A note on odd coloring 1-planar graphs

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Abstract

A proper coloring of a graph is odd if every non-isolated vertex has some color that appears an odd number of times on its neighborhood. This notion was recently introduced by Petruševski and Škrekovski, who proved that every planar graph admits an odd 9-coloring; they also conjectured that every planar graph admits an odd 5-coloring. Shortly after, this conjecture was confirmed for planar graphs of girth at least seven by Cranston; outerplanar graphs by Caro, Petruševski and Škrekovski. Building on the work of Caro, Petruševski and Škrekovski, Petr and Portier then further proved that every planar graph admits an odd 8-coloring. In this note we prove that every 1-planar graph admits an odd 47-coloring, where a graph is 1-planar if it can be drawn in the plane so that each edge is crossed by at most one other edge.

Keywords: proper coloring, odd coloring, 1-planar graph.

1 Introduction

All graphs considered in this paper are simple, finite and undirected. Let $G$ be a graph. We use $|G|$ to denote the number of vertices in $G$. A proper coloring of $G$ is odd if every non-isolated vertex has some color that appears an odd number of times on its neighborhood. The odd chromatic number of $G$, denoted $\chi_o(G)$, is the smallest number $k$ such that $G$ admits an odd $k$-coloring. Clearly, $\chi_o(G) \leq |G|$, since we can simply color each vertex with its own color. The notion of an odd coloring was recently introduced by Petruševski and Škrekovski [9]. Using the discharging method, they showed that $\chi_o(G) \leq 9$ holds for all planar graphs $G$. Furthermore, they conjectured that all planar graphs admit an odd coloring using 5 colors. Fewer colors cannot suffice since $\chi_o(C_5) = 5$.

Conjecture 1 (Petruševski and Škrekovski [9]). Every planar graph admits an odd 5-coloring.

Shortly after, Cranston [4] focused on the odd chromatic number of sparse graphs, obtaining, for instance, $\chi_o(G) \leq 6$ for planar graphs $G$ of girth at least 6 and $\chi_o(G) \leq 5$ for planar graphs $G$ of girth at least 7. Caro, Petruševski and Škrekovski [3] studied various properties of the odd chromatic number, including the following facts: every outerplanar graph admits an odd 5-coloring;
every graph of maximum degree three has an odd 4-coloring; and for every connected planar graph $G$, if $|G|$ is even, or $|G|$ is odd and has a vertex of degree 2 or any odd degree, then $\chi_o(G) \leq 8$ (an important step towards proving that 8 colors suffice for an odd coloring of any planar graph).

It is worth noting that their proof of this important step relies on Theorem 4 in Ashtab, Akbari, Ghanbari and Shidani [1] which uses the Four-Color Theorem [2, 10]. Building on the work of Caro, Petruševski and Škrekovski [3], Petr and Portier [8] then further proved that 8 colors suffice for all planar graphs, that is, every planar graph admits an odd 8-coloring.

In this paper, we focus on studying odd colorings of 1-planar graphs, where a graph is 1-planar if it can be drawn in the plane so that each edge is crossed by at most one other edge. We prove the following main result.

**Theorem 1.** Every 1-planar graph admits an odd 47-coloring.

It seems non-trivial to deduce a more general result for $k$-planar graphs for all $k \geq 1$. It is known [7] that every 1-planar graph is 7-degenerate. It would be interesting to see if one can prove that every 1-planar graph admits an odd 13-coloring. One obstacle is that the class of 1-planar graphs is not closed under edge-contraction. Following our ideas of Theorem 1, one can prove that every 1-planar graph $G$ admits an odd 14-coloring if every proper minor of $G$ is 1-planar.

We need to introduce more notation. For a proper odd coloring $\tau$ of some subgraph $H$ of $G$ and for each $v \in V(H)$, let $\tau_o(v)$ denote the unique color that appears an odd number of times in $N_H(v)$ if such a color exists; otherwise, $\tau_o(v)$ is undefined. Define $\tau(N_H(v)) := \{\tau(u) \mid u \in N_H(v)\}$ and $\tau_o(N_H(v)) := \{\tau_o(u) \mid u \in N_H(v)\}$. Note that, in any process that involves extending an odd coloring of $H$ to $G$ one vertex at a time, the existence and identity of $\tau_o(v)$ may change each time a new vertex is colored. A $k$-vertex is a vertex of degree $k$. A $k^+$-vertex is a vertex of degree at least $k$. Let $G$ be a plane graph and let $F$ be a face of $G$. We use $d(F)$ to denote the size of $F$; in addition, $F$ is a $k$-face of $G$ if $d(F) = k$ and a a $k^+$-face if $d(F) \geq k$. For any positive integer $k$, we define $[k] := \{1, 2, \ldots, k\}$.

