DECAY/GROWTH RATES FOR INHOMOGENEOUS HEAT EQUATIONS WITH MEMORY. THE CASE OF SMALL DIMENSIONS

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Abstract. We study the decay/growth rates in all $L^p$ norms of solutions to an inhomogeneous nonlocal heat equation in $\mathbb{R}^N$ involving a Caputo $\alpha$-time derivative and a power $\beta$ of the Laplacian when the spatial dimension is small, $1 \leq N \leq 4\beta$, thus completing the already available results for large spatial dimensions. Rates depend not only on $p$, but also on the space-time scale and on the time behavior of the spatial $L^1$ norm of the forcing term.

1. Introduction

1.1. Goal. This paper is devoted to study the large-time behavior of the $L^p$-norms, $p \in [1, \infty]$, of the solution to the fully nonlocal problem

\begin{equation}
\frac{\partial_t^\alpha u}{\Gamma(1-\alpha)} + (-\Delta)^\beta u = f \quad \text{in } Q := \mathbb{R}^N \times (0, \infty), \quad u(\cdot, 0) = 0 \quad \text{in } \mathbb{R}^N,
\end{equation}

in the case of small dimensions, $1 \leq N \leq 4\beta$. It may be regarded as a companion to [6], where the case of large dimensions, $N > 4\beta$, was treated. Here, $\partial_t^\alpha$, $\alpha \in (0,1)$, denotes the so-called Caputo $\alpha$-derivative, introduced independently by many authors using different points of view, see for instance [1, 10, 12, 13, 15, 18], which is defined for smooth functions by

\begin{equation}
\partial_t^\alpha u(x, t) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial_t}{\partial_t} \int_0^t \left( u(x, \tau) - u(x, 0) \right) (t-\tau)^{\alpha-1} d\tau,
\end{equation}

and $(-\Delta)^\beta$, $\beta \in (0,1]$, is the usual $\beta$ power of the Laplacian, defined for smooth functions by $(-\Delta)^s = \mathcal{F}^{-1}(|\cdot|^{2s}\mathcal{F})$, where $\mathcal{F}$ denotes Fourier transform; see for instance [19]. Equations of this kind, nonlocal both in space and time, have been proposed recently to model situations with long-range interactions and memory effects; see for example [2, 3, 8, 9, 16, 20].

Problem (1.1) does not have in general a classical solution, unless the forcing term $f$ is required to satisfy certain smoothness assumptions. However, if $f \in L^\infty_{\text{loc}}([0, \infty) : L^1(\mathbb{R}^N))$, it has a solution in a generalized sense, defined by Duhamel’s type formula

\begin{equation}
u(x, t) = \int_0^t \int_{\mathbb{R}^N} Y(x - y, t - s) f(y, s) \, dy \, ds,
\end{equation}

with $Y = \partial_t^{1-\alpha} Z$, where $Z$ is the fundamental solution of the Cauchy problem; see [11,14]. Throughout the paper we assume the integral size condition

\begin{equation}
\|f(\cdot, t)\|_{L^1(\mathbb{R}^N)} \leq \frac{C}{(1 + t)^\gamma} \quad \text{for some } \gamma \in \mathbb{R},
\end{equation}

and deal with solutions of this kind, denoted in the literature as mild solutions [14,17].

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Notation. As is common in asymptotic analysis, \( g \approx h \) will mean that there are constants \( \nu, \mu > 0 \) such that \( \nu h \leq g \leq \mu h \).

1.2. The kernel \( Y \). Critical exponent. Our study is based in a careful analysis of the singular integral (1.2) defining the mild solution. Hence, having good estimates for the kernel \( Y \) is essential. Such estimates are fortunately available in [14], and are recalled next.

The kernel \( Y \) has a self-similar form,
\[
Y(x, t) = t^{-\sigma_*}G(\xi), \quad \xi = xt^{-\theta}, \quad \sigma_* := 1 - \alpha + N\theta, \quad \theta := \frac{\alpha}{2\beta}.
\]
Its profile \( G \) is positive, radially symmetric, and smooth outside the origin, and if \( N \leq 4\beta \) satisfies close to the origin the sharp estimates
\[
G(\xi) \asymp |\xi|^{4\beta - N}, \quad \beta \in (0, 1], \quad |\xi| \leq 1.
\]
It is here that we find the main difference with respect to the case of high dimensions, \( N > 4\beta \), for which
\[
G(\xi) \leq C|\xi|^{4\beta - N}, \quad \beta \in (0, 1], \quad |\xi| \leq 1.
\]
Away from the origin we have the same (sharp) behavior as in high dimensions,
\[
G(\xi) \asymp |\xi|^{(N-2\beta)(\alpha-1)/(2-\alpha)} e^{-c\sigma|\xi|^{-\gamma}}, \quad \beta = 1,
\]
In particular, since \( |\xi|^{(N-2\beta)(\alpha-1)/(2-\alpha)} \leq C_\nu|\xi|^{-(N+2\beta)} \) if \( |\xi| \geq \nu \), we have the exterior bound
\[
0 \leq Y(x, t) \leq C_\nu t^{2\alpha-1}|x|^{-(N+2\beta)} \quad \text{if } |x| \geq \nu t^\theta, \ t > 0, \ \beta \in (0, 1].
\]
Notice that \( Y(\cdot, t) \in L^p(\mathbb{R}^N) \) for all \( p \in [1, \infty] \) if \( N < 4\beta \) and for all \( p \in [1, \infty) \) if \( N = 4\beta \). Moreover, in all these cases we have
\[
\|Y(\cdot, t)\|_{L^p(\mathbb{R}^N)} = Ct^{-\sigma(p)} \quad \text{for all } t > 0, \quad \text{where } \sigma(p) := \sigma_* - \frac{N\theta}{p}, \ p \in [1, \infty].
\]
We remark that \( \sigma(p) < 1 \), and hence \( Y \in L^1_{\text{loc}}([0, \infty) : L^p(\mathbb{R}^N)) \), if and only if \( p \) is subcritical,
\[
\text{S} \quad p \in [1, \infty] \text{ if } N < 2\beta, \quad p \in [1, p_c) \text{ if } N \geq 2\beta, \quad \text{where } p_c = \begin{cases} N/(N-2\beta) & \text{if } N > 2\beta, \\ \infty & \text{if } N = 2\beta. \end{cases}
\]
If \( p \) is not subcritical we need some extra assumption on the forcing term to guarantee that \( u(\cdot, t) \in L^p(\mathbb{R}^N) \). In the present paper we will use two different such extra hypotheses, the pointwise condition
\[
|f(x, t)| \leq C|x|^{-N(1+t)^{-\gamma}} \quad \text{for } |x| \text{ large},
\]
and the integral condition
\[
\|f(\cdot, t)\|_{L^q(\mathbb{R}^N)} \leq C(1+t)^{-\gamma} \quad \text{for some } q > q_c(p) := \frac{N}{2\beta + \frac{\gamma}{p}}.
\]
Note that \( q_c(p_c) = 1 \). We do not claim that these conditions are optimal; but they are not too restrictive, and are easy enough to keep the proofs simple.
1.3. Precedents and statement of results. As a first precedent we have [14], where the authors study the problem in the integrable in time case $\gamma > 1$ and prove, when $p$ is subcritical, that

$$\lim_{t \to \infty} t^{\sigma(p)} \|u(\cdot,t) - M_\infty Y(\cdot,t)\|_{L^p(\mathbb{R}^N)} = 0,$$

where $M_\infty := \int_0^\infty \int_{\mathbb{R}^N} f(x,t) \, dx \, dt < \infty$,

which, as long as $M_\infty \neq 0$, implies in particular the sharp decay rate

$$\|u(\cdot,t)\|_{L^p(\mathbb{R}^N)} \asymp t^{-\sigma(p)}.$$ (1.11)

