AN $O(n)$ ALGORITHM FOR GENERATING UNIFORM RANDOM VECTORS IN $n$ DIMENSIONAL CONES

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Abstract. Unbiased random vectors i.e. distributed uniformly in $n$-dimensional space, are widely applied and the computational cost of generating a vector increases only linearly with $n$. On the other hand, generating uniformly distributed random vectors in its subspaces typically involves the inefficiency of rejecting vectors falling outside, or re-weighting a non-uniformly distributed set of samples. Both approaches become severely ineffective as $n$ increases. We present an efficient algorithm to generate uniform random directions in $n$-dimensional cones, to aid searching and sampling tasks in high dimensions.

The problem of generating unbiased random vectors appears widely, and as described in subsection 4.1, has a relatively trivial solution that scales as $O(n)$ arithmetic operations where $n$ is the dimension of the space [7]. This problem can be reduced to an accumulation of random points distributed uniformly on the surface of the unit sphere. This description using the surface of the unit sphere allows us to effectively use elementary geometrical concepts in describing the proposed algorithms, for uniformly sampling the region of interest given by a part of the unit sphere.

We may simply generate samples that are uniformly distributed on the entire unit sphere, but only accept those that are within a region. This gives us our desired uniform distribution within the region of interest, and the number of rejections is determined by the fraction of the surface (solid angle) of the sphere that we wish to sample. For a region on the unit sphere bound using planar angles between the position vectors, one observes that the fraction of the total solid angle represented by this region rapidly decreases with the dimension. This makes rejection sampling prohibitively expensive in high dimensions and we demonstrate this quantitatively in section 3. We may also re-weight a non-uniformly distributed set of samples if the probability distribution is known. But, the re-weighting errors are known to increase significantly in higher dimensions[3, 8] and we demonstrate this with examples in Appendix A.

Note that many such naive methods are not effective for large $n$ as they do not generate the required uniform distribution, or do so at a prohibitive cost. While other preferred methods such as Markov-Chain-Monte-Carlo (MCMC) can be significantly more efficient than the above naive methods in generating uniformly distributed points in an arbitrary volume, they may nevertheless scale as poorly as $O(n^5)$ in the required computing effort [4, 5, 6]. $O(n^3)$ and $O(n^2)$ methods that use linear transformations for uniformly sampling certain regularly shaped surfaces are described in the literature [2]. In this work, we present an $O(n)$ method that uses a non-linear transformation to generate random points uniformly distributed on a section of the surface of the sphere.

1. Problem statement. It is required to generate random points uniformly distributed on a fraction $\frac{\Omega_0}{s_n}$ of the total solid angle $s_n$ of the unit sphere. In two dimensions, this corresponds to generating random points uniformly on an arc of the unit circle. In general, for $n$ dimensions, this corresponds to generating unit vectors in an $n$-dimensional cone with a spherical cap at its base and its apex at the center of the unit sphere of reference; we denote the central axis of the cone from the apex

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to the centre of the base as $\hat{\mu}$ (see Figure 1.1). In other words, the desired cone is the set of all unit vectors that fall within a planar angle $\theta$ of the central axis $\hat{\mu}$.

If $\Omega$ is the set of all points given by such unit vectors, then this set is given by

$$\{\hat{x} \in \Omega; \hat{x} \cdot \hat{\mu} \geq \cos \theta\}$$

where $0 \leq \theta \leq \pi$. Note that the proposed solution also extends to vectors contained in a hollow cone bound by two planar angles.

2. Map from planar angle to solid angle. If $\Theta : \mathbb{R} \rightarrow \mathbb{R}$ is the function mapping planar cross sectional angle $\theta$ to the rotated solid angle in $n$ dimensions, and $\phi, \theta_1, \theta_2, \ldots, \theta_{n-2}$ are the $n-1$ angles of the spherical coordinate system, then $\Theta (\theta)$ is given by the following integral where all angles except $\theta_{n-2}$ are integrated over their full range, whereas $\theta_{n-2}$ is integrated over $[0, \theta]$.

