Probes of axion-like particles in vector boson scattering at a muon collider

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Abstract
We have examined the sensitivity to the axion-like particle (ALP) couplings to electroweak gauge bosons in the diphoton production at a future muon collider. The collisions at the $\mu^+\mu^-$ energies of 3 TeV, 14 TeV, and 100 TeV are addressed. The differential cross sections versus the invariant mass of the final photons and total cross section versus minimal diphoton invariant mass are presented. We have derived the exclusion regions for the ALP-gauge boson coupling. The obtained bounds are much stronger than the current experimental bounds in the ALP mass region 10 GeV – 10 TeV. The partial-wave unitarity constraints on the ALP-gauge boson coupling are estimated. We have shown that the unitarity is not violated in the region of the ALP coupling studied in the present paper.

1 Introduction
The strong CP problem of the Standard Model (SM) can be solved by introducing a spontaneously broken Peccei-Quinn symmetry. As a result, a

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light pseudo-Nambu-Goldstone boson, QCD axion, arises \[3, 4\]. The QCD axion is a well-motivated candidate for the DM \[5]-[9] which can be produced via the vacuum misalignment mechanism \[5, 10\] or as the decay of topological defects \[11\].

The axion-like particle (ALPs) are particles having interactions similar to the axion. The origin of the ALP is expected to be similar but without the relationship between its coupling constant and mass. It means that the ALP mass can be treated independently of its couplings to the SM fields. Since the ALPs are not directly relevant for the QCD axion, heavy ALPs can be detected at colliders. The production of the ALPs was studied in the \(pp\) \[12]-[19] and heavy-ion \[13], [20]-[22] collisions at the LHC, as well as at future colliders \[23]-[32], including electron-ion scattering \[33]-[34]. For a review on the axions and ALPs, see \[9], [35]-[40] and references therein.

Many ALP searches assume their strong couplings to the electromagnetic term \(F_{\mu\nu}\tilde{F}^{\mu\nu}\). One of the most preferred processes to probe the ALP-photon coupling is a light-by-light (LBL) scattering. The first evidence of the subprocess \(\gamma\gamma \rightarrow \gamma\gamma\) was observed by the ATLAS and CMS collaborations in high-energy ultra-peripheral PbPb collisions \[41]-[43]. The phenomenology of the LBL scattering at the LHC was examined in \[44]-[48]. In a number of papers \[24, 25], [49, 50] a phenomenology of the LBL collisions at future \(e^+e^-\) colliders were presented. The search for ALPs in the \(\gamma\gamma \rightarrow a \rightarrow \gamma\gamma\) collision with proton tagging at the LHC was given in \[12\].

It is a muon collider that could provide the simplest, but the most striking signature of the existence of the ALPs \[51]-[54]. Muon colliders were proposed by F. Tikhonin and G. Budker in the late 1960’s \[55, 56\]. Then they were actively discussed in the early 1980’s \[57, 58\]. Muon colliders have a great potential for high-energy physics since they can offer collisions of elementary particles at very high energies. The point is that muons can be accelerated in a ring without limitation from synchrotron radiation compared to linear or circular electron-positron colliders \[59]-[61\]. Note, however, that getting high luminosity needs to solve a technical problem related to the short muon lifetime at rest and the difficulty of producing large numbers of muons in bunches with small emittance \[62]-[65].

The muon collider could provide a determination of the electroweak couplings of the Higgs boson which is significantly better than what is considered attainable at other future colliders \[66]-[72]. Interest in designing and building a muon collider is also based on its capability of probing the physics beyond the SM. In a number of recent papers searches for SUSY particles
WIMPs, vector boson fusion, leptoquarks, lepton flavor violation, and physics of $(g - 2)_\mu$ at the muon colliders are presented.

In the present paper, we study the high energy production of the ALP in the $\mu^+\mu^- \rightarrow \mu^+\gamma\gamma\mu^-$ process which goes via vector boson fusion subprocess $V_1V_2 \rightarrow a \rightarrow \gamma\gamma$, where $V_{1,2}$ is $\gamma$ or $Z$, and $a$ is a heavy ALP. The main goal is to obtain constraints on the ALP-vector boson coupling as a function of the ALP mass at TeV and multi-TeV muon colliders.

2 ALP in gauge boson scattering

The interaction of the ALP $a$ with SM gauge bosons is described by the Lagrangian

$$L_{\text{int}} = \frac{1}{2} \partial_\mu a \partial_\mu a - \frac{1}{2} m_a^2 a^2 + g^2 C_{BB} \frac{a}{\Lambda} B_{\mu\nu} \tilde{B}^{\mu\nu} + g^2 C_{WW} \frac{a}{\Lambda} W_{\mu\nu}^c \tilde{W}_{\mu\nu}^c, \quad (1)$$

where $B_{\mu\nu}$ and $W_{\mu\nu}^c$ are the field strength of $U(1)_Y$ and $SU(2)_L$, respectively, while $\tilde{B}_{\mu\nu}$ and $\tilde{W}_{\mu\nu}^c$ are dual field strength tensors. As was already mentioned above, the ALP mass $m_a$ and coupling $f_a$ can be regarded as independent parameters. After electroweak symmetry breaking, the ALP couples to the photon and $Z$ boson as

$$L_a = \frac{1}{2} \partial_\mu a \partial_\mu a - \frac{1}{2} m_a^2 a^2 + g_{a\gamma\gamma} a F_{\mu\nu} \tilde{F}^{\mu\nu} + g_{a\gamma Z} a F_{\mu\nu} \tilde{Z}^{\mu\nu} + g_{aZZ} a Z_{\mu\nu} \tilde{Z}^{\mu\nu}, \quad (2)$$

Here $\tilde{F}_{\mu\nu} = (1/2) \varepsilon_{\mu\nu\alpha\beta} F_{\alpha\beta}$ and $\tilde{Z}_{\mu\nu} = (1/2) \varepsilon_{\mu\nu\alpha\beta} Z_{\alpha\beta}$ are the dual tensors, and

$$g_{a\gamma\gamma} = \frac{e^2}{\Lambda} [C_{WW} + C_{BB}],$$

$$g_{a\gamma Z} = \frac{2e^2}{\Lambda s_w c_w} [c_w^2 C_{WW} - s_w^2 C_{BB}],$$

$$g_{aZZ} = \frac{e^2}{\Lambda s_w^2 c_w^2} [c_w^4 C_{WW} + s_w^4 C_{BB}], \quad (3)$$

where $s_w$ and $c_w$ are sine and cosine of the Weinberg angle, respectively.

