The $L$-functions of Witt coverings

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Abstract. Results on $L$-functions of Artin-Schreier coverings by Dwork, Bombieri and Adolphson-Sperber are generalized to $L$-functions of Witt coverings.

Key words: $L$-functions, exponential sums, Newton polygon

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1 Introduction

We shall state our main results after recalling the notion of $L$-functions of Witt coverings.

Let $F_q$ be the finite field of characteristic $p$ with $q$ elements, and $W_m$ the ring scheme of Witt vectors of length $m$ over $F_q$. Let $f \in W_m(F_q[x_1,\ldots,x_n])$ with its first coordinate non-constant.

Let $T^n$ be the $n$-dimensional torus over $F_q$, and $F$ the Frobenius morphism of $W_m$. The fibre product over $W_m$ of $W_m F^{-1} \to W_m$ and $T^n f \to W_m$ is a $W_m(F_p)$-covering of $T^n$, with group action $g(y,x) = (y + g, x)$. The Frobenius element of the Galois group $W_m(F_p)$ at a closed $x$ of $X$ with degree $k$ is $\text{Tr}_{W_m(F_q)/W_m(F_p)}(f(x))$. So the Artin $L$-function of $T^n$ determined by that $W_m(F_p)$-covering and a fixed character $\psi : W_m(F_p) \to \mathbb{Q}^\times$ of exact order $p^m$ is

$$L_f(t) = \prod_{x \in |T^n|} (1 - \psi(\text{Tr}_{W_m(F_q)/W_m(F_p)}(f(x)))t)^{-1}^n,$$

where $|T^n|$ is the set of closed points of $T^n$. By a well known theorem of Deligne [De],

$$L_f(t) = \frac{\prod (1 - \alpha t)}{\prod (1 - \beta t)},$$

where $\alpha$ and $\beta$ are algebraic integers such that $q^n\alpha^{-1}$ and $q^n\beta^{-1}$ are also algebraic integers. It implies, as observed by Bombieri [Bo2], $\text{ord}_q(\alpha), \text{ord}_q(\beta) \leq n$, where $\text{ord}_q$ is the $q$-order function of $\overline{\mathbb{Q}}_p$ such that $\text{ord}_q(q) = 1$. ($\overline{\mathbb{Q}}_p$ is the algebraic closure of $\mathbb{Q}_p$, the field of $p$-adic numbers.)

By logarithmic differentiation, we get

$$L_f(t) = \exp\left(\sum_{k=1}^{\infty} S_k(f) t^k \right),$$
where

$$S_k(f) = (-1)^{n-1} \sum_{x \in (\mathbb{F}_{q^k})^n} \psi(\text{Tr}_{W_m(\mathbb{F}_{q^k})/W_m(\mathbb{F}_p)}(f(x)))$$

are exponential sums associated to characters of $p$-power order. To have a look at these exponential sums, we denote by $\lambda_i : A^1 \to W_m$, $i = 0, \ldots, m - 1$, the embedding which maps $A^1$ onto the $i$-th axis of $W_m$, and write

$$f = \sum_{i=0}^{m-1} \sum_{u \in I_i} \lambda_i(a_{iu}x^u),$$

where $I_i \subset \mathbb{Z}^n$ and $a_{iu} \in \mathbb{F}_{q^k}^\times$ are uniquely determined. That decomposition can be obtained by solving the congruences

$$f \equiv \lambda_0(\sum_u a_{0u}x^u)(\mod V)$$
$$f - \sum_u \lambda_0(a_{0u}x^u) \equiv \lambda_1(\sum_u a_{1u}x^u)(\mod V^2)$$
$$\vdots$$
$$f - \sum_{i=0}^{m-2} \sum_u \lambda_i(a_{0u}x^u) \equiv \lambda_{m-1}(\sum_u a_{(m-1)u}x^u)(\mod V^m)$$
successively, where $V$ is the shift operator on $W_m$.

Let $\mathbb{F}_{q^k}$ be the algebraic closure of $\mathbb{F}_q$, and $\omega$ the Teichmüller lifting from $\mathbb{F}_q$ to $\mathbb{Q}_p$. We define $\omega(f) = \sum_{i=0}^{m-1} p^i \sum_{w \in I_i} \omega(a_{iw})x^u$. Let $\mathbb{Z}_p$ be the ring of $p$-adic integers, and $\mu_l$ ($l \geq 1$) be the set of $l$-th roots of unity in $\mathbb{Q}_p$. Identifying $W_m(\mathbb{F}_{q^k})$ with $\mathbb{Z}_p[\mu_{q^k-1}]/(p^m)$ under the isomorphism

$$(a_0, \ldots, a_{m-1}) \mapsto \sum_{j=0}^{m-1} \omega(a_{i}^{p^{-i}})p^i \pmod{p^m},$$

one finds, for $x \in (\mathbb{F}_{q^k})^n$, that

$$\psi(\text{Tr}_{W_m(\mathbb{F}_{q^k})/W_m(\mathbb{F}_p)}(f(x))) = \psi(\text{Tr}_{\mathbb{Q}_p[\mu_{q^k-1}]/\mathbb{Q}_p}(\sum_{i=0}^{m-1} \sum_{u} p^i \omega(a_{iu}^{p^{-i}}x^u)))$$
$$= \psi(\text{Tr}_{\mathbb{Q}_p[\mu_{q^k-1}]/\mathbb{Q}_p}(\sum_{i=0}^{m-1} \sum_{u} p^i \omega(a_{iu}x^u))) = \psi(\text{Tr}_{\mathbb{Q}_p[\mu_{q^k-1}]/\mathbb{Q}_p}(\omega(f)(\omega(x))))).$$

Therefore, we have

Lemma 1.1 For $k = 1, 2, \ldots$, we have

$$S_k(f) = \sum_{x \in \mathbb{Q}_p[\mu_{q^k-1}]} \psi(\text{Tr}_{\mathbb{Q}_p[\mu_{q^k-1}]/\mathbb{Q}_p}(\omega(f)(x))).$$
We define the Newton polyhedron $\Delta_\infty(f)$ of $f$ at infinity to be the convex hull in $\mathbb{Q}^n$ of \(\{p^{m-i-1}u : 0 \leq i \leq m-1, u \in I_i \cup \{0\}\}\). Recall that, for a convex polyhedron $\Delta$ of dimension $n$ in $\mathbb{Q}^n$ that contains the origin, there is a $\mathbb{R}_{\geq 0}$-linear degree function $u \mapsto \deg(u)$ on $L(\Delta)$, the set of integral points in the cone $\bigcup_{k=1}^\infty k\Delta$, such that $\deg(u) = 1$ when $u$ lies on a face of $\Delta$ that does not contain the origin. That degree function may take on non-integral values. But there is a positive integer $D$ such that $\deg L(\Delta) \subset D^{-1}\mathbb{Z}$. We denote the least positive integer with this property by $D(\Delta)$. For $k = 0, 1, \cdots$, we denote by $W_\Delta(k)$ the number of points of degree $\frac{k}{D(\Delta)}$ in $L(\Delta)$. We define $P_\Delta(t) = (1 - t^{D(\Delta)})^n \sum_{k=0}^{+\infty} W_\Delta(k)t^k$ for later use. Our first result is an upper bound for the total degree of $L_f(t)$.

**Theorem 1.2** The total degree of $L_f(t)$ is bounded by $\sum_{i=0}^{n} \binom{n}{i} \sum_{k=0}^{D(n-i+1)} W_\Delta(k)$ with $D = D(\Delta)$ and $\Delta = \Delta_\infty(f)$.

For $j = 1, \cdots, n$, we write

$$jf^j = \sum_{i=0}^{m-1} \sum_{p^{m-i-1}u \in \tau} u_j a_i^{p^{m-i-1}} x^{p^{m-i-1}u},$$

where $u_j$ is the $j$-th coordinate of $u$. We call $f$ non-degenerate with respect to $\Delta_\infty(f)$ if $\Delta_\infty(f)$ is of dimension $n$, and for every face $\tau$ of $\Delta_\infty(f)$ that does not contain 0, the system $\frac{1}{f^j} = \cdots = \frac{n}{f^j}$ has no common solution in $(\mathbb{F}_q^n)^n$. Our second result is on $L$-functions from non-degenerate Witt vectors.

