G-Homotopy Invariance of the Analytic Signature of Proper Co-compact G-manifolds and Equivariant Novikov Conjecture

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Abstract

The main result of this paper is the G-homotopy invariance of the G-index of signature operator of proper co-compact G-manifolds. If proper co-compact G manifolds X and Y are G-homotopy equivalent, then we prove that the images of their signature operators by the G-index map are the same in the K-theory of the C*-algebra of the group G. Neither discreteness of the locally compact group G nor freeness of the action of G on X are required, so this is a generalization of the classical case of closed manifolds. Using this result we can deduce the equivariant version of Novikov conjecture for proper co-compact G-manifolds from the Strong Novikov conjecture for G.

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Introduction

Before discussing on our case of proper $G$-action, let us review the classical case of closed manifolds. For even dimensional oriented closed manifold $M$, the ordinary Fredholm index of the signature operator $\partial_M$ is equal to the signature of the manifold $M$ which is defined using the cup product of the ordinary cohomology of $M$. In particular it follows that $\text{ind}(\partial_M)$ is invariant under orientation preserving homotopy. We have the following classical and important result;

Theorem 0.1 [Kas75], [KM], [HiSk] Let $M$ and $N$ be even dimensional oriented closed manifolds with fundamental group $\Gamma = \pi_1(M) = \pi_1(N)$. Assume that $M$ and $N$ are orientation preserving homotopy equivalent to each other. Then $\text{ind}_\Gamma(\partial_M) = \text{ind}_\Gamma(\partial_N) \in K_0(\Gamma)$.

Notice that we can deduce the Novikov conjecture from the Strong Novikov conjecture by using this theorem. Moreover, we also have a more generalized result;

Theorem 0.2 [RW, 3.3. PROPOSITION and 3.6. THEOREM] Let a finite group $\Gamma$ acts on $M$ and $N$ and let $\Gamma = \pi_1(M) = \pi_1(N)$. Let $\text{ind}_{\Gamma}^G$ be the $G$-equivariant $\Gamma$-index map with value in $K_0^G(C^*_\text{red}(\Gamma)) \simeq K_0(C^*_\text{red}(\Gamma))$, where $G^\Gamma$ denotes the group extension $\{1\} \to \Gamma \to G^\Gamma \to G \to \{1\}$. Assume that $M$ and $N$ are orientation preserving $\Gamma$-equivariantly homotopy equivalent. Then $\text{ind}_{\Gamma}^G(\partial_M) = \text{ind}_{\Gamma}^G(\partial_N) \in K_0(C^*_\text{red}(G^\Gamma))$.

Our main theorem is a generalization of them. Let us fix the settings. Let $X$ and $Y$ be oriented even-dimensional complete Riemannian manifolds and let $G$ be a second countable locally compact Hausdorff group acting on $X$ and $Y$ isometrically, properly and co-compactly.

Theorem A Let $X$ and $Y$ be oriented even-dimensional complete Riemannian manifolds and let $G$ be a second countable locally compact Hausdorff group acting on $X$ and $Y$ isometrically, properly and co-compactly. Let $\partial_X$ and $\partial_Y$ be the signature operators. Assume we have a $G$-equivariant orientation preserving homotopy equivalent map $f: Y \to X$.

Then $\text{ind}_G(\partial_X) = \text{ind}_G(\partial_Y) \in K_0(C^*(G))$.

This claim is also stated in [BCH] without proofs and here we will give a proof for it to obtain Corollary B. The method we use in this paper is based on [HiSk], so we will construct a map that sends $\text{ind}_G(\partial_X)$ to $\text{ind}_G(\partial_Y)$. Our group $C^*$-algebras can be either maximal one or reduced one.

Theorem 0.1 is the case when $X$ and $Y$ are the universal covering of closed manifolds $M$ and $N$. Thus, analogously to the case of closed manifolds, the equivariant version of the Novikov conjecture can be deduced from the Strong Novikov conjecture for the acting group $G$. In particular, by using this theorem and the result discussed in [F], we obtain the following equivariant version of Novikov conjecture for low dimensional cohomologies:

Corollary B Let $X$, $Y$ and $G$ as above and let $L$ be a $G$-hermitian line bundles over $X$ which is induced from a $G$-line bundle over $\mathcal{E}G$, or more generally, $G$-hermitian line bundle $L$ over $X$ satisfying $c_1(L) = 0 \in H^2(X; \mathbb{R})$. Suppose, in addition, that $G$ is unimodular and $H_1(X; \mathbb{R}) = H_1(Y; \mathbb{R}) = \{0\}$. Then,

$$\int_X c_X(x)\mathcal{L}(TX) \wedge \text{ch}(L) = \int_Y c_Y(y)\mathcal{L}(TY) \wedge \text{ch}(f^*L),$$

where $c_X$ denotes the cut-off function, that is, $c_X$ is a $\mathbb{R}_{\geq 0}$-valued compactly supported function on $X$ satisfying $\int_X c(x^{-1}x)\,dy = 1$ for any $x \in X$. In the case of the closed manifold, that is, when $X$ is obtained as the universal covering of a closed manifold $M$, and the acting group is the fundamental group, the above value is equal to the ordinary, so called, higher signature $(\mathcal{L}(TX) \cup \text{ch}(L), [M])$.

The same result in this case of closed manifolds was obtained in [Ma] and [HaSc].
Moreover in Section , we will prove the \(G\)-homotopy invariance of the analytic signature twisted by almost flat bundles as in [HiSk, Section 4.]. However we will use a different method from [HiSk] to deal with general \(G\)-invariant elliptic operators. To be specific, we will prove the following Theorem C to obtain Corollary D.

