THE HEUN EQUATION AND THE CALOGERO-MOSER-SUTHERLAND SYSTEM V: GENERALIZED DARBOUX TRANSFORMATIONS

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Abstract. We obtain isomonodromic transformations for Heun’s equation by generalizing Darboux transformation, and we find pairs and triplets of Heun’s equation which have the same monodromy structure. By composing generalized Darboux transformations, we establish a new construction of the commuting operator which ensures finite-gap property. As an application, we prove conjectures in part III.

1. Introduction

It was shown in [9] that some pairs of Schrödinger operators are isomonodromic. Set

$$H_1 = -\frac{d^2}{dx^2} + 2\varphi(x) + 2\varphi(x + \omega_1) + 2\varphi(x + \omega_2),$$

where $\varphi(x)$ is the Weierstrass $\wp$-function with periods $(2\omega_1, 2\omega_3)$ and $\omega_2 = -\omega_1 - \omega_3$. Now we consider eigenfunctions of $H_1$ (resp. $H_2$) with the eigenvalue $E$. Set

$$\Xi_1(x, E) = 9\varphi(x)^2 + 3E\varphi(x) + E^2 - 9g_2/4,$$
$$\Xi_2(x, E) = (E - 3e_3)\varphi(x) + (E - 3e_2)\varphi(x + \omega_1) + (E - 3e_1)\varphi(x + \omega_2) + E^2 - 3g_2/2,$$
$$Q(E) = (E^2 - 3g_2)(E - 3e_1)(E - 3e_2)(E - 3e_3),$$
$$\Lambda_k(x, E) = \sqrt{\Xi_k(x, E)} \exp \left( \int \frac{\sqrt{-Q(E)} dx}{\Xi_k(x, E)} \right), \quad (k = 1, 2),$$

where $e_i = \varphi(\omega_i)$ ($i = 1, 2, 3$) and $g_2 = -4(e_1e_2 + e_2e_3 + e_3e_1)$. Then it was shown that the functions $\Lambda_1(x, E)$ and $\Lambda_1(-x, E)$ (resp. $\Lambda_2(x, E)$ and $\Lambda_2(-x, E)$) are eigenfunctions of $H_1$ (resp. $H_2$) with the eigenvalue $E$, and they satisfy

$$\Lambda_k(\pm(x + 2\omega_i), E) = \Lambda_k(x, E) \exp \left( \pm \frac{1}{2} \int_{\sqrt{3g_2}}^{E} \frac{-6E\eta_i + (2E^2 - 3g_2)\omega_i dE}{\sqrt{-Q(E)}} \right),$$

for $k = 1, 2$ and $i = 1, 2, 3$, where $\eta_i = \zeta(\omega_i)$ and $\zeta(x)$ is the Weierstrass zeta function. Hence the monodromy of eigenfunctions of $H_1$ with the eigenvalue $E$ coincides with that of eigenfunctions of $H_2$.

In this paper, we investigate this phenomena by Darboux transformation and generalized Darboux transformation. Let $\phi_0(x)$ be an eigenfunction of the operator.
\[ H = -\frac{d^2}{dx^2} + q(x) \] with an eigenvalue \( E_0 \), i.e.
\[ \left( -\frac{d^2}{dx^2} + q(x) \right) \phi_0(x) = E_0 \phi_0(x). \]

For this case, the potential \( q(x) \) is written as \( q(x) = (\phi'_0(x)/\phi_0(x))^2 + (\phi'_0(x)/\phi_0(x))^2 + E_0 \). Set \( L = d/dx - \phi'_0(x)/\phi_0(x) \) and \( \tilde{H} = -\frac{d^2}{dx^2} + q(x) - 2(\phi'_0(x)/\phi_0(x))^1 \). Then we have
\[ \tilde{H}L = LH. \]

Hence, if \( \phi(x) \) is an eigenfunction of the operator \( H \) with the eigenvalue \( E \), then \( L\phi(x) \) is an eigenfunction of the operator \( \tilde{H} \) with the eigenvalue \( E \). This transformation is called the Darboux transformation. We generalize the operator \( L \) to be the differential operator of higher order, and we call it the generalized Darboux transformation.

The Schrödinger operator we consider in this paper is the Hamiltonian of the BC\(_1\) Inozemtsev model, which is written as
\[ H^{(l_0,l_1,l_2,l_3)} = -\frac{d^2}{dx^2} + \sum_{i=0}^{3} l_i(l_i + 1)\varphi(x + \omega_i), \]
where \( \omega_0 = 0 \). The potential of this operator is called the Treibich-Verdier potential, because Treibich and Verdier [13] found and showed that, if \( l_i \in \mathbb{Z}_{>0} \) for all \( i \in \{0, 1, 2, 3\} \), then it is an algebro-geometric finite-gap potential. For further results on this subject, see [2, 5, 6, 8, 12]. The algebro-geometric finite-gap property cause the possibility for calculation of eigenfunction and monodromy of the operator \( H^{(l_0,l_1,l_2,l_3)} \).

Let \( f(x) \) be an eigenfunction of the operator \( H^{(l_0,l_1,l_2,l_3)} \) with the eigenvalue \( E \), namely,
\[ \left( -\frac{d^2}{dx^2} + \sum_{i=0}^{3} l_i(l_i + 1)\varphi(x + \omega_i) \right) f(x) = Ef(x). \]

Then this equation is an elliptic representation of Heun’s equation. Here Heun’s equation is the standard canonical form of a Fuchsian equation with four singularities (see [4]). Thus, solving Heun’s equation is equivalent to studying eigenvalues and eigenfunctions of the Hamiltonian of the BC\(_1\) Inozemtsev model.

We now describe the main result of this paper. Let \( \alpha_i \) be a number such that \( \alpha_i = -l_i \) or \( \alpha_i = l_i + 1 \) for each \( i \in \{0, 1, 2, 3\} \). Set \( d = -\sum_{i=0}^{3} \alpha_i/2 \) and assume \( d \in \mathbb{Z}_{\geq 0} \). Then there exists a differential operator \( L \) of order \( d + 1 \) which satisfies
\[ H^{(\alpha_0+d,\alpha_1+d,\alpha_2+d,\alpha_3+d)} L = LH^{(l_0,l_1,l_2,l_3)}. \]

Note that Khare and Sukhatme [3] essentially established this result for the case \( d = 0 \), that is the case of original Darboux transformation. It follows immediately that, if \( \phi(x) \) is an eigenfunction of the operator \( H^{(l_0,l_1,l_2,l_3)} \) with an eigenvalue \( E \), then \( L\phi(x) \) is an eigenfunction of the operator \( H^{(\alpha_0+d,\alpha_1+d,\alpha_2+d,\alpha_3+d)} \) with the eigenvalue \( E \). Since all coefficients of the operator \( L \) with respect to the differential \((d/dx)^k\) \((k = 0, \ldots, d + 1)\) is shown to be doubly-periodic, the operator \( L \) preserves the data of monodromy. Hence the operators \( H^{(l_0,l_1,l_2,l_3)} \) and \( H^{(\alpha_0+d,\alpha_1+d,\alpha_2+d,\alpha_3+d)} \) are isomonodromic, and isospectral, because boundary condition for spectral problem is characterized by monodromy. Note that the condition \( d \in \mathbb{Z}_{\geq 0} \) corresponds to quasi-solvability of the operator \( H^{(l_0,l_1,l_2,l_3)} \).
For the case that \( l_0, l_1, l_2, l_3 \) are all integers, there exists an operator \( H^{(l_0,1,1,3)} \) such that the pair \( H^{(l_0,1,2,3)} \) and \( H^{(l_0,1,1,3)} \) is connected by isomonodromic transformation. In some cases, they are self-dual. For example, the operator \( H_1(= H^{(2,0,0,0)}) \) in Eq. (1) is connected to the operator \( H_2(= H^{(1,1,0)}) \) in Eq. (2) by the transformation 
\[
L = d/dx - \varphi'(x)/(2(\varphi(x) - e_1)) - \varphi'(x)/(2(\varphi(x) - e_2)),
\]
i.e. we have 
\[
H^{(1,1,0)} = L H^{(2,0,0)}.
\]

For the case that \( l_0 + 1/2, l_1 + 1/2, l_2 + 1/2, l_3 + 1/2 \) are integers, there exists two operators \( H^{(l_0,0,1,1,3)} \) and \( H^{(l_0,0,2,1,2,3)} \) such that the triplet \( H^{(l_0,1,1,3)} \), 
\( H^{(l_0,1,1,3)} \), and \( H^{(l_0,2,1,2,3)} \) is connected by isomonodromic transformations.

In the paper [8], finite-gap property of the operator \( H^{(l_0,1,1,3)} \) for the case \( l_0, l_1, l_2, l_3 \in \mathbb{Z}_{\geq 0} \) is studied (see also [13, 12, 2, 5]). Especially, a differential operator \( A \) of odd order which commutes with \( H^{(l_0,1,1,3)} \) is constructed. In this paper, we propose a new method for construction of the commuting operator by composing four generalized Darboux transformations. Note that each generalized Darboux transformation is written explicitly. To show that the commuting operator constructed by composing four generalized Darboux transformations coincides with the one defined in [8], we need a discussion which will be done in section 6. As an application, we prove conjectures in [8]. Namely, we establish that the polynomial defined by quasi-solvability coincides with the one defined by using the doubly-periodic function \( \Xi(x, E) \) which is written as a product of two eigenfunctions. We also prove that the commuting operator is characterized by annihilating the spaces of quasi-solvability. As for the isomonodromic pair \( H^{(l_0,1,1,3)} \) and \( H^{(l_0,1,1,3)} \), it is shown that some functions related to the monodromy of \( H^{(l_0,1,1,3)} \) coincide with that of \( H^{(l_0,1,1,3)} \).

This paper is organized as follows. In section 2, we review the connection between quasi-solvability and generalized Darboux transformation which was essentially done in [1]. In section 3, we consider generalized Darboux transformations for the case of Heun’s equation. In section 4 (resp. section 5), we investigate isomonodromic transformations for the case \( l_0, l_1, l_2, l_3 \in \mathbb{Z} \) (resp. \( l_0, l_1, l_2, l_3 \in \mathbb{Z} + 1/2 \)). In section 6, we construct a differential operator of odd order which commutes with \( H^{(l_0,1,1,3)} \) and investigate it.

2. QUASI-SOLVABILITY AND GENERALIZED DARBOUX TRANSFORMATION

We review the relationship between the quasi-solvability and the generalized Darboux transformation.

We set \( H = -d^2/dx^2 + q(x) \). Let \( n \) be a positive integer. If the operator \( H \) preserve a \( n \)-dimensional space \( U \) of functions, then the operator \( H \) is called quasi-solvable. Then there exists a basis \( \{ f_1(x), \ldots, f_n(x) \} \) of the invariant space such that \( H f_j(x) = \sum a_{i,j} f_i(x) \) for some constants \( a_{i,j} \) (\( 1 \leq i, j \leq n \)). Let \( P_{H,U}(t) \) be the characteristic polynomial of the operator \( H \) on the space \( U \). Then the set \( \{ E | P_{H,U}(E) = 0 \} \) coincides with the set of eigenvalues of the operator \( H \) on the space \( U \). Then the model is partially solved, and this is an origin of “quasi-solvability”. For the space \( U \), there exists a monic differential operator of order \( n \)

\[
L = \left( \frac{d}{dx} \right)^n + \sum_{i=1}^{n} c_i(x) \left( \frac{d}{dx} \right)^{n-i},
\]
such that \( Lf(x) = 0 \) for all \( f(x) \in U \). It is determined uniquely and written as

\[
L = \begin{pmatrix}
    f_1(x) & \frac{d}{dx}f_1(x) & \ldots & (\frac{d}{dx})^{n-1}f_1(x) \\
    f_2(x) & \frac{d}{dx}f_2(x) & \ldots & (\frac{d}{dx})^{n-1}f_2(x) \\
    \vdots & \vdots & \ddots & \vdots \\
    f_n(x) & \frac{d}{dx}f_n(x) & \ldots & (\frac{d}{dx})^{n-1}f_n(x)
\end{pmatrix}^{-1} \begin{pmatrix}
    f_1(x) & \frac{d}{dx}f_1(x) & \ldots & (\frac{d}{dx})^{n}f_1(x) \\
    f_2(x) & \frac{d}{dx}f_2(x) & \ldots & (\frac{d}{dx})^{n}f_2(x) \\
    \vdots & \vdots & \ddots & \vdots \\
    f_n(x) & \frac{d}{dx}f_n(x) & \ldots & (\frac{d}{dx})^{n}f_n(x)
\end{pmatrix}.
\]

**Proposition 2.1.** (c.f. [1]) Assume that the operator \( H = -d^2/dx^2 + q(x) \) preserve a \( n \)-dimensional space \( U \) of functions. Let \( L \) be the differential operator written as Eq. (2.1) which annihilate the functions in \( U \). Set \( \tilde{H} = -d^2/dx^2 + q(x) + 2c_1(x) \). Then we have

\[
\tilde{H}L = LH.
\]

**Proof.** By a direct calculation, it follows that the order of the differential operator \( \tilde{H}L - LH \) is at most \( n - 1 \). Assume that \( \tilde{H}L - LH \neq 0 \). We denote the order by \( k \). Then the dimension of solutions to the differential equation \((\tilde{H}L - LH)f(x) = 0\) is \( k \). Let \( g(x) \in U \). Since the operator \( H \) preserve the space \( U \), we have \( Hg(x) \in U \). By definition of the space \( U \), we have \( Lg(x) = 0 \) and \( LHg(x) = 0 \). Hence we have \((\tilde{H}L - LH)g(x) = 0\) for \( g(x) \in U \), but it contradicts to that the dimension of solutions is \( k(\leq n - 1) \). Therefore we obtain \( \tilde{H}L = LH \). \( \square \)

We consider the case \( n = 1 \), Let \( \phi_0(x) \) be a non-zero function in \( U \). Then \( U = \mathbb{C}\phi_0(x) \), and the operator which annihilate \( \phi_0(x) \) is written as \( L = d/dx - \phi_0(x)/\phi_0(x) \). The operator \( \tilde{H} \) is written as \( \tilde{H} = H - 2(\phi'_0(x)/\phi_0(x))' \). Hence the proposition reproduce the Darboux transformation. In this sense, the transformation in the proposition may be called the generalized Darboux transformation.

### 3. Generalized Darboux transformation for Heun’s equation

In this section, we apply Proposition 2.1 for Heun’s equation. For this purpose, we recall quasi-solvability of Heun’s equation.

**Proposition 3.1.** [7 Proposition 5.1] Let \( \alpha_i \) be a number such that \( \alpha_i = -l_i \) or \( \alpha_i = l_i + 1 \) for each \( i \in \{0, 1, 2, 3\} \). Set \( d = -\sum_{i=0}^{3} \alpha_i/2 \) and assume \( d \in \mathbb{Z}_{\geq 0} \). Let \( V_{\alpha_0, \alpha_1, \alpha_2, \alpha_3} \) be the \( d \)-1-dimensional space spanned by

\[
\{ (\varphi(x) - e_1)^{\alpha_1/2}(\varphi(x) - e_2)^{\alpha_2/2}(\varphi(x) - e_3)^{\alpha_3/2}\varphi(x)^n \}_{n=0, \ldots, d}.
\]

Then the operator \( H^{(l_0, l_1, l_2, l_3)} \) (see Eq. (1.3)) preserves the space \( V_{\alpha_0, \alpha_1, \alpha_2, \alpha_3} \).