**Theorem 1.3** Suppose that $f$ is non-degenerate with respect to $\Delta := \Delta_\infty(f)$. Then the $L$-function $L_f(t)$ is a polynomial, and its Newton polygon with respect to $\text{ord}_q$ lies above the Hodge polygon of $P_\Delta(t)$ of degree $D(\Delta)$ with the same endpoints. In particular, $L_f(t)$ is of degree $n!\text{Vol}(\Delta)$.

Recall that the Newton polygon of $\prod(1 - at) \in \overline{\mathbb{Q}}_p[[t]]$ with respect to $\text{ord}_q$ is the polygon with vertices at points

$$\left(\sum_{\text{ord}_q(\alpha) \leq y} 1, \sum_{\text{ord}_q(\alpha) \leq y} \text{ord}_q(\alpha)\right), \ y \in \mathbb{Q}.$$ 

And the Hodge polygon of $\sum_{k=0}^{+\infty} a_k t^k$ of degree $D$ is the polygon with vertices at the points $(0,0)$ and

$$(\sum_{i=0}^{k} a_i, \frac{1}{D} \sum_{i=0}^{k} ia_i), \ k = 0, 1, \cdots.$$

Theorem 1.3 was proved by Dwork [Dw] when $m = 1$, and $f(x_1, \cdots, x_n) = x_n h(x_1, \cdots, x_{n-1})$ for some polynomial $h$ with coefficients in $\mathbb{F}_q$. In that case, the $L$-function $L_f(t)$, by the orthogonality of characters, is related to the zeta function of the hypersurface defined by $h = 0$ in the $(n-1)$-dimensional affine space defined over $\mathbb{F}_q$. It was completely proved by Adolphson-Sperber [AS2] in the case $m = 1$. In the case $n = 1$, the degree of $L_f(t)$ was determined by Kumar-Helleseth-Calderbank [KHC] with applications to coding theory, and by W.-C. W. Li [Li], who read the $p = 2$ version of [KHC].
Our proof of the main results is based on the $p$-adic method set up by Dwork [Dw, Dw2] and developed by Bombieri [Bo, Bo2], Monsky [Mo], Adolphson-Sperber [AS, AS2], Wan [Wn], and others. The innovation lies in the use of the Artin-Hasse exponential series to produce roots of unity of $p$-power order.

One can infer the following theorem from Theorem 1.3.

**Theorem 1.4** If $f$ is non-degenerate with respect to $\Delta_\infty(f)$, and the origin lies in the interior of $\Delta_\infty(f)$, then the reciprocal roots of $L_f(t)$ are of absolute value $q^{n/2}$.

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**2 The Artin-Hasse exponential series**

Let

$$E(t) = \exp\left(\sum_{i=0}^{\infty} \frac{t^{p^i}}{p^i}\right) \in \mathbb{Z}_p[[t]]$$

be the Artin-Hasse exponential series. We shall use it to produce roots of unity of $p$-power order.

**Lemma 2.1** If $l$ is a positive integer, and $\pi$ is a root of $\sum_{i=0}^{\infty} \frac{t^{p^i}}{p^i} = 0$ in $\overline{\mathbb{Q}}_p$ with order $\frac{1}{p^{l-1}(p-1)}$, $E(\pi)$ is a primitive $p^l$-th root of unity.

**Proof.** First, $\exp(p^l \pi^{p^i})$ exists as $\text{ord}_p(p^l \pi^{p^i}) \geq \frac{p^l}{p-1}$. So

$$E(\pi)^{p^l} = E(p^l t)|_{t=\pi} = \prod_{i=0}^{\infty} \exp(p^l \pi^{p^i}) = \exp\left(\sum_{i=0}^{\infty} p^l \pi^{p^i}\right) = \exp(0) = 1.$$

Secondly, as $E(t) \in 1 + t + t^2 \mathbb{Z}_p[[t]]$,

$$E(\pi)^{p^l-1} \equiv (1 + \pi)^{p^l-1} \equiv 1 + \pi^{p^l-1} \pmod{\pi^{p^{l-1}+1}}.$$

The lemma is proved.

**Lemma 2.2** Let $l$ be a positive integer. Then the Artin-Hasse exponential series induces a bijection $\pi \mapsto E(\pi)$ from the set of roots of $\sum_{i=0}^{\infty} \frac{t^{p^i}}{p^i} = 0$ in $\overline{\mathbb{Q}}_p$ with order $\frac{1}{p^{l-1}(p-1)}$ to the set of all primitive $p^l$-th roots of unity in $\overline{\mathbb{Q}}_p$.

**Proof.** The field generated over $\mathbb{Q}_p$ by the set of roots of $\sum_{i=0}^{\infty} \frac{t^{p^i}}{p^i} = 0$ in $\overline{\mathbb{Q}}_p$ with order $\frac{1}{p^{l-1}(p-1)}$ is precisely $\mathbb{Q}_p(\mu_{p^l})$ since it contains $\mathbb{Q}_p(\mu_{p^l})$ by the preceding lemma, and is of degree no greater than
Proof. As Lemma 2.5, if \( \psi \) is an automorphism of the field \( \mathbb{Q}_p \) and \( \psi(\mu_p) = \mu_p \) for all \( p > 1 \). Let \( \Delta = \Delta_\infty(f) \), \( D = D(\Delta) \), and \( \pi \) a \( D \)-th root of \( \pi_m \). For \( b \geq 0 \), we write

\[
L(b) = \{ \sum_{u \in L(\Delta)} a_u x^u : a_u \in \mathbb{Z}_p[\mu_{q-1}, \pi, \pi], \text{ord}_p(a_u) \geq b \deg(u) \}.
\]

The Galois group \( \text{Gal}(\mathbb{Q}_p[\mu_{q-1}, \pi, \pi]/\mathbb{Q}_p) \) acts on \( L(b) \) coefficientwise. Define

\[
E_f(x) = \prod_{i=0}^{m-1} \prod_{u \in I_i} E(\pi_m - i \omega(a_{iu})x^u).
\]

Lemma 2.5 We have \( E_f(x) \in L(\frac{1}{p^m-1}) \).

Proof. Suppose that \( 0 \leq i \leq m - 1 \) and \( u \in I_i \). We have \( p^{m-i-1}u \in \Delta \). So \( \deg(p^{m-i-1}u) \leq 1 \), and

\[
\text{ord}_p(\pi_m - i) = \frac{1}{p^{m-i-1}(p-1)} \geq \frac{\deg(p^{m-i-1}u)}{p^{m-i-1}(p-1)} = \frac{\deg(u)}{p-1}.
\]
It follows that $\pi_{m-i} \omega(a_{1u}) x^u \in L(\frac{1}{p^{m-1}})$. Since $E(t) \in \mathbb{Z}_p[[t]]$, we have $E(\pi_{m-i} \omega(a_{1u}) x^u) \in L(\frac{1}{p^{m-1}})$. The lemma now follows.

Let $\sigma$ be the Frobenius element of $\text{Gal}(\mathbb{Q}_p[\mu_{q-1}, \pi_m, \pi]/\mathbb{Q}_p)$ fixing $\pi_m$ and $\pi$. The following lemma follows from Corollary 2.4.

**Lemma 2.6** If $k$ is a positive integer, and $x \in \mu_{q^{k-1}}$, then

$$\psi(\text{Tr}_{\mathbb{Q}_p[\mu_{q^{k-1}}]/\mathbb{Q}_p}(\omega(f)(x))) = \prod_{i=0}^{a_k-1} E_f^x(x^{p^j}).$$

**Corollary 2.7** We have

$$S_k(f) = (-1)^{n-1} \sum_{x \in \mu_{q^{k-1}}} \prod_{i=0}^{a_k-1} E_f^x(x^{p^j}), \ k = 1, 2, \ldots.$$  

### 3 Functions from the Artin-Hasse exponential series

We shall study the growth of the coefficients of $\hat{f}(k = 1, \ldots, n)$, which are defined by

$$d \log \hat{E}(x) = \sum_{k=1}^{n} \sum_{i=0}^{m-1} \frac{\hat{f}_k dx_k}{x_k}, \ \hat{E}(x) = \prod_{j=0}^{\infty} E_f^x(x^{p^j}).$$

**Lemma 3.1** We have

$$\hat{f} = \sum_{k=1}^{m-1} \sum_{j=0}^{\infty} x_j \gamma_{i,j} \sum_{u \in I_i} u_k \omega(a_{1u}) x^{p^j u}, \ k = 1, \ldots, n,$$

where $\gamma_{i,j} = \sum_{l=0}^{n} \frac{p^{m-1}}{p^l}$.

