HOMOLOGICAL STABILITY AND STABLE MODULI OF FLAT MANIFOLD BUNDLES

SAM NARIMAN

Abstract. We prove that group homology of the diffeomorphism group of \( \#^g S^n \times S^n \) as a discrete group is independent of \( g \) in a range, provided that \( n > 2 \). This answers the high dimensional version of a question posed by Morita about surface diffeomorphism groups made discrete. The stable homology is isomorphic to the homology of a certain infinite loop space related to the Haefliger’s classifying space of foliations. One geometric consequence of this description of the stable homology is a splitting theorem that implies certain classes called generalized Mumford-Morita-Miller classes can be detected on flat \((\#^g S^n \times S^n)\)-bundles for \( g \gg 0 \).

1. Introduction

1.1. Statements of the main results. We begin by fixing some notations appearing in this paper. For a manifold \( M \), \( \text{Diff}(M) \) denotes the group of \( C^\infty \)-diffeomorphisms that are the identity near \( \partial M \) equipped with \( C^\infty \)-topology. The same underlying group equipped with \textit{discrete} topology is denoted by \( \text{Diff}_d(M) \). Let \( \Sigma_{g,1} \) denote an orientable surface of genus \( g \) with one boundary component. In
view of the fact that all known cohomology classes in $H^*(\text{BDiff}^\delta(\Sigma_g,1); \mathbb{Z})$ are stable with respect to the genus, Morita [Mor06] conjectured that the group homology of surface diffeomorphism groups should stabilize in a range. J. Bowden [Bow12] answered Morita’s problem affirmatively in low homological degrees, namely he showed $H_k(\text{BDiff}^\delta(\Sigma_g,1); \mathbb{Z})$ is independent of $g$, provided $k \leq 3$ and $g \geq 8$. The purpose of this paper is to study a high dimensional version of Morita’s problem.

We set up Quillen’s stability machine to prove homological stability for diffeomorphism groups of $W_{g,1} := \#^g S^n \times S^n - \text{int}(D^{2n})$ made discrete for $n \geq 3$. We study the surface diffeomorphism groups in a separate paper.

In order to precisely state main results, let us fix more notations. For a manifold $W$ with a boundary and $Z$ a subspace of $W$ containing the boundary, let $\text{Diff}^\delta_1(W, \partial Z)$ denote compactly supported diffeomorphisms of $W - Z$. In particular, we write $\text{Diff}^\delta_1(W, \partial)$ to denote compactly supported diffeomorphisms of the interior of $W$. In fact, if we choose once and for all a collar $[0,1) \times \partial W \to W$ and let the discrete group $\text{Diff}^\delta_{c,c}(W, \partial)$ be those diffeomorphisms of $W$ that fix pointwise the $c$-collar, then we have

$$\text{Diff}^\delta_1(W, \partial) = \text{colim}_{c \to 0} \text{Diff}^\delta_{c,c}(W, \partial)$$

Let $j : W_{g,1} \to W_{g,1+1}$ be an embedding such that the complement of $j(W_{g,1})$ in $W_{g,1+1}$ is diffeomorphic to $W_{1,2} \cong S^n \times S^n - \text{int}(D^{2n} \cup D^{2n})$. Having fixed such an embedding, we define a homomorphism $s : \text{Diff}^\delta_1(W_{g,1}, \partial) \to \text{Diff}^\delta_1(W_{g,1+1}, \partial)$ by extending diffeomorphisms via identity on the complement of $j(W_{g,1})$. Although the homomorphism $s$ depends on $j$, any two choices of embedding lead to conjugate homomorphisms; therefore, we obtain a well-defined map up to homotopy between classifying spaces as $s : \text{BDiff}^\delta_1(W_{g,1}, \partial) \to \text{BDiff}^\delta_1(W_{g,1+1}, \partial)$.

Our first main theorem is the following:

**Theorem 1.1.** For $n > 2$ the stabilization map

$$H_k(\text{BDiff}^\delta_1(W_{g,1}, \partial); \mathbb{Z}) \to H_k(\text{BDiff}^\delta_1(W_{g,1+1}, \partial); \mathbb{Z})$$

is an isomorphism as long as $k \leq (g - 4)/2$.

If we denote $C^1$-diffeomorphisms of $W_{g,1}$ by $\text{Diff}^1_1(W_{g,1}, \partial)$, one consequence of Tsuboi’s remarkable theorem [Tsu89] is that $\text{BDiff}^\delta_1(W_{g,1}, \partial)$ is homology equivalent to $\text{BDiff}^1_1(W_{g,1}, \partial) \cong \text{BDiff}^\delta_1(W_{g,1}, \partial)$. Hence, the homological stability for $C^1$-diffeomorphisms with discrete topology, $\text{Diff}^\delta_{1,1}(W_{g,1}, \partial)$ is already implied by the homological stability of $\text{BDiff}(W_{g,1}, \partial)$ [GRW12a].

As always, when proving homological stability, the bulk of the work is the construction of highly connected simplicial complexes on which the groups act with “nice” stabilizer subgroups. In this case, we use a simplicial complex arising from certain germs of embeddings into $W_{g,1}$; this simplicial complex is a slight modification of the simplicial space introduced by Galatius and Randal-Williams in [GRW12a].

Theorem 1.2 below describes the stable homology of these diffeomorphism groups with discrete topology as the homology of an infinite loop space, which we now describe. Let $\Gamma_{2n}$ be the Haefliger category, i.e. the topological category whose objects are points in $\mathbb{R}^{2n}$ with its usual topology and morphisms between two points, say $x$ and $y$, are germs of diffeomorphisms that send $x$ to $y$. The classifying space of this category plays an important role in classifying foliations up to concordance (for details see [Mil], [Law77], [Hae71]). By $\text{ST}_n$, we mean the subcategory of $\Gamma_{2n}$ with the same objects, but the morphisms are orientation preserving diffeomorphisms. The classifying space of the Haefliger category classifies Haefliger structures up to concordance and the normal bundle to the Haefliger structure induces the following
and Theorem 1.2

1.1

Let $GL_{2n}(\mathbb{R})^+$ be the group of real matrices with positive determinants. Let $BGL_{2n}(\mathbb{R})^+(n)$ be the $n$-connected cover of $BGL_{2n}(\mathbb{R})^+$. Given that $\nu$ is a $(2n+2)$-connected map [Hae71], we have the following pullback diagram:

$$
\begin{array}{ccc}
B^*\Gamma_{2n} & \xrightarrow{\theta} & B^{*}\Gamma_{2n} \\
\downarrow{\nu(n)} & & \downarrow{\nu} \\
BGL_{2n}(\mathbb{R})^+(n) & \xrightarrow{\theta} & BGL_{2n}(\mathbb{R})^+.
\end{array}
$$

Take the inverse of the tautological bundle, $-\gamma$, on $BGL_{2n}(\mathbb{R})^+$ and pull it back to $B^*\Gamma_{2n}(n)$ via $\theta \circ \nu$. We denote the Thom spectrum of this virtual bundle by $B^*\Gamma_{2n}(n)\nu^*(-\gamma)$. (For a definition of the Thom spectrum of a virtual bundle see e.g. [Swi75]). For our convenience, we write $\gamma \circ \nu$ for the associated infinite loop space and $\Omega^\infty(X^{-\gamma})$ for the base point component. In section 5, we show that there exists a map

$$
\alpha : BDiff^*_c(W_{g,1}, \partial) \rightarrow \Omega^\infty_0 X^{-\gamma}
$$

such that:

**Theorem 1.2.** The map $\alpha$ induces an isomorphism in integral homology

$$
H_k(BDiff^*_c(W_{g,1}, \partial); \mathbb{Z}) \xrightarrow{\cong} H_k(\Omega^\infty_0 X^{-\gamma}; \mathbb{Z}),
$$

as long as $k \leq (g-4)/2$.

1.2. Applications. As applications of Theorem 1.1 and Theorem 1.2, two results are presented about the map induced by $BDiff^*_c(W_{g,1}, \partial) \rightarrow BDiff(W_{g,1}, \partial)$ on the cohomology in the stable range. Let $MT\theta$ be the Thom spectrum of the virtual bundle $\theta^*(-\gamma)$ over $BGL_{2n}(\mathbb{R})^+(n)$ and $\Omega^\infty_0 MT\theta$ be the base point component of the infinite loop space associated to this spectrum. Galatius and Randal-Williams showed in [GRW12a] that the stable rational cohomology of topologized diffeomorphisms of $W_{g,1}$ is isomorphic to the rational cohomology of $\Omega^\infty_0 MT\theta$ and the latter can be easily described; for each class $c \in H^k(BGL_{2n}(\mathbb{R})^+(n))$, there are corresponding “generalized Mumford-Morita-Miller” classes $\kappa_c \in H^{k+2n}(\Omega^\infty_0 MT\theta)$ which we recall the definition of these classes in section 6, and $H^{k+2n}(\Omega^\infty_0 MT\theta; \mathbb{Q})$ is the free graded-commutative algebra on the classes $\kappa_c$, where $c$ runs through monomials generated by the classes $c, p_{n-1}, \ldots, p_{[n+1/4]}$ with a degree larger than $2n$.

Unlike the description of the stable cohomology of topologized diffeomorphisms, it is not easy to compute $H^*(\Omega^\infty_0 X^{-\gamma}; \mathbb{Q})$. In order to construct nontrivial classes in the stable cohomology of $Diff^*_c(W_{g,1}, \partial)$, one might attempt to pull back generalized Miller-Morita-Mumford classes from the cohomology of $BDiff(W_{g,1}, \partial)$. On the one hand, because of Bott’s vanishing theorem (see section 6), the pull-back of MMM-classes with cohomological degrees larger than $4n$ to $H^*(BDiff^*_c(W_{g,1}, \partial); \mathbb{Q})$ are zero under the map

$$
BDiff^*_c(W_{g,1}, \partial) \rightarrow BDiff(W_{g,1}, \partial)
$$
that is induced by identity on the level of groups. On the other hand, the situation is surprisingly different with finite coefficients.

**Theorem 1.3.** For any prime \( p \), the natural map
\[
H^*(\Omega^\infty_0 BGL_{2n}(n)^-; \mathbb{F}_p) \hookrightarrow H^*(\Omega^\infty_0 X^-; \mathbb{F}_p)
\]
is split injective.

**Corollary 1.4.** For any odd prime \( p \), the natural map
\[
H^*(B\text{Diff}(W_{g,1},\partial); \mathbb{F}_p) \hookrightarrow H^*(B\text{Diff}^c(W_{g,1},\partial); \mathbb{F}_p)
\]
is injective, provided that \( * \leq (g - 4)/2 \).

One geometric consequence of Theorem 1.3 is that all nontrivial MMM-classes in the stable cohomology of \( B\text{Diff}(W_{g,1},\partial) \) are detected on flat \( W_{g,1} \)-bundles, meaning that for any nontrivial MMM-class \( \kappa_c \in H^*(B\text{Diff}(W_{g,1},\partial); \mathbb{Z}) \) that lives in the stable range, there exists a flat \( W_{g,1} \)-bundle whose \( \kappa_c \) class is nonzero. The group homology of \( \text{Diff}^c(W_{g,1},\partial) \) with integer or rational coefficients is believed to be gigantic and although there is not much known about the cohomology of \( \Omega^\infty_0 X^- \), Theorem 1.2 implies that there are nontrivial cohomology classes arising from secondary characteristic classes of foliations known as Godbilon-Vey classes that vary continuously. More precisely, in section 6 we show:

**Corollary 1.5.** There is a surjection
\[
H_{2n+1}(B\text{Diff}^c(W_{g,1},\partial); \mathbb{Q}) \twoheadrightarrow \mathbb{R}
\]
as long as \( 4n + 6 \leq g \).

This paper is organized as follows: in section 2, we discuss various models of the stabilization map \( s \) from Theorem 1.1 and prove that although not homotopic, they do induce the same map in homology. In section 3, we construct a highly connected semisimplicial set on which \( \text{Diff}^c(W_{g,1},\partial) \) acts and we determine the set of orbits of this action. In section 4, we use the “relative” spectral sequence argument in the sense of [Cha87] to establish homological stability. In section 5, we apply Thurston’s theorem about classifying foliations to prove Theorem 1.2 and we use a transfer argument to prove our splitting Theorem 1.3. In section 6, we discuss various applications of Theorem 1.1, Theorem 1.2, and Theorem 1.3 to obtain partial results about characteristic classes of flat \( W_{g,1} \)-bundles.

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2. **Stabilization maps induce same map on homology**

For reasons that will become clear in section 3 and section 4, it is convenient to work with a stabilization map that is different from the one defined in the introduction. In this section, we describe this convenient nonstandard stabilization map and prove it induces the same map on homology as the standard stabilization.
map. We borrow some notations from [GRW12a], which has inspired much of our work. Throughout the paper $n$ is fixed and is larger than 2. We write:

$$W_{g,k} = \#^g S^n \times S^n - \frac{k}{k} D^{2n}$$

which is a manifold with a boundary obtained from the $g$-fold connected sum $\#^g S^n \times S^n$ by cutting out the interior of $k$ disjoint disks. It should be noted that this manifold with a boundary is well-defined up to diffeomorphism i.e. it does not depend on choices of where to perform connected sum and which disks to cut out.

We make our choices once and for all and let the notation $W_{g,k}$ denote the actual abstract manifold instead of a diffeomorphism class.

**The standard stabilization map.** Because of the choices we have made, the boundaries of $W_{g,k}$ are parametrized spheres. If we glue $W_{1,2}$ to $W_{g,1}$ respecting the parametrization of the boundaries, or rather parametrization of a collar of boundaries, we obtain a manifold that is diffeomorphic to $W_{g+1,1}$. A choice of diffeomorphism to $W_{g+1,1}$ provides us with an embedding $j : W_{g,1} \mapsto W_{g+1,1}$. Extending diffeomorphisms via the identity on the complement of $j(W_{g,1})$ induces a map from $\text{Diff}_c^d(W_{g,1})$ to $\text{Diff}_c^d(W_{g+1,1})$, which is an injection. It is important to note that this injection is unique up to conjugation. We call this injection the standard stabilization map and denote it by $\iota(g)$, which induces a well-defined map up to homotopy from $\text{BDiff}_c^d(W_{g,1})$ to $\text{BDiff}_c^d(W_{g+1,1})$.

