Loose Laplacian Spectra of Random Hypergraphs

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ABSTRACT: Let \( H = (V, E) \) be an \( r \)-uniform hypergraph with the vertex set \( V \) and the edge set \( E \). For \( 1 \leq s \leq r/2 \), we define a weighted graph \( G(s) \) on the vertex set \( \binom{V}{s} \) as follows. Every pair of \( s \)-sets \( I \) and \( J \) is associated with a weight \( w(I, J) \), which is the number of edges in \( H \) containing \( I \) and \( J \) if \( I \cap J = \emptyset \), and 0 if \( I \cap J \neq \emptyset \). The \( s \)-th Laplacian \( L(s) \) of \( H \) is defined to be the normalized Laplacian of \( G(s) \). The eigenvalues of \( L(s) \) are listed as \( \lambda_0(s), \lambda_1(s), \ldots, \lambda_{\binom{n}{s}-1}(s) \) in non-decreasing order.

Let \( \bar{\lambda}(s)(H) = \max_{i \neq 0} \{|1 - \lambda_i(s)|\} \). The parameters \( \bar{\lambda}(s)(H) \) and \( \lambda_1(s)(H) \), which were introduced in our previous paper, have a number of connections to the mixing rate of high-ordered random walks, the generalized distances/diameters, and the edge expansions.

For \( 0 < p < 1 \), let \( H^r(n, p) \) be a random \( r \)-uniform hypergraph over \( [n] := \{1, 2, \ldots, n\} \), where each \( r \)-set of \( [n] \) has probability \( p \) to be an edge independently. For \( 1 \leq s \leq r/2 \), \( p(1 - p) \gg \frac{\log n}{n^2} \), and \( 1 - p \gg \frac{\log n}{n^2} \), we prove that almost surely

\[
\bar{\lambda}(s)(H^r(n, p)) \leq \frac{s}{n - s} + (3 + o(1)) \sqrt{\frac{1 - p}{n - s} \frac{n}{r - s} p}.
\]

We also prove that the empirical distribution of the eigenvalues of \( L(s) \) for \( H^r(n, p) \) follows the Semicircle Law if \( p(1 - p) \gg \frac{\log^4 n}{n^2} \) and \( 1 - p \gg \frac{\log n}{n^{3/2}} \). © 2012 Wiley Periodicals, Inc. Random Struct. Alg., 41, 521–545, 2012

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1. INTRODUCTION

The spectrum of the adjacency matrix (and/or the Laplacian matrix) of a random graph was well-studied in the literature \([2, 11, 12, 14, 15, 17, 19, 20, 23]\). Given a graph \( G \), let
\(\mu_1(G), \ldots, \mu_n(G)\) be the eigenvalues of the adjacency matrix of \(G\) in the non-decreasing order, and \(\lambda_0(G), \ldots, \lambda_{n-1}(G)\) be the eigenvalues of (normalized) Laplacian matrix of \(G\) respectively. Let \(G(n, p)\) be the Erdős-Rényi random graph model. Füredi and Komlós [23] showed that almost surely \(\mu_n = (1 + o(1))np\) and \(\mu_{n-1} \leq (2 + o(1))\sqrt{np(1-p)}\) provided \(np(1-p) \gg \log^6 n\). The results are extended to sparse random graphs [19, 28] and general random matrices [17, 23]. Alon, Krivelevich, and Vu [2] proved the concentration of the \(s\)-th largest eigenvalue of a random symmetric matrix with independent random entries of absolute value at most one. Friedman (in a series of papers [20–22]) proved that the second largest eigenvalue of random \(d\)-regular graphs is almost surely \((2 + o(1))\sqrt{d-1}\) for any \(d \geq 4\). Chung, Lu, and Vu [12] studied the Laplacian eigenvalues of random graphs with given expected degrees; their results were supplemented by Coja-Oghlan [14, 15] for much sparser random graphs.

In this paper, we study the spectra of the Laplacians of random hypergraphs. Laplacians for regular hypergraphs were first introduced by Chung [6] using the homology approach. Rodríguez [33, 34] viewed a hypergraph as a multi-edge graph and then defined its Laplacian to be the Laplacian of the corresponding multi-edge graph. Inspired by these work, we [29] introduced the generalized Laplacian eigenvalues of hypergraphs through high-ordered random walks. Let \(H = (V, E)\) be an \(r\)-uniform hypergraph on \(n\) vertices. We can associate \(r-1\) Laplacians \(L^{(s)} (1 \leq s \leq r-1)\) to \(H\); roughly speaking, \(L^{(s)}\) captures the incidence relations between \(s\)-sets and edges in \(H\). Our definition of the Laplacian at the spatial case \(s = 1\) is the same as the Laplacian considered by Rodríguez [33, 34]. The \(s\)-th Laplacian is loose if \(1 \leq s \leq r/2\), and is tight if \(r/2 < s \leq r-1\). Here we consider only the spectra of loose Laplacians.

For \(1 \leq s \leq r/2\), we consider an auxiliary weighted graph \(G^{(s)}\), defined as follows: the vertex set of \(G^{(s)}\) is \(\binom{V}{s}\) while the weight function \(W: \binom{V}{s} \times \binom{V}{s} \rightarrow \mathbb{Z}\) is defined as

\[
W(S, T) = \begin{cases} 
|\{F \in E(H) : S \cup T \subset F\}| & \text{if } S \cap T = \emptyset; \\
0 & \text{otherwise.}
\end{cases} 
\]  

(1)

The \(s\)-th Laplacian of \(H\), denoted by \(L^{(s)}\), is the normalized Laplacian of \(G^{(s)}\). For any \(s\)-set \(S\), let \(d_S\) be the number of edges in \(H\) containing \(S\); the degree of \(S\) in \(G^{(s)}\) is \((\frac{r}{s})d_S\). Let \(D\) be the diagonal matrix whose \((S,S)\)-entry is the degree \(d_S\), and \(W\) be the weight matrix whose \((S, T)\)-entry is \(w(S, T)\). Note that \(T := (\frac{r}{s})D\) is the diagonal matrix of degrees in \(G^{(s)}\). We have

\[
L^{(s)} = I - T^{-1/2}WT^{-1/2}. 
\]  

(2)

The eigenvalues of \(L^{(s)}\) are listed as \(\lambda_0^{(s)}, \lambda_1^{(s)}, \ldots, \lambda_{\binom{r}{s}-1}^{(s)}\) in non-decreasing order. We have

\[
0 = \lambda_0^{(s)} \leq \lambda_1^{(s)} \leq \cdots \leq \lambda_{\binom{r}{s}-1}^{(s)} \leq 2. 
\]  

(3)

The first non-trivial eigenvalue \(\lambda_1^{(s)} > 0\) if and only if \(G^{(s)}\) is connected. When this occurs, we say \(H\) is \(s\)-connected. The diameter of \(G^{(s)}\) is called the \(s\)-th diameter of \(H\). The largest eigenvalue \(\lambda_{\binom{r}{s}-1}^{(s)}\) is also denoted by \(\lambda_{\max}^{(s)}\). The (Laplacian) spectral radius, denoted by \(\lambda^{(s)}\), is the maximum of \(1 - \lambda_1^{(s)}\) and \(\lambda_{\max}^{(s)} - 1\).

This definition differs slightly with the one in [29], where the vertex set of the auxiliary graph (denoted by \(G^{(s)}\)) is the set of all \(s\)-tuples of distinct coordinates instead. Note that

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$G^{(s)}$ is the blow-up of $G^{(o)}$. Their Laplacian spectra differ only by the multiplicity of 1. Therefore, two different definitions give the same values of $\lambda_1^{(o)}$, $\lambda_{\max}^{(o)}$, and $\bar{\lambda}^{(o)}$.

For different $s$, the following inequalities were proved in [29].

\begin{align*}
\lambda_1^{(1)} &\geq \lambda_1^{(2)} \geq \cdots \geq \lambda_1^{(\lfloor r/2 \rfloor)}; \\
\lambda_{\max}^{(1)} &\leq \lambda_{\max}^{(2)} \leq \cdots \leq \lambda_{\max}^{(\lfloor r/2 \rfloor)}. 
\end{align*}

The $s$-th Laplacian has a number of connections to the mixing rate of high-ordered random walks, the generalized distances/diameters, and the edge expansions. Here we list some applications, which are similar to results in [29], and results for graphs [5, 7–10, 13].

Random $s$-Walks: The mixing rate of the random $s$-walk on $H$ is at most $\bar{\lambda}^{(s)}$.

The $s$-Diameter: The $s$-diameter of $H$ is at most $\left\lceil \log \frac{|E(H)(n\choose s)|}{\delta} \frac{\lambda_{\max}^{(s)} + \lambda_1^{(s)}}{\lambda_{\max}^{(s)} - \lambda_1^{(s)}} \right\rceil$.

Here $\delta = \min_{S \in (n\choose s)} d_S$ is the minimum degree among all $s$-sets.

Edge expansion: For $1 \leq t \leq s \leq \frac{r}{2}$, $S \subset {n\choose t}$, and $T \subset {n\choose r}$, define

\[ E(S, T) = \{ F \in E(H) : \exists S \in S, \exists T \in T \text{ such that } S \cap T = \emptyset, \text{ and } S \cup T \subset F \}, \]

\[ e(S, T) = \frac{|E(S, T)|}{|E({n\choose t}, {n\choose r})|}, \quad e(S) = \frac{\sum_{S \in S} d_S}{\sum_{S \in {n\choose t}} d_S}, \quad e(T) = \frac{\sum_{T \in T} d_T}{\sum_{T \in {n\choose r}} d_T}. \]

Then we have

\[ |e(S, T) - e(S)e(T)| \leq \bar{\lambda}^{(s)} \sqrt{e(S)e(T)e(\bar{S})e(\bar{T})}. \]

The proofs of these claims are very similar to those in [29] and are omitted here.

Our first result is the eigenvalues of the $s$-th Laplacian of the complete $r$-uniform hypergraph $K^r_n$.

