Sharp constants in weighted trace inequalities on Riemannian manifolds

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Abstract

We establish some sharp weighted trace inequalities $W^{1,2}(\rho^{1-2\sigma}, M) \hookrightarrow L^{\frac{2n}{n-2\sigma}}(\partial M)$ on $n+1$ dimensional compact smooth manifolds with smooth boundaries, where $\rho$ is a defining function of $M$ and $\sigma \in (0,1)$. This is stimulated by some recent work on fractional (conformal) Laplacians and related problems in conformal geometry, and also motivated by a conjecture of Aubin.

1 Introduction

Let $\Omega$ be an open set in $\mathbb{R}^n$, $n \geq 1$, and $\rho(x) = \text{dist}(x, \partial \Omega)$ for $x \in \Omega$. There have been much work devoted to the structures of weighted Sobolev spaces of the type $W^{k,p}(\rho^\alpha, \Omega)$ where $\alpha \in \mathbb{R}$, $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, as well as to their applications in different areas such as (stochastic) partial differential equations and Riemannian manifolds with fractal boundaries or boundary singularities. We refer to the book [36] of Maz’ya and references therein for these topics.

In this paper, we would like to study sharp constants in weighted trace type inequalities $W^{1,2}(\rho^{1-2\sigma}) \hookrightarrow L^{\frac{2n}{n-2\sigma}}(\partial M)$ on Riemannian manifolds $M$ with boundaries $\partial M$. Let us start from Euclidean spaces. Denote $\dot{H}^\sigma(\mathbb{R}^n)$ as the $\sigma$-order homogeneous Sobolev space on $\mathbb{R}^n$, $n \geq 2$, which is the closure of $C_0^\infty(\mathbb{R}^n)$ under the norm

$$\|f\|_{\dot{H}^\sigma(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |(-\Delta)^{\sigma/2} f(x)|^2 \, dx \right)^{1/2}.$$  

The sharp $\sigma$-order Sobolev inequality asserts that

$$\|f\|_{L^{\frac{2n}{n-2\sigma}}(\mathbb{R}^n)} \leq c(n, \sigma) \|f\|_{\dot{H}^\sigma(\mathbb{R}^n)}^2$$

1
for all $f \in \dot{H}^\sigma(\mathbb{R}^n)$, where
\[
c(n, \sigma) = 2^{-2\sigma\pi^{-\sigma}} \left( \frac{\Gamma((n - 2\sigma)/2)}{\Gamma((n + 2\sigma)/2)} \right) \left( \frac{\Gamma(n)}{\Gamma(n/2)} \right)^{\frac{2\sigma}{n}},
\]
and the equality holds if and only if $f(x)$ takes the form
\[
c \left( \frac{\lambda}{1 + \lambda^2 |x - x_0|^2} \right)^{\frac{n-2\sigma}{2}}
\]
for some $c \in \mathbb{R}$, $\lambda > 0$ and $x_0 \in \mathbb{R}^n$. These have been proved by Lieb in [34]. Set $x = (x', x_{n+1}) \in \mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0, \infty)$ and
\[
F(x', x_{n+1}) = \int_{\mathbb{R}^n} \mathcal{P}_\sigma(x' - \xi, x_{n+1}) f(\xi) \, d\xi,
\]
where
\[
\mathcal{P}_\sigma(x', x_{n+1}) = \beta(n, \sigma) \frac{x_{n+1}^{2\sigma}}{(|x'|^2 + x_{n+1}^2)^{\frac{n+2\sigma}{2}}}
\]
with the normalization constant $\beta(n, \sigma) > 0$ such that $\int_{\mathbb{R}^n} \mathcal{P}_\sigma(x', 1) \, dx' = 1$. Then one has (see, e.g., [9])
\[
N_\sigma \int_{\mathbb{R}^{n+1}_+} x_{n+1}^{1-2\sigma} |\nabla F(x', x_{n+1})|^2 \, dx = \|f\|^2_{\dot{H}^\sigma(\mathbb{R}^n)},
\]
where $N_\sigma = 2^{2\sigma-1} \Gamma(\sigma)/\Gamma(1 - \sigma)$. Hence, we have
\[
\|f\|^2_{L^{2\pi-2\sigma}(\mathbb{R}^n)} \leq S(n, \sigma) \int_{\mathbb{R}^{n+1}_+} x_{n+1}^{1-2\sigma} |\nabla F(x', x_{n+1})|^2 \, dx
\]
for all $f \in \dot{H}^\sigma(\mathbb{R}^n)$, where $S(n, \sigma) = N_\sigma \cdot c(n, \sigma)$. Consequently, one can show (see, e.g., Proposition 2.1 below together with a density argument) that
\[
\|U(\cdot, 0)\|^2_{L^{2\pi-2\sigma}(\mathbb{R}^n)} \leq S(n, \sigma) \int_{\mathbb{R}^{n+1}_+} x_{n+1}^{1-2\sigma} |\nabla U(x', x_{n+1})|^2 \, dx
\]
for all $U \in W^{1,2}(x_{n+1}^{1-2\sigma}, \mathbb{R}^{n+1}_+)$, which is the closure of $C^\infty_c(\mathbb{R}^{n+1}_+)$ under the norm
\[
\|U\|_{W^{1,2}(x_{n+1}^{1-2\sigma}, \mathbb{R}^{n+1}_+)} = \sqrt{\int_{\mathbb{R}^{n+1}_+} x_{n+1}^{1-2\sigma} (|U|^2 + |\nabla U|^2) \, dx}.
\]

Stimulated by several recent work on fractional (conformal) Laplacians and related problems in conformal geometry (see, e.g., [22, 10, 21, 26] and a conjecture of Aubin [2], we study
weighted Sobolev trace inequalities of type (3) on Riemannian manifolds with boundaries. For \( n \geq 2 \), let \((M, g)\) be an \( n + 1 \) dimensional, compact, smooth Riemannian manifold with smooth boundary \( \partial M \). We say a function \( \rho \in C^\infty(M) \) is a defining function of \( M \) if

\[
\rho > 0 \quad \text{in } M, \quad \rho = 0 \text{ and } \nabla_g \rho \neq 0 \quad \text{on } \partial M.
\]

Since \( \rho^{1-2\sigma} \), where \( \sigma \in (0, 1) \) is a constant, belongs to the Muckenhoupt \( A_2 \) class, we define the weighted Sobolev space \( H^1(\rho^{1-2\sigma}, M) \) as the closure of \( C^\infty(M) \) under the norm

\[
\| u \|_{H^1(\rho^{1-2\sigma}, M)} = \left( \int_M \rho^{1-2\sigma} (|u|^2 + |\nabla u|^2) \, dv_g \right)^{\frac{1}{2}},
\]

where \( dv_g \) denote the volume form of \((M, g)\). \( H^1(\rho^{1-2\sigma}, M) \) is a Hilbert space and it has a well-defined trace operator \( T \) (see, e.g., [36] or [39]) which continuously maps \( H^1(\rho^{1-2\sigma}, M) \) to \( H^\sigma(\partial M) \), where \( H^\sigma(\partial M) \) is the \( \sigma \)-order Sobolev space on \( \partial M \).

**Theorem 1.1.** For \( n \geq 2 \), let \((M, g)\) be an \( n + 1 \) dimensional, compact, smooth Riemannian manifold with smooth boundary \( \partial M \). Let \( \sigma \in (0, \frac{1}{2}) \), and \( \rho \) be a defining function of \( M \) satisfying \( |\nabla_g \rho| = 1 \) on \( \partial M \). Then there exists a positive constant \( A = A(M, g, n, \rho, \sigma) \) such that

\[
\left( \int_{\partial M} |u|^{\frac{2n}{n-2\sigma}} \, ds_g \right)^{\frac{n-2\sigma}{n}} \leq S(n, \sigma) \int_M \rho^{1-2\sigma} |\nabla_g u|^2 \, dv_g + A \int_{\partial M} u^2 \, ds_g, \tag{4}
\]

for all \( u \in H^1(\rho^{1-2\sigma}, M) \), where \( ds_g \) denotes the induced volume form on \( \partial M \).

For \( \sigma \in (\frac{1}{2}, 1) \), we have

**Theorem 1.2.** Let \( \sigma \in (\frac{1}{2}, 1) \), \( n \geq 4 \) and \((M, g)\) be an \( n + 1 \) dimensional, compact, smooth Riemannian manifold with smooth boundary \( \partial M \). Suppose in addition that \( \partial M \) is totally geodesic. Let \( \rho \) be a defining function of \( M \) satisfying \( \rho(x) = d(x) + O(d(x)^3) \) as \( d(x) \to 0 \), where \( d(x) \) denotes the distance between \( x \) and \( \partial M \) with respect to the metric \( g \). Then there exists a positive constant \( A = A(M, g, n, \rho, \sigma) \) such that (4) holds for all \( u \in H^1(\rho^{1-2\sigma}, M) \).

**Remark 1.1.** The constant \( S(n, \sigma) \) in (4) is optimal for all \( \sigma \in (0, 1) \), see Proposition 2.2

**Remark 1.2.** Theorem 1.2 may fail without any geometric assumption on \( \partial M \). For example, it is the case when the mean curvature of \( \partial M \) is positive somewhere. In particular, (4) is false on any bounded smooth domain in \( \mathbb{R}^{n+1} \) when \( \sigma \in (1/2, 1) \). However, Theorem 1.1 holds for all \( \sigma \in (0, 1) \) if \( S(n, \sigma) \) is replaced by any \( S > S(n, \sigma) \), see Proposition 2.5

**Remark 1.3.** It is clear that we only need to consider the case when \( M \) is connected. Throughout the paper, we assume this.

3
When $\sigma = 1/2$, (3) is a standard Sobolev trace inequality which has been extensively studied, see, e.g., Lions [35], Escobar [14], Beckner [5], Adimurthi-Yadava [1], Li-Zhu [32, 33] and many others. In particular, Li-Zhu [32] established Theorem 1.1 for $\sigma = 1/2$. The sharp inequality (4) is in the same spirit of a conjecture posed by Aubin [2] which concerns the best constants in Sobolev embedding theorems on Riemannian manifolds. Aubin’s conjecture had been confirmed through the work of Hebe-Vaugon [25], Aubin-Li [4] and Druet [11, 12]. Besides, various refinements of Aubin’s conjecture were obtained in Druet-Hebey [13], Li-Ricardi [31] and etc. These sharp Sobolev type inequalities play important roles in the study of nonlinear partial differential equations, see, e.g., Lions [35], Escobar [14], Beckner [5], Adimurthi-Yadava [1], Li-Zhu [32, 33] and many others. In particular, Li-Zhu [32] established Theorem 1.1 for $\sigma = 1/2$. For the defining function in the above theorems, $I_{\alpha}$ does not contain terms like $\int M \rho^{1-2\alpha}|\nabla u|^2 \, dv_g$, we adapt a global argument from Li-Zhu [32, 33]. By contradiction, we assume that for any $\alpha > 0$,

$$I_{\alpha} := \frac{\int M \rho^{1-2\alpha}|\nabla u|^2 \, dv_g + \alpha \int_{\partial M} |u|^{2\alpha} \, ds_g}{(\int_{\partial M} |u|^{2\alpha} \, ds_g)^{\frac{2\alpha}{n-2\alpha}}} < \frac{1}{S(n, \sigma)}$$

for some $u \in H^1(\rho^{1-2\alpha}, M)$ with that $u \not\equiv 0$ on $\partial M$. It follows that there exists a minimizer $u_{\alpha}$ of $I_{\alpha}$, and $u_{\alpha}$ blows up at exactly one point as $\alpha \to \infty$. One key step is the asymptotical analysis of $u_{\alpha}$ near its blow up point. Here we have to overcome difficulties from the degeneracy and the lack of conformal invariance of the Euler-Lagrange equation of $I_{\alpha}$ satisfied by $u_{\alpha}$. Another difference from [32] (the case $\sigma = 1/2$) is that some Sobolev embedding theorems for $H^1(\rho^{1-2\alpha}, M)$,
which play important roles in establishing the blow-up profile of $u_\alpha$ in the interior of $M$ in \cite{32} in the case $\sigma = \frac{1}{2}$, fail when $\sigma > \frac{1}{2}$ (see, e.g., Theorem 1 in page 135 or Corollary 2 in page 193 of \cite{36}). However, we succeeded in establishing the optimal asymptotical behavior of $u_\alpha$ on the boundary $\partial M$ (Proposition 3.3). In this step, a Liouville type theorem in Jin-Li-Xiong \cite{26} and Neumann functions for degenerate equations in Theorem 1.3 are used. The last step is to derive a contradiction by checking balance via a Pohozaev type inequality in some proper region, where a Harnack inequality established by Cabre-Sire \cite{8} or Tan-Xiong \cite{43} is used to obtain the asymptotical behavior of $u_\alpha$ near it blowup point in $M$ from that on $\partial M$. Some extra arguments on $\partial M$ are needed for $\sigma > \frac{1}{2}$.

**Theorem 1.3.** Let $f \in L^1(\partial M)$ with mean value zero, i.e., $\int_{\partial M} f = 0$. Then there exists a weak solution $u \in W^{1,1+\varepsilon_0}(\rho^{1-2\sigma}, M)$ of \eqref{59} where $\varepsilon_0 > 0$ depending only on $n$ and $\sigma$. Consequently, if $f = \delta_{x_0} - \frac{1}{|\rho(x_0)|}$ for some $x_0 \in \partial M$, where $\delta_{x_0}$ is the delta function at $x_0$ and $|\partial M|_g$ is the area of $\partial M$ with respect to the induced metric $g$, then there exists a weak solution $u \in W^{1,1+\varepsilon_0}(\rho^{1-2\sigma}, M) \cap H^1_{\text{loc}}(\rho^{1-2\sigma}, M \setminus \{x_0\})$ of \eqref{59} with mean value zero. Moreover, for all $x \in \overline{M} \setminus \{x_0\}$,

$$A_1 \text{dist}_g(x, x_0)^{2\sigma-n} - A_0 \leq u(x) \leq A_2 \text{dist}_g(x, x_0)^{2\sigma-n}$$

where $A_0, A_1, A_2$ are positive constants depending only on $M, g, n, \sigma, \rho$.

