A Local Central Limit Theorem and Loss of Rotational Symmetry of Planar Simple Random Walk

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Abstract

We use elementary methods to derive a sharp local central limit theorem for simple random walk on the square lattice in dimensions one and two that is an improvement of existing results for points that are particularly distant from the starting point. More specifically, we give explicit asymptotic expressions in terms of $n$ and $x$ in dimensions one and two for $P(S(n) = x)$, the probability that simple random walk $S$ is at some point $x$ at time $n$, that are valid for all $|x| \leq n/\log^2(n)$. We also show that the behavior of planar simple random walk differs radically from that of planar standard Brownian motion outside of the disk of radius $n^{3/4}$, where simple random walk ceases to be approximately rotationally symmetric. Indeed, if $n^{3/4} = o(|S(n)|)$, $S(n)$ is more likely to be found along the coordinate axes. This loss of rotational symmetry is not surprising, since if $|S(n)| = n$, there are only four possible locations for $S(n)$. In this paper, we show how the transition from approximate rotational symmetry to complete concentration of $S$ along the coordinate axes occurs.

1 Introduction and Statement of Results

Simple random walk $S$ in dimension $d \in \mathbb{N}$ is defined by $S(0) = 0$ and for $n \in \mathbb{N}$, by $S(n) = \sum_{k=1}^{n} X_k$, where $\{X_k\}_{k \in \mathbb{N}}$ are independent random vectors satisfying $\mathbb{P}\{X_k = \pm e_i\} = \frac{1}{2d}, i = 1, \ldots, d$ and $\{e_i\}_{i \in \{1, \ldots, d\}}$ is the standard orthonormal basis of $\mathbb{R}^d$.

Donsker’s invariance principle (see [6]) tells us that a large class of rescaled random walks converge in distribution to standard Brownian motion. This suggests that Brownian motion and random walk have similar behavior “at large scales”. This is in many cases a good way of thinking about these processes and one can show using a number of methods that random walk and Brownian motion share many properties.

While the transition density for $d$-dimensional Brownian motion has the well-known exact expression $p_t(x) = (2\pi t)^{-d/2}e^{-|x|^2/2t}$, where $x \in \mathbb{R}^d, t \in \mathbb{R}$, and $|\cdot|$ denotes $d$-dimensional Euclidean norm (see for instance [9]), the expression for the probability $P(S(n) = x)$ that $d$-dimensional simple random walk $S$ started at the origin is at a given location $x \in \mathbb{Z}^d$ at time $n \in \mathbb{N}$ cannot immediately be expressed as a convenient function of $x$ and $n$. Coupling arguments (see for instance Section 3.5 in [8]) suggest that this probability is more or less the same as that of Brownian motion being in a ball centered at $x \in \mathbb{Z}^d$, of volume 2 (the factor 2 accounts for the fact that simple random walk has period 2) at time $n/d$, which is approximately $2(d/2\pi n)^{d/2} \exp\{-d|x|^2/2n\}$. As the reader can see below, this argument is correct for points in and somewhat beyond the typical range of $S(n)$, that is, the disk of radius $\sqrt{n}$. However, it fails for a large set of points that are attainable by $S(n)$.

Local central limit theorems estimating $P(S(n) = x)$ have been obtained by numerous authors and for very general classes of random walks $S$. This probability is closely related to the large
deviations probability $P(|S(n)| \geq r), r \in \mathbb{R}_+$, as a key quantity in both of them is the large deviations rate function (also called the deviation function). For an extensive survey, see [2], in particular Chapter 6 for the case discussed in this paper. Some of the local central limit theorems available in the literature (as in [2], [5],[3], and [4]) are very powerful, as they apply to large classes of random walks and are valid for all points in the range of the random walk. However, they are abstract results from which the rate function is difficult to extract and to express explicitly in terms of $x$ and $n$, particularly in dimensions greater than 1. Other local central limit theorems (see [11], [7], and [8]) give explicit asymptotic expressions but only for points within and a bit beyond the typical range of the random walks.

The aim of this paper is to reconcile the advantages of the two types of results mentioned above in the simple random walk case by providing explicit asymptotic expressions for $P(S(n) = x)$ in dimensions 1 and 2 for more or less all $x$ with $|x| = o(n)$. We point out that our one-dimensional result, Theorem 1.3, can also be obtained from Theorem 6.1.6 in [2] by computing the rate function explicitly. However, in the two-dimensional case, it appears to be considerably more difficult to derive this paper’s local central limit theorem, Theorem 1.4, from the multi-dimensional analogue (see [3]) of Theorem 6.1.6 in [2]. Theorem 1.4 can be used to show that in dimension 2, an interesting phenomenon arises: For points $x \in \mathbb{Z}^2$ with $|x| >> n^{3/4}$, some points along a same discrete circle are much more likely to be hit by $S$ than others. More precisely, for any $r = r(n) \in \mathbb{R}$ with $n^{3/4} = o(r)$, there exist points $x_1, x_2 \in \mathbb{Z}^2$ with $|x_1| \in [r, r+1]$ such that $\lim_{n \to \infty} P(S(n) = x_1)/P(S(n) = x_2) = 0$. In other words, while simple random walk $S[0, n]$ is essentially rotationally symmetric (we make this precise below) in the disk of radius $n^{3/4}$, it loses this symmetry outside of that disk. This is the content of Corollary 1.1 below.

