COMPUTING THE AUTOMORPHISM GROUPS OF HYPERELLIPTIC FUNCTION FIELDS

NORBERT GÖB

1. Introduction

The purpose of this paper is to propose an efficient method to compute the automorphism group of an arbitrary hyperelliptic function field over a given ground field of characteristic $> 2$ as well as over its algebraic extensions. Beside theoretical applications, knowing the automorphism group of a hyperelliptic function field also is useful in cryptography:

The Jacobians of hyperelliptic curves have been suggested by Koblitz as groups for cryptographic purposes, because the computation of the discrete logarithm is believed to be hard in this kind of groups ([Kob89]). In order to obtain “secure” Jacobians it is necessary to prevent attacks like Pohlig/Hellman’s ([PH78]), Frey/Rück’s ([FR94]) and Duursma/Gaudry/Morain’s ([DGM99]). The latter attack is only feasible, if the corresponding function field has an automorphism of large order. To forestall the Pohlig-Hellman attack, one needs to assert that the group order is almost prime, i.e. it ought to contain a large prime factor $p_0$. To prevent the Frey-Rück attack, $p_0$ needs to possess additional properties.

Therefore, one needs to know both the automorphism group of the function field and the order of the Jacobian. Unfortunately, there is no efficient algorithm known to compute this order for arbitrary hyperelliptic curves. Only for specific types of curves, divisor class counting\(^1\) is feasible for cryptographically relevant group sizes (e.g. [SSI98], [GH00]).

A theorem by Madan ([Mad70]) implies that $|J_F|$ divides $|J_{F'}|$ whenever $F \subseteq F'$ is a (hyper-)elliptic subfield of a hyperelliptic function field s.th. $[F' : F] < \infty$. Thus, a hyperelliptic function field with secure Jacobian will most likely have a trivial automorphism group, i.e. one consisting of the hyperelliptic involution, only. Therefore, the proposed technique provides a quick test to check whether a given hyperelliptic curve may yield a secure Jacobian, i.e. whether it is worthwhile to apply expensive divisor class counting algorithms.

Let us outline the afore mentioned algorithm briefly. It is well known that the automorphism group of a hyperelliptic function field is finite (cf. [Sch38]). For each finite group, which can occur as subgroup of such an automorphism group, Brandt gave a normal form for the corresponding hyperelliptic function fields and explicit formulas for these automorphisms (cf. [Bra88]). Brandt’s results only apply to function fields over algebraically closed constant fields, but this is no hindrance as we will see later. For now, we suppose the constant field to be algebraically closed.

Hence, computing the automorphism group reduces to the problem of deciding, whether a given hyperelliptic function field has a defining equation of the form given by Brandt’s theorems. This can be checked using theorem 10, which states that two hyperelliptic function fields $k(t, u), k(x, y)$ with $u^2 = D_t, y^2 = D_x$ are equal iff $x = \frac{\alpha_0 t^3 + \alpha_1}{\alpha_2 t + \alpha_3}$ for some $\alpha_i \in k$ and $y = \varphi u$, where $\varphi \in k(t)$ can be determined

\(^1\)i.e. computation of the order of the Jacobian
from the \( \alpha_i \). Hence, we substitute \( x = \frac{\alpha t + \alpha^2}{\alpha t + \alpha^3} \) symbolically into \( D_x \). Computing \( \varphi \) according to the theorem and comparing coefficients of \( D_t \) and \( \varphi^{-2} D_x \left( \frac{\alpha t + \alpha^2}{\alpha t + \alpha^3} \right) = \varphi^{-2} y^2 = u^2 = D_t \), we obtain a system of polynomial equations for the \( \alpha_i \). These can be tested for solvability or even solved using Gröbner basis methods.

If the constant field \( k \) is algebraically closed, algorithm 7 seems to be the only efficient possibility known to compute the automorphism group of an arbitrary hyperelliptic function field. For finite \( k \), the method described in section 4.2 is an alternative approach to the \texttt{AutomorphismGroup} function in \cite{Sto01}.

### 2. Notation and Fundamental Facts

Throughout this paper, we use the notations from \cite{Sti93}. For the reader’s convenience, we recall the essential notations: The natural numbers \( \mathbb{N} \) start at \( 0 \), \( \mathbb{N}_+ := \mathbb{N} \setminus \{0\} \). The greatest common divisor of two integers or polynomials \( p, q \) is denoted by \( (p, q) \). The unit group of a field \( k \) is denoted by \( k^* := k \setminus \{0\} \). Let \( k \) be some field of characteristic \( p > 2 \), and \( g \in \mathbb{N}, g > 1 \). A \textbf{hyperelliptic function field} of genus \( g \) over \( k \) is defined to be a field \( F := k(x, y) \) s.th. \( x \) is transcendental over \( k \) and \( y^2 = D(x) \), where \( D \in k[x] \) is a monic separable polynomial of degree \( 2g + 1 \) or \( 2g + 2 \). The \textbf{automorphism group} of \( F \) is the group \( \text{Aut}(F/k) \) of field automorphisms of \( F \) fixing \( k \). If \( U \leq \text{Aut}(F/k) \), we denote the fixed field of \( U \) by \( F^U \). The algebraic closure of \( k \) is denoted by \( \overline{k} \). If \( P \) is a place of \( F \), \( v_P \) denotes the valuation corresponding to \( P \). For \( t \in F \) we denote the principal divisor of \( t \) by \( (t)_0 \) and its pole divisor by \( (t)_\infty \). If \( (t)_\infty \) is a place, we also denote it by \( \infty_t := (t)_\infty \) and call it the \textbf{infinite place w.r.t.} \( t \).

Our aim is to compute the automorphism group of any given hyperelliptic function field \( k(x, y) \), \( y^2 = D \). As mentioned above, Brandt gives normal forms of hyperelliptic function fields for each possible finite subgroup of the automorphism group (cf. Brandt’s Ph.D. thesis, \cite{Bra88}). Since the automorphism group of such a field is a central extension of \( \text{Aut}(k(x)/k) \) by \( \mathbb{C}_2 \) generated by the hyperelliptic involution, Brandt rather investigates the possible subgroups of \( \text{Aut}(k(x,y)/k)/\mathbb{C}_2 \), i.e. he characterizes the fields by their “type” which is defined as follows.

**Definition 1.** Type of field \( F[G,k] \) Let \( F/k \) be a hyperelliptic function field and \( G \) some finite group. \( F \) is called a function field of type \( F[G,k] \), if there are finite groups \( C, U, \) s.th. \( U \leq \text{Aut}(F/k), C \trianglelefteq U \), \( C \cong \mathbb{C}_2 \). \( F^C \) is a rational function field over \( k \), and \( U/C \cong G \).

We denote such a group \( U \) by \( U(G) \) or \( U_F(G) \), although \( U \) needs not to be uniquely determined by \( F, G \) and \( k \). We will only use this notation to state that a specific group can be used as \( U \) in this definition.

For extension fields \( k'/k \geq k \), we call \( F \) to be of type \( F[G,k'] \) iff the constant field extension \( Fk'/k' \) is of type \( F[G,k] \).

The following types can occur for hyperelliptic function fields over algebraically closed constant fields of characteristic \( p > 2 \): \( F[C_n,k], \) where \( (n,p) = 1 \), \( F[\mathbb{Z}_p^m,k] \) for some \( m \in \mathbb{N}_+, F[\mathbb{D}_n,k], \) where \( (n,p) = 1 \) or \( n = p, F[\mathbb{A}_4,k], F[\mathbb{A}_5,k], F[\mathbb{S}_4,k], F[\mathbb{C}_p^m \times \mathbb{C}_n,k], \) where \( (n,p) = 1 \) and \( m \in \mathbb{N}_+, F[\text{PSL}_2(p^m),k], \) where \( m \in \mathbb{N}_+ \) and \( F[\text{PGL}_2(p^m),k], \) where \( m \in \mathbb{N}_+ \). As one needs to consider several cases for some of these types, the theorem stating Brandt’s normal forms contains 14 case distinctions. For brevity, we only consider the types with the smallest and largest possible subgroups as well as a subgroup which we will need in our examples. Hence, we restrict ourselves to the types \( F[C_n,k], \) where \( (n,p) = 1 \), \( F[\mathbb{C}_p^m,k] \) for some \( m \in \mathbb{N}_+, F[\mathbb{D}_n,k], \) where \( (n,p) = 1 \), and \( F[\text{PGL}_2(p^m),k], \) where \( m \in \mathbb{N}_+ \). The remaining cases are similar to these and can be found in \cite{Göbon} or directly in \cite{Bra88}.
Theorem 1 (Brandt). Let $F$ be a hyperelliptic function field over an algebraically closed constant field $k$ of characteristic $p \geq 3$. Then the types of $F$ are characterized as follows

1. $F$ is of type $F[C_n,k]$ for $n \in \mathbb{N}_+$ with $(n,p) = 1$ iff there are $t,u \in F$, s.th. $F = k(t,u)$, $u^2 = t^n \prod_{j=1}^{s} (x^n - a_j)$, where $\nu \in \{0,1\}$, $s \in \mathbb{N}$ and the $a_j \in k^*$ are pairwise distinct.

