**GENERIC HAMILTONIAN DYNAMICS**

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**ABSTRACT.** In this paper we contribute to the generic theory of Hamiltonians by proving that there is a $C^2$-residual $\mathcal{R}$ in the set of $C^2$ Hamiltonians on a closed symplectic manifold $M$, such that, for any $H \in \mathcal{R}$, there is an open and dense set $\mathcal{S}(H)$ in $H(M)$ such that, for every $e \in \mathcal{S}(H)$, the Hamiltonian level $(H, e)$ is topologically mixing.

**Keywords:** Hamiltonian vector field, topological transitivity, pseudo-orbits.

1. **Introduction**

1.1. **Hamiltonians and their applications.** The Hamiltonian systems form a fundamental subclass of all dynamical systems generated by differential equations. Their importance follows from the vast range of applications throughout different branches of science. In fact, the laws of physics are mostly expressed in terms of differential equations, and a well understood and successful subclass of these differential equations, which leave invariant a symplectic structure, are the Hamiltonian equations (see [3]).

Generic properties of such continuous-time systems are thus of great importance and interest since they give us the typical behavior in an appropriate sense that one could expect from the class of models at hand (cf. [25] [13] [14]).

The main result of this paper (Theorem 1) is the generalization of [17] [8] for Hamiltonians and states that “most” Hamiltonians have indecomposable energy levels in the sense that we cannot split the energy level or, in other words, there is some orbit that winds around the whole energy level. If we weaken the topology, and inspired on the Oxtoby and Ulam theorem\(^1\) (see [23]), we expect to obtain ergodicity in the energy level. On the other hand, for stronger topologies, KAM theorem (see [27]) makes impossible to obtain the same result (due to the persistence of invariant tori).

There are, of course, considerable limitations to the amount of information we can extract from a specific system by looking at generic cases. Nevertheless, it is of great utility to learn that a selected model can be slightly perturbed in order to obtain dynamics we understand in a reasonable way.

Finally, our results are essentially $C^2$-type results and therefore the information it provides holds only in that topology. For instance, in mathematical physics the forces are $C^s$ ($s \geq 3$) objects in its essence. Let us be more precise, a Hamiltonian of class $C^s$.

\(^1\)As far as we know there is not available yet a version for Hamiltonians for this theorem.
generates a Hamiltonian flow of class $C^{s-1}$, and forces involve a second derivative, thus, the natural environment are $C^2$ flows (or $C^3$-Hamiltonians). Hopefully, it is evident that this is not a weakness of our $C^2$-generic results but the counterweight to its undeniable generality.

1.2. The Hamiltonian framework. Let $(M^{2d}, \omega)$ be a symplectic manifold, where $M = M^{2d}$ ($d \geq 2$) is an even-dimensional, compact, boundaryless, connected and smooth Riemannian manifold, endowed with a symplectic form $\omega$. Denote by $C^s(M, \mathbb{R})$ the set of $C^s$-real-valued functions on $M$ and call $H \in C^s(M, \mathbb{R})$ a $C^s$-Hamiltonian, for $s \geq 2$. From now on, we set $s = 2$. Given a Hamiltonian $H$, we can define the Hamiltonian vector field $X_H$ by

$$\omega(X_H(p), u) = d_p H(u), \forall u \in T_p M,$$

which generates the Hamiltonian flow $X_H^t$. Observe that $H$ is $C^2$ if and only if $X_H$ is $C^1$ and that, since $H$ is smooth and $M$ is closed, $\text{Sing}(X_H) \neq \emptyset$, where $\text{Sing}(X_H)$ stands for the singularities of $X_H$ or, in other words, the critical points of $H$.

A scalar $e \in H(M) \subset \mathbb{R}$ is called an energy of $H$. An energy hypersurface $\mathcal{E}_{H,e}$ is a connected component of $H^{-1}(\{e\})$, called energy level set.

The energy level set $H^{-1}(\{e\})$ is said to be regular if any energy hypersurface of $H^{-1}(\{e\})$ is regular, i.e, does not contain any singularity. In this case, we can also say that the energy $e$ is regular. Observe that a regular energy hypersurface is a $X_H^t$-invariant, compact and $(2d-1)$-dimensional manifold. Consider a Hamiltonian $H \in C^2(M, \mathbb{R})$, an energy $e \in H(M)$ and a regular energy hypersurface $\mathcal{E}_{H,e}$. The triplet $(H, e, \mathcal{E}_{H,e})$ is called a Hamiltonian system and the pair $(H, e)$ is called a Hamiltonian level. If $(H, e)$ is regular then $H^{-1}(\{e\})$ corresponds to the union of a finite number of closed connected components, that is, $H^{-1}(\{e\}) = \bigsqcup_{i=1}^{\ell_e} \mathcal{E}_{H,e,i}$, for $\ell_e \in \mathbb{N}$.

Fixing a small neighborhood $\mathcal{W}$ of a regular energy hypersurface $\mathcal{E}_{H,e}$, there exist a small neighborhood $\mathcal{U}$ of the Hamiltonian $H$ and $\epsilon > 0$ such that, for any $\tilde{H} \in \mathcal{U}$ and for any $\tilde{e} \in (e - \epsilon, e + \epsilon)$, we have $\tilde{H}^{-1}(\{\tilde{e}\}) \cap \mathcal{W} = \mathcal{E}_{\tilde{H},\tilde{e}}$. The energy hypersurface $\mathcal{E}_{\tilde{H},\tilde{e}}$ is called the analytic continuation of $\mathcal{E}_{H,e}$.

The next definition states when a Hamiltonian system is Anosov (see Definition [2,3]): a Hamiltonian system $(H, e, \mathcal{E}_{H,e})$ is Anosov if $\mathcal{E}_{H,e}$ is uniformly hyperbolic for the Hamiltonian flow $X_H^t$ associated to $H$ (see [12] for further details on Anosov Hamiltonian systems).

1.3. Topological transitivity. The topological transitivity is a global property of a dynamical system. As a motivation for this notion, we may think of a real physical system, where a state is never measured exactly. Thus, instead of points, we should study (small) open subsets of the phase space and describe how they move in that space. If each one of these open subsets meet each other by the action of the system after some time, then we say that the system is\textit{ topologically transitive}. Equivalently, if we take a compact phase space, we may say that the system has a\textit{ dense orbit}. However, if the open subsets remain inseparable after some time, by the iteration of the system, then we say
that the system is topologically mixing. Obviously, a topologically mixing system is also a topologically transitive system.

The concept of transitivity goes back to Birkhoff \[15, 16\]. Throughout in this paper transitive will always mean topologically transitive.

There exist a lot of transitive systems, as the irrational rotations of \(S^1\), the shift maps and the basic sets (see \[19\]). It is also well-known that \(C^{1+\alpha}\)-Anosov conservative systems \((\alpha > 0)\) are ergodic and so transitive (see \[3\]). In fact, the same holds for \(C^{1}\)-Anosov conservative systems because, by Poincaré recurrence, its non-wandering set equals the whole manifold and by the Anosov closing lemma the periodic orbits are dense in the non-wandering set (\[26\]). Then, using Smale spectral decomposition (\[24\]), we get only one piece which is transitive. Nevertheless, transitivity is not an open property.

**Question 1.1.** Can the transitivity property be generic?

Some authors have been working on this question. The first remarkable result on this subject is due to Bonatti and Crovisier, in \[17\]. They show that, \(C^{1}\)-generically, a \(C^{1}\)-conservative diffeomorphism is transitive. Later, jointly with Arnaud, Bonatti and Crovisier extend this result for \(C^{1}\)-symplectic diffeomorphisms defined on a symplectic manifold (see \[8\]). Adapting the techniques used to prove these results to the continuous-time case, one of the authors proved an analogous result for \(C^{1}\)-divergence-free vector fields. In fact, by a result due to Abdenur et al. (see \[1\]), the first author was able to show that, \(C^{1}\)-generically, a divergence-free vector field is topologically mixing (see \[10\]). Recently, the results in \[17, 8\] get an upgrading in \[2\]. In the direction against the abundance of transitivity (ergodicity), but with a much more exigent smoothness hypothesis, we recall the results of \[20\].

Our contribution to this issue is the statement and the proof of a result that is an answer to Question 1.1 for Hamiltonian systems.

**Definition 1.1.** A compact energy hypersurface \(E_{H,e}\) is topologically mixing if, for any open and non-empty subsets of \(E_{H,e}\), say \(U\) and \(V\), there is \(\tau \in \mathbb{R}\) such that \(X^t_{H}(U) \cap V \neq \emptyset\), for any \(t \geq \tau\). A regular Hamiltonian level \((H,e)\) is topologically mixing if each one of the energy hypersurfaces of \(H^{-1}(\{e\})\) is topologically mixing.

Accordingly with this definition, we prove the following result.

**Theorem 1.** There is a residual \(\mathcal{R}\) in \(C^2(M,\mathbb{R})\) such that, for any \(H \in \mathcal{R}\), there is an open and dense set \(S(H)\) in \(H(M)\) such that, for every \(e \in S(H)\), the Hamiltonian level \((H,e)\) is topologically mixing.

The main tool to prove the previous result is a version for Hamiltonians of the Connecting Lemma for pseudo-orbits developed in \[8\] by Arnaud et al.. To state it, we need the notions of resonance relations and of pseudo-orbits, which we postpone to Section 2.3.

