Orthogonality Catastrophe for Vortices in \textit{d}-Wave Superconductors

Akakii Melkidze

Kavli Institute for Theoretical Physics, University of California, Santa Barbara CA 93106, USA

(Dated: December 29, 2002)

The dynamics of magnetic vortices in the mixed state of \textit{d}-wave superconductors is affected by interaction with quasiparticles near the gap nodes. We study this effect by computing the overlap of the ground state wave functions of the nodal quasiparticles in a two-dimensional \textit{d}-wave superconductor as the ground state changes in response to the motion of the vortex. We find that the overlap is strongly suppressed. This orthogonality catastrophe is specific to a half-quantum vortex and disappears in the case of a doubly-quantized vortex. This implies strong inhibition of the motion of half-quantum vortices. The results allow us to develop a simple description of the quantum vortex creep at low temperatures.

PACS numbers: 74.25.Qt

Magnetic vortices in Cuprate superconductors \cite{1, 2} behave rather differently compared to their counterparts in conventional materials. One of the characteristic features of the superconducting state of the Cuprates is the existence of the low-energy quasiparticles near the nodes of the \textit{d}-wave gap. One is naturally led to ask: what is the influence of the interaction between the vortices and nodal quasiparticles on vortex dynamics?

Before answering this question, let us briefly describe our current knowledge. Vortex creep at low temperatures is believed to proceed by quantum tunneling: the Magnus force \cite{3} and the friction force. Their ratio determines the “cleanness” of the material. The creep in the dirty, \cite{4} super-clean, \cite{5} and intermediate regimes \cite{6} has been studied theoretically.

Quasiparticles play prominent role in the vortex dynamics. In \textit{s}-wave superconductors, finite gap in the bulk precludes the existence of low-energy extended states, however quasiparticle states localized in the cores of vortices, where the gap vanishes, exist. \cite{7} Transitions between these states give rise to the friction force and the renormalization of the Magnus force. \cite{8} The situation with quasiparticle states localized inside vortex cores in \textit{d}-wave superconductors is far less clear. \cite{9} On the other hand, extended states exist and are expected to strongly interact with vortices. Indeed, they have been argued \cite{10, 11} to give a quantized value of thermal Hall conductance in the mixed state at low temperatures. This prediction is hard to verify since current experiments \cite{12} are performed at relatively high temperatures. \cite{13} Quantum vortex creep, which is also expected to be sensitive to the interaction with quasiparticles, occurs at much lower temperatures and thus offers experimental information about the quasiparticle spectrum at a much lower energy scale.

To study the contribution of nodal quasiparticles to vortex action we calculate the overlap of the ground state wave functions of the nodal fermions as the ground state changes in response to the motion of the vortex. We find that this overlap is extremely small in the case of a half-quantum vortex. This implies that the motion of half-quantum vortices is strongly suppressed due to the interaction with nodal quasiparticles. Indeed, we argue that this orthogonality catastrophe provides the dominant contribution to the vortex action at temperatures relevant to quantum vortex creep. Based on this, we present a simple phenomenological description of the available experimental data.

Let us consider a vortex in a 2D \textit{d}-wave superconductor. We use the Bogoliubov-de Gennes (BdG) equation to treat the interaction of quasiparticles with the vortex. \cite{10, 14}. We first take advantage of the conservation of spin in the BCS theory by changing from the original electronic operators $c$ to the new fermions: $\psi_{1k} = e_{\uparrow} k$, $\psi_{2k} = e_{\downarrow} k$.

The new operators are combined to form a two-component spinor: $\psi = (\psi_1, \psi_2)$. The density of the $\psi$-particles $d\psi$ is the $z$-component of the spin density of the original electrons, and visa versa. Next, we concentrate on the variation of the phase of the order parameter $\psi(\vec{r})$ around the vortex and neglect the variation of the amplitude. We then eliminate the phase winding by making a single-valued gauge transformation: $\psi_1 \rightarrow \exp[-i\phi(\vec{r})] \psi_1$, $\psi_2 \rightarrow \psi_2$. The BdG equation takes the form $H \psi = \varepsilon \psi$ with the following Hamiltonian (we use units with $\hbar = c = 1$):

$$H = E(\vec{p} + \vec{a} + j_s \sigma_z) \sigma_z + \Delta(\vec{p} + \vec{a}) \sigma_x.$$ (1)

