hk + Hk - k(\Delta_v \delta + 2)f,$$

whose integral over $S^2 = \partial M$ vanishes in view of the assumption that $k$ belongs to the kernel of the symmetric operator $\Delta_v \delta + 2$. Referring to (1.14), this confirms that the value of $\tilde{m}$ is independent of the choice of $f$ in (1.17). \qed

Now we solve the linearized static vacuum boundary-value problem (2.7).

**Proposition 2.16** (BVP with homogeneous interior data). Let $\alpha \in (0, 1)$. Given any boundary data $\tilde{\gamma} \in C^{3,\alpha}(\partial M)$ and $\tilde{H} \in C^{2,\alpha}(\partial M)$ the linearized static vacuum boundary-value problem (2.7) has a solution $(h_{ab}, u)$ with $\|h\|_{3,\alpha} + \|u\|_{3,\alpha,1} \leq C \left(\|\tilde{\gamma}\|_{3,\alpha} + \|\tilde{H}\|_{2,\alpha}\right)$ for some $C > 0$ independent of the data $\tilde{\gamma}$ and $\tilde{H}$. The function $u$ is uniquely determined by the data, and any two choices for $h_{ab}$ differ by the Lie derivative of the flat metric $\delta$ along a vector field whose restriction to $\partial M$ (with values in $T\mathbb{R}^3$) is the restriction to $\partial M$ of some Killing field on $\mathbb{R}^3$.

**Proof.** For existence we repeat the proof in [57]. The static vacuum conditions (2.1) are obviously preserved by diffeomorphisms, but diffeomorphisms deforming the geometry of $\partial M$ will alter the boundary data. At the linearized level this means (keeping in mind that the background potential 1 is constant) that for any vector field $\xi^a$ on $M$ the pair $(h_{ab}, 1u) = (\xi_{ab} + \xi_{b,a}, 0)$ will satisfy $\dot{S}(\delta, 1)(1h, 1u) = (0, 0)$. Meanwhile $\mathcal{B}$ will be sensitive only to $\xi_{\partial M}$ specifically, if $\xi_{\partial M} = \epsilon_x F + x \partial_r$ for some function $f$ and vector field $X^a$ on $S^2 = \partial M$, then

$$\label{2.17} \mathcal{B}(\delta)(1h) = (X_{\alpha,\beta} + X_{\beta,\alpha} + 2f \epsilon_x^* \delta, (\Delta + 2)f).$$
On the other hand, it is also easy to use conformal changes with harmonic conformal factor to adjust the boundary data while maintaining the static vacuum conditions. Specifically, if \( v \) is a harmonic function on \( M \), then examination of (2.5) and (2.6) confirms that the pair \((2h_{ab},2u) = (v\delta_{ab},-v/2)\) satisfies \( \tilde{S}[\delta,1](2h,2u) = (0,0) \), and at the boundary we have

\[
(2.18) \quad \tilde{S}[\delta](2h) = (\iota^*(v\delta), \iota^*(v-v_r)).
\]

Summing the contributions of (2.17) and (2.18), it is clear that Lemma 2.9 delivers a solution to (2.7) satisfying the asserted estimate.

For uniqueness suppose \((h_{ab},u)\) is a solution to (2.7) with \( \tilde{\gamma}_{\alpha\beta} = 0 \) and \( \tilde{H} = 0 \). It follows from (2.5) and (2.6) that \( \tilde{R}_{ab}[\delta](h-2u\delta) = 0 \). By Proposition A.7 \( h_{ab} - 2u\delta_{ab} = \xi_{a;1}\dot{\xi} + \xi_{b;\alpha} \) for some \( \xi^a \), but by items (ii) and (iii) of Lemma 2.9 \( \iota^*(u) = 0 \) and \( \iota^*(v-v_r) \) is the restriction to \( \partial M \) of some Killing field. \( \square \)

To complete the analysis of the linearized problem we now solve (2.8).

**Proposition 2.19** (Inhomogenous interior data). Let \( \alpha \in (0,1) \) and \( \beta \in (1/2,1) \). Given \( S_{ab} \in C^{1,\alpha,3+\beta}(T^*M;\mathbb{R}^2) \) and \( \sigma \in C^{1,\alpha,3+\beta}(M) \), system (2.8) has a solution \((h_{ab},u,\chi)\) so that for some \( C > 0 \) independent of the data we have (i) \( \|h\|_{3,\alpha,1} + \|u\|_{3,\alpha,1+\beta} \leq C(\|S\|_{1,\alpha,3+\beta} + \|\sigma\|_{1,\alpha,3+\beta}) \), (ii) \( h_{ab} = \frac{1}{2} h^{c}_{ca;\alpha} \) and (iii) \( \|\chi\|_{2,\alpha,2+\beta} \leq C \|\Delta_\delta \chi\|_{0,\alpha,4+\beta} \).

**Proof.** First pick \( u \) satisfying \( u_{c;\alpha} = \sigma \) and the estimate required by item (i) of the statement. To solve for \( h \) (and \( \chi \) in the process) the proof of the corresponding Lemma 2.8 in [57] made use of the representation (A.6) and an associated result in elasticity theory (see for example Section 17 of [29].) Alternatively we can apply Proposition E.9 with \( F_{ab} := u_{;ab} - S_{ab} \) to select \( \chi \), satisfying the estimate (iii), and then conclude by applying Proposition E.7 to obtain \( h \) satisfying (i) and (ii). \( \square \)

### 3. Solving to first order

With assumptions as in Theorem 1.13 we start by taking \( f, v, X \) as in Lemma 2.9 with \( \tilde{\gamma}^1 = \gamma \) and \( \tilde{H} = H \), so that in particular

\[
(3.1) \quad \|v\|_{3,\alpha,1} + \|f\|_{4,\alpha} = O(\epsilon).
\]

Next let \( \xi^a \) be the unique radially constant (\( \delta \)-parallel along rays extending from the origin) vector field on \( M \) satisfying \( \xi^a \circ \iota = \iota^a_{\mu}X^\mu \). (Here \( \iota^a_{\mu} \) is \( dt_i \), so that the right-hand side is \( X \) pushed forward by \( \iota \), while the left-hand side is \( \xi \) pulled back by \( \iota \).) Now set

\[
(3.2) \quad \Xi^a(\vec{x}) := \psi(\psi(\vec{x})) \left[ f \frac{(\vec{x})}{|\vec{x}|} \vec{x} + \vec{\xi}(\vec{x}) \right],
\]

where \( \psi : [1, \infty) \rightarrow [0,1] \) is a smooth function taking the value 1 constantly on \([1,2]\) and taking the value 0 constantly on \([3, \infty)\). Thus, by Lemma 2.9 (cf. the proof of Proposition 2.16), \((h_{ab}, u) := \Xi_{a;b} + \Xi_{b;a} + v\delta_{ab}, -v/2)\) is a solution to 2.7 with data \((\gamma^1, \tilde{H})\).

We now define the function \( \rho : M \rightarrow \mathbb{R} \) and the map \( \phi : M \rightarrow \mathbb{R}^3 \) by

\[
(3.3) \quad \rho := 1 + v \quad \text{and} \quad \phi(\vec{x}) = \vec{x} + \Xi(\vec{x}).
\]

For \( \epsilon \) sufficiently small \( \rho \) is strictly positive and \( \phi \) is a diffeomorphism onto its image \( \phi(M) \). From these we define on \( M \) the Riemannian metric and real-valued function

\[
(3.4) \quad \gamma_{ab} := \frac{1}{2} \rho \phi^* \delta_{ab} \quad \text{and} \quad \phi := 1 - \frac{1}{2} v.
\]
Clearly

\[(3.5) \quad \| g - \delta \|_{3, \alpha, 1} + \| \Phi - 1 \|_{3, \alpha, 1} = O(\epsilon) \]

and by design

\[(3.6) \quad \| S \left[ 1 \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right] \|_{1, \alpha, 3+\beta} = O(\epsilon^2) \quad \text{and} \quad \| B \left[ 1 \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right] \|_{1, \alpha} = (\gamma, H) + O(\epsilon^2), \]

where we fix now and henceforth some \( \beta \in (1/2, 1) \). Here we use the facts (as we shall do repeatedly in the sequel) that, working in any coordinates, \( B[g] \) is pointwise a smooth function of the derivatives of \( g \) from order zero to one and that, working in Cartesian coordinates, \( S[g, \Phi] \) is pointwise a smooth function of the derivatives of \( \Phi \) from order zero to two and of the derivatives of \( g \) from order zero to two. Consequently we obtain (3.6) by pointwise Taylor expansion to first order with a quadratic estimate for the remainder; for the decay estimate in particular we exploit the specific structure of \( S \).

4. ACHIEVING THE STATIC VACUUM CONDITION TO SECOND ORDER

We next seek higher-order corrections \( (2S) g \in C^{3, \alpha, \beta}(T^*M^2) \) and \( (2S) \Phi \in C^{3, \alpha, \beta}(M) \) making the pair

\[(4.1) \quad \left( \frac{2S}{g}, \frac{2S}{\Phi} \right) := \left( \frac{1}{g} + \frac{(2S) 1}{g}, \frac{(2S) 1}{g} + \frac{(2S)}{\Phi} \right) \]

static vacuum to second order in \( \epsilon \) while preserving the boundary conditions to first order: we require

\[(4.2) \quad \| S \left[ \frac{(2S)}{g}, \frac{(2S)}{\Phi} \right] \|_{1, \alpha, 3+\beta} = O(\epsilon^3) \]

and

\[(4.3) \quad \| \frac{(2S)}{g} \|_{3, \alpha, \beta} + \| \frac{(2S)}{\Phi} \|_{3, \alpha, \beta} = O(\epsilon^2). \]

Assuming we can arrange (4.3), we will have

\[(4.4) \quad S \left[ \frac{(2S)}{g}, \frac{(2S)}{\Phi} \right] = S \left[ \frac{1}{g}, \frac{1}{\Phi} \right] + S[\delta, 1] \left( \frac{(2S)}{g}, \frac{(2S)}{\Phi} \right) + O(\epsilon^3) \]

(having also used (3.5) to justify writing \( S[\delta, 1] \) rather than \( S[1 \frac{\partial}{\partial x}, 1 \frac{\partial}{\partial y}] \)), so to achieve (4.2) we intend to solve

\[(4.5) \quad S[\delta, 1] \left( \frac{(2S)}{g}, \frac{(2S)}{\Phi} \right) = -S \left[ \frac{1}{g}, \frac{1}{\Phi} \right] + O(\epsilon^3) \]

or equivalently

\[(4.6) \quad D_{ab}[\delta] \frac{(2S)}{g} \Phi - \delta_{ab}R_{g} \frac{(2S)}{g} = \frac{1}{g} \Phi R_{g} \frac{1}{g} \frac{(2S)}{g} - D_{ab}[\frac{(2S)}{g} 1] \frac{1}{g} + O(\epsilon^3) \]

\[ \Delta[\delta] \frac{(2S)}{\Phi} = -\Delta \left[ \frac{1}{g} \right] \frac{1}{g} + O(\epsilon^3) \]

for \( \frac{(2S)}{g} \) and \( \frac{(2S)}{\Phi} \).
We conclude that from the construction of which completes the verification of the system (4.6), the validity of whose second equation is obvious

\[ S_{ab} := \frac{1}{2} \Phi R_{ab} \left[ g \right] - D_{ab}^2 \left[ g \right] \frac{1}{2} \Phi, \quad \sigma := -\Delta \left[ g \right] \frac{1}{2} \Phi, \]

(4.7)

\[ g_{ab} := h_{ab}, \quad \text{and} \quad \Phi := u. \]

Note that since \( \hat{R}_{ab}[\delta] g \) itself satisfies the linearization at \( \delta \) of the twice contracted differential Bianchi identity, we have

\[ \chi_{a;b} := \frac{1}{2} \sigma_{a} - S_{ab} ; b + \frac{1}{2} S_{b,a}, \]

