KATO'S CONDUCTOR AND GENERIC RESIDUAL PERFECTION

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Abstract. Let $A$ be a complete discrete valuation ring with possibly imperfect residue field, and let $\chi$ be a rank-one Galois representation over $A$. I show that the non-logarithmic variant of Kato's Swan conductor is the same for $\chi$ and the pullback of $\chi$ to the generic residual perfection of $A$. This implies the conductor from "Conductors and the moduli of residual perfection" extends the non-logarithmic variant of Kato's.

Introduction

Let $A$ be a complete (or even henselian) discrete valuation ring with fraction field $K$. When the residue field $k$ of $A$ is perfect, there is [17,IV,VI] a well-known and satisfactory theory of wild ramification over $A$. We understand much less, though, when we do not require that $k$ be perfect. In this context, Kato [11] has developed a good abelian theory: to every one-dimensional Galois representation $\chi$ over $A$, he gives a non-negative integer, the Kato-Swan conductor $sw_K(\chi)$ of $\chi$, that measures the extent to which $\chi$ is wildly ramified. It is the pole order, in the logarithmic sense, of a certain differential form—his refined Swan conductor. We can just as well, however, consider the order in the usual, non-logarithmic sense. I will call this the Kato-Artin conductor of $\chi$ and denote it by $ar_K(\chi)$. When $k$ is perfect, it agrees with the usual Artin conductor, and $sw_K(\chi)$ agrees with the usual Swan conductor.

There have been several recent attempts [1,3,4,19] to give a general approach to non-abelian wild ramification over such rings $A$. In one of them, I showed how to associate a non-negative integer $ar(\rho)$ to any Galois representation $\rho$ over $A$. The purpose of this paper is to show that when $\rho$ is one-dimensional, $ar(\rho)$ agrees with $ar_K(\rho)$.

Let $A^g$ be the generic residual perfection [4] of $A$. It is a residually perfect complete discrete valuation ring of ramification index one over $A$ and is, in a certain sense, universally generic with respect to these properties. Let $K^g$ be its fraction field. Most of this paper is devoted to the proof of the following result.

Theorem. If $\chi$ is a class in $H^1(K,Q/Z)$ and $\chi'$ is its image in $H^1(K^g,Q/Z)$, then $ar_K(\chi) = ar_K(\chi')$.

The intuitive reason why this should be true is that the order of any differential form on $A$ should remain unchanged when the form is pulled back to $A^g$. When $A$ is of equal characteristic, this has meaning and, once the necessary foundations are laid, is essentially a proof. In fact, the observation that there are residually perfect extensions with this property is what led to the definition of the general Artin conductor. In mixed characteristic, however, this provides little more than motivation, and most of this paper is spent pushing it through to a real proof.
In section 2, I recall Kato’s theory, prove some basic results, and define the Kato-Artin conductor. The proof of the theorem when $A$ is of equal characteristic is in section 3. It uses Matsuda’s refinement \cite{12} of Kato’s refined Swan conductor. Because the proof in equal characteristic is so much simpler than the proof in mixed characteristic, I encourage the reader to read it first. The basic technique in mixed characteristic is to use Kato’s description \cite[4.1]{11} (following Bloch-Kato \cite{2}) of certain graded pieces of cohomology groups in terms of explicit $K$-theoretic symbols and then understand how these symbols behave under pullback to $A^\otimes$. Section 5 contains a commutative diagram that encodes this behavior, and section 6 gives the proof in mixed characteristic. In the final section, I show how the theorem implies that $\ar(\rho)$ and $\ar_K(\rho)$ agree for one-dimensional Galois representations $\rho$.

Conventions

If $A$ is a discrete valuation ring, $p_A$ will denote its maximal ideal, and $U_A^\bullet$ will denote the filtration of $A^\ast$ with $U_A^0 = A^\ast$ and $U_A^i = 1 + p_A^i$ for positive integers $i$. If its residue field $A/p_A$ is perfect and has positive characteristic, then $s_A$ will denote the unique multiplicative section \cite[II §4 Prop. 8]{17} of the reduction map $A \to A/p_A$. If $x$ is an element of $A$, its image in $A/p_A$ will be denoted $\bar{x}$. An extension of $A$ is a discrete valuation ring $B$ equipped with an injective local homomorphism $A \to B$. We will denote its ramification index by $e_{B/A}$.

Throughout, $A$ will denote a henselian \cite{14} discrete valuation ring, held fixed within each subsection, with fraction field $K$ and residue field $k$. We will assume for simplicity of exposition that $k$ always has characteristic $p$, where $p$ is a fixed prime number. We will say $A$ is of equal characteristic if $K$ has characteristic $p$ and is of mixed characteristic if $K$ has characteristic 0.

1. The generic residual perfection

The purpose of this section is to recall the main facts \cite[1.14, 2.4]{4} about the generic residual perfection we will need later.

1.1. The generic residual perfection of (the completion of) $A$ is the $A$-algebra $A^\otimes$ corresponding to the generic point of the moduli space of residual perfections of $A$. It is a complete discrete valuation ring with $e_{A^\otimes/A} = 1$ whose residue field $k^\otimes$ is perfect. Its fraction field will be denoted $K^\otimes$.

1.2. Theorem. Let $T$ be a lift to $A$ of a $p$-basis of its residue field $k$, and let $\pi \in A$ be a uniformizer. For each element $t \in T$, let $u_{t,1}, u_{t,2}, \cdots \in k^\otimes$ be the unique sequence such that the image of $t$ in $A^\otimes$ is $s_{A^\otimes}(\bar{t}) + s_{A^\otimes}(u_{t,1})\pi + s_{A^\otimes}(u_{t,2})\pi^2 + \cdots$. Let $R$ be the free polynomial algebra $k[T \times \mathbb{Z}_{>0}]$. Then the map $R \to k^\otimes$ determined by $(t,j) \mapsto u_{t,j}$ induces an isomorphism from the fraction field of $R^{\otimes\sim}$ to $k^\otimes$.

1.3. Theorem. Fix a separable closure of $K^\otimes$. Then the map $G_{K^\otimes} \to G_K$ of the corresponding absolute Galois groups is surjective. The induced maps of inertia groups and wild inertia groups are also surjective.

2. Kato’s theory

The purpose of this section is to collect some results in Kato’s theory \cite{11}. Let us first recall the basics.
2.1. Let $F$ be a field and let $n > 0$ and $r$ be integers. If $n$ is invertible in $F^*$, let $\mathbf{Z}/n(r)$ be the $r$-th Tate twist of the constant sheaf $\mathbf{Z}/n$ on the étale topology (of Grothendieck \cite{G}) of $F$. If the characteristic of $F$ is $p > 0$, write $n = mp^r$, where $p \nmid m$, and let $\mathbf{Z}/n(r)$ be the complex

$$\mathbf{Z}/m(r) \oplus W_s \Omega^r_{F, \log}[-r]$$

of abelian sheaves on $\text{Spec}(F)_{\text{ét}}$. Here, $W_s \Omega^r_{F, \log}$ is the piece of degree $r$ of the logarithmic part \cite[I 5.7] of Deligne and Illusie’s de Rham-Witt complex $W_s \Omega^n_F$ on $\text{Spec}(F)_{\text{ét}}$.

