The influence of time-varying mesh stiffness on natural characteristics of planetary gear system

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Abstract. Modal property is the crucial factor that can influence the vibration characteristics of the planetary gear system. Previous studies used time invariant mesh stiffness (TIMS) to study the natural characteristics. However, with the rotation of the gear, the meshing stiffness will change periodically which will lead to changes of modal properties. Based on these properties which are ignored in previous studies, a translational-torsional free vibration model base on time-varying mesh stiffness (TVMS) rather than time invariant mesh stiffness (TIMS) is established in this paper to study the modal properties of the planetary gear system, the effects of TVMS on the natural characteristics are investigated. The results show that the meshing phase difference and the change regulation of the time varying natural frequencies are correlated, and there is a dramatic variation of higher order natural frequencies at the contact teeth change points. Furthermore, a useful conclusion can be drawn that takes TVMS to calculate higher order frequencies has certain advantages over traditional methods.

1. Introduction

Planetary gear set is commonly used in mechanical transmission due to its higher transmission ratio and highly compact structure. The characteristics of natural frequencies and vibration modes have an important meaning in designing and optimizing of planetary gear set, precise modal properties are conducive to noise and vibration reduction. On the other hand, the time varying mesh stiffness is a crucial factor that can significantly influence the vibration of gear, as the mesh stiffness change periodically with time, this parametric excitation is primary source of noise and vibration and it will cause severe vibration under a certain operating condition[1].So it is important to characterize the influences of TVMS on the natural frequencies and vibration modes, while the majority of researches used time invariant mesh stiffness, which are considered to be constant and equal to their average stiffness over a mesh period, to calculate natural characteristics of planetary gear system[1-9], little attention has been paid to studying the influence of TVMS on natural characteristics. However, with the rotation of gear, the meshing states of different meshing points are distinctive with each other, especially the changing pairs of teeth in contact will result in a dramatic variation of mesh stiffness, accordingly, the change number of teeth in contact should has large effect on the characteristics of the system. Also, with the existence of meshing phase, there exist certain mesh stiffness combination for planetary gears mesh over a meshing cycle. So, the characteristics of natural frequencies and vibration modes are more
complicated than the former studies considered, simply taking time invariant mesh stiffness to calculate
the natural characteristics of the system will lose some critical information of the system.

Taking advantage of the discussion above, in a bid to precisely study the detailed characteristics of
natural frequencies and vibration modes and systematically analyse the effect of TVMS bring to the
modal properties of the system, this paper proposes a method that take TVMS into characteristics
equation to investigate the time varying characteristic properties, the mesh stiffness of each planet at
different meshing time are separately considered. A translational-torsional free vibration model base on
TVMS with meshing phase is introduced firstly, based on the proposed model, the time varying
frequencies are obtained, by comparison with the results of the traditional method, the difference of two
analyses is investigated. Then the influence of the change number of teeth in contact and meshing phase
on the natural characteristics is studied. Finally, sensitivities of natural frequencies to mesh stiffness
over one meshing period are presented to further study the influence of TVMS and the change of
meshing teeth brings to the natural characteristics of the system.

2. System model

The present work considered TVMS to investigate the free vibration characteristics of single stage
planetary gear set, a lumped-parameter dynamic model for spur planetary gear set is the basis for this
study. A planetary gear system with N planets is given in Figure 1.

![Figure 1. Equivalent dynamics model of planetary gear set.](image)

In this model, each component admits one rotational and two translational motions. Which indicates
the system has $3(N+3)$ degrees of freedom. The free vibration equation of this system with consideration
of TVMS at $t$ is:

$$Mq + [K_b + K_{m}^t]q = 0$$

(1)

where, $K_b$ denotes bearing support stiffness, $K_{m}^t$ is the stiffness matrix of tooth meshes at time $t$, $M$
stands for the mass matrix. $K_b$, $K_{m}^t$ and $M$ with the forms of:

$$K_b = \begin{bmatrix} K_{b1} & \ldots & K_{bN} \\ \vdots & \ddots & \vdots \\ K_{bN} & \ldots & K_{b1} \end{bmatrix}$$

symmetric

$$K_{m}^t = \begin{bmatrix} \sum(K_{m1}^t) & \sum(K_{m2}^t) & \ldots & \sum(K_{mN}^t) \\ \sum(K_{mN}^t) & \sum(K_{m1}^t) & \ldots & \sum(K_{m2}^t) \\ \vdots & \vdots & \ddots & \vdots \\ \sum(K_{m2}^t) & \sum(K_{mN}^t) & \ldots & \sum(K_{m1}^t) \end{bmatrix}$$

