Small Uncolored and Colored Choice Dictionaries

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Abstract. A choice dictionary is a data structure that can be initialized with a parameter \( n \in \mathbb{N} = \{1, 2, \ldots\} \) and subsequently maintains an initially empty subset \( S \) of \( \{1, \ldots, n\} \) under insertion, deletion, membership queries and an operation choice that returns an arbitrary element of \( S \). The choice dictionary is fundamental in space-efficient computing and has numerous applications. The best previous choice dictionary can be initialized with \( n \) and a second parameter \( t \in \mathbb{N} \) in constant time and subsequently executes all operations in \( O(t) \) time and occupies \( n + O(n(t/w)^2 + \log n) \) bits on a word RAM with a word length of \( w = \Omega(\log n) \) bits. We describe a new choice dictionary that, following a constant-time initialization, executes all operations in constant time and, in addition to the space needed to store the integer \( n \), occupies only \( n + 1 \) bits, which is shown to be optimal if \( w = o(n) \).

A generalization of the choice dictionary called a colored choice dictionary is initialized with a second parameter \( c \in \mathbb{N} \) in addition to \( n \) and subsequently maintains a semipartition \((S_0, \ldots, S_{c-1})\) of \( U = \{1, \ldots, n\} \), i.e., a sequence of \( c \) (possibly empty) disjoint subsets of \( U \) whose union is \( U \), under the operations setcolor\((j, \ell)\) \((j \in \{0, \ldots, c-1\} \text{ and } \ell \in U)\), which moves \( \ell \) from its current subset to \( S_j \), color\((\ell)\) \((\ell \in U)\), which returns the unique \( j \in \{0, \ldots, c-1\} \) with \( \ell \in S_j \), and choice\((j)\) \((j \in \{0, \ldots, c-1\})\), which returns an arbitrary element of \( S_j \). We describe new colored choice dictionaries that, if initialized with constant \( c \), execute setcolor, color and choice in constant time and occupy \( n \log_2 c + 1 \) bits plus the space needed to store \( n \) if \( c \) is a power of 2, and at most \( n \log_2 c + n' \) bits in general, for arbitrary fixed \( \epsilon > 0 \). We also study the possibility of iterating over the set \( S \) or over \( S_j \) for given \( j \in \{0, \ldots, c-1\} \). This allows us to derive new results for space-efficient breadth-first search (BFS). On a directed or undirected graph with \( n \) vertices and \( m \) edges, we can carry out a BFS either in \( O(n \log n + m \log \log n) \) time with \( n \log_3 3 + O((\log n)^2 + 1) \) bits of working memory or in \( O(n \log n + m) \) time with at most \( n \log_3 3 + n' \) bits for arbitrary fixed \( \epsilon > 0 \). The best previous algorithm is faster, \( O(n + m) \) time, but needs more space, \( n \log_3 3 + O(n/(\log n)^3) \) bits for arbitrary fixed \( t \in \mathbb{N} \).

Keywords. Data structures, space efficiency, choice dictionaries, bounded universes, constant-time initialization, breadth-first search (BFS).

1 Introduction

Concurrently with the extreme growth in the size of data sets, there is a trend towards the complete data not being stored locally on a user’s computer. The data may be provided by a remote server, it may be in a “cloud”, or it may even exist only in the form of an interface that can answer queries. In such scenarios, and also if the “computer” is in fact a small mobile device, it may be important to use only (relatively) little local memory and small data structures. Space efficiency may be even more generally beneficial in view of the ubiquitous memory hierarchies that operate according to the tradeoff “the bigger, the slower”. This paper deals with one particular class of space-efficient data structures and their applications.

Following similar earlier definitions [30] and concurrently with that of [2], the choice-dictionary data type was introduced by Hagerup and Kammer [10] as a basic tool in space-efficient computing and is known to have numerous applications [2,6,10,12,13]. Its precise characterization is as follows:

Definition 1.1. A choice dictionary is a data type that can be initialized with an arbitrary integer \( n \in \mathbb{N} = \{1, 2, \ldots\} \), subsequently maintains an initially empty subset \( S \) of \( U = \{1, \ldots, n\} \) and supports the following operations, whose preconditions are indicated in parentheses:
Immediately after the initialization, the latter must force each of Ω
enumerate twice, and the iteration is constant-time only in an amortized sense.

If the only changes to S are insertions, the iteration is robust, except that an integer may be
iterate; if all elements have already been enumerated, 0 is returned),
and iterate.next, which returns 1 if one or more elements of S remain to be enumerated, and
0 otherwise. When stating that a choice dictionary allows iteration in a certain time t, what
we mean is that each of the operations iterate.init, iterate.next and iterate.more runs in time
bounded by t. The order in which the elements of S are enumerated may be chosen arbitrarily
by the data structure. In applications it is sometimes important to be able to allow changes
to S while it is being iterated over. The main choice dictionaries of [10] provide robust iteration:
Every integer that is a member of S during the whole iteration is enumerated, while no integer
is enumerated more than once or at a time when it does not belong to S. Robust iteration is an
ideal that we do not know how to attain for the new and very space-efficient choice dictionaries
presented here. In Section 5, however, we design a weaker form of iteration that is still useful:
If the only changes to S during an iteration are deletions, the iteration is robust and constant-
time. If the only changes are insertions, the iteration is robust, except that an integer may be enumerated twice, and the iteration is constant-time only in an amortized sense.

A generalization of the choice dictionary called a colored choice dictionary, rather than maintaining a single subset of \( U = \{1, \ldots, n\} \), maintains a semipartition \( (S_0, \ldots, S_{c-1}) \) of \( U \), i.e., a sequence of (possibly empty) disjoint subsets of \( U \) whose union is \( U \), called its client vector.
Viewing the elements of $S_j$ as having color $j$, for $j = 0, \ldots, c-1$, we speak of a $c$-color choice dictionary or a choice dictionary for $c$ colors. The number $c$ of colors is fixed, together with the universe size $n$, during the initialization of an instance of the data structure, and we now take "externally sized" to mean that both $n$ and $c$ are available without being stored in the instance. For emphasis, the original choice dictionary may be characterized as uncolored. Its operations $\text{insert}$, $\text{delete}$ and $\text{contains}$ are replaced by

\begin{align*}
\text{setcolor}(j, \ell) \ (j \in \{0, \ldots, c-1\} \text{ and } \ell \in U): & \text{ Changes the color of } \ell \text{ to } j, \text{i.e., moves } \ell \text{ to } S_j \text{ (if it is not already there).} \\
\text{color}(\ell) \ (\ell \in U): & \text{ Returns the color of } \ell, \text{i.e., the unique } j \in \{0, \ldots, c-1\} \text{ with } \ell \in S_j.
\end{align*}

Moreover, the operations $\text{choice}$ and $\text{iterate}$ (with its three suboperations) take an additional (first) argument $j \in \{0, \ldots, c-1\}$ that indicates the set $S_j$ to which the operations are to apply; e.g., $\text{choice}(j)$ returns an arbitrary element of $S_j \ (0 \text{ if } S_j = \emptyset)$. Initially, all elements of $U$ belong to $S_0$.

Sections 3 and 4 describe new externally sized $c$-color choice dictionaries. Provided that $c$ is a constant—to date the most useful case in applications—the new choice dictionaries are atomic and occupy $n \log_2c + 1$ bits if $c$ is a power of 2 (again, this is optimal), and at most $n \log_2c + n^\epsilon$ bits in general for arbitrary fixed $\epsilon > 0$. As an alternative, still with constant-time initialization, we can implement $\text{setcolor}$, $\text{color}$ and $\text{choice}$ in $O(\log \log n)$ time using $n \log_2c + O((\log n)^2 + 1)$ bits. Except as concerns iteration, the 2-color choice dictionary subsumes the uncolored choice dictionary of Section 2. We still provide a self-contained description of the latter both because it is particularly simple and may be suited for practical use and classroom teaching and because all of the techniques developed for the uncolored case are used again, now in a more complex setting, in the colored case.

For the colored choice dictionaries, it is easy to support constant-time iteration over a set $S_j$ in the client vector for the static case, i.e., when no $\text{setcolor}$ operations are executed on the choice dictionary during the iteration. For the dynamic case we can provide a weak form of iteration over $S_j$ that enumerates only integers that belong to $S_j$ when they are enumerated and that enumerates all integers present in $S_j$ during the whole iteration, but that may enumerate an integer repeatedly and for which we can bound the total iteration time only by $O(n + k \log n)$, where $m$ is the number of elements present in $S_j$ at the start of the iteration and $k$ is the number of calls of $\text{setcolor}$ executed during the iteration (this bound does not by itself ensure that the iteration will terminate—we may have $k = \infty$). Iteration over a colored choice dictionary is relevant to an algorithm of [10] for a problem known loosely as breadth-first search (BFS) and more precisely as the computation of a shortest-path spanning forest of a directed or undirected graph $G$ consistent with a given vertex ordering. If $G$ has $n$ vertices and $m$ edges, the algorithm needs $O(n + m)$ time in addition to the time needed to execute $O(n + m)$ operations on a 3-color dictionary with universe size $n$ that supports iteration. Plugging in the new colored choice dictionaries, we can solve the problem in $O(m + n \log n)$ time with at most $n \log_23 + n^\epsilon$ bits, for arbitrary fixed $\epsilon > 0$, or in $O(n \log n + m \log n)$ time with $n \log_23 + O((\log n)^2 + 1)$ bits. The best previous algorithm [10] (Theorem 8.5) is faster, $O(n + m)$ time, but needs more space, $n \log_23 + O(n/(\log n)^t)$ bits for arbitrary fixed $t \in \mathbb{N}$.

Our new results were obtained by combining techniques of Katoh and Goto [15] and Hagerup and Kammer [10] with new ideas. Katoh and Goto used a new so-called in-place chain technique to obtain improved initializable arrays, arrays with an additional operation to store the same given value in every cell. A connection between choice dictionaries and initializable arrays was first noted by Hagerup and Kammer [11], who observed that the light-path technique, invented in [10] in the context of choice dictionaries, also yields initializable arrays better than those known at the time. Here we use the in-place chain technique, slightly modified and extended with new operations, to derive the uncolored choice dictionary of Section 2. Parts of the present paper that draw heavily on techniques and results of [10] include Subsections 5.1 and 5.2 and there, in particular, a method of storing external information in regularly spaced free bits in the representation of a choice dictionary during times when its client vector is deficient, i.e., contains one or more empty sets. A difference is that whereas isolated free bits are sufficient in [10], here
a new step had to be introduced that aggregates free bits into groups of \( \Omega(\log n) \) contiguous bits.

Our space bounds for a data structure apply at times when the data structure is in a *quiescent* state, i.e., between the execution of operations. During the execution of an operation, the data structure may temporarily need more space—we speak of *transient* space requirements. By definition of the word RAM, the transient space requirements are always at least \( \Theta(w) \) bits, and we will mention them only if they exceed this bound. Similarly, if no space bound is indicated for an algorithm, it gets by with \( O(w) \) bits in addition to the space needed for its input and output.

The present text is an expanded version of a preliminary report [9] and essentially repeats the material of the latter for uncolored choice dictionaries as the main part of Section 2. A different generalization of the approach of [9] to \( c \geq 2 \) colors was found recently by Kammer and Sajenko [14], but only for the case in which \( c \) is a power of 2. The colored choice dictionaries of [14] are simpler than ours and allow a slightly more general form of iteration. This paper, on the other hand, allows general values of \( c \) and offers faster operations for nonconstant \( c \), especially if \( w = \omega(\log n) \), and better support for iteration in the uncolored case.

## 2 An Uncolored Choice Dictionary

This section describes the very simple new atomic uncolored choice dictionary and provides fairly detailed pseudo-code for its operations. The addition of support for iteration to the data structure is postponed to Section 5.

**Theorem 2.1.** There is an externally sized atomic (uncolored) choice dictionary that, when initialized for universe size \( n \), occupies \( n + 1 \) bits.

Throughout the paper we make use of the natural bijections, for each given \( n \in \mathbb{N} \), that relate a subset \( S \) of \( \{0, \ldots, n - 1\} \), the integer \( \sum_{\ell \in S} 2^\ell \) in \( \{0, \ldots, 2^n - 1\} \), and the sequence \( (b_0, \ldots, b_{n-1}) \) of \( n \) bits with \( b_\ell = 1 \iff \ell \in S \), for \( \ell = 0, \ldots, n - 1 \). When speaking about the finite subset of \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), the nonnegative integer or the bit sequence corresponding to an object \( X \) one of the two other kinds we mean the object obtained from \( X \) by an application of the relevant bijection. Already the definition of the word RAM makes use of this correspondence by viewing the contents of memory words as integers when arithmetic operations are applied to them and as bit sequences when the operations AND, OR and XOR are used. Similarly, we may view a sequence of bits stored in memory as representing a nonnegative integer (given via its binary representation) or a finite subset of \( \mathbb{N}_0 \) (given via its bit-vector representation). When the inverse bijections are used to derive a bit sequence from a finite subset of \( \mathbb{N}_0 \) or a nonnegative integer, the length of the bit sequence must be supplied either explicitly or by context, since a bit sequence can always be extended by additional zeros without any change to the set or the integer that it represents; it will usually be clear that a bit sequence of a particular length is called for. For given integers \( a \) and \( n \in \mathbb{N} \), we can extend the bijections under consideration to subsets of \( \{a, \ldots, a + n - 1\} \) by mapping \( S \subseteq \{a, \ldots, a + n - 1\} \) to \( \{\ell - a \mid \ell \in S\} \subseteq \{0, \ldots, n - 1\} \). Thus such a set may also be represented via a bit sequence or a nonnegative integer.

