Lehn-Sorger example via Cox ring and torus action

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Abstract. We present a new approach to effective construction of Cox rings of crepant resolutions of quotient singularities based on the action of an algebraic torus. In the present paper we deal with a symplectic resolution of a quotient of \( \mathbb{C}^4 \) by a reducible symplectic representation of the binary tetrahedral group which admits an action of rank two algebraic torus \((\mathbb{C}^*)^2\). We use the Lefschetz-Riemann-Roch theorem for the equivariant Euler characteristic as well as the Kawamata-Viehweg vanishing and the multigraded Castelnuovo-Mumford regularity for a verification that the construction method established in \([15, 13, 24]\) leads effectively to the Cox ring of a resolution of the singularity under the consideration.

1. Introduction

In this article we inspect the interaction between three themes – torus actions, crepant resolutions of quotient singularities and Cox rings – in the study of crepant resolutions of a quotient variety \( \mathbb{C}^4/G \) for a reducible symplectic representation \( G \) of a binary tetrahedral group generated by the matrices:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}, \quad -\frac{1}{2} \begin{pmatrix}
(1+i)\epsilon & (-1+i)\epsilon & 0 & 0 \\
(1+i)\epsilon & (1-i)\epsilon & 0 & 0 \\
0 & 0 & (1+i)\epsilon^2 & (-1+i)\epsilon^2 \\
0 & 0 & (1+i)\epsilon^2 & (1-i)\epsilon^2
\end{pmatrix}.
\]

By the classical theorem of Hilbert and Noether, an invariant ring \( \mathbb{C}[x_1, \ldots, x_n]^G \), where \( G \) is a finite group acting linearly on polynomial ring, is a finitely generated algebra. Quotient singularities are the corresponding singular varieties of the form \( \mathbb{C}^n/G = \text{Spec} \mathbb{C}[x_1, \ldots, x_n]^G \). The study of their crepant resolutions generalizes the theory of minimal resolutions of du Val singularities. It shows an interplay between geometry and the theory of finite groups celebrated in the McKay correspondence \([22]\).

The Cox ring of a normal algebraic variety \( X \) with a finitely generated class group is a \( \text{Cl}(X) \)-graded ring (see \([4]\) for a precise construction and a detailed exposition):

\[
\mathcal{R}(X) = \bigoplus_{D \in \text{Cl}(X)} H^0(X, D).
\]

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If $\mathcal{R}(X)$ is a finitely generated $\mathbb{C}$-algebra it gives a powerful tool to study the geometry of $X$ and its small modifications. In particular one may recover $X$ as a GIT quotient of $\text{Spec } \mathcal{R}(X)$ by the action of the Picard (quasi)torus $T = \text{Hom} (\text{Cl}(X), \mathbb{C}^*)$. This is the case in our setting, i.e. if $X$ is a crepant resolution of a four-dimensional symplectic quotient singularity, then $\mathcal{R}(X)$ is a finitely generated algebra, see [3, Theorem 3.2] and [23, Theorem 1.2].

Varieties with an action of an algebraic torus $(\mathbb{C}^*)^r$ form an important class of objects in algebraic geometry due to their relations with combinatorics and convex geometry. Well-studied examples include toric varieties [10], but not only, see also [2]. Of the most importance for us will be the recent methods presented in [8]. They enable us to describe combinatorially local data of the torus action around fixed points and to connect them with the cohomology of line bundles via the Lefschetz-Riemann-Roch theorem.

The quotient $\mathbb{C}^4/G$ was an object of study in [6], where it was shown that the crepant resolution exists, and in [21], where such resolution was constructed as a sequence of two blow-ups. Here however we take a different approach, originated in [12] and [15], and developed in [13] and [24], aiming at generalizations to the study of crepant resolutions of other quotient singularities. The idea is to study the geometry of crepant resolution via the algebraic and combinatorial properties encoded by its Cox ring. In particular one would like to find the presentation of the Cox ring and construct a resolution as a GIT quotient.

One of the previous attempts to give a general framework for computing Cox rings of crepant resolutions of quotient singularities was based on the embedding of the Cox ring into the ring of Laurent polynomials over the invariant ring of the commutator subgroup. Then, using the McKay correspondence one constructs a finitely generated subring $\mathcal{R}$ which is a candidate to be the whole Cox ring of a resolution. The nontrivial step is to check if this ring is the actual Cox ring. See [12], [15], [13], [24] for examples of this approach. In fact the (non-minimal) list of generators of the Cox ring of the crepant resolution of $\mathbb{C}^4/G$ were originally found by Yamagishi by an algorithm presented in [24, §4]. In this paper we present an alternative, more geometric method to deal with this difficulty, using a torus action on the resolution.

We first study the geometry of a GIT quotient $\text{Spec } \mathcal{R}/T$. By a general result of Kaledin [19, § 4] a crepant resolution of a four-dimensional symplectic quotient singularity admit a stratification whose closed strata are the exceptional divisors in dimension 3 and a fibre over point 0 in dimension 2. This fibre encodes some important geometric properties of the resolution, in particular in our case it contains all the fixed points of the torus action. In section 3 assuming that the GIT quotient $\text{Spec } \mathcal{R}/T$ is a resolution we describe in detail its central fibre, see theorems 3.4, 3.5 and 3.7. Then, using methods of [8] we give conditionally the local description of a torus action around the fixed points, see theorem 4.5. This gives us an open cover by smooth affine varieties. We then prove unconditionally that the elements of this open cover are indeed smooth in theorem 5.1 As a consequence the GIT quotient $X = \text{Spec } \mathcal{R}/T$ is a crepant resolution and the results of sections 3 and 4 are unconditionally valid see 5.3.

As the study of toric varieties and $T$-varieties of complexity one shows, a torus action on a variety may have significant consequences for the structure of its Cox ring. Inspired by the methods of [8] we use the Lefschetz-Riemann-Roch formula [5].
for the equivariant Euler characteristic and the Kawamata-Viehweg vanishing to calculate the generating function for dimensions of movable linear systems on a resolution $X \to \mathbb{C}^4/G$ in theorem \[6,3\]. Finally, in section \[7\] we show that one can use the multigraded Castelnuovo-Mumford regularity and the Kawamata-Viehweg vanishing to reduce the proof of equality $\mathcal{R}(X) = \mathcal{R}$ to checking equalities between finite number of graded pieces. Having a formula like one of theorem \[6,3\] and the ideal of relations between the generators of the candidate ring $\mathcal{R}$ this can be done by a computer calculation as we do in the case of singularity $\mathbb{C}^4/G$.

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2. The singularity, its resolutions and their Cox ring

In this section we collect known facts on the group $G \subset \text{Sp}_4(\mathbb{C})$ investigated by Bellamy and Schedler in \[6\] and by Lehn and Sorger in \[21\], the corresponding symplectic quotient singularity $\mathbb{C}^4/G$ and its symplectic resolution $X$. In particular we introduce two $\mathbb{Z}^2$-gradings on the Cox ring of $X$, one induced by the Picard torus action and one induced by the action of a two-dimensional torus on $X$.

Let $G \subset \text{Sp}_4(\mathbb{C})$ be the symplectic representation of binary tetrahedral group generated by the matrices:

$$
\begin{pmatrix}
i & 0 & 0 & 0 \\
0 & -i & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -i
\end{pmatrix},
\quad
-\frac{1}{2}
\begin{pmatrix}
(1+i)\epsilon & (1+i)\epsilon & 0 & 0 \\
(1+i)\epsilon & (1-i)\epsilon & 0 & 0 \\
0 & 0 & (1+i)\epsilon^2 & (1-i)\epsilon^2 \\
0 & 0 & (1+i)\epsilon^2 & (1-i)\epsilon^2
\end{pmatrix},
$$

where $\epsilon = e^{2\pi i/3}$ is a third root of unity.

Proposition 2.1.

1. There are 7 conjugacy classes of elements of $G$ among which two consist of symplectic reflections.
2. The commutator subgroup $[G, G]$ has order 8 and it is isomorphic to the quaternion group. In particular the abelianization $G/[G, G]$ is cyclic of order 3.
3. The representation $G$ defined above is reducible. It decomposes into two two-dimensional representations $V_1 \oplus V_2$. In particular there is a natural $(\mathbb{C}^*)^2$-action on $\mathbb{C}^4/G$ induced by the multiplication by scalars on $V_i$.

