Hausdorff metric BV discontinuity of sweeping processes

To cite this article: Olaf Klein and Vincenzo Recupero 2016 J. Phys.: Conf. Ser. 727 012006

View the article online for updates and enhancements.

Related content
- On the remainders of compactifications of locally compact mappings
  Elvira N Belyanova
- On Hausdorff ordered structures
  Vladimir G Kanovei
- An upper bound for the Hausdorff dimension of a hyperbolic set
  Xiaoping Gu
Hausdorff metric BV discontinuity of sweeping processes

Olaf Klein
Weierstrass Institute for Applied Analysis and Stochastics,
Leibniz Institute in Forschungsverbund Berlin e.V.
Mohrenstr. 39, 10117 Berlin, Germany
E-mail: Olaf.Klein@wias-berlin.de

Vincenzo Recupero
Dipartimento di Scienze Matematiche, Politecnico di Torino
C.so Duca degli Abruzzi 24, 10129 Torino, Italy
E-mail: vincenzo.recupero@polito.it

Abstract. Sweeping processes are a class of evolution differential inclusions arising in elastoplasticity and were introduced by J.J. Moreau in the early seventies. The solution operator of the sweeping processes represents a relevant example of rate independent operator. As a particular case we get the so called play operator, which is a typical example of a hysteresis operator. The continuity properties of these operators were studied in several works. In this note we address the continuity with respect to the strict metric in the space of functions of bounded variation with values in the metric space of closed convex subsets of a Hilbert space. We provide counterexamples showing that for all BV-formulations of the sweeping process the corresponding solution operator is not continuous when its domain is endowed with the strict topology of BV and its codomain is endowed with the $L^1$-topology. This is at variance with the play operator which has a BV-extension that is continuous in this case.

1. Introduction

A sweeping process is an evolution problem arising in elastoplasticity that can be described in the following way. Let $\mathcal{H}$ be a real Hilbert space and let $\mathcal{C}(t) \subseteq \mathcal{H}$ be a moving closed convex set, i.e. a family of closed convex sets indexed by the time parameter $t \in [0,T]$, $T$ being the final time of the evolution. One has to find a function $y : [0,T] \rightarrow \mathcal{H}$ such that

$$y(t) \in \mathcal{C}(t) \quad \forall t \in [0,T],$$
$$-y'(t) \in N_{\mathcal{C}(t)}(y(t)) \quad \text{for } L^1\text{-a.e. } t \in [0,T],$$
$$y(0) = \text{Proj}_{\mathcal{C}(0)}(y_0),$$

where $L^1$ is the one dimensional Lebesgue measure, $y_0 \in \mathcal{H}$ is prescribed, $\text{Proj}_{\mathcal{C}(0)}(y_0)$ is its projection on $\mathcal{C}(0)$, and $N_{\mathcal{C}(t)}(y(t))$ is the normal cone to $\mathcal{C}(t)$ at $y(t)$ (cf. the definition in formula (2.12) below: precise definitions will be given in the next Section 2). The sweeping process formulated as in (1.1)–(1.3) is well posed in the framework of Lipschitz continuous functions,
Indeed it can be shown that there exists a unique Lipschitz continuous function \( y : [0, T] \rightarrow \mathcal{H} \) satisfying (1.1)–(1.3), once \( y_0 \in \mathcal{H} \) and the mapping \( t \mapsto C(t) \) is Lipschitz continuous when the class of closed convex subsets of \( \mathcal{H} \) is endowed with the Hausdorff metric (see (2.13) below). The proof of this fact can be found in [13]. For an overview on sweeping processes we refer the reader to [12].

If one wants to deal with more general movements of \( C(t) \), for instance when \( t \mapsto C(t) \) is of bounded variation, then the above formulation has to be modified. In [14] the following generalized formulation is proposed. One has to look for a function \( y : [0, T] \rightarrow \mathcal{H} \) of bounded variation and a positive measure \( \mu \) such that the distributional derivative \( D_y \) of \( y \) satisfies the equality \( D_y = w\mu \) for some \( w \in L^1([0, T], \mu; \mathcal{H}) \) and the condition

\[
-w(t) \in NC(y(t)) \quad \text{for } \mu\text{-a.e. } t \tag{1.4}
\]

holds together with (1.3). Such a function \( y \) is also called a solution of the sweeping process in the sense of the differential measures. We will call \( S : C \rightarrow y \) the solution operator of the sweeping process, associating with \( C(t) \) the solution \( y(t) \).

The operator \( S \) is a very relevant example of a rate independent operator, i.e. an operator \( S \) such that

\[
S(C \circ \phi)(t) = (S(C) \circ \phi)(t) \quad \forall t \in [0, T]. \tag{1.5}
\]

whenever \( \phi : [0, T] \rightarrow [0, T] \) is an increasing surjective function. This fact was already observed by J.J. Moreau in (cf. [14, Proposition 2i]), even if he did not use the term “rate independence”.

A relevant particular case of sweeping processes is obtained when \( C(t) = u(t) - \mathcal{Z} \), \( \mathcal{Z} \) being a fixed closed convex set in \( \mathcal{H} \) and \( u : [0, T] \rightarrow \mathcal{H} \) a given function. In this case the solution operator \( P \) mapping \( u \) to the solution of the corresponding sweeping process can be equivalently defined on the space of \( \mathcal{H} \)-valued functions of bounded variation. The operator \( P \) is usually called play operator and plays an important role in elastoplasticity (cf. e.g. [7, 21, 4, 8, 11]).