For the purpose of comparison with the results obtained in this paper, we state two result from [7] and [8] which, to our knowledge, are the strongest “explicit” local central limit theorems in the literature. The first result is weaker but valid in all dimensions, while the second is stronger, but only valid in dimension 1.

**Theorem 1.1.** If $S$ is a $d$-dimensional simple random walk, $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$, $\sum_{i=1}^d x_i + n$ is even and $\tilde{p}_n(x) = 2 \left( \frac{d}{2\pi n} \right)^{d/2} \exp\left(-\frac{d|x|^2}{2n}\right)$, then

$$|P(S(n) = x) - \tilde{p}_n(x)| \leq O \left(n^{-d/2}\right), P(S(n) = x_i) - \tilde{p}_n(x) \leq |x^{-2}|O \left(n^{-d/2}\right).$$

This theorem implies that there exists a constant $C = C_d > 0$ such that for all $x \in \mathbb{Z}^d$ with $|x| \leq \sqrt{n}$ and $\sum_{i=1}^n x_i$ even,

$$C^{-1}n^{-d/2} \leq P(S(n) = x) \leq Cn^{-d/2}.$$

However, if $|x|$ is of larger order of magnitude than $\sqrt{n}$, the error term is greater than $\tilde{p}_n(x)$, so Theorem 1.1 cannot be used for such $x$. In [8], a sharper version of this result is given in the one-dimensional case:

**Theorem 1.2.** For simple random walk in $\mathbb{Z}$, if $n \in \mathbb{N}$ and $x \in \mathbb{Z}$ with $|x| \leq n$ and $x + n$ even,

$$P(S(n) = x) = \sqrt{\frac{2}{\pi n}} e^{-x^2/2n} \exp\left(O \left(\frac{1}{n} + \frac{x^4}{n^3}\right)\right).$$

(1)

In particular, if $|x| \leq n^{3/4}$, then

$$P(S(n) = x) = \sqrt{\frac{2}{\pi n}} e^{-x^2/2n} \left(1 + O \left(\frac{1}{n} + \frac{x^4}{n^3}\right)\right).$$
This theorem gives asymptotics for \( P(S(n) = x) \) for points such that \( x = o(n^{3/4}) \). However, if \( |x| \gg n^{3/4} \), the multiplicative error term in (1) doesn’t go to 1, so the function \( (2/\pi n)^{1/2} e^{-x^2/2n} \) ceases to describe the asymptotic behavior of \( P(S(n) = x) \) for such points. The present paper extends these results to essentially all points \( x \) with \( |x| = o(n) \) in dimensions 1 and 2. In the one-dimensional case, we obtain exact asymptotics for all such points. In the two-dimensional case, we also obtain exact asymptotics for points \((x, y)\) with \( |y^2 - x^2| \leq n^{3/2}/\log n \) — in particular, this includes all points inside the disk of radius \( n^{3/4}/\log^{1/2} n \), but only logarithmic asymptotics for other points which satisfy \(|(x, y)| = o(n)\).

Throughout this paper, the notation \( f(n) = O(g(n)) \) will mean that there is a universal constant \( C \) such that \( f(n) \leq C g(n) \) for all \( n \). If the constant \( C \) depends on some other quantity, this will be made explicit inside the \( O(\cdot) \). We will write \( f(n) \sim g(n) \) if \( \lim_{n \to \infty} f(n)/g(n) = 1 \) and \( f(n) \approx g(n) \) if \( \lim_{n \to \infty} \log f(n)/\log g(n) = 1 \). For notational simplicity, multiplicative constants will generally be denoted by \( C \). They may be different from one line to the next.

The first result of this paper is a one-dimensional local central limit theorem which gives asymptotics for \( P(S(n) = x) \) which are exact when \( \lim_{n \to \infty} \frac{x}{n} < 1 \) and up to a multiplicative constant for which we provide bounds when \( \lim_{n \to \infty} \frac{x}{n} = 1 \):

**Theorem 1.3.** If \( S \) is one-dimensional simple random walk, then for every \( n \in \mathbb{N}, x \in \mathbb{Z} \) with \( |x| \leq n \) and \( x + n \) even,

\[
P(S(n) = x) = \exp \left( \phi(n, x) \right) \cdot \begin{cases} 
\sqrt{\frac{2}{\pi n}} \left( 1 + O \left( \frac{x^2}{n^2} + \frac{1}{n} \right) \right), & x = \ell(n) \\
\sqrt{\frac{2}{\pi n}} \frac{1}{1-a} \left( 1 + O \left( \frac{\ell(n)}{n(1-a^2)} \right) \right), & x = an + \ell(n), \ 0 < a < 1, \\
\frac{c_x(n)}{\sqrt{\pi \ell(n)}} \left( 1 + O \left( \frac{\ell(n)}{n} \right) \right), & x = n - \ell(n), \ell(n) > 0, \\
1, & x = n,
\end{cases}
\]

where \( \ell(n) \) is any integer-valued function satisfying \( \ell(n) = o(n) \),

\[
\phi(n, x) = -\sum_{\ell=1}^{\infty} \frac{1}{2\ell(2\ell - 1)} \frac{x^{2\ell}}{n^{2\ell-1}},
\]

\( c_x(n) \in [e^{-10/39}, 1] \), and the constants in the \( O(\cdot) \) terms are universal. Moreover, for all \( n \geq 1, \ |x| < n \),

\[
P(S(n) = x) \leq \sqrt{\frac{2}{\pi}} \exp \left( \phi(n, x) \right).
\]