Set \( z = \varphi(x) \), \( \widehat{\Phi}(z) = (z - e_1)^{\alpha_1/2}(z - e_2)^{\alpha_2/2}(z - e_3)^{\alpha_3/2} \), and \( \widehat{H}^{(l_0, l_1, l_2, l_3)} = \widehat{\Phi}(z)^{-1} \circ H^{(l_0, l_1, l_2, l_3)} \circ \widehat{\Phi}(z) \). Proposition 3.1 is proved by showing that the operator \( \widehat{H}^{(l_0, l_1, l_2, l_3)} \) preserve the space spanned by \((z - e_2)^r \) (\( r = 0, \ldots, d \)). For details, see the proof of [7 Proposition 5.1].

Now we calculate the differential operator which annihilate the space \( V_{\alpha_0, \alpha_1, \alpha_2, \alpha_3} \).

**Proposition 3.2.** The monic differential operator of order \( d + 1 \) which annihilate the space \( V_{\alpha_0, \alpha_1, \alpha_2, \alpha_3} \) is written as

\[
L_{\alpha_0, \alpha_1, \alpha_2, \alpha_3} = \varphi'(x)^{d+1}\widehat{\Phi}(\varphi(x)) \circ \left( \frac{1}{\varphi'(x)} \frac{d}{dx} \right)^{d+1} \circ \widehat{\Phi}(\varphi(x))^{-1}.
\]
We write the operator \( L_{\alpha_0, \alpha_1, \alpha_2, \alpha_3} \) as

\[
L_{\alpha_0, \alpha_1, \alpha_2, \alpha_3} = \left( \frac{d}{dx} \right)^{d+1} + \sum_{i=1}^{d+1} c_i(x) \left( \frac{d}{dx} \right)^{d+1-i}.
\]

Then

\[
c_1(x) = -\frac{d+1}{4} \left( \sum_{i=1}^{3} \frac{2\alpha_i + d}{\varphi(x) - e_i} \right) \varphi'(x).
\]

If \( i \) is even (resp. odd), then \( c_i(x) \) is expressed as \( c_i(x) = R_i(\varphi(x)) \) (resp. \( c_i(x) = R_i(\varphi(x))\varphi'(x) \)), where \( R_i(z) \) is a rational function in \( z \).

\[\text{Proof.}\] It is trivial that the operator \((d/dz)^{d+1}\) annihilate the space spanned by \( z^r \) \((r = 0, \ldots, d)\), and the operator \( \hat{\Phi}(z) \circ (d/dz)^{d+1} \circ \hat{\Phi}(z)^{-1} \) annihilate the space spanned by \( \hat{\Phi}(z)z^r \) \((r = 0, \ldots, d)\). Write

\[
\hat{\Phi}(z) \circ \left( \frac{d}{dz} \right)^{d+1} \circ \hat{\Phi}(z)^{-1} = \left( \frac{d}{dz} \right)^{d+1} + \sum_{i=1}^{d+1} \hat{c}_i(z) \left( \frac{d}{dz} \right)^{d+1-i}.
\]

Then \( \hat{c}_1(z) = -\sum_{i=1}^{3} ((d+1)\alpha_i)/(2(\varphi(x) - e_i)) \), and all coefficients \( \hat{c}_i(z) \) are rational functions in \( z \). By the transformation \( z = \varphi(x) \), the monic operator

\[
L_{\alpha_0, \alpha_1, \alpha_2, \alpha_3} = \varphi'(x)^{d+1} \hat{\Phi}(\varphi(x)) \circ \left( \frac{1}{\varphi'(x)} \frac{d}{dx} \right)^{d+1} \circ \hat{\Phi}(\varphi(x))^{-1}
\]

\[
= \varphi'(x)^{d+1} \left( \frac{1}{\varphi'(x)} \frac{d}{dx} \right)^{d+1} + \sum_{i=1}^{d+1} \hat{c}_i(\varphi(x))\varphi'(x)^{d+1} \left( \frac{1}{\varphi'(x)} \frac{d}{dx} \right)^{d+1-i}
\]

annihilate the space \( V_{\alpha_0, \alpha_1, \alpha_2, \alpha_3} \). Write \((1/\varphi'(x))(d/dx))^i = \sum_{j=0}^{i} \hat{b}_j(x)(d/dx)^j\). If \( j \) is even (resp. odd), then \( \hat{b}_j(x) \) is expressed as \( r_j(\varphi(x)) \) (resp. \( r_j(\varphi(x))\varphi'(x) \)), where \( r_j(z) \) is a rational function in \( z \). Combining with the relation \( \varphi'(x)^2 = 4(\varphi(x) - e_1)(\varphi(x) - e_2)(\varphi(x) - e_3) \), it follows that, if \( i \) is even (resp. odd), then \( c_i(x) \) is expressed as \( c_i(x) = R_i(\varphi(x)) \) (resp. \( c_i(x) = R_i(\varphi(x))\varphi'(x) \)), where \( R_i(z) \) is a rational function in \( z \). We now calculate \( c_1(x) \). By Eqs. \((3.5)\) \((A.2)\), we have

\[
c_1(x) = -\frac{(d+1)d \varphi''(x)}{2 \varphi'(x)} + \hat{c}_1(\varphi(x))\varphi'(x) = -\frac{d+1}{4} \left( \sum_{i=1}^{3} \frac{2\alpha_i + d}{\varphi(x) - e_i} \right) \varphi'(x).
\]

\[\square\]

**Theorem 3.3.** Let \( \alpha_i \) be a number such that \( \alpha_i = -l_i \) or \( \alpha_i = l_i + 1 \) for each \( i \in \{0, 1, 2, 3\} \). Set \( d = -\sum_{i=0}^{3} \alpha_i/2 \) and assume \( d \in \mathbb{Z}_{\geq 0} \). Let \( L_{\alpha_0, \alpha_1, \alpha_2, \alpha_3} \) be the operator defined in Proposition \(5.2\). Then we have

\[
H^{(\alpha_0+d, \alpha_1+d, \alpha_2+d, \alpha_3+d)} L_{\alpha_0, \alpha_1, \alpha_2, \alpha_3} = L_{\alpha_0, \alpha_1, \alpha_2, \alpha_3} H^{(l_0, l_1, l_2, l_3)}.
\]

**Proof.** It follows from Propositions \(2.1\) and \(3.2\) that

\[
(H^{(l_0, l_1, l_2, l_3)} + 2c'_1(x)) L_{\alpha_0, \alpha_1, \alpha_2, \alpha_3} = L_{\alpha_0, \alpha_1, \alpha_2, \alpha_3} H^{(l_0, l_1, l_2, l_3)},
\]
where $c_1(x)$ is defined in Eq.\,(3.2). By Eq.\,(3.2) we have
\[
2c_1'(x) = \frac{d+1}{2} \left( \sum_{i=1}^{3} \frac{2\alpha_i + d}{(\varphi(x) - e_i)^2} \right) \varphi'(x)^2 - \frac{d+1}{2} \left( \sum_{i=1}^{3} \frac{2\alpha_i + d}{\varphi(x) - e_i} \right) \varphi''(x)
\]
\[= -(d+1)(2(\alpha_1 + \alpha_2 + \alpha_3) + 3d)\varphi(x) + \sum_{i=1}^{3} (d+1)(2\alpha_1 + d)\varphi(x + \omega_i).
\]
Since $\alpha_i = -l_i$ or $\alpha_i = l_i + 1$, we have $l_i(l_i + 1) = -\alpha_i(-\alpha_i + 1)$. Hence we obtain $H^{(l_0,l_1,l_2,l_3)} + 2c_1'(x) = H^{(\alpha_0-1,\alpha_1-1,\alpha_2-1,\alpha_3-1)} + 2c_1'(x) = H^{(\alpha_0+d,\alpha_1+d,\alpha_2+d,\alpha_3+d)}$ and Eq.\,(3.6).

We consider the converse relation to Eq.\,(3.6). The operator $H^{(\alpha_0+d,\alpha_1+d,\alpha_2+d,\alpha_3+d)}$ preserve the $d+1$-dimensional space $V_{-\alpha_0-d,-\alpha_1-d,-\alpha_2-d,-\alpha_3-d}$, because $-\alpha_i - d \in \{-\alpha_i + d, \alpha_i + d + 1\}$ and $-\sum_{i=0}^{3}(-\alpha_i - d)/2 = d$.

**Proposition 3.4.** \(\text{(i)}\) We have
\[
H^{(l_0,l_1,l_2,l_3)} L_{-\alpha_0-d,-\alpha_1-d,-\alpha_2-d,-\alpha_3-d} = L_{-\alpha_0-d,-\alpha_1-d,-\alpha_2-d,-\alpha_3-d} H^{(\alpha_0+d,\alpha_1+d,\alpha_2+d,\alpha_3+d)}.
\]
\(\text{(ii)}\) The characteristic polynomial of the operator $H^{(l_0,l_1,l_2,l_3)}$ on the space $V_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$ coincides with that of the operator $H^{(\alpha_0+d,\alpha_1+d,\alpha_2+d,\alpha_3+d)}$ on the space $V_{-\alpha_0-d,-\alpha_1-d,-\alpha_2-d,-\alpha_3-d}$.

**Proof.** Set $\tilde{l}_i = \alpha_i + d$ \((i = 0, 1, 2, 3)\). Then $-\sum_{i=0}^{3}(\tilde{l}_i - l_i)/2 = d$. By Theorem 3.3 we have
\[
H^{(-\tilde{l}_0+1,\alpha_1-1,\alpha_2-1,\alpha_3-1)} L_{-\tilde{l}_0,-\tilde{l}_1,-\tilde{l}_2,-\tilde{l}_3} = L_{-\tilde{l}_0,-\tilde{l}_1,-\tilde{l}_2,-\tilde{l}_3} H^{(\tilde{l}_0,\tilde{l}_1,\tilde{l}_2,\tilde{l}_3)}.
\]
Hence we obtain Eq.\,(3.7). We will prove (ii) in the appendix.

Let $f_1(x, E)$ and $f_2(x, E)$ be a basis of solutions to the differential equation \((H^{(l_0,l_1,l_2,l_3)} - E)f(x) = 0\). Since the operator $H^{(l_0,l_1,l_2,l_3)}$ is doubly-periodic, the functions $f_1(x + 2\omega_k, E)$ and $f_2(x + 2\omega_k, E)$ \((k = 1, 3)\) are also solutions to \((H^{(l_0,l_1,l_2,l_3)} - E)f(x) = 0\), and they are written as linear combinations of $f_1(x, E)$ and $f_2(x, E)$. Hence we have monodromy matrices
\[
(f_1(x + 2\omega_k, E), f_2(x + 2\omega_k, E)) = (f_1(x, E), f_2(x, E)) \begin{bmatrix} a_{11}(k) & a_{12}(k) \\ a_{21}(k) & a_{22}(k) \end{bmatrix}.
\]
Now assume that $\alpha_0, \ldots, \alpha_3, d$ satisfy the assumption of Theorem 3.3. Set $\tilde{f}_i(x, E) = L_{\alpha_0,\alpha_1,\alpha_2,\alpha_3} f_i(x, E)$ \((i = 1, 2)\). It follows from Eq.\,(3.6) that $\tilde{f}_1(x, E)$ and $\tilde{f}_2(x, E)$ are eigenfunctions of the operator $H^{(\alpha_0+d,\alpha_1+d,\alpha_2+d,\alpha_3+d)}$ with the eigenvalue $E$. If $E$ is not an eigenvalue of $H^{(l_0,l_1,l_2,l_3)}$ on the space $V_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$, then $\{\tilde{f}_1(x, E), \tilde{f}_2(x, E)\}$ is a basis of solutions to the differential equation \((H^{(\alpha_0+d,\alpha_1+d,\alpha_2+d,\alpha_3+d)} - E)f(x) = 0\). It is shown in Proposition 3.4 that operator $L_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$ is doubly-periodic, and it follows from Eq.\,(3.8) that
\[
(f_1(x + 2\omega_k, E), f_2(x + 2\omega_k, E)) = (f_1(x, E), f_2(x, E)) \begin{bmatrix} a_{11}(k) & a_{12}(k) \\ a_{21}(k) & a_{22}(k) \end{bmatrix}.
\]
Hence the monodromy structure of $H^{(l_0,l_1,l_2,l_3)}$ coincides with the one of $H^{(\alpha_0+d,\alpha_1+d,\alpha_2+d,\alpha_3+d)}$ for the case that $E$ is not an eigenvalue of $H^{(l_0,l_1,l_2,l_3)}$ on the space $V_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$. If $E$ is an eigenvalue of $H^{(l_0,l_1,l_2,l_3)}$ on the space $V_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$, it is also the eigenvalue of the operator $H^{(\alpha_0+d,\alpha_1+d,\alpha_2+d,\alpha_3+d)}$ on the space $V_{-\alpha_0-d,-\alpha_1-d,-\alpha_2-d,-\alpha_3-d}$, which follows from
Proposition 3.5. Thus the operator $L_{\alpha_0, \alpha_1, \alpha_2, \alpha_3}$ defines an isomonodromic transformation from $H^{(l_0, l_1, l_2, l_3)}$ to $H^{(\alpha_0 + d, \alpha_1 + d, \alpha_2 + d, \alpha_3 + d)}$. Similarly $L_{-\alpha_0 - d, -\alpha_1 - d, -\alpha_2 - d, -\alpha_3 - d}$ defines an isomonodromic transformation from $H^{(\alpha_0 + d, \alpha_1 + d, \alpha_2 + d, \alpha_3 + d)}$ to $H^{(l_0, l_1, l_2, l_3)}$.

On the multiplicity of eigenvalues of the operator $H^{(l_0, l_1, l_2, l_3)}$ on the space $V_{\alpha_0, \alpha_1, \alpha_2, \alpha_3}$, we have the following proposition:

**Proposition 3.5.** (c.f. [11]) Let $\alpha_i$ be a number such that $\alpha_i = -l_i$ or $\alpha_i = l_i + 1$ for each $i \in \{0, 1, 2, 3\}$. Set $d = -\sum_{i=0}^{3} \alpha_i / 2$ and assume $d \in \mathbb{Z}_{\geq 0}$ and $\alpha_i \neq \alpha_j$ for some $i, j \in \{0, 1, 2, 3\}$. Then zeros of the characteristic polynomial of the operator $H^{(l_0, l_1, l_2, l_3)}$ on the space $V_{\alpha_0, \alpha_1, \alpha_2, \alpha_3}$ are distinct for generic periods $(2\omega_1, 2\omega_3)$.

