INTEGRATION OF PARTICLE-GAS SYSTEMS WITH STIFF MUTUAL DRAG INTERACTION

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ABSTRACT

Numerical simulation of numerous mm/cm-sized particles embedded in a gaseous disk has become an important tool in the study of planet formation, both theoretically and observationally. On the one hand, planetary cores must be built up from micron-sized interstellar dust grains, which need to gradually grow in size within a gaseous environment (Safronov 1969). Especially challenging is how mm/cm-sized particles can effectively sediment toward the mid-plane of a protoplanetary disk and coalesce to form km-sized planetesimals (Johansen et al. 2014, and references therein). On the other hand, thanks to advances in telescopes, it has become possible for observations to resolve the distribution of mm/cm-sized particles in nearby protoplanetary disks from their thermal emission in sub-millimeter and radio wavelengths or polarized scattered light in the near-infrared. For instance, van der Marel et al. (2013) used the Atacama Large Millimeter/centimeter Array (ALMA) to locate a large-scale, lopsided concentration of mm-sized pebbles in the transition disk around Oph IRS 48. ALMA Partnership et al. (2015) found an almost perfectly axisymmetric, ring-like distribution of pebbles around HL Tau. The Strategic Explorations of Exoplanets and Disks with Subaru (SEEDS) project (Tamura & SEEDS Team 2009) found a wealth of morphologically diverse structures like spiral arms and rings in an array of protoplanetary disks. These structures have also been detected with the Very Large Telescope (VLT; see, e.g., Benisty et al. 2015).

To gain more insight into the physical processes at work in these protoplanetary disks, numerical simulations modeling both gas and solid particles are often enlisted, given the complexity of such a system. For example, numerical simulations were used to demonstrate how pebbles/boulders could spontaneously concentrate themselves in a gaseous disk via the streaming instability (Johansen & Youdin 2007; Bai & Stone 2010a; Yang & Johansen 2014) to the extent that the planetesimal formation is triggered (Johansen et al. 2009, 2015; Simon et al. 2016, U. Schäfer et al. 2016, in preparation). Simulations have also been used to study how dust is trapped within vortices, which could explain the observed lopsided dust concentration in some transition disks (Lyra et al. 2009; Zhu et al. 2014; Baruteau & Zhu 2016; Surville et al. 2016). Using simulations of a protoplanetary disk with planet-induced gaps, the axisymmetric dust distribution observed in the HL Tau disk can be reproduced (Dipierro et al. 2015; Dong et al. 2015b; Jin et al. 2016). By matching simulation models with the spiral structures in an observed protoplanetary disk, the mass and orbit of any potentially unseen planet can also be inferred (Dong et al. 2015a; Fung & Dong 2015).

For a solid particle that is less than \( \sim 100 \) m in a protoplanetary disk, the dominant interaction between the particle and its surrounding gas is via mutual drag force instead of mutual gravity (see, e.g., Oishi et al. 2007; Nelson & Gressel 2010). The main tendency of the drag force is to reduce the relative velocity between the gas and the particle exponentially with time, the strength of which can be characterized by the stopping time \( t_s \) (Whipple 1972; Weidenschilling 1977). When the collective motion of a swarm of identical solid particles is considered, the time constant of the exponential decay due to this mutual drag force is given by \( t_s/(1 + \epsilon) \), where \( \epsilon \) is the local solid-to-gas density ratio (see Sections 2.1 and 3.2). Therefore, for any numerical method explicitly integrating a particle-gas system with mutual drag interaction, the stability criterion requires that the time step be less than this time constant at all times.

Depending on the value of this time constant, \( t_s/(1 + \epsilon) \), the mutual drag force can become extremely stiff in particular regimes. First, for particles of sizes less than about the mean free path of the surrounding gas, the stopping time \( t_s \) is linearly...
proportional to the size of the particles (Epstein 1924; Weidenschilling 1977). Hence the smaller the particles, the stiffer the mutual drag force becomes. As a reference point, \(t_s\) is on the order of \(10^{-5}\) local Keplerian orbital period for mm-sized particles embedded at 1 au in the mid-plane of a Minimum Mass Solar Nebula model (e.g., Youdin 2010). Second, the stronger the local concentration of solid particles, the higher the maximum solid-to-gas density ratio \(\epsilon\) and yet again the stiffer the mutual drag force is. It has been suggested that the mm-sized chondrules ubiquitous in the Solar System were formed in a solid-rich environment, with a density of solids roughly 100 times the background gas density or more (Cuzzi & Alexander 2006; Alexander et al. 2008). Moreover, for sedimented layer of particles marginally coupled to the surrounding gas, the maximum local solid-to-gas density ratio in the saturated state of the streaming turbulence can be as high as \(\epsilon \sim 10^5-10^6\) (Johansen et al. 2007; Bai & Stone 2010a; Yang & Johansen 2014). Therefore, either effect or a combination of both can render the time step so short that the computational cost of the simulation model becomes intractable.

A classic approach to treat stiff source terms is to operator split these terms out and use a dedicated method to integrate them with reasonable accuracy an efficiency (e.g., Inoue & Inutsuka 2008; Yang & Krumholz 2012). When considering two-fluid approximation for a particle-gas system with mutual drag interaction, this leads to a system of ordinary differential equations for the drag force without any spatial coupling. It is then relatively straightforward to integrate this system strictly locally, either with analytical formulas or with numerical methods. This approach and similar ones have been implemented in Eulerian grid-based schemes (Paardekooper & Mellema 2006; Balsara et al. 2009; Miniati 2010; Kowalik et al. 2013) and in smoothed particle hydrodynamics (Laibe & Price 2012; Løren-Aguilar & Bate 2014, 2015; Booth et al. 2015).

Instead of the two-fluid approximation, however, it is typically preferable to use Lagrangian solid particles to model a particle-gas system (Youdin & Johansen 2007; Balsara et al. 2009; Bai & Stone 2010b; Miniati 2010; Zhu et al. 2014). The exact information of position and velocity carried by each particle better samples the distribution of particles in phase space. The ability to sample the velocity distribution of particles is especially important in the study of the collisional evolution of solids, both large (e.g., Kokubo & Ida 1996) and small (e.g., Zsom & Dullemond 2008). Lagrangian particles not only help measure their velocity dispersion, but can also be directly used to predict the collision parameters, neither of which are readily available in the fluid approximation. Moreover, in the drag-dominated regime, particles do not dynamically equilibrate among themselves due to a lack of direct interaction; thus the effective pressure of particles is virtually zero. This implies that any spatial variation in the fluid description of particles is connected by a contact discontinuity, which is not trivial to numerically accurately trace, especially when the contrast is significant. This issue is absent when using Lagrangian particles.

Despite these advantages of employing Lagrangian solid particles along with Eulerian gas, major difficulties arise for the direct integration of the mutual drag force between gas and particles (see Section 2). The most difficult of these is that the presence of the particles can make the system of equations globally coupled, much like a diffusion equation, with which the temporal solution for any cell depends on the initial conditions for all other cells. With the diffusion equation, an initial delta function is immediately broadened in time and becomes a Gaussian distribution that is nonzero for the whole domain. Surprisingly, using the standard particle-mesh approach to compute the mutual drag force in a particle-gas system also induces this property for the connected domain covered by the particles, as we show in Appendix A. Consider, for example, a one-dimensional grid of gas with uniformly distributed particles. Suddenly pushing one particle on one side of the domain would drag not only its surrounding gas, but also all the other particles via their intermediate gas cells, up to the particles on the other side of the domain. Although this effect of global propagation of information diminishes exponentially with distance, it indicates that when using the particle-mesh method, the coupling between the gas and the particles is more complex than simple local particle-gas pairs. While standard numerical techniques exist to treat a diffusion equation efficiently (e.g., the Crank–Nicolson method and the spectral method), the incongruence between the Lagrangian and the Eulerian descriptions makes these methods not applicable to particle-gas systems with mutual drag force. Therefore, the focus has been only on the direct integration of the drag force on the particles without operator splitting the drag force on the gas, as in Balsara et al. (2009), Bai & Stone (2010b), and Miniati (2010). This approach only relieves the time-step constraint due to small particles, whereas it remains problematic when strong solid concentration occurs. Note also that in Bai & Stone (2010b) and in Johansen et al. (2007), an artificial increase of the stopping time \(t_s\) is implemented for those cells with high local solid-to-gas density ratios in order to circumvent the time-step constraint, and the numerical accuracy of this approach has not been systematically demonstrated yet.

In this work, we devise a numerical algorithm that effectively disentangles the system of equations for the mutual drag force and allows for its direct integration on a cell-by-cell basis. For each cell we use an analytical solution to assist in predicting the velocities of the gas and the particles at the next time step, so the time-step constraint posed by the mutual drag force is lifted. This algorithm is described in detail in Section 2. To validate our algorithm, we use an extensive suite of benchmarks with known solutions in one, two, and three dimensions, presented in Sections 3–5, respectively. Finally, we discuss the generality and possible applications of our algorithm in Section 6.

