Multi-colored Links From 3-Strand Braids Carrying Arbitrary Symmetric Representations

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Abstract. Obtaining HOMFLY-PT polynomials $H_{R_1,\ldots,R_l}$ for arbitrary links with $l$ components colored by arbitrary $SU(N)$ representations $R_1,\ldots,R_l$ is a very complicated problem. For a class of rank $r$ symmetric representations, the $[r]$-colored HOMFLY-PT polynomial $H_{[r_1],[r_2],\ldots,[r_l]}$ evaluation becomes simpler, but the general answer lies far beyond our current capabilities. To simplify the situation even more, one can consider links that can be realized as a 3-strand closed braid. Recently (Itoyama et al. in Int J Mod Phys A28:1340009, 2013. arXiv:1209.6304), it was shown that $H_{[r]}$ for knots realized by 3-strand braids can be constructed using the quantum Racah coefficients (6j-symbols) of $U_q(sl_2)$, which makes easy not only to evaluate such invariants, but also to construct analytical formulas for $H_{[r]}$ of various families of 3-strand knots. In this paper, we generalize this approach to links whose components carry arbitrary symmetric representations. We illustrate the technique by evaluating multi-colored link polynomials $H_{[r_1],[r_2]}$ for the two-component link $L7a3$ whose components carry $[r_1]$ and $[r_2]$ colors. Using our results for exclusive Racah matrices, it is possible to calculate symmetric-colored HOMFLY-PT polynomials of links for the so-called one-looped links, which are obtained from arborescent links by adding a loop. This is a huge class of links that contains the entire Rolfsen table, all 3-strand links, all arborescent links, and, for example, all mutant knots with 11 intersections.

1. Introduction

For any knot colored by an arbitrary $SU(N)$ representation, one can formally construct colored invariants basing on Chern–Simons field theory. Equivalently, these invariants can also be obtained within the Reshetikhin–Turaev approach based on the theory of quantum groups and quantum $\mathcal{R}$-matrix. These invariants involve braiding eigenvalues of the $\mathcal{R}$-matrix and quantum Racah matrices.
of $U_q(sl_N)$. In order to see the explicit polynomial form of these invariants, one needs to know the matrix elements of quantum Racah, which are not known for arbitrary $SU(N)$ representations.

Recall that the $U_q(sl_2)$ Racah coefficients or Wigner 6j-symbols are very important in many theoretical and mathematical physics. These coefficients appear as transformation matrix elements between two equivalent bases obtained from combining three angular momenta [2]. They can also be viewed as duality matrices between two equivalent $SU(2)_k$ Wess–Zumino–Witten conformal blocks [3–8] where the quantum deformation $q$ is taken as the $(k + 2)$th root of unity. Interestingly, there is a closed-form analytic expression for the $U_q(sl_2)$ Racah coefficients [9], which can also be described in terms of the $q$-hypergeometric function $4\Phi_3$ [10].

The Racah coefficients are well defined for both finite-dimensional [10] and infinite-dimensional representations [11,12] of classical Lie groups as well as for quantum groups [10]. In this paper, we consider only irreducible finite-dimensional representations of the quantum group $U_q(sl_N)$. Such representations are enumerated by the Young diagrams and can be separated into two groups: with and without multiplicities. A representation $V$ of $U_q(sl_N)$ is called multiplicity-free if the decomposition of its tensor square $V \otimes V$ into irreducible components has no repeated summands. Multiplicity-free representations are enumerated by the rectangular Young diagrams. Unfortunately, an analytic expression for the $U_q(sl_N)$ Racah matrix relating two equivalent bases involving the tensor product of three $U_q(sl_N)$ representations still remains an open problem.

Let us note that there are two types of the Racah matrices: the Racah matrices that intertwine the map from a tensor product of three representations into a single one, we call them inclusive Racah or mixing matrices [13], and the Racah matrices that intertwine the map from a tensor product of two representations into a tensor product of two possibly different representations, we call them exclusive Racah matrices [14]. The crucial difference between them is that the inclusive Racah matrices are the same for any $U_q(sl_N)$ group at sufficiently large $N$, while the exclusive Racah matrices essentially depend on $N$ (or on $A := q^N$).

For a special class of $U_q(sl_N)$ representations whose Young diagram has a single row (known as symmetric representations), which belong to the multiplicity-free class, the exclusive Racah matrices have been evaluated [15,16]. The inclusive Racah matrices for these representations have been evaluated so far only in some examples [17–19]. Hence our aim in this paper is to determine the inclusive $U_q(sl_N)$ Racah matrix elements arising in the tensor product of three arbitrary symmetric representations of $U_q(sl_N)$. Particularly, we establish an equivalence between such $U_q(sl_N)$ Racah coefficients with Racah coefficients for $U_q(sl_2)$. In fact, this equivalence is based on the eigenvalue hypothesis supplemented with the multiplicity-free property which we will elaborate.
According to the eigenvalue hypothesis, all the inclusive Racah matrices for knots can be given in terms of normalized eigenvalues of the braiding $\mathcal{R}$-matrix. As a consequence [1], the Racah matrices for 3-strand braids carrying same $[r]$ symmetric representations turn out to be same as $U_q(sl_2)$ Racah matrix corresponding to the same set of eigenvalues. In [20], the eigenvalue hypothesis is extended to links realized as closed 3-strand braids. Note that the generalization from braids to links will now involve many possible $\mathcal{R}$ matrices with different normalized braiding eigenvalues. However, the identification of $U_q(sl_N)$ Racah matrices appearing in the link invariant computation has not been related to $U_q(sl_2)$ Racah matrices. Hence, the main subject in the present paper is to consider three different symmetric representations $[r_1]$, $[r_2]$ and $[r_3]$ on a 3-strand braid and to identify the $U_q(sl_N)$ Racah matrices with the corresponding $U_q(sl_2)$ Racah matrices. This result will enable computation of multi-colored link invariants $H_{[r_1],[r_2],[r_3]}$ for the three-component links and $H_{[r_1],[r_2]}$ for the two-component links. In fact, the two-component links are even more interesting, since the three-component link realized as a closed 3-strand braid is always an entangling of three unknots, while the two-component link can be more sophisticated.

