ON THE DE RHAM COMPLEX OF MIXED TWISTOR $\mathcal{D}$-MODULES

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Abstract. Given a complex manifold $S$, we introduce for each complex manifold $X$ a $t$-structure on the bounded derived category of $\mathbb{C}$-constructible complexes of $\mathcal{O}_S$-modules on $X \times S$. We prove that the de Rham complex of a holonomic $\mathcal{D}_{X \times S/S}$-module which is $\mathcal{O}_S$-flat as well as its dual object is perverse relatively to this $t$-structure. This result applies to mixed twistor $\mathcal{D}$-modules.

1. Introduction

Given a vector bundle $V$ of rank $d \geq 1$ with an integrable connection $\nabla : V \to \Omega^1_X \otimes V$ on a complex manifold $X$ of complex dimension $n$, the sheaf of horizontal sections $V^h = \ker \nabla$ is a locally constant sheaf of $d$-dimensional $\mathbb{C}$-vector spaces, and is the only nonzero cohomology sheaf of the de Rham complex $\text{DR}_{X}(V, \nabla) = (\Omega^\bullet_X \otimes V, \nabla)$. Assume moreover that $(V, \nabla)$ is equipped with a harmonic metric in the sense of [19, p. 16]. The twistor construction of [20] produces then a holomorphic bundle $\mathcal{V}$ on the product space $\mathcal{X} = X \times \mathbb{C}$, where the factor $\mathbb{C}$ has coordinate $z$, together with a holomorphic flat $z$-connection. By restricting to $\mathcal{X}^* := X \times \mathbb{C}^*$, giving such a $z$-connection on $\mathcal{V}^* := \mathcal{V}|_{\mathcal{X}^*}$ is equivalent to giving a flat relative connection $\nabla$ with respect to the projection $p : \mathcal{X}^* \to \mathbb{C}^*$. Similarly, the relative de Rham complex $\text{DR}_{\mathcal{X}^*/\mathbb{C}^*}(\mathcal{V}^*, \nabla)$ has cohomology in degree zero at most, and $(\mathcal{V}^*)^h := \ker \nabla$ is a locally constant sheaf of locally free $p^{-1}\mathcal{O}_{\mathbb{C}^*}$-modules of rank $d$.

Holonomic $\mathcal{D}_X$-modules generalize the notion of a holomorphic bundle with flat connection to objects having (possibly wild) singularities, and a well-known theorem of Kashiwara [2] shows that the solution complex of such a holonomic $\mathcal{D}_X$-module has $\mathbb{C}$-constructible cohomology, from which one can deduce that the de Rham complex is of the same kind and more precisely that both are $\mathbb{C}$-perverse sheaves on $X$ up to a shift by $\dim X$.

The notion of a holonomic $\mathcal{D}_X$-module with a harmonic metric has been formalized in [14] and [10] under the name of pure twistor $\mathcal{D}$-module (this
generalizes holonomic $\mathcal{D}_X$-modules with regular singularities), and then in [13] and [11] under the name of wild twistor $\mathcal{D}$-modules (this takes into account arbitrary irregular singularities). More recently, Mochizuki [12] has fully developed the notion of a mixed (possibly wild) twistor $\mathcal{D}$-module. When restricted to $X^*$, such an object contains in its definition two holonomic $\mathcal{D}_{X^*/\mathbb{C}}$-modules, and we say that both underlie a mixed twistor $\mathcal{D}$-module.

The main result of this article concerns the de Rham complex and the solution complex of such objects.

**Theorem 1.1.** The de Rham complex and the solution complex of a $\mathcal{D}_{X^*/\mathbb{C}}$-module underlying a mixed twistor $\mathcal{D}$-module are perverse sheaves of $p^{-1}\mathcal{O}_{\mathbb{C}}$-modules (up to a shift by $\dim X$).

In Section 2, we define the notion of relative constructibility and perversity. This applies to the more general setting where $p : X^* \to \mathbb{C}^*$ is replaced by a projection $p_X : X = X \times S \to S$, where $S$ is any complex manifold. We usually set $p = p_X$ when $X$ is fixed. On the other hand, we call holonomic any coherent $\mathcal{D}_{X \times S/S}$-module whose relative characteristic variety in $T^*(X \times S/S) = (T^*X) \times S$ is contained in a variety $\Lambda \times S$, where $\Lambda$ is a conic Lagrangian variety in $T^*X$. We say that a $\mathcal{D}_{X \times S/S}$-module is strict if it is $p^{-1}\mathcal{O}_S$-flat.

**Theorem 1.2.** The de Rham complex and the solution complex of a strict holonomic $\mathcal{D}_{X \times S/S}$-module whose dual is also strict are perverse sheaves of $p^{-1}\mathcal{O}_S$-modules (up to a shift by $\dim X$).

A $\mathcal{D}_{X^*/\mathbb{C}}$-module $\mathcal{M}$ underlying a mixed twistor $\mathcal{D}$-module is strict and holonomic (see [12]). Moreover, Mochizuki has defined a duality functor on the category of mixed twistor $\mathcal{D}$-modules, proving in particular that the dual of $\mathcal{M}$ as a $\mathcal{D}_{X^*/\mathbb{C}}$-module is also strict holonomic. Therefore, these results together with Theorem 1.2 imply Theorem 1.1.

Note that, while our definition of perverse objects in the bounded derived category $\mathbb{D}^b(p^{-1}\mathcal{O}_S)$ intends to supply a notion of holomorphic family of perverse sheaves, we are not able, in the case of twistor $\mathcal{D}$-modules, to extend this notion to the case when the parameter $z \in \mathbb{C}^* = S$ also achieves the value zero, and to define a perversity property in the Dolbeault setting of [19] for the associated Higgs module.

2. **Relative constructibility in the case of a projection**

We keep the setting as above, but $X$ is only assumed to be a real analytic manifold. Given a real analytic map $f : Y \to X$ between real analytic manifolds, we will denote by $f_!S$ (or $f$ if the context is clear) the map $f \times \text{id}_S : Y \times S \to X \times S$.

2.1. **Sheaves of $\mathbb{C}$-vector spaces and of $p^{-1}\mathcal{O}_S$-modules.** Let $f : Y \to X$ be such a map. There are functors $f^{-1}, f_!, Rf_!, Rf_!$ between $\mathbb{D}^b(C_{X \times S})$ and $\mathbb{D}^b(C_{Y \times S})$, and functors $f_S^{-1}, f_S_!, Rf_S_!, Rf_S_!$ between $\mathbb{D}^b(p^{-1}\mathcal{O}_S)$ and
D^b(p^{-1}_X \mathcal{O}_S). These functors correspond pairwise through the forgetful functor $D^b(p^{-1}_X \mathcal{O}_S) \to D^b(C_{X \times S})$. Indeed, this is clear except for $f'_S$ and $f'$. To check it, one decomposes $f$ as a closed immersion and a projection. In the first case, the compatibility follows from the fact that both are equal to $f^{-1}Rf_!(\mathcal{F})$ (see [5 Prop. 3.1.12]) and for the case of a projection one uses [5 Prop. 3.1.11 & 3.3.2]. We note also that the Poincaré-Verdier duality theorem [5 Prop. 3.1.10] holds on $D^b(p^{-1}_X \mathcal{O}_S)$ (see [5 Rem. 3.1.6(i)]). From now on, we will write $f^{-1}$, etc. instead of $f_S^{-1}$, etc.

The ring $p^{-1}_X \mathcal{O}_S$ is Noetherian, hence coherent (see [3 Prop. A.14]). For each $s_0 \in S$ let us denote by $\mathfrak{m}_{s_0}$ the ideal of sections of $\mathcal{O}_S$ vanishing at $s_0$ and by $i^*_{s_0}$ the functor

$$\text{Mod}(p^{-1}_X \mathcal{O}_S) \longrightarrow \text{Mod}(\mathbb{C}_X)$$

$$F \longrightarrow F \otimes_{p^{-1}_X \mathcal{O}_S} p^{-1}_X(\mathcal{O}_S/\mathfrak{m}_{s_0}).$$

This functor will be useful for getting properties of $D^b(p^{-1}_X \mathcal{O}_S)$ from well-known properties of $D^b(\mathbb{C}_X)$.

**Proposition 2.1.** Let $F$ and $F'$ belong to $D^b(p^{-1}_X \mathcal{O}_S)$. Then, for each $s_0 \in S$ there is a well-defined natural morphism

$$Li^*_{s_0}(R\mathcal{H}om_{p^{-1}(\mathcal{O}_S)}(F, F')) \to R\mathcal{H}om_{\mathbb{C}_X}(Li^*_{s_0}(F), Li^*_{s_0}(F'))$$

which is an isomorphism in $D^b(\mathbb{C}_X)$.

**Proof.** Let us fix $s_0 \in S$. The existence of the morphism follows from [3] (A.10). Moreover, since $p^{-1}_X \mathcal{O}_S$ is a coherent ring as remarked above and $p^{-1}_X(\mathcal{O}_S/\mathfrak{m}_{s_0})$ is $p^{-1}_X \mathcal{O}_S$-coherent, we can apply the argument given after (A.10) in loc. cit. to show that it is an isomorphism. q.e.d.

**Proposition 2.2.** Let $F$ and $F'$ belong to $D^b(p^{-1}_X \mathcal{O}_S)$ and let $\phi : F \to F'$ be a morphism. Assume the following conditions:

1. for all $j \in \mathbb{Z}$ and $(x, s) \in X \times S$, $\mathcal{H}^j(F)_{(x, s)}$ and $\mathcal{H}^j(F')_{(x, s)}$ are of finite type over $\mathcal{O}_{S,s}$,
2. for all $s_0 \in S$, the natural morphism

$$Li^*_{s_0}(\phi) : Li^*_{s_0}(F) \to Li^*_{s_0}(F')$$

is an isomorphism in $D^b(\mathbb{C}_X)$.

Then $\phi$ is an isomorphism.

**Proof.** It is enough to prove that the mapping cone of $\phi$ is quasi-isomorphic to 0. So we are led to proving that for $F \in D^b(p^{-1}_X \mathcal{O}_S)$, if $\mathcal{H}^j(F)_{(x, s)}$ are of finite type over $\mathcal{O}_{S,s}$ for all $(x, s) \in X \times S$, and $Li^*_{s_0}(F)$ is quasi-isomorphic to 0 for each $s_0 \in S$, then $F$ is quasi-isomorphic to 0.