2. THE ALGORITHM

We first consider a system of Eulerian gas and Lagrangian solid particles moving in a differentially rotating disk in a rotating frame. In addition to the Coriolis force, the centrifugal force, and the external axisymmetric gravitational potential, the gas and each of the particles interact with their mutual drag force. We further adopt the local-shearing-sheet approximation (Goldreich & Lynden-Bell 1965), in which the origin of the frame is located at an arbitrary distance \(R\) away from the rotation axis with the \(x-, y-,\) and \(z-\)axes of the frame that are constantly in the radial, azimuthal, and vertical directions, respectively, and the frame co-rotates with the disk at the local angular frequency at its origin. We include also a constant \(x-\)acceleration \(a_x\) to the gas from, for example, a background
pressure gradient, and a vertically varying z-acceleration $g_z(z)$ on both the gas and the particles due to, for example, the vertical component of the central gravity. Without loss of generality, we assume an isothermal equation of state for the gas with $c_s$ being the speed of sound. Then the governing equations for the gas read

\[
\frac{\partial \rho_g}{\partial t} - q\Omega \frac{\partial \rho_g}{\partial y} + \mathbf{u} \cdot \nabla \rho_g + \rho_g \nabla \cdot \mathbf{u} = 0, \tag{1}
\]

\[
\frac{\partial \mathbf{u}}{\partial t} - q\Omega \frac{\partial \mathbf{u}}{\partial y} + \mathbf{u} \cdot \nabla \mathbf{u} = a_s \hat{e}_z + g_z(z_k)\hat{e}_z - c_s^2 \nabla \ln \rho_g \]

\[
-2\Omega \times \mathbf{u} + q \Omega \mathbf{u}, \hat{e}_z \sum_j \frac{\hat{e}_j \cdot (\mathbf{v}_j - \mathbf{u})}{t_s}, \tag{2}
\]

while the equations of motion for the particles read

\[
\frac{d\mathbf{r}_{p,j}}{dt} = -q\Omega \mathbf{r}_{p,j}, \hat{e}_z + \mathbf{v}_j, \tag{3}
\]

\[
\frac{d\mathbf{v}_j}{dt} = g_z(z_{p,j})\hat{e}_z - 2\Omega \times \mathbf{v}_j + q \Omega \mathbf{v}_{j,k}, \hat{e}_z + \frac{\mathbf{u}_j - \mathbf{v}_j}{t_s}. \tag{4}
\]

In the above equations, $\rho_g$ and $\mathbf{u}$ are, respectively, the density and the velocity of the gas at the grid point $\mathbf{r}_k = (x_k, y_k, z_k)$, and $\mathbf{v}_j$ is the velocity of the $j$th particle, which is located at $\mathbf{r}_{p,j} = (x_{p,j}, y_{p,j}, z_{p,j})$; see Figure 1(a). The constant local angular velocity $\Omega = \Omega(R)$ is parallel to the z-axis, and $q = -d \ln \Omega/d \ln R$ is the dimensionless shear parameter at radial distance $R$ from the rotation axis, which is $3/2$ for a Keplerian potential. Both $\mathbf{u}$ and $\mathbf{v}_j$ are measured relative to the background shear velocity $-q\Omega \hat{e}_z$, at their respective locations. The parameter $t_s$ is the stopping time of the mutual drag force between the gas and each of the particles. For simplicity, and more importantly, for a clean demonstration of our algorithm, we assume $t_s$ is a constant in this work and discuss the possibility of its generalization in Section 6. The remaining variables are $\hat{e}_j$ and $\mathbf{u}_j$: the former is some averaged particle-to-gas density ratio "perceived" by the gas in cell $k$ from the $j$th particle, while the latter is some averaged velocity of the surrounding gas "perceived" by the $j$th particle.

We then operator split out the source terms for the mutual drag force, the rotation/shear, the external radial acceleration for the gas, and the external vertical acceleration for the particles from the full system of Equations (1)–(4) (see Appendix B). As a result, the two independent systems of equations now read\(^1\)

\[
\frac{\partial \rho_g}{\partial t} - q\Omega \frac{\partial \rho_g}{\partial y} + \mathbf{u} \cdot \nabla \rho_g + \rho_g \nabla \cdot \mathbf{u} = 0, \tag{5a}
\]

\[
\frac{\partial \mathbf{u}}{\partial t} - q\Omega \frac{\partial \mathbf{u}}{\partial y} + \mathbf{u} \cdot \nabla \mathbf{u} = g_z(z_k)\hat{e}_z - c_s^2 \nabla \ln \rho_g \]

\[
-2\Omega \times \mathbf{u} + q \Omega \mathbf{u}, \hat{e}_z \sum_j \frac{\hat{e}_j \cdot (\mathbf{v}_j - \mathbf{u})}{t_s}, \tag{5b}
\]

\[
\frac{d\mathbf{r}_{p,j}}{dt} = -q\Omega \mathbf{r}_{p,j}, \hat{e}_z + \mathbf{v}_j, \tag{5c}
\]

\[\text{and}\]

\[
\frac{d\mathbf{u}}{dt} = a_s \hat{e}_z - 2\Omega \times \mathbf{u} + q \Omega \mathbf{u}, \hat{e}_z + \frac{\hat{e}_j \cdot (\mathbf{v}_j - \mathbf{u})}{t_s}, \tag{6a}
\]

\[
\frac{d\mathbf{v}_j}{dt} = g_z(z_{p,j})\hat{e}_z - 2\Omega \times \mathbf{v}_j + q \Omega \mathbf{v}_{j,k}, \hat{e}_z + \frac{\mathbf{u}_j - \mathbf{v}_j}{t_s}. \tag{6b}
\]

The system of Equations (5) consists of the usual Euler equations for fluid dynamics and an ordinary differential equation for the positions of the particles, both of which can be integrated with any standard technique. Our task is therefore to devise a numerical algorithm to solve the system of Equations (6) without any time-step constraint due to the mutual drag force between the gas and the particles.

Even after the operator splitting, the system of Equations (6) still presents several major difficulties, as mentioned in Section 1. First, the rather loose definitions of $\hat{e}_j$ and $\mathbf{u}_j$ are a manifestation of the distinctly different formalisms of the gas and the solid particles (i.e., Eulerian versus Lagrangian). Therefore, their design is one of the central factors that determine the accuracy and efficiency of the algorithm concerning the mutual drag force. Second, to relieve the time-step constraint posed by both small stopping time $t_s$ and large solid-to-gas density ratio $\epsilon$, the characteristic velocity curves followed by each cell of gas and each particle need to be accurately captured by the numerical method. This is best achieved by solving the system of Equations (6) simultaneously in the same integrator, given the particle-gas coupling via the mutual drag force. Worst of all, the density ratio $\hat{e}_j$ usually includes all particles within a definite distance to cell $k$, while the velocity $\mathbf{u}_j$ is usually a weighted average from the surrounding cells around the $j$th particle. This implies that all cells by Equation (6a) and all particles by Equation (6b) are more than likely to be completely coupled via $\hat{e}_j$ and $\mathbf{u}_j$, particularly when the particles are roughly uniformly distributed throughout the computational domain. These couplings make the solutions for the whole velocity field of the gas and all the particle velocities as a function of time dependent on one another (see also Appendix A). To solve this system efficiently, an implicit numerical method is usually employed, because the method is unconditionally stable against time step. However, even with an implicit method, the matrix representing the system of Equations (6) is huge, with the total number of cells plus particles on each side, and the inversion of this matrix can be a prohibitive computational task, because it cannot be organized in simple band diagonal form.\(^2\) In the following subsections, we construct our numerical algorithm with these difficulties in mind.

2.1. Analytical Solutions to the System of Equations

In order to proceed, we first conceive the simplest possible scenario to solve the system of Equations (6), which leads to the nearest-grid-point (NGP) scheme of the particle-mesh method. In this scheme, each individual cell containing some number of particles is considered independently; the gas in one
Figure 1. Schematic diagrams in two dimensions, demonstrating the particle-mesh scheme we use to integrate the mutual drag force. (a) A grid of gas cells with several Lagrangian solid particles. The jth particle (shown in blue) has a mass of \( m_{pj} \) and a velocity of \( v_j \). The green dashed line shows the boundary of cell \( k \), in which the gas has a density of \( \rho_{gk} \) and a velocity of \( u_k \). (b) Each particle is interpreted as a distributed cloud, with its boundary shown by a blue or red dashed line. The white dotted lines are the density contours for the jth particle. Each particle cloud is further split by the grid cells into sub-clouds, each of which has the same initial velocity \( v_j \) as the parent particle. The mass of the sub-cloud from the jth particle that is enclosed in cell \( k \) is \( m_{pj} W_{jk} \), where \( W_{jk} \equiv W (\mathbf{r}_{pj} - \mathbf{r}_k) \). (c) Within cell \( k \), the enclosed sub-clouds are allowed to interact with the gas as well as each other via the mutual drag force, independent of the other sub-clouds in other cells. The sub-cloud from the jth particle thus receives a change in velocity \( \Delta v^h_j \) over a finite time step \( \Delta t \), similarly for the other sub-clouds in the same cell. (d) The same operation is applied to every cell, and thus all sub-clouds undergo a change in velocity. (e) The total change in velocity for the jth particle \( \Delta v_j \) is then the weighted summation of all \( \Delta v^h_j \)'s of its sub-clouds. As a result, the velocity of the jth particle becomes \( v'_j \) at the next time step, similarly for all other particles. (f) To find the change in the velocity field of the gas, we once again apply the interpretation of the sub-particle clouds, except that each sub-cloud from the jth particle now has the same change of velocity \( \Delta v_j \) as its parent particle cloud. Because the change in total momentum for cell \( k \) is known, the change of gas velocity \( \Delta u_k \) in the cell can be computed from the momentum changes of the sub-particle clouds in the cell. We dub this last procedure “particle-mesh back reaction.”
cell interacts only with the particles inside and vice versa. As a result, the gas and the particles in a cell do not couple with those in any of the neighboring cells. This implies that \( \tilde{v}_{ij} = m_{pj}/\rho_j V \) and \( \tilde{u}_j = u \), where \( m_{pj} \) is the mass of the \( j \)th particle and \( V \) is the volume of the cell. With these simplifications, the system of Equations (6) can be solved analytically.