**Lemma 3.2** We have $\pi_l \equiv \pi_{m-1}^{p^{m-1}} \ (\mod \pi_{m-1}^{2m-1})$.

Since $E(t) \in 1 + t + t^2 \mathbb{Z}_p[[t]]$, we have $E(\pi_l) \equiv 1 + \pi_l \ (\mod \pi_l^2)$. So we have

$$E(\pi_l)^{p^{m-1}} \equiv (1 + \pi_l)^{p^{m-1}} \equiv 1 + \pi_l^{p^{m-1}} \ (\mod \pi_{m-1}^{2m-1}),$$

which, combined with the equality $E(\pi_l) = E(\pi_m)^{p^{m-1}}$, implies that $\pi_l \equiv \pi_{m-1}^{p^{m-1}} \ (\mod \pi_{m-1}^{2m-1})$.

**Corollary 3.3** We have $\pi_{m-i}^{p^j} \equiv \pi_m^{p^{i+j}} \ (\mod \pi_m^{p^{i+j}+1})$.

**Lemma 3.4** If $j \leq m - i - 1$ and $l < j$, we have have

$$\text{ord}_p(\frac{\pi_{m-i}^{p^j}}{p^j}) > \text{ord}_p(\frac{\pi_{m-i}^{p^j}}{p^j}).$$

**Corollary 3.5** If $j \leq m - i - 1$, we have $p^j \gamma_{i,j} \equiv \pi_{m-i}^{p^{i+j}} \ (\mod \pi_{m-i}^{p^{i+j}+1})$.  

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Corollary 3.6 Suppose that $j \leq m - i - 1$. Then $\text{ord}_p(p^j \gamma_{i,j}) > \frac{\deg(p^j u)}{p-1}$ if $\deg(p^{m-i-1} u) \leq 1$, and $\text{ord}_p(p^j \gamma_{i,j} - \pi^D \deg(p^j u)) > \frac{\deg(p^j u)}{p-1}$ if $\deg(p^{m-i-1} u) = 1$.

Lemma 3.7 If $j \geq m - i$, we have

$$\text{ord}_p(p^j \gamma_{i,j}) - \frac{\deg(p^j u)}{p-1} \geq p^{j-(m-i)+1} - 1.$$ 

Proof. Since $\gamma_{i,j} = -\sum_{l=j+1}^{\infty} \frac{a^l}{p^l}$, and $\text{ord}_p(a^l) \geq \frac{p^{l+1}}{p^{m-i-1}(p-1)} - j + 1$ when $j \geq m - i$ and $l \geq j + 1$, we have $\text{ord}_p(p^j \gamma_{i,j}) \geq \frac{p^{j+1}}{p^{m-i-1}(p-1)} - 1$ if $j \geq m - i$. The lemma now follows from the fact that $\deg(p^{m-i-1} u) \leq 1$.

Write

$$B = \{ \sum_{u \in L(\Delta)} a_u x^u \in L(\frac{1}{p-1}) : 0 \leq \text{ord}_p(a_u) - \frac{\deg(u)}{p-1} \to +\infty \text{ as } \deg(u) \to \infty \}.$$ 

Corollary 3.8 For $k = 1, \cdots, n$, we have $\widehat{k f} \in B$, and

$$\widehat{k f} \equiv \sum_{i=0}^{m-1} \sum_{j=0}^{m-i-1} \sum_{\deg(p^{m-i-1} u) = 1} u_{k \omega}(a^p) \pi^{D \deg(p^j u)} x^{p^j u} \pmod{\pi B}.$$ 

4 The $p$-adic trace formula

We shall relate the $L$-function $L_f(t)$ to the characteristic polynomials of an operator $(p^n F^{-1})^a$ on $p$-adic spaces.

Since $E_f(x) \in L(\frac{1}{p-1})$ (Lemma 3.1), and $\psi_p : \sum_{u \in L(\Delta)} a_u x^u \mapsto \sum_{u \in L(\Delta)} a_{pu} x^u$ maps $L(b)$ to $L(pb)$, we have the following lemma.

Lemma 4.1 The map $p^n F^{-1} : g \mapsto \sigma^{-1} \circ \psi_p(E_f(x) g)$ sends $L(\frac{1}{p-1})$ to $L(\frac{p}{p-1})$. In particular, $p^n F^{-1}$ acts on $B$.

Note that $p^n F^{-1}$ is $\sigma^{-1}$-linear, and $(p^n F^{-1})^a = \psi^{-1}_p \circ \prod_{i=0}^{a-1} E_f^{|\sigma|}(x^{p^i})$ is $\mathbb{Z}_p[\mu_{q-1}, \pi_m, \pi]$-linear. Write

$$\prod_{i=0}^{a-1} E_f^{|\sigma|}(x^{p^i}) = \sum_{u \in L(\Delta)} a_u x^u.$$ 

Then the trace of $(p^n F^{-1})^a_k$ on $B$ is $\sum_{u \in L(\Delta)} a_{(q^k-1)u}$. And

$$S_k(f) = (-1)^{n-1} (q^k - 1)^n \sum_{u \in L(\Delta)} a_{(q^k-1)u}.$$ 

So we have the following preliminary trace formula.
Proposition 4.2 For \( k = 1, 2, \cdots \), we have
\[
S_k(f) = -(1 - q^k)^n \text{tr}(p^n F^{-1})^{ak}; B).
\]
Equivalently,
\[
L_f(t) = \prod_{i=0}^{n} \det(1 - (p^n F^{-1})^a q^i t; B)^{(-1)^i \binom{n}{i}}.
\]
Let \( e_1 = (1, 0, \cdots, 0), \cdots, e_n = (0, \cdots, 0, 1) \). For \( l = 0, 1, \cdots, n \), we write
\[
K_l = \bigoplus_{1 \leq i_1 < \cdots < i_l \leq n} B e_{i_1} \wedge \cdots \wedge e_{i_l}
\]
and define
\[
p^n F^{-1} : K_l \to K_l, \ g e_{i_1} \wedge \cdots \wedge e_{i_l} \mapsto p^{l+n} F^{-1}(g) e_{i_1} \wedge \cdots \wedge e_{i_l}.
\]
Then the preliminary trace formula takes the following form.

Proposition 4.3 For \( k = 1, 2, \cdots \), we have
\[
S_k(f) = \sum_{l=0}^{n} (-1)^{l+1} \text{tr}(p^n F^{-1})^{ak}; K_l).
\]

By Corollary 3.9, \( \hat{D}_j : g \mapsto (x_j \frac{\partial}{\partial x_j} + \hat{f}) g, j = 1, \cdots, n \), operate on \( B \). Obviously, they commute with each other. So, for \( l = 1, \cdots, n \),
\[
\hat{\partial} : K_l \to K_{l-1}, \ g e_{i_1} \wedge \cdots \wedge e_{i_l} \mapsto \sum_{k=1}^{l} (-1)^{k-1} \hat{D}_{i_k}(g) e_{i_1} \wedge \cdots \wedge \hat{e}_{i_k} \wedge \cdots \wedge e_{i_l}, \ i_1 < \cdots < i_l
\]
are well-defined, and satisfy \( \hat{\partial}^2 = 0 \). Thus we get a complex
\[
K_n \xrightarrow{\hat{\partial}} K_{n-1} \xrightarrow{\hat{\partial}} \cdots \xrightarrow{\hat{\partial}} K_0.
\]
It is easy to check that \( p^n F^{-1} \circ \hat{\partial} = \hat{\partial} \circ p^n F^{-1} \). That is, \( p^n F^{-1} \) operates on the complex \((K_\bullet, \hat{\partial})\). Therefore we have the following homological trace formula.

Proposition 4.4 For \( k = 1, 2, \cdots \), we have
\[
S_k(f) = \sum_{l=0}^{n} (-1)^{l+1} \text{tr}(p^n F^{-1})^{ak}; H_l(K_\bullet, \hat{\partial})).
\]
Equivalently,
\[
L_f(t) = \prod_{l=0}^{n} \det(1 - (p^n F^{-1})^a t; H_l(K_\bullet, \hat{\partial}))^{(-1)^{l+1}}.
\]
5 The total degree of the $L$-function

We shall study the Newton polygon of $\det(1 - (p^n F^{-1})^a; B)$, and then prove Theorem 1.2.