In what follows, we assume that the ALP couples to hypercharge $U(1)_Y$, not to $SU(2)_L$, that corresponds to $C_{WW} = 0$. Let us define $e^2 C_{BB}/\Lambda = 1/f_a$, ...
then a set of the ALP couplings takes the form
\begin{align}
g_{a\gamma\gamma} &= \frac{1}{f_a}, \quad g_{a\gamma Z} = \frac{-2s_w}{c_w f_a}, \quad g_{aZZ} = \frac{s_w^2}{c_w^2 f_a}. \tag{4}
\end{align}

We also assume that the ALP has a nonzero total width
\begin{align}
\Gamma_a &= \frac{\Gamma(a \to \gamma\gamma)}{\text{Br}(a \to \gamma\gamma)}, \tag{5}
\end{align}
where
\begin{align}
\Gamma(a \to \gamma\gamma) &= \frac{m_a^3}{4\pi f_a^2}. \tag{6}
\end{align}
is the ALP decay width into two photons. In general, the ALP can also couple to fermions as \( \partial^\mu a \bar{\psi} \gamma_\mu \gamma_5 \psi \). But for \( m_a \gg m_\psi \) the full width of the ALP decay should be mainly defined by its decay to two photons. In our upcoming calculations the ALP branching \( \text{Br}(a \to \gamma\gamma) \) is considered as a free parameter that is equal to (or less than) 1.

The differential cross section of the subprocess \( V_1 V_2 \to \gamma\gamma \), where \( V_1,2 = \gamma \) or \( Z \), is a sum of helicity amplitudes squared
\begin{align}
\frac{d\hat{\sigma}}{d\Omega} &= \frac{1}{64\pi^2 \hat{s}} \sum_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} |M_{V_1 V_2}^{\lambda_1 \lambda_2 \lambda_3 \lambda_4}|^2, \tag{7}
\end{align}
where \( \sqrt{\hat{s}} \) is a collision energy of this subprocess, and \( \lambda_i \) are boson helicities. In its turn, each of the helicity amplitudes in (7) is a sum of the ALP and SM (electroweak) terms,
\begin{align}
M &= M_a + M_{\text{ew}}. \tag{8}
\end{align}
The Feynman diagrams describing \( M_a \) are shown in Fig. 1. The explicit expressions of the ALP helicity amplitudes of the \( \gamma\gamma \to \gamma\gamma \) process can be found in [12] (see also [21]). The results of our calculations of the ALP helicity amplitudes \( M_a \) of the processes \( Z\gamma \to \gamma\gamma \) and \( ZZ \to \gamma\gamma \) are presented in Appendix A. Each of the SM amplitudes is a sum of the fermion and \( W \) boson one-loop amplitudes
\begin{align}
M_{\text{ew}} &= M_{\text{ew}}^f + M_{\text{ew}}^W. \tag{9}
\end{align}
The SM helicity amplitudes \( M_{\text{ew}}^f \) and \( M_{\text{ew}}^W \) have been calculated for the processes \( \gamma\gamma \to \gamma\gamma \) [79]-[81] (see also [82]), \( \gamma\gamma \to \gamma Z \) [83], and \( \gamma\gamma \to ZZ \) [84].
Figure 1: The Feynman diagrams describing virtual production of the axion-like particle \( a \) in the collision of two vector bosons \( V_1, V_2 = \gamma \) or \( Z \), with two outgoing photons.

2.1 Differential cross section of diphoton production

We consider the process shown in Figure 1. In the equivalent photon approximation (EPA) [85]-[90], the photon has the following leading logarithmic approximation spectrum [89]:

\[
f_{\gamma/\mu}^{x}(x, Q^2) = \frac{\alpha}{2\pi} \frac{1 + (1 - x)^2}{x} \ln \frac{Q^2}{m_{\mu}^2},
\]

where \( x = E_{\gamma}/E_{\mu} \) is the ratio of the photon energy \( E_{\gamma} \) and energy of the incoming muon \( E_{\mu} \), \( m_{\mu} \) is the muon mass. To examine the collisions of massive vector bosons \((W^{\pm} \text{ and } Z)\), the effective \( W \) approximation (EWA) is applied [91, 92] which allows to treat massive vector bosons as partons inside the colliding beams (see also [93]-[101]). In this scheme the \( Z \) boson has different distributions for its transverse \((T)\) and longitudinal \((L)\) polarizations. The leading order distributions of the \( Z \) boson inside the colliding muon are the following [98, 101]:

\[
f_{ZT/\mu}^{x}(x, Q^2) = \frac{\alpha_{Z}^{\pm}}{2\pi} \frac{1 + (1 - x)^2}{x} \ln \frac{Q^2}{m_{Z}^2},
\]

\[
f_{ZL/\mu}^{x}(x, Q^2) = \frac{\alpha_{Z}^{\pm}}{\pi} \frac{1 - x}{x},
\]

where \( x = E_{Z}/E_{\mu} \) is the fraction of the muon energy held by \( Z \), and

\[
\alpha_{Z}^{\pm} = \frac{\alpha}{(\cos \theta_{W} \sin \theta_{W})^2} \left[ (g_{V}^{\pm})^2 + (g_{A}^{\pm})^2 \right],
\]

\[(12)\]
with \( g_V^\pm = -1/4 \mp \sin^2 \theta_W, \ g_A^\pm = 1/4. \)

Remember that we examine the *exclusive* process \( \mu^+\mu^- \to \mu^+\mu^- + \gamma\gamma. \) Should one study the *inclusive* process \( \mu^+\mu^- \to \gamma\gamma + X, \) where \( X \) is an unspecified remnant, he has to use electroweak parton distribution functions (see [102] and references therein).