We may assume that $S$ is an open subset of $\mathbb{C}^n$ with coordinates $s^1, \ldots, s^n$ and we will argue by induction on $n$. Assume $n = 1$. For such an $F$, for each $s_0 \in S$ and any $j \in \mathbb{Z}$ the morphism $(s^1 - s^1_0) : \mathcal{H}^j(F) \to \mathcal{H}^j(F)$ is an isomorphism, hence $\mathcal{H}^j(F)/(s^1 - s^1_0)\mathcal{H}^j(F) = 0$ and by Nakayama’s Lemma, for any $x \in X$, $\mathcal{H}^j(F)_{(x, s_1)} = 0$ and the result follows. For $n \geq 2$,
the cone $F'$ of the morphism $(s^n - s^n_o) : F \to F$ also satisfies $Li_{s_o}^* F' = 0$ for any $s'_o = (s^1_o, \ldots, s^{n-1}_o)$, hence is zero by induction, so we can argue as in the case $n = 1$. q.e.d.

2.2. $S$-locally constant sheaves. We say that a sheaf $F$ of $\mathbb{C}$-vector spaces (resp. $p_X^{-1} \mathcal{O}_S$-modules) on $X \times S$ is $S$-locally constant if, for each point $(x, s) \in X \times S$, there exists a neighbourhood $U = V_x \times T_s$ of $(x, s)$ and a sheaf $G^{(x, s)}$ of $\mathbb{C}$-vector spaces (resp. $\mathcal{O}_S$-modules) on $T_s$, such that $F|_U \simeq p_U^{-1} G^{(x, s)}$. The category of $S$-locally constant sheaves is an abelian full subcategory of that of sheaves of $\mathbb{C}_{X \times S}$-vector spaces (resp. $p^{-1} \mathcal{O}_S$-modules), which is stable by extensions in the respective categories, by $\mathcal{H}om$ and tensor products. Moreover, if $\pi : Y \times X \times S \to Y \times S$ is the projection, with $X$ contractible, then, if $F'$ is $S$-locally constant on $Y \times X \times S$,

- $\pi_* F'$ is $S$-locally constant on $Y \times S$,
- $R^k \pi_* F' = 0$ if $k > 0$,
- $F' \simeq \pi_* \pi^* F'$.

Applying this to $Y = \{pt\}$, we find that, if $F$ is $S$-locally constant, then for each $x \in X$ there exists a connected neighbourhood $V_x$ of $x$ and a $\mathcal{O}_S$-module (resp. $\mathcal{O}_S$-module) $G^{(x)}$ such that $F = p_{V_x}^{-1} G^{(x)}$, and one has $G^{(x)} = p_{V_x} F|_{V_x \times S} = F|_{\{x\} \times S}$. We shall also denote by $\mathcal{D}^{b}_c(p_X^{-1} \mathcal{O}_S)$ (resp. $\mathcal{D}^{b}_c(p_X^{-1} \mathcal{O}_S)$) the bounded triangulated category whose objects are the complexes having $S$-locally constant cohomology sheaves. Similarly, for such a complex $F$ we have $F|_{V_x \times S} \simeq p_{V_x}^{-1} R p_{V_x} F|_{V_x \times S} \simeq p_{V_x}^{-1} F|_{\{x\} \times S}$.

We conclude from the previous remarks, by using the natural forgetful functor $\mathcal{D}^{b}(p_X^{-1} \mathcal{O}_S) \to \mathcal{D}^{b}(\mathbb{C}_{X \times S})$:

Lemma 2.3.

1. An object $F$ of $\mathcal{D}^{b}(p_X^{-1} \mathcal{O}_S)$ belongs to $\mathcal{D}^{b}_c(p_X^{-1} \mathcal{O}_S)$ if and only if, when regarded as an object of $\mathcal{D}^{b}(\mathbb{C}_{X \times S})$, it belongs to $\mathcal{D}^{b}_c(p_X^{-1} \mathcal{O}_S)$.

2. For any object $F$ of $\mathcal{D}^{b}_c(p_X^{-1} \mathcal{O}_S)$ and for any $s_0 \in S$, $Li_{s_0}^* F$ belongs to $\mathcal{D}^{b}_c(\mathcal{O}_X)$.

2.3. $S$-weakly $\mathbb{R}$-constructible sheaves. As long as the manifold $X$ is fixed, we shall write $p$ instead of $p_X$.

Definition 2.4. Let $F \in \mathcal{D}^{b}(\mathbb{C}_{X \times S})$ (resp. $F \in \mathcal{D}^{b}(p^{-1} \mathcal{O}_S)$). We shall say that $F$ is $S$-weakly $\mathbb{R}$-constructible if there exists a subanalytic $\mu$-stratification $(X_\alpha)$ of $X$ (see [2] Def. 8.3.19]) such that, for all $j \in \mathbb{Z}$, $\mathcal{H}^j(F)|_{X_\alpha \times S}$ is $S$-locally constant.

This condition is independent of the choice of the $\mu$-stratification and characterizes a full triangulated subcategory $\mathcal{D}^{b}_{w, \mathbb{R}}(p^{-1} \mathcal{C}_S)$ (resp. $\mathcal{D}^{b}_{w, \mathbb{R}}(p^{-1} \mathcal{O}_S)$) of $\mathcal{D}^{b}(\mathbb{C}_{X \times S})$ (resp. $\mathcal{D}^{b}(p^{-1} \mathcal{O}_S)$). Due to Lemma 2.3, an object $F$ of $\mathcal{D}^{b}(p^{-1} \mathcal{O}_S)$ is in $\mathcal{D}^{b}_{w, \mathbb{R}}(p^{-1} \mathcal{O}_S)$ if and only if $F$ is $\mathcal{D}^{b}_{w, \mathbb{R}}(p^{-1} \mathcal{O}_S)$ when considered as an object of $\mathcal{D}^{b}(\mathbb{C}_{X \times S})$. By mimicking for $\mathcal{D}^{b}_{w, \mathbb{R}}(p^{-1} \mathcal{C}_S)$ the proof of [1] Prop. 8.4.1] and according to the previous remark for $\mathcal{D}^{b}_{w, \mathbb{R}}(p^{-1} \mathcal{O}_S)$, we obtain:
Proposition 2.5. Let $F$ be $S$-weakly $\mathbb{R}$-constructible on $X$ and let $X = \bigsqcup_{\alpha} X_{\alpha}$ be a $\mu$-stratification of $X$ adapted to $F$. Then the following conditions are equivalent:

1. for all $j \in \mathbb{Z}$ and for all $\alpha$, $\mathcal{H}^j(F)|_{X_{\alpha} \times S}$ is $S$-locally constant.
2. $SS(F) \subset (\bigsqcup_{\alpha} T^*_{X_{\alpha}} X) \times T^* S$.
3. There exists a closed conic subanalytic Lagrangian subset $\Lambda$ of $T^* X$ such that $SS(F) \subset \Lambda \times T^* S$.

Proposition 2.6. Let $F \in D^b_{w-R-c}(p^{-1}\Theta_S)$ and let $s_0 \in S$. Then $L_i^* s_0(F) \in D^b_{w-R-c}(\mathbb{C}_X)$.

Proof. Let $i_\alpha : X_{\alpha} \hookrightarrow X$ denote the locally closed inclusion of a stratum of an adapted stratification $(X_{\alpha})$. It is enough to observe that, for each $\alpha$, we have $i_\alpha^{-1} L_i^* s_0(F) \simeq L_i^* s_0(i_\alpha^{-1} F)$, and to apply Lemma 2.3(2) q.e.d.

Let now $Y$ be another real analytic manifold and consider a real analytic map $f : Y \to X$. The following statements for objects of $D^b_{w-R-c}(p^{-1}\Theta_S)$ are easily deduced from Proposition 2.5 similarly to the absolute case treated in [3], as consequences of Theorem 8.3.17, Proposition 8.3.11, Corollary 6.4.4 and Proposition 5.4.4 of loc.cit. In order to get the same statements for objects of $D^b_{w-R-c}(p^{-1}\Theta_S)$, one uses Lemma 2.3(1) together with [2.4]. We will not distinguish between $f$ and $f_S$.

Proposition 2.7.

1. If $F$ is $S$-weakly $\mathbb{R}$-constructible on $X$, then so are $f^{-1}(F)$ and $f^!(F)$.
2. Assume that $F^*$ is $S$-weakly $\mathbb{R}$-constructible on $Y$ and that $f$ is proper on $\text{Supp}(F^*)$. Then $Rf_*(F^*)$ is $S$-weakly $\mathbb{R}$-constructible on $X$.

Given a closed subanalytic subset $Y \subset X$, we will denote by $i : Y \times S \hookrightarrow X \times S$ the closed inclusion and by $j$ the complementary open inclusion.

Corollary 2.8. Assume that $F^*$ is $S$-weakly $\mathbb{R}$-constructible on $X \setminus Y$. Then the objects $Rj_!F^*$ and $Rj_*F^*$ are also $S$-weakly $\mathbb{R}$-constructible on $X$.

Proof. The statement for $Rj_*F^*$ is obvious. Then Proposition 2.7 implies that $i^!Rj_!F^*$ is $S$-weakly $\mathbb{R}$-constructible. Conclude by using the distinguished triangle

$$Ri_*i^!Rj_!F^* \to Rj_!F^* \to Rj_*F^* \xrightarrow{+1}$$

and the $S$-weak $\mathbb{R}$-constructibility of the first two terms. q.e.d.

Proposition 2.9. An object $F \in D^b(\mathbb{C}_{X \times S})$ (resp. $F \in D^b(p^{-1}(\Theta_S))$) is $S$-weakly $\mathbb{R}$-constructible with respect to a $\mu$-stratification $(X_{\alpha})$ if and only if, for each $\alpha$, $i_\alpha^! F$ has $S$-locally constant cohomology on $X_{\alpha}$.