### 2.1 A Simple Reduction

If we represent the client set of a choice dictionary with universe size \( n \) via its bit-vector representation \( B \) of length \( n \), the choice-dictionary operations translate into the reading and writing of individual bits in \( B \) and the operation *choice*, which now returns the position of a nonzero bit in \( B \) (0 if all bits in \( B \) are 0). It is trivial to carry out all operations other than the initialization and *choice* in constant time. In the special case \( n = O(w) \), the latter operations can also be supported in constant time. This is a consequence of part (a) of the following lemma, used with \( f = 1 \).
Let \( m \) and \( f \) be given integers with \( 1 \leq m, f < 2^w \) and suppose that a sequence \( (a_1, \ldots, a_m) \) with \( a_i \in \{0, \ldots, 2^f - 1\} \) for \( i = 1, \ldots, m \) is given in the form of the \((mf)\)-bit binary representation of the integer \( x = \sum_{i=0}^{m-1} 2^i a_{i+1} \). Then the following holds:

(a) Let \( I_{>0} = \{ i \in \mathbb{N} \mid 1 \leq i \leq m \text{ and } a_i > 0 \} \). Then, in \( O([mf/w]) \) time, we can test whether \( I_{>0} = \emptyset \) and, if not, compute \( \min I_{>0} \) and \( \max I_{>0} \).

(b) Let \( I_0 = \{ i \in \mathbb{N} \mid 1 \leq i \leq m \text{ and } a_i = 0 \} \). Then, in \( O([mf/w]) \) time, we can test whether \( I_0 = \emptyset \) and, if not, compute \( \min I_0 \).

(c) If an additional integer \( k \in \{0, \ldots, 2^f - 1\} \) is given, then \( O([mf/w]) \) time suffices to compute the integer \( \sum_{i=0}^{m-1} 2^i b_{i+1} \), where \( b_i = 1 \) if \( k \geq a_i \) and \( b_i = 0 \) otherwise for \( i = 1, \ldots, m \).

We use the externally sized atomic choice dictionary for universe sizes of \( O(w) \) implied by these considerations to handle the few bits left over when we divide a bit-vector representation of \( n \) bits into pieces of a fixed size. The details are as follows:

Let \( b \) be a positive integer that can be computed from \( w \) and \( n \) in constant time using \( O(w) \) bits (and therefore need not be stored) and that satisfies \( b \geq \log_2 n \), but \( b = O(w) \). In order to realize an externally sized choice dictionary \( D \) with universe size \( n \) and client set \( S \), partition the bit-vector representation \( B \) of \( S \) into \( N = \lfloor n/(2b) \rfloor \) segments \( B_1, \ldots, B_N \) of exactly \( 2b \) bits each, with \( n' = n \mod (2b) \) bits left over. If \( n' \neq 0 \), maintain (the set corresponding to) the last \( n' \) bits of \( B \) in an externally sized atomic choice dictionary \( D_2 \) realized as discussed above. Assume without loss of generality that \( N \geq 1 \). The following lemma is proved in the remainder of this section:

**Lemma 2.3.** There is a data structure that, if given access to \( b \) and \( N \), can be initialized in constant time and subsequently occupies \( 2bN + 1 \) bits and maintains a sequence \( (a_1, \ldots, a_N) \in \{0, \ldots, 2^{2b} - 1\}^N \), initially \( (0, \ldots, 0) \), under the following operations, all of which execute in constant time: \( \text{read}(k) \) \((k \in \{1, \ldots, N\}) \), which returns \( a_k \); \( \text{write}(k, x) \) \((k \in \{1, \ldots, N\} \text{ and } x \in \{0, \ldots, 2^{2b} - 1\}) \), which sets \( a_k \) to \( x \); and \( \text{ nonzero } \), which returns a \( k \in \{1, \ldots, N\} \) with \( a_k \neq 0 \) if there is such a \( k \), and \( 0 \) otherwise.

For \( k = 1, \ldots, N \), view \( B_k \) as the binary representation of an integer and maintain that integer as \( a_k \) in an instance of the data structure of Lemma 2.2. This yields an externally sized atomic choice dictionary \( D_1 \) for \( (\text{the set corresponding to}) \) the first \( 2bN \) bits of \( B \). To carry out \( \text{insert} \), \( \text{delete} \) or \( \text{contains} \), update or inspect the relevant bit in one of \( a_1, \ldots, a_N \), and to execute \( \text{choice} \), call \( \text{ nonzero } \) and, if the return value \( k \) is positive, apply an algorithm of Lemma 2.2 (a) to \( a_k \) and add \( 2b(k-1) \). It is obvious how to realize the full choice dictionary \( D \) through a combination of \( D_1 \) and \( D_2 \). The only nontrivial case is that of the operation \( \text{choice} \). To execute \( \text{choice} \) in \( D \), first call \( \text{choice} \) in \( D_1 \) (say). If the return value is positive, it is a suitable return value for the parent call. Otherwise call \( \text{choice} \) in \( D_2 \), increase the return value by \( 2bN \) if it is positive, and return the resulting integer. \( D \) is atomic because \( D_1 \) and \( D_2 \) are, and the total number of bits used by \( D \) is \( 2bN + 1 + n' = n + 1 \). Theorem 2.1 follows.

### 2.2 The Storage Scheme

To prove Lemma 2.3 we first show a slightly weaker form of the lemma in which the space bound is relaxed to allow \( 2bN + w \) bits instead of \( 2bN + 1 \) bits. This subsection describes how the sequence \( (a_1, \ldots, a_N) \) is represented in memory in \( 2bN + w \) bits. Most of the available memory stores an array \( A \) of \( N \) cells \( A[1], \ldots, A[N] \) of \( 2b \) bits each. In addition, a \( w \)-bit word is used to hold an integer \( \mu \in \{0, \ldots, N\} \) best thought of as a “barrier” that divides \( V = \{1, \ldots, N\} \) into a part to the left of the barrier, \( \{1, \ldots, \mu\} \), and a part to its right, \( \{\mu + 1, \ldots, N\} \). We often consider a \( (2b) \)-bit quantity \( x \) to consist of a lower half, denoted by \( x \) and composed of the \( b \) least significant bits of \( x \) (i.e., \( \lfloor x = x \text{ AND } (2^b - 1) \rfloor \)), and an upper half, \( \lceil x \rceil = x \gg b \), and we may write \( x = (\lfloor x \rfloor, \lceil x \rceil) \). A central idea, due to Katoh and Goto [15], is that the upper halves of \( A[1], \ldots, A[N] \) are used to implement a matching on \( V \) according to the following convention: Elements \( k \) and \( \ell \) of \( V \) are matched exactly if \( A[k] = \ell \) and \( A[\ell] = k \), and precisely one of \( k \) and \( \ell \) lies to the left of the barrier, i.e., \( k \leq \mu < \ell \) or \( \ell \leq \mu < k \). In this case we call \( \ell \) the mate of \( k \).
and vice versa. The assumption $b \geq \log_2 \mu$ ensures that the upper half of each cell in $A$ can hold an arbitrary element of $V$. A function that inputs an element $k$ of $V$ and returns the mate of $k$ if $k$ is matched and $k$ itself if not is easily coded as follows:

$mate(k)$:
\[
\begin{align*}
k' &:= A[k]; \\
\text{if } (1 \leq k \leq \mu < k' \leq N \text{ or } 1 \leq k' \leq \mu < k \leq N) \text{ and } A[k'] = k \text{ then return } k'; \\
\text{return } k;
\end{align*}
\]

For all $k \in V$, call $k$ strong if $k$ is matched and $k \leq \mu$ or $k$ is unmatched and $k > \mu$, and call $k$ weak if it is not strong. The integers $A[1], \ldots, A[N]$ and $\mu$ represent the sequence $(a_1, \ldots, a_N)$ according to the following storage invariant: For all $k \in V$,

- $a_k = 0$ exactly if $k$ is weak;
- if $k$ is strong and $k > \mu$, then $a_k = A[k]$;
- if $k$ is strong and $k \leq \mu$, then $a_k = (A[k], A[mate(k)])$.

The storage invariant is illustrated in Fig. 1. The following drawing conventions are used here and in subsequent figures: The barrier is shown as a thick vertical line segment with a triangular base. Each pair of mates is connected with a double arrow, and a cell $A[k]$ of $A$ is shown in a darker hue if $k$ is strong. A question mark indicates an entry that can be completely arbitrary, except that it may not give rise to a matching edge, and the upper and lower halves of some cells of $A$ are shown separated by a dashed line segment.

Fig. 1. The storage scheme. Above: The array $A$. Below: The sequence $a$ represented by $A$.

2.3 The Easy Operations

The data structure is initialized by setting $\mu = N$, i.e., by placing the barrier at the right end. Then the matching is empty, and all elements of $V$ are to the left of the barrier and weak. Thus the initial value of $(a_1, \ldots, a_N)$ is $(0, \ldots, 0)$, as required. The implementation of $read$ closely reflects the storage invariant:

$read(k)$:
\[
\begin{align*}
\text{if } mate(k) \leq \mu \text{ then return } 0; \text{ (* $k$ is weak exactly if $mate(k) \leq \mu$ *)} \\
\text{if } k > \mu \text{ then return } A[k]; \text{ else return } (A[k], A[mate(k)]);
\end{align*}
\]

The code for $nonzero$ is short but a little tricky:

$nonzero$:
\[
\begin{align*}
\text{if } \mu = N \text{ then return } 0; \text{ else return } mate(N);
\end{align*}
\]

The implementation of $write(k, x)$ is easy if $k$ is weak and $x = 0$ (then nothing needs to be done) or $k$ is strong and $x \neq 0$. In the latter case the procedure $simple\_write$ shown below can be used. The only point worth noting is that writing to $A[k]$ when $k$ is strong and $k > \mu$ may create a spurious matching edge that must be eliminated.
A := µ deletion, so assume that it gives rise to an insertion or a deletion, since in the remaining cases the procedure usually the matching. In fact, µ on the choice dictionary. Insertions and deletions are the operations that change the barrier and are triggered by (certain) insertions and deletions, respectively, executed on the choice dictionary. Insertions and deletions are the operations that change the barrier and usually the matching. In fact, µ is decreased by 1 in every insertion and increased by 1 in every deletion, so µ is always the number of k ∈ V with ak = 0.

The various different forms that an insertion may take are illustrated in Figs. 2 and 3. The situation before the insertion is always shown above the situation after the insertion. A “1” outside of the “stripes” indicates the position of an insertion and symbolizes the nonzero value to be written, while a “1” inside the stripes symbolizes that value after it has been written. The various forms of a deletion are illustrated in Figs. 4 and 5. Here a “0” indicates the position of a deletion, while a “1” symbolizes the nonzero value that is to be replaced by zero.

There are many somewhat different cases, but for each it is easy to see that the storage invariant is preserved and that the sequence (a1, . . . , aN) changes as required. It is also easy to turn the figures into a write procedure that branches into as many cases. Here we propose the following realization of write that is terser, but needs a more careful justification.

write(k, x):
  x0 := read(k); (* the value to be replaced by x *)
  k′ := mate(k);
  if x 0 then (* an insertion *)
      µ′ := mate(µ); (* µ = µ will cross the barrier *)
      u := read(µ); (* save aµ *)
      µ := µ − 1; (* move the barrier left *)
      simple_write(µ + 1, u); (* reestablish the value of aµ *)
      if k 0 then { A[k′] := µ′; A[µ′] := k′; A[µ′] := A[k]; } (* match k’ and µ’ *)
  else (* x = 0 *)
      if x0 0 then (* a deletion *)
        µ′ := mate(µ + 1); (* µ = µ + 1 will cross the barrier *)
        v := read(µ′); (* save aµ′ *)
        µ := µ + 1; (* move the barrier right *)
        A[k′] := µ′; A[µ′] := k′; (* match k’ and µ’ *)
        if µ′ 0 then simple_write(µ′, v); (* reestablish the value of aµ′ *)

To see the correctness of the procedure write given above, consider a call write(k, x) and assume that it gives rise to an insertion or a deletion, since in the remaining cases the procedure is easily seen to perform correctly. Let µ0 be the value of µ (immediately) before the call. Because the call changes the value of µ by 1, a single element µ of V crosses the barrier, i.e., is to the left of the barrier before or after the call, but not both. In the case of an insertion, µ = µ0; in that of a deletion, µ = µ0 + 1.

Assume that k does not cross the barrier, i.e., that k µ. Because the call changes ak from zero to a nonzero value or vice versa, k must change its matching status, i.e., be matched before or after the call, but not both. In detail, if k is matched before the call, its mate at that
Fig. 2. Insertion to the left of the barrier.

Fig. 3. Insertion to the right of the barrier.

Fig. 4. Deletion to the left of the barrier.

Fig. 5. Deletion to the right of the barrier.
time, if different from $\tilde{\mu}$, must find a new mate, which automatically leaves $k$ unmatched. If $k$ is unmatched before the call, $k$ itself must find a mate. We can unify the two cases by saying that if $k' = \text{mate}(k)$ (evaluated before the call under consideration has changed $\mu$ and $A$) is not $\tilde{\mu}$, then $k'$ must find a (new) mate. If $k' \neq \tilde{\mu}$, moreover, $k'$ is to the left of the barrier in the case of an insertion and to the right of it in the case of a deletion.

