$dE/dx$ to Subleading Order in the Plasma Density

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Abstract. Dimensional continuation is employed to compute the energy loss rate for a non-relativistic particle moving through a highly ionized plasma. No restriction is made on the charge, mass, or speed of this particle, but it is assumed that the plasma is not strongly coupled in that the dimensionless plasma coupling parameter $g = e^2 \kappa_D / 4\pi T$ is small, where $\kappa_D$ is the Debye wave number. To leading order in this coupling, $dE/dx$ is of the generic form $g^2 \ln[g^2C]$. The prefactor of the logarithm is well known. We compute the constant $C$ under the logarithm exactly. Our result differs from approximations given in the literature, with differences in the range of about 20% for cases relevant to inertial confinement fusion experiments.

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1. $dE/dx$ and the Coulomb Log

The stopping power of plasma component $b$ for projectile $p$ is of the form

$$\frac{dE_b}{dx} = \frac{e_p^2 e_b^2}{4\pi} \frac{m_b}{m_b v_p^2} \ln \Lambda_b = \frac{e_p^2}{4\pi} \frac{\kappa_b^2}{\beta_b m_b v_p^2} \ln \Lambda_b,$$

where the Coulomb logarithm $\ln \Lambda_b$ involves a ratio of short- and long-distance length scales. To compute $\ln \Lambda_b$, we employ the method of dimensional continuation [1]. To introduce this method, we consider the Coulomb potential $\phi_\nu(r)$ of a point source in $\nu$ spatial dimensions: $\phi_\nu(r) \sim 1/r^{\nu-2}$. Clearly the long and short distance behavior of $\phi_\nu$ depends on the spatial dimensionality $\nu$. In high $\nu$, short distance (hard) interactions are accentuated, while in low $\nu$ the large distance (soft) physics predominates.

For Coulomb interactions, $\nu = 3$ is special in that neither hard nor soft processes are dominant. For $\nu < 3$, the soft physics is predominant, and for $\nu > 3$ the hard processes are dominant. The energy loss for $\nu > 3$, $dE^B/dx$, is obtained from the Boltzmann (B) equation, and it contains a pole $(\nu - 3)^{-1}$ that reflects an infrared divergence in the scattering process when $\nu \to 3^+$. The energy loss for $\nu < 3$, $dE^{LB}/dx$, is obtained from the Lenard-Balescu (LB) kinetic equation, and it contains a pole $(3 - \nu)^{-1}$ that reflects an ultraviolet divergence when $\nu \to 3^-$. The stopping power to subleading order is therefore

$$\frac{dE}{dx} = \lim_{\nu \to 3} \left( \frac{dE^{LB}}{dx} + \frac{dE^B}{dx} \right),$$

(2)
and it is completely finite. Hence the two poles must cancel. The dependence of the residues of the poles on \( \nu \) brings in a logarithm of the ratio of the relevant short- and long-distance length scales, which is precisely the Coulomb logarithm.

2. Collective Excitations: Lenard-Balescu Equation for \( \nu < 3 \)

The soft physics is described to leading order in the plasma density by

\[
\frac{\partial}{\partial t} f_a(p_a) = -\sum_b \frac{\partial}{\partial p_a} \cdot J_{ab},
\]

which is the Lenard–Balescu kinetic equation for plasma species \( a \) and \( b \), where

\[
J_{ab} = e_a^2 e_b^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{\epsilon(k, v_a, k)} \int \frac{d^3p_b}{(2\pi \hbar)^3} \delta(k \cdot (v_a - v_b)) \frac{k}{\epsilon(k, v_a, k)} \frac{\partial}{\partial p_b} f_a(p_a) f_b(p_b),
\]

with \( v_a = p_a/m_a \) and \( v_b = p_b/m_b \). The collective behavior of the plasma enters through its dielectric function

\[
\epsilon(k, \omega) = 1 + \sum_e \frac{e^2}{k^2} \int \frac{d^3p_e}{(2\pi \hbar)^3} \frac{1}{\omega - k \cdot v_e + i\eta} \frac{k}{\epsilon(k, v_e, k)} \frac{\partial}{\partial p_e} f_e(p_e), \quad \eta \to 0^+.
\]