**Theorem C** Let \(X\) be a complete oriented Riemannian manifold and let \(G\) be a locally compact Hausdorff group acting on \(X\) isometrically, properly and co-compactly. Moreover we assume that \(X\) is simply connected. Let \(D\) be a \(G\)-invariant properly supported elliptic operator of order 0 on \(G\)-Hermitian vector bundle over \(X\).

Then there exists \(\varepsilon > 0\) satisfying the following: for any finitely generated projective Hilbert \(B\)-module \(G\)-bundle \(E\) over \(X\) equipped with a \(G\)-invariant Hermitian connection such that \(\|R^E\| < \varepsilon\), we have

\[
\text{ind}_G \left( [E] \otimes_{C_0(X)} [D] \right) = 0 \quad \in K_0 \left( C^*_\text{Max}(G) \otimes_{\text{Max}} B \right)
\]

if \(\text{ind}_G([D]) = 0 \in K_0(C^*_\text{Max}(G))\). If we only consider commutative \(C^*\)-algebras for \(B\), then the same conclusion is also valid for \(C^*_\text{red}(G)\).

**Corollary D** Consider the same conditions as Theorem A on \(X, Y\) and \(G\) and assume additionally that \(X\) and \(Y\) are simply connected.

Then there exists \(\varepsilon > 0\) satisfying the following: for any finitely generated projective Hilbert \(B\)-module \(G\)-bundle \(E\) over \(X\) equipped with a \(G\)-invariant Hermitian connection such that \(\|R^E\| < \varepsilon\), we have

\[
\text{ind}_G([E] \otimes_{\partial X}) = \text{ind}_G([f^*E] \otimes_{\partial Y}) \quad \in K_0(C^*_\text{Max}(G) \otimes_{\text{Max}} B).
\]

If we only consider commutative \(C^*\)-algebras for \(B\), then the same conclusion is also valid for \(C^*_\text{red}(G)\).

\section{Preliminaries on proper actions}

**Definition 1.1** Let \(G\) be a second countable locally compact Hausdorff group. Let \(X\) be a complete Riemannian manifold.

- \(X\) is called a \(G\)-Riemannian manifold if \(G\) acts on \(X\) isometrically.
- The action of \(G\) on \(X\) is said to be proper or \(X\) is called a proper \(G\)-space if the following continuous map is proper: \(X \times G \to X \times X, \ (x, \gamma) \mapsto (x, \gamma x)\).
- The action of \(G\) on \(X\) is said to be co-compact or \(X\) is called \(G\)-compact space if the quotient space \(X/G\) is compact.

**Definition 1.2** The action of \(G\) on \(X\) induces actions on \(TX\) and \(T^*X\) given by

\[
\gamma: T_xX \to T_{\gamma x}X \quad \text{and} \quad \gamma: T^*_xX \to T^*_{\gamma x}X
\]

\[
\gamma(v) := \gamma_*v \quad \text{and} \quad \gamma(\xi) := (\gamma^{-1})^*\xi.
\]

The action on \(\mathfrak{X}(X)\) and \(\Omega^*(X)\) is given by

\[
\gamma[V] := \gamma_*V \quad \text{and} \quad \gamma[\omega] := (\gamma^{-1})^*\omega
\]

for \(\gamma \in G, V \in \mathfrak{X}(X)\) and \(\omega \in \Omega^*(X)\). Obviously, \(\gamma[\omega \wedge \eta] = \gamma[\omega] \wedge \gamma[\eta]\) and \(d(\gamma[\omega]) = \gamma[d\omega]\).

**Proposition 1.3 (Slice theorem)** Let \(G\) be a second countable locally compact Hausdorff group and act properly and isometrically on \(X\). Then for any neighborhood \(O\) of any point \(x \in X\) there exists a compact subgroup \(K \subset G\) including the stabilizer at \(x\), \(K \supset G_x := \{ \gamma \in G \mid \gamma x = x \}\) and there exists a \(K\)-slice \(\{ x \} \subset S \subset O\).
Here $S \subset X$ is called $K$-slice if the followings are satisfied;

- $S$ is $K$-invariant; $K(S) = S$,
- the tubular subset $G(S) \subset X$ is open,
- there exists a $G$-equivariant map $\psi: G(S) \to G/K$ satisfying $\psi^{-1}([e]) = S$, called a slice map.

**Corollary 1.4** We additionally assume that $X$ is $G$-compact. Then for any open covering $X = \bigcup_{x \in X} O_x$, there exists a sub-family of finitely many open subsets $\{O_{x_1}, \ldots, O_{x_N}\}$ such that

$$\bigcup_{\gamma \in G} \bigcup_{i=1}^N \gamma(O_{x_i}) = X.$$  

In particular, $X$ is of bounded geometry, namely, the injective radius is bounded below and the norm of Riemannian curvature is bounded. \qed

**Lemma 1.5** Let $X$ and $Y$ be manifolds on which $G$ acts properly. Suppose that the action on $Y$ is co-compact. Let $f: Y \to X$ be a $G$-equivariant continuous map. Then $f$ is a proper map.