   In this case, $U_F(C_n)$ is generated by $\varphi : t \mapsto t$, $u \mapsto -u$ and $\psi : t \mapsto \eta^t$, $u \mapsto \eta^t u$, where $\eta$ is a primitive $2n$-th root of unity.

2. $F$ is of type $F[C_m,k]$ with $m \in \mathbb{N}_+$ iff there are $t,u \in F$ and a subgroup $A$ of the additive group of $k$ of order $|A| = p^m$, s.th. $F = k(t,u)$, $u^2 = \prod_{j=1}^{s} (\prod_{a \in A} (x + a) - a_j)$, where $s \in \mathbb{N}$ and the $a_j \in k$ are pairwise distinct.

   In this case, $U_F(C_m)$ is generated by $\varphi : t \mapsto t$, $u \mapsto -u$ and all $\psi_a : t \mapsto t + a$, $u \mapsto u$ with $a \in A$.

3. $F$ is of type $F[D_n,k]$, where $n \in \mathbb{N}_+$, $(n,p) = 1$ iff there are $t,u \in F$, s.th. $F = k(t,u)$, $u^2 = t^n \prod_{j=1}^{s} (t^{n} - 1)^\nu_j (t^{n+1} + 1) \prod_{j=1}^{s} (t^{2n - a_j t^n + 1})$, where $\nu_j \in \{0,1\}$, $s \in \mathbb{N}$ and the $a_j \in k \setminus \{\pm 2\}$ are pairwise distinct. If $n = 2$ or $n \equiv 1 \mod 2$, we need to have $\nu_1 = \nu_2$.

   In this case, $U_F(D_n)$ is generated by $\varphi : t \mapsto t$, $u \mapsto -u$, $\psi : t \mapsto \eta^t$, $u \mapsto \eta^t u$ and $\sigma : t \mapsto \frac{1}{t}$, $u \mapsto \frac{u}{\eta^t}$, where $\eta$ is a primitive $2n$-th root of unity, $\eta^{2} = -1$ and $m = \frac{1}{2}(\nu_1 + \nu_2) + 2\nu_0 + 2\nu_1$.

4. $F$ is of type $F[PGL_2(p^m),k]$ iff there are $t,u \in F$, s.th. $F = k(t,u)$, $u^2 = (t' - t)^{r+1} + 1$, where $\nu_j \in \{0,1\}$, $s \in \mathbb{N}$, $r = p^m$ and the $a_j \in k^*$ are pairwise distinct.

   In this case, $U_F(PGL_2(p^m))$ is generated by $\varphi : t \mapsto t$, $u \mapsto -u$, $\psi : t \mapsto \eta^t$, $u \mapsto \eta^t u$, $\sigma : t \mapsto t + 1$, $u \mapsto u$ and $\tau : t \mapsto \frac{1}{t}$, $u \mapsto \frac{u}{\eta^t}$, where $\eta$ is a primitive $2(p^m - 1)$-th root of unity and

   $$n = \frac{1}{2} \left((p^m + 1)\nu_0 + p^m(p^m - 1)\nu_1 + p^m(p^{2m} - 1)s\right).$$

Proof. A slightly more general theorem is proved by Rolf Brandt in his Ph.D. thesis [Bra88]: He characterizes the types of cyclic extensions of rational function fields over algebraically closed constant fields. We list the references for each of the stated facts, citing the proof that a function field of the given type has the given normal form, first. The proof of the inverse implication and the generators are given thereafter.

1. [Bra88, Satz 5.1], [Bra88, Satz 5.6] and [Bra88, Lemma 5.5].
2. [Bra88, Satz 6.3] and its proof.
3. Cf. [Bra88, Satz 7.3], [Bra88, Satz 7.5] and [Bra88, Lemma 7.4], in the case $n \equiv 0 \mod 2$. Otherwise, we apply [Bra88, Satz 7.9], as $p \geq 3$ and $(n,p) = 1$ obviously imply $(2n,p) = 1$. The generators and the inverse implication are proved analogously to [Bra88, Satz 7.5] and [Bra88, Lemma 7.4].
4. [Bra88, Satz 13.1], [Bra88, Satz 13.6] and [Bra88, Lemma 13.2].

Let us illustrate this theorem and the related problems with an example.

Example 2. We consider $F := \mathbb{F}_7(x,y)$,

$$y^2 = x^5 + x^3 + x = x(x + 2)(x - 2)(x + 3)(x - 3) = x(x^2 - 4)(x^2 - 2).$$
Obviously $F$ is of type $\mathbb{F}[C_2, \mathbb{F}_7]$. The basis $x, y$ of $F$ is not uniquely determined by $F$, neither is the defining equation. Therefore, we cannot immediately see if $F$ is of any other types.

In the following section, we solve this problem, i.e. we propose an efficient possibility to find out, if a hyperelliptic function field has a given normal form.

3. Relations Between Bases

In this section we show the connection between different bases of a hyperelliptic function field (cf. theorem 10): If $k(t, u) = k(x, y)$ is a hyperelliptic function field, then $x$ needs to be a fraction of linear polynomials in $t$ and the relation between $u$ and $y$ can be computed easily from these polynomials. In contrast to theorem 1, we do not need to have an algebraically closed constant field, here; theorem 10 applies to hyperelliptic function fields over arbitrary constant fields of characteristic $\neq 2$.

This theorem is one of the core components of our algorithm for computing the automorphism group of a hyperelliptic function field, as we will see in section 4.

3.1. Relations Between the Variable Symbols. Here, we show that $x$ can be represented as a fraction of linear polynomials in $t$. We start our proof by citing the following lemma:

Lemma 2. Let $k(t, u) = k(x, y)$ be a hyperelliptic function field, $u^2 = D_t$, $y^2 = D_x$, where $D_t \in k[t]$ and $D_x \in k[x]$ are separable monic polynomials. Then $k(t) = k(x)$.

Proof. [Sti93, Proposition VI.2.4].

Lemma 2 means, that the following proposition can be applied to our situation, i.e. in hyperelliptic function fields with two given bases, we always have $k(x) = k(t)$.

We see that $x$ is a fraction of linear polynomials in $t$ in this case:

Proposition 3. Let $k(t)$ be a rational function field and $x \in k(t)$ s.th. $k(t) = k(x)$. Then there are $\alpha_0, \ldots, \alpha_3 \in k$ with $x = \frac{\alpha_0 t + \alpha_3}{\alpha_2 t + \alpha_1} + \alpha_0 \alpha_3 - \alpha_1 \alpha_2 \neq 0$.

Proof. As $x \in k(t)$, there are polynomials $\varphi, \psi \in k[t]$, s.th. $x = \frac{\varphi}{\psi}$ and $(\varphi, \psi) \in k$. We consider the principal divisor of $x$. [Sti93, Theorem I.4.11] implies

$$\deg((x)_0) = \deg((x)_{\infty}) = [k(t) : k(x)] = 1.$$ 

Let us consider the case $\infty \notin \text{supp}(x)$, first. Then $0 = v_{\infty}(x) = \deg_t(\varphi) - \deg_t(\varphi)$, i.e. $\deg_t(\varphi) = \deg_t(\psi)$. As $\varphi, \psi \in k[t]$, we get $(\varphi)_{\infty} = \deg_t(\varphi) \infty_t = \deg_t(\psi) \infty_t = (\psi)_{\infty}$. We have $(x) = (\varphi) - (\psi) = (\varphi)_0 - (\varphi)_{\infty} - ((\psi)_0 - (\psi)_{\infty}) = (\varphi)_0 - (\psi)_0$, i.e. $(x)_0 = (\varphi)_0$ and $(x)_{\infty} = (\psi)_0$. Thus,

$$\deg_t(\varphi) = \deg((\varphi)_{\infty}) = \deg((\varphi)_0) = \deg((x)_0)$$

$$= 1 = \deg((x)_{\infty}) = \deg((\psi)_0) = \deg((\psi)_{\infty}) = \deg_t(\psi).$$

Thus there are $\alpha_i \in k$ s.th. $\varphi = \alpha_0 t + \alpha_1$, $\psi = \alpha_2 t + \alpha_3$ and $\alpha_0 \alpha_3 - \alpha_1 \alpha_2 \neq 0$ as claimed.