The rate of kinetic energy loss of species \( a \) to species \( b \) is given by

\[
\frac{dE_{ab}^{\text{kin}}}{dt} = \int \frac{d^3p_a}{(2\pi \hbar)^3} \frac{p_a^2}{2m_a} \frac{\partial}{\partial p_a} \cdot J_{ab}.
\]

We evaluate this for the case in which species \( a \) is a single projectile of mass \( m_p \) and velocity \( v_p \), \( f_a(p_a) = (2\pi \hbar)^3 \delta^{(3)}(p_a - m_p v_p) \), and the distribution function \( f_b(p_b) \) for plasma species \( b \) is Maxwell-Boltzmann at temperature \( T_b = 1/\beta_b \). With \( dx = v_p dt \),

\[
\frac{dE_{ab}^{\text{kin}}}{dx} = \frac{e^2}{4\pi} \frac{1}{\beta_b m_p v_p^2} \frac{\Omega_{\nu - 2}}{2\pi} \left( \frac{K}{2\pi} \right)^{\nu - 3} \frac{1}{3 - \nu} \int_0^1 du (1 - u)^{(\nu - 3)/2} \rho_b(v_p u^{1/2}) \left[ \beta_b M_{pb} v_p^2 - \frac{1}{u} \right] \\
+ \frac{e^2}{4\pi} \frac{i}{2\pi} \int_{-1}^{+1} d\cos \theta \cos \theta \rho_b(v_p \cos \theta) \sum_c \rho_c(v_p \cos \theta) \left[ F(v_p \cos \theta) \ln \left( \frac{F(v_p \cos \theta)}{K^2} \right) \\
- \frac{e^2}{4\pi} \frac{1}{2\pi} \beta_b m_p v_p^2 \sum_c \rho_c(v_p) \left[ F(v_p) \ln \left( \frac{F(v_p)}{K^2} \right) - F^*(v_p) \ln \left( \frac{F^*(v_p)}{K^2} \right) \right] \right],
\]

where \( M_{pb} = m_p + m_b \) is the total mass, \( F(v) = k^2 [\epsilon(k, kv) - 1] \), and \( \rho_b(v) = \kappa^2 \sqrt{\beta_b m_b / 2\pi \nu} \exp(-\beta_b m_b v^2 / 2) \). Here \( \Omega_{\nu} \) is the area of a unit sphere in \( \nu \) dimensions and \( K \) is an arbitrary wave number whose dependence cancels in the limit \( 2 \).

3. Hard Collisions: Boltzmann Equation for \( \nu > 3 \)

Hard collisions in the plasma density are correctly described by the Boltzmann equation, which gives

\[
\frac{dE_b}{dx} = \frac{1}{v_p} \int \frac{d^3p_b}{(2\pi \hbar)^3} f_b(p_b) v_p \int d\sigma_{pb} \frac{1}{2} m_p \left[ v_p^2 - v_b^2 \right],
\]

where \( v_{pb} = |v_p - v_b| \), and \( d\sigma_{pb} \) is the full quantum-mechanical differential cross section for scattering of the projectile \( p \) from the initial velocity \( v_p = p_p/m_p \) to the final velocity.
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\( v'_p \) off a plasma particle \((b)\). Straightforward kinematical manipulations exploiting the axial symmetry of the scattering produce the form

\[
\frac{dE_b^{\nu}}{dx} = \frac{1}{v_p} \int \frac{d^\nu \mathbf{p}_b}{(2\pi \hbar)^\nu} f_b(\mathbf{p}_b) \frac{\mathbf{P} \cdot \mathbf{p}}{2p^2 M_{pb} v_{pb}} \int d\sigma_{pb} q^2,
\]

in which \( \mathbf{P} \) is the total momentum of the center of mass, \( \mathbf{p} \) is the relative momentum in the center of mass, and \( q \) is the momentum transfer.