Proof. Points (1) and (2) can be quickly verified with GAP Computer Algebra System \[16\]. Point (3) follows directly by the definition of $G$. \hfill \square

Let $\Sigma \subset \mathbb{C}^4/G$ be the singular locus of the quotient. It can be described as follows.
Proposition 2.2 ([21] §1). The preimage of $\Sigma$ via the quotient map $\mathbb{C}^4 \to \mathbb{C}^4/G$ consists of four planes, each of which maps onto $\Sigma$. Outside the image of 0 the singular locus is a transversal $A_2$-singularity.

Let $\pi : X \to \mathbb{C}^4/G$ be a projective symplectic resolution which exists by [6] or by [21]. Using the symplectic McKay correspondence [20] and propositions 2.1(1) and 2.2 we obtain the following facts about the geometry of $X$.

Proposition 2.3. There are two exceptional divisors $E_1, E_2$ of $X$ each of which is mapped onto $\Sigma$. The central fibre $\pi^{-1}(0)$ consist of four surfaces. The fibre of $\pi$ over any point in $\Sigma \setminus \{0\}$ consists of two curves isomorphic to $\mathbb{P}^1$ intersecting in one point and each of which is contained in exactly one of two exceptional divisors.

By a theorem of Kaledin [19] Thm 1.3 and by proposition 2.1(3) we have also:

Corollary 2.4. There is a natural $T := (\mathbb{C}^*)^2$-action on $X$ making $\pi$ an equivariant map.

Remark 2.5. Beware that throughout this article we consider two different two-dimensional tori – one is the Picard torus $\mathbb{T} := \text{Hom}(\text{Cl}(X), \mathbb{C}^*)$ and the other one is the torus $T = \mathbb{C}^*$ acting on $\mathbb{C}^4 = \mathbb{C}^2 \times \mathbb{C}^2$ by multiplication of scalars on each component $\mathbb{C}^2$.

Let $C_1$ be the numerical class of a complete curve which is a generic fibre of the morphism $\pi|_{E_1} : E_1 \to \Sigma$. We may describe the generators of the Picard group of $X$ in terms of their intersections with curves $C_1$.

Proposition 2.6 (cf. [15] 2.D). The Picard group of $X$ is a free rank two abelian group generated by line bundles $L_1, L_2$ such that the intersection matrix $(L_i, C_j)_{i,j}$ is equal to identity matrix.

One may see the Cox ring $\mathcal{R}(X)$ of $X$ as a subring of $\mathbb{C}[x_1, x_2, x_3, x_4]^{[G,G]}[t_1^{\pm 1}, t_2^{\pm 1}]$ (see [13] § 2.1 for details). It follows that the action of the two-dimensional torus $T = (\mathbb{C}^*)^2$ on $X$ induces the action on Spec $\mathcal{R}(X)$. The algorithm given in [24] § 4 allows one to compute the presentation of $\mathcal{R}(X)$. Denote the following elements of the (Laurent) polynomial ring $\mathbb{C}[x_1, y_1, x_2, y_2][t_1^{\pm 1}, t_2^{\pm 1}]$:

\begin{align*}
w_{01} &= y_1 x_2^2 - x_1 y_2^2, \\
w_{02} &= x_2^2 y_2 - x_2 y_2^2, \\
w_{03} &= x_1^2 y_1 - x_1 y_1^2, \\
w_{04} &= x_1^2 + (-6b + 2)x_1 y_1^2 + y_1^4, \\
w_{05} &= x_1 x_2^2 + (4b - 2)x_2^2 y_1^2 + y_1^4, \\
w_{06} &= x_1 x_2^2 + (-2b + 1)x_1 x_2 y_2^2 + (-6b + 1)x_1 x_2 y_2^2 + y_1 y_2^2, \\
w_{07} &= x_1 x_2^2 + (2b - 1)x_1 y_2^2 (x_2 + (2b - 1)x_1 y_1 y_2 + y_1 y_2^2), \\
w_{11} &= (-3b x_1 y_2^2 + (-b + 2)y_2^2 x_2^2 + (4b + 8)x_1 y_1 y_2 y_2 - (2b + 3) x_1^2 y_2^3 - 30 y_1 y_2^4) t_1, \\
w_{12} &= (x_2^2 + (4b + 2)x_2^2 y_2^2 + y_2^4) t_1, \\
w_{13} &= (x_1 x_2^2 + (-2b + 1)x_1 y_1 y_2^2 - (2b - 1)x_2 y_1 y_2^2 + y_1^2 y_2^2) t_1, \\
w_{14} &= (-5x_1^2 x_2 y_2 + 9y_2^2 - x_1^2 y_1 y_2 + 5x_1 y_2 y_2 y_1) t_1, \\
w_{15} &= (x_1 x_2 y_2^2 + 2x_2^2 x_2 y_2^2 - 2x_1^2 x_2 y_2^2 - x_1 y_1 y_2^2) t_1, \\
w_{21} &= (3b - 3)x_1 x_2^2 + (b + 1)y_1^2 x_2^2 + (4b + 4)x_1 y_1 y_2 y_2 + (b + 1)x_1^2 y_2^2 + (3b - 3)y_1 y_2^2) t_2, \\
w_{22} &= (x_1^2 + (4b - 2)x_1 y_1^2 + y_1^4) t_2, \\
w_{23} &= (x_1 x_2^2 + (4b - 1)y_1 x_2 y_2 + (2b - 1)x_1 x_2 y_2 + y_1 y_2^2) t_2, \\
w_{24} &= (y_1 y_2^2 + 5x_1 x_2 y_2 - 9y_1^2 x_2^2 + y_1^2 y_2^2) t_2, \\
w_{25} &= (-2y_1^2 x_2 y_2 + x_1^2 y_2^2 + 9y_2^2 - 2x_1 x_2 y_2 + 2x_1 y_1 y_2^2) t_2, \\
w_{3} &= (9x_1^2 y_2^2 + (-2b + 1)y_2^2 x_2^2 + 9x_1^2 x_2^2 y_2 + (6b - 3)x_1 y_1 x_2^2 + 2b - 9y_1^2 x_2^2 + (2b - 1)x_2^2 y_1^2 + 9x_1 y_1 y_2^2) t_1 t_2, \\
s &= t_1 t_2, \\
t &= t_1^2 t_2^2.
\end{align*}
where $b$ is a primitive root of unity of order 6.

**Theorem 2.7.** The Cox ring $R(X)$ of $X$ is generated by 20 generators:

$$w_{01}, \ldots, w_{07}, w_{11}, \ldots, w_{15}, w_{21}, \ldots, w_{25}, w_3, s, t.$$  

The degree matrix of this generators with respect to the generators $L_1, L_2$ of Pic($X$) (first two rows) and with respect to the $T$-action (remaining two rows) is:

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 6 & 4 & 0 & 1 & 3 & 2 & 0 & 3 & 5 & 2 & 4 & 2 & 4 & 1 & 1 & 4 & 3 & 0 & 0 \\
1 & 6 & 0 & 0 & 4 & 3 & 1 & 2 & 4 & 0 & 1 & 1 & 4 & 2 & 0 & 3 & 5 & 2 & 3 & 0 & 0 \\
\end{pmatrix}
$$

We will reprove this theorem in section 7. From now on we will denote the ring generated by elements from the statement of theorem 2.7 by $R$.

It is a general principle that using the degrees of generators of the Cox ring of $X$ we can describe the movable cone $\text{Mov}(X)$ of $X$ and find the number of resolutions and the corresponding subdivision of $\text{Mov}(X)$ into the nef cones of resolutions of $X$ (see [4], Prop. 3.3.2.9). However, since we will be using this two results to prove that $R$ is a Cox ring of $X$ we give independent proofs.

**Proposition 2.8.**

(1) The cone $\text{Mov}(X)$ of movable divisors of $X$ is the cone generated by the line bundles $L_1$ and $L_2$.

(2) There are two symplectic resolutions of $\mathbb{C}^4/G$. The chambers in $\text{Mov}(X)$ corresponding to the nef cones of these resolutions are $\text{cone}(L_1, L_1 + L_2)$ and $\text{cone}(L_2, L_1 + L_2)$. The Mori cones of corresponding resolutions are $\text{cone}(C_2, C_1 - C_2)$ and $\text{cone}(C_1, C_2 - C_1)$.

**Proof.**

(1) This follows from [3, Theorem 3.5].

(2) We will prove the first part of the claim in section 6 as proposition 6.2.

The part on Mori cones then follows by taking dual cones.

