Open level lines of a superposition of periodic potentials on a plane

A.Ya. Maltsev$^1$, S.P. Novikov$^{1,2}$

$^1$ L.D. Landau Institute for Theoretical Physics of Russian Academy of Sciences
142432 Chernogolovka, pr. Ak. Semenova 1A
$^2$ V.A. Steklov Mathematical Institute of Russian Academy of Sciences
119991 Moscow, Gubkina str. 8

We consider here open level lines of potentials resulting from the superposition of two different periodic potentials on the plane. This problem can be considered as a particular case of the Novikov problem on the behavior of open level lines of quasi-periodic potentials on the plane with four quasi-periods. At the same time, the formulation of this problem may have many additional features that arise in important physical systems related to it. Here we will try to give a general description of the emerging picture both in the most general case and in the presence of additional restrictions. The main approach to describing the possible behavior of the open level lines will be based on their division into topologically regular and chaotic level lines.

I. INTRODUCTION

In this paper, we consider applications of the theory of quasi-periodic functions to two-dimensional systems formed by a superposition of periodic systems on a plane. More precisely, we will consider applications of the latest results obtained in the study of the Novikov problem with four quasi-periods to potentials generated by a superposition of two arbitrary periodic potentials with their possible influence on each other.

The beginning of the theory of quasi-periodic functions goes back to the works of H. Bohr and A.S. Besikovich (see e.g. [1, 2]). Generally speaking, there are a number of definitions of a quasi-periodic function in the literature. Here we will call a quasi-periodic function $f(x_1, \ldots, x^n)$ in $\mathbb{R}^n$ the restriction of a periodic function $F(z_1, \ldots, z^N)$, defined in space $\mathbb{R}^N$, under an affine embedding $\mathbb{R}^n \rightarrow \mathbb{R}^N$. The dimension $N$ of the embedding space is then called the number of quasi-periods of the function $f(x_1, \ldots, x^n)$. Let us also note here, that this definition is well known, for instance, in the theory of quasicrystals.

The general problem of Novikov is to describe the geometry of the level lines of quasi-periodic functions on the plane with an arbitrary number of quasi-periods. It must be said that this problem was originally set for functions with three quasi-periods (3). The problem in this formulation is directly related to the theory of galvano-magnetic phenomena in metals with complex Fermi surfaces. The Novikov problem with three quasi-periods has now been studied in most detail. In particular, a complete classification of all types of level lines of the corresponding functions on the plane has been obtained, and a number of important consequences of such a classification for the theory of transport phenomena have been described. Here, however, we will not consider the case of three quasi-periods and give just some references to reviews on this topic (see e.g. [4, 5]).

In this paper, the central role will be played by the Novikov problem with four quasi-periods. The case of four quasi-periods has not been studied in as much detail as the case of three quasi-periods, however, for this case there are also deep analytical results ([6, 7]). As can be shown, the potentials formed by the superposition of two periodic potentials in the plane are potentials with four quasi-periods, and the results obtained in the works [4, 5] can thus be applied to them. It should be noted, however, that the results of the papers [4, 5] are formulated in terms of the original parameter space of the problem (i.e., parameters describing the embedding of a two-dimensional plane in the space $\mathbb{R}^4$) that differs from the space of parameters we are considering here. Therefore, we will have to specifically consider here the connection between the initial parameter space and the parameters of the problem we are studying. As we will see, this connection strongly depends in fact on the specifics of the problem under consideration, which directly affects the description of the geometry of the level lines of the corresponding potentials.

Here we will be interested mainly in open (non-closed) level lines of potentials $f(x, y)$. The main thing that we would like to present here is the division of open level lines into topologically regular and chaotic ones. Lines of both the first and second types correspond to the general situation and appear in complementary sets in the parameter space of the problem. The lines of these two types are distinguished by their global geometry, which is relatively simple for lines of the first type and rather complex for lines of the second type. In addition to the simplicity of the global geometry, topologically regular
level lines are stable with respect to small variations of the problem parameters, while chaotic lines are unstable. As a consequence, the sets in the parameter space that correspond to the appearance of level lines of different types have a rather nontrivial structure. In the general case, the set corresponding to topologically regular level lines consists of an infinite number of Stability Zones corresponding to the appearance of open level lines with different values of certain topological invariants. The set corresponding to the appearance of chaotic trajectories is the addition to the set of Stability Zones and has a non-trivial fractal structure (except for special cases).

