On flavor conservation in weak interaction decays 

involving mixed neutrinos

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In the context of quantum field theory (QFT), we compute the amplitudes of weak interaction processes such as \( W^+ \rightarrow e^+ + \nu_e \) and \( W^+ \rightarrow e^+ + \nu_\mu \) by using different representations of flavor states for mixed neutrinos. Analyzing the short time limit of the above amplitudes, we find that the neutrino states defined in QFT as eigenstates of the flavor charges lead to results consistent with lepton charge conservation. On the contrary, the Pontecorvo flavor states produce a violation of lepton charge in the vertex, which is in contrast with what expected at tree level in the Standard Model.

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1. Introduction

Given the importance of neutrino mixing and oscillations\(^1\) in elementary particle physics, a great deal of work has been devoted to the related theoretical issues. For example, in the definition of flavor states, it has emerged\(^2\) that the vacuum for the mass eigenstates of neutrinos turns out to be unitarily inequivalent to the vacuum for the flavor eigenstates of neutrinos. The vacuum structure associated with the field mixing\(^3\),\(^4\),\(^5\),\(^6\),\(^7\),\(^8\),\(^9\),\(^10\),\(^11\) leads to a modification of the flavor oscillation formulas\(^3\),\(^7\),\(^8\) and exhibits new features with respect to the quantum mechanical ones\(^1\).

The theoretical understanding of the mixing phenomena in the framework of the quantum field theory (QFT) has also been confirmed by mathematically rigorous analysis\(^12\). One of the offsprings from the QFT treatment consists in the fact that it has led also to consider, from the perspective of particle mixing, other physically relevant problems which would have not been possible to handle by resorting to the Pontecorvo quantum mechanical (QM) approximation. For example, we quote the particle mixing contribution to the dark energy of the Universe\(^13\),\(^14\),\(^15\).

In this paper, we consider the concrete problem of verifying the lepton flavor conservation in the processes that produce the (mixed) neutrino. Here, we analyze the amplitudes of weak interaction processes such as \( W^+ \rightarrow e^+ + \nu_e \) and \( W^+ \rightarrow e^+ + \nu_\mu \) at tree level in the context of QFT. Although it is well known that the flavor changing loop induced processes, such as \( \mu \rightarrow e\gamma \), are possible, they are not relevant
to our discussion since they have very low branching ratios, e.g. $Br(\mu \rightarrow e\gamma) < 10^{-50}$

We carry out our calculations by resorting to two different representations of flavor neutrino states: 1) the ones defined in Ref.\textsuperscript{[2]} hereon denoted as “exact flavor states” and 2) the quantum mechanical (Pontecorvo) flavor states. In particular, we consider the amplitudes in the short time range, i.e. at very small distances from the production vertex. We find that the use of the exact flavor states leads to results consistent with the lepton charge conservation as expected in the Standard Model (SM) at tree level, whereas the Pontecorvo states yield a violation of the lepton charge in the vertex.

Although obtained in different context, a similar violation has been found in Ref.\textsuperscript{[17]} where it has been shown that the processes such as $\pi \rightarrow \mu \bar{\nu}_e$ are possible with a branching ratio much greater than the loop induced processes as the one mentioned above. The conclusion of Ref.\textsuperscript{[17]} was that an intrinsic flavor violation for massive neutrinos would be present in the Standard Model. We show that such a violation arises as a consequence of an incorrect choice for the (mixed) neutrino flavor states.

In Section II, we compute the amplitudes of the weak interaction processes $W^+ \rightarrow e^+ + \nu_e$ and $W^+ \rightarrow e^+ + \nu_\mu$ by using the exact flavor states and the Pontecorvo states. In Section III, we consider the explicit form of the above amplitudes for short time intervals. The long time limit is studied in Appendix B. Section IV is devoted to conclusions. A brief summary of the vacuum structure for Dirac neutrino mixing is presented in the Appendix A.

2. Amplitudes of weak interaction processes containing mixed neutrinos

In this Section, we compute the amplitudes of the following two decays at tree level:

$$W^+ \rightarrow e^+ + \nu_e,$$  \hspace{1cm} (1)

$$W^+ \rightarrow e^+ + \nu_\mu,$$  \hspace{1cm} (2)

where neutrinos are produced through charged current processes. Although our computations are specific for these decay processes, our conclusions are general and hold for all the different neutrino production processes. We perform the calculations by means of standard QFT techniques.

In Section II.A, we use the exact flavor neutrino states defined as the eigenstates of flavor charges (see the Appendix for notations and definitions). They are generated by the action of the flavor neutrino creation operators on the flavor vacuum $|0\rangle_f$ as follows:

$$|\nu^r_{k,\sigma}\rangle \equiv \alpha^r_{k,\sigma} |0\rangle_f, \quad \sigma = e, \mu,$$  \hspace{1cm} (3)

and $|\nu^r_{k,\sigma}(t)\rangle = e^{iH_0t} \alpha^r_{k,\sigma} |0\rangle_f$. 

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In Section II.B, we perform the same calculations of Section II.A by using the quantum mechanical Pontecorvo states

$$|\nu_{r,k}^\mu\rangle_P = \cos \theta |\nu_{r,1}\rangle + \sin \theta |\nu_{r,2}\rangle,$$

$$|\nu_{r,k,\mu}^e\rangle_P = -\sin \theta |\nu_{r,1}\rangle + \cos \theta |\nu_{r,2}\rangle,$$

where the neutrino states with definite masses are defined by the action of the creation operators for the free fields $\nu_i$ on the vacuum $|0\rangle_m$ (see Appendix A):

$$|\nu_{r,k,i}\rangle \equiv \alpha_{r,k,i}^\dagger |0\rangle_m,\quad i = 1, 2.$$  

Note that the Pontecorvo neutrino states Eqs.(4),(5) are not eigenstates of flavor neutrino charges. In the scattering theory for finite range potentials, it is assumed that the interaction Hamiltonian $H_{\text{int}}(x)$ can be switched off adiabatically as $x^0_{\text{in}} \to -\infty$ and $x^0_{\text{out}} \to +\infty$ so that the initial and final states can be represented by the eigenstates of the free Hamiltonian. However, in the present case and more generally in the decay processes where the mixed neutrinos are produced, the application of the adiabatic hypothesis leads to erroneous conclusions (as made in Ref.[19]). Indeed, the flavor neutrino field operators do not have the mathematical characterization necessary to be defined as asymptotic field operators acting on the massive neutrino vacuum. Moreover, the flavor states $|\nu_{r,k,\sigma}\rangle$ are not eigenstates of the free Hamiltonian.

