On the cohomology of the mapping class group of the punctured projective plane

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Abstract

The mapping class group $\Gamma^k(N_g)$ of a non-orientable surface with punctures is studied via classical homotopy theory of configuration spaces. In particular, we obtain a non-orientable version of the Birman exact sequence. In the case of $\mathbb{R}P^2$, we analyze the Serre spectral sequence of a fiber bundle $F_k(\mathbb{R}P^2)/\Sigma_k \to X_k \to BSO(3)$ where $X_k$ is a $K(\Gamma^k(\mathbb{R}P^2),1)$ and $F_k(\mathbb{R}P^2)/\Sigma_k$ denotes the configuration space of unordered $k$-tuples of distinct points in $\mathbb{R}P^2$. As a consequence, we express the mod-2 cohomology of $\Gamma^k(\mathbb{R}P^2)$ in terms of that of $F_k(\mathbb{R}P^2)/\Sigma_k$.

1 Introduction

Let $S_g$ be a closed orientable surface of genus $g$ and let $\text{Diff}^+(S_g)$ denote the group of orientation preserving diffeomorphisms of $S_g$, under the compact-open topology. The mapping class group $\Gamma_g$ is defined to be $\pi_0 \text{Diff}^+(S_g)$, the group of path components for $\text{Diff}^+(S_g)$. Equivalently, $\Gamma_g$ can be defined as
$\text{Diff}^+(S_g)/\text{Diff}_0(S_g)$, the group of isotopy classes of orientation preserving diffeomorphisms of $S_g$, where $\text{Diff}_0(S_g)$ is the identity component of $\text{Diff}^+(S_g)$. This group plays an important role in the theory of Teichmüller spaces since $\Gamma_g$ acts on the space $T_g$ of complex structures on $S_g$ and the quotient $M_g$ is the moduli space of Riemann surfaces of genus $g$, see [16], [11]. Moreover, its cohomology is closely related to the theory of characteristic classes of $S_g$-bundles, [19], [10]. There exist some variations on the definition above, for instance, consider the subgroup $\text{Diff}^+(S_g;k)$ of orientation preserving diffeomorphisms of $S_g$ which leave a fixed set of $k$ distinct points invariant. The mapping class group of $S_g$ with $k$ marked points $\Gamma^k_g$ is defined to be $\pi_0\text{Diff}^+(S_g;k)$, the group of path components for $\text{Diff}^+(S_g;k)$. Similarly, if $N_g$ is a non-orientable surface of genus $g$, $\Gamma^k(N_g)$ is defined to be $\pi_0\text{Diff}(N_g;k)$ where $\text{Diff}(N_g;k)$ is the group of diffeomorphisms of $N_g$ which leave a set of $k$ points invariant. In the special case when $k = 0$ one recovers the classical non-orientable mapping class group $\Gamma(N_g) = \text{Diff}(N_g)/\text{Diff}_0(N_g)$.

The purpose of this work is to use classical homotopy theory to compute the mod-2 cohomology of $\Gamma^k(\mathbb{R}P^2)$. Let $F_k(M)$ denote the configuration space of ordered $k$-tuples of distinct points in $M$, equipped with the natural action of the symmetric group $\Sigma_k$. Notice that $\text{Diff}(\mathbb{R}P^2)$ acts diagonally on the space $F_k(\mathbb{R}P^2)$ and thus it acts on the unordered configuration space $F_k(\mathbb{R}P^2)/\Sigma_k$. So one is led to consider the Borel construction

$$E\text{Diff}(\mathbb{R}P^2) \times_{\text{Diff}(\mathbb{R}P^2)} F_k(\mathbb{R}P^2)/\Sigma_k.$$  

Using the classical result that $\text{Diff}(\mathbb{R}P^2)$ has $SO(3)$ as a deformation retract [14], [9], the construction above is homotopy equivalent to

$$ESO(3) \times_{SO(3)} F_k(\mathbb{R}P^2)/\Sigma_k.$$  

Moreover, it was proven in [23] that this space is an Eilenberg-MacLane space $K(\pi, 1)$ for $k \geq 2$, and we show here that its fundamental group is isomorphic to $\Gamma^k(\mathbb{R}P^2)$. Therefore, one may use the universal fibration

$$F_k(\mathbb{R}P^2)/\Sigma_k \longrightarrow ESO(3) \times_{SO(3)} F_k(\mathbb{R}P^2)/\Sigma_k \longrightarrow BSO(3)$$  

(1)

to study the cohomology of $\Gamma^k(\mathbb{R}P^2)$. In fact we show in Section 5 that the
associated Serre spectral sequence in mod-2 cohomology collapses at the $E_2$-term. As a consequence, there is an isomorphism of $H^*(BSO(3); F_2)$–modules

$$H^*(\Gamma_k(\mathbb{R}P^2); F_2) \cong H^*(BSO(3); F_2) \otimes H^*(F_k(\mathbb{R}P^2)/\Sigma_k; F_2).$$

The additive structure of the mod-2 cohomology of $F_k(M)/\Sigma_k$ is well understood for $M$ a compact smooth manifold and it is determined by the dimension of $M$ and its Betti numbers, see [4]. In the case of the projective plane, one can also express the mod-2 (co)homology of $F_k(\mathbb{R}P^2)/\Sigma_k$ in terms of that of the classical braid groups $B_k$, see section 6.