Assume now that the call does not change $a_k$, i.e., that $\tilde{\mu} \neq k$. Then, because $\tilde{\mu}$ crosses the barrier, it must also change its matching status: If $\tilde{\mu}$ is matched before the call, its mate at that time, if different from $k$, must find a new mate, and otherwise $\tilde{\mu}$ itself must find a mate. As above, this can be expressed by saying that if $\mu' = \text{mate}(\tilde{\mu})$ (evaluated before the call has changed $\mu$ and $A$) is not $k$, then $\mu'$ must find a (new) mate. Moreover, after the call $\mu'$ is to the right of the barrier in the case of an insertion and to the left of it in the case of a deletion.

Exclude the special cases identified above by assuming that $\{k,k'\} \cap \{\tilde{\mu},\mu'\} = \emptyset$. Then it can be seen that all required changes to the matching can be effectuated by matching $k'$ and $\mu'$, which is what the procedure write does. In the case of an insertion, this makes $k$ strong, which implies that $a_k$ can be set to $x$ simply by executing simple_write($k$, $x$) at the very end.

In addition, with $\ell = \min\{k',\mu'\}$, it must be ensured that the call does not change $a_\ell$ except if $\ell = k$. In the case of an insertion, $\ell = k'$, and if $k' \neq k$, the mate of $k'$ switches from being $k$ to being $\mu'$, so that it suffices to execute $A[\mu'] := A[k]$, which happens in the procedure. The same assignment is executed if $k' = k$, in which case it is useless but harmless. In the case of a deletion, $\ell = \mu'$. Here the procedure plays it safe by remembering the value of $a_{\mu'}$ before the call in a variable $v$ and restoring $a_{\mu'}$ to that value at the end, unless $\mu' = k$, via the call simple_write($\mu'$, $v$). This is convenient because $a_{\mu'}$ is not stored in a unique way before the call.

At this point, $k$, $k'$ and $\mu'$ have been “taken care of”, but $\tilde{\mu}$ still needs attention. In the case of a deletion, either $\tilde{\mu} = \mu'$ or $\tilde{\mu}$ is weak, so nothing more needs to be done. In an insertion, the procedure saves the original value of $\tilde{\mu}$ in $u$ and restores it afterwards through the statement simple_write($\mu$ + 1, $u$). This is necessary and meaningful only if $\tilde{\mu}$ is strong. If $\tilde{\mu}$ is weak, however, the effect of the statement—except for the harmless possible elimination of a spurious matching edge—is canceled through the subsequent assignment to $A[\mu']$ and $\bar{A}[\mu']$.

We still need to consider the special cases that were ignored above, namely calls with $\{k,k'\} \cap \{\tilde{\mu},\mu'\} \neq \emptyset$. These form part (b) of Figs. 2–4. In fact, the number of special cases is quite limited. If $\tilde{\mu}$ is weak before an insertion or strong before a deletion, it is unmatched. Thus if $k = \tilde{\mu}$, we have $k = k' = \tilde{\mu} = \mu'$, and $k' = \mu'$ implies $k = \tilde{\mu}$. On the other hand, each of the statements $k = \mu'$ and $k' = \tilde{\mu}$ implies the other one. Thus there are two cases to consider: (1) $k = k' = \tilde{\mu} = \mu'$ and (2) $k = \mu' \neq \tilde{\mu} = k'$.

In case (1), all writing to $A$ happens to $A[k]$. For insertion (Fig. 2(b)), the execution of simple_write($k$, $x$) at the very end ensures the correctness of the call. For deletion (Fig. 3(b)), the execution of $\bar{A}[\mu'] := k'$ at the end ensures that $k$ is unmatched, which is all that is required. In case (2), after an insertion (Fig. 3(b)), $k$ and $\tilde{\mu}$ are both to the right of the barrier, $a_k$ and $a_{\tilde{\mu}}$ are both nonzero, and the execution of simple_write($k$, $x$) and simple_write($\mu$ + 1, $u$) ensures that $A[k]$ and $\bar{A}[\tilde{\mu}]$ have the correct values after the call. After a deletion (Fig. 4(b)), $k$ and $\tilde{\mu}$ are both to the left of the barrier and $a_k = a_{\tilde{\mu}} = 0$, and the execution of $\bar{A}[k'] := \mu'$ and $\bar{A}[\mu'] := k'$ in fact ensures that $k$ and $\tilde{\mu}$ are both unmatched, which is all that is required.

Since all operations of the data structure have been formulated as pieces of code without loops and $b = \Theta(w)$, it is clear that the operations execute in constant time.

### 2.5 Reducing the Space Requirements

The space requirements of the data structure of Subsections 2.2–2.4 can be reduced from $2bN + w$ bits to $2bN + 1$ bits, as promised in Lemma 2.3 by a method of [11,15]. First, $b$ is chosen to satisfy not only $b \geq \log_2n$, but $b \geq 2\log_2n$, which is clearly still compatible with $b = \Theta(w)$. As a result, for each $k \in V$ to the left of the barrier, $A[k]$ has at least $2\log_2n - \log_2N \geq \lceil \log_2N \rceil$ unused bits. If $\mu \geq 1$, we store $\mu$ in the unused bits of $A[1]$ (the unused bits of $A[2], \ldots, A[\mu]$ continue to be unused). When $\mu = 0$, even $A[1]$ is to the right of the barrier and there are no unused bits in $A$, so we use a single bit outside of $A$ to indicate whether $\mu$ is nonzero. The resulting data structure occupies exactly $2bN + 1$ bits.
2.6 The Choice of $b$

A practical choice dictionary based on the ideas of this section is likely to content itself with the main construction of Subsections 2.2–2.4 and refrain from applying the method of Subsection 2.5 to squeeze out the last few bits. Then there is no reason to choose $b$ larger than $w$, and $b = w$ seems the best choice. This yields a self-contained atomic choice dictionary that occupies $n + 2w$ bits when initialized for universe size $n$.

If $w$ is even and $w \geq 2\log_2 n$, another plausible choice is $b = w/2$, which allows an entry in the array $A$ to be manipulated with a single instruction and simplifies the access to cells of $A$. It seems, however, that the gains in certain scenarios from choosing $b = w/2$ instead of $b = w$ are small and can be reduced still further through an optimization of the case $b = w$ that omits superfluous operations on upper or lower halves of cells in $A$.

If the space needed for an externally sized choice dictionary is to be reduced all the way to $n + 1$ bits for universe size $n$, $b = 2w$ seems the best choice.

2.7 A Self-Contained Choice Dictionary

In order to convert the externally sized atomic choice dictionary of Theorem 2.1 to a self-contained one, we must augment the data structure with an indication of the universe size $n$. This can clearly always be done with $w$ additional bits. If a space bound is desired that depends only on $n$, $n$ must be stored as a so-called self-delimiting numeric value. Assume first that the most significant bits in a word are considered to be its “first” bits, i.e., the ones to be occupied by a data structure of fewer than $w$ bits (the “big-endian” convention). Then one possibility is to use the code $\gamma'$ of Elias [5]: With $\text{bin}(n)$ denoting the usual binary representation of $n \in \mathbb{N}$ (e.g., $\text{bin}(13) = 1101$), store $n$ in the form of the string $0^{\text{bin}(n) - 1}\text{bin}(n)$, which can be decoded in constant time with an algorithm of Lemma 2.2(a). Since $|\text{bin}(n)| = \lceil \log(n + 1) \rceil$, this yields a space bound for the self-contained choice dictionary of $n + 2\lceil \log(n + 1) \rceil$ bits. If instead the least significant bits of a word are considered to be its first bits (the “little-endian” convention), the scheme needs to be changed slightly: The string $0^{\text{bin}(n) - 1}\text{bin}(n)$ is replaced by $\hat{\text{bin}}(n)0^{\text{bin}(n) - 1}$, where $\hat{\text{bin}}(n)$ is the same as $\text{bin}(n)$, except that the leading 1 is moved to the end.

Incidentally, if an application can guarantee that $\mu$ never becomes zero, the method of Subsection 2.5 can be used to “hide” $n$ as well as $\mu$ in the array $A$ if we choose $b \geq 4\lceil \log(n + 1) \rceil$. This yields a restricted self-contained atomic choice dictionary that occupies $n + 1$ bits. The restriction is satisfied, e.g., if the universe $\{1, \ldots, n\}$ always contains $4b - 1$ consecutive elements that do not belong to the client set.

2.8 Making the Choice Dictionary Dynamic

It is easy to extend the choice dictionary of Theorem 2.1 to allow gradual changes to the universe size, i.e., to support the following two additional operations, where $n$ is the universe size and $S$ is the client set:

- $\text{expand}(b)$ ($b \in \{0, 1\}$): Increases $n$ by 1 and subsequently, if $b = 1$, replaces $S$ by $S \cup \{n\}$.
- $\text{contract}$ ($n > 0$): Replaces $S$ by $S \setminus \{n\}$ and subsequently decreases $n$ by 1.

We call the resulting data structure a dynamic externally sized (uncolored) choice dictionary. We allow the universe size of a dynamic choice dictionary to be 0. When it is, $\text{choice}$ should return 0, and calls of $\text{insert}$, $\text{delete}$ and $\text{contains}$ are illegal.

**Theorem 2.4.** There is an atomic dynamic externally sized (uncolored) choice dictionary that occupies $n + 1$ bits when its universe size is $n$, for all $n \in \mathbb{N}_0$.

**Proof.** We use the same construction as for the choice dictionary of Theorem 2.1 except that $b$ should now be chosen as a function of $w$ alone. Apart from changing the externally stored universe size in the obvious way, the operations $\text{expand}$ and $\text{contract}$ carry out the steps described in the following. First, $\text{expand}(b)$ stores $b$ in the new bit that becomes available to the data structure.
Unless the call of expand or contract changes \(|n/(2b)|\), it need not do anything else—the number \(n'\) of bits in the trivial choice dictionary \(D_2\) of Subsection 2.1 simply increases or decreases by 1 (allow \(n' = 0\)). Assume now that the call of expand or contract changes \(|n/(2b)|\), so that \(n'\) jumps from \(2b - 1\) to 0 or vice versa and the array \(A\) acquires a new cell or loses one. If the call in question is expand(1) and the universe size after the call is \(n\), let \(x\) be the integer formed by the last \(2b\) of the \(n + 1\) bits that represent the data structure and execute simple_write\((n, x)\), which serves exclusively to eliminate a possible spurious matching edge. If the call is expand(0), simulate it by expand(1) followed by delete\((n)\). If the call in question is contract and the universe size before the call is \(n\), simply execute insert\((n)\) before the change to the universe size. This implementation of expand and contract works because the rest of the representation of the data structure, except for the issue of a spurious matching edge, is independent of the presence or absence of a last cell in \(A\) whose index is strong.

□

3 Power-of-2-Colored Choice Dictionaries

This section describes colored choice dictionaries that can be used only in the simpler case in which the number \(c\) of colors is a power of 2. Let us call such choice dictionaries power-of-2-colored. The case of general values of \(c\) is considered in the next section, and the discussion of iteration is again postponed to Section 5. To avoid trivialities, we always assume that the number \(c\) of colors is at least 2.

The following lemma, based on the fast integer-multiplication algorithm of Schönhage and Strassen [16], bounds the complexity of multiple-precision multiplication and division on the \(w\)-bit word RAM. Except for the space bound, it was observed in [8].

Lemma 3.1. For all integers \(m\) and \(n\) with \(1 \leq m \leq n\), if \(x\) and \(y\) are given integers with \(0 \leq x < 2^m\) and \(0 \leq y < 2^m\), then \(x/y\) can be computed in \(O(|n \log(2 + m/w)/w|)\) time with \(O(n + m + w)\) bits of working memory.

For \(m, f \in \mathbb{N}\), let \(1_{m,f} = \sum_{i=0}^{m-1} 2^i = (2^m - 1)/(2^f - 1)\). If the \((mf)\)-bit binary representation of \(1_{m,f}\) is divided into \(m\) fields of \(f\) bits each, each field contains the value 1. As follows from [8] Theorem 2.5, the possibly multword integer \(1_{m,f}\) can be computed in \(O([mf/w])\) time, and an integer in \(\{0, \ldots, 2^m - 1\}\) can be multiplied by \(1_{m,f}\) in \(O([n + mf]/w)\) time with \(O(n + mf + w)\) bits of working memory.

3.1 Changing Base

In this subsection we study the following problem, which plays a central role for all of our colored dictionaries: Given integers \(c\) and \(d\) with \(c, d \geq 2\) and an integer \(x\) of the form \(x = \sum_{j=0}^{s-1} a_j c^j\), where \(s \in \mathbb{N}\) and \(a_0, \ldots, a_{s-1}\) are integers with \(0 \leq a_j < \min\{c, d\}\) for \(j = 0, \ldots, s - 1\), compute \(y = \sum_{j=0}^{s-1} a_j d^j\). From the perspective of positional numeral systems, the problem can be viewed as one of changing the base from \(c\) to \(d\), but in the peculiar sense of leaving the digits unchanged while interpreting them according to a new base. Alternatively, the problem can be seen as the evaluation of a polynomial on the argument \(d\), but in a situation in which the coefficients \(a_0, \ldots, a_{s-1}\) of the polynomial are available only in the form of the integer \(x\).

First one can observe that the problem is well-defined: Because \(c, d \geq 2\) and \(0 \leq a_j < c\) for \(j = 0, \ldots, s - 1\), the mapping from \((a_0, \ldots, a_{s-1})\) to \(\sum_{j=0}^{s-1} a_j c^j\) is injective, except that the mapping is insensitive to trailing zeros in its argument, so \(y\) is uniquely determined by \(x\).

