AN INVERSE CARTIER TRANSFORM VIA EXPONENTIAL IN POSITIVE CHARACTERISTIC

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Abstract. Let\(k\) be a perfect field of odd characteristic \(p\) and \(X_0\) a smooth connected algebraic variety over \(k\) which is assumed to be \(W_2(k)\)-liftable. In this short note we associate a flat bundle to a nilpotent Higgs bundle over \(X_0\) of exponent \(n \leq p - 1\) via the exponential function. Presumably, the association is equivalent to the inverse Cartier transform in \([12]\) for these Higgs bundles. However this point has not been verified in the note. Instead, we show the equivalence of the association with that of \([10]\) in the geometric case. The construction relies on the cocycle property of the difference of different Frobenius liftings over \(W_2(k)\), which plays the key role in the proof of \(E_1\)-degeneration of the Hodge to de Rham spectral sequence of \(X_0\) in \([3]\).

A grand theory in complex geometry is the nonabelian Hodge theory connecting flat bundles with Higgs bundles. It has been achieved after works of many people, notably Narasimhan and Seshadri, Uhlenbeck and Yau, Donaldson, Hitchin, Corlette, Simpson (see \([6], [13], [3], [7], [2], [11]\)). Recently, Ogus and Vologodsky (see \([12]\)) has established an analogue of the theory in positive characteristic. In this short note, we intend to use some rudiments in differential geometry to exhibit a construction of flat bundles for an important class of Higgs bundles, which is believed to be equivalent to the inverse Cartier transform in loc. cit.

Let \(k\) be a perfect field with odd characteristic \(p\) and \(X_0\) a smooth connected algebraic variety over \(k\) of dimension \(d\). We assume that there is a \(W_2 := W_2(k)\)-lifting of \(X_0\). Namely, there exists a smooth \(W_2\)-scheme \(X_1\) whose closed fiber is \(X_0\).

Let \((E, \theta)\) be a nilpotent Higgs bundle over \(X_0\) of exponent \(n \leq p - 1\). Locally over an open affine subset \(U \subset X_0\) with local coordinates \(\{t_1, \cdots, t_d\}\), the Higgs field \(\theta\) under a local basis of sections of \(E\) over \(U\) is written as \(\theta = \sum \theta_idt_i\), where \(\theta_i = \partial t_i \cdot \theta\) is a matrix of elements in \(\mathcal{O}_U\). The integrability of the Higgs field is then equivalent to the commuting relations \([\theta_i, \theta_j] = 0, 1 \leq i < j \leq d\), and the nilpotent condition put on the Higgs field means that \(\prod_{j=1}^{d} \theta_j^{i_j} = 0\) once \(\sum_{j=1}^{d} i_j \geq n + 1\).

Before carrying out our construction, we first recall some basics on connection and curvature that could be found in any standard book in differential geometry (see e.g. \([9]\)). Let \(H\) be a vector bundle over \(X_0\) together with a \((k)\)-connection \(\nabla : H \rightarrow H \otimes \Omega_{X_0}\).

Take a local basis \(e_U\) of sections of \(H\) over \(U\). Then over \(U\), \(\nabla = d + A_U\), where \(A_U\) is a matrix-valued one form given by \(\nabla(e_U) = A_U e_U\). More precisely, the formula means that for a local section \(s = s_U e_U\) of \(H\) over \(U\), one has \(\nabla(s) = ds_U e_U + s_U A_U e_U\). Let \(A'_U\) be the connection one form of \(\nabla\) under another local basis \(e'_U = M e_U\) of \(H\) over \(U\), where \(M\) is an invertible matrix with entries in \(\mathcal{O}_U\). Then one has the transformation formula of connection forms:

\[ A'_U = dMM^{-1} + MA_U M^{-1} \]

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Finally, the curvature $K$ of $\nabla$ is defined to the composite

$$H \xrightarrow{\nabla} H \otimes \Omega_{X_0} \xrightarrow{\nabla^i} H \otimes \Omega_{X_0}^2,$$

where $\nabla^1(h \otimes \omega) = \nabla(h) \otimes \omega + h \otimes d\omega$. It is $\mathcal{O}_{X_0}$-linear and under the local basis $e_U$ of $H$ over $U$ expressed by the formula $K_U = d^A_U + A_U \wedge A_U$. We say that the connection is integrable if its curvature is zero. A vector bundle with an integrable connection is called a flat bundle.