Proof. Assume that $F$ is $S$-weakly $\mathbb{R}$-constructible with respect to a $\mu$-stratification $(X_{\alpha})$ of $X$. Then $i_\alpha^! F$ has $S$-locally constant cohomology on $X_{\alpha}$. Indeed the estimation of the micro-support of [5] Cor. 6.4.4(ii)] implies that $SS(i_\alpha^! F)$ (like $SS(i_\alpha^* F)$) is contained in $T^*_{X_{\alpha}} X_{\alpha} \times T^* S$, so $i_\alpha^! F$ has locally constant cohomology on $X_{\alpha}$ for each $\alpha$, according to Proposition 2.5.
Conversely, if \( \iota^{i}_{\alpha}F \) is locally constant for each \( \alpha \), then \( F \) is \( S \)-weakly \( \mathbb{R} \)-constructible. Indeed, we argue by induction and we denote by \( X_k \) the union of strata of codimension \( \leq k \) in \( X \). Assume we have proved that \( F_{|X_{k-1} \times S} \) is \( S \)-weakly \( \mathbb{R} \)-constructible with respect to the stratification \((X_{\alpha})\) with codim \( X_{\alpha} \leq k - 1 \). We denote by \( j_{k} : X_{k-1} \hookrightarrow X_k \) the open inclusion and by \( i_{\alpha} \) the complementary closed inclusion. According to Corollary 2.8, \( Rj_{k*}j_{k}^{-1}F \) is \( S \)-weakly \( \mathbb{R} \)-constructible with respect to \( (X_{\alpha})|X_{k} \). Now, by using the exact triangle \( \iota^{1}_{k}F \to \iota^{-1}_{k}F \to i^{-1}_{k}Rj_{k*}j_{k}^{-1}F \overset{\alpha}{\longrightarrow} \), we conclude that \( \iota^{-1}_{k}F \) is locally constant, hence \( F_{|X_k \times S} \) is \( S \)-weakly \( \mathbb{R} \)-constructible.\[ \text{q.e.d.} \]

**Corollary 2.10.** Let \( F, F' \in D^{b}_{w,\mathbb{R},c}(p^{-1}_{X}\mathcal{O}_{S}) \). Then \( R\mathcal{H}\text{om}_{p^{-1}_{X}\mathcal{O}_{S}}(F, F') \) also belongs to \( D^{b}_{w,\mathbb{R},c}(p^{-1}_{X}\mathcal{O}_{S}) \).

**Proof.** In view of Proposition 2.9, it is sufficient to prove that for each \( \alpha \), \( \iota^{i}_{\alpha}R\mathcal{H}\text{om}_{p^{-1}_{X}\mathcal{O}_{S}}(F, F') \) belongs to \( D^{b}_{\mathbb{R}}(p^{-1}_{X}\mathcal{O}_{S}) \). We have:

\[
i^{i}_{\alpha}R\mathcal{H}\text{om}_{p^{-1}_{X}\mathcal{O}_{S}}(F, F') \simeq R\mathcal{H}\text{om}_{p^{-1}_{X}\mathcal{O}_{S}}(\iota^{-1}_{\alpha}F, \iota^{i}_{\alpha}F').
\]

Since both \( \iota^{-1}_{\alpha}F \) and \( \iota^{i}_{\alpha}F' \) belong to \( D^{b}_{\mathbb{R}}(p^{-1}_{X}\mathcal{O}_{S}) \), according to Proposition 2.9, we have locally on \( X_{\alpha} \) isomorphisms \( \iota^{-1}_{\alpha}F = p^{-1}_{\alpha}G_{\alpha} \) and \( \iota^{i}_{\alpha}F' = p^{-1}_{\alpha}G'_{\alpha} = p^{1}_{\alpha}G'_{\alpha}(-\dim_{\mathbb{R}}X_{\alpha}) \) for some \( \mathcal{O}_{S} \)-modules \( G_{\alpha} \) and \( G'_{\alpha} \). Then

\[
R\mathcal{H}\text{om}_{p^{-1}_{\alpha}\mathcal{O}_{S}}(\iota^{-1}_{\alpha}F, \iota^{i}_{\alpha}F') = R\mathcal{H}\text{om}_{p^{-1}_{\alpha}\mathcal{O}_{S}}(p^{-1}_{\alpha}G_{\alpha}, p^{1}_{\alpha}G'_{\alpha}(-\dim_{\mathbb{R}}X_{\alpha}))
\]
\[
\simeq p^{i}_{\alpha}R\mathcal{H}\text{om}_{\mathcal{O}_{S}}(G_{\alpha}, G'_{\alpha})(-\dim_{\mathbb{R}}X_{\alpha})
\]
\[
= p^{-1}_{\alpha}R\mathcal{H}\text{om}_{\mathcal{O}_{S}}(G_{\alpha}, G'_{\alpha}).
\]

The following lemma will be useful in the next section. Assume that \( X = Y \times Z \) and that the \( \mu \)-stratification \((X_{\alpha})\) of \( X \) takes the form \( X_{\alpha} = Y \times Z_{\alpha} \), where \( (Z_{\alpha}) \) is a \( \mu \)-stratification of \( Z \). We denote by \( q : X \to Y \) the projection. Let \( z_{o} \in Z \), let \( U \supseteq z_{o} \) be a coordinate neighbourhood of \( z_{o} \) in \( Z \) and, for each \( \varepsilon > 0 \) small enough, let \( B_{\varepsilon} \subseteq U \) be the open ball of radius \( \varepsilon \) centered at \( z_{o} \) and let \( \overline{B}_{\varepsilon} \) be the closed ball and \( S_{\varepsilon} \) its boundary. For the sake of simplicity, we denote by \( q_{\varepsilon}, q_{\varepsilon}, q_{\varepsilon} \) the corresponding projections.

We set \( Z^{*} = Z \setminus \{z_{o}\} \) and \( X^{*} = Y \times Z^{*} \). We denote by \( i : Y \times \{z_{o}\} \hookrightarrow Y \times Z \) and by \( j : Y \times Z^{*} \to Y \times Z \) the complementary closed and open inclusions.

**Lemma 2.11.** Let \( F^{*} \in D^{b}_{w,\mathbb{R},c}(p^{-1}_{X}\mathcal{O}_{S}) \) (resp. \( F^{*} \in D^{b}_{w,\mathbb{R},c}(p^{-1}_{X}\mathcal{O}_{S}) \)) be adapted to the previous stratification. Then there exists \( \varepsilon_{o} > 0 \) such that, for each \( \varepsilon \in (0, \varepsilon_{o}) \), the natural morphisms

\[
Rq_{\varepsilon,!*}F^{*}_{Y \times S_{\varepsilon} \times S} \leftarrow Rq_{\varepsilon,!*}Rj_{*}F^{*} \to Rq_{\varepsilon,!*}Rj_{*}F^{*} \to i^{-1}_{*}Rj_{*}F^{*}
\]

are isomorphisms.

**Proof.** We note that, according to Corollary 2.8, \( F := Rj_{*}F^{*} \) is \( S \)-weakly \( \mathbb{R} \)-constructible, and is adapted to the stratification \((Y \times Z_{\alpha})\). On the other hand, according to (2.7), it is enough to consider the case where \( F^{*} \) is an object of \( D^{b}_{w,\mathbb{R},c}(p^{-1}_{X}\mathcal{O}_{S}) \).

Let us start with the right morphisms. We can argue with any object \( F \in D^{b}_{w,\mathbb{R},c}(p^{-1}_{X}\mathcal{O}_{S}) \), not necessarily of the form \( Rj_{*}F^{*} \). Recall that we have
an adjunction morphism $\varepsilon_{\varepsilon}^{-1}Rq_{\varepsilon,*} \rightarrow \text{id}$ and thus $i^{-1}q_{\varepsilon}^{-1}Rq_{\varepsilon,*} \rightarrow i^{-1}$. Since $q_s \circ i = \text{id}_{Y \times S}$, we get the second right morphism. The first one is the restriction morphism.

According to [5, Prop. 8.3.12 and 5.4.17], there exists $\varepsilon_o > 0$ such that, for $\varepsilon^i < \varepsilon$ in $(0, \varepsilon_o)$, the restriction morphisms $Rq_{\varepsilon^i,*}F \rightarrow Rq_{\varepsilon,*}F \rightarrow Rq_{\varepsilon^i,*}F$ are isomorphisms. In particular, the first right morphism is an isomorphism.

Let us take a $q$-soft representative of $F$, that we still denote by $F$. The inductive system $q_{\varepsilon,*}F (\varepsilon \rightarrow 0)$ has limit $\hat{i}^{-1}F$ and all morphisms of this system are quasi-isomorphisms. Hence the second right morphism is a quasi-isomorphism.

**Remark 2.12.** A similar argument gives an isomorphism $\hat{i}F \sim Rq_{\varepsilon^i}F$, by using [5, Prop. 5.4.17(c)].

For the left morphism, we take a $q$-soft representative of $F^*$ that we still denote by $F^*$. For $\varepsilon_- < \varepsilon < \varepsilon_+$, we denote by $B_{\varepsilon_- \varepsilon_+}$ the open set $B_{\varepsilon_- \varepsilon_+}$ and by $q_{\varepsilon_- \varepsilon_+}$ the corresponding projection. We have $q_{\varepsilon_- \varepsilon_+}F^* = \lim_{\rightarrow} q_{\varepsilon_- \varepsilon_+}^*F^*$. On the other hand, the morphisms of this inductive system are all quasi-isomorphisms, according to [5, Prop. 5.4.17]. Fixing $\varepsilon^i \in (\varepsilon, \varepsilon_o)$ we find a quasi-isomorphism $q_{\varepsilon^i,*}F^* \rightarrow q_{\varepsilon,*}F^*$. On the other hand, from the first part we have $q_{\varepsilon^i,*}F^* \sim q_{\varepsilon,*}F^*$, hence the result.

**q.e.d.**

2.4. S-coherent local systems and $S$-$\mathbb{R}$-constructible sheaves.

**Notation 2.13.** We shall denote by $D^b_{\text{lc coh}}(p_X^{-1}\mathcal{O}_S)$ the full triangulated subcategory of $D^b(\mathcal{O}_S)$ whose objects satisfy, locally on $X$, $F \simeq p_X^{-1}G$ with $G \in D^b_{\text{coh}}(\mathcal{O}_S))$. Equivalently, for each $x \in X$, $F|_{(x)\times S} \in D^b_{\text{coh}}(\mathcal{O}_S)$ (see the remarks before Lemma 2.3).