**Lemma 1** (Connecting Lemma for pseudo-orbits of Hamiltonians). Take \(H \in C^2(M,\mathbb{R})\) and a regular energy \(e \in H(M)\), such that the eigenvalues of any closed orbit of \(H\) do
not satisfy non-trivial resonances. Then, for any $C^2$-neighborhood $U$ of $H$, for any energy hypersurface $E_{H,e} \subset H^{-1}(\{e\})$ and for any $x, y \in E_{H,e}$ connected by an $\epsilon$-pseudo-orbit, for $\epsilon > 0$, there exist $\tilde{H} \in U$ and $t > 0$ such that $e = \tilde{H}(x)$ and $X^t_{\tilde{H}}(x) = y$ on the analytic continuation $E_{\tilde{H},e}$ of $E_{H,e}$.

To prove these results, we have to resume the arguments used by Arnaud et al. \cite{8,17} and by one of the authors in \cite{10} and to adapt it to the Hamiltonian setting. Besides the perturbation techniques, the core of the proofs is the need to restrict our attention to the energy hypersurface, in order to perturb the Hamiltonian and keep the energy, when analyzing the perturbations and their supports.

From Theorem 1 we can derive the following result concerning on the homoclinic class of a hyperbolic closed orbit $\gamma$ of $H$, which is the closure of the set of transversal intersections between the stable and unstable manifolds of all points $p$ in $\gamma$ (see Section 2.3, for more details).

**Corollary 1.** There is a residual set $\mathcal{R}$ in $C^2(M, \mathbb{R})$ such that, for any $H \in \mathcal{R}$, there is an open and dense set $\mathcal{S}(H)$ in $H(M)$ such that if $e \in \mathcal{S}(H)$ then any energy hypersurface of $H^{-1}(\{e\})$ is a homoclinic class.

If any energy hypersurface of $H^{-1}(\{e\})$ is a homoclinic class, we say that $H^{-1}(\{e\})$ is a homoclinic class.

We end this section with an overview of the remaining sections of this paper. This paper is organized in two additional sections. In Section 2 we include some notes on Hamiltonian dynamics and in Section 3 we concern about the proof of Theorem 1 by proving the connecting lemma for pseudo orbits. In each section we also include extra definitions and useful auxiliary results.

### 2. Hamiltonian dynamics

#### 2.1. More definitions

Recall that $(M, \omega)$ denotes a symplectic manifold, where $M$ is an even-dimensional Riemannian manifold endowed with a symplectic form $\omega$. Recall that a symplectic form is a skew-symmetric and non-degenerate 2-form on the tangent bundle $TM$. These properties, on the symplectic form, play an important role in the characterization of the Hamiltonian dynamics. The non-degeneracy of the form $\omega$ guarantees that a Hamiltonian vector field is well-defined, while the skew-symmetry of $\omega$ leads to conservative properties for the Hamiltonian vector field. Once more, since $\omega$ is non-degenerate, given $H \in C^2(M, \mathbb{R})$ and $p \in M$, we know that $d_p H = 0$ is equivalent to $X_H(p) = 0$, where $d_p H$ stands for the gradient of $H$ in $p \in M$. Therefore, the extreme values of a Hamiltonian $H$ are exactly the singularities of the associated Hamiltonian vector field $X_H$. Let $\text{Per}(H)$ denote the set of closed orbits of $X_H$ and $\text{Sing}(H)$ denote the set of singularities of $X_H$.

We say that $\tilde{H}$ is $\epsilon$-$C^2$-close to $H$, for $\epsilon > 0$ fixed, if $\|H - \tilde{H}\|_{C^2} < \epsilon$, where $\|H - \tilde{H}\|_{C^2}$ denotes the $C^2$-distance between $H$ and $\tilde{H}$. 
Given a Hamiltonian level \((H, e)\), let \(\Omega(H|_E, e)\) be the set of non-wandering points of \(H\) on the energy hypersurface \(E_{H,e}\), that is, the points \(x \in E_{H,e}\) such that, for every neighborhood \(U\) of \(x\) in \(E_{H,e}\), there is \(\tau > 0\) such that \(X^\tau_H(U) \cap U \neq \emptyset\).

Fix a Hamiltonian level \((H, e)\). We want \(H^{-1}\{e\}\) to decompose into a finite number of connected components, say \(H^{-1}\{e\} = \bigsqcup_{i=1}^{I_e} E_{H,e,i}\), for \(I_e \in \mathbb{N}\). Let us look at the following example.

**Example 1:** Write \(H : \mathbb{R}^2 \to \mathbb{R}\) such that

\[
H(x, y) = \begin{cases} 
  x^7 \sin \left( \frac{1}{x} \right), & x \neq 0 \\
  0, & x = 0.
\end{cases}
\]

It is immediate to see that, for the value of energy \(e = 0\), \(H^{-1}\{e\}\) corresponds to an infinite number of connected components. This construction can be made local (torus, annulus). A direct consequence of the Implicit Function Theorem ensures that the absence of singularities is enough to ensure a finite decomposition of \(H^{-1}\{e\}\).

By Liouville’s Theorem, the symplectic manifold \((M, \omega)\) is also a volume manifold (see, for example, [3]). This means that the volume form \(\omega^2 = \omega \wedge \omega\) induces a measure \(\mu\) on \(M\), which is the Lebesgue measure associated to \(\omega^2\). Notice that the measure \(\mu\) on \(M\) is preserved by the Hamiltonian flow. So, given a regular Hamiltonian level \((H, e)\), we induce a volume form \(\omega_{E_{H,e}}\) on each energy hypersurface \(E_{H,e} \subset H^{-1}\{e\}\), where for all \(p \in E_{H,e}\):

\[
\omega_{E_{H,e}} : T_pE_{H,e} \times T_pE_{H,e} \times T_pE_{H,e} \to \mathbb{R}
\]

\[
(u, v, w) \mapsto \omega^2(d_pH, u, v, w)
\]

The volume form \(\omega_{E_{H,e}}\) is \(X^*_H\)-invariant. Hence, it induces an invariant volume measure \(\mu_{E_{H,e}}\) on \(E_{H,e}\) that is finite, since any energy hypersurface is compact. Observe that, under these conditions, we have that \(\mu_{E_{H,e}}\)-a.e. \(x \in E_{H,e}\) is recurrent, by the Poincaré Recurrence Theorem.

Now we state the definition of transitive Hamiltonian level, which is weaker than the definition of topologically mixing Hamiltonian level (Definition 1.1).

**Definition 2.1.** A Hamiltonian vector field \(X_H\), restricted to a energy hypersurface \(E_{H,e}\), is transitive if, for any open and non-empty subsets \(U\) and \(V\) of \(E_{H,e}\), there is \(\tau \in \mathbb{R}\) such that \(X^\tau_H(U) \cap V \neq \emptyset\). A regular Hamiltonian level \((H, e)\) is transitive if the Hamiltonian vector field \(X_H\) restricted to any energy hypersurface of \(H^{-1}\{e\}\) is transitive.

2.2. Transversal linear Poincaré flow and hyperbolicity. Let us begin with the definition of the transversal linear Poincaré flow. After, we state some results using this linear flow.
Consider a Hamiltonian vector field \( X_H \) and a regular point \( x \) in \( M \) and let \( e = H(x) \). Define \( N_x := N_x \cap T_x H^{-1}(\{e\}) \), where \( T_x H^{-1}(\{e\}) = \text{Ker} dH(x) \) is the tangent space to the energy level set. Thus, \( N_x \) is a \((\dim(M) - 2)\)-dimensional bundle.

**Definition 2.2.** The transversal linear Poincaré flow associated to \( H \) is given by

\[
\Phi_t^H (x) : N_x \to N_{X_H^t(x)} \\
v \mapsto \Pi_{X_H^t(x)} \circ DX_H^t(v),
\]

where \( \Pi_{X_H^t(x)} : T_{X_H^t(x)} M \to N_{X_H^t(x)} \) denotes the canonical orthogonal projection.

Observe that \( N_x \) is \( \Phi_t^H(x) \)-invariant.

It is well-known (see e.g. [3]) that, given a regular point \( x \in \mathcal{E}_{H,e} \), then \( \Phi_t^H(x) \) is a linear symplectomorphism for the symplectic form \( \omega_{\mathcal{E}_{H,e}} \), that is,

\[
\omega_{\mathcal{E}_{H,e}}(u,v) = \omega_{\mathcal{E}_{H,e}}(\Phi_t^H(x) u, \Phi_t^H(x) v) \text{ for any } u, v \in N_x.
\]

We recall that the set of symplectomorphisms forms a group under composition, denoted by \( \text{Sp}(M, \omega) \), called symplectic group.

For any symplectomorphism, in particular for \( \Phi_t^H(x) \), we have the following result.

**Theorem 2.1.** (Symplectic eigenvalue theorem, [3]) Let \( f \in \text{Sp}(M, \omega) \), \( p \in M \) and \( \sigma \) an eigenvalue of \( Df_p \) of multiplicity \( k \). Then \( 1/\sigma \) is an eigenvalue of \( Df_p \) of multiplicity \( k \).

Moreover, the multiplicity of the eigenvalues \(+1\) and \(-1\), if they occur, is even.

The proof of the following result can be found in [13, Section 2.3].

**Lemma 2.2.** Take a Hamiltonian \( H \in C^2(M, \mathbb{R}) \) and let \( \Lambda \) be a \( X_H^t \)-invariant, regular and compact subset of \( M \). Then \( \Lambda \) is uniformly hyperbolic for \( X_H^t \) if and only if the induced transversal linear Poincaré flow \( \Phi_t^H \) is uniformly hyperbolic on \( \Lambda \).