**Theorem 1.** Let $K^r_n$ be the complete $r$-uniform hypergraph on $n$ vertices. For $1 \leq s \leq r/2$, the eigenvalues of $s$-th Laplacian of $K^r_n$ are given by

\[ 1 - \frac{(-1)^i (\begin{array}{c} n-s-i \\ t-i \end{array})}{\begin{array}{c} n \\ s \end{array}} \text{ with multiplicity } \left( \begin{array}{c} n \\ i \end{array} \right) - \left( \begin{array}{c} n \\ i-1 \end{array} \right) \text{ for } 0 \leq i \leq s. \]

Here we point out an application of this theorem to the celebrated Erdős-Ko-Rado Theorem, which states “if $n \geq 2s$, then the maximum size of the intersecting family of $s$-sets in $[n]$ is $\binom{n-1}{s-1}$.” (The theorem was originally proved by Erdős-Ko-Rado [18] for sufficiently large $n$; the simplest proof was due to Katona [27].) Here we present a proof adapted
from Calderbank-Frankl [3], where they use the eigenvalues of Kneser graph instead. (The relation between $L^{(i)}(K'_n)$ and the Laplacian of the Kneser graph is explained in section 2.)

It suffices to show for any intersecting family $U$ of $s$-sets, $|U| \leq \binom{n-1}{s}$. Note that $U$ is an independent set of $G^{(i)}(K'_n)$. Restricting to $U$, $L^{(i)}(K'_n)$ becomes an identity matrix; whose eigenvalues are all equal to 1. By Cauchy’s interlace theorem, we have

$$\lambda_k^{(i)} \leq 1 \leq \lambda_{(U)+k}^{(i)}$$ \hspace{1cm} (6)

for $0 \leq k \leq |U| - 1$. Let $N^+$ (or $N^-$) be the number of eigenvalues of $L^{(i)}(K'_n)$ which is $\geq 1$ (or $\leq 1$) respectively. Inequality (6) implies that $|U| \leq N^+$ and $|U| \leq N^-$. By Theorem 1, $N^+ = \sum_{i=0}^{[s/(s-1)/2]} \left( \binom{n}{2i+1} - \binom{i}{2} \right)$ and $N^- = \sum_{i=0}^{[s/(s-1)]} \left( \binom{n}{i} - \binom{n-i}{2i-1} \right)$. We have

$$|U| \leq \min\{N^+, N^-\} = \sum_{i=0}^{s-1} (-1)^{s-1-i} \binom{n}{i} = \binom{n-1}{s-1}.$$  

For $0 < p < 1$, let $H' (n, p)$ be a random $r$-uniform hypergraph over $[n] = \{1, 2, \ldots, n\}$, where each $r$-set of $[n]$ has probability $p$ to be an edge independently. We can estimate the Laplacian spectrum of $H' (n, p)$ using the Laplacian spectrum of $K'_n$ as follows. Here for two functions $f(n)$ and $g(n)$, the notation $f(n) \gg g(n)$ (or $g(n) \ll f(n)$) means $\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty$.

**Theorem 2.** Let $H' (n, p)$ be a random $r$-uniform hypergraph. For $1 \leq s \leq r/2$, if $p(1-p) \gg \frac{\log^6 n}{n^2}$ and $1-p \gg \frac{\log n}{n^2}$, then almost surely the $s$-th spectral radius $\tilde{\lambda}^{(i)}(H' (n, p))$ satisfies

$$\tilde{\lambda}^{(i)}(H' (n, p)) \leq \frac{s}{n-s} + (3 + o(1)) \sqrt{\frac{1-p}{r-1}p}.$$ \hspace{1cm} (7)

Moreover, for $1 \leq k \leq \binom{n}{s} - 1$, almost surely we have

$$|\lambda_k^{(i)}(H' (n, p)) - \lambda_k^{(i)}(K'_n)| \leq (3 + o(1)) \sqrt{\frac{1-p}{n-s}p}.$$ \hspace{1cm} (8)

Note that $\lambda_k^{(i)}(K'_n)$ is completely determined by Theorem 1. When $H' (n, p)$ gets denser, more information on “significant” Laplacian eigenvalues is revealed by Eq. (8). Here “significant” means the distance to 1 is as large as possible. For example, when $s < r - 2$ and $p = \Omega(1/n^{r-s-2})$, almost surely the most $(n - 1)$ significant Laplacian eigenvalues (except for 0) of $H' (n, p)$ are $(1 + o(1))n/(n - s)$.

Note that $G(n, p)$ is a special case of $H' (n, p)$ with $r = 2$. By choosing $s = 1$, Theorem 2 implies that

$$\tilde{\lambda}(G(n, p)) \leq (3 + o(1)) \sqrt{\frac{1-p}{n-1}}p \hspace{0.3cm} \text{for} \hspace{0.3cm} p(1-p) \gg \frac{\log^4 n}{n}.$$ \hspace{1cm} (9)

Chung, Lu, and Vu’s result [12], when restricted to $G(n, p)$, implies

$$\tilde{\lambda}(G(n, p)) \leq (4 + o(1)) \frac{1}{\sqrt{np}} \hspace{0.3cm} \text{for} \hspace{0.3cm} 1 - \epsilon \geq p \gg \frac{\log^6 n}{n}.$$ \hspace{1cm} (10)

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Inequality (9) has a smaller constant and works for a larger range of \( p \) than inequality (10).

Füredi and Komlós [23] proved the empirical distribution of the eigenvalues of \( G(n, p) \) follows the Semicircle Law. Chung, Lu, and Vu [12] showed a similar result for the random graphs with given expected degrees. Here we establish a similar result for random hypergraphs.

**Theorem 3.** For \( 1 \leq s \leq r/2 \), if \( p(1 - p) \gg \frac{\log^{1/3} n}{n^{2/3}} \) and \( 1 - p \gg \frac{\log n}{n^{2/3} + 2r - 2s} \), then almost surely the empirical distribution of eigenvalues of the \( s \)-th Laplacian of \( H^r(n, p) \) follows the Semicircle Law centered at 1 and with radius \( (2 + o(1)) \sqrt{\frac{1 - p}{(r - s)(r - 2s)}} \sqrt{1 - p} \).

**Remark 1.** The proof of Theorem 3 actually implies the eigenvalues of \( \mathcal{L}^{(s)}(H^r(n,p)) - \mathcal{L}^{(s)}(K_n^{r}) \) follows the Semicircle Law centered at 0 and with radius \( (2 + o(1)) \sqrt{\frac{1 - p}{(r - s)(r - 2s)}} \sqrt{1 - p} \). Thus we have

\[
\max_{1 \leq k \leq \binom{s}{2} - 1} |\lambda_k^{(s)}(H^r(n,p)) - \lambda_k^{(s)}(K_n^{r})| \geq \left( \frac{2}{\sqrt{(r - s)}} + o(1) \right) \sqrt{\frac{1 - p}{(r - s)(r - 2s)}} \sqrt{1 - p}.
\]

(11)

This shows that the upper bound of \( |\lambda_k^{(s)}(H^r(n,p)) - \lambda_k^{(s)}(K_n^{r})| \) in Theorem 2 is best up to a constant multiplicative factor.

**Remark 2.** To study the spectral bound of random matrices, several matrix concentration inequalities are recently derived by many authors including Ahlswede-Winter [1], Chung-Radcliff [13], Cristofides-Markström [16], Gross [25], Oliveira [30], Recht [31], and Tropp [32]. These matrix concentration inequalities can be used to prove similar upper bounds like the one in Eq. (8). Although with simpler proofs, the results seem to be worse than our bound in Eq. (8) by a multiplicative factor of \( O(\sqrt{\ln n}) \). This is the main reason why we stick to the classical trace method.

The rest of the paper is organized as follows. The notation and some basic lemmas are introduced in Section 2. Theorem 1 is also proved in Section 2. Theorem 2 is proved in section 3 while Theorem 3 is proved in section 4.

## 2. NOTATION AND LEMMAS

### 2.1. Laplacian Eigenvalues of Hypergraphs

Let \( H = (V, E) \) be an \( r \)-uniform hypergraph. For any subset \( S(|S| < r) \), the degree of \( S \), denoted by \( d_S \), is the number of edges containing \( S \). For each \( 1 \leq s \leq r/2 \), we associate a weighted graph \( G^{(s)} \) on the vertex set \( \binom{r}{s} \) to \( H \) as follows. Every pair of \( s \)-sets \( S \) and \( T \) is associated with a weight \( w(S, T) \), which is given by

\[
w(S, T) = \begin{cases} 
d_{S,T} & \text{if } S \cap T = \emptyset, \\
0 & \text{otherwise}.
\end{cases}
\]

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The $s$-th Laplacian $\mathcal{L}^{(s)}$ of $H$ is defined to be the normalized Laplacian of $G^{(s)}$. The degree of $S$ in $G^{(s)}$ is $\sum_{T} w(S, T) = \binom{n}{s} d_S$.

We assume that the $s$-sets in $\binom{[n]}{s}$ are ordered alphabetically. Let $N := \binom{n}{s}$; all square matrices considered in the paper have the dimension $N \times N$ and all vectors have dimension $N$. Let $W := (W(S, T))$ be the weight matrix, $D$ be the diagonal matrix with diagonal entries $D(S, S) = d_S$, $\mathbf{d}$ be the column vector with entries $d_S$ at position $S \in \binom{[n]}{s}$, $J$ be the square matrix of all 1’s, and $\mathbf{1}$ be the column vector of all 1’s. Let $T := \binom{[n]}{s-1} D$; here $T$ is the diagonal matrix of degrees in $G^{(s)}$. Then, we have

$$\mathcal{L}^{(s)} = I - T^{-1/2} W T^{-1/2}.$$

We list the eigenvalues of $\mathcal{L}^{(s)}$ as

$$0 = \lambda_0^{(s)} \leq \lambda_1^{(s)} \leq \ldots \leq \lambda_{\binom{n}{s}-1}^{(s)} \leq 2.$$

We aim to compute the spectral radius $\bar{\lambda}^{(s)}(H) = \max_{\rho \neq 0} |1 - \lambda_{\rho}^{(s)}|$.

We are ready to prove theorem 1.