Therefore, the integration limits in the amplitudes of decay processes where mixed neutrinos are produced must be chosen so that the time interval $\Delta t = x^0_{\text{out}} - x^0_{\text{in}}$ is much shorter than the characteristic neutrino oscillation time $t_{\text{osc}}$: $\Delta t \ll t_{\text{osc}}$.

In this paper we consider at the first order of the perturbation theory the amplitudes of the decays (1) and (2).

In general, if $|\psi_i\rangle$ and $|\psi_f\rangle$ denote initial and final states, the probability amplitude $\langle \psi_f | e^{-iHt} | \psi_i \rangle$ is given by

$$\langle \psi_f | e^{-iHt} | \psi_i \rangle = \langle \psi_f | e^{iH_{\text{free}}t} e^{-iH_{\text{free}}t} | \psi_i \rangle = \langle \psi_f | U_{H_{\text{free}}} U_{H_{\text{int}}} (t) | \psi_i \rangle. \quad (7)$$

Here the time evolution operator $U_{H_{\text{free}}} (t)$ in the interaction picture is given approximatively by

$$U_{H_{\text{free}}} (t) \simeq 1 - i \int_0^t dt' H_{\text{free}} (t'), \quad (8)$$

with $H_{\text{free}} (t) = e^{iH_{\text{free}}t} H_{\text{free}} e^{-iH_{\text{free}}t}$ interaction hamiltonian in the interaction picture. In the following $H_0$ is the free part of the Hamiltonian for the fields involved in the decays (1) and (2) and the relevant interaction Hamiltonian is given by [20]

$$H_{\text{int}}(x) = -\frac{g}{\sqrt{2}} W^\mu_\mu(x) J^\mu(x) + \text{h.c.} = -\frac{g}{2\sqrt{2}} W^\mu_\mu(x) \overline{\nu}_e(x) \gamma^\mu (1 - \gamma^5) e(x) + \text{h.c.}, \quad (9)$$

where $W^\mu(x)$, $e(x)$ and $\nu_e(x)$ are the fields of the boson $W^\mu$, the electron and the flavor (electron) neutrino, respectively.
2.1. Exact flavor states

Let us first consider the process $W^+ \rightarrow e^+ + \nu_e$ and the states defined in Eq. (3). The amplitude of the decay at first order in perturbation theory is given by

$$A_{W^+ \rightarrow e^+ + \nu_e} = \langle \bar{v}_{\nu_e}^c, e^+_q | -i \int_{x_{0}^{\mathrm{out}}}^{x_{0}^{\mathrm{in}}} d^4x H_{\mathrm{int}}(x) | W_{\mu,\lambda}^+ \rangle$$

where $W_{\mu,\lambda}^+ = (g/2) \langle \bar{v}_{\nu_e}^c, e^+_q | \bar{v}_{\nu_e}^{\mu,\lambda} e^+_p \rangle e^{i(p \cdot x - E^0_p x^0)}$, and $E^0_p = \sqrt{q^2 + M_W^2}$.

The terms involving the expectation values of the vector boson and electron fields are given by

$$w \langle 0 | W_{\mu,\lambda}^+(x) | W_{\mu,\lambda}^+ \rangle = \frac{1}{(2\pi)^{3/2}} \frac{2}{\omega_p} e^{i(p \cdot x - E^0_p x^0)}$$

where $E^0_p = \sqrt{p^2 + M_V^2}$ and $E^0_q = \sqrt{q^2 + M_e^2}$.

The terms involving the expectation values of the neutrino field yields

$$\langle \bar{v}_{\nu_e}^c, e^+_q | \bar{v}_{\nu_e}^{\mu,\lambda} e^+_p \rangle = \frac{|U_{\nu_e}^\mu|}{(2\pi)^{3/2}} \left\{ \begin{array}{l}
\cos^2 \theta \left[ e^{i\omega_{k,1}(x^0 - x_{0}^{\mathrm{in}})} +|V_{k}^1|^2 e^{-i\omega_{k,1}(x^0 - x_{0}^{\mathrm{in}})} \right]
\sin^2 \theta \left[ e^{-i\omega_{k,1}(x^0 - x_{0}^{\mathrm{in}})} - e^{i\omega_{k,1}(x^0 - x_{0}^{\mathrm{in}})} \right] \end{array} \right\}$$

where $\omega_{k,i} = \sqrt{k^2 + m_i^2}$, $i = 1, 2$. Here we have utilized the explicit form of the flavor annihilation/creation operators given in Appendix A. Notice the presence in Eq. (13) of the Bogoliubov coefficients $U_k$ and $V_k$.

It is also convenient to rewrite Eq. (13), by means of the relations (A.11) and (A.12) given in Appendix A, as

$$\langle \bar{v}_{\nu_e}^c, e^+_q | \bar{v}_{\nu_e}^{\mu,\lambda} e^+_p \rangle = \frac{|U_{\nu_e}^\mu|}{(2\pi)^{3/2}} \left\{ \begin{array}{l}
\cos^2 \theta \left[ \prod_{k,1} e^{i\omega_{k,1}(x^0 - x_{0}^{\mathrm{in}})} 
\sin^2 \theta \left[ e^{-i\omega_{k,1}(x^0 - x_{0}^{\mathrm{in}})} - e^{i\omega_{k,1}(x^0 - x_{0}^{\mathrm{in}})} \right] \end{array} \right\}$$

Note that in the case of the flavor states, because of the orthogonality of the Hilbert spaces at different times (see Appendix A), instead of Eq. (4) the amplitude should be defined as $\langle \bar{v}_{\nu_e}^c(x_{0}^{\mathrm{out}}) | e^{-iH(x_{0}^{\mathrm{out}} - x_{0}^{\mathrm{in}})} | v_{\nu_e}(x_{0}^{\mathrm{in}}) \rangle = \langle \bar{v}_{\nu_e}^c(x_{0}^{\mathrm{in}}) | U_f(x_{0}^{\mathrm{out}} - x_{0}^{\mathrm{in}}) | \bar{v}_{\nu_e}(x_{0}^{\mathrm{in}}) \rangle$. 

