Counting the Chain Records: The Product Case

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Abstract

Chain records is a new type of multidimensional record. We discuss how often the chain records are broken when the background sampling is from the unit cube with uniform distribution (or, more generally, from an arbitrary continuous product distribution).

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1 Introduction

Consider independent marks $X_1, X_2, \ldots$ sampled from the uniform distribution in $Q_d = [0,1]^d$. We define a mark $X_n$ to be a chain record if $X_n$ beats the last chain record in $X_1, \ldots, X_{n-1}$. More precisely, record values and record indices are introduced recursively, by setting $T_1 = 1, R_1 = X_1$ and

$$T_k = \min\{n > T_{k-1} : X_n < R_{k-1}\}, \quad R_k = X_{T_k}, \quad k > 1.$$ 

Here, $<$ denotes the standard strict partial order on $\mathbb{R}^d$ defined in terms of component-wise orders by

$$x = (x^{(1)}, \ldots, x^{(d)}) < y = (y^{(1)}, \ldots, y^{(d)}) \iff x \neq y \text{ and } x^{(i)} \leq y^{(i)} \text{ for } i = 1, \ldots, d.$$ 

It is easy to see that, in any dimension $d$, the terms of $(T_k)$ are indeed well defined for all $k$, that is the chain records occur infinitely many times.

Although the definition is an obvious restatement of the classical definition of lower record, this notion of multidimensional record has not been explored so far. The chain records interpolate between two other types of multidimensional records which have been studied in some depth 1 2 4 13 14 15 18 19 20 21 22. We say that a strong record occurs at index $n$ if either $n = 1$, or $n > 1$ and

$$X_n < X_j \text{ for } j = 1, \ldots, n - 1.$$ 

In the terminology of partially ordered sets, a strong record $X_n$ is the least element in the point set $\{X_1, \ldots, X_n\}$. Since repetitions in each component have probability zero, $X_n$ is a strong record if and only if there are $d$ marginal strict lower records at index $n$ simultaneously. We say that a weak record occurs at index $n$ if either $n = 1$, or $n > 1$ and

$$X_j \neq X_n \text{ for } j = 1, \ldots, n - 1.$$
A weak record $X_n$ is a minimal element in the set $\{X_1, \ldots, X_n\}$. Obviously, each strong record is a chain record. Also, each chain record is a weak record, as follows easily by induction from transitivity of the relation $\prec$. To illustrate, for the two-dimensional configuration of points in Figure 1 the weak records occur at times 1, 2, 3, 5, 6, 7, 8, 9, a sole strong record occurs at 1, the marginal records occur at 1, 2, 3, 5, 6, and the chain records occur at indices 1, 5, 8. Notably, the chain records are more sensible to arrangement of marks in sequence: a permutation of $X_1, \ldots, X_{n-1}$ may destroy or create a chain record at index $n$.

Denote $N_n, \overline{N}_n$ and $\underline{N}_n$, respectively, the counts of strong, weak and chain records among the first $n$ marks. Thus

$$N_n \leq \overline{N}_n \leq \underline{N}_n.$$ 

To underscore concretely the extent of compromise between weak and strong records, we need some estimates of how often the records of different kinds may occur.

Recall that in the case $d=1$ the occurrences of records are independent, with probability $1/n$ for index $n$; this basic fact (known as the Dwass-Rényi lemma [24, 26]) implies that the number of classical records is asymptotically Gaussian with both mean and variance about $\log n$. This translates easily to the marginal records in $d$ dimensions, since the marginal rankings are independent. The latter kind of independence is characteristic for sampling from product distributions in $\mathbb{R}^d$ with continuous marginals, hence the instance of $Q_d$ with uniform distribution covers the general product case.

Properties of the strong-record counts for sampling from $Q_d$ are also rather simple. By independence of marginal rankings we have a representation $\overline{N}_n = I_1 + \ldots + I_n$ with independent Bernoulli indicators and $p_n := P(I_n = 1) = n^{-d}$. Thus

$$E N_n = \sum_{j=1}^n \frac{1}{j^d}.$$ 

Since for $d > 1$ the series $\sum p_n$ converges, the total number of strong records in the infinite sequence of marks is almost surely finite.

