Weight Structures Cogenerated by Weak Cocompact Objects

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Abstract
We study t-structures generated by sets of objects which satisfy a condition weaker than the compactness. We also study weight structures cogenerated by sets of objects satisfying the dual condition. Under some appropriate hypothesis, it turns out that the weight structure is right adjacent to the t-structure.

Keywords Weak compact object · Weak cocompact object · T-structure · Weight structure · Brown representability

Mathematics Subject Classification 18E30 · 16D90 · 55U35

1 Introduction
Denote by $\mathcal{T}$ a triangulated category, and by $\mathcal{T}^\circ$ its dual. If necessary, in this Introduction we assume implicitly that $\mathcal{T}$ has (co)products. An object $G \in \mathcal{T}$ is compact if the covariant hom-functor $\mathcal{T}(G, -)$ preserves coproducts. Let $G \subseteq \mathcal{T}$ be a set of compact objects. Then one can construct the smallest localizing subcategory containing $G$. Recall that a triangulated subcategory of $\mathcal{T}$ is a subcategory which is closed under extensions and shifts, and a (co)localizing subcategory is a triangulated subcategory which is also closed under arbitrary coproducts (products) in $\mathcal{T}$ (all subcategories we consider in this paper are full). The construction of the smallest localizing subcategory can be generalized in various ways. For example, we can consider the smallest aisle containing $G$, that is the smallest subcategory which is closed under extensions, coproducts and only positive shifts (see for example [20, Theorem 6.1]). This aisle is a component of (and determines) a t-structure, as this concept was defined in [2] (the definition is recalled in Sect. 2 below). In this paper we try to formalize and dualize this approach. Since duals of compact objects, that is cocompact ones, rarely exist in triangulated category, in [20] are defined 0-cocompact objects, notion which is obtained by weakening the dual of the definition of compact objects above. Inspired by this approach, we define...
weak compact and weak cocompact objects in \( T \), and consider t-structures generated by a set of weak compact objects, respectively weight structures cogenerated by a set of weak cocompact objects. Note that weight structure are in some sense duals of t-structures; they were independently defined in [4,21], in this last paper being called co-t-structures.

The paper is organized as follows: In Sect. 2 we take an object \( X \in T \), a set of objects \( \mathcal{C} \subseteq T \) and we construct inductively two towers of objects, called a \( \mathcal{C} \)-cophantom respectively a \( \mathcal{C} \)-cocellular tower associated to \( X \). We learned this construction from Beligiannis’s paper [3], but here we performed it in the dual setting, since this case seems to be less known. In this way we “approximate” \( X \) with objects which are constructed inductively from objects in \( \mathcal{C} \). Theorems 2.4 and 2.5 are similar to [20, Theorems 6.6 and 6.1], but in addition they are completed with results concerning Brown representability for \( T^0 \), respectively \( T \); these completions are based on the approximation of an object \( X \) described above (compare with [14, Theorem 1.1]). The main result from Sect. 3 is Theorem 3.5, where there are given hypotheses.

The homotopy category of flat modules considered by Neeman in [19] fulfill some of our symmetric pairs in arbitrary triangulated categories defined in the spirit of Krause’s [8], and (co)compact objects, in order to apply the results of the previous ones. We show that both above (for example the subcategory consisting of all weak compact and weak cocompact objects in \( \mathcal{C} \) in [3], but here we performed it in the dual setting, since this case seems to be less known. In this way we “approximate” \( X \) with objects which are constructed inductively from objects in \( \mathcal{C} \). Theorems 2.4 and 2.5 are similar to [20, Theorems 6.6 and 6.1], but in addition they are completed with results concerning Brown representability for \( T^0 \), respectively \( T \); these completions are based on the approximation of an object \( X \) described above (compare with [14, Theorem 1.1]). The main result from Sect. 3 is Theorem 3.5, where there are given conditions under which the t-structure generated by a set of weak compact objects is left adjacent to the weight structure cogenerated by a set of weak cocompact objects. The key observation here is that if \( G \) is any object, then the covariant hom-functors \( T(G, -) \) behaves well with respect to inverse towers and homotopy limits involved in the construction of \( \mathcal{C} \)-cophantom and \( \mathcal{C} \)-cocellular towers. In Sect. 4 we focus on finding examples of weak (co)compact objects, in order to apply the results of the previous ones. We show that both symmetric pairs in arbitrary triangulated categories defined in the spirit of Krause’s [8], and the set of complexes which generates the homotopy category of projective modules inside the homotopy category of flat modules considered by Neeman in [19] fulfill some of our hypotheses.

2 Cophantom and Cocellular Towers

In this paper \( T \) will denote always a triangulated category, whose suspension functor is denoted by \( \Sigma \). If it is convenient, we write a triangle \( X \to Y \to Z \to \Sigma X \), shortly as \( X \to Y \to Z \to \Sigma X \). Let \( S \subseteq T \) be a subcategory of \( T \); its left and right orthogonal \( \perp S \) and \( S^\perp \) are by definition the (full) subcategories of \( T \) having as objects those \( X \) for which \( T(X, S) = 0 \), respectively \( T(S, X) = 0 \) for all \( S \in S \). We denote by \( \Sigma^\geq 0 S = \bigcup_{n \geq 0} \Sigma^n S \), \( \Sigma^\leq 0 S = \bigcup_{n \leq 0} \Sigma^n S \), respectively \( \Sigma^\geq 0 S = \bigcup_{n \geq 0} \Sigma^n S \), \( \Sigma^\leq 0 S = \bigcup_{n \leq 0} \Sigma^n S \) the completion of \( S \) under positive shifts, negative shifts respectively all shifts. More generally, if \( K \subseteq \mathbb{Z} \) then we denote \( \Sigma^K S = \bigcup_{k \in K} \Sigma^k S \). Note that if \( S \) is a small set, the same is true for these completions.

Among others, these notations allow us to use only the left and right orthogonal defined above (for example the subcategory consisting of all \( X \in T \) for which \( T(\Sigma^n X, S) = 0 \) for all \( n \leq 0 \) and all \( S \in S \) is equal to \( \Sigma^\leq 0 S \), so we don’t need to index the symbol \( \perp \), how is the usage in other papers, e. g. [20]). It is also clear that \( \Sigma^\geq 0 S \subseteq \Sigma^\geq 0 S \), \( \Sigma^\leq 0 S \subseteq \Sigma^\leq 0 S \) and \( \Sigma^\geq 0 S \subseteq \Sigma^\geq 0 S \). We say that \( S \) (co)generates \( T \) if for an object \( X \in T \) the equality \( T(S, X) = 0 \) (respectively \( T(S, X) = 0 \) for all \( S \in \Sigma^\geq 0 S \)) implies \( X = 0 \). Recall also that a \( S \)-precover of an object \( X \in T \) is a map \( S_X \to X \) with \( S_X \in S \), such that the induced map \( T(S, S_X) \to T(S, X) \) is surjective for all \( S \in S \). Dually we define an \( S \)-preenvelope.

