A Design of a Dynamic Sparse Circulant Measurement Matrix Based on a New Compound Sine Chaotic Map

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ABSTRACT The performance of the measurement matrix is always the key to affecting the application of compressed sensing in engineering practice. The measurement matrix designed based on the chaotic map is easy to implement in physical circuits, but the weak chaotic behavior and small chaotic interval of the common one-dimensional chaotic map directly affects the signal reconstruction accuracy. To solve this problem, this paper uses the ratio form of the logistic chaotic map to the simple quadratic chaotic map to improve the sine chaotic map, and obtain a new type of compound sine (NC-sine) chaotic map. Its good chaotic behavior and chaotic interval expansion characteristics are verified by the bifurcation diagram, the Lyapunov exponent, and the complexity analysis. Based on the NC-sine chaotic map, a dynamic sparse circulant (DSC) measurement matrix with adaptive zero-setting elements is designed. The simulation results show that compared with the sine measurement matrix, the reconstruction success rate of the DSC measurement matrix is increased by 5\% and 9.69\% on average for a one-dimensional signal when the measurements and sparsity change, respectively. The peak signal-to-noise ratio of the reconstructed two-dimensional signals under different compression rates is improved by more than 0.92 dB on average, and the reconstruction efficiency is higher. The average structural similarity of the reconstructed signals at different initial values is improved by more than 0.027 compared to the Gaussian measurement matrix. This can then be utilized for the promotion of signal transmission in rate and accuracy.

INDEX TERMS Chaotic map, compressed sensing, Lyapunov exponent, measurement matrix

I. INTRODUCTION Compressive sensing (CS) [1], which breaks through the Nyquist sampling theorem, has outstanding advantages for reducing signal storage space, as well as improving signal transmission in real-time and high efficiency. As a result, CS is widely used in image processing [2]-[4], spectrum sensing [5], [6], channel estimation [7], [8], and other fields. The implementation of CS for signals involves three main components: sparse representation, compression, and reconstruction [9]. Among them, the performance of the measurement matrix directly affects the compression rate, the reconstruction efficiency, and the reconstruction accuracy of the signal. However, from the perspective of engineering applications, the existing random measurement matrices, such as the Gaussian matrix and Bernoulli matrix, generally have problems, such as large resource occupation, difficulty in modularization, and poor adaptability to engineering applications. Relatively speaking, deterministic measurement matrices, which are highly reproducible and easy to implement on physical circuits, have more engineering applicability.

However, the generation method and complexity of the pseudorandom sequence determine the engineering applicability of the deterministic matrix, which is a key factor that affects the reconstruction accuracy. In [10]-[12], researchers used m-sequences with a strong correlation, Berlekamp-Justesen codes, etc. to construct circulant or binary deterministic measurement matrices. This approach...
requires multiple parameters to be adjusted simultaneously in each sequence update. Compared with the above pseudorandom sequence generation method, the chaotic map-based generation method avoids the complicated multiparameter adjustment process. In [13], the high-order correlation of Chebyshev chaotic sequence elements was analyzed, and a Chebyshev chaotic matrix was constructed by sampling the eight-order Chebyshev chaotic map with a sampling step size of five. In [14]-[17], researchers constructed different types of deterministic matrices based on logistic chaotic map, cat chaotic map, and tent chaotic map. However, the basic chaotic map used in the above method has a small chaotic interval [18] or is obviously restricted by the calculation accuracy, which often degenerates the chaotic rule into a regular behavior [19], directly affecting the performance of the pseudorandom sequence. References [20], [21] used the special nonlinearity of the discrete memristor to construct two-dimensional chaotic maps with a high-quality performance and verified that the generated sequence has good pseudo randomness, which provides a new possibility for generating deterministic measurement matrices. In [22]-[24], the common different types of chaotic maps were combined to form new chaotic maps that were more sensitive to initial values and had stronger chaotic characteristics. Then, based on these new chaotic maps, different forms of deterministic measurement matrices with superior performances were designed to improve the reconstruction accuracy.

Throughout the current research, it can be seen that constructing a map with better chaotic characteristics and generating a sequence with strong pseudorandom characteristics is the key to optimizing the performance of the measurement matrix. Inspired by this, we combine a sine chaotic map, simple quadratic chaotic map, and logistic chaotic map to construct a new type of compound sine (NC-sine) chaotic map based on the idea of a functional composite. From the perspective of expanding the chaotic interval, increasing the complexity of the chaotic map and the sensitivity to the initial value, the characteristics of the chaotic map are improved. Then, based on the NC-sine chaotic map, the dynamic sparse circulant (DSC) measurement matrix is designed by combining the advantages of a sparse matrix to reduce the signal reconstruction time and a circulant measurement matrix to improve the signal reconstruction quality.

