Long Nonbinary Codes Exceeding the Gilbert-Varshamov bound for Any Fixed Distance

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Abstract—Let $A(q, n, d)$ denote the maximum size of a $q$-ary code of length $n$ and distance $d$. We study the asymptotic redundancy $\rho(q, n, d)$ as $n$ grows while $q$ and $d$ are fixed. For any $d$ and $q \geq d - 1$, long algebraic codes are designed that improve on the BCH codes and have the lowest asymptotic redundancy

$$\rho(q, n, d) \lesssim \frac{n - \log_q A(q, n, d)}{\log_q n}$$

known to date. Prior to this work, codes of fixed distance that asymptotically surpass BCH codes and the Gilbert-Varshamov bound were designed only for distances 4, 5, and 6.

Index Terms—affine lines, BCH code, Bezout’s theorem, norm.

I. INTRODUCTION

Let $A(q, n, d)$ denote the maximum size of a $q$-ary code of length $n$ and distance $d$. We study the asymptotic size $A(q, n, d)$ if $q$ and $d$ are fixed as $n \to \infty$, and introduce a related quantity

$$c(q, d) = \lim_{n \to \infty} \frac{n - \log_q A(q, n, d)}{\log_q n},$$

which we call the redundancy coefficient. The Hamming upper bound

$$A(q, n, d) \leq q^n \sum_{i=0}^{[(d-1)/2]} (q-1)^i \binom{n}{i}$$

leads to the lower bound

$$c(q, d) \geq [(d-1)/2],$$

which is the best bound on $c(q, d)$ known to date for arbitrary values of $q$ and $d$. On the other hand, the Varshamov existence bound admits any linear $[n, k, q^d]$ code of dimension

$$k \leq n - 1 - \left\lfloor \log_q \sum_{i=0}^{d-2} (q-1)^i \binom{n-1}{i} \right\rfloor.$$

This leads to the redundancy coefficient

$$c(q, d) \leq d - 2.$$  

(Note that the Gilbert bound results in a weaker inequality $c(q, d) \leq d - 1$.)

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Let $e$ be a primitive element of the Galois field $F_q$. Consider (see [20]) the narrow-sense BCH code defined by the generator polynomial with zeros $e^1, ..., e^{q-2}$. Let $C^m_q(d)$ denote the extended BCH code obtained by adding the overall parity check. Code $C^m_q(d)$ has length $q^m$, constructive distance $d$, and redundancy coefficient

$$c(q, d) \leq \left\lceil \frac{(d - 2)(q - 1)}{q} \right\rceil.$$

Note that the above BCH bound [3] is better than the Varshamov bound [4] for $q < d - 1$ and coincides with [4] for $q \geq d - 1$. Note also that [3] meets the Hamming bound [1] if $q = 2$ or $d = 3$. Therefore

$$c(2, d) = \lfloor (d - 1)/2 \rfloor$$

and $c(q, 3) = 1$.

For distances 4, 5, and 6, infinite families of nonbinary linear codes are constructed in [5] and [6] that reduce asymptotic redundancy [3]. Open Problem 2 from [6] also raises the question if the BCH bound [4] can be improved for larger values of $d$. Our main result is an algebraic construction of codes that gives an affirmative answer to this problem for all $q \geq d - 1$. In terms of redundancy, the new bound is expressed by

**Theorem 1**: For all $q$ and $d \geq 3$,

$$c(q, d) \leq (d - 3) + 1/(d - 2).$$

Combining [3] and [4], we obtain

$$c(q, d) \leq \min \left( \frac{(d - 2)(q - 1)}{q}, (d - 3) + \frac{1}{(d - 2)} \right).$$

Note that the above bound is better than the Varshamov existence bound for arbitrary $q$ and $d \geq 4$.

The rest of the paper is organized as follows. In Section II, we review the upper bounds for $c(q, d)$ that surpass the BCH bound [4] for small values of $d$. In Section III, we present our code construction and prove the new bound [4]. This proof rests on important Theorem [4] which is proven in Section IV. Finally, we make some concluding remarks in Section V.

