Ordinals in Frobenius Monads

KOSTA ĐOŠEN and ZORAN PETRIĆ

Mathematical Institute, SANU
Knez Mihailova 36, p.f. 367, 11001 Belgrade, Serbia
email: {kosta, zpetric}@mi.sanu.ac.yu

Abstract

This paper provides geometrical descriptions of the Frobenius monad freely generated by a single object. These descriptions are related to results connecting Frobenius algebras and topological quantum field theories. In these descriptions, which are based on coherence results for self-adjunctions (adjunctions where an endofunctor is adjoint to itself), ordinals in $\varepsilon_0$ play a prominent role. The paper ends by considering how the notion of Frobenius algebra induces the collapse of the hierarchy of ordinals in $\varepsilon_0$, and by raising the question of the exact categorial abstraction of the notion of Frobenius algebra.

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1 Introduction

The purpose of this paper is to connect two seemingly distant and unrelated topics: Frobenius algebras and ordinals contained in the infinite denumerable ordinal $\varepsilon_0$ (namely, the least ordinal $\xi$ such that $\omega^\xi = \xi$). Frobenius algebras play an important role in topology, mathematical physics and algebra (see [18] and references therein), while $\varepsilon_0$ is usually deemed interesting only for set-theorists and proof-theorists.

The categorial abstraction of the notion of Frobenius algebra leads to the notion of Frobenius monad (for some more details, see below). The structure of a Frobenius monad is given by a category with an endofunctor that bears both the structure of a monad (or triple) and a comonad, and satisfies moreover additional conditions called Frobenius equations (see the next section).
The notion of Frobenius monad is closely related to a special kind of adjoint situation where two functors (not necessarily distinct) are both left and right adjoint to each other (see [28], [3], [17], [29], [33], [22], [6], and further references in these papers). Adjunction is a central notion in category theory, in logic, and perhaps in mathematics in general (see [27] and [25]), and the connection of this notion with the notion of Frobenius monad may serve to explain the importance of the latter.

One of the goals of this paper is to show that the notion of adjunction where two functors are both left and right adjoint to each other amounts, in a sense that we will make precise, to the notion of self-adjunction, which we have investigated in [10]. A self-adjunction is an adjoint situation where an endofunctor is both left and right adjoint to itself. So we find a close relationship between Frobenius monads and self-adjunctions. Through this relationship, we can prove coherence results for Frobenius monads, by relying on a coherence result that we have previously established for self-adjunctions. (That self-adjunction arises in the context of Frobenius monads was noted in [30], Note after Definition 2.9; this is however implicit already in [24], pp. 151-152, in [5], Theorem 2.4, and in [18], Chapter 2.)

These coherence results assert that there is a faithful functor from a freely generated Frobenius monad to manageable model categories, which we will consider in this paper. This faithful functor is here an isomorphism. With our model categories we can easily decide whether a diagram of arrows commutes. In logical terms, this is like proving completeness with respect to a manageable model, which helps us to solve the decision problem. Coherence here is analogous to the isomorphism that exists between the syntactically constructed freely generated monad and the simplicial category (see [13], Section 3, [9], Section 4, and references therein).

The coherence we establish is also the gist of the connection between the notions of Frobenius monad and two-dimensional topological quantum field theory (2TQFT). A 2TQFT may be understood as a functor from the category $2Cob$, whose arrows are cobordisms in dimension 2, to the category $Vect_K$ of finite-dimensional vector spaces over the field $K$. In terms of category theory, a Frobenius algebra is characterized by a monoidal functor from the Frobenius monad freely generated by a single object to $Vect_K$, modulo the strictification of $Vect_K$ with respect to its monoidal structure given by the tensor product and $K$ (cf. the beginning of Section 7). A Frobenius algebra is the image of the object 1 of the Frobenius monad. The main result here is that 2TQFTs correspond bijectively, modulo a skeletization of $2Cob$, to commutative Frobenius algebras. This result is stated officially as a result about equivalence of categories (see [18], Section 3.3).

An alternative result with the same mathematical content is that the free commutative Frobenius monad is isomorphic to the skeleton of $2Cob$. From that alternative result, the former result follows immediately. This alternative result may be conceived as a coherence result for commutative Frobenius monads.
Our coherence results for Frobenius monads mentioned above are more general. They deal with Frobenius monads in general, and not only commutative ones. Because of that, infinite ordinals contained in $\varepsilon_0$ enter into the picture. They arise naturally in our principal model category, which bears some similarity to $2\text{Cob}$. It is a kind of planar version of $2\text{Cob}$. Something related to this model category has been described topologically in a 2-categorical context in [16] (Appendix C; see also [23]). The infinite ordinal structure of the model category is however mentioned neither in this book, nor in the papers mentioned in the third paragraph, nor in [18]. In [18] (Section 3.6.20) we find only the vague conclusion that this ordinal structure, with which we want to deal, is “nearly about any possible drawing you can imagine”. This structure is the main novelty we obtain when we reject commutativity and pass to Frobenius monads in general.

This structure could be described by other means than by the ordinals in $\varepsilon_0$. What we need is the commutative monoid with one unary operation freely generated by the empty set of generators (see Section 6). This monoid can be isomorphically represented in the positive integers too, but we believe its isomorphic representation in $\varepsilon_0$, which is quite natural, is worth investigating.

Towards the end of his book [18] (Sections 3.6.16-27), J. Kock discusses heuristically a project to describe geometrically the freely generated Frobenius monad, and leaves the matter as a challenge to the reader (Section 3.6.26). In this paper, one can find an answer to this challenge.

To make the hierarchy of ordinals in $\varepsilon_0$ collapse, and pass to something that amounts to $2\text{Cob}$, we need not assume commutativity. In the last two sections of this paper, we show how the notion of Frobenius algebra requires that the notion of Frobenius monad be extended with further assumptions, which produce the collapse of the hierarchy. The culprit for this collapse is the symmetry of $\text{Vect}_K$, without assuming that the Frobenius algebra is commutative (the Frobenius objects in symmetric monoidal categories of [14], Section 2, involve such a collapse too). We know that such a collapse must take place, but we do not know what should be its exact extent. In that context, we consider the collapse brought by the assumption of separability in Frobenius algebras, for which the exact categorial abstraction is the notion of separable matrix Frobenius monad in the last section of the paper. We leave however as an open question what is the exact categorial abstraction of the notion of Frobenius algebra.

This paper is a companion to [13], but, except for some side comments, an acquaintance with that paper is not indispensable. We rely however, as we said above, on the results of [10], and we assume an acquaintance with parts of that earlier paper, though some of the essential matters we need are reviewed in Section 6. We assume also the reader is acquainted with some basic notions of category theory, which may all be found in [27], but, for the sake of notation, we define some of these basic notions below.
2 The free Frobenius monad

A *Frobenius monad* is a structure made of a category $A$, an endofunctor $M$ of $A$ (i.e., a functor from $A$ to $A$) and the natural transformations

\[ \varepsilon \square : M \xrightarrow{\sim} I_A, \quad \varepsilon \lozenge : I_A \xrightarrow{\sim} M, \]
\[ \delta \square : M \xrightarrow{\sim} MM, \quad \delta \lozenge : MM \xrightarrow{\sim} M, \]

for $I_A$ being the identity functor of $A$, such that $\langle A, M, \varepsilon \square, \delta \square \rangle$ is a monad, $\langle A, M, \varepsilon \lozenge, \delta \lozenge \rangle$ is a comonad, and, moreover, for every object $A$ of $A$, the following *Frobenius* equations hold:

\[ M \delta \lozenge A \circ \delta \square M A = \delta \lozenge M A \circ M \delta \square A = \delta \square A \circ \delta \lozenge M A = \varepsilon \square M A \circ \delta \lozenge M A = \delta \lozenge M A \circ M \delta \square A = \delta \square A. \]

(For easier comparison, we use here, with slight modifications, the notation with the modal superscripts $\square$ and $\lozenge$, which was introduced in [13].)