So, we can define a uniformly hyperbolic set as follows.

**Definition 2.3.** Let \( H \in C^2(M, \mathbb{R}) \). An \( X_H^t \)-invariant, compact and regular set \( \Lambda \subset M \) is uniformly hyperbolic if \( N_{\Lambda} \) admits a \( \Phi_t^H \)-invariant splitting \( N_{\Lambda}^s \oplus N_{\Lambda}^u \) such that there is \( \ell > 0 \) satisfying

\[
\|\Phi_{\ell}^H(x)|_{N_x^s}\| \leq \frac{1}{2} \text{ and } \|\Phi_{-\ell}^H(X_{\ell}(x))|_{N_{X_H^\ell(x)}^u}\| \leq \frac{1}{2}, \text{ for any } x \in \Lambda.
\]

We remark that the constant \( \frac{1}{2} \) can be replaced by any constant \( \theta \in (0, 1) \).

### 2.3. Homoclinic classes, resonance relations and pseudo-orbits.

Given a hyperbolic closed orbit of saddle-type \( \gamma \) of a Hamiltonian \( H \), with period \( \pi \), and \( p \in \gamma \). We define the stable and unstable manifolds of \( \gamma \) by

\[
W_{H}^{s,u}(\gamma) = \bigcup_{0 \leq \ell \leq \pi} X_H^\ell(W_{H}^{s,u}(p)).
\]
The homoclinic class of $\gamma$ is defined by
\[
\mathcal{H}_{\gamma,H} = \overline{W^s_H(\gamma) \cap W^u_H(\gamma)},
\]
where $\overline{\mathcal{S}}$ stands for the closure of the set $\mathcal{S}$ and $\cap$ denotes the transversal intersection of manifolds.

It is well-known that a non-empty homoclinic class is invariant by the flow, has a dense orbit, contains a dense set of closed orbits and is transitive. Moreover, the hyperbolic closed orbits of some index are dense in the homoclinic class.

Consider $H \in C^2(M,\mathbb{R})$ and recall that $\text{dim}(M) = 2d$. Let \( \{\sigma_1,\ldots,\sigma_{2d}\} \) denote the eigenvalues of $DX_H(p)$, if $p \in \text{Sing}(H)$, or of $DX^\pi_H(q)$, if $q \in \text{Per}(H)$ has period $\pi$. A resonance relation between $\{\sigma_1,\ldots,\sigma_{2d}\}$ is an equality of the type
\[
\sigma_i = \prod_{j=1}^{2d} \sigma_j^{k_j},
\]
for some $i \in \{1,\ldots,2d\}$ and $k_1,\ldots,k_{2d}$ natural numbers such that either $k_i \neq 1$, or else there exists $j \neq i$ such that $k_j \neq 0$.

Since $\Phi^\pi_H(q)$ is a symplectomorphism, the following trivial resonance relations are satisfied:
\[
\sigma_i = \sigma_i \prod_{k=1}^{d} (\sigma_k \sigma_{d+k})^{\alpha_k},
\]
for naturals $\alpha_k$. A resonance relation different from these ones is called a non-trivial resonance relation. Robinson proved in [25] that, $C^2$-generically, there are not non-trivial resonance relations.

Theorem 2.3. [25, Theorem 1] There is a residual $\mathcal{R}$ in $C^2(M,\mathbb{R})$ such that, for any $H \in \mathcal{R}$, any $p \in \text{Sing}(H)$ and any $q \in \text{Per}(H)$ with period $\pi$, the eigenvalues of $DX_H(p)$ and of $DX^\pi_H(q)$ do not satisfy non-trivial resonance relations.

We observe that, if we fix $H$ in the previous residual set $\mathcal{R}$, sometimes we say that $\text{Sing}(H)$ and $\text{Per}(H)$ do not satisfy non-trivial resonances.

Now, we state the definition of pseudo-orbit for Hamiltonians.

Definition 2.4. Consider a Hamiltonian system $(H,e,\mathcal{E}_{H,e})$ and $\epsilon > 0$. A sequence $\{x_i\}_{i=0}^n$ on $\mathcal{E}_{H,e}$, with $n \in \mathbb{N}$, is an $\epsilon$-pseudo-orbit on $\mathcal{E}_{H,e}$ if $d(X_H^1(x_i),x_{i+1}) < \epsilon$, for any $i \in \{0,\ldots,n-1\}$, where $d(\cdot,\cdot)$ denotes the distance inherited by the Riemannian structure.

The length of the pseudo-orbit is equal to $n$.

Remark 2.1. For divergence-free vector fields, and so for Hamiltonian vector fields, we have that $\Omega(H|_{\mathcal{E}_{H,e}}) = \mathcal{E}_{H,e}$. Therefore, any $x, y \in \mathcal{E}_{H,e}$ are connected by an $\epsilon$-pseudo-orbit, for any $\epsilon > 0$. 
2.4. Lift axiom. Fix $p \in Per(H)$ and a small neighborhood $U_p$ of $p$. By the Darboux Theorem (see, for example, [22, Theorem 1.18]), there is a smooth symplectic change of coordinates $\varphi_p : U_p \to T_pM$, such that $\varphi_p(p) = 0$. Denote by $N_{p,\delta}$ the ball centered in $0$ at the normal fiber at $p$ and with radius $\delta$. For a given $\delta > 0$ depending on $p$ we let $f_H : \varphi_p^{-1}(N_{p,\delta}) \to \varphi_p^{-1}(N_{\varphi_p(p)}(N_{\varphi_p(p)}))$ be the canonical Poincaré time-one arrival associated to $H$.

In [24], when proving the closing lemma for Hamiltonians, Pugh and Robinson show that the lift axiom is satisfied for Hamiltonians, and they obtain the closing from the lifting. In rough terms, lifting is a way of pushing the orbit along a given direction by a small Hamiltonian perturbation $C^2$-close to the identity. We point out that we never have to push in the direction of increasing energies.

Furthermore, we recall the key point on the using of the $C^1$ topology of the Hamiltonian vector field: “...one can lift points $p$ in prescribed directions $v$ with results proportional to the support radius” ([24, pp. 266]).

Lift Axiom for Hamiltonians. (cf. [24, §9 (a)]) Consider a Hamiltonian $H \in C^2(M, \mathbb{R})$ and let $\mathcal{U}$ be a $C^2$-neighborhood of $H$. Then there are $0 < \epsilon \leq 1$ and a continuous function $\delta : M \setminus Sing(X_H) \to (0, 1)$, both depending on $H$ and on $\mathcal{U}$, such that, for any $p$ and $v \in N_{p,\delta} \cap \varphi_p(H^{-1}(H(p)))$, there exists $\tilde{H} \in \mathcal{U}$ satisfying:

- $f_H^{-1} \circ f_{\tilde{H}}(p) = \varphi_p^{-1}(\epsilon v)$;
- $supp(X_{\tilde{H}} - X_H)$ is contained in the flowbox $\mathcal{T} = \bigcup_{t \in (0,T)} X_H^t(B_{\|v\|}(p))$, where $B_{\|v\|}(p)$ is taken in a transversal section of $p$ and $T = T(y)$ is such that $T(p) = 1$ and $X_H^T(y) \in B_{\|v\|}(X_H(p))$, for any $y \in B_{\|v\|}(p)$;
- If several such perturbations are made in disjoint flowboxes, then their union-perturbation is also realizable by a Hamiltonian.

2.5. Perturbation flowboxes. Consider the standard cube $\mathbb{R}^{2d}$, tiled by smaller cubes by homotheties and translations. Given a symplectic chart $\varphi : U \to \mathbb{R}^{2d}$, for $U \subset \mathcal{E}_{H,e}$, the $\varphi$-pre-image of any tilled cube in $\varphi(U)$ is called a tiled cube of the chart $(U, \varphi)$ and it is denoted by $\mathcal{C}$. Note that $\mathcal{C} = \bigcup_{k=1}^m \mathcal{T}_k$, with $m \in \mathbb{N}$, where each $\mathcal{T}_k$ is called a tile of $\mathcal{C}$.

Definition 2.5. Consider a Hamiltonian system $(H,e,\mathcal{E}_{H,e})$, a tiled cube of a chart $\mathcal{C} = \bigcup_{k=1}^m \mathcal{T}_k$ and a constant $T > 0$. We say that the pseudo-orbit $\{x_i\}_{i=0}^n$ on $\mathcal{E}_{H,e}$, with $n \in \mathbb{N}$, preserves the tiling in the injective flowbox

$$\mathcal{F}_H(\mathcal{C}, T) = \bigcup_{t \in [0,T]} X_H^t(\mathcal{C})$$

\(^2\)In fact, given a regular point $p$, we can chose any $\tau > 0$ less than its period, if $p$ is periodic.
Figure 1. Representation of a tiled cube of the chart \((U, \varphi)\).

This definition asserts that the intersection of the pseudo-orbit \(\{x_i\}_{i=0}^n\) with the flowbox \(F_H(C, T)\) is an union of segments \(\{x_j, ..., x_j+T\}\) such that \(x_j \in C\) and \(x_{j+k} = X_H^k(y_j)\), for every \(k \in \{1, ..., T\}\), where \(y_j\) is a point in the same tile of \(x_j\). Observe that if a pseudo-orbit preserves the tiling then we just have to take care about the jumps of the pseudo-orbit outside \(\bigcup_{t \in [1, T-1]} X_H^t(C)\).

As Pugh and Robinson explained in [24 §9 (a)], local perturbations on \(H\) do not change the energy hypersurfaces in the bottom and top of the flowboxes where the perturbations take place. So, we are allowed to push along the energy levels. This property motivates the following definition of perturbation flowbox.

