SEMI-EXACT SOLUTIONS AND PULSATING FRONTS FOR LOTKA-VOLTERRA SYSTEMS OF TWO COMPETING SPECIES IN SPATIALLY PERIODIC HABITATS

CHIUN-CHUAN CHEN
Department of Mathematics, National Taiwan University
National Center for Theoretical Sciences, Taipei, Taiwan

YIN-LIANG HUANG
Department of Applied Mathematics
National University of Tainan, Tainan, Taiwan

LI-CHANG HUNG*
Department of Mathematics
National Taiwan University, Taipei, Taiwan

CHANG-HONG WU
Department of Applied Mathematics
National Chiao Tung University, Taiwan

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Abstract. We are concerned with the coexistence states of the diffusive Lotka-Volterra system of two competing species when the growth rates of the two species depend periodically on the spatial variable. For the one-dimensional problem, we employ the generalized Jacobi elliptic function method to find semi-exact solutions under certain conditions on the parameters. In addition, we use the sine function to construct a pair of upper and lower solutions and obtain a solution of the above-mentioned system. Next, we provide a sufficient condition for the existence of pulsating fronts connecting two semi-trivial states by applying the abstract theory regarding monotone semiflows. Some numerical simulations are also included.

1. Introduction. In this paper, our main purpose is to study the coexistence states of the following diffusive Lotka-Volterra system of two competing species in spatially heterogeneous habitats:

\[
\begin{align*}
    u_t &= d_1 \Delta u + u \left( m_1(x) - c_{11} u - c_{12} v \right), \quad x \in \mathbb{R}^N, \quad t > 0, \\
    v_t &= d_2 \Delta v + v \left( m_2(x) - c_{21} u - c_{22} v \right), \quad x \in \mathbb{R}^N, \quad t > 0,
\end{align*}
\]

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* Corresponding author.
where the diffusion rates \( d_1 \) and \( d_2 \) are positive constants; the pair of solutions \((u,v) = (u(x,t), v(x,t))\) represents the density of the two species \( u \) and \( v \) which depends on the spatial variable \( x \) and time \( t \); the positive constants \( c_{11} \) and \( c_{22} \) stand for self-regulation of each species; the positive constants \( c_{12} \) and \( c_{21} \) measure the competition between the two species; the spatial functions \( m_1(x) \) and \( m_2(x) \) are the growth rates of the two species \( u \) and \( v \), respectively, exhibiting the effect of spatial heterogeneity of the environment.

In particular, we consider the growth rates \( m_1(x) \) and \( m_2(x) \) which depend on the spatial variable \( x \) represent the situation when the resources competed by the two species are heterogeneously distributed. Under such circumstance of heterogeneity, when the two species are restricted in a bounded region and are assumed to be identical except for their dispersal strategies, i.e., \( m_1(x) \equiv m_2(x) \) and \( c_{ij} = 1 \) for \( i, j = 1, 2 \), the joint effect of spatial heterogeneity and diffusion under the Neumann condition has been investigated and the phenomenon ”slower diffuser always prevails” has been found in [6]. In this case, non-existence of coexistence states can be shown. By a coexistence state, we mean both components of a solution \((u(x), v(x))\) to

\[
\begin{align*}
0 &= d_1 \Delta u + u (m_1(x) - c_{11} u - c_{12} v), \quad x \in \mathbb{R}^N, \\
0 &= d_2 \Delta v + v (m_2(x) - c_{21} u - c_{22} v), \quad x \in \mathbb{R}^N,
\end{align*}
\]

are positive for \( x \in \mathbb{R}^N \).

In 1979, Freidlin and Gartner were the first to investigate the heterogeneous equation in [10]. They studied the case of space-periodic coefficients and generalized the spreading properties. The effects of dispersal and spatial heterogeneity for (1) with the zero Neumann boundary condition have been widely studied in the last two decades. In 2006, Lou [17] considered (1) the case where \( m_1(x) \equiv m_2(x) \), \( c_{ii} \) is normalized to 1 for \( i = 1, 2 \) and \( 0 < c_{12}, c_{21} < 1 \) (weak competition case) under the zero Neumann boundary condition. He found that, due to the spatially heterogeneous environment, two weakly competing species may not always coexist. Later, Lam and Ni [15] solved a conjecture proposed in [17]. Very recently, He and Ni [12] have completely determined the global dynamics of (1) under the zero Neumann boundary condition. For related works devoted to the model (1) on bounded domains, we also refer to [3, 6, 14, 18, 19] and the references therein.

Much attention seems to have been paid to the case where spatially bounded domains are considered. However, when the domain is extended to the entire space \( \mathbb{R}^N \) (\( N \geq 1 \)), very little is known about the dynamics of (1). As a consequence, the following questions related to (2) naturally arise:

- As discussed for bounded domains, how is the joint effect of spatial heterogeneity and diffusion ([19]) for unbounded domains?
- What is the sufficient condition for the two species to coexist?
- What is the condition under which the two species cannot coexist and then one species can invade the other one (the wave propagation occurs)?

These questions of interest lead us to propose the issue of the present paper. To be more specific, we shall investigate the coexistence states of (1), i.e., find a pair of solutions \((u(x), v(x))\) satisfying (2). For simplicity, throughout this paper we consider the heterogeneously distributed resources \( m_1(x) \) and \( m_2(x) \) which are spatially periodic, i.e., \( m_i(x + L_j e_j) = m_i(x) \) for all \( i, j = 1, 2 \) and \( j = 1, ..., N \), where \( \{e_j\} \) is an orthonormal basis in \( \mathbb{R}^N \). The spatial heterogeneity in \( x \) stands for the domain under consideration consisting of the region which is inferior to the species
and the region which is superior to the species. For instance, the inhabited domain may contain desert and oasis.

Our motivation for studying the heterogeneous problem related to (2) is inspired by the work of Berestycki, Hamel and Roques [1], where a complete investigation of the following periodic heterogeneous model

\[ u_t = \nabla \cdot (A(x) \nabla u) + f(x, u), \quad x \in \mathbb{R}^N, \quad t > 0 \]  

(3)

is given. More precisely, the existence, uniqueness and stability results are established and criteria under which the species persists is given in [1].