The colored HOMFLY-PT polynomials are interesting primarily because they connect various fields of mathematics and physics. Examples include the well-known 3d topological quantum field theory and two-dimensional conformal field theory, the Kontsevich integral and quantum groups. Similar new connections continue opening up today, and new interesting hypotheses are being put forward. These are: the AJ conjecture [21,22], the integrality conjectures proposed within topological string context [23–32], the knot/quiver correspondence [27,28], the topological string versus tangle calculus [29,30], etc. To study these connections further, to test and further develop the conjectures, it is necessary to have an ample set of examples, which would include very different knots in order to provide a confidence in applicability of the conjectures not only within a subclass of knots: say, the torus knots being simplest to deal with have a lot of specific properties that are not generalized to other knots. In addition, knowledge of the HOMFLY-PT polynomials is typically required not for a specific knot, but for a whole family of knots. In the present paper, we develop a technique that allows one to fulfill just this task. Indeed, according to [33,34], combining a 3-strand braid and arborescent braids [35,36], one can build up a large class of knots and links, which we call fingered 3 strand or 1-looped links. This class definitely includes all arborescent links and the links that have 3-strand braid realizations. Note that these two classes cover more than a half of the Rolfsen table, while the class of 1-looped links covers the whole table.

Another interesting result of our work is the evaluating of the quantum 6-symbols. It is amazing that the eigenvalues hypothesis allows us, in some cases beyond $N = 2$, to write down the answer in an analytic form (in this particular case, through a q-hypergeometric function). Thus, we can use it as a tool for calculating 6j-symbols and quantum knot invariants. One can also invert the logic: obtaining previously known invariants using the eigenvalue hypothesis,
one can thereby obtain an evidence in favor of its validity. Note also that the obtained analytic formulas can be further investigated. For example, one can explore the symmetries (beyond the tetrahedral ones) of 6j-symbols in the spirit of recent work [37].

Our paper is organized as follows. In Sect. 2, we define link invariants with the help of quantum $\mathcal{R}$-matrices. Particularly, we discuss the finite-dimensional symmetric representations of $U_q(sl_N)$ and their relations with the quantum Racah coefficients. We also formulate an eigenvalue conjecture for links whose components are colored by different $U_q(sl_N)$ symmetric representations. In Sect. 3, we consider three-strand braids colored by symmetric representations. We indicate that the normalized eigenvalues of $\mathcal{R}$ matrices are same when we reduce the rank of the different symmetric representations on all the three strands by the same integer $n$. Further, assuming eigenvalue hypothesis, we show that $U_q(sl_N)$ Racah matrix is equal to $U_q(sl_2)$ Racah matrix. We work out the multi-colored HOMFLY-PT polynomial for a two-component link $L7a3$ in Sect. 4. Interestingly, we give a closed-form expression for the multi-colored link invariant. In the concluding section, we summarize and suggest future directions toward generalizations to higher strand braids.

2. Link Invariants from Quantum Groups

We will focus in this section on obtaining multi-colored link invariants from $m$-strand braids where the component knots could carry different representations. Particularly, we will follow Reshetikhin–Turaev approach based on the theory of quantum groups and the quantum $\mathcal{R}$-matrix.

2.1. $\mathcal{R}$-Matrix and Multi-Colored Link Invariants

First of all, let us define quantum $\mathcal{R}$-matrices associated with the multi-colored $m$-strand braid. We associate a finite-dimensional representation $\mathcal{R}_i$, of the quantized universal enveloping algebra $U_q(sl_N)$, with $i$th strand where we assume the quantum deformation parameter $q$ to be root of unity. In fact, all finite-dimensional representations are representations of highest weights which can be enumerated using Young diagrams. Hence from now on we will identify these representations using the Young diagrams and follow the conventional notation. For example, the notation $[l,m,n,\ldots]$ denotes Young diagram with $l$-boxes in the first row, $m$-boxes in the second row and so on.

- For the $U_q(sl_N)$ algebra defined using generators $\{H_i, E_i, F_i\}$, there exists a universal $\mathcal{R}$-matrix:

$$\mathcal{R} = q^{\sum C_{ij}^{-1}H_i \otimes H_j} \prod_{\text{positive root } \beta} \exp_q[(1-q^{-1})E_\beta \otimes F_\beta],$$

where $(C_{ij})$ is the Cartan matrix.

- The action of quantum $\mathcal{R}_i$ on the $U_q(sl_N)$ modules $V_i$ and $V_{i+1}$ involves the above universal $\mathcal{R}$-matrix as well as a permutation operator as shown below:
\[ R_i = 1_{V_1} \otimes 1_{V_2} \otimes \cdots \otimes P \tilde{R}_{i,i+1} \otimes \cdots \otimes 1_{V_m} \in \text{End}(V_1 \otimes \cdots \otimes V_m), \]  
where the permutation operation is \( P(x \otimes y) = y \otimes x \) and \( \tilde{R} \) acts only on \( V_i \) and \( V_{i+1} \) and the identity operation on the rest of the modules \( V_j \neq i, i+1 \). It is well known [9, 38–40] that \( R_1, \ldots, R_{m-1} \) define a representation of the Artin’s braid group \( B_m \) on \( m \) strands: 

\[ \pi : B_m \rightarrow \text{End}(V_1 \otimes \cdots \otimes V_m) \]

\[ \pi(\sigma_i) = R_i, \]  
where \( \sigma_1, \ldots, \sigma_{m-1} \) are generators of the braid group \( B_m \). Graphically, we can represent \( R_i \) as follows:

\[ \begin{array}{c}
V_1 \\
\vdots \\
R_i = \begin{array}{c}
V_i \\
\bigtriangledown \\
V_{i+1} \\
\vdots \\
V_m
\end{array}
\end{array} \]

Clearly, inverse crossing is given by \( R_i^{-1} \). Operators \( R_i \) satisfy relations of the braid group \( B_m \):

- **commutativity property** \( R_i R_j = R_j R_i \), for \( |i - j| \neq 1 \)
- **braiding property** \( R_i R_{i+1} R_i = R_{i+1} R_i R_{i+1} \), for \( i = 1, \ldots, m - 2 \).

Graphically, the braiding relation is equivalent to the third Reidemeister move, whereas algebraically, it is a well-known quantum Yang–Baxter equation on quantum \( R \)-matrix.

- According to Alexander’s theorem, any link in \( \mathbb{R}^3 \) can be constructed from closure of \( m \)-strand braid. Let \( L \) be an oriented link with \( L \) components \( \mathcal{K}_1, \ldots, \mathcal{K}_L \) colored by representations \( [r_1], \ldots, [r_L] \), and \( \beta_L \in B_m \) is a some braid whose closure gives \( L \). Then according to Reshetikhin–Turaev approach [41, 42] the quantum group invariant, which is also known as colored HOMFLY polynomial, of the link \( L \) is defined as follows\(^1\):

\[ H_{[r_1],\ldots,[r_L]}^L = q^{\text{tr} V_1 \otimes \cdots \otimes V_m} (\pi(\beta_L)), \]

where \( q^{\text{tr}} \) is a quantum trace whose definition in the theory of quantum groups [10] for every \( z \in \text{End}(V) \) is a follows:

\[ q^{\text{tr}}(z) = \text{tr}_V(zK_{2\rho}), \]

\(^1\)The usual framing factor in front of quantum trace, which provide the invariance under the first Reidemeister move, we incorporate in \( R \)-matrix by modifying its eigenvalues (10).
where $\vec{\rho}$ is the Weyl vector (half the sum of positive roots) whose relation in terms of simple roots $\vec{\alpha}_i$ is $2\vec{\rho} = \sum_i n_i \vec{\alpha}_i$. The explicit form of $K_{2\rho}$ is

$$K_{2\rho} = K_1^{n_1} K_2^{n_2} \cdots K_{N-1}^{n_{N-1}}$$

where $K_i = q^{\vec{\alpha}_i \cdot H}$ involving Cartan generators $H_1, H_2, \ldots H_{N-1}$.

