DEGREE VERSIONS OF THEOREMS ON INTERSECTING FAMILIES VIA STABILITY

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Abstract. The matching number of a family of subsets of an \( n \)-element set is the maximum number of pairwise disjoint sets. The families with matching number 1 are called intersecting. The famous Erdős–Ko–Rado theorem determines the size of the largest intersecting family of \( k \)-sets. Its generalization to the families with larger matching numbers, known under the name of the Erdős Matching Conjecture, is still open for a wide range of parameters. In this paper, we address the degree versions of both theorems.

More precisely, we give degree and \( t \)-degree versions of the Erdős–Ko–Rado and the Hilton–Milner theorems, extending the results of Huang and Zhao, and Frankl, Han, Huang and Zhao. We also extend the range in which the degree version of the Erdős Matching conjecture holds.

1. Introduction

For integers \( a \leq b \), put \( [a, b] := \{a, a+1, \ldots, b\} \) and \( [n] := [1, n] \). For a set \( X \) and an integer \( k \geq 0 \), let \( 2^X \) and \( \binom{X}{k} \) stand for the collections of all subsets and of all \( k \)-element subsets (\( k \)-sets) of \( X \), respectively. Any collection of sets is called a family. We call a family intersecting, if any two sets from it intersect. A “trivial” example of such family is all sets containing a fixed element. A family \( \mathcal{F} \) is called non-trivial, if \( \bigcap_{F \in \mathcal{F}} F = \emptyset \).

The following theorem is one of the classic results in extremal combinatorics.

Theorem 1.1 (Erdős, Ko, Rado [9]). Let \( n \geq 2k > 0 \) and consider an intersecting family \( \mathcal{F} \subset \binom{[n]}{k} \). Then \( |\mathcal{F}| \leq \binom{n-1}{k-1} \).

Answering a question of Erdős, Ko, and Rado, Hilton and Milner [24] determined the largest non-trivial intersecting family of \( k \)-sets. If \( k > 3 \) then, up to a permutation of the ground set, the unique largest non-trivial intersecting family is \( \mathcal{H}_k \), where for any \( u \in [2, k+1] \)

\[
\mathcal{H}_u := \left\{ A \in \binom{[n]}{k} : [2, u+1] \subset A \right\} \cup \left\{ A \in \binom{[n]}{k} : 1 \in A, [2, u+1] \cap A \neq \emptyset \right\}.
\]

For \( k = 3 \), \( \mathcal{H}_3 \) is the largest but not unique: \( \mathcal{H}_2 \) has the same size.

Erdős–Ko–Rado theorem spurred the development of extremal combinatorics, and by now there are numerous variations and extensions of Theorem 1.1 and the Hilton–Milner theorem (cf. [2, 4, 5, 6, 7, 14, 15, 22, 27, 28, 31, 33, 34, 35, 36, 37] to name a few recent ones). We refer the reader to a recent survey by Frankl and Tokushige [21].

For \( \mathcal{F} \subset \binom{X}{k} \) and \( i \in X \), the degree \( d_i(\mathcal{F}) \) of an element \( i \) is the number of sets from \( \mathcal{F} \) containing it. We denote by \( \delta(\mathcal{F}) \) and \( \Delta(\mathcal{F}) \) the minimum degree and maximum degree of an element in \( \mathcal{F} \). Recently, Huang and Zhao [26] gave an elegant proof of the following theorem using a linear-algebraic approach:

Theorem 1.2 (26). Let \( n > 2k > 0 \). Then any intersecting family \( \mathcal{F} \subset \binom{[n]}{k} \) satisfies \( \delta(\mathcal{F}) \leq \binom{n-2}{k-2} \).

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The bound in the theorem is tight because of the trivial intersecting family, and the condition \( n > 2k \) is necessary: the authors of \([26]\) provide an example of such family for \( n = 2k \) which has larger minimum degree. In fact, for most values of \( k \) there are regular intersecting families in \( \binom{[2k]}{k} \) of maximum possible size \( \binom{2k-1}{k-1} \) (see \([27]\)). In the follow-up paper, Frankl, Han, Huang, and Zhao \([13]\) proved the following theorem.

**Theorem 1.3 \([13]\).** Let \( k \geq 4 \) and \( n \geq ck^2 \), where \( c = 30 \) for \( k = 4, 5 \), and \( c = 4 \) for \( k \geq 6 \). Then any non-trivial intersecting family \( \mathcal{F} \subset \binom{[n]}{k} \) satisfies \( \delta(\mathcal{F}) \leq \binom{n-2}{k-2} - \binom{n-k-2}{k-2} \).

This theorem is again tight: the lower bound is provided by the Hilton–Milner family \( \mathcal{H}_k \).

