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A mathematical model for COVID-19 transmission by using the Caputo fractional derivative

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\textbf{A B S T R A C T}

We present a mathematical model for the transmission of COVID-19 by the Caputo fractional-order derivative. We calculate the equilibrium points and the reproduction number for the model and obtain the region of the feasibility of system. By fixed point theory, we prove the existence of a unique solution. Using the generalized Adams-Bashforth-Moulton method, we solve the system and obtain the approximate solutions. We present a numerical simulation for the transmission of COVID-19 in the world, and in this simulation, the reproduction number is obtained as $R_0 = 1 : 61007996$, which shows that the epidemic continues.

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1. Introduction

Coronaviruses (CoV) have a distinctive corona or 'crown' of sugary-proteins and because of their appearance, were called Corona Viruses in 1960. Viruses that cause common cold diseases and fatal diseases such as Middle East Respiratory Syndrome (MERS-CoV) and Severe Acute Respiratory Syndrome (SARS-CoV), are from the coronaviruses family. Detailed investigations found that coronaviruses are transmitted between animals and people. Such as SARS-CoV and MERS-CoV that were transmitted from civet cats and dromedary camels to humans respectively. Also, Several known coronaviruses are circulating in animals that have not yet infected humans.

A coronavirus (COVID-19) that was first identified in the Chinese city of Wuhan in 2019, is a new strain that has not been previously identified in humans. Snakes or bats have been suspected as a potential source for the outbreak, though other experts currently consider this as unlikely. Fever, cough, shortness of breath and breathing difficulties are initial symptoms of this infection. In the next steps, the infection can cause pneumonia, severe acute respiratory syndrome, kidney failure and even death.

The study of diseases dynamics is a dominating theme for many biologists and mathematicians (see for example, Haq et al. [13], Koca [14], Rida et al. [22], Singh et al. [24], Tchuenche et al. [25], Upadhyay and Roy [28]). It has been studied by many researchers that fractional extensions of mathematical models of integer order represent the natural fact in a very systematic way such as in the approach of Akbari et al. [1], Baleanu et al. [3–5], Talaei et al. [26] and Qureshi et al. [21]. In recent years, many papers have been published on the subject of Caputo-Fabrizio fractional derivative [18] (see for example, Dokuyucu et al. [12], Koca [14], Khan et al. [15], Ucar et al. [27]). Mathematical models are used to simulate the transmission of Coronavirus (see for example, Atangana [2], Dighe et al. [11], Khan and Atangana [16], Zhou et al. [31]).

Given that the fractional-order derivative is the generalization of the integer-order derivative, and in recent years the fractional-order derivative has produced better results in modelling the real phenomena, so we investigate the transmission model of COVID-19 using the Caputo fractional-order derivative. Also, due to the
prevalence of COVID-19 in most countries, we intend to present a prediction about the spread of COVID-19 by mathematical simulation and using published statistical information. Since one of the solutions in controlling epidemics is to limit the relationships of people in the community, we investigate the impact of the rate of contact between the infected and healthy people on the spread of COVID-19.

The structure of the paper is as follows. In Section 2 some basic definitions and concepts of fractional calculus are recalled. The fractional model for COVID-19 transmission is presented in Section 3. Existence and uniqueness of solution for system is proved in Section 4. In Section 5, numerical method for solving of model is described and numerical results is presented.

2. Preliminary results and definitions

In this section, we recall some of the fundamental concepts of fractional differential calculus, which are found in many books and papers. We introduce three types of fractional derivatives and fractional integrals, the first of which is Caputo.

Definition 2.1. Samko et al. [23] For a integrable function g, the Caputo derivative of fractional order η ∈ (0, 1) is given by

\[ \mathcal{D}_t^\eta g(t) = \frac{1}{\Gamma(1-\eta)} \int_0^t (t-s)^{-\eta} g'(s) ds, \quad m = [\eta] + 1. \]

Also, the corresponding fractional integral of order η with Re(η) > 0 is given by

\[ \mathcal{I}_t^\eta g(t) = \frac{1}{\Gamma(1+\eta)} \int_0^t (t-s)^{\eta-1} g(s) ds. \]

Definition 2.2. Caputo and Fabrizio [6], Losada and Nieto [18] For \( g \in H^d(c, d) \) and \( d > c \), The Caputo-Fabrizio derivative of fractional order \( \eta \in (0, 1) \) for g is given by

\[ \mathcal{D}_t^\eta g(t) = \frac{M(\eta)}{(1-\eta)} \int_0^t (t-s)^{\eta-1} g'(s) ds, \]

where \( t \geq 0, M(\eta) \) is a normalization function that depends on \( t \) and \( M(0) = M(1) = 1. \) If \( g \notin H^d(c, d) \) and \( 0 < \eta < 1, \) this derivative for \( g \in L^1(-\infty, d) \) as given by

\[ \mathcal{D}_t^\eta g(t) = \frac{\eta M(\eta)}{(1-\eta)} \int_{-\infty}^t (g(t) - g(s))ds \cdot \frac{1}{1-\eta} \]

Also, the corresponding Caputo-Fabrizio fractional integral is presented by

\[ \mathcal{I}_t^\eta g(t) = \frac{2(1-\eta)}{2-\eta} g(t) + \frac{2\eta}{(2-\eta)M(\eta)} \int_0^t \frac{g(s) ds}{(t-s)^{\eta}}. \]

The Laplace transform is one of the important tools in the solving of differential equations that are defined below for two kind of fractional derivative.

