The discrete optimization problems with interval objective function on graphs and hypergraphs and the interval greedy algorithm

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Abstract. We consider the discrete optimization problems with interval objective function on graphs and hypergraphs. For the problems, we need to find either a strong optimal solution or a set of all possible weak solutions. A strong solution of the problem is a solution that is optimal for all possible values of the objective function’s coefficients that belong to predefined intervals. A weak solution is a solution that is optimal for some of the possible values of the coefficients. We characterize the strong solutions for considered problems. We give a generalization of the greedy algorithm for the case of interval objective function. For the discrete optimization problems we consider, the algorithm gives a set of all possible greedy solutions and the set of all possible values of the objective function for the solutions. For a given probability distribution that is defined on coefficients’ intervals, we compute probabilities of the weak solutions, expected values of the objective function for them, etc.

Keywords: discrete optimization, interval uncertainty, greedy algorithm.

Introduction

Discrete optimization problems with inexact input data has been investigated in many directions by many researchers. Linear programming problems with inexact input data have been considered, in particular, in [1234]. As usual, the presented approaches search for some unique solution of the problem. Another approaches search for some predefined set of the possible solutions that corresponds to some possible values of inexact parameters [6789].

We consider the discrete optimization problems that may be formulated in the following way. Let \( E = \{e_1, \ldots, e_n\} \). Let \( w(e) > 0 \) be the weight of \( e \in E \), \( w_i = w(e_i) \). A binary vector \( x = (x_1, \ldots, x_n) \) determines the set \( E_x \subseteq E: x_i = 1 \) if \( e_i \in E_x \) and \( x_i = 0 \) if \( e_i \in E \setminus E_x \). The set of feasible solution \( D \) may be considered as a some set of vectors of this form that may be associated with subgraphs of some predefined specific form, i.e., set of the paths that connects the two graph’s vertices, set of spanning trees, set of hamiltonian cycles, etc.

Optimization problem (I). We need to find such \( x \in D \) that gives minimum of the objective function

\[
\tag{1}
f(x, w) = \sum_{e \in E_x} w(e).
\]
For example, the following optimization problems on graphs may be stated in a such way if the set \( E \) is the set of a graph edges: the shortest path problem, the minimum spanning tree problem, the traveling salesman problem, the minimum edge cover problem. If the set \( E \) be a set of graph’s vertices, we may state in a such way the minimum vertex cover problem. If \( E \) is a set of edges of a hypergraph, we may state the set cover problem that we shall consider further.

A lot of applied problems may be formulated as discrete optimization problems. There may be uncertainties in the input data for an applied problem, so the application of discrete optimization methods that operates with exact values of weights will not give us any more information than information on some of many possible solutions that corresponds to some possible values of the input data. It is not always a reliable way to use the mean values of inexact parameters since they may be unrepresentative. For different possible values of inexact parameters, we may have different optimal or approximate solutions and different values of objective function for them, and these differences may be big enough. The uncertainties on the input data may be caused by various reasons. It may be measurements errors or the values of some parameters may be varied with time. Thus, for example, the amount of fuel that is needed to take the same load to the same point by vehicle is different for different weather conditions and different fuel quality.

We consider the discrete optimization problems of the form (I) but with interval uncertainties on the objective function’s coefficients. Using the approach we present, the person that make a decision in the situation of the uncertainty may obtain the information on the possible approximate solutions, the possible values of objective function for them and another information that may be used to analyze the possible scenarios for the situation.

It is often the case that the interval of possible values is the only known information on uncertain value. In addition, we may have an information on the probability distribution of the parameter’s values on the interval. We shall denote interval values using bold font. Let \( \mathbb{I} \) denotes the set of intervals on \( \mathbb{R} \). For an interval \( \mathbf{a} \in \mathbb{I} \), its lower and upper bounds are denoted as \( \underline{\mathbf{a}} \) and \( \overline{\mathbf{a}} \) respectively: \( \mathbf{a} = [\underline{\mathbf{a}}, \overline{\mathbf{a}}] \). The sum of intervals \( \mathbf{a} \) and \( \mathbf{b} \) is defined as follows: \( \mathbf{a} + \mathbf{b} = [\underline{\mathbf{a}} + \underline{\mathbf{b}}, \overline{\mathbf{a}} + \overline{\mathbf{b}}] \). The multiplication of an interval by \( \alpha \in \mathbb{R}^+ \) is the interval \( \alpha \mathbf{a} = [\alpha \underline{\mathbf{a}}, \alpha \overline{\mathbf{a}}] \). Let \( \mathbb{I}^n \) denotes the set of interval vectors of dimension \( n \).

We consider the discrete optimization problems with interval objective functions of the form

\[
 f(x, w) = \sum_{e \in E_x} w(e),
\]

where the values of weights are intervals: \( w(e) \in w(e) = [\underline{w}(e), \overline{w}(e)] \subset \mathbb{R} \) that is to say the possible weights of \( e \) are belong to \( w(e) \). Let \( w = (w_1, \ldots, w_n) \in \mathbb{I}^n \), where \( w_i = w(e_i) \), \( w_i > 0 \). To state the formulation of the discrete optimization problem with interval objective function, we need to define a concept of an optimal solution for the problem. One of the possible way to do this is to use the Pareto set of possible solutions [8,9,10,11] considering the problem as a two-
criteria optimization problem, where the criteria are
\[ f_1(x, w) = \sum_{e \in E_x} w(e) \to \min, \quad f_2(x, w) = \sum_{e \in E_x} \overline{w}(e) \to \min. \]

The other way is to use the notions of weak and strong solutions for discrete optimization problem with interval weights [6, 7, 8, 10, 12].