For positive integers $q$, write $H^n_q(F) = H^n(F, \mathbf{Z}/n(q - 1))$, and let $H^n_q(F)$ be the colimit of $H^n_q(F)$ over the integers $n$ (ordered by divisibility). The natural map $H^n_q(F) \to H^n_q(F)$ is an isomorphism of $H^n_q(F)$ with the $n$-torsion of $H^n_q(F)$. I will usually identify the two without comment.

Let

$$h_F : F^* \longrightarrow H^1(F, \mathbf{Z}/n(1))$$

be the connecting homomorphism of the Kummer triangle

$$\mathbf{Z}/n(1) \longrightarrow G_m \overset{n}{\longrightarrow} G_m \longrightarrow \mathbf{Z}/n(1)[1].$$

(When $n$ is a power of the characteristic of $F$, the existence of such a triangle follows from the theory of the de Rham-Witt complex \cite[I 3.23.2, I 5.7.1]{GI}.) Let $K^r_M(F)$ denote the $r$-th Milnor $K$-group \cite{K} of $F$. Then there is a homomorphism

$$K^r_M(F) \longrightarrow H^r(F, \mathbf{Z}/n(r)),$$

also denoted $h_F$, sending $\{x_1, \ldots, x_r\}$ (for $x_1, \ldots, x_r \in F^*$) to the cup product

$$h_F(x_1) \cup \cdots \cup h_F(x_r).$$

For $\chi \in H^n_q(F)$, let $\{\chi, x_1, \ldots, x_r\}$ denote $\chi \cup h_F(\{x_1, \ldots, x_r\})$. Taking the colimit over integers $n$, we get a pairing

$$H^n_q(F) \otimes K^r_M(F) \longrightarrow H^{q+r}_p(F).$$

We will use $\{\chi, x_1, \ldots, x_r\}$ to denote the image of $\chi \otimes \{x_1, \ldots, x_r\}$ under this pairing as well.

If the characteristic of $F$ is $p$, let $\xi_s : W_s \Omega^{q-1}_F \to H^q_p(F)$ denote the higher Artin-Schreier maps \cite[1.3]{K}.

2.2. For any non-negative integer $n$, let $\text{fil}_n H^q(K)$ be the subgroup of classes $\chi$ that have the property $\{\chi|_{A'}, 1 + p^np_{A'}\} = 0$ for every henselian extension $A'$ of $A$. This gives \cite[2.2, 6.3]{K} an exhaustive increasing filtration of $H^q(K)$. The Kato-Swan conductor (or logarithmic Kato conductor) of a class $\chi \in H^q(K)$ is the smallest integer $n$ such that $\chi \in \text{fil}_n H^q(K)$. It is denoted $\text{sw}_n(\chi)$.

2.3. Let $\hat{A}$ denote the henselization of the localization of the polynomial algebra $A[T]$ at the ideal generated by $p_A$. Then $\hat{A}$ is a henselian discrete valuation ring, and for any uniformizer $\pi$ of $A$, we have \cite[6.3]{K}

$$\chi \in \text{fil}_n H^q(K) \iff \{\chi|_{A'}, 1 + \pi^{n+1}T\} = 0.$$

We will denote the fraction field of $\hat{A}$ by $\hat{K}$ and the residue field by $\hat{k}$.
2.4. The map $H^q(k) \to H^q(K)$ extends naturally to an exact sequence
\[
0 \to H^q(k) \to \fil_0 H^q(K) \to H^{q-1}(k) \to 0.
\]
Given a uniformizer $\pi$ of $A$, the map $\psi \mapsto \{\psi, \pi\}$ is a splitting \[6.1\].

2.5. The reduction map
\[
(A - \{0\})/U_A \to k
\]
gives $k$ the structure of a log ring \[11, 1.1\]. Let $\omega^1_k = \omega^1_{k/\mathbb{Z}}$ denote the $k$-module of absolute Kähler differentials with respect to this log structure \[10, 1.7\]. For $q \in \mathbb{N}$, let $\omega^q_k$ denote the $q$-th exterior power of $\omega^1_k$. There is a natural exact sequence
\[
0 \to \Omega^q_k \to \omega^q_k \to \Omega^{q-1}_k \to 0.
\]
(Of course, $\Omega^*_k$ means $\Omega^*_k/\mathbb{Z}$.) Given a uniformizer $\pi$ of $A$, the map $\eta \mapsto \eta \wedge \dlog(\pi)$ is a splitting.

2.6. There is a unique map $\lambda_A : \omega^{q-1}_k \to \fil_0 H^q_p(K)$ that gives rise to a map of sequences
\[
0 \to \Omega^{q-1}_k \to \omega^{q-1}_k \to \Omega^{q-2}_k \to 0
\]
respecting the splittings in \[2.4\] and \[2.5\] (for every uniformizer $\pi$ of $A$).

The following theorem \[11, 5.1, 5.2, 5.3\] is fundamental.

2.7. Theorem. Let $n$ be a positive integer. Then for any class $\chi \in \fil_0 H^q(K)$, there is a unique element $\eta \in p^{-n}_A \otimes_A \omega^q_k$ such that for every henselian extension $A'$ of $A$ and every element $z$ of $p^{-n}_A A'$, we have
\[
\{\chi, 1 + z\} = \lambda_{A'}(z\eta).
\]
Furthermore, the function $\chi \mapsto \eta$ induces an injective homomorphism
\[
\kappa_n : \gr_p H^q(K) \to p^{-n}_A \otimes \omega^q_k.
\]

I will typically write $\kappa_n(\chi)$ for the image under $\kappa_n$ of the graded class of $\chi$. It is called the refined Swan conductor of $\chi$.

2.8. Let us now consider the non-logarithmic analogue of the Kato-Swan conductor. Let $\chi$ be a class in $H^1(K, \mathbb{Q}/\mathbb{Z})$ and put $n = \sw_k(\chi)$. Define
\[
\ar_k(\chi) = \begin{cases} 0 & \text{if } \chi \text{ is unramified} \\ 1 & \text{if } \chi \text{ is tame and ramified} \\ n & \text{if } \chi \text{ is ramified and } \kappa_n(\chi) \in p^{-n}_A \otimes \Omega^1_k \\ n + 1 & \text{if } \chi \text{ is ramified and } \kappa_n(\chi) \notin p^{-n}_A \otimes \Omega^1_k \end{cases}
\]
We call $\ar_k(\chi)$ the Kato-artin conductor of $\chi$. It is the natural non-logarithmic analogue of Kato’s Swan conductor. As mentioned in the introduction, when $\sw_k(\chi)$ is not zero, $\ar_k(\chi)$ can be viewed as the order of the pole of $\kappa_n(\chi)$ in the usual sense and $\sw_k(\chi)$ can be viewed as the order in the logarithmic sense.

Matsuda \[12, 3.2.5\] has used what is essentially the same conductor. His is one less than $\ar_k(\chi)$ except when $\chi$ is unramified, in which case both are zero.