Counting the weak records is a more delicate matter since their occurrences are not independent. However, we may exploit a correspondence between weak records in $Q_d$
and the minimal elements in $Q_{d+1}$ (depending on the context these points are also called Pareto, admissible, efficient, etc.). The correspondence is established by arranging the marks in $d+1$ dimensions by increase in one fixed component. By induction in $d$ one can show that

$$
\mathbb{E} \mathcal{N}_n = \sum_{1 \leq j_1 \leq \ldots \leq j_d \leq n} \frac{1}{j_1 \cdots j_d} \sim \frac{1}{d!} (\log n)^d,
$$

see [1]. From further known results (see [2] and references therein) follows that the variance $\text{Var} [\mathcal{N}_n]$ is of the same order $(\log n)^d$, and that $\mathcal{N}_n$ is asymptotically Gaussian.

Thus the strong records are much more rare and the weak records are much more frequent than the classical records. In this note we show that, as far as the frequency is concerned, the chain records in any dimension $d$ are more in line with the classical records:

**Proposition 1.** For sampling from $Q_d$ with uniform distribution the number of chain records $N_n$ is approximately Gaussian with moments

$$
\mathbb{E} [N_n] \sim d^{-1} \log n, \quad \text{Var} [N_n] \sim d^{-2} \log n.
$$

The CLT will be proved in Section 3. Above that, we will derive exact and asymptotic formulas for the probability of a chain record and discuss some scaling limits.

The chain records comprise a ‘greedy’ chain in $\prec$, meaning that a mark is joined each time the chain constraint is not violated. More efficient nonanticipating algorithms for constructing long chains were designed in [3], and the length of the longest possible chain on $n$ random marks was estimated in [9]. From yet another perspective, the sequence of chain records corresponds to a particular path in a random data structure called quad-tree [10, 12].

## 2 The heights at records

For $x \in Q_d$ the quadrant $L_x := \{y \in Q_d : y \prec x\}$ is the lower section of the partial order at $x$. The height $h(x)$ is the product of coordinates, which in the case of uniform distribution under focus is equal to the value of the multidimensional distribution function at $x$, i.e. the measure of $L_x$. The height is a key quantity to look at, because the heights at chain records determine the sojourns. Let $H_k = h(R_k)$.

**Lemma 2.** Given $(H_k)$ the sojourns $T_{k+1} - T_k$ are conditionally independent, geometric with parameters $H_k$, $k = 1, 2, \ldots$

**Proof.** A new chain record $R_{k+1}$ occurs as soon as $L_{R_k}$ is hit by some mark. \qed

The lemma has the following elementary but important consequence.

**Corollary 3.** Given $(H_k)$, the conditional law for occurrences of the chain records for any $d$ is the same as in the classical case $d = 1$.

The heights at records undergo a multiplicative renewal process, sometimes called stick-breaking. Let $W, W_1, W_2, W_3, \ldots$ be i.i.d. copies of $H_1 = h(X_1)$.

**Lemma 4.** The heights $(H_k)$ have the same law as the sequence of products $(W_1 \cdots W_k, \ k = 1, 2, \ldots)$.
Proof. Each lower section $L_k$, viewed as a partially ordered probability space with normalised Lebesgue measure is isomorphic to $Q_d$ (via a coordinate-wise scale transformation). Hence all ratios $H_{k+1}/H_k$ are i.i.d., with the same law as $H_1$. □

Explicitly, the density of $W$ is

$$\mathbb{P}(W \in ds) = \frac{(\log s)^{d-1}}{(d-1)!} ds, \quad s \in [0,1],$$

and its Mellin transform is

$$g(\lambda) := \mathbb{E}[W^\lambda] = (\lambda + 1)^{-d},$$

as follows by noting that $H_1$ is the product of $d$ independent uniform variables.

The distinction with the classical $d = 1$ case is seen already at this early stage of our discussion. In the classical case $H_1$ has uniform distribution, hence the stick-breaking sequence $(W_1 \cdots W_k, \ k = 1, 2, \ldots)$ is the sequence of points of a self-similar (i.e. invariant under homotheties) Poisson process with intensity $ds/s$, $s \in [0,1]$. For $d > 1$ the point process $(H_k)$ is neither Poisson nor self-similar, which is a major source of difficulties leading, e.g., to dependencies in the occurrences of chain records at distinct $n$.

