On spherical 4-distance 7-designs

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Abstract—We investigate spherical 4-distance 7-designs by studying their distance distributions. We compute these distance distributions and use their product (an integer) to derive certain divisibility conditions relating the dimension $n$ and the cardinality $M$ of our designs. It follows that $n$ divides $12M$ and $n+1$ divides $4M^2$. This result provides a good base for computer experiments to support the folklore conjecture that the only spherical 4-distance 7-designs are the tight spherical 7-designs. We then proceed with a computer assisted proof of this conjecture in all dimensions $n \leq 1000$.

Keywords—Spherical designs, few distance sets, distance distribution.

I. INTRODUCTION

Let $S^{n-1}$ be the unit sphere in $\mathbb{R}^n$. A finite set $C \subset S^{n-1}$ is called a spherical code. Spherical designs were introduced by Delsarte, Goethals and Seidel [4] as a special class of spherical codes with good combinatorial and integration properties.

Definition 1. [4] A spherical code $C \subset S^{n-1}$ is called a spherical $\tau$-design if the quadrature formula

$$\int_{S^{n-1}} f(x) d\mu(x) = \frac{1}{|C|} \sum_{x \in C} f(x)$$

is exact for all polynomials $f(x) = f(x_1, x_2, \ldots, x_n)$ of degree at most $\tau$. Here the measure $\mu$ is normalized, i.e. $\mu(S^{n-1}) = 1$. The maximal positive integer $\tau$ such that $C$ is a spherical $\tau$-design is called strength of $C$.

A spherical $\tau$-design $C \subset S^{n-1}$ is called tight if it attain the Delsarte-Goethals-Seidel bound [4]

$$|C| \geq \binom{n+k-1-\varepsilon}{n-1} + \binom{n+k-2}{n-1},$$

where $\tau = 2k-\varepsilon$, $\varepsilon \in \{0, 1\}$.

For a spherical code $C$ we consider the set of distinct inner products of points of $C$,

$$A(C) := \{\langle x, y \rangle : x, y \in C, x \neq y\},$$

and denote by $s := |A(C)|$ their number. The code $C$ is called then an $s$-distance set.

Designs with large strength $\tau$ and small number of distances $s$ are clearly interesting. It is well known [4] that $\tau \leq 2s$ or even $\tau \leq 2s-1$ if $A(C) \cup \{1\}$ is symmetric with respect to 0.

Levenshtein [6] proved that if $\tau \geq 2s-1$ or $\tau \geq 2s-2$ and $-1 \in A(C)$, then $C$ is maximal (and attains what is called now Levenshtein bounds).

In this paper we consider the case $(s, \tau) = (4, 7)$. We prove that all 4-distance 7-designs have cardinality which has some good divisibility properties. We prove that the dimension $n$ divides $12M$ and, in some cases even better, $n$ divides $M$.

All these results are collected towards the following conjecture which could be quite old but we have not seen it explicitly written.

Conjecture 2. Let $C \subset S^{n-1}$, $n \geq 2$, be a spherical 4-distance 7-design. Then $C$ is a tight spherical 7-design. In particular, $|C| = 2\binom{n+2}{3}$.

Tight spherical 7-designs are possible only in dimensions $n = 3k^2 - 4$, $k \geq 2$, and are known only for $k = 2$ and $k = 3$, where the corresponding designs are unique up to isometry of $S^{n-1}$. Further nonexistence results for tight spherical 7-designs were obtained in [11, 8].

In this paper we provide a computer assisted proof of Conjecture 2 in all dimensions $n \leq 1000$ based on the results from Section 3. This result is confirmed in dimensions $n \leq 215$ by a brute force approach based on formulas from Section 2.

II. PRELIMINARIES

We collect some facts about 4-distance 7-designs which follow from general results in [4], [6, 2]. The exposition is adapted to [3], where we considered 3-distance 5-designs.

Let $C \subset S^{n-1}$ be a spherical 4-distance 7-design with cardinality $M = |C|$ and

$$A(C) = \{a, b, c, d\},$$

where $-1 \leq a < b < c < d < 1$. For $x \in C$ let

$$(X, Y, Z, T) = (A_a(x), A_b(x), A_c(x), A_d(x))$$

be the distance distribution of $C$ with respect of $x$, i.e.,

$$A_a(x) = |\{y \in C : \langle x, y \rangle = a\}|,$$

e etc. It is well known [4] that the distance distribution does not depend on $x$ (this fact follows whenever $s-1$ does not exceed $\tau$).