The rest of the paper is organized as follows: In the second section, CS theory is introduced, and an NC-sine chaotic map is constructed. The bifurcation diagram, the Lyapunov exponent, and the $C_0$ complexity are used to qualitatively analyze the NC-sine chaotic map. In the third section, the DSC measurement matrix is designed based on the NC-sine chaotic map, and its restricted isometric property (RIP) is analyzed. In the fourth section, we use the DSC measurement matrix to process the one-dimensional signals and the two-dimensional signals to verify the performance of the DSC measurement matrix. The conclusion is given in the last section.

II. INTRODUCTION OF THE CS THEORY AND PROPOSAL OF NC-SINE CHAOTIC MAP

In this section, CS theory is first introduced in detail. Then, the NC-sine chaotic map is proposed based on the logistic chaotic map, the simple quadratic chaotic map, and the sine chaotic map. Finally, the performance of the proposed chaotic map is analyzed.

A. CS THEORY

A signal $D \in R^{N \times 1}$ is projected into the sparse domain through a set of orthogonal sparse bases $\Psi \in R^{N \times N}$.

$$D = \Psi \zeta$$

(1)

where, $\zeta \in R^{N \times 1}$ is the sparse coefficient and $\zeta$ has only a finite number of nonzero elements. We use the sparsity $K$ to represent the number of nonzero elements in $\zeta$, and we also assert that the signal $D$ is $K$ sparse in the $\Psi$ domain. After sparse conversion, the signal $D$ can be sampled by the measurement vector $\Phi \in R^{M \times N}$ to obtain the measurement vector $y \in R^{M \times 1}$ as follows:

$$y = \Phi D = \Phi \Psi \zeta$$

(2)

The design of the measurement matrix $\Phi$ in CS is an extremely important part. The measurement matrix with a superior performance must not only improve the accuracy of the reconstructed signal but also reduce the complexity of the hardware implementation. However, if the signal sampled and compressed by the measurement matrix can be accurately reconstructed by the reconstruction algorithm, the measurement matrix must satisfy the RIP [25], i.e., for any $K$ sparse signal, if there is always a constant $\delta \in (0, 1)$, this is defined by

$$(1 - \delta)\|D\|_2^2 \leq \|\Phi D\|_2^2 \leq (1 + \delta)\|D\|_2^2$$

(3)

then the measurement matrix $\Phi$ satisfies the $K$-order RIP with $\delta$.

When the measurement matrix satisfies the RIP condition, the sparse coefficient can be obtained by solving the $l_0$ norm optimization of (4), and thus, the original signal can be obtained by inverting the sparse coefficients.

$$\min \| \zeta \|_0 \quad s.t. \quad y = \Theta \zeta$$

(4)

Greedy algorithms have both reconstruction accuracy and reconstruction efficiency, and are widely used to solve (4) [26]. In this paper, the orthogonal matching pursuit (OMP) [27] reconstruction algorithm of the greedy algorithm is used for reconstructing the original signal.

B. THE PROPOSAL OF THE NC-SINE CHAOTIC MAP

Chaos is a type of nonlinear dynamics system. It has characteristics, such as external randomness and internal
certainty, sensitivity to initial values, ergodicity, etc., and it is widely used in communication encryption, physics, and in other fields [28]. Chaotic maps can be divided into one-dimensional chaotic maps and high-dimensional chaotic maps. Commonly used one-dimensional chaotic maps, such as logistic chaotic map and simple quadratic chaotic map, have simple structures. However, their chaotic interval is narrow, which affects the performance of the chaotic sequence, and thus, the performance of the measurement matrix. To improve the accuracy of the signal reconstruction, a chaotic map with stronger chaotic characteristics must be constructed. We use the ratio form of the logistic chaotic map to the simple quadratic chaotic map to improve the sine chaotic map, obtain the NC-sine chaotic map, and analyze its chaotic performance through the bifurcation diagram, the Lyapunov exponent and the \( C_0 \) complexity.

The logistic chaotic map is defined by

\[
L_{n+1} = \alpha L_n (1 - L_n) \quad \alpha \in [0, 4]
\]

(5)

The simple quadratic chaotic map can be expressed as follows:

\[
Q_{n+1} = \beta \left( Q_n^2 - 5 \right) \quad \beta \in [0.1, 0.61]
\]

(6)

The sine chaotic map is given as

\[
S_{n+1} = \mu \sin(\pi S_n) \quad \mu \in [0, 1]
\]

(7)

The abovementioned \( \alpha \), \( \beta \) and \( \mu \) are chaotic map control parameters. Inspired by the idea of functional composition, we construct the NC-sine map according to the process shown in Fig. 1.