Based on the periodic frameworks as in [1], we may expect that under some conditions, there exist spatially periodic coexistence states of (1). It is not hard to obtain some results regarding the existence of stable periodic coexistence states of (1) by using the method of upper and lower solutions with the help of the sign of the principal eigenvalue associated with the linearized operator about the semi-trivial states, and these results may be similar to those reported in the literature that considered problems with the Neumann boundary problems, which will not be our concern here. Instead, we shall make it by using semi-exact solutions as well as employing the method of upper and lower solutions. In [4], the authors show that under certain conditions, traveling wave solutions for Lotka-Volterra competitive systems of three-species exist by giving semi-exact solutions. By a semi-exact solution which seems, to the best of our knowledge, to be the first in the literature, we mean that each component of the solution \((u, v, w)\) can be expressed in terms of a polynomial of an implicit function that satisfies an autonomous differential equation. For the one-dimensional problem, we employ the generalized Jacobi elliptic function method to find the coexistence states of (2) for the case where \(m_1(x)\) are \(m_2(x)\) are polynomials of generalized Jacobi elliptic functions.

Another interesting feature of problem (3) is the wave propagation. In [2], the authors showed the existence of pulsating fronts connecting 0 and the positive steady state. To capture this phenomenon for our system, we shall give a criterion for the non-existence of coexistence states of (1). With this, one can apply the abstract theory theory regarding monotone semiflows established by Yu and Zhao [21] to show the existence of pulsating fronts of system (1) connecting two semi-trivial states in the strong-weak competition case. We also refer to the work of Girardin [11] regarding the existence of pulsating fronts in strong competition case.

This paper is organized as follows. In Section 2 we apply the Jacobi elliptic function method to study (2). Under certain conditions, semi-exact solutions as well as upper and lower solutions are given. Section 3 is devoted to the proof of the existence of pulsating fronts. Finally, some numerical results are presented in Section 4.

2. One-dimensional problem: semi-exact solutions and method of upper and lower solutions. We consider in this section system (2) in one spatial dimension:

\[
\begin{align*}
0 &= d_1 u_{xx} + u \left( m_1(x) - c_{11} u - c_{12} v \right), \quad x \in \mathbb{R}, \\
0 &= d_2 v_{xx} + v \left( m_2(x) - c_{21} u - c_{22} v \right), \quad x \in \mathbb{R}.
\end{align*}
\]

(4)

The generalized Jacobi elliptic function method will be introduced in Section 2.1. Applying this method, a semi-exact solution of (4) will be given in Theorem 2.1. Periodic solutions \((u(x), v(x))\) of (4) will be constructed in Section 2.2 by the method of upper and lower solutions.
2.1. Generalized Jacobi elliptic function method: semi-exact solutions. In this subsection, we introduce the generalized Jacobi elliptic function method ([7, 22]) and its algorithm. By making an appropriate ansatz for solutions, the spirit of this method lies in the fact that through the Jacobi elliptic function method a problem of solving differential equations is transformed into a problem of solving algebraic equations. Applying this method, semi-exact solutions of (4) with \( m_i \) (i=1,2) being polynomials of \( \phi \) which is the generalized Jacobi elliptic function satisfying (9) can be given under certain constraints on the parameters.

To explain the generalized Jacobi elliptic function method, let us consider a second-order partial differential equation (PDE) with the solution \( u = u(\varsigma, t) \) in the two independent variables \( \varsigma \) and \( t \):

\[
F(u, u_t, u_{\varsigma}, u_{\varsigma\varsigma}, u_{\varsigma t}, u_{tt}, \cdots) = 0.
\]

(5)

The generalized Jacobi elliptic function method is as follows:

- **Step 1.** Using the wave transformation

\[
u(\varsigma, t) = u(x), \quad x = \varsigma - \theta t,
\]

where \( \theta \) is the wave speed to be determined, (5) becomes an ordinary differential equation (ODE):

\[
G(u, u', u'', \cdots) = 0,
\]

(7)

where \( u' \) means \( \frac{du}{dx} \) and \( u'' \) means \( \frac{d^2u}{dx^2} \).

- **Step 2.** Apply a judicious ansatz for solutions which is expressed in terms of a polynomial of the generalized Jacobi elliptic function \( \phi = \phi(x) \)

\[
u(x) = \sum_{m=0}^{n} a_m \phi^m(x),
\]

(8)

where \( a_0, a_1, a_2, \cdots, a_n \) are constants to be determined. Here \( \phi(x) \) satisfies the following second-order ODE:

\[
\phi''(x) = k_3 \phi^3(x) + k_1 \phi(x)
\]

(9)

for some constants \( k_1 \) and \( k_3 \). From (9) it follows that

\[
\phi'^2(x) = \frac{k_3}{2} \phi^4(x) + k_1 \phi^2(x) + k_0,
\]

(10)

where \( k_0 \) is an integrating constant.

- **Step 3.** Substituting the ansatz (8) into (7) and setting the coefficients of the powers of \( \phi \) to 0 lead to a system of algebraic equations.

- **Step 4.** Using software for symbolic computation (e.g. Mathematica), solve the resulting system of algebraic equations in the previous step.

We remark that when \( k_0 = 0 \) and \( k_3 = 2, k_1 = -2 \), the generalized Jacobi elliptic function \( \phi = \phi(x) \) which solves (9) is reduced to the hyperbolic function \( \tanh \ z \). In this case, exact traveling wave solutions for Lotka-Volterra competitive systems of two-species ([20]) and three-species ([4, 5]) have been found by the hyperbolic function method. However, exact solutions in [4, 5, 20] are given only for the case where the growth rates are constants. In this paper, under certain conditions on the parameters, semi-exact solutions of (4) can be constructed when spatial heterogeneity of growth rates is allowed.

Following the steps of the generalized Jacobi elliptic function method introduced above, we give semi-exact solutions to (4) or standing wave solutions to (1) with \( N = \cdots \)
1 when the growth rates are polynomials of generalized Jacobi elliptic functions. With the aid of software Mathematica, which allows us to perform complicated and tedious algebraic calculations on a computer, we are led to the result as follows (see also Section 5).

**Theorem 2.1.** Suppose that \( l_i, n_i, \gamma_i \) (\( i = 0, 1 \)), \( \rho_j \) (\( j = 0, 1, 2 \)), \( k_1, k_3 \) and \( \phi_0 \neq 0 \) are constants. Let \( m_1(x) = l_0 + l_1 \phi(x) \) and \( m_2(x) = n_0 + n_1 \phi(x) \), where \( \phi = \phi(x) \) is the solution of the following problem

\[
\begin{align*}
\phi''(x) &= k_3 \phi^3(x) + k_1 \phi(x), \quad x \in \mathbb{R}, \\
\phi(0) &= 0, \quad \phi'(0) = \phi_0.
\end{align*}
\]

(11)

Assume that either (49) or (50) (see Section 5) holds. Then (4) admits a solution \((u(x), v(x))\) of the form

\[
\begin{align*}
u(x) &= \rho_0 + \rho_1 \phi(x) + \rho_2 \phi^2(x), \\
u(x) &= \gamma_0 + \gamma_1 \phi(x).
\end{align*}
\]

(12)

We note that the existence of semi-exact solutions expressed in terms of elliptic functions to (4) has been shown for the homogeneous case, i.e. for the case when \( m_1(x) \) and \( m_2(x) \) are constants independent of \( x \) in [20]. Our Theorem 2.1 asserts that the method of semi-exact solutions can be applied successfully in the framework of heterogeneous competitive Lotka-Volterra equations.

