BIFURCATION OF LIMIT CYCLES FOR A FAMILY OF PERTURBED KUKLES DIFFERENTIAL SYSTEMS

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Abstract. We consider an integrable non-Hamiltonian system, which belongs to the quadratic Kukles differential systems. It has a center surrounded by a bounded period annulus. We study polynomial perturbations of such a Kukles system inside the Kukles family. We apply averaging theory to study the limit cycles that bifurcate from the period annulus and from the center of the unperturbed system. First, we show that the periodic orbits of the period annulus can be parametrized explicitly through the Lambert function. Later, we prove that at most one limit cycle bifurcates from the period annulus, under quadratic perturbations. Moreover, we give conditions for the non-existence, existence, and stability of the bifurcated limit cycles. Finally, by using averaging theory of seventh order, we prove that there are cubic systems, close to the unperturbed system, with 1 and 2 small limit cycles.

1. Introduction. In this paper we will focus on the limit cycles that can appear under polynomial perturbations of the following real quadratic system

\[
\begin{align*}
\dot{x} &= -y, \\
\dot{y} &= x + x^2 + y^2.
\end{align*}
\]  

(1)

This system has two finite critical points: \((0,0)\) and \((-1,0)\). It is easy to see that the first one is a weak focus and the second is a saddle. Since the system is invariant under the transformation \((x,y,t) \rightarrow (x,-y,-t)\), \((0,0)\) is a center and the period annulus \(P\) surrounding the origin is bounded by a homoclinic loop \(\Gamma\) that joins the stable and the unstable manifolds of \((-1,0)\). Moreover, a simple computation shows that the analytic function

\[ H(x,y) = (x^2 + y^2) e^{2x}. \]

is a first integral of (1) with integrating factor \(2e^{2x}\). Thus, (1) is an integrable non-Hamiltonian reversible system, which belongs to the family of Kukles systems. We recall that a Kukles system is a planar differential system of the form

\[
\begin{align*}
\dot{x} &= -y, \\
\dot{y} &= Q(x,y),
\end{align*}
\]

2010 Mathematics Subject Classification. Primary: 34C07, 34C29, 37G15; Secondary: 34C25.

Key words and phrases. Limit cycle, Kukles system, averaging theory, quadratic system, periodic orbit.

The authors are supported by Universidad de Bío-Bío grant DIUBB 167208 2/R.

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where $Q(x, y)$ is a real polynomial of degree at least two and without $y$ as a divisor.

The quadratic Kukles system (1) appears in [11] as a particular integrable system, which has a transversal to infinity invariant quadratic algebraic curve.

In this paper we will consider polynomial perturbations of system (1) inside the Kukles family, that is, systems of the form
\[
\begin{align*}
\dot{x} &= -y, \\
\dot{y} &= x + x^2 + y^2 + \varepsilon Q(x, y, \varepsilon),
\end{align*}
\]  
(2)

where $Q$ is a polynomial in the variables $x$ and $y$, whose coefficients are analytic in the parameter $\varepsilon$ which we will assume real and small enough. Thus (2) with $\varepsilon = 0$ is the original system (1).

We ask about the maximum number of small and medium amplitude limit cycles of system (2) (with $\varepsilon \neq 0$), that is, the maximum number of limit cycles of (2) that bifurcate from the origin and from the period annulus $P$ of (1), respectively.

The motivation in the Kukles family is because it is one of most important families related to the Hilbert 16th problem [4]. Moreover, some classes of Kukles systems appear in applied sciences: For example, they are used as predator-prey models [5]. Bifurcation of limit cycles in Kukles systems have been tackled by several authors and by using different approaches. See for example [1, 6, 12, 13, 15].

The Kukles systems studied in [1, 6, 13] are of arbitrary degree. However, they come essentially from polynomial perturbations of the harmonic oscillator. The paper [15] considers cubic perturbed Kukles systems whose unperturbed system is also cubic. The aim of this work is to apply averaging methods to find lower and upper estimations for the number of small and medium limit cycles of (2).

Concerning the medium limit cycles of (2) our main result is the following.

**Theorem 1.1.** The perturbed quadratic Kukles system
\[
\begin{align*}
\dot{x} &= -y, \\
\dot{y} &= x + x^2 + y^2 + \varepsilon(a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2),
\end{align*}
\]
where $a_i \in \mathbb{R}$, for $i = 0, 1, \ldots, 5$, has at most one medium limit cycle $\gamma_{\varepsilon}$. Moreover,

a) if $a_4 > a_2 > 0$, then $\gamma_{\varepsilon}$, if it exists, is asymptotically stable;

b) if $a_4 < a_2 < 0$, then $\gamma_{\varepsilon}$, if it exists, is asymptotically unstable;

c) if $0 < a_2$ and $a_4 < a_2$, then the system does not have medium limit cycles;

d) if $a_2 < 0$ and $a_4 < a_2$, then the system does not have medium limit cycles.

Regarding the research of small limit cycles of system (2), we will consider special cubic perturbations of sixth order in $\varepsilon$ of (1). Our result is the following.

**Theorem 1.2.** Consider the cubic perturbed Kukles system
\[
\begin{align*}
\dot{x} &= -y, \\
\dot{y} &= x + x^2 + y^2 + \varepsilon Q_1(x, y) + \varepsilon^2 Q_2(x, y) + \cdots + \varepsilon^6 Q_6(x, y),
\end{align*}
\]
where $Q_j(x, y)$, with $j = 1, \ldots, 6$, is a polynomial of degree three vanishing at $(0, 0)$.

The maximum number of small limit cycles of this system, by using the averaging theory of order

a) one and two, is 0;

b) three and four, is 1;

c) five, six, and seven, is 2.

Moreover, we characterize the stability of the bifurcated limit cycles in terms of the coefficients of the perturbation and we show that there are suitable choices of
$Q_1(x,y),\ldots,Q_6(x,y)$ such that the perturbed system has either one or two hyperbolic small limit cycles.

As far as we know, this is the first work where the theory of averaging up to seventh order is used to get limit cycles in the problem under study.

The organization of the paper is as follows. Section 2 is devoted to recall the main tools of the averaging theory that we will use along the paper. The transformation of system (2), in order to apply the averaging methods, as well as the parametrization of the solutions of the unperturbed system are developed in Section 3. Moreover, in such a section we will give some elementary but essential results for proving our first result. In Section 4 we will provide the proof of Theorem 1.1. Finally, in Section 5 we will prove Theorem 1.2.