Basic facts
The rest of this section contains some propositions we will need in the proof of the theorem in mixed characteristic. All the proofs are straightforward.

**2.9. Proposition.** Let $\chi$ be a class in $H^1(K)$. If $ar_K(\chi)$ and $sw_K(\chi)$ have the same value, then it is a multiple of $p$.

Proof. [11, 5.4]

Let $A'$ be a finite extension of $A$ of ramification index $e$, let $K'$ denote its fraction field, and let $k'$ denote its residue field. Let $n$, $q$, and $s$ be positive integers.

**2.10. Proposition.** The following diagram commutes:

\[
\begin{array}{ccc}
\text{gr}_{\kappa n} H^q(K') & \xrightarrow{\kappa n} & \text{gr}_{\kappa n} H^q(K) \\
\uparrow & & \uparrow \\
\text{gr}_{\kappa n} H^q(k') & \xrightarrow{\kappa n} & \text{gr}_{\kappa n} H^q(k)
\end{array}
\]

Proof. Use the uniqueness statement in 2.7.

**2.11. Corollary.** If the extension $A'/A$ is tame, then for any class $\chi \in H^q(K)$, we have $sw_K(\chi|_{A'}) = e sw_K(\chi)$.

Proof. The map $\Omega^*_k \rightarrow \Omega^*_{k'}$ is injective.

**2.12. Proposition.** If the extension $K'/K$ is Galois with group $G$ and its degree is not a multiple of $p$, the natural maps

\[
H^i_{p^s}(K') \rightarrow H^i_{p^s}(K')^G \\
\text{fil}_{n-1} H^q_{p^s}(K) \rightarrow \text{fil}_{n-1} H^q_{p^s}(K')^G
\]

are isomorphisms.

Proof. Since the order of $G$ is relatively prime to $p$, the groups $H^i(G, H^j_{p^s}(K'))$ are zero for $i > 0$. The existence of a spectral sequence

\[
H^i(G, H^j_{p^s}(K')) \Rightarrow H^{i+j}_{p^s}(K),
\]

implies the first map is an isomorphism. The second then is by 2.11.

**2.13. Proposition.** The exact sequences of 2.4 form a commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & H^0_{p^s}(k') & \rightarrow & \text{fil}_0 H^0_{p^s}(K') & \rightarrow & H^0_{p^s-1}(k') & \rightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow e & & \\
0 & \rightarrow & H^0_{p^s}(k) & \rightarrow & \text{fil}_0 H^0_{p^s}(K) & \rightarrow & H^0_{p^s-1}(k) & \rightarrow & 0
\end{array}
\]

(2.13.1)

where the left two vertical maps are the canonical maps and the rightmost vertical map is the canonical map multiplied by $e$.

Proof. Apply 2.4.

**2.14. Proposition.** Suppose the extension $A'/A$ is tame and generically Galois with group $G$. Then its inertia group $G_0$ acts trivially on $\text{fil}_0 H^0_{p^s}(K')$. 

Proof. Apply 2.3.
Proof. Assume, as we may, that $A'/A$ is residually trivial, and consider (2.13.1). Since $A'/A$ is tame and $k' = k$, the outer vertical morphisms are isomorphisms. Therefore the inner vertical map is, too. Since $G$ acts trivially on $\text{fil}_0 H^q_{p'}(K)$, it acts trivially on $\text{fil}_0 H^q_{p'}(K')$.

3. The proof: equal characteristic

In this section, we use Matsuda’s refinement [12] of Kato’s refined Swan conductor to prove the theorem in equal characteristic. Again, let us first recall the basics.

Assume $A$ is of equal characteristic.

3.1. For any positive integer $s$, let $W_s(K)$ denote the group of Witt vectors of $K$ of length $s$. The Verschiebung maps form an inductive system of abelian groups

$$W_1(K) \rightarrow W_2(K) \rightarrow W_3(K) \rightarrow \cdots$$

As in Fontaine [3], let $CW(K)$ be its colimit. For example, if $\mathbf{F}_p$ denotes the finite field of $p$ elements, then $CW(\mathbf{F}_p) = \mathbb{Q}_p/\mathbb{Z}_p$. The maps $W_s(K) \rightarrow \prod_{-N}^\infty K$ defined by

$$(a_{-s+1}, \ldots, a_0) \mapsto (\ldots, 0, a_{-s+1}, \ldots, a_0)$$

induce a bijection between $CW(K)$ and the set of elements $(\ldots, a_{-i}, a_0)$ such that $a_{-i} = 0$ for sufficiently large $i$. I will typically use this identification without comment.

3.2. The Frobenius endomorphisms of the groups $W_s(K)$ extend to an endomorphism of $CW(K)$. Call it $F$. For any separable closure $K^{\text{sep}}$ of $K$, we have an Artin-Schreier sequence of $\text{Gal}(K^{\text{sep}}/K)$-modules

$$0 \rightarrow CW(\mathbf{F}_p) \rightarrow CW(K^{\text{sep}}) \rightarrow F^{-1} CW(K^{\text{sep}}) \rightarrow 0.$$ 

It is easy to show that this sequence induces a surjection

$$\xi : CW(K) \rightarrow H^1(K, CW(\mathbf{F}_p)) = H^1(K, \mathbb{Q}_p/\mathbb{Z}_p)$$

with kernel $(F - 1) \cdot CW(K)$.

3.3. Let $\varphi : CW(K) \rightarrow \Omega^1_K$ denote the homomorphism defined by

$$(\ldots, a_{-1}, a_0) \mapsto - \sum_{i \in \mathbb{N}} a_{-i}^{p^i - 1} a_{-i}.$$ 

It appears that this map was first considered by Serre [16]; it has a nice interpretation in terms of the de Rham-Witt complex [7, I 3.12].

3.4. Define the following filtrations indexed by non-negative integers $n$:

$$\text{fil}_n CW(K) = \{ (\ldots, a_{-1}, a_0) \mid \forall i \in \mathbb{N} \ p^i v_A(a_{-i}) \geq -n \},$$

$$\text{fil}_n \Omega^1_K = p^{-n} \cdot \text{dlog}(K^*), \text{ and}$$

$$\text{fil}_n H^1(K, \mathbb{Q}_p/\mathbb{Z}_p) = \xi(\text{fil}_n CW(K)).$$

The filtration on fixed length Witt vectors was apparently first considered by Schmid [15] in the residually perfect case and (independently) Brylinski [7] in the residually imperfect case. It is immediate that $\xi$ and $\varphi$ preserve these filtrations. It is easy to check that $\varphi$ does not factor through $\xi$ but that $\text{gr} \varphi$ does factor through $\text{gr} \xi$. Matsuda remarked [12, 3.2.2] that we have even more:
3.5. Proposition. Let \( n \) be a non-negative integer. Then \( \text{fil}_n \varphi / \text{fil}_{[n/p]} \varphi \) factors through \( \text{fil}_n \xi / \text{fil}_{[n/p]} \xi \).

(Here, \([x]\) is the greatest integer that is at most \( x \).)

