FREE MINIMAL ACTIONS OF SOLVABLE LIE GROUPS WHICH ARE NOT AFFABLE

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Dedicated to Jean Renault on his 70th birthday

Abstract. We construct an uncountable family of transversely Cantor laminations of compact spaces defined by free minimal actions of solvable groups, which are not affable and whose orbits are not quasi-isometric to Cayley graphs.

1. Introduction

This paper is motivated by two different issues put forward by G. Hector in [17], and T. Giordano, I. Putnam, and C. Skau in [16], for which we give partial negative results.

(1) Hector’s claim asserts that generic leaves of compact laminations are quasi-isometric to Cayley graphs. From [3] Theorem 2 combined with [13] Theorem C (see also [3] Theorem 4 for transversely Cantor laminations), we know that this holds when the generic leaves have two ends. However, we will prove that the claim is not true when the generic leaves have one end. The question when they have a Cantor set of ends (which was the case that really interested Hector) remains open.

(2) Giordano, Putnam, and Skau conjectured that any minimal and free continuous action of an amenable countable group on the Cantor set is orbit equivalent to a Cantor minimal system (that is, a minimal \( \mathbb{Z} \)-action on the Cantor set). From [16] Theorems 4.8 and 4.16, this happens if and only if the orbital equivalence relation is affable – namely, the union of an increasing sequence of compact open equivalence subrelations, which turns into an AF-equivalence relation endowed with the inductive limit topology (see [16] and [24] for detailed definitions). We will exhibit examples of amenable equivalence relations on the Cantor set which are not affable, and therefore not orbit equivalent to a minimal \( \mathbb{Z} \)-action. This proves that there is no analogue to the famous Connes-Feldman-Weiss theorem [7] in the topological setting.

To address these issues, we consider an uncountable family of 3-dimensional solvable Lie groups \( Sol(a, b) \), with \( a, b > 0 \), which are not quasi-isometric to Cayley graphs unless \( a = b \). These groups have been introduced and studied by A. Eskin, D. Fisher and K. Whyte in [8] and [9]. For those groups with \( 2^{b/a} \) being an integer, we construct a repetitive and aperiodic tiling \( \hat{T} \) inspired by the Penrose construction of an aperiodic tiling of the hyperbolic plane [21]. As a byproduct, we obtain a repetitive and aperiodic tiling of the solvable group \( Sol^3 = Sol(1, 1) \). Theorem 1 states that the continuous hull of \( \hat{T} \) – that is, the closure of the set of all its translates –
is a compact metric space $M(a,b)$ endowed with a minimal free action of $\text{Sol}(a,b)$. This defines a transversely Cantor lamination $\mathcal{F}(a,b)$ on $M(a,b)$, which satisfies:

(i) the leaves of $\mathcal{F}(a,b)$ and $\mathcal{F}(a',b')$ are quasi-isometric if and only if $b/a = b'/a'$,
(ii) the leaves of $\mathcal{F}(a,b)$ are quasi-isometric to Cayley graphs if and only if $a = b$,

in which case $\text{Sol}(a,b)$ is isomorphic to the unimodular solvable group $\text{Sol}^3$.

(iii) the lamination $\mathcal{F}(a,b)$ induces an equivalence relation $\mathcal{R}(a,b)$ on a complete transversal homeomorphic to the Cantor set which is not affable if $a \neq b$.

In this foliated context, a transversely Cantor lamination is said to be affable if the equivalence relation induced on some (every) complete transversal is affable. So property (iii) can be rephrased as the lamination $\mathcal{F}(a,b)$ is not affable if $a \neq b$.

The problem of whether $\mathcal{F}(a,b)$ is affable involves studying it from an ergodic point of view, and associating to it two classes of measures which are well known in foliation dynamics: transverse invariant measures and harmonic measures. Theorem 2 asserts that harmonic measures for $\mathcal{F}(a,b)$ (having harmonic densities when they are locally desintegrated on flow boxes according to [12]) coincide with measures $\text{Sol}(a,b)$-invariant (which remain invariant when they are translated by any element of $\text{Sol}(a,b)$). Both kinds of measures provide quasi-invariant measures on transversals, only defined up to equivalence, with respect to which the equivalence relation $\mathcal{R}(a,b)$ is amenable. A harmonic measure for $\mathcal{F}(a,b)$ is said to be completely invariant when the transverse measure is $\mathcal{R}(a,b)$-invariant, that is, preserved by partial transformations whose graphs are contained in $\mathcal{R}(a,b)$. Unless $a = b$, the lamination $\mathcal{F}(a,b)$ does not admit transverse invariant measures, and this is the obstruction we use to see that $\mathcal{R}(a,b)$ is not affable.

