PRINCIPAL PARTS OF OPERATORS IN THE $\bar{\partial}$-NEUMANN PROBLEM ON STRICTLY PSEUDOCONVEX NON-SMOOTH DOMAINS

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1. INTRODUCTION

Let $D$ be a bounded domain in $\mathbb{C}^n$. We consider the operator of the Cauchy-Riemann equations

$$\bar{\partial} : L^2_{0,q}(D) \to L^2_{0,q+1}(D),$$

where the $L^2$-spaces on $D$ are defined in terms of a Hermitian metric given on $\mathbb{C}^n$, its Hilbert space adjoint,

$$\bar{\partial}^* : L^2_{0,q+1}(D) \to L^2_{0,q}(D),$$

and the associated complex Laplacian

$$\Box_q = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} : L^2_{0,q}(D) \to L^2_{0,q}(D),$$

whose domain of definition is singled out by the conditions

$$u \in Dom(\bar{\partial}) \cap Dom(\bar{\partial}^*), \quad \bar{\partial}u \in Dom(\bar{\partial}^*), \quad \bar{\partial}^*u \in Dom(\bar{\partial}).$$

The $\bar{\partial}$-Neumann problem asks to solve the equation

$$\Box u = f \quad \text{for } f \perp \ker(\Box),$$

with $u \in Dom(\Box)$. We will study this problem on strictly pseudoconvex domains which may have singularities at the boundary. More precisely:

**Definition 1.1.** $D \subseteq \mathbb{C}^n$ is a Henkin-Leiterer (HL) domain if there is a strictly plurisubharmonic smooth function $r$ on a neighborhood $U$ of the boundary $\partial D$ such that

$$U \cap D = \{ \zeta \in U : r(\zeta) < 0 \}.$$
We shall make the additional assumption that $r$ is a Morse function. Then $\partial D = \{ \zeta : r(\zeta) = 0 \}$ and we may assume that $r$ has finitely many critical points on the boundary, and none on $U \setminus \partial D$.

Under these - and even more general - conditions the $\bar{\partial}$-Neumann problem is solvable in the following sense: there is a linear operator

$$N : L^2_{0,q}(D) \to Dom(\Box)$$

such that one has the orthogonal decomposition

$$L^2_{0,q}(D) = \text{ker}(\Box) \oplus \Box N (L^2_{0,q}(D))$$

$$= \text{ker}(\Box) \oplus \bar{\partial}^* N (L^2_{0,q}(D)) \oplus \bar{\partial} \bar{\partial}^* N (L^2_{0,q}(D)).$$

$N$ is called the $\bar{\partial}$-Neumann operator. For $q > 0$, one knows that the harmonic space $\text{ker}(\Box)$ is zero; this is no longer true on more general manifolds. For $q = 0$, $\text{ker}(\Box)$ is the space of square integrable holomorphic functions.

We can now formulate our aim: to express the abstract operators, $N, \bar{\partial}N, \bar{\partial}^* N$, as integral operators with explicit (in terms of the defining function and the metric) kernels. The expression should be valid up to error terms which have stronger smoothing properties than the explicit terms; consequently, the boundedness properties of the above operators in various function spaces ($L^p$-spaces, for instance) can be read off the corresponding properties of the integral operators (which have to be established, of course).

This program has been implemented in the case of smoothly bounded domains by the work of many people - see [4] for historical comments; we carry it over to the non-smooth HL case.

In order to state our results we now describe some needed conventions and notations which will be kept fixed throughout this paper. The metric on $\mathbb{C}^n$ will be chosen to coincide, near the boundary of $D$, with the Levi form of $r$:

$$ds^2 = \sum r_{ij}(\zeta) d\zeta_i d\bar{\zeta}_j.$$  

Any such metric is called a Levi metric; the $\bar{\partial}$-Neumann problem is formulated in terms of this metric.

We set

$$\gamma(\zeta) = |\partial r(\zeta)|,$$

where the length is measured by the metric in (1.1).

For a double differential form $K(\zeta, z)$ on $D \times D$ we define the corresponding integral operator, $K$ by the formula

$$Kf(z) = \int_{\zeta \in D} f(\zeta) \wedge *_{\zeta} \overline{K(\zeta, z)}$$

and call $K$ the kernel of $K$. Here $*_\zeta$ is the Hodge operator for the metric (1.1), and $f$ is a differential form. If the types of $f$ and $K$ do not match, the integral is 0 by definition. We finally set

$$K^*(\zeta, z) = \overline{K(z, \zeta)};$$

in particular $\gamma^*(\zeta) = \gamma(z)$.

The first step in our program has already been done by the first author in [2]:
Theorem 1.2. Let $D \subset X$ be a HL domain in a complex manifold $X$, given by a Morse defining function $r$. There are integral operators of type 1, $T_q : L^2_{0,q+1}(D) \to L^2_{0,q}(D)$ such that for $f \in L^2_{0,q}(D) \cap \text{Dom}(\bar{\partial}) \cap \text{Dom}(\bar{\partial}^*)$

$$f = T_q \bar{\partial} f + T^*_q \bar{\partial}^* f + \text{error terms} \quad \text{for } 1 \leq q < n = \dim X,$$

where the error terms, after multiplication with suitable powers of $\gamma$, involve only operators with better smoothing properties than the principal terms, with $f, \bar{\partial} f,$ and $\bar{\partial}^* f$ arguments.

The error terms will be explicitly described in Section 7, in Theorem 7.1. The type will be defined in Section 3: it describes the continuity properties of the operators in question. A detailed description of the operators $T_q$ is given [2]; we will resume it in Section 4. Although $X$ above can be an arbitrary complex manifold we shall restrict attention, in this paper, to $X = \mathbb{C}^n$. The necessary adjustments in the general case can be made as in [4] or [5].

In order to express our main results, we introduce the notion of principal part. We only indicate here what we mean and refer to Section 6 for the precise definition. Let $A : L^2(D) \to L^2(D)$. $B$ is called a principal part of $A$ if for all large $L$

$$\gamma^L A = \gamma^L B + C,$$

where now $\gamma^L B$ is an admissible integral operator (to be precise: a Z-operator) and where $C$ has better continuity properties than $\gamma^L B$. We recall the definition of "admissible" in Section 3, the definition of a Z-operator in Section 6.

Our main results are the calculations of the principal parts of the operators $N_q, \bar{\partial} N_q, \text{ and } \bar{\partial}^* N_q$. For $N_q$ we calculate explicitly an integral operator, $N^0_q$, with kernel, $N_q$, such that we have

**Main Theorem 1.** For $1 \leq q \leq n-3$:

a) $N^0_q$ is a principal part of the Neumann operator, $N_q$

b) $T_{q-1}$ is a principal part of $\bar{\partial}^* N_q$

c) $T^*_q$ is a principal part of $\bar{\partial} N_q$.

Similar results hold for $q = n-2$, but will not be proven here. See [3] for details.

We can interpret some of our earlier results on the Bergman projector in terms of principal parts:

**Main Theorem 2.** The admissible operator $P^*_0$ of [3] is a principal part of the Bergman projector $P$.

See also [1] for the Bergman projection in the setting of domains in complex manifolds.

From the above representation we get, in view of the known continuity properties of admissible operators, estimates for the Neuman operator, which we express as follows:

**Main Theorem 3.** For $q$ as in Main Theorem 1 and for all $p \geq 2$ and $s$ such that

$$\frac{1}{s} > \frac{1}{p} - \frac{1}{n+1}$$
we have the estimates
\[ a\|\gamma^{3(n+2)}N_qf\|_{L^p} \lesssim \|\gamma^2 f\|_{L^p} + \|f\|_{L^2}, \]
and for \( f \in \text{dom} \Box \subset L^2_{0,q}(D) \)
\[ \|\gamma^{3(n+2)}f\|_{L^p} \lesssim \|\gamma^2 \Box f\|_{L^p} + \|f\|_{L^2}. \]

These \( L^p \) estimates are of course only a typical example of the use of our analysis of the Neumann operator; it is possible to obtain estimates in other norms (see \[1\] or \[4\]); they will all invoke the \( \gamma \) weights.

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2. Geometric data

We need the following data, all of which are given by the defining function, \( r \):

1. \( \rho^2(\zeta, z) \), (square of) the geodesic distance for the metric (1.1),

2. \( P(\zeta, z) = \rho^2(\zeta, z) + 2\frac{r(\zeta)}{\gamma(\zeta)} \frac{r(z)}{\gamma(z)} \),

the extended (squared) distance function,

3. \( F(\zeta, z) \), the Ramírez-Henkin function of the domain (resp. of \( r \)). Its definition and properties can be found in \[4\] or \[6\]. Let us only note that it is holomorphic in \( z \), smooth in both variables, and that it gives rise to

4. \( \Phi(\zeta, z) = F(\zeta, z) - r(\zeta) \),

the extended Ramírez-Henkin function, which satisfies the crucial estimates

\[ \text{Re } \Phi(\zeta, z) > 0 \text{ on } \partial D \times D, \]

\[ \Phi(\zeta, \zeta) = 0 \text{ } \text{for } \zeta \in \partial D, \]

\[ |\Phi(\zeta, z)| \gtrsim \rho^2(\zeta, z) + |r(\zeta)| + |r(z)| + |\text{Im } \Phi(\zeta, z)|. \]

It is from these data that all the following integral kernels will be constructed.

3. Admissible operators

We write \( \xi_k(\zeta) \) for a function with the property

\[ |\gamma^\alpha D^\alpha_\xi \xi_k(\zeta)| \lesssim \gamma^k. \]

We shall write \( \mathcal{E}_j \) for those double forms on open sets \( U \subset \mathbb{C}^n \times \mathbb{C}^n \) such that \( \mathcal{E}_j \) is smooth and satisfies

\[ \mathcal{E}_j(\zeta, z) \lesssim \rho^j(\zeta, z), \]

where \( \rho \) coincides with the geodesic distance. In many cases we work with similar forms which are not necessarily smooth up to the boundary, and for such forms we
define $\sigma_j$, $j \geq 0$, for those double forms which are smooth on open sets $U \subset D \times D$ such that

$$|\sigma_j| \lesssim \mathcal{E}_j,$$

$$D_\zeta \sigma_j = \xi_0 \sigma_{j-1},$$

$$D_z \sigma_j = \xi_0^* \sigma_{j-1}.$$

Here and below $\xi_k^* = \overline{\xi_k}(z)$, the $*$ having a similar meaning for other functions of one variable.

**Definition 3.1.** A double differential form $A(\zeta, z)$ on $D \times D$ is an admissible kernel, if it has the following properties:

i) $A$ is continuous on $D \times D - \Lambda$, where $\Lambda$ is the boundary diagonal, and smooth except possibly at the singular points $(\zeta, z)$ with $\gamma(\zeta) = 0$ or $\gamma(z) = 0$.

ii) For each point $(\zeta_0, \zeta_0) \in \Lambda$ there is a neighborhood $U \times U$ of $(\zeta_0, \zeta_0)$ on which $A$ or $A^*$ has the representation

$$(3.1) \quad \xi_N \xi_M \sigma_j P^{-t_0} \Phi^{t_1} \Phi^* \sigma_j \Phi^{t_3} \Phi^{*t_4} r^l r^m$$

with $N, M, j, t_0, \ldots, m$ integers and $j, t_0, l, m \geq 0$, $-t = t_1 + \cdots + t_4 \leq 0$, $N, M \geq 0$, $l + m \leq t + 1$, and $N + M \geq 0$.

We define the type of $A(\zeta, z)$ to be

$$\tau = 2n + j + \min\{2, t - l - m, N + M\} - 2(t_0 + t - l - m).$$

The type controls the regularity properties of the operator: the larger the type, the better the regularity - see Proposition 6.2. Type 0 is at the edge of integrability - see [3]; here we only work with positive type kernels.

Double forms $\mathcal{E}_{j-2n}$ will be called isotropic kernels of type $j$. Operators with the corresponding kernels will be called isotropic (resp. admissible) of the corresponding type.