Several questions and problems in this context were asked in \([26]\) and \([13]\), as well as in personal communication with Hao Huang and his presentation on the Recent Advances in Extremal Combinatorics Workshop at TSIMF, Sanya, (May 2017). Some of them are as follows:

1. Can one find a combinatorial proof of Theorem \([12]\)? This question was partially answered by Frankl and Tokushige \([20]\), who proved it under the additional assumption \( n \geq 3k \). Huang claims that their proof can be made to work for \( n \geq 2k + 3 \), provided that one applies their approach more carefully. However, the cases \( n = 2k + 2 \) and \( n = 2k + 1 \) remained open.

2. Extend Theorem \([3]\) to the case \( n \geq ck \) for large \( k \). Ultimately, prove Theorem \([3]\) for all values \( n \geq 2k + 1 \) for which it is valid.

3. Extend Theorems \([2]\) and \([3]\) to \( t \)-degrees. The degree of a subset \( S \subset [n] \) is the number of sets from the family containing \( S \). We denote by \( \delta_t(\mathcal{F}) \) the minimal degree of an \( t \)-element subset \( S \subset [n] \) (minimal \( t \)-degree).

In this note, we partially answer these three questions. Our first theorem provides a \( t \)-degree version of Theorem \([2]\). Its proof is combinatorial and works, in particular, for \( t = 1 \) and \( n \geq 2k + 2 \).

**Theorem 1.4.** If \( n \geq 2k + 2 > 2 \), then for any intersecting family \( \mathcal{F} \) of \( k \)-subsets of \([n]\) we have \( \delta(\mathcal{F}) \leq \binom{n-2}{k-2} \). More generally, if \( n \geq 2k + \frac{2n}{1-t} \) and \( 1 \leq t < k \), then \( \delta_t(\mathcal{F}) \leq \binom{n-t-1}{k-t-1} \).

After writing a preliminary version of the paper, we read the paper \([20]\), where Theorem \([4]\) is proved for \( s = 1 \) and \( n \geq 3k \). It turned out that the approach the authors took is very similar to the approach we use to prove Theorem \([4]\). However, it seems that their proof, unlike ours, does not work for \( n = 2k + 2 \), which is probably due to the fact that they use the original Frankl's degree theorem \([10]\) instead of Theorem \([2]\) (cf. Section 2).

Our main theorem is a \( t \)-degree version of Theorem \([3]\) with much weaker restrictions on \( n \) (for moderately large \( k \)).

**Theorem 1.5.** If \( t = 1, n \geq 2k + 5 \), and \( k \geq 28 \), or \( 1 < t \leq \frac{k}{k-1} - 2, n \geq 2k + 14t \), then for any non-trivial intersecting family \( \mathcal{F} \) of \( k \)-subsets of \([n]\) we have \( \delta_t(\mathcal{F}) \leq \binom{n-t-1}{k-t-1} - \binom{n-t-k-1}{k-t-1} \).

The dependencies in the theorem are not optimal. One reason is that there is a tradeoff between different parameters. (E.g., it follows from the proof that the theorem holds for \( n \geq 2k + 6 \) and \( k \geq 15 \).) Most importantly, pushing farther the bounds on \( n, k \) would require much more complicated calculations, which we decided to avoid. It is very much possible that one can prove the validity of Theorem \([5]\) for \( n \geq 2k + 3 \) and, say, \( k \geq 100 \), using a refinement of our approach. We made even less effort to optimize the dependencies than the tradeoff that in Theorem \([3]\) and are very close to the best possible ones.

The matching number \( \nu(\mathcal{F}) \) of a family \( \mathcal{F} \) is the maximum number of pairwise disjoint sets from \( \mathcal{F} \). That is, intersecting families are exactly the families with matching number 1. It is a natural question to ask, what is the largest family with matching number (at most) \( s \). Let
Erdős conjectured [8] that known as the Erdős Matching Conjecture. It was studied quite extensively over the last 50 years, but it remains unsolved in general. It is known to be true for \( k \leq 3 \) [12], for \( n \geq (2s + 1)k - s \) [11], as well as for \( n \geq \frac{2s^2k - 3s}{3} \) for \( s \geq s_0 \) [18]. We note that \( A_0(n, k, s) \) is bigger than \( A_k(k, s) \) already for relatively small \( n \): the condition \( n > (k + 1)(s + 1) \) suffices.

Finding degree versions of the Erdős Matching Conjecture falls into a more general recent trend to study the so-called Dirac thresholds. See, e.g., [11], [18], [23], [30]. The following theorem was proved in [20].

