On a problem of J. Nakagawa, K. Sakamoto, M. Yamamoto

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Abstract
In this paper, we give a positive answer to a problem posed by Nakagawa, Sakamoto and Yamamoto concerning a nonlinear equation with a fractional derivative.

Keywords: Fractional differential equation, global existence, asymptotic behavior, blow-up time, blow-up profile.

1. Introduction

In their overview paper concerning the mathematical analysis of fractional equations, Nakagawa, Sakamoto and Yamamoto [1] posed the problem concerning global solutions and blowing-up in a finite time of solutions to the equation

\[
\begin{align*}
&C_D^\alpha_0 u(t) = -u(t)(1 - u(t)), \quad t > 0, \\
&u(0) = u_0,
\end{align*}
\]

where \(C_D^\alpha_0\) is the Caputo derivative defined for \(g \in C^1[0, T]\) by

\[
C_D^\alpha_0 g(t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - \tau)^{-\alpha} g'(\tau) \, d\tau,
\]

for \(0 < \alpha < 1\).

Let us recall, in the case \(\alpha = 1\), the results concerning solutions of (1):

- For \(0 < u(0) < 1\), the solution exists globally. Moreover,
  \[|u(t)| \leq \frac{1}{e'(1 - u_0)} \longrightarrow 0, \quad \text{as} \quad t \longrightarrow +\infty.\]

- For \(u(0) > 1\), the solution can not exist globally.

Here, we show that the same conclusions are valid for equation (1). Moreover we analyse:

1. The large time behavior of the global solution.
2. The blow-up time and profile of the blowing-up solutions.

Note that if we set \(w = u - 1\), then (1) reads

\[
C_D^\alpha_0 w(t) = w(t)(1 + w(t)),
\]

which describes the evolution of a certain species; the reaction term \(w(1 + w)\) describes the law of increase of the species.
2. Preliminaries

In this section, we present some definitions and results concerning fractional calculus that will be used in the sequel. For more information see [2].

The Riemann-Liouville fractional integral of order \(0 < \alpha < 1\) of the integrable function \(f : \mathbb{R}^+ \rightarrow \mathbb{R}\) is

\[
J_0^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) \, d\tau, \quad t > 0,
\]

where \(\Gamma(\alpha)\) is the Euler Gamma function.

The Riemann-Liouville fractional derivative of an absolutely continuous function \(f(t)\) of order \(0 < \alpha < 1\) is

\[
D_0^\alpha f(t) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} f(\tau) \, d\tau.
\]

The Caputo fractional derivative of an absolutely continuous function \(f(t)\) of order \(0 < \alpha < 1\) is defined by

\[
C^{D_0^\alpha} f(t) := J_0^{1-\alpha} f'(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} f'(\tau) \, d\tau.
\]

Both derivatives present a drawback :
- The Riemann-Liouville derivative of a constant is different from zero,
  \(D_0^\alpha C \neq 0\),
while the Caputo derivative require \(f'(t)\) to calculate \(C^{D_0^\alpha} f(t)\), for \(0 < \alpha < 1\).
- We know that the Riemann-Liouville derivative of the Weierstrass function exists for any \(0 < \alpha < 1\), but not for \(\alpha = 1\).

But for regular function with \(f(0) = 0\), both definitions coincide.

Next, we recall a lemma that will be used hereafter.

**Lemma 2.1.** (see [3]). Let \(a, b, K, \psi\) be non-negative continuous functions on the interval \(I = (0, T_1)(0 < T \leq \infty)\), let \(\omega : (0, \infty) \rightarrow \mathbb{R}\) be a continuous, non-negative and non-decreasing function with \(\omega(0) = 0\) and \(\omega(u) > 0\) for \(u > 0\), and let \(A(t) = \max_{0 \leq s \leq t} a(s)\) and \(B(t) = \max_{0 \leq s \leq t} b(s)\). Assume that

\[
\psi(t) \leq a(t) + b(t) \int_0^t K(s) \omega(\psi(s)) \, ds, \quad t \in I.
\]

Then

\[
\psi(t) \leq H^{-1}\left[H(A(t)) + B(t) \int_0^t K(s) ds\right], \quad t \in (0, T_1),
\]

where \(H(v) = \int_{v_0}^v \frac{dv}{\omega(v)}\) (\(v \geq v_0 > 0\)), \(H^{-1}\) is the inverse of \(H\) and \(T_1 > 0\) is such that \(H(A(t)) + B(t) \int_0^t K(s) ds \in D(H^{-1})\) for all \(t \in (0, T_1)\).