The cross section of our process \( \mu^-\mu^+ \to \mu^-V_1V_2\mu^+ \to \mu^-\gamma\gamma\mu^+ \) is defined by the formula

\[
d\sigma = \int_{\tau_{\text{min}}}^{\tau_{\text{max}}} d\tau \int_{x_{\text{min}}}^{x_{\text{max}}} dx \sum_{V_1,V_2 = \gamma,Z_T,Z_L} f_{V_1/\mu^+(x,Q^2)} f_{V_2/\mu^-(\tau/x,Q^2)} d\hat{\sigma}(V_1V_2 \to \gamma\gamma) .
\]

Here

\[
x_{\text{max}} = 1 - \frac{m_\mu}{E_\mu} , \ \tau_{\text{max}} = \left( 1 - \frac{m_\mu}{E_\mu} \right)^2 , \ x_{\text{min}} = \tau / x_{\text{max}} , \ \tau_{\text{min}} = \frac{p_\perp^2}{E_\mu^2} ,
\]

and \( p_\perp \) is the transverse momenta of the outgoing photons. The boson distributions inside the muon beam, \( f_{\gamma/\mu^\pm}(x,Q^2), \) \( f_{Z_T/\mu^\pm}(x,Q^2), \) and \( f_{Z_L/\mu^\pm}(x,Q^2) \) are given by eqs. (10) and (11), respectively, and the subprocess cross section \( d\hat{\sigma}(V_1V_2 \to \gamma\gamma) \) is defined by eq. (7). We take \( Q^2 = \hat{s}, \) where \( \sqrt{\hat{s}} = 2E_\mu \sqrt{\tau} \) is the invariant energy of the VBF subprocess \( V_1V_2 \to \gamma\gamma. \)
2.2 Numerical analysis

The results of our calculations of the differential cross section for the $\mu^+\mu^- \rightarrow \mu^+\gamma\gamma\mu^-$ collision at the future muon collider are presented in Fig. 3. The collision energies $\sqrt{s}$ of 3 TeV, 14 TeV and 100 TeV, and two values of the ALP branching are addressed. For comparison, the SM predictions are shown. As one can see, a discrepancy between the cross section and its SM part rises significantly as the collision energy grows. The same tendency takes place for the total cross section $\sigma(m_{\gamma\gamma} > m_{\gamma\gamma,\text{min}})$, where $m_{\gamma\gamma,\text{min}}$ is the minimal invariant mass of the final photons $m_{\gamma\gamma}$, see Fig. 4. Moreover, the total cross section becomes larger with the increase of $s$ in the whole region of $m_{\gamma\gamma,\text{min}}$. We have used the cut on the rapidity and transverse momentum of the outgoing photons, $|\eta| < 2.5$ and $p_t > 30$ GeV respectively in all calculations.

Figure 3: The differential cross sections for the $\mu^+\mu^- \rightarrow \mu^+\gamma\gamma\mu^-$ scattering at the future muon collider versus diphoton invariant mass $m_{\gamma\gamma}$. The curves correspond to the ALP mass $m_a = 1$ TeV and ALP-gauge boson coupling $f_a = 10$ TeV.

To derive the exclusion region, we apply the following formula for the statistical significance $SS$ [103]

$$SS = \sqrt{2[(S - B \ln(1 + S/B))]},$$

(15)
where $S$ is the number of signal events and $B$ is the number of background (SM) events. We define the regions $SS \leq 1.645$ as the regions that can be excluded at the 95% C.L. To reduce the SM background, we used the cut $m_{\gamma\gamma} > 800$ GeV. The results are shown in Fig. 5. Following [63] (see also [74]), we consider the integrated luminosities of $1 \text{ ab}^{-1}$, $20 \text{ ab}^{-1}$, and $1000 \text{ ab}^{-1}$ for the muon collider energies of 3 TeV, 14 TeV, and 100 TeV, respectively.

As one can see in Fig. 5 for $\sqrt{s} = 3$ TeV the best sensitivity region is limited to a rather sharp region 800 GeV – 2 TeV. In this region the ALP term dominates, while outside it the contributions from the ALP and SM terms are comparable and they partially cancel each other. For $\sqrt{s} = 14$ TeV and $\sqrt{s} = 100$ TeV the ALP contributions dominate in wider regions of the ALP mass ($\geq 800$ GeV). A similar effect was shown to take place for the ALP production in the LBL scattering [24] (for details, see Appendix A in [24]). For comparison, in Fig. 6 we present previously obtained 95% C.L. exclusion region for the ALP-photon coupling coming from the polarized LBL scattering at the 3 TeV CLIC [24]. Other current exclusion regions for this
Figure 5: 95% C.L. exclusion regions for the ALP-gauge boson coupling $f_a$ coming from the $\mu^+\mu^- \to \mu^+\gamma\gamma\mu^-$ scattering at the future muon collider. The curves are obtained with the use of the cut on diphoton invariant mass, $m_{\gamma\gamma} > 800$ GeV.

coupling are also shown [12]. As seen from the Figs. 5 and 6, the excluded areas that we have found from studying diphoton production at the muon collider extends to wider regions, especially for $\sqrt{s} = 14$ TeV and $\sqrt{s} = 100$ TeV.

3 Unitarity constraints on ALP coupling

Let us study bounds imposed by partial-wave unitarity. The partial-wave expansion of the helicity amplitude in the center-of-mass system was derived
Figure 6: Our previous 95% C.L. exclusion region for the ALP-photon coupling in the polarized light-by-light scattering at the 3 TeV CLIC induced by ALPs (green area) \cite{24} in comparison with other current exclusion regions \cite{12}.

