ON GRADINGS MODULO 2 OF SIMPLE LIE ALGEBRAS IN CHARACTERISTIC 2

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ABSTRACT. The ground field in the text is of characteristic 2. The classification of modulo 2 gradings of simple Lie algebras is vital for the classification of simple finite-dimensional Lie superalgebras: with each grading, a simple Lie superalgebra is associated, see arXiv:1407.1695. No classification of gradings was known for any type of simple Lie algebras, bar restricted Zassenhaus algebras (a.k.a. Witt algebras, i.e., Lie algebras of vector fields with truncated polynomials as coefficients) on not less than 3 indeterminates.

Here (under a natural assumption about their algebras of derivations) we completely describe gradings modulo 2 for several series of simple (relatives of the) Lie algebras: of special linear series (except for \( \text{psl}(4) \) for which a conjecture is given), two inequivalent orthogonal, and projectivizations of their derived Lie algebras. The classification of gradings is new but all of the corresponding superizations are known.

For the Zassenhaus algebras on one indeterminate of height \( n > 1 \), there is an \((n - 2)\)-parametric family of modulo 2 gradings; all but one of the corresponding simple Lie superalgebras are new. Our classification yields non-triviality of a deformation of a simple \((3|2)\)-dimensional Lie superalgebra (new result).

1. Introduction

1.1. Basic definitions. Hereafter \( K \) is an algebraically closed field of characteristic \( p = 2 \), unless otherwise specified; all algebras are finite-dimensional; for a review of simple vectorial Lie (super)algebras over \( K \) and basic background, see arXiv:1407.1695.

1.1.1. Lie superalgebras in characteristic 2. A Lie superalgebra is a superspace \( g = g_0 \oplus g_1 \) such that the even part \( g_0 \) is a Lie algebra, the odd part \( g_1 \) is a \( g_0 \)-module, and on \( g_1 \), a squaring (roughly speaking, the halved bracket) is defined as a map

\[
x \mapsto x^2 \quad \text{such that} \quad (ax)^2 = a^2 x^2 \quad \text{for any} \quad x \in g_1 \quad \text{and} \quad a \in K, \quad \text{and}
\]

\[(x + y)^2 - x^2 - y^2 \quad \text{is a bilinear form on} \quad g_1 \quad \text{with values in} \quad g_0.
\]

Then the bracket of odd elements is defined to be

\[
[x, y] := (x + y)^2 - x^2 - y^2.
\]

The Jacobi identity involving odd elements takes the following form:

\[
[x^2, y] = [x, [x, y]] \quad \text{for any} \quad x \in g_1, y \in g_0, \\
[x^2, x] = 0 \quad \text{for any} \quad x \in g_1.
\]

1.1.2. Divided powers. There are two natural integer bases of the commutative algebra \( \mathbb{C}[x] \) of polynomials in \( m \) indeterminates \( x = (x_1, \ldots, x_m) \): the monomial one and the basis of

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\]

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divided powers constructed as follows. For any multi-index \( \underline{r} = (r_1, \ldots, r_m) \), where \( r_1, \ldots, r_m \) are non-negative integers, we set
\[
u_i^{(r_i)} := \frac{x_i^{r_i}}{r_i!} \quad \text{and} \quad u^{(\underline{r})} := \prod_{1 \leq i \leq m} \nu_i^{(r_i)}.
\]
These \( u^{(\underline{r})} \) form an integer basis of \( \mathbb{C}[x] \). Their multiplication relations are
\[
u^{(\underline{r})} \cdot \nu^{(\underline{s})} = \left( \underline{r} + \underline{s} \right) \nu^{(\underline{r} + \underline{s})}, \quad \text{where} \quad \left( \underline{r} + \underline{s} \right) = \prod_{1 \leq i \leq m} \left( r_i + s_i \right).
\]
For an arbitrary field \( \mathbb{K} \) of characteristic \( p > 0 \), we can consider the commutative algebra \( \mathbb{K}[u] \) spanned by the elements \( u^{(\underline{r})} \) with multiplication relations written above. For any \( m \)-tuple \( \underline{N} = (N_1, \ldots, N_m) \), where \( N_i \) are either positive integers or infinity, denote (we set \( p^\infty := \infty \))
\[
\mathcal{O}(m; \underline{N}) := \mathbb{K}[u] = \text{Span}_\mathbb{K} \left( u^{(\underline{r})} \mid r_i < p^{N_i} \right).
\]
The algebra \( \mathbb{K}[u] \) and its subalgebras \( \mathbb{K}[u; \underline{N}] \) are called the \textit{algebras of divided powers}; they are analogs of the polynomial algebra. Let \( \underline{1} := (1, \ldots, 1) \) denote the shearing vector \( \underline{N} \) with the smallest values of heights \( N_i \) of the indeterminates.

Since any derivation \( D \) of a given algebra is determined by the values of \( D \) on the generators, we see that \( \text{det}(\mathcal{O}(m; \underline{N})) \) has more than \( m \) functional parameters if \( N_i \neq 1 \) for at least one \( i \). Define \textit{distinguished partial derivatives} \( \partial_i := \partial_{u_i} \) by setting
\[
\partial_i(u_j^{(k)}) = \delta_{ij} u_j^{(k-1)} \quad \text{for any} \ k.
\]
The Lie algebra of all derivations \( \text{det}(\mathcal{O}(m; \underline{N})) \) turns out to be not so interesting (it is not transitive, and hence can not be simple) as the \textit{general vectorial Lie algebra of distinguished derivations} \( \mathfrak{vect}(m; \underline{N}) = \text{Span}_\mathbb{K} \left( f \partial_k \mid f \in \mathcal{O}(m; \underline{N}), \ k = 1, \ldots, m \right) \).

1.1.3. \textbf{p-structure}. For every \( x \in \mathfrak{g} \), the operator \( (\text{ad}_x)^p \) is a derivation of \( \mathfrak{g} \). If this derivation is an inner one for every \( x \in \mathfrak{g} \), then the Lie algebra \( \mathfrak{g} \) is said to be \textit{restricted} or having a \textit{p-structure}. More specifically, a \textit{p-structure} on \( \mathfrak{g} \) is a map \( [p] : \mathfrak{g} \to \mathfrak{g} \), \( x \mapsto x^{[p]} \) such that
\[
\begin{align*}
[x^{[p]}, y] &= (\text{ad}_x)^p(y) \quad \text{for any} \ x, y \in \mathfrak{g}, \\
(ax)^{[p]} &= a^p x^{[p]} \quad \text{for any} \ a \in \mathbb{K}, \ x \in \mathfrak{g}; \\
(x + y)^{[p]} &= x^{[p]} + y^{[p]} + \sum_{1 \leq i \leq p-1} s_i(x, y) \quad \text{for any} \ x, y \in \mathfrak{g},
\end{align*}
\]
where \( s_i(x, y) \) is the coefficient of \( \lambda^{i-1} \) in \((\text{ad}_{\lambda x + y})^{p-1}(x)\).

1.1.4. \textbf{Remarks}. 1) If the Lie algebra \( \mathfrak{g} \) is without center, then the last two conditions of (2) follow from the first one. There might be several \( p \)-structures on one Lie algebra; all of them are equal modulo center. Hence, on any simple Lie algebra, there is at most one \( p \)-structure.

2) For every \( x \in \mathfrak{g} \), the operator \( (\text{ad}_x)^p \) is a derivation of \( \mathfrak{g} \). If this derivation is an inner one for every \( x \in \mathfrak{g} \), then the following condition is sufficient for a Lie algebra \( \mathfrak{g} \) to possess a \( p \)-structure: for a basis \( \{g_i\}_{i \in I} \) of \( \mathfrak{g} \), there exist elements \( g_i^{[p]} \) such that
\[
[g_i^{[p]}, y] = (\text{ad}_{g_i})^p(y) \quad \text{for any} \ y \in \mathfrak{g}.
\]

\footnote{For \( p > 0 \), the Lie algebra \( \mathfrak{vect}(m; \underline{N}) \) is called \textit{Zassenhaus algebra} if \( \underline{N} \neq \underline{1} \); it is usually denoted \( W(m; \underline{N}) \) for any \( m \) and \( \underline{N} \); if \( \underline{N} = \underline{1} \), it is called \textit{Witt algebra}. Zassenhaus algebras are simple if \( m > 1 \); there is no special name for the simple derived \( \mathfrak{vect}^{(1)}(1; \underline{N}) \); see a discussion in [GZ]. The Lie algebra of \textit{divergence-free}, or \textit{“special”}, vector fields is denoted \( \mathfrak{svect}(m; \underline{N}) \), usually abbreviated to \( S(m; \underline{N}) \).}
1.2. \textbf{Problem.} We consider the following problem, whose solution is well-known for $p = 0$; for a clear exposition of the solution, see the book [H] around p.500:

(3) \hspace{1cm} \text{For any finitely generated commutative group $G$, classify $G$-gradings of simple finite-dimensional Lie algebras over $\mathbb{K}$.}

\textbf{Lie algebras for which a solution of Problem (3) is known.} Although we are interested in a particular case of Problem (3), let us briefly review the known general results.

For $p \neq 2$, [Ko] is a very clear review of the cases where the solution of Problem (3) has been found; for further details, see [BK] [KPS]; for examples of applications of certain $G$-gradings, see [Kos].

For $p = 2$, the result of [BK] is as follows: all $G$-gradings of the Witt algebra $W(m; 1)$ (it is $\text{vecl}(m; 1)$ in our notation) for $m \geq 3$ are given by $G$-gradings of the corresponding algebra of divided powers $\mathcal{O}(m; 1)$ due to an isomorphism of their automorphism group schemes [Sk01]. The classification of such gradings is given. Any $G$-grading of $\mathcal{O}(m; 1)$ is equivalent, up to an algebra automorphism, to one that can be described as follows. For a given $s$ such that $0 \leq s \leq m$, and $a_1, \ldots, a_m \in G$, set

(4) \hspace{1cm} \mathcal{O}_g = \text{Span} \left\{ (1 + x_1)^{j_1} \ldots (1 + x_s)^{j_s} x_{s+1}^{j_{s+1}} \ldots x_m^{j_m} \mid j_i = 0 \text{ or } 1, \sum_{1 \leq i \leq m} j_i a_i = g \right\},

where $a_1, \ldots, a_m \in G$ are the respective degrees of the indeterminates generating $\mathcal{O}(m; 1)$, i.e.,

$1 + x_1, \ldots, 1 + x_s, x_{s+1}, \ldots, x_m$.

For $p > 0$ and the \textbf{restricted} Lie algebras considered in [BK] (for $p = 2$: Witt algebras; for $p > 2$: Witt algebras, simple relatives of Hamiltonian algebras and algebras of divergence-free vector fields), the only possible $\mathbb{Z}/2$-gradings are generated by those described in eq. (4). For \textbf{non-restricted} Lie algebras, we know several examples of $\mathbb{Z}/2$-gradings of Kaplansky algebras, see [BLLS].

1.3. \textbf{Problem (3)} for $G = \mathbb{Z}/2$ and $p = 2$: an application of the solution. In [BLLS1], there are offered two methods for constructing a simple finite-dimensional Lie superalgebra from each simple finite-dimensional Lie algebra \textbf{over a field of characteristic 2}; it is proved that every simple finite-dimensional Lie superalgebra can be obtained by one of these two methods (queerification and “method 2”). The “method 2” depends on the $\mathbb{Z}/2$-gradings of the simple Lie algebra which is being superized. Let us recall the method:

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a simple Lie algebra with a $\mathbb{Z}/2$-grading $\text{gr}$. Let the 1-step \textit{restricted closure} of $\mathfrak{g}$ associated with the grading $\text{gr}$ be

(5) \hspace{1cm} \mathfrak{g}^{<1>} := \text{the minimal Lie subalgebra of the restricted closure } \bar{\mathfrak{g}} \text{ containing } \mathfrak{g}

and all the elements $x^{[2]}$, where $x \in \mathfrak{g}_1$.