We end this section with some easy results about graphs with forbidden minors. Let $\mathcal{F}$ denote a family of graphs that are closed under taking minors, that is, if $G \in \mathcal{F}$, then every minor of $G$ also belongs to $\mathcal{F}$. We say $\mathcal{F}$ is $d$-degenerate for some integer $d \geq 0$ if every graph in $\mathcal{F}$ is $d$-degenerate. As observed in [9], for every $n \in \mathbb{N}$, there exists a 2-degenerate graph $G$ with $\chi_o(G) = n$. However, for minor-closed families, we prove the following.

**Theorem 2.** If $\mathcal{F}$ is $d$-degenerate, then every graph in $\mathcal{F}$ admits an odd $(2d + 1)$-coloring.

**Proof.** Suppose the statement is false. Let $G \in \mathcal{F}$ be a graph with $\chi_o(G) \geq 2d + 2$. We choose $G$ with $|G|$ minimum. Then $G$ is connected and $\delta(G) \leq d$. Let $x \in V(G)$ with $d(x) = \delta(G)$. Let $H := G/xy$ and $w$ be the new vertex in $H$. Then $H \in \mathcal{F}$. By the minimality of $G$, let $\tau : V(H) \rightarrow [2d + 1]$ be an odd coloring of $H$. We extend $\tau$ to $G$ by coloring $y$ with the color $\tau(w)$ and $x$ with a color in $[2d + 1] \setminus (\tau(N(x)) \cup \tau_o(N(x)))$, forbidding at most $2d$ colors. It is simple
to check that $\tau$ is an odd $(2d + 1)$-coloring of $G$ because $\tau(y)$ appears exactly once on $N(x)$, a contradiction.

Theorem 3 below follows directly from Theorem 2 and the fact that every graph with no $K_4$ minor is 2-degenerate; it extends the result of Caro, Petruševski and Škrekovskii \[3\] on outerplanar graphs. The sharpness of Theorem 3 is witnessed by $C_5$.

**Theorem 3.** Every graph with no $K_4$ minor admits an odd 5-coloring.

For each integer $p$ with $4 \leq p \leq 9$, it is known \[5, 6, 11\] that every graph with no $K_p$ minor is $(2p - 5)$-degenerate, and so has an odd $(4t - 9)$-coloring by Theorem 2.

2 Proof of Theorem 1

Suppose the statement is false. Let $G$ be a minimal counter-example. Then $G$ is connected and $\chi_o(G) \geq 48$. We may assume that $G$ is a 1-plane graph with as few edge crossings as possible. Throughout the proof, we use $N(v)$ and $d(v)$ to denote the neighborhood and degree of a vertex $v$ in $G$, respectively. We next prove several claims.

Claim 1. $G$ is 2-edge-connected and so $\delta(G) \geq 2$.

**Proof.** Suppose to the contrary that $xy \in E(G)$ is a bridge in $G$. Let $G_1, G_2$ be the components of $G \setminus xy$ such that $x \in V(G_1)$ and $y \in V(G_2)$. By the choice of $G$, let $\tau_1 : V(G_1) \to [47]$ be an odd coloring of $G_1$ such that $\tau_1(x) = 1$ and color 2 appears an odd number of times on $N(x) \setminus y$ if $x$ is a $2^+$-vertex; let $\tau_2 : V(G_2) \to [47]$ be an odd coloring of $G_2$ such that $\tau_2(y) = 3$ and color 4 appears an odd number of times on $N(y) \setminus x$ if $y$ is a $2^+$-vertex. It is simple to see that we can combine $\tau_1$ and $\tau_2$ into an odd 47-coloring of $G$, a contradiction. Thus $G$ is 2-edge-connected and so $\delta(G) \geq 2$. \[\square\]

Claim 2. Every odd vertex in $G$ has degree at least 25.

**Proof.** Suppose not, and let $v \in V(G)$ be an odd vertex with degree at most 23. Then $G \setminus v$ has an odd 47-coloring $\tau$ by the minimality of $G$. But then $\tau$ can be extended to an odd 47-coloring of $G$ by assigning to $v$ a color that does not occur in $\tau(N(v)) \cup \tau_o(N(v))$, since there are at most 46 such forbidden colors. (Note that some color is guaranteed to occur an odd number of times on $N(v)$, since $d(v)$ is odd.) \[\square\]

For the rest of the proof, a vertex $v$ in $G$ is big if $v$ is a $24^+$-vertex, and small otherwise.