**Proof.** Since the action on $Y$ is co-compact, there exists a compact subset $F \subset Y$ satisfying $G(F) = Y$. Fix a compact subset $C \subset X$ and assume that the closed set $f^{-1}C \subset Y$ is not compact. Then there exists a sequence $\{y_j\} \subset f^{-1}C$ tending to the infinity, that is, any compact subset in $Y$ contains only finitely many points of $\{y_j\}$. Since the action on $Y$ is proper, there exists a sequence $\{\gamma_j\} \subset G$ tending to the infinity satisfying $y_j \in \gamma_j F$. Then it follows that $f(y_j) \in f(\gamma_j F) = \gamma_j f(F)$. Due to the compactness of $f(F) \subset X$ and the properness of the action on $X$, the sequence $\{f(y_j)\} \subset X$ tends to the infinity. However, the compact subset $C$ cannot contain such a sequence. So, $f^{-1}C$ is compact. \qed

## 2 Perturbation arguments

In this section, we will discuss on some technical method introduced in [HiSk, Section 1 and 2]. For now, we will forget about the manifolds and group actions. Let $A$ be a $C^*$-algebra, which may not be unital. Especially we will consider $A = C^*(G)$. Let $\mathcal{E}$ be a Hilbert $A$-module equipped with $A$-valued scalar product $\langle \cdot, \cdot \rangle$. Let us fix some notations:

- $\mathbb{L}(\mathcal{E}_1, \mathcal{E}_2)$ denotes a space consisting of adjointable $A$-linear operators, and we also use $\mathbb{L}(\mathcal{E}) := \mathbb{L}(\mathcal{E}, \mathcal{E})$.
- $\mathbb{K}(\mathcal{E}_1, \mathcal{E}_2)$ denotes a sub-space of $\mathbb{L}(\mathcal{E}_1, \mathcal{E}_2)$ consisting of compact $A$-linear operators, namely, the norm closure of the space of operators whose $A$-rank are finite. We also use $\mathbb{K}(\mathcal{E}) := \mathbb{K}(\mathcal{E}, \mathcal{E})$.

### 2.1 Quadratic forms and graded modules

**Definition 2.1** (Regular quadratic forms) $Q: \mathcal{E} \times \mathcal{E} \to A$ is called a quadratic form on $\mathcal{E}$ if it satisfies

$$Q(\xi, \nu) = Q(\nu, \xi)^* \quad \text{and} \quad Q(\nu, \xi a) = Q(\nu, \xi)a \quad \text{for} \quad \nu, \xi \in \mathcal{E}, a \in A. \quad (2.1)$$

A quadratic form $Q$ is said to be regular if there exists an invertible operator $B \in \mathbb{L}(\mathcal{E})$ satisfying that $Q(\xi, B\nu) = \langle \xi, \nu \rangle$.

For an operator $T \in \mathbb{L}(\mathcal{E})$, let $T'$ denote the adjoint with respect to $Q$, that is, an operator satisfying that $Q(T\xi, \nu) = Q(\xi, T'\nu)$. Using $B$, it is written as $T' = BT^*B^{-1}$. 


Definition 2.2 (Compatible scalar product) Another scalar product $\langle \cdot, \cdot \rangle_1 : \mathcal{E} \times \mathcal{E} \to A$ is called compatible with $\langle \cdot, \cdot \rangle$ if there exists a linear bijection $P : \mathcal{E} \to \mathcal{E}$ satisfying that $\langle \nu, \xi \rangle_1 = \langle \nu, P\xi \rangle$.

Note that $P$ is a positive operator with respect to both of the scalar product and $\sqrt{P} : (\mathcal{E}, \langle \cdot, \cdot \rangle_1) \to (\mathcal{E}, \langle \cdot, \cdot \rangle)$ is a unitary isomorphism. In particular, neither the spaces $L(\mathcal{E})$ nor $\mathbb{K}(\mathcal{E})$ depends on the choice of compatible scalar product.

Lemma 2.3 Let $Q$ be a regular quadratic form on $\mathcal{E}$. then there exist a compatible scalar product $\langle \cdot, \cdot \rangle_Q$ with the initial scalar product of $\mathcal{E}$ and $U \in L(\mathcal{E})$ satisfying that $Q(\xi, U\nu) = \langle \xi, \nu \rangle_Q$ and $U^2 = 1$. Moreover they are unique.

Proof. With respect to the initial scalar product $\langle \cdot, \cdot \rangle$, we have that
\[ \langle \nu, B^{-1}\xi \rangle = Q(\nu, \xi) = Q(\xi, \nu)^* = \langle \xi, B^{-1}\nu \rangle^* = \langle B^{-1}\nu, \xi \rangle = \langle \nu, (B^{-1})^*\xi \rangle, \]
which implies that $B^{-1}$ is an invertible self-adjoint operator. Thus, it has the polar decomposition $B^{-1} = UP$ in which $B^{-1}, U$ and $P$ commute one another, here $U$ is unitary and $P$ is positive. To be specific, $U$ and $P$ are given by the continuous functional calculus. Let $f$ and $g$ be continuous functions given by $f(x) := \frac{x}{|x|}$ and $g(x) := |x|$ on the spectrum of $B^{-1}$, which is contained in $\mathbb{R} \setminus \{0\}$, and set $U := f(B^{-1})$ and $P := g(B^{-1})$. Note that
\[ U = P^{-1}B^{-1} = P^{-1}(B^{-1})^* = P^{-1}PU^* = U^*, \]
so it follows that $U^2 = U^*U = 1$. Let us set $\langle \nu, \xi \rangle_Q := \langle \nu, P\xi \rangle$. Then,
\[ Q(\nu, U\xi) = Q(\nu, U^{-1}\xi) = Q(\nu, BP\xi) = \langle \nu, P\xi \rangle = \langle \nu, \xi \rangle_Q. \]
If there is another such operator $U_1$ satisfying that $U_1^2 = 1$ and that $Q(\nu, U_1\xi)$ is another scalar product, then $U_1^{-1}U$ is a positive unitary operator, which implies that $U_1^{-1}U = 1$. Thus we obtained the uniqueness.