Definition 2.3. Samko et al. [23] The Laplace transform of Caputo fractional differential operator of order \( \eta \) is given by

\[ \mathcal{L}[\mathcal{D}_t^\eta g(t)](s) = s^n g(t) - \sum_{i=0}^{m-1} s^{n-i-1} g^{(i)}(0), \quad m - 1 < \eta \leq m \in N. \]

Which can also be obtain in the form

\[ \mathcal{L}[\mathcal{D}_t^\eta g(t)] = \frac{s^n L[g(t)] - s^{n-1} g(0) - \cdots - s^{n-m} g^{(m-1)}(0)}{s^{\eta-n}}. \]

3. A mathematical model for the transmission of COVID-19 with Caputo fractional derivative

Chen and colleagues have proposed a transmission model to simulate possible transmission from the source of infection (possibly bats) to human infection [7]. To model the transmission of COVID-19, they assumed that the virus has transmitted to a host by transferring between bats, and then by hunting the host, the virus has reached the seafood market, which is defined as the source of the virus. In this model, people are divided into five categories: susceptible individuals (S), exposed individuals (E), symptomatic infected individuals (I), asymptomatic infected individuals (A), removed individuals (R) including recovered and death individuals. The total population is denoted by \( N, N = S + E + I + A + R \) and the COVID-19 in the reservoir was denoted as \( W \). This model was presented as follows

\[ \begin{align*}
\frac{dS}{dt} &= \Lambda - sM - \beta_S(t)(I(t) + \kappa A(t)) - \beta_S(t) S(t), \\
\frac{dE}{dt} &= \beta_S(t)(I(t) + \kappa A(t)) + \beta_S(t) S(t) - W(t), \\
\frac{dI}{dt} &= (1 - \delta) W(t) - (\gamma + m) I(t), \\
\frac{dA}{dt} &= \delta W(t) - (\gamma + m) A(t), \\
\frac{dR}{dt} &= \gamma I(t) + \gamma A(t) - mR(t), \\
\frac{d\rho}{dt} &= \mu I(t) + \mu A(t) - \epsilon W(t),
\end{align*} \]

where \( \Lambda = n \times N \), \( N \) is the total number of individuals and \( n \) is the birth rate, \( m : \text{the death rate of individuals}, \)

\( \beta_S : \text{The transmission rate from I to S} \),

\( \kappa : \text{Transfer coefficient from A to I} \),

\( \beta_H : \text{The transfer rate from W to S} \),

\( \delta : \text{The proportion of asymptomatic infection rate of individuals} \),

\( \frac{1}{\gamma} : \text{The incubation period of individuals} \),

\( \frac{1}{\tau} : \text{The latent period of individuals} \),

\( \frac{1}{\gamma} : \text{The asymptomatic infection period of individuals} \),

\( \mu : \text{The shedding coefficients from I to W} \),

\( \mu : \text{The shedding coefficients from A to W} \),

\( \frac{1}{\tau} : \text{The lifetime of the virus in W} \).

Also, \( S(0) = S_0, E(0) = E_0, I(0) = I_0, A(0) = A_0, W(0) = W_0 \), are the initial conditions.

In this section, we moderate the system by substituting the time-derivative by the Caputo fractional derivative. With this change, the right and left sides will not have the same dimension. To solve this problem, we use an auxiliary parameter \( \lambda \), having the dimension of sec., to change the fractional operator so that the sides have the same dimension [29]. According to the explanation presented, the corona virus transmission fractional model for \( t > 0 \) and \( \eta \in (0, 1) \) is given by

\[ \begin{align*}
\lambda^{\eta-1} D_{+}^\eta S(t) &= \frac{\Lambda - sM - \beta_S(t)(I(t) + \kappa A(t)) - \beta_S(t) S(t)}{\rho(t - t_0) + \rho(t - t_0)}, \\
\lambda^{\eta-1} D_{+}^\eta E(t) &= \frac{\beta_S(t)(I(t) + \kappa A(t)) + \beta_S(t) S(t)}{\rho(t - t_0) + \rho(t - t_0)}, \\
\lambda^{\eta-1} D_{+}^\eta I(t) &= \frac{(1 - \delta) W(t)}{\rho(t - t_0) + \rho(t - t_0)} - (\gamma + m) I(t), \\
\lambda^{\eta-1} D_{+}^\eta A(t) &= \frac{\delta W(t)}{\rho(t - t_0) + \rho(t - t_0)} - (\gamma + m) A(t), \\
\lambda^{\eta-1} D_{+}^\eta R(t) &= \frac{\gamma I(t) + \gamma A(t)}{\rho(t - t_0) + \rho(t - t_0)} - mR(t), \\
\lambda^{\eta-1} D_{+}^\eta W(t) &= \frac{\mu I(t) + \mu A(t) - \epsilon W(t)}{\rho(t - t_0) + \rho(t - t_0)},
\end{align*} \]

\( \text{(1)} \)

where initial conditions are \( S(0) = S_0, E(0) = E_0, I(0) = I_0, A(0) = A_0, W(0) = W_0 \).

3.1. Non-negative solution

Consider \( \Phi = \left\{ S, E, I, A, R \in R^5_+ : S + E + I + A + R \leq \frac{\rho}{\lambda}, W \in R_+ : W \leq \frac{\mu}{\lambda} \right\} \), we show that the closed set \( \Phi \) is the region of the feasibility of system (1).
Lemma 3.1. The closed set $\Phi$ is positively invariant with respect to fractional system (1).