2. Averaging theory. We consider the perturbed differential equation

$$\dot{x} = F_0(t,x) + \varepsilon F_1(t,x) + \ldots + \varepsilon^k F_k(t,x) + \varepsilon^{k+1} R(t,x,\varepsilon),$$

with $\varepsilon \in (-\varepsilon_0,\varepsilon_0)$ and $\varepsilon_0 > 0$ small enough. The functions $F_0,\ldots,F_k : \mathbb{R} \times I \to \mathbb{R}$ and $R : \mathbb{R} \times I \times (-\varepsilon_0,\varepsilon_0) \to \mathbb{R}$ are $C^1$ functions and $T$-periodic in the variable $t$, where $I$ is an open subset of $\mathbb{R}$.

To analyze the periodic orbits of (3) that bifurcate from the unperturbed differential equation

$$\dot{x} = F_0(t,x),$$

we will consider two different cases: $F_0(t,x) \neq 0$ and $F_0(t,x) \equiv 0$.

First, we assume that $F_0(t,x) \neq 0$ and that the unperturbed differential equation (4) has the following property: There exists a subset $J \subset I$ such that for each $p \in J$ its solution $\varphi(t,p)$ with $\varphi(0,p) = p$ is $T$-periodic in the variable $t$.

The variational equation of (4) along the solution $\varphi(t,p)$ is the $T$-periodic linear differential equation

$$\dot{y}(t) = D_x F_0(t,\varphi(t,p)) y.$$

Let $Y(t,p)$ be a solution of (5) such that $Y(0,p) = 1$. Then, the first order averaged function is defined as

$$\mathcal{F}_1(p) = \int_0^T Y^{-1}(t,p) F_1(t,\varphi(t,p)) \, dt.$$

The following result is well-known from the averaging theory.

**Theorem 2.1.** Suppose $\mathcal{F}_1(p) \neq 0$. If $p_0 \in J$ satisfies $\mathcal{F}_1(p_0) = 0$ and $\mathcal{F}_1'(p_0) \neq 0$, then for sufficiently small $|\varepsilon| > 0$ there exists a unique $T$-periodic solution $x(\cdot,\varepsilon)$ of (3) such that $x(0,\varepsilon) \to p_0$ as $\varepsilon \to 0$.

There are several versions of this result, in particular for higher dimensions. See for instance [7, 8, 9]. Here, we have stated a simplified version according to the purposes of this work. This version will be used in the proof of Theorem 1.1.

To prove Theorem 1.2 we will use averaging theory of higher order under the condition $F_0(t,x) \equiv 0$. In such a case, the solution $\varphi(t,p)$ of (4) with $\varphi(0,p) = p$ is periodic, since $\varphi(t,p) = p$ for all $p \in J = I$. The following definition and result come from [3, 7, 8] and they are adapted to the one-dimensional differential equation (3).

For $i = 2,3,\ldots,k$, it is defined the $i$-th order averaged function as

$$\mathcal{F}_i : J \to \mathbb{R}$$

$$p \mapsto y_i(T,p)$$

(6)
where \( y_i : \mathbb{R} \times J \to \mathbb{R} \), for \( i = 1, \ldots, k - 1 \), are defined recurrently by the integral equation:

\[
y_i(t, p) = \int_0^t \left( F_i(s, \varphi(s, p)) + \sum_{l=1}^i \sum_{s_l} \partial^L F_{i-l}(s, \varphi(s, p)) \prod_{j=1}^{i-l} y_j(s, p)^{b_j} \right) ds,
\]

where \( \partial^L F(s, u) \) denotes the derivative of order \( L \) of \( F \) with respect to the variable \( u \), \( S_l \) is the set of all \( l \)-tuples of non-negative integers \((b_1, b_2, \ldots, b_l)\) that satisfy \( b_1 + 2b_2 + \cdots + lb_l = l \), and \( L = b_1 + b_2 + \cdots + b_l \).

**Theorem 2.2.** Suppose that \( F_i(p) \equiv 0 \), for \( i = 1, 2, \ldots, \nu - 1 \), and \( F_\nu(p) \neq 0 \) with \( \nu \in \{2, 3, \ldots, k\} \). If \( p_0 \) satisfies \( F_\nu(p_0) = 0 \) and \( F_\nu'(p_0) \neq 0 \), then for sufficiently small \( |\varepsilon| > 0 \) there exists a unique \( T \)-periodic solution \( x(\cdot, \varepsilon) \) of (3) such that \( x(0, \varepsilon) \to p_0 \) as \( \varepsilon \to 0 \).

Moreover, for the one-dimensional differential equation (3) it is well-known the following result.

**Theorem 2.3.** Suppose that \( F_\nu(p) \), with \( \nu \in \{1, 2, \ldots, k\} \), is the first non-vanishing averaged function associated with (3). If \( p_0 \in J \) is a zero of \( F_\nu(p_0) \) with \( F_\nu'(p_0) < 0 \) (\( F_\nu'(p_0) > 0 \)), then for sufficiently small \( |\varepsilon| > 0 \) there exists a stable (unstable) \( T \)-periodic solution \( x(\cdot, \varepsilon) \) of (3) such that \( x(0, \varepsilon) \to p_0 \) as \( \varepsilon \to 0 \).

### 3. Preliminary results.

The period annulus \( P \) of (1) is contained in the region bounded by the unit circle with center at the origin. Indeed, we have

\[
\Gamma \subset \{(x, y) \in \mathbb{R}^2 \mid (x^2 + y^2)e^{2x} = H(-1, 0) = e^{-2} \},
\]

then the intersection points of \( \Gamma \) with the \( x \)-axis are \((-1, 0)\) and \((x_0, 0)\), where \( x_0 \) is positive and satisfies the equation \( x^2e^{2x} = 1/e^2 \). Thus, if \((x, y) \in \Gamma\), then \( -1 \leq x \leq x_0 \) and

\[
x^2 + y^2 = e^{-(2+2x)}.
\]

As \( e^{-(2+2x)} \) is a strictly decreasing function we have \( e^{-(2+2x)} \leq 1 \) for \(-1 \leq x \leq x_0\). Thus, \( x^2 + y^2 \leq 1 \). This implies that \( \Gamma \) is contained in the unitary disc \( \mathbb{D}^1 \) with center at the origin. Hence \( \mathbb{P} \subseteq \mathbb{D}^1 \).