For a discrete optimization problem with interval objective functions of the form (2), a scenario is a vector \( w \in w \). A scenario \( w \in w \) sets the discrete optimization problem of the form (1) with real-valued coefficients \( w \) of its objective function. We shall call an optimal solution of a such problem as an optimal solution for scenario \( w \). A weak optimal solution is a solution that is optimal for some scenario \( w \in w \). For a discrete optimization problem with an interval objective function, a strong optimal solution is the solution that is optimal for any scenario \( w \in w \). Note that a strong solution is a weak one too. Using the concept of a strong optimal solution, we may state the following formulation of the discrete optimization problem with interval objective function.

**Optimization problem (II).** For given \( w \), we need to find a strong optimal solution of the optimization problem with given set \( D \) of feasible solutions and the objective function of the form (2).

The united solution set is a set of all weak solutions. We may state the discrete optimization problems in the following form.

**Optimization problem (III).** We need to find a united solution set \( \Xi \):
\[ \Xi = \{ x \in D \mid \exists w \in w \forall y \in D : f(x, w) \leq f(y, w) \} \].

The problem (III) may be too hard computationally even for low-dimensional cases, e.g., if the corresponded problem of the form (1) with real-valued function is \( \mathbf{NP} \)-hard. So, we may try to solve the problem (III) approximately, searching for the united approximate solution set that contains the approximate solutions for all of the possible scenarios. Methods for solving such problems goes beyond the exhaustive search on \( w \in w \) and may try to solve the problem by less costly means.

Using the introduced concepts of optimal solutions, we give the characterization of a strong solution for considered problems. We give a generalization of the greedy algorithm for the case of interval objective function. The algorithm gives a united approximate solution set to a problem instance. Also, it gives a set of all possible values of of the objective function for the solutions. For a given probability distribution that is defined on intervals of coefficients, we compute probabilities of the weak solutions, expected values of the objective function for them, etc.

### 1 Characterization of strong solutions

The worst scenario for \( x \in D \) is a such scenario \( w \in w \) that \( w(e) = \overline{w}(e) \) for \( e \in E_x \) while \( w(e) = \underline{w}(e) \) for \( e \in E \setminus E_x \). It was shown in [6] that, for the longest path
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problem, a weak solution is a strong solution if only it is an optimal solution for its worst scenario. The same result was obtained for the minimum spanning tree problem [7]. Indeed, the same result may be obtained for any problem of the form (II).

**Theorem 1.** A weak solution of the problem (II) is a strong solution if and only if it is an optimal solution for its worst scenario.

**Proof.** Let \( x \in D \) be an optimal solution for its worst scenario. For any \( y \in D \), we have

\[
    f(x, w_x) = \sum_{e \in E_x \backslash E_y} w(e) + \sum_{e \in E_x \cap E_y} w(e) \leq \sum_{e \in E_y \backslash E_x} w(e) + \sum_{e \in E_x \cap E_y} w(e),
\]

where \( w_x \) is the worst scenario for \( x \). For arbitrary weights \( w(e) \in w(e) \),

\[
    \sum_{e \in E_x \backslash E_y} w(e) + \sum_{e \in E_x \cap E_y} w(e) \leq \sum_{e \in E_y \backslash E_x} w(e) + \sum_{e \in E_x \cap E_y} w(e).
\]

Since

\[
    \sum_{e \in E_x \backslash E_y} w(e) \leq \sum_{e \in E_x \cap E_y} w(e), \quad \sum_{e \in E_y \backslash E_x} w(e) \leq \sum_{e \in E_x \cap E_y} w(e),
\]

for any scenario \( w \in w \), we have

\[
    f(x, w) = \sum_{e \in E_x \backslash E_y} w(e) + \sum_{e \in E_x \cap E_y} w(e) \leq \sum_{e \in E_y \backslash E_x} w(e) + \sum_{e \in E_x \cap E_y} w(e) = f(y, w).
\]

Thus \( f(x, w) \leq f(y, w) \) for any \( w \in w \), i.e., \( x \) is a strong optimal solution.

2 The generalization of the greedy algorithm for the case of interval objective function

Using the Theorem 1, we may check whether a weak solution is a strong one. The weak solutions may be obtained by some algorithm for some scenarios. But it is often the case that there is no strong solution for the problem instance, i.e., there is no solution of a problem of the form (II). So, instead of it, we may try to approximately solve the problem (III).

2.1 Greedy algorithms for the problem (I)

A rather common approach to solution of the computationally hard problems of the form (I) is to use an appropriate greedy algorithm to get an optimal or an approximate solution of the problem. Applying the greedy algorithm for the problem, we obtain the solution \( x \in D \) taking the elements of \( e \in E \) into \( E_x \) one after another in accordance with a selection function that is specifically defined.
for a problem of the form (I). The selection function depends on the weights \( w(e) \) and probably on some other parameters of the problem instance that relate to \( e \). The algorithm stops when a feasible solution \( x \in D \) of the problem instance is obtained this way.

Let \( \varphi(w_i, e_i) \) be a selection function, e.g., for the set cover problem that we shall consider further, \( E \) is a collection of sets that may be selected in the cover that the algorithm builds and the value \( \varphi(w_i, e_i) \) depends on \( w_i \) and on cardinality of the set at the current iteration of the algorithm. For the considered problems, the basic scheme of the greedy algorithm is the following one.