We prove this proposition in the appendix. Now we give a remark on the meaning of “generic”. In the appendix, it is shown that there exists a period $(2\omega_1, 2\omega_3)$ such that zeros of the characteristic polynomial are distinct. Hence the discriminant of the characteristic polynomial is not identically zero, and the set of periods such that the discriminant of the characteristic polynomial is not equal to zero is open-dense.

4. INTEGER CASE

We investigate quasi-solvability and generalized Darboux transformation for the case $l_i \in \mathbb{Z}_{\geq 0}$ ($i = 0, 1, 2, 3$). Throughout this section, assume $l_i \in \mathbb{Z}_{\geq 0}$ for $i = 0, 1, 2, 3$.

Let $\mathcal{F}$ be the space spanned by meromorphic doubly periodic functions up to signs, namely

$$
\mathcal{F} = \bigoplus_{\epsilon_1, \epsilon_3 = \pm 1} \mathcal{F}_{\epsilon_1, \epsilon_3},
$$

$$
\mathcal{F}_{\epsilon_1, \epsilon_3} = \{ f(x) : \text{meromorphic } |f(x + 2\omega_1) = \epsilon_1 f(x), f(x + 2\omega_3) = \epsilon_3 f(x) \},
$$

If $\alpha_i \in \mathbb{Z}$ ($i = 0, 1, 2, 3$) and $-\sum_{i=0}^{3} \alpha_i / 2 \in \mathbb{Z}_{\geq 0}$, then $V_{\alpha_0, \alpha_1, \alpha_2, \alpha_3}$ is a subspace of $\mathcal{F}_{\epsilon_1, \epsilon_3}$ for suitable $\mathcal{F}_{\epsilon_1, \epsilon_3}$ ($\epsilon_1, \epsilon_3 \in \{\pm 1\}$), because $(\varphi(x) - \epsilon_i)^{1/2} = \varphi_i(x)$ ($i = 1, 2, 3$), where $\varphi_i(x)$ is the co-$\varphi$ function. Invariant subspaces of $\mathcal{F}$ with respect to the operator $H^{(l_0, l_1, l_2, l_3)}$ is studied in [6] (see also [2, 7, 8]).

Let $\alpha_i \in \{-l_i, l_i + 1\}$ ($i = 0, 1, 2, 3$),

$$
U_{\alpha_0, \alpha_1, \alpha_2, \alpha_3} = \begin{cases} 
V_{\alpha_0, \alpha_1, \alpha_2, \alpha_3}, & \sum_{i=0}^{3} \alpha_i / 2 \in \mathbb{Z}_{\leq 0}; \\
V_{-\alpha_0, 1-\alpha_1, 1-\alpha_2, 1-\alpha_3}, & \sum_{i=0}^{3} \alpha_i / 2 \in \mathbb{Z}_{\geq 2}; \\
\{0\}, & \text{otherwise},
\end{cases}
$$

and

$$
L_{\alpha_0, \alpha_1, \alpha_2, \alpha_3} = \begin{cases} 
L_{\alpha_0, \alpha_1, \alpha_2, \alpha_3}, & \sum_{i=0}^{3} \alpha_i / 2 \in \mathbb{Z}_{\leq 0}; \\
L_{-\alpha_0, 1-\alpha_1, 1-\alpha_2, 1-\alpha_3}, & \sum_{i=0}^{3} \alpha_i / 2 \in \mathbb{Z}_{\geq 2}; \\
1, & \text{otherwise}.
\end{cases}
$$

If $l_0 + l_1 + l_2 + l_3$ is even, then the operator $H^{(l_0, l_1, l_2, l_3)}$ (see Eq. (13)) preserves the spaces

$$
(4.3) \quad U_{-l_0, -l_1, -l_2, -l_3}, \ U_{-l_0, -l_1, l_2 + 1, l_3 + 1}, \ U_{-l_0, l_1 + 1, -l_2, l_3 + 1}, \ U_{-l_0, l_1 + 1, l_2 + 1, -l_3}.
$$

Each space is contained in $\mathcal{F}_{\epsilon_1, \epsilon_3}$ for some $\epsilon_1, \epsilon_3 \in \{\pm 1\}$, and the correspondence between the spaces and the signs of $(\epsilon_1, \epsilon_3)$ is one-to-one. Let $V$ be the sum of these spaces. Then $V$ is written as the direct sum of these spaces, i.e.

$$
(4.4) \quad V = U_{-l_0, -l_1, -l_2, -l_3} \oplus U_{-l_0, -l_1, l_2 + 1, l_3 + 1} \oplus U_{-l_0, l_1 + 1, -l_2, l_3 + 1} \oplus U_{-l_0, l_1 + 1, l_2 + 1, -l_3}.
$$
and it is the maximal finite-dimensional invariant subspace in $\mathcal{F}$ with respect to the action of the operator $H^{(l_0,l_1,l_2,l_3)}$. Let $k_i$ be the rearrangement of $l_i$ such that $k_0 \geq k_1 \geq k_2 \geq k_3(\geq 0)$. If $k_0 + k_3 \geq k_1 + k_2$ (resp. $k_0 + k_3 < k_1 + k_2$), then the dimension of the space $V$ is equal to $2k_0 + 1$ (resp. $k_0 + k_1 + k_2 - k_3 + 1$). Set

$$(4.5) \quad g = \begin{cases} k_0, & k_0 + k_3 \geq k_1 + k_2; \\ (k_0 + k_1 + k_2 - k_3)/2, & k_0 + k_3 < k_1 + k_2. \end{cases}$$

Then $g \in \mathbb{Z}_{\geq 0}$ and $\dim V = 2g + 1$. We set

$$(4.6) \quad l_0^e = (-l_0 + l_1 + l_2 + l_3)/2, \quad l_1^e = (l_0 - l_1 + l_2 + l_3)/2, \quad l_2^e = (l_0 + l_1 - l_2 + l_3)/2, \quad l_3^e = (l_0 + l_1 + l_2 - l_3)/2.$$  

Note that $l_0^e + l_1^e + l_2^e + l_3^e = l_0 + l_1 + l_2 + l_3 \in 2\mathbb{Z}$. It follows directly from Proposition 4.6 that

$$H^{(l_0^e,l_1^e,l_2^e,l_3^e)} \tilde{L}_{-l_0,-l_1,-l_2,-l_3} = \tilde{L}_{-l_0,-l_1,-l_2,-l_3}H^{(l_0,l_1,l_2,l_3)},$$

$$H^{(l_0,l_1,l_2,l_3)} \tilde{L}_{-l_0,-l_1,l_2+1,l_3+1} = \tilde{L}_{-l_0,-l_1,l_2+1,l_3+1}H^{(l_0,l_1,l_2,l_3)},$$

$$H^{(l_0,l_1,l_2,l_3)} \tilde{L}_{-l_0,l_1+1,-l_2,l_3+1} = \tilde{L}_{-l_0,l_1+1,-l_2,l_3+1}H^{(l_0,l_1,l_2,l_3)},$$

$$H^{(l_0,l_1,l_2,l_3)} \tilde{L}_{-l_0,l_1+1,l_2+1,-l_3} = \tilde{L}_{-l_0,l_1+1,l_2+1,-l_3}H^{(l_0,l_1,l_2,l_3)}.$$  

Thus it is shown that the operators which are linked by generalized Darboux transformations from $H^{(l_0,l_1,l_2,l_3)}$ are

$$(4.7) \quad H^{(l_0,l_1,l_2,l_3)}H^{(l_0^e,l_1^e,l_2^e,l_3^e)}, \quad H^{(l_0,l_1,l_2+1,l_3)}, \quad H^{(l_0,l_1,l_2,l_3+1)},$$

$$H^{(l_0,l_1,l_2,l_3)}H^{(l_0,l_1,l_2,l_3)}, \quad H^{(l_0,l_1,l_2+1,l_3)}, \quad H^{(l_0,l_1,l_2,l_3+1)}.$$  

As is discussed in section 3 these eight operators are isomonodromic. Note that the operators $H^{(l_0,l_1,l_2,l_3)}$, $H^{(l_0,l_1,l_2+1,l_3)}$, $H^{(l_0,l_1,l_2,l_3+1)}$ are obtained by the shift $x \rightarrow x + \omega_i$ ($i = 1, 2, 3$) from the operator $H^{(i_0,i_1,i_2,i_3)}$.

Assume that $l_0$, $l_1$, $l_2$, $l_3$ $\in \mathbb{Z}$, $l_0 \geq l_1 \geq l_2 \geq l_3 \geq 0$ and $l_0 + l_1 + l_2 + l_3$ is even. Set $\tilde{l}_0 = l_0^e$, $\tilde{l}_1 = l_1^e$, $\tilde{l}_2 = l_2^e$ and $\tilde{l}_3 = \max(l_0^e, -l_0^e - 1)$. Then $\tilde{l}_0 \geq \tilde{l}_1 \geq \tilde{l}_2 \geq \tilde{l}_3 \geq 0$, and the operator $H^{(l_0,l_1,l_2,l_3)}$ is isomonodromic to $H^{(\tilde{l}_0,\tilde{l}_1,\tilde{l}_2,\tilde{l}_3)}$. Note that, if $l_0 + l_3 \neq l_1 + l_2$ (resp. $l_0 + l_3 = l_1 + l_2$), then we have $(l_0, l_1, l_2, l_3) \neq (\tilde{l}_0, \tilde{l}_1, \tilde{l}_2, \tilde{l}_3)$ (resp. $(l_0, l_1, l_2, l_3) = (\tilde{l}_0, \tilde{l}_1, \tilde{l}_2, \tilde{l}_3)$).

We consider the case $l_0 + l_1 + l_2 + l_3$: odd. If $l_0 + l_1 + l_2 + l_3$ is odd, then the operator $H^{(l_0,l_1,l_2,l_3)}$ preserves the spaces

$$(4.8) \quad U_{-l_0,-l_1,-l_2,-l_3}, \quad U_{-l_0,-l_1,l_2+1,-l_3}, \quad U_{-l_0,l_1+1,-l_2,-l_3}, \quad U_{l_0+1,-l_1,-l_2,-l_3}.$$  

Each space is contained in $\mathcal{F}_{\epsilon_1,\epsilon_3}$ for some $\epsilon_1, \epsilon_3 \in \{\pm 1\}$, and the correspondence between the spaces and the signs of $(\epsilon_1, \epsilon_3)$ is one-to-one. Let $V$ be the sum of these spaces. Then $V$ is written as

$$(4.9) \quad V = U_{-l_0,-l_1,-l_2,-l_3+1} \oplus U_{-l_0,-l_1,l_2+1,-l_3} \oplus U_{-l_0,l_1+1,-l_2,-l_3} \oplus U_{l_0+1,-l_1,-l_2,-l_3},$$  

and it is the maximal finite-dimensional invariant subspace in $\mathcal{F}$ with respect to the action of the operator $H^{(l_0,l_1,l_2,l_3)}$. Let $k_i$ be the rearrangement of $l_i$ such that $k_0 \geq k_1 \geq k_2 \geq k_3(\geq 0)$. If $k_0 \geq k_1 + k_2 + k_3 + 1$ (resp. $k_0 < k_1 + k_2 + k_3 + 1$), then the dimension of the space $V$ is equal to $2k_0 + 1$ (resp. $k_0 + k_1 + k_2 - k_3 + 2$). Set

$$(4.10) \quad g = \begin{cases} k_0, & k_0 \geq k_1 + k_2 + k_3 + 1; \\ (k_0 + k_1 + k_2 + k_3 + 1)/2, & k_0 < k_1 + k_2 + k_3 + 1. \end{cases}$$
Then \( g \in \mathbb{Z}_{\geq 0} \) and \( \dim V = 2g + 1 \). We set

\[
\begin{align*}
\tilde{l}_0 &= \frac{(l_0 + l_1 + l_2 + l_3 + 1)}{2}, \\
\tilde{l}_1 &= \frac{(l_0 + l_1 - l_2 - l_3 - 1)}{2}, \\
\tilde{l}_2 &= \frac{(l_0 - l_1 + l_2 - l_3 - 1)}{2}, \\
\tilde{l}_3 &= \frac{(l_0 - l_1 - l_2 + l_3 - 1)}{2}.
\end{align*}
\]

It follows from Proposition 4.6 that

\[
\begin{align*}
H^{(l_0, l_1, l_2, l_3)}_{(l_0, l_1, l_2, l_3)}(L_{l_0+1, -l_1, -l_2, -l_3}) = \tilde{L}_{l_0+1, -l_1, -l_2, -l_3} H^{(l_0, l_1, l_2, l_3)}_{(l_0, l_1, l_2, l_3)}, \\
H^{(l_0, l_1, l_2, l_3)}_{(l_0, l_1, l_2, l_3)}(L_{l_0, l_1+1, -l_2, -l_3}) = \tilde{L}_{l_0, l_1+1, -l_2, -l_3} H^{(l_0, l_1, l_2, l_3)}_{(l_0, l_1, l_2, l_3)}, \\
H^{(l_0, l_1, l_2, l_3)}_{(l_0, l_1, l_2, l_3)}(L_{l_0, -l_1, l_2+1, -l_3}) = \tilde{L}_{l_0, -l_1, l_2+1, -l_3} H^{(l_0, l_1, l_2, l_3)}_{(l_0, l_1, l_2, l_3)}, \\
H^{(l_0, l_1, l_2, l_3)}_{(l_0, l_1, l_2, l_3)}(L_{l_0, -l_1, -l_2, l_3+1}) = \tilde{L}_{l_0, -l_1, -l_2, l_3+1} H^{(l_0, l_1, l_2, l_3)}_{(l_0, l_1, l_2, l_3)}.
\end{align*}
\]

It is shown that the operators which are linked by generalized Darboux transformations from \((l_0, l_1, l_2, l_3)\) are

\[
\begin{align*}
H^{(l_0, l_1, l_2, l_3)}_{(l_0, l_1, l_2, l_3)}, & \quad H^{(l_0, l_1, l_2, l_3)}_{(l_0, l_1, l_2, l_3)}, & \quad H^{(l_0, l_1, l_2, l_3)}_{(l_0, l_1, l_2, l_3)}, & \quad H^{(l_0, l_1, l_2, l_3)}_{(l_0, l_1, l_2, l_3)}, \\
H^{(l_0, l_1, l_2, l_3)}_{(l_0, l_1, l_2, l_3)}, & \quad H^{(l_0, l_1, l_2, l_3)}_{(l_0, l_1, l_2, l_3)}, & \quad H^{(l_0, l_1, l_2, l_3)}_{(l_0, l_1, l_2, l_3)}, & \quad H^{(l_0, l_1, l_2, l_3)}_{(l_0, l_1, l_2, l_3)},
\end{align*}
\]

and these eight operators are isomonodromic.