Proof. To obtain the fractional derivative of the total population, we add all the relations in the system (1). So
\[
\lambda^{\eta-1} D_\tau^\eta N(t) = \Lambda - mN(t),
\]
where $N(t) = S(t) + E(t) + I(t) + A(t) + R(t)$. Using the Laplace transform, we obtain the initial population size as follows
\[
N(t) = N(0)E_\eta(\xi_{\eta,\theta}(-m\lambda^{1-\eta}t^\eta)) + \int_0^t \Lambda \lambda^{1-\eta} \eta^{-1} E_{\eta,\theta}(-m\lambda^{1-\eta} \theta^\eta)d\theta,
\]
where $N(0)$ is the initial population size. With some calculations, we obtain
\[
N(t) = N(0)E_\eta(-m\lambda^{1-\eta}t^\eta) + \int_0^t \Lambda \lambda^{1-\eta} \eta^{-1} \sum_{i=0}^\infty \frac{(-1)^i m \lambda^{1-\eta}}{i! (i+\eta)} d\theta,
\]
\[
= \frac{\Lambda \lambda^{1-\eta}}{m\lambda^{1-\eta}} + E_\eta(-m\lambda^{1-\eta}t^\eta) \left( N(0) - \frac{\Lambda \lambda^{1-\eta}}{m\lambda^{1-\eta}} \right),
\]
\[
= \frac{\Lambda}{m} + E_\eta(-m\lambda^{1-\eta}t^\eta) \left( N(0) - \frac{\Lambda}{m} \right).
\]
Thus, if $N(0) \leq \frac{\Lambda}{m}$, then for $t > 0$, $N(t) \leq \frac{\Lambda}{m}$. As the same way, for $W$ we obtain
\[
\lambda^{\eta-1} D_\tau^\eta W(t) = \mu I(t) + \mu' A(t) - \epsilon W(t),
\]
\[
\leq (\mu + \mu') N - \epsilon W(t),
\]
\[
\leq (\mu + \mu') N - \epsilon W(t),
\]
then
\[
W(t) \leq \frac{(\mu + \mu') \frac{\Lambda}{m}}{\epsilon} + E_\eta(-\epsilon \lambda^{1-\eta}t^\eta) \left( W(0) - \frac{(\mu + \mu') \frac{\Lambda}{m}}{\epsilon} \right).
\]
So, if $W(0) \leq \frac{(\mu + \mu') \frac{\Lambda}{m}}{\epsilon}$, then $W(t) \leq \frac{(\mu + \mu') \frac{\Lambda}{m}}{\epsilon}$. Consequently, the closed set $\Phi$ is positively invariant with respect to fractional model (1). \(\square\)

3.2. Equilibrium points

To determine the equilibrium points of the fractional order system (3), we solve the following equations
\[
D^\eta S(t) = \xi^\eta E(t) = \xi^\eta I(t) = \xi^\eta A(t) = \xi^\eta R(t) = \xi^\eta W(t) = 0.
\]
By solving the algebraic equations we obtain equilibrium points of system. The disease-free equilibrium point, the point where there is no disease, given by $E_0 = (\frac{\Lambda}{m}, 0, 0, 0, 0)$. In addition, if $R_0 > 1$, then the system (1) has a positive endemic equilibrium $E_1 = (S^*, E^*, I^*, A^*, R^*, W^*)$.

$$S^* = \frac{\gamma \delta \epsilon k \beta p y + \gamma' \mu' \beta w \delta y - \beta w \delta \mu oz - \delta \epsilon \omega \beta p z + \beta w \mu oz + \epsilon \omega \beta p \delta y}{x(y' \delta \epsilon k \beta p y + \gamma' \mu' \beta w \delta y - \beta w \delta \mu oz - \delta \epsilon \omega \beta p z + \beta w \mu oz + \epsilon \omega \beta p \delta y)}$$

$$E^* = \frac{\gamma' \delta \epsilon k \beta p y + \gamma' \mu' \beta w \delta y - \beta w \delta \mu oz - \delta \epsilon \omega \beta p z + \beta w \mu oz + \epsilon \omega \beta p \delta y}{x(y' \delta \epsilon k \beta p y + \gamma' \mu' \beta w \delta y - \beta w \delta \mu oz - \delta \epsilon \omega \beta p z + \beta w \mu oz + \epsilon \omega \beta p \delta y)}$$

$$I^* = \frac{\omega (y' \delta \epsilon k \beta p y + \gamma' \mu' \beta w \delta y - \beta w \delta \mu oz - \delta \epsilon \omega \beta p z + \beta w \mu oz + \epsilon \omega \beta p \delta y)}{x(y' \delta \epsilon k \beta p y + \gamma' \mu' \beta w \delta y - \beta w \delta \mu oz - \delta \epsilon \omega \beta p z + \beta w \mu oz + \epsilon \omega \beta p \delta y)}$$

$$A^* = \frac{\delta \gamma (y' \delta \epsilon k \beta p y + \gamma' \mu' \beta w \delta y - \beta w \delta \mu oz - \delta \epsilon \omega \beta p z + \beta w \mu oz + \epsilon \omega \beta p \delta y)}{x(y' \delta \epsilon k \beta p y + \gamma' \mu' \beta w \delta y - \beta w \delta \mu oz - \delta \epsilon \omega \beta p z + \beta w \mu oz + \epsilon \omega \beta p \delta y)}$$

$$R^* = \frac{(y' \delta \epsilon k \beta p y + \gamma' \mu' \beta w \delta y - \beta w \delta \mu oz - \delta \epsilon \omega \beta p z + \beta w \mu oz + \epsilon \omega \beta p \delta y)}{x(y' \delta \epsilon k \beta p y + \gamma' \mu' \beta w \delta y - \beta w \delta \mu oz - \delta \epsilon \omega \beta p z + \beta w \mu oz + \epsilon \omega \beta p \delta y)}$$

$$W^* = \frac{(y' \delta \epsilon k \beta p y + \gamma' \mu' \beta w \delta y - \beta w \delta \mu oz - \delta \epsilon \omega \beta p z + \beta w \mu oz + \epsilon \omega \beta p \delta y)}{x(y' \delta \epsilon k \beta p y + \gamma' \mu' \beta w \delta y - \beta w \delta \mu oz - \delta \epsilon \omega \beta p z + \beta w \mu oz + \epsilon \omega \beta p \delta y)}$$

where $y = (y + m), x = (1 - \delta) a + \delta a + m, z = y' + m$. Also, $R_0$ is the basic reproduction number and is obtained using the next generation method [30]. To find $R_0$, we first consider the system as follows
\[
D^\eta (\Psi(t)) = F(\Psi(t)) - V(\Psi(t)),
\]
where
\[
F(\Psi(t)) = \lambda^{-\eta}
\]
\[
\begin{bmatrix}
\beta p S(t)(I(t) + \kappa A(t)) + \beta w S(t)W(t) \\
0 \\
0 \\
0
\end{bmatrix}
\]
and

\[ V(\Psi(t)) = \lambda^{1-\eta} \begin{bmatrix} (1 - \delta)\omega E(t) + \delta \omega' E(t) + mE(t) \\ -(1 - \delta)\omega E(t) + (\gamma + m)I(t) \\ -\delta \omega' E(t) + (\gamma' + m)A(t) \\ -\mu I(t) - \mu' A(t) + \epsilon W(t) \end{bmatrix} \]