By using polar coordinates

\[
x = r \cos \theta, \quad y = r \sin \theta,
\]

the system (2) becomes

\[
\dot{r} = r^2 \sin \theta + \varepsilon \sin \theta Q(r \cos \theta, r \sin \theta, \varepsilon),
\]

\[
\dot{\theta} = 1 + r \cos \theta + \varepsilon \cos \theta Q(r \cos \theta, r \sin \theta, \varepsilon)/r.
\]

Moreover we know that

\[
Q(x, y, \varepsilon) = Q_0(x, y) + \varepsilon Q_1(x, y) + O(\varepsilon^2).
\]

Then, by changing the independent variable \( t \) of system (8) by the variable \( \theta \), we obtain the equivalent one-dimensional differential equation

\[
r' = F_0(\theta, r) + \varepsilon F_1(\theta, r) + O(\varepsilon^2),
\]

where prime denotes differentiation with respect to the variable \( \theta \),

\[
F_0(\theta, r) = \frac{r^2 \sin \theta}{1 + r \cos \theta}, \quad F_1(\theta, r) = \frac{Q_0(\theta, r) \sin \theta}{(1 + r \cos \theta)^2},
\]
and $Q_0(\theta, r) := Q_0(r \cos \theta, r \sin \theta)$. Thus, each medium limit cycle of (2) corresponds to an isolated periodic orbit of (9) that bifurcates from the unperturbed differential equation $r' = F_0(\theta, r)$.

The first integral for (9) with $\varepsilon = 0$ is

$$H(\theta, r) = r^2 e^{2r \cos \theta}.$$  

Hence, the level curve of $H$ through $(0, r_0)$ contains the solution $r(\theta; r_0)$ of (9), with $\varepsilon = 0$ and initial condition $r(0) = r_0$, that is,

$$(r(\theta; r_0))^2 e^{2r(\theta; r_0) \cos \theta} = (r_0)^2 e^{2r_0}$$  

for all $\theta \in [0, 2\pi]$, and since $H(\theta, r) \geq 0$,

$$r(\theta; r_0)e^{r(\theta; r_0) \cos \theta} = r_0 e^{r_0}, \quad (10)$$  

for all $\theta \in [0, 2\pi]$. In particular the homoclinic loop $\Gamma$ corresponds to the level curve

$$r(\theta; r_0)e^{r(\theta; r_0) \cos \theta} = \frac{1}{e}. \quad (11)$$  

We now multiply (10) by $\cos \theta$ to get

$$Xe^X = r_0 e^{r_0} \cos \theta, \quad (11)$$  

where $X = r(\theta; r_0) \cos \theta$. Moreover, as we are interested in the orbits of the unperturbed system living in $\mathcal{P} \subseteq \mathbb{D}^1$, then we can assume $-1 < X < 1$.

A simple computation shows that the real function $se^s$ is strictly increasing for $s > -1$, therefore for $s > -1$ there exists the inverse: if $se^s = u$ with $s > -1$, then $s = W(u)$, where $W(\cdot)$ is the so-called Lambert $W$-function (see [2], [10] or [14] for details and properties of the Lambert $W$-function). Hence, applying this property to equation (11) we get

$$r(\theta; r_0) \cos \theta = W(r_0 e^{r_0} \cos \theta),$$  

whence

$$r(\theta; r_0) = \frac{W(r_0 e^{r_0} \cos \theta)}{\cos \theta}. \quad (12)$$  

A straightforward computation gives us that $r_0 \in [0, W(1/e))$ and that $r(\theta; r_0)$ is an analytic function for all $\theta \in \mathbb{R}$. Moreover, it is an even and $2\pi$-periodic function. This proves that equation (9) with $\varepsilon = 0$ has a submanifold foliated by $2\pi$-periodic orbits. To study the limit cycles bifurcating from the periodic orbits of this submanifold we will apply methods of averaging theory.

The variational equation of (9) with $\varepsilon = 0$ along the solution (12) is given by

$$y' = \frac{r(\theta; r_0) (r(\theta; r_0) \cos \theta + 2) \sin \theta}{(r(\theta; r_0) \cos \theta + 1)^2} y.$$  

By applying separation of variables we get

$$\frac{dy}{y} = \frac{r(\theta; r_0) \sin \theta}{r(\theta; r_0) \cos \theta + 1} d\theta + \frac{r(\theta; r_0) \sin \theta}{(r(\theta; r_0) \cos \theta + 1)^2} d\theta.$$  

Then, by using (12) and multiplying by $r_0 e^{r_0}/r_0 e^{r_0}$ the right-hand side of the previous equation we have

$$\frac{dy}{y} = \frac{W(r_0 e^{r_0} \cos \theta)r_0 e^{r_0} \sin \theta}{r_0 e^{r_0} \cos \theta (W(r_0 e^{r_0} \cos \theta) + 1)} d\theta + \frac{W(r_0 e^{r_0} \cos \theta)r_0 e^{r_0} \sin \theta}{r_0 e^{r_0} \cos \theta (W(r_0 e^{r_0} \cos \theta) + 1)^2} d\theta,$$
which, by means of the change of variable \( z = r_0 e^{r_0} \cos \theta \), can be written as

\[
\frac{dy}{y} = -\frac{W(z)}{z(W(z) + 1)} dz - \frac{W(z)}{z(W(z) + 1)^2} dz.
\]

The first term in the right-hand side of the previous equation is the derivative of \( W(z) \) and the second one can be integrated by using the change of variable \( v = W(z) + 1 \). Thus, we have

\[
\ln(y) = -W(r_0 e^{r_0} \cos \theta) - \ln(W(r_0 e^{r_0} \cos \theta) + 1) - \ln(K)
\]

or equivalently,

\[
y(\theta; r_0) = \frac{e^{-W(r_0 e^{r_0} \cos \theta)}}{K(W(r_0 e^{r_0} \cos \theta) + 1)} = \frac{W(r_0 e^{r_0} \cos \theta)}{K r_0 e^{r_0} (W(r_0 e^{r_0} \cos \theta) + 1) \cos \theta}.
\]

Finally, from the condition \( y(0; r_0) = 1 \) we obtain the particular solution

\[
Y(\theta; r_0) = \frac{(W(r_0 e^{r_0}) + 1) W(r_0 e^{r_0} \cos \theta)}{W(r_0 e^{r_0}) (W(r_0 e^{r_0} \cos \theta) + 1) \cos \theta}.
\]

Hence, the first order averaged function for our system (8) is

\[
\mathcal{F}_1(r_0) = \int_0^{2\pi} Y^{-1}(\theta; r_0) F_1(\theta, r(\theta; r_0)) \, d\theta,
\]

where

\[
Y^{-1}(\theta; r_0) = \frac{W(r_0 e^{r_0}) (W(r_0 e^{r_0} \cos \theta) + 1) \cos \theta}{(W(r_0 e^{r_0}) + 1) W(r_0 e^{r_0} \cos \theta)}
\]

and

\[
F_1(\theta, r(\theta; r_0)) = \frac{Q_0(r(\theta; r_0) \cos \theta, r(\theta; r_0) \sin \theta) \sin \theta}{(1 + r(\theta; r_0) \cos \theta)^2}.
\]

The proof of Theorem 1.1 will be based in the research of the number of zeros of the function \( \mathcal{F}_1(r_0) \). For such investigation it will be useful the following two technical lemmas.

