Lagrangian Equations of Motion of Classical Many Body Systems on Shape Space obeying the Modified Newtonian Theory

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Abstract

The equivalence class of absolute configurations of a system under the group of similarity transformations Sim(3) is called the shape of the system. It is explained in this paper how the direct application of the Principle of Relationalism leads to a new theory whose Lagrangian is Sim(3)-invariant, which in turn ensures the existence of the theory's law of motion on shape space. To find out the equations of motion for a system's shape degrees of freedom, the Boltzman-Hammel equations of motion in an anholonomic frame on the tangent space $T(Q)$ to the system's absolute configuration space $Q$, is adapted to the Sim(3)-fiber-bundle structure of the configuration space. The derived equations of motion on shape space enable us, among others, to predict the evolution of the shape of a classical system governed by this new theory without any reference to its absolute position, orientation, or size in space. We will explain by treating the measuring instruments as part of the matter in the theory, how the mass metric $M$ on the configuration space $Q$ uniquely defines a metric on the reduced tangent bundle $T(Q)/\text{Sim}(3)$, and how the unique metric structure on shape space $S$ can be derived. After treating the general N-body system, the shape equations of motion of a three-body system are derived explicitly as an illustration of the general method. Some cosmological implications of this theory are also worked out. In particular, we explain how the observed universe's accelerated expansion follows from the conservation of the dilational momentum in the modified Newtonian theory. Finally, we compare the present work with two other approaches to relational physics and discuss their essential differences.

"Dedicated to the good memories and countless sacrifices of my mother, Sholeh Shamsakhzari."

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Local expression of the Connection form and the metric in orientational and internal coordinates . . . . . . . . . . . . . . . . 12
In the first part (Section (II)), we review the Principle of Relationalism and its consequences as explained in [1]. Direct attempts to implement this principle in classical physics lead us to a new theory, which we call the modified Newtonian theory. At the end of this section, we will present a more mathematical formulation of the principle of relationalism and explain its difference with Barbour-Bertotti’s postulate (principle) of relational mechanics [2] and with Barbour’s fundamental postulate of relational mechanics [3].

In the second part (Sections (III), (IV), and (V)), following [4], [5], and [6] to a large extent, we first review in Section (III) the geometric setting on the center of mass configuration space $Q_{cm} \cong \mathbb{R}^{3N-3}$ of a $N$-particle system as a $SO(3)$-fiber-bundle. Then, in section (IV), we expand this setting to scale transformations $Sc$ and discuss the construction of $Sim(3)$-fiber-bundle. Under the action of the group of scale transformations $Sc$, the internal configuration space $Q_{int} := \frac{Q}{E(3)}$ becomes a fiber bundle, whose base space is known as shape space. We show that the connection form on $Q_{int}$ (considered as a $Sc$-fiber-bundle) is flat by an explicit calculation. Subsequently, we explain how with the help of the principle of relationalism the mass metric $M$ on absolute configuration space $Q \cong \mathbb{R}^{3N}$ induces a unique metric $N$ on the reduced tangent bundle $\frac{T(Q)}{Sim(3)}$. We also present a way to solve the non-uniqueness problem in the metric $N$ on shape space, which arises in the DGZ-approach [15], by explaining the physical origin of the conformal factor.

In Section (V), after reviewing the Lagrangian formulation of mechanics in anholonomic frames, and their Boltzmann-Hamel equations of motion, we aim to derive the equation of motion for the shape degrees of freedom of a $N$-particle system, whose behavior in abso-
lute space and time is given by the *modified Newtonian theory*.

In the third part (Sections (VI) and (VII)), we explicitly derive the shape equations of motion of a three-body system. Then we discuss some cosmological consequences of the modified classical mechanics developed in this paper and [1]. Scale invariance of the modified classical mechanics leads naturally to a new conserved quantity. In Section (VII.i), we will explain how this new conservation law explains the observed accelerated expansion of the universe. We also show that the known central collision singularity (of the Newtonian gravity) will not be reached in any finite time in an imploding $N$-body system evolving according to this theory. In other words, seen from the Newtonian world-view, the nearer the system gets to the central collision, the slower it moves towards it.

Research in relational physics has a rich and long history, and there are many important attempts at implementing relational ideas in physics. See for instance [7],[8],[9] for more information. In the last part (Section (VIII)) of this paper, we will give a quick comparison of our work (as explained in this paper and [1]) with two of the other approaches in relational physics. The first alternative approach (denoted here by BKM) is based on the mechanical similarities in Newtonian mechanics, as is developed and elaborated in [10],[11],[12],[13]. The second approach (denoted here by BDGZ) is based on the geodesic dynamics on shape space, as is developed and expanded in [14], [15], [16]. In particular, we will explain how the BKM-approach and the BDGZ-approach differ from our work but still both satisfy the Principle of Relationalism formulated in Section (II). Moreover, in contrast to the BDGZ-approach, we will see that once the behaviour of the measuring units made out of matter is theoretically included, the *unique metric structure on Shape space* reveals itself. In particular, we will explain the relationship between the choice of a unit of length and the choice of a conformal factor, and elaborate that all reasonable choices of length units lead to the same metric on shape space. Furthermore, we will discuss how the group of scale transformations $Sc \subset Sim(3)$ acts on the absolute phase space according to these different approaches, and we will see that three different actions are engaged. We are of the opinion that one has to argue for the choice of the action (on the absolute phase space) one makes. In Section (II), we will see how the principle of relationalism leads to an action of the group $Sc$ on the phase space of the modified Newtonian theory, which turns out to also coincide with the mechanical similarity transformation of the modified Newtonian theory (MNT).
II. PRINCIPLE OF RELATIONALISM

It is explained in [1] that the ideas of Gottfried Wilhelm Leibniz, and Ernst Mach on the foundation of mechanics, and in particular, about the absolute 3-dimensional space and the absolute time, as two of the building blocks of the classical physics ([17],[18]), can partly be expressed, and made more concise in terms of the Principle of Relationalism, which amounts to the following statement:

Two possible universes, differing from each other just by the action of a global similarity transformation Sim(3), are observationally indistinguishable.

This statement includes, among others, the invariance of the laws of physics under any Sim(3) transformation of the universe. The dynamical indistinguishably of the two alternative universes mentioned above requires the introduction of a particular action of the group of scale transformations Sc on particles’ velocities (or on phase space), as will be explained in more detail in this paper. Different absolute theories may require different actions. For some classes of absolute theories, there may exist no action of Sc on the universe’s phase space for which the considered theory could satisfy the mentioned principle. In this case, the nonexistence of a relational reformulation of the absolute theory under consideration follows immediately. We explained in [1] that a direct implementation of this principle requires the gravitational coupling G appearing in Newton’s theory of gravitation to be a homogeneous function of degree one on the configuration space, and not a constant. Hence in a given absolute frame of reference, after a global scale transformation of the universe by a factor $b \in \mathbb{R}^+$

\[(x_1, \ldots, x_N) \rightarrow (bx_1, \ldots, bx_N)\]

the gravitational coupling must transform as

\[G \rightarrow G' = bG\]

This makes the Newtonian gravitational potential $V = \sum_{i,j=1}^{N} \frac{Gm_i m_j}{|x_i - x_j|}$ scale-invariant, and together, because of its rotation and translation invariance, $V : Q \rightarrow \mathbb{R}$ uniquely projects down to a function $V_s : S \rightarrow \mathbb{R}$ on shape space. The new theory, which promotes the gravitational coupling from a constant (as is the case in the Newtonian theory) to a function with the mentioned properties, is called the Modified Newtonian Theory.

The above construction alone is, however, not sufficient for implementing the Principle of Relationalism, as it would make the strength of the gravitational force (measured, for instance, with respect to the strong nuclear force) scale-dependent. This would cause a clear violation of the mentioned principle. In order to avoid this and other violations, Planck’s constant $\hbar$, and the vacuum permittivity $\varepsilon_0$, have also been promoted to homogeneous functions on Q of degrees 1 and −1 respectively. These, in particular, ensure the scale independence of the relative strength of the four known forces in nature [1].

We want to emphasize that while the value of $G$ depends directly on the units chosen to measure distance and duration, once a set of units is chosen (e.g., the SI units), there is no justification in Newtonian theory for why the value of $G$ should be what it is (in these chosen units). We have called this problem a "foundational gap" in Newtonian theory (see [1]) and provided a partial filling of it in a way that the principle of relationalism can be directly implemented in modern physics (which undoubtedly includes more than gravitational interaction alone). Completing the present absolute theories in line with the idea that $G$, $\hbar$, $\varepsilon_0$ are homogeneous functions.

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1With observationally indistinguishable we mean kinematically and dynamically indistinguishable.
2Whose exact form is yet unknown, but not needed for the aim of this paper.
3A hypothetical immaterial frame attached to the absolute space. One can call it god’s frame of reference if one wishes so.

4We use the word partial because the exact expressions of these functions are not determined and are yet to be discovered only in a future complete physical theory. Only then will it be permissible to omit the word "partial".
of the mentioned degrees on the configuration space of the universe not only partially fills the foundational gap in Newtonian mechanics, but also directly addresses Einstein’s legitimate concern about finely tuned dimensionless constants of nature\(^5\) e.g., relative strength of the forces in nature. Namely, if in the ultimate absolute theory \(G, \hbar, \epsilon_0\) turn out to be certain homogeneous functions on the configuration space (as expected by the direct implementation of the principle of relationalism), an immediate justification is provided for the (otherwise surprisingly fine-tuned) value of the dimensionless constants characteristic of our universe, since the latter are ratios of the former.

It is also essential to take the transformations of the *measuring units* under the global scale transformations into account. Max Planck has derived units of duration, length, and mass from \(G, \hbar, \epsilon_0\), and light’s average velocity \(c\) by dimensional analysis, i.e.

\[
L_p = \sqrt{\frac{G\hbar}{c^3}} \\
M_p = \sqrt{\frac{h\epsilon}{G}} \tag{1} \\
T_p = \sqrt{\frac{G\hbar}{c^2}}
\]

One of the main advantages of this set of units is their accessibility in all regions of the universe in which the same laws of physics as ours (i.e., the laws of quantum mechanics, electromagnetism, and gravity) hold. Seen from the absolute frame of reference and measured with some hypothetical fixed absolute units of duration and length\(^6\), Planck’s units will change under a global scale transformation of the universe as follows

\[
L_p \rightarrow L'_p = bL_p \\
M_p \rightarrow M'_p = M_p \tag{2} \\
T_p \rightarrow T'_p = bT_p
\]

These transformations follow immediately from the expressions (1) by taking the mentioned degree of homogeneity of the functions \(G, \epsilon_0, \hbar\) into account. It shows the expected behavior for Planck’s length unit. Additionally, it shows an increase in Planck’s unit of time, mimicking a time dilation for internal observers (which keep measuring with the new Planck’s units).

Given the above information, we want to discuss now how the velocity of different objects transform under a similarity transformation of the universe. In other words, given the initial state of a classical \(N\)-body system (universe) by the \(3N\) position variables \((x_1, \ldots, x_N)\) and the \(3N\) velocity variables \((v_1, \ldots, v_N)\), what will the new velocities \((v'_1, \ldots, v'_N)\) become, after we have changed (jumped) the system’s configuration to \((x'_1, \ldots, x'_N) = (gx_1, \ldots, gx_N)\) for any \(g \in Sim(3)\)?

If the performed transformation belongs to the euclidean subgroup, i.e., \(g \in E(3) \subset Sim(3)\), the transformation of the velocities is already well-known\(^7\). So we need just to discuss the transformation of the velocities under the group of scale transformations \(Sc \subset Sim(3)\).

As the measurement of a velocity is basically an experimental task, the transformation law of the velocities under scale transformation of the system (or, in fact, any other transformation of the system) must also include some experimental reasoning. We will explain here existence of these absolute units in Newtonian Mechanics can be inferred from the existence of absolute space and time.

\(^5\)Mentioned in his correspondence with Isle Rosenthal-Schneider\([7]\).

\(^6\)God’s immaterial rods and clocks if you wish. They can also be called the absolute SI units, coinciding with the real (experimental) SI units, just for a specific scale of the universe. For instance, the absolute SI units and real SI units coincide for the universe now (at this moment of time), and before performing any scale transformation of the universe. The performance of such a transformation changes the internal units, but not the absolute ones. The existence of these absolute units in Newtonian Mechanics can be inferred from the existence of absolute space and time.

\(^7\)Velocities are invariant under the action of the group of spatial translations \(T(3)\). Under the action of the group of spatial rotations \(SO(3)\), the velocities transform to \((v'_1, \ldots, v'_N) = (g v_1, \ldots, g v_N)\), where \(g\) is the rotation connecting the old configuration to the new one.
that based on the Principle of Relationalism, the behavior of rods and clocks of the modified Newtonian theory under a universe’s scale transformation is such that the measured velocities of objects (or parts of the system) remain invariant under such transformations. This invariance is a natural consequence of the simultaneous expansion of the measuring rod and the dilation of the unit of time we came across previously. To be more precise, we have already argued above that a global scale transformation of the universe by a factor \( b \in \mathbb{R}^+ \)

\[
x = (x_1, \ldots, x_N) \rightarrow x' = (bx_1, \ldots, bx_N)
\]
causes the following transformation of the gravitational coupling \( G \), Planck’s \( \hbar \), and the vacuum permittivity \( \varepsilon_0 \)

\[
G \rightarrow bG
\]

\[
\hbar \rightarrow b\hbar
\]

\[
\varepsilon_0 \rightarrow \frac{\varepsilon_0}{b}
\]

These transformations, in turn, changed the behavior of rods and clocks through a change of their (Planck) units \(^1\) as given in \(^2\). The measured speed \( v \) of an object, e.g., a particle, gets transformed under a global scale transformation as follows

\[
v = \frac{\Delta x}{\Delta t} \rightarrow v' = \frac{\Delta x'}{\Delta t'} = \frac{b\Delta x}{b\Delta t} = v
\]  

(3)

where \( \Delta x \) stands for instance for the distance between two other objects (which are needed to define the start and end point of any interval in space), and \( \Delta t \) stands for the time duration (measured in Planck unit) that the object needs to travel between those two reference objects. The primed versions are the same quantities after scale transforming the universe and measuring everything in the new Planck units. So, the measured velocities of the \(^3\) objects become invariant under the scale transformations. In the relation, \( \Delta t' = b\Delta t \), which is used in \(^3\), the Principle of Relationalism is silently invoked in equating the number of ticks (or steps) of our new clock in the scaled universe for the duration of a physical phenomenon (in this example the passage of an object between the two reference objects), and the number of ticks of the old clock in the old (smaller) universe while the same phenomenon is taking place. So the measured speed of any object in universes before and after global scale transformations remains the same. The Principle of Relationalism and the characteristics of the Planck units together are responsible for this result. Analogously, the system’s measured configuration velocity, which is the collection of all the velocities of its constituent particles, is also invariant under a global scale transformation, i.e.

\[
v_x = (v_1, \ldots, v_N) \in T_xQ
\]

\[
\downarrow
\]

\[
v_{bx} = (v_1, \ldots, v_N) \in T_{bx}Q
\]

In other words, the group of scale transformations \( Sc \subset Sim(3) \) acts on the phase space of \( \mathbb{N} \)-body systems as follows

\[
\begin{pmatrix}
  x_1 \\
  \vdots \\
  x_N
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
  bx_1 \\
  \vdots \\
  bx_N
\end{pmatrix}
\]

\[
\begin{pmatrix}
  p_1 \\
  \vdots \\
  p_N
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
  px_1 \\
  \vdots \\
  px_N
\end{pmatrix}
\]

(5)

where \( p_i := m_i v_i \) and \( v_i \)'s are the relative velocities measured either in absolute or relational units.\(^8\)

Note that the absolute velocities of the Newtonian world-view are not related in any way to the observable relative motions we were dealing with so far (e.g., in \(^3\)). Instead, they are derived from the unobservable motion in absolute space, i.e., the motion between two "space points" instead of two reference objects. As the

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\(^8\)As explained before, the value of the relative velocities in absolute units and relational units are the same according to the modified Newtonian mechanics.
The true homogeneous function of the first degree on $Q \sim \mathbb{R}^3$ naturally leads us to the kinetic metric on shape space. For example, a relation of this kind for the current state of the universe allows the existence of Kepler pairs now\[11\]. The unit of length and time defined as the semi-major axis, and the orbital period of a Kepler pair, change under a global scale transformation $S_c$ exactly in the same way as their Planck counterparts in the modified Newtonian theory. It is because of (6). In other words, the homogeneity of the potential function of zero’s degree in the modified Newtonian theory guarantees that the Kepler pair’s orbital period becomes longer by a factor $b \in \mathbb{R}^+$ after a mechanical similarity transformation\[91\] of the universe (by the factor $b$) has been performed.

For clarity, we give here a more mathematical formulation of the principle of relationalism for particle-based mechanical theories and compare it with two other principles (postulates) in relational mechanics.

Consider an absolute physical theory, built upon the notions of absolute space $\mathbb{R}^3$ and absolute time $\mathbb{R}$ of Newton. Denote the absolute time by the letter $t(n)$. The state space $Q$ of a system according to a theory can be specified by the systems configuration $\{x\}$, and its time derivatives of sufficient order $\dot{x}, \ddot{x}, \ldots$, depending on the theory. The (absolute) theory’s law of motion is usually given by a system of coupled differential equations for the $3N$ configuration variables\[3\] with the absolute time $t$ being the independent variable. A time-dependent isomorphism can express the solutions of this system of equations (law of motion)

$$O_{t(n)}^{n,0} : Q \to Q$$

In other words, the (absolute time) parameterized curves $O_{t(n)}^{n,0}(x_0)$ with $-\infty < t(n) < \infty$, all satisfy the system of differential equations constituting the (absolute)theory’s law of motion. As the law of motion of the (modified or original) Newtonian mechanics is a second-order differential equation, just the explicit

$$x_i \rightarrow \frac{b}{p_i}$$

where $p_i := m_i v_i$. In two of the other approaches to relational physics (BKM and BDGZ\[10\]), the action of the group $Sc$ on absolute phase space is taken to be different from (6), i.e., being (102) and (108) respectively. We are led to (5) or (6) directly by the Principle of Relationalism. However, one can alternatively arrive at (6) by mechanical similarity transformations in the modified Newtonian theory. In Section (VIII) this is explained in more details.

Using (5), we have shown (in Section (IV.ii) or (11)) that the mass metric $M$ on $Q$ uniquely induces a metric $N$ on shape space $S$. In the modified Newtonian theory the invariance of the kinetic energy under scale transformations $Sc$ naturally leads us to the kinetic metric on shape space.

It is worth mentioning that the discussed property (5) of the modified Newtonian theory is not keen on using the Planck units. The true homogeneous function of the first degree on $Q$ which must lead to the value $6.6743 \times 10^{-11} \frac{m^3}{kg^2 s^2}$ for the current state of the universe governed by the modified Newtonian theory is a second-order differential equation, just the explicit

\[11\] Whether or not these Kepler pairs form in a typical universe governed by the modified Newtonian theory is a separate question to be addressed in future.

\[10\] Considered in Section (VIII) for the purpose of comparison with our work.

\[12\] Or by the theory’s “beables”.

\[13\] For particle-based mechanical theories.
specification of the velocities suffices for the determination of the solution emerging from an arbitrary configuration \( x \), and hence for the construction of the time-dependent isomorphism \( O_{t(\mathfrak{a})}^p \).

The mathematical expression of the Principle of Relationalism is as follows:

\[
\forall x_0 \in Q, \forall g \in \text{Sim}(3) : O_{t(\mathfrak{a})}^p(gx_0) = gO_{t(\mathfrak{a})}^p x_0
\]

(7)

where \( A_x v \) is the transformed velocity under the action of \( g \in \text{Sim}(3) \). As it is well known that velocities in classical mechanics are invariant under group \( T^3 \) of spatial translations and covariant under the group \( SO(3) \) of spatial rotations, it suffices to specify only the action of the group \( Sc \) of scale transformations.

As the Newtonian world view is quite central to the above formulation of the principle, some comments on its relation to the Leibnizian world view are in order. One can define the infinitesimal increment of a relational time variable as a monotonically increasing positive function of the infinitesimal increment of the system’s actual configuration \( x \), i.e.,

\[
\delta t := f( | \delta x_1 |, | \delta x_2 |, ..., | \delta x_N | )
\]

where \(| \cdot |\) denotes the Euclidean norm on the absolute space \( \mathbb{R}^3 \). For instance, one may take \( f \) as the function for the arc length on the configuration space \( Q \). It is explained in [19] that the ephemeris time \( t^{(\text{e})} \), which corresponds to a specific choice \( f_e \) for the function \( f \), replaces perfectly the absolute time \( t^{(\text{a})} \) in the Newtonian mechanics. As total speeding up or slowing down of all universe’s particles by the same factor would not lead to any observable effect without loss of generality one can set \( E = 0 \) in the denominator of \( f_e \). In this way, all possible universes

\[
A_{x} v = v_x \quad \text{where} \quad A_{x} v \in T_{x}(Q) \quad \text{and} \quad v_x \in T_{x}(Q)
\]

\[
A_{x} v = g v_x \quad \text{where} \quad A_{x} v \in T_{gx}(Q) \quad \text{and} \quad v_x \in T_{x}(Q)
\]

14\[g \in SO(3) \quad \text{where} \quad A_{x} v \in T_{x}(Q) \quad \text{and} \quad v_x \in T_{x}(Q)
\]

15See also [19], and [19] to better understand the concept of relational time.

16For any internal observer, which by definition must be a subsystem of the universe.

can be considered as having vanishing total energy. Identify Newton’s absolute time \( t^{(\text{a})} \) with the ephemeris time of a universe with vanishing total energy and pathing through a configuration \( x_0 \). Call this universe the reference universe. Its role is to provide a relational representation of Newton’s absolute time. After performing any transformation on the universe, a relational time variable may run differently compared to the absolute time \( t^{(\text{a})} \), i.e., compared to the ephemeris time of the reference universe.