As an example to illustrate Theorem 2.1, we give the following remark. We note that the initial condition \( \phi(0) = 0 \) and (10) lead to \((\phi'(0))^2 = \phi_0^2 = k_0\).

**Remark 1** (Semi-exact solution with periodic \( m_1(x) \) and \( m_2(x) \)). The following example shows that there do exist parameters which satisfy either (49) or (50).

Indeed, letting

\[
d_2 = 1, c_{11} = 1, c_{12} = 1, c_{21} = \frac{1}{2}, c_{22} = 1, \gamma_1 = \frac{\sqrt{3}}{8}, \rho_0 = 1, \rho_1 = 1, \rho_2 = \frac{3}{16}
\]

in (49) leads to

\[
d_1 = 3, k_0 = \frac{8}{81}, k_1 = -\frac{17}{27}, k_3 = \frac{1}{16}, \rho_0 = 1, \rho_1 = 1, \rho_2 = \frac{3}{16}
\]

(13a)

\[
 l_0 = 1 + \frac{\sqrt{17}}{6}, l_1 = 1 + \frac{\sqrt{3}}{8} + \frac{\sqrt{51}}{12},
\]

(13b)

\[
 n_0 = \frac{26}{27} + \frac{\sqrt{17}}{12}, n_1 = \frac{5}{3} + \frac{\sqrt{3}}{16}, \gamma_0 = \frac{\sqrt{17}}{6}.
\]

(13c)

By Theorem 2.1, the following system

\[
\begin{align*}
3u_{xx} + u \left( 1 + \frac{\sqrt{17}}{6} + \left( 1 + \frac{\sqrt{3}}{8} + \frac{\sqrt{51}}{12} \right) \phi(x) - u - v \right) &= 0, \quad x \in \mathbb{R}, \\
v_{xx} + v \left( \frac{26}{27} + \frac{\sqrt{17}}{12} + \left( \frac{5}{3} + \frac{\sqrt{3}}{16} \right) \phi(x) - \frac{1}{2} u - v \right) &= 0, \quad x \in \mathbb{R},
\end{align*}
\]

(15)

has a solution

\[
\begin{align*}
u(x) &= \frac{\sqrt{17}}{6} + \frac{\sqrt{3}}{8} \phi(x), \\
u(x) &= 1 + \phi(x) + \frac{3}{16} \phi^2(x).
\end{align*}
\]

(16)
where \( \phi = \phi(x) \) is the solution of the problem
\[
\begin{cases}
\phi''(x) = \frac{1}{16} \phi^3(x) - \frac{17}{27} \phi(x), 
& x \in \mathbb{R}, \\
\phi(0) = 0, \phi'(0) = \frac{2\sqrt{2}}{9}.
\end{cases}
\] (17)

Here we choose \( \phi'(0) = \sqrt{k_0} = \frac{2\sqrt{2}}{9} \). The profiles of \( \phi(x) \), \( u(x) \), \( v(x) \), \( m_1(x) \), and \( m_2(x) \) are shown in Figure 1 and Figure 2, respectively.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{The profile of \( \phi(x) \).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{The profiles of \( u \) (red), \( v \) (green), \( m_1 \) (blue) and \( m_2 \) (cyan).}
\end{figure}

**Remark 2** (Strong competition). We shall show in Section 2.2 (for more details see Theorem 2.3) that when \( c_{12} \) and \( c_{21} \) are sufficiently small, which corresponds to the case of weak competition, (4) admits a periodic solution by the method of upper and lower solutions. Under weak competition, our solution of (4) constructed by the method of upper and lower solutions can be shown to be stable.

To the best of the authors’ knowledge, there seems to be no study on the semi-exact solutions of (4) when strong competition occurs between the two species. One reason is due to the fact that a solution of (4) under strong competition is often unstable. We say (4) is under strong competition if the following inequalities hold:
\[
\frac{c_{22}}{c_{12}} < \frac{m_2(x)}{m_1(x)} < \frac{c_{21}}{c_{11}}, \quad x \in \mathbb{R}.
\] (18)
We use Theorem 2.1 to show that (4) admits a solution under strong competition given by (18). Indeed, we let
\[ d_2 = 1, c_{11} = 1, c_{12} = 3, c_{21} = 2, c_{22} = 1, \gamma_1 = \frac{\sqrt{3}}{8}, \rho_0 = 1, \rho_1 = 1, \rho_2 = \frac{3}{16} \] (19)
in (49) to obtain
\[ d_1 = 9, k_0 = \frac{8}{81}, k_1 = -\frac{17}{27}, k_3 = \frac{1}{16}, \]
\[ l_0 = 3 + \frac{\sqrt{17}}{6}, l_1 = 3 + \frac{\sqrt{3}}{8} + \frac{\sqrt{51}}{4}, \]
\[ n_0 = \frac{26}{27} + \frac{\sqrt{17}}{3}, n_1 = \frac{5}{3} + \frac{\sqrt{3}}{4}, n_2 = \frac{\sqrt{17}}{6}, \] (20)
which results in the following system
\[
\begin{cases}
9 u_{xx} + u \left( 3 + \frac{\sqrt{17}}{6} + \left( 3 + \frac{\sqrt{3}}{8} + \frac{\sqrt{51}}{4} \right) \phi(x) - u - 3 v \right) = 0, & x \in \mathbb{R}, \\
v_{xx} + v \left( \frac{26}{27} + \frac{\sqrt{17}}{3} + \left( \frac{5}{3} + \frac{\sqrt{3}}{4} \right) \phi(x) - 2 u - v \right) = 0, & x \in \mathbb{R}.
\end{cases}
\] (21)
According to Theorem 2.1, (21) has a solution
\[
\begin{cases}
u(x) = \frac{\sqrt{17}}{6} + \frac{\sqrt{3}}{8} \phi(x), \\
v(x) = 1 + \phi(x) + \frac{3}{16} \phi^2(x),
\end{cases}
\] (22)
where \( \phi = \phi(x) \) is the solution of (17) with \( \phi'(0) = \frac{2\sqrt{3}}{3} \). The profiles of \( u(x) \), \( v(x) \), \( m_1(x) \), and \( m_2(x) \) are shown in Figure 3. We see from Figure 4 that the condition of strong competition (18) is satisfied.

