ULTRAVIOLET PROPERTIES OF THE SPINLESS, ONE-PARTICLE YUKAWA MODEL

D.-A. Deckert, A. Pizzo

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Abstract

We consider the one-particle sector of the spinless Yukawa model, which describes the interaction of a nucleon with a real field of scalar massive bosons (neutral mesons). The nucleon as well as the mesons have relativistic dispersion relations. In this model we study the dependence of the nucleon mass shell on the ultraviolet cut-off $\Lambda$. For any finite ultraviolet cut-off the nucleon one-particle states are constructed in a bounded region of the energy-momentum space. We identify the dependence of the ground state energy on $\Lambda$ and the coupling constant. More importantly, we show that the model considered here becomes essentially trivial in the limit $\Lambda \to \infty$ regardless of any (nucleon) mass and self-energy renormalization. Our results hold in the small coupling regime.

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1 Introduction and Definition of the Model

The Yukawa theory provides an effective description of the strong nuclear forces between massive nucleons which are mediated by mesons. The nucleons as well as the mesons have relativistic dispersion relations. It is well-known that the Yukawa theory is plagued by ultraviolet divergences, and so far the fully relativistic model has only been constructed in $1 + 1$ dimensions; see [11] and references therein for the details.
In this paper we consider a toy model of the Yukawa theory, referred to as spinless, one-particle Yukawa model, obtained by neglecting pair-creation and spin, and we restrict the analysis to the one-nucleon sector. In order to yield a well-defined Hamiltonian for this model one usually introduces a cut-off which removes the problematic meson momenta from the interaction term above a finite threshold energy \( \Lambda \). While for non-relativistic situations one may argue that a cut-off \( \Lambda \) of the order of the nucleon rest mass should render a satisfying predictive power of the model, a finite cut-off is not justified in the relativistic regime. Though the model we deal with is a caricature of the relativistic interaction between nucleons and mesons, we address the mathematical problem how to control the model uniformly in \( \Lambda \) beyond perturbation theory.

More specifically, we analyze the effect of self-energy and mass renormalization in the limit \( \Lambda \to \infty \). It is a common hope that at least for non-relativistic QED, i.e., for the Pauli-Fierz Hamiltonian, the ultraviolet cut-off can possibly be removed by introducing a suitable mass and energy renormalization; see [13]. The believe is that, in contrast to classical electrodynamics where the electron bare mass is sent to negative infinity, in non-relativistic QED the bare mass should tend to zero as \( \Lambda \to \infty \) to compensate for the growing electrodynamic mass. Our results show that because of the relativistic dispersion relation of the nucleon this is not the case for the spinless, one-particle Yukawa model. Namely, in a neighborhood of the origin of the (total) momentum space and for small values of the coupling constant, we establish two goals:

1. We identify the dependence of the ground state energy on \( \Lambda \) and the coupling constant \( g \).

2. We show that the nucleon mass shell becomes flat in the limit \( \Lambda \to \infty \) up to corrections estimated to be \( O_{g \to 0}(|g|^{1/2}) \), irrespectively of any scaling of the (nucleon) bare mass \( m \), i.e., \( m \equiv m(\Lambda) > 0 \).

Our analysis is based on a multi-scale technique which was developed in [12] to treat the infrared divergence of the Nelson model, and which was recently refined in [1] to simultaneously control the infrared and ultraviolet divergences of the same model. We extend this multi-scale technique further and apply it to the spinless, one-particle Yukawa model.

It is interesting to note that for this model the self-energy diverges linearly for \( \Lambda \to \infty \) as it is the case for its classical analogue.

**Definition of the Model.** The Hilbert space of the model is

\[
\mathcal{H} := L^2(\mathbb{R}^3, \mathbb{C}; dx) \otimes \mathcal{F}(h),
\]

where \( \mathcal{F}(h) \) is the Fock space of scalar bosons

\[
\mathcal{F}(h) := \bigoplus_{j=0}^{\infty} \mathcal{F}^{(j)}, \quad \mathcal{F}^{(0)} := \mathbb{C}, \quad \mathcal{F}^{\geq 1} := \bigotimes_{l=1}^{j} h, \quad h := L^2(\mathbb{R}^3, \mathbb{C}; dk)
\]

where \( \otimes \) denotes the symmetric tensor product. Let \( a(k), a^*(k) \) be the usual Fock space annihilation and creation operators satisfying the canonical commutation relations (CCR)

\[
[a(k), a(q)^*] = \delta(k - q), \quad [a(k), a(q)] = 0 = [a(k)^*, a^*(q)], \quad \forall k, q \in \mathbb{R}^3.
\]
The kinematics of the system is described by: (a) The position $x$ and the momentum $p$ of the nucleon that satisfy the Heisenberg commutation relations. (b) The real scalar field $\Phi$ and its conjugate momentum.

The dynamics is generated by the Hamiltonian

$$H_\kappa^\Lambda := \sqrt{p^2 + m^2} + H^f + g\Phi_\kappa^\Lambda(x)$$

where:

- $m$ is the nucleon mass;
- $g \in \mathbb{R}$ is the coupling constant;
- $H^f := \int dk \omega(k)a^\dagger(k)a(k)$, $\omega(k) \equiv \omega(|k|) := \sqrt{|k|^2 + \mu^2}$, is the free field Hamiltonian with $\mu$ being the meson mass;
- the interaction term is given by
  $$\Phi_\kappa^\Lambda(x) := \phi_\kappa^\Lambda(x) + \phi^\dagger_\kappa^\Lambda(x), \quad \phi_\kappa^\Lambda(x) := \int_{B_\kappa \setminus B_k} dk \rho(k)a(k)e^{ikx}, \quad \rho(k) := \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega(k)}}$$
  for $0 \leq \kappa < \Lambda$, and for the domain of integration we use the notation $B_{\sigma} := \{k \in \mathbb{R}^3 | |k| < \sigma\}$ for any $\sigma > 0$;
- we use units such that $\hbar = c = 1$.

Note that for $\Lambda = \infty$ the formal expression of the interaction $\Phi_\kappa^\Lambda(x)$ is not a well-defined operator on $\mathcal{H}$ because the form factor $\rho(k)$ is not square integrable. It is well-known (see also Proposition 1.1 below) that for $0 \leq \kappa < \Lambda < \infty$ the operator $H_\kappa^\Lambda$ is self-adjoint and its domain coincides with the one of $H^{(0)} := \sqrt{p^2 + m^2} + H^f$

We briefly recall some well-known facts about this model. The total momentum operator of the system is

$$P := p + P^f := p + \int dk a^\dagger(k)a(k)$$

where $P^f$ is the field momentum. Due to translational invariance of the system the Hamiltonian and the total momentum operator commute. Hence, the Hilbert space $\mathcal{H}$ can be decomposed on the joint spectrum of the three components of the total momentum operator, i.e.,

$$\mathcal{H} = \int dP \mathcal{H}_P$$

here $\mathcal{H}_P$ is a copy of the Fock space $\mathcal{F}$ carrying the (Fock) representation corresponding to annihilation and creation operators

$$b(k) := a(k)e^{ikx}, \quad b^\dagger(k) := a^\dagger(k)e^{-ikx}.$$
We will use the same symbol $\mathcal{F}$ for all Fock spaces. The fiber Hamiltonian can be expressed as

$$H_{P|\Lambda} := \sqrt{(P - P_f)^2 + m^2} + H_f + g\Phi|_{\Lambda}$$

where

$$\Phi|_{\Lambda} := \phi|_{\Lambda} + \phi^*|_{\Lambda}, \quad \phi|_{k} := \int_{\mathbb{R}^3} dk \rho(k) b(k),$$

and

$$H_f = \int dk \omega(k) b^*(k) b(k), \quad P_f = \int dk k b^*(k) b(k).$$

By construction, the fiber Hamiltonian maps its domain in $\mathcal{H}_P$ into $\mathcal{H}_P$. Finally, for later use we define

$$H_{P(0)} := H_{P\text{uc}} + H_f, \quad H_{P\text{uc}} := \sqrt{(P - P_f)^2 + m^2}.\]$$

We restrict our study to the model parameters:

$$m > 0, \quad \mu > 1, \quad 0 < |g| \leq 1, \quad 0 < \kappa \leq 1 < \Lambda < \infty, \quad 0 < P_{\text{max}} < \frac{1}{2}, \quad |P| < P_{\text{max}}.$$

The choice $\mu > 1$ and $P_{\text{max}}$ less than one is only a technical artifact of the crude estimate (14) in the proof of Lemma 3.1 which provides an easy spectral gap estimate in Lemma 3.3 that we employ in the multi-scale analysis.

Concerning previous results on the spinless, one-particle Yukawa model we refer the reader to [2, 3, 4, 14]. In [2] Eckmann considers the spinless Yukawa model without pair-creation with a regularization of the meson form factor. In contrast to our choice given in (2) the interaction term in his Hamiltonian is given by

$$\int dp \int dk \frac{n^*(p - k) a^*(k) n(p)}{\sqrt{(p - k)^2 + \mu^2}^{1/2}(k^2 + \mu^2)^{1/2}(p^2 + \mu^2)^{1/2}} + \text{h.c.}$$

where $n^*(p)$ and $n(p)$ denote the nucleon creation and annihilation operators. This implies that the Hamiltonian renormalized by means of a mass operator (for details see [2]) converges in the norm resolvent sense as $\Lambda \rightarrow \infty$. Furthermore, in [2] the one-particle scattering states are constructed in the small coupling regime. Also Fröhlich [4] studied the spinless, one-particle Yukawa model but with the meson form factor $\frac{\omega_{\Lambda}(k)}{|k|^{1/2}}$, for which he showed that the Hamiltonian including a logarithmically divergent self-energy renormalization constant is well defined in the limit $\Lambda \rightarrow \infty$ and that the nucleon mass shell is non-trivial.

The behavior of the ground state energy for $\Lambda \rightarrow \infty$ has been addressed in [10] and [6] for non-relativistic and pseudo-relativistic QED models. In particular, in [10], for the relativistic dispersion relation the electron self-energy has been proven to obey the same type of dependence on $\Lambda$ as in our model, but without the restriction to the small coupling regime. Perturbative mass renormalization in non-relativistic QED has been addressed in [7]. Furthermore, mass renormalization based on the binding energy of hydrogen has been discussed in models of quantum electrodynamics in [9].

We also want to mention [8] for a recent application of the iterative analytic perturbation theory to the so-called semi-relativistic Pauli-Fierz model that focusses on the infrared corrections to the electron mass shell.
Notation.

1. The symbol \( C \) denotes any positive universal constant and may change its value from line to line.

2. The components of a vector \( v \in \mathbb{R}^3 \) are denoted by \( v = (v_1, v_2, v_3) \).

3. The bars \(|·|, ∥·∥\) denote the euclidean and the Fock space norm, respectively.

4. The brackets \( ⟨·, ·⟩ \) denote the scalar product of vectors in \( \mathcal{F} \). Given a subspace \( \mathcal{K} \subseteq \mathcal{F} \) and an operator \( A \) on \( \mathcal{F} \) we use the notation
\[
∥A∥_\mathcal{K} ≡ ∥A \restriction_\mathcal{K}∥_{\mathcal{F}}.
\]

5. A hat over a vector means that the vector is of unit length, i.e., \( \hat{\Psi} := \frac{\Psi}{∥\Psi∥} \).

6. For two vectors \( \psi, \chi \) we write \( ψ \parallel \chi \) if they are parallel and \( ψ \perp \chi \) if they are perpendicular.

7. We denote the spectral gap of a self-adjoint operator \( H \) restricted to a subspace \( \mathcal{K} \subseteq \mathcal{F} \) with unique ground state \( \Psi \) and corresponding ground state energy \( E \) by
\[
\text{Gap}(H \restriction_\mathcal{K}) := \inf \text{spec}(H \restriction_\mathcal{K}) \setminus \{E\} - E = \inf_{\psi \perp \Psi} ⟨\tilde{\psi}, (H - E)\tilde{\psi}⟩
\]
where the infimum is taken over the domain of \( H \restriction_\mathcal{K} \).