**Theorem 1.6** ([20]). Given \( n, k, s \) with \( n \geq 3k^2(s + 1) \), if for a family \( F \subset \binom{[n]}{k} \) with \( \nu(F) \leq s \) we have \( \delta_i(F) \leq \binom{n - 1}{k - 1} - \binom{n - s - 1}{k - s - 1} = \delta_1(A_0(n, k, s)) \).

This improved the result of Bollobás, Daykin, and Erdős [8], who arrived at the same conclusion for \( n \geq 2k^3s \). The authors of [20] conjectured that the same should hold for any \( n > k(s + 1) \). Note that the family \( A_k(k, s) \) does not appear in the degree version since its minimum t-degree is 0 for \( n \geq k(s + 1) \) and \( t \geq 1 \). Note that, for integer \( t \geq 1 \), we have \( \delta_t(A_0(n, k, s)) = \binom{n - t}{k - t} - \binom{n - s - t}{k - s - t} \).

In this paper we improve and generalize Theorem 1.6 for \( k \) large in comparison to \( s \).

**Theorem 1.7.** Fix \( n, s, k \) and \( t \geq 1 \), such that \( n \geq 2k^2 \), and \( k \geq 5st \) \( (k \geq 3s \text{ for } t = 1) \). For any family \( F \subset \binom{[n]}{k} \) with \( \nu(F) \leq s \) we have \( \delta_i(F) \leq \delta(A_0(n, k, s)) \), with equality only in the case \( F = A_0(n, k, s) \).

The constants in the statements of Theorems 1.5 and 1.7 are not optimal, and chosen in this way to simplify the calculations. All theorems are proved using stability results for the corresponding problems (cf. Section 2). We also note that one can obtain asymptotic analogues of Theorems 1.6 and 1.7 for fixed \( k \) and \( n > \frac{2}{3}ks \) from the results on the EMC that we mentioned (cf. [18] for details).

### 2. Preliminaries

For a family \( F \), the **diversity** \( \gamma(F) \) is the quantity \( |F| - \Delta(F) \). In particular, a family \( F \) is non-trivial if and only if \( \gamma(F) \geq 1 \). The following far-reaching generalization of the Hilton–Milner theorem was proved by Frankl [10] and further strengthened by Kupavskii and Zakharov [33].

**Theorem 2.1** ([33]). Let \( n > 2k > 0 \) and \( F \subset \binom{[n]}{k} \) be an intersecting family. If \( \gamma(F) \geq \binom{n - u - 1}{k - 1} \) for some real \( 3 \leq u \leq k \), then

\[
|F| \leq \binom{n - 1}{k - 1} + \binom{n - u - 1}{k - 1} - \binom{n - u - 1}{k - 1}.
\]

The bound from Theorem 2.1 is sharp for integer \( u \), as witnessed by \( H_u \).

In [32], the author derived the following handy corollary of some general results in the spirit of Theorem 2.1.

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1 By analogy with the famous degree criterion for the existence of a Hamilton cycle in a graph.
Corollary 2.2. Let \( n > 2k \geq 6 \). For any intersecting family \( \mathcal{F} \subset \binom{[n]}{k} \), \( \gamma(\mathcal{F}) \leq \binom{n-4}{k-3} \), we have \( |\Delta(\mathcal{F})| + \frac{n-k-2}{k-3} \gamma(\mathcal{F}) \leq \binom{n-1}{k-1} \).

If, additionally, \( \mathcal{F} \) is non-trivial, then \( |\Delta(\mathcal{F})| + \frac{n-k-3}{k-3} \gamma(\mathcal{F}) \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + \frac{n-k-3}{k-3} \).

In the proof of Theorem 1.7, we use the following stability theorem, proved by Frankl and the author [16, 17]. Recall that covering number \( \tau(\mathcal{F}) \) is the minimal size of a set \( S \subset [n] \), such that \( S \cap F \neq \emptyset \) for any \( F \in \mathcal{F} \). Among the families \( \mathcal{F} \subset \binom{[n]}{k} \) that satisfy \( \nu(\mathcal{F}) \leq s \), the families with \( \tau(\mathcal{F}) > s \) are exactly the ones that are not isomorphic to a subfamily of \( \mathcal{A}_0(n, k, s) \).