**Proposition 5.1** The Newton polygon of $\det(1 - (p^n F^{-1})^a; B)$ with respect to $\text{ord}_q$ lies above the Hodge polygon of $\sum_{i=0}^{+\infty} W_\Delta(k) t^k$ of degree $D$.

Write $E_f(x) = \sum_{u \in L(\Delta)} a_u \pi^D \deg(u) x^u$, $a_u \in \mathbb{Z}[\mu_{q-1}, \pi_m, \pi]$. Then the matrix of $p^n F^{-1}$ with respect to the orthonormal basis $\{\pi^D \deg(u) x^u\}_{u \in L(\Delta)}$, written as a column vector, is

$$A^{\sigma^{-1}} = (a_{pw-u} \pi^D ((p-1) \deg(w)+c(w,u)))_{w,u}, c(w,u) = \deg(pw - u) + \deg(u) - p \deg(w) \geq 0.$$ 

So, the matrix of $(p^n F^{-1})^a$ with respect to that orthonormal basis is $AA^{\sigma} \cdots A^{\sigma^{a-1}}$. Obviously, the Newton polygon of $\det(1 - A t)$ with respect to $\text{ord}_p$ lies above the polygon with vertices at points $(0,0)$ and

$$\left(\sum_{i=0}^k W_\Delta(i), \sum_{i=0}^k W_\Delta(i) \frac{i}{D}\right), \quad k = 0, 1, \ldots.$$ 

It follows that the Newton polygon of $\det(1 - (p^n F^{-1})^a; B) = \det(1 - AA^{\sigma} \cdots A^{\sigma^{a-1}} t)$ with respect to $\text{ord}_q$ lies above the polygon with vertices at points $(0,0)$ and

$$\left(\sum_{i=0}^k W_\Delta(i), \sum_{i=0}^k W_\Delta(i) \frac{i}{D}\right), \quad k = 0, 1, \ldots.$$ 

The proposition is proved.

**Corollary 5.2** If $j \leq n + 1$, then $\det(1 - (p^n F^{-1})^a; B)$ has at most $\sum_{k=0}^{D_j} W_\Delta(k)$ zeros of $q$-order $\leq j - 1$.

**Proof.** Define

$$\sum_{k=0}^{+\infty} h_\Delta(k) t^k = (1 - t)^n \sum_{k=0}^{+\infty} W_\Delta(k) t^k.$$ 

Since $\sum_{k=0}^{+\infty} h_\Delta(k) t^k$ is a polynomial of degree $\leq n$ with nonnegative coefficients by a lemma of Kouchnirenko [Ko, Lemma 2.9], and

$$\sum_{k=0}^{jD-i} \binom{n-1+k}{n-1}(k+i) = \binom{n+Dj-i}{n}(\frac{n(Dj-i)}{n+1} + i) \geq \binom{n+Dj-i}{n} D(j-1),$$

we have

$$\frac{1}{D} \sum_{k=0}^{jD} kW_\Delta(k) = \frac{1}{D} \sum_{k=0}^{jD} \sum_{i=0}^k h_\Delta(i) \binom{n-1+k-i}{n-1}$$

9
\[
\frac{1}{D} \sum_{i=0}^{n} h_\Delta(i) \sum_{k=i}^{jD} \binom{n-1+k-i}{n-1}(k+i) = \frac{1}{D} \sum_{i=0}^{n} h_\Delta(i) \sum_{k=0}^{jD-i} \left( \binom{n-1+k}{n-1} \right)(k+i)
\]

\[
\geq (j-1) \sum_{i=0}^{n} h_\Delta(i) \sum_{k=0}^{jD-i} \binom{n-1+k}{n-1} \geq (j-1) \sum_{k=0}^{jD} W_\Delta(k).
\]

The corollary now follows from the above inequality by Proposition 5.1.

We now prove Theorem 1.2. Since the reciprocal zeros and reciprocal poles of \( L_f(t) \) are of \( q \)-order \( \leq n \), its total number, by the preliminary trace formula, is bounded by the number of reciprocal zeros of \( \prod_{i=0}^{n} \det(1 - (p^n F^{-1})^q t; B)^{\binom{n}{i}} \). By Corollary 5.2, that number is bounded by

\[
\sum_{i=0}^{n} \binom{n}{i} D(n-i+1) \sum_{k=0}^{jD} W_\Delta(k).
\]

Theorem 1.2 is proved.

6 The acyclicity of the \( p \)-adic complex

In this section we shall prove the following proposition, which implies the first statement of Theorem 1.3.

**Proposition 6.1** If \( f \) non-degenerate with respect to \( \Delta_\infty(f) \), then \((K_\bullet, \hat{\partial})\) is acyclic at positive dimensions, and \( H_0(K_\bullet, \hat{\partial}) \) is a \( \mathbb{Z}_p[\mu_{q-1}, \pi_m, \pi] \)-module free of rank \( n! \text{Vol}(\Delta_\infty(f)) \).

Write

\[
\bar{B} := \mathbb{F}_q[x^{L(\Delta)}] := \left\{ \sum_{u \in L(\Delta)} a_u x^u : a_u \in \mathbb{F}_q \right\}.
\]

It is a ring with the multiplication rule

\[
x^u x^{u'} = \begin{cases} x^{u+u'}, & \text{if } u \text{ and } u' \text{ are cofacial,} \\ 0, & \text{otherwise.} \end{cases}
\]

Define

\[
B \to \bar{B}, \quad \sum_{u \in L(\Delta)} a_u \pi^{D\deg(u)} x^u \mapsto \sum_{u \in L(\Delta)} \bar{a}_u x^u,
\]

where \( \bar{a}_u \) is the residue class of \( a_u \) modulo the maximal ideal of \( \mathbb{Z}_p[\mu_{q-1}, \pi_m, \pi] \).

**Lemma 6.2** The map \( B \to \bar{B} \) is a ring homomorphism. And the sequence

\[
0 \to B \to B \to \bar{B} \to 0
\]

is exact.
For $j = 1, \cdots, n$, we define
\[ \bar{D}_j : \bar{B} \to \bar{B}, \; g \mapsto (x_j \frac{\partial}{\partial x_j} + \bar{j}f)g, \]
where
\[ \bar{j}f = \sum_{i=0}^{m-1} \sum_{j=0}^{m-i-1} \sum_{\deg(p^{m-i-1}u) = 1} u_k a_{ij}^p x^{p^{m-i-1}u}. \]

By Corollary 3.8, we have the following lemma.

**Lemma 6.3** For $j = 1, \cdots, n$, the diagram
\[
\begin{array}{ccc}
B & \to & \bar{B} \\
\bar{D}_j & \downarrow & \bar{D}_j \\
B & \to & \bar{B}
\end{array}
\]
is commutative.

For $l = 0, \cdots, n$, we define
\[ \bar{K}_l = \bigoplus_{1 \leq i_1 < \cdots < i_l \leq n} \bar{B} e_{i_1} \wedge \cdots \wedge e_{i_l}. \]

For $l = 1, \cdots, n$, we define
\[ \bar{\partial} : \bar{K}_l \to \bar{K}_{l-1}, \; ge_{i_1} \wedge \cdots \wedge e_{i_l} \mapsto \sum_{k=1}^{l} (-1)^{k-1} \bar{D}_{i_k}(g) e_{i_1} \wedge \cdots \wedge \hat{e}_{i_k} \wedge \cdots \wedge e_{i_l}, \; i_1 < \cdots < i_l. \]

It is easy to see that the sequence
\[ \bar{K}_n \to \bar{K}_{n-1} \to \cdots \to \bar{K}_0 \]
is a complex.

**Proposition 6.4** The map $B \to \bar{B}$ induces a morphism of complexes from $(K_\bullet, \bar{\partial})$ to $(\bar{K}_\bullet, \bar{\partial})$. Moreover, the sequence
\[ 0 \to (K_\bullet, \bar{\partial}) \to (K_\bullet, \bar{\partial}) \to (\bar{K}_\bullet, \bar{\partial}) \to 0 \]
is exact.

