GALOIS CORRESPONDENCE FOR AUGMENTED MONADS

JOHAN FELIPE GARCÍA VARGAS

Abstract. We establish a Galois connection between sub-monads of an augmented monad and sub-functors of the forgetful functor from its Eilenberg-Moore category. This connection is given in terms of invariants and stabilizers defined through universal properties. An explicit procedure for the computation of invariants is given assuming the existence of suitable right adjoints. Additionally, in the context of monoidal closed categories, a characterization of stabilizers is made in terms of Tannakian reconstruction.

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1. Introduction

The structural approach to Galois theory, as develop by Emil Artin and taught in most algebra textbooks, place as fundamental theorem the correspondence, given by invariants and stabilizers, between sub-groups of symmetries and intermediate structures.

In [4, Exposé V], Grothendieck reframed Galois theory as an equivalence between a category of interest and the category of actions of a group. Since the actions of a group $G$ form the Eilenberg-Moore category for the monad $G \times ?$, Grothendieck’s formulation is a monadicity result. This article proposes a path for recovering the Galois correspondence from a monadicity result.

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Our main hypothesis is that the monad possess an augmentation. Where, an augmentation is a monad homomorphism to the identity of the base category. This hypothesis is strong, because in the case of Hopf monads it implies that the monad is $\otimes$-representable, as shown in [2, Section 5].

Morally, having an augmentation distinguish groups from groupoids. More precisely, the monad induced by a groupoid has an augmentation only when every morphism in the groupoid is an endomorphism. However, this makes sense, because there is no clear meaning for invariants of a groupoid action when it moves elements from a set to another.

Before presenting the details, let me exhibit the intuition behind them. Imagine the monad $T$ as an algebraic gadget useful for “capturing the symmetries” of some kind of structure defined over a category $C$. With this in mind, we regard the forgetful functor $U$ as the “universal $T$-structure”. A Galois correspondence will emerge from comparing the action of a part of the symmetries, parametrized by a monad homomorphism $h : S \Rightarrow T$, over a part of the of the structure, parametrized by a natural transformation $\alpha : V \Rightarrow U$, against a trivial action induced by an augmentation of the monad.

The next example illustrates the notions that we aim to generalize and might be considered a trailer for the paper:

**Example 1.1.** Let $G$ be a group. Consider the augmented monad $G \times ?$ in $\text{Sets}$, where the augmentation is the projection to the second component. The Eilenberg-Moore category $\text{Sets}^G$ is the category of $G$-sets and equivariant maps.

For any group homomorphism $H \xrightarrow{\phi} G$ the $H$-invariants define a subfunctor of the forgetful $U : \text{Sets}^G \rightarrow \text{Sets}$, mapping an action $(X, G \times X \xrightarrow{r} X)$ to the set inclusion

$$\text{Inv} H_\phi(X, r) = \{x \in X : \phi(h)x = x \text{ for all } h \in H\} \subseteq X.$$

We will dissect this construction in the following way:

The action $G \times X \xrightarrow{r} X$ and the augmentation $G \times X \xrightarrow{e \times X} X$ (the $e$ stands for erase, or for the constant function to the group identity) parametrize for every $x \in X$ functions from $G$ to $X$, mapping $g \mapsto (r(g, x) = gx)$ and $g \mapsto (e(g)x = x)$, respectively.

Denoting by $[G, X] \xrightarrow{[\phi, X]} [H, X]$ the function precomposing by $H \xrightarrow{\phi} G$, we can present $\text{Inv} H(X, r)$ as the equalizer of $X \xrightarrow{r^{-1}} [G, X] \xrightarrow{[\phi, X]} [H, X]$. This presentation of invariants is generalized in Lemma 4.1.
On the other hand, for every natural transformation \( \alpha : V \Rightarrow U \) the stabilizer of \( V_\alpha \) is the subgroup of \( G \) defined by

\[
\text{Stab} V_\alpha = \{ g \in G : g\alpha_M(v) = \alpha_M(v) \text{ for every } M \in \text{Sets}^G \text{ and every } v \in VM \}.
\]

Inspired by Tannakian reconstruction, we can also dissect the construction of stabilizers. Notice that each \( g \in G \) gives a natural endomorphism of \( U \) obtained by acting, explicitly \( \rho(g) : U \Rightarrow U \) is defined by \( \rho(g)_M(x) = r_M(g, x) = gx \) for every \( G \)-set \( M = (X, r_M) \) and every \( x \in X \). The augmentation similarly defines a natural endomorphism of \( U \) for each \( g \in G \), \( \epsilon(g) : U \Rightarrow U \) defined by \( \epsilon(g)_M(x) = \epsilon(g)x = x \) for every \( G \)-set \( M = (X, r_M) \) and every \( x \in X \), which is clearly the identity of \( U \).

Denoting by \( \text{End}(U) \xrightarrow{\text{Nat}(\alpha, U)} \text{Nat}(V, U) \) the function precomposing by \( \alpha : V \Rightarrow U \), we can present \( \text{Stab} V_\alpha \) as the equalizer of \( G \xrightarrow{\rho} \text{End}(U) \xrightarrow{\text{Nat}(\alpha, V)} \text{Nat}(V, U) \).

This presentation of stabilizers is generalized in Lemma 5.10.

The well known Galois connection between invariants and stabilizers is generalized in Corollary 3.7.

The structure of the document is straightforward. Section 2 contains three preliminars one about lifting functors to Eilenberg-Moore categories, one about monoidal closed categories and Hopf monads and one about Galois connections. Section 3 introduces invariants and stabilizers as universal representants for the fix relation. Section 4 is about computing invariants and Section 5 about computing stabilizers in monoidal closed categories.

2. Preliminars

**Notation 2.1.** We write compositions from right to left. Given natural transformations \( \alpha : F \Rightarrow F' \) and \( \gamma : F' \Rightarrow F'' \), where \( F, F', F'' : C \rightarrow D \), we denote their vertical composition by \( \gamma \cdot \alpha : F \Rightarrow F'' \). If additionally given \( \beta : G \Rightarrow G' \), with \( G, G' : D \rightarrow E \), we denote the horizontal composition by \( \beta \star \alpha : GF \Rightarrow G'F' \). We will abbreviate identities of functors (and of objects) as \( \text{id}_G = G \). We will also omit the \( \star \) when whiskering. For instance, \( \beta \star \alpha = (G'' \alpha) \cdot (\beta F) = (\beta F') \cdot (G \alpha) \).

2.1. **Monadic lifting to Eilenberg-Moore categories.** It is well known, see [5, Chapter VI] for instance, that every adjunction yields a monad and that the Eilenberg-Moore construction allow to decompose a monad into an adjunction in a “universal” way. Where by universal, we mean that every other adjunction, yielding the same monad, is included in it. It is less well known, that the lift of functors to Eilenberg-Moore categories can be understood in terms of natural transformations.
of the base categories. We will make a quick review of this and refer the reader to [1, Chapter 2] for the details.