The equations defining the notions of monad and comonad are given below. For the Frobenius equations the reader may consult [18] (in particular, Lemma 2.3.19, and [13], Sections 6-7; as far as we know, and according to [19], the first appearance of these equations is in [5]). Lawvere introduced in [24] (pp. 151-152) the notion of Frobenius monad with the equations

\[ M \varepsilon \square A \circ M \varepsilon \lozenge M A = \varepsilon \square M A \circ M \delta \lozenge A = \delta \square A, \]

or, alternatively, the dual equations

\[ \delta \lozenge M A \circ M \varepsilon \square A = M \varepsilon \square M A \circ \delta \lozenge M A = \delta \lozenge A, \]

which can replace the Frobenius equations. (In the terminology of [13], Section 8, a Frobenius monad is a dyad, or codyad, where $\square$ and $\lozenge$ coincide.)

The category $\text{Frob}$ of the Frobenius monad freely generated by a single object, denoted by 0, has as objects the natural numbers $n \geq 0$, where $n$ stands for a sequence of $n$ occurrences of $M$; so $Mn$ is $n+1$. The arrows of this category are defined syntactically as equivalence classes of *arrow terms*, which are defined inductively as follows. The primitive arrow terms of $\text{Frob}$ are

\[ 1_n : n \rightarrow n, \]
\[ \varepsilon \square : n+1 \rightarrow n, \quad \varepsilon \lozenge : n \rightarrow n+1, \]
\[ \delta \square : n+1 \rightarrow n+2, \quad \delta \lozenge : n+2 \rightarrow n+1. \]

The remaining arrow terms of $\text{Frob}$ are defined inductively out of these with the clauses:

- if $f : n \rightarrow m$ and $g : m \rightarrow k$ are arrow terms, then so is $(g \circ f) : n \rightarrow k$;
- if $f : n \rightarrow m$ is an arrow term, then so is $Mf : n+1 \rightarrow m+1$. 


We take for granted the outermost parentheses of arrow terms, and omit them. (Further omissions of parentheses will be permitted by the associativity of \( \circ \).)

The least equivalence relation, congruent with respect to \( \circ \) and \( M \), by which we obtain the arrows of \( \text{Frob} \) is such that, first, we have the \textit{categorial} equations of composition with \( 1 \) and associativity of composition \( \circ \), and the \textit{functorial} equations for \( M \) (see \[13\], Section 2). We have next the \textit{naturality} equations:

\begin{align}
(\varepsilon^\Box \text{ nat}) & \quad f \circ \varepsilon^\Box_n = \varepsilon^\Box_m \circ Mf, & (\varepsilon^\Box \text{ nat}) & \quad \varepsilon^\Box_m \circ f = Mf \circ \varepsilon^\Box_n, \\
(\delta^\Box \text{ nat}) & \quad MMf \circ \delta^\Box_n = \delta^\Box_m \circ Mf, & (\delta^\Box \text{ nat}) & \quad \delta^\Box_m \circ MMf = Mf \circ \delta^\Box_n,
\end{align}

the \textit{comonad} and \textit{monad} equations:

\begin{align}
(\delta^\Box) & \quad M\delta^\Box_n \circ \delta^\Box_n = \delta^\Box_{n+1} \circ \delta^\Box_n, & (\delta^\Box) & \quad \delta^\Box_m \circ M\delta^\Box_n = \delta^\Box_n \circ \delta^\Box_{n+1}, \\
(\Box\beta) & \quad \varepsilon^\Box_{n+1} \circ \delta^\Box_n = 1_{n+1}, & (\Box\beta) & \quad \delta^\Box_n \circ \varepsilon^\Box_{n+1} = 1_{n+1}, \\
(\Box\eta) & \quad M\varepsilon^\Box_n \circ \delta^\Box_n = 1_{n+1}, & (\Box\eta) & \quad \delta^\Box_n \circ M\varepsilon^\Box_n = 1_{n+1},
\end{align}

and, finally, the \textit{Frobenius} equations where \( A \) is replaced by \( n \). The equations \((\delta^\Box)\) and \((\delta^\Box)\) are redundant in this axiomatization (see \[18\], Proposition 2.3.24, and \[13\], Section 6; they do not seem however to be redundant when the Frobenius equations are replaced by Lawvere’s equations).

The category \( \text{Frob} \) has a strict monoidal structure. The \( \otimes \) of this monoidal structure is addition on objects. We define \( 1_n \otimes f \) as \( M^n f \), where \( M^n \) is a sequence of \( n \geq 0 \) occurrences of \( M \), while \( f \otimes 1_n \) is defined by increasing the subscripts of \( f \) by the natural number \( n \). Then for \( f_1 : n_1 \rightarrow m_1 \) and \( f_2 : n_2 \rightarrow m_2 \) we have

\[ f_1 \otimes f_2 = (f_1 \otimes 1_{m_2}) \circ (1_{n_1} \otimes f_2). \]

The category \( \text{Frob} \) was envisaged as a monoidal category in \[18\] (Section 3.6.16).

The category \( \mathcal{M} \) of the monad freely generated by a single object 0 is defined like \( \text{Frob} \) save that we omit the arrow terms \( \varepsilon^\Box_n \) and \( \delta^\Box_n \), and whatever involves them. By omitting \( \varepsilon^\Box_n \) and \( \delta^\Box_n \), we define analogously the comonad freely generated by 0.

### 3 Free adjunctions and monads

An adjunction is given by two categories \( \mathcal{A} \) and \( \mathcal{B} \), a functor \( F \) from \( \mathcal{B} \) to \( \mathcal{A} \), the \textit{left adjoint}, a functor \( G \) from \( \mathcal{A} \) to \( \mathcal{B} \), the \textit{right adjoint}, a natural transformation \( \gamma : I_B \rightarrow GF \), the \textit{unit} of the adjunction, and a natural transformation \( \varphi : FG \rightarrow I_A \), the \textit{counit} of the adjunction, which satisfy the following \textit{triangular} equations for every object \( B \) of \( \mathcal{B} \) and every object \( A \) of \( \mathcal{A} \):

\[ \varphi_{FB} \circ F\gamma_B = 1_{FB}, \quad G\varphi_A \circ \gamma_GA = 1_{GA}. \]

The adjunction freely generated by a single object 0 on the \( \mathcal{B} \) side is defined in syntactical terms analogously to \( \text{Frob} \) (see \[8\], Chapter 4, for a detailed exposition). In this free adjunction, the objects of \( \mathcal{B} \) are 0, \( GF0 \), \( GFGF0 \), etc.,
while those of $A$ are $F0$, $FG0$, $FGFG0$, etc. This notion of freely generated adjunction is essentially the same as a 2-categorical notion that may be found in [2], [31] (cf. also [15]) and [23]. If we consider the sub-2-category of the 2-category $\mathcal{Cat}$ of categories whose only 0-cells are $A$ and $B$, whose 1-cells are made of $F$ and $G$, and whose 2-cells are made of $\varphi, \gamma, F$ and $G$, we obtain a 2-category isomorphic in the 2-categorial sense to the free category $\mathcal{Ad}$ of [2] (called $\mathcal{Adj}$ in [31]). This does not depend on the number of generators of our free adjunction, provided it is not zero, and they may be either on the $A$ side or on the $B$ side.

The connection of our notion of free adjunction with the 2-category $\mathcal{Ad}$ may also be construed as follows. In addition to what we have above, we should consider the adjunction freely generated by a different object on the $A$ side, which altogether gives us four disjoint categories. These four categories are isomorphic respectively to the categories $\text{Hom}(A,A)$, $\text{Hom}(A,B)$, $\text{Hom}(B,B)$ and $\text{Hom}(B,A)$ that may be found in the 2-categorial approach of [2] and the other references above. Roughly speaking, one has only to understand our freely generated objects as 1-cells, and add 0-cells, to pass to the 2-categorial approach. In contradistinction to that approach, we restrict ourselves to syntactically constructed free adjunctions within the category $\mathcal{Cat}$, and we make explicit the free generators, but the mathematical content is essentially the same. (The mathematical content changes by moving to a new level of categorification with the pseudoadjunctions of [35] and [21].)