**Lemma 3.1.** The function \( g(x) := x - W(x) \) defined in \((-1/e, 1/e)\) is non negative and has a global minimum at the origin. In particular, for \( x \in (-1/e, 1/e) \setminus \{0\} \) it holds that \( W(x) + W(-x) < 0 \).

**Proof.** It is clear that \( g(0) = 0 \). Easy computations show that \( g'(0) = 0 \) and \( g''(0) = 2 \). Thus, \( g \) has a minimum at the origin. Moreover, it is the unique critical point. Therefore, \( g(x) \geq 0 \) in \((-1/e, 1/e)\).

In order to prove the second part of the lemma, we note that the first part of the lemma implies \( x > W(x) \) and \( -x > W(-x) \). Thus, \( -x < -W(x) \), whence \( W(-x) < -W(x) \). Hence \( W(x) + W(-x) < 0 \). \(\square\)

**Lemma 3.2.** For each \( r_0 \in [0, W(1/e)] \), it holds that

\[
S_1(r_0) = \int_0^{2\pi} \frac{W(r_0 e^{r_0} \cos \theta) \sin^2 \theta}{(1 + W(r_0 e^{r_0} \cos \theta))^3} \, d\theta < 0.
\]

**Proof.** For fixed \( r_0 \), the function

\[
S(\theta; r_0) = \frac{W(r_0 e^{r_0} \cos \theta) \sin^2 \theta}{(1 + W(r_0 e^{r_0} \cos \theta))^3}
\]
is symmetric with respect to $\pi$, that is, $S(\pi - \theta; r_0) = S(\pi + \theta; r_0)$. Thus,

$$S_1(r_0) = 2 \int_0^\pi S(\theta; r_0) \, d\theta$$

which can be written as $S_1(r_0) = 2 \left( I_{11}(r_0) + I_{12}(r_0) \right)$, where

$$I_{11}(r_0) = \int_0^{\pi/2} S(\theta; r_0) \, d\theta \quad \text{and} \quad I_{12}(r_0) = \int_{\pi/2}^\pi S(\theta; r_0) \, d\theta.$$

By interchanging the integration limits and by using the change of variable $\tilde{\theta} = \pi - \theta$, the integral $I_{12}(r_0)$ is equivalent to

$$\int_0^{\pi/2} S(\pi - \theta; r_0) \, d\theta,$$

hence

$$S_1(r_0) = 2 \int_0^{\pi/2} \left[ S(\theta; r_0) + S(\pi - \theta; r_0) \right] \, d\theta. \quad (14)$$

Moreover, a simple computation gives us

$$S(\pi - \theta; r_0) = \frac{W(-r_0e^{r_0} \cos \theta) \sin^2 \theta}{(1 + W(-r_0e^{r_0} \cos \theta))^3}.$$

Thus, we get

$$S(\theta; r_0) + S(\pi - \theta; r_0) = \left( \frac{W(x)}{(1 + W(x))^3} + \frac{W(-x)}{(1 + W(-x))^3} \right) \sin^2 \theta, \quad (15)$$

with $x = r_0e^{r_0} \cos \theta$ and $\theta \in [0, \pi/2)$.

Since $\theta \in [0, \pi/2)$ and $r_0 \in [0, W(1/e))$, $0 < x < 1/e$ and $-1/e < -x < 0$. Thus, $W(x) > 0$ and $-1 < W(-x) < 0$. This implies that $1 < (1 + W(x))^2$ and $0 < 1 + W(-x) < 1$, then

$$\frac{1}{(1 + W(x))^2} < 1 \quad \text{and} \quad \frac{W(-x)}{(1 + W(-x))^2} < W(-x),$$

whence we have

$$\frac{W(x)}{(1 + W(x))^3} + \frac{W(-x)}{(1 + W(-x))^3} < \frac{W(x)}{1 + W(x)} + \frac{W(-x)}{1 + W(-x)}.$$

The right-hand side of the previous inequality can be written as

$$\frac{2W(x)W(-x) + W(x) + W(-x)}{(1 + W(x))(1 + W(-x))}$$

whose denominator is positive, and $W(x)W(-x)$ is negative. Moreover, from Lemma 3.1, we have that $W(x) + W(-x)$ is negative, so by using this property and (15) we have that

$$\int_0^{\pi/2} \left[ S(\theta; r_0) + S(\pi - \theta; r_0) \right] \, d\theta < 0.$$

Finally, the assertion follows from (14).
4. Proof of Theorem 1.1. Recall that Theorem 1.1 is related to the perturbed differential system

\[
\begin{align*}
\dot{x} &= -y, \\
\dot{y} &= x + x^2 + y^2 + \varepsilon(a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 x y + a_5 y^2),
\end{align*}
\]

with \(a_i \in \mathbb{R}\), for \(i = 0, 1, \ldots, 5\). In this section we will give the proof of Theorem 1.1, whose assertion is that this quadratic system has at most one medium amplitude limit cycle.

Proof of Theorem 1.1. By using polar coordinates, system (16) can be transformed in the form (8), with \(Q(x, y, \varepsilon) = Q_0(x, y) = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 x y + a_5 y^2\). Moreover, the transformed system can be written in the form (9) by changing the independent variable \(t\) by the variable \(\theta\). Since \(r(\theta; r_0)\), given in (12), is the periodic solution of the unperturbed system, the first order averaged function (13) becomes

\[
F_1(r_0) = G(r_0) \sum_{i=0}^{5} a_i I_i(r_0),
\]

where

\[
G(r_0) = \frac{W(r_0 e^{\rho_0})}{1 + W(r_0 e^{\rho_0})}, \quad I_i(r_0) = \int_0^{2\pi} J_i(r_0, \theta) d\theta,
\]

and

\[
J_0(r_0, \theta) = \frac{\sin \theta \cos \theta}{W(x)(1 + W(x))}, \quad J_1(r_0, \theta) = \frac{\sin \theta \cos \theta}{1 + W(x)},
\]

\[
J_2(r_0, \theta) = \frac{\sin^2 \theta}{1 + W(x)}, \quad J_3(r_0, \theta) = \frac{\sin \theta \cos \theta W(x)}{1 + W(x)},
\]

\[
J_4(r_0, \theta) = \frac{\sin^2 \theta W(x)}{1 + W(x)}, \quad J_5(r_0, \theta) = \frac{\sin^3 \theta W(x)}{1 + W(x)},
\]

with \(x = r_0 e^{\rho_0} \cos \theta\).