II. PREVIOUS WORK

Prior to this work, codes that asymptotically exceed the BCH bound [4] were known only for $d \leq 6$. We start with the bounds for $c(q, 4)$. Linear $[n, n - \rho, 4]$ codes are equivalent to caps in projective geometries $PG(\rho - 1, q)$ and have been studied extensively under this name. See [18] for a review. However, the exact values of $c(q, 4)$ remain unknown for all $q \geq 3$, and the gaps between the upper and the lower bounds are still large.
The Hamming bound yields $c(q, 4) \leq 1$. Mukhopadhyay [22] obtained the upper bound $c(q, 4) \leq 1.5$. For all values of $q$, this was later improved by Edel and Bierbrauer [7] to

$$c(q, 4) \leq \frac{6}{\log_q (q^4 + q^2 - 1)}.$$  

(5)

Note that for large values of $q$ the right hand side of (5) tends to 1.5. The case of $q = 3$ has been of special interest, and general bound (5) has been improved in a few papers (see [17], [11], [2], [8]). The current record

$$c(3, 4) \leq 1.3796$$

due to Edel [8] slightly improves on the previous record $c(3, 4) \leq 1.3855$ obtained by Calderbank and Fishburn [2]. For $q = 4$, the construction of [14] also improves (5). Namely, $c(4, 4) \leq 1.45$.

Now we proceed to the bounds for $c(q, 5)$. The Hamming bound yields $c(q, 5) \geq 2$. Several families of linear codes constructed in [6] reach the bound

$$c(q, 5) \leq 7/3$$

(6)

for all values of $q$. Later, alternative constructions of codes with the same asymptotic redundancy were considered in [9]. Similarly to the case of $d = 4$, there exist better bounds for small alphabets. Namely, Goppa pointed out that ternary double error-correcting BCH codes asymptotically meet the Hamming bound (11). For $q = 4$ and $d = 5$, two different constructions that asymptotically meet the Hamming bound were proposed in [12] and [4]. Thus,

$$c(3, 5) = c(4, 5) = 2.$$  

For $d = 6$, the infinite families of linear codes designed in [5] and [6] reach the upper bound

$$c(q, 6) \leq 3$$

(7)

for all $q$. The constructions are rather complex and the resulting linear codes are not cyclic. Later, a simpler construction of a cyclic code with the same asymptotic redundancy was proposed in [3]. Again, better bounds exist for small values of $q$. Namely, $c(3, 6) \leq 2.5$ [6] and $c(4, 6) \leq 17/6$ [10].

We summarize the bounds described so far in Figure 1.

The following Lemma 2 due to Gevorkyan [13] shows that redundancy $c(q, d)$ cannot increase when the alphabet size is reduced.

**Lemma 2:** For arbitrary value of distance $d,$

$$q_1 \leq q_2 \Rightarrow c(q_1, d) \leq c(q_2, d).$$

**Proof:** Given a code $V$ of length $n$ over the $q_2$-ary alphabet we prove the existence of a code $V'$ of the same length over $q_1$-ary alphabet with the same redundancy coefficient. Let $q_2$-ary alphabet be an additive group $E_{q_2},$ and $q_1$-ary alphabet form a subset $E_{q_1} \subseteq E_{q_2}.$ Define the componentwise shift $V_v = V + v$ of code $V$ by an arbitrary vector $v \in E_{q_2}.$ Note that any vector $f \in E_{q_1}^n$ belongs to exactly $|V|/q_2^n$ codes among all $q_2^n$ codes $V_v,$ as $v$ runs through $E_{q_2}^n.$ Hence, codes $V_v$ include on average $q_1^n |V|/q_2^n$ vectors of the subset $E_{q_1} \subseteq E_{q_2}.$ Therefore, some set $V_v \cap E_{q_1}^n$ has at least this average size. Denote this set by $V'.$ Clearly, $V'$ is a $q_1$-ary code with the same distance as code $V.$ It remains to note that

$$n - \log_{q_1} (q_1^n |V|/q_2^n) = n - \log_{q_2} |V| = \log_{q_1} n - \log_{q_2} n.$$  

The proof is completed.

**Corollary 3:** Let $\{q_i\}$ be an infinite sequence of growing alphabet sizes. Assume there exist $c^*$ and $d$ such that for all $i,$ $c(q_i, d) \leq c^*.$ Then $c(q, d) \leq c^*$ for all values of $q.$

**Proof:** This follows trivially from Lemma 2.

**III. CODE CONSTRUCTION**

In the sequel, the elements of the field $F_q$ are denoted by Greek letters, while the elements of extension fields $F_{q^m}$ are denoted by Latin letters.