A second and very recent precedent is [6], where we study the case of large dimensions, $N > 4\beta$. In sharp contrast with the case in which the time derivative is local, and due to the effect of memory, we proved there that the decay/growth rates are not the same in different space-time scales. We already found this phenomenon for the homogeneous Cauchy problem (with nontrivial initial datum) in [4, 5]. In the present paper we have to face the same difficulty, which forces us to study separately the rates in exterior regions, $|x| \geq \nu t^\theta$ with $\nu > 0$, compact sets or intermediate regions $|x| \asymp g(t)$ with $g(t) \to \infty$ and $g(t) = o(t^\theta)$. The main difference in comparison with the case of large dimensions is that now we have to deal with additional critical behaviors associated with the critical dimensions $N = 2\beta$ and $N = 4\beta$.

Our results are summarized as follows:

**Theorem 1.1 (Exterior regions).** Let $1 \leq N \leq 4\beta$. Let $f$ satisfy (1.3) and also (1.9) if $p$ is not subcritical. Let $u$ be the mild solution to (1.1). For all $\nu > 0$ there is a constant $C$ such that

$$\|u(\cdot,t)\|_{L^p(|x| \geq \nu t^\theta)} \leq C \begin{cases} t^{-\sigma(p)+1-\gamma}, & \gamma < 1, \\
\sigma^* \log t, & \gamma = 1, \\
\sigma^* t, & \gamma > 1.\end{cases}$$

These estimates are sharp.

When we say that the estimates are sharp we mean that there are forcing functions $f$ such that the estimates from below hold.

**Remark.** For $p \in [1,p_c)$ and $\gamma > 1$ the result follows from (1.11).

It is in compact regions where we find critical behaviors associated with the dimension. **Theorem 1.2 (Compact sets).** Let $f$ satisfy (1.3). If $p$ is not subcritical, assume also (1.10) with $\gamma$ as in (1.3). Let $u$ be the mild solution to (1.1). For every compact set $K$ there exists a constant $C$ such that

$$\|u(\cdot,t)\|_{L^p(K)} \leq C \begin{cases} t^{-\sigma_*+1-\gamma}, & \gamma < 1, \\
\sigma_* \log t, & \gamma = 1, \\
\sigma_* t, & \gamma > 1,\end{cases}$$

$$\|u(\cdot,t)\|_{L^p(K)} \leq C \begin{cases} t^{-\gamma \log t}, & \gamma \leq \sigma_* = 1, \\
\sigma_* t, & \gamma > \sigma_* = 1, \quad N = 2\beta, \\
\gamma \leq \sigma_* = 1, & 2\beta < N < 4\beta, \\
\sigma_* t, & \gamma \geq \sigma_* = 1, \\
\gamma \leq \sigma_* = 1 + \alpha, & N = 4\beta, \\
\sigma_* \log t, & \gamma \geq \sigma_* = 1 + \alpha.\end{cases}$$
These estimates are sharp.

Remark. Though the decay rates in compact sets are the same in all $L^p$ norms, the value of $p$ still plays a role in our proofs, since we will need an extra decay assumption when $p$ is not subcritical.

As expected, the rates in intermediate regions, between compact sets and exterior regions, are intermediate between the ones in such scales, and hence will also exhibit critical behaviors associated with the dimension.

**Theorem 1.3 (Intermediate regions).** Let $f$ satisfy (1.3) and also (1.9) if $p$ is not subcritical. Let $g(t) \to \infty$ be such that $g(t) = o(t^\theta)$. Let $u$ be the mild solution to (1.1). For all $0 < \nu < \mu < \infty$ there exists a constant $C$ such that

$$
\|u(\cdot, t)\|_{L^p(\nu < |x|/g(t) < \mu)} \leq C g(t)^{N/p} \left\{ t^{-\sigma_*+1-\gamma}, \begin{array}{l}
\gamma < 1, \\
t^{-\sigma_*} \log t, \quad \gamma = 1, \quad N < 2\beta, \\
t^{-\sigma_*}, \quad \gamma > 1, \\
\log \left( t^\theta/g(t) \right) t^{-\gamma}, \quad \gamma < 1, \\
t^{-\sigma_*} \log t, \quad \gamma = 1, \quad N = 2\beta, \\
t^{-\sigma_*}, \quad \gamma > 1, \\
g(t)(1-\sigma_*)/\theta t^{-\gamma}, \quad \gamma < 1, \\
\max \left\{ g(t)(1-\sigma_*)/\theta t^{-1}, t^{-\sigma_*} \log t \right\}, \quad \gamma = 1, \quad 2\beta < N < 4\beta, \\
\max \left\{ g(t)(1-\sigma_*)/\theta t^{-\gamma}, t^{-\sigma_*} \right\}, \quad \gamma > 1, \\
\max \left\{ g(t)(1-\sigma_*)/\theta t^{-\gamma}, \log \left( t^\theta/g(t) \right) t^{-\sigma_*+1-\gamma} \right\}, \quad \gamma < 1, \\
\max \left\{ g(t)(1-\sigma_*)/\theta t^{-1}, \log \left( t^\theta/g(t) \right) t^{-\sigma_*} \log t \right\}, \quad \gamma = 1, \quad N = 4\beta, \\
\max \left\{ g(t)(1-\sigma_*)/\theta t^{-\gamma}, \log \left( t^\theta/g(t) \right) t^{-\sigma_*} \right\}, \quad \gamma > 1, \\
\gamma < 1, \\
\gamma = 1, \quad N < 2\beta, \\
\gamma > 1, \\
\gamma = 1, \quad 2\beta < N < 4\beta, \\
\gamma > 1, \\
\gamma = 1, \quad N = 4\beta, \\
\gamma > 1,
\end{array} \right.
\right.

These estimates are sharp.

We also obtain results that connect the behavior in compact sets and exterior regions thus getting the decay rate in $L^p(\mathbb{R}^N)$.

**Theorem 1.4 (Global results).** Assume (1.3), and also (1.9) and (1.10) with $\gamma$ as in (1.3) if $p$ is not subcritical. Let $u$ be the mild solution to (1.1).

(i) If $p$ is subcritical, there is a constant $C$ such that

$$
\|u(\cdot, t)\|_{L^p(\mathbb{R}^N)} \leq C \left\{ t^{-\sigma(p)+1-\gamma}, \begin{array}{l}
\gamma < 1, \\
t^{-\sigma(p)} \log t, \quad \gamma = 1, \\
t^{-\sigma(p)}, \quad \gamma > 1.
\end{array} \right.
\right.