Let \(f = \lceil \log_2 \max\{c, d\} \rceil\) and take \(q = 2 + sf/w\). Thus \(q\) is essentially the number of \(w\)-bit words occupied by \(x\) and \(y\). Compute \(t\) as the smallest positive integer with \(c^t > x\). Using repeated squaring, \(t\) can be obtained in \(O(\sum_{k=0}^{t-1} 2^{-kq} \log(2 + 2^{-kq})) = O(t + q \log q)\) time according to Lemma 3.1. By adding or dropping trailing zeros as appropriate, we can assume that \(s = 2^f\). We will convert from \(c\) to \(d\) via \(2^f\) in the sense of first computing \(z = \sum_{j=0}^{s-1} a_j 2^{fj}\) from \(x\) and subsequently obtaining \(y\) from \(z\). The significance of \(f\) is that in the binary representation of \(z\) the coefficients \(a_0, \ldots, a_{s-1}\) are readily available as the contents of \(s\) fields of \(f\) bits each.
This makes it easy to compute $y$ from $z$ via a word-parallel version of a straightforward divide-and-conquer procedure. For $k = 0, \ldots, t-1$, the procedure partitions $s/2^k$ digits to base $b = d^{2^k}$ into pairs of consecutive digits and replaces each pair $(a', a'')$ by a single digit to base $b^2 = d^{2^{k+1}}$ with the same value, namely $a' + ba''$. Using word parallelism, this can be formulated as follows:

\[
\begin{align*}
    b &:= d; \quad (\text{the current base}) \\
    r &:= f; \quad (\text{the current region size}) \\
    \bar{y} &:= z; \quad (\text{interpreted according to } b \text{ and } r, \text{ always has the value } y) \\
    \text{for } k := 0 \text{ to } t-1 \text{ do} \\
    \quad (\ast b = d^{2^k} \text{ and } r = 2^k f) \\
    \quad u := (2^r - 1) \cdot 1_{sf/(2r)2r}; \quad (\ast \text{bit mask for keeping every other region}) \\
    \quad \bar{y} := (\bar{y} \text{ AND } u) + ((\bar{y} \gg r) \text{ AND } u); \quad (\ast \text{two digits are combined into one}) \\
    \quad b := b^2; \\
    \quad r := 2r;
\end{align*}
\]

The final value of $\bar{y}$ is $y$. To understand the code, note that $u$ is computed as a bit mask with the property that forming the conjunction with $u$ picks out every other region of $r = 2^k f$ bits, starting with the least significant one. Regions hold the digits to base $b = d^{2^k}$ alluded to above. They start as single fields of $f$ bits and double in size in every iteration. At the end, there is only a single region of $(sf/f)$ bits that contains the integer $y$. The computation of $y$ takes $O(q \sum_{k=0}^{t-1} \log(2 + 2^{-k}q)) = O(q(t + (\log q)^2))$ time.

Suppose that $p \in \mathbb{N}$ and that $p$ instances of the problem just solved have the same value of $d$ but different values of $z_1, \ldots, z_p$ of $z$ and that $z_1, \ldots, z_p$ are presented as a single sequence of $psf$ bits in consecutive regions of $sf$ bits each, where $s$ and $f$ are now given and $f$ is at least $[\log_2 \max\{c, d\}]$. If we again take $q = 2 + sf/w$, the procedure is easily modified to solve all the instances simultaneously in $O((psf/w)(t + (\log q)^2))$ time. Only two aspects need attention. First, the mask $u$ must be extended to “cover” all instances. Second, the assumption that $s$ is a power of 2 may cause an instance to “encroach on” its left neighbor. To counter this, one can simply solve the even-numbered instances in a first round and the odd-numbered instances in a subsequent round.

The procedure for obtaining $z$ from $x$ is essentially the reverse of the procedure for obtaining $y$ from $z$: For $k = t-1, \ldots, 0$, each of $s/2^{k+1}$ digits $a$ to base $b^2 = c^{2^{k+1}}$ is split into the two digits $a' = a \mod b$ and $a'' = [a/b]$ to base $b = d^2$ and replaced by the pair $(a', a'')$. Of course, it is easy to obtain $a'$ from $a''$, even in a word-parallel setting, as $a' = a - ba''$, but the formula for $a''$ involves division, which is not in general readily amenable to word parallelism. If all divisors are the same integer $b \geq 1$, however, as is the case here, division by $b$ can be replaced by multiplication by its approximate inverse. The details are worked out in the following lemma.

\textbf{Lemma 3.2.} Given integers $b, r \geq 1$ and an integer $x$ of the form $x = \sum_{j=0}^{p-1} a_j 2^j$, where $p \in \mathbb{N}$ and $a_0, \ldots, a_{p-1}$ are integers with $0 \leq a_j < 2^r$ for $j = 0, \ldots, p-1$, the quantity $\sum_{j=0}^{p-1} [a_j/b] 2^j$ can be computed in $O([pr/w] \log(2 + r/w))$ time with $O(pr + w)$ bits of working memory.

\textbf{Proof.} We first argue that for all integers $a \geq 0$ and $t > a^2$, $[a/b] = [a \cdot [t/b]/t]$. If $b > a$, the left-hand size is zero, and the right-hand size is also zero, since

\[
\frac{a}{t} \cdot \frac{\lceil t/b \rceil}{\left\lfloor \frac{t}{a+1} \right\rfloor} \leq \frac{a}{t} \cdot \left( \frac{t}{a+1} + 1 \right) = \frac{a + at + a^2}{at + t} < 1.
\]

If $b \leq a$ and hence $a/t \leq 1/b$,

\[
\frac{a}{b} \leq \frac{a}{t} \cdot \frac{\lceil t/b \rceil}{\left\lfloor \frac{t}{b} \rceil} < \frac{a}{t} \left( \frac{t}{b} + 1 \right) = \frac{a}{b} + \frac{a}{t} \leq \frac{a + 1}{b},
\]

and there are no integers strictly between $a/b$ and $(a + 1)/b$.

If there would be no interference between regions of $r$ bits, the regionwise division by $b$ could be carried out according to the formula $[a/b] = [a \cdot [t/b]/t]$, used with $t = 2^{2r}$, simply by
Lemma 3.3. Given positive integers $c$, $d$, $f$ and $s$ with $c, d \geq 2$ and $f \geq \lceil \log_2 \max \{c, d\} \rceil$ and an integer of the form $\sum_{i=0}^{p-1} \left( \sum_{j=0}^{s-1} a_{i,j} c^j \right) \cdot 2^{isf}$, where $p \in \mathbb{N}$ and $0 \leq a_{i,j} < \min \{c, d\}$ for $i = 0, \ldots, p-1$ and $j = 0, \ldots, s-1$, the integer $\sum_{i=0}^{p-1} \left( \sum_{j=0}^{s-1} a_{i,j} d^j \right) \cdot 2^{isf}$ can be computed in $O([psf/w](\log s + (2 + sf/w)^2))$ time with $O(sf + w)$ bits of working memory.

3.2 A Small Power-of-2-Colored Choice Dictionary

This subsection describes a power-of-2-colored choice dictionary $D$ that is very slow for all but the smallest universe sizes. It is the core building block of the more generally useful power-of-2-colored choice dictionary presented in the next subsection. We always assume that $c = 2^{O(w)}$, so that colors can be manipulated in constant time. Instead of choice, $D$ supports the operation successor, defined as follows, where $(S_0, \ldots, S_{c-1})$ is $D$'s client vector.

successor$(j, \ell)$ $(j \in \{0, \ldots, c-1\}$ and $\ell$ is an integer): With $I = \{i \in S_j \mid i > \ell\}$, returns min $I$ if $I \neq \emptyset$, and 0 otherwise.

Recall that we call $D$'s client vector deficient if $S_j = \emptyset$ for at least one $j \in \{0, \ldots, c-1\}$, i.e., if some color is entirely absent. If the client vector is not deficient, it is full. The main feature of $D$ is that it needs less space when its client vector is deficient. Although Lemma 3.4 below mentions successor instead of choice, we still speak of a choice dictionary because choice reduces to successor (instead of choice$(j)$, execute successor$(j, 0)$).

Lemma 3.4. There is an externally sized power-of-2-colored choice dictionary $D$ that, for arbitrary given $m, f \in \mathbb{N}$, can be initialized for universe size $m$ and $c = 2^f$ colors in $O([mf/w])$ time and subsequently needs to store in an external fullness bit whether its client vector is full, executes color in $O([mf/w + |cf/w|A])$ time and setcolor and successor in $O([mf/w+A])$ time, where $A = f + (\log(2 + cf/w))^2$, and occupies at most $mf$ bits and at most $mf - m/c + 2f$ bits when its client vector is deficient.

Alternatively, if initialized with an additional parameter $t \in \mathbb{N}$ and given access to suitable tables of at most $c^t/t$ bits that can be computed in $O(c^t/t)$ time and depend only on $c$ and $t$, $D$ can execute color in $O([mf/w + |cf/w|t])$ time.

The transient space needed by $D$ is $O(mf + w)$ bits.

Proof. We view $D$'s task as that of maintaining a sequence of $m$ color values or digits drawn from the alphabet $\Sigma = \{0, \ldots, c-1\}$, where $c = 2^f$. When its client vector is full, $D$ employs a standard representation that stores the $m$ color values as the concatenation of their binary representations, each of which is given in an $f$-bit field. Assume first that $D$ is in the standard representation. The operation successor must locate the first occurrence, if any, of a particular color $j$ after a certain position. This can be carried out in $O([mf/w])$ time with the algorithm of Lemma 3.2(b) after forming the xor with $j \cdot 1_{mf}$. It is trivial to execute color in constant time by inspecting the value of a single field. Similarly, setcolor updates the value of a single field. If this makes the client vector deficient, however, $D$ is converted to a compact representation described in the following.

We will assume that $m$ is a multiple of $c^f$, noting that up to $c^f - 1$ “surplus” digits, always kept in the standard representation, can be handled within the time bounds of the
lemma, as argued in the previous paragraph. The compact representation partitions the $m$

digits into groups of $c$ consecutive digits each. The compact representation also stores the bit

tensor $Z = (z_0, \ldots, z_{c-1})$, where $z_j = 1$, for $j = 0, \ldots, c - 1$, exactly if $S_j \neq \emptyset$. Since $Z$ is easy to

initialize when $D$ is converted to the compact representation and subsequently can change only in

called a setcolor, and then only in at most two bits whose values can be tested with successor,

maintaining $Z$ is not a bottleneck. As already mentioned, the compact representation is used only when $J_0 = \{j \in \{0, \ldots, c - 1\} | z_j = 0\}$ is nonempty. When this is the case, $j_0 = \min J_0$ can

be computed in $O(c/w)$ time with the algorithm of Lemma 2.2(b). Let $skip_{j_0}$ be the increasing

bijection from $\Sigma \setminus \{j_0\}$ to $\Sigma' = \{0, \ldots, c - 2\}$.

The conversion from the standard representation to the compact representation is done in three successive steps: excision, base change, and compaction. The excision excludes the unused color $j_0$ from the alphabet by applying $skip_{j_0}$ independently to each of the $m$ digits. This can be done in $O([mf/w])$ time as described in [10]: First the algorithm of Lemma 2.2(c) is used to compute an integer $y$, each of whose fields—with $y$ viewed as composed of $m$ fields of $f$ bits each—stores 1 if the corresponding digit is $\leq j_0$, and 0 otherwise. The application of $skip_{j_0}$ to all digits is finished by subtracting $1_{m,f} - y$ from the $(mf)$-bit standard representation, viewed as a single integer. Now we have a sequence of $m$ transformed digits drawn from the smaller alphabet $\Sigma'$, but still stored in $f$-bit fields.

Each group, composed of the (transformed) digits $a_0, \ldots, a_{c-1}$, say, can be viewed as representing the integer $\sum_{j=0}^{c-1} a_j c^j$. The base change uses the algorithm of Lemma 3.3 with $s = c$ to replace $\sum_{j=0}^{c-1} a_j c^j$ by $\sum_{j=0}^{c-1} a_j (c-1)^j$ independently within each group. This takes $O([mf/w]A)$ time, where $A = f + (\log(2 + cf/w))^2$, and, informally, encodes each group more economically. Indeed, since $c \log_2(c-1) = c \log_2 c + c \log_2 (1 - 1/c) \leq cf + c \ln(1 - 1/c) \leq cf - 1$, within each group of $cf$ bits that hold a group the most significant bit is 0.

At this point the entire representation, viewed as an integer $u$, consists of $h = m/c$ repetitions of a pattern consisting of $g - 1 = cf - 1$ bits considered to be in use followed by a single bit that is unused, and $h$ is a multiple of $g$. In order to satisfy the space bound of the lemma, the compaction reorders the $h(g-1)$ used bits in $u$ and stores them tightly in the $h(g-1) = mf - m/c$ least significant bit positions. This can be done in a way illustrated in Fig. 6. The part $v$ of $u$ consisting of its least significant $r = h - 1$ bits, whose $s = h - h/g$ used bits are labeled 1, . . . , 12 in the figure, is broken off and replicated $g - 1$ times through a multiplication with $1_{g-1,r}$ to yield an integer $x$. The remaining larger part of $u$ is shifted right by $r$ bits to yield an integer $y$.