The study of the continuity properties of the solution operators \( S \) and \( P \) has been addressed in several works. For instance in [14] it is shown that \( S \) is continuous with respect to the topology of the uniform convergence. Instead in [8] it is proved that \( P \) is continuous with respect to the \( BV \) strict topology when \( P \) is restricted to the space of continuous functions of bounded variation provided \( \mathcal{Z} \) is a bounded convex closed set whose interior contains 0, for every boundary point \( x \) of \( \mathcal{Z} \) there exists a unique outward normal \( n(x) \), and the mapping \( x \rightarrow n(x) \) is continuous. All these assumptions are dropped in [17], where \( \mathcal{Z} \) is allowed to be an arbitrary closed convex set. Geometric conditions on \( \mathcal{Z} \) are given in Section 3 in order to characterize when \( P \) is continuous from the space of left continuous functions of bounded variation into itself when the domain is endowed with the strict topology and the codomain is endowed with the \( L^1 \)-topology.

The aim of the present note is to show that this continuity property does not hold for the general solution operator \( S \). This is achieved by exhibiting a concrete example in the one dimensional case \( \mathcal{H} = \mathbb{R} \).

Here is a brief plan of the paper. In the following Section 2 we present all the technical tools in order to deal with the sweeping processes: \( BV \) functions with values in a metric space and convex sets in a Hilbert space. In Section 3 we state the main known existence and continuity results about the sweeping process and in Section 4 we present our counterexample showing the \( BV \)-discontinuity of its solution operator. In the final section we make some remarks connecting the \( BV \)-discontinuity with the existence of multiple geodesics in the space of closed convex subsets of a Hilbert space.

2. Preliminaries

From now on \( T \) will be a fixed strictly positive number and \( \mathbb{N} \) is the set of integers that are greater or equal than one. The family of Borel sets in \([0, T]\) will be denoted by \( \mathcal{B}([0, T]) \).
2.1. Functions with bounded variation

We assume that 

\( (\mathcal{X}, d) \) is a complete metric space. \hspace{1cm} (2.1)

If \( x \in \mathcal{X} \) and \( S \subseteq \mathcal{X} \), \( S \neq \emptyset \), we set \( d(x,S) := \inf_{y \in S} d(x,y) \).

We will mainly deal with spaces of \( \mathcal{X} \)-valued functions defined on \([0, T]\). As usual the space of continuous functions is denoted by \( C([0, T]; \mathcal{X}) \). In the next definition we recall the most simple space containing discontinuous functions.

**Definition 2.1.** Given a function \( u : [0, T] \rightarrow \mathcal{X} \) and a subinterval \( J \subseteq [0, T] \), the (pointwise) variation of \( u \) on \( J \) is defined by

\[
V(u, J) := \sup \left\{ \sum_{j=1}^{m} d(u(t_{j-1}), u(t_{j})) : m \in \mathbb{N}, \ t_{j} \in J \ \forall j, \ t_{0} < \cdots < t_{m} \right\}. \hspace{1cm} (2.2)
\]

If \( V(u, [0, T]) < \infty \) we say that \( u \) is of bounded variation on \([0, T]\) and we set \( BV([0, T]; \mathcal{X}) := \{ u : [0, T] \rightarrow \mathcal{X} : V(u, [0, T]) < \infty \} \).

It is well-known that every \( u \in BV([0, T]; \mathcal{X}) \) admits one sided limits \( u(t-) \), \( u(t+) \) at every point \( t \in [0, T] \), with the convention that \( u(0-) := u(0) \) and \( u(T+) := u(T) \). In this note we will limit ourselves to left continuous functions, i.e. we will deal with the space \( BV_{L}([0, T]; \mathcal{X}) := \{ u \in BV([0, T]; \mathcal{X}) : u(t-) = u(t) \ \forall t \in [0, T] \} \). \hspace{1cm} (2.3)

When we consider left continuous functions we are essentially dealing with Lebesgue equivalence classes of functions with a special view on the initial point 0, allowing us to take into account Dirac masses at 0. In the next definition we introduce some natural metrics in \( BV_{L}([0, T]; \mathcal{X}) \).

**Definition 2.2.** For every \( u, v \in BV_{L}([0, T]; \mathcal{X}) \) we set

\[
d_{\infty}(u,v) := \sup_{t \in [0,T]} d(u(t), v(t)), \hspace{1cm} (2.4)
\]

\[
d_{s}(u,v) := \int_{0}^{T} d(u(t), v(t)) \, dt + d(u(0), v(0)) + | V(u, [0,T]) - V(v, [0,T]) |. \hspace{1cm} (2.5)
\]

We call \( d_{s} \) strict metric and we say that \( u_{n} \rightarrow u \) strictly on \([0, T]\) if \( d_{s}(u_{n}, u) \rightarrow 0 \) as \( n \rightarrow \infty \). The topology induced by \( d_{s} \) is called strict topology.

Of course \( d_{\infty} \) is the distance inducing the topology of uniform convergence. Observe that if \( u, v \in BV_{L}([0, T]; \mathcal{X}) \) then \( t \mapsto d(u(t), v(t)) \) is a measurable integrable function (cf. [6, Section 4.5.10, p. 505]), thus formula (2.5) makes sense.

The strict metric \( d_{s} \) is the natural metric in \( BV \) in the metric framework. It is also used when one deals with approximation procedures (see, e.g., [2]). In connection with hysteresis, it has been studied in [21, 4, 8, 16, 19]. Notice that the strict topology is different from the strong (or norm) \( BV \)-topology (see (2.9) below): the norm topology is usually too strong for applications (indeed it is often called the \( W^{1,1} \)-topology, that cannot be adapted to the metric framework).

Usually in the definition of strict metric the term \( d(u(0), v(0)) \) is missing. The reason why we insert it, is that we are considering left continuous functions on the closed interval \([0, T]\) and we want to take into account the value of these functions at the point \( t = 0 \). This is equivalent to artificially extend any function \( u : [0, T] \rightarrow \mathcal{X} \) from \([0, T]\) to \([-1, T]\) by setting \( u(t) = u(0) \) for every \( t < 0 \). If we write down the classical notion of strict metric for these extended functions, we get exactly our \( d_{s} \) of Definition 2.2.
Let us recall that the space of Lipschitz continuous functions is defined by
\[
\text{Lip}([0, T] ; \mathcal{X}) := \left\{ u : [0, T] \rightarrow \mathcal{X} : \sup_{s \neq t} \frac{d(u(s), u(t))}{|t-s|} < \infty \right\}.
\]
(2.6)
It is clear that Lip([0, T] ; \mathcal{X}) \subseteq C([0, T] ; \mathcal{X}) \cap BV([0, T] ; \mathcal{X}). In the following definition we recall the notion of geodesic in a metric space.