- Technically, more convenient is to use a modified version of the Reshetikhin–Turaev approach [13–44]. Let us expand tensor product of symmetric representations (denoted as Young diagram notation) $[r_1] \otimes [r_2] \otimes \cdots \otimes [r_m]$ into a direct sum of irreducible representations as shown

$$\bigotimes_i [r_i] = \bigoplus (\dim \mathcal{M}_{Q_{\nu}}^{1,2,\ldots,m}) Q_{\nu},$$

where $Q_{\nu}$ denote the irreducible representations. If any irreducible representation occurs more than once (called multiplicity), then $\mathcal{M}_{Q_{\nu}}^{1,2,\ldots,m}$ indicates the subspace of the highest weight vectors sharing same highest weights\(^2\) corresponding to Young diagram $Q_{\nu} \vdash \sum_i |r_i|$. The dimension of the space $\mathcal{M}_{Q_{\nu}}^{1,2,\ldots,m}$ is called multiplicity of representation $Q_{\nu}$ which we indicate as $Q_{\nu,s}$ where $s$ takes values $1, 2, \ldots \dim \mathcal{M}_{Q_{\nu}}^{1,2,\ldots,m}$.

To evaluate quantum trace (6), we need to write the states in weight space incorporating the multiplicity as well. We have many equivalent ways of denoting the state corresponding to the irreducible representations $Q_{\nu}$ (7). One such state in the weight space denoted by the following diagram

![Diagram](https://via.placeholder.com/150)

is written algebraically as

$$[(\ldots ([r_1] \otimes [r_2])_{X_{\alpha}} \otimes [r_3]_{Q_{1,\beta}} \cdots [r_m]_{Q_{\nu,\gamma}})^{(s)}] = |Q_{\nu,\gamma}; Q_{\nu,s}, X_{\alpha} \rangle \cong |Q_{\nu,s}, X_{\alpha} \rangle \otimes |Q_{\nu}, \rangle,$$

where $[r_1] \otimes [r_2] = \oplus_{\alpha=0}^{r_1-r_2} X_{\alpha} \equiv [r_1 + r_2 - \alpha, \alpha]$ and $Q_{\nu,s} \in \mathcal{M}_{Q_{\nu}}^{1,2,\ldots,m}$ with the index $s$ to keep track of the different highest weight vectors sharing the same highest weight $\vec{\omega}_{Q_{\nu}}$.

Incidentally, the choice of state (8) is an eigenstate of quantum $\mathcal{R}_1$ matrix:

$$\mathcal{R}_1[(\ldots ([r_1] \otimes [r_2])_{X_{\alpha}} \otimes [r_3]_{Q_{1,\beta}} \cdots [r_m]_{Q_{\nu}})^{(s)}] = \lambda_{X_{\alpha},s}(\ldots ([r_1] \otimes [r_2])_{X_{\alpha}} \otimes [r_3]_{Q_{1,\beta}} \cdots [r_m]_{Q_{\nu}})^{(s)}.$$  

\(^2\)Recall, that if $Q_{\nu}$ is a Young diagram $Q_{\nu} = \{Q_{\nu_1} \geq Q_{\nu_2} \geq \cdots Q_{\nu_l} > 0\}$, then the highest weights $\omega_{Q_{\nu}}$ of the corresponding representation are $\omega_i = Q_{\nu_i} - Q_{\nu_{i+1}} \forall i = 1, \ldots, l$, and vice versa $Q_{\nu_i} = \sum_{k=i}^{l} \omega_k$. 
Hence we will denote the $R_1$ matrix which is diagonal in the above basis as $\Lambda_{R_1}$ involving diagonal matrix elements $\lambda_{X,\alpha,s}([r_1],[r_2])$. These elements are the braiding eigenvalues whose explicit form is $[10,45]$

$$\lambda_{X,\alpha,s}([r_1],[r_2]) = \begin{cases} 
\epsilon_{X,\alpha,s}q^{x(X_{\alpha})-x([r_1])-x([r_2])} & \text{if } [r_1] \neq [r_2] \\
\epsilon_{X,\alpha,s}q^{x(X_{\alpha})-4x([r_1])-|r_1|N} & \text{if } [r_1] = [r_2].
\end{cases} \tag{10}$$

For a representation $X_\alpha$ whose Young diagram is denoted by $\alpha_1 \geq \alpha_2 \cdots \geq \alpha_{N-1}$, $\lambda_{X_\alpha} = \frac{1}{2} \sum_j \alpha_j (\alpha_j + 1 - 2j)$ and $\epsilon_{X,\alpha,s}$ will be $\pm 1$ if $X_\alpha$ is connected to the multiplicity subspace state $Q_{\nu,s}$ and zero otherwise. We will scale $\lambda_{X,\alpha,s}([r_1],[r_2]) \rightarrow \text{const } \tilde{\lambda}_{X,\alpha,s}([r_1],[r_2])$ such that

$$\prod_s \tilde{\lambda}_{X,\alpha,s}([r_1],[r_2]) = 1 \tag{11}$$

and use these normalized values $\tilde{\lambda}_{X,\alpha,s}([r_1],[r_2])$ in the definition of $\Lambda_{R_1}([r_1],[r_2])$ in the rest of the paper.