Theorem 2.3 ([16]). Fix integers \( s, k \geq 2 \). Let \( n = (u + s - 1)(k - 1) + s + k, u \geq s + 1 \). Then for any family \( \mathcal{G} \subset \binom{[n]}{k} \) with \( \nu(\mathcal{G}) = s \) and \( \tau(\mathcal{G}) \geq s + 1 \) we have

\[
|\mathcal{G}| \leq \binom{n-1}{k} - \binom{n-s}{k} - \frac{u-s-1}{u} \binom{n-s-k}{k-1}.
\]

2.1. Calculations. In this section we do some of the calculations used in the proofs of Theorems 1.3 and 1.5. Substituting \( u = 3 \) in (2.1), we get that

\[
|\mathcal{F}| \leq \binom{n-1}{k} - \binom{n-4}{k-3} + \binom{n-4}{k-1} = \sum_{i=2}^{4} \binom{n-i}{k-2} + \binom{n-4}{k-3} = \binom{n-2}{k-2} + 2 \binom{n-3}{k-2} = \left(1 + \frac{2(n-k)}{n-2}\right) \binom{n-2}{k-2} = \frac{k(k-1)(3n-2k-2)}{n(n-1)(n-2)} \binom{n}{k}.
\]

We also have

\[
\frac{n-u-1}{n-k-1} \binom{n-u-1}{k-1} = \prod_{i=1}^{u-1} \frac{n-u-i}{i} = \prod_{i=k}^{n-k-1} \frac{n-u-i}{i} = \prod_{i=k}^{n-k-1} \frac{n-u-i}{n-1-i}.
\]

Clearly, in the range \( 3 \leq u \leq k \) the last expression is maximized for \( u = 3 \), and we get the following equality, provided \( n \geq 2k + l \):

\[
\frac{n-u-1}{n-k-1} \binom{n-u-1}{k-1} \leq \prod_{i=k}^{n-k-1} \frac{n-3-i}{n-1-i} = \prod_{i=1}^{k-1} \frac{k-3+i}{n-k-i} \quad \text{for } 3 \leq u \leq k.
\]

We will also use the following formula:

\[
\frac{n-u-1}{n-k-1} \binom{n-u-1}{k-1} = \frac{n-2k+1}{n-t-k} \cdot \frac{n-t-k}{n-1-t-k} \cdot \ldots = \prod_{i=1}^{k} \frac{n-k+1-i}{n-t-i}.
\]

3. Proofs

3.1. Proof of Theorem 1.4. Take an intersecting family \( \mathcal{F} \) with maximum degree \( \Delta \) and diversity \( \gamma \). Then, by definition, \( |\mathcal{F}| = \Delta + \gamma \). W.l.o.g., we suppose that \( \Delta(\mathcal{F}) = d_1(\mathcal{F}) \). The statement is vacuously true for trivial intersecting families, so we may assume that \( \gamma \geq 1 \). We have two cases to distinguish.

Case 1. \( \gamma \leq \binom{n-4}{k-3} \). In this case we use the following proposition.

Proposition 3.1. Fix some \( n, t, k \). If for an intersecting family of \( k \)-sets \( \mathcal{F} \subset 2^{[n]} \) with maximum degree \( \Delta \) and diversity \( \gamma \) we have

\[
\Delta + \frac{k}{k-t} \gamma \leq \binom{n-1}{k-1},
\]
then $\delta_t(F) \leq \binom{n-t-1}{k-t-1}$.

Proof. The sum of $t$-degrees of all $t$-subsets of $[2,n]$ is $\gamma^t(\binom{k}{t}) + \Delta^t(\binom{k-1}{t})$. Therefore, there is a $t$-tuple $T$ of elements in $[2,n]$, such that

$$\delta_t(T) \leq \frac{\gamma^t(\binom{k}{t}) + \Delta^t(\binom{k-1}{t})}{\binom{n-t}{t}} = \frac{(k-1)(n-1)}{t} \gamma + \Delta \leq \frac{(k-1)(n-1)}{t} = \binom{n-t-1}{k-t-1}. \tag{3.2}$$

To prove the theorem in Case 1, it is sufficient to verify (3.1) for all intersecting families. We may apply Corollary 2.12 to $F$ (otherwise, it is not difficult to obtain via direct calculations from (2.1)). We only have to check that

$$\frac{k}{k-t} \leq \frac{n-k-2}{k-2} \iff \frac{t}{k-t} \leq \frac{s}{k-2}, \tag{3.3}$$

where $n = 2k + s$ and $s \geq 1$. We see that if $t = 1$, then (3.3) holds for any $s \geq 1$. If $t > 1$, then we must have

$$s \geq \frac{k-2}{k-t} \iff s \geq \frac{t}{1-t/k}. \tag{3.4}$$

Case 2. $\gamma \geq \binom{n-4}{k-3}$. We use the following bound on $\delta_t(F)$:

$$\delta_t(F) \leq \binom{\frac{k}{t}}{t} |F|. \tag{3.5}$$

Thus, it is sufficient for us to check that the following inequality holds:

$$|F| \leq \frac{\binom{k}{t} \binom{n-t-1}{k-t-1}}{\binom{k}{t}} = \frac{k-t}{n-t} \binom{n}{k}. \tag{3.6}$$