**Definition 2.14.** Given $F \in D^b_{w,\mathbb{R}-c}(p_X^{-1}\mathcal{O}_S)$, we say that $F$ is $\mathbb{R}$-constructible if, for some $\mu$-stratification of $X$, $X = \bigsqcup X_\alpha$, for all $j \in \mathbb{Z}$, $\mathcal{H}^j(F)|_{X_\alpha \times S} \in D^b_{\text{coh}}(p_{X_\alpha}^{-1}\mathcal{O}_S)$. This condition characterizes a full triangulated subcategory of $D^b_{w,\mathbb{R}-c}(p_X^{-1}\mathcal{O}_S)$ which we denote by $D^b_{\mathbb{R}-c}(p_X^{-1}\mathcal{O}_S)$.

Similarly to Proposition 2.6 we have:

**Proposition 2.15.** Let $F \in D^b_{\mathbb{R}-c}(p_X^{-1}\mathcal{O}_S)$ and let $s_0 \in S$. Then $Li^{s_0}_*(F) \in D^b_{\mathbb{R}-c}(\mathbb{C}_X)$.

**Remark 2.16.** An object of $D^b_{w,\mathbb{R}-c}(p_X^{-1}\mathcal{O}_S)$ is in $D^b_{\mathbb{R}-c}(p_X^{-1}\mathcal{O}_S)$ if and only if, for any $x \in X$, $F|_{(x)\times S}$ belongs to $D^b_{\text{coh}}(\mathcal{O}_S)$.

A straightforward adaptation of [5, Prop. 8.4.8] gives:

**Proposition 2.17.** Let $f : Y \rightarrow X$ be a a morphism of manifolds and let $F \in D^b_{\mathbb{R}-c}(p_Y^{-1}\mathcal{O}_S)$. Assume that $f_S$ is proper on $\text{Supp}(F)$. Then

$$Rf_{S,*}F \in D^b_{\mathbb{R}-c}(p_X^{-1}\mathcal{O}_S).$$

We can also characterize $D^b_{\mathbb{R}-c}(p_X^{-1}\mathcal{O}_S)$ as in Corollary 2.9.
Corollary 2.18. An object $F \in D^b(p_X^{-1}O_S)$ is in $D^b_{\mathbb{R}-c}(p_X^{-1}O_S)$ if and only if, for some subanalytic Whitney stratification $(X_\alpha)$ of $X$, the complexes $i_*^! F$ belong to $D^b_{\mathbb{R}-coho}(p_X^{-1}O_S)$.

Proof. Assume $F$ is in $D^b_{\mathbb{R}-c}(p_X^{-1}O_S)$. We need to prove the coherence of $i_*^! F$. We argue by induction as in Corollary 2.10, with the same notation. Since the question is local on $X_k$, by the Whitney property of the stratification $(X_\alpha)$ we can assume that $X_{k-1} = Z \times Y_k$ and there exists a Whitney stratification $(Z_\alpha)$ of $Z$ such that $X_\alpha = Z_\alpha \times Y_k$ for each $\alpha$ such that $X_\alpha \subset X_{k-1}$ (see e.g. [11 §1.4]). Proving that $i_*^! F$ is $p^{-1}O_S$-coherent is equivalent to proving that $i_*^{-1}Rj_k^*j_k^{-1}F$ is so, since we already know that $i^{-1} F$ is so. According to Lemma 2.11, $i_*^{-1}Rj_k^*j_k^{-1}F$ is computed as $Rq_\partial^*F$, and since $q_\partial$ is proper, we can apply Proposition 2.17 to get the coherence.

Conversely, Corollary 2.9 already implies that $F$ is an object of $D^b_{w-\mathbb{R}-c}(p_X^{-1}O_S)$. We argue then as above: since we know by assumption that $i_*^! F$ is coherent, it suffices to prove that $i_*^{-1}Rj_k^*j_k^{-1}F$ is so, and the previous argument applies. q.e.d.

2.5. $S$-weakly $\mathbb{C}$-constructible sheaves and $S$-$\mathbb{C}$-constructible sheaves. Let now assume that $X$ is a complex analytic manifold.

Definition 2.19.

(1) Let $F \in D^b_{w-\mathbb{R}-c}(p_X^{-1}C_S)$ (resp. $F \in D^b_{w-\mathbb{R}-c}(p_X^{-1}O_S)$). We shall say that $F$ is $S$-weakly $\mathbb{C}$-constructible if $SS(F)$ is $\mathbb{C}^*$-conic. The corresponding categories are denoted by $D^b_{w-\mathbb{R}-c}(p_X^{-1}C_S)$ (resp. $F \in D^b_{w-\mathbb{R}-c}(p_X^{-1}O_S)$).

(2) If $F$ belongs to $D^b_{w-\mathbb{R}-c}(p_X^{-1}O_S)$, we say that $F$ is $S$-$\mathbb{C}$-constructible if $F \in D^b_{w-\mathbb{R}-c}(p_X^{-1}C_S)$, and we denote by $D^b_{\mathbb{C}-c}(p_X^{-1}O_S)$ the corresponding category, which is full triangulated sub-category of $D^b(p_X^{-1}O_S)$.

The following properties are obtained in a straightforward way, by using [3 Th. 8.5.5] in a way similar to [3 Prop. 8.5.7].

Properties 2.20.

(1) An object $F$ of $D^b(p_X^{-1}O_S)$ belongs to $D^b_{w-\mathbb{R}-c}(p_X^{-1}O_S)$ if and only if it belongs to $D^b_{w-\mathbb{R}-c}(p_X^{-1}C_S)$.

(2) Remark 2.16 applies to $D^b_{w-\mathbb{R}-c}(p_X^{-1}O_S)$ and $D^b_{\mathbb{C}-c}(p_X^{-1}O_S)$.

(3) Proposition 2.7 applies to $D^b_{w-\mathbb{R}-c}$.

(4) Propositions 2.15, 2.17 and Corollary 2.18 apply to $D^b_{\mathbb{C}-c}(p_X^{-1}O_S)$.

(5) Corollary 2.10 applies to $D^b_{w-\mathbb{C}-c}$, $D^b_{\mathbb{R}-c}$ and $D^b_{\mathbb{C}-c}$.

2.6. Duality. According to the syzygy theorem for the regular local ring $O_{S,s}$ (for any $s \in S$) and e.g. [3 Prop. 13.2.2(ii)] (for the opposite category), any object of $D^b_{coho}(O_S)$ is locally quasi-isomorphic to a bounded complex of locally free $O_S$-modules of finite rank $L^\star$. As a consequence, the local duality functor

$$D : D^b_{coho}(O_S) \to D^b_{coho}(O_S), \quad D(\mathcal{F}) := R\mathcal{H}om_{O_S}(\mathcal{F}, O_S)$$
Lemma 2.21. Let \( G \) be an object of \( \mathcal{D}^{b,c}_{\text{coh}}(\mathcal{O}_S) \). Assume that \( \mathcal{D}G \) belongs to \( \mathcal{D}^{b,c}_{\text{coh}}(\mathcal{O}_S) \). Then \( G \) belongs to \( \mathcal{D}^{b,\geq 0}_{\text{coh}}(\mathcal{O}_S) \).

Proof. Setting \( G' = \mathcal{D}G \), the biduality isomorphism makes it equivalent to proving that \( \mathcal{D}G' \) belongs to \( \mathcal{D}^{b,c}_{\text{coh}}(\mathcal{O}_S) \). The question is local on \( S \) and we may therefore replace \( G' \) with a bounded complex \( L' \) as above. Moreover, \( L' \) is quasi-isomorphic to such a bounded complex, still denoted by \( L' \), such that \( L^k = 0 \) for \( k > 0 \). Indeed, note first that the kernel \( K \) of a surjective morphism of locally free \( \mathcal{O}_S \)-modules of finite rank is also locally free of finite rank (being \( \mathcal{O}_S \)-coherent and having all its germs \( K_s \) free over \( \mathcal{O}_{S,s} \), because they are projective and \( \mathcal{O}_{S,s} \) is a regular local ring). By assumption, we have \( \mathcal{R}^j(L') = 0 \) for \( j > 0 \). Let \( k > 0 \) be such that \( L^k \neq 0 \) and \( L^\ell = 0 \) for \( \ell > k \), and let \( L^{k-1} = \ker[L^k \rightarrow L^k] \). Then \( L' \) is quasi-isomorphic to \( L' \) defined by \( L'_j = L_j \) for \( j < k-1 \) and \( L'_{j+k} = 0 \) for \( j \geq k \). We conclude by induction on \( k \).

Now it is clear that \( \mathcal{D}G' \simeq \mathcal{D}L' \) is a bounded complex having terms in nonnegative degrees at most, and thus is an object of \( \mathcal{D}^{b,\geq 0}_{\text{coh}}(\mathcal{O}_S) \). q.e.d.

Remark 2.22. Let \( G \) be an object of \( \mathcal{D}^{b,c}_{\text{coh}}(\mathcal{O}_S) \). Assume that \( G \) and \( \mathcal{D}G \) belong to \( \mathcal{D}^{b,c}_{\text{coh}}(\mathcal{O}_S) \). Then \( G \) and \( \mathcal{D}G \) are \( \mathcal{O}_S \)-coherent sheaves, hence \( G \) and \( \mathcal{D}G \) are \( \mathcal{O}_S \)-locally free.

We now set \( \omega_{X,S} = p_{X}^{-1} \mathcal{O}_S[2 \dim X] = p_{X}^{-1} \mathcal{O}_S \).

Proposition 2.23. The functor \( \mathcal{D} : \mathcal{D}^{b}(p_{X}^{-1} \mathcal{O}_S) \rightarrow \mathcal{D}^{+}(p_{X}^{-1} \mathcal{O}_S) \) defined by \( \mathcal{D}F = R\mathcal{H}\text{om}_{p_{X}^{-1} \mathcal{O}_S}(F, \omega_{X,S}) \) induces an involution \( \mathcal{D}^{b,c}(p_{X}^{-1} \mathcal{O}_S) \rightarrow \mathcal{D}_{c}(p_{X}^{-1} \mathcal{O}_S) \) and \( \mathcal{D}_{c}(p_{X}^{-1} \mathcal{O}_S) \rightarrow \mathcal{D}_{c}(p_{X}^{-1} \mathcal{O}_S) \).