The greedy algorithm for the problem (I)

1. \( E_x \leftarrow \emptyset \).
2. If \( E_x \) such that \( x \in D \), output \( x \).
3. Else select such \( e_{\text{min}} \in E \) that \( \varphi(e_{\text{min}}) = \min \{ \varphi(w(e), e) \mid e \in E \setminus E_x \} \), \( E_x \leftarrow E_x \cup \{ e_{\text{min}} \} \),
5. Go to step 2.

It was shown in [13], that, using the greedy algorithms, we obtain an optimal solution for the problem of the form (I) if its set \( D \) has a matroidal structure. The minimum spanning tree is an example of such a problem. For some of the problems of the form (I), e.g., for the set cover problem, the greedy algorithms are asymptotically best possible approximation algorithms.

2.2 The interval greedy algorithm

At first, for a detailed consideration, let us consider the interval greedy algorithm for the set cover problem (we shall abbreviate it further as SCP) as an example of the presented approach application.

In the weighted set cover problem, we are given set \( U \) (a ground set). Let \( m = |U| \). There is a collection \( S \) of its subsets \( S_i \subseteq S \), \( S = \{ S_1, \ldots, S_n \} \), such that \( \cup_{i=1}^n S_i = U \). A collection of sets \( S' = \{ S_{i_1}, \ldots, S_{i_k} \} \), \( S_{i_j} \in S \), is called a cover of \( U \) if \( \cup_{j=1}^k S_{i_j} = U \). For \( S_i \in S \), there are given weights \( w_i = w(S_i) \), \( w_i > 0 \). For a collection of sets \( S' = \{ S_{i_1}, \ldots, S_{i_k} \} \), the weight \( w(S') \) is equal to the sum of the weights of the subsets that belong to \( S' \): \( w(S') = \sum_{j=1}^k w(S_{i_j}) \). We need to find an optimal cover of \( U \), i.e., the cover of minimum weight.

The problem may be stated in the form (III) if we associate a hypergraph with it. Elements of \( U \) are associated with vertices of the hypergraph and the sets from \( S \) are associated with the hypergraph’s edges, i.e., the set \( S \) is considered as the set \( E \). The set \( D \), for the problem, is a set of all possible covers for the SCP instance.

Let us consider the greedy algorithm for the set cover problem with non-interval weights. In the course of its operating, we select the sets in the cover depending on the values of their relative weights \( w_i/|S_i| \) until all of the elements of \( U \) are covered. Here, the ratio \( w_i/|S_i| \) is the value of selection function for \( S_i \).
THE GREEDY ALGORITHM FOR SCP

1. $E_x \leftarrow \emptyset$;
2. if $E_x$ is a cover of $U$
   3. output $x$.
4. else select such $S_q \in S$ that $w_q/|S_q| = \min \left\{ w_i/|S_i| \mid S_i \in S \text{ and } S_i \not\subset \bigcup_{S_j \in E_x} S_j \right\}$;
5. $E_x \leftarrow E_x \cup \{S_q\}$;
6. \forall i : $S_i \leftarrow S_i \setminus S_q$;
7. Go to step 2.

For SCP with non-interval weights, the greedy algorithm consists of iterations of the following form.

**An iteration of the greedy algorithm:**

For an SCP instance $\mathcal{P}$,
1) select $S_q$ such that
$$w_q/|S_q| = \min \left\{ w_i/|S_i| \mid S_i \in S \text{ and } S_i \not\subset \bigcup_{S_j \in E_x} S_j \right\};$$
2) add $S_q$ into $E_x$: $E_x \leftarrow E_x \cup \{S_q\}$;
3) obtain the the problem instance $\mathcal{P}'$: $U' \leftarrow U \setminus S_q$, $S_j' \leftarrow S_j \setminus S_q$,
$$w' \leftarrow w.$$

As a result of the greedy algorithm’s iteration, we take some set $S_q \in S$ into $E_x$ and make the transition from the SCP instance $\mathcal{P}$ with given $U$, $S$, $w$ to the instance $\mathcal{P}'$ with $U'$, $S'$, $w'$.

We call a **weak approximate solution** of SCP the solution that the greedy algorithm gives for some scenario $w \in \mathbf{w}$. We consider an approximate solution as an ordered set of elements of $S$. Let us denote as $\mathbf{w}[x]$, $\mathbf{w}[x] \subseteq \mathbf{w}$, the set of scenarios for which the cover $x$ is obtained by the non-interval greedy algorithm. $\mathbf{w}[x] = ((\mathbf{w}[x])_1, \ldots, (\mathbf{w}[x])_n)$. A **united approximate solution set** of SCP with interval weights is a such set $\mathfrak{X}$ of its covers that, for every scenario $w \in \mathbf{w}$, there is $\bar{x} \in \mathfrak{X}$ such that $\bar{x}$ is an approximate solution that the non-interval greedy algorithm gives for the weights that the scenario $w$ specifies for the problem. The interval greedy algorithm takes an instance of SCP with interval objective function and, using backtracking scheme, gives a united approximate solution. We search for all possible weak solutions and, as a result, we obtain the united approximate solution set $\mathfrak{X}$ and $\mathbf{w}[x]$ for all of $\bar{x} \in \mathfrak{X}$. For an SCP instance with interval weights $\mathcal{P}$, let us denote as $U_\mathcal{P}$ the set we need to cover. Let $S_\mathcal{P}$ denotes the collection of sets that we may use to build a cover for the problem instance and let the vector $\mathbf{w}_\mathcal{P} \in \mathbb{R}^n$ be the vector of interval weights of the sets.