**Proof of Theorem 1.** We can express $\mathcal{L}^{(s)}(K'_n)$ using the following notation. The Kneser graph $K(n, s)$ is a graph over the vertex set $\binom{[n]}{r}$; two $s$-sets $S$ and $T$ form an edge of $K(n, s)$ if and only if $S \cap T = \emptyset$. Let $K$ be the adjacency matrix of $K(n, s)$; the eigenvalues of $K$ are $(-1)^i \binom{n-s-i}{r-i}$ with multiplicity $\binom{n}{i} - \binom{n}{i-1}$ for $0 \leq i \leq s$ (see [24]). Note that $K(n, s)$ is a regular graph; so the Laplacian eigenvalues can be determined from the eigenvalues of its adjacency matrix. We observe that the associated weighted graph $G^{(s)}$ for the complete $r$-uniform hypergraph $K'_n$ is essentially the Kneser graph with the same weight $\binom{n-s}{r-2s}$ for each edge. Note that the multiplicative factor $\binom{n-2s}{r-2s}$ is canceled after normalization. The $\mathcal{L}^{(s)}$ (for $K'_n$) is exactly the Laplacian of Kneser graph. Hence,

$$\mathcal{L}^{(s)}(K'_n) = I - \frac{1}{\binom{n}{s}} K.$$

Thus, the eigenvalues of $s$-th Laplacian of $K'_n$ are given by

$$1 - \frac{(-1)^i \binom{n-s-i}{r-i}}{\binom{n}{s}} \text{ with multiplicity } \binom{n}{i} - \binom{n}{i-1} \text{ for } 0 \leq i \leq s.$$

**Remark 3.** For $1 \leq s \leq r/2$, we have

$$\lambda_1^{(s)}(K'_n) = 1 - \frac{s(s-1)}{(n-s)(n-s-1)}, \quad (12)$$

$$\lambda_{\max}^{(s)}(K'_n) = 1 + \frac{s}{n-s}, \quad (13)$$

$$\bar{\lambda}^{(s)}(K'_n) = \frac{s}{n-s}. \quad (14)$$

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2.2. Random Hypergraphs

Let \( H'(n, p) \) be a random \( r \)-uniform hypergraph over the vertex set \( V = [n] \) and each \( r \)-set has probability \( p \) to be an edge independently. We would like to bound the spectral radius of the \( s \)-th Laplacian of \( H'(n, p) \) for \( 1 \leq s \leq r/2 \).

For any \( F \in \binom{V}{r} \), let \( X_F \) be the random indicator variable for \( F \) being an edge in \( H'(n, p) \); all \( X_F \)'s are independent to each other. We would like to bound the spectral radius of the difference of \( T - 1 \).

By Weyl's Theorem, to prove Theorem 2, it is sufficient to bound the spectral norm of \( X_F \). For any \( S, T \in \binom{V}{s} \), we have

\[
W(S, T) = \begin{cases} 
\sum_{F \in \binom{V}{r}} X_F & \text{if } S \cap T = \emptyset; \\
0 & \text{otherwise.}
\end{cases}
\]

Thus,

\[
E(W(S, T)) = \begin{cases} 
\binom{n-s}{r-s} p & \text{if } S \cap T = \emptyset; \\
0 & \text{otherwise.}
\end{cases}
\] (15)

The degree \( d_S = \sum_{S \subseteq F \in \binom{V}{r}} X_F \); we have \( E(d_S) = \binom{n-s}{r-s} p \). For simplicity, let \( d := \binom{n-s}{r-s} p \).

By Weyl's Theorem, we have

\[
M_{1} = \frac{1}{\binom{n-s}{r-s}} (D^{-1/2} (W - E(W)) D^{-1/2} - d^{-1} (W - E(W))),
\]

\[
M_{2} = \frac{1}{\binom{n-s}{r-s} d} (W - E(W)),
\]

\[
M_{3} = \frac{1}{\binom{n-s}{r-s} d} D^{-1/2} E(W) D^{-1/2} - \frac{d}{\binom{n-s}{r-s} d} D^{-1/2} J D^{-1/2} - \frac{1}{\binom{n-s}{r-s} d} K + \frac{1}{\binom{n-s}{r-s} d} J,
\]

\[
M_{4} = \frac{1}{\binom{n-s}{r-s} d} (d D^{-1/2} J D^{-1/2} - J).
\]

By the triangular inequality of matrix norms, we have

\[
\|M\| \leq \|M_1\| + \|M_2\| + \|M_3\| + \|M_4\|.
\]

Through this paper, the norm of any square matrix is the spectral norm. We would like to bound \( \|M_i\| \) for \( i = 1, 2, 3, 4 \). We use the following Chernoff inequality.

**Theorem 4** ([4]). Let \( X_1, \ldots, X_n \) be independent random variables with

\[
\Pr(X_i = 1) = p, \quad \Pr(X_i = 0) = 1 - p.
\]

We consider the sum \( X = \sum_{i=1}^n X_i \), with expectation \( E(X) = np \). Then we have

(Lower tail) \( \Pr(X \leq E(X) - \lambda) \leq e^{-\lambda^2/2E(X)}, \)

(Upper tail) \( \Pr(X \geq E(X) + \lambda) \leq e^{-\lambda^2/(2np)} \).
Lemma 1. Suppose \(d \geq \log N\). With probability at least \(1 - \frac{1}{N^2}\), we have \(d_s \in (d - 3\sqrt{d \log N}, d + 3\sqrt{d \log N})\) for all \(S \in \binom{\mathcal{V}}{s}\).

Proof. Note \(d_s = \sum_{F \in \mathcal{F}} \mathcal{X}_F\) and \(\mathbb{E}(d_s) = d\). Applying the lower tail of Chernoff’s inequality with \(\lambda = 3\sqrt{\mathbb{E}(X) \log N}\), we have

\[
\Pr(X - \mathbb{E}(X) \leq -\lambda) \leq e^{-\lambda^2/2\mathbb{E}(X)} \leq \frac{1}{N^{9/2}}.
\]

Applying the upper tail of Chernoff’s inequality with \(\lambda = 3\sqrt{\mathbb{E}(X) \log N}\), we have

\[
\Pr(X - \mathbb{E}(X) \geq \lambda) \leq e^{-\lambda^2/2\mathbb{E}(X)} \leq \frac{1}{N^{9/2}}.
\]

The probability that \(d_s \notin (d - 3\sqrt{d \log N}, d + 3\sqrt{d \log N})\) is at most \(\frac{1}{N^3}\). Thus, with probability at least \(1 - \frac{1}{N^2}\), we have \(d_s \in (d - 3\sqrt{d \log N}, d + 3\sqrt{d \log N})\) for all \(S \in \binom{\mathcal{V}}{s}\). \(\blacksquare\)

For convenience, let \(d_{\min} := d - 3\sqrt{d \log N}, d_{\max} := d + 3\sqrt{d \log N}\); almost surely we have \(d_{\min} \leq d_s \leq d_{\max}\) for all \(S\).

Lemma 2. If \(d \geq \log N\), then almost surely \(\|M_3\| = O\left(\frac{\sqrt{\log N}}{n \sqrt{d}}\right)\).

Proof. Note \(\mathbb{E}(W) = \binom{n - s}{r - s}pK\), where \(K\) is the adjacency matrix of the Kneser graph \(K(n, s)\). Let \(M_0 := \frac{1}{\binom{\mathcal{V}}{s}}K - \frac{1}{\binom{\mathcal{V}}{s}}J\). We can rewrite \(M_3\) as

\[
M_3 = dD^{-1/2}M_0D^{-1/2} - M_0.
\]

Note \(\|M_0\| = \mathbb{E}(\ell)^{(1)}(K^*_0) = \frac{\lambda}{n^s}\). We have

\[
\|M_3\| = \|dD^{-1/2}M_0D^{-1/2} - M_0\|
\leq \|dD^{-1/2} - d^{1/2}I\|M_0\|D^{-1/2}\| + \|M_0(\infty^{1/2}D^{1/2} - I)\|
\leq \|dD^{-1/2} - d^{1/2}I\|\|M_0\||D^{-1/2}\| + \|M_0\|\|d^{1/2}D^{1/2} - I\|
\leq \left|d^{1/2} - dd_{\min}^{-1/2}\right| \frac{s}{n - s}d_{\min}^{-1/2} + \frac{s}{n - s}d_{\min}^{-1/2} - 1
\]

\[
= O\left(\frac{\sqrt{\log N}}{n \sqrt{d}}\right).
\]

Lemma 3. If \(p(1 - p) \gg \frac{\log n}{n^2}\), then almost surely

\[
\sum_{S \in \binom{\mathcal{V}}{s}} (d_s - d)^2 = (1 + o(1))\binom{n}{s} d(1 - p).
\]

Proof. For \(S \in \binom{\mathcal{V}}{s}\), let \(X_S = (d_s - d)^2\). We have

\[
\mathbb{E}(X_S) = \mathbb{E}((d_s - d)^2) = \text{Var}(d_s) = \binom{n - s}{r - s}p(1 - p) = d(1 - p).
\]
We use the second moment method to prove that \( \sum_S X_s \) concentrates around its expectation \( \binom{n}{r} d(1 - p) \). For any \( S, T \in \binom{V}{r} \), the covariance can be calculated as follows.

\[
\text{Cov}(X_S, X_T) = E(X_S X_T) - E(X_S)E(X_T) = E((d_S - d)(d_T - d)) - d^2(1 - p)^2.
\]

For \( F \in \binom{V}{r} \), let \( Y_F = X_F - E(X_F) \). Then we have \( d_S - d = \sum_{S \subseteq F} Y_F \).

\[
E((d_S - d)^2(d_T - d)^2) = \sum_{F_1, F_2, S \subseteq F_1 \cap F_2 \subseteq F, F_3, F_4, T \subseteq F_1 \cup F_2} E(Y_{F_1} Y_{F_2} Y_{F_3} Y_{F_4}).
\]

Since \( E(Y_{F_1}) = 0 \), the non-zero terms occur only if

1. \( F_1 = F_2 = F_3 = F_4 \). In this case, we have
   \[
   E(Y_{F_1} Y_{F_2} Y_{F_3} Y_{F_4}) = E(Y_{F_1}^4) = (1 - p)^4p + (-p)^4(1 - p) = p(1 - p)(1 - 3p + 3p^2).
   \]
   The number of choices is \( \binom{n - |S \cup T|}{r - |S \cup T|} \).

2. \( F_1 = F_2 \neq F_3 = F_4 \). In this case, we have
   \[
   E(Y_{F_1} Y_{F_2} Y_{F_3} Y_{F_4}) = E(Y_{F_1}^2)E(Y_{F_3}^2) = p^2(1 - p)^2.
   \]
   The number of choices is \( \binom{n - s}{r - |S \cup T|} \binom{n - s}{r - |S \cup T|} - \binom{n - |S \cup T|}{r - |S \cup T|} \binom{n - |S \cup T|}{r - |S \cup T|} \).

