M theory Branes : U duality properties
and a class of new Static Solutions

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ABSTRACT
We obtain the most general static intersecting brane solutions by
directly solving the relevant equations of motion analytically and
in complete generality. These solutions reduce to the known ones
in special cases, and contain further a class of new static solutions
which are horizonless. We describe their properties and discuss
their physical relevance. Along the way, we also describe the
features of the brane energy momentum tensors, the equations
of motion, and their solutions which arise as consequences of the
intersection rules and the U duality symmetries of M theory.
1. Introduction

Consider a static spherically symmetric star. If it is sufficiently massive then it is likely to collapse and to finally form a black hole or, possibly, some other static object. It is self evident that, generically, the fields outside such stars and the final collapsed objects must be described by the most general static solutions to the relevant equations of motion. In Einstein’s general theory of relativity, for example, the most general static spherically symmetric solution is given by the Schwarzschild solution. In Brans–Dicke theory, the most general static spherically symmetric solutions are given by the Janis–Newman–Winicour–Wyman (JNWW) solutions [1], the Schwarzschild solution now being a special case.

In M theory (or equivalently string theory), the black holes are described by various intersecting brane configurations where the branes wrap the compact internal spaces, taken to be toroidal. For example, the four dimensional black holes are described in M theory by two stacks of $M_2$ branes and two stacks of $M_5$ branes which wrap the seven dimensional compact toroidal space and intersect according to the BPS rules. The standard intersecting brane solutions are given in [2, 3, 4, 5, 6, 7]. They are static solutions independent of compact toroidal coordinates and are spherically symmetric in the non compact transverse space.

It is natural to expect that such black holes in M theory must have been formed by the collapse of sufficiently massive stars. In order to study such collapses, one first needs to construct a static star in M theory. This requires, among other things, a knowledge of the most general static solutions to the relevant equations of motion.

The standard brane solutions given in [2, 3, 4, 5, 6, 7] are not the most general ones. These solutions can be thought of as analogous to the Schwarzschild solutions rather than to the JNWW solutions. Brane solutions of the JNWW type also exist. They have been found by several groups in different forms and using different methods [8, 9], see [10] also. However, in all these works, while solving the equations of motion, some ansatz or the other is made re-

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1 According to the BPS rules, two stacks of five branes intersect along three common spatial directions; a stack each of two branes and five branes intersect along one common spatial direction; and two stacks of two branes intersect along zero common spatial direction. Also, the branes are taken to be uniformly smeared along the remaining compact directions [2, 3].
Regarding the form of the brane solutions which thereby limits their generality. Therefore, these solutions are not the most general ones.

In this paper, we find the most general static intersecting brane solutions. We directly solve the relevant equations of motion analytically and in complete generality. No ansatz is made regarding the form of the solutions. The resulting solutions are thus completely general. In special cases, they reduce to the standard solutions of [2, 3, 4, 5, 6, 7] and to the more general ones of [8, 9]. The general solutions we find contain further a class of new static solutions which are horizonless and, to the best of our knowledge, have not appeared in the literature.  

We point out here that, except the Schwarzschild and the standard brane solutions, the general solutions mentioned above – namely, JNWW solutions, those given in [8, 9], and the new ones found in this paper – all have, generically, a singularity either at a finite radius or at a vanishing radius. These singularities are not covered by any horizon and, hence, are naked. Also, they are present even though the solutions all have positive ADM mass. In the case of a static star, the interior solutions get modified and, hence, these singularities are not relevant since the general solutions are applicable only outside the star. The general solutions can then be used to explore the observational consequences by studying the motions of various probes outside such a static star, as in [12, 13, 14, 15] for example.

Also, such general solutions are essential ingredients in the study of collapse of stars, and in determining the nature and the properties of the final collapsed objects; in particular, in determining whether a naked singularity appears or not during and after the collapse. In the context of four dimensional JNWW solutions, such collapse processes and various issues involving naked singularities have been extensively studied in [14, 16]. Similar studies are also needed in the present context.

In this paper, we describe our general static intersecting brane solutions. We start with the eleven dimensional low energy effective action for the intersecting branes of M theory, and write down the equations of motion. These equations, their solutions, and the energy momentum tensors for the branes all exhibit several features which arise as consequences of the BPS intersec-

\footnote{In our earlier reports [11], we obtained the most general vacuum solutions first and then used it to generate the brane solutions by suitable boosting and U duality operations [6]. The brane solutions thus generated turn out to be the same as the ones obtained in this paper by directly solving the equations of motion.}
tion rules and the underlying U duality symmetries of M theory. We describe these features along our way to solving the equations of motion. We first solve the equations and obtain the general analytical solutions in terms of a variable $\tau$ which is suggested naturally by the equations themselves. We then obtain the general relation between $\tau$ and the standard radial coordinate $r$, which completes the solutions. All along, we take particular care in ensuring that no generality is lost at any stage. We then describe the properties of the resulting solutions.

The paper is organised as follows. In section 2, we present the action and the metric which can describe the static intersecting branes. In section 3, we write down the equations of motion in several convenient forms and discuss the structure of the solutions. In section 4, we describe the features which are consequences of the intersection rules and the U duality symmetries of M theory. In section 5, we solve the equations of motion in complete generality in terms of $\tau$. In section 6, we obtain the general relation between $\tau$ and the radial coordinate $r$. In section 7, we describe the properties of the solutions pointing out the special cases which lead to the solutions known before, and the general cases which lead to the new solutions. In section 8, we make several remarks which illustrate various features of the present solutions. In section 9, we conclude with a brief summary and by mentioning a few issues for further studies.

Three Appendices contain some useful expressions and formulae. In Appendix A, we give the non-vanishing components of the Riemann tensor for the metric used here. In Appendix B, following the analysis of [17], we give the general expressions for the ADM mass. In Appendix C, for the sake of completeness, we give the general vacuum solutions obtained earlier [11] directly in terms of $r$.

A Note to the reader: The paper is somewhat long and contains many technical details which we have presented in order to solve the equations analytically and, at the same time, to make the generality of our approach obvious. Much of these details may be familiar to the reader. Hence, for the sake of the reader’s convenience, we have collected the main results and the expressions for the general solutions together in one place in the beginning of section 7 before proceeding to discuss their properties in sections 7 and 8. So those readers who are mainly interested in the solutions may go straight to these sections at a first reading.
2. General Set Up

We study the static solutions describing \( N \) stacks of M theory branes, intersecting according to the BPS rules [2, 3] whereby two stacks of five branes intersect along three common spatial directions; a stack each of two branes and five branes intersect along one common spatial direction; and two stacks of two branes intersect along zero common spatial direction. We start with the low energy effective action for M theory branes and an appropriate static ansatz for the metric.

The relevant part of the eleven dimensional low energy effective action describing \( N \) stacks of M theory branes mentioned above may be written in standard notation as

\[
S = \frac{1}{2\kappa^2} \int d^{11}x \sqrt{-g} \left( \mathcal{R} - \sum_I \frac{F_{(I)}^2}{2(n_I + 2)!} \right)
\]

where \( I = 1, 2, \ldots, N \),

\[
F_{(I)}^2 = \sum_{M_1 \cdots M_{n_I + 2}} F_{M_1 M_2 \cdots M_{n_I + 2}} F^{M_1 M_2 \cdots M_{n_I + 2}}
\]

\( F_{M_1 \cdots M_{n_I + 2}} \) is the \((n_I + 2)\)-form field strength for the \( I^{th} \) stack of \( n_I \)-branes, and \( n_I = 2 \) or \( 5 \). We write the summation indices explicitly since not all the sums in this paper are over repeated indices. The above form of the brane action can be shown to follow as a consequence of the BPS intersection rules, see [7, 10]. Setting \( \kappa^2 = 1 \), the resulting equations of motion are given by

\[
\mathcal{R}_{MN} - \frac{1}{2} g_{MN} \mathcal{R} = T_{MN} = \sum_I T_{MN(I)}
\]

\[
\sum_M \partial_M \left( \sqrt{-g} F^{MM_2 \cdots M_{n_I + 2}} \right) = 0
\]

\[
\leftrightarrow \sum_M \nabla_M T_{N(I)}^M = 0
\]

where \( T_{MN} \) is the total energy momentum tensor and

\[
T_{MN(I)} = \frac{1}{2(n_I + 1)!} \left( \sum_{M_2 \cdots M_{n_I + 2}} F_{M_2 \cdots M_{n_I + 2}} F^{M_2 \cdots M_{n_I + 2}}_N - \frac{g_{MN} F_{(I)}^2}{2(n_I + 2)} \right)
\]
is the energy momentum tensor for the \( I^{th} \) stack of branes.

We take the spatial directions of the brane worldvolumes to be toroidal and assume necessary isometries. Let the spacetime coordinates be given by \( x^M = (t, x^i, r, \theta^a) \) where \( i = 1, 2, \ldots, n_c \) and \( a = 1, 2, \ldots, m \). The total spacetime dimension \( D = n_c + m + 2 = 11 \). The coordinates \( x^i \) describe the compact, \( n_c \) dimensional, toroidal space; and the radial coordinate \( r \) and the spherical coordinates \( \theta^a \) describe the non compact, \((m + 1)\) dimensional, transverse space. In the following, we will assume that \( m \geq 2 \) and study the static solutions. The line element \( ds \) which can describe the static intersecting branes may now be written as

\[
d s^2 = g_{MN} \, d x^M \, d x^N = -e^{2\lambda_0} \, d t^2 + e^{2\lambda} \, d r^2 + \sum_i e^{2\lambda_i} \, (d x^i)^2 + e^{2\sigma} \, d \Omega_m^2
\]

where \( d \Omega_m \) is the standard line element on an \( m \) dimensional unit sphere. The non vanishing components of the field strengths are

\[
F^i_{01\cdots i_n I} \, r = \partial_r A^I \quad \text{where} \quad i_1, \cdots, i_n I \quad \text{and} \quad A^I \quad \text{denote, respectively, the spatial worldvolume directions and the \((n_I + 1)\)-form gauge field for the \( I^{th} \) stack of branes. The fields \((\lambda^0, \lambda^i, \lambda, \sigma) \) and \( A^I \) are functions of \( r \) only.}

It can now be seen that the energy momentum tensors \( T_{MN(I)} \) given in equation (5) are all diagonal. We denote these diagonal elements as

\[
\begin{pmatrix}
T^0_{0(I)}, & T^i_{i(I)}, & T^r_{r(I)}, & T^a_{a(I)}
\end{pmatrix} = (-\rho_I, \, p_{iI}, \, \Pi_I, \, p_{aI})
\]

where \( p_{aI} = p_{\Omega I} \) for all \( a \). The total energy momentum tensor is now given by \( T^M_N = \text{diag} \, \begin{pmatrix} -\rho, & p_i, \, \Pi, \, p_a \end{pmatrix} \) where

\[
\rho = \sum_I \rho_I \quad , \quad p_i = \sum_I p_{iI} \quad , \quad \Pi = \sum_I \Pi_I \quad , \quad p_a = p_{\Omega I} = \sum_I p_{\Omega I} .
\]

From the expression for \( T_{MN(I)} \) in equation (5), it follows that \( p_{iI} = p_{\| I} \) or \( p_{\perp I} \) according to whether the \( x^i \) direction is parallel or transverse to the worldvolume of the \( I^{th} \) stack of branes, and further that

\[
-\rho_I = p_{\| I} = -p_{\perp I} = \Pi_I = -p_{\Omega I} = \frac{1}{4} \, F_{0i_1\cdots i_n I} \, F^{0i_1\cdots i_n I} \quad .
\]

Note that, in the studies of stars or of cosmological evolution, one often assumes that the total energy momentum tensor is the sum of \( N \) individual energy momentum tensors each of which is diagonal and seperately conserved;
and then supplements the equations of motion with equations of state for each individual $I$ which determine the diagonal components $T^M_{M(I)}$ in terms of $\rho_I$, or equivalently $\Pi_I$. In our case here, these properties of the energy momentum tensors follow from an underlying action for branes and from the BPS intersection rules, and equation (9) provides the equations of state for each individual $I$.

3. Equations of motion

We first write down the equations of motion for the fields $\Pi_I$ and $(\lambda^0, \lambda^i, \lambda, \sigma)$ using equations (2), (4), (6), (7), and (8) only. In particular, we will not use equations (5) or (9). The resulting equations are, therefore, valid quite generally wherever the metric is given by equation (6) and where the total energy momentum tensor is the sum of $N$ energy momentum tensors each of which is diagonal and separately conserved.