An equivalent expression of the Principle of Relationalism is as follows:

\[
\forall x_0 \in Q, \forall g \in \text{Sim}(3) : O_{t'}^p(gx_0) = gO_{t'}^p x_0
\]

(8)

where \( t \) and \( t' \) denote the two alternative universe’s internal times, corresponding to the same shape, which can be moreover assumed to be initially synchronized

\[
t \mid x_0 = t' \mid x'_0 = g x_0 = 0
\]

The specification of the initial velocity of the transformed universe

\[
u' := \frac{dx'}{dt'} \mid x' = gx_0
\]

from the previous universe’s initial velocity

\[
u := \frac{dx}{dt} \mid x_0
\]

depends partly on the choice of the new internal time variable \( t' \) (so the choice of \( f \) for the new universe), and partly on the action of the group \( Sc \) on the absolute phase space.

The mentioned principle moreover entails the nonexistence of any experiment carried out by the subsystems of the two alternative universes (emerging from \( x_0 \) and \( gx_0 \) respectively), which could signify a difference between the two alternative universes, and hence be conclusive on the question: “In which one of the two alternative universes am I finding myself?”. This point, among others, specifies an action of the group of scale transformations \( Sc \) on the phase space.
as explained before (see [3] or [4]).

A theory satisfying Barbour’s fundamental postulate of relational mechanics [3]:

"An initial point and an undirected line through it in shape space should uniquely determine a solution."

or satisfying the Barbour-Bertotti’s postulate of relational mechanics [2]:

"A point and a direction in shape space should uniquely determine a solution."

known also as the Mach-Poincare’s criterion [20], certainly satisfies the above principle of relationalism, but the other way around is not always true.

The less appropriately chosen word "initial" in the expression of Barbour’s fundamental postulate refers, in fact, to a special shape on the curve achieved by projecting the solution of a Newtonian $N$-body problem to shape space $S$. This special point is defined as the minimum of the complexity (97) along the solution curve. Complexity is a $\text{Sim}(3)$-invariant function on $Q$; hence it is also a function on $S$. At this special shape, the dilational momentum $D$ vanishes, and the system’s moment of inertia is at its absolute minimum along the considered $N$-body’s solution curve on $Q$. It is just at this special shape that the specification of a direction (or undirected line) on shape space would uniquely determine a whole solution. These data would not be sufficient for a generic shape, and the additional specification of $D$ at a generic shape is needed to determine a solution. Equivalently one can say [19] that the resulting dynamics on shape space do not deal with plain geometrical paths on $S$, but centered paths (Paths with the extra structure of a "central point"); and a dynamics which depends appropriately on the "distance" from the central point.

In contrast, Barbour-Bertotti’s postulate is a statement about every point in shape space and is, therefore, a stronger (more stringent) criterion than Barbour’s fundamental postulate.

For the purpose of comparison, consider an absolute theory that contains some interaction potentials depending on the particles’ relative velocities or relative accelerations or ... . Such a theory (under some circumstances) can still perfectly satisfy the principle of relationalism. However, the reformulation of its law of motion on shape space would certainly neither satisfy the fundamental postulate of Barbour nor Barbour-Bertotti’s postulate. Hence, the principle of relationalism is a broader statement and a less stringent requirement than the other two postulates mentioned above.

The BKM-approach satisfies Barbour’s fundamental postulate but does not satisfy Barbour-Bertotti’s postulate. This last criterion is, however, satisfied by the BDGZ-approach. In fact, in the DGZ-approach [15], it has been shown that even a similarity invariant interaction potential can be explained by a geodesic motion on shape space; for a suitable choice of conformal factor [20], and hence still satisfies the Barbour-Bertotti’s postulate.

18 Which is a dynamical variable in the Newtonian theory of gravitation but a constant of motion of the modified Newtonian theory.
19 As was pointed out to us by Sheldon Goldstein in private correspondence.

20 Namely multiplying the conformal factor used for the non-interacting theory on shape space with the scale-invariant potential.
III. GEOMETRY OF $Q_{cm}$ AS $SO(3)$-FIBER-BUNDLE

This section reviews the geometry of $Q_{cm}$ as a principal-fiber-bundle, with $G = SO(3)$ being its structure group. It provides a preparation for the content of the next sections. We discuss the notion of connection form on $SO(3)$-fiber-bundle, decomposition of the kinetic energy (or metric) in rotational and vibrations parts, and its expressions in terms of coordinates adopted to the fiber-bundle structure. As references, one can find more information regarding these topics in [4], [5], [6].

i. Fiber-bundle structure and definition of the connection form

The absolute configuration space of a $N$-particle system is the set

$$Q = \{ x = (x_1, ..., x_N) \mid x_i \in \mathbb{R}^3 \} \cong \mathbb{R}^{3N}$$

The center of mass system

$$Q_{cm} = \{ x = (x_1, ..., x_N) \mid \sum_{j=1}^{N} m_j x_j = 0 \} \cong \mathbb{R}^{3(N-1)}$$

with $x_j \in \mathbb{R}^3$, forms a stratified fiber-bundle by the action of the group $G = SO(3)$. A new coordinate system on $Q$ adapted to the projection $Q \rightarrow Q_{cm}$ is given by the following linear transformation

$$
\begin{pmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_N \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
  r_1 \\
  r_2 \\
  \vdots \\
  r_{N-1} \\
\end{pmatrix}
$$

where $R_{cm} = \frac{1}{\sum_{i=1}^{N} m_i} \sum_{i=1}^{N} m_i r_i$ is the center of mass of the system, and the $N - 1$ vectors $r_i$ are the mass weighted Jacobi vectors defined as follows

$$r_j := \left( \frac{1}{\mu_j} + \frac{1}{m_{j+1}} \right)^{-\frac{1}{2}} (x_{j+1} - \frac{1}{\mu_j} \sum_{i=1}^{j} m_i x_i) \quad (9)$$

with $\mu_j := \sum_{i=1}^{j} m_i$. Then, the center of mass configuration space $Q_{cm}$ can be expressed in these coordinates by

$$Q_{cm} \cong \{ x = (r_1, ..., r_{N-1}) \mid r_j \in \mathbb{R}^3, j = 1, ..., N - 1 \} \quad (10)$$

By the introduction of the coordinate transformation $(x_1, ..., x_N) \rightarrow (r_1, ..., r_{N-1}, R_{cm})$ on the absolute configuration space $Q$, the system’s total kinetic energy naturally separates in the center of mass kinetic energy and the center of mass rotational energy naturally. Specifically system’s total kinetic energy

$$T = \frac{1}{2} \sum_{a=1}^{N} m_a | \dot{x}_a |^2 = \sum_{a, \beta=1}^{N} K_{a\beta}(\dot{x}_a, \dot{x}_\beta)$$

with $K_{a\beta} = \frac{m_a}{m_\beta} \epsilon_{a\beta}^{(3)}$ being the $3N \times 3N$ kinetic tensor (110), transforms to

$$T = \frac{1}{2} \sum_{a=1}^{N-1} | \dot{r}_a |^2 + \sum_{i=1}^{N} \frac{m_i}{2} | \dot{R}_{cm} |^2$$

(11)

One can view each element of $Q_{cm}$ as a $3 \times (N - 1)$-matrix. The rank of this matrix can suitably describe the stratification of $Q_{cm}$. Namely,

$$Q_{cm} = Q_{cm0} \cup Q_{cm1} \cup Q_{cm2} \cup Q_{cm3} \quad (12a)$$

$$Q_{cmk} := \{ x \in Q_{cm} \mid \text{rank}(x) = k \} \quad (12b)$$

For instance, $Q_{cm0}$ denotes the simultaneous total collisions of all the particles, $Q_{cm1}$ denotes the linear configurations, and $Q_{cm2}$ the planar configurations (all the particles are located on a single plane in $\mathbb{R}^3$). Members of $Q_{cm0}$ and $Q_{cm1}$ are called singular configurations.

Action of the rotation group $SO(3)$ on $Q_{cm}$, is as follows

$$x \rightarrow gx = (gr_1, ..., gr_{N-1})$$

for all $g \in SO(3)$. It is well known that the above $SO(3)$ action can be used to define an equivalence relation on $Q_{cm}$ with help of which one defines a quotient space $Q_{int} := Q_{cm} / SO(3)$ as the space of the equivalence classes. We denote the corresponding projection map
by \( \pi : Q_{cn} \to Q_{cm} \). 

The isotropy group at \( x \in Q_{cm} \) is defined as the following subgroup of the structure group \( SO(3) \)

\[
G_x := \{ g \in SO(3) \mid gx = x \} \tag{13}
\]

and the \( SO(3) \) orbit through \( x \) is defined as follows

\[
O_x := \{ gx \mid g \in SO(3) \} \tag{14}
\]

One can easily verify the following equalities

\[
G_x = \begin{cases} 
  e & : x \in Q_{cm2} \cup Q_{cm3} \\
  SO(2) & : x \in Q_{cm1} \\
  SO(3) & : x \in Q_{cm0}
\end{cases}
\]

and

\[
O_x \cong \frac{SO(3)}{G_x} \cong \begin{cases} 
  SO(3) & : x \in Q_{cm2} \cup Q_{cm3} \\
  S^2 & : x \in Q_{cm1} \\
  0 & : x \in Q_{cm0}
\end{cases}
\]

In this sense, \( Q_{cm} \) is stratified into a union of strata, i.e. \( Q_{cm} = Q_{cmns} \cup Q_{cm1} \cup Q_{cm0} \). Where \( ns \) stands for non-singular. So all \( x \in \partial Q_{cmns} = Q_{cm} \setminus Q_{cmns} \) are singular configurations. \( Q_{cmns} \) contains all the two and three dimensional configurations in the absolute space.

Since each stratum is \( SO(3) \) invariant, we can consider \( Q \) as a stratified fiber-bundle where each stratum has its own projection map

\[
\begin{align*}
Q_{cmns} & \to \frac{Q_{cmns}}{SO(3)} \\
Q_{cm1} & \to \frac{Q_{cm1}}{SO(3)} \\
Q_{cm0} & \to \frac{Q_{cm0}}{SO(3)}
\end{align*}
\]

whose fibers are \( O_x \cong \frac{SO(3)}{e_x} \). In particular, \( Q_{cmns} \) is a \( SO(3) \) principal-fiber-bundle, as the structure group has a free action on \( Q_{cmns} \).

Associated with the group action of \( SO(3) \) on \( Q_{cm} \), a moving frame

\[
\{ e'_1, e'_2, e'_3 \}
\]

of the absolute space \( \mathbb{R}^3 \) given by

\[
e'_1 = g e_1, e'_2 = g e_2, e'_3 = g e_3
\]

can be attached to the mechanical system, where \( \{ e_1, e_2, e_3 \} \) stands for the (fixed) space frame. The Euler angles \( \phi^1, \phi^2, \phi^3 = \alpha, \beta, \gamma \) are used to specify the orientation of the moving frame w.r.t. the space frame. Hence, any point on \( Q_{cm} \) can be specified by \( 3N - 6 \) internal coordinates \( q^i \) relative to the body frame and the three Euler angles \( \phi^i \). The components of a vector in \( \mathbb{R}^3 \) with respect to the body frame will be denoted from now on by a subscript ‘b’. The relation between expression of a vector in the space frame and the body frame is as follows

\[
v_b = g^{-1} v_s
\]

where \( g \), as already mentioned, is the rotation that takes the space frame to the body frame. For notational simplicity, from now on, we drop the subscript \( s \) whenever we mean the expression of vectors w.r.t. the space frame.

The moment of inertia tensor expressed in Jacobi coordinates, is the following map

\[
A_x : \mathbb{R}^3 \to \mathbb{R}^3 \\
A_x(v) = \sum_{j=1}^{N-1} r_j \times (v \times r_j) \tag{15}
\]

for any \( v \in \mathbb{R}^3 \) and \( x \in Q_{cmns} \). You can parallel transport the lab frame \( \{ e_1, e_2, e_3 \} \) in absolute space to the center of mass of the system under study. Denote the vectors connecting the center of mass to the particle \( a \) by \( x_a \). In these coordinates, the moment of inertia tensor takes the more familiar form

\[
A_x(v) = \sum_{a=1}^{N} m_a x_a \times (v \times x_a)
\]

Note the operator \( A_x^{-1} \) just exists for \( x \in Q_{cmns} \). The reason is that \( A_x \) sends all vectors to zero if any \( x \) belongs to \( Q_{cm1} \cup Q_{cm0} \), hence can not be inverted.

For a velocity vector \( \dot{x} \in T_x(Q_{cm}) \)

\[
\dot{x} = \begin{bmatrix} 
  \dot{x}_1 \\
  \dot{x}_2 \\
  \vdots \\
  \dot{x}_{3N}
\end{bmatrix} = \begin{bmatrix} 
  \dot{x}_1 \\
  \dot{x}_2 \\
  \vdots \\
  \dot{x}_N
\end{bmatrix}
\]
denote $dx_i$ as the 1-form defined as follows
\[ dx_i(x) := \delta_i \]
Similarly, the 1-forms $dr_i$'s are defined for any $1 \leq i \leq N - 1$. For any $x \in Q_{cmons}$, a connection form \[ \omega_x : T_x(M_{ns}) \to \text{so}(3) \]
is known to be defined with the help of $A$, as follows
\[ \omega_x := R(A_x^{-1}(\sum_{j=1}^{N-1} r_j \times dr_j)) \quad (16) \]
Here $R : \mathbb{R}^3 \to \text{so}(3)$ is defined as
\[ R(a) = \begin{bmatrix} 0 & -a^3 & a^2 \\ a^3 & 0 & -a^1 \\ -a^2 & a^1 & 0 \end{bmatrix} \quad (17) \]
for any $a \in \mathbb{R}^3$ (see appendix (iii)) and the expression of one form $dr_i$ in orientational and internal coordinates is given by
\[ dr_i = \sum_{i=1}^{3N-6} \frac{\partial r_i}{\partial q^i} dq^i + \sum_{a=1}^{3} \frac{\partial r_i}{\partial \phi^a} d\phi^a \]

ii. Local expression of the Connection form and the metric in orientational and internal coordinates

In this subsection we review (based on [4] and [5]) the expression of the connection form [16] and the kinetic metric in terms of some new coordinates, which shall be introduced on the center of mass configuration space $Q_{cmns}$. Consider an open subset $U \subset \frac{Q_{cmons}}{SO(3)}$, and define a local section
\[ \sigma : U \to Q_{cmons} \]
Then, any point $x \in \pi^{-1}(U)$ can be expressed as
\[ x = g\sigma(q) = (g\sigma_1(q), ..., g\sigma_{N-1}(q)) \quad (18) \]
with $g \in SO(3)$, and $q \in U \subset \frac{Q_{cmons}}{SO(3)}$. So, $q$ denotes a point on the internal configuration space $Q_{int}$. Note that
\[ \sigma(q) = \begin{pmatrix} \sigma_1(q) \\ \vdots \\ \sigma_{N-1}(q) \end{pmatrix} = \begin{pmatrix} \sum_{a=1}^{3} C_a^q e_a \\ \vdots \\ \sum_{a=1}^{3} C_{N-1}^q e_a \end{pmatrix} \]
determines a way to put the multi-particle system with internal configuration (or shape in the next section) $q \in U$, in the absolute space $\mathbb{R}^3$. This allocation is achieved by choosing the point $x \in Q_{cm}$ on the fiber above $q$, to which the internal configuration $q$ is meant to be lifted as follows
\[ r_{3(i-1)+a} := C_i^q \]
So specifying a section $\sigma$, comes down to the specification of a set of $3(N-1)$ real valued functions on $Q_{int}$, i.e.
\[ C_i^q : Q_{int} \to \mathbb{R} \]
For another subset $V \subset Q_{int}$ with $V \cap U \neq \emptyset$, one can define another local section
\[ \tau : V \to Q_{cm} \]
such that $\sigma(q) = h\tau(q)$ for $(q, h) \in V \times SO(3)$. The two local sections $\sigma$ and $\tau$ on $V \cap U$ are then related by $\tau = k\sigma$, where $k$ is a $SO(3)$-valued function on $U \cap V$.

The vertical and horizontal vectors (and one forms) can be expressed with respect to a local trivialization of the center of mass configuration space $^{22}$
\[ Q_{cm} \cong \mathbb{R}^{3N-3} \cong U \times SO(3) \]
To this end, denote the components of the connection form
\[ \omega = R(A_x^{-1} \sum_{a=1}^{N} m_a x_a \times dx_a) \quad (19) \]

$^{21}$In this case, they are known as rotational and vibrational vectors in the molecular physics literature.
$^{22}$Technically speaking, we mean the non-singular stratum of configuration space.
\[ R(A_x^{-1}(\sum_{j=1}^{N-1} r_j \times dr_j)) \]

with respect to the laboratory frame \( e_a \) as follows

\[ \omega = \sum_{a=1}^{3} R(e_a) \omega^a = \sum_{a=1}^{3} R(e'_a) \omega'^a \]

where \( \omega^a : = \omega . R(e_a) \), \( \omega'^a := \omega' . R(e'_a) \). Denote also the components of the total angular momentum operator

\[ J = \sum_{a=1}^{N} n_a x_a \times dx_a = \sum_{a=1}^{N-1} r_a \times dr_a \]

in the fixed laboratory frame and the moving frame as

\[ J = \sum_{a=1}^{3} e_a J_a = \sum_{a=1}^{3} e'_a L_a \]

where \( J_a = (e_a \mid J) \), and \( L_a = (e'_a \mid J) \).

Remember that

\[ J_a r_j = e_a \times r_j \]

\[ L_a r_j = e'_a \times r_j = g(e_a \times \sigma_j(q)) \]

The forms \( \{ \omega^a, dq^i \} \) with \( a = 1, 2, 3 \) and \( i = 1, \ldots, 3N - 6 \) constitute a local basis of the space of 1-forms on \( T^* (Q_{cm}) \). Moreover, the vector fields \( L_a \) and

\[ (\frac{\partial}{\partial q^a})^\ast := \frac{\partial}{\partial q^a} - \sum_{a} \wedge_j L_a \]  

form a local basis of the space of vector fields \( T_x(Q_{cm}) \). Here \( (\frac{\partial}{\partial q^a})^\ast \) is called the “horizontal lift” of \( \frac{\partial}{\partial q^a} \) from the internal configuration space \( Q_{int} \) to \( Q_{cm} \). It is defined by the conditions

\[ \omega((\frac{\partial}{\partial q^a})^\ast) = 0 \]

\[ \pi_x(\frac{\partial}{\partial q^a})^\ast = \frac{\partial}{\partial q^a} \]

which explain its name. These conditions lead to the following expression for the functions \( \wedge_i^a \) in (20)

\[ \wedge_i^a = \langle A_x^{-1}(\sum_{a=1}^{N-1} r_a \times \frac{\partial}{\partial q^a} \mid e_a) \rangle \]  

So, we have the following identities

\[ \omega^a(J_b) = \delta^a_b, \omega^a((\frac{\partial}{\partial q^a})^\ast) = 0 \]

\[ dq^i(J_b) = 0, dq^i((\frac{\partial}{\partial q^a})^\ast) = \delta^i_j \]

\[ \omega'^a(L_b) = \delta^a_b, \omega'^a((\frac{\partial}{\partial q^a})^\ast) = 0 \]

\[ dq^i(L_b) = 0, dq^i((\frac{\partial}{\partial q^a})^\ast) = \delta^i_j \]

By writing out the connection form in the local coordinates \((q, g)\), one can calculate the local expression of \( \omega^a \)

\[ \omega^a = \psi^a + \sum_{i=1}^{3N-6} \wedge_i^a dq^i \]

(22)

\[ \omega'^a = \theta^a + \sum_{i=1}^{3N-6} \wedge_i'^a dq^i \]  

(23)

where \( \psi^a \) and \( \theta^a \) are respectively the three right and left invariant one-forms on \( SO(3) \) defined through

\[ dg g^{-1} =: \sum_{a=1}^{3} \psi^a R(e_a) \]

(24)

\[ g^{-1} dg =: \sum_{a=1}^{3} \theta^a R(e_a) \]  

(25)

The \( \psi^a \) and \( J_a \) can be expressed in terms of Euler angles, and are dual to each other, i.e.,

\[ \psi^a(J_b) = \delta^a_b \]

\[ \theta^a(L_b) = \delta^a_b \]

\[ 23 \text{Mathematically speaking one should write } \pi^* dq^i \text{ instead of } dq^i \text{ there.} \]
It can be shown that the rotational vector fields $J_a$ and the vibrational vector fields $(\frac{\partial}{\partial q_i})^\alpha$ satisfy the following commutation relations [5]

$$[J_a, J_b] = -\sum_{c=1}^3 \epsilon_{abc} J_c \quad (26a)$$

$$[(\frac{\partial}{\partial q_i})^\alpha, (\frac{\partial}{\partial q_j})^\beta] = -\sum_{c=1}^3 F_{ij}^c J_c \quad (26b)$$

$$[(\frac{\partial}{\partial q_i})^\alpha, J_a] = 0 \quad (26c)$$

The second equation implies that two independent vibrational vectors are coupled in a way to lead to an infinitesimal rotation, and because of this, vibrations and rotations are inseparable. Another fact is that the distribution spanned by the vectors $(\frac{\partial}{\partial q_i})^\alpha$ is not completely integrable. If it was so, there existed a submanifold to which $(\frac{\partial}{\partial q_i})^\alpha$ are tangential, and this surface could be identified with the internal space $Q_{int}$. The constraint of constancy of total angular momentum is realized by selecting the distribution spanned by $(\frac{\partial}{\partial q_i})^\alpha$. This constraint is an anholonomic one in the case of non-vanishing angular momentum and a holonomic one in the case of vanishing angular momentum.

