Counterexamples to the $L^p$-Calderón–Zygmund estimate on open manifolds

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Abstract
We prove that for every $m \in \mathbb{N}_{\geq 2}$ there exists a Riemannian manifold $M$ of dimension $m$ on which the $L^p$-Calderón–Zygmund estimate fails for all $p \in [1, \infty[$.

Keywords
Calderón–Zygmund estimate · Counterexample · Riemannian manifold · Open manifold · Warped product · Laplace–Beltrami · Hessian

Mathematics Subject Classification
Primary 58J05 · 53C21; Secondary 35J05 · 35J08

1 Introduction

Given $p \in [1, \infty[$, a Riemannian manifold $M$ is said to satisfy the $L^p$-Calderón–Zygmund inequality if there exist constants $C_1$, $C_2$ such that for all smooth, compactly supported functions $f$ on $M$, one has

$$\|\nabla^2 f\|_{L^p} \leq C_1 \|\Delta f\|_{L^p} + C_2 \|f\|_{L^p}. \quad (CZ(p))$$

In this note, we prove the following:

Theorem 1.1 For any $p \in [1, \infty[$ and any $m \in \mathbb{N}_{\geq 2}$, there exists an open Riemannian manifold $M$ of dimension $m$ on which the $L^p$-Calderón–Zygmund inequality $CZ(p)$ fails.

Throughout, a Riemannian manifold (without boundary) $(M, g)$ is said to be open if it is non-compact and geodesically complete. The 2-tensor field $\nabla_g \nabla_g f$ denotes the Hessian of $f$. Its trace under the metric is the Laplace–Beltrami operator, denoted by $\Delta_g f$. In this note, all the Hessian and Laplace–Beltrami operators shall be taken with respect to a fixed metric $g$ (the one in (2.1) below); let us abbreviate by $\nabla^2 f := \nabla_g \nabla_g f$ and $\Delta f := \Delta_g f$. 

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In the Euclidean space $\mathbb{R}^m$, $CZ(p)$ was established by Calderón–Zygmund in the seminal paper [1] for $1 < p < \infty$ with constants $C_1, C_2$ depending only on $p$ and $m$. A natural question is the validity of $CZ(p)$ on a Riemannian manifold $(M, g)$. Many works are devoted to proving $CZ(p)$ on $(M, g)$ that satisfies certain geometric assumptions, e.g., the boundedness of Ricci or sectional curvatures, the boundedness of the injectivity radius away from zero, and the doubling property of the Riemannian volume measure. We refer to Cheeger et al. [2], Strichartz [10], Taylor [11], Wang [12], and to the currently most general results by Güneysu and Pigola [4] and Impera et al. [7]. Güneysu–Pigola have also proved nonlinear versions of $CZ(p)$ in [5], which play a fundamental role in precompactness results for isometric immersions.

On the other hand, in [4], Güneysu–Pigola constructed a two-dimensional complete manifold $(M, g)$ on which $CZ(2)$ is invalid. To the author’s knowledge, this is among the first negative results for Calderón–Zygmund estimates.

Before starting the proof of Theorem 1.1, let us make three remarks:

1. Throughout this paper, $\| \bullet \|_{L^p}$ denotes the $L^p$-norm of tensor fields on $M$, taken with respect to the metric $g$ in (2.1) below. For the definition and discussions on Sobolev spaces over manifolds, see, e.g., Hebey [6].
2. $CZ(p)$ has been known to be false for $p = 1$ and $p = \infty$ on Euclidean spaces; see Ornstein [9] and McMullen [8]. So we focus on $1 < p < \infty$ in this paper.
3. Our proof is crucially based on the construction in Theorem B, [4] by Güneysu–Pigola.

### 2 Construction of the manifold $(M, g)$

In this section, we construct the manifold $(M, g)$ which leads eventually to the proof of Theorem 1.1. The presentation in this section works for all $m \in \mathbb{N}_{\geq 2}$. It involves the choice of several parameters, which will be specified in the subsequent sections.

