Poisson–Lie transformations and Generalized Supergravity Equations

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Abstract

In this paper we investigate Poisson–Lie transformation of dilaton and vector field $J$ appearing in Generalized Supergravity Equations. While the formulas appearing in literature work well for isometric sigma models, we present examples for which Generalized Supergravity Equations are not preserved. Therefore, we suggest modification of these formulas.

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1 Introduction

Formula for Poisson–Lie transformation [I] of dilaton field accompanying sigma model background was given long ago in [2]. Its limitations discussed in [3] concern the problem of possible appearance of unwanted "dual" coordinates of Drinfel’d double in the transformed dilaton. The problem was partially solved in [4, 5] for special cases where transformations of relevant coordinates of Drinfel’d double are linear. The price we had to pay was that in order to keep invariance with respect to Poisson–Lie transformations we had to replace the beta function equations by Generalized Supergravity Equations [6] containing not only dilaton, but also additional vector fields for which correct transformation formulas need to be found. Unfortunately, there are cases for which the transformation of relevant coordinates is not linear or the Poisson–Lie formulas do not provide solutions of Generalized Supergravity Equation.

In the following we have chosen several examples for which the problem of unwanted coordinates in the transformed dilatons does not appear, and still, it turns out that the original formula [5, 7, 8, 9] for Killing vector field \( J \), which works well for isometric initial sigma models, fails. These are the cases where the initial sigma models are constructed from Manin triple \((\mathfrak{d}, \mathfrak{g}, \tilde{\mathfrak{g}})\) where \(\tilde{\mathfrak{g}}\) is neither Abelian nor unimodular. The purpose of this note is to extend the validity of Poisson–Lie formulas to these cases. Beside that
for NS-NS Generalized Supergravity Equations it is not necessary to have formulas for both dilaton and Killing vector \( J \) and one only needs formula for Poisson–Lie transformation of X-form that combines these two. It is given as well.

## 2 Basics of Poisson–Lie T-plurality

Here we shall recapitulate well known basics of Poisson–Lie T-plurality with spectators [1][2][10] to establish notation.

Sigma models in curved background are given by Lagrangian

\[
\mathcal{L} = \partial_\mu \phi^\mu \mathcal{F}_{\mu\nu}(\phi) \partial_\nu \phi^\nu, \quad \phi^\mu = \phi^\mu(\sigma_+, \sigma_-), \quad \mu = 1, \ldots, n + d
\]

where tensor field \( \mathcal{F} = \mathcal{G} + \mathcal{B} \) defines metric and torsion potential (Kalb–Ramond field) of the target manifold \( \mathcal{M} \).

Assume that there is \( d \)-dimensional Lie group \( \mathcal{G} \) with free action on \( \mathcal{M} \). The action of \( \mathcal{G} \) is transitive on its orbits, hence we may locally consider \( \mathcal{M} \approx (\mathcal{M}/\mathcal{G}) \times \mathcal{G} = \mathcal{N} \times \mathcal{G} \), and introduce adapted coordinates

\[
\{x^\mu\} = \{s^\alpha, x^a\}, \quad \alpha = 1, \ldots, n = \text{dim } \mathcal{N}, \quad a = 1, \ldots, d = \text{dim } \mathcal{G}
\]

where \( x^a \) are group coordinates and \( s^\alpha \) label the orbits of \( \mathcal{G} \). \( s^\alpha \) are treated as spectators in Poisson–Lie transformations.

Poisson–Lie duality/plurality is based on the possibility to pass between various decompositions of Lie algebra of Drinfel’d double \( D = (\mathcal{G} | \tilde{\mathcal{G}}) \) into Manin triples \((\mathfrak{d}, \mathfrak{g}, \tilde{\mathfrak{g}})\). Poisson–Lie dualizable sigma models on \( \mathcal{N} \times \mathcal{G} \) are given by tensor field \( \mathcal{F} \) of the form

\[
\mathcal{F}(s, x) = \mathcal{E}(x) \cdot (1_{n+d} + E(s) \cdot \Pi(x))^{-1} \cdot E(s) \cdot \mathcal{E}^T(x).
\]

\( E(s) \) is spectator-dependent \((n + d) \times (n + d)\) matrix, \( \Pi(x) \) is given by submatrices \( a(x) \) and \( b(x) \) of the adjoint representation as

\[
ad_{\mathfrak{g}^{-1}}(\tilde{T}) = b(x) \cdot T + a^{-1}(x) \cdot \tilde{T},
\]

\[
\Pi(x) = \begin{pmatrix}
0_n & 0 \\
0 & b(x) \cdot a^{-1}(x)
\end{pmatrix},
\]

and matrix \( \mathcal{E}(x) \) reads

\[
\mathcal{E}(x) = \begin{pmatrix}
1_n & 0 \\
0 & e(x)
\end{pmatrix}
\]
where $e(x)$ is $d \times d$ matrix of components of right-invariant Maurer–Cartan form $(dg)g^{-1}$ on $\mathcal{G}$.