Recall that a torsion pair in \( T \) is a pair \((\mathcal{X}, \mathcal{Y})\) of (full) subcategories, such that \( T(X, Y) = 0 \) for all \( X \in \mathcal{X} \) and all \( Y \in \mathcal{Y} \) and for all \( T \in T \) there is a triangle \( X \to T \to Y \to \Sigma X \), with \( X \in \mathcal{X} \) and \( Y \in \mathcal{Y} \). Further, this torsion pair is called a t-structure, respectively a weight structure (or w-structure for short), provided that \( \Sigma \mathcal{X} \subseteq \mathcal{X} \) (or, equivalently, \( \mathcal{Y} \subseteq \Sigma \mathcal{Y} \)), respectively \( \mathcal{X} \subseteq \Sigma \mathcal{X} \) (or, equivalently, \( \Sigma \mathcal{Y} \subseteq \mathcal{Y} \)). If \((\mathcal{X}, \mathcal{Y})\) is a t-structure (w-structure),
then the triangle \( X \to T \to Y \to \) , with \( X \in \mathcal{X} \) and \( Y \in \mathcal{Y} \) is called the \textit{t-decomposition} (respectively \textit{a w-decomposition}) of \( T \in T \). The difference between the definite, respectively indefinite article used before is explained by the fact that in the case of a \( t \)-structure this triangle is functorial in \( T \in T \) (see [2, Proposition 1.3.3]), but is not at all unique in the case of a \( w \)-structure (see [5, Remark 2.2.3]). Moreover if \((\mathcal{X}, \mathcal{Y})\) is a \( t \)-structure, then the inclusion functors \( \mathcal{X} \to T \) and \( \mathcal{Y} \to T \) have a right, respectively left adjoint which are defined by the assignments \( T \mapsto X \) and \( T \mapsto Y \), where \( X \to T \to Y \to \) is the t-decomposition of \( T \). A \( t \)-structure \((\mathcal{X}, \mathcal{Y})\) is is called \textit{stable} if \( \Sigma \mathcal{X} = \mathcal{X} \) (or, equivalently, \( \mathcal{Y} = \Sigma \mathcal{Y} \)). In this case \( \mathcal{X} \) is a localizing subcategory, and \( \mathcal{Y} \) is a colocalizing subcategory of \( T \). Note that the notions of stable \( t \)-structure and stable \( w \)-structure coincide. Given two torsion pairs \((\mathcal{X}, \mathcal{Y})\) to \(\mathcal{Y} \to T \) functors \( X \to \), where the \( j \)-th component of \( X \to \mathcal{X} \) consists of all direct factors of products of objects in \( \mathcal{X} \), we denote by Prod \( C \in C \). Observe that a map in \( T \) is \( C \)-cphantom if and only if it is Prod \( C \)-cphantom. Denote

\[
\Psi(C) = \{ \psi : X \to Y \mid \psi \text{ is a } C \text{-cphantom} \},
\]

and observe that \( \Psi(C) \) is an ideal in \( T \), that is closed under sums (of maps), and under compositions with arbitrary maps.

A diagram in \( T \) (indexed over \( \mathbb{N} \)) of the form

\[
X_0 \xleftarrow{\xi_0} X_1 \xleftarrow{\xi_1} X_2 \leftarrow \cdots
\]

is called an \textit{inverse tower}. A dual diagram is called a \textit{direct tower}. Sometimes we skip the adjectives “inverse” or “direct” when they are clear from the context. An inverse tower is called \textit{C-cphantom tower} if all its connecting maps are \( C \)-cphantoms. Recall that the homotopy limit of the inverse tower above is defined (up to a non–canonical isomorphism) by the triangle

\[
\text{holim } X_i \longrightarrow \prod_{i \in \mathbb{N}} X_i \xrightarrow{1-\xi} \prod_{i \in \mathbb{N}} X_i \xrightarrow{+},
\]

where the \( j \)-th component of \( \xi \) is \( \prod_{i \in \mathbb{N}} X_i \to X_{j+1} \xrightarrow{\xi_j} X_j \) (see [15, dual of Definition 1.6.4]).

\textbf{Lemma 2.1} \textit{If} \( C \) \textit{is a set of objects, then every object} \( X \in T \) \textit{has a Prod}(\( C \))-\textit{preenvelope}. \hfill \Box

\textbf{Proof} \ We put \( C_X = \prod_{C \in C} C^{\mathcal{T}(C, X)} \) and the Prod\( (C)\)-preenvelope of \( X \) is the unique map \( X \to C_X \) whose composition with the \( \alpha \)-th projection \( C_X \to C \) is equal to \( \alpha \), for all \( \alpha \in \mathcal{T}(C, X) \) and all \( C \in C \).
Construction 2.2 [3, Dual of 5.1]. Let $C$ be a set of objects, and let $X \in \mathcal{T}$. Associated to this data we will construct an inverse tower as following: Put $Y_0 = X$, and if $Y_i$ with $i \geq 0$ is given, then we construct inductively $Y_{i+1}$ via the triangle

$$Y_{i+1} \xrightarrow{\psi_i} Y_i \rightarrow C_i \rightarrow$$

where $Y_i \rightarrow C_i$ is the Prod($C$)-preenvelope of $Y_i$. Note that provided that $Y_i$ is given, even if we construct the preenvelope in a fixed manner as in Lemma 2.1, the object $Y_{i+1}$ is determined only up to a non-canonical isomorphism. By the very definition of a preenvelope, we know that for all $C \in C$ the second morphism in the following exact sequence

$$T(Y_{i+1}, C) \leftarrow T(Y_i, C) \leftarrow T(C_i, C)$$

is surjective, therefore $- \circ \psi_i : T(Y_i, C) \rightarrow T(Y_{i+1}, C)$ vanishes. Therefore all connecting maps of the tower

$$Y_0 \xrightarrow{\psi_0} Y_1 \xleftarrow{\psi_1} Y_2 \leftarrow \cdots$$

are $C$-cophantoms, justifying the name we will give to it: a $C$-cphantom tower associated to $X$. Denote $Y_\infty = \text{holim} Y_i$.

For a set of objects $C \subseteq \mathcal{T}$ we define inductively $\text{Prod}_0(C) = \{0\}$ and $\text{Prod}_{n+1}(C)$ is the full subcategory of $\mathcal{T}$ which consists of all objects $Y$ lying in a triangle $X \rightarrow Y \rightarrow Z \rightarrow$ with $X \in \text{Prod}(C)$ and $Z \in \text{Prod}_n(C)$. Clearly $\text{Prod}_1(C) = \text{Prod}(C)$ and the construction leads to an ascending chain of subcategories $\{0\} = \text{Prod}_0(C) \subseteq \text{Prod}_1(C) \subseteq \text{Prod}_2(C) \subseteq \cdots$.

From [17, Remark 07] we learn that if $X \rightarrow Y \rightarrow Z \rightarrow$ is a triangle with $X \in \text{Prod}_n(C)$ and $Z \in \text{Prod}_m(C)$ then $Y \in \text{Prod}_{n+m}(C)$. Moreover if $C$ is supposed to be closed under (de)suspensions, then the same is true for $\text{Prod}_n(C)$, (see [17, Remark 07] again). An object $X \in \mathcal{T}$ is called $C$-cofiltered if it can be written as a homotopy limit $X \cong \text{holim} X_n$ of an inverse tower $X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots$, such that $X_0 = 0$ and the completion of $X_{n+1} \rightarrow X_n$ leads to a triangle $C_n \rightarrow X_{n+1} \rightarrow X_n \rightarrow$, with $C_n \in \text{Prod}(C)$. If this is the case, then it is easy to see inductively that $X_n \in \text{Prod}_n(C)$, for all $n \in \mathbb{N}$, and the inverse tower above is called a $C$-cofiltration of $X$.

Construction 2.3 [3, Dual of 5.2]. Fix a $C$-cphantom tower associated to $X$, and use the notations from Construction 2.2. Put $\psi^0 = 1_X : Y_0 \rightarrow X$ and define inductively $\psi^{i+1} = \psi^i \psi_i : Y_{i+1} \rightarrow X$, for all $i \geq 0$. Use the octahedral axiom in order to complete commutatively the diagram
with the dotted triangle in the second column. Observe that \( Y^0 = 0 \) and the objects \( Y^i \) for \( i > 0 \) are determined again only up to a non-canonical isomorphism. Call

\[
0 = Y^0 \xleftarrow{\delta_0} Y^1 \xleftarrow{\delta_1} Y^2 \xleftarrow{\delta_2} \cdots
\]

a \( C \)-cocellular tower associated to \( X \) (and to the initial \( C \)-cophantom tower). Denote \( Y^\infty = \underleftarrow{\text{holim}} Y^i \). The existence of the dotted triangle in the diagram above implies that the \( C \)-cocellular tower associated to \( X \) is a \( C \)-cofiltration of \( Y^\infty \).