As these quantum $R$-matrices commute with any element from $U_q(sl_N)$, these $R$-matrices get a block structure corresponding to the decomposition on irreducible components $(7, 8)$, that is, it does not mix vectors from different representations. Furthermore, $R$-matrix acts on $[Q_\nu]$ as an identity operator and non-trivially on the subspace $M^{1,2\ldots,m}_\nu$. The element $K_{2\rho}$ acts diagonally on $[Q_\nu]$ but as identity operator on subspace $M^{1,2\ldots,m}_\rho$ as this space represents all possible highest weight vectors $Q_{\nu,s}$ with the same weight $\omega_{Q_\nu}$. Incorporating these facts and the decomposition of states $(8)$, the colored HOMFLY $(5)$ will become

$$H^{\nu}_{[r_1],\ldots,[r_L]}(q, A = q^N) = \text{tr}_{V_1 \otimes \cdots \otimes V_m} (\pi(\beta_\nu) K_{2\rho})$$

$$= \sum_{\nu} \text{tr}_{M^{1,2\ldots,m}_\nu} (\pi(\beta_\nu)) \cdot \text{tr}_{Q_\nu} (K_{2\rho})$$

$$= \sum_{\nu} \text{tr}_{M^{1,2\ldots,m}_\nu} (\pi(\beta_\nu)) \cdot \text{qd}im Q_\nu$$

$$= \sum_{\nu,s} (Q_{\nu,s}, X_\alpha | \pi(\beta_\nu) Q_{\nu,s}, X_\alpha) \text{qd}im Q_\nu, \tag{12}$$

where $\text{qd}im Q_\nu$ is a quantum dimension of the representation $Q_\nu$ explicitly given in terms of Schur polynomials:

$$\text{qd}im Q_\nu = s_{Q_\nu}(x_1, \ldots, x_N) \bigg|_{x_i=q^{N+1-2i}}$$

$$= s_{Q_\nu}(p_1, \ldots, p_N) \bigg|_{p_k=p_k^*} \equiv s^*_{Q_\nu}(A, q), \tag{13}$$

where $p_k = \sum_{i=1}^N x_i^k$ and $p_k^* = A^{k-A_k}/q^{k-q-A_k}$. For $s^*_{Q_\nu}(A, q)$ there exists a very simple and straightforward hook formula:

$$s^*_{Q_\nu}(A, q) = \prod_{(i,j) \in R} \frac{Aq^{j-i} - (Aq^{j-i})^{-1}}{q^{\kappa(i,j)+\lambda(i,j)+1} - (q^{\kappa(i,j)+\lambda(i,j)+1})^{-1}}, \tag{14}$$
where \( \kappa(i, j) = R_i - j - 1 \) and \( \lambda(i, j) = R'_j - i - 1 \). Here \( R_i \) denotes the number of boxes in the \( i \)th row of the Young diagram of \( Q_\nu \) and \( R'_j \) is the number of boxes of the \( j \)th row of transpose of the Young diagram of \( Q_\nu \). Using Eqs. (9, 10), we can obtain multi-colored HOMFLY-PT \( H^L_{[r_1], [r_2]}(q, A = q^N) \) for braid words \( \mathcal{R}^2_{2^n} \) belonging to 2-strand braid. Note that \( X_\alpha \) will be \( Q_\nu \) which are multiplicity-free. Hence

\[
H^L_{[r_1], [r_2]}(q, A = q^N) = \sum_{\nu, s} \tilde{\lambda}_{X_\alpha, s}([r_1], [r_2])^{2n} \delta_{X_\alpha, Q_\nu} \delta_{s, 1} \text{qdim} Q_\nu
\]

However, to go beyond 2-strand braid, we need to deal with elements of \( \mathcal{R}_{i \neq 1} \) as well. Hence, to work out the trace of any braid word in the multiplicity subspaces, we need to perform transformation of the states (8), using quantum Racah coefficients, so that quantum \( \mathcal{R}_{i \neq 1} \) matrices are diagonal in the transformed states. In the following subsection, we will focus on 3-strand braid and indicate the transformation between two possible states through \( U_q(sl_N) \) Racah matrices and obtain the matrix form for \( \mathcal{R}_2 \).

### 2.2. Quantum Racah Matrix for 3-Strand Braid

Consider highest weight states of the three finite-dimensional irreducible representations \([r_1], [r_2], [r_3]\) of \( U_q(sl_N) \). As the tensor product of these representations is associative, one has a natural isomorphism between two equivalent bases states. Hence we can unitary relate the two equivalent basis states as follows [9,10,46]:

\[
|(([r_1] \otimes [r_2])X_\alpha \otimes [r_3])Q_\nu\rangle^{(s_1)} \overset{U}{\rightarrow} |([r_1] \otimes ([r_2] \otimes [r_3])Y_\beta)Q_\nu\rangle^{(s_2)},
\]

where the elements of the transformation matrix \( U \) (known as quantum Racah coefficients) are

\[
U \begin{bmatrix} [r_1] & [r_2] & X_\alpha \\ [r_3] & Q_\nu & Y_\beta \end{bmatrix}.
\]

Remember that \( X_\alpha \otimes [r_3] \in Q_\nu \) and \( [r_1] \otimes Y_\beta \in Q_\nu \) and the notation of quantum Racah coefficient means that the representation \( X_\alpha \in [r_1] \otimes [r_2] \cap [r_3] \otimes Q_\nu \) and the representation \( Y_\beta \in [r_2] \otimes [r_3] \cap [r_2] \otimes Q_\nu \). The corresponding Racah matrix \( U \) is denoted as

\[
U \begin{bmatrix} [r_1] & [r_2] \\ [r_3] & Q_\nu \end{bmatrix}.
\]

Pictorially the above Racah matrix is drawn with three ingoing external lines \([r_i]\)’s and an outgoing external line \( Q_\nu \) (implying its conjugate representation):
Similar to Eq. (9), the basis state of $\mathcal{R}_2$ obeys

$$\mathcal{R}_2\left([r_1] \otimes ([r_2] \otimes [r_3])_{Y_{\beta}}\right)^{s_2} = \lambda_{Y_{\beta},s_1}(\mathcal{R}_2)\left([r_1] \otimes ([r_3] \otimes [r_2])_{Y_{\beta}}\right)^{s_2},$$

(19)

whose diagonal matrix form will be $\Lambda_{\mathcal{R}_2}$. The matrix form in the basis (8) can be deduced using Eqs. (19,16) as

$$\mathcal{R}_2 = U^\dagger \begin{bmatrix} [r_1] & [r_3] \\ [r_2] & Q_{\nu} \end{bmatrix} \cdot (\Lambda_{\mathcal{R}_2}(\mathcal{R}_2)) \cdot U \begin{bmatrix} [r_1] & [r_2] \\ [r_3] & Q_{\nu} \end{bmatrix}.$$  

(20)

Using the systematic approach of highest weight method [13,47–51], the quantum Racah coefficients have been explicitly calculated for some $[r_i]$‘s. However, our aim in this paper is to obtain the Racah matrix when the symmetric representations on the 3-strands are arbitrary $[r_i]$‘s. We shall present in the next section the proofs and details relating these $U$ matrices to an equivalent $U_q(sl_2)$ Racah matrices. Once the $U$ matrix elements are known, the trace in the multiplicity subspace for any braid word $\prod_i \mathcal{R}_1^{a_i} \mathcal{R}_2^{b_i}$ belonging to 3-strand braid can be determined.