To obtain the united approximate solution set $\mathfrak{X}$, we perform iterations of the following form.
An iteration of the interval greedy algorithm for SCP

For the SCP instance $\mathcal{P}$.

1) get the set $Q = \{S_{i_1}, \ldots, S_{i_t}\}$ ($|Q| = t$), where $S_{i_j} \in Q$; if $\exists w \in w_\mathcal{P}$ such that

$$
\frac{w_{i_j} / |S_{i_j}|}{w_{i_j}} = \min \left\{ \frac{w_i / |S_i|}{|S_i|} \mid S_i \in S_\mathcal{P} \text{ and } S_i \not\subset \bigcup_{S_j \in E_x} S_j \right\};
$$

2) for scenarios $w \in w_\mathcal{P}$, obtain the possible variants of $E_x$:

$E_x \leftarrow E_x \cup \{S_{i_1}\}, \ldots, E_x \leftarrow E_x \cup \{S_{i_t}\}$;

3) obtain the SCP instances $\mathcal{P}^{(i_1)}, \ldots, \mathcal{P}^{(i_t)}$ with the sets $U_{\mathcal{P}^{(i_j)}}, S_{\mathcal{P}^{(i_j)}}$, $w_{\mathcal{P}^{(i_j)}}, j = 1, \ldots, t$.

On every iteration of the interval greedy algorithm, having an SCP instance $\mathcal{P}$ and the set $Q$, we obtain a collection of SCP instances $\mathcal{P}^{(i_1)}, \ldots, \mathcal{P}^{(i_t)}$. For all of these problems, we perform the iterations of the form that presented above.

We call a weak approximate solution of SCP the solution that the greedy algorithm gives for some scenario $w \in w$. We consider an approximate solution as an ordered set of elements of $S$. Let us denote as $w[x], w[x] \subseteq w$, the set of scenarios for which the cover $x$ is obtained by the non-interval greedy algorithm. $w[x] = ((w[x])_1, \ldots, (w[x])_n)$. A united approximate solution set of SCP with interval weights is a such set $\Xi$ of its covers that, for every scenario $w \in w$, there is $\hat{x} \in \Xi$ such that $\hat{x}$ is an approximate solution that the non-interval greedy algorithm gives for the weights that the scenario $w$ specifies for the problem.

The interval greedy algorithm takes an instance of SCP with interval objective function and, using backtracking scheme, gives a united approximate solution. We search for all possible weak solutions and, as a result, we obtain the united approximate solution set $\Xi$ and $w[x]$ for all of $\hat{x} \in \Xi$.

Let us give the procedures that the algorithm uses. The procedure Selection has an SCP instance $\mathcal{P}$ as an input and it gives the set $Q$ as an output. $S_i \in Q$ only if there is a such scenario $w \in w_\mathcal{P}$ that the set $S_i$ has a minimum relative weight for the sets in $S_\mathcal{P}$.

**Selection (\(\mathcal{P}\)) : Q;**

1. for $\forall S_i \in S_\mathcal{P}$:
2. $v \leftarrow w_i / |S_i|$;
3. $Q \leftarrow \emptyset$;
4. $v \leftarrow \min \{w_i \mid S_i \in S_\mathcal{P}, S_i \not\subset \emptyset\}$;
5. for $\forall S_i \in S_\mathcal{P}$:
6. if $v \leq v$
7. $Q \leftarrow Q \cup \{S_i\}$;
8. output $Q$. 

Let \( v_i \) denotes the interval of relative weights for the set \( S_i \) for the given \( w_i \): \( v_i = w_i / |S_i| \). Let \( S = \{ S_1, S_2, S_3 \} \) and the intervals \( v_i \) are \( v_1 = [1, 5] \), \( v_2 = [3, 7] \), \( v_3 = [6, 11] \). In the course of operating of the procedure \( \text{SELECTION} \), we select the sets \( S_1 \) and \( S_2 \) into \( Q \), while the set \( S_3 \) we do not select into \( Q \).

The procedure **Possible weights of a selected set** takes as an input an SCP instance \( P \) and an index \( q \) of a some set that belongs to \( Q \). As a result of its implementation, we have the modified weight of the set \( (w[x])_q \). Modifying the interval \( w_q \), we exclude the scenarios that are incompatible with selection of \( S_q \) into \( Q \), i.e., \( (w[x])_q \) is the interval that contains the possible weights of \( S_q \) for which the set may be selected into \( E_x \).

Possible weights of a selected set \((P, Q, q) : (w[x])_q\):

1. \( v \leftarrow \min\{\frac{w_i}{|S_i|} \mid S_i \in Q, i \neq q\} \);
2. if \( \frac{w_q}{|S_q|} > v \)
3. \( w_q \leftarrow |S_q| \cdot v \).
4. \( (w[x])_q \leftarrow w_q \).
5. output \((w[x])_q\).

For the relative weights \( v_i \) are \( v_1 = [1, 5] \), \( v_2 = [3, 7] \), \( v_3 = [6, 11] \), if we take \( S_2 \) into \( E_x \), we do not include into \((w[x])_2\) the part of \( w_2 \) that contains the values of \( w_2 \) which are greater than \( |S_2| \cdot v \), i.e., we exclude the values for which \( v_2 > v_1 \).