In particular, for every \( n \in \mathbb{N} \), every \( N \geq 2, a < 2N - 1 \), and \( |x| \leq n^a \),

\[
P(S(n) = x) = \sqrt{\frac{2}{\pi n}} \exp \left( -\sum_{\ell=1}^{N-1} \frac{1}{2\ell(2\ell - 1)} \frac{x^{2\ell}}{n^{2\ell-1}} \right) \left( 1 + O \left( \frac{x^{2N}}{n^{2N-1}} + \frac{1}{n} + \frac{x^2}{n^2} \right) \right).
\]

As mentioned in the introduction, one can verify that the sum in (2) is equal to \( n\Lambda(x/n) \) where \( \Lambda \) is the rate function for simple random walk.

One interesting consequence of Theorem 1.3 is that for one-dimensional simple random walk, the exponent in the probability \( P(S(n) = x) \) is the same as for the corresponding Brownian motion probability of being in a ball of volume 2 (recall that the ball is taken to have volume 2 to take into account the periodicity of the random walk) around \( x \), namely \(-|x|^2/2\), as long as \( |x| = O(n^{3/4}) \).

However as soon as \( |x| \gg n^{3/4} \), the random walk probability has an additional exponential term
which makes the random walk probability smaller than the corresponding Brownian motion probability. This should of course not be surprising, since for $x$ with $|x| > n$, $P(S(n) = x) = 0$, while $P(B(n) = x) > 0$. Theorem 1.3 describes precisely the transition from the exponent $-|x|^2/2$ to a 0 probability as the magnitude of $x$ goes from $n^{3/4}$ to $n$. In particular, (3) gives the exact asymptotic behavior of $P(S(n) = x)$ for all $x \leq n^{a}$, $a < 1$.

This difference in behavior between planar simple random walk and standard Brownian motion in $\mathbb{R}^2$ is also true, though another interesting phenomenon arises in that case: If $|x| >> n^{3/4}$, the asymptotic behavior $P(S(n) = x)$ depends on the location of $x$, not just on $|x|$:

**Theorem 1.4.** If $S$ is planar simple random walk, then for all $(x, y) \in \mathbb{Z}^2$, $n \in \mathbb{N}$ with $x + y + n$ even, $x^2 + y^2 \leq n^2/\log^2(n)$, and $|y^2 - x^2| \leq n^{3/2}/\log(n)$, we have

$$P(S(n) = (x, y)) = \frac{4}{\pi n} e^{-s_0} \left(1 + O\left(\frac{1}{\log^2(n)}\right)\right),$$

where

$$s_0 = \sum_{\ell \geq 1} \frac{1}{4\ell(2\ell - 1)} \frac{(2x)^{2\ell} + (2y)^{2\ell}}{n^{2\ell - 1}}.$$

In particular, for all $n \in \mathbb{N}$, $N \geq 2$, $a < (2N - 1)/2N$, and $(x, y)$ such that $\sqrt{x^2 + y^2} \leq n^a$ and $|y^2 - x^2| \leq n^{3/2}/\log(n)$, we have

$$P(S(n) = (x, y)) = \frac{4}{\pi n} e^{-\sum_{\ell = 1}^{N-1} \frac{1}{\ell(2\ell - 1)} \frac{(2x)^{2\ell} + (2y)^{2\ell}}{n^{2\ell - 1}}} \left(1 + O\left(\frac{1}{\log^2(n)}\right) + O\left(\frac{(x^2 + y^2)^N}{N^{2N-1}}\right)\right).$$

Moreover, for all $(x, y) \in \mathbb{Z}^2$ with $x^2 + y^2 = o(n^2)$, we have

$$P(S(n) = (x, y)) \approx \exp\left\{-\sum_{\ell \geq 1} \frac{1}{2\ell(2\ell - 1)} \frac{(y^2 - x^2)^{2\ell}}{n^{4\ell - 1}} - \sum_{k \geq 0} \left(-\frac{|y^2 - x^2|}{2n}\right)^k |s_k|\right\},$$

where for $k \in \mathbb{N} \cup \{0\}$,

$$s_k = s_k(x, y, n) = \sum_{\ell \geq 1} \frac{2^k}{4\ell(2\ell - 1)} \left(1 - \frac{2\ell}{k}\right) \frac{(-1)^k(2y)^{2\ell} + (2x)^{2\ell}}{n^{2\ell + k - 1}}.$$

We will say that planar simple random walk is approximately rotationally symmetric in the disk of radius $r = r(n)$ if for all $s \leq r$ and all points $x, x'$ with $s - 1 \leq |x|, |x'| \leq s$, we have $P(S(n) = x) \sim P(S(n) = x')$ as $n \to \infty$. Donsker’s invariance principle says that if $S$ is planar simple random walk interpolated linearly between integer times and for $n \in \mathbb{N}$, $0 \leq t \leq 1$, one defines $\hat{S}_n(t) = 1/\sqrt{n} S(2nt)$, then the sequence $\hat{S}_n$ converges weakly to planar standard Brownian motion $\{B(t) : 0 \leq t \leq 1\}$ on $\mathcal{C}[0, 1]$. Corollary 1.1 shows that this principle misses some of the subtle differences between $B$ and $S$. Indeed, on the rare events where $\hat{S}$ goes beyond the circle of radius $n^{1/4}$, the distribution of the paths of $\hat{S}$ differs radically from that of its scaling limit, since one process is rotationally symmetric, while the other is much more likely to be along the coordinate axes than on the diagonals of $(1/\sqrt{n})\mathbb{Z}^2$:

**Corollary 1.1.** Planar simple random walk is approximately rotationally symmetric in the disk of radius $n^{3/4}/(\log n)^{1/2}$. However, for every $r = r(n)$ such that $n^{3/4} = o(r)$ and $r \leq n/\log^2(n)$, there are points $x, x' \in \mathbb{Z}^2$ with $|x|, |x'| \in [r - 1, r + 1]$, but

$$\lim_{n \to \infty} \frac{P(S(n) = x)}{P(S(n) = x')} = 0.$$
To prove (6), suppose \( n \) now concludes the proof.

First suppose a term.

Proof. First suppose \( x = (x_1, y_1) \) and \( x' = (x_2, y_2) \) satisfy \( |x| = s \leq n^{3/4}/(\log n)^{1/2} \) and \( |x'| \in [s - 1, s] \). Then by (4),

\[
P(S(n) = (x_1, y_1)) = 1 + O\left(\frac{1}{\log^2(n)}\right),
\]

which implies that \( S \) is approximately rotationally symmetric in the disk of radius \( n^{3/4}/(\log n)^{1/2} \).

To prove (6), suppose \( n^{3/4} = o(r) \) and assume that \( r \in \mathbb{N} \). Then, by (5),

\[
P(S(n) = (r, 0)) \approx \exp \left\{ -s_0 - \sum_{\ell \geq 1} \frac{1}{2\ell(2\ell - 1)} \frac{r^{4\ell}}{n^{4\ell - 1}} - \sum_{k \geq 1} \left( -\frac{r^2}{2n} \right)^k s_k \right\},
\]

where for \( k \in \mathbb{N} \cup \{0\},

\[
s_k = s_k(r, n) = \sum_{\ell \geq 1} \frac{2^k}{4\ell(2\ell - 1)} \frac{1 - 2\ell}{k} (-1)^k \frac{(2r)^{2\ell}}{n^{2\ell+k-1}}.
\]

Since \( r = o(n) \), for any \( k \in \mathbb{N} \cup \{0\}, s_k = \frac{2^k}{4\ell(2\ell - 1)} \frac{1 - 2\ell}{k} (-1)^k \frac{(2r)^{2\ell}}{n^{2\ell+k-1}} + O\left(\frac{r^4}{n^{4+k-1}}\right) \). Assembling the two highest-order terms in (7), we obtain

\[
P(S(n) = (r, 0)) \approx \exp \left\{ -r^2/n - r^4/6n^3 + O\left(\frac{r^6}{n^5}\right) \right\}.
\]

Now we can note that one of the points \( \left(\left\lfloor \frac{r}{\sqrt{2}} \right\rfloor, \left\lfloor \frac{r}{\sqrt{2}} \right\rfloor \right) \) and \( \left(\left\lceil \frac{r}{\sqrt{2}} \right\rceil, \left\lceil \frac{r}{\sqrt{2}} \right\rceil \right) \) is at a distance from the origin which is in the interval \([r - 1, r + 1]\). By (4),

\[
P\left(S(n) = \left(\left\lfloor \frac{r}{\sqrt{2}} \right\rfloor, \left\lfloor \frac{r}{\sqrt{2}} \right\rfloor \right)\right) \approx \exp\{-r^2/n - r^4/3n^3 + O\left(\frac{r^6}{n^5}\right)\}
\]

and the same holds for \( P\left(S(n) = \left(\left\lceil \frac{r}{\sqrt{2}} \right\rceil, \left\lceil \frac{r}{\sqrt{2}} \right\rceil \right)\right) \). Taking the ratio of the expressions in (8) and (9) now concludes the proof.

\[
\square
\]
2 Proof of Theorem 1.3

The proof of Theorem 1.3 is purely combinatorial and uses nothing more than Stirling’s formula with an error estimate and Taylor’s theorem. We begin by stating the three lemmas needed in our proof. We will use the following version of Stirling’s formula (see [10] for a proof):

Lemma 2.1. For all \( n \in \mathbb{N} \),

\[
  n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n e^{r_n}
\]

with \( 1/(2n + 1) < r_n < 1/12n \).

The following Lemma will prove to be useful when trying to obtain precise estimates for the factorial terms appearing in the expression for \( P(S(n) = x) \). Recall that \( \ell(n) \) represents any integer-valued function such that \( \ell(n) = o(n) \).

Lemma 2.2. If \( \alpha \) is a function such that for all \( n \in \mathbb{N} \), \( e^{1/(12n+1)} < \alpha(n) < e^{1/12n} \), then for all \( n \in \mathbb{N}, x \in \mathbb{Z} \) with \( |x| < n \),

\[
  \beta_x(n) := \frac{\alpha(n)}{\alpha((n + x)/2)\alpha((n - x)/2)} = \begin{cases} 1 - \frac{1}{4n} + \mathcal{O}\left(\frac{x^2}{n^2} + \frac{1}{n}\right), & \text{if } x = \ell(n) \geq 0 \\ 1 - \frac{1}{n} \left(\frac{3+a^2}{12(1-a^2)}\right) + \mathcal{O}\left(\frac{\ell(n)}{(1-a^2)^2 n^2}\right), & \text{if } x = an + \ell(n), 0 < a < 1. \end{cases}
\]

Moreover, for all \( n \in \mathbb{N}, x \in \mathbb{Z} \) with \( |x| < n \), \( \beta_x(n) \in [e^{-10/39}, 1] \).