For many Drinfel’d doubles several decompositions may exist. Suppose that we have sigma model on $\mathcal{N} \times \mathcal{G}$ and the Drinfel’d double splits into another pair of subgroups $\hat{\mathcal{G}}$ and $\bar{\mathcal{G}}$. Then we can apply the full framework of Poisson–Lie T-plurality [1, 2] and find background for sigma model on $\mathcal{N} \times \hat{\mathcal{G}}$.

Let Manin triples $(\mathfrak{d}, \mathfrak{g}, \tilde{\mathfrak{g}})$ and $(\mathfrak{d}, \hat{\mathfrak{g}}, \bar{\mathfrak{g}})$ be two decompositions of $\mathfrak{d}$ into double cross sum of subalgebras that are maximally isotropic with respect to non-degenerate symmetric bilinear form $\langle ., . \rangle$ on the Lie algebra of Drinfel’d double. Pairs of mutually dual bases $T_a \in \mathfrak{g}$, $\tilde{T}_a \in \tilde{\mathfrak{g}}$, $\hat{T}_a \in \hat{\mathfrak{g}}$, $\bar{T}_a \in \bar{\mathfrak{g}}$, $a = 1, \ldots, d$, then must be related by transformation

\[
\begin{pmatrix} \hat{T} \\ \bar{T} \end{pmatrix} = C \begin{pmatrix} T \\ \tilde{T} \end{pmatrix} \quad (4)
\]

where $C$ is an invertible $2d \times 2d$ matrix. For the following formulas it will be convenient to introduce $d \times d$ matrices $P, Q, R, S$ as

\[
\begin{pmatrix} T \\ \tilde{T} \end{pmatrix} = C^{-1} \begin{pmatrix} \tilde{T} \\ \hat{T} \end{pmatrix} = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} \hat{T} \\ \bar{T} \end{pmatrix} \quad (5)
\]

and extend these to $(n + d) \times (n + d)$ matrices

\[P = \begin{pmatrix} 1_n & 0 \\ 0 & P \end{pmatrix}, \quad Q = \begin{pmatrix} 0_n & 0 \\ 0 & Q \end{pmatrix}, \quad R = \begin{pmatrix} 0_n & 0 \\ 0 & R \end{pmatrix}, \quad S = \begin{pmatrix} 1_n & 0 \\ 0 & S \end{pmatrix}\]

to accommodate the spectator fields.

The sigma model on $\mathcal{N} \times \hat{\mathcal{G}}$ obtained from (3) via Poisson–Lie T-plurality is given by tensor field

\[
\hat{\mathcal{F}}(s, \hat{x}) = \hat{\mathcal{E}}(\hat{x}) \cdot \hat{\mathcal{E}}(s, \hat{x}) \cdot \hat{\mathcal{E}}^T(\hat{x}), \quad \hat{\mathcal{E}}(\hat{x}) = \begin{pmatrix} 1_n & 0 \\ 0 & \hat{e}(\hat{x}) \end{pmatrix}, \quad (6)
\]

where $\hat{e}(\hat{x})$ is $d \times d$ matrix of components of right-invariant Maurer–Cartan form $(d\hat{g})\hat{g}^{-1}$ on $\hat{\mathcal{G}}$ and

\[
\hat{\mathcal{E}}(s, \hat{x}) = \begin{pmatrix} 1_{n + d} + \hat{\mathcal{E}}(s) \cdot \hat{\Pi}(\hat{x}) \end{pmatrix}^{-1} \cdot \hat{\mathcal{E}}(s) = \begin{pmatrix} \hat{E}^{-1}(s) + \hat{\Pi}(\hat{x}) \end{pmatrix}^{-1}. \quad (7)
\]
The matrix $\hat{E}(s)$ is obtained from $E(s)$ in (8) by formula

$$\hat{E}(s) = (P + E(s) \cdot R)^{-1} \cdot (Q + E(s) \cdot S), \quad (8)$$

and

$$\hat{\Pi}(\hat{x}) = \begin{pmatrix} 0_n & 0 \\ 0 & \hat{b}(\hat{x}) \cdot \hat{a}^{-1}(\hat{x}) \end{pmatrix},$$

$$ad_{\hat{g}^{-1}}(\hat{T}) = \hat{b}(\hat{x}) \cdot \hat{\Phi} + \hat{a}^{-1}(\hat{x}) \cdot \hat{R}.$$ 