![FIGURE 1. The construction process of the NC-sine chaotic map.](image)

As shown in Fig. 1, at the input side, \( C_n \) is simultaneously input into the logistic chaotic map and the simple quadratic chaotic map. Then, the ratio form of the two types of chaos is multiplied by a control parameter \( \eta \) to obtain \( w \). Finally, the NC-sine chaotic map is obtained by replacing the part in the brackets of the sine chaotic map with \( w \). The NC-sine chaotic map can be expressed as follows:

\[
C_{n+1} = \sigma \sin \left( \eta \frac{C_n (1 - C_n)}{C_n^2 - 5} \right)
\]

(8)

where, \( C_n \in [-1, 1] \), \( n = 1, 2, \ldots \) and the control parameter, \( \sigma \in [0, 1] \), \( \eta > 0 \).

It is verified that the NC-sine chaotic map performance is strong when the control parameter \( \sigma \) is taken as one. Therefore, we fix \( \sigma \) as one in the subsequent analysis, as well as in the experimental process.

1) A PERFORMANCE ANALYSIS OF THE NC-SINE CHAOTIC MAP

The performances of chaotic maps can be qualitatively analyzed by using the Lyapunov exponents, the bifurcation diagrams and the \( C_0 \) complexity [22]. The bifurcation diagram can visually show the stability boundary of chaos and quantify the sensitive dependence of the chaotic map on the control parameters. The Lyapunov exponent can accurately reflect the relationship between the adjacent space orbits of the chaotic map and the sensitivity of the chaotic map to the initial value [29]. The chaotic map must have at least one Lyapunov exponent greater than zero. If the Lyapunov exponents of a chaotic map are less than zero in a certain region, the chaos map is in a stable state, and the chaotic behavior is interrupted. The \( C_0 \) complexity can measure the complexity of chaotic maps [30] very well. The closer the \( C_0 \) complexity of a chaotic map is to one, the more complex its chaotic behavior is.

When letting \( f(x) \) be a chaotic map, the Lyapunov exponents of \( f(x) \) can be obtained by (9).

\[
\lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|
\]

(9)

where, \( f'(x_i) \) is the derivative of \( f(x_i) \). We draw on [31] to analyze the chaotic behavior of the NC-sine map. The expression (8) of the NC-sine can be transformed into (10).

\[
C_{n+1} = \sin \left( \frac{\pi \alpha C_n (1 - C_n)}{\beta (C_n^2 - 5)} \right)
\]

(10)

We let \( \frac{\alpha C_n (1 - C_n)}{\beta (C_n^2 - 5)} = A(C_n) \) and the sine map with a control parameter of one is marked as \( \bar{S}(C_n) \). Then (10) can be transformed into (11).

\[
\sin \left( \pi A(C_n) \right) = \bar{S}(A(C_n))
\]

(11)

According to the definition, the Lyapunov exponents of the NC-sine chaotic map can be derived by (12).

\[
\lambda_c = \lim_{n \to \infty} \left\{ \frac{1}{n} \sum_{j=0}^{n-1} \ln \left\| \bar{S} (A(C_n)) \right\| \right\}
\]

\[
= \lim_{n \to \infty} \left\{ \frac{1}{n} \sum_{j=0}^{n-1} \ln \left\| \bar{S} (A(C_n)) \right\| \right\}
\]

(12)

\[
= \lim_{n \to \infty} \left\{ \frac{1}{n} \sum_{j=0}^{n-1} \ln \left\| \bar{S} (A(C_n)) \right\| + \ln |A'(C_n)| \right\}
\]

\[
= \lambda_\sigma + \lambda_\alpha
\]
The sine map with a control parameter of one has chaotic behavior, i.e., $\lambda > 0$. We discuss the Lyapunov exponent of the NC-sine chaotic map in the following three categories.

- $\lambda_3 > 0$, i.e., equation $A(C_n)$ has chaotic behavior. At this point, $\lambda > 0$ and $\lambda = \max \{\lambda_2, \lambda_3\}$, indicating that the NC-sine map has better chaotic performance than sine chaotic map.

- $-\lambda_4 < \lambda_4 < 0$, in which case the equation $A(C_n)$ does not have chaotic behavior, but $\lambda_4 = \lambda_2 + \lambda_3 > 0$, i.e., NC-sine still has chaotic behavior.

- $\lambda_5 < -\lambda_5$, then $\lambda_5 = \lambda_2 + \lambda_3 < 0$, i.e., the NC-sine chaotic map no longer has chaotic behavior.

If the equation $A(C_n)$ has chaotic behavior, NC-sine is always in a chaotic state, and it has better chaotic performance than the sine chaotic map when the control parameter is one. If the $A(C_n)$ does not have chaotic behavior, NC-sine can also be in a chaotic state. Therefore, it can be initially seen that NC-sine has better chaotic characteristics and a wider chaotic interval. Next, we demonstrate the advantages of NC-sine chaotic map more explicitly through bifurcation diagram, Lyapunov exponents curve, and $C_0$ complexity.