An important example of a type 2 isotropic kernel is

$$\Gamma_{0q} = \Gamma_{0q}(\zeta, z),$$

a parametrix of the complex Laplacian; its derivatives $\overline{\partial}_\zeta \Gamma_{0q}$ and $\partial_z \Gamma_{0q}$ are of type 1. They are part of the $T_q$-kernels and the corresponding $T_q$ operators mentioned above and defined in the next paragraph.

In the next theorem and throughout this paper we shall denote the kernels of the operators $T_q$ by $T_q$; the adjoint operator has the kernel $T_q^*$ - see [1,2].

We will use $A_l$ for a generic admissible kernel of type $l$ and $\mathcal{E}_{l-2n}$ for a generic isotropic kernel of type $l$. More elaborate and more general concepts (the ”$Z$-operators”) will be defined in Section 6.

Our main theorems follow from the following result whose proof takes up the bulk of the present paper:

**Theorem 3.2.** There are explicit kernels, $\mathcal{N}_q$, $0 \leq q \leq n - 1$, which satisfy

$$\mathcal{N}_q = \mathcal{N}_q^* + \frac{1}{\gamma} A_3 + \frac{1}{\gamma^*} A_3$$

$$\overline{\partial}_\zeta \mathcal{N}_q = \mathcal{T}_q + \frac{1}{\gamma \gamma^*} A_2$$

$$* \mathcal{N}_q |_{\partial D} = 0.$$
For \(1 \leq q \leq n-1\) we have
\[ N_q = \frac{1}{\gamma} A_2 + \Gamma_0 q. \]
Moreover, if \(1 \leq q \leq n-2\), we have
\[ \partial \zeta^* N_q = T_{q-1}^* + \frac{1}{\gamma} A_2. \]
The kernels \(A_2\) above satisfy
\[ \partial \zeta A_2 = A_1 + \frac{1}{\gamma} A_2. \]

4. Preliminary calculations

In the next few lemmas we will often refer to a particular choice of local coordinates. We work in coordinate patch near a boundary point of \(D\) and define orthonormal frame of \((1,0)\)-forms on a neighborhood \(U \cap D\) with \(\omega^1, \ldots, \omega^n\) where \(\partial r = \gamma \omega^n\) as the orthonormal frame, and \(L_1, \ldots, L_n\) comprising the dual frame. These operators refer to the variable \(\zeta\). When they are to refer to the variable \(z\), they will be denoted by \(\Theta^j\) and \(\Lambda_j\), respectively.

We fix the point \(\zeta\) and choose local coordinates \(z\) such that
\[ dz_j(\zeta) = \Theta^j(\zeta). \]
From the Morse Lemma, near the critical points of \(r\), denoted by \(p_1, \ldots, p_k\), we can take \(\varepsilon\) small enough so that in each
\[ U_{2\varepsilon}(p_j) = \{ \zeta : D \cap |\zeta - p_j| < 2\varepsilon \}, \]
for \(j = 1, \ldots, k\), there are coordinates \(u_1, \ldots, u_m, v_{m+1}, \ldots, v_{2n}\) such that
\[ - r(\zeta) = u_1^2 + \cdots + u_m^2 - v_{m+1}^2 + \cdots - v_{2n}^2, \]
with \(u_{j\alpha}(p_j) = v_{j\beta}(p_j) = 0\) for all \(1 \leq \alpha \leq m\) and \(m + 1 \leq \beta \leq 2n\). Working in a neighborhood of a singularity in the boundary and using such coordinates, we see first that for \(j = 1, \ldots, n\), \(L_j\) is a sum of terms of the form \(\xi_0 \Lambda\), where \(\Lambda\) here and below denotes any smooth first order differential operator. Similarly, \(\Lambda_j\) is a sum of terms of the form \(\xi_0^* \Lambda\).

Working now with the case \(j = n\), the other cases being handled similarly, we see \(\Lambda_n - \frac{\partial}{\partial z_n}\) is a sum of terms of the form
\[ (\frac{a(z)}{\gamma(z)} - \frac{a(\zeta)}{\gamma(\zeta)}) \Lambda = (\xi_{-1} \xi_0^* \sigma_1) \Lambda, \]
where \(a\) is a smooth function such that \(|a(\zeta)| \lesssim \gamma\). \[\text{(4.3)}\] follows from
\[ \frac{a(z)}{\gamma(z)} - \frac{a(\zeta)}{\gamma(\zeta)} = a(z) \frac{\gamma(\zeta) - \gamma(z)}{\gamma(\zeta) \gamma(z)} + \frac{a(z) - a(\zeta)}{\gamma(\zeta)} \]
\[ = \xi_0^* \frac{\sigma_1}{\gamma(\zeta)} + \xi_{-1} \xi_1 \]
\[ = \xi_{-1} \xi_0^* \sigma_1. \]
We thus use repeatedly
\[ \Lambda_j - \frac{\partial}{\partial z_j} = (\xi_{-1} \xi_0^* \sigma_1) \Lambda. \]
\[\text{(4.4)}\]
By symmetry, we also have
\[ L_j - \frac{\partial}{\partial \zeta_j} = (\xi_j - 1) \zeta_0 \sigma_1 \Lambda. \]

Coordinates taken as in (4.1) give us the following
\[ \zeta_j = \xi_1 \]
\[ \zeta_j - z_j = \sigma_1. \]

We now collect various properties of functions comprising the integral kernels.

**Lemma 4.1.**

i. \( r(\zeta) = -\Phi(\zeta, z) + \mathcal{E}_1 \)

ii. \( \Phi(\zeta, z) = \Phi^*(\zeta, z) + \mathcal{E}_3. \)

**Proof.** i. follows as an immediate consequence of the definition of the function \( \Phi. \)

ii. follows as in the smooth case (see [4]). \( \square \)

**Lemma 4.2.**

i. \( \Lambda_n \Phi = -\gamma + \xi_0 \zeta_0 \sigma_1, \)
\( \mathcal{T}_n \Phi = -\gamma^* + \xi_0 \zeta_0 \sigma_1. \)

ii. \( \forall j \)
\( \overline{\Lambda}_j \Phi = \xi_0^* \mathcal{E}_2, \)
\( L_j \Phi = \xi_0 \mathcal{E}_2. \)

iii. For \( j < n \)
\( \Lambda_j \Phi = \xi_0 \zeta_0^* \sigma_1, \)
\( L_j \overline{\Phi} = \xi_0 \zeta_0^* \sigma_1. \)

**Proof.** i.
\[ \Lambda_n \Phi = \sum_{j=1}^{n} \frac{\partial r}{\partial \zeta_j} \Lambda_n (\zeta_j - z_j) + \xi_0^* \mathcal{E}_1 \]
\[ = \sum_{j<n} \frac{\partial r}{\partial \zeta_j} (\xi_j - 1) \zeta_0^* \sigma_1 \Lambda (\zeta_j - z_j) + \xi_0^* \mathcal{E}_1 \]
\[ + \frac{\partial r}{\partial \zeta_n} (-1 + (\xi_0 \zeta_0^* \sigma_1) \Lambda (\zeta_n - z_n)) \]
\[ = -\gamma + \xi_0 \zeta_0^* \sigma_1. \]

The second relation in i. follows by taking conjugates, and by Lemma 4.1.

ii. That \( \overline{\Lambda}_j \Phi = \xi_0^* \mathcal{E}_2 \) is clear. \( L_j \Phi = \xi_0 \mathcal{E}_2 \) then follows by taking the adjoint and using the fact that \( \Phi - \Phi^* = \mathcal{E}_3. \)
iii. As we wrote in the proof of i., we write
\[ \Lambda_j \Phi = \sum_{k=1}^{n} \frac{\partial r}{\partial \zeta_k} \Lambda_j (\zeta_k - z_k) + \xi_0^* \xi_1 \]
\[ = \sum_{k \neq j} \frac{\partial r}{\partial \zeta_k} (\xi_0^* \zeta_1) \Lambda_j (\zeta_k - z_k) + \xi_0^* \xi_1 \]
\[ + \frac{\partial r}{\partial \zeta_j} (-1 + (\xi_0^* \zeta_1) \Lambda_j (\zeta_j - z_j)) \]
\[ = \xi_0^* \xi_1. \]
\[ L_j \Phi = \xi_0^* \xi_1 \]
follows similarly. □

**Lemma 4.3.**

i.\[ \gamma \Lambda_n P = -2 \Phi + \xi_1^* \xi_1 (P + \mathcal{E}_2) + \xi_0^* \xi_0^* \sigma_2 \]
\[ \gamma^* L_n P = -2 \Phi^* + \xi_1^* \xi_1 (P + \mathcal{E}_2) + \xi_0^* \xi_0^* \sigma_2. \]

ii. For $j < n$
\[ L_j \Phi = L_j \rho^2 + \xi_1^* \xi_1 \sigma_2 \]
\[ = L_j P + \xi_0^* \xi_1^* r^* + \xi_1^* \xi_0^* \sigma_2. \]

iii.\[ \gamma \gamma^* \left( 2P - \sum_{j<n} |L_j \rho^2|^2 \right) = 4 |\Phi|^2 + r \xi_0^* \xi_1^* \sigma_2 + \xi_0^* \xi_1^* \sigma_3 + \xi_0^* \xi_1^* \sigma_4. \]

**Proof.** Variants of i. and iii. were proved in [2], and we will follow those proofs here.

i. We prove the first relation in i., the second being a consequence of the first. We have
\[ \Lambda_n P = \Lambda_n \rho^2 + \frac{2r}{\gamma} - \frac{1}{\gamma^*} \xi_0^* (z) \frac{rr^*}{\gamma^*}. \]

With $\zeta$ fixed, we choose coordinates $z_j$, as in (4.1), so that $dz_j(\zeta) = \theta_j(\zeta)$, and we let
\[ R^2(\zeta, z) = \sum g_{jk}(\zeta)(\zeta_j - z_j)(\overline{\zeta_k} - \overline{z_k}), \]
where the $g_{jk}$ are determined by the metric, $ds^2 = \sum g_{jk} dz_j dz_k$.

With the metric chosen as the Levi metric, we write
\[ g_{jk}(\zeta) = g_{jk}(z) + \sigma_1. \]

This gives us the relation
\[ \rho^2 = R^2 + \sigma_3, \]
and thus
\[ \Lambda_n \rho^2 = \frac{\partial}{\partial z_n} R^2 + (\xi_1^* \sigma_1) (\Lambda R^2) + \xi_0^* \sigma_2 \]
\[ = \frac{\partial}{\partial z_n} R^2 + (\xi_1^* \sigma_1) (\xi_0^* \sigma_1) + \xi_0^* \sigma_2 \]
\[ = -2(\overline{\zeta_n} - \overline{z_n}) + \xi_1^* \xi_0^* \sigma_2, \]
where the last line follows from $g_{jk}(\zeta) = 2 \delta_{jk}$ due to the orthonormality of the $\Theta_j$. 
Finally, this gives

\[ \Lambda_n P = -2(\bar{\zeta}_n - \bar{z}_n) + 2 \frac{r}{\gamma} + \xi_{-1}^* \frac{rr^*}{\gamma \gamma^*} + \xi_1 \zeta_0 \sigma_2 \]

where we use

\[ \frac{rr^*}{\gamma \gamma^*} = P + \epsilon_2 \]

in the last line.

