CYCLIC GROUP ACTIONS AND EMBEDDED SPHERES IN 4-MANIFOLDS

M. J. D. HAMILTON

ABSTRACT. In this note we derive an upper bound on the number of spheres in the fixed point set of a smooth and homologically trivial cyclic group action of prime order on a simply-connected 4-manifold. This improves the a priori bound which is given by one half of the Euler characteristic of the 4-manifold. The result also shows that in some cases the 4-manifold does not admit such actions of a certain order or that any such action has to be pseudofree.

1. INTRODUCTION

Actions of finite groups, in particular cyclic groups $\mathbb{Z}_p$ of prime order $p$, on simply-connected 4-manifolds have been studied in numerous places in the literature. An interesting subclass are those actions which act trivially on homology. In the topological setting, Edmonds has shown [7, Theorem 6.4] that every closed, simply-connected, topological 4-manifold admits for every $p > 3$ a (non-trivial) homologically trivial action which is locally linear. However, it is an open question from the Kirby list if such actions exist in the smooth setting for 4-manifolds like the $K3$ surface (it is known that there is no such action of $\mathbb{Z}_2$ [19, 22] on $K3$ and no such action of $\mathbb{Z}_p$ which is holomorphic [5, 21] or symplectic [6]).

The actions in the theorem of Edmonds can be assumed to be pseudofree, i.e. the fixed point set consists of isolated points. In general, if the action is homologically trivial, the fixed point set will consist of isolated points and disjoint embedded spheres. We recall this fact in Proposition 2.3. If $m$ is the number of points and $n$ the number of spheres, then $m + 2n$ is equal to the Euler characteristic $\chi(M)$ of the 4-manifold. This implies an a priori upper bound on the number of spheres:

$$n \leq \frac{\chi(M)}{2}.$$  

A natural question is whether all cases of possible values for $n$ can occur. We will show that this upper bound can indeed be improved, for example, by a factor of roughly $\frac{1}{2}$ if the 4-manifold $M$ and the action are smooth, $M$ is smoothly minimal and the Seiberg-Witten invariants of $M$ are non-vanishing. The proof uses the $G$-signature theorem together with an estimate on the signature defects at the fixed points and a result from Seiberg-Witten theory, which implies that under the assumption of non-vanishing Seiberg-Witten invariants the spheres in the fixed point set have negative self-intersection number. The proof is elementary.

Date: June 30, 2014.

Key words and phrases. 4-manifold, group action, fixed point set, $G$-signature theorem.
and perhaps known to the experts. However, it seems worthwhile to record this fact together with a number of corollaries. One consequence of the main theorem, for instance, is that a simply-connected 4-manifold with positive signature and non-trivial Seiberg-Witten invariants does not admit homologically trivial, smooth involutions. This generalizes a theorem of Ruberman for spin 4-manifolds and contrasts a theorem of Edmonds, who has shown that every smooth, simply-connected, non-spin 4-manifold admits a homologically trivial, locally linear involution.

**Convention.** All 4-manifolds in the following will be closed, oriented and connected. All group actions will be non-trivial and orientation-preserving.

**Acknowledgements.** I would like to thank Dieter Kotschick for helpful comments regarding reference [13].

2. **SPHERES IN THE FIXED POINT SET AND THE G-SIGNATURE THEOREM**

Let $M$ denote a simply-connected, topological 4-manifold with a locally linear action of a cyclic group $G = \mathbb{Z}_p$, with $p \geq 2$ a prime. The group action is generated by a locally linear homeomorphism $\tau: M \to M$ of order $p$, such that $\tau$ is not equal to the identity. There is an induced action of $G$ on $H^2(M; \mathbb{Z})$ preserving the intersection form. According to [9, 15] this action decomposes over the integers into $t$ copies of the trivial action of rank 1, $c$ copies of the cyclotomic action of rank $p - 1$ and $r$ copies of the regular action of rank $p$, where $t, c, r$ are certain non-negative integers. As a consequence, the second Betti number of $M$ is equal to

$$b_2(M) = t + c(p - 1) + rp.$$ 

In particular we have:

**Lemma 2.1.** If $p > b_2(M) + 1$, then $G$ acts trivially on homology.

Let $F$ denote the fixed point set of the locally linear homeomorphism $\tau$. Since $G$ is of prime order, the set $F$ is the fixed point set of every group element in $G$ different from the identity. The fixed point set $F$ is a closed topological submanifold of $M$ [4, p. 171]. The action is locally linear and hence given by an orthogonal action in a neighbourhood of a fixed point. Since the action preserves orientation, the fixed point set $F$ has even codimension [23]. It consists of a disjoint union of finitely many isolated points and finitely many closed surfaces. If $p$ is odd, then every surface in the fixed point set is orientable [4, p. 175].