We denote monads by \((T, \mu, \eta)\) and adjunctions by \((\eta, L \dashv R, \varepsilon)\) as displayed in the next figure:

\[
\begin{array}{ccc}
\text{C} & \xrightarrow{T} & \text{C} \\
\text{Id}_C & \downarrow{\eta} & \downarrow{\mu} & \text{C} & \xleftarrow{L} & \text{D} & \hookrightarrow & LR \\
\end{array}
\]

(A) Monad. 

\[
\begin{array}{ccc}
\text{C} & \xrightarrow{L} & \text{C} & \xleftarrow{R} & \text{D} & \hookrightarrow & LR \\
\end{array}
\]

(B) Adjunction.

**Figure 1.** Structure of monads and adjunctions.

**Definition 2.2.** Let \((T, \mu, \eta)\) be a monad over a category \(C\). The Eilenberg-Moore category of \(T\), denoted \(\text{C}_T\), has \(T\)-actions as objects and \(T\)-morphisms as morphisms.

(a) A \(T\)-action is a pair \(M = (X, TX \xrightarrow{r_M} X)\), where \(X\) is in \(C\) and \(r\) satisfies:

- associativity \quad \(r_M \mu_X = r_T r_M\),
- and unity \quad \(r_M \eta_X = X\).

(b) A \(T\)-morphism \(M \xrightarrow{f} N\), between \(T\)-actions \(M = (X, r)\) and \(N = (Y, r')\), is a morphism \(X \xrightarrow{f} Y\) satisfying \(f r = r'T f\).

(c) The forgetful functor \(U^T : \text{C}_T \to C\), defined by

\[
U^T((X, r) \xrightarrow{f} (Y, r')) = (X \xrightarrow{f} Y),
\]

has a left adjoint \(F^T : C \to \text{C}_T\). This functor is known as the free functor and it is defined by

\[
F^T X = (TX, TX \xrightarrow{\mu_X} TX) \quad \text{and} \quad F^T (X \xrightarrow{f} Y) = (TX, \mu_X) \xrightarrow{Tf} (TY, \mu_Y).
\]

Notice that \(T = U^T F^T\), hence the unit of the adjunction \(\eta : \text{Id}_C \Rightarrow U^T F^T\) is equal to the unit of the monad \(\eta : \text{Id}_C \Rightarrow T\). Moreover, the evaluation \(\varepsilon : F^T U^T \Rightarrow \text{Id}_{\text{C}_T}\) is given by \(\varepsilon_M = (TX, \mu_X) \xrightarrow{\mu} (X, r)\), for any \(T\)-action \(M = (X, TX \xrightarrow{r} X)\).

(d) Suppose that there is another adjunction \((\eta, L \dashv R, \varepsilon)\) with \(R : \text{D} \to \text{C}\) such that \(T = RL\) and \(\mu = R\varepsilon L\). The comparison functor \(K : \text{D} \to \text{C}_T\) is defined by

\[
KA = (RA, RLRA \xrightarrow{R\varepsilon A} RA),
\]
for every object $A$ in $\mathcal{D}$. It is easy to verify that $KA$ is a $T$-action and obviously $U^T K = R$.

(e) A functor $R : \mathcal{D} \to \mathcal{C}$ is called monadic if it has a left adjoint $L \dashv R$, and the comparison functor is an equivalence from $\mathcal{D}$ to the Eilenberg-Moore category $\mathcal{C}^T$ of the induced monad $T = RL$.

**Definition 2.3.** Let $(T, \mu, \eta)$ and $(T', \mu', \eta')$ be monads over $\mathcal{C}$ and $\mathcal{C}'$, respectively. Given a functor $G : \mathcal{C} \to \mathcal{C}'$, we say that:

- A lift of $G$ is a functor $\overline{G} : \mathcal{C}^T \to \mathcal{C}'^{T'}$ such that $U^T \overline{G} = GU^T$.
- A lifting data for $G$ is a natural transformation $\gamma : T' G \Rightarrow GT$ such that

$$\gamma \cdot (\mu' G) = (G \mu) \cdot (\gamma T) \cdot (T' \gamma),$$

and

$$\gamma \cdot (\eta' G) = G \eta.$$

- When $\mathcal{C} = \mathcal{C}'$, an homomorphism of monads $h : T' \Rightarrow T$ is a lifting data for $\text{Id}_\mathcal{C}$. In that case, the above axioms simplify to $h \cdot \mu' = \mu \cdot (h \ast h)$ and $h \cdot \eta' = \eta$.

**Theorem 2.4.** [1, Theorem and definition 2.27] Let $T$, $T'$ and $G$ as in the previous definition. There is a bijection between lifts of $G$ and lifting data for $G$.

As a special case of the above theorem, taking the identity as base functor (i.e $\mathcal{C} = \mathcal{C}'$ and $G = \text{Id}_\mathcal{C}$), we obtain the following corollary:

**Corollary 2.5.** Let $T$ and $T'$ be monads over $\mathcal{C}$. There is a bijection between monad homomorphisms $h : T' \Rightarrow T$ and functors $H : \mathcal{C}^T \to \mathcal{C}'^{T'}$ such that $U^{T'} H = U^T$.

The next proposition gives a criterion for lifting natural transformations:

**Proposition 2.6.** [1, Theorem and definition 2.30] Given a natural transformation $\omega : G \Rightarrow H$ between functors from $\mathcal{C}$ to $\mathcal{C}'$ with lifting data $\alpha : T' G \Rightarrow GT$ and $\beta : T' H \Rightarrow HT$ along the monads $T$ and $T'$. There is an unique $\overline{\omega} : G^\alpha \Rightarrow H^\beta$ which lifts $\omega$ (i.e. $U^{T'} \overline{\omega} = \omega U^T$), if and only if, $(\omega T) \cdot \alpha = \beta \cdot (T' \omega)$.

2.2. Monoidal closed categories and Hopf monads. A monoidal category is a category $\mathcal{C}$ endowed with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called monoidal product, an object $1$ in $\mathcal{C}$ called unit object, and natural isomorphisms called associativity and unit constrains which are subject to well known axioms of coherence. In virtue of the strictness theorems, — see [1, Lemma 3.7 and 3.8], [5, Section VII.2] or [3, Section 2.8]
— it is unnecessary to explicitly keep track of the associative or unit constraints in any monoidal category, as long as you are only interested in properties up to monoidal equivalence of categories.

One key feature of monoidal categories is that an object \( X \), can be regarded as an endofunctor \( X \otimes ? \); this kind of functors are called \( \otimes \)-representable. The same idea applies to morphisms and natural transformations. For instance, \( \otimes \)-representable monads correspond to monoids. Furthermore, the Eilenberg-Moore category of such a monad is the category of actions of the monoid.