Proof. The proof is a bit tedious but straightforward, so we just outline it: The asymptotic behavior of \( \beta_x(n) \) when \( x/n \sim c < 1 \) is obtained by computing upper and lower bounds for \( \beta_x(n) \) and directly substituting the appropriate expressions for \( x \). Moreover, \( \beta_x(n) \leq \exp\{1/12n - 1/(6(n + x) + 1) - 1/(6(n - x) + 1)\} \leq 1 \), since the expression in the exponent is \( \leq 0 \). On the other hand, \( \beta_x(n) \geq \exp\{1/12n + 1 - 1/(6(n + x) - 1/6(n - x))\} \), which can be shown to be minimal if \( x = 0 \) and \( n = 1 \) and equal to \( e^{-10/39} \) in that case.

The last lemma we need in the proof of Theorem 1.3 is an easy application of Taylor’s theorem. Some details of this straightforward calculation can be found in [1].

Lemma 2.3. For all \( z \in \mathbb{R} \) with \( |z| < 1 \),

\[
  \log(1 + z) + \log(1 - z) + z (\log(1 + z) - \log(1 - z)) = \sum_{\ell \geq 1} \frac{z^{2\ell}}{2\ell(2\ell - 1)}.
\]

Moreover,

\[
  \sum_{\ell \geq 1} \frac{1}{2\ell(2\ell - 1)} = \log 2. \tag{10}
\]

Proof of Theorem 1.3. Assume \( x \sim n \). \( \mathbb{P}\{S(n) = \pm n\} = \left(\frac{1}{2}\right)^n \) and for \( |x| < n \), Lemma 2.1 yields

\[
  P(S(n) = x) = \frac{\sqrt{2\pi n}}{\sqrt{(n+x)\pi}\sqrt{(n-x)\pi}} \frac{1}{\Gamma\left(\frac{n+x}{2}\right)\Gamma\left(\frac{n-x}{2}\right)} \frac{(n/e)^n}{\Gamma\left(\frac{n}{2}\right)} \beta_x(n)
\]

\[
  = \frac{\sqrt{2\pi n}}{\sqrt{(n+x)\pi}\sqrt{(n-x)\pi}} \exp(\phi(n,x)) \beta_x(n), \tag{11}
\]
where \( \phi(n, x) = \log \left( \frac{n^n}{(n+x)^{n+x}/2(n-x)^{n-x}/2} \right) \) and \( \beta \) is as defined in Lemma 2.2. Lemma 2.3 gives, for \( |x| < n \),

\[
\phi(n, x) = n \log n - n \frac{x}{2} \log(n + x) - n \frac{x}{2} \log(n - x)
\]

\[
= -n \frac{x}{2} \left( \log(1 + \frac{x}{n}) + \log(1 - \frac{x}{n}) \right)
\]

\[
= -n \frac{x}{2} \left( \log(1 + \frac{x}{n}) + \frac{x}{n} \left( \log(1 + \frac{x}{n}) - \log(1 - \frac{x}{n}) \right) \right)
\]

\[
= -\sum_{\ell=1}^{\infty} \frac{1}{2\ell(2\ell-1)} x^{2\ell} (\frac{n}{2})^{2\ell-1}.
\]

(12)

Also, one can easily see that

\[
\sqrt{2\pi n} \sqrt{n+x} \frac{\pi}{\sqrt{(n-x)}} = \sqrt{\frac{2}{\pi}} \sqrt{\frac{1}{n(1-x^2/n^2)}}
\]

\[
= \begin{cases} 
\sqrt{\frac{2}{\pi n}} \left( 1 + O \left( \frac{x^2}{n^2} \right) \right), & x = \ell(n) \\
\sqrt{\frac{2}{\pi n}} \sqrt{1-x^2} \left( 1 + O \left( \frac{\ell(n)}{1-a^2} \right) \right), & x = an + \ell(n), 0 < a < 1 \\
\frac{1}{\sqrt{\pi n}} \left( 1 + O \left( \frac{\ell(n)}{n} \right) \right), & x = n - \ell(n), \ell(n) > 0
\end{cases}
\]

(13)

and that for all \( n \geq 1 \) and \( x \) with \( |x| \leq n - 1 \),

\[
\sqrt{2\pi n} \sqrt{n+x} \frac{\pi}{\sqrt{(n-x)}} \leq \sqrt{\frac{2}{\pi}}.
\]

Using Lemma 2.2, (12), and (13) to rewrite (11) and noting that (10) yields \( \exp(\phi(n, n)) = (\frac{1}{2})^{2n} \), concludes the proof of the theorem.