3.6. Denote by \( \phi_n \) the resulting homomorphism
\[
\text{fil}_n H^1(K, \mathbb{Q}_p / \mathbb{Z}_p) / \text{fil}_{[n/p]} H^1(K, \mathbb{Q}_p / \mathbb{Z}_p) \to \text{fil}_n \Omega^1_K / \text{fil}_{[n/p]} \Omega^1_K
\]
and by \( \text{gr} \) the induced map \( \text{gr} H^1(K, \mathbb{Q}_p / \mathbb{Z}_p) \to \text{gr} \Omega^1_K \) of associated graded modules. The following theorem of Kato’s \([11, 2.5, 3.2, 3.7]\) shows how to use \( \text{gr} \phi \) to compute the refined Swan conductor of a Galois character given a representation of it as a Witt vector.

3.7. Theorem. For every positive integer \( n \), we have
\[
\text{fil}_n H^1(K, \mathbb{Q} / \mathbb{Z}) \cap H^1(K, \mathbb{Q}_p / \mathbb{Z}_p) = \text{fil}_n H^1(K, \mathbb{Q}_p / \mathbb{Z}_p),
\]
where \( H^1(K, \mathbb{Q}_p / \mathbb{Z}_p) \) is viewed as the \( p^n \)-torsion subgroup of \( H^1(K, \mathbb{Q} / \mathbb{Z}) \). Furthermore, under the natural identification \( \text{gr}_n \Omega^1_K = \mathbb{Z}_p \otimes_{\mathbb{A}_g} \mathfrak{ω}_K^1 \), the restriction of \( \kappa_n \) to \( \text{gr}_n H^1(K, \mathbb{Q}_p / \mathbb{Z}_p) \subseteq \text{gr}_n H^1(K, \mathbb{Q} / \mathbb{Z}) \) coincides with \( \text{gr}_n \phi \).

3.8. Proposition. For non-negative integers \( m \leq n \), the diagram
\[
\begin{array}{ccc}
\text{fil}_n H^1(K, \mathbb{Q}_p / \mathbb{Z}_p) & \xrightarrow{\phi_n} & \text{fil}_n \Omega^1_K / \text{fil}_{[n/p]} \Omega^1_K \\
\downarrow & & \downarrow \\
\text{fil}_m H^1(K, \mathbb{Q}_p / \mathbb{Z}_p) & \xrightarrow{\phi_m} & \text{fil}_m \Omega^1_K / \text{fil}_{[m/p]} \Omega^1_K
\end{array}
\]

commutes.

Proof. Clear. \( \square \)

3.9. Proposition. Let \( A' \) be an extension of \( A \) of ramification index \( e \), and let \( K' \) denote its fraction field. If \( \chi \in H^1(K, \mathbb{Q}_p / \mathbb{Z}_p) \), then
\[
\phi_{\text{swk}(\chi)}(\chi|\mathfrak{A}') = \phi_{\text{swk}(\chi|\mathfrak{A})}(\chi|\mathfrak{A}) \text{ mod } \text{fil}_{[e \text{swk}(\chi)/p]} \Omega^1_{K'}.
\]

Proof. By 3.8. \( \square \)

3.10. Finally, for any non-negative integer \( n \), put \( \text{fil}_n' \Omega^1_K = \mathbb{P}^{-n}_A \cdot \Omega^1_K \subseteq \Omega^1_K \). This filtration measures the order of the pole in the usual sense, whereas \( \text{fil}_n \Omega^1_K \) measures it in the logarithmic sense. The two filtrations are intertwined:
\[
\cdots \subseteq \text{fil}_n \Omega^1_K \subseteq \text{fil}_{n+1} \Omega^1_K \subseteq \text{fil}_{n+1} \Omega^1_K \subseteq \cdots.
\]

Matsuda \([12, 3.1]\) has given non-logarithmic variants of the other filtrations in 3.4, but we will not need them here. (Note, however, that our indexing of \( \text{fil}_n \Omega^1_K \) differs from Matsuda’s by one.)

3.11. Proposition. Let \( \chi \) be a class in \( H^1(K, \mathbb{Q}_p / \mathbb{Z}_p) \). Then \( \text{ar}_K(\chi) \) is the smallest integer \( n \) satisfying \( \chi \in \text{fil}_n H^1(K, \mathbb{Q}_p / \mathbb{Z}_p) \).

Proof. By 3.7. \( \square \)

3.12. Lemma. For \( n \geq 1 \), the natural map \( \text{gr}_n' \Omega^1_K \to \text{gr}_n' \Omega^1_{K'} \) is injective.
Proof. Since we have $\text{gr}_n^i \Omega^1_k = \Omega_A \otimes_A p_A^n / p_A^{n-1}$, it is enough to show the map

$$k \otimes_A \Omega^1_A \rightarrow k \otimes_A \Omega^1_{A^s}$$

is injective.

Let $T$ be a lift to $A$ of a $p$-basis for $k$, and let $\pi$ be a uniformizer of $A$. Then the set $dT \cup \{d\pi\}$ is a basis for the $k$-module $k \otimes_A \Omega^1_A$. To show injectivity, it is enough to check that the image of $dT \cup \{d\pi\}$ is $k$-linearly independent. But $\Omega^1_{A^s} = A^s d\pi$, and so it is enough to show, in the notation of [2], that $\{1\} \cup \{u_t \mid t \in T\}$ is $k$-linearly independent in $k^s$. This follows from [3]. \qed

We can now prove the theorem in equal characteristic.

Proof. First suppose $\chi$ is in $H^1(K, \mathbb{Q}_p / \mathbb{Z}_p)$. Put $n = sw_K(\chi)$ and $m = sw_K(\chi|_{K^s})$. Then $ar_K(\chi)$ is either $n$ or $n+1$. We will treat these two subcases separately.

If $ar_K(\chi) = n + 1$, then $\kappa_n(\chi) \notin p_A^n \otimes \Omega^1_k$. The naturality (2.10) of the maps $\kappa_n$ therefore implies $\kappa_n(\chi|_{A^s}) \notin p_A^n \otimes \Omega^1_{A^s}$, and so, $ar_K(\chi|_{A^s}) = n + 1$.

Now consider the second subcase, when $ar_K(\chi)$ is $n$. By [2.3] we have $n \geq 2$; so, for $n = 2$, it is enough to show $\chi|_{A^s}$ is not tame. This follows from [3.3]. If $n \geq 3$, we have $[n/p] \leq n - 2$ and, hence,

$$\text{fil}_{[n/p]} \Omega^1_{K^s} \subseteq \text{fil}_{n-2} \Omega^1_{K^s} \subseteq \text{fil}'_{n-1} \Omega^1_{K^s}.$$  

Then, by [3.9] and [3.12], we have

$$\phi_m(\chi|_{A^s}) = \phi_n(\chi|_{A^s}) \neq 0 \mod \text{fil}'_{n-1} \Omega^1_{K^s}.$$  

Therefore, $\chi \notin \text{fil}'_{n-1} \Omega^1_{K^s}$, and so [3.11] implies $ar_K(\chi) = n$.