Claim 3. No two small vertices are adjacent in $G$.

**Proof.** Suppose not, and let $v$ and $w$ be two adjacent small vertices in $G$. By Claim 2, $d(v) \leq 22$ and $d(w) \leq 22$. Let $\tau$ be an odd 47-coloring of $H := G \setminus \{v, w\}$. Then $|\tau(N_H(x)) \cup \tau_o(N_H(x))| \leq 42$ for each $x \in \{v, w\}$. But then $\tau$ can be extended to an odd 47-coloring of $G$ as follows: first assign
\[ v \text{ a color that does not belong to } \{ \tau_o(w) \} \cup \tau(N_H(v)) \cup \tau_o(N_H(v)); \text{ then assign } w \text{ a color that does not belong to } \{ \tau(v), \tau_o(v) \} \cup \tau(N_H(w)) \cup \tau_o(N_H(w)). \]

Claim 4. Every edge incident to a small vertex in \( G \) has a crossing.

Proof. Suppose to the contrary that some edge \( xy \in E(G) \) has no crossing. Let \( G' \) be the graph obtained from \( G \) by first deleting the edge \( xy \) and then adding an edge \( yz \) for each \( z \in N(x) \setminus N(y) \); since the edge \( xy \) had no crossing, we can guarantee that each new edge \( yz \) has a crossing if and only if \( xz \) has a crossing. Thus \( G' \) is 1-planar. By the minimality of \( G \), let \( \tau \) be an odd 47-coloring of \( G' \). We can then extend \( \tau \) to be an odd 47-coloring of \( G \) by coloring \( x \) with any color that does not occur in \( \tau(N(v)) \cup \tau_o(N(v)) \); there are at most 44 such forbidden colors, and \( \tau(y) \) is a color that occurs exactly once on \( N(x) \), a contradiction.

Now, we let \( H \) be the plane graph obtained from \( G \) by replacing each crossing with a “virtual” 4-vertex. Then \( H \) is 2-edge-connected by Claim 1. Note that no two virtual 4-vertices are adjacent in \( H \). Moreover, by Claim 4, \( N_H(v) \) consists entirely of virtual 4-vertices for each small vertex \( v \) in \( G \).

Claim 5. Every 3-face in \( H \) is incident to either three big vertices or two big vertices and one virtual 4-vertex.

Proof. Let \( v \) be a small vertex in \( G \). As observed earlier, every edge incident to \( v \) has a crossing by Claim 4 which implies that every neighbor of \( v \) in \( H \) is a virtual 4-vertex. Thus all the neighbors of \( v \) are pairwise non-adjacent in \( H \) and so \( v \) is not incident to any 3-face in \( H \). It follows that each 3-face in \( H \) is incident to either three big vertices or two big vertices and one virtual 4-vertex.

Claim 6. Every 2-vertex in \( H \) is incident to a 5\(^+\)-face.

Proof. Let \( x \) be a 2-vertex in \( H \) and let \( y, z \) be the neighbors of \( x \). Then both \( y, z \) are virtual 4-vertices by Claim 4. Recall that \( H \) is 2-edge-connected. Let \( F_1 \) and \( F_2 \) be the two distinct faces that contain the vertex \( x \). Then \( x, y, z \in V(F_1) \cap V(F_2) \). Since \( yz \notin E(H) \), we see that both \( F_1 \) and \( F_2 \) are 4\(^+\)-faces. Suppose both \( F_1 \) and \( F_2 \) are 4-faces. Let \( u, w \) be the remaining vertex of \( F_1 \) and \( F_2 \), respectively. Then each vertex in \( \{ u, w \} \) is adjacent to both \( y, z \) in \( H \). But this implies that there is a multiple edge between \( u \) and \( w \) in \( G \), a contradiction.

We shall apply the discharging method to \( H \) obtain a contradiction. Let \( \mathcal{F} \) denote the set of all faces of \( H \). For each vertex \( v \in V(H) \) and each face \( F \in \mathcal{F} \), let \( \omega(v) := d_H(v) - 4 \) be the initial charge of \( v \) and \( \omega(F) := d(F) - 4 \) be the initial charge of \( F \). Then the total charge of \( H \) is
\[
\omega(H) := \sum_{v \in V(H)} (d(v) - 4) + \sum_{F \in \mathcal{F}} (d(F) - 4) \\
= -4(|H| - e(H) + |\mathcal{F}|) \\
= -8.
\]

Note that \( \omega(v) < 0 \) if and only if \( d_H(v) = 2 \); \( \omega(F) < 0 \) if and only if \( F \) is a 3-face. We will redistribute the charges of \( H \) according to the following discharging rules:

(R1) Every \( 5^+ \)-face sends \( \frac{2}{3} \) to each incident 2-vertex.