The proof of Theorem 1.3 follows from Lemma A.5, Theorem A.5 and some approximation arguments. When $\sigma = 1/2$, Theorem 1.3 follows directly from Brezis-Strauss \cite{7} and Kenig-Pipher \cite{29}.

**Notations.** We collect below a list of the main notations used throughout the paper.

- We always assume that $n \geq 2, \sigma \in (0, 1)$, and $\rho$ is a smooth defining function as in Theorem 1.1 without otherwise stated. Denote $q = \frac{2n}{n-2\sigma}$.

- For a domain $D \subset \mathbb{R}^{n+1}$ with boundary $\partial D$, we denote $\partial' D$ as the interior of $\overline{D} \cap \partial \mathbb{R}^{n+1}$ in $\mathbb{R}^n = \partial \mathbb{R}^{n+1}$ and $\partial'' D = \partial D \setminus \partial' D$.

- For $\bar{x} \in \mathbb{R}^{n+1}$, $B_r(\bar{x}) := \{x \in \mathbb{R}^{n+1}: |x - \bar{x}| = \sqrt{(x_1 - \bar{x}_1)^2 + \cdots + (x_{n+1} - \bar{x}_{n+1})^2} < r\}$, $B^+_r(\bar{x}) := B_r(\bar{x}) \cap \mathbb{R}^n_+$. If $\bar{x} \in \partial \mathbb{R}^{n+1}$, $B_r(\bar{x}) := \{x = (x', 0): |x' - \bar{x}'| < r\}$. Hence $\partial' B^+_r(\bar{x}) = B_r(\bar{x})$ if $\bar{x} \in \partial \mathbb{R}^{n+1}$. We will not keep writing the center $\bar{x}$ if $\bar{x} = 0$.

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2 Preliminaries

Proposition 2.1. For any \( u \in C_c^\infty(\mathbb{R}_+^{n+1}) \), we have

\[
\left( \int_{\mathbb{R}^n} |u(x',0)|^q \, dx' \right)^{\frac{2}{q}} \leq S(n, \sigma) \int_{\mathbb{R}^{n+1}_+} x_{n+1}^{1-2\sigma} |\nabla u(x)|^2 \, dx.
\]

Moreover, the above inequality fails if \( S(n, \sigma) \) is replaced by any smaller constant.

Proof. It follows from (3) and Lemma A.3 of [25]. See also Corollary 5.3 of [21]. \( \square \)

Proposition 2.2. Let \( M \) be as in Theorem 1.1. Let \( \sigma \in (0, 1) \), and \( \rho \) be a defining function of \( \partial M \) with \( |\nabla g \rho| = 1 \) on \( \partial M \). Suppose there exist some positive constants \( \tilde{S} \) and \( \tilde{A} \) such that, for all \( u \in H^1(\rho^{1-2\sigma}, M) \),

\[
\left( \int_{\partial M} |u|^q \, ds_g \right)^{\frac{2}{q}} \leq \tilde{S} \int_M \rho^{1-2\sigma} |\nabla g u|^2 \, dv_g + \tilde{A} \int_{\partial M} |u|^2 \, ds_g.
\]

Then \( \tilde{S} \geq S(n, \sigma) \).

Proof. Given Proposition 2.1, the proof is standard (see, e.g., Proposition 4.2 of [24]). We include it here for completeness and to illustrate the role of \( |\nabla \rho| = 1 \). We argue by contradiction. Suppose there exists a Riemannian manifold \( (M, g) \), a defining function \( \rho \) of \( \partial M \) with \( |\nabla g \rho| = 1 \) on \( \partial M \), \( \sigma \in (0, 1) \), \( \tilde{S} < S(n, \sigma) \) and \( \tilde{A} > 0 \) such that for all \( u \in H^1(\rho^{1-2\sigma}, M) \),

\[
\left( \int_{\partial M} |u|^q \, ds_g \right)^{\frac{2}{q}} \leq \tilde{S} \int_M \rho^{1-2\sigma} |\nabla g u|^2 \, dv_g + \tilde{A} \int_{\partial M} |u|^2 \, ds_g.
\] (6)

Let \( x \in \partial M \). For any \( \varepsilon > 0 \), which will be chosen sufficiently small, there exists a chart \( (\Omega, \varphi) \) of \( M \) at \( x \) and \( \delta > 0 \) such that \( \varphi(\Omega) = B_+^\delta(0) \) the upper half Euclidean ball of center 0 and radius \( \delta \) in \( \mathbb{R}_+^{n+1} \), and

\[
(1 - \varepsilon)\delta_{ij} \leq g_{ij} \leq (1 + \varepsilon)\delta_{ij}.
\] (7)

By assumption, (6) holds for any \( u \in C_c^\infty(\Omega \cup (\partial \Omega \cap \partial M)) \), i.e.,

\[
\left( \int_{B_{\delta}(0)} |u|^q \sqrt{|\det(g_{ij})|} \, dx' \right)^{\frac{2}{q}} \leq \tilde{S} \int_{B_{\delta}^+(0)} \rho^{1-2\sigma} g^{ij} u_i u_j \sqrt{|\det(g_{ij})|} \, dx
\]

\[
+ \tilde{A} \int_{B_\delta(0)} |u|^2 \sqrt{|\det(g_{ij})|} \, dx'.
\]
It follows from (7), $|\nabla g\rho| = 1$ and $\rho = 0$ on $\partial M$ that there exists $\delta_0 > 0$, $\tilde{S}' < S(n, \sigma)$, $\tilde{A}' > 0$ such that for all $\delta \in (0, \delta_0)$ and $u \in C^\infty_c(B_\delta(0) \cup B_\delta(0))$, i.e.,

$$
\left( \int_{B_\delta(0)} |u|^q \, dx' \right)^{\frac{2}{q}} \leq \tilde{S}' \int_{B_\delta^+ (0)} x_{n+1}^{1-2\sigma} |\nabla u|^2 \, dx + \tilde{A}' \int_{B_\delta(0)} |u|^2 \, dx'.
$$

By H"older’s inequality, $\int_{B_\delta(x)} |u|^2 \, dx' \leq |B_\delta(0)|^{\frac{q-2}{q}} \left( \int_{B_\delta(0)} |u|^q \, dx' \right)^{\frac{2}{q}}$. By choosing $\delta$ sufficiently small, we have that there exists $\tilde{S}'' < S(n, \sigma)$ such that for all $u \in C^\infty_c(B_\delta(0) \cup B_\delta(0))$

$$
\left( \int_{B_\delta(0)} |u|^q \, dx' \right)^{\frac{2}{q}} \leq \tilde{S}'' \int_{B_\delta^+ (0)} x_{n+1}^{1-2\sigma} |\nabla u|^2 \, dx.
$$

Consequently, by a scaling argument, we have

$$
\left( \int_{R^n} |u(x', 0)|^q \, dx' \right)^{\frac{2}{q}} \leq \tilde{S}'' \int_{R^{n+1}_+} x_{n+1}^{1-2\sigma} |\nabla u(x)|^2 \, dx.
$$

for any $u \in C^\infty(R^{n+1}_+)$, which contradicts Proposition 2.1.

**Proposition 2.3.** Assume the assumptions in Proposition 2.2. Then for any $\epsilon > 0$ there exists a positive constant $B_\epsilon$ such that

$$
\left( \int_{\partial M} |u|^q \, ds_g \right)^{\frac{2}{q}} \leq (S(n, \sigma) + \epsilon) \int_M \rho^{1-2\sigma} |\nabla g u|^2 \, dv_g + B_\epsilon \int_M \rho^{1-2\sigma} |u|^2 \, dv_g.
$$

**Proof.** It also follows from Proposition 2.1 and a standard partition of unity argument, see, e.g., Theorem 4.5 of [24] on page 95.

For every $\alpha > 0$, consider the functional

$$
I_\alpha[u] = \frac{\int_M \rho^{1-2\sigma} |\nabla g u|^2 \, dv_g + \alpha \int_{\partial M} |u|^2 \, ds_g}{\left( \int_{\partial M} |u|^q \, ds_g \right)^{2/q}}, \quad u \in H^1(\rho^{1-2\sigma}, M), \quad u \not\equiv 0 \quad \text{on} \quad \partial M.
$$

**Proposition 2.4.** Suppose that for some $\alpha > 0$,

$$
\xi_\alpha := \inf_{u \in H^1(\rho^{1-2\sigma}, M), \ u|_{\partial M} \not\equiv 0} I_\alpha[u] < \frac{1}{S(n, \sigma)},
$$

then $\xi_\alpha$ is achieved by a nonnegative function $u_\alpha \in H^1(\rho^{1-2\sigma}, M)$ with

$$
\int_{\partial M} u_\alpha^q \, ds_g = 1.
$$
Proof. Given Proposition 2.3, the Proposition follows from standard calculus of variations, see page 452 of [32].

Proposition 2.5. Assume the assumptions in Proposition 2.2. For any \( \varepsilon > 0 \), there exists a positive constant \( A_\varepsilon \) such that
\[
\left( \int_{\partial M} |u|^q \, ds_g \right)^{\frac{1}{q}} \leq \left( S(n, \sigma) + \varepsilon \right) \int_M \rho^{1-2\sigma} |\nabla g u|^2 \, dv_g + A_\varepsilon \int_{\partial M} |u|^2 \, ds_g.
\]

Proof. Given Propositions 2.3 and 2.4 and Corollary A.1, the proof of Proposition 2.5 is similar to Proposition 1.2 of [32] and we omit it here.

3 Asymptotic analysis

For brevity, from now on we write \( S \) instead of \( S(n, \sigma) \). We prove Theorem 1.1 by contradiction. Namely, assume that for any \( \alpha \geq 1 \),
\[
\xi_\alpha < \frac{1}{S},
\]
where \( \xi_\alpha \) is defined as in Proposition 2.4. Let \( u_\alpha \) be some nonnegative minimizer of \( I_\alpha \) obtained in Proposition 2.4 which satisfies
\[
\xi_\alpha = \int_M \rho^{1-2\sigma} |\nabla g u_\alpha|^2 \, dv_g + \alpha \int_{\partial M} u_\alpha^2 \, ds_g, \quad \int_{\partial M} u_\alpha^q \, ds_g = 1,
\]
and for any \( \varphi \in H^1(\rho^{1-2\sigma}, M) \),
\[
\int_M \rho^{1-2\sigma} \langle \nabla g u_\alpha, \nabla g \varphi \rangle_g \, dv_g + \alpha \int_{\partial M} \varphi \, ds_g = \xi_\alpha \int_{\partial M} \varphi \, ds_g.
\]

The geodesic distance function \( d(x) := \text{dist}(x, \partial M) \) determines for some \( \varepsilon_0 > 0 \) an identification of \( \partial M \times [0, \varepsilon_0) \) with a neighborhood of \( \partial M \) in \( M \): \( (x', d) \in \partial M \times [0, \varepsilon_0) \) corresponds to the point obtained by following the integral curve of \( \nabla g d \) emanating from \( x' \) for \( d \) units of time. Furthermore, \( \nabla g d \) is orthogonal to the slices \( \partial M \times \{d\} \). Define \( \nu := -\nabla g d \) for \( d < \varepsilon_0 \). It follows from Theorem A.2, Theorem A.3 and Proposition A.1 that \( u_\alpha \in C^{\gamma}(M) \cap C^\infty(M) \cap C^\infty(\partial M) \) for some \( \gamma \in (0, 1) \) and \( \rho^{1-2\sigma} \frac{\partial u_\alpha}{\partial \nu} \in C(\partial M \times [0, \varepsilon_0/2]) \). Hence, \( u_\alpha \) satisfies the Euler-Lagrange equation
\[
\begin{cases}
\text{div}_g \left( \rho^{1-2\sigma} \nabla g u_\alpha \right) = 0, & \text{in } M, \\
\lim_{d \to 0} \rho^{1-2\sigma} (x', d) \frac{\partial g u_\alpha}{\partial \nu} (x', \rho) = \xi_\alpha u_\alpha^{q-1}(x') - \alpha u_\alpha(x'), & \text{on } \partial M.
\end{cases}
\]
in the pointwise sense.
It follows from the maximum principle that \( \max_M u_\alpha = \max_{\partial M} u_\alpha \). Let \( u_\alpha(x_\alpha) = \max_M u_\alpha \), where \( x_\alpha \in \partial M \), and \( \mu_\alpha = u_\alpha(x_\alpha) - \frac{1}{\alpha^{2\sigma}} \). By a Hopf Lemma (see, e.g., Proposition 4.11 in \cite{8}), we have \( \xi_\alpha u_\alpha(x_\alpha)^{q-1} - \alpha u_\alpha(x_\alpha) > 0 \), that is

\[
\alpha \mu_\alpha^{2\sigma} < \xi_\alpha. \tag{14}
\]

Hence, \( \lim_{\alpha \to \infty} \mu_\alpha^{2\sigma} = 0 \).