Now let $\chi$ be an arbitrary class in $H^1(K)$. If $\chi$ is tame, the result follows from [1.3]. If $\chi$ is wild, then write $\chi = \chi' + \chi''$, where $\chi'$ is in $H^1(K, \mathbb{Q}_p / \mathbb{Z}_p)$ and $\chi''$ is tame. Since $\chi''$ is tame, $\chi$ and $\chi'$ have the same refined Swan conductor and, hence, the same Kato-Artin conductor. Similarly, $\chi'|_{A^s}$ is wild (again by [1.3]), and so $\chi|_{A^s}$ and $\chi'|_{A^s}$ have the same Kato-Artin conductor. The work above then implies $ar_K(\chi') = ar_K(\chi'|_{A^s})$, and this completes the proof. \qed

4. SOME LEMMAS

The purpose of this section is to prove some lemmas needed in the proof of the theorem in mixed characteristic. All the results in this section are, however, still valid in equal characteristic. Let $\pi$ be a uniformizer of $A$.

4.1. Let $U_i^i K^M_2(K)$ (for $i \geq 1$) denote the subgroup of $K^M_2(K)$ (see [2.1]) generated by the set $\{U_i^i, K^s\}$. This filtration satisfies [2.4]--[4.1]

$$\{U_i^i, U_A^j\} \subseteq U^{i+j} K^M_2(K).$$  

4.2. Lemma. Let $x$ and $y$ be non-zero elements of $p_A$. Then

$$\{1 + x, 1 + y\} \equiv \{1 + xy, -y\} \mod U^{v_A(xy) + 1} K^M_2(K).$$  

Proof. We have

$$\{1 + x, 1 + y\} = \{-y(1 + x), 1 + y\} = \{-y(1 + x), 1 + xy(1 + y)^{-1}\} \equiv \{-y(1 + x), 1 + xy\} \mod U^{v_A(xy) + 1} K^M_2(K) \equiv \{1 + xy, -y\} \mod U^{v_A(xy) + 1} K^M_2(K).$$
4.3. Lemma. Suppose $A$ is residually perfect. Let $x \neq 0$ be an element of $p_A$, and let $z \in A^*$ be such that the element $z' = z - s_A(\bar{z})$ is non-zero. Then

\[ \{1 + x, z\} \equiv v_A(z')\{1 + xz', \pi\} \mod U^{v_A(z') + 1}K_2^M(K) + DK_2^M(K), \]

where $DK_2^M(K)$ is the infinitely $p$-divisible subgroup of $K_2^M(K)$.

Proof. The defining property of the lift $s_A(\bar{z})$ of $\bar{z}$ is that it is infinitely $p$-divisible in $K^*$. Therefore, it suffices to assume $\bar{z} = 1$. By (4.2), we have

\[ \{1 + x, 1 + z'\} \equiv v_A(z')\{1 + xz', \pi\} + \{1 + xz', -z'/\pi^{v_A(z')}\} \mod U^{v_A(z') + 1}K_2^M(K). \]

Since $-z'/\pi^m$ is in $A^* = s_A(k^*)U_1^A$, we also have

\[ \{1 + xz', -z'/\pi^m\} \in U^{v_A(z') + 1}K_2^M(K) + DK_2^M(K). \]

\[ \square \]

4.4. Proposition. If $F$ is a field of characteristic $p$, the $p^\infty$-torsion subgroup of $H^1(F)$ is $p$-divisible.

Proof. By the surjectivity of the higher Artin-Schreier maps $\xi_q$ and the natural maps $\Omega^n_{W_s(F)} \to W_s\Omega^*_F$ and $W_{s+1}(F) \to W_s(F)$.

4.5. Lemma. Let $\chi \in H^1(K)$ be a class such that $ar_k(\chi) = sw_k(\chi) \neq 0$. Then, in the notation of (2.3), the element $\{\chi|_{\bar{A}}, 1 + \pi^{sw_k(\chi) - 1}T\}$ is in $fil\ H^2_p(\bar{K}) + H^2_p(\bar{k})$ but is not in $fil\ H^2_p(K)$.

Proof. Write $n = sw_k(\chi)$. Then by (2.9), we have $n > 1$. Let $\psi = \{\chi|_{\bar{A}}, 1 + \pi^{n-1}T\}$. Since $n > 1$, we have

\[ (1 + \pi^{n-1}T)p^\psi \in U_{n+1}^\eta \quad \text{and} \quad (1 + \pi^{n-1}T)p^\psi \in U_{n+1}^\eta. \]

Applying (4.1), we get $\psi \in fil\ H^2_p(\bar{K})$ and $p^\psi \in fil\ H^2_p(\bar{k})$.

Let $A'$ be a henselian extension of $A$ whose residue field $k'$ is perfect and has the property that $k$ is separably closed in it. (Take $A' = A^\theta$, for example.) Since ar$_k(\chi) = sw_k(\chi)$ and since $k'$ is perfect, (2.10) implies $sw_k(\chi|_{A'}) \leq n - 1$. Therefore, $p^\psi|_{\bar{A}'}$ is zero. Since $k$ is separably closed in $k'$, the field $\bar{k} = k(T)$ is separably closed in $\bar{k}' = k'(T)$. The natural map $H^1_p(\bar{k}) \to H^1_p(\bar{k}')$ is therefore an injection. So, by chasing diagram (2.13.1) applied to the extension $A'/\bar{A}$, we conclude $p^\psi \in H^2_p(\bar{k})$. By (4.2), there is a class $\psi'' \in H^2_p(\bar{k})$ such that $p^\psi'' = p^\psi$. Then, $\psi - \psi''$ is in $fil\ H^2_p(K)$.

On the other hand, putting $\eta = \kappa_n(\chi)$, we have

\[ \{\psi, 1 - \pi\} = \{\chi|_{\bar{A}}, 1 + \pi^{n-1}T, 1 - \pi\} \]
\[ = \{\chi|_{\bar{A}}, 1 - \pi^nT, \pi\} \quad \text{by (4.2)} \]
\[ = \{\lambda_{\bar{A}}(-\eta\pi^nT), \pi\}. \quad \text{by (2.7)} \]

But, by assumption, $\eta$ is in $p_A^{-n} \otimes \Omega^1_A$, and so $\lambda_{\bar{A}}(-\eta\pi^nT)$ equals $\xi_1(-\eta\pi^nT)$, which is non-zero (3.8). Because of this, $\{\psi, 1 - \pi\}$ is non-zero (2.4) and, hence, $\psi$ is not in $fil\ H^2_p(K)$.
5. A Diagram

Assume $A$ is of mixed characteristic. Let $k_2(K)$ denote $K^M_2(K)/pK^M_2(K)$, and let $U^*k_2(K')$ denote the image of the filtration $U^*K^M_2(K)$.