If $\infty \in \text{supp}(x)$, we obviously have $\deg_t(\varphi) \neq \deg_t(\psi)$. W.l.o.g. we assume $v_{\infty}(x) < 0$ (consider $\frac{1}{x}$ in the other case). As $\deg((x)_0) = 1$, we need to have $v_{\infty}(x) = -1$. Thus

$$-1 = v_{\infty}(x) = \deg_t(\psi) - \deg_t(\varphi),$$

i.e. $\deg_t(\psi) = \deg_t(\varphi) - 1$. As $(\varphi)_{\infty} = \deg_t(\varphi) \infty_t$ and $(\psi)_{\infty} = \deg_t(\psi) \infty_t$, we infer

$$(x) = (\varphi) - (\psi) = (\varphi)_0 - (\varphi)_{\infty} - (\psi)_0 + (\psi)_{\infty} = (\varphi)_0 - (\psi)_0 - \infty_t.$$ 

Thus, we have $(x)_0 = (\varphi)_0$, i.e.

$$\deg_t(\varphi) = \deg((\varphi)_{\infty}) = \deg((\varphi)_0) = \deg((x)_0) = 1.$$
Furthermore $\deg_{t}(\psi) = \deg_{t}(\varphi) - 1 = 0$. We obtain $x = \hat{\varphi} = \alpha_{t}t + \alpha_{s}$ with $\alpha_{t} \in k$ and $\alpha_{0}\alpha_{3} \neq 0$ as claimed. \hfill \qed

Summing up these facts, we obtain, that $x$ is a fraction of linear polynomials in $t$, if $k(x,y) = k(t,u)$:

**Corollary 4.** Let $k(t,u) = k(x,y)$ be a hyperelliptic function field, $u^{2} = D_{t}$, $y^{2} = D_{x}$, where $D_{t} \in k[t]$ and $D_{x} \in k[x]$ are separable monic polynomials. Then there are $\alpha_{0}, \ldots, \alpha_{3} \in k$ with $x = \frac{\alpha_{0}t + \alpha_{3}}{\alpha_{2}t + \alpha_{3}}$ and $\alpha_{0}\alpha_{3} - \alpha_{1}\alpha_{2} \neq 0$.

**Proof.** By lemma 2, we have $k(t) = k(x)$. Thus proposition 3 implies the existence of the $\alpha_{i}$. \hfill \qed

### 3.2. Relation Between the Square Roots.

Since we know, how $t$ and $x$ are related in a hyperelliptic function field for which we have two bases $k(t,u) = k(x,y)$, we proceed studying the relationship between $u$ and $y$. The next lemma tells us, that $y$ is a multiple of $u$ over $k(t)$:

**Lemma 5.** Let $F = k(t,u) = k(x,y)$, $u^{2} = D_{t}$, $y^{2} = D_{x}$ be a hyperelliptic function field over a constant field $k$ of characteristic $\neq 2$, where both $D_{t} \in k[t]$ and $D_{x} \in k[x]$ are monic separable polynomials. Then there is some $\varphi \in k(t) \setminus \{0\}$, s.th. $y = \varphi u$.

**Proof.** As $y \in F = k(t,u)$ and $[k(t,u) : k(t)] = 2$, there are $\varphi, \psi \in k(t)$ s.th. $y = \varphi u + \psi$. Let us suppose $\varphi = 0$. Then we had $y \in k(t)$. From lemma 2 we know that $k(x) = k(t)$. Thus we had $y \in k(x)$, i.e. $k(x,y) = k(x)$ implying $[k(x,y) : k(x)] = 1$, which contradicts $[k(x,y) : k(x)] = 2$. Therefore $\varphi \neq 0$.

Substituting our representation of $y$ into its minimal polynomial we get

$$D_{x} = y^{2} = (\varphi u + \psi)^{2} = \varphi^{2}u^{2} + 2\varphi\psi u + \psi^{2} = \varphi^{2}D_{t} + 2\varphi\psi u + \psi^{2}.$$  

Thus $2\varphi\psi u \in k(t) = k(x)$. As $u \notin k(t)$, this leads to $2\varphi\psi = 0$, from with we conclude $\psi = 0$ because char($k$) $\neq 2$ and $\varphi \neq 0$. \hfill \qed

Knowing that $y = \varphi u$, we will examine $\varphi$ more closely. We start by the following lemma, which is quite technical, but will be useful in the subsequent proofs.

**Lemma 6.** Let $k(t,u) = k(x,y)$, $u^{2} = D_{t}$, $y^{2} = D_{x}$ be a hyperelliptic function field over a constant field $k$ of characteristic $\neq 2$, where both $D_{t} \in k[t]$ and $D_{x} \in k[x]$ are monic separable polynomials. Let $x = \frac{\alpha_{0}t + \alpha_{3}}{\alpha_{2}t + \alpha_{3}}$, $\alpha_{t} \in k$, $\alpha_{0}\alpha_{3} - \alpha_{1}\alpha_{2} \neq 0$ as stated in corollary 4 and $y = \varphi u$, $\varphi \in k(t) \setminus \{0\}$ as in lemma 5. Then there are $d_{x} := \deg_{x}(D_{x})$ pairwise relatively prime $p_{i} \in \overline{k}[t]$, $\deg_{t}(p_{i}) \leq 1$ s.th.

$$D_{t} = \varphi^{-2}(\alpha_{2}t + \alpha_{3})^{-d_{x}}\prod_{i=1}^{d_{x}} p_{i}.$$  

Furthermore we have

1. $p_{i} = (\alpha_{0} + \alpha_{2}\eta_{i})t + \alpha_{1} - \alpha_{3}\eta_{i}$, where $\eta_{i} \in \overline{k}$, $i = 1, \ldots, d_{x}$ are the zeroes of $D_{x}$.
2. $d_{x} - 1 \leq \deg_{t}\left(\prod_{i=1}^{d_{x}} p_{i}\right) \leq d_{x}$.
3. Let $q \in \overline{k}[t]$ be linear. Then $q^{2} \mid \prod_{i=1}^{d_{x}} p_{i}$. In particular, $(\alpha_{2}t + \alpha_{3})^{2} \mid \prod_{i=1}^{d_{x}} p_{i}$.
Proof. We factor $D_x$ over $\overline{k}$ into $D_x = \prod_{i=1}^{d_x}(x - \eta_i)$, $\eta_i \in \overline{k}$, $\eta_i \neq \eta_j$ for all $i \neq j$. This yields

$$D_t = u^2 = \varphi^{-2}y^2 = \varphi^{-2}D_x = \varphi^{-2}\prod_{i=1}^{d_x}(x - \eta_i)$$

$$= \varphi^{-2}\prod_{i=1}^{d_x}\left(\frac{\alpha_0t + \alpha_1 + \alpha_3}{\alpha_2t + \alpha_3} - \eta_i\right)$$

$$= \varphi^{-2}\prod_{i=1}^{d_x}\left(\frac{\alpha_0(\alpha_0 - \alpha_2\eta_i)t + \alpha_1 - \alpha_3\eta_i}{\alpha_2t + \alpha_3}\right)$$

$$= \varphi^{-2}\prod_{i=1}^{d_x}\frac{p_i}{\alpha_2t + \alpha_3}$$

$$= \varphi^{-2}(\alpha_2t + \alpha_3)^{-d_x}\prod_{i=1}^{d_x}p_i \in k[t],$$

where $p_i := (\alpha_0 - \alpha_2\eta_i)t + \alpha_1 - \alpha_3\eta_i$. Suppose there were indices $i \neq j$ s.th. $p_i$ and $p_j$ had a common divisor of nonzero degree w.r.t. $t$. Then we had $p_i = \beta p_j$ for some $\beta \in \overline{k} \setminus \{0\}$, i.e. $(x - \eta_i) = \frac{p_i}{\alpha_2t + \alpha_3} = \beta\frac{p_j}{\alpha_2t + \alpha_3} = \beta(x - \eta_j)$. Thus, $D_x$ were not separable. Contradiction. Therefore, the $p_i$ are pairwise relatively prime, which proves our main claim.