(R2) Every big vertex sends \( \frac{1}{2} \) to each incident 3-face.

(R3) Every big vertex \( v \) sends \( \frac{2}{3} \) to each 2-vertex in \( N_G(v) \).

Let \( \omega^* \) be the new charge of \( H \) after applying the above discharging rules. Then \( \omega^*(H) = \omega(H) = -8 \). We obtain a contradiction by showing that \( \omega^*(v) \geq 0 \) and \( \omega^*(F) \geq 0 \) for each vertex \( v \in V(H) \) and each face \( F \in \mathcal{F} \).

**Claim 7.** If \( F \) is a \( 5^+ \)-face, then \( \omega^*(F) \geq 0 \).

**Proof.** Let \( F \in \mathcal{F} \) be a \( 5^+ \)-face. By Claim 4, every 2-vertex in \( H \) is adjacent only to virtual 4-vertices. Recall that no two virtual 4-vertices are adjacent in \( H \). It follows that \( F \) is incident to at most \( \lfloor d(F)/2 \rfloor \) 2-vertices; moreover, if \( F \) is a 5-face, then \( F \) is incident to at most one 2-vertex.

By rule (R1), \( \omega^*(F) \geq \omega(F) - \frac{2}{3} = \frac{1}{3} \) when \( F \) is a 5-face, and

\[
\omega^*(F) \geq \omega(F) - \left\lfloor \frac{d(F)}{2} \right\rfloor \cdot \frac{2}{3} \geq \frac{2d(F)}{3} - 4 \geq 0
\]

when \( F \) is a \( 6^+ \)-face. \( \square \)

![Figure 1: The worst-case scenario when \( d(v) = 8 \) or 9, where vertices in red are 2-vertices.](image)

**Claim 8.** If \( v \) is a big vertex, then \( \omega^*(v) \geq 0 \).
Proof. Let \( v \) be a big vertex. To determine the worst-case scenario, we note that each edge incident to \( v \) in \( G \) can contribute to the charge sent out by \( v \) in \( H \) by either crossing over another edge to reach a 2-vertex (thereby requiring a charge of \( \frac{2}{3} \) to be sent out) or by creating up to two 3-faces (thereby requiring a charge of up to 1 to be sent out). Since \( v \) can only be incident to \( d(v) \) faces, at most \( \left\lfloor \frac{d(v)}{2} \right\rfloor \) edges incident to \( v \) can create two new 3-faces; thereafter, the amount of charge sent out is maximized by having every other edge incident to \( v \) in \( G \) be incident to a 2-vertex. (See Figure 1 for what the worst-case scenario would be if we were to consider a vertex \( v \) such that \( d(v) = 8 \) or 9.) Thus, in the worst-case scenario, \( v \) is incident to

\[
2 \left\lfloor \frac{d(v)}{2} \right\rfloor
\]

3-faces and adjacent to

\[
d(v) - \left\lfloor \frac{d(v)}{2} \right\rfloor = \left\lceil \frac{d(v)}{2} \right\rceil
\]

virtual 4-vertices which are adjacent to 2-vertices. Therefore,

\[
\omega^*(v) \geq \omega(v) - \frac{1}{2} \cdot 2 \cdot \left\lfloor \frac{d(v)}{2} \right\rfloor - \frac{2}{3} \cdot \left\lceil \frac{d(v)}{2} \right\rceil \geq \frac{d(v)}{2} - \frac{24}{6} \geq 0,
\]

since \( d(v) \geq 24 \).

It remains to check that \( \omega^*(F) \geq 0 \) for each 3-face \( F \) and \( \omega^*(v) \geq 0 \) for each 2-vertex \( v \). Since each 3-face \( F \) is incident to at least two big vertices, by rule (R2), \( \omega^*(F) = \omega(F) + \frac{1}{2} + \frac{1}{2} \geq 0 \). Note that each 2-vertex \( v \) is incident to a \( 5^+ \)-face by Claim 6 in addition, each neighbor of \( v \) in \( G \) is a big vertex by Claim 3. By rules (R1) and (R3), \( \omega^*(F) = \omega(F) + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} \geq 0 \). It follows that

\[-8 = \omega(H) = \omega^*(H) \geq 0, \]

a contradiction. \( \square \)

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