**Lemma 3.1.** As \( \alpha \to \infty \), we have

\[
\xi_\alpha \to \frac{1}{S}, \tag{15a}
\]

\[
\alpha \|u_\alpha\|_{L^2(\partial M)} \to 0. \tag{15b}
\]

**Proof.** For all small \( \varepsilon > 0 \), it follows from Proposition 2.5 that

\[
1 \leq (S + \varepsilon) \int_M \rho^{1 - 2\sigma} |\nabla_g u_\alpha|^2 \, dv_g + A_\varepsilon \int_{\partial M} u_\alpha^2 \, ds_g
\]

\[
= (S + \varepsilon) \xi_\alpha + (A_\varepsilon - (S + \varepsilon)\alpha) \int_{\partial M} u_\alpha^2 \, ds_g.
\]

Hence, for every \( \alpha \geq \frac{2A_\varepsilon}{S + \varepsilon} \) we have

\[
\frac{1}{S + \varepsilon} \leq \xi_\alpha \leq \frac{1}{S}, \quad \frac{S}{2} \alpha \int_{\partial M} u_\alpha^2 \, ds_g < \frac{\varepsilon}{S}.
\]

(15a) and (15b) follow immediately. \qed

Let \( x = (x_1, \cdots, x_n, x_{n+1}) = (x', x_{n+1}) \) be Fermi coordinates (see, e.g., \cite{15}) at \( x_\alpha \), where \( (x_1, \cdots, x_n) \) are normal coordinates on \( \partial M \) at \( x_\alpha \) and \( \gamma(x_{n+1}) \) is the geodesic leaving from \( (x_1, \cdots, x_n) \) in the orthogonal direction to \( \partial M \) and parametrized by arc length. In this coordinate system,

\[
\sum_{1 \leq i,j \leq n+1} g_{ij}(x)dx_idx_j = dx_{n+1}^2 + \sum_{1 \leq i,j \leq n} g_{ij}(x)dx_idx_j.
\]

Moreover, \( g^{ij} \) has the following Taylor expansion near \( \partial M \):

**Lemma 3.2** (Lemma 3.2 in \cite{15}). For \( \{x_k\}_{k=1,\cdots,n+1} \) are small,

\[
g^{ij}(x) = \delta^{ij} + 2h^{ij}(x', 0)x_{n+1} + O(|x|^2), \tag{16}
\]

where \( i, j = 1, \cdots, n \) and \( h_{ij} \) is the second fundamental form of \( \partial M \).
For suitably small \( \delta_0 > 0 \) (independent of \( \alpha \)), we define \( v_\alpha \) in a neighborhood of \( x_\alpha = 0 \) by
\[
v_\alpha(x) = \mu_\alpha^{(n-2\sigma)}/2 u_\alpha(\mu_\alpha x), \quad x \in B^+_{\delta_0/\mu_\alpha}.
\]
It follows that
\[
\begin{align*}
\text{div} g_\alpha \left( \rho^{1-2\sigma}_\alpha \nabla g_\alpha v_\alpha \right) &= 0, \quad \text{in } B^+_{\delta_0/\mu_\alpha} \\
\lim_{x_{n+1} \to 0^+} \rho^{1-2\sigma}_\alpha \frac{\partial g_\alpha v_\alpha}{\partial x_{n+1}} &= \xi_\alpha v_\alpha^{q-1} - \alpha \mu_\alpha^{2\sigma} v_\alpha, \quad \text{on } \partial B^+_{\delta_0/\mu_\alpha} = B_{\delta_0/\mu_\alpha}
\end{align*}
\] (17)
where \( g_\alpha(x) = g_{ij}(\mu_\alpha x)dx_i dx_j, \rho_\alpha(x) = \rho(\mu_\alpha x)/\mu_\alpha \). It follows from (14) and Theorem A.2 in the Appendix that for all \( R > 1 \),
\[
\|v_\alpha\|_{C^{\gamma}(B^+_R)} + \|v_\alpha\|_{H^1(\rho^{1-2\sigma}_\alpha B^+_R)} \leq C(R), \quad \text{for all sufficiently large } \alpha,
\] (18)
where \( \gamma \in (0, 1) \) is independent of \( R \) and \( \alpha \). It follows that there exists \( v \in C_{\text{loc}}^{\gamma}(\mathbb{R}_{n+1}^+) \cap H^1_{\text{loc}}(x_{n+1}^{1-2\sigma}, \mathbb{R}^+_{n+1}) \) such that along some subsequence,
\[
\begin{align*}
v_\alpha &\to v \text{ in } C^{\gamma/2}(B^+_R) \\
v_\alpha &\to v \text{ weakly in } H^1(x_{n+1}^{1-2\sigma}, B^+_R)
\end{align*}
\] (19)
for any \( R > 0 \) as \( \alpha \to \infty \). Since \( v_\alpha(0) = 1 \), we have
\[
\begin{align*}
\int_{B_1} v_\alpha^q \, ds_\alpha &\geq 1/C > 0, \\
\int_{B_1} v_\alpha^2 \, ds_\alpha &\geq 1/C > 0.
\end{align*}
\] (20)
On the other hand,
\[
\alpha\|u_\alpha\|_{L^2(\partial M)}^2 \geq \alpha \int_{B_{\mu_\alpha}(x_\alpha)} u_\alpha^2 = \alpha \mu_\alpha^{2\sigma} \int_{B_1} v_\alpha^2,
\]
where we abused notation by denoting \( B_r(x_\alpha) \) as the geodesic ball on \( \partial M \) centered at \( x_\alpha \) with radius \( r \). It follows from (15b) and (20) that
\[
\lim_{\alpha \to \infty} \alpha \mu_\alpha^{2\sigma} = 0.
\] (21)
From (17), (21) and (15a), we conclude that \( v \) is a weak solution (see Section A.2 for the definition of weak solutions) of
\[
\begin{align*}
\text{div}(x_{n+1}^{1-2\sigma} \nabla v) &= 0, \quad \text{in } \mathbb{R}^{n+1}_+, \\
- \lim_{x_{n+1} \to 0^+} x_{n+1}^{1-2\sigma} \partial_{x_{n+1}} v &= \frac{1}{S} v^{q-1}, \quad \text{on } \partial \mathbb{R}^{n+1}_+, \\
v(0) &= 1, \quad 0 \leq v \leq 1.
\end{align*}
\] (22)
By a Liouville type theorem, Theorem 1.5 in [26],

\[ v(x', 0) = \left( \frac{1}{1 + \tilde{c}(n, \sigma)|x'|^2} \right)^{\frac{n-2\sigma}{2}}, \quad v(x', x_{n+1}) = \int_{\mathbb{R}^n} \mathcal{P}_\sigma(x' - y', x_{n+1}) v(y', 0) \, dy', \]

where \( \tilde{c}(n, \sigma) \) is a positive constant such that \( \int_{\mathbb{R}^n} v^q(z) \, dz = 1 \), and \( \mathcal{P}_\sigma(x) \) is given in (1). Due to the uniqueness of the limit function \( v \), we know that (19) holds for all \( \alpha \to \infty \).

**Proposition 3.1.** For \( \delta_0 = \delta_0(M, g) > 0 \) small enough,

\[ \lim_{\alpha \to \infty} \int_{B_{\delta_0(\mu_\alpha)} x} |v_\alpha - v|^q = 0. \]

**Proof.** Note that \( v_\alpha \geq 0 \) and

\[ \int_{B_{\delta_0(\mu_\alpha)} x} v_\alpha^q \leq \int_{\partial M} u_\alpha^q = 1. \]  \( \tag{23} \)

For any \( \varepsilon > 0 \), choose \( R > 0 \) such that \( \int_{\mathbb{R}^n \setminus B_R} v^q(x', 0) \, dx' \leq \varepsilon \). It follows from (19) that \( \int_{B_R} |v_\alpha - v|^q \leq \varepsilon \) and \( 1 - \int_{B_R} v_\alpha^q < 2\varepsilon \) for all \( \alpha \) sufficiently large. Then

\[
\int_{B_{\delta_0(\mu_\alpha)} x} |v_\alpha - v|^q \\
= \int_{B_{\delta_0(\mu_\alpha)} x \cap B_R} |v_\alpha - v|^q + \int_{B_{\delta_0(\mu_\alpha)} x \cap B_R^c} |v_\alpha - v|^q \\
\leq \int_{B_{\delta_0(\mu_\alpha)} x \cap B_R} |v_\alpha - v|^q + 2\varepsilon \int_{B_{\delta_0(\mu_\alpha)} x \cap B_R^c} v_\alpha^q + 2\varepsilon \int_{B_{\delta_0(\mu_\alpha)} x \cap B_R^c} v^q \\
\leq \varepsilon + 2\varepsilon (1 - \int_{B_R} v_\alpha^q) + 2\varepsilon (1 - \int_{B_R} v^q) \leq \varepsilon (1 + 3 \cdot 2^q),
\]

which finishes the proof. \( \square \)

**Corollary 3.1.** For all \( \delta_1 > 0 \) we have

\[ \lim_{\alpha \to \infty} \int_{B_{\delta_1(\mu_\alpha)} x \cap \partial M} u_\alpha^q = 1. \]

**Proof.** It follows immediately from Proposition 3.1. \( \square \)

Let \( \tilde{G}_\alpha \) be the weak solution of

\[
\begin{aligned}
- \text{div}_g \left( \rho^{1-2\sigma} \nabla_g \tilde{G}_\alpha \right) &= 0, \\ \\
\lim_{x \to x_{n+1}} \rho^{1-2\sigma} (y) \frac{\partial}{\partial \nu} \tilde{G}_\alpha (y) &= \delta_{x_{n+1}} - \frac{1}{|\partial M|},
\end{aligned}
\]

constructed in Theorem [A.5]. We can find a positive constant \( C > 0 \) sufficiently large depending only on \( M, g, n, \sigma, \rho \) such that \( G_\alpha := \tilde{G}_\alpha + C \geq 1 \) on \( \overline{M} \).
Proposition 3.2. Let \( \varphi(x) = \frac{n-2\sigma}{n-2\sigma} G_\alpha(x) \), \( \bar{g}_{ij} = \varphi_\alpha^{-\frac{4}{n-2\sigma}} g_{ij} \) and \( a = 2 - \frac{2(n-1)}{n-2\sigma} \). Then \( w_\alpha := \frac{u_\alpha}{\varphi_\alpha} \) satisfies

\[
\begin{aligned}
& \text{div}_{\bar{g}} \left( \varphi_\alpha^a \rho^{1-2\sigma} \nabla_{\bar{g}} w_\alpha \right) = 0, \quad \text{in } M, \\
& \lim_{y \to \bar{x} \in \partial M} \varphi_\alpha^a \rho^{1-2\sigma} \frac{\partial w_\alpha(y)}{\partial \nu} \leq \xi_\alpha w_\alpha^a - 1(\bar{x}), \quad \bar{x} \in \partial M \setminus \{x_\alpha\},
\end{aligned}
\tag{24}
\]

for \( \alpha \geq \frac{1}{|\partial M|} \).

Proof. The proof follows from some direct computations. For brevity, we drop the subscript \( \alpha \) of \( \varphi_\alpha \) and \( u_\alpha \). First of all,

\[
\begin{aligned}
\text{div}_{\bar{g}} \left( \varphi_\alpha^a \rho^{1-2\sigma} \nabla_{\bar{g}} \frac{u}{\varphi} \right) &= \varphi_\alpha^{-1-\frac{4}{n-2\sigma}} \text{div}_{g} \left( \rho^{1-2\sigma} \nabla_{g} u \right) - u \varphi_\alpha^{-2-\frac{4}{n-2\sigma}} \text{div}_{g} \left( \rho^{1-2\sigma} \nabla_{g} \varphi \right) \\
&\quad + \left( a - 2 + \frac{2(n-1)}{n-2\sigma} \right) \rho^{1-2\sigma} \varphi_\alpha^{-2-\frac{4}{n-2\sigma}} ((\nabla_{g} u, \nabla_{g} \varphi)_{g} - u \varphi |\nabla_{g} \varphi|_g^2) \\
&= 0.
\end{aligned}
\]

On the other hand, in Fermi coordinate system centered at \( \bar{x} \),

\[
\begin{aligned}
&\lim_{x_{n+1} \to 0} \varphi_\alpha^a \rho^{1-2\sigma} \frac{\partial g}{\partial \nu} \left( \frac{u}{\varphi} \right) \\
&= \lim_{x_{n+1} \to 0} \varphi_\alpha^a \rho^{1-2\sigma} \left( \frac{1}{\varphi} \frac{\partial u}{\partial x_{n+1}} - \frac{u}{\varphi^2} \frac{\partial \varphi}{\partial x_{n+1}} \right) \tilde{g}^{n+1,n+1} \left( \frac{\partial}{\partial x_{n+1}}, \tilde{\nu} \right) \\
&= \varphi_\alpha^{-1-\frac{2}{n-2\sigma}} \left( \xi_\alpha u \frac{a+2\sigma}{n-2\sigma} - \alpha u \right) + \varphi_\alpha^{-2-\frac{2}{n-2\sigma}} u \mu_\alpha \frac{n-2\sigma}{|\partial M|} \left( \frac{1}{|\partial M|} - \alpha \right) \\
&\leq \xi_\alpha \left( \frac{u}{\varphi} \right)^{\frac{n+2\sigma}{n-2\sigma}} + \varphi_\alpha^{-2-\frac{2}{n-2\sigma}} u \mu_\alpha \frac{n-2\sigma}{|\partial M|} \left( \frac{1}{|\partial M|} - \alpha \right) \\
&\leq \xi_\alpha \left( \frac{u}{\varphi} \right)^{\frac{n+2\sigma}{n-2\sigma}},
\end{aligned}
\]

provided \( \alpha \geq \frac{1}{|\partial M|} \). \( \square \)

Proposition 3.3. Suppose the assumptions in Proposition 3.2. Then there exists some constant \( C \) depending only on \( M, g, n, \rho, \sigma \) such that for all \( \alpha \geq 1 \),

\[
w_\alpha \leq C, \quad \text{on } \partial M.
\]
Proof. In the following, $C$ denotes some constant which may depend on $M, g, n, \rho, \sigma$ but not on $\alpha$ and may vary from line to line.

It suffices to prove the proposition for large $\alpha$, in particular, say, $\alpha \geq \max\{\frac{1}{|\partial M|g}, 1\}$. Let \( \tilde{\rho} := \varphi_\alpha^{-\frac{2}{\alpha - 2\sigma}} \rho \). Then (24) can be rewritten as

\[
\begin{cases}
\text{div}_{\tilde{g}} \left( \tilde{\rho}^{1-2\sigma} \nabla_{\tilde{g}} w_{\alpha} \right) = 0, & \text{in } M, \\
\lim_{y \to \tilde{x}} \tilde{\rho}^{1-2\sigma} \frac{\partial_{\tilde{g}} w_{\alpha}(y)}{\partial \nu} \leq \xi_{\alpha} w_{\alpha}^{q-1}(\tilde{x}), & \text{for } \tilde{x} \in \partial M \setminus \{x_{\alpha}\},
\end{cases}
\]  

where the limit is taken in the sense explained in the paragraph above (13). In the following, we shall abuse notation a little by writing $\psi^{-1}(B_0^+(0))$ as $B_0^+(0)$ where $(\psi^{-1}(B_0^+(0)), \psi)$ is a Fermi coordinate of $M$ at $x_{\alpha}$, and denoting $B_0(x_{\alpha})$ as the geodesic ball on $\partial M$ centered at $x_{\alpha}$ with radius $\delta$ as before. Note that the interior of $B_0^+(0) \cap \partial M$ is $B_0(x_{\alpha})$.