Now the positions in $x$ of the $s$ unused bits in $y$ in least significant positions (i.e., ignore the leading unused bit of $y$) together hold copies of all $s$ used bits in $v$, so that applying a suitable mask to $x$ and adding the result to $y$ finishes the computation. The steps just described consist in evaluating the expression $(u \gg r) + ((u \text{ AND } (2^r - 1)) \cdot 1_{g-1,r}) \text{ AND } 1_{s,g}$, which can be done in $O([mf/w])$ time.

Fig. 6. The compaction of the used bits in an example with $g = 4$ and $h = 16$.

In order to convert the compact representation back to the standard representation, we reverse the steps described above. The integer $v$ can be restored by extracting its $s$ used bits with a mask, multiplying them by $1_{g-1,r}$, shifting the result right by $(g - 2)r$ bits and forming the conjunction with $2^r - 1$. After concatenating $v$ with $y$ to obtain $u$, we use the algorithm of Lemma 3.3 to replace $\sum_{j=0}^{c-1} a_j (c-1)^j$ by $\sum_{j=0}^{c-1} a_j c^j$ within each group. Finally $skip_{j_0}$ can be
applied independently to each digit much as \textit{skip} was. The entire conversion from the standard to the compact representation or back takes \(O([mf/w]A)\) time.

When \(D\) is in the compact representation, we can execute \textit{color} by undoing the compaction (\(O([mf/w])\) time), undoing the base change and the excision for the single relevant group (\(O([cf/w]A)\) time), and finally reading out the \(f\) bits of interest. To execute \textit{setcolor} and \textit{successor}, we carry out the complete conversion from the compact to the standard representation, apply the corresponding algorithm for the standard representation and, in the case of \textit{setcolor} and unless an inspection of (a saved copy of) \(Z\) shows that \(D\)'s client set has become full, convert \(D\) back to the compact representation. This takes \(O([mf/w]A)\) time.

Because \(m\) may not actually be a multiple of \(c^2f\), the number of bits saved by the compact representation relative to the standard representation is not necessarily \(m/c\), but still at least \((m - c^2f)/c\). Moreover, \(c\) bits are needed for the bit vector \(Z\). Hence the number of bits occupied by the compact representation is at most \(mf - m/c + 2cf\), as indicated in the lemma.

For given \(t \in \mathbb{N}\), suitable tables of at most \(c^2/t\) bits that can be computed in \(O(c^2/t)\) time and depend only on \(c\) and \(t\) allow us to carry out a base change in a group in \(O(t)\) time via table lookup (say, \([c/(2t)]\) digits at a time). The alternative time bound for \textit{color} follows easily. \(\Box\)

### 3.3 An Unrestricted Power-of-2-Colored Choice Dictionary

The top-level idea behind the colored choice dictionary of this section is to keep the overall organization of the uncolored choice dictionary of Section 2, but now letting weak indices correspond to deficient client vectors implemented with the data structure of Lemma 3.4 in order to gain the space needed for pointers to mates.

**Theorem 3.5.** There is an externally sized power-of-2-colored choice dictionary \(D\) that, for arbitrary given \(n, f \in \mathbb{N}\), can be initialized for universe size \(n\) and \(c = 2^f\) colors in constant time and subsequently occupies \(nf + 1\) bits and executes \textit{color} in \(O(cf + log n)/f + [cf/w]A\) time and \textit{setcolor} and \textit{choice} in \(O(cf + log n)/f + [cf/w]A\) time, where \(A = f + (log(2 + cf/w))^2\).

In particular, if \(cf = O(w)\), \textit{color} runs in \(O(cf log n)/w + f) = O(cf)\) time and \textit{setcolor} and \textit{choice} run in \(O(cf^2)\) time. For constant \(c\) the choice dictionary is atomic.

Alternatively, if initialized with an additional parameter \(t \in \mathbb{N}\) and given access to suitable tables of at most \(c^2/t\) bits that can be computed in \(O(c^2/t)\) time and depend only on \(c\) and \(t\), \(D\) can execute \textit{color} in \(O(cf + log n)/f + [cf/w]t\) time.

The transient space needed by \(D\) is \(O(cf + log n)/f + w)\) bits.

**Proof.** For the time being ignore the claim about constant-time initialization. Concerning many aspects, described in this paragraph, the colored choice dictionary \(D\) of Theorem 3.3 is similar to the uncolored choice dictionary of Section 2. Consider the situation following an initialization of \(D\) for universe size \(n\) and \(c\) colors and assume first that \(n\) is a multiple of an integer \(N \in \mathbb{N}\) that will be chosen later. Most of \(D\)'s information is kept in an array \(A\) of \(N\) cells \(A[1], \ldots, A[N]\) that we now call \textit{containers}. Correspondingly, we view \(D\)'s task as that of maintaining a sequence \((a_1, \ldots, a_N)\), where \(a_k\) is a sequence of \(m = n/N\) color values drawn from \(\{0, \ldots, c - 1\}\), for all \(k \in V = \{1, \ldots, N\}\). Again \(D\) stores an integer \textit{barrier} \(\mu\) with \(0 \leq \mu \leq N\), and an integer \(k \in V\) is said to be to the left of the barrier if \(k \leq \mu\) and to its right otherwise. An integer \(k\) to the left of the barrier is \textit{matched} to an integer \(\ell\) to the right of the barrier exactly if a designated field in \(A[k]\), called \(A[k].mate\), contains \(\ell\) and \(A[\ell].mate\) contains \(k\), and then \(k\) and \(\ell\) are \textit{mates}. As in Section 2, let \(mate(k)\) be the mate of \(k\) if \(k\) is matched and \(k\) itself if not, for all \(k \in V\). Again, \(k \in V\) is \textit{strong} if \(k\) is matched and \(k \leq \mu\) or \(k\) is unmatched and \(k \geq \mu\), and \(k\) is \textit{weak} if it is not strong. For convenience, we will apply the terms of being to the left or right of the barrier, matched, mates, strong, and weak also to containers, saying that \(A[k]\) is to the left of the barrier exactly if \(k\) is, for all \(k \in V\), etc. If a container \(A[k]\) is strong and unmatched, it simply stores \(a_k\) as a sequence of \(mf\) bits. If \(A[k]\) is strong and matched to \(A[\ell]\), \(A[k]\) stores the biggest part, \(a_k\), of \(a_k\) and the rest of \(a_k\), \(\overline{a_k}\), is stored in a field \(A[k].top\) of \(A[\ell]\). The part of \(A[k]\) not taken up by \(a_k\) holds \(A[k].mate\) as well as an \textit{auxiliary field} \(A[k].aux\) of \(O(c + log n)\) bits. Similarly as in the construction of Subsection 2.2, we store the barrier \(\mu\) in \(A[1].aux\) (except if \(\mu = 0\)). Again, the similarity to the data organization of the uncolored choice dictionary is pronounced.
The most significant difference to the situation in Section 2 is that if \( k \in V \) is weak, we can no longer conclude that \( a_k \) is zero (or a sequence of zeros) and hence that no information must be stored about \( a_k \) beyond the fact that \( k \) is weak. Instead the convention here is that \( A[k] \) is weak exactly if (the client vector corresponding to) \( a_k \) is deficient, i.e., if some color does not occur in \( a_k \). Thus even if \( A[k] \) is weak, it must store information “of its own”, but the deficiency of \( a_k \) makes it possible to do this in less space.

We realize each container \( A[k] \) as an instance of the data structure of Lemma 3.3 initialized for universe size \( m \) and \( c \) colors. Thus a container \( A[k] \) needs at least \( m/c - 2cf \) fewer bits when its client vector is deficient than when it is full. Informally, we can express this by saying that \( A[k] \) can carry a payload of at least \( m/c - 2cf \) bits when its client vector is deficient—so many unrelated bits can be stored within the space reserved for \( A[k] \). We need containers to have a payload of at least \( K(cf + \log n) \) bits for some constant \( K \in \mathbb{N} \) and achieve this by choosing \( m = \Theta(cf + \log n) \) appropriately.

If \( k \in V \) is strong, \( A[k] \) is in the standard representation. It has no payload, but \( O(c + \log n) \) of its bits form the fields \( A[k].mate \) and \( A[k].aux \). As mentioned above, if \( k \) is strong and matched, the rest of \( A[k] \) stores \( a_k \). If \( k \) is weak, \( A[k].mate, A[k].aux \) and \( A[k].top \) constitute the payload of \( A[k] \). The logical realization of the fields \( mate \) and \( aux \) can be seen to be different in weak and strong containers, but we ensure that they are located in the same bits in the two cases so that, in particular, it can be determined in constant time whether a container is weak or strong. This realizes in a procedural way the external fullness bit required by Lemma 3.3.

Each container also contributes \( c \) special bits, one to each of \( c \) dynamic uncolored choice dictionaries \( D_0, \ldots, D_{c-1} \) whose universe sizes are kept equal to \( \mu \) at all times. One may think of the special bits of a container as located in the container, but in fact \( D_0, \ldots, D_{c-1} \) are stored in contiguous memory locations, and whenever a container wants to inspect or change one of its \( c \) special bits, it must call the appropriate operation in one of \( D_0, \ldots, D_{c-1} \). We shall say that \( D_0, \ldots, D_{c-1} \) are distributed over \( A[1], \ldots, A[N] \). Each of \( D_0, \ldots, D_{c-1} \) needs one additional bit, which is stored in \( A[1].aux \) (except if \( \mu = 0 \), in which case the states of \( D_0, \ldots, D_{c-1} \) can be arbitrary). The positions of the \( c \) special bits in a container are chosen within the payload of the compact representation, but outside of the parts of the payload used for other purposes, and such that in the standard representation of the container no single color value is stored in bits that include two or more special bits; this is possible with a payload of \( cf \) bits reserved for this purpose. The operations are extended to maintain as an invariant for \( j = 0, \ldots, c - 1 \) that an integer \( k \in \{1, \ldots, \mu \} \) belongs to the client set of \( D_j \) exactly if \( j \) occurs as a color value in \( a_k \). Correspondingly, the client sets of \( D_1, \ldots, D_{c-1} \) are initialized to be empty, whereas the initial client set of \( D_0 \) is the entire set \( V \).

The information of its own that \( A[k] \) stores when \( k \) is weak (i.e., what “pays for” the entire payload) is (the deficient) \( a_k \). An unmatched weak container \( A[k] \) has the same structure as a matched weak container, except that \( A[k].mate \) and \( A[k].top \) are arbitrary—\( A[k].mate \) may not give rise to a spurious matching edge, though.

To execute color, we must determine a single color value in \( a_k \) for some \( k \in V \). Comparing \( k \) to \( \mu \) and inspecting \( A[k].mate \) and possibly \( A[\ell].mate \) for some \( \ell \in V \), we can discover in constant time whether \( A[k] \) is matched and, if so, its mate. This allows us to identify the container that contains the relevant color value, and we finish by returning the value obtained by calling color for that container with an appropriate argument, which may require us to retrieve a special bit.

Since \( m = \Theta(c(cf + \log n)) \), the operation takes \( O(c(cf + \log n)jf/w + [cf/w]|A) \) time.

To execute choice \((j)\), we distinguish between two cases. If \( \mu = N \), we compute \( k = D_j.choice \) and return 0 if \( k = 0 \) and \((k−1)m + A[k].successor(j, 0) \) otherwise. If \( \mu < N \), the color \( j \) occurs in \( a_k \), where \( k = mate(N) \), and a suitable return value can be obtained by executing \( successor(j, 0) \) in \( A[k] \) and possibly in the mate of \( A[k] \). We may have to retrieve up to \( 2c \) special bits, so the total time comes to \( O(c(cf + \log n)jf/w)|A| \).

To execute setcolor \((j, \ell)\), first read out the old color \( j_0 = color(\ell) \) of \( \ell \). Assume that \( j \neq j_0 \). Determine the \( k \in V \) such that the color of \( \ell \) is a component of \( a_k \) and take \( k' = mate(k) \). In the following, by “eliminating a possible spurious matching edge at \( \ell \)'s position, where \( i \in V \), we mean the following: If \( i \) has a mate \( \ell \)'s position, then change this fact by setting \( A[i'].mate \) to \( i' \).
If \( k \) is weak, save the payload of \( A[k] \) before changing the color of \( \ell \) from \( j_0 \) to \( j \) through an appropriate call of \textit{setcolor} in \( A[k] \). If the client set of \( A[k] \) continues to be deficient, i.e., if \( A[k] \) does not attempt to change its fullness bit, nothing more needs to be done, except that if \( k \leq \mu \), \( D_{j_0} \) and \( D_j \) should be updated appropriately with respect to \( k \) (again, the necessary tests can be carried out with \textit{successor}). If the client set of \( A[k] \) becomes full, proceed to carry out what corresponds to an insertion in Section 2. Take \( \tilde{\mu} = \mu \), compute \( \mu' = \text{mate}(\mu) \) and decrease \( \mu \) by 1. If \( \tilde{\mu} \neq \mu' \), i.e., if \( \tilde{\mu} \) is strong, then restore \( a_{\tilde{\mu}} \) by storing \( \overline{\text{top}} = A[\mu'], \text{top} \) in the appropriate bits of \( A[\mu] \) (if \( \mu' = k \), this involves the saved payload of \( A[k] \)) and eliminate a possible spurious matching edge at \( \tilde{\mu} \). If \( k \neq \mu' \), set \( A[\mu'], \text{top} := A[k], \text{top} \) and match \( k' \) and \( \mu' \) by executing \( A[k'], \text{mate} := \mu' \) and \( A[\mu'], \text{mate} := k' \). Finally if \( k > \mu \), eliminate a possible spurious matching edge at \( k \).