II. TOPOLOGICALLY REGULAR AND CHAOTIC LEVEL LINES OF QUASI-PERIODIC POTENTIALS

First of all, we describe here the difference between topologically regular and chaotic open level lines of a quasiperiodic function \( f(x, y) \). The main feature of topologically regular open level lines is that each such line lies in a straight strip of finite width and passes through it (Fig. 1). Chaotic level lines, on the contrary, cannot be enclosed in straight strips of finite width and wander around the plane in some pseudo-random way (Fig. 2). Let us also note here that topologically regular level lines are not periodic in the case of general position.

The second distinguishing feature of topologically regular level lines is their stability with respect to small variations in the parameters of the problem, while chaotic level lines are unstable with respect to arbitrarily small variations in the parameters.

The third important feature of topologically regular level lines is their connection with topological numbers that define their mean direction. The topological numbers for the Novikov problem with four quasiperiods have the form of (irreducible) integer quadruples \((m^1, m^2, m^3, m^4)\) and, as applied to the situation under consideration, are described as follows.

Let there be a quasi-periodic pattern on the plane, which arises as a result of the superposition (possibly with mutual influence) of two periodic potentials \( V(x, y) \) and \( U(x, y) \) with periods \((v_1, v_2)\) and \((u_1, u_2)\) respectively. Consider in the plane the basis \((v_1', v_2')\) reciprocal to the basis \((v_1, v_2)\) and the basis \((u_1', u_2')\) reciprocal to the basis \((u_1, u_2)\)

\[ v'_1 \cdot v_j = \delta_{ij}, \quad u'_1 \cdot u_j = \delta_{ij} \]

Then the mean direction \( l \) of topologically regular level lines of the resulting potential is determined by the relation

\[ (m^1v'_1 + m^2v'_2 + m^3u'_1 + m^4u'_2) \cdot l = 0 \] (II.1)

with some irreducible integer quadruple \((m^1, m^2, m^3, m^4)\).

The numbers \((m^1, m^2, m^3, m^4)\) are locally stable and do not change with small variations of parameters of the problem. Thus, the set of existence of topologically regular open potential level lines in the parameter space is actually divided into Stability Zones, each of which has its own values of \((m^1, m^2, m^3, m^4)\). In general case the quadruples \((m^1, m^2, m^3, m^4)\) form a certain subset in \( \mathbb{Z}^4 \) and the number of Stability Zones in the parameter space may be either finite or infinite. We also note here that with increasing values of \((m^1, m^2, m^3, m^4)\) the corresponding level lines become more and more complex. In particular, the width of the strips containing such level lines increases, and the level lines themselves acquire a chaotic shape on small scales (Fig. 3).

Among other parameters, topologically regular level lines are also stable under small variations of the energy value \( f(x, y) = \epsilon \). Thus, topologically regular level lines in general position exist in some finite connected energy interval. The exception is the level lines that appear at the boundaries of the Stability Zones, where this interval collapses into a single energy level. In the case of general position we have also a finite energy interval, for which the geometry of the regions \( f(x, y) \leq \epsilon \) is similar to the geometry of topologically regular open level lines. The topologically regular open level lines are also stable in this case with respect to sufficiently small perturbations of the potential \( f(x, y) \) of an arbitrary form.
It is easy to see that the specific properties of topologically regular trajectories may manifest themselves, in particular, in transport phenomena in the corresponding two-dimensional systems.

As for the behavior of chaotic level lines, it is actually very diverse. Let us say here that the behavior of chaotic level lines of quasi-periodic potentials has been fairly well studied by now for the Novikov problem with three quasi-periods (see, for example, [21]). In this case, the chaotic level lines should actually be divided into certain types that have their own specific features.

In the case of four quasi-periods, the behavior of chaotic open level lines is more complex, and is still very far from its complete description. Here we will classify as chaotic open level lines all non-periodic open level lines that do not satisfy the properties listed above for topologically regular level lines. It is easy to see that no quasi-periodic potential can have simultaneously topologically regular and chaotic open level lines. Thus, each of the quasi-periodic potentials can be attributed to one of two types in accordance with the geometry of its open level lines.