![Figure 3. The profiles of u (red), v (green), m_1 (blue) and m_2 (cyan).](image)

Remark 3 (Open problem). Theorem 2.1 motivates us to propose an open problem about the existence of semi-exact solutions when (4) is generalized to the three-species equations:

\[
\begin{cases}
0 = d_1 u_{xx} + u \left( m_1(x) - c_{11} u - c_{12} v - c_{13} w \right), & x \in \mathbb{R}, \\
0 = d_2 v_{xx} + v \left( m_2(x) - c_{21} u - c_{22} v - c_{23} w \right), & x \in \mathbb{R}, \\
0 = d_3 w_{xx} + v \left( m_3(x) - c_{31} u - c_{32} v - c_{33} w \right), & x \in \mathbb{R}.
\end{cases}
\] (23)
the solution of (17), and make the following ansatz for solving (23):

\[ \begin{align*}
&u(x) = \gamma_0 + \gamma_1 \phi(x) + \gamma_2 \phi^2(x), \\
v(x) = \rho_0 + \rho_1 \phi(x) + \rho_2 \phi^2(x), \\
w(x) = v_0 + v_1 \phi(x) + v_2 \phi^2(x),
\end{align*} \]

(24)

where \( \gamma_i, \rho_i \) and \( v_i \) (\( i = 0, 1, 2 \)) are constants to be determined. Substituting the ansatz into (23) and equating the coefficients of powers of \( \phi(x) \) to zero lead to a system of algebraic equations (cf. (47) in Section 5). We carry out a straightforward but tedious calculation and obtained detailed results by using Mathematica.

2.2. Method of upper and lower solutions. The idea of the proof of the main Theorem 2.3 in this subsection is as follows. Let \( \phi(x) = \sin(k x) \), where \( k \neq 0 \) is a constant. We insert the form \( (\bar{u}(x), \bar{v}(x)) = (M_1, a_1 \phi(x) + b_1) \) and \( (\bar{y}(x), \bar{v}(x)) = (a_2 \phi(x) + b_2, M_2) \) into (4). The assumptions (H1)∼(H4) then allow us to construct a pair of upper and lower solutions of (4) and obtain a solution of it. The lemma below is elementary.

Lemma 2.2. Let \( g(y) = \sum_{k=0}^{2} \eta_k \sin^k(y) \), where \( \eta_k \) (\( k = 0, 1, 2 \)) is a constant. Then

for \( y \in \mathbb{R} \),

(i) \( g(y) \geq 0 \) if \( \eta_0 \geq |\eta_1| - \min(0, \eta_2) \);

(ii) \( g(y) \leq 0 \) if \( -\eta_0 \geq |\eta_1| + \max(0, \eta_2) \).

Proof. The desired results follow from the properties that \( |\sin z| \leq 1 \) for \( z \in \mathbb{R} \).

Theorem 2.3 (Existence of coexistence states). Let \( M_1, M_2, b_1 \) and \( b_2 \) be positive constants and \( a_1, a_2 \) be constants. Assume that the following hypotheses are satisfied:

(H1) \( m_1(x) \) is periodic in \( x \) and satisfies \( c_{11} b_2 < m_1(x) \leq c_{11} M_1 \) for \( x \in \mathbb{R} \), where \( M_1 \geq b_2 + |a_2| \);

(H2) \( m_2(x) \) is periodic in \( x \) and satisfies \( c_{22} b_1 < m_2(x) \leq c_{22} M_2 \) for \( x \in \mathbb{R} \), where \( M_2 \geq b_1 + |a_1| \).

When \( c_{12}, c_{21}, |a_1| \) and \( |a_2| \) are sufficiently small, (4) admits a pair of upper and lower solutions \((\bar{u}(x), \bar{v}(x)) = (M_1, a_1 \phi(x) + b_1) \) and \((\bar{y}(x), \bar{v}(x)) = (a_2 \phi(x) + b_2, M_2) \) into (4). The assumptions (H1)∼(H4) then allow us to construct a pair of upper and lower solutions of (4) and obtain a solution of it. The lemma below is elementary.
Due to the existence of pulsating fronts.

Proof. For \( x \in \mathbb{R} \), a straightforward calculation gives

\[
dl_1 \bar{u}_{xx} + \bar{u} \left( m_1(x) - c_{11} \bar{u} - c_{12} \bar{v} \right)
= M_1 \left( m_1(x) - c_{11} M_1 - c_{12} b_1 - c_{12} a_1 \phi \right) \leq 0,
\]

\[
dl_2 \bar{v}_{xx} + \bar{v} \left( m_2(x) - c_{21} \bar{u} - c_{22} \bar{v} \right)
= -d_2 a_1 k^2 \phi + (a_1 \phi + b_1)(m_2 - c_{21} M_1 - c_{22}(a_1 \phi + b_1))
= -a_1^2 c_{22} \phi^2 + a_1 (-d_2 k^2 + m_2 - c_{21} M_1 - 2c_{22} b_1) \phi
+ b_1 (m_2 - c_{21} M_1 - c_{22} b_1) \geq 0,
\]

\[
dl_1 \bar{u}_{xx} + \bar{u} \left( m_1(x) - c_{11} \bar{u} - c_{12} \bar{v} \right)
= -d_1 a_2 k^2 \phi + (a_2 \phi + b_2)(m_1 - c_{12} M_2 - c_{11} (a_2 \phi + b_2))
= -a_2^2 c_{11} \phi^2 + a_2 (-d_1 k^2 + m_1 - c_{12} M_2 - 2c_{11} b_2) \phi
+ b_2 (m_1 - c_{12} M_2 - c_{11} b_2) \geq 0,
\]

Due to (H1), (H2) and Lemma 2.2, it is readily seen that the above four inequalities hold when \( c_{12}, c_{21}, |a_1| \) and \( |a_2| \) are sufficiently small. Therefore, \((\bar{u}(x), \bar{v}(x))\) and \((\bar{u}(x), \bar{v}(x))\) form a pair of upper and lower solutions for (4) with \( u(x) \leq \bar{u}(x) \) and \( v(x) \leq \bar{v}(x) \).