**Warped product** The manifold $M$ we choose is the Euclidean space $\mathbb{R}^m$ equipped with the warped product metric:

$$g = dr \otimes dr + \sigma^2(r) \text{can}^{m-1},$$

(2.1)

where $\text{can}^{m-1}$ is the canonical round metric on the $(m - 1)$-dimensional Euclidean sphere. Such $M$ is known as a warped product manifold. Note that the space forms are special examples of warped products: $M = \mathbb{S}^m$ if $\sigma = \sin$, $M = \mathbb{R}^m$ if $\sigma = \text{Id}$, and $M = \mathbb{H}^m$ if $\sigma = \sinh$. We shall choose the warping function $\sigma$ to be non-negative, smooth, and growing to infinity as the radial coordinate $r \to +\infty$. Thus, $(M, g)$ is an open manifold. The warped products are central objects of many recent works on geometric analysis; cf. e.g., Guan and Lu [3].

**Green’s function** Let $\tilde{G}(x)$ be the Green’s function of the Laplace–Beltrami operator on $M$ as above. Since $g$ in (2.1) is rotationally symmetric, there is a function $G : [0, \infty[ \to \mathbb{R}$ such that $\tilde{G}(x) = G(r)$ for $r := |x|$. Writing $\Delta$ in polar coordinates, we find that

$$\Delta \tilde{G} = 0 \iff G'' + (m - 1) \frac{\sigma'}{\sigma} G = 0 \quad \text{on } M \sim \{0\}.$$  

(2.2)

**Hessian and Laplacian** The key idea of the construction, as in [4], is to take $f$ to be a localised version of the Green’s function $G$. For $k \in \mathbb{N}$ and $[\alpha_k, \beta_k] \subset \mathbb{R}$, let $\phi_k \in C^\infty_c([\alpha_k, \beta_k])$ be a cut-off function. Here, $\phi_k$, $\alpha_k$, and $\beta_k$ will be specified later. Then, define

$$u_k(r) := \phi_k \circ G(r).$$  

(2.3)
In the end, one shall set \( f := u_k \) for some sufficiently large \( k \).

Direct computations in [4] lead to the following formulae for the Hessian and the Laplace–Beltrami of \( u_k \), as well as the volume density of \( g \):

\[
\nabla \nabla u_k = u_k'' \, dr \otimes dr + \sigma \sigma' u_k' \text{can}^{m-1},
\]

(2.4)

\[
\Delta u_k = u_k'' + (m - 1) \frac{\sigma' u_k'}{\sigma},
\]

(2.5)

\[
\sqrt{\det g} = \sigma^{m-1}.
\]

(2.6)

Throughout, \( \sigma \), \( G \), and \( u_k \) are functions of \( r \) only; \( \sigma', \sigma'' \) denote the derivatives in \( r \).

In the rest of this section, fixing a \( p \in ]1, \infty[ \), we collect some preliminary estimates for the \( L^p \)-norm of \( u_k \), \( \Delta u_k \), and \( \nabla \nabla u_k \). First of all, neglecting the radial components in (2.4), we have

\[
|\nabla \nabla u_k|^p \geq \left| u_k' \frac{\sigma'}{\sigma} \right|^p.
\]

Hence, denoting by \( \gamma_m := \text{Vol}_{\text{can}}^{m-1}(S^{m-1}) \) the area of the unit sphere, we deduce

\[
\|\nabla \nabla u_k(r)\|_{L^p} = \gamma_m \left\{ \int_0^\infty |\nabla \nabla u_k|^p \sigma^{m-1}(r) \, dr \right\}^{\frac{1}{p}} \geq \gamma_m \left\{ \int_0^\infty \left| \phi_k'(G(r)) \, G'(r) \left( \frac{\sigma'}{\sigma} \right)(r) \right|^p \sigma^{m-1}(r) \, dr \right\}^{\frac{1}{p}} = \gamma_m \left\{ \int_0^\infty \left| \phi_k'(G(r)) \left| \frac{\sigma'}{\sigma}(r) \right|^p \sigma^{(1-m)(p-2)}(r) G'(r) \, dr \right\}^{\frac{1}{p}}.
\]

(2.7)

In the last line, we used the identity

\[
G'(r) = \sigma^{-m}(r).
\]