Clearly, there is a single way to extend the grading $\text{gr}$ from $\mathfrak{g}$ to $\mathfrak{g}^{<1>}$; we assume this extension performed. On the space of $\mathfrak{g}^{<1>}$, define the structure of a Lie superalgebra denoted (in what follows, we often omit indicating the grading $\text{gr}$ since it enters the definition of $\mathfrak{g}^{<1>}$)

(6) \hspace{1cm} S(\mathfrak{g}^{<1>}, \text{gr}) \text{ by setting } x^2 := x^{[2]} \text{ for any } x \in \mathfrak{g}_1,

and retaining the bracket of any even elements with any other element. The Lie superalgebra $S(\mathfrak{g}^{<1>})$ is simple, see [BLLS1]. Our strategic goal is classification of simple Lie (super)algebras, so we formulate our description of $\mathbb{Z}/2$-gradings gr of $\mathfrak{g}$ in terms of superizations $S(\mathfrak{g}^{<1>})$ whenever we can.
1.4. How we seek \( \mathbb{Z}/2 \)-gradings. Let \( \mathfrak{g} \) be a Lie algebra, and \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) its \( \mathbb{Z}/2 \)-grading. For an arbitrary \( x \in \mathfrak{g} \), we denote by \( x_0 \) its even part and by \( x_1 \) its odd part. We have
\[
[x, y] = [x_0 + x_1, y_0 + y_1] = [x_0, y_0] + [x_1, y_1] + [x_0, y_1] + [x_1, y_0]
\]
for any \( x, y \in \mathfrak{g} \).
This implies
\[
([x, y])_0 = [x_0, y_0] + [x_1, y_1] \quad \text{and} \quad ([x, y])_1 = [x_0, y_1] + [x_1, y_0].
\]
Finally, let \( U \in \text{End}(\mathfrak{g}) \), where \( \mathfrak{g} \) is considered as a vector space, be the projection on the odd part, i.e., \( Ux = x_1 \) and \((I - U)x = x_0\), where \( I \) is the identity operator. A linear operator \( U \) is such a projection on the odd part of \( \mathfrak{g} \) in some \( \mathbb{Z}/2 \)-grading if and only if it satisfies the following conditions for all \( x, y \in \mathfrak{g} \):
\[
U^2 = U, \tag{7a}
\]
\[
U[x, y] = [Ux, (I - U)y] + [(I - U)x, Uy] = -2[Ux, Uy] + [Ux, y] + [x, Uy]. \tag{7b}
\]
Having fixed a basis in \( \mathfrak{g} \), we express these conditions in terms of structure constants \( c_{ij}^k \) as (summation over repeated indices is assumed)
\[
U_k^{ij} U_k^l = U_k^{ij}; \tag{8a}
\]
\[
c_{ij}^k U_k^l = c_{ij}^k U_k^l + c_{ij}^l U_k^l - 2c_{lm}^k U_l^i U_l^j \tag{8b}.
\]
For \( p = 2 \), eq. (8b) becomes linear and easy (for example, for \textit{Mathematica}-based computer package \textit{SuperLie}, see [Gr]) to solve. There remains, however, a problem to be solved:
\[
\text{(9a) condition (8a) is still quadratic;}
\]
\[
\text{(9b) we need equivalence classes of solutions } U \bmod \text{ Aut}(\mathfrak{g}), \text{ not individual operators.}
\]
1.4.1. A comment: gradings and derivations. For a general discussion of relation between gradings and derivations, and interesting examples for \( p = 2 \), see [BLLS]. For \( p = 3 \), an example illustrating the said discussion is the \( \mathbb{Z}/4 \)-grading of the Skryabin algebra \( \mathfrak{b} \), see [GL].
The Lie algebra \( \text{der vect}(m;\underline{N}) \) of all derivations of \( \text{vect}(m;\underline{N}) \) coincides with the \( p \)-envelop of \( \text{vect}(m;\underline{N}) \). Hence, the problem of classification of equivalence classes of \( \mathbb{Z}/p \)-gradings reduces to the classification of equivalence classes of toral elements in \( \text{der vect}(m;\underline{N}) \). The equivalence classes of maximal tori with respect to the group of automorphisms are described in [Kuz, T] and the answer depends on \( N_1 - 2 \) parameters for \( m = 1 \).

1.5. Our results. 1) We classify the \( \mathbb{Z}/2 \)-gradings of the Lie algebras of series \( \mathfrak{sl} \), their simple subquotients, and simple derived or subquotients of both orthogonal series \( \mathfrak{o} \) and \( \mathfrak{o}_I \), (under an assumption of the structure of Lie algebras of their derivations in general case; this assumption is proved for ranks \( \leq 8 \) in [BGLL2] except for \( \mathfrak{psl}(4) \), see subsec. 1.6.1) In all cases considered, these gradings yield the known superizations of the corresponding Lie algebras.

Note that one of the gradings of \( \mathfrak{s}(1;3) \) has no analogs among the \( \mathbb{Z}/2 \)-gradings of the simple 3-dimensional Lie algebras for \( p \neq 2 \). This unusual grading has analogs among \( \mathbb{Z}/2 \)-gradings of the Zassenhaus algebras \( \text{vect}(1;\underline{n}) \).

2) The classification of \( \mathbb{Z}/2 \)-gradings of \( \mathfrak{s}(1;3) \) has one more application: it gives the shortest known proof of the fact that the deform of \( \mathfrak{so}_{10}^{(1)}(1|2) \), one of the two superizations of \( \mathfrak{s}^{(1)}(3) \), found in [BGL1], is a “true” one, not “semitrivial”, see Remark 5.3.3; this fact is new and unexpected (although the deform itself, \( \mathfrak{so}_{10}^{(1)}(1|2) \), is known).

3) We describe \((n-2)\)-parametric collection of \( \mathbb{Z}/2 \)-gradings of \( \text{vect}(1;\underline{n}) \). For \( n = 2 \), these gradings yield three non-isomorphic Lie superalgebras.

For \( n > 2 \), these \( \mathbb{Z}/2 \)-grading yield
a) purely even Lie superalgebra \( \text{vect}(1;\underline{n}|0) \);
b) \((n - 2)\)-parametric family of filtered deforms of \(\mathfrak{t}(1; n - 1|1)\);

c) the filtered deform of \(\mathfrak{q}(\text{vect}(1; n - 1))\).

1.5.1. **Examples where \(\mathbb{Z}/2\)-gradings of “the same algebra” for \(p \neq 2\) and \(p = 2\) differ or where previously unknown \(\mathbb{Z}/2\)-gradings have been found.** These are the most interesting of our results.

1) For any \(p\), there is only one simple Lie algebra of dimension 3: for \(p \neq 2\), it is \(\mathfrak{o}(3) \simeq \mathfrak{sl}(2)\); for \(p = 2\), it is \(\mathfrak{o}^{(1)}(3)\), the derived of \(\mathfrak{o}(3)\).

For \(p \neq 2\), there is only one nontrivial \(\mathbb{Z}/2\)-grading (as always, up to an automorphism), and hence if the superization procedure had been defined for \(p \neq 2\), the superization of \(\mathfrak{sl}(2)\) would have been unique, \(\mathfrak{sl}(1|1)\).

For \(p = 2\), there are 2 inequivalent nontrivial \(\mathbb{Z}/2\)-gradings of \(\mathfrak{o}^{(1)}(3)\), as we will see below. The corresponding non-isomorphic, see [Leb], Lie superalgebras are \(\mathfrak{oo}^{(1)}_{II}(1|2)\) and \(\mathfrak{oo}^{(1)}_{II}(1|2)\).

1.6. **Open questions.** For \(G = \mathbb{Z}/2\), the gradings \(|\) with \(s = 0\) of \(\text{vect}(m; 1)\) for \(m > 2\) yield \(\text{vect}(k; 1|m - k)\), where the vectors \(1\) have \(m\) and \(k\) coordinates, respectively, i.e., known superizations. If \(s \neq 0\) in \(|\) and \(a_i = 1\) for some of the indices \(i \leq s\), then the corresponding indeterminates \(1 + x_i\) are odd; at the moment we are unable to identify the resulting Lie superalgebra.

The list of simple Lie algebras for which Problem \(|\) is open:

- For \(p > 3\), the simple vectorial Lie algebras for the shearing vector \(N \neq 1\), and the deforms (results of deformations) thereof (for their classification and description, see [Sk [Kos, SkH]]);

- For \(p = 3\), the simple vectorial Lie algebras; in particular, exceptional ones, mainly discovered by Skryabin and lucidly described in [GL], and their deforms to be described (for a review, see [BGLII]); the deforms of \(\mathfrak{o}(5)\) and Brown algebras described in [BLW];

- For \(p = 2\), there are many examples of simple Lie algebras of types not existing for \(p \neq 2\), see [SkT1, Eil, BGLL2, BLS1, GZ, BGLLS1, BGLLS2], and BGL, BGL1.

1.6.1. **Two conjectures (D. Leites).**

1) For (the simple derived of) Lie algebras with indecomposable Cartan matrix not considered in this note, i.e., parametric families of Weisfeiler-Kac algebras, all their \(\mathbb{Z}/2\)-gradings correspond, conjecturally, to their known superizations listed in [BGL].

2) Since \(\mathfrak{psl}(4) \simeq \mathfrak{h}^{(1)}_{II}(4; 1) \simeq \text{vect}^{(1)}(3; 1)\), see [ChKu], it is natural to conjecture that all \(\mathbb{Z}/2\)-gradings of \(\mathfrak{psl}(4)\) correspond to superizations of superdimension 8|6, namely

\[
\text{vect}^{(1)}(0|3) \simeq \mathfrak{psl}(3|1), \quad \text{vect}^{(1)}(2; 1|1), \quad \mathfrak{h}^{(1)}_{II}(1; 1|3), \quad \mathfrak{h}^{(1)}_{II}(3; 1|1),
\]

and of superdimension 6|8, namely

\[
\mathfrak{h}^{(1)}_{II}(0|4) \simeq \mathfrak{psl}(2|2), \quad \mathfrak{h}^{(1)}_{II}(2; 1|2), \quad \text{vect}^{(1)}(1; 1|2),
\]

and two more, most interesting, superizations of the same type as described, for any simple vectorial Lie algebra and its standard \(\mathbb{Z}\)-grading, in §6 of [BGLLS2]; the natural \(\mathbb{Z}\)-gradings of these two superizations are supported in degrees \(-2\) to 1, and their superdimensions are 10|8 for \(\mathfrak{h}^{(1)}_{II}(4; 1)\) and 11|6 for \(\text{vect}^{(1)}(3; 1)\).
1.6.2. A technical hypothesis. In §23 and §34 we use the conjectural description of all derivations of Lie algebras \( \mathfrak{sl}(n) \), \( \mathfrak{psl}(2n) \), as well as \( \mathfrak{o}_{1}(2n) \) and \( \mathfrak{o}_{1}(n) \) and their simple relatives, see [BGLL2]. This description is proved if \( \text{rk} \mathfrak{g} \leq 8 \), and sometimes for any \( n \).

2. The \( \mathfrak{sl}(n) \) and \( \mathfrak{psl}(2n) \) for \( n > 2 \): same answer as for \( p \neq 2 \)

Clearly, \( \mathfrak{sl}(0|n) = \mathfrak{sl}(n) \) and \( \mathfrak{psl}(0|2n) = \mathfrak{psl}(2n) \).

2.1. Theorem. All \( \mathbb{Z}/2 \)-gradings of the Lie algebras \( \mathfrak{sl}(n) \) for \( n > 2 \) and \( n \neq 4 \) and \( \mathfrak{psl}(2n) \) for \( n > 2 \) are analogous to those for \( p = 0 \), i.e., correspond to \( \mathfrak{sl}(k|n-k) \) for \( k = 0, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \) and \( \mathfrak{psl}(k|2n-k) \) for \( k = 0, \ldots, n \).

2.1.1. Proof: the \( \mathfrak{sl}(2n+1) \) series. All derivations of \( \mathfrak{sl}(2n+1) \) are inner ones, see [BGLL2]. Then the general solution of the linear equation (7b) is \( U = \text{ad}_A \), where \( A \in \mathfrak{sl}(2n+1) \). The nonlinear equation (7a) reduces then to \( \text{ad}_A = (\text{ad}_A)^2 \). Now recall that on \( \mathfrak{sl}(2n+1) \) there is the 2-structure given by \( A^{[2]} = A^2 \), see [BGL, BLLS1]. Therefore

\[
\text{ad}_A = (\text{ad}_A)^2 = \text{ad}_{A|2} = \text{ad}_{A^2};
\]

and since \( \mathfrak{sl}(2n+1) \) has no center, we have \( A = A^2 \), i.e., \( A \) is a projection. Since its trace is equal to the dimension of the subspace onto which the projection projects (just look at the normal form of the matrix of the projection), it follows that \( A \) satisfies the condition (7a)

\[
\text{a given projection } A \text{ belongs to } \mathfrak{sl} \text{ if and only if it is a projection onto a subspace of even dimension.}
\]

If \( A \in \mathfrak{sl}(2n+1) \) is a projection, then the Lie superalgebra obtained from it by “method 2” of [BLLS1] is isomorphic to

\[
\mathfrak{sl}(\dim \text{Ker } A | \dim \text{Im } A) \cong \mathfrak{sl}(\dim \text{Im } A | \dim \text{Ker } A).
\]

So we see there are \( n + 1 \) equivalence classes of \( \mathbb{Z}/2 \)-gradings of \( \mathfrak{sl}(2n+1) \) and \( n + 1 \) of its nonisomorphic superizations (including the trivial purely even one). We can enumerate them either as

\[
\mathfrak{sl}(2n+1 - 2k|2k) \cong \mathfrak{sl}(2k|2n+1 - 2k), \quad \text{where } k = 0, \ldots, n,
\]

or, more simply, as \( \mathfrak{sl}(k|2n+1 - k) \), where \( k = 0, \ldots, n \). So the answer is the same as for \( p \neq 2 \).

2.1.2. Proof: the \( \mathfrak{sl}(2n) \) series, \( n > 2 \). The Lie algebra \( \text{der } \mathfrak{sl}(2n) \) can be identified, see [BGLL2], with \( \mathfrak{psl}(2n) \) in the sense that for any \( D \in \text{der } \mathfrak{sl}(2n) \), there is \( A_D \in \mathfrak{gl}(2n) \) such that \( D \) coincides with the restriction of \( \text{ad}_{A_D} \) on \( \mathfrak{sl}(2n) \). These elements \( A_D \) are defined modulo center; in particular, one can take \( A_{D^2} \) to be \( (A_D)^2 \). Therefore \( D \) satisfies the condition (7a) if and only if \( (A_D)^2 = A_D + cI_{2n} \) for some \( c \in \mathbb{K} \). Let \( d \in \mathbb{K} \) be a root of the equation \( d^2 = d + c \) and set \( A_D = A_D + dI_{2n} \), then \( (A_D')^2 = A_D' \). So we see that an operator \( U \) on \( \mathfrak{sl}(2n) \) satisfies the conditions (7a) if and only if it can be represented as a restriction of \( \text{ad}_{A_D} \) on \( \mathfrak{sl}(2n) \) for some projection \( A \in \mathfrak{gl}(2n) \), and then the Lie superalgebra obtained from \( U \) by “method 2” is isomorphic to \( \mathfrak{sl}(\dim \text{Ker } A | \dim \text{Im } A) \cong \mathfrak{sl}(\dim \text{Im } A | \dim \text{Ker } A) \). So there are \( n + 1 \) nonisomorphic superizations of \( \mathfrak{sl}(2n) \) (including the trivial purely even one): \( \mathfrak{sl}(k|2n-k) \), where \( k = 0, \ldots, n \). Again, the answer is the same as for \( p \neq 2 \).

