Lie Algebras/Differential Geometry

Exponential map and $L_\infty$ algebra associated to a Lie pair

Application exponentielle et algèbre $L_\infty$ associée à une paire de Lie

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Abstrait

In this Note, we unveil homotopy-rich algebraic structures generated by the Atiyah classes relative to a Lie pair $(L, A)$ of algebroids. In particular, we prove that the quotient $L/A$ of such a pair admits an essentially canonical homotopy module structure over the Lie algebroid $A$, which we call Kapranov module.

Résumé

Dans cette note, nous dévoilons des structures algébriques riches en homotopies, engendrées par les classes d’Atiyah relatives à une paire de Lie $(L, A)$ d’algebroides. En particulier, nous prouvons que le quotient $L/A$ d’une telle paire admet une structure essentiellement canonique de module à homotopie près sur l’algebroïde de Lie $A$ que nous appelons module de Kapranov.

Version française abrégée

Étant donnée une paire de Lie $(L, A)$, c.a.d. une algébroïde de Lie $L$ et une sous-algébroïde de Lie $A$, la classe d’Atiyah $\alpha_E$ d’un $A$-module $E$ relative à la paire de Lie $(L, A)$ est définie comme l’obstruction à l’existence d’une $L$-connexion $A$-compatible sur $E$. Le lecteur pourrait souhaiter consulter les deux premiers paragraphes de la Section 1 ou la première section de [3] pour un rappel des définitions. Cette classe, dont la définition est fort récente [3], a pour double origine les classes d’Atiyah des fibrés vectoriels holomorphes et les classes de Molino des feuilletages qu’elle généralise. Le quotient $L/A$ d’une paire de Lie est un $A$-module [3]. Voici une description de sa classe d’Atiyah $\alpha_{L/A}$. La courbure d’une $L$-connexion $\nabla$ sur $L/A$ (choix de façon arbitraire) est le morphisme de fibrés $R^\nabla : \wedge^2 L \rightarrow \text{End}(L/A)$ défini par la relation $R^\nabla(l_1, l_2) = \nabla_l_1 \nabla_l_2 - \nabla_l_2 \nabla_l_1 - \nabla_{[l_1, l_2]}$, pour tous $l_1, l_2 \in \Gamma(L)$. Puisque $L/A$ est un $A$-module, $R^\nabla$ s’annule sur $\wedge^2 A$ et, par conséquent, détermine une section $R^\nabla_{L/A}$ du fibré vectoriel $A^* \otimes (L/A)^* \otimes \text{End}(L/A)$. Il fut établi dans [3] que $R^\nabla_{L/A}$ est un 1-cocycle pour l’algébroïde de Lie $A$ à valeurs dans le $A$-module $(L/A)^* \otimes \text{End}(L/A)$ et que sa classe de cohomologie $\alpha_{L/A} \in H^1(A; (L/A)^* \otimes \text{End}(L/A))$ est indépendante du choix de la connexion.

Définition 0.1. Soit $A$ une algébroïde de Lie sur une variété différentiable $M$. Un fibré vectoriel $E \rightarrow M$ est un module de Kapranov sur $A$ si $\Gamma(\wedge^* \otimes E)$ est une $L_\infty[1]$-algébre définie par une suite d’applications $\lambda_k : \otimes^k \Gamma((\wedge^* \otimes E) \rightarrow \Gamma((\wedge^* \otimes E)[1])$ pour $k \in \mathbb{N}$) dont la première $\lambda_1 : \Gamma((\wedge^1 \otimes E) \rightarrow \Gamma((\wedge^1 \otimes E)$ est la différentielle de Chevalley-Eilenberg.
associée à une action infinitésimale de \( A \) sur \( E \) et les suivantes sont \( \Gamma(\wedge^\bullet A^\bullet) \)-multilinéaires. Nous appellerons \( k \)-ième crochet de Kapranov l’application \( \lambda_k \).