The methodology of formally writing the trace in multiplicity subspace presented for braid words belonging to 2-strand and 3-strand braids can be generalized to higher strand braids. The matrix form of quantum $\mathcal{R}_3, \mathcal{R}_4, \ldots$ in the basis (8) will involve many Racah matrices. Since our focus in this paper is to compute the multi-colored link invariants from 3-strand braids carrying different symmetric representations, we leave $m$-strand quantum $\mathcal{R}_{i \geq 3}$ matrices and the corresponding computation of the Racah matrices for a future publication.

Before we proceed to provide a neat correspondence of the quantum Racah matrix (18) to $U_q(sl_2)$ Racah matrix, we will highlight the eigenvalue hypothesis implications for strands carrying different symmetric representations.

### 2.3. Eigenvalue Hypothesis

In [1], the eigenvalue hypothesis states that quantum Racah coefficients can be written in terms of the eigenvalues of $\mathcal{R}$-matrices. In fact for knots, in the absence of coinciding eigenvalues of $\mathcal{R}$-matrices, the Racah matrices of sizes up to $6 \times 6$ were explicitly written using the set of normalized eigenvalues of
the corresponding $\mathcal{R}$-matrix \cite{1,19} confirming the eigenvalue hypothesis (see also \cite{52}).

For describing multi-colored links from 3-strand braid, carrying three different representations $([r_1],[r_2],[r_3])$, there are three different quantum $\mathcal{R}$ eigenvalues $\tilde{\lambda}_{X,\alpha,s}([r_i],[r_j])$ where $i \leq j$. Hence we conjecture a generalization of the eigenvalue hypothesis applicable for links.

**Conjecture.** The Racah matrix $U = \begin{bmatrix} [r_1] & [r_2] & Q_\nu \\ [r_3] \end{bmatrix}$ is expressed through 3 sets of the normalized eigenvalues of the corresponding three possible $\mathcal{R}$ matrices whose diagonal form will be $\Lambda_{\mathcal{R}}([r_i],[r_j])$ where $i \leq j = 1, 2, 3$.

Let us clarify the status of the eigenvalue hypothesis.

1. It was proposed in paper \cite{1}. We refer the reader to that paper and to the review \cite{20} for a detailed explanation of an origin of this conjecture and the precise formulation for arbitrary representations.
2. It is proven in the case of arbitrary 3-strand braid when the size of Racah matrices $\leq 5$ and there are no coinciding eigenvalues. Actually, it immediately follows from the Yang–Baxter relations \cite{1,53}.
3. For size 6, it was validated rather recently within the framework of the knot universality of \cite{19} and by application to advanced Racah calculus in \cite{47–49}.
4. In paper \cite{52} from the eigenvalue hypothesis, it was derived the 1-hook scaling property of colored Alexander polynomials \cite{55,56}. Since this property can be verified for many particular cases (in particular for torus knots it is proven in \cite{54}), it provides evidence in favor of the validity of the eigenvalue hypothesis.
5. Some explicit checks of the conjecture were done for 3-strand braids in \cite{51} and for multi-strand braids in \cite{60}.
6. In paper \cite{37}, it was proven that for $U_q(sl_2)$ representations, the eigenvalue hypothesis is provided by the Regge symmetry when three representations coincide. Moreover, for $U_q(sl_N)$ representations the eigenvalue hypothesis provides new symmetries, which can be treated as generalizations of the Regge symmetry.

Thus, we see that the eigenvalue hypothesis not only significantly simplifies the calculation of the Racah coefficients, but also leads to various interesting consequences. It will be desirable to have a rigorous proof of the eigenvalue hypothesis which we hope to attempt in the future.

### 2.4. Signs of the Eigenvalues

Formula (10) defines the eigenvalues of the $\mathcal{R}$-matrices up to a sign $\epsilon_{X,\alpha,s}$. These signs also exist in the classical limit $q = 1$ when $\mathcal{R}$-matrix is just a permutation operator. $\mathcal{R}$-matrix is an operator acting from the space $[r_1] \otimes [r_2]$ to the space $[r_2] \otimes [r_1]$:

$$\mathcal{R} : [r_1] \otimes [r_2] \rightarrow [r_2] \otimes [r_1].$$

(21)
If we study knots, then \([r_1] = [r_2] = [r]\) and both spaces are the same ones. Here signs of the eigenvalues depend on whether highest weight vectors of the representations are symmetric or antisymmetric under permutation of two representations \([r_1]\) and \([r_2]\). For symmetric representation \([r]\), we can place sign as \((-1)^\alpha\) for irreducible representation \(X_\alpha = [2r - \alpha, \alpha]\). Equivalently, we associate + sign for that \(X_\alpha\) whose eigenvalue has the highest power of \(q\) and alternating signs for the other \(X_\alpha\) in descending powers of \(q\). For example, we take \(R = [2]\). There are three eigenvalues, namely \(\epsilon_{[4]}q^4\), \(\epsilon_{[3,1]}q^0\) and \(\epsilon_{[2,2]}q^{-2}\). Then the signs are \(\epsilon_{[4]} = +1\), \(\epsilon_{[3,1]} = -1\) and \(\epsilon_{[2,2]} = +1\).

It appears that the same sign convention is applicable for eigenvalues of \(\mathcal{R}\)-matrices when \([r_1] \neq [r_2]\). In fact, this sign convention determines the signs of the Racah coefficients obtained from the highest weight method. These are the signs we will use for eigenvalues of \(\mathcal{R}\) applicable for multi-colored links obtained from 3-strand braid.

3. Three-Strand Braid Colored by Symmetric Representations

In this section, we consider 3-strand braids colored by arbitrary symmetric representations: \([r_1], [r_2], [r_3]\). As mentioned earlier, there will be three possible \(\mathcal{R}\) matrices with three sets of normalized eigenvalues (10) constituting the diagonal matrices \(\text{diag}\{\Lambda_{\mathcal{R}}([r_1], [r_2])\}, \text{diag}\{\Lambda_{\mathcal{R}}([r_1], [r_3])\}, \text{diag}\{\Lambda_{\mathcal{R}}([r_2], [r_3])\}\). The eigenvalues of these matrices depend on the irreducible representation obtained from following tensor products:

\[
[r_1] \otimes [r_2] = \bigoplus_{\alpha} X_\alpha = \bigoplus_{i_{12}=0}^{\text{min}(r_1,r_2)} [r_1 + r_2 - i_{12}, i_{12}];
\]

\[
[r_2] \otimes [r_3] = \bigoplus_{\beta} Y_\beta = \bigoplus_{i_{23}=0}^{\text{min}(r_2,r_3)} [r_2 + r_3 - i_{23}, i_{23}];
\]

\[
[r_1] \otimes [r_3] = \bigoplus_{\gamma} Z_\gamma = \bigoplus_{i_{13}=0}^{\text{min}(r_1,r_3)} [r_1 + r_3 - i_{13}, i_{13}].
\]  

(22)

Taking tensor product of these possible irreducible representations \(X_\alpha, Y_\beta, Z_\gamma\) with the representation placed on third strand, we can compactly write

\[
([r_i] \otimes [r_j]) \otimes [r_k] = \bigoplus_{\nu} \dim M^{ijk}_{\nu} Q_\nu = \bigoplus_{\nu} \dim M^{ijk}_{\nu}[\ell_\nu, m_\nu, n_\nu]
\]  

(23)

where \(i, j, k \in 1, 2, 3\) and \(\ell_\nu + m_\nu + n_\nu = r_1 + r_2 + r_3\). In order to relate the range of \(i_{ij}\) in terms of \(\ell_\nu, m_\nu, n_\nu\), we highlight the relevant logical steps in the following subsection.