Assume that \( l_0, l_1, l_2, l_3 \in \mathbb{Z}, l_0 \geq l_1 \geq l_2 \geq l_3 \geq 0 \) and \( l_0 + l_1 + l_2 + l_3 \) is odd. Set \( \tilde{l}_0 = l_0, \tilde{l}_1 = l_1, \tilde{l}_2 = l_2 \) and \( \tilde{l}_3 = \max(l_3, -l_3 - 1) \). Then \( \tilde{l}_0 \geq \tilde{l}_1 \geq \tilde{l}_2 \geq \tilde{l}_3 \geq 0 \), and the operator \( H^{(l_0, l_1, l_2, l_3)}_{(l_0, l_1, l_2, l_3)} \) is isomonodromic to \( H^{(\tilde{l}_0, \tilde{l}_1, \tilde{l}_2, \tilde{l}_3)}_{(\tilde{l}_0, \tilde{l}_1, \tilde{l}_2, \tilde{l}_3)} \). Note that, if \( l_0 \neq l_1 + l_2 + l_3 + 1 \) (resp. \( l_0 = l_1 + l_2 + l_3 + 1 \)), then we have \( (l_0, l_1, l_2, l_3) \neq (\tilde{l}_0, \tilde{l}_1, \tilde{l}_2, \tilde{l}_3) \) (resp. \( (l_0, l_1, l_2, l_3) = (\tilde{l}_0, \tilde{l}_1, \tilde{l}_2, \tilde{l}_3) \)).

Now we reproduce results in this section for the case of Lamé equation and associated Lamé equation. If three (resp. two) of \( l_i \) \((i = 0, 1, 2, 3)\) are zero, then Eq. (124) is called the Lamé equation (resp. the associated Lamé equation). For simplicity, we consider the case \( l_2 = l_3 = 0, l_0, l_1 \in \mathbb{Z} \) and \( l_0 \geq l_1 \geq 0 \).

If \( l_0 > l_1 \), then the operator \( H^{(l_0, l_1, 0, 0)}_{(l_0, l_1, 0, 0)} \) is isomonodromic to \( H^{(l_0+1, l_1, 0, 0)}_{(l_0+1, l_1, 0, 0)} \) by the transformation \( L_{-l_0, l_1+1, 0, 0} \). Especially, if \( l_0 \) is even and \( l_1 = 0 \) (the case of Lamé equation), then \( H^{(l_0, 0, 0, 0)}_{(l_0, 0, 0, 0)} \) is isomonodromic to \( H^{(l_0/2, l_0/2, l_0/2, l_0/2)}_{(l_0/2, l_0/2, l_0/2, l_0/2)} \). Note that, if \( l_0 + l_1 \) is odd and \( l_0 = l_1 \), then the operator \( H^{(l_0, l_1, 0, 0)}_{(l_0, l_1, 0, 0)} \) is self-dual.

If \( l_0 + l_1 \) is odd and \( l_0 > l_1 - 1 \), then the operator \( H^{(l_0, l_1, 0, 0)}_{(l_0, l_1, 0, 0)} \) is isomonodromic to \( H^{(l_0+1, l_1, 0, 0)}_{(l_0+1, l_1, 0, 0)} \) by the transformation \( L_{-l_0+1, l_1+1, l_1, l_1} \). Especially, if \( l_0 \) is odd and \( l_1 = 0 \) (the case of Lamé equation), then \( H^{(l_0, 0, 0, 0)}_{(l_0, 0, 0, 0)} \) is isomonodromic to \( H^{(l_0+1, l_0-1, l_0-1, l_0-1)}_{(l_0+1, l_0-1, l_0-1, l_0-1)} \). Note that, if \( l_0 + l_1 \) is odd and \( l_0 = l_1 - 1 \), then the operator \( H^{(l_0, l_1, 0, 0)}_{(l_0, l_1, 0, 0)} \) is self-dual.

Thus we confirmed the conjecture of Khare and Sukhatme [3] on isospectral partner of Lamé equation and associated Lamé equation.

5. Half-integer case

We investigate quasi-solvability and generalized Darboux transformation for the case \( l_i + 1/2 \in \mathbb{Z} \) \((i = 0, 1, 2, 3)\). Write \( l_i = n_i - 1/2 \) and assume \( n_i \in \mathbb{Z}_{\geq 0} \) for \( i = 0, 1, 2, 3 \).

Candidates for the space related to quasi-solvability written as Eq. (3.1) are described as \( V_{\bar{n}_0 + \frac{1}{2}, \bar{n}_1 + \frac{1}{2}, \bar{n}_2 + \frac{1}{2}, \bar{n}_3 + \frac{1}{2}} \), where \( \bar{n}_i \in \{ \pm n_i \} \) \((i = 0, 1, 2, 3)\). For the existence of non-zero space \( V_{\tilde{n}_0 + \frac{1}{2}, \tilde{n}_1 + \frac{1}{2}, \tilde{n}_2 + \frac{1}{2}, \tilde{n}_3 + \frac{1}{2}} \), the condition \( \sum_{i=0}^{3} \tilde{n}_i \in 2\mathbb{Z} \) is necessary. In
other words, if $\sum_{i=0}^{3} n_i$ is odd, then the space related to quasi-solvability written as Eq. (3.1) does not exist.

On the rest of this section, we assume that $\sum_{i=0}^{3} n_i$ is even. Then the operator $H^{(n_0, n_1, n_2, n_3, \frac{1}{2})}$ preserves the spaces

\[
U_{-n_0+\frac{1}{2}, n_1+\frac{1}{2}} U_{-n_0+\frac{1}{2}, n_1+\frac{1}{2}, n_2+\frac{1}{2}} U_{-n_0+\frac{1}{2}, n_1+\frac{1}{2}, n_2+\frac{1}{2}, n_3+\frac{1}{2}},
\]

where $U_{\alpha_0, \alpha_1, \alpha_2, \alpha_3}$ is defined in Eq. (4.11). Unlike the case $l_0, l_1, l_2, l_3 \in \mathbb{Z}$, these eight spaces are not disjoint. In fact, these eight spaces are subspaces of the space $U_{-n_0+\frac{1}{2}, n_1+\frac{1}{2}, n_2+\frac{1}{2}, n_3+\frac{1}{2}}$. Set

\[
n_0^{(1)} = \frac{(n_0 + n_1 + n_2 + n_3)}{2}, \quad n_1^{(1)} = \frac{(n_0 + n_1 - n_2 - n_3)}{2},
\]

\[
n_2^{(1)} = \frac{(n_0 - n_1 + n_2 - n_3)}{2}, \quad n_3^{(1)} = \frac{(n_0 - n_1 - n_2 + n_3)}{2},
\]

\[
n_0^{(2)} = \frac{(-n_0 + n_1 + n_2 + n_3)}{2}, \quad n_1^{(2)} = \frac{(n_0 - n_1 + n_2 + n_3)}{2},
\]

\[
n_2^{(2)} = \frac{(n_0 + n_1 - n_2 + n_3)}{2}, \quad n_3^{(2)} = \frac{(n_0 + n_1 + n_2 - n_3)}{2}.
\]

It follows from Proposition 3.6 that

\[
H^{(n_0^{(1)}, n_1^{(1)}, n_2^{(1)}, n_3^{(1)}, \frac{1}{2})} \tilde{L}_{-n_0+\frac{1}{2}, -n_1+\frac{1}{2}, -n_2+\frac{1}{2}, -n_3+\frac{1}{2}} \tilde{L}_{n_0+\frac{1}{2}, n_1+\frac{1}{2}, n_2+\frac{1}{2}, n_3+\frac{1}{2}} = H^{(n_0, n_1, n_2, n_3, \frac{1}{2})},
\]

\[
H^{(n_0^{(1)}, n_1^{(1)}, n_2^{(1)}, n_3^{(1)}, \frac{1}{2})} \tilde{L}_{-n_0+\frac{1}{2}, -n_1+\frac{1}{2}, -n_2+\frac{1}{2}, -n_3+\frac{1}{2}} \tilde{L}_{n_0+\frac{1}{2}, n_1+\frac{1}{2}, n_2+\frac{1}{2}, n_3+\frac{1}{2}} = H^{(n_0, n_1, n_2, n_3, \frac{1}{2})},
\]

\[
H^{(n_0^{(1)}, n_1^{(1)}, n_2^{(1)}, n_3^{(1)}, \frac{1}{2})} \tilde{L}_{-n_0+\frac{1}{2}, -n_1+\frac{1}{2}, -n_2+\frac{1}{2}, -n_3+\frac{1}{2}} \tilde{L}_{n_0+\frac{1}{2}, n_1+\frac{1}{2}, n_2+\frac{1}{2}, n_3+\frac{1}{2}} = H^{(n_0, n_1, n_2, n_3, \frac{1}{2})},
\]

\[
H^{(n_0^{(1)}, n_1^{(1)}, n_2^{(1)}, n_3^{(1)}, \frac{1}{2})} \tilde{L}_{-n_0+\frac{1}{2}, -n_1+\frac{1}{2}, -n_2+\frac{1}{2}, -n_3+\frac{1}{2}} \tilde{L}_{n_0+\frac{1}{2}, n_1+\frac{1}{2}, n_2+\frac{1}{2}, n_3+\frac{1}{2}} = H^{(n_0, n_1, n_2, n_3, \frac{1}{2})},
\]

\[
H^{(n_0^{(1)}, n_1^{(1)}, n_2^{(1)}, n_3^{(1)}, \frac{1}{2})} \tilde{L}_{-n_0+\frac{1}{2}, -n_1+\frac{1}{2}, -n_2+\frac{1}{2}, -n_3+\frac{1}{2}} \tilde{L}_{n_0+\frac{1}{2}, n_1+\frac{1}{2}, n_2+\frac{1}{2}, n_3+\frac{1}{2}} = H^{(n_0, n_1, n_2, n_3, \frac{1}{2})}.
\]
where $\tilde{L}_{\alpha_0, \alpha_1, \alpha_2, \alpha_3}$ is defined in Eq. (1.2). We also obtain that

$$
H(n_0^{(2)} - \frac{1}{2}, n_1^{(2)} - \frac{1}{2}, n_2^{(2)} - \frac{1}{2}, n_3^{(2)} - \frac{1}{2}) \tilde{L}_{-n_0 + \frac{1}{2}, -n_1 + \frac{1}{2}, -n_2 + \frac{1}{2}, -n_3 + \frac{1}{2} \\
= \tilde{L}_{-n_0 + \frac{1}{2}, -n_1 + \frac{1}{2}, -n_2 + \frac{1}{2}, -n_3 + \frac{1}{2}} H^{n_0^{(2)} - \frac{1}{2}, n_1^{(2)} - \frac{1}{2}, n_2^{(2)} - \frac{1}{2}, n_3^{(2)} - \frac{1}{2})
$$

$$
H(n_1^{(2)} - \frac{1}{2}, n_2^{(2)} - \frac{1}{2}, n_3^{(2)} - \frac{1}{2}) \tilde{L}_{-n_0 + \frac{1}{2}, -n_1 + \frac{1}{2}, -n_2 + \frac{1}{2}, -n_3 + \frac{1}{2} \\
= \tilde{L}_{-n_0 + \frac{1}{2}, -n_1 + \frac{1}{2}, -n_2 + \frac{1}{2}, -n_3 + \frac{1}{2}} H^{n_0^{(2)} - \frac{1}{2}, n_1^{(2)} - \frac{1}{2}, n_2^{(2)} - \frac{1}{2}, n_3^{(2)} - \frac{1}{2})
$$

Thus it is shown that the operators which are linked by generalized Darboux transformations from $H^{(n_0^{(2)} - \frac{1}{2}, n_1^{(2)} - \frac{1}{2}, n_2^{(2)} - \frac{1}{2}, n_3^{(2)} - \frac{1}{2})}$ are

$$
H^{(n_0^{(1)} - \frac{1}{2}, n_1^{(1)} - \frac{1}{2}, n_2^{(1)} - \frac{1}{2}, n_3^{(1)} - \frac{1}{2})}, \quad H^{(n_0^{(1)} - \frac{1}{2}, n_1^{(1)} - \frac{1}{2}, n_2^{(1)} - \frac{1}{2}, n_3^{(1)} - \frac{1}{2})}, \\
H^{(n_0^{(2)} - \frac{1}{2}, n_1^{(2)} - \frac{1}{2}, n_2^{(2)} - \frac{1}{2}, n_3^{(2)} - \frac{1}{2})}, \quad H^{(n_0^{(2)} - \frac{1}{2}, n_1^{(2)} - \frac{1}{2}, n_2^{(2)} - \frac{1}{2}, n_3^{(2)} - \frac{1}{2})}, \\
H^{(n_0^{(1)} - \frac{1}{2}, n_1^{(1)} - \frac{1}{2}, n_2^{(1)} - \frac{1}{2}, n_3^{(1)} - \frac{1}{2})}, \quad H^{(n_0^{(1)} - \frac{1}{2}, n_1^{(1)} - \frac{1}{2}, n_2^{(1)} - \frac{1}{2}, n_3^{(1)} - \frac{1}{2})}, \\
H^{(n_0^{(2)} - \frac{1}{2}, n_1^{(2)} - \frac{1}{2}, n_2^{(2)} - \frac{1}{2}, n_3^{(2)} - \frac{1}{2})}, \quad H^{(n_0^{(2)} - \frac{1}{2}, n_1^{(2)} - \frac{1}{2}, n_2^{(2)} - \frac{1}{2}, n_3^{(2)} - \frac{1}{2})}
$$

Now we investigate the case $n_0$, $n_1$, $n_2$, $n_3 \in \mathbb{Z}_{\geq 0}$, $n_0 + n_1 + n_2 + n_3 \in 2\mathbb{Z}_{\geq 0}$ and $n_0 \geq n_1 \geq n_2 \geq n_3$. Recall that $l_i = n_i - 1/2$ ($i = 0, 1, 2, 3$). We divide into three cases, the case $n_0 \geq n_1 + n_2 + n_3$, the case $n_1 + n_2 + n_3 \leq n_0 \leq n_1 + n_2 + n_3$ and the case $n_0 < n_1 + n_2 + n_3$. if $\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 2$, then we set $V(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = \{0\}$.