Here, $\sigma_z, \sigma_x$ are Pauli matrices, $\vec{p} = (p_x, p_y)$ is the 2D momentum, $E(\vec{p})$ is the bare electron energy relative to the Fermi energy and $\Delta(\vec{p})$ is the gap function. The presence of the vortex produces two contributions to the Hamiltonian: 1) There appears a “Doppler shift”. The superflow $j_s = \partial \phi/2 - eA$ shifts the energies of the nodal quasiparticles similar to the classical Doppler effect. 2) The other contribution is of topological nature. It requires that a parallel transport of a quasiparticle around the vortex should lead to the change of its phase by $\pi \nu$, where $\nu$ is the vortex quantum number. This is enforced by the Chern-Simons gauge field $\vec{a} = \partial \phi/2$ which satisfies...
\hat{\mathbf{d}} \times \hat{a} = \pi \nu \hbar \delta(r) \) (assuming the vortex is at the origin).

First, we would like to calculate the overlap between the ground state wave functions of the superconductor with and without the half-quantum vortex. The two terms in the Hamiltonian that were discussed above both perturb the system shifting the single quasiparticle eigenstates. An important difference between them is their spatial range. The superflow \( \hat{j}_x(r) \) is screened at distances larger than the penetration depth. On the contrary, the field \( a(r) \) decays as \( 1/r \) at all distances. Let us consider the Doppler shift term first. The solution of the orthogonality catastrophe problem \[15\] in the case of a short-range perturbing potential is given by:

\[
|\langle f | i \rangle| \sim N^{-A},
\]

\[
A = \frac{1}{2} \text{Tr}[\hat{\delta}(\varepsilon_F)/\pi]^2.
\]

Here, \(|i\rangle\) and \(|f\rangle\) are the ground states of the system before and after the perturbing potential is switched on, \(N\) is the total number of fermions in the system, and \(\hat{\delta}(\varepsilon) = (1/2i)\ln \hat{S}(\varepsilon)\) is the phase shift matrix defined as the logarithm of the scattering matrix. The trace in the Eq. (3) is over all states at the Fermi energy. In our case the Fermi energy is zero. The density of states of nodal quasiparticles vanishes linearly with energy: there are no states at zero energy. This property of the phase space implies that the orthogonality exponent \(A\) is zero for any scattering potential of finite range. This means that the Doppler shift term causes possibly quite large but finite reduction of the overlap. Absence of infrared divergence in this case has been found previously \[16\] in a different context.

Based on what was said above, we set in the following \( \hat{j}_x = 0 \) in Eq. (1). \[14\] The remaining perturbation of the system is the long-range vector Chern-Simons field \( \hat{a}(r) \). Because the field decays slowly at infinity, it is not clear whether the above result for the orthogonality catastrophe for short-range potentials can be generalized to this case. It is also not clear whether at all one can use plane waves as asymptotic states to calculate the \( \hat{S} \) matrix: the asymptotic completeness has been proven only for short-range potentials in the Dirac equation. \[17\]

For these reasons we shall adopt another approach. The field \( \hat{a}(r) \) can be gauged away by a gauge transformation: \( \psi \rightarrow \psi \exp(i\nu \theta/2) \), where \( \nu \) is the vortex quantum number and \( \theta \) is the polar angle. The BdG Hamiltonian Eq. (1) reduces to a free one:

\[
H = \mathcal{E}(\mathbf{p})\sigma_z + \Delta(\mathbf{p})\sigma_x.
\]

For a doubly-quantized vortex, \( \nu = 2 \), this implies no orthogonality catastrophe between the ground states with and without such a vortex. On the other hand, for a singly-quantized vortex this transformation makes \( \psi \) non-singular valued. To take this into account, we introduce a cut \( C \) in the \((x,y)\) plane as shown in Fig. 1. The vortex is assumed to be at point \( A \) at the origin. Since the cut should end on another half-quantum vortex, we introduce one at point \( B \) for definiteness. We then impose a \( \pi \)-phase shift across the cut.

The distance \( L \) between the two vortices can be interpreted in various ways: First, \( L \) may be thought of as a long-distance cut-off. In another setting, one may consider an isolated vortex being moved from point \( A \) to point \( B \). This can be done by removing a vortex at point \( A \) (inserting a vortex with negative vorticity) and creating one at point \( B \). In this situation \( L \) is the distance by which the vortex has been moved.