Mapping class groups have also been exhaustively studied due to its relations to low dimensional topology. One of these relations is the classical result that the mapping class group of the 2-disk with $k$ marked points is isomorphic to $B_k$, the Artin’s braid group on $k$-strands ([11]). In the case of the sphere $S^2$, the group $\Gamma^k_0$ is isomorphic to the quotient of the braid group $B_k(S^2)$ by its center ([2]). In section 3 we restrict attention to non-orientable surfaces $N_g$ and consider the $Diff_0(N_g)$-Borel construction on the unordered configuration space

$$EDiff_0(N_g) \times_{Diff_0(N_g)} F_k(N_g)/\Sigma_k.$$ 

We define the reduced mapping class group $\tilde{\Gamma}^k(N_g)$ to be the fundamental group of this space. We show that $\tilde{\Gamma}^k(N_g)$ is naturally a subgroup of $\Gamma^k(N_g)$ and fits into the following exact sequence

$$1 \longrightarrow \tilde{\Gamma}^k(N_g) \longrightarrow \Gamma^k(N_g) \longrightarrow \Gamma(N_g) \longrightarrow 1.$$ 

It is easy to show that for $g \geq 3$, the group $\tilde{\Gamma}^k(N_g)$ is isomorphic to the surface braid group $B_k(N_g)$. Thus, the exact sequence above recovers the Birman exact sequence in the non-orientable case. In the case of the real projective plane $\mathbb{R}P^2$, the center of $B_k(\mathbb{R}P^2)$ is $\mathbb{Z}_2$ and we show there is an isomorphism

$$\Gamma^k(\mathbb{R}P^2) \cong B_k(\mathbb{R}P^2)/\mathbb{Z}_2.$$ 

Similarly, in the case of the Klein bottle $\mathbb{K}$, we show the group $\tilde{\Gamma}^k(\mathbb{K})$ is isomorphic to $B_k(\mathbb{K})$ modulo a central $\mathbb{Z}$. 

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The article is organized as follows. In section 2 we show how to construct concrete $K(\pi, 1)$ spaces for mapping class groups using configuration spaces. In section 3 we study the group $\tilde{\Gamma}^k(M) = \pi_0(\text{Diff}_0(M) \cap \text{Diff}(M; k))$ and derive the Birman exact sequence in the non-orientable case. In section 4 we show the Serre spectral sequence of bundle (1) collapses and prove the main theorem. In section 5 we recall the main results from [4] on the mod-2 homology of configuration spaces for compact smooth manifolds, and finally, in section 6 we carry out some explicit calculations for the homology of $F_k(M)/\Sigma_k$ in the case when $M = \mathbb{R}P^2$.

2 Configuration spaces and mapping class groups

Given a manifold $M$ and $k \geq 1$, define the configuration space of ordered $k$-tuples of distinct points in $M$ by

$$F_k(M) = \{(m_1, \ldots, m_k) \in M^k \mid m_i \neq m_j \text{ if } i \neq j\}.$$  

The symmetric group on $k$ letters, $\Sigma_k$, acts naturally on $F_k(M)$ by permutation of coordinates and the orbit space $F_k(M)/\Sigma_k$ is the unordered configuration space. It is a classical result that when $M$ is a surface $\neq S^2, \mathbb{R}P^2$ the spaces $F_k(M)/\Sigma_k$ are Eilenberg-MacLane spaces $K(\pi, 1)$ for the corresponding surface braid groups on $k$ strands, $B_k(M) = \pi_1 F_k(M)/\Sigma_k$. In the exceptional cases when $M = S^2$ or $\mathbb{R}P^2$, one needs to consider the Borel construction with respect to the natural $S^3$ action. Namely, let $G$ be a topological group, $EG$ a contractible space with a right free $G$-action and $X$ a left $G$-space. The associated homotopy orbit space (the Borel construction) $EG \times_G X$ is defined as the quotient of $EG \times X$ under the action of $G$ given by $g(x, m) = (x g^{-1}, g m)$. The projection onto the first coordinate induces a fiber bundle

$$X \longrightarrow EG \times_G X \longrightarrow BG$$

where $BG$ denotes the classifying space of $G$.

Now, in the case when $M = S^2$ or $\mathbb{R}P^2$, the corresponding $K(\pi, 1)$'s for the braid groups $B_k(S^2)$ and $B_k(\mathbb{R}P^2)$ are given as the Borel constructions.
$ES^3 \times _{S^3} F_k(S^2)/\Sigma_k$ and $ES^3 \times _{S^3} F_k(\mathbb{R}P^2)/\Sigma_k$, respectively, where $S^3$ acts on the configuration spaces $F_k(S^2)/\Sigma_k$ and $F_k(\mathbb{R}P^2)/\Sigma_k$ via the double cover $S^3 \to SO(3)$, see [7], [23]. Some further Borel constructions provide $K(\pi, 1)$ spaces for groups related to the mapping class groups as shown next. A proof of the following result can be found in [22], Theorem 1.8.6.

**Lemma 2.1** Let $G$ be a locally compact Hausdorff topological group with a countable basis acting transitively on a locally compact Hausdorff space $X$. Then for each $x \in X$ the map $G \to X$, given by $g \mapsto gx$ is open and the induced map

$$G/G_x \to X$$

is a homeomorphism, where $G_x$ is the isotropy group of $x$.

**Corollary 2.2** Under the assumptions of Lemma 2.1 above, for every $x \in X$ there is a homotopy equivalence $EG \times_G X \simeq BG_x$.

**Proof:** Notice that $EG \times_G X \approx EG \times_G G/G_x = EG/G_x \simeq BG_x$. □

Now, let $M$ be a closed orientable surface and notice that $\text{Diff}^+(M)$ acts diagonally on $F_k(M)$ and thus on the unordered configuration space $F_k(M)/\Sigma_k$. The isotropy group of a fixed configuration $Q_k = \{m_1, \ldots, m_k\}$ in $F_k(M)/\Sigma_k$ is $\text{Diff}^+(M; k)$, the group of orientation preserving diffeomorphisms of $M$ which leave the set $Q_k$ invariant. Thus, by the corollary above, there is a homotopy equivalence

$$E\text{Diff}^+(M) \times_{\text{Diff}^+(M)} F_k(M)/\Sigma_k \simeq B\text{Diff}^+(M; k)$$

which induces an isomorphism of fundamental groups

$$\pi_1\left(E\text{Diff}^+(M) \times_{\text{Diff}^+(M)} F_k(M)/\Sigma_k\right) \cong \pi_1 B\text{Diff}^+(M; k) \cong \pi_0 \text{Diff}^+(M; k).$$
Notice the last group is $\Gamma^k(M)$, the mapping class group of $M$ with $k$ marked points.