**Proposition 0.2.** Soit \( A \) une algèbre de Lie sur une variété différentiable \( M \). Un fibré vectoriel \( E \) sur \( M \) est un module de Kapranov sur \( A \) si, et seulement si, \( A \) agit infinitésimalement sur \( E \) et il existe une suite de morphismes de fibrés vectoriels \( R_k : S^k(E) \rightarrow A^\bullet \otimes E \) (\( k \geq 2 \)) dont la somme \( R = \sum_{k=2}^\infty R_k \in \Gamma(A^\bullet \otimes S^k(E) \otimes E) \) satisfait l’équation de Maurer–Cartan

\[
d_A R + \frac{1}{2} [R, R] = 0. \tag{1}
\]

(ici, on considère les sections de \( S(E) \otimes E \) comme des champs de vecteurs formels sur \( E \) le long de la section nulle et on en déduit un crochet de Lie naturel sur l’espace vectoriel gradué \( \Gamma(A^\bullet \otimes \hat{S}(E) \otimes E) \).) Pour tout \( k \geq 2 \), le \( k \)-ième crochet de Kapranov \( \lambda_k \) est lié à la \( k \)-ième composante \( R_k \in \Gamma(A^\bullet \otimes S^k(E) \otimes E) \) de l’élément de Maurer–Cartan \( R \) au travers de l’identité

\[
\lambda_k(\xi_1 \otimes b_1, \ldots, \xi_k \otimes b_k) = (-1)^{|\xi_1|+\ldots+|b_k|} \xi_1 \wedge \ldots \wedge \xi_k \wedge R_k(b_1, \ldots, b_k),
\]

valide pour tous \( b_1, \ldots, b_k \in \Gamma(E) \) et tous éléments homogènes \( \xi_1, \ldots, \xi_k \in \Gamma(\wedge^\bullet A^\bullet) \).

Le théorème qui suit résume notre principal résultat :

**Théorème 0.3.** Le quotient \( L/A \) d’une paire de Lie \((L, A)\) admet une structure de module de Kapranov sur l’algèbre de Lie \( A \), canonique à isomorphisme près, dont le \( R_2 \in \Gamma(A^\bullet \otimes S^2(L/A)^\bullet \otimes L/A) \) (voir Proposition 1.1) est un cocycle représentant la classe d’Atiyah de \( L/A \) relative à la paire \((L, A)\).

De surcroît, si l’algèbre de Lie \( L \) est le fruit de l’accouplement \( A \mapsto B \) de l’algèbre de Lie \( A \) avec une autre algèbre de Lie \( B \) telle qu’il existe une \( B \)-connexion \( \nabla \) sur \( B \) sans torsion ni courbure, alors les composantes de l’élément de Maurer–Cartan \( R \) sont liées entre elles par la relation de récurrence \( R_{k+1} = \hat{\nabla}^k R_k \) où le symbole \( \hat{\nabla} \) désigne la différentielle covariante associée à la connexion \( \nabla \).

Comme corollaires, nous retrouvons deux résultats de [3] \( (\text{cf. Corollaires 3.2 et 3.3}) \).

1. Kapranov modules

   Let \( A \) be a Lie algebroid (either real or complex) over a manifold \( M \) with anchor \( \rho \). By an \( A \)-module, we mean a module of the corresponding Lie–Rinehart algebra \( \Gamma(A) \) over the associative algebra \( C^\infty(M) \). An \( A \)-connection on a smooth vector bundle \( E \) over \( M \) is a bilinear map \( \mathcal{V} : \Gamma(A) \otimes \Gamma(E) \rightarrow \Gamma(E) \) satisfying \( \mathcal{V}_{fe} = f \mathcal{V}_{e} \) and \( \mathcal{V}_{(f\alpha)e} = (\rho(\alpha)f)e + f \mathcal{V}_{\alpha}e \), for all \( a \in \Gamma(A) \), \( e \in \Gamma(E) \), and \( f \in C^\infty(M) \). A vector bundle \( E \) endowed with a flat \( A \)-connection (also known as an infinitesimal \( A \)-action) is an \( A \)-module; more precisely, its space of smooth sections \( \Gamma(E) \) is one.