**Step 1.** We claim that there exist some constants $0 < \delta_2 < 1, s_0 > q$ independent of $\alpha$ such that

\[
\int_{\partial M \setminus B_{\mu_{\alpha}/\delta_2}(x_{\alpha})} w_{\alpha}^{s_0} \, ds_{\tilde{g}} \leq C. \tag{26}
\]

For any $\varepsilon > 0$, it follows from Proposition 3.1 that there exists a small $\delta_2$ such that

\[
\int_{\partial M \setminus B_{\mu_{\alpha}/\delta_2}(x_{\alpha})} w_{\alpha}^{q} \, ds_{\tilde{g}} = \int_{\partial M \setminus B_{\mu_{\alpha}/\delta_2}(x_{\alpha})} w_{\alpha}^{q} \, ds_{g} = 1 - \int_{\partial B_{\delta_2}(x_{\alpha})} v_{\alpha}^{q} \leq \varepsilon. \tag{27}
\]

Without loss of generality, we may assume $10 \mu_{\alpha}/\delta_2 < \delta_0$ where $\delta_0$ is the constant such that the Fermi coordinate system centered at $x_{\alpha}$ exists in $B_{\delta_0}(x_{\alpha})$.

We choose $\eta$ to be some cutoff function satisfying

\[
\eta(x) = 1 \text{ if } |x| \geq \mu_{\alpha}/\delta_2, \quad \eta(x) = 0 \text{ if } |x| \leq \mu_{\alpha}/(2\delta_2),
\]

and $\eta = \eta(|x|)$ in the Fermi coordinate system centered at $x_{\alpha}$.

Multiplying (25) by $w_{\alpha}^{k} \eta^2$ for $k > 1$ and integrating by parts, we obtain

\[
\int_{M} \tilde{\rho}^{1-2\sigma} \nabla_{\tilde{g}} w_{\alpha} \nabla_{\tilde{g}} (w_{\alpha}^{k} \eta^2) \, dv_{\tilde{g}} \leq \xi_{\alpha} \int_{\partial M} w_{\alpha}^{q-1+k} \eta^2 \, ds_{\tilde{g}}.
\]
By a direct computation, we see that
\[
\int_M \tilde{\rho}^{1-2\sigma} \nabla_{\tilde{g}} w_\alpha \nabla_{\tilde{g}} (w_\alpha^k \eta^2) \, \mathrm{d}v_{\tilde{g}} = \frac{4k}{(k+1)^2} \int_M \tilde{\rho}^{1-2\sigma} |\nabla_{\tilde{g}} (w_\alpha^k \eta^2)|^2 \, \mathrm{d}v_{\tilde{g}} + \frac{k-1}{(k+1)^2} \int_M w_\alpha^{k+1} \text{div}_{\tilde{g}} \left( \tilde{\rho}^{1-2\sigma} \nabla_{\tilde{g}} \eta^2 \right) \, \mathrm{d}v_{\tilde{g}}
\]
\[
- \frac{4k}{(k+1)^2} \int_M \tilde{\rho}^{1-2\sigma} w_\alpha^{k+1} |\nabla_{\tilde{g}} \eta|^2 \, \mathrm{d}v_{\tilde{g}},
\]
where we have used that \( \lim_{\rho \to 0} \rho^{1-2\sigma} \frac{\partial^2 \rho}{\partial \eta^2} = 0 \) since \( \eta \) is radial. In conclusion, we obtain
\[
\int_M \tilde{\rho}^{1-2\sigma} |\nabla_{\tilde{g}} (w_\alpha^k \eta^2)|^2 \, \mathrm{d}v_{\tilde{g}} \leq \frac{k-1}{4k} \int_M w_\alpha^{k+1} \text{div}_{\tilde{g}} \left( \tilde{\rho}^{1-2\sigma} \nabla_{\tilde{g}} \eta^2 \right) \, \mathrm{d}v_{\tilde{g}} + \frac{\xi_\alpha (k+1)^2}{4k} \int_{\partial M} w_\alpha^{q-1+k} \eta^2 \, \mathrm{d}s_{\tilde{g}}.
\]

Since \( \tilde{g}^{ij} \sim \mu_\alpha^2 \delta^{ij} \) in \( B^+_2 (x_\alpha) \setminus B^+_2 (4\delta_2) (x_\alpha) \), we have
\[
|\nabla_{\tilde{g}} \eta| + |\nabla^2_{\tilde{g}} \eta| \leq C.
\]

Since \( \eta \) is radial in the Fermi coordinate system, using (65a), (65b) and (65c), we have
\[
|\text{div}_{\tilde{g}} (\tilde{\rho}^{1-2\sigma} \nabla_{\tilde{g}} \eta^2)| \leq C \tilde{\rho}^{1-2\sigma}.
\]

Taking \( 1 < k \leq q-1 \) in (28) and using Theorem A.1 and Theorem A.5 it follows that
\[
\int_M \tilde{\rho}^{1-2\sigma} |\nabla_{\tilde{g}} (w_\alpha^k \eta^2)|^2 \, \mathrm{d}v_{\tilde{g}} \leq C(k, \delta_2) + \frac{\xi_\alpha (k+1)^2}{4k} \int_{\partial M} w_\alpha^{q-1+k} \eta^2 \, \mathrm{d}s_{\tilde{g}} \leq C(k, \delta_2) + \frac{\xi_\alpha (k+1)^2}{4k} \varepsilon^{(q-2)/q} \left( \int_{\partial M} (w_\alpha^{(1+k)/2})^q \, \mathrm{d}s_{\tilde{g}} \right)^{2/q} \leq C(k, \delta_2) + C \varepsilon^{(q-2)/q} \int_M \tilde{\rho}^{1-2\sigma} |\nabla_{\tilde{g}} (w_\alpha^k \eta^2)|^2 \, \mathrm{d}v_{\tilde{g}},
\]
where we used
\[
\int_{M \cap (B_{\mu_\alpha}^+ \setminus B_{\mu_\alpha}^+)} \tilde{\rho}^{1-2\sigma} w^{k+1}_\alpha \, dv_{\tilde{g}} \\
\leq C(\delta) \int_{M \cap (B_{\mu_\alpha}^+ \setminus B_{\mu_\alpha}^+)} \left( \frac{\rho}{\mu_\alpha} \right)^{1-2\sigma} (\mu_\alpha^{n-2\sigma}/2 u_\alpha)^{k+1} u_\alpha^{-(n+1)} \, dv_g \\
\leq C(\delta) \int_{1/(2\delta) \leq |z| \leq 1/\delta} \rho_\alpha(z)^{1-2\sigma} v_\alpha(z)^{k+1} \, dv_{g_\alpha} \quad \text{by changing variables} \\
\leq C(k, \delta_2),
\] (29)

and \(\rho_\alpha(z), v_\alpha(z)\) are those in (17).

Taking \(\varepsilon > 0\) sufficiently small, we have
\[
\int_{M \cap (B_{\mu_\alpha}^+ \setminus B_{\mu_\alpha}^+)} \tilde{\rho}^{1-2\sigma} \left( w^{(k+1)/2} \eta_\alpha^2 \right) \, dv_{\tilde{g}} \leq C.
\]

The claim follows immediately from Theorem A.1 in the Appendix.

**Step 2.** We shall complete the proof by Moser’s iterations. Set, for \(\delta = \delta_2/10\),
\[
R_l = \mu_\alpha \left( 2 - \frac{2^{-(l-1)}}{\delta} \right), \quad l = 1, 2, 3, \ldots
\]

We choose \(\eta_l\) to be some cutoff function satisfying
\[
\eta_l(x) = 1 \text{ if } |x| \geq R_{l+1}, \quad \eta_l(x) = 0 \text{ if } |x| \leq R_l,
\]
and \(\eta_l = \eta_l(|x|)\) in the Fermi coordinate system centered at \(x_\alpha\).

Since \(\tilde{g}_{ij} \sim \mu_\alpha^2 g_{ij}^\alpha\) in \(B_{2\mu_\alpha/\delta_2}^+(x_\alpha) \setminus B_{\mu_\alpha/(2\delta_2)}^+(x_\alpha)\) and \(\eta_l\) is radial in the Fermi coordinate system, we have
\[
|\nabla_{\tilde{g}} \eta_l| \leq C 2^l, \quad |\text{div}_{\tilde{g}}(\rho^{1-2\sigma} \nabla_{\tilde{g}} \eta_l^2)| \leq C 4^l \rho^{1-2\sigma}, \quad \text{and } \lim_{\rho \to 0} \rho^{1-2\sigma} \frac{\partial g_{ij}^\alpha}{\partial \rho} = 0.
\]

In view of (28), we have
\[
\int_{M} \rho^{1-2\sigma} |\nabla_{\tilde{g}}(w^{(k+1)/2} \eta_l)|^2 \, dv_{\tilde{g}} \\
\leq C 4^l \int_{M \cap (B_{R_{l+1}^+}(x_\alpha) \setminus B_{R_l^+}(x_\alpha))} \rho^{1-2\sigma} u^{k+1}_\alpha \, dv_{\tilde{g}} + \frac{C(k+1)^2}{k} \int_{\partial M \setminus B_{R_l}(x_\alpha)} w_{\alpha}^{q-1+k} \, ds_{\tilde{g}}.
\] (30)
Set $r_0 = s_0 / (q - 2)$, where $s_0$ is given in the step 1. It follows Hölder inequality and (26) that
$$\int_{\partial M \setminus B_{R_l}(x_0)} w_{\alpha}^{q-1+k} \, ds_{\tilde{g}} = \int_{\partial M \setminus B_{R_l}(x_0)} w_{\alpha}^{q-2} w_{\alpha}^{k+1} \, ds_{\tilde{g}}$$
$$\leq C \left( \int_{\partial M \setminus B_{R_l}(x_0)} w_{\alpha}^{(k+1)r_0 / (r_0 - 1)} \, ds_{\tilde{g}} \right)^{(r_0 - 1) / r_0} \tag{31}$$
Computing as (29), we see that
$$\int_{M \cap (B_{R_l+1}(x_0) \setminus B_{R_l}(x_0))} \tilde{\rho}^{1-2\sigma} w_{\alpha}^{k+1} \, dv_{\tilde{g}}$$
$$\leq C^{k+1} \int_{2^{2-2(k-1)} \leq |z| \leq 2^{2-1}} \rho_{\alpha}(z)^{1-2\sigma} v_{\alpha}(z)^{k+1} \, dv_{g_{\alpha}}$$
$$\leq C^{k+1} \delta^{-2} \max_{B^+_2} v_{\alpha}^{k+1},$$
and
$$\left( \int_{\partial M \setminus B_{R_l}(x_0)} u_{\alpha}^{(k+1)r_0 / (r_0 - 1)} \, ds_{\tilde{g}} \right)^{(r_0 - 1) / r_0} \geq C^{-(k+1)} \left( \int_{1 \leq |z'| \leq 2} \rho_{\alpha}(z', 0)^{1-2\sigma} v_{\alpha}(z)^{[k+1]r_0 / (r_0 - 1)} \, ds_{g_{\alpha}} \right)^{(r_0 - 1) / r_0}$$
$$\geq C^{-(k+1)} \min_{\partial B^+_2} v_{\alpha}^{k+1}.$$ Hence, it follows from (19) that
$$\left( \int_{M \cap (B_{R_l+1}(x_0) \setminus B_{R_l}(x_0))} \tilde{\rho}^{1-2\sigma} w_{\alpha}^{k+1} \, dv_{\tilde{g}} \right)^{1/(k+1)}$$
$$\leq C \left( \int_{\partial M \setminus B_{R_l}(x_0)} u_{\alpha}^{(k+1)r_0 / (r_0 - 1)} \, ds_{\tilde{g}} \right)^{(r_0 - 1) / r_0 (k+1)} \tag{32}$$
It follows from Theorem [A.1] (30), (31) and (32) that
$$\left( \int_{\partial M \setminus B_{R_l+1}(x_0)} w_{\alpha}^{(k+1)q / 2} \, ds_{\tilde{g}} \right)^{2/(k+1)q}$$
$$\leq \left( C^4 + \frac{C(k+1)}{k} \right)^{1/(k+1)} \left( \int_{\partial M \setminus B_{R_l}(x_0)} u_{\alpha}^{(k+1)r_0 / (r_0 - 1)} \, ds_{\tilde{g}} \right)^{(r_0 - 1) / r_0 (k+1)} \tag{33}$$
Set \( \chi := \frac{r_0 - 1}{r_0} \cdot \frac{q}{2} = 1 + \frac{(s_0 - q)(q - 2)}{2s_0} > 1, \) \( q_0 = \frac{2r_0}{r_0 - 1}, \) \( q_l = q_{l-1} \cdot \chi = \chi^{l-1} q \) and \( p_l = q_l(r_0 - 1)/r_0 = 2\chi^l \) where \( l \geq 1. \) Taking \( k = p_l - 1 \) in (33), we obtain

\[
\|w_\alpha\|_{L^q(\partial M \setminus B_{R_{l+1}})} \leq \left( C 4^l + \frac{C p_l^2}{p_l - 1} \right)^{1/p_l} \|w_\alpha\|_{L^q(\partial M \setminus B_{R_l})}.
\]