(ii) If $p$ is critical, $N \geq 2\beta$, $p = p_c$, there is a constant $C$ such that

$$
\|u(\cdot, t)\|_{L^p(\mathbb{R}^N)} \leq C \left\{ t^{-\gamma} \log t, \quad \gamma \leq 1, \\
\gamma > 1.
\right.
\right.$$
(iii) If $p$ is supercritical, $N > 2\beta$, $p > p_c$, there is a constant $C$ such that

$$\|u(\cdot, t)\|_{L^p(\mathbb{R}^N)} \leq C \begin{cases} t^{-\gamma}, & \gamma < \sigma(p), \\ t^{-\sigma(p)}, & \gamma \geq \sigma(p), \\ t^{-\sigma_* \log t}, & \gamma \geq \sigma_* = 1 + \alpha, \quad p = \infty, \quad N = 4\beta. \end{cases}$$

These estimates are sharp.

Thus, if $p$ is subcritical or if $p$ is supercritical and $\gamma$ is large, the global decay rate is determined by the behavior in exterior domains, while if $p$ is supercritical and $\gamma$ is small it is dictated by the behavior in compact sets. In the critical case with $\gamma$ large, compact sets rule the game. The same is true for $\gamma$ small in the critical dimension $N = 2\beta$. However, if $\gamma$ is small and $2\beta < N \leq 4\beta$, then the whole intermediate region plays a role in the determination of the global rate.

Once sharp decay/growth rates are available, the next natural step is to obtain the asymptotic profile, in each space-time scale, after correcting the solution with the rate. Such goal is the content of the forthcoming paper [7].

2. Exterior region

This section is devoted to the proof of Theorem 1.1, which gives the decay/growth rates in all $L^p$ norms of the mild solution $u$ to (1.1) in exterior regions, $\{(x, t) \in Q : |x| \geq \nu t^\theta\}$, $\nu > 0$.

**Proof of Theorem 1.1.** The starting point will be always Duhamel’s type formula (1.2).

If $p$ is subcritical, that is, if (S) holds, using the size condition (1.3) and (1.8) we get the global estimate

$$\|u(\cdot, t)\|_{L^p(\mathbb{R}^N)} \leq C \int_0^t \|Y(\cdot, t-s)\|_{L^p(\mathbb{R}^N)} \|f(\cdot, s)\|_{L^1(\mathbb{R}^N)} ds \leq C \int_0^t (t-s)^{-\sigma(p)} (1+s)^{-\gamma} ds$$

$$\leq C t^{-\sigma(p)} \int_0^{\frac{t}{2}} (1+s)^{-\gamma} ds + C t^{-\gamma} \int_\frac{t}{2}^t (t-s)^{-\sigma(p)} ds,$$

which yields the desired bound in exterior regions, since

$$\int_0^{\frac{t}{2}} (1+s)^{-\gamma} ds \approx \begin{cases} t^{1-\gamma}, & \gamma < 1, \\ \log t, & \gamma = 1, \\ 1, & \gamma > 1, \end{cases}$$

Let us consider then the critical and supercritical cases, which only occur if $N \geq 2\beta$. We make the decomposition $|u| \leq I + II$, where

$$I(x, t) = \int_0^t \int_{\{|y| \leq \frac{|x|}{2t}\}} Y(x-y,t-s) |f(y,s)| dy ds,$$

$$II(x, t) = \int_0^t \int_{\{|y| > \frac{|x|}{2t}\}} Y(x-y,t-s) |f(y,s)| dy ds.$$
If $|y| < |x|/2$, then $|x - y| > |x|/2$. Thus, if moreover $|x| \geq \nu t^\theta$, then $|x - y|(t - s)^{-\theta} > \nu/2$. Hence, the exterior bound \cite{17} together with the size hypothesis \cite{13} yield

$$I(x, t) \leq C \int_0^t (t - s)^{2\alpha - 1} \int_{\{|y| < \frac{|x|}{2}\}} |x - y|^{-(N+2\beta)} |f(y, s)| \, dy \, ds$$

$$\leq C|x|^{-(N+2\beta)} \int_0^t (t - s)^{2\alpha - 1} \int_{\{|y| < \frac{|x|}{2}\}} |f(y, s)| \, dy \, ds$$

$$\leq C|x|^{-(N+2\beta)} \int_0^t (t - s)^{2\alpha - 1}(1 + s)^{-\gamma} \, ds,$$

and therefore

$$||I(\cdot, t)||_{L^p(|x| \geq \nu t^\theta)}) \leq C t^{-\sigma(p)} \int_0^{\frac{t}{2}} (1 + s)^{-\gamma} \, ds + C t^{-\sigma(p)+1-\gamma-2\alpha} \int_0^t (t - s)^{2\alpha - 1} \, ds$$

$$\leq C t^{-\sigma(p)} \int_0^{\frac{t}{2}} (1 + s)^{-\gamma} \, ds + C \cdot C t^{-\sigma(p)+1-\gamma},$$

from where the desired bound for $I$ follows using \cite{22}.

In order to estimate $\Pi$ we use the decay condition \cite{19} on $f$ and the integrability estimate \cite{18} for the kernel with $p = 1$. Since $\sigma(1) = 1 - \alpha$,

$$\Pi(x, t) \leq C \int_0^t (1 + s)^{-\gamma} \int_{\{|y| < \frac{|x|}{2}\}} Y(x - y, t - s)|y|^{-N} \, dy \, ds$$

$$\leq C|x|^{-N} \int_0^t (1 + s)^{-\gamma} \int_{\mathbb{R}^N} Y(x - y, t - s) \, dy \, ds = C|x|^{-N} \int_0^t (1 + s)^{-\gamma}(t - s)^{\alpha - 1} \, ds,$$

and therefore, remembering that $p > 1$ in the critical and supercritical cases,

$$||\Pi(\cdot, t)||_{L^p(|x| \geq \nu t^\theta)}) \leq C t^{-\sigma(p)} \int_0^{\frac{t}{2}} (1 + s)^{-\gamma} \, ds + C t^{-\sigma(p)+1-\gamma-\alpha} \int_0^t (t - s)^{\alpha - 1} \, ds$$

$$\leq C t^{-\sigma(p)} \int_0^{\frac{t}{2}} (1 + s)^{-\gamma} \, ds + C t^{-\sigma(p)+1-\gamma},$$

from where the desired bound for $\Pi$ follows using \cite{22}.

**Optimality.** We choose $f(x, t) = (1 + t)^{-\gamma} \chi_{B_t}(x)$.

Let $t \geq 1$, $s \in (0, t/2)$, $|x| < \mu t^\theta$, with $\mu > \nu$, and $|y| < 1$. Then

$$\frac{|x - y|}{(t - s)^\theta} \leq \frac{1 + \mu t^\theta}{(t/2)^\theta} \leq C,$$

and hence $G((x - y)(t - s)^{-\theta}) \geq c > 0$, since $G$ is positive. Therefore,

$$u(x, t) \geq C \int_0^{\frac{t}{2}} (1 + s)^{-\gamma}(t - s)^{-\sigma^*} \, ds \geq C t^{-\sigma^*} \int_0^{\frac{t}{2}} (1 + s)^{-\gamma} \, ds.$$
Thus,
\[
\|u(\cdot,t)\|_{L^p(|x|\geq \nu t^\theta)} \geq \|u(\cdot,t)\|_{L^p(\{\mu>|x|/t^\theta|>\nu\})} \geq C|\{\nu < |x|/t^\theta < \mu\}|^{1/p} t^{-\sigma_*} \int_0^t (1 + s)^{-\gamma} \, ds
\]
\[
= Ct^{-\sigma(p)} \int_0^t (1 + s)^{-\gamma} \, ds,
\]
which implies the desired lower bound. \(\square\)

3. Compact regions

In this section we study the decay/growth rate of the \(L^p\) norm of the mild solution to (1.3) in compact sets.