Atiyah class Given a Lie pair \((L, A)\) of algebroids, i.e., a Lie algebroid \( L \) with a Lie subalgebroid \( A \), the Atiyah class \( \alpha_E \) of an \( A \)-module \( E \) relative to the pair \((L, A)\) is defined as the obstruction to the existence of an \( A \)-compatible \( L \)-connection on \( E \). An \( L \)-connection \( \nabla \) is \( A \)-compatible if its restriction to \( \Gamma(L) \otimes \Gamma(E) \) is the given infinitesimal \( A \)-action on \( E \) and \( \nabla_a \nabla_b \xi = \nabla_b \nabla_a \xi = [\xi, \{a, b\}]_{L/A} \) for all \( a \in \Gamma(L) \) and \( b \in \Gamma(L) \). This fairly recently defined class \( \alpha_E \) has as double origin, which it generalizes, the Atiyah class of holomorphic vector bundles and the Molino class of foliations. The quotient \( L/A \) of the Lie pair \((L, A)\) is an \( A \)-module \( \alpha_E \). Its Atiyah class \( \alpha_{L/A} \) can be described as follows. Choose an \( L \)-connection \( \nabla \) on \( L/A \) extending the \( A \)-action. Its curvature is the vector bundle map \( R^\nabla : \wedge^2 L \rightarrow \text{End}(L/A) \) defined by \( R^\nabla(l_1, l_2) = \nabla_{l_1} l_2 - \nabla_l_2 l_1 - \nabla_{[l_1, l_2]} \) for all \( l_1, l_2 \in \Gamma(L/A) \). Since \( L/A \) is an \( A \)-module, \( R^\nabla \) vanishes on \( \wedge^2 A \) and, therefore, determines a section \( R^\nabla_{L/A} \) of \( \wedge^2 \otimes (L/A)^* \otimes \text{End}(L/A) \). It was proved in [3] \( R^\nabla_{L/A} \) is a 1-cocycle for the Lie algebra \( A \) with values in the \( A \)-module \( (L/A)^* \otimes \text{End}(L/A) \) and that its cohomology class \( \alpha_{L/A} \in H^1(A; (L/A)^* \otimes \text{End}(L/A)) \) is independent of the choice of the connection.

**Kapranov modules over a Lie algebroid** Let \( M \) be a smooth manifold, and let \( L \) be the algebra of smooth functions on \( M \) valued in \( \mathbb{R} \) (or \( \mathbb{C} \)). Let \( A \) be a Lie algebroid over \( M \). The Chevalley–Eilenberg differential \( d_A \) and the exterior product make \( \wedge^\bullet A^\bullet \) into a differential graded commutative \( \mathbb{R} \)-algebra.

Now let \( E \) be a smooth vector bundle over \( M \). Deconcatenation defines an \( R \)-coalgebra structure on \( \Gamma(S^\bullet E) \). Let \( \mathcal{E} \) denote the ideal of \( \Gamma(S^\bullet (E^\bullet)) \) generated by \( \Gamma(E^\bullet) \). The algebra \( \text{Hom}_\mathbb{R}(\Gamma(S^\bullet E), R) \) dual to the coalgebra \( \Gamma(S^\bullet E) \) is the \( \mathbb{C} \)-adic completion of \( \Gamma(S^\bullet (E^\bullet)) \). It will be denoted by \( \Gamma(S^\bullet (E^\bullet)) \). Equivalently, one can think of the completion \( S^\bullet (E^\bullet) \) as a bundle of algebroids over \( M \). Note that \( \Gamma(\wedge^\bullet A^\bullet \otimes S^\bullet E^\bullet) \) is an \( A \)-module.