**Example:** If $M = S^2$, a classical result of Smale [20] states that the inclusion $SO(3) \rightarrow \text{Diff}^+(S^2)$ is a homotopy equivalence. Thus the natural map

$$ESO(3) \times_{SO(3)} F_k(S^2)/\Sigma_k \rightarrow E\text{Diff}^+(S^2) \times_{\text{Diff}^+(S^2)} F_k(S^2)/\Sigma_k$$

is a homotopy equivalence and the fundamental group of this space is the mapping class group $\Gamma^k(S^2)$. Moreover, F. Cohen proved in [7] that for $k \geq 3$, the above construction is an Eilenberg-MacLane space $K(\Gamma^k(S^2), 1)$ and the cohomology $H^*(\Gamma^k(S^2); \mathbb{F}_2)$ with mod-2 coefficients was described by C.F. Bödigheimer, F. Cohen and D. Peim [5].

### 3 The reduced mapping class group

In this section $M$ will denote a non-orientable surface, although the same arguments apply to the orientable case with the obvious modifications. In the non-orientable case, the mapping class groups $\Gamma(M)$ and $\Gamma^k(M)$ are defined using the group $\text{Diff}(M)$ of all diffeomorphisms of $M$. The homotopy type of $\text{Diff}_0(M)$, the group of diffeomorphisms isotopic to the identity, is also known in this case, see [9] and [14]. We recall the result here:

**Theorem 3.1 ([9], [14])** Let $N$ be a closed non-orientable surface, and let $\text{Diff}_0(N)$ be the group of diffeomorphisms isotopic to the identity. Then,

1. If $N = \mathbb{RP}^2$, then $\text{Diff}_0(N) = \text{Diff}(N)$ is homotopy equivalent to $SO(3)$.
2. If $N$ is the Klein bottle, then $\text{Diff}_0(N)$ is homotopy equivalent to $SO(2)$.
3. If $N = N_g$, a closed non-orientable surface of genus $g \geq 3$, $\text{Diff}_0(N)$ is contractible.
Next we consider the $\text{Diff}_0(M)$-Borel construction. Notice $\text{Diff}_0(M)$ acts on the configuration space $F_k(M)/\Sigma_k$ with isotropy subgroup $\text{Diff}_0(M) \cap \text{Diff}(M; k)$. Therefore, there is a homotopy equivalence

$$E\text{Diff}_0(M) \times_{\text{Diff}_0(M)} F_k(M)/\Sigma_k \simeq B(\text{Diff}_0(M) \cap \text{Diff}(M; k))$$

and thus an isomorphism of groups

$$\pi_1 \left( E\text{Diff}_0(M) \times_{\text{Diff}_0(M)} F_k(M)/\Sigma_k \right) \cong \pi_1 B(\text{Diff}_0(M) \cap \text{Diff}(M; k)) \cong \pi_0 (\text{Diff}_0(M) \cap \text{Diff}(M; k)).$$

Inspired by this isomorphism, we define the reduced mapping class group of $M$ with $k$ marked points by

$$\tilde{\Gamma}^k(M) = \pi_0(\text{Diff}_0(M) \cap \text{Diff}(M; k))$$

This group is closely related to the extended and punctured mapping class group, as shown next.

**Theorem 3.2** Let $M$ be a compact connected surface. The reduced mapping class group $\tilde{\Gamma}^k(M)$ is naturally a subgroup of $\Gamma^k(M)$ and fits into the following exact sequence:

$$1 \longrightarrow \tilde{\Gamma}^k(M) \longrightarrow \Gamma^k(M) \longrightarrow \Gamma(M) \longrightarrow 1.$$ 

To prove this result, we will need a couple of simple lemmas.

**Lemma 3.3** For every $k$, $\text{Diff}(M; k) \cdot \text{Diff}_0(M) = \text{Diff}(M)$.

*Proof:* Let $\varphi \in \text{Diff}(M)$ and let $\{x_1, \ldots, x_k\}$ be a fixed configuration of $k$ distinct points in $M$. It is easy to see there is diffeomorphism $\psi : M \to M$, isotopic to the identity, sending the points $\varphi^{-1}(x_1), \ldots, \varphi^{-1}(x_k)$ to $x_1, \ldots, x_k$. Then $\varphi = (\varphi \circ \psi^{-1}) \circ \psi$, where $\varphi \circ \psi^{-1} \in \text{Diff}(M; k)$ and $\psi \in \text{Diff}_0(M)$. $\square$
Lemma 3.4 For $k \geq 0$, the quotient group $\text{Diff}(M; k)/\text{Diff}(M; k) \cap \text{Diff}_0(M)$ is isomorphic to $\Gamma(M)$.

Proof: Consider the natural projection $p : \text{Diff}(M) \to \text{Diff}(M)/\text{Diff}_0(M)$ and let $p_k$ be the restriction to the subgroup $\text{Diff}(M; k)$. Notice that the image of $\text{Diff}(M; k)$ is isomorphic to $\text{Diff}(M; k) \cdot \text{Diff}_0(M)$ and the kernel of $p_k$ is $\text{Diff}(M; k) \cap \text{Diff}_0(M)$. Thus $p_k$ induces an isomorphism

$$\frac{\text{Diff}(M; k) \cdot \text{Diff}_0(M)}{\text{Diff}_0(M)} \cong \frac{\text{Diff}(M; k) \cap \text{Diff}_0(M)}{\text{Diff}_0(M)} = \frac{\text{Diff}(M; k) \cap \text{Diff}_0(M)}{\text{Diff}_0(M)}.$$