**Definition 2.3.** If \( x_0, y_0 \in \mathcal{X} \) and there is a curve \( g \in \text{Lip}([0, 1] ; \mathcal{X}) \) such that \( g(0) = x_0 \), \( g(1) = y_0 \) and \( d(x_0, y_0) = \mathcal{V}(g, [0, 1]) \), then \( g \) is called a geodesic connecting \( x_0 \) and \( y_0 \).

2.2. Convex sets in Hilbert spaces
Let us assume that
\[
\begin{align*}
\mathcal{H} & \text{ is a real Hilbert space with inner product } \langle \cdot, \cdot \rangle, \\
\|x\|_\mathcal{H} & := (\langle x, x \rangle)^{1/2}.
\end{align*}
\]
(2.7)
If \( \mu : \mathcal{B}([0, T]) \rightarrow [0, +\infty] \) is a measure and \( p \in [1, \infty] \), then the space of functions \( u : [0, T] \rightarrow \mathcal{H} \) such that \( t \mapsto \|u(t)\|_\mathcal{H}^p \) is integrable with respect to \( \mu \) will be denoted by \( L^p([0, T], \mu; \mathcal{H}) \) or by \( L^p(\mu; \mathcal{H}) \) if no confusion may arise. For the theory of integration of vector valued functions we refer to [3, Appendix]. When \( \mu = \mathcal{L}^1 \), the one dimensional Lebesgue measure, we will simply write \( L^p(0, T; \mathcal{H}) := L^p([0, T], \mu; \mathcal{H}) \). We warn the reader that we do not identify two functions which are equal \( \mathcal{L}^1 \)-almost everywhere (\( \mathcal{L}^1 \) a.e.). For example, two functions \( u \) and \( v \) that are identical on \( [0, T] \) but satisfy \( u(0) \neq v(0) \) are not considered as two representation function of the same equivalence class. We recall that the Sobolev space \( W^{1,p}(0, T; \mathcal{H}) \) is the space of functions \( u \in L^p(0, T; \mathcal{H}) \) such that there exists \( v \in L^p(0, T; \mathcal{H}) \) satisfying the equality \( u(t) = u(0) + \int_0^t v(s) \, ds \) for every \( t \in [0, T] \) (cf., e.g., the Appendix of [3]). We also have that \( \text{Lip}([0, T]; \mathcal{H}) = W^{1,\infty}([0, T]; \mathcal{H}) \). Moreover if \( u \in W^{1,1}([0, T]; \mathcal{H}) \), then there exists the derivative \( u'(t) \) for \( \mathcal{L}^1 \) a.e. \( t \in [0, T] \), and we have the equality
\[
\mathcal{V}(u, [0, T]) = \int_0^T \|u'(t)\|_\mathcal{H} \, dt, \quad \forall u \in W^{1,1}([0, T]; \mathcal{H}).
\]
(2.8)
Since \( \mathcal{H} \) has a linear structure we can also consider the so called strong BV-metric (or \( W^{1,1} \)-metric) in \( BV_L([0, T]; \mathcal{H}) \):
\[
d_{BV}(u, v) := \int_0^T \|u(t) - v(t)\|_\mathcal{H} \, dt + \|u(0) - v(0)\|_\mathcal{H} + \mathcal{V}(u-v, [0, T]),
\]
\( u, v \in BV_L([0, T]; \mathcal{H}), \)
(2.9)
In the regular case we obtain the topology of the Sobolev space \( W^{1,1}([0, T]; \mathcal{H}) \) induced by the norm
\[
\|u\|_{W^{1,1}([0, T]; \mathcal{H})} := \int_0^T \|u(t)\|_\mathcal{H} \, dt + \int_0^T \|u'(t)\|_\mathcal{H} \, dt, \quad u \in W^{1,1}([0, T]; \mathcal{H}).
\]
(2.10)
Now we set
\[
\mathcal{G}_\mathcal{H} := \{ \mathcal{K} \subseteq \mathcal{H} : \mathcal{K} \text{ nonempty, bounded, closed and convex} \}.
\]
(2.11)
If \( \mathcal{K} \in \mathcal{G}_\mathcal{H} \) and \( x \in \mathcal{H} \), then the projection on \( \mathcal{K} \) of \( x \) is denoted by \( \text{Proj}_\mathcal{K}(x) \). If \( \mathcal{K} \in \mathcal{G}_\mathcal{H} \) and \( x \in \mathcal{K} \), we recall that the (exterior) normal cone to \( \mathcal{K} \) at \( x \) is defined by
\[
\mathcal{N}_\mathcal{K}(x) := \{ y \in \mathcal{H} : \langle y, v-x \rangle \leq 0 \ \forall v \in \mathcal{K} \}.
\]
(2.12)
We endow the set \( \mathcal{G}_\mathcal{H} \) with the Hausdorff distance. Here is the definition.
Definition 2.4. The Hausdorff distance $d_H : \mathcal{C}_H \times \mathcal{C}_H \rightarrow [0, \infty]$ is defined by

$$d_H(A, B) := \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}, \quad A, B \subseteq \mathcal{C}_H. \quad (2.13)$$

The distance $d_H$ makes $\mathcal{C}_H$ a complete metric space (see, e.g., [1, Theorem 3.85, Section 3.17, p. 116]).

For the sake of simplicity we assumed that the elements of $\mathcal{C}_H$ are bounded. If this assumption is dropped, then $d_H$ is a metric that may take on the value $\infty$, thus some supplementary technical details have to be added. However for our purposes this assumption is not restrictive and in order to prove the BV discontinuity of the sweeping process we can limit ourselves to the bounded case.