A change of variable \( r \mapsto s = G(r) \) yields that

\[
\|\nabla \nabla u_k\|_{L^p} \geq \gamma_m \left\{ \int_{\alpha_k}^{\beta_k} |\phi_k'(s)|^p \left| \frac{\sigma'}{\sigma} \circ G^{-1}(s) \right|^p [\sigma \circ G^{-1}(s)]^{(1-m)(p-2)} \, ds \right\}^{\frac{1}{p}}.
\]

(2.8)

For the Laplace–Beltrami, it is crucial to observe that

\[
\Delta u_k(r) = \phi_k'' \circ G(r) \sigma^{2-2m}(r),
\]

thanks to the defining property (2.2) of the Green’s function. Thus, we have

\[
\|\Delta u_k\|_{L^p} = \gamma_m \left\{ \int_{\alpha_k}^{\beta_k} |\phi_k''(s)|^p [\sigma \circ G^{-1}(s)]^{2(p-1)(1-m)} \, ds \right\}^{\frac{1}{p}}.
\]

(2.9)

Finally, note that

\[
\|u_k\|_{L^p} = \gamma_m \left\{ \int_{\alpha_k}^{\beta_k} |\phi_k(s)|^p [\sigma \circ G^{-1}(s)]^{2(m-1)} \, ds \right\}^{\frac{1}{p}}.
\]

(2.10)

The key observation: Only the norm of \( \sigma \) is involved in the upper bounds for \( \|\Delta u_k\|_{L^p} \) and \( \|u_k\|_{L^p} \), while \( \sigma' \) is present in the lower bound for \( \|\nabla \nabla u_k\|_{L^p} \); see (2.7), (2.9), and (2.10). As a consequence, by carefully choosing a highly oscillatory profile for \( \sigma \), we may force \( \|\nabla \nabla u_k\|_{L^p} \) to be much larger than \( \|\Delta u_k\|_{L^p} \) and \( \|u_k\|_{L^p} \), thus contradicting CZ \((p)\).
3 Proof for $m = 2$

In this section, we prove Theorem 1.1 for $m = 2$ by specifying the warping function $\sigma$. The proof is essentially an adaptation of the arguments for Theorem B in [4] by Güneysu–Pigola, which corresponds to the case $m = 2$, $p = 2$. For the sake of completeness, we shall explain in detail why the construction in [4] works for all $p \in [1, \infty[$.

First, we set $\alpha_k = k$ and $\beta_k = k + 1$ for each $k \in \mathbb{N}$.

Next, let us require the warping function $\sigma$ to satisfy the following:

\[
\begin{cases}
\sigma^{(2k)}(0) = 0 \text{ for each } k \in \mathbb{N}; \\
\sigma'(0) = 1; \\
\sigma(t) > 0 \text{ for any } t > 0; \\
t \leq \sigma(t) \leq t + 1 \text{ for any } t \geq 1.
\end{cases}
\]

When $m = 2$, one has the simple comparison results (see p. 377 in [4]):

\[
\log \left( \frac{t + 1}{2} \right) \leq G(t) \leq \log t \quad \text{for all } t > 1
\]

and

\[
es \leq \sigma \circ G^{-1}(s) \leq 2e^s \quad \text{for all } s > 0.
\]

Moreover, there exists a universal constant $\delta > 0$ such that for all sufficiently large $k$, we can find $h = h(k) > k$ such that

\[
[h, h + 1] \subset \left[ G^{-1}(k + \delta), G^{-1}(k + 1 - \delta) \right].
\]

In addition, we choose the cut-off function $\phi_k$ in (2.3) as follows: Fix some $\phi \in C^\infty_c([0, 1])$ such that $\phi \equiv \text{Id}$ on $[\delta, 1 - \delta]$ and $\phi \leq 1$, and then set

\[
\phi_k(t) := \phi(t - k)
\]

for each $k \in \mathbb{N}$. Here, $\delta > 0$ is the same constant as in (3.4). We shall fix $\phi$ once and for all; in particular, $\|\phi\|_{C^2([0, 1])}$ is bounded by a universal constant.