Let

$$\alpha = (0, i, a), \quad \lambda^\alpha = (\lambda^0, \lambda^i, \lambda^a), \quad p_\alpha I = (p_{0I}, p_{iI}, p_{aI})$$

where $p_{0I} = -\rho_I$, and $\lambda^a = \sigma$ and $p_{aI} = p_{0I}$ for all $a$. Define

$$\Lambda = \sum \lambda^\alpha = \lambda^0 + \sum_i \lambda^i + m \sigma$$

$$T_I = \sum_M T^M_{M(I)} = \Pi_I + \sum_\alpha p_\alpha I.$$

It is straightforward to calculate the Riemann tensor corresponding to the metric given by equation (6). We have listed its non vanishing components in Appendix A. Using these components and the above definitions, it follows after some algebra that equations (2), (4), (6), (7), and (8) give

$$\left(\Pi_I\right)_r = -\Pi_I \Lambda_r + \sum_\alpha p_{aI} \lambda^a_r$$

$$\Lambda_r^2 - \sum_\alpha (\lambda^\alpha_r)^2 = 2 \sum_I \Pi_I e^{2 \lambda} + m(m-1) e^{2 \lambda - 2 \sigma}$$

$$\lambda^\alpha_{rr} + (\Lambda_r - \lambda_r) \lambda^\alpha_r = \sum_I \left(-p_{aI} + \frac{T_I}{D-2}\right) e^{2 \lambda} + \delta^{\alpha\alpha} (m-1) e^{2 \lambda - 2 \sigma}$$

where the subscripts $r$ denote $r-$derivatives.
Equation (15) now suggests a change of variable from $r$ to $\tau$ given by

$$e^{\lambda} \, dr = e^{\Lambda} \, d\tau \, .$$

(16)

Then, for any function $X(r(\tau))$, we have

$$X_\tau = e^{\Lambda - \lambda} \, X_r \, , \quad X_{\tau\tau} = e^{2(\Lambda - \lambda)} \left( X_{rr} + (\Lambda_r - \lambda_r) \, X_r \right)$$

(17)

where the subscripts $\tau$ denote $\tau-$derivatives. The line element for the $(m+1)$ dimensional transverse space given in equation (6) now becomes

$$e^{2\lambda} \, dr^2 + e^{2\sigma} \, d\Omega_m^2 = e^{2\sigma} \left( e^{2\chi} \, d\tau^2 + d\Omega_m^2 \right)$$

(18)

where the field $\chi$ is defined by

$$\chi = \Lambda - \sigma = \lambda^0 + \sum_i \lambda^i + (m - 1)\sigma \, .$$

(19)

Equations (13)–(15), expressed in terms of $\tau$, become

$$(\Pi_I)_\tau = -\Pi_I \, \Lambda_\tau + \sum_\alpha p_{\alpha I} \lambda^\alpha_\tau$$

(20)

$$\Lambda_\tau^2 - \sum_\alpha (\lambda^\alpha_\tau)^2 = 2 \sum_I \Pi_I \, e^{2\Lambda} + m(m-1) \, e^{2\chi}$$

(21)

$$\lambda^\alpha_{\tau\tau} = \sum_I \left( -p_{\alpha I} + \frac{T_I}{D-2} \right) \, e^{2\Lambda} + \delta^{\alpha\alpha} \, (m - 1) \, e^{2\chi} \, .$$

(22)

Equation for the field $\chi$ defined in equation (19) can now be obtained. It is given by

$$\chi_{\tau\tau} = \sum_I \left( \Pi_I + p_{\Omega I} \right) \, e^{2\Lambda} + (m-1)^2 \, e^{2\chi} \, .$$

(23)

The above equations may be written more compactly upon defining $G_{\alpha\beta}$ and $G^{\alpha\beta}$ by

$$G_{\alpha\beta} = 1 - \delta_{\alpha\beta} \, , \quad G^{\alpha\beta} = \frac{1}{D-2} - \delta^{\alpha\beta} \, .$$

(24)

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3We note in passing that this variable $\tau$ and the solutions obtained in terms of $\tau$ bear a striking similarity to those which appear in the ‘H-FGK formalism’ [18]. Upon substitution of the solution for $\chi(\tau)$, see later in the following, the form of the $(m+1)$ dimensional line element given in equation (18) becomes an example of such a similarity; the $M_P$ brane solutions given in equation (80) below is another example.
and upon defining $h_{\alpha I}$ by
\[ h_{\alpha I} = \sum_{\beta} G_{\alpha \beta} \left( -p_{\beta I} + \frac{T_I}{D - 2} \right) = \Pi_I + p_{\alpha I} . \tag{25} \]

Then we get
\[ (\Pi_I)_\tau = -2 \Pi_I \Lambda_\tau + \sum_{\alpha} h_{\alpha I} \lambda^\alpha_\tau \tag{26} \]
\[ \sum_{\alpha \beta} G_{\alpha \beta} \lambda^\alpha_\tau \lambda^\beta_\tau = 2 \sum_I \Pi_I e^{2\Lambda} + m(m - 1) e^{2\chi} \tag{27} \]
\[ \lambda^\alpha_\tau = \sum_{\beta I} G^{\alpha \beta} h_{\beta I} e^{2\Lambda} + \delta^{\alpha a} (m - 1) e^{2\chi} . \tag{28} \]
\[ \chi_\tau = \sum_I h_{\alpha I} e^{2\Lambda} + (m - 1)^2 e^{2\chi} . \tag{29} \]

We require the solutions to correspond to asymptotically flat spacetime in the limit $r \to \infty$. Therefore, the above equations of motion are to be solved with the asymptotic condition that, in the limit $r \to \infty$, the fields
\[ \left( e^{\lambda^0}, e^{\lambda^i}, e^{\lambda}, e^{\sigma}; A_{(I)} \right) \to \left( 1, 1, 1, r; \frac{Q_I}{r^{m-1}} \right) \tag{30} \]
where $Q_I$ are the charges. The definitions of $(\Lambda, \chi)$ and equation (16) then imply that, in the limit $r \to \infty$,
\[ e^{\Lambda} \to r^m , \quad e^{\chi} \to r^{m-1} , \quad (\tau_0 - \tau) \to \frac{1}{(m - 1) r^{m-1}} \tag{31} \]
where $\tau_0$ is a constant. Thus $\tau \to \tau_0$ from below in the limit $r \to \infty$. Furthermore, since $\frac{dr}{d\tau}$ is positive, it follows that $\tau$ decreases from $\tau_0$ as $r$ decreases from $\infty$. Hence $\tau \leq \tau_0$ for $r \leq \infty$ and, incorporating this feature, equation (16) may be written as
\[ \int_{\tau}^{\tau_0} d\tau = \int_r^{\infty} dr \ e^{-\Lambda} . \]

The constant $\tau_0$ can be set to zero with no loss of generality. But we will not do so in the following in order to avoid the seemingly spurious negative signs in expressions involving $(\tau_0 - \tau)$.
Note that, for a metric of the form given in equation (6) and which corresponds to asymptotically flat spacetime, one can obtain the ADM mass. In Appendix B, following the analysis of [17], we have given the corresponding general expressions for the ADM mass.

Linear equations of state

To solve equations (26) – (29), one requires the equations of state which give, for example, \( p_{\alpha I} = (p_{0I}, p_{aI}, p_{aI}) \) as functions of \( \Lambda_I \). Consider the case where these equations of state are linear and are given by

\[
h_{\alpha I} = \Pi_I + p_{\alpha I} = z^I_{\alpha} \Pi_I
\]

(32)

where \( z_{\alpha}^I \) are constants and \( z_a^I = z^I_\Omega \) for all \( a \). Equation (26) can now be solved to give

\[
\Pi_I = \Pi_{I0} e^{-2\Lambda_I + l^I} \quad \rightarrow \quad h_{\alpha I} e^{2\Lambda} = z_{\alpha}^I \left( \Pi_{I0} e^{l^I} \right)
\]

(33)

where \( \Pi_{I0} \) are constants and \( l^I \) are given by

\[
l^I = \sum_{\alpha} z^I_{\alpha} \lambda^\alpha
\]

(34)

Using these expressions, equations for \( \lambda^\alpha \), \( l^I \), and \( \chi \) become

\[
\lambda^\alpha_{\tau \tau} = \sum_I z^{I\alpha} \left( \Pi_{I0} e^{l^I} \right) + \delta^{\alpha a} (m - 1) e^{2\chi}
\]

(35)

\[
l^I_{\tau \tau} = \sum_J G^{IJ} \left( \Pi_{J0} e^{l^J} \right) + z^I_\Omega (m - 1) e^{2\chi}
\]

(36)

\[
\chi_{\tau \tau} = \sum_I z^I_\Omega \left( \Pi_{I0} e^{l^I} \right) + (m - 1)^2 e^{2\chi}
\]

(37)

where

\[
z^{I\alpha} = \sum_{\beta} G^{\alpha \beta} z^I_{\beta} \quad , \quad G^{IJ} = \sum_{\alpha \beta} G^{\alpha \beta} z^I_{\alpha} z^J_{\beta}
\]

(38)

One may now express \( \lambda^\alpha \) in terms of \( l^I \) and \( \chi \) as follows. Solve equations (36) and (37) as simultaneous equations to obtain \( \Pi_{I0} e^{l^I} \) and \( e^{2\chi} \) in terms of \( l^I_{\tau \tau} \) and \( \chi_{\tau \tau} \). Substituting these expressions into equation (35) gives an equation of the form

\[
\lambda^\alpha_{\tau \tau} = X_{\tau \tau}^\alpha \quad , \quad X^\alpha = \sum_I a^\alpha_I l^I + a^\alpha \chi
\]
where \( a^\alpha_I \) and \( a^\alpha_\alpha \) are constants. It then follows that the most general solution for \( \lambda^\alpha \) is given by \( \lambda^\alpha = X^\alpha + L^\alpha \tau + a^\alpha_\alpha \) where \( L^\alpha \) and \( a^\alpha_\alpha \) are integration constants. Substituting this expression for \( \lambda^\alpha \) into the definitions \( l^I = \sum_\alpha z^I_\alpha \lambda^\alpha \) and \( \chi = \Lambda - \sigma \) will lead to \( N + 1 \) consistency constraints each on \( L^\alpha \) and \( a^\alpha_\alpha \). A complete solution thus follows once equations (36) and (37) are solved for \( l^I(\tau) \) and \( \chi(\tau) \). Finally, equation (27) will lead to one further constraint among various integration constants appearing in the complete solutions for \( (l^I, \chi, \lambda^\alpha) \).

As pointed out earlier, the various forms of equations of motion above have been obtained without using the expression for \( T_{MN(I)} \) given in equation (5) or (9). Hence, these equations of motion are all valid quite generally wherever the metric is given by equation (6) and the total energy momentum tensor is the sum of \( N \) energy momentum tensors, each of which is diagonal and separately conserved. Thus, for example, these equations may also be used to study stars made up of M theory branes; similarly, with a few straightforward modifications involving \( t \) and \( r \), and \( \rho_I \) and \( \Pi_I \), they may also be used to study the cosmological evolution of a universe made up of intersecting M theory branes [19, 20, 21, 22, 23].

4. M theory branes

Whether or not the equations for \( l^I(\tau) \) and \( \chi(\tau) \) can be solved explicitly depends on the values of the constants \( z^I_\alpha \) and the consequent structure of \( G^{IJ} \). It does not seem to be possible to obtain explicit solutions for general arbitrary values of \( z^I_\alpha \). Hence we now specialise to M theory branes and first incorporate the U duality symmetries of M theory which lead to a relation among \( z^I_\alpha \) and to an elegant form for \( G^{IJ} \).

U duality relation among \( (\Pi_I, p_{\alpha I}) \)

We use the U duality symmetries here the same way as in our earlier works [20, 21, 22]. We note that, for a given \( N \), different intersecting brane configurations of M theory can be related to each other by suitable U duality operations – namely, by suitable dimensional reduction and uplifting to and from type IIA string theory, and T and S dualities in type IIA/B string theories. For a metric of the form given in equation (6), such an operation
leads to relations among $\lambda^a$ which in turn, through their equations of motion, imply relations among $(\Pi, p_\alpha)$. Although only time dependent cases in early universe were studied in [20, 21, 22], the U duality details of these works carry over and are applicable to the static cases also with only a few minor changes. The relevant details may be found in [20] and in Appendix A of [22]. Hence, we present only the main results here without repeating the details.