2.3. Differential measures

We recall that a $(\mathcal{H}\text{-valued Borel})$ vector measure on $[0, T]$ is a map $\mu : \mathcal{B}([0, T]) \rightarrow \mathcal{H}$ such that $\mu(\bigcup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} \mu(B_n)$ whenever $(B_n)$ is a sequence of mutually disjoint sets in $\mathcal{B}([0, T])$. Let us also recall that if $\mu : \mathcal{B}([0, T]) \rightarrow \mathcal{H}$ is a vector measure, then $|\mu| : \mathcal{B}([0, T]) \rightarrow [0, \infty]$ is defined by

$$|\mu|(B) := \sup \left\{ \sum_{n=1}^{\infty} ||\mu(B_n)||_H : B = \bigcup_{n=1}^{\infty} B_n, B_n \in \mathcal{B}([0, T]), B_h \cap B_k = \emptyset \text{ if } h \neq k \right\}.$$

The map $|\mu|$ is a positive measure which is called total variation of $\mu$ and the vector measure $\mu$ is said to be with bounded variation if $|\mu|(\{0, T\}) < \infty$ (see, e.g., [5, Chapter I, Section 3]).

The following proposition (cf. [5, Theorem 1, section III.17.2, p. 358]) provides a connection between functions with bounded variation and vector measures.

Theorem 2.1. If $f \in BV_L([0, T]; \mathcal{H})$ then there exists a unique vector measure of bounded variation $\mu_f : \mathcal{B}([0, T]) \rightarrow \mathcal{H}$ such that for every $c, d \in [0, T]$ with $c < d$ we have

$$\mu_f([c, d]) = f(d) - f(c), \quad \mu_f([c, d]) = f(d) - f(c).$$

Vice versa if $\mu : \mathcal{B}([0, T]) \rightarrow \mathcal{H}$ is a vector measure with bounded variation, then the map $f_\mu : [0, T] \rightarrow \mathcal{H}$ defined by $f_\mu(t) := \mu([a, t])$ is such that $f_\mu \in BV_L([0, T]; \mathcal{H})$ and $\mu_{f_\mu} = \mu$.

The measure $\mu_f$ is called Lebesgue-Stieltjes measure or differential measure of $f$. Like in the scalar case if $f \in BV([0, T]; \mathcal{H})$ then $\mu_f = Df$, with $Df$ being the distributional derivative of $f$, i.e.

$$-\int_0^T \varphi'(t)f(t) \, dt = \int_0^T \varphi \, dDf \quad \forall \varphi \in C^1_c([0, T]).$$

(cf. [17, Section 2]).

3. The sweeping processes and the play operator

Now we can present the general existence result for the sweeping processes in $BV_L$. This is the main result in [14]: the existence theorem for right continuous data is [14, Propositions 2a and 3a] and the left continuous case can be deduced from [14, Section 2d].

Theorem 3.1. Let $C \in BV_L([0, T]; \mathcal{C}_H)$ and $y_0 \in \mathcal{H}$ be given. There exists a unique $y = \mathcal{S}_{y_0}(\mathcal{C}) \in BV_L([0, T]; \mathcal{H})$ such that there exist a measure $\mu : \mathcal{B}([0, T]) \rightarrow [0, \infty]$ and
a function \( w \in L^1(\mu; \mathcal{H}) \) satisfying

\[
\begin{align*}
Dy &= w\mu, \\
y(t) &\in \mathcal{C}(t) \quad \forall t \in [0, T], \\
- u(t) &\in N_{\mathcal{C}(t)}(y(t)) \quad \text{for } \mu\text{-a.e. } t \in [0, T], \\
y(0) &= \text{Proj}_{\mathcal{C}(0)}(y_0),
\end{align*}
\tag{3.1}
\]

where \( w\mu \) is the vector measure defined by \( w\mu(B) := \int_B w \, d\mu \), \( B \in \mathcal{B}([0, T]) \). If \( \mathcal{C} \in BV([0, T]; \mathcal{H}) \cap C([0, T]; \mathcal{H}) \), then \( S_{y_0}(\mathcal{C}) \in BV([0, T]; \mathcal{H}) \cap C([0, T]; \mathcal{H}) \). Finally if \( \mathcal{C} \in Lip([0, T]; \mathcal{H}) \) then \( S_{y_0}(\mathcal{C}) \in Lip([0, T]; \mathcal{H}) \) and problem (3.1)–(3.4) reads as (1.1)–(1.3).

The two last statements of the previous theorem can be easily deduced by the representation formula of [18, Theorem 3.1] for the continuous case. Thanks to the previous theorem the following solution operator is defined

\[ S_{y_0} : BV_L([0, T]; \mathcal{H}) \rightarrow BV_L([0, T]; \mathcal{H}). \]

If \( Z \in \mathcal{H}, 0 \in Z \) and \( z_0 \in Z \), we consider the play operator

\[ P_{z_0} : BV_L([0, T]; \mathcal{H}) \rightarrow BV_L([0, T]; \mathcal{H}) \]

which is defined by

\[ P_{z_0}(u) := S_{u(0)-z_0}(u - Z), \quad u \in BV_L([0, T]; \mathcal{H}). \]

The restrictions of \( S_{y_0} \) and \( P_{z_0} \) to the various spaces subspaces of \( BV \) will be denoted by the same symbols \( S_{y_0} \) and \( P_{z_0} \). Using this notation, it holds that \( P_{z_0} : Lip([0, T]; \mathcal{H}) \rightarrow Lip([0, T]; \mathcal{H}) \) is the classical play operator, which is the operator associating with \( u \in Lip([0, T]; \mathcal{H}) \) the only function \( y \in Lip([0, T]; \mathcal{H}) \) satisfying (1.1)–(1.3) with \( \mathcal{C}(t) = u(t) - Z \) and \( y_0 = u(0) - z_0 \).