The next lemma follows from [9, Proposition 2.5]:

**Lemma 2.2.** Suppose that the fixed point set $F$ has more than one component. Then every surface component of $F$ represents a non-zero class in $H_2(M; \mathbb{Z}_p)$.

If the action is not free, then according to [9, Proposition 2.4] the $\mathbb{Z}_p$-Betti numbers of the fixed point $F$ satisfy

$$b_1(F; \mathbb{Z}_p) = c$$

$$b_0(F; \mathbb{Z}_p) + b_2(F; \mathbb{Z}_p) = t + 2.$$
Let $\chi(M) = b_2(M) + 2$ denote the Euler characteristic of $M$. If $G$ acts trivially on homology, then $\chi(F) = \chi(M)$ by the Lefschetz fixed point theorem. Hence the action is not free and we get:

**Proposition 2.3.** Suppose that $G$ acts trivially on the homology of $M$. Then $F$ consists of a disjoint union of $m$ isolated points and $n$ spheres, with $m + 2n = \chi(M)$. If $b_2(M) \neq 0$, then after a choice of orientation every sphere in $F$ represents a non-zero class in $H_2(M; \mathbb{Z})$.

From now on we assume that the action of $G$ is trivial on homology. We want to improve the upper bound $\frac{1}{2} \chi(M)$ on the number $n$ of spheres. We can use the $G$-signature theorem \cite{[3]}, which is valid not only for smooth, but also for locally linear actions in dimension 4, cf. \cite{[25]} and a remark in \cite{[7], p. 164} (all our applications will be for smooth actions). Let $S_1, \ldots, S_n$ denote the spherical components of the fixed point set $F$ and $P$ the set of isolated fixed points. Note that the signature satisfies

$$\text{sign}(M/G) = \text{sign}(M),$$

since the action of $G$ is trivial on homology. The $G$-signature theorem implies \cite{[11], p. 14–17]:

$$(p - 1)\text{sign}(M) = \sum_{x \in P} \text{def}_x + \frac{p^2 - 1}{3} \sum_{i=1}^n [S_i]^2.$$

Here $[S_i]^2$ denotes the self-intersection number of the sphere $S_i$. The numbers $\text{def}_x$ are equal, in Hirzebruch’s notation, to $\text{def}(p; q, 1)$ for certain integers $q$ coprime to $p$ and depending on $x$. We have

$$\text{def}(p; q, 1) = -\frac{2}{3}(q, p) = -4p \sum_{k=0}^{p-1} \left( \left( \frac{k}{p} \right) \left( \frac{qk}{p} \right) \right).$$

In this equation $(q, p)$ denotes the Dedekind symbol, while $((\cdot)) : \mathbb{R} \to \mathbb{R}$ is a certain function introduced by Rademacher and given by

$$((x)) = x - [x] - \frac{1}{2}, \quad \text{if } x \text{ is not an integer}$$
$$((x)) = 0, \quad \text{if } x \text{ is an integer}.$$ 

Here $[x]$ denotes the greatest integer less than or equal to $x$. We want to prove the following estimate:

**Lemma 2.4.** For all prime numbers $p$ and integers $q$ coprime to $p$ we have

$$|\text{def}(p; q, 1)| \leq |\text{def}(p; 1, 1)| = \frac{1}{3}(p - 1)(p - 2).$$
Proof. We have by Cauchy-Schwarz
\[
\left| \sum_{k=0}^{p-1} \left( \left( \frac{k}{p} \right) \left( \frac{qk}{p} \right) \right) \right| \leq \left( \sum_{k=1}^{p-1} \left( \left( \frac{k}{p} \right)^2 \right) \right)^{\frac{1}{2}} \cdot \left( \sum_{k=1}^{p-1} \left( \left( \frac{qk}{p} \right)^2 \right) \right)^{\frac{1}{2}}
\]
\[
= \sum_{k=1}^{p-1} \left( \frac{1}{p} \right)^2,
\]
because \( q \) generates \( \mathbb{Z}_p \) and \( ((0)) = 0 \). Since \( 0 < \frac{k}{p} < 1 \) for all \( k = 1, \ldots, p-1 \) we have
\[
\sum_{k=1}^{p-1} \left( \frac{k}{p} \right)^2 = \sum_{k=1}^{p-1} \left( \frac{k}{p} - \frac{1}{2} \right)^2
\]
\[
= \sum_{k=1}^{p-1} \left( \frac{k^2}{p^2} - \frac{k}{p} + \frac{1}{4} \right)
\]
\[
= \frac{1}{6p^2}(p-1)p(2p-1) - \frac{1}{2p}(p-1)p + \frac{p-1}{4}
\]
\[
= \frac{1}{6p}(2p^2 - 3p + 1) - \frac{1}{2p}(p^2 - p) + \frac{p-1}{4}
\]
\[
= \frac{1}{12p}(4p^2 - 6p + 2 - 6p^2 + 6p + 3p^2 - 3p)
\]
\[
= \frac{1}{12p}(p^2 - 3p + 2)
\]
\[
= \frac{1}{12p}(p-1)(p-2).
\]
This implies the claim. The number \( \text{def}(p; 1, 1) \) has also been calculated in equation (28) in [11].

We can now prove the main theorem. We use the standard notation
\[
c^2(M) = 2\chi(M) + 3\text{sign}(M)
\]
for every 4-manifold \( M \). We abbreviate the following conditions on the action and the manifold by simply saying that “\( \mathbb{Z}_p \) acts homologically trivially on a simply-connected 4-manifold \( M \)”: The group \( \mathbb{Z}_p \), with \( p \geq 2 \) prime, acts locally linearly and homologically trivially on a simply-connected, topological 4-manifold \( M \).

We consider in the following only actions of this kind.

Theorem 2.5. Let \( \mathbb{Z}_p \) act homologically trivially on a simply-connected 4-manifold \( M \). Suppose that all spheres \( S \) in the fixed point set of the action satisfy an a priori bound \( [S]^2 \leq s < 0 \) for some integer \( s \). Then the number \( n \) of spheres in the fixed
point set satisfies the upper bound
\[ n \leq \frac{p\chi(M) - c_1^2(M)}{p(2-s)-(4+s)}. \]
For all possible values of \( c_1^2(M) \) we have the bound
\[ n < \frac{\chi(M)}{2-s} \left( 1 + \frac{6}{p(2-s)-(4+s)} \right). \]

**Proof.** By Proposition 2.3 the number of isolated fixed points in \( F \) is \( \chi(M) - 2n \).
By the \( G \)-signature theorem and Lemma 2.4 we have
\[(p - 1)\text{sign}(M) \leq \frac{1}{3}(p - 1)(p - 2)(\chi(M) - 2n) + \frac{1}{3}sn(p^2 - 1).\]
This implies the first claim (note that the denominator is positive under our assumption \( s < 0 \)). The second claim follows from the estimate \( \text{sign}(M) > -\chi(M) \), which is true for all oriented 4-manifolds with \( b_1(M) = 0 \). \( \Box \)

3. SMOOTHLY EMBEDDED SPHERES

We say that a smooth 4-manifold \( M \) satisfies property \((*)\) if the following holds:

Every smoothly embedded sphere \( S \) in \( M \) that represents a non-zero homology class \( [S] \in H_2(M; \mathbb{Q}) \) has negative self-intersection number.

We are interested under which conditions a 4-manifold \( M \) satisfies property \((*)\). The following is clear:

**Proposition 3.1.** Let \( M \) be a smooth 4-manifold. Assume that \( b_2(M) = 0 \). Then \( M \) satisfies property \((*)\).

The next theorem is well-known, cf. [13, Proposition 1]. The statement also follows from the adjunction inequality [14, 10].

**Proposition 3.2.** Let \( M \) be a smooth 4-manifold. Assume that \( b_2^+ (M) > 1 \) and the Seiberg-Witten invariants of \( M \) do not vanish identically. Then \( M \) satisfies property \((*)\).

We did not find in the literature a similarly general theorem in the case of 4-manifolds \( M \) with \( b_2^+ (M) = 1 \). However, we can prove the following.