Let us proceed by examining the supplementary statements. Obviously,

$$\deg_t \left(\prod_{i=1}^{d_x}p_i\right) \leq d_x.$$

If $\deg_t \left(\prod_{i=1}^{d_x}p_i\right) < d_x - 1$, there were two indices $i \neq j$ s.th. $p_i, p_j \in \overline{k}$, thus $\alpha_0 - \alpha_2\eta_i = \alpha_0 - \alpha_2\eta_j = 0$, i.e. $\alpha_0 = \alpha_2\eta_i = \alpha_2\eta_j$. Hence, $\alpha_2(\eta_i - \eta_j) = 0$ which yields $\alpha_2 = 0$ since $\eta_i \neq \eta_j$. Now we can easily deduce $\alpha_0 = 0$ from $\alpha_0 - \alpha_2\eta_i = 0$.

Since $\alpha_0\alpha_3 - \alpha_1\alpha_2 \neq 0$, this is not possible. Thus $\deg_t \left(\prod_{i=1}^{d_x}p_i\right) \geq d_x - 1$.

Finally, let $q^\nu \mid \prod_{i=1}^{d_x}p_i$ for some linear $q \in k[t]$ and $\nu \in \mathbb{N}_+$. As $\deg_t(p_i) \leq 1$, there are $\nu$ factors $p_{i_1}, \ldots, p_{i_{\nu}}$, which are multiples of $q$. Thus $p_{i_1}, \ldots, p_{i_{\nu}}$ are scalar multiples of each other. If $\nu > 1$, this contradicts the relative primality of the $p_i$. This proves the last claim.

The following lemma states, that $\varphi^{-1}$ is a non-zero polynomial in $t$:

**Lemma 7.** Let $k(t, u) = k(x, y)$, $u^2 = D_t$, $y^2 = D_x$ be a hyperelliptic function field over a constant field $k$ of characteristic $\neq 2$, where both $D_t \in k[t]$ and $D_x \in k[x]$ are monic separable polynomials. Let $x = \frac{\alpha_0t + \alpha_1}{\alpha_2t + \alpha_3}$, $\alpha_i \in k$, $\alpha_0\alpha_3 - \alpha_1\alpha_2 \neq 0$ as stated in corollary 4 and $y = \varphi u$ as in lemma 5. Then we have $\varphi^{-1} \in k[t] \setminus \{0\}$.

**Proof.** Lemma 6 implies $D_t = \varphi^{-2}(\alpha_2t + \alpha_3)^{-d_x}\prod_{i=1}^{d_x}p_i$. Suppose $\varphi^{-1} = \frac{x}{\varphi_0} \notin k[t]$. As $D_t \in k[t]$, $\varphi_0$ needs to be canceled by $\prod_{i=1}^{d_x}p_i$. Let $q \in k[t]$ be a linear factor of $\varphi_0$. Thus $q^2 \mid \prod_{i=1}^{d_x}p_i$, which contradicts lemma 6. Thus we need to have $\varphi^{-1} \in k[t]$.

We will prove now, that $\varphi^{-1}$ is a power of the denominator of $x$, multiplied by some constant from $k$.

**Lemma 8.** Let $k(t, u) = k(x, y)$, $u^2 = D_t$, $y^2 = D_x$ be a hyperelliptic function field over a constant field $k$ of characteristic $\neq 2$, where both $D_t \in k[t]$ and $D_x \in k[x]$ are monic separable polynomials. Let $x = \frac{\alpha_0t + \alpha_1}{\alpha_2t + \alpha_3}$, $\alpha_i \in k$, $\alpha_0\alpha_3 - \alpha_1\alpha_2 \neq 0$ as
stated in corollary 4 and \( y = \varphi u \) as in lemma 5. Then there are \( \gamma \in k^* \) and \( m \in \mathbb{N} \) s.th.
\[
\varphi^{-1} = \gamma (\alpha_2 t + \alpha_3)^m.
\]

**Proof.** By lemma 7 we know \( \varphi^{-1} \in k[t] \setminus \{0\} \). Factoring it over \( k \) yields \( \varphi^{-1} = \gamma \cdot (\alpha_2 t + \alpha_3)^m \), where \( \gamma \in k[t] \setminus \{0\} \) s.th. \((\alpha_2 t + \alpha_3) \not\mid \gamma \) (\( \gamma \) does not need to be irreducible). By lemma 6 we have
\[
D_t = \varphi^{-2}(\alpha_2 t + \alpha_3)^{-d_x} \prod_{i=1}^{d_x} p_i = \gamma^2(\alpha_2 t + \alpha_3)^{m-d_x} \prod_{i=1}^{d_x} p_i.
\]

As \( D_t \) is separable, we need to have \( \gamma \in k^* \) which proves our claim. \( \Box \)

Computing the degree of \( \varphi^{-1} \), we see that it is a scalar multiple of the \((g+1)\)-th power of the denominator of \( x \).

**Lemma 9.** Let \( k(t, u) = k(x, y) \), \( u^2 = D_t \), \( y^2 = D_x \) be a hyperelliptic function field over a constant field \( k \) of characteristic \( \neq 2 \), where both \( D_t \in k[t] \) and \( D_x \in k[x] \) are monic separable polynomials. Let \( x = \frac{\alpha_2 t + \alpha_3}{\alpha_2 t + \alpha_3} \), \( y = \varphi u \) as stated in corollary 4 and lemma 5. Then we have

1. If \( x \in k[t] \), then \( \varphi \in k^* \).
2. If \( x \notin k[t] \), we assume w.l.o.g. \( \alpha_2 = 1 \). Then there exists some \( \gamma \in k^* \) s.th. \( \varphi^{-1} = \gamma(t+\alpha_3)^{g+1} \).

**Proof.** By lemma 6, there are \( p_i \in k[t] \), s.th. \( D_t = \varphi^{-2}(\alpha_2 t + \alpha_3)^{-d_x} \prod_{i=1}^{d_x} p_i \) and \( d_x - 1 \leq \deg_t \left( \prod_{i=1}^{d_x} p_i \right) \leq d_x \). Let us consider the given cases, separately.

1. Let us assume \( x \in k[t] \), first. We already know \( \varphi^{-1} \in k[t] \setminus \{0\} \) (cf. lemma 7) and \( D_t = \varphi^{-2} \prod_{i=1}^{d_x} p_i \). If \( \varphi^{-1} \notin k \), then \( \varphi^{-2} \) were a non trivial square polynomial in \( t \) dividing \( D_t \). This contradicts the separability of \( D_t \). Thus \( \varphi^{-1} \in k \), which immediately implies \( \varphi \in k^* \).
2. We proceed with the case \( x \notin k[t] \), i.e. \( \alpha_2 \neq 0 \). By reducing the fraction \( x = \frac{\alpha_2 t + \alpha_3}{\alpha_2 t + \alpha_3} \), we can assume \( \alpha_2 = 1 \) without loss of generality. As \( \varphi^{-1} \in k[t] \), we get

\[
\deg_t(D_t) = \deg_t(\varphi^{-1}) - d_x \deg_t(t + \alpha_3) + \deg_t \left( \prod_{i=1}^{d_x} p_i \right)
\]

\[
= 2 \deg_t(\varphi^{-1}) - d_x + \deg_t \left( \prod_{i=1}^{d_x} p_i \right),
\]

which implies

\[
2 \deg_t(\varphi^{-1}) = \deg_t(D_t) + d_x - \deg_t \left( \prod_{i=1}^{d_x} p_i \right).
\]

Thus, the inequality \( d_x - 1 \leq \deg_t \left( \prod_{i=1}^{d} p_i \right) \leq d_x \) yields

\[
\deg_t(D_t) = \deg_t(D_t) + d_x - d_x
\]

\[
\leq \deg_t(D_t) + d_x - \deg_t \left( \prod_{i=1}^{d_x} p_i \right)
\]

\[
= 2 \deg_t(\varphi^{-1})
\]

\[
\leq \deg_t(D_t) + d_x - d_x + 1
\]

\[
= \deg_t(D_t) + 1.
\]
As \( \deg_t(D_t) \in \{2g+1, 2g+2\} \) we conclude \( \deg_t(\varphi^{-1}) = g+1 \). From lemma 8 we know that there is some \( \gamma \in k^* \) and some \( m \in \mathbb{N} \) s.th. \( \varphi^{-1} = \gamma(t + \alpha_3)^m \).