$$\Theta (\theta) = \int_0^{\theta} \int_0^{\pi} \cdots \int_0^{\pi} \sin^{n-2} \theta_{n-2} \sin^{n-3} \theta_{n-3} \cdots \sin^2 \theta_2 \sin \theta_1 \, d\theta_{n-2} \, d\theta_{n-3} \cdots d\theta_2 \, d\theta_1 \, d\phi$$

Separating out the multiple integral into a product of one dimensional integrals over each angle,

$$\Theta (\theta) = \int_0^{\theta} \sin^{n-2} \theta_{n-2} \, d\theta_{n-2} \int_0^{\pi} \sin^{n-3} \theta_{n-3} \, d\theta_{n-3} \cdots \int_0^{\pi} \sin^2 \theta_2 \, d\theta_2 \int_0^{\pi} \sin \theta_1 \, d\theta_1 \int_0^{2\pi} \, d\phi$$

All but the first and last one of these integrals are Wallis’ integrals $W_m$ with $m = 1, 2, 3, \ldots, n-3$.

$$\int_0^{\pi} \sin^m x \, dx = 2 \int_0^{\frac{\pi}{2}} \sin^m x \, dx = 2W_m$$

Substituting $u = \sin^2 x$, we obtain the following relation between the Wallis’ integral and the beta function.

$$W_m = \int_0^{\frac{\pi}{2}} \sin^m x \, dx = \frac{1}{2} B \left( \frac{m+1}{2}, \frac{1}{2} \right)$$

Thus,

$$\int_0^{\pi} \sin^m x \, dx = B \left( \frac{m+1}{2}, \frac{1}{2} \right)$$
Substituting this integral back into $\Theta (\theta)$,

$$\Theta (\theta) = 2\pi B \left( \frac{n - 2}{2}, \frac{1}{2} \right) \cdots B \left( \frac{3}{2}, \frac{1}{2} \right) B \left( \frac{1}{2}, \frac{1}{2} \right) \int_0^\theta \sin^{n-2}\theta_n \, d\theta_n$$

Expanding the beta function in terms of the gamma function $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, the product of beta functions telescopically cancels out to give

$$\Theta (\theta) = \frac{2\pi^{n-1}}{\Gamma \left( \frac{n-1}{2} \right)} \int_0^\theta \sin^{n-2}\theta_n \, d\theta_n$$

We can relate this integral to the incomplete Beta function. In order to do so, we need to handle it as two separate cases—one when $\theta \in \left[ 0, \frac{\pi}{2} \right]$ and another when $\theta \in \left[ \frac{\pi}{2}, \pi \right]$.

2.1. Case (i): $\theta \in \left[ 0, \frac{\pi}{2} \right]$. For $\theta \in \left[ 0, \frac{\pi}{2} \right]$, the integral is an incomplete form of the Wallis’ integral, which we denote by $W_m(\theta)$.

$$W_m(\theta) = \int_0^\theta \sin^m x \, dx$$

Substituting $u = \sin^2 x$, we obtain the following relation between the incomplete Wallis’ integral and the incomplete Beta function.

$$W_m(\theta) = \frac{1}{2} B \left( \frac{m+1}{2}, \frac{1}{2} \right) \quad \text{for} \quad \theta \in \left[ 0, \frac{\pi}{2} \right]$$

2.2. Case (ii): $\theta \in \left[ \frac{\pi}{2}, \pi \right]$. When $\theta \in \left[ \frac{\pi}{2}, \pi \right]$, we split the integral at $\theta = \frac{\pi}{2}$.

$$W_m(\theta) = \int_0^{\frac{\pi}{2}} \sin^m x \, dx + \int_{\frac{\pi}{2}}^\theta \sin^m x \, dx$$

The first integral is the complete Wallis’ integral $W_m$.

$$W_m(\theta) = W_m + \int_{\frac{\pi}{2}}^\theta \sin^m x \, dx$$

Due to the symmetry of the integrand $\sin^m x$ about $\theta = \frac{\pi}{2}$,

$$\int_{\frac{\pi}{2}}^\theta \sin^m x \, dx = \int_{\pi - \theta}^{\frac{\pi}{2}} \sin^m x \, dx = W_m - W_m(\pi - \theta)$$