Definition 2.6. Fix a Hamiltonian system \((H,e, \mathcal{E}_{H,e})\), \(\epsilon > 0\) and an \(\epsilon\)-\(C^2\)-neighborhood \(U\) of \(H\). A tiled cube \(C\) is an \(\epsilon\)-perturbation flowbox of length \(T\) for \((H, U)\) if, for any
pseudo-orbit \( \{ x_i \}_{i=0}^n \) on \( \mathcal{E}_{H,e} \) preserving the tiling in \( \mathcal{F}_H(\mathcal{C}, T) \), there is \( \tilde{H} \in \mathcal{U} \), such that \( \tilde{H} = H \) outside \( \mathcal{F}_H(\mathcal{C}, T - 1) \), and a pseudo-orbit \( \{ y_j \}_{j=0}^m \) on \( \mathcal{E}_{\tilde{H},e} \), with \( m \in \mathbb{N} \), such that:

- \( y_0 = x_0 \) and \( y_m = x_n \);
- \( \tilde{H}(y_j) = e \), for any \( j \in \{0, ..., m\} \);
- the intersection of the pseudo-orbit \( \{ y_j \}_{j=0}^m \) with \( \mathcal{F}_H(\mathcal{C}, T) \) is an union of segments \( \{ y_i, ..., y_i+T \} \) such that \( y_i \in \mathcal{C} \) and \( y_i+k = X^k_H(y_i) \), for every \( k \in \{1, ..., T\} \). Moreover, the segments of \( \{ y_j \}_{j=0}^m \) that do not intersect \( \bigcup_{t \in [1,T-1]} X^t_H(\mathcal{C}) \) are segments of the initial pseudo-orbit \( \{ x_i \}_{i=0}^n \), where the starting point belongs to \( X^T_H(\mathcal{C}) \) or coincides with \( x_0 \) and the ending point belongs to \( \mathcal{C} \) or coincides with \( x_n \).

\[ \text{Figure 3. Perturbation in a tiled cube.} \]

The set \( \text{supp}(\mathcal{C}) = \bigcup_{t \in [0,T]} X^t_H(\mathcal{C}) \) is called the support of the perturbation flowbox \( \mathcal{C} \).

The Hayashi Connecting Lemma is a key ingredient to prove the Connecting Lemma for pseudo-orbits of Hamiltonians (Lemma 1) and, as stated in [28], it can be adapted for Hamiltonians. From Definition 2.6, we can extract a slightly stronger statement of the Connecting Lemma for Hamiltonians in [28, Theorem E], which can be seen as a theorem of existence of perturbation flowboxes.

**Theorem 2.4.** Given a Hamiltonian system \((H,e,\mathcal{E}_{H,e})\) and \( \epsilon > 0 \), there exists \( T > 0 \) such that if any tiled cube \( \mathcal{C} \) on \( \mathcal{E}_{H,e} \) is a flowbox of length \( T \) then \( \mathcal{C} \) is an \( \epsilon \)-perturbation flowbox of length \( T \).

From the previous definitions and theorem, the following proposition follows immediately.

**Proposition 2.5.** Consider a Hamiltonian system \((H,e,\mathcal{E}_{H,e})\) and let \( \mathcal{U} \) be a \( C^2 \)-neighborhood of \( H \). For any pseudo-orbit \( \{ x_i \}_{i=0}^n \) on \( \mathcal{E}_{H,e} \) preserving the tiling in a flowbox, there exist \( \tilde{H} \in \mathcal{U} \) and \( t > 0 \), such that \( \tilde{H}(x_0) = e \) and \( X^t_H(x_0) = x_n \) on \( \mathcal{E}_{\tilde{H},e} \).

In fact, flowbox after flowbox, the Connecting Lemma for pseudo-orbits of Hamiltonians (Lemma 1) erases all the jumps of the pseudo-orbit. However, notice that the jumps of a
pseudo-orbit have no reason to respect the tiling of some perturbation flowbox. To deal with this difficulty, we introduce the concept of covering families and of avoidable closed orbits.

2.6. Covering families. Given a Hamiltonian system \((H, e, \mathcal{E}_{H,e})\), we want to cover the orbits on \(\mathcal{E}_{H,e}\) by a family of perturbation flowboxes, with pairwise disjoint supports. Let \(\mathcal{U}\) be a \(C^2\)-neighborhood of \(H\) and let \(\mathcal{C}\) denote a family of perturbation flowboxes for \((H, \mathcal{U})\), with pairwise disjoint supports, and \(\mathcal{V}\) denote a family of non-empty open subsets of \(\mathcal{E}_{H,e}\) with pairwise disjoint supports.

Definition 2.7. The family \(\mathcal{C} = \bigcup_{k=1}^{m} T_k\), for \(m \in \mathbb{N}\), is a covering family of \(\mathcal{E}_{H,e}\) if, for any \(x \in \mathcal{E}_{H,e}\), there exist \(t > 0\) and \(1 \leq k \leq m\) such that \(X^t_H(x) \in \text{int}(T_k)\).

![Figure 4. Representation of a covering family of \(\mathcal{E}_{H,e}\).](image)

In general, if \(\mathcal{E}_{H,e}\) contains closed orbits with small period then \(\mathcal{E}_{H,e}\) has not a covering family. In fact, this kind of closed orbits is disjoint from the perturbation flowboxes. This motivates the definition of covering families outside \(\mathcal{V} = \bigcup_{j=1}^{r} V_j\). The sets \(V_j\) \((1 \leq j \leq r)\) are, in fact, neighborhoods of these closed orbits with small period.

The following definition is an adaption of [8, Definition 3.2] for Hamiltonians.

Definition 2.8. Fix a Hamiltonian system \((H, e, \mathcal{E}_{H,e})\), \(\epsilon > 0\) and an \(\epsilon\)-\(C^2\)-neighborhood \(\mathcal{U}\) of \(H\). A perturbation flowbox \(\mathcal{C}\) for \((H, \mathcal{U})\) is a covering family of \(\mathcal{E}_{H,e}\) outside \(\mathcal{V}\) if there are

- \(t > 0\) and \(\epsilon > 0\);
- an open set \(W_j\) and a compact set \(F_j\), such that \(F_j \subset W_j \subset V_j\), for every \(j \in \{1, \ldots, r\}\);
- a finite family of compacts \(D = \bigcup_{i=1}^{s} D_i\) on \(\mathcal{E}_{H,e}\), such that every \(D_i\) is contained in the interior of a tile of \(\mathcal{C}\);
- two parts \(D_{o,j}\) and \(D_{a,j}\) of \(D\) such that the support of the tiles of \(\mathcal{C}\) containing this compacts is contained in \(V_j\), for any \(j \in \{1, \ldots, r\}\),
such that

a) any segment of any $\epsilon$-pseudo-orbit on $\mathcal{E}_{H,e}$ with length greater or equal than $t$ meets a compact $F_j$ or a compact of $\mathcal{D}$;

b) any segment of any $\epsilon$-pseudo-orbit on $\mathcal{E}_{H,e}$ starting outside $V_j$ and ending inside $W_j$ meets a compact of $\mathcal{D}_{a,j}$, for any $j \in \{1, \ldots, r\}$;

c) any segment of any $\epsilon$-pseudo-orbit on $\mathcal{E}_{H,e}$ starting inside $W_j$ and ending outside $V_j$ meets a compact of $\mathcal{D}_{o,j}$, for any $j \in \{1, \ldots, r\}$;

d) for any $j \in \{1, \ldots, r\}$ and for any compact sets $D_a \subset \mathcal{D}_{a,j}$ and $D_o \subset \mathcal{D}_{o,j}$, there exists a pseudo-orbit with jumps inside the tiles of $\mathcal{C}$, with starting point inside $D_a$ and ending point inside $D_o$.

Figure 5. Covering family of $\mathcal{E}_{H,e}$ outside $\mathcal{V}$.

Roughly speaking, $\mathcal{C}$ is a covering family of $\mathcal{E}_{H,e}$ outside $\mathcal{V}$ if any pseudo-orbit returns regularly to a compact $\mathcal{D} \subset \text{int}(\mathcal{T}_k)$, for some $1 \leq k \leq m$, during the time it passes out of $\mathcal{V}$. If the pseudo-orbit takes a long time to return to another compact set $\tilde{D} \subset \mathcal{D}$, it approaches some compacts $F_j \subset V_j$. For this, the pseudo-orbit must go through an entrance compact $D_a \subset \mathcal{D}$ and then through an exit compact $D_o \subset \mathcal{D}$. Moreover, we can even switch the segment of the pseudo-orbit between $D_a$ and $D_o$ by a pseudo-orbit with jumps inside the tiles of $\mathcal{C}$.

2.7. Avoidable closed orbits. Consider a Hamiltonian system $(H, e, \mathcal{E}_{H,e})$ and a closed orbit $\gamma$ of $H$ on $\mathcal{E}_{H,e}$. Let $\mathcal{U}$ be a $C^2$-neighborhood of $H$ and fix $T > 0$ and $p \in \gamma$. The next definition is adapted from [8, Definition 3.10] for Hamiltonians.