As \( \deg_t(\varphi^{-1}) = g + 1 \), this implies our claim.

\[ \square \]

3.3. Putting Both Relations Together. The following theorem completely characterizes the relation between any two bases of a hyperelliptic function field of characteristic \( \neq 2 \). This can be used to check whether a given function field has a specific kind of defining equation. It is the key ingredient of algorithm 7, which computes automorphism groups.

Using the facts proved above, it remains to compute the scalar factor of \( \varphi \) in order to know the relation between two bases:

**Theorem 10.** Let \( k(t, u) = k(x, y) \), \( u^2 = D_t \), \( y^2 = D_x \) be a hyperelliptic function field over a constant field \( k \) of characteristic \( \neq 2 \), where both \( D_t \in k[t] \) and \( D_x \in k[x] \) are monic separable polynomials. Let \( d_x := \deg_x(D_x) \).

1. If \( x \in k[t] \), then there are \( \alpha_0, \alpha_1 \in k \) s.th. \( x = \alpha_0 t + \alpha_1 \) and \( \alpha_0 \neq 0 \). Furthermore we have \( y = \varphi u \) with \( \varphi \in k^* \),

   \[ \varphi^2 = \alpha_0^{d_x}. \]

2. If \( x \notin k[t] \), then there are \( \alpha_0, \alpha_1, \alpha_3 \in k \), s.th. \( x = \frac{\alpha_1 + \alpha_3}{t + \alpha_3} \), \( \alpha_0 \alpha_3 - \alpha_1 \neq 0 \). Furthermore we have \( y = \varphi u \), where

   \[ \varphi = \frac{\beta}{(t + \alpha_3)^{g+1}}, \]

   with \( \beta \in k^* \). For \( \beta \) we have the formula

   \[ \beta^2 = \begin{cases} D_x(\alpha_0) & \text{, if } D_x(\alpha_0) \neq 0 \\ (\alpha_1 - \alpha_0 \alpha_3) \tilde{D}_x(\alpha_0) & \text{, if } D_x(\alpha_0) = 0, \end{cases} \]

   where \( \tilde{D}_x(x) := \frac{D_x(x)}{x + \alpha_0} \).

**Proof.** Corollary 4 gives the existence of \( \alpha_0, \ldots, \alpha_3 \in k \) s.th. \( x = \frac{\alpha_1 + \alpha_3}{t + \alpha_3} \) and \( \alpha_0 \alpha_3 - \alpha_1 \alpha_2 \neq 0 \). Lemma 5 yields some \( \varphi \in k(t) \setminus \{0\} \) s.th. \( y = \varphi u \). By lemma 9, we know \( \varphi \in k^* \) if \( x \in k[t] \) and \( \varphi^{-1} = \gamma(t + \alpha_3)^{g+1} \) with \( \gamma \in k^* \) otherwise. Lemma 6 implies

\[ D_t = \varphi^{-2}(t + \alpha_3)^{-d_t} \prod_{i=1}^{d_x} p_i, \quad (1) \]

where \( p_i = (\alpha_0 - \alpha_2 \eta_i) t + \alpha_1 - \alpha_3 \eta_i \) and the \( \eta_i \in k \) are the zeroes of \( D_x \).

Let us consider the different cases, now:

1. If \( x \in k[t] \), we have \( \alpha_2 = 0 \). Reducing the fraction \( \frac{\alpha_1 + \alpha_3}{\alpha_3} \), we may w.l.o.g. assume \( \alpha_3 = 1 \). Thus equation (1) becomes

   \[ D_t = \varphi^{-2} \prod_{i=1}^{d_x} (\alpha_0 t + \alpha_1 - \eta_i). \]

   As \( \alpha_0 \neq 0 \) (which we conclude from \( \alpha_0 \alpha_3 - \alpha_1 \alpha_2 = \alpha_0 \neq 0 \)) and \( \varphi \in k \), the leading coefficient of \( D_t \) is

   \[ 1 = \text{lcm}(D_t) = \varphi^{-2} \alpha_0^{d_x}, \]

   because \( D_t \) is monic by assumption. This implies \( \varphi^2 = \alpha_0^{d_x} \).
(2) If \( x \notin k[t] \), we have \( \alpha_2 \neq 0 \). Reducing the fraction \( \frac{a(t) + b}{a(t) + c} \), we may assume \( \alpha_2 = 1 \). We already know \( \nu^{-1} = \gamma(t + \alpha_3)^{g+1} \). Setting \( \beta := \gamma^{-1} \), it remains to compute \( \beta^2 \). From equation (1), we get

\[ D_t = \beta^{-2}(t + \alpha_3)^{2g+2-d_x} \prod_{i=1}^{d_x} p_i. \]

As before, we compute the leading coefficients:

\[ 1 = \text{lc}_t(D_t) = \text{lc}_t \left( \beta^{-2}(t + \alpha_3)^{2g+2-d_x} \prod_{i=1}^{d_x} p_i \right) = \beta^{-2} \text{lc}_t \left( \prod_{i=1}^{d_x} p_i \right). \]

We obtain

\[ \beta^2 = \text{lc}_t \left( \prod_{i=1}^{d_x} p_i \right). \tag{2} \]

From Lemma 6, we know \( d_x - 1 \leq \deg_t(\prod_{i=1}^{d_x} p_i) \leq d_x \). Thus, there are two cases: \( \deg_t(\prod_{i=1}^{d_x} p_i) = d_x \) and \( \deg_t(\prod_{i=1}^{d_x} p_i) = d_x - 1 \). In the latter case, there is some index \( j \) s.th. \( p_j = (\alpha_0 - \eta_j)t + \alpha_1 - \alpha_3 \eta_j \in k \), i.e. \( \alpha_0 - \eta_j = 0 \). Hence, \( \alpha_0 = \eta_j \), which implies \( D_x(\alpha_0) = 0 \). In the former case, there is no such index, i.e. we have \( D_x(\alpha_0) \neq 0 \).

(a) If \( D_x(\alpha_0) \neq 0 \), we have \( \alpha_0 - \eta_i \neq 0 \) for all \( i \). Thus we get

\[ \beta^2 = \text{lc}_t \left( \prod_{i=1}^{d_x} p_i \right) = \text{lc}_t \left( \prod_{i=1}^{d_x} (\alpha_0 - \eta_i)t + \alpha_1 - \alpha_3 \eta_i \right) = \prod_{i=1}^{d_x}(\alpha_0 - \eta_i) = D_x(\alpha_0) \]

as claimed.

(b) If \( D_x(\alpha_0) = 0 \), there is exactly one index \( j \) s.th. \( \eta_j = \alpha_0 \). W.l.o.g. we assume \( j = d_x \). Then we have \( p_{d_x} = \alpha_1 - \alpha_0 \alpha_3 \). Thus equation (2) implies

\[ \beta^2 = \text{lc}_t \left( \prod_{i=1}^{d_x} p_i \right) = \text{lc}_t \left( \prod_{i=1}^{d_x} (\alpha_1 - \alpha_0 \alpha_3) \right) = \text{lc}_t \left( \prod_{i=1}^{d_x-1} (\alpha_0 - \eta_i) \right) = (\alpha_1 - \alpha_0 \alpha_3) \tilde{D}_x(\alpha_0), \]

as \( \tilde{D}_x(\alpha) = \frac{D_x(\alpha)}{x-\alpha} = \frac{D_x(\alpha)}{x-\eta_{d_x}} = \prod_{i=1}^{d_x-1} (x - \eta_i). \)

\[ \square \]

**Corollary 11.** Let \( k(x, y) \), \( y^2 = D_x \) be a hyperelliptic function field over a constant field \( k \) of characteristic \( \neq 2 \), where \( D_x \in k[x] \) is a monic separable polynomial. Let \( D_t \in k[T] \) be another monic separable polynomial. There exists a basis \( t, u \in k(x, y) \) s.th. \( k(x, y) = k(t, u) \), \( u^2 = D_t(t) \) iff there exist \( t, u \in k(x, y) \) for which \( u^2 = D_t(t) \) and the relations \( x = \frac{a(t)+\alpha_1}{\alpha_2 t + \alpha_3} \), \( y = \nu u \) given in theorem 10 hold.
Proof. It remains to show that the existence of \( t, u, u^2 = D_t(t) \) s.t. \( x = \frac{a_0 t + a_1}{a_2 t + a_3}, y = \varphi u \) as given in theorem 10 implies \( k(x, y) = k(t, u) \). It is obvious, that \( k(x) \subseteq k(t) \) and \( k(t)(u) = k(t)(y) \). Solving \( x = \frac{a_0 t + a_1}{a_2 t + a_3} \) for \( t \), we see \( k(t) \subseteq k(x) \). Thus \( k(t) = k(x) \), i.e. \( k(t, u) = k(t)(u) = k(t)(y) = k(x)(y) = k(x, y) \).