It looks like

\[ M_{\lambda_1\lambda_2\lambda_3\lambda_4}(s, \theta, \varphi) = 16\pi \sum_J (2J + 1) \sqrt{(1 + \delta_{\lambda_1\lambda_2})(1 + \delta_{\lambda_3\lambda_4})} \]
\[ \times e^{i(\lambda - \mu)\varphi} d_{\lambda\mu}^J(\theta) T_{\lambda_1\lambda_2\lambda_3\lambda_4}^J(s) , \]

(16)

where \( \lambda = \lambda_1 - \lambda_2 \), \( \mu = \lambda_3 - \lambda_4 \), \( \theta(\phi) \) is the polar (azimuth) scattering angle, and \( d_{\lambda\mu}^J(\theta) \) is the Wigner (small) \( d \)-function \cite{105}. Relevant formulas for the \( d \)-functions can be found in \cite{106}. If we choose the plane \((x - z)\) as a scattering plane, then \( \phi = 0 \) in (16). Parity conservation means that

\[ T_{\lambda_1\lambda_2\lambda_3\lambda_4}^J(s) = (-1)^{\lambda_1+\lambda_2+\lambda_3+\lambda_4} T_{-\lambda_1-\lambda_2-\lambda_3-\lambda_4}^J(s) . \]

(17)

Partial-wave unitarity in the limit \( s \gg (m_1 + m_2)^2 \) requires that

\[ |T_{\lambda_1\lambda_2\lambda_3\lambda_4}^J(s)| \leq 1 . \]

(18)
Using orthogonality of the $d$-functions,

\[
\int_{-1}^{1} d_{\lambda\lambda}^{J}(z) d_{\lambda\lambda}^{J'}(z) \, dz = \frac{2}{2J + 1} \delta_{JJ'} ,
\]

we find from (16) that the partial-wave amplitude is defined as

\[
T_{\lambda_1\lambda_2\lambda_3\lambda_4}^{J}(s) = \frac{1}{32\pi} \frac{1}{\sqrt{(1 + \delta_{\lambda_1\lambda_2})(1 + \delta_{\lambda_3\lambda_4})}} \int_{-1}^{1} M_{\lambda_1\lambda_2\lambda_3\lambda_4}(s, z) d_{\lambda\mu}^{J}(z) \, dz .
\]

(20)

Here and in what follows, $z = \cos \theta$. The helicity amplitudes $M_{\lambda_1\lambda_2\lambda_3\lambda_4}$ are given in Appendix A.

The $d$-functions obey, inter alia, the relation

\[
d_{\lambda\mu}^{J}(z) = (-1)^{J-\lambda} d_{\lambda\mu}^{J}(z).
\]

In particular, we have ($J \geq 0$)

\[
d_{00}^{J}(z) = P_{J}(z) ,
\]

(21)

$P_{J}(z)$ being the Legendre polynomial, and [106]

\[
d_{2-2}^{J}(z) = (-1)^{J} \left( \frac{1-z}{2} \right)^{2} \frac{\Gamma(2-J, J+3; 1; \frac{1+z}{2})}{\Gamma(2-J, J+3; 1; \frac{1-z}{2})},
\]

(22)

\[
d_{22}^{J}(z) = \left( \frac{1+z}{2} \right)^{2} \frac{\Gamma(2-J, J+3; 1; \frac{1-z}{2})}{\Gamma(2-J, J+3; 1; \frac{1+z}{2})},
\]

(23)

where $_2F_1(a, b; c; x)$ is the hypergeometric function [107], and $J \geq 2$.

1. Consider the helicity amplitude $M_{\lambda_1\lambda_2\lambda_3\lambda_4}^{\gamma\gamma+++}(A.3)$. Then $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1$, and $J = \mu = 0$. Since $s, m^2_a \gg m_0\Gamma_a$, we can write

\[
M_{+++}^{\gamma\gamma+++}(s, z) = -\frac{4}{f_a^2} \frac{s^2}{s - m^2_a} .
\]

(24)

The partial-wave amplitude with $J = 0$ is the only non-zero amplitude, since

\[
T_{+++}^{J}(s) = -\frac{1}{16\pi f_a^2} \frac{s^2}{s - m^2_a} \int_{-1}^{1} P_{J}(z) \, dz = -\frac{1}{8\pi f_a^2} \frac{s^2}{s - m^2_a} \delta_{J0} .
\]

(25)
Then we obtain from (18), (25) the unitarity bound on the the ALP-gauge boson coupling

\[ f_a^2 \geq \frac{1}{8\pi} \frac{s}{|1 - \varepsilon|}, \]

(26)

where \( \varepsilon = m_a^2/s \).

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(26)

where \( \varepsilon = m_a^2/s \).

2. For the helicity amplitude \( M_{++-}^{\gamma\gamma} \) we have \( \lambda_1 = \lambda_2 = 1, \lambda_3 = \lambda_4 = -1 \), and, consequently, \( \lambda = 0, \mu = 0 \). It looks like \( \text{(A.7)} \)

\[ M_{++-}^{\gamma\gamma}(s, z) = \frac{2s}{f_a^2} \left[ \frac{2}{1 - \varepsilon} - \frac{(1 - z)^2}{1 - z + 2\varepsilon} - \frac{(1 + z)^2}{1 + z + 2\varepsilon} \right]. \]

(27)

As a result, we obtain

\[ T_{++-}^J(s) = \frac{s}{32\pi f_a^2} \left[ \frac{2}{1 - \varepsilon} \int_{-1}^{1} P_J(z) \, dz - (1 + (-1)^J) \int_{-1}^{1} \frac{(1 - z)^2}{1 - z + 2\varepsilon} P_J(z) \, dz \right] \]

\[ = \frac{s}{8\pi f_a^2} \left[ \varepsilon(3 - 2\varepsilon) \frac{\delta J_0 - (1 + (-1)^J) \varepsilon^2 Q_J(1 + 2\varepsilon)}{1 - \varepsilon} \right], \]