(4.8)

but by (3.5) and (3.6)

\[ \left\| S_{ab} ; b - g D_{c} \left[ g \right] S_{ab} \right\|_{0, \alpha, 4+\beta} + \left\| S_{b,a} - \left( \frac{1}{2} \Phi S_{bc} \right) , a \right\|_{0, \alpha, 4+\beta} = O (\varepsilon^3), \]

(4.9)

so, applying the twice contracted differential Bianchi identity for \( R_{ab}[g_1] \),

\[ \chi_{a;b} = -\frac{1}{2} \left( \Delta \left[ g \right] \frac{1}{2} \Phi \right) , a - g^{bc} \frac{1}{2} \Phi R_{ab} \left[ g \right] - \frac{1}{2} \Phi R_{a} \left[ g \right] + \left( \Delta \left[ g \right] \frac{1}{2} \Phi \right) , a + g^{bc} \frac{1}{2} \Phi R_{ab} \left[ g \right] \]

\[ + \frac{1}{2} \Phi R_{a} \left[ g \right] + \frac{1}{2} \Phi R_{a} \left[ g \right] - \frac{1}{2} \left( \Delta \left[ g \right] \frac{1}{2} \Phi \right) , a + O(\varepsilon^3) \]

(4.10)

\[ = \frac{1}{2} \Phi R_{a} \left[ g \right] + O(\varepsilon^3). \]

On the other hand, (3.5) and (3.6) imply that \( \left\| R \left[ g \right] \right\|_{1, \alpha, 4} = O (\varepsilon^2) \), so in fact

\[ \left\| \chi_{a;b} \right\|_{0, \alpha, 4+\beta} = O (\varepsilon^3), \]

(4.11)

and therefore by item (iii) of Proposition 2.19

\[ \left\| \chi \right\|_{2, \alpha, 2+\beta} = O (\varepsilon^3), \]

(4.12)

so in turn

\[ \left\| \chi_{a;b} + \chi_{b,a} \right\|_{1, \alpha, 3+\beta} = O (\varepsilon^3). \]

(4.13)

We conclude that

\[ \Phi^{(2S)}_{ab} - \hat{R}_{ab}[\delta] g = \frac{1}{2} \Phi R_{ab} \left[ g \right] - D_{ab}^2 \left[ g \right] \frac{1}{2} \Phi + O (\varepsilon^3), \]

(4.14)

which completes the verification of the system (4.6), the validity of whose second equation is obvious from the construction of \( \Phi^{(2S)} \) via Proposition 2.19. We have now established (4.2).

**An estimate for future use.** Before proceeding to correct the boundary metric and mean curvature to second order, we pause to derive an estimate for \( \int_M g^{(2S)} \frac{d}{c} \) that will be useful later. It follows from item (ii) of Proposition 2.19 that \( \hat{R}_{ab}[\delta] g = -\frac{1}{2} g^{(2S)}_{ab, c} \), whereby, in view of (4.6),

\[ g^{(2S)}_{ab, c} = 2 \Phi R_{ab} \left[ g \right] - 2 D_{ab}^2 \left[ g \right] \Phi - 2 \Phi \frac{1}{2} \Phi_{ab} + O (\varepsilon^3). \]

(4.15)
Appealing again to (3.5) and (3.6) and contracting the above equation against $\delta^{ab} = \delta^{ab} + O(\epsilon)$, we find

\begin{equation}
\frac{(28)^{c}}{g^{c}_{\phantom{c}d}} = 2 \Phi R \left[ \frac{1}{g} \right] - 2 \Delta \left[ \frac{1}{g} \right] \Phi - 2 \Delta[\delta] \Phi + O(\epsilon^3) = 2 \Phi R \left[ \frac{1}{g} \right] + O(\epsilon^3),
\end{equation}

where to get the second estimate we have used (4.6).

Now

\begin{equation}
R_{ab} \left[ \frac{1}{g} \right] = R_{ab}[\rho \phi^* \delta]
\end{equation}

\begin{align*}
&= \phi^* R_{ab}[\delta] + \frac{3}{4} \rho^{-2} \rho_{\alpha\beta} - \frac{1}{2} \rho^{-1} D_{ab}[\phi^* \delta] \rho \\
&\quad - \frac{1}{2} \rho^{-1} (\Delta \phi^* \delta \rho) (\phi^* \delta)_{ab} + \frac{1}{4} \rho^{-2} |d\rho|^2_{\phi^* \delta} (\phi^* \delta)_{ab},
\end{align*}

so

\begin{align*}
R \left[ \frac{1}{g} \right] &= \frac{3}{4} \rho^{-3} |d\rho|^2_{\phi^* \delta} - \frac{1}{2} \rho^{-2} \Delta \phi^* \delta \rho - \frac{3}{2} \rho^{-2} \Delta \phi^* \delta \rho + \frac{3}{4} \rho^{-3} |d\rho|^2_{\phi^* \delta} \\
&= \frac{3}{2} \rho^{-3} |d\rho|^2_{\phi^* \delta} - 2 \rho^{-2} \Delta \phi^* \delta \rho \\
&= \frac{3}{2} (1 + v)^{-3} v_{,c} v_{,d} (\phi^* \delta)^{cd} - 2 (1 - v)^2 \Delta \phi^* \delta v + O(\epsilon^3)
\end{align*}

\begin{align*}
&= \frac{3}{2} |d\nu|_\delta^2 - 2 \Delta \phi^* \delta v + 4 \nu \Delta \phi^* \delta v + O(\epsilon^3) \\
&= \frac{3}{2} |d\nu|_\delta^2 - 2 \Delta \phi^* \delta v + O(\epsilon^3),
\end{align*}

having made use of (2.10), (3.1), and (3.3).

Continuing (4.16) with this last estimate, we get

\begin{align*}
\frac{(28)^{c}}{g^{c}_{\phantom{c}d}} &= 2 \left( 1 - \frac{1}{2} v \right) \left( \frac{3}{2} |d\nu|_\delta^2 - 2 \Delta \phi^* \delta v \right) + O(\epsilon^3) \\
&= 3 |d\nu|_\delta^2 - 4 \Delta \phi^* \delta v + 2 \nu \Delta \phi^* \delta v + O(\epsilon^3) \\
&= 3 |d\nu|_\delta^2 - 4 \Delta \phi^* \delta v + 2 \nu \Delta \delta v + O(\epsilon^3) \\
&= 3 |d\nu|_\delta^2 - 4 \Delta \phi^* \delta v + O(\epsilon^3).
\end{align*}

Now

\begin{align*}
\int_M |d\nu|_\delta^2 &= \int_M v^c v_{,c} = \int_M (vv^c)_{,c} - \int_M vv^c_{,c} \\
&= - \int_\partial_M vv_{,r} + \lim_{R \to \infty} \int_{\{r = R\}} vv_{,r} - 0 \\
&= - \int_\partial_M vv_{,r},
\end{align*}

since $\|vv_r\|_{0,0,3} = O(1)$ as a consequence of (3.1).
Additionally, writing \( N_{\phi^s \delta} \) for the \( \phi^s \delta \) \( \infty \)-directed unit normal to \( \iota \),
\[
\int_M \Delta_{\phi^s \delta} v = \int_{\{1 \leq r \leq 3\}} \Delta_{\phi^s \delta} v = \int_{\{1 \leq r \leq 3\}} \Delta_{\phi^s \delta} v \sqrt{|\phi^s \delta|} + \int_{\{1 \leq r \leq 3\}} \Delta_{\phi^s \delta} v \left( \sqrt{|\delta|} - \sqrt{|\phi^s \delta|} \right) \\
= \int_{\{r=3\}} v_r - \int_{\partial M} N_{\phi^s \delta} v \sqrt{|\iota^* \phi^s \delta|} + \int_{\{1 \leq r \leq 3\}} \Delta_{\phi^s \delta} v \left( \sqrt{|\delta|} - \sqrt{|\iota^* \phi^s \delta|} \right) + O(\varepsilon^3) \\
= \int_{\partial M} (\partial_r - N_{\phi^s \delta}) v \sqrt{|\iota^* \phi^s \delta|} + \int_{\partial M} v_r \left( \sqrt{|\iota^* \delta|} - \sqrt{|\iota^* \phi^s \delta|} \right) + O(\varepsilon^3) \\
= -\int_{\partial M} (N_{\phi^s \delta} - \partial_r) v + \int_{\partial M} v_r \left( \sqrt{|\iota^* \delta|} - \sqrt{|\iota^* \phi^s \delta|} \right) + O(\varepsilon^3),
\] 
(4.21)
where in particular we have used the facts that \( N_{\phi^s \delta} - \partial_r = O(\varepsilon), v = O(\varepsilon), \) and \( \sqrt{|\iota^* \phi^s \delta|} = \sqrt{|\iota^* \delta|} + O(\varepsilon) \) to conclude that the first integral on the penultimate line and the first integral on the ultimate line have \( O(\varepsilon^3) \) difference. More precisely, we compute (for example by appealing to (D.7) and the first line of (D.15))
\[
N_{\phi^s \delta} = \partial_r + dt(\nabla_{\iota^s \delta} f - X) + O(\varepsilon^2),
\]
\[
\sqrt{|\iota^* \phi^s \delta|} = \sqrt{|\iota^* \delta|} + 2 f + \text{div}_{\iota^s \delta} X + O(\varepsilon^2).
\]
Picking up where we left off in (4.21), we proceed to estimate
\[
\int_M \Delta_{\phi^s \delta} v = \int_{\partial M} v_r (f^\sigma - X^\sigma) - \int_{\partial M} v_r (2 f + X^\sigma, r) + O(\varepsilon^3) \\
= -\int_{\partial M} v (f^\sigma, r - X^\sigma, r) - \int_{\partial M} v_r (2 f + X^\sigma, r) + O(\varepsilon^3) \\
= -\int_{\partial M} v \left( f^\sigma, r + v + 2 f - \frac{1}{2} (\gamma^\sigma, r) \right) - \int_{\partial M} v_r \left( \frac{1}{2} (\gamma^\sigma, r) - v \right) + O(\varepsilon^3) \\
= -\int_{\partial M} v \left( \frac{1}{2} (H - \frac{1}{2} \gamma^\sigma, r) + v, r \right) - \int_{\partial M} v_r \left( \frac{1}{2} (\gamma^\sigma, r) - v \right) + O(\varepsilon^3) \\
= -\int_{\partial M} \left[ v \frac{1}{2} (H - \frac{1}{2} (\gamma^\sigma, r) + v, r \right) \gamma^\sigma, r \right] + O(\varepsilon^3).
\] 
(4.23)
From (4.19), (4.20), and (4.23) we arrive at
\[
\int_M \frac{\partial \delta^c_d}{\partial g} = \int_{\partial M} \left[ 4 v^{\gamma, r} (H - \frac{1}{2} (\gamma^\sigma, r) - 3 v v, r \right] + O(\varepsilon^3).
\] 
(4.24)
5. Solving to second order

It remains to enforce the boundary data to second order while maintaining the static vacuum condition also to second order: we seek corrections \( g^{(2B)} \in C^{3, \alpha, \beta}(T^* M \odot 2) \) and \( \Phi \in C^{3, \alpha, \beta}(M) \) such that the pair
\[
\left( \frac{2}{g}, \Phi \right):= \left( \frac{2}{g^{(2B)}} g^{(2B)} \Phi + \Phi \right)
\]
satisfies
\[
S \left[ \frac{2}{g}, \Phi \right] = O(\varepsilon^3) \quad \text{and} \quad B \left[ \frac{2}{g}, \Phi \right] = (\gamma, H) + O(\varepsilon^3).
\]
(5.1)
5. Mass estimate