The procedure **Modification of an SCP instance** as an input takes the SCP instance \( P \) and the index \( q \) of the set \( S_q \in Q \). For the procedure’s output \( P' \), we have \( U_{P'} = U_P \setminus S_q \). For the sets \( S_i \not\subset \bigcup_{j \in E_x} S_j \), we set \( S_i' = S_i \setminus S_q \). Also, we modify weights of the sets \( S_i \in S_P \) excluding the weights that are incompatible with selection of \( S_q \) into \( E_x \).

Modification of an SCP instance \((P, q) : P'\):

1. \( S_{P'} \leftarrow \emptyset \);
2. for \( \forall S_i \in S \)
3. if \( i \neq q \) and \( S_i \not\subset \emptyset \)
4. if \( \frac{w_i}{|S_i|} < \frac{w_q}{|S_q|} \)
5. \( w_i' \leftarrow |S_i| \cdot \frac{w_q}{|S_q|}, w_i \leftarrow w_i' \);
6. else \( w_i' \leftarrow w_i \);
7. \( S_i' \leftarrow S_i \setminus S_q \);
8. \( S_{P'} \leftarrow S_{P'} \cup \{S_i'\} \);
9. \( U_{P'} \leftarrow U_{P} \setminus S_q \);
10. \( w_{P'} \leftarrow (w'_1, \ldots, w'_n) \);
11. output \( P' \).

For the situation that presented on Fig. 1, taking the set \( S_2 \) into \( E_x \), we exclude the values \( w_1 \) which are less than \( |S_1| \cdot \frac{w_q}{|S_2|} \) from \( w_1 \).

Note that some sets in \( S_{P'} \) may become empty at some iteration of the algorithm. To have the same enumeration of the sets in \( S \) for the course of the algorithm’s operating, these sets are not excluded from \( S_{P'} \) in such situations.

The interval greedy algorithm is implemented by the following procedure.
The interval greedy algorithm for SCP (P) : ˜Ξ:

1. ˜Ξ ← ∅; x ← ∅;
2. ˜Ξ ← United approximate solution (P, x, ˜Ξ);
3. output ˜Ξ.

Here, the procedure United approximate solution set is a backtracking procedure that use the procedures which were presented above. Implementing the procedure’s for an SCP instance P with interval weights, we obtain a weak approximate solutions which we include into united approximate solution ˜Ξ. x and ˜Ξ are the procedure’s arguments that are alterable in the course of the procedure operating.

United approximate solution set (P, x, ˜Ξ):

1. if U = ∅
   2. save the pair (Ex, w[x]);
   3. ˜Ξ ← ˜Ξ ∪ {x};
   4. return.
5. else
   6. Q ← Select (P);
   7. for ∀S ∈ Q:
   8.   x' ← x; w' ← w[x];
   9.   Ex ← Ex ∪ {S};
10. (w[x])i ← Possible weights of a selected set (P, Q, i);
11. P' ← Modification of an SCP instance (P, i);
12. United approximate solution (P', x, ˜Ξ);
13. x ← x'; w[x] ← w'.

Since we obtain the ordered sets Ex as a result, after the algorithm’s implementation, we must unite the saved sets of scenarios w[x] obtained for the different ordered Ex that correspond to the same weak approximate solution x.

The presented algorithm is a generalization of the greedy algorithm for interval weights. If all of the intervals’ weights are degenerated, i.e., wi = w for all S ∈ S, the interval greedy algorithm operates like the non-interval greedy algorithm for SCP except the fact that it searches for not one but all possible greedy solutions if the minimum value at the step 4 of The greedy algorithm for SCP is shared by several sets in S.

2.3 Accuracy of the solutions that are obtained by the interval greedy algorithm

SCP is NP-hard. The complexity of the greedy algorithm for SCP with real-valued weights is equal to \(O(m^2n)\). For the general case of the problem, it holds [14] that

\[
w(\hat{x}) \leq H(m)w(\hat{x}) \leq (\ln m + 1)w(\hat{x}), \tag{3}\]
where \( H(m) = \sum_{k=1}^{m} 1/k \), \( \hat{x} \) is a cover that is obtained by the greedy algorithm, \( \hat{x} \) is an optimal cover. It is shown that, whenever \( P \neq NP \), there is no polynomial algorithm for the set cover problem with approximation ratio \((1 - \varepsilon) \ln m\) for \( \varepsilon > 0 \) [15]. There are another inapproximability results for SCP which exclude the possibility of a polynomial time approximation with better than logarithmic approximation ratio. For all \( \hat{x} \in \hat{\Xi} \), (3) holds for all of scenarios in \( w[x] \).

### 2.4 The interval greedy algorithm for discrete optimization problems on graphs and hypergraphs

Let us, for an arbitrary discrete optimization problem of the considered form, formulate the common scheme of the interval greedy algorithm and the procedures that it uses. For such problems, the formulations differs only by its interval selection function \( \varphi(w_i, e_i) = [\varphi(w_i, e_i), \varphi(w_i, e_i)] \) that are obtained using \( \varphi(w_i, e_i) \), where \( \varphi(w_i, e_i) \) is the real-valued selection function that we use to select the elements of \( E \) performing corresponded non-interval greedy algorithm. For SCP, we used \( \varphi(w_i, e_i) = [v_i, v_i] \).