It can be shown that the relations among $(\Pi, p_\alpha)$ for intersecting branes of M theory, obtained by applying U duality operations as described above, are all satisfied if the individual $(\Pi_I, p_{\alpha I})$ obey the relation

$$p_{\parallel I} = \Pi_I + p_{0I} + p_{\perp I} + m \left( p_{\Omega I} - p_{\perp I} \right)$$

(39)

where $p_{\parallel I}$ and $p_{\perp I}$ are the pressures along the directions that are parallel and transverse to the worldvolume of the $I^{th}$ stack of branes. The above relation is a consequence of U duality symmetries and, therefore, must always be valid independent of the details of the equations of state. We further take $p_{\Omega I} = p_{\perp I}$ which is natural since the sphere directions are transverse to the branes.  

It can further be argued [20] that the equations of state may be written in terms of one single function only. For example, they may be written as

$$h_{\alpha I} = \Pi_I + p_{\alpha I} = z_\alpha \mathcal{F}(\{\ast\}_I), \quad \alpha = \{0, \parallel, \perp, \Omega\}$$

(40)

where $\{\ast\}_I$ denote brane quantities such as the number and the nett charge of the branes in $I^{th}$ stack, the constant coefficients $z_\alpha$ and the functional dependence of $\mathcal{F}$ on the brane quantities $\{\ast\}$ are same for all $I$, and

$$z_\Omega = z_\perp, \quad z_\parallel = z_0 + z_\perp$$

(41)

as follows from $p_{\Omega I} = p_{\perp I}$ and from equation (39). Note that equations (40) may still lead to equations of state of the type considered in equation (32) if $\mathcal{F}$ depends on $\Pi$ only and if the function $\mathcal{F}(\Pi)$ is such that, for example, $\mathcal{F}(\Pi) = u^{(lo)}\Pi$ or $u^{(hi)}\Pi$ for low or high magnitudes of $\Pi$. Equation (32)
then follows, with $z^I_{\alpha} = z_{\alpha} u^I$ where $u^I = u^{(lo)}$ or $u^{(hi)}$ according to whether the magnitude of $\Pi_I$ is low or high.

Consider the case where $z^I_{\alpha} = z_{\alpha} u^I$ and $z_{\alpha}$ obey equations (41). It then follows from equations (38) and from BPS intersection rules that

$$z^{\alpha I} = \left( \frac{(n_I + 1)}{9} z_0 + z_\perp - z_{\alpha} \right) u^I$$

$$G^{IJ} = 2 z_0 \left( z_\perp - z_0 \delta^{IJ} \right) u^I u^J.$$ (43)

The form of $z^{\alpha I}$ and the relations among them are consequences of U duality symmetries of M theory. The relations among $z^{\alpha I}$ will be reflected in the solutions for $\lambda^a$ when written in terms of $l^I$, see equation (35) and subsequent comments. Note the elegant structure of $G^{IJ}$ and its independence on $n_I$. These features of $G^{IJ}$ are consequences of both the U duality symmetries and the BPS intersection rules. Similar $G^{IJ}$ was also present in our time dependent early universe studies.

**Equations of state from action $S$**

Now consider the equations of state for M theory branes which are given in equation (9). They follow from the energy momentum tensor in equation (5), which in turn follows from the action $S$ given in equation (1). Setting $p_{0I} = -\rho_I$, it follows from equation (9) that $h_{\alpha I} = \Pi_I + p_{\alpha I}$ are given by

$$\left( h_{0I}, h_{\parallel I}, h_{\perp I}, h_{\Omega I} \right) = (2, 2, 0, 0) \Pi_I .$$

Clearly, these $h_{\alpha I}$ satisfy the U duality relations in equation (39). They are of the form given in equation (40) with $F(\{*\}_I) = \Pi_I$ and with $z_{\alpha}$ given by

$$(z_0, z_\parallel, z_\perp, z_\Omega) = (2, 2, 0, 0)$$ (44)

which satisfy equations (41). Writing $h_{\alpha I} = z^I_{\alpha} \Pi_I$ with $z^I_{\alpha} = z_{\alpha} u^I$ and $u^I = 1$, it follows from equations (42) and (43) that $z^{\alpha I}$ and $G^{IJ}$ for M theory branes are given by

$$z^{0I} = z^{\parallel I} = \frac{2(n_I - 8)}{9}, \quad z^{\perp I} = z^{\Omega I} = \frac{2(n_I + 1)}{9}, \quad G^{IJ} = -8 \delta^{IJ} .$$ (45)
Thus \((z^\parallel, z^\perp) = \frac{2}{3} (-2, 1)\) for \(M2\) branes and \((z^\parallel, z^\perp) = \frac{2}{3} (-1, 2)\) for \(M5\) branes.

5. Solutions in terms of \(\tau\)

We now solve the equations of motion in complete generality for the case where the equations of state for M theory branes are given by equations (9). The fact that \(z^I_\Omega = z^I_\perp = 0\) in this case leads to crucial simplifications which in turn lead to explicit analytical solutions.

**Expressions for \(\lambda^\alpha\) and \(A_{(I)}\) in terms of \(l^I\) and \(\chi\)**

Using \((z^I_0, z^I_\parallel, z^I_\perp, z^I_\Omega) = (2, 2, 0, 0)\) and \(G^{IJ} = -8 \delta^{IJ}\) in equations (34), (36), and (37), we obtain

\[ l^I = 2 \left( \lambda^0 + \sum_{i \in \parallel I} \lambda^i \right) \]  

(46)

where \(i \in \parallel I\) denote spatial directions of the \(I^{th}\) stack of branes; and

\[ l^I_{\tau \tau} = 8 \rho_{I0} e^{l^I}, \quad \chi_{\tau \tau} = (m - 1)^2 e^{2\chi} \]  

(47)

where \(\rho_{I0} = -\Pi_{I0}\). These simplified equations for \(l^I\) and \(\chi\) can now be solved explicitly. Integrating the above equations once, we get

\[ (l^I_{\tau})^2 - 16 \rho_{I0} e^{l^I} = 2 c^I, \quad (\chi_{\tau})^2 - (m - 1)^2 e^{2\chi} = \frac{c_\chi}{2} \]  

(48)

where \(c^I\) and \(c_\chi\) are integration constants. Furthermore, it follows from the asymptotic conditions given in equations (30) and (31) that, in the limit \(\tau \to \tau_0\),

\[ e^{l^I} \to 1, \quad e^{\chi} \to \frac{1}{(m - 1) (\tau_0 - \tau)} . \]  

(49)

Using equations (47) for \(e^{l^I}\) and \(e^{2\chi}\), equation (35) for \(\lambda^\alpha\) can now be integrated to give

\[ \lambda^\alpha = -\frac{1}{8} \sum_I z^{\alpha I} l^I + \frac{\delta^{\alpha a}}{m - 1} \chi + L^\alpha (\tau - \tau_0) \]  

(50)
where $L^\alpha$ are constants, $L^a = L^\Omega$ for all $a$, and the asymptotic conditions given in equation (30) have been incorporated. Substituting the above $\lambda^\alpha$ into the definitions of $l^I$ and $\chi$ leads to $N + 1$ consistency constraints on $L^\alpha$ given by

$$\sum_\alpha z^I_\alpha L^\alpha = 0, \quad \sum_\alpha L^\alpha = L^\Omega.$$ (51)

Substituting $\lambda^\alpha$ into equation (27), and using equations (48) and (51), leads to one further constraint among $(c^I, c_\chi, L^\alpha)$ given by

$$\frac{m}{2(m-1)} c_\chi = \frac{1}{4} \sum_I c^I - \sum_{\alpha\beta} G_{\alpha\beta} L^\alpha L^\beta.$$ (52)

Note that the constraint $\sum_\alpha L^\alpha = L^\Omega$ implies that

$$- \sum_{\alpha\beta} G_{\alpha\beta} L^\alpha L^\beta = - (\sum_\alpha L^\alpha)^2 + \sum_\alpha (L^\alpha)^2$$

$$= (L^\Omega)^2 + \sum_i (L^i)^2 + (m-1)(L^\Omega)^2 \geq 0.$$ (53)

Hence $c_\chi \geq 0$ if $\sum_I c^I \geq 0$.

An expression for the field strength $F_{0i_1\cdots i_n}$ can also be obtained. Using $\rho_{I_0} = -\Pi_{I_0}$, equation (33) for $\Pi_I$, and equation (46) for $l^I$, it follows from equation (9) that

$$\rho_I = -\Pi_I = \rho_{I_0} e^{-2\Lambda + l^I} = \frac{1}{4} e^{-2\lambda - l^I} (F_{0i_1\cdots i_n})^2,$$

and hence that $\rho_{I_0} \geq 0$. Furthermore, since $F_{0i_1\cdots i_n} = \partial_\tau A_{(I)}$, it follows that

$$e^{2\Lambda - 2\lambda} (\partial_\tau A_{(I)})^2 = (\partial_\tau A_{(I)})^2 = 4 \rho_{I_0} e^{2l^I}$$

where the first equality follows from equation (17). Using the asymptotic form for $A_{(I)}$ given in equation (30), we then get

$$e^{\Lambda - \lambda} \partial_\tau A_{(I)} = \partial_\tau A_{(I)} = -2 \sqrt{\rho_{I_0}} e^{l^I}$$ (53)

and, taking $Q_I$ to be positive with no loss of generality,

$$2 \sqrt{\rho_{I_0}} = (m-1) Q_I.$$ (54)
Solutions for $l_I$, $\chi$, and $A_{(I)}$

We now solve equations (47) for $l$ and $\chi$. Let $X(\tau)$ satisfy the equation

$$X_{\tau\tau} = b \ e^X \implies (X_\tau)^2 - 2b \ e^X = 2c$$

where $b$ is a given constant and $c$ an integration constant. $X$ may be thought of as the coordinate of a particle moving in a ‘potential’ ($-be^X$) with ‘energy’ $c$. The second equation above implies that if $b$ is negative then $c$ must be positive. Thus, there are three cases depending on the signs of $b$ and $c$. The corresponding general solutions may be written as

$$e^X = \frac{2 \gamma^2}{\alpha^2 \sinh^2 y}, \quad b = \alpha^2, \quad c = 2 \gamma^2$$

$$e^X = \frac{2 \gamma^2}{\alpha^2 \sin^2 y}, \quad b = \alpha^2, \quad c = -2 \gamma^2$$

$$e^X = \frac{2 \gamma^2}{\alpha^2 \cosh^2 y}, \quad b = -\alpha^2, \quad c = 2 \gamma^2$$

where $y = \gamma(\tau_\ast - \tau)$, $\alpha$ is obtained from $b$, and $\gamma$ and $\tau_\ast$ are two integration constants. Note that, in the above three cases, $X$ is unbounded above and below, bounded from below, or bounded from above respectively.

In the following, we take $\alpha$ and $\gamma$ to be positive with no loss of generality. The constant $b$ is given, hence it determines the sign of $b$ and the value of $\alpha$. The constants $\gamma$ and $\tau_\ast$ may be determined, for example, as follows. Let $(x, v)$ be the initial values of $(X, X_\tau)$ at $\tau = \tau_{in}$. Then $v^2 - 2be^x = 2c$ determines the sign of $c$ and the value of $\gamma$. The solution for $e^X$ can now be selected from above according to the signs of $b$ and $c$. This solution evaluated at $\tau = \tau_{in}$ then determines $y_{in} = \gamma(\tau_\ast - \tau_{in})$, and thus $\tau_\ast$.