The primary purpose of this section is to prove a certain diagram \textbf{\[5.0\]} commutes. This will allow us to understand in terms of symbols how some classes in $H^2_p(K)$ change when pulled back to $A^g$. The relation between symbols and cohomology classes is provided by a theorem from Bloch-Kato \textbf{\[2, 5.12\]} and Kato \textbf{\[11, 4.1(6)\]}:

**5.1. Theorem.** If $A$ contains the group $\mu_p$ of all $p$-th roots of unity, then $h_K$ (of \textbf{\[2.3\]}) induces isomorphisms

$$k_2(K) \xrightarrow{\sim} H^2_p(K) \otimes \mu_p \quad \text{and} \quad U^e-n k_2(K) \otimes \mu_p \xrightarrow{\sim} \fil_m H^2_p(K),$$

where $e = e_{A/p}(p-1)^{-1}$ and $0 \leq n < e$.

**5.2.** Fix an extension $A_\mu$ of $A$ that is generically generated by a primitive $p$-th root of unity. We will assume throughout this section that the extension $A_\mu/A$ is residually trivial. Let $A'$ be a henselian residually perfect extension of $A$ of ramification index one. Because we will use this only when $A' = A^g$, the reader is free to assume it (even though it does not simplify anything).

As so fix the following notation: $A'_\mu$ is $A' \otimes_A A_\mu$ (a residually perfect henselian discrete valuation ring); $K', K_\mu, K'_\mu$ are the fraction fields of $A', A_\mu, A'_\mu$, and $k'$ is the residue field of $A'$; $\Gamma$ is $\Gal(K_\mu/K)$, and $m$ is the degree of $K_\mu/K$; $\mu_p$ is the group of all $p$-th roots of unity in $K_\mu$, and $\mu_p$ is $\Hom(\mu_p, Z/p)$; $\zeta$ is some non-trivial element of $\mu_p$, and $\bar{\zeta}$ is the element in $\mu_p$ such that $\bar{\zeta}(\zeta) = 1$. (None of the constructions below will depend on the choice of $\zeta$.)

**5.3. Proposition.** There is a map $\gamma_{A'/A}$ that makes the following diagram commute:

\[
\begin{array}{ccc}
\fil_m H^2_p(K_\mu) & \xrightarrow{\sim} & \fil_0 H^2_p(K'_\mu) \\
\downarrow & & \downarrow \\
H^2_p(K_\mu) & \xrightarrow{\sim} & \fil_0 H^2_p(K'_\mu) \\
\downarrow & & \downarrow \\
H^1_p(k') & \xrightarrow{\sim} & \coker[H^1_p(k) \to H^1_p(k')] \\
\end{array}
\]

(Note that $\fil_0 H^2_p(K'_\mu) = H^2_p(K'_\mu)$ since $A'_\mu$ is residually perfect \textbf{\[11, 6.1\]}.)

**Proof.** Let $\chi$ be a class in $\fil_{m-1} H^2_p(K_\mu)$, and put $\chi' = m^{-1} \sum_{\sigma \in \Gamma} \sigma(\chi)$. It is clear that $\chi'$ lies in $\fil_{m-1} H^2_p(K_\mu)^\Gamma$, which by \textbf{\[2.13\]} agrees with $\fil_0 H^2_p(K)$. On the other hand, \textbf{\[2.14\]} implies $\Gamma$ acts trivially on $\fil_0 H^2_p(K_\mu') = H^2_p(K'_\mu)$, and so $\chi$ and $\chi'$ have the same image in $H^2_p(K'_\mu)$. The image of $\chi$ in $H^1_p(k')$ is therefore contained in the image of $H^1(k)$.

\[\square\]
5.4. Proposition. The composite map

$$A \to p_{\mathcal{A}'}/(p_{\mathcal{A}'}^2 + p_{\mathcal{A}}) = (k'/k) \otimes_A p_{\mathcal{A}} \to (k'/k^{p^{-1}}) \otimes_A p_{\mathcal{A}},$$

where the leftmost map sends $x$ to the class of $x - s_{\mathcal{A}'}(x)$, is a derivation that vanishes on $p_{\mathcal{A}}$.

Proof. It is immediate that it vanishes on $p_{\mathcal{A}}$ and a short computation shows it satisfies the Leibniz rule. To see it is additive, it suffices to show that for all $x, y \in k'$,

$$s_{\mathcal{A}'}(x + y) \equiv s_{\mathcal{A}'}(x) + s_{\mathcal{A}'}(y) \mod p_{\mathcal{A}'}^2 + s_{\mathcal{A}'}(k^{p^{-1}})p_{\mathcal{A}}.$$

Let $W$ be the ring of Witt vectors $[8, 0.1]$ with entries in $k''$. By the definition of addition in $W$, there is an element $z \in k'$ such that

$$(x, 0, \ldots) + (y, 0, \ldots) = (x + y, z, \ldots), \quad \text{i.e.,}$$

$$s_W(x) + s_W(y) \equiv s_W(x + y) + ps_W(z^{p^{-1}}) \mod p^2.$$

Since there is a map $W \to \mathcal{A}'$ that is compatible with multiplicative sections, the congruence above holds. \hfill \Box

If $p$ does not generate $p_{\mathcal{A}}$, even the map $A \to (k'/k) \otimes_A p_{\mathcal{A}}$ is a derivation (in both equal and mixed characteristic).

5.5. Construction. Morphisms $\partial_{\mathcal{A}'/A}, \nu$, and $\theta$

Let $\partial_{\mathcal{A}'/A} : p_{\mathcal{A}}^{-1} \otimes_A \Omega^1_k \to k'/k^{p^{-1}}$ denote the $k$-linear homomorphism induced by the derivation in 5.4. Put

$$\hat{e} = e_{\mathcal{A}_u}/z_p p(p - 1)^{-1} = v_{A_{\mu}}(p(\zeta - 1)),$$

and let (in the notation of [4.1])

$$\nu : \text{gr}^{\hat{e} - m} k_2(K_{\mu}) \otimes \hat{\mu}_p \to p_{\mathcal{A}}^{-1} \otimes \Omega^1_k$$

denote the map determined by

$$\{1 - p(\zeta - 1)x, y\} \otimes \zeta \mapsto mx \otimes d\log(\hat{g}),$$

where $x \in p_{\mathcal{A}}^{-1} A_{\mu}$, $y \in A_{\mu}$. Bloch and Kato show it is (well-defined and) an isomorphism [2, 4.3, 5.2]. Because $U_{\mathcal{A}_{\mu}}^{\hat{e} + 1} \subseteq (K_{\mu}^*)^p$, we have $U_{\mathcal{A}_{\mu}}^{\hat{e}} k_2(K_{\mu}) = \text{gr}^{\hat{e}} k_2(K_{\mu})$. We can therefore define a map

$$\theta : k' \to U_{\mathcal{A}_{\mu}}^{\hat{e}} k_2(K_{\mu}) \otimes \hat{\mu}_p$$

by $x \mapsto \{1 - p(\zeta - 1)x, \pi_{\mu}\}$, where $\pi_{\mu}$ is a uniformizer of $A_{\mu}$. (The map is independent of the choice.)
5.6. Proposition. The following diagram commutes:

\[
\begin{array}{cccccc}
\pi^{-1} & \rightarrow & k'/k^{p^{-1}} & \rightarrow & \xi_1 & \rightarrow & \text{coker } [H^1_p(k^{p^{-1}}) \rightarrow H^1_p(k')] \\
\partial_{A'/A} & \rightarrow & & & & & \\
\nu \cong & \rightarrow & & & & & \\
\text{gr}^{e-m}k_2(K_{\mu}) \otimes \mu_p & \rightarrow & \gamma_{A'/A} & \rightarrow & H^1_p(k') \\
\kappa & \rightarrow & & & & & \\
\text{gr}^mH^2_p(K_{\mu}) & \rightarrow & & & & & \\
\nu \cong & \rightarrow & & & & & \\
\text{fil}_mH^2_p(K_{\mu}). & \rightarrow & & & & & \\
\end{array}
\]

Proof. The commutativity of the rear lower face follows from the splittings of 2.4 and the usual compatibility between Artin-Schreier theory and Kummer theory. It is clear the other three faces for which it makes sense to ask the question commute. Therefore, it only remains to check that the perimeter commutes.