According to the orthogonal decomposition

$$T_x(M_{ns}) = V_x \oplus H_x$$

it is known that the kinetic metric can be expressed in terms of the one-forms $dq^a$, and $\omega^a$ as follows

$$ds^2 = \sum_{a, b=1}^{3N-6} a_{ab} dq^a dq^b + \sum_{a, b} A_{ab} \omega^a \omega^b \quad (27)$$

where

$$\left\{ \begin{array}{l}
a_{ab} := ds^2((\frac{\partial}{\partial q_i})^\alpha, (\frac{\partial}{\partial q_j})^\beta) \\
A_{ab} := ds^2(J_a, J_b) = e_a \cdot A_{\sigma(q)}(e_b)
\end{array} \right.$$  

Remember that $\omega^a(J_b) = \theta^a(J_b) = \delta^a_b$, and notice that $(a_{ab})$ defines a Riemannian metric on the internal space $Q_{int}$ of $SO(3)$, and $A$ is the inertia tensor. One way to see why the coefficients $A_{ab}$ appearing in (27) coincide with the coefficients of inertia tensor (15) is through the following calculation

$$ds^2(J_a, J_b) = \sum_{a=1}^{N-1} (dr_a, J_a) \cdot (dr_b, J_b)$$

$$= \sum_{a=1}^{N-1} (e_a, r_a) \cdot (e_b, r_a)$$

$$= A_{ab}$$

To clarify the notation, we add here that $dr_a$ is a 1-form which tells us how much (infinitesimally) the position of the a’s particle, i.e., $r_a$, would change if we let it act on an infinitesimal displacement (a tangent vector). In this particular case, we have chosen the infinitesimal generator of rotation about the a’s axis, i.e., $J_a$. So, from this it should be clear how the substitution $dr_a, J_a = e_a \times r_a$ used above, is justified.

It is worth mentioning that we have so far seen two different kinds of vector fields defined on $SO(3)$. Namely $L_a$ and $J_a$, for $a = 1, 2, 3$. The first set of vector fields $L_a$, are duals to the one-forms $\theta^a$ defined by (23). $L_a$ coincides with the direction of movement along the fiber (which is itself part of $T(Q_{cm})$), if we rotate the system in $\mathbb{R}^3$ around the a’th axis of body frame ($e'_a = ge_a$), without changing it’s shape. The second set of vector fields $J_a$, are, however, dual to the one-forms $\psi^a$ defined by (22). They coincide with the direction of movement along the fiber if we rotate the system in $\mathbb{R}^3$ around the a’th axis of space frame $e_a$, without changing its shape. Because changing the shape causes extra total rotation and makes the direction of the movement of the configuration point deviate from $L_a$ or $J_a$, depending on how fast and in which way the system’s shape is changing, one needs the second terms in (22) and (23).

A local expression of the connection form is given in terms of the left-invariant one forms on $SO(3)$ and local section $\sigma$ of the fiber-bundle. Let’s take the Euler angles
\[(a, \beta, \gamma)\) as coordinates on \(SO(3)\), and \(q^a\) with \(a = 1, \ldots, 3N - 6\), as local coordinates on \(U \subset Q_{\text{cm}}\). So, \(g\) and \(q\) can be expressed in terms of them, respectively. A point \(x\) on the center of mass configuration system, can hence be expressed by the coordinates \(x = (q^1, \ldots, q^{3N-6}, \phi, \theta, \psi)\). The connection form at the point \(g \sigma(q)\), can then be written as follows

\[
\omega_{g \sigma(q)} = dg g^{-1} + g \omega_{\nu(q)} g^{-1}
\]

where

\[
\omega_{\nu(q)} := R \left( A^{-1}_{\nu(q)} \left( \sum_{j=1}^{N-1} \sigma_j(q) \times d \sigma_j(q) \right) \right)
\]

Now take the fixed laboratory basis \(e_a\) of \(R^3\), with \(a = 1, 2, 3\). We introduced three left invariant one-forms \(\theta^a\) on \(SO(3)\), and \(3 \times (3N - 6)\) functions \(\wedge^a\) by the following equations (4) and (5)

\[
g^{-1} dg := \sum_{a=1}^{3} \theta^a R(e_a)
\]

\[
\omega_{\nu(q)=x} := \sum_{a=1}^{3N-6} \sum_{a=1}^{3N-6} \wedge^a(x) dq^a R(e_a)
\]

\[
= \sum_{a=1}^{3N-6} \wedge^a dq_a = \sum_{a=1}^{3N-6} R(\lambda_a) dq_a
\]

in which the following is used

\[
R(\lambda_a) = \wedge_a(q) = \sum_{a=1}^{3N-6} \wedge^a(q) R(e_a)
\]

to express \(\omega\) in a compacter form. The coefficients \(\wedge^a\) in the last equation were given by (21), which are also known as gauge potentials.

Now we rewrite the connection form (28) in terms of \(\theta^a\) and \(\wedge^a\)

\[
\omega_{g \sigma(q)} = g^{-1} dg + g \omega_{\nu(q)} g^{-1}
\]

\[
= g \left( \sum_{a=1}^{3N-6} \theta^a R(e_a) + \sum_{a=1}^{3N-6} \wedge^a(x) dq^a R(e_a) \right) g^{-1}
\]

\[
= g \left( \sum_{a=1}^{3N-6} \theta^a + \sum_{a=1}^{3N-6} \wedge^a(q) \right) R(e_a) g^{-1}
\]

in the third line of the above calculation, we could move \(g\) from the behind of the term \(\sum_{a=1}^{3N-6} \wedge^a(x) dq^a\) to the front of it, cause the forms \(dq^a\) by definition with \(g\), and the forms \(\theta^a\) are left invariant. Thus, the connection form (16) expressed in local coordinates \((q^1, \ldots, q^{3N-6}, a, \beta, \gamma)\), takes the following form

\[
\omega_{g \sigma(q)} = \sum_{a=1}^{3N-6} \theta^a R(e_a) = \sum_{a=1}^{3N-6} \omega^a R(e_a)
\]

As mentioned before, from the fixed space frame \(e_a\), a moving frame \(e'_a\) can be defined as \(e'_a = g e_a\). One can think of \(\omega^a\) as components of \(\omega\) in the moving frame, cause \(\omega R(e'_a) = \omega^a\).

The horizontal lift\(^{21}\) \((\frac{\partial}{\partial \rho^a})^*\), of a local vector field \(\frac{\partial}{\partial \rho^a}\) on \(U\) to a point \(x \in Q_{\text{cm}}\) can be shown to be given by (see\([^5\])

\[
\left( \frac{\partial}{\partial \rho^a} \right)^* = \frac{\partial}{\partial \rho^a} - \sum_{a=1}^{3N-6} \wedge^a(x) L_a
\]

with

\[
\wedge^a(x) = \left( A^{-1}_{\nu(q)} \left( \sum_{i=1}^{N-1} \frac{\partial r_i}{\partial \rho^a} \right) e'_a \right)
\]

In the above expression as usual \(L_a\) denote the left invariant vector fields on \(SO(3)\), which are dual to \(\theta^a\), i.e., \(\theta^a(L_b) = \delta^a_b\). So

\[
\frac{\partial}{\partial \rho^a} , \omega^a
\]
and 

$$\left( \frac{\partial}{\partial q^a} \right)^*, L_a$$

form local bases of the 1-forms, and of the vector fields on $\pi^{-1}(U) \cong U \times SO(3)$, respectively. They are in accordance with the decomposition $T_x(Q_{cm}) = V_x \oplus H_x$. As mentioned before, $L_a$ can be identified with the infinitesimal rotation with respect to axis $e'_a$ of the body frame. Technically speaking, we have to use $\pi^* dq^a$, the pullback of $dq^a$ under the bundle’s projection map, but for the sake of notational simplicity, we still used $dq^a$.

Expression (32) can be derived by requiring $\omega_x \left( \left( \frac{\partial}{\partial q^a} \right)^* \right) = 0$, which is of course how a horizontal lift w.r.t. a connection should be.

### IV. Ingredients for the Reduction with respect to the Similarity Group

Which equations of motion do the evolution of the shape of a classical system behaving according to the modified Newtonian theory satisfy is the central question to be answered in this and next section. To this end, we explain in this section how the necessary ingredients, i.e., the metric $N$ on the reduced tangent-bundle $T(Q_{cm})$ and the connection form $\omega$ for the $Sim(3)$-fiber-bundle $Q_{cm}$ can be derived.

The mass metric $M$ of the absolute configuration space $Q \cong \mathbb{R}^N$ induces metrics on the reduced spaces, like the internal configuration space $Q_{int} = \frac{Q}{E(3)}$, shape space $S = \frac{Q}{Sim(3)}$ and $Sim(3)$-reduced tangent-bundle in a natural fashion. We first review the derivation of the metric on $Q_{int}$, following [6]. Then, using the principle of relationalism, we explain a new way to derive a metric $N$ on the $Sim(3)$-reduced tangent bundle $T(Q_{cm})$ from the mass metric $M$ on the absolute configuration space in a unique way. We also show how to derive the metric structure $N$ on shape space $S$ in a unique way. On the other hand, the direct implementation of the principle of relationalism leads to scale-invariant interaction potentials, which is the key to the complete decoupling of dynamics on shape space from the gauge degrees of freedom, i.e., the $Sim(3)$ degrees of freedom.

#### i. Metric on the internal space

Let us review how the metric $B$ on the internal space $Q_{int} = \frac{Q_{cm}}{SO(3)}$ can be derived from the $SO(3)$-invariant mass metric $M$ on the the center of mass configuration space $Q_{cm}$, i.e.,

$$M_x(u, v) = \sum m_k < u_k | v_k > \quad (33a)$$
$$M_x(u, v) = M_{gx}(g[u, gv]) \quad (33b)$$

where $u = (u_1, ..., u_N)$ and $v = (v_1, ..., v_N)$ are members of $T_x(Q_{cm})$, so being any two tangent vectors of $Q_{cm}$ at the point $x \in Q_{cm}$.
Given two internal vectors 

\[ v', u' \in T_q(Q_{int}) \]

there are unique vectors \( u, v \in T_x(Q_{cm}) \) \(^{26}\) so that

\[
\begin{align*}
\pi(x) &= q \\
\pi_x(u) &= u' \\
\pi_x(v) &= v'
\end{align*}
\]

Now, the metric \( B \) on \( Q_{int} \) can be defined by the following equation:

\[
B_q(v', u') := M_q(v, u).
\] (34)

As the metric \( M \) is \( SO(3) \)-invariant, to which \( x \in \pi^{-1}(q) \) the internal vectors \( v, u \) are lifted, would not make any difference for the value assigned by \( B_q \). Hence, the derived metric is well-defined.

The kinetic energy of a \( N \)-particle system in the center of mass frame, coordinatized by the \( N - 1 \) Jacobi vectors \( r_a \) is as follows

\[
K = 0.5 \sum_{a=1}^{N-1} |\dot{r}_a|^2
\]

Now, consider the Jacobi vectors \( r_{ba} \) in body frame, and denote the system’s angular velocity with respect to the body frame by \( \omega \). Using

\[
r_{ba} = \omega \times r_{ba} + \frac{\partial r_{ba}}{\partial q^\mu} \dot{q}^\mu
\]

and the expression for the so called gauge potential

\[
A_{\mu}(q) := A^{-1} a_{\mu}
\]

with

\[
a_{\mu} = a_{\mu}(q) := \sum_{a=1}^{N-1} r_{ba} \times \frac{\partial r_{ba}}{\partial q^\mu}
\] (35)

and \( A \) being the moment of inertia tensor

\[
A_{ij} = A_{ij}(q) := \sum_{a=1}^{N-1} (r_{ba} \delta_{ij} - r_{ba i} r_{ba j})
\]

one can write down the kinetic energy as follows (see \(^6\))

\[
K = 0.5 < \omega \mid A \mid \omega > + < \omega \mid A_{\mu} \mid \dot{q}^\mu
\] (36)

with

\[
h_{\mu\nu} = h_{\mu\nu}(q) = \sum_{a=1}^{N-1} \frac{\partial r_a}{\partial q^\mu} \frac{\partial r_a}{\partial q^\nu}
\] (37)

The velocity of a system’s configuration in Jacobi coordinates is given by a vector

\[
|v> = |\dot{r}_1, ..., \dot{r}_{N-1}>
\]

and in orientational and internal coordinates by the vector

\[
|v> = |\dot{\theta}^i, \dot{q}^\mu>
\]

with \( 1 \leq i \leq 3 \), and \( 1 \leq \theta \leq 3N - 6 \). The \( \theta^i \)’s are the Euler angles, which turn the space frame to the body frame of a configuration. If one decides to use the components of body angular velocity \( \omega \) instead of the time derivatives of Euler angles for denoting vectors in \( T(SO(3)) \), the configuration’s velocity can alternatively be expressed as follows

\[
|v> = |\omega, \dot{q}^\mu>
\]

in angular velocity and internal basis. This last combination form an anholonomic frame or a vielbein on \( T(Q_{cm}) \). Remember the relation between the body components of angular velocity and derivatives of Euler angles

\[
\begin{bmatrix}
\omega_1 \\
\omega_2 \\
\omega_3
\end{bmatrix} =
\begin{bmatrix}
-sin\beta cos\gamma & sin\gamma & 0 \\
-sin\beta sin\gamma & cos\gamma & 0 \\
cos\beta & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\dot{\alpha} \\
\dot{\beta} \\
\dot{\gamma}
\end{bmatrix}
\]

So, the expression for the mass metric \( M \) in angular and internal coordinates \( \{\omega^\nu, \dot{q}^\mu\} \) becomes as follows

\[
< |v, v> = |\omega^T \dot{q}^\nu|
\begin{bmatrix}
A & AA_{\mu} \\
A_{\nu} A & h_{\mu\nu}
\end{bmatrix}
|\omega^\nu
\]

\[
= M_{ab} v^a v^b
\]
With $1 \leq a, b \leq 3N - 3$. In other words, the metric on $Q_{cm}$ in angular and internal basis vectors $[\omega, \dot{q}^\mu]$ is given as follows

$$M_{ab} = \begin{bmatrix}
A & A A_v \\
A_a^\mu A & h_{\mu\nu}
\end{bmatrix}$$  \hspace{1cm} (38)

$h_{\mu\nu}$ can be considered as the restriction of the mass metric metric $M$ of $Q_{cm}$, on the section determined by the choice made for the body frame for each internal configuration $q = (q_1, ..., q_{3N-6})$ (see [6]).

Decomposition of an arbitrary configuration velocity, in horizontal and vertical parts, becomes as follows

$$| v \rangle = | v_v \rangle + | v_h \rangle$$

$$[\omega, \dot{q}^\mu] = [\omega + A_\mu q^\nu, 0] + [-A_v q^\nu, \dot{q}^\mu]$$

Correspondingly, the kinetic energy of the system can also be thought of as the addition of two separate vertical and horizontal kinetic energies

$$K = K_v + K_h = \frac{1}{2} (\omega + A_\mu q^\nu) A (\omega + A_v q^\nu) + \frac{1}{2} B_{\mu\nu} \dot{q}^\mu \dot{q}^\nu$$

where $B_{\mu\nu}$ is a new metric on internal space, which is (contrary to $h_{\mu\nu}$) invariant under changing the choice of the body frame [6]

$$B_{\mu\nu} = h_{\mu\nu} - A_\mu A A_v$$

So, in summary to a vector

$$| v' \rangle = \dot{q}^\mu$$

on the internal space $Q_{int}$, we associate a vector $| v_h \rangle$ on $Q_{cm}$, which is called its horizontal lift, cutting the two neighbouring $\text{SO}(3)$-fibers orthogonally. In angular velocity and shape basis, horizontal lift of $v'$ takes the form

$$| v_h \rangle = [-A_\mu q^\nu, \dot{q}^\mu]$$

Then, the metric $B_{\mu\nu}$ on the internal space $Q_{int}$ can be found by the following defining equation

$$\langle v'_1 | v'_2 \rangle = B_{\mu\nu} q^\nu_1 q'^\nu_2 := \langle v_{1h} | v_{2h} \rangle$$

and this leads directly to

$$B_{\mu\nu} = h_{\mu\nu} - A_\mu A A_v$$ \hspace{1cm} (39)

For more detailed explanations, we suggest the reader look at [6], where a clear and complete presentation of this topic is given. Later we will see in an explicit example of three particle system what the metric $B$ becomes explicitly.

ii. Unique metric on $\text{Sim}(3)$-reduced tangent-bundle and on shape space

Bearing in mind that measurements of the velocities are, in essence, an experimental task, the transformation law of the velocities under scale transformations of the system (or any other transformation of the system) must also include experimental reasoning. Based on the principle of relationalism, we showed that the behavior of rods and clocks under scale transformations of the system is such that the measured velocities of objects (subsystems) are invariant under scale transformations. It is a natural consequence of the simultaneous expansion of the measuring rod and the corresponding dilation of the unit of time (See Section (II) for an explanation of this fact). Hence, a velocity vector

$$v_x = (v_1, ..., v_N) \in T_x(Q_{cm})$$

of a N-particle system transforms under scale transformations of the system as follows

$$x \rightarrow cx$$

$$v_x = (v_1, ..., v_N) \in T_x(Q_{cm}) \rightarrow v_{cx} = (v_1, ..., v_N) \in T_{cx}(Q_{cm})$$

Given the above action $A_c$ of $c \in \text{Sc} \subset \text{Sim}(3)$ on velocities(or on $T(Q)$); the mass metric is a $A_{Sc}$-invariant metric on $Q$, as can be seen by a short calculation:

$$M_x(v_x, u_x) \rightarrow M_{cx}(A_c v_x, A_c u_x) = M_x(v_{cx}, u_{cx})$$ \hspace{1cm} (40)

where the equalities $M_x = M_{cx} = M$ and $A_c v_x = v_{cx}$ has been used. Considering
\[ T(Q_{int}) = T\left(\frac{Q}{\text{Sim}(3)}\right) \] as a \(A_{Sc}\)-fiber-bundle, the mass metric \(B\) on \(Q_{int} = \frac{Q}{\text{Sim}(3)}\) (defined previously by expression (34)) induces a unique metric

\[ N_s : T(Q_{int}) / A_{Sc} \times T(Q_{int}) / A_{Sc} \rightarrow \mathbb{R} \]
as follows

\[ N_s(v', u') := B_q(v, u) \tag{41} \]

where

\[
\begin{cases}
\pi(q) = s \\
\pi'(u) = u' \\
\pi'(v) = v'
\end{cases}
\]

with the projection maps defined as follows

\[ \pi : Q_{int} \rightarrow S \]
\[ \pi' : T(Q_{int}) \rightarrow T(Q_{int}) / A_{Sc} \]

Because the above construction is \(A_{Sc}\)-invariant, to which \(q \in \pi^{-1}(s)\) the pair of shape vectors \(v', u' \subset T_q(Q_{int}) / A_{Sc}\) are lifted, does not make any difference for the value assigned by \(N_s\) to them. Hence, the metric \(N\) is also well defined. This method brings one uniquely to the shape (kinetic)metric involved with it.

Since the mass metric \(M\) is not scale invariant, i.e.,

\[ M_{cx}(Sc_s u, Sc_s v) = M_{cx}(cu, cv) = c^2 M_{cx}(v, u) \tag{42} \]

it is generally believed that, contrary to \(Q_{int}\), it does not uniquely induce a metric on shape space \(S\). However, once one uses measuring units built from matter instead of absolute measuring units, one sees that the mass metric uniquely induces a metric on Shape space. We first review how a metric on shape space is derived with the introduction of a conformal factor and then give our derivation of the unique metric on shape space, and explain the metric’s uniqueness, and the relationship between realistic units of length and conformal factors.

As is explained in [5] one can introduce a new \(\text{Sim}(3)\)-invariant metric on \(Q\), which subsequently induces a metric on shape space in a natural way. As the mass metric \(M\) is already rotation- and translation-invariant, the easiest way to arrive at a similarity-invariant metric is to multiply the mass metric by a function \(f(x)\) (the so-called conformal factor) so that the whole expression

\[ M'_x := f(x)M_x \]

becomes scale invariant, i.e.,

\[ \forall c \in \mathbb{R}^+ \forall u, v \in T_x(Q) : \]
\[ M'_{cx}(Sc_s u, Sc_s v) = f(cx)M_{cx}(Sc_s u, Sc_s v) = f(x)M_x(u, v) = M'_x(u, v). \]

Note that the function \(f\) must be translation- and rotation-invariant so that it does not spoil the Euclidean invariance of the mass metric. As \(M'_x = f(x)M_x\) is now a metric invariant under the whole similarity group we are ready to write down the metric \(N\) on shape space:

\[ N_s(v', u') := M'_x(v, u) = f(x)M_x(v, u), \tag{43} \]

where

\[
\begin{cases}
\pi(x) = s \\
\pi_x(u) = u' \\
\pi_x(v) = v'
\end{cases}
\]

with the projection map \(\pi : Q_{cm} \rightarrow S = \frac{Q}{\text{Sim}(3)}\).

When the action of scale transformation on \(T(Q)\) is defined by the differential of the scale transformations, i.e., \(Sc_s\), from the behavior of the mass metric \(M\) under scale transformation (42) one sees that any rotation- and translation-invariant homogeneous function\(^{[27]}\) of degree \(-2\) perfectly meets all the requirements of a conformal factor. For instance

\[ f(x) = \sum_{i < j} ||x_i - x_j||^{-2} \tag{44} \]

\(^{[27]}\) A function of \(r\) variables \(x_1, ..., x_r\) is being called homogeneous of degree \(n\) if \(f(cx_1, ..., cx_r) = c^n f(x_1, ..., x_r), \forall c\)
or

\[ f(x) = I_{cm}^{-1} \quad (45) \]

where

\[ I_{cm}(x) = \sum_{i} m_i \| x - x_{cm} \|^2 \]

\[ = \sum_{i \neq j} m_i m_j \| x_j - x_i \|^2 \]

are two legitimate examples of conformal factors (as suggested in [3]). However, as the introduction of different conformal factors leads to different metrics on shape space and leads also to the appearance of unphysical forces (see appendix A for clarification), this treatment is unsatisfactory to us.