Therefore,

\[
\|w_\alpha\|_{L^q(\partial M \setminus B_{R_{l+1}})} \leq \|w_\alpha\|_{L^q(\partial M \setminus B_{R_l})} \prod_{l=1}^{\infty} \left( C 4^l + \frac{C p_l^2}{p_l - 1} \right)^{1/p_l}
\leq \|w_\alpha\|_{L^q(\partial M \setminus B_{R_l})} \prod_{l=1}^{\infty} C^{1/(2\chi^l)} (4 + \chi)^{l/(2\chi^l)}
\leq C \|w_\alpha\|_{L^q(\partial M \setminus B_{R_l})}.
\]

Sending \( l \) to \( \infty \), we have

\[
\|w_\alpha\|_{L^q(\partial M \setminus B_{2\mu_\alpha/\delta}(x_\alpha))} \leq C.
\] \((34)\)

By the choice of \( G_\alpha \), \( \varphi_\alpha(x) \geq C^{-1} \mu_\alpha^{(n-2\sigma)/2} \) for \( x \in B_{2\mu_\alpha/\delta}(x_\alpha) \). Hence, for \( x \in B_{2\mu_\alpha/\delta}(x_\alpha) \),

\[
w_\alpha(x) = \frac{u_\alpha(x)}{\varphi_\alpha(x)} \leq C \mu_\alpha^{(n-2\sigma)/2} u_\alpha(x) \leq C.
\] \((35)\)

In view of (34) and (35), we completed the proof of the proposition. \( \square \)

**Corollary 3.2.** There exists a positive constant \( C \) depending only on \( M, g, n, \rho, \sigma \) such that

\[
u_\alpha(x) \leq C u_\alpha(x)^{-1} \text{dist}_{\partial M, g}(x, x_\alpha)^{2\sigma - n}, \quad \text{for all } x \in \partial M.
\]

**Proof.** It follows immediately from Proposition 3.3 \( \square \)

## 4 Proofs of the main theorems

Let \( u_\alpha \) and \( x_\alpha \) be as in Section 3. We will still use Fermi coordinates \( x = (x_1, \cdots, x_{n+1}) \) centered at \( x_\alpha \). In this coordinate system,

\[
\sum_{1 \leq i, j \leq n+1} g_{ij}(x)dx_idx_j = dx_{n+1}^2 + \sum_{1 \leq i, j \leq n} g_{ij}(x)dx_idx_j, \quad \text{for } |x| \leq \delta_0,
\]

where \( \delta_0 > 0 \) is independent of \( \alpha \). Then we have

\[
\left\{ \begin{array}{ll}
div_g \left( \rho(x)^{1-2\sigma} \nabla_g u_\alpha(x) \right) = 0, & \text{in } B^+_{\delta_0}, \\
- \lim_{x_{n+1} \to 0^+} \rho(x)^{1-2\sigma} \frac{\partial u_\alpha}{\partial x_{n+1}} = \xi_\alpha u_\alpha^{g_{\alpha}^{-1}}(x', 0) - \alpha u_\alpha(x', 0), & \text{on } \partial' B^+_{\delta_0}.
\end{array} \right.
\] \((36)\)
Proposition 4.1. There exists a positive constant $C$ independent of $\alpha$ such that

$$u_\alpha(x) \leq Cu_\alpha(0)^{-1}|x|^{2\sigma-n}, \quad B^-_{10\alpha^{-1/2\sigma}}(0).$$

Proof. By Corollary 3.2

$$u_\alpha(x',0) \leq Cu_\alpha(0)^{-1}|x'|^{2\sigma-n}, \quad |x'| \leq \delta_0. \tag{37}$$

Let $r := |\tau| < 10\alpha^{-1/2\sigma}, \phi_\alpha(x) = r^{n-2\sigma}u_\alpha(rx)$. Then $\phi_\alpha$ satisfies

$$\begin{cases}
\text{div}_g\left(\hat{\rho}(x)^{1-2\sigma}\nabla_g \phi_\alpha(x)\right) = 0, & \text{in } B^+_{\delta_0/r}, \\
- \lim_{x_{n+1} \to 0^+} \hat{\rho}(x)^{1-2\sigma} \frac{\partial \phi_\alpha}{\partial x_{n+1}} = \xi_\alpha \phi_\alpha^{-1}(x',0) - \alpha r^{2\sigma} \phi_\alpha(x',0), & \text{on } \partial' B^+_{\delta_0/r}, \tag{38}
\end{cases}$$

where $\hat{\rho}(x) = \rho(rx)/r$, $\hat{g}(x) = g_{ij}(rx)dx_idx_j$. Since $x_\alpha = 0$ is a maximum point of $u_\alpha$, it follows from (37) that

$$\phi_\alpha(x',0) = r^{n-2\sigma}u_\alpha(rx',0) \leq Cr^{n-2\sigma}(r|x'|)^{-n-2\sigma} \leq C, \quad \frac{1}{2} < |x'| < 2. \tag{39}$$

Applying the Harnack inequality in [8] or [43] and standard Harnack inequality for uniformly elliptic equations to $\phi_\alpha$ in $\{x : \frac{1}{2} < |x| < 2, x_{n+1} > 0\}$, we conclude that

$$\max_{B^+_{3/2}\backslash B^+_{3/4}} \phi_\alpha \leq C \min_{B^+_{3/2}\backslash B^+_{3/4}} \phi_\alpha.$$ 

Hence, by (37)

$$u_\alpha(\bar{x}) \leq Cu(\bar{x}',0) \leq Cu_\alpha(0)^{-1}|\bar{x}|^{2\sigma-n},$$

where $|\bar{x}'| = |\bar{x}|$. By the arbitrary choice of $\tau$, the proposition follows immediately. $\square$

Let $\mu_\alpha = u_\alpha(0)^{-\frac{\sigma}{n-2\sigma}}, R_\alpha = (\alpha^{1/2\sigma} \mu_\alpha)^{-1}, g_\alpha = g_{ij}(\mu_\alpha x)dx_idx_j$ and $\rho_\alpha(x) = \frac{\rho(\mu_\alpha x)}{\mu_\alpha}$ in $B^+_{10R_\alpha}$. Set $v_\alpha(x) = \mu_\alpha^{-1} u_\alpha(\mu_\alpha x)$ for $x \in B^+_{10R_\alpha}$. It follows that

$$\begin{cases}
\text{div}_{g_\alpha}\left(\rho_\alpha^{1-2\sigma}\nabla_{g_\alpha} v_\alpha\right) = 0, & \text{in } B^+_{10R_\alpha} \\
- \lim_{x_{n+1} \to 0} \rho_\alpha^{1-2\sigma} \frac{\partial v_\alpha}{\partial x_{n+1}} = \xi_\alpha v_\alpha^{-1} - \alpha \mu_\alpha v_\alpha, & \text{on } \partial' B^+_{10R_\alpha} = B_{10R_\alpha}, \tag{40}
v_\alpha(0) = 1, \quad 0 < \alpha \leq 1.
\end{cases}$$

By Proposition 4.1

$$v_\alpha(x) \leq \frac{C}{1 + |x|^{n-2\sigma}}, \quad x \in B^+_{10R_\alpha}. \tag{41}$$
Proposition 4.2. For all $\alpha \geq 1$, $x \in B_{R_{\alpha}}^+(0)$, we have
\[
|\nabla x\nabla v_\alpha(x', x_{n+1})| \leq \frac{C}{1 + |x|^{n+1-2\sigma}},
\]
\[
|\nabla^2 x\nabla v_\alpha(x', x_{n+1})| \leq \frac{C}{1 + |x|^{n+2-2\sigma}},
\]
\[
|\partial_{n+1} v_\alpha(x', x_{n+1})| \leq \frac{C|x|^{2\sigma - 1}}{1 + |x|^n}.
\]

Proof. Given Theorem A.3 and Proposition A.1, the proofs follow from (41) and standard rescaling arguments (see, e.g., Proposition 3.1 of [32]).

Proof of Theorem 1.1. We complete the proof of Theorem 1.1 by checking balance via a Pohozaev type inequality.

It follows from direct computations that
\[
\begin{align*}
2 \text{div}(x_{n+1}^{1-2\sigma} \nabla v_\alpha)(\nabla v_\alpha \cdot x) \\
= \text{div}(2x_{n+1}^{1-2\sigma} \nabla v_\alpha \cdot x) \nabla v_\alpha - x_{n+1}^{1-2\sigma} |\nabla v_\alpha|^2 x + (n - 2\sigma)x_{n+1}^{1-2\sigma} |\nabla v_\alpha|^2.
\end{align*}
\]
(42)

Integrating both sides of (42) over $B_{R_{\alpha}}^+$, we have
\[
\begin{align*}
\int_{B_{R_{\alpha}}^+} \text{div}(x_{n+1}^{1-2\sigma} \nabla v_\alpha)(\nabla v_\alpha \cdot x) \, d x = & \frac{n - 2\sigma}{2} \int_{B_{R_{\alpha}}^+} x_{n+1}^{1-2\sigma} |\nabla v_\alpha|^2 \, d x \\
= & \frac{1}{2} \int_{\partial B_{R_{\alpha}}^+} \text{div}(2x_{n+1}^{1-2\sigma} \nabla v_\alpha \cdot x) \nabla v_\alpha - x_{n+1}^{1-2\sigma} |\nabla v_\alpha|^2 x \, d x.
\end{align*}
\]
(43)

Integrating by parts, we obtain
\[
\begin{align*}
\frac{1}{2} \int_{B_{R_{\alpha}}^+} \text{div}(2x_{n+1}^{1-2\sigma} \nabla v_\alpha \cdot x) \nabla v_\alpha - x_{n+1}^{1-2\sigma} |\nabla v_\alpha|^2 x \, d x \\
= & - \int_{\partial B_{R_{\alpha}}^+} \left( \sum_{i=1}^n x_i \frac{\partial v_\alpha}{\partial x_i} \right) \frac{\partial v_\alpha}{\partial x_{n+1}} \, d S + \int_{\partial B_{R_{\alpha}}^+} |x|x_{n+1}^{1-2\sigma} \left( \left( \frac{\partial v_\alpha}{\partial v} \right)^2 - \frac{1}{2} |\nabla v_\alpha|^2 \right) \, d S \\
= & - \int_{\partial B_{R_{\alpha}}^+} \left( \sum_{i=1}^n x_i \frac{\partial v_\alpha}{\partial x_i} \right) \frac{\partial v_\alpha}{\partial x_{n+1}} \, d S + \int_{\partial B_{R_{\alpha}}^+} |x|x_{n+1}^{1-2\sigma} \left( \left( \frac{\partial v_\alpha}{\partial v} \right)^2 - |\partial_{\text{tan}} v_\alpha|^2 \right) \, d S,
\end{align*}
\]
where $\frac{\partial v_\alpha}{\partial x_{n+1}} := \lim_{x_{n+1} \to 0^+} x_{n+1}^{-2\sigma} \frac{\partial v_\alpha}{\partial x_{n+1}}$ and $\partial_{\text{tan}}$ denotes the tangential differentiation on $\partial B_{R_{\alpha}}^+$.

On the other hand,
\[
\int_{B_{R_{\alpha}}^+} x_{n+1}^{1-2\sigma} |\nabla v_\alpha|^2 \, d x = - \int_{B_{R_{\alpha}}^+} \text{div}(x_{n+1}^{1-2\sigma} \nabla v_\alpha) v_\alpha \, d x \\
= - \int_{\partial B_{R_{\alpha}}^+} v_\alpha \frac{\partial v_\alpha}{\partial x_{n+1}} \, d S + \int_{\partial B_{R_{\alpha}}^+} x_{n+1}^{1-2\sigma} v_\alpha \frac{\partial v_\alpha}{\partial v} \, d S.
\]