We apply the standard iteration method to establish the existence of solutions \((u(x), v(x))\) for (4) with \( u(x) \leq \bar{u}(x) \) and \( v(x) \leq \bar{v}(x) \) for \( x \in \mathbb{R} \). From the periodicity of the upper and lower solutions, the periodicity of the constructed solution \((u(x), v(x))\) follows. The proof of the theorem is completed. \( \square \)

3. The existence of pulsating fronts. In this section, we consider

\[
\begin{align*}
\begin{cases}
u_t &= \nabla \cdot (A_1(x) \nabla u) + u \left( m_1(x) - c_{11} u - c_{12} v \right), & x \in \mathbb{R}^N, \ t > 0, \\
v_t &= \nabla \cdot (A_2(x) \nabla v) + v \left( m_2(x) - c_{21} u - c_{22} v \right), & x \in \mathbb{R}^N, \ t > 0,
\end{cases}
\end{align*}
\]

where \( A_i = (a_{i,j,k}(x))_{1 \leq j,k \leq N} : \mathbb{R}^N \to \mathbb{R}^{N \times N} \) is a symmetric matrix field and \( m_i : \mathbb{R}^N \to \mathbb{R} \) is non-constant \( (i = 1, 2) \).

It can be shown that periodic coexistence states exist when \( c_{12} \) and \( c_{21} \) are small enough via the upper-lower solution method (see [13], where advection term was taken into account). Such results may be more or less similar to problems with Neumann boundary conditions (see, e.g., [14, 12]) and thus it shall not be our aim here. Here we are interested in the non-existence of periodic coexistence states, which allows us to establish the existence of pulsating fronts of (29) for \( N = 1 \) by applying the abstract theory regarding monotone semiflows established by Yu and Zhao [21] (based on the works [8, 16]).

Define

\[ C^\beta_{per} := \{ u \in C^\beta(\mathbb{R}^N) | u \text{ is } L\text{-periodic in } x \}, \beta \geq 0. \]

Hereafter we always impose the assumptions on \( m_i \) and \( A_i \) \( (i = 1, 2) \):
\[ H \] \[ m_{i} \in C_{\text{per}}^{\alpha}, m_{i} \neq \text{constant}, m_{i} \text{ is positive somewhere and } A_{i} \in [C_{\text{per}}^{1+\alpha}]^{N \times N} \]
for some \( \alpha \in (0, 1) \) and for \( i = 1, 2 \). Furthermore, the symmetric matrix field 
\[ A_{i}(x) = (a_{ij}(x))_{1 \leq j, k \leq N} \text{ is uniformly elliptic, i.e., there exists } a > 0 \text{ such that} \]
\[ \sum_{1 \leq j, k \leq N} a_{ij}(x) \xi_{j} \xi_{k} \geq g |\xi|^{2} \text{ for all } x, \xi \in \mathbb{R}^{N}. \]

Define \( \lambda(A, m) \) as the principal eigenvalue of the linearized problem
\[
\begin{aligned}
&\left\{ -\nabla \cdot (A(x) \nabla \phi) - m(x) \phi = \lambda(A, m) \phi, \quad x \in \mathbb{R}^{N}, \\
&\phi \text{ is } L\text{-periodic in } x, \quad \phi > 0 \text{ in } \mathbb{R}^{N}, \quad \|\phi\|_{\infty} = 1.
\end{aligned}
\]
(30)

It is well-known (see [1]) that
\[
\lambda(A, m) = \min_{\psi \in C_{\text{per}}^{1}, \psi > 0} \frac{\int_{\mathcal{C}} \left[ \nabla \psi \cdot (A(x) \nabla \psi) - m(x) \psi^{2} \right] dx}{\int_{\mathcal{C}} \psi^{2} dx}.
\]
(31)

From [1, Theorem 2.1], system (29) has a semi-trivial state \( (p_{1}(x), 0) \) (resp., \( (0, p_{2}(x)) \)) if and only if \( \lambda(A_{1}, m_{1}) < 0 \) (resp., \( \lambda(A_{2}, m_{2}) < 0 \)). Moreover, \( p_{i} \) is \( L\)-periodic and unique as long as it exists.

**Lemma 3.1.** Assume [H]. Furthermore, assume that
\[
\int_{\mathcal{C}} m_{1}(x) dx > 0.
\]
(32)

Then there exists \( \eta_{1} = \eta_{1}(A_{1}, A_{2}, m_{1}, m_{2}) > 0 \) such that (29) has no periodic coexistence as long as
\[
0 < c_{12} < \eta_{1} \quad \text{and} \quad c_{21} > \max_{x \in \mathcal{C}} \frac{m_{2}(x)}{p_{1}(x)}.
\]
(33)

**Proof.** Due to (32), one can apply [1, Theorem 2.12] to insure that \( \lambda(A_{1}, m_{1}) < 0 \) and then system (29) has a unique semi-trivial state \( (p_{1}(x), 0) \).

Inspired by [19, Lemma 4.8], we prove by contradiction that such \( \eta_{1} \) exists.

Suppose that, contrary to our claim, there exists no such \( \eta_{1} \). Then there exists \( c_{12}^{n} \downarrow 0 \) as \( n \to \infty \) and \( (29) \) with \( c_{12} = c_{12}^{n} \) has a positive and periodic steady state \( (u_{n}, \bar{v}_{n}) \). Since \( u_{n} \) and \( v_{n} \) are uniformly bounded, by standard elliptic regularity and Sobolev embedding theorem, we have \( (u_{n}, v_{n}) \to (\bar{u}, \bar{v}) \) in \( C^{2}(\overline{\mathcal{C}}) \) (up to passing a subsequence), where \( (\bar{u}, \bar{v}) \in C_{\text{per}}^{2} \times C_{\text{per}}^{2} \) satisfies
\[
\begin{aligned}
\nabla \cdot (A_{1}(x) \nabla \bar{u}) + \bar{u} (m_{1}(x) - c_{11} \bar{u}) &= 0, \quad x \in \mathbb{R}^{N}, \\
\nabla \cdot (A_{2}(x) \nabla \bar{v}) + \bar{v} (m_{2}(x) - c_{21} \bar{u} - c_{22} \bar{v}) &= 0, \quad x \in \mathbb{R}^{N}.
\end{aligned}
\]
(34)
(35)

Because \( \lambda(A_{1}, m_{1}) < 0 \), we see that either \( \bar{u} \equiv 0 \) or \( \bar{u} \equiv p_{1} \). We show that \( \bar{u} \equiv p_{1} \).