A simple computation shows that \(J_0(r_0, \theta), J_1(r_0, \theta), J_3(r_0, \theta),\) and \(J_5(r_0, \theta)\) are odd functions respect to \(\pi\), i.e., \(J_i(r_0, \pi - \theta) = -J_i(r_0, \pi + \theta)\) for all \(\theta \in [0, \pi]\) and \(i = 0, 1, 3, 5\). Thus,

\[
I_0(r_0) = I_1(r_0) = I_3(r_0) = I_5(r_0) \equiv 0.
\]

Hence, the first order averaged function reduces to

\[
F_1(r_0) = G(r_0)(a_2 I_2(r_0) + a_4 I_4(r_0)).
\]

Furthermore, it is not difficult to see that the derivative of \(a_2 I_2(r_0) + a_4 I_4(r_0)\) is

\[
\frac{(a_4 - a_2)(1 + r_0)}{r_0} \int_0^{2\pi} W(x) \sin^2 \theta \frac{W(x)}{(1 + W(x))} d\theta,
\]

which has a defined sign if \(a_4 \neq a_2\), according to Lemma 3.2, or it vanishes identically if \(a_4 = a_2\).

Hence, in the case \(a_4 \neq a_2\), the function \(a_2 I_2(r_0) + a_4 I_4(r_0)\) is strictly monotone, which implies that \(F_1(r_0)\) has at most one zero in the interval \((0, W(1/e))\) because \(G(r_0) > 0\) on such interval. For the case \(a_4 = a_2\), we have to consider two possibilities: \(a_2 \neq 0\) and \(a_2 = 0\). In the former, the function \(a_2 (I_2(r_0) + I_4(r_0))\) is a non-zero constant, because \(a_2 (I_2(0) + I_4(0)) = a_2 \neq 0\), whence \(F_1(r_0)\) does not have any zero in \((0, W(1/e))\). In the later, the function \(a_2 (I_2(r_0) + I_4(r_0))\) vanishes identically, which implies that \(F_1(r_0)\) vanishes identically on \((0, W(1/e))\). In addition, if \(a_2 = a_4 = 0\), then the perturbed differential system is reversible under the
transformation \((x, y, t) \rightarrow (x, -y, -t)\). Thus, in such a case and for \(\varepsilon\) small enough, the perturbed system has no limit cycles bifurcating from \(\mathcal{P}\).

Therefore, either \(\mathcal{F}_1(r_0) \neq 0\) and has at most one zero in \((0, W(1/e))\) or the perturbed system has no medium limit cycles. Thus, by applying Theorem 2.1 we complete the proof of the main part of the theorem.

Finally, we will prove the four items of the theorem. If \(a_4 > a_2 > 0\), then from (19) it follows that the derivative of \(a_2 I_2(r_0) + a_4 I_4(r_0)\) is negative, whence the function \(a_2 I_2(r_0) + a_4 I_4(r_0)\) is strictly monotone decreasing on \((0, W(1/e))\). Thus, if \(\mathcal{F}_1(r_0)\) has a zero \(r^*\) in \((0, W(1/e))\), then \(\mathcal{F}_1'(r^*) < 0\). Hence, Theorem 2.3 implies that the medium limit cycle \(\gamma_\varepsilon\) bifurcating from \(r(\theta; r^*)\) is asymptotically stable. This proves the first item. The second item follows from the same idea.

If \(0 < a_2 < a_4 < a_2\), then from (19) it follows that the derivative of \(a_2 I_2(r_0) + a_4 I_4(r_0)\) is positive. Moreover, since \(a_2 I_2(0) + a_4 I_4(0) = \pi a_2 > 0\), \(a_2 I_2(r_0) + a_4 I_4(r_0)\) is positive. Hence, \(\mathcal{F}_1(r_0)\) has no zeros in \((0, W(1/e))\). Thus the system does not have medium limit cycles. This proves the third item. The proof of the fourth one is analogous.

We finish this section by providing an example and a remark of our Theorem 1.1.

**Example 1.** Consider the perturbed Kukles system

\[
\begin{align*}
\dot{x} &= -y, \\
\dot{y} &= x + x^2 + y^2 + \varepsilon(2y + 50xy).
\end{align*}
\]

(20)

A numerical routine gives the graph of the first averaged function \(\mathcal{F}_1(r_0)\) of the system with \(r_0 \in [0, 27/100] \subset [0, W(1/e)]\). See a) in Figure 1. Thus, \(\mathcal{F}_1(r_0)\) has a zero \(r^*\), close to \(r_0 = 0.21\), with \(\mathcal{F}_1'(r^*) < 0\). Hence, for small enough \(\varepsilon\) the system (20) has a stable limit cycle bifurcating from the periodic orbit \(\{r(\theta; r^*) | \theta \in [0, 2\pi]\}\) of \(\mathcal{P}\). The local phase portrait of (20), with \(\varepsilon = 1/50\), is given in b) of Figure 1, where we have considered the initial conditions \((1.0, 0.0), (0.211, 0.0),\) and \((0.25, 0.0)\).

![Figure 1. a) Graph of \(\mathcal{F}_1(r_0)\) for (20). b) Phase portrait of (20) with \(\varepsilon = 1/50\).](image-url)
Remark 1. The assertion of Theorem 1.1 also holds by considering \(Q(x, y, \varepsilon)\) of the form
\[
Q(x, y, \varepsilon) = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2 + \tilde{Q}(x, y^2),
\]
with \(\tilde{Q}(x, y^2)\) an arbitrary polynomial function.