We will also set \( \mathcal{D}^{+} = R\mathcal{H}\text{om}_{p_{X}^{-1} \mathcal{O}_S}(F, p_{X}^{-1} \mathcal{O}_S) \).

Proof. Let us first show that, for \( F \) in \( \mathcal{D}^{b,c}_{\text{w-\text{str}}}(p_{X}^{-1} \mathcal{O}_S) \), the dual \( \mathcal{D}F \) also belongs to \( \mathcal{D}^{b,c}_{\text{w-\text{str}}}(p_{X}^{-1} \mathcal{O}_S) \). Let \( (X_\alpha) \) be a \( \mu \)-stratification adapted to \( F \). According to Corollary 2.29, it is enough to show that \( i_\alpha^{i} \mathcal{D}F \) has locally constant cohomology for each \( \alpha \). One can use [3] Prop. 3.1.13] in our setting and get

\[
i_\alpha^{i} \mathcal{D}F = R\mathcal{H}\text{om}_{p_{X}^{-1} \mathcal{O}_S}(i_\alpha^{i} F, \omega_{X_\alpha,S}).
\]

Locally on \( X_\alpha \), \( i_\alpha^{i} F = p_{X}^{-1} G \) for some \( G \) in \( \mathcal{D}^{b}(\mathcal{O}_S) \) or \( \mathcal{D}^{b}(\mathcal{O}_S) \). Then, locally on \( X_\alpha \),

\[
i_\alpha^{i} \mathcal{D}F \simeq R\mathcal{H}\text{om}_{p_{X}^{-1} \mathcal{O}_S}(p_{X}^{-1} G, p_{X}^{-1} \mathcal{O}_S) = p_{X}^{i} R\mathcal{H}\text{om}_{\mathcal{O}_S}(G, \mathcal{O}_S) = p_{X}^{i} (\mathcal{D}G)[2 \dim X_\alpha].
\]
The proof for $F$ in $D_{w-c}^{b}(p_{X}^{-1} \mathcal{O}_{S})$ is similar. Moreover, by using Corollary 2.18 instead of Corollary 2.9 one shows that $D$ sends $D_{w-c}^{b}(p_{X}^{-1} \mathcal{O}_{S})$ to itself and, according to Properties 2.20, $D_{c-c}^{b}(p_{X}^{-1} \mathcal{O}_{S})$ to itself.

Let us prove the involution property. We have a natural morphism of functors $\text{id} \to DD$. It is enough to prove the isomorphism property after applying $Li_{s_{0}^{*}}$ for each $s_{0} \in S$, according to Proposition 2.22. On the other hand, Proposition 2.14 implies that $Li_{s_{0}^{*}}$ commutes with $D$, so we are reduced to applying the involution property on $D_{c-c}^{b}(C_{X})$, according to the $\mathbb{C}$-analogue of Proposition 2.15 which is known to be true (see e.g. [5]). q.e.d.

**Remark 2.24.** By using the biduality isomorphism and the isomorphism $i_{x}^{1*}DF \simeq D_{x}^{-1}F$ for $F$ in $D_{w-c}^{b}(p_{X}^{-1} \mathcal{O}_{S})$ or $D_{c-c}^{b}(p_{X}^{-1} \mathcal{O}_{S})$, where $i_{x} : \{x\} \times S \hookrightarrow X \times S$ denotes the inclusion, we find a functorial isomorphism $i_{x}^{1*}DF \simeq D_{x}^{1}F$.

### 2.7. Perversity

We will now restrict to the case of $S-$\textbf{C}-constructible complexes, which is the only case which will be of interest for us, although one could consider the case of $S-$\textbf{R}-constructible complexes as in [5 §10.2].

We define the category $pD_{c-c}^{\leq 0}(p_{X}^{-1} \mathcal{O}_{S})$ as the full subcategory of $D_{c-c}^{b}(p_{X}^{-1} \mathcal{O}_{S})$ whose objects are the $S-$\textbf{C}-constructible bounded complexes $F$ such that, for some adapted $\mu$-stratification $(X_{\alpha})$ ($i_{z}$ is as above),

$$(\text{Supp}) \quad \forall \alpha, \forall x \in X_{\alpha}, \forall j > - \dim X_{\alpha}, \quad H^{j}i_{x}^{-1}F = 0.$$  

Similarly, $pD_{c-c}^{\geq 0}(p_{X}^{-1} \mathcal{O}_{S})$ consists of objects $F$ such that

$$(\text{Cosupp}) \quad \forall \alpha, \forall x \in X_{\alpha}, \forall j < - \dim X_{\alpha}, \quad H^{j}i_{x}^{1}F = 0.$$  

In the preceding situation in view of Corollary 2.14 we have, similarly to [3 Prop.10.2.4]:

**Lemma 2.25.**

1. $F \in pD_{c-c}^{\leq 0}(p_{X}^{-1} \mathcal{O}_{S})$ if and only if for any $\alpha$ and $j > - \dim X_{\alpha}$,

   $$H^{j}(i_{\alpha}^{-1}F) = 0.$$  

2. $F \in pD_{c-c}^{\geq 0}(p_{X}^{-1} \mathcal{O}_{S})$ if and only if for any $\alpha$ and $j < - \dim X_{\alpha}$,

   $$H^{j}(i_{\alpha}^{1}F) = 0.$$  

Namely, if $F \in pD_{c-c}^{\leq 0}(p_{X}^{-1} \mathcal{O}_{S})$ and $Z$ is a closed analytic subset of $X$ such that $\dim Z = k$, then $i_{Z}^{-1}F$ is concentrated in degrees $\leq -k$, and if $F' \in pD_{c-c}^{\geq 0}(p_{X}^{-1} \mathcal{O}_{S})$, then $i_{Z}^{1}F'$ is concentrated in degrees $\geq -k$. We have the following variant of [3 Prop.10.2.7]:

**Proposition 2.26.** Let $F$ be an object of $pD_{w-c}^{\leq 0}(p_{X}^{-1} \mathcal{O}_{S})$ and $F'$ an object of $pD_{w-c}^{\geq 0}(p_{X}^{-1} \mathcal{O}_{S})$. Then

$$H^{j}R\mathcal{H}om_{p_{X}^{-1} \mathcal{O}_{S}}(F, F') = 0, \quad \text{for } j < 0.$$  

**Proof.** Let $(X_{\alpha})$ be a $\mu$-stratification of $X$ adapted to $F$ and $F'$. By assumption, for each $\alpha$, $i_{\alpha}^{-1}H^{j}F = H^{j}i_{\alpha}^{-1}F = 0$ for $j > - \dim X_{\alpha}$. Similarly, $H^{j}i_{\alpha}^{1}F' = 0$ for $j < - \dim X_{\alpha}$.
Let $X_\alpha$ be a stratum of maximal dimension such that
\[ i_\alpha^{-1} \mathcal{H}^j \mathcal{H}\text{om}_{p_X^{-1}\mathcal{O}_S}(F, F') \neq 0 \text{ for some } j < 0. \]

Let $V$ be an open neighbourhood of $X_\alpha$ in $X$ such that $V \smallsetminus X_\alpha$ intersects only strata of dimension $\geq \dim X_\alpha$, and let $j_\alpha : (V \smallsetminus X_\alpha) \times S \hookrightarrow V \times S$ be the inclusion. Then the complex $i_\alpha^{-1} R j_\alpha_* j_\alpha^{-1} R \mathcal{H}\text{om}_{p_X^{-1}\mathcal{O}_S}(F, F')$ has nonzero cohomology in nonnegative degrees only: indeed, by the definition of $X_\alpha$, this property holds for $j_\alpha^{-1} R \mathcal{H}\text{om}_{p_X^{-1}\mathcal{O}_S}(F, F')$, hence it holds for $R j_\alpha_* j_\alpha^{-1} R \mathcal{H}\text{om}_{p_X^{-1}\mathcal{O}_S}(F, F')$, and then clearly for the complex $i_\alpha^{-1} R j_\alpha_* j_\alpha^{-1} R \mathcal{H}\text{om}_{p_X^{-1}\mathcal{O}_S}(F, F')$. From the distinguished triangle
\[ i_\alpha^{-1} \mathcal{H}^j \mathcal{H}\text{om}_{p_X^{-1}\mathcal{O}_S}(F, F') \to i_\alpha^{-1} R \mathcal{H}\text{om}_{p_X^{-1}\mathcal{O}_S}(F, F') \to i_\alpha^{-1} R j_\alpha_* j_\alpha^{-1} R \mathcal{H}\text{om}_{p_X^{-1}\mathcal{O}_S}(F, F') \to i_\alpha^{-1} \mathcal{H}^j \mathcal{H}\text{om}_{p_X^{-1}\mathcal{O}_S}(F, F') \]
we conclude that $\mathcal{H}^j i_\alpha^{-1} \mathcal{H}\text{om}_{p_X^{-1}\mathcal{O}_S}(F, F') \to \mathcal{H}^j i_\alpha^{-1} \mathcal{H}\text{om}_{p_X^{-1}\mathcal{O}_S}(F, F') = i_\alpha^{-1} \mathcal{H}^j \mathcal{H}\text{om}_{p_X^{-1}\mathcal{O}_S}(F, F')$ is an isomorphism for all $j < 0$. Therefore, we obtain, for this stratum $X_\alpha$ and for any $j < 0$,
\[ i_\alpha^{-1} \mathcal{H}^j \mathcal{H}\text{om}_{p_X^{-1}\mathcal{O}_S}(F, F') \simeq \mathcal{H}^j i_\alpha^{-1} \mathcal{H}\text{om}_{p_X^{-1}\mathcal{O}_S}(F, F') \]
\[ \simeq \mathcal{H}^j \mathcal{H}\text{om}_{p_X^{-1}\mathcal{O}_S}(i_\alpha^{-1} F, i_\alpha^{-1} F'). \]

Since $i_\alpha^{-1} F$ has nonzero cohomology in degrees $\leq - \dim X_\alpha$ at most and $i_\alpha^{-1} F'$ in degrees $\geq - \dim X_\alpha$ at most, $\mathcal{H}^j \mathcal{H}\text{om}_{p_X^{-1}\mathcal{O}_S}(i_\alpha^{-1} F, i_\alpha^{-1} F') = 0$ for $j < 0$, a contradiction with the definition of $X_\alpha$. q.e.d.