Definition 2.9. A closed orbit $\gamma$ is avoidable for $(\mathcal{U}, T)$ if, for any neighborhood $V_0$ of $\gamma$ and for any $t > 0$, there exist $\epsilon > 0$, open neighborhoods $W$ and $V$ of $\gamma$, such that $W \subset V \subset V_0$, and a perturbation flowbox $\mathcal{C}$ for $(H, \mathcal{U})$ of length $T$ with disjoint supports, such that:

a) the support of $\mathcal{C}$ is contained in $V$;

b) there exist two families of compacts $\mathcal{D}_a$ and $\mathcal{D}_o$ contained in the interior of the tiles of $\mathcal{C}$ such that
any segment of any $\epsilon$-pseudo-orbit on $E_{H,e}$ starting outside $V$ and ending inside $W$ has a point in a compact of $D_a$;
• any segment of any $\epsilon$-pseudo-orbit on $E_{H,e}$ starting inside $W$ and ending outside $V$ has a point in a compact of $D_0$;

c) for any compacts $D_a \in D_a$ and $D_o \in D_o$, there exist a pseudo-orbit on $E_{H,e}$, with jumps inside the tiles of $C$, starting in $D_a$ and ending in $D_o$.

d) for any $x$ in $\mathcal{C}$, the time taking by $X_H^T(x)$ to return to $\text{supp}(\mathcal{C})$ is bigger than $t$.

**Figure 6. Representation of an avoidable closed orbit $\gamma$.**

Therefore, a closed orbit $\gamma$ is avoidable for $(\mathcal{U}, T)$, for fixed $T > 0$, if, for any $t > 0$, there exists a family $\mathcal{C}$ of perturbation flowboxes for $(H, U)$ of length $T$ such that, given a pseudo-orbit with starting and ending points far from $\gamma$, but passing very close of $\gamma$, we can exchange the segments of the pseudo-orbit passing close of $\gamma$ by segments of another pseudo-orbit with jumps inside the tiles $T_k$ ($1 \leq k \leq m$).

A closed orbit can be even characterized as uniformly avoidable.

**Definition 2.10.** Let $(H, e, E_{H,e})$ be a Hamiltonian system and $\mathcal{U}$ a $C^2$-neighborhood of $H$. The closed orbits of $H$ on $E_{H,e}$ are called uniformly avoidable if they are isolated and there exists a constant $T > 0$ such that any closed orbit of $H$ on $E_{H,e}$ is avoidable for $(\mathcal{U}, T)$.

This kind of orbits is used to derive perturbation flowboxes with disjoint supports, in such a way that the pseudo-orbits stay away from closed orbits with small period. We anticipate that, if $E_{H,e}$ has no orbits with small period and has all the closed orbits uniformly avoidable then we will be able to build a covering family of perturbation flowboxes for $E_{H,e}$, as shown in Proposition 3.4 in Section 3.

**2.8. Perturbation results in the $C^2$-topology.** In this section, we state two perturbation lemmas for the Hamiltonian setting, namely the Closing Lemma and the Pasting Lemma.

The first perturbation result is a version of the Closing Lemma for Hamiltonians that we obtain by combining Arnaud’s Closing Lemma (see [7]) with Pugh and Robinson’s Closing
Lemma for Hamiltonians (see [24]). It states that the orbit of a non-wandering point can be approximated, for a very long time, by a closed orbit of a nearby Hamiltonian.

Lemma 2.6. (Closing Lemma for Hamiltonians) Fix \( H_1 \in C^2(M, \mathbb{R}) \). Let \( x \in M \) be a non-wandering point and \( \epsilon, r \) and \( \tau \) positive constants. Then, there exist \( H_2 \in C^2(M, \mathbb{R}) \), a closed orbit \( \gamma \) of \( H_2 \) with period \( \pi \), \( p \in \gamma \) and a map \( g : [0, \tau] \to [0, \pi] \), close to the identity, such that:

- \( H_2 \) is \( \epsilon \)-\( C^2 \)-close to \( H_1 \);
- \( d\left(X_{H_1}^t(x), X_{H_2}^g(t)(p)\right) < r \), \( 0 \leq t \leq \tau \);
- \( H_2 = H_1 \) on \( M \setminus A \) where \( A = \bigcup_{0 \leq t \leq \tau} B_r(X_{H_1}^t(p)) \).

The next lemma is a version of the \( C^1 \)-Pasting Lemma ([6, Theorem 3.1]) for Hamiltonians. Actually, in the Hamiltonian setting, the proof of this result is much more simple.

Lemma 2.7. (Pasting Lemma for Hamiltonians) Fix \( H_1 \in C^r(M, \mathbb{R}), 2 \leq r \leq \infty \), and let \( K \) be a compact subset of \( M \) and \( U \) a small neighborhood of \( K \). Given \( \epsilon > 0 \), there exists \( \delta > 0 \) such that if \( H_2 \in C^s(M, \mathbb{R}) \), for \( 2 \leq s \leq \infty \), is \( \delta \)-\( C^{\min(r,s)} \)-close to \( H_1 \) on \( U \) then there exist \( H_3 \in C^s(M, \mathbb{R}) \) and a closed set \( V \) such that:

- \( K \subset V \subset U \);
- \( H_3 = H_2 \) on \( V \);
- \( H_3 = H_1 \) on \( U^c \);
- \( H_3 \) is \( \epsilon \)-\( C^{\min(r,s)} \)-close to \( H_1 \).

Figure 7. Perturbation given by the Pasting Lemma for Hamiltonians.

Proof. Consider \( \{U_1, U_2\} \) an open cover of \( M \), such that \( U_1 := U \) and \( U_2 \) does not contain \( K \). Then, there is a smooth partition of unity \( \{\alpha_1, \alpha_2\} \), subordinate to \( \{U_1, U_2\} \), such that \( \alpha_i : M \to [0, 1] \) satisfies \( \text{supp}(\alpha_i) \subseteq U_i \), for \( i = 1, 2 \), and \( \alpha_1(x) + \alpha_2(x) = 1 \), for any \( x \in M \).

Letting \( V := U_2^c \) and \( H_3 := \alpha_1 H_2 + (1 - \alpha_1) H_1 \), we have that:

- \( K \subset V \subset U \);
- \( H_3 = H_2 \) on \( V \), since \( \alpha_1(x) = 1 \) and \( \alpha_2(x) = 0 \), for any \( x \in V \);
- \( H_3 = H_1 \) on \( U^c \), since \( \alpha_1(x) = 0 \) and \( \alpha_2(x) = 1 \), for any \( x \in U^c \);
- \( \| H_3 - H_1 \|_{C^{\text{min}(r,s)}} \leq \max \{ \alpha_1(x) \} \| H_2 - H_1 \|_{C^{\text{min}(r,s)}} = \| H_2 - H_1 \|_{C^{\text{min}(r,s)}} < \delta \),

since, by hypothesis, \( H_2 \) and \( H_1 \) are \( \delta-C^{\text{min}(r,s)} \)-close. So, for \( \delta > 0 \) sufficiently small, we are done. \( \square \)

3. Proof of Theorem 1

3.1. Proof of the Connecting Lemma for pseudo-orbits. This section contains the proof of the Connecting Lemma for pseudo-orbits of Hamiltonians. Let \((M, \omega)\) denote a closed, symplectic \(2d\)-manifold, for \( d \geq 2 \). Take \( H \in C^2(M, \mathbb{R}) \) and a regular energy \( e \in H(M) \), such that the eigenvalues of any closed orbit of \( H \) do not satisfy non-trivial resonances. Then, for any \( C^2\)-neighborhood \( U \) of \( H \), for any energy hypersurface \( \mathcal{E}_{H,e} \subset H^{-1}(\{e\}) \) and for any \( x, y \in \mathcal{E}_{H,e} \) connected by an \( \epsilon \)-pseudo-orbit, for \( \epsilon > 0 \), there exist \( \bar{H} \in U \) and \( t > 0 \) such that \( e = \bar{H}(x) \) and \( X^t_{\bar{H}}(x) = y \), on the analytic continuation \( \mathcal{E}_{\bar{H},e} \) of \( \mathcal{E}_{H,e} \).

![Diagram of Connecting Lemma for pseudo-orbits](image)

**Figure 8.** Perturbation given by the Connecting Lemma for pseudo-orbits.

As explained in [8, 17] and in [10], the proof of the Connecting Lemma for pseudo-orbits is split into three main parts. The first step to prove Lemma 1 concerns on local perturbations. These perturbations motivate the definition of perturbation boxes whose support must be in the interior of small open sets, pairwise disjoint till a sufficiently large number of iterates. Separately, we need to analyze the dynamics near closed orbits with small period because these orbits are not contained in any perturbation box. Finally, we must analyze the global dynamics, in order to cover any orbit with perturbation flowboxes.

This strategy was firstly followed by Bonatti and Crovisier for diffeomorphisms (see [17]). Later, jointly with Arnaud (see [8]), these authors proceeded with this methodology...
to get the proof of the Connecting Lemma for pseudo-orbits of symplectomorphisms. The main novelties in the symplectomorphisms context are the need for the perturbations to be symplectic and also that the closed orbits can be stably elliptic. This means that the symplectomorphisms case cannot be reduced to the one treated in [17], where the closed orbits are assumed to be hyperbolic. That is why, in [8], the authors prove this result for symplectomorphisms, by doing the necessary changes.