Combining (2.1) and (2.3), we get that (3.5) is satisfied if

$$\frac{k(k-1)(3n-2k-2)}{n(n-1)(n-2)} \leq \frac{k-t}{n-t}. \tag{3.7}$$

If $t = 1$, then, knowing that $n = k$ is a root of the expression above, it simplifies to $\frac{k(3n-2k-2)}{n(n-2)} \leq 1$, which holds for any $n \geq 2k + 2$. If $t > 1$, then it simplifies to the following quadratic inequality in $n$: $(k-t)n^2 - (2k^2 + (k-3)t)n + 2(k^2 - t) \geq 0$, which is definitely valid if

$$n \geq \frac{2k^2 + kt}{k-t} = 2k + \frac{3t}{1-t/k}. \tag{3.8}$$

3.2 Proof of Theorem 1.5. The strategy of the proof is very similar to that of Theorem 1.4. Fix a non-trivial intersecting family $F$ with maximum degree $\Delta$ and diversity $\gamma \geq 2$ (if $\gamma \leq 1$ then $F$ is a subfamily of either an Erős–Ko–Rado or a Hilton–Milner family). W.l.o.g., suppose that $d_1(F) = \Delta(F)$ and that $F$ contains the set $[2,k+1]$. Then any other set containing 1 must intersect $U$. We compare $F$ with the Hilton-Milner family $H_k$. We consider cases depending on $\gamma$. The case analysis, however, is more complicated, as compared to the previous case. Notably, we get a new non-trivial Case 1.

Case 1. $1 < \gamma < \binom{n-k+1}{t+2}$. We have $\gamma \geq 2$, and thus there is $U' \in F$, such that $1 \notin U$ and $U' \neq [2,k+1]$. Then it is easy to see that $^2 d_1(F) \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} - \binom{n-k-2}{k-2}$. To put
it differently, we have $d_1(H_k) - d_1(F) \geq \binom{n-k-2}{k-2}$. Let us denote by $\alpha$ the number of sets containing 1 and intersecting $[k+2,n]$ in at most $t-1$ elements. Then

\begin{equation}
\delta_t(H_k) - \delta_t(F) \geq \frac{\binom{n-k-2}{k-2} - \alpha - \binom{k}{t} \gamma}{\binom{n-k-1}{t}}.
\end{equation}

The right hand side actually provides a lower bound for the difference of the average degree in $[k+2,n]$ in $H_k$ and $F$. Since the average $t$-degree on $[k+2,n]$ in $H_k$ is equal to the minimum $t$-degree, the right hand side is also a lower bound for the difference of the minimum $t$-degree.

Let us show that the right hand side is a lower bound for the difference in the average $t$-degree on $[k+2,n]$. Indeed, the first two terms in the denominator on the right hand side gives a lower bound on the average loss in the $t$-degree of $t$-sets in $[k+2,n]$ due to the missing sets that contain 1 and intersect $[k+2,n]$ in at least $t$ elements. Each of them contribute at least 1 to the $t$-degree of at least one $t$-set in $[k+2,n]$. At the same time, each set contributing to $\gamma$ can contribute at most $\binom{k}{t}/\binom{n-k-1}{t}$ to the average $t$-degree on $[k+2,n]$, which explains the third term.

We have

\[\alpha = \sum_{i=k-t}^{k-1} \binom{k}{i} \binom{n-k-1}{k-i-1}.
\]

We have $\binom{n-k-1}{t-j} > 2\binom{n-k-1}{t-j-1}$ for any $j \geq 0$, since $n > 2k \geq 6t$. Therefore,

\[\alpha \leq \binom{k}{k-t} \binom{n-k-1}{t} = \binom{k}{t} \binom{n-k-1}{t}.
\]

To show that the RHS in (3.6) is always nonnegative and thus to conclude the proof in Case 1, it is sufficient to show the following inequality:

\begin{equation}
\frac{n-k-2}{k-2} \geq \binom{k}{t} \left( \binom{n-k-1}{t} + \binom{n-k+t+1}{t+2} \right).
\end{equation}

Note that we used the assumption $\gamma < \frac{(n-k+t+1)}{t+2}$. In what follows, we verify (3.7).

If $t = 1$, $n \geq 2k + 5$, then in the worst case for (3.7) is $n = 2k + 5$. This reduces to

\[\binom{k+3}{5} \geq k\left( k+4 + \binom{k+6}{3} \right),
\]

which holds for $k \geq 28$.