Conformal invariance up to the first loop requires introduction of dilaton field $\Phi$ satisfying beta function equations

$$0 = R_{\mu \nu} - \frac{1}{4} H_{\mu \rho \sigma} H^{\rho \sigma}_{\nu} + 2 \nabla_{\mu} \nabla_{\nu} \Phi, \quad (9)$$

$$0 = -\frac{1}{2} \nabla^{\rho} H_{\rho \mu \nu} + \nabla^{\rho} \Phi H_{\rho \mu \nu}, \quad (10)$$

$$0 = R - \frac{1}{12} H_{\rho \sigma \tau} H^{\rho \sigma \tau} + 4 \nabla_{\mu} \nabla_{\mu} \Phi - 4 \nabla_{\mu} \Phi \nabla_{\mu} \Phi \quad (11)$$

where

$$H_{\rho \mu \nu} = \frac{1}{2} (\partial_{\rho} B_{\mu \nu} + \partial_{\mu} B_{\nu \rho} + \partial_{\nu} B_{\rho \mu}),$$

and $\nabla_{\mu}$ are covariant derivatives with respect to metric $G$.

Formula for transformation of dilaton under Poisson–Lie T-plurality was given in [2] as

$$\hat{\Phi}(\hat{x}) = \Phi(y) + \frac{1}{2} L(y) - \frac{1}{2} \hat{L}(\hat{x}) \quad (12)$$

where $y$ represent coordinates of group $G$, $\Phi(y)$ is the dilaton of the initial model, and terms $L(y)$, $\hat{L}(\hat{x})$ read

$$L(y) = \ln \left| \det \left[ (1 + \Pi(y)E(s)) a(y) \right] \right|,$$

$$\hat{L}(\hat{x}) = \ln \left| \det \left[ (1 + \hat{\Pi}(\hat{x})\hat{E}(s)) N^{-1} \hat{a}(\hat{x}) \right] \right|,$$

with

$$N = P^T - R^T E(s).$$
The relation between original and new dilaton can be equivalently written as in [7, 8] as

\[ \exp(-2\Phi^{(0)}(y)) := \exp(-2\Phi(y)) \frac{(\det G(y))^{1/2}}{\det u(y)} \]

\[ = \exp(-2\tilde{\Phi}^{(0)}(\hat{x})) := \exp(-2\tilde{\Phi}(\hat{x})) \frac{(\det \tilde{G}(\hat{x}))^{1/2}}{\det \hat{u}(\hat{x})} \]  \hspace{1cm} (13)

where \( G \) and \( \tilde{G} \) are metrics of sigma models on \( \mathcal{G} \) resp. \( \hat{\mathcal{G}} \) and \( u, \hat{u} \) are corresponding matrices of components of left-invariant forms.

3 Poisson–Lie transformations and Generalized Supergravity Equations

In the case that the initial dilaton \( \Phi(y) \) depends on coordinates \( y_k \), we have to express these in terms of \( \hat{x}, \bar{x} \) to get explicit form of transformed dilaton. This can be done using relation between two different decompositions of Drinfel’d double elements

\[ g(y)\hat{h}(\tilde{y}) = \tilde{g}(\hat{x})\bar{h}(\bar{x}), \quad g \in \mathcal{G}, \tilde{h} \in \tilde{\mathcal{G}}, \tilde{g} \in \tilde{\mathcal{G}}, \bar{h} \in \bar{\mathcal{G}}. \]  \hspace{1cm} (14)

The origin of the puzzle discussed in [3] is that if

\[ \frac{\partial y^k}{\partial \bar{x}_j} \neq 0, \]

formulas (12), (13) give \( \tilde{\Phi} \) that may depend not only on coordinates \( \hat{x} \) of the group \( \hat{\mathcal{G}} \) but also on coordinates \( \bar{x} \) of \( \bar{\mathcal{G}} \).

Partial solution of this problem was given in [5] for the case of linear dependence

\[ y^k(\hat{x}, \bar{x}) = \hat{d}^k_j \hat{x}^j + \bar{d}^{kj} \bar{x}_j. \]  \hspace{1cm} (15)

It was suggested that in this case we can set \( y^k = \hat{d}^m_m \hat{x}^m \) in the formula (12) and extend the beta function equations to (NS-NS) Generalized Supergravity.
Equations [6, 11]

\[ 0 = R_{\mu\nu} - \frac{1}{4} H_{\rho\sigma} H^\rho_H^\sigma + \nabla_\mu X_\nu + \nabla_\nu X_\mu, \quad (16) \]

\[ 0 = -\frac{1}{2} \nabla^\rho H_{\rho\mu\nu} + X^\rho H_{\rho\mu\nu} + \nabla_\mu X_\nu - \nabla_\nu X_\mu, \quad (17) \]

\[ 0 = R - \frac{1}{12} H_{\rho\sigma\tau} H^\rho_H^\sigma_H^\tau + 4 \nabla_\mu X^\mu - 4 X_\mu X^\mu \quad (18) \]

where

\[ X_\mu = \partial_\mu \Phi + J^\nu F_{\nu\mu}, \quad (19) \]

\( \Phi \) is the dilaton and \( J \) is a vector field. For vanishing \( J \) the usual beta function equations are recovered.