We can deduce from (2.9), (2.10), and (3.3) the following bounds:

\[
\|u_k\|_{L^p} \leq 2(\gamma_2)^p e^{2k+2}
\]

and

\[
\|\Delta u_k\|_{L^p} \leq \frac{(\gamma_2)^p}{2(p - 1)4^{p-1}}\|\phi''\|_{L^\infty([0, 1])} e^{-2(p-1)k}.
\]

So it remains bound $\|\nabla u_k\|_{L^p}$ from below.

For this purpose, we shall further specify $\sigma$. Consider the cube

\[
Q_k := [k, k + 1] \times [k, k + 1]
\]

for each $k \in \mathbb{N}$; from the previous constructions, the graph of $\sigma$ is contained in $\bigcup_{k=0}^{\infty} Q_k$ (in fact, in the union of the upper-left corners of $Q_k$). For certain sequence $\{n_k\} \subset \mathbb{N}$ increasing to $+\infty$ as $k$ grows, we take

\[
\epsilon_k := \frac{1}{2n_k}.
\]
Define $\mathcal{G}_k$ on $[k, k+1]$ by the “sawtooth” function on p.378, [4]:

$$
\mathcal{G}_k(t) := \begin{cases} 
  k + 2j\epsilon_k + \frac{\epsilon_{k+1}}{\epsilon_k}(t - k - 2j\epsilon_k) & \text{on } [k + 2j\epsilon_k, k + (2j + 1)\epsilon_k] \text{ for each } j \in \{0, 1, \ldots, n_k\}, \\
  k + (2j + 1)\epsilon_k + \frac{1-\epsilon_k}{\epsilon_k}(k + (2j + 1)\epsilon_k - t) & \text{on } [k + (2j + 1)\epsilon_k, k + 2j + 1\epsilon_k] \text{ for each } j \in \{0, 1, \ldots, n_k\}.
\end{cases}
$$

Then, one defines $\sigma|[k, k+1]$ by smoothing out the corners of $\mathcal{G}_k$. More precisely, for each $k \in \mathbb{N}$ we can take $\sigma \in C^\infty([k, k+1])$ such that

$$
\sigma = \mathcal{G}_k \text{ on } [k, k+1] \sim \bigcup_{j=0}^{n_k} \left[k + 2j\epsilon_k - \epsilon_k^{10}, k + 2j\epsilon_k + \epsilon_k^{10}\right]
$$

and that $\|\sigma\|_{C^3} \leq 2$ in each of the small intervals removed.

The idea for the construction of $\mathcal{G}_k$ is clear: its graph (lying in the upper-left corner of the cube $Q_k$) is obtained by continuously concatenating $n_k$ copies of the following “sawtooth unit” with step-length $(2\epsilon_k)$—in the first $\epsilon_k$ it grows with constant gradient $(\epsilon_{k+1})/\epsilon_k$, and in the second $\epsilon_k$ it decreases with constant gradient $(1-\epsilon_k)/\epsilon_k$. In particular, in the second half of each sawtooth unit, the norm of the gradient is large.

With the above choice of $\sigma$, we can continue the lower bound (2.7) for the Hessian of $u_k$ as in below. First, by the definition of $\mathcal{G}_k$ and (3.3), we have

$$
\|\nabla^2 u_k\|_{L^p} \geq (\gamma_2)^p 2^{-p} e^{-p(k+1)} \int_{k+\delta}^{k+1-\delta} |\sigma' \circ G^{-1}(s)|^p |\sigma \circ G^{-1}(s)|^{2-p} \ ds.
$$

Considering separately $p \geq 2$ and $p < 2$ and using again (3.3), one deduces

$$
\|\nabla^2 u_k\|_{L^p}^p \geq \min \left\{1, 2^{2-p}\right\} (\gamma_2)^p 2^{-p} e^{-p(k+1)} \int_{k+\delta}^{k+1-\delta} |\sigma' \circ G^{-1}(s)|^p \ ds.
$$