The classical cross section in \( \nu \) dimensions is \( d\sigma_{pb}^C = \Omega_{\nu-2} B^{\nu-2} dB \), where \( B \) is the classical impact parameter. Some calculation gives

\[
\int d\sigma_{pb}^C q^2 = \frac{\Omega_{\nu-2}}{2\pi} \frac{\eta_b m_p v_p^2}{p^2} \left[ \frac{p^{2(\nu-3)}}{\nu-3} - \ln \left( \frac{\eta_b m_p}{4} \right) - \gamma \right],
\]

with \( m_{pb} = m_p m_b/M_{pb} \) being the reduced mass. Placing the result (10) in Eq. (9) yields

\[
\frac{dE_b^{\nu}}{dx} = \frac{\eta_b^2 \kappa_b^2}{4\pi} \frac{2v_p}{\sqrt{2\pi \alpha_b}} e^{-\alpha_b u/2} \int_0^\infty du \, \frac{e^{-\alpha_b u/2}}{\sqrt{2\pi \alpha_b}} \left[ \ln(\eta_b/u) - \Re \psi(1 + i\eta_b/u) \right] \left[ \frac{M_{pb}}{m_p u} \left( \cosh \alpha_b u - \frac{\sinh \alpha_b u}{\alpha_b u} \right) - \frac{m_b}{m_p} \sinh \alpha_b u \right],
\]

where \( \psi \) is the logarithmic derivative of the gamma function, \( \Re \) denotes the real part, \( \alpha_b \equiv \beta_b m_b v_p^2 \), and \( \eta_b \equiv \epsilon_{p} v_{b} / 4\pi \hbar v_{p} \).

**4. Results**

The total stopping power is the sum of the contributions from large-distance collective excitations \( dE^{L}/dx \) and from short-distance hard collisions \( dE^{H}/dx \), that is, the sum over species \( b \) of Eqs. (7), (11), and (12). The poles at \( \nu = 3 \) and the \( \ln K \) terms cancel. Our result for \( dE/dx \) is generically of the form \( n \ln n + C \) in the plasma density \( n \), and it is accurate to all orders in the quantum parameter \( \eta_b \). Figures 1 and 2 illustrate our result with an example that is relevant to the DT plasmas in laser fusion capsules.

**References**

[1] Brown L S, Preston D L and Singleton Jr. R L 2005 *Phys. Rep.* 410 237, arXiv:physics/0501084

[2] Li C-K and Petrasso R D 1993 *Phys. Rev. Lett* 70 3059
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Figure 1. The energy $E(x)$ (in MeV) of an $\alpha$ particle with initial energy $E_0 = 3.54$ MeV vs. the distance $x$ (in $\mu$m) traveled through an equal molal $DT$ plasma. Note that $E(x)$ is obtained by inverting $x = \int_{E_0}^{E} dE/(dE/dx)^{-1}$, where the stopping power $dE/dx$ has been expressed as a function of energy. The plasma temperature is $T = 3$ keV and the electron number density is $n_e = 10^{25}$ cm$^{-3}$. The plasma coupling is small, $q = 0.011$, and so our calculation (BPS) is essentially exact. Our result is shown by the solid curve. The work of Li and Petrasso [2] is often used in laser fusion simulations. Their result (LP) is shown by the dashed line. Note that the difference in the total ranges between our result and that of Li and Petrasso of about 5 $\mu$m is a little larger than 20%.

Figure 2. The $\alpha$ particle $dE(x)/dx$ (in MeV/$\mu$m) vs. $x$ (in $\mu$m) split into separate ion (spiked curves) and electron components (softly decreasing curves). The area under each curve gives the corresponding energy partition into electrons and ions. For our results (BPS), the total energy deposited into electrons is $E_e = 3.16$ MeV and into ions is $E_I = 0.38$ MeV, while LP give $E^{LP}_e = 3.11$ MeV and $E^{LP}_I = 0.43$ MeV. These energies sum to the initial $\alpha$ particle energy of $E_0 = 3.54$ MeV. Note that BPS has a longer $\alpha$ particle range and deposits less energy into ions than LP. Both observations would tend to make fusion more difficult to achieve for BPS than for LP.