Let us observe that the operator \( P_{z_0} \) here defined coincides with the one introduced in [9]. Indeed, as observed in [9], the two operators coincide on the set of left continuous step functions (cf. formula (3.8)) and they are both continuous with respect to the \( d_{\infty} \)-metric (cf. Theorem 3.2 (iii) below and [9, Theorem 2.3]).

Another possible extension of the play operator on \( Lip([0, T]; \mathcal{H}) \) to the space of functions of bounded variation is provided in [17], where an operator \( \overline{P}_{z_0} : BV_L([0, T]; \mathcal{H}) \rightarrow BV_L([0, T]; \mathcal{H}) \) is obtained as the continuous extension of the classical play operator \( P_{z_0} : Lip([0, T]; \mathcal{H}) \rightarrow Lip([0, T]; \mathcal{H}) \) with respect to the \( d_{\ast} \)-topology in the domain and the \( L^1 \)-topology in the codomain. As shown in [17], the two operators coincide on \( BV([0, T]; \mathcal{H}) \cap C([0, T]; \mathcal{H}) \), but in general they are different on \( BV_L([0, T]; \mathcal{H}) \).

A main feature of \( S_{y_0} \) and \( P_{z_0} \) is rate independence, i.e. if \( \phi : [0, T] \rightarrow [0, T] \) is an increasing surjective function, then

\[
\begin{align*}
S_{y_0}(\mathcal{C} \circ \phi) &= S_{y_0}(\mathcal{C}) \circ \phi, \\
P_{z_0}(u \circ \phi) &= P_{z_0}(u) \circ \phi,
\end{align*}
\tag{3.5}
\]

whenever \( \mathcal{C} \in BV_L([0, T]; \mathcal{H}) \) and \( u \in BV_L([0, T]; \mathcal{H}) \) (cf. [14, Proposition 2i]). Now let \( m \in \mathbb{N}, t_0 = 0 < t_1 < \cdots < t_m = T \), and \( C_0, C_1, \ldots, C_m \in \mathcal{H} \). If \( \mathcal{C} \in BV_L([0, T]; \mathcal{H}) \) is the step function defined by

\[
\mathcal{C}(t) := \begin{cases} C_0 & \text{if } t = 0, \\
C_k & \text{if } t \in [t_{k-1}, t_k], k \in 1, \ldots, m,
\end{cases}
\tag{3.7}
\]
then for every $y_0 \in {\mathcal{H}}$ it turns out that (cf. [14, Formulas (1.13)-(1.14)])

$$S_{y_0}(C)(t) = \begin{cases} 
\text{Proj}_{L^\infty}(y_0) & \text{if } t = 0, \\
\text{Proj}_{L^\infty}(y_{k-1}) & \text{if } t \in [t_{k-1}, t_k], k = 1, \ldots, m.
\end{cases} \quad (3.8)$$

Of course the same kind of result holds for the play operator $P_{z_0}$. Here we list some of the main continuity properties of $S_{y_0}$, of $P_{z_0}$ and of the restrictions of these operators to various spaces subspaces of $BV$. We recall that these restrictions are denoted by the same symbols $S_{y_0}$ and $P_{z_0}$.

**Theorem 3.2.** Assume that $y_0 \in {\mathcal{H}}$, $Z \in {\mathcal{C}}_{\mathcal{H}}$, and $0, z_0 \in Z$. The following statements holds true.

(i) $S_{y_0} : BV_L([0, T]; {\mathcal{C}}_{\mathcal{H}}) \to BV_L([0, T]; {\mathcal{H}})$ is continuous with respect to the $d_{\infty}$-topology.

(ii) $S_{y_0} : BV([0, T]; {\mathcal{C}}_{\mathcal{H}}) \cap C([0, T]; {\mathcal{C}}_{\mathcal{H}}) \to BV([0, T]; {\mathcal{H}}) \cap C([0, T]; {\mathcal{H}})$ is continuous if its domain is endowed with the $d_{\infty}$-topology and its codomain is endowed with the $d_{\infty}$-topology.

(iii) $P_{z_0} : BV_L([0, T]; {\mathcal{H}}) \to BV_L([0, T]; {\mathcal{H}})$ is continuous with respect to the $d_{\infty}$-topology.

(iv) $P_{z_0} : BV([0, T]; {\mathcal{H}}) \cap C([0, T]; {\mathcal{H}}) \to BV([0, T]; {\mathcal{H}}) \cap C([0, T]; {\mathcal{H}})$ is continuous with respect to the $d_1$-topology.

(v) $P_{z_0} : BV_L([0, T]; {\mathcal{H}}) \to BV_L([0, T]; {\mathcal{H}})$ is continuous if the domain is endowed with the strict topology and the codomain with the $L^1$-topology.

(vi) If $H$ is of finite dimension and $Z$ is a non-obtuse polyhedron, i.e. $Z = \{x \in H : \langle n_j, x \rangle \leq c_j, j = 1, \ldots, p\}$ for some $p \in \mathbb{N}$, $c_j \geq 0$ and $n_j \in H$ with $|n_j|_H = 1$ and $\langle n_j, n_k \rangle \leq 0$ whenever $1 \leq i \leq j \leq p$, then it follows that $P_{z_0} : BV_L([0, T]; H) \to BV_L([0, T]; H)$ is continuous if its domain is endowed with the strict topology and the codomain with the $L^1$-topology. This extension $P_{z_0}$ is however not necessarily equal to the sweeping process $P_{z_0} = S_{u(0) - Z}(u - Z)$ on discontinuous functions considered in (vi): in [10] it is shown for $H$ being finite dimensional that $P_{z_0} = P_{z_0}$ on the whole $BV_L([0, T]; H)$ if and only if $Z$ is a non-obtuse polyhedron.