It is an easy observation to see that if \( C \in T \) is a compact object then the covariant hom functor \( T(C,-) \) sends homotopy limits (of direct towers) into ordinary colimits. If we want to dualize the compactness we should ask that the contravariant hom functor \( T(-,C) \) sends products into coproducts and, consequently, it sends homotopy limits (of inverse towers) to colimits. But this condition is happen to be too restrictive, in many usual triangulated categories the only cocompact object being 0. Therefore we will work with a weaker notion as follows: We say that an object \( C \in T \) is weak cocompact with respect to an inverse tower

\[
X_0 \xleftarrow{\xi_0} X_1 \xleftarrow{\xi_1} X_2 \xleftarrow{\xi_2} \cdots
\]

if the equalities \( T(\xi_i, C) = 0 \), for all \( i \geq 0 \), imply \( T(X^\infty, C) = 0 \), where \( X^\infty = \underleftarrow{\text{holim}} X_i \).

Remark that if \( T(\xi_i, C) = 0 \), for all \( i \geq 0 \), then obviously \( \text{colim} T(X_i, C) = 0 \), so for this particular tower the contravariant hom functor \( T(-,C) \) sends its homotopy limit \( X^\infty \) into the colimit of the induced direct tower of abelian groups. Remark also that if the tower above is supposed to be \( \Sigma^K C \)-cocellular, with \( 0 \in K \subseteq \mathbb{Z} \), then automatically \( T(\xi_i, C) = 0 \), for all \( i \geq 0 \), hence \( C \) is weak compact with respect to that tower means exactly that \( T(X^\infty, C) = 0 \).

In particular, if \( 0 \in K \subseteq K' \subseteq \mathbb{Z} \), then the condition \( C \) is weak cocompact with respect to any \( \Sigma^K C \)-cophantom tower is stronger that the hypothesis \( C \) is weak cocompact with respect to any \( \Sigma^K' C \)-cophantom tower, since every \( \Sigma^K' C \)-cophantom tower is a \( \Sigma^K C \)-cophantom tower, but the converse is not true in general.

We are in position to state the main result of this section. As we will see in Lemma 4.1 below, if the objects \( C \) are 0-cocompact, as in the hypothesis of [20, Theorem 6.6], then they are weak cocompact with respect to any \( \Sigma^{\geq 0} C \)-cophantom tower, therefore the first two conclusions of the next Theorem are (slight) generalizations of (i) and (ii) in the cited result (even the proof is almost identical, with a small exception, explained below). The real improvement seems to be conclusion (3), where we learn that we can apply a result from [11] in order to deduce Brown representability for the dual of the triangulated category \( T \).

**Theorem 2.4** Let \( C \) be a set of objects in a triangulated category with products \( T \), with the property every \( C \in C \) is weak cocompact with respect to any \( \Sigma^{\geq 0} C \)-cophantom tower. Then:

1. \( (\bot \Sigma^{\geq 0} C, (\bot \Sigma^{\geq 0} C)^\perp) \) is a w-structure in \( T \).
2. \( (\bot \Sigma^Z C, (\bot \Sigma^Z C)^\perp) \) is a stable w-structure (=stable t-structure) in \( T \) and \( (\bot \Sigma^Z C)^\perp \) is the smallest colocalizing subcategory of \( T \) containing \( C \).
3. If \( T \) is the base of a strong stable derivator and \( C \) cogenerates \( T \), then \( T^o \) satisfies Brown representability.

**Proof** For proving (1) let \( Y_0 \xleftarrow{\psi_0} Y_1 \xleftarrow{\psi_1} Y_2 \xleftarrow{\psi_2} \cdots \) be a \( \Sigma^{\geq 0} C \)-cophantom tower associated to an object \( X \in T \), and let \( Y^\infty = \underleftarrow{\text{holim}} Y_i \). Then its connecting maps are \( \Sigma^{\geq 0} C \)-cophantoms, and the same is true for all negative shifts of this tower. Therefore by hypotheses we get \( T(\Sigma^n Y^\infty, C) = 0 \), for all \( n \leq 0 \) and all \( C \in C \). Next we apply the same argument as
for (i) of [20, Theorem 6.6], the only modification being the use of the argument above for showing that $Y_\infty \in \perp^\perp$. For (2) we replace in (1) the set $C$ by $\Sigma^{\leq 0}C$. Then obviously $\Sigma^{\leq 0}(\Sigma^{\leq 0}C) = \Sigma^{\leq 0}C$. Noting that if $C$ is weak cocompact with respect to a given tower, then for all $n \in \mathbb{Z}$, the object $\Sigma^n C$ is weak compact with respect to that tower shifted by $n$, it is routine to check that every object in $\Sigma^{\leq 0}C$ is weak cocompact with respect to any inverse tower whose connecting maps are $\Sigma^{\leq 0}C$-cophantoms. The last statement of (2) is proved in (ii) of [20, Theorem 6.6].

For the proof of (3) we use the stable w-structure constructed in (2). Note that in the proof of above cited result, the w-decomposition associated to an $X \in T$ (with respect to the w-structure whose existence is stated in (2) or, mutatis mutandis, in (1) too) is obtained as follows: The maps $\psi : Y_i \to X$ considered in Construction 2.3 induce a map $Y_\infty \to X$. Complete this map to a triangle

$$Y_\infty \to X \to \overline{Y} \to .$$

One can show that $\overline{Y}$ lies in the right side term of the obtained w-structure, that is $\overline{Y} \in (\perp \Sigma^{\leq 0}C)^{\perp}$ in our case, respectively in $\overline{Y} \in (\perp \Sigma^{\leq 0}C)^{\perp}$ in the case (1). We do not recall precisely what a strong stable derivator is, since the unique feature of this notion we need is the result in [6, Corollary 11.4]: If $T$ is the base of a strong stable derivator, then taking homotopy limits of the middle two rows of the diagram in Construction 2.3, we can choose the maps such that we get a triangle

$$Y_\infty \to X \to \Sigma Y_\infty \to$$

therefore $\overline{Y} \cong \Sigma Y_\infty$. We have by construction $T(Y_\infty, C) = 0$ for all $C \in \Sigma^{\leq 0}C$, and using the hypothesis that $C$ cogenerates $T$, it follows that $Y_\infty = 0$. Thus the map $X \to \Sigma Y_\infty$ is an isomorphism. By construction, $Y_\infty$ is $C$ cofiltered, therefore every object in $T$ is $C$-cofiltered. Finally we get the conclusion by applying [11, Theorem 8] which says that $T^0$ satisfies Brown representability provided that every object is cofiltered by a fixed set. $\square$

We say that the w-structure $(\perp \Sigma^{\leq 0}C, (\perp \Sigma^{\leq 0}C)^{\perp})$ from Theorem 2.4 is cogenerated by $C$. In particular we can reformulate the fact that $C$ cogenerates $T$ by saying the $(0, T)$ is the stable t-structure cogenerated by $C$.  