8. We use the short-hand notation (\( γ \) is defined in (4))
\[
H_{P, n} := H_{P, n}^{\Lambda^m}, \quad \ldots |_{\Lambda^0} = \ldots |_{\Lambda^m}, \quad \int_{\Lambda^0}^{\Lambda^m} dk = \int_{\mathcal{B}_b \setminus \mathcal{B}_a} dk.
\]

## 2 Strategy and Main Results

Our computations are based on von Neumann expansion formulas of the ground state of the Hamiltonians \( H_{P, n}^{\Lambda^m} \) by iterative analytic perturbation theory, that means by a multi-scale procedure that relies on analytic perturbation theory. Indeed, in order to study the \( \Lambda \)-dependence of the mass shell, we need to construct the ground states for a fixed and non-zero value of \( g \) that is independent of the cut-off \( \Lambda \). Note however that unless the coupling constant \( g \) is of order \( \left(\frac{1}{\Lambda}\right)^{\frac{1}{2}} \) one cannot add the full interaction \( g\Phi_{n}^{\Lambda} \) to the free Hamiltonian \( H_{P}^{(0)} \) in a single shot of perturbation theory. Therefore, instead of adding the interaction in one shot we shall do many intermediate steps in the expansion by slicing up the interaction term of the Hamiltonian into smaller pieces, namely slices corresponding to momentum ranges \( [\Lambda^{m-1}, \Lambda^{m}] \) that can be made arbitrarily thin by adjusting a fineness parameter \( γ \)
\[
\frac{1}{2} < γ < 1.
\]
It turns out that in this way one can maintain control over the convergence radius of the von Neumann expansions uniformly in \( \Lambda \). With respect to this slicing we define the Fock spaces:
**Definition 2.1.** For \( n \in \{0\} \cup \mathbb{N} \), we define the Fock spaces
\[
\mathcal{F} := \mathcal{F} \left( L^2 \left( \mathbb{R}^3, \mathbb{C}; dk \right) \right),
\]
\[
\mathcal{F}_n := \mathcal{F} \left( L^2 \left( \mathbb{R}^3 \setminus B_{\Lambda \gamma^n}, \mathbb{C}; dk \right) \right),
\]
\[
\mathcal{F}^{m-1}_n := \mathcal{F} \left( L^2 \left( B_{\Lambda \gamma^{n-1}} \setminus B_{\Lambda \gamma^n}, \mathbb{C}; dk \right) \right).
\]
In all these Fock spaces we shall use the same symbol \( \Omega \) to denote the vacuum. For a vector \( \psi \) in \( \mathcal{F}_{n-1} \) and an operator \( O \) on \( \mathcal{F}_{n-1} \) we shall use the same symbol to denote the vector \( \psi \otimes \Omega \) in \( \mathcal{F}_n \) and the operator \( O \otimes 1_{\mathcal{F}^{n-1}} \) on \( \mathcal{F}_n \), respectively, where \( 1_{\mathcal{F}^{n-1}} \) is the identity operator on \( \mathcal{F}^{n-1} \).

Furthermore, for simplicity of our presentation we keep an infrared cut-off
\[
\kappa = \Lambda \gamma^N = 1,
\]
and in the following, for fixed \( \Lambda \), the finenness parameter \( \gamma \) will be chosen in such a way that
\[
N = \frac{\ln \Lambda}{-\ln \gamma}
\]
is an integer. Note that by construction \( 1 \leq \Lambda \gamma^n \leq \Lambda \) for \( 0 \leq n \leq N \).

**Remark 2.2.** We warn the reader that, though it is not explicit in the notation, the definitions of \( \mathcal{F}_n \) and \( H_{P,n} \) are \( \Lambda \)-dependent as well as for other quantities introduced later on (e.g., \( E_{P,n}, \Psi_{P,n} \)).

We introduce:

**Definition 2.3.** For \( P \in \mathbb{R}^3 \) and integers \( 0 \leq n \leq N \) we define the ground state energies
\[
E_{P,n} := \inf \text{spec} \left( H_{P,n} \upharpoonright \mathcal{F}_n \right).
\]

The desired expansion formulas are a byproduct of the construction of the ground states of the Hamiltonians \( H_{P,N} \upharpoonright \mathcal{F}_N, \left| P \right| < P_{\text{max}} \). At the heart of this construction lies an induction argument. Suppose that:

(i) At step \( (n - 1) \) the vector \( \Psi_{P,n-1} \) is the unique ground state of the Hamiltonian \( H_{P,n-1} \upharpoonright \mathcal{F}_{n-1} \) with corresponding ground state energy \( E_{P,n-1} \).

(ii) For some \( \zeta > 0 \) the spectral gap can be bounded from below by
\[
\text{Gap} \left( H_{P,n-1} \upharpoonright \mathcal{F}_{n-1} \right) \geq \zeta \omega \left( \Lambda \gamma^n \right).
\]
Given the assumptions (i) and (ii) we can derive the implications reported below.

1. In Lemma 3.3 we show through a variational argument that

\[ \text{Gap} (H_{P,n-1} \upharpoonright \mathcal{F}_n) \geq \zeta \omega (\Lambda \gamma^n) . \]

2. Next, we justify the Neumann expansion of the resolvent \( \frac{1}{H_{P,n}} \) in terms of \( \frac{1}{H_{P,n-1}} \) and the slice interaction \( H_{P,n} - H_{P,n-1} \) for \( z \in \mathbb{C} \) in the domain defined by

\[ \frac{1}{2} \zeta \omega (\Lambda \gamma^{n+1}) \leq |E_{P,n-1} - z| \leq \zeta \omega (\Lambda \gamma^{n+1}) \]

by a direct computation; see Lemma 3.4. We find

\[ \left\| \left( \frac{1}{H_{P,n-1} - z} \right)^{1/2} g \Phi_{n-1}^{(n)} \left( \frac{1}{H_{P,n-1} - z} \right)^{1/2} \right\|_{\mathcal{F}_n} \leq C |g| \]

uniformly in \( n \) and in \( \Lambda \). The reason for this is that we add interaction slices starting from \( \Lambda \) down to \( \Lambda \gamma^n = 1 \) in decreasing order so that the contribution of

\[ \left\| g \Phi_n^{(n-1)} \left( \frac{1}{H_{P,n-1} - z} \right)^{1/2} \right\|_{\mathcal{F}_n} \leq C |g| (\Lambda \gamma^{n-1} (1 - \gamma))^{1/2} \]

is compensated thanks to the spectral gap estimate and the chosen domain for \( z \) which gives

\[ \left\| \left( \frac{1}{H_{P,n-1} - z} \right)^{1/2} \right\|_{\mathcal{F}_n} \leq C \left( \frac{1}{\Lambda \gamma^n (1 - \gamma)} \right)^{1/2} . \]

3. Finally, Theorem 3.6 ensures the existence of a unique ground state

\[ \Psi_{P,n} := -\frac{1}{2\pi i} \oint_{\Gamma_{P,n}} \frac{dz}{H_{P,n} - z} \Psi_{P,n-1} \]

\[ = -\frac{1}{2\pi i} \sum_{j=0}^{\infty} \oint_{\Gamma_{P,n}} \frac{dz}{H_{P,n-1} - z} \left[ -(H_{P,n} - H_{P,n-1}) \frac{1}{H_{P,n-1} - z} \right]^j \Psi_{P,n-1} \tag{6} \]

of the Hamiltonian \( H_{P,n} \upharpoonright \mathcal{F}_n \) by analytic perturbation theory for sufficiently small \( |g| \) uniformly in \( n \) and \( \Lambda < \infty \), where the contour \( \Gamma_{P,n} \) is appropriately chosen around \( E_{P,n-1} \); see Definition 3.5.

4. Furthermore, another variational argument guarantees \( E_{P,n} \leq E_{P,n-1} \) and, hence, by Kato’s theorem

\[ \text{Gap} (H_{P,n} \upharpoonright \mathcal{F}_n) \geq \zeta \omega (\Lambda \gamma^{n+1}) . \]

Along this construction we gain the expansion formula (6) of the ground state \( \Psi_{P,n} \) in terms of the previous ground state \( \Psi_{P,n-1} \) for each induction step. The above induction is based on the following well-known results:
Proposition 2.4. For $P \in \mathbb{R}^3$ and any integer $0 \leq n < \infty$ the Hamiltonians $H_p^{\text{nuc}}, H_f^{(0)}, H_p, n$ acting on $\mathcal{F}$ are essentially self-adjoint on the domain $D(H_p^{(0)})$ and bounded from below.

Theorem 2.5. For $P \in \mathbb{R}^3$ and integers $0 \leq n < \infty$ the ground state energies fulfill

$$E_{P,n} \geq E_{0,n}. \quad (7)$$

The inequality in (7) is due to [5].

Remark 2.6. We remark that the construction of the ground state can be implemented for $\gamma$ arbitrarily close to 1. This feature of our technique will be crucial to derive the results on the limiting regime, as $\Lambda \to \infty$, of the ground state energy and of the effective velocity stated in Theorems (2.7) and (2.8), respectively. Indeed, by (5) it allows us to control any error term that can be bounded by $O(N(1 - \gamma)^{1+\epsilon})$ with $\epsilon > 0$.

Main Results. As a direct application of the established expansion formulas we can bound the ground state energy from above and from below. The bounds are sharp in the sense that they identify the order of dependence of the ground state energy on the ultraviolet cut-off $\Lambda$ and the coupling constant $g$:

Theorem 2.7. Let $|g|$ be sufficiently small and $|P| < P_{\text{max}}$. Define $E_{P,\Lambda} := \inf \text{spec} \left( H_p^{(0)} \right)$. There exist universal constants $a, b > 0$ such that for all $1 < \Lambda < \infty$ it holds

$$\sqrt{P^2 + m^2 - g^2 b \Lambda} \leq E_{P,\Lambda} \leq \sqrt{P^2 + m^2 - g^2 a \Lambda} \quad (8)$$

The proof will be given in the end of Section 3.

In our second main result we give an estimate of the effective velocity of the nucleon in a one-particle state:

Theorem 2.8. Let $|g|$ be sufficiently small and $|P| < P_{\text{max}}$. Then, there exist universal constants $c_1, C_1 > 0$ such that the following estimate holds true

$$\limsup_{\gamma \to 1} | \frac{\partial E_{P,N}}{\partial P_i} | \leq \Lambda^{-g^2 c_1} \frac{|P|}{[P^2 + m^2]^{1/2}} + C_1 |g|^{1/2}, \quad i = 1, 2, 3. \quad (9)$$

The proof will be given in Section (4). A direct consequence of the bound in (9) is

$$\limsup_{\Lambda \to \infty} | \frac{\partial E_{P,\Lambda}}{\partial P_i} | \leq C|g|^{1/2}. \quad (10)$$

In order to interpret this result consider that in the free case, i.e., $g = 0$, one finds

$$| \frac{\partial E_{P,\Lambda}}{\partial P_i} | = \frac{|P|}{\sqrt{P^2 + m^2}}.$$

Therefore, Theorem 2.8 states that if the interaction is turned on, even for an arbitrarily small but non-zero $|g|$, the absolute value of the gradient of the ground state energy decreases to an order smaller or equal to $|g|^{1/2}$ in the limit $\Lambda \to \infty$. The physical interpretation of this result is that the mass shell essentially becomes flat and the theory trivial in the limit $\Lambda \to \infty$. Moreover, our proof
shows that not even a suitable scaling of the bare mass, i.e., \( m \equiv m(\Lambda) > 0 \), may prevent the mass shell from becoming essentially flat.

A crucial tool for the above results comes from the non-perturbative estimates that we derive in Theorem (3.7) and Theorem (3.8), respectively:

\[
a\Lambda \gamma^{\mu-1}(1 - \gamma) \leq \left( \hat{\psi}_{P,n-1}, \phi_{n-1}^{\mu-1} \frac{1}{H_{P,n-1} - E_{P,n-1}} \phi_{n-1}^{\mu-1} \hat{\psi}_{P,n-1} \right) \leq b\Lambda \gamma^{\mu-1}(1 - \gamma),
\]

(11)

\[
c_l(1 - \gamma) \leq \alpha c_{n-1} := \left( \hat{\psi}_{P,n-1}, \phi_{n-1}^{\mu-1} \left( \frac{1}{H_{P,n-1} - E_{P,n-1}} \right)^2 \phi_{n-1}^{\mu-1} \hat{\psi}_{P,n-1} \right) \leq c_2(1 - \gamma)
\]

(12)

which hold for some universal constants \( 0 < a \leq b < \infty, 0 < c_1 \leq c_2 < \infty \). In order to get the bounds in (11)-(12) we make use of the spectral information obtained during the construction of the ground states.