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Combining the above results, Eq. (10) can be written as

\[ A_{W^+\rightarrow e^+\nu_e} = \frac{i g}{2\sqrt{2}(2\pi)^{3/2}} \delta^3(p - q - k) \int_{x_{in}^0}^{x_{out}^0} dx_0 \frac{\varepsilon_{p,\mu,\lambda}}{\sqrt{2E_p}} \]

\[ \times \left\{ \frac{\bar{\nu}_{k,1}\gamma^\mu(1 - \gamma^5)v_{q,e}^s}{U_{k,1}} \left[ \cos^2\theta e^{-i(E_p^W - E_q^W - \omega_{k,1})x_0} e^{-i\omega_{k,1}x_{in}^0} + \sin^2\theta \left( |U_{k,2}|^2 e^{-i(E_p^W - E_q^W - \omega_{k,2})x_0} e^{-i\omega_{k,2}x_{in}^0} + |V_k|^2 e^{-i(E_p^W - E_q^W + \omega_{k,2})x_0} e^{i\omega_{k,2}x_{in}^0} \right) \right] + \varepsilon^* |U_k||V_{k,1}| \bar{\nu}_{k,1}\gamma^\mu(1 - \gamma^5)v_{q,e}^s \sin^2\theta \times \left[ e^{-i(E_p^W - E_q^W - \omega_{k,2})x_0} e^{i\omega_{k,2}x_{in}^0} - e^{-i(E_p^W - E_q^W + \omega_{k,2})x_0} e^{-i\omega_{k,2}x_{in}^0} \right] \right\}, \] (15)

when Eq. (13) is used, or equivalently as:

\[ A_{W^+\rightarrow e^+\nu_e} = \frac{i g}{2\sqrt{2}(2\pi)^{3/2}} \delta^3(p - q - k) \int_{x_{in}^0}^{x_{out}^0} dx_0 \frac{\varepsilon_{p,\mu,\lambda}}{\sqrt{2E_p}} \]

\[ \times \left\{ \cos^2\theta e^{-i\omega_{k,1}x_{in}^0} \bar{\nu}_{k,1}\gamma^\mu(1 - \gamma^5)v_{q,e}^s e^{-i(E_p^W - E_q^W - \omega_{k,1})x_0} + \sin^2\theta \left[ |U_{k,2}|^2 e^{-i(E_p^W - E_q^W - \omega_{k,2})x_0} e^{-i\omega_{k,2}x_{in}^0} + |V_k|^2 e^{-i(E_p^W - E_q^W + \omega_{k,2})x_0} e^{i\omega_{k,2}x_{in}^0} \right] \right\}, \] (16)

when the term involving the expectation value of the neutrino field is expressed in the form of Eq. (13).

Next we consider the process \( W^+ \rightarrow e^+\nu_e \). By using the Hamiltonian (9), we have now

\[ A_{W^+\rightarrow e^+\nu_e} = \langle \nu_{k,\mu}^e, e_{q}^s \rangle \left[ -i \int_{x_{in}^0}^{x_{out}^0} d^4x \; H_{int}(x) \right] |W_{p,\lambda}^+\rangle \]

\[ = W(0) \langle \nu_{k,\mu}^e(x_{in}^0) | e_{q}^s \rangle \left\{ \frac{i g}{2\sqrt{2}} \right\} \int_{x_{in}^0}^{x_{out}^0} d^4x \left[ W_{q,e}^+(x) \bar{\nu}_{k,1}\gamma^\mu(1 - \gamma^5)e(x) \right] |W_{p,\lambda}^+\rangle |0\rangle_e |0(x_{in}^0)\rangle_f. \] (17)

The term involving the expectation value of the neutrino field is now

\[ \langle \nu_{k,\mu}(x_{in}^0) | \bar{\nu}_{e}(x) |0(x_{in}^0)\rangle_f = e^{-i k \cdot x} \frac{2^3}{(2\pi)^{3/2}} \sin \theta \cos \theta \left\{ |U_{k,1}| e^{i\omega_{k,2}(x^0 - x_{in}^0)} - e^{i\omega_{k,1}(x^0 - x_{in}^0)} + \varepsilon^* |U_k||V_{k,1}| \left[ e^{i\omega_{k,2}(x^0 - x_{in}^0)} + e^{-i\omega_{k,1}(x^0 - x_{in}^0)} \right] \right\}, \] (18)
which, using the relations (A.11) and (A.12) in Appendix A, can be also written as

\[ \langle \nu_{k,\mu}^c(x_0^0)|\nu_e(x_0^0)\rangle = \frac{e^{-i\mathbf{k} \cdot \mathbf{x}}}{(2\pi)^{3/2}} \sin \theta \cos \theta \left\{ \mathbf{p}_{k,2} e^{i\omega_{k,2}(x^0 - x_{in}^0)} - |U_k| \mathbf{p}_{k,1} e^{i\omega_{k,1}(x^0 - x_{in}^0)} + \varepsilon^r |V_k| \mathbf{p}_{-k,1} e^{-i\omega_{k,1}(x^0 - x_{in}^0)} \right\}. \]

Thus, the amplitude (Eq.(17)) can be expressed as

\[ A_{W^+ \to e^+ + \nu_e} = \frac{i g}{2\sqrt{2}(2\pi)^{3/2}} \delta^3(\mathbf{p} - \mathbf{q} - \mathbf{k}) \sin \theta \cos \theta \int_{x_{in}^0}^{x_{out}^0} dx^0 \frac{\varepsilon_{p,\mu,\lambda}}{\sqrt{2E_p^W}} \times \left\{ |U_k| \mathbf{p}_{k,1} \gamma^\mu(1 - \gamma^5)\varepsilon^\lambda_{q,e} \left[ e^{-i(E_p^W - E_q^W - \omega_{k,2})x^0} e^{-i\omega_{k,2}x_{in}^0} \right.ight. \]
\[ \left. - e^{-i(E_p^W - E_q^W - \omega_{k,1})x^0} e^{-i\omega_{k,1}x_{in}^0} \right] + \varepsilon^r |V_k| \mathbf{p}_{-k,1} \gamma^\mu(1 - \gamma^5)\varepsilon^\lambda_{q,e} \left[ e^{-i(E_p^W - E_q^W - \omega_{k,2})x^0} e^{-i\omega_{k,2}x_{in}^0} \right. \]
\[ \left. + e^{-i(E_p^W - E_q^W + \omega_{k,1})x^0} e^{i\omega_{k,1}x_{in}^0} \right\}, \]

when Eq.(18) is utilized, or equivalently as

\[ A_{W^+ \to e^+ + \nu_e} = \frac{i g}{2\sqrt{2}(2\pi)^{3/2}} \delta^3(\mathbf{p} - \mathbf{q} - \mathbf{k}) \sin \theta \cos \theta \int_{x_{in}^0}^{x_{out}^0} dx^0 \frac{\varepsilon_{p,\mu,\lambda}}{\sqrt{2E_p^W}} \times \left[ e^{-i\omega_{k,2}x_{in}^0} \mathbf{p}_{k,2} \gamma^\mu(1 - \gamma^5)\varepsilon^\lambda_{q,e} e^{-i(E_p^W - E_q^W - \omega_{k,2})x^0} \right. \]
\[ \left. - e^{-i\omega_{k,1}x_{in}^0} |U_k| \mathbf{p}_{k,1} \gamma^\mu(1 - \gamma^5)\varepsilon^\lambda_{q,e} e^{-i(E_p^W - E_q^W - \omega_{k,1})x^0} \right] \]
\[ + e^{i\omega_{k,1}x_{in}^0} \varepsilon^r |V_k| \mathbf{p}_{-k,1} \gamma^\mu(1 - \gamma^5)\varepsilon^\lambda_{q,e} e^{-i(E_p^W - E_q^W + \omega_{k,1})x^0} \right\}, \]

when Eq.(19) is used.

