Positive knots have negative signature

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Abstract. We show that if a nontrivial link in $\mathbb{R}^3$ has a diagram with all crossings positive ($\bigtriangledown$) then the signature of the link is negative. It settles the old folklore conjecture.

It was asked by Birman, Williams, and Rudolph whether nontrivial Lorentz knots [B-W] have always positive signature. Lorentz knots are examples of positive braids (in our convention they have all crossings negative so they are negative links). It was shown by Rudolph [R] that positive braids have positive signature (if they represent nontrivial links). Murasugi has shown that nontrivial, alternating, positive links have negative signature. Here we solve the conjecture in general.

Theorem 1. Let $L$ be a nontrivial link which has a diagram with all crossings positive (i.e. $L$ is positive), then the signature of $L$, $\sigma(L) < 0$.

Proof: Our main tool is the result of Murasugi (compare [P]) which says that if two links $L_+$ and $L_-$ have identical diagrams except near one crossing where they look as on Fig. 1

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{}
\end{figure}

then $\sigma(L_+) \leq \sigma(L_-)$. First assume that $L$ is a knot. Consider a positive diagram of $L$ (also denoted by $L$) with minimal number of crossings. Consider an innermost 1-gon in $L$. Now move along $L$ starting from $b$ (Fig. 2). Let $p_1$ be the last enter of $L$ into the 1-gon. Now change the overcrossing to undercrossing in $L$ in such a way that, starting from $b$,

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Fig. 2.

the new diagram, $L'$, is descending except the crossing $p_2$. Therefore the crossings $p_0$, $p_1$ and $p_2$ are not changed (Fig. 2). By the Murasugi result $\sigma(L') \geq \sigma(L)$. We claim the $L'$ is a diagram of the right handed trefoil knot (Fig. 3) and therefore $\sigma(L') = -2$, so $\sigma(L) < 0$.

It remains to prove the above claim.

Let us assume that the point $b$ is on the level 3 and that the diagram descends to level 2 just before $p_2$ (say in $p_2^-$) then the underscrossing reach the lever $-1$ and ascends back to level 1 at $c$, finally reaching the level 0 just before $b$. From this point of we ascend quickly to the level 3 at $b$. We can assume that the 1-gon is convex. Now we can easily deform $L'$ by isotopy so that the part $b,p_2^-$ is the straight line and the part $p_2^+,c$, is a simple arc which is descending and disjoint from other parts of the diagram (except ends). Therefore $L'$ is isotopic to the diagram of Fig. 3 which represents the right handed trefoil knot.

This completes the proof of Theorem 1. in the case of a knot. The case of a link is analogous. The only difference is that we have to consider an innermost 1-gon or 0-gon and reduce diagram to the right handed trefoil or Hopf link (with, possibly, some additional trivial components).

Theorem 1 is stronger than that of Rudolph or Murasugi because there are positive knots which are neither alternating nor have a presentation as positive braids.

**Corollary 2** A nontrivial positive link is neither slice link nor amphicheiral link.
Proof: Amphicheiral links and slice links have signature equal to 0 \([M-1]\). □

**Corollary 3 (Murasugi)** The following inequalities hold for the Jones polynomial of a non-split link:

(a) \(d_{\min} V_L(t) > 0\) for a nontrivial positive link \(L\),

(b) \(d_{\max} V_L(t) < 0\) for a nontrivial negative link \(L\),

where \(d_{\max}\) (resp. \(d_{\min}\)) denotes the highest (resp. lowest) power of \(t\) in \(V_L(t)\).

Proof: By Murasugi \([M-5]\), Theorem 13.3, the following holds for any non-split link diagram \(\tilde{L}\) of a link \(L\):

\[
\begin{align*}
    &d_{\max} V_L(t) \leq c_+((\tilde{L})) - \frac{1}{2} \sigma(L), \\
    &d_{\min} V_L(t) \geq -c_-((\tilde{L})) - \frac{1}{2} \sigma(L),
\end{align*}
\]

where \(c_+\) (resp. \(c_-\)) is the number of positive (resp. negative) crossings of \(\tilde{L}\). Now Corollary 3 follows from Theorem 1. □

Corollary 3 was first proven by Murasugi \([M-3]\, \text{Theorem 2.1}\) in implicit form. A different proof has been found by Traczyk.

Theorem 1 can be extended to other Tristram–Levine signatures as long as it holds for the Hopf link and the trefoil knot.

We use the notation of \([G]\) (see also \([P]\)). We assume also (without loss of generality) that \(|1 - \xi| = 1\) in the Tristram–Levine signature \(\sigma_\xi\).

**Theorem 4** If \(L\) is a nontrivial positive link then for \(\Re \xi < \frac{1}{2}\), \(\sigma_\xi < 0\).

Proof: For \(\Re \xi < 1/2\), \(\sigma_\xi\) is negative for the right handed trefoil knot and Hopf link. Furthermore, by \([P-T]\) (see also \([P]\, \text{Lemma 4.13(b)}\) for \(\Re \xi < 1\), \(\sigma_\xi(L_+) \leq \sigma_\xi(L_-)\) so the proof of Theorem 1 can be repeated without changes here too. □

**Conjecture 5** If a nontrivial link has a diagram with at most one negative crossing then the link has negative signature.\(^1\)

I have been informed, after completing this manuscript that the Theorem 1 has been proven independently by P. Traczyk (“Non-trivial negative links have positive signature”, preprint, Summer 1987) and, in the case of knots by R. Gompf and T. Cochran (“Applications of Donaldson’s theorems to classical knot concordance. Homology 3-spheres and property P”, preprint 1987).

\(^1\)Added for e-print: This conjecture with its generalizations was proved in a joint paper with K. Taniyama \([P-T]\).
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