(28)

where \( Q_J(x) \) is the Legendre function of the second kind \[108]. If \( x > 1 \), \( Q_J(x) \) is a real strictly decreasing function of \( J \), and it decreases exponentially as \( J \to \infty \). Note that \( Q_J(1 + 2\varepsilon) \simeq -(\ln \varepsilon)/2 \) for \( \varepsilon \ll 1, J \geq 0 \). The term with \( J = 0 \) is a leading one,

\[ T_{++-}^0(s) = \frac{s}{8\pi f_a^2} \varepsilon \left[ \frac{3 - 2\varepsilon}{1 - \varepsilon} - \varepsilon \ln \frac{1 + \varepsilon}{\varepsilon} \right]. \]

(29)

It results in the following unitarity bound

\[ f_a^2 \geq \frac{s}{8\pi} \varepsilon \left[ \frac{3 - 2\varepsilon}{1 - \varepsilon} - \varepsilon \ln \frac{1 + \varepsilon}{\varepsilon} \right]. \]

(30)

3. Now consider the helicity amplitude \( M_{++-}^{\gamma\gamma} \) \( \text{(A.8)} \). Then \( \lambda_1 = \lambda_4 = 1, \lambda_2 = \lambda_3 = -1 \), and \( \lambda = 2, \mu = -2 \). The helicity amplitude is given by eq. \( \text{(A.8)} \),

\[ M_{++-}^{\gamma\gamma}(s, z) = \frac{2}{f_a^2} \frac{s(1 - z)^2}{1 - z + 2\varepsilon}. \]

(31)
Then we get from (20), (22), (31)

\[ T_{+--}^J(s) = (-1)^J \frac{s}{64\pi f_a^2} \int_{-1}^{1} \frac{(1-z)^4}{1-z+2\varepsilon} 2F_1 \left( 2 - J, J + 3; 1; \frac{1+z}{2} \right) dz \]

\[ = (-1)^J \frac{s}{4\pi f_a^2} I(J, \varepsilon), \] (32)

where the notation

\[ I(J, \varepsilon) = \int_{0}^{1} \frac{x^4}{x+\varepsilon} 2F_1(2 - J, J + 3; 1 - x) \, dx \] (33)

is introduced. Using formula 2.21.1.26 in [109], we obtain a sequence of two equalities (recall that \( J \geq 2 \))

\[ I(J, \varepsilon) = \frac{\Gamma(5)}{\varepsilon \Gamma(3-J) \Gamma(J+4)} 3F_2 \left( 1; 1; 5; 3 - J; J + 4; -\frac{1}{\varepsilon} \right) \]

\[ = (-1)^J \frac{\Gamma(J-1) \Gamma(J+3)}{\varepsilon^{J-1} \Gamma(2J+2)} 2F_1 \left( J - 1, J + 3; 2J + 2; -\frac{1}{\varepsilon} \right), \] (34)

where \( \Gamma(x) \) denotes the gamma function [107]. In (34) we have reduced a generalized hypergeometric function \( 3F_2(a, b, c; d, e; x) \) to a traditional hypergeometric function. With a help of equation 2.10(6) in [107] we find the final analytic expression for \( I(J, \varepsilon) \),

\[ I(J, \varepsilon) = (-1)^J (1 + \varepsilon)^{1-J} \frac{\Gamma(J-1) \Gamma(J+3)}{\Gamma(2J+2)} \times 2F_1 \left( J - 1, J - 1; 2J + 2; \frac{1}{1 + \varepsilon} \right). \] (35)

Using integral representation for the hypergeometric function (see formula 2.12(1) in [107]), one can show that for \( \varepsilon > 0 \) the right-hand side of eq. (35) is a strictly decreasing function of \( J \). Moreover, it falls off exponentially at large \( J \). Thus, the most stringent unitarity bound comes from the partial-wave amplitude with \( J = 2 \) that looks like

\[ T_{+--}^2(s) = \frac{s}{20\pi f_a^2} \frac{1}{1 + \varepsilon} 2F_1 \left( 1, 1; 6; \frac{1}{1 + \varepsilon} \right) \]

\[ = \frac{s}{16\pi f_a^2} \left[ 1 - \frac{4\varepsilon}{3} + 2\varepsilon^2 - 4\varepsilon^3 + 4\varepsilon^4 \ln \frac{1 + \varepsilon}{\varepsilon} \right]. \] (36)
As a result, we come to the unitarity bound

\[ f_a^2 \geq \frac{s}{16\pi} \left| 1 - \frac{4\varepsilon}{3} + 2\varepsilon^2 - 4\varepsilon^3 + 4\varepsilon^4 \ln \frac{1 + \varepsilon}{\varepsilon} \right|. \] (37)

Note that the right-hand side of this equation does not exceed \( s/(16\pi) \).

The analogous examination of the amplitude \( M_{\gamma++}^{++} \), using eqs. (A.9) and (23), results in just the same bound (37).

The unitarity constraints for the amplitudes \( M_{z_1z_2z_3z_4}^{\gamma+} \) and \( M_{z_1z_2z_3z_4}^{zz} \) differ from the above presented bounds for \( M_{\gamma++}^{++} \) (with the same helicities) by the factors \( 2s_w/c_w \approx 1.1 \) and \( s_w^2/c_w^2 \approx 0.3 \), respectively, neglecting small corrections of \( O(m_Z/\sqrt{s}) \) or \( O(m_Z^2/s) \). In particular, imposing unitarity constraint on the amplitude \( M_{z_1z_2z_3z_4}^{++} \), we get the lower bound (up to small corrections \( O(m_Z^2/s) \))

\[ f_a^2 \geq \frac{1}{4\pi c_w} \frac{s}{\left| 1 - \varepsilon \right|}. \] (38)

This constraint is slightly stronger than (26). Unitarity bounds for the most of the other helicity amplitudes with the \( Z \) boson(s) are suppressed by small factors \( m_Z/\sqrt{s} \) or \( m_Z^2/s \).