We start with an extended BCH code $C = C_q^m(d - 1)$ of length $n = q^m$ and constructive distance $d - 1.$ Here for any position $j \in [1, q^m], $ we define its locator $e_j,$ where $e_j = e^j$ for $j < n$ and $e_n = 0.$ Then the parity check matrix of code $C$ has the form

$$H_q^m(d - 1) =\begin{pmatrix} 1 & \ldots & 1 & 1 \\ e_1 & \ldots & e_{n-1} & 0 \\ \vdots & \ldots & \vdots & \vdots \\ e_1^{d-3} & \ldots & e_n^{d-3} & 0 \end{pmatrix}.$$  

(8)

Here the powers of locators $e_j$ are represented with respect to some basis of $F_{q^m}$ over $F_q.$ Note that the redundancy of $C$ is at most $(d - 3)m + 1.$ Also, we assume in the sequel that $q$ does not divide $d - 2$, since code $C$ has constructive distance $d$ instead of $d - 1$ otherwise.

Consider any nonzero codeword $c \in C$ of weight $w$ with nonzero symbols in positions $j_1, \ldots, j_w.$ Let $X(c) = \{x_1, \ldots, x_w\}$ denote its locator set, where we use notation $x_i = e_{j_i}$ for all $i = 1, \ldots, w.$ We say that $X(c)$ lies on an affine line $L(a, b)$ over $F_q$ if there exist $a, b \in F_{q^m}$ such that

$$x_i = a + \lambda_i b$$

(9)
where $\lambda_i \in F_q$ for all values of $i = 1, \ldots, w$.

The key observation underlining our code construction is that under some restrictions on extension $m$ and characteristic char$F_q$ of the field $F_q$, any code vector $c \in C$ of weight $d - 1$ has its locator set $X(c)$ lying on some affine line. Formally, this is expressed by

**Theorem 4:** Let $m$ be a prime, $m > (d - 3)!$ and char$F_q > d - 3$. Consider the extended BCH code $C'_m(d-1)$ of constructive distance $d - 1$. Then any codeword $c$ of minimum weight $d - 1$ has its locator set $X(c)$ lying on some affine line $L(a, b)$ over $F_q$.

We defer the proof of Theorem 4 till section IV and proceed with the code construction. Let

$$s = \lfloor m/(d-2) \rfloor, \quad \mu = s(d-2).$$

Consider the field $F_{q^\mu}$ and its subfield $F_{q^\mu}$. Let $g = \{g_1, \ldots, g_\mu\}$ be the basis of $F_{q^\mu}$ over $F_q$ such that $F_{q^\mu}$ is spanned by $\{g_1, \ldots, g_\mu\}$. Let $h = \{h_1, \ldots, h_m\}$ be an arbitrary basis of $F_{q^m}$ over $F_q$. Below we map each element $x = \sum_{i=1}^m \alpha_i h_i$ of the field $F_{q^m}$ onto the element

$$\hat{x} = \sum_{i=1}^m \alpha_i g_i$$

of the field $F_{q^\mu}$. It is readily seen that for arbitrary $a, b \in F_{q^m}$ and $\lambda \in F_q$

$$a + \lambda b = \hat{a} + \hat{\lambda},$$

Recall that the norm [15] of $\hat{x} \in F_{q^\mu}$

$$N_{F_{q^\mu}/F_{q^\mu}}(\hat{x}) = N_{d-2}(\hat{x}) = \hat{x}^{q^{(d-3)s} + \ldots + q^s + 1}$$

is a classical mapping from $F_{q^\mu}$ to $F_{q^\mu}$.

Now we are ready to present our code construction. Consider the $q$-ary code $C'(n, k', d')$ of length $n = q^m$ with the parity check matrix

$$\hat{H}^m_q = \begin{pmatrix} 1 & \ldots & 1 & 1 \\ e_1 & \ldots & e_{n-1} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ e_{d-3} & \ldots & e_{d-1} & 0 \\ N_{d-2}(\hat{e}_1) & \ldots & N_{d-2}(\hat{e}_{n-1}) & 0 \end{pmatrix}$$

where the locators $e_j$ and their powers are represented in $F_q$ with respect to the basis $h$ and values of $N_{d-2}$ are represented in $F_q$ with respect to $g$. Recall that $N_{d-2}(\hat{x})$ takes values in $F_{q^\mu}$. Therefore the redundancy of $C'$ does not exceed $(d - 3)m + s + 1$.

Below is the main theorem of the paper.