□

Proof of Theorem 3.2: Let $\text{Diff}_0(M; k)$ be the connected component of the identity of $\text{Diff}(M; k)$ and consider the following diagram of short exact sequences of topological groups:

$$\begin{array}{cccccccccccccc}
\text{Diff}_0(M; k) & \longrightarrow & \text{Diff}_0(M; k) & \longrightarrow & \{id\} & \longrightarrow & \\
\downarrow & & \downarrow & & \downarrow & & \\
\text{Diff}(M; k) \cap \text{Diff}_0(M) & \longrightarrow & \text{Diff}(M; k) & \longrightarrow & \text{Diff}(M; k) \cap \text{Diff}_0(M) & \longrightarrow & \\
\downarrow & & \downarrow & & \downarrow & & \\
\text{Diff}(M; k) \cap \text{Diff}_0(M) & \longrightarrow & \text{Diff}(M; k) \cap \text{Diff}_0(M) & \longrightarrow & \text{Diff}(M; k) \cap \text{Diff}_0(M) & \longrightarrow & \\
\downarrow & & \downarrow & & \downarrow & & \\
\text{Diff}_0(M; k) & \longrightarrow & \text{Diff}_0(M; k) & \longrightarrow & \text{Diff}_0(M; k) & \longrightarrow & \\
\end{array}$$

where the bottom row is an exact sequence of discrete groups. By Lemma 3.4 the cokernel is isomorphic to $\Gamma(M)$. On the other hand, notice the path component of the identity of the group $\text{Diff}_0(M) \cap \text{Diff}(M; k)$ is precisely $\text{Diff}_0(M; k)$. Then the kernel is given by

$$\frac{\text{Diff}(M; k) \cap \text{Diff}_0(M)}{\text{Diff}_0(M; k)} \cong \pi_0(\text{Diff}(M; k) \cap \text{Diff}_0(M)) = \tilde{\Gamma}^k(M)$$

and the theorem follows. □
We proceed to give some examples in low genus.

**Example:** Let $M = \mathbb{R}P^2$ and consider the natural action of $SO(3)$ on $\mathbb{R}P^2$ given by rotation of lines through the origin in $\mathbb{R}^3$. It was shown by Gramain [14] that the natural inclusion $SO(3) \rightarrow Diff(\mathbb{R}P^2)$ is a homotopy equivalence, and thus $Diff_0(\mathbb{R}P^2) = Diff(\mathbb{R}P^2)$ and $\tilde{\Gamma}^k(\mathbb{R}P^2) = \Gamma^k(\mathbb{R}P^2)$. Moreover, the natural map

$$ESO(3) \times_{SO(3)} F_k(\mathbb{R}P^2)/\Sigma_k \xrightarrow{\sim} EDiff(\mathbb{R}P^2) \times_{Diff(\mathbb{R}P^2)} F_k(\mathbb{R}P^2)/\Sigma_k$$

is a homotopy equivalence and the fundamental group of this space is $\Gamma^k(\mathbb{R}P^2)$. It was shown in [23] that for $k \geq 2$ the Borel construction above is a $K(\pi, 1)$. Thus:

**Theorem 3.5** If $k \geq 2$, the $SO(3)$-Borel construction

$$ESO(3) \times_{SO(3)} F_k(\mathbb{R}P^2)/\Sigma_k$$

is an Eilenberg-MacLane space $K(\pi, 1)$ where $\pi = \Gamma^k(\mathbb{R}P^2)$.

On the other hand, notice there is a fibration

$$B\mathbb{Z}_2 \longrightarrow ES^3 \times_{S^3} F_k(\mathbb{R}P^2)/\Sigma_k \longrightarrow ESO(3) \times_{SO(3)} F_k(\mathbb{R}P^2)/\Sigma_k$$

of $K(\pi, 1)$ spaces and thus one gets a short exact sequence of groups

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow B_k(\mathbb{R}P^2) \longrightarrow \Gamma^k(\mathbb{R}P^2) \longrightarrow 1$$

Moreover, recall that the center of $B_k(\mathbb{R}P^2)$ can be described as the image of the full twist braid $\Delta$ in $B_k(D^2)$ under the homomorphism $B_k(D^2) \rightarrow B_k(\mathbb{R}P^2)$ relative to an embedded disc $D^2 \subset \mathbb{R}P^2$. This braid is known to be the only element of order 2, see [12].

**Corollary 3.6** There is an isomorphism $\Gamma^k(\mathbb{R}P^2) \cong B_k(\mathbb{R}P^2)/\mathbb{Z}_2$, where $\mathbb{Z}_2$ is the center of $B_k(\mathbb{R}P^2)$.
Remark: Consider the matrices $\alpha, \beta, \gamma \in SO(3)$ given by

$$
\alpha = \begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix}, \quad
\beta = \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{pmatrix}, \quad
\gamma = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{pmatrix}.
$$

It can be shown that the group generated by $\alpha, \beta$ and $\gamma$ is isomorphic to $D_8$, the dihedral group of order 8. Moreover, it is proven in [23] that $SO(3)/D_8$ is homotopy equivalent to the unordered configuration space $F_2(\mathbb{R}P^2)/\Sigma_2$. Thus, there is a homotopy equivalence

$$
ESO(3) \times_{SO(3)} SO(3)/D_8 \simeq ESO(3) \times_{SO(3)} F_2(\mathbb{R}P^2)/\Sigma_2.
$$

Since the space on the left is homotopy equivalent to the classifying space $BD_8$, we have that $ESO(3) \times_{SO(3)} F_2(\mathbb{R}P^2)/\Sigma_2$ is a $K(D_8, 1)$ space.

Example: If $M = \mathbb{K}$ is the Klein bottle, it is well known that $\Gamma(\mathbb{K}) = \mathbb{Z}_2 \times \mathbb{Z}_2$, see [15]. Then the exact sequence in Theorem 3.2 is given by

$$
1 \longrightarrow \widetilde{\Gamma}^k(\mathbb{K}) \longrightarrow \Gamma^k(\mathbb{K}) \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 1
$$

and this exhibits $\widetilde{\Gamma}^k(\mathbb{K})$ as a normal subgroup of index 4 of $\Gamma^k(\mathbb{K})$. In this case $SO(2) \simeq \text{Diff}_0(\mathbb{K})$ and the natural map

$$
ESO(2) \times_{SO(2)} F_k(\mathbb{K})/\Sigma_k \xrightarrow{\sim} E\text{Diff}_0(\mathbb{K}) \times_{\text{Diff}_0(\mathbb{K})} F_k(\mathbb{K})/\Sigma_k
$$

is a homotopy equivalence. Also, it is easy to show this space is a $K(\pi, 1)$.