Bimonads are monads which lift the monoidal product to the Eilenberg-Moore category. More precisely, a monad \( T \) over \((C, \otimes, \mathbb{1})\) is a bimonad if and only if \( C^T \) is also monoidal and the forgetful functor \( U^T : C^T \to C \) is strong monoidal. By means of Theorem 2.4, Bimonads are characterized as opmonoidal monads, see [1, Theorem 3.19].

A monoidal category is left closed if every \( \otimes \)-representable functor \( ? \otimes X \) has a right adjoint, denoted \([X, ?]^\ell\). In this case, we call \([X, Z]^\ell\) the left internal Hom from \( X \) to \( Z \). Since the adjunction \((\eta^X, ? \otimes X \dashv [X, ?]^\ell, \varepsilon^X)\), gives a bijection between \( \text{Hom}(Y \otimes X, Z) \) and \( \text{Hom}(Y, [X, Z]^\ell) \). Borrowing the terminology from computer science, we define the currying of any \( Y \otimes X \xrightarrow{f} Z \) as \( f^\ell = \eta_Y^X f \eta_X^Y \), and the uncurrying of \( Y \xrightarrow{g} [X, Z]^\ell \) as \( g \cdot \varepsilon_Z^X (g \otimes X) \). It is worth noticing that by the theorem on adjunctions with a parameter — see [5, Theorem 3 in IV.7] — \([?, ?]^\ell\) is functor from \( C^{op} \otimes C \) to \( C \), when \( C \) is left monoidal closed.

When the functor \([X, ?]^\ell\) is \( \otimes \)-representable, i.e. \([X, ?]^\ell = ? \otimes X^\ell\) we call \( X^\ell \) the left dual of \( X \). A monoidal category is left rigid (also called compact or autonomous) if every object has a left dual.

Since a bimonad \( T \) lifts the monoidal product, for every \( T \)-action \( M = (X, r_M) \) the functor \( ? \otimes M : C^T \to C^T \) is a lift of \( ? \otimes X : C \to C \). The bimonad \( T \) is a left Hopf monad if whenever \( ? \otimes X \) has a right adjoint, the adjoint lifts to an adjoint of \( ? \otimes M \). Consequently, if \( C \) is left monoidal closed (resp. rigid) and \( T \) is a left Hopf monad, then \( C^T \) is also left closed (resp. rigid) and the forgetful functor preserves internal Homs. Left Hopf monads are characterized by the invertibility of the left fusion operator, see [2, Theorem 3.6] or [1, Theorem 3.27].

Another key feature of monoidal categories is that, in addition to the usual duality between a category and its opposite, there is a left-right duality obtained by exchanging the order in the monoidal product. Implying that there are right variants for each of the above defined concepts. When the laterality is omitted it is tacitly understood that
both variants holds. For instance, a rigid category is both left and right rigid.

A gadget for comparing left and right \( \otimes \)-representable functors is called a braiding. More precisely, a lax half-braiding on a monoidal category \( \mathcal{C} \) is a pair \((A, \sigma)\) where \( A \) is an object and \( \sigma : A \otimes ? \Rightarrow ? \otimes A \) is a \( \otimes \)-multiplicative natural transformation. Where \( \otimes \)-multiplicative means that
\[
\sigma_{\otimes Y} = (X \otimes \sigma_Y)(\sigma_X \otimes Y),
\]
for arbitrary \( X \) and \( Y \) in \( \mathcal{C} \). The adjective lax is dropped when \( \sigma \) is an isomorphism. A braiding is a \( \otimes \)-multiplicative choice of a half braiding for every object in \( \mathcal{C} \).

A lax central bimonoid is simultaneously a lax central comonoid and monoid such that the multiplication and unit are comonoid homomorphisms. Explicitly, for a lax central bimonoid \( A = (A, \sigma, \delta, \mu, \varepsilon, \eta) \):
\( (A, \sigma) \) is a lax half-braiding, \( (A, \delta, \varepsilon) \) is a comonoid, \( (A, \mu, \eta) \) is a monoid, and they satisfy
\[
(\mu \otimes \mu)(A \otimes \sigma_A \otimes A)(\delta \otimes \delta) = \delta \mu, \varepsilon \otimes \varepsilon = \varepsilon \mu, \eta \otimes \eta = \delta \eta
\]
and \( \varepsilon \eta = 1 \). Every lax central bimonoid defines a bimonad given by the functor \( A \otimes ? \), see \[2, Lemma 5.6\].

A lax central bimonoid \( H = (H, \sigma, \delta, \mu, \varepsilon, \eta) \) is called lax Hopf monoid if the fusion morphism \( (H \otimes \mu)(\delta \otimes H) \) is an isomorphism. By the the relation between convolution and fusion morphism, a lax Hopf monoid has an antipode \( H \xrightarrow{s} H \) such that
\[
\mu(s \otimes H)\delta = \eta \varepsilon = \mu(H \otimes s)\delta.
\]
The adjective lax is dropped when both \( \sigma \) and \( s \) are isomorphisms.

Since this paper is about augmented monads is very significant for us that in the Hopf case every augmented monad is representable.

**Theorem 2.7.** \[2, Corollary 5.18\] There is an equivalence between the categories of augmented left Hopf monads and lax central Hopf monoids, which restricts to an equivalence between augmented Hopf monads and central Hopf monoids.

### 2.3. Galois connections induced by functorial relations.

This is slight generalization of the Galois connection induced by a relation relationship on sets. It is completely elementary and should be folklore, but the author didn’t find any appropriate reference and decided to include brief but self contained exposition of this.

**Definition 2.8.** We call functorial relation between the categories \( \mathcal{X} \) and \( \mathcal{Y} \) to a functor \( R : \mathcal{X} \times \mathcal{Y} \rightarrow 2 \), where \( 2 = \{ \text{true} \rightarrow \text{false} \} \) is thought as the truth values of the relation. In other words, for objects
and $Y$ in $X$ and $\mathcal{Y}$, respectively, we say that $X$ is related to $Y$ when $R(X,Y) = \text{true}$.

Notice that asserting that $R$ is a functor simply means that for any pair of morphisms, $X' \xrightarrow{f} X$ and $Y' \xrightarrow{g} Y$, if $X$ is related to $Y$ then $X'$ is related $Y'$.

**Definition 2.9.** The $R$-representant for an object $X$ in $X$ is an object, denoted $R^*X$, such that for every $Y$ in $\mathcal{Y}$, $X$ is related to $Y$ if and only if there is an unique morphism $Y \rightarrow R^*X$. The $R$-representant for a $Y$ in $\mathcal{Y}$, denoted $R_*Y$ is analogously defined.

The next definition generalizes the case of a Galois connection between partially ordered sets, to adjunctions with a pre-order as its core. Remember that a pre-order is a category with at most one morphism among any pair of objects.

**Definition 2.10.** A Galois connection, between the categories $X$ and $\mathcal{Y}$, is a contravariant adjunction $G : \mathcal{Y} \rightarrow X^{\text{op}} \dashv F : X^{\text{op}} \rightarrow \mathcal{Y}$ such that $\text{Hom}_\mathcal{Y}(Y, F X) \approx \text{Hom}_X(X, G Y)$ has at most one element, for arbitrary $X$ and $Y$.

