Two roles of relativistic spin operators

Daniel R. Terno
Department of Physics, Technion—Israel Institute of Technology, 32000 Haifa, Israel

Spin degrees of freedom appear in a variety of applications in quantum mechanics and foundations of quantum information theory and usually are analyzed non-relativistically. In a relativistic domain an observable of choice is the helicity \( S \cdot p \), which is well defined for particles with sharp momentum (for beams in accelerators typical spread to energy ratios are about \( 10^{-3} - 10^{-4} \)). Nevertheless, there is also an interest in spin operators in general.

In this paper we consider two standard spin operators for massive spin-\( \frac{1}{2} \) particles, the rest frame spin and the Dirac spin operator \( \Sigma \) that is associated with the spin of moving particles as seen by a stationary observer. These two quantities can serve as prototypes for various alternative 'spin operators' that appear in the literature.

Spin and many other operators play a double role: they are symmetry generators and 'observables.' The analysis of different splittings of angular momentum into ‘spin’ and ‘orbital’ parts reveals the difference between these two roles. We also discuss a relation of different choices of spin observables to the violation of Bell inequalities.

Following Wigner, the Hilbert space is

\[
\mathcal{H} = \mathbb{C}^2 \otimes L^2(\mathbb{R}^3, d\mu(p)), \quad d\mu(p) = \frac{1}{(2\pi)^3} \frac{d^3p}{(2p^0)},
\]

where \( p^0 = \sqrt{m^2 + p^2} \). The generators of the Poincaré group are represented by

\[
P^\mu = p^\mu, \quad J = -ip^0 \nabla_p - \frac{p \times S}{m + p^0}, \quad K = -ip \times \nabla_p + S,
\]

where the angular momentum is split into orbital and spin parts, respectively. We label basis states \( |\sigma, p\rangle \). Pure state of definite momentum and arbitrary spin will be labeled as \( \left( \begin{array}{c} \sigma \\ p \end{array} \right) \).

Lorentz transformation \( \Lambda \), \( y^\mu = \Lambda^\mu_{\nu} x^\nu \), induces a unitary transformation of states. In particular,

\[
U(\Lambda)|\sigma, p\rangle = \sum_\xi D_{\xi\sigma}|W(\Lambda, p)||\xi, \Lambda p\rangle,
\]

where \( D_{\xi\sigma} \) are the matrix elements of a unitary operator \( D \) which corresponds to a Wigner rotation \( W(\Lambda, p) \). The Wigner rotation itself is given by

\[
W(\Lambda, p) := L^\dagger_{\Lambda p} A L_p,
\]

where \( L_p \) is a standard pure boost that takes a standard momentum \( k_R = (m, 0, 0, 0) \) to a given momentum \( p \). Explicit formulas of \( L_p \) are given, e.g., in [4, 14].

It is well known that for a pure rotation \( R \) the three-dimensional Wigner rotation matrix is the rotation itself,

\[
W(R, p) = R, \quad \forall p = (E(p), p).
\]

As a result, Wigner’s spin operators are nothing else but halves of Pauli matrices (tensored with the identity of \( L^2 \)).

A useful corollary of Eqs. (3) and (4) is a property of the rest frame spin. If an initial state (in the rest frame) is

\[
|\Psi\rangle = \alpha|\frac{1}{2}, k_R\rangle + \beta|\frac{-1}{2}, k_R\rangle,
\]

with \( |\alpha|^2 + |\beta|^2 = 1 \), then a pure boost \( \Lambda \) leads to

\[
U(\Lambda)|\Psi\rangle = \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) |\Lambda k_R\rangle.
\]

*Electronic address: terno@physics.technion.ac.il

\[\text{arXiv:quant-ph/0208074v2  28 Nov 2002}\]
A Pauli-Lubanski vector is an important quantity that is constructed from the group generators [14].

\[ w_{\rho} = \frac{i}{2} \epsilon_{\lambda \mu \nu \rho} P^\lambda M^{\mu \nu}, \quad w_0 = -P \cdot J, \quad w = P^0 J + P \times K, \]

where \( M^{12} = J^3, M^{01} = K^1 \), etc. In particular, it helps in splitting spin out of the angular momentum,

\[ S = \frac{1}{m} \left( w - \frac{u^0 P}{P^0 + m} \right). \tag{9} \]

For a particle with definite 4-momentum \( p \), this formula is just saying that the components of a spin operator are three spacelike components of the Pauli-Lubanski operator at the rest frame,

\[ S_k = (L_p^{-1} w)_k. \tag{10} \]

We take three following properties as defining a natural relativistic extension of the “spin observable”.