Theorem 3.7 If $k \geq 1$, the $SO(2)$-Borel construction

$$
ESO(2) \times_{SO(2)} F_k(\mathbb{K})/\Sigma_k
$$

is an Eilenberg-MacLane space $K(\pi, 1)$ where $\pi = \widetilde{\Gamma}^k(\mathbb{K})$. 

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Finally, consider the natural fibration

\[ F_k(K)/\Sigma_k \longrightarrow ESO(2) \times_{SO(2)} F_k(K)/\Sigma_k \longrightarrow BSO(2) \]

whose homotopy exact sequence is given by

\[ 1 \longrightarrow \mathbb{Z} \longrightarrow B_k(K) \longrightarrow \tilde{\Gamma}^k(K) \longrightarrow 1. \]

Then we have

**Corollary 3.8** There is an isomorphism \( \tilde{\Gamma}^k(K) \cong B_k(K)/\mathbb{Z}. \)

**Remark:** The subgroup \( \mathbb{Z} \) also corresponds to the center of the braid group of \( K \), a fact that can be checked by two different methods. The first one considers an inductive argument using the Fadell-Neuwirth fibrations to relate the center of the pure braid group to the center \( B_2(K) \); then one uses the fact that any free group on at least two generators has trivial center. The other method consists of a direct calculation of the center from a given finite presentation of \( B_k(K) \). We thank F. Cohen and D. Gonçalves for nice talks discussing these methods.

**Example:** If \( M = N_3 \) it also well known that \( \Gamma(N_3) = SL(2,\mathbb{Z}) \), see [13]. Therefore, we have an exact sequence

\[ 1 \longrightarrow \tilde{\Gamma}^k(N_3) \longrightarrow \Gamma^k(N_3) \longrightarrow SL(2,\mathbb{Z}) \longrightarrow 1 \]

which shows that \( \tilde{\Gamma}^k(N_3) \) is a much smaller group than \( \Gamma^k(N_3) \). In fact, we can prove the following result.

**Theorem 3.9** For \( g \geq 3 \), the reduced mapping class group \( \tilde{\Gamma}^k(N_g) \) is isomorphic to the braid group \( B_k(N_g) \).

**Proof:** Recall the group \( \tilde{\Gamma}^k(N_g) = \pi_0(\text{Diff}_0(N_g) \cap \text{Diff}(N_g; k)) \) is the fundamental group of the Borel construction

\[ E\text{Diff}_0(N_g) \times_{\text{Diff}_0(N_g)} F_k(N_g)/\Sigma_g \]
On the other hand, projection onto the first coordinate induces a universal bundle of the form

$$F_k(N_g) / \Sigma_g \longrightarrow E \text{Diff}_0(N_g) \times_{\text{Diff}_0(N_g)} F_k(N_g) / \Sigma_g \longrightarrow B \text{Diff}_0(N_g)$$

But for $g \geq 3$, Theorem 3.1 implies the classifying space $B \text{Diff}_0(N_g)$ is contractible and the result follows. □

Thus, Theorem 3.2 recovers a version of the Birman exact sequence on the non-orientable case, see [1], [11].

4 The mod-2 cohomology of $\Gamma^k(\mathbb{R}P^2)$

Recall that $SO(3)$ acts on $\mathbb{R}P^2$ by rotating lines in $\mathbb{R}^3$:

$$\mu : SO(3) \times \mathbb{R}P^2 \longrightarrow \mathbb{R}P^2$$

$$(A, [x, y, z]) \mapsto [A(x, y, z)]$$

and thus, it acts diagonally on $F_k(\mathbb{R}P^2) / \Sigma_k$. So one may consider the $SO(3)$-Borel construction and the associated fibration

$$F_k(\mathbb{R}P^2) / \Sigma_k \longrightarrow ESO(3) \times_{SO(3)} F_k(\mathbb{R}P^2) / \Sigma_k \longrightarrow BSO(3).$$

**Theorem 4.1** The Serre spectral sequence for the fibration above collapses at the $E_2$-term, in mod 2 cohomology.

*Proof:* Notice the $SO(3)$ action on $\mathbb{R}P^2$ can be extended to an action on $\mathbb{R}P^\infty$ by setting: $\mu(A, [x_0, x_1, x_2, x_3, \ldots]) = [A(x_0, x_1, x_2, x_3, \ldots)]$. Thus the inclusion $F_k(\mathbb{R}P^2) / \Sigma_k \hookrightarrow F_k(\mathbb{R}P^\infty) / \Sigma_k$ is $SO(3)$-equivariant and gives rise to a map of fibrations.
which induces a map between the corresponding spectral sequences. We will show in Theorem 5.3 that the induced map on the fibers is an epimorphism in mod-2 cohomology. Then the desired spectral sequence collapses provided the spectral sequence for the fibration on the right collapses. Secondly, notice the natural maps

\[
F_k(\mathbb{RP}^\infty)/\Sigma_k \xleftarrow{\Sigma_k} E \Sigma_k \times F_k(\mathbb{RP}^\infty) \xrightarrow{\Sigma_k} E \Sigma_k \times (\mathbb{RP}^\infty)^k
\]

are $SO(3)$-equivariant and also homotopy equivalences. Therefore we get equivalences of fibrations