Remark 2.4 A regular quadratic form $Q$ on a Hilbert $A$-module $\mathcal{E}$ determines the renewed compatible scalar product $\langle \cdot, \cdot \rangle_Q$ associated to $Q$ and the $(\mathbb{Z}/2\mathbb{Z})$-grading given by the $\pm 1$-eigen spaces of $U$. Conversely, if a Hilbert $A$-module $\mathcal{E}$ is equipped with a $(\mathbb{Z}/2\mathbb{Z})$-grading, then it determines a regular quadratic form $Q$ given by $Q(\nu, \xi) = \langle \nu, (-1)^{\text{deg}(\xi)}\xi \rangle$ for homogeneous elements.

Definition 2.5 Let $A$ be a $C^*$-algebra. $\mathcal{J}(A)$ denotes the space consisting of unitary equivalent classes of triples $(\mathcal{E}, Q, \delta)$, where $\mathcal{E}$ is a Hilbert $A$-module, $Q$ is a regular quadratic form on $\mathcal{E}$ and $\delta : \text{dom}(\delta) \to \mathcal{E}$ is a densely defined closed operator satisfying the following conditions;

1. $\delta' = -\delta$, namely, $Q(-\delta(\nu), \xi) = Q(\nu, \delta(\xi))$ for $\nu, \xi \in \text{dom}(\delta)$.
2. $\text{Im}(\delta) \subset \text{dom}(\delta)$ and $\delta^2 = 0$.
3. There exists $\sigma, \tau \in \mathbb{K}(\mathcal{E})$ satisfying $\sigma\delta + \delta\tau - 1 \in \mathbb{K}(\mathcal{E})$.

The typical example, which we will use for dealing with the signature, is given by Definition 3.7. Roughly speaking, $\mathcal{E}$ is a completion of the space of compactly supported differential forms $\Omega^*_c$, $Q$ is given by the Hodge $*$-operation and $\delta$ is the exterior derivative.

Remark 2.6 This definition is slightly different from $L_{ab}(A)$ in [HiSk, 1.5 Définition] and our $\mathcal{J}(A)$ is smaller. However it is sufficient for our purpose.
Lemma 2.7 If a closed operator $\delta$ satisfies the condition (3), then both operators $(\delta + \delta^* \pm i)^{-1}$ can be defined and they belong to $\mathbb{K}(\mathcal{E})$. Here, $\delta^*$ denotes the adjoint of $\delta$ with respect to a certain scalar product on $\mathcal{E}$.

Proof. Since $\delta$ is a closed operator, $\delta + \delta^*$ is self-adjoint. Thus $\text{Im}(\delta + \delta^* \pm i)$ are equal to $\mathcal{E}$ and both operators $\delta + \delta^* \pm i$ are invertible. We now claim that both $(\delta + \delta^* \pm i)^{-1} \in \mathbb{L}(\mathcal{E})$ are compact operators. Since $\text{Im}((\delta + \delta^* \pm i)^{-1}) = \text{dom}(\delta + \delta^* \pm i) = \text{dom}(\delta) \cap \text{dom}(\delta^*)$ and $\delta$ and $\delta^*$ are closed operators, the following operators

$$ \alpha_{\pm} := (\delta + \delta^* \pm i)^{-1} \quad \text{and} \quad \beta_{\pm} := (\delta^* + \delta \pm i)^{-1} $$

are closed operators defined on entire $\mathcal{E}$, which implies that they are bounded; $\alpha, \beta \in \mathbb{L}(\mathcal{E})$.

On the other hand, note that $(\sigma \delta)^2 = (\sigma \delta)(1 - \delta \tau) = \sigma \delta$ and $(\delta^* \delta)^2 = \delta^* \delta$ modulo $\mathbb{K}(\mathcal{E})$. Let $p$ be the orthogonal projection onto $\text{Im}(\delta \tau)$ and let $q = 1 - p$. Then we have that $p(\delta \tau) = \delta \tau$ and $(\delta^* \delta)p = p$ modulo $\mathbb{K}(\mathcal{E})$. Moreover,

$$ (\sigma \delta)q = (1 - \delta \tau)(1 - p) = 1 - \delta \tau - p + (\delta \tau)p = 1 - \delta \tau = \sigma \delta, $$

$$ q(\sigma \delta) = (1 - p)(1 - \delta \tau) = 1 - p - \delta \tau + p(\delta \tau) = 1 - p = q, $$

$$ 1 - (\delta^* \delta^*)q - (\delta^* \delta)p = 1 - (q \sigma \delta^*) - p $$

$$ 1 - q^* - p = 1 - q - p = 0 \quad \text{modulo } \mathbb{K}(\mathcal{E}). $$

Then, set $\ell := 1 - (\delta^* \delta^*)q - (\delta \tau)p \in \mathbb{K}(\mathcal{E})$. Now we conclude that

$$ 1 = \ell + (\delta^* \delta^*)q - (\delta \tau)p, $$

$$ (\delta + \delta^* \pm i)^{-1} = (\delta + \delta^* \pm i)^{-1} \ell + (\alpha_{\pm}^\ast q - \beta_{\pm}^\ast p) \in \mathbb{K}(\mathcal{E}) $$

because $\ell, \sigma$ and $\tau$ belong to $\mathbb{K}(\mathcal{E})$ and $\alpha_{\pm}$ and $\beta_{\pm}$ belong to $\mathbb{L}(\mathcal{E})$. \hfill $\square$

Definition 2.8 For $(\mathcal{E}, Q, \delta) \in \mathbb{J}(A)$, we define the $K$-homology class $\Psi(\mathcal{E}, Q, \delta) \in K_0(A)$ as follows. As in Lemma 2.3, let $\mathcal{E}$ be equipped with the compatible scalar product $\langle \cdot, \cdot \rangle_Q$ and $(\mathbb{Z}/2\mathbb{Z})$-grading associated to $Q$. Next, put

$$ F_{\delta} := (\delta + \delta^*) (1 + (\delta + \delta^*)^2)^{-\frac{1}{2}} \in \mathbb{L}(\mathcal{E}), $$

where $\delta^*$ is the adjoint of $\delta$ with respect to the scalar product $\langle \cdot, \cdot \rangle_Q$. Obviously $F_{\delta}$ is self-adjoint and $F_{\delta}$ is an odd operator since $U \delta U = \delta^t = -\delta$. Moreover it follows that

$$ 1 - F_{\delta}^2 = (1 + (\delta + \delta^*)^2)^{-1} \in \mathbb{K}(\mathcal{E}) $$

by the previous lemma. Then we define $\Psi(\mathcal{E}, Q, \delta) := (\mathcal{E}, F_{\delta}) \in KK(C, A) \cong K_0(A)$. The action of $C$ on $\mathcal{E}$ is the natural multiplication.