If $1 < t \leq \frac{k}{4} - 2$, $n \geq 2k + 14t$, then the right hand side of (3.7) is at most $\binom{k}{t} \binom{n-k+t+2}{t+2}$, and we have

\[\binom{k}{t} \binom{n-k+t+2}{t+2} = \frac{k!(n-k+t+2)!}{(k-t)!t!(t+2)!(n-k)!} \leq \frac{k!(n-k-2)!}{(n-k-t-4)!} \cdot \frac{2t+2}{(k-t)!} \cdot \frac{2^{t+4}}{t}
\]

\[\leq \left( \frac{5}{4} \right)^{t+4} \frac{(n-k-2)!}{(2t+2)!(n-k-2t-4)!} . 2^{2t}
\]

\begin{equation}
(3.8)
\end{equation}

\[\leq 5^{t+1} \binom{n-k-2}{2t+2}.
\]

\[\text{If } n \geq 2k + 6, \text{ then (3.7) holds already for } k \geq 15, \text{ and if } n \geq 2k + 8, \text{ then it holds for } k \geq 10.
\]
On the other hand, in the same assumptions, we have
\[
{n - k - 2 \choose 2t + 2} \leq \left( \frac{4t + 4}{n - k - 4t - 6} \right)^{2t+2} \left( \frac{n - k - 2}{4t + 4} \right)^{2t+2} \leq \left( \frac{2}{5} \right)^{2t+2} \left( \frac{n - k - 2}{2t + 2} \right)^{2t+2} 
\]
(3.9)

Comparing (3.8) and (3.9), we conclude that (3.7) holds.

**Case 2.** \((\frac{n-k+i+1}{t+2}) \leq \gamma \leq (\frac{n-k}{k-3})\). We can get the following analogue of Statement 3.1.

**Statement 1.** If a family \(\mathcal{F} \subset \binom{n}{k}\) with maximum degree \(\Delta\) and diversity \(\gamma\) satisfies
\[
\Delta + \frac{k}{k-t} \gamma \leq \left(1 - \prod_{i=1}^{k} \frac{n-k+1-i}{n-t-i} \right) \binom{n-1}{k-1},
\]
then \(\delta_t(\mathcal{F}) \leq \binom{n-t-1}{k-t} - \binom{n-t-k-1}{k-1}\).

**Proof.** We may repeat the calculation in (3.2), and get that
\[
\delta_t(\mathcal{F}) \leq \left(1 - \prod_{i=1}^{k} \frac{n-k+1-i}{n-t-i} \right) \binom{n-t-1}{k-t} - \binom{n-t-k-1}{k-1}.
\]

We apply (2.1). The bounds on \(\gamma\) defining Case 2 correspond to the range \(3 \leq u \leq k-t - 2\) in (2.1). Then (3.10) is implied by the following inequality.
\[
\binom{n-u-1}{k-1} - \frac{k}{k-t} \binom{n-u-1}{k-1} - \prod_{i=1}^{k} \frac{n-k+1-i}{n-t-i} \binom{n-1}{k-1} \geq 0.
\]
(3.11)

We have
\[
\binom{n-u-1}{k-1} - \frac{k}{k-t} \binom{n-u-1}{k-1} \geq (1 - \frac{k(k-1)(k-2)}{(k-t)(n-k-1)(n-k-2)} \prod_{i=1}^{k-1} \frac{n-u-i}{n-i} \binom{n-1}{k-1}.
\]
The last expression is minimized when \(u = k-t - 2\). Comparing the product above with the product in (3.11), we get
\[
\prod_{i=1}^{k} \frac{n-k+1-i}{n-t-i} \leq \prod_{i=1}^{k-1} \frac{n-k+1-i}{n-t-i} \prod_{i=1}^{k-2} \frac{n-k+1-i}{n-t-i} \leq \prod_{i=1}^{k-2} \frac{n-k+1-i}{n-t-i} = 1 - \frac{k}{n-k+1}.
\]
Therefore, to prove (3.11), it is sufficient for us to show that
\[
1 - \frac{k(k-1)(k-2)}{(k-t)(n-k-1)(n-k-2)} \geq 1 - \frac{k}{n-k+1}.
\]
(3.12)

For the fraction on the left hand side, we use the following property: if we add 1 to one of the multiples in the numerator and 1 to one of the multiples in the denominator, then the fraction will only increase, and the expression in the left hand side will decrease. If \(t = 1\), then the LHS of (3.12) is
\[
1 - \frac{k(k-2)}{(n-k-1)(n-k-2)} \geq 1 - \frac{k^2}{(n-k+1)(n-k-2)} > 1 - \frac{k}{n-k+1},
\]
where the last inequality holds so long as \(n \geq 2k+2\). Therefore, (3.11) is satisfied for any \(k\) and \(n \geq 2k+2\).
If $1 < t \leq \frac{k}{4}$ and $n \geq 2k + 4t$, then $(k - t)(n - k - 1) \geq (k - t)(k + 4t - 1) = k^2 + 3kt - 4t^2 - k + t > k^2$, and, therefore, the LHS of (3.12) is at least
\[
1 - \frac{k^3}{(k - t)(n - k - 1)(n - k + 1)} > 1 - \frac{k}{n - k + 1},
\]
and (3.11) is satisfied again.