If $n_0 \geq n_1 + n_2 + n_3$, then the operator $H^{(n_0^{(2)} - \frac{1}{2}, n_1^{(2)} - \frac{1}{2}, n_2^{(2)} - \frac{1}{2}, n_3^{(2)} - \frac{1}{2})}$ preserves the spaces

$$
V_{\frac{1}{2} - n_0 + \frac{1}{2}, \frac{1}{2} - n_1 + \frac{1}{2}, \frac{1}{2} - n_2 + \frac{1}{2}} \rightarrow V_{\frac{1}{2} - n_0 + \frac{1}{2}, \frac{1}{2} - n_1 + \frac{1}{2}, \frac{1}{2} - n_2 + \frac{1}{2}}, \\
V_{\frac{1}{2} - n_0 + \frac{1}{2}, \frac{1}{2} - n_1 + \frac{1}{2}, \frac{1}{2} - n_2 + \frac{1}{2}} \rightarrow V_{\frac{1}{2} - n_0 + \frac{1}{2}, \frac{1}{2} - n_1 + \frac{1}{2}, \frac{1}{2} - n_2 + \frac{1}{2}}, \\
V_{\frac{1}{2} - n_0 + \frac{1}{2}, \frac{1}{2} - n_1 + \frac{1}{2}, \frac{1}{2} - n_2 + \frac{1}{2}} \rightarrow V_{\frac{1}{2} - n_0 + \frac{1}{2}, \frac{1}{2} - n_1 + \frac{1}{2}, \frac{1}{2} - n_2 + \frac{1}{2}}, \\
V_{\frac{1}{2} - n_0 + \frac{1}{2}, \frac{1}{2} - n_1 + \frac{1}{2}, \frac{1}{2} - n_2 + \frac{1}{2}} \rightarrow V_{\frac{1}{2} - n_0 + \frac{1}{2}, \frac{1}{2} - n_1 + \frac{1}{2}, \frac{1}{2} - n_2 + \frac{1}{2}}, \\
V_{\frac{1}{2} - n_0 + \frac{1}{2}, \frac{1}{2} - n_1 + \frac{1}{2}, \frac{1}{2} - n_2 + \frac{1}{2}} \rightarrow V_{\frac{1}{2} - n_0 + \frac{1}{2}, \frac{1}{2} - n_1 + \frac{1}{2}, \frac{1}{2} - n_2 + \frac{1}{2}}.
$$

Here $V(\alpha_0, \alpha_1, \alpha_2, \alpha_3) - V_{\alpha_0, \alpha_1, \alpha_2, \alpha_3}$ means that $V_{\alpha_0, \alpha_1, \alpha_2, \alpha_3}$ is a subspace of $V(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$. On this case, the operator $H^{(n_0^{(2)} - \frac{1}{2}, n_1^{(2)} - \frac{1}{2}, n_2^{(2)} - \frac{1}{2}, n_3^{(2)} - \frac{1}{2})}$ is isomonodromic to $H^{(n_0^{(1)} - \frac{1}{2}, n_1^{(1)} - \frac{1}{2}, n_2^{(1)} - \frac{1}{2}, n_3^{(1)} - \frac{1}{2})}$ and $H^{(n_0^{(2)} - \frac{1}{2}, n_1^{(2)} - \frac{1}{2}, n_2^{(2)} - \frac{1}{2}, n_3^{(2)} - \frac{1}{2})}$ by the transformations $L_{-n_0 + \frac{1}{2}, n_1 + \frac{1}{2}, n_2 + \frac{1}{2}, n_3 + \frac{1}{2}}$ and $L_{-n_0 + \frac{1}{2}, n_1 + \frac{1}{2}, n_2 + \frac{1}{2}, n_3 + \frac{1}{2}}$.

If $n_1 + n_2 + n_3 \leq n_0 < n_1 + n_2 + n_3$, then the operator $H^{(n_0^{(2)} - \frac{1}{2}, n_1^{(2)} - \frac{1}{2}, n_2^{(2)} - \frac{1}{2}, n_3^{(2)} - \frac{1}{2})}$ preserves the spaces

$$
H^{(n_0^{(2)} - \frac{1}{2}, n_1^{(2)} - \frac{1}{2}, n_2^{(2)} - \frac{1}{2}, n_3^{(2)} - \frac{1}{2})} = H^{(n_0^{(2)} - \frac{1}{2}, n_1^{(2)} - \frac{1}{2}, n_2^{(2)} - \frac{1}{2}, n_3^{(2)} - \frac{1}{2})}.
$$
In this section, we construct an odd-order differential operator \( \tilde{\mathcal{A}} \) and show that it commutes with \( H^{(n_0, n_1, n_2, n_3)} \) by composing generalized Darboux transformations for the case \( l_0, l_1, l_2, l_3 \in \mathbb{Z}_{\geq 0} \). We also show that the commuting operator coincides with the one constructed in [3]. We use notations in section 4.

**Proposition 6.1.** If \( l_0 + l_1 + l_2 + l_3 \) is even, then we set

\[
\tilde{\mathcal{A}} = \tilde{L}_{l_0, l_1, l_2, l_3} + \tilde{L}_{l_0, l_1, l_2, l_3} + \tilde{L}_{l_0, l_1, l_2, l_3} + \tilde{L}_{l_0, l_1, l_2, l_3}.
\]

If \( l_0 + l_1 + l_2 + l_3 \) is odd, then we set

\[
\tilde{\mathcal{A}} = \tilde{L}_{l_0, l_1, l_2, l_3} + \tilde{L}_{l_0, l_1, l_2, l_3} + \tilde{L}_{l_0, l_1, l_2, l_3} + \tilde{L}_{l_0, l_1, l_2, l_3}.
\]

The operator \( \tilde{\mathcal{A}} \) commutes with \( H^{(l_0, l_1, l_2, l_3)} \), i.e.,

\[
\tilde{\mathcal{A}} H^{(l_0, l_1, l_2, l_3)} = H^{(l_0, l_1, l_2, l_3)} \tilde{\mathcal{A}}.
\]

**Proof.** If \( l_0 + l_1 + l_2 + l_3 \) is even, then Eq. (3.3) is shown by applying the following relations:

\[
H^{(l_0, l_1, l_2, l_3)} \tilde{L}_{l_0, l_1, l_2, l_3} = \tilde{L}_{l_0, l_1, l_2, l_3} H^{(l_0, l_1, l_2, l_3)},
\]

If \( l_0 + l_1 + l_2 + l_3 \) is odd, then Eq. (6.3) is shown by applying the following relations:

\[
H^{(l_0, l_1, l_2, l_3)} \tilde{L}_{l_0, l_1, l_2, l_3} = \tilde{L}_{l_0, l_1, l_2, l_3} H^{(l_0, l_1, l_2, l_3)}.
\]
Note that the order of the differential operator $\tilde{A}$ is equal to $2g + 1$, where $g$ is defined in Eq. (4.5) or Eq. (4.10). Recall that the space $V$ is written as Eq. (4.1) or Eq. (4.9), $\dim V = 2g + 1$ and it is the maximal finite-dimensional invariant subspace in $\mathcal{F}$ with respect to the action of the operator $H^{(l_0, l_1, l_2, l_3)}$.

**Proposition 6.2.** The operator $\tilde{A}$ is the monic differential operator of minimum order which vanishes all elements in $V$.

**Proof.** We show this proposition for the case that $l_0 + l_1 + l_2 + l_3$ is even. For the case that $l_0 + l_1 + l_2 + l_3$ is odd, it is shown similarly.

Since $V$ is written as Eq. (4.1) and $\deg \tilde{A} = \dim V$, it is enough to show that $\tilde{A}\phi(x) = 0$ for $\phi(x) \in U_{-l_0, -l_1, -l_2, -l_3}, U_{-l_0, -l_1, l_2 + 1, l_3 + 1}, U_{-l_0, -l_1, l_2 + 1, l_3 + 1}$. The operator $\tilde{A}$ is written as Eq. (6.1), and the operator $\tilde{L}_{-l_0, -l_1, -l_2, -l_3}$ vanishes any element in the space $U_{-l_0, -l_1, -l_2, -l_3}$. Hence we have $\tilde{A}\phi(x) = 0$ for $\phi(x) \in U_{-l_0, -l_1, -l_2, -l_3}$. Let $(f_1(x), \ldots, f_n(x))$ be a basis of the space $U_{-l_0, -l_1, l_2 + 1, l_3 + 1}$. Since the operator $H^{(l_0, l_1, l_2, l_3)}$ preserves the space $U_{-l_0, -l_1, l_2 + 1, l_3 + 1}$, the function $H^{(l_0, l_1, l_2, l_3)} f_j(x)$ is written as $\sum^n_{i=1} a_{i,j} f_i(x)$ for some constants $a_{i,j}$. It follows from Eq. (6.4) that

$$H^{(\ell_0, \ell_1, \ell_2, \ell_3)} \tilde{L}_{-l_0, -l_1, -l_2, -l_3} f_j(x) = \tilde{L}_{-l_0, -l_1, -l_2, -l_3} H^{(l_0, l_1, l_2, l_3)} f_j(x) = \sum^n_{i=1} a_{i,j} \tilde{L}_{-l_0, -l_1, -l_2, -l_3} f_i(x).$$

Set $\tilde{U}_{-l_0, -l_1, l_2 + 1, l_3 + 1} = \tilde{L}_{-l_0, -l_1, -l_2, -l_3} U_{-l_0, -l_1, l_2 + 1, l_3 + 1}$. Then it follows from Eq. (6.6) that the space $\tilde{U}_{-l_0, -l_1, l_2 + 1, l_3 + 1}$ is invariant under the action of $H^{(\ell_0, \ell_1, \ell_2, \ell_3)}$. Let $(\epsilon_1, \epsilon_3) \in \{\pm 1\}$ be the numbers such that $U_{-l_0, -l_1, l_2 + 1, l_3 + 1} \subset \mathcal{F}_{\epsilon_1, \epsilon_3}$. Then we have $\tilde{U}_{-l_0, -l_1, l_2 + 1, l_3 + 1} \subset U_{-l_0, -l_1, l_2 + 1, l_3 + 1}$ and $U_{-l_0, -l_1, l_2 + 1, l_3 + 1} \subset \mathcal{F}_{\epsilon_1, \epsilon_3}$. As is shown in [8] Theorem 3.1, the space $\tilde{U}_{-l_0, -l_1, l_2 + 1, l_3 + 1}$ is the maximum subspace of $\mathcal{F}_{\epsilon_1, \epsilon_3}$ which is invariant under the action of $H^{(\ell_0, \ell_1, \ell_2, \ell_3)}$. Thus we have

$$\tilde{U}_{-l_0, -l_1, l_2 + 1, l_3 + 1} = \tilde{L}_{-l_0, -l_1, -l_2, -l_3} U_{-l_0, -l_1, l_2 + 1, l_3 + 1} \subset U_{-l_0, -l_1, l_2 + 1, l_3 + 1}.$$ 

Similarly it is shown that

$$\tilde{L}_{-l_0, -l_1, l_2 + 1, l_3 + 1} = \tilde{L}_{-l_0, -l_1, l_2 + 1, l_3 + 1} U_{-l_0, -l_1, -l_2, -l_3} \subset U_{-l_0, -l_1, l_2 + 1, l_3 + 1},$$

$$\tilde{L}_{-l_0, -l_1, l_2 + 1, l_3 + 1} = \tilde{L}_{-l_0, -l_1, l_2 + 1, l_3 + 1} U_{-l_0, -l_1, l_2 + 1, l_3 + 1} \subset U_{-l_0, -l_1, l_2 + 1, l_3 + 1}.$$ 

Since the operator $\tilde{L}_{-l_0, -l_1, l_2 + 1, l_3 + 1}$ (resp. $\tilde{L}_{-l_0, -l_1, l_2 + 1, l_3 + 1}$) vanishes any element in the space $U_{-l_0, -l_1, l_2 + 1, l_3 + 1}$ (resp. $U_{-l_0, -l_1, l_2 + 1, l_3 + 1}$), we have $\tilde{A}\phi(x) = 0$ for $\phi(x) \in U_{-l_0, -l_1, l_2 + 1, l_3 + 1}$ (resp. $\phi(x) \in U_{-l_0, -l_1, l_2 + 1, l_3 + 1}$).

It follows from Proposition 6.2 that the kernel of the operator $\tilde{A}$ coincides with the space $V$. We denote the monic characteristic polynomial of the operator $H^{(l_0, l_1, l_2, l_3)}$ on the space $V$ by $P(E)$. For simplicity, we set $u(x) = \sum^3_{i=0} l_i(i + 1)\varphi(x + \omega_i)$ and $H = H^{(l_0, l_1, l_2, l_3)} = -d^2/dx^2 + u(x)$.

**Proposition 6.3.** Set $\tilde{a}_0(x) = 1$ and $\tilde{a}_g+1(x) = 0$. The operator $\tilde{A}$ is written as

$$(6.7) \quad \tilde{A} = (-1)^g \left[ \sum_{j=0}^{g} \tilde{a}_j(x) \frac{d}{dx} - \frac{1}{2} \left( \frac{d}{dx} \tilde{a}_j(x) \right) \right] H^{g-j} + \sum_{j=0}^{g} c_j H^{g-j},$$
for even doubly-periodic functions \( \tilde{a}_j(x) \) (\( j = 1, \ldots, g \)) and constants \( c_j \) (\( j = 0, \ldots, g \)), and \( \tilde{a}_j(x) \) (\( j = 0, \ldots, g \)) satisfies

\[
\tilde{a}_j''(x) - 4u(x)\tilde{a}_j'(x) + 4\tilde{a}_{j+1}'(x) - 2u'(x)\tilde{a}_j(x) = 0. 
\]

Proof. Since \( \tilde{A} \) is a monic differential operator of order \( 2g + 1 \), it is written as

\[
\tilde{A} = (-1)^g \left[ \sum_{j=0}^{g} \left( \tilde{a}_j(x) \frac{d}{dx} + \tilde{b}_j(x) \right) H^{g-j} \right], 
\]

where \( \tilde{a}_0(x) = 1 \). We have

\[
[(-1)^g \tilde{A}, H] = \sum_{j=0}^{g} \left[ \tilde{a}_j(x) \frac{d}{dx} + \tilde{b}_j(x), -\frac{d^2}{dx^2} + u(x) \right] H^{g-j} 
\]

\[
= \sum_{j=0}^{g} \left( \tilde{a}_j(x)u'(x) + 2\tilde{a}_j'(x)\frac{d^2}{dx^2} + (\tilde{a}_j''(x) + 2\tilde{b}_j'(x))\frac{d}{dx} + \tilde{b}_j''(x) \right) H^{g-j} 
\]

\[
= \sum_{j=0}^{g} \left( 2\tilde{a}_j'(x)(-H + u(x)) + (\tilde{a}_j''(x) + 2\tilde{b}_j'(x))\frac{d}{dx} + \tilde{a}_j(x)u'(x) + \tilde{b}_j''(x) \right) H^{g-j} 
\]

\[
= \sum_{j=0}^{g} \left( (\tilde{a}_j''(x) + 2\tilde{b}_j'(x))\frac{d}{dx} - 2\tilde{a}_{j+1}'(x) + 2\tilde{a}_j'(x)u(x) + \tilde{a}_j(x)u'(x) + \tilde{b}_j''(x) \right) H^{g-j} 
\]

\[
= 0. 
\]

Hence we have

\[
\tilde{a}_j''(x) + 2\tilde{b}_j'(x) = 0, \quad -2\tilde{a}_{j+1}'(x) + 2\tilde{a}_j'(x)u(x) + \tilde{a}_j(x)u'(x) + \tilde{b}_j''(x) = 0. 
\]

Therefore

\[
\tilde{b}_j(x) = -\tilde{a}_j'(x)/2 + c_j, \quad \tilde{a}_j''(x) - 4u(x)\tilde{a}_j'(x) + 4\tilde{a}_{j+1}'(x) - 2u'(x)\tilde{a}_j(x) = 0 
\]

for some constants \( c_j \) (\( j = 0, \ldots, g \)), and we obtain the proposition. \(\square\)

Set

\[
\tilde{\Xi}(x, E) = \sum_{i=0}^{g} \tilde{a}_{g-i}(x)E^i. 
\]

It follows from Eq.(6.8) that

\[
\left( \frac{d^3}{dx^3} - 4(u(x) - E)\frac{d}{dx} - 2u'(x) \right) \tilde{\Xi}(x, E) = 0. 
\]