Fig. 2 shows the bifurcation diagrams of the logistic chaotic map, the simple quadratic chaotic map, the sine chaotic map, and the NC-sine chaotic map. The chaotic control parameter of the NC-sine chaotic map we proposed is set to $[0,30]$.

![Figure 2. Bifurcation diagram: (a) Logistic; (b) Simple quadratic; (c) Sine; (d) NC-sine.](image1)

It can be seen that the logistic chaotic map is uniformly distributed when the parameter $\alpha \in [3.86,4]$ showing completely chaotic characteristic. The uniform distribution intervals of the simple quadratic chaotic map and the sine map are $\beta \in [0.55,0.61]$ and $\mu \in [0.89,1]$, respectively. It is obvious that the chaotic intervals of the above three common maps are narrow. The NC-sine chaotic map we proposed has periodicity in the $\eta \in [4.8,7.64]$ time, $\eta \in [7.65,8.37]$ time, and $\eta \in [8.38,8.56]$ intervals, while the periods are 2, 4, and 8, respectively. The map enters a chaotic state when $\eta > 8.56$.

![Figure 3. Lyapunov exponents curves: (a) Logistic; (b) Simple quadratic; (c) Sine; (d) NC-sine.](image2)

Fig. 3 shows the changed trend of the Lyapunov exponents curves of the abovementioned chaotic maps with respect to their control parameters. The interval where the Lyapunov exponents of each chaotic map are greater than zero corresponds to the intervals where the bifurcation diagram starts to be uniformly distributed. The Lyapunov exponents of the NC-sine chaotic map we proposed are greater than zero in the interval of $\eta \in [8.56,30]$. The maximum Lyapunov exponents of the logistic chaos, the simple quadratic chaos, and the sine chaos map in Fig. 3 are 0.89, 0.50, and 0.74, respectively. However, the NC-sine chaotic map starts to be greater than one when the control parameter reaches 17. This shows that the NC-sine chaotic map is more sensitive to the initial value.

![Figure 4. $C_0$ complexity of chaotic maps: (a) Logistic; (b) Simple quadratic; (c) Sine; (d) NC-sine.](image3)

In Fig. 4, we can see that the maximum $C_0$ complexity of the logistic chaos, the simple quadratic chaos, and the sine chaos are 0.34, 0.68, and 0.35, respectively. The NC-sine chaotic map designed in this paper has a greater complexity than the above three chaotic maps in a wide range of intervals, and its maximum $C_0$ complexity exceeds 0.9. In
summary, the performance of the NC-sine chaotic map constructed in this paper is greatly improved compared to the logistic chaotic map, the simple quadratic chaotic map, and the sine chaotic map. It has a wider chaotic interval, is more sensitive to initial values, and has a higher complexity. Thus, it is suitable for generating the sequences needed to design the measurement matrix.

III. THE DESIGN AND ANALYSIS OF THE DSC MEASUREMENT MATRIX

The design of the measurement matrix is a key link for the application of CS. The measurement matrix with a superior performance can obtain more information about the signal under the same compression rate to ensure that the accuracy of the signal reconstruction is higher.

A. SEQUENCE CORRELATION ANALYSIS

To reduce the correlation between the matrix elements, and thus, ensure the high-quality performance of the measurement matrix, when constructing the measurement matrix using chaotic sequences, the sequences are usually sampled at certain intervals. The elements of the measurement matrix satisfy statistical independence when the sampling interval is large enough [32].

In [33], the researchers used the Pearson correlation coefficient to evaluate the sampling interval of the sequence generated based on the designed chaotic map. Therefore, we evaluate the correlation of the sequences generated by the NC-sine chaos map and the sine chaos map at different sampling intervals based on this coefficient. The Pearson coefficient is expressed as follows:

$$\rho = \frac{\sum_{i} X_i Y_i - \frac{1}{N} \sum_{i} X_i \sum_{i} Y_i}{\sqrt{\left(\sum_{i} X_i^2 - \frac{1}{N} \left(\sum_{i} X_i\right)^2\right) \left(\sum_{i} Y_i^2 - \frac{1}{N} \left(\sum_{i} Y_i\right)^2\right)}}$$ (13)

where, $\rho \in [-1, 1]$, $X$ and $Y$ are chaotic sequences. Usually, sequences are considered uncorrelated if $|\rho| < 0.2$. The sequence correlation coefficients generated by the NC-sine chaotic map and the sine chaotic map at different intervals are shown in Fig. 5.

![Image](image_url)

**FIGURE 5.** The trend of the Pearson coefficient.

The correlation coefficients of the NC-sine chaotic sequence and sine chaotic sequence are relatively large at the beginning. The correlation coefficients of the two types of chaotic sequences float between $[-0.1, 0.1]$ when the sampling interval is greater than two, i.e., the sequences are irrelevant.

Therefore, the measurement matrix constructed based on the sequence generated by the NC-sine chaotic map and the sine chaotic map only needs a sampling interval greater than two to make the elements of the matrix satisfy independence.