To prove the assertion of the theorem we will actually show the right column is a trivial fibration. Let $(\mathbb{RP}^\infty)_t$ denote the space $\mathbb{RP}^\infty$ endowed with the
trivial $SO(3)$-action and consider the map $s: (\mathbb{R}P^\infty)_l \to \mathbb{R}P^\infty$ given by shifting of coordinates: $s[x_0, x_1, x_2, \ldots] = [0, 0, 0, x_0, x_1, x_2, \ldots]$. Notice the map $s$ is $SO(3)$-equivariant and also a homotopy equivalence, since it is non-trivial on fundamental groups. Thus we get an equivalence of fibrations

$$
\begin{align*}
E\Sigma_k \times_{\Sigma_k} (\mathbb{R}P^\infty)_l^k & \xrightarrow{1 \times s^k} E\Sigma_k \times_{\Sigma_k} (\mathbb{R}P^\infty)^k \\
\cong & \\
ESO(3) \times_{SO(3)} E\Sigma_k \times_{\Sigma_k} (\mathbb{R}P^\infty)_l^k & \cong ESO(3) \times_{SO(3)} E\Sigma_k \times_{\Sigma_k} (\mathbb{R}P^\infty)^k \\
BSO(3) & \cong BSO(3)
\end{align*}
$$

But the left column of the previous diagram is clearly a trivial fibration, since the $SO(3)$-action on the fiber was trivial, and the statement of the theorem follows. □

As a consequence of Theorem 4.1 we have

**Theorem 4.2** For $k \geq 2$, there is an isomorphism of $H^*(BSO(3); F_2)$-modules

$$
H^*(ESO(3) \times_{SO(3)} F_k(\mathbb{R}P^2)/\Sigma_k; F_2) \cong F_2[w_2, w_3] \otimes H^*(F_k(\mathbb{R}P^2)/\Sigma_k; F_2)
$$

where $w_2, w_3$ are the Stiefel-Whitney classes in the cohomology of $BSO(3)$.

It follows from here that the mod-2 cohomology of $\Gamma^k(\mathbb{R}P^2)$ is determined by the mod-2 cohomology of $F_k(\mathbb{R}P^2)/\Sigma_k$.
5 Homology of configuration spaces

In this section we describe the homology of the space $F_k(\mathbb{R}P^2)/\Sigma_k$ in terms of the homology of a much larger space. Let $M$ be manifold of dimension $m$ and $X$ a connected space with base point $*$ and recall the labelled configuration space $C(M; X)$ is given by

$$C(M; X) = \left( \prod_{k \geq 0} F_k(M) \times X^k / \Sigma_k \right) / \approx$$

where the relation $\approx$ is generated by

$$[m_1, \ldots, m_k; x_1, \ldots, x_k] \approx [m_1, \ldots, m_{k-1}; x_1, \ldots, x_{k-1}]$$

if $x_k = *$. Such spaces of labelled configurations occur in [3], [4], [18] and [17] as models for mapping spaces. The space $C(M; X)$ is naturally filtered by length of configurations

$$* = C_0(M; X) \subseteq C_1(M; X) \subseteq \ldots \subseteq C(M; X)$$

and the filtration quotients $C_k(M; X)/C_{k-1}(M; X)$ are denoted by $D_k(M; X)$. The basic properties of this construction are given next, see [17], [4], [21]:

1. If $M = \mathbb{R}^n$ then $C(\mathbb{R}^n; X) \simeq \Omega^n \Sigma^n X$.

2. There is an analogue of Snaith’s stable splitting

$$C(M; X) \simeq \bigvee_{k=1}^{\infty} D_k(M; X).$$

3. There is a natural vector bundle over the configuration space

$$\eta_k : F_k(M) \times \Sigma_k \mathbb{R}^k \to F_k(M)/\Sigma_k$$

and the Thom space of its $n$-fold sum is homeomorphic to $D_k(M; S^n)$. Therefore, by the Thom isomorphism

$$H_q(F_k(M)/\Sigma_k) \simeq \tilde{H}_{q+kn}D_k(M; S^n)$$
Theorem 5.1 ([4]) For $M$ a smooth, compact manifold of dimension $m$ and $X = S^n$, there is an isomorphism of graded vector spaces

$$\theta : H_*(C(M; S^n); \mathbb{F}_2) \cong \bigotimes_{q=0}^m H_*(\Omega^{m-q}S^{m+n}; \mathbb{F}_2)^{\otimes \beta_q},$$

where $\beta_q$ is the $q$-th Betti number of $M$.

Each factor $H_*(\Omega^{m-q}S^{m+n})$ is an algebra with weights associated to its generators. This yields a filtration on the tensor product and $\bar{H}_*D_k(M; S^n)$ corresponds, via the isomorphism $\theta$, to the vector space generated by the elements of weight $k$. For completeness, we record here the mod 2 homology of the iterated loop spaces $\Omega^kS^{n+k}$, see [6]. Throughout the rest of this and next section, all homology groups are taken with mod-2 coefficients.

Recall there are mod 2 homology operations which are natural for $n$-fold loop maps

$$Q_i : H_*(\Omega^n X) \longrightarrow H_{2q+i}(\Omega^n X), \quad 0 \leq i \leq n-1$$

which are linear if $0 \leq i < n - 1$, known as the Dyer-Lashof operations [8]. Let $x_n \in H_n(\Omega^kS^{n+k})$ be the fundamental class and let $Q_I x_n$ denote the composition $Q_{i_1}Q_{i_2} \cdots Q_{i_r} x_n$ if $I = (i_1, i_2, \ldots, i_r)$. The sequence $I$ is admissible if $0 < i_1 \leq i_2 \leq \cdots \leq i_r$; write $\lambda(I) \leq q$ if $i_r \leq q$ and $\ell(I) = r$.

Theorem 5.2 ([6]) There is and isomorphism of Hopf algebras

$$H_*(\Omega^kS^{n+k}) \cong \mathbb{F}_2[Q_I x_n], \quad n \geq 1,$$

for admissible $I$ with $\lambda(I) \leq k - 1$ and $Q_I x_n$ is primitive.