Thus,

$$W_m(\theta) = 2W_m - W_m(\pi - \theta)$$

$$W_m(\theta) = B \left( \frac{m+1}{2}, \frac{1}{2} \right) - \frac{1}{2} B \left( \sin^2 \theta; \frac{m+1}{2}, \frac{1}{2} \right) \quad \text{for} \quad \theta \in \left[ \frac{\pi}{2}, \pi \right]$$

Thus,

$$\int_0^\theta \sin^m x \, dx = \begin{cases} \frac{1}{2} B \left( \sin^2 \theta; \frac{m+1}{2}, \frac{1}{2} \right) & \theta \in \left[ 0, \frac{\pi}{2} \right] \\ B \left( \frac{m+1}{2}, \frac{1}{2} \right) - \frac{1}{2} B \left( \sin^2 \theta; \frac{m+1}{2}, \frac{1}{2} \right) & \theta \in \left[ \frac{\pi}{2}, \pi \right] \end{cases}$$

And, the complete expression for $\Theta (\theta)$ is

$$\Theta (\theta) = \frac{2\pi^{n-1}}{\Gamma \left( \frac{n-1}{2} \right)} \begin{cases} \frac{1}{2} B \left( \sin^2 \theta; \frac{n-1}{2}, \frac{1}{2} \right) & \theta \in \left[ 0, \frac{\pi}{2} \right] \\ B \left( \frac{n-1}{2}, \frac{1}{2} \right) - \frac{1}{2} B \left( \sin^2 \theta; \frac{n-1}{2}, \frac{1}{2} \right) & \theta \in \left[ \frac{\pi}{2}, \pi \right] \end{cases}$$
3. Cost of rejection sampling. The number of samples required to produce one accepted sample follows a geometric distribution with the probability of acceptance given by

\[ p = \frac{\Theta (\theta)}{s_n} = \frac{\Theta (\theta)}{\Theta (\pi)} \]  

From (2.7) and (2.16),

\[ p = \frac{1}{B \left( \frac{n-1}{2}, \frac{1}{2} \right)} \int_0^\theta \sin^{n-2} \theta \, d\theta_{n-2} \]

Thus, the average number of samples required to produce one accepted sample is

\[ \frac{1}{p} = \frac{s_n}{\Theta (\theta)} = \frac{B \left( \frac{n-1}{2}, \frac{1}{2} \right)}{\int_0^\theta \sin^{n-2} \theta \, d\theta_{n-2}} \]

This average number of samples, for various planar angles \( \theta \), is shown plotted against dimension \( n \) in Figure 3.1. As can be seen from the figure, the average number of samples required increases exponentially with dimension. We justify this further using the following analytical approximation. For a small planar angle \( \theta \), \( \theta_{n-2} \) only takes on small values, and hence \( \sin \theta_{n-2} \approx \theta_{n-2} \). Therefore,

\[ \frac{1}{p} \approx \frac{B \left( \frac{n-1}{2}, \frac{1}{2} \right)}{\theta^{n-1}} \]

Integrating and applying the limits,

\[ \frac{1}{p} \approx \frac{(n-1)B \left( \frac{n-1}{2}, \frac{1}{2} \right)}{\theta^{n-1}} \]

For large \( n \), we use Stirling’s approximation for the beta function

\[ B \left( \frac{n-1}{2}, \frac{1}{2} \right) \approx \sqrt{\frac{2\pi e}{n-1}} \]

to get

\[ \frac{1}{p} \approx \frac{\sqrt{2\pi e(n-1)}}{\theta^{n-1}} \]

Thus, for small planar angles \( \theta \), the average number of samples required to produce one accepted sample increases exponentially with dimension \( n \). Therefore, rejection sampling is prohibitively expensive especially in high dimensions. Note that while this approximation applies only to small \( \theta \), the cost of rejection sampling rises exponentially even for large \( \theta \). This follows from (3.3) and is observed in Figure 3.1.