We compare (4.5) to \( \Phi \) by calculating the Levi polynomial, \( F(\zeta, z) \) in the above coordinates:

\[ \Phi(\zeta, z) = F(\zeta, z) - r(\zeta) + \sigma_2 \]

(4.6)

\[ = \gamma(\zeta)(\bar{\zeta}_n - \bar{z}_n) - r(\zeta) + \xi_0 \xi^*_0 \sigma_2. \]

ii. Again we use

\[ \rho^2 = R^2 + \sigma_3 \]

below to obtain

\[ L_j \bar{\Phi} = \sum_k \left( L_j \frac{\partial r}{\partial \zeta_k} \right) (\bar{\zeta}_k - \bar{z}_k) + \xi_0 \sigma_2 \]

\[ = \sum_k \left[ \left( \frac{\partial}{\partial \zeta_j} + (\xi_{-1} \xi^*_0 \sigma_1) \Lambda \right) \frac{\partial r}{\partial \zeta_k} \right] (\bar{\zeta}_k - \bar{z}_k) + \xi_0 \sigma_2 \]

\[ = \sum_k \frac{\partial^2 r}{\partial \zeta_j \partial \zeta_k} (\bar{\zeta}_k - \bar{z}_k) + \xi_{-1} \xi^*_0 \sigma_2 \]

\[ = \frac{\partial}{\partial \zeta_j} \left( \sum_{k,l} \frac{\partial^2 r}{\partial \zeta_k \partial \zeta_l} (\zeta_l - z_l)(\bar{\zeta}_k - \bar{z}_k) \right) + \xi_{-1} \xi^*_0 \sigma_2 \]

\[ = \frac{\partial}{\partial \zeta_j} \left( \rho^2 + \sigma_3 \right) + \xi_{-1} \xi^*_0 \sigma_2 \]

\[ = (L_j + (\xi_{-1} \xi^*_0 \sigma_1) \Lambda) \rho^2 + \xi_{-1} \xi^*_0 \sigma_2 \]

\[ = L_j \rho^2 + \xi_{-1} \xi^*_0 \sigma_2. \]

We now use the relation

\[ L_j P = L_j \rho^2 + \xi_{-1} \frac{rr^*}{\gamma \gamma^*} \]

\[ = L_j \rho^2 + \xi_0 \xi^*_0 \rho^* + \xi_{-1} \xi^*_0 \sigma_2 \]
to finish the proof of \( ii. \)

\( iii. \) We have

\[
|L_j \rho^2|^2 = (L_j \rho^2)^2 + (\xi_{-1} \xi_0^* \sigma_3) \rho^2
\]

\[
= (L_j \rho^2)^2 + \xi_{-1} \xi_0^* \sigma_3
\]

\[
= \frac{\partial}{\partial \zeta_j} \rho^2 + \xi_{-1} \xi_0^* \sigma_3
\]

\[
= 4|\zeta_j - z_j|^2 + \xi_{-1} \xi_0^* \sigma_3
\]

\[
= 4|\zeta_j - z_j|^2 + \xi_{-1} \xi_0^* \sigma_3.
\]

We can then, with the relation

\[
P = 2 \sum_j |\zeta_j - z_j|^2 + \sigma_3 + 2 \frac{r r^*}{\gamma \gamma^*},
\]

write

\[
2P - \sum_{j<n} |L_j \rho^2|^2 = 4|\zeta_n - z_n|^2 + 4 \frac{r r^*}{\gamma \gamma^*} + \xi_{-1} \xi_0^* \sigma_3.
\]

The calculations in (4.6) also give

\[
\Phi = \gamma(\zeta_n - z_n) - r + \xi_0 \xi_0^* \sigma_2;
\]

which we use in

\[
\Phi \bar{\Phi} = (\gamma(\zeta_n - z_n) - r(\zeta) + \xi_0 \xi_0^* \sigma_2) \bar{\Phi}
\]

\[
= \gamma(\zeta_n - z_n)[\gamma(\zeta_n - \zeta_n) - r(\zeta) + \xi_0 \xi_0^* \sigma_2] - r(\zeta) \bar{\Phi} + (\xi_0 \xi_0^* \sigma_2) \bar{\Phi}
\]

\[
= \gamma \gamma^* |\zeta_n - z_n|^2 - r(\zeta) \gamma(\zeta_n - z_n) + \bar{\Phi} + r(\zeta) \xi_0 \xi_0^* \sigma_2 + \xi_1 \xi_0^* \sigma_3 + \xi_0 \xi_0^* \sigma_4,
\]

where we use \( \gamma(\zeta) = \gamma(z) + \sigma_1 \) and \( \Phi = \xi_1 \sigma_1 + \sigma_2 - r \) in the last step.

From Lemma 4.1 we have

\[
\gamma(\zeta_n - z_n) + \bar{\Phi} = \gamma(\zeta_n - z_n) + \bar{\Phi} + \xi_3
\]

\[
= \gamma(\zeta_n - z_n) + \gamma^*(z_n - \zeta_n) - r(z) + \xi_0 \xi_0^* \sigma_2
\]

\[
= -r(z) + \xi_0 \xi_0^* \sigma_2,
\]

and so we can write

\[
\Phi \bar{\Phi} = \gamma \gamma^* |\zeta_n - z_n|^2 + r r^* + r \xi_0 \xi_0^* \sigma_2 + \xi_1 \xi_0^* \sigma_3 + \xi_0 \xi_0^* \sigma_4.
\]

\( iii. \) now easily follows.

\[
\square
\]

We want to compute the principal parts of the kernels \( T_q \) occurring in the integral representation (1.2). From [2] we have

(4.7) \[
T_q = \partial_z L_q - \partial_z L_{q-1} + \partial_z \Gamma_{0q}, \quad q \geq 1
\]

\[
T_0 = \partial_z L_0 - \partial_z \Gamma_{00},
\]

where the various kernels are defined below.
We start with the differential forms
\[ \beta(\zeta, z) = \frac{\partial_{\zeta} \rho^2(\zeta, z)}{\rho^2(\zeta, z)} \]
\[ \alpha(\zeta, z) = \xi(\zeta) \frac{\partial r(\zeta)}{\phi(\zeta, z)}, \]
where \( \xi(\zeta) \) is a smooth patching function which is equivalently 1 for \(|r(\zeta)| < \delta\) and 0 for \(|r(\zeta)| > \frac{3}{2}\delta\), and \( \delta > 0 \) is sufficiently small. We define
\[ C_q = C_q(\alpha, \beta) = \sum_{\mu=0}^{n-q-2} \sum_{\nu=0}^{q} a_{q\mu\nu} C_{q\mu\nu}(\alpha, \beta), \]
where
\[ a_{q\mu\nu} = \left( \frac{1}{2\pi i} \right)^n \binom{\mu + \nu}{\mu} \binom{n - 2 - \mu - \nu}{q - \mu} \]
and
\[ C_{q\mu\nu}(\alpha, \beta) = \alpha \wedge \beta \wedge (\bar{\partial}_{\zeta} \alpha)^\mu \wedge (\bar{\partial}_{\zeta} \beta)^{n-q-\mu-2} \wedge (\bar{\partial}_z \alpha)^\nu \wedge (\bar{\partial}_z \beta)^{q-\nu}. \]
Denoting the Hodge *-operator by *, we then define
\[ (4.8) \]
\[ L_q(\zeta, z) = (-1)^{q+1} * \xi C_q(\zeta, z). \]
We also write
\[ K_q(\zeta, z) = (-1)^{q(q-1)/2} \left( \frac{n-1}{q} \right) \frac{1}{(2\pi i)^n} \alpha \wedge (\bar{\partial}_{\zeta} \alpha)^{n-q-1} \wedge (\bar{\partial}_z \alpha)^q \]
and
\[ \Gamma_{0,q}(\zeta, z) = \frac{(n-2)!}{2\pi n} \frac{1}{\rho^{2n-2}} (\bar{\partial}_{\zeta} \partial_z \rho^2)^q. \]
For ease of notation we will drop here the superscripts \( \epsilon \), which were used in \[2\] to do calculations on the smooth subdomains, \( D_\epsilon = \{ r < -\epsilon \} \) noting that the following calculations also hold when the kernels on \( D \times D \) are replaced with the corresponding kernels on \( D_\epsilon \times D_\epsilon \). All formulas remain the same and make sense when one looks at the appropriate weighted \( L^p \) spaces.

**Lemma 4.4.** The kernels \( L_q \) given in (4.8) can be represented, for \( 0 \leq q \leq n-2 \), in the following form:
\[ (4.9) \]
\[ L_q = c_{nq} \sum_{0 \leq \mu \leq n-q-2 \atop \mu \leq 0} \left( \frac{n-2-\mu}{q} \right) \frac{T_{j} \rho^2}{\Phi^* \mu + 1 \mu_{n-1}} \alpha \wedge (\bar{\partial}_{\zeta} \alpha)^{n-q-1} \wedge (\bar{\partial}_z \alpha)^q \wedge \Theta^L + A_3, \]
where
\[ c_{nq} = 2^{n-2} \left( \frac{1}{2\pi} \right)^n q!(n-q-2)! \]
and the terms \( A_3 \) satisfy
\[ \partial_{\zeta} A_3 = A_2, \quad \partial_z A_3 = A_2, \]
\[ \bar{\partial}_{\zeta} \partial_{\zeta} A_3 = A_1 + \frac{1}{\gamma} A_2, \quad \bar{\partial}_{z} \partial_{z} A_3 = A_1. \]
Alternatively, we can use
\[ \bar{\partial}_{\zeta} \partial_{\zeta} A_3 = A_1 + \frac{1}{\gamma^*} A_2. \]
Proof. (4.9) is given in [2]. To see the error terms have the property we write from [2]

\[ L_q = \sum_{\mu=0}^{n-q-2} \left( g_{q\mu} C_{q\mu} + \frac{\mathcal{E}_2 \wedge \bar{\partial}r}{\Phi^{\mu+1} p_{n-\mu-1}} \right) + \mathcal{E}_0, \]

where

\[ g_{q\mu} = c_{nq} \binom{n-2-\mu}{q} \]

and

\[ C_{q\mu} = \sum_{j<n} \frac{\bar{T}_j \rho^2}{\Phi^{\mu+1} p_{n-\mu-1}} \gamma^{\tau j L} \wedge \Theta^L. \]

We have

\[ \partial \varepsilon \left( \frac{\mathcal{E}_2 \wedge \bar{\partial}r}{\Phi^{\mu+1} p_{n-\mu-1}} \right) = \frac{\mathcal{E}_2 + \xi \mathcal{E}_1}{\Phi^{\mu+1} p_{n-\mu-1}} + \frac{\mathcal{E}_2 \wedge \bar{\partial}r}{\Phi^{\mu+2} p_{n-\mu-1}} \mathcal{E}_1 + \frac{\mathcal{E}_2 \wedge \bar{\partial}r}{\Phi^{\mu+1} p_{n-\mu}} \left( \mathcal{E}_1 + \xi \mathcal{E}_0 \right) = A_2. \]

Similarly,

\[ \partial \varepsilon \left( \frac{\mathcal{E}_2 \wedge \bar{\partial}r}{\Phi^{\mu+1} p_{n-\mu-1}} \right) = A_2. \]

We use (4.10) and Lemma 4.2 to calculate

\[ \bar{\partial} \varepsilon \left( \frac{\mathcal{E}_2 \wedge \bar{\partial}r}{\Phi^{\mu+1} p_{n-\mu-1}} \right). \]

We note \( \partial \bar{\partial} \bar{\partial} = 0 \), and calculate

\[ \bar{\partial} \varepsilon \left( \frac{\mathcal{E}_2 \wedge \bar{\partial}r}{\Phi^{\mu+1} p_{n-\mu-1}} \right) = \frac{\mathcal{E}_1 + \xi \mathcal{E}_0}{\Phi^{\mu+1} p_{n-\mu-1}} + \frac{\mathcal{E}_2 + \xi \mathcal{E}_1}{\Phi^{\mu+2} p_{n-\mu-1}} \mathcal{E}_2 + \frac{\mathcal{E}_2 + \xi \mathcal{E}_1}{\Phi^{\mu+1} p_{n-\mu}} \left( \mathcal{E}_1 + \xi \mathcal{E}_0 \right) = A_1. \]