Indeed, if \( \bar{u} \equiv 0 \), by (32), we can choose \( j \gg 1 \) large enough such that
\[
\int_{\mathcal{C}} (m_{1} - c_{11} u_{j} - c_{12}^{j} v_{j}) dx > 0.
\]
(36)

Using the equation of \( u_{j} \), dividing it by \( u_{j} \) and integrating it over \( \mathcal{C} \) give
\[
\int_{\mathcal{C}} \frac{\nabla \cdot (A_{1}(x) \nabla u_{j})}{u_{j}} dx + \int_{\mathcal{C}} (m_{1}(x) - c_{11} u_{j} - c_{12}^{j} v_{j}) dx = 0.
\]
Integrating by parts and using the periodicity of \( u_{j} \) and the uniform ellipticity of \( A_{1} \) yield
\[
0 \geq - \int_{\mathcal{C}} \frac{\nabla u_{j} \cdot (A_{1}(x) \nabla u_{j})}{u_{j}^{2}} dx = \int_{\mathcal{C}} (m_{1}(x) - c_{11} u_{j} - c_{12}^{j} v_{j}) dx,
\]
which contradicts (36). Thus, \( \bar{u} \equiv p_1 \). Put it into (35), we have

\[
\nabla \cdot (A_2(x) \nabla \bar{v}) + \bar{v}(\mu(x) - c_{22} \bar{v}) = 0, \quad x \in \mathbb{R}^N,
\]

where \( \mu(x) := m_2(x) - c_{21} p_1(x) \). Using the hypothesis of \( c_{21} \) in (33), we have \( \mu(x) \leq 0 \) for \( x \in \mathcal{C} \), which together with the uniform ellipticity of \( A_1 \) yields

\[
\lambda(A_2, m_2) = \min_{\psi \in \mathcal{C}^2_{p_1, \mu} \psi > 0} \frac{\int_{\mathcal{C}} (\nabla \psi \cdot (A_2(x) \nabla \psi) - \mu(x) \psi^2) \, dx}{\int_{\mathcal{C}} \psi^2 \, dx} \geq 0.
\]

Using [1, Theorem 2.1], we have \( \bar{v} \equiv 0 \). On the other hand, let \( V_n = v_n/\|v_n\|_{L^\infty} \). Then \( \|V_n\|_{L^\infty} = 1 \) for each \( n \). Considering the equation of \( V_n \) and passing \( n \to \infty \) (up to passing a subsequence), we obtain \( V_n \to \bar{V} \) as \( n \to \infty \), where \( \bar{V} \geq 0 \), \( \|\bar{V}\|_{L^\infty} = 1 \) and

\[
\nabla \cdot (A_2(x) \nabla \bar{V}) + \bar{v}(\mu(x) \bar{V}) = 0, \quad x \in \mathbb{R}^N.
\]

Since \( \mu(x) \leq 0 \) in \( \mathbb{R}^N \) and \( \mu(x) \) is \( L \)-periodic, the strong maximum principle gives \( \bar{V} \equiv 1 \), which implies that \( m_2 \equiv c_{21} p_1 \) in \( \mathbb{R}^N \). This contradicts (33) and the proof is completed.

Note that our competition system can be reduced into a monotone system by a suitable transformation \( w := p_2(x) - v \). Then, following the work of Yu and Zhao [21], the existence of rightward pulsating fronts of system (29) for \( N = 1 \) connecting \((p_1(x), 0)\) and \((0, p_2(x))\) can be established (which means species \( u \) invades species \( v \) eventually) if the following conditions hold:

(A1) \( \lambda(A_i, m_i) < 0 \) for \( i = 1, 2 \). This implies that two semi-trivial states \((p_1(x), 0)\) and \((0, p_2(x))\) exist.

(A2) \( \lambda(A_1, m_1 - c_{12} p_2) < 0 \). This means that \((0, p_2(x))\) is unstable.

(A3) System (29) has no positive periodic coexistence states.

(A4) \( c_+^* + c_-^* > 0 \), where \( c_+^* \) is the rightward spreading speed of

\[
u_t = (A_1(x) u_x)_x + u (m_1(x) - c_{11} u), \quad x \in \mathbb{R}^1, \quad t > 0,
\]

and \( c_-^* \) is the leftward spreading speed of

\[
u_t = (A_2(x) w_x)_x + w (m_2(x) - c_{22} w), \quad x \in \mathbb{R}^1, \quad t > 0.
\]

We state the following result on pulsating fronts of system (29) for \( N = 1 \).

**Theorem 3.2.** Suppose that \([H]\) holds, the spatial dimension \( N = 1 \) and

\[
\int_{\mathcal{C}} m_i(x) \, dx > 0, \quad i = 1, 2
\]

and

\[
0 < c_{12} < \min \left\{ \eta_1, \frac{c_{22} \int_{\mathcal{C}} m_1(x) \, dx}{L \max_{x \in \mathcal{C}} m_2(x)} \right\} \quad \text{and} \quad c_{21} > \max_{x \in \mathcal{C}} \frac{m_2(x)}{p_1(x)},
\]

where \( \eta_1 \) is given in Lemma 3.1. Then there exists \( c_0 > 0 \) such that if \( c \geq c_0 \), system (29) admits a pulsating front \((U, V)(x - ct, x)\) connecting \((p_1(x), 0)\) and \((0, p_2(x))\) satisfying the following:

(i) \( U \) and \( V \) are non-increasing and non-decreasing in their first argument, respectively,

(ii) \( U \) and \( V \) are \( L \)-periodic with respect to their second argument.
(iii) it holds that
\[
\lim_{\xi \to -\infty} |U(\xi, x) - p_1(x)| = \lim_{\xi \to -\infty} |V(\xi, x)| = 0,
\]
\[
\lim_{\xi \to \infty} |U(\xi, x)| = \lim_{\xi \to \infty} |V(\xi, x) - p_2(x)| = 0 \quad \text{uniformly for } x \in \mathbb{R}.
\]
Moreover, if \( c \in (0, c_0) \), system (29) has no pulsating front connecting \((p_1(x), 0)\) and \((0, p_2(x))\).

Proof. To prove this result, it suffices to show that (A1)-(A4) mentioned above hold. Thanks to (38), from [1, Theorem 2.12] we have \( \lambda(A_i, m_i) < 0 \) for \( i = 1, 2 \) and so system (29) has two semi-trivial states \((p_1(x), 0)\) and \((0, p_2(x))\). Thus, (A1) holds.

We see from (39) that
\[
\frac{1}{L} \int_C m_1(x) dx > c_{12} \max_{x \in C} \frac{m_2(x)}{c_{22}} \geq c_{12} \max_{x \in C} p_2(x), \tag{40}
\]
where the last inequality follows from the maximum principle. Then using the fact that \(|C| = L\), we have
\[
\int_C (m_1(x) - c_{12} p_2(x)) dx \geq \int_C m_1(x) dx - c_{12} \max_{x \in C} p_2(x) L > 0.
\]
From [1, Theorem 2.12] we have \( \lambda(A_1, m_1 - c_{12} p_2) < 0 \) and (A2) thus holds. In view of Lemma 3.1, (A3) is valid.