At \( E^0 \), the Jacobian matrix for \( F \) and \( V \) are obtained as

\[ J_F(E^0) = \lambda^{1-\eta} \begin{bmatrix} 0 & N_{3 \times 3} \\ 0 & 0 \end{bmatrix}, \quad N_{3 \times 3} = \begin{bmatrix} \beta_p S & \beta_p kS & \beta_w S \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

and

\[ J_V(E^0) = \lambda^{1-\eta} \begin{bmatrix} (1 - \delta)\omega + \delta \omega' + m & 0 & 0 & 0 \\ -(1 - \delta)\omega & \gamma + m & 0 & 0 \\ -\delta \omega' & 0 & \gamma' + m & 0 \\ 0 & -\mu & -\mu' & \epsilon \end{bmatrix} \]

\( FV^{-1} \) is the next generation matrix for system (1), then the basic reproduction number is

\[ R_0 = \rho(FV^{-1}) = \frac{\beta_p \Lambda (1 - \delta)\omega (m + \gamma') + \beta_p k \Lambda \delta \omega (m + \gamma')}{m(m + \omega)(m + \gamma')(m + \gamma')} + \frac{\beta_w \Lambda \mu \omega (1 - \delta)(m + \gamma') + \beta_w \Lambda \delta \omega \mu'(m + \gamma')}{m(e(m + \omega)(m + \gamma)(m + \gamma')} \]

This basic reproduction number \( R_0 \), is an epidemiologic metric used to describe the contagiousness or transmissibility of infectious agents.

3.3. \( R_0 \) sensitivity analysis

To check the \( R_0 \) sensitivity, we have

\[ \frac{\partial R_0}{\partial \mu} = \frac{\beta_p \Lambda \omega (1 - \delta)}{m(m + \omega)(m + \gamma')} \]

\[ \frac{\partial R_0}{\partial \mu'} = \frac{\beta_p \Lambda \omega \delta}{m(m + \omega)(m + \gamma')} \]

\[ \frac{\partial R_0}{\partial \lambda} = \frac{\Lambda e(1 - \delta)\omega (m + \gamma') + \kappa e \Lambda \delta \omega (m + \gamma')}{m(m + \omega)(m + \gamma')(m + \gamma')} \]

\[ \frac{\partial R_0}{\partial \beta_p} = \frac{\beta_p e (1 - \delta)\omega (m + \gamma') + \beta_p k \Lambda \delta \omega (m + \gamma') + \beta_p \mu \omega (1 - \delta)(m + \gamma') + \beta_p \omega \delta \mu'(m + \gamma')}{m(m + \omega)(m + \gamma')(m + \gamma')} \]

\[ \frac{\partial R_0}{\partial \beta_w} = \frac{\beta_p \Lambda (1 - \delta)\omega (m + \gamma') + \beta_p k \Lambda \delta \omega (m + \gamma')}{m(m + \omega)(m + \gamma')(m + \gamma')} \]

\[ \frac{\partial R_0}{\partial \omega} = \frac{\beta_p \Lambda \epsilon (m + \gamma') + \beta_p k \Lambda \delta \omega (m + \gamma') - \beta_w \Lambda \mu \omega (m + \gamma') + \beta_w \Lambda \omega \mu'(m + \gamma')}{m(m + \omega)(m + \gamma')(m + \gamma')} \]

\[ \frac{\partial R_0}{\partial \delta} = \frac{\beta_p \Lambda \epsilon (m + \gamma') + \beta_p k \Lambda \delta \omega (m + \gamma') - \beta_w \Lambda \mu \omega (m + \gamma') + \beta_w \Lambda \omega \mu'(m + \gamma')}{m(m + \omega)(m + \gamma')(m + \gamma')} \]

\[ \frac{\partial R_0}{\partial \omega} = \frac{\beta_p \Lambda \epsilon (m + \gamma') + \beta_p k \Lambda \delta \omega (m + \gamma') - \beta_w \Lambda \mu \omega (m + \gamma') + \beta_w \Lambda \omega \mu'(m + \gamma')}{m(m + \omega)(m + \gamma')(m + \gamma')} \]

\[ \frac{\partial R_0}{\partial \gamma'} = \frac{\beta_p \Lambda \epsilon (m + \gamma') + \beta_p k \Lambda \delta \omega (m + \gamma') - \beta_w \Lambda \mu \omega (m + \gamma') + \beta_w \Lambda \omega \mu'(m + \gamma')}{m(m + \omega)(m + \gamma')(m + \gamma')} \]

\[ \frac{\partial R_0}{\partial m} = \frac{\beta_p \Lambda \epsilon (m + \gamma') + \beta_p k \Lambda \delta \omega (m + \gamma') - \beta_w \Lambda \mu \omega (m + \gamma') + \beta_w \Lambda \omega \mu'(m + \gamma')}{m^2(m + \omega)(m + \gamma')(m + \gamma')^2} \]
Suppose that $b_1 \leq b_2, \| I(t) \| \leq I_3, \| A(t) \| \leq I_4, \| W(t) \| \leq I_5$. Then $G_1(t, S(t))$ is bounded and $G_1(t, S(t)) = b_1 \| I(t) \| + b_2 \| A(t) \| + b_3 \| W(t) \|$, and $G_1(t, S(t))$ is a Lipschitz function.