The strategy of proof in Theorem 2.8 consists in re-expanding back the vectors in the matrix element yielding the effective velocity. This means that, iteratively, the matrix element

\[
\left( \hat{\psi}_{P,n}, V_i(P) \hat{\psi}_{P,n} \right) \equiv \frac{\partial E_{P,n}}{\partial P_i}, \quad V_i(P) := \frac{P_i - P_i^f}{[(P - P^f)^2 + \gamma^2]^{1/2}}
\]

will be expressed in terms of:

1. The analogous quantity on scale \( n - 1 \), i.e.,

\[
\left( \hat{\psi}_{P,n-1}, V_i(P) \hat{\psi}_{P,n-1} \right)
\]

(13)

2. The scalar products

\[
A_{P,n-1} := g^2 \left( \frac{1}{H_{P,n-1} - E_{P,n-1}} \phi_{n-1}^{\mu-1} \hat{\psi}_{P,n-1}, V_i(P) \frac{1}{H_{P,n-1} - E_{P,n-1}} \phi_{n-1}^{\mu-1} \hat{\psi}_{P,n-1} \right)
\]

and

\[
B_{P,n-1} := 2g^2 \Re \left( \frac{1}{H_{P,n-1} - E_{P,n-1}} \phi_{n-1}^{\mu-1} \hat{Q}_{P,n-1}, V_i(P) \frac{1}{H_{P,n-1} - E_{P,n-1}} \phi_{n-1}^{\mu-1} \hat{\psi}_{P,n-1} \right)
\]

where \( \hat{Q}_{P,n-1} \) is defined in equation (24).

3. A remainder that can be estimated to be \( O(|g|^4(1 - \gamma)^\frac{5}{2}) \).

The hard part of our proof is showing that some a priori estimates on \( A_{P,n-1} \) and \( B_{P,n-1} \) hold so that they shall not be re-expanded like the leading term (13) but their cumulative contribution can be estimated to be of order \( |g|^\frac{5}{2} \) as in (9). Two substantially different arguments are devised to control \( A_{P,n-1} \) and \( B_{P,n-1} \):

- As for \( A_{P,n-1} \), due to the velocity operator \( V_i(P) \) we can show summability in \( n \) after contracting the boson operators \( \phi_{n-1}^{\mu-1} \).
• As for $B_{P,n-1}$, by exploiting the presence of the orthogonal projection $\tilde{Q}^\perp_{P,n-1}$ and a suitable one-step, $g$-dependent backwards expansion, we can improve the crude estimate, $O(g^2(1 - \gamma))$, that follows from the operator bounds derived in Section 3 by, at least, an extra factor $|g|^{\frac{1}{2}}$.

The product of the coefficients $\{(1 - g^2\alpha_P|\tilde{P}|_n^{-1})\}_{1 \leq n \leq N}$ that are generated in front of the leading term (13) at each step of the re-expansion gives rise to a damping factor bounded above by $\Lambda^{-\xi^2c}$ as $\gamma$ tends to 1.

3 Construction of the One-Particle States

We begin our discussion with the construction of the ground states corresponding to the Hamiltonians $H_{P,n} \upharpoonright F_n$, $0 \leq n \leq N$. This construction is based on an induction completed in Theorem 3.6. Next, we collect helpful estimates and expansion formulas which also will be used frequently in Section 4. This section ends with Lemma 3.8 where we derive some upper and lower bounds on the ground state energies.

The first lemma provides some a priori estimates on the groundstate energies. In particular claim (iii) of Lemma 3.1 will be crucial for the gap estimate in Lemma 3.3.

**Lemma 3.1.** For $P \in \mathbb{R}^3$ and any integer $0 \leq n < N$ suppose $\Psi_{P,n}$ is the ground state of $H_{P,n} \upharpoonright F_n$ and $E_{P,n}$ is the corresponding ground state energy. Then:

(i) $E_{P,n+1} \leq E_{P,n}$

(ii) $-g^2CA \leq E_{P,n} \leq \sqrt{P^2 + m^2}$.

(iii) $\forall k \in \mathbb{R}^3$, $E_{P-k,n} - E_{P,n} \geq -|P|\omega(k)$.

**Proof.**

(i) By definition of the ground state energy we can estimate

$$E_{P,n+1} - E_{P,n} \leq \frac{\langle \Psi_{P,n}, [H_{P,n+1} - H_{P,n}] \Psi_{P,n} \rangle}{\langle \Psi_{P,n}, \Psi_{P,n} \rangle} = \frac{\langle \Psi_{P,n}, g\Phi^{|\tilde{P}|_{n+1}}_{P,n} \Psi_{P,n} \rangle}{\langle \Psi_{P,n}, \Psi_{P,n} \rangle} = 0.$$

(ii) It suffices to observe that

$$E_{P,n} \leq \langle \Psi_{P,0}, H_{P,n} \tilde{\Psi}_{P,0} \rangle = \sqrt{P^2 + m^2}$$

and

$$0 \leq \sqrt{(P - Pf)^2 + m^2} + \int_{\Lambda^n} dk \omega(k) \left( b_k^+ + g\frac{\rho(k)}{\omega(k)} \right) \left( b_k + g\frac{\rho(k)}{\omega(k)} \right) = H_{P,n} + g^2 \int_{\Lambda^n} dk \frac{\rho(k)^2}{\omega(k)}$$

where

$$g^2 \int_{\Lambda^n} dk \frac{\rho(k)^2}{\omega(k)} \leq g^2CA.$$
(iii) Inequality (7) implies
\[ E_{P-k,n} - E_{P,n} = E_{P-k,n} - E_{0,n} + E_{0,n} - E_{P,n} \geq E_{0,n} - E_{P,n} \]
and
\[ E_{0,n} - E_{P,n} \geq \frac{\langle \Psi_{0,n}, [H_{0,n} - H_{P,n}] \Psi_{0,n} \rangle}{\langle \Psi_{0,n}, \Psi_{0,n} \rangle} = \frac{\langle \Psi_{0,n}, [H_{0,n}^{\text{nuc}} - H_{P,n}^{\text{nuc}}] \Psi_{0,n} \rangle}{\langle \Psi_{0,n}, \Psi_{0,n} \rangle} \geq -|P| \geq -|P| \omega(k) \]
(14)
because
\[ \left\| \sqrt{P_f^2 + m^2} - \sqrt{(P - P_f)^2 + m^2} \right\| \leq |P| \]
and \( \omega(k) = \sqrt{k^2 + \mu^2} \) with \( \mu > 1 \).

In our construction we shall single out two parameters needed to control the gap of the Hamiltonians \( H_{P,n} \uparrow \mathcal{F}_n, 0 \leq n \leq N \):

\textbf{Definition 3.2.} Define \( \frac{1}{8} < \theta < \frac{1}{4} \) and \( \zeta > \frac{1}{4} \) such that
\[ 1 - \theta - P_{\text{max}} \geq \zeta. \]

Later the following lemma will be invoked from the main induction in Theorem 3.6 to provide the gap estimate that is used in the inductive scheme.

\textbf{Lemma 3.3.} Let \( |P| < P_{\text{max}} \) and \( 1 \leq n \leq N \). Assume:

A(i) \( E_{P,n-1} \) is the non-degenerate ground state energy of \( H_{P,n-1} \uparrow \mathcal{F}_{n-1} \) corresponding to the ground state vector \( \Psi_{P,n-1} \).

A(ii) \( \text{Gap} (H_{P,n-1} \uparrow \mathcal{F}_{n-1}) \geq \zeta \omega (\Lambda \gamma^n) \).

Then:

C(i) \( E_{P,n-1} \) is the non-degenerate ground state energy of \( H_{P,n-1} \uparrow \mathcal{F}_n \) corresponding to the ground state vector \( \Psi_{P,n-1} \otimes \Omega \).

C(ii)
\[ \text{Gap} (H_{P,n-1} \uparrow \mathcal{F}_n), \quad \inf_{\varphi = \psi \otimes \eta} \langle \varphi, (H_{P,n-1} - \theta H_{f,n-1}^{\text{f}}) \varphi \rangle \geq \zeta \omega (\Lambda \gamma^n) \]
where the infimum is taken over \( \varphi \in D(H_{P,n-1}^{(0)}) \) such that \( \psi \in \mathcal{F}_{n-1} \) and \( \eta \in \mathcal{F}_{n-1}^{\text{f}} \) contains a strictly positive number of bosons.
Proof. A direct computation using A(i) shows that $\Psi_{P_{n-1}} \otimes \Omega$ is eigenvector of $H_{P_{n-1}} \uparrow \mathcal{F}_n$ with corresponding eigenvalue $E_{P_{n-1}}$. Since $H_{P_{n-1}}^{\uparrow \mathcal{F}_n}$ is a positive operator one has
\[
\inf_{\varphi \in \mathcal{F}_n} \bAn{\varphi, (H_{P_{n-1}} - E_{P_{n-1}}) \varphi} \geq \inf_{\varphi \in \mathcal{F}_n} \bAn{\varphi, (H_{P_{n-1}} - \theta H_{P_{n-1}}^{\uparrow \mathcal{F}_n} - E_{P_{n-1}}) \varphi};
\]
we subtract the term $\theta H_{P_{n-1}}^{\uparrow \mathcal{F}_n}$ for a technical reason which will become clear in Lemma 3.4. Now, the right-hand side of (15) is bounded from below by
\[
\min \left\{ \text{Gap} (H_{P_{n-1}} \uparrow \mathcal{F}_n), \inf_{\varphi = \psi \otimes \eta} \bAn{\varphi, (H_{P_{n-1}} - \theta H_{P_{n-1}}^{\uparrow \mathcal{F}_n} - E_{P_{n-1}}) \varphi} \right\},
\]
where $\psi \in \mathcal{F}_{n-1}, \eta \in \mathcal{F}_n^{n-1}, \psi \otimes \eta$ belongs to $D(H_p^{(0)})$, and $\eta$ is a vector with definite, strictly positive number of bosons. For a vector $\eta$ with $l \geq 1$ bosons we compute
\[
\inf_{\varphi = \psi \otimes \eta} \bAn{\varphi, (H_{P_{n-1}} - \theta H_{P_{n-1}}^{\uparrow \mathcal{F}_n} - E_{P_{n-1}}) \varphi}
\geq \inf_{\psi, \Lambda \gamma^\rho \leq |k| \leq \Lambda \gamma^\nu} \bAn{\psi, \left( H_{P_{n-1}} - E_{P_{n-1}} + \theta \sum_{j=1}^l \omega(k_j) \right) \psi}
\geq \inf_{\psi, \Lambda \gamma^\rho \leq |k| \leq \Lambda \gamma^\nu} \left( E_{P_{n-1}} - E_{P_{n-1}} + (1 - \theta) \sum_{j=1}^l \omega(k_j) \right).
\]
Furthermore, Lemma 3.1 implies
\[
E_{P_{n-1}} - E_{P_{n-1}} \geq -P_{\text{max}} \sum_{j=1}^l \omega(k_j).
\]
Hence, by Definition 3.2 the inequality
\[
\inf_{\varphi = \psi \otimes \eta} \bAn{\varphi, (H_{P_{n-1}} - \theta H_{P_{n-1}}^{\uparrow \mathcal{F}_n} - E_{P_{n-1}}) \varphi} \geq \zeta \omega (\Lambda \gamma^\nu)
\]
holds. Now by A(ii) we also get
\[
(16) \geq \zeta \omega (\Lambda \gamma^\nu).
\]
From the estimate in equation (17) we can conclude that $\Psi_{P_{n-1}} \otimes \Omega$ is the unique ground state of $H_{P_{n-1}} \uparrow \mathcal{F}_n$ with eigenvalue $E_{P_{n-1}}$ and
\[
\text{Gap} (H_{P_{n-1}} \uparrow \mathcal{F}_n) \geq \zeta \omega (\Lambda \gamma^\nu).
\]
This proves C(i) and C(ii). □