Set

\[
\tilde{Q}(E) = \tilde{\Xi}(x, E)^2(E - u(x)) + \frac{1}{2} \tilde{\Xi}(x, E)\frac{d^2\tilde{\Xi}(x, E)}{dx^2} - \frac{1}{4} \left( \frac{d\tilde{\Xi}(x, E)}{dx} \right)^2. 
\]

It is shown by differentiating the right hand side of Eq.(6.11) and applying Eq.(6.10) that \( \tilde{Q}(E) \) is independent of \( x \). \( \tilde{Q}(E) \) is a monic polynomial in \( E \) of degree \( 2g + 1 \).
1, which follows from the expression for $\tilde{\Xi}(x, E)$ given by Eq. (6.9). Similarly to Proposition 3.2 in [8], we can show
\begin{equation}
(-1)^g A - \sum_{j=0}^{g} c_j H^{g-j} = \left( \sum_{j=0}^{g} \left\{ \tilde{a}_j(x) \frac{d}{dx} - \frac{1}{2} \left( \frac{d}{dx} \tilde{a}_j(x) \right) \right\} H^{g-j} \right)^2 = \tilde{Q}(H)^2.
\end{equation}

Let us recall the function $\Xi(x, E)$ defined in [6]. It satisfies Eq. (6.10) and has an expression
\begin{equation}
\Xi(x, E) = c_0(E) + \sum_{i=0}^{3} \sum_{j=0}^{l_i-1} b^{(i)}_j(E) \varphi(x + \omega_i)^{l_i-j},
\end{equation}
where the coefficients $c_0(E)$ and $b^{(i)}_j(E)$ are polynomials in $E$, they do not have common divisors and the polynomial $c_0(E)$ is monic. It is shown that the dimension of the functions which are doubly-periodic and satisfy Eq. (6.10) is one (see [10, Proposition 3.9]). Since $\tilde{\Xi}(x, E)$ is a polynomial with respect to the variable $E$ and coefficients of $\Xi(x, E)$ are coprime, we have
\begin{equation}
\tilde{\Xi}(x, E) = \Xi(x, E) \left( \sum_{i=0}^{k} d_i E^{k-i} \right)
\end{equation}
for non-negative integer $k$ and constants $d_i$ ($i = 0, \ldots, k$) such that $d_0 = 1$. Write
\begin{equation}
\Xi(x, E) = \sum_{i=0}^{g-k} a_{g-k-i}(x) E^i.
\end{equation}
Then we have $a_0(x) = 1$ and
\begin{equation}
\tilde{a}_{g-i}(x) = \sum_{j=0}^{k} a_{g-i-k+j}(x) d_{k-j}.
\end{equation}
The functions $a_i(x)$ ($i = 0, \ldots, g - k$) also satisfy Eq. (6.8), because $\Xi(x, E)$ satisfies Eq. (6.10). Set
\begin{equation}
A = \sum_{j=0}^{g-k} \left\{ a_j(x) \frac{d}{dx} - \frac{1}{2} \left( \frac{d}{dx} a_j(x) \right) \right\} H^{g-k-j},
\end{equation}
It follows from Eq. (6.8) for $a_i(x)$ that $[A, H] = 0$.

**Proposition 6.4.**

\begin{equation}
(-1)^g A - \sum_{l=0}^{g} c_{g-l} H^l = A \left( \sum_{j=0}^{k} d_{k-j} H^j \right).
\end{equation}
Proof.

\[ (-1)^g \tilde{A} - \sum_{i=0}^{g} c_{g-i} H^i = \sum_{i=0}^{g} \left\{ \tilde{a}_i(x) \frac{d}{dx} - \frac{1}{2} \left( \frac{d}{dx} \tilde{a}_i(x) \right) \right\} H^{9-i} \]

\[ \sum_{i=0}^{g} \left\{ \left( \sum_{j=0}^{k} a_{g-i-k+j}(x) d_{k-j} \right) \frac{d}{dx} - \frac{1}{2} \left( \frac{d}{dx} \sum_{j=0}^{k} a_{g-i-k+j}(x) d_{k-j} \right) \right\} H^{9-i} \]

\[ = \sum_{j=0}^{k} \left\{ a_{g-i-k+j}(x) \frac{d}{dx} - \frac{1}{2} \left( \frac{d}{dx} a_{g-i-k+j}(x) \right) \right\} H^{9-i-j} d_{k-j} H^j \]

\[ = A \left( \sum_{j=0}^{k} d_{k-j} H^j \right). \]

Set

\[ (6.17) \quad Q(E) = \Xi(x, E)^2 (E - u(x)) + \frac{1}{2} \Xi(x, E) \frac{d^2 \Xi(x, E)}{dx^2} - \frac{1}{4} \left( \frac{d \Xi(x, E)}{dx} \right)^2. \]

Then the right hand side of Eq. (6.17) is independent of \( x \), and \( Q(E) \) is a monic polynomial in \( E \) of degree \( 2(g - k) + 1 \). It is shown in \([8, Proposition 3.2]\) that \( A^2 = Q(H) \). By Eq. (6.13) and definitions of \( Q(E) \) and \( \tilde{Q}(E) \), we have

\[ (6.18) \quad \tilde{Q}(E) = Q(E) \left( \sum_{i=0}^{k} d_i E^{k-i} \right)^2, \quad \deg Q(E) \leq \deg \tilde{Q}(E) = 2g + 1. \]

On zeros of \( Q(E) \) and \( P(E) \), the following proposition is shown in \([8, Theorem 3.8]\) (see also \([8, Proposition 2.4]\)):

Proposition 6.5. (c.f. \([8, Theorem 3.8]\)) The set of zeros of \( Q(E) \) coincides with the set of zeros of \( P(E) \).

Proposition 6.6. (c.f. \([11]\)) Roots of the equation \( P(E) = 0 \) are distinct for generic periods \( (2\omega_1, 2\omega_3) \).

We prove Proposition 6.6 in the appendix.

Proposition 6.7. For the periods \( (2\omega_1, 2\omega_3) \) that roots of the equation \( P(E) = 0 \) are distinct, we have \( P(E) = Q(E) = \tilde{Q}(E) \) and \( \Xi(x, E) = \tilde{\Xi}(x, E) \).

Proof. By assumption, the equation \( P(E) = 0 \) has \( 2g + 1 \) roots which are distinct. It follows from Proposition 6.3 that the number of roots of the equation \( Q(E) = 0 \) is equal to or more than \( 2g + 1 \). Then we have \( \deg Q(E) \geq 2g + 1 \). Combining with Eq. (6.18) and \( \deg P(E) = 2g + 1 \), we have \( \tilde{Q}(E) = Q(E) = P(E) \), and the value \( k \) in Eq. (6.13) is zero. Hence \( \Xi(x, E) = \tilde{\Xi}(x, E) \).

Proposition 6.8. All constants \( c_j \) \( (j = 1, \ldots, g) \) are zero. Namely we have

\[ (6.19) \quad \tilde{A} = (-1)^g \sum_{j=0}^{g} \left\{ \tilde{a}_j(x) \frac{d}{dx} - \frac{1}{2} \left( \frac{d}{dx} \tilde{a}_j(x) \right) \right\} H^{9-j}. \]
Proof. First, we assume that roots of the equation $P(E) = 0$ are distinct. Let \( \{E_i\}_{i=1, \ldots, 2g+1} \) be the roots of the equation $P(E) = 0$. Then $E_i$ ($i = 1, \ldots, 2g+1$) is an eigenvalue of the operator $H$ on the space $V$. Let $\phi_i(x) \in V$ be the eigenfunction of the operator $H$ with the eigenvalue $E_i$. It follows from Proposition 6.1 that
due to Proposition 6.1.

Hence we have periodic up to signs, which is essentially shown in [6, Theorem 3.8] (see also the
due to Proposition 6.1)

But it contradicts that all solutions to ($6.2$) that the operator ($6.2$) annihilates the space $V$, we have \( \sum_{l=0}^g c_{g-l} H^l \) = $P(H)$. Hence ($\tilde{A} - 2(-1)^g \sum_{l=0}^g c_{g-l} H^l \) $+$ \sum_{l=0}^g c_{g-l} H^l \) = $P(H)$. We apply this operator to $\phi_i(x)$. Since the operator $\tilde{A}$ annihilates the space $V$, we have \( \sum_{l=0}^g c_{g-l} (E_i)^l \) = $P(E_i)\phi_i(x) = 0$. Hence \( \sum_{l=0}^g c_{g-l} (E_i)^l \) = 0 for $i = 1, \ldots, g$. Since the polynomial of degree less that $g + 1$
cannot have $2g + 1$ zeros, we have $c_i = 0$ ($i = 0, \ldots, g$) for the case that roots of the
equation $P(E) = 0$ are distinct.

Now consider the case that roots of the equation $P(E) = 0$ are not distinct. Since the operator $\tilde{A}$ is defined by Eq. ($6.1$) or Eq. ($6.2$), the coefficients $c_i$ ($i = 0, \ldots, g$) are continuous as a function of periods ($2\omega_1, 2\omega_3$). Since $c_i = 0$ ($i = 0, \ldots, g$) for a dense set of periods, we have $c_i = 0$ ($i = 0, \ldots, g$) for all periods.

For the case that roots of the equation $P(E) = 0$ are distinct, it is shown that $P(E) = Q(E)$, the value $k$ in Eq. ($6.13$) is zero, $\Xi(x, E) = \tilde{\Xi}(x, E)$ and $\tilde{A} = (-1)^g A$.

We now consider the case that roots of the equation $P(E) = 0$ are not distinct. It is already shown that $P(E) = \tilde{Q}(E)$ for a dense set of periods, $\text{deg } P(E) = \text{deg } \tilde{Q}(E)$ for all periods, and coefficients of $P(E)$ and $\tilde{Q}(E)$ are continuous with respect to the periods. Hence we have $P(E) = \tilde{Q}(E)$ for all periods. Assume that the value $k$ in Eq. ($6.13$) is positive. Combining with Eq. ($6.18$) and Proposition 6.5, all zeros of ($\sum_{i=0}^k d_i E^n$) are zeros of $Q(E)$. Let $E_0$ be a zero of $\sum_{i=0}^k d_i E^n$, i.e. $\sum_{i=0}^k d_i (E_0)^n = 0$. By Eq. ($6.16$) and Proposition 6.8, it is shown that, if $f(x)$ satisfies $Hf(x) = E_0 f(x)$, then $\tilde{A} f(x) = 0$. Hence all solutions to $(H - E_0) f(x) = 0$ are contained in the space $V$. But it contradicts that all solutions to $(H - E_0) f(x) = 0$ cannot be doubly-periodic up to signs, which is essentially shown in [6, Theorem 3.8] (see also the proof of [10] Proposition 3.9). Hence we have $k = 0$. Therefore $A = (-1)^g A$, $\text{deg } P(E) = \text{deg } \tilde{Q}(E)$ and $P(E) = Q(E)$ for all periods. It follows from Proposition ($6.2$) that the operator $(-1)^g A(\tilde{A})$ is characterized by the monic operator of order $2g + 1$ which annihilates all elements in the space $V$. Summarizing, we obtain the following theorem:

**Theorem 6.9.** Let $\tilde{A}$ be the operator defined by composing generalized Darboux transformation (see Eq. ($6.1$) or Eq. ($6.2$)) and $A$ be the operator defined from the even doubly-periodic function $\Xi(x, E)$ (see Eqs. ($6.14$, $6.12$)). Let $P(E)$ be the characteristic polynomial of the operator $H$ on the space $V$, and $Q(E)$ be the polynomial defined from $\Xi(x, E)$ (see Eq. ($6.17$)).

(i) We have $A = (-1)^g A$ and $P(E) = Q(E)$.

(ii) The operator $(-1)^g A$ is also characterized by the monic operator of order $2g + 1$ which annihilates all elements in the space $V$.

Note that we proved Conjecture 1 in [8] by (i) and Conjecture 2 in [8] by (ii). Since $P(E) = Q(E)$, the genus of the spectral curve $\nu^2 = -Q(E)$ is $g$, where $g$ is defined in Eq. ($1.13$) ($l_0 + l_1 + l_2 + l_3$: even) or Eq. ($1.10$) ($l_0 + l_1 + l_2 + l_3$: odd), and it agrees with the results in [2] [12].

We consider isomonodromic property again. Let $l_i^\varepsilon$ ($i = 0, 1, 2, 3$) be the numbers defined in Eqs. ($1.6$, $1.11$). We have shown in section 4 that, if $l_0 + l_1 + l_2 + l_3$ is even (resp. odd), then the operator $H(l_0, l_1, l_2, l_3)$ is linked to $H(l_0, l_1, l_2, l_3)$ by generalized
Darboux transformation, where \( H^{(l_0,l_1,l_2,l_3)} \) is any operator listed in Eq. (4.17) (resp. Eq. (4.12)). We now show that functions related to monodromy for the operator \( H^{(l_0,l_1,l_2,l_3)} \) coincides to ones for the operator \( H^{(\tilde{l}_0,\tilde{l}_1,\tilde{l}_2,\tilde{l}_3)} \). For this purpose, we recall functions defined in [6, 8, 9].

Let \( \Xi^{(l_0,l_1,l_2,l_3)}(x, E) \) be the function defined in Eq. (6.12), \( Q^{(l_0,l_1,l_2,l_3)}(E) \) be the polynomial defined in Eq. (6.17) and \( P^{(l_0,l_1,l_2,l_3)}(E) \) be the polynomial \( P(E) \) defined in this section. Set

\[
\Lambda^{(l_0,l_1,l_2,l_3)}(x, E) = \sqrt{\Xi^{(l_0,l_1,l_2,l_3)}(x, E)} \exp \int \frac{\sqrt{-Q^{(l_0,l_1,l_2,l_3)}(E)} \, dx}{\Xi^{(l_0,l_1,l_2,l_3)}(x, E)}.
\]

Then the function \( \Lambda^{(l_0,l_1,l_2,l_3)}(x, E) \) is a solution to the differential equation

\[
(H^{(l_0,l_1,l_2,l_3)} - E) f(x) = 0.
\]

Set

\[
\Phi_i(x, \alpha) = \frac{\sigma(x + \omega_i - \alpha)}{\sigma(x + \omega_i)} \exp(\zeta(\alpha)x), \quad (i = 0, 1, 2, 3),
\]

where \( \sigma(x) \) is the Weierstrass sigma function. The function \( \Lambda^{(l_0,l_1,l_2,l_3)}(x, E) \) admits an expression in terms of Hermite-Krichever Ansatz. Namely, we have the following proposition:

**Proposition 6.10.** (c.f. [9, Theorem 2.3]) The function \( \Lambda^{(l_0,l_1,l_2,l_3)}(x, E) \) is expressed as

\[
(6.20) \quad \Lambda^{(l_0,l_1,l_2,l_3)}(x, E) = \exp(\kappa x) \left( \sum_{i=0}^{3} \sum_{j=0}^{l_i-1} \tilde{b}_j^{(i)} \left( \frac{d}{dx} \right)^j \Phi_i(x, \alpha) \right)
\]

for some values \( \tilde{b}_j^{(i)} \) (\( i = 0, \ldots, 3, j = 0, \ldots, l_i-1 \)) except for finitely-many eigenvalues \( E \). The values \( \alpha \) and \( \kappa \) are expressed as

\[
\varphi(\alpha) = R_1^{(l_0,l_1,l_2,l_3)}(E), \quad \varphi'(\alpha) = R_2^{(l_0,l_1,l_2,l_3)}(E) \sqrt{-Q^{(l_0,l_1,l_2,l_3)}(E)}, \quad 
\kappa = R_3^{(l_0,l_1,l_2,l_3)}(E) \sqrt{-Q^{(l_0,l_1,l_2,l_3)}(E)},
\]

where \( R_1^{(l_0,l_1,l_2,l_3)}(E), R_2^{(l_0,l_1,l_2,l_3)}(E), R_3^{(l_0,l_1,l_2,l_3)}(E) \) are rational functions in \( E \).