Proof of Theorem 1.2. We have the bound \(|u| \leq I + II\), where
\[
I(x,t) = \int_0^{t-1} \int Y(x-y, t-s)|f(y,s)| \, dy \, ds,
\]
\[
II(x,t) = \int_{t-1}^t \int Y(x-y, t-s)|f(y,s)| \, dy \, ds.
\]

In order to estimate I we have to consider two cases, to take into account the different behaviors of the profile \(G\) at the origin, depending on the dimension.

If \(1 \leq N < 4\beta\), \(G \in L^\infty(\mathbb{R}^N)\). Hence, using the decay assumption (1.3), for all \(t \geq 2\) we have
\[
I(x,t) \leq C \int_0^{t-1} (t-s)^{-\sigma_*} (1 + s)^{-\gamma} \, ds \leq Ct^{-\sigma_*} \int_0^t (1 + s)^{-\gamma} \, ds + Ct^{-\gamma} \int_{t-1}^1 (t-s)^{-\sigma_*} \, ds.
\]

Using (2.2) and
\[
\int_{t-1}^t (t-s)^{-\sigma_*} \, ds = \int_1^{t-1} t^{-\sigma_*} \, d\tau \propto \begin{cases} \log t, & N = 2\beta, \\ 1, & 2\beta < N < 4\beta, \end{cases}
\]
we get the desired bound for I for \(p = \infty\), and therefore for all values of \(p\), because we are considering compact sets.

If \(N = 4\beta\), we make the decomposition \(I = I_1 + I_2 + I_3\), where
\[
I_1(x,t) = \int_0^{t-1} \int_{|x-y|<1} Y(x-y, t-s)|f(y,s)| \, dy \, ds,
\]
\[
I_2(x,t) = \int_0^{t-1} \int_{1<|x-y|<(t-s)\theta} Y(x-y, t-s)|f(y,s)| \, dy \, ds,
\]
\[
I_3(x,t) = \int_0^{t-1} \int_{|x-y|>(t-s)\theta} Y(x-y, t-s)|f(y,s)| \, dy \, ds.
\]
Let $q = 1$ if $p \in [1, p_c)$, $q > q_c(p)$ as in (1.11) if $p \geq p_c$. Let $r$ satisfy $1 + \frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Then $r \in [1, p_c)$ and, using the behavior at the origin (1.5) of the profile $G$, if $s \leq t - 1$ we have
\[
\|Y(\cdot, t - s)\|_{L^r(B_1)} \leq C(t - s)^{-\sigma} \left( \int_{B_1} (1 - \log |x| + \log(t - s))^r \, dx \right)^{1/r} \\
\leq C(t - s)^{-\sigma}(1 + \log(t - s)).
\]
Therefore, the integral condition (1.10) implies, for all $t \geq 2$,
\[
\|I_1(\cdot, t)\|_{L^p(\mathbb{R}^N)} \leq C \int_0^{t-1} \|Y(\cdot, t - s)\|_{L^r(B_1)} \|f(\cdot, s)\|_{L^q(\mathbb{R}^N)} \, ds \\
\leq Ct^{-\sigma} \log t \int_0^{\frac{t}{2}} (1 + s)^{-\gamma} \, ds + Ct^{-\gamma} \int_{\frac{t}{2}}^{t-1} (t - s)^{-\sigma}(1 + \log(t - s)) \, ds.
\]
Since $\sigma_s = 1 + \alpha$ in this case, $\int_{\frac{t}{2}}^{t-1} (t - s)^{-\sigma_s}(1 + \log(t - s)) \, ds \leq C$ for all $t \geq 2$, and using also (2.2), we finally get
\[
\|I_1(\cdot, t)\|_{L^p(\mathbb{R}^N)} \leq C \begin{cases} \\
\begin{array}{ll}
t^{-\gamma}, & \gamma < \sigma_s, \\
t^{-\sigma_s} \log t, & \gamma \geq \sigma_s.
\end{array}
\end{cases}
\]
In order to bound $I_2$, we observe that $(t-s)^{-\theta} < |x| |t-s|^{-\theta} < 1$ in the region under consideration. Hence, using the size condition (1.3) and the behavior at the origin (1.5) of the profile $G$,
\[
I_2(x, t) \leq C \int_0^{t-1} (t - s)^{-\sigma} \int_{\{1<|x-y|<(t-s)\}} (1 + \log(t - s)) |f(y, s)| \, dy \, ds \\
\leq Ct^{-\sigma} \log t \int_0^{\frac{t}{2}} (1 + s)^{-\gamma} \, ds + Ct^{-\gamma} \int_{\frac{t}{2}}^{t-1} (t - s)^{-\sigma_s}(1 + \log(t - s)) \, ds \\
\leq C \begin{cases} \\
\begin{array}{ll}
t^{-\gamma}, & \gamma < \sigma_s, \\
t^{-\sigma_s} \log t, & \gamma \geq \sigma_s.
\end{array}
\end{cases}
\]
Finally, since $G$ is bounded outside the origin, using the size condition (1.3),
\[
I_3(x, t) \leq C \int_0^{t-1} (t - s)^{-\sigma_s}(1 + s)^{-\gamma} \, ds \\
\leq Ct^{-\sigma_s} \int_0^{\frac{t}{2}} (1 + s)^{-\gamma} \, ds + Ct^{-\gamma} \int_{\frac{t}{2}}^{t-1} (t - s)^{-\sigma_s} \, ds \leq Ct^{-\min(\gamma, \sigma_s)}.
\]
In order to bound $II$, for all small dimensions, $1 \leq N \leq 4\beta$, we take $r \in [1, p_c)$ as we did when we obtained the bound for $I_1$. Notice that $\sigma(r) < 1$. Hence, using (1.8) and (1.10) we get, for all $t \geq 2$,
\[
\|II(\cdot, t)\|_{L^p(\mathbb{R}^N)} \leq C \int_0^{t} \|Y(\cdot, t - s)\|_{L^r(\mathbb{R}^N)} \|f(\cdot, s)\|_{L^q(\mathbb{R}^N)} \, ds \\
\leq C \int_0^{t} (t - s)^{-\sigma(r)}(1 + s)^{-\gamma} \, ds \leq Ct^{-\gamma} \int_0^{1} r^{-\sigma(r)} \, ds = Ct^{-\gamma},
\]
which combined with the estimate for $I$ yields the result.
Optimality. We consider \( f(x, t) = (1 + t)^{-\gamma} \chi_{K + B_1}(x) \), where \( K \) is any compact set with measure different from 0. If \( x \in K \) and \( |x - y| < 1 \), then \( y \in K + B_1 \). Therefore, for all \( x \in K \),

\[
    u(x, t) \geq \int_0^{t-1} \int_{\{|x-y|<1\}} Y(x-y, t-s) f(y, s) \, dy \, ds
    = \int_0^{t-1} (1 + s)^{-\gamma} \int_{\{|x-y|<1\}} Y(x-y, t-s) \, dy \, ds.
\]