The vertical direction is independent of the other directions, and the solutions for it read

\[
\begin{align*}
    u_z(t) &= u_z(0) e^{-(1 + \epsilon_{tot})t} + \alpha t \\
    &+ \left( U_z - \frac{\alpha \tau_{st}}{1 + \epsilon_{tot}} \right) \left[ 1 - e^{-(1 + \epsilon_{tot})t} \right], \\
    v_{jz}(t) &= \left[ v_{jz}(0) + u_z(0) \left( \frac{1 - e^{-\epsilon_{tot}t}}{\epsilon_{tot}} \right) \right] e^{-\tau} + \alpha t \\
    &+ \left( U_z - \frac{\alpha \tau_{st}}{1 + \epsilon_{tot}} \right) \left[ 1 - e^{-\epsilon_{tot}t} \left( 1 - \frac{1 - e^{-\epsilon_{tot}t}}{\epsilon_{tot}} \right) \right] \\
    &+ [g_z(\tilde{v}_{pj}) - \alpha] t \left( 1 - e^{-\tau} \right),
\end{align*}
\]

where

\[ \epsilon_{tot} \equiv \sum_j \tilde{v}_{kj} \]

is the total solid-to-gas density ratio in the cell,

\[ \tau \equiv \frac{t}{\tau_s} \]

is the number of e-folding times for the drag force at \( t \),

\[ \alpha \equiv \sum_j \tilde{v}_{kj} \frac{g_z(\tilde{v}_{pj})}{1 + \epsilon_{tot}} \]

is the vertical center-of-mass acceleration, and

\[ U_z \equiv \frac{u_z(0) + \sum_j \tilde{v}_{kj} v_{jz}(0)}{1 + \epsilon_{tot}} \]

is the vertical component of the initial center-of-mass velocity of the particle-gas system. The first term in Equations (7) and (8) is the decaying mode from the initial velocities of the gas and the particles. The second term represents the center-of-mass motion of the system. The remaining terms stem from the coupling between the gas and the particles due to the mutual drag force, and determine the terminal velocities relative to the center-of-mass motion.

Solving the system of Equations (6) for the horizontal directions is more involved because the Coriolis force and the shear couple the \( x \)- and \( y \)-components of the velocities. Nevertheless, the analytical solutions can still be found:

\[
\begin{align*}
    u_x(t) &= \bar{u}_x + (U_x \cos \omega t + j \beta U_y \sin \omega t) \\
    &+ \epsilon_{tot} e^{-(1 + \epsilon_{tot})t} (V_x \cos \omega t + j \beta V_y \sin \omega t), \\
    u_y(t) &= \bar{u}_y + (U_y \cos \omega t - j \beta U_x \sin \omega t) \\
    &+ \epsilon_{tot} e^{-(1 + \epsilon_{tot})t} (V_y \cos \omega t - j \beta V_x \sin \omega t), \\
    v_{jx}(t) &= \bar{v}_x + (U_x \cos \omega t + j \beta U_y \sin \omega t) \\
    &- e^{-(1 + \epsilon_{tot})t} (V_x \cos \omega t + j \beta V_y \sin \omega t) \\
    &+ e^{-\tau} (w_{jx} \cos \omega t + j \beta w_{jy} \sin \omega t), \\
    v_{jy}(t) &= \bar{v}_y + (U_y \cos \omega t - j \beta U_x \sin \omega t) \\
    &- e^{-(1 + \epsilon_{tot})t} (V_y \cos \omega t - j \beta V_x \sin \omega t) \\
    &+ e^{-\tau} (w_{jy} \cos \omega t + j \beta w_{jx} \sin \omega t).
\end{align*}
\]

The first term in the above equations denotes the equilibrium velocities, which are calculated by

\[ \bar{u}_x \equiv \frac{2 \epsilon_{tot} \tau_s}{(1 + \epsilon_{tot})^2 + 2(2 - q) \tau_s^2} \left( \frac{a_x}{2\Omega} \right), \]

\[ \bar{u}_y \equiv \frac{-1 + \epsilon_{tot} + 2(2 - q) \tau_s^2}{(1 + \epsilon_{tot})^2 + 2(2 - q) \tau_s^2} \left( \frac{a_y}{2\Omega} \right), \]

\[ \bar{v}_x \equiv - \frac{2 \epsilon_{tot}}{(1 + \epsilon_{tot})^2 + 2(2 - q) \tau_s^2} \left( \frac{a_x}{2\Omega} \right), \]

\[ \bar{v}_y \equiv \frac{1 + \epsilon_{tot}}{(1 + \epsilon_{tot})^2 + 2(2 - q) \tau_s^2} \left( \frac{a_y}{2\Omega} \right), \]

where \( \tau_s \equiv \Omega \tau_s \) is the dimensionless stopping time, or the Stokes number in the context of a rotating disk. When the shear parameter \( q = 3/2 \) and \( \Omega = \Omega_k \), the Keplerian frequency, these are simply the Nakagawa–Sekiya–Hayashi (NSH) equilibrium solutions (Nakagawa et al. 1986). The vectors \( U \) and \( V \) are defined by

\[ U \equiv \frac{[u(0) - \bar{u}] + \sum_j \tilde{v}_{kj} [v_j(0) - \bar{v}]}{1 + \epsilon_{tot}}, \]

\[ V \equiv \frac{[u(0) - \bar{u}] - \epsilon_{tot}^{-1} \sum_j \tilde{v}_{kj} [v_j(0) - \bar{v}]}{1 + \epsilon_{tot}}, \]

which are, respectively, the initial center-of-mass velocity of the system and the weighted velocity difference between the gas and the particles, measured with respect to the equilibrium state. The constant \( \beta \) is defined by

\[ \beta \equiv \sqrt{\frac{2}{2 - q}} \]

and

\[ \omega \equiv \sqrt{2(2 - q)} \Omega \]

is the epicyclic frequency. Hence the second term in Equations (13)–(16) describes the constant, in-phase, epicyclic motion of the bulk system, while the third term depicts the decaying, out-of-phase, epicyclic mode of the velocity difference between the gas and the particles. Finally, the vectors \( w_j \) are defined by

\[ w_j \equiv v_j(0) - \frac{1}{\epsilon_{tot}} \sum_j \tilde{v}_{kj} [v_j(0) - \bar{v}], \]

and thus the last term in Equations (15) and (16) denotes the decaying, epicyclic mode of the velocity difference of each individual particle relative to the center-of-mass of the particle system.

It should be clear now what the advantages are to operator split not only the mutual drag force but also the other source terms, as done in Equations (6). These source terms do not depend on any spatial derivative of the field variables or differences between particles, and an equilibrium state should be preserved numerically when they cooperate in balance.
Hence the equilibrium velocities $\bar{u}$ and $\bar{v}$ inherent in Equations (13)–(16) are important in guaranteeing that the equilibrium state can be maintained down to machine precision.\footnote{In practice, the round-off errors can serve as the seeds to “physical” instabilities in multidimensional models. However, given that many more $e$-folding times are required for these seeds to grow to appreciable amplitude compared with the dynamical timescale of many models of interest, these errors do not pose a serious issue.} Note that the hydrostatic equilibrium is determined by the system of Equations (5a)–(5b), or the like, which should be independently maintained by the other integrator, given the operator split. Also with Equations (13)–(16), all the epicyclic modes due to the coupling between the gas and the particles are accurately followed in time. Most importantly, the velocity damping due to mutual drag force can be predicted with a time step of arbitrary size using the solutions, and thus the time-step constraint posed by the drag timescale in either direction can be relieved.

2.2. Update of the Particle Velocities

Even though the NGP scheme is simple and easy to implement, it is not suitable in many circumstances and a higher-order particle-mesh scheme is usually desirable. With the NGP scheme, each particle only interacts with the gas at the nearest grid point, so the particle behaves as if it is sitting at the grid point and the significance of its positional information within the cell is lost. Moreover, the mass and the momentum density fields sampled via the particles are prone to be overwhelmed by the Poisson noise. To have a 1% sensitivity, at least $10^4$ particles are required in each cell. On the other hand, any higher-order interpolation scheme that utilizes the positional information of the particles drastically over-turns the situation; as little as one particle per cell is sufficient to describe a signal of arbitrarily low amplitude (Youdin & Johansen 2007). Therefore, it is necessary to incorporate any particle-mesh scheme of choice into the solutions discussed in Section 2.1.

To achieve this, the solid-to-gas density ratio contributed by the $j$th particle in the cell at $\mathbf{r}_k$ should be generalized to

$$\tilde{\varepsilon}_{kj} = \frac{m_{pj} W (\mathbf{r}_k - \mathbf{r}_{pj})}{\rho_g W}, \tag{26}$$

where $W(r)$ is the weight function of the particle-mesh scheme in question. More often than not, the weight function $W(r)$ has a physical interpretation (e.g., Hockney & Eastwood 1988); see Figure 1(b). For example, in the cloud-in-cell (CIC) scheme, the mass of each particle is distributed uniformly in a rectangular box of the cell size that is centered at the particle (i.e., a uniform rectangular “particle cloud”). In the triangular-shaped-cloud (TSC) scheme, each dimension of the particle cloud is doubled and the mass of the particle is distributed non-uniformly with a density peak at the cloud center and zero density at the cloud boundary. This scheme is so named because the density profile of the particle cloud resembles an isosceles triangle when viewed at the cloud center along any of the coordinate directions. The density function of the NGP scheme in this interpretation is simply a delta function. Hence the term $m_{pj} W (\mathbf{r}_k - \mathbf{r}_{pj})$ in Equation (26) can be interpreted as the fraction of the mass of the $j$th “particle cloud” that is enclosed by the cell at $\mathbf{r}_k$ and is treated as part of the “particle fluid” in the cell.

Guided by this interpretation of Equation (26), we have the following proposition to update the velocities of the particles.