Authors of [5, 9] give formula allowing to find components of \( \hat{J} \) for Poisson–Lie transformed sigma model as

\[ \hat{J}^\alpha = 0, \quad \alpha = 1, \ldots, n = \dim \mathcal{N}, \]

\[ \hat{J}^{\dim \mathcal{N} + m}(\hat{x}) = \left( \frac{1}{2} \tilde{f}_{ab}^c b - \frac{\partial \Phi^0(y)}{\partial y_k} |_{y=\hat{D} \cdot \hat{x}} \tilde{f}_{ka} \right) \hat{V}_a^m(\hat{x}), \quad (20) \]

where \( a, b, k, m = 1, \ldots, \dim \mathcal{G} \),

\[ \hat{D} = \begin{pmatrix} 1_n & 0 \\ 0 & \hat{d} \end{pmatrix}, \quad (21) \]

\( \hat{V}_a \) are left-invariant fields of the group \( \hat{G} \), and \( \tilde{f}_{ba}^c \) are structure constants of the Lie algebra of \( \hat{G} \).

From the form of equations (16)–(18) one can see that only one-form \( X \) is important for their satisfaction, not separately dilaton \( \Phi \) and vector field \( J \). Therefore, for Poisson–Lie transformation of Generalized Supergravity Equations it would be sufficient to know only Poisson–Lie transformation of the one-form \( X \) and of the tensor \( F \).

Note that \( \Phi \) and \( J \) are not defined uniquely as the form \( X \) is invariant with respect to gauge transformation

\[ \Phi(x) \mapsto \Phi(x) + \lambda(x), \quad J^\kappa \mapsto J^\kappa - \partial_\nu \lambda F^{\nu \kappa}. \quad (22) \]

This means that we can always choose dilaton vanishing. On the other hand, if \( X \) is closed, we can choose \( X = d\Phi \) and \( J \) vanishing in which case Generalized Supergravity Equations of Motion become usual beta function equations.
Moreover, note that even the form $X$ satisfying equations (16)–(18) is not unique. Namely, if $X_\mu$ satisfy the Generalized Supergravity Equations, then

$$X'_\mu := X_\mu + \chi_\mu, \quad (23)$$

where

$$\nabla_\nu \chi_\mu = 0, \quad (X_\mu + 2 \chi_\mu) \chi^\mu = 0, \quad (24)$$

satisfy the equations as well. Simple example that was mentioned in [13] is sigma model given by flat Minkowski metric and

$$X_\mu = (0, 0, 0, 0), \quad \chi_\mu = \left(0, 0, 0, \frac{1}{1 - x_3^2}\right). \quad (24)$$

In the following we verify whether backgrounds obtained by Poisson–Lie T-plurality supported by $\hat{\Phi}$ and $\hat{J}$ obtained from (12) and (20) satisfy Generalized Supergravity Equations and present examples where this is not true. In all our examples $\partial y^k/\partial \hat{x}_j = 0$ so it is not necessary to solve the equation (14). It turns out that formula (20) works well for isometric initial sigma models, but not always as we will show in the following examples. Therefore, it is desirable to modify the prescription for vector fields $\hat{J}$. It seems that appropriate formula is

$$\hat{J}^{\text{dim.} N + m}(\hat{x}) = \frac{1}{2} \tilde{f}^{ab}_{c} \left( \frac{\partial y_a}{\partial \hat{x}_k} \hat{V}^m_k(\hat{x}) - \frac{\partial y_k}{\partial \hat{x}^a} \hat{F}^{km} \right) + \left( \frac{1}{2} \tilde{f}^{ab}_{c} \frac{\partial \Phi^0}{\partial y^k} \bigg|_{y = \hat{D}^{-1} \hat{x}} \right) \hat{V}^m_a(\hat{x}) \quad (25)$$

where $\hat{V}_a$ are left-invariant fields of the group $\hat{G}$, $\tilde{f}^{ba}_{c}$ and $\tilde{f}^{ba}_{c}$ are structure constants of Lie algebras of $\hat{G}$, $\tilde{G}$ and $\hat{D}$ is given by (21). This modification does not change results of [5] and [9] because those papers deal with groups for which $\tilde{f}^{ab}_{c} = 0$. Finally let us mention that $\hat{J}$ obtained from (25) is not always Killing of $\hat{F}$ (which is not necessary for satisfaction of the NS-NS Generalized Supergravity Equations) but we can use the gauge transformation (23) to acquire this property.