**Case 3.** $\gamma \geq \frac{(n - 4)}{k - 3}$. We again use the bound (3.3). Thus, we have to verify that
\[
|\mathcal{F}| \leq \binom{n}{t} \left( \begin{array}{c} n-t-1 \\ \frac{k}{2} \end{array} \right) \leq \frac{k^t t^t}{(n - t)!} \leq k^t \frac{(n - k + 1 - i) (n - k + 1)}{n - t - i} \binom{n}{k}.
\]

Using (2.3), the inequality (3.13) is implied by
\[
\frac{k - t}{n - t} \left( 1 - \prod_{i=1}^{k} \frac{n - k + 1 - i}{n - t - i} \right) \geq \frac{k(k - 1) (3n - 2k - 2)}{n(n - 1)(n - 2)}.
\]

In what follows, we verify (3.13). If $t = 1$, then it simplifies to
\[
\left( 1 - \prod_{i=1}^{k} \frac{n - k + 1 - i}{n - 1 - i} \right) \geq \frac{k(3n - 2k - 2)}{n(n - 2)} \iff \frac{(n - k) (n - 2k - 2)}{n(n - 2)} \geq \prod_{i=1}^{k} \frac{n - k + 1 - i}{n - 1 - i}.
\]

This is equivalent to
\[
\frac{n - 2k - 2}{n} \geq \prod_{i=2}^{k} \frac{n - k + 1 - i}{n - 1 - i}.
\]
The right hand side is at most
\[
\frac{(n - 2k + 1)(n - 2k + 2)}{(n - 3)(n - 4)} \leq \frac{(n - 2k + 4)(n - 2k + 2)}{n(n - 4)}.
\]

Therefore (3.13) follows from
\[
\frac{n - 2k - 2}{n} \geq \frac{(n - 2k + 4)(n - 2k + 2)}{n(n - 4)} \iff (n - 2k - 2)(n - 4) \geq (n - 2k + 4)(n - 2k + 2),
\]
which holds for any $n \geq 2k + 4$ and $k \geq 12$.

If $1 < t \leq \frac{k}{4} - 2$, then $\frac{k+4t}{k} \geq \frac{k}{k-1}$. We use this in the second inequality below:
\[
1 - \frac{n - t}{k - t} \cdot \frac{k(k - 1) (3n - 2k - 2)}{n(n - 1)(n - 2)} \geq 1 - \frac{k^2 (3n - 2k)}{(k - t)n^2} \geq \frac{(n - k)(n - 2k - 6t)}{n^2}.
\]

On the other hand,
\[
\prod_{i=1}^{k} \frac{n - k + 1 - i}{n - t - i} = \prod_{i=1}^{k-1} \frac{n - t - k - i}{n - t - i} \leq \left( \frac{n - k}{n} \right)^{k-1} \leq \left( \frac{n - k}{n} \right)^{3k/4 + 1}.
\]

Therefore, combining these calculations, the inequality (3.14) would follow from the inequality
\[
1 - \frac{2k + 6t}{n} \geq \left( 1 - \frac{k}{n} \right)^{3k/4}.
\]

If $2k + 14t \leq n \leq 7k$, then the right hand side of the inequality above is at most $e^{-\frac{3k^2}{4n}} < e^{-\frac{k}{16}}$, while the left hand side is at least \( \frac{8t}{2k+14t} > \frac{16}{2k+28} \). It is easy to see that, say, for $k \geq 15$, we have $\frac{16}{2k+28} > e^{-\frac{k}{16}}$. 
If \( n > 7k \) and \( k \geq 10 \), then
\[
\left(1 - \frac{k}{n}\right)^{3k/4} < \left(1 - \frac{k}{n}\right)^\frac{7}{4} < 1 - \frac{7k}{n} + \frac{21k^2}{n^2} \leq 1 - \frac{4k}{n} < 1 - \frac{2k}{n}.
\]
Therefore, the inequality (3.16) is verified.

To conclude, we remark that the only conditions on \( k \) that we used for \( t \geq 2 \) were \( k \geq 4t + 8 \) and \( k \geq 10 \). The later one is implied by the former one.