2.2 The t-Structure Induced by a Set

In this subsection we dualize all constructions and all results from the preceding one. Thus starting with a set of objects $G$ in a triangulated category with coproducts $T$, we consider the ideal of $G$-phantom maps

$$\Phi(G) = \{ \phi : X \to Y \mid T(G, X) \xrightarrow{\phi} T(G, Y) \text{ vanishes for all } G \in G \}.$$ 

A direct tower whose connecting maps are $G$-phantoms is called a $G$-phantom tower.

We can construct two (direct) towers associated to an object $X$ as follows: For $X \in T$ we construct inductively a tower of the form

$$Z_0 \xrightarrow{\phi_0} Z_1 \xrightarrow{\phi_1} Z_2 \xrightarrow{} \cdots ,$$

by $Z_0 = X$, and $G_i \to Z_i \xrightarrow{\phi_i} Z_{i+1}$ is obtained by the completion to a triangle of an Add($G$)-precover of $Z_i$, for $i \geq 0$. Here Add($G$) is the subcategory consisting of all direct summands of arbitrary direct sums of objects in $G$. This is called a $G$-phantom tower.
associated to $X$. Further we construct a $G$-cellular tower by completion of the following diagram with the triangle in the dotted line:

\[
\begin{array}{ccc}
G_i & \xrightarrow{\phi^i} & G_i \\
\downarrow & & \downarrow \\
X & \xrightarrow{\phi_i} & Z_i \\
\downarrow & \downarrow & \downarrow \\
X & \xrightarrow{\phi_{i+1}} & Z_{i+1} \\
\downarrow & \downarrow & \downarrow \\
\Sigma G_i & \xrightarrow{\gamma_i} & \Sigma G_i \\
\end{array}
\]

Denote by $Z_\infty = \hocolim Z_i$ and $Z^\infty = \hocolim Z^i$. Then $Z^\infty$ is $G$-filtered, (in the dual sense of above, when we said that $Y^\infty$ is $C$-cofiltered).

An object $G \in T$ is called weak compact with respect to a direct tower

\[
X_0 \xrightarrow{\xi_0} X_1 \xrightarrow{\xi_1} X_2 \to \ldots
\]

if $T(G, \hocolim X_i) = 0$, provided that all $\xi_i$ are $\{G\}$-phantoms. As one can expect, compactness implies weak compactness, since if the object $G$ is compact, then for the tower above we have $\colim T(G, X_i) \cong T(G, \hocolim X_i)$. Finally we record the dual of Theorem 2.4:

**Theorem 2.5** Let $G$ be a set of objects in a triangulated category with coproducts $T$ with the property every $G \in G$ is weak compact with respect to any $\Sigma^{\geq 0} G$-phantom tower. Then:

1. $\left(\perp (\Sigma^{\geq 0} G^\perp), \Sigma^{\geq 0} G^\perp\right)$ is a t-structure in $T$.
2. If $\left(\perp (\Sigma G^\perp), \Sigma G^\perp\right)$ is a stable t-structure in $T$ and $\left(\perp \Sigma G^\perp\right)$ is the smallest localizing subcategory of $T$ containing $G$.
3. If $T$ is the base of a strong stable derivator and $G$ generates $T$, then $T$ satisfies Brown representability.

The t-structure $\left(\perp (\Sigma^{\geq 0} G^\perp), \Sigma^{\geq 0} G^\perp\right)$ from the previous Theorem is said to be generated by $G$. For avoiding unnecessary brackets, we adopt the convention that $\Sigma$ is granted a higher precedence than $\perp$, that is $\Sigma^{\geq 0} G^\perp$ should be automatically read as $(\Sigma^{\geq 0} G)^\perp$.

### 3 When the $C$-Cogenerated $w$-Structure is Right Adjacent to the $G$-Generated t-Structure

In the beginning of this section, recall that an abelian category is said to satisfy condition AB3 if it has coproducts (or equivalently it is cocomplete). That category satisfies AB4 if, in addition, coproducts are exact. Dually an abelian category satisfies AB3$^*$ (AB4$^*$) if it has (exact) products.

**Remark 3.1** [15, Remark A.3.6] Let

\[
A_0 \xleftarrow{\alpha_0} A_1 \xleftarrow{\alpha_1} A_2 \leftarrow \ldots
\]
be an inverse tower in an abelian \( \text{AB}4^* \) category then denote by \( \lim_{\leftarrow}^{(n)} A_i \) the \( n \)-th derived functor of the inverse limit functor. Then \( \lim_{\leftarrow}^{(n)} A_i = 0 \) for \( n > 1 \) and for \( n \in \{0, 1\} \) the respective functors are computed by the short exact sequence:

\[
0 \to \lim A_i \to \prod A_i \xrightarrow{1-\alpha} \prod A_i \to \lim_{\rightarrow}^{(1)} A_i \to 0,
\]

where \( \alpha \) is induced by \( \alpha_i \) with \( i \geq 0 \).

Recall that a functor \( H : \mathcal{T} \to \mathcal{A} \) defined on a triangulated category \( \mathcal{T} \) with values in an abelian category \( \mathcal{A} \) is called homological if it sends triangles into (long) exact sequences. We call cohomological a contravariant homological functor. The next Lemma is a technical one and it express the fact that a homological product preserving functor, as for example the representable functor \( \mathcal{T}(G, -) \) for some \( G \in \mathcal{T} \), behaves well with respect to the diagram used in Construction 2.3 of the cocellular tower associated to an object \( X \in \mathcal{T} \).

**Lemma 3.2** Let \( H : \mathcal{T} \to \mathcal{A} \) be a homological functor into an abelian \( \text{AB}4^* \) category, and let

\[
\begin{array}{ccc}
\Sigma^{-1}X & \longrightarrow & Y^{i+1} \\
\downarrow & \delta_i & \downarrow \psi_i \\
\Sigma^{-1}X & \longrightarrow & Y^i \\
\end{array}
\]

be a diagram in \( \mathcal{T} \) whose rows are triangles. Denote by \( Y^\infty = \mathrm{holim} \ Y^i \).

(a) If \( H(\psi_i) = 0 = H(\Sigma^{-1} \psi_i) = 0 \) for all \( i \geq 0 \), then \( \lim_{\leftarrow} H(Y^i) \cong H(\Sigma^{-1} X) \) and \( \lim_{\rightarrow}^{(1)} H(Y^i) = 0 \).

(b) If, in addition, \( H(\Sigma^{-2} \psi_i) = 0 \) for all \( i \geq 0 \) and \( H \) preserves products, then the map \( H(\Sigma^{-1} X) \to H(Y^\infty) \) is an isomorphism.

**Proof** (a). Applying the homological functor \( H \) to the given diagram we get a commutative diagram with exact rows (consisting of solid arrows):

\[
\begin{array}{ccc}
H(\Sigma^{-1} Y_{i+1}) & \longrightarrow & H(\Sigma^{-1} X) \\
0 & \downarrow & \delta_i \downarrow \psi_i \\
H(\Sigma^{-1} Y_i) & \longrightarrow & H(\Sigma^{-1} X) \\
\end{array}
\]

\[
\begin{array}{ccc}
H(Y^{i+1}) & \longrightarrow & H(Y^i) \\
H(\Sigma^{-1} Y_{i+1}) & \longrightarrow & H(Y^i) \\
0 & \downarrow & H(\psi_i) \\
H(\Sigma^{-1} Y_i) & \longrightarrow & H(Y^i) \\
\end{array}
\]

\[
\begin{array}{ccc}
H(X) & \longrightarrow & H(Y^i) \\
H(\Sigma^{-1} Y_{i+1}) & \longrightarrow & H(Y^i) \\
0 & \downarrow & H(\psi_i) \\
H(\Sigma^{-1} Y_i) & \longrightarrow & H(Y^i) \\
\end{array}
\]