In this paper, we describe the picture that arises in the problem we are considering only from the most general point of view. A more precise description depends on the specific situation and requires additional clarification of the conditions of the problem.

III. THE EMERGENCE OF TOPOLOGICALLY REGULAR AND CHAOTIC LEVEL LINES IN THE GENERAL SITUATION AND SPECIAL CASES

It is easy to see that in the Novikov problem with four quasi-periods, a part of the parameters is given by the parameters of the affine embedding of the two-dimensional plane into the four-dimensional space. These parameters, in turn, include the parameters of the linear map $\mathbb{R}^2 \rightarrow \mathbb{R}^4$ as well as the shift of the origin for the full affine embedding. Here, in fact, we will be mainly interested in the division of open potential level lines into regular and chaotic ones, the type of which does not change under affine transformations of the plane $\mathbb{R}^2$ itself. For us, therefore, the essential parameters will be represented by a two-dimensional direction in four-dimensional space, i.e. a point of the Grassmann manifold $G_{4,2}$, and the values of shifts in one-dimensional directions orthogonal to it. It is easy to see that for a fixed two-dimensional direction we have a two-parameter family of two-dimensional planes representing essentially different affine embeddings $\mathbb{R}^2 \rightarrow \mathbb{R}^4$.

In addition, the Novikov problem naturally contains the space of 4-periodic smooth functions, i.e., space of smooth functions on the four-dimensional torus $\mathbb{T}^4$. Let us formulate here the main results obtained so far for the Novikov problem with four quasi-periods ([7]).

There is an open everywhere dense subset $S \subset C^\infty(\mathbb{T}^4)$ of 4-periodic functions $F$ and an open everywhere dense subset $X_F \subset G_{4,2}$, depending on $F$, such that for any $\Pi \in X_F$ any level $M^4 = \{F = c\}$ contains only stable topologically regular level lines of functions $f$ (or does not contain open level lines of $f$).

In addition, for any regular level line, the width of the strip bounding it, as well as the diameter of any compact level line of the functions $f$, are bounded from above by one constant $C$, which does not depend on the choice of the affine plane (of the direction $\Pi$) containing the level line, provided that the level $c$ and direction $\Pi \in X_F$ are fixed.

The above statements are formulated in our original setting, where the function $f(x,y)$ is obtained by restricting of a 4-periodic function $F(z^1, z^2, z^3, z^4)$ under the affine embedding $\mathbb{R}^2 \rightarrow \mathbb{R}^4$. It is easy to see that the level lines of the functions $f(x, y)$ are given then by the intersections of the corresponding two-dimensional planes with the periodic three-dimensional manifolds $F(z^1, z^2, z^3, z^4) = \text{const}$.

As we have already said, the mean direction of topologically regular level lines of the function $f(x, y)$ is determined by some quadruple of integers $(m^1, m^2, m^3, m^4)$. In the general case, assuming that the function $F(z^1, z^2, z^3, z^4)$ is periodic with respect to the standard integer lattice, the mean direction of the topologically regular level lines of $f(x, y)$ is given by the intersection of the corresponding two-dimensional plane with an integral plane in $\mathbb{R}^4$ given by the equation

$$m^1z^1 + m^2z^2 + m^3z^3 + m^4z^4 = 0 \quad (\text{III.1})$$

The set corresponding to the appearance of topologically regular open level lines, therefore, consists of Stability Zones with different values of $(m^1, m^2, m^3, m^4)$, the union of which is everywhere dense in the described parameter space. In the general case, the number of such Zones is infinite, and the corresponding numbers $(m^1, m^2, m^3, m^4)$ can be arbitrarily large.
Here we would like to pay special attention to the fact that in the case of the presence of topologically regular open level lines, the above statements apply immediately to the entire family of parallel two-dimensional planes embedded in four-dimensional space. Thus, in the case of the appearance of such level lines, we immediately obtain their description for a whole family of periodic potentials corresponding to all embeddings of the same direction.

It should also be added here that, as follows from the results of \[1 - 2\], whenever the potentials we consider are not periodic, the mean directions of their topologically regular open level lines are the same in all indicated planes. At the same time, if such potentials are periodic, the mean direction of their open level lines can be different in different planes of the family (but not necessarily).