If \( k \) is strong, first change the color of \( \ell \) from \( j_0 \) to \( j \) through an appropriate call of \textit{setcolor} in a container. Then determine with one or two calls of \textit{successor}(\( j_0, 0 \)) whether the color \( j_0 \) still occurs in \( a_k \). If it does, nothing more needs to be done. Otherwise proceed to carry out what corresponds to a deletion in Section 2. If \( k \leq \mu \), overwrite the appropriate part of \( A[k] \) with \( A[k'], \text{top} \). Let \( \mu' = \text{mate}(\mu + 1) \) and increase \( \mu \) by 1. If \( k \leq \mu \), remove \( k \) from \( D_{j_0} \). If \( \mu' \neq k \), observe that \( \overline{\text{top}} \) is stored in \( A[\mu], \text{top} \) if \( \mu \neq \mu' \) and as part of the value of \( A[\mu'] \) otherwise and save \( \overline{\text{top}} \) in a variable \( v \). Determine for each \( j' \in \{0, \ldots, c - 1\} \) with a call of \( A[\mu], \text{successor}(j, 0) \) whether the color \( j' \) occurs in \( a_{\mu} \) and ensure that \( \mu \) belongs to the client set of \( D_{j'} \) if and only if this is the case. Then execute \( A[k'], \text{mate} := \mu' \) and \( A[\mu'], \text{mate} := k' \), which matches \( k' \) and \( \mu' \) except if \( k' \) and \( \mu' \) are on the same side of the barrier. Finally, if \( \mu' \neq k \), restore \( a_{\mu'} \) by executing \( A[k'], \text{top} := v \).

Since the total number of operations executed on containers and the number of special bits that need to be retrieved and written back are both \( O(c) \), \textit{setcolor} can be seen to operate within the time bound of \( O(c(cf + \log n)f/w/A) \) indicated in the theorem. In order to satisfy the assumption that \( n \) is a multiple of \( N \), we maintain \( O(cf + \log n)f \) surplus digits in a single instance of the data structure of Lemma 3.3 that is always kept in the standard representation. The time bounds of the theorem can still be guaranteed.

A final issue to be addressed is the constant-time initialization of \( D \). Viewing \( D \) as composed of \( w \)-bit words (plus possibly one incomplete word that can be initialized in constant time), we can almost provide the initialization using the initializable arrays of Katoh and Goto 15, but need to modify them in two ways. First, Katoh and Goto consider the initialization of all array entries to the same value \( v \), but here we need an initialization of \( D \) to a bit pattern in which all colors are 0, the client sets of \( D_1, \ldots, D_{c-1} \) are empty, and the client set of \( D_0 \) is \( \{1, \ldots, N\} \). We can handle this issue using a simple mechanism, described by Hagerup and Kammer 11, that consists in setting a fixed value \( v \) represent a “word-sized slice” of the desired initial bit pattern, while conversely using the slice to represent \( v \). The slice depends on the position in the array, but is easy to compute from that position. Second, the data structure of Katoh and Goto needs a bit \( \text{flag} \) in addition to the bits of the array that it maintains, \( \text{flag} = 1 \) signifying that all positions in the array have been written to, much as a special bit is used in Subsection 2.5 to signify that \( \mu = 0 \). Here we have already used all of the \( nf + 1 \) bits allowed by Theorem 3.5 and have no bit to spare. As long as \( \mu > 0 \), however, \( \text{flag} \) can be stored in \( A[1], \text{aux} \). Moreover, without violating the time bound of \textit{setcolor} we can easily ensure that all words of all containers to the right of the barrier have been written to, which implies that \( \text{flag} \) is superfluous (its value is known to be 1) whenever \( \mu = 0 \). Thus maintaining \( \text{flag} \) does not cost any extra space.

The alternative bound for \textit{color} is obtained simply by appealing to the corresponding part of Lemma 3.3. All containers can share the same tables. \( \square \)

The alternative time bounds of Theorem 3.5 depend on an external table. As expressed in the following theorem, we can also incorporate the table into the data structure itself.

\textbf{Theorem 3.6.} There is an externally sized power-of-2-colored choice dictionary that, for arbitrary given \( n, f, t \in \mathbb{N} \), can be initialized for universe size \( n \) and \( c = 2^f \) colors in constant time and subsequently occupies at most \( nf + c^f t + 1 \) bits and executes \textit{color} in \( O(cf + \log n)f/w + (cf/w)t \) time and \textit{setcolor} and \textit{choice} in \( O((cf + \log n)f/w/A) \) time, where \( A = f + (\log(2 + cf/w))^2 \). The transient space needed by \( D \) is \( O(w + cf \log n) \) bits.
Proof. The theorem follows immediately from Theorem 3.5 except that we must show how to achieve a constant initialization time despite the use of a table $Y$ of nontrivial size.

The main observation is that before an entry in $Y$ is first needed, with one exception, it can be computed from an earlier entry that it resembles. The reason is that $Y$ is used to map between $\sum_{i=0}^{u-1}a_i c^i$ and $\sum_{i=0}^{u-1}a_i(c-1)^i$ for some $s \leq c$, where $a_0, \ldots, a_{s-1}$ are consecutive color values. At the point where a new tuple $(a_0, \ldots, a_{s-1})$ of color values arises, it does so in a call of $\text{setcolor}$, and it is derived from a tuple $(a'_0, \ldots, a'_{s-1})$ that differs from $(a_0, \ldots, a_{s-1})$ in only one component. The two entries for $(a_0, \ldots, a_{s-1})$ (one for each direction of the mapping) can be computed from those of $(a'_0, \ldots, a'_{s-1})$ with a constant number of multiplications and divisions by numbers of the form $(c-1)^i$, with $i \in \{1, \ldots, s-1\}$. Since two integers of at most $cf$ bits each can be multiplied and divided in $O([cf^2/w])$ time, it can be seen that the computation can be accomplished within the time bound for $\text{setcolor}$ indicated in the theorem. The first entries in $Y$, those corresponding to the tuple $(0, 0, \ldots, 0)$, are trivial and can be filled in in constant time during the initialization. □

4 General Colored Choice Dictionaries

We now turn from the case in which the number $c$ of colors is a power of 2 to the case of general $c \geq 2$, the immediate difficulty being that a single color value cannot be stored in a number of bits without an unacceptable waste of space.

4.1 Compaction

When $c$ is not a power of 2, we need a compaction algorithm more general than the one illustrated in Fig. 4. It is characterized in the following lemma.

Lemma 4.1. Given positive integers $n$, $m$ and $u$ with $u \leq m$ and an integer of the form $\sum_{i=0}^{nu-1}2^i b_{nu+j}$, where $b_k \in \{0, 1\}$ for $k = 0, \ldots, nu - 1$, for a certain permutation $\sigma$ of $\{0, \ldots, nu - 1\}$ the integer $\sum_{k=0}^{nu-1}2^k b_{\sigma(k)}$ can be computed in $O([nm/w] A)$ time, where $A = \log{\min\{u, m-u\} + 2}$, using $O(nm)$ bits of working memory. Moreover, given $n$, $m$, $u$ and $\sum_{k=0}^{nu-1}2^k b_{\sigma(k)}$, the original integer $\sum_{i=0}^{u-1} \sum_{j=0}^{nu-1} 2^i b_{nu+j}$ can be reconstructed within the same time and space bounds.

Proof. We provide only an informal proof sketch based mostly on figures. The positions of the bits $b_0, \ldots, b_{nu-1}$ can be visualized as what we will call a group arithmetic progression with period $m$, $n$ groups, group size $u$, weight $nu$ and range $nm$ (see Fig. 5). Every group arithmetic progression considered in the following has period $m$ and range at most $nm$ without this being stated explicitly. If a group arithmetic progression has group size $u$, we call it a $u$-sequence. In Fig. 7, the vertical bar is placed in position $nu$, i.e., with $nu$ positions to its right. The task at hand can therefore be viewed as that of mapping the balls to the left of the bar bijectively to the holes to the right of the bar.

![Fig. 7. The compaction problem: Mapping balls to holes.](image-url)

Imagine that the vertical bar splits into two copies. One copy, the left bar, moves left until it hits the first position that is a multiple of $m$. The other copy, the right bar, moves right until it hits the first position that is a multiple of $m$ and has the property that the number of holes to
its right is bounded by the number of balls to the left of the left bar. Ignoring for the time being the subproblem represented by the balls and holes between the two bars, we are faced with the problem of mapping a subset of the \( u_0 \)-sequence of balls to the left of the left bar bijectively to the \( v_0 \)-sequence of holes to the right of the right bar, where \( u_0 = u \) and \( v_0 = m - u_0 \). More generally, we consider the problem of mapping a subset of a left \( u \)-sequence of balls bijectively to a right \( v \)-sequence of bins, where \( u \) and \( v \) are positive integers with \( u + v \leq m \) and the weight of the right sequence is bounded by that of the left sequence (see Fig. 8).

![Fig. 8. Mapping one group arithmetic progression to another of no larger weight.](image)

If \( u \leq v \) (a group of balls fits in a group of holes), we place some of the balls in some of the holes as illustrated in Fig. 9 where groups of balls are shown labeled consecutively in the order, from right to left, in which they occur in the left sequence. Similarly as in the procedure of Fig. 6, this can be carried out in \( O(\lceil nm/w \rceil) \) time, mainly with a multiplication and a constant number of bitwise Boolean operations and shifts.

![Fig. 9. A partial mapping of smaller groups of balls to larger groups of holes.](image)

If \( u \) divides \( v \), all holes are filled, and this part of the computation is finished. Otherwise what remains of the left sequence is still a \( u \)-sequence (but with fewer groups), whereas the remaining holes form a \( (v \mod u) \)-sequence (with the same number of groups). We can therefore say that the computation of Fig. 9 reduces a \( (u, v) \)-instance of the problem to a \( (u, v \mod u) \)-instance.

If \( u > v \) (a group of balls is larger than a group of holes), we place some of the balls in some of the holes with the alternative procedure shown in Fig. 10, which can again be carried out in \( O(\lceil nm/w \rceil) \) time. Here the condition \( u + v \leq m \) is essential, as it prevents overlap between the different shifted copies of the sequence of balls. Sequences of \( v \) balls are shown labeled by an integer that indicates the group (of size \( u \)) of the left sequence from which they originate and a letter that indicates their position within that group.

![Fig. 10. A partial mapping of larger groups of balls to smaller groups of holes.](image)

If \( v \) divides \( u \), all holes are filled, and this part of the computation is finished. Otherwise what remains of the right sequence is still a \( v \)-sequence (but with fewer groups), whereas the balls that remain unplaced form a \( (u \mod v) \)-sequence (with the same number of groups). Thus the \( (u, v) \)-instance is reduced to a \( (u \mod v, v) \)-sequence. It is well-known that the mapping that
takes \((u, v)\) to \((u, v \mod u)\) if \(u \leq v\) and to \((u \mod v, v)\) if \(u > v\), if started at \((u_0, v_0)\), reaches a pair with a zero component after \(O(\log(\min(u_0, v_0) + 2))\) repeated applications. Indeed, if \(u > v > 0\), \(u \mod v \leq u/2\). This shows the running time claimed for the compaction, except that we still have to consider the “middle” instance ignored above.

Consider the balls of the middle instance that are to the left of the original vertical bar in Fig. 7 i.e., that are to be placed. If their number is \(p\), by construction, the remaining holes form a \(v_0\)-sequence of weight at most \(p + v_0\). Letting \(p'\) be the largest integer bounded by \(p\) that is a multiple of \(v_0\), we can use the method of Fig. 10 to place \(p'\) consecutive of the \(p\) balls in holes so that the remaining holes form at most two groups (that may not be of the same size). In the same manner, the number of groups of unplaced balls can be reduced below a constant with the method of Fig. 9 without increasing the number of groups of remaining holes. The remaining instance has \(O(1)\) groups of balls and holes and can therefore obviously be solved in constant time.

It is not difficult to see that the steps represented by Figs. 9 and 10 are reversible in the sense that the balls placed in holes can be returned to their original positions in \(O(nm/w)\) time. Therefore the whole computation is reversible within the time bound of the lemma. For the “forward” computation we must keep track of a constant number of nonnegative integer parameters that are functions of \(n, m\) and \(u\) and bounded by \(nm\). In order to reverse the computation, we need these parameters for every stage of the computation. They can be obtained in \(O((nm/w) \Lambda)\) time by simulating the forward computation and take up \(O(1)\) bits of memory.