- The triple of operators \( S \) reduces in the rest frame to a non-relativistic expression \( w_R/m \);
- It is a three-vector
  \[ [J_j, S_k] = i \epsilon_{jkl} S_l; \tag{11} \]
- It satisfies spin commutation relation
  \[ [S_j, S_k] = i \epsilon_{jkl} S_l. \tag{12} \]

A simple lemma (the proof can be found in [14]) shows that this operator is unique, under one technical assumption.

**Lemma 1** The only triple of operators \( S \) that satisfies the above assumptions, and in addition is a linear combination of the operators \( w^\alpha \) is given by Eq. (10).

To discuss Dirac spin operators we need more elements of field-theoretical formalism. States of definite spin and momentum are created from the vacuum by creation operators \( \{ \sigma, p \} = \hat{a}_{\sigma p}(0) \), while antiparticles are created by \( \hat{b}^\dagger_{\sigma p} \). We use the following normalisation convention:

\[ \langle \sigma, p | \xi, q \rangle = (2\pi)^3 (2p^0)^2 \delta_{\sigma \sigma} \delta(3)(p - q). \tag{13} \]

Field operators are usually written with Dirac spinors. A Hilbert space and unitary representation on it can be constructed from the bispinorial representation of the Poincaré group. To this end we take positive-energy solutions of Dirac equation that form a subspace of the space of all four-component spinor functions \( \Psi = \Psi^{\lambda}(p) \), \( \lambda = 1, \ldots, 4 \). A Lorentz-invariant inner product becomes positive-definite and the subspace of positive energy solutions becomes a Hilbert space. The generators in this representation are

\[ P^\mu = p^\mu, \quad J^{\mu \nu} = L^{\mu \nu} + S^{\mu \nu}. \tag{14} \]

where

\[ L^{lm} = i \left( P^l \frac{\partial}{\partial p_m} - p_m \frac{\partial}{\partial p_l} \right), \tag{15a} \]

\[ L^{0m} = ip^0 \frac{\partial}{\partial p_m} \quad S^{\mu \nu} = \frac{i}{4} [\gamma^\mu, \gamma^\nu]. \tag{15b} \]

A double infinity of plane-wave positive energy solutions of the Dirac equation (functions \( u^\lambda_{p, (1/2)} \) and \( u^\lambda_{p, (-1/2)} \) that are proportional to \( e^{-ip \cdot x} \)) is a basis of this space. Basis vectors of Wigner and Dirac Hilbert spaces are related by [14]

\[ u^\lambda_{p, (1/2)} \leftrightarrow \left( \frac{1}{2}, p \right), \quad u^\lambda_{p, (-1/2)} \leftrightarrow \left( -\frac{1}{2}, p \right). \tag{16} \]

A discrete part of \( S^{\mu \nu} \) \((\Sigma^1/2 \equiv S^{23}, \text{etc})\) is a Dirac spin operator. In standard or Weyl representations it looks like

\[ \Sigma = \left( \begin{array}{cc} \sigma & 0 \\ 0 & -\sigma \end{array} \right). \tag{17} \]

It is possible to say that different propositions for spin operators are different ways to split \( J \). However, a momentum-dependent Foldy-Wouthuysen transformation takes \( \sigma \) to the spinor representation of \( S \). Hence we see that the difference is essentially in a covariant treatment.

A field operator is constructed with the help of plane wave solutions of Dirac equations [3, 14],

\[ \hat{\phi}(x) = \sum_{\sigma = \frac{1}{2}, -\frac{1}{2}} \int \! \! d\mu(p) (e^{ix \cdot p} \epsilon^\sigma_{\rho \nu} b^\dagger_{\sigma p} + e^{-ix \cdot p} \nu^\rho_{\sigma} \hat{a}_{\sigma p}), \tag{18} \]

where \( \nu^\rho_{\sigma} \) are negative energy plane wave solutions of Dirac equation.