Then the nine numbers $a, b, c, d, X, Y, Z, T$, and $|C| = M$ satisfy the following system of eight equations

$$a'X + b'Y + c'Z + d'T = f_iM - 1, \ i = 0, 1, \ldots, 7, \ (1)$$

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where \( f_i = 0 \) for odd \( i \), \( f_0 = 1 \), \( f_2 = 1/n \), \( f_4 = 3/(n+2) \) and \( f_6 = 15/(n+2)(n+4) \). The cardinality \( M \) is restricted by
\[
2 \binom{n+2}{3} \leq M \leq \frac{n+3}{4} + \binom{n+2}{3}.
\]
(2)
The lower bound in (2) comes from \( C \) being a 7-design, and the upper bound (so-called absolute bound) follows from \( C \) being a 4-distance set. Both bounds were proved in [4]. We will assume that \( C \) is not a tight spherical 7-design, i.e.
\[
M > 2 \binom{n+2}{3} = \frac{n(n+1)(n+2)}{3}.
\]
In what follows we will use for short the notations
\[
A = 6M - n(n+1)(n+5),
\]
\[
B = 3M - n(n+1)(n+2).
\]
The numbers \( a, b, c, d \) are the roots of the equation
\[
f(t) = 0,
\]
where
\[
f(t) = (n+2)(n+4)A t^4 + n(n-1)(n+1)(n+2)(n+4)t^3
\]
\[
- 9(n+2)(4M - n(n+1)(n+3))t^2
\]
\[
- 3n(n-1)(n+1)(n+2)t + 6B.
\]
This fact already involves the Levenshtein framework from [6, 7] and the polynomial \( f(t) \) is, in the notations from [6], equal to
\[
(t - \alpha_0)(t - \alpha_1)(t - \alpha_2)(t - \alpha_3)
\]
up to a multiplicative constant, where \( \alpha_0 = a, \alpha_1 = b, \alpha_2 = c, \alpha_3 = d \), in a turn to our notations. We also note that \( b < 0 < c \) and
\[
|a| > |d| > |b| > |c|
\]
(see [2] for the general case of \( s \)-distance (2\( s \) - 1)-designs).

It is straightforward to resolve the Vandermonde-like system of equations from (1) with odd \( i = 1, 3, 5, 7 \). The solution
\[
X = -\frac{(1-b^2)(1-c^2)(1-d^2)}{a(a^2-b^2)(a^2-c^2)(a^2-d^2)},
\]
\[
Y = -\frac{b(b^2-a^2)(b^2-c^2)(b^2-d^2)}{(1-a^2)(1-b^2)(1-c^2)(1-d^2)},
\]
\[
Z = -\frac{c(c^2-a^2)(c^2-b^2)(c^2-d^2)}{(1-a^2)(1-b^2)(1-d^2)}
\]
\[
T = -\frac{d(d^2-a^2)(d^2-b^2)(d^2-c^2)}{(1-a^2)(1-b^2)(1-c^2)}
\]
is unique because of the inequalities (4) (see [2] for a treatment of the general case of \( s \)-distance (2\( s \) - 1)-designs).

III. DIVERSIBILITY RESULTS FOR \( M \)

A. Computation of \( XYZT \) in terms of \( n \) and \( M \)

The integers \( X, Y, Z, \) and \( T \) can be involved in different symmetric expressions. We tried many reasonable combinations but succeeded to get results only from the investigation of \( XYZT \) as shown below.