The second ingredient needed for the main induction in Theorem 3.6 is a control of the resolvent expansion of the Hamiltonians:
Lemma 3.4. Let $|g|$ be sufficiently small and $|P| < P_{\text{max}}$. Suppose further that for $1 \leq n \leq N$ $E_{P_n}$ is the non-degenerate ground state energy of $H_{P_n}$ corresponding to the ground state vector $\Psi_{P_n}$ and that

$$\text{Gap} (H_{P_n} \upharpoonright \mathcal{F}_n) \geq \zeta \omega (\Lambda \gamma^n).$$

(18)

Then, for $z \in \mathbb{C}$ such that

$$\frac{1}{2} \zeta \omega (\Lambda \gamma^{n+1}) \leq |E_{P_n} - z| \leq \zeta \omega (\Lambda \gamma^{n+1}),$$

the resolvent $\frac{1}{H_{P_n} - z}$ is a well-defined operator on $\mathcal{F}_n$ which equals to

$$\frac{1}{H_{P_n} - z} \sum_{j=0}^{\infty} \left[ -g \Phi_{\mathcal{F}_n} - 1 \right]^j.$$

(19)

Proof. We start with the estimate

$$\left\| \frac{1}{\sqrt{\text{dist} (z, \text{spec} (H_{P_n} \upharpoonright \mathcal{F}_n))}} \right\| \leq \left( \max \left\{ \frac{2}{\zeta \omega (\Lambda \gamma^{n+1})}, \frac{C}{\zeta \omega (\Lambda \gamma^n) - \zeta \omega (\Lambda \gamma^{n+1})} \right\} \right)^{1/2} \leq \left( \frac{C}{\zeta \Lambda \gamma^{n+1} (1 - \gamma)} \right)^{1/2}.$$

where we made use of the assumption in (18). Next, we estimate

$$\left\| g \Phi_{\mathcal{F}_n} \left( \frac{1}{H_{P_n} - z} \right)^{1/2} \right\| \leq |g| C \left[ \Lambda \gamma^{n-1} (1 - \gamma) \right]^{1/2} \left\| \left( \frac{1}{H_{P_n} - z} \right) \left( \frac{1}{H_{P_n} - z} \right) \right\|.$$n

(20)

The operators $H_{P_n}^{\mathcal{F}_n}$ and $H_{P_n}$ commute, and we may apply the spectral theorem and Lemma 3.3 in order to get

$$\left\| \left( H_{P_n}^{\mathcal{F}_n} \right)^{1/2} \left( \frac{1}{H_{P_n} - z} \right)^{1/2} \right\| \leq \left\| \left( H_{P_n}^{\mathcal{F}_n} - \theta H_{P_n}^{\mathcal{F}_n} - z + \theta H_{P_n}^{\mathcal{F}_n} \right) \right\| \leq \theta^{-1/2}.$$

In consequence, we can estimate

$$\left\| g \left( \frac{1}{H_{P_n} - z} \right) \Phi_{\mathcal{F}_n} \left( \frac{1}{H_{P_n} - z} \right)^{1/2} \right\| \leq |g| C (\zeta \gamma^2)^{-1/2} \theta^{-1/2}.$$n

Since $\gamma > \frac{1}{2}$, $\zeta > \frac{1}{4}$, and $\theta > \frac{1}{8}$ the coupling constant $|g|$ can be chosen independently of $n$ (and of $\Lambda$) such that

$$|g| C (\zeta \gamma^2)^{-1/2} \theta^{-1/2} < 1$$

which implies the convergence of the power series on the right-hand side of (19) and, thus, the claim.

We will now prove that the vectors in the following definition are the unique, non-zero ground states of the Hamiltonians $H_{P_n} \upharpoonright \mathcal{F}_n$, $0 \leq n \leq N$. (We warn the reader that the spectral projection in (21) will be shown to be well defined in Theorem 3.6.)
**Definition 3.5.** For $1 \leq n \leq N$ we define

$$Q_{P,n} := -\frac{1}{2\pi i} \oint_{\mathcal{F}_n} \frac{dz}{H_{P,n} - z} \uparrow \mathcal{F}_n \quad \Gamma_{P,n} := \left\{ z \in \mathbb{C} \mid |E_{P,n-1} - z| = \frac{1}{2} \zeta \omega \left( \Lambda \gamma^{n+1} \right) \right\}$$

(21)

and recursively

$$\Psi_{P,n} := Q_{P,n} \Psi_{P,n-1}, \quad \Psi_{P,0} := \Omega.$$  

(22)

Note that $\Psi_{P,n}$ are in general unnormalized vectors with $\|\Psi_{P,n}\| \leq 1$.

**Theorem 3.6.** Let $|g|$ be sufficiently small and $|P| < P_{\text{max}}$. For $0 \leq n \leq N$ it holds:

(i) $\Psi_{P,n}$ is well-defined, non-zero, and the unique ground state vector of $H_{P,n} \uparrow \mathcal{F}_n$ with corresponding eigenvalue

$$E_{P,n} := \inf \text{spec} (H_{P,n} \uparrow \mathcal{F}_n).$$

(ii) $\text{Gap} (H_{P,n} \uparrow \mathcal{F}_n) \geq \zeta \omega \left( \Lambda \gamma^{n+1} \right).$

**Proof.** A direct computation shows that the claim holds for $n = 0$. Let us assume it holds for $n - 1$ with $0 \leq n - 1 < N - 1$:

1. The assumptions allow to apply Lemma 3.3 which states that

$$\text{Gap} (H_{P,n-1} \uparrow \mathcal{F}_n) \geq \zeta \omega \left( \Lambda \gamma^{n} \right).$$

2. Hence, Lemma 3.4 ensures that for $|g|$ small enough but uniform in $n$ (and in $\Lambda$) the resolvent

$$\frac{1}{H_{P,n} - z} \uparrow \mathcal{F}_n = \frac{1}{H_{P,n-1} - z} \sum_{j=0}^{\infty} \left[ -g \Phi_n^{n-1} \frac{1}{H_{P,n-1} - z} \right]^j \uparrow \mathcal{F}_n$$

is well-defined for

$$\frac{1}{2} \zeta \omega \left( \Lambda \gamma^{n+1} \right) \leq |E_{P,n-1} - z| \leq \zeta \omega \left( \Lambda \gamma^{n+1} \right).$$

(23)

3. For $|g|$ small enough but uniform in $n$ (and in $\Lambda$), $\Psi_{P,n}$ defined in (22) is non-zero. Indeed for $0 \leq n \leq N$ and $z \in \Gamma_{P,n}$ we have

$$\left\| \left( \frac{1}{H_{P,n-1} - z} \right)^{1/2} g \Phi_n^{n-1} \left( \frac{1}{H_{P,n-1} - z} \right)^{1/2} \right\|_{\mathcal{F}_n} \leq C|g|(1 - \gamma)^{1/2}$$

because for $z$ in the domain $\Gamma_{P,n}$ defined in (21) we get

$$\left\| \left( \frac{1}{H_{P,n-1} - z} \right)^{1/2} \right\|_{\mathcal{F}_n} \leq \left( \frac{C}{\Lambda \gamma^n} \right)^{1/2}$$

that we can combine with the bound in (20). By Kato’s theorem we can conclude that it is the unique ground state of $H_{P,n} \uparrow \mathcal{F}_n$ with corresponding ground state energy $E_{P,n}$.
4. Lemma 3.1(i), Kato’s theorem, and the domain of \( z \) given in (23) provide the estimate
\[
\text{Gap} \left( H_{P,n} \upharpoonright \mathcal{F}_n \right) \geq \zeta \omega \left( \Lambda \gamma^{n+1} \right).
\]

Next we provide expansion formulas which will be used frequently in our computations in Section 4.

**Theorem 3.7.** Let \( |g| \) be sufficiently small and \( |P| < P_{\text{max}} \). For \( 0 \leq n \leq N \) the following statements hold:

(i) The following equality is satisfied:
\[
\Psi_{P,n} = \Psi_{P,n-1} - g \frac{1}{H_{P,n-1} - E_{P,n-1}} \phi_{n}^{(n-1)} \Psi_{P,n-1} + \sum_{i=1}^{3} \mathcal{Q}_{P,n-1} \phi_{n}^{(n-1)} \left( \frac{1}{H_{P,n-1} - E_{P,n-1}} \right) \phi_{n}^{(n-1)} \Psi_{P,n-1} + O \left( |g|^3(1 - \gamma)^{3/2} \right)
\]

for
\[
\mathcal{Q}_{P,n-1} := -\frac{1}{2\pi i} \oint_{\Gamma_{P,n}} dz \frac{1}{H_{P,n-1} - z} \upharpoonright \mathcal{F}_n, \quad \mathcal{Q}_{P,n-1}^\dagger := 1_{\mathcal{F}_n} - \mathcal{Q}_{P,n-1} \tag{24}
\]

where \( 1_{\mathcal{F}_n} \) is the identity operator on \( \mathcal{F}_n \).

(ii) The norm of the ground state vectors fulfills the relation
\[
\| \Psi_{P,n} \|^2 = \langle \Psi_{P,n}, \Psi_{P,n} \rangle = \left( 1 - g^2 \alpha_{P,n-1} + O \left( |g|^4(1 - \gamma)^{4/2} \right) \right) \| \Psi_{P,n-1} \|^2 \tag{25}
\]

where
\[
\alpha_{P,n-1} := \left( \Psi_{P,n-1}, \phi_{n}^{(n-1)} \left( \frac{1}{H_{P,n-1} - E_{P,n-1}} \right) \phi_{n}^{(n-1)} \Psi_{P,n-1} \right).
\]

(iii) There exist universal constants \( 0 < c_1 \leq c_2 < \infty \) such that
\[
c_1(1 - \gamma) \leq \alpha_{P,n-1} \leq c_2(1 - \gamma).
\]

**Proof.** Claim (i) can be shown by a direct computation using Definition 3.5. Likewise claim (ii) follows from Definition 3.5 by exploiting the relation
\[
\Psi_{P,n} = \mathcal{Q}_{P,n} \Psi_{P,n-1} = \frac{\langle \Psi_{P,n}, \Psi_{P,n-1} \rangle}{\langle \Psi_{P,n}, \Psi_{P,n} \rangle} \Psi_{P,n}
\]
that holds by construction.

Next, we prove claim (iii). The bound from above is obtained by using the pull-through formula and Lemma 3.1 (iii), i.e.,

\[
\alpha_{\rho_{1,n}}^{\mu-1} = \left\langle \hat{\Psi}_{P_{n-1}}, \phi_{\rho_{1,n}}^{\mu-1} \left( \frac{1}{H_{P_{n-1}} - E_{P_{n-1}}} \right)^2 \phi_{\rho_{1,n}}^{\mu-1} \hat{\Psi}_{P_{n-1}} \right \rangle
= \int_{\lambda_{\rho_{1}}}^{\lambda_{\rho_{n}}} dk \rho(k)^2 \left( \hat{\Psi}_{P_{n-1}}, \left( \frac{1}{H_{P_{n-1}} + \omega(k) - E_{P_{n-1}}} \right)^2 \hat{\Psi}_{P_{n-1}} \right) \leq -C \ln \gamma \leq c_2(1 - \gamma) \quad (26)
\]

for an appropriately chosen constant \(c_2\); recall that \(\frac{1}{2} < \gamma < 1\).