Now we proceed to the construction of the association. First of all we take an affine covering $\mathcal{U} = \{U'_\alpha\}$ of $X_1$. Note that over each $U'_\alpha$ we can take a lifting $F_\alpha : U'_\alpha \to U'_\alpha$ of the absolute Frobenius $F_0 : U_\alpha \to U_\alpha$ determined by the power $p$ map, where $U_\alpha$ is the closed fiber of $U'_\alpha$. The composite of $\mathcal{O}_{U'_\alpha}$-morphisms

$$F_\alpha^* \Omega_{U'_\alpha} \xrightarrow{dF_\alpha|_{\mathcal{U}_\alpha}} p\Omega_{U_\alpha} \cong \Omega_{U_\alpha}$$

descends clearly to an $\mathcal{O}_{U_\alpha}$-morphism $dF_\alpha : F_0^* \Omega_{U_\alpha} \to \Omega_{U_\alpha}$. We consider then the vector bundle $H_\alpha := F_0^* E|_{U_\alpha}$ over $U_\alpha$, where $E|_{U_\alpha}$ denotes for the restriction of $E$ over $U_\alpha$. Choose a local basis $e_\alpha$ of $E|_{U_\alpha}$ and define a connection on $H_\alpha$ by the formula

$$\nabla_\alpha = d + dF_\alpha|_{\mathcal{U}_\alpha} (F_\alpha^* \theta_\alpha),$$

where $\theta_\alpha$ is the Higgs field over $U_\alpha$ under the local basis $e_\alpha$ of $E|_{U_\alpha}$.

**Proposition 1.** The connection $\nabla_\alpha$ on $H_\alpha$ is well defined and integrable.

**Proof.** First we verify the well-definedness. Take another local basis $e'_\alpha = M e_\alpha$ of $E|_{U_\alpha}$ and put $\theta'_\alpha$ to be the Higgs field under this basis. According to the transformation formula for connection forms, we shall check that

$$\frac{dF_\alpha}{[p]} (F_0^* \theta'_\alpha) = (dF_0|M) F_0^* M^{-1} + F_0^* M \frac{dF_\alpha}{[p]} (F_0^* \theta_\alpha) F_0^* M^{-1}.$$

As $\theta'_\alpha = M \theta_\alpha M^{-1}$ and $dF_0|M = 0$, the above equality holds. Next we verify the integrability of $\nabla_\alpha$. As $\theta$ is integrable, i.e. $\theta_\alpha \wedge \theta_\alpha = 0$, it follows that $F_\alpha^* (\theta_\alpha \wedge \theta_\alpha) = F_0^* (\theta_\alpha \wedge \theta_\alpha) = 0$ and furthermore

$$\frac{dF_\alpha}{[p]} (F_0^* \theta_\alpha) \wedge \frac{dF_\alpha}{[p]} (F_0^* \theta_\alpha) = (\bigwedge^2 \frac{dF_\alpha}{[p]})(F_0^* \theta_\alpha \wedge F_0^* \theta_\alpha) = 0.$$

It is left to show that $d(\frac{dF_\alpha}{[p]} (F_0^* \theta_\alpha)) = 0$. This is done by a local computation: by definition, for $\omega \in \Omega_{U_\alpha}$, $\frac{dF_\alpha}{[p]} (F_0^* \omega) = \frac{1}{[p]} (dF_\alpha(F_0^* \omega'))$, where $\omega' \in \Omega_{U'_\alpha}$ is any lifting of $\omega$. Then

$$d \circ \frac{dF_\alpha}{[p]} (F_0^* \omega) = d \circ \frac{1}{[p]} (dF_\alpha(F_0^* \omega')) = \frac{1}{[p]} (d \circ dF_\alpha(F_0^* \omega')).$$