\( \square \)

3 Proof of Theorem 1.4

In this section, \( S \) will denote planar simple random walk. In order to prove Theorem 1.4, we will need the following standard estimate (see [12], Section 14.8):

Lemma 3.1. If \( \phi \) is the standard normal density function,

\[
\int_{c}^{\infty} \phi(x) \, dx = \frac{1}{c} \phi(c) \left( 1 + O \left( \frac{1}{c^2} \right) \right).
\]

Proof of Theorem 1.4. In order to estimate \( P(S(n) = (x, y)) \) when \( (x, y) \in \mathbb{Z}^2, x + y \equiv n \text{(mod 2)}, \) and \( x^2 + y^2 \leq n^2 / \log^4 n \), we will assume without loss of generality that \( 0 \leq x \leq y \leq n / \log^2 n \). We let \( N_1 \) be the number of steps taken by \( S[0, n] \) in the horizontal direction and write \( S(n) = (S^{(1)}(n), S^{(2)}(n)) \). Then for \( (x, y) \in \mathbb{Z}^2 \),

\[
P(S(n) = (x, y)) = \sum_{j=-[n/2]+x}^{[n/2]-y} P_{n,x,y,j}, \quad (14)
\]
where for $-\lfloor \frac{n}{2} \rfloor \leq j \leq \lfloor \frac{n}{2} \rfloor$,

$$P_{n,x,y,j} = P(N_1 = \lfloor \frac{n}{2} \rfloor + j) P(S^{(1)}(\lfloor \frac{n}{2} \rfloor + j) = x) P(S^{(2)}(\lfloor \frac{n}{2} \rfloor - j) = y).$$

One can show as in the proof of Theorem 1.3 that for all $j = o(n)$,

$$P(N_1 = \lfloor \frac{n}{2} \rfloor + j) = \sqrt{\frac{2}{\pi n}} \exp \left\{ - \sum_{\ell \geq 1} \frac{4^\ell}{2\ell(2\ell - 1) n^{2\ell - 1}} \right\} \left( 1 + O\left( \frac{1 + |j|}{n} \right) \right). \quad (15)$$

and that if $j \neq o(n)$, there is a constant $C$ such that

$$P(N_1 = \lfloor \frac{n}{2} \rfloor + j) \leq C \exp \left\{ - \sum_{\ell \geq 1} \frac{4^\ell}{2\ell(2\ell - 1) n^{2\ell - 1}} \right\}, \quad (16)$$

and use directly Theorem 1.3 and the binomial expansion of $(1 + x)^{-2\ell}$ to see that if $\lfloor \frac{n}{2} \rfloor + j + x$ is even, $\lfloor \frac{n}{2} \rfloor + j \geq x$, and $j = o(n)$,

$$P(S^{(1)}(\lfloor \frac{n}{2} \rfloor + j) = x) = \exp \left\{ - \sum_{k \geq 0} \sum_{\ell \geq 1} \frac{1}{4\ell(2\ell - 1)} \left( 1 - 2\ell \right) \binom{2x}{k} \frac{(2j)^{2\ell}(-2j)^k}{n^{2\ell + k - 1}} \right\} E_{x,j,n}, \quad (17)$$

where $E_{x,j,n} = \left( 1 + O\left( \frac{|j| + 1}{n} \right) + O\left( \frac{x^2}{n^2} \right) \right)$, and if $\lfloor \frac{n}{2} \rfloor - j + y$ is even, $\lfloor \frac{n}{2} \rfloor - j \geq y$, and $j = o(n)$,

$$P(S^{(2)}(\lfloor \frac{n}{2} \rfloor - j) = y) = \frac{2}{\sqrt{\pi n}} \exp \left\{ - \sum_{k \geq 0} \sum_{\ell \geq 1} \frac{1}{4\ell(2\ell - 1)} \left( 1 - 2\ell \right) \binom{2y}{k} \frac{(2j)^{2\ell}(2j)^k}{n^{2\ell + k - 1}} \right\} E_{y,j,n}. \quad (18)$$

Moreover, there exists a constant $C$ such that for all $n, x, y, j$,

$$P(S^{(1)}(\lfloor \frac{n}{2} \rfloor + j) = x) \leq C \exp \left\{ - \sum_{k \geq 0} \sum_{\ell \geq 1} \frac{1}{4\ell(2\ell - 1)} \left( 1 - 2\ell \right) \binom{2x}{k} \frac{(2j)^{2\ell}(2j)^k}{n^{2\ell + k - 1}} \right\} \quad (19)$$

and

$$P(S^{(2)}(\lfloor \frac{n}{2} \rfloor - j) = y) \leq C \exp \left\{ - \sum_{k \geq 0} \sum_{\ell \geq 1} \frac{1}{4\ell(2\ell - 1)} \left( 1 - 2\ell \right) \binom{2y}{k} \frac{(2j)^{2\ell}(-2j)^k}{n^{2\ell + k - 1}} \right\}. \quad (20)$$

If we define for $k \geq 0$,

$$s_k = \sum_{\ell \geq 1} \frac{2^k}{4\ell(2\ell - 1)} \binom{1 - 2\ell}{k} \frac{(-1)^k(2y)^{2\ell} + (2x)^{2\ell}}{n^{2\ell + k - 1}},$$

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combining (15), (17), and (18) and using in the error terms our assumption that $0 \leq x \leq y$ yields for $j = o(n)$

\[ P_{n,x,y,j} = \frac{4\sqrt{2}}{(\pi n)^{3/2}} \exp \left\{ - \sum_{\ell \geq 1} \frac{4^\ell}{2\ell(2\ell - 1)} \frac{j^{2\ell}}{n^{2\ell - 1}} - \sum_{k \geq 0} j^k s_k \right\} \left(1 + O \left( \frac{|j| + 1}{n} \right) + O \left( \frac{y^2}{n^2} \right) \right) \] (21)

and (16), (19), and (20) imply that there is a constant $C$ such that for all $n, x, y, j$,

\[ P_{n,x,y,j} \leq C \exp \left\{ - \sum_{\ell \geq 1} \frac{4^\ell}{2\ell(2\ell - 1)} \frac{j^{2\ell}}{n^{2\ell - 1}} - \sum_{k \geq 0} j^k s_k \right\} \] (22)