2.1.3. Proof: the \( \mathfrak{psl}(2n) \) series, \( n > 2 \). The algebra \( \text{der } \mathfrak{psl}(2n) \) is isomorphic to \( \text{der } \mathfrak{sl}(2n) \), see [BGLL2]. So the arguments from the previous subsection apply. There are \( n + 1 \) nonisomorphic superizations of \( \mathfrak{psl}(2n) \) (including the purely even one), which can be enumerated as \( \mathfrak{psl}(k|2n-k) \), where \( k = 0, \ldots, n \).
2.1.4. **Remark.** A \( \mathbb{Z}/2 \)-grading of \( \mathfrak{gl}(2n+1) \) for \( n > 2 \) can produce a Lie superalgebra which is not isomorphic to any superalgebra of the form \( \mathfrak{gl}(k|2n+1-k) \). More specifically, such a superalgebra would be isomorphic to the direct sum of a Lie superalgebra of the form \( \mathfrak{sl}(k|2n+1-k) \), where \( 0 \leq k \leq n \), and \( 0 \)\(|1\)-dimensional center (i.e., 1-dimensional odd center). But since it does not result in any new *simple* Lie superalgebra, we do not consider gradings of \( \mathfrak{gl} \) here.

3. **The \( \mathfrak{o}^{(1)}(2n+1) \) Series**

Clearly, \( \mathfrak{o}_{B_1B_2}(0)n = \mathfrak{o}_{B_2}(n) \) and \( \mathfrak{o}_{B_1B_2}(n)0 = \mathfrak{o}_{B_1}(n) \).

3.1. **Example:** \( \mathfrak{o}^{(1)}(3) \). Observe the occasional isomorphism \( \mathfrak{o}^{(1)}(3) \simeq \mathfrak{vect}^{(1)}(1;2) \), so this case is considered in Subsec. 6.3. In [BGL], the deforms are described; but until now it was unclear if these deformations are true ones, i.e., the deformed algebra is not isomorphic to the initial algebra, as is the case for semitrivial deformations corresponding to certain integrable cocycles from nontrivial cohomology class, for examples, see [BLLS, Ri]. The results of Subsec. 6.3 prove that one of the deformations found in [BGL] is a true one.

3.2. **Theorem.** For any \( n \geq 1 \), all \( \mathbb{Z}/2 \)-gradings of the Lie algebra \( \mathfrak{o}^{(1)}(2n+1) \) correspond to Lie superalgebras \( \mathfrak{o}^{(1)}_{II}(2n + 1 - 2k|2k) \), where \( 0 \leq k \leq n \) (or, which is the same but looks simpler, \( \mathfrak{o}^{(1)}_{II}(k|2n + 1 - k) \), where \( 0 \leq k \leq n \)), and \( \mathfrak{o}^{(1)}_{III}(2n + 1 - 2k|2k) \), where \( 1 \leq k \leq n \).

**Proof.** The algebra of derivations of \( \mathfrak{o}^{(1)}(2n + 1) \) is \( \mathfrak{o}(2n+1)/\mathfrak{c} \), the quotient of \( \mathfrak{o}(2n+1) \) modulo center, or the subalgebra of traceless elements of \( \mathfrak{o}(2n+1) \), see [BGLL2].

Again, an element \( A \in \mathfrak{der} \mathfrak{o}^{(1)}(2n+1) \) describes a \( \mathbb{Z}/2 \)-grading of \( \mathfrak{o}^{(1)}(2n+1) \) if and only if \( A^2 = A \), i.e., \( A \) is a projection.

By definition, an operator \( A \in \mathfrak{gl}(V) \) belongs to \( \mathfrak{o}_B(V) \), where \( B \) is a nondegenerate symmetric bilinear form, if and only if

\[
B(Ax, y) + B(x, Ay) = 0 \quad \text{for all} \quad x, y \in V.
\]

In what follows, we assume that \( A \) is a projection. It is easy to check that \( (11) \) is automatically satisfied if \( x, y \in \text{Im} A \) or if \( x, y \in \text{Ker} A \). If \( x \in \text{Im} A \) and \( y \in \text{Ker} A \), then \( (11) \) is equivalent to \( B(x, y) = 0 \). So \( A \in \mathfrak{o}_B(V) \) if and only if \( \text{Im} A \) and \( \text{Ker} A \) are orthogonal with respect to \( B \). For \( A \) to be traceless, \( \text{dim} \text{Im} A \) must be even, see condition \( (10) \).

Let \( A \) be a projection belonging to \( \mathfrak{der} \mathfrak{o}^{(1)}_{B}(2n+1) \) with \( \text{dim} \text{Im} A = 2k > 0 \). Denote restrictions of \( B \) to \( \text{Im} A \) and \( \text{Ker} A \) by \( B_{\text{Im} A} \) and \( B_{\text{Ker} A} \), respectively.

Since \( B \) is nondegenerate, \( \text{Im} A \perp \text{Ker} A = V \) and \( \text{Im} A \perp \text{Ker} A \) with respect to \( B \), it follows that these restrictions are nondegenerate.

Since \( \text{dim} \text{Ker} A = 2n + 1 - 2k \) is odd, \( B_{\text{Ker} A} \) is equivalent to \( I_{2n+1-2k} \). As \( \text{dim} \text{Im} A = 2k \) is even, \( B_{\text{Im} A} \) may be equivalent to either \( I_{2k} \) or \( \Pi_{2k} \). In the latter case, the resulting Lie superalgebra is isomorphic to \( \mathfrak{o}^{(1)}_{II}(2n+1-2k|2k) \); in the former case, it is isomorphic to \( \mathfrak{o}^{(1)}_{II}(2k|2n+1-2k) \). Note that the collection \( \{\mathfrak{o}^{(1)}_{II}(2k|2n+1-2k), \text{where} \ 1 \leq k \leq n\} \) can be described more simply as \( \{\mathfrak{o}^{(1)}_{II}(k|2n+1-k), \text{where} \ 1 \leq k \leq n\} \).

So we get \( 2n + 1 \) nonisomorphic Lie superalgebras from \( \mathfrak{o}^{(1)}(2n+1) \), including the purely even case. To see that all those cases are really realizable with some projection \( A \), the following approach can be used. Recall that in an odd-dimensional space, all nondegenerate symmetric forms are equivalent, so for any such form \( B \) and any symmetric invertible matrix, there is a basis in which the matrix of \( B \) is equal to the given one. In particular, for a given \( k \), there are bases in which the Gram matrix of \( B \) is either \( I_{2n+1} \) (call it the first basis) or \( \text{diag}(I_{2n+1-2k}, \Pi_{2k}) \) (call it the second basis for this value of \( k \)). Take the operator \( A \) whose matrix in the first basis
is $\text{diag}(0_{2n+1-2k}, I_{2k})$; clearly, $A^2 = A$ and $\dim \text{Im } A = 2k$. Relative the form $B$, the space $\text{Im } A$, which is spanned by the last $2k$ vectors of the basis, is, clearly, orthogonal to $\text{Ker } A$, which is spanned by the first $2n + 1 - 2k$ vectors of the basis. So $A \in \text{der } o_B$, and hence determines a grading of $o_B^{(1)}$. It is easy to see that the Lie superalgebra obtained by “method 2” of \cite{BLLS1} is $o_{II}^{(1)}(2n + 1 - 2k|2k)$.

Analogously, an operator $A$ with the same matrix $\text{diag}(0_{2n+1-2k}, I_{2k})$ in the second basis (it does not matter what is the matrix of $A$ in the first basis) determines a grading that corresponds to the Lie superalgebra $o_{II}^{(1)}(2n + 1 - 2k|2k)$.

It is possible to show that there are $2n + 1$ equivalence classes of $\mathbb{Z}/2$-gradings. It could have happened that there were more gradings than nonisomorphic superizations if two inequivalent gradings yielded isomorphic Lie superalgebras. This, however, does not happen; we skip the details.

3.2.1. Remark. Similarly to $\mathfrak{gl}(2n + 1)$, it is clear that $\mathbb{Z}/2$-gradings of $o(2n + 1)$ can produce Lie superalgebras not isomorphic to any superalgebras of the form $o_{II}^{(1)}(2n + 1 - 2k|2k)$ or $o_{II}^{(1)}(2n + 1 - 2k|2k)$: for example, a direct sum of $o_{II}^{(1)}(2n + 1 - 2k|2k)$ and $0|1$-dimensional center. But again, these gradings do not produce any new simple Lie superalgebras, so we do not consider such gradings of $o(2n + 1)$ here.

4. The $o_I^{(1)}(2n)$ series

4.1. Theorem. For $n > 2$, all $\mathbb{Z}/2$-gradings of the Lie algebra $o_I^{(1)}(2n)$ correspond to Lie superalgebras $o_{II}^{(1)}(m|2n - m)$, where $1 \leq m \leq n$, and $o_{II}^{(1)}(2n - 2k|2k)$, where $0 \leq k \leq n$.

Proof. If a bilinear form $B$ on vector space $V$ of dimension $2n$ is equivalent to $I_{2n}$, then the algebra $\text{der } o_B^{(1)}(2n)$ can be identified, see \cite{BGLL2}, with $o_B(2n)/\mathcal{C}$ in the sense that for any $D \in \text{der } o_B^{(1)}(2n)$, there is $A_D \in o_B(2n)$ such that $D$ coincides with the restriction of $\text{ad}_{A_D}$ on $o_B^{(1)}(2n)$. These $A_D$ are defined up to a central element; in particular, one can take $A_{D^2}$ to be $(A_D)^2$. By arguments similar to the ones we used in the case of $\mathfrak{sl}(2n)$, one can show that an operator $U$ on $o_B^{(1)}(2n)$ satisfies the conditions \eqref{eq:7a} if and only if it can be represented as a restriction of $\text{ad}_A$ on $o_B^{(1)}(2n)$ for some projection $A \in o_B(2n)$.

By arguments we used in the previous section, a projection $A$ belongs to $o_B(2n)$ if and only if $\text{Im } A$ and $\text{Ker } A$ are orthogonal with respect to $B$. We denote restrictions of $B$ to $\text{Im } A$ and $\text{Ker } A$ as $B_{\text{Im } A}$ and $B_{\text{Ker } A}$, respectively. These restrictions have to be nondegenerate.

If $\dim \text{Im } A = 2k + 1$, where $0 \leq k \leq n - 1$, then these restrictions are equivalent to $I_{2k+1}$ and $I_{2n-2k-1}$, respectively, and the resulting Lie superalgebra is isomorphic to $o_{II}^{(1)}(2k+1|2n - 2k - 1) \simeq o_{II}^{(1)}(2n - 2k - 1|2k + 1)$.

If $\dim \text{Im } A = 2k$, where $1 \leq k \leq n - 1$, then $B_{\text{Im } A}$ can be equivalent to either $I_{2k}$ or $\Pi_{2k}$, and $B_{\text{Ker } A}$ can be equivalent to either $I_{2n-2k}$ or $\Pi_{2n-2k}$. However, it is impossible for $B_{\text{Im } A}$ to be equivalent to $\Pi_{2k}$ while $B_{\text{Ker } A}$ is equivalent to $\Pi_{2n-2k}$ at the same time, because the direct sum of these two forms is equivalent to $\Pi_{2n}$. All the other combinations are possible, and, depending on them, the resulting Lie superalgebra can be isomorphic to either $o_{II}^{(1)}(2k|2n - 2k)$, or $o_{II}^{(1)}(2k|2n - 2k)$, or $o_{II}^{(1)}(2k|2n - 2k) \simeq o_{II}^{(1)}(2n - 2k|2k)$.

If $\dim \text{Im } A = 0$ or $2n$, then the resulting Lie superalgebra is purely even. So, as described above, $o_B^{(1)}(2n)$ has $2n + 1$ nonisomorphic superizations.

4.1.1. Remark. The algebras $o_I^{(1)}(2)$ and $o_I^{(1)}(4)$ are not simple, so we do not consider them here.
5. The simple relatives of $\mathfrak{o}_\Pi(2n)$ series

The algebras $\mathfrak{o}_\Pi(2)$ and $\mathfrak{o}_\Pi(4)$ do not have simple relatives, so we do not consider them.

5.1. Theorem. For $n > 2$, all $\mathbb{Z}/2$-gradings of the simple relative of $\mathfrak{o}_\Pi(2n)$ (that is, of $\mathfrak{o}_\Pi(2)(2n)$, if $n$ is odd, or of $\mathfrak{o}_\Pi(2)(2n)/c$, if $n$ is even) correspond to the simple relatives of the corresponding superizations, i.e., $\mathfrak{o}_\Pi(2)(2k|2n - 2k)$, where $0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$, and $\mathfrak{pe}(2)(n)$, if $n$ is odd, or $\mathfrak{o}_\Pi(2)(2k|2n - 2k)/c$, where $0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$, and $\mathfrak{pe}(2)(n)/c$, if $n$ is even.

Proof. If $n$ is odd, then the center of $\mathfrak{o}_\Pi(2)(2n)$ is trivial, so in this case $\mathfrak{o}_\Pi(2)(2n)/c \equiv \mathfrak{o}_\Pi(2)(2n)$. Keeping this in mind, we will denote the simple relative of $\mathfrak{o}_\Pi(2n)$ as $\mathfrak{o}_\Pi(2n)/c$ for both even and odd values of $n$. Similarly, we will denote the simple relatives of $\mathfrak{o}_\Pi(2k|2n - 2k)$ and $\mathfrak{pe}(2)(n)$ as $\mathfrak{o}_\Pi(2k|2n - 2k)/c$ and $\mathfrak{pe}(2)(n)/c$ for both even and odd values of $n$.