Lemma 2.9 For $(\mathcal{E}, Q, \delta) \in \mathbb{J}(A)$ satisfying $\text{Im}(\delta) = \text{Ker}(\delta)$, $\Psi(\mathcal{E}, Q, \delta) = 0 \in K_0(A)$.

Proof. First, remark that $\text{Im}(\delta)$ and $\text{Ker}(\delta^*)$ are orthogonal to each other, and hence, $\text{Im}(\delta) \cap \text{Ker}(\delta^*) = \{0\}$. Indeed, for $\delta(\eta) \in \text{Im}(\delta)$ and $\nu \in \text{Ker}(\delta^*)$, it follows that $\langle \delta(\eta), \nu \rangle = \langle \eta, \delta^* (\nu) \rangle = 0$. Now let $\xi \in \text{Ker}(\delta + \delta^*)$. Then

$$ 0 = \langle \xi, (\delta + \delta^*)^2(\xi) \rangle = \langle \xi, \delta^* \delta(\xi) + \delta \delta^*(\xi) \rangle = \langle \delta(\xi), \delta(\xi) \rangle + \langle \delta^*(\xi), \delta^*(\xi) \rangle, $$

which implies that $\xi \in \text{Ker}(\delta) \cap \text{Ker}(\delta^*) = \text{Im}(\delta) \cap \text{Ker}(\delta^*) = \{0\}$. Therefore, $\text{Ker}(F_{\delta}) = \{0\}$. Since $F_{\delta}$ is a bounded self-adjoint operator, it is invertible. To conclude, $(\mathcal{E}, F_{\delta}) = 0 \in KK(C, A)$. \hfill $\square$
2.2 Perturbation arguments

Lemma 2.10 [HiSk, 2.1, Lemma] Let \((E_X, Q_X, \delta_X), (E_Y, Q_Y, \delta_Y) \in \mathcal{J}(A)\). Suppose that we have...

1. \(T \in \mathbb{L}(E_X, E_Y)\) satisfying \(T(\text{dom}(\delta_X)) \subset \text{dom}(\delta_Y)\), \(T\delta_X = \delta_Y T\) and \(T\) induces an isomorphism \([T]: \text{Ker}(\delta_X)/\text{Im}(\delta_X) \to \text{Ker}(\delta_Y)/\text{Im}(\delta_Y)\);

2. \(\phi \in \mathbb{L}(E_X)\) satisfying \(\phi(\text{dom}(\delta_X)) \subset \text{dom}(\delta_X)\) and \(1 - T'T = \delta_X \phi + \phi \delta_X\);

3. \(\varepsilon \in \mathbb{L}(E_X)\) satisfying \(\varepsilon^2 = 1, \varepsilon' = \varepsilon, \varepsilon \delta_X = -\delta_X \varepsilon\) and \(\varepsilon(1 - T'T) = (1 - T'T)\varepsilon\).

Then, \(\Psi(E_X, Q_X, \delta_X) = \Psi(E_Y, Q_Y, \delta_Y) \in K_0(A)\).

Proof. First, we may assume that \(\phi' = -\phi\). Indeed, since \(1 - T'T = (1 - T'T)' = (\delta_X \phi + \phi \delta_X)' = -((\delta_X \phi' + \phi' \delta_X)\), we may replace \(\phi\) by \(\frac{1}{2}(\phi + \phi')\) which satisfies the same assumption.

Set \(E := E_X \oplus E_Y\), \(Q := Q_X \oplus (-Q_Y)\) and \(\nabla := \left[ \begin{array}{cc} \delta_X & 0 \\ 0 & -\delta_Y \end{array} \right]\). Note that the replacing of \(Q_Y\) by \(-Q_Y\) means the reversing of the grading of \(E_Y\). Then it is easy to see that \(\Psi(E, Q, \nabla) = \Psi(E_X, Q_X, \delta_X) - \Psi(E_Y, Q_Y, \delta_Y)\). Therefore it is sufficient to verify that \(\Psi(E, Q, \nabla) = 0\).

Let us introduce invertible operators \(R_t \in \mathbb{L}(E)\) and a quadratic form \(B_t\) on \(E\) given by the formula:

\[
R_t := \begin{bmatrix} 1 & 0 \\ it\varepsilon & 1 \end{bmatrix} \quad \text{and} \quad B_t(\nu, \xi) := Q(R_t\nu, R_t\xi) = Q(R_t' R_t \nu, \xi)
\]

for \(t \in [0, 1]\). We claim that \((E, B_t, \nabla) \in \mathcal{J}(A)\).

It is easy to see that \(\nabla R_t = R_t^\dagger \nabla\), and hence, \(B_t(\nu, \nabla \xi) = B_t(-\nabla \nu, \xi)\). Clearly the scalar products associated to \(B_t\) and \(Q\) are compatible with each other, also the condition (2) and (3) in the definition of \(\mathcal{J}(A)\) are satisfied. Therefore \((E, B_t, \nabla) \in \mathcal{J}(A)\) and \(\Psi(E, B_t, \nabla) = \Psi(E, Q, \nabla)\).