symmetric

$$(K_{mp}^t)^\top = (K_{mp}^t)^\top + (K_{mp}^t)^\top$$

$M = \text{diag}[M_1, M_2, \ldots, M_N]$
The detailed submatrices in $K_b$, $K^t_m$ and $M$ refer to Ref.[3]. Noticed that the matrix with subscript $t$ because of the containment of TVMS. $q$ is the coordinates for the motion of each component:

$$q = \begin{pmatrix} x_c, y_c, w_c, x_r, y_r, w_r, x_s, y_s, w_s, u_1, v_1, w_1, \ldots, u_N, v_N, w_N \end{pmatrix}^T$$

where, $x_i$ and $y_i$ ($i=s,r,c$) are assigned to represent the translational motion of central components, subscript $l$ denotes s(sun gear), r(ring gear) and c(carrier), and $u_i, v_i, (i=1,\ldots,N)$ stand for the equilibrium deflections of planetary gear in radial direction and circumferential direction, $w_i = \theta_i r_i$, ($i=s,r,c,1,\ldots,N$) are rotational coordinates, $\theta_i$ is the rotational angular deflection of component $l$, $r_i$ is the base circle radius of ring, sun, planet gear, and the circle passing through the center of planet gear for $l=c$ (carrier). Using eigenvalue problem to determine the natural frequencies and vibration modes, substituting of $q = \phi^t e^{\omega^t t}$ into Equation (1) yields:

$$\begin{pmatrix} -\omega^t \lambda^t & M + K_b + K^t_m \end{pmatrix} \phi^t = 0$$

where $\omega^t$ and $\phi^t$ indicate the natural frequency and vibration mode of the system at $t$th order at time $t=t_0$ respectively, introducing $\lambda^t$ denotes ($\omega^t \phi^t$), Equation (2) reduces to:

$$\begin{pmatrix} K_b + K^t_m - \lambda^t M \end{pmatrix} \phi^t = 0$$

The difference between the proposed method and the traditional method is the proposed method considered mesh stiffness $K_m$ is time variant which indicates that the characteristics of natural frequencies and vibration modes are no more identical in one meshing period, they are time dependent, in different meshing state, the natural characteristics are distinct.

3. Mesh stiffness of each planet mesh

In order to investigate the influence of the TVMS on the modal properties of the system, a system with equally spaced planet gears is proposed.

Using $\gamma_{sn}$ denotes the meshing phase difference between sun/nth planet mesh and the sun/1st planet mesh and $\gamma_{rn}$ denotes the meshing phase difference between the ring/nth planet mesh and the ring/1st planet mesh. With planet gear counter-clockwise rotation, $\gamma_{sn}$ and $\gamma_{rn}$ yields:

$$\gamma_{sn} = (1-n)\frac{z_s}{N}, \gamma_{rn} = (n-1)\frac{z_r}{N}$$

While based on assembly condition of equally spaced planetary gear system:

$$(z_r + z_s) / N = \text{integer}$$

Substituting Equation (5) into Equation (4) gives:

$$\begin{cases} \gamma_{sn} = (1-n)\frac{z_s}{N} \\ \gamma_{rn} = (n-1)\frac{z_r}{N} = (n-1)\frac{\text{integer} * N - z_s}{N} = (1-n)\frac{z_r}{N} \end{cases}$$

Which indicates: $\gamma_{sn} = \gamma_{rn} = \gamma_{sn}$. The TVMS of sun-planet mesh $k_{sn}(t)$ and ring-planet mesh $k_{rn}(t)$ can be expressed as:

$$\begin{cases} k_{sn}(t) = k_{spt} (t - \gamma_{sn} T) \\ k_{rn}(t) = k_{rpt} (t - (\gamma_{rn} + \gamma_{sn}) T) \end{cases}$$

where $T$ denotes time of one meshing cycle, $\gamma_{rn}$ stands for the relative meshing phase difference between the sun/planet and the ring/planet mesh at a given planetary gear.
With the parameter given in Tab. (1) and the expression of meshing phase, using potential energy to calculate the TVMS [9], the mesh stiffness of each planets in two meshing cycles can be obtained which are shown in Figure 2, in this figure, the blue dash line denotes stiffness of sun/$n^{th}$ planet mesh while the red dash line denotes stiffness of ring/$n^{th}$ planet mesh.