Thus, we consider the product
\[
XYZT = \frac{f(1)^3 f(-1)^3}{abcdDF^2},
\]
where
\[
D = a_0^6 ((a-b)(a-c)(a-d)(b-c)(b-d)(c-d))^2
\]
is the discriminant of \( f \),
\[
a_0 = (n+2)(n+4)A
\]
is its leading coefficient and
\[
F = (a+b)(a+c)(a+d)(b+c)(b+d)(c+d)
\]
(3)
Expressing the necessary symmetric functions of \( a, b, c, d \) obtained via (3) we find that
\[
\frac{abcd}{n^2(n+1)(n+2)} = \frac{6B}{(n+2)(n+4)A}
\]
\[
\frac{f(1)f(-1)}{F} = \frac{a_0^2}{n^2(n+1)(n+2)} \frac{MA}{(n+2)(n+4)A^2}
\]
(4)
(the last expression was computed by subtracting the equations with \( i = 2 \) and \( i = 4 \) of the system (1), and the most complicated
\[
D = a_0^6 \cdot \frac{108(n+1)^2R(n, M)}{(n+2)^3(n+4)^3A^6}
\]
where
\[
R(n, M) = 2^{14} 3^5 6^6 - 2^{13} 3^7 n(n+1)(n+3) M^5
\]
\[
+ 2^2 3^2 n^2(n+1)(n^4 + 93n^3 + 629n^2 + 1339n + 818)M^4
\]
\[
- 2^7 3^n(n+1)^2(13n^5 + 436n^4 + 3688n^3 + 12782n^2
\]
\[
+ 19163n + 9998)M^3
\]
\[
+ 2^2 3^4 n^2(n+1)^3(n+2)(n+7)^2(5n^4 + 447n^3 + 3303n^2
\]
\[
+ 7873n + 5652)M^2
\]
\[
- 2^2 3^2 n^2(n+1)^4(n+2)^2(n+5)^2(n+7)^3(3n+5)M
\]
\[
+ n^6(n+1)^5(n+2)^3(n+5)^3(n+7)^4.
\]
Combining these we obtain
\[
XYZT = \frac{M^3(n-1)^2(n+4)^4A^7}{54n(n+1)^2R(n, M)}.
\]
(5)
We investigate below the integrality of \( XYZT \).
B. Divisibility properties of $M$

For a prime $p$ and a nonzero integer $x$, we will denote by $v_p(x)$ the largest power of $p$ which divides $x$ (i.e., the power of $p$ in the canonical representation of $x$; for example $v_2(24) = 3$, $v_3(24) = 1$ and $v_5(24) = 0$).

The power of $p$ in the numerator and denominator of $XYZT$ in (5) will be denoted for short by $v_p(\text{num})$ and $v_p(\text{den})$, respectively. Since $XYZT$ is a positive integer, the inequality $v_p(\text{den}) \leq v_p(\text{num})$ holds true for every $p$.

Lemma 3. If $XYZT$ is an integer, then the following statements hold true:

a) if $\gcd(n, 6) = 1$, then $n$ divides $M$;
b) if $\gcd(n, 3) = 1$, $n$ is even but $v_2(n) \not\in \{2, 1 + v_2(M)\}$, then $n$ divides $M$;
c) if $\gcd(n, 3) = 1$, and $v_2(n) = 2$, then $n$ divides $4M$;
d) if $\gcd(n, 3) = 1$, and $v_2(n) = 1 + v_2(M)$, then $n$ divides $2M$;

e) if $\gcd(n, 2) = 1$, 3 divides $n$ but $v_3(n) \neq 1 + v_3(M)$, then $n$ divides $M$;
f) if $\gcd(n, 2) = 1$ and $v_3(n) = 1 + v_3(M)$, then $n$ divides $3M$;
g) if 6 divides $n$, $v_3(n) \neq 1 + v_3(M)$, and $v_2(n) \not\in \{2, 1 + v_2(M)\}$, then $n$ divides $M$;
h) if 6 divides $n$, $v_3(n) = 1 + v_3(M)$, and $v_2(n) \not\in \{2, 1 + v_2(M)\}$, then $n$ divides $3M$;
i) if 6 divides $n$, $v_3(n) = 1 + v_3(M)$, and $v_2(n) = 1 + v_2(M)$, then $n$ divides $6M$;
j) if 6 divides $n$, $v_3(n) = 1 + v_3(M)$, and $v_2(n) = 2$, then $n$ divides $12M$.

Proof. Considering the numerator of $XYZT$ from (5) modulo $n$ we find that $n$ divides $2^{15}3^7M^{10}$. Hence $v_2(n) \leq 10v_2(M) + 15$, $v_3(n) \leq 10v_3(M) + 7$ and $v_p(n) \leq 10v_p(M)$ for every prime $p > 3$.

Assume that $v_p(n) > v_p(M)$ for some prime $p > 3$. Then it follows that $p$ divides $M$, $v_p(A) = v_p(M)$, and we have $v_p(\text{num}) = 10v_p(M)$. For the power of $p$ in the denominator we see that $v_p(R(n, M)) = 6v_p(M)$, whence $v_p(\text{den}) = 4v_p(n) + 6v_p(M)$. Thus

$$10v_p(M) \geq 4v_p(n) + 6v_p(M) > 10v_p(M),$$

a contradiction. Therefore $v_p(M) \geq v_p(n)$ for all primes $p > 3$. This proves a) and makes obvious the statements in b2), b3), c2), d3) and d4). Note also that the conclusions modulo 2 in d1) will imply d2). Thus, it remains to prove b1), c1), and d1), where the primes $p = 2$ and $p = 3$ are only relevant.