**Proof.** Part (i) of the previous theorem is proved in [14] while part (ii) is proved in [18]. Concerning the play operator $P_{z_0}$, part (iii) follows directly from part (i), since $d_{\infty}(u_n - Z, u - Z) \to 0$ whenever $u_n \to u$ uniformly on $[0, T]$. Part (iv) is instead proved in [17], where our extra-term $d(u(0), v(0))$ of the strict metric in (2.5) can be handled by means of the reduction method presented in [17, Section 4.4] taking into account the Lipschitz continuity of the projection. In order to prove (v) we recall that in [15] it is proved that $P_{z_0} : BV([0, T]; H) \cap C([0, T]; H) \to BV([0, T]; H) \cap C([0, T]; H)$ admits a continuous extension to the above mentioned operator $\overline{P}_{z_0} : BV_L([0, T]; H) \to BV_L([0, T]; H)$ if the domain is endowed with the strict topology and the codomain with the $L^1$-topology. This extension $\overline{P}_{z_0}$ is however not necessarily equal to the sweeping process $P_{z_0} = S_{u(0) - Z}(u - Z)$ on discontinuous functions considered in (vi): in [10] it is shown for $H$ being finite dimensional that $P_{z_0} = P_{z_0}$ on the whole $BV_L([0, T]; H)$ if and only if $Z$ is a non-obtuse polyhedron. \[\Box\]

In the one dimensional case the play operator $P_{z_0}$ is continuous from $BV_L([0, T]; \mathbb{R})$ into itself endowed with strict topology, this is proved in [15]. In the vector case this property is in general false: this can be deduced from [17, Theorem 3.7] where it is shown that the restriction $P_{z_0} : BV([0, T]; H) \cap C([0, T]; H) \to BV([0, T]; H) \cap C([0, T]; H)$ can be $d_{\infty}$-continuously extended to the whole $BV_L([0, T]; H)$ if and only if $Z$ is a closed vector subspace or $Z = \{x \in H : -\alpha \leq \langle f, x \rangle \leq \beta\}$ for some $\alpha, \beta \in [0, \infty]$ and some $f \in H \setminus \{0\}$, hence for bounded $Z$ the operator $P_{z_0}$ is never continuous from the whole $BV$ into itself, both domain and codomain endowed with the strict metric.
4. Metric BV discontinuity

From Theorem 3.2 we can infer that if \( Z \) is a non-obtuse polyhedron and \( \mathcal{H} \) has finite dimension, then \( P_{z_0} : BV_L([0,T];\mathcal{H}) \rightarrow BV_L([0,T];\mathcal{H}) \) is continuous if the domain is endowed with the strict topology and the codomain with the \( L^1 \)-topology, while the operator \( \mathbb{P}_{z_0} : BV_L([0,T];\mathcal{H}) \rightarrow BV_L([0,T];\mathcal{H}) \) is continuous in this sense even if these assumptions on \( Z \) and \( \mathcal{H} \) are not satisfied.

Now we provide the counterexample showing that this last continuity property stated is in general not true for the solution operator \( S_{z_0} \) of the sweeping process.

Let us consider \( \mathcal{H} = \mathbb{R} \) so that

\[
\mathcal{C}_\mathbb{R} = \{ I \subseteq \mathbb{R} : I \text{ is a bounded closed interval, } I \neq \emptyset \}. \tag{4.1}
\]

If \( a, b, c, d \in \mathbb{R} \) and \( a \leq b, c \leq d \), then

\[
d(\mathcal{C})([a,b], [c,d]) = \max\{|a-c|, |b-d|\}. \tag{4.2}
\]

Let us set

\[
K_0 := [0,2], \quad K_1 := [1,4] \tag{4.3}
\]

and fix \( t_0 \in ]0,T[. \) So \( K_0, K_1 \in \mathcal{C}_\mathbb{R} \) and we can define \( \mathcal{C} : [0, T] \rightarrow \mathcal{C}_\mathbb{R} \) by setting

\[
\mathcal{C}(t) := \begin{cases} 
K_0 & \text{if } 0 \leq t \leq t_0, \\
K_1 & \text{if } t_0 < t \leq T,
\end{cases} \quad t \in [0, T]. \tag{4.4}
\]

Hence \( \mathcal{C} \in BV([0,T];\mathcal{C}_\mathbb{R}) \) and \( V(\mathcal{C}, [0,T]) = 2 \). Let us now define \( B : [0,1] \rightarrow \mathcal{C}_\mathbb{R} \) by

\[
B(t) := \begin{cases} 
[2t, 2+2t] & \text{if } 0 \leq t \leq 2/3, \\
[2-t, 2+2t] & \text{if } 2/3 < t \leq 1.
\end{cases} \tag{4.5}
\]

Observe that \( B \in Lip([0,1];\mathcal{C}_\mathbb{R}) \) and thanks to [18, Proposition 6.1] we have that, up to reparametrization, the restriction of \( B \) to \([0,2/3]\) is a geodesic connecting \( B(0) = K_0 \) to \( B(2/3) = [4/3,10/3] \). On the other hand, up to reparametization, the restriction of \( B \) to \([2/3,1]\) is a geodesic connecting \( B(2/3) = [4/3,10/3] \) and \( B(1) = K_1 \). Hence

\[
V(B, [0,1]) = V(B, [0,2/3]) + V(B, [2/3,1]) = 4/3 + 2/3 = 2, \tag{4.6}
\]

thus \( B \) is a geodesic connecting \( K_0 \) and \( K_1 \). Let \( B_n \in Lip([0,T];\mathcal{C}_\mathbb{R}) \) be the sequence defined (for \( n \) large enough) by

\[
B_n(t) := \begin{cases} 
K_0 & \text{if } 0 \leq t \leq t_0, \\
B(n(t-t_0)) & \text{if } t_0 < t \leq t_0 + 1/n, \\
K_1 & \text{if } t_0 + 1/n < t \leq T.
\end{cases} \tag{4.7}
\]