The hypothesis says that the first and the fourth vertical maps are 0, therefore both maps \( H(\Sigma^{-1} Y_{i+1}) \to H(\Sigma^{-1} X) \) and \( H(Y_{i+1}) \to H(X) \) vanish. Thus for \( i \geq 1 \) the diagram above is actually a map between short exact sequences, hence the factorization through the kernel produces the dotted arrow which makes the whole diagram commutative. This implies that the inverse tower

\[
H(Y^2) \leftarrow H(Y^3) \leftarrow H(Y^4) \leftarrow \cdots,
\]

starting with \( i = 2 \), is the direct sum between the towers

\[
H(Y_2) \xleftarrow{0} H(Y_3) \xleftarrow{0} H(Y_4) \leftarrow \cdots \text{ and } H(\Sigma^{-1} X) \xleftarrow{0} H(\Sigma^{-1} X) \xleftarrow{0} \cdots
\]

and this proves the conclusion.
(b). Now \( H(\Sigma^n \phi_i) = 0 \) for \( n = 0, -1, -2 \), therefore we can apply (a), both to \( H \) and to \( H \circ \Sigma^{-1} \). We get \( \text{lim} H(Y^i) \cong H(\Sigma^{-1} X) \), \( \text{lim} H(\Sigma^{-1} Y^i) \cong H(\Sigma^{-2} X) \) and \( \text{lim}^{(1)} H(Y^i) = 0 \), \( \text{lim}^{(1)} H(\Sigma^{-1} Y^i) \). Since \( \mathcal{A} \) is AB4\(^*\) the exact sequence mentioned in Remark 3.1 becomes:

\[
0 \to \text{lim} H(Y^i) \to \prod_i H(Y^i) \to \prod_i H(Y^i) \to 0,
\]

and similar for \( \text{lim} H(\Sigma^{-1} Y^i) \). On the other hand, \( H \) preserves products, therefore applying it to the triangle which define the homotopy limit \( Y^\infty \) we get a commutative diagram

\[
\begin{array}{c}
\prod_i H(\Sigma^{-1} Y^i) \to \prod_i H(\Sigma^{-1} Y^i) \to 0 \\
\downarrow \cong \quad \quad \downarrow \cong \\
H(\Sigma^{-1} \prod_i Y^i) \to H(\Sigma^{-1} \prod_i Y^i) \to H(Y^\infty) \to H(\prod_i Y^i) \to H(\prod_i Y^i) \\
\downarrow \cong \quad \quad \downarrow \cong \\
0 \quad \quad \quad \text{lim} H(Y^i) \to \prod_i H(Y^i) \to \prod_i H(Y^i)
\end{array}
\]

which proves the conclusion. \( \square \)

**Remark 3.3** As we already mentioned, we are interested mainly in representable functors. With this in mind, we can dualize Lemma 3.2 as follows: If \( T \) has coproducts, \( H : T \to \mathcal{A} \) is a homological coproduct preserving functor into an abelian AB4 category (e.g. \( H = \mathcal{T}(G, -) \)) for a compact object \( G \in T \), then \( H^o : T^o \to A^o \) preserves products, therefore if

\[
\begin{array}{c}
X \rightarrow Z_i \rightarrow Z^i \rightarrow \Sigma X \\
\downarrow \phi_i \quad \quad \downarrow \gamma_i \\
X \rightarrow Z_{i+1} \rightarrow Z^{i+1} \rightarrow \Sigma X
\end{array}
\]

is a commutative diagram such that \( H(\Sigma^n \phi_i) = 0 \) for all \( i \geq 0 \) and for \( n \in \{-1, 0, 1\} \), then \( H(X) \cong \text{colim} H(\Sigma^{-1} Z^i) \cong H(\Sigma^{-1} Z^\infty) \), where \( Z^\infty = \text{hocolim} Z^i \). A similar conclusion can be drawn if we start with a cohomological (contravariant) functor \( H : T \to \mathcal{A} \), which sends coproducts into products (e.g. \( H = \mathcal{T}(\cdot, C) \) for some \( C \in T \)); in this case \( T \) has to have coproducts, and \( \mathcal{A} \) has to be AB4\(^*\): If \( H(\Sigma^n \phi_i) = 0 \) for \( n \in \{-1, 0, 1\} \) and all \( i \geq 0 \), then \( H(X) \cong \text{lim} H(\Sigma^{-1} Z^i) \cong H(\Sigma^{-1} Z^\infty) \). Sure, if we apply a cohomological functor which sends products into coproducts to the original diagram considered in Lemma 3.2, then the argument works formally. But we don’t have concrete examples of such functors.

The idea behind the next lemma is that if \( G \) and \( C \) are a two set of objects, such that every \( C \)-cophantom is also \( \Sigma^n G \)-phantom for some integers \( n \) around 0, then the representable functors \( \mathcal{T}(G, -) \) satisfies the hypothesis of Lemma 3.2. Using the fact that the homotopy limit of a tower coincide with the homotopy limit of a subtower, we can weaken the hypothesis, by asking that only a composition of \( s \in \mathbb{N}^* \) successive \( C \)-cophantoms is a \( \Sigma^n G \)-phantom. For an ideal \( I \) in \( T \), denote by \( I^s \) the ideal generated by all compositions of \( s \) successive maps in \( I \).

**Lemma 3.4** Let \( T \) be the base of a strong stable derivator and let \( G \) and \( C \) be two set of objects.
If \( T \) has products, all objects in \( C \) are weak cocompact with respect to any \( \Sigma^{\geq 0}C \)-cophantom tower and \( \Psi(C)^s \subseteq \Phi(\Sigma^{[-1,0,1,2]} G) \) for some \( s \geq 1 \), then \( \perp \Sigma^{\geq 0}C \subseteq \Sigma^{\geq 0}G^{\perp} \).

(2) If \( T \) has coproducts, all objects in \( G \) are weak compact with respect to any \( \Sigma^{\geq 0}G \)-phantom tower and \( \Phi(G)^t \subseteq \Psi(\Sigma^{[-2,1,0,1]} C) \) for some \( t \geq 1 \), then \( \Sigma^{\geq 0}G^{\perp} \subseteq \perp \Sigma^{\geq 0}C \).

**Proof** (1). First we claim that if \( Y_{\infty} \) is the homotopy colimit of a \( \Sigma^{\geq 0}C \)-cophantom tower associated to an arbitrary object \( X \in T \), then \( Y \in \Sigma^{\geq 0}G^{\perp} \). In order to show this, we start by pasting together \( s \) successive diagrams obtained by Construction 2.2 in order to get the commutative diagram

\[
\begin{array}{c}
\Sigma^{-1}X \twoheadrightarrow Y_{i+s} \twoheadrightarrow Y_{i+s} \twoheadrightarrow X \\
\downarrow \downarrow \downarrow \\
\Sigma^{-1}X \twoheadrightarrow Y_i \twoheadrightarrow Y_i \twoheadrightarrow X
\end{array}
\]

Then \( \phi \in \Psi(C)^s \subseteq \Phi(\Sigma^{[-1,0,1,2]} G) \), therefore \( T(\Sigma^{-n}G, \phi) = T(G, \Sigma^n \phi) = 0 \) for \( n \in \{-1,0,1,2\} \) and \( G \in G \) arbitrary chosen. This means precisely that both functors

\[ T(G, -), T(\Sigma^{-1}G, -) : T \to Ab \]

fulfill the hypothesis of Lemma 3.2 relative to the above diagram. Since the homotopy limit of a subtower is the same as those of the whole tower, see \([15, \text{Lemma 1.7.1}]\), we get isomorphisms \( T(G, \Sigma^{-1}X) \cong T(G, Y_{\infty}) \) and \( T(\Sigma^{-1}G, \Sigma^{-1}X) \cong T(\Sigma^{-1}G, Y_{\infty}) \), therefore \( T(G, \Sigma^{-1}Y_{\infty}) = 0 \). Now the replacement of \( G \) by \( \Sigma^n G \) \( (n \geq 0) \) in the argument above is the same as we would shift with \(-n\) the diagram, hence we get the same conclusion \( T(\Sigma^n G, Y_{\infty}) = 0 \) for all \( G \in G \) and all \( n \geq 0 \), which proves our claim.