Assume now that $n$ is even, $v_2(n) \not\in \{1 + v_2(M), 2\}$, and $v_3(n) \neq 1 + v_3(M)$. We will argue by contradiction, assuming that $v_2(n) > 1 + v_2(M)$. Then it follows that $v_2(n + 4) = 2$ and, therefore,

$$v_2(\text{num}) = 10v_2(M) + 15.$$  

For the denominator we first see that

$$v_2(R(n, M)) \geq \min\{6v_2(M) + 14, 5v_2(M) + v_2(n) + 13, 4v_2(M) + 2v_2(n) + 10, 3v_2(M) + 3v_2(n) + 8, 2v_2(M) + 4v_2(n) + 5, 5v_2(n) + v_2(M) + 4, 6v_2(n) + 9\}.$$

It is easy to see that $v_2(n) > 1 + v_2(M)$ implies that each term in the minimum is greater than or equal to $6v_2(M) + 13$. We conclude that

$$v_2(\text{den}) \geq 4v_2(n) + 1 + 6v_2(M) + 13 > 10v_2(M) + 18.$$  

Therefore,

$$10v_2(M) + 18 < v_2(\text{num}) = 10v_2(M) + 15,$$

a contradiction. Hence $v_2(n) \leq v_2(M)$, which completes, in particular, the proof of b1).

Finally, assume that $v_3(n) > 1 + v_3(M)$ for the proofs of c1) and d1). Then

$$v_3(\text{num}) = 10v_3(M) + 7.$$  

On the other hand,

$$v_3(R(n, M)) \geq \min\{6v_3(M) + 6, 5v_3(M) + v_3(n) + 7, 4v_3(M) + 2v_3(n) + 5, 3v_3(M) + 3v_3(n) + 4, 2v_3(M) + 4v_3(n) + 3, v_3(M) + 5v_3(n) + 3, 6v_3(n)\}.$$  

It is easy to deduce that all terms in the last minimum are greater than or equal to $6v_3(M) + 6$. Therefore,

$$v_3(\text{den}) \geq 3 + 4v_3(n) + 6v_3(M) + 7 > 10v_3(M) + 14,$$

whence

$$10v_3(M) + 14 < v_3(\text{num}) = 10v_3(M) + 7,$$

which is impossible. This completes the proof. 

We can bring some parts of the above lemma together. For example we can combine parts c1), c2) as follows: if $\gcd(n, 2) = 1$ and $3|n$, then $n|3M$. More general, we present a reformulation that is focused on the divisibility properties of $M$.

Corollary 4. Let $C \subset \mathbb{Z}^{n-1}$ be a spherical 4-distance 7-design of cardinality $|C| = M$. Then $n$ divides $M$ if one of the following is fulfilled:

a) $\gcd(n, 6) = 1$;

b) $2|n, \gcd(n, 3) = 1$, and $v_2(n) \not\in \{2, 1 + v_2(M)\}$;

c) $\gcd(n, 2) = 1, 3|n$, and $v_3(n) \neq 1 + v_3(M)$;

d) $6|n, v_3(n) \neq 1 + v_3(M)$, and $v_2(n) \not\in \{2, 1 + v_2(M)\}$.

Further, $n$ divides $2M$ if $\gcd(n, 3) = 1$ and $v_2(n) = 1 + v_2(M)$; $n$ divides $4M$ if $\gcd(n, 3) = 1$ and $v_2(n) = 2$; $n$ divides $3M$ if $\gcd(n, 2) = 1$ and $v_3(n) = 1 + v_3(M)$ or $v_3(n) = 1 + v_3(M)$ and $v_2(n) = 1 + v_2(M)$; $n$ divides $6M$ if $v_3(n) = 1 + v_3(M)$ and $v_2(n) = 1 + v_2(M)$; and $n$ divides $12M$ if $v_3(n) = 1 + v_3(M)$ and $v_2(n) = 2$. In particular, $n$ divides $12M$.