Consider the Lie algebra

$$\tilde{\mathfrak{g}}(V) := \{ M \in \mathfrak{gl}(V) | \text{there is } c_M \in \mathbb{K} \text{ such that } B(Mx, y) + B(x, My) = c_M B(x, y) \text{ for all } x, y \in V \}. \tag{12}$$

As is not difficult to see, for $B \sim I_{2n}$ we have

$$\tilde{\mathfrak{o}}_{\Pi_{2n}}(V) = \mathfrak{o}_{\Pi_{2n}}(V) \ltimes \mathbb{K} d_n, \text{ where } d_n = \text{diag}(0_n, 1_n) \text{ or diag}(1_n, 0_n).$$

Then, according to [BGLL2]

$$\mathfrak{der} \left( \mathfrak{o}_{\Pi_{2n}}(V)/c \right) \simeq \tilde{\mathfrak{o}}_{\Pi_{2n}}(V)/c. \tag{13}$$

(To compare with our previous results, observe that if $B \sim I_{2n}$, then $\tilde{\mathfrak{g}}_{I_{2n}}(V) = \mathfrak{o}_{I_{2n}}(V)$.)

As before, a given operator $U \in \mathfrak{der} \left( \mathfrak{o}_B(V)/c \right) \simeq \tilde{\mathfrak{g}}(V)/c$ satisfies the condition $U^2 = U$ if and only if the corresponding equivalence class of $\tilde{\mathfrak{g}}(V)$ contains a projection $A$. Let us consider the values $c_A$ can take.

$c_A = 0$: In this case, the definition (12) is equivalent to the statement that $\text{Im} A$ and $\text{Ker} A$ are orthogonal with respect to $B$. It means that the restrictions of $B$ on $\text{Im} A$ and $\text{Ker} A$ have to be nondegenerate. Since $B$ is anti-symmetric (i.e., $B(x, x) = 0$ for any $x \in V$), these restrictions are anti-symmetric as well, which means that dimensions of $\text{Im} A$ and $\text{Ker} A$ are even. The resulting Lie superalgebra in this case is isomorphic to

$$\mathfrak{o}_{\Pi_{2n}}(\dim \text{Im} A | \dim \text{Ker} A)/c \simeq \mathfrak{o}_{\Pi_{2n}}(\dim \text{Ker} A | \dim \text{Im} A)/c.$$ 

$c_A = \bar{1}$: In this case, the condition (12) is equivalent to the statement that both $\text{Im} A$ and $\text{Ker} A$ are isotropic with respect to $B$. Since $\text{Im} A \oplus \text{Ker} A = V$, this means that

$$\dim \text{Im} A = \dim \text{Ker} A = n$$

and there is an invertible linear map $f : \text{Ker} A \rightarrow (\text{Im} A)^*$ such that

$$B(x, y) = (f(y))(x) \text{ for all } x \in \text{Im} A, y \in \text{Ker} A.$$ 

The resulting Lie superalgebra in this case is isomorphic to $\mathfrak{pe}(2)(n)/c$.

$c_A \neq 0, \bar{1}$: This is impossible, because in this case, (12) would mean that $\text{Im} A$ and $\text{Ker} A$ are both isotropic and orthogonal to each other with respect to $B$, i.e., $B(x, y) = 0$ for all $x, y \in V$.

So we get $\left\lfloor \frac{n}{2} \right\rfloor + 2$ nonisomorphic superizations of $\mathfrak{o}_{\Pi_{2n}}(V)/c$ (including the purely even one), which are $\mathfrak{o}_{\Pi_{2n}}(2k|2n - 2k)/c$, where $0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$, and $\mathfrak{pe}(2)(n)/c$. \qed
6. The $\text{vect}^{(1)}(1; \frac{n}{n-1})$ Series

We describe the $\mathbb{Z}/2$-gradings of $\text{vect}^{(1)}(1; \frac{n}{n-1})$ in Theorem 6.2. First, we need some definitions.

Let $v_n$ be the minimal Lie subalgebra of the restricted closure of $\text{vect}^{(1)}(1; \frac{n}{n-1})$ that contains $\text{vect}^{(1)}(1; \frac{n}{n-1})$ and all the elements $\chi^{[2]}$, where $\chi \in \text{vect}^{(1)}(1; \frac{n}{n-1})$. As a vector space, $v_n$ can be written as a direct sum

$$v_n = \text{vect}(1; \frac{n}{n-1}) \oplus K\partial^2.$$  

Thus, $\dim v_n = 2^{n-1} + 1$. Observe that $v_n = (\text{vect}^{(1)}(1; \frac{n}{n-1}))^{<1>}$, see eq. (5), associated with the $\mathbb{Z}/2$-grading induced by the standard $\mathbb{Z}$-grading of $\text{vect}^{(1)}(1; \frac{n}{n-1})$.

For $n > 2$, consider the Lie superalgebra $q(\text{vect}(1; \frac{n}{n-1}))$. As a vector space, this Lie superalgebra is isomorphic to $v_n \oplus \Pi(\text{vect}(1; \frac{n}{n-1}))$, where $\Pi$ is the change of parity functor. Consider the Lie algebra $\text{vect}(1; \frac{n}{n})$ in indeterminate $z$ in order to distinguish from indeterminate $x$ in eq. (16). Let

$X_{-2} = \partial^2$, $X_{-1} = \partial$, $X_{0} = z\partial$, $X_{2^{m-1} - 1} = z(2^{m-1})\partial,$

be the basis in the even part $v_n$ of $q(\text{vect}(1; \frac{n}{n-1}))$ and

$Y_{-1} = \Pi\partial$, $Y_{0} = \Pi(z\partial)$, $Y_{2^{m-1} - 2} = \Pi(z(2^{m-1})\partial),$

be the basis in the odd part of $q(\text{vect}(1; \frac{n}{n-1}))$. The Lie superalgebra structure in $q(\text{vect}(1; \frac{n}{n-1}))$ is given by following formulas

$$[X_{-2}, X_k] = [\partial^2, z^{(k-1)}\partial] = X_{k-2}$$ for $k = 1, \ldots, 2^{n-1} - 1,$

$$[X_{-2}, Y_k] = \Pi[\partial^2, z^{(k-1)}\partial] = Y_{k-2}$$ for $k = 1, \ldots, 2^{n-1} - 1,$

$$[X_{-2}, X_{-1}] = [X_{-2}, X_{0}] = [X_{-2}, Y_{-1}] = [X_{-2}, Y_{0}] = [X_{-1}, Y_{-1}] = 0,$$

$$[X_{-1}, X_{m}] = X_{m-1}$$ and $[X_{-1}, Y_{m}] = Y_{m-1}$ for $m = 0, \ldots, 2^{n-1} - 1,$

$$[X_k, X_m] = [z^{(k-1)}\partial, z^{(m-1)}\partial] = \left(\frac{k+m+2}{k+1}\right)X_{k+m}$$ for $k, m = 0, \ldots, 2^{n-1} - 1,$

$$[X_k, Y_m] = [z^{(k-1)}\partial, z^{(m-1)}\partial] = \left(\frac{k+m+2}{k+1}\right)Y_{k+m}$$ for $k, m = 0, \ldots, 2^{n-1} - 1, s = 0, \ldots, 2^{n-1} - 2$

$$[Y_t, Y_s] = [z^{(t-1)}\partial, z^{(s-1)}\partial] = \left(\frac{t+s+2}{k+1}\right)X_{t+s}$$ for $t, s = 0, \ldots, 2^{n-1} - 2,$

$$(Y_{-1})^2 = X_{-2}, \quad (Y_{k})^2 = (z^{(k-1)}\partial)^2 = \left(\frac{2k+1}{k}\right)X_{2k}$$ for $k = 0, \ldots, 2^{n-1} - 2.$

On the superspace of $q(\text{vect}(1; \frac{n}{n-1}))$, define another Lie superalgebra structure, call it $q\tilde{v}_{n-1}$, by the following formulas:

$$[X_{-2}, X_{-1}] := 0, \quad [X_{-2}, X_{0}] := 0,$$

$$[X_{-2}, X_{m}] := X_{m-2}$$ for $m = 0, \ldots, 2^{n-1} - 1,$

$$[X_k, X_m] := \left(\frac{k+m+2}{k+1}\right)X_{k+m}$$ for $m, k = -2, \ldots, 2^{n-1} - 1$, except for $m = k = -1,$

$$[Y_k, Y_m] := \left(\frac{k+m+2}{k+1}\right)X_{k+m}$$ for $m, k = -2, \ldots, 2^{n-1} - 2$, except for $m = k = -1,$

$$[X_{-1}, Y_{-1}] := Y_{-1},$$

$$[X_k, Y_m] := \left(\frac{k+m+2}{k+1}\right)Y_{k+m} + \left(\frac{k+m+2}{k+1}\right)Y_{k+m+1}$$ for $m = -2, \ldots, 2^{n-1} - 1$

and $k = -2, \ldots, 2^{n-1} - 2$, except for $m = k = -1,$

$$(Y_{-1})^2 := X_{-2} + X_0, \quad (Y_{k})^2 := \left(\frac{2k+1}{k}\right)X_{2k}$$ for $k = 0, \ldots, 2^{n-1} - 2.$
I. Shchepochkina observed that the Lie superalgebra $\tilde{\mathfrak{g}}_{n-1}$ is a filtered deform of the Lie superalgebra $\mathfrak{q}(\text{vect}(1; n-1))$.

6.1. **Lemma.** Lie superalgebras $\tilde{\mathfrak{g}}_{n-1}$ and $\mathfrak{q}(\text{vect}(1; n-1))$ are not isomorphic.

**Proof.** Let us prove that $\tilde{\mathfrak{g}}_{n-1}$ is not isomorphic to any Lie superalgebra of the form $\mathfrak{q}(\mathfrak{g})$, where $\mathfrak{g}$ is a Lie algebra. A superalgebra of the form $\mathfrak{q}(\mathfrak{g})$ possesses the following property: if $x \in \mathfrak{q}(\mathfrak{g})_0$ acts nilpotently on $\mathfrak{q}(\mathfrak{g})_0$, then it acts nilpotently on $\mathfrak{q}(\mathfrak{g})_1$ as well. Indeed, if $(\text{ad}_x)^k|_{\mathfrak{q}(\mathfrak{g})_0} = 0$ for some positive integer $k$, then $(\text{ad}_x)^k|_{\mathfrak{q}(\mathfrak{g})_0} = 0$ for any $y \in \mathfrak{g}$.

On the other hand, one can see from the relations above that $X_1$ acts nilpotently on $(\tilde{\mathfrak{g}}_{n-1})_0$ (more specifically, $(\text{ad}_x)^{2n-1+1}|_{(\tilde{\mathfrak{g}}_{n-1})_0} = 0$), but $[X_{-1}, Y_{-1}] = Y_{-1}$, so the action of $X_{-1}$ on $(\tilde{\mathfrak{g}}_{n-1})_1$ is not nilpotent.

Consider the Lie algebra $\text{vect}(1; \underline{n})$ in indeterminate $x$. For a basis of $\text{vect}(1; \underline{n})$ we take $e_i := x^{(i+1)}\partial$ for $i = -1, 0, \ldots, 2^n - 2$.

6.2. **Theorem.** All $\mathbb{Z}/2$-gradings of the Lie algebra $\text{vect}(1; \underline{n})$ correspond to the one of the following Lie superalgebras:

1) purely even $\text{vect}(1; \underline{n})(0)$;

2) $\mathfrak{t}(1; n-1][1)$, described in [BGLLS2], and its $(n-2)$-parametric family of filtered deformations described below;

3) for $n > 2$, the Lie superalgebra $\tilde{\mathfrak{g}}_{n-1}$.

**Proof.** Let $u \in \mathcal{O}(1; \underline{n})$ be a linear combination of only even powers of the indeterminate $x$, and let $a$ be its constant term. In other words, let $u, a \in \mathcal{O}(1; \underline{n})$ be such that

\begin{equation}
(15) \quad x \cdot (\partial u) = 0; \quad \partial a = 0; \quad u^2 = a^2.
\end{equation}

Consider derivation $D_u \in \text{der} \mathcal{O}(1; \underline{n})$ of the form

\[ D_u = \left( u + xu \sum_{1 \leq i \leq n-1} a^{2i-2} (\partial^2 u) \right) \partial + \sum_{1 \leq i \leq n-1} a^{2i} \partial^2 i. \]

Properties (15) imply (after rather lengthy calculations we omit) that $(D_u)^2 = D_u$, so $D_u$ describes a $\mathbb{Z}/2$-grading of $\mathcal{O}(1; \underline{n})$.

Consider the linear map $\text{ad}_{D_u}$ on the Lie algebra $\text{der} \mathcal{O}(1; \underline{n})$ given by $D \mapsto [D_u, D]$. Due to the Jacobi identity, we see that $\text{ad}_{D_u} \in \text{der} \text{der} \mathcal{O}(1; \underline{n}))$, where the outer $\text{der}$ is for derivations of the Lie algebra $\text{der} \mathcal{O}(1; \underline{n})$. Since the Lie algebra $\text{der} \mathcal{O}(1; \underline{n})$ has a 2-structure given by $D^{[2]} = D^2$, we have $(\text{ad}_{D_u})^{[2]} = \text{ad}_{D_u}$. Thus, $\text{ad}_{D_u}$ describes a $\mathbb{Z}/2$-grading of $\text{der} \mathcal{O}(1; \underline{n})$.