**Theorem 4.1.** The kernel $G_1$ satisfies the Lipschitz condition and contraction if the inequality given below hold

$$0 \leq (m + \beta_p (l_1 + \kappa l_4) + \beta_w l_6) < 1.$$ 

**Proof.** For $S$ and $S_1$ we have

\[
\| G_1(t, S) - G_1(t, S_1) \| = \| -m(S(t) - S_1(t)) - \beta_p(I(t) + \kappa A(t))(S(t) - S_1(t)) - \beta_w W(t)(S(t) - S_1(t)) \|
\]

\[
\leq m \| S(t) - S_1(t) \| + \beta_p \| I(t) + \kappa A(t) \| \| S(t) - S_1(t) \| + \beta_w \| W(t) \| \| S(t) - S_1(t) \|
\]

\[
\leq (m + \beta_p (l_3 + \kappa l_4) + \beta_w l_6) \| S - S_1 \|
\]

Suppose that $b_1 = (m + \beta_p (l_3 + \kappa l_4) + \beta_w l_6)$. Then $G_1(t, S(t))$ is bounded and $G_1(t, S(t)) = b_1 \| I(t) \| + b_2 \| A(t) \| + b_3 \| W(t) \| \| S(t) - S_1(t) \|$. Thus, for $G_1$ the Lipschitz condition is obtained and if $0 \leq (m + \beta_p (l_3 + \kappa l_4) + \beta_w l_6) < 1$ then $G_1$ is a contraction. \qed
Similarly, the Lipschitz condition for $G_i$, $i = 2, 3, 4, 5, 6$ given as follows
\[
\begin{align*}
&\|G_2(t, E) - G_2(t, E_1)\| \leq b_2 \| (E(t) - E_1(t)) \|, \\
&\|G_3(t, I) - G_3(t, I_1)\| \leq b_3 \| (I(t) - I_1(t)) \|, \\
&\|G_4(t, A) - G_4(t, A_1)\| \leq b_4 \| (A(t) - A_1(t)) \|, \\
&\|G_5(t, R) - G_5(t, R_1)\| \leq b_5 \| (R(t) - R_1(t)) \|, \\
&\|G_6(t, W) - G_6(t, W_1)\| \leq b_6 \| (W(t) - W_1(t)) \|,
\end{align*}
\]
where $\|S(t)\| \leq l_1$, $\|E(t)\| \leq l_2$, $\|R(t)\| \leq l_3$ and $b_2 = ((1 - \delta_1)\omega + \delta_4)\omega + m$, $b_3 = (\delta_4 + m)$, $b_4 = (\delta_2 + m)$, $b_5 = \epsilon$ are bounded functions, if $0 \leq b_i < 1, i = 2, 3, 4, 5, 6$ then $G_i, i = 2, 3, 4, 5, 6$ are contraction. According to System (2), consider the following recursive forms
\[
\begin{align*}
H_{1n}(t) &= S_{0n}(t) - S_{-1n}(t) = \frac{\lambda^{1-\eta}}{\Gamma(\eta)} \int_0^t (G_1(\tau, S_{-1n} - G_1(\tau, S_{0n}))(t - \tau)^{\eta-1}d\tau, \\
H_{2n}(t) &= E_{0n}(t) - E_{-1n}(t) = \frac{\lambda^{1-\eta}}{\Gamma(\eta)} \int_0^t (G_2(\tau, E_{-1n} - G_2(\tau, E_{0n}))(t - \tau)^{\eta-1}d\tau, \\
H_{3n}(t) &= I_{0n}(t) - I_{-1n}(t) = \frac{\lambda^{1-\eta}}{\Gamma(\eta)} \int_0^t (G_3(\tau, I_{-1n} - G_3(\tau, I_{0n}))(t - \tau)^{\eta-1}d\tau, \\
H_{4n}(t) &= A_{0n}(t) - A_{-1n}(t) = \frac{\lambda^{1-\eta}}{\Gamma(\eta)} \int_0^t (G_4(\tau, A_{-1n} - G_4(\tau, A_{0n}))(t - \tau)^{\eta-1}d\tau, \\
H_{5n}(t) &= R_{0n}(t) - R_{-1n}(t) = \frac{\lambda^{1-\eta}}{\Gamma(\eta)} \int_0^t (G_5(\tau, R_{-1n} - G_5(\tau, R_{0n}))(t - \tau)^{\eta-1}d\tau, \\
H_{6n}(t) &= W_{0n}(t) - W_{-1n}(t) = \frac{\lambda^{1-\eta}}{\Gamma(\eta)} \int_0^t (G_6(\tau, W_{-1n} - G_6(\tau, W_{0n}))(t - \tau)^{\eta-1}d\tau,
\end{align*}
\]
with initial conditions $S_0(t) = S(0), E_0(t) = E(0), I_0(t) = I(0), A_0(t) = A(0), R_0(t) = R(0)$ and $W_0(t) = W(0)$. We take the norm of first equation in the above system, then
\[
\begin{align*}
\|H_{1n}(t)\| &= \|S_{0n}(t) - S_{-1n}(t)\| \\
&= \frac{\lambda^{1-\eta}}{\Gamma(\eta)} \int_0^t \|G_1(\tau, S_{-1n} - G_1(\tau, S_{0n}))(t - \tau)^{\eta-1}d\tau\| \\
&\leq \frac{\lambda^{1-\eta}}{\Gamma(\eta)} \int_0^t \|G_1(\tau, S_{-1n} - G_1(\tau, S_{0n}))(t - \tau)^{\eta-1}d\tau\|.
\end{align*}
\]
with Lipschitz condition (3), we have
\[
\|H_{1n}(t)\| \leq \frac{\lambda^{1-\eta}}{\Gamma(\eta)} b_1 \int_0^t \|H_{1n-1}(\tau)\|d\tau.
\]
(4)
As a similar way, we obtained
\[
\begin{align*}
\|H_{2n}(t)\| &\leq \frac{\lambda^{1-\eta}}{\Gamma(\eta)} b_2 \int_0^t \|H_{2n-1}(\tau)\|d\tau, \\
\|H_{3n}(t)\| &\leq \frac{\lambda^{1-\eta}}{\Gamma(\eta)} b_3 \int_0^t \|H_{3n-1}(\tau)\|d\tau, \\
\|H_{4n}(t)\| &\leq \frac{\lambda^{1-\eta}}{\Gamma(\eta)} b_4 \int_0^t \|H_{4n-1}(\tau)\|d\tau, \\
\|H_{5n}(t)\| &\leq \frac{\lambda^{1-\eta}}{\Gamma(\eta)} b_5 \int_0^t \|H_{5n-1}(\tau)\|d\tau, \\
\|H_{6n}(t)\| &\leq \frac{\lambda^{1-\eta}}{\Gamma(\eta)} b_6 \int_0^t \|H_{6n-1}(\tau)\|d\tau.
\end{align*}
\]
(5)
Thus, we can write that
\[
\begin{align*}
S_n(t) &= \sum_{j=1}^n H_{1j}(t), E_n(t) = \sum_{j=1}^n H_{2j}(t), I_n(t) = \sum_{j=1}^n H_{3j}(t), \\
A_n(t) &= \sum_{j=1}^n H_{4j}(t), R_n(t) = \sum_{j=1}^n H_{5j}(t), W_n(t) = \sum_{j=1}^n H_{6j}(t).
\end{align*}
\]
In the next theorem, we prove the existence of a solution.