Now consider equations (47) and (48). Let $\rho_{I0} > 0$ and $c^I$ be positive for all $I$. (We will later comment on the case where one or more of the $c^I$ may be negative.) Then $c_\chi > 0$ as follows from equation (52). Hence let $c^I = 2(\gamma^I)^2$ and $c_\chi = 2\beta^2$, and take $\gamma^I$ and $\beta$ to be positive with no loss of generality. Then, comparing with the equations and solutions for $X$ given above, it follows that the general solutions for $l_I$ and $\chi$ may be written as

$$e^{l_I} = \frac{(\gamma^I)^2}{4 \rho_{I0} \sinh^2 y_I}, \quad y_I = \gamma^I (\tau_I + \tau_0 - \tau)$$ (55)
where $\tau_I$ are integration constants, and we have incorporated the asymptotic condition given in equation (49) for $e^{\chi}$ in the limit $\tau \to \tau_0$. It follows from equation (52) that $\beta$, $\gamma_I$, and $L^\alpha$ must satisfy the constraint

$$
\frac{m}{m-1} \beta^2 = \frac{1}{2} \sum_I (\gamma_I)^2 - \sum_{a\beta} G_{a\beta} L^a L^\beta .
$$

We now express $e^{l_I}$ in a different form that facilitates comparisons with the brane solutions as presented commonly in the literature. Since $e^{l_I} \to 1$ in the limit $\tau \to \tau_0$, we have

$$
2 \sqrt{\rho_I} \gamma_I \sinh (\gamma_I \tau_I) = 1 ,
$$

which may be satisfied identically upon setting

$$
\frac{\sqrt{\rho_I}}{\gamma_I} e^{\gamma_I \tau_I} = C_I^2 , \quad \frac{\sqrt{\rho_I}}{\gamma_I} e^{-\gamma_I \tau_I} = S_I^2
$$

where $C_I = \cosh \Theta_I$ and $S_I = \sinh \Theta_I$. It then follows that

$$
\frac{\sqrt{\rho_I}}{\gamma_I} = C_I S_I , \quad e^{\gamma_I \tau_I} = \frac{C_I}{S_I} .
$$

Using the above expressions in equation (55), we now obtain

$$
e^{l_I} = \frac{e^{2\gamma_I (\tau - \tau_0)}}{H_I^2} , \quad H_I = C_I^2 - e^{2\gamma_I (\tau - \tau_0)} S_I^2 . \tag{58}
$$

Using the expression for $e^{l_I}$ given above, equation (53) can now be integrated to obtain $A(I)$. Noting that $H_I \to 1$ and $A(I) \to 0$ in the limit $\tau \to \tau_0$, and using $\frac{\sqrt{\rho_I}}{\gamma_I} = C_I S_I$, it follows from equations (53) and (58) that

$$
A(I) = \frac{C_I S_I}{H_I} \left( 1 - e^{2\gamma_I (\tau - \tau_0)} \right) ; \tag{59}
$$

and from equation (54) for the charges $Q_I$ that

$$(m - 1) Q_I = 2 \gamma_I C_I S_I . \tag{60}$$
Also, using $l^I = 2 \gamma^I (\tau - \tau_0) - 2 \ln H_I$, the solution for $\lambda^\alpha$ given in equation (50) can be written in terms of $H_I$ as

$$\lambda^\alpha = \sum_I \frac{z^{\alpha I}}{4} \ln H_I + \frac{\delta^{\alpha I}}{m-1} \chi + \left( L^\alpha - \sum_I \frac{z^{\alpha I}}{4} \gamma^I \right) (\tau - \tau_0). \quad (61)$$

We note here that the functions $H_I$ and $A_{(I)}$ and the form of $\lambda^\alpha$ given above turn out to be the generalisations of the corresponding harmonic functions, the $(n_I + 1)$—form gauge fields, and the metric components which appear in the standard intersecting brane solutions as given, for example, in [2, 3, 5, 6, 7].

6. Solutions in terms of $r$: obtaining $r(\tau)$

We now have the complete general solutions for static intersecting M theory branes, obtained by directly solving the equations of motion in terms of the variable $\tau$. To express these solutions in terms of the more familiar radial coordinate $r$, we need to obtain the function $r(\tau)$ in complete generality using equation (16). We place an emphasis here on the generality of $r(\tau)$ because of the following. General solutions in terms of $\tau$ have been obtained by several groups for systems which, at different levels, are more general than ours [10]. But, while translating such solutions to the $r$ variable, invariably some ansatz is made which thereby limits the generality of the final solutions given in terms of $r$.

Consider the line element $ds$ given in equation (6). With no loss of generality, we exchange the function $e^\lambda$ for another function $f$ by taking

$$e^{2\lambda} dr^2 + e^{2\sigma} d\Omega_m^2 = e^{2\sigma} \left( \frac{dr^2}{r^2 f} + d\Omega_m^2 \right),$$

thus $e^{\sigma - \lambda} = r \sqrt{f}$. Using $e^{\lambda} dr = e^\Lambda d\tau$ and $e^{\Lambda - \sigma} = e^\chi$, it now follows that

$$r_{\tau} = \frac{dr}{d\tau} = e^{\Lambda - \lambda} = r \sqrt{f} e^\chi. \quad (62)$$

Since $e^\chi \to r^{m-1}$ in the limit $\tau \to \tau_0$, see equation (31), the general relation between $r$ and $\tau$ can be expressed as

$$e^{\chi(\tau)} = r^{m-1} e^{h(\tau)} \quad (63)$$
where the function $h(\tau) \to 0$ in the limit $\tau \to \tau_0$. Using equation (56) for $e^\chi$, it now follows that $r(\tau)$ is given by

$$r^{m-1} = e^{\chi-h} = \frac{\beta}{m-1} \frac{e^{-h}}{\sinh \beta (\tau_0 - \tau)} . \quad (64)$$

Calculating $r_\tau$ from this expression and substituting it into equation (62) gives

$$\sqrt{f} = \frac{r_\tau}{r} e^{-\chi} = \cosh \beta (\tau_0 - \tau) - \frac{h_\tau}{\beta} \sinh \beta (\tau_0 - \tau) . \quad (65)$$

Thus $r(\tau)$ and $f(\tau)$ are determined once $h(\tau)$ is known.

**Obtaining $h(\tau)$**

We now obtain $h(\tau)$ first for vacuum solutions, thereby also illustrating our method in a simpler context; and then for brane solutions. The function $h(\tau)$ turns out to be the same in both of these cases.

Consider first the vacuum equations of motion in terms of $r$, namely equations (14) and (15) with $\Pi_I = p_{\alpha I} = 0$. Equation (15) gives

$$\lambda^{i'} = b^{i'} F$$

where $i' = (0, i)$, $\lambda^{i'} = (\lambda^0, \lambda^i)$, $b^{i'} = (b^0, b^i)$ are constants, and the function $F$ is the solution to the homogeneous equation

$$F_{rr} + (\Lambda_r - \lambda_r) F_r = 0 . \quad (66)$$

This equation for $F$ implies that

$$e^{\Lambda - \lambda} F_r = F_{\tau} = \mathcal{M} \implies F = \mathcal{M} (\tau - \tau_0) \quad (67)$$

where $\mathcal{M} = (m-1) r_0^{m-1}$ is an integration constant and we have incorporated the condition that $F \to 0$ in the limit $\tau \to \tau_0$, see equations (30) and (31).

As for $\sigma$, note that there is a freedom in defining the $r$ coordinate. Using this freedom, we set

$$\sigma = b^0 F + \ln r$$

with no loss of generality.\footnote{Upon substitution of $\sigma(r)$, equations (14) and (15) will now give an equation for the function $f(r)$. See Appendix C.} Thus, for vacuum solutions, we have

$$\lambda^\alpha = b^\alpha F + \delta^\alpha_0 \ln r \quad (68)$$
where \( b^\alpha = (b^0, b^i, b^a) \), and \( \lambda^\alpha = \sigma \) and \( b^\alpha = b^\Omega \) for all \( a \). One can further set \( b^\Omega = 0 \) with no loss of generality, so that \( r \) becomes the conventional radial coordinate. However, we keep \( b^\Omega \) non vanishing since the underlying structure is then clearer. Also, define

\[
B = \sum_\alpha b^\alpha, \quad K = -\sum_{\alpha\beta} G_{\alpha\beta} b^\alpha b^\beta .
\]  

(69)

Using \( \lambda^\alpha \) given in equations (68), we now get

\[
\chi = \Lambda - \sigma = (m - 1) \ln r + (B - b^\Omega) F .
\]

Comparing with equation (63) then gives \( h = (B - b^\Omega) F \) for vacuum solutions.

Consider now the brane equations of motion in terms of \( r \), namely equations (13) – (15) with \( \Pi_I \) and \( p_{aI} \) given by equations (9). Following the comments made below equation (38) regarding expressing \( \lambda^\alpha \) in terms of \( U \), and following the analysis of section 5, it can be seen that \( \lambda^I = (\lambda^0, \lambda^i) \) are now given in terms of \( U \) by equation (50), or equivalently in terms of \( H_I \) by equation (61), where \( z^{\alpha I} \) are given in equation (45) and \( \tau - \tau_0 \) is now to be replaced by the function \( F \) satisfying the homogeneous equation (66). Thus, we write

\[
\lambda^I = \sum_I \frac{z^{\alpha I}}{4} \ln H_I + b^I F .
\]

As for \( \sigma \), note that \( p_{aI} = p_{aI} = p_{\perp I} \) for the branes. It therefore follows that, using the freedom in defining the \( r \) coordinate, we can set

\[
\sigma = \sum_I \frac{z^{\perp I}}{4} \ln H_I + b^\Omega F + \ln r
\]

with no loss of generality. Thus, for brane solutions, we have

\[
\lambda^\alpha = \sum_I \frac{z^{\alpha I}}{4} \ln H_I + b^\alpha F + \delta^{\alpha} \ln r .
\]

(70)

Using \( \lambda^\alpha \) given above, we again get

\[
\chi = \Lambda - \sigma = (m - 1) \ln r + (B - b^\Omega) F
\]

because it follows from equation (45) that

\[
\sum_\alpha z^{\alpha I} - z^\Omega = (n_I + 1) z^{\parallel I} + (8 - n_I) z^{\perp I} = 0
\]

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and, hence, the coefficient of the $\ln H_1$ term in $\chi$ vanishes. Comparing with equation (63) then gives $h = (B - b\Omega) F$ for brane solutions also.

Writing $F$ in terms of $\tau$ using equation (67), we now have

$$h(\tau) = k (\tau - \tau_0) , \quad k = \mathcal{M} (B - b\Omega)$$

(71)

for both vacuum and brane solutions. The functions $r(\tau)$ and $f(\tau)$ then follow from equations (64) and (65), and are given by

$$r^{m-1} = \frac{\beta}{m-1} \frac{e^{k (\tau_0 - \tau)}}{\sinh \beta (\tau_0 - \tau)}$$

(72)

and

$$\sqrt{f} = \cosh \beta (\tau_0 - \tau) - \frac{k}{\beta} \sinh \beta (\tau_0 - \tau) .$$

(73)

Note that, in principle, one can now obtain the functions $F$ and $f$, and thereby all other functions also, in terms of radial coordinate $r$ by using equations (67), (72), and (73). However, it turns out that the functions $F(r)$ and $f(r)$ can be obtained more directly. It is straightforward to obtain their equations of motion in terms of $r$. We have earlier solved these equations analytically and in complete generality, and reported the results in [11]. For the sake of completeness here, we briefly describe these analytical solutions and their derivation in Appendix C.

**Relations between $(b^\alpha, \mathcal{M})$ and $(L^\alpha, \gamma^I)$**

The brane solutions given in terms of $\tau$ are parametrised by the set of constants $(L^\alpha, \gamma^I)$ , whereas those given in terms of $r$ are parametrised by the set $(b^\alpha, \mathcal{M})$ . We now obtain the relations between these two sets.

Consider the expression for $\lambda^\alpha$ given in equation (70). Using equation (63) to write $\ln r = \frac{\chi - h}{m-1}$ in this expression, substituting $F = \mathcal{M} (\tau - \tau_0)$ and $h = k (\tau - \tau_0)$ into it, and then comparing it with equation (61), we get

$$\mathcal{M} b^\alpha - \frac{\delta^{\alpha a}}{m-1} k = L^\alpha - \sum_I \frac{z^0_I \gamma^I}{4} .$$

(74)

Using the expressions for $z^I_\alpha$ and $z^0_I$ given in equations (44) and (45), the $\mathcal{N} + 1$ constraints on $L^\alpha$ given in equations (51) can be shown to imply that
\[ k = \mathcal{M} \left( B - b^\Omega \right) , \text{ which is already known, and that} \]
\[ \gamma^I = \mathcal{M} w^I , \quad w^I = \frac{1}{2} \sum_\alpha z_\alpha^I b^\alpha = b^0 + \sum_{i \in \| I} b^i \]  
(75)

where \( i \in \| I \) denotes the spatial directions of the \( I^{th} \) stack of branes. Furthermore, using the \( b^\alpha \) given in equation (74), it can be shown that \( K \) defined in equation (69) is now given by
\[ \mathcal{M}^2 K = \frac{1}{2} \sum_I (\gamma^I)^2 - \sum_{\alpha\beta} G_{\alpha\beta} L^\alpha L^\beta - \frac{m}{m-1} k^2 . \]

Then the constraint given in equation (57) becomes
\[ \beta^2 = k^2 + \frac{m-1}{m} \mathcal{M}^2 K . \]  
(76)

Note that equation (75) gives \( \gamma^I (\tau - \tau_0) = w^I F \). Equations (58) and (59) then give \( \epsilon^I = \frac{2 w^IF}{H_I} \) and
\[ H_I = C_i^2 - \epsilon^{2w^IF} S_I^2 , \quad A_{(I)} = \frac{C_i S_I}{H_I} \left( 1 - \epsilon^{2w^IF} \right) . \]  
(77)

Using \( \gamma^I = (m-1) r_0^{m-1} w^I \), equation (60) for the charges \( Q_I \) gives
\[ Q_I = (2 w^I C_i S_I) r_0^{m-1} . \]  
(78)

Note also that there is a scaling freedom, namely
\[ (F, \mathcal{M}, b^\alpha) \rightarrow (c F, c \mathcal{M}, c^{-1} b^\alpha) \]

where \( c \) is a constant, under which \( (b^\alpha F, \mathcal{M} b^\alpha) \) and hence the solutions \( (\lambda^\alpha, H_I, A_{(I)}) \) remain unchanged. Using this scaling freedom, we can set, for example, \( 2 (B - b^\Omega) = 1 \) with no loss of generality.