For $m = 2$, we have $G'(r) = \sigma^{-1}(r)$, hence

$$(G^{-1})'(s) = \frac{1}{G'[G^{-1}(s)]} = \sigma[G^{-1}(s)].$$

It follows that

$$
\|\nabla^2 u_k\|_{L^p}^p \geq \min \left\{1, 2^{2-p}\right\} (\gamma_2)^p 2^{-p} e^{-p(k+1)} \int_{k+\delta}^{k+1-\delta} \left|\sigma' \circ G^{-1}(s)\right|^p \left|\frac{1}{\sigma \circ G^{-1}(s)} \left(G^{-1}(s)\right)'(s)\right| \ ds
$$

$$
\geq \min \left\{1, 2^{2-p}\right\} (\gamma_2)^p 2^{-p-1} e^{-p(k+1)} e^{-k+1+\delta} \int_{G^{-1}(k+\delta)}^{G^{-1}(k+1-\delta)} |\sigma'(r)|^p \ dr.
$$

Here, we have used (3.2) once more.

Recall that the universal constant $\delta$ is chosen right beneath (3.3). For $k$ sufficiently large, we have selected $h = h(k) > k$ in (3.4) so that

$$
\|\nabla^2 u_k\|_{L^p}^p \geq \min \left\{1, 2^{2-p}\right\} (\gamma_2)^p 2^{-p-1} e^{-(p+1)(k+1)+\delta} \int_{h}^{h+1} |\sigma'(r)|^p \ dr.
$$

Thanks to the choice of $\sigma$, on $[h, h+1]$ the norm of the gradient $|\sigma'|$ is larger than $(2n_k - 1)$ on more than $n_k$ subintervals longer than $(\epsilon_k - \epsilon_k^{10})$, where $2n_k\epsilon_k = 1$. Thus,
\[ \| \nabla \nabla u_k \|_{L^p}^p \geq \min \{ 1, 2^{2-p} \} (\gamma_2)^p 4^{-p} e^{-(p+1)(k+1)+\delta} (2n_k - 1) e_k (\epsilon_k - e_k^0)^p. \]  

(3.7)

We may now derive the contradiction by comparing (3.7) with (3.5) and (3.6). Note that
\[ \| u_k \|_{L^p} \lesssim e^{2k+2} \quad \text{and} \quad \| \Delta u_k \|_{L^p} \lesssim e^{-2(p-1)k} \lesssim 1, \]
where the constants in \( \lesssim \) depend on \( p, C_1, C_2, \) and \( \| \phi'' \|_{L^\infty([0,1])} \).

On the other hand,
\[ \| \nabla \nabla u_k \|_{L^p} \gtrsim e^{-(p+1)k} (1 - \epsilon_k^0) e_k^{-p}. \]

By further requiring for large \( k \in \mathbb{N} \) that \( \epsilon_k \leq 100^{-1} \), we get
\[ \| \nabla \nabla u_k \|_{L^p} \gtrsim e^{-(p+1)k} e_k^{-p}, \]
with the constant in \( \gtrsim \) depends only on \( p \). Therefore, we can contradict \( CZ(p) \) by choosing, e.g.
\[ \epsilon_k := C e^{-\epsilon_k^0} \]
for a suitable constant \( C = C(p, C_1, C_2, \| \phi \|_{C^2([0,1])}) \). Thus, choosing a sufficiently large \( k \) completes the proof of Theorem 1.1 for \( m = 2 \).

4 Proof for \( m \geq 3 \)

In this section, we prove Theorem 1.1 for arbitrary \( m \geq 3 \).

The new feature is that the cubes \( Q_k \) in § 3 are not available, since we cannot choose the warping function to satisfy \( t \leq \sigma(t) \leq t + 1 \) for all \( t \geq 1 \). Instead, we shall choose an infinite sparse family of cubes \( \{ Q'_k \} \) sandwiched between the graphs of \( t \mapsto t^{1/m} \) and \( t \mapsto (t+1)^{1/m} \). Necessarily, the size of the \( Q'_k \) will shrink to zero as \( k \to \infty \); nevertheless, we can prescribe the rate of oscillation of \( \sigma \) to be much larger than the shrinking rate of \( Q'_k \). This is enough to conclude Theorem 1.1 for \( m \geq 3 \).