**Remark 2.11.** Let $G : \mathcal{Y} \rightarrow X^{\text{op}} \dashv F : X^{\text{op}} \rightarrow \mathcal{Y}$ be a Galois connection.

i. For any $X$ in $X$, there is a unique morphism $X \rightarrow GFX$. We say that $X$ is Galois closed if this is an isomorphism. The Galois closed objects in $X$ form a pre-order.

ii. Since evaluating $G$ at $Y \rightarrow FGY$ gives a morphism $GFGY \rightarrow GY$, for any $Y$ in $\mathcal{Y}$, $GY$ is Galois closed in $X$. Analogously, for any $X$ in $X$, $FX$ is closed in $\mathcal{Y}$.

iii. In conclusion, $F$ and $G$ induce an order-reversing equivalence between Galois closed objects. This is known as the induced Galois correspondance.

Now we are able to define the Galois connection induced by $R$.

**Proposition 2.12.** Let $R$ be a functorial relationship between $X$ and $Y$.

i. If every object in $X$ has a representant, then mapping $X \mapsto R^*X$ defines a functor $R^* : X^{\text{op}} \rightarrow \mathcal{Y}$.

Analogously, when every object in $\mathcal{Y}$ has a representant there is $R_* : \mathcal{Y} \rightarrow X^{\text{op}}$.

ii. If both $R^*$ and $R_*$ are defined, then they form a Galois connection.

*Proof.* Given a morphism $X' \xrightarrow{f} X$ in $X$, since $R$ is functorial and $X$ is related to $R^*X$, then $X'$ is related to $R^*X$; hence there is an unique $R^*X \rightarrow R^*X'$.
The adjunction plainly recalls that the existence of maps $Y \to R^*X$ and $X \to R_*Y$ are both equivalent to $X$ is related to $Y$. \qed

3. INVARIANTS AND STABILIZERS FOR AUGMENTED MONADS

This section contains the main result of this paper. It states that in an augmented monad it is possible to define invariants and stabilizers, in a way that naturally establish a Galois correspondence when they exist.

**Definition 3.1.** Let $T = (T, \mu, \eta)$ be a monad over $C$. An augmentation of $T$ is a monad homomorphism $e : T \to \text{Id}_C$. This is a natural transformation which satisfies $e \cdot \eta = \text{Id}_C$ and $e \cdot \mu = e \star e$. A monad endowed with an augmentation is called augmented monad.

The intuition for an augmentation, is that the morphism $TX \xrightarrow{e_X} X$ is the trivial action for any $X$ in $C$; the action which forgets $T$ and leaves $X$ fixed. During this whole section we will work in an augmented monad $T = (T, \mu, \eta, e)$ over a category $C$ and denote by $(\eta, F \dashv U, \varepsilon)$ the Eilenberg-Moore decomposition of $T$; where $U : C^T \to C$ is the forgetful functor from the category of $T$-actions, and its left adjoint $F$ sends every object in $C$ to the free action on it. See definition 2.2 for the details.

3.1. **The fix relation.** Let $\text{Monad}_C$ be the category of monads over $C$. For brevity, we denote by $S_h$ a monad homomorphism $h : S \to T$ considered as object of $\text{Monad}_C/T$ and by $V_\alpha$ a natural transformation $\alpha : V \to U$ from a functor $V : C^T \to C$ considered as object of $\text{Fun}(C^T, C)/U$.

Since the augmentation provides us with a notion of trivial action, we can compare any action with the trivial one over the same object. Moreover, we can restrict this comparison to a part of the monad or to a part of the object. The next definition does both restrictions simultaneously and then performs the comparison.

**Definition 3.2.** We say that $S_h$ fixes $V_\alpha$ when $(U \varepsilon) \cdot (h \star \alpha) = (eU) \cdot (h \star \alpha)$.

Let’s expand the above definition for a given $T$-action $M = (X, TX \xrightarrow{r_M} X)$:

Since $X = UM$ and the action morphism $r_M$ equals $U \varepsilon_M$, then the left hand side is obtained by composing $r_M$ with $(h \star \alpha)_M = (T_\alpha M) h_{VM} = h_X(S_\alpha M)$.

On the other hand, $(eU)_M = e_X$ is the trivial action; therefore, when $S_h$ fixes $V_\alpha$ the restriction of the trivial action equals the restricted
action, this is
\[ r_M h_X(S\alpha_M) = e_X h_X(S\alpha_M). \]

**Remark 3.3.** By Corollary 2.5, every monad homomorphism \( h : S \Rightarrow T \) corresponds to a functor \( H : C^T \rightarrow C^S \) such that \( U^S H = U \), where \( U^S : C^S \rightarrow C \) is the forgetful functor from the category of \( S \)-actions. In particular, the augmentation \( e : T \Rightarrow \text{Id}_C \) corresponds to a functor \( E : C \rightarrow C^T \) which is a section of the forgetful functor, i.e. \( UE = \text{Id}_C \).

Recalling the definitions, any \( X \) in \( C \) gets mapped to \( EX = (X, TX \xrightarrow{e_X} X) \), and any \( M = (X, TX \xrightarrow{r_M} X) \) in \( C^T \) gets mapped to \( HM = (X, SX \xrightarrow{r_M h_X} X) \).

The next lemma allow us to reinterpret the definition of acting trivially in terms of the functors defined in the above remark.

**Lemma 3.4.** The homomorphism \( h : S \Rightarrow T \) fixes \( \alpha : V \Rightarrow U \) if and only if there is a natural transformation \( \hat{\alpha} : HEV \Rightarrow H \) such that \( U^S \hat{\alpha} = \alpha \).

**Proof.** First notice that \( H \) and \( HEV \) are lifts of \( U \) and \( V \), respectively, along the monads \( \text{Id}_{C^T} \) and \( S \).

\[
\begin{array}{ccc}
C^T & \xrightarrow{HEV} & C^S \\
\downarrow{\alpha} & \circlearrowright H & \downarrow{U^S} \\
C^T & \xrightarrow{V} & C \\
\downarrow{\alpha} & \circlearrowright U & \\
\text{Id}_{C^T} & \xrightarrow{U} & C
\end{array}
\]

The lifting data for \( H \) over \( U \) is \( (U\varepsilon) \cdot (hU) \) and for \( HEV \) over \( V \) is \( (e \cdot h)V \).

As an application of Proposition 2.6, there is a lift \( \hat{\alpha} \) if and only if \( \alpha \cdot ((e \cdot h)V) = (eU) \cdot (h \ast \alpha) \) equals \( (U\varepsilon) \cdot (hU) \cdot (S\alpha) = (U\varepsilon) \cdot (h \ast \alpha) \); which is the definition of \( S_h \) fixes \( V_\alpha \). \( \square \)

Now we will check that the fix relationship is preserved whenever we restrict further. In other words that fixing defines a functorial relation between \( \text{Monad}_{C/T} \) and \( \text{Fun}(C^T, C)/U \).