19
In summary, we obtain
\begin{equation}
\int_{B_{R_0}^+} \text{div}(x_{n+1}^{1-2\sigma} \nabla v_\alpha)(\nabla v_\alpha \cdot x) \, dx + \frac{n-2\sigma}{2} \int_{B_{R_0}^+} \text{div}(x_{n+1}^{1-2\sigma} \nabla v_\alpha) v_\alpha \, dx = B'(R_\alpha, v_\alpha) + B''(R_\alpha, v_\alpha), \tag{44}
\end{equation}
where
\begin{align*}
B'(R_\alpha, v_\alpha) &= -\frac{1}{2} \int_{\partial^+ B_{R_0}^+} \left( \sum_{i=1}^{n} x_i \frac{\partial v_\alpha}{\partial x_i} \right) \frac{\partial v_\alpha}{\partial x_{n+1}^\sigma} + (n-2\sigma) v_\alpha \frac{\partial v_\alpha}{\partial x_{n+1}^\sigma} \, dx',
B''(R_\alpha, v_\alpha) &= \frac{1}{2} \int_{\partial^+ B_{R_0}^+} |x|^2 x_{n+1}^{1-2\sigma} \left( \frac{\partial v_\alpha}{\partial \nu} \right)^2 - |\partial_{\tan} v_\alpha|^2 \right) + (n-2\sigma) x_{n+1}^{1-2\sigma} v_\alpha \frac{\partial v_\alpha}{\partial \nu} \, dS.
\end{align*}
Note that
\begin{align*}
\text{div}_{g_\alpha}(\rho_\alpha^{1-2\sigma} \nabla g_\alpha v_\alpha) &= g_\alpha^{ij} \frac{\partial v_\alpha}{\partial x_i} \frac{\partial \rho_\alpha^{1-2\sigma}}{\partial x_j} + \rho_\alpha^{1-2\sigma} g_\alpha^{ij} \left( \frac{\partial^2 v_\alpha}{\partial x_i \partial x_j} - \Gamma^k_{ij} \frac{\partial v_\alpha}{\partial x_k} \right) \\
&= \text{div}(x_{n+1}^{1-2\sigma} \nabla v_\alpha) + \sum_{1 \leq i,j \leq n} g_\alpha^{ij} \frac{\partial v_\alpha}{\partial x_i} \frac{\partial \rho_\alpha^{1-2\sigma}}{\partial x_j} + \left( \frac{\partial \rho_\alpha^{1-2\sigma}}{\partial x_{n+1}} - \frac{\partial x_{n+1}^{1-2\sigma}}{\partial x_{n+1}} \right) \frac{\partial v_\alpha}{\partial x_{n+1}^\sigma} \\
&+ \rho_\alpha^{1-2\sigma} (g_\alpha^{ij} - \delta^{ij}) \frac{\partial^2 v_\alpha}{\partial x_i \partial x_j} + \delta^{ij} \rho_\alpha^{1-2\sigma} (\rho_\alpha^{1-2\sigma} - x_{n+1}^{1-2\sigma}) \Delta v_\alpha - \rho_\alpha^{1-2\sigma} g_\alpha^{ij} \Gamma^k_{ij} \frac{\partial v_\alpha}{\partial x_k}, \tag{45}
\end{align*}
where $\Gamma^k_{ij}$ is the Christoffel symbol of $g_\alpha$. It is easy to see that
\begin{align*}
|h_\alpha^{ij}(x) - \delta^{ij}| &\leq C \mu_\alpha |x|, \tag{46a} \\
|\Gamma^k_{ij}| &\leq C \mu_\alpha, \tag{46b} \\
|\rho_\alpha(x)^{1-2\sigma} - x_{n+1}^{1-2\sigma}| &\leq C \mu_\alpha x_{n+1}^{2-2\sigma}, \tag{46c} \\
\left| \frac{\partial \rho_\alpha(x)^{1-2\sigma}}{\partial x_i} \right| &\leq C \mu_\alpha x_{n+1}^{1-2\sigma} \text{ for } i < n + 1, \tag{46d} \\
\left| \frac{\partial \rho_\alpha(x)^{1-2\sigma}}{\partial x_{n+1}} - \frac{\partial x_{n+1}^{1-2\sigma}}{\partial x_{n+1}} \right| &\leq C \mu_\alpha x_{n+1}^{1-2\sigma}. \tag{46e}
\end{align*}
Indeed,
\begin{align*}
|\rho_\alpha(x)^{1-2\sigma} - x_{n+1}^{1-2\sigma}| &= x_{n+1}^{1-2\sigma} \left| \frac{\rho(\mu_\alpha x_n + O(\mu_\alpha x_{n+1})^{2})^{1-2\sigma}}{\mu_\alpha x_{n+1}} - 1 \right| \\
&= x_{n+1}^{1-2\sigma} \left| \left( \frac{(\mu_\alpha x_{n+1} + O(\mu_\alpha x_{n+1})^{2})}{\mu_\alpha x_{n+1}} \right)^{1-2\sigma} - 1 \right| \\
&\leq C \mu_\alpha x_{n+1}^{2-2\sigma}.
\end{align*}
\[
\frac{\partial \rho_\alpha(x)^{1-2\sigma}}{\partial x_i} = (1 - 2\sigma) \rho_\alpha(x)^{-2\sigma} \left( \frac{\partial \rho_\alpha(x)}{\partial x_i} - \frac{\partial \rho_\alpha(x', 0)}{\partial x_i} \right) \\
= O(1) \mu_\alpha \rho_\alpha^{1-2\sigma} \\
\leq C \mu_\alpha x_n^{1-2\sigma}.
\]

It follows from (40), (44), (45) and (46a)-(46e) that

\[
B'(R_\alpha, v_\alpha) + B''(R_\alpha, v_\alpha) \\
\leq C \mu_\alpha \int_{\partial B^+_R_{\alpha}} x_n^{1-2\sigma} (v_\alpha + |\nabla v_\alpha \cdot x|)(|\nabla v_\alpha| + |x||\nabla^2 v_\alpha| + x_{n+1} |\Delta v_\alpha|) \, dx.
\]

Since \(\lim_{x_{n+1} \to 0} \rho_\alpha^{1-2\sigma} \frac{\partial g_\alpha v_\alpha}{\partial \nu} = -\frac{\partial v_\alpha}{\partial x_{n+1}}\) on \(\partial B^+_R_{\alpha}\),

\[
B'(R_\alpha, v_\alpha) = \int_{\partial B^+_R_{\alpha}} \left( \sum_{i=1}^{n} x_i \frac{\partial v_\alpha}{\partial x_i} \right) (\xi_\alpha v_\alpha^{q-1} - \alpha \mu_\alpha^{2\sigma} v_\alpha) + \frac{(n - 2\sigma)}{2} (\xi_\alpha v_\alpha^q - \alpha \mu_\alpha^{2\sigma} v_\alpha^2) \, dx' \\
= \sigma \alpha \mu_\alpha^{2\sigma} \int_{\partial B^+_R_{\alpha}} v_\alpha^{2} \, dx' + \int_{\partial B_{R_\alpha}} (\frac{\xi_\alpha}{q} v_\alpha^q - \frac{\alpha \mu_\alpha^{2\sigma}}{2} v_\alpha^2) R_\alpha \, dS,
\]

where integrations by parts were used in the second equality. Clearly,

\[
B''(R_\alpha, v_\alpha) = O \left( \int_{\partial B^+_R_{\alpha}} x_n^{1-2\sigma} (|x| |\nabla v_\alpha|^2 + v_\alpha |\nabla v_\alpha|) \, dS \right).
\]

Therefore, we obtain

\[
\alpha \mu_\alpha^{2\sigma} \int_{\partial B^+_R_{\alpha}} v_\alpha^{2} \, dx' \\
\leq C \mu_\alpha \int_{B^+_R_{\alpha}} x_n^{1-2\sigma} (v_\alpha + |\nabla v_\alpha \cdot x|)(|\nabla v_\alpha| + |x||\nabla^2 v_\alpha| + x_{n+1} |\Delta v_\alpha|) \, dx
\]

\[
+ C \int_{\partial B^+_R_{\alpha}} x_n^{1-2\sigma} |x||\nabla v_\alpha|^2 + v_\alpha |\nabla v_\alpha| \, dS + C \int_{\partial B_{R_\alpha}} \alpha \mu_\alpha^{2\sigma} v_\alpha^{2} R_\alpha \, dS.
\]

Since \(\text{div}_{g_\alpha} (\rho_\alpha^{1-2\sigma} \nabla g_\alpha v_\alpha) = 0\) and \(g_\alpha^{i,n+1} = 0\) for \(i < n + 1\),

\[
|\partial_{x_{n+1}}^2 v_\alpha(x', x_{n+1})| \leq C (\mu_\alpha |\nabla v_\alpha| + |\partial_{x+1} v_\alpha| x^{-1}_{n+1} + |\nabla^2 v_\alpha|).
\]
It follows from (48), (49) and Proposition 4.2 that

\[
\alpha \mu^2 \int_{\partial \mathcal{B}'_R} v^2 \, dx' \leq C \mu \int_{\mathcal{B}'_R} x_{n+1}^{1-2\sigma} (v_\alpha + |\nabla v_\alpha \cdot x|) (|\nabla v_\alpha| + |x| |\nabla^2 v_\alpha|) \, dx
\]

\[
+ C \int_{\partial' \mathcal{B}'_{R_\alpha}} x_{n+1}^{1-2\sigma} \left( \frac{1}{R_\alpha^{n+1-4\sigma}} + \frac{x_{n+1}^{2\sigma-1}}{R_\alpha^{2\sigma-2\sigma}} + \frac{x_{n+1}^{4\sigma-2}}{R_\alpha^{4n-1}} \right) \, dS + C \frac{\alpha \mu^2}{R_\alpha^{n-4\sigma}}
\]

\[
\leq C \mu \int_{\mathcal{B}'_R} \left( \frac{x_{n+1}^{1-2\sigma}}{(1 + |x|)^{2n+1-4\sigma}} + \frac{1}{(1 + |x|)^{2n-2\sigma}} \right) \, dx
\]

\[
+ CR_\alpha^{2\sigma-n} \int_{\partial' \mathcal{B}_1} (y_{n+1}^{1-2\sigma} + 1 + y_{n+1}^{2\sigma-1}) \, dS + C \frac{\alpha \mu^2}{R_\alpha^{n-4\sigma}}
\]

\[
\leq \left\{ \begin{array}{ll}
C \mu \ln R_\alpha + C(\alpha \mu^2 \alpha^\frac{n-2\sigma}{2\sigma}) + C \alpha \mu^2 R_\alpha^{2\sigma-n}, & n = 2\sigma + 1 \\
C \mu + C(\alpha \mu^2 \alpha^\frac{n-2\sigma}{2\sigma}) + C \alpha \mu^2 R_\alpha^{2\sigma-n}, & n > 2\sigma + 1.
\end{array} \right.
\]

For \( \sigma = 1/2 \) and \( n = 2 \), Theorem 1.1 was proved in [32]. Hence, we may assume that \( n > 2\sigma + 1 \).

Since \( \sigma \in (0, 1/2] \), \( n > 2\sigma + 1 \geq 4\sigma \). Therefore,

\[
0 < \frac{1}{C} \leq \int_{\partial' \mathcal{B}'_R} v^2 \, dx' \to 0, \text{ as } \alpha \to \infty
\]

which is a contradiction. \( \square \)

**Proof of Theorem 1.2.** Since \( \partial M \) is totally geodesic, Lemma 3.2 implies that

\[
|h^{ij}_\alpha(x) - \delta^{ij}| \leq C \mu^2 |x|^2, \quad (50a)
\]

\[
|\Gamma^{k}_{ij}| \leq C \mu^2 |x|. \quad (50b)
\]

Since \( \rho = d(x) + O(d(x)^3) \), it follows that

\[
|h^{ij}_\alpha(x)^{1-2\sigma} - x_{n+1}^{1-2\sigma}| \leq C \mu^2 |x|^{3-2\sigma}, \quad (51a)
\]

\[
|\frac{\partial h^{ij}_\alpha(x)}{\partial x_i}^{1-2\sigma}| \leq C \mu^2 |x|^{2-2\sigma}, \quad i < n + 1, \quad (51b)
\]

\[
|\frac{\partial h^{ij}_\alpha(x)}{\partial x_{n+1}}^{1-2\sigma} - \frac{\partial x_{n+1}^{1-2\sigma}}{\partial x_{n+1}}| \leq C \mu^2 |x|^{2-2\sigma}. \quad (51c)
\]

22
Similar to (48), we have

\[
\alpha \mu^2 v^2 \int_{\partial B^+_{R_\alpha}} \leq C \mu^2 \int_{B^+_{R_\alpha}} x_{n+1}^{-2\sigma} (v_{\alpha} + |\nabla v_{\alpha} \cdot x|)(|x||\nabla v_{\alpha}| + |x|^2|\nabla^2 v_{\alpha}| + x_{n+1}^2 |\Delta v_{\alpha}|) \, dx \\
+ C \int_{\partial B^+_{R_\alpha}} x_{n+1}^{-2\sigma} (|x||\nabla v_{\alpha}|^2 + v_{\alpha} |\nabla v_{\alpha}|) \, dS + C \int_{\partial B^+_{R_\alpha}} \alpha \mu^2 v_{\alpha}^2 R_{\alpha} \, dS.
\]  

(52)

It follows from (49), (52) and Proposition 4.2 that

\[
\alpha \mu^2 \int_{\partial B^+_{R_\alpha}} v^2 \, dx' \\
\leq C \mu^2 \int_{B^+_{R_\alpha}} x_{n+1}^{-2\sigma} (v_{\alpha} + |\nabla v_{\alpha} \cdot x|)(|x||\nabla v_{\alpha}| + |x|^2|\nabla^2 v_{\alpha}| + x_{n+1}^2 |\Delta v_{\alpha}|) \, dx \\
+ C \int_{\partial B^+_{R_\alpha}} x_{n+1}^{-2\sigma} (|x||\nabla v_{\alpha}|^2 + v_{\alpha} |\nabla v_{\alpha}|) \, dS + C \int_{\partial B^+_{R_\alpha}} \alpha \mu^2 v_{\alpha}^2 R_{\alpha} \, dS \\
\leq C \mu^2 \int_{B^+_{R_\alpha}} \frac{x_{n+1}^{-2\sigma}}{(1 + |x|^{2n-4\sigma})} \, dx + C(\alpha \mu^2_{\alpha})^{\frac{a-2\sigma}{2\sigma}} + C\alpha \mu^2_{\alpha} R_{\alpha}^{4\sigma-n} \\
\leq C \mu^2_{\alpha} + C(\alpha \mu^2_{\alpha})^{\frac{a-2\sigma}{2\sigma}} + C\alpha \mu^2_{\alpha} R_{\alpha}^{4\sigma-n},
\]

provided \( n > 2 + 2\sigma \) (i.e., \( n \geq 4 \)). Therefore,

\[
0 < \frac{1}{C} \leq \int_{\partial B^+_{R_\alpha}} v^2 \, dx' \to 0 \quad \text{as} \ \alpha \to \infty,
\]

which is a contradiction. \( \square \)

\section{Appendix}

\subsection{A trace inequality}

Let \((M, g)\) be a smooth, compact Riemannian manifold of dimension \( n + 1 \) \((n \geq 2)\) with boundary.