3 Proving the CLT

Corollary 3 suggests to focus on properties of a univariate sequence of random variables modified by conditioning and then mixing over some given distribution for its subsequence of record values.

Let $(U_j)$ be a sequence of $[0, 1]$ uniform points, independent of $(H_k)$. We shall produce a transformed sequence $(U_j) \mid (H_k)$ by replacing some of the terms in $(U_j)$ by the $H_k$’s. Replace $U_1$ by $H_1$. Do not alter $U_2, U_3, \ldots$ as long as they do not hit $[0, H_1[$; then replace the first uniform point hitting the interval $[0, H_1[$ by $H_2$. Inductively, as $H_1, \ldots, H_k$ got inserted, keep on screening uniforms until first hitting $[0, H_k[$, then insert $H_{k+1}$ in place of the uniform point that caused the hit, and so on. Eventually all $H_k$’s will enter the resulting sequence. It is easy to see that given $(H_k)$ the distribution of $(U_j) \mid (H_k)$ is the same as the conditional distribution of $(U_j)$ given the subsequence of record values $(H_k)$. In the classical case, $(H_k)$ is the stick-breaking sequence with uniform factors, and we have $(U_j) \mid (H_k) \overset{d}{=} (U_j)$, so the insertion does not alter the law of the sequence.

By Corollary 3, $N_n$ can be identified with the number of points among $U_1, \ldots, U_n$ that get replaced by some $H_k$’s.

There is yet another related interpretation in terms of partially exchangeable partitions, as introduced in [25]. The unit interval $[0,1]$ is divided by $(H_k)$ in infinitely many disjoint subintervals $[H_1, H_0[, [H_2, H_1[, \ldots$ (where $H_0 = 0$). A random partition $\Pi$ of the set $\mathbb{N}$ into disjoint nonempty blocks is defined by assigning two generic integers $m$ and $n$ to the same block if and only if the $m$th and the $n$th terms of $(U_j) \mid (H_k)$ hit the same subinterval. The same partition $\Pi$ can be defined directly in terms of $(X_n)$, by decomposing $Q_d$ in disjoint layers $Q \setminus L_{R_1}, L_{R_1} \setminus L_{R_2}, \ldots$. Clearly, $T_1, T_2, \ldots$ are the minimal integers in the blocks of $\Pi$, and $N_n$ is the number of blocks represented on the first $n$ integers.

The construction of $(U_j) \mid (H_k)$ does not impose any constraints on the law of the sequence $(H_k)$, which can be an arbitrary nonincreasing sequence (the induced $\Pi$ is then
the most general partially exchangeable partition \cite{25}). With this in mind, we shall take for a while a more general approach and assume (as in \cite{17}) that \( H_k = W_1 \cdots W_k, \ k = 1, 2, \ldots \) where \( W_1, W_2, \ldots \) are independent copies of a random variable \( W \in [0, 1] \) with finite logarithmic moments

\[
\mu = \mathbb{E}[-\log W], \quad \sigma^2 = \text{Var}[-\log W].
\]

**Proposition 5.** For \( n \to \infty \), the variable \( N_n \) is asymptotically Gaussian with moments

\[
\mathbb{E}[N_n] \sim \frac{1}{\mu} \log n, \quad \text{Var}[N_n] \sim \frac{\sigma^2}{\mu^3} \log n.
\]

We also have the strong law

\[
N_n \sim \frac{1}{\mu} \log n \quad \text{a.s.}
\]

**Proof.** Our strategy is to show that \( N_n \) is close to \( K_n := \max\{k : H_k > 1/n\} \). By the renewal theorem \cite{11} \( K_n \) is asymptotically Gaussian with the mean \( \mu^{-1} \log n \) and the variance \( \sigma^2 \mu^{-3} \log n \) because \( K_n \) is just the number of epochs on \([0, \log n]\) of the renewal process with steps \(-\log W_j\).

By the construction of \( (U_j) | (H_k) \), we have a dichotomy: \( U_n \in [H_k, H_{k-1}] \) implies that either \( U_n \) will enter the transformed sequence or will get replaced by some \( H_i \geq H_k \). Let \( U_{n1} < \ldots < U_{nn} \) be the order statistics of \( U_1, \ldots, U_n \). It follows that

(i) if \( U_{nj} > H_k \) then \( N_n \leq k + j \),

(ii) if \( U_{nk} < H_k \) then \( N_n \geq k \).