We give a new simple proof of the following result of [2] (Corollaire 2.8), which connects the category $\mathcal{M}$ of the free monad defined at the end of the preceding section with the category $\mathcal{B}$ of the adjunction freely generated by 0 on the $B$ side. This result is interesting for us, because it is at the base of a more complicated result concerning $\mathcal{Frob}$ that we establish in Section 5.

**Proposition.** The categories $\mathcal{M}$ and $\mathcal{B}$ are isomorphic.

**Proof.** This isomorphism is proved syntactically by defining first by induction a functor $I$ from $\mathcal{M}$ to $\mathcal{B}$ for which we have

\[
\begin{align*}
I0 &= 0, \\
I(n+1) &= GFIn, \\
I\varepsilon^n_B &= \gamma^n_B, \\
I\delta^n_B &= G\varphi^n_A, \\
I(h_2 \circ h_1) &= Ih_2 \circ Ih_1, \\
IMh &= GFIh.
\end{align*}
\]

We verify that $I$ is indeed a functor by induction on the length of derivation of an equation of $\mathcal{M}$.

Next we define by induction a functor $J$ from the category $\mathcal{B} + \mathcal{A}$, which is the disjoint union of the categories $\mathcal{B}$ and $\mathcal{A}$ of the free adjunction, to the category $\mathcal{M}$. For $J$ we have

\[
\begin{align*}
J0 &= 0, \\
JGFB &= JFB = JB + 1, \\
J\varepsilon_B &= \varepsilon^n_B, \\
J\delta_A &= \delta^n_A, \\
JI_C &= IJC.
\end{align*}
\]
\[ J(h_2 \cdot h_1) = J h_2 \cdot J h_1, \quad JGf = Jf, \quad JFg = MJg. \]

To verify that \( J \) is indeed a functor, which is done by induction on the length of derivation of an equation, we had to define it from \( B + A \), but there is an obvious functor \( J_B \) from \( B \) to \( M \) obtained by restricting \( J \).

It is straightforward to verify by induction on the complexity of objects and arrow terms that \( I \) and \( J_B \) are inverse to each other. So the categories \( M \) and \( B \) are isomorphic.

A more involved, graphical, proof of this proposition may be found in [9] (Sections 6-8).

If our free adjunction is generated by a single object on the \( A \) side, then we establish the isomorphism of \( A \) with the category of the comonad freely generated by a single object (see the end of Section 2).

4 Bijunctions and self-adjunctions

We call \textit{trijunction} a structure made of the categories \( A \) and \( B \), the functor \( U \) from \( A \) to \( B \), and the functors \( L \) and \( R \) from \( B \) to \( A \), such that \( L \) is left adjoint to \( U \), with the unit \( \gamma^B : I_B \to UL \) and counit \( \varphi^A : LU \to I_A \), and \( R \) is right adjoint to \( U \), with the unit \( \gamma^A : I_A \to RU \) and counit \( \varphi^B : UR \to I_B \). This notion plays an important role in [13].

We call \textit{bijunction} a trijunction where the functors \( L \) and \( R \) are equal. We write in this context \( P \) instead of \( L \) and \( R \). (Various terms have been used for bijunctions and related notions: in [28] one finds \textit{strongly adjoint} pairs of functors, in [3] \textit{Frobenius functors}, in [17] \textit{biadjunction}, which has already been introduced for something else in [32], in [33] one finds \textit{autonomous category} and \textit{Frobenius pseudomonoid}, and in [22] \textit{ambidextrous adjunction}.)

A \textit{self-adjunction} is an adjunction where the categories \( A \) and \( B \) are equal, and the functors \( F \) and \( G \), which are now endofunctors, are also equal. We write in this context \( S \) for \( A \) and \( B \), and \( F \) for both \( F \) and \( G \). So the unit and counit of a self-adjunction are respectively \( \gamma : I_S \to FF \) and \( \varphi : FF \to I_S \). Every self-adjunction is a bijunction. (Self-adjunction is not often mentioned in textbooks of category theory—an exception is [1], Chapter 5, Exercise 19G; the notion of self-adjoint functor of [34] is a related but different notion.)

The bijunction freely generated by a single object 0 on the \( A \) side is defined in syntactical terms analogously to \textit{Frob} in Section 2. The objects of the category \( A \) are here 0, \( PU0 \), \( PUPU0 \), etc., while those of \( B \) are \( U0 \), \( UPU0 \), \( UPUPU0 \), etc.

We define analogously the free self-adjunction generated by a single object 0. An object of the category \( S \) of this self-adjunction is of the form \( F^n0 \), where \( F^n \) is a sequence of \( n \geq 0 \) occurrences of \( F \). We identify this object with \( n \), so that \( F^n \) is \( n+1 \). (One can find in [10] a more detailed construction of \( S \), which
is there called \( \mathcal{L}_c \). The category \( \mathcal{S} \) is the disjoint union of the categories \( \mathcal{S}_A \), whose objects are even, and \( \mathcal{S}_B \), whose objects are odd.

For \( \mathcal{C} \) being one of the categories \( \mathcal{A} \) and \( \mathcal{B} \) of the penultimate paragraph, and a subscript of one of the categories \( \mathcal{S}_A \) and \( \mathcal{S}_B \) of the preceding paragraph, we can prove the following.

**Proposition.** The categories \( \mathcal{C} \) and \( \mathcal{S}_C \) are isomorphic.

**Proof.** We define first by induction the functors \( H_C \) from \( \mathcal{S}_C \) to \( \mathcal{C} \), for \( \alpha \) being \( \varphi \) or \( \gamma \):

\[
\begin{align*}
H_A 0 &= 0, & H_A(2n+2) &= PH_B(2n+1), & H_B(2n+1) &= UH_A 2n, \\
H_A \alpha 2n &= \alpha H_A 2n, & H_B \alpha 2n+1 &= \alpha B H_B (2n+1), \\
H_C 1_n &= 1_{H_C n}, & H_C(h_2 \circ h_1) &= H_C h_2 \circ H_C h_1, \\
H_A Fg &= PH_B g, & H_B Ff &= UH_A f.
\end{align*}
\]

Next we define by induction the functors \( K_C \) from \( \mathcal{C} \) to \( \mathcal{S}_C \):

\[
\begin{align*}
K_A 0 &= 0, & K_A PB &= K_B B+1, & K_B UA &= K_A A+1, \\
K_A C &= \alpha K_C C, & K_C(h_2 \circ h_1) &= K_C h_2 \circ K_C h_1, \\
K_A Pg &= FK_B g, & K_B Uf &= FK_A f.
\end{align*}
\]

We verify by induction on the length of derivation of an equation that \( H_C \) and \( K_C \) are indeed functors. Next we verify by induction on the complexity of objects and arrow terms that \( H_C \) and \( K_C \) are inverse to each other. So the categories \( \mathcal{C} \) and \( \mathcal{S}_C \) are isomorphic.

\( \dashv \)

### 5 Frobenius monads and self-adjunctions

We want to prove the following result concerning the category \( \text{Frob} \) of the free Frobenius monad of Section 2 and the category \( \mathcal{S}_A \) of the free self-adjunction of the preceding section.