4. Proposed sampling method. As a prerequisite, we will first look at generating points distributed inside a sphere according to some specified distribution. Then, we will describe the special case of generating points uniformly distributed on the spherical cap of an \( n \)-dimensional cone aligned along the \( n \)-th canonical axis. Finally, we will describe a rotation to reorient the random points on the spherical cap to the desired direction \( \hat{\mu} \).
4.1. Generating random points in a sphere. We can generate points uniformly distributed on the surface of the unit sphere using the Box-Muller transform \([1]\).

If \(Z_1, Z_2, \ldots, Z_n\) are standard normal random variable and \(\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n\) are canonical basis vectors, then the vector \(\hat{s}\) uniformly distributed on the surface of the sphere is given by

\[
\hat{s} = \frac{Z_1\hat{e}_1 + Z_2\hat{e}_2 + Z_3\hat{e}_3 + \cdots + Z_n\hat{e}_n}{\sqrt{Z_1^2 + Z_2^2 + Z_3^2 + \cdots + Z_n^2}}
\]

To generate points \(\vec{x}\) distributed according to some distribution in a sphere, we scale \(\hat{s}\) uniformly distributed on the surface of the sphere by some random variable \(R\) representing the magnitude in the radial direction.

\[
\vec{x} = R\hat{s}
\]

For example, to obtain points uniformly distributed in a sphere of radius \(r_0\), the cumulative distribution function of \(R\) should be the fraction of the sphere enclosed upto that radius. In other words, larger radii should be more probable since most of the volume is on the periphery of the sphere.

\[
F_R(r) = \left(\frac{r}{r_0}\right)^n
\]

Using the universality of the uniform random variable (also known as the probability integral transform), if \(U\) is the standard uniform random variable distributed uniformly between 0 and 1, then

\[
F_R(R) = U
\]

\[
\left(\frac{R}{r_0}\right)^n = U
\]
It is required to generate random points that are uniformly distributed on the spherical cap. The spherical cap subtends a solid angle $\Omega_0$ at the center of the sphere and is aligned along the $n$-th canonical axis $\hat{e}_n$. We generate points $P$ that are non-uniformly distributed on the disk such that their projections $Q$ on to the sphere are uniformly distributed on the spherical cap. To achieve this, $R$ has to be generated randomly as follows.

\begin{equation}
R = r_0 U_n^\frac{1}{n}
\end{equation}

Thus, to generate points uniformly distributed in a sphere, we use

\begin{equation}
\tilde{x} = r_0 U \frac{Z_1 \hat{e}_1 + Z_2 \hat{e}_2 + Z_3 \hat{e}_3 + \cdots + Z_n \hat{e}_n}{\sqrt{Z_1^2 + Z_2^2 + Z_3^2 + \cdots + Z_n^2}}
\end{equation}

Likewise, any isometric distribution of points inside a sphere can be obtained if the appropriate cumulative distribution function $F_R$ is known.

**4.2. Generating points along the $n$-th canonical axis.** It is required to generate random points that are uniformly distributed on the spherical cap of Figure 4.1. The spherical cap subtends a solid angle $\Omega_0$ at the center of the sphere and is aligned along the $n$-th canonical axis $\hat{e}_n$. We generate points $P$ that are non-uniformly distributed on the disk such that their projections $Q$ on to the sphere are uniformly distributed on the spherical cap. The disk is a sphere in $n - 1$ dimensions. So, given the distribution for $R$, we may sample it as described in subsection 4.1.

The cumulative distribution function of $R$ should be proportional to the solid angular width $\Theta(\theta)$ it corresponds to. Hence,

\begin{equation}
F_R(r) = \frac{\Theta(\theta)}{\Omega_0}
\end{equation}

Using the universality of the uniform random variable (also known as the probability integral transform), if $U$ is the standard uniform random variable distributed uniformly between 0 and 1, then

\begin{equation}
U = F_R(R) = \frac{1}{\Omega_0} \Theta \left( \sin^{-1} \left( \frac{R}{\sqrt{R^2 + \cos^2 \theta_0}} \right) \right)
\end{equation}
Solving for $R$, we get an expression for it in terms of $U$.

\[(4.10) \quad R = \cos \theta_0 \tan \left( \Theta^{-1} (U \Omega_0) \right) \]