We now count the nett number of constants among each of the sets \( (L^\alpha, \gamma^I) \) and \( (b^\alpha, \mathcal{M}) \). The constants \( (L^\alpha, \gamma^I) \) are \( (n_c + 2 + N) \) in number, and must obey \( (N + 1) \) constraints given in equations (51). Hence, there is a nett total of \( (n_c + 1) \) constants among \( (L^\alpha, \gamma^I) \). Similarly, \( (b^\alpha, \mathcal{M}) \) are \( (n_c + 2 + 1) \) in number but with no constraints. However, one can set
\( b^\Omega = 0 \) using the freedom in defining the \( r \) coordinate; and set, for example, 
\( 2 \ (B - b^\Omega) = 1 \) using the scaling freedom. Hence, there is a nett total of 
\((n_c + 1)\) constants among \((b^\alpha, \mathcal{M})\) also. This equality of the nett totals is a 
进一步 check that there has been no loss of generality in our derivation of \( r(\tau) \).

7. Properties of the solutions

We have now completed the derivation of the general solutions for static intersecting M theory branes, expressed in terms of \( \tau \) or \( r \). Below, for the 
sake of the reader’s convenience, we first collect the main results and the 
expressions for the general solutions together in one place; and then discuss 
the properties of the solutions.

**Main results in terms of \( \tau \)**

In terms of the variable \( \tau \), the general intersecting brane solutions are 
given by

\[
ds^2 = -e^{2\lambda^0} dt^2 + \sum_i e^{2\lambda^i} (dx^i)^2 + e^{2\sigma} \left( e^{2\chi} d\tau^2 + d\Omega_2^2 \right) \tag{79}
\]

and \( \mathcal{A}_{(I)} \) where

\[
\chi^\alpha = \sum_I \frac{z^{\alpha I}}{4} \ln H_I + \frac{\delta^{0\alpha}}{m - 1} \chi + \tilde{L}^\alpha (\tau - \tau_0),
\]

\[
e^{\chi} = \frac{\beta}{(m - 1) \sinh \beta (\tau_0 - \tau)}, \quad \tilde{L}^\alpha = L^\alpha - \sum_I \frac{z^{\alpha I} \gamma^I}{4},
\]

\[
H_I = C_I^2 - e^{2\gamma^I (\tau - \tau_0)} S_I^2, \quad \mathcal{A}_{(I)} = \frac{C_I S_I}{H_I} \left( 1 - e^{2\gamma^I (\tau - \tau_0)} \right),
\]

and the constants \( z^{\alpha I} \) are given in equation (45). In the above expressions, 
\((\beta, \gamma^I)\) are taken to be positive, \( \beta \) is given by equation (57), and \( L^\alpha \) obey the 
constraints given in equation (51). The general \( Mp \) brane solutions follow
upon setting $\mathcal{N} = 1$. Thus, for example, the line element for the general $M_p$ brane solutions is given by

$$ds^2_{(M_p)} = H^{A^I} ds^2_{p+1} + H^{A^L} e^{2\tilde{\sigma}} \left( e^{2\chi} d\tau^2 + d\Omega^2_{p-3} \right)$$  \hspace{1cm} (80)

where the $I$ subscripts are omitted,

$$ds^2_{p+1} = -e^{2\tilde{\lambda}_0} d\tau^2 + \sum_{i=1}^p e^{2\tilde{\lambda}_i} (dx^i)^2,$$

$$e^{2\tilde{\sigma}} = e^{2\tilde{\chi}_p} e^{2\tilde{L}^0 (\tau - \tau_0)} , \quad \tilde{L}^\alpha = L^\alpha - \frac{2\alpha \gamma}{4},$$

and $(p, A^\parallel, A^\perp) = (2, -\frac{2}{3}, \frac{1}{3})$ for $M2$ branes and $(5, -\frac{1}{3}, \frac{2}{3})$ for $M5$ branes. The general vacuum solutions follow upon taking $S_I = 0$, thus $H_I = 1$ and $A_{(I)} = 0$, and formally setting $\gamma^I = 0$ in the expression for $\tilde{L}^\alpha$.

**Main results in terms of $r$**

In terms of the radial coordinate $r$, the general intersecting brane solutions are given by

$$ds^2 = -e^{2\lambda_0} d\tau^2 + \sum_{i} e^{2\lambda_i} (dx^i)^2 + e^{2\sigma} \left( \frac{dr^2}{r^2 f} + d\Omega^2_m \right)$$  \hspace{1cm} (81)

and $A_{(I)}$ where

$$\lambda^\alpha = \sum_I \frac{z^{\alpha I}}{4} \ln H_I + b^\alpha F + \delta^{\alpha a} \ln r ,$$

and

$$H_I = C_I^2 - e^{2w^l F} S_I^2 , \quad A_{(I)} = \frac{C_I S_I}{H_I} \left( 1 - e^{2w^I F} \right).$$

The functions $F$ and $f$ in these expressions are not obtained in terms $r$. Instead $r$, $F$, and $f$ are all obtained in terms of $\tau$ and are given by

$$r^{m-1} = \frac{\beta e^k (\tau - \tau_0)}{(m - 1) \sinh \beta (\tau_0 - \tau)} , \quad F = M (\tau - \tau_0) ,$$  \hspace{1cm} (82)
\[ \sqrt{f} = \cosh \beta (\tau_0 - \tau) - \frac{k}{\beta} \sinh \beta (\tau_0 - \tau) , \]  
\[ (83) \]

using which one can obtain \( F(r) \) and \( f(r) \), see the comments below equation (73). In the above expressions, \((\beta, w^I, k)\) are taken to be positive, 
\[ \mathcal{M} = (m - 1) r_0^{m-1} , \quad \beta^2 = k^2 + \frac{m-1}{m} \mathcal{M}^2 K , \]
and \((w^I, k, K)\) are defined in equations (75), (71), and (69). The general \( Mp \) brane solutions follow upon setting \( N = 1 \). Thus, for example, the line element for the general \( Mp \) brane solutions is given by 
\[ ds^2_{(Mp)} = H^{\parallel} ds^2_{p+1} + H^{\perp} e^{2b_0 F} \left( \frac{dr^2}{f} + r^2 d\Omega^2_{3-p} \right) \]  
\[ (84) \]

where the \( I \) subscripts are omitted, 
\[ ds^2_{p+1} = -e^{2b_0 F} dt^2 + \sum_{i=1}^{p} e^{2b_i F} (dx^i)^2 , \]
and \((p, A^{\parallel}, A^{\perp}) = (2, \frac{2}{3}, \frac{1}{3})\) for \( M2 \) branes and \((5, \frac{1}{3}, \frac{2}{3})\) for \( M5 \) branes. The general vacuum solutions follow upon taking \( S_I = 0 \), thus \( H_I = 1 \) and \( A_{(I)} = 0 \). 

**Behaviour of \((F, f, r)\) and \((H_I, A_{(I)})\)** 

We now describe the behaviour of the functions \((F, f, r)\) and \((H_I, A_{(I)})\) as \( \tau \) decreases from \( \tau_0 \) to \(-\infty\). The behaviour of \( \lambda^\alpha \) then follows. It is clear from their expressions that these functions remain finite and vary smoothly in the open range \(-\infty < \tau < \tau_0\). Therefore, we will focus on their behaviours in the limits where \( \tau \to \tau_0 \) and where \( \tau \to -\infty \). In the following, we set \( b^I = 0 \) and \( 2 (B - b^I) = 1 \) with no loss of generality. Then 
\[ B = b^0 + \sum_i b_i = \frac{1}{2} , \quad 2 k = \mathcal{M} , \quad K = (b^0)^2 + \sum_i (b_i)^2 - \frac{1}{4} . \]

**Asymptotic limit \( \tau \to \tau_0 \)**
By construction, the limit $\tau \to \tau_0$ corresponds to asymptotically flat spacetime. In this limit, it follows from equation (31) that

$$r^{m-1} \to \frac{1}{(m-1)(\tau_0 - \tau)} \to \infty .$$

Hence $\mathcal{M}(\tau_0 - \tau) \to \frac{r^{m-1}_0}{r^{m-1}}$. It then follows that

$$\sqrt{f} \to 1 - k(\tau_0 - \tau) \implies f \to 1 - \frac{r^{m-1}_0}{r^{m-1}}$$

$$e^F \to 1 + \mathcal{M}(\tau - \tau_0) \to 1 - \frac{r^{m-1}_0}{r^{m-1}}$$

$$H_I \to 1 + (2w^I S_I^2) \frac{r^{m-1}_0}{r^{m-1}} , \quad A(I) \to \frac{Q_I}{r^{m-1}}$$

where $Q_I = (2w^I C_I S_I) r^{m-1}_0$, see equation (78). Thus, the general brane solutions here all behave the same way as the standard ones in the asymptotic limit where $r \to \infty$.

**Functions $e^F$, $H_I$, and $A(I)$**

We now describe the behaviour of the functions $e^F$, $H_I$, and $A(I)$ as $\tau$ decreases from $\tau_0$ and approaches the limit $\tau \to -\infty$.

The behaviour of the function $e^F$ is straightforward to see. Since $F = \mathcal{M}(\tau - \tau_0)$, it follows that $e^F$ varies smoothly and monotonically as $\tau$ decreases, and that $e^F \to 0$ in the limit $\tau \to -\infty$. The function $e^{2w^F}$ behaves the same way as $e^F$ since $w^I$ is positive. Therefore, it follows that the functions $H_I$ and $A(I)$ vary smoothly and monotonically as $\tau$ decreases and, further, that $H_I \to C_I^2$ and $A(I) \to \frac{S_I^2}{C^2_I}$ in the limit $\tau \to -\infty$.

**Functions $r(\tau)$ and $f(\tau)$**

We now describe the behaviour of the functions $r(\tau)$ and $f(\tau)$ as $\tau$ decreases away from $\tau_0$ and approaches the limit $\tau \to -\infty$. Their behaviour
depends on whether $K = 0$ or $K > 0$ or $K < 0$. We describe these cases separately.

**K = 0**

Note that $\beta = k$ when $K = 0$. Then, equations (82) and (83) give

\[ r_{m-1} = \frac{2k}{(m-1)(1 - e^{-2k(\tau_0 - \tau)})}, \quad \sqrt{f} = e^{-k(\tau_0 - \tau)}. \]

Hence, $f \to 0$ and $r \to r_0$ as $\tau \to -\infty$. Indeed, using $2k = \mathcal{M} = (m-1)r_0^{m-1}$, it follows that

\[ f = e^{2k(\tau - \tau_0)} = e^F = 1 - \frac{r_0^{m-1}}{r^{m-1}}. \]

The standard intersecting brane solutions as given, for example, in [2, 3, 5, 6, 7] all follow from this $K = 0$ case upon further choosing $2b^0 = 1$ and $b^i = 0$. One then has $2w^I = 1$ for all $I$, see equation (75), and hence $e^{2w^IF} = e^F$ and $Q_I = (C_I S_I) r_0^{m-1}$. It then follows that $H_I$ and $A_{(I)}$ are given by

\[ H_I = 1 + \frac{r_0^{m-1} S_I^2}{r^{m-1}} \quad \text{and} \quad A_{(I)} = \frac{Q_I}{H_I r^{m-1}}. \]

More general static intersecting brane solutions have been obtained earlier by several groups in different forms and using different methods [8]. They can all be shown [9] to follow for other choices of $b^\alpha$, but still with $K = 0$. The solutions in [9] were obtained by taking $f = e^F = 1 - \frac{r_0^{m-1}}{r^{m-1}}$ as an ansatz; $K = 0$ was then found as a constraint. The present solutions are completely general. They are obtained without making any ansatz and, hence, are applicable when $K \neq 0$ also.