By iteratively correcting the boundary and interior geometry as in the above stages (or applying the contraction mapping lemma as in the proof of Proposition 2.51 in [57]) we obtain a static vacuum metric \((g, \Phi)\) with the prescribed boundary data,

\[
S[g, \Phi] = (0, 0) \quad \text{and} \quad B[g] = (\gamma, H),
\]

and satisfying

\[
g = g + \frac{(2S)}{g} + \frac{(2B)}{g} + O(\epsilon^3).
\]

Consequently,

\[
m_{\text{ADM}}[g] = m_{\text{ADM}}[g] + m_{\text{ADM}}[\delta + \frac{(2S)}{g} + \frac{(2B)}{g}] + O(\epsilon^3).
\]

From (2.10), (3.3), and (3.4) we get (as in [57])

\[
m^{(1)} := m_{\text{ADM}}[g] = \frac{1}{16\pi} \int_{\partial M} -2v_{\tau} = \frac{1}{16\pi} \int_{\partial M} \left(2H - (1)\sigma\right).
\]
Similarly, recalling (5.7) and setting
\[
(6.5) \quad \frac{(2B)_m}{m} := \frac{1}{16\pi} \int_{\partial M} \left( (2B)_\gamma \sigma - 2H \right) \quad \text{and} \quad \frac{(2S)_m}{m} := \frac{1}{16\pi} \int_{\partial M} \left( (2S)_\gamma \sigma - 2H \right),
\]
we also have
\[
(6.6) \quad m_{ADM} \left[ \delta + \frac{(2B)}{m} \right] = \frac{(2B)}{m} + \frac{(2S)}{m}.
\]
Using (D.17) and (D.18) (and integration by parts) we obtain
\[
(6.7) \quad \frac{(2B)}{m} = \frac{1}{16\pi} \int_{\partial M} \left[ 3(f + v)(\Delta + 2)f + \frac{3}{2}v^2 - 3v\nu_r + |DX|^2 - |X|^2 + 2X^{\alpha;\beta} f_{\alpha;\beta} + 2fX^{\sigma;\sigma} \right].
\]
By (2.10)(d) we have
\[
(6.8) \quad X^{\alpha;\beta} + X^\alpha = (1)_{\gamma \alpha \beta}^{\gamma} - \frac{1}{2} (1)_{\gamma \sigma}^{\gamma \sigma} X^\alpha = (1)_{\alpha \beta}^{\alpha \beta},
\]
where
\[
(6.9) \quad (1)_{\gamma}^{\gamma} := \gamma - \frac{1}{2} \gamma^\sigma \epsilon_{\sigma}^* \delta.
\]
Thus further integration by parts and another appeal to (2.10)(d) yields
\[
(6.10) \quad \int_{\partial M} |DX|^2 - |X|^2 = - \int_{\partial M} \left[ X^{\alpha;\beta} + X^\alpha \right] X^\alpha = \int_{\partial M} (1)_{\gamma}^{\gamma} X^\alpha = \frac{1}{2} \int_{\partial M} (1)_{\gamma}^{\gamma}. \quad \text{Additionally, noting that} \quad \partial M \text{\ is the round unit sphere with Ricci curvature} \quad \epsilon^* \delta,
\]
\[
(6.11) \quad \int_{\partial M} X^{\alpha;\beta} f_{\alpha;\beta} = - \int_{\partial M} X^{\alpha;\beta} f^\beta = - \int_{\partial M} \left[ X^{\alpha;\beta} f^\beta + X^\beta f^\beta \right] = \int_{\partial M} X^\sigma \sigma (\Delta + 1)f.
\]
It follows that
\[
(6.12) \quad \frac{(2B)}{m} = \frac{1}{16\pi} \int_{\partial M} \left[ (3f + 3v + 2X^{\sigma;\sigma})(\Delta + 2)f + \frac{3}{2}v^2 - 3v\nu_r + \frac{1}{2} (1)_{\gamma}^{\gamma} \right].
\]
Using (2.10) we conclude
\[
(6.13) \quad \frac{(2B)}{m} = \frac{1}{16\pi} \int_{\partial M} \left[ (v - f + (1)_{\gamma}^{\gamma} \sigma) \left( (1)_{\gamma}^{\gamma} H - (v - \nu_r) \right) + \frac{3}{2}v^2 - 3v\nu_r + \frac{1}{2} (1)_{\gamma}^{\gamma} \right].
\]
On the other hand, by the definition of \((2S)_\gamma, (2S)_H\) in terms of \((2S)_g\) in (5.3), the definition of \((2S)_g\) in (4.7) via Proposition 2.19 (particularly item (ii)), and the result of (C.4) (in Appendix C)
\[
16\pi \frac{(2S)_m}{m} + O(\epsilon^4) = - \int_{\partial M} \left( \frac{(2S)_g_{ij}}{g} - \frac{(2S)_g}{g_{ij}} \right) x^i = \frac{1}{2} \int_{\partial M} \left( \frac{(2S)_g}{g_{ij};j} \right) x^i
\]
\[
(16.14) \quad = -16\pi m_{ADM} \left[ \delta + \frac{(2S)_g}{g} \right] - \frac{1}{2} \int_{M} \frac{(2S)_\sigma}{g_{cd}} \delta_{ad}.
\]
In turn (4.24) yields
\[
(6.15) \quad \frac{(2S)}{m} = m_{ADM} \left[ \delta + \frac{(2S)_g}{g} \right] = \frac{1}{16\pi} \int_{\partial M} \left[ \frac{3}{2}v\nu_r + (v - \nu_r) (1)_{\gamma}^{\gamma} \sigma - 2(vH) \right] + O(\epsilon^3).
\]
Summing, we obtain the estimate \(m\) of Theorem 1.13.
7. Application to small spheres

We will now apply our estimate to the case that the data \((\gamma, H)\) correspond to the boundary of a small metric ball, appropriately scaled. Specifically, fix a Riemannian 3-manifold \((N, h)\) and a point \(p \in N\); identify \(S^2 = \partial M\) with the unit sphere in \(T_p N\) and for each \(\tau \in \mathbb{R}\) define \(\varphi_{\tau} : S^2 \rightarrow N\) by \(\varphi_{\tau}(v) := \exp_p^{(N,h)} \tau v\) (\(\exp_p^{(N,h)} : T_p N \rightarrow N\) being the exponential map of \((N, h)\) at \(p\)); and (for \(\tau > 0\) sufficiently small) set

\[
(\gamma, H) = (\gamma[\tau], H[\tau]) := (\varphi_{\tau}^{*} \tau^{-2} h, \mathcal{H}[\varphi_{\tau}, \tau^{-2} h]),
\]

so that the metric ball in \((N, h)\) of center \(p\) and radius \(\tau\) has induced metric \(\tau^2 \gamma[\tau]\) and mean curvature \(\tau H[\tau]\), both pulled back to \(\partial M\).

We then have (see for example [57]) the well known Taylor expansions

\[
\begin{align*}
(\gamma)^{(1)}_{\alpha\beta} &= \frac{1}{3} \tau^2 R^a_{\alpha \beta r} + \frac{1}{6} \tau^3 R_{\alpha \beta r | r} + \tau^4 \left( \frac{1}{20} R_{\alpha \beta r | r r} + \frac{2}{45} R_{\alpha r c r} R^c_{\tau \beta r} \right) + O(\tau^5) \quad \text{and} \\
(\mathcal{H})^{(1)} &= \frac{1}{3} \tau^2 R_{r r} + \frac{1}{4} \tau^3 R_{r r | r} + \tau^4 \left( \frac{1}{10} R_{r r | r r} + \frac{4}{45} R_{c r d r} R^c_{r | d r} \right) + O(\tau^5),
\end{align*}
\]

where \(R_{abcd}\) is the Riemann curvature tensor (and \(R_{ab}\) the Ricci curvature) of \((N, h)\) (following the curvature conventions declared in (A.1)) evaluated at \(p\), the vertical bar \(|\) indicates differentiation via the Levi-Civita connection induced by \(h\), also evaluated at \(p\), and each \(r\) index indicates contraction with \(\nu := \partial_r \circ \iota\), the unit normal of \(\partial M\) directed into \(M\) and regarded as an element of \(T_p N\). Alternatively, we may regard the curvature tensors (and their contractions and derivatives) in the above expansions as parallel tensors over \(S^2 = \partial M\), writing for example

\[
\begin{align*}
(\gamma)^{(1)}_{\alpha\beta} &= \frac{1}{3} \tau^2 R_{\alpha \beta r} + \tau^3 R_{\alpha \beta r | r} + \tau^4 \left( \frac{1}{20} R_{\alpha \beta r | r r} + \frac{2}{45} R_{\alpha r c r} R^c_{\beta r} \right) + O(\tau^5) \\
(\mathcal{H})^{(1)} &= \frac{1}{3} \tau^2 R_{r r} + \tau^3 R_{r r | r} + \tau^4 \left( \frac{1}{10} R_{r r | r r} + \frac{4}{45} R_{c r d r} R^c_{r | d r} \right) + O(\tau^5),
\end{align*}
\]

etc. We may also choose to specify a point on \(\partial M\) by its standard Euclidean coordinates, or position vector, in \(\mathbb{R}^3\) and thereby write

\[
\begin{align*}
(\gamma)^{(1)}_{\alpha\beta}(x) &= \frac{1}{3} \tau^2 R^a_{\alpha \beta j} x^i x^j + \cdots, \quad \text{etc.}
\end{align*}
\]

In the following we will take the liberty of applying whichever of these notational options best suits our purposes at a given step.

We wish to estimate the ADM mass \(m = m[\tau] = m[\gamma, H]\) of the unique small static vacuum extension of \((\gamma, H)\), within an error of order \(\tau^5\). Since \(\|(\gamma)^{(1)}_{\alpha \beta}, H^{(1)}\|_B = O(\tau^2)\), we have

\[
\begin{align*}
(\gamma)^{(1)}_{\alpha\beta} &= \frac{1}{3} \tau^2 R^a_{\alpha \beta r} + \tau^3 R_{\alpha \beta r | r} + \tau^4 \left( \frac{1}{20} R_{\alpha \beta r | r r} + \frac{2}{45} R_{\alpha r c r} R^c_{\beta r} \right) + O(\tau^5) \\
(\mathcal{H})^{(1)} &= \frac{1}{3} \tau^2 R_{r r} + \tau^3 R_{r r | r} + \tau^4 \left( \frac{1}{10} R_{r r | r r} + \frac{4}{45} R_{c r d r} R^c_{r | d r} \right) + O(\tau^5),
\end{align*}
\]

recall (1.14) From (7.2) it follows readily (as in [57]) that

\[
\begin{align*}
(\gamma)^{(1)}_{\alpha\beta} &= \frac{1}{12} R_{\tau r} + \frac{1}{120} \Delta R_{\tau r} + O(\tau^4) \\
(\mathcal{H})^{(1)} &= \frac{1}{3} \tau^2 R_{r r} + \tau^3 R_{r r | r} + \tau^4 \left( \frac{1}{10} R_{r r | r r} + \frac{4}{45} R_{c r d r} R^c_{r | d r} \right) + O(\tau^5),
\end{align*}
\]

(whose \(\mathcal{H}\) is the scalar curvature of \((N, h)\) at \(p\)). To achieve our goal we will now estimate \(\mathcal{H}\) up to order \(\tau^4\), which means we need only keep track of the leading terms in (7.2).