Recall that an \( \mathcal{L}_{\infty}[1] \) algebra is a \( \mathbb{Z} \)-graded vector space \( V = \bigoplus_{n \in \mathbb{Z}} V_n \) endowed with a sequence \( (\lambda_k)_{k=1}^\infty \) of skew-symmetric multilinear maps \( \lambda_k : \otimes^k V \rightarrow V \) of degree 1 satisfying the generalized Jacobi identity

\[
\sum_{k=1}^n \sum_{\sigma \in \mathfrak{S}_{n-k}} \mathcal{E}(\sigma) \cdot v_{\sigma(1)}, \ldots, v_{\sigma(n)} \lambda_{1+n-k}(\lambda_k(v_{\sigma(1)}, \ldots, v_{\sigma(k)}), v_{\sigma(k+1)}, \ldots, v_{\sigma(n)}) = 0
\]
for each \( n \in \mathbb{N} \) and for any homogeneous vectors \( v_1, v_2, \ldots, v_n \in V \). Here \( \mathcal{S}_q^n \) denotes the set of \((p, q)\)-shuffles\(^1\) and \( \varepsilon(\sigma; v_1, \ldots, v_n) \) the Koszul sign\(^2\) of the permutation \( \sigma \) of the (homogeneous) vectors \( v_1, v_2, \ldots, v_n \).

**Definition 1.1.** A Kapranov module over a Lie algebroid \( A \to M \) is a vector bundle \( E \to M \) together with an \( L_\infty \) algebra structure on \( \Gamma(\wedge^*A^* \otimes E) \) defined by a sequence \((\lambda_k)_{k \in \mathbb{N}}\) of multibrackets (called Kapranov multibrackets) such that (1) the unary bracket \( \lambda_1 : \Gamma(\wedge^*A^* \otimes E) \to \Gamma(\wedge^2A^* \otimes E) \) is the Chevalley–Eilenberg differential associated to an infinitesimal \( A \)-action on \( E \), and (2) all multibrackets \( \lambda_k : \otimes^k \Gamma(\wedge^*A^* \otimes E) \to \Gamma(\wedge^*A^* \otimes E) \) with \( k \geq 2 \) are \( \Gamma(\wedge^*A^*) \)-multilinear.

**Proposition 1.1.** Let \( A \) be a Lie algebroid over a smooth manifold \( M \) and let \( E \) be a smooth vector bundle over \( M \). Each of the following four data is equivalent to a Kapranov \( A \)-module structure on \( E \):

(i) A degree 1 derivation \( D \) of the graded algebra \( \Gamma(\wedge^*A^* \otimes \tilde{S}(E^*)) \), which preserves the filtration \( \tilde{S}(\wedge^*A^* \otimes \tilde{S}^{\geq n}(E^*)) \), satisfies \( D^2 = 0 \), and whose restriction to \( \Gamma(\wedge^*A^* \otimes \tilde{S}(E^*)) \) is the Chevalley–Eilenberg differential of the Lie algebroid \( A \). (Here, by convention, all elements of \( \tilde{S}(E^*) \) have degree 0.)

(ii) An infinitesimal action of \( A \) on \( \tilde{S}(E^*) \) by derivations which preserve the decreasing filtration \( \tilde{S}^{\geq n}(E^*) \).

(iii) An infinitesimal action of \( A \) on \( S(E) \) by coderivations which preserve \( S^{\geq 1}(E) \) and the increasing filtration \( S^{\leq n}(E) \).

(iv) An infinitesimal action of \( A \) on \( E \) together with a sequence of morphisms of vector bundles \( R_k : S^k(E) \to A^* \otimes E \) (\( k \geq 2 \)) whose sum \( R = \sum_{k=2}^{\infty} R_k \in \Gamma(A^* \otimes \tilde{S}(E^*) \otimes E) \) is a solution of the Maurer–Cartan equation \( d_A R + \frac{1}{2}[R, R] = 0 \). (Here, we consider \( \Gamma(\tilde{S}(E^*) \otimes E) \) as the space of formal vertical vector fields on \( E \) along the zero section and derive a natural Lie bracket on the graded vector space \( \Gamma(\wedge^*A^* \otimes \tilde{S}(E^*) \otimes E) \).)

Characterizations (i) and (iv) are related by the identity \( D = d_A^{\tilde{S}(E^*)} + R \), where \( d_A^{\tilde{S}(E^*)} \) denotes the Chevalley–Eilenberg differential associated to the infinitesimal \( A \)-action on \( E \), and \( R \) denotes its own action on \( \Gamma(\wedge^*A^* \otimes \tilde{S}(E^*)) \) by contraction.