\textbf{Lemma A.1.} For \( n \geq 2 \), there exists some positive constant \( C = C(n, \sigma) \) such that for all \( u \in H^1(x_{n+1}^{-2\sigma}, B_1^+) \), \( u \equiv 0 \) in an open neighborhood of \( x = 0 \), we have

\[
\left( \int_{\partial B^+_{R_\alpha}} \frac{|u(x', 0)|^q}{|x'|^{2n}} \, dx' \right)^{2/q} \leq C \int_{B^+_{R_\alpha}} \frac{x_{n+1}^{-2\sigma} |\nabla u|^2}{|x|^{2n-4\sigma}} \, dx.
\]
Proof. By the assumption of \( u \), there exists a positive constant \( \mu = \mu(u) > 0 \) such that \( u \equiv 0 \) for \( |x| < \mu \) with \( x_{n+1} > 0 \). Consider

\[
v(y) = u \left( \frac{y}{|y|^2} \right), \quad |y| > 1, y_{n+1} > 0.
\]

It is easy to see that

\[
v(y) \equiv 0, \quad \text{for all } |y| > 1/\mu, \ y_{n+1} > 0,
\]

and for some \( C(n) > 0 \),

\[
\int_{\partial B_{1}^+(x)} \frac{|u(x', 0)|^q}{|x'|^{2n}} \, dx' = C(n) \int_{|y| \geq 1} |v(y', 0)|^q \, dy',
\]

and

\[
\int_{B_{1}^+(x)} \frac{x_{n+1}}{|x|^{2n-4\sigma}} \, dx = C(n) \int_{|y| \geq 1, y_{n+1} > 0} y_{n+1}^{1-2\sigma} \, |\nabla v(y)|^2 \, dy.
\]

By some appropriate extension of \( v \) to \( |y| < 1 \), it follows from (3) that

\[
\int_{|y| \geq 1} |v(y', 0)|^q \, dy' \leq C(n, \sigma) \int_{|y| \geq 1, y_{n+1} > 0} y_{n+1}^{1-2\sigma} \, |\nabla v(y)|^2 \, dy.
\]

The proof is completed. \( \square \)

Lemma A.2. For \( \delta > 0, \) there exists \( C = C(M, g, n, \sigma, \delta, \rho) > 0 \) such that for all \( x_0 \in \partial M, \ u \in H^1(\rho^{1-2\sigma}, M \setminus B_{\delta/2}(x_0)) \), we have

\[
\left( \int_{\partial M \setminus B_{\delta}(x_0)} |u(x)|^q \right)^{2/q} + \int_{M \setminus B_{\delta}^+(x_0)} \rho^{1-2\sigma} |u(x)|^2 \\
\leq C \left\{ \int_{M \setminus B_{\delta}^+(x_0)} \rho^{1-2\sigma} |\nabla u|^2 + \int_{\partial M \cap (B_{\delta}(x_0) \setminus B_{\delta/2}(x_0))} |u(x)|^2 \right\}.
\]  

(53)

Proof. We prove (53) by contradiction. Suppose the contrary of (53) that for some \( \delta > 0 \), there exists a sequence of points \( \{x_i\} \in \partial M, \ \{u_i\} \in H^1(\rho^{1-2\sigma}, M \setminus B_{\delta/2}^+(x_i)) \) satisfying

\[
\left( \int_{\partial M \setminus B_{\delta}(x_i)} |u_i(x)|^q \right)^{2/q} + \int_{M \setminus B_{\delta}^+(x_i)} \rho^{1-2\sigma} |u_i(x)|^2 = 1,
\]  

(54)

but

\[
\int_{M \setminus B_{\delta/2}^+(x_i)} \rho^{1-2\sigma} |\nabla u_i|^2 + \int_{\partial M \cap (B_{\delta}(x_i) \setminus B_{\delta/2}(x_i))} |u_i(x)|^2 \leq \frac{1}{i}.
\]  

(55)
After passing to some subsequence, \( \{u_i\} \) converges weakly to \( u \) in \( H^1(\rho_1^{1-2\sigma}, M \setminus B_0^+(x_i)) \). By (55), \( u \equiv 0 \). It follows from a compact Sobolev embedding in Proposition A.2 that

\[
\int_{M \setminus B_0^+(x_i)} \rho_1^{1-2\sigma} |u_i(x)|^2 \to 0.
\]

By a trace embedding in Proposition 2.3, we also conclude that

\[
\left( \int_{\partial M \setminus B_0^+(x_i)} |u(x)|^q \right)^{2/q} \to 0.
\]

Therefore, we reach a contradiction to (54).

**Theorem A.1.** There exists some constant \( C = C(M, g, \rho, n, \sigma) \) such that for all \( x_0 \in \partial M, \mu > 0, u \in H^1(\rho_1^{1-2\sigma}, M), u \equiv 0 \) in \( \{x \in M : \text{dist}(x, x_0) < \mu\} \), we have

\[
\int_{\partial M \setminus B_0^+(x_i)} |u(x)|^q \text{dist}(x, x_0)^{2n} \leq C \int_{M} \rho_1^{1-2\sigma} |\nabla g u|^2 \text{dist}(x, x_0)^{2n-4\sigma} \text{d}v_g.
\]

**Proof.** The theorem follows clearly from Lemma A.1 and Lemma A.2.

**A.2 Regularity results for degenerate elliptic equations**

Suppose that \( a^{ij}(x), 1 \leq i, j \leq n+1 \), is a smooth positive definite matrix-valued in \( B_2^+ \) and there exists a positive constant \( \Lambda \geq 1 \) such that

\[
\frac{1}{\Lambda} |\xi|^2 \leq a^{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2, \quad \forall \xi \in \mathbb{R}^{n+1}
\]

Suppose also that

\[
a^{i n+1} = a^{n+1 i} = 0 \quad \text{for} \quad i < n + 1.
\]

Consider

\[
\begin{cases}
\frac{\partial}{\partial x_i} (x_{n+1}^{1-2\sigma} a^{ij}(x) \frac{\partial}{\partial x_j} u(x)) = 0, & \text{in } B_2^+, \\
- \lim_{x_{n+1} \to 0^+} x_{n+1}^{1-2\sigma} a^{n+1,n+1} \frac{\partial u(x)}{\partial x_{n+1}} = b(x') u + f(x'), & \text{on } \partial B_2^+.
\end{cases}
\]

(56)

We say \( u \in H^1(\rho_1^{1-2\sigma}, B_2^+) \) is a weak solution of (56) if

\[
\int_{B_2^+} x_{n+1}^{1-2\sigma} a^{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial \varphi}{\partial x_i} = \int_{\partial B_2^+} b(x') u(x', 0) \varphi(x', 0) + f(x') \varphi(x', 0)
\]

for all \( \varphi \in C^\infty_c (B_2^+ \cup \partial B_2^+) \).
Theorem A.2. Suppose that \( b, f \in L^p(B_2) \) for some \( p > \frac{n}{2\sigma} \). Let \( u \in H^1(x_{n+1}^{1-2\sigma}, B_2^+) \) be a weak solution of (56). Then there exist constants \( \gamma \in (0,1) \), \( C > 0 \) depending only on \( n, \sigma, \Lambda, p, \|b\|_{L^p(B_2)} \) such that \( u \in C^\gamma(B_1^+) \) and

\[
\|u\|_{C^\gamma(B_1^+)} \leq C(\|u\|_{L^1(x_{n+1}^{1-2\sigma}, B_2^+)} + \|f\|_{L^p(B_2)}).
\]

Proof. It follows from a modification of the proof of Proposition 2.4 in [26], which uses standard Moser iteration techniques.

Theorem A.3. Suppose that \( b, f \in C^\beta(B_2) \) for some \( 0 < \beta \notin \mathbb{N} \). Let \( u \in H^1(x_{n+1}^{1-2\sigma}, B_2^+) \) be a weak solution of (56). Suppose that \( 2\sigma + \beta \) is not an integer. Then \( x_{n+1}^{1-2\sigma} \partial u(x) \partial x_{n+1} \in C(B_1^+) \), and \( u(\cdot, 0) \in C^{2\sigma + \beta}(B_1) \). Moreover,

\[
\|x_{n+1}^{1-2\sigma} \partial u(x) \partial x_{n+1} \|_{C^\beta(B_1^+)} + \|u(\cdot, 0)\|_{C^{2\sigma + \beta}(B_1)} \leq C(\|u\|_{L^2(x_{n+1}^{1-2\sigma}, B_2^+)} + \|f\|_{C^\beta(B_2)}),
\]

where \( C > 0 \) depending only on \( n, \sigma, \Lambda, \beta, \|b\|_{C^\beta(B_2)} \).

Proof. It follows from modifications of the proofs of Theorem 2.3 and Lemma 2.3 in [26].

Proposition A.1. Let \( b, f \in C^k(B_2) \), \( u \in H^1(x_{n+1}^{1-2\sigma}, B_2^+) \) be a weak solution of (56), where \( k \) is a positive integer. Then we have

\[
\sum_{j=1}^{k} \|\nabla_j^x u\|_{L^\infty(B_1^+)} \leq C(\|u\|_{L^2(x_{n+1}^{1-2\sigma}, B_2^+)} + \|f\|_{C^k(B_2)}),
\]

where \( C > 0 \) depending only on \( n, \sigma, \Lambda, \beta, \|b\|_{C^k(B_2)} \).

Proof. It follows from a modification of the proof of Proposition 2.5 in [26].

A.3 Degenerate elliptic equations with conormal boundary conditions involving measures

We start with some Sobolev embeddings. For every \( p \in [1, +\infty) \), we define \( W^{1,p}(\rho^{1-2\sigma}, M) \) as the closure of \( C^\infty(M) \) under the norm

\[
\|u\|_{W^{1,p}(\rho^{1-2\sigma}, M)} = \left( \int_M \rho^{1-2\sigma}(|u|^p + |\nabla u|^p) \, dv_g \right)^{\frac{1}{p}},
\]

where \( dv_g \) denote the volume form of \( (M, g) \). \( W^{1,p}(\rho^{1-2\sigma}, M) \) is a Banach space for all \( p \in [1, +\infty) \) (see [30]). The following Proposition follows directly from Theorem 8.8 and Theorem 8.12 in [23].
Proposition A.2. Let $\Omega$ be a bounded domain in $\mathbb{R}^{n+1}$ with Lipschitz boundary $\partial \Omega$. Let $\sigma \in (0,1)$, $1 \leq p \leq q < \infty$ with $\frac{1}{p+1} > \frac{1}{p} - \frac{1}{q}$ and $d(x)$ be the distance from $x$ to $\partial \Omega$.

(i) Suppose that $2 - 2\sigma \leq p$. Then $W^{1,p}(d^{1-2\sigma}, \Omega)$ is compactly embedded in $L^q(d^{1-2\sigma}, \Omega)$ if

$$\frac{2 - 2\sigma}{p(n + 2 - 2\sigma)} > \frac{1}{p} - \frac{1}{q}.$$ 

(ii) Suppose that $2 - 2\sigma > p$. Then $W^{1,p}(d^{1-2\sigma}, \Omega)$ is compactly embedded in $L^q(d^{1-2\sigma}, \Omega)$ if and only if

$$\frac{1}{n + 2 - 2\sigma} > \frac{1}{p} - \frac{1}{q}.$$ 

Corollary A.1. For $n \geq 2$, let $(M,g)$ be an $n + 1$ dimensional, compact, smooth Riemannian manifold with smooth boundary $\partial M$. Let $\sigma \in (0,1)$, and $\rho$ be a defining function of $M$ with $|\nabla g \rho| = 1$ on $\partial M$. Let $1 \leq p \leq q < \infty$ with $\frac{1}{p+1} > \frac{1}{p} - \frac{1}{q}$.

(i) Suppose that $2 - 2\sigma \leq p$. Then $W^{1,p}(\rho^{1-2\sigma}, M)$ is compactly embedded in $L^q(\rho^{1-2\sigma}, M)$ if

$$\frac{2 - 2\sigma}{p(n + 2 - 2\sigma)} > \frac{1}{p} - \frac{1}{q}.$$ 

(ii) Suppose that $2 - 2\sigma > p$. Then $W^{1,p}(\rho^{1-2\sigma}, M)$ is compactly embedded in $L^q(\rho^{1-2\sigma}, M)$ if and only if

$$\frac{1}{n + 2 - 2\sigma} > \frac{1}{p} - \frac{1}{q}.$$ 

Proof. It follows from Proposition A.2 and partition of unity. \qed

Proposition A.3. For $n \geq 2$, let $(M,g)$ be an $n + 1$ dimensional, compact, smooth Riemannian manifold with smooth boundary $\partial M$. Let $\sigma \in (0,1)$, $\rho$ be a defining function of $M$ with $|\nabla g \rho| = 1$ on $\partial M$, and $(u)_{M,\rho} = \int_M \rho^{1-2\sigma} u \, dV_g / \int_M \rho^{1-2\sigma} \, dV_g$. Let $1 < p < \infty$. Then there exists a constant $C$, depending only on $M, g, p, n, \sigma$ and $\rho$, such that

$$\|u - (u)_{M,\rho}\|_{L^p(\rho^{1-2\sigma}, M)} \leq C \|
abla g u\|_{L^p(\rho^{1-2\sigma}, M)}$$ 

(57)

for every function $u \in W^{1,p}(\rho^{1-2\sigma}, M)$.

Proof. We argue by contradiction. Were the stated estimate false, there would exist for each integer $k = 1, 2, \cdots$ a function $u_k \in W^{1,p}(\rho^{1-2\sigma}, M)$ satisfying

$$\|u_k - (u_k)_{M,\rho}\|_{L^p(\rho^{1-2\sigma}, M)} > k \|
abla g u_k\|_{L^p(\rho^{1-2\sigma}, M)}.$$ 

For each $k$, define

$$v_k := \frac{u - (u)_{M,\rho}}{\|u - (u)_{M,\rho}\|_{L^p(\rho^{1-2\sigma}, M)}}.$$
Then
\[ (v_k)_{M,\rho} = 0, \quad \|v_k\|_{L^p(\rho^{1-2\sigma}, M)} = 1, \quad \|\nabla g v_k\|_{L^p(\rho^{1-2\sigma}, M)} < 1/k. \]

By Corollary \[A.1\] there exists a subsequence of \( \{v_k\} \), which is still denoted as \( \{v_k\} \), and a function \( v \in L^p(\rho^{1-2\sigma}, M) \) such that
\[ v_k \to v \text{ in } L^p(\rho^{1-2\sigma}, M), \quad v_k \to v \text{ in } W^{1,p}(\rho^{1-2\sigma}, M). \]

Consequently,
\[ (v)_{M,\rho} = 0, \quad \|v\|_{L^p(\rho^{1-2\sigma}, M)} = 1, \quad \|\nabla g v\|_{L^p(\rho^{1-2\sigma}, M)} \leq \liminf_{k \to \infty} \|\nabla g v_k\|_{L^p(\rho^{1-2\sigma}, M)} = 0. \]

We reach a contradiction. \( \square \)

**Corollary A.2.** For \( n \geq 2 \), let \( (M, g) \) be an \( n+1 \) dimensional, compact, smooth Riemannian manifold with smooth boundary \( \partial M \). Let \( \sigma \in (0,1), \rho \) be a defining function of \( M \) with \( |\nabla g \rho| = 1 \) on \( \partial M \), and \( (u)_{M,\rho} = \int_M \rho^{1-2\sigma} u \, dV_g / \int_M \rho^{1-2\sigma} \, dV_g \). Let \( 1 < p < \infty \). Then there exists a constant \( \delta_0 \) depending only on \( n, \sigma, p \) such that for any \( 1 \leq k \leq 1 + \delta_0 \),
\[ \|u - (u)_{M,\rho}\|_{L^k(\rho^{1-2\sigma}, M)} \leq C \|\nabla g u\|_{L^p(\rho^{1-2\sigma}, M)} \tag{58} \]
for every function \( u \in W^{1,p}(\rho^{1-2\sigma}, M) \), where \( C \) is a positive constant depending only on \( M, g, p, n, \sigma, \rho \).