We also have

\[ \bar{\partial} \varepsilon \left( \frac{\mathcal{E}_3 \wedge \bar{\partial}r}{\Phi^{\mu+2} p_{n-\mu-1}} \right) = \frac{\xi \mathcal{E}_2}{\Phi^{\mu+2} p_{n-\mu-1}} + \frac{\mathcal{E}_3 \wedge \bar{\partial}r}{\Phi^{\mu+3} p_{n-\mu-1}} \mathcal{E}_2 + \frac{\mathcal{E}_3 \wedge \bar{\partial}r}{\Phi^{\mu+2} p_{n-\mu}} \left( \mathcal{E}_1 + \xi \mathcal{E}_0 \right) = A_1. \]
and

\[ (4.11) \quad \bar{\partial}_\zeta \left( \frac{\mathcal{E}_2 \wedge \bar{\partial} r}{p^{n-\mu}} \left( \mathcal{E}_1 + \xi_0 \frac{r^*}{\gamma^*} \right) \right) \]
\[ = \bar{\partial}_\zeta \left( \frac{\xi_1 \mathcal{E}_1}{p^{n-\mu}} \left( \mathcal{E}_1 + \xi_0 \frac{r^*}{\gamma^*} \right) + \frac{\mathcal{E}_2 \wedge \bar{\partial} r}{p^{n-\mu}} (\mathcal{E}_0 + \xi_0 \xi_1^* + \xi_1 \xi_0^* + \xi_0 \xi_1 + \xi_1 \xi_0) \right) + \mathcal{E}_2 \wedge \bar{\partial} r \left( \mathcal{E}_1 + \xi_0 \frac{r^*}{\gamma^*} \right) \]
\[ = \sigma_1 + \xi_0 \frac{r}{\gamma}. \]

We can now write

\[ \mathcal{E}_1 + \xi_0 \frac{r^*}{\gamma^*} = \sigma_1 + \xi_0 \frac{r}{\gamma} \]

in (4.11) and

\[ \bar{\partial}_\zeta P = \mathcal{E}_1 + \xi_0 \frac{r^*}{\gamma^*} \]
\[ = \sigma_1 + \xi_0 \frac{r}{\gamma}. \]

We then use

\[ \left( \frac{r}{\gamma} \right)^2 = \frac{rr^*}{\gamma^*} + \sigma_1 \frac{r}{\gamma} \]
\[ = P + \mathcal{E}_2 + \sigma_1 \frac{r}{\gamma} \]

to write

\[ \frac{\mathcal{E}_2 \wedge \bar{\partial} r}{p^{n-\mu+1}} \left( \xi_0 \frac{r}{\gamma} \right)^2 = \frac{\mathcal{E}_2 \wedge \bar{\partial} r}{p^{n-\mu+1}} (P + \mathcal{E}_2) + \frac{\mathcal{E}_2 \wedge \bar{\partial} r}{p^{n-\mu+1}} \sigma_1 \frac{r}{\gamma} \]
\[ = \mathcal{A}_1 + \frac{1}{\gamma} \mathcal{A}_2. \]

Thus (4.11) gives terms of the form

\[ \mathcal{A}_1 + \frac{1}{\gamma} \mathcal{A}_2. \]

Alternatively, using

\[ \bar{\partial}_z P = \mathcal{E}_1 + \xi_0 \frac{r^*}{\gamma^*} \]

directly in (4.11) leads to terms of the form

\[ \mathcal{A}_1 + \frac{1}{\gamma^*} \mathcal{A}_2. \]

Similar calculations hold for \( \bar{\partial} \zeta \partial_z \mathcal{A}_3. \)
In order to calculate the derivations of $L_q$ which turn up in our formula (4.7), we set

$$\mathcal{M}_{k_j}^\mu = \Lambda_k \left( \frac{T_j \rho^2}{\Phi^{\mu+1} p_{n-\mu-1}} \right)$$

$$\mathcal{M}_{k_j}^\mu = -(\mu + 1) T_j \rho^2 \Lambda_k \Phi + \frac{1}{\Phi^{\mu+1}} \Lambda_k \left( \frac{T_j \rho^2}{p_{n-\mu-1}} \right)$$

$$\tilde{\mathcal{M}}_{k_j}^\mu = L_k \left( \frac{T_j \rho^2}{\Phi^{\mu+1} p_{n-\mu-1}} \right)$$

$$\tilde{\mathcal{M}}_{k_j}^\mu = -(\mu + 1) T_j \rho^2 L_k \Phi + \frac{1}{\Phi^{\mu+1}} L_k \left( \frac{T_j \rho^2}{p_{n-\mu-1}} \right).$$

From

$$\Lambda_k \bar{T}_j \rho^2 = -2\delta_{jk} + \xi_{-1} \xi^*_0 \sigma_1 + \xi_{-1} \xi^*_1 \sigma_2$$

we have, for $k < n$,

$$\frac{1}{\Phi^{\mu+1}} \Lambda_k \left( \frac{T_j \rho^2}{p_{n-\mu-1}} \right) = \frac{1}{\Phi^{\mu+1}} \left[ -2\delta_{jk} + \frac{n - \mu - 1}{p_{n-\mu}} (L_k \rho^2)(\bar{T}_j \rho^2) \right] + \frac{1}{\gamma \gamma^*} A_2,$$

and for $k = n$ and $j < n$, using Lemma 4.3, we have

$$\frac{1}{\Phi^{\mu+1}} \Lambda_n \left( \frac{T_j \rho^2}{p_{n-\mu-1}} \right) = \frac{2(n - \mu - 1) \bar{T}_j \rho^2}{\Phi^{\mu} p_{n-\mu}} + \frac{1}{\gamma \gamma^*} A_2.$$

Thus, for $k < n$,

$$\mathcal{M}_{k_j}^\mu = \frac{1}{\Phi^{\mu+1}} \left[ -2\delta_{jk} + \frac{n - \mu - 1}{p_{n-\mu}} (L_k \rho^2)(\bar{T}_j \rho^2) \right] + \frac{1}{\gamma \gamma^*} A_2$$

$$\mathcal{M}_{n_j}^\mu = \frac{1}{\gamma} \frac{2(n - \mu - 1) \bar{T}_j \rho^2}{\Phi^{\mu} p_{n-\mu}} + \frac{1}{\gamma \gamma^*} A_2,$$

taking into account calculations such as multiplying and the dividing by a factor of $\gamma$ in order to obtain a type two operator divided by a factor of $\gamma$.

We calculate in a similar manner the $\tilde{\mathcal{M}}_{k_j}^\mu$ terms. For these terms we use the symmetry of (4.3) to write

$$\bar{T}_j = \frac{\partial}{\partial \xi_{k_j}} + (\xi_0 \xi^*_1 \sigma_1) \Lambda,$$

and as a consequence

$$L_k \bar{T}_j \rho^2 = 2\delta_{jk} + \xi_{-1} \xi^*_0 \sigma_1 + \xi_{-1} \xi^*_1 \sigma_2.$$

For $k < n$,

$$\tilde{\mathcal{M}}_{k_j}^\mu = \frac{1}{\Phi^{\mu+1}} L_k \left( \frac{T_j \rho^2}{p_{n-\mu-1}} \right) + \frac{1}{\gamma \gamma^*} A_2$$

$$\tilde{\mathcal{M}}_{k_j}^\mu = \frac{1}{\Phi^{\mu+1}} \frac{L_k \bar{T}_j \rho^2}{p_{n-\mu-1}} - \frac{n - \mu - 1}{\Phi^{\mu+1} p_{n-\mu} L_k P} + \frac{1}{\gamma \gamma^*} A_2$$

$$\tilde{\mathcal{M}}_{k_j}^\mu = \frac{2\delta_{jk}}{\Phi^{\mu+1} p_{n-\mu-1}} - \frac{(n - \mu - 1)(\bar{T}_j \rho^2)(L_k \rho^2)}{\Phi^{\mu+1} p_{n-\mu}} + \frac{1}{\gamma \gamma^*} A_2.$$
For $k = n$ and $j < n$, we calculate
\[
\widehat{M}_{nj}^n = \frac{1}{\Phi^{n+1}} L_n \left( \frac{T_{j\rho}^2}{P^{n+\mu-1}} \right) - (q + 1) \Phi \frac{T_{j\rho}^2}{P^{\mu+1} P^{n+\mu-1}} L_n \Phi.
\]
Using Lemma 4.4 we can write
\[
\begin{aligned}
(4.14) \quad \widehat{M}_{nj}^n &= \gamma^*(\mu + 1) \frac{T_{j\rho}^2}{\Phi^{n+1} P^{n-\mu-1}} - (n - \mu - 1) \frac{T_{j\rho}^2}{\Phi^{n+1} P^{n+\mu-1}} L_n P + \frac{1}{\gamma^*} A_2.
\end{aligned}
\]
We now use Lemma 4.3 to write the second term on the right side of (4.14) as
\[
\frac{2(n - \mu - 1) \Phi T_{j\rho}^2}{\gamma^* \Phi^{n+1} P^{n-\mu}} + \frac{1}{\gamma^*} A_2,
\]
and we can then write
\[
\begin{aligned}
\widehat{M}_{nj}^n &= \gamma^*(\mu + 1) \frac{T_{j\rho}^2}{\Phi^{n+1} P^{n-\mu-1}} + \frac{2(n - \mu - 1) \Phi T_{j\rho}^2}{\gamma^* \Phi^{n+1} P^{n+\mu-1}} + \frac{1}{\gamma^*} A_2.
\end{aligned}
\]
From Lemma 4.3, we have
\[
\partial_n L_q = -c n_q \sum_{\frac{\mu}{\gamma^*} q} \mathcal{M}_{nj}^n \gamma e_{njL} \Phi \frac{1}{K} \wedge \Theta - \frac{1}{\gamma^*} A_2
\]
\[
\partial_n L_{q-1} = c n_q - 1 \sum_{\frac{\mu}{\gamma^*} q} \mathcal{M}_{nj}^n \gamma e_{jQ} \Phi \frac{1}{K} \wedge \Theta + \frac{1}{\gamma^*} A_2.
\]
We separate the terms with $n \in K$ from those with $n \notin K$.
\[
\begin{aligned}
\partial_n L_q - \partial_n L_{q-1} &= - \sum_{\frac{\mu}{\gamma^*} K} c n_q \sum_{\frac{\mu}{\gamma^*} L} \mathcal{M}_{nj}^n \gamma e_{jL} \Phi \frac{1}{K} \wedge \Theta - \frac{1}{\gamma^*} A_2
\end{aligned}
\]
\[
\begin{aligned}
&+ \sum_{\frac{\mu}{\gamma^*} L} \left( c n_q \sum_{\frac{\mu}{\gamma^*} K} \mathcal{M}_{nj}^n \gamma e_{jL} \Phi \frac{1}{K} \wedge \Theta - c n_q \sum_{\frac{\mu}{\gamma^*} L} \mathcal{M}_{nj}^n \gamma e_{jL} \Phi \frac{1}{K} \wedge \Theta + \frac{1}{\gamma^*} A_2.
\end{aligned}
\]
(4.15)
We write
\[
\begin{aligned}
\partial_n L_q - \partial_n L_{q-1} &= \sum_{\frac{\mu}{\gamma^*} L} \mathcal{H} \wedge \Theta - \frac{1}{\gamma^*} A_2.
\end{aligned}
\]
and compute the $\mathcal{H}_L$ terms.
For $n \in L$ we have
\[
\mathcal{H}_{nQ} = -2^{n-1} \left( \frac{1}{2\pi} \right) (n - 1)! \frac{1}{P^n} \sum_{\frac{\mu}{\gamma^*} Q} \mathcal{T}_{j\rho}^2 \Phi^{nQ} + \frac{1}{\gamma^*} A_2.
\]
For $n \notin L$ we distinguish the following different cases for the exponent, $K$ of $\Phi$ in (4.15).

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a) $K = lL$ with $l < n$

b) $K = nL$

c) $K = nJ$, $J \neq L$.