2.2. Pontecorvo flavor states

We now repeat the computations using the Pontecorvo states (4) and (5) instead of the exact flavor states (3).

For the decay $W^+ \to e^+ + \nu_e$, we have (the superscript $P$ denotes the amplitude computed with Pontecorvo states)

\[ A_{W^+ \to e^+ + \nu_e}^P = W|0\rangle\langle \nu_e^P(x_0^0)|\langle \nu_e^P|\frac{i g}{2\sqrt{2}} \times \int_{x_{in}^0}^{x_{out}^0} d^4x \left[ W_{\mu}^+ (x) \nu_e(x) \gamma^\mu (1 - \gamma^5) e(x) \right] |W_{p,\lambda}\rangle |0\rangle e |0\rangle_m. \]

In the above expressions, notice that the vacuum $|0\rangle_m$ appears for the fields with definite masses $\nu_i$. 
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In the amplitude (22), the term involving the expectation value of the neutrino fields given in Eq. (18) (or equivalently in Eq. (19)) is replaced by

\[ P\langle \nu_{k,e}(x_{out}^0)|\nu_e(x)\rangle|0\rangle_m = \cos \theta \ e^{-i\omega_k \cdot x_{out}^0} \langle \nu_{k,1}^c|\nu_e(x)\rangle|0\rangle_m + \sin \theta \ e^{-i\omega_k \cdot x_{out}^0} \langle \nu_{k,2}^c|\nu_e(x)\rangle|0\rangle_m \]

\[ = \frac{e^{-i k \cdot x}}{(2\pi)^{3/2}} \left[ \cos^2 \theta \ \overline{u}_{k,1} e^{-i\omega_k \cdot (x_{out}^0 - x^0)} + \sin^2 \theta \ \overline{u}_{k,2} e^{-i\omega_k \cdot (x_{out}^0 - x^0)} \right], \]

with respect to the amplitude (10) computed with the exact flavor states. Thus, the amplitude \( A_{W^+ \rightarrow e^+ + \nu_e}^P \) becomes

\[ A_{W^+ \rightarrow e^+ + \nu_e}^P = \frac{i g}{2\sqrt{2}(2\pi)^{1/2}} \frac{\varepsilon_{\mu,\nu,\lambda}}{\sqrt{2E_p^W}} \delta^3(p - q - k) \]

\[ \times \int x_{out}^0 \ dx_0 \left[ \cos^2 \theta \ e^{-i\omega_k \cdot x_{out}^0} \overline{u}_{k,1} \gamma^\mu (1 - \gamma^5) v_{q,e} e^{-i(E_p^W - E_q^W - \omega_k,1)x^0} \right. \]

\[ + \left. \sin^2 \theta \ e^{-i\omega_k \cdot x_{out}^0} \overline{u}_{k,2} \gamma^\mu (1 - \gamma^5) v_{q,e} e^{-i(E_p^W - E_q^W - \omega_k,2)x^0} \right]. \] \hfill (24)

In a similar way, when we consider the decay \( W^+ \rightarrow e^+ + \nu_\mu \), the expectation value given in Eq. (19) (or equivalently in Eq. (19)) is replaced by

\[ P\langle \nu_{k,\mu}(x_{out}^0)|\nu_\mu(x)\rangle|0\rangle_m = \cos \theta \ e^{-i\omega_k \cdot x_{out}^0} \langle \nu_{k,1}^c|\nu_\mu(x)\rangle|0\rangle_m + \sin \theta \ e^{-i\omega_k \cdot x_{out}^0} \langle \nu_{k,2}^c|\nu_\mu(x)\rangle|0\rangle_m \]

\[ = \frac{e^{-i k \cdot x}}{(2\pi)^{3/2}} \left[ \cos^2 \theta \ \overline{u}_{k,1} e^{-i\omega_k \cdot (x_{out}^0 - x^0)} - \overline{u}_{k,2} e^{-i\omega_k \cdot (x_{out}^0 - x^0)} \right], \]

and the amplitude \( A_{W^+ \rightarrow e^+ + \nu_\mu}^P \) is now given by

\[ A_{W^+ \rightarrow e^+ + \nu_\mu}^P = \frac{i g}{2\sqrt{2}(2\pi)^{1/2}} \frac{\varepsilon_{\mu,\nu,\lambda}}{\sqrt{2E_p^W}} \sin \theta \ \cos \theta \delta^3(p - q - k) \]

\[ \times \int x_{out}^0 \ dx_0 \left[ e^{-i\omega_k \cdot x_{out}^0} \overline{u}_{k,1} \gamma^\mu (1 - \gamma^5) v_{q,\mu} e^{-i(E_p^W - E_q^W - \omega_k,1)x^0} \right. \]

\[ - \left. e^{-i\omega_k \cdot x_{out}^0} \overline{u}_{k,2} \gamma^\mu (1 - \gamma^5) v_{q,\mu} e^{-i(E_p^W - E_q^W - \omega_k,2)x^0} \right]. \] \hfill (26)

In the relativistic limit, |\( V_k | \rightarrow 0 \) and |\( U_k | \rightarrow 1 \), and, regardless phase factors, Eqs. (16) and (21) coincide with Eqs. (24) and (20), respectively, obtained by using the Pontecorvo states. These are indeed the approximation of the exact flavor states in such a relativistic limit [2].

The general expressions given by Eqs. (16) and (21) as well as those given by Eqs. (24) and (20) will be the basis for our analysis of lepton charge conservation for the processes \( W^+ \rightarrow e^+ + \nu_e \) and \( W^+ \rightarrow e^+ + \nu_\mu \). For this purpose, in next Section, we study the detailed structure of the amplitudes of these processes in the short time limit.
In the Appendix B, we also comment on the above amplitudes in the long time limit.