**Corollary 2.9.** Taking a GIT quotient of $\text{Spec } R$ by the Picard torus action with respect to the linearization given by character $(a, b)$ with $a > b > 0$ and with $b > a > 0$ one obtains the two symplectic resolutions of $\mathbb{C}^4/G$.

**Remark 2.10.** The weights of the $T$-action on global sections of the fixed line bundle $L$ on $X$ are lattice points in $\mathbb{Z}^2$. Taking a convex hull one obtains a lattice polyhedron in $\mathbb{R}^2$. For example fixing a line bundle $L = 2L_1 + L_2$ one gets a polyhedron with tail equal to the positive quadrant of $\mathbb{R}^2$ and with a head spanned by the lattice points from the picture below:
By [8 Lemma 2.4(c)] if \( L \) is globally generated, then marked vertices of this polyhedron correspond to \( T \)-fixed points of \( X \) where \( X \) is the resolution on which \( L \) is relatively ample. We will see in [1,2] that indeed fixed points of this polytope correspond to points in \( X^T \) and in lemma 3.8 that \( L \) is globally generated.

3. The structure of the central fibre

In this section we will study the structure of the central fibre \( \pi^{-1}([0]) \) of such a resolution \( \pi: X \to \mathbb{C}^4/G \) using the ideal of relation of generators of the ring \( R \), under the assumption that \( X = \text{Spec } R//T \). The results of this section will be useful in the next one, where we will be investigating the action of the two-dimensional torus \( T \) on \( X \) with the fixed point locus \( X^T \) contained in the central fibre. The additional assumption that \( X = \text{Spec } R//T \) will be dealt with in section 5.

**Lemma 3.1.** We have an isomorphism \( \text{Spec } R^T \cong \mathbb{C}^4/G \). In particular the inclusion of invariants \( R^T \subset R \) induce map \( p: \text{Spec } R \to \mathbb{C}^4/G \).

**Proof.** This follows from the fact that \( R^T = \mathbb{C}[x_1, x_2, x_3, x_4]^G \) via the embedding \( R \subset \mathbb{C}[x_1, x_2, x_3, x_4]^{[G,G]}[t_1^{\pm 1}, t_2^{\pm 1}] \). \( \square \)

Let \( Z = p^{-1}([0]) \). Decomposing the ideal of relations from the presentation of Spec \( R \) one obtains the decomposition of \( Z \) into irreducible components. We consider the closed embedding Spec \( R \subset \mathbb{C}^{20} \) given by the generators of \( R \) from statement of theorem 2.7.
PROPOSITION 3.2. The components of $Z$ are the following subvarieties of $\mathbb{C}^{20}$:

- $Z_{0} = \{ (w_{3}, w_{1} ) | (i, j) \in \{ (0, 1), \ldots, (0, 7), (1, 1), \ldots, (1, 5), (2, 1), \ldots, (2, 5) \} \}$
- $Z_{0} = \{ (x, t, w_{25}, w_{24}, w_{15}, w_{14}, w_{07}, w_{06}, w_{05}, w_{04}, w_{03}, w_{02}, w_{01}) | w_{12}w_{22} = w_{13}w_{23}, w_{11}w_{21} = 9w_{13}w_{23}, w_{02} - 27w_{22}w_{23} - 3w_{11}w_{22}w_{23}, w_{12}w_{21} = 3w_{11}w_{22}, w_{13}w_{21} = w_{11}w_{22}w_{23}, w_{13}w_{21} - 3w_{11}w_{22}w_{23} \}$

where $\zeta_{3}, \zeta_{6}, \zeta_{12}$ are primitive 3rd, 6th and 12th roots of unity. The component $Z_{u}$ is contained in the locus of unstable points with respect to any linearization of Picard torus via character from movable cone. Points in the component $Z_{p}$ are unstable with respect to any linearization by a character $(2, 1)$ and points in the component $Z_{r}$ are unstable with respect to any linearization by a character $(1, 2)$.

LEMMA 3.3. The unstable locus of Spec $\mathcal{R}$ with respect to a linearization of the trivial line bundle by a character $(2, 1)$ is cut out by equations:

$$w_{12}w_{21} = w_{11}w_{21} = w_{12}w_{21} = w_{13}w_{21} = w_{12}w_{21} = w_{22} = 0.$$

Moreover all the semistable points of $Z$ are stable and have trivial isotropy groups.

PROOF. This can be done by a computer calculation, using the Singular library gitcomp by Maria Donten-Bury (see [url](www.mimuw.edu.pl/marysia/gitcomp.lib)).

The following theorem gives a description of components of the central fibre. Let $W$ be the locus of stable points of Spec $\mathcal{R}$ with respect to the $T$-action linearized by a character $(2, 1)$ (the case $(1, 2)$ is analogous) and consider the quotient map $W \to X$. Denote by $S_{0}, S_{1}, S_{2}, P$ the images of sets of stable points of the components $Z_{0}, Z_{1}, Z_{2}, Z_{P}$. Note that these are precisely the components of the central fibre of $X$.

THEOREM 3.4.

(a) $S_{0}$ is a non-normal toric surface whose normalization is isomorphic to the Hirzebruch surface $\mathbb{H}_{6}$. The action of $T$ on the normalization of $S_{0}$ is given by characters in the columns of the matrix $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

(b) $S_{1}$ is a non-normal toric surface whose normalization is the toric surface of a fan spanned by rays: $(0, 1), (1, 0), (1, -1), (-1, -2)$. The action of $T$ on the normalization of $S_{1}$ is given by characters in the columns of the matrix $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.

(c) $S_{2}$ is a non-normal toric surface whose normalization is the toric surface of a fan spanned by rays: $(0, 1), (1, -1), (-1, -2)$. The action of $T$ on the normalization of $S_{2}$ is given by characters in the columns of the matrix $\begin{pmatrix} -1 & -3 \\ 0 & 0 \end{pmatrix}$. 


(d) $P$ is isomorphic to $\mathbb{P}^2$. The action of $T$ on $P$ in homogeneous coordinates is given by the matrix \( \begin{pmatrix} \frac{2}{3} & 0 & \frac{1}{3} \end{pmatrix} \).

Proofs of (a)–(c) follow a fairly standard procedure of division of a toric variety by a torus action. We outline an argument in the case of $S_0$, and then we comment on (d).

**Proof of (A).**

**Claim 1.** By rescaling variables we may assume that $Z_0$ is the toric variety embedded into $\mathbb{C}^7$ with coordinates $w_{11}, w_{12}, w_{13}, w_{21}, w_{22}, w_{23}, w_3$ defined by the toric ideal generated by binomials:

\[
\begin{align*}
&w_{12}w_{22} - w_{13}w_{23}, \quad w_{11}w_{21} - w_{13}w_{23}, \quad w_{21}^3 - w_{22}w_{23}, \quad w_{13}w_3^2 - w_{11}w_{22}w_{23}, \\
&w_{12}w_{21}^2 - w_{11}w_{23}^2, \quad w_{13}w_{21} - w_{11}w_{22}, \quad w_{12}w_{13}w_{21} - w_{11}w_{23}, \quad w_{11}^3 - w_{12}w_{13}^2.
\end{align*}
\]

**Proof of the claim.** This is a general argument (over an algebraically closed field) to reduce a prime binomial ideal that does not contain monomials to a toric ideal: one takes a point $(a_{ij}, a_3)$ with all $a_{ij} \neq 0 \neq a_3$ in the zero set of the original ideal and then set new coordinates $w_{ij}' = \frac{1}{a_{ij}} w_{ij}$, $w_3' = \frac{1}{a_3} w_3$. □

**Claim 2.** $\tilde{S}$ is the affine variety of the cone $\sigma^\vee = \text{cone}(-2e_2 + 3e_3, e_1, e_1 + 3e_2 - 3e_3, e_2, e_4)$ in space $\mathbb{R}^4$ spanned by the character lattice $M = \mathbb{Z}^4$ of four-dimensional torus.