θ₀ is the cross sectional planar angle subtended by the spherical cap at the center of the sphere.

\[(4.11) \quad \Omega_0 = \Theta(\theta_0) \]

In the case where $\Omega_0$ is given and $\theta_0$ is not, we obtain the required $\Theta^{-1}$ from $\Theta$ using the bisection root finding method. Any other suitable root finding method could also have been used. $\hat{x}$, the uniformly distributed random points in a disk are easily generated using (4.7), and hence the required non-uniform distribution of random points $R\hat{x}$ in the disk can be evaluated as well. The vectors from the origin of reference in the unit sphere to these points in the $n-1$ disk, when normalized, provide the required vectors $\hat{OQ}$ as illustrated in Figure 4.1. This $O(n)$ arithmetic procedure for random vectors in a cone along the $n$-th canonical axis $\hat{e}_n$, can be extended to an arbitrary direction $\hat{\mu}$ as shown in the next section.

### 4.3. Generating points along an arbitrary direction

We have generated points along the $n$-th canonical axis $\hat{e}_n$, but it is required to generate points along a given arbitrary direction $\hat{\mu}$. To do this, we simply rotate the vectors to align along $\hat{\mu}$. More precisely, we rotate vectors $\hat{x}$ by the angle between $\hat{e}_n$ and $\hat{\mu}$ with the plane containing $\hat{e}_n$ and $\hat{\mu}$ being the plane of rotation. If $P$ is an orthonormal matrix whose columns form a basis for the plane containing $\hat{e}_n$ and $\hat{\mu}$, and $G$ is the two dimensional Given’s rotation matrix for the required rotation, the rotated vector $\hat{y}$ is

\[(4.12) \quad \hat{y} = \hat{x} + PGP^T \hat{x} - PP^T \hat{x} \]

Rewriting using the $2 \times 2$ identity matrix $I_2$,

\[(4.13) \quad \hat{y} = \hat{x} + P(G - I_2)P^T \hat{x} \]

$P$, $G$ and $\mu_n$ are given by

\[(4.14) \quad P = \begin{bmatrix} \hat{e}_n & \hat{\mu} - \mu_n \hat{e}_n \\ \frac{\hat{\mu} - \mu_n \hat{e}_n}{\|\hat{\mu} - \mu_n \hat{e}_n\|} & 1 \end{bmatrix} \]

\[(4.15) \quad G = \begin{bmatrix} \mu_n & -\sqrt{1-\mu_n^2} \\ +\sqrt{1-\mu_n^2} & \mu_n \end{bmatrix} \]

\[(4.16) \quad \mu_n = \hat{e}_n^T \hat{\mu} \]

A general $n$-dimensional rotation of vectors costs $O(n^2)$ operations. But, the above is a simple rotation and hence costs only $O(n)$ operations. Thus, we have generated random points aligned along an arbitrary direction without an increase in the order of the arithmetic complexity.

The overall algorithm discussed in subsections 4.1 to 4.3 is shown in Algorithm 4.1.

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1A simple rotation is a rotation with only one plane of rotation.
Algorithm 4.1: Generate random points uniformly distributed on a spherical cap

procedure GENERATE POINT ON SPHERICAL CAP($\bar{\mu}, \theta_0$)
\begin{align*}
\Omega_0 & \leftarrow \Theta(\theta_0) \\
U & \leftarrow \text{random number uniformly distributed between 0 and } \Omega_0 \\
R & \leftarrow \cos \theta_0 \tan \left(\Theta^{-1}(U)\right) \\
& \text{for } i = 1 : n - 1 \text{ do} \\
& \quad x[i] \leftarrow \text{normally distributed random number with } (0, 1) \text{ as (mean, variance)} \\
& \text{end for} \\
& h \leftarrow \|\bar{x}\| \\
& \text{for } i = 1 : n - 1 \text{ do} \\
& \quad x[i] \leftarrow \frac{Rx[i]}{h} \\
& \text{end for} \\
& x[n] \leftarrow \cos \theta_0 \\
& \bar{x} \leftarrow \frac{x}{\|x\|} \\
& \text{for } i = 1 : n - 1 \text{ do} \\
& \quad P[i][1] \leftarrow 0 \\
& \text{end for} \\
& P[n][1] \leftarrow 1 \\
& \text{for } i = 1 : n - 1 \text{ do} \\
& \quad P[i][2] \leftarrow \frac{\bar{\mu}[i]}{\sqrt{1 - \{\bar{\mu}[n]\}^2}} \\
& \text{end for} \\
& P[n][2] \leftarrow 0 \\
& G \leftarrow \begin{bmatrix} \mu[n] & -\sqrt{1 - \{\mu[n]\}^2} \\ +\sqrt{1 - \{\mu[n]\}^2} & \mu[n] \end{bmatrix} \\
& I_2 \leftarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
& \bar{y} \leftarrow \bar{x} + P(G - I_2)P^T\bar{x} \\
& \text{return } \bar{y} \\
\end{align*}
end procedure