Thus, to compute $H_*(C(M; S^n))$, we first introduce for every basis element $\alpha \in H_q(M)$ a generator, namely the fundamental class $u_\alpha \in H_*(\Omega^{m-q}S^{m+n})$ of degree $|u_\alpha| = q + n$ and weight $\omega(u_\alpha) = 1$. Secondly, for each $u_\alpha$ and index $I = (i_1, i_2, \ldots, i_r)$ there is an additional generator $Q_I u_\alpha = Q_{i_1}Q_{i_2} \cdots Q_{i_r} u_\alpha$ if the condition $0 < i_1 \leq i_2 \leq \cdots \leq i_r < m - q$ holds. We have:
\begin{itemize}
  \item $|Q_I u_\alpha| = i_1 + 2i_2 + 4i_3 + \cdots + 2^{r-1}i_r + 2^r(q+n)$
  \item $\omega(Q_I u_\alpha) = 2^{\ell(I)} = 2^r$
\end{itemize}

and $u_\alpha^2 = 0$ if $|\alpha| = q = m$.

The isomorphism $\theta$ in Theorem 5.1 depends on the choice of a handle decomposition for $M$ and it is natural for embeddings which respect the handle decompositions. Recall that a manifold $\bar{M}$ is obtained from a submanifold $M \subset \bar{M}$ of codimension 0 by attaching a handle of index $q$ if $\bar{M} = M \cup D$ with $D \approx [0,1]^m$ and $M \cap D \approx [0,1]^{m-q} \times \partial[0,1]^q$. A handle decomposition for a manifold $M$ is a filtration by submanifolds

$$M_0 \subset M_1 \subset \ldots \subset M_{m-1} \subset M_m = M,$$

where $M_q$ is obtained from $M_{q-1}$ by attaching handles of index $q$ and $M_0$ is a disjoint union of closed $m$-dimensional discs.

As an example, consider the usual CW structure on $S^m$ with two antipodal $q$-cells in every dimension, for $q = 0, 1, \ldots, m$, so that the $q$-skeleton of $S^m$ is the sphere $S^q$. It is clear that the cell structure above can be thickened to induce a handle decomposition for $S^m$ with two antipodal handles of index $q$, for $q = 0, \ldots, m$, such that the natural embedding $S^m \subset S^{m+1}$ respects the handle decomposition.

Moreover, passing to the quotient by the antipodal $\mathbb{Z}_2$-action, we get a handle decomposition for $\mathbb{R}P^m$ with one handle of index $q$, for $q = 0, \ldots, m$, as follows:

\begin{center}
\begin{figure}
\centering
\includegraphics[width=\textwidth]{handle_decompositions.png}
\caption{Handle decompositions of $S^1$ and $S^2$}
\end{figure}
\end{center}
so that the natural embedding $\mathbb{R}P^m \subset \mathbb{R}P^{m+1}$ respects the corresponding handle decompositions. Thus, as a direct application of Theorem 5.1 we get

**Theorem 5.3** The natural inclusion $\mathbb{R}P^2 \subset \mathbb{R}P^\infty$ induces a map of unordered configuration spaces, which is an epimorphism in mod-2 cohomology:

$$H^*(F_k(\mathbb{R}P^\infty)/\Sigma_k; \mathbb{F}_2) \longrightarrow H^*(F_k(\mathbb{R}P^2)/\Sigma_k; \mathbb{F}_2).$$

*Proof:* Let $m \geq 2$ and consider the induced inclusion at the level of labelled configuration spaces $C(\mathbb{R}P^2; S^n) \hookrightarrow C(\mathbb{R}P^m; S^n)$. By Theorem 5.1 we have isomorphisms in mod-2 cohomology:

$$H_*(\mathbb{R}P^2; S^n) \cong H_*(\Omega^2 S^{n+2}) \otimes H_*(\Omega S^{n+2}) \otimes H_*(S^{n+2})$$

and

$$H_*C(\mathbb{R}P^m; S^n) \cong H_*(\Omega^m S^{m+n}) \otimes H_*(\Omega^{m-1} S^{n+m}) \otimes \ldots \otimes H_*(S^{m+n}).$$

Moreover, the embedding $\mathbb{R}P^2 \subset \mathbb{R}P^m$ respects the handle decompositions and the induced map in homology $H_*C(\mathbb{R}P^2; S^n) \to H_*C(\mathbb{R}P^m; S^n)$ is given by the usual adjunction maps:

$$H_*(\Omega^2 S^{n+2}) \to H_*(\Omega^m S^{m+n})$$

$$H_*(\Omega S^{n+2}) \to H_*(\Omega^{m-1} S^{n+m})$$

$$H_*(S^{n+2}) \to H_*(\Omega^{m-2} S^{n+m})$$

which are monomorphisms and preserve the weight of the generators. Therefore, the inclusion $\mathbb{R}P^2 \subset \mathbb{R}P^\infty$ induces a monomorphism in mod-2 homology

$$H_*(F_k(\mathbb{R}P^2)/\Sigma_k; \mathbb{F}_2) \longrightarrow H_*(F_k(\mathbb{R}P^\infty)/\Sigma_k; \mathbb{F}_2)$$

and thus by duality, an epimorphism in mod-2 cohomology. □
6 Explicit calculations

Let us specialize to the case when $M$ is a surface. By Theorem 5.1, the mod-2 homology of the labelled configuration space $C(M; S^n)$ is given by the tensor product

$$H_*(\Omega^2 S^{n+2}) \otimes \beta_0 \otimes H_*(\Omega S^n) \otimes \beta_1 \otimes H_*(S^{n+2}) \otimes \beta_2,$$

where $\beta_0, \beta_1$ and $\beta_2$ are the mod-2 Betti numbers of $M$. In the case of the closed non-orientable surface $N_g$ of genus $g$ we have

$$H_* C(N_g; S^n) \cong F_2[y_0, y_1, \ldots] \otimes F_2[x_1, x_2, \ldots, x_g] \otimes F_2[u]/u^2,$$

where $y_0, x_1, \ldots, x_g$ and $u$ are the fundamental classes on degrees $n, n+1$ and $n+2$, respectively, and $y_j = Q_j^1 y_0 = Q_j Q_1 \ldots Q_1 y_0$. Here $Q_1$ is the first Dyer-Lashof operation, so notice $|y_j| = (2^j - 1) + 2^j n$. The weights of all generators are given by

$$\omega(u) = 1,$$
$$\omega(x_i) = 1, \text{ for } i = 1, \ldots, g,$$
$$\omega(y_j) = 2^j,$$

and a basis for $\overline{H}_q D_k(N_g; S^n)$ consists of all monomials of the form

$$h = u^e x_1^{a_1} \ldots x_g^{a_g} y_0^{b_0} y_1^{b_1} \ldots y_r^{b_r},$$

for some $r \geq 0, e = 0, 1$ and $a_i, b_j \geq 0$, such that

$$\omega(h) = e + a_1 + \ldots + a_g + b_0 + 2b_1 + \ldots + 2^r b_r = k.$$

For example, $\overline{H}_q D_2(N_g; S^n)$ is determined by the following table.