**Proof.** The first statement follows from Lemma 6.3, and the second follows from Lemma 6.2.

By Proposition 6.4, and a lemma of Monsky [Mo, Theorem 8.5], the proof of Proposition 6.1 is reduced to the proof of the following proposition.

**Proposition 6.5** If $f$ is non-degenerate with respect to $\Delta_\infty(f)$, then $(\bar{K}_\bullet, \bar{\partial})$ is acyclic at positive dimensions, and $H_0((\bar{K}_\bullet, \bar{\partial}))$ is a $\mathbb{F}_q$-vector space of dimension $n! \text{Vol}(\Delta_\infty(f))$.

For $j = 1, \cdots, n$, we define
\[ \bar{j}f^0 = \sum_{i=0}^{m-1} \sum_{\deg(p^{m-i-1}u) = 1} u_k a_{ij}^p x^{p^{m-i-1}u}. \]
For $l = 1, \ldots, n$, we define

$$
\partial^0 : \bar{K}_l \to \bar{K}_{l-1}, \quad ge_{i_1} \wedge \cdots \wedge e_{i_l} \mapsto \sum_{k=1}^l (-1)^{k-1} \alpha_{i_k} ge_{i_1} \wedge \cdots \wedge \hat{e}_{i_k} \wedge \cdots \wedge e_{i_l}, \quad i_1 < \cdots < i_l.
$$

Then

$$
\bar{K}_n \xrightarrow{\partial^0} \bar{K}_{n-1} \xrightarrow{\partial^0} \cdots \xrightarrow{\partial^0} \bar{K}_0
$$

is a complex. In the next section, we shall prove the following proposition.

**Proposition 6.6** If $f$ is non-degenerate with respect to $\Delta := \Delta_\infty(f)$, then the complex $(\bar{K}_*, \partial^0)$ is acyclic at positive dimensions, and the Poincaré series of $H_0((\bar{K}_*, \partial^0))$ is $P_\Delta(t)$. In particular, $H_0((\bar{K}_*, \partial^0))$ is a $\mathbb{F}_q$-vector space of dimension $n!Vol(\Delta)$.

We now deduce the first statement of Proposition 6.5 from Proposition 6.6. In a given a homology class of positive dimension, we choose one representative $\xi$ of lowest degree. We claim that $\xi = 0$. Otherwise, let $\xi^0$ be the leading term of $\xi$. We have $\partial^0(\xi^0) = 0$ since it is the leading term of $\partial(\xi) = 0$. By the acyclicity of $(\bar{K}_*, \partial^0)$, $\xi^0 = \partial^0(\eta)$ for some $\eta$. The form $\xi - \partial(\eta)$ is now of lower degree than $\xi$, contradicting to our choice of $\xi$. The proposition is proved.

The second statement of Proposition 6.5 follows the following proposition.

**Proposition 6.7** Let $V$ be a basis of $\bar{K}_0$ modulo $\partial^0(\bar{K}_1)$ consisting of homogeneous elements. Then $V$ is also a basis of $\bar{K}_0$ modulo $\partial(\bar{K}_1)$.

**Proof.** First, we show that $\bar{K}_0$ is generated by $V$ and $\partial(\bar{K}_1)$. Otherwise, among elements of $\bar{K}_0$ which are non-linear combinations of elements of $V$ and $\partial(\bar{K}_1)$, we choose one of lowest degree. We may suppose that it is of form $\partial^0(\xi)$. Let $\xi^0$ be the leading term of $\xi$. Then $\partial^0(\xi) = \partial(\xi^0)$ is not a linear combination of elements of $V$ and $\partial(\bar{K}_1)$, and is of lower degree than $\partial^0(\xi)$. This is a contradiction. Therefore $\bar{K}_0$ is generated by $E$ and $\partial(\bar{K}_1)$. It remains to show that $\xi = 0$ whenever $\xi$ belongs to $\partial(\bar{K}_1)$ and is a linear combination of elements of $V$. Otherwise, we may choose one element $\zeta$ of lowest degree such that $\xi = \partial(\zeta)$. Let $\zeta^0$ be the leading term of $\zeta$. Then $\partial^0(\zeta^0)$ is a linear combination of elements of $V$ since it is the leading term of $\partial(\zeta)$. So we have $\partial^0(\zeta^0) = 0$. By the acyclicity of $(\bar{K}_*, \partial^0)$, $\zeta^0 = \partial^0(\eta)$ for some $\eta$. The form $\zeta - \partial(\eta)$ is now of lower degree than $\zeta$, contradicting to our choice of $\zeta$. This completes the proof of the proposition.

## 7 The complex obtained by reduction

In this section, we shall prove Proposition 6.6. The second statement follows from the first, and the last follows from the second and a lemma of Kouchnirenko [Ko, Lemma 2.9]. So it remains to prove the acyclicity of the complex $(\bar{K}_*, \partial^0)$.

Let $\tau$ be a face of $\Delta$ that does not contain the origin, and $\bar{\tau}$ is the convex hull in $\mathbb{Q}^n$ generated by $\tau$ and the origin. For $\alpha_1, \ldots, \alpha_s \in \mathbb{F}_q[x^L(\tau)]$, we define $\bar{K}_s(\bar{\tau}, \{\alpha_j\}_{j=1}^s)$ to be the complex

$$
\bar{K}_l(\bar{\tau}, \{\alpha_j\}_{j=1}^s) = \bigoplus_{1 \leq i_1 < \cdots < i_l \leq s} \mathbb{F}_q[x^L(\tau)] e_{i_1} \wedge \cdots \wedge e_{i_l}, \quad l = 0, \ldots, s
$$

with derivation

$$
ge_{i_1} \wedge \cdots \wedge e_{i_l} \mapsto \sum_{k=1}^l (-1)^{k-1} \alpha_{i_k} e_{i_1} \wedge \cdots \wedge \hat{e}_{i_k} \wedge \cdots \wedge e_{i_l}, \quad 1 \leq i_1 < \cdots < i_l \leq s.
$$
By a proposition of Kouchnirenko [Ko, Proposition 2.6] and the argument of Adolphson-Sperber [AS2, p379], the sequence
\[
0 \to \bar{K}_s^0(f) \to \bigoplus_{\dim \tau = n-1} \bar{K}_s(\bar{\tau}, \{\bar{\tau}_j^1\}_{j=1}^n) \to \cdots \to \bigoplus_{\dim \tau = 0} \bar{K}_s(\bar{\tau}, \{\bar{\tau}_j^1\}_{j=1}^n) \to \bar{K}_s^{-1} \to 0
\]
is exact, where \(\tau\) denotes a face of \(\Delta\) that does not contain the origin, and
\[
\bar{K}_s^{-1} = \begin{cases} 
\binom{n}{l}, & \text{if the origin is in the interior of } \Delta \text{ and } 1 \leq l \leq n, \\
0, & \text{otherwise.}
\end{cases}
\]
By the exactness of that sequence, the acyclicity of the complex \((\bar{K}_s, \bar{\partial}^0)\) follows from the following lemma.

**Lemma 7.1** Let \(f\) be a Witt vector of length \(m\) with coefficients in \(\mathbb{F}_q[x_1^{\pm1}, \cdots, x_n^{\pm1}]\). Suppose that \(f\) is non-degenerate with respect to \(\Delta := \Delta_\infty(f)\) and \(\dim \Delta = n\). Let \(\tau\) be a face of \(\Delta\) of dimension \(s-1\) that does not contain the origin. Then the complex \(\bar{K}_s(\bar{\tau}, \{\bar{\tau}_j^1\}_{j=1}^n)\) is acyclic at dimensions \(n-s\).

Since the sequence
\[
0 \to \bar{K}_s(\bar{\tau}, \{\alpha_j\}_{j=1}^{s-1}) \to \bar{K}_s(\bar{\tau}, \{\alpha_j\}_{j=1}^s) \to \bar{K}_s(\bar{\tau}, \{\alpha_j\}_{j=1}^{s-1})[-1] \to 0
\]
is exact, Lemma 7.1 follows from the following one.

**Lemma 7.2** Suppose that \(f\) is non-degenerate with respect to \(\Delta_\infty(f)\). If \(\tau\) is a face of \(\Delta\) of dimension \(r-1\) that does not contain the origin, then there are \(1 \leq i_1 < \cdots < i_r \leq n\) such that the complex \(\bar{K}_s(\bar{\tau}, \{\bar{\tau}_j^l\}_{j=1}^r)\) is acyclic at positive dimensions.