To clarify this point, recall that Penrose’s tiling has been used in [22] to construct free minimal actions of the affine group on the Cantor set. All these actions also give a negative answer to question (2) as they do not admit transverse invariant measures according to [22, Proposition 3.1]. However, their orbits are always quasi-isometric to Cayley graphs. We extend here Petite’s remark by proving in Proposition 3 that $\mathcal{F}(a,b)$ admits a transverse invariant measure if and only if $\text{Sol}(a,b)$ is unimodular, that is, $a = b$. In fact, this result applies to (locally) free actions of Lie groups and transformational groupoids as detailed in Appendix A. Thus, any (locally) free minimal action of an amenable Lie group $G$ is not affable if $G$ is non unimodular, in contrast with the abelian case solved in [15]. However, it is still an open question if $\text{Sol}(1,1)$-solenoids are affable or not.

2. A FAMILY OF NONUNIMODULAR SOLVABLE LIE GROUPS OF DIMENSION 3

As established in [9, Theorem 1.6] (see also [8, Theorem 5.9]) and proved in [10] and [11], there are Lie groups which are not quasi-isometric to any finitely generated group. Indeed, if $a, b > 0$ with $a \neq b$, the semi-direct product $\text{Sol}(a,b) = \mathbb{R}^2 \rtimes \mathbb{R}$ defined by the $\mathbb{R}$-action

\[ z \in \mathbb{R} \mapsto \begin{pmatrix} e^{az} & 0 \\ 0 & e^{-bz} \end{pmatrix} \in GL(2, \mathbb{R}) \]

is a nonunimodular solvable group that does not admit any quasi-isometric finitely generated group. If $a = b$, then $\text{Sol}(a,b)$ is isomorphic to the unimodular Lie group $\text{Sol}^3$, which defines one of the eight Thurston geometries of closed 3-manifolds. Moreover, two Lie groups $\text{Sol}(a,b)$ and $\text{Sol}(a',b')$ are quasi-isometric if and only if $b/a = b'/a'$.

Each solvable group $\text{Sol}(a,b)$ contains two transverse fields of hyperbolic planes $H$, obtained as orbits of two transverse affine actions. In the next section, we shall use this idea to construct an aperiodic tiling of $\text{Sol}(a,b)$ in a similar way as
R. Penrose constructed an aperiodic tiling of $\mathbb{H}$ [21]. Identifying the hyperbolic plane $\mathbb{H} = \{ \alpha + \beta i \mid \beta > 0 \}$ with the semi-direct product $\mathbb{R} \rtimes \mathbb{R}_+^*$, we can consider the natural inclusion into $\mathbb{R}^2 \rtimes \mathbb{R}$ which sends $(\alpha, \beta)$ in $(\alpha, \beta, \log \beta a)$. This is an integral surface of the foliation defined by the invariant vector fields

$$X = e^{az} \frac{\partial}{\partial x} \quad \text{and} \quad Z = -\frac{\partial}{\partial z},$$

with flows

$$h^+_s(x, y, z) = (x, y, z) \cdot (s, 0, 0) = (x + e^{az} s, y, z)$$

and

$$g_t(x, y, z) = (x, y, z) \cdot (0, 0, t) = (x, y, z + t)$$

respectively. Since the Lie bracket $[X, Z] = aX$, the foliation is actually given by a locally free affine action. Equivalently both flows are related by

$$h^+_s \circ g_t = g_t \circ h^+_s. \quad (2.1)$$

The flow $h^+_s$ restricts to the horocycle flow of $\mathbb{H}$, but the geodesic flow is not obtained from the flow $g_t$ but from the reparametrized flow $g_t/a$. Indeed, the flow generated by $X$ and $Z$ in restriction to $\mathbb{H}$ are given by

$$h^+_s(\alpha + \beta i) = \alpha + \beta s + \beta i \quad \text{and} \quad g_t(\alpha + \beta i) = \alpha + e^{at} \beta i.$$

The left invariant Riemannian metric on $\text{Sol}(a, b)$ is

$$e^{-2az} dx^2 + e^{2bz} dy^2 + dz^2,$$

and then its restriction to $\mathbb{H}$ is conformally equivalent (by a homothety) to the Poincaré metric (up to the coordinate change $\gamma = \beta/a$). Thus, the inclusion of $\mathbb{H}$ into $\text{Sol}(a, b)$ sends geodesics and horocycles into orbits of $g_t$ and $h^+_s$ respectively.

The flow of the third left invariant vector field

$$Y = e^{-bz} \frac{\partial}{\partial y}$$

is given by

$$h^+_s(x, y, z) = (x, y, z) \cdot (0, s, 0) = (x, y + e^{-bz} s, z),$$

satisfying

$$h^+_s \circ g_t = g_t \circ h^+_{e^{-bt}s}. \quad (2.2)$$

This equality can be also deduced from the Lie bracket equality $[Y, Z] = -bY$.