On the other hand, if \( |x - y| < 1 \) and \( s < t - 1 \), then \( |x - y|(t - s)^{-\theta} < 1 \), and we have, using the behavior \( (1.5) \) for \( G \) close to the origin, that there exists a constant \( c > 0 \) such that

\[
    G((x-y)(t-s)^{-\theta}) \geq c \begin{cases} 1, & 1 \leq N < 4\beta, \\ \log(t-s), & N = 4\beta. \end{cases}
\]

Hence, for all \( x \in K \) and \( t \geq 2 \),

\[
    u(x, t) \geq c \begin{cases} t^{-\sigma*} \int_0^{t/2} (1 + s)^{-\gamma} \, ds + t^{-\gamma} \int_{t/2}^{t-1} (t-s)^{-\sigma*} \, ds, & 1 \leq N < 4\beta, \\ t^{-\sigma*} \log t \int_0^{t/2} (1 + s)^{-\gamma} \, ds + t^{-\gamma} \int_{t/2}^{t-1} (t-s)^{-\sigma*} \log(t-s) \, ds, & N = 4\beta, \end{cases}
\]

from where the desired lower bounds follow using \( (2.2) \) and \( (3.1) \), since

\[
    \int_{t/2}^{t-1} (t-s)^{-\sigma*} \log(t-s) \, ds \geq \int_1^2 \tau^{-\sigma*} \log \tau \, d\tau = c > 0
\]

for all \( t \geq 4 \) if \( N = 4\beta \).

\[\square\]

4. Intermediate scales

In this section we study the decay/growth rate of the \( L^p \) norm of the mild solution to \( (1.3) \) in regions where \( |x| \simeq g(t) \) with \( g(t) \to \infty \) such that \( g(t) = o(t^{\theta}) \).

Proof of Theorem \( (1.3) \) We have \( |u| \leq I + II \), where

\[
    I(x, t) = \int_0^t \int_{\{|y|>|x|/2\}} |f(x-y, t-s)| Y(y, s) \, dy \, ds,
\]

\[
    II(x, t) = \int_0^t \int_{\{|y|<|x|/2\}} |f(x-y, t-s)| Y(y, s) \, dy \, ds.
\]

We make the decomposition \( I = I_1 + I_2 + I_3 \), where

\[
    I_1(x, t) = \int_0^{(|x|/2)^{1/\theta}} \int_{\{|y|>|x|/2\}} |f(x-y, t-s)| Y(y, s) \, dy \, ds,
\]

\[
    I_2(x, t) = \int_{(|x|/2)^{1/\theta}}^{t} \int_{\{|y|>|x|/2\}} |f(x-y, t-s)| Y(y, s) \, dy \, ds,
\]

\[
    I_3(x, t) = \int_{t/2}^{t} \int_{\{|y|>|x|/2\}} |f(x-y, t-s)| Y(y, s) \, dy \, ds.
\]
We start by estimating $I_1$. In the region under consideration we have $|y| s^{-\theta} > 1$. Hence we can use the estimate (1.7) for $Y$ in exterior domains, which combined with the decay condition (1.3) yields, for $|x| \leq \delta t^\theta$ with $\delta < 2$,

$$
I_1(x, t) \leq C |x|^{-(N+2\beta)} \int_0^{\left(\frac{|y|}{s}\right)^{1/\theta}} s^{2\alpha-1} (1 + t - s)^{-\gamma} \, ds
$$

(4.3)

$$
\leq C |x|^{-(N+2\beta)} t^{-\gamma} \int_0^{\left(\frac{|y|}{s}\right)^{1/\theta}} s^{2\alpha-1} \, ds = C |x|^{(1-\sigma_*)/\theta} t^{-\gamma}.
$$

We conclude that $\|I_1(\cdot, t)\|_{L^p(|v|<|x|/g(t)<\mu))} \leq C t^{-\gamma} g(t)^{(1-\sigma_*)/\theta}$.

Let us now consider $I_j$, $j = 2, 3$. If $N < 4\beta$, the profile $G$ is bounded. Hence, using the decay condition (1.3), if $|x| \leq \delta t^\theta$ with $\delta < \min\{1, 2^{1-\theta}\}$ we have

$$
I_2(x, t) \leq C t^{-\gamma} \int_{\left(\frac{|y|}{s}\right)^{1/\theta}} s^{-\sigma_+} \, ds \leq C t^{-\gamma} \begin{cases} 1/(t-s), & 1 \leq N < 2\beta, \\ \log(t^\theta/|x|), & N = 2\beta, \\ |x|^{(1-\sigma_*)/\theta}, & 2\beta < N < 4\beta, \end{cases}
$$

(4.4)

$$
I_3(x, t) \leq C t^{-\sigma_+} \int_{\left(\frac{|y|}{s}\right)^{1/\theta}} (1 + t - s)^{-\gamma} \, ds.
$$

Therefore, using also (3.1),

$$
\|I_2(\cdot, t)\|_{L^p(|v|<|x|/g(t)<\mu))} \leq C g(t)^{-N/p} \begin{cases} t^{-\sigma_+ + 1 - \gamma}, & 1 \leq N < 2\beta, \\ \log(t^\theta/g(t) t^{-\gamma}), & N = 2\beta, \\ g(t)^{(1-\sigma_*)/\theta} t^{-\gamma}, & 2\beta < N < 4\beta, \end{cases}
$$

$$
\|I_3(\cdot, t)\|_{L^p(|v|<|x|/g(t)<\mu))} \leq C g(t)^{-N/p} \begin{cases} t^{-\sigma_+ + 1 - \gamma}, & \gamma < 1, \\ t^{-\sigma_+} \log t, & \gamma = 1, \\ t^{-\sigma_*}, & \gamma > 1. \end{cases}
$$

When $N = 4\beta$, using the inner and outer behaviors (1.5)–(1.6) of the profile $G$ (the latter one implying boundedness), and the decay condition (1.3), and remembering that $\sigma_* = 1 + \alpha > 1$ in this case, if $|x| \leq \delta t^\theta$ with $\delta < \min\{1, 2^{1-\theta}\}$,

$$
I_2(x, t) \leq C \int_{\left(\frac{|y|}{s}\right)^{1/\theta}} \int_{\left\{\frac{|y|}{s} < |y| < \delta t^\theta\right\}} |f(x - y, t - s)| s^{-\sigma_+} (1 + \log(s^\theta/|y|)) \, dy \, ds
$$

$$
+ C \int_{\left(\frac{|y|}{s}\right)^{1/\theta}} \int_{\{|y| > \delta t^\theta\}} |f(x - y, t - s)| s^{-\sigma_*} \, dy \, ds
$$

(4.5)

$$
\leq C \int_{\left(\frac{|y|}{s}\right)^{1/\theta}} (1 + t - s)^{-\gamma} s^{-\sigma_*} (1 + \log(2s^\theta/|x|)) \, ds
$$

$$
\leq C t^{-\gamma} |x|^{(1-\sigma_*)/\theta} \int_{\left(\frac{|y|}{s}\right)^{1/\theta}} \tau^{-(1+\frac{2\alpha-1}{\theta})} (1 + \log \tau) \, d\tau \leq C t^{-\gamma} |x|^{(1-\sigma_*)/\theta},
$$
\[ \text{I}_3(x,t) \leq C \int_2^t \int_{\{\frac{|y|}{2} < |y| < s^\beta\}} |f(x-y, t-s)| s^{-\sigma^*} \left(1 + \log(s^\theta/|y|)\right) \, dy \, ds \]