**Theorem 5:** Suppose $m > (d - 3)!$ is a prime, and char$F_q > d - 3$; then code $C'(n, k', d')$ defined by [13] has parameters

$$[q^m, k' \geq q^m - (d - 3)m - \lfloor m/(d-2) \rfloor - 1, d' \geq d_q].$$

**Proof:** Note that $d' \geq d - 1$, since $C'$ is a subcode of the extended BCH code $C$ defined in [5]. Let $C_{d-1} \subseteq C$ be the set of all codewords of weight exactly $d - 1$. It remains to prove that $C' \cap C_{d-1} = \emptyset$.

Assume the converse. Let $c \in C'$ be a codeword of weight $d - 1$ with locator set $X(c) = (x_1, \ldots, x_{d-1})$. This implies that for some nonzero symbols $\xi_1, \ldots, \xi_{d-1} \in F_q$:

$$\begin{cases} \xi_i x_i^t = 0, & t = 0, \ldots, d - 3; \\ \sum_{i=1}^{d-1} \xi_i N_{d-2}(\hat{x}_i) = 0. \end{cases}$$

Note that $c \in C_{d-1}$. Therefore according to Theorem 4 there exist $a, b \neq 0$ from $F_{q^\mu}$ and pairwise distinct $\{\lambda_i\} \in F_q$ such that $x_i = a + \lambda_i b$. Consider the affine permutation $\pi(x) = A + Bx$ of the entire locator set $F_{q^m}$, where $A = -ab^{-1}$ and $B = b^{-1}$. Clearly, $\pi$ maps each $x_i$ onto $\lambda_i$, i.e.

$$\lambda_i = A + Bx_i.$$

It is well known ([1], [20]) that the extended BCH code $C$ is invariant under any affine permutation of the locators, so that $\{\lambda_i\}$ is also a locator set in $C_{d-1}$. Indeed, for any $t \in [0, d-3]$, we have an equality

$$\sum_{i=1}^{d-1} \xi_i \lambda_i^{d-2t} = 0.$$

Indeed, we use [11] and [12] to obtain

$$N_{d-2}(A + B\hat{x}) = \left(\hat{a} + \hat{\lambda} \hat{b}\right)^{q^{(d-3)s} + \ldots + q^s + 1}$$

$$= \prod_{t=0}^{d-3} \left(\hat{a}\hat{q}^s + \hat{\lambda}\hat{b}\hat{q}^s\right)$$

$$= \sum_{t=0}^{d-2} C_t(\hat{a}, \hat{b}) \xi_i^t,$$

where $C_t$ are some polynomials in $\hat{a}$ and $\hat{b}$. Now the last equation in [14] can be rewritten as

$$\sum_{i=1}^{d-1} \xi_i C_t(\hat{a}, \hat{b}) \lambda_i^t = \sum_{i=0}^{d-2} C_t(\hat{a}, \hat{b}) \sum_{i=1}^{d-1} \xi_i \lambda_i^t = 0.$$
these \( d - 1 \) equations hold only if \( \xi_i = 0 \) simultaneously. Thus, our initial assumption that \( c \) has weight \( d - 1 \) does not hold. This completes the proof.

**Lemma 6**: Suppose \( \text{char} F_q > d - 3 \); then
\[
c(q, d) \leq (d - 3) + 1/(d - 2).
\]

**Proof**: We estimate the asymptotic redundancy of the family of codes presented in Theorem 5. Here \( q \) and \( d \) are fixed, while \( m > (d - 3)! \) runs to infinity over primes. Then
\[
c(q, d) \leq \lim_{m \to \infty} \frac{(d - 3)m + [m/(d - 2)] + 1}{m} = (d - 3) + 1/(d - 2).
\]

(18)
The proof is completed.

It is obvious that for every \( d \geq 3 \) there exists an infinite family \( \{q_i\} \) of growing alphabets such that \( \text{char} F_{q_i} > d - 3 \). Combining Lemma 6 with Corollary 5 we get Theorem 1.

The proof is completed.

To conclude, we would like to note that our construction of code \( C' \) (13) generalizes the construction of nonbinary double error-correcting codes from Theorem 7 in [6].

**IV. AFFINE LINES**

Before we proceed to the proof of Theorem 4 let us introduce some standard concepts and theorems of algebraic geometry. Let \( F \) be an algebraically closed field and \( r, t \) be two positive integers. Let \( f_1, \ldots, f_r \in F[x_1, \ldots, x_t] \). For any \( x = (a_1, \ldots, a_t) \in F^t \), the matrix
\[
J_x(f_1, \ldots, f_r) = \left( \begin{array}{ccc}
\frac{\partial f_1}{\partial x_1} |_x & \cdots & \frac{\partial f_1}{\partial x_t} |_x \\
\vdots & \ddots & \vdots \\
\frac{\partial f_r}{\partial x_1} |_x & \cdots & \frac{\partial f_r}{\partial x_t} |_x 
\end{array} \right)
\]
is called the Jacobian of functions \( f_i \) at point \( x \).