Afterwards, we will apply Definition 2.9 and Proposition 2.12, to this relation, in order to define a Galois connection.

**Proposition 3.5.** Let \( g : S' \Rightarrow S \) be any monad homomorphism and \( \beta : V' \Rightarrow V \) be any natural transformation. If \( h : S \Rightarrow T \) fixes \( \alpha : V \Rightarrow U \), then \( h \cdot g : S' \Rightarrow T \) fixes \( \alpha \cdot \beta : V' \Rightarrow U \).
Proof. This is obvious from the facts that \((h \cdot g) \star (\alpha \cdot \beta) = (h \star \alpha) \cdot (g \star \beta)\) and the definition of fixing \((U \varepsilon) \cdot (h \star \alpha) = (eU) \cdot (h \star \alpha)\). □

Definition 3.6. Let \(h : S \Rightarrow T\) be a monad homomorphism and \(\alpha : V \Rightarrow U\) a natural transformation, where \(U : C^T \rightarrow C\) is the forgetful functor.

(a). We say that \(h\) is the stabilizer of \(\alpha\) (or \(S\) is the stabilizer of \(V\)) if:

it fixes \(\alpha\) and for every monad homomorphism \(h' : S' \Rightarrow T\) that also fixes \(\alpha\) there is an unique \(g : S' \Rightarrow S\) such that \(h' = h \cdot g\).

(b). We say that \(\alpha\) is the \(h\)-invariant inclusion (or \(V\) is the functor of \(S\)-invariants) if:

it is fixed by \(h\) and for every functor \(\alpha' : V' \rightarrow U\) also fixed by \(h\) there is an unique \(\beta : V' \rightarrow V\) such that \(\alpha' = \alpha \cdot \beta\).

The next corollary is a straightforward application of Proposition 2.12, to the fixing relationship.

Corollary 3.7. Let \(\mathcal{M}\) and \(\mathcal{N}\) be subcategories of \(\text{Monad}(C)/T\) and \(\text{Fun}(C^T, C)/U\), respectively.

i. If every \(h\) in \(\mathcal{M}\) has an \(h\)-invariant inclusion in \(\mathcal{N}\), then the mapping \(h \mapsto \text{Inv}_h\) defines a functor \(\text{Inv} : \mathcal{M}^{\text{op}} \rightarrow \mathcal{N}\). Analogously, if every \(\alpha\) in \(\mathcal{N}\) has a stabilizer in \(\mathcal{M}\), then the mapping \(\alpha \mapsto \text{Stab}_\alpha\) defines a functor \(\text{Stab} : \mathcal{N} \rightarrow \mathcal{M}^{\text{op}}\).

ii. If both \(\text{Stab}\) and \(\text{Inv}\) are defined, then they form a Galois connection.

4. Computing invariants via right adjoints

Now that we have an abstract definition of invariants and stabilizers, we face the concrete — or slightly less abstract — questions of determining when they exist and how can we compute it. This segment offers an answer for the case of invariants, by linking them to right adjoints. Remember that \((T, \mu, \eta, e)\) is an augmented monad over \(C\).

First we observe that when the domain monad has a right adjoint we compute invariants as an equalizer.

Lemma 4.1. Let \(h : S \Rightarrow T\) be a monad homomorphism and assume that the monad \(S : C \rightarrow C\) has a right adjoint, \((\theta, S \dashv R, v)\).

For any \(\alpha : V \Rightarrow U\), \(S_h\) fixes \(V_\alpha\) if and only if for each \(M = (X, r_M)\) in \(C^T\), \((R(eh)_X)(\theta_X)\alpha_M = (R(r_Mh_X))(\theta_X)\alpha_M\).

In consequence, when the pair \((R(eh)_X)(\theta_X)\) and \((R(r_Mh_X))(\theta_X)\) has an equalizer for every \(M\), this equalizer defines the functor of \(S_h\)-invariants.
Proof. Just recall that $S_h$ fixes $V_{α}$ when $r_{M}h_{X}(Sα_{M}) = e_{X}h_{X}(Sα_{M})$ for every $M$, and bend $S$ to $R$ in both sides of the equation. In other words,
\[
(R((Uε) \cdot (hU))) \cdot (θU) \cdot α = (R(e \cdot h)U) \cdot (θU) \cdot α.
\]
Since equalizers are funtorial, the last sentence is evident. □

The next example illustrate the above lemma in the case of a $⊗$-representable monad.

Example 4.2. Let $T = A ⊗ ?$ for an augmented monoid
\[
A = (A, A ⊗ A \xrightarrow{α} A, 1 \xrightarrow{1} A, A \xrightarrow{0})
\]
in a monoidal category $(C, ⊗, 1)$. Assume that $A$ is right closed in $C$ and that $C$ has equalizers. The functor of $A$-invariants $Inv A := InvT$ sends each action $M = (X, A ⊗ X \xrightarrow{r_{M}} X)$ to the equalizer of
\[
X \xrightarrow{(r_{M})_{ε⊗X}} [A, X]^r.
\]
Moreover, given a monoid homomorphism $B \xrightarrow{h} A$, with $B$ also right closed in $C$.

The functor of $B$-invariants $Inv B := Inv(B ⊗ ?)_{h⊗?}$ sends each action to the equalizer of
\[
X \xrightarrow{(r_{M})_{ε⊗X}} [A, X]^r \xrightarrow{[h,X]^r} [B, X]^r.
\]

4.1. Invariants are right adjoints. Right adjoints not only allow us to compute invariants. In fact, the $T$-invariants, i.e the invariants for $id_{T} : T \Rightarrow T$, are right adjoints of the trivial action functor $E$. Where $E : C \rightarrow C^{T}$ is the functor corresponding to the augmentation $e : T \Rightarrow id_{C}$, remember that it endows every object in $C$ with the trivial action given by the augmentation.

Lemma 4.3. If $E$ has a right adjoint functor $Γ : C^{T} \rightarrow C$, then $Γ$ is the functor of $T$-invariants.

Proof. Take the adjunction $(ζ, E \rightleftarrows Γ, ℓ)$ and define $γ : Γ \Rightarrow U$ as $γ = Uℓ$. Hence by Lemma 3.4, $T$ fixes $Γ_{γ}$. Moreover by the same lemma, if $T$ fixes $α : V \Rightarrow U$ there is $α : EV \Rightarrow id_{C^{r}}$ such that $α = Uα$. We need a $β : V \Rightarrow Γ$ such that $α = γ \cdot β = (Uℓ) \cdot (UEβ)$ since $U$ is faithful $α = γ \cdot (Eβ)$, after bending the $E$ we obtain that $(Γα) \cdot (V) = β$.