Because \( \nu \) is an isomorphism, it is enough to consider elements

\[
x = \nu^{-1}(\pi^{-1} \otimes dz) = \{1 - p(\zeta - 1)\pi^{-1}z, z\} \otimes \zeta \in \text{gr}^{e-m}k_2(K_{\mu}) \otimes \mu_p.
\]

where \( \pi \) is a uniformizer of \( A \) and \( z \in A^* \). Write \( z|_{A'} = s_{A'}(\bar{z}) + \pi y \), where \( y \in A' \). Then, letting \( [x] \) denote the graded class of \( x \), we have

\[
\partial_{A'/A} \circ \nu([x]) = y \mod k^{p^{-1}}.
\]

On the other hand, by 4.3, we have

\[
x|_{K_{\mu}} = \{1 - p(\zeta - 1)yzs(\bar{z})^{-1}, \pi_{\mu}\} \otimes \zeta = \theta(\bar{y}).
\]

Since the front lower and rear faces commute, the proof is complete.

6. THE PROOF: MIXED CHARACTERISTIC

Assume in this section that \( A \) is of mixed characteristic, and let \( A_{\mu} \) be an extension of \( A \) that is generically generated by a primitive \( p \)-th root of unity. (The extension \( A_{\mu}/A \) is no longer assumed to be residually trivial.)

6.1. Proposition. The composite map

\[
\pi^{-1} \otimes_A \Omega^1_k \xrightarrow{\partial_{A'/A}} k^e/k^{p^{-1}} \xrightarrow{\xi_1} \text{coker } [H^1_p(k^{p^{-1}}) \rightarrow H^1_p(k^e)]
\]

is injective.

(Compare with 3.12)
Proof. Let $T$ be a $p$-basis for $k$. Then $dT$ is a basis for $\Omega^1_k$. Let $\eta$ be an element of the kernel of $\xi_1 \circ \partial_{A^s/A}$ and write
\[
\eta = \pi^{-1} \otimes \sum_{t \in T} a_t dt, \ a_t \in k
\]
where $\pi$ is a uniformizer of $A$ and $a_t$ is zero for all but finitely many $t \in T$. By 1.2, we have
\[
\partial_{A^s/A}(\eta) = \sum_{t \in T} a_t u_{t,1} \mod k^{p^{-1}}.
\]
Since the image of this in
\[
\operatorname{coker}\left[H^1_p(k^{p^{-1}}) \to H^1_p(k^s)\right]
\]
is assumed to be zero, there are elements $x \in k^s$ and $y \in k$ such that
\[
x^{p^s} - x = y + \sum_{t \in T} a_t u_{t,1}.
\]
Suppose for a contradiction that there is an element $s \in T$ such that $a_s \neq 0$. Put
\[
F = k(u_{t,1} \mid t \neq s)^{p^{-\infty}}.
\]
Then 1.2 implies $F(u_{s,1})$ is separably closed in $k^s$. Therefore, we have $x \in F(u_{s,1})$ and $x^{p^s} - x = y' + a_su_{s,1}$, where $y' = y + \sum_{t \neq s} a_t u_{t,1} \in F$. But valuation considerations in the completion $F((u_{s,1}^{-1}))$ show this is impossible.

6.2. Corollary. If $A_\mu/A$ is residually trivial, the map $\gamma_{A^s/A}$ is injective.

Proof. By 5.6 and 6.1.

6.3. Proposition. Let $s \geq 1$ be an integer. Then the following sequences are exact:
\[
\begin{align*}
0 & \to H^2_p(k) \to \fil_0 H^2_p(K) \to H^2_p(K^s), & (6.3.1) \\
0 & \to H^2_p(k) \to \fil_1 H^2_p(K) \to H^2_p(K^s). & (6.3.2)
\end{align*}
\]

Proof. Because $k^s$ is perfect, we have $H^2_p(k^s) = 0$ (by, say, higher Artin-Schreier theory [1, 1.3]). Now, 1.3 implies the map $H^1_p(k) \to H^1_p(k^s)$ is an injection. The exactness of (6.3.1) then follows from 2.13.

Let $A_0$ be the maximal unramified subextension of $A_\mu/A$. Applying 6.2 to the extension $A_\mu/A_0$, we conclude that the map $\gamma_{A^s_{\mu}/A_0}$ is injective, and therefore the kernel of the map
\[
\fil_m H^2_p(K_\mu) \to H^2_p(K^s_{\mu} \otimes K_0 K_\mu)
\]
is contained in $\fil_{m-1} H^2_p(K_\mu)$. By 2.11, we have
\[
\fil_{m-1} H^2_p(K_\mu) \cap H^2_p(K) = \fil_0 H^2_p(K).
\]
Because there is [1, 2.1] a map $K^s \to K^s_0$, we then have
\[
\ker [\fil_1 H^2_p(K) \to H^2_p(K^s)] \subseteq \ker [\fil_m H^2_p(K_\mu) \to H^2_p(K^s_{\mu} \otimes K_0 K_\mu)] \cap H^2_p(K)
\]
\[
\subseteq \ker [\fil_0 H^2_p(K) \to H^2_p(K^s)],
\]
which proves the exactness of (6.3.2).
We can finally prove the theorem from the introduction in mixed characteristic.

**Proof.** Let \( \pi \in A \) be a uniformizer. Put \( n = \text{sw}_K(\chi) \). If \( n = 0 \), then \( \chi \) is tame, and the result follows immediately from \[1.3\]. Now assume \( n > 0 \).

By definition, \( \text{ar}_K(\chi) \) is either \( n \) or \( n + 1 \). If it is \( n + 1 \), then \( \kappa_n(\chi) \notin p_A^{-n} \otimes_A \Omega^1_K \). Therefore, \[2.10\] implies \( \kappa_n(\chi|_{A^s}) \notin p_{A^s}^{-n} \otimes_A \omega^1_{A^s} \), and so \( \text{ar}_K(\chi|_{A^s}) = n + 1 \), as desired.