For the Hamiltonian case, recall that the transversal linear Poincaré flow is, in fact, a symplectomorphism and observe that we are assuming the absence of singularities on the energy hypersurfaces. Keeping in mind the strategy described in [8], the novelties in the proof of the Connecting Lemma for pseudo-orbits of Hamiltonian are the statement of adequate definitions and, since the energy hypersurfaces are invariant by the Hamiltonian flow, the need for the pseudo-orbit being completely contained in the same energy hypersurface. Hence, we have to ensure the creation of symplectic perturbations without leaving the initial energy hypersurface. Recall that the energy hypersurface is indexed to the Hamiltonian. Thus, it may change when we perturb the Hamiltonian. That is why, in the statement of Lemma [1] we want the energy of the points in the pseudo-orbit to be kept constant, even if we $C^2$-perturb the Hamiltonian. However, since we are allowed to push along the energy levels (see [24, §9(a)]), the arguments stated in [8] can be adapted to the Hamiltonian case. At the end, we have a version of the Connecting Lemma for pseudo-orbits of Hamiltonians, where the condition on the persistence of the energy of the pseudo-orbit is trivially satisfied. Let us briefly explain how to prove Lemma [1].

Arnaud et al. proved, in [8, Proposition 4.2], that if the eigenvalues of any closed orbit of a symplectomorphism do not satisfy non-trivial resonance relations, then the closed orbits are uniformly avoidable. Therefore, since the transversal linear Poincaré flow is a symplectomorphism, the following proposition follows directly for Hamiltonians.

**Proposition 3.1.** Consider a Hamiltonian $H \in C^2(M,\mathbb{R})$. If, for any closed orbit $p$ of $H$ with period $\pi$, the eigenvalues of $\Phi_H(\pi)$ do not satisfy non-trivial resonances then the closed orbits of $H$ are uniformly avoidable.

As explained before, to prove this proposition, the authors take into account that the closed orbits can be hyperbolic (case analyzed in [17]) but also completely elliptic or elliptic.

Observe that, by the previous proposition, Theorem [2.3] implies that the closed orbits of a $C^2$-generic Hamiltonian are uniformly avoidable.

Now, by Proposition [3.1] to prove the Connecting Lemma for pseudo-orbits of Hamiltonians it is enough to show the following result.

**Theorem 3.2.** Consider a Hamiltonian system $(H,e,E_{H,e})$ such that the closed orbits of $H$ on $E_{H,e}$ are uniformly avoidable. Then, for any $C^2$-neighborhood $U$ of $H$ and for any $x,y \in E_{H,e}$, there is $\tilde{H} \in U$ and $t > 0$, such that $\tilde{H}(x) = e$ and $X^t_{\tilde{H}}(x) = y$, on the analytic continuation $E_{H,e}$ of $E_{H,e}$.
It is obvious that Theorem 3.2 follows immediately if \( y \in \mathcal{O}_H(x) \). In fact, to prove Lemma 1, it is enough to show Theorem 3.2 for some kind of points \( x, y \in \mathcal{E}_{H,e} \).

Lemma 3.3. Consider a Hamiltonian system \((H,e,\mathcal{E}_{H,e})\) such that the closed orbits on \( \mathcal{E}_{H,e} \) are isolated. Take any \( x, y \in \mathcal{E}_{H,e} \) such that \( y \notin \mathcal{O}_H(x) \). Then, there exist \( \bar{x} \) and \( \bar{y} \), arbitrarily close to \( x \) and \( y \), such that either \( \bar{y} \in \mathcal{O}_H(\bar{x}) \), or else \( \bar{x} \) and \( \bar{y} \) are not closed orbits.

Recall that a uniformly avoidable closed orbit is indeed isolated. So, by the previous lemma, the proof of Lemma 1 is reduced to the proof of Theorem 3.2, when \( x, y \) are not closed orbits. In fact, if \( y \notin \mathcal{O}_H(x) \) and \( x \) or \( y \) are closed orbits, we just have to apply Theorem 3.2 to \( \bar{x} \) and \( \bar{y} \), given by Lemma 3.3. Then, a Hamiltonian perturbation of the identity sends \( x, y \) into \( \bar{x}, \bar{y} \), and it allows us to conclude the result for any \( x \) and \( y \) in \( \mathcal{E}_{H,e} \).

Recall that \( H \) satisfies the lift axiom and that any two distinct points \( x, y \in \mathcal{E}_{H,e} \) are connected by an \( \epsilon \)-pseudo-orbit, for any \( \epsilon > 0 \). Therefore, by Lemma 3.3, we can reduce the proof of Theorem 3.2 and so of the Connecting Lemma for pseudo-orbits of Hamiltonians, to the proof of Proposition 3.4 and Proposition 3.5 below.

Proposition 3.4. Take a Hamiltonian system \((H,e,\mathcal{E}_{H,e})\), such that \( H \) satisfies the lift axiom and any closed orbit of \( H \) on \( \mathcal{E}_{H,e} \) is uniformly avoidable. Let \( \mathcal{U}_0 \) be a \( C^2 \)-neighborhood of \( H \) and \( x, y \in \mathcal{E}_{H,e} \) be such that \( x, y \notin \text{Per}(H) \) and \( y \notin \mathcal{O}_H(x) \). Then there exist a neighborhood \( \mathcal{U} \subset \mathcal{U}_0 \) of \( H \), a family of disjoint open sets \( \mathcal{V} \) and a family of perturbation flowboxes \( \mathcal{C} \) for \((H,\mathcal{U})\) with disjoint supports, both \( \mathcal{V} \) and \( \mathcal{C} \) not containing \( x \) nor \( y \), such that \( \mathcal{C} \) is covering \( \mathcal{E}_{H,e} \) outside \( \mathcal{V} \).

In this case, we want to build a family of perturbation flowboxes in a neighborhood of closed orbits. Let us sketch the proof of this proposition, adapting the ideas of the proof in [8, Proposition 3.13].

We want to construct finitely many disjoint perturbation flowboxes, whose union meets every orbit of \( \mathcal{E}_{H,e} \), called topological tower of order \( T \). Clearly, the existence of closed orbits with small period, even in a finite number, goes against the existence of a topological tower. However, if we construct a perturbation flowbox \( \mathcal{C} \), covering \( \mathcal{E}_{H,e} \) outside a finite family of disjoint open sets \( \mathcal{V} = \bigcup_{i=1}^j V_i \), we can include any closed orbit with small period in the interior of some \( V_i \). In this case, we have a finite family of disjoint perturbation flowboxes \( \mathcal{C} \) far from closed orbits with small period. Now, it remains to show how can we build these disjoint perturbation flowboxes with length \( T \).

Remark 3.1. We state the definition of a flow, built under a ceiling function \( h \). Consider a measure space \( \Sigma \), a map \( R : \Sigma \to \Sigma \), a measure \( \bar{\mu} \) in \( \Sigma \) and an integrable function
$h : \Sigma \rightarrow [c, +\infty]$, with $c > 0$ and $\int_{\Sigma} h(x) d\bar{\mu}(x) = 1$. The flow

$$S^s : \Sigma \times \mathbb{R} \longrightarrow \Sigma \times \mathbb{R}$$

$$(x, r) \mapsto \left( R^k(x), r + s - \sum_{i=0}^{k-1} h(R^i(x)) \right),$$

where $k \in \mathbb{Z}$ is uniquely defined by

$$\sum_{i=0}^{k-1} h(R^i(x)) \leq r + s < \sum_{i=0}^{k} h(R^i(x)),$$

is called a special flow. In fact, the flow $S^s$ moves the point $(x, r)$ to $(x, r + s)$ at velocity one, until it hits the graph of $h$. After this, the point returns to $\Sigma$ and continues its journey.

The Ambrose-Kakutani Theorem states that a flow having the set of critical points with zero Lebesgue measure is isomorphic to a special flow (see [4]).

Recall that any closed orbit of $H$ on the regular energy hypersurface $E_{H,e}$ is uniformly avoidable, and so isolated. Then, $H$ has a finite number of closed orbits with small period. Therefore, by Ambrose-Kakutani’s Theorem in [4], $X^t_H$ is equivalent to a special flow. Now, following [11, Section 3.6.1], with the obvious changes, we can build a topological tower with very high towers in order to have enough time to perform a lot of small non-overlapped perturbations.

The next proposition, jointly with Proposition 3.4, finishes the proof of Lemma 1.

**Proposition 3.5.** Consider a Hamiltonian system $(H, e, E_{H,e})$ and a neighborhood $U$ of $H$. Let $C$ denote a family of perturbation flowboxes for $(H, U)$ covering $E_{H,e}$ outside a family of open sets $\mathcal{V}$. Take any $x, y \in E_{H,e}$ outside the support of $C$ and outside of any $V \in \mathcal{V}$. Then there exist $\tilde{H} \in U$ and $t > 0$, such that $\tilde{H}(x) = e$ and $X^t_H(x) = y$, on the analytic continuation $E_{\tilde{H},e}$ of $E_{H,e}$.

By Proposition 2.5, if the hypothesis of the previous proposition ensure that a pseudo-orbit connecting $x$ and $y$ preserves the tiling of $C$, then we are done. In fact, as explained in Section 2.6, given that the perturbation flowbox $C$ covers $E_{H,e}$ outside $\mathcal{V}$, every orbit on $E_{H,e}$ spends a uniformly bounded time to return to the interior of any tile of $C$. It is straightforward to see that the same holds for any $\epsilon$-pseudo-orbit, with small $\epsilon > 0$. Moreover, if we choose $\epsilon > 0$ even smaller, we can modify the pseudo-orbit in such a way that, whenever the pseudo-orbit returns to the interior of some tile, we add at this time all the next jumps of the pseudo-orbit until the next return to a tile, defining, in this way, a new jump. The final jump respects the tile and is small, because the number of grouped jumps is uniformly bounded. In this way, we construct a pseudo-orbit preserving the tiling of $C$.