**K > 0**

Note that $\beta > k$ when $K > 0$. Then, equation (82) implies that, in the limit $\tau \to -\infty$,

\[ r_{m-1} \to \frac{2\beta}{m-1} e^{(k-\beta)(\tau_0 - \tau)} \to 0. \]

Consider equation (83). It implies that, as $\tau$ decreases from $\tau_0$ to $-\infty$, $\sqrt{f}$ begins to decrease from 1, but remains strictly positive since $\cosh(*) > 0$.
\( \frac{k}{\beta} \sinh(\kappa) \) when \( \beta > k \). It also implies that, in the limit \( \tau \to -\infty \),

\[
\sqrt{f} \to \left( 1 - \frac{k}{\beta} \right) \frac{e^{\beta(\tau_0 - \tau)}}{2} \to \infty .
\]

It therefore follows that \( f(\tau) \) decreases from 1; remains strictly positive; reaches a non zero, positive minimum; and then increases to \( \infty \) as \( \tau \) decreases from \( \tau_0 \) to \( -\infty \), equivalently as \( r(\tau) \) decreases from \( \infty \) to 0. It can be shown easily that \( f \) reaches its minimum \( f_{\text{min}} = 1 - \frac{k^2}{\beta^2} \) at \( \tau_{f_{\text{min}}} \) where \( \tau_{f_{\text{min}}} \) is given by \( \tanh \beta (\tau_0 - \tau_{f_{\text{min}}}) = \frac{k}{\beta} \).

Thus, for \( K > 0 \), we have that \( f \) remains strictly positive, \( H_I \) remain > 1 and finite, and \( 0 \leq e^F \leq \infty \) in the interval \( 0 \leq r \leq \infty \). Therefore, it follows that the \( K > 0 \) solutions have no horizons. These general, horizonless, \( K > 0 \) solutions are new and, to the best of our knowledge, have not appeared in the literature. These are the ones referred to as ‘a class of new static solutions’ in the title of this paper.

\( K < 0 \)

Note that \( \beta < k \) when \( K < 0 \). Then, equation (82) implies that, in the limit \( \tau \to -\infty \),

\[
r^{m-1} \to \frac{2 \beta}{m-1} e^{(k-\beta)(\tau_0 - \tau)} \to \infty .
\]

Since \( r^{m-1} \to \infty \) in the limit \( \tau \to \tau_0 \), it now follows that \( r(\tau) \) decreases from \( \infty \); reaches a non zero, positive minimum; and then increases to \( \infty \) again as \( \tau \) decreases from \( \tau_0 \) to \( -\infty \). It can be seen from equation (62), or by a direct calculation, that \( r \) reaches its minimum \( r_{\text{min}} \) at \( \tau_{r_{\text{min}}} \) where \( f(\tau_{r_{\text{min}}}) = 0 \).

Consider equation (83). It implies that, as \( \tau \) decreases from \( \tau_0 \) towards \( -\infty \), \( \sqrt{f} \) decreases and must reach zero at a finite value \( \tau_{r_{\text{min}}} \) given by \( \tanh \beta (\tau_0 - \tau_{r_{\text{min}}}) = \frac{\beta}{k} \). The right hand side of equation (83) then becomes negative for \( \tau < \tau_{r_{\text{min}}} \). However, we do not fully understand the solution for \( \tau < \tau_{r_{\text{min}}} \). We mention this \( K < 0 \) case here only for the sake of completeness. Since it is not clear to us how to continue the solution for \( \tau < \tau_{r_{\text{min}}} \), nor how to analyse the equations of motion for the corresponding parameter values, we will not discuss the \( K < 0 \) case any further in this paper.
8. Remarks on the solutions

We will now make several remarks, mostly in the context of vacuum solutions. Since the functions $H_I$ and $A_{(I)}$ are finite for $\tau \leq \tau_0$, these remarks may easily be adapted for the general intersecting brane solutions also.

Consider general static vacuum solutions. The corresponding line element is given by

$$ds^2_{(M_P)} = -e^{2b^0 F} dt^2 + \sum_{i=1}^{n_c} e^{2b^i F} (dx^i)^2 + e^{2b^\Omega F} \left( \frac{dr^2}{f} + r^2 d\Omega_m^2 \right)$$

(85)

where $r$, $F$, and $f$ are given by equations (82) and (83).

(1) Note that $f = e^F = 1 - \frac{r^{m-1}}{\rho^m}$ for $K = 0$, and that the standard black $n_c$-brane solution is obtained upon further choosing $2b^0 = 1$ and $b^i = b^{\Omega} = 0$. It is well known [6] that one can start with this black brane solution, boost it along a compact direction to first generate a momentum charge, and then use the boosted black brane to generate various $\mathcal{N} = 1$ brane solutions by suitable U duality operations – namely, by suitable dimensional reduction and uplifting, and T and S dualities. Further repeats of boosting and U duality operations will generate other $\mathcal{N} > 1$ intersecting brane solutions. The standard static intersecting brane solutions of string theory can also be generated in this way.

More general brane solutions have been obtained earlier by several groups in different forms and using different methods [8]. We have shown in [9] that these solutions can all be obtained by suitable boosting and U duality operations, where now the starting vacuum solution is as given in equation (85), and still with $K = 0$, but with otherwise arbitrary values of $b^\alpha$.

It turns out that the general static intersecting brane solutions presented here can also be obtained by suitable boosting and U duality operations. The starting vacuum solution is as given in equation (85), but with arbitrary values of $b^\alpha$ irrespective of whether $K = 0$ or not. We have presented in [11] the solutions generated by this method, and it is easy to verify that they are same as the ones given in this paper which have been obtained by directly solving the equations of motion in complete generality. Also, as shown in [11], the expressions for $(B - b^{\Omega})$ and $K$ remain invariant under U duality operations. Hence, their values remain unchanged for all the solutions related by U duality operations.
(2) The appearance of hyperbolic functions in the solutions for $e^{l'}$, and thus for $H_{l'}$, can now be related to the underlying boost operations mentioned above. In section 5 here, the $Sinh$ functions arise because the integration constants $c^I$ in equation (48) are taken to be positive. If $c^I$ were negative then trigonometric functions would have appeared in the solutions.

Such solutions involving trigonometric functions can also be generated by U duality methods. Instead of a boost along a compact $x^I$ direction, one now performs a rotation in the $(x^i, x^j)$ directions, and proceeds with other U duality operations which will generate ‘tilted brane’ solutions, see [24] for example. The underlying rotation will lead to the appearance of trigonometric functions in the solutions. In the present context, they correspond to choosing one or more of the integration constants $c^I$ to be negative.

(3) Upon compactifying on the $n_c$ dimensional torus described by the coordinates $x^i$, one obtains an effective $d = m + 2$ dimensional non compact spacetime described by the coordinates $x^\mu = (t, r, \theta^a)$. The corresponding effective action and the line element may be written, symbolically and in the standard notation, as

$$S_{(d)} \sim \int d^d x \sqrt{-g_{(d)}} \ e^{\Lambda^c} \left( R_{(d)} + \sum_i (\partial_\mu \lambda^i)^2 + \cdots \right), \quad \Lambda^c = \sum_i \lambda^i$$

and

$$ds^2_{(d)} = g_{(d) \mu \nu} \, dx^\mu \, dx^\nu = -e^{2\lambda_0} \, dt^2 + e^{2\lambda} \, dr^2 + e^{2\sigma} \, d\Omega^2_m$$

where $g_{(d) \mu \nu}$ and $R_{(d)}$ are the $d$ dimensional metric and the corresponding Ricci scalar in the ‘physical frame’, and $\hat{g}_{(d) \mu \nu} = e^{2\Lambda^c / 2} \, g_{(d) \mu \nu}$ and $\hat{R}_{(d)}$ are those in the ‘Einstein frame’. The $d$ dimensional action $S_{(d)}$ now contains $n_c$ number of scalars descending from the sizes of the compact dimensions, and $N$ number of 1–form gauge fields descending from the $(n_l + 1)$–form gauge fields $A_{(I)}$. The couplings of the scalars among themselves and with the gauge fields are not the most general possible ones, but are dictated by the eleven dimensional parent action $S$ given in equation (1).

Clearly, the solutions presented in this paper lead to the most general static spherically symmetric solutions of the above $d$ dimensional system with
scalars and 1–form gauge fields. For example, Janis–Newman–Winicour–Wyman (JNWW) solutions [1] follow now as a special case. These JNWW solutions can be viewed as Einstein frame solutions for the four dimensional Brans–Dicke theory or, equivalently, as solutions for a five dimensional theory with one compact coordinate; and, can be shown to follow from the present vacuum solutions upon setting \( m = 2, \ n_c = 1, \ b^\Omega \neq 0 \) and \( K = 0 \), see [9] also.

(4) For \( K > 0 \), we have that \( f \) remains strictly positive, \( H_\ell \) remain \( > 1 \) and finite, and \( 0 \leq e^F \leq \infty \) in the interval \( 0 \leq r \leq \infty \). Thus, the \( K > 0 \) solutions have no horizons. We note here that similar horizonless solutions have also appeared recently in several works [25]. The set ups and the contexts of these works are totally different from the present ones, and the reasons for the similarities of the solutions are not clear to us.

(5) The ADM mass \( M_{ADM} \) for the solutions can be obtained from the expressions given in Appendix B. It is easy to see that \( M_{ADM} \) is positive for all the general intersecting brane solutions if \( b^\alpha \) satisfy the inequality

\[
2 (mb^\Omega + \sum_i b^i) < \frac{m}{m-1} .
\]

Using \( 2(B - b^\Omega) = 1 \), the above inequality becomes \( b^\Omega < b^0 + \frac{1}{2(m-1)} \). We will assume that \( b^\alpha \) are choosen, e.g. \( b^\Omega = 0, \ 2B = 1, \) and \( b^0 > 0 \), such that ADM mass is positive.

Consider now the curvature tensors. Note that the Riemann tensor components \( \hat{R}_{ABCD} \), calculated in the local tangent frame with indices \( A, B, \cdots \), correspond to tidal forces. The non vanishing components of \( \hat{R}_{ABCD} \) for the metric in equation (6) are given in Appendix A. If any of these components diverges then we assume conservatively that there will generically be a curvature singularity. Now, by construction, the limit \( \tau \to \tau_0 \) corresponds to asymptotically flat spacetime. Therefore, the components of \( \hat{R}_{ABCD} \) will all vanish in this limit. Furthermore, the fields behave smoothly and remain finite for all finite values of \( \tau < \tau_0 \); hence, so will the \( \hat{R}_{ABCD} \) components for these values of \( \tau \).

Consider the limit \( \tau \to -\infty \). Let the radial coordinate \( r \to r_S \) in this limit, hence \( r_S = r_0 \) if \( K = 0 \) and \( r_S = 0 \) if \( K > 0 \). Using the expressions given in Appendix A, it can be seen after some algebra that,
for the general static intersecting brane solutions presented here, except for
those with $2b^0 = 1$ and $b^i = b^\Omega = 0$, at least one of the components of $\hat{R}_{ABCD}$
diverges in the limit $r \to r_S$. Therefore, for generic values of $b^\alpha$, there is
likely to be a curvature singularity at $r_S$. Note that this singularity is not
covered by any horizon, is naked, and is present even though the ADM mass
is positive. This feature, that naked singularity is present even though ADM
mass is positive, is also exhibited, for example, by JNWW solutions, by those
in [8, 9], and by the recent ones found in [25].

(6) We will list a few reasons as to why the general static solutions given
in this paper are useful and are also likely to be physically relevant despite
the singularities at $r_S$ mentioned above. The present solutions are obtained
by solving the equations of motion in complete generality. Also, they have
positive ADM mass. Therefore, it is physically reasonable to expect that
these solutions describe the general static configuration of the fields outside
a static star in a $D = n_c + m + 2$ dimensional theory with compact $n_c$
dimensional toroidal space or, equivalently, in a $d = m + 2$ dimensional
theory with $n_c$ number of scalars.

Some of the reasons for the usefulness and the physical relevance of the
present solutions are as follows.

(i) In the studies of the static or dynamical properties of a star, one must
assume a most general initial field configuration in the exterior of the star,
which corresponds to the most general solutions to the equations of motion;
otherwise, one is likely to miss some crucial properties. Any restriction of the
parameter ranges must then come from physical criteria, such as stability of
the resulting static configuration of a star or as an attractor basin during its
dynamical evolution; or from requiring consistency with observations.