The linear map $\text{ad}_{D_u}$ sends elements of $\text{vect}(1; \underline{n})$ to $\text{vect}(1; \underline{n})$ because $D_u$ is a linear combination of a vector field and elements of the form $\partial^2 i$, and

\[ [\partial^2 i, f \partial] = (\partial^2 f) \partial = [\partial, (\partial^2 - 1 f) \partial] \in \text{vect}(1; \underline{n}) \text{ for any } f \in \mathcal{O}(1; \underline{n}). \]

This means that the restriction $(\text{ad}_{D_u})|_{\text{vect}(1; \underline{n})}$ describes a $\mathbb{Z}/2$-grading of $\text{vect}(1; \underline{n})$ and the restriction $(\text{ad}_{D_u})|_{\text{vect}(1; \underline{n})}$ describes a $\mathbb{Z}/2$-grading of $\text{vect}(1; \underline{n})$. In what follows, we will denote the latter restriction by $D_u$ as well. Different polynomials $u$ correspond to different gradings, as $[D_u, x \partial] = u \partial$. We need, however, not individual gradings, but their equivalence classes. We were unable to solve this problem completely so far.

\footnote{The notation $u(0)$ we use for brevity in what follows is meaningless, strictly speaking, because the divided power polynomials can not be evaluated at any $x \in \mathbb{K}$ for $p > 0$; by writing like this we mean $a \in \mathbb{K} \cdot 1$ which is the value of $u$ modulo the maximal ideal of $\mathcal{O}(1; \underline{n})$; this is a standard abuse of notation.}
In notation of [11], any torus of the restricted closure of $\text{vect}(1; \underline{n})$ lying in the maximal subalgebra of elements of non-negative degree (assuming $\deg x = 1$) is called an inner one, the other tori are called outer ones.

If $u(0) = a \neq 0$, the derivation $D_u$ is an outer toroidal derivation of $\mathcal{O}(1; \underline{n})$ (i.e., it spans an outer torus of 2-closure of $\text{vect}(1; \underline{n})$). The automorphism group of $\text{vect}^{(1)}(1; \underline{n})$ acts transitively on the set of outer toroidal derivations, see [13]. Hence, any derivation $D_u$ with $a \neq 0$ is conjugate by an automorphism of $\text{vect}^{(1)}(1; \underline{n})$ to the derivation

$$D_1 = (1 + x)\partial + \sum_{1 \leq i \leq n-1} \partial^2.$$

Recall that

$$(u + v)^{(k)} = \sum_{0 \leq i \leq k} u^{(k-i)}v^{(i)}$$

and for $k = -2, \ldots, 2^{n-1} - 1$, define $e_k \in (\text{vect}^{(1)}(1; \underline{n}); D_1)^{<1>}$ by setting

$$e_{-2} = \partial^2 + (1 + x)\partial,$$

$$e_k = \partial ((x + x(2))^{(k+2)}) \partial = (1 + x)(x + x(2))^{(k+1)}\partial \text{ for } k > -2,$$

and for $k = -1, \ldots, 2^{n-1} - 2$, define $o_k$ by formulas

$$o_k = (x + x(2))^{(k+1)}\partial, \text{ where } k = -1, \ldots, 2^{n-1} - 2.$$ 

Direct computations show that $D_1(e_k) = 0$ and $D_1(o_m) = o_n$. Let $e_{-2}, \ldots, e_{2^{n-1}-2}$ and $\Pi o_{-1}, \ldots, \Pi o_{2^{n}-2}$ form a basis in the Lie superalgebra corresponding to the $\mathbb{Z}/2$-grading defined by $D_1$. The formulas

$$e_k \mapsto X_k, \text{ where } k = -2, \ldots, 2^{n-1} - 2,$$

$$o_m \mapsto Y_m, \text{ where } m = -1, \ldots, 2^{n-1} - 2,$$

establish an isomorphism between this Lie superalgebra, spanned by $e_i$ and $o_j$, and $\text{vect}_{n-1}$.

If $u(0) = a = 0$, the derivation $D_u$ spans an inner torus. Hence, see [11], the derivation $D_u$ is conjugate to the derivation $D_f$, where $f = \sum_{1 \leq i \leq n-1} c_i x^{(2)}$ and the summands $c_i x^{(2)}$ are the corresponding terms of $u$. This gives us an $(n - 1)$-parametric family of $\mathbb{Z}/2$-gradings. Let the automorphism $\sigma_\varepsilon$ of $\mathcal{O}(1; \underline{n})$ be given on its generators by the formulas

$$\sigma_\varepsilon(x) = \varepsilon x, \quad \sigma_\varepsilon(x^{(2)}) = \varepsilon^2 x^{(2)}, \quad \ldots, \quad \sigma_\varepsilon(x^{(2^{n-1})}) = \varepsilon^{2^{n-1}} x^{(2^{n-1})},$$

where $\varepsilon \in \mathbb{K}$ and $\varepsilon \neq 0$. Then

$$\sigma_\varepsilon \left( \left( \sum_{1 \leq i \leq n-1} c_i x^{(2)} \right) \partial \right) = \left( \sum_{1 \leq i \leq n-1} c_i \varepsilon^{2^{i-1}} x^{(2)} \right) \partial.$$

Hence, the conjugacy class of the derivation $D_u$ is defined by a tuple of parameters $(c_1, c_2, \ldots, c_{n-1})$ up to equivalence

$$\left( c_1, c_2, \ldots, c_{n-1} \right) \sim (\varepsilon c_1, \varepsilon^3 c_2, \ldots, \varepsilon^{2^{n-1}-1} c_{n-1}).$$

---

3In [11], Tyrunin proved that, for any two outer toroidal derivations $O_1$ and $O_2$ of $\text{vect}(1; \underline{n})$, there exists an $A \in \text{Aut} \text{vect}(1; \underline{n})$ such that $O_2 = A^{-1}O_1A$. Instead we need “for any two outer toroidal derivations $O_1$ and $O_2$ of $\text{vect}^{(1)}(1; \underline{n})$, there exists an $A \in \text{Aut} \text{vect}^{(1)}(1; \underline{n})$ such that $O_2 = A^{-1}O_1A$”. This is so because “our” $D_u = (D_u^T)|_{\text{vect}^{(1)}} \in \text{der} \text{vect}^{(1)}(1; \underline{n})$ is the restriction of the corresponding “Tyrunin’s” $D_u^T \in \text{der} \text{vect}(1; \underline{n})$ to $\text{vect}^{(1)}(1; \underline{n})$.

If $D_u$ and $D_v$ are outer toroidal derivations of $\text{vect}^{(1)}(1; \underline{n})$, then $D_u^T$ and $D_v^T$ are outer toroidal derivations of $\text{vect}(1; \underline{n})$, and therefore there exists an $A \in \text{Aut} \text{vect}(1; \underline{n})$ such that $D_v^T = A^{-1}D_u^T A$. Any automorphism of $\text{vect}(1; \underline{n})$ preserves $\text{vect}^{(1)}(1; \underline{n})$, and hence $D_v = (\tilde{A})^{-1}D_u \tilde{A}$, where $\tilde{A} := A|_{\text{vect}^{(1)}}(1; \underline{n})$ is an automorphism of $\text{vect}^{(1)}(1; \underline{n})$. 

Therefore, the number of parameters of $\mathbb{Z}/2$-gradings reduces to $n - 2$. And, indeed, for $n = 2$, we do not have parametric families of $\mathbb{Z}/2$-gradings, see Subsection 6.3.

Observe that $x \cdot \partial u = 0$, so

\[ 0 = \partial(x \partial u) = \partial u + x \partial^2 u, \]

which implies $x \partial^2 u = \partial u$.

Then

\[ D_u = (u + x + xu\partial^2 u)\partial = (u + x + u\partial u)\partial. \]

Desuperization of any superization of $\mathfrak{vect}(1; n)$ can be considered as a Lie subalgebra in $\mathfrak{v}_{n+1} := (\mathfrak{vect}(1; n))^{<1>}$, see [14]. The operator $\text{ad}_{D_u}$ preserves $\mathfrak{v}_{n+1}$, so the grading it defines on $\mathfrak{vect}(1; n)$ can be extended to $\mathfrak{v}_{n+1}$ with the same definition: $D \mapsto [D_u, D]$. This extended operator describes a $\mathbb{Z}/2$-grading of $\mathfrak{v}_{n+1}$. The elements of $\mathfrak{v}_{n+1}$ are said to be $u$-even or $u$-odd if they are, respectively, even or odd in this grading.

In particular, for the grading on $\mathfrak{v}_{n+1}$ given by the function $u = 0$, we have $D_0 = x\partial$, and the resulting superization is isomorphic to $\mathfrak{sl}(1; n-1)$, see [BLLSII §7].

Consider the linear maps $T_u, A_u : \mathfrak{v}_{n+1} \to \mathfrak{v}_{n+1}$ defined as follows (their action on other elements being extended by linearity):

\begin{align*}
T_u(f\partial) &= (f + u(1 + \partial u)\partial f)(1 + \partial u + u\partial^2 u)\partial \text{ for any } f \in \mathcal{O}(1; n); \quad \text{(19)} \\
T_u\partial^2 &= (T_u\partial)^2 = \partial^2 + (\partial^2 u + u\partial^3 u + u(\partial^2 u)^2 + u\partial u(\partial^2 u)\partial); \\
A_u(f\partial) &= (f + u\partial u\partial f)\partial \text{ for any } f \in \mathcal{O}(1; n); \\
A_u\partial^2 &= \partial^2.
\end{align*}

For any $X \in \mathfrak{v}_{n+1}$, one can check that $T_uX$ is $u$-even (resp. $u$-odd) if and only if $X$ is $0$-even (resp. $0$-odd). We omit the calculations that show it, but the idea is as follows: $\partial$ is $0$-odd, while $(1 + \partial u + u\partial^2 u)\partial$ is $u$-odd. Besides, if we extend the concept of $u$-evenness/$u$-oddness to $\mathcal{O}(1; n)$, then $f + u(1 + \partial u)\partial f$ is $u$-even (resp. $u$-odd) for any $f \in \mathcal{O}(1; n)$ if and only if $f$ is $0$-even (resp. $0$-odd).

Also, the following is true for any $X, Y \in \mathfrak{vect}(1; n)$:

\[ [T_uX, T_uY] = T_uA_u[X, Y]; \]
\[ (T_uX)^2 = T_uA_uX^2 \text{ for any } 0\text{-odd } X; \]
\[ [T_u\partial^2, T_uX] = T_u((1 + u\partial^2 u) \cdot [\partial^2, X]). \]

Note also that the operator $T_u$ is invertible, so its image is the whole $\mathfrak{v}_{n+1}$. The invertibility follows from the fact that the matrix of $T_u$ is upper triangular with $1$'s on the main diagonal in the basis $\partial^2, \partial, x\partial, \ldots, x^{(2^n-1)}\partial$.

This means that the superization given by any function $u$ such that $u(0) = 0$ can be considered as a deform of the superization given by $u = 0$ with the deformation parametrized by the polynomial $u$ or, which is the same, the coefficients of $u$ as follows:

\begin{align*}
[X, Y]_u = \begin{cases} 
A_u[X, Y] & \text{if } X, Y \in \mathfrak{vect}(1; n); \\
(1 + u\partial^2 u) \cdot [X, Y] & \text{if } X = \partial^2 \text{ and } Y \in \mathfrak{vect}(1; n); \\
\text{defined by linearity and anti-symmetry} & \text{in other cases;}
\end{cases}
\end{align*}

\[ (X^2)_u = A_uX^2 \text{ for } 0\text{-odd } X. \]

This deformation is filtered relative the decreasing filtration in which $L_{-2} = S(\mathfrak{v}_{n+1})$, see (6), while $L_{k-1}$ for $k \geq -1$ consists of vector fields of the form $f\partial$, where $f \in \mathcal{O}(1; n)$ does not contain term of degree $< k$. Such superizations are listed in heading 2) of Theorem.

Finally, observe that these formulas do not capture the trivial $\mathbb{Z}/2$-grading given by the derivation $U = 0$; the even part $\mathfrak{g}_0$ of this grading is the whole Lie algebra $\mathfrak{vect}(1; n)$. Since
any torus in $\text{Der}(\text{vect}^{(1)}(1; n))$ has a form $D_u$, see [Kuz 1] and footnote 3 on p. 12, we completely described all $\mathbb{Z}/2$-gradings of $\text{vect}^{(1)}(1; n)$ and the corresponding superizations. 

6.2.1. Remark. Observe an unpredicted fact: according to (20), if $X, Y \in \text{vect}(1; n)$, then $[X, Y]_u$ can be expressed in terms of $[X, Y]$ and $u$. In particular, concerning the 0-even part of $\text{vect}(1; n)$, this implies the following fact:

6.2.2. Corollary. If $u(0) = 0$, then the even part of the superization corresponding to $D_u$ is a solvable Lie algebra.

For $n = 2$, any outer torus is conjugate to an inner one, see (22). This is not so for $n > 2$.

6.2.3. Lemma. Let $n > 2$, consider the superization $\mathfrak{g}^{n-1} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ corresponding to the outer torus $D_1$. Its even part $\mathfrak{g}_0 \simeq \mathfrak{m}_n$ is spanned by the elements (10), and $\mathfrak{g}_1$ is spanned by the elements (17). The odd part $\mathfrak{g}_1$ is a reducible $\mathfrak{g}_0$-module with no lowest weight vector and with the two highest weight vectors $\Pi_{2n-1}$ and $\Pi_{2n-2}$ with respect to the standard $\mathbb{Z}$-grading of $\mathfrak{g}_0$, namely $(\mathfrak{g}_0)_k = \mathbb{K}e_k$ for $k = -2, \ldots, 2n - 1$.