Since $E = (ℓE) \cdot (Eζ)$ then
\[
α = α \cdot (ℓE) \cdot (EζV) = γ \cdot (ETα) \cdot (EζV).
\]
Therefore \( \alpha = U\bar{\alpha} = \gamma \cdot (\Gamma\bar{\alpha}) \cdot (\zeta V). \)

Right adjoints also allow to change the category where one is calculating. Let \( h : S \Rightarrow T \) be a monad homomorphism, with correspondent functor \( H : C^T \rightarrow C^S \) as in Remark 3.3; in particular \( U = U^S H \).

Notice that \((S, \mu', \eta', eh)\) is also an augmented monad and instead of computing \( h \)-invariants in \( C^T \), we can try to compute \( S \)-invariants in \( C^S \) and pull them back. The next two lemmas show that this strategy is successful when \( H \) has a right adjoint.

**Lemma 4.4.** Let \((\zeta, H \dashv K, \omega)\) be an adjunction with \( H : C^T \rightarrow C^S \), coming from a monad homomorphism \( h : S \Rightarrow T \). For any \( \alpha : V \Rightarrow U \), \( S_h \) fixes \( V_\alpha \) is equivalent to \( S \) fixes \( \hat{\alpha} : V K \Rightarrow U S \), where \( \hat{\alpha} = (U^S \omega) \cdot (\alpha K) \).

**Proof.** We will use Lemma 3.4. If \( S_h \) fixes \( V_\alpha \), there is \( \bar{\alpha} : HEV \Rightarrow H \) such that \( \alpha = U^S \bar{\alpha} \). Therefore

\[
\hat{\alpha} = (U^S \omega) \cdot (\alpha K) = U^S(\omega \cdot (\alpha K)).
\]

Hence, by the same lemma \( S \) fixes \( (V K)_{\hat{\alpha}} \).

For the reciprocal, notice that \( \alpha = (\hat{\alpha} H) \cdot (V \zeta) \) by bending \( K \) to \( H \).

Now the proof is completely analogous:

If \( S \) fixes \( \hat{\alpha} \), there is \( \bar{\alpha} : HEVK \Rightarrow \text{Id}_{C^S} \) such that \( \hat{\alpha} = U^S \bar{\alpha} \). Therefore

\[
\alpha = (\hat{\alpha} H) \cdot (V \zeta) = U^S((\hat{\alpha} H) \cdot (HEV \zeta)),
\]

since \( U^S HE = UE = \text{Id}_C \). The same lemma tells us that \( S_h \) fixes \( V_\alpha \). \( \square \)

**Corollary 4.5.** Let \( h : S \Rightarrow T \) be a monad homomorphism and suppose that \( H : C^T \rightarrow C^S \) has a right adjoint. If \( \gamma^S : \Gamma^S \Rightarrow U^S \) is the \( S \)-invariant inclusion, then \( \gamma^S H : \Gamma^S H \Rightarrow U \) is the \( h \)-invariant inclusion.

**Proof.** With the notations from the previous lemma, \( S_h \) fixes \( V_\alpha \) if and only if \( S \) fixes \( \hat{\alpha} \). Since \( \gamma^S \) is the \( S \)-invariant inclusion, there is an unique \( \hat{\beta} : \Gamma^S \Rightarrow VK \) such that \( \hat{\alpha} = \gamma^S \cdot \hat{\beta} \). Remembering that \( \alpha = (\hat{\alpha} H) \cdot (V \omega) \) we obtain \( \alpha = ((\gamma^S \cdot \hat{\beta}) H) \cdot (V \omega) = (\gamma^S H) \cdot \beta \) where \( \beta = (\hat{\beta} H) \cdot (V \omega) \). \( \square \)

We end this segment with a summary of the computation of invariants by means of right adjoints.

**Corollary 4.6.** Let \((T, \mu, \eta, e)\) be an augmented monad, and \( h : S \Rightarrow T \) a monad homomorphism. Take \( E : C \rightarrow C^T \) and \( H : C^S \rightarrow C^T \) the functors corresponding to \( e \) and \( h \), respectively; and assume that they are part of adjunctions \((\zeta, E \dashv \Gamma, \bar{\gamma})\) and \((\zeta, H \dashv K, \omega)\).
Then \( \text{Inv} T = \Gamma \gamma \) with \( \gamma = U \gamma \), \( \text{Inv} S = \Gamma K \hat{\gamma} \) with \( \hat{\gamma} = (U^S \omega) \cdot (U \gamma K) \), and \( \text{Inv} S_h = \Gamma KH \hat{\gamma}_H \) with \( \hat{\gamma}_H = (U^S \omega H) \cdot (U \gamma KH) \).

Proof. This is Lemma 4.3 applied twice to \( E \vdash \Gamma \) and to \( HE \vdash \Gamma K \), and Corollary 4.5. Notice that when \( H \) has also a left adjoint \( L \) then \( LHE \vdash \Gamma KH \).

\[ \square \]

5. Computing stabilizers in monoidal closed categories

Currently we don’t know of any procedure for explicitly describing the stabilizers for augmented monads in categories without additional structure. We imagine a description in terms of Kan extensions, but we have not succeeded in formalizing it. However, in the case of a \( \otimes \)-representable monad over a monoidal closed category, we can use ends as a tool for the computation of stabilizers. This is a procedure inspired by Tannakian reconstruction, encompass augmented Hopf monads in monoidal closed categories.

5.1. The end enrichment of functor categories. For this section, we assume familiarity with the basic properties of ends, as presented for instance in [5, Sections IX. 5-8]. For fixing notation, we briefly recall the definition.

**Definition 5.1.** A wedge for a functor \( S : \mathcal{D}^{op} \times \mathcal{D} \to \mathcal{C} \) is a pair \( (W, \xi_X)_{X \in \mathcal{D}} \) consisting of an object \( W \) in \( \mathcal{C} \) and a family of morphisms \( W \xrightarrow{\xi_X} S(X, X) \) indexed by the objects in \( \mathcal{C} \) such that for any morphism \( X \xrightarrow{f} Y \) in \( \mathcal{D} \) the diagram

\[
\begin{array}{ccc}
W & \xrightarrow{\xi_X} & S(X, X) \\
\downarrow{\xi_Y} & & \downarrow{\xi_Y} \\
S(Y, Y) & \xrightarrow{S(f, f)} & S(X, Y)
\end{array}
\]

commutes.

The end of \( S \) denoted

\[
\int_{\mathcal{D}} S = \int_{X \in \mathcal{D}} S(X, X) = (L, \lambda_X)_{X \in \mathcal{D}}
\]

is the universal wedge. This means that for any other wedge, \( (W, \xi_X)_{X \in \mathcal{D}} \) there is an unique morphism, denoted \( W \xrightarrow{f_X \xi_X} L \), such that \( \lambda_Y \int_X \xi_X = \xi_Y \) for any \( Y \) in \( \mathcal{C} \).