In case a) we have

$$\mathcal{H}_L = -\sum_{j \leq n} c_{nq} \left( \sum_{\mu} \left( \frac{n - \mu - 2}{q} \right) \gamma^2 (\mu + 1) \frac{\tilde{T}_j \rho^2}{\Phi^{2n-\mu+1}} \right) + 2(n-1) \frac{\gamma \Phi \tilde{T}_j \rho^2}{\Phi^{2n}} \omega^j + \frac{1}{\gamma^*} A_2.$$ 

For case b), we have

$$\mathcal{H}_L = \sum_{j \leq n \in L} c_{nq} \left( \sum_{\mu} \left( \frac{n - 2 - \mu}{q} \right) \tilde{M}_{jj}^\mu - c_{n,q-1} \sum_{\mu} \left( \frac{n - 2 - \mu}{q - 1} \right) M_{jj}^\mu \right) \gamma^\omega n L = 2^{n-2} \left( \frac{1}{2\pi} \right) (n-2)! \left( \frac{2}{\Phi^{2n-1}} - (n-1) \frac{|L_j \rho^2|^2}{\Phi^{2n}} \right) \gamma^\omega n L + \frac{1}{\gamma^*} A_2$$

$$= 2^{n-2} \left( \frac{1}{2\pi} \right) (n-1)! \frac{1}{\Phi^{2n}} \sum_{n \notin L} \left( 2P - \sum_{j < n} |L_j \rho^2|^2 \right) \gamma^\omega n L + \frac{1}{\gamma^*} A_2$$

$$= \frac{1}{\gamma^*} \frac{2^{n-2}}{(2\pi)^{n-1}} (n-1)! \frac{4\Phi}{P^{2n}} \omega^{n L} + \frac{1}{\gamma^*} A_2$$

For case c) we write

$$\mathcal{H}_L = \sum_{j \neq k \in L} c_{nq} \left( \sum_{\mu} \left( \frac{n - 2 - \mu}{q} \right) \tilde{M}_{kj}^\mu \gamma^{jL} \epsilon_{jL}^{kL} - c_{n,q-1} \sum_{\mu} \left( \frac{n - 2 - \mu}{q - 1} \right) M_{kj}^\mu \gamma^{jL} \epsilon_{jL}^{kL} \right) \omega^{n L}$$

$$= -\sum_{j \neq k \in L} c_{nq} \left( \sum_{\mu} \left( \frac{n - 2 - \mu}{q} \right) \tilde{M}_{kj}^\mu + c_{n,q-1} \sum_{\mu} \left( \frac{n - 2 - \mu}{q - 1} \right) M_{kj}^\mu \right) \gamma^{jL} \epsilon_{jL}^{kL} \omega^{n L},$$

since $\epsilon^{jL}_{kj} = -\epsilon^{kL}_{jL}$ when $k \neq j$. Thus from (4.12) and (4.13) we can write

$$\frac{2^{n-2}}{(2\pi)^n} (n-1)! \frac{1}{\Phi^{2n}} \sum_{j \neq k \in L} \left( -\left( \frac{L_j \rho^2}{\Phi^{2n}} \right) + \left( \frac{L_k \rho^2}{\Phi^{2n}} \right) \right) \gamma^{jL} \epsilon_{jL}^{kL} \omega^{n L} + \frac{1}{\gamma^*} A_2$$

$$= \frac{1}{\gamma^*} A_2.$$

We have therefore established the
Lemma 4.5. \( \partial_\zeta L_q - \partial_\zeta L_{q-1} = \sum_{|L|=q} \mathcal{K}_L \wedge \Theta^L \) with

\[
\mathcal{K}_{nQ} = -2^{n-1} \left( \frac{1}{2\pi} \right)^n (n-1)! \frac{1}{P^n} \sum_{j<n \notin Q} T_{ij} \rho^2 \psi^j Q + \frac{1}{\gamma^s} A_2,
\]

and for \( n \notin L \):

\[
\mathcal{K}_L = - \sum_{j<n \notin L} \epsilon_{nq} \left( \sum_\mu \frac{n-\mu-2}{q} \right) \gamma^2 (\mu + 1) T_{ij} \rho^2 \frac{\Phi^{\mu+2}}{Pn-\mu-1} \]

\[
+ 2 \left( \frac{n-2}{q} \right) (n-1) \frac{\gamma \Phi T_{ij} \rho^2}{\Phi Pn} \right) \psi^j L + \frac{1}{\gamma^s} \left( 2^{n-2} (2\pi)^n (n-1)! \frac{4\Phi}{Pn} \frac{\psi^n L + 1}{\gamma^s} A_2. \right.
\]

5. The structure of the kernels \( T_q \)

In this section we prove Theorem 3.2 which expresses the kernels, \( T_q \) and \( T_q^* \) as derivatives of explicitly computed simpler kernels. We solve

\[
\partial_\zeta N_q = T_q + \frac{1}{\gamma^s} A_2
\]

\[
\partial^*_\zeta N_q = T_q^* - 1 + \frac{1}{\gamma^s} A_2
\]

\[
N_q = N^*_q + \frac{1}{\gamma} A_3 + \frac{1}{\gamma^s} A_3
\]

\[
*N_q |_{\partial D} \equiv 0.
\]

We set

\[
N_q = G_q + \Gamma_{0q}
\]

and determine \( G_q \). With

\[
G_q = \sum_{|L|=q} G_L \wedge \Theta^L, \quad G_q = G_q^* + \frac{1}{\gamma} A_3 + \frac{1}{\gamma^s} A_3
\]

we solve

\[
(5.1) \quad \partial_\zeta G_L = \mathcal{K}_L + \frac{1}{\gamma^s} A_2
\]

where \( \mathcal{K}_L \) is as in Lemma 4.5.

If \( q < n-1 \) then we obtain (5.1) by choosing

\[
(5.2) \quad G_{nQ} = -\frac{2^{n-1}(n-2)!}{(2\pi)^n} \frac{1}{P^n-1} \psi^m Q, \quad L = nQ,
\]

\[
G_L = \epsilon_{nq} \left( \sum_\mu \frac{n-\mu-2}{q} \right) \gamma^2 (\mu + 1) T_{ij} \rho^2 \frac{1}{n-\mu-2} \frac{\Phi^{\mu+2}}{Pn-\mu-2} \]

\[
+ \left( \frac{n-2}{q} \right) \frac{\gamma}{\gamma^s} \frac{2\Phi}{\Phi Pn-1} \psi^j L \quad n \notin L.
\]
We verify (5.1) for the case \( q < n - 1 \) by calculating \( \bar{\partial}_s G_L \). That \( \bar{\partial}_s G_{nQ} = \mathcal{H}_{nQ} \) is easy to see, and we turn to (5.1) in the case \( n \notin L \). We have

\[
\bar{\partial}_s G_L = \sum_{j \notin L} c_{nq} T_j \left( \sum_{\mu} \binom{n - \mu - 2}{q} \gamma^2 \frac{\mu + 1}{n - \mu - 2} \frac{1}{\Phi^{\mu+2} P_{n-\mu-2}} \right) + \left( \frac{n-2}{q} \right) \frac{\gamma^2}{\Phi P_{n-1}} \bar{\omega}^L
\]

\[
= \sum_{j \notin L} c_{nq} \left( \sum_{\mu} \binom{n - \mu - 2}{q} \gamma^2 \frac{\mu + 1}{n - \mu - 2} T_j \left( \frac{1}{\Phi^{\mu+2} P_{n-\mu-2}} \right) \right) + \left( \frac{n-2}{q} \right) \frac{\gamma^2}{\gamma^s} T_j \left( \frac{2F}{\Phi P_{n-1}} \right) \bar{\omega}^L + \frac{1}{\gamma^s} A_2.
\]

We consider separately the cases \( j < n \) and \( j = n \).

In view of Lemma [4.2], we have, for \( j < n \),

\[
\gamma^2 T_j \left( \frac{1}{\Phi^{\mu+2} P_{n-\mu-2}} \right) = -(n - \mu - 2) \gamma^2 \frac{1}{\Phi^{\mu+2} P_{n-\mu-1}} T_j P + A_2
\]

\[
= -(n - \mu - 2) \gamma^2 \frac{1}{\Phi^{\mu+2} P_{n-\mu-1}} T_j P + A_2,
\]

where the last line follows from Lemma [4.3] ii. Similarly, we have

\[
\frac{\gamma^2}{\gamma^s} T_j \left( \frac{2F}{\Phi P_{n-1}} \right) = -(n-1) \frac{\gamma^2}{\gamma^s} \frac{2F}{\Phi P_{n-1}} T_j P + A_2
\]

\[
= -(n-1) \frac{\gamma^2}{\gamma^s} \frac{2F}{\Phi P_{n-1}} T_j P + A_2.
\]

We thus far can write

\[
\bar{\partial}_s G_L = - \sum_{j \notin L} c_{nq} \left( \sum_{\mu} \binom{n - \mu - 2}{q} \gamma^2 \frac{(\mu + 1) T_j \rho^2}{\Phi^{\mu+2} P_{n-\mu-1}} \right) + 2 \left( \frac{n-2}{q} \right) (n-1) \frac{\gamma^{\mu+2} T_j \mu^2}{\Phi^{\mu+2} P_{n-\mu-1}} \bar{\omega}^L
\]

\[
+ c_{nq} \left( \sum_{\mu} \binom{n - \mu - 2}{q} \gamma^2 \frac{\mu + 1}{n - \mu - 2} T_n \left( \frac{1}{\Phi^{\mu+2} P_{n-\mu-2}} \right) \right) + \left( \frac{n-2}{q} \right) \frac{\gamma^2}{\gamma^s} T_n \left( \frac{2F}{\Phi P_{n-1}} \right) \bar{\omega}^L + \frac{1}{\gamma^s} A_2.
\]

In dealing with \( j = n \) we have

\[
\gamma^2 T_n \left( \frac{1}{\Phi^{\mu+2} P_{n-\mu-2}} \right) = -(n - \mu - 2) \gamma^2 \frac{1}{\Phi^{\mu+2} P_{n-\mu-1}} T_n P + A_2
\]

\[
= -2(n - \mu - 2) \gamma^2 \frac{1}{\Phi^{\mu+1} P_{n-\mu-1}} + \frac{1}{\gamma} A_2 + \frac{1}{\gamma^s} A_2
\]

\[
= -2(n - \mu - 2) \gamma^2 \frac{1}{\Phi^{\mu+1} P_{n-\mu-1}} + \frac{1}{\gamma} A_2 + \frac{1}{\gamma^s} A_2.
\]
by Lemma 4.3 i. Also, we have

\[ (5.6) \quad \gamma^* L_n \left( \frac{2\Phi}{\Phi p^{n-1}} \right) = -2\gamma \frac{1}{\Phi p^{n-1}} - 2(n - 1) \gamma^* \frac{\Phi}{\Phi p^n} L_n P + \frac{1}{\gamma^*} A_2 \]

by Lemma 4.2 i. We now use a variation of Lemma 4.3 i. which follows. First, using the symmetry involved in (4.3) we can write

\[ \gamma^* L_n P = -2\Phi^* + \xi_{-1}\xi_1^*(P + \xi_2) + \xi_{-1}\xi_1^* \sigma_2. \]

Now using \( \gamma - \gamma^* = \sigma_1 \) we have

\[ \gamma L_n P = \gamma^* L_n P + \sigma_1 L_n P \]

\[ = -2\Phi + \xi_{-1}\xi_1^*(P + \xi_2) + \xi_{-1}\xi_1^* \sigma_2 + \sigma_1 L_n P. \]

And so (5.6) becomes

\[ \gamma^* L_n \left( \frac{2\Phi}{\Phi p^{n-1}} \right) = -2\gamma \frac{1}{\Phi p^{n-1}} + (n - 1) \frac{1}{\gamma^*} \frac{4\Phi}{\Phi p^n} + \frac{1}{\gamma^*} \frac{\Phi}{\Phi} \sigma_1 L_n P + \frac{1}{\gamma^*} A_2 + \frac{1}{\gamma^*} \sigma_1 A_2 \]

\[ = -2\gamma \frac{1}{\Phi p^{n-1}} + (n - 1) \frac{1}{\gamma^*} \frac{4\Phi}{\Phi p^n} + \frac{1}{\gamma^*} \frac{\Phi}{\Phi} \sigma_1 L_n P + \frac{1}{\gamma^*} A_2 + \frac{1}{\gamma^*} \sigma_1 A_2. \]

We now show the third term on the right can be written as

\[ (5.7) \quad \gamma \frac{\Phi}{\gamma^*} \sigma_1 L_n P = \frac{1}{\gamma^*} \sigma_1 A_2. \]

\( \Phi \) is a sum of terms of the form

\[ \xi_0 \xi_1 + \xi_2 - r, \]

and so we consider separately

\[ (5.8) \quad \gamma^* \sigma_2 \]

\[ (5.9) \quad \frac{1}{\gamma^*} \sigma_3 \]

\[ (5.10) \quad \frac{1}{\gamma^*} \sigma_1 r \]

(5.9) leads to the desired error terms with the substitution \( \mathcal{T}_n P = \xi_1^* + \xi_1^* \).