**Proposition 3.3.** Let \( M \) be a smooth 4-manifold. Assume that \( b_2^+ (M) = 1 \), \( b_3 (M) \leq 9 \), \( b_1(M) = 0 \) and \( M \) is not diffeomorphic to a rational surface. If \( M \) admits a symplectic form, then \( M \) satisfies property \((*)\).

**Remark 3.4.** In this situation, the assumption \( b_3 (M) \leq 9 \) is equivalent to \( K^2 \geq 0 \), where \( K \) denotes the canonical class of the symplectic form.

For the proof recall the following theorem of Liu [18, Theorem B] (slightly adapted to make the statement more precise):
**Theorem 3.5.** Let $M$ be a symplectic 4-manifold with $b_2^+(M) = 1$. If $K \cdot \omega < 0$, then $M$ must be either rational or (a blow-up of) an irrational ruled 4-manifold.

We also need an adjunction inequality of Li and Liu [17, p. 467]:

**Theorem 3.6.** Suppose $M$ is a symplectic 4-manifold with $b_2^+(M) = 1$ and $\omega$ is the symplectic form. Let $C$ be a smooth, connected, embedded surface with non-negative self-intersection. If $[C] \cdot \omega > 0$, then the genus of $C$ satisfies $2g(C) - 2 \geq K \cdot [C] + [C]^2$.

We have the following general light cone lemma, compare with [17, Lemma 2.6]:

**Lemma 3.7.** Let $M$ be a 4-manifold with $b_2^+(M) = 1$. The forward cone is one of the two connected components of $\{ a \in H_2(M; \mathbb{R}) \mid a^2 > 0 \}$. Then the following holds for elements $a, b \in H_2(M; \mathbb{R})$:

(a) If $a$ is in the forward cone and $b$ in the closure of the forward cone with $b \neq 0$, then $a \cdot b > 0$.

(b) If $a$ and $b$ are in the closure of the forward cone, then $a \cdot b \geq 0$.

(c) If $a$ is in the forward cone and $b$ satisfies $b^2 \geq 0$ and $a \cdot b \geq 0$, then $b$ is in the closure of the forward cone.

**Proof.** All claims follow by applying the Cauchy-Schwarz inequality:

$$\pm \sum a_i b_i \leq \sqrt{\sum a_i^2} \sqrt{\sum b_i^2}.$$ 

We can now prove Proposition 3.3.

**Proof.** Let the forward cone be defined by the class of $\omega$. Our assumptions together with Theorem 3.5 and Lemma 3.7 imply that the canonical class $K$ is in the closure of the forward cone. Suppose that the class $[S]$ of a sphere $S$ satisfies $[S] \neq 0$ and $[S]^2 \geq 0$. Choose the orientation on $S$ such that $[S]$ is in the closure of the forward cone. By Lemma 3.7, $[S] \cdot \omega > 0$. Then Theorem 3.6 applies and shows that $-2 \geq K \cdot [S] + [S]^2$. However, Lemma 3.7 implies that $K \cdot [S] \geq 0$. This is a contradiction.

We conjecture the following:

**Conjecture 3.8.** Let $M$ be a smooth 4-manifold. Assume that $b_2^+(M) = 1$, $b_2^-(M) \leq 9$, $H_1(M; \mathbb{Z}) = 0$ and $M$ has non-trivial small perturbation Seiberg-Witten invariants. Then $M$ satisfies property $(\ast)$.

For a definition of the small perturbation Seiberg-Witten invariants see [24].

4. The main corollaries

Recall that an oriented 4-manifold is called (smoothly) minimal if it does not contain smoothly embedded spheres of self-intersection $-1$. 
Corollary 4.1. Let the group $\mathbb{Z}_p$ act homologically trivially and smoothly on a simply-connected, smooth 4-manifold $M$ that satisfies property $(\ast)$. Then

$$n \leq \frac{p\chi(M) - c_2^2(M)}{3(p-1)}.$$ 

If in addition $M$ is smoothly minimal, then

$$n \leq \frac{p\chi(M) - c_2^2(M)}{2(2p-1)}.$$ 

Independently of $c_2^2(M)$ we have in these cases the bounds

$$n < \frac{\chi(M)}{3} \left(1 + \frac{2}{p-1}\right)$$

and

$$n < \frac{\chi(M)}{4} \left(1 + \frac{3}{2p-1}\right),$$

respectively.