Having formulas (12) and (25) for dilatons and vector fields $\hat{J}$ it is easy to write down formula for Poisson–Lie transformation of the form $X$

$$\hat{X}_\mu(\hat{x}) = \frac{\partial \Phi^0}{\partial y^\nu} \bigg|_{y = \hat{D}^{-1} \hat{x}} \frac{\partial y^\nu}{\partial \hat{x}_\mu} + \frac{1}{2} \frac{\partial \hat{L}(\hat{x})}{\partial \hat{x}_\mu} + \hat{J}^\nu(\hat{x}) \hat{F}_{\nu \mu}(\hat{x}) \quad (26)$$

8
where $\Phi^0(y) = \Phi(y) + \frac{1}{2} L(y)$, and $\widehat{J}^{\nu}(\hat{x})$ are given by (25). Advantage of the formula (26) is that $\hat{X}$, differently from $\hat{\Phi}$ and $\hat{J}$, is invariant with respect to the gauge transformation (22).

4 Examples

In this paper the groups $\mathcal{G}$ will be non-semisimple Bianchi groups. Their elements will be parametrized as $g = e^{x_1T_1}e^{x_2T_2}e^{x_3T_3}$ where $e^{x_2T_2}e^{x_3T_3}$ and $e^{x_3T_3}$ parametrize their normal subgroups. We will deal with backgrounds on four-dimensional manifolds, hence $\dim N = 1$ and we denote the spectator $s_1$ as $t$.

4.1 Poisson–Lie plurality on Drinfel’d double $(1|5)$

We shall start our discussion with tensor field

\[
\mathcal{F}(t, y) = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & \frac{-t^2}{t^4 + y_2^2 + y_3^2} & \frac{-y_2}{t^4 + y_2^2 + y_3^2} & \frac{-y_3}{t^4 + y_2^2 + y_3^2} \\
0 & -\frac{y_2}{t^4 + y_2^2 + y_3^2} & \frac{-t^2}{t^4 + y_2^2 + y_3^2} & -\frac{y_2 y_3}{t^4 + y_2^2 + y_3^2} \\
0 & -\frac{y_3}{t^4 + y_2^2 + y_3^2} & -\frac{y_2 y_3}{t^4 + y_2^2 + y_3^2} & \frac{-t^2}{t^4 + y_2^2 + y_3^2}
\end{pmatrix}
\]

specifying sigma model on Abelian group $\mathcal{G}$ with corresponding Drinfel’d double $\mathcal{D} = (\mathcal{G}|\hat{\mathcal{G}}) = (1|5)$\footnote{For typographic reasons we write coordinate indices as subscripts.} whose non-trivial commutation relations read

\[
[\hat{T}^1, \hat{T}^2] = \hat{T}^2, \quad [\hat{T}^1, \hat{T}^3] = \hat{T}^3, \quad [\hat{T}^1, T_2] = -T_2, \quad [\hat{T}^1, T_3] = -T_3, \quad [\hat{T}^2, T_2] = T_1, \quad [\hat{T}^3, T_3] = T_1.
\]

Background (27), dilaton

\[
\Phi(t, y) = -\frac{1}{2} \ln (-t^2 (t^4 + y_2^2 + y_3^2))
\]

and Killing vector $\mathcal{J} = 2\partial_{\hat{x}}$, satisfy Generalized Supergravity Equations. Corresponding X-form with components

\[
X_\mu(t, y) = \frac{1}{t^4 + y_2^2 + y_3^2} \left( -\frac{3t^4 + y_2^2 + y_3^2}{t}, 2t^2, y_2, y_3 \right)
\]
is not closed and we cannot get rid of the vector $\mathcal{J}$ by gauge transformations, so reduction of Generalized Supergravity Equations to beta function equations is not possible.

Background (27) is actually non-Abelian dual of flat background

$$\hat{\mathcal{F}}(t, \hat{x}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & t^2 & 0 & 0 \\ 0 & 0 & e^{2\hat{x}_1}t^2 & 0 \\ 0 & 0 & 0 & e^{2\hat{x}_1}t^2 \end{pmatrix}$$ (31)

studied frequently in the literature [14, 15, 16]. $\hat{\mathcal{F}}$ is invariant with respect to action of Bianchi 5 group and dilaton (29) and $\mathcal{J}$ were obtained via (12) and (20).