In addition, the open level lines of every function \(f(x, y)\) appear in some connected energy interval \([\varepsilon_1, \varepsilon_2]\) (which can shrink to a single point \(\varepsilon_0\)). When topologically regular open level lines appear at a non-periodic potential, the corresponding interval also coincides for all planes of the family we have indicated. At the same time, if such potentials are periodic, these intervals may differ in different planes of this family.

It is easy to see that all open level lines of periodic potentials are periodic, and these potentials themselves are not generic potentials. In our situation here, such potentials will appear rather as exceptional potentials.

It also follows from the above statements that chaotic level lines cannot be stable in the described parameter space, since each such situation is a point of accumulation of Stability Zones corresponding to the presence of topologically regular open level lines. As we noted above, the corresponding topologically regular level lines in this situation acquire chaotic properties on finite scales.

The above statements describe the structure of the set of appearance of regular level lines in the most general parameter space corresponding to the Novikov problem with four quasi-periods. When considering a specific problem, we must additionally consider the embedding of the parameter space of this problem into the complete parameter space described above.

In our case, we are dealing with the superimposition of two periodic pictures on top of each other. It is natural to assume that the position of one of the pictures is fixed. The parameters of the problem are then two independent shifts and a rotation of the second picture relative to the first.

Denoting the corresponding Euclidean transformation of the plane by \(A(x, y)\)

\[
A(x, y) = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \end{pmatrix},
\]

we can define the embedding \(\mathbb{R}^2 \rightarrow \mathbb{R}^4\) according to the formula

\[
(x, y) \rightarrow (x, y, A(x, y)) \quad (\text{III.2})
\]

Thus, the transformation \(A(x, y)\) will play a double role for us. On the one hand, it determines the shift of one of the periodic potentials relative to the other in the plane. On the other hand, we will also use it to define an embedding \(\mathbb{R}^2 \rightarrow \mathbb{R}^4\) in order to define the resulting potential as the restriction of some 4-periodic function \(F(z^1, z^2, z^3, z^4)\) to the embedded plane. As we have already said, we assume here that one periodic potential \(V(x, y)\) is stationary in the plane, while the second potential \(U(x, y)\) undergoes some rotation and shift before being superimposed.

For all the transformations \(A(x, y)\) the projection of the embedding \(\text{III.2}\) on the plane \((z^1, z^2)\) is the identical transformation, while the parameters of the transformation \(A(x, y)\) determine the shifts and rotation of the projection on the plane \((z^3, z^4)\). As it is easy to see, the rotation in the transformation \(A(x, y)\) corresponds to the rotation of this projection while the shifts in \(A(x, y)\) correspond to parallel shifts of the nested plane in the four-dimensional space. It is also easy to see that for any fixed value of the rotation angle in the transformation \(A(x, y)\) all such shifts fill the entire four-dimensional space.

It is easy to see that if the resulting potential \(f(x, y)\) in the plane is given by the simple sum of periodic potentials \(f_1(x, y) = V(x, y)\) and \(f_2(x, y) = U(A(x, y))\), the corresponding periodic function \(F(z^1, z^2, z^3, z^4)\) in four-dimensional space can be defined as

\[
F(z^1, z^2, z^3, z^4) = V(z^1, z^2) + U(z^3, z^4)
\]

As we can see, the function \(F(z^1, z^2, z^3, z^4)\) is fixed in this case and the problem parameters affect only the parameters of the embedding \(\mathbb{R}^2 \rightarrow \mathbb{R}^4\). The same situation actually takes place also if the resulting potential \(f(x, y)\) is a local function of the values \(f_1(x, y)\) and \(f_2(x, y)\) taken at the same point \((x, y)\): \(f = Q(f_1, f_2)\). In this case, the function \(F(z^1, z^2, z^3, z^4)\) is given by the expression

\[
F(z^1, z^2, z^3, z^4) = Q(V(z^1, z^2), U(z^3, z^4))
\]

In both cases, the function \(F(z^1, z^2, z^3, z^4)\) is obviously periodic in \(\mathbb{R}^4\), with periods

\[
(\nu_1, 0, 0), \quad (\nu_2, 0, 0), \quad (0, 0, \nu_1), \quad (0, 0, \nu_2), \quad \text{(III.3)}
\]

where \((\nu_1, \nu_2), (\nu_1, \nu_2)\) represent the periods of the potentials \(V(x, y)\) and \(U(x, y)\) respectively.