#### Selection (\( P \)) : \( Q \):
1. \( Q \leftarrow \emptyset; \varphi_{\text{min}} \leftarrow \min\{\varphi(e_i) \mid e_i \in E \setminus E_x\} \);
2. for \( \forall e_i \in E \):
   3. if \( \varphi(w_i, e_i) \leq \varphi_{\text{min}} \)
   4. \( Q \leftarrow Q \cup \{e_i\} \);
5. output \( Q \).

#### Possible weights of a selected element (\( P, Q, q \)) : \( (w[x])_q \):
1. \( \varphi_{\text{min}} \leftarrow \min\{\varphi(e_i) \mid e_i \in E \setminus E_x\} \);
2. if \( \varphi(w_q, e_q) > \varphi_{\text{min}} \)
3. Get \( (w[x])_q \) excluding from \( w_q \) such \( w_q \) that \( \varphi(w_q, e_q) > \varphi_{\text{min}} \);
4. output \( (w[x])_q \).

#### Modification of the problem instance (\( P, q \)) : \( P' \):
1. \( E' \leftarrow \emptyset \);
2. for \( \forall e_i \in E \)
3. if \( i \neq q \)
4. if \( \varphi(w_i, e_i) < \varphi(w_q, e_q) \)
5. Get \( w'_i \) excluding from \( w_i \) such \( w_i \) that \( \varphi(w_q, e_q) > \varphi(w_i, e_i) \);
6. else \( w'_i \leftarrow w_i \);
7. \( E' \leftarrow E' \cup \{e_i\} \);
8. \( w_{P'} \leftarrow (w'_1, \ldots, w'_n) \);
9. output \( P' \).

#### The interval greedy algorithm (\( P \)) : \( \hat{\Xi} \):
1. \( E_x \leftarrow \emptyset; \hat{\Xi} \leftarrow \emptyset; x \leftarrow (0, \ldots, 0) \);
2. \( \check{\Xi} \leftarrow \text{UNITED SOLUTION SET} (P, x, \hat{\Xi}) \);
3. output \( \hat{\Xi} \).

\[ H(m) = \sum_{k=1}^{m} 1/k, \]
United approximate solution set \((\mathcal{P}, x, \Xi)\):

1. If \(x \in \mathcal{D}\)
   2. \(x\) is obtained solution, \(x \in \Xi\); save the pair \((E_x, w[x])\); \(\Xi \leftarrow \Xi \cup \{x\}\); return.
3. Else
   4. \(Q \leftarrow \text{SELECT} (\mathcal{P})\);
   5. For \(\forall e_i \in Q\):
      6. \(x' \leftarrow x; w' \leftarrow w[x]; E_x \leftarrow E_x \cup \{e_i\}\);
      7. \((w[x])_i \leftarrow \text{POSSIBLE WEIGHTS OF A SELECTED ELEMENT} (\mathcal{P}, Q, i)\);
      8. \(\mathcal{P}' \leftarrow \text{MODIFICATION OF THE PROBLEM INSTANCE} (\mathcal{P}, i)\);
      9. \(E\) \(\leftarrow \text{UNITED APPROXIMATE SOLUTION SET} (\mathcal{P}', x, \Xi)\);
      10. \(x \leftarrow x'; w[x] \leftarrow w'\).

2.5 Computation of the weak solution’s probabilities

Suppose a probability distribution is given for the values of weights \(w_i \in \mathcal{W}_i, i = 1, \ldots, n\). Let it be a uniform distribution that is least informative distribution of all possible distributions \([10]\). So we assume that the values \(w_i\) of \(e_i \in E\) are random variables which are uniformly distributed on the intervals \(w_i\). Further, for ease of description, let \(\varphi(w_i, e_i) = w_i\).

The probability of the weak approximate solution \(P(\hat{x})\) is the probability of obtaining of a such scenario \(w \in \mathcal{W}\) that the non-interval greedy algorithm gives the solution \(\hat{x}\). For the ordered set \(E_x = \{e_{i_1}, \ldots, e_{i_k}\}\), the probability \(P(\hat{x})\) may be computed as \(P(\hat{x}) = P(e_{i_1}) \cdots P(e_{i_k})\), where \(P(e_{i_j})\) is the probability that we take \(e_{i_j}\) into \(E_x\) as the \(j\)-th set in it computing \(\hat{x}\) by the interval greedy algorithm.

The procedure \textsc{Probability Of A Selection} takes the instance \(\mathcal{P}\) and the index \(q\) of \(e_q \in Q\) for which we compute the probability of obtaining such \(w_q \in \mathcal{W}_q\) that we take \(e_q\) into \(x\) for the computed set \(Q\) at iteration of the algorithm. The output of the procedure is the probability \(P(e_q)\) of obtaining of a such weight. To compute \(P(e_q)\), we use the procedures \textsc{Partition} and \textsc{Probability}. Implementing the procedure \textsc{Partition}, we get the partition \(P\) of the weights' intervals for the elements that belong to \(Q\). We use the partition to compute the probability of a such weight \(w_q \in \mathcal{W}_q\) that \(e_q\) is selected into \(E_{\hat{x}}\).