With respect to the bound from below we consider the spectral representation for the self-adjoint operator \(H_{P_{-k,n-1}} + \omega(k) - E_{P_{n-1}}\) and define the spectral projections

\[
\chi^+(k) := \chi_{(5\omega(k),+\infty)}(H_{P_{-k,n-1}} + \omega(k) - E_{P_{n-1}}), \quad \chi^-(q) := \mathbb{1}_{\mathbb{R}_{\rho_{1}}} - \chi^+(q)
\]

where \(\chi_{(5\omega(k),+\infty)}\) is the characteristic function being one on the interval \((5\omega(k), +\infty)\) and zero otherwise. We also define the function

\[
f(k) := \rho(k)^2 \left( \hat{\Psi}_{P_{n-1}}, \left( \frac{1}{H_{P_{n-1}} + \omega(k) - E_{P_{n-1}}} \right)^2 (\chi^+(k) + \chi^-(k)) \hat{\Psi}_{P_{n-1}} \right)\]

that we study for two complementary cases:

(a) In the case \(\left\| \chi^+(k) \hat{\Psi}_{P_{n-1}} \right\|^2 < \frac{1}{2}\) we get

\[
f(k) \geq \rho(k)^2 \left( \hat{\Psi}_{P_{n-1}}, \left( \frac{1}{H_{P_{n-1}} + \omega(k) - E_{P_{n-1}}} \right)^2 \chi^-(k) \hat{\Psi}_{P_{n-1}} \right) \geq \frac{\rho(k)^2}{50\omega(k)^2}. \quad (27)
\]

(b) In the other case, i.e., \(\left\| \chi^+(k) \hat{\Psi}_{P_{n-1}} \right\|^2 \geq \frac{1}{2}\), we start with the trivial inequality

\[
f(k) \geq \rho(k)^2 \left( \frac{1}{H_{P_{n-1}} + \omega(k) - E_{P_{n-1}}} \right)^2 \hat{\Psi}_{P_{n-1}}, \chi^+(k) \left( \frac{1}{H_{P_{n-1}} + \omega(k) - E_{P_{n-1}}} \hat{\Psi}_{P_{n-1}} \right) \quad (28)
\]

and consider the resolvent formulas

\[
\frac{1}{H_{P_{n-1}} + \omega(k) - E_{P_{n-1}}} = \frac{1}{H_{P_{n-1}} + \omega(k) - E_{P_{n-1}}} - \frac{1}{H_{P_{n-1}} + \omega(k) - E_{P_{n-1}}} \Delta_{\rho(k)} \frac{1}{H_{P_{n-1}} + \omega(k) - E_{P_{n-1}}} \quad (29)
\]

and

\[
\frac{1}{H_{P_{n-1}} + \omega(k) - E_{P_{n-1}}} = \frac{1}{H_{P_{n-1}} + \omega(k) - E_{P_{n-1}}} - \frac{1}{H_{P_{n-1}} + \omega(k) - E_{P_{n-1}}} \Delta_{\rho(k)} \frac{1}{H_{P_{n-1}} + \omega(k) - E_{P_{n-1}}} \quad (30)
\]
where
\[ \Delta_p(k) := \sqrt{(P - k - P')^2 + m^2} - \sqrt{(P - P')^2 + m^2}. \]

Then we apply the expansions in (29) and in (30) to the resolvents on the left and on the right in the scalar product of (28), respectively, and get
\[
\begin{align*}
f(k) & \geq \rho(k)^2 \left( \frac{1}{H_{P,n-1} + \omega(k) - E_{P,n-1}} \chi^+(k) \frac{1}{H_{P,n-1} + \omega(k) - E_{P,n-1}} - \frac{2 \mathcal{R} \rho(k)^2}{H_{P,n-1} + \omega(k) - E_{P,n-1}} \Delta_p(k) \frac{1}{H_{P,n-1} + \omega(k) - E_{P,n-1}} \times \chi^+(k) \frac{1}{H_{P,n-1} + \omega(k) - E_{P,n-1}} \right) \\
& \quad + \rho(k)^2 \left( \frac{1}{H_{P,n-1} + \omega(k) - E_{P,n-1}} \Delta_p(k) \frac{1}{H_{P,n-1} + \omega(k) - E_{P,n-1}} \right)^2 \Psi_{P,n-1}. \\
\end{align*}
\]

Note that
\[ \|\Delta_p(k)\| \leq |k| \]
so that neglecting the last positive term in (31) we get the estimate
\[
\begin{align*}
f(k) & \geq \frac{\rho(k)^2}{\omega(k)^2} \left\| \chi^+(k) \Psi_{P,n-1} \right\|^2 - \frac{2 \rho(k)^2 |k|}{5 \omega(k)^2} \left\| \chi^+(k) \Psi_{P,n-1} \right\| \\
& \quad \geq \frac{\rho(k)^2}{\omega(k)^2} \left\| \chi^+(k) \Psi_{P,n-1} \right\| \left( \frac{1}{\sqrt{2}} - \frac{2}{5} \right) \geq \frac{5 - 2 \sqrt{2}}{10} \rho(k)^2.
\end{align*}
\]
Combining the bounds (27) and (32) we obtain
\[
\int_{\Lambda \gamma^{n-1}} dk \rho(k)^2 \left\| \chi^+(k) \Psi_{P,n-1} \right\|^2 \geq -C \ln \gamma \geq c_1 (1 - \gamma)
\]
that gives the bound from below on \( \alpha_p |n|^{-1} \) for an appropriately chosen constant \( c_1 \). This together with the bound from above (26) proves the claim. \( \square \)

With the help of these expansion formulas we get upper and lower bounds on the ground state energy shifts:

**Lemma 3.8.** Let \( |g| \) be sufficiently small and \( |P| < P_{\text{max}} \). For \( 1 \leq n \leq N \) the following holds:

(i) \[
E_{P,n} - E_{P,n-1} = -\Delta E_{P,n-1} + O \left( |g|^4 \Lambda (1 - \gamma)^{4/2} \right),
\]
\[
\Delta E_{P,n-1} := g^2 \left( \left\| \Psi_{P,n-1}, \phi_{P,n-1} \right\| \frac{1}{H_{P,n-1} - E_{P,n-1}} \phi_{P,n-1} \psi_{P,n-1} \right).
\]

(ii) There exist universal constants \( a, b > 0 \) such that
\[
g^2 a \Lambda \gamma^{n-1} (1 - \gamma) \leq \Delta E_{P,n-1} \leq g^2 b \Lambda \gamma^{n-1} (1 - \gamma).
\]

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Proof. Claim (i) follows from the expansion formula of Theorem 3.7 applied to
\[ E_{P,n} - E_{P,n-1} = \langle \Psi_{P,n}, [H_{P,n} - H_{P,n-1}] \Psi_{P,n-1} \rangle = \langle \Psi_{P,n}, g \Phi_{n-1} \Psi_{P,n-1} \rangle. \]

Next, we show claim (ii). The bound from above follows by using the pull-through formula, i.e.,
\[ \Delta E_{P,n} = g^2 \int_{\gamma^n}^\Lambda dk \rho(k)^2 \left( \Psi_{P,n-1}, \frac{1}{H_{P,k,n-1} + \omega(k) - E_{P,n-1}} \Psi_{P,n-1} \right) \]
and the estimate
\[ g^2 \int_{\gamma^n}^\Lambda dk \rho(k)^2 \left( \Psi_{P,n-1}, \frac{1}{H_{P,k,n-1} + \omega(k) - E_{P,n-1}} \Psi_{P,n-1} \right) \leq g^2 b \Lambda \gamma^{n-1}(1 - \gamma). \]
that uses Lemma 3.1 (iii). The bound from below of (34) can be shown by a similar argument as in (iii) of Theorem 3.7. Therefore we omit the proof.

The established upper and lower bounds given in Lemma 3.8 enable us to prove the first main result.

Proof of Theorem 2.7. Using (i) of Lemma 3.8 we find
\[ E_{P,N} = E_{P,0} - \sum_{n=1}^N \Delta E_{P,n} + O \left( N|g|^4 \Lambda(1 - \gamma)^{3/2} \right), \]
where by construction \( E_{P,0} = \sqrt{P^2 + m^2} \).

The inequalities in (ii) of Lemma 3.8 imply
\[ E_{P,N} \leq \sqrt{P^2 + m^2} - g^2 a \Lambda (1 - \gamma) \sum_{n=1}^N \gamma^{n-1} + |g|^4 C \Lambda N(1 - \gamma) \]
(36)
as well as
\[ E_{P,N} \geq \sqrt{P^2 + m^2} - g^2 b \Lambda (1 - \gamma) \sum_{n=1}^N \gamma^{n-1} - |g|^4 C \Lambda \ln \Lambda (1 - \gamma). \]
(37)
Notice that by the same argument used in Lemma 3.3 one can conclude that \( E_{P,N} = \inf \text{spec} \left( H_{P,N} \upharpoonright F_j \right) \) for all \( j \geq N \). Since \( N = \frac{\ln \Lambda}{\ln \gamma} \) and the estimates in (36) and (37) hold for \( \gamma \) arbitrarily close to 1, they imply the inequalities in (8).

4 The Effective Velocity and the Mass Shell

In this last section we provide the proof of Theorem 2.8, the starting point of which is the expression of the first derivative of the ground state energies \( E_{P,n} \) that follows from analytic perturbation theory in \( P \) as stated in the proposition below:
Proposition 4.1. Suppose $E_{P,n}$ is the non-degenerate isolated eigenvalue corresponding to the ground state $\Psi_{P,n}$. Then, the equation
\[
\frac{\partial E_{P,n}}{\partial P_i} = \langle \overline{\Psi}_{P,n}, V_i(P) \overline{\Psi}_{P,n} \rangle, \quad V_i(P) := \frac{P_i - P_i^f}{[(P - P^f)^2 + m^2]^{1/2}}
\] (38)
holds true for components $i = 1, 2, 3$.

Proof. See Lemma 3.7 in [4].

In order to control the scalar product in (38) the following definition will be convenient:

Definition 4.2. For each $\Lambda \gamma^{n-1}$ we consider the energy level
\[
\min \left\{ \Lambda, \frac{\Lambda \gamma^{n-1}}{g^\epsilon} \right\}, \quad 0 < \epsilon \leq 1/2,
\] (39)
and $l \in \mathbb{N} \cup \{0\}$ such that
\[
\Lambda \gamma^l \leq \min \left\{ \Lambda, \frac{\Lambda \gamma^{n-1}}{g^\epsilon} \right\} < \Lambda \gamma^{l-1}.
\]
We define
\[
\Xi_{n-1} := \Lambda \gamma^l.
\] (40)

The energy scale $\Xi_{n-1}$ will be used in a convenient backwards expansion to gain a certain power of $|g|$ in some estimates. From now on, we use the notation
\[
H_{P,\Xi_{n-1}} := H_P|_{\Xi_{n-1}}, \quad \Psi_{P,\Xi_{n-1}} := \Psi_{P,l}.
\]
The following lemma gives a justification for this type of expansion:

Lemma 4.3. Let $|g|$ be sufficiently small, $|P| < P_{\text{max}}$, and $0 < \epsilon \leq 1/2$. For $z \in \Gamma_{P,n-1}$ the bound
\[
\left\| \left( \frac{1}{H_{P,\Xi_{n-1}} - z} \right)^{1/2} g \Phi|_{\Lambda \gamma^{n-1}} \left( \frac{1}{H_{P,\Xi_{n-1}} - z} \right)^{1/2} \right\|_{\mathcal{F}_{n-1}} \leq |g|^\delta C, \quad \delta := 1 - \frac{\epsilon}{2},
\] (41)
holds true. Consequently, the expansion formulas
\[
\Psi_{P,\Xi_{n-1}} := \Psi_{P,\Xi_{n-1}},
\]
\[
Q_{P,n-1} := -\frac{1}{2\pi i} \int_{\Gamma_{P,n-1}} \frac{dz}{H_{P,\Xi_{n-1}} - z} \uparrow \mathcal{F}_{n-1}
\]
\[
= -\frac{1}{2\pi i} \int_{\Gamma_{P,n-1}} \frac{dz}{H_{P,\Xi_{n-1}} - z} \sum_{j=0}^{\infty} \left[ -g \Phi|_{\Lambda \gamma^{n-1}} \left( \frac{1}{H_{P,\Xi_{n-1}} - z} \right)^j \right] \uparrow \mathcal{F}_{n-1}
\] (42)
hold true and
\[
\|\Psi_{P,\Xi_{n-1}}\|^2 \geq (1 - O(|g|^\delta)) \|\Psi_{P,\Xi_{n-1}}\|^2.
\] (43)
\textbf{Proof.} With the help of Lemma 3.8 we infer the bound 
\[ |E_{P,n-1} - E_{P,\Xi_{n-1}}| \leq C g^2 \Xi_{n-1}. \] 
(44)