Proof. Using (16) and (17), for positive generators $e_{2k-1}$, where $k = 1, \ldots, n - 2$, of $\mathfrak{g}_0$ we have

$$[e_{2k-1}, \Pi_{2n-2}] = \Pi[(\partial w)w^{(2k)}\partial, w^{(2n-1)}\partial] = \Pi(w^{(2k)}w^{(2n-2)} + w^{(2k)}w^{(2n-1)}w^{(2k-1)}w^{(2n-1)})\partial,$$

where $w = (x + x^{(2)})$. Note that

$$2^{n-1} - 1 = 1 \cdot 2^{n-2} + 1 \cdot 2^{n-3} + \ldots + 1 \cdot 2 + 1,$$

$$2^{n-1} - 2 = 1 \cdot 2^{n-2} + 1 \cdot 2^{n-3} + \ldots + 1 \cdot 2 + 0.$$

Then by Lucas’s Theorem for $k < n - 1$, we have

$$\left(\begin{array}{c}2^{n-1} - 2 + 2k \\ 2k\end{array}\right) \mod 2 = \left(\begin{array}{c}2^{n-1} - 1 + 2k \\ 2k\end{array}\right) \mod 2 = \left(\begin{array}{c}2^{n-1} - 1 + 2k \\ 2k - 1\end{array}\right) \mod 2 = 0,$$

at $k$th place

Thus, we have $w^{(2k)}w^{(2n-2)} = 0$, $w^{(2k)}w^{(2n-1)} = 0$, and $w^{(2k-1)}w^{(2n-1)} = 0$. Finally, we see that $[e_{2k-1}, \Pi_{2n-2}] = 0$ for $k = 1, \ldots, n - 2$. Computations of the same kind show that $[e_{2k-1}, \Pi_{2n-3}] = 0$ for $k = 1, \ldots, n - 2$. Therefore, $o_{2n-1,2}$ and $o_{2n-1,3}$ are highest weight vectors in $\mathfrak{g}_1$.

Now, let us prove that there are no lowest weight vectors in $\mathfrak{g}_1$. Consider the action of $e_{-1}$ on elements of $\mathfrak{g}_1$. Using (16) and (17), we obtain

$$[e_{-1}, \Pi o_{-1}] = o_1, \quad \text{and} \quad [e_{-1}, \Pi o_k] = \Pi o_k + \Pi o_{k-1} \text{ for } k = 0, \ldots, 2n - 2.$$

Therefore, the equation

$$[e_{-1}, \sum_{-1 \leq k \leq 2n-1-2} q_k \Pi o_k] = 0$$

for lowest weight vectors reduces to the following system of linear equations

$$q_{-1} + q_0 = 0, \quad q_0 + q_1 = 0, \quad \ldots, \quad q_{2n-1-3} + q_{2n-1-2} = 0, \quad q_{2n-1-2} = 0.$$
Clearly, this system has only one solution: 0. Hence, there are no lowest weight vectors in \( \mathfrak{g}_1 \).

Observe that the highest weight vector \( \Pi_0^{2n-1} \) generates the whole \( \mathfrak{g}_0 \)-module \( \mathfrak{g}_1 \) and the highest weight vector \( \Pi_0^{2n-1-1} \) generates a \( \mathfrak{g}_0 \)-submodule of codimension 1 in \( \mathfrak{g}_1 \), as follows from (21).

6.2.3a. Remark. Put \( v = \alpha \Pi_0^{2n-1-3} + \beta \Pi_0^{2n-1-2} \), where \( \alpha, \beta \in \mathbb{K} \). Construct a new basis in \( \mathfrak{g}_1 \) by the rule

\[ \begin{align*}
\phi'_1 &= v, \\
\phi'_k &= [e'_{-1}, \phi'_{k-1}].
\end{align*} \]

For \( n = 4 \), the coordinates of \( \phi'_k \) in the old basis \( \phi_k \) are defined by the following matrix

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \alpha & \alpha + \beta \\
0 & 0 & 0 & 0 & \alpha & \beta & \alpha + \beta \\
0 & 0 & 0 & \alpha & \beta & \beta & \alpha + \beta \\
0 & \alpha & \alpha + \beta & 0 & 0 & \alpha & \alpha + \beta \\
\alpha & \alpha + \beta & \alpha & \alpha + \beta & \alpha & \alpha + \beta & \alpha + \beta
\end{pmatrix}.
\]

Notice that \([e'_{-1}, \phi'_k] = \phi'_1\). For \( \beta = 0 \), we have \( \phi'_k = \phi'_1 + \phi'_2 + \ldots + \phi'_{7} \). For \( \alpha = 0 \), the \( \mathfrak{g}_0 \)-action on \( \mathfrak{g}_1 \) in the new basis is defined by the following table

| \( e'_{-1} \) | \( e_0 \) | \( e_1 \) | \( e_2 \) | \( e_3 \) | \( e_4 \) | \( e_5 \) | \( e_6 \) | \( e_7 \) | \( e_8 \) | \( e_9 \) | \( e_{10} \) | \( e_{11} \) | \( e_{12} \) | \( e_{13} \) | \( e_{14} \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( \phi'_2 \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( \phi'_3 \) | \( \phi'_2 \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( \phi'_4 \) | \( \phi'_2 \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( \phi'_5 \) | \( \phi'_2 \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( \phi'_6 \) | \( \phi'_2 \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( \phi'_7 \) | \( \phi'_2 \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( \phi'_8 \) | \( \phi'_2 \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( \phi'_9 \) | \( \phi'_2 \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( \phi'_{10} \) | \( \phi'_2 \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( \phi'_{11} \) | \( \phi'_2 \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( \phi'_{12} \) | \( \phi'_2 \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( \phi'_{13} \) | \( \phi'_2 \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( \phi'_{14} \) | \( \phi'_2 \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( \phi'_{15} \) | \( \phi'_2 \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( \phi'_{16} \) | \( \phi'_2 \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

In what follows we consider only generating functions of the form

\[ u(c) = c_0 + \sum_{1 \leq k \leq n-1} c_k x^{(2)} \]

where \( c = (c_0, c_1, \ldots, c_{n-1}) \) is a set of free parameters.

Let us consider the classification of \( \mathbb{Z}/2 \)-gradings of \( \text{vect}^{(1)}(1; 2) \) in more detail.

6.3. \( \mathfrak{g} = \text{vect}^{(1)}(1; 2) \). The general solution \( U_{lin} \) of linear equations (8b), where \( c_{1,1}, c_{1,2}, c_{1,3}, c_{2,1}, c_{3,1} \) are free parameters, and the general solution \( U \) of eq. (8), where \( c_0, c_1 \) are free parameters, and the schematic form of \( U \) are as follows:

\[
U_{lin} = \begin{pmatrix}
c_{1,1} & c_{1,2} & c_{1,3} \\
c_{2,1} & 0 & c_{1,2} \\
c_{3,1} & c_{2,1} & c_{1,1}
\end{pmatrix}, \quad
U = \begin{pmatrix}
c_0 c_1 + 1 & c_0 & c_0^2 \\
c_1 & 0 & c_0 \\
c_1^2 & c_1 & c_0 c_1 + 1
\end{pmatrix}.
\]

This solution corresponds to the derivation \( D_u \) with \( u(c) = c_0 + c_1 x^{(2)} \).
6.3.1. Parameters and Aut \( \mathfrak{g} \). We denote by \( \mathfrak{g}_0(c) \) the kernel of the projection \( D_{u(c)} \), the even part of the corresponding \( \mathbb{Z}/2 \)-grading, e.g., \( \mathfrak{g}_0(c_0, c_1) \), where \( u = c_0 + c_1 x^{(2)} \).

Certain automorphisms and their actions:

\[
\begin{pmatrix}
\frac{1}{c_0} & 0 & 0 \\
0 & 1 & 0 \\
\frac{c_1 + c_0 c_1}{c_0} & 0 & c_0
\end{pmatrix}
\]

maps \( \mathfrak{g}_0(c_0, c_1) \) to \( \mathfrak{g}_0(1, c_1) \) for all \( c_0 \neq 0 \).

\[
\begin{pmatrix}
1 + c_1 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}
\]

maps \( \mathfrak{g}_0(1, c_1) \) to \( \mathfrak{g}_0(1, 1) \) for all \( c_1 \in \mathbb{K} \).

\[
\begin{pmatrix}
c_1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \frac{1}{c_1}
\end{pmatrix}
\]

maps \( \mathfrak{g}_0(0, c_1) \) to \( \mathfrak{g}_0(0, 1) \) for all \( c_1 \neq 0 \).

(22)

\[
\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

establishes equivalence of \( \mathfrak{g}_0(1, 1) \) and \( \mathfrak{g}_0(0, 1) \).

Observe that the automorphism (22) sends the outer derivation \( D_{1+x^{(2)}} \) to the inner derivation \( D_{x^{(2)}} \).

The final answer: there are three inequivalent \( \mathbb{Z}/2 \)-gradings; their \( \mathfrak{g}_0 \)'s are:

(23)

\[
\mathfrak{g}_0(0, 0) = \mathbb{K}\mathfrak{e}_0 \quad \text{and} \quad \mathfrak{g}_0(1, 1) = \mathbb{K}(\mathfrak{e}_{-1} + \mathfrak{e}_0 + \mathfrak{e}_1),
\]

and the trivial \( \mathbb{Z}/2 \)-grading with \( \mathfrak{g}_0 = \text{vect}(1; 1) \),

not two as is the case for \( \mathbb{Z}/2 \)-gradings of \( \mathfrak{o}^{(1)}(3) = \mathfrak{o}(3) \) for any \( p \neq 2 \).

6.3.2. Lie superalgebras corresponding to the gradings (23).

6.3.2a. Grading \((0, 0)\). The minimal Lie subsuperalgebra

(24)

\[
\mathfrak{g}^{<1>} := (\mathfrak{g}^{<1>})_0 \oplus (\mathfrak{g}^{<1>})_1
\]

of the restricted closure \( \bar{\mathfrak{g}} \) containing all elements \( v^{[2]} \), where \( v \in \mathfrak{g}_1 = (\mathfrak{g}^{<1>})_1 \), see eq. (5), is as follows:

\[
(\mathfrak{g}^{<1>})_0 = \text{Span}(\mathfrak{e}_{-2} = \partial^2, \mathfrak{e}_0 = x \partial, \mathfrak{e}_2 = x^{(3)} \partial), \quad (\mathfrak{g}^{<1>})_1 = \text{Span}(\mathfrak{o}_{-1} = \partial, \mathfrak{o}_1 = x^{(2)} \partial).
\]

The squaring in \( S(\mathfrak{g}^{<1>}) \) is given by the following formulas:

\[
o_{-1}^2 = (\partial)^2 = e_{-2} = \partial^2, \quad o_1^2 = (x^{(2)} \partial)^2 = e_2 = x^{(3)} \partial.
\]

The commutation relations in \((\mathfrak{g}^{<1>})_0\) are defined by the following formulas:

\[
[e_0, e_{-2}] = 0, \quad [e_0, e_2] = 0, \quad [e_{-2}, e_2] = e_0.
\]

Observe that \((\mathfrak{g}^{<1>})_0\) is the Heisenberg algebra. The \((\mathfrak{g}^{<1>})_0\)-module structure in \((\mathfrak{g}^{<1>})_1\) is given by the following formulas:

\[
[e_{-2}, o_{-1}] = 0, \quad [e_0, o_{-1}] = o_{-1}, \quad [e_2, o_{-1}] = o_1, \quad [e_0, o_1] = o_1, \quad [e_2, o_1] = 0.
\]

It is easy to see that \( S(\mathfrak{g}^{<1>}) \simeq \mathfrak{o}^{(1)}_{\bar{m}}(1|2) \simeq \mathfrak{k}(1; 1|1) \); for the (non-obvious) definition of the latter, see [BGLLS]. The Lie superalgebra \( \mathfrak{o}^{(1)}_{\bar{m}}(1|2) \) is given by supermatrices of the form

\[
\begin{pmatrix}
0 & b_2 & b_1 \\
b_1 & a_1 & a_2 \\
b_2 & a_3 & a_1
\end{pmatrix}, \quad \text{where} \quad a_i, b_i \in \mathbb{K}.
\]
The correspondence between the abstract and matrix representations of the elements of $\mathfrak{oo}^{(1)}_{III}(1|2)$ is as follows
\[
e_{-2} = e_{2,3}, \quad e_{0} = e_{2,2} + e_{3,3}, \quad e_{2} = e_{3,2},
\]
\[
o_{-1} = e_{1,3} + e_{2,1}, \quad o_{1} = e_{1,2} + e_{3,1}.
\]

6.3.2b. Grading $(1, 1) \simeq (1, 0)$. The Lie subsuperalgebra $\mathfrak{g}^{<1>}$, see eq. (24), is as follows:
\[
(\mathfrak{g}^{<1>})_{0} = \text{Span}(e_{1} = (1 + x + x^{(2)})\partial, \ e_{2} = \partial^{2} + (1 + x)\partial, \ e_{3} = \partial^{2} + (x + x^{(3)})\partial),
\]
\[
(\mathfrak{g}^{<1>})_{1} = \text{Span}(o_{1} = (1 + x)\partial, \ o_{2} = (1 + x^{(2)})\partial).
\]

The squaring in $S(\mathfrak{g}^{<1>})$ is given by the following formulas:
\[
o_{1}^{2} = e_{2}, \quad o_{2}^{2} = e_{3}.
\]

The commutation relations in $(\mathfrak{g}^{<1>})_{0}$ are defined by the following formulas:
\[
[e_{1}, e_{2}] = e_{1}, \quad [e_{1}, e_{3}] = 0, \quad [e_{2}, e_{3}] = e_{1}.
\]

The $(\mathfrak{g}^{<1>})_{1}$-module structure in $(\mathfrak{g}^{<1>})_{1}$ is given by the following formulas:
\[
[e_{1}, o_{1}] = o_{1} + o_{2}, \quad [e_{2}, o_{1}] = 0, \quad [e_{3}, o_{1}] = o_{2},
\]
\[
[e_{1}, o_{2}] = o_{2}, \quad [e_{2}, o_{2}] = o_{1} + o_{2}, \quad [e_{3}, o_{2}] = 0.
\]