Now, we start the proof. First of all, let us observe that the estimates (2.7), (2.9), and (2.10) are valid for all \( m \in \mathbb{Z}_{\geq 2} \), and that the radial Green’s function again verifies
\[ G(r) = \int_1^r \sigma^{1-m}(t) \, dt. \]

We shall pick a \( \sigma \) satisfying \( G(+\infty) = +\infty \), which ensures the parabolicity of \( (M, g) \). For brevity, we write
\[ \alpha \equiv \alpha_m := \frac{1}{m - 1}. \]

Then, we choose \( \sigma \) to satisfy a set of properties similar to those in (3.1):
\[
\begin{align*}
\sigma^{(2k)}(0) &= 0 \text{ for each } k \in \mathbb{N};
\sigma'(0) &= 1; \\
\sigma(t) &= 0 \text{ for any } t > 0; \\
\sigma'(t) &\leq \sigma(t) \leq (t + 1)^\alpha \text{ for any } t \geq 1.
\end{align*}
\]

(4.1)
The motivation is to require the norm of \( \sigma \) to be comparable to \( t^\alpha \) without introducing a singularity at the origin. This can be achieved, e.g. by gluing \( \sigma |[1, \infty[ \) to \( \sinh \) or \( \sin \) near \( r = 0 \).

Notice that (3.2) and (3.3) in the case of \( m = 2 \) are still valid, namely that
\[
\log \left( \frac{t + 1}{2} \right) \leq G(t) \leq \log t \quad \text{for all } t > 1,
\]
and that
\[
e^s \leq G^{-1}(s) \leq 2e^s - 1 \quad \text{for all } s > 0.
\]
Applying to (4.3), the last property in (4.1), we may infer:
\[
e^{\alpha s} \leq \sigma \circ G^{-1}(s) \leq 2^\alpha e^{\alpha s} \quad \text{for all } s > 0.
\]
In addition, note that (3.4) still holds true. In fact, there exists a universal constant \( \delta > 0 \) such that for all \( k \geq 1 \), we can find \( h = h(k) > k \) satisfying
\[
[h, h + 1] \subset \left[ G^{-1}(k + \delta), G^{-1}(k + 1 - \delta) \right].
\]
For example, \( \delta := 4^{-1}(1 - \log 2) \) ensures that the length of the interval on the right-hand side of (4.5) is greater than 2.

Let the choices for \( \phi_k, \alpha_k, \beta_k, \) and \( u_k \) be the same as in the \( m = 2 \) case. It follows from (2.9) and (2.10) that
\[
\| u_k \|_{L^p}^p \leq 4(\gamma_m)^pe^{2(k+1)}
\]
and
\[
\| \Delta u_k \|_{L^p}^p \leq (\gamma_m)^p\| \phi'' \|_{L^\infty([0,1])}^p e^{-2(p-1)(k+1)},
\]
which are similar to (3.5) and (3.6) for \( m = 2 \).

Now, we shall specify the warping function. Again, the idea is to introduce high-frequency oscillations to \( \sigma \). In view of the final line in (4.1), the graph of \( \sigma |[1, \infty[ \) lies in the strip
\[
S := \left\{ (t, y) \in \mathbb{R}^2 : t \geq 1, \ t^\alpha \leq y \leq (t + 1)^\alpha \right\}.
\]
Let us denote by
\[
S_k := S \cap \left\{ k \leq t \leq k + 1 \right\} \quad \text{for each } k \in \mathbb{N}.
\]
Note that the height of the window \( S_k \) shrinks to 0 as \( k \searrow \infty \). We introduce the parameter:
\[
\eta_k := \min_{t \in [k, k+1]} \frac{(t + 1)^\alpha - t^\alpha}{10}.
\]
As discussed above, \( \eta_k \searrow 0 \) as \( k \searrow \infty \). Moreover, it is easy to see that one can place a cube \( Q'_k \), whose sides are parallel to the Cartesian axes and have length \( \eta_k \), inside the window \( S_k \).

For \( k \in \mathbb{N} \) fixed momentarily, let us define \( \sigma \) on part of \([k, k+1]\). More precisely, we shall require that the graph of \( \sigma \) over the horizontal projection of the cube \( Q'_k \) is contained in \( Q'_k \). For this purpose, we can carry out a construction slightly simpler that in [4] for the \( m = 2 \) case.