**Theorem 2.27.** $p \mathcal{D}^{<0}_{C_c}(p_X^{-1}\mathcal{O}_S)$ and $p \mathcal{D}^{\geq0}_{C_c}(p_X^{-1}\mathcal{O}_S)$ form a t-structure of $\mathcal{D}^{0}_{C_c}(p_X^{-1}\mathcal{O}_S)$, whose heart is denoted by $\text{Perv}(p_X^{-1}\mathcal{O}_S)$.

**Sketch of proof.** We have to prove:

1. $p \mathcal{D}^{<0}_{C_c}(p_X^{-1}\mathcal{O}_S) \subseteq p \mathcal{D}^{<1}_{C_c}(p_X^{-1}\mathcal{O}_S)$ and $p \mathcal{D}^{\geq0}_{C_c}(p_X^{-1}\mathcal{O}_S) \supset p \mathcal{D}^{\geq1}_{C_c}(p_X^{-1}\mathcal{O}_S)$
2. For $F \in p \mathcal{D}^{<1}_{C_c}(p_X^{-1}\mathcal{O}_S)$ and $F' \in p \mathcal{D}^{\geq1}_{C_c}(p_X^{-1}\mathcal{O}_S)$, $\text{Hom}_{p \mathcal{D}^{<0}_{C_c}(p_X^{-1}\mathcal{O}_S)}(F, F') = 0$.
3. For any $F \in p \mathcal{D}^{\geq0}_{C_c}(p_X^{-1}\mathcal{O}_S)$ there exist $F' \in p \mathcal{D}^{<0}_{C_c}(p_X^{-1}\mathcal{O}_S)$ and $F'' \in p \mathcal{D}^{<1}_{C_c}(p_X^{-1}\mathcal{O}_S)$, giving rise to a distinguished triangle $F' \to F \to F'' \oplus 1$.

Then, following the line of the proof of \[\text{Lemma 2.21}\], one cannot expect that the previous t-structure is interchanged by duality when $\dim S \geq 1$. However we have:

**Proposition 2.28.** Let $F$ be an object of $p \mathcal{D}^{<0}_{C_c}(p_X^{-1}\mathcal{O}_S)$ such that $DF$ also belongs to $p \mathcal{D}^{<0}_{C_c}(p_X^{-1}\mathcal{O}_S)$. Then $F$ and $DF$ are objects of $\text{Perv}(p_X^{-1}\mathcal{O}_S)$. 
Proof. Let us fix $x \in X_\alpha$. We have $i_x^* F \simeq D(i_x^{-1} DF)$, as already observed in Remark 2.24. By assumption $G := i_x^{-1} DF$ belongs to $D^{b, \leq -\dim X_\alpha}(\mathcal{O}_S)$, and Lemma 2.21 suitably shifted and applied to $DG$ implies that $DG$ belongs to $D^{b, \geq \dim X_\alpha}(\mathcal{O}_S)$, which is the cosupport condition (Cosupp) for $F$. q.e.d.

Assume $F \in \Perv(p_X^{-1}\mathcal{O}_S)$. The description of the dual standard $t$-structure on $D^{b, \bullet}_{\coh}(\mathcal{O}_S)$ given in [4] §4 supplies the following refinement to (Supp) and (Cosupp) when $DF$ is also perverse.

Corollary 2.29. Let $F \in \Perv(p_X^{-1}\mathcal{O}_S)$ and assume that $DF \in \Perv(p_X^{-1}\mathcal{O}_S)$. Let $(X_\alpha)$ be a stratification adapted to $F$. Then for each $\alpha$, each $x \in X_\alpha$ and each closed analytic subset $Z \subset S$, we have

\[(\text{Cosupp}+) \quad \mathcal{H}^k(i_Z^* (i_x^* F)) = 0, \quad \forall k < \text{codim}_S Z + \dim X_\alpha.\]

(The perversity of $F$ only gives the previous property when $Z = S$.)

3. The de Rham complex of a holonomic $\mathcal{D}_{X \times S/S}$-module

In what follows $X$ and $S$ denote complex manifolds and we set $n = \dim X$, $\ell = \dim S$. We shall keep the notation of the preceding section. Let $\pi : T^*(X \times S) \to T^* X \times S$ denote the projection and let $\mathcal{D}_{X \times S/S}$ denote the subsheaf of $\mathcal{D}_{X \times S}$ of relative differential operators with respect to $p_X$ (see [18, §2.1 & 2.2]).

Recall that $p_X^{-1}\mathcal{O}_S$ is contained in the center of $\mathcal{D}_{X \times S/S}$. With the same proof as for Proposition 2.1 we obtain:

Proposition 3.1. Let $s_\alpha \in S$ be given. Let $\mathcal{M}$ and $\mathcal{N}$ be objects of $D^b(\mathcal{D}_{X \times S/S})$. Then, there is a well-defined natural morphism

\[Li_{s_\alpha}^!(R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(\mathcal{M}, \mathcal{N})) \to R\mathcal{H}om_i(i_{s_\alpha}^!(\mathcal{D}_{X \times S/S}))(Li_{s_\alpha}^!(\mathcal{M}), Li_{s_\alpha}^!(\mathcal{N}))\]

which is an isomorphism in $D^b(\mathcal{C}_X)$.

3.1. Duality for coherent $\mathcal{D}_{X \times S/S}$-modules. We refer for instance to [3] Appendix for the coherence properties of the ring $\mathcal{D}_{X \times S/S}$. The classical methods used in the absolute case, i.e., for coherent $\mathcal{D}_X$-objects (see for instance [3] Prop. 2.1.16), [9] Prop. 2.7-3]) apply here:

Proposition 3.2. Let $\mathcal{M}$ be a coherent $\mathcal{D}_{X \times S/S}$-module. Then $\mathcal{M}$ locally admits a resolution of length at most $2n + \ell$ by free $\mathcal{D}_{X \times S/S}$-modules of finite rank.

Proposition 3.2 and [6] Prop. 13.2.2(ii)] (for the opposite category) imply:

Corollary 3.3. Let $\mathcal{M} \in D^{b, \bullet}_{\coh}(\mathcal{D}_{X \times S/S}).$ Let us assume that $\mathcal{M}$ is concentrated in degrees $[a, b]$. Then, in a neighborhood of each $(x, z) \in X \times S$, there exist a complex $\mathcal{L}^\bullet$ of free $\mathcal{D}_{X \times S/S}$-modules of finite rank concentrated in degrees $[a - 2n - \ell, b]$ and a quasi-isomorphism $\mathcal{L}^\bullet \to \mathcal{M}$.

We set $\Omega^n_{X \times S/S} = \Omega^n_{X \times S/S}$, where $\Omega^n_{X \times S/S}$ denotes the sheaf of relative differential forms of degree $n = \dim X$. 

Definition 3.4. The duality functor $D(\cdot) : D^b(\mathcal{D}_{X \times S}) \to D^b(\mathcal{D}_{X \times S})$ is defined as:

$$M \mapsto D M = R\mathcal{H}om_{\mathcal{D}_{X \times S}}(M, \mathcal{D}_{X \times S} \otimes_{\mathcal{O}_{X \times S}} \Omega_{X \times S}^{\leq -1})[n].$$

We also set $D' M := R\mathcal{H}om_{\mathcal{D}_{X \times S}}(M, \mathcal{D}_{X \times S}) \in D^b(\mathcal{D}_{X \times S}^{\text{opp}})$.

By Proposition 3.2, $\mathcal{D}_{X \times S}$ has finite cohomological dimension, so (A.11) gives a natural morphism in $D^b(\mathcal{D}_{X \times S})$:

$$M \mapsto D' D M \simeq D D M.$$  

Moreover, in view of Corollary 3.3, if $M \in D^b_{\text{coh}}(\mathcal{D}_{X \times S})$, then $D' M \in D_{\text{coh}}^b(\mathcal{D}_{X \times S}^{\text{opp}})$. Indeed, we may choose a local free finite resolution $L^\bullet$ of $M$, so that $D' M$ is quasi isomorphic to the transposed complex $(L^\bullet)^t$ whose entries are free.

By the same argument we deduce that (1) is an isomorphism whenever $M \in D^b_{\text{coh}}(\mathcal{D}_{X \times S})$.

Again by Proposition 3.2, $\mathcal{D}_{X \times S}$ has finite flat dimension so we are in conditions to apply [3, (A.10)]: given $M, N \in D^b(\mathcal{D}_{X \times S})$ there is a natural morphism:

$$D'M \otimes_{\mathcal{D}_{X \times S}} N \to R\mathcal{H}om_{\mathcal{D}_{X \times S}}(M, N)$$

which an isomorphism provided that $M$ or $N$ belong to $D^b_{\text{coh}}(\mathcal{D}_{X \times S})$. When $M, N \in D^b_{\text{coh}}(\mathcal{D}_{X \times S})$, composing (2) with the biduality isomorphism (1) gives a natural isomorphism

$$R\mathcal{H}om_{\mathcal{D}_{X \times S}}(M, N) \simeq R\mathcal{H}om_{\mathcal{D}_{X \times S}}(D N, D M).$$

3.2. Characteristic variety. Recall (see [17, §III.1.3]) that the characteristic variety $\text{Char} M$ of a coherent $\mathcal{D}_{X \times S}$-module $M$ is the support in $T^*X \times S$ of its graded module with respect to any (local) good filtration. One has (see [17, Prop. III.1.3.2])

$$\text{Char}(\mathcal{D}_{X \times S} \otimes_{\mathcal{D}_{X \times S}} M) = \pi^{-1} \text{Char} M,$$

$$\text{Char} M = \pi(\text{Char}(\mathcal{D}_{X \times S} \otimes_{\mathcal{D}_{X \times S}} M)).$$

One may as well define the characteristic variety of an object $M \in D^b_{\text{coh}}(\mathcal{D}_{X \times S})$ as the union of the characteristic varieties of its cohomology modules. By the flatness of $\mathcal{D}_{X \times S}$ over $\mathcal{D}_{X \times S}$, (4) holds for any object of $D^b_{\text{coh}}(\mathcal{D}_{X \times S})$.