The constraint (38) appears to be the strongest unitary bound. We find from it that for \( m_a = 800 \) GeV the inverse ALP coupling \( f_a^{-1} \) must be smaller than 1.54 TeV\(^{-1} \), 0.34 TeV\(^{-1} \), and 0.048 TeV\(^{-1} \), for the collision energies of 3 TeV, 14 TeV, and 100 TeV, respectively. Correspondingly, for \( m_a = 10 \) TeV the inverse coupling should be less than 5.01 TeV\(^{-1} \), 0.24 TeV\(^{-1} \), and 0.048 TeV\(^{-1} \). By comparing these constraints with the curves presented in Fig. 5, we conclude that the unitarity is not violated in the region of the ALP coupling \( f_a \) studied in the present paper.

### 4 Conclusions

We have examined the possibility to search for heavy axion-like particles in the \( \mu^+\mu^- \rightarrow \mu^+\gamma\gamma\mu^- \) scattering at the future muon collider. The studies are presented for the collision energies of 3 TeV, 14 TeV, and 100 TeV and integrated luminosities of 1 ab\(^{-1} \), 20 ab\(^{-1} \), and 1000 ab\(^{-1} \), respectively. We have obtained the explicit expressions for the helicity amplitudes for the \( Z\gamma \rightarrow \gamma\gamma \) and \( ZZ \rightarrow \gamma\gamma \) collisions. Using these amplitudes (as well as known helicity amplitudes for the \( \gamma\gamma \rightarrow \gamma\gamma \) collision), the differential cross sections versus invariant mass of the final photons and total cross section versus minimal
diphoton invariant mass are calculated. As a result, the 95% C.L. exclusion regions for the ALP-gauge boson coupling coming from the $\mu^+\mu^- \rightarrow \mu^+\gamma\gamma\mu^-$ scattering at the high energy muon collider are obtained. The excluded areas extend to wider regions in comparison to the region obtained previously for the polarized light-by-light scattering at the 3 TeV CLIC. Our constraints are also much stronger than the current experimental bounds presented in Fig. 6. The partial-wave unitarity bounds on the ALP-gauge boson coupling are estimated. We have shown that the unitarity is not violated in the region of the ALP coupling which has been studied in our paper. We can conclude that the future muon collider has a great physical potential in searching for axion-like particle couplings to the SM gauge bosons.

5 Appendix A. Helicity amplitudes

5.1 $\gamma\gamma \rightarrow \gamma\gamma$ scattering

The Mandelstam variables for the $\gamma\gamma \rightarrow \gamma\gamma$ collision satisfy the relation $s + t + u = 0$, and we get

$$\cos \theta = \frac{u - t}{u + t}, \quad \sin \theta = -\frac{2\sqrt{tu}}{t + u}, \quad (A.1)$$

$$t = -\frac{s}{2} (1 - \cos \theta), \quad u = -\frac{s}{2} (1 + \cos \theta). \quad (A.2)$$

The helicity amplitudes of the LBL scattering are known to be [12]

$$M_{++++}^{\gamma\gamma} = -\frac{4}{f_a^2} \frac{s^2}{s - m_a^2 + im_a \Gamma_a}, \quad (A.3)$$

$$M_{+++ -}^{\gamma\gamma} = 0, \quad (A.4)$$

$$M_{++-+}^{\gamma\gamma} = 0, \quad (A.5)$$

$$M_{+-++}^{\gamma\gamma} = 0, \quad (A.6)$$
\[ M_{++--}^{\gamma\gamma} = \frac{4}{f_a^2} \left( \frac{s^2}{s - m_a^2 + im_a \Gamma_a} + \frac{s^2}{t - m_a^2 + im_a \Gamma_a} \right), \]  
\[ M_{+-+-}^{\gamma\gamma} = -1 \frac{s^2(1 - \cos \theta)^2}{f_a^2 (s - m_a^2 + im_a \Gamma_a)}, \]  
\[ M_{+-+-}^{\gamma\gamma} = -1 \frac{s^2(1 + \cos \theta)^2}{f_a^2 (u - m_a^2 + im_a \Gamma_a)}, \]  
\[ M_{+-+-}^{\gamma\gamma} = 0. \]  

Other helicity amplitudes \( M_{\lambda_1\lambda_2\lambda_3\lambda_4}^{\gamma\gamma} \) can be obtained by the \( P \)-parity relation
\[ M_{\lambda_1\lambda_2\lambda_3\lambda_4}^{\gamma\gamma} = M_{-\lambda_1-\lambda_2-\lambda_3-\lambda_4}^{\gamma\gamma}. \]  

5.2 \( Z\gamma \rightarrow \gamma\gamma \) scattering

The Mandelstam variables for this process obey the relation \( s + t + u = m_Z^2 \), variables \( \cos \theta, \sin \theta \) are given by eq. (A.1), and
\[ t = -\frac{s - m_Z^2}{2} (1 - \cos \theta), \quad u = -\frac{s - m_Z^2}{2} (1 + \cos \theta). \]  