Finally, since $[X, Y] = 0$, the flows $h^+_s$ and $h^-_s$ commute.

3. The hyperbolic Penrose tiling

According to [21], the Poincaré half-plane $\mathbb{H}$ admits a tiling $\mathcal{T}$ constructed from a single tile $P$ (see Figure 1), which is neither periodic nor aperiodic. Let us explain the meaning of both notions (see also [18]). If we consider the isometries $R$ and $S$ given by

$$R(Z) = 2Z \quad \text{and} \quad S(Z) = Z + 1$$

for every $Z = \alpha + i \beta \in \mathbb{H}$, then

$$\mathcal{T} = \{ R^i \cdot S^j(P) \mid i, j \in \mathbb{Z} \}.$$
isometries preserving $\mathcal{T}$ is not trivial. In fact, both notions can be formulated in terms of affine transformations instead of isometries, that is, we can replace the group of orientation-preserving isometries $\text{PSL}(2, \mathbb{R})$ by the affine subgroup

$$B^+ = \left\{ \begin{pmatrix} \sqrt{3} & \alpha/\sqrt{3} \\ 0 & 1/\sqrt{3} \end{pmatrix} \middle| \alpha, \beta \in \mathbb{R}, \beta > 0 \right\}$$

acting freely and transitively on $\mathbb{H}$.

But as explained in [22] (and detailed below), we can decorate the tiles $R^i(P)$ to break down this symmetry by using a repetitive sequence $\{\omega_i\}_{i \in \mathbb{Z}}$ of 0’s and 1’s (as shown in Figure 2). Then $\mathcal{T}$ becomes aperiodic, that is, $\mathcal{T}$ is not preserved by any non-trivial element of $B^+$. We actually obtain a set of decorated prototiles, which is not longer a singleton but finite, allowing to construct both aperiodic and non aperiodic tilings.

The compact metric space $\mathcal{M} \left( \mathbb{H} \right)$ made up of these hyperbolic tilings (marked with a fixed base point) is equipped with a natural right $B^+$-action where each tiling is translated by the inverse of each isometry in $B^+$, that is,

$$\mathcal{T} \cdot g = g^{-1}(\mathcal{T})$$

for each $g \in B^+$. The orbital equivalence relation $\mathcal{R}$ coincides with the natural equivalence relation that consists of moving the base point of each tiling. Recall also that two tilings in $\mathcal{M} \left( \mathbb{H} \right)$ are close if they agree on a large ball in $\mathbb{H}$ centered at the base point, up to an affine transformation close to the identity (see [14] and [22] for details).

The closure of the orbit $\mathcal{R}[\mathcal{T}]$ is a nonempty closed invariant subset of $\mathcal{M} \left( \mathbb{H} \right)$, called the continuous hull of $\mathcal{T}$, which contains a nonempty minimal subset $\mathcal{M}_0$. In fact, the tiling $\mathcal{T}$ is repetitive. This means that for each patch $\mathcal{P}$, there is a positive number $R > 0$ such that every ball in $\mathbb{H}$ of radius $R$ contains a translation copy of $\mathcal{P}$. It is a general fact (see for example [18]) for any repetitive and aperiodic tiling that the minimal set $\mathcal{M}_0$ coincides with continuous hull of $\mathcal{T}$, and that any tiling in $\mathcal{M}_0$ is also aperiodic. Then $\mathcal{M}_0$ is equipped with a minimal free affine action.
Figure 2. Penrose’s tiling decorated with a repetitive sequence.

4. Constructing an aperiodic tiling of $\text{Sol}(a,b)$

To obtain a similar tiling of $\text{Sol}(a,b)$, we will use the isometries

$$\hat{T}_s(x,y,z) = (0,s,0) \cdot (x,y,z) = (x,y+s,z)$$

to thicken the tile $P$ into a tile

$$\hat{P} = \bigcup_{s \in [0,1]} \hat{T}_s(P).$$

First, we extend the isometries $R$ and $S$ into isometries of $\text{Sol}(a,b)$ and we see their effect on $\hat{P}$. Thus, we consider the left translations

$$\hat{R}(x,y,z) = (0,0,\frac{\log 2}{a}) \cdot (x,y,z) = (2x,\Delta y,z + \frac{\log 2}{a})$$