By paying attention to the important role of the measurement units in determination of the geometry of space, we propose below another way to derive the metric on shape space which does not have the problem just mentioned. In other words, one can complement the DGZ-derivation of metric on shape space and remove the involved arbitrariness in it as follows.

As mentioned before, Mathematically, a metric \( G \) on a manifold \( Q \) is called scale-invariant if and only if

\[ \forall v_1, v_2 \in T_q(Q) : G_q(v_1, v_2) = G_q(Sc_q v_1, Sc_q v_2) \quad (46) \]

where \( Sc_q : T(Q) \rightarrow T(Q) \) denotes the push forward of vectors along the scale transformations \( Sc : q \rightarrow cq \) on \( Q \). Since \( Sc_q v = cv \), we saw that the mass metric \( M \) is not scale-invariant in this sense.[12] However, what one physically measures and is relevant is not \( M \) but

\[ M_q^{(m)}(v_1, v_2) = \frac{M_q(v_1, v_2)}{M_q(q_i - q_j, q_i - q_j)} \quad (47) \]

where \( 1 < i, j < N \) are two particles that are used to define the unit of length. This is another way to realize how the arbitrariness of the metric on shape space criticised before disappears by the usage of real measuring units instead of “inaccessible absolute units”.

The measured mass metric is on its own scale invariant in the mathematical sense mentioned above. One could say that part of the arbitrariness of the conformal factor is now in fact shifted to the arbitrariness in the choice of a length unit, i.e., which particles \( i \) and \( j \) one chooses to define the length unit. However, one should realize that all reasonable choices of length unit will lead to the same metric \( N \) on shape space. A reasonable choice of length unit would be such that leads to no fictitious forces. For instance the choice of two particles forming a harmonic oscillator (w.r.t. the gross background structure of the universe) as the unit of length would be a very bad choice. This also explains the appearance of nonphysical forces encountered by the choice of different conformal factors in the DGZ-approach.

It is worth noting that \( N \) is a metric on \( T(S) = T(Q_{ini})/Sc_s \), while \( N \) is a metric on \( T(Q)/A_S \). Thus, these are metrics on two different vector bundles. Although we intuitively expect them to represent the same physical entity, their mathematical equivalence is not obvious to us. Throughout the rest of this text we will always work with \( T(Q)/A_S \) and use \( N \). With some abuse of notation, we denote both bundles by \( T(S) \), but it is clear from the context which bundle is meant.

iii. Connection form for \( Sim(3) \)-fiber-bundle

For deriving the Lagrangian equations of motion on \( S \) (which is the topic of the next section), besides having a similarity invariant potential function on absolute configuration space (see Section (II)) and a metric \( N \) on shape space \( S \), we need to have the suitable connection form \( \omega \) on the absolute configuration space \( Q \), compatible with the \( Sim(3) \)-fiber-bundle structure.

Here, we first discuss some features of the similarity group, and we present two repre-
sentations of this group and its Lie-algebra \( \text{sim}(3) \). We will subsequently use these in constructing the connection form of the \( \text{Sim}(3) \)-fiber-bundle. At the end of this section, we will show by an explicit calculation that shape space \( S \) has the same curvature as the internal configuration space \( Q_{\text{int}} := \frac{O(3)}{\mathbb{Z}(3)} \).

Similarity group \( \text{Sim}(3) \) acts on any point \( x \in \mathbb{R}^3 \) of absolute space as follows

\[
x \to x' = cRx + t
\]

where

\[
c \in \mathbb{R}^+\]

stands for the spatial scale transformations,

\[
R \in SO(3)
\]

for the \( 3 \times 3 \) matrix representation of spatial rotations\(^{28}\) and

\[
t = (t_1, t_2, t_3)^T \in \mathbb{R}^3
\]

for spatial translations.

The group of rotations \( SO(3) \) does not form a normal subgroup of \( \text{Sim}(3) \), while the groups of translations \( T(3) \cong \mathbb{R}^3 \) and scale transformations \( Sc \cong \mathbb{R}^+ \) both do. As a result, one recognizes a semi-direct product structure in \( \text{Sim}(3) \)\(^{29}\), i.e.,

\[
\text{Sim}(3) = Sc \times T(3) \times SO(3)
\]

If one thinks of absolute space as a section \( \mathbb{R}^3 \times \{1\} \in \mathbb{R}^4 \), one can give a representation of the similarity group \( \text{Sim}(3) \) in terms of the \( 4 \times 4 \) matrices of the form

\[
\begin{bmatrix}
cR_{11} & cR_{12} & cR_{13} & t_1 \\
cR_{21} & cR_{22} & cR_{23} & t_2 \\
cR_{31} & cR_{32} & cR_{33} & t_3 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

(48)

The Lie-algebra \( \text{sim}(3) \) of the similarity group is then given by the matrices

\[
\begin{bmatrix}
\dot{c} & \omega_3 & -\omega_2 & v_1 \\
-\omega_3 & \dot{c} & \omega_1 & v_2 \\
\omega_2 & -\omega_1 & \dot{c} & v_3 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

(49)

where \( \omega = (\omega_1, \omega_2, \omega_3), v = (v_1, v_2, v_3) \in \mathbb{R}^3 \) are possible angular and linear velocity vectors. The so called similarity velocity can be characterized by the triple

\[
\delta = (v, \omega, \dot{c})
\]

Alternatively, we can give a representation of the similarity group by expressing the position of the particle in absolute space \( \mathbb{R}^3 \), on real projective space \( \mathbb{RP}^4 \) as

\[
\begin{pmatrix}
x \\
y \\
z \\
1
\end{pmatrix} \rightarrow \begin{pmatrix}
x \\
y \\
z \\
1
\end{pmatrix}
\]

Matrices representing \( \text{Sim}(3) \) on \( \mathbb{RP}^4 \) are then of the following form

\[
\begin{bmatrix}
R_{11} & R_{12} & R_{13} & t_1 \\
R_{21} & R_{22} & R_{23} & t_2 \\
R_{31} & R_{32} & R_{33} & t_3 \\
0 & 0 & 0 & c^{-1}
\end{bmatrix}
\]

(50)

A simple calculation shows indeed

\[
\begin{bmatrix}
R & t \\
0 & c^{-1}
\end{bmatrix} \begin{pmatrix}
x \\
1
\end{pmatrix} = \begin{pmatrix}
Rx + t \\
c
\end{pmatrix}
\]

In this representation, in contrary to the previous one, any \( g \in \text{Sim}(3) \) is thought of as a translation and rotation followed by a dilatation. Correspondingly, the matrix representation of the Lie-algebra \( \text{sim}(3) \) is given by

\[
\begin{bmatrix}
0 & \omega_3 & -\omega_2 & v_1 \\
-\omega_3 & 0 & \omega_1 & v_2 \\
\omega_2 & -\omega_1 & 0 & v_3 \\
0 & 0 & 0 & -\dot{c}
\end{bmatrix}
\]

(51)

The action of \( \text{Sim}(3) \) on the configuration space \( Q \) of a multiparticle system, can consequently be given from the previous actions in

\(\text{footnote}{28}\)Which can be parameterized for instance by the three Euler angels.

\(\text{footnote}{29}\)Applying a translation and then a rotation is equivalent to applying the rotation and then a translation by the rotated translation vector. Hence \( E(3) \) is a semi-direct product of \( T(3) \) and \( O(3) \), i.e., \( E(3) = T(3) \ltimes O(3) \).
a straight forward way.

We use the upper left $3 \times 3$ block of (48) to construct a representation of the group

$$G_{rs} := \frac{\text{Sim}(3)}{\text{trans}(3)} = SO(3) \times \mathbb{R}^+$$

comprising all the rotations and scale transformations, on the centre of mass configuration space $Q_{cm}$. This group has a direct product structure and acts on $\mathbb{R}^3$ as follows

$$\begin{bmatrix} cR_{11} & cR_{12} & cR_{13} \\ cR_{21} & cR_{22} & cR_{23} \\ cR_{31} & cR_{32} & cR_{33} \end{bmatrix}$$

This action introduces a $(SO(3) \times \mathbb{R}^+)$-fiber-bundle structure on $Q_{cm}$. The Lie-algebra $g_{rs}$ of $G_{rs}$ consists of the matrices of the following form

$$g_{rs} = \text{so}(3) + I_3 \dot{c} \quad (52)$$

The letter $I_3$ stands for the $3 \times 3$ identity matrix, and $\dot{c} \in \mathbb{R}$ for the generator of scale transformations. One can arrive at the expression for the connection form

$$\omega = T(Q_{cm}) \rightarrow g_{rs}$$

of the $G_{rs}$-fiber-bundle, by modifying (16) in the following way

$$\omega = \omega_r + \omega_s$$

$$= R \left( A_x^{-1} \left( \sum_{j=1}^{N-1} r_j \times dr_j \right) \right) + I_3 D_x^{-1} \left( \sum_{j=1}^{N-1} r_j dr_j \right) \quad (53)$$

Here, we have defined the operator

$$D_x : \mathbb{R} \rightarrow \mathbb{R}$$

as follows

$$D_x(\lambda) := \sum_{j=1}^{N-1} r_j^2 \dot{\lambda} \quad (54)$$

and it can be called the "dilational tensor"\(^{33}\). The letter $\dot{\lambda}$ stands for the rate of change of scale of the system (scale velocity)\(^{34}\)

$$\dot{\lambda} := \frac{\dot{\lambda}}{\lambda} \quad (55)$$

where

$$\lambda := \max | x_i - x_j | \quad (56)$$

with $i,j$ varying between $1,2,..,N$; being one choice for the system’s scale variable.

We have constructed this operator in direct analogy with the moment of inertia tensor $A_x$. Inertia tensor sends an angular velocity (which can be represented as a vector in $\mathbb{R}^3$) to another vector in $\mathbb{R}^3$ representing the total angular momentum of the whole system. In the same way, the dilational tensor $D_x$ takes an expansion velocity, which in turn can be represented by just a number in $\mathbb{R}$, to a measure of the total expansion of the system (dilational momentum $D$), which again can be represented by another number in $\mathbb{R}$. As the Lie-algebra of $G_{rs}$ can be represented by the matrices (52), one recognizes the correct structure in the connection form (53), for the $(SO(3) \times \mathbb{R}^+)$-fiber-bundle. If one takes any vector of $T_x(Q_{cm})$ and acts on it with this connection form, the first term gives you a member of $\text{so}(3)$. The second term gives a number, which is multiplied by the identity matrix. It results in a matrix of the above form (hence a member of the Lie-algebra of the bundle’s structure group). So, it does what it is expected to do.

Last but not least, we want to investigate

\(^{33}\)It is compatible with the definition $D = \sum_{i=1}^{N-1} x_i \cdot p_i$ introduced by Barbour et al. much earlier than us.

\(^{34}\)Here, we assume all measurements are conducted using special Newtonian rods and clocks, which are isolated from the material universe and do not get affected by them in any way, or by any transformations, we perform on the material universe. Practically, such measuring instruments, of course, do not exist. However, the existence of absolute space and absolute time in the Newtonian world-view justifies their hypothetical existence in the context of this view.
the curvature $C$ of the connection form

$$\omega_s = D_x^{-1} \left( \sum_{i=1}^{3N} m_{i,1} x_i dx_i \right)$$

where for every rational number $p$, the largest integer smaller than $p$ is denoted by $\lfloor p \rfloor$. Given two arbitrary horizontal vectors $v, v' \in T_x(Q)$ as the input of the curvature 2-form, it is known \([21]\) that

$$C(\omega_s) = -\frac{1}{2} \omega_s([v, v'])$$

where $[,]$ is the Lie-bracket of the extension of horizontal vectors $v$ and $v'$ to horizontal vector fields. Choosing a basis $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{3N}}$ of the tangent space $T_x(Q)$, Lie-bracket of the two vector fields $v = \sum_{i=1}^{3N} v_i \frac{\partial}{\partial x_i}$ and $v' = \sum_{i=1}^{3N} v'_i \frac{\partial}{\partial x_i}$ can be computed by

$$[v, v'] = \sum_i \sum_j (v'_j \frac{\partial v_i}{\partial x_j} - v'_i \frac{\partial v_j}{\partial x_j}) \frac{\partial}{\partial x_i}$$

As both vectors $v, v' \in T_x(Q)$ are horizontal, they satisfy the following conditions

$$\omega_s(v) = \omega_s(v') = 0$$

Using (58) the above conditions can be translated into relations (or constraint equations) in the variables $x_1, \ldots, x_{3N}, v_1, \ldots, v_{3N}$ which after some rearrangement of terms become as follows

$$v_{3N} = \frac{1}{m_N x_{3N}} \sum_{i=1}^{3N-1} m_{i,1} \lfloor \frac{i-1}{N} \rfloor + 1 x_i v_i$$

$$v'_{3N} = \frac{1}{m_N x_{3N}} \sum_{i=1}^{3N-1} m_{i,1} \lfloor \frac{i-1}{N} \rfloor + 1 x_i v'_i$$

As all other $6N-1$ variables involved are independent, for all $j = 1, \ldots, 3N$, and $i = 1, \ldots, 3N-1$; all the following derivatives vanish

$$\frac{\partial v_i}{\partial x_j} = \frac{\partial v'_i}{\partial x_j} = 0$$

which simplify the above expression for $[v, v']$ greatly

$$[v, v'] = \sum_{j=1}^{3N} (v_j \frac{\partial v_{3N}}{\partial x_j} - v'_j \frac{\partial v'_{3N}}{\partial x_j}) \frac{\partial}{\partial x_j} = 0$$

In the last equality $\frac{\partial v_i}{\partial x_j} = \frac{m_{i,1} \lfloor \frac{i-1}{N} \rfloor + 1}{m_N x_{3N}} v_j$ is used. Hence, from (59) it follows that the connection form $\omega_s$ has a vanishing curvature. This means that shape space $S = \frac{Q}{Sim(3)}$ is exactly as curved as $Q_{int} := \frac{Q}{\mathbb{T}(3)}$. That $Q_{int}$ is curved, or in other words, that the curvature of the connection form $\omega_r$ given by (16) is non-vanishing, has been shown in \([21]\). So, in the process of quotiening out the flat configuration space $Q$ by the action of the similarity group $Sim(3)$, the only stage which causes curvature in the final base space, is the quotiening with respect to the group of rotations $SO(3)$.

For the reduction of the classical mechanics w.r.t. the scale transformations, we will use the geometric setting explained in this section and Section (III). One of the reasons why reduction w.r.t. the similarity group has not been studied as extensively as the euclidean group is that the potential function defined on the absolute configuration space, though being manifestly rotational and translational invariant, obviously is not scale invariant (take Newtonian gravity potential as an example). However, as explained in Section (II), as a consequence of the direct implementation of principle of relatinism, scale transformations become an additional symmetry in (modified) classical physics. This additional symmetry immediately enables the shape degrees of freedom to have an autonomous evolution, entirely decoupled from the system’s transnational, orientational, and scale degrees of freedom.
V. REDUCED EQUATIONS OF MOTION IN SHAPE COORDINATES

In this section, we seek the equations of motion of a $N$ particle system in shape, orientation, and scale coordinates and velocities. To this end, first, the Lagrangian of the system needs to be expressed in the new coordinates and velocities, and subsequently, the equations of motion can be derived. As the angular velocities used to quantify the rate of rotations of a system are not derivatives of the three Euler angles (or any other variables), the Lagrange, or Euler-Lagrange equations of motion, cannot be used. For such cases, the Boltzmann-Hammel equations of motion have to be used. In the first subsection, following [22], we shortly review the formulation of mechanics in quasi-coordinates and quasi-velocities, and then based on [4] and [5], we derive the equations of motion for classical systems behaving according to the modified Newtonian theory in shape, orientation, and scale coordinates, and shape, angular, and scale velocities.

i. Equations of motion in quasi-coordinates

The generalized coordinates are the set of coordinates defining the degrees of freedom of a system. For instance for a rigid body moving in $\mathbb{R}^3$, there are six generalized coordinates (three specifying the position of the body and three the orientation of it), i.e.

$$q = [q_1, ..., q_6] := [x, y, z, \alpha, \beta, \gamma]$$

The generalized speeds are obviously the derivatives of the generalized coordinates $\dot{q} = [\dot{x}, \dot{y}, \dot{z}, \dot{\alpha}, \dot{\beta}, \dot{\gamma}]$. These coordinates $\dot{q}_k$ can be called true coordinates, in a sense that if the velocities $\dot{q}_k$ are known functions of time, integration with respect to time determines the respective coordinates, and hence the state of the system.

On the other hand, one may define generalized speeds, which cannot be written as
the time derivative of any coordinates; for instance, they are defined as linear combinations of the time derivatives of generalized coordinates. Such generalized speeds are called *quasi-velocities*, and the generalized coordinates corresponding to these velocities are called *quasi-coordinates*. The word "quasi" in the last expression should be understood as nonexistent. As the most famous example of quasi-velocities, one can mention angular velocity components of a rigid body, which are called coordinates corresponding to these velocities quasi-velocities. Such generalized speeds are linear combinations of derivatives of Euler angles. However, they are themselves not time derivatives of any coordinates. Quasi-coordinates and quasi-velocities were first introduced to derive the so-called Boltzmann-Hammel equations of motion, which we will shortly discuss below.

In analyzing anholonomic systems, quasi-velocity formulation casts the dynamical equations of motion in a form requiring fewer equations. For a system possessing \( n \) degrees of freedom with \( m \) anholonomic constraints, the usage of Lagrangian formalism leads to \( 2n + m \) equations of motion \( (2n \) equations for the system’s state and \( m \) algebraic relations that must be solved for the multipliers). However, if quasi-velocity formalism is used, the same problem can be described by a system of \( 2n - m \) degrees of freedom (see [23],[24]).

Equations of motion for classical mechanics in true coordinates are the known Lagrange equations

\[
\frac{d}{dt} \frac{\partial K}{\partial \dot{q}_i} - \frac{\partial K}{\partial q_i} = F_i
\]

with \( K \) being the system’s kinetic energy, and \( F_i \) being the generalized force associated with the generalized coordinate \( q_i \), for \( i \in [1,n] \).

Consider now the usage of the quasi-velocities \( Y_i \), which are defined as \( n \) independent linear combinations of the \( \dot{q}_k \)'s, i.e.,

\[
Y_i := a_{i1}\dot{q}_1 + a_{i2}\dot{q}_2 + \ldots + a_{in}\dot{q}_n = \sum_{r=1}^{n} a_{ir}\dot{q}_r
\]

with \( a_{ir} \) being known functions of the generalized coordinates \( q_k \). Constructing a \( n \times n \) matrix \( \alpha \) from \( a_{ij}'s \), one can write the definition of quasi-velocities \( Y_k \)'s more compactly as follows

\[
\begin{bmatrix}
Y_1 \\
Y_2 \\
\vdots \\
Y_n
\end{bmatrix} = \alpha
\begin{bmatrix}
\dot{q}_1 \\
\dot{q}_2 \\
\vdots \\
\dot{q}_n
\end{bmatrix}
\]

Having the above relations between the quasi-velocities \( Y_i \)'s, and the generalized (true) velocities \( \dot{q}_i \)'s in mind, one can define a set of differential forms \( dy_k \) as follows

\[
dy_k := \sum_{r=1}^{n} a_{rk}dq_r
\]

The above equations cannot always be integrated to obtain the variable \( y_k \). In such cases, the differential form \( dy_k \) are naturally called “nonintegrable” and cannot be thought of as differential of some configuration variable \( y_k \). The quantities \( dy_k \) are called *differentials of quasi-coordinates*, with some abuse of words because they are not really differentials, and the variables \( y_k \) are undefined.

If the quasi-velocities are known, the true velocities can be calculated using

\[
\dot{q}_k = \beta_{kl}Y_l
\]

where \( \alpha_{sk}\beta_{kl} = \delta_{kl} \). Here \( \delta_{kl} \) is the Kronecker Delta. It is easy to check that

\[
\frac{\partial \dot{q}_k}{\partial Y_l} = \frac{\partial \dot{q}_k}{\partial q_l} = b_{kl}
\]

For a function \( f(q_1, \ldots, q_n, t) \), with the partial derivative with respect to a quasi-coordinate \( y_l \) one means the following

\[
\frac{\partial f}{\partial y_l} := \frac{\partial f}{\partial \dot{q}_k} \frac{\partial \dot{q}_k}{\partial y_l} = \frac{\partial f}{\partial \dot{q}_k} b_{kl}
\]

The kinetic energy \( K(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n) \) can be expressed in the new variables, i.e., \( K'(q_1, \ldots, q_n, Y_1, \ldots, Y_n) \), where the prime here indicates just the difference in the variables of the function. It can be shown (see [22]) that the equations of motion expressed in these
new coordinates \(q_1, \ldots, q_n, Y_1, \ldots, Y_n\) become of the following form
\[
\frac{d}{dt} K'_k - \frac{\partial K'_i}{\partial Y_k} + \gamma_{kl} \frac{\partial K'_l}{\partial Y_k} = F'_k \tag{61}
\]
for \(k = 1, 2, \ldots, N\) where
\[
F'_k = \sum_{s=1}^{n} F_s b_{sk}
\]
are the generalized forces corresponding to the virtual displacements \(\delta y_k\), and
\[
\gamma_{kij} := b_{sk} b_{lj} \left( \frac{a_{ij}}{q_l} - \frac{\partial a_{i\ell}}{\partial q_s} \right)
\]

These are known as Boltzmann-Hammel equations of motion. As already mentioned, the partial derivative with respect to the quasi-coordinate appearing in second term of (61) should be understood as follows
\[
\frac{\partial K'_i}{\partial y_k} := \frac{\partial f}{\partial q_l} \frac{\partial y_k}{\partial q_i} = \frac{\partial f}{\partial q_l} b_{lk}
\]
The \(n\) equations of (61) are equations of motion in quasi-coordinates. If \(y_1, \ldots, y_n\) are true coordinates the coefficients \(\gamma_{kij}\) all vanish, and the Boltzmann-Hammel equations (61) take back the form of Lagrange equations (60).