Next let us introduce

\[
L_t := \begin{bmatrix} 1 - T'T & (i \varepsilon + t \phi) T' \\ T(i \varepsilon + t \phi) & 1 \end{bmatrix} \quad \text{and} \quad C_t(\nu, \xi) := Q(L_t\nu, \xi)
\]

Notice that since \(Q = Q_X \oplus (-Q_Y)\) and that \(T'\) denotes the adjoint of \(T\) with respect to \(Q_X\) and \(Q_Y\), the adjoint of the matrix \(\begin{bmatrix} 0 & 0 \\ T & 0 \end{bmatrix}\) with respect to \(Q\) is equal to \(\begin{bmatrix} 0 & -T' \\ 0 & 0 \end{bmatrix}\). Thus we have that

\[
R_t' \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and that}
\]

\[
R_t' R_t = \begin{bmatrix} 1 - \varepsilon T'T & i \varepsilon T' \\ iT\varepsilon & 1 \end{bmatrix} = \begin{bmatrix} (1 - T'T) \varepsilon & i \varepsilon T' \\ iT\varepsilon & 1 \end{bmatrix} = L_t.
\]

In particular, \(B_1 = C_0\). Since \(L_t\) is invertible at \(t = 0\), there exists \(t_0 > 0\) such that \(L_t\) is invertible for \(t \in [0, t_0]\). Besides it is clear that \(L'_t = L_t\), so \(C_t\) is a regular quadratic form for \(t \in [0, t_0]\).

Moreover consider the operator \(\nabla_t := \left[ \begin{array}{cc} \delta_X & tT' \\ 0 & -\delta_Y \end{array} \right]\), and we claim that \((E, C_t, \nabla_t) \in \mathcal{J}(A)\) for \(t \in [0, t_0]\). The adjoint of \(\nabla_t\) with respect to the quadratic form \(C_t\) is equal to \(L_t^{-1} \nabla'_t \nabla_t\) so in order to check that it is equal to \(-\nabla_t\), we should check that \(L_t \nabla_t = -\nabla'_t \nabla_t\).

\[
L_t \nabla_t = \begin{bmatrix} (1 - T'T)\delta_X & t(1 - T'T)T' - (i \varepsilon + t \phi) T' \delta_Y \\ T(i \varepsilon + t \phi) \delta_X & tT(i \varepsilon + t \phi) T' - \delta_Y \end{bmatrix}
\]

\[
\nabla'_t \nabla_t = \begin{bmatrix} -\delta_X (1 - T'T) & -\delta_X (i \varepsilon + t \phi) T' \\ -tT(1 - T'T) - \delta_Y T(i \varepsilon + t \phi) & -tT(i \varepsilon + t \phi) T' + \delta_Y \end{bmatrix}
\]
Obviously the (1,1) and (2,2)-entries are the negative of each other. Besides we can see that

\[
[(1,2)\text{-entry of } L_t\nabla_t] = t(\delta_X\phi + \phi\delta_X)T' - (i\varepsilon + t\phi)\delta_X T' \\
= t\delta_X\phi T' - i\varepsilon\delta_X T' \\
= \delta_X(i\varepsilon + t\phi)T' = -[(1,2)\text{-entry of } \nabla'_t L_t].
\]

Since \((L_t\nabla_t)' = \nabla'_t L_t\), it automatically follows that \([\text{(2,1)-entry of } L_t\nabla_t] = -[\text{(1,2)-entry of } \nabla'_t L_t]\) as well, and now we obtained that \(L_t\nabla_t = -\nabla'_t L_t\). It is easy to see that \((\nabla'_t)^2 = 0\). If \(\sigma_X, \tau_X \in \mathbb{K}(E_X)\) and \(\sigma_Y, \tau_Y \in \mathbb{K}(E_Y)\) satisfy \(\sigma_X\delta_X + \delta_X\tau_X - 1 \in \mathbb{K}(E_X)\) and \(\sigma_Y\delta_Y + \delta_Y\tau_Y - 1 \in \mathbb{K}(E_Y)\), then it follows that \(\begin{bmatrix} \sigma_X & 0 \\ 0 & -\sigma_Y \end{bmatrix} \nabla_t + \nabla_t \begin{bmatrix} \tau_X & 0 \\ 0 & -\tau_Y \end{bmatrix} - 1 \in \mathbb{K}(E)\) since \(T \in \mathbb{L}(E_X, E_Y)\). Thus we obtained that \((E, C_t, \nabla_t) \in \mathcal{J}(E)\) and \(\Psi(E, C_t, \nabla_t) = \Psi(E, B_1, \nabla) = \Psi(E, Q, \nabla)\).

Finally check that \(\operatorname{Ker}(\nabla_t) = \operatorname{Im}(\nabla_t)\) for any \(t \in (0, t_0)\). \(\operatorname{Ker}(\nabla_t) \supset \operatorname{Im}(\nabla_t)\) is implied by \((\nabla_t)^2 = 0\), so let \(\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \in \operatorname{Ker}(\nabla_t)\). Then \(\theta_2 \in \operatorname{Ker}(\delta_Y)\) and \(tT\theta_2 = -\delta_X\theta_1 \in \operatorname{Im}(\delta_X)\). Since \(T\) induces an isomorphism \([T']\colon \operatorname{Ker}(\delta_Y)/\operatorname{Im}(\delta_Y) \to \operatorname{Ker}(\delta_X)/\operatorname{Im}(\delta_X)\), it follows from the injectivity that \(\theta_2 \in \operatorname{Im}(\delta_Y)\). There exists \(\eta \in E_2\) such that \(\delta_Y\eta = \theta_2\). On the other hand, \(\theta_1 + tT'\eta \in \operatorname{Ker}(\delta_X)\) and the surjectivity of \([T']\) imply that there exists \(\zeta \in \operatorname{Ker}(\delta_Y)\) such that \(T'\zeta = \frac{1}{t}(\theta_1 + tT'\eta)\). Therefore \(\operatorname{Im}(\nabla_t) \supset \nabla_t \begin{bmatrix} 0 \\ \zeta - \eta \end{bmatrix} = \begin{bmatrix} tT'(\zeta - \eta) \\ -\delta_Y(\eta) \end{bmatrix} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}\), which concludes that \(\operatorname{Ker}(\nabla_t) \supset \operatorname{Im}(\nabla_t)\).