**Non-standard stabilization maps.** Another model for a stabilization map that is not conjugate to $\iota(g)$ but, as we shall prove, does induce the same map on homology is described as follows. Instead of gluing $W_{1,2}$ to the boundary of $W_{g,1}$, we glue $W_{1,1}$ to the part of the boundary of $W_{g,1}$. More precisely, let $H$ be the manifold obtained from $W_{1,1}$ by gluing $[0, 1] \times D^{2n-1}$ onto $\partial W_{1,1}$ along an oriented embedding

$$\{1\} \times D^{2n-1} \to \partial W_{1,1}$$

which we also choose once and for all. The point of working with $H$ is, although it is diffeomorphic to $W_{1,1}$, it has a standard embedded $[0, 1] \times D^{2n-1} \subset H$. We choose an embedding $\epsilon : [0, .5] \times D^{2n-1} \mapsto \partial W_{g,1} \times [0, \epsilon) \subset W_{g,1}$ where $\epsilon : \{0\} \times D^{2n-1} \mapsto \partial W_{g,1}$. Then we attach $H$ to $W_{g,1}$ via identifying $[0, .5] \times D^{2n-1} \subset H$ to $\epsilon([0, .5] \times D^{2n-1})$,
therefore, we obtain $W_{g,1} \cup_{(0,5) \times D^{2n-1}} H$, whose interior is diffeomorphic to the interior of $W_{g+1,1}$. If we pick such a diffeomorphism $f$, it gives an isomorphism of following two groups $\text{Diff}^\delta_c(W_{g,1} \cup H, \partial) \xrightarrow{\sim} \text{Diff}^\delta_c(W_{g+1,1}, \partial)$. Also note that extending diffeomorphisms of $W_{g,1}$ by $\text{id}_H$ gives an injection $\text{Diff}^\delta_c(W_{g,1}, \partial) \to \text{Diff}^\delta_c(W_{g,1} \cup H, \partial)$; the composition of these two maps is a nonstandard stabilization associated to $f$ and we denote it by $i'(g)$.

In section 3 and section 4, we prove homological stability for the nonstandard stabilization map for some choice of $f$. Therefore, we need to show that we can choose $f$ in such a way that $i(g)$ and $i'(g)$ induce the same map on homology.

**Lemma 2.1 (Pushing Collar).** Let $W$ be a manifold with a boundary. The complement of the closure of the $\epsilon$-collar of the boundary is diffeomorphic to the interior of the manifold; if we choose a diffeomorphism, we obtain a map, $s$, from $\text{Diff}^\delta_c(W, \partial)$ to $\text{Diff}^\delta_c(W, \text{rel } \epsilon\text{-collar})$. The following composition

$$\text{Diff}^\delta_c(W, \partial) \xrightarrow{s} \text{Diff}^\delta_c(W, \text{rel } \epsilon\text{-collar}) \xrightarrow{i} \text{Diff}^\delta_c(W, \partial)$$

acts as the identity in $H_\ast(\text{Diff}^\delta_c(W, \partial); \mathbb{Z})$.

**Proof.** In order to show that the composition is the identity on homology, we invoke a lemma from [McD80, Lemma 3.7]. In this lemma, McDuff showed if $K$ is a discrete group and $c$ is an endomorphism of $K$ such that the restriction of $c$ to any finite subset of $K$ is equal to a conjugation by some element of the group that may depend on the finite subset that $c$ is restricted to, then $c$ acts as identity on group homology. Hence, to prove $i \circ s$ acts as an identity on homology, it suffices to prove that for any finite set of elements $\{x_1, x_2, \ldots, x_n\}$ in $\text{Diff}^\delta_c(W, \partial)$, there is a group element $h$ such that for all $1 \leq i \leq n$, we have $h(i \circ s(x_i))h^{-1} = x_i$. We choose a positive $\delta \leq \epsilon$ such that the $\delta$-collar is fixed by all $x_i$'s; such $\delta$ exists because every element in $\text{Diff}^\delta_c(W, \partial)$ is fixing some neighborhood of the boundary by definition. Choose a diffeomorphism $h \in \text{Diff}^\delta_c(W, \partial)$ that maps the $\epsilon$-collar diffeomorphically to the $\delta$-collar and maps the $\delta$-collar into itself. It is easy to check that such a diffeomorphism as $h$ exists and satisfies $h(i \circ s(x_i))h^{-1} = x_i$ for $i = 1, \ldots, n$. \(\square\)

**Theorem 2.2.** There exists a diffeomorphism $f$ from $\text{int}(W_{g,1} \cup_{(0,5) \times D^{2n-1}} H)$ to $\text{int}(W_{g+1,1})$ such that the nonstandard stabilization associated to $f$ induces the same map on homology as the standard stabilization map, i.e. the following two injections

$$\text{Diff}^\delta_c(W_{g,1}, \partial) \xrightarrow{i(g)} \text{Diff}^\delta_c(W_{g+1,1}, \partial)$$

induce the same map on group homology.

**Proof.** Let $i'$ be the map induced by gluing $H$

$$\text{Diff}^\delta_c(W_{g,1}, \partial) \xleftarrow{i'} \text{Diff}^\delta_c(W_{g,1} \cup_{(0,5) \times D^{2n-1}} H, \partial)$$

By Lemma 2.1, we deduce that $i'$ induces the same map on group homology as the following composition

$$\text{Diff}^\delta_c(W_{g,1}, \partial) \xleftarrow{i \circ s} \text{Diff}^\delta_c(W_{g,1}, \partial) \xrightarrow{i'} \text{Diff}^\delta_c(W_{g,1} \cup_{(0,5) \times D^{2n-1}} H, \partial)$$

therefore, this injection of $\text{Diff}^\delta_c(W_{g,1}, \partial)$ into $\text{Diff}^\delta_c(W_{g,1} \cup_{(0,5) \times D^{2n-1}} H, \partial)$ is the identity on the $\epsilon$-collar of the boundary of $W_{g,1}$ and induces the same map on
homology as $\iota_*$. Choose a diffeomorphism $f$ that maps diffeomorphically the complement of the collar in $W_{g,1}$ to $W_{g,1}$ itself and the remaining to the interior of $W_{1,2}$.

In this section, we describe a simplicial resolution $\xi$ in $W_g,1$ to $W_{g,1}$, which we now recall its definition. Let $\Delta$ denote the category whose objects are non-empty totally ordered finite sets and whose morphisms are monotone maps. Let $\Delta_{0,1}$ be the full subcategory of $\Delta$ with the same objects but only the injective maps as morphisms. A semi-simplicial space is a contravariant functor from $\Delta_{0,1}$ to the category of topological spaces. More concretely, we denote a semi-simplicial space by $X_* = \{X_n|n = 0, 1, \ldots\}$, which is a collection of spaces for each $n \geq 0$ and face maps $d_i : X_n \to X_{n-1}$ defined for $i = 0, 1, \ldots, n$ satisfying $d_id_j = d_{j-1}d_i$ for $i < j$.

The geometric realization of a semi-simplicial space $X_*$ is

$$|X_*| = \coprod_{n \geq 0} X_n \times \Delta^n / \sim$$

where the equivalence relation is $(d_i(x), y) \sim (x, d^i(y))$, for $d^i : \Delta^n \to \Delta^{n+1}$ the inclusion of the $i$-th face where $i = 0, \ldots, n$. An augmented semi-simplicial space $X_* \to X_{-1}$ is a semi-simplicial space $X_*$ with a map $\epsilon : X_0 \to X_{-1}$ called augmentation, which equalizes the face maps $d_0 : X_0 \to X_{-1}$ and $d_1 : X_0 \to X_{-1}$. The augmentation induces a map $|X_*| \to X_{-1}$, from the geometric realization of $X_*$ to the $(-1)$-th space. If this map is $n$-connected, we call $X_*$ a $n$-resolution for $X_{-1}$.

In this section, we describe a simplicial resolution $X_* \to \text{BDiff}^\delta(W_{g,1}, \partial)$ such that $|X_*| \to \text{BDiff}^\delta(W_{g,1}, \partial)$ is $[(g-3)/2]$-connected.

Recall $H$ is the manifold obtained from $W_{1,1}$ by gluing $[0,1] \times D^{2n-1}$ onto $\partial W_{1,1}$. Assume that $(S^n, \ast)$ is a sphere with a chosen base point, then $S^n \vee S^n = (S^n \times \{\ast\} \cup (\ast) \times S^n) \subset S^n \times S^n$. We may choose it so that it is contained in $\text{int}(W_{1,1})$.

**Definition 3.1.** Let $\gamma$ be a path in $\text{int}(H)$ from $(0,0) \in [0,1] \times D^{2n-1}$ to some chosen point on $S^n \vee S^n$ that is not $(\ast, \ast) \in S^n \times S^n$ such that the interior of $\gamma$ in $H$ does not intersect $S^n \vee S^n$ and the image of $\gamma$ agrees with $[0,1] \times \{0\}$ inside $[0,1] \times D^{2n-1}$. We define the core $C \subset H$ to be:

$$C = (S^n \vee S^n) \cup \gamma([0,1]) \subset H$$

The core in $H$ is depicted in fig. 2.

We make our choice of $\gamma$ once and for all to choose a fixed core $C$ in $H$. Note that $H$ is homotopy equivalent to its core. By embedding $H$ into a manifold $W$, we mean $\{0\} \times D^{2n-1}$ is sent to $\partial W$ and the rest of $H$ is sent to the interior of $W$. A germ near $C \subset H$ of an embedding $H \hookrightarrow W$ is an equivalence class of the following

$$\begin{align*}
\begin{array}{ccc}
\text{int}(H) \cup \text{int}(\text{collar}) & \to & \text{int}(W_{1,2}) \\
\int & & \\
\end{array}
\end{align*}$$

the first part that sends $W_{g,1} \setminus \text{collar}$ to $W_{g,1}$ is the inverse of the embedding that is used to define $s$. For such an $f$, it is easy to see $\iota^f(g) \circ i \circ s = \iota(g)$, hence by Lemma 2.1, we deduce $\iota^f(g)_* = \iota(g)_*$.

3. A simplicial resolution of $\text{BDiff}^\delta(W_{g,1})$

In this section, we construct a semi-simplicial resolution for $\text{BDiff}^\delta(W_{g,1}, \partial)$, which we now recall its definition
data, $[\phi] := (U, \phi)$, such that $U$ is a neighborhood of the core $C \subset H$ and $\phi : U \to W$ is an embedding of $U$ into $W$. We say $(U, \phi)$ is equivalent to $(U', \phi')$ if and only if an open neighborhood of the core $U'' \subset U \cap U'$ exists, such that $\phi|_{U''} = \phi'|_{U''}$.

To define the semisimplicial resolution for a manifold $W$ with a boundary, we only consider those “collared” embeddings of the core into $W$ that behave in a certain way near the boundary. To define a collared embedding of the core, we need to fix the data of an embedding $c : \mathbb{R} = \mathbb{R} \times \mathbb{R}^{2n-1} \to \partial W$ where $c^{-1}(\partial W) = \partial \mathbb{R}$, even if we don’t write $c$ along with $W$, we assume that we have chosen the chart $c$ once and for all to define the resolution for $\text{BDiff}_c^r(Wx, \partial)$.

Let $(W, c)$ be a pair where $W$ is a manifold with a boundary, $c$ is the fixed chart on the boundary and let $B_x(0)$ denote the ball of radius $r$ around the origin in $\mathbb{R}^{2n-1}$. For any neighborhood $U$ of $C \subset H$, a small enough $r$ exists such that $[0,1] \times B_x(0) \subset U$, where $[0,1] \times B_x(0)$ is contained in the standard embedded $[0,1] \times D^{2n-1} \subset H$.

**Definition 3.2.** A data $(t, [\phi])$, where $t \in \mathbb{R}$ and $[\phi]$ is a germ of an embedding of the core into $W$, is called collared if for some $\epsilon, \eta > 0$ and $(U, \phi)$ a representative of the germ $[\phi]$, the restriction of $\phi$ to $[0,1] \times B_\eta(0) \subset U$ satisfies

$$\phi(x, p) = c(x, p + t\epsilon_1)$$

for all $p \in B_\eta(0)$ and $x < \epsilon$. Here, $\epsilon_1$ is the first standard basis vector in $\mathbb{R}^{2n-1}$.

For a pair $(W, c)$, we define a semisimplicial space $E_\bullet(W)$ as follows:

- $E_0(W)$ consists of collared tuples $(t_0, [\phi_0]), (t_1, [\phi_1]), \ldots, (t_k, [\phi_k])$ satisfying that $t_0 < t_1 < \cdots < t_k$ and for all distinct $i, j$ we have $\phi_i(C) \cap \phi_j(C) = \emptyset$.
- Topologize $E_k(W)$ as a discrete set.
- The face map $d_i$ is given by forgetting $(t_i, [\phi_i])$.

Let $E(W)$ be the simplicial complex with vertices $E_0(W)$ and the unordered set $\{(t_0, [\phi_0]), (t_1, [\phi_1]), \ldots, (t_k, [\phi_k])\}$ is a $k$-simplex if, assuming $t_i$’s are ordered as $t_0 < t_1 < \cdots < t_k$, the $k$-tuple $((t_1, [\phi_1]), \ldots, (t_k, [\phi_k]))$ is in $E_k(W)$. Note that for a vertex $(t, [\phi])$, $t$ is determined by $[\phi]$; therefore, there is a well-defined ordering on the vertices of $k$-simplex of $E(W)$. Consequently, there is a natural homeomorphism $|E_\bullet(W)| = |E(W)|$.

**Remark 3.3.** We can actually consider germs of all embedded cores which are “flat” near the boundary instead of those embedded cores that end in a chart $c$ satisfying Definition 3.2. Thus, we obtain a simplicial complex instead of a semisimplicial space. The point of considering only collared embeddings is to have a shorter spectral sequence argument, because it is easier to work with a spectral sequence that is naturally associated to semisimplicial space [Seg68] rather than a spectral sequence associated to a group action on a simplicial complex [Iva93]. Wherefore we impose the collared condition to get a semisimplicial set.
We equip $\pi_n(W_{g,1})$ with a unimodular hermitian quadratic form\(^1\) $\langle \pi_n(W_{g,1}), \lambda, q \rangle$, where $\lambda$ is a hermitian bilinear form on $\pi_n(W_{g,1}) \cong \mathbb{Z}^{2n}$, $\lambda : \pi_n(W_{g,1}) \otimes \pi_n(W_{g,1}) \to \mathbb{Z}$ induced by the intersection form and $q : \pi_n(W_{g,1}) \to \mathbb{Z}$ is the quadratic refinement induced by the self intersection for $\Lambda = 0, 2\mathbb{Z}$ or $\mathbb{Z}$ [Wal62, Lemma 2]. Note that every germ of an embedding of the core of $H$ into $W_{g,1}$ induces a map of quadratic modules from the hyperbolic form $H$ to $(\pi_n(W_{g,1}), \lambda, q)$. The hyperbolic form is the data $(\mathcal{H}, \lambda, q)$ where $\mathcal{H} = \mathbb{Z}^2$ is generated by $e_1$ and $e_2$ satisfying $\lambda(e_1, e_1) = \lambda(e_2, e_2) = 0$, $\lambda(e_1, e_2) = 1$, $\lambda(e_2, e_1) = (-1)^n$ and $q(e_1) = q(e_2) = 0$. Analogous to [GRW12a, (4.1)], this algebraic structure on $\pi_n(W_{g,1})$ induces a map between simplicial complexes
\[ (3.4) \quad \theta : E(W_{g,1}) \to K^n(\pi_n(W_{g,1}), \lambda, q) \]
where $K^n(\pi_n(W_{g,1}), \lambda, q)$ is the algebraic simplicial complex (see Definition A.3) associated to the unimodular hermitian quadratic form $(\pi_n(W_{g,1}), \lambda, q)$. We will use $(3.4)$ to compute the connectivity of $|E_\bullet(W_{g,1})|$. For brevity, we write $E(W_{g,1}) \to K^n(\pi_n(W_{g,1}))$ for this map in the proof of the following lemma.