5. **Proof of Theorem 1.2.** Recall that Theorem 1.2 deals with the small limit cycles of the cubic perturbed differential system:
\[
\dot{x} = -y, \\
\dot{y} = x + x^2 + y^2 + \varepsilon Q_1(x, y) + \varepsilon^2 Q_2(x, y) + \cdots + \varepsilon^6 Q_6(x, y),
\]
(21)
where the polynomials \(Q_j\) are
\[
Q_1(x, y) = a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2 + a_6 x^3 + a_7 x^2 y + a_8 xy^2 + a_9 y^3,
\]
\[
Q_2(x, y) = b_1 x + b_2 y + b_3 x^2 + b_4 xy + b_5 y^2 + b_6 x^3 + b_7 x^2 y + b_8 xy^2 + b_9 y^3,
\]
\[
Q_3(x, y) = c_1 x + c_2 y + c_3 x^2 + c_4 xy + c_5 y^2 + c_6 x^3 + c_7 x^2 y + c_8 xy^2 + c_9 y^3,
\]
\[
Q_4(x, y) = d_1 x + d_2 y + d_3 x^2 + d_4 xy + d_5 y^2 + d_6 x^3 + d_7 x^2 y + d_8 xy^2 + d_9 y^3,
\]
\[
Q_5(x, y) = e_1 x + e_2 y + e_3 x^2 + e_4 xy + e_5 y^2 + e_6 x^3 + e_7 x^2 y + e_8 xy^2 + e_9 y^3,
\]
\[
Q_6(x, y) = f_1 x + f_2 y + f_3 x^2 + f_4 xy + f_5 y^2 + f_6 x^3 + f_7 x^2 y + f_8 xy^2 + f_9 y^3,
\]
with \(a_k, b_k, c_k, d_k, e_k, f_k \in \mathbb{R}\).

Proof of Theorem 1.2. By using the change of variables \((x, y) \rightarrow (x/\varepsilon, y/\varepsilon) = (u, v)\) and renaming the variables, system (21) becomes
\[
\dot{x} = -y, \\
\dot{y} = x + \sum_{i=1}^8 \varepsilon^i \tilde{Q}_i(x, y),
\]
(22)
where
\[
\tilde{Q}_1(x, y) = a_1 x + a_2 y + x^2 + y^2,
\]
\[
\tilde{Q}_2(x, y) = b_1 x + b_2 y + a_3 x^2 + a_4 xy + a_5 y^2,
\]
\[
\tilde{Q}_3(x, y) = c_1 x + c_2 y + b_3 x^2 + b_4 xy + b_5 y^2 + a_6 x^3 + a_7 x^2 y + a_8 xy^2 + a_9 y^2,
\]
\[
\tilde{Q}_4(x, y) = d_1 x + d_2 y + c_3 x^2 + c_4 xy + c_5 y^2 + b_6 x^3 + b_7 x^2 y + b_8 xy^2 + b_9 y^3,
\]
\[
\tilde{Q}_5(x, y) = e_1 x + e_2 y + d_3 x^2 + d_4 xy + d_5 y^2 + c_6 x^3 + c_7 x^2 y + c_8 xy^2 + c_9 y^3,
\]
\[
\tilde{Q}_6(x, y) = f_1 x + f_2 y + e_3 x^2 + e_4 xy + e_5 y^2 + d_6 x^3 + d_7 x^2 y + d_8 xy^2 + d_9 y^3,
\]
\[
\tilde{Q}_7(x, y) = f_3 x^2 + f_4 xy + f_5 y^2 + e_6 x^3 + e_7 x^2 y + e_8 xy^2 + e_9 y^3,
\]
\[
\tilde{Q}_8(x, y) = f_6 x^3 + f_7 x^2 y + f_8 xy^2 + f_9 y^3.
\]
Thus, each medium limit cycle of (22) corresponds to a small limit cycle of the original system (21). As in previous sections, we use polar coordinates to transform system (22) in a one-dimensional differential equation of the form (3). A straightforward computation gives
\[
r' = \sum_{i=1}^7 \varepsilon^i F_i(\theta, r) + O(\varepsilon^8),
\]
(23)
where
\[ F_1(\theta, r) = r \sin \theta (a_1 \cos \theta + a_2 \sin \theta) + r^2 \sin \theta, \]
\[ F_2(\theta, r) = r \sin \theta \left[ (b_2 - 2a_1a_2 \cos \theta) \sin \theta + (b_1 - a_2^2 \sin^2 \theta) \cos \theta - a_1^2 \cos^3 \theta \right] \\
+ r^2 \sin \theta \left[ (a_3 - 2a_1) \cos^2 \theta + (a_4 - 2a_2) \sin \theta \cos \theta + a_5 \sin^2 \theta \right] \\
- r^3 \sin \theta \cos \theta, \]
\[ F_3(\theta, r) = \frac{1}{2} r^3 \sin \theta \left[ (a_1 - a_2 - 3a_5) \cos \theta + (4a_1 - 2a_3 + 2a_5 + 2a_6) \cos^3 \theta \\
2a_8 \cos \theta \sin^2 \theta + (6a_2 - 4a_4 + 2a_7) \sin \theta \cos^2 \theta + 2a_9 \sin^3 \theta \right] \\
+ r^4 \sin \theta \cos^2 \theta, \]
\[ F_4(\theta, r) = A_{41} r + A_{42} r^2 + A_{43} r^3 + A_{44} r^4 + A_{45} r^5 + A_{46} r^6, \]
\[ F_5(\theta, r) = A_{51} r + A_{52} r^2 + A_{53} r^3 + A_{54} r^4 + A_{55} r^5 + A_{56} r^6, \]
\[ F_6(\theta, r) = A_{61} r + A_{62} r^2 + A_{63} r^3 + A_{64} r^4 + A_{65} r^5 + A_{66} r^6 + A_{67} r^7, \]
\[ F_7(\theta, r) = A_{71} r + A_{72} r^2 + A_{73} r^3 + A_{74} r^4 + A_{75} r^5 + A_{76} r^6 + A_{77} r^7 + A_{78} r^8, \]
and the coefficients \( A_{ij} \) depend on \( \theta \) and the parameters \( a_k, b_k, c_k, d_k, e_k \) and \( f_k \), whose expressions have been omitted because they are very large. Thus, (23) has the form (3) with \( F_0 \equiv 0 \) and the solution \( \varphi(\theta, r_0) \) of the unperturbed equation \( r' = 0 \) with \( \varphi(0, r_0) = r_0 \) is the 2π-periodic function \( \varphi(\theta, r_0) = r_0 \).

Hence, we have reduced the study of the limit cycles of the perturbed differential system (22) to study the isolated periodic orbits of the differential equation (23), which will be analyzed by using averaging theory.

First, we will apply the averaging theory of first order. By Theorem 2.2 we must study the simple positive zeros of the first order averaged function \( F_1(r_0) \), which in our case has the form
\[ \mathcal{F}_1(r_0) = \int_0^{2\pi} F_1(s, r_0) ds = \pi a_2 r_0. \]
Thus, if \( a_2 \neq 0 \), then \( \mathcal{F}_1(r_0) \) has no positive zeros, which implies that (23) has no periodic orbits bifurcating from the unperturbed equation.