Next remark that the hypotheses on \( C \) assure us that \((\perp \Sigma^{\geq 0}C, (\perp \Sigma^{\geq 0}C)^{\perp})\) is a w-structure on \( T \) (cf. Theorem 2.4), and for \( X \in T \) a w-decomposition is \( Y_{\infty} \to X \to \Sigma Y_{\infty} \to \). Therefore if we start with \( X \in \perp \Sigma^{\geq 0}C \), then \( X \) is a direct summand of \( Y_{\infty} \) and, according to the claim above, \( X \in \Sigma^{\geq 0}G^{\perp} \).

(2). This statement is the dual of (1). \( \square \)

Next we keep the hypothesis made in the previous lemma and we want to apply it order to get a cosuspended TTF triple in \( T \) having at the left the t-structure generated by \( G \) and at the right the w-structure generated by \( C \) (in Theorem 3.5 (1)). Note that the same conclusion follows from other hypotheses, but we don’t know the precise relationship between the two sets of conditions (in Theorem 3.5 (2)). However in both cases, the hypotheses are inspired from the examples contained in the next section (see Theorem 4.5 and Example 4.9):

**Theorem 3.5** Let \( T \) be the base of a strong stable derivator and let \( G \) and \( C \) be two set of objects. Suppose that \( T \) has both products and coproducts, and all objects in \( G \) are weak compact with respect to any \( \Sigma^{\geq 0}G \)-phantom tower.

(1) If, in addition, all objects in \( C \) are weak cocompact with respect to any \( \Sigma^{\geq 0}C \)-cophantom tower, \( \Psi(C)^s \subseteq \Phi(\Sigma^{[-1,0,1,2]} G) \) for some \( s \geq 1 \) and \( \Phi(G)^t \subseteq \Psi(\Sigma^{[-2,1,0,1]} C) \) for some \( t \geq 1 \), then

\[
(\perp (\Sigma^{\geq 0}G^{\perp}), \Sigma^{\geq 0}G^{\perp} = \perp \Sigma^{\geq 0}C, (\perp \Sigma^{\geq 0}C)^{\perp})
\]

is a cosuspended TTF triple in \( T \).
(2) If, in addition, $\Psi(C)^s \subseteq \Phi(\Sigma^{s-1} G)$ for some $s \geq 1$ and $C \subseteq (\Sigma^{\geq 0} G)^\perp$, then $C$ cogenerates a $w$-structure which is the right adjacent of the $G$-generated $t$-structure, as in the case (1) above.

**Proof** The statement (1) follows immediately from of Theorems 2.4, 2.5 and Lemma 3.4. (2). Note that the hypothesis on $G$ assures us that $(\perp (\Sigma^{\geq 0} G), \Sigma^{\geq 0} G^\perp)$ is a $t$-structure (cf. Theorem 2.5). Moreover $(\Sigma^{\geq 0} G^\perp)$ is closed under positive shifts, therefore $C \subseteq (\Sigma^{\geq 0} G^\perp)$ implies $\Sigma^{\geq 0} C \subseteq (\Sigma^{\geq 0} G^\perp)^\perp$, and we deduce $\Sigma^{\geq 0} G^\perp \subseteq \perp \Sigma^{\geq 0} C$. If $Y_\infty$ is the homotopy colimit of a $\Sigma^{\geq 0} C$-cophantom tower associated to an arbitrary object $X \in T$, then we have showed in the proof of Lemma 3.4 that $Y_\infty \in \Sigma^{\geq 0} G^\perp$, thus $Y_\infty \in \perp \Sigma^{\geq 0} C$. By the argument of [20, Theorem 6.6] already used in Theorem 2.4, we learn that $(\perp \Sigma^{\geq 0} C, (\perp \Sigma^{\geq 0} C)^\perp)$ is a $w$-structure. We also know that a $w$-decomposition of $X \in T$ with respect to this $w$-structure is $Y_\infty \rightarrow X \rightarrow \Sigma Y_\infty \rightarrow \ldots$. Thus if $X \in \perp \Sigma^{\geq 0} C$ then $X \in \Sigma^{\geq 0} G^\perp$ as a direct summand of $Y_\infty$, which proves the converse inclusion $\perp \Sigma^{\geq 0} C \subseteq \Sigma^{\geq 0} G^\perp$. □

### 4 Weak Compact and Weak Cocompact Objects

The aim of this section is to produce examples of sets of objects $G$ and $C$ which are weak compact, respectively weak cocompact with respect to appropriate direct, respective inverse towers. The first example is taken from [20], which is one of the most important source of inspiration for the present paper. We can see that the $0$-cocompact objects defined in [20] are weak cocompact (with respect to certain towers) in our sense. Dualizing the well-known notion of a Mittag-Leffler inverse tower, we say that a direct tower in an abelian category

$$A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \rightarrow \cdots$$

is called co-Mittag-Leffler if for each $i$ the increasing sequence

$$0 \subseteq \text{Ker}(\alpha_i) \subseteq \text{Ker}(\alpha_{i+1}\alpha_i) \subseteq \cdots$$

stabilizes. An object $C$ in a triangulated category with products $T$ is called $0$-cocompact, provided that the equality $T(\text{holim} X_i, C) = 0$ holds for each inverse tower $X_0 \xleftarrow{\xi_0} X_1 \xleftarrow{\xi_1} X_2 \leftarrow \cdots$ with the properties that the direct tower

$$T(X_0, \Sigma C) \rightarrow T(X_1, \Sigma C) \rightarrow T(X_2, \Sigma C) \rightarrow \cdots$$

is co-Mittag-Leffler and $\text{colim} T(X_i, C) = 0$. The proof of the following Lemma is implicitly contained in [20, Proof of Theorem 6.6]; despite this fact we include the argument for the sake of completeness:

**Lemma 4.1** Let $C$ be a set of $0$-cocompact objects, in a triangulated category with products $T$. Then objects in $C$ are weak cocompact with respect to any $\Sigma^{\{0,1\}} C$-cophantom (consequently to any $\Sigma^{\geq 0} C$-cophantom) tower.

**Proof** Let $C \subseteq C$ and let $Y_0 \xrightarrow{\psi_0} Y_1 \xrightarrow{\psi_1} Y_2 \leftarrow \cdots$ be a tower such that $\psi_i \in \Psi(\Sigma^{\{0,1\}} C)$. Put $Y_\infty = \text{holim} Y_i$. Both towers of abelian groups

$$T(Y_0, C) \rightarrow T(Y_1, C) \rightarrow T(Y_2, C) \rightarrow \cdots$$

and

$$T(Y_0, \Sigma C) \rightarrow T(Y_1, \Sigma C) \rightarrow T(Y_2, \Sigma C) \rightarrow \cdots$$
have zero connecting maps. Among others this implies for first one that \( \text{colim} T(Y_i, C) = 0 \) and for the second one that it is co-Mittag-Leffler. Since \( C \) is 0-cocompact we conclude \( T(Y_\infty, C) = 0 \).