4.4. Generating random vectors uniformly distributed in a hollow cone.

The proposed method can be generalized easily to generate points on the surface of the unit sphere formed by the difference of two cones corresponding to solid angles $\Omega_1$ and $\Omega_2$ where $\Omega_1 < \Omega_2$. Such a hollow cone is shown in Figure 4.2. If $\Omega$ is the set of all points given by such unit vectors, it is given by

$$\{\hat{x} \in \Omega; \cos \theta_1 \geq \hat{x} \cdot \hat{\mu} \geq \cos \theta_2\} \quad (4.17)$$

If $U$ is the standard uniform random variable distributed uniformly between 0 and 1, the required random variable $R$ for such a distribution of points is

$$R = \cos \theta_2 \tan \left(\Theta^{-1} \{U(\Omega_2 - \Omega_1) + \Omega_1\}\right) \quad (4.18)$$

$\theta_1$ and $\theta_2$ are the cross sectional planar angles corresponding to solid angles $\Omega_1$ and $\Omega_2$.

$$\Omega_1 = \Theta(\theta_1) \quad (4.19)$$
Fig. 4.2. It is required to generate points uniformly distributed on the spherical cap of the hollow cone $\Delta$ along $\hat{\mu}$. The cross sectional planar angles $\theta_1$ and $\theta_2$ corresponding to the solid angles $\Omega_1$ and $\Omega_2$ are indicated.

$$\Omega_2 = \Theta(\theta_2)$$

5. Numerical Results. The uniform distribution of the generated random points on the spherical cap is verified by comparing the distribution of $\theta$, the angle between the generated direction vector and the reference direction $\hat{\mu}$, with the exact analytically known distribution.

An empirical probability density of $\theta$ for a dimension $n = 10$ and $\theta_0 = \frac{\pi}{4}$ is shown in Figure 5.1 with the exact probability density function overlaid. The empirical probability density was constructed using a histogram of 100 bins and 10000 direction vector samples. $\theta$ is the angle between the generated direction vector and the reference direction $\hat{\mu}$.

$$f(\theta) = \begin{cases} \frac{8n-1}{n(n-1)} \sin^{n-2} \theta & 0 \leq \theta \leq \theta_0 \\ 0 & \theta > \theta_0 \end{cases}$$
The Kolmogorov-Smirnov statistic comparing the empirical cumulative distribution function of $\theta$ and the exact cumulative distribution is plotted against an increasing number of samples. The Kolmogorov-Smirnov statistic decreases with an increasing number of samples indicating convergence of the empirical distribution with the exact distribution. $\theta$ is the angle between the generated direction vector and the reference direction $\hat{\mu}$. The maximum $\theta$ is given by the cone angle $\theta_0 = \frac{\pi}{4}$.