From (2.10), making use of (6.8) and the identity \(t^* \Delta_\delta = \Delta_t \delta t^* + 2 t^* \partial_r + t^* \partial_r^2\), we find

\[
\begin{align*}
t^* v_{r r} &= 2(\gamma)^{(1)}_{\alpha\beta} \cdot \delta^\alpha_\beta - \frac{1}{2}(\Delta + 2)(\gamma)^{(1)}_\sigma \sigma.
\end{align*}
\]
Thus, from the above,
\[ v \]
where in the last step we have decomposed the leading terms of \( v_{rr} \) into spherical harmonics. Recalling that \( v \) is a harmonic function on \( M \) vanishing at infinity, we conclude that
\[ \tau^2 v + O(\tau^3) = \frac{1}{6} R\tau^2 + \frac{1}{12} \left( \frac{2}{3} R - 2 R_{ij} x^i x^j \right) \tau^2 = \left( \frac{2}{9} R - \frac{1}{6} R_{rr} \right) \tau^2, \]

and so
\[ \tau^2 f + O(\tau^3) = \left( \frac{5}{9} R - \frac{2}{3} R_{rr} \right) \tau^2. \]

Next we will solve for \( f \). Referring again to (2.10), we see that
\[ (\Delta + 2) f = \tau^2 (v - v_r) \]

Thus, from the above,
\[ (\Delta + 2) f + O(\tau^3) = \left( \frac{5}{9} R + R_{rr} \right) \tau^2 = \frac{2}{9} R\tau^2 + \left( \frac{1}{3} R + R_{ij} x^i x^j \right) \tau^2; \]

and so
\[ f + O(\tau^3) = \frac{1}{9} R\tau^2 - \frac{1}{4} \left( \frac{1}{3} R + R_{rr} \right) \tau^2 = \left( \frac{1}{36} R - \frac{1}{4} R_{rr} \right) \tau^2. \]

With the preceding in place we can now compute
\[ \gamma^\sigma_{\sigma} - f - \tau^2 v = \left( \frac{7}{36} R + \frac{1}{12} R_{rr} \right) \tau^2 + O(\tau^3), \]
\[ \tau^2 v + 2f \]
\[ \frac{1}{2} (v - v_r)(v + 2f) = \left( \frac{5}{108} R^2 - \frac{13}{36} R_{rr} + \frac{2}{9} R_{rr}^2 \right) \tau^4 + O(\tau^5), \]
and
\[ \frac{1}{2} \gamma \gamma = \frac{1}{2} (\gamma) \gamma - \frac{1}{4} (\gamma) \gamma = \left( \frac{1}{18} R_{arbr} R_{arbr} - \frac{1}{36} R_{rr}^2 \right) \tau^4 + O(\tau^5). \]
In turn we find
\begin{equation}
\tag{7.15}
\frac{(2)}{m} = \frac{\tau^4}{16\pi} \int_{\mathbb{S}^2} \left[ \left( \frac{1}{18} R_{abcd} R^{abcd} + \frac{2}{9} R_{ij} R_{ij} \right) x^i x^j x^k x^\ell + \frac{5}{108} R^2 - \frac{11}{36} R R_{ij} x^i x^j \right] + O(\tau^5).
\end{equation}
Using
\begin{equation}
\tag{7.16}
\frac{1}{16\pi} \int_{\mathbb{S}^2} 1 = \frac{1}{4}, \quad \frac{1}{16\pi} \int_{\mathbb{S}^2} x^i x^j = \frac{1}{12} \delta^{ij}, \quad \text{and}
\end{equation}
we obtain
\begin{equation}
\tag{7.17}
\frac{(2)}{m} = \frac{1}{1080} |\mathrm{Riem}|^2 \tau^4 + \frac{1}{1080} R_{abcd} R^{abcd} \tau^4 + \frac{1}{120} |\mathrm{Ric}|^2 \tau^4 - \frac{11}{1080} R^2 \tau^4 + O(\tau^5).
\end{equation}
Applying (A.5) we conclude
\begin{equation}
\tag{7.18}
\frac{(2)}{m} = \frac{1}{72} |\mathrm{Ric}|^2 \tau^4 - \frac{5}{432} R^2 \tau^4 + O(\tau^5) = \frac{1}{432} \left( 6 |\mathrm{Ric}|^2 - 5 R^2 \right) \tau^4 + O(\tau^5),
\end{equation}
completing the proof of Corollary 1.18.

**Appendix A. Riemann Curvature Conventions and Identities**

We adopt the convention that the Riemann curvature tensor $R_{abcd} = \mathrm{Riem}$ of a Riemannian manifold $(M,g)$ satisfies
\begin{equation}
\tag{A.1}
X^d |_a - X^d |_{ab} = X^c R_{abc}^d
\end{equation}
for all smooth vector fields $X$ (the vertical bar indicating differentiation via the Levi-Civita connection with respect to the indices following it). Then $R_{abcd} = -R_{bacd} = -R_{abdc} = R_{cdab}$. We denote the corresponding Ricci curvature by $\mathrm{Ric} = R_{ab} = R_{cab}^c = R_{ac}^c b$ and the scalar curvature by $R$. For any given twice-differentiable symmetric tensor $h_{ab}$ one readily computes the linearized Riemann curvature
\begin{equation}
\tag{A.2}
\hat{R}_{abc}^d [g] h := \left. \frac{d}{dt} \right|_{t=0} R_{abcd} d^g + th
= \frac{1}{2} \left( h^{d}_{|bc} + h^{d}_{ac} |_b - h^{d}_{a|cb} - h^{d}_{b|ca} + R_{abk}^f h_{c}^f + R_{abf} h_{df}^f \right).
\end{equation}

**Dimension 3.** Now assume that $M$ has dimension 3 and let $\epsilon_{abc}$ be a choice of orientation form (at least locally defined). A short calculation reveals the identity
\begin{equation}
\tag{A.3}
\epsilon_{abc} \epsilon_{def} R_{abcd} = 4 G_{xy},
\end{equation}
where $G_{ab} := R_{ab} - \frac{1}{2} R g_{ab}$ is the Einstein tensor, and in turn we have
\begin{equation}
\tag{A.4}
R_{abcd} = \epsilon_{abc} \epsilon_{def} G^{xy} = R_{ad} g_{bc} + R_{bc} g_{ad} - R_{ac} g_{bd} - R_{bd} g_{ac} + \frac{1}{2} R g_{ac} g_{bd} - \frac{1}{2} R g_{ad} g_{bc}.
\end{equation}
In particular
\begin{equation}
\tag{A.5}
|Riem|^2 = 4 |Ric|^2 - R^2 \quad \text{and} \quad R_{abcd} R^{abcd} = 2 |Ric|^2 - \frac{1}{2} R^2.
\end{equation}
Furthermore, it follows from (A.2) and (A.3) that for $M \subset \mathbb{R}^3$ the linearization $\hat{G}[\delta]$ of the Einstein tensor about the Euclidean metric $\delta$ is given by
\begin{equation}
\tag{A.6}
\hat{G}_{xy} [\delta] h = \frac{1}{2} \epsilon_{abc} \epsilon_{def} h^{bd} |ca.
\end{equation}
Proposition A.7 (Triviality of first-order Ricci-flat deformations). Let $\Omega$ be a simply connected open subset of $\mathbb{R}^3$. If $h_{ab}$ is a symmetric tensor on $\Omega$ such that $\tilde{R}_{ab}[\delta]h = 0$, then there is a vector field $X^a$ on $\Omega$ such that $h_{ab} = X_{a;b} + X_{b;a} = L_X \delta_{ab}$.

Proof. The claim can be established using the representation (A.6) and the Poincaré lemma; see for example Section 14 of [29]. Alternatively, for any given $h_{ab} \in C^2_\text{loc}(T^\odot 2 \Omega)$ and $p \in \Omega$ there is a sufficiently small origin-centered open ball $B$ in $\mathbb{R}^3$ such that for all sufficiently small $t$ each exponential map $\phi_t$ of $g_t := \delta + th$ at $p$ is a diffeomorphism of $B$ onto its image. Assuming $\tilde{R}_{ab}[\delta]h = 0$, we have also (in dimension 3) $\tilde{R}_{abcd}[\delta]h = 0$, but then $\frac{d}{dt}|_{t=0} \phi_t^* g_t = 0$, meaning that $h = L_X \delta$ on $B$ with $X = -\frac{d}{dt}|_{t=0} \phi_t$. Furthermore, if $Y$ and $Z$ are vector fields on a connected open subset $U \subset \Omega$ with $h = L_Y \delta$ on $U$, then $h = L_Z \delta$ on $U$ as well if and only if $Y - Z$ is the restriction of a Killing field on $\mathbb{R}^3$. The existence of a global $X$ on $\Omega$ such that $h = L_X \delta$ now follows from the assumption that $\Omega$ is simply connected.

Dimension 2. Note that, since the Einstein tensor vanishes in dimension 2, it follows directly from (A.2) that any twice differentiable transverse (that is having vanishing divergence) traceless symmetric tensor $\eta_{ab}$ on $(M, g)$ satisfies

\[(A.8) \quad \eta_{ab|c}^{|c} - 2R\eta_{ab} = 0.\]

APPENDIX B. RICCI CURVATURE UNDER CONFORMAL CHANGE OF METRIC

We will derive the well-known expression for the transformation of Ricci curvature under conformal change of metric by applying the standard formula for the first variation of mean curvature under normal deformation. Let $(M, g)$ be a Riemannian manifold of dimension $\dim M = n + 1$, and let $\rho \in C^2_\text{loc}(M)$ be strictly positive. We will make use of the identities

\[\Delta_{\rho g} u = \rho^{-1} \Delta_g u + \frac{\dim M - 2}{2} \rho^{-2} g^{cd} \rho_{[c} u_{d]},\]

\[(\Delta_g u)|_{\Sigma} = \Delta_{\nu \rho g} u|_{\Sigma} + u_{[ab]} \nabla^a \nabla^b - H u_{[a] \nabla^a},\]

\[A_\nu = (\nu|\rho)^{1/2} A(t, \rho g)|_{\alpha\beta} - \frac{1}{2} (\nu|\rho)^{-1/2} \nu^c (\rho_{[c} \circ t)(\nu|\rho_{\cdot})_{\alpha\beta}, \text{ and}\]

\[H_\nu = (\nu|\rho)^{-1/2} H(t, g) - \frac{n}{2} (\nu|\rho)^{-3/2} \nu^c (\rho_{[c} \circ t),\]

where $u \in C^2_\text{loc}(M)$, vertical bars before indices indicate covariant differentiation defined by $g$, and $\Sigma$ is an embedded hypersurface of $M$, with inclusion map $\iota$, $\nu$ a local choice of unit normal, $A = A(t, \rho g) = -\frac{1}{2} \nu^a L_a \rho g$ (for any extension of $\nu$) and $H = H(t, \rho g) = \tr_g A$ the corresponding second fundamental form and mean curvature of $\Sigma$ in $(M, g)$, and $A_\nu$ and $H_\nu$ the analogously defined second fundamental form and mean curvature of $\Sigma$ in $(M, \rho g)$.

Given any $p \in M$ and unit vector $U \in T_p M$, define $\Sigma$ to be the intersection of a sufficiently small open neighborhood of $p$ with the union of geodesics through $p$ orthogonal to $U$, so that $\Sigma$ is an embedded, two-sided hypersurface, with $\nu$ the unit normal satisfying $\nu|_p = U$. We write $\iota : \Sigma \to M$ for the inclusion map, and for each $t \in \mathbb{R}$ we define the deformed inclusion $\iota_t : \Sigma \to M$ by $\iota_t(q) := \exp^g_{\iota(q)} \nu(t)(q)$, where $\exp^g$ is the exponential map for $(M, g)$. Then $\iota_t$ is a $C^2_\text{loc}$ embedding for sufficiently small $t$, and we write $\nu(t)$ for the choice of unit normal for $\iota_t$, which is continuous in $t$ and agrees with $\nu$ at $t = 0$.