(ii) Let $r_*$ be the equilibrium radius of a static star. If $r_* > r_S$, which
is trivially valid for $K > 0$ case since $r_S = 0$, then the singularities at $r_S$
mentioned above are not relevant since the equations of motion are different
inside the star, namely for $r < r_*$, and the present solutions are not appli-
cable there. For $r \leq r_*$, one then has to solve a different set of equations
of motion which includes the stellar material, and then match the interior
solutions onto the most general exterior ones. Further stability studies of
the resulting static configuration of the star may put an upper bound on its
mass, but they may or may not further restrict the allowed values of $b^\alpha$.

(iii) If a star is sufficiently massive then it is likely to collapse. Let us
assume that a static configuration is reached at the end of such a collapse, and that the \( r = r_S \) surface is exposed. Then the present solutions are applicable for \( r \geq r_S \) and, generically, a naked singularity is likely to be present at \( r_S \).

It is now possible that the collapse dynamics is such that, irrespective of the initial values of \( b^\alpha \) prior to the collapse, the final values after the collapse are always given by an ‘attractor basin’ where \( 2b^0 = 1 \) and \( b^i = b^\Omega = 0 \) so that the \( r = r_S \) surface is a standard non-singular horizon. This may be accomplished, for example, if all the non-trivial field configurations are ‘radiated away’ during the collapse. A singularity, similar to the one mentioned above or a different dynamical one, may or may not occur and it may or may not become naked during the process of collapse itself.

In the context of four dimensional JNWW solutions, such collapse processes and various issues involving naked singularities have been extensively studied in [14, 16]. Similar studies are needed in the present context also.

(iv) Note that the present solutions are obtained from a low energy effective action involving only two derivative terms. Such an action is unlikely to be applicable in the strong curvature regime; then, one has to invoke the underlying fundamental theory, namely string/M theory here. It is then reasonable to assume that singularities will be resolved within such a theory. If this is the case then the present solutions are indeed physically relevant, but are applicable only until the strong curvature regime is reached.

(v) On the other hand, and more practically, one may simply assume that our universe is \((n_c + 4)\) dimensional with compact \(n_c\) dimensional toroidal space, or that it is four dimensional with \(n_c\) number of scalars; and that the present vacuum solutions describe the general static configuration of the fields outside a static star. It is then possible to study the motions of various probes around such stars, their orbitals, et cetera and compare them with observations. Such studies may lead to novel phenomena or other observational consequences. See [15] for such studies in the context of higher dimensions and \(Dp\) branes, and [13, 14] in the context of four dimensional JNWW solutions.

9. Conclusions

We now give a brief summary and conclude by mentioning a few issues for further studies. In this paper, we obtained analytically the most general
static intersecting brane solutions. These solutions reduce to the known ones in special cases, and contain further a class of new static solutions. We described the properties of these general solutions and discussed their usefulness and physical relevance. Along the way, we also described the features of the brane energy momentum tensors, the equations of motion, and their solutions which arise as consequences of the BPS intersection rules and the U duality symmetries of M theory.

We now mention a few issues which can be studied further.

Assuming that the present general solutions describe the fields outside the stars in our universe, one may calculate the motions of probes around such stars and study the observational consequences, as in [12, 13, 14, 15] for example.

As mentioned in the Introduction, it is natural to expect that the black holes described in M theory have been formed by the collapse of sufficiently massive stars. It is desireable to study such collapse processes. Towards this goal, one may first construct a static star in M theory where the present general solutions describe its exterior fields.

Once the static solutions for stars in M theory are obtained, their stability properties can then be studied along the lines of [26]. It is likely that a sufficiently massive star will collapse. Such a collapse may also be studied. In these studies of collapse, the M theoretic cosmological solutions we had obtained earlier in [20, 21, 22] are likely to be useful.

Appendix A : Riemann Tensor components

Consider the metric given in equation (6) where \( i = 1, 2, \cdots, n_c \) and the fields \((\lambda^0, \lambda^i, \lambda, \sigma)\) depend on \( r \) only. Let

\[
d\Omega_m^2 = h_{ab} \, d\theta^a \, d\theta^b
\]

where \( a = 1, 2, \cdots, m \) and \( h_{ab} \) depend on \( \theta_s \) only. Let \( e^M_A \) denote the \( D \)-bein fields and let

\[
\hat{R}_{ABCD} = e^M_A \, e^N_B \, e^P_C \, e^Q_D \, R_{MNPQ}
\]

where \( R_{MNPQ} \) is the Riemann tensor. It follows after a straightforward algebra that the non vanishing components of \( \hat{R}_{ABCD} \) for the metric in equation (6) are given by

\[
\hat{R}_{i'i'j'} = -\delta_{i'j'} \, e^{-2\lambda} \left( \lambda_{i'i'} + (\lambda^i - \lambda_r) \, \lambda_{r'} \right)
\]
\[ \hat{R}_{\tau ab} = -h_{ab} \, e^{-2\lambda} (\sigma_{\tau\tau} + (\sigma_{\tau} - \lambda_{\tau}) \sigma_{\tau}) \]
\[ \hat{R}_{\tau'j'k'l'} = (\delta_{\tau'\tau'} \delta_{j'k'l'} - \delta_{\tau'k'l'} \delta_{j'\tau'}) \, e^{-2\lambda} \left( \lambda_{\tau'}^{j'} \lambda_{\tau'}^{l'} \right) \]
\[ \hat{R}_{\tau'aj'b} = -\delta_{\tau'j'} h_{ab} \, e^{-2\lambda} \lambda_{\tau}^{i'} \sigma_{\tau} \]
\[ \hat{R}_{abcd} = e^{-2\sigma} \rho_{abcd}(h) + (h_{ad} h_{bc} - h_{ac} h_{bd}) \, e^{-2\lambda} \sigma_{\tau}^2 \]

where \( i' = (0, i) \), \( \lambda^{i'} = (\lambda^0, \lambda^i) \), and \( \rho_{abcd}(h) \) is the Riemann tensor corresponding to the metric \( h_{ab} \).

The Riemann tensor components in terms of the variable \( \tau \) defined in equation (16) may be obtained from the above expressions by replacing \((r, \lambda)\) with \((\tau, \Lambda)\). Thus,

\[ \hat{R}_{\tau\tau'j'k'} = -\delta_{\tau\tau'} \, e^{-2\Lambda} \left( \lambda_{\tau\tau}^{j'} + (\lambda_{\tau}^{j'} - \Lambda_{\tau}) \lambda_{\tau}^{l'} \right) \]
\[ \hat{R}_{\tau ab} = -h_{ab} \, e^{-2\Lambda} (\sigma_{\tau\tau} + (\sigma_{\tau} - \Lambda_{\tau}) \sigma_{\tau}) \]
\[ \hat{R}_{\tau'j'k'l'} = (\delta_{\tau'\tau'} \delta_{j'k'l'} - \delta_{\tau'k'l'} \delta_{j'\tau'}) \, e^{-2\Lambda} \left( \lambda_{\tau'}^{j'} \lambda_{\tau'}^{l'} \right) \]
\[ \hat{R}_{\tau'aj'b} = -\delta_{\tau'j'} h_{ab} \, e^{-2\Lambda} \lambda_{\tau}^{i'} \sigma_{\tau} \]
\[ \hat{R}_{abcd} = e^{-2\sigma} \rho_{abcd}(h) + (h_{ad} h_{bc} - h_{ac} h_{bd}) \, e^{-2\Lambda} \sigma_{\tau}^2 \].

**Appendix B : ADM Mass**

Consider the metric given in equation (6) where the fields \((\lambda^0, \lambda^i, \lambda, \sigma)\) depend on \( r \) only and obey the boundary conditions given in equation (30) which correspond to asymptotically flat spacetime. The ADM mass \( M_{ADM} \) for such a metric is given, following the analysis of [17], by

\[ M_{ADM} = \frac{\omega_m V_{nc}}{2\kappa^2} \left( m \, r^{m-1} (e^{2\lambda} - \frac{e^{2\sigma}}{r^2}) - r^{m} \partial_r (m \, \frac{e^{2\sigma}}{r^2} + \sum_i e^{2\lambda^i}) \right)_{r \to \infty} \]

where \( \omega_m \) and \( V_{nc} \) are the volumes of the \( m \) dimensional unit sphere and the \( n_c \) dimensional torus. Let the asymptotic behaviour of \((\lambda^0, \lambda^i, \lambda, \sigma)\) in the limit \( r \to \infty \) be given by

\[ \left( e^{2\lambda^0}, e^{2\lambda^i}, \frac{e^{2\sigma}}{r^2}, \sum_i e^{2\lambda^i} \right) \to 1 - (e^0, e^i, e^r, e^{\Omega}) \frac{r^m}{r^{m-1}} \]
where \((c^0, c^i, c^r, c^\Omega)\) and \(r_0\) are constants. Evaluating the above expression for the ADM mass then gives

\[
M_{ADM} = \frac{\omega_m V_n}{2\kappa^2} \left( r_0^{m-1} C_M \right)
\]

where

\[
C_M = m(c^\Omega - c^r) - (m - 1) \left( mc^\Omega + \sum_i c^i \right).
\]

(86)

We now write down the constants \((c^0, c^i, c^r, c^\Omega)\) and \(C_M\) for vacuum solutions, for \(p\)-brane solutions, and for intersecting brane solutions.

**Vacuum solutions**

For vacuum solutions, we have

\[
\left( e^{2\lambda^0}, e^{2\lambda^i}, e^{2\lambda^r}, \frac{e^{2\sigma}}{r^2} \right) = \left( e^{2b^0 F}, e^{2b^i F}, \frac{e^{2b^\Omega F}}{f}, e^{2b^\Omega F} \right).
\]

From the behaviour of the solutions described in section 7 in the asymptotic limit \(\tau \to \tau_0\), it follows that

\[
f \to 1 - \frac{r_0}{r m-1}, \quad e^F \to 1 - \frac{r_0}{r m-1}
\]

in the limit \(r \to \infty\). Hence, \((c^0, c^i, c^r) = 2 (b^0, b^i, b^\Omega)\) and \(c^r = (2b^\Omega - 1)\).

It then follows from equation (87) that

\[
C_M^{(\text{vac})} = m - 2 (m - 1) \left( mb^\Omega + \sum_i b^i \right)
\]

(88)

for vacuum solutions. In the standard case, we have

\[
2 (b^0, b^i, b^\Omega) = (1, 0, 0), \quad C_M^{(\text{vac})} = m.
\]

**\(p\)-branes**

For M (string) theory \(p\)-brane solutions, the metric (in Einstein frame) is described by

\[
\left( e^{2\lambda^0}, e^{2\lambda^i}, e^{2\lambda^r}, \frac{e^{2\sigma}}{r^2} \right) = \left( H^A_{\parallel} e^{2b^0 F}, H^A_{\parallel} \text{ or } A_{\perp} e^{2b^i F}, H^A_{\perp} e^{2b^\Omega F}, \frac{e^{2b^\Omega F}}{f}, H^A_{\perp} e^{2b^\Omega F} \right)
\]

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where \( H = 1 + (1 - e^{2wF}) S^2 \) with \( w = b^0 + \sum_{i \parallel} b^i \) and \( S = \text{Sinh} \Theta \), and the exponents \( A^\parallel \) is for directions parallel to the brane and \( A^\perp \) is for transverse directions. See equation (81) with \( N = 1 \) and with \( n_c \geq p \) in general. In the limit \( r \to \infty \), the functions \( f \) and \( e^{f} \) are as given earlier, and hence

\[
H \to 1 + (2 w S^2) \frac{r_0^{m-1}}{r^{m-1}}.
\]

Thus \( (e^{2\lambda_0}, e^{2\lambda_1}, e^{2\lambda_2}, \frac{e^{2\sigma}}{r}) \) $\to$ \( 1 - (c^0, c^i, c^r, c^\Omega) \frac{r_0^{m-1}}{r^{m-1}} \) in the limit \( r \to \infty \) where, for a \( p \)-brane along \((x^1, \cdots, x^p)\) directions,

\[
\begin{align*}
c^0 &= 2b^0 - 2wA^\parallel S^2 \\
c^i &= 2b^i - 2wA^\parallel S^2 \quad \text{for} \quad i = 1, \cdots, p \\
&= 2b^i - 2wA^\perp S^2 \quad \text{for} \quad i = p + 1, \cdots, n_c \\
c^\Omega &= 2b^\Omega - 2wA^\perp S^2 \\
c^r &= 2b^r - 1 - 2wA^\perp S^2.
\end{align*}
\]

Noting that \( c^\Omega - c^r = 1 \), and defining \( X_H = pA^\parallel + (m + n_c - p)A^\perp \), it then follows from equation (87) that

\[
C_{M (p)} = m - 2 (m - 1) (mb^\Omega + \sum_i b^i - wX_H S^2).
\]