(4.6)

\[ + C \int_2^t \int_{\{|y| > s^\theta\}} |f(x-y, t-s)| s^{-\sigma^*} \, dy \, ds \]

\[ \leq C \left(1 + \log(t^\theta/|x|)\right) t^{-\sigma^*} \int_2^t (1 + t-s)^{-\gamma} \, ds. \]

Hence, using (3.1),

\[ ||\text{I}_2(\cdot, t)||_{L^p(\nu < |x|/g(t) < \mu)} \leq C t^{-\gamma} g(t)^{(1-\sigma(p))/\theta} \]

\[ ||\text{I}_3(\cdot, t)||_{L^p(\nu < |x|/g(t) < \mu)} \leq C g(t)^{N/p} \log(t^\theta/g(t)) \begin{cases} t^{-\sigma^*+1-\gamma}, & \gamma < 1, \\ t^{-\sigma^*} \log t, & \gamma = 1, \\ t^{-\sigma^*}, & \gamma > 1. \end{cases} \]

Let us now turn to II. We decompose it as II = II_1 + II_2, where

\[ \text{II}_1(x,t) = \int_0^{t/2} \int_{\{|y| < \frac{|x|}{2}\}} |f(x-y, t-s)| Y(y, s) \, dy \, ds, \]

\[ \text{II}_2(x,t) = \int_{t/2}^t \int_{\{|y| < \frac{|x|}{2}\}} |f(x-y, t-s)| Y(y, s) \, dy \, ds. \]

We start with \( \text{II}_1 \). If \( N < 2\beta \), then \( G \) is bounded and \( \sigma^* < 1 \). Hence, using the decay assumption (1.3),

\[ \text{II}_1(x,t) \leq C \int_0^{t/2} \int_{\{|y| < \frac{|x|}{2}\}} |f(x-y, t-s)| s^{-\sigma^*} \, dy \, ds \]

\[ \leq C \int_0^{t/2} (1 + t-s)^{-\gamma} s^{-\sigma^*} \, ds \leq C t^{-\gamma} \int_0^{t/2} s^{-\sigma^*} \, ds = C t^{-\sigma^*+1-\gamma}, \]

so that \( ||\text{II}_1(\cdot, t)||_{L^p(\nu < |x|/g(t) < \mu)} \leq C g(t)^{N/p} t^{-\sigma^*+1-\gamma}. \)

For \( 2\beta \leq N \leq 4\beta \) we make the decomposition \( \text{II}_1 = \text{II}_{11} + \text{II}_{12} \), where

\[ \text{II}_{11}(x,t) = \int_0^{t/4} \int_{\{|y| < \frac{|x|}{2}\}} |f(x-y, t-s)| Y(y, s) \, dy \, ds, \]

\[ \text{II}_{12}(x,t) = \int_{t/4}^{t/2} \int_{\{|y| < \frac{|x|}{2}\}} |f(x-y, t-s)| Y(y, s) \, dy \, ds. \]

If \( p \) is subcritical, using the decay assumption (1.3) we get (remember that \( \sigma(p) < 1 \) in this case)

\[ ||\text{II}_{11}(\cdot, t)||_{L^p(\nu < |x|/g(t) < \mu)} \leq C \int_0^{t/4} \left(\frac{g(t)}{1/\theta}\right)^{1/p} \|f(\cdot, t-s)\|_{L^1(\mathbb{R}^N)} s^{-\sigma^*} \left(\int_{\mathbb{R}^N} G^p(y s^{-\theta}) \, dy\right)^{1/p} \, ds \]

\[ \leq C \int_0^{t/4} \left(\frac{g(t)}{1/\theta}\right)^{1/p} (1 + t-s)^{-\gamma} s^{-\sigma(p)} \, ds \leq C t^{-\gamma} g(t)^{(1-\sigma(p))/\theta}, \]

since \( G \in L^p(\mathbb{R}^N) \) in this case.
If \( p \) is not subcritical, since \( |x - y| > |x|/2 \) when \( |y| < |x|/2 \), using now the decay assumption \( \| \cdot \|_{L^p(N)} \) we get, for \( |x| \leq \delta t^\theta \) with \( \delta < 2^{1-\theta} \),

\[
\Pi_{11}(x, t) \leq C|x|^{-N} \int_0^{1/\theta} \int_{\mathbb{R}^N} (1 + t - s)^{-\gamma} s^{-\sigma} G(ys^{-\theta}) \, dy \, ds \leq C|x|^{-N} t^{-\gamma} \int_0^{1/\theta} s^{-\sigma + N\theta} \, ds \leq Ct^{-\gamma} |x|^{(1-\sigma)/\theta},
\]

(4.7)

since \( G \in L^1(\mathbb{R}^N) \), and we get the same \( L^p \) estimate as in the subcritical case.

As for \( \Pi_{12} \), since \( G \) is bounded if \( N < 4\beta \), using the decay assumption \( \| \cdot \|_{L^p(N)} \),

\[
\Pi_{12}(x, t) \leq C \int_0^{1/\theta} \int_{\mathbb{R}^N} (1 + t - s)^{-\gamma} s^{-\sigma} \, ds \leq C \begin{cases} t^{-\gamma} \log(t^\theta/|x|), & N = 2\beta, \\ t^{-\gamma}|x|^{(1-\sigma)/\theta}, & 2\beta < N < 4\beta, \end{cases}
\]

if \( |x| \leq \delta t^\theta \) with \( \delta < 2^{1-\theta} \), so that

\[
\|\Pi_{12}(\cdot, t)\|_{L^p(\{\nu < |x|/g(t) < \mu\})} \leq C \begin{cases} t^{-\gamma} g(t)^{N/p} \log(t^\theta/g(t)), & N = 2\beta, \\ t^{-\gamma} g(t)^{(1-\sigma(p))/\theta}, & 2\beta < N < 4\beta. \end{cases}
\]

(4.8)

On the other hand, if \( N = 4\beta \), since \( |y| \leq s^\theta \) in the region under consideration, using the behavior \( \| \cdot \|_{L^p(N)} \) of \( G \) close to the origin,

\[
\Pi_{12}(\cdot, t) \leq C \int_0^{1/\theta} f(x - y, t - s) s^{-\sigma} \int_{\{y < \frac{1}{2}\}} \left( 1 + \log(|y|/s^\theta) \right) \, dy \, ds.
\]

(4.9)

It is easy to check that for each \( p \in [1, \infty) \) there exists a constant \( C \) such that

\[
\int_{\{|z| < \rho\}} \left( 1 + \log|z| \right)^p \, dy \leq C \rho^N \left( 1 + \log \rho \right)^p \quad \text{for every } \rho \geq 0.
\]

(4.10)