Using (B.3) we find

\[A_\nu = 0, \quad H_\nu = 0, \quad A(t, \rho g) = -\frac{1}{2} \rho|\Sigma|^{-1/2} \nu^c (\rho_{[c} \circ t)(\nu|\rho_{\cdot})_{\alpha\beta}, \text{ and}\]

\[H(t, \rho g) = (\rho \circ t)^{-1/2} H(t, g) - \frac{n}{2} (\rho \circ t)^{-3/2} \nu^c (\rho_{[c} \circ t(\nu|\rho_{\cdot})_{\alpha\beta}.\]

\[(B.5) \quad A_\nu = 0, \quad H_\nu = 0, \quad A(t, \rho g) = -\frac{1}{2} \rho|\Sigma|^{-1/2} \nu^c (\rho_{[c} \circ t)(\nu|\rho_{\cdot})_{\alpha\beta}, \text{ and}\]

\[H(t, \rho g) = (\rho \circ t)^{-1/2} H(t, g) - \frac{n}{2} (\rho \circ t)^{-3/2} \nu^c (\rho_{[c} \circ t(\nu|\rho_{\cdot})_{\alpha\beta}.\]
and differentiation then yields

\[
\frac{d}{dt} \bigg|_{t=0} H[\nu, \rho] = \rho^{-1/2} \frac{dH[\nu, \rho]}{dt} \bigg|_{t=0} + \frac{3n}{4} \rho^{-5/2} \left[ \nu^\rho(\rho_{\nu} \circ \nu) \right]^2 - \frac{n}{2} \rho^{-3/2} \nu^\rho \rho^\rho(\rho_{\nu} \circ \nu).
\]

On the other hand from the standard expression for the first variation of mean curvature we have of course \( \frac{d}{dt} \bigg|_{t=0} H[\nu, \rho] = R_{ab}[\nu^a \nu^b] \) and, using also the expression for \( A[\nu, \rho] \) in (B.5) again,

\[
\frac{d}{dt} \bigg|_{t=0} H[\nu, \rho] = \Delta_{\rho[\nu]} \rho[\nu]^2 + n \frac{n}{4} \rho^{-5/2} \left[ \nu^\rho(\rho_{\nu} \circ \nu) \right]^2 + \rho^{-1/2} R_{ab}[\nu^a \nu^b].
\]

Comparing, we obtain

\[
R_{ab}[\nu^a \nu^b] = R_{ab}[\nu^a \nu^b] + n \frac{n}{2} \rho^{-5/2} \left[ \nu^\rho(\rho_{\nu} \circ \nu) \right]^2 - \frac{n}{2} \rho^{-3/2} \nu^\rho \rho^\rho(\rho_{\nu} \circ \nu) - \rho^{-1/2} \Delta_{\rho[\nu]} \rho[\nu]^2.
\]

From (B.1) and (B.2), bearing in mind that \( H[\nu, \rho] = 0 \),

\[
\Delta_{\rho[\nu]} \rho[\nu]^2 = \rho^{-1} \left( \Delta_{\rho} \rho[\nu]^2 \right) - \rho^{-1} \rho[\nu] D^2[\rho] \left( \rho[\nu] \right)_{\nu} - \frac{n}{4} \rho^{-5/2} \left| \nabla_{g} \rho[\nu] \right|^2.
\]

Since \( U = \nu \) was arbitrary and \( R_{ab} \) is symmetric, by polarization we conclude, after some simplification,

\[
R_{ab}[\nu^a \nu^b] = R_{ab}[\nu^a \nu^b] + \frac{3(\dim M - 2)}{4} \rho^{-2} \rho_{\nu \nu} - \frac{\dim M - 2}{2} \rho^{-1} \rho_{ab} - \frac{1}{2} \rho^{-1} \left( \Delta_{\rho} \rho \right) g_{ab} - \frac{\dim M - 4}{4} \rho^{-2} \left| \nabla_{g} \rho \right|^2 g_{ab}.
\]

In particular, in three dimensions

\[
R_{ab}[M^3, \rho] = R_{ab}[M^3, \rho] + \frac{3}{4} \rho^{-2} \rho_{\nu \nu} - \frac{1}{2} \rho^{-1} \rho_{ab} - \frac{1}{2} \rho^{-1} \left( \Delta_{\rho} \rho \right) g_{ab} + \frac{1}{4} \rho^{-2} \left| \nabla_{g} \rho \right|^2 g_{ab}.
\]

**Appendix C. Mass and Variation of Mean Curvature**

Let \( S \) and \( M \) be smooth manifolds, \( \phi : S \rightarrow M \) a two-sided \( C^2_{\text{loc}} \) codimension-one immersion, and \( \{g(t)\}_{t \in \mathbb{R}} \) a smooth one-parameter family of \( C^1_{\text{loc}} \) Riemannian metrics on \( M \). Pick a corresponding smooth one-parameter family \( \{\nu(t)\} \) of unit normals on \( S \), inducing corresponding second fundamental form \( A_{\alpha \beta}(t) := D[g(t)]_{\alpha} \phi^\nu, \beta \nu^\nu(g(t) \circ \phi) \) and mean curvature \( H(t) := \phi^* g(t) \) and mean curvature \( H(t) := \phi^* g(t) \). Set \( \gamma := \phi^* g(0), A_{\alpha \beta} := A_{\alpha \beta}(0), H := H(0), \hat{g} := \partial_t |_{t=0} g(t) \circ \phi, \hat{\gamma} := \partial_t |_{t=0} \phi^* \hat{g}(t), \) and \( \hat{H} := \partial_t |_{t=0} H(t) \). Then it is easy to compute (as for example in Appendix B of [57]) that

\[
\hat{H} = -A^{\alpha \beta} \hat{g}_{\alpha \beta} + \frac{1}{2} H \hat{g}_{ab} \nu^a \nu^b + \frac{1}{2} \nu^\nu \left( \hat{g}_{cd} \nu^a + \hat{g}_{cd} \nu^a - \hat{g}_{cd} \nu^a \right),
\]

where vertical bars indicate differentiation relative to \( g(0) \), Roman (Greek) indices are raised and lowered via \( g(0) (\gamma) \) and a Greek index on \( \hat{g} \) indicates (partial) pullback by \( d\phi \), so that for example \( \hat{g}_{ab} = \hat{g}_{\alpha \beta} \phi^\alpha, \beta \).
Of course
\[ \nu^c \dot{g}_{c|a} = \dot{g}_{cd} \nu^d - \dot{g}_{ab} \nu^a \nu^b \nu^c, \]
\[ \nu^c \dot{g}_{\sigma|c} = \dot{g}_{dc|c} - \dot{g}_{ab} \nu^a \nu^b \nu^c, \]
and
\[ \nu^c \dot{g}_{c|\sigma} = (\nu^c \dot{g}_{\sigma|c})_{\rho} - A^{\rho\sigma} \dot{\gamma}_{\rho|\sigma} = (\nu^c \dot{g}_{c|\sigma})_{\rho} + \dot{H} \dot{\gamma}_{\rho|\sigma} - A^{\rho\sigma} \dot{\gamma}_{\rho|\sigma}, \]
where colons indicate differentiation relative to \( \gamma \). It thus follows that
\[ 2 \dot{H} = -A^{\alpha\beta} \dot{\gamma}_{\alpha\beta} + \nu^d \left( \dot{g}_{cd} |_{\rho} - \dot{g}_{c|d} \right) - (\nu^c \dot{g}_{c|\sigma})_{\rho}. \]
If \( S \) is closed, then
\[ \int_S \left( 2 \dot{H} + A^{\alpha\beta} \dot{\gamma}_{\alpha\beta} \right) \sqrt{|\gamma|} = \int_S \left( \dot{g}_{cd} |_{\rho} - \dot{g}_{c|d} \right) \nu^d \sqrt{|\gamma|}. \]

**APPENDIX D. SECOND VARIATION OF \( B \)**

**Variation with respect to \( f \) and \( X \).** Given any \( C^1 \) map (not necessarily an immersion) \( \varphi : P \to M \) from a smooth manifold \( P \) into a smooth Riemannian manifold \((M, g)\) with corresponding Levi-Civita connection \( D[TM, g] \), we write \( D[\varphi^*TM, g] \) for the unique connection on \( \varphi^*TM \) satisfying the chain rule
\[ D[\varphi^*TM] \alpha (Z^c \circ \varphi) = (D[TM, g]_{\alpha} Z^c \circ \varphi^\alpha) \circ \varphi. \]
Then \( D[\varphi^*TM] \) is torsion-free and metric-compatible in the obvious senses. We reserve the right to write simply \( D \) in instances when context suffices to identify the connection we have in mind.

Now let \( \phi : S \to M \) be a smooth two-sided (codimension-one) immersion of a smooth manifold \( S \) into a complete smooth Riemannian manifold \((M, g)\) and write \( \gamma_{\alpha\beta} := (\phi^*g)_{\alpha\beta} \) for the corresponding induced metric on \( S \). Let \( \nu \in \phi^*(TM) \) be a global unit normal for \( \phi \) and write \( A_{\alpha\beta} := -D_{\alpha} \nu^e \phi^d_{d,\beta} (g_{cd} \circ \phi) \) for the corresponding scalar-valued second fundamental form and \( H := \gamma_{\alpha\beta} A_{\alpha\beta} \) for the corresponding mean curvature.

Suppose also that \( X^a \in C^2(TS) \) and \( f \in C^2(S) \) and define the vector fields \( \xi^a, Z^a \in \phi^*(TM) \) by
\[ \xi := \phi_* X \quad \text{and} \quad Z := \xi + f \nu. \]
In turn define the map the map \( \Phi : S \times \mathbb{R} \to M \) and, for each \( t \in \mathbb{R} \) the map \( \phi[t] : S \to M \) by
\[ \Phi(p, t) := \exp^{(M, g)}_{\phi(p)} tZ \quad \text{and} \quad \phi[t] := \Phi(., t), \]
\( \exp^{(M, g)} : TM \to M \) being the exponential map on \((M, g)\). For \( |t| \) sufficiently small the map \( \phi[t] \) is an immersion with continuous unit normal \( \nu[t] \in \phi[t]^*(TM) \) chosen so that \( \nu[0] = \nu \) and so as to make continuous the vector field \( N \in \Phi^*TM \) given by \( N(p, t) := \nu[t](p) \). We set \( \gamma[t] := \phi[t]^* g, \)
\[ A[t]_{\alpha\beta} := -D_{\alpha} \nu[t]^e \phi[t]^d_{d,\beta} (g_{cd} \circ \phi[t]), \quad \text{and} \quad H[t] := \gamma[t]_{\alpha\beta} A[t]_{\alpha\beta} \]
so that \( \gamma[0] = \gamma, \ A[0] = A, \ \text{and} \ H[0] = H). \] We also set \( \dot{\gamma} := \partial_t \gamma[t] |_{t=0} \) and \( \ddot{\gamma} := \partial^2_t \gamma[t] |_{t=0} \) and adopt like notation for \( t \)-derivatives of \( A \) and \( H \) at \( t = 0 \).

Now we compute
\[ \partial_t (g \circ \Phi)(\Phi_* V, \Phi_* W) = (g \circ \Phi)(D_V \Phi_* \partial_t, \Phi_* W) + (g \circ \Phi)(\Phi_* V, D_W \Phi_* \partial_t) \]
and in turn
\[ \partial^2_t (g \circ \Phi)(\Phi_* V, \Phi_* W) = 2(g \circ \Phi)((R \circ \Phi)(\Phi_* \partial_t, \Phi_* V) \Phi_* \partial_t, \Phi_* W) + 2(g \circ \Phi)(D_V \Phi_* \partial_t, D_W \Phi_* \partial_t), \]
\( R \) being the Riemann curvature of \((M, g)\), our conventions specified by (A.1). It follows that
\[ \dot{\gamma}_{\alpha\beta} = Z_{\alpha|\alpha} \phi^c_{c,\beta} + Z_{\alpha|\beta} \phi^c_{c,\alpha} \quad \text{and} \quad \ddot{\gamma}_{\alpha\beta} = 2Z_{\alpha|\alpha} Z_{\beta|\beta} + 2(R_{abcd} \circ \phi) Z^a \phi^b_{c,\alpha} \phi^d_{c,\beta}, \]
where the bar $|$ indicates differentiation via $D[\phi^*TM, g]$ and Roman indices are raised and lowered via $g \circ \Phi$.