As said before, theorem 10 can be applied to check if a hyperelliptic function field \( k(x, y) \) has a basis \( t, u \) satisfying a given equation \( u^2 = D_t \). According to corollary 11, we can decide this question by checking, if there are \( \alpha_i \in k \), and \( \varphi \in k(t) \) as given in theorem 10 s.t. \( u^2 = D_t \), which is equivalent to \( D_t = u^2 = \varphi^{-2} y^2 = \varphi^{-2} D_x \), here. This can be done using the following algorithm:

**Algorithm 3.** Let \( k(x, y), y^2 = D_x, D_x \in k[x] \) monic and separable, be some hyperelliptic function field of genus \( g \) with \( \text{char}(k) \not\equiv 2 \) and let \( D_t \in k[t] \) be some monic, separable polynomial of \( \deg(D_t) \in \{ 2g + 1, 2g + 2 \} \). Let \( d_x := \deg(D_x) \).

1. We compute \( \varphi^2 \in k(t) \) symbolically from the \( \alpha_i \) according to theorem 10. Since we do not know the \( \alpha_i \) in advance, we cannot tell which of the cases of our theorem applies. Thus we have to compute \( \varphi^2 \) and do the following steps in each of these cases:
   - If \( x \in k[t] \), we have to use \( x = \alpha_0 t + \alpha_1, \varphi^2 = \alpha_0^d \).
   - If \( x \not\in k[t] \), i.e. \( x = \frac{\alpha_0 t + \alpha_1}{a_2 t + a_3} \), we consider both \( D_x(\alpha_0) \neq 0 \) and \( D_x(\alpha_0) = 0 \). In the former case we have \( \varphi^2 = D_x(\alpha_0)(t + \alpha_3)^{-2g+2} \). If \( D_x(\alpha_0) = 0 \), we know that \( x - \alpha_0 \) is a divisor of \( D_x \). Thus we can find all possible \( \alpha_0 \) explicitly by factoring \( D_x \) over \( k \). For each such \( \alpha_0 \), we compute \( D_x := \frac{D_x(\alpha_0)}{x - \alpha_0} \) obtaining \( \varphi^2 = (\alpha_1 - \alpha_0 \alpha_3)D_x(\alpha_0)(t + \alpha_3)^{-2g-2} \).

2. After multiplying by the denominators, our condition \( D_t = \varphi^{-2} D_x \) becomes an equation of polynomials in \( t \) and the \( \alpha_i \). We compare coefficients of \( t \). The resulting system of polynomial equations for the \( \alpha_i \) is denoted by \((*)\).

3. Let the ideal \( I \) be generated by \((*)\) and the polynomial \( 1 - (\alpha_0 \alpha_3 - \alpha_1 \alpha_2)T \), where \( T \) is new variable symbol and the \( \alpha_i \) satisfy \( x = \frac{\alpha_0 t + \alpha_1}{a_2 t + a_3} \) according to the case we are considering. Using Gröbner basis methods, we check \( I \) for solvability and construct a solution, if it exists.

Thus we can construct a basis \( k(x, y) = k(t, u) \), \( u^2 = D_t \) iff there are \( \alpha_i, T \) in the variety of \( I \over k \) for any of the cases mentioned in step (1).

Let us illustrate this algorithm with an example:

**Example 4.** Let \( k = \mathbb{F}_{11}, F = k(x, y), y^2 = D_x := x^5 + x^4 + 4x^3 + 5x^2 + 10x + 7 \). We would like to know, if there is a basis \( F = k(t, u) \) s.t. \( u^2 = D_t := t^5 + 7t^3 + 9t^2 + 9t + 6 \).

We start with the easiest case \( x \in k[t] \). From theorem 10, we get \( x = \alpha_0 t + \alpha_1, y = \varphi u \) and \( \varphi^2 = \alpha_0^5 \).

Substituting, we get

\[
D_x = D_x(\alpha_0 t + \alpha_1) = \alpha_0^5 t^5 + (5\alpha_0^4 \alpha_1 + \alpha_0^4) t^4 + (10\alpha_0^3 \alpha_1^2 + 4\alpha_0^3 \alpha_1 + 4\alpha_0^3) t^3 + (10\alpha_0^2 \alpha_1^3 + 6\alpha_0^2 \alpha_1^2 + 2\alpha_0^2 \alpha_1 + 5\alpha_0^2) t^2 + (5\alpha_0 \alpha_1^4 + 4\alpha_0 \alpha_1^3 + \alpha_0 \alpha_1^2 + 10\alpha_0 \alpha_1 + 10\alpha_0) t + \alpha_1^5 + \alpha_1^4 + 4\alpha_1^3 + 5\alpha_1^2 + 10\alpha_1 + 7
\]
As in algorithm 3, we infer the existence of a basis analogous to algorithm 3. We only note the differences:

\[ 5\alpha_0^4\alpha_1 + \alpha_0^4 = 0, \]
\[ 10\alpha_0^3\alpha_1^2 + 4\alpha_0^3\alpha_1 + 4\alpha_0^3 = 7\alpha_0^5, \]
\[ 10\alpha_0^2\alpha_1^3 + 6\alpha_0^2\alpha_1^2 + 2\alpha_0^2\alpha_1 + 5\alpha_0^2 = 9\alpha_0^5, \]
\[ 5\alpha_0\alpha_1^4 + 4\alpha_0\alpha_1^3 + \alpha_0\alpha_1^2 + 10\alpha_0\alpha_1 + 10\alpha_0 = 9\alpha_0^5, \]
\[ \alpha_1^5 + 4\alpha_1^3 + 5\alpha_1^2 + 10\alpha_1 + 7 = 6\alpha_0^5.\]

Augmenting (*) by \( 1 - \alpha_0 T \), we get the ideal \( I \). Singular ([GPS+02]) computes the following Gröbner basis of \( I \) w.r.t. the lexicographical ordering:

\[ T - 4 = 0 \]
\[ \alpha_0 - 3 = 0 \]
\[ \alpha_1 - 2\alpha_0^5 T^2 - 3\alpha_0^2 T^2 = 0.\]

This implies \( T = 4, \alpha_0 = 3 \). Substituting these values into the remaining equation, we obtain \( \alpha_1 = 2 \). Thus, setting \( t := 4x - 3, u := y \), i.e. \( x = 3t + 2, \varphi = 3^2 = 1 \), we get a basis \( F = k(t, u) \), with \( u^2 = D_t \).

In order to compute the automorphism group \( \text{Aut}(\overline{\kappa(x,y)}/\overline{\kappa}) \) over an algebraically closed constant field, it suffices to check \( \overline{\kappa(x,y)} \) for normal forms, as we will see in section 4. This simplifies the Gröbner basis step of algorithm 3, giving the following modified algorithm:

**Algorithm 5.** Let \( k(x, y), y^2 = D_x, D_x \in k[x] \) monic and separable, be some hyperelliptic function field of genus \( g \) with \( \text{char}(k) \neq 2 \) and \( D_t \in k[t] \) be some monic, separable polynomial of \( \deg(D_t) \in \{2g + 1, 2g + 2\} \). Let \( d_x := \deg_x(D_x) \).

Whether there exists a basis \( \overline{\kappa(x,y)} = \overline{\kappa(t,u)} \) with \( u^2 = D_t \), can be checked analogous to algorithm 3. We only note the differences:

1. In order to compute \( \varphi^2 \) in the case \( x \notin k[t] \), \( D_x(\alpha_0) = 0 \), we have to consider all zeroes \( \alpha_0 \) of \( D_x \) over \( \overline{\kappa} \), i.e. we have to factor \( D_x \) over its splitting field.
2. Instead of constructing an element of the variety of \( I \), we only need to check if it’s empty. To do so, we compute a Gröbner basis \( B \) of \( I \) (e.g. w.r.t. the degree reverse lexicographical ordering). There exists a solution \( \alpha_i, T \in \overline{\kappa} \), i.e. \( I \neq \{1\} \), iff \( B \neq \{1\} \).

As in algorithm 3, we infer the existence of a basis \( \overline{\kappa(x,y)} = \overline{\kappa(t,u)} \), \( u^2 = D_t \) iff \( B \neq \{1\} \).