3. \( F_1 = F_3 \neq F_2 = F_4 \). In this case, we have
   \[
   E(Y_{F_1} Y_{F_2} Y_{F_3} Y_{F_4}) = E(Y_{F_1}^2)E(Y_{F_2}^2) = p^2(1 - p)^2.
   \]
   The number of choices is \( \binom{n - |S \cup T|}{r - |S \cup T|} \binom{n - |S \cup T|}{r - |S \cup T|} - \binom{n - |S \cup T|}{r - |S \cup T|} \binom{n - |S \cup T|}{r - |S \cup T|} \).

4. \( F_1 = F_4 \neq F_2 = F_3 \). This is the same as item 3.

Thus, we have

\[
E(X_S X_T) = \binom{n - |S \cup T|}{r - |S \cup T|} p(1 - p)(1 - 3p + 3p^2)
+ \left( \binom{n - s}{r - s}^2 + 2 \binom{n - |S \cup T|}{r - |S \cup T|}^2 - 3 \binom{n - |S \cup T|}{r - |S \cup T|}^2 \right) p^2(1 - p)^2
= \binom{n - |S \cup T|}{r - |S \cup T|} p(1 - p)(1 - 6p + 6p^2)
+ \left( \binom{n - s}{r - s}^2 + 2 \binom{n - |S \cup T|}{r - |S \cup T|}^2 \right) p^2(1 - p)^2.
\]
This expression on the right depends only on the size of $S \cup T$. Putting together, we get

$$\text{Var} \left( \sum_{S \in \binom{T}{r}} X_S \right) = \sum_{S \in \binom{T}{r}} \text{Cov}(X_S, X_T)$$

$$= \sum_{S \in \binom{T}{r}} (\text{E}(X_S X_T) - d^2 (1-p)^2)$$

$$= \sum_{S \in \binom{T}{r}} \left( \text{E}(X_S X_T) - \left( \frac{n-s}{r-s} \right)^2 p^2 (1-p)^2 \right)$$

$$= \sum_{i=1}^{2s} \sum_{|S \cup T|=i} \left( \frac{n-i}{r-i} \right) p(1-p) \left( 1 - 6p + 6p^2 \right) + 2 \left( \frac{n-i}{r-i} \right)^2 p^2 (1-p)^2$$

$$\leq \sum_{i=1}^{2s} \sum_{|S \cup T|=i} \left( \frac{n-i}{r-i} \right) p(1-p) \left( 1 - 6p + 6p^2 \right)$$

$$\leq \sum_{i=1}^{2s} \sum_{|S \cup T|=i} \left( \frac{n-i}{r-i} \right) 3dp(1-p)^2$$

$$= \binom{n}{r} 3dp(1-p)^2 \sum_{i=1}^{2s} \frac{r!}{(i-s)! (2s-i)! (r-i)!}$$

$$< 3 \cdot 4 \binom{n}{r} dp(1-p)^2$$

$$= O \left( \left( \frac{n}{s} \right) d^2 (1-p)^2 \right).$$

Let $X = \sum_s X_S$. We have $\text{E}[X] = \binom{n}{r} d(1-p)$ and $\text{Var}(X) = O \left( \binom{n}{r} d^2 (1-p)^2 \right)$. Applying Chebyshev’s inequality to $X = \sum_{S \in \binom{T}{r}} X_S$, we have

$$\text{Pr}(\left| X - \text{E}(X) \right| \geq \log n \sqrt{\text{Var}(X)}) \leq \frac{1}{\log^2 n}.$$
Here $\alpha := d^{1/2}D^{-1/2}1 - 1$. Note that the spectral norm of a vector is the same as the $L_2$-norm. We have

$$\|\alpha\| = \|d^{1/2}D^{-1/2}1 - 1\| = \sqrt{\sum_{S \in \binom{V}{s}} \left( \frac{\sqrt{d}}{\sqrt{d_S}} - 1 \right)^2}$$

$$= \sqrt{\sum_{S \in \binom{V}{s}} \left( d_S - d \right)^2 d_S \sqrt{d + \sqrt{d_S}}^2} \leq \sqrt{\sum_{S \in \binom{V}{s}} \left( d_S - d \right)^2 d_S} \frac{\sqrt{\sqrt{d} + \sqrt{d_S}}}{\sqrt{d_{\min}} \sqrt{\sqrt{d} + \sqrt{d_{\min}}}}$$

$$= \left( \frac{1}{2} + o(1) \right) \sqrt{\frac{(1 - p)\binom{n}{s}}{d}}.$$

In the last step, we applied Lemma 3. Therefore, we have

$$\|M_4\| = \left\| \frac{1}{\binom{n}{s}} \left( \alpha^t D^{-1/2} d^{1/2} + 1 \alpha \right) \right\|$$

$$= \frac{1}{\binom{n}{s}} \left( \|\alpha^t D^{-1/2} d^{1/2}\| + \|1 \alpha\| \right) \leq \frac{1}{\binom{n}{s}} \|\alpha\| \left( \|1 \|\left(\|D^{-1/2} d^{1/2}\| + \|1\| \right) \right)$$

$$= \frac{1}{\binom{n}{s}} \|\alpha\| \left( \sum_{S \in \binom{V}{s}} \frac{d}{d_S} + \sqrt{n} \right)$$

$$\leq \frac{1}{\binom{n}{s}} \left( \frac{1}{2} + o(1) \right) \sqrt{\frac{(1 - p)\binom{n}{s}}{d}} \left( 2 + o(1) \right) \left( \sqrt{n} \right)$$

$$= (1 + o(1)) \sqrt{\frac{1 - p}{d}}.$$

\[ \square \]

3. PROOF OF THEOREM 2

To estimate the spectral norm of $M_1$ and $M_2$, we need consider the matrix $C := W - E(W)$. We have the following estimate on the expectation of the trace of $C$.

**Lemma 5.** For any $k$ satisfying $k \ll \sqrt{n}^{-1}p(1 - p)$, we have

$$E(\text{Trace}(C^{2k})) \leq (1 + o(1)) \frac{n^{s+k(r-s)}(r-s)^k}{(k+1)(s!)^{k+1}((r-2s)!)} \left( \frac{2k}{k} \right)^p (1 - p)^k, \quad (16)$$

$$E(\text{Trace}(C^{2k+1})) = O \left( \frac{2k^2 n^{s+k(r-s)}(r-s)^{k+1}}{(k+1)(s!)^{k+1}((r-2s)!)} \left( \frac{2k}{k} \right)^p (1 - p)^k \right). \quad (17)$$

If further $k = o(\log(n^{-1}p(1 - p)))$, then we have

$$E(\text{Trace}(C^{2k})) = (1 + o(1)) \frac{n^{s+k(r-s)}(r-s)^k}{(k+1)(s!)^{k+1}((r-2s)!)} \left( \frac{2k}{k} \right)^p (1 - p)^k. \quad (18)$$
To prove Theorem 2, we only need Inequality (16) (for \( k = \Omega(\log n) \)) Equations (17) and (18) will be used to prove Theorem 3, where \( k \) is a big constant. The proof of this technical Lemma is quite long. We will delay its proof until the end of this section.

**Lemma 6.** If \( p(1 - p) \gg \frac{\log^4 n}{n^2} \), then we have \( \| C \| \leq (2^{(r_s)} + o(1)) \sqrt{d(1 - p)} \) almost surely.

**Proof.** By Lemma 5, we have \( \mathbb{E}[\text{Trace}(C^{2k})] \leq (1 + o(1)) \frac{n^{s+k(1-s)(r_s)}(r_s)^k}{(k+1)(s!)^k((r-2s)!)^k} \frac{2^k}{k} p^k (1 - p)^k \).

As \( \mathbb{E}[\| C \|^{2k}] \leq \mathbb{E}[\text{Trace}(C^{2k})] \), we have

\[
\mathbb{E}[\| C \|^{2k}] \leq (1 + o(1)) \frac{n^{s+k(1-s)(r_s)}(r_s)^k}{(k+1)(s!)^k((r-2s)!)^k} \frac{2^k}{k} p^k (1 - p)^k.
\]

Let \( U := \frac{n^{s+k(1-s)(r_s)}(r_s)^k}{(k+1)(s!)^k((r-2s)!)^k} \frac{2^k}{k} p^k (1 - p)^k \). By Markov’s inequality,

\[
\Pr(\| C \| \geq (1 + \epsilon)^{2k} \sqrt{U}) = \Pr(\| C \|^{2k} \geq (1 + \epsilon)^{2k} U) \leq \frac{\mathbb{E}[\| C \|^{2k}]}{(1 + \epsilon)^{2k} U} \leq \frac{(1 + o(1))U}{(1 + \epsilon)^{2k} U} = \frac{1 + o(1)}{(1 + \epsilon)^{2k}}.
\]

Let \( g(n) \) be a slowly growing function such that \( g(n) \to \infty \) as \( n \) approaches the infinity and \( g(n) \ll \frac{(n^{r-s}p(1-p))^{1/4}}{s \log n} \). This is possible because \( n^{r-s}p(1-p) \gg \log^4 n \). Choose \( k = \frac{s g(n) \log n}{\log d} \) and \( \epsilon = 1/g(n) \). We have \( k \ll (n^{r-s}p(1-p))^{1/4} \) and \( \epsilon \to 0 \). Then we have \( (1 + o(1))/(1 + \epsilon)^{2k} = O(n^{-s}) \), which implies that almost surely

\[
\| C \| \leq (1 + o(1))^{2k} \sqrt{U} = (1 + o(1))^{2k} \left( \frac{n^{s+k(1-s)(r_s)}(r_s)^k}{(k+1)(s!)^k((r-2s)!)^k} \frac{2^k}{k} p^k (1 - p)^k \right)^{\frac{1}{2k}} < n^{\frac{k}{2k}} 2^{\frac{1}{2k}} \sqrt{\frac{n^{r-s}(r_s)p(1-p)}{s!(r-2s)!}} = \left( 2^{\left( \frac{r-s}{s} \right)} + o(1) \right) \sqrt{d(1-p)}.
\]

Recall \( M_2 = \frac{1}{(r_s)^d} C \). We have

**Corollary 1.** If \( p(1 - p) \gg \frac{\log^4 n}{n^2} \), then we have \( \| M_2 \| \leq (2 + o(1)) \sqrt{\frac{d}{d^2}} \) almost surely.