**Theorem 4.2.** A system of solutions given by the fractional COVID-19 model (1) exists if there exist $t_1$ such that
\[
\frac{\lambda^{1-\eta}}{\Gamma(\eta)} t_1 b_i < 1.
\]

**Proof.** From recursive technique, and Eqs. (4) and (5) we conclude that
\[
\begin{align*}
\|H_{1n}(t)\| &\leq \|S_n(t)\| \|H_{1n-1}(t)\| \\
&\leq \frac{\lambda^{1-\eta}}{\Gamma(\eta)} \int_0^t \|G_1(\tau, S_{-1n} - G_1(\tau, S_{0n}))(t - \tau)^{\eta-1}d\tau\| \\
&\leq \frac{\lambda^{1-\eta}}{\Gamma(\eta)} \int_0^t \|G_1(\tau, S_{-1n} - G_1(\tau, S_{0n}))(t - \tau)^{\eta-1}d\tau\|.
\end{align*}
\]
Thus, the system has a solution and also it is continuous. Now we show that the above functions construct solution for the model (2), we assume that
\[
\begin{align*}
S(t) &= S(0) = S_n(t) - B_{1n}(t), \\
E(t) &= E(0) = E_n(t) - B_{2n}(t), \\
I(t) &= I(0) = I_n(t) - B_{3n}(t), \\
A(t) &= A(0) = A_n(t) - B_{4n}(t), \\
R(t) &= R(0) = R_n(t) - B_{5n}(t), \\
W(t) &= W(0) = W_n(t) - B_{6n}(t).
\end{align*}
\]
So
\[
\begin{align*}
\|B_{1n}(t)\| &= \|G_1(\tau, S) - G_1(\tau, S_{-1n})(t - \tau)^{\eta-1}d\tau\| \\
&\leq \frac{\lambda^{1-\eta}}{\Gamma(\eta)} \int_0^t \|G_1(\tau, S) - G_1(\tau, S_{-1n})(t - \tau)^{\eta-1}d\tau\| \\
&\leq \frac{\lambda^{1-\eta}}{\Gamma(\eta)} b_1 \|S - S_{-1n}\|.
\end{align*}
\]
By repeating the method we obtain
\[
\|B_{1n}(t)\| \leq \left(\frac{\lambda^{1-\eta}}{\Gamma(\eta)} \right)^{n+1} b_1^{n+1} t.
\]
At $t_i$, we get
\[ \|B_{tn}(t)\| \leq \frac{\lambda^{1-n}}{\Gamma(n)} t_i \|b_t^{n+1}\| h. \]

Taking limit on recent equation as $n$ approaches to $\infty$, we obtain
\[ \|B_{tn}(t)\| \to 0. \]
A same way, we can show that $\|B_{tn}(t)\| \to 0. i = 2, 3, 4, 5, 6. \] This complete the proof. □

To show the uniqueness of the solution, we suppose that the system has another solution such as $S_1(t), E_1(t), I_1(t), A_1(t), R_1(t)$ and $W_1(t)$, then we have
\[ S(t) - S_1(t) = \frac{\lambda^{1-n}}{\Gamma(n)} \int_0^t (G_1(\tau, S) - G_1(\tau, S_1)) d\tau. \]

We take norm from this equation
\[ \|S(t) - S_1(t)\| = \frac{\lambda^{1-n}}{\Gamma(n)} b_t \|S(t) - S_1(t)\|. \]
It follows from Lipschitz condition (3) that
\[ \|S(t) - S_1(t)\| \leq \frac{\lambda^{1-n}}{\Gamma(n)} b_t \|S(t) - S_1(t)\|. \]
Thus
\[ \|S(t) - S_1(t)\| (1 - \frac{\lambda^{1-n}}{\Gamma(n)} b_t t) \leq 0. \] (6)

Theorem 4.3. The solution of COVID-19 model (1) is unique if below condition hold
\[ 1 - \frac{\lambda^{1-n}}{\Gamma(n)} b_t t > 0. \]

Proof. Suppose that condition (6) hold
\[ \|S(t) - S_1(t)\| (1 - \frac{\lambda^{1-n}}{\Gamma(n)} b_t t) \leq 0. \]
Then $\|S(t) - S_1(t)\| = 0$. So, we obtain $S(t) = S_1(t)$. Similarly we can show the same equality for $E, I, A, R, W$. □

5. Numerical results and discussion

In this section, we present the numerical results for the COVID-19 model (1).