Therefore, using the decay assumption \( \| \cdot \|_{L^p(N)} \), for all \( p \in [1, p_c) \) we have

\[
\|\Pi_{12}(\cdot, t)\|_{L^p(\{\nu < |x|/g(t) < \mu\})}
\leq C \int_0^{1/\theta} \int_{\{\frac{1}{2}\}} \left( \int_{\{|y| < \frac{1}{2}\} g(t)} \right) \left( 1 + \log(|y|/s^\theta) \right)^p \, dy \, ds
\]

\[
\leq Ct^{-\gamma} \int_0^{1/\theta} \frac{1}{s^{\sigma(p)}} \left( \int_{\{|z| < \frac{1}{2}\} \mu g(t)} \right) \left( 1 + \log|z| \right)^p \, dy \, ds
\]

\[
\leq Ct^{-\gamma} \int_0^{1/\theta} \frac{1}{s^{\sigma(p)}} \left( 1 + \log(2s^\theta/\mu g(t)) \right) \, ds
\]

\[
= Ct^{-\gamma} g(t)^{(1-\sigma(p))/\theta} \int_{\nu/\mu} \tau^{-1+\sigma_*/\theta} \log \tau \, d\tau \leq Ct^{-\gamma} g(t)^{(1-\sigma(p))/\theta}
\]

since \( \sigma_* = 1 + \alpha > 1 \) in this case.
If $N = 4\beta$ and $p \geq p_c$, we plug the decay assumption (1.9) in (4.9). Therefore, since $|x - y| > |x|/2$ if $|y| < |x|/2$, using (4.11) with $p = 1$,

$$\Pi_{12}(x, t) \leq C|x|^{-N} \int_{(\frac{|x|}{2})^{1/\theta}}^{\frac{1}{2}} (1 + t - s)^{-\gamma} s^{-\sigma^*} \int_{\{|y| < \frac{|x|}{2}\}} (1 + \log(s^\theta/|y|)) \, dy \, ds$$

$$\leq Ct^{-\gamma} \int_{(\frac{|x|}{2})^{1/\theta}}^{\frac{1}{2}} s^{-\sigma^* + N\theta} \int_{\{|z| < \frac{|x|}{2}\}} (1 + \log |z|) \, dz \, ds$$

(4.11)

$$\leq Ct^{-\gamma} \int_{(\frac{|x|}{2})^{1/\theta}}^{\frac{1}{2}} s^{-\sigma^*} (1 + \log(2s^\theta/|x|)) \, ds$$

$$\leq Ct^{-\gamma} |x|^{(1-\sigma^*)/\theta} \int_{1}^{\frac{1}{2}} \tau^{-1+\gamma} \log \tau \, d\tau \leq Ct^{-\gamma} |x|^{(1-\sigma^*)/\theta},$$

and we get the same $L^p$ estimate as in the subcritical case.

We finally consider $\Pi_2$. If $N < 4\beta$, using that $G$ is bounded and the decay assumption (1.3),

$$\Pi_2(x, t) \leq C \int_{\frac{1}{2}}^{1} \int_{\{|y| < \frac{|x|}{2}\}} |f(x - y, t - s)| s^{-\sigma^*} \, dy \, ds \leq Ct^{-\sigma^*} \int_{\frac{1}{2}}^{1} (1 + t - s)^{-\gamma} \, ds,$$

so that, using also (2.2)

$$\|\Pi_2(\cdot, t)\|_{L^p(\nu < |x|/g(t) < \mu)} \leq Cg(t)^{N/p} \begin{cases} t^{-\sigma^* + 1 - \gamma}, & \gamma < 1, \\ t^{-\sigma^*} \log t, & \gamma = 1, \\ t^{-\sigma^*}, & \gamma > 1. \end{cases}$$

Let now $N = 4\beta$. Since $|y| < s^\theta$ in the region under consideration, using the behavior (1.5) of $G$ close to the origin,

$$\Pi_2(x, t) \leq C \int_{\frac{1}{2}}^{1} \int_{\{|y| < \frac{|x|}{2}\}} |f(x - y, t - s)| s^{-\sigma^*} (1 + \log(|y|/s^\theta)) \, dy \, ds.$$

Thus, using the decay assumption (1.3) and the estimates (2.2) and (4.10), for $p \in [1, \infty)$ we have

$$\|\Pi_2(\cdot, t)\|_{L^p(\nu < |x|/g(t) < \mu)} \leq C \int_{\frac{1}{2}}^{1} \|f(\cdot, t - s)\|_{L^1(\mathbb{R}^N)} s^{-\sigma^*} \left( \int_{\{|y| < \frac{|x|}{2}\}} (1 + \log(|y|/s^\theta)) \, dy \right)^{1/p} \, ds \leq Cg(t)^{N/p} \log \left( t^\theta/g(t) \right) t^{-\sigma^*} \int_{\frac{1}{2}}^{1} (1 + t - s)^{-\gamma} \, ds \leq Cg(t)^{N/p} \log \left( t^\theta/g(t) \right) \begin{cases} t^{-\sigma^* + 1 - \gamma}, & \gamma < 1, \\ t^{-\sigma^*} \log t, & \gamma = 1, \\ t^{-\sigma^*}, & \gamma > 1. \end{cases}$$
If \( N = 4\beta \) and \( p = \infty \), we use the decay assumption \([1,9]\) instead, so that, as \( |x - y| > |x|/2 \) in the region under consideration, if \( |x| \leq t^\theta \) we have

\[
\Pi_2(x, t) \leq C|x|^{-N} \int_{t}^{\infty} \int_{\{y < \frac{|x|}{2}\}} (1 + t - s)^{-\gamma} s^{-\sigma*} (1 + |\log(|y|/s^\theta)|) \, dy \, ds \\
\leq Ct^{-\sigma* + N\theta} |x|^{-N} \int_{t}^{\infty} \int_{\{|z| < \frac{|x|}{2}\}} (1 + t - s)^{-\gamma} (1 + |\log|z||) \, dz \, ds \\
\leq Ct^{-\sigma*} \log(t^\theta/|x|) \int_{t}^{\infty} (1 + t - s)^{-\gamma} \, ds.
\]

(4.14)

Therefore, if \( |x| \approx g(t) \), \( \Pi_2(x, t) \leq C \log(t^\theta/g(t)) \left\{
\begin{aligned}
t^{-\sigma* + 1 - \gamma}, & \quad \gamma < 1, \\
t^{-\sigma*} \log t, & \quad \gamma = 1, \\
t^{-\sigma*}, & \quad \gamma > 1.
\end{aligned}
\right.