### 3.2. Auxiliary lemmas.

In this section, we state the proof of some auxiliary results for Hamiltonian systems defined on a $2d$-dimensional symplectic manifold, for $d \geq 2$. The first one (Lemma 3.6) asserts that, $C^2$-generically, the quotient between the period of two
distinct closed orbits of a Hamiltonian is irrational. After, in Lemma 3.7, we state that, given a $C^2$-generic Hamiltonian $H$, there exists an open and dense set in $H(M)$ such that every energy taken in such a set is regular. Afterwards, we show that, given a $C^2$-generic Hamiltonian $H$, there exists an open and dense set in $H(M)$ such that, for every energy $e$ taken in such a set, the Hamiltonian level $(H,e)$ is transitive (Lemma 3.8).

**Lemma 3.6.** There is a residual $\mathcal{R}$ in $C^2(M,\mathbb{R})$ such that, for any $H \in \mathcal{R}$, any distinct $p,q \in \text{Per}(H)$, with periods $\pi_p$ and $\pi_q$, satisfy $\frac{\pi_p}{\pi_q} \in \mathbb{R}\setminus\mathbb{Q}$.

**Proof.** Fix $n \in \mathbb{N}$. By Robinson’s results [23], the following set

$$\mathcal{A}_n := \{ H \in C^2(M,\mathbb{R}) : \text{Sing}(H) \text{ is hyperbolic and } \text{Per}_n(H) \text{ is hyperbolic or elliptic} \}$$

is open and dense in $C^2(M,\mathbb{R})$. Also, define the open set

$$\mathcal{B}_n := \left\{ H \in \mathcal{A}_n : \text{ if } p,q \in \text{Per}_n(H) \text{ and } p \neq q \text{ then } \frac{\pi_p}{\pi_q} \notin \{ r_i \}_{i=1}^n \right\},$$

where $\{r_i\}_{i=1}^\infty$ denote the positive rational numbers, with a fixed order.

Now, this proof follows the ideas stated in the proof of [10, Lemma 2.2], but using the version of the Pasting Lemma for Hamiltonians, proved in Lemma 2.7.

Fix $\epsilon > 0$ and $H_1 \in C^2(M,\mathbb{R})$. By density of $\mathcal{A}_n$, there is $H_2 \in \mathcal{A}_n$, $\epsilon$-$C^2$-close to $H_1$. Recall that, by Proposition 3.1, the closed orbits with period less or equal than $n$ of $H_2$ are uniformly hyperbolic, and so isolated. So, $\text{Per}_n(H_2)$ has a finite number of elements, say $\{ p_i \}_{i=1}^m$, for fixed $m \in \mathbb{N}$.

Given a positive sequence $\{s_i\}_{i=1}^m$, the vector field $X_{\overline{H}_i} = \frac{1}{s_{i+1}} X_{H_2}$ is also a Hamiltonian vector field, for any $1 \leq i \leq m$. Actually, by [11], $X_{\overline{H}_i}$ is associated to the Hamiltonian $\frac{1}{s_{i+1}} H_2$. Observe that if we choose $s_i$ arbitrarily close to 0 then $\overline{H}_i$ is $\epsilon$-$C^2$-close to $H_2$.

For any $1 \leq i \leq m$, consider tubular compact neighborhoods $K_i$ of $p_i$, sufficiently small such that some open neighborhoods $\mathcal{W}_i$ of $K_i$ are pairwise disjoint. The idea now is to apply, recursively, Lemma 2.7 in order to define $\tilde{H}_m \in C^2(M,\mathbb{R})$ such that:

- $\tilde{H}_m$ converges to $H_2$ in the $C^2$-sense, as $s_i$ converges to 0;
- $\pi_{\tilde{H}_m,p_i} = (1 + s_i) \pi_{H_2,p_i}$, for $1 \leq i \leq m$.

By a good small choice of the sequence $\{s_i\}_{i=1}^m$, we have that $\tilde{H}_m \in \mathcal{A}_n$ and that $\frac{\pi_{\tilde{H}_m,p_i}}{\pi_{\tilde{H}_m,p_j}} \notin \{ r_i \}_{i=1}^n$, for $i \neq j$. Thus, $\tilde{H}_m \in \mathcal{B}_n$.

Since $\mathcal{B}_n$ is open and dense in $C^2(M,\mathbb{R})$, for any $n \in \mathbb{N}$, the desired residual subset of $C^2(M,\mathbb{R})$ is given by $\mathcal{R} := \cap_{n \in \mathbb{N}} \mathcal{B}_n$. □

The following result is an immediate consequence of the fact that Morse functions are $C^2$-open and dense among $C^2(M,\mathbb{R})$ ([21]).
Lemma 3.7. There is a $C^2$-open and dense subset $\mathcal{O}$ in $C^2(M, \mathbb{R})$ such that, for any $H \in \mathcal{O}$ there exists an open and dense set $\mathcal{S}(H)$ of energies such that any energy $e \in \mathcal{S}(H)$ the Hamiltonian level $(H, e)$ is regular.

Lemma 3.8. There is a residual set $\mathcal{R}$ in $C^2(M, \mathbb{R})$ such that, for any $H \in \mathcal{R}$, there is an open and dense set $\mathcal{S}(H)$ in $H(M)$ such that, for every $e \in \mathcal{S}(H)$,

- $H^{-1}(\{e\})$ is regular;
- the closed orbits of $H$ in $H^{-1}(\{e\})$ do not satisfy non-trivial resonances;
- the Hamiltonian level $(H, e)$ is transitive.

Proof. Let $\mathcal{R}_0$ be the residual set given by Theorem 2.3 and consider $\mathcal{O}$ and $\mathcal{S}(H)$, for $H \in \mathcal{O}$, as in Lemma 3.7. Observe that, if $e \in \mathcal{S}(H)$ then $H^{-1}(\{e\}) = \bigcup_{i=1}^t \mathcal{E}_{H,e,i}$. In this case, let $\{U_n\}_n$ be a countable basis of open sets on $M$. Fix $1 \leq i \leq I_e$ and define $U_n = U_n \cap \mathcal{E}_{H,e,i}$, whenever non-empty. So, $\{U_n\}_n$ is a countable basis of open sets on $\mathcal{E}_{H,e,i}$. We say that $H \in \mathcal{P}_{n,m,i,e}$ if

$$\left[\bigcup_{t>0} X_H^t(U_n^i)\right] \cap U_m^i \neq \emptyset.$$

Now, we define the residual set

$$\mathcal{R} := \mathcal{R}_0 \cap \mathcal{O} \cap \bigcap_{n,m} \left(\mathcal{P}_{n,m,i,e} \cup \overline{(\mathcal{P}_{n,m,i,e})^c}\right),$$

where, given a set $S$, $\overline{S}$ stands for its closure and $S^c$ for its complementary.

Fix $H \in \mathcal{R}$, $e \in \mathcal{S}(H)$ and $1 \leq i \leq I_e$. Thus, $H^{-1}(\{e\})$ is regular and any closed orbit of $H$ in $\mathcal{E}_{H,e,i}$ do not satisfy non-trivial resonances. Moreover, for all integers $n$ and $m$, we have that $H \in \mathcal{P}_{n,m,i,e}$ or $H \in (\mathcal{P}_{n,m,i,e})^c$. Observe that if $H \in \mathcal{P}_{n,m,i,e}$, for all integers $n$ and $m$ and any $1 \leq i \leq I_e$, then $(H, e)$ is transitive.

So, by contradiction, assume that there are some integers $n$ and $m$ and $1 \leq i \leq I_e$ such that $H \in \overline{\mathcal{P}_{n,m,i,e}}^c$. Choose $x \in U_n^i$ and $y \in U_m^i$. By Remark 2.1, all points $x, y \in \mathcal{E}_{H,e,i}$ are connected by an $\epsilon$-pseudo-orbit, for any $\epsilon > 0$. Moreover, since $H \in \mathcal{R}_0$, we can apply the Connecting Lemma for pseudo-orbits of Hamiltonians (Lemma 1). So, for any $C^2$-neighborhood $\mathcal{U}$ of $H$, there exists $\tilde{H} \in \mathcal{U} \cap \mathcal{R}_0 \cap \mathcal{O} \cap \overline{\mathcal{P}_{n,m,i,e}}^c$ such that $e = \tilde{H}(x)$, where $U_n^i$ and $U_m^i$ are elements of the basis of the well-defined analytic continuation $\mathcal{E}_{H,e,i}$ of $\mathcal{E}_{H,e,i}$ such that $x \in U_n^i$ and $y \in U_m^i$, and there is $T > 0$ such that $X_{\tilde{H}}^T(x) = y$ on $\mathcal{E}_{H,e,i}$. Then $\tilde{H} \in \mathcal{P}_{n,m,i,e}$, which is a contradiction. Hence $H \in \mathcal{P}_{n,m,i,e}$, for all integers $n$ and $m$ and for any $1 \leq i \leq I_e$. Therefore, $(H, e)$ is transitive, for any $H \in \mathcal{R}$ and any $e \in \mathcal{S}(H)$.

3.3. Energy hypersurfaces as homoclinic classes. In this section, we want to prove the following corollary of Lemma 3.8.
Corollary 3.9. There is a residual set $\mathcal{R}$ in $C^2(M, \mathbb{R})$ such that, for any $H \in \mathcal{R}$, there is an open and dense set $\mathcal{S}(H)$ in $H(M)$ such that if $e \in \mathcal{S}(H)$ then any energy hypersurface of $H^{-1}(\{e\})$ is a homoclinic class.