B. DSC MEASUREMENT MATRIX

The DSC measurement matrix is designed based on the NC-sine chaotic map, and the specific steps are as follows:

1. The first step is to use the NC-sine map to generate a chaotic sequence $Z_n$ of length $200 + (M - 1) d$.

2. The second step is to start sampling from the 200th element of $Z_n$ with a sampling interval of $d = 5$ to form the sequence $H$ of length $N$.

3. The third step is to calculate the dynamic coefficient $\omega$ according to (14), then randomly select $\left\lfloor \omega N + 1/2 \right\rfloor$ elements in $H$ and set them to zero to obtain the first row of the measurement matrix as $H_1$.

$$\omega = d_1 r^2 + d_2 r + d_3$$ (14)

where $d_1 = 0.0935$, $d_2 = 0.1349$, $d_3 = -0.0125$, and $r = M / N$ is the compression rate.

4. The fourth step is to circulate $H_1$ to generate matrix $\Phi_1$ and then to normalize the columns of $\Phi_1$ to obtain the DSC measurement matrix $\Phi$.

The DSC measurement matrix designed according to the above steps has the advantages of both a circulant matrix and a sparse matrix. Moreover, the coefficient changes dynamically with the compression rate, and the number of matrix zero-setting elements is different under different compression rates (the greater the compression rate is, the more redundant information is collected, and the more elements need to be zeroed in the measurement matrix), enabling the storage resources to be used more reasonably.

C. RIP ANALYSIS

The measurement matrix generated based on the chaotic map is a pseudorandom matrix, which satisfies the Johnson-Lindenstrauss lemma [34].

**Johnson-Lindenstrauss lemma**: We give the parameter $\varepsilon \in (0, 1)$ and there is a point set $E$ belonging to $R^N$. As long as $M > O(\log N) / \varepsilon^2$, the equation (15) holds for any $p, q \in E$. The equation is as follows:

$$(1 - \varepsilon)\|p - q\|_2 \leq \|f(p) - f(q)\|_2 \leq (1 + \varepsilon)\|p - q\|_2$$ (15)

Baraniuk et al. [35] obtained a conclusion by using the Johnson-Lindenstrauss lemma after strict proof, i.e., as long as $M > (a_0 \cdot K \cdot \log(N / K))$ and all random or pseudorandom
matrices could satisfy RIP with a high probability. The probability is expressed by (16).

\[ P \geq 1 - 2 \left( \frac{12}{\varepsilon} \right)^K \exp \left( -a_{\varepsilon} \left( \frac{\varepsilon}{2} \right) M \right) \]  

where, \( a_{\varepsilon} \) and \( a_{\delta} \) are only related to the constant \( \varepsilon \) and \( K \) is the sparsity. Obviously, the smaller \( a_{\varepsilon} \) is, the closer \( p \) is to one.

We learn from Baraniuk’s proof idea to prove that DSC satisfies RIP. For any subspace \( R^K \) of \( R^N \), the probability that the measurement matrix satisfies RIP is \( p_1 \). However, space \( R^K \) has \( sp \leq (\lambda N / K)^K \) subspaces, meaning that the probability that the measurement matrix satisfies RIP on \( R^K \) increases as follows:

\[ P \geq 1 - 2sp \left( 12 \right)^K \exp \left( -l_2 \left( \frac{\delta}{2} \right) M \right) \]  

If \( l_2 \) is a constant, whether \( K \) satisfies \( \lambda_2 = l_2 \cdot M \cdot \log(N / K) \) or not, as long as \( l_2 \leq l_1 \left[ 1 + \left( \frac{1 + \log(12 \delta)}{\log(N / K)} \right) \right] \), the maximum value of the exponential part of \( p_2 \) is always guaranteed to be \( -l_2 \cdot M \). Therefore, we can choose a very small \( l_2 \) to ensure that \( l_2 \geq 0 \). Additionally, for any space \( R^K \), the probability that the measurement matrix satisfies RIP is at least the following:

\[ P \geq 1 - 2\exp \left( -l_2 \cdot M \right) \]

It is possible to choose a sufficiently small \( l_1 \) to ensure a sufficiently large \( P(P \leq 1) \), and to ensure that the DSC measurement matrix satisfies the RIP with a high probability.

IV. EXPERIMENTS

In this section, to verify the performance of the DSC measurement matrix, we use it to process one-dimensional and two-dimensional signals, and compare the results with the sine measurement matrix, the sine circulant measurement matrix, the Gaussian measurement matrix, and the Bernoulli measurement matrix. For processing one-dimensional signals, we first compare the reconstruction success rates of the abovementioned measurement matrices in the noise-free case with measurements and sparsity as variables. Then, we add noise to the signal and compare the output signal-to-noise ratio (SNR) of the reconstructed signal of the abovementioned measurement matrices. When DSC is used to process two-dimensional signals, the peak signal-to-noise ratio (PSNR) and reconstruction time of the reconstructed signals at different compression rates are compared with other measurement matrices used in the experiments. The structural similarity (SSIM) of the reconstructed signals of each measurement matrix at different initial values is also compared.