**Lemma 3.5.** The geometric realization $|E_\bullet(W_{g,1})|$ is $[(g - 5)/2]$-connected.

**Proof.** This is almost the same proof as [GRW12a, Lemma 4.3.]. For convenience, we rephrase it in our context. For every $k < \frac{2g - 5}{2}$, we need to solve the following lifting problem
\[
\begin{array}{ccc}
S^k & \xrightarrow{f} & |E_\bullet(W_{g,1})| \\
\downarrow & & \downarrow \theta \\
D^{k+1} & \xrightarrow{h} & K^n(\pi_n(W_{g,1}))
\end{array}
\]

Since $|E_\bullet(W_{g,1})| = |E(W_{g,1})|$, we have a PL structure on $|E_\bullet(W_{g,1})|$; therefore, we can arrange $f$ and $h$ to be simplicial maps with respect to some choice of PL triangulation for $D^{k+1}$. Using one of Charney’s theorems, it can be shown [GRW12a, Theorem 3.2] that $|K^n(\pi_n(W_{g,1}), \lambda, \alpha)|$ is $[(g - 5)/2]$-connected\(^2\). By Theorem A.4, we know $K^n(\pi_n(W_{g,1}))$ is weakly Cohen-Macaulay of dimension at least $[(g - 3)/2]$, therefore by the generalized coloring lemma\(^3\) as $k + 1 \leq [(g - 3)/2]$, we can assume that $h$ is simplexwise injective on the interior of $D^{k+1}$. We pick a total ordering on the interior vertices and inductively lift each vertex to $E_0(W_{g,1})$. Note that each vertex is given by a morphism of quadratic modules $J : \mathcal{H} \to \pi_n(W_{g,1})$. The element $J(e_1)$ is represented by an embedding $S^n \to W_{g,1}$ [Hae62]. Since $J$ respects the quadratic structure, the self intersection of this embedded sphere must be zero. Thus $J(e_1)$ can be represented by an embedding $r : S^n \to W_{g,1}$ [Hae62]. Similarly, $J(e_2)$ can be represented by an embedding $s : S^n \to W_{g,1}$.

Because $J$ is a quadratic map, $r$ and $s$ must have 1 as the algebraic intersection number. Since $W_{g,1}$ is simply-connected and has a dimension of at least 6, we use the Whitney trick to isotope these embeddings so that their cores $S^n \times \{0\}$ intersect transversally in precisely one point; therefore, we obtain an embedding of the plumbing $S^n \times D^n$ and $D^n \times S^n$ which is diffeomorphic to $W_{1,1} \subset H$. We need to extend this embedding to a neighborhood of $\{0\} \times D^{2n-1} \cup [0, 1] \times \{0\}$. We choose an embedding of $\{0\} \times D^{2n-1}$ into $\partial W_{g,1}$ disjoint from previous embeddings, then we extend this embedding to a neighborhood of an embedded core such that it

---

\(^1\) See the appendix A.2 for definitions.

\(^2\) This connectivity is improved to $[(g - 4)/2]$ in [GRW14].

\(^3\) See appendix A.1 for definition of weakly Cohen Macaulay and the statement of the coloring lemma.
makes the embedding collared. Since J is orthogonal to all previously lifted vertices adjacent to it, we use the Whitney trick to isotope the germ of an embedded core representing J, so that its core is disjoint from all previously chosen vertices that are adjacent to it. By repeating this procedure to interior vertices, we obtain \( h \). □

3.1. Orbits of the action of \( \text{Diff}^s_\ast(W_{g,1}, \partial) \) on \( E_\ast(W_{g,1}) \). As we shall see, the action of \( \text{Diff}^s_\ast(W_{g,1}, \partial) \) on \( E_\ast(W_{g,1}) \) is not transitive, but the set of orbits, which is uncountable, does not depend on \( g \) in a certain sense. Let us describe the set of the orbits. Let \( C_p(\mathbb{R}) \) be the configuration space of \( p \) points in \( \mathbb{R} \) that inherits the natural order on \( \mathbb{R} \). To every \( p \)-simplex \((t_0, [\phi_0]), (t_1, [\phi_1]), \ldots, (t_p, [\phi_p])\) in \( E_p(W_{g,1}) \), we associate \((t_0, t_1, \ldots, t_p) \in C_{p+1}(\mathbb{R}) \) where \( t_0 < t_1 < \cdots < t_p \), since every \( p \)-simplex gives \((p + 1)\) distinct points in \( \mathbb{R} \). The diffeomorphism group \( \text{Diff}^s_\ast(W_{g,1}, \partial) \) fixes the boundary, hence it does not change \((t_0, t_1, \ldots, t_p) \) associated to \((t_0, [\phi_0]), (t_1, [\phi_1]), \ldots, (t_p, [\phi_p])\) and if two \( p \)-simplices give two different elements in \( C_{p+1}(\mathbb{R}) \), they are obviously in different orbits, so we have a well defined map

\[
I : E_\ast(W_{g,1})/\text{Diff}^s_\ast(W_{g,1}, \partial) \to C_{p+1}(\mathbb{R})
\]

We claim that \( I \) is a bijection, i.e. orbits of the action of \( \text{Diff}^s_\ast(W_{g,1}, \partial) \) on \( E_p(W_{g,1}) \) are indexed over \( C_{p+1}(\mathbb{R}) \). For every \( \sigma \in C_{p+1}(\mathbb{R}) \), there exist \( p \)-simplices over \( \sigma \), we choose a fixed \( p \)-simplex \( \phi_\sigma \) over \( \sigma \) in \( E_p(W) \). By virtue of Kreck’s cancellation theorem [Kre99, Theorem D] or [GRW12a, Corollary 4.5], the complement of an open neighborhood of \((p + 1)\) embedded cores given by \( \phi_\sigma \) is diffeomorphic to \( W_{g-p-1,1} \). Furthermore, for every \( \sigma \), we choose a fixed diffeomorphism \( f_\sigma \) that sends the complement of embedded cores to the standard \( W_{g-p-1,1} \). The stabilizer of \( \phi_\sigma \) is a subgroup of \( \text{Diff}^s_\ast(W_{g,1}, \partial) \) and is isomorphic to \( \text{Diff}^s_\ast(W_{g-p-1,1}, \partial) \). The choices of \( \phi_\sigma \) and \( f_\sigma \) determine the stabilizer of \( \phi_\sigma \) as a subgroup of \( \text{Diff}^s_\ast(W_{g,1}, \partial) \), which we denote it by \( \text{Diff}^s_\ast(W_{g-p-1,1}, \partial)_\sigma \) that is decorated by \( \sigma \).

Lemma 3.6. Every two \( p \)-simplices with the same image in \( C_{p+1}(\mathbb{R}) \) are in the same orbit as \( g - p \geq 5 \). Therefore, the orbit decomposition is

\[
E_p(W_{g,1}) = \bigsqcup_{\sigma \in C_{p+1}(\mathbb{R})} \text{Diff}^s_\ast(W_{g,1}, \partial)/\text{Diff}^s_\ast(W_{g-p-1,1}, \partial)_\sigma
\]

Proof. First, we prove the claim for 0-simplices, so assume that \( p = 0 \) and let \((t, [\phi_1]) \) and \((t, [\phi_2]) \) be 0-simplices with the same index. Since \( \phi_1 \) and \( \phi_2 \) are germas of collared embedding of the core, there exist a tubular neighborhood of the core \( C \subset U \) and positive numbers \( \epsilon, \eta \) such that \((0, \epsilon) \times B_\eta(0) \subset U \) and \( \phi_1((0, \epsilon) \times B_\eta(0)) = \phi_2((0, \epsilon) \times B_\eta(0)) \). Let \( W^s_{g,1} \) denote \( W_{g,1} \setminus \text{collar} \). The intersections of \( \phi_1(U)'s \) and \( W^s_{g,1} \) are collared embeddings of the core in \( W^s_{g,1} \). These intersections provide us with embeddings \( \psi_1 : H \to W^s_{g,1} \) satisfying \( \psi_1([0, \epsilon/3) \times D^{2n-1}) = \phi_1([2\epsilon/3, \epsilon) \times B_\eta(0)) \). Using [GRW12a, Corollary 4.4], we can find a diffeomorphism \( l \in \text{Diff}^s_\ast(W^s_{g,1}, \partial) \) that is isotopic to the identity on the boundary \( \partial W^s_{g,1} \times \{2\epsilon/3\} \) such that \( l \circ \psi_1 = \psi_2 \). Note that \( l \) has to fix \( \{2\epsilon/3\} \times B_\eta(0) \). Let \( h \) be an isotopy from \( l \) to the identity.

\[
h : \partial W^s_{g,1} \times [\epsilon/3, 2\epsilon/3] \to \partial W^s_{g,1}
\]

where for all \( x \in \partial W^s_{g,1} \), \( h \) satisfies \( h(x, t) = x \) for \( t \in [\epsilon/3, 4\epsilon/9) \) and \( h(x, t) = l(x) \) for \( t \in (5\epsilon/9, 2\epsilon/3) \). Since \( l \) fixes \( \{2\epsilon/3\} \times B_\eta(0) \), by isotopy extension theorem we can choose \( h \) so that it fixes \([\epsilon/3, 2\epsilon/3] \times B_\eta(0) \). Hence, by gluing \( l \), \( h \) and the identity on \( \epsilon/3 \)-collar together, we obtain a diffeomorphism \( f \in \text{Diff}^s_\ast(W_{g,1}, \partial) \) that sends \( \phi_1(V) \) to \( \phi_2(V) \) for some open neighborhood of \( C \subset V \subset U \).
Now for $p$-simplices in general, assume $((t_{0},[\phi_{0}^{i}]),(t_{1},[\phi_{1}^{i}]),\ldots,(t_{p},[\phi_{p}^{i}]))$ for $i = 1, 2$ are two $p$-simplices with the same index. There exists a diffeomorphism $f_{0} \in \text{Diff}^{\delta}_{c}(W_{g,1}, \partial)$ such that for a neighborhood of the core $C \subset U_0$, $f_0$ satisfies $f_{0} \circ \phi_{1}^{i}(U_{0}) = \phi_{2}^{i}(U_{0})$. We can choose $U_0$ so that $W_{g,1}\backslash \psi_{0}^{i}(U_{0})$’s become manifolds with boundaries (without corners). By the cancellation theorem [GRW12a, Corollary 4.5] the manifold $W_{g,1}\backslash \psi_{0}^{i}(U_{0})$ is diffeomorphic to $W_{g-1,1}$. Note that since $f_{0} \circ \phi_{1}^{i}(C), \phi_{2}^{i}(C) \subset W_{g,1}\backslash \psi_{0}^{i}(U_{0})$, by the same argument as above, a diffeomorphism $f_{1} \in \text{Diff}^{\delta}_{c}(W_{g,1}\backslash \psi_{0}^{i}(U_{0}), \partial)$ exists such that $f_{1} \circ \phi_{1}^{i}(U_{1}) = \phi_{2}^{i}(U_{1})$ for some neighborhood of the core $U_1$. We extend $f_1$ via identity to a diffeomorphism of $W_{g,1}$. By repeating this argument, we obtain a diffeomorphism $f = f_{p} \circ f_{p-1} \circ \cdots \circ f_{0}$ that sends the first $p$-simplex to the other.

\[\]

Fix once for all a coordinate patch near the boundary $c : \mathbb{H} \to W_{g,1}$, which is disjoint from the embedding $e : [0, 5) \times D^{2n-1} \to \partial W_{g,1} \times [0, \epsilon) \subset W_{g,1}$ that we used to define the nonstandard stabilization map.

**Corollary 3.7.** For the pair $(W_{g,1}, c)$, the stabilization map $\iota^{\delta}(g) : E_{*}(W_{g,1}) \to E_{*}(W_{g+1,1})$ is a bijection on orbits of the action of $\text{Diff}^{\delta}_{c}(W_{g,1}, \partial)$ on $E_{*}(W_{g,1})$ and the action of $\text{Diff}^{\delta}_{c}(W_{g+1,1}, \partial)$ on $E_{*}(W_{g+1,1})$.

We now use the high connectivity of $E_{*}(W_{g,1})$ to construct a semisimplicial resolution for $\text{BDiff}^{\delta}_{c}(W_{g,1}, \partial)$, meaning a semisimplicial space $X_{*}$ with an augmentation $X_{*} \to \text{BDiff}^{\delta}_{c}(W_{g,1}, \partial)$ such that the map $|X_{*}| \to \text{BDiff}^{\delta}_{c}(W_{g,1}, \partial)$ is highly connected.

**Construction 3.8.** Recall from Definition 3.2 that for the pair $(W_{g,1}, c)$, we defined $E_{p}(W_{g,1})$ to be the set of $p$-tuples of disjoint collared embedded cores. Let

$$X_{p} = (\text{EDiff}^{\delta}_{c}(W_{g,1}, \partial) \times E_{p}(W_{g,1}))/\text{Diff}^{\delta}_{c}(W_{g,1}, \partial)$$

Note that $X_{*}$ is a semisimplicial space augmented over $\text{BDiff}^{\delta}_{c}(W_{g,1}, \partial)$.

**Proposition 3.9.** $X_{*}$ is a $[(g - 3)/2]$-resolution for $\text{BDiff}^{\delta}_{c}(W_{g,1}, \partial)$, i.e. the map $|X_{*}| \to \text{BDiff}^{\delta}_{c}(W_{g,1}, \partial)$ induced by the augmentation is $[(g - 3)/2]$-connected.

**Proof.** It is useful to know that for an augmented semisimplicial space $\ast : X_{*} \to X_{-1}$, the homotopy fiber of $|X_{*}| \to X_{-1}$ can be computed levelwise [RV09, Lemma 2.1], meaning that the following square is weakly homotopy pullback square

\[
\begin{array}{ccc}
|\text{hofib}(\ast)| & \longrightarrow & |X_{*}| \\
\downarrow & & \downarrow \\
\ast & \longrightarrow & X_{-1}
\end{array}
\]

therefore, from the construction above, we obtain the semi-simplicial space $X_{*}$ augmented over $\text{BDiff}^{\delta}_{c}(W_{g,1}, \partial)$. The levelwise fiber of the augmentation map $X_{*} \to \text{BDiff}^{\delta}_{c}(W_{g,1}, \partial)$ is the semisimplicial space $E_{*}(W_{g,1})$ whose geometric realization $|E_{*}(W_{g,1})|$ by Lemma 3.5 is $[(g - 5)/2]$-connected. Hence, $X_{*}$ is a $[(g - 3)/2]$-resolution for $\text{BDiff}^{\delta}_{c}(W_{g,1}, \partial)$.

**Proposition 3.10.** There are homotopy equivalences for each $k$:

$$X_{k} = \coprod_{c \in C_{p+1}(\mathbb{R})} \text{BDiff}^{\delta}_{c}(W_{g-k-1,1}, \partial)_{\ast}$$
under these identifications all face maps \( d_i : X_k \rightarrow X_{k-1} \) and the augmentation map \( X_0 \rightarrow \text{BDiff}^D_c(W_1, \partial) \) induce the same map on each component on the level of homology as the stabilization map.