1. Split each particle cloud into multiple sub-clouds and distribute them into the surrounding cells according to Equation (26). Each sub-cloud has the same initial velocity as its parent particle. We denote the velocity of the sub-cloud in cell $k$ from the $j$th particle as $v_{kj}$, and thus $v_{kj}^0(0) = v_j(0)$ for all $k$. See Figure 1(b).
2. For each cell $k$, treat the gas and all the sub-particle clouds inside as a multi-fluid system. Hence, identify $v_j$ as $v_{kj}$ and $\bar{u}_j$ as $\mathbf{u}$ in Equations (6).
3. Find the velocity changes $\Delta v_j^{(k)} \equiv v_j^{(k)}(t) - v_j^{(k)}(0)$ at time $t$ from the analytical solutions in Equations (8), (15), and (16). See Figures 1(c) and (d).
4. After all cells are integrated, collect the momentum changes of the sub-clouds back to the parent particles:

$$\Delta v_j \equiv v_j(t) - v_j(0) = \sum_k W (\mathbf{r}_k - \mathbf{r}_{pj}) \Delta v_{kj}, \tag{27}$$

5. Update the velocities of the particles by $v_j(t) = v_j(0) + \Delta v_j$. See Figure 1(e).

Essentially, this procedure decouples the coupled system of Equations (6) and makes it possible to conduct the integrations on a cell-by-cell basis, and hence substantially reduces the amount of computational work required, as discussed in the introduction of this section.

We note that the procedure outlined above is consistent with the standard particle-mesh interpolation. Because the momentum change of an individual particle is the sum of the momentum changes of its sub-clouds,

$$\frac{dv_j}{dt} = \sum_k W (\mathbf{r}_k - \mathbf{r}_{pj}) \frac{dv_j^{(k)}}{dt} = \sum_k W (\mathbf{r}_k - \mathbf{r}_{pj}) \left[ g_x (z_{pj}) \hat{e}_x - 2 \Omega \times v_{kj} + q \Omega v_{kj, x} \hat{e}_y + \frac{u_k - v_{kj}}{t_s} \right] \hat{e}_x + g_x (z_{pj}) \hat{e}_x - 2 \Omega \times v_j + q \Omega v_{j, x} \hat{e}_y + \sum_k W (\mathbf{r}_k - \mathbf{r}_{pj}) \frac{u_k - v_j}{t_s}, \tag{28}$$

where $u_k$ is the gas velocity at $\mathbf{r}_k$. By comparing this with Equation (6b), it can be seen that $\bar{u}_j = \sum_k W (\mathbf{r}_k - \mathbf{r}_{pj}) u_k$, which proves that the gas velocity experienced by the particle is the standard particle-mesh interpolation from its surrounding cells.

In principle, the gas velocity in cell $k$ at time $t$ can be similarly obtained, along with Step 3 above, via the analytical solutions in Equations (7), (13), and (14), and thus can be updated directly. As will be shown in Section 4.2.1, the growth rate of a linear mode for the streaming instability does converge with resolution using this approach. However, the convergence rate is relatively poor compared with that using the explicit integration. In the next subsection, we devise further steps for
the update of the gas velocity that significantly improve the benchmarks.

2.3. Update of the Gas Velocity

The reason for the relatively poor performance of directly using the analytical solutions to update the gas velocity after the steps described in Section 2.2 is that this approach remains local from the perspective of the gas. Even though the gas in a cell receives sub-particle clouds from the surrounding cells, it does not interact with the neighboring gas. This can be seen in the system of Equations (6) with the substitutions $v_j \rightarrow v_j^{(i)}$ and $\tilde{u}_j \rightarrow \tilde{u}$, where the gas velocity $u$ in cell $k$ is the sole state variable for the gas, and no coupling for gas velocities between different cells exists. In reality, however, the gas in neighboring cells should couple via the drag force with the interpenetrating particle clouds as interpreted in the particle-mesh method.

This missing coupling can be remedied, as inspired by the algorithm suggested in Youdin & Johansen (2007) for distributing the back reaction of the drag forces from particles to gas. We note that the velocity change of each particle $\Delta v_j$ acquired by the steps in Section 2.2 contains all the information of the mutual drag force between the particle and the surrounding gas. That is, the particle has sampled the spatial variation in the velocity field of the gas to determine its own velocity change, as shown in Equation (28). This process can then be reversed because the mutual drag force forms an action-reaction pair. Each particle can now be considered a unified particle cloud undergoing a momentum change $m_{p,j}/\Delta v_j$ instead of a group of independent sub-clouds. This momentum change can then be redistributed onto the grid by standard particle-mesh assignment. See Figure 1(f).

One more difficulty remains, though, because additional source terms are included in our system and thus the total momentum of the gas and particles in each cell is not conserved. This difficulty can be resolved with the center-of-mass frame approach, in which the mutual drag force cancels out. Combining Equations (6a), (6b), and (11) gives

$$\frac{d\tilde{u}}{dt} = \frac{ax}{1 + \epsilon_{total}} \dot{e}_x + a\dot{e}_z - 2\Omega \times \tilde{u} + q\Omega \tilde{e}_z \dot{e}_x, \quad (29)$$

where

$$\tilde{u}(t) = \frac{u(t) + \sum_j \dot{e}_j v_j(t)}{1 + \epsilon_{total}} \quad (30)$$

is the center-of-mass velocity. Equation (29) can be analytically integrated and gives the change at time $t$ as

$$\Delta \tilde{u} = \frac{\Delta u + \sum_j \dot{e}_j \Delta v_j}{1 + \epsilon_{total}} \quad (31)$$

with

$$\Delta \tilde{u}_x = -U_x(1 - \cos \omega t) + \beta U_y \sin \omega t, \quad (32)$$
$$\Delta \tilde{u}_y = -U_y(1 - \cos \omega t) - \beta^{-1} U_x \sin \omega t, \quad (33)$$
$$\Delta \tilde{u}_z = \alpha t. \quad (34)$$

Because $\Delta v_j$’s are known, Equation (31) can be rearranged to find the velocity change of the gas $\Delta u$ (Figure 1(f)). This completes our algorithm.

2.4. Boundary Conditions and Implementation

As a final note, the sheared periodic boundary conditions for the local-shearing-sheet approximation (Brandenburg et al. 1995; Hawley et al. 1995) require some attention in our algorithm. For the Eulerian description, they state that $f(x, y, z) = f(x + L_x, y - q\Omega L_z t, z)$, where $f$ is any of the dynamical fields, $L_x$ is the radial dimension of the computational domain, and $t$ is the time at the beginning of each time step instead of the size of a time step used liberally in the previous subsections. Note, however, that our algorithm completely decouples the gas fields and no simultaneous information in any pair of adjacent cells is needed in any of our steps. On the other hand, all the coupling is achieved via the splitting of the particle “clouds.” Thus the only place that the boundary conditions (as well as domain decomposition in parallel computing) are required is in the particle-mesh weight function $W(r)$, specifically in Equations (26), (27), and (31). Therefore, the radial boundary conditions can simply be executed by shifting the positions of the particles near the radial boundaries by

$$x_{p,j} = x_{p,j} \pm L_x, \quad (35a)$$
$$y_{p,j} = y_{p,j} \pm q\Omega L_z t, \quad (35b)$$

when these particles are cast into the other side of the boundary, in which the upper/lower sign is taken for the left/right boundary. The azimuthal and the vertical boundary conditions for our algorithm can be similarly implemented by revising Equations (35). All the other properties of the particles remain unchanged. We note that this approach also eliminates the need of interpolation as required in the implementation of Youdin & Johansen (2007), the latter of which introduces additional numerical errors near the radial boundaries in the particle-mesh assignment.

We have implemented our algorithm in the Pencil Code, which is a high-order finite-difference simulation code for astrophysical fluids and particles (Brandenburg & Dobler 2002). The code employs sixth-order centered differences in space and a third-order Runge–Kutta integration in time. The system of Equations (6) is operator split out of the Runge–Kutta steps and thus separately integrated by the algorithm described in this section. Throughout this work, we restrict ourselves to the TSC weight function, with which the interpolation error is of second order in cell size (Youdin & Johansen 2007). In the following, we validate the algorithm and our implementation on several systems with known solutions.

3. ONE-DIMENSIONAL TESTS

3.1. Sedimentation—Damped Harmonic Oscillation

Sedimentation of solid particles toward the mid-plane of a gas disk is one of the most important topics in the theory of planet formation. It is considered to be the first process in the core accretion scenario, creating a dense layer of solid materials that can later concentrate and become seeds of planetary cores

4 Special care needs to be taken in Equation (35b) to fold the $y$-coordinate into the limits of the ghost cell a particle is sent to, by applying the azimuthal periodicity $f(x, y, z) = f(x, y + L_z, z)$, where $L_z$ is the azimuthal size of the computational domain.

5 The Pencil Code is publicly available at http://pencil-code.nordita.org/.

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(Safronov 1969; Goldreich & Ward 1973). The degree of sedimentation intimately couples with the gas dynamics, especially in turbulent disks, and thus numerical simulations are often required (Carballido et al. 2006; Johansen et al. 2006; Tilley et al. 2010). We hereby use a simple form of the sedimentation process as the first benchmark against our integrator.

We consider a single test particle moving vertically through a stationary gas. The particle undergoes a gravitational acceleration of the form $g(z_p) = -\omega_0^2 z_p$, where $\omega_0$ is the vertical natural frequency, and the gas drag of stopping time $t_s$. The equation of motion for the particle is then

$$\frac{d^2z_p}{dt^2} + \frac{1}{t_s} \frac{dz_p}{dt} + \omega_0^2 z_p = 0. \quad (36)$$

This system is the well-known damped harmonic oscillator, and its analytical solutions are readily available. Using our algorithm, Equation (36) is equivalently being operator split into two separate equations as

$$\frac{dz_p}{dt} = v_z, \quad (37)$$

$$\frac{dv_z}{dt} = -\omega_0^2 z_p - \frac{v_z}{t_s}, \quad (38)$$

the first of which is in the Runge–Kutta integrator,$^6$ while the second is in the split integrator of Section 2. Note that in this case the mass of the particle $m_p$ is effectively zero so that the gas is unaffected and remains stationary. The particle is released at a height of $z_{p,0}$ at rest (i.e., $z_p(0) = z_{p,0}$ and $v_z(0) = 0$). We integrate this system with both the Godunov and the Strang splitting methods, which are formally first- and second-order accurate, respectively (see Appendix B).