**Remark 6.** An essential feature of algorithms 3 and 5 is, that \( D_t \) does not need to be known completely. It may contain some parameters for which we can also solve. Therefore, we can use our algorithms to check, whether a given hyperelliptic function field \( \overline{\kappa(x,y)} \) has some of Brandt’s normal forms (cf. theorem 1). We will see how to do this, in the following section.

### 4. Computing the Automorphism Group

#### 4.1. Algebraically Closed Constant Fields

Algorithm 5 can be applied to compute the automorphism group of a hyperelliptic function field over an algebraically closed constant field:

**Algorithm 7.** Let \( k(x, y), y^2 = D_x, D_x \in k[x] \) monic and separable, be a hyperelliptic function field of genus \( g \) and \( \text{char}(k) \neq 2 \). We denote \( F := \overline{\kappa(x,y)} \).

1. For each possible type \( F[\overline{G}, \overline{\kappa}] \), we look up the corresponding normal form \( u^2 = D_t \) in theorem 1.
(2) For each normal form found in step (1), we check, for which parameter sets $D_t$ has degree $2g + 1$ or $2g + 2$. This yields the set $N$ of all polynomials $D_t$, s.th. $u^2 = D_t$ is a normal form for a field of genus $g$ and type $F[G, \mathbb{k}]$. The integer parameters in each $D_t \in N$ are fixed, while the $D_t$ may still contain parameters from $\mathbb{k}$.

(3) For each $G$, $N$ and each $D_t \in N$, we check if $F$ has a basis $F = \mathbb{k}(t, u)$ satisfying $u^2 = D_t$ as well as the additional conditions from theorem 1. To do so, we use a slight modification of algorithm 5:

Let $C_0$ and $C_1$ be the sets of polynomials that according to theorem 1 have to be $= 0$ and $\neq 0$, respectively. Let

$$c := (\alpha_0 \alpha_3 - \alpha_1 \alpha_2) \prod_{f \in C_1} f.$$ 

We define the ideal $I$ to be generated by $(*)$, $C_0$ and $1 - c \cdot T$ rather than just by $(*)$ and $1 - (\alpha_0 \alpha_3 - \alpha_1 \alpha_2)T$. Note that the polynomial ring $R \supseteq I$ may contain more variables than just the $\alpha_i$ and $T$, now.

We apply the rest of algorithm 5 without any changes.

The variety of $I$ is non-empty iff $k(x, y)$ is of type $F[G, \mathbb{k}]$.

(4) Let $G$ be the largest group $G$ s.th. $k(x, y)$ is of type $F[G, \mathbb{k}]$. Then

$$\text{Aut}(\mathbb{k}(x, y)/\mathbb{k})/C_2 \cong G,$$

and the generators of $U(G) = \text{Aut}(\mathbb{k}(x, y)/\mathbb{k})$ are given in theorem 1.

Thus, we are able to compute the structure as well as the generators of the automorphism group $\text{Aut}(\mathbb{k}(x, y)/\mathbb{k})$ for each hyperelliptic function field $k(x, y)$.

Example 8. Let $F := \mathbb{F}_7(x, y)$ with $y^2 = x^3 + x^3 + x$ as in example 2. The above algorithm yields that $F$ is of the types $F[\mathbb{F}_2, \mathbb{F}_7]$, $F[\mathbb{F}_3, \mathbb{F}_7]$, $F[\mathbb{F}_6, \mathbb{F}_7]$, $F[\mathbb{F}_9, \mathbb{F}_7]$ and $F[\mathbb{F}_6, \mathbb{F}_7]$.

To see, how the algorithm works, we consider parts of the proof that $F$ is of type $F[\mathbb{F}_3, \mathbb{F}_7]$:

1. The normal form for fields of type $F[\mathbb{F}_3, \mathbb{F}_7]$ is given by

$$y^2 = t^\nu (t^3 - 1)^\nu_1 (t^3 + 1)^\nu_2 \prod_{j=1}^8 (t^6 - a_j t^3 + 1),$$

where $\nu_1 \in \{0, 1\}$, $\nu_2 \in \mathbb{N}$ and the $a_j \in \mathbb{F}_7 \setminus \{\pm 2\}$ are pairwise distinct.

2. As $g = 2$, we need to have $\deg_4(D_t) \in \{5, 6\}$, from which we get

$$N = \{(t^3 - 1)(t^3 + 1), t^6 - a_1 t^3 + 1\}$$

3. Algorithm 5 finds out that $F$ possesses a basis $F = \mathbb{F}_7(t, u)$, $u^2 = (t^3 - 1)(t^3 + 1)$. Thus, $F$ is of type $F[\mathbb{F}_3, \mathbb{F}_7]$. The corresponding system $(*)$ of equations and inequalities is not given, as it looks quite ugly and does not help in understanding this step of the algorithm. It is similar to the one given in example 4. Simplifying $(t^3 - 1)(t^3 + 1) = t^6 - 1$, theorem 1 immediately implies that $F$ is of type $F[\mathbb{F}_6, \mathbb{F}_7]$.

Furthermore, the second element of $N$ can also be used to find a basis of $F$: Setting $a_1 := 0$ and $a_1 := i$, where $i^2 = -1$, implies $a_0 = 1$ and $a_3 = -i$. Thus $F = \mathbb{F}_7(v, w)$ with $w^2 = v^6 + 1$.

4. From the list above, we know that $D_9$ is the largest group $G$ s.th. $F$ is of type $F[\mathbb{F}_7, \mathbb{F}_7]$. Thus, $\text{Aut}(\mathbb{F}_7(x, y)/\mathbb{F}_7)$ is a central extension of $D_9$ by the $C_2$, generated by the hyperelliptic involution. From the normal form $u^2 = (t^6 + 1)$, we know $\nu_0 = \nu_1 = s = 0$, $\nu_2 = 1$. According to theorem 1, a
set of generators of $\text{Aut}(\overline{F}_7(x, y)/\overline{F}_7)$ is given by $\{\varphi, \psi, \sigma\}$, where $\varphi : t \mapsto t$, $u \mapsto -u$, $\psi : t \mapsto \zeta t$, $u \mapsto u$ and $\sigma : t \mapsto \frac{1}{t}$, $u \mapsto \frac{u}{t^2}$ with a primitive 6-th root $\zeta$ of unity.

Looking at these generators, we conclude $\text{Aut}(\overline{F}_7(x, y)/\overline{F}_7) \cong D_6 \times C_2$.

4.2. Arbitrary Constant Fields. Using algorithm 7, it is also possible to compute $\text{Aut}(k(x, y)/k)$ for a hyperelliptic function field $k(x, y)$, where $k$ needs not to be algebraically closed. A similar application is the computation of the smallest algebraic extension $k' \supseteq k$ s.th. $\text{Aut}(k'(x, y)/k') = \text{Aut}(\overline{F}(x, y)/\overline{F})$.

Let $k$ be any field of characteristic $p > 2$ and $k(x, y)$ be a hyperelliptic function field. We use algorithm 7 to compute the types of $\overline{F}(x, y)$. Let $k(x, y)$ be of type $F[G, \overline{F}]$ and let $\overline{F}(x, y) = \overline{F}(t, u)$, $u^2 = D_t$ be the corresponding normal form. Solving the ideal $I$ for the $\alpha_i$ and the parameters of $D_t$, we obtain explicit formulas for the generators of $\text{U}(G)$. Using these, it is easy to find the smallest field $k' \supseteq k$, s.th. all automorphisms from $\text{U}(G)$ define automorphisms of $k'(x, y)$. Then, $k' \supseteq k$ is the smallest field extension s.th. $k(x, y)$ is of type $F[G, k']$.

This method is used to solve the two problems given above: In order to compute $\text{Aut}(k(x, y)/k)$, we construct $k'$ for each $G$ s.th. $k(x, y)$ is of type $F[G, \overline{F}]$. The largest $G$ with $k' = k$ yields $\text{U}(G) = \text{Aut}(k(x, y)/k)$.

To find the smallest $k' \supseteq k$ s.th. $\text{Aut}(k'(x, y)/k') = \text{Aut}(\overline{F}(x, y)/\overline{F})$, we compute $\text{Aut}(\overline{F}(x, y)/\overline{F})$ and construct $k'$ for $G = \text{Aut}(\overline{F}(x, y)/\overline{F})/C_2$ as explained above. We show how to apply this method in the following example.