The next definition allow us to view the natural transformations as an object in the co-domain category.
Definition 5.2. Let $V$ and $W$ be two functors from an arbitrary category $\mathcal{D}$ to a monoidal category $(\mathcal{C}, \otimes, 1)$. Assume that $\mathcal{C}$ is left closed over $V$, i.e. for every $D$ in $\mathcal{D}$ the functor $? \otimes VD : \mathcal{C} \to \mathcal{C}$ has a right adjoint $[VD, ?]^\ell$.

The left internal Nat from $V$ to $W$ is defined as the end of the functor $[V?, W?]_\ell : \mathcal{D}^{op} \times \mathcal{D} \to \mathcal{C}$, and denote it by $[V, W]_\ell := \int_{D \in \mathcal{D}} [V D, W D]_\ell = ([V, W]^\ell, [V, W]^\ell \lambda_D, [VD, WD]^\ell)_{D \in \mathcal{D}}$.

Remark 5.3. If the internal Nat $[V, W]^\ell$ exists for any pair of functors, — for instance when $\mathcal{C}$ is complete, left closed and $\mathcal{D}$ is small — then $\text{Fun}(\mathcal{D}, \mathcal{C})$ is a left $\mathcal{C}$-enriched category. The next two propositions are consequences of this fact.

Proposition 5.4. Let $V$, $V'$, $W$ and $W'$ be functors from $\mathcal{D}$ to $\mathcal{C}$ with $\omega : W \Rightarrow W'$ and $\upsilon : V' \Rightarrow V$ natural transformations.

- If both internal Nats $[V, W]^\ell$ and $[V, W']^\ell$ exist, then there is a morphism $[V, W]^\ell \xrightarrow{[V, \omega]^\ell} [V, W']^\ell$.

- Analogously, when $[V, W]^\ell$ and $[V', W]^\ell$ exist, there is $[V, W]^\ell \xrightarrow{[\upsilon, W]^\ell} [V', W]^\ell$.

- Whenever the four internal Nats in the following diagram exist, the arrows also exist and the diagram is commutative.

\[
\begin{array}{ccc}
[V, W]^\ell & \xrightarrow{[V, \omega]^\ell} & [V, W']^\ell \\
\downarrow_{[V, W]^\ell} & & \downarrow_{[\upsilon, W']^\ell} \\
[V', W]^\ell & \xrightarrow{[\upsilon, \omega]^\ell} & [V', W']^\ell
\end{array}
\]

Proof. In the first case, for any $D$ in $\mathcal{D}$ the composition of $[V, W]^\ell \xrightarrow{\lambda_D} [VD, WD]^\ell$ and $[VD, WD]^\ell \xrightarrow{[VD, \omega_D]^\ell} [VD, W'D]^\ell$ defines a wedge for the functor $[V?, W'?]^\ell$. Define $[V, \omega]^\ell := \int_{D \in \mathcal{D}} [VD, \omega_D]^\ell \lambda_D$. □

Proposition 5.5. Take $W : \mathcal{D} \to \mathcal{C}$ such that the internal Nat $[W, W]^\ell$ exist. There is a canonical monoid structure on $[W, W]^\ell$ such that for any $D$ in $\mathcal{D}$ the morphism $[W, W]^\ell \xrightarrow{\lambda_D} [WD, WD]^\ell$ is a monoid homomorphism.
Proof. Remember that for any \( D \in \mathcal{D} \), \([WD, WD]^{\ell}\) is a monoid denoted End \([WD]^{\ell}\). The units \((1 \overset{\text{id}_{WD}}{\longrightarrow} WD)_{D \in \mathcal{D}}\) make a wedge, therefore take \(\eta = \int_{D \in \mathcal{D}} \text{id}_{WD}^{\ell}\) as unit for \([W, W]^{\ell}\).

For the multiplication, compose \([W, W]^{\ell} \otimes [W, W]^{\ell}\) with \(\text{End}[WD]^{\ell} \otimes \text{End}[WD]^{\ell} \overset{\mu_D}{\longrightarrow} \text{End}[WD]^{\ell}\), verify that this conforms a wedge and define \(\mu = \int_{D \in \mathcal{D}} \mu_D(\lambda_D \otimes \lambda_D)\). Finally, the monoid axioms are verified component-wise. \(\square\)

Following the usual terminology we will denote \([W, W]^{\ell}\) by \(\text{End}[W]^{\ell}\) and call it the left internal End of the functor \(W\). The next proposition formalizes the idea of restricting the endomorphisms of \(W\) along a functor \(G : \mathcal{D}' \rightarrow \mathcal{D}\).

**Proposition 5.6.** Take \(W : \mathcal{D} \rightarrow \mathcal{C}\) and \(G : \mathcal{D}' \rightarrow \mathcal{D}\) such that \(\text{End}[W]^{\ell}\) and \(\text{End}[WG]^{\ell}\) both exist, then there is a canonical monoid homomorphism \(\text{End}[W]^{\ell} \overset{|G|}{\longrightarrow} \text{End}[WG]^{\ell}\).

**Proof.** The map \(|G|\) is defined by \(|G := \int_{X \in \mathcal{D}'} \lambda_{GX}\) when \(\text{End}[W]^{\ell} = (\text{End}[W]^{\ell}, \lambda_D)_{D \in \mathcal{D}}\). \(\square\)

**Lemma 5.7.** Take \(W : \mathcal{D} \rightarrow \mathcal{C}\) such that \(\text{End}[W]^{\ell}\) exists. There is a unique functor \(\overline{W} : \mathcal{D} \rightarrow \mathcal{C}^{\text{End}[W]^{\ell}}\) which lifts \(W\) to the category of actions of \(\text{End}[W]^{\ell}\). This means that \(W = U\overline{W}\) where \(U : \mathcal{C}^{\text{End}[W]^{\ell}} \rightarrow \mathcal{C}\) is the forgetful functor.

**Proof.** For \(D\) in \(\mathcal{D}\) define \(\overline{WD} = (WD, \downarrow \lambda_D, \uparrow)\) where \(\text{End}[W]^{\ell} \overset{\lambda_D}{\longrightarrow} [WD, WD]^{\ell}\) and \(\text{End}[W]^{\ell} \otimes WD \overset{\lambda_D}{\longrightarrow} WD\) is obtained by un-currying. \(\square\)

### 5.2. Tannakian reconstruction of stabilizers.

The above construction of internal Nats gives us a path for reconstruction. Let

Let \((\widetilde{\mathcal{C}}, \otimes, \mathbb{1})\) be a monoidal category, with all small limits and left closed over a small subcategory \(\mathcal{C} \hookrightarrow \widetilde{\mathcal{C}}\), i.e. for every \(X\) in \(\mathcal{C}\) the functor \(? \otimes X : \widetilde{\mathcal{C}} \rightarrow \widetilde{\mathcal{C}}\) has a right adjoint \([X, ?]^{\ell}\). In this case, we denote by \(\text{End}[^{\ell}]\) the internal End for the inclusion functor \(\mathcal{C} \hookrightarrow \widetilde{\mathcal{C}}\) which exists in \(\widetilde{\mathcal{C}}\). Moreover, for a small category \(D\) every pair of functors \(V, W : \mathcal{D} \rightarrow \mathcal{C}\), after composing with the inclusion \(\mathcal{C} \hookrightarrow \widetilde{\mathcal{C}}\) has an internal Nat in \(\widetilde{\mathcal{C}}\), which abusing a little of notation we call \([V, W]^{\ell}\).