The set \( V \) of common roots to the system of equations
\[
\begin{align*}
f_1(x_1, \ldots, x_t) &= 0, \\
\vdots & \ \\
f_r(x_1, \ldots, x_t) &= 0,
\end{align*}
\]
is called an affine variety. The ideal \( I(V) \) is the set of all polynomials \( f \in F[x_1, \ldots, x_t] \) such that \( f(x) = 0 \) for all \( x \in V \). One important characteristic of a variety is its dimension \( \dim V \). Dimension of a non-empty variety is a non-negative integer. Let \( x = (a_1, \ldots, a_t) \in V \) be an arbitrary point on \( V \). The dimension of a variety \( V \) at a point \( x \), denoted \( \dim_x V \), is the maximum dimension of an irreducible component of \( V \) containing \( x \). A point \( x \in V \) such that \( \dim_x V = 0 \) is called an isolated point.

We shall need the following lemma ([19], p.166).

**Lemma 7**: Let \( V \) be an affine variety with the ideal \( I(V) \subset F[x_1, \ldots, x_t] \).

Then for any \( x = (a_1, \ldots, a_t) \in V \) and \( f_1, \ldots, f_r \in I(V) \)
\[
\text{rank } J_x(f) \leq t - \dim_x V.
\]

The next lemma is a corollary to the classical Bezout’s theorem ([16], p.53).

**Lemma 8**: Let \( V \) be an affine variety defined by \( \{f_i\} \). Then the number of isolated points on \( V \) does not exceed
\[
\prod_{i=1}^{r} \deg f_i.
\]

Let \( \xi_1, \ldots, \xi_{t+1} \) be fixed non-zero elements of some finite field \( F_q \). Consider a variety \( V \) in the algebraic closure of \( F_q \) defined by the following system of equations.
\[
\begin{align*}
\xi_1 x_1 + \cdots + \xi_t x_t + \xi_{t+1} &= 0, \\
\xi_1 x_1^2 + \cdots + \xi_t x_t^2 + \xi_{t+1} &= 0, \\
\vdots & \ \\
\xi_1 x_1^{t+1} + \cdots + \xi_t x_t^{t+1} + \xi_{t+1} &= 0.
\end{align*}
\]

(21)

Let \( x = (a_1, \ldots, a_t) \) be an arbitrary point on \( V \). We say that \( x \) is an interesting point if \( a_i \neq a_j \) for all \( i \neq j \).

**Lemma 9**: Let \( V \) be the variety defined by (21). Suppose \( \text{char} F_q > t \); then every interesting point on \( V \) is isolated.

**Proof**: Let \( x = (a_1, \ldots, a_t) \) be an arbitrary interesting point on \( V \). Let \( f_i(x_1, \ldots, x_t) \) denote the left hand side of the \( i \)-th equation of (21). Consider the Jacobian of \( \{f_i\} \) at point \( x \).

\[
J_x(f_1, \ldots, f_t) = \begin{pmatrix}
\xi_1 & \cdots & \xi_t \\
2\xi_1 a_1 & \cdots & 2\xi_t a_t \\
\vdots & \ddots & \vdots \\
(t-1)\xi_1 a_1^{t-1} & \cdots & t\xi_t a_t^{t-1}
\end{pmatrix}
\]

Thus we have
\[
\det J_x(f_1, \ldots, f_t) = t! \prod_{i=1}^{t} \xi_i a_1 \cdots a_t 
\]

Using standard properties of the Vandermonde determinant and the facts that \( \xi_i \) are non-zero and \( \text{char} F_q > t \), we get
\[
\text{rank } J_x(f_1, \ldots, f_t) = t.
\]

(22)

It is easy to see that \( f_1, \ldots, f_t \in I(V) \). Combining 22 with Lemma 7 we obtain \( \dim_x V = 0 \). The proof is completed.

**Lemma 10**: Let \( m \) be a prime \( m > t \). Assume \( \text{char} F_q > t \).

Let \( V \) be the variety defined by (21). Suppose \( x \in F_q^m \) is an interesting point on \( V \); then \( x \in F_q^t \). In other words, every interesting point on \( V \) that is rational over \( F_q \) is rational over \( F_q^m \).