Using field transformation properties it is a standard exercise to get the following expression for Dirac spin operator [15]

\[ \hat{\Sigma} = :: \int \! \! d^4x \hat{\phi}^\dagger(x) \frac{\Sigma}{2} \hat{\phi}(x) :: \tag{19} \]

where :: designates a normal ordering. Wigner spin is given by

\[ \hat{S} = \frac{i}{2} \sum_{\eta \xi} \langle \eta | \xi \rangle \int \! \! d\mu(p) (\hat{a}_{\eta p}^\dagger \hat{a}_{\xi p} + \hat{b}_{\eta p}^\dagger \hat{b}_{\xi p}). \tag{20} \]

An interpretation of \( \hat{\Sigma} \) and \( \hat{S} \) as observables is based on the analysis of one-particle states with well-defined momentum and an arbitrary spin, such as \( |\Psi\rangle = \left( \begin{array}{c} \psi \\ \sigma \psi \end{array} \right) \equiv \psi |p\rangle \). A corresponding Dirac spinor for this state is \( \Psi_p = \alpha u_{p,(1/2)} + \beta u_{p,(-1/2)} \).

An expectation value of Wigner spin operator is just a non-relativistic rest-frame expression

\[ \langle S \rangle = \frac{\langle \Psi | \hat{S} | \Psi \rangle}{\langle \Psi | \Psi \rangle} = \psi^\dagger \frac{\sigma}{2} \psi. \tag{21} \]
The transformation properties of momentum eigenstates Eq. \[ \text{(11)} \]. Lemma 1 and the fact that Wigner spin operator commutes with the Hamiltonian lead to the association of $\mathbf{S}$ with a conserved quantity ‘rest frame spin’.

Dirac spin $\mathbf{\Sigma}$ is associated with the spin of a moving particle. A quantity

$$
\mathbf{s}^D = \frac{\langle \Psi | \mathbf{\Sigma} | \Psi \rangle}{\langle \Psi | \Psi \rangle} = \sum_{k} \frac{1}{4E(p)} \Psi^\dagger \mathbf{\Sigma} \Psi,
$$

is interpreted as an expectation value of the spin of a moving particle with momentum $\mathbf{p}$ \[ \text{(12)} \]. It reduces to its non-relativistic value for $\mathbf{p} \to 0$ and particle’s helicity can be calculated with either of the operators.

While from the Lemma 1 it is clear that $\mathbf{\Sigma}$ does not define operators on the one-particle Hilbert space, it is instructive to see how it fails to do so. En route we construct $\mathbf{S}$, an one-particle Hilbert space restriction of $\mathbf{\Sigma}$. To this end we derive a necessary and sufficient condition for three expectation values to be derivable from the three operators that satisfy spin commutation relations.

Consider six $2 \times 2$ spin density matrices with Bloch vectors $\pm \mathbf{\hat{z}}, \pm \mathbf{\hat{x}}$ and $\pm \mathbf{\hat{y}}$. These density matrices are

$$
\rho_z = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \rho_x = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \quad \rho_y = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix},
$$

etc. We are looking for three Hermitian $2 \times 2$ matrices $\mathbf{S}_k$ the expectation values of which on the above states are the prescribed numbers $\bar{s}_k(\rho_l)$,

$$
\bar{s}_k(\rho_l) = \text{tr}(S_k \rho_l), \quad k = 1, 2, 3; \quad \pm l = \pm x, \pm y, \pm z.
$$

We decompose these matrices in terms of Pauli matrices,

$$
\mathbf{S}_k = \sum_{n=0}^{3} \bar{s}_{kn} \sigma_n,
$$

and analogous expressions for other states. From Eq. \[ \text{(27)} \] we see that $\mathbf{S}$ is given by

$$
\mathbf{S}_k = \sum_{l} \bar{s}_k^D(\rho_l) \sigma_l
$$

However, a simple calculation reveals that, e.g.,

$$
[\mathbf{S}_x(\mathbf{p}), \mathbf{S}_y(\mathbf{p})] \neq i\mathbf{S}_z(\mathbf{p}),
$$

and the equality is recovered only in the non-relativistic limit. It is not just a problem of a combining three operators into $3$-vector. If we write $n_z = \cos \theta$ we find that where $\sigma_0$ is an identity. It is easy to see that