We may write $\omega' = \sum_i f_i d g_i$ for $f_i, g_i \in \mathcal{O}_{U'_\alpha}$. Then $d(F_\alpha(F_0^* \omega')) = \sum_i d(F_\alpha(F_0^* f_i)) \wedge d(F_\alpha(F_0^* g_i)) \in p^2 \Omega_{U'_\alpha}$. Thus $d(\frac{dF_\alpha}{[p]} (F_0^* \omega)) = 0$. Clearly, it follows that $d(\frac{dF_\alpha}{[p]} (F_0^* \theta_\alpha)) = 0$. The proposition is proved. \hfill \Box

Thus we have defined a local flat bundle $(H_\alpha, \nabla_\alpha)$ over each $U_\alpha \in \mathcal{U}$. In order to glue them into a global one, we have to provide a set of invertible matrices $\{G_{\alpha\beta}\}$ with $G_{\alpha\beta}$ defined over $U_{\alpha\beta} := U_\alpha \cap U_\beta$ satisfying

(i) the bundle gluing condition over $U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma$

$$G_{\alpha\beta} \cdot G_{\beta\gamma} = G_{\alpha\gamma},$$

$\mathcal{O}_{U_{\alpha\beta}}$. \hfill \Box
(ii) the connection gluing condition over $U_{\alpha\beta}$

$$\frac{dF_\alpha}{[p]}(F_0^*\theta_\alpha) = dG_{\alpha\beta}G_{\alpha\beta}^{-1} + G_{\alpha\beta}\frac{dF_\beta}{[p]}(F_0^*\theta_\beta)G_{\alpha\beta}^{-1}. $$

First a lemma:

**Lemma 2.** There are homomorphisms $h_{\alpha\beta} : F_0^*\Omega_{U_{\alpha\beta}} \to \mathcal{O}_{U_{\alpha\beta}}$, satisfying the following two properties:

(i) $\frac{dF_\alpha}{[p]} - \frac{dF_\beta}{[p]} = dh_{\alpha\beta}$,

(ii) the cocyle condition over $U_{\alpha\beta\gamma}$: $h_{\alpha\beta} + h_{\beta\gamma} = h_{\alpha\gamma}$.

**Proof.** Consider the $W_2$-morphism $G_\alpha : Z'_\alpha \to U'_\alpha := U'_\alpha \cap U'_{\beta}$ sitting in the following Cartesian diagram:

$$\begin{array}{ccc}
Z'_\alpha & \xrightarrow{G_\alpha} & U'_\alpha \\
\downarrow j'_\alpha & & \downarrow i'_\alpha \\
Z_\alpha & \xrightarrow{G_\alpha} & U_{\alpha\beta} \\
\downarrow j_\alpha & & \downarrow i_\alpha \\
U_\alpha & \xrightarrow{F_0} & U_{\alpha\beta}
\end{array}$$

where $i'_\alpha$ is the natural inclusion. By reduction modulo $p$, we obtain the following Cartesian square

$$\begin{array}{ccc}
Z_\alpha & \xrightarrow{G_\alpha} & U_{\alpha\beta} \\
\downarrow j_\alpha & & \downarrow i_\alpha \\
U_\alpha & \xrightarrow{F_0} & U_{\alpha\beta}
\end{array}$$

Thus we see that $Z_\alpha$ is $U_{\alpha\beta}$ and $G_\alpha : Z'_\alpha \to U'_\alpha$ is a lifting of the absolute Frobenius $F_0$ over $U_{\alpha\beta}$. Similarly for $(U'_\beta, F_\beta)$, we have $G_\beta : Z'_\beta \to U'_\beta$ which is also a lifting of $F_0 : U_{\alpha\beta} \to U_{\alpha\beta}$. Now we apply Lemma 5.4 to the pair $(G_\alpha : Z'_\alpha \to U'_\alpha, G_\beta : Z'_\beta \to U'_\beta)$ of Frobenis lifttings of the absolute Frobenius $F_0$ on $U_{\alpha\beta}$, we get the homomorphisms $h_{\alpha\beta} : F_0^*\Omega_{U_{\alpha\beta}} \to \mathcal{O}_{U_{\alpha\beta}}$ such that $\frac{dF_\alpha}{[p]} - \frac{dF_\beta}{[p]} = dh_{\alpha\beta}$ and $h_{\alpha\beta} + h_{\beta\gamma} = h_{\alpha\gamma}$. □