It follows from Eq. (6.20) that

\[
(6.21) \quad \Lambda^{(l_0,l_1,l_2,l_3)}(x + 2\omega_k, E) = \exp(-2\eta_k \alpha + 2\omega_k \zeta(\alpha) + 2\kappa \omega_k) \Lambda^{(l_0,l_1,l_2,l_3)}(x, E)
\]

for \( k = 1, 2, 3 \). A monodromy formula in terms of hyperelliptic integral was obtained in [8].

**Proposition 6.11.** (c.f. [8, Theorem 3.7]) Let \( k \in \{1, 2, 3\}, q_k \in \{0, 1\} \) and \( E_0 \) be the eigenvalue such that \( \Lambda^{(l_0,l_1,l_2,l_3)}(x + 2\omega_k, E_0) = (-1)^{q_k} \Lambda^{(l_0,l_1,l_2,l_3)}(x, E_0) \). Then

\[
(6.22) \quad \Lambda^{(l_0,l_1,l_2,l_3)}(x + 2\omega_k, E) = (-1)^{q_k} \Lambda^{(l_0,l_1,l_2,l_3)}(x, E).
\]

\[
\exp \left( -\frac{1}{2} \int_{E_0}^{E} \frac{2\eta_k a^{(l_0,l_1,l_2,l_3)}(\tilde{E}) + 2\omega_k c^{(l_0,l_1,l_2,l_3)}(\tilde{E})}{\sqrt{-Q^{(l_0,l_1,l_2,l_3)}(\tilde{E})}} \, d\tilde{E} \right),
\]

where \( a^{(l_0,l_1,l_2,l_3)}(E) \) (resp. \( c^{(l_0,l_1,l_2,l_3)}(E) \)) is a polynomial defined in [9].

The following proposition states the coincidence of functions.
Proposition 6.12. Let $H^{(l_0, l_1, l_2, l_3)}$ be any operator listed in Eq. (4.14) ($l_0 + l_1 + l_2 + l_3$: even) or Eq. (4.12) ($l_0 + l_1 + l_2 + l_3$: odd). Then we have

$$P^{(l_0, l_1, l_2, l_3)}(E) = P^{(l_0, l_1, l_2, l_3)}(E), \quad Q^{(l_0, l_1, l_2, l_3)}(E) = Q^{(l_0, l_1, l_2, l_3)}(E),$$

$$R_i^{(l_0, l_1, l_2, l_3)}(E) = R_i^{(l_0, l_1, l_2, l_3)}(E), \quad (i = 1, 2, 3),$$

$$a^{(l_0, l_1, l_2, l_3)}(E) = a^{(l_0, l_1, l_2, l_3)}(E), \quad c^{(l_0, l_1, l_2, l_3)}(E) = c^{(l_0, l_1, l_2, l_3)}(E).$$

Proof. We show the proposition for the case that $l_0 + l_1 + l_2 + l_3$ is even and $(l_0, l_1, l_2, l_3) = (l_0^e, l_2^e, l_1^e, l_0^e)$. For the other cases, it is shown similarly.

The space $V$ for the operator $H^{(l_0, l_1, l_2, l_3)}$ is written as

$$U_{-l_0, -l_1, -l_2, -l_3} \oplus U_{-l_0, -l_1, l_2 + 1, l_3 + 1} \oplus U_{-l_0, l_1 + 1, l_2 - 1, l_3} \oplus U_{-l_0, l_1 + 1, l_2 + 1, -l_3},$$

and the corresponding space for the operator $H^{(l_0^e, l_2^e, l_1^e, l_0^e)}$ is written as

$$U_{-l_0^e, -l_1, -l_2^e, -l_3^e} \oplus U_{-l_0^e, l_1, l_2^e + 1, l_3^e + 1} \oplus U_{-l_0^e, l_1 + 1, l_2^e - 1, l_3^e} \oplus U_{-l_0^e, l_1 + 1, l_2^e + 1, -l_3^e}.$$

It is seen that the space $U_{-l_0, -l_1, -l_2, -l_3}$ (resp. $U_{-l_0, -l_1, l_2 + 1, l_3 + 1}$, $U_{-l_0, l_1 + 1, -l_2, l_3}$, $U_{-l_0, l_1 + 1, l_2 + 1, -l_3}$) is linked to the space $U_{-l_0^e, -l_1^e, -l_2^e, -l_3^e}$ (resp. $U_{-l_0^e, l_1^e, -l_2^e, l_3^e}$, $U_{-l_0^e, l_1^e + 1, -l_2^e, l_3^e}$, $U_{-l_0^e, l_1^e, l_2^e + 1, -l_3^e}$) by the generalized Darboux transformation $\tilde{L}_{-l_0, -l_1, -l_2, -l_3}$ (resp. $\tilde{L}_{-l_0, -l_1, l_2 + 1, l_3 + 1}$, $\tilde{L}_{-l_0, l_1 + 1, -l_2, l_3}$, $\tilde{L}_{-l_0, l_1 + 1, l_2 + 1, -l_3}$) and the shift $x \to x + \omega_3$ (resp. $x \to x + \omega_1$, $x \to x + \omega_2$, $x \to x$). It follows from Proposition 3.3 that the characteristic polynomial of $H^{(l_0, l_1, l_2, l_3)}$ on the space $U_{-l_0, -l_1, -l_2, -l_3}$ (resp. $U_{-l_0, -l_1, l_2 + 1, l_3 + 1}$, $U_{-l_0, l_1 + 1, -l_2, l_3}$, $U_{-l_0, l_1 + 1, l_2 + 1, -l_3}$) is equal to that of $H^{(l_0^e, l_2^e, l_1^e, l_0^e)}$ on the space $U_{-l_0^e, -l_1^e, -l_2^e, -l_3^e}$ (resp. $U_{-l_0^e, l_1^e, -l_2^e, l_3^e}$, $U_{-l_0^e, l_1^e + 1, -l_2^e, l_3^e}$, $U_{-l_0^e, l_1^e, l_2^e + 1, -l_3^e}$). Since the polynomial $P(E)$ is written as the product of characteristic polynomials of invariant subspaces, we have

$$P^{(l_0, l_1, l_2, l_3)}(E) = P^{(l_0^e, l_2^e, l_1^e, l_0^e)}(E).$$

It follows from Theorem 6.9 that $Q^{(l_0, l_1, l_2, l_3)}(E) = Q^{(l_0^e, l_2^e, l_1^e, l_0^e)}(E)$.

Let $\tilde{\alpha}$ and $\tilde{\kappa}$ be the corresponding values in Eq. (6.21) for the parameters $(l_0^e, l_2^e, l_1^e, l_0^e)$. Namely, they satisfy

$$A^{(l_0^e, l_2^e, l_1^e, l_0^e)}(x + 2\omega_k, E) = \exp(-2\eta_k \tilde{\alpha} + 2\omega_k \zeta(\tilde{\alpha}) + 2\tilde{\kappa}\omega_k)A^{(l_0^e, l_2^e, l_1^e, l_0^e)}(x, E)$$

for $k = 1, 2, 3$. The monodromy matrix is preserved by the generalized Darboux transformation for almost eigenvalues $E$. Since the values $\exp(\pm (-2\eta_k \alpha + 2\omega_k \zeta(\alpha) + 2\kappa\omega_k))$ (resp. $\exp(\pm (-2\eta_k \tilde{\alpha} + 2\omega_k \zeta(\tilde{\alpha}) + 2\tilde{\kappa}\omega_k))$) are eigenvalues of the monodromy matrix, we have

$$-2\eta_1 \alpha + 2\omega_1(\zeta(\alpha) + \kappa) = \pm (-2\eta_1 \tilde{\alpha} + 2\omega_1(\zeta(\tilde{\alpha}) + \tilde{\kappa})), \quad 2n_1 \pi \sqrt{-1},$$

$$-2\eta_3 \alpha + 2\omega_3(\zeta(\alpha) + \kappa) = \pm (-2\eta_3 \tilde{\alpha} + 2\omega_3(\zeta(\tilde{\alpha}) + \tilde{\kappa})), \quad 2n_3 \pi \sqrt{-1},$$

for some integers $n_1$ and $n_3$ and almost eigenvalues $E$. By the Legendre's relation $\eta_1\omega_3 - \eta_3\omega_1 = \pi \sqrt{-1}/2$ and the relation $\zeta(x + 2\omega_k) = \zeta(x) + 2\eta_k$ ($k = 1, 3$), it follows that

$$\alpha = \pm \tilde{\alpha} - (2n_1\omega_3 - 2n_3\omega_1), \quad \kappa = \pm \tilde{\kappa}.$$

It follows from the asymptotic of $\kappa$ (see [3, Proposition 3.2]) that the sign $\pm$ is plus. Hence $\phi(\alpha) = \phi(\tilde{\alpha})$, $\phi'(\alpha) = \phi'(\tilde{\alpha})$ and $\kappa = \tilde{\kappa}$ for almost $E$. Since $R_i^{(l_0, l_1, l_2, l_3)}(E)$ and $R_i^{(l_0^e, l_2^e, l_1^e, l_0^e)}(E)$ ($i = 1, 2, 3$) are rational functions, we have $R_i^{(l_0, l_1, l_2, l_3)}(E) = R_i^{(l_0^e, l_2^e, l_1^e, l_0^e)}(E)$ for $i = 1, 2, 3$. 
Let $\tilde{E}_0$ be the corresponding values in Eq. (6.22) for the parameters $(l_0, l_1, l_2, l_3)$. By applying a similar discussion for Eq. (6.22), we obtain that the integrals

$$
\int_{E_0}^{E} \frac{a(l_0, l_1, l_2, l_3)(\tilde{E})}{\sqrt{Q(l_0, l_1, l_2, l_3)}(\tilde{E})} \, d\tilde{E} - \int_{E_0}^{E} \frac{a(l_3, l_2, l_1, l_0)(\tilde{E})}{\sqrt{Q(l_3, l_2, l_1, l_0)}(\tilde{E})} \, d\tilde{E},
$$

are constants. By differentiating them in the variable $E$ and using the relation $Q(l_0, l_1, l_2, l_3)(E) = Q(l_3, l_2, l_1, l_0)(E)$, we have $a(l_0, l_1, l_2, l_3)(E) = a(l_3, l_2, l_1, l_0)(E)$ and $c(l_0, l_1, l_2, l_3)(E) = c(l_3, l_2, l_1, l_0)(E)$. □

It follows from Proposition 6.12 that the hyperelliptic-to-elliptic integral formulae obtained in [9] for the parameters $(l_0, l_1, l_2, l_3)$ coincide with that for the parameters $(\tilde{l}_0, \tilde{l}_1, \tilde{l}_2, \tilde{l}_3)$.

We calculate the commuting operator $A$ explicitly for the case $l_0 = l_1 = l_2 = l_3 = g \in \mathbb{Z}_{\geq 1}$. The commuting operator $A$ of $H^{(g, g, g, g)}$ is written as

$$
A = (-1)^g L_{-g, -g, -g, -g} = (-1)^g \varphi'(x)^{2g+1} \tilde{\Phi}(\varphi(x)) \circ \left( \frac{1}{\varphi'(x)} \frac{d}{dx} \right)^{2g+1} \circ \tilde{\Phi}(\varphi(x))^{-1},
$$

(see Eq. (5.21)), and $\tilde{\Phi}(\varphi(x)) = ((\varphi(x) - e_1)^{-g/2}(\varphi(x) - e_2)^{-g/2}(\varphi(x) - e_3)^{-g/2} = (\varphi'(x)/2)^{-g}$. Hence we have

$$
A = (-1)^g \varphi'(x)^{g+1} \circ \left( \frac{1}{\varphi'(x)} \frac{d}{dx} \right)^{2g+1} \circ \varphi'(x)^g.
$$

On the other hand, we have $\sum_{i=0}^{3} \varphi(x + \omega_i) = 4\varphi(2x)$. By the change $2x \rightarrow x$, we recover the Lamé operator $4(-d^2/dx^2 + g(g+1)\varphi(x))$. Hence the commuting operator $A$ of $H^{(g, g, 0, 0)}$ is written as

$$
A = (-1)^g \varphi'(x/2)^{g+1} \circ \left( \frac{1}{\varphi'(x/2)} \frac{d}{dx} \right)^{2g+1} \circ \varphi'(x/2)^g.
$$

Therefore we obtain the following proposition:

**Proposition 6.13.** The commuting operator $A$ for the Lamé operator $H^{(g, g, 0, 0)}$ ($g \in \mathbb{Z}_{\geq 1}$) is written as Eq. (6.24).

**Appendix A. Elliptic functions**

This appendix presents the definitions of and the formulas for the elliptic functions. The Weierstrass $\wp$-function is defined by

$$
\wp(x) = \frac{1}{x^2} + \sum_{(m,n)\in \mathbb{Z} \times \mathbb{Z} \setminus \{(0,0)\}} \frac{1}{(x - 2m\omega_1 - 2n\omega_3)^2} - \frac{1}{(2m\omega_1 + 2n\omega_3)^2}.
$$

(A.1)
Setting $\omega_2 = -\omega_1 - \omega_3$ and $e_k = \varphi(\omega_k)$ \((k = 1, 2, 3)\) yields the relations

$$e_1 + e_2 + e_3 = 0,$$

$$\frac{\varphi''(x)}{(\varphi'(x))^2} = \frac{1}{2} \left( \frac{1}{\varphi(x) - e_1} + \frac{1}{\varphi(x) - e_2} + \frac{1}{\varphi(x) - e_3} \right),$$

$$\varphi(x + \omega_i) = e_i + \frac{(e_i - e_{i'})(e_i - e_{i''})}{\varphi(x) - e_i} \quad (i = 1, 2, 3),$$

where $i', i'' \in \{1, 2, 3\}$ with $i' < i''$, $i \neq i'$ and $i \neq i''$.