3.1. Restrictions on the Representations

Let us explicitly understand the restriction of range for \(i_{12}\) before we generalize for \(i_{ij}\) in the tensor product (23). From the relation

\[
\bigoplus_{\alpha} X_\alpha \otimes [r_3] \equiv \bigoplus_{i_{12}=0}^{\text{min}(r_1,r_2)} [r_1 + r_2 - i_{12}, i_{12}] \otimes [r_3]
\]

\[
= \bigoplus_{\nu} \dim M_{\nu}^{123}[\ell_\nu, m_\nu, n_\nu],
\]  

(24)
we can infer the following inequalities between the Young diagrams corresponding to $X_\alpha \equiv [r_1 + r_2 - i_{12}, i_{12}]$ and $Q_\nu \equiv [\ell_\nu, m_\nu, n_\nu]

\begin{align}
\ell_\nu - (r_1 + r_2 - i_{12}) &\leq r_3, & \ell_\nu &\geq r_1 + r_2 - i, \\
m_\nu - i_{12} &\leq r_3, & m_\nu &\geq i_{12}, \\
r_1 + r_2 - i_{12} &\geq m_\nu, & i_{12} &\geq n_\nu, \\
n_\nu &\leq \min(r_1, r_2, r_3), & \ell_\nu &\geq \max(r_1, r_2, r_3).
\end{align}

The above exercise for restriction of range for $i_{12}$ can be generalized for $i_{ij}$ in the tensor product (24) as follows:

\begin{align}
j_{ij,k} &\leq i_{ij} \leq J_{ij,k} \quad \text{where } j_{ij,k} = \max(r_i, r_j - \ell_\nu, m_\nu - r_k, n_\nu); \\
J_{ij,k} &\equiv \min(r_i + r_j + k - \ell_\nu, r_i + r_j - m_\nu, m_\nu, r_i, r_j).
\end{align}

Using the fact $\ell_\nu + m_\nu + n_\nu = r_i + r_j + r_k$, we observe

\begin{align}
(r_i + r_j - \ell_\nu) - (m_\nu - r_k) = n_\nu &\geq 0 \Rightarrow (r_i + r_j - \ell_\nu) \geq (m_\nu - r_k) \\
\Rightarrow (r_i + r_j + r_k - \ell_\nu) &\geq m_\nu.
\end{align}

The above inequalities imply more stringent restriction on $j_{ij,k}$ and $J_{ij,k}$:

\begin{align}
&j_{ij,k} = \max(r_i + r_j - \ell_\nu, n_\nu), & J_{ij,k} = \min(r_i + r_j - m_\nu, m_\nu, r_i, r_j).
\end{align}

In the following subsection, we will see that the normalized eigenvalues of $\mathcal{R}$ are identical whenever the number of boxes in the Young diagrams of $[r_1], [r_2], [r_3], Q_\nu$ is reduced in a systematic way eventually modifying three-row Young diagram of $Q_\nu$ to two-row Young diagram.

3.2. Eigenvalues of the Link $\mathcal{R}$-Matrices

Eigenvalues of the $\mathcal{R}$-matrix (10) for $X_\alpha \equiv [r_1 + r_2 - i, i]$ are

\begin{align}
\lambda_{X_\alpha} = \epsilon_{X_\alpha} q^{r_1 r_2 + i^2 - i(r_1 + r_2 + 1)} = (-1)^i q^{r_1 r_2 + i^2 - i(r_1 + r_2 + 1)}.
\end{align}

Consider an irreducible representation $Q_\nu$ in the decomposition (24) such that the Young diagram has non-trivial third row. That is, $Q_\nu \equiv [\ell_\nu, m_\nu, n_\nu > 0]$. Interestingly, the multiplicity of subspace $Q_{\nu, s}$ remains same when we reduce the rank of all the symmetric representations $[r_i]$'s by one as well as change $Q_{\nu,s} \rightarrow Q_{\nu', s}$ such that the irreducible representation $Q_{\nu'} \equiv [\ell_\nu - 1, m_\nu - 1, n_\nu - 1]$. Furthermore, the eigenvalues of the $\mathcal{R}$ matrices under the shift of $[r_1], [r_2] \rightarrow [r_1 - 1], [r_2 - 1]$ give new eigenvalues

\begin{align}
\lambda_{X_{\nu'}} = -(-1)^i q^{r_1 r_2 + i^2 - i(r_1 + r_2 + 1) + 1}.
\end{align}

Note that ratio of the eigenvalues $\lambda_{X_{\nu'}/\lambda_{X_\alpha}} = q$ which implies that the normalized eigenvalues are indeed same. If the eigenvalue conjecture from Sect. (2.3) is correct, then the Racah matrix must obey

\begin{align}
U \begin{bmatrix} [r_1] & [r_2] & [r_3] \\ [\ell_\nu, m_\nu, n_\nu] & [r_1 - 1] & [r_2 - 1] \\ [r_3 - 1] & [r_3 - 1] & [r_3 - 1]
\end{bmatrix} = U \begin{bmatrix} [r_1 - 1] & [r_2 - 1] & [r_3 - 1] \\ [\ell_\nu - 1, m_\nu - 1, n_\nu - 1] & [r_1 - 1] & [r_2 - 1] \\ [r_3 - 1] & [r_3 - 1] & [r_3 - 1]
\end{bmatrix}.
\end{align}
Following the above steps iteratively \( n_{\nu} \) times, the modified \( Q_{\nu} \) Young diagram has only two rows. The Racah matrix must remain same under such an iteration implying

\[
U \begin{bmatrix}
[r_1] & [r_2] \\
[r_3] & [\ell_{\nu}, m_{\nu}, n_{\nu}]
\end{bmatrix} = U \begin{bmatrix}
[r_1 - n_{\nu}] & [r_2 - n_{\nu}] \\
[r_3 - n_{\nu}] & [\ell_{\nu} - n_{\nu}, m_{\nu} - n_{\nu}]
\end{bmatrix} . \tag{32}
\]