**Proof**: Assume the converse. Let \( x = (a_1, \ldots, a_t) \) be an interesting point on \( V \) such that \( x \in F_q^m \setminus F_q^t \). Consider the following \( m \) conjugate points
\[
p_i = (a_1^i, \ldots, a_t^i), \text{ for all } 0 \leq i \leq m - 1.
\]

Each of the above points is interesting. Since \( m \) is a prime, the points are pairwise distinct. However, according to Lemma 9 every interesting point on \( V \) is isolated. Thus, we have \( m > t \) isolated point on \( V \). This contradicts Lemma 5.

**Remark 11**: Note that we can slightly weaken the condition of Lemma 10 replacing
\[
m \text{ prime and } m > t.
\]
with condition: \( \forall s \neq 1, s|m \) implies \( s > t! \).

Now we are ready to prove Theorem 4.

**Proof:** Assume \( C_{d-1} \) is nonempty (this fact will be proven later) and consider the locator set \( X(c) = \{x_1, \ldots, x_{d-1}\} \) for any \( c \in C_{d-1} \). Recall that \( X(c) \) satisfies the first \( d - 2 \) equations in (24) where \( \xi_i \neq 0 \) for all \( i \). Consider an affine permutation \( \pi(x) = a + bx \) of the locator set \( F_q^m \) of the code \( C \). Let \( a, b \neq 0 \) in \( F_q^m \) be such that

\[
\pi(x_{d-2}) = 1 \quad \text{and} \quad \pi(x_{d-1}) = 0.
\]

Let \( y_i \) denote \( \pi(x_i) \). Now we again use the fact that code \( C \) is invariant under affine permutations. Therefore the new locator set \( y(c) = \{y_1, \ldots, y_{d-3}, 1, 0\} \) satisfies similar equations

\[
\begin{align*}
\xi_1 + \cdots + \xi_{d-3} + \xi_{d-2} &= -\xi_{d-1}, \\
\xi_1 y_1 + \cdots + \xi_{d-3} y_{d-3} + \xi_{d-2} &= 0, \\
\xi_1 y_1^2 + \cdots + \xi_{d-3} y_{d-3}^2 + \xi_{d-2} &= 0, \\
&\vdots \\
\xi_1 y_1^{d-3} + \cdots + \xi_{d-3} y_{d-3}^{d-3} + \xi_{d-2} &= 0.
\end{align*}
\]

(24)

Now we remove the first equation (which does not include variables \( y_i \)) from (24), and obtain the system of equations, which is identical to system (21) for \( t = d - 3 \). Recall that \( x_1, \ldots, x_{d-1} \) are pairwise distinct elements of \( F_q^m \). Therefore \( y_1, \ldots, y_{d-3}, 1, 0 \) are also pairwise distinct. Thus \( y_1, \ldots, y_{d-3} \) is an interesting solution to the above system.

It is straightforward to verify that all the conditions of Lemma 10 hold. This yields

\[
y_i = a + bx_i = \lambda_i \in F_q, \quad \forall i \in [1, d - 1].
\]

Thus, we obtain all locators \( x_i \) on the affine line

\[
x_i = \frac{a}{b} + \frac{\lambda_i}{b}, \quad \lambda_i \in F_q.
\]

Finally, we prove that \( C_{d-1} \) is nonempty. Note that \( \text{char} F_q \geq d - 2 \). Also, recall that we consider codes \( C(n, d - 1) \) with constructive distance \( d - 1 \), in which case \( q \) does not divide \( d - 2 \). Thus, we now assume that \( q \geq d - 1 \). Then we consider the equations taking \( \xi_{d-1} = 1 \) and arbitrarily choosing \( d - 3 \) different locators \( y_1, \ldots, y_{d-3} \) from \( F_q \setminus \{0, 1\} \). Obviously, the resulting system of linear equations has nonzero solution \( \xi_1, \ldots, \xi_{d-2} \). This gives the codeword of weight \( d - 1 \) and completes the proof of Theorem 4. \( \square \)

**V. CONCLUSION**

We have constructed an infinite family of nonbinary codes that reduce the asymptotic redundancy of BCH codes for any given alphabet size \( q \) and distance \( d \) if \( q \geq d - 1 \). Families with such a property were earlier known only for distances 4, 5, and 6 [6]. Even the shortest codes in our family have very big length \( n \approx q^{(d-3)} \), therefore the construction is of theoretical interest.

The main question (i.e. the determination of the exact values of \( c(q, d) \)) remains open.

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