Next, using the product rule, we find

$$(g \circ \Phi)(D_{\partial t} N, N)|_{t=0} = -Z_{c[\alpha} \nu^c V^{\alpha}, \text{ and}$$

$$(g \circ \Phi)(D_{\partial t} N, \Phi_* V)|_{t=0} = -(g \circ \Phi)(D_{\partial t} N, D_{\partial t} N)|_{t=0} = -Z_{c[\alpha} Z_d]\nu^c V^d,$$

where Greek indices are raised and lowered via $\gamma$. We also compute

$$\partial_t(g \circ \Phi)(D_V \Phi_* W, N) = (g \circ \Phi)((R \circ \Phi)(\Phi_* \partial_t, \Phi_* V)\Phi_* W, N)$$

$$+ (g \circ \Phi)(D_V D_W \Phi_* \partial_t, N) + (g \circ \Phi)(D_V \Phi_* W, D_{\partial t} N)$$

and, at this point specializing to the flat case $R_{abcd} = 0$,

$$(g \circ \Phi)(D_V \Phi_* W, N) = 2(g \circ \Phi)(D_V D_W \Phi_* \partial_t, N) + (g \circ \Phi)(D_V \Phi_* W, D_{\partial t} D_{\partial t} N).$$

Applying (D.7) to (D.8) and (D.9), we obtain

$$(D.10) \quad \dot{A}_{\alpha \beta} = Z_{c[\alpha \beta} \nu^c \text{ and } \ddot{A}_{\alpha \beta} = -2Z_{c[\alpha \beta} Z_d]\nu^c V^d - A_{\alpha \beta} Z_{c[\rho} Z_d]\nu^c V^d.$$

From (D.2)

$$Z_{c[\sigma}^c = X^\rho : : \nu^c - f_{\sigma} \nu^c - f A^\rho \phi^c, \text{ and so}$$

$$Z_{c[\alpha \beta}^c = X^\rho : : \nu^c - f_{\alpha \beta} \nu^c + f_{\alpha} \phi^c - f A^\rho \phi^c, \text{ and so}$$

Specializing further to the case when $\phi = t : S \to M$ is the inclusion map of the standard unit sphere $S = S^2$ in $M = \mathbb{R}^3$, we have $A = -\gamma$, whence

$$Z_{c[\sigma}^c = X^\rho : : \nu^c + f \phi^c, \text{ and } (f_{\sigma} - X_\sigma) \nu^c \text{ and}$$

$$Z_{c[\alpha \beta}^c = X^\rho : : \nu^c - X_\alpha \phi^c, \text{ and } (f_{\alpha \beta} - f \gamma_{\alpha \beta} - X_{\alpha \beta} - X_{\beta \alpha}) \nu^c.$$

Applying these last expressions in (D.6) and (D.10), we conclude

$$\dot{\gamma}_{\alpha \beta} = 2f_{\gamma_{\alpha \beta}} + X_{\alpha \beta} + X_{\beta \alpha},$$

$$\dot{A}_{\alpha \beta} = f_{\alpha \beta} - f_{\gamma_{\alpha \beta}} - X_{\alpha \beta} - X_{\beta \alpha},$$

$$\ddot{\gamma}_{\alpha \beta} = 2X^{\sigma : \alpha} X_{\sigma : \beta} + 2f^2 \gamma_{\alpha \beta} + 2f_{\alpha} f_{\beta} + 2X_{\alpha} X_{\beta} - 2f_{\alpha} X_{\beta} - 2f_{\beta} X_{\alpha} + 2f_{\alpha} f_{\beta} + 2f_{\beta} f_{\alpha},$$

$$\ddot{A}_{\alpha \beta} = (|X|^2)_{\alpha \beta} - 2X^{\sigma : \alpha} X_{\sigma : \beta} - 2X_{\alpha} X_{\beta} - 2f^{\sigma} X_{\sigma : (\alpha \beta)} + 6X_{(\alpha} f_{\beta)} - 4f_{\alpha} f_{\beta}$$

whence it follows, setting

$$(D.14) \quad (KX)_{\alpha \beta} := X_{\alpha \beta} + X_{\beta \alpha} = 2X_{(\alpha \beta)}.$$


that
\[\dot{\gamma}_\sigma = 4f + 2 \text{div } X,\]
\[\ddot{\gamma}_\sigma = 2 |DX|^2 + 4f^2 + 2 |df|^2 + 2 |X|^2 - 4Xf + 4f \text{ div } X,\]
\[\ddot{\gamma}_{\alpha\beta} = 8f^2 + 8f \text{ div } X + |KX|^2,\]
\[\ddot{\gamma}_{\alpha\beta} \dot{A}_{\alpha\beta} = 2f \Delta f - 4f^2 - 6f \text{ div } X + 2X_{\alpha;\beta} f_{\alpha;\beta} - |KX|^2,\]
\[\dot{A}_\sigma = \Delta |X|^2 - 2 |DX|^2 - 2 |df|^2 + 2Xf + 2X_{\alpha;\beta} f_{\alpha;\beta} - 2(X_{\alpha;\beta} f_{\alpha;\beta}),\]
\[\dot{H} = \Delta f + 2f, \quad \text{and} \]
\[\ddot{H} = (\Delta + 2) |X|^2 - 4f(\Delta + 1)f - 2Xf - 2X_{\alpha;\beta} f_{\alpha;\beta} - 2(X_{\alpha;\beta} f_{\alpha;\beta}).\]

**Total variation.** Now, given \( f, v \in C^2(\partial M), X \in C^2(TM), \) and \( t \in \mathbb{R} \) we define \( \rho_t : M \to \mathbb{R} \) and \( \phi_t : M \to \mathbb{R} \) by
\[\rho_t := 1 + tv \quad \text{and} \quad \phi_t(x) := x + tv \left( \|x\| \right) \left[ f \left( \frac{x}{\|x\|} \right) + \xi(x) \right].\]

We have just computed, as summarized in (D.15), the first and second derivatives at \( t = 0 \) of \((\iota^* \phi_t^* \delta)^\sigma\) and \( \mathcal{H}[t, \phi_t^* \delta] \). Making use of these calculations we next compute
\[
\frac{d^2}{dt^2} \left|_{t=0} \right. (\iota^* \rho_t \phi_t^* \delta)^\sigma = 8vf + 4v \text{ div } X + 2 |DX|^2 + 4f^2
\[+ 2 |df|^2 + 2 |X|^2 - 4Xf + 4f \text{ div } X\]

and, using also the identity
\[
\mathcal{H}[t, \rho_t \phi_t^* \delta] = \left( \iota^* \rho_t^{-1/2} \right) \mathcal{H}[t, \phi_t^* \delta] - \iota^* \rho_t^{-3/2} N \phi_t^* \delta) \rho_t
\]
(see for example (B.27) in [57]) along with (D.7) and (D.12),
\[
\frac{d^2}{dt^2} \left|_{t=0} \right. \mathcal{H}[t, \rho_t \phi_t^* \delta] = (\Delta + 2) |X|^2 - 4f(\Delta + 1)f - v(\Delta + 2)f - 2X(f + v)
\[+ 3vv_r - \frac{3}{2} v^2 + 2f_{\alpha^2} v_{\alpha} - 2X_{\alpha;\beta} f_{\alpha;\beta} - 2 \left( X_{\alpha;\beta} f_{\alpha;\beta} \right)_{\beta}.
\]

**Appendix E. Divergence and Ricci deformation on Euclidean domains**

Assume \( n \in \mathbb{Z} \cap [3, \infty), k \in \mathbb{Z} \cap [0, \infty), \alpha \in (0, 1), \) and \( \beta \in (0, n - 2) \setminus \mathbb{Z}. \) Let \( \Omega \) be a (not necessarily bounded) connected, open subset of \( \mathbb{R}^n \) with smoothly embedded, compact (but not necessarily connected) boundary. For any symmetric tensor \( F_{ab} \) on \( \Omega \) we set \( \tilde{F}_{ab} := F_{ab} - \frac{1}{n-2} F^c e_{ab} \) and \( \tilde{F} := F - \frac{1}{n-2} F^c e_{ab}, \) so that \( \tilde{F} = \tilde{F} = F, \) and we call \( F \) self-equilibrated on \( \Omega \) if for each closed embedded hypersurface \( S \subset \Omega \) and each Killing field \( X \) of \( \mathbb{R}^n \) we have \( \int_S F_{ab}X^a N^b = 0, \) where \( N \) is a global unit normal for \( S. \) If \( F \) is \( C^1 \) in \( \Omega \) and continuous up to the boundary, \( F \) is then self-equilibrated on \( \Omega \) if and only if \( F_{ab}^b = 0 \) on \( \Omega \) and \( \int_S F_{ab}X^a N^b = 0 \) for each Killing field \( X \) and each component \( S \) of \( \partial \Omega. \)

Writing \( \dot{R}_{ab} \) as above for the linearization at \( \delta \) of the Ricci operator, note that \( \tilde{F}_{ab} \) is self-equilibrated on \( \Omega \) for \( F_{ab} := \dot{R}_{ab} \delta h \) with \( h_{ab} \) any given \( C^3 \) symmetric tensor on \( \Omega \) (by virtue of the Bianchi identity applied to \( \dot{R}_{ab} \delta \tilde{h} \) for any compactly supported \( C^3 \) symmetric extension \( \tilde{h}_{ab} \) of \( h \) to all \( \mathbb{R}^n \)). In fact self-equilibration of a symmetric tensor (assuming appropriate decay if \( \Omega \) is unbounded) is also a sufficient condition for it to be the linearized Einstein tensor of some deformation of \( \delta \) on \( \Omega, \) as shown below. (Sufficiency holds also in dimension 1, trivially, but fails
in dimension 2, where every $\hat{R}[\delta]_{ab}$ is scalar and yet on the unit disc, for example, there is a two-dimensional space of constant trace-free symmetric tensors, all of which are self-equilibrated.

We will solve the problem $\hat{R}_{ab}[\delta]h = F_{ab}$ by first extending $F$ to a self-equilibrated tensor on $\mathbb{R}^n$, but the extension we specify loses regularity, so we first solve the problem modulo error smoother than $F$. Throughout we will make use of the identities

(E.1)  \[ \hat{R}_{ab}[\delta]h = \frac{1}{2} \hat{h}_{c(a}|_{b)} - \frac{1}{2} h_{ab}|_{c} \quad \text{and} \]

(E.2)  \[ \text{div}(T + V \otimes N + N \otimes V + \xi N \otimes N) = \left[ \text{div}^T T + (\overline{D}_N + S - H)V \right] \]

where we assume (locally) a foliation by hypersurfaces with $N^a$ unit and normal to each leaf, $V^a$ and $T_{ab} = T_{ba}$ purely tangential to leaves, $\text{div}^T$ the intrinsic (vector or symmetric tensor) divergence on each leaf, $\overline{D}$ the ambient connection, and $S$ and $H$ the shape operator and mean curvature respectively of the leaves.

**Lemma E.3** (Right parametrix for linearized Ricci). There exists a constant $C = C(\alpha, \beta, k, \Omega) > 0$ such that if $F^a \in C^{k+1, \alpha, 2+\beta}(T\Omega \otimes 2)$ is self-equilibrated on $\Omega$, then there exist $h_{ab}$, $G_{ab}$ on $\Omega$ such that $\hat{R}_{ab}[\delta]h = \tilde{F}_{ab} + \tilde{G}_{ab}$, $\|h\|_{k+3, \alpha, \beta} + \|G\|_{k+2, \alpha, 2+\beta} \leq C\|F\|_{k+1, \alpha, 2+\beta}$, and $G$ is self-equilibrated on $\Omega$.