Proof. If the action is smooth, then every sphere in $F$ is smoothly embedded [4, p. 309]. The first claim follows with Theorem 2.5, since $[S]^2 \leq -1$ for every embedded sphere $S$ representing a non-zero homology class if $M$ satisfies property $(\ast)$. If $M$ is smoothly minimal, $(-1)$-spheres do not exist in $M$, hence $[S]^2 \leq -2$. □

This improves the a priori bound $n \leq \frac{1}{2}\chi(M)$ by a factor of approximately $\frac{2}{3}$ and $\frac{1}{2}$, at least for large $p$.

Example 4.2. Let $M = E(k)_{a,b}$ be a simply-connected, minimal elliptic surface with multiple fibres of coprime indices $a, b$. Assume that either $k \geq 2$, or $k = 1$ and both $a, b \neq 1$. Then $M$ is smoothly minimal and satisfies property $(\ast)$. We have $c_2^2(M) = 0$ and $\chi(M) = 12k$. Therefore

$$n \leq \frac{\chi(M)}{4} \left(1 + \frac{1}{2p-1}\right).$$

This rules out some of the possible $\mathbb{Z}_3$-actions on elliptic surfaces in [16].

Since the integer $n$ has to be non-negative, we get:

Proposition 4.3. Let the group $\mathbb{Z}_p$ act homologically trivially on a simply-connected 4-manifold $M$. Suppose that all spheres $S$ in the fixed point set of the action satisfy an a priori bound $[S]^2 \leq s < 0$ for some integer $s$. Then

$$p\chi(M) \geq c_2^2(M).$$

Corollary 4.4. Let the group $\mathbb{Z}_p$ act homologically trivially and smoothly on a simply-connected, smooth 4-manifold $M$ that satisfies property $(\ast)$. If $p = 2$, then $\text{sign}(M) \leq 0$. If $p = 3$, then $c_2^2(M) \leq 3\chi(M)$.
Remark 4.5. Ruberman [22] has shown that if $\mathbb{Z}_2$ acts homologically trivially on a simply-connected spin 4-manifold, then $\text{sign}(M) = 0$. The first part of Corollary 4.4 is a partial extension of this result to non-spin 4-manifolds. Edmonds [8] has shown that every smooth, non-spin 4-manifold $M$ admits a homologically trivial, locally linear $\mathbb{Z}_2$-action whose fixed point set consists of a single sphere and a collection of isolated points, the sphere having self-intersection number equal to the signature of $M$. Under our assumptions there does not exist a smooth, homologically trivial $\mathbb{Z}_2$-action. It is not known if there exist simply-connected, smooth 4-manifolds with non-trivial Seiberg-Witten invariants and $c_1^2(M) > 3\chi(M)$. Note that any simply-connected 4-manifold satisfies a priori $c_1^2(M) \leq 5\chi(M)$.

Example 4.6. Let $M$ be a simply-connected, complex algebraic surface of general type and positive signature with $b_2^+(M) > 1$. Then $M$ satisfies property $(\ast)$ by Proposition 3.2. Hence $M$ does not admit a homologically trivial, smooth $\mathbb{Z}_2$-action.

We can also study the case $n = 0$:

Proposition 4.7. Let the group $\mathbb{Z}_p$ act homologically trivially on a simply-connected 4-manifold $M$. Suppose that all spheres $S$ in the fixed point set of the action satisfy an a priori bound $|S|^2 \leq s < 0$ for some integer $s$ and that $M$ satisfies 

$$p\chi(M) - c_1^2(M) < p(2 - s) - (4 + s).$$

Then $n = 0$, hence the fixed point set consists only of isolated points, i.e. the action is pseudofree.

The following is an application to $\mathbb{Z}_2$-actions on 4-manifolds close to the line $\text{sign}(M) = 0$ in the $\chi-c_1^2$-plane:

Corollary 4.8. Let the group $\mathbb{Z}_2$ act homologically trivially and smoothly on a simply-connected, smooth 4-manifold $M$ that satisfies property $(\ast)$. Assume that either $\text{sign}(M) = 0$, or $M$ is minimal and $\text{sign}(M) = -1$. Then the action is pseudofree. In particular, every smooth, homologically trivial $\mathbb{Z}_2$-action on a simply-connected, smooth, spin 4-manifold that satisfies property $(\ast)$ is pseudofree.