and

$$\hat{S}(x,y,z) = (1,0,0) \cdot (x,y,z) = (x+1,y,z)$$

where

$$\Delta = e^{-\frac{\log x}{a}} = 2^{-b/a}. \quad (4.1)$$

They extend $R$ and $S$ since

$$\hat{R}(\alpha,0,\frac{\log \beta}{a}) = (2\alpha,0,\frac{\log 2\beta}{a}) \quad \text{and} \quad \hat{S}(\alpha,0,\frac{\log \beta}{a}) = (\alpha+1,0,\frac{\log \beta}{a})$$

for each $\alpha+i\beta \in H$. Now, we have

$$\hat{T}_s \cdot \hat{R}(x,y,z) = \hat{T}_s(2x,\Delta y,z + \frac{\log 2}{a}) = (2x,\Delta y+s,z + \frac{\log 2}{a})$$

$$= \hat{R}(x,y+\Delta^{-1}s,z)$$

$$= \hat{R} \cdot \hat{T}_{\Delta^{-1}}(x,y,z)$$

for any $(x,y,z) \in \text{Sol}(a,b)$. Therefore, we obtain:

$$\hat{T}_s \cdot \hat{R}^i = \hat{R}^i \cdot \hat{T}_{s \Delta^{-i}} \quad (4.2)$$

for any $i \in \mathbb{Z}$. Similarly, we have

$$\hat{T}_s \cdot \hat{S}(x,y,z) = \hat{T}_s(x+1,y,z) = (x+1,y+s,z)$$

$$= \hat{S}(x,y+s,z) = \hat{S} \cdot \hat{T}_s(x,y,z)$$
for any \((x, y, z) \in \text{Sol}(a, b)\), and therefore

\[ \hat{T}_s \hat{S}^j = \hat{S}^j \hat{T}_s \]

(4.3)

for any \(m \in \mathbb{Z}\). It follows that:

**Proposition 1.** The family

\[ \hat{T} = \{ \hat{R}^i \hat{S}^j \hat{T}_k(\hat{P}) \mid i, j, k \in \mathbb{Z} \} \]

is a tiling of \(\text{Sol}(a, b)\), by which we mean that \(\text{Sol}(a, b)\) is the union of the tiles \(\hat{R}^i \hat{S}^j \hat{T}_k(\hat{P})\) and that the intersection of any two tiles has empty interior.

**Proof.** From (4.2) and (4.3), we deduce that

\[ \hat{R}^i(\hat{T}_k(\hat{P})) = \bigcup_{s \in [k, k+1]} \hat{R}^i \hat{T}_s(P) = \bigcup_{s \in [k \Delta', (k+1)\Delta']} \hat{T}_s(\hat{R}^i(P)) \]

and

\[ \hat{S}^j(\hat{T}_k(\hat{P})) = \bigcup_{s \in [k, k+1]} \hat{S}^j \hat{T}_s(P) = \bigcup_{s \in [k, k+1]} \hat{T}_s(\hat{S}^j(P)) \]

for \(i, j \in \mathbb{Z}\). The real line \(\mathbb{R}\) is covered by the intervals \([k, k+1]\) without overlaps, and similarly by the intervals \([k \Delta', (k+1)\Delta']\) if we apply a homothety of (fixed) ratio \(\Delta'\). Therefore, since \(\mathcal{T} = \{ \hat{R}^i \hat{S}^j(P) \mid i, j \in \mathbb{Z} \}\) is a tiling of \(\mathbb{R}\) without gaps or overlaps, the family \(\hat{T} = \{ \hat{R}^i \hat{S}^j \hat{T}_k(\hat{P}) \mid i, j, k \in \mathbb{Z} \}\) also covers \(\text{Sol}(a, b)\) without gaps or overlaps. \(\square\)

Similarly to \(\mathcal{T}\), the tiling \(\hat{T}\) is neither periodic, nor aperiodic. It cannot be periodic because in that case \(\mathcal{T}\) would be also periodic. In fact, if \(a \neq b\), this property can be deduced from the non-unimodularity of \(\text{Sol}(a, b)\). On the other hand, \(\mathcal{T}\) remains invariant by \(\hat{R}\). In fact, accordingly to (4.2), the group of isometries preserving \(\mathcal{T}\) is reduced to the subgroup of \(\text{Sol}(a, b)\) generated by \(\hat{R}\).

Moreover, in the previous construction, the tiles of \(\hat{T}\) do not meet face-to-face in the \(y\) direction. We will impose an additional condition, the condition that \(\Delta^{-1} \in \mathbb{Z}^+\).

This is a condition on \(b/a\), allowing us to choose countably many values of it.

**Proposition 2.** If \(b/a = \log n / \log 2\) for some integer \(n \geq 2\), then the tiling \(\hat{T}\) is face-to-face and repetitive. Moreover, the prototile \(\hat{P}\) admits a finite number of decorations such that \(\hat{T}\) also becomes aperiodic.