On the other hand, for any \( k \geq 2 \), the \( k \)-th Kapranov multibracket \( \lambda_k \) is related to the \( k \)-th component \( R_k \in \Gamma(A^* \otimes S^kE^* \otimes E) \) of the Maurer–Cartan–Cattaneo module \( R \) through the equation

\[
\lambda_k(\xi_1 \otimes e_1, \ldots, \xi_k \otimes e_k) = (-1)^{|\xi_1|+\cdots+|\xi_k|} \xi_1 \wedge \cdots \wedge \xi_k \wedge R_k(e_1, \ldots, e_k),
\]

which is valid for any \( e_1, \ldots, e_k \in \Gamma(E) \) and any homogeneous elements \( \xi_1, \ldots, \xi_k \) of \( \Gamma(\wedge^*A^*) \).

The algebraic structure described in the above proposition is related to Costello’s \( L_\infty \) algebras over the differential graded algebra \( (\Gamma(\wedge^*A^*), d_A) \) [4], and to Yu’s \( \mathcal{L} \)-algebras [9].

Two Kapranov \( A \)-modules \( E_1 \) and \( E_2 \) over \( M \) are isomorphic if there exists an isomorphism \( \Phi : S(E_1) \to S(E_2) \) of bundles of coalgebras over \( M \), which intertwines the infinitesimal \( A \)-actions.

2. Exponential map and Poincaré–Birkhoff–Witt isomorphism

Assume \( \mathcal{A} \) is a Lie subgroupoid of a Lie groupoid \( \mathcal{L} \) (over the same unit space), and let \( A \) and \( L \) denote the corresponding Lie algebroids. The source map \( s : \mathcal{L} \to M \) factors through the quotient of the action of \( \mathcal{A} \) on \( \mathcal{L} \) by multiplication from the right. Therefore, it induces a surjective submersion \( s : \mathcal{L} / \mathcal{A} \to M \). Note that the zero section \( 0 : M \to L/A \) and the unit section \( 1 : M \to \mathcal{L} / \mathcal{A} \) are both embeddings of \( M \).

**Proposition 2.1.** Each choice of a splitting of the short exact sequence of vector bundles \( 0 \to A \to L \to L/A \to 0 \) and of an \( L \)-connection \( \nabla \) on \( L/A \) extending the \( A \)-action determines an exponential map, i.e. a fiber bundle map \( \exp^\nabla : L/A \to \mathcal{L} / \mathcal{A} \), which identifies the zero section of \( L/A \) to the unit section of \( \mathcal{L} / \mathcal{A} \), whose differential along the zero section of \( L/A \) is the canonical isomorphism between \( L/A \) and the tangent bundle to the s-foliation of \( \mathcal{L} / \mathcal{A} \) along the unit section, and which is locally diffeomorphic around \( M \).

Let \( \mathcal{N}(L/A) \) denote the space of all functions on \( L/A \) which, together with their derivatives of all degrees in the direction of the \( \pi \)-fibers, vanish along the zero section. The space of \( \pi \)-fiberwise differential operators on \( L/A \) along the zero section is canonically identified to the symmetric \( R \)-algebra \( \Gamma(S(L/A)) \). Therefore, we have the short exact sequence of \( R \)-algebras

\[
0 \to \mathcal{N}(L/A) \to C^\infty(L/A) \to \text{Hom}_R \left( \Gamma(S(L/A)), R \right) \to 0.
\]

Likewise, let \( \mathcal{N}(\mathcal{L} / \mathcal{A}) \) denote the space of all functions on \( \mathcal{L} / \mathcal{A} \) which, together with their derivatives of all degrees in the direction of the \( s \)-fibers, vanish along the unit section. The space of \( s \)-fiberwise differential operators on \( \mathcal{L} / \mathcal{A} \) along

---

1 A \((p, q)\)-shuffle is a permutation \( \sigma \) of the set \( \{1, 2, \ldots, p + q\} \) such that \( \sigma(1) \leq \sigma(2) \leq \cdots \leq \sigma(p) \) and \( \sigma(p + 1) \leq \sigma(p + 2) \leq \cdots \leq \sigma(p + q) \).