Having made the gauge transformation we can reformulate our original problem as that of finding the overlap of the ground state wave functions of the system before and after a \( \pi \)-phase shift across the cut shown in Fig. 1 was imposed. To this end, consider the dynamical overlap function which is defined as follows: Let \(|i\rangle\) be the ground state wave function of the unperturbed system. We shall work in imaginary time to simplify calculations and avoid problems with time-ordering. At time \( \tau = 0 \) the phase shift \( \pi \) across the cut \( C \) is imposed. Let \(|\tau\rangle = U(\tau)|i\rangle\), where \(U(\tau)\) is the imaginary time evolution operator. Since in the limit \( \tau \rightarrow \infty \) only the contribution from the new ground state to \(U(\tau)\) survives, we have: \( \lim_{\tau \rightarrow \infty} \langle \tau |i\rangle = \langle f |i\rangle \), where \(|f\rangle\) is the new ground state. Since \( \langle \tau |i\rangle = \langle i|U(\tau)|i\rangle \), the problem reduces to that of finding the expectation value of the evolution operator in the old ground state. We have:

\[
U(\tau) = \exp\{i\pi Q(\tau)\},
\]

\[
Q(\tau) = \int_0^\tau J(\tau') d\tau'.
\]

Here, \( J(\tau) \) is the total current through the cut \( C \) and \( Q(\tau) \) is the net charge that has flowed through the cut between times 0 and \( \tau \) (here and below, unless mentioned otherwise, the charges and currents are those of the \( \psi \)-fermions). The meaning of the above expressions is quite transparent: Each fermion contributes phase \( \pi \) to the path integral every time it crosses the cut. The operator \( U(\tau) \) can thus be thought as a generator of counting
To proceed, we need to find the current-current correlation function. We linearize the Hamiltonian Eq. (9) around one of the nodes (e.g. one at \((k_F, 0)\)): \(E(\vec{p}) = v_F p_x, \Delta(\vec{p}) = v_F \Delta p_y\). The anisotropy parameter \(\alpha = v_F / v_\Delta \sim E_F / \Delta\) is quite large (\(\alpha \approx 14\) for YBCO). We then make \(\alpha = 1\) by rescaling the distances: \(p_x \rightarrow p_x / v_F, \quad p_y \rightarrow p_y / v_\Delta\). This allows us to exploit the full “relativistic” invariance of the problem. Indeed, the imaginary time action (in the second-quantized notation) becomes:

\[
S = \sum_p \bar{\psi}_\mu p_{\mu} \psi.
\]

(9)

Here, \(p_{\mu} = (w, p_x, p_y)\) is the 2+1 momentum, \(\bar{\psi} = -i\psi^\dagger \sigma_3\). Pauli matrices \(\sigma_\mu\) form a Clifford algebra: \(\{\sigma_\mu, \sigma_\nu\} = 2\delta_{\mu\nu}\). The “relativistic” invariance is now explicit since both \(\sigma_\mu\) and \(p_{\mu}\) transform as 2+1 vectors under rotations of the Euclidean space-time and the action \(S\) is therefore a scalar. The conserved current is:

\[
\tilde{j}_\mu = \bar{\psi} \sigma_\mu \psi.
\]

(10)

The current-current correlator is given (in the free theory) by the so-called bubble or vacuum polarization diagram:

\[
\langle j_\mu j_\nu \rangle_k = \int \frac{d^3q}{(2\pi)^3} \text{Tr} [\sigma_\mu G(q) \sigma_\nu G(q + k)].
\]

(11)

Here, \(G(q) = \sigma_\mu q_\mu / q^2\) is the Green function of the Dirac equation. The correlator, as written, is given by an ultraviolet divergent expression. This divergence can be eliminated by an appropriate high-energy cut-off. Introduction of such a cut-off, however, will break the gauge invariance. A more physical way is to explicitly impose the current conservation condition: \(k_\mu \langle j_\mu j_\nu \rangle_k = 0\). This way one obtains the well-known result (see [13] for a related calculation):

\[
\langle j_\mu j_\nu \rangle_k = \frac{1}{16k} (k^2 \delta_{\mu\nu} - k_\mu k_\nu).
\]

(12)

The inverse Fourier transform of this correlator is:

\[
\langle j_\mu(r) j_\nu(0) \rangle = -\langle \partial^2 \delta_{\mu\nu} - \partial_\mu \partial_\nu \rangle \frac{1}{32\pi^2 r^2}.
\]