**Proof.** Denote by \(\tau\) the top curve of the prototile \(P\) and set \(\hat{\tau} = \bigcup_{s \in [0,1]} \hat{T}_s(\tau)\). If \(\Delta^{-1}\) is a positive integer \(n \geq 2\), we can divide the face \(\hat{\tau}\) into \(n\) equal faces

\[ \bigcup_{s \in \left[0, \frac{n-1}{\Delta}, \frac{1}{\Delta} \right]} \hat{T}_s(\tau) \]

for \(l \in \{1, 2, \ldots, n\}\) as depicted in Figure \(3\). In this way, the tiles of \(\hat{T}\) meet face to face.

Moreover, if we denote by \(\omega = \{\omega_i\}_{i \in \mathbb{Z}}\) the bilateral Morse sequence \([1]\) or an Oxtoby sequence \([20]\), we can use the terms \(\omega_i\) to decorate the tiles of \(\hat{T}\) as we decorated \(\mathcal{T}\) in Figure \([2]\). In both cases, two colors are enough to break down the
initial symmetry of $\mathcal{T}$ and $\hat{\mathcal{T}}$, although the continuous hulls $\mathcal{M}(a, b, \omega)$ will have different ergodic properties depending on the sequence used (see [22] for details in the case of $\mathcal{T}$). Finally, since both $\omega$ and $\hat{\mathcal{T}}$ are repetitive, so the decorated tiling $\hat{\mathcal{T}}(\omega)$ is.

Remark 1. As explained before, the groups $\text{Sol}(a, b)$ considered by Eskin, Fisher and White verify $a, b > 0$. If $a > 0$ and $b < 0$, the groups $\text{Sol}(a, b)$ are examples of Heintze groups having negative curvature. In this case, the fact that $\text{Sol}(a, b)$ is not quasi-isometric to a Cayley graph was previously proved by B. Kleiner as pointed in [8] (see also [9] and [10]). Now, Proposition 2 remains valid if $\Delta \in \mathbb{Z}^+$, or equivalently if

$$-\frac{b}{a} = \frac{\log n}{\log 2}$$

for some integer $n \geq 2$.

5. A $\text{Sol}(a, b)$-solenoid

Consider the space of tilings of $\text{Sol}(a, b)$ constructed from the decorated prototiles constructed in Proposition 2. It admits an action by left translations of $\text{Sol}(a, b)$, which is of course an action by isometries for any left-invariant metric. Two tilings of $\text{Sol}(a, b)$ remain close if they agree on a large ball in $\text{Sol}(a, b)$ centered at the identity element, up to an isometry close to the identity map. The orbital equivalence relation still coincides with the natural equivalence relation that consists of translating the base point of each tiling. Restricted to the continuous hull $\mathcal{M}(a, b, \omega)$ of the decorated tiling $\hat{\mathcal{T}}(\omega)$, the action of $\text{Sol}(a, b)$ is free and minimal. Finally, as $\hat{\mathcal{T}}(\omega)$ satisfies the finiteness condition of [18] Theorem 2.2, $\mathcal{M}(a, b, \omega)$ is compact. Therefore, we get the result announced in the introduction:

Theorem 1. If $\pm b/a = \log n / \log 2$ for some integer $n \geq 2$, the convex hull $\mathcal{M}(a, b, \omega)$ of $\hat{\mathcal{T}}(\omega)$ is a nonempty compact metric space that has a free minimal action of $\text{Sol}(a, b)$. With the foliation given by the orbits, it has the structure of a transversely Cantor lamination.
Notice that $\mathcal{M}(a,b,\omega)$ is a $\text{Sol}(a,b)$-solenoid in the sense of [2] since tilings are translated by isometries (or equivalently the right $\text{Sol}(a,b)$-action on $\mathcal{M}(a,b,\omega)$ is derived from the natural left action of $\text{Sol}(a,b)$ on itself) although other free minimal actions of $\text{Sol}(a,b)$ on compact spaces can be constructed using similar ideas.

Since $\text{Sol}(a,b)$ is amenable, there is always a probability measure $\mu$ on $\mathcal{M}(a,b,\omega)$ which is invariant under the action of $\text{Sol}(a,b)$. If we use the Morse sequence to construct the tiling $T(\omega)$, this action is uniquely ergodic, whereas $\mathcal{M}(a,b,\omega)$ admits many ergodic invariant measures when we use an Oxtoby sequence $\omega$ to decorate the tiles of $T$ similarly to [22].

If $a = b$, the Lie group $\text{Sol}(a,b)$ is unimodular, isomorphic to $\text{Sol}^1$, and then $\mu$ is completely invariant (see the proof of Theorem 5.2 of [3]). In fact, using the natural foliated structure of $\mathcal{M}(a,b,\omega)$, we can directly prove the following result which is the analogue of [22, Proposition 3.1] in our context. A more general version valid for any lamination defined by a locally free action is given in Appendix A.