**Proof.** First let $\eta_{ab}$ be the solution to the Poisson equation $\eta_{ab}|_{c} = -2\tilde{F}_{ab}$ with $\eta|_{\partial\Omega} = 0$. In light of (E.1) any symmetric tensor $\theta_{ab}$ satisfying $\theta_{ab}|_{c} = 0$ and $\theta_{ac}|_{c} = \zeta := -\eta_{ac}|_{c}$ yields an exact solution to $\hat{R}_{ab}[\delta](\eta + \tilde{\theta}) = \tilde{F}_{ab}$. It is easy though to construct instead a harmonic, symmetric tensor $\tilde{\theta}_{ab}$ satisfying the divergence condition to first order, as follows. Writing $\mathcal{P}$ for the operator taking tensors on $\partial \Omega$ to their bounded (Cartesian componentwise) harmonic extensions on $\Omega$ and $\xi$ for the inward unit normal on $\partial \Omega$, let $X^a = V^a + \xi \eta^a$ (with $V \perp N$) and $\xi$ be the solutions on $\partial \Omega$ to $(\overline{D}_N - 1)\mathcal{P}X = \xi - \xi_c N^c N$ and $(N - 1)\mathcal{P} \xi = \xi_c N - \text{div}^T V - 2N \xi$. Then, referring to (E.2) (and using the regularity of $\partial \Omega$), on $\partial \Omega$ the symmetric tensor $(\mathcal{P}X) \otimes N + N \otimes (\mathcal{P}X) + (\mathcal{P}\xi) N \otimes N$ has divergence $\xi$ plus a $C^{k+3, \alpha}$ function, and consequently (since $[\mathcal{P}(-\partial_{\Omega}), \cdot \otimes Z]$ has order $-1$ for any smooth tensor $Z$) so does $\tilde{\theta} := \mathcal{P}(X \otimes N + N \otimes X + \xi N \otimes N)$. Finally note that, by its definition above, $\xi$ is harmonic, as of course is $\tilde{\theta}$, so $\tilde{h} := \eta + \tilde{\theta}$ satisfies $\text{div} \tilde{h} \in C^{k+3, \alpha}(\Omega)$. The proof is now completed by taking $\tilde{G}$ to be the first term in (E.1), its self-equilibration following from that of $F$ and $\hat{R}_{ab}[\delta]h$; the estimates are clear from the construction. \[\square\]

**Lemma E.4** (Right inverse for divergence with support contained in a hypercube). Suppose $d \in \mathbb{Z} \cap [1, \infty)$ and set $Q := [-1, 1]^d \subset \mathbb{R}^d$. There exists a constant $C = C(d, k, \alpha) > 0$ such that the following hold.

(i) If $f \in C^{k+1, \alpha}(\mathbb{R}^d)$ has support contained in $Q$ and $\int_{\mathbb{R}^d} f = 0$, then there exists $X^a \in C^{k+1, \alpha}(T\mathbb{R}^d)$ with support contained in $Q$, $X^c|_{c} = f$, and $\|X\|_{k+1, \alpha} \leq C\|f\|_{k+1, \alpha}$.

(ii) If $F^a \in C^{k+1, \alpha}(T\mathbb{R}^d)$ has support contained in $Q$ and $\int_{\mathbb{R}^d} F^c X_c = 0$ for every Killing field $X^a$ on $\mathbb{R}^d$, then there exists $S_{ab} \in C^{k+1, \alpha}(T\mathbb{R}^d \otimes T\mathbb{R}^d)$ with support contained in $Q$, $S_{ab}|_{b} = F_a$, and $\|S\|_{k+1, \alpha} \leq C\|F\|_{k+1, \alpha}$.

(iii) If $F^a \in C^{k+1, \alpha}(T\mathbb{R}^d)$ has support contained in $Q$ and $B$ is any open ball contained in the interior of $Q$, then there exists a smooth vector field $\rho^a$ such that $\rho$ has support contained in $B$ and there exists $S_{ab} \in C^{k+1, \alpha}(T\mathbb{R}^d \otimes T\mathbb{R}^d)$ with support contained in $Q$, $S_{ab}|_{b} = F_a + \rho_a$, and $\|S\|_{k+1, \alpha} \leq C\|F\|_{k+1, \alpha}$.

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Proof. We prove (i) and (ii) by induction on $d$, the case $d = 1$ following immediately for both by integration. Assume $d \geq 2$. For the inductive step we will apply (E.2) and the analogous decomposition \( \text{div}(V + \xi N) = \text{div}^\top V + (N - H)\xi \), in the same notation, for vector fields to the foliation of \( \mathbb{R}^n \) by hyperplanes orthogonal to the \( x^d \)-axis. To simplify the notation slightly we set \( t := x^d \) and \( x := (x^1, \ldots, x^{d-1}) \). Fix a \( \psi \in C_c^\infty(\mathbb{R}) \) with support contained in \([-1/2, 1/2]\) and \( \int \psi = 1 \).

For (i) define functions \( f_\perp \) and \( f_\top \) on \( \mathbb{R}^d \) by \( f_\perp(x, t) := \psi(t) \int_{\mathbb{R}} f(x, \tau) \, d\tau \) and \( f_\perp := f - f_\top \), so that both \( f_\perp \) and \( f_\top \) have support contained in \( Q \), \( \int_{\mathbb{R}} f_\perp(x, t) \, dt = 0 \) for each \( x \in \mathbb{R}^{d-1} \) and \( \int_{\mathbb{R}^{d-1}} f_\top(x, t) \, dx \mid = 0 \) (since \( \int_{\mathbb{R}} f = 0 \)) for each \( t \in \mathbb{R} \). Now we define \( \xi \) on \( \mathbb{R}^d \) by \( \xi(x, t) := f_{-\infty}^t f_\perp(x, \tau) \, d\tau \), and by the inductive hypothesis there exists a vector field \( V \) on \( \mathbb{R}^d \) with values orthogonal to \( N = \partial_t = \partial_d \) and satisfying \( \text{div} V(x, t) = f_\top(x, t) \) for each \( t \in \mathbb{R} \). Taking \( X := V + \xi N \) concludes the proof of (i).

For (ii) we first reduce to the case that \( F \) is everywhere orthogonal to \( N \), as follows. Set \( f := F^c N_c \), and define \( V \) and \( \xi \) exactly as in the previous paragraph, with \( \int_{\mathbb{R}^{d-1}} f_\top(x, t) \, dx \mid = 0 \) now because \( \int_{\mathbb{R}} f = \int_{\mathbb{R}^d} F^c N_c = 0 \), \( N = \partial_t = \partial_d \) being a Killing field. Then \( \text{div}(V \otimes N + N \otimes V + \xi N \otimes N) \) is orthogonal to the Killing fields and, referring to (E.2), equals \( D_N V + f N \), where the first term is orthogonal to \( N \).

Thus we now assume \( F \) is purely tangential to the leaves. Fix a \( \phi \in C_c^\infty([0, \infty)) \) with support contained in \([0, 1], \phi \geq 0, \) and \( \int_{\mathbb{R}^{d-1}} \phi(|x|) \, d|x| = 1 \). For each \( i \in \mathbb{Z} \cap [1, d-1] \) define the vector fields \( iK := \partial_i \) and \( i\tilde{K} := \phi(|x|) \, iK \) and, setting \( a := 1/\int_{\mathbb{R}^{d-1}} 2(x^i)^2 \phi(|x|) \, d|x| \), for each \( i < j \in \mathbb{Z} \cap [1, d-1] \) define the vector fields \( i,jK := x^i \partial_j - x^j \partial_i \) and \( i,j\tilde{K} := a \phi(|x|) \, i,jK \). The \( iK \) and \( i,jK \) form a basis for the Killing fields on \( \mathbb{R}^{d-1} \), and the \( i\tilde{K} \) and \( i,j\tilde{K} \) restrict (acting by inner product and integration) to the dual basis. Note that each \( K_{ij} \) is divergence-free, despite the cutoff.

Next set
\[
\begin{align*}
F_{\rightarrow}(x, t) &:= \sum_{i=1}^{d-1} \left( \int_{\mathbb{R}^{d-1}} iK^c F_c(x, t) \, d|x| \right) i\tilde{K}(x, t), \\
F_{\circ}(x, t) &:= \sum_{1 \leq i < j \leq d-1} \left( \int_{\mathbb{R}^{d-1}} i,jK^c F_c(x, t) \, d|x| \right) i,j\tilde{K}(x, t), \quad \text{and} \\
F_{\leftarrow}(x, t) &:= F - F_{\rightarrow} - F_{\circ},
\end{align*}
\]
(E.5)
so that \( F_{\rightarrow}, F_{\circ}, \) and \( F_{\leftarrow} \) all have support contained in \( Q, \) \( F_{\leftarrow}(\cdot, t) \) is orthogonal to the Killing fields on \( \mathbb{R}^{d-1} \) for each \( t \in \mathbb{R}, \) and \( \int_{\mathbb{R}} F_{\circ}(x, t) \, dt = \int_{\mathbb{R}} F_{\rightarrow}(x, t) \, dt = 0 \) for each \( x \in \mathbb{R}^{d-1} \). In particular, by the inductive hypothesis, there exists a symmetric tensor \( T_{ab} \) with \( T_{ab} N^b = 0 \) everywhere, support contained in \( Q, \) and \( \text{div}^\top T = F_{\leftarrow} \). Furthermore \( V_{\circ}(x, t) := \int_{-\infty}^t F_{\circ}(x, \tau) \, d\tau \) has support contained in \( Q \) and \( \text{div}^\top V_{\circ}(x, t) = 0 \) for each \( t \in \mathbb{R} \) (since each \( i,j\tilde{K} \) has vanishing divergence, as noted above).

Referring again to (E.2), we therefore have \( \text{div}(T + V_{\circ} \otimes N + N \otimes V_{\circ}) = F_{\leftarrow} + F_{\circ} \).

Finally we will use the orthogonality of \( F \) to \( t \partial_t - x^i \partial_i = x^d \partial_t - x^d \partial_d \) for each \( i \in \mathbb{Z} \cap [1, d-1] \). Setting \( c_i(t) := \int_{\mathbb{R}^{d-1}} iK^c F_c(x, t) \, d|x|, \) so that \( F_{\rightarrow}(x, t) = \sum_{i=1}^{d-1} c_i(t) i\tilde{K}(x, t), \) we then have \( \int_{\mathbb{R}} c_i(t) \, dt = 0 \) for each \( i, \) but orthogonality of \( F \) to each \( iK \) gives \( \int_{\mathbb{R}} c_i(t) \, dt = 0, \) so \( \int_{\mathbb{R}} \int_{-\infty}^t c_i(\tau) \, d\tau \, dt = 0 \). This follows not only that \( V_{\rightarrow}(x, t) := \int_{-\infty}^t F_{\rightarrow}(x, \tau) \, d\tau \) has support contained in \( Q \) but that \( \xi_{\rightarrow}(x, t) := - \int_{-\infty}^t \text{div}^\top V_{\rightarrow}(x, \tau) \, d\tau \) does too. Since \( \text{div}(V_{\rightarrow} \otimes N + N \otimes V_{\rightarrow} + \xi_{\rightarrow} N \otimes N) = F_{\rightarrow}, \) the proof of (ii) is complete, the estimates being clear from the construction. For (iii) we construct \( \rho \) by cutting off a basis for the Killing fields on \( \mathbb{R}^d, \) as done for \( \mathbb{R}^{d-1} \) above, but “centered” on \( p, \) and choosing coefficients so that \( F^a - \rho^a \) is orthogonal to the Killing fields; then we apply (ii).
For alternative approaches to constructing compactly supported symmetric tensors of prescribed divergence see [27, 37] (though neither states results in a form we can directly apply to our ends here.)