3. Amplitudes in the short time limit

In this Section we consider the explicit form of the amplitudes given by Eqs. (16), (21), (24) and (26) for short time intervals $\Delta t$. The physical meaning of such a time scale $\Delta t$ is represented by the relation $\frac{1}{\Gamma} \ll \Delta t \ll L_{osc}$, where $\Gamma$ is the $W^+$ decay width and $L_{osc}$ is the typical flavor oscillation length. Given the experimental values of $\Gamma$ and $L_{osc}$, this interval is well defined. A similar assumption has been made in Ref.[17] where the decay $\pi \rightarrow \mu \nu_e$ has been considered. In the following, when we use the expression “short time limit”, we refer to the time scale defined above. Of course, energy fluctuations are constrained by the Heisenberg uncertainty relation, where $\Delta t$ is the one given above.

We will see that the use of the exact flavor states gives results which agree with lepton charge conservation in the production vertex, as predicted (at tree level) by the SM. On the other hand, we observe a clear violation of the lepton charge when the Pontecorvo states are used. Our calculation shows that the origin of such a violation is due to the fact that the Pontecorvo flavor states are defined by use of the vacuum state $|0\rangle_m$ for the massive neutrino states.

3.1. Exact flavor states

Let us first consider the decay $W^+ \rightarrow e^+ + \nu_e$. Taking the limit of integrations in Eq. (10) as $x_{in}^0 = -\Delta t/2$ and $x_{out}^0 = \Delta t/2$, the amplitude $A_{W^+ \rightarrow e^+ + \nu_e}$ becomes

$$A_{W^+ \rightarrow e^+ + \nu_e} = \frac{i g}{\sqrt{2}(2\pi)^{3/2}} \frac{\varepsilon_{p,\mu,\lambda}}{\sqrt{2E_p^W}} \delta^3(p - q - k)$$

$$\times \left\{ \cos^2 \theta e^{i\omega_{k,1}\Delta t/2} \frac{\sin((E_p^W - E_q^e - \omega_{k,1})\Delta t/2)}{E_p^W - E_q^e - \omega_{k,1}} \right.$$

$$\left. + \sin^2 \theta e^{i\omega_{k,2}\Delta t/2} \frac{\sin((E_p^W - E_q^e - \omega_{k,2})\Delta t/2)}{E_p^W - E_q^e - \omega_{k,2}} \right. $$

$$\left. + e^{-i\omega_{k,2}\Delta t/2} e^{\varepsilon^*} \frac{\sin((E_p^W - E_q^e + \omega_{k,2})\Delta t/2)}{E_p^W - E_q^e + \omega_{k,2}} \right\} \gamma^\mu (1 - \gamma^5) \nu_{q,e}^c.$$ 

We now consider the short time limit of the above expression. It is clear that the dominant contributions in Eq. (27) are those for which $E_p^W - E_q^e \approx 0$. For such dominant terms, it is then safe to perform the expansion $\sin x \approx x$. Moreover we perform the expansion $e^{\pm i\omega_{k,i}\Delta t/2} \approx 1$, with $i = 1, 2$. We thus obtain the following result at first order in $\Delta t$: 

$$A_{W^+ \rightarrow e^+ + \nu_e} \approx \frac{i g}{\sqrt{2}(2\pi)^{3/2}} \frac{\varepsilon_{p,\mu,\lambda}}{\sqrt{2E_p^W}} \delta^3(p - q - k) \Delta t \times$$

$$\times \left\{ \cos^2 \theta \bar{U}_{k,1} + \sin^2 \theta \left[ \bar{U}_{k,2} + \varepsilon^* \bar{V}_{k,2} \right] \right\} \gamma^\mu (1 - \gamma^5) \nu_{q,e}^c.$$
The quantity in the curly brackets can be evaluated by means of the identity given by Eq. (A.13) among the Bogoliubov coefficients. The result is

\[ A_{W^+ \rightarrow e^+ + \nu_\mu} \simeq \frac{ig}{2\sqrt{2}(2\pi)^{3/2}} \frac{\varepsilon_{p,\mu,\lambda}}{\sqrt{2E_p}} \delta^3(p - q - k) \Delta t \pi_{k,1}^c \gamma^\mu (1 - \gamma^5) v_{q,e}^\mu \] (29)

This amplitude resembles the one for the production of a free neutrino with mass \( m_1 \).

Let us now turn to the process \( W^+ \rightarrow e^+ + \nu_\mu \). Proceeding in a similar way as above, taking \( x_{in}^0 = -\Delta t/2 \) and \( x_{out}^0 = \Delta t/2 \) in Eq. (21), we get

\[ A_{W^+ \rightarrow e^+ + \nu_\mu} = \frac{ig}{2\sqrt{2}(2\pi)^{3/2}} \frac{\varepsilon_{p,\mu,\lambda}}{\sqrt{2E_p}} \delta^3(p - q - k) \sin 2\theta \]

\[ \times \left[ e^{i\omega_{e,2} \Delta t/2} \pi_{k,2} \sin[(E_p^W - E_q^e - \omega_{k,2})\Delta t/2] \right. \]

\[- e^{i\omega_{\mu,1} \Delta t/2} \left| U_k \right| \pi_{k,1} \sin[(E_p^W - E_q^e - \omega_{k,1})\Delta t/2] \]

\[ + e^{-i\omega_{e,2} \Delta t/2} e^{-\nu_e} |V_k| \pi_{-k,1} \sin[(E_p^W - E_q^e + \omega_{k,1})\Delta t/2] \gamma^\mu (1 - \gamma^5) v_{q,e}^\mu , \] (30)

which becomes

\[ A_{W^+ \rightarrow e^+ + \nu_\mu} \simeq \frac{ig}{4\sqrt{2}(2\pi)^{3/2}} \frac{\varepsilon_{p,\mu,\lambda}}{\sqrt{2E_p}} \Delta t \sin 2\theta \]

\[ \times \left[ \pi_{k,2} - |U_k| \pi_{k,1} + e^{-\nu_e} |V_k| \pi_{-k,1} \right] \gamma^\mu (1 - \gamma^5) v_{q,e}^\mu , \] (31)

in the short time limit.

We now observe that the quantity in square bracket vanishes identically due to the relation given by Eq. (A.11); i.e.

\[ A_{W^+ \rightarrow e^+ + \nu_\mu} \simeq 0 . \] (32)

This proves that, in the short time limit, the use of the exact flavor states leads to the conservation of lepton charge in the production vertex in agreement with what we expected from the Standard Model.

### 3.2. Pontecorvo states

It is now straightforward to analyze the short time limit of the amplitudes \( A_{W^+ \rightarrow e^+ + \nu_\mu}^P \) and \( A_{W^+ \rightarrow e^+ + \nu_\mu}^P \), defined by means of the Pontecorvo flavor states.