Starting on the configuration space \(Q_{cm}\) with \(n = \dim(Q_{cm}) = 3N - 3\), and a system with the lagrangian \(L: T(Q) \to \mathbb{R}\), which usually is of the following form
\[
L(x, \dot{x}) = \frac{1}{2} M_{ij} \dot{x}^i \dot{x}^j + V(x)
\]
we impose \(m\) linear scleronomic (time independent) nonholonomic constraints, i.e., constraints of the form
\[
a^\sigma_i(x) \dot{x}^i = 0 \tag{62}
\]
where \(1 \leq \sigma \leq m\). Define a vector space isomorphism \(\Psi^i_j\) on the tangent space \(T(Q_{cm})\). Set the first \(m\) rows of \(\Psi^i_j\) to be identical with the constraint matrix, i.e.,
\[
\Psi^i_j(q) = a^\sigma_i(q)
\]
and the remaining rows can be chosen freely as long as the resulting matrix \(\Psi\) is invertible. This transformation \(\Psi\) can be viewed as change of basis of \(T(Q_{cm})\)
\[
\Psi : \left( \frac{\partial}{\partial x^i} \right)_{i=1}^{n} \to \left( \frac{\partial}{\partial \dot{x}^i} \right)_{i=1}^{n} \tag{63}
\]
This new basis is called the quasi-basis. Hence a vector \(v \in T(Q_{cm})\) can be expressed in either bases, i.e.,
\[
v = x^i \frac{\partial}{\partial x^i} = \dot{u}^i \frac{\partial}{\partial \dot{x}^i}
\]
where \(u^i = \Psi^i_j(x)\dot{x}^j\) are the components of the quasi-velocities.

As is well known, the basis vectors transform like \(\frac{\partial}{\partial x^i} = \Psi^i_j \frac{\partial}{\partial y^j}\) and \(\frac{\partial}{\partial y^j} = (\Psi^{-1})^i_j \frac{\partial}{\partial x^i}\), and the set of \(n\) one-forms dual to the quasi basis (i.e., the quasi coordinate forms) are
\[
d\psi^i = \Psi^i_j dx^j.
\]
Bear in mind that the one-forms \(d\psi^i\) are not exact.

**ii. Lagrangian in quasi-coordinates**

Take the local coordinates
\[
(s, g, \lambda, \dot{s}, \dot{g}, \dot{\lambda})
\]
on
\[
T(\pi^{-1}(U)) \subset T(Q_{cm})
\]
with \(U \subset S\). They are adopted coordinates to the bundle’s projection map
\[
\pi : Q_{cm} \to S
\]
So \((s, g, \lambda) \in \pi^{-1}(U)\), and \((\dot{s}, \dot{g}, \dot{\lambda}) \in T_{c(s)}(\pi^{-1}(U))\). Here \(s = (x^a)\) for \(a \in [1, \ldots, 3N - 7]\), are for instance the \(3N - 7\) independent angles between the \(N - 1\) Jacobi vectors \(r_i\), and \(\lambda := \frac{\dot{\lambda}}{\lambda}\) is the scale velocity.

Having the connection form (31) in mind, one can introduce a \(so(3)\)-valued variable (41, 51)
\[
\Pi = e + \sum_{a=1}^{3N-6} \Lambda_a(x) q^a \tag{64}
\]
where
\[
e = g^{-1} \dot{g}
\]
and
\[ \land_{\alpha}(x) = \sum_{a=1}^{3} \land_{\alpha}^a(x) R(e_a) \]
The vectors associated with \( \Pi \) and \( \epsilon \) will be denoted by \( \Omega' \) and \( \Omega \), respectively, i.e.,
\[ R(\Omega') = \Pi \]
and
\[ R(\Omega) = \epsilon \]
Thus, the tuple
\[ (s, g, \lambda, \dot{s}, \Omega', \dot{\lambda}) \]
constitutes a local (quasi)coordinate system on \( T(\pi^{-1}(U)) \). Bear in mind that \( \Omega' \) denotes the angular velocity of the system in the body frame, and hence the angular momentum of the system in the space frame would become \( L = gA_{\tau(s)\Omega'} \). Therefore, the angular momentum vector expressed in the body frame becomes \( A_{\tau(s)\Omega'} \). As usual, \( g \) stands for the rotation, which brings the space frame to the body frame.

As discussed in Section (III.ii), the moment of inertia tensor can be expressed as follows
\[ A_{ab} = d^2(L_a, L_b) \]
with \( L_a \) being the left invariant vector fields on \( SO(3) \) (which are dual to \( \theta^a \)). The mass metric (27) on \( Q_{cm} \) expressed in coordinates \( (s, \lambda, \alpha, \beta, \gamma) \) takes subsequently the following form:
\[
ds^2 = \sum_{a,b=1}^{3N-7} N_{ab} ds^a ds^b + \sum_{i=1}^{N-1} | \mathbf{r}_i |^2 (d\lambda)^2 + \sum_{a,b=1}^{3} A_{ab} \omega^a \omega^b \tag{65}\]
where \( \mathbf{r}_i \)'s are as before the Jacobi vectors of the system, and hence are unique functions of the \( s_a \)'s and \( \lambda \). The \( \omega^{ab} \)s are components of the connection form of rotations \( \Omega^a \) in body frame \( \Omega \). By setting the instantaneous unit of length equal to the system’s scale variable \( \lambda \), one gets the following expression for the metric
\[
ds^2 = \sum_{a,b=1}^{3N-7} N_{ab} ds^a ds^b + \sum_{i=1}^{N-1} | \mathbf{r}_i |^2 (d\lambda)^2 + \sum_{a,b=1}^{3} A_{ab} \omega^a \omega^b \tag{66}\]
where now all the \( \mathbf{r}_i \)'s and \( A \) are expressed in the internal (expanding or contracting or stationary) length unit. It is worth mentioning that as the independent angles \( s_i \) and the scale coordinate bring the metric tensor in a diagonal form, they form an orthogonal coordinate system on \( Q_{int} = \frac{Q_{cm}}{SO(3)} \).

Consider a system with a similarity invariant Lagrangian \( \mathcal{L}'(s, g, \lambda, \dot{s}, \Omega', \dot{\lambda}) \), i.e.,
\[
\mathcal{L}'(s, \mathbf{hg}, c\lambda, \dot{s}, \Omega', \dot{\lambda}) = \mathcal{L}(s, g, \lambda, \dot{s}, \Omega', \dot{\lambda}) \tag{67}
\]
\( \forall h \in SO(3), \quad \forall c \in \mathbb{R}^+ \). Note that \( \Omega' \) is left \( SO(3) \)-invariant. Such a function \( \mathcal{L}' \) on \( T(Q_{cm}) \) reduces naturally to a function \( \mathcal{L}' \) on \( T(Q_{Sim}) \). The similarity invariance of the Lagrangian in classical mechanics is a consequence of the similarity invariance of kinetic and potential energies, as already explained based on the principle of relationalism. It is important to remember that the units (of time and spatial distance) in which the Lagrangian has the property (67) are internal units.

We write the similarity invariant Lagrangian of classical mechanics as follows
\[
\mathcal{L}' = \frac{1}{2} \sum_{a,b=1}^{3N-7} N_{ab} \dot{s}^a \dot{s}^b + \frac{1}{2} \sum_{i=1}^{N-1} (\mathbf{r}_i \dot{\lambda})^2 + \frac{1}{2} \sum_{a,b} A_{ab} \Omega^a \Omega^b - V(s) \tag{68}
\]
where \( V \) is a similarity invariant potential function, hence depending only on the \( 3N - 7 \) coordinates \( s_i \). Notice that our previous expression for dilational momentum (54) is derivable from the above Lagrangian:
\[
D = \frac{\partial L}{\partial \dot{\lambda}} = \sum | \mathbf{r}_j |^2 \lambda \]

\( ^{32} \text{Compared with absolute length unit.} \)

\( ^{33} \text{For the non-singular configurations.} \)
As the lagrangian \((68)\) is scale independent, \(D\) is a constant of motion. Bear again in mind, that the units in which \(D\) is constant, are all internal units. In absolute(external) units if at a given instant of time \(D > 0\), then it will monotonically increase with (absolute)time. This has been explained in details in Section (VII.i).

iii. Reduced Euler-Lagrange equations of motion

In this section, finally, we discuss the equations of motion in the anholonomic frame \((s, g, \lambda, \dot{\lambda}, Q, \Omega)\) on \(T(Q_{cm})\) introduced in the last section.

Let \(x^\lambda, \lambda = 1, ..., 3N - 3\) be a local coordinate system on \(W \subset Q_{cm}\). From this coordinate system, one can derive a basis for the vector fields, i.e., \(\frac{\partial}{\partial x^\lambda}\), and a basis for the 1-forms, i.e. \(dx^\lambda\) on \(T(Q_{cm})\). Let \(Z_\lambda\) and \(Z^\lambda\) be another local basis of the vector fields and 1-forms (dual to each other) on \(W\). The later vector fields and one forms are related to the former ones \(^{34}\) by

\[
Z_\lambda = \sum_{\mu=1}^{3N-3} B^\lambda_\mu \frac{\partial}{\partial x^\mu} \\
Z^\lambda = \sum_{\mu=1}^{3N-3} A^\lambda_\mu dx^\mu
\]

where their duality requires \(\sum_\lambda A^\lambda_\mu B^\lambda_\nu = \delta^\mu_\nu\). If the above relations for \(Z^\lambda\)'s are integrable, there exists true coordinates \(z^\lambda\) on \(Q_{cm}\), for which \(Z_\lambda = \frac{\partial}{\partial z^\lambda}\), and \(Z^\lambda = dz^\lambda\). Otherwise, the \(Z_\lambda\)'s form an anholonomic basis of \(T(Q_{cm})\), and the \(z^\lambda\)'s become the corresponding quasi-coordinates on \(Q_{cm}\).

Differentiation of \(Z^\lambda\) leads to \(^{35}\)

\[
dZ^\lambda = \sum_{\sigma < \kappa} \gamma^\lambda_{\sigma \kappa} Z^\sigma \wedge Z^\kappa
\]

with

\[
\gamma^\lambda_{\sigma \kappa} := \sum_{\mu \nu} \left( \frac{\partial A^\lambda_\mu}{\partial x^\nu} - \frac{\partial A^\lambda_\nu}{\partial x^\mu} \right) B^\mu_\sigma B^\nu_\kappa
\]

\(^{34}\)Which were derived from the coordinate system \(x^\lambda\)

In the expression of the Lagrangian function \(L'(x, \dot{x})\), one can replace the coordinate velocities \(\dot{x}^\lambda\) with a new set of velocities \(^{35}\)

\[
z^\lambda = \sum_{\mu} A^\lambda_\mu(x) \dot{x}^\mu
\]

In these new variables the Lagrangian is denoted by \(L'\), i.e.,

\[
L'(x, \dot{z}) = \mathcal{L}(x, \dot{x})
\]

As reviewed in Section (V.i), the Euler-Lagrange equations in terms of \((x^\lambda, \dot{z}^\lambda)\)

\[
\frac{d}{dt} \left( \frac{\partial \mathcal{L'}}{\partial \dot{z}^\lambda} \right) - \frac{\partial \mathcal{L'}}{\partial z^\lambda} = 0
\]

for \(\lambda = 1, ..., 3N - 3\); takes the following form of Boltzmann-Hammel equations of motion in terms of \((x^\lambda, \dot{z}^\lambda)\)

\[
\frac{d}{dt} \left( \frac{\partial \mathcal{L'}}{\partial \dot{z}^\sigma} \right) - \sum_{\mu, \kappa} \gamma^\sigma_{\mu \kappa} \frac{d \mathcal{L'}}{d z^\kappa} z^\mu = 0 \quad (69)
\]

for \(\sigma = 1, ..., 3N - 3\).

A derivation of the reduced Lagrangian equation of motion on \(Q_{int} = \frac{Q_{cm}}{SO(3)}\) is explained in \([5]\). In the rest of this section, we will present an extension of this work to find the reduced Lagrangian equations of motion on shape space \(S = \frac{Q_{cm}}{SO(3) \times SO(3)}\).

So we consider \(Q_{cm}\) as a \(SO(3) \times SO(3)\) fiber bundle, and make a coordinate transformation on \(Q_{cm}\) from the Euler angles and Jacobi coordinates to the Euler angles, scale, and the shape coordinates as follows

\[
(r_1, r_2, ..., r_{3N-1}, \alpha, \beta, \gamma) \rightarrow (s_1, ..., s_{3N-7}, \lambda, \alpha, \beta, \gamma)
\]

A basis of 1-forms on \(Q_{cm}\) in the new coordinates is given as follows

\[
Z^\alpha := \omega^\alpha \\
Z^4 := \omega_s \\
Z^{4+i} := ds^i
\]
where \( a = 1, 2, 3 \) and \( i = 1, 2, ..., 3N - 7 \). The \( \omega^a \)'s and \( \omega_s \) can be read from the new connection form \((53)\). In particular, the \( \omega^a \)'s are components of the rotations connection form \((22)\) form (w.r.t. the fixed space frame). Their dual vector fields are as follows

\[
Z_a = f_a \\
Z_4 = \frac{\partial}{\partial \lambda}
\]

\[
Z_{4+i} = \partial^s_{4+i} = (\frac{\partial}{\partial q^s})^s
\]

Given a section (or a lifting map) \( \sigma : S \rightarrow Q_{cm} \), the horizontal lift of a vector \( \frac{\partial}{\partial q^s} \in T_{q_0}(S) \) at the point \( q_0 \) of shape space \( S \), to the point \( \sigma(q_0) \in Q_{cm} \) is given as follows

\[
(\frac{\partial}{\partial s^a})^s = \frac{\partial}{\partial s^a} - D_r^{-1} \left( \sum_{j=1}^{N-1} r_j \frac{d}{ds^a} \frac{\partial}{\partial s^j} \right) \frac{\partial}{\partial \lambda} \quad (70)
\]

where analogous to \((32)\) one has

\[
\beta^a_s := \langle A_{r(q_0)}^{-1} \left( \sum_{i=1}^{N-1} r_i \times \frac{\partial r_i}{\partial s^a} \right) | \dot{e}'_a \rangle
\]

Taking the exterior derivatives of \( Z^\lambda \) leads to the factors \( \gamma_a^\lambda_k \). They become

\[
\gamma^a_{bc} = -\epsilon_{cba} \\
\gamma^a_{4+i,4+j} = -k^a_{ij}
\]

with \( k^a_{ij} \) are the components of the curvature tensor of shape space

\[
k^a_{ij} = \frac{\partial \beta^a \dot{r}_i}{\partial \lambda} - \frac{\partial \beta^a \dot{r}_j}{\partial \lambda} - 3 \epsilon_{abc} \beta^b \dot{r}^c
\]

and all other \( \gamma^a_{cd} \) vanishing.

\( k^a_{ij} \) are the components of the curvature tensor of shape space

\[
k^a = d\omega^a - 3 \epsilon_{abc} \omega^b \wedge \omega^c = \sum_{i<j} k^a_{ij} ds^i \wedge ds^j
\]

for the connection form \( \omega \) of \((SO(3) \times Sc)\)-fiber-bundle given by \((53)\). As at the end of the Section (VI.iii), we had calculated that the connection form \( \omega_s \) on \( Q_{int} \) considered as \( Sc \)-fiber-bundle is flat \((57)\), the above curvature tensor is the same as the curvature tensor \((16)\) on \( Q_{int} = \frac{Q_{cm}}{SO(3)} \).

Note that besides the well-known interconnection of changes in Euler angles, which manifest themselves in the structure constants \( \gamma^a_{bc} = -\epsilon_{abc} \), the only non-vanishing couplings are the coupling of shape variables \( s^i \), to the orientational variables (Euler angles). Intuitively, as the scale of a mechanical system can be changed without leading to any changes in either total orientation, or shape of the system, one expects the vanishing of corresponding \( \gamma \) factors which is just another expression of the flatness of \( \omega_s \) given at \((57)\).

Consider the coordinates

\[
(\alpha, \beta, \gamma, \lambda, s^i; \Omega^1, \Omega^2, \Omega^3, \lambda, s^i)
\]

on \( T(Q_{cm}) \), where the quasi-velocities \( \Omega^a \)'s are defined using \((22)\), i.e.

\[
\Omega^a := \omega^a \left( \frac{d}{dt} \right) = \dot{q}_a + \sum_i \beta^a s^i = \dot{q}_a + \sum_i \beta^a s^i
\]

These are the components of angular velocity with respect to the fixed space frame. As seen before \((63)\), the Lagrangian describing a mechanical \( N \)-particle system expressed in the above coordinates becomes as follows

\[
\mathcal{L} = \frac{1}{2} \sum_{a,b=1}^{3N-7} N_{ab} s^a s^b + \frac{1}{2} \sum_{i=1}^{N-1} (r_i \dot{\lambda})^2 \\
+ \frac{1}{2} \sum_{a,b} A_{ab} \Omega^a \Omega^b - V(s)
\]

The Boltzmann-Hammel equations of motion \((69)\) in this new coordinate system then take the following form:

\[
\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_a} \right) - \frac{\partial \mathcal{L}}{\partial q_a} - \sum_{i<j} k^a_{ij} \frac{\partial \mathcal{L}}{\partial \Omega^i} \frac{\partial \Omega^j}{\partial \Omega^i} = 0 \\
\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \Omega^a} \right) - I_a \mathcal{L} - \sum_{b,c} \epsilon_{abc} \frac{\partial \mathcal{L}}{\partial \Omega^b} \Omega^c = 0
\]
\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\lambda}} \right) - \frac{\partial L}{\partial \lambda} = 0 \]  
(73)

Specifically, (73) expresses the conservation of the dilational momentum.

As we will see explicitly in the next section, for the most general case, in the above equations of motion of the modified Newtonian theory on shape space both \( D \) and \( L \) will appear as two single numbers. If one likes, one can interpret these numbers as the conserved total angular momentum and the conserved total dilatational momentum of the universe in absolute space.\(^{36}\) This interpretation is, however, meaningless in the Leibnizian world-view. There one has to consider these two constants just being there as part of the fundamental law of motion (of the modified Newtonian theory) on shape space. It may be the case that in the future some explanation or argumentation for their values based on shape space could be found. According to the (modified) Newtonian theory, because of the absence of the Coriolis and centrifugal forces in the cosmological frame of reference,\(^{37}\) the total \( L \) of our universe is zero, and hence it does not appear in the evolution on the universe’s shape space neither. Contrary to this, there is a \( D \) (or \( \lambda \)) involved in the universe’s shape equations of motion. Again, a true relationalist, as mentioned above, should consider this constant number \( D \) appearing in the equation of motion primarily as part of the law of motion on shape space. Contrary to this, an absolutist sees the origin of \( D \) in the dilatational momentum of our universe in absolute space.

The Couchy data for the modified Newtonian theory consists of the shape, shape velocity, total \( L \), and total \( D \). By absorbing the last two in the law of motion, specification of a point and a velocity on shape space would suffice for the determination of a whole history(solution).

---

\(^{36}\)In the language of the DGZ-approach, this corresponds to a particular choice of gauge.

\(^{37}\)Built from distant galaxies of the universe.

### VI. Three Body System

Take three particles located at the positions

\[ x_1 = (x_1, y_1, z_1) \]
\[ x_2 = (x_2, y_2, z_2) \]
\[ x_3 = (x_3, y_3, z_3) \]

The configuration of this system in centre of mass frame can be characterized by two Jacobi vectors

\[ r_1 = \left( \frac{1}{m_1} + \frac{1}{m_2} \right)^{-1/2}(x_2 - x_1) \]
\[ r_2 = \left( \frac{1}{m_1 + m_2} + \frac{1}{m_3} \right)^{-1/2}(x_3 - m_1 x_1 + m_2 x_2) \]

As shape variables, we introduce the two angles formed by the inter-particle vectors, i.e.,

\[ s_1 := \cos^{-1}\left( \frac{(x_2 - x_1) \cdot (x_3 - x_1)}{|x_2 - x_1| \cdot |x_3 - x_1|} \right) \]
\[ s_2 := \cos^{-1}\left( \frac{(x_3 - x_2) \cdot (x_1 - x_2)}{|x_3 - x_2| \cdot |x_1 - x_2|} \right) \]

and the scale variable of the system is chosen like in (59) as follows

\[ \lambda := \max |x_i - x_j| \]

As explained before, the system’s rotational degrees of freedom can be taken care of by three Euler angles \( \alpha, \beta, \gamma \), which connect the space frame and the body frame.\(^{38}\)

So we have the following coordinate transformation on absolute configuration space \( Q \) of the three particle system

\[
\begin{pmatrix}
  x_1 \\
  y_1 \\
  z_1 \\
  x_2 \\
  y_2 \\
  z_2 \\
  x_3 \\
  y_3 \\
  z_3 \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
  x_{cm} \\
  y_{cm} \\
  z_{cm} \\
  \alpha \\
  \beta \\
  \gamma \\
  \lambda \\
  s_1 \\
  s_2 \\
\end{pmatrix}
\]

\(^{38}\)Define the space frame simply equal to body frame at some specific time, for instance, at the initial time.
From our previous discussions and with some calculations, we can express the mass metric $M$ on $Q$ in these new coordinates. It becomes as follows

$$dl^2 = \frac{m_3(m_1 + m_2)\lambda^2 \sin^2(s_2)}{m_1 m_2 \sin^2(s_1 + s_2)} ds_1^2$$

$$+ \frac{m_3(m_1 + m_2)\lambda^2 \sin^2(s_1)}{m_1 m_2 \sin^2(s_1 + s_2)} ds_2^2$$

$$+ \left(1 + \frac{m_2}{m_1} + \frac{m_3(m_1 + m_2)\sin^2(s_2)}{m_1 m_2 \sin^2(s_1 + s_2)}\right) d\lambda^2$$

$$+ \sum_{a,b} A_{ab} \omega^a \omega^b$$

$$+ (m_1 + m_2 + m_3)(dx_{cm}^2 + dy_{cm}^2 + dz_{cm}^2)$$

Using the above line element, one can express the infinitesimal increment of Newton’s absolute time $dt$ in terms of the system’s motion (infinitesimal increments of particles spatial positions), i.e. (see also [1],[19])

$$dt = \frac{dl}{\sqrt{E - V}}$$

It is the increment of the ephemeris time for this three-particle universe. Now it becomes clear that for a spatially larger three-particle system by a factor $b > 1$, the same amount of relational motion $(ds_1, ds_2)$ leads to a longer increment of ephemeris time $b dt$, if distances are measured with the fixed (non-scalable) absolute rod (unit of length) attached to Newton’s absolute space (w.r.t. which the way changes of $\lambda$ can be measured and communicated). Considering ticks of clocks as a specific amount of relational motion of the system (where the clock itself is part of), the relation between the seconds of clocks after and before the system’s spatial scale transformation $x_i \rightarrow c\mathbf{x}_i$ becomes $T \rightarrow T' = cT$. Of course, again, this difference in the rate of clock ticking can only have meaning if we use the absolute Newtonian clock, which is unaffected by whatever happens with the matter in the universe. This is also in complete agreement with our previous discussions (see Section (II) and [1]) about relation between behaviour of Planck’s time unit under systems global spatial scale transformations $x_i \rightarrow c\mathbf{x}_i$, namely $T_p \rightarrow T_p' = cT_p$. This relation was there derived directly from the Principle of Relationalism.