Due to Lemma 2.9, it follows that \(\Psi(E, C_t, \nabla_t) = 0 \in KK(C, A)\) and we conclude that \(\Psi(E, Q, \delta_X) - \Psi(E, Q, \delta_Y) = \Psi(E, Q, \nabla) = 0\).

## 3 \(G\text{-signature}\)

### 3.1 Description of the Analytic \(G\text{-index}\)

Let \(G\) be a second countable locally compact Hausdorff group. Let \(X\) be a \(G\)-compact proper complete \(G\)-Riemannian manifold. And let \(\mathcal{V}\) be a \(G\)-Hermitian vector bundle over \(X\). In this section, we will define and investigate a \(\mathcal{C}^*(G)\)-module denoted by \(\mathcal{E}(\mathcal{V})\) obtained by completing \(C_c(X; \mathcal{V})\). This will be used for the definition of the index of \(G\)-invariant elliptic operators, in particular, the signature operator.

**Definition 3.1** [Kas16, Section 5] First we define on \(C_c(X; \mathcal{V})\) the structure of a pre-Hilbert module over \(C_c(G)\) using the action of \(G\) on \(C_c(X; \mathcal{V})\) given by \(\gamma[s](x) = \gamma(s(\gamma^{-1}x))\) for \(\gamma \in G\).

- The action of \(C_c(G)\) on \(C_c(X; \mathcal{V})\) from the right is given by

  \[
  s \cdot b = \int_G \gamma[s] \cdot b(\gamma^{-1}) \Delta(\gamma)^{-\frac{1}{2}} d\gamma \in C_c(X; \mathcal{V})
  \]

  for \(s \in C_c(X; \mathcal{V})\) and \(b \in C_c(G)\). Here, \(\Delta\) denotes the modular function.

- The scalar product valued in \(C_c(G)\) is given by

  \[
  \langle s_1, s_2 \rangle = \Delta(\gamma)^{-\frac{1}{2}} \langle s_1, \gamma[s_2] \rangle_{L^2(\mathcal{V})}
  \]

  for \(s_i \in C_c(\mathcal{V})\).

Define \(\mathcal{E}(\mathcal{V})\) as the completion of \(C_c(\mathcal{V})\) in the norm \(\| (s, s) \|_{\mathcal{C}^*(G)}\).
**Theorem 3.2** [Kas16, Theorem 5.8] Let $G$ be a second countable locally compact Hausdorff group. Let $X$ be a $G$-compact proper complete $G$-Riemannian manifold. Let $D: C_c^\infty(X; \mathbb{V}) \to C_c^\infty(X; \mathbb{V})$ be a formally self-adjoint $G$-invariant first-order elliptic operator on a $G$-Hermitian vector bundle $\mathbb{V}$. Then both operators $D \pm i$ have dense range as operators on $\mathcal{E}(\mathbb{V})$ and $(D \pm i)^{-1}$ belong to $\mathbb{K}(\mathcal{E}(\mathbb{V}))$. The operator $D(1 + D^2)^{-1/2} \in \mathbb{L}(\mathcal{E}(\mathbb{V}))$ is a Fredholm and determines an element $\text{ind}_G(D) \in K_0(\mathbb{C}^*(G))$.

In this paper, mainly we consider $\mathbb{V}$ as $\Lambda^* T^* X$ equipped with the $\mathbb{Z}/2\mathbb{Z}$-grading given by the Hodge $*$-operation and $D$ as a signature operator.

**Definition 3.3** Let $X$ and $Y$ be proper and co-compact Riemannian $G$-manifolds and let $\mathbb{V}$ and $\mathbb{W}$ be $G$-Hermitian vector bundles over $X$ and $Y$ respectively. Let $T: C_c^\infty(X; \mathbb{V}) \to C(Y; \mathbb{W})$ be a linear operator. The support of the distributional kernel of $T$ is given by the closure of the complement of the following union of all subsets $K_X \times K_Y \subset X \times Y$:

$$
\bigcup_{\substack{\langle T s_1, s_2 \rangle = 0 \text{ for any sections} \\ s_1 \in C_c(X; \mathbb{V}) \text{ and } s_2 \in C_c(Y; \mathbb{W}) \text{ satisfying} \\ \text{supp}(s_1) \subset K_X, \text{ supp}(s_2) \subset K_Y}} K_X \times K_Y.
$$

$T$ is said to be properly supported if both

$$
\text{supp}(k_T) \cup (K_X \times Y) \quad \text{and} \quad \text{supp}(k_T) \cup (X \times K_Y) \quad \subset X \times Y
$$

are compact for any compact subset $K_X \subset X$ and $K_Y \subset Y$.

$T$ is said to be compactly supported if $\text{supp}(k_T) \subset X \times Y$ is compact.

The following proposition is used for the construction of the bounded operators on $\mathcal{E}(\mathbb{V})$.