We have the following upper bound on \( \| M_1 \| \).
Lemma 7. If \( p(1 - p) \gg \frac{\log^2 n}{n^2} \), then we have \( \|M_1\| = O\left(\frac{\sqrt{(1-p) \log N}}{d}\right) \) almost surely.

Proof. We have

\[
M_1 = \frac{1}{(r-s)} \left( D^{-1/2} CD^{-1/2} - d^{-1} C \right)
\]

\[
= \frac{1}{(r-s)} \left( (D^{-1/2} - d^{-1/2} I) CD^{-1/2} + d^{-1/2} C(D^{-1/2} - d^{-1/2} I) \right).
\]

Note \( \|D^{-1/2} - d^{-1/2} I\| \leq |d|^{-1/2} - d^{-1/2}| = O\left(\frac{\log N}{d}\right) \), \( \|D^{-1/2}\| \leq d_{\min}^{-1/2} = (1 + o(1))d^{-1/2} \), and \( \|C\| \leq (2\left(\frac{r}{s}\right) + o(1))\sqrt{d(1-p)} \). We have

\[
\|M_1\| = \frac{1}{(r-s)} \|D^{-1/2} - d^{-1/2} I\| C \|D^{-1/2} + d^{-1/2} C\| D^{-1/2} - d^{-1/2} I \|
\]

\[
= \frac{1}{(r-s)} \|D^{-1/2} - d^{-1/2} I\| \|C\| \|D^{-1/2}\| \|D^{-1/2} - d^{-1/2} I\|
\]

\[
= \frac{1}{(r-s)} \|D^{-1/2} - d^{-1/2} I\| \|C\| \left(\frac{\|D^{-1/2}\|}{d^{-1/2}}\right)
\]

\[
= O\left(\frac{\sqrt{(1-p) \log N}}{d}\right).
\]

Proof of Theorem 2. Combining Lemmas 2, 4, 7, and Corollary 1, we have

\[
\|M\| = \|M_1 + M_2 + M_3 + M_4\|
\]

\[
\leq \|M_1\| + \|M_2\| + \|M_3\| + \|M_4\|
\]

\[
\leq O\left(\frac{\sqrt{(1-p) \log N}}{d}\right) + \frac{(2 + o(1))\sqrt{1-p}}{\sqrt{d}} + O\left(\frac{\sqrt{\log N}}{n\sqrt{d}}\right) + (1 + o(1))\sqrt{\frac{1-p}{d}}
\]

\[
= (3 + o(1))\sqrt{\frac{1-p}{d}}.
\]

In the last step, we use the fact \( \frac{\sqrt{\log N}}{n\sqrt{d}} = o\left(\frac{\sqrt{1-p}}{d}\right) \) since \( 1 - p \gg \frac{\log n}{n^2} \).

By Weyl’s Theorem, for \( 1 \leq k \leq \binom{n}{t} - 1 \), we have

\[
\left| \lambda^{(s)}_k(H'(n,p)) - \lambda^{(s)}_k(K'_n) \right| \leq \|M\| \leq (3 + o(1))\sqrt{\frac{1-p}{d}}.
\]

Reall that \( X_F \) is the random indicator variable for \( F \) being an edge in \( H'(n,p) \). For any fixed positive integer \( t \), the terms in Trace\( (C') \) are of the form

\[
c_{S_1}c_{S_2}c_{S_3} \cdots c_{S_t}.
\]
Here \( c_{ST} = W(S, T) - E(W(S, T)) = \sum_{F \in \binom{V}{r}} (X_F - E(X_F)) \) if \( S \cap T = \emptyset; c_{ST} = 0 \) otherwise.

Note \( c_{S,S_i} = 0 \) if \( S_i \cap S \neq \emptyset \). Thus we need only to consider the sequence \( S_1, S_2, \ldots, S_i, S_1 \) such that \( S_i \cap S_{i+1} = \emptyset \) for each \( 1 \leq i \leq t \), here \( S_{i+1} = S_1 \).

For \( F \in \binom{V}{r} \) and \( S, T \in \binom{V}{s} \), we define a random variable \( c^F_{ST} \) as follows.

\[
c^F_{ST} = \begin{cases} 
  X_F - E(X_F) & \text{if } S \cap T = \emptyset \text{ and } S \cup T \subseteq F; \\
  0 & \text{otherwise.}
\end{cases}
\]

The sequence \( w := S_1 F_1 S_2 F_2 S_3 \ldots S_i F_i S_1 \) is called a closed \( s \)-walk of length \( t \) if

1. \( S_1, \ldots, S_i \in \binom{V}{s} \),
2. \( F_1, \ldots, F_i \in \binom{V}{r} \),
3. \( S_i \cap S_{i+1} = \emptyset \), for \( i = 1, 2, \ldots, t \),
4. \( S_i \cup S_{i+1} \subseteq F_i \), for \( i = 1, 2, \ldots, t \).

Here we use the convention \( S_{i+1} = S_1 \). Those \( r \)-sets \( F_i \)'s are referred as edges while those \( s \)-sets \( S_i \)'s are referred as stops. For \( 1 \leq i \leq t \), we say \( w \) walks from \( S_i \) to \( S_{i+1} \) at step \( i \) via the edge \( F_i \).

Using the notation above, we rewrite the trace as

\[
\text{Trace}(C') = \sum_{\text{closed } s\text{-walks}} c^F_{S_1 S_2 S_3} \cdots c^F_{S_i S_1},
\]

where the summation is over all possible closed \( s \)-walks of length \( t \).

Taking expectation on both sides, we get

\[
E(\text{Trace}(C')) = \sum_{\text{closed } s\text{-walks}} E(c^F_{S_1 S_2 S_3} \cdots c^F_{S_i S_1}).
\]

The terms in the product above can be regrouped according to the values of \( F_i \)'s; those terms with distinct \( F_i \)'s are independent to each other. Since \( E(c^F_{S,T}) = 0 \), the contribution of a closed walk is 0 if some \( F \) appears only once. Thus we need only to consider the set of closed walks where each edge appears at least twice; we call these closed walks as good closed walks. A good closed walk can contain at most \( \lfloor \frac{t}{2} \rfloor \) distinct edges.

Let \( G_i \) be the set of good closed walks of length \( t \) with \( i \) distinct edges. For \( 1 \leq i \leq \lfloor \frac{t}{2} \rfloor \), let \( G'_i \) be the set of good closed walks with exactly \( i \) distinct edges and \( j \) distinct vertices; we have \( G_i := \cup_{j} G'_i \).

We consider a good closed walk in \( G_i \). When a new edge comes in the walk, it can bring in at most \( (r-s) \) new vertices. Thus such a good closed walk covers at most \( m_i := s + i(r-s) \) vertices. Any walk contains at least one edge. Hence, the number of vertices in a walk from \( G_i \) is in the interval \([r, m_i]\).

We have

\[
E(\text{Trace}(C')) = \sum_{i=1}^{\lfloor \frac{t}{2} \rfloor} \sum_{S_1 F_1 S_2 \ldots S_i S_1 \in G'_i} E(c^F_{S_1 S_2 S_3} \cdots c^F_{S_i S_1}).
\]
Assume that an edge $F$ occurs $q$ times in a good closed walk and $T := \{i : 1 \leq i \leq t$ and $F_i = F\}$. We have $\Pr(\prod_{i \in T} c_{S_i S_{i+1}}^F = (1 - p)^q) = p$ and $\Pr(\prod_{i \in T} c_{S_i S_{i+1}}^F = (-p)^q) = 1 - p$. Thus, for each positive integer $q \geq 2$, we have

$$E \left( \prod_{i \in T} c_{S_i S_{i+1}}^F \right) = (1 - p)^q p + (-p)^q (1 - p) \leq p(1 - p).$$

The equality holds for $q = 2$.

Pick a good closed walk $w := S_1 F_1 S_2 F_2 S_3 \ldots S_l F_l S_1$ in $G$. Let $F^1, \ldots, F^l$ be the list of distinct edges in the order as they appear in $w$.

For each $1 \leq l \leq i$, let $T_l := \{1 \leq j \leq t : F_j = F^l\}$; then $\sum_{l=1}^i |T_l| = t$. We have

$$E(\varepsilon_{S_1 S_2}^F \varepsilon_{S_2 S_3}^F \ldots \varepsilon_{S_i S_1}^F) = \prod_{l=1}^i \prod_{j \in T_l} E(\varepsilon_{S_j S_{j+1}}^F) \leq \prod_{l=1}^i p(1 - p) = p^i(1 - p)^l.$$

This implies

$$\sum_{S_1 F_1 S_2 \ldots S_l F_l S_1 \in G} E(\varepsilon_{S_1 S_2}^F \varepsilon_{S_2 S_3}^F \ldots \varepsilon_{S_i S_1}^F) \leq |G| p^i(1 - p)^l \tag{20}$$

for all $1 \leq i \leq \lfloor \frac{t}{2} \rfloor$. In particular, the equality holds when $t = 2i$. Combining Eq. (19) and inequality (20), we get

$$E(\text{Trace}(C^i)) \leq \sum_{i=1}^{\lfloor \frac{t}{2} \rfloor} |G_i| p^i(1 - p)^i. \tag{21}$$

Now we estimate the value of $|G_i^l|$, the number of good closed walks of length $t$ on $i$ edges and $j$ vertices. Let $w$ be a good closed walk in $G_i^l$ and $F^1, \ldots, F^l$ be the list of $i$ distinct edges in the order of their first occurrence in $w$. For $2 \leq k \leq i$, let $\cdots SF^k S \cdots$ be a piece of sequence in $w$ where the edge $F^k$ occurs first time; $S$ is called the in-stop of $F^k$ and $S'$ is called the out-stop of $F^k$.

The following lemma will state the hypergraph structure of these $i$ edges. We will use the following notation. Let $S = \cup_{l=1}^i (F^l)$. For any $s$-set $S \in S$, let $f_S$ be the number of edges in $\{F^1, F^2, \ldots, F^l\}$ containing $S$; here $f_S$ can be viewed as the degree of $S$ in the hypergraph with the edge set $\{F^1, F^2, \ldots, F^l\}$.