5.1. Numerical method

To solve the system (1), we use the generalized Adams-Bashforth Moulton method [20]. To explain the method, consider the following nonlinear equation
\[ Q_t^p u(t) = f(t, u(t)). \quad 0 \leq t \leq T. \]

\[ u^{(q)}(0) = u_0^{(q)}, \quad q = 0, 1, 2, \ldots, \nu, \quad \nu = \lceil \eta \rceil. \]

The above equation is equivalent to the following Volterra integral equation
\[ u(t) = \sum_{q=0}^{\nu-1} u_0^{(q)} t^q \frac{q!}{q!} + \frac{1}{\Gamma(\eta)} \int_0^t (t-s)^{\eta-1} f(s, u(s)) ds. \] (7)

To integrate [7], Diethelm et al. used the Adams-Bashforth Moulton scheme [8–10]. Set $h = \frac{T}{n}$, $t_n = nh$, $n = 0, 1, 2, \ldots, N \in Z^*$. we can write the system (1) as follows [20]
\[ S_{n+1} = S_0 + \frac{h^n \lambda^{1-n}}{\Gamma(n+2)} [\Lambda - mS_0 - \beta_p S_0 (\psi_{n+1}^p + \kappa A_{n+1}^p) + \beta_a S_0 W_0] + \frac{h^n \lambda^{1-n}}{\Gamma(n+2)} \sum_{i=0}^n a_{n+1,i} \Lambda - mS_i - \beta_p S_i (l_i + \kappa A_i) - \beta_a S_i W_i, \]
\[ E_{n+1} = E_0 + \frac{h^n \lambda^{1-n}}{\Gamma(n+2)} [\beta_p S_0 (\psi_{n+1}^p + \kappa A_{n+1}^p) + \beta_a S_0 W_0] - ((1 - \delta) \omega + \delta \omega') m E_{n+1}] + \frac{h^n \lambda^{1-n}}{\Gamma(n+2)} \sum_{i=0}^n a_{n+1,i} [\beta_p S_i (l_i + \kappa A_i) + \beta_a S_i W_i] - ((1 - \delta) \omega + \delta \omega' + m E_i], \]
\[ I_{n+1} = I_0 + \frac{h^n \lambda^{1-n}}{\Gamma(n+2)} [(1 - \delta) \omega E_{n+1} - (\gamma + m) E_{n+1}] + \frac{h^n \lambda^{1-n}}{\Gamma(n+2)} \sum_{i=0}^n a_{n+1,i} [(1 - \delta) \omega E_i - (\gamma + m) I_i], \]
\[ A_{n+1} = A_0 + \frac{h^n \lambda^{1-n}}{\Gamma(n+2)} [(\gamma + m) A_{n+1}^p] + \frac{h^n \lambda^{1-n}}{\Gamma(n+2)} \sum_{i=0}^n a_{n+1,i} [\gamma I_i + \gamma A_i - m R_i], \]
\[ R_{n+1} = R_0 + \frac{h^n \lambda^{1-n}}{\Gamma(n+2)} [(\gamma + m) A_{n+1}^p - m R_{n+1}] + \frac{h^n \lambda^{1-n}}{\Gamma(n+2)} \sum_{i=0}^n a_{n+1,i} [\gamma I_i + \gamma A_i - m R_i], \]
\[ W_{n+1} = W_0 + \frac{h^n \lambda^{1-n}}{\Gamma(n+2)} [(\mu + \mu') A_{n+1}^p - m W_{n+1}] + \frac{h^n \lambda^{1-n}}{\Gamma(n+2)} \sum_{i=0}^n a_{n+1,i} [\mu I_i + \mu A_i - m W_i], \]
where
\[ \psi_{n+1}^p = S_0 + \frac{\lambda^{1-n}}{\Gamma(n)} \sum_{i=0}^n \psi_{n+1,i} (\Lambda - mS_i - \beta_p S_i (l_i + \kappa A_i) - \beta_a S_i W_i). \]
\[ E_{n+1} = E_0 + \frac{h^n \lambda^{1-n}}{\Gamma(n+2)} \sum_{i=0}^n \psi_{n+1,i} [\beta_p S_i (l_i + \kappa A_i) + \beta_a S_i W_i] - ((1 - \delta) \omega + \delta \omega' + m E_i]. \]
\[ I_{n+1} = I_0 + \frac{\lambda^{1-n}}{\Gamma(n)} \sum_{i=0}^n \psi_{n+1,i} [(1 - \delta) \omega E_{n+1} - (\gamma + m) I_i]. \]
\[ A_{n+1} = A_0 + \frac{\lambda^{1-n}}{\Gamma(n)} \sum_{i=0}^n \psi_{n+1,i} [\delta \omega' E_i - (\gamma' + m) A_i]. \]
\[ R_{n+1} = R_0 + \frac{\lambda^{1-n}}{\Gamma(n)} \sum_{i=0}^n \psi_{n+1,i} [\gamma I_i + \gamma A_i - m R_i]. \]
\[ W_{n+1} = W_0 + \frac{\lambda^{1-n}}{\Gamma(n)} \sum_{i=0}^n \psi_{n+1,i} [\mu I_i + \mu A_i - m W_i], \]
in which
\[ a_{n+1,i} = \begin{cases} (n+1) \eta^n, & 0, \\ (n-i+2) \eta^{n-1} + (n-i) \eta^{n-1} - 2(n-i+1) \eta^{n-1}, & 1 \leq i \leq n, \\ 1, & i = n + 1. \end{cases} \]
and
\[ \psi_{n+1,i} = \frac{h^n \eta_j}{\eta_j} ((n-i+1) \eta_j - (n-i) \eta_j), \quad 0 \leq i \leq n, \] with $j = 1, 2, 3.$
The minimization model predicted cumulative infected cases and the model predicted cumulative infected data at each day. Using this method, we obtain the parameters as follows

\[ \Lambda = 379726.0273, m = 20.85 \times 10^{-6}, \beta_p = 2.6 \times 10^{-8}, \]

\[ \kappa = 0.0001, \beta_w = 1 \times 10^{-6}, \delta = 0.075, \]

\[ \omega = 0.000058, \omega' = 0.000007, \gamma = 0.02, \gamma' = 0.009, \]

\[ \mu = 1 \times 10^{-6}, \mu' = 1 \times 10^{-6}, \epsilon = 0.01 \]