Our calculations result in the following analytic expressions for the helicity amplitudes of the \( Z\gamma \rightarrow \gamma\gamma \) process:
\[ M_{++--}^{Z\gamma} = \frac{8s_w}{c_w f_a^2} \frac{1}{s - m_Z^2} \frac{s(s - m_Z^2)}{s - m_a^2 + im_a \Gamma_a}, \]  
\[ M_{+-+-}^{Z\gamma} = \frac{2s_w}{c_w f_a^2} \frac{m_Z^2(s - m_Z^2)(\sin \theta)^2}{t - m_a^2 + im_a \Gamma_a}, \]  
\[ M_{+-+-}^{Z\gamma} = \frac{2s_w}{c_w f_a^2} \frac{m_Z^2(s - m_Z^2)(\sin \theta)^2}{u - m_a^2 + im_a \Gamma_a}, \]  
\[ M_{+-+-}^{Z\gamma} = 0. \]
\begin{align}
M^{Z\gamma}_{++--} &= -\frac{2s_w}{c_w f_a^2} \frac{m_Z^2(s - m_Z^2)(\sin \theta)^2}{t - m_a^2 + im_a \Gamma_a + u - m_a^2 + im_a \Gamma_a}, \\
M^{Z\gamma}_{++--} &= \frac{8s_w}{c_w f_a^2} \frac{s(s - m_Z^2)}{s - m_a^2 + im_a \Gamma_a} - \frac{2s_w}{c_w f_a^2} \frac{s(s - m_Z^2)}{u - m_a^2 + im_a \Gamma_a} \\
M^{Z\gamma}_{++--} &= \frac{2s_w}{c_w f_a^2} \frac{s(s - m_Z^2)(1 - \cos \theta)^2}{t - m_a^2 + im_a \Gamma_a} \\
M^{Z\gamma}_{++--} &= \frac{2s_w}{c_w f_a^2} \frac{s(s - m_Z^2)(1 + \cos \theta)^2}{u - m_a^2 + im_a \Gamma_a}, \\
M^{Z\gamma}_{++--} &= 0, \\
M^{Z\gamma}_{++--} &= 0, \\
M^{Z\gamma}_{0++] &= \frac{4is_w}{\sqrt{2} c_w f_a^2} \frac{m_Z \sqrt{s(s - m_Z^2)(1 - \cos \theta)\sin \theta}}{t - m_a^2 + im_a \Gamma_a}, \\
M^{Z\gamma}_{0++] &= -\frac{4is_w}{\sqrt{2} c_w f_a^2} \frac{m_Z \sqrt{s(s - m_Z^2)(1 + \cos \theta)\sin \theta}}{u - m_a^2 + im_a \Gamma_a}, \\
M^{Z\gamma}_{0++] &= \frac{4is_w}{\sqrt{2} c_w f_a^2} \frac{m_Z \sqrt{s(s - m_Z^2)\sin \theta}}{t - m_a^2 + im_a \Gamma_a} \\
&\times \left[ \frac{(1 - \cos \theta)}{t - m_a^2 + im_a \Gamma_a} - \frac{(1 + \cos \theta)}{u - m_a^2 + im_a \Gamma_a} \right],
\end{align}
\[ M_{0+-}^{Z\gamma} = -\frac{4i}{\sqrt{2}} s_{w} c_{w} f_{a}^{2} \sqrt{s} \left( s - m_{Z}^{2} \right) \sin \theta \times \left[ \frac{(1 - \cos \theta)}{t - m_{a}^{2} + im_{a} \Gamma_{a}} - \frac{(1 + \cos \theta)}{u - m_{a}^{2} + im_{a} \Gamma_{a}} \right], \quad (A.25) \]

\[ M_{0--}^{Z\gamma} = \frac{4i}{\sqrt{2}} s_{w} c_{w} f_{a}^{2} \sqrt{s} \left( s - m_{Z}^{2} \right)(1 - \cos \theta) \sin \theta \frac{(1 - \cos \theta)}{t - m_{a}^{2} + im_{a} \Gamma_{a}}. \quad (A.26) \]

Other amplitudes \( M_{\lambda_{1}\lambda_{2}\lambda_{3}\lambda_{4}}^{Z\gamma} \) can be obtained by relation \[83\]

\[ M_{\lambda_{1}\lambda_{2}\lambda_{3}\lambda_{4}}^{Z\gamma} = (-1)^{1-\lambda_{1}} M_{-\lambda_{1}-\lambda_{2}-\lambda_{3}-\lambda_{4}}^{Z\gamma}, \quad (A.27) \]

where \( \lambda_{1} \) is a helicity of the \( Z \) boson.

### 5.3 \( ZZ \rightarrow \gamma\gamma \) scattering

The Mandelstam variables obey the relation \( s + t + u = 2m_{Z}^{2} \), and we obtain

\[ \cos \theta = \frac{t - u}{\sqrt{(t + u)^{2} - 4m_{Z}^{2}}}; \quad \sin \theta = \frac{2\sqrt{tu - m_{Z}^{4}}}{\sqrt{(t + u)^{2} - 4m_{Z}^{2}}}, \quad (A.28) \]

\[ t = -\frac{1}{2} \left[ (s - 2m_{Z}^{2}) - \sqrt{s(s - 4m_{Z}^{2})} \cos \theta \right], \]

\[ u = -\frac{1}{2} \left[ (s - 2m_{Z}^{2}) + \sqrt{s(s - 4m_{Z}^{2})} \cos \theta \right]. \quad (A.29) \]

We have derived the following helicity amplitudes of the \( ZZ \rightarrow \gamma\gamma \) process:

\[ M_{++++}^{ZZ} = -\frac{4s_{w}^{2}}{c_{w}^{2} f_{a}^{2}} s^{3/2} \sqrt{s - 4m_{Z}^{2}} + \frac{s_{w}^{2}}{c_{w}^{2} f_{a}^{2}} s \left( \sqrt{s} - \sqrt{s - 4m_{Z}^{2}} \right)^{2} \times \left[ \frac{(1 + \cos \theta)^{2}}{t - m_{a}^{2} + im_{a} \Gamma_{a}} + \frac{(1 - \cos \theta)^{2}}{u - m_{a}^{2} + im_{a} \Gamma_{a}} \right], \quad (A.30) \]