**Sketch.** The first part of the proposition is a direct consequence of Lemma 3.6 and the fact that \( \text{EG} \times_G G/H \simeq BH \) where \( H \) is a subgroup of a group \( G \). We do not use that all face maps induce the same map as the stabilization map, so we just hint at the proof. First, we show that all face maps are the same on the level of homology and then, using the same idea as Theorem 2.2, it is easy to see that the induced map on homology should be the same as the stabilization map. Let us focus on a case where \( k = 1 \), the general case is similar.

\[
\begin{array}{c c c}
X_1 & d_1 & X_0 \\
\downarrow{d_0} & & \downarrow{d_0} \\
\end{array}
\]

Suppose \( \sigma = ([\phi_0], [\phi_1]) \in C_2(\mathbb{R}) \) (we omitted \( t_i \)'s since they are determined by the *collared* condition of \( \phi_i \)'s). Then, under the above identification, \( d_1 \) and \( d_0 \) become

\[
\begin{array}{c c c}
\text{BDiff}^D_c(W_{g-2,1}, \partial)([\phi_0],[\phi_1]) & d_0 & \text{BDiff}^D_c(W_{g-1,1}, \partial)([\phi_1]) \\
\text{BDiff}^D_c(W_{g-2,1}, \partial)([\phi_0],[\phi_1]) & d_1 & \text{BDiff}^D_c(W_{g-1,1}, \partial)([\phi_0]) \\
\end{array}
\]

here \( \text{Diff}^D_c(W_{g-1,1}, \partial)([\phi_1]) \) and \( \text{Diff}^D_c(W_{g-1,1}, \partial)([\phi_0]) \) are two different subgroups of \( \text{Diff}^D_c(W_{g,1}, \partial) \), which are abstractly isomorphic to \( \text{Diff}^D_c(W_{g-1,1}, \partial) \). Note that if \( [\phi_0] \) and \( [\phi_1] \) were over the same element in \( C_1(\mathbb{R}) \), then, because of the cancellation theorem, these two subgroups were actually conjugate by an element in \( \text{Diff}^{+}_{c}(W_{g-1,1}, \partial) \). In that case, \( d_0 \) and \( d_1 \) were even homotopic. However, similar to Lemma 2.1, it can be shown that after picking an isomorphism between \( \text{Diff}^D_c(W_{g-1,1}, \partial)([\phi_0]) \) and \( \text{Diff}^D_c(W_{g,1}, \partial) \), the face maps \( d_0 \) and \( d_1 \) induce the same map in homology although they are not homotopic. 

\[\square\]

4. **Proof of Theorem 1.1**

**Proof.** We use the “relative” spectral sequence argument in the sense of [Cha87, Proposition 4.2] to prove homological stability by induction. There is an augmented semisimplicial object in the category of pairs of spaces

\[
\begin{array}{c c c}
(X_*, (W_{g+1,1})), (X_*(W_{g,1})) & \longrightarrow & (\text{BDiff}^D_c(W_{g+1,1}, \partial), \text{BDiff}^D_c(W_{g,1}, \partial)) \\
\end{array}
\]

and this augmented semisimplicial is a \((g-3)/2\)-resolution. There is a spectral sequence for this pair of semisimplicial spaces

\[
E^1_{p,q} = H_q(X_p(W_{g+1,1}), X_p(W_{g,1})) \Rightarrow H_{p+q}([X_p(W_{g+1,1})], [X_p(W_{g,1})])
\]

The fact that \((X_*(W_{g+1,1}), X_*(W_{g,1}))\) is a \((g-3)/2\)-resolution implies the relative spectral sequence converges to

\[
H_{p+q}([X_p(W_{g+1,1})], [X_p(W_{g,1})]) = H_{p+q}(\text{BDiff}^D_c(W_{g+1,1}, \partial), \text{BDiff}^D_c(W_{g,1}, \partial))
\]

as long as \( p + q \leq (g - 3)/2 \). In order to prove Theorem 1.1, we need to show that \( H_k(\text{BDiff}^D_c(W_{g+1,1}, \partial), \text{BDiff}^D_c(W_{g,1}, \partial)) = 0 \) as long as \( k \leq (g - 4)/2 \). By induction and Lemma 3.6, we know for \( p \geq 0 \) and \( q \leq (g - p - 5)/2 \)

\[
E^1_{p,q} = \bigoplus_{\sigma} H_q(\text{BDiff}^D_c(W_{g-p,1}, \partial), \text{BDiff}^D_c(W_{g-p-1,1}, \partial)) = 0
\]
The first page of the spectral sequence in the range that we are interested in looks like

![First page of spectral sequence](image)

Figure 3. First page of spectral sequence

On the first page of the spectral sequence, everything below the thick line that is given by \( p + 2q = g - 5 \) in Figure 4, is zero by induction. If \( g \) is an odd number, then \( E^1_{p,q} = 0 \) for \( p + q \leq \lfloor (g - 4)/2 \rfloor \). If \( g \) is even, then everything except \( E^1_{0,\lfloor (g-4)/2 \rfloor} \) on the dashed line in Figure 4 and below is zero. In order to finish the proof, we need to show that the image of \( E^1_{0,\lfloor (g-4)/2 \rfloor} \) in \( E^\infty_{0,\lfloor (g-4)/2 \rfloor} \) vanishes. Given Lemma 4.1 below, all cycles in

\[
E^1_{0,\lfloor (g-4)/2 \rfloor} = \bigoplus_{\sigma \in C_1(R)} H_{\lfloor (g-4)/2 \rfloor}(BDiff^g_c(W_{g,1}, \partial)_{\sigma}, BDiff^g_c(W_{g-1,1}, \partial)_{\sigma})
\]
die in \( E^\infty \)-page which completes the proof of Theorem 1.1.

□

Lemma 4.1. For every \( \sigma \in C_1(R) \), the following commutative diagram

\[
\begin{array}{ccc}
\text{BDiff}^g_c(W_{g-1,1}, \partial)_{\sigma} & \overset{x}{\longrightarrow} & \text{BDiff}^g_c(W_{g,1}, \partial) \\
\downarrow & & \downarrow \\
\text{BDiff}^g_c(W_{g,1}, \partial) & \overset{y}{\longrightarrow} & \text{BDiff}^g_c(W_{g+1,1}, \partial)
\end{array}
\]

induces a map of pairs from \((\text{BDiff}^g_c(W_{g,1}, \partial)_{\sigma}, \text{BDiff}^g_c(W_{g-1,1}, \partial)_{\sigma})\) to \((\text{BDiff}^g_c(W_{g+1,1}, \partial), \text{BDiff}^g_c(W_{g,1}, \partial))\), which is homologically trivial in degrees less than \( (g-3)/2 \).

Proof. Let \( x \) be a class in \( H_* (\text{Diff}^g_c(W_{g,1}, \partial)_{\sigma}, \text{Diff}^g_c(W_{g-1,1}, \partial)_{\sigma}) \); in order to show that the image of \( x \) in \( H_* (\text{Diff}^g_c(W_{g+1,1}, \partial), \text{Diff}^g_c(W_{g,1}, \partial)) \) is zero, first, we show its image non-canonically comes from a class in \( H_{*-1} (\text{Diff}^g_c(W_{g-1,1}, \partial)_{\sigma}) \). Randal-Williams proved a factorization theorem for spaces [RW09, Prop 6.7] that is reformulated for groups in [Wah10, Lemma 2.5]. It says that if \( G_0, G_1, G_2 \) are subgroups of some group \( G \) fitting into a similar diagram

\[
\begin{array}{ccc}
G_0 & \xrightarrow{t} & G_1 \\
\downarrow & & \downarrow \\
G_2 & \xleftarrow{t} & G
\end{array}
\]

such that \( G_1 < t \cdot G_2 \cdot t^{-1} \) and \( G_0 \) is fixed under the conjugation, then the map \( H_k(G_1, G_0) \to H_k(G, G_2) \) canonically factors through \( H_{k-1}(G_0) \). However, in our case there is no choice for \( t \) and the factorization is not canonical either. Using Lemma 2.1, we next discuss how to construct a particular diffeomorphism
Recall we fix an embedding of \( e : \{0\} \times D^{2n-1} \to \partial W_{g,1} \) which does not intersect our chosen chart \( c : \{0\} \times \mathbb{R}^{2n-1} \to \partial W_{g,1} \), then attach \( H \) to \( W_{g,1} \) via gluing \( \{0\} \times D^{2n-1} \subset H \) to \( e(\{0\} \times D^{2n-1}) \) so we get \( W_{g,1} \cup_{\partial(0) \times D^{2n-1}} H \) whose interior is diffeomorphic to the interior of \( W_{g+1,1} \). To get a smooth manifold, we smooth out creases or we glue \( H \) to a collar of the boundary like the description of non-standard stabilization maps in fig. 1. We pick a diffeomorphism that satisfies Lemma 2.1 and it gives a non-standard stabilization map. Assume that two horizontal maps in the commutative diagram of the statement of the lemma are this variant of the stabilization map. If \( \sigma = [\varphi] \) is a germ of an embedding of the core to \( W_{g,1} \), then \( \text{Diff}^e_c(W_{g-1,1}, \partial) \sigma = \text{Stab}(\sigma) \). Choose an embedding of \([0,1] \times D^{2n-1}\) into \( W_{g+1,1} \)

\[
d : [0,1] \times D^{2n-1} \to W_{g+1,1} = W_{g,1} \cup_{\partial(0) \times D^{2n-1}} H
\]

such that \( d(0, D^{2n-1}) = e(D^{2n-1}) \) and \( d(1, D^{2n-1}) = e(0, D^{2n-1}) \). We will choose \( d \) such that its image is in \( \epsilon \)-collar neighborhood of the boundary. If we fix a small neighborhood \( N \) of \( \varphi(C) \) such that \( N \) is diffeomorphic to \( H \) then we can find a diffeomorphism \( t \in \text{Diff}^e_c(W_{g+1,1}, \partial) \) such that it swaps \( N \) with \( d([0,1] \times D^{2n-1}) \cup_{\partial(0) \times D^{2n-1}} H \). Let \( \text{Diff}^e_c(W_{g-1,1}, \text{rel} d([0,1] \times D^{2n-1}) \cup \partial)_\sigma \) denote the subgroup of \( \text{Diff}^e_c(W_{g-1,1}, \partial) \sigma \) that fixes a neighborhood of \( d([0,1] \times D^{2n-1}) \). Note that conjugating by \( t \) maps a subgroup of \( \text{Diff}^e_c(W_{g-1,1}, \partial) \sigma \) that fixes the chosen neighborhood \( N \) around the core, into \( \text{Diff}^e_c(W_{g,1}, \partial) \). Also note that conjugating by \( t \) is identity on \( \text{Diff}^e_c(W_{g-1,1}, \text{rel} \epsilon\text{-collar}) \).

![Cartoon of a diagram that induces the maps in the diagram of Lemma 4.1 for \( n = 1 \)](image)

In Lemma 2.1 we showed

(4.2) \[ H_* (\text{Diff}^e_c(W_{g-1,1}, \text{rel} \epsilon\text{-collar}) \sigma; \mathbb{Z}) \xrightarrow{\sim} H_* (\text{Diff}^e_c(W_{g-1,1}, \partial) \sigma; \mathbb{Z}) \]

For brevity, we shall write \( G_0, G_1, G_2 \) and \( G \) to denote \( \text{Diff}^e_c(W_{g-1,1}, \text{rel} \epsilon\text{-collar}) \sigma, \text{Diff}^e_c(W_{g,1}, \partial) \sigma, \text{Diff}^e_c(W_{g,1}, \partial) \) and \( \text{Diff}^e_c(W_{g+1,1}, \partial) \) respectively. These subgroups of \( \text{Diff}^e_c(W_{g+1,1}, \partial) \) fit into the following diagram

![Diagram](image)
A class $x$ in $H_k(G_1, G_0)$ is represented in homogenous chain as a finite sum
\[ \sum_i a_i(x_{0,i}, x_{1,i}, \ldots, x_{k,i}) \text{ where } x_{j,i} \in G_1, a_i \in \mathbb{Z} \text{ and } dx \text{ is a chain in } G_0. \]
Following the notation of [Wah10, Lemma 2.5.], we write $(x_{0,i}, x_{1,i}, \ldots, x_{k,i}) \times t$ to denote the
$(k+1)$-chain given by the following linear combination
\[
(x_{0,i}, x_{1,i}, \ldots, x_{k,i}, t) + \sum_{j=0}^k (x_{0,i}, x_{1,i}, \ldots, t, t^{-1}x_{j,i}, t^{-1}x_{j+1,i}, t, \ldots, t^{-1}x_{k,i}, t)
\]

It is easy to compute $d(x \times t) = (-1)^k x + dx \times t + (-1)^{k+1} tx$. Similar to the
discussion at the beginning of the proof, we can find $\tau \in G$ depending on $x$ such that
conjugation by $\tau$ maps all $x_{j,i}$'s to $G_2$. Thus, we have $[\tau x \tau^{-1} x] = 0$ in $H_k(G, G_2)$
that implies the image of the class $[x]$ in $H_k(G, G_2)$ is equal to $(-1)^k [dx \times \tau]$. We
obtain non-canonical factorization of the relative map $H_k(G_1, G_0) \rightarrow H_k(G, G_2)$ via
$H_{k-1}(G_0)$, meaning the image of $[x]$ in $H_k(G, G_2)$ is hit by an element in $H_{k-1}(G_0)$,
but the map from $H_{k-1}(G_0)$ to $H_k(G, G_2)$ depends on $x$ itself.

\[ H_k(G_1, G_0) \xrightarrow{\partial} H_k(G, G_2) \]

\[ H_{k-1}(G_0) \]

Recall that we want to prove that
\[ H_k(G_1, G_0) \rightarrow H_k(G, G_2) \]
is zero as long as $k \leq (g-4)/2$. Let $[x] \in H_k(G_1, G_0)$, then $dx \in H_{k-1}(G_0)$. By
induction we know
\[ H_{k-1}(\text{Diff}^c_\epsilon(W_{g-1,1}, \partial)|_{[\phi], [\phi']} \rightarrow H_{k-1}(\text{Diff}^c_\epsilon(W_{g-1,1}, \partial)|_{[\phi]}) = H_{k-1}(G_0) \]
where $[\phi'] \neq [\phi] \in \text{Eq}(W_{g,1})$, is an isomorphism as long as $k \leq (g-4)/2$. So $dx$ comes
from a class $y \in H_{k-1}(\text{Diff}^c_\epsilon(W_{g-1,1}, \partial)|_{[\phi], [\phi']})$ meaning that it can be represented
by a linear combination of elements in $\text{Diff}^c_\epsilon(W_{g,1}, \partial)$ that fixes a neighborhood of
$\phi(C)$ and $\phi'(C)$. Let $N'$ be a neighborhood of $\phi'(C)$. Again by the same argument
as the beginning of the proof, there exists a diffeomorphism $t' \in \text{Diff}^c_\epsilon(W_{g,1}, \partial)$,
swapping $\overline{N'}$ with $d([0,1] \times D^{2n-1} \cup_{\{0\}} D^{2n-1} H)$. Let us compute $d(y \times t \times t')$:
\[
d(y \times t \times t') = (-1)^{k+1} (y \times t) + d(y \times t) \times t' + (-1)^k t^{-1}(y \times t) t'
\]
\[
= (-1)^{k+1} (y \times t) + (-1)^k (y \times t') + (dy \times t \times t' + (-1)^{k+1}(t^{-1} y t) t' + (-1)^k t^{-1}(y \times t) t'
\]
\[
= (-1)^{k+1} (y \times t) + (-1)^k (y \times t') + (-1)^{k+1}(t^{-1} y t) t' + (-1)^k t^{-1}(y \times t) t'
\]
\[
= (-1)^{k+1} (y \times t) + (-1)^k t^{-1}(y \times t) t'
\]

Note that we have used the fact that $dy = 0$ and $t^{-1} y t = y$. So $[x] = [y \times t] = [t^{-1} (y \times t) t'] \in H_k(G, G_2)$. To finish the proof, we need to show $[t^{-1} (y \times t) t'] = 0$
in $H_k(G, G_2)$. Suppose $y = \sum_i b_i(y_{0,i}, \ldots, y_{k-1,i})$ then by definition $t^{-1} (y \times t) t'$ is
\[
\sum_i (t^{-1} x_{0,i} t', \ldots, t^{-1} x_{k-1,i} t', t^{-1} t t') + \sum_{j=0}^{k-1} (t^{-1} x_{0,i} t', \ldots, t^{-1} t t', t^{-1} t^{-1} x_{j,i} t' t, t^{-1} t^{-1} x_{j+1,i} t', \ldots, t^{-1} t^{-1} x_{k,i} t')
\]
Using the way we chose $t'$, it is easy to see that $t'^{-1}x_{j,t'} = x_{j,i}$ for all $i,j$ and $t'^{-1}t' \in G_2$ that implies all the group elements in the above chain are in $G_2$. Hence, $[t'^{-1}(g \times t)t'] = 0$ in $H_k(G, G_2)$. □

**Remark 4.3.** Very recently, Galatius and Randal Williams improved Charney’s theorem [GRW14]. They showed that the algebraic complex in Theorem A.4 is at least $[(g - 4)/2]$-connected which is 1/2 better than the original bound $[(g - 5)/2]$. Thus, the range of stability actually can be improved by 1/2.