According to the values of parameters and \( \lambda = 0.99 \), we have \( R_0 = 1.61007996 > 1 \). Then, the system (1) has a positive endemic equilibrium point \( E_1^* \).

\[ E_1^* = (1.14 \times 10^6, 7.41 \times 10^6, 2.04 \times 10^7, 4.32 \times 10^5, 11.4 \times 10^8, 1.5 \times 10^3) \]

In Figs. 2–4, we plotted the results of the system of COVID-19 transmission (1). As you can see in Fig. 2–4, the variables have different results in different amounts of \( \eta \) but exhibit the same behavior. Fig. 2 shows that over time, \( S(t) \) decreases and reaches equilibrium point \( S^* = 1.14 \times 10^6 \), and \( E(t) \) increases and reaches equilibrium point \( E^* = 7.41 \times 10^6 \), in fact, over time, almost all people are at risk for the virus. Fig. 3 shows that both \( I(t) \) and \( A(t) \) increase over time and reach equilibrium points \( I^* = 2.04 \times 10^7 \) and \( A^* = 4.32 \times 10^5 \), respectively. As you can see with this simulation in about two years about 20 million people will infected with the COVID-19. Fig. 4 shows that \( R(t) \) increases over time, also \( W(t) \) decreases to equilibrium point \( W^* = 1.5 \times 10^3 \).

5.3. Effects of parameters

The value of each parameter of the model affects the spread of the disease. The most important issue in controlling epidemics is to create quarantine to reduce the relationship between individuals. In this model the rate of transmission of the disease from infected people to susceptible people is shown with \( \beta_p \). In Fig. 5, we reduce the amount of \( \beta_p \) from \( 2.6 \times 10^{-8} \) to \( 2.6 \times 10^{-10} \), and by comparing the plot of \( I(t) \) in Figs. 3 and 5, we see that with this decrease in \( \beta_p \), the number of infected people decreases sharply. In Fig. 3, number of infected people reaches \( 2.04 \times 10^7 \) in 200 days.
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Fig. 3. Plots of $I(t)$ and $A(t)$ for different values of $\eta = 0.9, 0.8, 0.7, 0.6, 0.5$.

Fig. 4. Plots of $R(t)$ and $W(t)$ for different values of $\eta = 0.9, 0.8, 0.7, 0.6, 0.5$.

Table 1: Results of fractional model for $I(t)$ with various values of $\kappa$.

| $\kappa$  | $8 \times 10^{-5}$ | $9 \times 10^{-5}$ | $1 \times 10^{-4}$ | $2 \times 10^{-4}$ | $3 \times 10^{-4}$ |
|-----------|-------------------|-------------------|-------------------|-------------------|-------------------|
| $I(t)$    | 83,421            | 83,653            | 83,802            | 84,107            | 84,522            |

Table 2: Results of fractional model for $I(t)$ with various values of $\beta_w$.

| $\beta_w$   | $8 \times 10^{-10}$ | $9 \times 10^{-10}$ | $1 \times 10^{-9}$ | $2 \times 10^{-9}$ | $3 \times 10^{-9}$ |
|-------------|---------------------|---------------------|---------------------|---------------------|---------------------|
| $I(t)$      | 82,817              | 83,201              | 83,802              | 84,100              | 84,925              |

Table 3: Results of fractional model for $I(t)$ with various values of $\delta$.

| $\delta$  | 0.073  | 0.074  | 0.075  | 0.076  | 0.077  |
|-----------|--------|--------|--------|--------|--------|
| $I(t)$    | 84,514 | 84,122 | 83,802 | 83,521 | 83,170 |

while, Fig. 5 shows this number after 2000 days, and this is very important because it provides opportunity for the treatment system provides an to manage the disease. Tables 1–3 examines the effect of other parameters on the spread of COVID-19, and as the results show, $\kappa$ and $\beta_w$ have a direct effect on the amount of $I(t)$, by increasing or decreasing each of them $I(t)$ increases or decreases respectively, while $\delta$ has an inverse effect and increasing the delta value reduces $I(t)$.

5.4. Comparison of results

In recent years, much research has been done on the use of fractional derivatives in the modelling of natural phenomena, and the results indicate that the fractional derivative performs better.
To further investigate this issue, we compared the data reported for the number of infected people in 10 time periods (4-days) with the results of the fractional-order model of COVID-19 and its integer-order model, the results of which are shown in Table 4. Comparing the results, we see that the behavior of both models is the same, but the results are different. Comparing the relative error obtained is show that the results of the fractional-order Caputo derivative have less relative error and are closer to the real data.

6. Conclusion

In this work, a mathematical model for transmitting the COVID-19 with fractional-order Caputo derivative has been investigated. The region of the feasibility of system, equilibrium points and $R_0$ are calculated. Also, the existence of a unique solution to the model has been proven using fixed point theory. Using the Adams-Bashforth scheme, approximate solutions are provided for System (1). Using COVID-19 reports from around the world from January 22, to April 11, simulations have been performed to predict COVID-19 transmission over the next three years. In the simulation, the equilibrium points and the reproduction number have calculated, and the results show that the pandemic will continue. We also have compared the results of the fractional-order model and the integer-order model with the real data, and the calculation of the relative error has shown that the values derived from the fractional derivative are closer to the real data, and have less relative error. As the COVID-19 pandemic persists and is not controlled, and the spread information of the disease is constantly changing, further simulations could be presented in future research. The presentation of mathematical models based on the effect of drug and vaccination on the spread of COVID-19 could be the subject of further research in this field.

### Availability of data and materials

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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Not applicable.

### Authors’ contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

### Declaration of Competing Interest

The authors declare that they have no competing interests.

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