$$
\text{tr}(\rho_l \sigma_n) = \delta_{ln}, \quad n = 1, 2, 3.
$$

Therefore, $\bar{s}_k(\rho_l) = s_{k0} + s_{kl}$. If instead of spin states $\rho_l$ we take their orthogonal complements $\rho_{-l}$ we see that all $s_{k0} = 0$, so

$$
\mathbf{S}_k = \sum_{l} \bar{s}_k(\rho_l) \sigma_l.
$$

We want these operators to satisfy spin commutation relations $[\mathbf{S}_j, \mathbf{S}_k] = i\epsilon_{jkl} \mathbf{S}_l$. Therefore,

$$
\bar{s}_j(\rho_m) \bar{s}_k(\rho_n) [\sigma_m, \sigma_n] = 2i \epsilon_{jkl} \bar{s}_l(\rho_p) \sigma_p,
$$

holds and the summation is understood over the repeated indices. As a result we establish the following

**Lemma 2** A necessary and sufficient condition for a triple of probability distributions on spin-$\frac{1}{2}$ states with the expectation values $\mathbf{s} = (s_1, s_2, s_3)$ to be derivable from a triple of matrices that satisfy spin commutation relations is

$$
\bar{s}_j(\rho_m) \bar{s}_k(\rho_n) \epsilon_{mnp} = \epsilon_{jkl} \bar{s}_l(\rho_p)
$$

where the three states $\rho_p$ are the pure states with Bloch vectors $\mathbf{\hat{x}}, \mathbf{\hat{y}}$ and $\mathbf{\hat{z}}$, respectively.

We apply this technique to a relativistic spin. Six states $\rho$ are taken to be spin parts of zero momentum states. Consider them in a frame where they have a momentum $\mathbf{p} = p (n_x, n_y, n_z)$. Expectation values of $\mathbf{s}^D$ in that frame are calculated according to Eq. \[ \text{(22)} \]. For $\rho_z$ we get

$$
\bar{s}_z(\rho_z) = \frac{1}{2} \frac{1}{p^2 + m(m + \sqrt{m^2 + p^2})} \left( n_x n_x p^2, n_y n_y p^2, m^2 + n_z^2 p^2 + m\sqrt{m^2 + p^2} \right).
$$

the eigenvalues of $\mathbf{S}_z$ are

$$
s_{\pm} = \pm \frac{1}{2} \sqrt{E^2(p) + m^2 + p^2 \cos 2\theta}. 
$$

A triple of operators $\mathbf{S}_x \otimes 1_{\mathcal{L}^2}$ is a restriction of $\hat{\mathbf{S}}$ that operates on the Fock space $\mathcal{F}(\mathcal{H}) = \oplus_n S^-(\mathcal{H}^\otimes n)$ to the one-particle space $\mathcal{H}$. In the process of restriction the essential spin operator properties are lost, even if the resulting operators are the legitimate observables, similarly to \[ \text{(11)} \]. Therefore, if one requires fixed outcomes $\pm \frac{1}{2}$ they are possible to achieve at the price of introducing a two-outcome pos-
We have two distinct representations of SU(2) algebra. This operator is also an observable with ‘usual’ properties. The expectation value \( \bar{E} \) is recovered only in the limit \( p \to 0 \). Since \( P_k^+ > 0 \) and \( P_k^+ + P_k^- = 1 \) we indeed have a two-outcome POVM.

From these results we learn that being a representation of a symmetry generator does not necessarily imply that this operator is also an observable with ‘usual’ properties. We have two distinct representations of SU(2) algebra on the spin-\( \frac{1}{2} \) Fock space, \( \hat{S} \) and \( \hat{\Sigma} \). However, only one of them preserves defining commutation relations when restricted to one-particle Hilbert space.

Now let us consider a relation of different spin operators to the maximal violation of Bell inequalities. Consider the Clauser-Horne version of Bell inequalities, where two pairs of operators describe pairs of possible tests (\( A_1 \) and \( A_2 \) for the first particle, \( B_1 \) and \( B_2 \) for the second). In each test two possible outcomes are conventionally labeled ‘+’ and ‘-’. Probabilities of these outcomes, e.g., for the first particle, are given as expectations \( p_i^\pm = \text{tr}(E_i^\pm \rho) \), where positive operators \( E_i^\pm \) form a two-outcome POVM, \( E_1^+ + E_1^- = 1 \). The four operators \( A_i, B_i \) are defined similarly to Eq. (35). In particular, \( A_i = 2E_i^+ - 1 \), and the absence of a factor \( \frac{1}{2} \) is conventional.