4.2 A Small \(c\)-Color Choice Dictionary for General \(c\)

When \(c\) is not a power of 2, we can no longer represent a color drawn from \(\{0, \ldots, c - 1\}\) in \(\log_c c\) bits, as in the standard representation of Lemma 3.4. Our core tool for coping with this complication is a result of Dodis, Pătrașcu and Thorup [4]. They demonstrate that a usual binary computer can simulate a \(C\)-ary computer, for arbitrary integer \(C \geq 2\), essentially without a loss in time or space in the sense that an array of \(n\) integers, each drawn from \(\{0, \ldots, C - 1\}\), can be represented in \(n \log_2 C + O(1)\) bits so as to support constant-time reading and writing of individual array entries. In order to be able to employ word parallelism, we use this not with \(C = c\), as would be most natural, but with \(C = c^m\) for some \(m\) chosen essentially to make \(C \approx 2^w\). Let us call elements of \(\{0, \ldots, c - 1\}\) and of \(\{0, \ldots, C - 1\}\) small digits and big digits, respectively. A big digit is shown symbolically in Fig. 11(a) as it is represented in the data structure of Dodis, Pătrașcu and Thorup; it may be thought of a composed of \(m\) \(c\)-ary digits, each of which is drawn as a triangle. Once the big digit is read out of the data structure of Dodis, Pătrașcu and Thorup, it is given by its usual binary representation as a sequence of \(\lceil m \log_2 c \rceil\) bits. In Fig. 11(b) each bit of the big digit is shown as a dot. If \(c\) is not a power of 2, certain patterns of values (namely those that represent the integers \(c^{m}, c^{m} + 1, \ldots, 2^{\lceil m \log_2 c \rceil} - 1\)) cannot occur. This is symbolized in Fig. 11(b) by the leftmost (most significant) dot being only partially drawn; put differently, the missing part of the leftmost bit corresponds to a fraction of a bit that is wasted. If we convert the big digit to the corresponding sequence of small digits, as shown in Fig. 11(c), we can operate efficiently on the small digits as on the standard representation in the proof of Lemma 3.4. Indeed, the sequence of small digits is in the standard representation of Lemma 3.4 only for a number of colors equal to \(2^f\), where \(f = \lceil \log c \rceil\)—the largest colors simply happen not to be present. The conversion can be carried out with the algorithm of Lemma 3.3 we shall express this by saying that we convert the big digit from base \(c\) to base \(2^f\).

Just as when \(c\) is a power of 2, we need a compact representation that can be used to encode deficient client vectors and essentially substitutes base \(c - 1\) for base \(c\). Since \(c - 1\) was never assumed to be a power of 2, here the differences are small. Once containers are available, the proof can proceed as in Subsection 3.3. We now describe the details and begin by providing an analogue of Lemma 3.4 for general values of \(c\).

Lemma 4.2. There is an externally sized colored choice dictionary \(D\) that, for arbitrary given \(m, c \in \mathbb{N}\), can be initialized for universe size \(m\) and \(c\) colors in \(O((m \log c)/w)\) time and
subsequently needs to record in an external bit whether its client vector is full, stores its state as an element of \(\{0, \ldots, c^m - 1\}\) and executes \textit{color}, \textit{setcolor} and \textit{successor} in \(O(\lceil (m \log c)/w \rceil A)\) time, where \(A = \log m + (\log(2 + (m \log c)/w))^2\). Moreover, at times when \(D\)'s client vector is deficient, \(D\)'s state is bounded by \(2^{\lceil m \log c/(m/2c + 3c) \rceil}\).

Alternatively, if initialized with an additional parameter \(t \in \mathbb{N}\) and given access to certain tables of at most \(c^{m/t}\) bits that can be computed in \(O(c^{m/t})\) time and depend only on \(c\), \(m\) and \(t\), \(D\) can execute \textit{color}, \textit{setcolor} and \textit{choice} in \(O(\lceil (m \log c)/w t \rceil)\) time.

The transient space needed by \(D\) is \(O(w + m \log c)\) bits.

**Proof.** We again view \(D\)'s task as that of maintaining a sequence of \(m\) color values or digits, each drawn from \(\{0, \ldots, c - 1\}\). When \(D\) is not in the compact representation, it simply stores its state as the single \(c\)-ary integer in \(\{0, \ldots, c^m - 1\}\) formed by the \(m\) digits. In order to execute \textit{color}, \textit{setcolor} or \textit{successor}, \(D\) converts its state from base \(2\) to base \(2^f\), where \(f = \lfloor \log c \rfloor\), obtaining what we call the \textit{loose} representation, executes the operation in question on the loose representation as described in the proof of Lemma 3.3 and reconverts its state from base \(2^f\) to base \(c\). The base conversion is carried out with the algorithm of Lemma 3.3 in \(O(\lceil (m \log c)/w \rceil)\) time, and the operations executed on the loose representation are no more expensive.

As in the proof of Lemma 3.4 in the compact representation a vector \(Z\) of \(c\) bits is used to keep track of the set of colors represented in \(D\). When required to convert between the standard and the compact representation, \(D\) carries out the conversion via the loose representation. The conversion between the loose representation and the compact representation is illustrated in Fig. 12. To go from the loose to the compact representation at a time when the client vector is deficient (Fig. 12(a), we first apply the excision operation to ensure that the unused color is \(c - 1\) (Fig. 12(b)). Ignoring rounding issues, we proceed as follows: Within groups, now of \(2c\) consecutive digits each, we convert from base \(2^f\) to base \(c - 1\) (Fig. 12(c)). Since \(2c \log(c - 1) \leq 2c \log c + 2c \ln(1 - 1/c) \leq [2c \log c] - 1\), each group can be stored in a field of \([2c \log c] - 1\) \textit{used} bits. Appealing to the algorithm of Lemma 1.1 we move the used bits of all groups to \(m/(2c))([2c \log c] - 1) \leq [m \log c] - m/(2c)\) consecutive positions. Since \([m \log c]\) unrestricted bits are available for storing the state of \(D\) (Fig. 11(b) has at least this many full dots), this yields \(m/(2c)\) free bits. Let us now take rounding into account. The formation of groups may leave up to \(2c - 1\) digits that are not part of any group. Even in the compact representation, we store such digits to base \(2^f\), which wastes less than one bit per digit and at most \(2c\) digits altogether. In addition, \(c\) bits are occupied by \(Z\). In summary, the number of free bits is at least \(m/(2c) - 3c\), as indicated in the lemma.

For given \(t \in \mathbb{N}\), suitable tables of at most \(c^{m/t}\) bits that can be computed in \(O(c^{m/t})\) time and depend only on \(c\) and \(m\) allow us to carry out all base conversions in \(O(\lceil (m \log c)/w t \rceil)\) time, which leads to the alternative time bounds.

\[\square\]

### 4.3 An Unrestricted \(c\)-Color Choice Dictionary for General \(c\)

In order to combine several instances of the data structure of the previous lemma, we need the data structure of Dodis, Pătraşcu and Thorup [4] as extended by Hagerup and Kammer [10] to support constant-time initialization. Lemma 4.3 below specializes [10] Theorem 6.5 to the case.
Fig. 12: The conversion between the loose and the compact representation of a container. A filled-in dot represents a bit that is in use, a circle represents a free bit, and a circle with a cross represents a bit that is part of the payload.

c_1 = \cdots = c_{n-1} \geq c_n. On the other hand, Lemma 4.3 incorporates a factor of $O(q \log q)$ not present in the formulation of [10], where $q = O(1)$ was assumed. The factor accounts for the time needed to multiply and divide integers of $O(q)$ $w$-bit words each according to Lemma 3.1

**Lemma 4.3 ([10]).** There is a data structure that, for all given $n, C, C' \in \mathbb{N}$ with $2 \leq C' \leq C$, can be initialized in constant time and subsequently occupies $(n-1) \log_2 C + \log_2 C' + O((\log n)^2 + 1)$ bits and maintains a sequence drawn from $\{0, \ldots, C-1\}^{n-1} \times \{0, \ldots, C'-1\}$ under reading and writing of individual elements in the sequence in $O(q \log q)$ time, where $q = 2 + (\log C)/w$. The data structure does not initialize the sequence. The parameter $C$ may be presented to the data structure in the form of a pair $(x, y)$ of positive integers with $C = x^y$ and $y = n^{O(1)}$, and the analogous statement holds for $C'$.

**Theorem 4.4.** There is an externally sized choice dictionary that, for arbitrary given $n, c \in \mathbb{N}$ with $c = n^{O(1)}$, can be initialized for universe size $n$ and $c$ colors in constant time and subsequently occupies $n \log_2 c + O((\log n)^2 + 1)$ bits and executes `color`, `setcolor` and `choice` in $O(q(\log(c + \log n) + (\log q)^2))$ time, where $q = 2 + c \log c + \log n)/w$. In particular, if $c = O(w)$, $q = O(1 + c \log c)$ and the operation times are $O((\log \log(n + 4) + (\log c)^2)c \log c)$.

Alternatively, if initialized with an additional parameter $t \in \mathbb{N}$ and given access to suitable tables of at most $c^{\log log n}/t$ bits that can be computed in $O(c^{(\log log n)/t})$ time and depend only on $c$, $n$ and $t$, $D$ can execute `color`, `setcolor` and `choice` in $O(q t)$ time. If $c = O(w)$, the operation times are $O(1 + t c \log c)$.

The transient space needed by $D$ is $O(w + c(c + \log n) \log c)$ bits.

**Proof.** We use the same overall organization as in the data structure of Theorem 3.5 but implement containers via Lemma 4.2 and store the state of each container as a “big digit” in a single global instance $D$ of the data structure of Lemma 4.3. Recall that each container should offer a payload of $\Theta(c + \log n)$ bits when its client vector is deficient. We choose $m \in \mathbb{N}$ just large enough to meet this goal, which for the data structure of Lemma 4.2 means that $m = \Theta(c + \log n))$. We then form $N = \lfloor n/m \rfloor$ containers, each with a universe size of $m$, and, if $m$ does not divide $n$, an additional instance of the data structure of Lemma 4.2 with a universe size of $m \mod n$ whose state is also stored in $D$ but that is otherwise handled separately, similarly to what happens to the surplus digits in the data structure of Theorem 4.4. The quantities $C = c^m$ and $C' = c^{m \mod n}$ are available in the form $x^y$ with $x = c$ and $y = n^{O(1)}$, as allowed by Lemma 4.3.

Each of the operations `color`, `setcolor` and `choice` carries out a constant number of operations on containers. To operate on a container, we first fetch its state in $D$, which takes $O(q \log q)$ time according to Lemma 4.3. By Lemma 4.2, the operation on the container itself can be carried out in $O(q A)$ time, where $A = \log m + (\log q)^2 = O(\log(c + \log n) + (\log q)^2)$. Storing the new state of the container after the operation in $D$ again takes $O(q \log q)$ time, and an overall time bound of $O(q(\log(c + \log n) + (\log q)^2))$ follows. The alternative time bounds are obtained simply by substituting $t$ for $A$. \qed
5 Iteration

This section discusses iteration for the choice dictionaries developed in the previous sections. If a choice dictionary is to support iteration, it must be supplied with additional space (informally, for storing how far the iteration has progressed), namely $O(\log n)$ bits for an uncolored choice dictionary and $O(c \log n)$ bits for a $c$-colored choice dictionary, where $n$ is the universe size. In the following discussion, such additional space is assumed to be available.

5.1 Iteration in the Uncolored Choice Dictionary

When the data structure of Lemma 2.3 is used to realize an (uncolored) choice dictionary with universe size $n$ and client set $S$, it is natural to associate the $k$th component of the sequence $(a_1, \ldots, a_N)$, for $k = 1, \ldots, N$, with the subset $U_k = \{2b(k-1)+1, \ldots, 2bk\}$ of the universe $U = \{1, \ldots, n\}$ and to view $a_k$ as a data structure that represents the subset $S_k = S \cap U_k$ of $S$, each $1$ in the binary representation of $a_k$ corresponding to an element of $S_k$. With an algorithm of Lemma 2.2(a), it is easy to support robust iteration over $S_k$ in constant time: If $S_k$ is nonempty when iterate.next is first called, enumerate min $S_k$. At every subsequent call of iterate.next, enumerate the smallest element of $S_k$ larger than the element most recently enumerated, ending the enumeration when there is no such larger element. Similarly, the dictionary $D_2$ of Subsection 2.1 supports constant-time robust iteration and can be ignored in what follows.

By the considerations of the previous paragraph, iteration in the choice dictionary of Section 2 essentially boils down to iterating over those $k \in V = \{1, \ldots, N\}$ with $a_k \neq 0$, i.e., over the strong $k \in V$. Noting that the set of strong $k \in V$ at all times is $\{\text{mate}(\ell) \mid \ell \in \{\mu+1, \ldots, N\}\}$, it is easy to achieve this in the static case, i.e., when no insertions or deletions take place during the iteration: For $\eta = N, N-1, \ldots, \mu+1$, enumerate $\text{mate}(\eta)$. We think of $\eta$ as carrying out a sweep from $N$ down to $\mu+1$, always enumerating $\text{mate}(\ell)$ for each $\ell$ encountered. Even in the general (not necessarily static) case, ideally, we should have $R_0 \subseteq L \subseteq R$, where $R$ is the set of strong integers in $V$ that have not been enumerated, $R_0 \subseteq R$ is the set of such integers that have been strong since the beginning of the iteration and $L = \{\text{mate}(\ell) \mid \ell \in \{\mu+1, \ldots, \eta\}\}$. A strong integer $k \in V$ is considered to belong to $R$ and possibly $R_0$ as long as not all elements represented by $a_k$ have been enumerated.

A deletion may cause an element $k$ of $R$ to the left of the barrier to drop out of $L$, namely if $k$ swaps a mate in front of the sweep (and therefore still to be swept over) for a mate behind the sweep (this may happen in the situations of Fig. 1(c) and Fig. 5(c)). In order to “rescue” such elements $k$, we collect them in a set $S^+$ and iterate also over $S^+$, as described below. On the other hand, an insertion may cause an element to the left of the barrier that has already been enumerated and left $R$ and $L$ at that time to reenter $L$ by acquiring a new mate in front of the sweep (see Figs. 3 parts (a) and (c)). In an attempt to prevent such elements from being enumerated again, we store them in a set $S^-$ of elements to be skipped. The details follow.