2 The Koszul sign of a permutation \( \sigma \) of the (homogeneous) vectors \( v_1, v_2, \ldots, v_n \) is determined by the relation \( v_{\sigma(1)} \circ v_{\sigma(2)} \circ \cdots \circ v_{\sigma(n)} = \varepsilon(\sigma; v_1, \ldots, v_n) v_1 \circ v_2 \circ \cdots \circ v_n \).
the unit section is canonically identified to the quotient of the enveloping algebra \( \mathcal{U}(L) \) by the left ideal generated by \( \Gamma^*(A) \).

Therefore, we have the short exact sequence of R-modules

\[
0 \to \mathcal{N}(\mathcal{L} / \mathcal{S}) \to C^\infty(\mathcal{L} / \mathcal{S}) \to \text{Hom}_R \left( \frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}, R \right) \to 0.
\]

(2)

Since the exponential (or more precisely its dual) maps \( \mathcal{N}(\mathcal{L} / \mathcal{S}) \) to \( \mathcal{N}(L/A) \), it induces an isomorphism of \( R \)-modules from \( \text{Hom}_R \left( \frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}, R \right) \) to \( \text{Hom}_R \left( \frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}, R \right) \).

**Proposition 2.2.** Each choice of a splitting of the short exact sequence of vector bundles \( 0 \to A \to L \to L/A \to 0 \) and of an \( L \)-connection \( \nabla \) on \( L/A \) extending the \( A \)-action determines an isomorphism of filtered \( R \)-modules \( \text{PBW} : \Gamma(S(L/A)) \to \frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)} \) called Poincaré–Birkhoff–Witt map.

**Remark 2.1.** In case \( L = A \cong B \) is the Lie algebroid sum of a matched pair of Lie algebroids \( (A, B) \), the \( L \)-connection \( \nabla \) on \( L/A \cong B \) extending the \( A \)-action determines a \( B \)-connection on \( B \), the coalgebras \( \frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)} \) and \( \mathcal{U}(B) \) are isomorphic, and the corresponding Poincaré–Birkhoff–Witt map \( \text{PBW} : \Gamma(S(B)) \to \mathcal{U}(B) \) is standard (see [7] for instance).

**Proposition 2.3.** The Poincaré–Birkhoff–Witt map associated to a splitting \( j : L/A \to L \) of the short exact sequence of vector bundles \( 0 \to A \to L \to L/A \to 0 \) and an \( L \)-connection \( \nabla \) on \( L/A \) satisfies \( \text{PBW}(1) = 1 \) and, for all \( b \in \Gamma(L/A) \) and \( n \in \mathbb{N} \), \( \text{PBW}(b^n) = j(b) \cdot \text{PBW}(b) - \text{PBW}(\nabla j(b)(b^n)) \), where \( b^k \) stands for the symmetric product of \( b \) copies of \( b \).

**Remark 2.2.** Although the construction of the Poincaré–Birkhoff–Witt map outlined above presupposes that \( L \) and \( A \) are integrable real Lie algebroids, PBW can be defined for any real (resp. complex) Lie pair provided one works with local (resp. formal) groupoids.

The infinitesimal actions of \( A \) on \( L/A \) and \( \mathcal{L} / \mathcal{S} \) induce infinitesimal actions of \( A \) by derivations on the algebras of functions \( C^\infty(L/A) \) and \( C^\infty(\mathcal{L} / \mathcal{S}) \) and, consequently, on the algebras of infinite jets \( \text{Hom}_R \left( \Gamma(S(L/A)), R \right) \) and \( \text{Hom}_R \left( \frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}, R \right) \).