Hence, by the definition of $\Xi_{n-1}$ in (39) and $0 < \varepsilon \leq 1/2$, $|g|$ can be chosen sufficiently small but uniformly in $n$ such that both ground state energies, $E_{P,n-1}$ and $E_{P,\Xi_{n-1}}$, lie inside the contour $\Gamma_{P,n-1}$. We estimate 
\[ \sup_{z \in \Gamma_{P,n-1}} \left\| \frac{1}{H_{P,\Xi_{n-1}} - z} \right\|_{F_{n-1}}^{1/2} \leq 2|g| \sup_{z \in \Gamma_{P,n-1}} \left\| \frac{1}{H_{P,\Xi_{n-1}} - z} \right\|_{F_{n-1}}^{1/2} \cdot \sup_{z \in \Gamma_{P,n-1}} \left\| \phi_{\Xi_{n-1}} \right\|_{F_{n-1}}^{1/2}. \]

A similar computation as in Lemma 3.3 gives 
\[ \text{Gap} (H_{P,\Xi_{n-1}} | \mathcal{F}_{n-1}) \geq \zeta \omega (\Lambda y^n) \] 
(45)
such that for sufficiently small $|g|$ one has the bound 
\[ \left\| \frac{1}{H_{P,\Xi_{n-1}} - z} \right\|_{F_{n-1}}^{1/2} \leq \left( \frac{C}{\Lambda y^n} \right)^{1/2} \] 
(46)
by using inequality (i) in Lemma 3.1. Furthermore, one can bound 
\[ \left\| \phi_{\Xi_{n-1}} \right\|_{F_{n-1}}^{1/2} \leq \Xi_{n-1}^{1/2} \left\| \frac{1}{H_{P,\Xi_{n-1}} - z} \right\|_{F_{n-1}}^{1/2} \] 
(47)
Hence, we may conclude that 
\[ \left\| \phi_{\Xi_{n-1}} \right\|_{F_{n-1}}^{1/2} \leq |g| C \left( \frac{\Lambda y^n}{\Lambda} \right)^{1/2} \] 
for $\frac{\Lambda y^n}{g} < \Lambda$, and 
\[ \left\| \phi_{\Xi_{n-1}} \right\|_{F_{n-1}}^{1/2} \leq |g| C \left( \frac{\Lambda y^n}{\Lambda} \right)^{1/2} \] 
for $\frac{\Lambda y^n}{g} \geq \Lambda$. 

This ensures the validity of the expansion formulas (42) as well as the relation in (43). \hfill \Box

We can now prove our second main result:

\textbf{Proof of Theorem 2.8.} The strategy of proof is an expansion using the formulas provided by Theorem 3.7. As a first observation we note that by the spectral theorem the bounds 
\[ \|V_i(P)\| \leq 1 \quad \forall P \in \mathbb{R}^3, \quad \left| \frac{\partial E_{P,n}}{\partial P_i} \right| \leq 1 \quad \text{for} \quad |P| < P_{\text{max}} \] 
(47)
hold. These inequalities will be employed frequently without further notice.
With the help of Theorem 3.7 we find the following expansion for all \( N \geq n \geq 1 \):

\[
\left\langle \tilde{\Psi}_{P,n}, V_i(P)\tilde{\Psi}_{P,n} \right\rangle = \frac{\left\langle \Psi_{P,n}, V_i(P)\Psi_{P,n} \right\rangle}{\left\langle \Psi_{P,n}, \Psi_{P,n} \right\rangle} = 1 + g^2 \alpha_p^{n-1} + O\left(|g|^4 (1 - \gamma)^{4/2}\right) + O\left(|g|^4 (1 - \gamma)^{4/2}\right) \tag{48}
\]

\[
+ g^2 \left( \frac{1}{H_{P,n-1} - E_{P,n-1}} \phi_{P,n-1}^{*} \Psi_{P,n-1}, V_i(P) \frac{1}{H_{P,n-1} - E_{P,n-1}} \phi_{P,n-1}^{*} \Psi_{P,n-1} \right) + h.c.
\]

\[
+ g^2 \left( \overline{Q}_{P,n-1}^{*} \phi_{P,n-1}^{*} \left( \frac{1}{H_{P,n-1} - E_{P,n-1}} \right)^2 \phi_{P,n-1}^{*} \Psi_{P,n-1}, V_i(P) \Psi_{P,n-1} \right) + h.c. \tag{49}
\]

\[
+ O\left(|g|^4 (1 - \gamma)^{4/2}\right) \right].
\]

We observe that

\[(49) = -2 g^2 \alpha_p^{n-1} \left\langle \Psi_{P,n-1}, V_i(P)\Psi_{P,n-1} \right\rangle
\]

because

\[
g^2 \left( \overline{Q}_{P,n-1}^{*} \phi_{P,n-1}^{*} \left( \frac{1}{H_{P,n-1} - E_{P,n-1}} \right)^2 \phi_{P,n-1}^{*} \Psi_{P,n-1}, V_i(P) \Psi_{P,n-1} \right) = g^2 \alpha_p^{n-1} \left\langle \Psi_{P,n-1}, V_i(P)\Psi_{P,n-1} \right\rangle.
\]

Hence, we can rewrite (48) as

\[
\left\langle \tilde{\Psi}_{P,n}, V_i(P)\tilde{\Psi}_{P,n} \right\rangle = \left(1 - g^2 \alpha_p^{n-1} + O\left(|g|^4 (1 - \gamma)^{4/2}\right)\right) \left\langle \tilde{\Psi}_{P,n-1}, V_i(P)\tilde{\Psi}_{P,n-1} \right\rangle \tag{50}
\]

\[
+ g^2 \left( \frac{1}{H_{P,n-1} - E_{P,n-1}} \phi_{P,n-1}^{*} \overline{\Psi}_{P,n-1}, V_i(P) \frac{1}{H_{P,n-1} - E_{P,n-1}} \phi_{P,n-1}^{*} \Psi_{P,n-1} \right) + h.c. \tag{51}
\]

\[
+ g^2 2 \mathcal{R} \left( \overline{Q}_{P,n-1}^{*} \phi_{P,n-1}^{*} \left( \frac{1}{H_{P,n-1} - E_{P,n-1}} \right)^2 \phi_{P,n-1}^{*} \overline{\Psi}_{P,n-1}, V_i(P)\overline{\Psi}_{P,n-1} \right) \tag{52}
\]

\[
+ O\left(|g|^4 (1 - \gamma)^{4/2}\right).
\]

Next, we proceed iteratively by expanding \( \left\langle \tilde{\Psi}_{P,n}, V_i(P)\tilde{\Psi}_{P,n} \right\rangle \) at each step from \( n = N \) to \( n = 0 \). Meanwhile, we define

\[
A_{P,n-1} := (51), \quad B_{P,n-1} := (52).
\]

As a result of the iteration we find the following expansion

\[
\left\langle \tilde{\Psi}_{P,N}, V_i(P)\tilde{\Psi}_{P,N} \right\rangle = \prod_{j=1}^{N} \left(1 - g^2 \alpha_p^{N-j} \right) \left\langle \tilde{\Psi}_{P,0}, V_i(P)\tilde{\Psi}_{P,0} \right\rangle
\]

\[
+ \sum_{j=2}^{N-1} \left(1 - g^2 \alpha_p^{N-j} \right) \ldots \left(1 - g^2 \alpha_p^{N-j+1} \right) \left[ A_{P,N-j-1} + B_{P,N-j-1} \right]
\]

\[
+ \left(1 - g^2 \alpha_p^{N-1} \right) \left[ A_{P,N-2} + B_{P,N-2} \right] + \left[ A_{P,N-1} + B_{P,N-1} \right] + O\left(|g|^4 N(1 - \gamma)^{4/2}\right). \tag{53}
\]
Let us assume one could show the bounds
\[
|A_{P,N-j}| \leq g^2 C \frac{1 - \gamma}{\Lambda \gamma^{N-j+1}}, \quad (54)
\]
\[
|B_{P,N-j}| \leq |g|^{5/2} C (1 - \gamma) \quad (55)
\]
where we stress that the universal constant \( C \) is independent of the mass \( m \). Then, using the following ingredients

- (iii) of Theorem 3.7,
- \( N = \frac{\ln \Lambda}{\ln \gamma} \),
- the basic estimates
\[
\prod_{j=1}^{N} (1 - g^2 \alpha_p^{N-j}) \leq \prod_{j=1}^{N} (1 - g^2 c_1 (1 - \gamma)) \leq \Lambda^{-g^2 c_1 (1-\epsilon) N},
\]
\[
\sum_{j=2}^{N-1} \left( 1 - g^2 \alpha_p^{N-j} \right) \left( 1 - g^2 \alpha_p^{N-j-1} \right) + 1 \leq \sum_{j=0}^{N-1} (1 - g^2 c_1 (1 - \gamma))^j \leq \frac{1}{g^2 c_1 (1 - \gamma)},
\]
and using \( \Lambda \gamma^N = 1 \)
\[
\sum_{j=2}^{N-1} \left( 1 - g^2 \alpha_p^{N-j} \right) \left( 1 - g^2 \alpha_p^{N-j-1} \right) \frac{1 - \gamma}{\Lambda \gamma^{N-j}} + (1 - g^2 \alpha_p^{N-1}) \frac{1 - \gamma}{\Lambda \gamma^N} + \frac{1 - \gamma}{\Lambda \gamma^N} \leq C \gamma \sum_{j=0}^{N-1} \gamma^j \leq C,
\]
the bounds in (54)-(55) are seen to imply
\[
\left| \langle \hat{\Psi}_{P,N}, V_i(P) \hat{\Psi}_{P,N} \rangle \right| \leq \Lambda^{-g^2 c_1 (1-\gamma)} \frac{|P|}{[P^2 + m^2]^{1/2}} + C|g|^{1/2} + C|g|^4 \ln (1 - \gamma), \quad (56)
\]
where we recall that \( \left| \langle \hat{\Psi}_{P,0}, V_i(P) \hat{\Psi}_{P,0} \rangle \right| \leq \frac{|P|}{(P^2 + m^2)^{1/2}} \).

As the fineness parameter \( \gamma \) can be chosen arbitrarily close to one the bound in (9) is proven. We show now that the bounds (54)-(55) hold true.

**Bound (54):** Defining \( P_A := \lambda P \) and its components \( P_{Ai} := \lambda P_i, 1 \leq i \leq 3 \), we start with the identity
\[
A_{P,n-1} = \int_0^1 \frac{d\lambda}{dA} \frac{d\gamma}{\gamma^2} \left\{ \frac{1}{H_{P_A,n-1} - E_{P_A,n-1}} \phi^* \psi_{P,A,n-1}, V_i(P_A) \frac{1}{H_{P_A,n-1} - E_{P_A,n-1}} \phi \psi_{P,A,n-1} \right\} \quad (57)
\]
that holds because of analytic perturbation theory in $P$ (see Lemma 3.7 in [4]) and

$$\left\langle \frac{1}{H_{0,n-1} - E_{0,n-1}} \phi^*_{n-1} \hat{\Psi}_{0,n-1}, V_i(0) \frac{1}{H_{0,n-1} - E_{0,n-1}} \phi^*_{n-1} \hat{\Psi}_{0,n-1} \right\rangle = 0$$

by symmetry under rotational invariance of $H_{0,n-1}$, $E_{0,n-1}$ and $\hat{\Psi}_{0,n-1}$. In order to estimate the integrand

$$g^2 \frac{d}{d\lambda} \left( \frac{1}{H_{P,\lambda,n-1} - E_{P,\lambda,n-1}} \phi^*_{n-1} \hat{\Psi}_{P,\lambda,n-1}, V_i(P) \frac{1}{H_{P,\lambda,n-1} - E_{P,\lambda,n-1}} \phi^*_{n-1} \hat{\Psi}_{P,\lambda,n-1} \right) =$$

$$= \lim_{h \to 0} \frac{g^2}{h} \left( \frac{1}{H_{P,\lambda,n-1} - E_{P,\lambda,n-1}} \phi^*_{n-1} \hat{\Psi}_{P,\lambda,n-1}, V_i(P + h) \frac{1}{H_{P,\lambda,n-1} - E_{P,\lambda,n-1}} \phi^*_{n-1} \hat{\Psi}_{P,\lambda,n-1} \right)$$

$$- \left( \frac{1}{H_{P,\lambda,n-1} - E_{P,\lambda,n-1}} \phi^*_{n-1} \hat{\Psi}_{P,\lambda,n-1}, V_i(P) \frac{1}{H_{P,\lambda,n-1} - E_{P,\lambda,n-1}} \phi^*_{n-1} \hat{\Psi}_{P,\lambda,n-1} \right) \right)$$