Here, we consider the problem

\[
\begin{cases}
C^{D_0^\alpha} u(t) = -u(t)(1-u(t)), \\ u(0) = u_0,
\end{cases}
\]

for \(0 < \alpha < 1\) and \(u_0 > 0\).
3. Main results

The local existence of solutions to (2) is assured by the

**Theorem 3.1. (see [2])**. We consider the fractional differential equation of Caputo’s type given by

\[
\begin{cases}
\mathcal{C}D_t^\alpha u(t) = f(t, u(t)), & t > 0, \\
u(0) = u_0,
\end{cases}
\]

(3)

For \(0 < \alpha < 1\), \(u_0 \in \mathbb{R}\), \(b > 0\) and \(T > 0\).

Assume that

1. \(f \in C(R_0, \mathbb{R})\) where \(R_0 = \{(t, u), \ 0 \leq t \leq T, \ |u - u_0| \leq b\}\) and \(|f(t, u)| \leq M\) on \(R_0\);
2. \(|f(t, u) - f(t, v)| \leq L|u - v|, \ L > 0, \ (t, u) \in R_0\).

Then there exists a unique solution \(u \in C([0, h])\) for (3), where \(h = \min \left\{ T, \left(\frac{b\Gamma(\alpha + 1)}{M} \right)^\frac{1}{\alpha}\right\}\).

**Theorem 3.2.** Let \(u\) be the solution of problem (2). We have:

- If \(0 < u_0 < 1\), the solution is global and it satisfies \(0 < u < 1\). Moreover, \(u\) is given by

\[
u(t) = E_{\alpha}(-t^\alpha)u_0 + \int_0^t (t - s)^{\alpha-1}E_{\alpha,\alpha}(-(t-s)^\alpha) \ u^2(s)ds,
\]

and for some constants \(c > 0\) and \(c_1 > 0\), we have

\[
0 < u(t) \leq \frac{1}{c - c_1t^\alpha}, \quad t > T_0 := \left(\frac{\alpha}{c_1u_0}\right)^\frac{1}{\alpha}.
\]

- If \(u_0 > 1\), the solution blows-up in a finite time \(T^*\):

\[
\lim_{t \to T^*} u(t) = +\infty.
\]

Moreover, we have the bilateral estimate:

\[
\tilde{w}(t) + 1 \leq u(t) \leq \tilde{w}(t) + 1,
\]

and

\[
\left(\frac{\Gamma(\alpha + 1)}{4(u_0 - \frac{1}{2})}\right)^\frac{1}{\alpha} \leq T^* \leq \left(\frac{\Gamma(\alpha + 1)}{u_0 - \frac{1}{2}}\right)^\frac{1}{\alpha},
\]

where

\[
\tilde{w}(t) + \frac{1}{2} \frac{\Gamma(2\alpha)}{\Gamma(\alpha)}(T_{\tilde{w}} - t)^{-\alpha}, \text{ as } t \to T_{\tilde{w}},
\]

\[
\tilde{w}(t) \sim \frac{\Gamma(2\alpha)}{\Gamma(\alpha)}(T_{\tilde{w}} - t)^{-\alpha}, \text{ as } t \to T_{\tilde{w}}.
\]

Here, \(T_{\tilde{w}}\) is the blow-up time of \(\tilde{w}\), which satisfies

\[
\left(\frac{\Gamma(\alpha + 1)}{4(u_0 - \frac{1}{2})}\right)^\frac{1}{\alpha} \leq T_{\tilde{w}} \leq \left(\frac{\Gamma(\alpha + 1)}{u_0 - \frac{1}{2}}\right)^\frac{1}{\alpha},
\]

and \(T_{\tilde{w}}\) is the blow-up time of \(\tilde{w}\), which satisfies

\[
\left(\frac{\Gamma(\alpha + 1)}{4(u_0 - \frac{1}{2})}\right)^\frac{1}{\alpha} \leq T_{\tilde{w}} \leq \left(\frac{\Gamma(\alpha + 1)}{u_0 - \frac{1}{2}}\right)^\frac{1}{\alpha}.
\]
Proof of Theorem 3.2.