By the lemma, we have locally over $U_{\alpha\beta}$ a matrix of functions $h_{\alpha\beta}(F_0^*\theta_\alpha)$. Notice that, because of the assumption on the exponent of the nilpotent Higgs field $\theta$, the matrix of functions

$$g_{\alpha\beta} := \exp[h_{\alpha\beta}(F_0^*\theta_\alpha)]$$

is well defined and in fact equal to the finite sum $\sum_{i=0}^{n} (h_{\alpha\beta}(F_0^*\theta_\alpha))^i$. It is clearly invertible with the inverse $\exp[-h_{\alpha\beta}(F_0^*\theta_\alpha)]$. Furthermore, over $U_{\alpha\beta}$ we have the transition matrix $M_{\alpha\beta}$ such that $e_{\alpha} = M_{\alpha\beta}e_{\beta}$. As $E$ is globally defined, one has the cocycle condition for $\{M_{\alpha\beta}\}$:

$$M_{\alpha\beta} \cdot M_{\beta\gamma} = M_{\alpha\gamma}. $$

Now we define our gluing matrices for $\{H_\alpha\}_{\alpha \in \mathcal{U}}$ by $G_{\alpha\beta} := g_{\alpha\beta} \cdot F_0^*M_{\alpha\beta}$. We have then the following:

**Theorem 3.** The local flat bundles $\{(H_\alpha, \nabla_\alpha)\}_{\alpha \in \mathcal{U}}$ glue into a global flat bundle $(H, \nabla)$ via the matrices $\{G_{\alpha\beta}\}$.

**Proof.** The proof is divided into two steps.

**Step 1: Bundle gluing.** It is to show the cocycle condition

$$G_{\alpha\beta} \cdot G_{\beta\gamma} = G_{\alpha\gamma}. $$
It is a direct computation. First of all,
\[
g_{\beta\gamma} = \exp[h_{\beta\gamma}(F_0^*\theta_\beta)] \\
= \exp[h_{\beta\gamma}(F_0^*(M^{-1}_{\alpha\beta}\theta_\alpha M_{\alpha\beta}))] \\
= \exp[F_0^*M^{-1}_{\alpha\beta}h_{\beta\gamma}(F_0^*\theta_\alpha)F_0^*M_{\alpha\beta}] \\
= F_0^*M^{-1}_{\alpha\beta}exp[h_{\beta\gamma}(F_0^*\theta_\alpha)]F_0^*M_{\alpha\beta}.
\]

So it follows that
\[
G_{\alpha\beta} \cdot G_{\beta\gamma} = g_{\alpha\beta}F_0^*M_{\alpha\beta} \cdot g_{\beta\gamma}F_0^*M_{\beta\gamma} \\
= \exp[h_{\alpha\beta}(F_0^*\theta_\alpha)]\exp[h_{\beta\gamma}(F_0^*\theta_\alpha)]F_0^*(M_{\alpha\beta} \cdot M_{\beta\gamma}) \\
= \exp[h_{\alpha\gamma}(F_0^*\theta_\alpha)]F_0^*M_{\alpha\gamma}.
\]

It follows from the integrability of Higgs field that the two matrices $h_{\alpha\beta}(F_0^*\theta_\alpha)$ and $h_{\beta\gamma}(F_0^*\theta_\alpha)$ commute with each other. Thus we compute further that
\[
G_{\alpha\beta} \cdot G_{\beta\gamma} = \exp[(h_{\alpha\beta} + h_{\beta\gamma}) (F_0^*\theta_\alpha)]F_0^*M_{\alpha\gamma} \\
= \exp[h_{\alpha\gamma}(F_0^*\theta_\alpha)]F_0^*M_{\alpha\gamma} \\
= G_{\alpha\gamma}.
\]

The second equality follows from Lemma 2 (ii).