Define

$$f'_S \begin{cases} f_S - 1 & \text{if there exists a unique } k \text{ such that } S = F^k \cap (\cup_{l=1}^{k-1} F^l), \\ f_S & \text{otherwise.} \end{cases}$$

**Lemma 8.** Assume that $F^1, \ldots, F^l$ is the list of distinct edges in the order as they appear in $w \in G_i^l$. Then we have

$$\sum_{S \in S} (f'_S - 1) \leq \left( 1 + \frac{2}{s} \binom{r}{s-1} \right) (m_i - j).$$

**Proof.** For $2 \leq k \leq i$, let $x_k = |F^k \setminus (\cup_{l=1}^{k-1} F^l)|$; we have

$$0 \leq x_k \leq r - s.$$
Thus,
\[ j = r + x_2 + x_3 + \cdots + x_i \leq r + (i - 1)(r - s) = m_i. \]

Since a new edge \( F^k \) can contribute at most \( \binom{r-x_k}{s} \) to \( \sum_{S \in S}(f_S - 1) \), we have
\[
\sum_{S \in S}(f_S - 1) \leq \sum_{k=2}^{i} \binom{r-x_k}{s}.
\]

Let \( K := \{ k : x_k = r-s, 2 \leq k \leq i \} \) and \( \overline{K} = [2, \ldots, i]\setminus K \). The edges in the set \( \{ F^k : k \in K \} \) are called forward edges while the edges in the set \( \{ F^k : k \in \overline{K} \} \) are called backward edges. Note each backward edge contribute at least one to \( m_i - j \); thus
\[ m_i - j \geq |\overline{K}|. \]

Note for each \( k \in K \), \( \binom{r-x_k}{s} = 1 \). We have
\[
\sum_{S \in S}(f_S - 1) \leq \sum_{k \in K} \binom{r-x_k}{s} = |K| + \sum_{k \in K} \frac{r-x_k-s+1}{s}(r-x_k) - z
\]
\[
\leq |K| + \sum_{k \in \overline{K}} \frac{2}{s} \left( \frac{r}{s-1} \right) (r-x_k-s)
\]
\[
= |K| + \frac{2}{s} \left( \frac{r}{s-1} \right) (m_i - j).
\]

For any \( k \in K \), let \( S(F^k) \) be the in-stop of \( F^k \). List the elements in \( K \) as \( k_1, k_2, \ldots, k_{|K|} \) in an increasing order. Consider the sequence of stops \( S(F^{k_1}), S(F^{k_2}), \ldots, S(F^{k_{|K|}}) \) (not necessarily distinct). Let \( z \) be the number of distinct stops in the sequence. If \( S(F^{k_l}) \) does not appear the first time in the sequence above, then we consider the partial walk \( S(F^{k_{l-1}})F^{k_{l-1}} \cdots S(F^{k_l})F^{k_l} \). We observe that there exists at least one backward edge \( F^{k_l} \) for some \( l' \in (k_{l-1}, k_l) \); otherwise, the stop \( S(F^{k_l}) \) has to occur first time in \( w \) when \( F^{k_l} \) is created. Thus,
\[ |K| \leq z + |\overline{K}| \leq z + m_i - j. \]

Hence,
\[
\sum_{S \in S}(f_S - 1) = \sum_{S \in S}(f_S - 1) - z
\]
\[
\leq |K| + \frac{2}{s} \left( \frac{r}{s-1} \right) (m_i - j) - z
\]
\[
\leq \frac{2}{s} \left( \frac{r}{s-1} \right) (m_i - j) + m_i - j
\]
\[
= \left( 1 + \frac{2}{s} \left( \frac{r}{s-1} \right) \right) (m_i - j).
\]

The proof of this Lemma is finished. \[ \blacksquare \]
Lemma 9. For $1 \leq i \leq \lfloor \frac{t}{2} \rfloor$ and $r \leq j \leq m$, we have
\[
|G_i| \leq \binom{t-2}{t-2i} \frac{1}{i+1} \binom{2i}{i} \left( \frac{r-s}{s} \right)^{t-i} \frac{n^{m_i}}{(s!)^i ((r-2s)!)^i} \frac{n^{m_j}}{(s!)^i ((r-2s)!)^i}.
\]

Here $C_1$ and $C_2$ depend only on $r$ and $s$, independent of $i$, $j$, and $n$.

Corollary 2. For $1 \leq i \leq \lfloor \frac{t}{2} \rfloor$ and $n \gg t^{c_2}$, we have
\[
|G_i| \leq (1 + o(1)) \binom{t-2}{t-2i} \frac{1}{i+1} \binom{2i}{i} \left( \frac{r-s}{s} \right)^{t-i} \frac{n^{m_i}}{(s!)^i ((r-2s)!)^i}.
\]

To prove Lemma 9, we use a similar approach due to Füredi and Komlós [23]. Namely, we will assign each walk a sequence of code from an alphabet of three symbols; but the assignment is slightly different from theirs. Füredi-Komlós’ code assignment is based on the fact that every connected graph has a spanning tree. We do not have the analogue of this fact in hypergraphs. As the result, we have to modify the code assignment; the analysis is much harder than the one in the graph case.

Proof of Lemma 9. We can associate a walk $w \in G_i^j$ with a code of length $t$ consisting of three symbols: ‘(’, ‘)’, and ‘∗’. We scan the edges of the walk $w$ from left to right; if an edge appears first time, then we assign the code ‘(’; if an edge appears second time, then we assign the code ‘)’; otherwise, we assign the code ‘∗’.

For example, consider the following good walk with $i = 3, j = 8$, and $t = 8$:
\[
w = S^1 F^1 S^2 F^2 S^3 F^3 S^4 F^4 S^5 F^5 S^6 F^6 S^7 F^7 S^8 F^8 S^9 F^9 S^10 F^10 S^11.
\]

Here edges are: $F^1 = (1, 2, 3, 4, 5), F^2 = (4, 5, 6, 7, 8), F^3 = (7, 8, 5, 3, 4)$. Stops are: $S^1 = (1, 2), S^2 = (4, 5), S^3 = (7, 8), S^4 = (3, 4), S^5 = (2, 5)$. The code for this walk is (((∗))∗∗).

Since $w$ has $i$ distinct edges, there are $i$ ‘(’, ‘)’, and $(t-2i)$ ‘∗’s. Note that the number of ‘(’ is always greater than or equal to the number of ‘)’ at any point when the sequence is read from left to right; each ‘(’ has a matched ‘)’ in the sequence. The symbol ‘∗’ starts at position three and up. There are $\binom{t-2}{t-2i}$ ways to choose the ‘∗’-positions and $\frac{1}{i+1} \binom{2i}{i}$ ways to choose $i$ matched parentheses (the Catalan number). The number of such codes is
\[
\binom{t-2}{t-2i} \frac{1}{i+1} \binom{2i}{i}.
\]

To construct a walk from a given code, we first select $j$ vertices and we have $\binom{n}{j}$ choices. For the fixed set of $j$ vertices, we aim to recover walks from $W_i^j$ for this fixed $j$ vertices and the given code. We scan the symbols from left to right. The first symbol is always ‘(’. There are $\binom{n}{j}$ ways to choose the first stop $S_1$. Assume that we already build a partial walk and need to decide the next edge and the next stop. We always select a edge first followed by selecting a stop. Each stop is a $s$-set of the current edge; there are at most $\binom{r-s}{s}$ ways to choose the next stop $S$ (we have $t$ stops left). The choices of selecting the next edge depends on the next symbol in the code sequence. Let $b_1, b_2,$ and $b_3$ be the product of the number of ways to choose the next edge at the ‘(’, ‘)’, and ‘∗’ positions respectively. We have
\[
|G_i| \leq \binom{t-2}{t-2i} \frac{1}{i+1} \binom{2i}{i} \frac{n^j}{j} \binom{r-s}{s} b_1 \cdot b_2 \cdot b_3.
\]
First we estimate $b_i$, the number of ways to choose new edges $F^1, \ldots, F^i$ given $j$ vertices and the first stop $S_1$. Recall that $F^1, \ldots, F^i$ is the list of distinct edges in the order as they first appear in $w$. For $2 \leq l \leq i$, let $\tilde{F}^l := F^l \setminus (\bigcup_{j=1}^{l-1} F^j)$, $x_l := |\tilde{F}^l|$, and $y_l := r - s - x_l$. We also define $\tilde{F}^1 := F^1 \setminus S_1; x_1 := |\tilde{F}^1| = r - s,$ and $y_1 = 0$. Note that $\bigcup_{j=1}^{l} \tilde{F}^j$ forms a partition of the remaining ($j - s$) selected vertices. The number of ways to choose such a partition is

$$\frac{(j - s)!}{x_1!x_2! \cdots x_i!}.$$

To choose $F^l$, we need select $x_l$ new vertices and $y_l$ old vertices; each old vertex has at most $j$ choices. We have

$$b_i \leq \sum_{x_2, \ldots, x_i} \frac{(j - s)!}{(r - s)!x_2! \cdots x_i!} \frac{\sum_{l=2}^{i} y_l}{j}.$$

Observe that $\sum_{l=2}^{i} y_l = m_i - j$ and

$$\frac{(j - s)!}{(r - s)!x_2! \cdots x_i!} \leq \frac{(j - s)}{(r - s)} \frac{(m_i - r)!}{((r - s)!)^{r-1}}.$$

The number of ways to choose $x_2, \ldots, x_i$ is the same as the number of ways to choose $y_2, \ldots, y_i$, which is $\binom{m_i - j + r - 2}{m_i - j} \leq (m_i - j + i - 2)^{m_i - j}$. Therefore,

$$b_i \leq \sum_{x_2, \ldots, x_i} \frac{(j - s)!}{(r - s)!x_2! \cdots x_i!} \frac{\sum_{l=2}^{i} y_l}{j} \leq \frac{(j - s)}{r - s} \frac{(m_i - r)!}{((r - s)!)^{r-1}} (m_i - j + i - 2)^{m_i - j}$$

$$\leq \frac{(j - s)}{r - s} \frac{(m_i - r)!}{((r - s)!)^{r-1}} \left( \frac{m_i + i - 2}{2} \right)^{2(m_i - j)}. \quad (24)$$