**Proposition.** The categories \( \text{Frob} \) and \( \mathcal{S}_A \) are isomorphic.

**Proof.** We define first by induction a functor \( I \) from \( \text{Frob} \) to \( \mathcal{S}_A \):

\[
\begin{align*}
I n &= 2n, & I \delta^\varphi_n &= \varphi 2n, & I \delta^\gamma_n &= \gamma 2n, \\
I (h_2 \circ h_1) &= I h_2 \circ I h_1, & I 1_n &= 1_{2n}, \\
IMh &= FFh.
\end{align*}
\]
Next we define by induction a functor $J$ from $\mathcal{S}$ to $\text{Frob}$:

\[
\begin{align*}
J2n &= n, & J(2n+1) &= n+1, \\
J\varphi 2n &= \varepsilon_n, & J\gamma 2n+1 &= \delta_n, \\
J\gamma 2n &= \varepsilon_n, & J\varphi 2n+1 &= \delta_n, \\
J(h_2 \cdot h_1) &= Jh_2 \cdot Jh_1, & JFg &= Jg, \text{ for } g \in \mathcal{S}_B, & JFf &= MJf, \text{ for } f \in \mathcal{S}_A.
\end{align*}
\]

We verify by induction on the length of derivation of an equation that $I$ and $J$ are indeed functors. We will not dwell on that verification for $I$, while for $J$ we have to verify first that

\[J(h \cdot \varphi_n) = J(\varphi_m \cdot FFh).\]

If $h$ is from $\mathcal{S}_A$, then we use the equation ($\varepsilon$ nat) of Section 2. If $h$ is from $\mathcal{S}_B$, then we proceed by induction on the complexity of $h$, by using the Frobenius equations and the equations ($\delta$ nat) of Section 2. Note that if $h$ is from $\mathcal{S}_B$, then $Jh$ can be neither $\varepsilon_k$ nor $\varepsilon_k'$. We proceed analogously for

\[J(\gamma_m \cdot h) = J(FFh \cdot \gamma_n).\]

To verify $Jh_1 = Jh_2$ for $h_1 = h_2$ a triangular equation, we use the equations $(\Box \beta)$, $(\Box \eta)$, $(\Diamond \beta)$ and $(\Diamond \eta)$ of Section 2.

There is an obvious functor $J_A$ from $\mathcal{S}_A$ to $\text{Frob}$ obtained by restricting $J$, and it is straightforward to verify by induction on the complexity of arrow terms that $I$ and $J_A$ are inverse to each other. So the categories $\text{Frob}$ and $\mathcal{S}_A$ are isomorphic. \(\dagger\)

From this proposition and from the Proposition of the preceding section we can conclude that $\text{Frob}$ is isomorphic to the category $\mathcal{A}$ of the bijunction freely generated by a single object on the $\mathcal{A}$ side, but the isomorphism we have established in this section is more interesting for us, as it will become clear in the next section.

### 6 Coherence for Frobenius monads

Out of the category $\mathcal{S}$ of the free self-adjunction of Section 4, we build a monoid $\mathcal{S}^*$ (which in [10], Section on $\mathcal{L}_c$ and $\mathcal{L}_\omega$, is called $\mathcal{L}^*_c$, while $\mathcal{S}$ is called $\mathcal{L}_c$). On the arrows of $\mathcal{S}$, we define a total binary operation $*$ based on composition in the following manner: for $f : m \to n$ and $g : k \to l$,

\[
g * f =_{df} \left\{ \begin{array}{ll}
g \cdot F^{k-n}f & \text{if } n \leq k \\
F^{n-k}g \cdot f & \text{if } k \leq n. \end{array} \right.
\]

Next, let $f \equiv g$ if and only if for some $k$ and $l$ we have $F^k f = F^l g$ in $\mathcal{S}$. It is easy to check that $\equiv$ is an equivalence relation on the arrows of $\mathcal{S}$, which
satisfies moreover

\[ f_1 \equiv f_2 \text{ and } g_1 \equiv g_2, \text{ then } g_1 * f_1 \equiv g_2 * f_2. \]

For every arrow \( f \) of \( S \), let \([f] \) be \( \{ g \mid f \equiv g \} \), and let \( S^* \) be \( \{ [f] \mid f \text{ is an arrow of } S \} \). With

\[
1 =_{df} [1_0], \\
[g][f] =_{df} [g * f],
\]

we can check that \( S^* \) is indeed a monoid, which in [10] is shown to be isomorphic to the monoid \( L_\omega \). The monoid \( S^* \) is built of syntactical material, coming from the category \( S \), which is presented by generators and equations. The monoid \( L_\omega \) is just another presentation by generators and equations of \( S^* \). A reason for introducing it in [10] was to make simpler reduction to normal form, without being encumbered by the sources and targets of arrow terms. We presuppose the reader is acquainted with \( L_\omega \), but indications about how this monoid is presented will be given below when we deal with composition in \( Frobse \). This monoid interests us here only as an auxiliary, leading towards the geometric categories \( Frz \) and \( Frobse \), which we will consider below.

Out of the material of the monoid \( S^* \) we return to \( S \), by building a category isomorphic to \( S \), in the following way. Let \( S^{st} \) be the category whose objects are the natural numbers, and whose arrows are the triples \( \langle [f], n, m \rangle \) such that there is an arrow \( g : n \rightarrow m \) in \([f]\); the arrow \( \langle [f], n, m \rangle \) is of type \( n \rightarrow m \) in \( S^{st} \), which means that its source is \( n \) and its target \( m \). The composition of \( \langle [f_1], n, m \rangle \) and \( \langle [f_2], m, k \rangle \) is defined as \( \langle [f_2][f_1], n, k \rangle \), and the identity arrow on \( n \) is \( \langle [1_0], n, n \rangle \). We can prove the following.

**Proposition 1.** The categories \( S^{st} \) and \( S \) are isomorphic.

The functor from \( S \) to \( S^{st} \) giving this isomorphism is identity on objects and an arrow \( f : n \rightarrow m \) is mapped to \( \langle [f], n, m \rangle \). To prove the proposition, we rely on the fact that for every element \( [f] \) of \( S^* \), and every arrow \( g : n \rightarrow m \) in \([f]\), the arrow \( g \) is the only arrow in \([f]\) of the type \( n \rightarrow m \). To establish this fact we need to establish the following.

**S Cancellation Lemma.** In \( S \), if \( Ff = Fg \), then \( f = g \).

An analogous lemma (called the \( L \) Cancellation Lemma) was proved in [10], but with a restriction on the type of \( f \) and \( g \). It was stated there that the restriction can be lifted, and it was suggested how to achieve that. The suggestion envisaged two ways, one of which is rather straightforward (both of these ways are however lengthy, and for lack of space a detailed exposition was omitted.) As a matter of fact, there is a direct way to prove the \( S \) Cancellation Lemma along the lines of the \( K_c \) Cancellation Lemma of [10]. Here are indications for this direct proof (which presuppose an acquaintance with [10]).
PROOF OF THE $S$ CANCELLATION LEMMA. We proceed as in the proof of the $\mathcal{K}_c$ Cancellation Lemma of [10] until we reach the case that $f = \varphi_0 \circ f'$ and $g = \varphi_0 \circ g'$. Then we must ensure that $\varphi_0$ in $\varphi_0 \circ f'$ and $\varphi_0$ in $\varphi_0 \circ g'$ are tied in $\delta(\psi(f))$, which is equal to $\delta(\psi(g))$, to circles encompassing the same circular forms. It is always possible to achieve that. We conclude that $\delta(\psi(f')) \cong \Delta \delta(\psi(g'))$. This is because $\delta(\psi(\varphi_0 \circ f'))$ and $\delta(\psi(\varphi_0 \circ g'))$ are both $\mathcal{L}$-equivalent to $\delta(c_1^\alpha)$, for some $\alpha \geq 1$, while $\delta(\psi(f'))$ and $\delta(\psi(g'))$ must both be $\mathcal{L}$-equivalent to the same $\delta(b_1^{\beta_1}c_1^\gamma)$, for $\beta$ and $\gamma$ lesser than $\alpha$. We conclude that $f' \equiv g'$, and since $f'$ and $g'$ are of type $0 \rightarrow 2$, by the $\mathcal{L}$ Cancellation Lemma of [10], we have that $f' = g'$ in $S$. From that we obtain that $f = g$ in $S$.

As $S$, the isomorphic category $S^{*k}$ is the disjoint union of two categories $S^{*k}_A$ and $S^{*k}_B$, isomorphic respectively to $S_A$ and $S_B$. If $\langle [f], n, m \rangle$ is in $S^{*k}_A$, then $n$ and $m$ are even, and if it is in $S^{*k}_B$, then they are odd. We will now define a category $\mathcal{L}^A_\omega$, isomorphic to $S^{*k}_A$. This category, made out of the material of the monoid $L_\omega$, interests us only as a stepping stone towards the isomorphic geometric categories Fr$z$ and Fro$bse$, which we will consider in a moment.