As can be seen from (68), the Lagrangian of the three-particle system in the center of mass frame is the following function on $Q_{cm}$

$$\mathcal{L} = \frac{1}{2} \frac{m_3(m_1 + m_2)\lambda^2 \sin^2(s_2)}{m_1 m_2 \sin^2(s_1 + s_2)} s_1^2$$

$$+ \frac{1}{2} \frac{m_3(m_1 + m_2)\lambda^2 \sin^2(s_1)}{m_1 m_2 \sin^2(s_1 + s_2)} s_2^2$$

$$+ \left(1 + \frac{m_2}{m_1} + \frac{m_3(m_1 + m_2)\sin^2(s_2)}{m_1 m_2 \sin^2(s_1 + s_2)}\right) \dot{\lambda}^2$$

$$+ \frac{1}{2} \sum_{a,b} A_{ab} \Omega^a \Omega^b - V$$

where as before $\dot{\lambda} = \frac{1}{\lambda}$ and

$$V = G\left(\frac{m_1 m_2}{|x_2 - x_1|} + \frac{m_1 m_3}{|x_3 - x_1|} + \frac{m_2 m_3}{|x_3 - x_2|}\right)$$

Even though the actual scale ($\lambda$) of the system appears explicitly in the above Lagrangian, it is, in fact, a scale-invariant Lagrangian if one takes the transformation of the time unit and gravitational constant after the performance of a global scale transformation into account. Here we elaborate a bit more on this point. Regarding the units used in (76), one has measured the lengths w.r.t the absolute rod of Newton (w.r.t. which the diameter of our system happens to be $\lambda$), and one uses the ephemeris unit of time for time measurements. Now, one sees the quantity $\lambda s = \lambda \frac{ds}{dt}$, where subscript $e$ stands for ephemeris, is an

39 Consider for simplicity the case where all collective momenta (linear, angular, and dilational) are vanishing.

40 Longer by the same factor with which the spatial size of the system has been multiplied.
invariant quantity under systems spatial scaling \( \mathbf{x}_i \rightarrow \mathbf{x}'_i = c \mathbf{x}_i \), as such a transformation leads to

\[
\lambda \rightarrow \lambda' = c \lambda
\]

and

\[
\delta t_x \rightarrow \delta t'_x = c \delta t_x
\]

So one gets

\[
\lambda \delta s = \lambda \frac{ds}{dt_x} \rightarrow \lambda' \delta s' = \lambda' \frac{ds'}{dt'_x} = c \lambda \frac{ds}{ct_x} = \lambda \frac{ds}{dt_x} = \lambda \delta s
\]

Now, if one wants to be realistic about length measurements and include this reality in physical theory, one has to use a length unit built out of the matter. So, an internal(or relational) length unit must be used instead of the invisible absolute Newton’s length unit, which had its origin and justification from the absolute space. Take, for instance, the system’s diameter as the internal length unit. That means \( \lambda = 1 \). For an increment of system’s shape \((ds_1, ds_2)\), calculate the increment of time by using the formula for increment of ephemeris time \((75)\) in which we also set \( \lambda = 1 \), i.e., \( dt = dt_x \) \(|\lambda = 1\). Then the expression of Lagrangian \((76)\) with the usage of relational units of time and length becomes as follows

\[
\mathcal{L} = \frac{1}{2} m_3 (m_1 + m_2) \sin^2(s_1) s_1^2 + \frac{1}{2} m_1 m_2 \sin^2(s_1 + s_2) s_2^2 + \left(1 + \frac{m_2}{m_1} + \frac{m_3 (m_1 + m_2) \sin^2(s_2)}{m_1 m_2 \sin^2(s_1 + s_2)} \right) \dot{\lambda}^2 + \frac{1}{2} \sum A_{ab} \Omega^a \Omega^b - V(s_1, s_2) \quad (78)
\]

where now its scale-invariance has become explicitly apparent.

One has to be careful with interpreting the third term here containing the scale velocity \( \dot{\lambda} \). It has the same origin as the fourth term in the lagrangian. The fourth term causes the Coriolis and centrifugal forces in a rotating frame of reference (like in a body frame of a rotating system). Similarly, the third term causes dilational forces in a spatially expanding frame of reference (like a body frame of an expanding system with an internal unit of length). Note also that the scale velocity \( \dot{\lambda} = \frac{\dot{\lambda}}{\lambda} \) has the dimension of the inverse of time, as it is the ratio of change of system’s scale \( \delta \lambda \) during one second of internal time \( \delta t_x \) to the system’s scale \( \lambda \). At an instant of time \( t_0 \), when one is viewing the expanding system from some point of the absolute space, one can manually set the absolute unit of length equal to the instantaneous scale of the system at that time and express \( \dot{\lambda} \) in this new absolute length unit. It becomes simply \( \dot{\lambda} = \frac{\dot{\lambda}}{\lambda} = \frac{\delta \lambda}{\delta t_x |_{\lambda = 1}} \), where during the small observation time interval \([t_0, t_0 + \delta t_x \ |_{\lambda = 1} \) we have fixed the absolute length unit to \( \lambda |_{t = t_0} \), and hence can measure the new scale of the system at the end of the time interval \( \lambda |_{t = t_0 + \delta t_x \ |_{\lambda = 1}} = \lambda |_{t = t_0} + \delta \lambda \).

The equations of motion of the two shape degrees of freedom \( s_1, s_2 \) for the case of vanishing total angular velocity \( \omega_t = 0 \) can be given using \((71)\)

\[
\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{s}} \right) - \left( \frac{\partial \mathcal{L}}{\partial s} \right) = 0
\]

After some lengthy calculations, we end up with the following coupled second order nonhomogeneous non-linear differential equations for the shape degrees of freedom of a nonrotating three-body system

\[
\begin{align*}
\sin^2(s_2) \sin(s_1 + s_2) \dot{s}_1 - 3 \sin^2(s_2) \cos(s_1 + s_2) s_1^2 + 2 \sin(s_2) \cos(s_2) \sin(s_1 + s_2) - \sin(s_2) \cos(s_1 + s_2) s_2 \dot{s}_1 & \dot{s}_1 + 2 \lambda \sin(s_1 + s_2) \sin^2(s_2) s_1 \\
+ \sin(s_1) \left( \cos(s_1) \sin(s_1 + s_2) - \sin(s_1) \cos(s_1 + s_2) \right) s_2^2 & + 2 \lambda^2 \sin^2(s_2) \cos(s_1 + s_2) + \frac{m_1 m_2}{m_3 (m_1 + m_2)} \frac{\partial V}{\partial s_1} = 0
\end{align*}
\]

\((79)\)

\[\text{This is always the only relevant physical situation if the system under consideration is supposed to represent our universe. It follows from the fact that the frame built from the background stars and galaxies is an inertial reference frame.}\]
and
\[\begin{align*}
\sin^2(s_2)\sin(s_1 + s_2) & s_2 \\
+ \cos(s_1 + s_2) (\sin^2(s_1 - 2\sin^2(s_2)) & s_2 \\
+ 2\sin(s_2) \left(\cos(s_2)\sin(s_1 + s_2)\right) & - \sin(s_2)\cos(s_1 + s_2) \right) s_1^2 \\
+ 2\lambda \sin^2(s_2) \sin(s_1 + s_2) & s_2 \\
+ \sin(s_2) \left(\cos(s_2)\sin(s_1 + s_2)\right) & - \sin(s_2)\cos(s_1 + s_2) \\
- \sin(s_2)\cos(s_1 + s_2) & s_1^2 \\
+ 2\lambda^2 \sin(s_2) \left(\cos(s_2)\sin(s_1 + s_2)\right) & - \sin(s_2)\cos(s_1 + s_2) \\
- \sin(s_2)\cos(s_1 + s_2) & s_1^2 \\
- \frac{m_1 m_2}{m_3(m_1 + m_2)} \frac{\partial V}{\partial s_2} & = 0
\end{align*}\]

Now we have to discuss the structure of potential (77) in some more detail. As discussed before, the potential function can be considered to be the product of two functions \( G \) and \( f \) on the absolute configuration space \( Q \), i.e., \( V = Gf \). The form of the function \( f \) is known from the time of Isaac Newton and in the special case of our three-body system, it can be read off from (77). Contrary to \( f \), the \( E(3) \)-invariant function \( G \) has remained unknown to this date. All we know about \( G \) is that it has now a value of about \( 6.67384(80) \times 10^{-11} m^3 kg^{-1} s^{-2} \) on and near the earth, with a relative uncertainty of \( 2 \times 10^{-5} \), which makes it by far the least precisely known natural constant. Different contradictory results have been achieved so far by different measurement methods of \( G \) at different times. Consequently, there is no consensus on its correct value, and the value mentioned above is just the average of results achieved by different methods[25].

Following our discussion on Natural constants in Section (II), one candidate function for \( G \) is
\[G := \sqrt{I_{cm}} = \left(\sum_{i=1}^{N} m_i \left| \mathbf{x}_i - \mathbf{x}_{cm} \right|^2 \right)^{\frac{1}{2}} \]

which clearly satisfies the requirement of being a Euclidean-invariant function and being homogeneous of degree 1 under scale transformation[44]. For the region
\[0 \leq \sin(s_1), \sin(s_2) < \sin(s_1 + s_2)\]

This is, however, not a realistic candidate as a simple estimation of its value for our universe would differ from the measured value of \( G \) by many orders of magnitude. If one wishes, one can in an ad hoc way divide the proposed function by some appropriate number to ensure its estimated value is compatible with the measured value without changing its degree of homogeneity.

\[\text{33} \]

[44]
on the 3-body shape space, one has

\[ \lambda = | \mathbf{x}_2 - \mathbf{x}_1 | \]

A lift of the shape \( s = (s_1, s_2) \) to the absolute configuration space can be given as follows

\[
\mathbf{x}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} \lambda \\ 0 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} \frac{\sin(s_2)\cos(s_1)}{\sin(s_2)\sin(s_1)} \lambda \\ \frac{\sin(s_1)\sin(s_2)}{\sin(s_1)\sin(s_2)} \lambda \\ 0 \end{bmatrix}
\]

and

\[
\mathbf{x}_{cm} = \left[ \frac{m_2 \lambda}{m_1 + m_2 + m_3} + \frac{m_3 \sin(s_1)\cos(s_1)}{m_1 + m_2 + m_3} \right] \lambda
\]

Now by putting all these back in (83), and after some calculations, one gets the following expression for \( G \)

\[
G = \lambda \left( \frac{\sin^2(s_2)}{\sin^2(s_1 + s_2)} (m_1 + m_2) \left( 1 - \frac{2}{M} \right) \right)
+ \frac{m_2 m_3 \sin(s_2) \cos(s_1)}{M \sin(s_1 + s_2)} + m_2 \left( 1 - \frac{m_2}{M} \right)^{1/2}
\]

and for \( f \)

\[
f = \lambda^{-1} \left( m_1 m_2 + m_1 m_3 \frac{\sin(s_1 + s_2)}{\sin(s_2)} \right)
+ \frac{m_2 m_3}{\sin^2(s_1 + s_2)} - \frac{2 \sin(s_2) \cos(s_1)}{\sin(s_1 + s_2)} + 1
\]

Finally, we write down the potential function \( V = \lambda f \) of our three body system as follows

\[
V = \left( m_1 m_2 + m_1 m_3 \frac{\sin(s_1 + s_2)}{\sin(s_2)} \right)
+ \frac{m_2 m_3}{\sin^2(s_1 + s_2)} - \frac{2 \sin(s_2) \cos(s_1)}{\sin(s_1 + s_2)} + 1
\]

\[
\times \left( \frac{\sin^2(s_2)}{\sin^2(s_1 + s_2)} \left( m_3 + m_2 \left( 1 - \frac{2}{M} \right) \right) \right)
- \frac{2 \sin(s_2) \cos(s_1)}{M \sin(s_1 + s_2)} + m_2 \left( 1 - \frac{m_2}{M} \right)^{1/2}
\]

which manifests its scale-invariance now explicitly.

By using this function in the two reduced Euler-Lagrange equations (81), (82), we finally obtain the reduced equations of motion on shape space of the three-body system.

It is worth mentioning in the general case that even though the potential function \( V \) is scale independent\(^4\), it turns out from (79), (82) that the rate of change of scale leaves its trace on the shape dynamics, just as the rate of change of system’s orientation (angular velocity) does, but contrary to the rate of change of system’s position (linear translational velocity).

For the most simple case where the system under consideration is neither rotating nor expanding w.r.t. the absolute space, the Lagrangian function becomes

\[
\mathcal{L} = \frac{1}{2} m_3 (m_1 + m_2 \sin^2(s_1) \sin(s_2))^2 
+ \frac{1}{2} m_1 m_2 \sin^2(s_1 + s_2) \left( m_3 + m_2 \left( 1 - \frac{2}{M} \right) \right) 
+ \frac{1}{2} m_1 m_2 \sin^2(s_1 + s_2) \left( 1 - \frac{m_2}{M} \right)^{1/2} 
\]

\( ^4 \) So, scale transformations are among the symmetry transformation of our mechanical system.
VII. Cosmological consequences of modified Newtonian mechanics

In this section, we consider two situations that highlight the empirical difference between the original and the modified Newtonian theories.

i. Accelerated expansion of the universe

Using internal units we have in Section (IV.ii) that the kinetic metric $K$ of configuration space is scale-invariant. This together with the scale-invariant potential function $V$ of the modified Newtonian theory constitutes a scale-invariant Lagrangian $L = K - V$ for this theory. In a $N$-particle system, the following quantity

$$D = \sum_{j=1}^{N-1} |r_j|^2 \lambda$$

has been called the dilational momentum of the system. We have also seen at the end of Section (V.iii) that the dilational momentum of a system in internal units is a constant of motion of the scale-invariant modified Newtonian theory. This is a result of the scale-invariant Lagrangian of the modified Newtonian theory in the internal units. Without loss of generality, we choose the absolute Newtonian units of time and length by setting them identical to the internal units at some instant of time, i.e., $t_0$. Moreover, set

$$t_0 = t_{internal} = t_{external}$$

so that both internal and external clocks are synchronized at this instant. One can prove the following statement

$$D_{absolute} = cD_{relational}$$

where $b$ is the scale factor of the system at each instant of time

$$b := \frac{\lambda |_{now}}{\lambda |_{t_0}}$$

With this notation one has $b = \lambda$ in the external units (when $\lambda |_{t_0}$ is chosen as the fixed external length unit).

In order to prove (85), one should remember that the scale velocity of the system for an internal observer is understood as follows

$$\dot{\lambda}_{internal} = \frac{\delta \lambda |_{during \ 1 \ internal \ second \ in \ internal \ length \ unit(\lambda |_{now})}}{\lambda |_{now} \ (measured \ in \ internal \ length \ unit)}$$

As $\lambda |_{now}$ itself is the internal length unit, the denominator of the last fraction is 1. The discussion after the equation (78) explains how this fraction’s numerator is understood. The dualism assumed(or sought) in this work gives two ways of understanding or interpreting $\dot{\lambda}_{internal}$. The Newtonian world-view (as explained before) allows us to keep the length of the internal length unit seen from absolute space just during the time interval $[now, now + 1(internal second)]$ constant by setting it equal to $\lambda |_{now}$ during the whole mentioned time interval. With respect to this momentary length unit, the internal observer then measures $\delta \lambda$. Now, as the internal observer cannot communicate with the external observer to perform this measurement, a new internal length unit built from a virialized subsystem (a stable non-expanding, non-contracting subsystems seen from absolute space) can be used by the internal observer just to measure $\delta \lambda$. Both previous methods lead to the same numerical value for $\dot{\lambda}_{internal}$. Now, if one insists on the Leibnizian world view, one can view $\dot{\lambda}_{internal}$ just as a variable appearing in the law of motion on shape space. There may (or may not) be some deeper reasoning behind its value and dynamics from the shape space(Leibnizian) point of view, which we do not speculate about in this work.

Analogously, the scale velocity of the system for the external observer is understood as follows

$$\dot{\lambda}_{external} =$$

46 Units built from matter.
As at any moment of time\textsuperscript{48}

\[ \text{external second} = \frac{1}{c} \times \text{internal second} \]

and

\[ \text{length unit}(\lambda |_{t_0}) = \frac{1}{b} \times \text{internal length unit}(\lambda |_{\text{now}}) \]

one immediately recognizes that the numerical value of the numerator of \( \dot{\lambda} \) for both observers is equal, but the numerical value of the denominator for the external observer is \( b \) times larger than the internal observer. Hence

\[ \dot{\lambda}_{\text{external}} = \frac{1}{c} \dot{\lambda}_{\text{internal}} \]  

(87)

which by using (86) can be rewritten as follows

\[ \dot{\lambda} = \lambda |_{t_0} \dot{\lambda}_{\text{internal}} \]

Equation (87) together with \( \mathbf{r}_{\text{external}} = c \mathbf{r}_{\text{internal}} \) can be used to calculate the dilational momentum \( D_{\text{external}} \) as follows

\[ D_{\text{external}} = \sum_{j=1}^{N-1} | \mathbf{r}_{j,\text{external}} |^2 \dot{\lambda}_{\text{external}} \]

\[ = \sum_{j=1}^{N-1} | c \mathbf{r}_{j,\text{internal}} |^2 \frac{1}{c} \lambda_{\text{internal}} \]

\[ = c \sum_{j=1}^{N-1} | \mathbf{r}_{j,\text{internal}} |^2 \dot{\lambda}_{\text{internal}} \]

\[ = c D_{\text{internal}} \]

and it completes the proof of the equation (85).

Knowing that the value of dilational momentum in internal units \( D_{\text{internal}} = D \) is a constant of motion (as discussed at the end of Section (V.ii)), one can derive the time evolution of scale variable in external units. To this end, remember the following expression of dilational momentum for the external observer, which is written down purely “in terms of external units of time and length” (which are identical to the respective internal units at time \( t_0 \))

\[ D_{\text{external}} = \sum_{j=1}^{N-1} | \mathbf{r}_j |^2 \dot{\lambda}_{\text{external}} = \sum_{j=1}^{N-1} | \mathbf{r}_j |^2 \frac{\dot{\lambda}}{\lambda} \]

after solving for \( \dot{\lambda} \) one has

\[ \dot{\lambda} = \frac{\lambda D_{\text{external}}}{\sum_{j=1}^{N-1} | \mathbf{r}_j |^2} = \frac{\lambda c D}{\sum_{j=1}^{N-1} | \mathbf{r}_j |^2} = \frac{\lambda \dot{\lambda}_{t_0}}{D} \]

\[ = \frac{\lambda^2 D}{\lambda |_{t_0} \sum_{j=1}^{N-1} | \mathbf{r}_j |^2} \]

This expression would be greatly simplified if one defines the scale variable \( \lambda \) as

\[ \lambda := \sqrt{\sum_{j=1}^{N-1} | \mathbf{r}_j |^2} \]

which also has a clear intuitive justification. Putting this back in the previous equation one ends up with

\[ \dot{\lambda} = \frac{D}{\lambda |_{t_0}} = D \]  

(88)

as both \( D \) and \( \lambda |_{t_0} \) are constants, this means that the external observer would see the scale variable (roughly size of the system with the above choice for \( \lambda \)) changing with a constant speed \( D \) (of course measured with respect to external units of time and length). One subsequently has

\[ \dot{\lambda}_{\text{external}} = \frac{\dot{\lambda}}{\lambda} = \frac{\lambda D}{\lambda |_{t_0}} = \frac{D}{\lambda} \]  

(89)

From (88) and (86) one sees that the scale factor \( b \) is a linear function of time for the external observer

\[ b = \lambda |_{\text{now}} - \lambda |_{t_0} = \lambda (t_{\text{external}} - t_0) + \lambda |_{t_0} \]

\[ = \frac{D}{\lambda |_{t_0}} (t_{\text{external}} - t_0) + \lambda |_{t_0} \]

\[ \lambda |_{t_0} \]

\[ \lambda |_{t_0} \]

\[ \lambda |_{t_0} \]
where in the last two equalities, we have set $t_0$ being the initial time, i.e., $t_0 = 0$, and have chosen $\lambda |_{t_0}$ as the absolute unit of length, respectively. One could also arrive at time evolution of $b$ immediately from (83).

From (88) it also becomes evident that in a contracting universe (where $D = 3$) the external observer would see after a finite amount of time $\lambda |_{t_0}$ measured by his external clock, the scale variable $\lambda = \sqrt{\sum_{j=1}^{\infty} |r_j|^2}$ becoming zero. How would an internal observer experience this?