\textit{Step 2: Connection gluing.} It is to show that the local connections $\{\nabla_{\alpha}\}$ coincide on the overlaps. Recall that by definition $\nabla_{\alpha}(F_0^*e_\alpha) = \frac{dF_0}{|p|}(F_0^*\theta_\alpha)F_0^*e_\alpha$. Over $U_{\alpha\beta}$, it holds that
\[
\nabla_{\beta}(F_0^*e_\alpha) = \nabla_{\beta}(G_{\alpha\beta}F_0^*e_\beta) \\
= dG_{\alpha\beta}F_0^*e_\beta + G_{\alpha\beta}\nabla_{\beta}(F_0^*e_\beta) \\
= [dG_{\alpha\beta}F_0^*e_\beta + G_{\alpha\beta}\frac{dF_0}{|p|}(F_0^*\theta_\beta)G^{-1}_{\alpha\beta}(F_0^*e_\alpha)].
\]

As $d(F_0^*M_{\alpha\beta}) = 0$ and $\theta_\beta = M^{-1}_{\alpha\beta}\theta_\alpha M_{\alpha\beta}$, we have further
\[
\nabla_{\beta}(F_0^*e_\alpha) = [dg_{\alpha\beta}g_{\alpha\beta}^{-1} + g_{\alpha\beta}\frac{dF_0}{|p|}(F_0^*\theta_\alpha)g_{\alpha\beta}^{-1}]F_0^*e_\alpha \\
= [dh_{\alpha\beta}(F_0^*\theta_\alpha) + \frac{dF_0}{|p|}(F_0^*\theta_\alpha)]F_0^*e_\alpha \\
= \frac{dF_0}{|p|}(F_0^*\theta_\alpha)F_0^*e_\alpha \\
= \nabla_{\alpha}(F_0^*e_\alpha).
\]

Here the second equality follows from the fact that $g_{\alpha\beta}$ commutes with $\frac{dF_0}{|p|}(F_0^*\theta_\alpha)$ which is again because of the integrability of the Higgs field, and the third equality uses Lemma 2 (i).

Presumably, the association is equivalent to the inverse Cartier transform in \[12\]. This has not been verified at the moment. However, we shall show that it is equivalent to that of \[10\] in the geometric case. For sake of convenience, we recall briefly the setup loc. cit. Let $(H, \nabla, Fil, \Phi)$ be an object in the category $\mathcal{M}F_{[0,n]}(X)$ with $n \leq p - 2$, where $X$ is a smooth scheme over $W_{n+1}$. Let $(E, \theta) = Gr_{Fil}(H, \nabla)$ be the associated Higgs bundle and $(H, \nabla)_0$ (resp. $(E, \theta)_0$) the flat bundle (resp. the Higgs bundle) in characteristic $p$. In loc. cit., a flat subbundle $(H_{(G, \theta)}, \nabla)$ of $(H, \nabla)_0$ is associated to any Higgs subbundle $(G, \theta)$ of $(E, \theta)$. To distinguish the notations, we denote by $(H_{\text{exp}}, \nabla_{\text{exp}})$ the flat bundle by applying the previous construction to $(G, \theta)$. We claim the following

\textbf{Proposition 4.} Notation as above. Then there is a natural isomorphism of flat bundles $(H_{\text{exp}}, \nabla_{\text{exp}}) \cong (H_{(G, \theta)}, \nabla)$ induced by the relative Frobenius.
For simplicity of notation, we denote simply by \( a \) multi-index \( U \) are going to show that over \( U \), so it suffices to show \( \Phi \). By Proposition 2.2 loc. cit. this isomorphism does not depend on the choice of liftings \( \tilde{F}_a \). Since \( H(G,\theta)|U_\alpha \) is defined to be \( \Phi_{F_\alpha}(F_0^*G|U_\alpha) \) by loc. cit., \( \Phi_{F_\alpha} \) induces an isomorphism from \( H_\alpha \) to \( H(G,\theta)|U_\alpha \), which by abuse of notation is denoted by \( \Phi_{F_\alpha} \).