Let us note that $\Phi^0(t, y) = -\frac{1}{2} \ln t^3$ and it is not necessary to solve equation (14) for $y$ to get transformed dilatons $\hat{\Phi}(\hat{x})$. Similar results can be obtained starting from Drinfel’d double (1|3).

4.1.1 Identity $(1|5) \to (1|5)$ and full duality $(1|5) \to (5|1)$

Let us check formulas (12) and (20) applying Poisson–Lie transformation with $C$ equal to identity matrix to (27) and (29). We recover the original background and dilaton, but vector field $\hat{\mathcal{J}} = \partial_{\hat{x}_1}$ obtained from (20) is different from the initial one and Generalized Supergravity Equations are not satisfied even in this simple case. Using (25) instead of (20) we get back Killing vector $\hat{\mathcal{J}} = 2\partial_{\hat{x}_1}$ and Generalized Supergravity Equations are satisfied.

By full duality given by

$$C = D_0 := \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix}$$ (32)

we get flat background (31), but dilaton

$$\hat{\Phi}(t, \hat{x}) = \frac{1}{2} \ln (e^{2\hat{x}_1})$$ (33)

and vanishing vector $\hat{\mathcal{J}}$ obtained from formulas (12) and (20) do not satisfy Generalized Supergravity Equations.
On the other hand, equations (16)–(18) are satisfied for \((31), (33)\) and

\[ \hat{J}(t, \hat{x}) = -\frac{1}{t^2} \partial_{\hat{x}_1} \] (34)

that follows from (25). Corresponding X-form \(\hat{X}\) vanishes, and by gauge transformation it is possible to eliminate \(\hat{J}\) while changing dilaton to

\[ \hat{\Phi}'(t, \hat{x}) = -c_1 t e^{-\hat{x}_1} + c_2 \] (35)

with \(c_1, c_2\) arbitrary constants. Dilaton (35) and flat background (31) satisfy beta function equations.

\subsection*{4.1.2 Plurality \((1|5) \rightarrow (6_{-1}|ii5)\)}

By Poisson–Lie plurality given by

\[ C_{(1|5) \rightarrow (6_{-1}|ii5)} = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \] (36)

we get background tensor

\[ \hat{F}(t, \hat{x}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{t^2 e^{2\hat{x}_1} (\hat{x}_3^2 + 1)}{t^4 + e^{2\hat{x}_1} (\hat{x}_3^2 + 3)} & \frac{t^4}{t^4 + e^{2\hat{x}_1} (\hat{x}_3^2 + 1)} \\ 0 & \frac{t^4}{t^4 + e^{2\hat{x}_1} (\hat{x}_3^2 + 1)} & \frac{t^2 e^{2\hat{x}_1} \hat{x}_3}{t^4 + e^{2\hat{x}_1} (\hat{x}_3^2 + 1)} \\ 0 & -\frac{t^2 e^{2\hat{x}_1} \hat{x}_3}{t^4 + e^{2\hat{x}_1} (\hat{x}_3^2 + 1)} & \frac{t^4}{t^4 + e^{2\hat{x}_1} (\hat{x}_3^2 + 1)} \end{pmatrix} \] (37)

and dilaton

\[ \hat{\Phi}(t, \hat{x}) = -\frac{1}{2} \ln \left( t^2 e^{-2\hat{x}_1} \Delta \right), \quad \Delta = t^4 + e^{2\hat{x}_1} (\hat{x}_3^2 + 1). \] (38)

Generalized Supergravity Equations are satisfied for

\[ \hat{J}(t, \hat{x}) = \frac{1}{t^2} \partial_{\hat{x}_1} \] (39)
calculated via (25). Vector field \( \tilde{J} = \partial_{\tilde{x}_2} \) obtained from formula (20) does not satisfy Generalized Supergravity Equations.

Correct X-form with components
\[
\hat{X}_\mu(t, \hat{x}) = \frac{1}{\Delta} \left( -\frac{3t^4 + e^{2\hat{x}_1}(\hat{x}_3^2 + 1)}{t}, e^{2\hat{x}_1}(\hat{x}_3^2 + 1), 2t^2, e^{2\hat{x}_1}\hat{x}_3 \right)
\] (40)
is not closed and Generalized Supergravity Equations cannot be reduced to beta function equations. Beside that, vector field (39) is not Killing of (37). However, using the gauge transformation (23) with \( \lambda = \hat{x}_1 \) we get
\[
\hat{\Phi}'(t, \hat{x}) = \hat{\Phi}(t, \hat{x}) + \hat{x}_1
\]
and
\[
\hat{J}'(t, \hat{x}) = 2\partial_{\hat{x}_2}
\]
that is Killing vector field of (37). X-form remains unchanged, of course.