**Proposition 2.4.** (1) The space \( \text{Hom}_R \left( \frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}, R \right) \) of infinite \( s \)-fiberwise jets along \( M \) of functions on \( \mathcal{L} / \mathcal{S} \) is an associative algebra on which the Lie algebroid \( A \) acts infinitesimally by derivations. (2) The dual of the exponential map \( \text{PBW}^* : \text{Hom}_R \left( \frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}, R \right) \to \text{Hom}_R \left( \Gamma(S(L/A)), R \right) \) is an isomorphism of associative algebras, which may or may not intertwine the infinitesimal \( A \)-actions.

### 3. \( L_\infty[1] \) algebra associated to a Lie pair

Our main result is the following:

**Theorem 3.1.** If \( (L, A) \) is a Lie pair, i.e. a Lie algebroid \( L \) together with a Lie subalgebroid \( A \), then \( L/A \) admits a Kapranov module structure, canonical up to isomorphism, over the Lie algebroid \( A \), whose \( R_2 \in \Gamma(A^* \otimes S^2(L/A)^*) \) (see Proposition 1.1) is a 1-cocycle representative of the Atiyah class of \( L/A \) relative to the pair \( (L, A) \).

Moreover, when \( L = A \cong B \) is the Lie algebroid sum of a matched pair \( (A, B) \) of Lie algebroids and there exists a torsion free flat \( B \)-connection \( \nabla \) on \( B \), the components of the Maurer–Cartan element \( R \) satisfy the recursive formula \( R_{k+1} = \partial^\nabla R_k \), where \( \partial^\nabla \) denotes the covariant differential associated to the connection.

**Sketch of proof.** Choose a splitting of the short exact sequence of vector bundles \( 0 \to A \to L \to L/A \to 0 \) and an \( L \)-connection \( \nabla \) on \( L/A \) extending the \( A \)-action. Identify \( \Gamma(S(L/A)^*) \) to \( \text{Hom}_R \left( \frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}, R \right) \) via the PBW map and pull back the infinitesimal \( A \)-action of the latter to the former. According to Proposition 1.1, the resulting \( A \)-action on \( \Gamma(S(L/A)^*) \) by derivations determines a Kapranov \( A \)-module structure on \( L/A \). Making use of Proposition 2.3, one can check directly that \( R_2 \) is a 1-cocycle representative of the Atiyah class \( \alpha_{L/A} \).

As immediate consequences, we recover the following results of [3]:

**Corollary 3.2.** Given a Lie algebroid pair \( (L, A) \), let \( \mathcal{U}(A) \) denote the universal enveloping algebra of the Lie algebroid \( A \) and let \( \mathcal{A} \) denote the category of \( \mathcal{U}(A) \)-modules. The Atiyah class of the quotient \( L/A \) makes \( L/A[-1] \) into a Lie algebra object in the derived category \( D^b(\mathcal{A}) \).

**Corollary 3.3.** Let \( (L, A) \) be a Lie pair and let \( \mathcal{E} \) be a bundle (of finite or infinite rank) of associative commutative algebras on which \( A \) acts by derivations. There exists an \( L_\infty[1] \) algebra structure on \( \Gamma(\mathcal{E}^* \bigotimes L/A \otimes \mathcal{E}) \), canonical up to \( L_\infty \) isomorphism. Moreover, \( H^{*-1}(A; L/A \otimes \mathcal{E}) \) is a graded Lie algebra whose Lie bracket only depends on the Atiyah class of \( L/A \).
4. An example due to Kapranov

Let $X$ be a Kähler manifold with real analytic metric. Recall that the eigenbundles $T^0_X$ and $T^1_X$ of the complex structure $J : TX \to TX$ ($J^2 = -\text{id}$) form a matched pair of Lie algebroids [6]. Fix a point $x \in X$. The exponential map $\exp_{\text{hol}} : T_x X \to X$ defined using the geodesics of the Levi-Civita connection $\nabla^\text{LC}$ originating from the point $x$ needs not be holomorphic.