There is at most $i$ choices of edges at each ‘*’ position. Thus

$$b_s \leq \hat{t}^{s-2}. \quad (25)$$

Now we give an upper bound on $b_j$. We first present an easy bound for $b_j$. Edge $F$ can be chosen at most one ‘*’-position. Recall $S = \bigcup_{l=1}^{j} (F^l)$. For any possible stop $S \in S$, $S$ can appear at the ‘*’-positions for at most $f_s$ times; each occurrence of $S$ involves different edges since we are considering the second occurrence of edges. Thus,

$$b_j \leq \prod_{S \in S} f_s! \leq \prod_{S \in S} f_s^{f_s-1} \leq \hat{t}^{\sum_{S \in S} (f_s-1)}.$$

We need a better upper bound for $b_j$. Consider a stop $S$ which is first chosen at a ‘*’-position. Let $F$ be the edge on the walk right before the ‘*’-position; i.e., the walk $w$ enters $S$ through $F$. If this $F$ occurred before, then the choices of edges at ‘*’-positions starting with $S$ is at most

$$(f_s - 1)! \leq \hat{t}^{s-2}.$$
If this $F$ occurs first time and $F$ is an forward edge, then there is only one choice for the next edge leaving $S$; namely $F$ itself. In this case, the choices of edges at ')-positions starting with $S$ is at most

$$(f_s - 1)! \leq i^{f_s - 2}.$$  

Otherwise, $F$ must be a backward edge. Each such a stop $S$ gives additional factor $f_s \leq i$. The number of backward edges is at most $m_i - j$. Since $F$ contains at most $\binom{j}{i}$ stops, the number of such $S$ is at most $\binom{j}{i}(m_i - j)$. A additional factor $i^{\binom{j}{i}(m_i - j)}$ is enough. We have

$$b_i \leq i^{\binom{j}{i}(m_i - j)} \prod_{S|f_s \geq 2} f_s^{f_s - 2}$$

$$\leq i^{\binom{j}{i}(m_i - j)} \prod_{S} f_s^{f_s - 1}$$

$$\leq i^{\binom{j}{i}(m_i - j)} \left(1 + \frac{2}{s} \binom{j}{i}(m_i - j)\right)$$

$$= i^{\binom{j}{i} + \frac{2}{s} \binom{j}{i}(m_i - j)}.$$  

(26)

Combining Eqs. (23), (24), (25), and (26), we get

$$|\mathcal{G}_k| \leq \left(\frac{t - 2}{t + 2} \right)^{\frac{1}{i + 1}} \binom{2i}{i} \binom{n}{s}^{t - i} \binom{r - s}{s}^{i - 1} \left(\frac{m_i + i - 2}{2} \right)^{\binom{j}{i} + \frac{2}{s} \binom{j}{i}(m_i - j)}$$

$$\leq \left(\frac{t - 2}{t + 2} \right)^{\frac{1}{i + 1}} \binom{2i}{i} \binom{n}{s}^{t - i} \binom{r - s}{s}^{i - 1} \left(\frac{C_1 C_2}{n} \right)^{m_i - j}.$$  

Here we set $C_1 = 4(r - s)^3$ and $C_2 = \binom{j}{i} + 4 + \frac{2}{s} \binom{j}{i}(m_i - j).$

Lemma 10. If $t = 2k$ is even, then we have

$$|\mathcal{G}_k| = \binom{n}{m_k} \frac{m_k!}{(k + 1)(s!)^{k+1}(r - 2s)!} \binom{2k}{k}.$$  

(27)

Proof. We will construct a bijection from $\mathcal{G}_k^{mk}$ to a triple $(U, \mathcal{P}, C)$, where $U$ is a set of $m_k$ vertices, $\mathcal{P}$ is a partition of $U$ into $(k + 1)$ $s$-sets and $k$ $(r - 2s)$-sets, and $C$ is a code consisting of $k$ pairs valid parentheses.

For any good walk $w \in \mathcal{G}_k^{mk}$, let $U$ be the set of vertices covered by $w$. Note each edge appears exactly twice. We define a graph $T$, whose vertices are the stops in $w$. Two stops are connected if they belong to one edge. Observe that $T$ is acyclic and connected; $T$ must be a tree. Since $T$ has exactly $k$ edges, $T$ must have $k + 1$ vertices. Hence $w$ has exactly $k + 1$ stops; we list them as $S^0, S^1, \ldots, S^k$. For $1 \leq i \leq k$, let $E_i$ be the set of $(r - 2s)$ vertices
in $F^i$ but not in any stops. We get a partition: $U = (\bigcup_{j=0}^{k} S_j) \cup (\bigcup_{j=1}^{k} E_j)$. A code consisting of $k$ ‘(‘ and $k$ ‘)’ is generated as follows. When we scan the walk from left to right, if an edge appears the first time, we append the code by a ‘(‘; otherwise, we append the code by a ‘)’. The code is a valid sequence of $k$ pairs of parentheses. (In this case, the number of ‘*’s is zero.) It suffices to recover a walk from a partition of $[m_k]$ and a sequence of valid parentheses.

Given a partition of $U$, namely

$$\left( \bigcup_{j=0}^{k} S_j \right) \cup \left( \bigcup_{j=1}^{k} E_j \right)$$

and a sequence of $k$ pairs valid parentheses, we first build a rooted tree $T$ as follows. At each time, we maintain a tree $T$, a current stop $S$, a set of unused stops $\mathbb{S}$. Initially $T$ contains nothing but the root stop $S_0$, $S := S_0$, and $\mathbb{S} = \{S_1, S_2, \ldots, S_k\}$. At each time, read a symbol from the sequence. If the symbol is an open parenthesis, then find an $S_i$ in $\mathbb{S}$ with index $i$ as small as possible, delete $S_i$ from $\mathbb{S}$, attach $S_i$ to $T$ as a child stop of $S$, and let $S := S_i$; if the symbol is “)”, then let $S$ point to the the parent stop of the current $S$. Repeat this process until all symbols from the sequence are processed.

Since every closed parenthesis has a matching open parenthesis, this process never gets stuck. When the process ends, a rooted tree $T$ on the vertex set $\{S_0, \ldots, S_k\}$ is created. For $1 \leq i \leq k$, let $F_i$ be the union of $E_i$ and two ends of $i$-th edge, which created in the process. For example, for $k = 3$, if the sequence is $(())$, then the corresponding good closed walk is

$$S_1F_1S_2F_2S_3F_3S_4F_4S_5S_1$$

where $F_1 = S_1 \cup S_2 \cup E_1$, $F_2 = S_2 \cup S_3 \cup E_2$, and $F_3 = S_3 \cup S_4 \cup E_3$.

Thus, this is a bijection from $\mathbb{G}_k^{m_k}$ to all triples $\{U, P, C\}$. The number of ways to choose $m_k$ vertices is $\binom{n}{m_k}$. The number of ways to choose these sets $S_0, S_1, \ldots, S_k, E_1, \ldots, E_k$ as a partition of $U$ is

$$\frac{m_k!}{(s!)^{k+1}((r-2s)!)^k}.$$ 

The number of sequences of $k$ pairs valid parentheses is the Catalan number $\frac{1}{k+1} \binom{2k}{k}$. By taking product of these three numbers, we get Eq. (27).

**Proof of Lemma 5.** By Eqs. (21) and (22), we have

$$E(\text{Trace}(C')) \leq \sum_{i=1}^{\lfloor \frac{s}{2} \rfloor} |G_i| p^i (1 - p)^j \leq \left(1 + o(1)\right) \sum_{i=1}^{\lfloor \frac{s}{2} \rfloor} a_i.$$ 

Here $a_i := \binom{t-2}{t-2i} \frac{1}{t+1} \frac{1}{t+1} \binom{2i}{i} \binom{r-2s}{s} \frac{n^{m_i} p^i (1 - p)^i}{(t-i)! (r-2s)!}$. We get

$$\frac{a_i}{a_{i+1}} = \frac{\binom{t-2}{t-2i} \frac{1}{t+1} \frac{1}{t+1} \binom{2i}{i} \binom{r-2s}{s} \frac{n^{m_i} p^i (1 - p)^i}{(t-i)! (r-2s)!}}{\binom{t-2}{t-2i-2} \frac{1}{t+2} \frac{1}{t+2} \binom{2i+2}{i+1} \binom{r-2s-2}{s} \frac{n^{m_i+1} p^{i+1} (1 - p)^{i+1}}{(t-i-1)! (r-2s-2)!}}.$$ 

$$= \frac{r}{2t+1} (2i-1) (i+2) \frac{(r-s)!}{(t+1)(t-2i)(t-2i+1)} \frac{n^{r-s} p (1 - p)}{n^{r-s} p (1 - p)} < \frac{3i^4 (r-s)!}{n^{r-s} p (1 - p)}.$$ 

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When \( n^{-s}p(1 - p) \gg t^4 \), we have \( a_i = o(a_{i+1}) \). Thus,

\[
E(\text{Trace}(C')) \leq (1 + o(1))a_{1/2}.
\]

When \( t = 2k \), we get

\[
E(\text{Trace}(C^{2k})) \leq (1 + o(1))\frac{n^{s+k(r-s)}(r-s)^k}{(k+1)(s!)^{k+1}((r-2s)!)^k} \left( \frac{2k}{k} \right)^k (1 - p)^k.
\]

For \( t = 2k + 1 \), we have

\[
E(\text{Trace}(C^{2k+1})) \leq (1 + o(1))a_k \leq (1 + o(1))\frac{2k^2n^{s+k(r-s)}(r-s)^{k+1}}{(k+1)(s!)^{k+1}((r-2s)!)^k} \left( \frac{2k}{k} \right)^k (1 - p)^k.
\]

Now we assume \( k = o(\log(n^{-s}p(1 - p))) \). For \( t = 2k \), let

\[
b_k := |G_{mk}|p^k(1 - p)^k = \left( \frac{n}{m_k} \right)^k \frac{m_k!}{(k+1)(s!)^{k+1}((r-2s)!)^k} \left( \frac{2k}{k} \right)^k (1 - p)^k.
\]

It is clear that \( E(\text{Trace}(C^{2k})) \geq b_k \). We also have

\[
E(\text{Trace}(C^{2k})) - b_k \leq \sum_{i=1}^{k-1} |G_i|p^i(1 - p)^i + \sum_{j=r}^{m_k-1} |G_j|p^k(1 - p)^k
\]

\[
= (1 + o(1)) \sum_{i=1}^{k-1} a_i + a_k \sum_{j=r}^{m_k-1} \left( \frac{C_k C_j}{n} \right)^{m_k-j}.
\]