**Proposition 3.** The space of tiling $\mathcal{M}(a,b,\omega)$ admits a completely invariant measure $\mu$ if and only if $a = b$.

**Proof.** Let $\mu$ be a probability measure on $\mathcal{M}(a,b,\omega)$ which is invariant by the right $\text{Sol}(a,b)$-action. The space $\mathcal{M}(a,b,\omega)$ is covered by a finite number of flow boxes $U \cong D \times T$ where $T$ is a clopen subset of the canonical transversal (obtained by fixing base points in the prototiles and homeomorphic to the Cantor set). In restriction to each flow box $U$, the measure $\mu$ disintegrates into a family of probability measures $\mu_t$ on the plaques $D \times \{t\}$ with respect to the push-forward measure $\nu$ on $T$, that is,

$$d\mu(p,t) = d\mu_t(p) d\nu(t)$$

for every $(p,t) \in U \cong D \times T$. Moreover, since $\mu$ is invariant under the right $\text{Sol}(a,b)$-action, for $\nu$-almost $t \in T$, $\mu_t$ is also invariant under the right $\text{Sol}(a,b)$-action, so $\mu_t$ is the restriction of the right Haar measure of $\text{Sol}(a,b)$. In fact, in our case, plaques are simply tiles and changes of coordinates are obtained from transformations $R^a \times S^m \times T_k$ used in the construction of $T$ in Proposition 1. In other words, the changes of coordinates are left translations in the group $\text{Sol}(a,b)$, and hence the measure $\mu$ is completely invariant if and only the right Haar measure on $\text{Sol}(a,b)$ is left invariant. The left Haar measure on $\text{Sol}(a,b)$ is given by

$$-e^{(b-a)}z dx \wedge dy \wedge dz$$

while the right Haar measure is given by $-dx \wedge dy \wedge dz$. So $\text{Sol}(a,b)$ is unimodular if and only if $a = b$. \hfill \Box

More generally, any measure $\text{Sol}(a,b)$-invariant can be interpreted as a harmonic measure, as in [22, Lemma 4.2]:

**Proposition 4.** Any $\text{Sol}(a,b)$-invariant probability measure $\mu$ on $\mathcal{M}(a,b,\omega)$ is harmonic.

**Proof.** Let $\mu$ be a probability measure on $\mathcal{M}(a,b,\omega)$ which is invariant under the right $\text{Sol}(a,b)$-action. As in the proof of Proposition 3 in restriction to each flow $U \cong D \times T$, the measure $\mu$ disintegrates into a family of probability measures $\mu_t$ on the plaques $D \times \{t\}$ with respect to the push-forward measure $\nu$ on subset $T$. Moreover, for $\nu$-almost $t \in T$, the measure $\mu_t$ is induced by the right Haar measure of $\text{Sol}(a,b)$. But this measure

$$-dx \wedge dy \wedge dz$$
is absolutely continuous with respect to the Riemannian volume
\[-e^{(b-a)z}dx \wedge dy \wedge dz\]
with harmonic density \(e^{(a-b)z}\). Then \(\mu\) is harmonic.

In fact, the study of brownian motion and harmonic functions on \(\text{Sol}(a, b)\) by S. Brofferio, M. Salvatori and W. Woess \cite{4} allows us to generalize \cite[Theorem 1.1]{22} to our context:

**Theorem 2.** A probability measure \(\mu\) on \(\mathcal{M}(a, b, \omega)\) is harmonic if and only if it is \(\text{Sol}(a, b)\)-invariant.

**Proof.** Assume \(\mu\) is a harmonic measure on \(\mathcal{M}(a, b, \omega)\). By \cite[Theorem 1]{12}, in restriction to each flow box \(U \cong D \times T\), the measure \(\mu\) disintegrates again into a family of measures \(\mu_t\) on the plaques \(D \times \{t\}\) with respect to the push-forward measure \(\nu\) on \(T\) where each measure \(\mu_t\) is absolutely continuous with respect to the Riemannian volume \(d\text{vol}\) having harmonic density \(h(-, t)\). More precisely, for any positive continuous function \(f : \mathcal{M}(a, b, \omega) \to \mathbb{R}\) with support contained in \(U\), we have:

\[
\int_M f d\mu = \int_T \int_D f(x, y, z, t)h(x, y, z, t)d\text{vol}(x, y, z) d\nu(t)
\]