We have obtained Racah matrices (18) for some symmetric representations using the highest weight method and eigenvalue hypothesis. Interestingly, these matrices (18) for \( Q_{\nu} = [\ell_{\nu}, m_{\nu}, n_{\nu} = 0] \) having only two rows are agreeing with the known \( U_q(sl_2) \) matrices. Hence from these results we deduce:

\[
U \begin{bmatrix}
[r_1 - n_{\nu}] & [r_2 - n_{\nu}] \\
[r_3 - n_{\nu}] & [\ell_{\nu} - n_{\nu}, m_{\nu} - n_{\nu}]
\end{bmatrix} = Uq_{(sl_2)} \begin{bmatrix}
(r_1 - n_{\nu})/2 & (r_2 - n_{\nu})/2 \\
(r_3 - n_{\nu})/2 & (\ell_{\nu} - m_{\nu})/2
\end{bmatrix} . \tag{33}
\]

From Eqs. (32,33), it is clear that the \( U_q(sl_N) \) Racah matrix involving \( Q_{\nu} \) (whose Young diagram has three rows) can be identified as \( U_q(sl_2) \) Racah matrix:

\[
U \begin{bmatrix}
[r_1] & [r_2] \\
[r_3] & [\ell_{\nu}, m_{\nu}, n_{\nu}]
\end{bmatrix} = Uq_{(sl_2)} \begin{bmatrix}
(r_1 - n_{\nu})/2 & (r_2 - n_{\nu})/2 \\
(r_3 - n_{\nu})/2 & (\ell_{\nu} - m_{\nu})/2
\end{bmatrix} . \tag{34}
\]

For completeness, we give the closed-form expression of \( U_q(sl_2) \) Racah coefficients [9]:

\[
U_{U_q(sl_2)} \begin{bmatrix}
s_1 & s_2 \\
s_3 & s_4
\end{bmatrix} = \sqrt{[2i + 1][2j + 1]} (-1)^\sum_{m=1}^{4} \theta(s_1, s_2, i) \theta(s_3, s_4, j) \sum_{k \geq 0} (-1)^k [k + 1]! \times \theta(s_3, s_4, i) \theta(s_4, s_1, j) \theta(s_2, s_3, j) \sum_{k \geq 0} (-1)^k [k + 1]! \left( [k - s_1 - s_2 - i][k - s_3 - s_4 - i][k - s_1 - s_4 - j][k - s_2 - s_3 - j]! [s_1 + s_2 + s_3 + s_4 - k]! [s_1 + s_3 + i + j - k] [s_2 + s_4 + i + j - k]! \right)^{-1} , \tag{35}
\]

where the number in square bracket is called quantum number defined as \([n] = (q^n - q^{-n})/(q - q^{-1}) \) and

\[
\theta(a, b, c) = \sqrt{[a - b + c]![b - a + c]![a + b - c]!} / [a + b + c + 1]! .
\]

Remember \( s_i \)'s can be integers or half-odd integers whose tensor product will require \( i, j \) to be accordingly integers or half-odd integers.

In the next section, we will explicitly work out \( H^L_{[r_1],[r_2]} \) for a two-component link from 3-strand braid.
4. Colored HOMFLY Polynomials for Links from 3-Strand Braid

As discussed in Sect. 2, 3-strand braid group $B_3$ is generated by 2 elements $\sigma_1$ and $\sigma_2$. In order to evaluate multi-colored HOMFLY-PT of the links $H_{[r_1],[r_2],[r_3]}^L$ (12) from 3-strand braid, we need to compute $R_1 = \pi(\sigma_1)$ and $R_2 = \pi(\sigma_2)$.

The eigenvalues of the diagonal matrix $\Lambda R_1$ are given in Eq. (29). We can now explicitly write the matrix $R_2$ by substituting Eqs. (34, 35) into (20). Hence we can evaluate $\pi(\beta_L) \forall \beta_L \in B_3$ colored by arbitrary symmetric representations. The braid word $\beta_L$ could give one-component link (knots), two-component link and three-component link. In fact, the two-component link contains torus knot and unknot as components. The three-component link from $\beta_L$ gives non-trivial entanglement between three unknots. As an illustration, we compute multi-colored HOMFLY-PT for a two-component link which is not just entangling of unknots in the following subsection.

4.1. Link $L7a3$

From the Thistlethwaite link table [57], we take the simplest two-component link referred to as $L7a3$ whose non-trivial component is trefoil as drawn below (Fig. 1):

The braid word $\beta_L \in B_3$ for such a link is

There are two diagonal $R$-matrices: $\Lambda R([r_2],[r_2])$, $\Lambda R_2([r_2],[r_2])$. $\Lambda R([r_1],[r_2])$ stands for crossings of representations $[r_1]$ and $[r_2]$, and

![Figure 1. A picture of the link L7a3 from Knotilus [58]]
Λ\(_{\mathcal{R}}([r_2],[r_2])\) stands for crossings of representations \([r_2]\) and \([r_2]\). The two Racah matrices

\[
U_1 := U \begin{bmatrix} [r_2] & [r_2] \\ [r_1] & Q \end{bmatrix} \quad \text{and} \quad U_2 := U \begin{bmatrix} [r_2] & [r_1] \\ [r_2] & Q \end{bmatrix}
\]

(36)
correspond accordingly to the placements of representations \([r_2],[r_2],[r_1]\) and \([r_2],[r_1],[r_2]\). The answer for the HOMFLY polynomial (12) is then given by:

\[
H_{[r_1],[r_2]}^{L7a3} = \sum_Q \text{tr} \left( \Lambda_{\mathcal{R}}([r_1],[r_2]) \cdot U_2^T \cdot \Lambda_{\mathcal{R}}^{-1}([r_1],[r_2]) \cdot U_1 \cdot \Lambda_{\mathcal{R}}([r_2],[r_2])^3 \cdot U_1^T \cdot \Lambda_{\mathcal{R}}^{-1}([r_1],[r_2]) \cdot U_2 \cdot \Lambda_{\mathcal{R}}([r_1],[r_2]) \right) \cdot s^*_{Q}(A,q),
\]

(37)
where \(Q \in [r_1] \otimes [r_2] \otimes [r_2]\). Note that the second component colored by representation \([r_2]\) is trefoil.