We now turn to showing (4) and therefore assume that $y^2 - x^2 \leq n^{3/2}/\log n$. The main idea is to separate the sum in (14) into on one hand a sum which contains the dominant terms and can be estimated precisely by comparing it to an integral and on the other hand a sum which is of smaller order of magnitude and which we can afford to bound crudely. Note that since we are assuming that $0 \leq x \leq y \leq n^{1/2}/\log n$, we have, for $k \geq 0$,

\[ s_{2k} = c_{2k} \frac{x^2 + y^2}{n^{2k+1}} \left(1 + O \left( \frac{y^2}{n^2} \right) \right) \quad \text{and} \quad s_{2k+1} = c_{2k+1} \frac{y^2 - x^2}{n^{2k+2}} \left(1 + O \left( \frac{y^2}{n^2} \right) \right), \] (23)

with $c_k \geq 0$ for all $k \geq 0$, so that if $|j| \leq n^{1/2}/\log(n)$,

\[ P_{n,x,y,j} = \frac{4\sqrt{2}}{(\pi n)^{3/2}} \exp \left\{ - s_0 - \frac{2j^2}{n} - js_1 \right\} \left(1 + O \left( \frac{|j| + 1}{n} \right) + O \left( \frac{j^2}{n \log^4(n)} \right) + O \left( \frac{y^2}{n^2} \right) \right), \]

with all $O(\cdot)$ terms actually being $o(1)$. We observe that $s_1 \leq c_1/n^{1/2}/\log(n)$, which implies that there is some $C > 0$ such that $ns_1^2/8 \leq C/\log^2(n)$. Therefore,

\[
\exp \left\{ - \frac{2j^2}{n} - js_1 \right\} = \exp \left\{ - \frac{2}{n} \left( j + \frac{ns_1}{4} \right)^2 \right\} e^{ns_1^2/8} \\
= \exp \left\{ - \frac{2}{n} \left( j + \frac{ns_1}{4} \right)^2 \right\} e^{ns_1^2/8} \left(1 + O \left( \frac{|j| + 1}{n} \right) + O \left( \frac{1}{n^{1/2}/\log(n)} \right) \right) \\
= \exp \left\{ - \frac{2}{n} \left( j + \frac{ns_1}{4} \right)^2 \right\} \left(1 + O \left( \frac{1}{\log^2(n)} \right) \right),
\]

which implies that

\[
\sum_{j = -[n^{1/2}/\log(n)]}^{[n^{1/2}/\log(n)]} P_{n,x,y,j} = \frac{4\sqrt{2}}{(\pi n)^{3/2}} e^{-s_0} \sum_{j = -[n^{1/2}/\log(n)]}^{[n^{1/2}/\log(n)]} e^{-\frac{2j^2}{n} - js_1} \left(1 + O \left( \frac{1}{\log^2(n)} \right) \right) \\
= \frac{4\sqrt{2}}{(\pi n)^{3/2}} e^{-s_0} \sum_{j = -[n^{1/2}/\log(n)] - \frac{ns_1}{4}}^{[n^{1/2}/\log(n)] - \frac{ns_1}{4}} e^{-\frac{2j^2}{n}} \left(1 + O \left( \frac{1}{\log^2(n)} \right) \right). \] (24)

Since $y^2 - x^2 \leq n^{3/2}/\log(n)$, $ns_1/4 \leq Cn^{1/2}/\log(n)$, so for $n$ large enough,

\[
\sum_{j = -[n^{1/2}/\log(n)]/2}^{[n^{1/2}/\log(n)]/2} e^{-\frac{2j^2}{n}} \leq \sum_{j = -[n^{1/2}/\log(n)] - \frac{ns_1}{4}}^{[n^{1/2}/\log(n)] - \frac{ns_1}{4}} e^{-\frac{2j^2}{n}} \leq \sum_{j = -\infty}^{[n^{1/2}/\log(n)]} e^{-\frac{2j^2}{n}}. \] (26)
Recognizing the outer sums of (26) to be Riemann sums and using the monotonicity properties of the function $e^{-2x^2}$, we can use Lemma 3.1 to write

\[
\sum_{j = -\lfloor n^{1/2} \log(n)/2 \rfloor}^{\lfloor n^{1/2} \log(n)/2 \rfloor} e^{-\frac{2j^2}{n}} = 2 \sum_{j = 0}^{\lfloor n^{1/2} \log(n)/2 \rfloor} e^{-\frac{2j^2}{n}} - 1
\]

\[
\geq 2\sqrt{n} \int_{0}^{\log(n)/2} e^{-2x^2} dx - 1 = \sqrt{n} \left( \sqrt{\frac{\pi}{2}} - \int_{\log(n)}^{\infty} e^{-x^2/2} dx \right) - 1
\]

\[
= \sqrt{\frac{\pi n}{2}} \left( 1 + O \left( \frac{1}{\sqrt{n}} \right) \right)
\]  \hspace{1cm} (27)

and similarly,

\[
\sum_{j = -\infty}^{\lfloor n^{1/2} \log(n)/2 \rfloor} e^{-\frac{2j^2}{n}} \leq \sqrt{n} \int_{-\infty}^{\log(n)} e^{-2x^2} dx + 1 = \sqrt{\frac{\pi n}{2}} \left( 1 + O \left( \frac{1}{\sqrt{n}} \right) \right)
\]  \hspace{1cm} (28)