We have that

\[
V(B_n, [0,T]) = V(C, [0,T]) = d(\mathcal{C}(K_0, K_1)) \quad \forall n \in \mathbb{N},
\]

and

\[
\lim_{n \to \infty} d(\mathcal{C}(B_n(t), C(t))) = 0 \quad \forall t \in [0,T].
\]

Hence, as \( d(\mathcal{C}(B_n(t), C(t))) \leq d(\mathcal{C}(K_0, K_1)) \), by the dominated convergence theorem we have that

\[
\lim_{n \to \infty} \int_0^T d(\mathcal{C}(B_n(t), C(t))) \, dt = 0.
\]
Hence $B_n \rightarrow C$ as $n \rightarrow \infty$ in the strict topology of $BV_L([0,T]; \mathcal{C}_R)$. If $y_0 := 0$, then $y_0 \in C(0) = B(0) = K_0$ and, thanks to (3.7)–(3.8) we have

$$S_0(C)(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq t_0, \\ \text{Proj}_{K_1}(0) = 1 & \text{if } t_0 < t \leq T. \end{cases} \quad (4.7)$$

It is also easy to check that

$$S_0(B)(t) = \begin{cases} 2t & \text{if } 0 \leq t \leq 2/3, \\ 4/3 & \text{if } 2/3 < t \leq 1, \end{cases}$$

therefore, using for instance rate independence, it follows that

$$S_0(B_n)(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq t_0, \\ 2n(t - t_0) & \text{if } t_0 < t \leq t_0 + 2/3n, \\ 4/3 & \text{if } t_0 + 2/3n < t \leq T. \end{cases}$$

Hence, if $z : [0,T] \rightarrow \mathbb{R}$ is defined by

$$z(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq t_0, \\ 4/3 & \text{if } t_0 < t \leq T, \end{cases} \quad (4.8)$$

we have that $\lim_{n \rightarrow \infty} S_0(B_n)(t) = z(t)$ for every $t \in [0,T]$, thus, thanks to the dominated convergence theorem, we infer that

$$S_0(B_n) \rightarrow z \quad \text{in } L^1(0,T; \mathcal{H}). \quad (4.9)$$

Therefore, as $S_0(C) \neq z$, the operator $S_0$ is not continuous when its domain $BV_L([0,T]; \mathcal{H})$ is endowed with the topology induced by the strict convergence and its codomain is endowed with any reasonable topology weaker than or equal to the $L^1(0,T; \mathcal{H})$ topology.

5. Multiple geodesics

The lack of metric $BV$ continuity of the solution operator of the sweeping process in the one dimensional case is somehow connected to the existence of more than one geodesic connecting points (sets) in $\mathcal{C}_R$. Indeed let us consider the curve $\mathcal{A} : [0,1] \rightarrow \mathcal{C}_R$ defined by

$$\mathcal{A}(t) := (1 - t)K_0 + tK_1, \quad t \in [0,1]. \quad (5.1)$$

Observe that $\mathcal{A} \in \text{Lip}([0,1]; \mathcal{C}_R)$ and that $\mathcal{A} \neq B$. In the case of the one dimensional play operator with characteristic $\mathcal{Z}$, the curve $\mathcal{A}$ corresponds to the input $\mathcal{A}(t) = a(t) - \mathcal{Z}$, $\mathcal{K}_j = \{k_j\}$ for some $k_j \in \mathcal{H}$, $j = 1, 2$, with $a(t) = (1 - t)k_1 + tk_2$, $t \in [0,1]$. Thanks to [18, Proposition 6.1] or [20, Prop. 1]: we have that $\mathcal{A}$ is a geodesic connecting $K_0$ and $K_1$. Let $\mathcal{A}_n \in \text{Lip}([0,T]; \mathcal{C}_R)$ be the sequence defined (for $n$ large enough) by

$$\mathcal{A}_n(t) := \begin{cases} K_0 & \text{if } 0 \leq t \leq t_0, \\ \mathcal{A}(n(t - t_0)) & \text{if } t_0 < t \leq t_0 + 1/n, \\ K_1 & \text{if } t_0 + 1/n < t \leq T. \end{cases}$$

We have that

$$V(\mathcal{A}_n, [0,T]) = V(\mathcal{C}, [0,T]) = d_H(C_0,C_1) \quad \forall n \in \mathbb{N},$$
with \( \rho \) for the sequence \( 1 \)

The same holds for the geodesic \( E \)

Hence, the sequence \( A \)

Thus \( y \)

Hence, as \( d_{\mathcal{G}}(A_n(t), C(t)) \leq d_{\mathcal{G}}(K_0, K_1) \), by the dominated convergence theorem we have that

\[
\lim_{n \to \infty} \int_0^T d_{\mathcal{G}}(A_n(t), C(t)) \, dt = 0.
\]

Hence \( A_n \to C \) as \( n \to \infty \) in the strict topology of \( BV_L([0,T]; \mathcal{C}_\mathbb{R}) \). Taking again \( y_0 := 0 \) we find \( y_0 \in C(0) = A(0) = K_0 \) and

\[
S_0(A)(t) = t,
\]

therefore

\[
S_0(A_n)(t) = \begin{cases} 
0 & \text{if } 0 \leq t \leq t_0, \\
(n(t - t_0) & \text{if } t_0 < t \leq t_0 + 1/n, \\
1 & \text{if } t_0 + 1/n < t \leq T.
\end{cases}
\]

Thus

\[
S_0(A_n) \to S_0(C) \quad \text{in } L^1(0,T; \mathbb{R}).
\]

Hence, the sequence \( A_n \) approximating \( C \) by using the geodesic \( A \) does not allow to prove that \( S_0 \) is not continuous. \(^1\) But, using this sequences, it follows for every extension \( E_{S,0} : BV_L([0,T]; \mathcal{C}_\mathbb{R}) \to BV_L([0,T]; \mathcal{H}) \) of the restriction of \( S_0 \) to \( Lip([0,1]; \mathcal{C}_\mathbb{R}) \) that \( E_{S,0} \) is \( \text{not} \) continuous if the domain is endowed with the strict topology and the codomain with the \( L^1 \)-topology: If \( E_{S,0} \) were continuous as requested it would follow by the computations for the sequence \( A_n \) that \( E_{S,0}(C) = S_0(C) \), while the computations for \( B_n \) would yield that \( E_{S,0}(C) = z \neq S_0(C) \).