It was shown by Summers and Werner that the inequalities are maximally violated only if each couple of operators generates spin commutation relations. In particular, the operators \( A_i \) have to satisfy \( A_i^2 = 1 \) and \( A_1 A_2 + A_2 A_1 = 0 \), and operators \( B_i \) are similarly constrained. Hence, defining \( A_3 := -\frac{1}{2}[A_1, A_2] \) one indeed reproduces commutation relations of Pauli matrices.

Now assume that these operators are realized as \( A_i = 2a_i \cdot \hat{S} \), etc., where \( a_i \) is a unit vector. Then Eq. (33) shows that generically there will be less than maximal violations of the inequalities.

Czachor considers a different spin operator, \( \hat{S} \), which is suitably normalized Pauli-Lubanski operator \( \mathbf{w} \). Then \( A_i = 2a_i \cdot \mathbf{w} \), so

\[
A_i = 2 \left[ \frac{m}{\rho^0} a_i + \left( 1 - \frac{m}{\rho^0} \right) (\mathbf{a} \cdot \mathbf{n}) \mathbf{n} \right] \cdot \mathbf{S} = 2\alpha(\mathbf{a}, \mathbf{p}) \cdot \mathbf{S},
\]

where \( \mathbf{S} \) is the Wigner spin operator and \( \mathbf{n} = \mathbf{p}/|\mathbf{p}| \). The length of the auxiliary vector \( \mathbf{a} \) is

\[
|\mathbf{a}| = \sqrt{(\mathbf{p} \cdot \mathbf{a})^2 + m^2},
\]

so we see that generically \( A_i^2 = \alpha^2 < 1 \). This provides a simple explanation of the reported in lower than maximal Einstein-Podolsky-Rosen (EPR) correlations (and, accordingly, weak or no violations of Bell-type inequalities).

This work is supported by a grant from the Technion Graduate School. It is a pleasure to acknowledge discussions with Israel Klich, Ady Mann, Asher Peres and Yoshua Zak. Comments and criticism of the anonymous referee significantly improved the presentation.

[1] A. Peres, Quantum theory, Concepts and Methods (Kluwer, Dordrecht, 1995).
[2] M. Nielsen and I. L. Chuang, Quantum Computation and Quantum information (Cambridge University Press, Cambridge, 2000).
[3] V. B. Berestetskii, E. M. Lifshits, and L. P. Pitaevskii, Quantum Electrodynamics. (Pergamon, Oxford, 1982).
[4] S. Weinberg, The Quantum Theory of Fields I (Cambridge University Press, Cambridge, 1995).
[5] Review of Particle Physics. Eur. Phys. J. C 3 (1998).
[6] I. V. Volovich, quant-ph/0012010 (2000).
[7] M. Czachor, Phys. Rev. A 55, 72 (1997).
[8] G. N. Fleming, Phys. Rev. 137 B963 (1965).
[9] S. J. van Enk and G. Nienhuis, J. Mod. Opt. 41, 963 (1994); S. J. van Enk and G. Nienhuis, Europhys. Lett. 25, 497 (1994).
[10] M. Kirchbach and D. V. Ahluwalia, Phys. Lett. B 229, 124 (2002).
[11] A. Peres and D. R. Terno, Phys. Rev. A 63, 022101 (2001).
[12] A. Peres, P. F. Scudo and D. R. Terno, Phys. Rev. Lett. 88, 230402 (2002).
[13] E. Wigner, Ann. Math. 40, 149 (1939).
[14] N. N. Bogolubov, A. A. Logunov, and I. T. Todorov, Introduction to Axiomatic Quantum Field Theory (Benjamin, New York, 1975).
[15] J. M. Jauch and F. Rohrlich, The theory of photons and electrons (Springer, New York, 1976).
[16] J. F. Clauser and M.A. Horne, Phys. Rev. D 10, 526 (1974).
[17] S. J. Summers and R. Werner, J. Math. Phys. 28, 2440 (1987); S. J. Summers, in Quantum Probability and Applications V, edited by L. Accardi and W. von Waldenfels, p. 393 (Springer, Berlin, 1990).