Example 9. We consider $F := \overline{F}_7(x, y)$, $y^2 = x^5 + x^3 + x$, i.e. we examine the curve from example 8 over $\overline{F}_7$. We already know, that $\text{Aut}(\overline{F}_7(x, y)/\overline{F}_7) \cong D_6 \times C_2$. Thus we set $G := D_6$. To find out, for which extension $k \supseteq F_7$ we have $\text{Aut}(k(x, y)/k) = \text{Aut}(\overline{F}_7(x, y)/\overline{F}_7) \cong D_6 \times C_2$, we have a closer look at the proof\(^2\) that $\overline{F}_7(x, y)$ is of type $F[D_6, \overline{F}_7]$. As seen in example 8, we have $k(x, y) = k(t, u)$, $u^2 = t^6 + 1$ and the automorphism group is generated by $\varphi : t \mapsto t$, $u \mapsto -u$, $\psi : t \mapsto \zeta t$, $u \mapsto u$ and $\sigma : t \mapsto \frac{1}{t}$, $u \mapsto \frac{u}{t^2}$, with a primitive 6-th root of unity $\zeta$. As 3 is such a 6-th root, we may set $\zeta := 3$. Thus, our automorphism are defined over the smallest extension $k \supseteq F_7$ s.th. $t, u \in k(x, y)$.

Hence, to compute $k$ we have to examine $t$ and $u$ more closely. They can be computed from $x$ and $y$ using the coefficients $\alpha_i$ from theorem 10. Therefore, $k$ is the smallest field s.th. $\alpha_i \in k$. Solving the corresponding equations and inequalities, we get that $x \in k[t]$, $\alpha_0 = 1$, $\alpha_3 = i$ with $i^2 = -1$ is a possible solution. Furthermore, there is no solution over $\overline{F}_7$. Thus, $t, u \in \mathbb{F}_{49}(x, y) := \overline{F}_7(x, y, i)$ which implies that $k := \mathbb{F}_{49}$ is the smallest constant field s.th. $Fk = k(x, y)$ has the automorphism group $D_6 \times C_2$.

5. Computational Aspects

The author implemented algorithm 7 for the computer algebra systems MuPAD ([Sci02]) and Singular ([GPS+02]). The Gröbner basis steps are implemented for Singular, while anything else—i.e. Brandt’s normal forms, computing $N$, substitution and the comparing of coefficients—is programmed for MuPAD. Both parts of the program are combined using shell scripts. It was decided to separate the Gröbner basis steps from the rest of the computation, since on the one hand, Singular has one of the most efficient Gröbner basis implementations. On the other hand, Singular is restricted to characteristic $p \leq 32003$, which is too small for many fields of cryptographic relevance.

As a proof of concept, the implementation is not optimized for speed at all. Therefore, a speedup by a factor of at least 10 ought to be possible using a “proper”

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\(^2\)i.e. the computations proving that $\overline{F}_7(x, y)$ is indeed of the specified type.
Then, Aut(\mathbb{F}_k) is computed using algorithm 7, while Stoll’s method is used to compute \text{Aut}(\mathbb{F}_k) on an Intel® Celeron® 1.7 GHz, ordered by genus and \text{Aut}(\mathbb{F}_k). The running times for some examples are given in table 5.

| \text{Defining Equation} | \text{Time to compute } \text{Aut}(\mathbb{F}_k) | \text{seconds} |
|---------------------------|-----------------------------------------------|---------------|
| \text{\textit{g} = 2:}  |                                               |               |
| \mathbb{F}_{949}         | \begin{align*} y^2 &= x^5 - 4608x + 1124 \end{align*} | 2             | 12.6          |
| \mathbb{F}_{10223}      | \begin{align*} y^2 &= x^6 - 4x^4 - 4x^2 + 1 \end{align*} | 4             | 52.5          |
| \mathbb{F}_{10711}      | \begin{align*} y^2 &= x^6 + 394x^3 - 3378 \end{align*} | 12            | 23.3          |
| \mathbb{F}_3            | \begin{align*} y^2 &= x^6 + x^4 + x^2 + 1 \end{align*} | 24            | 9.8           |
| \mathbb{F}_3            | \begin{align*} y^2 &= x^6 + x^4 + x^2 + 1 \end{align*} | 48            | 9.2           |
| \mathbb{F}_5            | \begin{align*} y^2 &= x^5 + 4x \end{align*} | 240           | 22.7          |
| \text{\textit{g} = 3:}  |                                               |               |
| \mathbb{F}_{11}         | \begin{align*} y^2 &= x^7 + 6x^6 + 5x^4 + 4x^3 + x + 3 \end{align*} | 2             | 67.0          |
| \mathbb{F}_3            | \begin{align*} y^2 &= x^8 + x^7 + 2x^5 + 2x + 2 \end{align*} | 8             | 30.3          |
| \mathbb{F}_7            | \begin{align*} y^2 &= x^7 + 6x^4 + 4x^3 + x^2 + 2 \end{align*} | 42            | 67.8          |
| \text{\textit{g} = 4:}  |                                               |               |
| \mathbb{F}_5            | \begin{align*} y^2 &= x^{10} + x^8 + 3x^6 + 4x^2 + 4 \end{align*} | 4             | 81.8          |
| \mathbb{F}_3            | \begin{align*} y^2 &= x^9 + 2x^7 + 2x^3 + 2x \end{align*} | 8             | 46.6          |

**Table 1.** Time to compute Aut(\mathbb{F}_k) on an Intel® Celeron® 1.7 GHz, ordered by genus and \text{Aut}(\mathbb{F}_k).

implementation. Nevertheless, the examples given in table 5 suggest that even this implementation computes the automorphism group Aut(\mathbb{F}_k) of an arbitrary hyperelliptic function field very efficiently. The performance seems to depend neither on the size of the constant field, nor on the order of Aut(\mathbb{F}_k). Even though increasing the genus increases the size of the systems of polynomials—the number of both the polynomials and the parameters increase linear with \text{g} for types like \mathbb{F}[\mathbb{C}_2, \overline{\mathbb{F}}]—, the examples indicate that even for fields of genus 4 and higher, the automorphism group computations are quite fast.

Let us discuss the cryptographic application, briefly. As explained in the introduction, the initial goal was to provide an algorithm to check, whether a given hyperelliptic curve promises to yield a secure Jacobian, i.e. whether it is worthwhile to apply more expensive algorithms to check a given curve for security. Because of the attacks mentioned in the introduction, secure curves have small automorphism groups Aut(\mathbb{F}_k). Since Aut(\mathbb{F}_k) ≤ Aut(\mathbb{F}_k), algorithm 7 can be used to assure this property. The timings of table 5 also apply to the set of relevant curves, as secure curves are of genus \text{g} because of the Adleman-DeMarrais-Huang attack ([ADH94]) and as characteristic of the constant field and the size of the automorphism group do not seem to influence the running time.

Even though a small automorphism group is necessary for a secure curve, it is not obvious, how much information concerning security can be deduced from knowing the automorphism group. A discussion of this topic can be found in [Göbon].

The methods described in section 4.2 were not implemented. Nevertheless, we will try to compare algorithm 7 to Michael Stoll’s \texttt{AutomorphismGroup} function (cf. [Sto01]) in some examples. To do so, we choose the smallest field \text{g} of the given characteristic, for which Aut(\mathbb{F}_k) = Aut(\mathbb{F}_k), \text{Aut}(\mathbb{F}_k(k,y)/\mathbb{F}_k) holds, in each example. Then, Aut(\mathbb{F}_k(k,y)/\mathbb{F}_k) is computed using algorithm 7, while Stoll’s method is used to compute Aut(\mathbb{F}_k(k,y)/\mathbb{F}_k). The running times for some examples are given in table 5.

From these examples, Stoll’s algorithm seems to be quite fast for small automorphism groups, while it is very slow for large ones. As stated above, our implementation does not seem to be influenced by the group size at all. Thus, if you are quite sure that the field you are investigating only has a small automorphism group, Stoll’s algorithm ought to be preferred. Even though the majority of hyperelliptic function fields has a small automorphism group, the remaining fields do not seem
to be suited for Stoll’s algorithm. Hence, in order to compute the automorphism
group of an arbitrary hyperelliptic function field, it might be sensible to use the
algorithms from section 4.2 as those at least seem to be more predictable w.r.t.
performance. Furthermore, Stoll’s algorithm returns every single automorphism,
while the methods presented here, give the structure as well as the generators of
the automorphism group. Thus, it also depends on the application, which of the
algorithms ought to be used.

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