3.3. **Proof of Theorem 1.7** Fix \( t \geq 1 \) and take a family \( \mathcal{F} \) satisfying the requirements of the theorem. If \( \mathcal{F} \) is isomorphic to a subfamily of \( \mathcal{A}_0(n, k, s) \) then we are done. Otherwise, \( \tau(\mathcal{F}) \geq s + 1 \) and thus \( |\mathcal{F}| \) satisfies (2.2). By simple double counting as in (3.4), we have
\[
\delta_t(\mathcal{F}) \leq \binom{k}{n} \left[ \binom{n}{k} - \binom{n-s}{k} - \frac{u-s-1}{u} \binom{n-s-k}{k-1} \right].
\]
Note that
\[
\delta_t(\mathcal{A}_0(n, k, s)) = \binom{n-t}{k-t} - \binom{n-s-t}{k-t} = \frac{k}{n} \left[ \binom{n}{k} - \binom{n-s}{k} \right].
\]
Therefore,
\[
\delta_t(\mathcal{A}_0(n, k, s)) - \delta_t(\mathcal{F}) \geq \frac{k}{n} \left[ \frac{u-s-1}{u} \binom{n-s-k}{k-1} - \left[ \frac{k}{n} - \frac{k}{n-s} \right] \binom{n-s}{k} \right] =: (\ast).
\]
We have \( \prod_{i=0}^{t-1} \frac{n-i}{n-s-i} - 1 \leq (1 + \frac{s}{n-s})^t - 1 \). It is not difficult to verify that for \( \theta < \frac{1}{2m} \) one has \( (1 + \theta)^m \leq 1 + 2m\theta \). Therefore, assuming that
\[
(3.17) \quad n \geq s + t + 2st,
\]
we have
\[
(3.18) \quad \prod_{i=0}^{t-1} \frac{n-i}{n-s-i} - 1 \leq \frac{2ts}{n-s-t}.
\]
On the other hand, we have \( (1 - \theta)^m \geq 1 - m\theta \) and \( t \leq k - 1 \), and thus
\[
\frac{k}{n-s-t} \left( 1 - \frac{(k-1)k}{n-s-k} \right) > \frac{k}{2(n-s-t)},
\]
provided
\[
(3.19) \quad n \geq s + k + 2k(k-1).
\]
We conclude that, provided that (3.17) and (3.19) hold, we get
\[
(\ast) > \frac{k}{n} \left[ \frac{u-s-1}{u} \frac{k}{2(n-s-t)} - \frac{2ts}{n-s-t} \right] \binom{n-s}{k},
\]
which is nonnegative provided \( k \geq 4ts \frac{u}{n-s-1} \). This inequality holds for \( k \geq 5ts \) and \( u \geq 9s \). The last inequality is satisfied for \( n \geq 2k^2 \), since then \( n \geq 2k^2 \geq (9s + s)k \geq (9s + s - 1)(k - 1) + s + k \). We note that, with this choice of \( n \) and \( k \), both (3.17) and (3.19) hold.

For \( t = 1 \) one may improve (3.18) to \( \frac{u}{n-s-1} \leq \frac{u}{n-s} \) and the condition on \( k \) may be relaxed to \( k \geq 3s \). The equality part of the statement follows easily from the fact that strict inequality is obtained in the case when \( \tau(\mathcal{F}) \geq s + 1 \).
4. Conclusion

In this paper we explored several question concerning degrees and $t$-degrees of intersecting families and families with small matchings. Some of these questions remain only partially resolved, and it would be highly desirable to settle them.

The first problem is to give a purely combinatorial proof of Theorem 1.2 for $n = 2k + 1$. Although not fully optimized, our approach for the degree version of the Hilton–Milner family should surely fail for $n \leq 2k + 2$. Thus, we ask the following question.

**Problem 1.** Is there an example for $n = 2k + 1$ ($n = 2k + ct, c$ is a small constant), such that there exists a non-trivial intersecting family $F$ with minimal $1$-degree ($t$-degree) higher than that of the Hilton-Milner family?

One reason to believe that the answer to this question may be positive is that the degrees of elements in the Hilton-Milner family are irregular, even if we exclude the element of the highest degree out of consideration.

The main difficulty here are to understand the structure families with large diversity for $n$ very close to $2k$.

Finally, the following question concerning cross-intersecting families seem interesting for us.

**Problem 2.** Consider two cross-intersecting families $A, B \subseteq \binom{[n]}{k}$ that are disjoint. Is it true that

$$\min\{|A|, |B|\} \leq \frac{1}{2} \binom{n-1}{k-1}?$$

In between the first and the current version of the manuscript, Hao Huang published a manuscript [25], in which he addressed Problem 2. In particular, he showed that the answer is positive as long as $n > 2k^2$, and is negative as long as $n < ck^2$ for some $c > 0$. Similar results were obtained by Frankl and the author [19].

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