In order to simplify the notation, from now on we will replace \( r_0 \) by \( r \). Thus, \( \mathcal{F}_1(r) = \pi a_2 r \). We now will apply averaging theory of second order by assuming that \( \mathcal{F}_1(r) \equiv 0 \), that is, \( a_2 = 0 \). Hence, by using (7) we get
\[ y_1(\theta, r) = r (a_1 + 2r + a_1 \cos \theta) \sin^2 (\theta/2), \]
and by equation (6) we obtain
\[ \mathcal{F}_2(r) = \pi b_2 r. \]
Therefore, if \( b_2 \neq 0 \), then \( \mathcal{F}_2(r) \) has no positive zeros. This completes the proof of the statement a) of the theorem.

Now we are going to apply averaging theory of third order, so we must suppose that \( \mathcal{F}_1(r) \equiv \mathcal{F}_2(r) \equiv 0 \), that is, \( a_2 = b_2 = 0 \). Thus, by (7) we get
\[ y_2(\theta, r) = \frac{1}{96} r \left[ 48 b_1 - 21 a_7^2 - 16 (a_1 + 4a_3 + 8a_5 + 15r) r + 48 a_4 r \sin \theta \\
- 48 (a_3 + 3a_5 + 8r) r \cos \theta + 12 (a_7^2 - 4b_1 - 4a_1 r + 12r^2) \cos 2\theta \\
- 16 a_4 r \sin 3\theta + 9a_7^2 \cos 4\theta + 16 (4a_1 - a_3 + a_5) r \cos 3\theta \right], \]
and by computing the third averaged function according (6), we arrive to
\[ F_3(r) = \frac{\pi}{4} r \left[ 4c_2 + (a_7 - 2a_4 + 3a_9)r^2 \right]. \]
Then, under the condition \( c_2(2a_4 - a_7 - 3a_9) > 0 \), the function \( F_3(r) \) has a simple positive zero at
\[ r_1^* = \frac{4c_2}{2a_4 - a_7 - 3a_9}, \]
because \( F_3(r_1^*) = -2c_2 \pi \). Hence, Theorem 2.2 ensures that system (22) has a medium limit cycle, which is stable if \( c_2 > 0 \) and it is unstable if \( c_2 < 0 \) according to Theorem 2.3. This implies that the Kukles system (21), for \( \varepsilon \) small enough, has a small limit cycle.

We will suppose now that \( F_1(r) \equiv F_2(r) \equiv F_3(r) \equiv 0 \) for studying the limit cycles of (22) by using averaging theory of fourth order. So we assume \( a_2 = b_2 = c_2 = 0 \) and \( a_7 = 2a_4 - 3a_9 \). Proceeding in a similar way as in the previous analysis, we get
\[ F_4(r) = \frac{\pi}{4} r \left[ 4d_2 + (a_1a_4 + 3a_1a_9 - a_3a_4 - a_4a_5 - 2b_4 + b_7 + 3b_9)r^2 \right]. \]
This polynomial has a positive zero at
\[ r_2^* = \sqrt{-a_1a_4 - 3a_1a_9 + a_3a_4 + a_4a_5 + 2b_4 - b_7 - 3b_9}, \]
whenever \( d_2(-a_1a_4 - 3a_1a_9 + a_3a_4 + a_4a_5 + 2b_4 - b_7 - 3b_9) > 0 \). Since \( F_4(r_2^*) = -2d_2 \pi \), we have the same conclusions as in the previous one. Thus, we have proved the statement b) of the theorem.

Now, we are going to apply averaging theory of fifth order. Then, we must assume that \( F_1(r) \equiv F_2(r) \equiv F_3(r) \equiv F_4(r) \equiv 0 \), which is equivalent to take \( a_2 = b_2 = c_2 = d_2 = 0 \), \( a_7 = 2a_4 - 3a_9 \), and \( b_7 = -a_1a_4 - 3a_1a_9 + a_3a_4 + a_4a_5 + 2b_4 - 3b_9 \). Using the equation (7) and after some computations we obtain
\[ F_5(r) = \frac{\pi}{12} r[C_5 + B_5 r^2 + A_5 r^4], \]
where \( A_5 = 8a_4 - 24a_9 \), \( B_5 = 3a_1a_4(a_1 - a_1) + 3a_1(b_1 + 3b_9) + 3a_1(b_1 - b_1 - b_5) - 3a_3b_4 - 3a_5b_4 + 9a_9b_1 - 6c_4 + 3c_7 + 9c_9 \) and \( C_5 = 12c_2 \). Analyzing this polynomial in the variable \( u = r^2 \), we must calculate the positive roots of the quadratic polynomial
\[ F_5(u) = C_5 + B_5 u + A_5 u^2. \]
Let \( \Delta_5 = B_5^2 - 4A_5 C_5 \) be the discriminant of this polynomial. Then, by averaging theory of fifth order, we have the following properties:

1. If \( \Delta_5 < 0 \), then the Kukles system (21) has no limit cycles.
2. If \( \Delta_5 = 0 \), then system (21) has one limit cycle whether \( A_5B_5 < 0 \).
3. If \( \Delta_5 > 0 \) and \( A_5C_5 < 0 \), then system (21) has one limit cycle.
4. If \( \Delta_5 > 0 \), \( A_5C_5 > 0 \), \( A_5 < 0 \), and \( B_5 > 0 \), then (21) has two limit cycles.
5. If \( \Delta_5 > 0 \), \( A_5C_5 > 0 \), \( A_5 > 0 \), and \( B_5 < 0 \), then (21) has no limit cycles.
6. If \( \Delta_5 > 0 \), \( A_5C_5 > 0 \), and \( B_5 < 0 \), then (21) has one limit cycle.
7. If \( A_5 = 0 \) and \( B_5C_5 < 0 \), then (21) has one limit cycle, in the opposite case there are not limit cycles.

The zeros, when they exist, are given by
\[ r_3^* = \frac{1}{\sqrt{2}} \sqrt{-\frac{\sqrt{\Delta_5} + B_5}{A_5}} \quad \text{and} \quad r_4^* = \frac{1}{\sqrt{2}} \sqrt{\frac{\sqrt{\Delta_5} - B_5}{A_5}}. \]
It is verified that $F'_5(r^*_5)$ (when it exists) is negative, so the limit cycle is stable, and $F'_5(r^*_3)$ (when it exists) is positive, so the limit cycle is unstable. On the other hand, for the case $A_5 = 0$, there is one equilibrium point given by
\[ r^*_5 = \sqrt{-\frac{C_5}{B_5}}, \]
it is verified that $F'_5(r^*_5)$ (when it exists) is $-(\pi C_5)/6$, so the limit cycle is stable if $C_5 > 0$ and it is unstable if $C_5 < 0$.