Part 1. If $0 < u_0 < 1$, then the solution is global. The solution to (2) is given by

$$u(t) = E_{\alpha}(-t^\alpha)u_0 + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-(t-s)^\alpha)u^2(s)\,ds.$$  \hfill (4)

Where the Mittag-Leffler functions $E_{\alpha}(-t^\alpha)$ and $E_{\alpha,\alpha}(-t^\alpha)$ are defined by:

$$E_{\alpha}(-t^\alpha) = \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(j\alpha + 1)} t^{j\alpha},$$

$$E_{\alpha,\alpha}(-t^\alpha) = \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(j\alpha + \alpha)} t^{j\alpha}.$$

If $u_0 > 0$, then $u(t) > 0$ as $E_{\alpha}(-t^\alpha) > 0$ and $E_{\alpha,\alpha}(-t^\alpha) > 0$.

Now, we set the function $\bar{u}(t) = 1, \ t > 0$. As $0 < u_0 < 1$, then $u_0 < \bar{u}(0)$. In addition, we have

$$c^\alpha D_0^\alpha \bar{u}(t) = 0 = -\bar{u}(t)(1 - \bar{u}(t)).$$

Hence $\bar{u}$ is an upper solution of the equation (2), and we have $u(t) < \bar{u}(t) = 1$, (see [4], Thm. 2.4.3, p. 32).

Now, we examine the large time behavior of the global solution $0 < u < 1$. For, let us recall the estimates (see [5]):

- For $0 < \alpha < 1$, there exists a constant $c > 0$ such that

  $$0 < E_{\alpha}(-t^\alpha) \leq \frac{c}{1 + t^\alpha} \leq c, \ t > 0.$$  \hfill (5)

- For $0 < \alpha < 1$, there exists a constant $c_1 > 0$ such that

  $$0 < t^{\alpha-1}E_{\alpha,\alpha}(-t^\alpha) \leq c_1 t^{\alpha-1}, \ t > 0.$$  \hfill (6)

From (4) and using the inequalities (5) and (6), we obtain

$$u(t) \leq cu_0 + c_1 \int_0^t (t-s)^{\alpha-1}u^2(s)\,ds.$$  \hfill (7)

We apply Lemma 2.1 to (7) with $\omega(x) = x^2, \ K(s) = (t-s)^{\alpha-1}, \ A(t) = cu_0, \ B(t) = c_1$.

For $t > T_0$, we have

$$H(cu_0) + \frac{c_1}{\alpha} t^\alpha \in D(H^{-1}),$$

where $H(v) = \frac{1}{v_0} - \frac{1}{v}$ and $H^{-1}(z) = \frac{1}{v_0 - z}, \ z \neq \frac{1}{v_0}$.

So we obtain,

$$u(t) \leq H^{-1}\left[H(cu_0) + \frac{c_1}{\alpha} t^\alpha\right].$$

Therefore

$$u(t) \leq \frac{1}{\frac{1}{v_0} - \frac{c_1}{\alpha} t^\alpha}, \ t > T_0.$$  \hfill (8)

Part 2. If $u_0 > 1$, then the solution blows-up in a finite time.
1. We show that \( u > 1 \). For, let us define the new unknown function \( w = u - 1 \). The function \( w \) satisfies

\[
\begin{align*}
\begin{cases}
^{\mathbb{C}D}_0^\alpha w(t) &= w(t)(1 + w(t)), \\
w(0) &= w_0 = u_0 - 1.
\end{cases}
\end{align*}
\]

As \( u_0 > 1 \), then \( w_0 > 0 \). Moreover, we have \((8)\)

\[
w(t) = E_\alpha(t^\alpha)w_0 + \int_0^t (t - s)^{\alpha - 1}E_{\alpha,\alpha}((t - s)^\alpha)w^2(s)ds.
\]

Therefore, \( w > 0 \); hence \( u > 1 \).

2. We prove that \( u \) blows-up in a finite time.

Since we have \( w(t) = u(t) - 1 \), it is seen that if \( u(t) \to \infty \) as \( t \to T^* \), then \( w(t) \to \infty \) as \( t \to T^* \) and vice versa. That is \( w \) and \( u \) will have the same blow-up time.