Note \( a_k = (1 + o(1))\left( \frac{r-s}{s} \right)^k b_k \) and \( a_i = O(a_k \left( \frac{k^4}{n^{r-s}p(1 - p)} \right)^{k-i}) \). We conclude

\[
E(\text{Trace}(C^{2k})) - b_k = O \left( b_k \left( \frac{r-s}{s} \right)^k \left( \frac{k^4}{n^{r-s}p(1 - p)} + \frac{C_k C_j}{n} \right) \right) = o(b_k).
\]

Here we use the fact \( \left( \frac{r-s}{s} \right)^k \left( \frac{k^4}{n^{r-s}p(1 - p)} + \frac{C_k C_j}{n} \right) = o(1) \) since \( k = o(\log(n^{-s}p(1 - p))) \).

\[\text{4. THE SEMICIRCLE LAW}\]

Let us review the definition of the Semicircle Law. Let \( F(x) \) be the continuous distribution function with density \( f(x) \) such that \( f(x) = (2/\pi)\sqrt{1 - x^2} \) when \( |x| \leq 1 \) and \( f(x) = 0 \) when \( |x| > 1 \). Let \( A \) be a Hermitian matrix of dimension \( N \times N \). The empirical distribution of the eigenvalues of \( A \) is

\[
F(A, x) := \frac{1}{N} \left| \text{eigenvalues of } A \text{ less than } x \right|.
\]

We say, the empirical distribution of the eigenvalues of \( A \) asymptotically follows the Semicircle Law centered at \( c \) with radius \( R \) if \( F(\frac{1}{R}(A - cI), x) \) tends to \( F(x) \) in probability.
as $N$ goes to infinity. (In this case, we write $F(\frac{1}{k}(A - cI), x) \overset{p}{\to} F(x).$ If $c$ is the center of
the Semicircle Law, then any $c' = c + o(R)$ is also the center of the Semicircle Law.

**Theorem 5.** If $n^{-\frac{1}{2}}p(1 - p) \to \infty$, then the empirical distribution of the eigenvalues of
$W - E(W)$ follows the semicircle law centered at 0 with radius $2\sqrt{(r - \frac{1}{s})(\frac{n}{s})}p(1 - p)$.

**Proof.** Let $R := 2\sqrt{(r - \frac{1}{s})(\frac{n}{s})}p(1 - p)$, $C := W - E(W)$, and $C_{nor} := \frac{1}{R}C$.

To prove the theorem, we need to show that for any fixed $t$, the $t$-th moment of $F(C_{nor}, x)$
(with $n$ goes to infinity) is asymptotically equal to the $t$-th moment of $F(x)$. We know the
$t$-th moment of $F(C_{nor}, x)$ equals $\binom{\frac{n}{s}}{t}^{-1}E(\text{Trace}(C_{nor}^{t})).$ For even $t = 2k$, the $t$-th moment of
$F(x)$ is $(2k)!/2^{2k}k!(k + 1)!$. For odd $t$, the $t$-th moment of $F(x)$ is 0.

In order to prove the theorem, we need to show for any fixed $k$,

$$\frac{1}{\binom{n}{t}}E(\text{Trace}(C_{nor}^{2k})) = (1 + o(1)) \frac{(2k)!}{2^{2k}k!(k + 1)!}$$

and

$$\frac{1}{\binom{n}{t}}E(\text{Trace}(C_{nor}^{2k+1})) = o(1).$$

We know

$$E(\text{Trace}(C_{nor}^{t})) = \frac{1}{R^{t}}E(\text{Trace}(C^{t}))$$

for any $t$. By Eq. (18) of Lemma 5, we have

$$E(\text{Trace}(C^{2k})) = (1 + o(1)) \frac{n^{r+k(r-s)}}{(k + 1)(s!)^{k+1}(r - 2s)!^{k+1}} \binom{2k}{k} \frac{1}{(k + 1)!}p^k(1 - p)^k.$$ 

Then

$$\frac{1}{\binom{n}{t}}E(\text{Trace}(C_{nor}^{2k})) = (1 + o(1)) \frac{(2k)!}{2^{2k}k!(k + 1)!}$$

as desired.

By Eq. (17) of Lemma 5, we have

$$E(\text{Trace}(C^{2k+1})) = O\left(\frac{2k^{2}n^{r+k(r-s)}p^k(1 - p)^k(r - 2s)!^{k+1}}{(k + 1)(s!)^{k+1}(r - 2s)!^{k+1}} \binom{2k}{k}(k + 1)!\right).$$

Thus

$$\frac{1}{\binom{n}{t}}E(\text{Trace}(C_{nor}^{2k+1})) = O\left(\frac{2k^{2}(2k)!^{k+1}}{2^{2k}(k + 1)!R}\right) = o(1).$$

Here $k$ is any constant but $R \to \infty$. The theorem is proved. 

The following Lemma is useful to derive the Semicircle Law from one matrix to the other.

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Lemma 11. Let $A$ and $B$ be two $(N \times N)$-Hermitian matrices. Suppose that the empirical distribution of the eigenvalues of $A$ follows the Semicircle Law centered at $c$ with radius $R$. If either $\|B\| = o(R)$ or the rank of $B$ is $o(N)$, then the empirical distribution of the eigenvalues of $A + B$ also follows the Semicircle Law centered at $c$ with radius $R$.

Proof. It suffices to show $F(\frac{1}{R}(A + B - cI), x) \xrightarrow{p} F(x)$. First we assume $\|B\| = o(R)$. By Weyl’s Theorem, for $1 \leq k \leq N$, we have

$$\left| \mu_k \left( \frac{1}{R}(A + B - cI) \right) - \mu_k \left( \frac{1}{R}(A - cI) \right) \right| \leq \frac{\|B\|}{R} = o(1).$$

Hence

$$F \left( \frac{1}{R}(A - cI), x - \frac{\|B\|}{R} \right) \leq F \left( \frac{1}{R}(A + B - cI), x \right) \leq F \left( \frac{1}{R}(A - cI), x + \frac{\|B\|}{R} \right).$$

Since $\|B\| = o(R)$, we have $F(\frac{1}{R}(A-cI), x - \frac{\|B\|}{R}) \xrightarrow{p} F(x)$ and $F(\frac{1}{R}(A-cI), x + \frac{\|B\|}{R}) \xrightarrow{p} F(x)$. By the Squeeze theorem, we have $F(\frac{1}{R}(A + B - cI), x) \xrightarrow{p} F(x)$.

Now we assume $\text{rank}(B) = o(N)$. Let $U$ be the kernel of $B$ (i.e. $B|_U = 0$); $U$ has co-dimension $\text{rank}(B)$. Let $Z := \frac{1}{R}(A - cI)|_U = \frac{1}{R}(A + B - cI)|_U$. By Cauchy’s interlace theorem [26], for $1 \leq j \leq N - \text{rank}(B)$, we have

$$\mu_j \left( \frac{1}{R}(A - cI) \right) \leq \mu_j(Z) \leq \mu_j + \text{rank}(B) \left( \frac{1}{R}(A - cI) \right),$$

$$\mu_j \left( \frac{1}{R}(A + B - cI) \right) \leq \mu_j(Z) \leq \mu_j + \text{rank}(B) \left( \frac{1}{R}(A + B - cI) \right).$$

Thus, for $\text{rank}(B) + 1 \leq j \leq N - \text{rank}(B)$, we have

$$\mu_j - \text{rank}(B) \left( \frac{1}{R}(A - cI) \right) \leq \mu_j \left( \frac{1}{R}(A + B - cI) \right) \leq \mu_j + \text{rank}(B) \left( \frac{1}{R}(A - cI) \right).$$

It implies

$$F \left( \frac{1}{R}(A - cI), x \right) - \frac{\text{rank}(B)}{N} \leq F \left( \frac{1}{R}(A + B - cI), x \right) \leq F \left( \frac{1}{R}(A - cI), x \right) + \frac{\text{rank}(B)}{N}.$$ 

Since $\text{rank}(B) = o(N)$, we have $F(\frac{1}{R}(A - cI), x) \pm \frac{\text{rank}(B)}{N} \xrightarrow{p} F(x)$. By the Squeeze theorem, we have $F(\frac{1}{R}(A + B - cI), x) \xrightarrow{p} F(x)$.

Proof of Theorem 3. Recall

$$L^{(s)}(K'_n) - L^{(s)}(H'(n, p)) = M_1 + M_2 + M_2 + M_4.$$ 

We rewrite

$$L^{(s)}(H'(n, p)) = -M_2 + \left( 1 - \frac{(-1)^s}{\binom{n}{s}} \right) I + B_1 - M_3 - M_4 - M_1,$$

where $B_1 = L^{(s)}(K'_n) - (1 - \frac{(-1)^s}{\binom{n}{s}}) I$.

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By Theorem 5, the empirical distribution of the spectrum of \( W - E(W) \) follows the Semicircle Law centered at 0 with radius
\[
2 + o(1) \sqrt{\binom{n-s}{r-s} p(1-p)}.
\]
Since
\[
M_2 = \frac{1}{(r-s)^d} (W - E(W)),
\]
the matrix \((1 - (-1)^r/\binom{n}{r})I - M_2\) follows the Semicircle Law centered at \( c := 1 - (-1)^r/\binom{n}{r} \) with radius \( R \), where
\[
R = (2 + o(1)) \sqrt{\binom{n-s}{r-s} p(1-p)}.
\]
Note \((-1)^r/\binom{n}{r} = o(R)\). We can change the center to 1.

By Theorem 1, \( L^{(o)(K_n)} \) has an eigenvalue \(1 - (-1)^r/\binom{n}{r}\) with multiplicity \(\binom{n}{r} - \binom{n-s}{r-s} \). Thus \( B_1 \) has rank \( \binom{n}{r-s} = o\left(\binom{n}{r}\right) \). We also observe that \( M_3 \) has rank at most 2, \( \|M_1\| = O(\sqrt{(1-p) \log N/d}) = o(R) \), and \( \|M_3\| = O(\sqrt{\log N/(n^2 d)}) = o(R) \). Here we notice \( d \gg \log^{1/3} n \) and \( 1 - p \gg \frac{\log n}{n^2 d^2} \).

By Lemma 11, the matrices \( B_1, M_1, M_3, \) and \( M_4 \) will not affect the Semicircle Law. The proof of this Theorem is finished.

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