A. ONE-DIMENSIONAL SIGNAL PROCESSING

1) NOISE-FREE CASE

We set a one-dimensional signal \( D \) with sparsity \( K = 10 \), sparse positions at random, and a length \( N = 256 \). The DSC measurement matrix is used to compress and sample it. We choose the typical OMP algorithm in the greedy algorithm to reconstruct the signal, and the result is shown in Fig. 6.

Fig. 6 shows that the sparse one-dimensional signal can be accurately reconstructed after sampling and compression by the DSC measurement matrix. To more specifically verify the performance of the DSC measurement matrix for processing one-dimensional signals, we conduct two independent experiments to compare the reconstruction success rates of different measurement matrices under different conditions. In Experiment 1, the sparsity \( K = 10 \) is fixed, and the variable \( M \) ranges from 10 to 110 in an interval of 10. In Experiment 2, the measurements \( M = 128 \) are fixed, and the variable \( K \) ranges from 10 to 70 in an interval of 5. The sparse positions of the one-dimensional signals used in the two experiments are random, and the signal lengths are 256. The matrices for comparison are the sine measurement matrix, the sine circulant measurement matrix, the sine sparse measurement matrix, the Gaussian measurement matrix, and the Bernoulli measurement matrix. When generating the DSC measurement matrix, the control parameter \( \eta \) of the NC-sine chaotic map is set to 25, and the initial value is 0.5001. When generating the abovementioned sine types of measurement matrices, the control parameter \( \mu \) of the sine chaotic map is set to one, the initial value is 0.5001, and the sampling interval is set to 15 according to [12] and [36]. The proportion of the randomly selected zeroed elements in each row of the sine sparse measurement matrix is fixed at 10%. The elements of the Gaussian measurement matrix obey a Gaussian distribution with a mean of zero and a variance of one. The signal reconstruction residual is calculated as (19).

\[ \sigma = \frac{\| D - \hat{D} \|_2}{\| D \|_2} \]  

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where, $D_r$ is the reconstructed signal and $D$ is the original signal. If $\sigma \leq 10^{-6}$, the signal reconstruction is considered successful. To make the results more reliable, we use each matrix to perform 100 reconstructions, and the success rates are shown in Fig. 7 and Fig. 8.

Fig. 7 and Fig. 8 show that the reconstruction success rates of the DSC measurement matrix are higher than those of the other measurement matrices in both situations.

![FIGURE 7. The reconstruction success rate when measurements change.](image)

![FIGURE 8. The reconstruction success rate when sparsity changes.](image)

In Fig. 7, under the premise that the sparsity is unchanged, when the measurements are less than 20, the reconstruction success rates of all measurement matrices are 0%. When the measurements are greater than 20, the reconstruction success rate of each measurement matrix gradually increases. The success rate of the DSC measurement matrix is obviously increasing the fastest, and thus, its success rate is the highest, while the sine sparse measurement matrix has the lowest success rate. When the measurements reach 90, the reconstruction success rate of each measurement matrix reaches 100%. The average reconstruction success rate of the DSC measurement matrix is 5% higher than that of the sine measurement matrix for the variation of measurements.

In Fig. 8, the measurements are unchanged. The reconstruction success rate of each measurement matrix gradually decreases. The reconstruction success rate of the DSC measurement matrix decreases the slowest. The performances of the sine measurement matrix and the sine sparse measurement matrix are relatively poor. When the sparsity is 60, the reconstruction success rate of other measurement matrices used in the experiment reaches 0%, but the DSC measurement matrix still has a certain probability of successfully reconstructing the original signal. In this situation, the average reconstruction success rate of the DSC measurement matrix is 9.69% higher than that of the sine measurement matrix.

In the noise-free case, regardless of the measurements or the sparsity change, the DSC measurement matrix has a higher reconstruction success rate than the measurement matrices used in the experiments.

2) **NOISY CASE**

We add Gaussian noise with a mean value of zero and a standard deviation of 0.2 to the one-dimensional signal with length $N = 256$, sparsity $K = 20$, and the sparse position of the signal is random. The measurements range from 64 to 128 at intervals of 10. To avoid randomness, we use each measurement matrix to perform 100 reconstructions at different measurements and then average the results. The output SNR of each measurement matrix is shown in Fig. 9.

![FIGURE 9. Output SNR of reconstructed noisy signal.](image)

As shown in Fig. 9, the SNR of the signal reconstructed from the Gaussian measurement matrix, the Bernoulli measurement matrix and the three measurement matrices generated based on the sine chaotic map is similar in the noisy environment. The output SNR of the noisy signal reconstructed by the DSC measurement matrix is considerably higher than that of the other matrices. The DSC measurement matrix that combines the advantages of circulant and sparse measurement matrices has better robustness.