**Example 4.2** [20, Theorem 6.8 and Corollary 6.10] Let \( A \) be an algebra over a commutative artinian ring \( k \), and let \( E \) be the injective envelope of the regular left \( A \)-module \( A \). Then for any finite complex \( X \) of finitely generated left \( A \)-modules, the complex of right \( A \)-modules \( \text{Hom}_A(X, E) \) is 0-cocompact in \( \mathbf{K}(\text{Mod}-A) \). In particular, if \( A \) is an Artin algebra, then any finite complex of finitely generated \( A \)-modules is 0-cocompact in \( \mathbf{K}(\text{Mod}-A) \), therefore it is weak cocompact with respect to any \( \Sigma^{[0,1]} \text{Hom}_A(X, E) \)-cophantom tower, according to Lemma 4.1.

We already noted that \( (\aleph_0) \)-compact objects are weak cocompact with respect to any tower. In what follow we want to study the weak compactness of \( \kappa \)-compact objects, with \( \kappa \) being a regular cardinal (possible bigger than \( \aleph_0 \)). In this context, the category \( T \) has to have coproducts. Recall from [8] that a set of objects \( \mathcal{G} \subseteq T \) is called perfect provided that the induced map \( T(G, \coprod X_i) \to T(G, \coprod Y_i) \), with \( G \in \mathcal{G} \), is surjective for any set of maps \( \{X_i \to Y_i \mid i \in I \} \) in \( T \) with the property that \( T(G, X_i) \to T(G, Y_i) \) are surjective for all \( G \in \mathcal{G} \) and all \( i \in I \). Observe that if \( \mathcal{G} \subseteq T \) is perfect, then the same is true for \( \Sigma^n \mathcal{G} \), for all \( n \in \mathbb{Z} \). Note that \( \alpha \)-cocompact objects in the sense of [15] have to be perfect, see [7]. In what follows, for an additive category \( \mathcal{A} \), we denote by \( \widehat{\mathcal{A}} \) the category of finitely presented contravariant functors from \( \mathcal{A} \) to \( \text{Ab} \), that is the category of functors \( F : \mathcal{A}^0 \to \text{Ab} \) admitting a presentation

\[
\mathcal{A}(-, Y) \to \mathcal{A}(-, X) \to F \to 0,
\]

for some \( X, Y \in \mathcal{A} \).

The next lemma is probably known to the experts. A variant appeared in [15, Lemma 6.3.2] combined with the equivalence of categories from [7, Lemma 2]. However, we give an argument for it based on (the proofs of) Krause’s results [8, Lemmas 1, 2, and 3], since we could not find it in the literature in the form we need.

**Lemma 4.3** Let \( T \) be a triangulated category with coproducts. The set of objects \( \mathcal{G} \subseteq T \) is perfect if and only if the restricted Yoneda functor

\[
H : T \to \text{Add}(\mathcal{G}), \quad H(X) = T(-, X)|_{\text{Add}(\mathcal{G})}
\]

preserves coproducts. Moreover, if this is the case then the category \( \text{Add}(\mathcal{G}) \) is cocomplete abelian with exact coproducts, that is it satisfies AB4.

**Proof** The existence of coproducts in \( \text{Add}(\mathcal{F}) \) is a consequence of the existence of coproducts in \( \text{Add}(\mathcal{G}) \), see [8, Lemma 1]. The characterization of the perfectness of \( \mathcal{G} \) by the property that the restricted Yoneda functor is coproduct preserving is established in [8, Lemma 3]. Further let \( f : \text{Add}(\mathcal{G}) \to T \) be the inclusion functor. By [8, Lemma 2] the restriction functor \( f_* : \widehat{T} \to \text{Add}(\mathcal{G}) \), \( F \mapsto F \circ f \) is exact and has a fully faithful left adjoint \( f^* : \text{Add}(\mathcal{G}) \to \widehat{T} \). Moreover \( f_* \) preserves coproducts (see the proof of [8, Lemma 3]). Thus if \( 0 \to M_i \to N_i \), with \( i \) running over an arbitrary index set \( I \) are monomorphisms in \( \text{Add}(\mathcal{G}) \), then we get exact sequences \( 0 \to K_i \to f^*(M_i) \to f^*(N_i) \) in \( \widehat{T} \), satisfying \( f_*(K_i) = 0 \), for all \( i \in I \). Note that \( T \) has coproducts, by [8, Lemma 1], and these coproducts are exact since \( \widehat{T} \) has enough injectives (see also [15, Lemma 7.1.3]). Forming the coproduct
of the exact sequences obtained above in $\widehat{T}$ and applying the coproduct preserving functor $f_*$, the conclusion is proved by the following commutative diagram in $\text{Add}(\mathcal{G})$

$$
\begin{array}{ccccccc}
0 & \longrightarrow & f_*(\coprod K_i) & \longrightarrow & f_*(\coprod f^*(M_i)) & \longrightarrow & f_*(\coprod f^*(N_i)) \\
\cong & & \cong & & \cong & & \\
0 & \longrightarrow & \coprod f_*(K_i) & \longrightarrow & \coprod (f_* \circ f^*)(M_i) & \longrightarrow & \coprod (f_* \circ f^*)(N_i) \\
\cong & & \cong & & \cong & & \\
0 & \longrightarrow & 0 & \longrightarrow & \coprod M_i & \longrightarrow & \coprod N_i \\
\end{array}
$$

with all vertical maps being isomorphisms. \qed

**Proposition 4.4** Let $T$ be a triangulated category with coproducts, and let $\mathcal{G} \subseteq T$ be a perfect set of objects. Then objects in $\mathcal{G}$ are weak compact with respect to any $\Sigma^{(-1,0)}\mathcal{G}$-phantom tower.

**Proof** Let $Z_0 \xrightarrow{\phi_0} Z_1 \xrightarrow{\phi_1} Z_2 \longrightarrow \cdots$ be a direct tower in $T$. Apply the restricted Yoneda functor $H : T \rightarrow \text{Add}(\mathcal{G})$ to the diagram which defines $Z_\infty = \text{hocolim} Z_i$. According to Lemma 4.3, this functor is coproduct preserving, thus we get an exact sequence of the form

$$
\coprod H(Z_i) \rightarrow \coprod H(Z_i) \rightarrow H(Z_\infty) \rightarrow \coprod H(\Sigma Z_i) \rightarrow \coprod H(\Sigma Z_i)
$$

Assuming that the above tower is $\Sigma^{[-1,0]}\mathcal{G}$-phantom as required, we get $\phi_i \in \Phi(\Sigma^{-1}\mathcal{G} \cup \mathcal{G})$, for all $i \geq 0$, hence the first and the last map of the above exact sequence are isomorphisms. This shows that $H(Z_\infty) = 0$, which means $T(G, Z_\infty) = 0$ for all $G \in \mathcal{G}$. \qed

A pair $(\mathcal{G}, C)$ consisting of two sets of objects in a triangulated category with products and coproducts $T$ is called a symmetric pair if $\Phi(\mathcal{G}) = \Psi(C)$. This definition is a reminiscent of the definition in [8] of a symmetric set of generators: A set of symmetric generators in $T$ is a set of generators such that there is a set $C \subseteq T$, such that $\Phi(\mathcal{G}) = \Psi(C)$.

**Theorem 4.5** Let $T$ be a triangulated category with products and coproducts, which is the base of a strong stable derivator, and let $(\mathcal{G}, C)$ be a symmetric pair in $T$. Then $\mathcal{G}$ is a perfect set and $\mathcal{G}^\perp = 1^\perp C$. Moreover:

1. If objects $G \in \mathcal{G}$ are weak compact with respect to any $\Sigma^{\geq 0}\mathcal{G}$-phantom tower then $C$ cogenerates a $w$-structure and

$$
\left( ^{\perp}(\Sigma^{\geq 0}\mathcal{G})^\perp, \Sigma^{\geq 0}\mathcal{G}^\perp = ^{\perp}\Sigma^{\geq 0}C, (^{\perp}\Sigma^{\geq 0}C)^\perp \right)
$$

is a cosuspended TTF triple in $T$.