However, Calabi constructed a holomorphic exponential map $\exp_{\text{hol}} : T_x X \to X$ as follows [2] (see also [1]). First, extend the Levi-Civita connection $C$-linearly to a $T_X \otimes C$-connection $\nabla^\text{C}$ on $T_X \otimes C$. Since $X$ is Kähler, $\nabla^\text{LC} J = 0$ and $\nabla^\text{C}$ restricts to a $T_X \otimes C$-connection on $T_X$. It is easy to check that the induced $T^0_X$-connection on $T^1_X$ is the canonical infinitesimal $T^0_X$-action on $T^1_X$ — a section of $T^1_X$ is $T^0_X$-horizontal iff it is holomorphic — while the induced $T^1_X$-connection $\nabla^\text{C}$ on $T^1_X$ is flat and torsion free. Now let $X'$ denote the manifold $X$ and let $X''$ denote $X$ with the opposite complex structure $-J$. The image of the diagonal embedding $X \hookrightarrow X' \times X''$ is totally real so $X' \times X''$ can be seen as a complexification of $X$. Let $T_{X'} \times X''$ (resp. its subbundle $T_{X'} \times X''$) along the diagonal $X$ is precisely the complexified tangent bundle $T_X \otimes C$ (resp. its subbundle $T^2_X$). (See [8] for a discussion on integration of complex Lie algebroids.) The analytic continuation of the $T^0_X$-connection $\nabla^\text{C}$ on $T^1_X$ in a neighborhood of the diagonal is a holomorphic $T_{X'} \times X''$-connection on the Lie algebroid $T_{X'} \times X''$, whose exponential map $\exp_{\text{hol}}$ at a diagonal point $(x, x)$ takes $T_x X' \times [x]$ (which is $(T^2_X)_x$ or $T_x X$) into $X' \times [x]$ (which is $X$).

Consider the Lie pair $(L = T_{X'} \times X'', A = X' \times T_{X''})$, the corresponding Lie groupoids $\mathcal{L} = (X' \times X'') \times (X' \times X'')$ and $\mathcal{A} = X' \times (X'' \times X'')$, and the associated quotients $L/A = T_{X'} \times X''$ and $\mathcal{L}/\mathcal{A} = (X' \times X') \times X''$. Calabi's holomorphic exponential map $\exp_{\text{hol}}$ is indeed the restriction along the diagonal of the exponential map $\exp_{\text{hol}} : L/A \to \mathcal{L}/\mathcal{A}$ associated to the $T_{X'} \times X''$-connection $\nabla^\text{C}$ on the Lie algebroid $T_{X'} \times X''$ as described in Proposition 2.1.

Taking the infinite jet of $\exp_{\text{hol}}$, we obtain, as in Proposition 2.2, a Poincaré–Birkhoff–Witt map PBW$_{\text{hol}} : \Gamma(S(T^0_X)) \to \mathcal{U}(T^1_X)$. Then, pulling back the infinitesimal $T^0_X$-action on $\mathcal{U}(T^1_X)$ to an infinitesimal $T^0_X$-action by coderivations on $\Gamma(S(T^0_X))$, we obtain, as in Theorem 3.1, a Kapranov $T^1_X$-module structure on $T^1_X$. In this context, the tensors $R_n \in \Omega^{0,1}(\text{Hom}(S^n T^1_X, T^1_X))$ are the curvature $R_2 \in \Omega^{1,1}(\text{End}(T^1_X))$ and its higher covariant derivatives. Hence we recover the following result of Kapranov:

Theorem 4.1. ([5]) The Dolbeault complex $\Omega^{0,*}(T^1_X)$ of a Kähler manifold is an $L_{\infty}[1]$ algebra. For $n \geq 2$, the $n$-th multibracket $\lambda_n : \Omega^{0,j_1}(T^1_X) \otimes \cdots \otimes \Omega^{0,j_n}(T^1_X) \to \Omega^{0,j_1+\cdots+j_n}(T^1_X)$ is the composition of the wedge product with the map associated to $R_n \in \Omega^{0,1}(\text{Hom}(\otimes^n T^1_X, T^1_X))$ in the obvious way, while $\lambda_1$ is the Dolbeault operator $\partial : \Omega^{0,1}(T^1_X) \to \Omega^{0,1+1}(T^1_X)$.

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