Let us show that $S(\mathfrak{g}^{<1>}) \simeq \mathfrak{oo}^{(1)}_{II}(1|2)$. The Lie superalgebra $\mathfrak{oo}^{(1)}_{II}(1|2)$ consists of symmetric supermatrices with supertrace 0. For a basis of its even part we take
\[
E_{1} = e_{1,1} + e_{2,2}, \quad E_{2} = e_{2,2} + e_{3,3}, \quad E_{3} = e_{2,3} + e_{3,2}.
\]

For a basis of its odd part we take $O_{1} = e_{1,2} + e_{2,1}, \ O_{2} = e_{1,3} + e_{3,1}$. The commutation relations are given by the following formulas:
\[
[E_{1}, E_{2}] = 0, \quad [E_{1}, E_{3}] = E_{3}, \quad [E_{2}, E_{3}] = 0.
\]

The squaring in $S(\mathfrak{g}^{<1>})$ is given by the following formulas:
\[
O_{1}^{2} = E_{1}, \quad O_{2}^{2} = E_{1} + E_{2}.
\]

The $(\mathfrak{g}^{<1>})_{0}$-module structure is as follows:
\[
[E_{2}, O_{1}] = O_{1}, \quad [E_{2}, O_{2}] = O_{2}.
\]

The multiplication by elements $E_{1}$ and $E_{3}$ is given by the following formulas:
\[
[E_{1}, O_{1}] = 0, \quad [E_{1}, O_{2}] = O_{2}, \quad [E_{3}, O_{1}] = O_{2}, \quad [E_{3}, O_{2}] = O_{1}.
\]

The isomorphism between $\mathfrak{oo}^{(1)}_{II}(1|2)$ and $\mathfrak{g}^{<1>}$ is given by the following formulas:
\[
e_{1} = E_{3}, \quad e_{2} = E_{1}, \quad e_{3} = E_{2} + E_{3}, \quad o_{1} = O_{1}, \quad o_{2} = O_{1} + O_{2}.
\]

6.3.3. Remarks. 1) Observe that the even parts of both superizations are solvable Lie algebras, but the corresponding Lie superalgebras are simple. For more details and examples of this phenomenon indigenous to $p = 2$, see Shchepochkina’s comment (the last section) in [BCL].

2) Observe that due to Theorem 6.2 the Lie superalgebra $\mathfrak{oo}^{(1)}_{II}(1|2)$ is a filtered deform of the Lie superalgebra $\mathfrak{oo}^{(1)}_{III}(1|2)$. 
6.4. Superizations corresponding to inner tori. Consider a superization of \( \mathfrak{vect}^{(1)}(1; \underline{n}) \) given by a generating function \( u \) such that \( u(0) = 0 \). For such a function \( u \), the derivation \( D_u \) spans an inner torus. According to [1], the derivation \( D_u \) is conjugate to the derivation \( D_f \) in \( \mathfrak{vect}^{(1)}(1; \underline{n}) \), where \( f = \sum_{1 \leq i \leq n-1} c_i x^{(2i)} \) and the \( c_i x^{(2i)} \) are the corresponding terms of \( u \), so the superizations corresponding to \( D_u \) and \( D_f \) are isomorphic. For this reason, in the rest of this Subsection we consider only generating functions of the form \( \sum_{1 \leq i \leq n-1} c_i x^{(2i)} \). This gives us an \((n-1)\)-parametric family of \( \mathbb{Z}_2 \)-gradings.

6.4.1. Conjecture. Let \( u_1 = \sum_{1 \leq k \leq n-1} c_k x^{(2k)} \) and \( u_2 = \sum_{1 \leq k \leq n-1} b_k x^{(2k)} \) be generating functions, then the superizations corresponding to \( D_{u_1} \) and \( D_{u_2} \) are isomorphic if and only if there exists \( \varepsilon \in \mathbb{K}^x \) such that \( c_k = \varepsilon^{2k-1} b_k \) for all \( k = 1, \ldots, n-1 \).

The following is a sketch of a proof of this conjecture, including a complete computer-aided proof for \( 2 \leq n \leq 6 \).

First of all, if there is an \( \varepsilon \neq 0 \) such that \( c_k = \varepsilon^{2k-1} b_k \) for all \( k = 1, \ldots, n-1 \), then \( D_{u_1} \) and \( D_{u_2} \) are conjugate by the automorphism of \( \mathfrak{vect}^{(1)}(1; \underline{n}) \) given by \( x^{(m)} \partial \mapsto \varepsilon^{m-1} x^{(m)} \partial \) (this automorphism is generated by the automorphism of \( \mathcal{O}(1; \underline{n}) \) given by \( x^{(m)} \mapsto \varepsilon^m x^{(m)} \)).

By Theorem [6.2], the superization given by \( u_1 \) is isomorphic to a deform of \( \mathfrak{t}(1; n-1|1) \). Consider the even part of of such a deform. According to (20), it contains a commutative subalgebra of codimension 1, which is the 0-even part of \( \mathfrak{vect}(1; \underline{n}) \). The element \( \partial^2 \) acts on this commutative subalgebra, and its action is given by

\[
(26) \quad v \partial \mapsto (1 + u \partial^2 u) \partial^2 v \partial \text{ for any } 0\text{-odd } v \in \mathcal{O}(1; \underline{n}).
\]

6.4.2. Conjecture (Proved by computer for \( n = 2, 3, 4, 5, 6 \)). The characteristic polynomial of the linear operator (26) on the 0-even part of \( \mathfrak{vect}(1; \underline{n}) \) for \( n \geq 2 \) is defined by the formula

\[
\lambda^{2n-1} + \sum_{0 \leq k \leq n-2} c_{n-1-k}^{k+1} \lambda^{2k}.
\]

For two deformations given by \( u_1 \) and \( u_2 \) to be isomorphic to each other, their even parts have to be isomorphic, which means that the two actions of \( \partial^2 \) on the 0-even part of \( \mathfrak{vect}(1; \underline{n}) \) have to be conjugate (similar) up to a non-zero scalar factor, i.e., if \( A \) and \( A' \) are such operators, then \( A' = \alpha \text{MAM}^{-1} \) for some non-zero \( \alpha \in \mathbb{K} \) and an invertible linear map \( M \). If two operators are conjugate up to a scalar multiple \( \alpha \), the roots of their characteristic polynomials differ by the same multiple, so if the dimension of the space they act on is \( d \) (in our case, \( d = 2^{n-1} \)) and one polynomial is equal to \( \sum z_i \lambda^i \), where \( z_i \in \mathbb{K} \) and \( z_d = 1 \), then the other one would have the form \( \sum z_i \alpha^{k-i} \lambda^i \). So, if Conjecture 6.4.2 is correct, then for the superizations given by \( u_1 \) and \( u_2 \) to be isomorphic, there must exist non-zero \( \alpha \in \mathbb{K} \) such that

\[
\lambda^{2n-1} + \sum_{0 \leq k \leq n-2} c_{n-1-k}^{k+1} \lambda^{2k} = \lambda^{2n-1} + \sum_{0 \leq k \leq n-2} \alpha^{2n-1-2k} b_{n-1-k}^{k+1} \lambda^{2k},
\]

or, equivalently, \( c_k = \varepsilon^{2k-1} b_k \), where \( \varepsilon = \sqrt{\alpha} \), for all \( k = 1, \ldots, n-1 \).

So the set of equivalence classes of superizations of \( \mathfrak{vect}^{(1)}(1; \underline{n}) \) corresponding to generating functions \( u \) such that \( u(0) = 0 \) consist of the following two types:

A) the superization corresponding to \( u = 0 \), which is isomorphic to \( \mathfrak{t}(1; n-1|1) \),

B) an \((n-2)\)-parametric family of its pairwise non-isomorphic deformations. Note, though, that it is not a result of \((n-2)\)-parametric deformation of \( \mathfrak{t}(1; n-1|1) \); to obtain all these deformations, an \((n-1)\)-parametric deformation is needed, but the deformations obtained from some sets of
parameters are isomorphic: parameters \((c_1, \ldots, c_{n-1})\) and \((b_1, \ldots, b_{n-1})\) produce isomorphic deformations if and only if there exists \(\varepsilon \in \mathbb{K}^\times\) such that \(c_k = \varepsilon^{2k-1} b_k\) for all \(k \in \overline{1, n-1}\).

6.5. **Superizations of** \(\text{vect}^{(1)}(1; \overline{3})\). Let \(\mathfrak{g} = \text{vect}^{(1)}(1; \overline{3})\).

6.5.1. **Grading** \((0, 0, 0)\). The Lie subsuperalgebra \(\mathfrak{g}^{<1>}\), see eq. (24), is as follows:

\[
\begin{align*}
\mathfrak{g}^{<1>}_0 &= \text{Span}(e_{-2} = \partial^2, e_0 = x\partial, e_2 = x^{(3)}\partial, e_4 = x^{(5)}\partial, e_6 = x^{(7)}\partial), \\
\mathfrak{g}^{<1>}_1 &= \text{Span}(o_{-1} = \partial, o_1 = x^{(2)}\partial, o_3 = x^{(4)}\partial, o_5 = x^{(6)}\partial).
\end{align*}
\]

The squaring in \(S(\mathfrak{g}^{<1>})\) is given by the following formulas:

\[
o_{-1}^2 = e_{-2}, \quad o_1^2 = e_2, \quad o_3^2 = e_6, \quad o_5^2 = 0.
\]

The nonzero commutation relations in \((\mathfrak{g}^{<1>})_0\) are given by the following formulas:

\[
[e_2, e_{-2}] = e_0, \quad [e_4, e_{-2}] = e_2, \quad [e_{-2}, e_6] = e_4.
\]

The \((\mathfrak{g}^{<1>})_0\)-module structure in \((\mathfrak{g}^{<1>})_1\) is given by the following formulas:

\[
\begin{align*}
[e_{-2}, o_{-1}] &= 0, & [e_0, o_{-1}] &= o_{-1}, & [e_2, o_{-1}] &= o_1, & [e_4, o_{-1}] &= o_3, & [e_6, o_{-1}] &= o_5, \\
[e_{-2}, o_1] &= o_{-1}, & [e_0, o_1] &= o_1, & [e_2, o_1] &= 0, & [e_4, o_1] &= o_5, & [e_6, o_1] &= 0, \\
[e_{-2}, o_3] &= o_1, & [e_0, o_3] &= o_3, & [e_2, o_3] &= o_5, & [e_4, o_3] &= 0, & [e_6, o_3] &= 0, \\
[e_{-2}, o_5] &= o_3, & [e_0, o_5] &= o_5, & [e_2, o_5] &= 0, & [e_4, o_5] &= 0, & [e_6, o_5] &= 0.
\end{align*}
\]

Let \(L_k\) and \(L^{(k)}\) be given by the formulas

\[
L_0 = L^0 = (\mathfrak{g}^{<1>})_0; \quad L_k = [L_0, L_{k-1}] \text{ and } L^{(k)} = [L^{(k-1)}, L^{(k-1)}] \text{ for } k > 0.
\]

We have

\[
L_1 = \text{Span}(e_0, e_2, e_4), \quad L_2 = \text{Span}(e_0, e_2), \quad L_3 = \mathbb{K}e_0, \quad L_4 = 0.
\]

6.5.2. **Grading** \((0, 1, 0)\). The Lie subsuperalgebra \(\mathfrak{g}^{<1>}\), see eq. (24), is as follows:

\[
\begin{align*}
\mathfrak{g}^{<1>}_0 &= \text{Span}(e_1 = (x + x^{(2)})\partial, e_2 = x^{(3)}\partial, e_3 = (x^{(5)} + x^{(6)})\partial, e_4 = \partial^2 + (1 + x)\partial, e_5 = x^{(7)}\partial), \\
\mathfrak{g}^{<1>}_1 &= \text{Span}(o_1 = (1 + x)\partial, o_2 = x^{(2)}\partial, o_3 = (x^{(4)} + x^{(5)})\partial, o_4 = x^{(6)}\partial).
\end{align*}
\]

The squaring in \(S(\mathfrak{g}^{<1>})\) is given by the following formulas:

\[
o_1^2 = e_4, \quad o_2^2 = e_2, \quad o_3^2 = e_5, \quad o_4^2 = 0.
\]

The commutation relations in \((\mathfrak{g}^{<1>})_0\) are defined by the following formulas:

\[
[e_1, e_2] = 0, \quad [e_1, e_3] = 0, \quad [e_1, e_4] = e_1, \quad [e_1, e_5] = 0, \quad [e_2, e_3] = 0, \quad [e_2, e_4] = e_1, \\
[e_2, e_5] = 0, \quad [e_3, e_4] = e_2 + e_3, \quad [e_3, e_5] = 0, \quad [e_4, e_5] = e_3.
\]