### 4.2 Poisson–Lie plurality on Drinfel’d double (1|4)

Next we shall investigate plural sigma models on Drinfel’d double (1|4) with commutation relations
\[
[\tilde{T}^1, \tilde{T}^2] = -\tilde{T}^3, \quad [\tilde{T}^1, \tilde{T}^3] = -\tilde{T}^3, \quad [\tilde{T}^1, T^2] = T^2, \quad [\tilde{T}^1, T^3] = T^3, \quad [\tilde{T}^2, T^3] = T^1.
\] (41)

Background
\[
\mathcal{F}(t, y) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1-y_3}{y_3-1} \\
0 & 0 & 1 & \frac{y_3-1}{y_3+1} \\
0 & \frac{y_3-y_2}{y_3+1} & \frac{(y_2-y_3)^2}{y_3^2-1} & 0
\end{pmatrix}
\] (42)
on Abelian group \( \mathcal{G} \) was obtained as non-abelian T-dual of flat background
\[
\hat{\mathcal{F}}(t, \hat{x}) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & e^{-\hat{x}_1} & 0 \\
0 & e^{-\hat{x}_1} & e^{-2\hat{x}_1} & 0 \\
0 & e^{-\hat{x}_1} & 0 & 0
\end{pmatrix}
\] (43)
that is invariant with respect to the action of Bianchi 4 group.

Background (42), dilaton
\[
\Phi(t, y) = -\frac{1}{2} \ln \left( 1 - y_3^2 \right)
\] (44)
and Killing vector \( J = -2\partial_{\hat{x}_1} \) satisfy Generalized Supergravity Equations. Since \( \Phi^0(y) = 0 \), we once again do not need to solve (14).
4.2.1 Identity $(1|4) \rightarrow (1|4)$

To check formulas (20) and (25) we apply Poisson–Lie transformation with $C$ equal to identity matrix to (42) and (44). We get the original background and dilaton. Formula (20) gives vector $-\partial_{\hat{x}_1}$, while from (25) we obtain Killing vector $\hat{J} = -2\partial_{\hat{x}_1}$. For the former one Generalized Supergravity Equations are not satisfied, for the latter they hold.

Corresponding X-form

$$\hat{X}(t, \hat{x}) = \hat{x}_3 + 2 \frac{d\hat{x}_3}{\hat{x}_3^2 - 1}$$

(45)

is closed and we can pass to dilaton

$$\hat{\Phi}'(t, \hat{x}) = \frac{3}{2} \ln(1 - \hat{x}_3) - \frac{1}{2} \ln(\hat{x}_3 + 1)$$

(46)

that together with (42) satisfies beta function equations.

4.2.2 Full duality $(1|4) \rightarrow (4|1)$

By full duality $(1|4) \rightarrow (4|1)$ we get flat background (43), but non-trivial dilaton

$$\hat{\Phi}(t, \hat{x}) = \frac{1}{2} \ln \left( e^{-2\hat{x}_1} \right)$$

(47)

obtained from (12) and vanishing vector field $\hat{J}$ obtained from (20) do not satisfy Generalized Supergravity Equations. Correct vector field for which these equations are satisfied is

$$\hat{J}(t, \hat{x}) = e^{\hat{x}_1} \partial_{\hat{x}_3}$$

(48)

and follows from (25). Corresponding X-form vanishes and using gauge transformation (22) we can get $\hat{J}' = 0$ and dilaton $\hat{\Phi}' = 0$ satisfying beta function equations.
4.2.3 Plurality \((1|4) \rightarrow (6_{-1} | ii4)\)

Changing the decomposition of Drinfel’d double to \((6_{-1} | ii4)\) using matrix

\[
C_{(1|4)\rightarrow(6_{-1} | ii4)} = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 \\
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -2 & 0 \\
0 & -\frac{1}{2} & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\]

we get background

\[
\hat{F}(t, \hat{x}) = \begin{pmatrix}
1 & 0 & 0 & -\frac{\hat{e}^{2\hat{x}_1}(\hat{x}_1 + 2\hat{x}_3)}{\hat{e}^{2\hat{x}_1} - 4} & 0 & 0 \\
0 & -\frac{\hat{e}^{2\hat{x}_1}}{\hat{e}^{2\hat{x}_1} - 4} & -\frac{\hat{e}^{2\hat{x}_1}}{\hat{e}^{2\hat{x}_1} - 4} & -\frac{\hat{e}^{2\hat{x}_1}(\hat{x}_1 + 2\hat{x}_3)}{\hat{e}^{2\hat{x}_1} - 4} & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & \frac{\hat{e}^{2\hat{x}_1}(\hat{x}_1 + 2\hat{x}_3)}{\hat{e}^{2\hat{x}_1} - 4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

and dilaton

\[
\hat{\Phi}(t, \hat{x}) = \frac{1}{2} \ln \left(-\frac{2\hat{e}^{2\hat{x}_1}}{\hat{e}^{2\hat{x}_1} - 4}\right).
\]