As internal second $= b \times$ external second for the case when $b = 1$ the internal clock speeds up compared to the external clock. The duration $\lambda |_{t_0}$ of external time, which was needed for the scale variable (or the scale factor) to reach the value zero from $\lambda |_{now}$ (or $b |_{now}$), would be measured by the internal clock to be

$$\int_{\lambda=\lambda|_{now}}^{\lambda=0} \frac{\lambda|_{now}}{s_{in}} d\lambda = \frac{\lambda|_{now}}{D} \int_{\lambda=\lambda|_{now}}^{\lambda=0} s_{ex} d\lambda$$

$$= \frac{\lambda|_{now}}{D} \int_{b=b|_{now}}^{b=0} s_{ex} db = \frac{\lambda|_{now}}{D} \int_{b=b|_{now}}^{b=0} \frac{1}{b} db = \infty$$

So for the internal observer, it would take forever to see what the external observer would see in just $\lambda|_{now}$ external seconds $s_{ex}$. It means in a contracting universe, a big crunch by a central collision would never be seen in any finite time by the inhabitants of that universe.

Now, we wish to consider an expanding universe, i.e., $D > 0$. From (88) it becomes evident that the external observer sees the scale of the universe increases for ever with a constant rate $\dot{\lambda} = D$. How would an internal observer experience this?

Using (87) and (89) one can calculate the rate of change of scale variable for internal observer

$$\dot{\lambda}_{internal} = c \frac{\lambda_{external}}{\lambda|_{t_0}} = \frac{D}{\lambda|_{t_0}}$$

As already discussed once, $\dot{\lambda}_{internal}$ means if the internal observer chooses a new unit of length momentarily, for instance, let the size of a virialized subsystem be the new internal length unit (e.g., astronomical unit: au), then the amount of change of universe’s scale variable (measured in au) during one internal second, divided by the length of the universes scale variable (measured again in au) is $D$. So

$$\frac{d\lambda(in \text{ au})}{d\lambda|_{t_0}} = \frac{D}{\lambda|_{t_0}}$$

$$\Rightarrow \frac{d\lambda(in \text{ au})}{d\lambda|_{t_0}} = \frac{D}{\lambda|_{t_0}} \times \lambda(in \text{ au})$$

$$\Rightarrow \lambda_{internal}(\text{in au}) = D \times \frac{\lambda(in \text{ au})}{\lambda|_{t_0}}$$

As in an expanding universe $\frac{\lambda(in \text{ au})}{\lambda|_{t_0}}$ increases, the conclusion is as follows

The internal observer sees an accelerated expanding universe.

ii. Increase in strength of gravity in regions far from matter concentration

Consider a three-body system with two heavy masses $m_1 = m_2 = M$ located near each other at positions $x_1$ and $x_2$, and the third body with a light mass $m_3 = m << M$ located at position $x_3$. Let the origin of the spatial coordinate system coincide with the system’s center of mass. As before (93), the so-called gravitational constant for this three-body universe can be calculated as follows

$$G = \sqrt{T_{cm}} = \sqrt{M(|x_1|^2 + |x_2|^2) + m |x_3|^2}$$

(90)

The gravitational potential of the third (light) body is given as usual by

$$V_3 = -Gf_3 = -\sqrt{M(|x_1|^2 + |x_2|^2) + m |x_3|^2}$$

\text{Note that the bold and normal version of scale velocity are identical for internal observer if the system’s scale variable $\lambda$ is chosen as internal unit of length.}
A short calculation then leads to the following result:

\[
F_3 = -\nabla_3 V_3 = \nabla_3 (Gf_3) = G\nabla_3 f_3 + f_3 \nabla_3 G
\]

where \( Ne \) stands for Newton, and \( \nabla_3 \) is gradient w.r.t. the coordinates of the third particle. A short calculation then leads to the following results:

\[
F_{Ne} = \sqrt{M(|x_1|^2 + |x_2|^2) + m |x_3|^2}
\]

\[
\times \left( \frac{m_1m_3}{|x_1 - x_3|^2} + \frac{m_2m_3}{|x_2 - x_3|^2} \right)
\]

and

\[
\delta F = \frac{m x_3}{2\sqrt{M(|x_2|^2 + |x_1|^2) + m |x_3|^2}}
\]

\[
\times \left( \frac{M}{|x_1 - x_3|} + \frac{M}{|x_2 - x_3|} \right)
\]

It is worthwhile mentioning that the gravitational force felt by the third particle has an attractive part \( F_{Ne} \) and a repulsive part \( \delta F \).

Consider now the case where \( \frac{m_3}{m} << 1 \), so that the system’s center of mass is located in the middle of the bodies 1 and 2. Consider the configuration where the third particle is located on the perpendicular bisector of the line passing through the two heavy masses, i.e., particles one and two. Furthermore, choose the axis of spatial frame such that

\[
x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix}
\]

with \( y >> 1 \). Then, the exerted force on the third body approximately becomes the sum of the following two attractive and repulsive terms

\[
F_{Ne} \approx -\sqrt{2M + my^2} \frac{2mM}{y^2} \hat{y}
\]

\[
\delta F \approx \frac{my}{\sqrt{2M + my^2}} \left( \frac{mM}{y} \right) \hat{y}
\]

If one moreover goes to sufficiently larger distances for the third particle, e.g., \( y >> M \), the above expressions get simplified further as follows

\[
F_{Ne} \approx -2m^{3/2}M \frac{\hat{y}}{y}
\]

\[
\delta F \approx m^{3/2}M \frac{\hat{y}}{y}
\]

So in this regime, the total force felt by the third particle

\[
F_3 = F_{Ne} + \delta F \approx -m^{3/2}M \frac{\hat{y}}{y}
\]

becomes effectively an attractive force decreasing as \( \frac{1}{y^2} \). This \( \frac{1}{y^2} \) decay of the gravitational force would make it possible for the velocity of the third particle orbiting the other two heavy masses to become independent of \( y \). At first sight, this fact seems to explain the flat rotation curves observed in the galaxies.

There are, however, two caveats in the preceding argumentation, which pose difficulties in making such a conclusion. The first caveat is that in the above analysis, we acted as if only one galaxy exists in the universe (whose behavior we approximated by a three-body system). In reality, however, the considered three-body system is located in a homogeneous, isotropic background distribution of galaxies. To account for this, we split \( G \) into a local and a global part, i.e.,

\[
G = G_{local} + G_{global}
\]

where as before

\[
G_{local} = \sqrt{M(|x_1|^2 + |x_2|^2) + m |x_3|^2}
\]

and

\[
G_{global} = \sum_{i=4}^{N} m_i \frac{1}{|x_i - x_{cm}|^2}
\]

Here, the first three particles model, for instance, our Milkyway galaxy, and the rest of the particles model all the other galaxies located at far greater distances than the interparticle distances of the first three particles. As before, \( x_{cm} \) denotes the center of mass of the first
three particles, which is located almost in the middle of the particles 1 and 2, as we assumed $M >> m$. In a fixed, more or less homogeneous, and isotropic background for this three-body system, the gravitational forces acted on the third particle originating from the global (background) structure of the universe, averages out to zero, and as the value of $G_{\text{global}}$ is a constant number, it does not change the functional form of the potential (or the force) experienced by the third particle due to the local structure around it (so due to the first and second particles). In other words, for the potential energy of the third particle, one can write

$$V_3 = -(G_{\text{local}} + G_{\text{global}}) \times$$

$$\left( \frac{m_1 m_3}{|x_1 - x_3|} + \frac{m_2 m_3}{|x_2 - x_3|} + \sum_{i=4}^{N} \frac{m_i m_3}{|x_i - x_{cm}|} \right)$$

where as the locations of all other galaxies are far greater than the inter-particle distances of our galaxy, we have used the approximation

$$\forall i > 3 : |x_i - x_3| \approx |x_i - x_{cm}|$$

Hence, for the force acting on the third particle one gets

$$F_3 = -\nabla_3 V_3 =$$

$$G \nabla_3 \left( \frac{m_1 m_3}{|x_1 - x_3|} + \frac{m_2 m_3}{|x_2 - x_3|} + \sum_{i=4}^{N} \frac{m_i m_3}{|x_i - x_{cm}|} \right) +$$

$$\left( \frac{m_1 m_3}{|x_1 - x_3|} + \frac{m_2 m_3}{|x_2 - x_3|} + \sum_{i=4}^{N} \frac{m_i m_3}{|x_i - x_{cm}|} \right) \nabla_3 (G)$$

$$= G \nabla_3 \left( \frac{m_1 m_3}{|x_1 - x_3|} + \frac{m_2 m_3}{|x_2 - x_3|} \right)$$

$$+ \left( \frac{m_1 m_3}{|x_1 - x_3|} + \frac{m_2 m_3}{|x_2 - x_3|} \right) \nabla_3 (G_{\text{local}}) = F_{Ne} + \delta F$$

As the dominant part of $G$ comes from $G_{\text{global}}$ which is more or less a constant number for the scales of the three-body system, one sees that $F_{Ne}$ would not anymore behave as $\frac{1}{y}$. Because of this, the possibility of finding an explanation of the flat rotation curves along these lines is less probable.

The second caveat lies behind the form of function $\Omega$ or $\Omega^3$ chosen for $G$. As the current value of $G$ measured on Earth in SI-units is of the order of $10^{-11}$, one should, in an ad hoc way, multiply the expression of $G$ by a very small factor to make the chosen $G$-function compatible with the measured value.

All we know from the true function $G$ is that it is a Euclidean invariant homogeneous function of degree one on $Q$, which we argued based on the principle of relationalism. For instance

$$G = \left( \sum_{i=1}^{N} m_i |x_i - x_{cm}|^{-2} \right)^{-\frac{1}{2}}$$

is another function that matches the mentioned requirements, which for the considered three-body system in the regime $M >> m$ and $y >> M$ leads to the following forces on the third particle

$$F_{Ne} \approx -\frac{m \sqrt{2M}}{y^2} \hat{y}$$

$$\delta F \approx \frac{m^2}{\sqrt{2M} y^4} \hat{y}$$

The true $G$-function needs yet to be found and explained, probably from some deeper theoretical reasoning.
VIII. COMPARISON WITH TWO OTHER APPROACHES IN RELATIONAL PHYSICS

Last but not least, to clarify the physical aspects of our work, we give a short comparison with some of the other main approaches in the literature. The notations in this section slightly differ from the rest of this paper and will be mentioned every time.

One of the established approaches in relational physics (see, for instance, [10] and the references in it) denoted here by the BKM-approach uses a property of the Newtonian mechanics known as "mechanical similarity". This property says that if

\[ x(t) = (x_1(t), ..., x_N(t)) \]

is a solution to the Newtonian N-body problem with a homogeneous potential function \( V \) of degree \( k \), i.e. \( x(t) \) satisfies

\[ \frac{d^2 x(t)}{dt^2} = \nabla V \big|_{x(t)} \]

then

\[ x'(t') = (c x_1(t'), ..., c x_N(t')) \]

is also a solution of the theory with

\[ t' = tc^{1-\frac{k}{2}} \tag{91} \]

i.e., \( x'(t') \) satisfies

\[ \frac{d^2 x'(t')}{dt'^2} = \nabla V \big|_{x'(t')} \]

For a Newtonian N-body system, denote \( x_a, p^a, \) and \( m_a \) for the position, momentum, and the mass of the particle \( a \). Systems with vanishing total energy are considered, where the energy is the well-known expression

\[ E_{tot} = \sum_{a=1}^{N} \frac{p^a_p^a}{2m_a} + V_{New} \tag{92} \]

\[ V_{New} = -\sum_{a<b} \frac{m_am_b}{r_{ab}} \tag{93} \]

with \( r_{ab} := || x_a - x_b || \). So the Newton’s gravitational coupling \( G \) is considered to be the constant 1 in the BKM-approach. It is then explained as the dilational momentum \( D \) of a Newtonian gravitational system is a monotonic function (along its solution curves); it can be used as the system’s time variable (instead of the absolute Newtonian time). The transformed Hamiltonian of the Newtonian system in the new coordinates and the new time variable is shown to be

\[ H(D) = \ln \left( \sum_{a=1}^{N} \pi^a \pi^a + D^2 \right) - \ln \left( \frac{I^{\frac{1}{2}}_{cm} | V_{New} |} {C_s} \right) \tag{94} \]

where \( \pi^a \) denote here the shape momenta, defined as

\[ \pi^a := \sqrt{\frac{I_{cm}}{m_a}} - D \sigma_a \tag{95} \]

and \( \sigma_a \) is the following choice for the pre-shape coordinates

\[ \sigma_a := \sqrt{\frac{m_a}{I_{cm}} r_{a cm}} \]

coordinatizing pre-shape space \( PS \), the quotient of configuration space by global translations and scale transformations. By restricting themselves to systems with vanishing angular momentum, the authors of [10] have left quotienting with respect to the group of rotations out of consideration.

After defining some new shape momenta as follows

\[ \omega^a := \frac{\pi^a}{D} \]

from their previous expressions (95), and introducing a new time variable

\[ \lambda := \ln(D) \]

the Hamiltonian (94) is cast into the following apparently time independent form

\[ H_0 = \ln \left( \sum_{a=1}^{N} \omega^a \omega^a + 1 \right) - \ln C_s \tag{96} \]

with

\[ C_s = \sqrt{\frac{I_{cm}}{m_{tot}} | V_{New} |} \tag{97} \]
called the complexity. The Hamiltonian $H_0$ leads to the following equations of motion for the (pre)shape coordinates and their momenta

$$\frac{d\mathbf{q}}{d\lambda} = \frac{\partial H_0}{\partial \mathbf{\dot{q}}}$$  \hspace{1cm} (98)

$$\frac{d\mathbf{\dot{q}}}{d\lambda} = -\frac{\partial H_0}{\partial \mathbf{q}} - \mathbf{\dot{q}}$$  \hspace{1cm} (99)

the second of which shows dissipative dynamics for the (pre)shape momenta $\mathbf{\dot{q}}$.

Here are some comments on this approach for the purpose of comparison.

First of all, as the starting theory for the description of the $N$-body system in the absolute phase space $T^*(Q)$ given by the Hamiltonian (92), (93) is clearly not scale invariant, one should at first glance not even hope to find some law of motion on shape space of this system. In other words, two solutions, $O_t |1\rangle$, and $O_t |2\rangle$ emerging from the following two initial absolute states of the system

$$|1\rangle = (\mathbf{x}_1, \ldots, \mathbf{x}_N, \mathbf{p}_1, \ldots, \mathbf{p}_N)$$

and

$$|2\rangle = (c \mathbf{x}_1, \ldots, c \mathbf{x}_N, c \mathbf{p}_1, \ldots, c \mathbf{p}_N)$$

project down to two different curves on shape space. At this point, Barbour et al. (see 10, 11) came up with a clever idea to transform the momenta $\mathbf{p}_a$ under the spatial scale transformations in such a way to force the second orbit to describe the same path on shape space as the first orbit, i.e.

$$\forall t: \pi(O_t |1\rangle) = \pi(O_t |2\rangle)$$

where as usual $\pi: Q \to S = \Omega^{Q/S_{Sim(3)}}$ is the fiber-bundle’s projection map. To this end, mechanical similarities in Newtonian mechanics have been invoked to find the required transformation law for the momenta. As the potential function (93) considered here is homogeneous of degree $k = -1$, the new time variable after performance of a scale transformation by the factor $b$ becomes

$$t' = b^{3/2} t$$ \hspace{1cm} (100)

So for $b > 1$, the initial velocities\(^5\) should slow down by a factor $b^{-3/2}$. Hence, the correct transformed state must be

$$|2\rangle' = (b \mathbf{x}_1, \ldots, b \mathbf{x}_N, b^{-3/2} \mathbf{p}_1, \ldots, b^{-3/2} \mathbf{p}_N)$$  \hspace{1cm} (101)

Therefore, according to this approach, the group of scale transformations $S^c$ acts on the absolute phase space as follows

$$\begin{pmatrix}
\mathbf{x}_1 \\
\vdots \\
\mathbf{x}_N \\
\mathbf{p}_1 \\
\vdots \\
\mathbf{p}_N
\end{pmatrix} \xrightarrow{S^c} \begin{pmatrix}
b \mathbf{x}_1 \\
\vdots \\
b \mathbf{x}_N \\
b^{-3/2} \mathbf{p}_1 \\
\vdots \\
b^{-3/2} \mathbf{p}_N
\end{pmatrix}$$  \hspace{1cm} (102)

This transformation means, by the way, that from the Newtonian absolute perspective\(^5\) a larger universe by a factor $b$ runs slower by a factor $b^{-3/2}$. Now the orbits of the theory (92), (93) emerging out of the initial states $|1\rangle$ and its scale transformed version $|2\rangle'$, indeed project down to the same curve on $S$ as sought. However, does this mean that the inhabitants(observers) of the two alternative Newtonian $N$-body universes will not be able to tell whether they are located along the orbit $O_t |1\rangle$ or $O_t |2\rangle'$?

A first test that may lead us to the answer is the investigation of the observed velocities(so in relational length and ephemeris time units) of particles or subsystems under mechanical similarity transformations. The observers of these Newtonian universes, which are basically some subsystems, have only access to some internal rods and clocks. As the rods are built from matter, they will change their size by a factor of $b$ after the system undergoes a mechanical similarity transformation. It is well-known\(^19\) that the ephemeris time de-

\(^5\)Expressed with respect to the absolute units of duration and length.

\(^5\)With the usage of the absolute immaterial rods and clocks.
for a $N$-body universe with total energy $E$, mimics perfectly the flow of the absolute time (provides the most accurate internal clock\footnote{The degree of accuracy of an internal time variable can be tested by the degree of accuracy of Newton’s second law for the chosen internal time variable. In Newtonian mechanics, bad choices of internal time will lead to the appearance of fictitious forces not originating from the gradient of the interaction potential $V$ considered in theory.}) in Newtonian mechanics. It makes the relation between the absolute motions in space and the increment of Newton’s absolute time evident. In other words, it reveals how the seconds of Newton’s absolute time is related to the absolute displacements(w.r.t. the immaterial absolute unit of length) of all the particles in the whole universe. However, this perfect matching between the absolute time, and the ephemeris time of the universe, gets destroyed (distorted) by a mechanical similarity transformation. It is because the system’s(universe’s) total energy is not an invariant of the mentioned transformation. So, the ephemeris time of the new universe achieved by a mechanical similarity transformation would no longer coincide with the absolute time of Newton(which is, according to Newton, unaffected by whatever transformation you are making on the material universe\footnote{Absolute, true and mathematical time, of itself, and from its own nature flows equably without regard to anything external\footnote{Abbreviation for the solar year unit}}). Because of this, one may get an impression that a violation of the principle of relationalism is facing the BKM-approach. In the following part, we explain why this is not the case.

If one wants to be realistic about spatial displacements $\delta \vec{x}$’s appearing in \footnote{Abbreviation for the solar year unit} one has to choose the distance between the $i$’th and $j$’th particle (for some $1 < i, j < N$) as unit of length. However, not all choices of $i$ and $j$ are good here. Careless choice of $i$ and $j$ would lead to a violation of energy conservation because of the appearance of fictitious forces. One careless choice, for example, would be two particles that (in the absolute space) are moving accelerated towards or away from each other. On the other hand, if the mass $m_i$ is much larger than the mass $m_j$, and the particle $j$ is moving on a circular orbit (in absolute space) around the particle $i$, then these $i$ and $j$ particles constitute a good choice for the length unit. An example of such a good choice of length unit for our universe is the well-known astronomical unit $au$. Analogously, if one wants to be realistic about time durations, one has to choose also a good unit of time. In the previous example, the period of the lighter particle $j$ around the much heavier particle $i$ would constitute such a good time unit which we denote by $su$\footnote{Abbreviation for the solar year unit}. Note also, by changing the unit of length or time, the unit of energy will also get changed, and with that, the denominator of \footnote{Abbreviation for the solar year unit} which is the square root of the system’s kinetic energy.

Imagine we have a Newtonian $N$-particle universe located in absolute space and changing its location with absolute time. Furthermore, imagine this universe finds itself in the following absolute state

$$|1\rangle = (x_1, ..., x_N, p_1, ..., p_N)$$

Also, imagine that the state is such that a good choice of length and time unit (e.g., $au, su$) can be made. Define the absolute length unit ($Alu$) of the absolute space and the absolute time unit ($Atu$) of the absolute time to be the $au$ and $su$ of this universe at state $|1\rangle$, i.e.,

$$Alu := au |1\rangle$$

$$Atu := su |1\rangle$$

The formula \footnote{Abbreviation for the solar year unit} then tells you how the absolute time is marching forward. Performance of a mechanical similarity transformation brings the universe from state $|1\rangle$ to state $|2\rangle'$ given in \footnote{Abbreviation for the solar year unit}. It is important to remember that the units in which $|2\rangle'$ is expressed are the absolute units. Under a mechanical similarity transformation, the ephemeris time \footnote{Abbreviation for the solar year unit}
transforms as
\[ \delta t'_{e} = b^{3/2} \delta t_{e} = b^{3/2} \delta t^{(n)} \]  
(104)
It is because the kinetic energy
\[ E - V = \sum_{a=1}^{N} \frac{p_{a}^2}{2m_{a}} \]
changes by a factor of \( b^{-3} \) under the mechanical similarity transformations, as can be seen, form (102). So the transformation of the ephemeris time of the two systems related to each other by a mechanical similarity transformation is compatible with the prescription (100). How would the new ephemeris time behave if one uses the relational length and time units of the new universe, i.e., \( au \mid_{(2)}', su \mid_{(2)}', \) instead of the old universe/absolute units (which we do not have direct access to anymore)? As the new relational units are related to the old relational units/absolute units by the following qualities
\[ au \mid_{(2)}' = b. au \mid_{(1)} = b.Alu \]
\[ su \mid_{(2)}' = b^{3/2}.su \mid_{(1)} = b^{3/2}.Atu \]
the new kinetic energy in new relational units becomes
\[ K' = b^{-3}K \left[ \frac{k.g. Alu^2}{Atu^2} \right] = b^{-3}K \left[ \frac{k.g. (b^{-1} au \mid_{(2)}')^2}{(b^{-3/2} su \mid_{(2)}')^2} \right] \]
\[ = b^{-2}K \left[ \frac{k.g. au^2}{su^2} \mid_{(2)'} \right] \]
So the denominator of (103) changes by a factor \( b^{-1} \). As the numerical value for particle displacements \( \delta \mathbf{x} \); in absolute space measured w.r.t. the new length unit, changes with the factor \( b^{-1} \), the numerator of (103) changes also with a factor of \( b^{-1} \). All these together mean that the new ephemeris time (after performing a mechanical similarity transformation of the universe) seen in new relational units would coincide with the absolute time (the old ephemeris time variable). So there is neither a kinematical nor a dynamical violation of the principle of relationalism to be expected in the BKM-approach, and the orbits (Newtonian universes) emanating from \([1]\) and \([2]'\) are for internal observers fully indistinguishable. This point is a very remarkable and unexpected feature of the original Newtonian Mechanics exploited by Julian Barbour, Tim Koslowski, Flavio Mercati, David Sloan, Sean Grybe and their collaborators in \([10],[11],[12],[13]\) (and the references in them).