**Proof.** There are \(\{i_1, \ldots, i_r\} \subset \{1, \ldots, n\}\) and \((\alpha_{kj}) \in \mathbb{Q} \cap \mathbb{Z}_p\) \((1 \leq k \leq n, 1 \leq j \leq r)\) such that 
\[u_k = \alpha_{1k}u_{i_1} + \cdots + \alpha_{rk}u_{i_r}\] for all \(u = (u_1, \ldots, u_n) \in \bar{L}(\bar{\tau})\). Let \(\sigma\) be any face of \(\tau\). We have 
\[\bar{\tau}_j^{l'} = \alpha_{1ji} \bar{\tau}_j^1 + \cdots + \alpha_{rji} \bar{\tau}_j^1.
\] So \(\bar{\tau}_j^{l'}, \cdots, \bar{\tau}_r^{l'}\) have no common zeros in \(\mathbb{P}_q^X\). By a theorem of Kouchnirenko [Ko, Theorem 6.2], \(\bar{\tau}_j^{l'}, \cdots, \bar{\tau}_r^{l'}\) generate in \(\mathbb{F}_q[x^{L(\bar{\tau})}]\) an ideal of finite codimension. Note that \(\mathbb{F}_q[x^{L(\bar{\tau})}]\) is Cohen-Macaulay by a theorem of Hochster [Ho, Theorem 1]. The complex \(\bar{K}_s(\bar{\tau}, \{\bar{\tau}_j^l\}_{j=1}^r)\) is acyclic at positive dimensions by a theorem of Serre [Se, Theorem 3, Chapter IV]. The lemma is proved.

## 8 The Newton polygon of the \(L\)-function

In this section we shall prove the second statement of Theorem 1.3. (The last statement follows from the second by a lemma of Kouchnirenko [Ko, Lemma 2.9].) By the argument of Dwork [Dw2, §7], it suffices to prove the following proposition.

**Proposition 8.1** If \(f\) is non-degenerate with respect to \(\Delta := \Delta_\infty(f)\), then the Newton polygon of \(
\det(1 - p^gF^{-1}t; H_0(K_s, \bar{\partial}))
\) with respect to ord \(p\) lies above the Hodge polygon of \(P_\Delta(t)\) of degree \(D(\Delta)\), and their endpoints coincide.
Let $\hat{V}$ be a basis of $\hat{K}_0$ modulo $\partial^0(K_1)$ consisting of homogeneous elements. By Proposition 6.7, it is also a basis of $\hat{K}_0$ modulo $\partial(K_1)$. Define

$$V = \{ \sum \omega(a_u)x^u : \sum a_u x^u \in \hat{V} \}.$$ 

It is a basis of $B$ modulo $\sum_{k=1}^{n} \hat{D}_kB$. For real numbers $b > \frac{1}{p-1}$ and $c$, we write

$$L(b, c) = \{ \sum_{u \in L(D)} a_u x^u : a_u \in \mathbb{Q}_p[\mu_{q-1}, \pi_m, \pi], \ \text{ord}_p(a_u) \geq b \deg(u) + c \}.$$ 

It is compact with respect to the topology of coefficientwise convergence. Let $V(b, c)$ be the subset of elements of $L(b, c)$ which are finite linear combinations of elements of $V$. In the next section we shall prove the following proposition.

**Proposition 8.2** If $\frac{1}{p-1} < b < \frac{p}{p-1}$, then

$$L(b, c) = V(b, c) + \sum_{k=1}^{n} \hat{D}_kL(b, c + b - \frac{1}{p-1}).$$

We now prove the first statement of Proposition 8.1. For each $\xi \in V$, we write

$$p^n F^{-1}(\pi^D\deg(\xi)) \equiv \sum_{\eta \in V} c_{\eta, \xi} \pi^{D\deg(\eta)} \eta \pmod{\sum_{k=1}^{n} \hat{D}_kB}, \ c_{\eta, \xi} \in \mathbb{Z}_p[\mu_{q-1}, \pi_m, \pi]$$

By Lemma 4.1, $p^n F^{-1}(\pi^D\deg(\xi))$ lies in the space $L(\frac{p}{p-1})$. So, by Proposition 8.2, $c_{\eta, \xi} \pi^{D\deg(\eta)} \eta$ lies in every $L(b)$ with $\frac{1}{p-1} < b < \frac{p}{p-1}$. That is, $\text{ord}_p(c_{\eta, \xi}) \geq (b - \frac{1}{p-1}) \deg(\eta)$ for every $\frac{1}{p-1} < b < \frac{p}{p-1}$. Thus we have $\text{ord}_p(c_{\eta, \xi}) \geq \deg(\eta)$. Therefore, the Newton polygon of the characteristic polynomial of $(c_{\eta, \xi})$, which is now the Newton polygon of $\det(1 - p^n F^{-1} t; H_0(K_*, \partial))$, lies above the Hodge polygon of $P_D(t)$ of degree $D$. In particular, $\text{ord}_p(\det(c_{\eta, \xi})) \geq \sum_{\xi \in V} \deg(\xi)$.

It remains to show that the Newton polygon of $\det(1 - p^n F^{-1} t; H_0(K_*, \partial))$ share the same endpoints with the Hodge polygon of $P_D(t)$ of degree $D$. Define

$$\phi : L(\frac{p}{p-1}) \to L(\frac{1}{p-1}), \ \sum_{u \in L(D)} a_u x^u \mapsto \sum_{u \in L(D)} a_u x^{pu}.$$ 

Obviously, $p^n F^{-1} \circ (E^{-1}_f \circ \phi \circ \sigma) = 1$ on $L(\frac{p}{p-1})$. At the end of this we shall prove the following proposition.

**Proposition 8.3** Modulo $\sum_{k=1}^{n} \hat{D}_kL(\frac{1}{p-1})$, the space $L(\frac{1}{p-1})$ is generated by \{ $\pi^{D\deg(\xi)} : \xi \in V \}.$

So, for each $\xi \in V$, we can find $b_{\eta, \xi} \in \mathbb{Z}_p[\mu_{q-1}, \pi_m, \pi]$

$$E^{-1}_f \circ \phi \circ \sigma(\pi^{D\deg(\xi)} \eta) \equiv \sum_{\eta \in V} b_{\eta, \xi} \pi^{D\deg(\eta)} \eta \pmod{\sum_{k=1}^{n} \hat{D}_kL(\frac{1}{p-1})}.$$
It follows that \((c_{\eta, \xi})(b_{\eta, \xi}) = \text{diag}\{\pi^{D_{\deg(\xi)}}(p^{-1}), \xi \in V}\)\). So \(\text{ord}_p(\det(c_{\eta, \xi})) \leq \sum_{\eta \in V} \deg(\eta)\). Therefore

\[
\text{ord}_p(\det(c_{\eta, \xi})) = \sum_{\eta \in V} \deg(\eta).
\]

That is, the Newton polygon of \(\det(1 - p^n F^{-1} t; H_0(K_\bullet, \hat{\partial}))\) share the same endpoints with the Hodge polygon of \(P_\Delta(t)\) of degree \(D\).

We now prove Proposition 8.3. Let \(\xi = \sum_{u \in L(\Delta)} a_u x^u \in L(\frac{\mathbb{R}}{p^{n-1}})\). For \(N = 0, 1, \cdots\), write

\[
\xi^{(N)} = \sum_{u \in L(\Delta), \deg(u) \leq N} a_u x^u \in B.\]

As \(\{\pi^{D_{\deg(\eta)}} \eta : \eta \in V\}\) is a basis of \(B\) modulo \(\sum_{k=1}^m \hat{D}_k B\), there are elements \(\xi_{k}^{(N)} \in B\) \((k = 1, \cdots, n)\) such that

\[
\xi^{(N)} - \sum_{k=1}^n \hat{D}_k \xi_k^{(N)} = \sum_{\eta \in V} a^{(N)}_{\eta} \pi^{D_{\deg(\eta)}} \eta.
\]

As \(L(\frac{1}{p^{n-1}})\) is compact with respect to the topology of coefficientwise convergence, the sequence \((\{\xi_{k}^{(N)}\}_{k=1}^n, \{a^{(N)}_{\eta}\}_{\eta \in V})\), \(N = 0, 1, \cdots\), has an adherent point \((\{\xi_k\}_{k=1}^n, \{a_{\eta}\}_{\eta \in V})\) in the space \(L(b)^n \times (\mathbb{Z}_p[\mu_{p-1}, \pi_m, \pi])^{|V|}\). Therefore we get

\[
\xi - \sum_{k=1}^n \hat{D}_k \xi_k = \sum_{\eta \in V} a_{\eta} \pi^{D_{\deg(\eta)}} p^{m-1} \eta.
\]

This completes the proof of Proposition 8.3.