\[ M_{+++-}^{ZZ} = -\frac{4s_{w}^{2}}{c_{w}^{2} f_{a}^{2}} m_{Z}^{2} s(\sin \theta)^{2} \times \left[ \frac{1}{t - m_{a}^{2} + im_{a} \Gamma_{a}} + \frac{1}{u - m_{a}^{2} + im_{a} \Gamma_{a}} \right], \quad (A.31) \]
\[ M_{++--}^{ZZ} = \frac{4s_w^2}{c_w^2 f_a^2} m_Z^2 s (\sin \theta)^2 \]
\[
\times \left[ \frac{1}{t - m_a^2 + i m_a \Gamma_a} + \frac{1}{u - m_a^2 + i m_a \Gamma_a} \right],
\]
(A.32)

\[ M_{+-+-}^{ZZ} = -\frac{s_w^2}{c_w^2 f_a^2} (1 + \cos \theta)^2 \]
\[
\times \left\{ \frac{[s - \sqrt{s(s - 4m_Z^2)}]}{t - m_a^2 + i m_a \Gamma_a} + \frac{[s + \sqrt{s(s - 4m_Z^2)}]}{u - m_a^2 + i m_a \Gamma_a} \right\},
\]
(A.33)

\[ M_{++--}^{ZZ} = \frac{4s_w^2}{c_w^2 f_a^2} m_Z^3 \left( \frac{s - \sqrt{s(s - 4m_Z^2)}}{s - m_a^2 + i m_a \Gamma_a} \right)^{3/2} \]
\[
+ \frac{s_w^2}{c_w^2 f_a^2} \left[ s + \sqrt{s(s - 4m_Z^2)} \right]^2 \left( \frac{1 - \cos \theta}{t - m_a^2 + i m_a \Gamma_a} + \frac{(1 + \cos \theta)^2}{u - m_a^2 + i m_a \Gamma_a} \right),
\]
(A.34)

\[ M_{+-+-}^{ZZ} = -\frac{s_w^2}{c_w^2 f_a^2} (1 - \cos \theta)^2 \]
\[
\times \left\{ \frac{[s + \sqrt{s(s - 4m_Z^2)}]}{t - m_a^2 + i m_a \Gamma_a} + \frac{[s - \sqrt{s(s - 4m_Z^2)}]}{u - m_a^2 + i m_a \Gamma_a} \right\},
\]
(A.35)

\[ M_{0+++}^{ZZ} = \frac{4i s_w^2}{\sqrt{2} c_w^2 f_a^2} m_Z s \left( \sqrt{s - \sqrt{s^2 - 4m_Z^2}} \right) \sin \theta \]
\[
\times \left[ \frac{(1 + \cos \theta)}{t - m_a^2 + i m_a \Gamma_a} - \frac{(1 - \cos \theta)}{u - m_a^2 + i m_a \Gamma_a} \right],
\]
(A.36)

\[ M_{0++-}^{ZZ} = -\frac{4i s_w^2}{\sqrt{2} c_w^2 f_a^2} m_Z \sqrt{s} (1 - \cos \theta) \sin \theta \]
\[
\times \left[ \frac{[s + \sqrt{s(s - 4m_Z^2)}]}{t - m_a^2 + i m_a \Gamma_a} + \frac{[s - \sqrt{s(s - 4m_Z^2)}]}{u - m_a^2 + i m_a \Gamma_a} \right],
\]
(A.37)
\[ M^{ZZ}_{0-++} = \frac{4i}{\sqrt{2}} \frac{s^2_w}{c_w f_a^2} m_Z \sqrt{s} \left[ s + \sqrt{s(s - 4m_Z^2)} \right] \sin \theta \]
\[ \times \left[ \frac{1 - \cos \theta}{t - m_a^2 + i m_a \Gamma_a} - \frac{1 + \cos \theta}{u - m_a^2 + i m_a \Gamma_a} \right], \quad (A.38) \]

\[ M^{ZZ}_{0++-} = \frac{4i}{\sqrt{2}} \frac{s^2_w}{c_w f_a^2} m_Z \sqrt{s(1 + \cos \theta)} \sin \theta \]
\[ \times \left[ \frac{s - \sqrt{s(s - 4m_Z^2)}}{t - m_a^2 + i m_a \Gamma_a} + \frac{s + \sqrt{s(s - 4m_Z^2)}}{u - m_a^2 + i m_a \Gamma_a} \right], \quad (A.39) \]

\[ M^{ZZ}_{0-+-} = -\frac{4i}{\sqrt{2}} \frac{s^2_w}{c_w f_a^2} m_Z \sqrt{s} \left[ s + \sqrt{s(s - 4m_Z^2)} \right] \sin \theta \]
\[ \times \left[ \frac{1 - \cos \theta}{t - m_a^2 + i m_a \Gamma_a} - \frac{1 + \cos \theta}{u - m_a^2 + i m_a \Gamma_a} \right], \quad (A.40) \]

\[ M^{ZZ}_{00++} = -\frac{8s^2_w}{c_w f_a^2} m_Z^2 s (\sin \theta)^2 \]
\[ \times \left[ \frac{1}{t - m_a^2 + i m_a \Gamma_a} + \frac{1}{u - m_a^2 + i m_a \Gamma_a} \right], \quad (A.41) \]

\[ M^{ZZ}_{00+-} = -\frac{8s^2_w}{c_w f_a^2} m_Z^2 s (\sin \theta)^2 \]
\[ \times \left[ \frac{1}{t - m_a^2 + i m_a \Gamma_a} + \frac{1}{u - m_a^2 + i m_a \Gamma_a} \right], \quad (A.42) \]

Other helicity amplitudes \( M^{ZZ}_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} \) can be obtained using relation 84

\[ M^{ZZ}_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} = (-1)^{\lambda_1 - \lambda_2} M^{ZZ}_{-\lambda_1 - \lambda_2 \lambda_3 \lambda_4}, \quad (A.43) \]

where \( \lambda_1, \lambda_2 \) are helicities of the colliding \( Z \) bosons.
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