Figure 2 compares the numerical and the analytical solutions for the cases of a simple harmonic ($\omega_0 t_s \to \infty$), underdamped ($\omega_0 t_s > 1/2$), critically damped ($\omega_0 t_s = 1/2$), and over-damped ($\omega_0 t_s < 1/2$) oscillator. We use a fixed and unusually large time step of $\Delta t = \omega_0^{-1}$ to highlight the numerical errors in the comparison. Even with such a large time step, the numerical solutions agree reasonably well with the analytical ones, especially for more highly damped systems. The Strang splitting does perform better than the Godunov splitting in particle position, but no appreciable difference appears in particle velocity between the two splitting methods. In any case, dispersive errors do exist in both position and velocity for the case of oscillatory systems. However, no diffusive errors exist in either variable, and having this property is important in accurately establishing the scale height of the particle layer, which is one of the critical factors in driving planet formation.

To test this system with our algorithm, we set up a one-dimensional periodic grid of gas and uniformly distributed Lagrangian particles. The computational domain has a length of $100c_s t_s$, so that a time step greater than the stopping time $t_s$ can be covered in the test. We allocate one particle at the center of each cell. The particles have an initial velocity of $v_0 = +c_s$,

in position, the Godunov splitting shows the expected first-order convergence. On the other hand, the Strang splitting shows the expected second-order convergence for small $\Delta t$ but only first-order convergence for large $\Delta t$, the latter of which might be due to the unresolved initial acceleration of the particle. The transition occurs at $\Delta t \approx t_s$ and it seems that the error approaches the same asymptote toward small $\Delta t$ irrespective of the stopping time. Nevertheless, the Strang splitting is more accurate in position than the Godunov splitting, albeit only slightly at large time steps. For the errors in velocity, there exists no difference between the Godunov and the Strang splittings, which is consistent with Figure 2, and this does not depend on the stopping time. Similar to the convergence in position with the Strang splitting, the convergence in velocity for both splittings shows first order for large $\Delta t$ and second order for small $\Delta t$, and the transition occurs at $\Delta t \approx t_s$. In any case, the smaller the stopping time, the more accurate the results are at any given time step. Note that it is never stable to integrate this system explicitly with $\Delta t \gtrsim t_s$, which is the regime of interest in this work. Furthermore, in a typical model, one usually operates on the range $10^{-3} \lesssim \omega_0 \Delta t \lesssim 10^{-1}$. Hence we consider these results to be fairly accurate.

3.2. Uniform Streaming

We next consider the interpenetrating streaming motions between uniform gas and uniformly distributed particles. This is the same test performed by Bai & Stone (2010b), who called it the particle–gas deceleration test, and it is also the linear-drag case of the DUSTYBOX suite presented by Laibe & Price (2011). In this scenario, the system of equations reads

$$\frac{du}{dt} = \epsilon \frac{v - u}{t_s}, \quad (39)$$

$$\frac{dv}{dt} = \frac{u - v}{t_s}, \quad (40)$$

where $u$ and $v$ are the velocities of the gas and the particles, respectively, and $\epsilon$ is the constant solid-to-gas density ratio. With the initial conditions $u(0) = u_0$ and $v(0) = v_0$, the solutions for the velocities are

$$u(t) = u_0 e^{-(1+\epsilon)u_0/\epsilon} + U_0 [1 - e^{-(1+\epsilon)u_0/\epsilon}], \quad (41)$$

$$v(t) = v_0 e^{-(1+\epsilon)v_0/\epsilon} + U_0 [1 - e^{-(1+\epsilon)v_0/\epsilon}], \quad (42)$$

where $U_0 \equiv (u_0 + \epsilon v_0)/(1 + \epsilon)$ is the center-of-mass velocity. The displacement for each particle is given by

$$S(t) = x_p(t) - x_p(0) = \frac{(v_0 - U_0)t_s}{1 + \epsilon} [1 - e^{-(1+\epsilon)v_0/\epsilon}] + U_0 t. \quad (43)$$

In the center-of-mass frame, $U_0 = 0$. The only relevant scales of time and velocity in this system are the stopping time $t_s$ and the speed of sound $c_s$, respectively. Hence the time and all the velocities can be normalized by these two scales, and this in turn fixes the length scale at $c_s t_s$.

To test this system with our algorithm, we set up a one-dimensional, periodic grid of gas and uniformly distributed Lagrangian particles. The computational domain has a length of $100c_s t_s$, so that a time step greater than the stopping time $t_s$ can be covered in the test. We allocate one particle at the center of each cell. The particles have an initial velocity of $v_0 = +c_s$,

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while the gas has a uniform initial velocity of \( u_0 = -c_0 \) (i.e., the gas and the particles move in opposite directions). The errors are measured at a fixed final time of \( t_f = 2t_s \), and the cell size \( \Delta x \) is varied to test the numerical convergence. We note that for \( \Delta x = 10c_0t_s \), it takes exactly one time step to reach \( t = t_f \). Finally, we experiment with a wide range of solid-to-gas density ratio \( \rho_s \), which is the only free parameter in the system.
We find that the final velocities in all cases are accurate to the analytical solutions, Equations (41) and (42), close to the machine precision. Although the velocities do not exactly remain uniform, the noise introduced is close to the machine-precision level. Only a slight increase in the noise level can be observed for high resolutions and thus small time steps, and this can be attributed to the round-off errors. The high degree of accuracy in velocities is not surprisingly due to our analytical integration of the velocities in Section 2.1, assisted by the high degree of uniform motion.

Therefore, the only error left to consider is the displacement of the particles. Similar to the velocities, the noise in the displacements of the particles is small and close to the machine-precision level, and thus the separation between adjacent particles remains highly constant. Figure 4 shows the absolute error in the mean displacement of the particles as a function of cell size \( \Delta x \) for the uniform streaming test. The displacements are measured at a fixed final time of \( t_f = 2t_s \) and are normalized by \( c_s t_s \). The diamond and the square symbols are the results using the Godunov and the Strang splitting methods, respectively. The blue, green, and red colors denote a solid-to-gas density ratio of \( \epsilon = 1 \), \( \epsilon = 10^{-3} \), and \( \epsilon = 10^3 \), respectively. The dashed line indicates the cell size for a relative comparison of the errors. Note that it takes exactly one time step to reach \( t_f \) for \( \Delta x = 10c_s t_s \). Note also that the accuracy of the gas and the particle velocities we obtain in this test problem is close to machine precision irrespective of the resolution, as discussed in Section 3.2.

4. TWO-DIMENSIONAL TESTS

4.1. Shear Waves

In the local-shearing-sheet approximation (see, e.g., Equation (1) or (2)), the existence of the shear advection terms \(-q \Omega x \partial f / \partial y\), where \( f \) is any dynamical field variable, makes \( e^{i(k_x x + k_y y)} \) with a time-dependent x-wavenumber
\[
k_x(t) = k_x(0) + q \Omega t
\]
and a constant y-wavenumber \( k_y \), a natural choice of the basis function. This basis depicts a two-dimensional wave, in which the power in the azimuthal direction feeds the power in the radial direction, winding up the structure into a tighter and tighter spiral wave toward trailing morphology. Substituting this basis into the two-fluid description of the particle-gas system without a background radial gas pressure gradient (i.e., \( a_s = 0 \) in Equation (2)), Youdin & Johansen (2007) derived a set of ordinary differential equations for the (complex) amplitudes of the wave:

\[
d\hat{\rho}_x / dt = -\rho_{x,0} [ik_x(t) \hat{u}_x + ik_y \hat{u}_y],
\]

\[
d\hat{u}_x / dt = \Omega \hat{u}_y - \hat{e}_0 (\hat{u}_x - \hat{v}_x) - ik_x(0) \hat{v}_y c_s^2 \hat{\rho}_g,
\]

\[
d\hat{u}_y / dt = -(2 - q) \Omega \hat{u}_x - \hat{e}_0 (\hat{u}_y - \hat{v}_y) - ik_y c_s^2 \hat{\rho}_g,
\]

\[
d\hat{\rho}_q / dt = -\rho_{q,0} [ik_x(t) \hat{v}_x + ik_y \hat{v}_y],
\]

\[
d\hat{v}_x / dt = 2\Omega \hat{v}_y - 1 / t_s (\hat{v}_x - \hat{u}_x),
\]

\[
d\hat{v}_y / dt = -(2 - q) \Omega \hat{v}_x - 1 / t_s (\hat{v}_y - \hat{u}_y),
\]

where \( \rho_{q,0} \) and \( \rho_{x,0} \) are the background uniform densities of the gas and the particles, respectively; \( \hat{e}_0 \equiv \Omega \rho_{q,0} / \rho_{x,0} \); and \( c_s \) is the isothermal speed of sound. This system of equations can be readily integrated numerically, and its solution serves as a convenient analytical benchmark to validate our implementation of the sheared periodic boundary conditions as well as the mutual drag force.

Applying our algorithm to this shear-wave test, we evolve the particle-gas system for a square domain \( L \times L \) in the xy-plane with a single mode \( k_x(0) = -2 \pi / L \) and \( k_y = 2 \pi / L \). We use a 64 \( \times \) 64 grid and a shear parameter of \( q = 3 / 2 \), and allocate one particle per cell. The initial conditions are such that \( \hat{\rho}_q(0) = \hat{u}_x(0) = \hat{\rho}_x(0) = \hat{v}_y(0) = \hat{v}_x(0) = 0 \). Note that in this case, it takes \( 22 \Omega^{-1} \) for the x-wavenumber to reach the Nyquist frequency, when the numerical diffusion and/or aliasing becomes significant. In what follows, we vary the values of \( \tau_s \) and \( \rho_{q,0} \) to assess the performance of our algorithm, where \( \tau_s \equiv \Omega t_s \). We only present the results with the Godunov splitting method, and note that the Strang splitting only slightly improves the accuracy.