**Remark 4.4.** If we denote $C^r$-diffeomorphisms of $W_{g,1}$ by $\text{Diff}_c^r(W_{g,1}, \partial)$ and the same underlying group equipped with discrete topology by $\text{Diff}_c^d(W_{g,1}, \partial)$, the same proof shows that $\text{Diff}_c^d(W_{g,1}, \partial)$ establishes homological stability for all $r \geq 1$. Although for $r = 1$, this is a consequence of Tsuboi’s theorem that $\text{BDiff}_c^1(W_{g,1}, \partial)$ is homology equivalent to $\text{BDiff}^1(W_{g,1}, \partial) \simeq \text{BDiff}(W_{g,1}, \partial)$. Hence, the homological stability for $C^1$-diffeomorphisms with discrete topology $\text{Diff}_c^1(W_{g,1}, \partial)$ is already implied by homological stability of $\text{BDiff}(W_{g,1}, \partial)$.

As always after proving homological stability for a family of groups, the next step is to study the limit. Consider the following space

$$\mathcal{M} := \prod_g \text{BDiff}_c^d(W_{g,1}, \partial)$$

It is not hard to see that this space is an H-space with an associative and commutative product up to homotopy, but it is not clear to the author whether $\mathcal{M}$ has an $A_\infty$-structure. We would like to understand

$$\text{hocolim} (\mathcal{M} \xrightarrow{W_{1,1}} \mathcal{M} \xrightarrow{W_{1,1}} \cdots)$$

where the product by $W_{1,1}$ is the standard model for stabilization map. To this end, in the next section we prove that there exists a certain infinite loop space whose homology groups compute the stable homology of $\text{BDiff}_c^d(W_{g,1}, \partial)$.

5. Stable moduli of flat bundles

In this section we will prove Theorem 1.2 and Theorem 1.3. Using one of Thurston’s theorems, we shall see that $\text{BDiff}_c^d(W_{g,1}, \partial)$ is homologically equivalent to the homotopy quotient of a space of certain tangential structures on $W_{g,1}$ by the action of $\text{Diff}(W_{g,1}, \partial)$. Let us briefly digress to explain Thurston’s theorem.

5.1. Recollection from foliation theory. The general idea of Thurston’s theorem is as follows. Let $W$ be an $n$-dimensional smooth manifold without boundary and $\text{Diff}_{g,0}(W)$ be the identity component of $\text{Diff}_c(W)$. For a topological group $G$, let $\overline{BG}$ be the homotopy fiber of the map $BG^s \to BG$, where $G^s$ is the same group equipped with discrete topology. It is easy to see $\overline{BG}_{g,0} \simeq BG$, where $G_{0}$ is the base point component of $G$. Note that the manifold $W$ has a unique codimension $n$ foliation, namely foliation by points. By Haeffiger’s theorem [Hae71], this foliation gives rise to the commutative diagram

\[
\begin{array}{ccc}
\pi & \downarrow \gamma & \nu \\
W & \rightarrow & BGL_n
\end{array}
\]

where $\tau$ classifies the tangent bundle. Let $L \rightarrow W$ be the homotopy pull back by $\tau$ of the Hurewicz fibration associated to $\nu$, and let $\mathcal{S}(W)$ be the space of continuous sections of $L \rightarrow W$ with compact open topology. The space of sections $\mathcal{S}(W)$ can
also be thought of as the space of all pairs \((g,h)\) where \(g : W \to B\Gamma_n\) and \(h\) is a homotopy from \(\tau\) to \(\nu \circ g\). Thus, it has a base point \(s_0 = (\gamma, h_0)\) where \(h_0\) is a trivial homotopy. Let \(S_n(W)\) be the space of sections over \(W\) that is equal to the base section \(s_0\), outside a compact subset of \(W\), with the direct limit topology.

The topological space \(BDiff_{c,0}(W) \times W\) has a natural codimension \(n\) Haefliger structure (note that although \(BDiff_{c,0}(W) \times W\) is not a manifold, Haefliger structure makes sense on topological spaces), that is obtained by pulling back the point foliation on \(W\) by the evaluation map from \(Diff_0(W) \times W\) to \(W\). This Haefliger structure is transverse to the spaces \(b \times W\), where \(b \in BDiff_{c,0}(W)\), so its normal bundle is the pull back of \(\tau(W)\) by the projection \(\pi : BDiff_{c,0}(W) \times W \to W\). Hence, the foliation is classified by a homotopy commutative diagram

\[
\begin{array}{ccc}
BDiff_{c,0}(W) \times W & \xrightarrow{F} & B\Gamma_n \\
\downarrow{\pi} & & \downarrow{\nu} \\
W & \xrightarrow{\tau = \nu \circ \gamma} & BGL_n
\end{array}
\]

(5.1)

where there is a canonical choice of homotopy, \(\tilde{\nu}\) say, from \(\tau \circ \pi\) to \(\nu \circ F\). The homotopy \(\tilde{\nu}\) determines a map \(f_W : BDiff_{c,0}(W) \to S_n(W)\). To each \(b \in BDiff_{c,0}(W)\), the map \(f_W\) assigns the pair \((F_b, \tilde{\nu}_b)\), where \(F_b(x) = F(b, x)\) and \(\tilde{\nu}_b\) is the homotopy induced by \(\tilde{\nu}\) from \(\tau\) to \(\nu \circ F_b\). We describe certain categorical models for \(BDiff_{c,0}(W)\), \(B\Gamma_n\), and \(BGL_n\) in order to have \(Diff_c(W)\)-equivariant map \(f_W\) from \(BDiff_{c,0}(W)\) to \(S_n(W)\).

If \(M\) is a topological monoid that acts on a space \(S\) from the left, we shall write \(C(M,S)\) for the topological category whose space of objects is \(S\) and whose space of morphisms is \(M \times S\), where \((m,s)\) corresponds to the morphism \(s \to ms\). Our model for \(BDiff_{c,0}(W)\) is the fat realization of \(C(Diff^c_\bullet(W) \backslash Diff_c(W))\) which admits an action of \(Diff_c(W)\) from the right. For a manifold \(W\) without a boundary, let \(\Gamma(W)\) be the category whose space of objects is \(W\) with its usual topology and whose space of morphisms from \(x\) to \(y\) is the discrete set of germs of local diffeomorphisms of \(W\) that take \(x\) to \(y\). The space \(Mor(\Gamma(W))\) is equipped with sheaf topology. By [McD79, Lemma 1], the realization of \(\Gamma(W)\) is a model for \(B\Gamma_n\).

Let \(GL(W)\) be the topological category which also has \(W\) as a set of objects with its usual topology, but its space of morphisms is the bundle over \(W \times W\), whose fiber over \((x,y)\) is the space of all linear isomorphisms with its usual topology from tangent space at \(x\), \(T_x\), to the tangent space at \(y\), \(T_y\). By [McD79, Lemma 2], the realization of \(GL(W)\) is a model for \(BGL_n\). Note that there exists a functor \(\tilde{\nu} : \Gamma(W) \to GL(W)\) that is identity on the space of objects and it sends the morphism \(f : x \to y\) to \(df_x : x \to y\).

The discrete group \(Diff^\circ_\bullet(W)\) acts on \(Diff_c(W) \times W\) from the left by \(p : (m,x) \to (pm,x)\). Hence, there exists a functor

\[
\tilde{F} : C(Diff^\circ_\bullet(W) \backslash Diff_c(W) \times W) \to \Gamma(W)
\]

that takes the object \((m,x)\) to \(m(x)\) and the morphism \(p : (m,x) \to (pm,x)\) to the germ of \(p\) at \(m(x)\). The following diagram is a model for the categorification of the
diagram 5.1.

\[
\begin{array}{ccc}
\mathcal{C}(\text{Diff}_c(W)\backslash\text{Diff}_c(W) \times W) & \xrightarrow{\bar{F}} & \Gamma(W) \\
\pi & \gamma & \nu \\
\mathcal{C}(\text{Diff}_c(W)) & \tau & GL(W)
\end{array}
\]

(5.2)

where \( \mathcal{C}(\text{Diff}_c(W)) \) is the category that arises from the action of the trivial group \( \{e\} \) on \( W \), \( \bar{\pi} \) is the obvious projection, and \( \gamma \) and \( \bar{\pi} \) are induced by the identity map on the space of objects \( W \). Note that \( \bar{\pi} \circ \bar{\pi} = \nu \circ \bar{F} \). However, there exists a natural transformation \( \bar{H} : \bar{\pi} \circ \bar{\pi} \rightarrow \nu \circ \bar{F} \) that takes the object \( (m,x) \) of \( \mathcal{C}(\text{Diff}_c(W)\backslash\text{Diff}_c(W) \times W) \) to the following morphism of \( GL(W) \)

\[
dm_{x} : x = \bar{\pi} \circ \bar{\pi}(m,x) \rightarrow m(x) = \nu \circ \bar{F}(m,x)
\]

The fat realization of the bottom triangle in the diagram 5.2 gives a functorial description of \( S_c(W) \).

\[
\begin{array}{ccc}
\text{BF}(W) & \xrightarrow{\gamma} & \nu \\
W & \tau & BGL(W)
\end{array}
\]

for \( f \in \text{Diff}_c(W,\partial) \), let \( f_{\Gamma} : \text{BF}(W) \rightarrow \text{BF}(W) \) and \( f_{GL} : BGL(W) \rightarrow BGL(W) \) be the maps induced by \( f \). Hence, the action of \( f \) on a pair \( (g,h) \in S_c(W) \) is given by \( (f_{\Gamma}^{-1} \circ g \circ f, f_{GL}^{-1} \circ h \circ (f \times id_{[0,1]}) \) \). Using the functoriality of the diagram, it is easy to see that \( f_{GL}^{-1} \circ h \circ (f \times id_{[0,1]}) \) gives a homotopy from \( \tau \) to \( \nu \circ f_{1}^{-1} \circ g \circ f \). Realization of the diagram 5.2 provides us with a map \( f_W \) from \( \text{Diff}_c^{\delta}(W)\backslash\text{Diff}_c(W) \) to \( S_c(W) \) that respects the action.

**Theorem 5.3 (Thurston [Thu74]).** The map \( f_W : B\text{Diff}_c(W) \rightarrow S_c(W) \) induces an isomorphism on homology with integer coefficients.

Another model for \( S(W) \) is \( \text{Bun}(TW,\nu^*\gamma) \) that is all bundle maps \( TW \rightarrow \nu^*\gamma \) equipped with compact-open topology. The group \( \text{Diff}(W) \) acts on \( \text{Bun}(TW,\nu^*\gamma) \) by precomposing a bundle map with the differential of a diffeomorphism. This model is used in [GRW12b] and since we want to use [GRW12b, Theorem 1.8], we have to justify why these two models have the same homotopy quotients.

**Lemma 5.4.** The natural map from \( S(W) \) to \( \text{Bun}(TW,\nu^*\gamma) \) induces a homotopy equivalence between \( S(W) / \text{Diff}(W) \) and \( \text{Bun}(TW,\nu^*\gamma) / \text{Diff}_c(W) \)

**Sketch.** Let \( : S(W) \rightarrow \text{Bun}(TW,\nu^*\gamma) \) be the map that sends a pair \( (g,h) \) to the bundle isomorphism over the map \( g \) that is induced by the homotopy \( h \). This map is not quite equivariant but it is equivariant up to homotopy in a sense that we now describe. By the Haefliger’s theorem [Hae71, Theorem 7] the map \( g \) induces \( \Gamma_n \)-structure on \( W \) up to concordance. Because \( g \) is a lifting of \( \tau \), the \( \Gamma_n \)-structure on \( W \) that \( g \) induces is concordant to a codimension \( n \) foliation which is the point foliation. Let \( f \in \text{Diff}(W) \), hence \( f \) is transversal to the point foliation and its pullback via \( f \) is again the point foliation. Thus \( g \circ f \) is homotopic to \( g \) and as a result \( f_{\Gamma} \circ g \) is homotopic to \( g \). This homotopy is given by the natural transformation that sends an object \( x \in \mathcal{C}(\text{Diff}_c(W)) \) to the morphism in \( \Gamma(X) \) given by the germ of \( f \) at \( x \). This homotopy induces a natural fiber homotopy between \( \iota(f_{\Gamma}^{-1} \circ g \circ f, f_{GL}^{-1} \circ h \circ (f \times id_{[0,1]}) \) and the action of \( f \) on \( \iota(g,h) \). Because these homotopies are defined
by the natural transformation that is induced by the germ of a diffeomorphism, they are coherent homotopies. Hence, by [Vog73, Theorem 1.4] the map $\iota$ induces a homotopy equivalence on homotopy quotients. □

Recall we would like to find the stable homology of $\text{BDiff}^\delta_c(W_{g,1})$. Consider the following fibration sequence

$$\text{BDiff}^\delta_c(W_{g,1}) \to \text{BDiff}_c(W_{g,1}) \to \text{BDiff}_c(W_{g,1})$$

Note that $\text{Diff}_c(W_{g,1})$ can be thought of as compactly supported diffeomorphism of the interior of $W_{g,1}$. Also note that $\text{BDiff}^\delta_c(W_{g,1})$ can be written as a homotopy quotient $\text{EDiff}_c(W_{g,1}) \times_{\text{Diff}_c(W_{g,1})} \text{BDiff}_c(W_{g,1})$. Thus, we have the following commutative diagram

$$\text{BDiff}_c(W_{g,1}) \to \text{S}_c(W_{g,1}) \to \text{BDiff}_c(W_{g,1})$$

Because the equivariant map between fibers is homology isomorphism, by Thurston’s theorem, we have the following corollary.