Proof. We have $c_1^2(M) = 2\chi(M) + 3\text{sign}(M)$. For the first part we can take $s = -1$ in Proposition 4.7 and the inequality is $0 < 3$, which is true. For the second part we take $s = -2$ and the inequality is $3 < 6$. The third part follows from Ruberman’s theorem [22] since under these assumptions $\text{sign}(M) = 0$. \qed

Remark 4.9. Atiyah-Bott [2], Proposition 8.46] have shown that all components of the fixed point set have the same dimension, so that the fixed point set consists either of isolated fixed points or of a collection of embedded surfaces if $\mathbb{Z}_2$ acts smoothly and orientation-preservingly on a simply-connected spin 4-manifold (there are generalizations to the locally linear and general case by Edmonds [9, Corollary 3.3] and Ruberman [22]). Under our additional assumptions that the action is homologically trivial and $M$ satisfies property $(\ast)$ the second case of a fixed point set of dimension 2 does not occur.
We can prove a similar statement for $\mathbb{Z}_3$-actions on 4-manifolds close to the Bogomolov-Miyaoka-Yau line $c_1^2(M) = 3\chi(M)$:

**Corollary 4.10.** Let the group $\mathbb{Z}_3$ act homologically trivially and smoothly on a simply-connected, smooth 4-manifold $M$ that satisfies property $(\ast)$. Assume that either $c_1^2(M) = 3\chi(M) - l$ with $0 \leq l \leq 5$, or $M$ is minimal and $c_1^2(M) = 3\chi(M) - l$ with $6 \leq l \leq 9$. Then the action is pseudofree.

**Proof.** The proof is similar to the proof of Corollary 4.8. For Proposition 4.7 to work, $l$ has to be less than 6 in the first case and less than 10 in the second case. □

**Remark 4.11.** Note that $l = 3\chi(M) - c_1^2(M) = \chi(M) - 3\text{sign}(M) = 2 - 2b_2^+(M) + 4b_2^-(M)$ is always an even number. If $b_1(M) = 0$, the Seiberg-Witten invariants can be non-zero or $M$ has a symplectic form only if $b_2^+(M)$ is odd. Then $l$ is divisible by 4. Hence if we want to apply Proposition 3.2 and Proposition 3.3, then $l = 0$ or $l = 4$ in the first case and $l = 8$ in the second case of Corollary 4.10.

**Example 4.12.** Let $M$ be a smooth, minimal 4-manifold homeomorphic, but not diffeomorphic to the manifold $\mathbb{C}P^2 \# 2\mathbb{C}P^2$, cf. [1]. Suppose that $M$ admits a symplectic form $\omega$. Then $M$ satisfies property $(\ast)$ according to Proposition 3.3. Hence every homologically trivial $\mathbb{Z}_p$-action on $M$ with $p = 2$ or $p = 3$ is pseudofree.

**Lemma 4.13.** Let $\mathbb{Z}_p$, with $p \geq 3$ prime, act on $M$, where $M$ is a 4-manifold homeomorphic to $S^2 \times S^2$ or $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. Then the action is homologically trivial.

**Proof.** This follows as in [12, Proposition 5.8] (it follows from Lemma 2.1 if $p \geq 5$). □

**Corollary 4.14.** Let $\mathbb{Z}_p$ act smoothly on $M$, where $M$ is a smooth, minimal 4-manifold homeomorphic, but not diffeomorphic to $S^2 \times S^2$ or $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ and satisfying property $(\ast)$. If $p = 2$, assume in addition that the action is homologically trivial. Then the action is pseudofree.

**Proof.** We have $\chi(M) = 4$ and $c_1^2(M) = 8$. Hence the inequality in Proposition 4.7 with $s = -2$ is

$$4p - 8 < 4p - 2.$$ 

Since this is true, the claim follows. □

**Remark 4.15.** All statements in this paper remain true if the assumption that $M$ is simply-connected is replaced by $H_1(M; \mathbb{Z}) = 0$. This follows from [20, Corollary 3.3, Proposition 3.5], since in this situation Proposition 2.3 above remains true.