2. If $\Sigma \mathcal{G} = \mathcal{G}$, then $\Sigma C = C$, and $(^{\perp}(G^\perp), G^\perp)$ and $(^{\perp}C, (^{\perp}C)^\perp)$ are stable $t$-structures in $T$, the second one being the right adjacent of the first.

**Proof** Since $(\mathcal{G}, C)$ is a symmetric pair in $T$, then clearly $\Phi(\mathcal{G})$ is closed under coproducts, what is equivalent by [10, Proposition 2.1] to $\mathcal{G}$ being perfect. Moreover $\mathcal{G}^\perp = 1^\perp C$ since for an object $X \in T$ whose identity map is denoted $1_X$ we have

$$
X \in \mathcal{G}^\perp \iff 1_X \in \Phi(\mathcal{G}) \iff 1_X \in \Psi(C) \iff X \in 1^\perp C.
$$

For getting the TTF-triple whose existence is stated in conclusions (1) and (2) we want to apply Theorem 3.5 (2). In order to verify the hypotheses there, observe that the equality
\( G^\perp = \perp C \) implies \( C \subseteq (\Sigma^{\geq 0}G^\perp)^\perp \). Note that in the hypotheses of (2), this inclusion is written simply \( C \subseteq (\perp G)^\perp \), and, because \( G \) is closed under all shifts, the same is true for \( \Phi(G) = \Psi(C) \) and, finally, for \( C \) too. The objects of \( G \) are weak compact with respect to any \( \Sigma^{\geq 0}G \)-phantom tower in the case (1) by hypothesis, and in the case (2) by Proposition 4.4. Now conclusion follows by Theorem 3.5. In the case (2) both the \( G \)-generated t-structure and the \( C \)-cogenerated w-structure are obviously stable, since both equalities \( G = \Sigma G \) and \( C = \Sigma C \) hold.

**Remark 4.6** In the previous theorem we wanted to apply Theorem 2.5, in order to get a t-structure generated by a set of objects \( G \), therefore we needed the following condition: The objects \( G \in G \) are weak compact with respect to any \( \Sigma^{\geq 0}G \)-phantom tower. In contrast with the case of a set of 0-cocompact objects which automatically cogenerates a w-structure, for a perfect set of objects \( G \) we only know that they are weak compact with respect to any \( \Sigma^{[-1,0]}G \)-phantom tower. If we consider a \( \Sigma^{\geq 0}G \)-phantom tower

\[
Z_0 \xrightarrow{\phi_0} Z_1 \xrightarrow{\phi_1} Z_2 \longrightarrow \cdots ,
\]

then it is not clear that \( T(G, \Sigma \phi) = T(\Sigma^{-1}G, \phi) = 0 \) and we can not use the argument from the proof of Proposition 4.4. This explains why in Theorem 4.5 we have to impose the condition above. Interesting enough, using different arguments Bondarko encountered in [5] the same issue: Starting with a set of perfect objects, it is not sure that they generate a t-structure (for details see [5, Remark 2.3.5]).

**Corollary 4.7** Let \( T \) be a triangulated category with products and coproducts, which is the base of a strong stable derivator. If \( T \) has a symmetric set of generators, then both \( T \) and \( T^\circ \) satisfy Brown representability theorem.

**Proof** Let \( G \) be a set of symmetric generators. Replacing \( G \) with \( \Sigma^2 G \) we can assume that \( G \) is closed under all shifts. Thus \( G^\perp = 0 \) and there is a set (closed under all shifts) \( C \) of objects such that \( \Phi(G) = \Psi(C) \). Then Theorem 4.5 implies that \( T = C^\perp \). Thus every object of \( T \) is both \( G \)-filtered and \( C \)-cofiltered, and we apply [12, Theorem 8] and its dual. \( \Box \)

Corollary 4.7 is a particular case of [8, Theorem A and Theorem B]. Our result is more restrictive since, in order to apply Theorem 4.5 we need to assume additionally that \( T \) has a derivator enhancement. It would be therefore interesting to find a proof of Theorem 4.5 without this additional assumption.

**Example 4.8** Let \( T \) be a triangulated category with coproducts, which is the base of a strong stable derivator. Suppose also that \( T \) satisfies Brown representability; in particular \( T \) has to have products. If \( G \in T \) is a compact object, then the contravariant functor \( \text{Hom}_\mathbb{Z}(T(G, -), \mathbb{Q}/\mathbb{Z}) : T \to \mathbb{A}b \) is cohomological and sends coproducts into products. By Brown representability theorem it has to be representable, and the object \( C \in T \) (unique, up to a canonical isomorphism) which represents it is called the Brown–Comenetz dual of \( G \). Clearly for every map \( \phi \) in \( T \) we have \( T(G, \phi) = 0 \) if and only if \( T(\phi, C) = 0 \). Thus Theorem 4.5 (1) tells us that \( (\Sigma^{\geq 0}G^\perp, \Sigma^{\geq 0}(\perp C)^\perp) \) is a t-structure whose right adjacent w-structure is \( (\perp \Sigma^{\geq 0}C, \perp \Sigma^{\geq 0}(C)^\perp) \). For a more general version of this result, which do not use the derivator enhancement, see [5, Theorem 2.4.2].

**Example 4.9** Let \( T = \mathbb{K}(\text{Flat-}R) \) be the homotopy category of cochain complexes of flat (right) \( R \)-modules, where \( R \) is a ring. Let \( G \) be the set of all bounded bellow complexes of
finely generated projective modules, up to homotopy. Then clearly $\Sigma G = G$. By [16, Lemma 4.6] the set $G$ is $\aleph_1$-perfect in the sense of Neeman, that is [15, Definition 3.3.1], therefore according to [7, Lemma 4] it is also perfect in the sense of Krause’s definition we already used before. Moreover $G$ generates $K(\text{Proj-}R)$, by [16, Theorem 5.8]. On the other hand, by [18, Theorem 3.2], the inclusion $K(\text{Flat-}R) \rightarrow K(\text{Mod-}R)$ has a right adjoint denoted by $J$. We call a test complex a bounded below complex $I$ of injective left $R$-modules, such that $H^i(I) = 0$ for all but finitely many $i \in \mathbb{Z}$, and for all $i \in \mathbb{Z}$, $H^i(I)$ must be isomorphic to a subquotient of a finitely generated, projective left $R$-module (here $H^i(I)$ is the $i$-th cohomology of the complex $I$). Denote

$$C = \{ J(\text{Hom}_\mathbb{Z}(I, \mathbb{Q}/\mathbb{Z})) \mid I \text{ runs over all test complexes} \}.$$  

By [19, Lemma 2.8] the ideal of $C$ cophantom maps $\Psi(C)$ consists exactly of so called tensor phantom maps, that is maps $f$ in $K(\text{Flat-}R)$ such that $f \otimes_R I$ induces zero map in all cohomologies, for any test complex $I$. Moreover, from [19, Lemma 1.9], we learn that the composition of two tensor phantom maps is a $G$-phantom. With our notations this means $\Psi(C)^2 \subseteq \Phi(G)$. Finally by [19, Lemma 2.2 and Lemma 2.6], we have the inclusion $C \subseteq (G^{-1})^\perp$. Now by Theorem 3.5 (2), we deduce that $(G^{-1})^\perp$ and $(C^{-1})^\perp$ are stable $t$-structures in $\mathcal{T}$ and the second is adjacent to the first. More about this example can be found in [12] and [13].

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