**Corollary 5.5.** There exists a map from $\text{BDiff}^\delta_c(W_{g,1})$ to $\text{S}_c(W_{g,1}) \times_{\text{Diff}_c(W_{g,1})} \text{BDiff}_c(W_{g,1})$ that induces a homology isomorphism with integer coefficients.

$\text{S}_c(W_{g,1}) \times_{\text{Diff}_c(W_{g,1})} \text{BDiff}_c(W_{g,1})$ is the moduli space of $\Gamma_{2n}$-tangential structures on $W_{g,1}$. Using the main theorem of [GRW12b, Theorem 1.8], we describe in the following section, the stable homology of $\text{BDiff}^\delta_c(W_{g,1})$ as the homology of the moduli space of $\Gamma_{2n}$-tangential structures on $W_{g,1}$ as $g$ increases.

### 5.2. On stable moduli of flat $W_{g,1}$-bundles.

In this subsection, we show Thurston’s theorem and Galatius and Randal-Williams’ main theorem in [GRW12b] imply Theorem 1.2. Equivalently, we reformulate Theorem 1.2 as follows.

**Theorem 5.6.** There exists a map which becomes a homology equivalent as $g \to \infty$

$$\text{S}_c(W_{g,1}) \times_{\text{Diff}_c(W_{g,1})} \Omega_0^\infty \text{MT} \nu^n$$

**Corollary 5.7.** There exists a map that induces the following isomorphism:

$$H_k(\text{BDiff}^\delta_c(W_{g,1}), \mathbb{Z}) \xrightarrow{\cong} H_k(\Omega_0^\infty \text{MT} \nu^n, \mathbb{Z})$$

as long as $k \leq (g-4)/2$

Let us briefly recall and adapt the main theorem of Galatius and Randal-Williams in [GRW12b]. A tangential structure $\theta : B \to BO(2n)$ is called spherical if $S^{2n}$ has a $\theta$-structure. Since we have the foliation by points on $S^{2n}$, the structure map $\nu : \Gamma_{2n} \to BO(2n)$ is spherical. If we fix a $\nu$-structure on $S^{2n-1}$, then we
can define \( \mathcal{N}^\nu(S^{2n-1}) \) the moduli space of highly connected bordism as [GRW12b, Definition 1.4]. With this notation, we have

\[
\mathcal{N}^\nu(S^{2n-1}) \cong \coprod_W S_c(W, \partial) \otimes \text{Diff}(W, \partial)
\]

where the disjoint union is over compact manifolds \( W \) with \( \partial W = S^{2n-1} \) such that \((W, S^{2n-1})\) is \((n-1)\)-connected, one in each diffeomorphism class.

If we choose an embedding \( W_{1,2} \subset [0,1] \times \mathbb{R}^n \) as a cobordism with collar boundary, since \( \nu \) is \((2n+2)\)-connected, we can choose a \( \nu \)-structure on \( W_{1,2} \) extending our chosen \( \nu \)-structures on \( \{0\} \times S^{2n-1} \) and \( \{1\} \times S^{2n-1} \). There exists an induced self-map \( \mathcal{N}^\nu(S^{2n-1}) \to \mathcal{N}^\nu(S^{2n-1}) \) defined by taking union with \( W_{1,2} \) and subtracting 1 from the first coordinate. Hence, we have the following commutative diagram

\[
\begin{array}{c}
\coprod g \text{BDiff}^2_c(W_{g,1}, \partial) \xrightarrow{f_{W_{g,1}}} \mathcal{N}^\nu(S^{2n-1}) \\
\downarrow \quad \downarrow \\
\coprod W_{1,2} \quad \coprod W_{1,2} \\
\coprod g \text{BDiff}^2_c(W_{g,1}, \partial) \xrightarrow{f_{W_{g,1}}} \mathcal{N}^\nu(S^{2n-1})
\end{array}
\]

where the left vertical map is the standard stabilization map and horizontal maps are induced by Thurston’s map, which is a homology isomorphism. With abuse of the notation we denote them by \( f_{W_{g,1}} \).

Assume that \( K \subset [0,\infty) \times \mathbb{R}^n \) is a submanifold with \( \nu \)-structure \( l_K \), such that \( x_1 : K \to [0,\infty) \) has the natural numbers as regular values and \( K|_{[i,i+1]} \) is a cobordism such that the pairs \((K|_{[i,i+1]}, K_i)\) and \((K|_{[i,i+1]}, K_{i+1})\) are \((n-1)\)-connected for all natural numbers \( i \). There exists a notion of universal \( \nu \)-end [GRW12b, Definition 1.4 (ii)] that can be practically checked by the following conditions:

- For each integer \( i \), the map \( \pi_n(K|_{[i,\infty)}) \to \pi_n(\Gamma_{2n}) \) is surjective, for all base points in \( K \).
- For each integer \( i \), the map \( \pi_{n-1}(K|_{[i,\infty)}) \to \pi_{n-1}(\Gamma_{2n}) \) is injective, for all base points in \( K \).
- For each integer \( i \), each path component of \( K|_{[i,\infty)} \) contains a submanifold diffeomorphic to \( S^n \times S^n \sim \text{int}(D^{2n}) \), which in addition has null-homotopic structure map to \( \Gamma_{2n} \).

Using the main theorem of [GRW12b, Theorem 1.8] for the map \( \nu : B\Gamma_{2n} \to BO(2n) \), we obtain

**Theorem 5.8.** Let \( 2n > 4 \) and \((K,l_K)\) be a universal \( \nu \)-end such that \( \mathcal{N}^\nu(K|_0,l_K|_0) \neq \emptyset \), then there is a homology equivalence

\[
hocolim_{i \to \infty} \mathcal{N}^\nu(K_i|_{l_K|i}) \to \Omega^\infty \text{MT}_\eta
\]

where \( \eta : B' \to B\Gamma_{2n} \to BO(2n) \) is the \( n \)th stage of the Moore-Postnikov tower for \( l_K : K \to B\Gamma_{2n} \) and \( \text{MT}_\eta \) is the Madsen-Tillman spectrum associated to \( \eta \).

For \( \theta^n : BO(2n)(n) \to BO(2n) \), a universal \( \theta^n \)-end can be constructed by letting each \( K|_{[i,i+1]} \) be \( W_{1,2} \). The notion of universal end is preserved under highly connected maps between structures, because \( B\Gamma_{2n}(n) \to BO(2n)(n) \) is at least \((2n+2)\)-connected, which is more than is needed, \( K \) is also universal \( \nu^n \)-end, where \( \nu^n : B\Gamma_{2n}(n) \to B\Gamma_{2n} \to BO(2n) \). Having fixed this specific \( K \) as a universal \( \nu^n \)-end, Theorem 5.6 is a formal consequence of Theorem 5.8.
5.3. **Stable splitting after \( p \)-adic completion.** In order to understand the effect of the \( \text{BDiff}_c^\delta(W_{g,1}, \partial) \to \text{BDiff}(W_{g,1}, \partial) \) on the level of cohomology in the stable range, we have to study the following map

\[
\Omega_0^\infty \text{MT} \nu^n \to \Omega_0^\infty \text{BSO}(2n)(n)^{\gamma}
\]

and we shall prove below that this map is a split surjection after \( p \)-adic completion, in the sense of [MP11]. Therefore, the split surjection implies that

\[
H^*(\text{BDiff}_c^\delta(W_{g,1}, \partial); \mathbb{F}_p) \to H^*(\text{BDiff}(W_{g,1}, \partial); \mathbb{F}_p)
\]

provided that \( * \leq (g-4)/2 \).

**Theorem 5.9.** The following natural map

\[
\Omega_0^\infty \text{MT} \nu^n \to \Omega_0^\infty \text{BSO}(2n)(n)^{-\theta\gamma}
\]

is a split surjection after \( p \)-adic completion for all prime \( p \).

Recall that the map

\[
\text{BS} \Gamma_{2n}^\nu \to \text{BGL}_{2n}(\mathbb{R})^+
\]

is induced by the continuous map of topological pseudogroups: \( \tilde{\nu} : \text{ST}_{2n} \to \text{GL}_{2n}(\mathbb{R})^+ \), where \( \tilde{\nu} \) sends a germ \( f \in \text{ST}_{2n} \), to \( df \) evaluated at the source. Furthermore, there is an obvious map \( \tilde{\iota} : \text{SO}(2n)^\delta \to \text{ST}_{2n} \), which assigns to a matrix its germ as a diffeomorphism of \( \mathbb{R}^{2n} \) at 0. Note that the image of the composite \( \tilde{\nu} \circ \tilde{\iota} \) is \( \text{SO}(2n) \). Thus, we have the following diagram:

\[
\text{BSO}(2n)^\delta \xrightarrow{\iota} \text{BS} \Gamma_{2n}^\nu \to \text{BGL}_{2n}(\mathbb{R})^+ \cong \text{BSO}(2n)
\]

where \( \nu \circ \iota \) is homotopic to the map induced by the identity from \( \text{SO}(2n)^\delta \) to \( \text{SO}(2n) \). Hence, we have the following maps between Thom spectra:

\[
(\text{BSO}(2n)^\delta)^{-(\nu \circ \iota)^\gamma} \xrightarrow{\nu'} \text{BS} \Gamma_{2n}^{\nu'' \gamma} \xrightarrow{\nu''} \text{BSO}(2n)^{\gamma}
\]

The Milnor conjecture says for a Lie group \( G \), the classifying space \( BG \) and \( BG^\delta \) are \( p \)-adically equivalent. If the Milnor conjecture were known for \( \text{SO}(2n) \), the proof of the theorem would be much shorter. Because then \( \text{BSO}(2n) \) and \( \text{BSO}(2n)^\delta \) would be equivalent after \( p \)-adic completion and by Thom isomorphism so were \( \text{BSO}(2n)^{\gamma} \) and \( (\text{BSO}(2n)^\delta)^{-(\nu \circ \iota)^\gamma} \). Hence, this equivalence implies \( \nu' \) splits after \( p \)-completion. The theorem is a formal consequence of \( \nu' \) having a section after \( p \)-completion. Because Milnor’s conjecture seems to be unknown for real Lie groups, we give a transfer argument to show \( \nu' \) admits a section after \( p \)-completion.

**Lemma 5.10.** The following map of spectra splits after \( p \)-completion

\[
(\text{BSO}(2n)^\delta)^{-(\nu \circ \iota)^\gamma} \to \text{BSO}(2n)^{\gamma}
\]

i.e. it admits a section after \( p \)-completion.
Proof. We denote the normalizer of the maximal torus in $SO(2n)$ by $N(T)$. Consider the following commutative diagram:

\[
\begin{array}{c}
BSO(2n)^\delta \\
\downarrow i^\delta \\
B(N(T)^\delta)
\end{array}
\quad
\begin{array}{c}
BSO(2n) \\
\downarrow i \\
B(N(T))
\end{array}
\]

where $i$ (respectively $i^\delta$) is induced by injection of $N(T)$ in $SO(2n)$ (respectively injection of $N(T)^\delta$, which is the same group as $N(T)$ equipped with discrete topology, in $SO(2n)^\delta$). There exists a Becker-Gottlieb transfer for the following map between two Thom spectra:

\[
B(N(T))^{-i^\gamma} \xrightarrow{i} BSO(2n)^{-i^\gamma}
\]

To explain this twisted transfer, we digress from the proof and suppose we have a fiber bundle $F \to E \xrightarrow{p} B$, where $F$ is a manifold. Assume that $\gamma$ is a stable bundle over $B$, then there exists a vector bundle $\gamma_n$ of dimension $n$ for a large enough $n$ that is stably equivalent to $\gamma$. We choose $n$ in such a way that we can find an embedding $j : E \to \gamma_n$ over $B$. If we denote the normal bundle of $E$ in $\gamma_n$ by $Nj$, then $Nj$ fits into the following diagram:

\[
\begin{array}{c}
Nj \\
\downarrow j \\
E \\
\downarrow p \\
B
\end{array}
\]

it follows from the above diagram that there is a natural map from $E^{Nj}$ Thom space of the normal bundle of $E$ in $\gamma_n$, to $E^p \gamma_n$ Thom space of the pullback bundle $p^* \gamma_n$. If we precompose this map with the Pontryagin-Thom collapse map, we obtain a transfer map $\tau : B^\gamma \to E^p \gamma$, thus stably we have the following transfer map:

\[
B^\gamma \xrightarrow{\tau} E^p \gamma
\]

satisfying $\tau^* \circ p^* = \chi(F)$ on cohomology.

Now recall that $\chi(SO(2n)/N(T)) = 1$ and by the above discussion, we have a transfer map

\[
BSO(2n)^{-\gamma} \xrightarrow{\tau} B(N(T))^{-i^\gamma}
\]

if we show that $p$-completion of $B(N(T))^{-i^\gamma}$, in the sense of [MP11], is weakly equivalent to the $p$-completion of $B(N(T)^\delta)^{(\nu \circ \iota^\delta)^* \gamma}$, then by virtue of the following commutative diagram, there exists a section for $\nu^\delta \circ \iota^\delta$.

\[
\begin{array}{c}
((BSO(2n)^\delta)^{(\nu \circ \iota^\delta)^* \gamma})^\wedge_p \\
\downarrow \\
(B(N(T)^\delta)^{(\nu \circ \iota^\delta)^* \gamma})^\wedge_p \\
\downarrow \tau \\
(B(N(T))^{-i^\gamma})^\wedge_p
\end{array}
\]

\[
((BSO(2n)^\delta)^{(\nu \circ \iota^\delta)^* \gamma})^\wedge_p \\
\downarrow \\
(B(N(T)^\delta)^{(\nu \circ \iota^\delta)^* \gamma})^\wedge_p \\
\downarrow \tau \\
(B(N(T))^{-i^\gamma})^\wedge_p
\]
in order to show that the bottom map is weak-equivalence, it is sufficient to prove that the map induces isomorphism on mod $p$ homology [MP11, Theorem 11.1.2]. But using the Thom isomorphism, it only needs to show that the middle map in the following diagram induces isomorphism on mod $p$ homology

$$\begin{array}{ccc}
BT^δ & \xrightarrow{p\text{-adic equivalence}} & BT \\
\downarrow & & \downarrow \\
B(N(T)^δ) & \to & B(N(T)) \\
\downarrow & = & \downarrow \\
BW & & BW
\end{array}$$

where $W$ is the Weyl group of $SO(2n)$. Note that $(BT^δ)^\wedge_p \simeq (BT)^\wedge_\wedge_p$ because $H^*(BT^δ; \mathbb{F}_p) = H^*(B(Z/p\infty)^n; \mathbb{F}_p) = H^*(BT; \mathbb{F}_p)$, hence, the top horizontal morphism is mod $p$ homology isomorphism. The actions of Weyl group $W$ on the cohomology of fibers with mod $p$ coefficient are the same so by the comparison theorem of Leray-Serre spectral sequences, the middle map becomes mod $p$ homology isomorphism. □