First, we consider the case of \( \tau_s = 1 \) and \( \rho_{q,0} = 1 \), which makes the test exactly the same as was done in Youdin & Johansen (2007). Figure 5 shows the comparison between the analytical solutions obtained from integrating Equations (45) and the measurements of the amplitudes on the simulation data.
obtained with our algorithm. All the amplitudes of the shear wave from the simulation agree well with the analytical solutions for several orbital periods, with some minor deviation in the velocity field of the particles for $t \approx 3.5 \Omega^{-1}$. Note that the time steps used here are significantly larger than those used in Youdin & Johansen (2007).

Next we probe the case of small particles with $\tau_s = 10^{-3}$ and $\epsilon_0 = 1$, the result of which is shown in Figure 6. In this case, there exists an initial abrupt jump in the $u_y$ and $v_y$ fields, followed by a smooth oscillatory evolution in the amplitudes as in the previous case. An integration with the explicit method would require an extremely small time step to resolve and accurately capture this initial jump; it is not even numerically stable if the time step is larger than the width of this feature. Our algorithm, in contrast, accurately predicts the velocity fields with a time step much longer than the timescale of this feature. In spite of some initial minor deviation of the density fields, the solutions with our algorithm closely follow the analytical ones up to $t \approx 3.5 \Omega^{-1}$, after which some noticeable deviation appears in both the gas and the particle fields.

Finally, we barge into the solid-dominated regime with $\tau_s = 10^{-3}$ and $\epsilon_0 = 10$, as shown in Figure 7. Similar to the previous case, a yet larger initial jump occurs in the $u_y$ field but a lesser one in the $v_y$ field, and our algorithm accurately finds the first velocity fields with one long time step. Although the

Figure 5. Field amplitudes of a particle-gas shear wave as a function of time. In this case, the system has a dimensionless stopping time of $\tau_s = 1$ and a solid-to-gas density ratio of $\epsilon_0 = 1$, and its initial conditions are described in Section 4.1. The solid lines are the analytical solutions obtained from integrating Equations (45), while the circles are the measurements from each and every time step of the simulation data obtained by our algorithm. The top panel shows the velocity and density fields of the gas, while the bottom panel shows those of the particles.

Figure 6. Same as Figure 5, except that $\tau_s = 10^{-3}$ and $\epsilon_0 = 1$. Note that there exists an initial jump in $u_y$ and $v_y$ in the analytical solution.

Figure 7. Same as Figure 5, except that $\tau_s = 10^{-3}$ and $\epsilon_0 = 10$. Note that there exists an initial jump in $u_y$ and $v_y$ in the analytical solution.
algorithm also captures the density field of the particles relatively accurately, a significant error exists in the density field of the gas at the very first time step. This error affects the accuracy of the subsequent evolution of the shear wave, the most prominent of which is in the frequency of the oscillation in the amplitude of each field. Nevertheless, the general evolution of this shear wave is still reproduced by our algorithm up to \( t \sim 4\Omega^{-1} \), especially in the maximum amplitudes achieved in each oscillation of the fields.

This last case highlights that some inaccuracy in density fields can occur when there exists unresolved transient behavior in velocity, a situation reminiscent of the sedimentation benchmark presented in Section 3.1. The magnitude of this numerical error depends on how strong the change in velocity is; in this test problem, the error increases with increasing background solid-to-gas density ratio. Note that this kind of transient behavior often stems from initial conditions that are significantly out of equilibrium, as in this case, or impulses that are imposed onto the system during its course of evolution. Once the impulse subsides and the system resumes a smooth evolution (i.e., one that is resolved by the numerical time steps), our algorithm should exhibit a high degree of accuracy, as demonstrated in Figure 8, simply resolving the initial transient jump by four time steps makes the evolution of all fields accurate up to \( t \sim 3.2\Omega^{-1} \), a similar performance to that achieved in Figures 5 and 6.

### 4.2. The Streaming Instability

An important discovery in the theory of planet formation was that of the streaming instability by Youdin & Goodman (2005). This instability efficiently concentrates centimeter/meter-sized solid particles in a protoplanetary gas disk to trigger gravitational collapse and form kilometer-sized planetesimals, circumventing the problematic fast radial drift of the solid particles (e.g., Johansen et al. 2007). The mutual drag force between the gas and the solid particles moving in a differentially rotating disk is an essential ingredient in this instability, and hence the results of the analysis carried out by Youdin & Goodman (2005), serendipitously, can be used as a rigorous touchstone to validate any numerical algorithm concerning the integration of these kinds of particle-gas systems. This is the goal of this section.

#### 4.2.1. Linear Modes

The streaming instability is a local, two-dimensional, axisymmetric instability with interpenetrating gas and particles under differential rotation. Ignoring vertical gravity and assuming an isothermal equation of state for the gas, Youdin & Goodman (2005) performed linear analysis on this system and found the growth rate and wave speed of this instability and the corresponding eigenvector as a function of the wavenumber of the perturbation, as well as all the relevant dimensionless parameters for the system. The eigensystem was expressed in the frame of the center-of-mass velocity of the particle-gas system, which was not convenient for direct comparison with numerical simulations. Thus, Youdin & Johansen (2007) transformed these solutions back into the local-shearing-sheet frame and expressed them as a standing wave in the vertical direction while propagating in the horizontal direction. The resulting eigenfunction is either even or odd:

\[
f_x(x, z) = \text{Re}(\tilde{f})\cos(k_x x - \omega t) - \text{Im}(\tilde{f})\sin(k_x x - \omega t)\text{e}^{\omega t}\cos k_z z, \quad (46)
\]

or odd:

\[
f_x(x, z) = -\text{Re}(\tilde{f})\cos(k_x x - \omega t) + \text{Im}(\tilde{f})\cos(k_x x - \omega t)\text{e}^{\omega t}\sin k_z z. \quad (47)
\]

in the vertical direction, where \( \tilde{f} \) is the complex amplitude, \( k = k_x \mathbf{\hat{e}}_x + k_z \mathbf{\hat{e}}_z \) is the wavenumber, and \( \omega = \omega_R + is \) is the complex angular frequency. The vertical component of the velocities of the gas and the particles assumes the odd parity, while all other dynamical fields assume the even parity. On top of the background equilibrium state as given in Equations (17)–(20) and with an initially small amplitude for the perturbations, Equations (46) and (47) then serve as both the initial conditions and the analytical solutions against which our algorithm is validated.

Four eigensystems have been published in the literature, all of which are the (nearly) fastest growing modes at the respective set of the dimensionless parameters. We first test the linA and the linB modes in Youdin & Johansen (2007). They have the same dimensionless stopping time of \( \tau_z = 0.1 \) as well as the same background radial gas pressure gradient of \( \alpha_z = 0.1c_s^{1/2} \), where \( c_s \) is the isothermal speed of sound and \( \Omega \) is the local Keplerian angular frequency. On the other hand, the linA mode has a background solid-to-gas density ratio of...
$\epsilon_0 = 3$, while the linB mode has $\epsilon_0 = 0.2$. In our tests, the computational domain is such that one wavelength of the mode is fit in each direction. We use one particle per cell, and use the method described in Appendix C of Youdin & Johansen (2007) to seed a perturbation of $\rho_g = 10^{-6}$ in the density field of the particles. Given their positions, the initial velocities of the particles are then set by the perturbation Equations (46) and (47), as well as the equilibrium drift velocity Equations (19) and (20). It is trivial to set the initial density and velocity fields of the gas using Equations (46) and (47), as well as Equations (17) and (18). We measure each complex amplitude of the Fourier mode ($k_x, k_z$) as a function of time in our simulation data, and then use the linear regression on the magnitude of the amplitude to find the (exponential) growth rate $s$. We vary the resolution to seek the convergence of our measured rate as compared to the analytical one. For comparison purposes, we use three different integration schemes for this test. One is the explicit integration, as originally used by Youdin & Johansen (2007). The other two are our algorithms either with or without particle-mesh back reaction (PMBR) to the gas velocity field, as described in Section 2.3; we only present the results with the Godunov splitting here.

The results are shown in Figures 9 and 10. As can be seen, the growth rates for all three schemes converge to the theoretical one with resolution. For the linA mode, our algorithm with PMBR achieves virtually the same performance of the explicit integration. While it requires a resolution of around 32–64 points per wavelength $\lambda$ for the growth rate of the $\rho_g$ field to approach within 5% of the theoretical one, it requires around 16–32 points per $\lambda$ for those of the $v_x$, $v_y$, and $u_y$ fields; around 16 points per $\lambda$ for that of the $v_x$ field; and only around 4–8 points per $\lambda$ for that of the $v_y$ and $u_y$ fields. Albeit convergent, our algorithm without PMBR has a much slower convergence rate, requiring a resolution of around 64–128 points per $\lambda$ for the growth rates of most fields, except around 32–64 points per $\lambda$ for those of the $v_y$ and $u_y$ fields and around 16–32 points per $\lambda$ for that of the $v_x$ field. As for the linB mode, our algorithm with PMBR has again similar growth rates as achieved by the explicit integration. A resolution of around 32–64 points per $\lambda$ is required, except for the $u_x$ field, which requires only a resolution of around 16–32 points per $\lambda$. Our algorithm without PMBR remains inferior, requiring a resolution of around 128–256 points per $\lambda$ in all fields except for the $v_y$ field, which requires a resolution of around 64–128 points per $\lambda$.