Proposition 3.5 ([18, Prop. 2.5]). For $M \in D^b_{\text{coh}}(\mathcal{D}_{X \times S})$ we have

$$\text{Char}(M) = \text{Char}(D M).$$

3.3. The de Rham and solution complexes. For an object $M$ of $D^b(\mathcal{D}_{X \times S})$ we define the functors

$$\text{DR} M := R\mathcal{H}om_{\mathcal{D}_{X \times S}}(\mathcal{D}_{X \times S}, M),$$

$$\text{Sol} M := R\mathcal{H}om_{\mathcal{D}_{X \times S}}(M, \mathcal{D}_{X \times S})$$
which take values in $D^b(p_X^{-1}\mathcal{O}_S)$. If $\mathcal{M}$ is a $\mathcal{D}_{X\times S/S}$-module, that is, a $\mathcal{O}_{X\times S}$-module equipped with an integrable relative connection $∇ : \mathcal{M} \rightarrow \Omega^1_{X\times S/S} \otimes \mathcal{M}$, the object $\text{DR}(\mathcal{M})$ is represented by the complex $(\Omega_{X\times S/S}^* \otimes \mathcal{O}_{X\times S}, \nabla)$. Noting that $R\text{Hom}_{\mathcal{D}_{X\times S/S}}(\mathcal{O}_{X\times S}, \mathcal{D}_{X\times S/S}) \simeq \Omega_{X\times S/S}[-\dim X]$ we get

$$D\mathcal{O}_{X\times S} \simeq \mathcal{O}_{X\times S}.$$ 

For $\mathcal{N} = \mathcal{O}_{X\times S}$, (3) implies a natural isomorphism, for $\mathcal{M} \in D^b_{\text{coh}}(\mathcal{D}_{X\times S/S})$:

$$\text{Sol.}\mathcal{M} \simeq \text{DR}\mathcal{M}.$$ 

3.4. Holonomic $\mathcal{D}_{X\times S/S}$-modules. Let $\mathcal{M}$ be a coherent $\mathcal{D}_{X\times S/S}$-module. We say that it is holonomic if its characteristic variety $\text{Char}\mathcal{M} \subset T^*X \times S$ is contained in $\Lambda \times S$ for some closed conic Lagrangian complex analytic subset of $T^*X$. We will say that a complex $\mu$-stratification $(X_\alpha)$ is adapted to $\mathcal{M}$ if $\Lambda \subset \bigcup_\alpha T_{X_\alpha}X$. Similar definitions hold for objects of $D^b_{\text{hol}}(\mathcal{D}_{X\times S/S})$.

An object $\mathcal{M} \in D^b_{\text{hol}}(\mathcal{D}_{X\times S/S})$ is said to be holonomic if its cohomology modules are holonomic. We denote the full triangulated category of holonomic complexes by $D^b_{\text{hol}}(\mathcal{D}_{X\times S/S})$.

**Corollary 3.6** (of Prop. 3.5). If $\mathcal{M}$ is an object of $D^b_{\text{hol}}(\mathcal{D}_{X\times S/S})$, then so is $D\mathcal{M}$.

**Theorem 3.7.** Let $\mathcal{M}$ be an object of $D^b_{\text{hol}}(\mathcal{D}_{X\times S/S})$. Then $\text{DR}(\mathcal{M})$ and $\text{Sol.}\mathcal{M}$ belong to $D^b_{\text{c-c}}(p_X^{-1}\mathcal{O}_S)$.

**Proof.** Firstly, it follows [5 Prop. 11.3.3], that $\text{Sol.}(\mathcal{M})$ and $\text{DR}(\mathcal{M})$ have their micro-support contained in $\Lambda \times T^*S$ (see [18 p. 11 & Th. 2.13]) and, according to Proposition 2.5 these complexes are objects of $D^b_{\text{w-c-c}}(p_X^{-1}\mathcal{O}_S)$.

Let $x \in X$. In order to prove that $i_x^{-1}\text{DR}\mathcal{M}$ has $\mathcal{O}_S$-coherent cohomology, we can assume that $x$ is a stratum of a stratification adapted to $\mathcal{D}\mathcal{M}$ and we use Lemma 2.11 to get $i_x^{-1}\text{DR}\mathcal{M} \simeq R\mathcal{p}_{S*}(\mathbb{C}_{B_x \times S} \otimes \mathcal{O}_S)\text{DR}(\mathcal{M})$ for $\varepsilon$ small enough, where $B_\varepsilon$ is a closed ball of radius $\varepsilon$ centered at $x$. One then remarks that $(\mathbb{C}_{B_x \times S}, \mathcal{M})$ forms a relative elliptic pair in the sense of [18], and Proposition 4.1 of loc. cit. gives the desired coherence.

The statement for $\text{Sol.}\mathcal{M}$ is proved similarly. q.e.d.

**Lemma 3.8** (see [14 Prop. 1.2.5]). For $\mathcal{M}$ in $D^b_{\text{hol}}(\mathcal{D}_{X\times S/S})$ with adapted stratification $(X_\alpha)$ and for any $s_0 \in S$, $L_i s_0\mathcal{M}$ is $\mathcal{D}_X$-holonomic and $(X_\alpha)$ is adapted to it.

**Corollary 3.9.** For $\mathcal{M} \in D^b_{\text{hol}}(\mathcal{D}_{X\times S/S})$, there is a natural isomorphism $D^b\text{Sol.}\mathcal{M} \simeq \text{DR}\mathcal{M}$.

**Proof.** We consider the canonical pairing

$$\text{DR}\mathcal{M} \otimes p_X^{-1}\mathcal{O}_S \rightarrow \text{Sol.}\mathcal{M} \rightarrow p_X^{-1}\mathcal{O}_S$$

which gives a natural morphism

$$\text{DR}\mathcal{M} \rightarrow D^b\text{Sol.}\mathcal{M}.$$
in $\mathcal{D}_{\mathbb{C},c}^b(p_X^{-1}\mathcal{O}_S)$. We have for each $s_o \in S$, by Proposition 3.11

$$L_i s_o(\text{DR.}\mathcal{M}) \simeq \text{DR} L_i s_o(\mathcal{M}),$$

$$L_i s_o(\text{Sol.}\mathcal{M}) \simeq \text{Sol} L_i s_o(\mathcal{M}).$$

Since $L_i s_o(\mathcal{M}) \in \mathcal{D}_{\mathbb{C}}^b(\mathcal{D}_X)$ by Lemma 3.8 we have

$$\text{DR} L_i s_o(\mathcal{M}) \simeq D' \text{Sol} L_i s_o(\mathcal{M}),$$

so by Proposition 3.11 and Proposition 2.2

$$D' \text{Sol} L_i s_o(\mathcal{M}) \simeq D' L_i s_o(\text{Sol.}\mathcal{M}) \simeq L_i s_o(D' \text{Sol.}\mathcal{M}).$$

The assertion then follows by Proposition 2.2. q.e.d.

In the following proposition, the main argument is that of strictness, which is essential. We will set $\mathcal{P}_{\text{DR.}\mathcal{M}} := \text{DR.}\mathcal{M}^{[\dim X]}$ and $\mathcal{P}_{\text{Sol.}\mathcal{M}} = \text{Sol.}\mathcal{M}^{[\dim X]}$.

**Proposition 3.10.** Let $\mathcal{M}$ be a holonomic $\mathcal{D}_{X \times S/S'}$-module which is strict, i.e., which is $p^{-1}\mathcal{O}_{S'}$-flat. Then $\mathcal{P}_{\text{DR.}\mathcal{M}}$ satisfies the support condition (Supp) with respect to a $\mu$-stratification adapted to $\mathcal{M}$.

**Proof.** We prove the result by induction on $\dim S$. Since it is local on $S$, we consider a local coordinate $s$ on $S$ and we set $S' = \{s = 0\}$. The strictness property implies that we have an exact sequence

$$0 \to \mathcal{M} \xrightarrow{\phi} \mathcal{M} \to i^*_{S'} \mathcal{M} \to 0,$$

and $i^*_{S'} \mathcal{M}$ is $\mathcal{D}_{X \times S'/S'}$-holonomic and $p^{-1}\mathcal{O}_{S'}$-flat. We deduce an exact sequence of complexes $0 \to \mathcal{P}_{\text{DR.}\mathcal{M}} \xrightarrow{\phi} \mathcal{P}_{\text{DR.}\mathcal{M}} \to \mathcal{P}_{i^*_{S'} \mathcal{M}} \to 0$.

Let $X_\alpha$ be a stratum of a $\mu$-stratification of $X$ adapted to $\mathcal{M}$ (hence to $i^*_{S'} \mathcal{M}$, after Lemma 3.8). For $x \in X_\alpha$, let $k$ be the maximum of the indices $j$ such that $\mathcal{H}^j i^{-1}_x \mathcal{P}_{\text{DR.}\mathcal{M}} \neq 0$. For any $S'$ as above, we have a long exact sequence

$$\cdots \to \mathcal{H}^k i^{-1}_x \mathcal{P}_{\text{DR.}\mathcal{M}} \xrightarrow{\phi} \mathcal{H}^k i^{-1}_x \mathcal{P}_{\text{DR.}\mathcal{M}} \to \mathcal{H}^k i^{-1}_x \mathcal{P}_{i^*_{S'} \mathcal{M}} \to 0.$$

If $k > -\dim X_\alpha$, we have $\mathcal{H}^k i^{-1}_x \mathcal{P}_{i^*_{S'} \mathcal{M}} = 0$, according to the support condition for $i^*_{S'} \mathcal{M}$ (inductive assumption), since $(X_\alpha)$ is adapted to it. Therefore, $s : \mathcal{H}^k i^{-1}_x \mathcal{P}_{\text{DR.}\mathcal{M}} \to \mathcal{H}^k i^{-1}_x \mathcal{P}_{\text{DR.}\mathcal{M}}$ is onto. On the other hand, by Theorem 3.11, $\mathcal{H}^k i^{-1}_x \mathcal{P}_{\text{DR.}\mathcal{M}}$ is $\mathcal{O}_{S'}$-coherent. Then Nakayama’s lemma implies that $\mathcal{H}^k i^{-1}_x \mathcal{P}_{\text{DR.}\mathcal{M}}$ is strict, hence the assertion.

q.e.d.

**Proof of Theorem 1.2.** It is a direct consequence of the following.

**Theorem 3.11.** Let $\mathcal{M}$ be an object of $\mathcal{D}_{\mathbb{C}}^b(\mathcal{D}_{X \times S/S})$ and let $\mathcal{D}_i \mathcal{M}$ be the dual object. Then there is an isomorphism $\mathcal{P}_{\text{DR.}\mathcal{M}} \simeq D^\mathcal{P}_{\text{DR.}\mathcal{M}}$.