The objects of the category $\mathcal{L}^A_\omega$ are again the natural numbers. An arrow of this category will be obtained from an arrow $\langle [f], n, m \rangle$ of $S^{*k}_A$ by replacing the class $[f]$ by the corresponding element $e$ of $L_\omega$, and $n$ and $m$ by respectively $n/2$ and $m/2$; the type of $\langle e, n/2, m/2 \rangle$ in $\mathcal{L}^A_\omega$ is $n/2 \rightarrow m/2$. We divide the numbers in the types by two to obtain types that will correspond to the types in Fro$b$ (see the preceding section).

In this way, with every element of $L_\omega$ we associate in $\mathcal{L}^A_\omega$ a denumerable infinity of types. The generator $a_{2n+1}^A$ of $L_\omega$ (see [10], Section on Normal forms in $L_\omega$) will have as associated types $n+l+1 \rightarrow n+l$, for every $l \geq 0$, while the generator $b_{2n+1}^A$ will have $n+l \rightarrow n+l+1$, and the generator $c_{2n+1}^A$ will have $n+l \rightarrow n+l$. (This typing is explained by the typing of the friezes below.) The generator $a_{2n+2}^A$ will have as associated types $n+l+2 \rightarrow n+l+1$, the generator $b_{2n+2}^A$ will have $n+l+1 \rightarrow n+l+2$, and the generator $c_{2n+2}^A$ will have $n+l+1 \rightarrow n+l+1$. Multiplication of terms now becomes composition, and takes the types into account. Two typed terms of $L_\omega$ stand for the same arrow of the category $\mathcal{L}^A_\omega$ if and only if they are of the same type and equal in $L_\omega$. We can then assert the following.

PROPOSITION 2. The categories Fro$b$ and $\mathcal{L}^A_\omega$ are isomorphic.

This follows immediately from the isomorphism of Fro$b$ and $S_A$, proved in the preceding section, the isomorphism of $S_A$ and $S^{*k}_A$, which follows from Proposition 1, and the isomorphism of $S^{*k}_A$ and $\mathcal{L}^A_\omega$.

From [10] (Section on $L_\omega$, $K_\omega$ and friezes) one can infer that the category $\mathcal{L}^A_\omega$ is isomorphic to a category Fr$z$ whose arrows are diagrams called friezes with associated types. Roughly speaking, a frieze is a tangle without crossings in whose regions we find circular forms that correspond bijectively to the ordinals.
contained in the infinite ordinal $\varepsilon_0$. In [10] one can found a proof that $\mathcal{L}_\omega$ is isomorphic to the monoid of friezes, and from that the isomorphism of the categories $\mathcal{L}_\omega^A$ and $\mathcal{F}_\mathbb{Z}$ follows. So, by Proposition 2, the categories $\mathcal{F}_\text{ob}$ and $\mathcal{F}_\mathbb{Z}$ are isomorphic. By this last isomorphism, the arrows on the left are mapped to the friezes on the right, with the type associated to the friezes being those of the arrows:

When $n = 0$, the vertical thread connecting 1 at the top with 1 at the bottom does not exist in the first and the third frieze. Note that our friezes are “thin” tangles that may be conceived as the boundaries of the corresponding thick tangles of [16].

A **circular form** is a finite collection of nonintersecting circles in the plane factored through homeomorphisms of the plane mapping one collection into another (see the definition of $\mathcal{L}$-equivalence of friezes in [10], Section on *Friezes*). The circular forms obtained by composing friezes are coded by the ordinals contained in $\varepsilon_0$ in the following way. The circular form consisting of no circles is coded by 0. If the circular forms $c_1$, $c_2$ and $c$ are coded by the ordinals $\alpha_1$, $\alpha_2$ and $\alpha$ respectively, then the circular form $c_1 \cup c_2$ (the disjoint union of $c_1$ and $c_2$) is coded by the natural sum $\alpha_1 \oplus \alpha_2$, and the circular form $\bigcirc$ ($c$ inside a new circle) is coded by $\omega^\alpha$. So a single circle is coded by $\omega^0$, which is equal to 1 (see [10], Section on *Finite multisets, circular forms and ordinals*).
Let $\mathcal{F}$ be the commutative monoid with one unary operation freely generated by the empty set of generators. The elements of $\mathcal{F}$ may be identified with the hierarchy of finite multisets obtained by starting from the empty multiset as the only urelement, or by finite nonplanar trees with arbitrary finite branching, or by circular forms. A monoid isomorphic to $\mathcal{F}$ is the commutative monoid $\langle \varepsilon_0, \sharp, 0, \omega^- \rangle$ where $\sharp$ is binary natural sum, and we have the additional unary operation $\omega^-$ (for more details on these matters, see [10]). Note that though the elements of $\varepsilon_0$ greater than or equal to $\omega$ are associated with infinite ordinals, they may be used to code finite objects, such as circular forms. Another monoid isomorphic to $\mathcal{F}$ is the commutative monoid $\langle \mathbb{N}^+, \cdot, 1, p_n \rangle$ where $\mathbb{N}^+$ is the set of natural numbers greater than 0, the operation $\cdot$ is multiplication, and $p_n$ is the $n$-th prime number (we are indebted for this remark to a suggestion of Marko Stošić).

The isomorphism of $\text{Frob}$ with $\text{Frz}$ may be understood as a geometrical description of $\text{Frob}$. Towards the end of his book [18] (Sections 3.6.20 ff), Kock was looking for such a description, but not exactly in the same direction. The category $\text{Frob}_{se}$, isomorphic to $\text{Frz}$, which we will consider below, gives another alternative approach to the geometrization of $\text{Frob}$ sought by Kock.

The isomorphism of $\text{Frob}$ and $\text{Frz}$ may be understood also as a coherence result, which provides a decision procedure for equality of arrows in $\text{Frob}$. This decision procedure involves a syntactical description of friezes given by the monoid $\mathcal{L}_\omega$ of [10], and a reduction to normal form.

Instead of the category $\text{Frz}$, one can use an alternative isomorphic category, which we will call $\text{Frob}_{se}$. In the arrows of this category, the regions of friezes stand for equivalence classes of an equivalence relation whose domain is split into a source part and a target part, which are both copies of $\mathbb{N}^+$. Such equivalence relations were called split equivalences in [11]. Split equivalences are related to cospans in the base category $\text{Set}$ (see [27], XII.7, and [30], Example 2.4), but unlike cospans they do not register the common target of the two arrows making the cospan.

The split equivalences we envisage for $\text{Frob}$ are nonintersecting in the following sense. Let the source and target elements be identified respectively with the positive and negative integers (so 0 does not correspond to any element). For $a, b, c, d \in \mathbb{Z} - \{0\}$, we say that $(a, b)$ intersects $(c, d)$ when either $a < c < b < d$ or $c < a < d < b$. An equivalence relation on $\mathbb{Z} - \{0\}$ is nonintersecting when if $a$ and $b$ are in one equivalence class, while $c$ and $d$ are in another equivalence class, then $(a, b)$ does not intersect $(c, d)$. (This is related to the nonoverlapping segments of [10], Section on Friezes.)

For example, instead of the frieze on the left-hand side, which is an arrow of $\text{Frz}$ of the type $2+l \rightarrow 1+l$, we have the nonintersecting split equivalence on the right-hand side, which is an arrow of $\text{Frob}_{se}$ of the same type:
The thick white regions on the left-hand side become thin black equivalence classes on the right-hand side, and the thin black threads on the left-hand side become white regions on the right-hand side. We will not obtain in this way on the right-hand side every nonintersecting split equivalence.

The equivalence classes of those we obtain satisfy some additional conditions. First, they are all finite, and all but finitely many of them are such that they have just two elements—one at the top and one at the bottom. Secondly, they are either even or odd, depending on whether their members are even or odd; we have only such even and odd equivalence classes. Finally, two classes of the same parity cannot be immediate neighbours in the following sense. The classes $A$ and $B$ are immediate neighbours when for every $a \in A$ and every $b \in B$ and every class $C$ and every $c_1, c_2 \in C$, if $(a, b)$ intersects $(c_1, c_2)$, then $C$ is either $A$ or $B$.