**Appendix B. Proofs of Propositions 3.4 (ii), 3.5 and 3.6**

Let $\alpha_i$ be a number such that $\alpha_i = -l_i$ or $\alpha_i = l_i + 1$ for each $i \in \{0, 1, 2, 3\}$. Set $a = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ and

$$v_r^a = (\varphi(x) - e_1)^{\alpha_1/2}(\varphi(x) - e_3)^{\alpha_3/2}(\varphi(x) - e_2)^{\alpha_2/2},$$

$$a_{r+1,r}^a = -4(\gamma_1^a(r + \gamma_2^a)(r + \gamma_2^a),$$

$$a_{r-1,r}^a = -4r(\varphi(x) - e_3)^3(r + \alpha_2 - 1/2)(e_2 - e_3)(e_2 - e_1),$$

$$a_{r,r}^a = -4r((e_2 - e_3)(r + \alpha_2 + \alpha_1) + (e_2 - e_1)(r + \alpha_2 + \alpha_3)) - 4e_2\gamma_1^a\gamma_2^a + a_{r,r}^a + e_2(\gamma_1^a + \gamma_2^a)^2 + e_2(\alpha_1 + \alpha_3)^2 + e_3(\alpha_1 + \alpha_2)^2,$$

$$\gamma_1^a = (\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3)/2, \quad \gamma_2^a = (-\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + 1)/2.$$

Then the action of the operator $H_{(l_0,l_1,l_2,l_3)}$ is written as

$$H_{(l_0,l_1,l_2,l_3)}v_r^a = a_{r+1,r}^a v_{r+1}^a + a_{r,r}^a v_r^a + a_{r-1,r}^a v_{r-1}^a,$$

(see [7]). Set $d = -\sum_{i=0}^3 \alpha_i/2$ and assume $d \in \mathbb{Z}_{\geq 0}$. Then the space $V_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$ is spanned by $v_0^a, v_1^a, \ldots, v_d^a$, and it follows from Eq. (B.1) that $H_{(l_0,l_1,l_2,l_3)}$ preserves the space $V_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$. The operator $H_{(\alpha_0+d,\alpha_1+d,\alpha_2+d,\alpha_3+d)}$ also preserves the $d + 1$-dimensional space $V_{-\alpha_0-d,\alpha_0-d,\alpha_2-d,\alpha_3-d}$.

**Proposition B.1. (Proposition 3.4 (ii))** Let $d = -\sum_{i=0}^3 \alpha_i/2$ and assume $d \in \mathbb{Z}_{\geq 0}$. Then the characteristic polynomial of the operator $H_{(l_0,l_1,l_2,l_3)}$ on the space $V_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$ coincides with that of the operator $H_{(\alpha_0+d,\alpha_1+d,\alpha_2+d,\alpha_3+d)}$ on the space $V_{-\alpha_0-d,\alpha_0-d,\alpha_2-d,\alpha_3-d}$.

**Proof.** Set $-a - d = (-\alpha_0 - d, -\alpha_1 - d, -\alpha_2 - d, -\alpha_3 - d)$. The action of the operator $H_{(\alpha_0+d,\alpha_1+d,\alpha_2+d,\alpha_3+d)}$ on the space $V_{-\alpha_0-d,\alpha_0-d,\alpha_2-d,\alpha_3-d}$ is written as

$$H_{(\alpha_0+d,\alpha_1+d,\alpha_2+d,\alpha_3+d)}v_r^a = a_{r+1,r}^a v_{r+1}^a + a_{r,r}^a v_r^a - a_{r-1,r}^a v_{r-1}^a$$

Set $u_r^a = ((-1)^r d!/(r! (d-r)!))v_r^a$. By a direct calculation, Eq. (B.1) is rewritten as

$$H_{(l_0,l_1,l_2,l_3)}u_r^a = a_{r+1,r}^a u_{r+1}^a + a_{r,r}^a u_r^a + a_{r-1,r}^a u_{r-1}^a.$$
Proposition B.2. (Proposition 3.2) Set \( d = -\sum_{i=0}^{3} \alpha_i/2 \) and assume \( d \in \mathbb{Z}_{\geq 0} \) and \( \alpha_i \neq \alpha_j \) for some \( i, j \in \{0, 1, 2, 3\} \). Then zeros of the characteristic polynomial of the operator \( H^{(b_0, l_1, l_2, l_3)} \) on the space \( V_{\alpha_0, \alpha_1, \alpha_2, \alpha_3} \) are distinct for generic periods \((2\omega_1, 2\omega_3)\).

Proof. The assumption \( \alpha_i \neq \alpha_j \) for some \( i, j \in \{0, 1, 2, 3\} \) is rewritten that \( \alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 \), and it is easy to show that there exists \( i_0, i_1, i_2 \in \{0, 1, 2, 3\} \) such that \( i_1 \neq i_2 \), \( \alpha_{i_0} \neq \alpha_{i_1} \) and \( \alpha_{i_0} \neq \alpha_{i_2} \). By permutation of periods \( \omega_1, \omega_2, \omega_3 \) and shifts \( x \rightarrow x + \omega_i \) (\( i = 1, 2, 3 \)), we can permute numbers \( \alpha_0, \alpha_1, \alpha_2, \alpha_3 \). Hence it is sufficient to show the proposition under the assumption \( \alpha_3 \neq \alpha_1 \) and \( \alpha_3 \neq \alpha_2 \).

Set \( \omega_1 = 1/2, \omega_3 = \tau/2 \) and \( p = \exp(\pi \sqrt{-1} \tau) \). Then \( e_1, e_2 \) and \( e_3 \) are expressed as power series in \( p \) and we have \( \epsilon_1 = \pi^2(2/3 + O(p^2)), \epsilon_2 = \pi^2(-1/3 + 8p + O(p^2)) \) and \( \epsilon_3 = \pi^2(-1/3 - 8p + O(p^2)) \). The matrix elements are expressed as

\[
\begin{align*}
\alpha_{r+1,r}^a &= \tilde{a}_{r+1,r}^{(0)} + \tilde{a}_{r+1,r}^{(1)}, \\
\alpha_{r-1,r}^a &= \tilde{a}_{r-1,r}^{(0)}(p + O(p^2)), \\
\tilde{a}_{r-1,r}^{(0)} &= -4(r + \gamma_1^a(r + \gamma_2^a)), \\
\tilde{a}_{r-1,r}^{(1)} &= 64r(r + \alpha_2 - 1/2), \\
\tilde{a}_{r,r}^{(0)} &= (2r + \alpha_2 + \alpha_3)^2 - \sum_{i=0}^{3} l_i(l_i + 1)/3, \\
\tilde{a}_{r,r}^{(1)} &= -8\{r(12r + 8\alpha_1 + 12\alpha_2 + 4\alpha_3) + 4\gamma_1^a \gamma_2^a - (\alpha_1 + \alpha_3)^2 + (\alpha_1 + \alpha_2)^2\}. 
\end{align*}
\]

If \( p = 0 \), then the operator \( H^{(b_0, l_1, l_2, l_3)} \) acts triagonally and eigenvalues are written as \( \tilde{a}_{r,r}^{(0)} \) (\( r = 0, \ldots, d \)). Since the eigenvalues are quadratic in \( r \), multiplicity of the eigenvalues is one or two. Hence it is sufficient to show that the eigenvalue with multiplicity two on the case \( p = 0 \) separates when \( p \) varies. Assume that \( \tilde{a}_{r,r}^{(0)} = \tilde{a}_{r',r'}^{(0)} \) is the eigenvalue with multiplicity two, \( 0 \leq r < r' \leq d \) and \( r, r' \in \mathbb{Z} \). Then we have \( r + r' = -(\alpha_2 + \alpha_3) \).

If \( r + 1 < r' \), then the eigenvalue around \( \tilde{a}_{r,r}^{(0)} \) is expanded as \( E = \tilde{a}_{r,r}^{(0)} + c_1p + \ldots \), and \( c_1 \) satisfies

\[\text{(B.2)}\]

\[
\left\{ (\tilde{a}_{r-1,r}^{(0)}, r-1 - \tilde{a}_{r,r}^{(0)}(\tilde{a}_{r,r}^{(0)} - c_1)(\tilde{a}_{r+1,r+1}^{(0)} - \tilde{a}_{r,r}^{(0)}) - (\tilde{a}_{r-1,r-1}^{(0)} - \tilde{a}_{r,r}^{(0)})(\tilde{a}_{r+1,r+1}^{(0)} - \tilde{a}_{r,r}^{(0)})^2 \right. \\
- \left. (\tilde{a}_{r-1,r-1}^{(1)} - \tilde{a}_{r,r}^{(1)})(\tilde{a}_{r+1,r+1}^{(0)} - \tilde{a}_{r,r}^{(0)}) \right\} = 0,
\]

which follows from expanding the characteristic polynomial of the matrix \( (a_{i,j}^{a})_{i,j=0,\ldots,d} \) in variable \( p \) and observing the coefficient of \( p^2 \). By a direct calculation, the condition that Eq. (B.2) for the variable \( c_1 \) has multiple roots is written as \( (\alpha_3 - \alpha_1)(2r + \alpha_2 + \alpha_3) = 0 \). If \( 2r + \alpha_2 + \alpha_3 = 0 \), then \( r = r' \) and it contradicts that \( r < r' \). By the assumption \( \alpha_3 \neq \alpha_1 \), it follows that Eq. (B.2) for the variable \( c_1 \) does not have multiple roots, and the solution \( E = \tilde{a}_{r,r}^{(0)} + c_1p + \ldots \) separates.

If \( r + 1 = r' \), then \( r = -(\alpha_2 + \alpha_3 + 1)/2 \), the eigenvalue around \( \tilde{a}_{r,r}^{(0)} \) is expanded as \( E = \tilde{a}_{r,r}^{(0)} + c_{1/2}p + \ldots \), and \( c_{1/2} \) is determined by

\[\text{(B.3)}\]

\[
c_{1/2}^2 = \tilde{a}_{r+1,r}^{(0)} \tilde{a}_{r+1,r+1}^{(1)} = -16(\alpha_0 - \alpha_1)(\alpha_2 - \alpha_3)(\alpha_0 + \alpha_1 - 1)(\alpha_2 + \alpha_3 - 1).
\]

Since \( 0 \leq r + r' = -(\alpha_2 + \alpha_3) \leq 2d = -(\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3) \), we have \( (1 - (\alpha_0 + \alpha_1))(1 - (\alpha_2 + \alpha_3)) > 0 \). Combining with \( \alpha_2 \neq \alpha_3 \), it follows that, if \( \alpha_0 \neq \alpha_1 \), then Eq. (B.3) for the variable \( c_{1/2} \) does not have multiple roots, and the solution \( E = \tilde{a}_{r,r}^{(0)} + c_{1/2}p + \ldots \).
separates. If \( \alpha_0 = \alpha_1 \), then we have \( a_{r+1,r}^n = \tilde{a}_{r+1,r}^{(0)} = 0 \) for \( r = -(\alpha_2 + \alpha_3 + 1)/2 \). The eigenvalue around \( \tilde{a}_{r,r}^{(0)} \) is expanded as \( E = \tilde{a}_{r,r}^{(0)} + c_1 p + \ldots \), and \( c_1 \) is determined by

\[
(B.4) \quad \{(\tilde{a}_{r-1,r-1}^{(0)} - \tilde{a}_{r,r}^{(0)})(\tilde{a}_{r,r}^{(1)} - c_1) - \tilde{a}_{r,r-1}^{(1)} \tilde{a}_{r-1,r}^{(1)} \} \\
\{(\tilde{a}_{r+2,r+2}^{(0)} - \tilde{a}_{r+1,r+1}^{(0)})(\tilde{a}_{r+1,r+1}^{(1)} - c_1) - \tilde{a}_{r+2,r+1}^{(1)} \tilde{a}_{r+1,r+2}^{(1)} \} = 0.
\]

The condition that Eq.\((B.4)\) for the variable \( c_1 \) has multiple roots is written as \( (\alpha_2 - \alpha_3)(\alpha_2 + \alpha_3 - 1)(2\alpha_0 - 1) = 0 \). But it is impossible, because \((1 - (\alpha_3 + \alpha_3))(1 - 2\alpha_0) > 0\) and \( \alpha_3 \neq \alpha_2 \). Hence, if \( \alpha_0 = \alpha_1 \), then Eq.\((B.4)\) for the variable \( c_1 \) does not have multiple roots, and the solution \( E = \tilde{a}_{r,r}^{(0)} + c_1 p + \ldots \) separates.

Thus we obtain that the multiple roots \( E = \tilde{a}_{r,r}^{(0)} \) at \( p = 0 \) separates by expanding the eigenvalue as a series in \( p \) or \( \sqrt{p} \).

Therefore the zeros of the characteristic polynomial equation are distinct for generic periods \((2\omega_1, 2\omega_3)\). \(\square\)

**Corollary B.3.** (Proposition \[6, Proposition 3.9\]) Let \( l_0, l_1, l_2, l_3 \) be non-negative integers and \( V \) be the vector space written as Eq.\((4.4)\) \((l_0 + l_1 + l_2 + l_3: \text{even})\) or Eq.\((4.3)\) \((l_0 + l_1 + l_2 + l_3: \text{odd})\). We denote the monic characteristic polynomial of the operator \( H^{(l_0,l_1,l_2,l_3)} \) on the space \( V \) by \( P(E) \). Then the roots of the equation \( P(E) = 0 \) are distinct for generic periods \((2\omega_1, 2\omega_3)\).

**Proof.** It is shown in the proof of \[6, Theorem 3.2\] (or \[10, Proposition 3.9\]) that any two characteristic polynomials of distinct subspaces in Eq.\((4.3)\) or Eq.\((4.8)\) do not have common roots. Hence it is sufficient to show that the characteristic polynomial of the operator \( H^{(l_0,l_1,l_2,l_3)} \) on any space listed in Eq.\((4.3)\) or Eq.\((4.8)\) does not have multiple zeros for generic periods \((2\omega_1, 2\omega_3)\). In Proposition \[B.2\] it is shown that, if \( \alpha_i \neq \alpha_j \) for some \( i, j \in \{0, 1, 2, 3\} \), then the characteristic polynomial does not have multiple zeros for generic periods. In Eqs.\((4.3)\), \((4.8)\), the case \( \alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 \) for \( V_{\alpha_0,\alpha_1,\alpha_2,\alpha_3} \) appears only for the case \( l_0 = l_1 = l_2 = l_3 \) and \( \alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = -l_0 \). We set \( g = l_0(= l_1 = l_2 = l_3) \).

For the case \( l_0 = l_1 = l_2 = l_3 = g \), the operator \( H^{(g,g,g,g)} \) is expressed as

\[
H^{(g,g,g,g)} = -\frac{d^2}{dx^2} + 4g(g + 1)\phi(2x),
\]

and the finite-dimensional space \( V = V_{-g,-g,-g,-g} \) for the case \( l_0 = l_1 = l_2 = l_3 = g \) coincides with the space \( V \) for the case \( l_0 = g \) and \( l_1 = l_2 = l_3 = 0 \) by replacing basic periods \((2\omega_1, 2\omega_3) \rightarrow (\omega_1, \omega_3)\). For the case \( l_0 \neq 0 \) and \( l_1 = l_2 = l_3 = 0 \), the corresponding proposition is proved in Proposition \[B.2\] or \[14, \S 23.4\]. Thus Corollary \[B.3\] is proved. \(\square\)

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