Generalized Supergravity Equations are satisfied for background \((50)\), dilaton \((51)\) and vector field

\[
\hat{J}(t, \hat{x}) = \left(1 - \frac{\hat{e}^{\hat{x}_1}}{2}\right) \partial_{\hat{x}_2}
\]

obtained from \((25)\). X-form corresponding to \((51)\) and \((52)\)

\[
\hat{X}(t, \hat{x}) = \left(\frac{\hat{e}^{\hat{x}_1}(\hat{e}^{\hat{x}_1} - 4)}{e^{2\hat{x}_1} - 4}\right) d\hat{x}_1
\]

is closed so we can eliminate \(\hat{J}\) by gauge transformation. Dilaton then reads

\[
\hat{\Phi}'(t, \hat{x}) = \frac{1}{2} \ln \left(\frac{(2 + e^{\hat{x}_1})^3}{2 - e^{\hat{x}_1}}\right)
\]

and satisfies beta function equations together with \((50)\).
4.2.4 Plurality \((1|4) \rightarrow (ii4|6_{-1})\)

Plurality given by matrix \(C_{(1|4)\rightarrow(ii4|6_{-1})} = D_0 \cdot C_{(1|4)\rightarrow(6_{-1}|ii4)}\) gives flat and torsionless background

\[
\tilde{\mathcal{F}}(t, \hat{x}) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 \\
0 & 1 & \frac{1-3e^{\hat{x}_2}}{e^{\hat{x}_2+1}} & 0 \\
0 & 0 & \frac{2e^{\hat{x}_2+3e^{\hat{x}_2}-1}}{2-6e^{\hat{x}_2}} & \frac{e^{\hat{x}_2}+2e^{3\hat{x}_2-\hat{x}_2}}{e^{2\hat{x}_2}}
\end{pmatrix}
\] (55)

and dilaton

\[
\tilde{\Phi}(t, \hat{x}) = \frac{1}{2} \ln \left( \frac{2e^{2\hat{x}_2}}{-2e^{\hat{x}_2} - 3e^{2\hat{x}_2} + 1} \right).
\] (56)

Generalized Supergravity Equations are satisfied for (55), (56) and Killing vector field

\[
\tilde{\mathcal{J}}(t, \hat{x}) = -\frac{1}{3} \partial_{\hat{x}_1}.
\] (57)

X-form corresponding to (56) and (57)

\[
\tilde{X}(t, \hat{x}) = \frac{2e^{\hat{x}_2}}{(e^{\hat{x}_2}+1)(3e^{\hat{x}_2}-1)} d\hat{x}_2
\]

is closed and by gauge transformation to dilaton

\[
\tilde{\Phi}'(t, \hat{x}) = \frac{1}{2} \ln \frac{1 - 3e^{\hat{x}_2}}{1 + e^{\hat{x}_2}}.
\] (58)

we get solution of beta function equations.

5 Conclusions

It follows from the examples in Sections 4.1 and 4.2 that formulas (12) and (20) for Poisson–Lie transformations of dilatons and Killing vectors [5, 7, 8, 9] are not universal in the sense that \(\tilde{\Phi}\) and \(\tilde{\mathcal{J}}\) together with transformed backgrounds in general do not satisfy beta function equations nor Generalized Supergravity Equations. They work properly for transformations of isometric sigma models based on semi-abelian Manin triples \((\mathfrak{d}, \mathfrak{g}, \mathfrak{a})\) but not in other cases.
We propose modification (25) of formula (20) giving vector fields which together with dilatons given by formula (12) satisfy Generalized Supergravity Equations for all presented examples (and many other).

From the form of NS-NS sector of Generalized Supergravity Equations of Motion it is clear that knowledge of one-form $X$ is important for their satisfaction, not separately dilaton $\Phi$ and vector field $J$. Therefore, beside the Poisson–Lie transformation of tensor $F$ it is sufficient to know only the transformation of the form $X$ to keep the Generalized Supergravity Equations satisfied. The corresponding formula (26) was checked as well.

In many examples the form $X$ is closed so we can choose $J$ vanishing by gauge transformation (22), and Generalized Supergravity Equations of Motion become usual beta function equations. The same transformation can be used to make $J$ Killing vector field. Resulting dilatons then differ from those obtained from (12).

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