**Proof of the claim.** By a calculation as in proof of [10] Thm 1.1.17 $Z_0$ is the affine toric variety defined by lattice points in $M = \mathbb{Z}^4$:

\[
\begin{align*}
v_{11} &= e_1 + e_2 - e_3, \quad v_{12} = e_1 + 3e_2 - 3e_3, \quad v_{13} = e_1, \\
v_{21} &= e_3, \quad v_{22} = -2e_2 + 3e_3, \quad v_{23} = e_2, \quad v_{08} = e_4.
\end{align*}
\]

In other words there is a short exact sequence $0 \to L \to \mathbb{Z}^7 \xrightarrow{\varphi} M = \mathbb{Z}^4 \to 0$, where $\varphi$ sends the canonical base $e_i$ to the vectors $v_{ij}$ in order written as above and $L$ is a subgroup of $\mathbb{Z}^7$ defined by the binomial ideal of $Z_0$, i.e. $L$ is generated by:

\[
\begin{align*}
(0, 1, -1, 0, 1, -1, 0), \quad (-1, 1, 0, 2, 0, -2, 0), \quad (1, 0, -1, 1, 0, -1, 0), \quad (-2, 0, 2, 1, -1, 0, 0), \\
(0, 0, 0, 3, -1, -2, 0), \quad (-2, 1, 1, 1, 0, -1, 0), \quad (-1, 0, 1, 2, -1, -1, 0), \quad (3, -1, -2, 0, 0, 0, 0).
\end{align*}
\]

One checks that the vectors $v_{11}$ and $v_{21}$ are not needed to generate $\sigma^\vee$. □

**Claim 3.** The Picard torus acting on $Z_0$ may be viewed as $(\mathbb{C}^*)^2$ embedded into $T_M = (\mathbb{C}^*)^4$ by characters in the columns of the matrix:

\[
\begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 1
\end{pmatrix}.
\]

**Proof of the claim.** The $i$-th column corresponds to $e_i \in M$, $i = 1, 2, 3, 4$ and these correspond to $w_{13}, w_{23}, w_{21}, w_3$ respectively. □

To obtain $\tilde{S} = S_0$ from $Z_0$ we have to remove unstable orbits of $Z_0$ and divide the remaining open subset by the action of the Picard torus. By lemma [3.3] we have:

**Claim 4.** The unstable locus of $Z_0$ is the union of the closures of two orbits:

\[
O_1 = \{w_{11} = w_{12} = w_{13} = 0\}, \quad O_2 = \{w_{21} = w_{22} = w_{23} = w_3 = 0\}.
\]
One checks by \cite[3.2.7]{10} that:

**Claim 5.** The orbits $O_1$, $O_2$ correspond respectively to the following faces of the cone $\sigma = \text{cone}(e_4, e_2 + e_3, 3e_2 + 2e_3, e_1, 3e_1 + e_3)$ dual to $\sigma'$:

$$
\tau_1 = \text{cone}(e_1), \quad \tau_2 = \text{cone}(e_4, e_2 + e_3).
$$

To obtain the fan of the toric variety $S = S_0$ we consider the fan of $Z_0$, remove cones $\tau_1$ and $\tau_2$ together with all the cones containing them and take the family of the images of the remaining cones via the dual of the kernel of the matrix from claim 3, i.e. by:

$$
Q := \begin{pmatrix}
0 & -1 & 1 & 0 \\
-1 & 0 & -1 & 0
\end{pmatrix}.
$$

Now $Q(e_4) = (0, 1)$, $Q(e_2 + e_3) = (0, -1)$, $Q(3e_2 + 2e_3) = (-1, -3)$, $Q(3e_1 + e_3) = (1, -3)$ and changing coordinates in $\mathbb{Z}^2$ by $(1, 0) \mapsto (1, -3)$, $(0, 1) \mapsto (0, -1)$ we obtain the standard fan for Hirzebruch surface $\mathcal{H}_6$.

The matrix of the $T$-action on $S_0$ is given by the product of the matrix $J$ of embedding of $T$ into the big torus of $Z_0$ with the transpose of the matrix $Q$. Here, inspecting the construction of the generators of $R$, we obtain:

$$
J = \begin{pmatrix}
3 & 1 & 2 & 3 \\
1 & 3 & 2 & 3
\end{pmatrix}.
$$

**Proof of (d).** Denote $x := w_{11}$, $y := w_{12}$, $z := w_{13}$, $w := w_3$. Rescaling coordinates we may assume that $Z_P$ is embedded in $\mathbb{C}^5$ as the hypersurface $x^3 - yz^2 + w^2t = 0$. Then the unstable locus is described by equations $xw = yw = zw = wt = 0$, see lemma \cite[3.3]{13}. This gives an open cover of the set of semistable points, given by the union of three open sets \{ $xw \neq 0$ \} $\cup$ \{ $yw \neq 0$ \} $\cup$ \{ $zw \neq 0$ \} (note that $wt \neq 0$ together with the equation $x^3 - yz^2 + w^2t = 0$ implies that one of the variables $x, y, z$ does not vanish). Gluing the quotients of these open sets coincides with the standard construction of $\mathbb{P}^2$ (with coordinates $x, y, z$) by gluing three affine planes. Finally, note that $T$ acts on $w_{11}, w_{12}, w_{13}$ with weights $(2, 2), (0, 4), (3, 1)$ respectively.

We can also describe the non-normal locus of components of the central fibre.

**Theorem 3.5.** If $\nu_i : \tilde{S}_i \rightarrow S_i$ is the normalization of the component $S_i$ of the central fibre $i = 0, 1, 2$ and $\mathcal{N}_i \subset \tilde{S}_i$ is the locus of the non-normal points of $S_i$, then

(a) $\nu_0^{-1}(N_0)$ is the sum of the closures of the orbits corresponding to $(-1, -3)$ and $(1, -3)$ i.e. it is the sum of invariant fibres of $\mathcal{H}_6$.

(b) $\nu_1^{-1}(N_1)$ is the sum of the closures of the orbits corresponding to $(-1, -2)$ and $(1, -1)$.

(c) $\nu_2^{-1}(N_2)$ is the sum of the closures of the orbits corresponding to $(-1, -2)$ and $(1, -1)$.

**Proof.** We use lemma \cite[3.4]{13} and the description of $T$-stable orbits of $Z_i$ analogous to the one in the proof of theorem \cite[3.4]{3.4} for $Z_0$. Altogether, $T$-stable orbits of $Z_i$ which consist of normal points correspond to the cones which do not contain the cones $\tau_j$ and are not contained in $\omega_k$, where $\tau_j$’s are defined analogously as in the proof of \cite[3.3(a)]{3.3} and $\omega_k$’s are:
For $i=0$: $\omega_1 = \text{cone}(e_4, e_2 + e_3)$ and $\omega_2 = \text{cone}(e_1, e_4)$.

For $i = 1, 2$: $\omega := \omega_1 = \text{cone}(e_2 + e_3, e_4)$.

In each case one easily finds all such cones and their images via the map $Q$ (again, notation after proof of 3.4) turn out to be the cones parametrizing orbits in the statement. We conclude since the non-normal points of $S_i = Z_i//T$ are precisely the images of the non-normal points of $Z_i$ which are $T$-stable. Here we use the fact that all semistable points of $Z_i$ are stable and the isotropy groups of the $T$-action are trivial by lemma 3.3 so that the quotient $Z_i^* \to Z_i//T$ is a torsor. □

**Theorem 3.7.**

(a) $S_0 \cap S_1$ is the curve corresponding to $(-1, -3)$ on the normalization of $S_0$ and to $(1, -1)$ on a normalization of $S_1$.

(b) $S_0 \cap S_2$ is the curve corresponding to $(1, -3)$ on the normalization of $S_0$ and to $(1, -1)$ on a normalization of $S_2$.

(c) $S_0 \cap P$ is the curve corresponding to $(0, -1)$ on the normalization of $S_0$ and the cuspidal cubic curve $x^3 - yz^2 = 0$ on $P$ with homogeneous coordinates $x, y, z$.

(d) $S_1 \cap P$ is the curve corresponding to $(1, 0)$ on the normalization of $S_1$ and to the line $y = 0$ on $P$ with homogeneous coordinates $x, y, z$ (note that this is the flex tangent of the cuspidal cubic curve $S_0 \cap P$).

(e) $S_2 \cap P$ is the point corresponding to the cone spanned by rays $(0, 1), (1, -1)$ on the normalization of $S_2$ and to the point $x = z = 0$ on $P$ with homogeneous coordinates $x, y, z$ (note that this is the cusp of the cubic curve $S_0 \cap P$).

We give an argument in the case (c) using the notation from the proof of theorem 3.4.
Proof of (c). On $Z_0$ the intersection $Z_0 \cap Z_P$ is cut out by equations $w_{21} = w_{22} = w_{23} = 0$. Hence it contains as a dense subset the orbit of the toric variety $Z_0$ which corresponds to the one-dimensional cone $\tau = cone(e_2 + e_3)$, since $\sigma^\vee \setminus \tau^\perp \ni v_{21}, v_{22}, v_{23}$. Then we have $Q(e_2 + e_3) = (0, -1)$.