| $q$   | basis           | rank       |
|-------|-----------------|------------|
| $2n$  | $y_0^2$         | $1$        |
| $2n+1$| $x_1 y_0, \ldots, x_g y_0, y_1$ | $g + 1$   |
| $2n+2$| $u y_0, x_1^2, \ldots, x_i x_j, \ldots, x_g^2$ | $1 + (g^2 + 1)/2$ |
| $2n+3$| $u x_1, \ldots, u x_g$ | $g$       |
Thus, the rank of \( H_q(F_2(N_g)/\Sigma_2; F_2) \) is \( 1, g + 1, 1 + (g^2 + g)/2 \) and \( g \) for \( q = 0, 1, 2, 3 \). Similarly, in the case of \( \mathbb{R}P^2 \), \( g = 1 \), and \( k = 3, 4 \) one has

\[
H_q(F_3(\mathbb{R}P^2)/\Sigma_3; F_2) = \begin{cases} 
F_2 & \text{if } q = 0 \\
F_2^2 & \text{if } q = 1 \\
F_2^3 & \text{if } q = 2 \\
F_2^3 & \text{if } q = 3 \\
F_2 & \text{if } q = 4 \\
0 & \text{otherwise}
\end{cases}
\]

and

\[
H_q(F_4(\mathbb{R}P^2)/\Sigma_4; F_2) = \begin{cases} 
F_2 & \text{if } q = 0 \\
F_2^2 & \text{if } q = 1 \\
F_2^4 & \text{if } q = 2 \\
F_2^5 & \text{if } q = 3 \\
F_2^3 & \text{if } q = 4 \\
F_2 & \text{if } q = 5 \\
0 & \text{otherwise.}
\end{cases}
\]

Finally, we express the mod-2 homology of \( F_k(\mathbb{R}P^2)/\Sigma_k \) in terms of the homology of the classical braid groups. Since the Betti numbers of \( \mathbb{R}P^2 \) are \( \beta_0 = \beta_1 = \beta_2 = 1 \), then by Theorem 5.3 there is an isomorphism

\[
H_*(\mathbb{R}P^2; S^n) \cong F_2[u]/u^2 \otimes F_2[x] \otimes H_*(\Omega^2 S^{n+2}),
\]

where \( |u| = n + 2 \), \( |x| = n + 1 \) and \( \omega(u) = \omega(x) = 1 \). Thus, a basis for \( H_*(\mathbb{R}P^2; S^n) \) is given by all the monomials of the form

\[
u^\epsilon \otimes x^\ell \otimes y, \quad \epsilon = 0, 1, \quad \ell = 0, 1, 2, \ldots
\]

where \( y \) runs over an additive basis for \( H_*(\Omega^2 S^{n+2}) \). Notice that basis elements of the form \( u^\epsilon \otimes x^\ell \) have degree and weight given by
\[ |u^\epsilon \otimes x^\ell| = \epsilon(n+2) + \ell(n+1) = \epsilon n + 2\epsilon + \ell n + \ell, \]
\[ \omega(u^\epsilon \otimes x^\ell) = \epsilon + \ell \]

Therefore for fixed \( q \) and \( k \), the monomial \( u^\epsilon \otimes x^\ell \otimes y \) represents a generator in
\[ H_q F_k(\mathbb{R}P^2)/\Sigma_k \cong H_{q + kn} D_k(\mathbb{R}P^2; S^n) \]
if and only if \( y \) has degree:
\[ |y| = (q + kn) - (\epsilon n + 2\epsilon + \ell n + \ell) \]
\[ = q + (k - \ell - \epsilon)n - 2\epsilon - \ell \]
and weight: \( \omega(y) = k - (\epsilon + \ell) = k - \epsilon - \ell. \)

**Case \( \epsilon = 0 \):** Elements in \( H_*(\Omega^2 S^{n+2}) \cong H_* C(\mathbb{R}^2; S^n) \) of degree \( q + (k - \ell)n - \ell \) and weight \( k - \ell \) generate the vector space
\[ H_{q + (k - \ell)n - \ell} D_{k - \ell}(\mathbb{R}^2; S^n) \cong H_{q - \ell} F_{k - \ell}(\mathbb{R}^2)/\Sigma_{k - \ell}) \]
\[ = H_{q - \ell}(B_{k - \ell}) \]

**Case \( \epsilon = 1 \):** Elements in \( H_* C(\mathbb{R}^2; S^n) \) of degree \( q + (k - \ell - 1)n - \ell - 2 \) and weight \( k - \ell - 1 \) generate the vector space
\[ H_{q + (k - \ell - 1)n - \ell - 2} D_{k - \ell - 1}(\mathbb{R}^2; S^n) \cong H_{q - \ell - 2} F_{k - \ell - 1}(\mathbb{R}^2)/\Sigma_{k - \ell - 1}) \]
\[ = H_{q - \ell - 2}(B_{k - \ell - 1}) \]

Thus there is an isomorphism of graded vector spaces:
\[ H_q F_k(\mathbb{R}P^2)/\Sigma_k \cong \bigoplus_{\ell=0}^{\min\{q,k\}} H_{q - \ell}(B_{k - \ell}) \oplus \bigoplus_{\ell=0}^{\min\{q-2,k-1\}} H_{q - \ell - 2}(B_{k - \ell - 1}) \]

It is worth to compare this result with the analog for \( S^2 \) which is obtained in [5] in a similar manner to the one exposed here.
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