Proceeding in the same way as done in the previous subsection, Eq. (24) becomes

\[ A_{W^+ \rightarrow e^+ + \nu_\mu}^P \simeq \frac{ig}{2\sqrt{2}(2\pi)^{3/2}} \frac{\varepsilon_{p,\mu,\lambda}}{\sqrt{2E_p}} \delta^3(p - q - k) \Delta t \]

\[ \times \left[ \cos^2 \theta \pi_{k,1} + \sin^2 \theta \pi_{k,2} \right] \gamma^\mu (1 - \gamma^5) v_{q,e}^\mu , \] (33)
where we performed the expansion $e^{-i\omega_{k,i}\Delta t/2} \simeq 1$, with $i = 1, 2$. The structure of this amplitude is clearly different from the one obtained in Eq. (29). Such a difference is more relevant in the non-relativistic limit.

However, observed neutrinos are relativistic and thus it is convenient to consider the relativistic limit of the above result. To this end, we rewrite Eq. (33) in a more convenient form by using the identity given by Eq. (A.11):

$$A_{W^+ \rightarrow e^+ + \nu_e} \simeq \frac{ig}{2\sqrt{2}(2\pi)^{3/2}} \frac{\varepsilon_{\rho\mu\lambda}}{\sqrt{2E_p}} \delta^3(p - q - k) \Delta t$$

$$\times \left[ \bar{\nu}_{k,1} (1 - \sin^2 \theta (1 - |U_k|)) - \sin^2 \theta \varepsilon^\tau \bar{\nu}_{-k,1} |V_k| \right] \gamma^\mu (1 - \gamma^5) v_{q,e}^\rho.$$  

In the relativistic limit, the Bogoliubov coefficient $|U_k|$ and $|V_k|$ can be expressed respectively as (see Appendix A):

$$|U_k| \sim 1 - \frac{(\Delta m)^2}{4k^2}, \quad |V_k| \sim \frac{\Delta m}{2k},$$

Equation (34) can then be written at the first order in $O\left(\frac{\Delta m^2}{2k}\right)$,

$$A_{W^+ \rightarrow e^+ + \nu_e} \simeq \frac{ig}{4\sqrt{2}(2\pi)^{3/2}} \frac{\varepsilon_{\rho\mu\lambda}}{\sqrt{2E_p}} \delta^3(p - q - k) \sin 2\theta \Delta t$$

$$\times \left[ \bar{\nu}_{k,1} - \sin^2 \theta \varepsilon^\tau \bar{\nu}_{-k,1} \frac{\Delta m}{2k} \right] \gamma^\mu (1 - \gamma^5) v_{q,e}^\rho,$$

which shows how the results (29) and (36) agree in the ultra-relativistic limit (i.e. when $\frac{\Delta m}{2k} \to 0$).

We now consider the short time limit of the amplitude given in Eq. (29). We have

$$A_{W^+ \rightarrow e^+ + \nu_e} \simeq \frac{ig}{4\sqrt{2}(2\pi)^{3/2}} \frac{\varepsilon_{\rho\mu\lambda}}{\sqrt{2E_p}} \delta^3(p - q - k) \sin 2\theta \Delta t$$

$$\times \left[ \bar{\nu}_{k,2} - \bar{\nu}_{k,1} \right] \gamma^\mu (1 - \gamma^5) v_{q,e}^\rho,$$

which signals a violation of lepton charge in the tree level vertex. We performed the expansion $e^{-i\omega_{k,i}\Delta t/2} \simeq 1$, with $i = 1, 2$.

Again, we consider the relativistic limit. We first rewrite Eq. (37) by means of the relation given by Eq. (A.11),

$$A_{W^+ \rightarrow e^+ + \nu_e} \simeq \frac{ig}{4\sqrt{2}(2\pi)^{3/2}} \frac{\varepsilon_{\rho\mu\lambda}}{\sqrt{2E_p}} \delta^3(p - q - k) \sin 2\theta \Delta t$$

$$\times \left[ \bar{\nu}_{k,1} (|U_k| - 1) - \varepsilon^\tau \bar{\nu}_{-k,1} |V_k| \right] \gamma^\mu (1 - \gamma^5) v_{q,e}^\rho,$$
and, by using Eq. (35), we obtain the following result at first order in $\Delta m^2_k$:

$$A_{W^+ \rightarrow e^+ + \nu_p}^P \approx -\frac{i\theta}{4\sqrt{2}} \frac{\bar{\epsilon}_{p,\mu,\lambda}}{2E_p^W} \delta^3(p - q - k) \sin 2\theta \Delta t \frac{\Delta m}{2k} \times \mathbf{r}_{-k,1}^\mu \gamma^\mu(1 - \gamma^5) u_{q,e}^\nu.$$  

(39)

Eqs. (36) and (39) can be combined to give the branching ratio

$$\frac{\Gamma(W^+ \rightarrow e^+ + \nu_p)}{\Gamma(W^+ \rightarrow e^+ + \nu_e)} \sim \sin^2 \frac{\theta}{2} \frac{(\Delta m)^2}{4k^2}.$$  

(40)

This result clearly shows that the use of Pontecorvo flavor states leads to a violation of the lepton charge in the production vertex. The result (40) is derived in the relativistic limit; however, the lepton charge violation effect is more significant in the non-relativistic region (see Eqs. (33) and (37)).

In the above treatment, we have not considered explicitly the $W^+$ decay width $\Gamma$. This should be taken into account when comparing our results with the ones of Ref. 17. However, the fact that the amplitude $A_{W^+ \rightarrow e^+ + \nu_p}$ calculated with the exact flavor states vanishes is independent of the inclusion of the decay width in the calculation.

4. Conclusions

In this paper, we have analyzed the amplitudes of the weak interaction processes where flavor neutrinos are generated. We have done explicit computations at tree level for the processes $W^+ \rightarrow e^+ + \nu_e$ and $W^+ \rightarrow e^+ + \nu_\mu$ using the exact flavor states and the Pontecorvo states. We have considered the above amplitudes in the short time limit, i.e. at very small distances from the production vertex. In this case, we found that the use of the exact flavor states in the computations leads to consistent results, whereas the Pontecorvo states yield a violation of the lepton charge in the vertex. Consistency with the SM phenomenology is thus attained only for the QFT exact flavor states.