\textsc{Partition} \((\mathcal{P}, Q)\) : \(P\):

1. \(w_r \leftarrow \text{min} \{\overline{w}_i \mid e_i \in Q\}\);
2. For \(\forall e_i \in Q\):
   3. \(\overline{w}_i \leftarrow w_r\);
4. \(M \leftarrow \{w_1, \ldots, w_l\}\), where \(w_j\) such that \(\exists e_i \in Q\) for which \(w_j = \overline{w}_j\) or \(w_j = \overline{w}_i\), \(w_l = w_r\). \(M\) is an ordered set and its elements are sorted in ascending order.
5. For \(\forall e_i \in Q\):
   6. \(M_i \leftarrow \{w_j \in M \mid w_j \in \overline{w}_i\}\), \(M_i\) is an ordered set and its elements are sorted in ascending order.
7. \(P_i \leftarrow \{w_{i_1}, \ldots, w_{i_2}\}\), where \(w_{ik} = [w_j, w_{j+1}], w_j, w_{j+1} \in M_i\).
8. \(P \leftarrow \{P_1, \ldots, P_{|Q|}\}\).
For the partition \( P \) that is obtained by procedure Partition, using the procedure Probability, we compute the probability \( \mathbb{P}(e_q) \) of inclusion of \( e_q \in Q \) into greedy cover for given \( w \). If \( w_q \) is a random weight of \( e_q \), we have

\[
\mathbb{P}(e_q) = \sum_{w_{qj} \in P_q} \mathbb{P}(e_q \mid w_q \in w_{qj}) \cdot \mathbb{P}(w_q \in w_{qj}).
\]

\[
\mathbb{P}(w_q \in w_{qj}) = \frac{(w_{qj} - w_q)}{(w_q - w_q)}.
\]

Let \( R = \{ i \mid e_i \in Q, i \neq q, w_{qj} \in P_i \} \) be a set of indices of the elements that belong to \( Q \) excluding \( q \) from it and the partition of weights for them contains the interval \( w_{qj} \). We compute the probability \( \mathbb{P}(e_q \mid w_q \in w_{qj}) \) using the formula

\[
\mathbb{P}(S_q \mid v_q \in v_{qj}) = \sum_{r \in R} 1/(|r| + 1) \mathbb{P}_r,
\]

where \( r \) is a subset of \( R \), \( \mathbb{P}_r \) is the probability of the selection of \( e_q \) if \( w_i \in w_{qj} \) for \( i \in r \) and \( w_i > w_{qj} \). Denoting as \( \mathbb{P}_{ij} \) probability of the event \( w_i \in w_{qj} \) and denoting as \( \mathbb{P}_{ij} \) the probability of the event \( w_i > w_{qj} \), we have:

\[
\mathbb{P}_{ij} = \frac{(w_{qj} - w_{qj})}{(w_i - w_i)}, \quad \mathbb{P}_{ij} = \frac{(w_i - w_{qj})}{(w_i - w_{qj})},
\]

\[
\mathbb{P}_r = \prod_{i \in r} \mathbb{P}_{ij} \prod_{i \in R \setminus r} \mathbb{P}_{ij},
\]

where \( r \) takes all possible values in \([5]\) between \( r = \emptyset \) and \( r = R \). If \( r = \emptyset \) or \( R \setminus r = \emptyset \), the corresponding product \((\prod_{i \in r} \mathbb{P}_{ij} \prod_{i \in R \setminus r} \mathbb{P}_{ij})\) we substitute it by 1 in \([5]\). The multiplier \( 1/(|r| + 1) \) is due to the fact that, for random variables \( \xi_1, \ldots, \xi_N \) which are uniformly distributed on the same interval, we have \( \mathbb{P}(\xi_i = \min\{\xi_1, \ldots, \xi_N\}) = 1/N \) for all \( i \). To compute the probability \( \mathbb{P}(e_q) \), we use the following procedures.

**Probability of a selection \( (P, Q, q) : \mathbb{P}(e_q) \):**

1. \( P \leftarrow \text{Partition} (P, Q); \)
2. \( \mathbb{P}(e_q) \leftarrow \text{Probability} (P, Q, q). \)

**Probability \( (P, P, Q, q) : \mathbb{P}(e_q) \):**

1. for \( j = 1 \) to \(|P_q|\)
2. \( \mathbb{P}(w_q \in w_{qj}) = \frac{(w_{qj} - w_{qj})}{(w_q - w_q)}; \)
3. \( R \leftarrow \{ i \in Q \mid w_{qj} \in P_i \}; \)
4. for \( \forall i \in R: \)
5. \( \mathbb{P}_{ij} = \frac{(w_{qj} - w_{qj})}{(w_i - w_i)}; \)
6. for \( \forall i \in R \setminus r: \)
7. \( \mathbb{P}_{ij} = \frac{(w_i - w_{qj})}{(w_i - w_i)}; \)
8. \( \mathbb{P}(e_q \mid w_q \in w_{qj}) \leftarrow \sum_{r \in R} 1/(|r| + 1) \prod_{i \in r} \mathbb{P}_{ij} \prod_{i \in R \setminus r} \mathbb{P}_{ij}; \)
9. \( \mathbb{P}(e_q) \leftarrow \sum_{j=1}^{|P_q|} \mathbb{P}(e_q \mid w_q \in w_{qj}) \cdot \mathbb{P}(w_q \in w_{qj}). \)
If there are sets with degenerated interval weights in \( Q \), we do the computation of \( P(e_q) \) using the formula
\[
P(e_q) = \prod_{i \in Q, i \neq q} [(w_i - w_q)/(w_i - w)].
\] (6)

For another \( i \in Q, i \neq q \), such that \( w_i \) is degenerated, we put \( (w_i - w_q)/(w_i - w) = (w_i - w)/w = 1 \) in (6).