We proceed with divisibility with respect to the prime divisors of $n + 1$ as instructed from (5).
Lemma 5. If $XYZT$ is an integer, then the following statements hold true:

a) if $gcd(n + 1, 6) = 1$, then $n + 1$ divides $M^2$;
b) if $n + 1$ is even and $gcd(n + 1, 3) = 1$, then $n + 1$ divides $16M^2$;
c) if $n + 1$ is odd and $3$ divides $n + 1$, then $n + 1$ divides $3M^2$;
d) if $6$ divides $n + 1$, then $(n + 1)^2$ divides $48M^4$.

Proof. Considering the numerator in (5) modulo $n + 1$ we conclude that there exists a prime $p > 3$ such that $v_p(n + 1) > 2v_p(M)$. Then the power of $p$ in the numerator is
\[ v_p(n) = 10v_p(M). \]

Further, it also follows from our assumption that
\[ v_p(R(n, M)) = 6v_p(M), \]
whence
\[ v_p(den) = 6v_p(M) + 2v_p(n + 1). \]

Therefore
\[ 6v_p(M) + 2v_p(n + 1) \leq 10v_p(M), \]
which is impossible when $v_p(n + 1) > 2v_p(M)$. This completes the proof of a) and leaves the proofs of b), c) and d) only up to considerations of the powers of 2 and 3.

Modulo 2 considerations are needed for b) and d) only. We assume $v_2(n + 1) \geq 2v_2(M) + 5$ for a contradiction. This implies that $v_2(A) = 1 + v_2(M)$, $v_2(n - 1) = 1$, and $n + 4$ is odd. Therefore
\[ v_2(num) = 10v_2(M) + 9. \]

To estimate the power of 2 in the denominator we observe that
\[ v_2(R(n, M)) \geq \min\{6v_2(M) + 14, v_2(n + 1) + 4v_2(M) + 9\} = 6v_2(M) + 14 \]
to conclude that
\[ v_2(den) \geq 6v_2(M) + 2v_2(n + 1) + 15. \]

Hence,
\[ 2v_2(n + 1) + 6v_2(M) + 15 \leq 10v_2(M) + 9, \]
i.e. $v_2(n + 1) + 3 \leq 2v_2(M)$, which is impossible when $v_2(n + 1) \geq 2v_2(M) + 5$. This completes the proof of b) and what concerns the power of 2 in d).

Finally, in our course to finish the proof in c) and d), we assume that $v_3(n + 1) > 1 + 2v_3(M)$ for a contradiction. Similarly to above we see that $v_3(A) = 1 + v_3(M)$, $v_3(n - 1) = 1$ and $v_3(n - 1) = 0$, whence
\[ v_3(num) = 3v_3(M) + 7(1 + v_3(M)) + 4 = 10v_3(M) + 11. \]

For the denominator we first see that
\[ v_3(R(n, M)) \geq 6v_3(M) + 6 \]
giving that
\[ v_3(den) \geq 6v_3(M) + 2v_3(n + 1) + 9, \]
which contradicts to our assumption.

In our computer investigation below we will use the following reformulation.

Corollary 6. Let $C \subset S^{n-1}$ be a spherical $4$-distance $7$-design of cardinality $|C| = M$. Then $n + 1$ divides $4M^2$.

C. Further divisibility conditions

In [9], Nozaki proved necessary conditions for existence of spherical $k$-distance sets as a generalization of the classical Larman-Roges-Seidel result on 2-distance sets [5]. We present his result here in our context to find another divisibility condition.

The Nozaki’s Theorem 5.1 [9] states that the number
\[ k_a := \frac{(1 - b)(1 - c)(1 - d)}{(a - b)(a - c)(a - d)} \]
and its analogs for $b, c$ and $d$ are integers which sum up to 1 and their absolute values do not exceed
\[ \left| \frac{1}{2} + \sqrt{\frac{N^2}{2N - 2} + \frac{1}{4}} \right|, \]
where
\[ N = \binom{n + 2}{3} + \binom{n + 1}{2} = \frac{n(n + 1)(n + 5)}{6}. \]

Applying our technique (i.e. employing the equation (3)) to this, we obtain
\[ k_a k_b k_c k_d = \frac{2M^3(n + 1)(n + 4)^2(n - 1)^3A^3}{R(n, M)}, \]
which should be integer. We were unable to get further divisibility conclusions from (6) but combinations of (5) and (6) could give something. However, the proofs of Lemmas 3 and 5 could be probably simplified by using (6) instead of (5).