In order to better understand the results presented above, we observe that the amplitudes in the short time limit give information on the decay process very close to the vertex. Thus, one can associate a wavefunction, say $u_{k,\nu_e}^r$, with the electron neutrino in the amplitudes given by Eqs. (29) and (33). In the case of exact flavor states, the amplitude given by Eq. (29) suggests that $u_{k,\nu_e}^r = u_{k,1}^r$, i.e. the wavefunction for $\nu_e$ is the same as the one for $\nu_1$, with $u_{k,1}^r u_{k,1}^\dagger = 1$. On the other hand, in the case of Pontecorvo states, the amplitude given by Eq. (33) leads to the identification:

$$u_{k,\nu_e}^r = \cos^2 \theta u_{k,1}^r + \sin^2 \theta u_{k,2}^r.$$  

(41)

Such a wavefunction, however, is not normalized properly as one can easily see:

$$u_{k,\nu_e}^\dagger u_{k,\nu_e}^r = \cos^4 \theta + \sin^4 \theta + 2 \sin^2 \theta \cos^2 \theta |U_k|,$$  

(42)
where we have used Eq. (A.8). Since $|U_k| < 1$ for $m_1 \neq m_2$, the above wavefunction is not normalized.

Note also that the amplitude Eq. (37) contains the combination $u_{k, \nu e}^r = (u_{k,2} - u_{k,1}) \sin \theta \cos \theta$, which is also not normalized:

$$u_{k, \Delta \nu e}^r = 2 \sin^2 \theta \cos^2 \theta \left(1 - |U_k|\right). \tag{43}$$

This is just the missing piece necessary for the normalization of $u_{k, \nu e}^r$ in Eq. (41):

$$u_{k, \Delta \nu e}^r u_{k, \Delta \nu e}^r + v_{k, \nu e}^r v_{k, \nu e}^r = 1. \tag{44}$$

In conclusion, a violation of lepton charge in the production vertex is due to the incorrect treatment of the flavor neutrino states. Defining them as the eigenstates of flavor charges results consistent with Standard Model are found.

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### Appendix A. The vacuum structure for fermion mixing

We briefly summarize the QFT formalism of the neutrino mixing. For a detailed review see [1]. The mixing transformations are

$$\nu_e(x) = \cos \theta \nu_1(x) + \sin \theta \nu_2(x) \tag{A.1}$$

$$\nu_\mu(x) = -\sin \theta \nu_1(x) + \cos \theta \nu_2(x),$$

where $\nu_e(x)$ and $\nu_\mu(x)$ are the Dirac neutrino fields with definite flavors. Here, $\nu_1(x)$ and $\nu_2(x)$ are the free neutrino fields with definite masses $m_1$ and $m_2$, respectively. The fields $\nu_1(x)$ and $\nu_2(x)$ can be written as

$$\nu_i(x) = \frac{1}{\sqrt{2}} \sum_{k, r} \left[u_{k,i}^r \alpha^r_{k,i}(t) + v_{-k,i}^r \beta^r_{-k,i}(t)\right] e^{i k \cdot x}, \quad i = 1, 2 \tag{A.2}$$

with $\alpha_{k,i}^r(t) = \alpha_{k,i}^r e^{-i \omega_{k,i} t}$, $\beta_{k,i}^r(t) = \beta_{k,i}^r e^{i \omega_{k,i} t}$, and $\omega_{k,i} = \sqrt{k^2 + m_i^2}$. The operator $\alpha_{k,i}^r$ and $\beta_{k,i}^r$, $i = 1, 2$, $r = 1, 2$ are the annihilation operators for the vacuum state $|0\rangle_m \equiv |0\rangle_1 \otimes |0\rangle_2$: $\alpha_{k,i}^r |0\rangle_m = \beta_{k,i}^r |0\rangle_m = 0$. The anticommutation relations are: \(\{\nu_i^l(x), \nu_j^{l'}(y)\}_{t=t'} = \delta^3(x-y) \delta_{\alpha\beta} \delta_{ij}\), with $\alpha, \beta = 1, \ldots, 4$, and \(\{\alpha_{k,i}^{l}, \alpha_{q,j}^{l'}\} = \delta_{kq} \delta_{\alpha\beta} \delta_{ij}; \{\beta_{k,i}^{l}, \beta_{q,j}^{l'}\} = \delta_{kq} \delta_{\alpha\beta} \delta_{ij}\), with $i, j = 1, 2$. All other anticommutators vanish. The orthonormality and completeness relations are given by $u_{k,1}^r u_{k,1}^{r'} = \delta_{rs}$, $u_{k,2}^r u_{-k,2}^{r'} = \delta_{rs}$, $u_{k,1}^r u_{k,2}^{r'} = 0$, and $\sum_r (u_{k,i}^r v_{k,j}^{r'} + v_{-k,i}^r u_{-k,j}^{r'}) = \delta_{ij}$.
The generator of the mixing transformations is given by\[^2\]
\[
G_\theta(t) = \exp \left[ \theta \int d^3x \left( \nu^1_i(x) \nu_2(x) - \nu^1_2(x) \nu_1(x) \right) \right]
\] (A.3)
and \(\nu^\sigma(x) = G_\theta^{-1}(t) \nu^\sigma_i(x) G_\theta(t)\) for \((\sigma, i) = (e, 1)\) and \((\mu, 2)\). At finite volume, this is a unitary operator, \(G_\theta^{-1}(t) G_\theta(t) = G_\theta^0(t)\), preserving the canonical anticommutation relations. The generator \(G_\theta^{-1}(t)\) maps the Hilbert space for free fields \(\mathcal{H}_m\) to the Hilbert space for mixed fields \(\mathcal{H}_f: G_\theta^{-1}(t) : \mathcal{H}_m \mapsto \mathcal{H}_f\). In particular, the flavor vacuum is given by \(|0(t)\rangle = G_\theta^{-1}(t) |0\rangle_m\) at finite volume \(V\). We denote by \(|0\rangle_f\) the flavor vacuum at \(t = 0\). In the infinite volume limit, the flavor and the mass vacua are unitarily inequivalent\[^{[10]}\]. Similarly, flavor vacua at different times are orthogonal\[^{[10]}\].