We now must examine the blow-up properties of \( w \), the solution of problem \((8)\). These are obtained by comparing \( w(t) \) with the solutions of the following problems:

\[
\begin{align*}
\begin{cases}
^{\mathbb{C}D}_0^\alpha \tilde{w}(t) &= \tilde{w}^2(t), \\
\tilde{w}(0) &= w_0,
\end{cases}
\end{align*}
\]

and

\[
\begin{align*}
\begin{cases}
^{\mathbb{C}D}_0^\alpha \tilde{w}(t) &= (\tilde{w}(t) + \frac{1}{2})^2, \\
\tilde{w}(0) &= w_0.
\end{cases}
\end{align*}
\]

We see by comparison \((4)\) that

\[
\tilde{w}(t) \leq w(t) \leq \tilde{w}(t), \quad 0 \leq t < \min\{T_{\tilde{w}}, T_w\}.
\]

Following the paper of Kirk, Olmstead and Roberts \([6]\), we may assert that the solution \( \tilde{w} \) (resp. \( \tilde{w} \)) blows-up in a finite time \( T_{\tilde{w}} \) (resp. \( T_{\tilde{w}} \)), such that

\[
\left( \frac{\Gamma(\alpha + 1)}{4w_0} \right)^{\frac{1}{2}} \leq T_{\tilde{w}} \leq \left( \frac{\Gamma(\alpha + 1)}{w_0} \right)^{\frac{1}{2}},
\]

and

\[
\left( \frac{\Gamma(\alpha + 1)}{4(w_0 + \frac{1}{2})} \right)^{\frac{1}{2}} \leq T_{\tilde{w}} \leq \left( \frac{\Gamma(\alpha + 1)}{w_0 + \frac{1}{2}} \right)^{\frac{1}{2}}.
\]

So we have the following estimates

\[
T_{\tilde{w}} \leq T^* \leq T_{\tilde{w}}.
\]

Whereupon

\[
\left( \frac{\Gamma(\alpha + 1)}{4(w_0 + \frac{1}{2})} \right)^{\frac{1}{2}} \leq T^* \leq \left( \frac{\Gamma(\alpha + 1)}{w_0} \right)^{\frac{1}{2}}.
\]

\(\square\)

4. Numerical implementation

In this section, we will approximate the solution \( u \) given by \((4)\). For, we need a numerical approximation of the convolution integral; this can be obtained using the convolution quadrature method.
As it has been explained in [7], a convolution quadrature approximates the continuous convolution
\[ \int_0^t K(t - s)f(s) \, ds, \quad t > 0, \]
by a discrete convolution with a step size \( h > 0 \). Then
\[ \int_0^t K(t_n - s)f(s) \, ds \sim \sum_{j=0}^{n} \omega_{n-j}f(t_j), \]
where \( t_j = jh, \ j = 0, 1, 2, ..., n \) and the convolution quadrature weights \( \omega_j \) are determined from their generating power series as
\[ \sum_{j=0}^{\infty} \omega_j \zeta^j = L\{K(t) : \frac{\delta(\zeta)}{h}\}. \]
Here \( L[K(t) : s] \) is the Laplace transform of \( K(t) \) and \( \delta(\zeta) \) is the generating polynomial for a linear multistep method.

Let \( u_n \) be the approximation of \( u(t_n) \) for \( n \geq 0 \). Using the convolution quadrature method we obtain
\[ u_n = (1 - \omega_0)^{-1}\left[ E_\alpha(-t^n)u_0 + \sum_{j=0}^{n-1} \omega_{n-j}u_j \right], \quad n = 1, 2, 3, ... \]

Now, we introduce the following algorithm which gives the numerical approximation of solution to equation (2).
Algorithm
Input : Give $0 < \alpha < 1$ and $u_0$, $u_0 > 1$.
Initializations : Discretize the time with a step size $h > 0$; $t_i = ih$, for all $i = 1, 2, ..., n$, $u_i^{appx} = u_0$, $u^1 = (u_0)^2$.
Step 1 : Approximate the Mittag-Leffler function $GML$.
Step 2 : Calculate convolution quadrature weights $W$ using the fast Fourier transform (FFT).
Step 3 : Calculate $u_i^{appx}$.
\[
\begin{align*}
\text{do} & \\
& u^i = GML * u_i^{appx} + W * u^{i-1}, \\
& u_i^{appx} = (1 - W(1))^{-1} * u^i, \\
& u^i = (u_i^{appx})^2, \\
& i = i + 1.
\end{align*}
\]
until ($u_i^{appx}$ blows up) or ($i > n$).
Output : Numerical approximation of $u$.

Example 1. For Figure 1, we set $\alpha = 0.5$; the initial conditions are respectively $u_0 = 5$, $u_0 = 3$ and $u_0 = 2$.
For Figure 2, we take the initial condition $u_0 = 5$ and we plot the solutions; the dotted curve is the solution for $\alpha = 0.3$ and the solid curve corresponds to the solution for $\alpha = 0.5$.

As it has been proved, the solution blows up in a finite time which depends on $u_0$ and $\alpha$.

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