Let \( \xi_n \) be the number of uniform order statistics smaller than \( 1/n \). By definition, \( H_{K_n+1} < 1/n < H_{K_n} \), hence \( K_n \) and \( \xi_n \) are independent and \( \xi_n \) is binomial \((n, 1/n)\). By (i), we have \( N_n \leq K_n + \xi_n \) where \( \xi_n \) is approximately Poisson \((1)\), which yields the desired upper bound.

Now consider the threshold \( s_n = (\log n)^2/n \) and let \( J_n := \max\{k : H_k > s_n\} \). By (ii), if the number of order statistics smaller than \( s_n \) is at least \( J_n \) then \( N_n \geq J_n \). Because \( \log n \sim \log n - 2 \log \log n \) the index \( J_n \) is still asymptotically Gaussian with the same moments as \( K_n \). On the other hand, the number of order statistics smaller than \( s_n \) is asymptotically Gaussian with moments about \((\log n)^2\). Hence elementary large deviation bounds imply that \( N_n \geq J_n \) with probability very close to one. This yields a suitable lower bound, hence the CLT. Along the same lines, the strong law of large numbers follows from \( N_n \sim K_n \). \( \square \)

Similar limit theorems have been proved by other methods for the number of blocks of exchangeable partition in \cite{17}, and for a random (size-biased) path in a quad-tree \cite{10}.

Proposition \( \square \) follows as an instance of Proposition \( \heartsuit \) by computing the logarithmic moments as

\[
\mu = \mathbb{E}[-\log W] = -g'(0) = d, \quad \sigma^2 = \text{Var}[-\log W] = g''(0) - g'(0)^2 = d.
\]

## 4 Poisson-paced records

The probability \( p_n \) of a chain record at index \( n \) is equal to the mean height of the last chain record before \( n \). Asymptotics for these quantities follow most easily by poissonisation.
Let \( (\tau_n) \) be the increasing sequence of points of a homogeneous Poisson point process (PPP) on \( \mathbb{R}_+ \), independent of the marks \( (X_n) \). The sequence \( ((X_n, \tau_n), n = 1, 2, \ldots) \) is then the sequence of points of a homogeneous PPP in \( Q_d \times \mathbb{R}_+ \) in the order of increase of the time component, which now assumes values in the continuous range \( \mathbb{R}_+ \). Let \( \hat{N}_t \) be the number of chain records and \( B_t \) the height of the last chain record on \([0, t]\), that is

\[
\hat{N}_t = \max\{k : \tau_k < t\}, \quad B_t = H_{\hat{N}_t}.
\]

Clearly, \((B_t)\) is the predictable compensator for \((\hat{N}_t)\), in particular

\[
\mathbb{E} \left[ \int_0^t B_s ds \right] = \mathbb{E} [\hat{N}_t].
\]

Proposition translates literally as a CLT for \( \hat{N}_t \) as \( t \to \infty \).

The process \((B_t)\) is Markov time-homogeneous with a very simple type of behaviour. Given \( B_t = b \) the process remains in state \( b \) for some rate-\( b \) exponential time and then jumps to a new state \( bW \), with \( W \) a stereotypical copy of \( H_1 \). Immediate from this description is the following self-similarity property: the law of \((B_t)\) with initial state \( B_0 = b \) is the same as the law of the process \((bB_t)\) with \( B_0 = 1 \). This kind of process is well defined for arbitrary initial state \( b > 0 \). See [16] for features of this process related to the classical records and [7] for more general self-similar (also called semi-stable) processes related to increasing Lévy processes. The process \((B_t)\) with \( B_0 = b \) is naturally associated with the chain records defined in terms of a homogeneous PPP in \( bQ_d \times \mathbb{R}_+ \), with \( bQ_d \) being the cube with side \([0, b]\).

By the self-similarity of \((B_t)\) the moments

\[
m_\beta(t) := \mathbb{E} [B_t^\beta]
\]

satisfy a renewal-type equation

\[
m_\beta'(t) = -m_\beta(t) + \mathbb{E} [W^\beta m_\beta(tW)].
\]

The series solution to this equation with the initial value \( m_\beta(0) = 1 \) is

\[
m_\beta(t) = \sum_{k=0}^\infty \frac{(-t)^k}{k!} \prod_{j=0}^{k-1} (1 - g(j + \beta)) \quad \text{with} \quad g(\lambda) = \frac{1}{(\lambda + 1)^d},
\]

as one can check by direct substitution (see e.g. [6]).