9 The space \(L(b, c)\)

In this section we shall prove Propositions 8.2.

For \(k = 1, \cdots, n\), we write

\[
\kappa_f^0 = \sum_{i=0}^{m-1} p^{m-i-1} \gamma_{i, m-i-1} \sum_{\deg(p^{m-i-1} u) = 1} u_k \omega(a_{i u}^{p^{m-i-1}}) x^{p^{m-i-1} u}.
\]

For \(l = 1, \cdots, n\), we define

\[
\hat{\partial}^0 : K_l \to K_{l-1}, \ g e_{i_1} \wedge \cdots \wedge e_{i_l} \mapsto \sum_{k=1}^{l} (-1)^{k-l} \kappa_f^0 \hat{\gamma}_{i_k} \wedge \cdots \wedge \hat{e}_{i_k} \wedge \cdots \wedge e_{i_1}, \ i_1 < \cdots < i_l.
\]

It is easy to see that

\[
0 \to (K_\bullet, \hat{\partial}^0) \to (K_\bullet, \hat{\partial}) \to (\overline{K}_\bullet, \hat{\partial}) \to 0
\]

is an exact sequence of complexes. So we have the following lemma.

Lemma 9.1 Modulo \(\sum_{k=1}^n \kappa_f^0 B\), the space \(B\) is generated by \(\{\pi^{D_{\deg(\xi)}} \xi : \xi \in V\}\).
Corollary 9.2 If \( b > \frac{1}{p-1} \), then

\[
L(b, c) = V(b, c) + \sum_{k=1}^{n} \hat{f}^0_k L(b, c + b - \frac{1}{p-1}).
\]

Proof. Let \( \xi \in L(b, c) \), \( \xi_v \) (\( v \in \text{deg} \, L(\Delta) \)) its homogeneous part of degree \( v \), and \( k_v \) the least integer such that \( \text{ord}_p(\pi^{k_v}) \geq bv + c \). Then \( \pi^{Dv-k_v} \xi_v \in B \). By the above lemma, we may write

\[
\pi^{Dv-k_v} \xi_v = \sum_{\eta \in V, \text{deg}(\eta) \leq v} a_{\eta}^{(v)} \pi^{D \text{deg}(\eta)} \eta + \sum_{i=1}^{n} \hat{f}^0_i \eta_i^{(v)},
\]

where \( a_{\eta}^{(v)} \in \mathbb{Z}_p[\mu_{q-1}, \pi_m, \pi] \), and \( \eta_i^{(v)} \in B \) is of degree \( \leq v - 1 \). It follows that

\[
\xi = \sum_{\eta \in V, \text{deg}(\eta) \leq v} \eta \pi^{D \text{deg}(\eta)} \sum_{v \geq \text{deg}(\eta)} a_{\eta}^{(v)} \pi^{k_v-Dv} + \sum_{i=1}^{n} \hat{f}^0_i \sum_{v \in \text{deg} \, L(\Delta)} \pi^{k_v-Dv} \eta_i^{(v)}.
\]

It is easy to see that the first term on the right-hand side converges to an element in \( V(b, c) \), and the inner sum in the second term converges to an element in \( L(b, c + b - \frac{1}{p-1}) \). The corollary is proved.

For \( k = 1, \ldots, n \), we define

\[
D_k : B \to B, \quad g \mapsto (x_k \frac{\partial}{\partial x_k} + \sum_{i=0}^{m-1} \sum_{j=0}^{m-i-1} p^j \gamma_{i,j} \sum_{u \in I_i} u_k \omega(a_{\eta}^{(v)} x^p u) g).
\]

For \( l = 1, \ldots, n \), we define

\[
\partial : K_l \to K_{l-1}, \quad g e_{i_1} \wedge \cdots \wedge e_{i_l} \mapsto \sum_{k=1}^{l} (-1)^{k-1} D_k(g) e_{i_1} \wedge \cdots \wedge \hat{e}_{i_k} \wedge \cdots \wedge e_{i_l}, \quad i_1 < \cdots < i_l.
\]

Corollary 9.3 If \( b > \frac{1}{p-1} \), then

\[
L(b, c) = V(b, c) + \sum_{k=1}^{n} D_k L(b, c + b - \frac{1}{p-1}).
\]

Proof. Note that \( \hat{f}^0 - D_k \) maps \( L(b, c) \) to \( L(b, c - (b - \frac{1}{p-1}) e) \) for some constant \( e < 1 \). Let \( \xi \in L(b, c) \). By the previous corollary and induction, we can find a sequence

\[
(\eta_0^{(i)}, \cdots, \eta_n^{(i)}) \in V(b, c + i(1-e)(b - \frac{1}{p-1})) \times L(b, c + (i(1-e) + 1)(b - \frac{1}{p-1}))^n, \quad i = 0, 1, \cdots
\]

such that

\[
\xi = \eta_0^{(0)} + \sum_{k=1}^{n} \hat{f}^0_k \eta_k^{(0)},
\]
and
\[ \sum_{k=1}^{n} (\hat{f}_k^0 - D_k) \eta_k^{(i)} = \eta_0^{(i+1)} + \sum_{k=1}^{n} \hat{f}_k^0 \eta_k^{(i+1)}. \]

One sees immediately that \( \sum_{i=0}^{\infty} \eta_0^{(i)} \) converges to an element \( \eta_0 \) in \( V(b, c) \), and \( \sum_{i=0}^{\infty} \eta_k^{(i)} \) converges to an element \( \eta_k \) in \( L(b, c + b - \frac{1}{p-1}) \). Moreover, we have \( \xi = \eta_0 + \sum_{k=1}^{n} D_k \eta_k \). The corollary is proved.

We now prove Proposition 8.2. Note that \( D_k - \hat{D}_k \in L(b, p\left(\frac{p}{p-1} - b\right) - 1) \) by Lemma 3.7. So it maps \( L(b, c) \) to the space \( L(b, c + p\left(\frac{p}{p-1} - b\right) - 1) \). Let \( \xi \in L(b, c) \). By the previous corollary and induction, we can find a sequence
\[ (\eta_0^{(i)}, \ldots, \eta_n^{(i)}) \in V(b, c + i(p - (p - 1)b)) \times L(b, c + i(p - (p - 1)b) + (b - \frac{1}{p-1}))^n, \quad i = 0, 1, \ldots \]
such that
\[ \xi = \eta_0^{(0)} + \sum_{k=1}^{n} D_k \eta_k^{(0)}, \]
and
\[ \sum_{k=1}^{n} (D_k - \hat{D}_k) \eta_k^{(i)} = \eta_0^{(i+1)} + \sum_{k=1}^{n} D_k \eta_k^{(i+1)}. \]

One sees immediately that \( \sum_{i=0}^{\infty} \eta_0^{(i)} \) converges to an element \( \eta_0 \) in \( V(b, c) \), and \( \sum_{i=0}^{\infty} \eta_k^{(i)} \) converges to an element \( \eta_k \) in \( L(b, c + b - \frac{1}{p-1}) \). Moreover, we have \( \xi = \eta_0 + \sum_{k=1}^{n} D_k \eta_k \). This completes the proof of Proposition 8.2.

10 The weights of the \( L \)-function

We shall prove Theorem 1.4.