On $Z_P$ the intersection $Z_0 \cap Z_P$ is cut out by the equation $t = 0$. By the construction of the isomorphism $P \cong \mathbb{P}^2$ this equation yields the curve $x^3 - yz^2 = 0$ on $\mathbb{P}^2$. □

The next lemma shows that all nef line bundles on $X$ are globally generated, which will be important in the next sections.

**Lemma 3.8.** $L_1 + L_2$ and $L_1$ are globally generated line bundles on $X$.

**Proof.** Since $L_1 + L_2$ and $L_1$ are invariant with respect to the $T$-action and the base point locus of a linear system is closed for both linear systems $|L_1 + L_2|$ and $|L_1|$ it either has to be empty or it has a nontrivial intersection with the central fibre. The assertion follows by inspecting the weights of the generators of the Cox ring with respect to the Picard torus action and the equations of components of the fibre $p^{-1}([0])$ where $p : \text{Spec} \mathcal{R} \to \mathbb{C}^4/G$ is as in proposition 3.2. It turns out that the intersections of the zero sets of elements of each of these two linear systems with $p^{-1}([0])$ are contained in the unstable locus. □

### 4. Compasses of fixed points

In this section we obtain a local description of the action of the two-dimensional torus $T$ on a symplectic resolution $X \to \mathbb{C}^4/G$ at fixed points of this action, under the assumption that $X = \text{Spec} \mathcal{R}/T$. In fact the conditional results of this section will serve as a guide in the section 5, where we will prove that indeed $X = \text{Spec} \mathcal{R}/T$.

Assume that the GIT quotient $X = \text{Spec} \mathcal{R}/T$ is a resolution on which the line bundle $L = 2L_1 + L_2$ is relatively ample (i.e. $X$ is a GIT quotient of $\text{Spec} \mathcal{R}$ with respect to a character $(2, 1)$, and we assume that it is smooth). The torus $T$ acts on $X$. Let $x$ be a fixed point of this action. Torus $T$ acts then also on the tangent space $T_x X$. This gives weight-space decomposition $T_x X = \bigoplus_{i=1}^4 V_{\nu_i}$, where $\nu_i$ are in the character lattice of $T$, which we will identify with $\mathbb{Z}^2$.

The following definition was first introduced in [S § 2.3]

**Definition 4.1.** The weights $\nu_1, \ldots, \nu_4$ are called the compass of $x$ in $X$ with respect to the action of $T$.

We will now work to find compasses of all fixed points of the action $T$ on $X$.

**Lemma 4.2.** The following diagram shows the weights of the action of $T$ on the space of sections of $H^0(X, L)$ for $L = 2L_1 + L_2$ which are nonzero after the restriction to some irreducible component of the central fibre.
The black dots correspond to the weights of sections of \( L \) restricted to \( P \), blue ones to \( S_0 \), green ones to \( S_1 \) and red ones to \( S_2 \) (note that lattice points marked by multiple colours correspond to weights occurring in restriction to more than one component).

**Proof.** This follows by a computer calculation (using Macaulay2) of dimensions of appropriate graded pieces of the coordinate rings of \( Z_i \) and \( Z_P \) from proposition 3.2.

**Remark 4.3.** Lattice points from lemma 4.2 are contained in the polyhedron from remark 2.10. Moreover their convex hull form a minimal head of this polyhedron.

**Remark 4.4.** Considering the polytope which is a convex hull of weights marked by one colour in lemma 4.2 we get the polytope of the line bundle \( L \) pulled back to the corresponding component \( S \) of the central fibre viewed as a toric variety. As \( L \) is globally generated (see lemma 5.3), the vertices correspond to the fixed points of the action of \( T \) on \( S \), cf. [8] Lemma 2.3(3)]. In particular one obtains the weights of the action of \( T \) on the tangent space to \( S \) at fixed points. Here \( P \) is an exception, since the \( T \)-action on \( P \) is not faithful – \( P \) consists of the fixed points for \( \mathbb{C}^* \)-action given by the homothety which coincides with the action of \( \mathbb{C}^* \) embedded into \( T \) with the weight \((1,1)\).

**Theorem 4.5.** The fixed points of the \( T \)-action correspond to the vertices of the polytopes which are convex hulls of weights marked by fixed colour in lemma 4.2. The compasses of the points corresponding to the vertices of these polytopes are as in the table below:

| Point  | Compass  |
|--------|----------|
| \( P_1 \leftrightarrow (0,16) \) | \( \nu_1 = (1,-3) \), \( \nu_2 = (1,-5) \), \( \nu_3 = (0,4) \), \( \nu_4 = (0,6) \) |
| \( P_2 \leftrightarrow (1,11) \) | \( \nu_1 = (-1,5) \), \( \nu_2 = (2,-4) \), \( \nu_3 = (1,-1) \), \( \nu_4 = (0,2) \) |
| \( P_3 \leftrightarrow (3,7) \) | \( \nu_1 = (-2,4) \), \( \nu_2 = (-1,3) \), \( \nu_3 = (2,-2) \), \( \nu_4 = (3,-3) \) |
| \( P_4 \leftrightarrow (5,5) \) | \( \nu_1 = (-2,2) \), \( \nu_2 = (1,-1) \), \( \nu_3 = (3,-1) \), \( \nu_4 = (0,2) \) |
| \( P_5 \leftrightarrow (6,4) \) | \( \nu_1 = (-1,1) \), \( \nu_2 = (4,-2) \), \( \nu_3 = (-3,3) \), \( \nu_4 = (0,2) \) |
| \( P_6 \leftrightarrow (10,2) \) | \( \nu_1 = (-4,2) \), \( \nu_2 = (5,-1) \), \( \nu_3 = (-1,1) \), \( \nu_4 = (0,2) \) |
| \( P_7 \leftrightarrow (20,0) \) | \( \nu_1 = (-3,1) \), \( \nu_2 = (-5,1) \), \( \nu_3 = (4,0) \), \( \nu_4 = (6,0) \) |

The following picture illustrates the weights of the \( T \)-action calculated in the theorem. It is a directed graph. The points correspond to the sections of \( H^0(X,L) \) for \( L = 2L_1 + L_2 \) which are nonzero after the restriction to the central fibre together.
with vectors, as in lemma 4.2. The directed edges are the vectors from the compasses attached to the points which correspond to fixed points of $T$-action. In case when two vertices are connected by the two edges pointing in both ways we depict them by a single edge without any arrow.

**Proof of theorem 4.5.** First we calculate the weights of the $T$-action. Taking into account the natural inclusions of the tangent spaces to the components of the central fibre into the tangent space of $X$ most of them can be deduced from the fact that the action of $T$ on all components except $P$ is toric. The polytopes spanned by the points marked by one colour in the lemma 4.2 are the polytopes of those toric varieties. This gives their affine covers and the weights of the $T$-action on the tangent spaces to their $T$-fixed points. In the case of weights at $(3, 7), (5, 5)$ and $(6, 4)$ which come from the action of $T$ on $P$ we use the explicit description of $P$ from the proof of theorem 3.4(d). Altogether the weights calculated up to this point are the ones from the assertion except $\nu_{1,3}, \nu_{1,4}, \nu_{3,4}, \nu_{7,3}, \nu_{7,4}$.

Now the calculation of all the weights for the homothety action is easy, since this action is compatible with the symplectic form in the sense of lemma 4.6 below, and we can compute at least two weights at each $T$-fixed point, by summing components of each known weight $\nu_{i,j}$. For the remaining weights of the $T$-action we combine claims 1 and 2 below.

For example we know that $\nu_{1,1} = (-1, 3)$ and $\nu_{1,2} = (1, -5)$, which gives the weights of the homothety action $-2$ and $-4$ at point corresponding to $(0,16)$. By lemma 4.6 the remaining weights for homothety are equal to 4 and 6. By the claim 1 the remaining weights $\nu_{1,3}$ is of the form $(4,0)$ or $(0,4)$ and $\nu_{1,4}$ is of the form $(6,0)$ or $(0,6)$. Since $\nu_{1,1}$ and $\nu_{1,2}$ yield two positive weights for the $\mathbb{C}^*$-action considered in the claim 2 and so do $(6,0)$ and $(4,0)$ we have $\nu_{1,3} = (0,4)$ and $\nu_{1,4} = (0,6)$. The remaining weights are computed analogously. □

**Claim 1.** The remaining weights for the $T$-action on the central fibre are of the form $(a,0)$ or $(0,a)$. 
PROOF OF THE CLAIM. First note that every orbit of the $T$-action on $\mathbb{C}^4 \setminus \{0\}$ is either two-dimensional or has the isotropy group equal to $\mathbb{C}^* \times 1$ or $1 \times \mathbb{C}^*$.