Since \( m_1(t) \) is the probability that the first arrival after \( t \) is a chain record, we have the poissonisation identity

\[
m_1(t) = e^{-t} \sum_{n=0}^\infty \frac{t^n}{n!} p_{n+1},
\]

which implies, upon equating coefficients of the series,

\[
p_n = \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \prod_{j=0}^{k-1} (1 - g(j + 1)). \quad (2)
\]

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This can be compared with the analogous formulas

\[ p_n = g(n-1), \quad \bar{p}_n = \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k g(k) \]

for the occurrences of strong and weak records, so it would be nice to have a direct combinatorial argument for (2).

For \( d = 1 \) we obtain from (2) the familiar \( p_n = 1/n \), and for \( d = 2 \) we obtain (surprisingly simple) \( p_n = 1/(2n) \) (for \( n > 1 \)). For \( d > 2 \) the formulas for \( p_n \) do not simplify.

Factoring

\[ 1 - g(j + \beta) = \prod_{r=1}^{d} \frac{j + \beta + 1 - e^{2\pi i r/d}}{j + \beta + 1} \]

we see that the series for \( m_{\beta} \) is a generalised hypergeometric function of the type \(_dF_d\). Exploiting the asymptotic properties of this class of functions, we determine the asymptotics as

\[ \lim_{t \to \infty} m_{\beta}(t) t^\beta = \frac{1}{-g'(0)} \prod_{r=1}^{\beta-1} \frac{r}{1 - g(r)} = \frac{(\beta!)^{d+1}}{\beta d} \prod_{r=2}^{\beta} \frac{1}{r^d - 1}. \]  

(3)

where \( \beta = 1, 2, \ldots \). Full asymptotic expansion is obtainable in a similar way, see [6] for details of the method and references. The depoissonisation of the \( \beta = 1 \) instance implies, quite expectedly,

\[ p_n \sim \frac{1}{dn}, \quad \text{as } n \to \infty. \]

The following asymptotics for \( B_t \) is also derived from (3) by application of the method of moments.

**Proposition 6.** The random variable \( t B_t \) converges, as \( t \to \infty \), in distribution and with all moments to a random variable \( Y \) whose moments are given by

\[ \mathbb{E} [Y^\beta] = \frac{(\beta!)^{d+1}}{\beta d} \prod_{r=2}^{\beta} \frac{1}{r^d - 1}, \quad \beta = 1, 2, \ldots \]

(4)

The law of \( Y \), determined uniquely by the moments (4), may be considered as a kind of extreme-value distribution. In the case \( d = 1 \) we recover well-known \( Y \overset{d}{=} E \) with \( E \) standard exponential, and for \( d = 2 \) we get \( Y \overset{d}{=} E U \) with \( E \) and \( U \) independent exponential and uniform random variables. In general, there is a series representation

\[ Y \overset{d}{=} E_0 W_0 + \sum_{k=1}^{\infty} E_k \prod_{j=0}^{k} W_j \]

where \( E_k \)’s are exponential, \( W_j \)’s for \( j > 0 \) are as before, \( W_0 \) has density

\[ \mathbb{P}(W_0 \in ds) = \frac{\mathbb{P}(W \leq s)}{sd} ds, \quad s \in [0, 1] \]

(5)

(which is density of the stationary distribution for the stick-breaking with factor \( W \)) and all variables are independent. Also, \( Y \) may be interpreted as an exponential functional.
of a stationary compound Poisson process with initial state $-\log W_0$ and a generic jump $-\log W$, see \[3\]. In the discrete-time setting, the same limit law applies to the height of the last chain record before $n$.

5 Scaling limits

Let $b > 0$ be a scaling parameter which we will send to $\infty$. In the case of one dimension the point process \{\(R_k\)\} of record values is a self-similar PPP on \(\mathbb{R}_+\) with intensity \(d x / x\) (restricted to $x \in [0, 1]$). The same limit appears also for the point process of record times \{\(T_k / b\)\}. The bivariate point process \{(\(bR_k, T_k / b\), $k = 1, 2, \ldots\)\} has a joint scaling limit which may be identified with the set of minimal points (the Pareto boundary) of the homogeneous PPP in \(\mathbb{R}_+^d\). See \[24][26\] for these classical results.