Now consider the case \( \text{ar}_K(\chi) = n \). Since \( A^s \) is residually perfect, \( \text{ar}_K(\chi|_{A^s}) = n \) if and only if \( \text{sw}_K(\chi|_{A^s}) = n - 1 \). By \[2.10\] we know \( \text{sw}_K(\chi|_{A^s}) \) is at most \( n - 1 \). Put \( \psi = \{\chi|_{A^s} + \pi^{n-1}T\} \in H^2(\tilde{K}) \). Then \( \text{sw}_K(\chi|_{A^s}) \geq n - 1 \) if and only if \( \psi|_{\tilde{A}^s} \) is not zero.

Now we will construct a map \( \tilde{A}^s \to \tilde{A}^s \). (See figure \[1\].) Since \( \tilde{A}/A \) is residually separable, there is \[4, 2.1\] an injection \( A^s \to \tilde{A}^s \). Sending \( T \to T \) yields another injection \( A^s[T] \to \tilde{A}^s \). By the universal properties of localization and henselization \[4, VIII\], this naturally induces a map \( A^s \to \tilde{A}^s \). It is therefore enough to show \( \psi|_{\tilde{A}^s} \neq 0 \).

By \[4.3\], we can write \( \psi = \psi' + \psi'' \), where \( \psi' \) is in \( \text{fil}_1 H^2_p(\tilde{K}) - \text{fil}_0 H^2_p(\tilde{K}) \) and \( \psi'' \) is in \( H^2_p(\tilde{k}) \). We then have \( \psi|_{\tilde{A}^s} = \psi'|_{\tilde{A}^s} \neq 0 \) by \[6.3\].

7. Comparison with Kato’s conductor

Let \( A \) be a field whose characteristic is not \( p \), and fix an injection from the torsion subgroup of \( \Lambda^* \) to \( \mathbb{Q}/\mathbb{Z} \). In each of the two results below, \( B \) is a finite extension of \( A \) that is generically Galois with group \( G \), \( \chi \) is a class in \( H^1(K, \mathbb{Q}/\mathbb{Z}) \), and \( \rho : G \to \Lambda^* \) is the corresponding homomorphism. Let \( G_* \) denote the lower ramification filtration \[17\] \text{IV Prop. 1} of \( G \), and let \( \text{ar}(\rho) \) denote the the Artin conductor \[4, 3.2\] of \( \rho \).

**7.1. Proposition.** If \( B/A \) is residually separable, we have

\[
\text{ar}_K(\chi) = e_{B/A}^{-1} \sum_{i \geq 0} |G_i| \text{codim} \Lambda^G_i = e_{B/A}^{-1} \sum_{\rho(G_i) \neq 1} |G_i|
\]

**Proof.** If \( \rho \) is tame, the result is clear. Now assume \( \rho \) is wild and let \( n = \text{sw}_K(\chi) \). Then Kato \[11, 6.8\] \[3.6(1),3.16\] shows \( \kappa_n(\chi) \) is not in \( p_A^{-n} \otimes \Omega^1_K \) and that \( n \) agrees with the naive Swan conductor \[11, 6.7.1\]. And this, in turn, agrees \[18, 2.1\] with

\[
eq e_{B/A}^{-1} \sum_{i \geq 1} |G_i| \text{codim} \Lambda^G_i
\]

Figure 1.
Since $a_r(\chi)$ is one more than $sw_r(\chi)$ and since the number $a_r(\rho) = a_r^B(\rho)$ is one more than $[7.1.1]$, the equality of conductors follows.

7.2. Corollary. $a_r(\rho) = a_r(\chi)$.

Proof. Since $A^e$ is residually perfect, $ar(\rho)$ agrees $[4, 3.3]$ with the sum in $[7.1]$ and, hence, with $a_r(\chi)$. Applying the theorem in the introduction completes the proof.

References

[1] Ahmed Abbes and Takeshi Saito. Ramification of local fields with imperfect residue fields I. To appear.
[2] Spencer Bloch and Kazuya Kato. $p$-adic étale cohomology. Inst. Hautes Études Sci. Publ. Math., (63):107–152, 1986.
[3] Robert Boltje, G.-Martin Cram, and V. P. Snaith. Conductors in the non-separable residue field case. In Algebraic $K$-theory and algebraic topology (Lake Louise, AB, 1991), pages 1–34. Kluwer Acad. Publ., Dordrecht, 1993.
[4] James Borger. Conductors and the moduli of residual perfection. To appear.
[5] Jean-Luc Brylinski. Théorie du corps de classes de Kato et revêtements abéliens de surfaces. Ann. Inst. Fourier (Grenoble), 33(3):23–38, 1983.
[6] Jean-Marc Fontaine. Sur la construction du module de Dieudonné d’un groupe formel. C. R. Acad. Sci. Paris Sér. A-B. 280:Aii, A1273–A1276, 1975.
[7] Alexander Grothendieck. Site et topos étales d’un schema. In Théorie des topos et cohomologie étale des schémas. Tome 2, pages 341–365. Springer-Verlag, Berlin, 1972.
[8] Luc Illusie. Complexe de de Rham-Witt et cohomologie cristalline. Ann. Sci. École Norm. Sup. (4), 12(4):501–661, 1979.
[9] Kazuya Kato. Swan conductors with differential values. In Galois representations and arithmetic algebraic geometry (Kyoto, 1985/Tokyo, 1986), pages 315–342. North-Holland, Amsterdam, 1987.
[10] Kazuya Kato. Logarithmic structures of Fontaine-Illusie. In Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), pages 191–224. Johns Hopkins Univ. Press, Baltimore, MD, 1989.
[11] Kazuya Kato. Swan conductors for characters of degree one in the imperfect residue field case. In Algebraic $K$-theory and algebraic number theory (Honolulu, HI, 1987), pages 101–131. Amer. Math. Soc., Providence, RI, 1989.
[12] Shigeki Matsuda. On the Swan conductor in positive characteristic. Amer. J. Math., 119(4):705–739, 1997.
[13] John Milnor. Introduction to algebraic $K$-theory. Princeton University Press, Princeton, N.J., 1971. Annals of Mathematics Studies, No. 72.
[14] Michel Raynaud. Anneaux locaux henséliens. Springer-Verlag, Berlin, 1970. Lecture Notes in Mathematics, Vol. 169.
[15] Hermann L. Schmid. Zur Arithmetik der zyklischen $p$-Körper. J. Reine Angew. Math., 176:161–167, 1936.
[16] Jean-Pierre Serre. Sur la topologie des variétés algébriques en caractéristique $p$. In Symposium internacional de topología algebraica International symposium on algebraic topology, pages 24–53. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958.
[17] Jean-Pierre Serre. Corps locaux. Hermann, Paris, 1968. Deuxième édition, Publications de l’Université de Nancago, No. VIII.
[18] Jean-Pierre Serre. Facteurs locaux des fonctions zêta des variétés algébriques (définitions et conjectures). In Toposes. Vol. II, pages 581–592. Springer-Verlag, 1986.
[19] Igor B. Zhukov. On ramification theory in the imperfect residue field case. Technical report, Nottingham University, May 1998. To appear in the proceedings of the Luminy conference on ramification theory for arithmetic schemes.

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