Now take any $x \in X^T$, any remaining weight $\lambda$ of the $T$-action on $T_xX$ and a one-dimensional eigenspace $V_\lambda$ with this weight which is not contained in the tangent space to the central fibre. By the Luna slice theorem such an eigenspace corresponds to the closure of an orbit $O$ of the $T$-action via an equivariant local étale map $U \to T_xX$, where $U$ is an invariant neighbourhood of $x$. In particular $\dim O = 1$, and $\dim \pi(O) = 1$, as $O$ is not contained in the central fibre of the resolution $\pi : X \to \mathbb{C}^4/G$. Therefore $\pi(O)$ as well as $O$ and $V_\lambda$ are stabilized by either $\mathbb{C}^* \times 1$ or $1 \times \mathbb{C}^*$ and the claim follows. \hfill $\Box$

For the next claim, consider $\mathbb{C}^*$ as a subtorus of $T$ embedded with the weight $(1, -1)$. Clearly it acts on $\mathbb{C}^4, \mathbb{C}^4/G$ and $X$.

CLAIM 2. Let $x \in X^{\mathbb{C}^*}$. Among the weights of the induced $\mathbb{C}^*$-action on $T_xX$ two weights are positive and two are negative.

PROOF OF THE CLAIM. Let

\[ X^+_x = \{ x \in X : \lim_{t \to 0} tx \in X_x \}, \]
\[ X^-_x = \{ x \in X : \lim_{t \to 0} t^{-1}x \in X_x \}, \]

where $X_x \subset X^{\mathbb{C}^*}$ is the connected component containing $x$. We will use the fact that $X^+_x$ are irreducible, locally closed subsets of $X$ (see [7] § 4 and [9] § 4.1)).

If at least three weights at $x$ were nonnegative then $\dim X^+_x \geq 3$ by the Luna slice theorem. Similarly if at least three weights at $x$ are nonpositive then $\dim X^-_x \geq 3$.

Suppose that $\dim X^+_x \geq 3$. On the other hand $(\mathbb{C}^4)^+_0 = \mathbb{C}^2 \times 0$ is two-dimensional and hence also $(\mathbb{C}^4/G)^+_0 = (\mathbb{C}^2 \times 0)/G$ is two-dimensional. Since $\dim \pi^{-1}([0]) = 2$ and the fibres of $\pi : X \setminus \pi^{-1}([0]) \to \mathbb{C}^4/G$ are of dimension at most one then $\dim \pi(X^+_x) \geq \dim X_x^+ - 1 = 2$. As $\pi(X^+_x) \subset (\mathbb{C}^4/G)^+_0$ we know that $\dim \pi(X^+_x) = 2$ and hence $X^+_x$ has to be an exceptional divisor of the resolution $\pi : X \to \mathbb{C}^4/G$. But the image of such an exceptional divisor is contained in the singular locus of $\mathbb{C}^4/G$ which consists of the image of four planes in $\mathbb{C}^4$ that have one-dimensional intersection with $\mathbb{C}^2 \times 0$ and so, $\dim \pi(X^+_x) = 1$, a contradiction. The case $\dim X^-_x \geq 3$ is completely analogous. \hfill $\Box$

LEMMA 4.6. The symplectic form $\omega$ on $X$ is of weight 2 with respect to the $\mathbb{C}^*$-action induced by the homothety. In particular at each fixed point of this action we may order weights $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ of the action on $T_xX$, so that $\lambda_1 + \lambda_2 = \lambda_3 + \lambda_4 = 2$.

5. Smoothness of the GIT quotient

Let $\mathcal{R}$ be a subring of the Cox ring of the crepant resolution generated by the elements from the statement of theorem 2.7. In this section we show that the GIT quotient $\text{Spec} \mathcal{R}/\mathbb{T}$ with respect to the linearization of the trivial line bundle by the character $2L_1 + L_2$. In consequence we will see that $\text{Spec} \mathcal{R}/\mathbb{T} \to \mathbb{C}^4$ is a crepant resolution. This will make the results on geometry of crepant resolutions of $\mathbb{C}^4/G$ in the previous sections unconditional and will help to conclude that $\mathcal{R}$ is the whole Cox ring in the final section [7].
We consider Spec $\mathcal{R}$ as a closed subvariety of $\mathbb{C}^{20}$ via the embedding given by generators from statement of Theorem 2.7. Let $\mathbb{T} = \text{Hom}(\text{Cl}(X), \mathbb{C}^*)$ be a Picard torus of $X$.

**Theorem 5.1.** The stable locus of Spec $\mathcal{R}$ with respect to a linearization of the trivial line bundle by a character $(a, b)$, $a > b > 0$ is covered by seven $T \times \mathbb{T}$-invariant open subsets $U_1, \ldots, U_7$ such that $U_i/\mathbb{T} \cong \mathbb{C}^4$. More precisely:

1. $U_1 = \{w_{12}s \neq 0\}$ and $(\mathcal{R}_{w_{12}s})^\mathbb{T} = \mathbb{C} \left[w_{03}, w_{05}, \frac{w_{21}}{w_{12}}, \frac{w_{22}}{w_{12}}\right]$.
2. $U_2 = \{w_{12}w_{23} \neq 0\}$ and $(\mathcal{R}_{w_{12}w_{23}})^\mathbb{T} = \mathbb{C} \left[w_{01}, w_{02}, \frac{w_{21}}{w_{12}}, \frac{w_{22}}{w_{12}}, \frac{w_{31}}{w_{12}}, \frac{w_{32}}{w_{12}}\right]$.
3. $U_3 = \{w_{12}w_3 \neq 0\}$ and $(\mathcal{R}_{w_{12}w_3})^\mathbb{T} = \mathbb{C} \left[w_{01}, w_{02}, \frac{w_{21}}{w_{12}}, \frac{w_{22}}{w_{12}}, \frac{w_{31}}{w_{12}}\right]$.
4. $U_4 = \{w_{11}w_3 \neq 0\}$ and $(\mathcal{R}_{w_{11}w_3})^\mathbb{T} = \mathbb{C} \left[w_{01}, \frac{w_{21}}{w_{11}}, \frac{w_{22}}{w_{11}}, \frac{w_{31}}{w_{11}}\right]$.
5. $U_5 = \{w_{13}w_3 \neq 0\}$ and $(\mathcal{R}_{w_{13}w_3})^\mathbb{T} = \mathbb{C} \left[w_{01}, \frac{w_{21}}{w_{13}}, \frac{w_{22}}{w_{13}}, \frac{w_{31}}{w_{13}}\right]$.
6. $U_6 = \{w_{13}w_{22} \neq 0\}$ and $(\mathcal{R}_{w_{13}w_{22}})^\mathbb{T} = \mathbb{C} \left[w_{01}, \frac{w_{21}}{w_{13}}, \frac{w_{22}}{w_{13}}, \frac{w_{31}}{w_{13}}\right]$.
7. $U_7 = \{w_{22}t \neq 0\}$ and $(\mathcal{R}_{w_{22}t})^\mathbb{T} = \mathbb{C} \left[w_{01}, \frac{w_{21}}{w_{22}}, \frac{w_{22}}{w_{22}}, \frac{w_{31}}{w_{22}}\right]$.

In particular the GIT quotient Spec $\mathcal{R}/\mathbb{T}$ with respect to the linearization of the trivial line bundle by a character $(a, b)$, $a > b > 0$ is smooth.

**Proof.** Lemma 3.3 implies that $\{U_i\}_{i=1}^7$ form an open cover of the quotient. It remains to prove equalities from points (1)–(7). Note that then in each case the four generators of the ring on the right-hand side of the equality have to be algebraically independent as the GIT quotient Spec $\mathcal{R}/\mathbb{T}$ is irreducible and of dimension four.