The most general case is when the resulting potential \(f(x, y)\) is a non-local functional of potentials \(f_1(x, y)\) and \(f_2(x, y)\)

\[
f(x, y) = Q[f_1, f_2](x, y) \quad \text{(III.4)}
\]

(for example, due to local deformation of atomic lattices as a result of their interaction). For translationally invariant functionals \(f[f_1, f_2]\) the corresponding function \(f(x, y)\) will also be a quasi-periodic function with four quasi-periods. The corresponding function \(F(z^1, z^2, z^3, z^4)\) for a given transformation \(A\) is defined as follows.
Let us fix the angle \( \alpha \) in the transformation \( A \) and consider all the transformations \( A' \) with the same \( \alpha \) and all possible values of \( a_1 \) and \( a_2 \). Each transformation \( A' \) defines a certain configuration of superposition of the potentials \( V(x, y) \) and \( U(x, y) \) in the plane, and thus determines the corresponding values of the functional \( f(x, y, a_1, a_2) \) in it. On the other hand, each transformation \( A' \) defines an embedding of the plane in \( \mathbb{R}^4 \) and, as we have already said, the complete family of corresponding parallel planes fills the entire four-dimensional space. The latter circumstance allows us to uniquely define the coordinates \( z^i \) as (linear) functions of \( x, y, a_1, \) and \( a_2 \)

\[
z^i = z^i(x, y, a_1, a_2)
\]

and vice versa

\[
x = x(z), \ y = y(z), \ a_1 = a_1(z), \ a_2 = a_2(z)
\]

The function \( F(z^1, z^2, z^3, z^4) \) is then naturally defined as

\[
F(z) = f(x(z), y(z), a_1(z), a_2(z)) \quad (III.5)
\]

It is not difficult to verify that the function \( F(z) \) constructed in this way has the same periods \((III.3)\) as in the previous cases. Indeed, returning to the formula \((III.2)\), we see that the shifts

\[
(z^1, z^2) \to (z^1, z^2) + v_i, \quad (z^3, z^4) \to (z^3, z^4)
\]

correspond to the transformation of the parameters \((x, y, a_1, a_2)\) such that

\[
(x, y) \to (x, y) + v_i, \quad A'(x, y) \to A'(x, y)
\]

In the same way, the shifts

\[
(z^1, z^2) \to (z^1, z^2), \quad (z^3, z^4) \to (z^3, z^4) + u_i
\]

correspond to

\[
(x, y) \to (x, y), \quad A'(x, y) \to A'(x, y) + u_i
\]

It is easy to see, therefore, that in both cases the initial potentials \( f_1(x, y) = V(x, y) \) and \( f_2(x, y) = U(A'(x, y)) \) in the plane remain unchanged, which also entails the invariance of the functional \((III.3)\).

Unlike the previous cases, however, the form of the function \( F(z) \) depends here on the value of the angle \( \alpha \) in the transformation \( A(x, y) \). Thus, we see that in the most general case, the parameters of our problem are mapped to the most complete space of parameters of the Novikov problem, including the space of periodic functions \( F(z) \) in \( \mathbb{R}^4 \) (with fixed periods).

Let us say here that we use the term potentials rather conditionally, and the functions \( V(x, y), U(x, y) \) and \( f(x, y) \) may actually be any functionals that describe the properties of the system under consideration. Also, the dependence \((III.4)\) may be a very complex functional dependence, depending on the type of functionals \( V(x, y), U(x, y) \) and \( f(x, y) \). In any case, however, all the assertions formulated above will hold. Let us note also, that if we introduce the new coordinate system \((z^1, z^2, z^3, z^4)\), where the periods of the function \( F(z) \) play the role of the basis vectors, we will directly obtain the relation \((III.1)\) from the relation \((III.3)\).

To describe the generic situation in the problem considered here, we can, for example, impose the condition that the potentials \( V(x, y) \) and \( U(x, y) \) do not have nontrivial rotational symmetry, and the lengths of the periods \( \nu_1, \nu_2, \nu_3, \nu_4 \) are linearly independent over the field of rational numbers (Fig. 3). In this formulation, in particular, periodic potentials do not arise at all in any superposition of the layers and the only parameter that specifies the type of open level lines is the rotation angle \( \alpha \).