(13)

where \(r_{\mu} = (r^\prime, x, y)\). The fluctuation of the charge \(\langle Q^2(\tau) \rangle\) that flowed through the cut \(C\) of length \(L\) during time \(\tau\) is then given by the integral over the segment \(\Omega = \{r: 0 < r^\prime < \tau, \quad x = 0, \quad 0 < y < L\}\) of the hyperplane \((\tau, y)\). Explicitly, one has:

\[
\langle Q^2(\tau) \rangle = \int_\Omega dr_1 \int_\Omega dr_2 \langle j_x(r_1) j_x(r_2) \rangle,
\]

(14)

Substituting Eq. (3) into Eq. (14) and using Gauss theorem twice, one obtains:

\[
\langle Q^2(\tau) \rangle = \frac{1}{32\pi^2} \int \int \int dS(r_1) dS(r_2) \frac{1}{(r_1 - r_2)^2}.
\]

(15)

Here, \(dS = \partial S\) is the “surface” element of the boundary \(\partial S\), while \(dl\) is the length element of the corresponding perimeter. The integral in Eq. (15) diverges as \(r_1 \rightarrow r_2\). This means that the exponent in the final result acquires an explicit cut-off dependence. A suitable way to introduce a short time cut-off is to replace:

\[
\frac{1}{(r_1 - r_2)^2} \rightarrow \frac{1}{(r_1 - r_2)^2 + a^2}.
\]

(16)

The cut-off \(a\) is, roughly, the inverse of the energy at which the linear Dirac dispersion breaks down. It would be justified to assume \(a \sim 1 / \Delta\). The integral in Eq. (15) can now be easily evaluated. Plugging the result in Eq. (8) in the case where \(L, \tau \gg \Delta^{-1}\) one obtains:

\[
A \sim \Delta (L + \tau).\n\]

(17)

This can be called a perimeter law for vortex action: the contribution of nodal fermions to the imaginary time action of a half-quantum vortex is proportional to the length of the vortex trajectory in the Euclidean space-time. The proportionality of the action to the perimeter of \(\Omega\) rather than its surface area is essentially due to the conservation of the number of \(\psi\)-particles. In terms of the original electrons, it is the conservation of the \(z\)-component of spin polarization.

Let us now discuss the implication of these results for the vortex dynamics in \(d\)-wave superconductors. First of all, Eq. (17) shows that the orthogonality catastrophe for a half-quantum vortex is very strong. Even when the length \(L\) is finite (e.g. when the vortex moves over a small distance) the dynamical overlap of the two states vanishes in the limit \(\tau \rightarrow \infty\). Certainly, finite temperature provides a cut-off: \(\tau_{\text{th}} \sim 1 / T\). This leads to the conclusion that the contribution of the nodal fermions to the imaginary time action at low temperatures is dominated by the second term in Eq. (17): \(A \sim \Delta / T\). This, in turn, means that the dynamical magnetization relaxation rate \(Q(T) = 1 / S\), where \(S\) is the total tunneling action, vanishes linearly with temperature in the limit \(T \rightarrow 0\). Indeed, the diverging contribution \(A\) to the action \(S\) becomes the dominant one in the same limit. The fact that experiments show a finite \(Q(0)\) leads us to conclude that there is some other low-energy cut-off besides temperature. We speculate that it may appear as...
a consequence of a mini-gap in the Dirac spectrum of nodal quasiparticles. Such a mini-gap is expected in finite magnetic field as a result of curvature terms in the dispersion around the nodal point. Its magnitude has been estimated [10] to be $\delta = \kappa H$, $\kappa \approx 0.5 K \text{Tesla}^{-1}$, where $H$ is the external magnetic field. Taking the mini-gap into account, we expect the following behavior of the magnetization relaxation rate at low temperatures:

$$Q(T) \sim \begin{cases} T/\Delta, & T \gg \delta \\ \delta/\Delta, & T \ll \delta \end{cases}$$ (18)