Combining (26), (27), and (28), we get

\[
\sum_{j = -\lfloor n^{1/2} \log(n)/2 \rfloor}^{\lfloor n^{1/2} \log(n)/2 \rfloor} e^{-\frac{2j^2}{n}} = \sqrt{\frac{\pi n}{2}} \left( 1 + O \left( n^{-\log(n)/2} \log^{-1}(n) \right) + O \left( 1/\sqrt{n} \right) \right)
\]  \hspace{1cm} (29)

Therefore, (24) and (29) give the estimate

\[
\sum_{j = -\lfloor n^{1/2} \log(n) \rfloor}^{\lfloor n^{1/2} \log(n) \rfloor} P_{n,x,y,j} = \frac{4}{\pi n} e^{-s_0} \left( 1 + O \left( \frac{1}{\log^2(n)} \right) \right)
\]  \hspace{1cm} (30)

We will now show that $\sum_{|j| \geq \lfloor n^{1/2} \log(n) \rfloor} P_{n,x,y,j}$ is of smaller order than the sum obtained in (30). Differentiating with respect to $j$ the exponent in the expression for $P_{n,x,y,j}$ in (21),

\[
f(j) = f_{n,x,y}(j) = -s_0 - \sum_{\ell \geq 1} \frac{4^{\ell}}{2^{2\ell}(2\ell - 1)n^{2\ell - 1}} - \sum_{k \geq 1} j^k s_k
\]  \hspace{1cm} (31)

shows that $f$ has a unique maximum which occurs at $j_0 = -\lfloor y^2 - x^2 \rfloor$ or $j'_0 = -\lfloor y^2 - x^2 \rfloor$.

Note that when $y^2 - x^2 \leq n^{3/2}/\log(n)$, $j_0$ and $j'_0$ belong to the index set of the sum in (30). It is easy to see that in that case, on the set $\{ j \in \mathbb{Z} : |j| \geq \lfloor n^{1/2} \log(n) \rfloor \}$, $f$ is maximal when $j = -\lfloor n^{1/2} \log(n) \rfloor$ and, using (23), that

\[
f(-\lfloor n^{1/2} \log(n) \rfloor) = -s_0 - 2\log^2(n) + O(1)
\]

Therefore, using (21), we see that there exists a constant $C > 0$ such that

\[
\sum_{j = -\lfloor n^{1/2} \log(n) \rfloor}^{n} P_{n,x,y,j} \leq nP_{n,x,y,-\lfloor n^{1/2} \log(n) \rfloor} \leq Cn^{-1/2} e^{-s_0 - 2\log(n)^2}
\]  \hspace{1cm} (32)

where the sum is over the set $\{ j \in \mathbb{Z} : -\lfloor n/2 \rfloor + x \leq j \leq -\lfloor n^{1/2} \log n \rfloor$ or $\lfloor n^{1/2} \log n \rfloor \leq j \leq \lfloor n/2 \rfloor - y \}$. Combining (14), (30), and (32) now yields (4).
We now turn to the case $y^2 - x^2 \geq n^{3/2} / \log(n)$. Plugging both $j_0 = -\lceil y^2 - x^2 \rceil / 2n$ and $j'_0 = -\lfloor y^2 - x^2 \rfloor / 2n$ into (31) yields the same upper bound for $f$:

$$f_{n,x,y}(j) \leq \left( -\sum_{\ell \geq 1} \frac{1}{2\ell(2\ell - 1)} \frac{(y^2 - x^2) 2\ell}{n^4\ell - 1} - \sum_{k \geq 0} \left( -\frac{y^2 - x^2}{2n} \right)^k s_k \right) \left( 1 + O \left( \frac{1}{\log^4(n)} \right) \right).$$

(33)

Here we have used the assumption that $x^2 + y^2 \leq n^2 / \log^4(n)$ and thus that $y^2 - x^2 \leq n^2 / \log^4(n)$.

Since the upper bound for the function $f$ defined in (31) is attained at one of the integers $j_0$ or $j'_0$, equations (33), (21), and (22) yield the obvious bounds

$$\sum_{j = -\lfloor n/2 \rfloor + x}^{\lceil n/2 \rceil - y} P_{n,x,y,j} \geq \frac{1}{Cn^{3/2}} \exp \left( -\sum_{\ell \geq 1} \frac{1}{2\ell(2\ell - 1)} \frac{(y^2 - x^2) 2\ell}{n^4\ell - 1} - \sum_{k \geq 0} \left( -\frac{y^2 - x^2}{2n} \right)^k s_k \right)$$

and

$$\sum_{j = -\lceil n/2 \rceil + x}^{\lceil n/2 \rceil - y} P_{n,x,y,j} \leq Cn \exp \left( -\sum_{\ell \geq 1} \frac{1}{2\ell(2\ell - 1)} \frac{(y^2 - x^2) 2\ell}{n^4\ell - 1} - \sum_{k \geq 0} \left( -\frac{y^2 - x^2}{2n} \right)^k s_k \right),$$

where $C$ is some positive constant, which concludes the proof of the theorem.

\[\square\]

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