**Proof of Theorem 5.9.** First, we show that on the level of spectra, the map $M\mathbb{T}^\nu \to BSO(2n)(n)\gamma_p$, $p$-adically splits i.e. after the $p$-adic completion, we find a section. Having splitting on the level of spectra, we then show splitting of $\Omega_{\infty}^n M\mathbb{T}^\nu$ formally follows from properties of $p$-completion. Consider the following diagram:

$$\begin{array}{ccc}
M\mathbb{T}^\nu & \xrightarrow{\nu''} & BSO(2n)(n)^{-\theta_{\mathbb{Z}}\gamma} \\
\downarrow & & \downarrow \\
B\mathbb{S}\Gamma_2n & \xrightarrow{\nu'} & BSO(2n)^{-\gamma}
\end{array}$$

**Step 1:** We want to prove if after $p$-adic completion, the bottom horizontal map has a section, then we obtain a section for the upper horizontal map. Let $T$ be the maximal torus in $SO(2n)$ and $N(T)$ be the normalizer of the torus in $SO(2n)$. By the same arguments in the proof of Lemma 5.10, we have the following commutative diagram:

$$\begin{array}{ccc}
&(B\mathbb{S}\Gamma_2n)^\wedge_p & \\
\downarrow & \downarrow \\
(BN(T))^\wedge_p & \to & (BSO(2n))^\wedge_p
\end{array}$$

Note that there exists the (twisted) Becker-Gottlieb transfer for the bottom horizontal map even before $p$-completion. Let $Y$ and $Y'$ be the homotopy pullbacks in
the following diagram:

\[
\begin{array}{ccc}
Y' & \longrightarrow & Y & \longrightarrow & BSO(2n)(n) \\
\downarrow & & \downarrow & & \downarrow \\
BN(T) & \longrightarrow & BN(T) & \longrightarrow & BSO(2n)
\end{array}
\]  
(5.11)

Hence, the map from \(Y\) to \(BSO(2n)(n)\) also admits a (twisted) Becker-Gottlieb transfer. We claim that there exists a map \(Y' \rightarrow B\Gamma_{2n}(n)\), making the following diagram commutative:

\[
\begin{array}{ccc}
Y' & \longrightarrow & B\Gamma_{2n}(n) \\
\downarrow & & \downarrow \\
BSO(2n)(n) & \longrightarrow & BSO(2n)(n)
\end{array}
\]

and this follows from the following commutative diagram:

\[
\begin{array}{ccc}
Y' & \longrightarrow & B\Gamma_{2n}(n) \\
\downarrow & & \downarrow \\
BSO(2n)(n) & \longrightarrow & BSO(2n)(n)
\end{array}
\]

where the left bent arrow is given by the composition \(Y' \rightarrow BN(T)^\delta \rightarrow B\Gamma_{2n}\). By Haefliger’s theorem, the square is a pullback square, so the dotted arrow exists.

Let us with abuse of notation \(\gamma\) also denote the pullbacks of tautological bundle over \(Y\) and \(Y'\). Since the right vertical map in diagram 5.11 is between simply connected spaces, if we take \(p\)-completion of diagram 5.11, all pullback squares remain pullback. Given that \((N(T))_p^\gamma = (N(T)^\delta)_p^\gamma\) and Thom isomorphism, we have \((Y^-)^\gamma_p = (Y'^-)^\gamma_p\). Hence, given that \(\nu'\) has a section after \(p\)-adic completion, \(\nu''\) also admits a section after \(p\)-adic completion by the following composition

\[
(BSO(2n)(n))^{-\theta\gamma}_p \longrightarrow (Y^-)^\gamma_p \longrightarrow (MT\nu^n)^\gamma_p
\]

**Step 2:** To prove that \(\nu'\) has a section after \(p\)-adic completion, it suffices to prove the following map has a section after \(p\)-adic completion:

\[
(5.12) \quad (BSO(2n)^\delta)^{-(\nu\circ\iota)^\gamma} \longrightarrow BSO(2n)^{-\gamma}
\]

which is followed by Lemma 5.10.

**Step 3:** The last step is to use this section on the level of spectra and prove that it induces a section on the corresponding infinite loop spaces, i.e. we want to show that the following map has a section:

\[
(\Omega^\infty_BSO(2n)^\delta)^{-(\nu\circ\iota)^\gamma}_p \longrightarrow \Omega^\infty_BSO(2n)^{-\gamma}_p
\]

but this is a consequence of the fact that if \(X\) is a spectrum, then \(\Omega^\infty_0(X)^p\) is a \(p\)-completed space and it is weakly equivalent to \((\Omega^\infty_0(X))^p\). Note that homotopy
groups of $\Omega^\infty_0(X_p^\gamma)$ are the positive homotopy groups of $X_p^\gamma$ and these groups can be computed by the following exact sequence

$$0 \longrightarrow \text{Ext}(\mathbb{Z}/p^\infty, \pi_*(X)) \longrightarrow \pi_*(X_p^\gamma) \longrightarrow \text{Hom}(\mathbb{Z}/p^\infty, \pi_{*-1}(X)) \longrightarrow 0$$

since the two ends are $p$-completed groups, so are homotopy groups $\pi_*(X_p^\gamma)$. Hence, the fact that the homotopy groups of $\Omega^\infty_0(X_p^\gamma)$ are $p$-completed groups, [MP11, Theorem 11.1.1] implies that $\Omega^\infty_0(X_p^\gamma)$ is a $p$-completed space. Thus, by universal property of $p$-completion, there exists a map $(\Omega^\infty_0(X))^\gamma_p \to \Omega^\infty_0(X_p^\gamma)$. Given that homotopy groups of $(\Omega^\infty_0(X))^\gamma_p$ can be obtained by the same exact sequence, we deduce that it has the same homotopy groups as $\Omega^\infty_0(X_p^\gamma)$, hence $(\Omega^\infty_0(X))^\gamma_p \simeq \Omega^\infty_0(X_p^\gamma)$. This weak equivalence finishes the proof by providing the following section:

$$\Omega^\infty_0(BSO(2n)^\gamma)^\wedge_p \longrightarrow \Omega^\infty_0((BSO(2n)^\delta)^{(\nu_{1^\wedge})\gamma})^\wedge_p$$

\[\Omega^\infty_0(BSO(2n)^\gamma)^\wedge_p \longrightarrow \Omega^\infty_0((BSO(2n)^\delta)^{(\nu_{1^\wedge})\gamma})^\wedge_p\]

\[\Omega^\infty_0(BSO(2n)^\gamma)^\wedge_p \longrightarrow \Omega^\infty_0((BSO(2n)^\delta)^{(\nu_{1^\wedge})\gamma})^\wedge_p\]

\[\Omega^\infty_0(BSO(2n)^\gamma)^\wedge_p \longrightarrow \Omega^\infty_0((BSO(2n)^\delta)^{(\nu_{1^\wedge})\gamma})^\wedge_p\]

\[\Omega^\infty_0(BSO(2n)^\gamma)^\wedge_p \longrightarrow \Omega^\infty_0((BSO(2n)^\delta)^{(\nu_{1^\wedge})\gamma})^\wedge_p\]

\[\Omega^\infty_0(BSO(2n)^\gamma)^\wedge_p \longrightarrow \Omega^\infty_0((BSO(2n)^\delta)^{(\nu_{1^\wedge})\gamma})^\wedge_p\]

\[\Omega^\infty_0(BSO(2n)^\gamma)^\wedge_p \longrightarrow \Omega^\infty_0((BSO(2n)^\delta)^{(\nu_{1^\wedge})\gamma})^\wedge_p\]

\[\Omega^\infty_0(BSO(2n)^\gamma)^\wedge_p \longrightarrow \Omega^\infty_0((BSO(2n)^\delta)^{(\nu_{1^\wedge})\gamma})^\wedge_p\]

\[\Omega^\infty_0(BSO(2n)^\gamma)^\wedge_p \longrightarrow \Omega^\infty_0((BSO(2n)^\delta)^{(\nu_{1^\wedge})\gamma})^\wedge_p\]

\[\Omega^\infty_0(BSO(2n)^\gamma)^\wedge_p \longrightarrow \Omega^\infty_0((BSO(2n)^\delta)^{(\nu_{1^\wedge})\gamma})^\wedge_p\]

\[\Omega^\infty_0(BSO(2n)^\gamma)^\wedge_p \longrightarrow \Omega^\infty_0((BSO(2n)^\delta)^{(\nu_{1^\wedge})\gamma})^\wedge_p\]

\[\Omega^\infty_0(BSO(2n)^\gamma)^\wedge_p \longrightarrow \Omega^\infty_0((BSO(2n)^\delta)^{(\nu_{1^\wedge})\gamma})^\wedge_p\]

\[\Omega^\infty_0(BSO(2n)^\gamma)^\wedge_p \longrightarrow \Omega^\infty_0((BSO(2n)^\delta)^{(\nu_{1^\wedge})\gamma})^\wedge_p\]

Corollary 5.13. For all odd prime $p$, the following map

$$(\Omega^\infty_0)^\wedge_p : H^*(\Omega^\infty_0 BSO(2n)(n)^\gamma; \mathbb{F}_p) \to H^*(\Omega^\infty_0 M\nu^n; \mathbb{F}_p)$$

is split injective.

Corollary 5.14. If $G$ is non-torsion subgroup of $H^*(\Omega^\infty_0 BSO(2n)(n)^\gamma; \mathbb{Z})$, then

$$(\Omega^\infty_0)^\wedge_p : H^*(\Omega^\infty_0 BSO(2n)(n)^\gamma; \mathbb{Z}) \to H^*(\Omega^\infty_0 M\nu^n; \mathbb{Z})$$

is injective on $G$.

Proof. Suppose the contrary, so for some nonzero element $a \in G$, $(\Omega^\infty_0)^\wedge_p(a) = 0$. Consider the following commutative diagram:

$$H^*(\Omega^\infty_0 BSO(2n)(n)^\gamma; \mathbb{Z}) \longrightarrow H^*(\Omega^\infty_0 M\nu^n; \mathbb{Z})$$

$$H^*(\Omega^\infty_0 BSO(2n)(n)^\gamma; \mathbb{F}_p) \longrightarrow H^*(\Omega^\infty_0 M\nu^n; \mathbb{F}_p)$$

since $H_*(\Omega^\infty_0 BSO(2n)(n)^\gamma; \mathbb{Z})$ is finitely generated in each degree and $a$ is non-torsion, $a \in \text{Hom}(H_*(\Omega^\infty_0 BSO(2n)(n)^\gamma; \mathbb{Z}), \mathbb{Z})$. Choose a prime $p$ so that $a \otimes 1$ is nonzero in $\text{Hom}(H_*(\Omega^\infty_0 BSO(2n)(n)^\gamma; \mathbb{Z}) \otimes \mathbb{F}_p$, Note that

$$\text{Hom}(H_*(\Omega^\infty_0 BSO(2n)(n)^\gamma; \mathbb{Z}), \mathbb{F}_p) \longrightarrow \text{Hom}(H_*(\Omega^\infty_0 BSO(2n)(n)^\gamma; \mathbb{F}_p)$$

is injective. Hence, $i(a)$ is nonzero in $H^*(\Omega^\infty_0 BSO(2n)(n)^\gamma; \mathbb{F}_p)$ which is a contradiction. □

6. Remarks on characteristic classes of flat $W_{g,1}$-bundles

The goal of this section is two fold. One fold is to show that high dimensional generalized MMM-classes rationally vanish on flat $W_{g,1}$-bundles. This vanishing phenomenon implies the existence of non vanishing secondary characteristic classes. The other is to try to use Corollary 5.7 and our knowledge about cohomology of classifying space of the Haefliger category to detect non-trivial cohomology classes of $BDiff^+(W_{g,1})$ which may continuously vary.
6.1. On generalized MMM-classes for flat $W_{g,1}$-bundles. As explained in [GRW12b], to each $c \in H^{k+2n}(\text{BSO}(2n))$ we can associate a cohomology class $\kappa_c$ in $H^k(\text{BDiff}(W_{g,1}))$ for all $g$, sometimes called “generalized MMM classes". These classes can be roughly defined as follows, take the universal $W$ map, to each $W$ we can associate a class $c$ stable rational cohomology of $\text{BDiff}$.

We need to prove that for any flat $W$-bundle, $E \xrightarrow{\pi} M$, its $\kappa_c$ vanishes as long as $c \in \mathcal{B}$ and $\deg(c) - 2n \leq (g - 4)/2$. Recall that the Bott vanishing theorem [Bot70] says for a foliation $\mathcal{F}$ on $E$ of codimension $q$, we have

$$\operatorname{Pon}(\nu_{\mathcal{F}})^{2q} = 0$$

where $\operatorname{Pon}^{2q}(\nu_{\mathcal{F}})$ is a ring generated by monomials of Pontryagin classes of the normal bundle of $\mathcal{F}$ of degree larger than $2q$. Any flat $W$-bundle structure on $E$ gives a foliation of codimension $2n$ such that the vertical tangent bundle is the normal bundle of the of the given foliation. Suppose

$$c = e(T_x E)^a p_i (T_x E)^{a_i} \cdots p_k (T_x E)^{a_k}$$
if \( a \leq 1 \) then we have \( \sum 4i_j a_j > 4n \); Since by the Bott vanishing theorem, the class \( p_n(T_xE)^{a_1} \cdots p_k(T_xE)^{a_k} \) has to vanish so does \( c \). If \( a > 1 \) then we have 
\[
4 : \left[ a / 2 \right] + \sum 4i_j a_j > 4n ,
\]
again by the Bott vanishing theorem the class 
\[
p_n(T_xE)^{a_1 / 2} p_n(T_xE)^{a_1} \cdots p_k(T_xE)^{a_k}
\]
has to vanish, so does \( c \).

Note that in the above discussion, we showed \( c \) as a characteristic class of the normal bundle of the foliation on \( E \) of codimension \( 2n \) vanishes in \( H^{deg(c)}(E; \mathbb{R}) \) provided the degree of \( c \) is larger than \( 6n \). There exists a natural secondary characteristic class called the Cheeger-Simons class associated to \( E \) normal bundle of the foliation on \( E \) and this class lives in \( H^{deg(c)-1}(E; \mathbb{R} / \mathbb{Z}) \) \cite[corollary 2.4.]{CS85}, so there is a universal class \( \hat{c} \in H^{deg(c)-1}(BF_2n; \mathbb{R} / \mathbb{Z}) \) associated to \( c \in H^{deg(c)}(BSO(2n); \mathbb{Z}) \) and this class lives in \( H^{deg(c)-2n-1}(MT\nu^n; \mathbb{R} / \mathbb{Z}) \). If we pullback this class to \( BF_2n(n) \) and use Thom isomorphism, we obtain a class in \( H^{deg(c)-2n-1}(MT\nu^n; \mathbb{R} / \mathbb{Z}) \). Let \( \kappa_c \) denote the image of this class under the cohomology suspension map

\[
\sigma^* : H^{deg(c)-2n-1}(MT\nu^n; \mathbb{R} / \mathbb{Z}) \rightarrow H^{deg(c)-2n-1}(\Omega^n_0 MT\nu^n; \mathbb{R} / \mathbb{Z})
\]

if \( deg(c) - 2n - 1 \) is in the stable range, we have

\[
H^{deg(c)-2n-1}(\Omega^n_0 MT\nu^n; \mathbb{R} / \mathbb{Z}) \cong H^{deg(c)-2n-1}(BDiff^i_{\nu}(W_{g,1}, \partial); \mathbb{R} / \mathbb{Z})
\]

Using naturality of these classes, it is easy to show \cite[corollary 2.4.]{CS85} that \( \kappa_c \) maps to \( -\kappa_c \) under the Bockstein map

\[
H^{deg(c)-2n-1}(BDiff^i_{\nu}(W_{g,1}, \partial); \mathbb{R} / \mathbb{Z}) \xrightarrow{\beta} H^{deg(c)-2n}(BDiff^i_{\nu}(W_{g,1}, \partial); \mathbb{Z})
\]

By virtue of Corollary 5.14, we know that those \( \kappa_c \)'s that live in the stable range are non-torsion classes in \( H^*(BDiff^i_{\nu}(W_{g,1}, \partial); \mathbb{Z}) \); thus, corresponding \( \kappa_c \)'s are nontrivial and non-torsion classes. They induce the following map

\[
H_{deg(c)-2n-1}(BDiff^i_{\nu}(W_{g,1}, \partial); \mathbb{Z}) \rightarrow \mathbb{R} / \mathbb{Z}
\]

hence for those \( c \) with \( deg(c) > 6n \), we have \( H_{deg(c)-2n-1}(BDiff^i_{\nu}(W_{g,1}, \partial); \mathbb{Z}) \) is nontrivial. But we can actually do better.