Optimality. To see that the estimates are sharp, we consider \( f(x, t) = (1 + t)^{-\gamma} \chi_{B_1}(x) \), so that

\[
u(x, t) = \int_{t}^{\infty} \int_{\{|y| < 1\}} (1 + t - s)^{-\gamma} Y(y, s) \, dy \, ds.
\]

If \( |x - y| < 1 \), then \( |y| \leq |x| + 1 \leq 2|x| \) if \( t \) is large enough, since \( |x| \geq \nu g(t) \) and \( g(t) \to \infty \). In particular, if \( s \geq t/2 \), then \( |y|/s^\theta \leq 2^{1+\theta} \nu g(t)/t^\theta \leq 1 \) for \( t \) large, since \( g(t) = o(t^\theta) \). Hence, using the behavior \([1,5]\) of \( G \) close to the origin,

\[
u(x, t) \geq c \left\{
\begin{aligned}
\int_{t}^{\infty} (1 + t - s)^{-\gamma} s^{-\sigma*} \, ds, & \quad N < 4\beta, \\
\int_{\{|y| < 1\}} (1 + t - s)^{-\gamma} s^{-\sigma*} (1 + \log(s^\theta/|y|)) \, dy \, ds, & \quad N = 4\beta, \\
\int_{t}^{\infty} (1 + t - s)^{-\gamma} \, ds \left\{
\begin{aligned}
1, & \quad N < 4\beta, \\
\log(t^\theta/|x|), & \quad N = 4\beta,
\end{aligned}
\right.
\right.
\]

so that

\[
u(\cdot, t) \|_{L^p\{|y| < \gamma/g(t)\}} \geq c g(t)^{N/p} \left\{
\begin{aligned}
t^{-\sigma* + 1 - \gamma}, & \quad \gamma < 1, \\
t^{-\sigma*} \log t, & \quad \gamma = 1, \\
t^{-\sigma*}, & \quad \gamma > 1, \\
\log(t^\theta/g(t)) t^{-\sigma* + 1 - \gamma}, & \quad \gamma < 1, \\
\log(t^\theta/g(t)) t^{-\sigma*} \log t, & \quad \gamma = 1, \\
\log(t^\theta/g(t)) t^{-\sigma*}, & \quad \gamma > 1.
\end{aligned}
\right.
\]

On the other hand, since \( G \) is positive, for \( t \) large so that \( |x| > 1 \),

\[
u(x, t) \geq c \int_{|x|/\theta}^{t} (1 + t - s)^{-\gamma} s^{-\sigma*} \, ds \geq ct^{-\gamma} \int_{|x|/\theta}^{t} s^{-\sigma*} \, ds \\
\geq ct^{-\gamma} \left\{
\begin{aligned}
\log(t^\theta/|x|), & \quad N = 2\beta, \\
|x|^{(1-\sigma*)/\theta}, & \quad 2\beta < N \leq 4\beta.
\end{aligned}
\right.
\]

(4.15)
Therefore, as $p_c = \infty$ when $N = 2\beta$,

$$\|u(\cdot, t)\|_{L^p(\{|x| < g(t)\})} \geq cg(t)^{N/p}t^{-\gamma} \begin{cases} \log(t^\theta/g(t)), & N = 2\beta, \\ g(t)^{(1-\sigma)/\theta}, & 2\beta < N \leq 4\beta. \end{cases}$$

Combining both estimates, we get the desired lower bounds.

5. **Global estimates**

In this section we establish the behavior of the *global* $L^p(\mathbb{R}^N)$ norms of the mild solution to (1.1).

**Proof of Theorem 1.4.** The behavior in the subcritical cases, with $p$ satisfying (S), was already established in the proof of Theorem 1.1; see estimate (2.1). Hence we turn our attention to the critical and supercritical cases. Due to the results of theorems 1.1 and 1.2, it is enough to show that the estimates are true in some region of the form $\{R \leq |x| \leq \delta t\}$ with $R, \delta > 0$.

We will follow the computations in the proof of Theorem 1.3.

We have

$$|u| \leq I + II, \text{ with } I \text{ and } II \text{ as in (4.1)}. \text{ The term } I \text{ is further decomposed as } I = I_1 + I_2 + I_3, \text{ with } I_j, j \in \{1, 2, 3\} \text{ as in (4.2). Since } |x| < \delta t^\theta \text{ in the region we are interested in, taking } \delta < \min\{1, 2^{1-\theta}\} \text{ we have (4.3), and hence, since } (1-\sigma)/\theta = 2\beta - N = -N/p_c, \text{ we get (5.1).}

We consider now $I_2$ and $I_3$. If $2\beta \leq N < 4\beta$ we have (4.4), from where we immediately obtain

$$\|I_1(\cdot, t)\|_{L^p(\{|x| < \delta t^\theta\})} \leq C \begin{cases} t^{-\gamma} \log t, & p = p_c, \\ t^{\gamma}, & p > p_c. \end{cases}$$

We consider now $I_2$ and $I_3$. If $2\beta \leq N < 4\beta$ we have (4.4), from where we immediately obtain

$$\|I_2(\cdot, t)\|_{L^p(\{|x| < \delta t^\theta\})} \leq C \begin{cases} t^{-\gamma} \log t, & p = p_c, \\ t^{\gamma}, & p > p_c, \end{cases}$$

If $N = 4\beta$, we have (4.5)–(4.6) instead, and we get again (5.1).

Let us now turn to $II$, always for values of $p$ which are not subcritical. Using (4.7), (4.8), and (4.11), we obtain, for $j = 1, 2$,

$$\|\Pi_1(\cdot, t)\|_{L^p(\{|x| < \delta t^\theta\})} \leq C \begin{cases} t^{-\gamma} \log t, & p = p_c, \\ t^{\gamma}, & p > p_c. \end{cases}$$

On the other hand, if $N < 4\beta$, using (4.12) we obtain

$$\|\Pi_2(\cdot, t)\|_{L^p(\{|x| < \delta t^\theta\})} \leq C \begin{cases} t^{-\sigma(p)+1-\gamma}, & \gamma < 1, \\ t^{-\sigma(p)} \log t, & \gamma = 1, \\ t^{-\sigma(p)}, & \gamma > 1. \end{cases}$$
If $N = 4\beta$, starting from (4.13), using the decay assumption (1.3) and the estimate (2.2), for $p \in [1, \infty)$ we have

$$\|II_2(\cdot, t)\|_{L^p\left(\{|x| < \delta t^\theta\}\right)} \leq C \int_t^\infty \|f(\cdot, t-s)\|_{L^1(\mathbb{R}^N)} s^{-\sigma} \left(\int_{|y| < \delta t^\theta} \left(1 + \left|\log |y|/s^\theta\right|^p\right) dy\right)^{1/p} ds$$

$$\leq C t^{-\sigma(p)} \int_t^\infty (1 + s)^{-\gamma} \left(\int_{|z| < \delta} \left(1 + \left|\log |z|\right|^p\right) dy\right)^{1/p} ds$$

$$\leq C \begin{cases} t^{-\sigma(p)+1-\gamma}, & \gamma < 1, \\ t^{-\sigma(p)} \log t, & \gamma = 1, \\ t^{-\sigma(p)}, & \gamma > 1. \end{cases}$$

On the other hand, if $p = \infty$, for $R < |x| < \delta t^\theta$ estimate (4.14) yields

$$II_2(x, t) \leq C \begin{cases} t^{-\sigma(\infty)+1-\gamma} \log t, & \gamma < 1, \\ t^{-\sigma(\infty)} \left(\log t\right)^2, & \gamma = 1, \\ t^{-\sigma(\infty)} log t, & \gamma > 1. \end{cases}$$

**Optimality.** Except when we have simultaneously $p = p_c$, $\gamma < 1$, and $N > 2\beta$, the global rate coincides either with the one in exterior regions or the one in compact sets, which we have already shown to be optimal. To show that our estimates are sharp also in this exceptional case, we consider, once more, $f(x, t) = (1 + t)^{-\gamma}\chi_{B_1}(x)$. The desired lower bound follows then immediately from (4.15).

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ASYMPTOTICS FOR INHOMOGENEOUS EQUATIONS WITH MEMORY. SMALL DIMENSIONS

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