Now, we are going to apply averaging theory of sixth order. We start by assuming that $a_2 = 0$, $a_9 = -4a_4/3$, and $c_7 = a_7^2a_4 - 2a_4b_1 + a_5b_4 + a_4(b_4 + b_5) - a_1(a_3a_4 + b_4 + 3b_5) + 2c_4 - 3c_9$. Using the equation (7) and after some computations we obtain
\[ F_6(r) = \frac{\pi}{12} r [C_6 + B_6r^2 + A_6r^4], \]
where $A_6 = -8a_1a_4 + 2a_4(4a_5 + 3a_6 + a_8) + 8(b_4 - 3b_9)$, $B_6 = 3(a_4^3a_4 + a_3a_4b_1 + b_1b_4 - b_4b_1 - a_1^2(a_4a_4 + b_1) - b_4b_5 + B_1b_9 + 2a_4c_1 - a_4c_4 - a_3c_4 - a_3c_4 - a_4c_4 + a_1(a_4b_1 + b_4) + a_4b_4 + c_4 + 3c_9) - 2d_4 + d_7 + 3d_9$ and $C_6 = 12f_2$ we obtain that $F_6$ gives us at most two limit cycles and the analysis is identical to those of the function $F_3(r)$.

Finally, we apply averaging theory of seventh order, so we must do $A_6 = B_6 = 0$ and $C_6 = 0$, then we can write
\[ b_4 = \frac{1}{8}(4a_4a_4 - 4a_4a_5 - 3a_4a_6 - 4a_4a_7 + 12b_9), \]
\[ d_4 = \frac{1}{8}(4a_4a_4 - 4a_4a_5 - 3a_4a_6 - 4a_4a_7 + 12b_9), \]
\[ f_2 = 0. \]

Under this consideration, we use (7), together with some manipulations, to get
\[ F_7(r) = \frac{\pi}{288} r^3[B_7 + A_7r^2], \]
where $A_7 = -8a_4^3 - 3a_4(64a_4^2 + 34a_4a_5 + 110a_5a_6 + 27a_7^2 + 14a_4a_5 + 34a_5a_5 + 18a_6a_8 + 3a_7^2 - 2a_1(32a_5 + 17a_5 + 15a_5) + 64b_4 - 64b_5 - 48b_9 - 16b_9) - 48(12a_1b_9 - 12a_5b_9 - 9a_5b_9 - 3a_8b_9 - 4c_4 + 12c_9), \]
\[ B_7 = -9(4a_4a_4b_1 - 4a_4a_5b_1 - a_4a_5b_1 + 4a_4b_4 + 4a_4b_3 + 3a_4a_5b_3 - 8a_4b_3 + 4a_4^2b_5 + 3a_4a_5b_5 + a_4a_5b_5 + a_1^2(a_4(4a_5 + 3a_6 + a_8) - 12b_9) + 24a_3b_9 - 12a_5b_9 + 12a_5b_9 + 16a_4a_5c_9 + 6a_4a_6c_9 + 2a_4a_5c_9 - 48b_9c_9 - 12a_4a_5c_9 - 6a_4a_6c_9 - 2a_4a_5c_9 + 24b_9c_9 + 4a_4b_1 - c_4) - 4a_4b_4 - 8b_1c_4 + 8b_3c_4 + 8b_5c_4 - 12a_4a_5c_9 - 6a_4a_6c_9 - 2a_4a_5c_9 + 24b_9c_9 + a_1^2(a_4(4a_5 + 3a_6 + a_8) - 12b_9) + 4a_4a_5b_1 - a_4b_4 + 3a_4(3b_1 - b_3 + b_5) + 4a_5(4b_1 - b_3 + 2b_5) - 4c_3 - 12c_5) + 2a_4(4a_4^2 + 3a_4a_5 + 4b_5) - 12(a_5b_9 + c_9) - 4(6b_9b_9 - 3b_3b_9 + 3b_9b_5 + 2a_5c_9 + 3a_5c_9 - 12b_9) + a_3(a_4a_5b_1 + a_4b_4 + 3a_4b_9 + 5a_4b_9 + 5a_5(2b_1 + b_3 + b_5) - 4c_3 - 4c_5) - 4(3b_9b_9 + 3b_9b_9 + 2a_5c_9 - 12b_9) + 16b_9 - 8c_9 - 24c_9). \]
Thus, the analysis is similar to the cases $F_3$ or $F_4$. Therefore, we complete the proof of statement c) of the theorem.

**Remark 2.** If for $i = 1, 2, \ldots, 9$ we consider $a_i = 0$ for $i \neq 4$, $b_i = 0$, $e_i = 0$ for $i \neq 2$, $d_i = f_i = 0$, $e_2 = 1$, $c_3 = c_4 = c_5 = c_6 = c_8 = 0$, $c_4$, $c_7 = 4$, $c_9 = -2$, and $a_4 = 1/8$, then system (21) has two small limit cycles. Indeed, it is verified
that the fifth order averaged function $F_5(r)$ has exactly two positive zeros, namely, $r = 1.0493$ and $r = 3.30136$.

**Remark 3.** If we consider the polynomials $Q_1, \ldots, Q_6$ of degree two, then some simplifications are obtained, in particular

$$F_3(r) = \frac{\pi}{2} r \left[ 2c_2 - a_4 r^2 \right],$$

$$F_4(r) = \frac{\pi}{4} r \left[ 4d_2 + (a_4(a_1 - a_3 - a_5) - 2b_4) r^2 \right],$$

$$F_5(r) = \frac{\pi}{4} r \left[ 4c_2 + (a_1 a_4(a_3 - a_1) + a_4(b_1 - b_3 - b_5) + b_4(a_1 - a_3 - a_5) - 2c_4)r^2 + \frac{8}{3}a_4 r^4 \right],$$

$$F_6(r) = \frac{\pi}{4} r \left[ 4f_2 - (b_4(a_1 - a_3 - b_1 + b_3 + b_5) - c_4(a_1 - a_3 - a_5) + 2d_4)r^2 + \frac{8}{3}b_4 r^4 \right],$$

and

$$F_7(r) = \frac{\pi}{8} r^3 (a_3 c_4(a_3 + 2a_5) - a_1 c_4(a_1 + 2a_5) + 2c_4(b_1 - b_3 - b_5) + a_5^2 c_4 - 4e_4 + \frac{16}{3}c_4 r^2).$$

Thus, the conclusions of Theorem 1.2 hold for this quadratic case.

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Received for publication December 2017.

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