Indeed, let \( \sigma([z_k, z_k + \eta_k]) \) be the juxtaposition of \( \ell_k \) sawtooth functions of step-length
\[
\delta_k := \frac{\eta_k}{2\ell_k}.
\]
Each sawtooth function (modulo an obvious translation) increases from 0 to \( \eta_k \) in step-length \( \delta_k \) and then decreases from \( \eta_k \) to 0 in another step-length \( \delta_k \). Finally, we smooth out the corners by modifying on \( (2\ell_k) \) intervals of the length \( \delta_k^{10} \). In this way, we complete the definition of \( \sigma \) inside \( Q_k' \); it is smooth and has gradient \( |\sigma'| = \delta_k^{-1} = 2\ell_k/\eta_k \) for a large portion of the domain, i.e. the horizontal projection of \( Q_k' \). We shall specify the small parameter \( \delta_k \) and the large parameter \( \ell_k \) later in the proof. In passing let us note that, roughly speaking, the parameters \( (\ell_k, \delta_k, \eta_k) \) play the role of \( (\eta_1, \epsilon_1, 1) \) as in § 3.

In the above paragraph, we defined \( \sigma \) in \( Q_k' \). Now, let us extend it globally. For this purpose, consider a sequence \( \{k_j\}_{j=1}^{\infty} \) which tends to \( \infty \) as \( j \nearrow \infty \). Let \( h_j = h(k_j) \) be defined as in (4.5). As the Green’s function \( G \) is monotonically increasing, in view of (4.5), one can choose \( \{k_j\} \) so that the cubes \( Q_{h_j} \) are disjoint. Let \( \sigma \) be defined in each \( Q_{h_j} \) as above. Outside these cubes, we take \( \sigma \) to be any smooth function satisfying the properties in (4.1), and by a simple glueing argument we can obtain \( \sigma \in C^\infty([0, \infty[) \). For notational convenience, in the sequel let us relabel \( k = k_j \) and \( Q_k' = Q_{h_j} \equiv Q_{h(k_j)} \).

It remains to bound \( \|\nabla \nabla u_k\|_{L^p} \) from below. First of all, by (2.7), the choice of \( \phi_k \), and the upper bound in (4.4), we have

\[
\|\nabla \nabla u_k\|_{L^p}^p \geq (\gamma_m)^p 2^{-\alpha p} e^{-\alpha p(k+1)} \int_{k+\delta}^{k+1-\delta} \left| \sigma' \circ G^{-1}(s) \right|^p \left| \sigma \circ G^{-1}(s) \right|^{(1-m)(p-2)} ds.
\]

Utilising once more (4.4), we get

\[
\|\nabla \nabla u_k\|_{L^p}^p \geq (\gamma_m)^p 2^{-\alpha(p-1)} e^{-(k+1)(\alpha p+p-2)} \int_{k+\delta}^{k+1-\delta} \left| \sigma' \circ G^{-1}(s) \right|^p \left| \sigma \circ G^{-1}(s) \right|^{(1-m)} ds.
\]

For \( \dim M = m \), there holds \( G'(r) = \sigma^1 - m(r) \), so

\[
(\sigma^1 G^{-1}(s)) = \frac{1}{G'[G^{-1}(s)]} = \sigma^m \left[ G^{-1}(s) \right].
\]

It yields that

\[
\|\nabla \nabla u_k\|_{L^p}^p \geq (\gamma_m)^p 2^{-\alpha(p-1)} e^{-(k+1)(\alpha p+p-2)} \int_{k+\delta}^{k+1-\delta} \left| \sigma' \circ G^{-1}(s) \right|^p \left| \sigma \right|^{1-m} \left[ G^{-1}(s) \right] (G^{-1})'(s) ds.
\]

Thus, changing the variables \( s \mapsto r = G^{-1}(s) \) and invoking (4.5), we arrive at

\[
\|\nabla \nabla u_k\|_{L^p}^p \geq (\gamma_m)^p 2^{-\alpha(p-1)} e^{-(k+1)(\alpha p+p-2)} \int_h^{h+1} \left| \sigma'(r) \right|^p \sigma^{1-m} \left( r \right) dr.
\]

By (4.4), one further gets

\[
\|\nabla \nabla u_k\|_{L^p}^p \geq (\gamma_m)^p 2^{-\alpha(p-1)} e^{-(k+1)(\alpha p+p-2) - k - \delta} \int_h^{h+1} \left| \sigma'(r) \right|^p dr,
\]

where \( h = h(k) > k \) is chosen as in (4.5).