The flavor fields are written as:
\[
\nu_\sigma(x, t) = \frac{1}{\sqrt{V}} \sum_{k, r} e^{ik \cdot x} \left[ u^r_{k, 1} \alpha^r_{k, \nu_\sigma}(t) + v^r_{-k, 1} \beta^r_{-k, \nu_\sigma}(t) \right],
\] (A.4)
with \((\sigma, i) = (e, 1), (\mu, 2)\). The flavor annihilation operators are\[^2\]
\[
\alpha^r_{k, \nu_\sigma}(t) = \cos \theta \alpha^r_{k, 1}(t) + \sin \theta \sum_s \left[ u^s_{k, 1} u^s_{k, 2} \alpha^s_{k, 2}(t) + u^r_{k, 1} v^s_{-k, 2} \beta^s_{-k, 2}(t) \right]
\]
\[
\beta^r_{-k, \nu_\sigma}(t) = \cos \theta \beta^r_{-k, 1}(t) + \sin \theta \sum_s \left[ v^s_{-k, 2} u^s_{-k, 1} \beta^s_{-k, 2}(t) + v^r_{k, 2} v^s_{-k, 1} \alpha^s_{k, 2}(t) \right]
\]
and similar expressions hold for muon neutrinos. In the reference frame where \(k = (0, 0, |k|)\), we have
\[
\alpha^r_{k, \nu_\sigma}(t) = \cos \theta \alpha^r_{k, 1}(t) + \sin \theta \left( |U_k| \alpha^r_{k, 2}(t) + e^r |V_k| \beta^r_{-k, 2}(t) \right),
\] (A.5)
\[
\beta^r_{-k, \nu_\sigma}(t) = \cos \theta \beta^r_{-k, 1}(t) + \sin \theta \left( |U_k| \beta^r_{-k, 2}(t) - e^r |V_k| \alpha^r_{k, 2}(t) \right),
\] (A.6)
and similar ones for \(\alpha^r_{k, \nu}\) and \(\beta^r_{-k, \nu}\). In Eq. (A.5), \(e^r = (-1)^r\) and
\[
|V_k| \equiv e^r |u^r_{k, 1} u^r_{-k, 2} - e^r |u^r_{k, 2} v^r_{-k, 1}| = \frac{(\omega_{k, 1} + m_1) - (\omega_{k, 2} + m_2)}{2 \sqrt{\omega_{k, 1} \omega_{k, 2} (\omega_{k, 1} + m_1) (\omega_{k, 2} + m_2)}} |k|;
\] (A.7)
\[
|U_k| \equiv v^r_{k, 1} u^r_{k, j} = v^r_{-k, 1} v^r_{-k, j} = \frac{|k|^2 + (\omega_{k, 1} + m_1) (\omega_{k, 2} + m_2)}{2 \sqrt{\omega_{k, 1} \omega_{k, 2} (\omega_{k, 1} + m_1) (\omega_{k, 2} + m_2)}},
\] (A.8)
with \(i, j = 1, 2, i \neq j\). We have: \(|U_k|^2 + |V_k|^2 = 1\). Note that the following relations hold:
\[
\overline{u}^r_{k, 1} \sum_s u^s_{k, 1} u^s_{k, 2} + \overline{v}^r_{-k, 1} \sum_s v^s_{-k, 1} u^s_{k, 2} = \overline{u}^r_{k, 2},
\] (A.9)
\[
\overline{u}^r_{k, 1} \sum_s u^s_{k, 1} v^s_{-k, 2} + \overline{v}^r_{-k, 1} \sum_s v^s_{-k, 1} v^s_{-k, 2} = \overline{v}^r_{-k, 2}.
\] (A.10)
In the reference frame where \( k = (0,0,|k|) \), they become
\[
\pi_{k,1} |U_k| - \varepsilon^r \pi_{-k,1} |V_k| = \pi_{k,2} \quad \text{(A.11)}
\]
\[
\pi_{k,1} |V_k| + \varepsilon^r \pi_{-k,1} |U_k| = \varepsilon^r \pi_{-k,2} . \quad \text{(A.12)}
\]
Moreover, we have
\[
\pi^{+}_{k,2} |U_k| + \varepsilon^r \pi^{-}_{k,2} |V_k| = \pi^{-}_{k,1} . \quad \text{(A.13)}
\]

**Appendix B. Comment on the amplitudes in the long time limit**

Let us now comment on the amplitudes for the decay processes (1) and (2) computed in the long time limit as done in Ref.\(^{19}\). There it was argued that the non zero result in such a limit for the amplitude Eq.(2) implies a flavor violation. We point out, however, that such a “problem” arises also with Pontecorvo states. Indeed, considering \( x_{out}^0 \to -\infty \) and \( x_{out}^0 \to +\infty \), the non-vanishing amplitude \( A_{W^+\to e^+\nu_\mu}^P \) is obtained from Eq.(26):
\[
A^P_{W^+\to e^+\nu_\mu} = \frac{i g}{2\sqrt{2}(2\pi)^{1/2}} \frac{\varepsilon_{\mu,\nu_\lambda}}{\sqrt{2E_p^W}} \sin \theta \cos \theta \delta^3(p - q - k)
\]
\[
\times \left[ e^{-iw_k,2s_{out}} \pi_{k,2}^\gamma \gamma^\mu (1 - \gamma^5) v_{q,e}^s \delta(E_p^W - E_q^e - \omega_{k,2})
\right.
\]
\[
\left. - e^{-iw_k,1s_{out}} \pi_{k,1}^\gamma \gamma^\mu (1 - \gamma^5) v_{q,e}^s \delta(E_p^W - E_q^e - \omega_{k,1}) \right] . \quad \text{(B.1)}
\]

In a similar way, the amplitude \( A^P_{W^+\to e^+\nu_\mu} \) Eq.(24) becomes
\[
A^P_{W^+\to e^+\nu_\mu} = \frac{i g}{2\sqrt{2}(2\pi)^{1/2}} \frac{\varepsilon_{\mu,\nu_\lambda}}{\sqrt{2E_p^W}} \delta^3(p - q - k)
\]
\[
\times \left[ \cos^2 \theta e^{-iw_k,1s_{out}} \pi_{k,1}^\gamma \gamma^\mu (1 - \gamma^5) v_{q,e}^s \delta(E_p^W - E_q^e - \omega_{k,1})
\right.
\]
\[
\left. + \sin^2 \theta e^{-iw_k,2s_{out}} \pi_{k,2}^\gamma \gamma^\mu (1 - \gamma^5) v_{q,e}^s \delta(E_p^W - E_q^e - \omega_{k,2}) \right] . \quad \text{(B.2)}
\]

For the exact flavor states, one obtains from Eq. (16) (and Eq. (21)) results which reproduce Eq. (15.2) (and Eq. (15.1)) in the relativistic limit.

As already observed, the mixed neutrinos cannot be considered as asymptotic fields. Considering then the long time limit amounts to average over the flavor oscillations. Thus it is not surprising that the amplitude \( A_{W^+\to e^+\nu_\mu} \) gives a non zero result. In the long time limit the energy conservation is made explicit by the presence of the delta functions.

For the case of exact flavor states, the obtained results reproduce Eqs.(3.9) and (3.13) of Ref.\(^{19}\). In such a case, terms due to the neutrino condensate are also present and are proportional to the \( |V_k| \) function. We point out that one should not be misled (as in Ref\(^{19}\)) by the sign of the corresponding energies in the delta functions, since the negative \( \omega_{k,2} \), appearing in Eqs.(3.9) and (3.13) of Ref.\(^{19}\), is associated to “hole” contributions in the flavor vacuum condensate. Contrary to the claim of the authors of\(^{19}\), there is nothing paradoxical or wrong in these signs.
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