Fig. 5.2. The Kolmogorov-Smirnov statistic comparing the empirical cumulative distribution function of $\theta$ and the exact cumulative distribution is plotted against an increasing number of samples in Figure 5.2. The Kolmogorov-Smirnov statistic decreases with an increasing number of samples indicating convergence of the empirical distribution with the exact distribution. The exact cumulative distribution function is

$$F(\theta) = \begin{cases} \frac{\Theta(\theta)}{\Theta(\theta_0)} & 0 \leq \theta \leq \theta_0 \\ 1 & \theta > \theta_0 \end{cases}$$ (5.2)

If $\theta_1, \theta_2, \ldots, \theta_N$ are $N$ samples and $\mathbb{1}_{(-\infty, \theta]}$ is the indicator function of $(-\infty, \theta]$, then the empirical cumulative distribution function is

$$F_N(\theta) = \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{(-\infty, \theta]}(\theta_i)$$ (5.3)

The Kolmogorov-Smirnov statistic $D_N$ is the supremum of the absolute difference between the empirical cumulative distribution function and the exact cumulative distribution function.

$$D_N = \sup_{\theta} |F_N(\theta) - F(\theta)|$$ (5.4)

Appendix A. A note on generating samples by re-weighting non-uniform distributions.

One could sample the desired region of interest using a non-uniform distribution, where the samples are re-weighted for uniformity. Below we present a few straightforward methods to generate the samples efficiently in a region of interest using $\mathcal{O}(n)$
We generate random points uniformly distributed in the smaller sphere and normalize the position vectors to get direction vectors that lie on the surface of the larger sphere. The smaller sphere is shifted from the origin by $\hat{\mu}$. The generated direction vectors lie within the cone of half-angle $\theta_0$.

When the non-uniform distribution of these samples is known, they can be re-weighted to achieve a semblance of uniform distribution in the region. However this approach produces poor results compared to the proposed method with the same order of arithmetic complexity.

### A.1. Shifting generated points to the regions of interest and re-weighting.

It is inexpensive to generate random points uniformly distributed in a sphere. So, we bound our desired region of the unit sphere in a second sphere, generate points inside that sphere and normalize the position vectors of the generated points to get direction vectors that lie on the surface of the first sphere. This geometry is shown graphically in Figure A.1. If $\vec{S}$ is a vector random variable distributed uniformly in the unit sphere centered at the origin, then the generated direction vector $\hat{x}$ is

$$\vec{x} = r_0 \vec{S} + \hat{\mu} \quad (A.1)$$

The radius $r_0$ depends on $\theta_0$ as follows.

$$r_0 = \|\hat{\mu}\| \sin \theta_0 \quad (A.2)$$

The generated direction vector at $\hat{x}$ is

$$\hat{x} = \frac{\vec{x}}{\|\vec{x}\|} \quad (A.3)$$

The probability density of the generated direction vector at $\hat{x}$ is

$$f(\hat{x}) = A (r_1^n - r_2^n) \quad (A.4)$$

where $A$ is the normalization constant. $r_1$ and $r_2$ are

$$r_1, r_2 = (\hat{x} \cdot \hat{\mu}) \pm \sqrt{(\hat{x} \cdot \hat{\mu})^2 - \|\hat{\mu}\|^2 + r_0^2} \quad (A.5)$$

Note that the choice of bounding the desired region in a sphere is arbitrary. We could also have bound the desired region in a cube or any other convenient shape.
The Kolmogorov-Smirnov statistic of the re-weighted distribution of shifted sphere random vectors for $\theta$ is compared with that of the proposed method for a dimension $n = 10$. $\theta$ is the angle between the generated direction vector and the reference direction $\hat{\mu}$. The maximum $\theta$ is given by the cone angle $\theta_0 = \frac{\pi}{4}$, and $|\hat{\mu}| = 1.0$. The larger value of $n=100$ is not shown here, as the re-weighted samples do not exhibit convergence.

A comparison of this method with the proposed method using the Kolmogorov-Smirnov statistic of $\theta$ is shown in Figure A.2. $\theta$ is the angle between the generated direction vector and the reference direction $\hat{\mu}$. If $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_N$ are $N$ samples and $\mathbb{I}_{(-\infty, \theta]}$ is the indicator function of $(-\infty, \theta]$, then the empirical cumulative distribution function of $\theta$ after re-weighting is given by

$$F_N(\theta) = \frac{1}{\sum_{i=1}^{N} f(\hat{x}_i)} \sum_{i=1}^{N} \mathbb{I}_{(-\infty, \theta]} \left( \cos^{-1} \{ \hat{x}_i \cdot \hat{\mu} \} \right)$$

(A.6)

A.2. Re-weighting a shifted multivariate normal distribution. If the probability density function of a non-uniformly distributed random vector generator is known, the non-uniformly distributed samples can be re-weighted to obtain a uniformly distributed random vector in the limit of a large number of samples. However, error introduced due to re-weighting can hinder performance. We demonstrate this here with a multivariate normal distribution.