**Lemma E.6** (Rough extension of self-equilibrated fields). There exists a constant \( C = C(\alpha, k, \Omega) > 0 \) such that every self-equilibrated \( F^{ab} \in C^{k+2,\alpha}(T\Omega \odot \mathbb{R}^n) \) admits a self-equilibrated extension \( \tilde{F}^{ab} \in C^{k+1,\alpha}(T\mathbb{R}^n \odot T\mathbb{R}^n) \) such that \( \tilde{F} \) has compact support and \( \| \tilde{F} \|_{k+1,\alpha} \leq C \| F \|_{\partial\Omega} \) where \( \partial\Omega_i \) is the set of all points in \( \Omega \) at distance at most 1 from \( \partial\Omega \).

**Proof.** For any \( G \subset \mathbb{R}^n \) write \( \overline{Q_i} \) for its closure and for any \( E \subset \partial\Omega \) and \( \epsilon > 0 \) write \( E^\epsilon \) for the set of all points in \( \mathbb{R}^n \) having distance less than \( \epsilon \) from \( E \). Since \( \partial\Omega \) is \( C^\infty \), compact, and embedded there exist \( \epsilon > 0 \) and open sets \( Q_1, \ldots, Q_N \) in \( \partial\Omega \) (with its induced metric) covering \( \partial\Omega \) such that for each \( Q_i \) we have that \( \overline{Q_i} \cap \partial\Omega = Q_i \) and moreover that item (iii) of Lemma E.4 holds with the hypercube \( Q \) replaced by \( \overline{Q_i} \). (To verify that this last part of the claim can be achieved note that at each \( p \in \partial\Omega \) we can find a local coordinate system \( \Phi : \mathcal{U} \ni p \to \mathbb{R}^n \) and an open neighborhood \( Q_p \subset \partial\Omega \) of \( p \) such that \( \Phi(Q_p) \) is a hypercube of edge length \( \epsilon \), and for any \( \ell > 0 \), after rescaling so that \( \Phi(Q_p) \) has unit size, we can make \( \Phi^{-1} \delta \) arbitrarily \( C^\ell \)-close to \( \delta \) on \( \mathcal{U} \) by taking \( \epsilon \) small. For \( \epsilon \) small enough Line of (iii) can then be applied iteratively to produce an exact solution.)

Now we extend \( F \) (continuously in \( F \)) to some \( \tilde{F}^{ab} \in C^{k+2,\alpha}(T\mathbb{R}^n \odot T\mathbb{R}^n) \) whose restriction to \( \mathbb{R}^n \setminus \Omega \) has support contained in \( \partial\Omega^\epsilon \). Using a partition of unity on \( \partial\Omega \) subordinate to \( \{ Q_i \} \) and extended constantly in the normal direction, we obtain the decomposition \( \tilde{F}^{ab} \mid = \sum_{i=1}^N iF_a \), where each \( iF_a \) has support contained in \( \overline{Q_i} \). Then, by E.4.(iii) with \( \overline{Q_i} \) in place of \( Q \), for each \( iF_a \) there exist a smooth vector field \( iF_a \) with support contained in the interior \( Q_i \) and \( \tilde{F}^{ab} \in C^{k+1,\alpha}(T\mathbb{R}^n \odot T\mathbb{R}^n) \) with support contained in \( \overline{Q_i} \) and satisfying \( iF_a = iF_a - \rho \). Note that \( \int_{\mathbb{R}^n} iF_a X = \int_{\mathbb{R}^n} iF_a X \) for every Killing field \( X \) on \( \mathbb{R}^n \). Summing, we obtain \( S_{ab} \) such that \( \tilde{F} - S = C^{k+1,\alpha}(T\mathbb{R}^n \odot T\mathbb{R}^n) \) has divergence \( \sum_{i=1}^N \rho \) and its restriction to \( \mathbb{R}^n \setminus \Omega \) has support contained in \( \partial\Omega^\epsilon \).

By applying Lemma E.4 repeatedly we can “move” each \( \rho \) freely within the component of \( \partial\Omega^\epsilon \) that contains it by adding to \( \tilde{F} - S C^{k+2,\alpha} \) symmetric tensor fields with support contained in \( \partial\Omega^\epsilon \), so that the resulting sum has divergence \( \sum_{i=1}^N \rho \), where each \( \rho \) is smooth and can be chosen to have support contained in any open ball in any hypercube contained in \( \partial\Omega^\epsilon \) and containing the support of \( \rho \) and with \( \rho \) - \( \rho \) orthogonal to the Killing fields. In this way we can consolidate all the \( \rho \) (iteratively replacing each by \( \rho \) as above) for a given component of \( \partial\Omega \) into a single hypercube contained in \( \partial\Omega^\epsilon \), but by the construction of the \( \rho \) and the self-equilibration assumption on the original \( F \), the sum of the \( \rho \) contained in a given component of \( \partial\Omega^\epsilon \) is orthogonal to all Killing fields. Therefore we can apply Lemma E.4.(i) (once for each component of \( \partial\Omega \)) to eliminate all the \( \rho \) by adding further \( C^{k+2,\alpha} \) symmetric tensor fields with support contained in \( \partial\Omega^\epsilon \). Writing \( S_{ab} \) for the sum of all symmetric tensors introduced in this paragraph to eliminate the support \( \rho \) we conclude by taking \( F_{ab} := \tilde{F}_{ab} - S_{ab} + \tilde{S}_{ab} \). \( \Box \)

**Proposition E.7** (Right inverse for linearized Ricci). There exists a constant \( C = C(\alpha, \beta, k, \Omega) > 0 \) such that if \( F^{ab} \in C^{k+1,\alpha,2+\beta}(T\Omega \odot \mathbb{R}^n) \) is self-equilibrated on \( \Omega \), then there exists a symmetric tensor \( h_{ab} \) on \( \Omega \) such that \( \tilde{h}_{ab} = \tilde{F}_{ab}, \tilde{h} \) is self-equilibrated, and \( \| h \|_{k+3,\alpha} \leq C \| F \|_{k+1,\alpha,2+\beta} \).

**Proof.** By Lemma E.3 we may assume that \( F \in C^{k+1,\alpha,2+\beta}(T\Omega \odot \mathbb{R}^n) \). By Lemma E.6 \( F \) has a self-equilibrated extension \( \tilde{F} \in C^{k+1,\alpha,2+\beta}(T\mathbb{R}^n \odot T\mathbb{R}^n) \). Let \( h_{ab} \) be the unique solution in \( C^{k+3,\alpha}(T\mathbb{R}^n \odot T\mathbb{R}^n) \) to \( h_{ab} = -2\tilde{F}_{ab} \) on \( \mathbb{R}^n \). Then \( h_{ab} \mid_{\Omega} = 0 \) on \( \mathbb{R}^n \), so, by uniqueness of solutions in \( C^{k+2,\alpha,\beta} \) to this Poisson equation, \( h \) is self-equilibrated on \( \mathbb{R}^n \), and therefore, in light of (E.1), satisfies \( \tilde{h}_{ab} = \tilde{F}_{ab} \) on \( \mathbb{R}^n \). Restriction to \( \Omega \) completes the proof. \( \Box \)
Conversely, we can use Proposition E.7 to solve the extension problem.

**Corollary E.8** (Smooth extension of self-equilibrated fields). There exists a constant $C = C(\alpha, k, \Omega) > 0$ so that every self-equilibrated $F_{ab} \in C^{k+1,\alpha,2+\beta}(T\Omega^{\leq2})$ admits a self-equilibrated extension $\hat{T}_{ab} \in C^{k+1,\alpha,2+\beta}(T\mathbb{R}^n \cap T\mathbb{R}^n)$ such that $\hat{T}|_{\mathbb{R}^n \setminus \Omega}$ has compact support and $\|\hat{T}\|_{k+1,\alpha,2+\beta} \leq C\|F\|_{k+1,\alpha,2+\beta}$.

**Proof.** By Proposition E.7 $\hat{T}_{ab}$ admits a “potential” $h_{ab} \in C^{k+3,\alpha,\beta}(T\Omega^{\leq2})$ such that $\hat{R}_{ab} = \hat{\partial}_{ab} h = \hat{T}$. Extend $h_{ab}$ to $\tilde{h}_{ab} \in C^{k+3,\alpha,\beta}(T\mathbb{R}^n \cap T\mathbb{R}^n)$ and take $\mathcal{F} = (\hat{R}_{ab} \tilde{h})^{-}$.

Finally we observe that self-equilibration can be achieved at the cost of a Lie derivative of the metric.

**Proposition E.9** (Self-equilibration modulo Lie derivatives). Assume that $\partial \Omega$ is connected with outward unit normal $N$ and let $\beta' \in \mathbb{R} \setminus \mathbb{Z}$. There exists $C = C(\Omega, k, \alpha, \beta') > 0$ such that for any $F_{ab} \in C^{k+1,\alpha,1+\beta'}(T\Omega^{\leq2})$ there is a vector field $X^a$ on $\Omega$ such that $(F_{ab} + X_{alb} + X_{bla})^a$ is self-equilibrated and $\|X\|_{k+2,\alpha,\beta'} \leq C\|\hat{\partial}_{ab} F_{ab} Y^a N_b\| : Y$ is Killing on $\mathbb{R}^n$.

**Proof.** Since $\hat{\partial}_{ab} \mathcal{F} X = X_{alb}^b$, we take $X \in C^{k+2,\alpha,\beta'+1}(T\Omega)$ to be a solution to $X_{alb}^b = -\hat{T}_{ab}^b$. We may now assume that $\Omega$ is unbounded and $\beta' < n - 1$, in which case, by the divergence theorem (and the at most linear growth of the Killing fields), we are already done. Write $K$ for the Killing operator taking a vector field $Y$ to $Y_{alb} + Y_{bla}$, and for any Killing field $Y$ on $\mathbb{R}^n$ write $\mathcal{Y}$ for the vector field agreeing with $Y$ on $\partial \Omega$ (including components normal to $\partial \Omega$) and having bounded harmonic Cartesian components on $\Omega$. For any two Killing fields $Y$ and $Z$ on $\mathbb{R}^n$ consider the pairing

$$A(Y, Z) := \int_{\partial \Omega} (\mathcal{K} \mathcal{Y})_{ab} Z^a N^b = \int_{\Omega} (\mathcal{K} \mathcal{Y})_{ab} Z^a = \frac{1}{2} \int_{\Omega} (\mathcal{K} \mathcal{Y})_{ab} (\mathcal{K} \mathcal{Z})^{ab},$$

where the integral at infinity vanishes because of the decay of $\mathcal{Y}$ and $\mathcal{Z}$. In particular $A$ is symmetric. Next, using the fact that $\mathcal{K} \mathcal{Y} = 0$, we find

$$A(Y, Z) = \int_{\Omega} Y_{alb} Z^a + \int_{\mathbb{R}^n \setminus \Omega} Y_{alb} Z^a,$$

and consequently

$$A(Y, Z) = \int_{\Omega} Y_{alb} Z^a + \int_{\mathbb{R}^n \setminus \Omega} Y_{alb} Z^a,$$

so that $A$ is positive definite. Thus we may adjust the $X$ chosen above by harmonic extensions of restrictions to $\partial \Omega$ of Killing fields to complete the arrangement of the self-equilibration condition. Note that if $\beta' > n - 2$, then prior to any such adjustments $(\mathcal{F} - \mathcal{K} X)_{ab} N^b$ on $\partial \Omega$ is already orthogonal to the translational Killing fields, ensuring the estimate in all cases. \(\square\)

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