Indeed, with the assumptions of Theorem 1.2, $\mathcal{D}_i \mathcal{M}$ is holonomic since $\mathcal{M}$ is so (see Corollary 3.6), and both $\mathcal{M}$ and $\mathcal{D}_i \mathcal{M}$ are strict. Then both $\mathcal{P}_{\text{DR.}\mathcal{M}}$ and $\mathcal{P}_{\text{DR.}\mathcal{M}}$ satisfy the support condition, according to Proposition 3.10. Hence, according to Theorem 3.11 and Proposition 2.2, $\mathcal{P}_{\text{DR.}\mathcal{M}}$ satisfies the cosupport condition.
Similarly, $\text{Sol}_M \simeq D^{\text{pDR}} M$ and $D(\text{Sol}_M) \simeq \text{pDR}_M$ both satisfy the support condition, hence $\text{Sol}_M[\dim X]$ is a perverse object. q.e.d.

Proof of Theorem 3.11 Combining (3) with [5] Ex.II.24 (iv) (with $f = \text{id}$, $\mathcal{A} = \mathcal{D}_{X\times S/S}$ and $\mathcal{B} = p_X^{-1}\mathcal{O}_S$) entails, for any $\mathcal{N} \in D^b_{\text{coh}}(\mathcal{D}_{X\times S/S})$, a natural morphism

$$R\mathcal{H}\text{om}_{\mathcal{D}_{X\times S/S}}(\mathcal{N}, M) \to R\mathcal{H}\text{om}_{p_X^{-1}\mathcal{O}_S}(D\mathcal{M}, D\mathcal{N}).$$

When $\mathcal{N} = \mathcal{O}_{X\times S}$, we obtain a natural morphism

$$D\text{DR}_M \to D' D\text{DR}_M, \quad \text{that is,} \quad \text{pDR}_M \to D\text{pDR}_M D_M.$$

Suppose now that $M \in D^b_{\text{hol}}(\mathcal{D}_{X\times S/S})$. Recall that $D\mathcal{M} \in D^b_{\text{hol}}(\mathcal{D}_{X\times S/S})$, so $\text{pDR}_M D\mathcal{M} \in D^b_{\text{c-c}}(p_X^{-1}\mathcal{O}_S)$.

Hence, by biduality, we get a morphism

$$D\text{pDR}_M \leftarrow \text{pDR}_M D_M. \quad (6)$$

On the other hand, since $L^{\ast}_{s_0}(M) \in D^b_{\text{hol}}(\mathcal{D}_X)$ for each $s_0 \in S$, the morphisms above induce isomorphisms

$$L^{\ast}_{s_0}(D\text{pDR}_M) \simeq \text{pDR} D L^{\ast}_{s_0}(M)$$

according to Proposition 2.4 and Proposition 3.11 where in the right hand side we consider the duality for holonomic $\mathcal{D}_X$-modules. Thus (3) is an isomorphism by Proposition 2.2 and the local duality theorem for holonomic $\mathcal{D}_X$-modules (see [13] and the references given there). q.e.d.

Example 3.12. Let $X$ be the open unit disc in $\mathbb{C}$ with coordinate $x$ and let $S$ be a connected open set of $\mathbb{C}$ with coordinate $s$. Let $\varphi : S \to \mathbb{C}$ be a non constant holomorphic function on $S$ and consider the holonomic $\mathcal{D}_{X\times S/S}$-module $\mathcal{M} = \mathcal{D}_{X\times S/S} / \mathcal{D}_{X\times S/S} \cdot P$, with $P = x\partial_x - \varphi(s)$. It is easy to check that $\mathcal{M}$ has no $\mathcal{O}_S$-torsion and admits the resolution

$$0 \to \mathcal{D}_{X\times S/S} \xrightarrow{P} \mathcal{D}_{X\times S/S} \to \mathcal{M} \to 0,$$

so that the dual module $D\mathcal{M}$ has a similar presentation and is also $\mathcal{O}_S$-flat. The complex $\text{pSol}_M$ is represented by

$$0 \to \mathcal{O}_{X\times S} \xrightarrow{P} \mathcal{O}_{X\times S} \to 0 \quad (\text{terms in degrees } -1 \text{ and } 0).$$

Consider the stratification $X_1 = X \setminus \{0\}$ and $X_0 = \{0\}$ of $X$. Then $\mathcal{H}^{\ast-1}\text{pSol}_M|_{X_1}$ is a locally constant sheaf of free $p_X^{-1}\mathcal{O}_S$-modules generated by a local determination of $x^{\varphi(s)}$, and $\mathcal{H}^{\ast 0}\text{pSol}_M|_{X_1} = 0$. On the other hand, $\mathcal{H}^{\ast-1}\text{pSol}_M|_{X_0} = 0$ and $\mathcal{H}^{\ast 0}\text{pSol}_M|_{X_0}$ is a skyscraper sheaf on $X_0 \times S$ supported on $\{s \in S \mid \varphi(s) \in \mathbb{Z}\}$.

For each $x_0$ we have

$$i_0^{-1}(\text{pSol}_M) \simeq i_0^{-1}R\mathcal{H}\text{om}_{\mathcal{D}_{X\times S}}(\mathcal{D}_{X\times S} \otimes_{\mathcal{O}_{X\times S/S}} \mathcal{M}, R\Gamma_{\{x_0\} \times S}|X \times S \mathcal{O}_{X\times S})[\dim X] \simeq i_0^{-1}R\mathcal{H}\text{om}_{\mathcal{D}_{X\times S}}(\mathcal{D}_{X\times S} \otimes_{\mathcal{O}_{X\times S/S}} \mathcal{M}, B_{\{x_0\} \times S}|X \times S)$$. 
where $B_{\{x_0\} \times S|X \times S} := \mathcal{M}_{\{x_0\} \times S}(\mathcal{O}_X \times S)$ denotes the sheaf of holomorphic hyperfunctions (of finite order) along $x = x_0$ (cf. [10]). The second isomorphism follows from the fact that $\mathcal{D}_X \otimes \mathcal{D}_{X/S} \mathcal{M}$ is regular specializable along the submanifold $x = x_0$ (cf. [7]).

Recall that the sheaves $B_{\{x_0\} \times S|X \times S}$ are flat over $p_X^{-1} \mathcal{O}_S$ because locally they are inductive limits of free $p_X^{-1} \mathcal{O}_S$-modules of finite rank.

Since $i_{x_0}^! (p\mathcal{S}ol \mathcal{M})$ is quasi isomorphic to the complex

$$0 \to B_{\{x_0\} \times S|X \times S} \xrightarrow{F} B_{\{x_0\} \times S|X \times S} \to 0$$

it follows that the flat dimension over $\mathcal{O}_S$ of $i_{x_0}^! (p\mathcal{S}ol \mathcal{M})$ in the sense of [4] §4 is $\leq 0$ for any $x_0$. Moreover, $\mathcal{H}^0 i_{x_0}^! (p\mathcal{S}ol \mathcal{M}) = 0$ and, if $x_0 \neq 0$, $\mathcal{H}^1 i_{x_0}^! (p\mathcal{S}ol \mathcal{M})$ is locally free $\mathcal{O}_S$-module of rank 1. Hence the flat dimension of $i_{x_0}^! (p\mathcal{S}ol \mathcal{M})$ is $\leq 1$. This shows explicitly that $p\mathcal{S}ol \mathcal{M}$ satisfies the condition (Cosupp+) of Corollary 2.29.

4. Application to mixed twistor $\mathcal{D}$-modules

Let $\mathcal{R}_{X \times \mathbb{C}}$ be the sheaf on $X \times \mathbb{C}$ of $z$-differential operators, locally generated by $\partial_x \otimes z \partial_z$, in local coordinates $(x_1, \ldots, x_n)$ on $X$. When restricted to $X \times \mathbb{C}^*$, the sheaf $\mathcal{R}_{X \times \mathbb{C}^*}$ is isomorphic to $\mathcal{D}_{X \times \mathbb{C}^*/\mathbb{C}^*}$.

A mixed twistor $\mathcal{D}$-module on $X$ (see [12]) is a triple $\mathcal{I} = (\mathcal{M}', \mathcal{M}'', C)$, where $\mathcal{M}', \mathcal{M}''$ are holonomic $\mathcal{R}_{X \times \mathbb{C}}$-modules and $C$ is a certain pairing with values in distributions, that we will not need to make precise here. Such a triple is subject to various conditions. We say that a $\mathcal{D}_{X \times \mathbb{C}^*/\mathbb{C}^*}$-module $\mathcal{M}$ underlies a mixed twistor $\mathcal{D}$-module $\mathcal{I}$ if $\mathcal{M}$ is the restriction to $X \times \mathbb{C}^*$ of $\mathcal{M}'$ or $\mathcal{M}''$.

Theorem 1.1 is now a direct consequence of the following properties of mixed twistor $\mathcal{D}$-modules, since they imply that $\mathcal{M}$ satisfies the assumptions of Theorem 1.2. If $\mathcal{M}$ underlies a mixed twistor $\mathcal{D}$-module, then

- there exists a locally finite filtration $W_\bullet \mathcal{M}$ indexed by $\mathbb{Z}$ by $\mathcal{R}_{X \times \mathbb{C}}$-submodules such that each graded module underlies a pure polarizable twistor $\mathcal{D}$-module; then each $gr^W_1 \mathcal{M}$ is strict and holonomic (see [14] Prop. 4.1.3] and [11] §17.1.1)], and thus so is $\mathcal{M}$;
- the dual of $\mathcal{M}$ as a $\mathcal{R}_{X \times \mathbb{C}^*}$-module also underlies a mixed twistor $\mathcal{D}$-module, hence is also strict holonomic (see [12] Th.12.9]); using the isomorphism $\mathcal{R}_{X \times \mathbb{C}^*} \simeq \mathcal{D}_{X \times \mathbb{C}^*/\mathbb{C}^*}$, we see that the dual $D_{\mathcal{M}}$ as a $\mathcal{D}_{X \times \mathbb{C}^*/\mathbb{C}^*}$-module is strict and holonomic.

q.e.d.

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