This behavior is shown in Fig. 2. Qualitatively, the following picture emerges: There is a small energy scale $\delta$ set by the size of the mini-gap in the Dirac spectrum of nodal fermions. This energy scale defines a cross-over temperature between two regimes: a) For $T \gg \delta$ the function $Q(T)$ is linear in $T$ with the coefficient $\sim 1/\Delta$ which is little dependent on the details; b) For $T \ll \delta$ the function $Q(T)$ saturates at a value $\sim \delta/\Delta$. Experimentally, [2] in the regime $T \gg \delta$ typical slopes of $Q(T)$ are $10^{-2} K^{-1}$ which is consistent with the typical value $\Delta \sim 10^3 K$ for the Cuprates, although the linearity in $T$ is difficult to establish. Typical values of the cross-over temperature are $\delta \sim 1 K$ (i.e. for magnetic fields $\sim 1 T$; in agreement with the theoretical estimate [10]). The arguments above then predict $Q(0) \sim 10^{-2}$, also in agreement with experiments. Note that in this simple picture we have omitted disorder effects on the spectrum of nodal quasiparticles, inter-vortex interactions etc. In particular, the Zeeman splitting is expected [10] to change the simple $\delta = \kappa H$ behavior.

As an end note, we would like to point out one interesting consequence of the obtained results. As we have seen, the orthogonality catastrophe strongly inhibits the motion of half-quantum vortices, whereas the effect disappears for doubly-quantized vortices. This suggests a possible tendency of vortices to pair which would increase their mobility and, effectively, decrease their energy. This scenario is similar to the one used in recent studies of fractionalized phases [20] and deserves further investigation.

In summary, we have found that the interaction between magnetic vortices and nodal quasiparticles in $d$-wave superconductors leads to a strong orthogonality catastrophe in response to the motion of half-quantum vortices. This effect strongly suppresses their mobility and is argued to give the dominant contribution to the vortex action at low temperatures. Based on these results, we have developed a simple description of the quantum creep of vortices in $d$-wave superconductors. We would like to thank A. Paramekanti, A. Ludwig and M. P. A. Fisher for discussions and criticism.

[1] G. Blatter et al., Rev. Mod. Phys. 66, 1125 (1994).
[2] Y. Yeshurun, A. P. Malozemoff, and A. Shaulov, Rev. Mod. Phys. 68, 911 (1996).
[3] P. Ao and D. J. Thouless, Phys. Rev. Lett. 70, 2158 (1993); D. J. Thouless, P. Ao and Q. Niu, Phys. Rev. Lett. 76, 3758 (1996); D. J. Thouless et al., cond-mat/9709127.
[4] G. Blatter, V. B. Geshkenbein, and V. M. Vinokur, Phys. Rev. Lett. 66, 3297 (1991).
[5] M. V. Feigel’man et al., Pis’ma Zh. Eksp. Teor. Fiz. 57, 699 (1993) [Sov. Phys. JETP Lett. 57, 711 (1993)].
[6] G.-H. Kim and M. Shin, Physica C 303, 73 (1998); Phys. Rev. B 66, 064515 (2002); A. Melikidze, Phys. Rev. B 64, 024515 (2001).
[7] C. Caroli, P. G. De Gennes and J. Matricon, Phys. Lett. 9, 307 (1964).
[8] N. B. Kopnin and V. E Kravtsov, Pis’ma Zh. Eksp. Teor. Fiz. 23, 631 (1976) [JETP Lett. 23, 578 (1976)]; Zh. Eksp. Teor. Fiz. 71, 1644 (1976) [Sov. Phys. JETP 44, 861 (1976)].
[9] See: M. Franz, Z. Tešanović, Phys. Rev. Lett. 80, 4763 (1998) and references therein.
[10] A. Vishwanath, Phys. Rev. B 66, 064504 (2002).
[11] O. Vafek, A. Melikyan, and Z. Tešanović, Phys. Rev. B 64, 224508 (2001).
[12] Y. Zhang et al., Phys. Rev. Lett. 86, 890 (2001).
[13] A. C. Durst, A. Vishwanath, and P. A. Lee, cond-mat/0206092 (unpublished).
[14] A. S. Melnikov, Phys. Rev. Lett. 86, 4108 (2001).
[15] P. W. Anderson, Phys. Rev. Lett 18, 1049 (1967); P. Nozières and C. T. De Dominicis, Phys. Rev. 178, 1097 (1969).
[16] C. R. Cassanello and E. Fradkin, Phys. Rev. B 56, 11246 (1997).
[17] B. Thaller, The Dirac Equation (Springer-Verlag, Berlin 1992).
[18] L. S. Levitov, H. Lee, and G. B. Lesovik, J. Math. Phys. 37, 4845 (1996).
[19] M. Franz, Z. Tešanović, and O. Vafek, Phys. Rev. B 66, 054535 (2002).
[20] A. Paramekanti, L. Balents, and M. P. A. Fisher, Phys. Rev. B 66, 054526 (2002).