By symmetry it suffices to consider only $U_i$ for $i = 1, 2, 3$. In each case we calculate the invariants of the localization of the coordinate ring of the ambient $\mathbb{C}^{20}$, with help of 4ti2 [1] obtaining in consequence:

1. $\text{Spec } \mathcal{R}_{w_{12}s}^\mathbb{T} = \mathbb{C} \left[w_{01}, \frac{w_{11}}{w_{12}}, \frac{w_{21}}{w_{12}}, \frac{w_{31}}{w_{12}}, \frac{w_{32}}{w_{12}}, \frac{w_{33}}{w_{12}}\right]_{i=1, \ldots, 7, j=1, \ldots, 5}$
2. $\text{Spec } \mathcal{R}_{w_{12}w_{23}}^\mathbb{T} = \mathbb{C} \left[w_{01}, \frac{w_{11}}{w_{12}}, \frac{w_{21}}{w_{12}}, \frac{w_{22}}{w_{12}}, \frac{w_{23}}{w_{12}}, \frac{w_{31}}{w_{12}}, \frac{w_{32}}{w_{12}}, \frac{w_{33}}{w_{12}}\right]_{i=1, \ldots, 7, j=1, \ldots, 5}$
3. $\text{Spec } \mathcal{R}_{w_{12}w_3}^\mathbb{T} = \mathbb{C} \left[w_{01}, \frac{w_{11}}{w_{12}}, \frac{w_{21}}{w_{12}}, \frac{w_{22}}{w_{12}}, \frac{w_{31}}{w_{12}}, \frac{w_{32}}{w_{12}}\right]_{i=1, \ldots, 7, j=1, \ldots, 5}$

Then, using the Gröbner basis of the ideal of relations among generators of $\mathcal{R}$ with respect to an appropriate lexicographic order, we verify with Singular [11] in each case that each of the generators can be expressed as a polynomial of the four generators from the assertion. □

**Remark 5.2.** To find isomorphisms $U_i \cong \mathbb{C}^4$ we used the argument from the section 2 with its a priori assumption on smoothness of the quotient Spec $\mathcal{R}/\mathbb{T}$ as a heuristic. To guess the coordinates on each invariant open subset $U_i$ we picked an element $f \in \mathcal{R}$ of a degree corresponding to $i$th fixed point and four elements of $(\mathcal{R}^\mathbb{T})^i$ of degrees equal to the predicted weights of the action on a tangent space. In fact theorem 4.5 follows immediately by theorem 5.1 when we note that $U_i \cong \mathbb{C}^4$ is a $T$-invariant neighbourhood of $i$th $T$-invariant point (we order points as in rows of the table from the assertion). Nevertheless, since to guess an open cover $U_i$ we...
used the statement of theorem 4.5, we decided to give its conditional proof before theorem 5.1 to preserve the logical consequence of our considerations.

By the inclusion of invariants $R^T \subset R$ (see lemma 3.1) we have the induced projective map $\text{Spec } R//T \to \text{Spec } R/T \cong C^4/G$.

**Corollary 5.3.** The map $\pi : \text{Spec } R//T \to \text{Spec } R/T \cong C^4/G$ is a crepant resolution.

**Proof.** Denote $E_1 = \{s = 0\} \subset \text{Spec } R//T$ and $E_2 = \{t = 0\} \subset \text{Spec } R//T$. These are irreducible divisors on $\text{Spec } R//T$. By the construction of $R$ the map $\pi$ is an isomorphism outside $E_1 \cup E_2$. Hence $\pi$ is a resolution and it has to be crepant since there are two crepant divisors over $C^4/G$ by symplectic McKay correspondence and they have to be present on each resolution. $\square$

## 6. Dimensions of movable linear systems

In this section we use a torus $T$ action on $X$ to give a formula for dimensions of these graded pieces of $R(X)$ which correspond to the movable linear systems on some of the resolutions.

Let $X \to C^4/G$ be the resolution corresponding to the linearization of the Picard torus action by a character $(2, 1)$. Let $X^* \to C^4/G$ be the resolution corresponding to the linearization $(1, 2)$. Let $T$ be the two-dimensional torus which acts naturally on $C^4$, $C^4/G$ and $X$ via homothety on each irreducible component of $C^4$ viewed as a representation of $G$.

Denote by $P_i$ the fixed points of the $T$-action on $X$ as in the table from theorem 4.5. Let $\{\nu_{i,j}\}_{j=1}^4$ denote the compass of $P_i$ in $X$. Let us also denote by $\mu_i(L)$ the weight of the $T$-action on the fibre of $L$ over $P_i$. Note that $\mu_i$ is linear i.e. $\mu_i(A + B) = \mu_i(A) + \mu_i(B)$.

**Remark 6.1.** By lemma 3.8 we may compute the weights $\mu_i$ for line bundles $L_1 + L_2$ and $L_1$ similarly as for $2L_1 + L_2$ in section 4 to obtain (the last column is calculated from the first two ones by the linearity of $\mu_i$):

| $i$ | $\mu_i(L_1)$ | $\mu_i(L_1 + L_2)$ | $\mu_i(L_2)$ |
|-----|---------------|---------------------|---------------|
| 1   | (0, 4)        | (0, 12)             | (0, 8)        |
| 2   | (0, 4)        | (1, 7)              | (1, 3)        |
| 3   | (0, 4)        | (3, 3)              | (3, −1)       |
| 4   | (2, 2)        | (3, 3)              | (1, 1)        |
| 5   | (3, 1)        | (3, 3)              | (0, 2)        |
| 6   | (3, 1)        | (7, 1)              | (4, 0)        |
| 7   | (8, 0)        | (12, 0)             | (4, 0)        |

We may now prove the promised observation on the subdivision of the cone of movable divisors on $X$, see proposition 2.8.

**Proposition 6.2.** There are two symplectic resolutions of $C^4/G$. The chambers in $\text{Mov}(X)$ corresponding to the nef cones of these resolutions are $\text{cone}(L_1, L_1 + L_2)$ and $\text{cone}(L_2, L_1 + L_2)$.

**Proof.** Consider the homomorphisms $\mu_i : N^1(X) \to \mathbb{R}^2$. The two walls of the chamber $\mathcal{C}$ containing $2L_1 + L_2$ are corresponding to the contractions of $X$,
in particular they identify some $T$-fixed points of $X$. Hence each wall has to be spanned by an element $v \in \text{Mov}(X)$ satisfying $\mu_i(v) = \mu_j(v)$ for some $i \neq j$. Now the only such elements in $\text{cone}(L_1, L_1 + L_2)$ are lying on the rays spanned by $L_1 + L_2$ and $L_1$. Therefore $C = \text{cone}(L_1, L_1 + L_2)$. The analogous argument, using the homomorphisms $\mu'_i : N^1(X') \to \mathbb{R}^2$ corresponding to the $T$-fixed points of $X'$, shows that the chamber containing $L_1 + 2L_2$ is equal to $\text{cone}(L_1 + L_2, L_2)$.

**Theorem 6.3.** If $h^0(X, pL_1 + qL_2)_{(a,b)}$ is the dimension of the subspace of sections $H^0(X, pL_1 + qL_2)$ on which $T$ acts with the weight $(a,b)$, then we have the following generating function for such dimensions for line bundles inside the movable cone:

$$
\sum_{a,b,p,q \geq 0} h^0(X, pL_1 + qL_2)_{(a,b)} y_1^p y_2^q t_1^a t_2^b = \sum_{i=1}^7 \frac{1}{(1 - t\mu_i(L_1)y_1)(1 - t\mu_i(L_2)y_2) \prod_{j=1}^4 (1 - t\nu_{i,j})}.
$$

**Proof.** By a corollary of Lefschetz-Riemann-Roch theorem [8, Corollary A.3] we have:

$$
\chi^T(X, L) = \sum_{i=1}^7 \frac{\mu_i(L)}{\prod_{j=1}^4 (1 - t\nu_{i,j})}.
$$

Using the linearity of $\mu_i$:

$$
\sum_{p,q \geq 0} \chi^T(X, pL_1 + qL_2) y_1^p y_2^q = \sum_{p,q \geq 0} \frac{\sum_{i=1}^7 (\mu_i(L_1) p + \mu_i(L_2) q) y_1^p y_2^q}{\prod_{j=1}^4 (1 - t\nu_{i,j})} = \sum_{i=1}^7 \frac{1}{(1 - t\mu_i(L_1)y_1)(1 - t\mu_i(L_2)y_2) \prod_{j=1}^4 (1 - t\nu_{i,j})}.
$$

The assertion follows now by Kawamata-Viehweg vanishing, which implies

$$
\chi^T(X, pL_1 + qL_2) = \sum_{a,b \geq 0} h^0(X, pL_1 + qL_2)_{(a,b)} t_1^a t_2^b = \sum_{a,b \geq 0} h^0(X', pL_1 + qL_2)_{(a,b)} t_1^a t_2^b
$$

if $p \geq q \geq 0$ and likewise for $q \geq p \geq 0$ on $X'$.