These facts can be generalised to chain records in $d > 1$ dimensions. Observe that for the values of chain records we have the component-wise representation

\[
R_k^{(j)} = U_1^{(j)} \ldots U_k^{(j)}, \quad j = 1, \ldots, d; \quad k = 1, 2, \ldots
\]

with independent uniform \(U_k^{(j)}\)'s. Therefore, each marginal process \{(\(bR_k^{(j)}\), $k = 1, 2, \ldots\)\} converges to the same self-similar PPP on \(\mathbb{R}_+\). The vector point process \{(\(bR_k\)\)} converges, as $b \to \infty$, to a degenerate limit in \(\mathbb{R}_+^d\) which lives on the union of the coordinate axis (this follows because any level $c/b$ is surpassed by one of the marginal \{\(R_k^{(j)}\)\}'s considerably before the others). More interestingly, there is a planar limit for the joint process of heights and record times.

Proposition 7. The scaled point process \{(\(bH_k, T_k / b\), $k = 1, 2, \ldots\)\} has a weak limit as $b \to \infty$. The limiting point process \(\mathcal{R}\) in \(\mathbb{R}_+^2\) is invariant under hyperbolic shifts $(s, t) \mapsto (bs, t/b)$ (with $b > 0$), and the coordinate projections of \(\mathcal{R}\) are self-similar point processes.

Proof. The existence of the limit follows from the analogous result for Poisson-paced marks, and in the latter setup the result follows from \[16\] Theorem 1 which, adapted in our framework, guarantees existence of the entrance law from $\infty$ for the process \((B_t)\) started at $B_0 = b$, as $b \to \infty$. The hyperbolic invariance follows from self-similarity of \((B_t)\).

A more explicit construction of \(\mathcal{R}\) is the following. Let \(\mathcal{H}\) be the multiplicatively stationary (that is, self-similar) multiplicative renewal process with a generic factor $W$. We may view \(\mathcal{H}\) as an extension to \(\mathbb{R}_+\) from \([0, 1]\) of the stick-breaking point process \{(\(W_0, W_0 W_1, W_0 W_1 W_2, \ldots\)\} where \(W_k \overset{d}{=} W\) and \(W_0\) has the stationary density \[3\]. Let \{\(\xi_k, k \in \mathbb{Z}\)\} be the points of \(\mathcal{H}\) which may be labelled so that \(\xi_0 = W_0\) is the maximum point of \(\mathcal{H} \cap [0, 1]\), and \(\xi_1 > 1\). Assign to each \(\xi_k\) an arrival time \(\sigma_k := \sum_{i=-\infty}^{k} E_i / \xi_i\) where the \(E_i\)'s are independent standard exponential variables, also independent of \(\mathcal{H}\). Then let \(\mathcal{R} := \{((\xi_k, \sigma_k), k \in \mathbb{Z}\}\). The hyperbolic invariance of \(\mathcal{R}\) is obvious from the construction and self-similarity of \(\mathcal{H}\).

The limit process of heights \(\mathcal{H}\) is not Poisson, since the law of $W$ is not beta($\theta, 1$) (for some $\theta > 0$). For a similar reason, the limit process of record times, which is the time-projection of \(\mathcal{R}\), is also different from a Poisson process. In the discrete-time setting, the
dependence of occurrences of chain records follows from our interpretation of chain records in terms of partition Π and a characterisation of the Ewens partitions in [23] (where it is shown that the independence would force $W$ to be beta($\theta, 1$)).

As noticed by Charles Goldie, the component-wise logarithmic transform

$\left( -\log(R_k^{(1)}), \ldots, -\log(R_k^{(d)}) \right), \quad k = 1, 2, \ldots$

sends the chain records in $Q_d$ to the sequence of sites visited by a $d$-dimensional random walk whose components are independent one-dimensional random walks with exponentially distributed increments. Equivalently, one can consider the upper chain records from the product exponential distribution in $d$ dimensions. In this regime, subject to a suitable normalisation, the values of chain records concentrate near the diagonal of the positive orthant.

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