where \(d\text{vol}(x, y, z) = -e^{(b-a)z}dx \wedge dy \wedge dz\) is invariant by left translations. Thus, for each point \(t \in T\), the harmonic density \(h(x, y, z, t)\) defined on the plaque \(D\) extends to a positive harmonic function \(h(-, t)\) defined on the whole Lie group \(\text{Sol}(a, b)\). If \(R_g : \mathcal{M}(a, b, \omega) \to \mathcal{M}(a, b, \omega)\) is the right translation by an element \(g = (\alpha, \beta, \gamma)\) of \(\text{Sol}(a, b)\), then \(f \circ R_{g^{-1}} : \mathcal{M}(a, b, \omega) \to \mathcal{M}(a, b, \omega)\) is a positive continuous function whose support in contained in \(U.g = R_g(U)\). Assuming that the support of \(f\) is contained in both flow boxes \(U\) and \(U.g\), we have:

\[
\int_M f d(R_{g^{-1}})_* \mu = \int_T \int_D f((x, y, z), g^{-1}, t)h((x, y, z), t)d\text{vol}(x, y, z) d\nu(t)
\]

\[
= \int_T \int_D f(x, y, z, t)h((x, y, z), g, t) \frac{d\text{vol}(x, y, z)}{e^{(a-b)\gamma}} d\nu(t).
\]

Our aim is to prove that this integral does not depend on \(g\), which enables us (by decomposing the support of \(f \circ R_{g^{-1}}\) into smaller pieces contained in the flow boxes \(U\)) to conclude that \((R_g)_* \mu = \mu\). Indeed, as \(h(x, y, z, t)\) is harmonic, the map

\[
g = (\alpha, \beta, \gamma) \in \text{Sol}(a, b) \mapsto \frac{h(x + e^{az}\alpha, y + e^{bz}\beta, z + \gamma)}{e^{(a-b)\gamma}}
\]

is also harmonic and therefore the bounded map which sends \(g = (\alpha, \beta, \gamma) \in \text{Sol}(a, b)\) onto the integral (5.1) can be written as

\[
g = (\alpha, \beta, \gamma) \in \text{Sol}(a, b) \mapsto \frac{H(\alpha, \beta, \gamma)}{e^{(a-b)\gamma}}
\]

where \(H\) is a positive harmonic function on \(\text{Sol}(a, b)\). But according to \cite[Corollary 6.3]{4}, such a harmonic function decomposes as

\[
H(x, y, z) = H_1(x, z) + H_2(y, -z)
\]

where \(H_1\) is a harmonic function on the hyperbolic plane \(\mathbb{H}(a)\) with respect to the Riemannian metric \(ds^2 = e^{-2az}dx^2 + dz^2\) and the Laplace-Beltrami operator \(\Delta_1 = e^{2az} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + (b - a) \frac{\partial}{\partial z}\) and \(H_2\) is the harmonic function on the hyperbolic plane \(\mathbb{H}(b)\) with respect to the Riemannian metric \(ds^2 = e^{2bz}dy^2 + dz^2\) and the Laplace-Beltrami operator \(\Delta_2 = e^{2bz} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + (a - b) \frac{\partial}{\partial z}\). By applying \cite[Lemma 4.1]{22} in the case of \(\mathbb{H}(a)\) and \(\mathbb{H}(b)\), we deduce that \(H_1(x, z)/e^{(a-b)z}\) and
$H_2(y, z)/e^{(a-b)z}$ are constant. We deduce that $H(x, y, z)/e^{(a-b)z}$ is constant and then the integral \[5.1\] does not depend on $g$. □

Consequently, if $a \neq b$, the amenable equivalence relation $\mathcal{R}(a, b, \omega)$ induced on the canonical transversal we described above cannot be affable, proving Property (iii). Indeed, as $\text{Sol}(a, b)$ is solvable, the lamination $\mathcal{F}(a, b, \omega)$ is amenable with respect to $\mu$. Therefore, the equivalence relation $\mathcal{R}(a, b, \omega)$ induced on the canonical transversal is amenable with respect to the quasi-invariant measure class $[\nu]$. On the other hand, by definition, the lamination $\mathcal{F}(a, b, \omega)$ is affable if and only if the equivalence relation $\mathcal{R}(a, b, \omega)$ is affable, given as the union of an increasing sequence of compact open equivalence subrelations. According to \[16\], any AF-equivalence relation is orbit equivalent to a Cantor minimal $\mathbb{Z}$-system. But such a minimal system always admits an invariant measure defining a transverse invariant measure on $\mathcal{M}(a, b, \omega)$. By Proposition \[3\] this only happens when $a = b$. Note however that $\mathcal{R}(a, b, \omega)$ is not Kakutani-equivalent to any free action of a finitely generated group as the orbits of the $\text{Sol}(a, b)$-action are not quasi-isometric to Cayley graphs (according to Property (ii) deduced from \[10\]). Finally, Property (i) follows from the quasi-isometric classification of solvable groups $\text{Sol}(a, b)$, also proved in \[10\].