To compute \( P(\tilde{x}) \) for \( \tilde{x} \in \tilde{\Xi} \), we need to do the following modifications of the stated above procedures. At the step 1 of procedure The interval greedy algorithm, we set \( P(x) \leftarrow 1 \). The procedure Probability of a selection is called before implementation of the procedure Modification of the problem instance at the course of operating of the procedure United solution set. Taking \( e_i \) in \( E_x \), before the step 10 of the procedure, we compute the current value \( P(x): P(x) \leftarrow P(x) \cdot \text{Probability of a selection}(P, Q, i) \).

The value \( P(x) \), that we compute before selection of \( e_i \) into \( E_x \), must be saved at the step 8 of the procedure United approximate solution set. It must be restored on the step 14 in order to compute the probabilities of another approximate solutions that we obtain taking another sets from \( Q \) into \( E_x \). The computed value \( P(\tilde{x}) \) must be saved before we quit the procedure on the step 5.

2.6 The probability distribution on the set of possible values of objective function

Suppose that, in the course of the greedy algorithm operating, we get the set of possible objective function’s values for approximate solutions in \( \tilde{\Xi} \):
\[
\mathbf{w}(\tilde{\Xi}) = \bigcup_{\tilde{x} \in \tilde{\Xi}} \mathbf{w}(\tilde{x}) = \bigcup_{\tilde{x} \in \tilde{\Xi}} \left( \sum_{s_i \in E_{\tilde{x}}} (\mathbf{w}[^{\tilde{x}}])_i \right).
\]

Note that \( \mathbf{w}(\tilde{\Xi}) \) may be a disjoint collection of intervals.

Having probabilities \( P(\tilde{x}) \) and intervals \( \mathbf{w}(\tilde{x}) \) for \( \tilde{x} \in \tilde{\Xi} \), we may obtain the probability distribution on \( \mathbf{w}(\tilde{\Xi}) \). Suppose that \( \tilde{x} \) is a result of the non-interval greedy algorithm operating on scenario \( w \in \mathbf{w} \). For density \( p(w) \) of the distribution, we have \( p(w) = \sum_{\tilde{x} \in \tilde{\Xi}} p(w|\tilde{x})P(\tilde{x}) \), where \( p(w|\tilde{x}) \) is a density of distribution in \( w \). By the central limit theorem, having the uniform distribution on intervals of the sets’ weights, \( p(w|\tilde{x}) \) tends to density of normal distribution as number of elements in \( E_{\tilde{x}} \) grows. The probability distribution on \( \mathbf{w}(\tilde{x}) \) is close to \( \mathcal{N}(\tilde{\mu}, \tilde{\sigma}^2) \), where \( \tilde{\mu} = \sum_{e_j \in E_{\tilde{x}}} w_j \), \( \tilde{\sigma}^2 = \sum_{e_j \in E_{\tilde{x}}} \sigma_j^2 \) for mean values \( w_j \) and standard deviations \( \sigma_j \) of the uniformly distributed weight \( w_j: w_j = (w_j + w_j)/2, \sigma_j = \sqrt{(w_j - w_j)^2/12} \). For the small values of \( |E_{\tilde{x}}| \), we may apply the convolution formula.
2.7 Computational complexity of the approach

The interval greedy algorithm is exponential at the worst case since its complexity depends on the number calls of the backtracking procedure UNITED APPROXIMATE SOLUTION set of searching $\tilde{\Xi}$ for the problem. I.e., the complexity depends on the values of $|Q|$ that we obtain performing the procedure SELECTION. An interval vector $\mathbf{w}_P$ and combinatorial structure of the problem instance $\mathcal{P}$ are determine the search tree and, consequently, they determine the computational complexity of solution of an instance $\mathcal{P}$ by the interval greedy algorithm. As it was shown in [12], the complexity is a non-decreasing step function on values of radii of the weights’ intervals for SCP. And, since the complexity of the algorithm depends only on mutual positions of the weights’ intervals [12], the result may be applied to the general case of discrete optimization problem on graphs and hypergraphs that we consider.

3 Conclusions

We consider the discrete optimization problems with interval objective functions on graphs and hypergraphs. We characterize the strong solutions for considered problems. We give a generalization of the greedy algorithm for the case of interval objective function. Applying the presented approach to the problem $\mathcal{P}$ of the form (III), we may obtain the following information. 1) The united approximate solution set $\tilde{\Xi}$. 2) The sets of scenarios $\mathbf{w}[\tilde{x}] \subseteq \mathbf{w}[\mathcal{P}]$ for $\tilde{x} \in \tilde{\Xi}$. 3) The intervals $\mathbf{w}(\tilde{x})$ of possible weights for $\tilde{x} \in \tilde{\Xi}$: $\mathbf{w}(\tilde{x}) = \sum_{S_i \in \mathcal{E}} (\mathbf{w}[\tilde{x}])_i$; 4) For a given probability distribution on weights’ intervals, we may get the probabilities of $\tilde{x} \in \tilde{\Xi}$. 5) The probability distribution on the set of possible objective function’ values $\mathbf{w}(\tilde{\Xi})$ for solutions that belongs to $\tilde{\Xi}$. Using the distribution, we may compute expected value of the objective function, the standard deviation of it, etc.

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