**Proof.** By Corollary \[A.1\] there exists a constant \( \delta_0 \) depending only on \( n, \sigma, p \) such that for any \( 1 \leq k \leq 1 + \delta_0 \),
\[ \|u - (u)_{M,\rho}\|_{L^k(\rho^{1-2\sigma}, M)} \leq C \|\nabla g u\|_{L^p(\rho^{1-2\sigma}, M)} + C \|u - (u)_{M,\rho}\|_{L^p(\rho^{1-2\sigma}, M)} \leq C \|\nabla g u\|_{L^p(\rho^{1-2\sigma}, M)} \tag{59} \]
where in the last inequality we have used Proposition \[A.3\] \( \square \)

Let \( (M, g), \rho \) be as in Theorem \[1.1\]. For \( \sigma \in (0,1) \), we consider
\[ \left\{ \begin{array}{l}
\text{div}_g(\rho^{1-2\sigma} \nabla g u) = 0, \quad \text{in } M \\
\lim_{y \to x \in \partial M} \rho(y)^{-1-2\sigma} \frac{\partial u}{\partial \nu} = f(x) \quad \text{on } \partial M.
\end{array} \right. \tag{59} \]

We say \( u \in W^{1,1}(\rho^{1-2\sigma}, M) \) is a weak solution of \( \text{(59)} \) if
\[ \int_M \rho^{1-2\sigma} \langle \nabla g u, \nabla g \varphi \rangle \, dV_g = \int_{\partial M} f \varphi \, d\sigma_g \tag{60} \]
for all \( \varphi \in C^\infty(M) \). Define \( \tilde{H}^1 := \{ u \in H^1(\rho^{1-2\sigma}, M) : \int_M \rho^{1-2\sigma} u \, dV_g = 0 \} \).

28
Lemma A.3. Let \( f \in H^{-\sigma}(\partial M) := (H^{\sigma}(\partial M))^* \), the dual of \( H^{-\sigma}(\partial M) \), such that \( \langle f, 1 \rangle = 0 \). Then \((59)\) admits a unique weak solution \( u \in \tilde{H}^1 \).

Proof. The lemma follows immediately from Proposition A.3 and the Lax-Milgram theorem. \(\square\)

Lemma A.4. Let \( f \in L^2(\partial M) \) with zero mean value, \( u \in \tilde{H}^1 \) be the weak solution of \((59)\). Then for any \( \theta > 1 \),

\[
\int_M \rho^{1-2\sigma} \frac{\left| \nabla_g u \right|^2}{(1 + |u|^\theta)^\theta} \, dv_g \leq \frac{1}{\theta - 1} \| f \|_{L^1(\partial M)}.
\]

Proof. In our proofs of this and the next lemma, we adapt some arguments from [69] and [18]. For \( \theta > 0 \), let \( \phi_\theta(r) = \int_0^r \frac{dt}{(t + \theta)^\theta} \) if \( r \geq 0 \) and \( \phi_\theta(-r) = -\phi_\theta(r) \) if \( r < 0 \). It is easy to see that \( \varphi_\theta := \phi_\theta(u) \in H^1(\rho^{1-2\sigma}, M) \) and \( |\varphi_\theta| \leq 1/(\theta - 1) \) on \( \overline{M} \) if \( \theta > 1 \). Hence, the Lemma follows from multiplying \((60)\) by letting \( \varphi = \varphi_\theta \). \(\square\)

Lemma A.5. Let \( f \in L^2(\partial M) \) with zero mean value, \( u \in \tilde{H}^1 \) be the weak solution of \((59)\). Then there exists \( \varepsilon_0 > 0 \) depending only on \( n \) and \( \sigma \) such that for any \( 1 \leq \tau \leq 1 + \varepsilon_0 \), we have

\[
\| u \|_{W^{1,\tau}(\rho^{1-2\sigma}, M)} \leq C,
\]

where \( C > 0 \) depends only on \( M, g, \sigma, \rho, \| f \|_{L^1(\partial M)} \).

Proof. By the Hölder inequality,

\[
\int_M \rho^{1-2\sigma} |\nabla_g u|^\tau \, dv_g \leq \left( \int_M \rho^{1-2\sigma} |\nabla_g u|^2 \, dv_g \right)^{\tau/2} \left( \int_M \rho^{1-2\sigma} (1 + |u|)^{2\theta} \, dv_g \right)^{(2-\tau)/2} \leq C(\theta) \left( \int_M \rho^{1-2\sigma} (1 + |u|)^{\frac{\theta}{2-\tau}} \, dv_g \right)^{(2-\tau)/2},
\]

where we used Lemma A.4 in the last inequality and \( \theta \in (1, 2) \) will be chosen later. Applying Corollary A.2 (see also [17]) to \( \varphi_{\theta/2} \) yields that for any \( 1 \leq k \leq 1 + \delta_0 \)

\[
\left( \int_M \rho^{1-2\sigma} |\varphi_{\theta/2}| - \int_M \rho^{1-2\sigma} \varphi_{\theta/2}^k \, dv_g \right)^{1/k} \leq C \int_M \rho^{1-2\sigma} \frac{|\nabla_g u|^2}{(1 + |u|)^\theta} \, dv_g,
\]

where \( \delta_0 > 0 \) depends only on \( n, \sigma \), and \( C \) depends only on \( M, g, \rho, k \). Since \( \phi_{\theta/2}(r) \approx |r|^{1-\frac{\theta}{2}} \) for \( |r| \) large, it follows from \((62)\) and Lemma A.4 that

\[
\left( \int_M \rho^{1-2\sigma} |u|^{k(2-\theta)} \right)^{1/2k} \, dv_g \leq C + C \int_M \rho^{1-2\sigma} |u|^{1-\frac{\theta}{2}} \, dv_g.
\]
Choosing \( \theta \) close to 1 such that \( k(2 - \theta) = \frac{\pi \theta}{2 - \pi} \) (this can be achieved as long as \( \pi \) is closed to 1) and inserting (63) to (61), we obtain

\[
\left( \int_M \rho^{1 - 2\sigma} |\nabla_g u|^\tau \, dv_g \right)^{1/\tau} \leq C \left( 1 + \int_M \rho^{1 - 2\sigma} |u|^{1 - \frac{\theta}{2}} \, dv_g \right)^{\frac{\theta}{2 - \pi}}
\]

\[
\leq C + C \left( \int_M \rho^{1 - 2\sigma} |u| \, dv_g \right)^{\frac{\theta}{2}} \tag{64}
\]

Since \( \int_M \rho^{1 - 2\sigma} u \, dv_g = 0 \), by the Poincaré-Sobolev inequality, Hölder inequality and (64), we have

\[\|u\|_{L^1(\rho^{1 - 2\sigma}, M)} \leq C \int_M \rho^{1 - 2\sigma} |\nabla_g u| \, dv_g \leq C(1 + \|u\|_{L^1(\rho^{1 - 2\sigma}, M)}) \]

Thus, \( \|u\|_{L^1(\rho^{1 - 2\sigma}, M)} \leq C \) because \( \frac{\theta}{2} < 1 \). Therefore, the lemma follows immediately from (64) and the Poincaré-Sobolev inequality.

\[\square\]

**Theorem A.4.** For any bounded radon measure \( f \) defined on \( \partial M \) with \( \langle f, 1 \rangle = 0 \), there exists a weak solution \( u \in W^{1,1+\varepsilon_0}(\rho^{1 - 2\sigma}, M) \) of (59).

**Proof.** The proof follows from Lemma A.3 and A.5 and some standard approximating procedure, see, e.g., [18]. We omit the details here.

\[\square\]

**Theorem A.5.** For \( x_0 \in \partial M \), let \( f = \delta_{x_0} \) where \( |\partial M|_g \) is the area of \( \partial M \) with respect to the induced metric \( g \). Then there exists a weak solution \( u \in W^{1,1+\varepsilon_0}(\rho^{1 - 2\sigma}, M) \) of (59) with mean value zero and for all \( x \in \overline{M} \setminus \{x_0\} \),

\[
A_1 \text{dist}_g(x, x_0)^{2\sigma - n} - A_0 \leq u(x) \leq A_2 \text{dist}_g(x, x_0)^{2\sigma - n}, \tag{65a}
\]

\[
|\nabla_{\text{tan}} u| \leq A_3 \text{dist}_g(x, x_0)^{2\sigma - n - 1}, \tag{65b}
\]

\[
\frac{\partial u}{\partial \nu} \leq A_4 \rho^{2\sigma - 1} \text{dist}_g(x, x_0)^{-n}, \tag{65c}
\]

where \( A_0, A_1, A_2, A_3, A_4 \) are positive constants depending only on \( M, g, n, \sigma, \rho \).

**Proof.** Let \( f_k \in C^1(\partial M) \) with \( \int_{\partial M} f_k \, d\sigma_g = 0 \), \( \|f_k\|_{L^1(\partial M)} \leq C \) independent of \( k \), such that \( f_k \to f \) in distribution sense as \( k \to \infty \). We can also assume that \( f_k \to f \) in \( C^0_{\text{loc}}(\partial M \setminus \{x_0\}) \).

By Lemma A.3 and Lemma A.5 there exists a unique solution \( u_k \in H^1 \) of (59) with \( f \) replaced by \( f_k \), and

\[
\|u_k\|_{W^{1,1+\varepsilon_0}(\rho^{1 - 2\sigma}, M)} \leq C(\|f_k\|_{L^1(\partial M)}) \leq C.
\]

Moreover, it follows from Moser’s iterations (see, e.g., the proof of Theorem A.2) that there exists some \( \alpha > 0 \) such that

\[
\|u_k\|_{C^\alpha(M \setminus B_r(x_0))} \leq C(r) \tag{66}
\]
for any $r > 0$. By standard compactness arguments, $u_k \rightharpoonup u$ in $W^{1,1+\varepsilon}(\rho^{1-2\sigma}, M)$ for some $u$, which is a weak solution of \eqref{59} and satisfies

$$\|u\|_{C^{\alpha/2}(M \setminus B_r(x_0))} \leq C(r).$$

Now, it suffices to establish the estimate \eqref{65a} for $x \in B_r(x_0)$. For $r$ suitably small, choose a Fermi coordinate system $\{y_1, \cdots, y_{n+1}\}$ centered at $x_0$. Then $u_k(y)$ satisfies

$$\begin{aligned}
\partial_t(\rho^{1-2\sigma} \sqrt{\det g} g^{ij} \partial_j u_k) &= 0, & \text{in } B^+_r, \\
- \lim_{y_{n+1} \to 0} \rho^{1-2\sigma} \sqrt{\det g} \frac{\partial u_k}{\partial y_{n+1}} &= f_k, & \text{on } \partial B^+_r.
\end{aligned}$$

Let $v_k$ be the unique weak solution of

$$\begin{aligned}
\partial_t(\rho^{1-2\sigma} \sqrt{\det g} g^{ij} \partial_j v_k) &= 0, & \text{in } B^+_r, \\
- \lim_{y_{n+1} \to 0} \rho^{1-2\sigma} \sqrt{\det g} \frac{\partial v_k}{\partial y_{n+1}} &= -\frac{1}{|\partial M|}, & \text{on } \partial B^+_r,
\end{aligned}$$

in $H^1(\rho^{1-2\sigma}, M)$. In view of \eqref{66}, $\|v_k\|_{L^\infty(B_{2r})} \leq C(r)$ and hence $\|v_k\|_{C^{\alpha}(B^+_r)} \leq C(r)$. Moreover, $w_k := u_k - v_k \in H^1(\rho^{1-2\sigma}, M)$ satisfies

$$\begin{aligned}
\partial_t(\rho^{1-2\sigma} \sqrt{\det g} g^{ij} \partial_j w_k) &= 0, & \text{in } B^+_r, \\
- \lim_{y_{n+1} \to 0} \rho^{1-2\sigma} \sqrt{\det g} \frac{\partial w_k}{\partial y_{n+1}} &= f_k + \frac{1}{|\partial M|}, & \text{on } \partial B^+_r,
\end{aligned}$$

$$w_k = 0,$$

in $\partial B^+_r$. Let $\bar{w}_k$ be the even extension of $w_k$ in $B_{2r}$, i.e.,

$$\bar{w}_k = \begin{cases}
  w_k(y', y_{n+1}), & y_{n+1} \geq 0, \\
  w_k(y', -y_{n+1}), & y_{n+1} \leq 0.
\end{cases}$$

We also evenly extend $g$ and $\rho$ to be $\bar{g}$ and $\bar{\rho}$, respectively. It is easy to verify that the weak limit $w$ of $\bar{w}_k$ in $L^{1+\varepsilon}(\rho^{1-2\sigma}, B_{2r})$ is the weak solution vanishing on $\partial B_{2r}$ (see page 162 of \cite{16}) of

$$\partial_t(\bar{\rho}^{1-2\sigma} \sqrt{\det \bar{g}} g^{ij} \partial_j w) = -2\delta_0 \in B_{2r}.$$ 

It follows from Theorem 3.3 of \cite{16} that $w$ satisfies the estimates \eqref{65a} in $B_r(x_0)$. Thus, $u$ satisfies \eqref{65a}. Finally, \eqref{65b} and \eqref{65c} follows from \eqref{65a}, Theorem A.3, Proposition A.1 and some scaling arguments. 

$\square$
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