The nonintersecting split equivalences that satisfy these additional conditions concerning their equivalence classes will be called maximal split equivalences.

Note that in maximal split equivalences the odd equivalence classes are completely determined by the even equivalence classes, and vice versa. We cannot however reject either of them because of the ordinals. In the regions of friezes one finds finitely many circular forms that correspond to ordinals in $\varepsilon_0$, and we will assign these ordinals to the equivalence classes of maximal split equivalences.

Maximal split equivalences together with a function assigning ordinals in $\varepsilon_0$ to the equivalence classes, so that all but finitely many have zero as value, will be called Frobenius split equivalences. Frobenius split equivalences with types associated to them are the arrows of Frobse. For example, to the frieze on the left-hand side we assign the Frobenius split equivalence on the right-hand side:

All the Frobenius split equivalences are generated by composition from the following generating Frobenius split equivalences, which are correlated with the elements of the monoid $L_\omega$ mentioned on the left of the following pictures (see [10], Section on Normal forms in $L_\omega$ and $K_\omega$), where we omit mentioning that an equivalence class bears 0; here, $k \geq 1$ and $\alpha, \beta \in \varepsilon_0$:
The composition of Frobenius split equivalences is made according to the following reductions, which are correlated with the equations of $L_\omega$ on the left of the following pictures, for $l \leq k$:

\begin{align*}
\text{(aa)} & \quad a_k^\alpha a_l^\beta = a_l^\beta a_k^\alpha \\
\text{(c2)} & \quad c_k^\alpha c_k^\beta = c_k^{\alpha \beta} \\
\text{(cc) for } l < k, & \quad c_k^\alpha c_l^\beta = c_l^\beta c_k^\alpha \\
\text{(ab 1)} & \quad a_l^\alpha b_{k+2} = b_{k+2}^\beta a_l^\alpha \\
\text{(ab 3.1)} & \quad a_k^\alpha c_{k+1} = c_{k+1}^\beta a_k^\alpha \\
\text{(ab 3.2)} & \quad a_{k+1}^\alpha b_k = c_k^\alpha c_{k+1}^\beta
\end{align*}
If we disregard the ordinals, then this is exactly like composition of split equivalences.

There are moreover reductions corresponding to the equations (bb), (ab 2), (bc 1), (bc 2) and (bc 3) of [10] (Section on Normal forms in $L_\omega$), which are analogous to (aa), (ab 1), (ac 1), (ac 2) and (ac 3). We do not mention here trivial reductions involving $c^\gamma_k$, which is equal to 1. As a limit case, where $l = k$, of the reduction corresponding to (aa) we have

and analogously in other limit cases. The limit case $l = k$ of (ab 1) corresponds to one of the Frobenius equations:
We believe that our Frobenius split equivalences are more handy than the diagrams that may be found in [16] (Appendix C), to which they should be equivalent. They are more handy because the circular forms are coded efficiently by ordinals, while in the diagrams of [16] they make complicated patterns that are defined in all possible ways in terms of the generators. What these diagrams miss essentially is the reduction corresponding to the equation (ab 3.3).

The friezes appropriate for trijunctions (see [13], Section 8) are such that circular components and circular forms do not arise. Such friezes can be replaced by maximal split equivalences, without ordinals. As we said above, in maximal split equivalences, the odd equivalence classes are completely determined by the even equivalence classes, and vice versa. By rejecting the odd equivalence classes, we obtain the split equivalences that correspond to the categories $S_{5\oplus 5}$ and $5S_{\oplus 5}$ by the functor $G$; by rejecting the even equivalence classes, we obtain those that come with the functor $G^d$ (see [13], Sections 6-7). Coherence for trijunction could be proved with respect to nonintersecting split equivalences for which either odd or even equivalence classes are rejected.

7 Frobenius monads and matrices

Let $Mat$ be the skeleton of the category $Vect_K$ of finite-dimensional vector spaces over the field $K$, with linear transformations as arrows. The objects of $Mat$ are the natural numbers, which are dimensions of the objects of $Vect_K$, and its arrows are matrices. The category $Mat$ is strictly monoidal (in it the canonical arrows of its monoidal structure are identity arrows).

In this section we will show how the requirement of having a faithful functor into $Mat$ induces a collapse of the ordinals of $Frob$. This means that the usual notion of Frobenius algebra is not exactly caught by the notion of Frobenius monad. There are further categorial equations implicit in the notion of Frobenius algebra, which do not hold in every Frobenius monad. We will describe in this section these equations, and show their necessity. We leave open the question whether they are also sufficient to describe categorially the notion of Frobenius algebra.

There is no faithful monoidal functor from the strictly monoidal category $Frob$ into $Mat$. A necessary condition to obtain such a functor would be to extend the definition of $Frob$ with some new equations, for whose formulation we need the following abbreviations:

\[
\begin{align*}
(\delta_n^\oplus)^0 & = 1_{n+1}, \\
(\delta_n^\oplus)^{k+1} & = \delta_n^\oplus \circ (\delta_n^\oplus)^k, \\
(\delta_n^\oslash)^0 & = 1_{n+1}, \\
(\delta_n^\oslash)^{k+1} & = (\delta_n^\oslash)^k \circ \delta_{n+k}^\oslash, \\
\Phi_n^k & = \varepsilon_n^\oslash \circ (\delta_n^\oslash)^k \circ (\delta_n^\oplus)^k \circ \varepsilon_n^\oslash.
\end{align*}
\]

Our new equations are then all of the following equations, for $k, n \geq 0$:
\( (\Phi) \quad \Phi^n_k = M^n \Phi^n_0, \)

where \( M^n \) is a sequence of \( n \geq 0 \) occurrences of \( M \). Equations with the same force as \((\Phi)\), which we will also call \((\Phi)\), are, for \( k, n \geq 0 \),

\[ \Phi_{n+1}^k = M \Phi_n^k. \]

These equations do not hold in \( \text{Frob} \), as can be seen with the help of the monoid \( L_\omega \), where the corresponding equations \((\Phi_c)\) do not hold. These equations hold in the monoid \( K_\omega \) of \([10]\).

Let the category \( \text{Frob}' \) be defined like \( \text{Frob} \) save that we have in addition all the equations \((\Phi)\), and let \( L'_\omega \) be the monoid defined like \( L_\omega \) save that we have in addition all the equations \((\Phi_c)\). If all the subscripts \( n \) that may be found in defining \( \Phi^n_k \) are replaced by \( A \), while \( n+1 \) and \( n+k \) are replaced respectively by \( MA \) and \( M^k A \), then the equations \((\Phi)\) become

\[ \Phi_{n+1}^k = M^n \Phi_n^k \]

which we will also call \((\Phi)\), and which are the equations characterizing the class of Frobenius monads in which \( \text{Frob}' \) is the free one generated with a single object.

In the language of the free self-adjunction of Section 4, let \( \kappa^0_{2n+1} \) stand for \( 1_{2n+1} \), and let \( \kappa^k_{2n+1} \) be \( \kappa^k_{2n+1} = \varphi_{2n+1} \circ \gamma_{2n+1} \). Consider then the category \( S' \) constructed like the category \( S \) of the free self-adjunction save that we have in addition for every \( k, n \geq 0 \) the equation

\[ \varphi_{2n} \circ F \kappa_{2n+1}^k \circ \gamma_{2n} = F^{2n} (\varphi_0 \circ F \kappa_{1}^k \circ \gamma_0), \]

where \( F^m \) is a sequence of \( m \geq 0 \) occurrences of \( F \). The category \( S' \) is related to \( \text{Frob}' \) as the category \( S \) is related to \( \text{Frob} \); this is shown as in Section 5. On the other hand, \( S' \) is related to \( L'_\omega \) as \( S \) is related to \( L_\omega \); this is shown as in \([10]\) (Section on \( L_c \) and \( L_\omega \)).