The \((\mathfrak{g}^{<1>})_0\)-module structure in \((\mathfrak{g}^{<1>})_1\) is given by the following formulas:

\[
\begin{align*}
[e_1, o_1] &= o_1 + o_2, & [e_2, o_1] &= o_2, & [e_3, o_1] &= o_3 + o_4, & [e_4, o_1] &= 0, & [e_5, o_1] &= o_1, \\
[e_1, o_2] &= o_2, & [e_2, o_2] &= 0, & [e_3, o_2] &= o_4, & [e_4, o_2] &= o_1 + o_2, & [e_5, o_2] &= 0, \\
[e_1, o_3] &= o_3 + o_4, & [e_2, o_3] &= o_4, & [e_3, o_3] &= 0, & [e_4, o_3] &= o_2, & [e_5, o_3] &= 0, \\
[e_1, o_4] &= o_4, & [e_2, o_4] &= 0, & [e_3, o_4] &= 0, & [e_4, o_4] &= o_3 + o_4, & [e_5, o_4] &= 0.
\end{align*}
\]
6.5.3. Grading \((0, 1, \beta)\), where \(\beta \neq 0\). The Lie subsuperalgebra \(\mathfrak{g}^{<1>}\), see eq. (24), is as follows:

\[
\begin{align*}
(\mathfrak{g}^{<1>})_0 &= \text{Span}(e_1 = (x + x^{(2)} + \beta x^{(4)})\partial, e_2 = (x^{(3)} + \beta x^{(5)})\partial, e_3 = (x^{(5)} + x^{(6)})\partial, \\
&\hspace{1cm} e_4 = \partial^2 + (1 + x + \beta x^{(2)})\partial, e_5 = (x^{(3)} + \beta x^{(5)} + \beta^2 x^{(7)})\partial), \\
(\mathfrak{g}^{<1>})_1 &= \text{Span}(o_1 = (1 + x + \beta x^{(3)})\partial, o_2 = (x^{(2)} + \beta x^{(4)})\partial, o_3 = (x^{(4)} + x^{(5)})\partial, o_4 = x^{(6)}\partial).
\end{align*}
\]

The squaring in \(S(\mathfrak{g}^{<1>})\) is given by the following formulas:

\[
\begin{align*}
o_1^2 &= e_4, & o_2^2 &= e_5, & o_3^2 &= \frac{1}{\beta^2}e_2 + \frac{1}{\beta^3}e_5, & o_4^2 &= 0.
\end{align*}
\]

The commutation relations in \((\mathfrak{g}^{<1>})_0\) are defined by the following formulas:

\[
\begin{align*}
[e_1, e_2] &= 0, & [e_1, e_3] &= 0, & [e_1, e_4] &= e_1 + \beta e_2, & [e_1, e_5] &= 0, \\
[e_2, e_3] &= 0, & [e_2, e_4] &= e_1 + \beta e_2 + \beta^2 e_3, & [e_2, e_5] &= 0, \\
[e_3, e_4] &= e_2 + (1 + \beta) e_3, & [e_3, e_5] &= 0, & [e_4, e_5] &= e_1 + \beta e_2.
\end{align*}
\]

The \((\mathfrak{g}^{<1>})_0\)-module structure in \((\mathfrak{g}^{<1>})_1\) is given by the following formulas:

\[
\begin{align*}
[e_1, o_1] &= o_1 + o_2 + \beta o_4, & [e_2, o_1] &= o_2, & [e_3, o_1] &= o_3 + o_4, & [e_4, o_1] &= 0, & [e_5, o_1] &= o_2 + \beta^2 o_4, \\
[e_1, o_2] &= o_2, & [e_2, o_2] &= o_2, & [e_3, o_2] &= o_4, & [e_4, o_2] &= o_1 + (1 + \beta) o_2 + \beta^2 o_3, & [e_5, o_2] &= 0, \\
[e_1, o_3] &= o_3 + o_4, & [e_2, o_3] &= o_4, & [e_3, o_3] &= 0, & [e_4, o_3] &= o_2 + \beta (o_3 + o_4), & [e_5, o_3] &= o_4, \\
[e_1, o_4] &= o_4, & [e_2, o_4] &= 0, & [e_3, o_4] &= 0, & [e_4, o_4] &= o_3 + o_4, & [e_5, o_4] &= 0.
\end{align*}
\]

6.5.4. Grading \((0, 0, 1)\). The Lie subsuperalgebra \(\mathfrak{g}^{<1>}\), see eq. (24), is as follows:

\[
\begin{align*}
(\mathfrak{g}^{<1>})_0 &= \text{Span}(e_1 = (x + x^{(4)})\partial, e_2 = x^{(5)}\partial, e_3 = (x^{(3)} + x^{(6)})\partial, e_4 = \partial^2 + x^{(2)}\partial, e_5 = x^{(7)}\partial), \\
(\mathfrak{g}^{<1>})_1 &= \text{Span}(o_1 = (1 + x^{(3)})\partial, o_2 = x^{(4)}\partial, o_3 = (x^{(2)} + x^{(5)})\partial, o_4 = x^{(6)}\partial).
\end{align*}
\]

The squaring in \(S(\mathfrak{g}^{<1>})\) is given by the following formulas:

\[
\begin{align*}
o_1^2 &= e_4, & o_2^2 &= e_5, & o_3^2 &= e_3, & o_4^2 &= 0.
\end{align*}
\]

The commutation relations in \((\mathfrak{g}^{<1>})_0\) are defined by the following formulas:

\[
\begin{align*}
[e_1, e_2] &= 0, & [e_1, e_3] &= 0, & [e_1, e_4] &= e_2, & [e_1, e_5] &= 0, & [e_2, e_3] &= 0, \\
[e_2, e_4] &= e_3, & [e_2, e_5] &= 0, & [e_3, e_4] &= e_1, & [e_3, e_5] &= 0, & [e_4, e_5] &= e_2.
\end{align*}
\]

The \((\mathfrak{g}^{<1>})_0\)-module structure in \((\mathfrak{g}^{<1>})_1\) is given by the following formulas:

\[
\begin{align*}
[e_1, o_1] &= o_1 + o_4, & [e_2, o_1] &= o_2, & [e_3, o_1] &= o_3, & [e_4, o_1] &= 0, & [e_5, o_1] &= o_4, \\
[e_1, o_2] &= o_2, & [e_2, o_2] &= o_2, & [e_3, o_2] &= o_4, & [e_4, o_2] &= o_3, & [e_5, o_2] &= 0, \\
[e_1, o_3] &= o_3, & [e_2, o_3] &= o_4, & [e_3, o_3] &= 0, & [e_4, o_3] &= o_1 + o_4, & [e_5, o_3] &= 0, \\
[e_1, o_4] &= o_4, & [e_2, o_4] &= 0, & [e_3, o_4] &= 0, & [e_4, o_4] &= o_2, & [e_5, o_4] &= 0.
\end{align*}
\]

6.5.5. Grading \((1, 0, 0)\). The Lie subsuperalgebra \(\mathfrak{g}^{<1>}\), see eq. (24), is as follows:

\[
\begin{align*}
(\mathfrak{g}^{<1>})_0 &= \text{Span}(e_{-2} = \partial^2 + (1 + x)\partial, e_{-1} = (1 + x)\partial, e_0 = (x + x^{(2)} + x^{(3)})\partial, \\
e_1 &= (x^{(2)} + x^{(4)} + x^{(5)})\partial, e_2 = (x^{(3)} + x^{(5)} + x^{(6)} + x^{(7)})\partial), \\
(\mathfrak{g}^{<1>})_1 &= \text{Span}(o_1 = \partial, o_2 = (x + x^{(2)})\partial, o_3 = (x^{(2)} + x^{(3)} + x^{(4)})\partial, o_4 = (x^{(3)} + x^{(5)} + x^{(6)})\partial).
\end{align*}
\]

The squaring in \(S(\mathfrak{g}^{<1>})\) is given by the following formulas:

\[
\begin{align*}
o_1^2 &= e_{-2}, & o_2^2 &= e_0, & o_3^2 &= e_2, & o_4^2 &= 0.
\end{align*}
\]
The commutation relations in \( (g^{<1>})_0 \simeq (\text{vect}(1; 2))^{<1>} \) are defined by the following formulas:

\[
\begin{align*}
[e_{-1}, e_0] &= e_{-1}, & [e_{-1}, e_1] &= e_0, & [e_{-1}, e_2] &= e_1, & [e_{-1}, e_{-2}] &= 0, & [e_0, e_1] &= e_1, \\
[e_0, e_2] &= 0, & [e_0, e_{-2}] &= 0, & [e_1, e_2] &= 0, & [e_1, e_{-2}] &= e_{-1}, & [e_2, e_{-2}] &= e_0
\end{align*}
\]

The \( (g^{<1>})_0 \)-module structure in \( (g^{<1>})_1 \) is given by the following formulas:

\[
\begin{align*}
[e_{-1}, o_1] &= o_1, & [e_{-1}, o_2] &= 0, & [e_{-1}, o_3] &= 0, & [e_{-1}, o_4] &= 0, \\
[e_{-1}, o_2] &= o_1 + o_2, & [e_{-1}, o_3] &= o_2 + o_3 + 2, & [e_{-1}, o_4] &= o_2 + o_3 + 3, \\
[e_{-1}, o_3] &= o_2 + o_3, & [e_{-1}, o_4] &= o_3 + 3, & [e_{-1}, o_4] &= o_3 + 4, & [e_{-2}, o_1] &= o_1.
\end{align*}
\]

We have

\[
[(g^{<1>})_0, (g^{<1>})_0] = \text{Span}(e_{-1}, e_0, e_1), \quad [(g^{<1>})_0, [(g^{<1>})_0, (g^{<1>})_0]] = [(g^{<1>})_0, (g^{<1>})_0].
\]

The ideal \([g^{<1>})_0, (g^{<1>})_0]\] is isomorphic to \( \mathfrak{o}^{(1)}(3) \). The \( \mathfrak{o}^{(1)}(3) \)-module \( (g^{<1>})_1 \) is irreducible and has no lowest weight vectors and with highest vectors \( o_3 \) and \( o_4 \), cf. [D0].

6.6. Summary of computer-aided experiments. Let \( L_k \) and \( L^{(k)} \) be given by eq. (27); in tables below, \( k = 1, 2, \ldots \) up to first stable term. Theorem 6.2 implies the following fact

\[
\text{sdim } S(g, D_{u(c)}) = (2^{n-1} + 1)2^{n-1} \text{ and dim}[(g^{<1>})_0, (g^{<1>})_0] = 2^{n-1} - 1.
\]

Computer-aided experiments show that (recall that \( v_n \) is defined by (14); the symbol “solv” means that the corresponding Lie algebra is solvable, see Corollary 6.2.2)

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{vect}^{(1)}(1; 2) & (g^{<1>})_0 & \text{dim } L_k & \text{dim } L^{(k)} & \text{Parameters} \\
\hline
\text{Heisenberg} & 1, 0 & 1, 0 & (0) & \\
\hline
\text{solv, see } 25 & 1 & 1, 0 & (01), (10), (11) & \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{vect}^{(1)}(1; 3) & (g^{<1>})_0 & \text{dim } L_k & \text{dim } L^{(k)} & \text{Parameters} \\
\hline
\mathfrak{o}(3)/\mathfrak{c} \simeq v_3 & 3 & 3 & (1ab) & \\
\hline
\text{solv} & 3, 2, 1, 0 & 3, 0 & (000) & \\
\hline
\text{solv} & 3, 2 & 3, 0 & (010), (010), (000), where \( \alpha \neq 0 \) & \\
\hline
\text{solv} & 3 & 3, 0 & (001), (011), (01\alpha\beta), where \( \beta \neq 0 \) & \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|}
\hline
\text{vect}^{(1)}(1; 4) & g^{<1>}_0 & \text{dim } L_k & \text{dim } L^{(k)} & \text{Parameters} \\
\hline
\text{v}_4 & 7 & 7 & (1000) \text{ and } (1abc) & \\
\hline
\text{solv} & 7, 6, 5, 4, 3, 2, 1, 0 & 7, 0 & (0000) & \\
\hline
\text{solv} & 7, 6, 4 & 7, 0 & (0100) & \\
\hline
\text{solv} & 7, 6 & 7, 0 & (0010), (0110) & \\
\hline
\text{solv} & 7 & 7, 0 & (0001), (0011), (0101), (0111) & \\
\hline
\end{array}
\]
Let $i$ be the last term of the sequence $L_k$, see (27), for the even part $\mathfrak{g}_0$ of the corresponding Lie superalgebra $\mathfrak{g}$. Set

$$c(i) := (0 \ldots 010 \ldots 0)$$

with a 1 in the $(i+1)$st slot.

6.6.1. Conjecture. Grading $c(k)$, where $k = 1, \ldots, n-1$, yields the Lie superalgebras such that $\dim i = \sum_{1 \leq i \leq k} 2^{n-1-i}$. The corresponding generating function $u$ is $x^{(2^k)}$.

6.6.2. Conjecture. Let $u = \sum_{1 \leq i \leq n-1} c_i x^{(2^i)}$, i.e., $c_0 = 0$. Let $k$ be the maximal number such that $c_k \neq 0$ and $c_{k+1} = \ldots = c_{n-1} = 0$. Then the grading given by $D_u$ yields the Lie superalgebras such that $\dim i = \sum_{1 \leq i \leq k} 2^{n-1-i}$.

6.7. The derivations of $\textbf{vect}(1; \mathbf{n})$. The result of this subsection is probably known, but we’d like to draw attention to Sierpiński sieves here. Observe that solutions of the linear equations (8b), i.e., derivations of $\textbf{vect}(1; \mathbf{n})$, form the Sierpiński sieve of order $\mathbf{n}$ under the main diagonal, i.e., for $i \geq j$, we have

$$c_{i,j} = \binom{i}{j-1} c_{i-j+1,1},$$

which corresponds to the derivation $\text{ad}_{x^{(i-j+1)}}\mathfrak{g}$.

and nonzero diagonals with parameters $c_{1,2}$, $c_{1,3}$, $c_{1,5}$, $c_{1,2n+1}$, e.g., for $i > j$, we have

$$c_{1,j} = 0 \quad \text{if } j \neq 2^k + 1, \text{ where } k \text{ is a non-negative integer},$$

$$c_{i,j} = c_{i-1,j-1} \text{ for } i > 1, \text{ which corresponds to the derivation } \text{ad}_{x^{(i)}}\mathfrak{g}.$$
For $n = 4$, we have the following schematic form of the solution of linear equations (8b):

\[
\begin{pmatrix}
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
\end{pmatrix}
\]

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