(58)

we first observe that in expression (58), at least for small $|h|$, the vector $\hat{\Psi}_{P,\lambda,n-1}$ can be replaced by the vector $\hat{\Upsilon}_{P,\lambda,n-1}$ where

$$\hat{\Upsilon}_{P,\lambda,n-1} := -\frac{1}{2\pi i} \oint_{\Gamma_{P,n-1}} \frac{dz}{H_{P,\lambda,n-1} - z} \Psi_{P,n-1}.$$ 

Notice that $\hat{\Upsilon}_{P,\lambda,n-1} \parallel \Psi_{P,\lambda,n-1}$ and $\hat{\Upsilon}_{P,\lambda,n-1}(h=0) = \Psi_{P,\lambda,n-1}$. Hence, we need to estimate three types of terms:

$$\lim_{h \to 0} \frac{g^2}{h} \left( \frac{1}{H_{P,\lambda,n-1} - E_{P,\lambda,n-1}} - \frac{1}{H_{P,\lambda,n-1} - E_{P,\lambda,n-1}} \right) \phi^*_{n-1} \hat{\Psi}_{P,\lambda,n-1}, V_i(P) \frac{1}{H_{P,\lambda,n-1} - E_{P,\lambda,n-1}} \phi^*_{n-1} \hat{\Psi}_{P,\lambda,n-1} \right),$$

(59)

$$\lim_{h \to 0} \frac{g^2}{h} \left( \frac{1}{H_{P,\lambda,n-1} - E_{P,\lambda,n-1}} \phi^*_{n-1} \hat{\Upsilon}_{P,\lambda,n-1} - \hat{\Upsilon}_{P,\lambda,n-1} \right), V_i(P) \frac{1}{H_{P,\lambda,n-1} - E_{P,\lambda,n-1}} \phi^*_{n-1} \hat{\Upsilon}_{P,\lambda,n-1} \right),$$

(60)

$$\lim_{h \to 0} \frac{g^2}{h} \left( \frac{1}{H_{P,\lambda,n-1} - E_{P,\lambda,n-1}} \phi^*_{n-1} \hat{\Upsilon}_{P,\lambda,n-1}, [V_i(P + h) - V_i(P)] \frac{1}{H_{P,\lambda,n-1} - E_{P,\lambda,n-1}} \phi^*_{n-1} \hat{\Upsilon}_{P,\lambda,n-1} \right),$$

(61)

In order to estimate term (59) we observe that the expression is well defined because the vector $\phi^*_{n-1} \hat{\Psi}_{P,\lambda,n-1}$ is orthogonal to the ground state vector of both the Hamiltonians $H_{P,\lambda,n-1}$ and $H_{P,\lambda,n-1}$. Hence, we verify that

$$\lim_{h \to 0} \frac{1}{h} \left( \frac{1}{H_{P,\lambda,n-1} - E_{P,\lambda,n-1}} - \frac{1}{H_{P,\lambda,n-1} - E_{P,\lambda,n-1}} \right)$$

$$= \lim_{h \to 0} \frac{1}{h} \left( \frac{1}{H_{P,\lambda,n-1} - E_{P,\lambda,n-1}} (H_{P,\lambda,n-1} - H_{P,\lambda,n-1} - E_{P,\lambda,n-1} + E_{P,\lambda,n-1}) \frac{1}{H_{P,\lambda,n-1} - E_{P,\lambda,n-1}} \right)$$

$$= \frac{1}{H_{P,\lambda,n-1} - E_{P,\lambda,n-1}} \left( -\frac{d}{d\lambda} \sqrt{(P_{\lambda} - P^f)^2 + m^2} + \frac{d}{d\lambda} E_{P,\lambda,n-1} \right) \frac{1}{H_{P,\lambda,n-1} - E_{P,\lambda,n-1}}$$

$$= \frac{1}{H_{P,\lambda,n-1} - E_{P,\lambda,n-1}} \sum_{i=1}^{3} P_{\lambda,i} \left( -V_i(P_{\lambda}) + \frac{\partial E_{P,\lambda,n-1}}{\partial P_i} \right) \frac{1}{H_{P,\lambda,n-1} - E_{P,\lambda,n-1}}$$

(62)
holds true when applied to the vector $\phi^{[n-1]}(P_{\lambda},n-1)$. At first we treat the term proportional to $\sum_{i=1}^{3} P_{\lambda i} V_i(P_{\lambda})$. Using (iii) in Lemma 3.1, the estimate in (47), and the pull-through formula, we get the estimate

$$g^2 \left| \frac{1}{H_{P_{\lambda},n-1} - E_{P_{\lambda},n-1}} \sum_{j=1}^{3} P_{\lambda j} V_j(P_{\lambda}) \right|$$

$$= \int_{-\infty}^{\infty} d\rho(k)^2 \left| \sum_{j=1}^{3} P_{\lambda j} V_j(P_{\lambda} - k) \right|$$

$$\leq g^2 C \int_{-\infty}^{\infty} dk \frac{1}{|k|^d} \leq g^2 \frac{C(1-\gamma)}{\Lambda \gamma^d}.$$ 

The remaining term in (59) being proportional to $\sum_{i=1}^{3} P_{\lambda i} \left( \frac{\partial F_{\rho_{\lambda},1}}{\partial P_{\rho_i}} \right) |_{P_{\rho_i}=P_{\lambda}}$ can be estimated in the same way. In consequence, we get

$$|\langle 59 \rangle| \leq g^2 \frac{C(1-\gamma)}{\Lambda \gamma^d}. \quad (63)$$

Next, we consider term (60). Using the differentiability in $\lambda$ again we find

$$\lim_{h \to 0} \frac{\nabla_{P_{\lambda+h},n-1} - \nabla_{P_{\lambda},n-1}}{h} = -\frac{1}{2\pi i} \lim_{h \to 0} \frac{1}{h} \int_{\Gamma_{P_{\lambda},n-1}} dz \left[ \frac{1}{H_{P_{\lambda+h},n-1} - z} - \frac{1}{H_{P_{\lambda},n-1} - z} \right] \nabla_{P_{\lambda},n-1}$$

$$= -\frac{1}{2\pi i} \lim_{h \to 0} \frac{1}{h} \int_{\Gamma_{P_{\lambda},n-1}} dz \left[ \frac{1}{H_{P_{\lambda},n-1} - z} \left( \frac{1}{H_{P_{\lambda+h},n-1} - H_{P_{\lambda},n-1}} - \frac{1}{H_{P_{\lambda},n-1} - z} \right) \right] \nabla_{P_{\lambda},n-1}$$

$$= -\frac{1}{2\pi i} \int_{\Gamma_{P_{\lambda},n-1}} dz \left[ \frac{1}{H_{P_{\lambda},n-1} - z} \left( -\sum_{i=1}^{3} P_{\lambda i} V_i(P_{\lambda}) \right) \frac{1}{H_{P_{\lambda},n-1} - z} \right] \nabla_{P_{\lambda},n-1}$$

$$= -Q^{\perp}_{P_{\lambda},n-1} \frac{1}{H_{P_{\lambda},n-1} - E_{P_{\lambda},n-1}} \sum_{i=1}^{3} P_{\lambda i} V_i(P_{\lambda}) \nabla_{P_{\lambda},n-1} \quad (64)$$

and

$$\lim_{h \to 0} \frac{1}{h} \left[ \frac{1}{\nabla_{P_{\lambda+h},n-1}} - \frac{1}{\nabla_{P_{\lambda},n-1}} \right] = -\frac{1}{\nabla_{P_{\lambda},n-1}} \lim_{h \to 0} \Re \left( \nabla_{P_{\lambda+h},n-1} - \nabla_{P_{\lambda},n-1}, \nabla_{P_{\lambda},n-1} \right) = 0. \quad (65)$$

Equations (64) and (65), the pull-through formula, and the gap estimate in Theorem 3.6 give

$$|\langle 60 \rangle| \leq g^2 \frac{C(1-\gamma)}{\Lambda \gamma^d}. \quad (66)$$

In the estimate of the third term, i.e., term (61), we exploit the additional decay which we gain through the derivative of $V_i(P_{\lambda})$, i.e.,

$$\lim_{h \to 0} \frac{1}{h} \left[ V_i(P_{\lambda+h}) - V_i(P_{\lambda}) \right] = \frac{P_{\lambda i} - V_i(P_{\lambda}) \sum_{j=1}^{3} V_j(P_{\lambda}) P_{\lambda j}}{\sqrt{(P_{\lambda} - P^j)^2 + m^2}}. \quad (67)$$

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Thus, we can rewrite and estimate (61) as follows

$$
\left| g^2 \int_{\Lambda^n} dk \rho(k)^2 \left[ \frac{1}{H_{P_i,k,n-1} + \omega(k) - E_{P_i,k,n-1}} \times
\left[ P_{\lambda} - V_i(P\lambda - k) \sum_{j=1}^3 V_j(P\lambda - k) P_{\lambda,j} \right] \right]^{\frac{1}{2}} \frac{1}{\sqrt{(P\lambda - P\lambda - k)^2 + m^2}} \frac{1}{H_{P_i,k,n-1} + \omega(k) - E_{P_i,k,n-1}} \left[ \frac{1}{H_{P_i,k,n-1} + \omega(k) - E_{P_i,k,n-1}} \Psi_{P_i,k,n-1} \right] \right| \leq C g^2 \max \int_{\Lambda^n} dk \rho(k)^2 \left[ \frac{1}{H_{P_i,k,n-1} + \omega(k) - E_{P_i,k,n-1}} \times \right]
\left[ \frac{1}{\sqrt{(P\lambda - P\lambda - k)^2 + m^2}} \frac{1}{H_{P_i,k,n-1} + \omega(k) - E_{P_i,k,n-1}} \Psi_{P_i,k,n-1} \right]
(67)
$$

where we have used the pull-through formula. Next we consider the spectral measure $d\mu_k(\xi) \equiv f_k(\xi)d\xi$ (where $f_k(\xi) \geq 0 \text{ a.e.}$) associated with the vector

$$
\frac{1}{H_{P_i,k,n-1} + \omega(k) - E_{P_i,k,n-1}} \Psi_{P_i,k,n-1}
$$

in the joint spectral representation of the components of the operator $P\lambda$ where $\xi$ is the spectral variable. The measure is defined by

$$
(0 \leq \left| \chi_{\Omega} \frac{1}{H_{P_i,k,n-1} + \omega(k) - E_{P_i,k,n-1}} \Psi_{P_i,k,n-1} \right|^2 = \int_{\sigma(P\lambda)} d\xi f_k(\xi) \chi_{\Omega}(\xi) \leq \frac{C}{|k|^2}
$$

for every measurable set $\Omega \subseteq \sigma(P\lambda)$ where $\chi_{\Omega}(\xi)$ is the characteristic function of the set $\Omega$ and $\chi_{\Omega}$ is the corresponding spectral projection. Thus we can write (67) as follows

$$
(67) = C g^2 \int_{\Lambda^n} dk \frac{1}{|k|} \left[ \frac{1}{\sqrt{(P\lambda - P\lambda - k)^2 + m^2}} \right] \frac{1}{H_{P_i,k,n-1} + \omega(k) - E_{P_i,k,n-1}} \left[ \frac{1}{H_{P_i,k,n-1} + \omega(k) - E_{P_i,k,n-1}} \right]
$$