Proof. Let $\mathcal{R}$ and $\mathcal{S}(H)$, for $H \in \mathcal{R}$, be as in Lemma 3.8. Recall that if $e \in \mathcal{S}(H)$ then $H^{-1}(\{e\}) = \bigcup_{i=1}^{I_e} \mathcal{E}_{H,e,i}$ and, fixing $1 \leq i \leq I_e$, we can define a countable basis of open sets $\{U_n^i\}_n$ on the energy hypersurface $\mathcal{E}_{H,e,i}$.

Take a $C^2$-neighborhood $\mathcal{U}$ of $H$ such that the analytic continuation $p_H$ of a hyperbolic closed orbit $p_H$ of $H$ is well-defined, for any $\tilde{H} \in \mathcal{U}$. So, for any integer $n$, define the open sets

$$\mathcal{W}_n := \{\tilde{H} \in \mathcal{U} : W_{\tilde{H}}^{s,u}(p_H) \cap U_n^i \neq \emptyset\}.$$

We want to show that $\mathcal{W}_n$ is a dense subset of $\mathcal{U}$, for any $n \in \mathbb{N}$. First, observe that $\mathcal{R}_\mathcal{U} = \mathcal{R} \cap \mathcal{U}$ is a dense subset of $\mathcal{U}$ such that, for any $\tilde{H} \in \mathcal{R}_\mathcal{U}$, there is an open and dense set $\mathcal{S}(\tilde{H}) \subset \tilde{H}(M)$ such that any $e \in \mathcal{S}(\tilde{H})$ is regular and $(H,e)$ is transitive. So, fixing $n \in \mathbb{N}$, for any $\tilde{H} \in \mathcal{R}_\mathcal{U}$ and any neighborhood $V$ of a hyperbolic closed orbit $p_H$ there exist $j,k > 0$ satisfying $X_H^j(V) \cap U_n^i \neq \emptyset$ and $X_H^{-k}(V) \cap U_n^i \neq \emptyset$, where $\{U_n^i\}_n$ is a countable basis of open sets on $\tilde{H}(M)$. By Hayashi’s Connecting Lemma of Hamiltonians (see [28]), there exists a Hamiltonian $\tilde{H}$, $C^2$-close to $\tilde{H}$, such that $\tilde{H} \in \mathcal{W}_n$. Hence, $\mathcal{W}_n$ is dense on $\mathcal{U}$, for any $n \in \mathbb{N}$. Therefore,

$$\mathcal{W} := \bigcap_{n \in \mathbb{N}} \mathcal{W}_n = \{\tilde{H} \in \mathcal{U} : W_{\tilde{H}}^{s,u}(p_H) = \mathcal{E}_{\tilde{H},e,i}\}$$

is a residual subset of $\mathcal{U}$.

Fix $\tilde{H} \in \mathcal{R} \cap \mathcal{W}$ and $e \in \mathcal{S}(\tilde{H})$. Let $\{U_n^i\}_n$ be a countable basis of open sets on the energy hypersurface $\mathcal{E}_{\tilde{H},e,i}$ of $\tilde{H}^{-1}(\{e\})$. Fix $n \in \mathbb{N}$ and a hyperbolic closed orbit $p_{\tilde{H}}$ of $\tilde{H}$. Observe that any non-periodic $x \in U_n^i$ is an accumulation point of $W_{\tilde{H}}^{s,u}(p_{\tilde{H}})$. By the Connecting Lemma for Hamiltonians (see [28]), we construct homoclinic intersections on $U_n^i$ and, by a small $C^2$-perturbation, we turn it transversal. So, the set

$$\mathcal{Z}_n := \{\tilde{H} \in \mathcal{U} \cap \mathcal{R} : p_{\tilde{H}} \text{ has a homoclinic transversal intersection on } U_n^i\}$$

is open and dense on $\mathcal{U}$, for any $n \in \mathbb{N}$. Therefore, the set

$$\mathcal{Z} := \bigcap_{n \in \mathbb{N}} \mathcal{Z}_n = \{\tilde{H} \in \mathcal{U} \cap \mathcal{R} : \mathcal{H}_{p_{\tilde{H}},\tilde{H}} = \mathcal{E}_{\tilde{H},e,i}\}$$

is residual in $\mathcal{U}$. Observe that this is valid for any small $C^2$-neighborhood $\mathcal{U}$ of $H \in \mathcal{R}$. So, the set

$$\mathcal{R}_1 := \{H \in C^2(M, \mathbb{R}) \cap \mathcal{R} : \mathcal{H}_{p_H,H} = \mathcal{E}_{H,e,i}\}$$

is residual in $C^2(M, \mathbb{R})$, for any $1 \leq i \leq I_e$. Thus, there is a residual set $\mathcal{R}_1$ in $C^2(M, \mathbb{R})$ such that, for any $H \in \mathcal{R}_1$, there is an open and dense set $\mathcal{S}(H)$ such that, for $e \in \mathcal{S}(H)$, any energy hypersurface of $H^{-1}(\{e\})$ is a homoclinic class. □
3.4. **Generic topological mixing.** In this section, we conclude the proof of Theorem 1.

**Theorem 3.10.** There is a residual $\mathcal{R}$ in $C^2(M, \mathbb{R})$ such that, for any $H \in \mathcal{R}$, there is an open and dense set $\mathcal{S}(H)$ in $H(M)$ such that, for every $e \in \mathcal{S}(H)$, the Hamiltonian level $(H,e)$ is topologically mixing.

**Proof.** Let $\mathcal{R}_0$ be the residual set given by Lemma 3.6, $\mathcal{R}_1$ be the residual set given by Lemma 3.8 and $\mathcal{R}_2$ be the residual set given by Corollary 1. Define $\mathcal{R} := \mathcal{R}_0 \cap \mathcal{R}_1 \cap \mathcal{R}_2$.

Now, we follow the ideas on the proof of [1, Theorem B], making the necessary adaptations to the Hamiltonian setting.

Fix $H \in \mathcal{R}$. Since $H \in \mathcal{R}_1$, by Lemma 3.8, there is an open and dense set $\mathcal{S}(H)$ such that, for any $e \in \mathcal{S}(H)$, the Hamiltonian level $(H,e)$ is transitive. So, to conclude the proof of Theorem 1, we just have to prove that, for any $e \in \mathcal{S}(H)$, the Hamiltonian level $(H,e)$ is topologically mixing.

Fix $e \in \mathcal{S}(H)$ and let $\mathcal{E}_{H,e,i}$ be an energy hypersurface of $H^{-1}(\{e\})$, for $1 \leq i \leq I_e$. Let us prove that $\mathcal{E}_{H,e,i}$ is topologically mixing, that is, for any open, nonempty subsets $U$ and $V$ of $\mathcal{E}_{H,e,i}$, there is $\tau \in \mathbb{R}$ such that $X^\tau_H(U) \cap V \neq \emptyset$, for any $t \geq \tau$.

Given that $H \in \mathcal{R}_2$ and $e \in \mathcal{S}(H)$, by Corollary 1, $\mathcal{E}_{H,e,i}$ is a homoclinic class. Since hyperbolic closed orbits are dense in the homoclinic class and the index is constant and equal to $d$, we can find two different hyperbolic closed orbits $\gamma_1$ and $\gamma_2$ of $H$, with period $\pi_p$ and $\pi_q$, where $p \in \gamma_1$ and $q \in \gamma_2$, such that $\text{ind}(\gamma_1) = \text{ind}(\gamma_2) = d$ and $\gamma_1 \cap U \neq \emptyset$ and $\gamma_2 \cap V \neq \emptyset$. Moreover, since $H \in \mathcal{R}_0$, we have that $\frac{\pi_p}{\pi_q} \in \mathbb{R} \setminus \mathbb{Q}$.

Fix $x \in \gamma_1 \cap U$, $y \in \gamma_2 \cap V$ and $z \in W^u(x) \cap W^s(y)$. Thus, there is $\tau_1 > 0$ such that

- $\{X^{-\tau_1}_{H,U}(z)\}_{\tau \in \mathbb{R}} \subset W^u(x)$;
- $\lim_{m \to +\infty} X^{-\tau_1}_{H,U}(z) = x$.

Then, there is $t_1 > 0$ such that $X^{-t_1}_{H,U}(z) \in U$ and, therefore, $z \in X^{t_1}_{H,U}(U)$, for every $m \in \mathbb{N}$. Similarly, there is $t_2 > 0$ and a small $\epsilon > 0$ such that $X^{t_2}_{H,U}(z) \in V$, for every $n \in \mathbb{N}$ and $|s| < \epsilon$.

Since $\frac{\pi_p}{\pi_q} \in \mathbb{R} \setminus \mathbb{Q}$, observe that the set $\{m\pi_p + n\pi_q + s : m, n \in \mathbb{Z}, |s| < \epsilon\}$ contains an interval of the form $(T, +\infty)$, for some $T > 0$. This follows from the transitivity of the future orbits of irrational rotations of the circle. Hence, for any $t \geq t_1 + t_2 + T$, there are $m, n \in \mathbb{N}$ and $|s| < \epsilon$ such that $t = t_1 + t_2 + m\pi_p + n\pi_q + s$. Then, $X^{t_2}_{H,U}(z) \in X^{t_2}_{H,U}(U) \cap V$, for any $t \geq t_1 + t_2 + T$. So, $\mathcal{E}_{H,e,i}$ is a topologically mixing energy hypersurface, for any $1 \leq i \leq I_e$. Therefore, the Hamiltonian level $(H,e)$ is topologically mixing. $\square$
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