We have performed calculations for some values of \([r_1]\) and \([r_2]\) using the eigenvalues of \(\mathcal{R}\) as well as Racah matrix elements discussed in Sect. 3. The multi-colored HOMFLY-PT (37) for arbitrary colors \([r_1]\) and \([r_2]\) seems to have a neat form involving quantum number factorial \([n]! = [n][n-1]...[1]\) as well as numbers in parenthesis defined as

\[
\{x\} = x - 1/\bar{x}; \quad D_k = \{Aq^k\}/\{q\}.
\]

(38)
We have evaluated (37) for the range \(r_1 = 1, \ldots, 4\) and \(r_2 = 1, \ldots, 4\). Based on the obtained explicit answers and on the assumption of the analytical dependence of formula (37) on the parameters \(r_1\) and \(r_2\), we guess that the explicit expression for this two-component link is as follows:

\[
\frac{H_{[r_1],[r_2]}^{L7a3}}{s^*_{[r_1]} \cdot s_{[r_2]}} = T_{[r_2]}(q,A) + \sum_{k=1}^{\min(r_1,r_2)} \frac{[r_1]! [r_2]!}{[r_1-k]! [r_2-k]!} \frac{\{q\}^{3k}}{A^{3r_2}} \frac{D_{r_1+1}}{D_{r_2-1}} \times \prod_{n=1}^{k} D_{r_1+n-1} \prod_{m=0}^{k-1} D_{2k+m} \cdot G_{k,r_2}(q,A),
\]

(39)
where \(T_{[r_2]}(q,A)\) refers to (reduced) colored HOMFLY polynomial of the trefoil in the topological framing colored with \([r_2]\) and \(G_{k,r_2}\) can be derived using

\[
G_{1,r_2}(q,A) = \sum_{i=1}^{r_2} Q_{i,r_2}(q,A),
\]

(40)
\[
Q_{i,r_2}(q,A) = \frac{2(i+1)}{[i+1]} \left[ \frac{[r_2-1]!}{[i-1]! [r_2-i]!} \right] A^i q^{2r_2^2 - i^2 / 2 - 5i / 2 + 1} \prod_{j=2}^{i} D_{r_2+j-1}
\]

as follows:

\[
G_{k,r_2}(q,A) = \sum_{i=1}^{r_2-k+1} \left( \prod_{n=1}^{k-1} \frac{[r_2-i+n+1]}{[r_2-n]} \right) q^{(1-k)(k+2i-2) / 2} \times [G_{1,r_2}(q,A)]_{1-2i}^{A^{r_2-k+2-2i}}
\]

(41)
where \( [G]_p^A \) denotes the coefficient of the \( p \)th degree of the (Laurent) polynomial \( G \) of \( A \). Equivalently, the form of \( G_{k,r_2}(q,A) \) can be written as double sum:

\[
G_{k,r_2}(q,A) = \sum_{i=1}^{r_2} P_{i,k,r_2}(q,A) \tag{42}
\]

\[
P_{i,k,r_2}(q,A) = \frac{q^{i(i+5)/2}}{q^{2r_2^2 + k(k-1)/2} + 1} A^{2i+k-r_2-2} \frac{[2i+2]}{[i+1]} \frac{[r_2-1]!}{[r_2-i]![i-1]!} \prod_{p=r_2+k}^{r_2+i-1} \tilde{D}_p \\
\times \sum_{\min(i,k) r_2 + \min(i,k) - 1}^{\infty} \prod_{m=r_2+n}^{\infty} \tilde{D}_m \prod_{j=1}^{n-1} \frac{[i-j][k-j]}{[j][r_2-j]} \frac{1}{q^{j+k+i-2}}.
\]

Here \( \tilde{D}_m = A\{q\} D_m = A^2 q^m - q^{-m} \). Remember \([0]! = 1\) and terms inside \( \prod_{j=a}^{b} \) involving \( b < a \) are set to one.

Obtained formula (39), we checked it for a big cluster of values of \( r_1 \) and \( r_2 \). Namely, we have evaluated formula (37) for \( r_1 \leq 10 \) and \( r_2 \leq 10 \) and checked that the answers coincide with formula (39) for these particular values of \( r_1 \) and \( r_2 \).

Another way to evaluate the HOMFLY-PT polynomial for this link is described in [59], where the calculation is done for \( L7a3 \) for \([r_1] = [1] \) and \([r_2] = [k] \). We can check that our conjectured form (39) agrees with the results in [59] when \([r_1] = [1] \) and \([r_2] = [k] \). However, the current approach is completely different and uses properties of Racah matrices unlike the tangle decomposition technique used in [59]. Also our current approach is much more systematic.

Similarly, one can calculate the colored HOMFLY-PT polynomials of any 3-strand links. Here is a list of all 3-strand (but not 2-strand) links with the number of minimal crossings not greater than 7 according to the Thistlethwaite classification, indicating the number of components in the brackets:

- \( L5a1(2), L6a4(3), L6n1(3), L7a1(2), L7a3(2), L7a6(2), L7n1(2), L7n2(2). \)

Their corresponding braid representation can be found in [57]. Writing a closed-form expression (39) for these two-component links is in principle can be derived looking at the multi-colored link invariants for some values of \([r_1]\) and \([r_2]\). We will update them in notebook.org website.

### 5. Conclusion

In this paper, we study invariants of links from 3-strand braids where the strands are colored by arbitrary symmetric representations. We have presented the construction of multi-colored invariants \( H_{[r_1],[r_2],[r_3]}^L \) (5) for these links, where the component knots are colored by symmetric \([r_1],[r_2],[r_3] \) representations, using the quantum \( \mathcal{R} \) as well as \( U_q(sl_N) \) Racah matrices. In the literature, the matrix elements of these Racah matrices are not available for arbitrary representations. So we cannot explicitly write the multi-colored HOMFLY-PT in terms of variables \( q,A \). Hence the main theme of this paper was to determine these Racah matrix elements.
We have given a systematic procedure, using eigenvalue hypothesis, to identify the Racah matrices with quantum $U_q(sl_2)$ Racah matrices. As the $U_q(sl_2)$ Racah matrix coefficients are known [9], we could evaluate the multi-colored HOMFLY-PT for two- and three-component links colored by different symmetric representations. For concreteness, we have presented the results for a two-component link $L7a3 (37)-(39)$.

The extension of our results to $m > 3$-strand braids looks plausible. It appears that these Racah matrices may be identified with $U_q(sl_{m-1})$. We hope to get insight into Racah matrices obtainable from the highest weight approach [60] for some low-rank symmetric representations of $U_q(sl_N)$ placed on component knots. The results could lead to writing closed-form expression for $U_q(sl_{N \geq 3})$ Racah coefficients for arbitrary symmetric representations. It was done recently in papers [15,16] for exclusive $6j$-symbols which allows to rewrite them in terms of $q$-Racah orthogonal polynomials [61,62]. Extension to multiplicity-free rectangular representations must also be addressed for 3-strand and $m > 3$-strand braids. Extension to cases with multiplicity is also extremely interesting, but it will require substantial development of the mathematical apparatus (see paper [63] for an attempt to describe Racah matrices of small size with multiplicities). We will pursue these issues in the future so that multi-colored HOMFLY-PT for links from $m$-strand braids becomes computable.

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