Exponential Integration of the Linear Assignment Flow

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1 Motivation and Preliminaries

Recently, the paper [2] introduced the assignment flow which provides a novel approach to the image labeling problem, i.e. the task of finding a function (called labeling) which maps each pixel \( i \) from a set of pixels \( I \) to a label \( j \) from an application-dependent prespecified set of labels \( J \). For most applications in the field of image analysis and beyond, a good labeling is an adequate compromise between adhering to the image data and being spatially coherent.

The assignment flow is defined as a dynamical system evolving on the assignment manifold \( \mathcal{W} = \{ W \in \mathbb{R}^{[I] \times [J]} : W_{ij} > 0, W\mathbb{1} = \mathbb{1} \} \), which is the set of row-stochastic matrices with full support. Endowed with the Fisher-Rao (information) metric on the tangent space \( \mathcal{T}_0 \), it becomes a Riemannian manifold. The replicator operator \( R_W \) and the similarity matrix \( S(W) \), both introduced in [2], define the assignment flow as

\[
\dot{W}(t) = R_W S(W(t)), \quad W(0) = W_0.
\]

After following the trajectory for some time, we round the current assignment to a vertex of the (closure of the) assignment manifold which corresponds to a labeling.

2 Linear Assignment Flow

By means of the exponential map corresponding to the geodesics of the affine \( e \)-connection from information geometry, the assignment flow can be represented by an ordinary differential equation (ODE) on the tangent space \( \mathcal{T}_0 \) at the barycenter of \( \mathcal{W} \). Linearizing this ODE yields the linear assignment flow which was introduced in [3]:

\[
\dot{V}(t) = AV(t) + a, \quad V(0) = 0, \quad V \in \mathcal{T}_0 \subset \mathbb{R}^{[I] \times [J]},
\]

where \( A \in \mathbb{R}^{[I] \times [J] \times [I] \times [J]} \) is a suitably chosen matrix that relates neighboring pixels in the image to each other. The vector \( a \in \mathbb{R}^{[I]} \) incorporates the image data.

3 Exponential Integration

Among various integration methods discussed in [3], we consider here exponential integration of the linear assignment flow. The solution of the autonomous linear ODE (2) can be written as (see [4])

\[
V(t) = \int_0^t \varphi_0((t - \tau)A) a \, d\tau = t \varphi_1(tA)a \quad \text{with} \quad \varphi_p(A) = \sum_{k=0}^{\infty} \frac{A^k}{(k + p)!}, \quad \forall p \in \mathbb{N}_0,
\]

where \( \varphi_0 \) denotes the matrix exponential. Even for small images the matrix \( A \) is too large to make the computation of \( \varphi_1(tA) \) feasible. However, we only need the action of \( \varphi_1(tA) \) on the vector \( a \). This action can be approximated with Krylov subspaces \( \mathcal{K}_m = \text{span}\{a, Aa, \ldots, A^{m-1}a\} \) of order \( m \).

Let \( V_m \) be the matrix consisting columnwise of orthonormal basis vectors of the Krylov subspace with the first column equal to \( a \). The matrices \( V_m \) and \( H_m = V_m^T A V_m \) are computed with the Arnoldi iteration [5]. With \( A \approx V_m H_m V_m^T \) and the first unit vector \( e_1 \), we get \( \varphi_1(tA)a \approx \varphi_1(V_m H_m V_m^T) a = V_m \varphi_1(H_m) V_m^T a = \|a\| V_m^T \varphi_1(H_m) e_1 \) and thus

\[
V(t) \approx t\|a\| V_m \varphi_1(tH_m) e_1.
\]

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For the $m \times m$ matrix $H_m$ we can compute $\varphi_1(tH_m)$ using the matrix exponential [6] of the $(m+1) \times (m+1)$ matrix [7]
\[
\varphi_0 \left( \begin{pmatrix} tH_m & e_1 \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} \varphi_0(tH_m) & t\varphi_1(tH_m)e_1 \\ 0 & 1 \end{pmatrix}.
\]
(5)

As the spectral radius of $A$ satisfies $\rho(A) \leq \frac{1}{2}$, we have the following error estimate [8] for the Krylov approximation (4)
\[
\left\| V(t) - \xi_0 \| a \| V_m \varphi_1(tH_m)e_1 \right\| \leq 2\| a \| \frac{\rho^{m+1}\epsilon}{(m+1)!} \leq 2\| a \| \frac{\epsilon}{(m+1)!}.
\]
(6)

For common evaluation times $t$ and vector sizes $\| a \|$ this estimate ensures small errors already for Krylov orders $m \ll |I||J|$. Thus the exponential integration with Krylov approximation produces a labeling which is close to the labeling returned by the linear assignment flow when using an alternative accurate (and more costly) numerical integration method [3]. In addition, an increase of the image size only has a small effect on $\| a \|$. Therefore, a constant Krylov order $m$ can be used for a broad range of images sizes. Due to the final rounding from assignments to labelings, the error does not need to be equal or close to zero to produce a labeling that is close to the labeling returned by the full nonlinear assignment flow.

4 Numerical Results and Outlook

Figure 1 displays a labeling determined by the full nonlinear assignment flow and a corresponding labeling returned by the linear assignment flow using the numerical method sketched above. The result illustrates the good approximation property of the linear assignment flow that already holds for small orders $m$ of the Krylov subspace approximation.

In addition, the exponential integration of the linear assignment flow can be carried out approximately two orders of magnitude faster than the iterative integration of the nonlinear assignment flow.

Fig. 1: For the mandrill image (left) we consider the labeling produced by the nonlinear (center) and the corresponding linear assignment flow with $m = 5$ (right). The almost identical labelings highlight the good approximation property of the linear assignment flow.

Possible future work includes the application of the linear assignment flow to the evaluation of graphical models [9] and to unsupervised labeling [10], and methods for representing and learning the spatial context of classes of images directly from data in terms of parameters that define the matrix $A$ of the linear assignment flow.

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