The other two modes are the linC and the linD modes from Bai & Stone (2010b). These modes push the limit of the dimensionless stopping time down to $\tau_s = 0.01$ and $\tau_s = 0.001$, respectively. They have the same solid-to-gas density ratio of $\epsilon_0 = 2$, as well as the same background radial gas pressure gradient $\alpha_s = 0.1c_s\Omega$ as the linA and the linB modes. Our test results are shown in Figures 11 and 12. Again, the growth rates for all three schemes demonstrate convergence toward the theoretical one. Moreover, the performance of our algorithm with PMBR remains essentially the same as that of the explicit integration for all fields. For the linC mode, a resolution of
around 32–64 points per wavelength $\lambda$ is required for all the velocity fields, while a resolution of around 32 points per $\lambda$ and that of around 128–256 points per $\lambda$ are required for the $\rho_p$ and the $\rho_k$ fields, respectively. For the linD mode, a resolution of around 64–128 points per $\lambda$ is required for all the velocity fields, while that of around 128–256 points per $\lambda$ is required for both the density fields. We note that the performance of these integrators for these modes is relatively better in comparison.
with that performed in Bai & Stone (2010b). As for our algorithm without PMBR, the convergence is again much slower as compared to the other two schemes, and it does not even converge to within 5% of the theoretical rate at a resolution of 512 points per $\lambda$ for the linD mode except for the density fields, which require a resolution of around 256 points per $\lambda$.

These results demonstrate the excellent performance of our algorithm in reproducing the linear modes of the streaming instability, especially with a wide range of the dimensionless stopping time $\tau_s$. Moreover, they justify the necessity of the implementation of PMBR to the gas velocity in our algorithm, as described in Section 2.3. We note here that there exists no noticeable improvement in the test results when using our algorithm with the Strang splitting method. In the next subsection, we continue to explore our algorithm in the nonlinear saturation of the streaming instability.

### 4.2.2. Nonlinear Saturation

In a companion paper to Youdin & Johansen (2007), Johansen & Youdin (2007) investigated the nonlinear saturation of the streaming instability and found two distinctive patterns that could develop in this stage. One is for marginally coupled solid particles with $\tau_s = 1$, while the other is for strongly coupled particles with $\tau_s = 0.1$. The $\tau_s = 1$ particles initially concentrate themselves into short stripes that are close to horizontal but tilt alternately in the vertical direction with a separation consistent with the wavelength of the fastest growing mode of the streaming instability. These stripes then undergo an inverse cascade and merge into larger and larger stripes that tilt more and more toward the vertical direction. In the fully saturated stage, long, strongly concentrated filaments with a wide separation float either upward or downward with much reduced radial drift motion. On the other hand, the $\tau_s = 0.1$ particles initially create numerous small voids by random, locally divergent motions. These voids then undergo an inverse cascade and merge into larger and larger voids, driving the particles into their rims. In the fully saturated stage, voids of various sizes move around in random directions, while the particles stream along the alleyways between these voids.

Using our algorithm with the Godunov splitting method, we rerun Model BA and Model AB in Johansen & Youdin (2007). Model BA contains marginally coupled particles with $\tau_s = 1$ and has a solid-to-gas density ratio of $\epsilon_0 = 0.2$. While Model AB contains strongly coupled particles with $\tau_s = 0.1$ and has a solid-to-gas density ratio of $\epsilon_0 = 1$. Both models have a background radial gas pressure gradient of $a_r = 0.1c_s\Omega$. Adopting the same computational domains and resolutions as used in Johansen & Youdin (2007), we construct a $256 \times 256$ grid with a domain of $2H \times 2H$ and $0.1H \times 0.1H$ for Model BA and Model AB, respectively, where $H$ is the scale height of the gas. We allocate on average one particle per cell, which is sufficient in capturing the overall density distribution function of particles at a fixed resolution, as has been shown by Bai & Stone (2010b). The initial velocities of the gas and the particles are those under mutual drag equilibrium, Equations (17)–(20). The initial density field of the gas is uniform, while the initial positions of the particles are random to seed the streaming instability at all scales. The resulting evolutions of the density field of the particles are shown in Figure 13. As can be seen, we have reproduced the two aforementioned distinct patterns in the saturated state of the streaming instability, with quantitative similarity as the same models in Johansen & Youdin (2007). This again illustrates the consistency between the original explicit integration and our algorithm in this work.

Figure 12. Measured growth rates as a function of resolution for the linD mode of the streaming instability from Bai & Stone (2010b). This mode is for particles of dimensionless stopping time $\tau_s = 10^{-3}$ and a background solid-to-gas density ratio of $\epsilon_0 = 2$. The arrangement of the panels and the line styles is the same as in Figure 9. The growth rates are measured over the time interval $0 \leq t \leq 0.01\nu_p$.
Finally, we explore our algorithm in its full three-dimensional glory. To the best of our knowledge, there exists no appropriate analytical model that contains all the physics considered in our algorithm. We therefore resort to our earlier nonlinear simulations published in Yang & Johansen (2014), which employed the technique of explicit integration, as our base for comparison.

5.1. Sedimentation and Streaming Turbulence

In Yang & Johansen (2014), we systematically studied the dependence of the streaming turbulence in a sedimented layer of solid particles on the computational domain of the simulations. With particles of dimensionless stopping time $\tau_s \approx 0.31$ and a background gas pressure gradient $a_s = 0.1 c_s \Omega$—where $c_s$ is the isothermal speed of sound and $\Omega$ is the local Keplerian angular frequency—we found that multiple radial filamentary concentrations of solids can be driven by the streaming instability, and that the characteristic separation between adjacent filaments is on the order of $0.2H$, with $H$ being the vertical scale height of the gas.

Using our algorithm with the Godunov splitting method, we rerun what is otherwise exactly the same model in Yang & Johansen (2014) that has a computational domain of $1.6H \times 1.6H \times 0.2H$ and a resolution of 160 points per $H$. Figure 14 compares the resulting radial concentrations of solids between the simulations in Yang & Johansen (2014) and this work. As can be seen in the comparison, we obtain excellent agreement between the two simulations. Despite minor differences in the stochastic erosion and accretion of solids from and onto the filaments, the total number of filaments, their radial drifts, and the magnitude of the solid density are all quantitatively and evolutionarily similar. This experiment demonstrates the consistency between the two techniques in three dimensions and yet again the robustness of our algorithm.

In Figure 13, we compare the time steps used in the two simulations. For the simulation in Yang & Johansen (2014), the time steps were initially determined by the hydrodynamic Courant condition but soon dropped by roughly one order of magnitude beginning at $t \approx 8P$, where $P$ is the local orbital period. This is predominantly limited by the drag time $T_{\text{drag}}$, which is the timescale for the exponential decay in the relative velocity between the gas and the particles due to their mutual drag interactions, as discussed in Section 1. In the PENCIL CODE, the drag time for each cell is approximated by $T_{\text{drag}} \approx (1 + N_p m_p / \rho_g V)^{-1} t_s$, where $N_p$ is the total number of (super-)particles in the cell, $m_p$ is the mass of a (super-)particle, $\rho_g$ is the gas density in the cell, $V$ is the volume of the cell, and $t_s$ is the physical stopping time. Note that we used 100% of $T_{\text{drag}}$ as our time-step limiter in Yang & Johansen (2014).

Figure 13. Nonlinear evolution of the streaming instability computed by our algorithm. The top and the bottom rows show the models BA and AB, respectively, taken from Johansen & Youdin (2007). The models are axisymmetric and in the $xz$-plane, where the radial axis $x$ and the vertical axis $z$ are measured in terms of the gas scale height $H$. The color images depict the density field of the particles $\rho_p$ in terms of the initial uniform gas density $\rho_{g,0}$. The top panels show from left to right the snapshots at $t = 4P$, $14P$, and $50P$, respectively, while the bottom panels show the snapshots at $t = 0.4P$, $1.4P$, and $5P$, respectively, where $P$ is the local orbital period.
which, while speeding up the simulations, could amount to a relative error of roughly 10% in relative velocity in the worst-case scenario. The PENCIL CODE defaults to use 20% of \( T_{\text{drag}} \) as a time-step limiter to guarantee better accuracy, and thus we also show the time steps that would have been used in this case in Figure 15. As can be seen, the drag time begins to dominate earlier at \( t \approx 3P \) and the time steps drop by another order of magnitude than those used in Yang & Johansen (2014).

On the other hand, our algorithm is not limited by the drag time \( T_{\text{drag}} \) at all. As shown in Figure 15, the time steps remain almost constant and are only determined by the hydrodynamic Courant condition.\(^8\) These time steps are in drastic contrast to those used in the simulation of Yang & Johansen (2014), the latter of which are more than one order of magnitude smaller and would have been more than two orders of magnitude smaller if 20% of \( T_{\text{drag}} \) were adopted. Therefore, this experiment also illustrates the exceptional efficiency of our algorithm, which we set out to achieve in this work.\(^9\)

6. CONCLUDING REMARKS

In summary, we have devised an accurate, efficient numerical algorithm to directly integrate the mutual drag force in a system of Eulerian gas and Lagrangian solid particles. Despite the entanglement between the gas and the particles due to the conventional particle-mesh construct, we have

\(^8\) In Yang & Johansen (2014) we used a Courant number of 0.4, while in this work we use that of 0.8 with our algorithm. Note that with the technique of explicit integration when the drag time \( T_{\text{drag}} \) dominates the time-step limiter, the Courant number used is irrelevant.

\(^9\) As of this writing, our implementation of this algorithm in three dimensions requires roughly three times longer wallclock time per time step than the explicit integration. The benchmark was conducted on the Alarik supercomputing system at Lunarc in Lund University, Sweden.
effectively decomposed the globally coupled system of equations for the mutual drag force and have been able to integrate this system on a cell-by-cell basis. Analytical solutions exist for the temporal evolution of each cell, which we use to achieve the highest degree of accuracy. This solution relieves the time-step constraint posed by the mutual drag force, making simulation models with small particles and/or strong local solid concentration significantly more amenable. We have used an extensive suite of benchmarks with known solutions—in one, two, and three dimensions—to validate our algorithm and found satisfactory consistency in all cases.