By knowing that

$$
\int_{\sigma(P\lambda)} d\xi f_k(\xi) \frac{1}{\sqrt{(P\lambda - \xi - k)^2 + m^2}} < +\infty
$$

we can interchange the integration in $d\xi$ with the angular integration in the variable $k$, i.e.,

$$
(68) = C g^2 \int_{\Lambda^n} |kd| \int_{\sigma(P\lambda)} d\xi \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin \theta f_k(\xi) \frac{1}{\sqrt{(P\lambda - \xi - k)^2 + k^2 - 2 \cos \theta |P\lambda - \xi| |k| + m^2}}
$$

where $\theta$ denotes the angle between the vector $k$ and the vector $P\lambda - \xi$ and $\varphi$ the azimuthal angle with respect to an arbitrarily chosen vector orthogonal to $P\lambda - \xi$. We split the integration in the variable $\theta$ into two regions: $\theta \in \left[ \frac{\pi}{4}, \pi \right]$ and $\theta \in \left[ 0, \frac{\pi}{4} \right]$. For $\theta \in \left[ \frac{\pi}{4}, \pi \right]$ being $\cos \theta \in [-1, \frac{1}{2}]$ we observe that

$$
(P\lambda - \xi)^2 + k^2 - 2 \cos \theta |P\lambda - \xi| |k| \geq (P\lambda - \xi)^2 + k^2 - |P\lambda - \xi||k| \geq \frac{3}{4}k^2
$$
and, consequently,
\[
\int_{\sigma(P')} d\xi \int_0^{2\pi} d\varphi \int_0^{\pi/3} d\theta \sin \theta f_k(\xi) \frac{1}{\sqrt{(P_\lambda - \xi)^2 + k^2 - 2\cos \theta |P_\lambda - \xi| |k| + m^2}}
\]
\[
\leq C \int_{\sigma(P')} d\xi \int_0^{2\pi} d\varphi \int_0^{\pi/3} d\theta \sin \theta f_k(\xi) \frac{1}{|k|}
\]
\[
\leq \frac{C}{|k|^3}
\]

Notice that the constant \( C \) in (72) can be chosen to be independent of the mass \( m \). Next, we treat the integration over \( \theta \in [0, \pi/3] \) where \( \cos \theta \in [1/2, 1] \) and

\[
(P_\lambda - \xi)^2 + k^2 - 2\cos \theta |P_\lambda - \xi| |k| \geq [(P_\lambda - \xi)^2 + k^2] (1 - \cos \theta)
\]

we find
\[
\int_{\sigma(P')} d\xi \int_0^{2\pi} d\varphi \int_0^{\pi/3} d\theta \sin \theta f_k(\xi) \frac{1}{\sqrt{[(P_\lambda - \xi)^2 + k^2] (1 - \cos \theta) + m^2}}
\]
\[
\leq \int_{\sigma(P')} d\xi \int_0^{2\pi} d\varphi \int_0^{\pi/3} d\theta \sin \theta \frac{1}{|k|} f_k(\xi) \frac{1}{\sqrt{(1 - \cos \theta)}}
\]
\[
\leq \frac{C}{|k|^3}
\]

Notice that also the constant \( C \) in (73) can be chosen to be independent of the mass \( m \). Combining the results for the two integration domains, i.e., (69) and (73), we arrive at

\[
(68) \leq g^2 C \int_{\Lambda \gamma^{n-1}} \frac{1}{|k|^2} \leq g^2 C \frac{1 - \gamma}{\Lambda \gamma^n}.
\]

Hence, we have proven the bound in (61).

With the three bounds in (63), (66) and (74) we can control the integrand (59)-(61), and hence, the integral given in (57) which proves the bound in (54).

**Bound (55):** As a next step we proceed with the bound of (55) where by using the pull-through
formula we get

$$\left| B_{P,n-1} \right|$$

$$\begin{align*}
&= g^2 \left| 2 \Re \int_{\Lambda^{\gamma_{n-1}}} dk \rho(k)^2 \left( \frac{1}{H_{P,n-1} - E_{P,n-1}} \Psi_{P,n-1} \right) \left( \frac{1}{H_{P,n-1} - E_{P,n-1}} \Psi_{P,n-1} \right) \right| \\
&\leq g^2 C \left( \int_{\Lambda^{\gamma_{n-1}}} dk \frac{1}{k^2} \left| \frac{1}{H_{P,n-1} - E_{P,n-1}} \Psi_{P,n-1} \right| \right) \left( \int_{\Lambda^{\gamma_{n-1}}} dk \frac{1}{k^2} \left| \frac{1}{H_{P,n-1} - E_{P,n-1}} \Psi_{P,n-1} \right| \right) \\
&\leq g^2 C \gamma^{\gamma_{n-1}} (1 - \gamma) \left( \frac{1}{H_{P,n-1} - E_{P,n-1}} \right) \left( \frac{1}{H_{P,n-1} - E_{P,n-1}} \right). \quad (75)
\end{align*}$$

We shall now show that

$$\left\| \frac{1}{H_{P,n-1} - E_{P,n-1}} \Psi_{P,n-1} \right\| \leq C \frac{|g|^{1/2} \Lambda^{\gamma_n}}{\Lambda^{\gamma_{n-1}}} \quad (76)$$

holds true, so that, by inserting this bound in (75), we get the desired $m$-independent estimate in (55).

In order to gain a certain power of $|g|$ we re-expand the left-hand side of (76) backwards from energy level $\Lambda^{\gamma_{n-1}}$ to $\Xi_{n-1}$, as defined in (40), with the help of Lemma 4.3 for an $\epsilon$, $0 < \epsilon \leq \frac{1}{2}$, and $\delta = 1 - \frac{\epsilon}{2}$ which will be fixed later. We know that

- $\left\| \frac{1}{H_{P,n-1} - E_{P,n-1}} \right\| \leq C \frac{|g|^{\delta}}{\Lambda^{\gamma_{n-1}}} \quad$ for $z \in \Gamma_{P,n-1}$ (see (41)),

- $\Psi_{P,n-1}$ and $\Psi_{P,n-1}$ are two vectors belonging to the same ray with $\|\Psi_{P,n-1}\|^2 \geq (1 - O(|g|^{\delta})) \|\Psi_{P,n-1}\|^2$ (see (42)).

Thus, denoting the length of the contour $\Gamma_{P,n-1}$ by $|\Gamma_{P,n-1}|$, we find for $|g|$ sufficiently small

$$\begin{align*}
&\left\| \frac{1}{H_{P,n-1} - E_{P,n-1}} \Psi_{P,n-1} \right\| = \left\| \frac{1}{H_{P,n-1} - E_{P,n-1}} \Psi_{P,n-1} \right\| \\
&\leq C \left\| \frac{1}{H_{P,n-1} - E_{P,n-1}} \Psi_{P,n-1} \right\| + C \left\| \frac{1}{H_{P,n-1} - E_{P,n-1}} \Psi_{P,n-1} \right\| \\
&\leq \left\| \frac{1}{H_{P,n-1} - E_{P,n-1}} \Psi_{P,n-1} \right\| + C \frac{|g|^{\delta}}{\Lambda^{\gamma_{n-1}}} \quad (77)
\end{align*}$$

where we have used the bound in (41), the inequality in (46), and the gap estimate given in Theorem 3.6. Using the same ingredients, we estimate

$$\left\| \frac{1}{H_{P,n-1} - E_{P,n-1}} \Psi_{P,n-1} \right\| \quad (78)$$

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by expanding the spectral projection on the right, i.e.,

\[
(78) \leq \left\| Q_{P,v}^+ - \frac{1}{2} Q_{P,v}^+ V_i(P) \widehat{\Psi}_{P,v} \right\| \\
+ C \left\| Q_{P,v}^+ \frac{1}{H_{P,v} - E_{P,v}} \sup_{\gamma \in \Gamma_{P,v}} \sum_{j=1}^{\infty} \left( \frac{1}{H_{P,v} - E_{P,v}} \right) \| V_i(P) \| \right. \\
\leq \left. Q_{P,v}^+ \frac{1}{H_{P,v} - E_{P,v}} Q_{P,v}^+ V_i(P) \widehat{\Psi}_{P,v} \right\| + C \frac{|g|^6}{\Lambda \gamma^n} (79)
\]

where \( Q_{P,v}^+ \equiv Q_{P,v}^+ \), for some \( l \leq N \) specified in (40). Next, we study

\[
\left\| Q_{P,v}^+ \frac{1}{H_{P,v} - E_{P,v}} Q_{P,v}^+ V_i(P) \widehat{\Psi}_{P,v} \right\| (80)
\]

by applying the resolvent formula

\[
(80) \leq \left\| Q_{P,v}^+ \frac{1}{H_{P,v} - E_{P,v}} Q_{P,v}^+ V_i(P) \widehat{\Psi}_{P,v} \right\| \\
+ \left\| Q_{P,v}^+ \frac{1}{H_{P,v} - E_{P,v}} g \phi_\gamma \frac{1}{H_{P,v} - E_{P,v}} Q_{P,v}^+ V_i(P) \widehat{\Psi}_{P,v} \right\| (81)
\]

In order to estimate (81) we make use of the following intermediate steps:

- \( \left\| Q_{P,v}^+ \left( \frac{1}{H_{P,v} - E_{P,v}} \right) \right\| \leq \frac{C}{\Lambda \gamma^n} \),

- \[
\left\| g \phi_\gamma \frac{1}{H_{P,v} - E_{P,v}} Q_{P,v}^+ V_i(P) \widehat{\Psi}_{P,v} \right\| \\
= |g| \left( \int_{\Lambda \gamma^n}^{Z_{P,v}} d\kappa \rho(k)^2 \right)^{1/2} \left\| \frac{1}{H_{P,v} - E_{P,v}} Q_{P,v}^+ V_i(P) \widehat{\Psi}_{P,v} \right\| \\
\leq C |g| \frac{1}{Z_{P,v}} (82)
\]

following from

\[
\left\| Q_{P,v}^+ \frac{1}{H_{P,v} - E_{P,v}} \right\| \leq \frac{C}{Z_{P,v}} = C \max \left( \frac{g^\epsilon}{\Lambda \gamma^n}, \frac{1}{\Lambda} \right) (82)
\]

that holds because of Theorem 3.6 and inequality (i) in Lemma 3.1.

This implies

\[
(80) \leq \left\| Q_{P,v}^+ \frac{1}{H_{P,v} - E_{P,v}} Q_{P,v}^+ V_i(P) \widehat{\Psi}_{P,v} \right\| + C \frac{|g|^6}{\Lambda \gamma^n} (83)
\]
Next we consider
\[ \left\| \frac{1}{Q_{P,n-1}^+ H_{P,\Xi_{n-1}} - E_{P,n-1}} Q_{P,\Xi_{n-1}}^2 V_i(P) \bar{\Psi}_{P,\Xi_{n-1}} \right\| \] (84)
and re-expand the first spectral projection. Hence, by using (41) and (82) we can conclude that
\[ (84) \leq \left\| \frac{1}{Q_{P,\Xi_{n-1}}^+ H_{P,\Xi_{n-1}} - E_{P,n-1}} Q_{P,\Xi_{n-1}}^+ V_i(P) \bar{\Psi}_{P,\Xi_{n-1}} \right\| + C \frac{|g|^6}{\Lambda \gamma^n}. \] (85)

As a last step, for the first term on the right-hand side of (85) we have to regard two cases:

1. Case $\Xi_{n-1} < \Lambda$. In this case we exploit
\[ \left\| \frac{1}{Q_{P,\Xi_{n-1}}^+ H_{P,\Xi_{n-1}} - E_{P,n-1}} \right\|_{F^\Lambda_{\Xi_{n-1}}} \leq \frac{g^\epsilon C}{\Lambda \gamma^n} \]

2. Case $\Xi_{n-1} = \Lambda$. In this case we have
\[ Q_{P,\Xi_{n-1}}^+ V_i(P) \bar{\Psi}_{P,\Xi_{n-1}} = \frac{P}{\sqrt{P^2 + m^2}} Q_{P,\Xi_{n-1}}^+ \bar{\Psi}_{P,\Xi_{n-1}} = 0. \]

For both cases the estimate
\[ \left\| \frac{1}{Q_{P,\Xi_{n-1}}^+ H_{P,\Xi_{n-1}} - E_{P,n-1}} Q_{P,\Xi_{n-1}}^+ V_i(P) \bar{\Psi}_{P,\Xi_{n-1}} \right\| \leq C \frac{g^\epsilon}{\Lambda \gamma^n} \]
holds true.
Choosing $\epsilon = \frac{1}{2}$ and collecting all the remainders the bound in (76) is seen to be true. Hence, we have also proven the inequality in (55). This concludes the proof of the bound in (56). \qed

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