We can infer that \( \text{Frob}' \) is isomorphic to a category whose arrows are the elements of the monoid \( L'_\omega \) with types associated to them (see the preceding section). This result may be understood as a coherence result, which provides a decision procedure for equality of arrows in \( \text{Frob}' \). The normal form involved in this decision procedure would serve also for the isomorphism with the category \( \text{Frz}' \), which we will consider in a moment. We will deal with this normal form later (see the second paragraph after the proof of Lemma 2m+2).

One could consider a category \( \text{Frz}' \) analogous to the category \( \text{Frz} \) of the previous section, which would be isomorphic to our category derived from \( L'_\omega \). We will not describe \( \text{Frz}' \) in detail, but just make a few indications. For the arrows of \( \text{Frz}' \) we would take, instead of friezes, two-manifolds made out of friezes in the following way. The regions of friezes may be chessboard-coloured by making the leftmost region white, and then alternating black and white for subsequent regions. For example, one of the friezes we had above is chessboard-coloured as follows:
Then consider the two-manifolds with boundary made of the compact black regions, which we will call black friezes, and on black friezes consider the equivalence relation based on homeomorphisms that preserve all the points on the top and bottom line (this is like the K-equivalence of [10], Section on Friezes). So the following black frieze would be equivalent to the black frieze above:

The category $Frz'$ is related to the category $2Cob$ of [18] (Section 1.4), whose arrows are cobordisms of dimension 2. An arrow of $Frz'$ may be conceived as a kind of “thin” cobordism.

As we associated the category $Frob$ to $Frz$, so we may look for a category $Frob'$ like $Frob$ to associate to $Frz'$. We will previously demonstrate however the necessity of the equations (Φ) for faithful monoidal functors into $Mat$, and consider the consequences for ordinals of having (Φ) and related equations.

The necessity of (Φ) follows from the fact that $Mat$ is a symmetric strictly monoidal category, which has a symmetry natural isomorphism $c_{n,m} : n \otimes m \to m \otimes n$ for which we have the equation

\[(c1) \quad c_{1,m} = c_{m,1} = 1_m \]

(where 1 in the subscripts of c is the unit object of $Mat$). Hence, for every arrow $f : 1 \to 1$ of $Mat$, we have

\[1_m \otimes f = (1_m \otimes f) \circ c_{1,m} = c_{1,m} \circ (f \otimes 1_m) = f \otimes 1_m.\]

Since for every monoidal functor $G$ from $Frob$ to $Mat$ we have $G0 = 1$ (where 0 is the unit object of $Frob$), and since $G\Phi^k_n$ is of the form $1_{n \cdot p} \otimes f$ for $f : 1 \to 1$, we have $G\Phi^k_n = G\Phi^k_0$. So, from the faithfulness of $G$, the equation (Φ) follows.

In the reasoning above $c$ can be a braiding natural isomorphism, instead of a symmetry natural isomorphism. We would have the equation (c1), and the equation (Φ) would again be imposed by the faithfulness of $G$. So we could replace $Mat$ by a braided strictly monoidal category (cf. [18], Section 3.6.27).

We defined above the monoid $L'_\omega$ as $L_\omega$ with the equation (Φc) added. In $L'_\omega$, the hierarchy of $\varepsilon_0$ collapses to $\omega^\omega$. This means that every element of $L'_\omega$ is definable in terms of $e^{\beta}_n$, for $e$ being $a$, $b$ or $c$, and $\beta \in \omega^\omega$. We can restrict
the terms \( e_n^\beta \) even further, to those in the following table, without altering the structure of the normal form for \( \mathcal{L}_\omega \) of [10] (Section on Normal forms in \( \mathcal{L}_\omega \)):

|   |   |   |
|---|---|---|
| \( e \) | \( n \) | \( \beta \) |
| \( e \) | 1 | \( \beta \in \omega^\omega \) |
| \( e \) | \( 2m+2 \) | \( \beta \in \omega \) |
| \( a \) and \( b \) | \( 2m+1 \) | \( \beta \in \omega \) |
| \( a \) and \( b \) | \( 2m+2 \) | \( \beta = 0 \) |

This is shown as follows.

By Cantor’s Normal Form Theorem (see, for example, [20], VII.7, Theorem 2, p. 248, or [26], IV.2, Theorem 2.14, p. 127), for every ordinal \( \alpha > 0 \) in \( \varepsilon_0 \) there is a unique finite ordinal \( n \geq 1 \) and a unique sequence of ordinals \( \alpha_1 \geq \ldots \geq \alpha_n \) contained in \( \alpha \), i.e. lesser than \( \alpha \), such that \( \alpha = \omega^{\alpha_1} \ldots \sharp \omega^{\alpha_n} \). So every ordinal in \( \varepsilon_0 \) can be named by using the operations of the monoid \( \langle \varepsilon_0, \sharp, 0, \omega^- \rangle \) mentioned in the previous section.

Let \( \beta_0 \) be \( \omega^0 \), which is equal to 1, and let \( \beta_k : \varepsilon_0^k \to \varepsilon_0 \), for \( k \geq 1 \), be defined by

\[
\beta_k(\alpha_1, \ldots, \alpha_k) = \omega^{\alpha_1} \ldots \sharp \omega^{\alpha_k}.
\]

By Cantor’s Normal Form Theorem, to name the ordinals in \( \varepsilon_0 \) we can replace the unary operation \( \omega^- \) by the operations \( \beta_k \) for every \( k \geq 0 \). So the name of every ordinal in \( \varepsilon_0 \) can be written in terms of 0, \( \sharp \) and \( \beta_k \). We proceed by induction on the complexity of such a name to define the map \( \beta ' \) from \( \varepsilon_0 \) to \( \omega^\omega \):

\[
0' = 0,
\]

\[
(\alpha_1 \sharp \alpha_2)' = \alpha_1' \sharp \alpha_2',
\]

\[
\beta_0' = \omega^0 = 1 = \beta_0,
\]

\[
\beta_k(\alpha_1, \ldots, \alpha_k)' = \omega^k \sharp \alpha_1' \ldots \sharp \alpha_k', \quad \text{for } k \geq 1.
\]

We can then prove the following lemmata.

**Lemma 2m+1.** In \( \mathcal{L}_\omega' \), for every \( m \geq 0 \), we have \( c_{2m+1}^\alpha = c_{1}^{\alpha'} \).

**Proof.** We proceed by induction on the size of \( \alpha \). If \( \alpha = 0 \), then we use the following equation of \( \mathcal{L}_\omega' \):

\[
c_{2m+1}^0 = c_{1}^0 = 1.
\]

In the induction step we have

\[
c_{2m+1}^{\alpha_1 \sharp \alpha_2} = c_{2m+1}^{\alpha_1' \sharp \alpha_2'}, \quad \text{by (e2) and the induction hypothesis},
\]

\[
c_{2m+1}^{\beta_0} = c_{1}^{\beta_0}, \quad \text{by (\Phi e)},
\]

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for $k \geq 1$,  
\[ c_{2m+1}^{\beta(\alpha_1, \ldots, \alpha_k)} = a_{2m+1}^0 \omega_{2m+1}^{\alpha_1} \cdots c_{2m+2}^{\alpha_k} b_{2m+1}^0, \]  
by (ab 3.3), (ac 3) and (c2).

For every $i \in \{1, \ldots, k\}$, we have, by the same equations,  
\[ c_{2m+2}^{\alpha_i} = a_{2m+2}^0 \omega_{2m+2}^{\alpha_i} b_{2m+2}^0. \]

Then, by the induction hypothesis and the equations (ac 1), (bc 1) and (c2), for $d^0$ being 1, and $d^{n+1}$ being $d^n a_{2m+2}^0 b_{2m+2}^0$, we obtain  
\[ c_{2m+1}^{\beta(\alpha_1, \ldots, \alpha_k)} = a_{2m+1}^0 \omega_{2m+1}^{\alpha_1} \cdots \omega_{2m+2}^{\alpha_k} a_1^0 b_0, \]  
by (ab 3.3) and (ac 3),  
\[ = c_1^{\beta(\alpha_1, \ldots, \alpha_k)}, \]  
by (c2).  