In summary, the DSC measurement based on the NC-sine chaotic map design has higher reconstruction accuracy and better robustness.

**B. TWO-DIMENSIONAL SIGNAL PROCESSING**

To further verify the performance of the DSC measurement matrix designed in this paper, we process two-dimensional signals. We select a Cameraman image with a size of $256 \times 256$ and a Lena image with a size of $512 \times 512$ for processing, which are most commonly used by researchers. In the experiments, the measurement matrices used for comparison are the same as those used when processing one-
dimensional signals. The sparse basis is a wavelet basis. The OMP algorithm is still used as the reconstruction algorithm. When the compression rate is 0.6, the results are shown in Fig. 10 and Fig. 11.

It can be seen in Fig. 10 and Fig. 11 that when the compression rate is 0.6, the reconstruction effect of the DSC measurement matrix on the two images is better than that of other measurement matrices used in the experiments. To more intuitively and concretely compare the advantages and disadvantages of the above measurement matrices in the reconstruction of two-dimensional image signals, we calculate the PSNR and reconstruction time of the two images reconstructed by the above measurement matrices under different compression rates. Considering that the evaluation result of PSNR is different from those perceived by human eyes, we also calculate the SSIM index of reconstruction to comprehensively evaluate the effect of each measurement matrix on image reconstruction; the evaluation result is consistent with the human subjective perception [37].

To make the results more reliable, each measurement matrix reconstructs the image ten times and averages the results. The PSNR and SSIM index can be calculated according to (20) and (21), respectively.

\[
MSE = \frac{1}{MN} \sum_{i=1}^{M} \sum_{j=1}^{N} (x(i, j) - \hat{x}(i, j))^2
\]  

\[
PSNR = 10 \log \frac{255^2}{MSE}
\]  

where, \(x(i, j)\) is the gray value of the original image, \(\hat{x}(i, j)\) is the gray value of the reconstructed image, \(MN\) is the size of the original image, and \(MSE\) represents the mean square error between the reconstructed image and the original image.

\[
SSIM(x, y) = \frac{[l(x, y)]^2 \cdot [c(x, y)]^2 \cdot [s(x, y)]^2}{\sigma_l^4 + \sigma_c^4 + \sigma_s^4}
\]  

\[
l(x, y) = \frac{2u_x u_y + C_1}{u_x^2 + u_y^2 + C_1}
\]

\[
c(x, y) = \frac{2\sigma_x \sigma_y + C_2}{\sigma_x^2 + \sigma_y^2 + C_2}
\]

\[
s(x, y) = \frac{\sigma_{xx} + C_3}{\sigma_{xx} \sigma_{yy} + C_3}
\]  

where, \(u_x\), \(u_y\), \(\sigma_x\), \(\sigma_y\), and \(\sigma_{xy}\) are the local means, standard deviations, and cross-covariance for the images.

The PSNR, reconstruction time and SSIM of the Cameraman image and the Lena image reconstructed by each measurement matrix are shown in Fig. 12, Fig. 13, Fig. 14, Fig. 15, Fig. 16, and Fig. 17.
In Fig. 12, when the compression rate is between 0.2 and 0.4, the PSNR of the reconstructed image of the DSC measurement matrix, the Gaussian measurement matrix, and the Bernoulli measurement matrix are similar. The PSNR of the reconstructed image of the DSC measurement matrix, the Gaussian measurement matrix, and the Bernoulli measurement matrix are similar. The PSNR of the reconstructed image of the sine circulant measurement matrix is the worst. When the compression rate is 0.4, the PSNR gap between the reconstructed images of the DSC measurement matrix and the sine circulant measurement matrix reaches a maximum of 9.93 dB. At the same time, the PSNR of the reconstructed image of the sine circulant measurement matrix begins to increase sharply. When the compression rate is between 0.6 and 0.8, the PSNR of the reconstructed image by the DSC measurement matrix is similar to the sine circulant measurement matrix, and is larger than other measurement matrices used in the experiment. The average PSNR of the reconstructed image at different compression rates for the DSC measurement matrix is 0.925 dB higher than that of the sine measurement matrix.

Fig. 14 shows the SSIM of the reconstructed Cameraman image for each measurement matrix at different initial values for a compression rate of 0.8. The SSIM of the reconstructed image of the sine measurement matrix and the sine sparse measurement matrix is relatively poor and lower than the Gaussian measurement matrix and the Bernoulli measurement matrix. Sine circulant measurement matrix has relatively large fluctuations in a reconstructed image SSIM under different initial values, the DSC measurement matrix reconstruction of the SSIM is the best. With an initial value of 0.5, the SSIM gap between the DSC measurement matrix and the sine circulant measurement matrix reconstruction reaches a maximum of 0.016. The average SSIM of the reconstructed images for the DSC measurement matrix at each initial value is higher than that of the sine measurement matrix and Gaussian measurement matrix by 0.031 and 0.027, respectively.