Finally, we verify (A4). It is well known that the rightward (resp., leftward) spreading speed \( c_1^* \) (resp., \( c_2^* \)) is also the rightward (resp., leftward) minimal wave speed. Thus, from [2, Theorem 1.2] we have
\[
c_1^* = \min_{\mu > 0} \frac{\Lambda_1(\mu)}{\mu}, \quad c_2^* = \min_{\mu > 0} \frac{\Lambda_2(-\mu)}{\mu},
\]
where \( \Lambda_i \) \((i = 1, 2)\) is the principal eigenvalue of
\[
\begin{cases}
\Lambda_i \phi = A_i(x) \phi'' - (2\mu A_i(x) - A_i'(x)) \phi' + (A_i(x) \mu^2 - A_i'(x) \mu + m_i(x)) \phi, & x \in \mathbb{R}, \\
\phi \text{ is } L\text{-periodic in } x, & \phi > 0 \text{ in } \mathbb{R}^N.
\end{cases}
\]
By [21, Lemma 5.1], we see that \( \Lambda_i \) is even and convex in \( \mu \in \mathbb{R} \), and \( \Lambda_i(0) > 0 \). It follows that \( c_1^* > 0 \) and \( c_2^* > 0 \). Thus, \( c_1^* + c_2^* > 0 \) and then (A4) holds.

From the above discussion, Theorem 3.2 follows from [21, Theorem 3.1]. This completes the proof. \( \Box \)

Remark 4. We remark that when \( A_i(x) = d_i > 0 \) for \( i = 1, 2 \) with \( d_1 < d_2 \), \( m_1(x) = m_2(x), c_{11} = c_{21} = c_{22} = 1 \) and \( c_{12} = c \in [0, 1] \), the existence of pulsating fronts has been established in [21]. Among them, the asymptotic spreading speed was also discussed therein. We also refer to [9] regarding pulsating fronts and spreading speeds for time-space periodic monotone systems.

4. Numerical simulations. In this section, we shall provide some numerical examples to investigate the long-time behavior of solutions of system (29). For simplicity, we consider the one dimensional spatial domain \( (n = 1) \) and take parameters as follows:
\[
c_{11} = c_{22} = 1, \quad A_i(x) = d_i \quad (i = 1, 2), \quad c_{12} = k, \quad c_{21} = h.
\]
Then we arrive at the following system:
\[
\begin{align*}
    u_t &= d_1 u_{xx} + u(m_1(x) - u - kv), \quad x \in \mathbb{R}, \ t > 0, \\
v_t &= d_2 v_{xx} + v(m_2(x) - hu - v), \quad x \in \mathbb{R}, \ t > 0,
\end{align*}
\]
with the initial data \((u, v)(x, 0) = (u_0, v_0)(x)\).

**Example 1.** Suppose that (19)∼(21) hold, and choose \((u_0(x), v_0(x)) = (u(x), v(x))\), where \((u(x), v(x))\) is given by (22). We expect that the long time behavior of the solution \((u(x, t), v(x, t))\) of (41) will converge to one of the semi-trivial states, i.e., either \((p_1(x), 0)\) or \((0, p_2(x))\). This suggests that the exact solution given in Remark 2 is unstable.

We first use the initial data with a small perturbation of (22), namely,
\[
\begin{align*}
u_0(x) &= u(x) + 0.0001 \sin(6\pi x/L), \\
v_0(x) &= v(x) + 0.0001 \sin(12\pi x/L),
\end{align*}
\]
where \(L\) is the period of \(u\) and \(v\) in Remark 2. Then the numerical simulation shows that the long time behavior of the solution converges to \((p_1(x), 0)\) for some periodic function \(p_1(x)\) as in the Figure 5. On the other hand, if we use the following initial data
\[
\begin{align*}
u_0(x) &= u(x) + 0.1 \sin(8\pi x/L), \\
v_0(x) &= v(x) + 0.1 \sin^2(10\pi x/L),
\end{align*}
\]
then the numerical simulation shows that the long time behavior of the solution converges to \((0, p_2(x))\) for some periodic function \(p_2(x)\) as in the Figure 6. Therefore, we suggest that the exact solution given in Remark 2 is unstable.

**Example 2.** Fix \(d_1 = 1, d_2 = 2, m_1(x) = \sin(2x) + 0.1, m_2(x) = 2 \cos(2x) + 0.3\). Also, consider
\[
u_0(x) = \begin{cases} 1/2 & \text{if } x \in (-0.2, 0.2), \\
0 & \text{if } x < -0.2 \text{ or } x > 0.2, \end{cases} \quad v_0(x) = 0.5 \cos(4x) + 0.7.
\]
Moreover, choose \( k = 0.01 \) and \( h = 5 \). Then the numerical simulation shows that a pulsating front occurs (see Figure 7) and \( u \) invades \( v \) eventually (see Figure 8), which confirm the main result in the previous section.

**Example 3.** Using the same parameters in the above example except re-picking \( h = 0.01 \), the numerical simulation shows that a coexistence state occurs. See Figure 9. It means that the conclusion of Theorem 3.2 may not happen when \( c_{21} \ll 1 \).

5. **Appendix.** Following the generalized Jacobi elliptic function method introduced in Section 2.1, we make the following ansatz for solving (4):

\[
\begin{cases}
  u(x) = \gamma_0 + \gamma_1 \phi(x), \\
  v(x) = \rho_0 + \rho_1 \phi(x) + \rho_2 \phi^2(x),
\end{cases}
\tag{44}
\]

and

\[
\begin{cases}
  m_1(x) = l_0 + l_1 \phi(x), \\
  m_2(x) = n_0 + n_1 \phi(x),
\end{cases}
\tag{45}
\]
Figure 8. The profile of the long time behavior of Example 2 shows \( u \) invades \( v \) eventually.

Figure 9. The profile of the long time behavior of Example 3 with \( h = 0.01 \).

where \( \phi = \phi(x) \) is the solution of the following problem

\[
\begin{cases}
\phi''(x) = k_3 \phi^3(x) + k_1 \phi(x), \ x \in \mathbb{R}, \\
\phi(0) = 0, \ \phi'(0) = \phi_0,
\end{cases}
\]  