Another approach is based on the simplest kind of dynamics on shape space, i.e., the geodesic evolution on \( S \), considered first in \([14]\) and further expanded in \([15]\). In this approach, the gravitational potential function is replaced with a homogeneous function of degree \(-2\), e.g.
\[ V = I_{cm}^{-\frac{1}{2}} V_{New} \]  
(105)
This approach partly relies on the following (presumably) transformation of velocities under scale transformations:
\[ v' = bv \]  
(106)
It leads in turn to the following behavior of the norm of a configuration’s velocity \( v \) under the global scale transformations
\[ \|v'\|^2 = M_{bx}(v', v) = M_{bx}(bv, bv) \]  
(107)
\[ = b^2 M_{bx}(v, v) = b^2 M_{x}(v, v) = b^2 \|v\|^2 \]
where \( M \) as usual denotes the mass metric on the absolute configuration space \( Q \). Hence, the multiplication of \( \|v'\|^2 \) (or of \( M \)) with a homogeneous function of degree \(-2\) on \( Q \), e.g. (105) makes the integrand of the (Jacobi’s) action \( S = \int \sqrt{-g}d^2 \) scale invariant. Then, reinterpretting the potential function (105) as a conformal factor, a new (similarity invariant) metric \( M' = VM \) on \( Q \) can be defined, whose geodesics coincide with the evolution of Newtonian systems with the potential (105).

\[ \text{Known with the name of conformal factor}\[15]. \]  
\[ d^2 = M_{j}dx^jdx^j \]  
\[ d^2 = M_{j}dx^jdx^j \]
We want to emphasize here that the specific (or momenta) are scaled with respect to absolute Newtonian units of time and length by a factor $b^{-\frac{3}{2}}$ (see (101)), and also from our work (see (6)). The fact that the DGZ velocity transformation differs both from the BKM-approach mentioned above, where velocities (or momenta) are scaled with respect to absolute Newtonian units of time and length by a factor $b^{-\frac{3}{2}}$ (see (101)), and also from our work (see (6)). The fact that the DGZ velocity transformation is mathematically compatible with (or follows from) the push forward of vectors under scale transformations in configuration space, i.e.,

$$Sc : Q \rightarrow Q$$

$$Sc : T(Q) \rightarrow T(Q)$$

$$v' = Sc v$$

do not give this transformation a physically natural or privileged place over other possible transformations of velocities previously considered. We believe that a lack of attention to the physical origin of the notion of velocity can lead to confusion here. In physics, velocity is a derived notion and it immediately depends on the way we measure space and time intervals, without which one cannot talk about velocities. So, any physically serious statement about the transformation of velocities under some transformation of the universe’s configuration must therefore include a discussion of the changes in the relevant measurement tools, such as that presented in our work on modified Newtonian theory. The arbitrariness in the metric of the shape space in the DGZ-approach\[15\], caused by the arbitrariness in the choice of a conformal factor, is in our opinion due to the forgotten connection of length measures with the real rulers. As explained at the end of the Section (IV.ii), the measured mass metric \[42\] is on its own scale-invariant. A conformal factor would be required if we had access to absolute rulers and could thus measure absolute lengths. Since all rulers are themselves subsystems of the universe, they are also subject to the transformations applied to the universe. Taking this physical fact into account resolves the mentioned arbitrariness and provides us with the unique metric $N$ on shape space.

Note also that (108) differs from the mechanical similarity transformations of an absolute theory with a homogeneous potential function of degree $-2$, according to which the momenta transform as $p_i \rightarrow c^{-2} p_i$. This discrepancy does not pose a serious problem in the DGZ-approach, since in the relational world-view the rate (with respect to absolute time) at which the system’s actual shape of the system moves along a “given path” in shape space is a gauge freedom. This is because if the actual shape of the universe traverses a “given curve” (a geodesic in the BDGZ-approach) on shape space at different speeds (with respect to absolute time), no objective (relational) change can be observed by the inhabitants of that universe. On the other hand, if one takes an absolute physical theory (like the Newtonian or the modified Newtonian theory) as the starting point for finding the the relational laws of motion, a change in the absolute velocities (by some factor) generically has dynamical effects on

\[57\] Derived from the more primitive notions of space and time. Considering time itself is a derived notion, as a relationalist would, velocity is merely a concept derived from space in a non-empty universe.
the shape space, i.e., leads to alternative universes moving along "different paths" on shape space. As mentioned before, mechanical similarities provide a way to prevent this problem, by introducing a suitable action of $\text{Sim}(3)$ on the theory’s absolute state space.

At this point, we want to emphasize that the action of scale transformations on phase space, as used in our work, is a direct consequence of the way we implemented the Principle of Relationalism in (absolute) modified Newtonian theory, which turns out also to be compatible with the mechanical similarities of this theory. It is in particular, neither an additional postulate nor a gauge fixing condition.

Another difference worth mentioning is the potential function (105) used in the BDGZ-approach, being a homogeneous function of degree $-2$, which clearly differs both from the BKM-approach (93), and our work, where they are homogeneous functions of degree $-1$, and 0 (hence scale-invariant) respectively. Our scale invariant potential function can be incorporated into the geodesic DGZ-approach by choosing a new conformal factor $f' := Vf$, which is the multiplication of the conformal factors introduced in (15) by the scale-invariant potential. Nevertheless, the presence of a potential function of the type (105), which keeps the system’s moment of inertia constant, is a characteristic of this approach and is absent in the other two approaches. Whether a specific gauge can be found in which the DGZ-approach coincides with the BKM-approach or with our work remains an open question of the DGZ-approach.

We think that the mathematical definition of the notion of scale-invariance in Riemannian geometry (see15) is less relevant from the physical point of view. This is due to the use of the differential of the scale transformation $Sc$ as the action of $Sc$ on $T(Q)$ and its decoupling from the physical theory. Here we define a new notion that is more relevant to physics. A metric $G$ on the configuration space $Q$ is called mechanical similarity invariant if and only if

$$\forall v_1, v_2 \in T_q(Q), G_q(v_1, v_2) = G_{bq}(b^k v_1, b^k v_2)$$

(109)

where $k$ is the degree of homogeneity of the potential function of the physical theory. The factor $b^k$ results from the combined effect of the time transformation required by the theory’s mechanical similarity (91) and ruler’s extension. Thus, instead of defining the action of $Sc$ on $T(Q)$ by push forward $Sc_*$, we define the action of $Sc$ by the mechanical similarity transformation on $T(Q)$. The mass metric $M$ is mechanical similarity invariant for the modified Newtonian theory, but not for the original Newtonian theory. A mechanical similarity invariant metric defines a unique metric $N$ on the $\text{Sim}(3)$-reduced tangent-bundle $T(Q)_{\text{Sim}(3)}$.

\[58\] This type of time transformation was required for the dynamical equivalence of the two alternative universes.
IX. Conclusion

In this paper, we first reviewed (following [4], [5] and [6]) the derivation of the Euler-Lagrange equations of motion in nonholonomic frames and the reduced equations of motion on the internal configuration space $Q_{\text{int}} = \frac{\mathbb{Q}}{E(3)}$ of classical mechanics. Then, using the Principle of Relationalism, we extended the discussed methods to the entire similarity group $\text{Sim}(3)$. In particular, we constructed representations of the group $\text{Sim}(3)$ and its Lie-algebra $\text{sim}(3)$ on $\mathbb{Q}$, discussed how a vector on shape space can be lifted horizontally to the center of mass configuration space $Q_{\text{cm}}$, constructed the connection form $\omega_s$ for the action of the group of scale transformations $S\text{c}$ on $Q$, and showed that this connection form is flat. As a consequence of the latter, quotienting out the configuration space w.r.t. the group of scale transformations $S\text{c}$ does not produce any additional curvature in the resulting base space. Thus, the curvature in shape space is caused solely by the quotienting w.r.t. the group of rotations $\text{SO}(3)$.

Using these new ingredients, we derived the reduced equations of motion of a $N$-particle system for its shape degrees of freedom, whose behaviour in absolute space and time is given by the modified Newtonian mechanics introduced in Section (II) and [1]. The principle of relationalism guaranteed, among others, the existence of the laws of motion on shape space in a straight forward way. As the simplest nontrivial example of the formalism, we have explicitly derived the equations of motion for the shape degrees of freedom of a three-particle system. We then discussed some cosmological consequences of the theory. In particular, we have shown that an expanding universe must necessarily be an accelerating expanding universe, and that the total central collision of all particles in a contracting system cannot occur in a finite amount of (internal) time. These effects are partly due to a new conservation law in the modified Newtonian theory, namely the conservation of the dilatational momentum $D$.

At the end, we presented a comparison of our work with two other approaches to relational physics. In particular, we compared the used action of the group $S\text{c}$ on the absolute phase space in each approach. We explained how the principle of relationalism (as formulated in Section (II) ) itself defines an action $A$ of $S\text{c}$ on the absolute phase space of the modified Newtonian theory, which in turn allowed us to find the unique metric $N$ of $T(Q)_{\text{Sim}(3)}$. Alternatively, we discussed that by taking the role of rulers in determining the geometry of space into account, the measured mass metric is itself scale-invariant, so again no arbitrariness in the metric of shape space is involved. In particular, we explained the relationship between the choice of a length unit and the choice of a conformal factor, and elaborated that all reasonable choices of length units lead to the same metric on shape space. We have also seen that Barbour’s fundamental postulate of relational mechanics, as expressed in [3], and Barbour-Bertotti’s postulate of relational mechanics [2], are special cases of the (more general) principle of relationalism introduced in Section (II).
X. Appendix

i. Mass tensor

Originally the Kinetic energy $K$ of a classical $N$-particle system is expressed (defined) in Cartesian coordinates as following

$$K = \frac{1}{2} \sum_{i=1}^{N} m_i \dot{x}_i^2 = \frac{1}{2} [x_1, ..., x_N]M \begin{bmatrix} \dot{x}_1 \\ ... \\ \dot{x}_N \end{bmatrix} \tag{110}$$

where $\dot{x}_i := \frac{dx_i}{dt}$ with $x_i = \begin{bmatrix} x_{3i-2} \\ x_{3i-1} \\ x_{3i} \end{bmatrix}$ and $M$ is the so-called mass matrix which is in this case just a block diagonal $3N \times 3N$ matrix with

$$M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_j & 0 \\ 0 & 0 & m_l \end{bmatrix}$$

as it’s $j$’s block.

Here as usual the configuration space is coordinatized by $x_1, x_2, ..., x_{3N}$ which are in turn the collection of Cartesian coordinates $x_{3i-2}, x_{3i-1}, x_{3i}$ used to denote the position (vector in $\mathbb{R}^3$) of the $i$’th particle $r_i$. Now if the system suffers a number of holonomic constraints, the generalized coordinates $q_1, q_2, ..., q_f$ (with $f < 3N$ standing for total number of remaining degrees of freedom), can be used for coordinatizing the new (generalized) configuration space.

Now if one rewrites the kinetic energy $K$ in terms of this new generalized coordinates $\dot{q}_j$ and their velocities $\dot{q}_j$ one ends up usually with a much more complicated expression than \[110\] where it was in fact coordinate independent, and a simple diagonal quadratic form in the velocities. In generalized coordinates, it is quadratic but not necessarily homogeneous in the velocities $\dot{q}_j$, and has an arbitrary dependence on the coordinates $q_j$ (through $M$).

If the coordinate transformation between the set of Cartesian coordinates $x_1, ..., x_{3N}$ and the generalized coordinates $q_1, ..., q_f$ is time-independent (see \[21\]), the kinetic energy is written as

$$K = \frac{1}{2} \sum_{k,l} M_{kl} \dot{q}_k \dot{q}_l = \frac{1}{2} [\dot{q}_1, ..., \dot{q}_f]M \begin{bmatrix} \dot{q}_1 \\ ... \\ \dot{q}_f \end{bmatrix} \tag{111}$$

Where $M_{kl} = \sum_{i=1}^{N} m_j \frac{dx_i}{dq_k} \frac{dx_i}{dq_l} = \sum_{j=1}^{N} \sum_{i=0}^{2} \frac{dx_{j-i}}{dq_k} \frac{dx_{j-i}}{dq_l}$ are elements of the $f \times f$ matrix $M$.

The Lagrangian of classical mechanics is shown to be $L = K - V$, where the potential $V$ is usually independent of the generalized velocities $\dot{q}_i$. The conjugate momentum to $q_i$ is defined as

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial K}{\partial \dot{q}_i} = \sum_{j=1}^{f} M_{ij} \dot{q}_j \tag{112}$$

thus the expression \[111\] for the kinetic energy which involved just the velocities can be rewritten as

$$K = \frac{1}{2} \sum_{i=1}^{f} p_i \dot{q}_i \tag{113}$$

ii. Adjoint and Coadjoint action of a Lie-Group

Let $G$ be a Lie group, and $G^*$ it’s Lie algebra, and $G^*$ be the dual vector space of $G$. The adjoint representation of $g \in G$ on $G$ is defined by

$$Ad_g(y) = \frac{d}{dt} |_{t=0} (ge^t g^{-1}) \tag{114}$$

for $Y \in G$.

The coadjoint action of $g \in G$ on $G^*$ is characterized by

$$< Ad^*_g(\xi), Y > = < \xi, Ad_g^{-1}(Y) > \tag{115}$$

for $\xi \in G^*$

Here, $<, > : g^* \times g \to \mathbb{R}$ is the dual pairing.

In summary: $ad_g(x) = gxg^{-1}, Ad_g = (ad_g)_* : G \to G$ being called the adjoint action, $Ad^*_g : G^* \to G^*$ being called the coadjoint action.
iii. Isomorphism $R$

There exists an isomorphism $R$ between Lie-algebra $\mathfrak{so}(3)$ of rotation group, and the linear space $\wedge^2\mathbb{R}^3$ of all antisymmetric tensors of order 2, which we want to explain shortly.

Take $e_1 ... e_3$ as an orthonormal basis of $\mathbb{R}^3$. Then $e_i \wedge e_j$ with $i < j$ constitutes an orthonormal basis of $\wedge^2\mathbb{R}^3$. The inner product in $\wedge^2\mathbb{R}^2$ is defined as the following

$$ (u \wedge v \cdot x \wedge y) = \begin{vmatrix} (u \cdot x) & (u \cdot y) \\ (v \cdot x) & (v \cdot y) \end{vmatrix} \quad (116) $$

One can easily check that for two two-vectors (or tensors of order 2) $\xi = \sum_{i<j} \xi_{ij} e_i \wedge e_j$ and $\zeta = \sum_{i<j} \zeta_{ij} e_i \wedge e_j$, definition [116] leads to the following

$$ (\xi | \zeta) = \sum_{i<j} \xi_{ij} \zeta_{ij} \quad (117) $$

Now we identify the Lie-algebra of the rotation group in 3 dimensions $\mathfrak{so}(3)$ with the space of two forms (anti-symmetric tensors) $\wedge^2\mathbb{R}^3$ by the isomorphism $R$

$$ R : \wedge^2\mathbb{R}^3 \to \mathfrak{so}(3) \quad (118) $$

So for $u,v,x \in \mathbb{R}^3$ we define the following

$$ R_{u \wedge v}(x) := (v \cdot x)u - (u \cdot x)v \quad (119) $$

$R_{u \wedge v}$ is in fact a 3 dimensional square matrix and its multiplication by a 3 dimensional vector $x$ is given by the last equation. For $\xi \in \wedge^2\mathbb{R}^3$ and $x = \sum \xi_i e_i \in \mathbb{R}^3$ One can also write the above formula as

$$ R_\xi(x) = \sum \left( \sum_{i<j} \xi_{ij} x_j \right) e_i \quad (120) $$

That is, $R_\xi$ is an antisymmetric matrix with entries $\xi_{ij}$.

Given the natural scalar product of the Lie algebra; $(a | b) = \frac{1}{2} \text{tr}(a b^T)$ for $a,b \in \mathfrak{so}(3)$ one can show that the identification $R$ is even an isometry from $\wedge^2\mathbb{R}^3$ to $\mathfrak{so}(d)$.

As explained in [27], space $\wedge^2\mathbb{R}^3$ can be identified with $\mathbb{R}^3$ by $e_1 \wedge e_2 \to e_3$ and the cyclic permutations. Hence if one sets

$$ \xi_{12} = \phi^3, \xi_{23} = \phi_1, \xi_{31} = \phi^2 $$

the vector

$$ \zeta = \sum_{i<j} \xi_{ij} e_i \wedge e_j $$

is identified with

$$ \phi = \sum \phi^i e_i $$

So in this case in effect $R$ becomes a linear isomorphism from $\mathbb{R}^3$ to $\mathfrak{so}(3)$ i.e.

$$ R : \mathbb{R}^3 \to \mathfrak{so}(3) $$

$$ R_\xi(x) = R_\phi(x) = -\phi \times x \quad (121) $$

for $x \in \mathbb{R}^3$.

Alternatively, $R_{\xi_1}$ is the matrix $(\xi_{ij})$ with the only nonzero elements $\xi_{23} = -\xi_{32} = 1$.

One can also show [27] that $R$ is $Ad$-equivariant i.e. $R_{g\phi} = Ad_g R(\phi) = g R(\phi) g^{-1}$.

There exists the following properties for the map $R$, and the inertial tensor $A_x$

$$ R_a b = a \times b \quad (122a) $$

$$ R_g a = g R_g a \quad (122b) $$

$$ R_{a \cdot b} = <a | b> \quad (122c) $$

$$ A_g(a) = g A_x(g^{-1} a) = Ad_g A_x(a) \quad (122d) $$

$$ (x | R_\phi y) = (x \wedge y | \xi) \quad (122e) $$

$$ (R_\xi x | R_\eta y) = (R_\eta x \wedge y | \eta) \quad (122f) $$

where $a,b \in \mathbb{R}^3$ and $g \in SO(3)$.  

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iv. Symbols

\( t^{(n)} \): Newton’s absolute time
\( x \): A point on \( Q_{cm} := \frac{\Omega}{\mathbb{R}} \)
\( q \): A point on \( Q_{\text{int}} = \frac{Q}{SO(3)} \)
\( s \): A point on shape space \( S := \frac{Q}{Sim(3)} \)
\( r_i \): i’s Jacobi vector of an \( N \)-particle system
\( \lambda \): Scale variable of a system
\( \lambda \): Scale velocity of a system
\( \dot{\lambda} \): Scale velocity of a system measured in internal units
\( a, b, \gamma \): Euler angles connecting a body frame and the space frame
\( \{ e_1, e_2, e_3 \} \): Fixed laboratory frame, or space frame
\( \{ e_1', e_2', e_3' \} \): Body frame
\( g \): Rotation which brings the space frame to the body frame
\( J = \sum_{a=1}^{N} m_a \dot{x}_a \times \partial_{x_a} \): Total angular momentum
\( \Omega^a \): Components of angular velocity in space frame
\( \Omega_i^{ab} \): Components of angular velocity in body frame
\( J := \sum_{a=1}^{3} e_a J_a = \sum_{a=1}^{3} e_a' L_a \)
\( J_a := (e_a | J) \): Left invariant vector fields on \( SO(3) \)
\( L_a r_i := e_a \times r_i \)
\( L_a L_b := e_a' \times r_i = g(e_a \times \sigma_i(q)) \)
\( \omega^a_j, \omega_i^a \): \( \theta^a, L_b, \theta_i^a \)
\( \theta^a \): Left invariant one forms on \( SO(3) \)
\( g^{-1} \theta^a := \sum_{a=1}^{3} \theta^a R(e_a) \)
\( \psi^a \): Right invariant 1-forms on \( SO(3) \)
\( d g^{-1} := \sum_{a=1}^{3} \psi^a R(e_a) \)
\( k \): Curvature tensor of shape space
\( c \): Speed of light
\( b \): Letter used to characterize scale transformations by a factor \( b \in \mathbb{R}^+ \)
\( Sc \): Group of spatial scale transformations of matter
\( Grs \): Group of spatial rotations and scale transformations
\( A \): Moment of inertia tensor of a \( N \)-particle system
\( A_g \): Gauge fields on \( Q_{\text{int}} \)
\( A^g \): Action of \( g \in \text{Sim}(3) \) on the tangent bundle \( T(Q) \)
\( D \): Dilational momentum operator
\( D \): Value of system’s dilational momentum measured in internal units
\( M \): Mass metric on \( Q_{cm} \) or \( Q \)
\( B \): Metric on \( Q_{\text{int}} \)
\( N_i \): Metric on the reduced tangent bundle \( A_{\text{Sim}(3)} \)
\( N \): Metric on the shape space
\( M_{\lambda \lambda} \): Scale component of the mass metric in shape and scale coordinates. \( a(s) \)
\( M^m \): The measured mass metric on \( Q \)

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