Table 1. The parameters of example spur planetary gear set.

|                  | Sun | Ring | Planet | Carrier |
|------------------|-----|------|--------|---------|
| Teeth            | 31  | 73   | 21     | --      |
| Gear module (mm) | 3   | 3    | 3      | --      |
| Tooth width (mm) | 20  | 20   | 20     | --      |
| Mass(kg)         | 0.79| 1.17 | 0.35   | 3.50    |
| $I/r^2$ (kg)     | 0.58| 1.46 | 0.27   | 3.29    |
| Pressure angle (°) | $\alpha_s=\alpha_r=20 \degree$ | | | |
| Number           | 4   |      |        |         |
| Translational support stiffness(N/m) | $k_p=k_r=k_s=10^9$ | | | |
| Torsional support stiffness(N/m)    | $k_{r_t}=10^9$, $k_{s_t}=k_{r_t}=0$ | | | |

Figure 2. Stiffness of sun/planet mesh and ring/planet mesh.

4. Modal properties

The traditional method considers all mesh stiffness are constant and equal to their mean stiffness to determine their modal properties. The characteristic equation of this time-invariant system is:

$$\omega^2 M \phi = (K_p + K_n) \phi$$  \hspace{1cm} (8)

With exemplary system list in Table.1, the modal properties are calculated by taking mesh stiffness as mean value

$$k_{p1} = k_{p2} = \ldots = k_{pN}, k_{r1} = k_{r2} = \ldots = k_{rN}$$

Results of natural frequencies and vibration modes are shown in Figure 3.
Because of the symmetry of the mesh stiffness, the vibration modes of the system are divided into three categories: rotational mode (R), translational mode (T) and planet mode (P) which are shown in Figure 3. But with the existence of meshing phase difference, the engaging states exist a certain lag or lead, which means that the symmetry of the system is broken, the natural characteristic should be no longer strict regularity like traditional results.

In bid to investigate the influence of TVMS on natural characteristics, 10001 time nodes ($t_n = t_0, t_1, \ldots, t_{10000}$) with equivalent interval ($t_n - t_{n-1} = \Delta t$) are taken in one meshing cycle, substitution $t = t_n$ into $K_{m}^{t}$, there are 10001 matrices of $K_{m}^{t}$ in these time nodes, repeatedly solving the characteristic Equation (3) with distinct $K_{m}^{t}$, the variation track of the time-varying natural frequencies in one meshing cycle can be fitted which is shown by gray curves in Figure 4. It is obvious that the higher-order of natural frequencies which considered TVMS change rapidly over one meshing cycle, while for low-order frequencies, the fluctuate of value is relatively small, which suggests that the time variant characteristic of mesh stiffness has large influence on higher-order frequencies of the system, while for low order frequencies, the alterations caused by TVMS are relatively small.

Figure 3. Natural frequency distributions and three typical modes.

Figure 4. Natural frequencies of each order after considering TVMS and TIME
To facilitate the investigation of goodness of fit between the proposed method and the traditional method, the results of constant natural frequencies which considered TIMS are introduced in Figure 4, which are illustrated by the red dash straight line. Partial enlarge natural frequencies of low-orders (2nd-12nd), it is obvious that the natural frequencies calculated by two methods are agreed quite well with each other at low order. The natural frequency of the 2nd order and 3rd order is taken as a typical example in a bid to pertinently investigate the effect of TVMS brings to the low order frequencies. The detailed results are presented in Figure 5. The natural frequencies of the 2nd and 3rd order ω2, ω3 which are calculated by the traditional method are identical, ω2 = ω3, their vibration modes are distinguished by translational modes [3], the vibration mode of 3rd order is shown in Figure 3, but under the case of TVMS is taken into consideration, the frequencies of 2nd and 3rd order are no more identical even though their values are highly close with each other and the error between two analyses is less than 2Hz (0Hz < ω2' - ω3' < 2Hz), with small rotational displacement of the central components, vibration modes φ2, φ3 over one meshing period are similar to translational mode. φ2(0.2T), φ3(0.2T) are taken as typical examples, which illustrated in Figure 5. Their vibration modes are correlated to translational mode calculated by the traditional method.

Figure 5. Vibration mode of 2nd (left) and 3rd (right) order at 0.2T.