In (5.10) we substitute

\[ \mathcal{L}_n P = \xi_1 + \frac{r^*}{\gamma} + \frac{1}{\gamma} \xi_0 (P + \xi_2) \]

\[ = \frac{\gamma}{r} (P + \xi_2) + \frac{1}{\gamma} \xi_0 (P + \xi_2) \]

and we obtain

\[ \frac{1}{\gamma} \frac{\sigma_1 r}{\Phi p^n} \mathcal{L}_n P = \frac{1}{\gamma^*} A_2. \]

Turning now to (5.8), we write

\[ \frac{\gamma}{\gamma^*} = \frac{\gamma^*}{\gamma} + \frac{\sigma_1}{\gamma} + \frac{1}{\gamma^*} \]
so as to obtain
\[
\frac{\gamma}{\bar{\gamma}^*} \frac{\sigma_2}{\bar{\Phi} P^m} T_n P = \frac{\gamma}{\bar{\gamma}^*} \frac{\sigma_2}{\bar{\Phi} P^m} T_n P + \frac{1}{\gamma \bar{\Phi} P^m} \frac{\sigma_3}{\bar{\Phi} P^m} T_n P + \frac{1}{\gamma \bar{\Phi} P^m} \frac{\sigma_3}{\bar{\Phi} P^m} T_n P.
\]
We can now substitute
\[
T_n P = \mathcal{E}_1 + \xi_0 \gamma^*\]
in the first term on the right hand side, and use
\[
T_n P = \mathcal{E}_1 + \xi_0 \xi^* 1
\]
in the last two terms on the right hand side, in order to complete the verification of (5.7).

Thus (5.6) becomes
\[(5.11) \quad \frac{\gamma}{\bar{\gamma}^*} L_n \left(\frac{2\Phi}{\Phi P^m}\right) = -2\gamma \frac{1}{\bar{\Phi} P^{m-1}} + (n-1) \frac{4\Phi}{\gamma P^n} + \frac{1}{\gamma^* A_2}.
\]
Together, (5.5) and (5.11), when inserted into (5.4) give the term
\[
\frac{1}{\gamma^*} n \left(2\Phi \Phi P^m - 1\right) = -2 \gamma \frac{1}{\bar{\Phi} P^n} + \frac{1}{\gamma^* A_2} - \frac{1}{(2\pi)^n} (n-1)! \frac{\gamma^*}{\gamma^* A_2}
\]
which is the remaining part of $H_L$ in (5.1).

The calculations leading to the expressions for $G_L$ were done in a special coordinate chart near the boundary. To globalize the expressions we note there are double forms $\sigma_1$ such that
\[
-\frac{1}{2} \bar{\xi} \partial_{\xi} \rho^2 = \bar{\omega}^n \wedge \Theta^n + \bar{\omega}^n \wedge \sigma_1 + \sigma_1 \wedge \Theta^n + \tau,
\]
where $\tau$ does not contain any $\bar{\omega}^n$ or $\Theta^n$ terms. If we set $\nu(\zeta, z) = \bar{\omega}^n \wedge \Theta^n + \bar{\omega}^n \wedge \sigma_1 + \sigma_1 \wedge \Theta^n$, we have
\[
\nu(\zeta, z) = \bar{\omega}^n \wedge \Theta^n + \sigma_1
\]
\[
\tau(\zeta, z) = \sum_{j<n} \bar{\omega}^j \wedge \Theta^j + \sigma_1.
\]
We thus have
\[
(\bar{\partial}_{\zeta} \partial_{\zeta} \rho^2)^q = (-2)^q (\tau^q + q\tau^{q-1} \wedge \nu) + \sigma_1
\]
\[
\tau^q = q! \sum_{\|L\| \leq q} \bar{\omega}^L \wedge \Theta^L + \sigma_1
\]
\[
\tau^{q-1} \wedge \nu = (q-1)! \sum_{\|Q\| = q-1} \bar{\omega}^Q \wedge \Theta^Q + \sigma_1,
\]
which we use in connection with (5.2) and (5.3) to write

**Proposition 5.1.** Let $n \geq 3$. For $1 \leq q \leq n - 2$ let the differential forms $N_q$ be given by
\[
N_q = 2^{n-2} \left(\frac{1}{2\pi}\right)^n (n - q - 2)! \left(\sum_{\theta \leq \mu \leq n-q-2} \gamma^2 \left(\frac{n - \mu - 2}{q}\right) \frac{\mu + 1}{n - \mu - 2} \frac{1}{\Phi^{m+2}} P_n - A_2\right)
\]
\[
+ \left(\frac{n - 2}{q}\right) \left(\frac{2\Phi}{\Phi P^m}\right)^q \tau^q - \frac{2^{n-1}(n-2)!}{(q-1)! (2\pi)^n} \frac{1}{P_{n-1}} \tau^{q-1} \wedge \nu + \Gamma_0 q.
\]
Then the $N_q$ fulfills the first set of equations of Theorem 3.2.

Remark 5.2. While $N_0$ can also be explicitly computed, such a term will involve logarithms and does not fit the definition of admissible operators. Nonetheless, similar mapping properties for such operators do exist. $N_{n-1}$ can also be explicitly given and can be handled as in [4] combined with the above methods. The estimates remain true but the principal part of $N_{n-1}$ changes.

We now verify - in order to complete the proof of Theorem 3.2 -

$$\bar{\partial}_* N_q = T_{q-1}^* + \frac{1}{\gamma \gamma^*} A_2$$

for $1 \leq q \leq n - 2$, by showing

$$(5.12) \quad \vartheta_\zeta G_q = 3H_q^* + \frac{1}{\gamma \gamma^*} A_2.$$  

From Lemma 4.5 we have

$$3H_q^* = \frac{2^{n-1}}{(2\pi)^n} (n-1)! \left( \sum_{n \in L | |L| = q-1} \frac{\Lambda_j \rho^2}{P_n} \omega^j \wedge \Theta^L + \sum_{n \notin L | |L| = q-1} \frac{1}{\gamma \gamma^*} \frac{2\Phi}{P_n} \omega^L \wedge \Theta^n L ight)$$

$$- \sum_{n \notin L | |L| = q-1} \frac{\gamma^* \Phi A_j \rho^2}{\gamma} \frac{\mu}{P_n} \omega^L \wedge \Theta^L$$

$$- \sum_{n \notin L | |L| = q-1} c_{n,q-1} \sum_{\mu} \binom{n - \mu - 2}{q - 1} (\gamma^*)^2 \frac{\mu + 1}{\Phi \mu+2} \frac{1}{P_{n-\mu-1}} \omega^L \wedge \Theta^L$$

$$(5.13) \quad + \frac{1}{\gamma \gamma^*} A_2$$

for $q \geq 2$. The case $q = 1$ has a similar expression.
From (5.2) and (5.3) we calculate
\[ \partial_\zeta G_q = \frac{2^{n-1}(n-2)!}{(2\pi)^n} \sum_{\nu \in \mathcal{J}} L_j \left( \frac{1}{P_{n-1}} \right) \varepsilon_j^{L-\mu} \wedge \Theta^j \]
\[ - \sum_{\nu \in j, L} c_{nq} \left( \sum_{\mu} \left( \frac{n - \mu - 2}{q} \right) \gamma^2 (n - \mu - 2) L_j \left( \frac{1}{\Phi^{\mu+2} P_{n-\mu-2}} \right) \right) \]
\[ + \frac{\gamma}{\gamma^*} \left( \frac{n - 2}{q} \right) L_j \left( \frac{2\Phi}{\Phi P_{n-1}} \right) \varepsilon_j^{L-\mu} \wedge \Theta^j + \frac{1}{\gamma^*} \rho_2 \]
\[ = \frac{2^{n-1}(n-1)!}{(2\pi)^n} \left( \sum_{\nu \in j, L} \left( \sum_{\mu \leq n} A_j \rho^2 P_{n-\mu} \wedge \varepsilon_j^{L-\mu} \wedge \Theta^j \right) \right) + \frac{1}{\gamma^*} \frac{2\Phi}{\Phi P_{n-1}} \varepsilon_j^{L-\mu} \wedge \Theta^j \]
\[ - \sum_{\nu \in j, L} c_{nq} \left[ \sum_{\mu} \left( \frac{n - \mu - 2}{q} \right) \gamma^2 \left( \frac{(\mu + 1)(\mu + 2)}{n - \mu - 2} A_j \rho^2 P_{n-\mu-2} + \frac{(\mu + 1)A_j \rho^2}{\Phi^{\mu+2} P_{n-\mu-2}} \right) \right] \]
\[ + \frac{\gamma}{\gamma^*} \left( \frac{n - 2}{q} \right) \left( n - 1 \right) \frac{2\Phi A_j \rho^2}{\Phi P_{n-1}} \varepsilon_j^{L-\mu} \wedge \Theta^j \right) \right] + \frac{1}{\gamma^*} \rho_2 + \frac{1}{\gamma^*} \rho_2. \]

We use
\[ \frac{\gamma}{\gamma^*} \frac{\Phi A_j \rho^2}{\Phi P_{n-1}} = \frac{\gamma}{\gamma^*} \frac{A_j \rho^2}{\Phi P_{n-1}} + \frac{1}{\gamma^*} \rho_2 + \frac{1}{\gamma^*} \rho_2 \]
to compare (5.13) and (5.14), and to show (5.12) holds if
\[ \sum_{n \in J_{\nu L}} c_{n,q-1} \sum_{0 \leq \mu \leq n-q-1} \left( \frac{n - \mu - 2}{q - 1} \right) (\gamma^*)^2 \frac{(\mu + 1)A_j \rho^2}{\Phi^{\mu+2} P_{n-\mu-1}} \wedge \Theta^j = \]
\[ \sum_{n \in J_{\nu L}} c_{nq} \sum_{0 \leq \mu \leq n-q-2} \left( \frac{n - \mu - 2}{q} \right) \gamma^2 \frac{(\mu + 1)(\mu + 2)}{n - \mu - 2} \frac{A_j \rho^2}{\Phi^{\mu+2} P_{n-\mu-2}} \]
\[ + \frac{(\mu + 1)A_j \rho^2}{\Phi^{\mu+2} P_{n-\mu-1}} \wedge \Theta^j + \frac{1}{\gamma^*} \rho_2, \]
which is an elementary computation. The proof of Theorem 3.2 is complete.

6. Z-operators and Principal Parts

We generalize (slightly) the notion of an isotropic kernel (resp. operator). We let \( E_{j-2n}(\zeta, z) \) be a kernel of the form
\[ E_{j-2n}(\zeta, z) = \frac{\sigma_m(\zeta, z)}{r^{2k}(\zeta, z)} \quad j \geq 1, \]
where \( m - 2k \geq j - 2n \). We denote by \( E_{j-2n} \) the corresponding operator. The following theorem follows from [4] (see Theorem VII.4.1).