From a more general perspective, the next proposition defines a unit for the Tannakian reconstruction adjunction.

---

1. This uses that the compositions are associative and that \(\lambda_D\) is a wedge.
Proposition 5.8. Let \((A, m, u)\) be a monoid in \(\bar{C}, \overline{C^A} \hookrightarrow \bar{C^A}\) the subcategory of \(A\)-actions on \(C\) and \(U : C^A \to C\) the forgetful functor. There is a monoid homomorphism \(A \overset{\rho}{\to} \text{End} [U]^{\ell}\) such that the functors \(\overline{U} : C^A \to \bar{C}^\text{End}[U]^{\ell}\), obtained by 5.7, and \(K : \bar{C}^\text{End}[U]^{\ell} \to \bar{C^A}\), corresponding to the monad homomorphism \(\rho \otimes \_\), compose to the inclusion \(\bar{C^A} \hookrightarrow \bar{C^A}\).

Proof. For every \(A\)-action \(M = (X, (A \otimes X \overset{r_M}{\to} X))\) curry the action. Notice that for any \(A\)-action morphism to obtain \(A \overset{\rho}{\to} \text{End} [X, X]^{\ell}\) homomorphism \(M \overset{\eta}{\to} N\) the diagram

\[
\begin{array}{ccc}
A & \overset{\rho}{\to} & [UM, UM]^{\ell} \\
\downarrow{r_N^{\gamma}} & & \downarrow{[UM, f]^\ell} \\
[UN, UM]^{\ell} & \overset{[f, UN]^{\ell}}{\to} & [UN, UM]^{\ell}
\end{array}
\]

commutes because \(\gamma f r_M^{\gamma} = [X, f]^{\ell} r_M^{\gamma}\) and \(\gamma r_N^{\gamma} (Z \otimes f)^{\gamma} = [f, Y]^{\ell} r_N^{\gamma}\).

Therefore \((A, \overset{\rho}{\to} \text{End} [U]^{\ell})_{M \in C^A}\) is a wedge, and we obtain a morphism \(A \overset{\rho}{\to} \text{End} [U]^{\ell}\) such that \(\rho \gamma = \lambda_M^{\rho}\). Since \(\overline{U}M = (X, \_\lambda_M\_)\) then \(K \overline{U}M = (X, \_\lambda_M\_ (\rho \otimes X)), \) and \(\_\lambda_M\_ (\rho \otimes X) = \_\lambda_M\_ \rho = r_M\). \(\square\)

Now we turn to the question of internal Ends which fix a natural transformation.

Proposition 5.9. Let \((A, m, u, \epsilon)\) be an augmented monoid in \(\bar{C}\). Then we have a diagram of monoid homomorphisms

\[
\begin{array}{ccc}
A & \overset{\rho}{\to} & \text{End} [U]^{\ell} \\
\downarrow{\epsilon} & & \downarrow{|E|} \\
1 & \overset{|U|}{\to} & \text{End} [C]^{\ell},
\end{array}
\]

where \(|E| ) (|U|) = \text{id}_{\text{End} [C]^{\ell}}\) and \((|U|) \eta = \int_{M \in C^A} \gamma \epsilon \otimes U M^{\gamma}\).

Proof. Notice that the functor \(E : C \to C^\Upsilon\), given by \(EX = (X, A \otimes X \overset{\epsilon \otimes X}{\to} X)\) is a section for \(U\). The morphisms \(|E|\) and \(|U|\) are given by Proposition 5.6, and \((|U|)(|E|) = (|UE|) = \text{id}_{\text{End} [C]^{\ell}}\). Let’s check that \((|E|) \rho = \eta:\)

For any \(X \in C\), consider the universal wedge components \(\text{End} [C]^{\ell} \overset{\xi_X}{\to} [X, X]^{\ell}\) and \(\text{End} [U]^{\ell} \overset{\lambda_M^{\gamma}}{\to} [UM, UM]^{\ell}\). On one hand \(\xi_X (|E|) \rho = \lambda_{EX} \rho = \gamma r_{EX}^{\gamma} = \gamma \epsilon \otimes X^{\gamma}\) since \(\rho = \int_M \gamma r_M^{\gamma}\). On the other hand \(\xi_X \eta = \gamma \text{id}_X^{\gamma} \epsilon = \gamma \epsilon \otimes X^{\gamma}\) since \(\eta = \int_X \gamma \text{id}_X^{\gamma} \). \(\square\)
Lemma 5.10. Take a monoid homomorphism $B \xrightarrow{h} A$ in $\mathcal{C}$ and a natural transformation $\alpha : V \Rightarrow U$, then $(B \otimes ?) \otimes ?$ fixes $V_\alpha$ if and only if $h$ equalizes the diagram $A \xrightarrow{\rho}{\text{End}} [U]^\ell \xrightarrow{[\alpha, U]^\ell}{[V, U]^\ell}$.

**Proof.** For any $M = (X, r_M)$ in $\mathcal{C}^T$, consider the universal wedge components $\lambda_M$ and $\xi_M$ displayed in the next diagram:

\[
\begin{array}{ccc}
B & \xrightarrow{h} & A \\
& \xrightarrow{\rho} & \xrightarrow{\lambda_M} \text{End} [U]^\ell \xrightarrow{[\alpha, U]^\ell} [UM, UM]^\ell \\
& & \downarrow{[\alpha_M, UM]^\ell} \\
& & [V, U]^\ell \xrightarrow{\xi_M} [VM, UM]^\ell
\end{array}
\]

On one hand, since $\rho = \int_M \gamma r_M \gamma$ we obtain
\[
\xi_M[\alpha, U]^\ell \rho h = [\alpha_M, UM]^\ell \lambda_M \rho h = [\alpha_M, UM]^\ell \gamma r_M \gamma h
\]

On the other hand, since $(|U|)\eta e = \int_{M \in \mathcal{C}^A} \gamma e \otimes UM \gamma$, we obtain
\[
\xi_M[\alpha, U]^\ell (|U|)\eta e h = [\alpha_M, UM]^\ell \lambda_M (|U|)\eta e h = [\alpha_M, UM]^\ell \gamma e \otimes X \gamma h
\]

The result follows because uncurrying the last term in each equation gives the definition of the fix relation. □

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**Departamento de Matemáticas, Universidad de los Andes, Bogotá, Colombia.**

*Email address:* jf.garcia14@uniandes.edu.co