As seen from Fig. 13, the advantage of less reconstruction time required by the DSC measurement matrix becomes gradually obvious as the compression rate becomes larger, i.e., increasingly more data is sampled. Although the PSNR of the reconstructed image of the sine circulant measurement matrix is close to that of the DSC measurement matrix when the compression rate is between 0.6 and 0.8, the time used for the DSC measurement matrix reconstruction is less.

Fig. 15 shows image processing with a larger amount of data. When the compression rate is 0.2, the PSNR of the reconstructed image of the DSC measurement matrix is higher than that of the other matrices. The PSNR of the reconstructed image by the sine circulant measurement matrix is only 5.68 dB, whereas the reconstructed PSNR of the DSC measurement matrix is 19.98 dB. When the compression rate is 0.3, the PSNR of the reconstructed image of the sine measurement matrix, the sine sparse measurement matrix, and the sine circulant measurement matrix rises rapidly, which is close to the reconstruction result of the DSC measurement matrix. When the compression rate is between 0.3 and 0.8, the PSNR of the reconstructed image by the
DSC measurement matrix is similar to the sine circulant measurement matrix and is higher than the other measurement matrices used in the experiment. The average PSNR of the reconstructed image at different compression rates for the DSC measurement matrix is 2.32 dB higher than that of the sine measurement matrix.

In Fig. 15, the PSNR of the reconstructed signal of the sine circulant measurement matrix is close to the DSC measurement matrix when the compression rate is 0.3 for a two-dimensional signal with a larger amount of data. However, it can be seen in Fig. 16 that for the Lena image with a larger amount of data, the advantage of less reconstruction time required for the reconstruction time of the DSC measurement matrix at various compression rates is more obvious. When the compression rate is 0.5, the reconstruction time of the DSC measurement matrix is 1.9 seconds less than that of the other measurement matrices used in the experiment.

As shown in Fig. 17, when the compression rate is 0.8, the SSIM of the Lena image reconstructed by Gaussian measurement matrix, Bernoulli measurement matrix, sine sparse measurement matrix, and sine measurement matrix generated at different initial values are similar. The SSIM of the DSC measurement matrix and the sine circulant measurement matrix reconstructed image is better. The average SSIM of the reconstructed images for the DSC measurement matrix at each initial value is higher than that of the sine measurement matrix and the Gaussian measurement matrix by 0.029 and 0.03, respectively.

Integrating the processing results of one-dimensional and two-dimensional signals, it is not difficult to find that the comprehensive performance of the DSC measurement matrix we design is superior. The DSC measurement matrix combines the advantages of less storage space occupied by a sparse measurement matrix and easy generation of a circulant measurement matrix, which not only ensures the reconstruction accuracy but also improves the reconstruction efficiency. Moreover, the DSC measurement matrix has good robustness.

V. CONCLUSION

In this paper, the chaos intervals, sensitivity to initial values, and the complexity of the logistic chaotic map, the simple quadratic chaotic map, the sine chaotic map, and the proposed NC-sine chaotic map are compared and analyzed by the bifurcation diagram, the Lyapunov exponent and, the \( C_\alpha \) complexity. The results show that the NC-sine chaotic map effectively overcomes the problems with a small chaotic interval, a low sensitivity to the initial value, and the low complexity of common one-dimensional chaotic maps. To verify the performance of the measurement matrix constructed based on the proposed chaotic map, we sample the NC-sine chaotic sequence with an initial value of 0.5001 at a sampling interval of 5 and design the DSC measurement matrix. Then, the DSC measurement matrix is used to process one-dimensional and two-dimensional signals. The effect is compared with the Gaussian measurement matrix, the Bernoulli measurement matrix, and the three measurement matrices that are generated from the sine chaotic sequence with an initial value of 0.5001 at a sampling interval of 15. For the one-dimensional signal, the average success rate of the DSC measurement matrix reconstruction is higher than that of the sine measurement matrix by 5%, and 9.7% when the measurements and sparsity change, respectively. The average PSNR of the reconstructed two-dimensional signal is higher than that of the sine measurement matrix by more than 0.92 dB, and the efficiency of reconstruction and the SSIM of the reconstructed signal are higher.

In summary, this paper improves the performance of the CS measurement matrix by improving the chaotic map. Furthermore, the reconstruction success rate, reconstruction accuracy, and efficiency of CS of one-dimensional and two-dimensional signals are improved. The paper provides a theoretical basis and experimental data support for optimizing the measurement matrix design from the perspective of improving the chaotic map, while providing an effective way to overcome the technical obstacles of CS applied to engineering practical applications.

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