**Lemma 2.** In $L'_\omega$, for every $m \geq 0$, we have $c_{2m+2}^{\alpha} = c_1^{\alpha} c_{2m+2}^{\alpha}$.

**Proof.** We have  
\[ c_{2m+2}^{\alpha} = a_{2m+2}^0 \omega_{2m+2}^{\alpha} b_{2m+2}^0, \]  
by (ab 3.3) and (ac 3),  
\[ = c_1^{\alpha} c_{2m+2}^{\alpha}, \]  
by the preceding lemma, (ac 1) and (ab 3.3).  

With these two lemmata, we can show that the terms $c_\beta^\alpha$ in the table above are sufficient to define every element of $L'_\omega$ without altering the structure of our normal form. This is clear for the terms $c_\alpha^\alpha$. We also have  
\[ a_{2m+2}^\alpha = a_{2m+2}^0 c_{2m+2}^\alpha, \]  
by (ac 3),  
\[ = c_1^{\alpha} a_{2m+2}^0, \]  
by Lemma 2 and (ac 1);  
\[ a_{2m+1}^{\alpha_1 \ldots \alpha_n} = a_{2m+1}^0 c_{2m+2}^{\alpha_1} \cdots c_{2m+2}^{\alpha_n}, \]  
by (ac 3) and (c2),  
\[ = c_1^{\alpha_1 \ldots \alpha_n} a_{2m+1}^0, \]  
by Lemma 2, (ac 1) and (c2),  
and analogous equations with $a$ replaced by $b$.

Consider terms of $L'_\omega$ in the form exactly like the normal form of $L'_\omega$ in [10] save that all the generators $a_\alpha^\alpha$, $b_\beta^\beta$ and $c_\gamma^\gamma$ are terms from our table. We say that such terms are in normal form. This is the normal form we mentioned previously, which we can use to decide equations in $L'_\omega$, and to prove the isomorphism with $Fr'_\omega$, along the lines of [10].

We can now sketch how the category $Frobse'$ analogous to $Frobes$ and isomorphic to $Fr'_\omega$ would look like. Its arrows will be based on Frobenius split equivalences where the function assigning ordinals will follow restrictions in accordance with our table:
(1) an even class is mapped to an ordinal in \( \omega \),
(2) an odd class containing 1 is mapped to an ordinal in \( \omega^\omega \),
(3) an odd class not containing 1 is mapped to 0.

Even classes correspond to the black regions of the black friezes and odd classes to the white regions; the odd class containing 1 corresponds to the leftmost white region. The ordinals of (1) register the number of white holes in the black regions, and those of (2) the number of black disks and the number of white holes in them.

Composition in \( \text{Frobse}' \) would be defined by reductions based on the equations of \( L_\omega' \), like those we gave for \( \text{Frobse} \). Essentially, we would have to change only the reductions corresponding to \((ab\ 3.1),\ (ab\ 3.2)\) and \((ab\ 3.3)\). We could have instead

\[
(ab\ 3.1) \quad a_{2m+1}^n b_{2m+2}^0 = c_{2m+2}^n
\]

\[
(ab\ 3.3) \quad a_{2m+1}^n b_{2m+1}^l = c_{1}^{\omega^m+1}
\]

and analogous reductions for \((ab\ 3.2)\).

8 Separable matrix Frobenius monads

In the preceding section, we saw how symmetry in the category \( Mat \) induces a collapse of the ordinals in \( \varepsilon_0 \) of \( \text{Frob} \) into the ordinals in \( \omega^{\omega} \). In all that, we have not considered commutative Frobenius monads, which play a central role in connection with topological quantum field theories. With commutative
Frobenius monads, our ordinals are still contained in $\omega^\omega$, as in the preceding section.

Another collapse of ordinals comes with separability (see [7], [4] and [30]). The separability equation for Frobenius monads is the equation

$$\delta^0_A \circ \delta^0_A = 1_{MA}.$$ 

If we consider extending Frob with this equation, we just replace $A$ by $n$. To state the consequence of the corresponding equation $c^1_{2n+2} = 1$ for $L_\omega$, we need some terminology.

Let the ordinal $0$ be of even height. If $\alpha_1, \ldots, \alpha_n$ are all of even (odd) height, then $\omega^{\alpha_1} \cdot \ldots \cdot \omega^{\alpha_n}$ is of odd (even) height. If an ordinal in $\varepsilon_0$ is of even or odd height, we say that it has a homogeneous height. Not all ordinals in $\varepsilon_0$ have a homogeneous height. The consequence of the separability equation for $L_\omega$ is that every $c^\alpha_n$ is equal to $c^{\alpha'}_n$ for $\alpha'$ an ordinal in $\varepsilon_0$ of homogeneous height; if $n$ is $2m+2$, then $\alpha'$ is of even height, and if $n$ is $2m+1$, then $\alpha'$ is of odd height.

If we combine the separability equation with the equation $(\Phi)$ of the preceding section, then the ordinals in $\varepsilon_0$ collapse to the ordinals in $\omega$. More precisely, the consequence for $L_\omega$ is that we could take as primitive only the terms $e^k_n$, for $e$ being $a$, $b$ or $c$, and $k \in \omega$, where only $c^k_n$ may have $k \geq 0$; in all other cases, $k = 0$. In the presence of the separability equation, the equation

$$(\Phi^0) \quad \Phi^0_A = M \Phi^0_A$$

has the same force as the equations $(\Phi)$. According to our definition, $\Phi^0_A$ is $\varepsilon^0_A \circ \varepsilon^0_A$.

We call Frobenius monads that satisfy $(\Phi^0)$ and the separability equation separable matrix Frobenius monads. For separable matrix Frobenius monads, we can answer positively the question of sufficiency left open at the beginning of the preceding section. Namely, there is a faithful monoidal functor $F$ from the separable matrix Frobenius monad generated by a single object into the category $\text{Mat}$. In fact, something stronger holds: for every natural number $p \geq 2$, there is a functor $F$ as above such that $F(1) = p$. We will not prove this in detail, but just give some indications.

Our task is to represent in $\text{Mat}$ an ordered pair made of a maximal split equivalence (see Section 6) and a natural number, which is the ordinal $k \in \omega$ tied to $c^k_1$. We may reject the odd equivalence classes from this maximal split equivalence, and then represent the remaining split equivalence in a Brauerian manner (see [10], [11] and [12]). The natural number $k$ will be mapped to the scalar $p^k$. This is analogous to representing $K_c$ in $\text{Mat}$ (in the section with that name in [10]), but is not exactly the same. In the free self-adjunction $\mathcal{K}_c$ of the $K$ type (corresponding to Temperley-Lieb algebras), the ordinals in $\varepsilon_0$ of $\mathcal{L}_\omega$ also collapse to natural numbers, and are not tied to particular regions of the frieze. This is analogous to what we have with separable matrix Frobenius monads, but is not exactly the same. The difference is that for $\mathcal{K}_c$ all circles
are counted, while here we count circles tied to $\varepsilon_A^\circ \cdot \varepsilon_A^\circ$, which may be moved according to the equation $(\Phi)$ or $(\Phi^0)$, and do not count circles tied to $\delta_A^\circ \cdot \delta_A^\circ$, according to the separability equation. We will deal with these matters in more detail on another occasion.

Let us sum up matters from the preceding section and the present one. We know that the equation $(\Phi)$ is necessary for the existence of a faithful monoidal functor $F$ into the category $Mat$. We do not know whether $(\Phi)$ is sufficient. If it were, then we could legitimately call Frobenius monads that satisfy $(\Phi)$ matrix Frobenius monads. We know on the other hand that $(\Phi)$ together with the separability equation is sufficient for the existence of such an $F$, but we do not know whether the separability equation is necessary, though this necessity does not seem likely. Since ordinals in separable matrix Frobenius monads have collapsed to natural numbers, with these monads we reach the boundary we set ourselves for this paper, where we wanted to investigate the role of bigger ordinals in Frobenius monads.

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