Even though the Strang splitting is formally higher order, we find that its use with our algorithm does not offer significant advantage over the Godunov splitting, especially in multidimensional models. In our one-dimensional benchmarks, both splittings predict virtually the same velocities, and hence they give the same accuracy in velocities and have the same behavior in numerical convergence. As for the accuracy in particle positions, although the Strang splitting demonstrates an expected second-order convergence for small time steps, it degrades to first-order convergence and does not give a noticeably improved accuracy over the Godunov splitting for large time steps. In our multidimensional benchmarks, on the other hand, no appreciable difference in either density or velocity fields for the gas and the particles exists between the two splitting methods. We emphasize that since our objective is to relieve the time-step constraint due to the mutual drag force, the use of large time steps is of more interest here. Moreover, note that because the Strang splitting requires one more run of either our algorithm or the other integrator than the Godunov splitting (see Appendix B), the former is significantly more expensive than the latter, let alone other operator-splitting methods of even higher orders. Therefore, we find that employing the Godunov splitting with our algorithm is sufficient and more economical for practical purposes.

The simulation models to which our algorithm can be applied are more general than what have been presented in this work. First of all, our algorithm only concerns the mutual drag force, the rotation/shear-related source terms, and the background accelerations. All other physical processes are consolidated in the other half of the operator-split system of equations, as in Equations (5), and are integrated independently of our algorithm. In this regard, the particle-gas system that can be considered using our algorithm is unconstrained and potentially arbitrary. Moreover, our algorithm is not limited to the local-shearing-sheet approximation. For instance, a nonrotating inertial frame, either rectangular or curvilinear, is simply a degenerate case of Equations (6) with the angular frequency \( \Omega = 0 \), and thus the same solution found in Section 2.1 applies, except that the equilibrium velocities (Equations (17)–(20)) need to be modified accordingly. As for a rotating cylindrical coordinate system, the shear acceleration terms in Equations (6) are replaced by the centrifugal force and the external gravity, but an analytical counterpart of the solution in Section 2.1 can still be obtained from the modified system of equations. Finally, note also that it is not required for the particles to have the same mass, as shown in Equation (26).

Some elaboration is needed when generalizing our procedure for a system with particles of various stopping times. A closed-form analytical solution, such as the one in Section 2.1, is only possible when the stopping time \( \tau \) is a constant for all particles. For the case of independent stopping time for each particle, nevertheless, it becomes merely a matter of adopting a numerical method that can accurately and efficiently approximate the solution to the corresponding system of Equations (6) in place of the analytical solution of Section 2.1. Given the stiffness of the mutual drag force, an implicit method will likely be required for this purpose. However, because we have effectively decoupled the system and made it possible to integrate it on a cell-by-cell basis, the individual integration task for each cell does not amount to an unmanageable size of matrix inversion, for instance. Therefore, we expect that this extra complexity will not lead to noxious reduction in computational efficiency with our algorithm.

Having the time-step constraint due to stiff mutual drag force removed, our algorithm presented in this work will prove useful in the study of dust dynamics in protoplanetary disks. Specifically, a model containing numerous mm/cm-sized pebbles and/or with high mass loading of solids can be simulated as efficiently as a model containing m-sized boulders without strong concentration of them. In other words, it becomes feasible to evolve small solid particles interacting with a gas disk in relatively high numerical resolution and long simulation time. This capability is particularly important in further studies of the streaming instability with mm/cm-sized pebbles and its connection to planetesimal formation, as pioneered by Carrera et al. (2015), because the wavelength of the fastest growing mode decreases with decreasing particle size and the corresponding growth rate is low when the initial solid abundance is low (Youdin & Goodman 2005; Youdin & Johansen 2007). For the same reasons, our algorithm may also find its use in investigating the dynamics of pebbles with the influence of turbulence, vortices, or other large-scale structures in the gas disk, and the corresponding observational consequences.

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APPENDIX A
NUMERICAL STRESSES INTRINSIC IN THE PARTICLE-MESH METHOD

In this appendix, we demonstrate by a simple example that the mutual drag force between the gas and the particles leads to numerical stresses under the particle-mesh construct. Consider a system of uniform gas and uniformly distributed particles. Suppose that all the velocities of gas and particles align in the y-direction and only depend on x. The evolution of this system then becomes a one-dimensional problem in the x-direction with all the velocities in the transverse direction, so the densities of both the gas and the particles remain uniform over time. We construct a regular grid for the gas with one particle at each cell interface, such that the center of cell \( k \) is at
\[ x_k = (k - 1/2) \Delta x \] and the \( j \)th particle is at \( x_{pj} = j \Delta x \), where \( \Delta x \) is the cell size. The system of equations for this problem reads, from Equations (6), (26), and (28),

\[
\frac{du_k}{dt} = \varepsilon \frac{1}{t_s} \sum_j W(x_k - x_{pj})(v_j - u_k),
\]

\[
\frac{dv_j}{dt} = \frac{1}{t_s} \sum_k W(x_k - x_{pj})(u_k - v_j),
\]

where \( u_k \) is the velocity of gas in cell \( k \), \( v_j \) is the velocity of the \( j \)th particle, \( \varepsilon \) is the (uniform) solid-to-gas density ratio, \( t_s \) is the stopping time, and \( W(x) \) is the particle-mesh weight function.

We now consider the special case of \( \varepsilon / t_s \gg 1 \), where \( t_s \) is the dimensionless stopping time (i.e., \( t_s \) normalized by the timescale of interest for the system). In this limit, Equation (48) implies that \( u_k \approx \sum_j W(x_k - x_{pj})v_j \) for all \( k \). We further adopt the CIC weight function, such that \( W(\pm 1/2) = 1/2 \) and \( W(x) = 0 \) for all \( |x| \geq \Delta x \). Equation (49) then becomes

\[
\frac{dv_j}{dt} \approx \frac{\Delta x^2}{4t_s} \left( v_{j+1} - 2v_j + v_{j-1} \right),
\]

for all \( j \). This equation is exactly the same as the diffusion equation with a diffusion coefficient of \( \Delta x^2/4t_s \) when spatially discretized by second-order accurate, centered finite differences. We note that using a higher-order weight function as TSC does not change this diffusion coefficient, but only increases the accuracy of the spatial derivatives.

Another special case we can consider is the limit of \( \varepsilon / t_s \ll 1 \) and \( \varepsilon / t_s \sim 1 \). In this limit, Equation (49) implies that \( v_j \approx \sum_k W(x_k - x_{pj})u_k \) for all \( j \). With the CIC weight function, Equation (48) then becomes

\[
\frac{du_k}{dt} \approx \varepsilon \frac{\Delta x^2}{4t_s} \left( u_{k+1} - 2u_k + u_{k-1} \right),
\]

for all \( k \). Once again, this equation is the same as the diffusion equation, with a diffusion coefficient of \( \varepsilon \Delta x^2/4t_s \), to second-order spatial accuracy.

Equations (50) and (51) indicate that using the particle-mesh method to treat the mutual drag force introduces numerical shear stresses to the system. Moreover, it can be seen by generalizing the argument outlined above that numerical normal stresses are also induced by the particle-mesh method. Fortunately, the diffusion coefficient associated with these stresses diminishes quadratically with increasing resolution. On the other hand, this property implies that the system of Equations (6) is indeed globally coupled, and special care needs to be taken to numerically solve this system both accurately and efficiently.

**APPENDIX B**

**OPERATOR-SPLITTING METHODS**

In this appendix, we briefly review the concept of operator splitting, and its two standard methods. Suppose that

\[
\frac{\partial \xi}{\partial t} = (\mathcal{A} + \mathcal{B}) \xi
\]

is the full partial differential equation in question, where \( \xi \) is the vector for the dynamical variables to be solved for and \( \mathcal{A} \) and \( \mathcal{B} \) are two differential operators. We can **operator split**

Equation (52) into two separate differential equations as

\[
\frac{\partial \xi}{\partial t} = \mathcal{A} \xi,
\]

\[
\frac{\partial \xi}{\partial t} = \mathcal{B} \xi,
\]

with \( f(t; \xi_0) \) and \( g(t; \xi_0) \) being the respective solutions to Equations (53) along with the initial conditions \( \xi(0) = \xi_0 \). Note that \( f(t; \xi_0) \) and \( g(t; \xi_0) \) can be either exact or approximate. Then by somehow combining \( f(t; \xi_0) \) and \( g(t; \xi_0) \) it is possible to approximate the real solution \( \xi(t) \) to the full Equation (52).

There are two standard operator-splitting methods to approximate the solution \( \xi(t_{n+1}) \) at time step \( n+1 \) with the initial conditions \( \xi(t_n) \equiv \xi_n \) at time step \( n \). The first is called the Godunov splitting, and \( \xi(t_{n+1}) \equiv \xi_{n+1} \) is given by a two-step method:

\[
\xi' = f(\Delta t; \xi_n),
\]

\[
\xi_{n+1} = g(\Delta t; \xi'),
\]

where \( \Delta t \equiv t_{n+1} - t_n \) is the size of the time step. The second is called the Strang splitting, and \( \xi_{n+1} \) is given by a three-step method (Strang 1968):

\[
\xi' = f\left( \frac{1}{2} \Delta t; \xi_n \right),
\]

\[
\xi'' = g(\Delta t; \xi'),
\]

\[
\xi_{n+1} = f\left( \frac{1}{2} \Delta t; \xi'' \right).
\]

The accuracies for the Godunov and the Strang splittings are formally first order and second order, respectively. For a proof of these properties and more information, the interested reader is referred to, for example, Chapter 17 of LeVeque (2002).

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