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APPENDIX A. TRANSVERSE INVARIANT MEASURES FOR LOCALLY FREE ACTIONS

Let $M$ be a compact metric space endowed with a lamination $F$ defined by a right locally free action $\varphi : M \times G \to M$ of a Lie group $G$. The leaf passing through $x \in M$ is the orbit $x.G = \{x.g | g \in G\}$ identified with the homogeneous manifold $G_x/G$ where $G_x = \{g \in G | x.g = x\}$. Given any left invariant Riemannian metric on $G$, the space $M$ can be covered by a finite number of flow boxes $U_i = T_i.B$ which are obtained by translating finitely many transversals $T_i \subset M$ by the elements of some ball $B$ in $G$ centered at the identity element. In particular, holonomy transformations are defined by the right action of $G$ on transversals. The change of coordinates from $U_i$ to $U_j$ is given by left translations in $G$, that is, by local isometries between the plaques of $U_i$ and $U_j$.

The leaves of $F$ are then naturally endowed with the Riemannian volume of $G$, which corresponds to the left Haar measure $m_\ell$. The modular function $\lambda : G \to \mathbb{R}_+$ is given by

$$\int_G f(gg_0) dm_\ell(g) = \lambda(g_0) \int_G f(g) dm_\ell(g)$$

for any positive measurable function $f : G \to \mathbb{R}$. In other words, the measure $m_\ell$ is right invariant (and therefore $G$ is unimodular) if and only if $\lambda = 1$.

**Theorem 3.** Let $(M, F)$ be a compact lamination defined by a locally free action of a Lie group $G$. If $F$ admits a transverse invariant measure, then $G$ is unimodular.

**Proof.** Let $\nu$ be a transverse invariant measure, and let $\mu$ be the completely invariant measure on $M$ obtained by integrating the left Haar measure $m_\ell$ with respect to $\nu$. If $U = T.B$ is a flow box and $f : M \to \mathbb{R}$ is a positive continuous function with support contained in $U$, then

$$\int_M f \, d\mu = \int_T \int_B f(x,g) \, dm_\ell(g) \, d\nu(x).$$

If $R_{g_0} : M \to M$ is the right translation by an element $g_0 \in G$, then $f \circ R_{g_0} : M \to \mathbb{R}$ is a positive continuous function whose support is contained in the flow box $U.g_0^{-1} = T.(B.g_0^{-1})$. Then

$$\int_M f \circ R_{g_0} \, d\mu = \int_T \int_{B.g_0^{-1}} f \circ R_{g_0}(x,g) \, dm_\ell(g) \, d\nu(x)$$

$$= \int_T \int_{B.g_0^{-1}} f(x, gg_0) \, dm_\ell(g) \, d\nu(x)$$

$$= \int_T \lambda(g_0) \int_B f(x,g) \, dm_\ell(g) \, d\nu(x) = \lambda(g_0) \int_M f \, d\mu.$$


and hence
\[(R_{g_0})_*\mu = \lambda(g_0)\mu.\]
Since both \(\mu\) and \((R_{g_0})_*\mu\) are probability measures, this implies that \(\lambda(g_0) = 1\).
Therefore, \(G\) is unimodular.

As pointed out in the introduction, this result also fits the theory of measured groupoids. Indeed, as shown by A. Connes in [6], see also [23], transverse measures are essentially the same as quasi-invariant measures in Mackey’s theory of virtual groups. In particular, an invariant transverse measure is a quasi-invariant measure of module 1. The terminology for such a measure in the foliated context is transverse holonomy-invariant measure, usually shorten as transverse invariant measure. In the case of the transformational groupoid \(M\rtimes G\), a quasi-invariant measure in the sense of Mackey is exactly a measure \(\mu\) on \(M\) which is quasi-invariant under the action of \(G\). The same computation we just did in the proof of Theorem 3 (see also [6, Corollaire I.7] and [23, Section I.3.21]) shows that the module \(\Delta\) of \(\mu\) satisfies
\[\Delta(x, g)D(x, g)\lambda(g) = 1\]
where \(\lambda\) is the modular function of \(G\) defined above (inverse of the usual definition) and \(D\) is the Radon-Nikodym cocycle defined as
\[(R_{g^{-1}})_*\mu = D(\cdot, g)\mu.\]
If \(\mu\) is a probability measure of module \(\Delta = 1\), this implies \(\lambda = 1\). Thus, if \(G\) is not unimodular, there are no finite invariant transverse measures. The authors would like to thank the referee for this remark, which is reproduced almost verbatim, and for a careful reading of the manuscript.