**References**

1. A. Akhmedov, B. D. Park, *Exotic smooth structures on small 4-manifolds with odd signatures*, Invent. Math. 181, no. 3, 577–603 (2010).
2. M. F. Atiyah, R. Bott, *A Lefschetz fixed point formula for elliptic complexes. II. Applications*, Ann. of Math. 88, 451–491 (1968).
3. M. F. Atiyah, I. M. Singer, *The index of elliptic operators: III*, Ann. of Math. 87, 546–604 (1968).
4. G. E. Bredon, *Introduction to compact transformation groups*, Academic Press, New York–London (1972).
5. D. Burns Jr., M. Rapoport, *On the Torelli problem for Kählerian K–3 surfaces*, Ann. Sci. École Norm. Sup. (4) 8, no. 2, 235–273 (1975).
6. W. Chen, S. Kwasik, *Symplectic symmetries of 4-manifolds*, Topology 46, no. 2, 103–128 (2007).
7. A. L. Edmonds, *Construction of group actions on four-manifolds*, Trans. Amer. Math. Soc. 299, no. 1, 155–170 (1987).
8. A. L. Edmonds, *Involutions on odd four-manifolds*, Topology Appl. 30, no. 1, 43–49 (1988).
9. A. L. Edmonds, *Aspects of group actions on four-manifolds*, Topology Appl. 31, no. 2, 109–124 (1989).
10. R. E. Gompf, A. I. Stipsicz, *4-manifolds and Kirby calculus*, Graduate Studies in Mathematics, 20, Providence, Rhode Island. American Mathematical Society 1999
11. F. Hirzebruch, *The signature theorem: reminiscences and recreation*, Prospects in mathematics (Proc. Sympos., Princeton Univ., Princeton, N.J., 1970), pp. 3–31, Ann. of Math. Studies, No. 70, Princeton Univ. Press, Princeton, N.J. 1971.
12. M. Klemm, *Finite Group Actions on Smooth 4-Manifolds with Indefinite Intersection Form* (1995). Open Access Dissertations and Theses. Paper 2369. [http://digitalcommons.mcmaster.ca/opendissertations/2369](http://digitalcommons.mcmaster.ca/opendissertations/2369)
13. D. Kotschick, *Orientations and geometrisations of compact complex surfaces*, Bull. London Math. Soc. 29, no. 2, 145–149 (1997).
14. P. B. Kronheimer, T. S. Mrowka, *The genus of embedded surfaces in the projective plane*, Math. Res. Lett. 1, no. 6, 797–808 (1994).
15. S. Kwasik, R. Schultz, *Homological properties of periodic homeomorphisms of 4-manifolds*, Duke Math. J. 58, 241–250 (1989).
16. H. Li, *Cyclic group actions on elliptic surfaces E(2n)*, J. Math. Comput. Sci. 2, no. 6, 1759–1765 (2012).
17. T.-J. Li, A. Liu, *Symplectic structure on ruled surfaces and a generalized adjunction formula*, Math. Res. Lett. 2, no. 4, 453–471 (1995).
18. A.-K. Liu, *Some new applications of general wall crossing formula, Gompf’s conjecture and its applications*, Math. Res. Lett. 3, no. 5, 569–585 (1996).
19. T. Matumoto, *Homologically trivial smooth involutions on K3 surfaces*, Aspects of low-dimensional manifolds, 365–376, Adv. Stud. Pure Math., 20, Kinokuniya, Tokyo, 1992.
20. M. McCooey, *Symmetry groups of non-simply connected four-manifolds*, preprint: [arXiv:0707.3835v2](http://arxiv.org/abs/0707.3835v2).
21. C. A. M. Peters, *On automorphisms of compact Kähler surfaces*, Journées de Géometrie Algébrique d’Angers, Juillet 1979/Algebraic Geometry, Angers, 1979, pp. 249–267, Sijthoff & Noordhoff, Alphen aan den Rijn–Germantown, Md., 1980.
22. D. Ruberman, *Involutions on spin 4-manifolds*, Proc. Amer. Math. Soc. 123, no. 2, 593–596 (1995).
23. P. A. Smith, *Transformations of finite period. IV. Dimensional parity*, Ann. of Math. (2) 46, 357–364 (1945).
24. Z. Szabó, *Exotic 4-manifolds with b+ 2 = 1*, Math. Res. Lett. 3, no. 6, 731–741 (1996).
25. C. T. C. Wall, *Surgery on Compact Manifolds*, Vol. 69 of Mathematical Surveys and Monographs, 2nd Edition, American Mathematical Society, Providence, RI, 1999