**Theorem 6.4.** \( H_k(BDiff^i_{\nu}(W_{g,1}, \partial); \mathbb{Z}) \) is not finitely generated if \( k = deg(c) - 2n - 1 \) as \( deg(c) > 6n \) and \( c \in \mathcal{B} \).

**Proof.** As we proved in the previous section \( \kappa_c \)'s for \( c \in \mathcal{B} \) are non-torsion classes in \( H^*(\Omega^n_0 MT\nu^n; \mathbb{Z}) \), but for \( c \) satisfying Proposition 6.3, \( \kappa_c \) lives in the kernel of the following natural map

\[
H^*(\Omega^n_0 MT\nu^n; \mathbb{Z}) \otimes \mathbb{Q} \rightarrow H^*(\Omega^n_0 MT\nu^n; \mathbb{Q})
\]

Then, the theorem follows from the following lemma \cite{BH72} which is itself straightforward consequence of the universal coefficient theorem. \( \square \)

**Lemma 6.5.** Let \( f : X \rightarrow Y \) be a map between CW complexes and let \( i : H^*(X; \mathbb{Z}) \rightarrow H^*(X; \mathbb{Q}) \) be the natural map. Suppose that there is an infinite cyclic subgroup \( G \subset H^k(Y; \mathbb{Z}) \) such that \( f^* \) is an injective on \( G \) and \( i \circ f^*(G) = 0 \). Then, \( H_{k-1}(X; \mathbb{Z}) \) is not finitely generated.
6.2. On non-vanishing characteristic classes for flat \( W_{g,1} \)-bundles. We don’t know how to compute rational cohomology of \( \Omega^\infty \text{MT} \nu^n \), but by Haefliger’s theorem, we know the following map is at least \((2n + 2)\)-connected

\[
\text{B} \Gamma_{2n}(n) \to \text{BSO}(2n)(n)
\]

Hence, \( H^{2n+2}(\text{BSO}(2n)(n); \mathbb{Q}) \) injects into \( H^{2n+2}(\text{B} \Gamma_{2n}(n); \mathbb{Q}) \). It is well-known from the basics of Hopf algebra that

\[
H^\ast(\Omega^\infty \text{MT} \nu^n; \mathbb{Q}) \cong \Lambda(H^{>2n}(\text{B} \Gamma_{2n}(n); \mathbb{Q})[-2n])
\]

where \( \Lambda(A) \) for an algebra \( A \), means the free graded commutative and unital algebra generated by \( A \). Suppose \( n \equiv 3(\text{mod } 4) \), then using the connectivity of \( \nu^n \), we have that \( c = p_2^{\Delta} \) is nonzero in \( H^{2n+2}(\text{B} \Gamma_{2n}(n); \mathbb{Q}) \). Thus the corresponding \( \kappa_c^n \) is nontrivial in \( H^2(\Omega^\infty \text{MT} \nu^n; \mathbb{Q}) \). Using this observation and Corollary 5.7, we have

**Theorem 6.6.** For \( n \equiv 3(\text{mod } 4) \), the generalized MMM class \( \kappa_c^n \) associated to \( c = p_2^{\Delta} \), is nonzero in \( H^2(\text{BDiff}^\mathbb{F}_c(W_{g,1}, \partial); \mathbb{Q}) \) as \( g \geq 8 \).

**Remark 6.7.** This result is analogous to the surface case which was proved by Koschick and Morita [KM05]. They showed that \( \kappa_c^n \) is nonzero in \( H^2(\text{BDiff}^\mathbb{F}_c(\Sigma_{g,1}, \partial); \mathbb{Q}) \). In the sequel paper [Nar] in which we treat the surface case, we give a nonconstructive proof of theirs.

Thurston [Thu] in an unpublished manuscript, proved the Godbillon-Vey class \( h_1c_1^n \in H^{2n+3}(\text{B} \Gamma_{2n}; \mathbb{Z}) \) (for definition of these secondary characteristic classes consult e.g. [Pit76],[Bot72]) varies continuously on a foliated trivial bundle with fiber dimension \( n \). Therefore, in codimension \( 2n \), we have

\[
H_{4n+1}(\text{B} \Gamma_{2n}; \mathbb{Z}) \xrightarrow{h_1c_1^{2n}} \mathbb{R}
\]

where \( \text{B} \Gamma_{2n} \) is homotopy fiber of \( \text{B} \Gamma_{2n} \to \text{BGL}(2n) \) and classifies foliated trivial bundles with fiber dimension \( 2n \). Note that there is an evaluation map from \( H_{2n+1}(\Omega^\infty_0 \text{MT} \nu^n; \mathbb{Q}) \to H_{2n+1}(\text{B} \Gamma_{2n}(n)^\ast; \mathbb{Q}) \) which is surjective. By Haefliger’s theorem, we know \( \text{B} \Gamma_{2n} \) is at least \((2n + 1)\)-connected, so there exists a map from \( \text{B} \Gamma_{2n} \) to \( \text{B} \Gamma_{2n}(n) \) that makes the following diagram commute

\[
\begin{array}{ccc}
\text{B} \Gamma_{2n}(n) & \text{B} \Gamma_{2n} & \text{B} \Gamma_{2n} \\
\downarrow & \downarrow & \downarrow \\
\text{B} \Gamma_{2n} & \text{B} \Gamma_{2n} & \text{B} \Gamma_{2n} \\
\end{array}
\]

Therefore, \( h_1c_1^{2n} \) is a nonzero class in \( H^{4n+1}(\text{B} \Gamma_{2n}(n); \mathbb{Q}) \) and varies continuously. Hence, we obtain the following surjective map

\[
H_{2n+1}(\Omega^\infty_0 \text{MT} \nu^n; \mathbb{Q}) \to H_{4n+1}(\text{B} \Gamma_{2n}(n); \mathbb{Q}) \xrightarrow{h_1c_1^{2n}} \mathbb{R}
\]

Using Corollary 5.7, one can summarize the above as

**Theorem 6.8.** The following map is surjective, provided \( g \geq 4n + 6 \)

\[
H_{2n+1}(\text{BDiff}^\mathbb{F}_c(W_{g,1}, \partial); \mathbb{Q}) \xrightarrow{h_1c_1^{2n}} \mathbb{R}
\]

i.e. \( H_{2n+1}(\text{BDiff}^\mathbb{F}_c(W_{g,1}, \partial); \mathbb{Q}) \) as a vector space over rationals has uncountable dimension.
Remark 6.9. We can apply Bowden’s idea to determine stable homology of $\text{Diff}^c_\partial(W_{g,1})$ in low homological degrees. There is a spectral sequence [Hal98, Theorem 2.3.4] whose $E^2_{p,q}$ page can be described for $q \leq 3$ as

$$E^2_{p,q} = \begin{cases} 
Z & \text{if } p = q = 0 \\
0 & \text{if } q = 0, p > 0 \\
H_p(W_{g,1}, H_q(\text{BDiff}_c(\mathbb{R}^{2n}))) & \text{if } 0 < q \leq 3
\end{cases}$$

and converges to $H_{p+q}(\text{BDiff}(W_{g,1}; \partial))$ for $p+q \leq 3$. Since we don’t have differentials in this range, we deduce

$$H_k(\text{BDiff}(W_{g,1}, \partial); Z) = H_k(\text{BDiff}_c(\mathbb{R}^{2n}); Z) \text{ as } k \leq 3$$

In particular, $H_0(\text{BDiff}(W_{g,1}, \partial); Z) = Z$, $H_1(\text{BDiff}(W_{g,1}, \partial); Z) = 0$. Using Serre spectral sequence for the following fibration sequence

$$\text{BDiff}(W_{g,1}, \partial) \longrightarrow \text{BDiff}_c^\delta(W_{g,1}, \partial) \longrightarrow \text{BDiff}(W_{g,1}, \partial)$$

we can compute the stable homology of $\text{BDiff}_c^\delta(W_{g,1}, \partial)$ in the low homological degree, using our knowledge [GRW12a] about the stable homology of $\text{BDiff}(W_{g,1}, \partial)$. Thus, it is straightforward to see for $g \geq 10$ and $n \geq 3$

$$H_1(\text{BDiff}_c^\delta(W_{g,1}, \partial); Z) = H_1(\text{BDiff}(W_{g,1}, \partial); Z)$$

$$H_1(\text{BDiff}_c^\delta(W_{g,1}, \partial); \mathbb{Q}) = 0$$

$$H_2(\text{BDiff}_c^\delta(W_{g,1}, \partial); \mathbb{Q}) = H_2(\text{BDiff}(W_{g,1}, \partial); \mathbb{Q}) \oplus H_2(\text{BDiff}_c(\mathbb{R}^{2n}); \mathbb{Q})$$

$$H_3(\text{BDiff}_c^\delta(W_{g,1}, \partial); \mathbb{Q}) = H_3(\text{BDiff}_c(\mathbb{R}^{2n}); \mathbb{Q})$$

## Appendix A

In this section for the convenience of the reader, we recall two technical results used in Lemma 3.5 to establish the high connectivity of $[E_\bullet(W_{g,1})]$.

### A.1. Generalized coloring lemma.**

Recall a simplicial complex $K$ is called \textit{weakly Cohen–Macaulay} of dimension $n$ and we denote it by $wCM(K) \geq n$ if it is $(n - 1)$-connected and the link of any $p$-simplex is $(n - p - 2)$-connected.

\textbf{Definition A.1.} Let us say that a simplicial map $f : X \to Y$ of simplicial complexes is simplexwise injective if its restriction to each simplex of $X$ is injective, i.e. the image of any $p$-simplex of $X$ is a non-degenerate $p$-simplex of $Y$.

The following generalization of the “coloring lemma” is proved in [GRW12a, Theorem 2.4].

\textbf{Theorem A.2.} Let $X$ be a simplicial complex and $f : \partial I^n \to |X|$ be a map which is simplicial with respect to some PL triangulation on $\partial I^n$. Then, if $wCM(X) \geq n$, the triangulation extends to a PL triangulation of $I^n$, and $f$ extends to a simplicial map $g : I^n \to |X|$ with the property that $g(Lk(v)) \subset Lk(g(v))$ for each interior vertex $v \in \text{int}(I^n)$. In particular, $g$ is simplexwise injective if $f$ is.

### A.2. Unimodular hermitian quadratic forms.

For a group $G$, we abbreviate the group ring $\mathbb{Z}[G]$ by $R$. Note that $R$ is equipped with anti-involution, namely the map that sends $g$ to $g^{-1}$. For a ring $R$ with anti-involution, we define \textit{unimodular hermitian quadratic forms}. It is a finitely generated free $R$-module, $A$ together with a map

$$\lambda : A \times A \to R$$

such that
i) for each $\lambda \in A$ the map

$$A \to R, \ x \mapsto \lambda(x, y)$$

is linear

ii) $\lambda(x, y) = \overline{\lambda(y, x)}$, where bar is the anti-involution map.

iii) the associated map

$$A \to A^*, \ y \mapsto (x \mapsto \lambda(x, y))$$

is an isomorphism, where $A^* = \text{Hom}_R(A, R)$.

We also need the concept of a *quadratic refinement* of hermitian unimodular for $\lambda$. This is a map

$$q : A \to R/\{a - \bar{a} | a \in R\}$$

such that

iv) $\lambda(x, x) = q(x) + \overline{q(x)} \in R$

v) $q(x + y) = q(x) + q(y) + [\lambda(x, y)] \in R/\{a - \bar{a}\}$

vi) $q(ax) = qa(x)\bar{a} \in R/\{a - \bar{a}\}$

Here we note that iv) has to be interpreted as follows. Choose a representative $b \in R$ for $q(x)$ and consider $b + b$. If we change $b$ by adding some element $a - \bar{a}$, then $b + b$ is replaced by $b + a - \bar{a} + b - \bar{a} = b + b$. Thus, although $b$ is not well defined if only $[b] \in R/\{a - \bar{a}\}$ is given, the sum $b + b$ is well defined in $R$, so that the equation iv) makes sense in $R$. For equation vi) we have to convince ourselves that, if $b \in R$ represents $q(x)$, then $[ab\bar{a}] \in R/\{a - \bar{a}\}$ is independent of the choice of the representative $b$, which the reader can easily check.

The hyperbolic form $H$, in which $A = R \oplus R$ with basis $e$ and $f$ and $\lambda$ is given by $\lambda(e, e) = \lambda(f, f) = 0$ and $\lambda(e, f) = 1$. The quadratic refinement is given by $q(e) = q(f) = 0$. We abbreviate unimodular hermitian bilinear forms together with a quadratic refinement to *unimodular hermitian quadratic form*.

**Definition A.3.** For a unimodular hermitian quadratic form $(M, \lambda, q)$, let $K^a(M)$ be the simplicial complex whose vertices are morphism $e : H \to M$ of quadratic modules. The set $\{e_0, \ldots, e_p\}$ is a $p$-simplex if the submodules $e_i(H) \subset M$ are orthogonal with respect to $\lambda$.

To the simplicial complex $K^a(M)$, we can naturally associate a semisimplicial space $K^a_*(M)$, whose $p$-simplices are ordered $(p + 1)$-tuples of vertices in $K^a(M)$ spanning a $p$-simplex. $[K^a(M)]$ is at least as connected as $[K^a_*(M)]$. Charney [Cha87, Corollary 3.3] proved

**Theorem A.4.** Let $M = H^q_{\ast\ast}$ be direct sum of $g$ hyperbolic forms over $R = \mathbb{Z}$, i.e. the group $G$ is trivial. Then $[K^a_*(M)]$ is $[(g - 5)/2]$-connected, so is $[K^a(M)]$.

**Remark A.5.** Galatius and Randal-Williams proved in [GRW14] that $[K^a(M)]$ is at least $[(g - 4)/2]$-connected.

Let $\lambda$ and $q$ be intersection form and quadratic refinement induced by selfintersection on $\pi_n(W_{g, 1})$, respectively. By the Charney’s theorem, $[K^a((\pi_n(W_{g, 1}), \lambda, q))]$ is $[(g - 5)/2]$-connected.

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E-mail address: nariman@math.stanford.edu

Department of Mathematics, Stanford University, Stanford CA, 94305