**Theorem 6.1.** The integral operators \( E_{1-2n} \) are continuous from \( \|E_{1-2n}\|_1 : L^p(D) \to L^s(D) \)
for any \( 1 \leq p \leq s \leq \infty \) with \( 1/s > 1/p - 1/2n \).

We denote by \( Z_1 \) those operators which are of the form
\[
Z_1 = A_1 + E_{1-2n}.
\]

\( Z_2 \) operators are defined similarly, and we define \( Z_j, j > 2 \), operators by induction to be those operators of the form
\[
Z_j = Z_{i_1} \circ \cdots \circ Z_{i_k} \quad i_1 + \cdots + i_k = j.
\]

We have the following mapping properties for \( Z_j \) operators:

**Proposition 6.2.** Let \( p \geq 2 \).

\[
Z_j : L^p(D) \to L^q(D)
\]

where
\[
\frac{1}{q} > \frac{1}{p} - \frac{j}{2n+2}.
\]

We will also use the following property which commutes factors of \( \gamma \) with \( Z \) operators.

**Lemma 6.3.** Let \( m \geq 0, k \geq 1 \).

\[
\gamma^m Z_k = Z_k \circ \gamma^m + Z_{k+1}.
\]

**Proof.** The proof follows from the relation
\[
\gamma^m(z) = \gamma^m(\zeta) + \sigma_1.
\]

Let \( A \) be one of the operators \( N_q, \bar{\partial}N_q \) and \( \bar{\partial}^*N_q \) which arise in the \( \bar{\partial} \)-Neumann problem. We are going to describe \( A \) in terms of \( Z \)-operators; this will show that its continuity properties in weighted \( L^p \) spaces coincide with the behavior of \( Z \)-operators (although \( A \) itself is not a \( Z \)-operator). To proceed we need some more definitions.

**Definition 6.4.** Let \( m \leq k \) be nonnegative integers. An operator
\[
C = Z_m + \sum_{\alpha} Z^\alpha_{k} \circ K_{\alpha},
\]

where the \( Z_j \) are \( Z \)-operators of type \( \geq j \), and the \( K_{\alpha} \) are \( L^2 \)-bounded, is called a \( k \)-asymptotic \( Z \)-operator of type \( \geq m \). We shall denote a generic \( k \)-asymptotic \( Z \)-operator of type \( m \) by \( C^m_{(k)} \).

**Definition 6.5.** Let \( A^0 \) be an \( L^2 \)-bounded operator. \( A^0 \) is a generalized \( Z \)-operator if there is an integer \( l \geq 0 \) such that \( \gamma^l A^0 \) is a \( Z \)-operator. If \( l \) is chosen minimal with that property, then the type of \( \gamma^l A^0 \) is called the type of \( A^0 \).

Note that a \( Z_m \)-operator is \( k \)-asymptotic \( Z \) for any \( k \) and is also a generalized \( Z \)-operator of type \( \geq m \): the integer \( l \) must be 0.

**Definition 6.6.** Let \( A \) be an \( L^2 \)-bounded operator. An \( L^2 \)-bounded operator \( A^0 \) is called a principal part of \( A \), if

i) \( A^0 \) is a generalized \( Z \)-operator of type \( m \),
ii) for each $l$, there is an $L$ and an $l$-asymptotic $Z$-operator of type $\geq m+1$, $C_{m+1}^{(l)}$, such that

$$\gamma^L A = \gamma^L A^0 + C_{m+1}^{(l)}.$$ 

It now follows

**Theorem 6.7.** Let $A$ be an operator with principal part $A^0$ of type $m = 1$ or 2. Let $p \geq 2$ be given. Then there is an $L$ such that

$$\gamma^L A : L^p(D) \to L^q(D),$$

continuously, with

$$\frac{1}{q} > \frac{1}{p} - \frac{m}{2n+2}.$$ 

This follows from the well-established properties of $Z$-operators of positive type.

In general, $A$ does not have a principal part. If $A$ itself is a generalized $Z$-operator, it is, naturally, its own principal part. We do not claim that principal parts are unique - in fact, they are not. However, we do have the following theorem regarding principal operators which tells that the type of a principal part of $A$ is a property of $A$.

**Theorem 6.8.** Let $A$ be an operator with principal part $A^0$ of type $m$. Then $A^0$ is unique modulo generalized $Z$-operators of type $m+1$.

**Proof.** By hypothesis, the type of $A^0$ is $m$. Suppose further that $A$ can also be written with a principal part $B^0$ of type $m' \geq m$. By definition, for each $k, l$, we can find an $K, L$ such that

$$\gamma^K A = \gamma^K A^0 + C_{m+1}^{(k)}$$

$$\gamma^L A = \gamma^L B^0 + C_{m'+1}^{(l)}.$$ 

Let $M = \max(K, L)$. Then

$$\gamma^M B^0 + C_{m'+1}^{(l)} = \gamma^M A^0 + C_{m+1}^{(k)}. $$

The terms $C_{m'+1}^{(l)}$ and $C_{m+1}^{(k)}$ above themselves may be written as

$$C_{m+1}^{(l)} = Z_{m'+1} + \sum_{\alpha} Z_{l}^\alpha \circ K_{\alpha},$$

$$C_{m+1}^{(k)} = Z_{m+1} + \sum_{\alpha} Z_{k}^\alpha \circ K_{\alpha}.$$ 

Insert (6.2) and (6.3) into (6.1) to get

$$\gamma^M B^0 + Z_{m'+1} + \sum_{\alpha} Z_{l}^\alpha \circ K_{\alpha} = \gamma^M A^0 + Z_{m+1} + \sum_{\alpha} Z_{k}^\alpha \circ K_{\alpha}$$

then rearrange to get

$$\gamma^M B^0 + Z_{m'+1} - \gamma^M A^0 = Z_{m+1} - \sum_{\alpha} Z_{l}^\alpha \circ K_{\alpha} - \sum_{\alpha} Z_{k}^\alpha \circ K_{\alpha}.$$ 

The left hand side is an operator of type $m$, and the right hand side is a $j = \min(k, l)$ - asymptotic operator.

Therefore (6.4) is of the form

$$\gamma^M A^0 = C^{(j)}.$$
where \( j = \min(k, l) \), and \( A^j_m \) is a \( Z \)-operator of type \( m \). We note that the operator \( A^j_m \) may change with different \( j \).

The idea is to show that \( \gamma^M B^0 - \gamma^M A^0 \), and therefore \( \gamma^M A^j_m \), is an operator of type \( m + 1 \). If we suppose that it is not, we arrive at a contradiction by showing some property of \( C^{(j)}_j \) is not exhibited by \( \gamma^M A^j_m \).

Now choose \( j \) sufficiently large, and an appropriately large \( M \) in (6.5), such that for \( s \leq m + 2 \) and all differential operators, \( D^s \) of order \( s \), we have

\[
D^s C^{(j)}_j : L^2(D) \rightarrow L^\infty(D),
\]

as can be seen by differentiating under the integral.

Under the assumption that \( \gamma^M A^j_m \) is not of type \( m + 1 \), we can show there is a differential operator, \( D^s \), of order \( s \leq m + 2 \), such that

(6.6)

\[
D^s A^j_m : L^2(D) \not\rightarrow L^\infty(D),
\]

contradicting (6.5).

Since we have the option of multiplying (6.5) by factors of \( \gamma \), we will ignore all factors of \( \gamma \) which arise in the kernels or by differentiating such kernels.

We first note that, modulo factors of \( \gamma \),

\[
|A^j_m| \lesssim \frac{1}{|\zeta - z|^t}
\]

for some integral \( t \). From (6.5) we must have, modulo factors of \( \gamma \),

\[
|A^j_m| \lesssim \frac{1}{|\zeta - z|^n}
\]

since, otherwise \( A^j_m \) applied to the function \( 1/|\zeta - \zeta_0|^{n-1} \), for \( \zeta_0 \in D \), does not land in \( L^\infty(D) \), whereas the right hand side of (6.5) applied to the same function is in \( L^\infty(D) \).

Also, from our assumption that \( \gamma^M A^j_m \) is not of type \( m + 1 \), and from the examination of operators of type \( m + 1 \), we have

(6.7)

\[
\frac{1}{|\zeta - z|^{n+1-(n+1)/2}} \lesssim |A^j_m|.
\]

If there is no such \( D^s \), for \( s \leq m + 2 \), for which (6.6) holds then, since \( D^s A^j_m \) is bounded by an integer power of \( |\zeta - z| \), we must have

(6.8)

\[
|D^s A^j_m| \lesssim \frac{1}{|\zeta - z|^n}
\]

for all differential operators of order \( s \leq m + 2 \). But then (6.8) implies

\[
|A^j_m| \lesssim \frac{1}{|\zeta - z|^{n-(m+2)}},
\]

which contradicts (6.7). \( \square \)

7. Integral representations

From [2] we have the explicit version of Theorem 1.2.
Theorem 7.1. Let \( f \in L^2_{\gamma, q}(D) \cap \text{Dom}(\partial^*) \cap \text{Dom}(\bar{\partial}) \). For \( 1 \leq q \leq n - 2 \),

\[
f(z) = (\partial f, \bar{T}_q) + (\partial^* f, (\bar{T}_{q-1})^*) + \left( \partial f, \frac{1}{\gamma^*} A_2 + \mathcal{E}_{2-2n} \right) + (\partial^* f, \mathcal{E}_{2-2n})
+ \left( f, \frac{1}{\gamma^*} A_1 + \frac{1}{\gamma^s} A_2 + \mathcal{E}_{1-2n} \right).
\]

From now on we work with the case \( q < n - 2 \). The case \( q = n - 2 \) is somewhat exceptional and can be handled as in [5]. We do not pursue that case here.

From Theorem 3.2 we can write this in terms of \( \mathcal{N}_q \) as

\[
f(z) = (\partial f, \bar{\partial} \mathcal{N}_q) + (\partial^* f, \bar{\partial}^* \mathcal{N}_q)
+ \left( \partial f, \frac{1}{\gamma^*} A_2 + \mathcal{E}_{2-2n} \right) + (\partial^* f, \frac{1}{\gamma^*} A_2 + \mathcal{E}_{2-2n})
+ \left( f, \frac{1}{\gamma^*} A_1 + \frac{1}{\gamma^s} A_2 + \mathcal{E}_{1-2n} \right),
\]

where the \( A_2 \) kernels are such that

\[
\bar{\partial} A_2 = A_1 + \frac{1}{\gamma} A_2.
\]

Lemma 7.2.

i) \( \gamma^3 f = \gamma^* (\gamma^2 \Box f, \mathcal{N}_q) + Z_2 \bar{\partial} f + Z_2 \bar{\partial}^* f + Z_1 f \)

ii) \( \gamma^3 \bar{\partial} f = Z_1 \gamma^2 \Box f + Z_1 \bar{\partial} f + Z_1 \bar{\partial}^* f \)

iii) \( \gamma^3 \bar{\partial}^* f = Z_1 \gamma^2 \Box f + Z_1 \bar{\partial} f + Z_1 \bar{\partial}^* f. \)

Proof. i). Start with the equation (7.1)

\[
f(z) = (\partial f, \bar{\partial} \mathcal{N}_q) + (\partial^* f, \bar{\partial}^* \mathcal{N}_q)
+ \left( \partial f, \frac{1}{\gamma^*} A_2 + \mathcal{E}_{2-2n} \right) + (\partial^* f, \frac{1}{\gamma^*} A_2 + \mathcal{E}_{2-2n})
+ \left( f, \frac{1}{\gamma^*} A_1 + \frac{1}{\gamma^s} A_2 + \mathcal{E}_{1-2n} \right).
\]