For generic mappings of the parameters of our problem to the above-described space of parameters of the general Novikov problem, we must observe in general the intersection of the image of a mapping both with the Stability Zones and with their complement in this space. In the generic situation, we must therefore observe the separation of the cases of topologically regular and chaotic open level lines of emerging potentials depending on the angle \( \alpha \). The set of angles \( \alpha \) corresponding to the topologically regular case represents then a (finite or infinite) set of intervals on the unit circle, each of which corresponds to its own values of \((m_1, m_2, m_3, m_4)\). The set of angles corresponding to the chaotic case is the addition to the union of such intervals and may have a fractal structure in the most general situation. On the whole, such a situation should arise for many systems satisfying the above conditions of the general position.

In fact, a similar situation should apparently also arise under weaker assumptions about the geometry of the potentials \( V(x, y) \) and \( U(x, y) \). Most likely, to observe such a situation, it is sufficient to require only the absence of rotational symmetries of the same order for the superimposed potentials. Pairs of corresponding materials also appear in many interesting systems. We would like to give here just one example of such a pair, which has been of great interest in recent times (see [22]). It should also be added here that in special cases when the periods of \( V(x, y) \) and \( U(x, y) \) are commensurate, in addition to the sets described above, special angles \( \alpha \) may also arise for which the resulting potential is purely periodic.

It is easy to see, however, that many important and interesting physical two-layer systems are not systems in general position in the sense described above. Many of them, in particular, have a nontrivial rotational symmetry common to both potentials \( V(x, y) \) and \( U(x, y) \). Probably the most important example of such a system is two-layer graphene, with layers arranged at arbitrary angles to each other. Both layers in this case are identical to each other and have sixth-order rotational symmetry. For almost all rotation angles \( \alpha \) the resulting potential is quasi-periodic, and only for some (magic) angles special
periodic potentials (see for example [23, 24]). It is easy to see that in the case of quasi-periodic (non-periodic) potentials we cannot observe here topologically regular open level lines. Indeed, topologically regular level lines must exist for potentials in the entire family of planes of a given direction in $\mathbb{R}^4$ (i.e., for all values of parameters $a_1$ and $a_2$). On the other hand, by choosing the parameters $a_1$ and $a_2$ in the right way, we can achieve the occurrence of a quasi-periodic potential which has rotational symmetry. It is easy to see then that for such a potential the appearance of topologically regular level lines is impossible.

It can be seen, therefore, that for almost all angles $\alpha$ the open level lines in two-layer graphene belong to the class of chaotic ones. The only exceptions are magic angles corresponding to the appearance of periodic potentials in the plane. In this case, the overall picture in the plane also depends on the shift parameters $a_1$ and $a_2$. Depending on their values, we can observe here either (unstable) open periodic level lines of different directions for different $a_1$ and $a_2$, or periodic nets of singular trajectories that appear only at a single energy level (for example, for periodic potentials with rotational symmetry).

Another important example of a nongeneric situation is when the superimposed potentials have rotational symmetry of the same order, however, they are not identical to each other. An example of this may be given by the system formed by graphene aligned with hexagonal boron nitride, where both layers have hexagonal symmetry, but the periods of the corresponding potentials are not exactly equal to each other (let us quote here just a few of the many works devoted to this important system [25–30]). The chaotic level lines arising in such a system can indeed have a very complex shape. We would like to especially refer here to the work [30] where the shape of such trajectories was studied in the percolation theory approximation. We also note here that the paper [30] considers the situation when the superimposed po-
tentials have a very strong interaction with each other, and the investigated functional is a complex functional of the resulting potential, being a parameter of the electronic spectrum in this system (averaged electron mass). The problem studied in the work is directly related to the description of the transport electronic properties in the corresponding two-dimensional system.

IV. CONCLUSIONS

We consider the geometry of open level lines of special potentials obtained as a result of a superposition of periodic potentials on the plane. The consideration is based on the connection of such potentials with the theory of quasi-periodic functions and, in particular, with the Novikov problem for potentials with four quasi-periods on the plane. As follows from the general results, each of these potentials has either topologically regular or chaotic open level lines that replace each other in a rather nontrivial way as the parameters of the potential change. At the same time, the paper considers both families of potentials that correspond to the most general situation, and special families that are important from the physical point of view.

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