Generate vectors $\hat{x}$ that are distributed as the multivariate normal distribution $\mathcal{N}(\mu, \sigma^2 I_n)$ with mean $\mu$ and covariance $\sigma^2 I_n$, where $I_n$ is the $n \times n$ identity matrix and $\hat{\mu}$ is a vector of arbitrary magnitude in the desired direction $\hat{\mu}$. Normalize $\hat{x}$ to get the direction vector $\hat{\mu}$. The probability density at $\hat{x}$ is given by

$$f(\hat{x}) = \frac{\phi \left( \frac{1}{\sigma} \sqrt{\|\hat{\mu}\|^2 - (\hat{x} \cdot \hat{\mu})^2} \right)}{\sigma^n (2\pi)^{-1}} \int_0^\infty r^{n-1} \phi \left( \frac{r - \hat{x} \cdot \hat{\mu}}{\sigma} \right) dr$$

(A.7)

where $\phi$ is the probability density of the standard univariate normal distribution.

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

(A.8)
The Kolmogorov-Smirnov statistic of a re-weighted multivariate normal distribution for \( \theta \) is compared with that of the proposed method for a dimension \( n = 100 \). \( \theta \) is the angle between the generated direction vector and the reference direction \( \hat{\mu} \). The maximum \( \theta \) is given by the cone angle \( \theta_0 = \frac{\pi}{4} \), and \( \|\hat{\mu}\| = 1.0 \). For the re-weighted normal distributions, the fraction of accepted samples was 0.9824 and 0.0981 for \( \sigma = 0.08 \) and \( \sigma = 0.12 \) respectively. Note the larger Kolmogorov-Smirnov statistic for the re-weighted normal distribution especially when \( \sigma \) is small.

If \( \hat{x} \) does not fall within the desired spherical cap, reject and repeat. The probability density at \( \hat{x} \) after rejection is proportional to the probability density of (A.7). The value of \( \sigma \) should be chosen to minimize the rejection, but that makes the distribution more non-uniform resulting in larger re-weighting errors. A comparison of this method with the proposed method using the Kolmogorov-Smirnov statistic is shown in Figure A.3.

REFERENCES

[1] G. E. P. BOX AND M. E. MULLER, A note on the generation of random normal deviates, Annals of Mathematical Statistics, 29 (1958), pp. 610–611, https://doi.org/10.1214/aoms/1177706645.
[2] L. DEVROYE, Nonuniform random variate generation, Handbooks in Operations Research and Management Science, 13 (2006), pp. 83–121.
[3] I. DIAKONIKOLAS, G. KAMATH, D. KANE, J. LI, A. MOITRA, AND A. STEWART, Robust estimators in high-dimensions without the computational intractability, SIAM Journal on Computing, 48 (2019), pp. 742–864.
[4] M. E. DYER AND A. M. FRIEZE, On the complexity of computing the volume of a polyhedron, SIAM Journal on Computing, 17 (1988), pp. 967–974.
[5] R. KANNAN, L. LOVÁSZ, AND M. SIMONOVITS, Random walks and an \( o^*(n^5) \) volume algorithm for convex bodies, Random Structures & Algorithms, 11 (1997), pp. 1–50.
[6] L. LOVÁSZ AND S. VEMPALA, Simulated annealing in convex bodies and an \( o^*(n^{4/3}) \) volume algorithm, in Foundations of Computer Science, 2003. Proceedings. 44th Annual IEEE Symposium on, IEEE, 2003, pp. 650–659.
[7] C. J. MODE, Random number generators and simulation (Istvan Deak), SIAM Review, 34 (1992), pp. 338–341.
[8] T. STROHMER, Numerical analysis of the non-uniform sampling problem, Journal of Computational and Applied Mathematics, 122 (2000), pp. 297–316.