Apply to \( \gamma^2 f \), using

\[
\bar{\partial} \mathcal{N}_q = \frac{\gamma}{\gamma^*} A_1 + \frac{1}{\gamma} A_2 + \frac{1}{\gamma^*} A_2 + \mathcal{E}_{1-2n}
\]

\[
\bar{\partial}^* \mathcal{N}_q = \frac{\gamma}{\gamma^s} A_1 + \frac{1}{\gamma^s} A_2 + \frac{1}{\gamma^s} A_2 + \mathcal{E}_{1-2n}:
\]

\[
\gamma^2 f(z) = (\gamma^2 \bar{\partial} f, \bar{\partial} \mathcal{N}_q) + (\gamma^2 \bar{\partial}^* f, \bar{\partial}^* \mathcal{N}_q)
+ \left( \partial f, \frac{\gamma}{\gamma^*} A_2 + \mathcal{E}_{2-2n} \right) + (\partial^* f, \frac{\gamma}{\gamma^*} A_2 + \mathcal{E}_{2-2n})
+ \left( f, \frac{\gamma^2}{\gamma^*} A_1 + \frac{1}{\gamma^s} A_2 + \mathcal{E}_{1-2n} \right).
\]

Multiplying (7.2) by \( \gamma^* \) gives us

\[
\gamma^3 f(z) = \gamma^* (\gamma^2 \bar{\partial} f, \bar{\partial} \mathcal{N}_q) + \gamma^* (\gamma^2 \bar{\partial}^* f, \bar{\partial}^* \mathcal{N}_q) + Z_2 \bar{\partial} f + Z_2 \bar{\partial}^* f + Z_1 f
= \gamma^* (\gamma^2 \Box f, \mathcal{N}_q) + Z_2 \bar{\partial} f + Z_2 \bar{\partial}^* f + Z_1 f.
\]
To establish $ii)$, we integrate by parts in the fourth term on the right of (7.2) and use the fact that the $A_2$ term satisfies
\[ \bar{\partial}A_2 = A_1 + \frac{1}{\gamma}A_2. \]

We obtain
\[ \gamma^2 f(z) = (\gamma^2 \bar{\partial}f, \bar{\partial}N_q) + (\gamma^2 \bar{\partial}^* f, \bar{\partial}^* N_q) \]
\[ + \left( \bar{\partial}f, \gamma \gamma A_2 + \mathcal{E}_{2-2n} \right) + \left( f, \gamma \frac{1}{\gamma}A_1 + \frac{1}{\gamma}A_2 + \mathcal{E}_{1-2n} \right). \]

Then after multiplying by $\gamma^*$ we have
\[ \gamma^3 f(z) = \gamma^* (\gamma^2 \bar{\partial}f, \bar{\partial}N_q) + \gamma^* (\gamma^2 \bar{\partial}^* f, \bar{\partial}^* N_q) + Z_2 \bar{\partial}f + Z_1 f. \]

We now apply (7.3) to $\bar{\partial}f$:
\[ \gamma^3 \bar{\partial}f(z) = \gamma^* (\gamma^2 \bar{\partial}^* \bar{\partial}f, \bar{\partial}^* N_{q+1}) + Z_1 \bar{\partial}f \]
\[ = \gamma^* (\gamma^2 \bar{\partial} f, \bar{\partial}^* N_{q+1}) + Z_1 \bar{\partial}f + Z_1 \bar{\partial}^* f. \]

Now use
\[ \bar{\partial}^* N_{q+1} = \gamma^* A_1 + \frac{1}{\gamma}A_2 + \frac{1}{\gamma^*}A_2 + \mathcal{E}_{1-2n}, \]
which follows from Proposition [5.1].

The term $\gamma^*(\gamma^2 \bar{\partial} f, \bar{\partial}^* N_{q+1})$ is then written
\[ \gamma^* (\gamma^2 \bar{\partial} f, \bar{\partial}^* N_{q+1}) = Z_1 \gamma^2 \bar{\partial} f + (\gamma \bar{\partial} f, A_2). \]

The last term can be written as
\[ (\gamma \bar{\partial} f, A_2) = (\bar{\partial} \bar{\partial}^* f, \gamma A_2) + (\bar{\partial}^* \bar{\partial} f, \gamma A_2) \]
\[ = (\bar{\partial}^* f, \gamma A_1 + A_2) + (\bar{\partial} f, \gamma A_1 + A_2) \]
\[ = Z_1 \bar{\partial} f + Z_1 \bar{\partial}^* f. \]

Putting everything together we write
\[ \gamma^* (\gamma^2 \bar{\partial} f, \bar{\partial}^* N_{q+1}) = Z_1 \gamma^2 \bar{\partial} f + Z_1 \bar{\partial} f + Z_1 \bar{\partial}^* f, \]
and so by (7.3) we have
\[ \gamma^3 \bar{\partial} f(z) = Z_1 \gamma^2 \bar{\partial} f + Z_1 \bar{\partial} f + Z_1 \bar{\partial}^* f. \]

In the same way, we obtain $iii)$.

\[ \square \]

**Theorem 7.3.**

\begin{enumerate}
\item \[ \gamma^{3j} f = \gamma^* (\gamma^{3j-1} \bar{\partial} f, N_q) + \sum_{k=3}^{j+1} Z_k \gamma^{3(j-k)+5} \bar{\partial} f \]
\[ + Z_{j+1} \bar{\partial} f + Z_{j+1} \bar{\partial}^* f + Z_j f \]
\item \[ \gamma^{3j} \bar{\partial} f = \sum_{k=1}^{j} Z_k \gamma^{3(j-k)+2} \bar{\partial} f + Z_j \bar{\partial} f + Z_j \bar{\partial}^* f \]
\item \[ \gamma^{3j} \bar{\partial}^* f = \sum_{k=1}^{j} Z_k \gamma^{3(j-k)+2} \bar{\partial} f + Z_j \bar{\partial} f + Z_j \bar{\partial}^* f \]
\end{enumerate}
Inserting the expressions for consideration of the projection in order that the proof of Theorem 8.1 below goes through also in the case of domains in complex manifolds.

Multiplying by $\gamma$ denote the orthogonal projection operator onto the harmonic space. We first note

Finally, inserting these into (7.5) we obtain $f$).

Proof. $i$). The proof is by induction, the first step being Lemma 7.2.

To prove $i$) for $j+1$ we multiply by $\gamma^3$ and commute the $\gamma$'s with the $Z_k$ operators using repeatedly $\gamma^3Z_k = Z_k\gamma^3 + Z_{k+1}$.

Since $\gamma^* N_q = Z_2$, in light of the above, we can write

Thus multiplying $i$) by $\gamma^3$ gives

Inserting the expressions for $f$, $\gamma^3\bar{\partial}f$ and $\gamma^3\bar{\partial}^*f$ from Lemma 7.2 we can write

Finally, inserting these into (7.5) we obtain $i$).

ii) and iii) are proved similarly. $\square$

8. Asymptotic development of the Neumann operator

Let

\[ H_q : L^2_{0,q} \to \mathbb{H}^q \quad q \geq 1 \]

denote the orthogonal projection operator onto the harmonic space. We first note that in $\mathbb{C}^n$, the projection $H_q$ is just the 0 operator, however, we include this consideration of the projection in order that the proof of Theorem 8.1 below goes through also in the case of domains in complex manifolds.

Since for $f \in \mathbb{H}^q$, $\bar{\partial}f$ and $\bar{\partial}^*f$ vanish, from Theorem 7.1 we conclude

\[ \gamma^2 H_q f = Z_1 H_q f. \]

Multiplying by $\gamma^2$ and commuting the $\gamma$ with the $Z_1$-operator leads to

\[ \gamma^4 H_q f = \gamma^2 Z_1 H_q f \]
\[ \gamma^4 H_q f = Z_1 \gamma^2 H_q f + Z_2 H_q f \]
\[ \gamma^4 H_q f = Z_2 H_q f. \]

An induction argument then yields

\[ \gamma^{2j} H_q f = Z_j H_q f. \]
Theorem 8.1.

\[ \begin{align*}
i) & \quad \gamma^{3j} N_q f = \gamma^*(\gamma^{3j-1} f, N_q) + Z_3 \gamma^2 f + C_j(f) \\
ii) & \quad \gamma^{3j} \bar{\partial}^* N_q f = \gamma^*(\gamma^{3j-1} f, T_{q-1}) + Z_2 \gamma^2 f + C_j(f) \\
iii) & \quad \gamma^{3j} \partial N_q f = \gamma^*(\gamma^{3j-1} f, T_q^*) + Z_x \gamma^2 f + + C_j(f). \end{align*} \]

Proof. i. We apply Theorem 7.3 to \( N_q f \):

\[ \gamma^{3j} N_q f = \gamma^*(\gamma^{3j-1} \Box N_q f, N_q) + \sum_{k=3}^{j+1} Z_k \gamma^{3(j-k)-5} \Box N_q f + Z_{j+1} \bar{\partial}^* N_q f + Z_j N_q f. \]

By definition

\[ \Box N_q f = f - H_q f, \]

and so we can write

\[ \gamma^{3j} N_q f = \gamma^*(\gamma^{3j-1} f, N_q) - \gamma^*(\gamma^{3j-1} H_q f, N_q) + \sum_{k=3}^{j+1} Z_k \gamma^{3(j-k)-5} (f - H_q f) + Z_{j+1} \bar{\partial}^* N_q f + Z_j N_q f, \]

where we use

\[ \gamma^*(\gamma^{3j-1} H_q f, N_q) = Z_j H_q f \]

and

\[ Z_k \gamma^{3(j-k)-5} H_q f = Z_k \circ Z_{j-k} H_q f = Z_j H_q f. \]

ii. To prove ii. we note the first \( Z_1 \) operator on the right hand side of Theorem 7.3 ii), in view of Theorem 5.2 is related to \( T_{q-1} \) by

\[ Z_1 \gamma = \gamma^* T_{q-1} \gamma + Z_2. \]

We therefore write Theorem 7.3 ii) as

\[ \gamma^{3j} \bar{\partial}^* f = \gamma^*(\gamma^{3j-1} \Box f, T_{q-1}) + \sum_{k=2}^{j} Z_k \gamma^{3(j-k)-5} \Box f + Z_j \bar{\partial}^* f = \gamma^*(\gamma^{3j-1} H_q f, T_{q-1}) \]

Replacing \( f \) with \( N_q f \), we obtain

\[ \gamma^{3j} \bar{\partial}^* N_q f = \gamma^*(\gamma^{3j-1} f, T_{q-1}) - \gamma^*(\gamma^{3j-1} H_q f, T_{q-1}) + Z_2 \gamma^2 f - \sum_{k=2}^{j} Z_k \gamma^{3(j-k)+2} H_q f + Z_j \bar{\partial} N_q f + Z_j \bar{\partial}^* N_q f \]

\[ = \gamma^*(\gamma^{3j-1} f, T_{q-1}) + Z_2 \gamma^{3(j-k)+2} f + Z_j H_q f + Z_j \bar{\partial} N_q f + Z_j \bar{\partial}^* N_q f. \]

iii) The proof of iii), in which we make use of the relation Theorem 7.3 iii), follows as does that of ii) \( \square \)

As mentioned above, in \( \mathbb{C}^n \) we would have \( H = 0 \), and the proof simplifies. In particular, a cruder version of Theorem 7.3 would suffice.
Our Main Theorems 1 and 3 now follow from Theorem 8.1 after taking into account Lemma 6.3 and the types of the various operators. For instance, setting
\[ N_{\gamma}^0 f = (f, N_q), \]
we have
\[ \gamma^3 N_q f = \gamma^3 N_{\gamma}^0 f + Z_3 \gamma^2 f + C_j^{(j)} f. \]

In particular, we can add to Main Theorem 1 Theorem 8.2.

**Theorem 8.2.** For \( 1 \leq q \leq n - 3, \)
\begin{align*}
  i) & \quad N_q \equiv N_{\gamma}^0 \text{ of type } 2 \\
  ii) & \quad \bar{\partial}N_q \equiv T_q^* \text{ of type } 1 \\
  iii) & \quad \bar{\partial}^* N_q \equiv T_{q-1} \text{ of type } 1.
\end{align*}

We note that in the smooth case the above theorems coincide with the known results (see [3]): just set \( \gamma \equiv 1. \)

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