**Step 1: Bundle isomorphism.** By choosing a local basis \( e \) of sections of \( G \) over \( U_{\alpha\beta} \), we are going to show that over \( U_{\alpha\beta} \),

\[
\Phi_{F_\alpha}(F_0^*e) = g_{\alpha\beta} \Phi_{F_\beta}(F_0^*e).
\]

For simplicity of notation, we denote simply by \( \theta \) the Higgs field under the basis \( e \). For a multi-index \( j = (j_1, \ldots, j_d) \), we put

\[
\theta^j = (\partial t_{j_1} \theta) \cdots (\partial t_{j_d} \theta)^{j_d}, \quad z_l = \left( \frac{F_\alpha - F_\beta}{|p|} \right) (F_0^*t_l), \quad z_l^j = \prod_{i=1}^d z_l^{j_i}.
\]

(Notice that \( z_l \) and \( z_l^j \) differ from the notation loc. cit..) According to Lemma 2.4 loc. cit., we have by abuse of notation

\[
\Phi_{F_\alpha}(F_0^*e) = \left( 1 + \sum_{|j|=1}^n F_0^* (\theta^j) \cdot \frac{z_l^j}{j!} \right) \cdot \Phi_{F_\beta}(F_0^*e).
\]

So it suffices to show \( g_{\alpha\beta} = 1 + \sum_{|j|=1}^n F_0^* (\theta^j) \cdot \frac{z_l^j}{j!} \). As

\[
h_{\alpha\beta}(F_0^*\theta) = \sum_{i=1}^d F_0^* (\partial t_{i\cdot} \theta) h_{\alpha\beta}(F_0^* dt_l)
\]

and

\[
h_{\alpha\beta}(F_0^* dt_l) = \left( \frac{F_\alpha - F_\beta}{|p|} \right) (F_0^* t_l) = z_l, \text{ it follows that }
\]

\[
\frac{h_{\alpha\beta}(F_0^*\theta)^i}{i!} = \left( \sum_{|j|=i}^d F_0^* (\partial t_{j\cdot} \theta) z_l^j \right)^i = \sum_{|j|=i}^d F_0^* (\theta^j) z_l^j.
\]

Recall that \( g_{\alpha\beta} = \sum_{i=0}^n \frac{(h_{\alpha\beta}(F_0^*\theta))^i}{i!} \), it follows then \( g_{\alpha\beta} = 1 + \sum_{|j|=1}^n F_0^* (\theta^j) \cdot \frac{z_l^j}{j!} \) as wanted.

**Step 2: Connection isomorphism.** We need to show that under the above isomorphism, the connection \( \nabla_{\exp} \) on \( H_{\exp} \) is equal to the connection \( \nabla \) on \( H(G,\theta) \). Put \( e_\alpha \) to be a local basis of \( G \) over \( U_\alpha \). By Lemma 2.5 and the proof of Proposition 2.6 loc. cit., we have

\[
\nabla[\Phi_{F_\alpha}(F_0^*e_\alpha)] = \sum_{l=1}^d F_0^* \theta_l \frac{dF_\alpha}{|p|} (F_0^* dt_l) \cdot \Phi_{F_\alpha}(F_0^*e_\alpha)
\]

As \( \nabla_{\exp}(F_0^*e_\alpha) = \sum_{l=1}^d F_0^* \theta_l \cdot \frac{dF_\alpha}{|p|} (F_0^* dt_l) \cdot F_0^* e_\alpha \), it follows that

\[
\Phi_{F_\alpha}(\nabla_{\exp}(F_0^*e_\alpha)) = \sum_{l=1}^d F_0^* \theta_l \frac{dF_\alpha}{|p|} (F_0^* dt_l) \cdot \Phi_{F_\alpha}(F_0^*e_\alpha) = \nabla[\Phi_{F_\alpha}(F_0^*e_\alpha)].
\]

\(\square\)
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