\(^1\) The same holds for the geodesic \( G \) connecting \( K_0 \) and \( K_1 \) that is defined as in [20, Theorem 1]:

\[
G(t) := \delta_{\rho}(K_1) \cap \delta_{(1-t)\rho}(K_2)
\]

with \( \rho := d_{\mathcal{G}}(K_0, K_1) = 2 \) and \( \delta_{\lambda}(K) := \bigcup_{x \in X} \{ y \in X : d(x, y) \leq \lambda \} \) for \( \lambda > 0 \) and \( K \in \mathcal{C}_\mathbb{R} \). We have

\[
G(t) = [-2t, 2t + 2t] \cap [1 - 2(1 - t), 4 + 2(1 - t)] = \begin{cases} 
[1 - 2t, 2t + 2] & \text{if } t \leq 1/4, \\
[1 - 2(1 - t), 2t + 2] & \text{if } t > 1/4.
\end{cases}
\]

Figure 1. The approximations \( B_2 \) (left) and \( A_2 \) (center) for \( C \) and the corresponding output of the sweeping process \( S_0 \) for \( T = 1 \) and \( t_0 = 0.25 \). On the right, the approximation \( G_2 \), being analogously derived by using the geodesic \( G \), and the corresponding output of the sweeping process are shown.
Hence, we see that considering different geodesics connecting two elements in $C_R$ yields that there is no extension of the restriction of the sweeping process with the requested continuity properties.

6. Conclusion
We have provided a one dimensional counterexample showing that the solution operator of the sweeping process is not continuous when its domain is endowed with the strict topology of $BV$ and its codomain is endowed with the $L^1$-topology. For a general Hilbert space $\mathcal{H}$ one obtains a corresponding discontinuity for any $BV$-formulation of the sweeping process by introducing for any $e \in \mathcal{H}$ with $e \neq 0$ the mapping $x \mapsto \langle x, e \rangle$ and considering the pre-images of the intervals used above intersected with some appropriate bounded convex set. This is at variance with the case of the play operator on $Lip([0,T];\mathcal{H})$ and its continuous $BV$-extension $P_z$ (cf. Theorem 3.2 (v)).

Acknowledgment
The authors would like the thank the referees for their valuable comments. The second author is a member of GNAMPA-INDAM, which supported the first author’s staying in Torino in April 2013.

References

[1] Aliprantis C D and Border K C, *Infinite Dimensional Analysis - Third Edition*, Springer, Berlin, 2006
[2] Ambrosio L, Fusco N and Pallara D, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 2000.
[3] Brezis H, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland Mathematical Studies, Vol. 5, North-Holland Publishing Company, Amsterdam, 1973.
[4] Brokate M and Sprekels J, *Hysteresis and Phase Transitions*, Applied Mathematical Sciences, Vol. 121, Springer-Verlag, New York, 1996.
[5] Dinculeanu N, *Vector Measures*, International Series of Monographs in Pure and Applied Mathematics, Vol. 95, Pergamon Press, Berlin, 1967.
[6] Federer H, *Geometric Measure Theory*, Springer-Verlag, Berlin-Heidelberg, 1969.
[7] Krasnosel’skii M A and Pokrovskii A V, *Systems with Hysteresis*, Springer-Verlag, Berlin Heidelberg, 1989.
[8] Krejčí P, *Hysteresis, Convexity and Dissipation in Hyperbolic Equations*, Gakuto International Series Mathematical Sciences and Applications, Vol. 8, Gakkotosho, Tokyo, 1997.
[9] Krejčí P and Laurenčík P, *Generalized variational inequalities*, J. Convex Anal. 9 (2002), 159–183.
[10] Krejčí P and Recupero V, *Comparing BV solutions of rate independent processes*, J. Convex Anal., 21 (2014), 121–146.
[11] Mielke A, *Evolution in rate-independent systems*, In “Handbook of Differential Equations, Evolutionary Equations, vol. 2”, C. Dafermos, E. Feireisl editors, Elsevier, 2005, 461–559.
[12] Monteiro Marques M D P, *Differential Inclusions in Nonsmooth Mechanical Problems - Shocks and Dry Friction*, Birkhauser Verlag, Basel, 1993.
[13] Moreau J J, *Rafle par un convexe variable*, II, Sém. d’Anal. Convexe, Montpellier, Vol. 2 (1972), Exposé. No. 3, 36 pp.
[14] Moreau J J, *Evolution problem associated with a moving convex set in a Hilbert space*, J. Differential Equations 26 (1977), 347–374.
[15] Recupero V, *BV-extension of rate independent operators*, Math. Nachr., 282 (2009), 86–98.
[16] Recupero V, *Sobolev and strict continuity of general hysteresis operators*, Math. Methods Appl. Sci., 32 (2009), 2003–2018.
[17] Recupero V, *BV solutions of rate independent variational inequalities*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 10 (2011), 269–315.
[18] Recupero V, *A continuity method for sweeping processes*, J. Differential Equations 251 (2011) 2125-2142.
[19] Recupero V, *Hysteresis operators in metric spaces*, Discrete Contin. Dyn. Syst. Ser. S, 8 (2015) 773-792.
[20] Serra J, *Hausdorff distances and interpolations*, Mathematical morphology and its applications to image and signal processing (Amsterdam, 1998), 107–114, Comput. Imaging Vision, 12, Kluwer Acad. Publ., Dordrecht, 1998. 68U10 (54E35)
[21] Visintin A, *Differential Models of Hysteresis*, Applied Mathematical Sciences, Vol. 111, Springer-Verlag, Berlin Heidelberg, 1994.