We introduce two dynamic uncolored choice dictionaries distributed over $A[1], \ldots, A[N]$, $D^+$ with client set $S^+$ and $D^-$ with client set $S^-$. As in the proof of Theorem 2.3, $D^+$ and $D^-$ are realized via bits in the auxiliary fields, and their universe sizes are kept equal to $\mu$ at all times. By means of a buffer of $\lceil \log(n+1) \rceil$ bits, initialized to the value $0$, we ensure that if $D^+.choice$ at some point returns a nonzero integer $k$, subsequent calls of $D^+.choice$ will return the same integer $k$ for as long as $k$ remains an element of $S^+$. The method is simple: When the value of the buffer is $0$, the integer returned by a call of $D^+.choice$ is also stored in the buffer, and as long as the value of the buffer remains nonzero, subsequent calls of $D^+.choice$ simply return that value instead of executing the normal steps. Finally, when the value in the buffer is deleted from $D^+$, the buffer is set to $0$. This mechanism helps to ensure that once the enumeration of the elements represented by an integer $a_k$ has started, where $k \in V$, it is completed without intervening enumeration of other elements. The iteration also uses an integer $\eta \in \{0, \ldots, n\}$ to keep track of the sweep.

To start an iteration (i.e., to execute iterate.init), set $\eta := N$ and initialize $D^+$ and $D^-(to S^+ = S^- = \emptyset)$ for a universe size of (the current value of) $\mu$. To test whether elements remain to be enumerated (i.e., to execute iterate.more), test whether $S^+ \neq \emptyset$ or $\eta > \mu$. Finally,
to enumerate the next element (i.e., to execute \texttt{iterate.next}), do the following: If $S^+ \neq \emptyset$, let $k = D^+.choice$ and enumerate the next element of $a_k$. If subsequently $a_k$ has no more elements to be enumerated, delete $k$ from $S^+$. If instead $S^+ = \emptyset$ but $\eta > \mu$, let $k = \text{mate}(\eta)$. If $k \notin S^-$, proceed to enumerate the next element of $a_k$. If subsequently $a_k$ has no more elements to be enumerated, decrease $\eta$ by 1. If $k \in S^-$, also decrease $\eta$ by 1. Since no element was enumerated in this case, however, restart \texttt{iterate.next} recursively.

When an insertion or a deletion causes an element of $V \setminus S^-$ to drop out of $L$ even though it continues to be strong, insert it in $D^+$—it must be to the left of the barrier. When a deletion causes an element of $S^+$ to become weak, delete it from $D^+$. Finally, when an insertion causes an element of $V$ to the left of the barrier to enter $L$ even though it was not in $L \cup S^+$ before the insertion, insert it in $D^-$; an inspection of Figs.\ref{fig:fig3} and \ref{fig:fig4} shows that this cannot happen in a deletion. In addition to these situations in which $S^+$ and $S^-$ are explicitly changed, one may note that if an insertion causes an integer $k$ to leave $\{1, \ldots, \mu\}$ ($k$ crosses the barrier from left to right), then $k$ automatically drops out of $S^+$ and $S^-$. It can be verified by induction that between operations, the following holds at all times during an iteration: $L \setminus S^-$ and $S^+$ are disjoint and contain only strong integers. Moreover, $R_0 \subseteq (L \setminus S^-) \cup S^+$, so that all elements that should be enumerated are actually enumerated. In the \textit{decremental} case, i.e., when there are no insertions during an iteration, $S^-$ is always empty, and the stronger invariant $R = R_0 = L \cup S^+$ holds. As a consequence, the iteration can be seen to be robust.

In the \textit{incremental} case, i.e., when there are no deletions during an iteration, an element $k \in V$ may drop out of $S^-$ implicitly, namely by virtue of crossing the barrier, as explained above. Informally, this causes the data structure to lose knowledge of the fact that $k$ was already enumerated, and as a result $k$ may be enumerated a second time. However, because the barrier moves only in one direction, this can happen at most once to each $k \in V$, so that no element is enumerated more than twice. The iteration no longer happens in constant worst-case time because of the need to skip elements of $L \cap S^-$. Skipping an element $k \in L \cap S^-$ takes constant time and removes $k$ from $L$. Even though \texttt{iterate.more} is only a query operation, it should update the state of the iteration to prevent elements from being skipped repeatedly. Since $L \cap S^-$ is empty at the beginning of an iteration and no insertion causes more than a constant number of integers to enter $L \cap S^-$, the total time wasted in skipping elements of $L \cap S^-$ up to a certain point of an iteration can then be seen to be within a constant factor of the number of insertions carried out since the beginning of the iteration.

In the general case, when insertions and deletions may be arbitrarily intermingled, we can still bound the time spent in skipping elements of $L \cap S^-$ by a constant times $O(n_a)$, where $n_a$ is the number of insertions and deletions carried out since the beginning of the iteration. The number of times that an element $k \in V$ is enumerated a second or later time is also $O(n_a)$, and each such enumeration causes at most $2b$ elements represented by $a_k$ to be enumerated again. Since we can choose $b = \Theta(\log n)$, this leads to a bound of $O(n_a \log n)$ on the time spent in unwanted enumerations.

The results of this subsection can be summarized as follows:

\textbf{Theorem 5.1.} There is a self-contained (uncolored) choice dictionary that, for arbitrary $n \in \mathbb{N}$, can be initialized for universe size $n$ in constant time and subsequently occupies $n + O(\log n)$ bits, executes \texttt{insert}, \texttt{delete}, \texttt{contains} and \texttt{choice} in constant time and supports the following forms of iteration: If during an iteration there are no calls of \texttt{insert} (the decremental case), the iteration is robust and works in constant time. If during an iteration there are no calls of \texttt{delete} (the incremental case), the iteration is robust, except that each integer that is enumerated may be enumerated a second time, and the iteration works in constant amortized time in the following sense: At a time when, since the start of an ongoing iteration, there has been $n_a$ calls of \texttt{iterate.next} and \texttt{iterate.more} and $n_i$ calls of \texttt{insert}, the total time spent in the single call of \texttt{iterate.init} and in the $n_u$ calls of \texttt{iterate.next} and \texttt{iterate.more} is $O(1 + n_u + n_i)$. If during an iteration insertions and deletions may be carried out in an arbitrary order, finally, only elements of the client set are enumerated, but an integer may be enumerated repeatedly. If an iteration starts with $n_0$ elements in the client set and there are $n_a$ calls of \texttt{insert} and \texttt{delete} during a
3.5, we carry out the following procedure to iterate over \( S \). Let the client vector of \( n \) tables of at most \( \mathcal{O} \) calls of \( D \), setcolor there are no calls of \( O \), we use the dynamic uncolored dictionary number of containers whose occurrences of the color \( j \) in \( D \) previous iteration to have been complete. Suppose that \( \eta \) \( \eta \) enumerate both \( \eta \) and \( \eta \prime \). The following more precise definitions were adapted from \([10]\). An application of Theorem 5.2 is to breadth-first search (BFS) and the computation of shortest-path forests. The following more precise definitions were adapted from \([10]\).

5.2 Iteration in the Colored Choice Dictionaries

For the colored choice dictionaries of Sections 3 and 4, we can support efficient iteration only if each iteration is complete, i.e., is allowed to enumerate all elements without being terminated early. Since complete iterations are common, this still leaves interesting applications. Complete iteration can be added to the choice dictionaries of Theorems 3.3, 3.6, and 4.1. We give only one example, corresponding to Theorem 4.4 and specialize the theorem to the case of constant \( c \).

**Theorem 5.2.** There is a self-contained choice dictionary \( D \) that, for arbitrary given \( n \in \mathbb{N} \) and constant \( c \in \mathbb{N} \), can be initialized for universe size \( n \) and \( c \) colors in constant time and subsequently occupies \( n \log_2 c + O((\log n)^2 + 1) \) bits, executes color, setcolor and choice in \( O(\log \log n) \) time and, for \( j = 0, \ldots, c - 1 \), supports complete iteration over \( S_j \), where the client vector of \( D \) is \((S_0, \ldots, S_{c-1})\), as follows: Only elements of \( S_j \) are enumerated, every integer that belongs to \( S_j \) during the entire iteration is enumerated, but an integer may be enumerated repeatedly. If an iteration over \( S_j \) starts with \( |S_j| = n_j \) and there are \( n_u \) calls of setcolor during a period of time from the start of the iteration, the total time spent in the single call of iterate.init and in calls of iterate.next and iterate.more during that period of time is \( \mathcal{O}(1 + n_j \log \log n + n_u \log n) \). Alternatively, if initialized with an additional parameter \( t \in \mathbb{N} \) and given access to suitable tables of at most \( n^{1/t} \) bits that can be computed in \( O(n^{1/t}) \) time and depend only on \( n \), \( c \) and \( t \), \( D \) can execute color, setcolor and choice in \( O(t) \) time, and the time bound above for iteration is replaced by \( O(1 + n_j t + n_u \log n) \).

**Proof.** Let the client vector of \( D \) be \((S_0, \ldots, S_{c-1})\). With notation as in the proof of Theorem 3.3, we carry out the following procedure to iterate over \( S_j \): For \( \eta = \eta, \eta \prime \), enumerate both \( \eta \) and \( \eta \prime \). The following more precise definitions were adapted from \([10]\). An application of Theorem 5.2 is to breadth-first search (BFS) and the computation of shortest-path forests. The following more precise definitions were adapted from \([10]\).

5.3 Breadth-First Search and Shortest-Path Forests

An application of Theorem 5.2 is to breadth-first search (BFS) and the computation of shortest-path forests. The following more precise definitions were adapted from \([10]\).

Given a directed or undirected \( n \)-vertex graph \( G = (V, E) \) and a permutation \( \pi \) of \( V \), i.e., a bijection from \( \{1, \ldots, n\} \) to \( V \), we define a spanning forest of \( G \) consistent with \( \pi \) to be a sequence \( F = (T_1, \ldots, T_q) \), where \( T_1, \ldots, T_q \) are vertex-disjoint outtrees that are subgraphs of \( G \) (if \( G \) is directed) or of the directed version of \( G \) (if \( G \) is undirected) and the union of whose vertex sets is \( V \), such that for each \( v \in V \), the root of the tree in \( \{T_1, \ldots, T_q\} \) that contains \( v \) is the first vertex in the sequence \( (\pi(1), \ldots, \pi(n)) \) from which \( v \) is reachable in \( G \). If, in addition, every path in the union of \( T_1, \ldots, T_q \) (if \( G \) is directed) or its undirected version (if \( G \) is undirected) is a shortest path in \( G \), \( F \) is a shortes-path spanning forest of \( G \) consistent with \( \pi \). A shortest-path
spanning forest of $G$ can be produced by a BFS that, whenever there are no vertices adjacent to those already processed, picks the next vertex to process as the first vertex, in the order given by $\pi$, that has not yet been processed.

By computing a shortest-path spanning forest $F = (T_1, \ldots, T_q)$ of an $n$-vertex graph $G = (V, E)$ consistent with a permutation $\pi$ of $G$ we mean producing a sequence $((u_1, v_1, k_1, d_1), \ldots, (u_n, v_n, k_n, d_n))$ of 4-tuples with $u_i \in V \cup \{0\}$, $v_i \in V$, $k_i \in \mathbb{N}$ and $d_i \in \mathbb{N}_0$ for $i = 1, \ldots, n$ such that $k_1 \leq \cdots \leq k_n$, $\{v_i \mid 1 \leq i \leq n \text{ and } k_i = j\}$ and $\{(u_i, v_i) \mid 1 \leq i \leq n \text{ and } k_i = j\}$ are precisely the vertex and edge sets of $T_j$, respectively, for $j = 1, \ldots, q$, and $d_i$ is the depth of $v_i$ in $T_{k_i}$, for $i = 1, \ldots, n$. If, in addition, for each $\ell \in \{1, \ldots, n\}$ with $u_\ell \neq 0$ there is an $i \in \{1, \ldots, \ell - 1\}$ with $v_i = u_\ell$, we say that $F$ is computed in top-down order. Thus for $j = 1, \ldots, q$, the root and the edges of $T_j$ are to be output (in a top-down order), each with the index $j$ of its tree $T_j$ and an indication of the depth in $T_j$.

Hagerup and Kammer describe an algorithm for computing a shortest-path spanning forest of a given graph $G = (V, E)$ that stores a color drawn from \{white, gray, black\} for each vertex in $V$ in a 3-color choice dictionary $D$ and needs linear time outside of a number of complete iterations over the set of gray vertices \cite{HagerupKammer99} Theorem 8.5. Suppose that $G$ has $n$ vertices and $m$ edges. As is easy to see from the proof, the sum, over all iterations, of the number of gray vertices present at the beginning of the iteration is $O(n)$, and repeated enumerations of a gray vertex do not jeopardize the correctness of the algorithm. Therefore Theorem 5.2 implies the following new result.

**Theorem 5.3.** Given a directed or undirected graph $G = (V, E)$ with $n$ vertices and $m$ edges and a permutation $\pi$ of $V$, a shortest-path spanning forest of $G$ consistent with $\pi$ can be computed in top-down order in $O(n \log n + m \log \log n)$ time with $n \log_2 3 + O((\log n)^2 + 1)$ bits of working memory. Alternatively, for every $t \in \mathbb{N}$, the problem can be solved in $O(n \log n + mt)$ time with $n \log_2 3 + O(n^{1/4} + (\log n)^2)$ bits. If $G$ is directed, its representation must allow iteration over the inneighbors and outneighbors of a given vertex in time proportional to their number plus a constant (in the terminology of \cite{DodisPatrascuThorup07}, $G$ must be given with in/out adjacency lists).

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