**Example 6.4.** The dimensions of the weight spaces corresponding to the lattice points in a head of the polyhedron spanned by weights for the line bundle $2L_1 + L_2$ considered in remark 2.10 and in section 4 can be depicted on the following diagram:
Lemma 7.1. The Cox ring of $X$ is generated by $s, t$ and by

$$\mathcal{R}(X)_{\geq 0} := \bigoplus_{p, q \geq 0} H^0(X, pL_1 + qL_2).$$

Proof. Every effective Weil divisor on $X$ is of the form $D + nE_1 + mE_2$ where $D$ is the strict transform of a Weil divisor on $\mathbb{C}^4/G$ and $m, n \geq 0$. Now note that $D \sim pL_1 + qL_2$ for some $p, q \geq 0$ as $D.C_i \geq 0$. Then the divisor $D$ corresponds to a section of $H^0(X, pL_1 + qL_2) \subset \mathcal{R}_{\geq 0}$ and $E_1, E_2$ correspond to $s, t$ respectively. \qed

Thus to prove that $\mathcal{R} = \mathcal{R}(X)$ it suffices to show that $\mathcal{R}$ contains $\mathcal{R}(X)_{\geq 0}$. The following lemma reduces the problem further, to finitely many graded pieces with respect to $\mathbb{Z}^2$-grading by characters of the Picard torus of $X$.

Lemma 7.2. $\mathcal{R}(X)_{\geq 0}$ is generated by the sections of all linear spaces $H^0(X, L)$ for $L \in S \cup S'$ where $S := \{O_X, L_1, L_1 + L_2, 2L_1, 2L_1 + L_2, 2L_1 + 2L_2, 3L_1 + L_2, 3L_1 + 2L_2, 4L_1 + 2L_2\}$ and $S' := \{L_2, 2L_2, L_1 + 2L_2, L_1 + L_2, 2L_1 + 3L_2, 2L_1 + 3L_2, 2L_1 + 4L_2\}$.

Proof. We use the Kawamata-Viehweg vanishing theorem to show that if $L$ is a line bundle and $L \not\in S \cup S'$ then $L$ is $O_X$-regular with respect to an appropriate finite family of globally generated line bundles on $X$. Then we use the surjectivity of the multiplication map of sections for Castelnuovo-Mumford regular sheaves in
the sense of Hering-Schenck-Smith [18] Theorem 2.1(2)] to show that the sections corresponding to the elements of linear systems of line bundles in $S \cup S'$ generate $\mathcal{R}(X)_{\geq 0}$.

First note that $H^i(X, L) = 0$ for every $i > 2$ and every line bundle $L$ on $X$ since $\mathbb{C}^4/G$ is affine and $\pi : X \to \mathbb{C}^4/G$ has fibres of dimension at most 2. We will use [18] Theorem 2.1(2)] several times for various families $\{B_1, \ldots, B_l\}$ of globally generated line bundles. Denote by $X$ relatively ample. Then the line bundles $L_1$ and $L_1 + L_2$ on $X$ are globally generated by lemma [3.8].

Let $l = 1$ and $B_1 = L_1$. By the Kawamata-Viehweg vanishing theorem on $X$ every line bundle $L = mL_1 + nL_2$ with $m \geq n + 2$ and $n = 0, 1$ is $O_X$-regular. By [18] Theorem 2.1(2)] the global sections of line bundles $L_1, 2L_1$ and $3L_1 + L_2$ generate all spaces $H^0(X, L)$ for such $L$.

Let $l = 1$ and $B_1 = L_1 + L_2$. By the Kawamata-Viehweg vanishing theorem on $X$ every line bundle $L = mL_1 + nL_2$ with $n \geq 2$ and $m = n, n + 1$ is $O_X$-regular. By [18] Theorem 2.1(2)] the global sections of line bundles $L_1 + L_2, 2L_1 + 2L_2$ and $3L_1 + 2L_2$ generate all spaces $H^0(X, L)$ for such $L$.

Let $l = 2$, $B_1 = L_1$ and $B_2 = L_1 + L_2$. By the Kawamata-Viehweg vanishing theorem on $X$ a line bundle $mL_1 + nL_2$ with $m \geq n + 2$ and $n \geq 2$ is $O_X$-regular. By [18] Theorem 2.1(2)] the global sections of line bundles $L_1, L_1 + L_2$ and $4L_1 + 2L_2$ generate all spaces $H^0(X, L)$ for such $L$.

Changing the roles of $L_1$ and $L_2$ in the argument above (in particular considering the resolution on which $L_1 + 2L_2$ is ample) we see similarly that all the remaining spaces $H^0(X, L)$ for $L \notin S \cup S'$ are generated by the global sections of line bundles in $S \cup S'$.

Hence we reduced the problem to showing that $\mathcal{R}$ contains spaces of global sections only for these finitely many line bundles which are elements of $S \cup S'$ in lemma [7.2].

This, with the help of computer algebra, can be done with the use of the previous section, namely by theorem [6.3] in which we computed the Hilbert function of $\mathcal{R}(X)_{\geq 0}$. Note that, by the symmetry of the generators of $\mathcal{R}$ with respect to the Picard torus action, we may restrict our attention to line bundles in $S$.

**Lemma 7.3.** $\mathcal{R}$ contains $H^0(X, L)$ for each $L \in S$, where $S$ is as in the lemma [7.2].

**Proof.** We calculate the Hilbert function of $\mathcal{R}$ in Macaulay2 [17]. Then, using Singular [11] we extract from it Hilbert functions for each of the vector spaces $\mathcal{R}_L$, $L \in S$, graded by characters of $T$. The main difficulty here is the fact that the $\mathbb{Z}^2$-grading of $\mathcal{R}$ corresponding to the Picard torus action is not a positive grading, i.e. it is not in $\mathbb{N}^2$. Then, we observe that for each $L \in S$ this function agrees with the Hilbert function for $\mathcal{R}(X)_L$ which we calculated in theorem [6.3]. Since $\mathcal{R} \subset \mathcal{R}(X)$ it means that $\mathcal{R} = \mathcal{R}(X)$. □

**Remark 7.4.** The above consideration about re-obtaining Cox ring suggests the following geometric approach to the verification whether the set of elements of $\mathcal{R}(X)$ of the form as in [14] Theorem 2.2] generates of the Cox ring of a crepant resolution $X$ for a quotient singularity with the use of a torus $H$ acting on the resolution.
Step 1 Calculate the fixed point locus $X^H$ and the invariants needed to compute the equivariant Euler characteristic of the resolution with Lefschetz-Riemman-Roch formula. In case when $X^H$ is discrete it suffices to know the compass at each point of $X^H$ and the weights of the $H$-action on fibres over fixed points of line bundles generating $\text{Pic}(X)$.

Step 2 Use the Lefschetz-Riemman-Roch formula combined with the Kawamata-Viehweg vanishing theorem to compute the Hilbert function of non-negatively graded part of the Cox ring $\mathcal{R}(X)_{\geq 0}$. For the case when $\dim X^H \leq 1$ one may use [8, Corollary A.3].

Step 3 Use the multigraded Castelnuovo-Mumford regularity and the Kawamata-Viehweg vanishing to obtain a finite set $\mathcal{S}$ of line bundles whose global sections generate the Cox ring.

Step 4 Compute the Hilbert function for $\mathcal{R}$ and extract from it Hilbert functions of the vector spaces $\mathcal{R}_L$, $L \in \mathcal{S}$, graded by characters of $H$. Check whether they are equal to Hilbert functions $\mathcal{R}(X)_L$ calculated in Step 2.

Remark 7.5. Note that if we know the Hilbert function for $\mathcal{R}(X)_{\geq 0}$ calculated in step 1 then the above procedure suggests also where to look for additional generators of $\mathcal{R}$ if we do not have equality in step 4.

Remark 7.6. The actual implementation of the procedure proposed above is challenging, since one has to obtain enough information on the geometry of a resolutions without knowing the Cox ring. The required data are of two kinds – in step 1, one has to find data associated with the torus action. And in step 3, to successfully use regularity, one has to find sufficiently many line bundles which are globally generated on one of the resolutions.

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