To continue, it is crucial to note that in some subinterval of \([h, h + 1]\) of length \( \eta_k \), \( \sigma \) is highly oscillatory. This is due to our choice of \( Q_k' \) and the definition of \( \sigma \) thereon. More precisely, we can deduce the bound

\[
\|\nabla \nabla u_k\|_{L^p}^p \geq (\gamma_m)^p 2^{-\alpha(p-1)} e^{-(k+1)(\alpha p+p-2) - k - \delta} (\eta_k - \delta_k^9) (\delta_k)^{-p}. \tag{4.9}
\]
where \( \delta > 0 \) is universal as before. Here, recall that \( \eta_k = 2\delta_k \ell_k \searrow 0 \) for \( \ell_k \nearrow \infty \) to be determined. We shall select some \( \delta_k \) that shrinks to 0 much more rapidly than \( \eta_k \sim (k + 1)^\alpha - k^\alpha = (k + 1)^{\frac{1}{m+1}} - k^{\frac{1}{m+1}} \) does. Indeed, let us require that

\[
\begin{align*}
\delta_k^9 &\leq \frac{\eta_k}{2}, \\
\delta_k &\leq \left( \frac{\eta_k}{2} e^{-e^k} \right)^{\frac{1}{p}}.
\end{align*}
\]

The above two conditions give us

\[
(\eta_k - \delta_k^9) (\delta_k)^{-p} \geq e^{ek}; \quad (4.10)
\]

while the other term on the right-hand side of (4.9) is

\[
(\gamma_m)^p 2^{-\alpha(p-1)} e^{-(k+1)(\alpha p+p-2) - k - \delta} = C_3 e^{-C_4 k},
\]

with \( C_3, C_4 \) being positive constants depending only on \( m \) and \( p \), and with \( \delta \) being a fixed universal constant as before.

To conclude the proof, we can deduce from (4.10) and (4.9) that for any sufficiently large \( k \in \mathbb{N} \), there holds

\[
\| \nabla \nabla u_k \|_{L^p} \gtrsim e^{k1000}
\]

with the constants involved in \( \gtrsim \) depending on \( m \) and \( p \). On the other hand, in (4.6) (4.7), we have already proved that

\[
\| u_k \|_{L^p} \lesssim e^k, \quad \| \Delta u_k \|_{L^p} \lesssim e^{-2(p-1)k};
\]

the constants in \( \lesssim \) depending on \( m, p \) and the \( C^2 \)-norm of \( \phi \). Finally, the choice of a large \( k \) gives us the contradiction to \( CZ(p) \); hence, the proof of Theorem 1.1 for \( m \geq 3 \) is complete.

5 Concluding remarks

The open manifold \((M, g)\) constructed in this note has no bound on the norm of the Riemann curvature, and its injectivity radius degenerates. On the other hand, if the Ricci curvature of \((M, g)\) is bounded from both above and below, and if the injectivity radius is strictly positive, then \( CZ(p) \) is valid on \((M, g)\) for any \( 1 < p < \infty \) (cf. [4], Theorem C). Also, in the recent work [7], Impera–Rimoldi–Veronelli established \( CZ(2) \) with weights on open manifolds, under the mild assumption of either a sub-quadratic growth of the norm of the Riemann curvature, or a sub-quadratic growth of both the norm of the Ricci curvature and the squared inverse of the injectivity radius. It is interesting to further seek the minimal geometric boundedness assumptions on \((M, g)\) that ensures the validity of the \( L^p \)-Calderón–Zygmund estimate \( CZ(p) \).

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