IV. COMPUTER ASSISTED INVESTIGATIONS OF THE INTEGRALITY CONDITIONS

We describe the computer investigations of the integrality of $XYZT$ based on the divisibility conditions from Lemmas 3 and 5 and general investigations of the integrality of $XYZT$ and $k_a k_b k_c k_d$ from (6).

For fixed $n \geq 3$ all positive integers $M$ in the range defined by (2) are checked (with a strict inequality $M > 2^{\binom{n+2}{2}}$) to skip the considerations of tight spherical 7-designs. We first determine whether the pair $(n, M)$ satisfies the conditions of Lemmas 3 and 5 (i.e., whether $n|12M$ and $n + 1|4M^2$). Then, for each such pair we check if $XYZT$ is integer. If the pair $(n, M)$ passes this test (i.e., if the number $XYZT$ is integer) we compute the values of $a, b, c, d$ as roots of the equation (2) and find the values of $X, Y, Z, T$ themselves. The last part can be realized via computation of $k_a, k_b, k_c, k_d$ as well.
This computation yields the following (computer-assisted) result.

**Theorem 7.** There exist no spherical 4-distance 7-designs in dimensions $3 \leq n \leq 1000$ if

$$M > 2 \left( \frac{n + 2}{3} \right).$$

In other words, the only possible spherical 4-distance 7-designs in dimensions $3 \leq n \leq 1000$ are the tight spherical 7-designs (see the comment after Conjecture 2).

In these dimensions, there are 321 pairs $(n, M)$ that pass the tests of Lemmas 3 and 5 and, furthermore, $XYZT$ is an integer. This was verified by a simple Python program. There is no dimension with more than two suitable values of $M$ and there are 12 dimensions with two possible values of $M$ each.

Further, in all 321 cases none of $X, Y, Z, T$ is an integer. This was independently verified by a simple Maple program. With the last check, the non-integrality of the parameters $k_a, k_b, k_c$, and $k_d$ was also verified.

It is up to computing power to proceed with this proof. We used only a general purpose laptop and stopped at $n = 1000$ as it already started to take about three hours of computation per dimension. It seems that dimensions $n \leq 2000$ are reachable by this approach and more powerful computers. All our programs and the most important intermediate data are available upon request.

**Example 8.** The data from the first case that passes the integrality test of $XYZT$ is as follows. For $(n, M) = (7, 196)$ we have that $n = 7$ divides $2M = 392$ and $n + 1 = 8$ divides $4M^2 = 153664$, i.e. the conditions from Lemmas 3 and 5 are satisfied. Then

$$XYZT = 1185921$$

from (5) and

$$k_a k_b k_c k_d = 121$$

from (6), i.e. our condition and the Nozaki’s condition are satisfied and there is no contradiction so far. Thus we find the polynomial (3) as

$$f(t) = 49896t^4 + 33264t^3 - 18144t^2 - 9072t + 504,$$

whence the inner products are computed,

$$a \approx -0.821721, \quad b \approx -0.442124,$$

$$c \approx 0.5008952, \quad d \approx 0.546284.$$

This gives (with enough precision) the numbers

$$X \approx 5.63, \quad Y \approx 41.57, \quad Z \approx 93.84, \quad T \approx 53.95,$$

clearly non-integer. Therefore the existence question for spherical 4-distance 7-designs on $S^6$ with 196 points is resolved in the negative.

We also completed a brute force investigation of both $XYZT$ and $k_a k_b k_c k_d$ in dimensions $n \leq 215$ with a Python program. There are 72 pairs that pass the $XYZT$ test and 27 pairs that pass the $k_a k_b k_c k_d$ test, the smallest one being $(n, M) = (7, 196)$ (it passes both tests, as mentioned in Example 8). It is clear that these 72 pairs are the same as those passing the test via Lemmas 3 and 5 since the lemmas are based on the investigation of the integrality of $XYZT$. In all these cases we compute $X, Y, Z, T$ to see that they are not integers, confirming this way Theorem 7. The integrality of $k_a k_b k_c k_d$ is usually stronger, despite there are two cases that pass the $k_a k_b k_c k_d$ test while failing to pass the $XYZT$ test. The disadvantage of the integrality criterion of (6) is that one can not start with prime divisors of $n$ and $n + 1$. Of course, the strongest test is the integrality of $X, Y, Z, T$ but we do not see other approaches than brute force investigation of the formulas from Section 2.

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