For M theory branes, \( m+n_c = 9 \) and \((A^\parallel, A^\perp) = \left(\frac{2}{3}, \frac{1}{3}\right) \) or \( \left(-\frac{1}{3}, \frac{2}{3}\right) \) for M2 or M5 branes. (For string theory \( p \)-branes \( m + n_c = 8 \) and it can be shown that \((A^\parallel, A^\perp) = \left(\frac{2}{3}, \frac{1}{3}\right) \) in Einstein frame.) In all these cases, \( A^\parallel - A^\perp = -1 \) and \( (m + n_c) A^\perp = p + 1 \). Hence, \( X_H = p(A^\parallel - A^\perp) + (m + n_c)A^\perp = 1 \), and we have

\[
C_{M (p)} = m - 2 (m - 1) (mb^\Omega + \sum_i b^i - w S^2) \quad (89)
\]

for \( p \)-brane solutions. In the standard case, we have

\[
2 (b^0, b^i, b^\Omega ; w) = (1, 0, 0 ; 1) \quad , \quad C_{M (p)} = m + (m - 1)S^2.
\]

**Intersecting branes**
For \( \mathcal{N} \) sets of branes intersecting according to the BPS rules, the \( p \)-brane analysis given above applies with only a few straightforward modifications. Performing the analysis, it can be seen that the final expression may be obtained upon replacing \( w S^2 \) in the \( p \)-brane expressions by \( \sum_I w^I S^2_I \) where \( I = 1, 2, \cdots, \mathcal{N} \) and \( w^I \) are given in equation (75). Thus,

\[
C_{M (\mathcal{N}p)} = m - 2 (m - 1) \left( mb^{0} + \sum_i b^i - \sum_I w^I S^2_I \right) \tag{90}
\]

for intersecting brane solutions. In the standard case, we have

\[
2 (b^0, b^i, b^\Omega; w^I) = (1, 0, 0; 1), \quad C_{M (\mathcal{N}p)} = m + (m - 1) \sum_I S^2_I .
\]

### Appendix C: Analytical solutions for the functions \( F \) and \( f \)

The equations of motion for \( F \) and \( f \) in terms of \( r \) may be obtained by substituting \( \lambda^a \) given in equation (68) into vacuum equations of motion (14) and (15) with \( \Pi_I = p_a I = 0 \); or, by substituting \( \lambda^a \) given in equation (70) into brane equations of motion (13) – (15). After some algebra, the resulting equations for \( F(r) \) and \( f(r) \) can be written as

\[
2 (B - b^0) f (r F_r) = (m - 1)(1 - f) + \frac{K}{m} f (r F_r)^2 \tag{91}
\]

\[
2 (B - b^0) f (r F_r) = 2(m - 1)(1 - f) - r f_r \tag{92}
\]

\[
e^{(B-b^0)F} \sqrt{f (r F_r)} = \frac{\mathcal{M}}{r^{m-1}} \tag{93}
\]

where \( \mathcal{M} = (m-1) r^0_{m-1} \) and \( B \) and \( K \) are defined in equation (69). Equation (93) follows directly from equation (67), namely from \( e^{\Lambda - \lambda} F_r = \mathcal{M} \). We now set \( 2(B - b^0) = 1 \) with no loss of generality. For \( K = 0 \), the above equations can be solved easily and the solutions are given by

\[
f = e^F = 1 - \frac{r^0_0}{r^{m-1}} .
\]

It turns out that equations (91) – (93) can be solved analytically for \( K > 0 \) also; and, that the properties of the functions \( F(r) \) and \( f(r) \) can be understood even without obtaining solutions. These properties and the
explicit analytical forms of the functions \( F \) and \( f \) have been described in our earlier reports [11]. For the sake of completeness, we present these solutions here along with a description of their derivation and some of their properties.

Define \( R = r^{m-1} \) and \( R_0 = r_0^{m-1} \). We then get

\[
\begin{align*}
  f (R F_R) & = 1 - f + \frac{(m-1)K}{m} f (R F_R)^2 \\
  f (R F_R) & = 2(1-f) - R f_R \\
  e^F f (R F_R)^2 & = \frac{R_0^2}{R^2}
\end{align*}
\]

where the subscripts \( R \) denote \( R \)-derivatives. It follows that

\[
(1-f) - R f_R = \frac{(m-1)K}{m} f (R F_R)^2 = \frac{(m-1)K}{m} \left( \frac{2(1-f) - R f_R}{f} \right)
\]

which gives a quadratic equation for \( R f_R \), and that

\[
e^F = \frac{R_0^2 f}{R^2 (2(1-f) - R f_R)^2}.
\]

Defining

\[
b = \frac{4(m-1)K}{m} , \quad \alpha = \frac{1}{1 + b} , \quad f_0 = 1 - \alpha
\]

and solving the quadratic equation for \( R f_R \) gives, after some algebra,

\[
R f_R = \frac{2 (1-f) (f - f_0)}{f - f_0 + \epsilon \sqrt{\alpha f (f - f_0)}} \tag{94}
\]

and

\[
e^F = \frac{R_0^2}{4 \alpha R^2 (1-f)^2} \left( \sqrt{f - f_0} + \epsilon \sqrt{\alpha f} \right)^2 \tag{95}
\]

where \( \epsilon = \pm 1 \) and the square roots are always to be taken with a positive sign.

We consider the case \( K > 0 \), hence \( b > 0 \), so that \( \alpha < 1 \) and \( f_0 > 0 \). Define \( R_{\min} \) by \( f(R_{\min}) = f_0 \), and a function \( g(R) \) by

\[
\sqrt{f - f_0} = \epsilon g \sqrt{\alpha g} \quad \Rightarrow \quad f = 1 - \alpha + \alpha g^2 \tag{96}
\]

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where $\epsilon_g = Sgn g$ and, further, choose $\epsilon_g = \epsilon$ with no loss of generality. Using the asymptotic behaviour of $f$, namely $f \to 1 - \frac{R_0}{R}$ as $R \to \infty$, it can be seen from equation (94) that $\epsilon = +1$, and from equation (96) that $g(R) \to 1 - \frac{R_0}{2R}$, in this limit. It can further be seen that $g > 0$ and $\epsilon_g = +1$ for $R_{min} < R < \infty$; that $g(R_{min}) = 0$; and that $g < 0$ and $\epsilon_g = -1$ for $0 < R < R_{min}$. Writing $f$ in terms of $g$ and after some algebra, equations (94) and (95) now become

$$Rg_R = \frac{1 - g^2}{g + \sqrt{f}} \implies \frac{dR}{R} = dg \left( \frac{g + \sqrt{f}}{1 - g^2} \right)^{1/2} \quad (97)$$

and

$$e^F = \frac{R_0^2}{4\alpha^2 R^2} \left( \frac{g + \sqrt{f}}{1 - g^2} \right)^2. \quad (98)$$

It turns out that equation (97) can be solved and an explicit analytical solution for $R$ in terms of $g$ can be obtained, because

$$\frac{g + \sqrt{f}}{1 - g^2} = \frac{g}{1 - g^2} - \frac{\alpha}{\sqrt{f}} + \frac{1}{2} \left( \frac{1}{1 - g} + \frac{1}{1 + g} \right) \frac{1}{\sqrt{f}}$$

and each term on the right hand side can be integrated in a closed analytical form. Incorporating $g(R_{min}) = 0$, equivalently $R(0) = R_{min}$, we get

$$ln \frac{R_{min}}{R} = \frac{1}{2} ln |1 - g^2| + \sqrt{\alpha} \sinh^{-1} \frac{g \sqrt{\alpha}}{\sqrt{1 - \alpha}}$$

$$- \frac{1}{2} \sinh^{-1} \frac{1 - \alpha + \alpha g}{(1 - g) \sqrt{\alpha(1 - \alpha)}} \quad + \frac{1}{2} \sinh^{-1} \frac{1 - \alpha - \alpha g}{(1 + g) \sqrt{\alpha(1 - \alpha)}}.$$

A more explicit expression for $R(g)$ can be obtained, and in many equivalent forms, by using identities such as

$$\sinh^{-1} x = \ln \left( x + \sqrt{1 + x^2} \right)$$

$$f - g^2 = (1 - \alpha) (1 - g^2)$$

$$\alpha (1 - \alpha) (1 \pm g)^2 + (1 - \alpha \mp \alpha g)^2 = f$$
\[
(g - \sqrt{f}) \left(1 - \alpha - \alpha g - \sqrt{f}\right) = (1 - \alpha)(1 + g) \left(1 - \sqrt{f}\right)
\]
\[
(1 - \alpha - \alpha g + \sqrt{f}) \left(1 - \alpha + \alpha g + \sqrt{f}\right) = (1 - \alpha) \left(1 + \sqrt{f}\right)^2.
\]

The expression we find convenient is given, after a series of manipulations, by
\[
\frac{R_{\text{min}}}{R} = \frac{\sqrt{1 - \alpha} \left(1 - g\right) \left(1 + \sqrt{f}\right)}{1 - \alpha + \alpha g + \sqrt{f}} \left(\frac{\sqrt{f} + g\sqrt{\alpha}}{\sqrt{1 - \alpha}}\right)^{\sqrt{\alpha}}.
\]  
Equation (98) now gives an expression for \(e^F\) in terms of \(g\).

We now describe some properties of the analytical solutions given above. These properties all follow straightforwardly, but after some algebra.

- The above expression for \(R(g)\) satisfies the differential equation (97).

- Consider \(\alpha = 1\). Noting that \(\left(\sqrt{1 - \alpha}\right)^{1 - \sqrt{\alpha}} = 1\) in the limit \(\alpha \to 1\), and taking \(\sqrt{f} = g\), we get \(\frac{R_0}{R} = 1 - g^2 = 1 - f\). It then follows from equations (97) and (98) that \(e^F = f = 1 - \frac{R_0}{R}\).

- Consider \(\alpha < 1\). As \(R\) decreases from \(\infty\) to \(R_{\text{min}}\) to \(R_1\) to 0, \(g\) decreases from 1 to 0 to \(-1\) to \(-\infty\); hence, \(f\) decreases from 1, reaches its minimum \(f_0\), then increases and reaches 1 again, and increases further to \(\infty\). Thus, \(R(g)\) for some select values of \(g\) are:

\[R(1-) = \infty, \quad R(0) = R_{\text{min}}, \quad R(-1) = R_1, \quad R(-\infty) = 0.\]

- As \(R\) decreases from \(\infty\) to 0, \(e^F\) decreases monotonically from 1 to 0, remaining \(< f\) always.

- Consider the limit \(R \to \infty\). Setting \(1 - g = \frac{R_0}{2\alpha R}\) and \(f = g = 1\) in equation (99) gives

\[
R_{\text{min}} = \frac{c(\alpha)}{\alpha} R_0, \quad c(\alpha) = \frac{1}{2} \left(1 + \sqrt{\alpha}\right)^{\frac{1+\sqrt{\alpha}}{\alpha}} \left(1 - \sqrt{\alpha}\right)^{\frac{1-\sqrt{\alpha}}{2}}.
\]

which thus expresses \(R_{\text{min}}\), where the function \(f\) reaches its minimum, in terms of \(R_0 = \frac{r_0^{m-1}}{\alpha}\). Note that \(\frac{1}{2} \leq c(\alpha) \leq 1\) for 0 \(\leq \alpha \leq 1\), as can be shown easily.
• Consider $R = R_1$. Then $g = -1$ and $f = 1$. Equation (99) now gives

$$R_1 = c(\alpha) R_{\text{min}}.$$  

which thus expresses $R_1$ in terms of $R_{\text{min}}$ and thereby in terms of $R_0$.

• Consider the limit $R \to 0$. Then $g \to -\infty$, $\sqrt{f} \to (-g)\sqrt{\alpha}$, and

$$\sqrt{f} + g\sqrt{\alpha} = \frac{f - \alpha g^2}{\sqrt{f - g^2}} \to \frac{1}{2\sqrt{\alpha}} \frac{1}{(-g)}.$$  

We then get

$$R^{-1} \sim (-g)^{1-\sqrt{\alpha}}, \quad f \sim g^2 \sim R^{2-\frac{2}{\sqrt{\alpha}}}.$$  

• In the main body of the paper, the general solutions for $R = r^{m-1}$, $F$, and $f$ have been obtained in terms of $\tau$, and are given in equations (82) and (83). Using equation (96), one can now obtain an expression for $g$ also in terms of $\tau$. It can then be shown, after a long but straightforward algebra, that these expressions for $R(\tau)$, $F(\tau)$, $f(\tau)$, and $g(\tau)$ satisfy equations (98) and (99).

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