Let \( \Delta \) be a convex polyhedron in \( \mathbb{Q}^n \) of dimension \( n \) that contains the origin, and \( S_\Delta \) the sum of the volumes of all its \( (n-1) \)-dimensional faces that contain 0. Write
\[ \left(1 - t^{D(\Delta)}\right)^n \sum_{i=0}^{\infty} W_\Delta(i) t^i = \sum_{i=0}^{D(\Delta)n} h_\Delta(i) t^i. \]

Lemma 10.1 We have
\[ \frac{1}{D(\Delta)} \sum_{i=0}^{nD(\Delta)} ih_\Delta(i) = \frac{n}{2} n! \text{Vol}(\Delta) - \frac{(n-1)!}{2} S_\Delta. \]

In particular, \( \frac{1}{D(\Delta)} \sum_{i=0}^{nD(\Delta)} ih_\Delta(i) = \frac{n}{2} n! \text{Vol}(\Delta) \) if the origin is an interior point of \( \Delta \).
Proof. Note that
\[ W_\Delta(i) = \sum_{k=0}^{i/D(\Delta)} h_\Delta(i - D(\Delta)k) \left( \begin{array}{c} n - 1 + k \\ n - 1 \end{array} \right). \]
So
\[ \sum_{i \leq D(\Delta)x} (x - \frac{i}{D(\Delta)}) W_\Delta(i) = \sum_{j=0}^{nD(\Delta)} h_\Delta(j) \sum_{0 \leq k \leq \frac{j}{D(\Delta)}} (x - \frac{j}{D(\Delta)} - k) \left( \begin{array}{c} n - 1 + k \\ n - 1 \end{array} \right). \]
On the other hand, by [AS, (4.12-13)],
\[ \sum_{i \leq D(\Delta)x} (x - \frac{i}{D(\Delta)}) W_\Delta(i) = n!Vol(\Delta) \frac{x^{n+1}}{(n+1)!} + \frac{(n-1)!}{2} S_\Delta x^n + O(x^{n-1}). \]
The lemma now follows.

We now prove Theorem 1.4. Let \( \alpha_i, i = 1, \cdots, n \) be the eigenvalues of \( q^n F^{-1} \) on \( H_0(K\bullet, \hat{\partial}) \). By Theorem 1.2 and Lemma 10.1,
\[ \text{ord}_q( \prod_{i=1}^{n!Vol(\Delta)} \alpha_i) = \frac{n}{2} n!Vol(\Delta). \]
It is known that the eigenvalues \( \alpha_i \) are \( l \)-adic units when \( l \) is a prime different from \( p \). So, by the product formula, we have
\[ \prod_{i=1}^{n!Vol(\Delta)} |\alpha_i| = q^{\frac{n}{2} n!Vol(\Delta)}. \]
By a theorem of Kedlaya [Ke, Theorem 5.6.2], the Frobenius \( F \) on \( H_0(K\bullet, \hat{\partial}) \otimes \mathbb{Q}_p[\mu_{q-1}, \pi_m, \pi] \) is of mixed weight \( \geq n \). So \( q^n F^{-1} \) on \( H_0(K\bullet, \hat{\partial}) \) is of mixed weight \( \leq 2n - n \leq n \). That is, \( |\alpha_i| \leq q^{n/2} \). It follows that all the eigenvalues \( \alpha_i \) must have absolute value \( q^{n/2} \). This completes the proof of Theorem 1.4.

11 Applications to other situations
Let \( J \) be a subset of \( \{1, \cdots, n\} \). For \( \{j_1, \cdots, j_s\} \subseteq J \), we write
\[ B_{\{j_1, \cdots, j_s\}} = \{ \sum_{u \in L(\Delta)} a_u x^u \in B : u_{j_1}, \cdots, u_{j_s} > 0 \}. \]
For \( l = 0, 1, \cdots, n \), we define
\[ K_l(f, J) = \bigoplus_{1 \leq i_1 < \cdots < i_l \leq n} B_{J \setminus \{i_1, \cdots, i_l\}} \cdot e_{i_1} \wedge \cdots \wedge e_{i_l}. \]
Then \((K\bullet(f, J), \hat{\partial})\) is a subcomplex of \((K\bullet(f, \emptyset), \hat{\partial})\). The latter is the complex \((K\bullet, \hat{\partial})\) we defined earlier.
Lemma 11.1 The sequence

\[ 0 \to K_\ast(f, J) \to K_\ast(f, J \setminus \{j\}) \to K_\ast(f^{\{j\}}, J \setminus \{j\}) \to 0 \]

is exact, where \(f^{\{j\}}\) is the Witt vector whose \(i\)-th coordinate is the sum of monomials of the \(i\)-th coordinate of \(f\) not divided by \(x_j\).

We define, for \(k = 1, 2, \ldots\),

\[ S_k(f, J) = \sum_{x^{q^n} = x_{i_1} \cdots x_{i_r} \neq 0} \psi(\text{Tr}_{\mathbb{Q}_p} \mu_{q^{k-1}}/\mathbb{Q}_p(\omega(f)(x))) \]

if \(\{1, \ldots, n\} \setminus J = \{i_1, \ldots, i_r\}\), and \(f \in W_m(\mathbb{F}_q[x_1, \ldots, x_n, (x_{i_1} \cdots x_{i_r})^{-1}])\). Here the equation \(x^{q^n} = x\) is solved in \((\mathbb{Q}_p)^n\). We write

\[ L_{f, J}(t) = \exp\left(\sum_{k=1}^{\infty} S_k(f, J) \frac{t^k}{k}\right). \]

By the above lemma we infer the following trace formula from the earlier one.

Proposition 11.2 For \(k = 1, 2, \ldots\), we have

\[ S_k(f, J) = \sum_{l=0}^{n} (-1)^{l+1} \text{Tr}(p^n F^{-1})^{ak} H_l(K_\ast(f, J), \hat{\partial}). \]

Equivalently,

\[ L_{f, J}(t) = \prod_{l=0}^{n} \det(1 - (p^n F^{-1})^{a_l} t; H_l(K_\ast(f, J), \hat{\partial}))^{(-1)^l}. \]

We call \(f\) commode with respect to \(J\) if \(\Delta_\infty(f)\) is commode with respect to \(J\). Recall that a convex polyhedron \(\Delta\) in \(\mathbb{Q}^n\) that contains the origin is commode with respect to \(J\) if it lies in \((\prod_{i=1, i \notin J} \mathbb{Q}) \times (\prod_{i \in J} \mathbb{Q}_{\geq 0})\) and \(\dim(\Delta_C) = n - |C|\) for all subset \(C\) of \(J\), where \(\Delta_C = \{(u_1, \ldots, u_n) \in \Delta : u_j = 0\text{ if } j \in C\}\). By Lemma 11.1 and Proposition 11.2, we infer the following proposition from Theorem 1.3.

Proposition 11.3 If \(f\) is commode with respect to \(J\) and non-degenerate with respect to \(\Delta_\infty(f)\), then \(L_{f, J}(t)\) is a polynomial, and its Newton polygon with respect to \(\text{ord}_q\) lies above the Hodge polygon of

\[ \sum_{C \subseteq J} (-1)^{|C|} P_{\Delta_C}(\frac{D(\Delta)}{n! \Delta_C}) \]

with the same endpoints. In particular, \(L_{f, J}(t)\) is of degree

\[ \sum_{C \subseteq J} (-1)^{|C|} (n - |C|)! \text{Vol}(\Delta_C). \]

From Lemma 10.1 we infer the following one.
Lemma 11.4 Let $\Delta$ be a convex polyhedron in $\mathbb{Q}^n$ of dimension $n$ that contains the origin and is commode with respect to $J$. Let $(V_{\Delta,J}, U_{\Delta,J})$ be the endpoint of the Hodge polygon of

$$
\sum_{C \subset J} (-1)^{|C|} P_{\Delta C}(t^{D(\Delta C)})
$$

other than $(0,0)$. Then

$$
U_{\Delta,J} = \frac{n}{2} V_{\Delta,J} + \sum_{l=1}^{[J]} (-1)^l \frac{(n-l)!}{2} \sum_{C \subset J, |C|=l-1} S_{\Delta C} - l \sum_{C \subset J, |C|=l} \text{Vol}(\Delta C)).
$$

In particular, $U_{\Delta,J} = \frac{n}{2} V_{\Delta,J}$ if the origin is an interior point of $\Delta_J$.

By Lemma 10.3 we infer the following proposition from Proposition 10.2.

Proposition 11.5 If $f$ is commode with respect to $J$ and non-degenerate with respect to $\Delta := \Delta_{\infty}(f)$, and the origin lies in the interior of $\Delta_J$, then the reciprocal roots of $L_{f,J}(t)$ are of absolute value $q^{n/2}$.

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