With further study of the vibration modes for other orders, the results show that the vibration modes of each order are distinct with each other at different meshing points, at low-order, the vibration modes are similar to the correlative traditional vibration mode type (R, T and P), but these properties will disappear in higher-order frequencies, the vibration modes of higher-frequency are no more like typical vibration modes. It can be concluded that it is appropriate to take mesh stiffness as a constant value to estimate the natural frequencies of the lower order, but when it involves higher-order, this method is no more applicable. In Figure 4, obviously, there is a ‘jump phenomenon’ for natural frequencies of higher order at a certain time node, a possible explanation for this phenomenon is that with the change number of teeth in contact, the mesh stiffness will be sharply changed, near these ‘critical points’, with a little perturbation of time, the stiffness matrix will radically change which result in the steep variation of higher-order natural frequencies. To vindicate this assumption, the contact teeth changing points in the meshing process of each planet gear are proposed in Figure 6. K_{xpn}(D-S) stands for the critical point for x gear engaging with n^{th} planet gear with the state of double teeth area meshing changing into single teeth-meshing area, while K_{xpn} (D-S) denotes the critical point for x gear with the state of single teeth meshing area changing into double teeth-meshing area.
Applying these contact pair change points, this information in Figure 6 supports the original hypothesis that the change number of teeth in contact can significantly influence the value fluctuation of higher-order natural frequencies. From these results, we can arrive at the conclusion that to accurately calculate the natural characteristics of the system, TVMS should be taken into account, because the changing meshing teeth pair can largely influence the result.

5. The variation regularity

Through synthetically analysing the simulation results illustrated in Figure 4, 6, it can be noticed that the natural frequencies vary regularly with time, and there exist four sub-periods over one meshing period, this phenomenon are understandable because of the existence of meshing phase between adjacent planet gear meshes, with this meshing phase and uniformly distributed along the circumferential direction of the planet gears, the meshing time difference $\Delta t$ of two neighboring planet meshes is identical, which indicates that the meshing state of $n^{th}$ planet at time node $t$ is coincident with the meshing state of $(n+1)^{th}$ planet at time node $t+\Delta t$, that means the natural frequencies of the system in the $\Delta t$ and $t+\Delta t$ should remain the same.

$\Delta t$ can be expressed as:

$$\Delta t = (\gamma_{np(n+1)} - \gamma_{np(n)}) \times T = \text{mod} \left( \frac{z_s}{N}, 1 \right) \times T = \text{mod} \left( \frac{z_r}{N+1}, 1 \right) \times T$$

Where $x$ denotes $s$ (sun gear) and $r$ (ring gear), $z_s$, $z_r$ are the sun and ring tooth numbers, and $n$ stand for the serial number of $n^{th}$ planet, $N$ denotes the number of the planetary gears, $T$ indicates the meshing period, $\text{mod}(a,1)$ stand for the remainder after division of $a$ by 1.

If $n^{th}$ planet meshing state lead $(n+1)^{th}$ by $\Delta t$, it can be considered that the $n^{th}$ planet meshing state lag $(n+1)^{th}$ by $1-\Delta t$, for the sake of facilitating analysis for change rule of natural frequencies, ordering:

$$\begin{align*}
\Delta t &= \Delta t & \text{if} \ 0 < \Delta t \leq 0.5T \\
\Delta t &= 1-\Delta t & \text{if} \ 0.5T < \Delta t < T
\end{align*}$$

The natural frequencies will vary $n' = T / \Delta t$ times in one meshing cycle, with parameters in Table 1 and Equation (9) and Equation (10),

$$\Delta t = \text{mod} \left( \frac{31}{4}, 1 \right) \times T = \text{mod} \left( \frac{73}{4}, 1 \right) \times T = \frac{T}{4}, n' = T / \Delta t = 4$$

Which means that the natural frequencies vary 4 times in one meshing cycle, there exist 4 sub-periods in one meshing period, this deduce is in accordance with the results shown in Figure 4, 6.
6. Conclusion
In this study, a new method is proposed to calculate the natural characteristics of planetary gear set which incorporates time-varying mesh stiffness, the variation of natural frequencies in a meshing cycle is investigated, main conclusions are summarized as follows:

(1) The natural characteristics of the planetary gear set considered TVMS are more explicit and detailed than that calculated by the traditional method. TVMS significantly affect the variation of higher-order natural frequencies, in the analysis of the natural characteristics for high-speed planetary gear transmission, the TVMS should be taken into consideration.

(2) The change of contact teeth is the primary cause of the steep variation of higher-order natural frequencies. And the natural frequencies vary periodically in one meshing cycle, the number of sub-periods is closely related to the meshing phase difference between two adjacent planet gears.

(3) Take mesh stiffness as a constant value is applicable to calculate the low-order frequencies.

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