(46)

and \( l_i, n_i, \gamma_i \ (i = 0, 1), \rho_j \ (j = 0, 1, 2), k_1, k_3 \) and \( \phi_0 \neq 0 \) are constants. Substituting the ansatz (44), (45) into (4) leads to

\[
d_1 u_{xx} + u \left( m_1(x) - c_{11} u - c_{12} v \right) = \sum_{i=0}^{3} \zeta_{1i} \phi'(x), \ x \in \mathbb{R}, \tag{47a}
\]

\[
d_2 v_{xx} + v \left( m_2(x) - c_{21} u - c_{22} v \right) = \sum_{i=0}^{4} \zeta_{2i} \phi'(x), \ x \in \mathbb{R}, \tag{47b}
\]
where
\[
\begin{align*}
\zeta_{10} &= -c_{12} \gamma_0 \rho_0 - c_{11} \gamma_0^2 + \gamma_0 l_0, \\
\zeta_{11} &= -c_{12} \gamma_0 \rho_1 - c_{12} \gamma_1 \rho_0 - 2 c_{11} \gamma_1 \gamma_0 + \gamma_1 d_1 k_1 + \gamma_0 l_1 + \gamma_1 l_0, \\
\zeta_{12} &= -c_{12} \gamma_1 \rho_1 - c_{12} \gamma_0 \rho_2 - c_{11} \gamma_1^2 + \gamma_1 l_1, \\
\zeta_{13} &= \gamma_1 d_1 k_3 - c_{12} \gamma_1 \rho_2, \\
\zeta_{20} &= -c_{21} \gamma_0 \rho_0 - c_{22} \rho_0^2 + 2 d_2 k_0 \rho_2 + n_0 \rho_0, \\
\zeta_{21} &= -c_{21} \gamma_0 \rho_1 - c_{21} \gamma_1 \rho_0 - 2 c_{22} \rho_1 \rho_0 + d_2 k_1 \rho_1 + n_0 \rho_1, \\
\zeta_{22} &= -c_{21} \gamma_1 \rho_1 - c_{21} \gamma_0 \rho_2 - c_{22} \rho_1^2 - 2 c_{22} \rho_0 \rho_2 + 4 d_2 k_1 \rho_2 + n_0 \rho_1 + n_0 \rho_2, \\
\zeta_{23} &= -c_{21} \gamma_1 \rho_2 - 2 c_{22} \rho_2 \rho_1 + d_2 k_3 \rho_1 + n_1 \rho_2, \\
\zeta_{24} &= 3 d_2 k_3 \rho_2 - c_{22} \rho_2^2.
\end{align*}
\]

We remark that (10) is used in finding \( v_{xx} \) or \( \frac{d^2}{dx^2} (\rho_0 + \rho_1 \phi(x) + \rho_2 \phi^2(x)) \). Equating the coefficients of powers of \( \phi(x) \) in (47) to zero yields a system of 9 algebraic equations:
\[
\begin{align*}
\zeta_{11} &= 0 \quad (i = 0, 1, 2, 3), \\
\zeta_{21} &= 0 \quad (i = 0, 1, 2, 3, 4).
\end{align*}
\]

We assume that \( d_i \), \( c_{ij} \) \( (i, j = 1, 2) \), \( l_i \), \( n_i \), \( \gamma_i \) \( (i = 0, 1) \), \( \rho_j \) \( (j = 0, 1, 2) \), \( k_1 \) and \( k_3 \) are nonzero constants and \( \gamma_1 > 0 \). With the aid of \texttt{Mathematica}, it turns out that we can solve \( k_1, k_3, d_1, \gamma_0, \rho_0, l_1, n_0 \) and \( n_1 \) in terms of the other parameters from the 9 equations above to give (49) or (50) as follows:

\[
\begin{align*}
k_0 &= \frac{c_{22} \rho_0 (11 \rho_0 \rho_2 - 2 \rho_0^2)}{18 \, d_2 \rho_2^2}, \quad k_1 = \frac{c_{22} (5 \rho_0 \rho_2 - 2 \rho_1^2)}{9 \, d_2 \rho_2}, \quad k_3 = \frac{c_{22} \rho_2}{3 \, d_2}, \quad (49a) \\
d_1 &= \frac{3 c_{12} d_2}{c_{22}}, \quad \gamma_0 = \frac{\gamma_1 2 \sqrt{2} 5 \rho_0 - 5 \rho_0 \rho_2}{\sqrt{3} \rho_2}, \quad (49b) \\
l_0 &= \frac{c_{11} \gamma_1 2 \sqrt{2} 5 \rho_0 - 5 \rho_0 \rho_2}{\sqrt{3} \rho_2} + c_{12} \rho_0, \quad (49c) \\
l_1 &= \frac{c_{12} \sqrt{2} 5 \rho_0 - 5 \rho_0 \rho_2}{\sqrt{3}} + c_{11} \gamma_1 + c_{12} \rho_1, \quad (49d) \\
n_0 &= \frac{1}{9} \left( 3 \sqrt{3} c_{21} \gamma_1 2 \sqrt{2} 5 \rho_0 - 5 \rho_0 \rho_2 + 2 c_{22} \rho_1^2 - 2 c_{22} \rho_0 \rho_2 \right), \quad (49e) \\
n_1 &= \frac{1}{3} \left( 3 c_{21} \gamma_1 + 5 c_{22} \rho_1 \right). \quad (49f)
\end{align*}
\]

\[
\begin{align*}
k_0 &= \frac{c_{22} \rho_0 (11 \rho_0 \rho_2 - 2 \rho_0^2)}{18 \, d_2 \rho_2^2}, \quad k_1 = \frac{c_{22} (5 \rho_0 \rho_2 - 2 \rho_1^2)}{9 \, d_2 \rho_2}, \quad k_3 = \frac{c_{22} \rho_2}{3 \, d_2}, \quad (50a) \\
d_1 &= \frac{3 c_{12} d_2}{c_{22}}, \quad \gamma_0 = \frac{\gamma_1 2 \sqrt{2} 5 \rho_0 - 5 \rho_0 \rho_2}{\sqrt{3} \rho_2}, \quad (50b) \\
l_0 &= \frac{\gamma_1 \gamma_1 2 \sqrt{2} 5 \rho_0 - 5 \rho_0 \rho_2}{\sqrt{3} \rho_2} + c_{12} \rho_0, \quad (50c) \\
l_1 &= \frac{\gamma_1 \gamma_1 2 \sqrt{2} 5 \rho_0 - 5 \rho_0 \rho_2}{\sqrt{3}} + c_{11} \gamma_1 + c_{12} \rho_1, \quad (50d)
\end{align*}
\]
\[ n_0 = \frac{1}{9} \left( \frac{3 \sqrt{3} c_{21} \gamma_1 \sqrt{2 \rho_1^2 + 5 \rho_0 \rho_2 + 2 c_{22} \rho_1^2}}{\rho_2} - 2 c_{22} \rho_0 \right), \]  
\[ n_1 = \frac{1}{3} (3 c_{21} \gamma_1 + 5 c_{22} \rho_1). \]  

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E-mail address: chchchen@math.ntu.edu.tw
E-mail address: liang@mail.nutn.edu.tw
E-mail address: lichang.hung@gmail.com
E-mail address: changhong@math.nctu.edu.tw