**Proposition 3.4** [Kas16, Proposition 5.4] Let $G$, $X$, $Y$, $\mathbb{V}$ and $\mathbb{W}$ be as above. Let $T: C_c^\infty(X; \mathbb{V}) \to C_c(Y; \mathbb{W})$ be a properly supported $G$-invariant operator which is $L^2$-bounded. Then $T$ defines an element of $\mathbb{L}(\mathcal{E}(\mathbb{V}), \mathcal{E}(\mathbb{W}))$.

For the proof, we will use the following Lemma 3.5 and Lemma 3.6.

**Lemma 3.5** Let $P \in \mathbb{L}(L^2(X; \mathbb{V}), L^2(Y; \mathbb{W}))$ be a compactly supported bounded operator. Then the operator

$$
\tilde{P} := \int_G \gamma |P| \, d\gamma
$$

is well defined as a bounded operator in $\mathbb{L}(L^2(X; \mathbb{V}), L^2(Y; \mathbb{W}))$ and the inequality $\|\tilde{P}\|_{\text{op}} \leq C \|P\|_{\text{op}}$ holds, where $C$ is a constant depending on its support.

**Proof.** Assume that the support of the distributional kernel of $P$ is contained in $K_X \times K_Y$ for some compact subsets $K_X \subset X$ and $K_Y \subset Y$. We will follow the proof of [CM, Lemma 1.4–1.5]. Fix an arbitrary smooth section with compact support $s \in C_c^\infty(X; \mathbb{V})$ and let us consider $F_s \in L^2 \left( G; L^2(Y; \mathbb{W}) \right)$ given by

$$
F_s(\gamma) := \gamma |P| s.
$$

Note that for any $\gamma \in G$ the support of the distributional kernel of $\gamma |P|$ is contained in $\gamma(K_X) \times \gamma(K_Y)$. This is because for any $s \in C_c^\infty(X; \mathbb{V})$, it follows that $\text{supp}(\gamma |P| s) \subset \gamma(K_Y)$ and $\gamma |P| s = 0$ whenever $\text{supp}(s) \cap \gamma(K_X) = \emptyset$. In particular, since the actions are proper, $F_s$ has compact support in $G$. In
addition, again since the actions are proper, \( \gamma(K_Y) \cap \eta(K_Y) = \gamma(K_Y) \cap \gamma^{-1} \eta(K_Y) = \emptyset \) if \( \gamma^{-1} \eta \in G \) is outside some compact neighborhood \( Z \subset G \) in particular,

\[
\|F_s(\gamma)\|_{L^2(Y;\mathbb{W})} \cdot \|F_s(\eta)\|_{L^2(Y;\mathbb{W})} = 0
\]

for such \( \gamma \) and \( \eta \in G \). Remind that \( Z \) is determined only by \( K_Y \) so independent of \( s \). Then,

\[
\left\| \int_G F_s(\gamma) \, d\gamma \right\|^2_{L^2(Y;\mathbb{W})} = \left\| \int_G F_s(\gamma) \, d\gamma \right\|_{L^2(Y;\mathbb{W})} \left\| \int_G F_s(\eta) \, d\eta \right\|_{L^2(Y;\mathbb{W})} \\
\leq \int_G \left\| F_s(\gamma) \right\|_{L^2(Y;\mathbb{W})} \left\| F_s(\eta) \right\|_{L^2(Y;\mathbb{W})} \, d\gamma \, d\eta \\
\leq \int_G \left\| F_s(\gamma) \right\|_{L^2(Y;\mathbb{W})} \left( \int_G \chi_Z(\gamma^{-1}) \left\| F_s(\eta) \right\|_{L^2(Y;\mathbb{W})} \, d\eta \right) \, d\gamma \\
\leq \left\| F_s \right\|_{L^2(G)} \left\| \chi_Z \right\|_{L^2(G)} \left\| F_s \right\|_{L^2(G)} \\
\leq |Z| \left\| F_s \right\|^2_{L^2(G)},
\]

where \( \chi_Z : G \to [0,1] \) is the characteristic function of \( C \), that is \( \chi_Z(\gamma) = 1 \) for \( \gamma \in Z \) and \( \chi_Z(\gamma) = 0 \) for \( \gamma \notin Z \).

Next, take a compactly supported smooth function \( c_1 \in C_c^\infty(X;[0,1]) \) such that \( c_1 = 1 \) on \( K_X \). Noting that \( P = P_{c_1} \), we obtain

\[
\left\| F_s \right\|^2_{L^2(G)} = \int_G \|F_s(\gamma)\|^2_{L^2(Y;\mathbb{W})} \, d\gamma = \int_G \left\| F_s(\gamma) \right\|^2_{L^2(Y;\mathbb{W})} \, d\gamma \\
\leq \int_G \left\| F_s \right\|^2_{op} \left\| c_1 \gamma^{-1}s \right\|^2_{L^2(X;\mathbb{V})} \, d\gamma \\
\leq \|P\|^2_{op} \int_G \int_X \left| c_1(x) \right|^2 \left\| \gamma^{-1}s(x) \right\|^2_{\mathbb{V}} \, dx \, d\gamma \\
\leq \|P\|^2_{op} \sup_{x \in X} \left( \int_G \left| c_1(\gamma^{-1}x) \right|^2 \, d\gamma \right) \left\| s \right\|^2_{L^2(X;\mathbb{V})}.
\]

Since the action of \( G \) is proper, \( \{ \gamma \in G \mid \gamma^{-1}x \in \text{supp}(c_1) \} \subset G \) is compact so the value \( \int_G |c_1(\gamma^{-1}x)|^2 \, d\gamma \) is always finite for any fixed \( x \in X \). Besides, since \( X/G \) is compact, this value is uniformly bounded;

\[
C := \sup_{x \in X} \left( \int_G \left| c_1(\gamma^{-1}x) \right|^2 \, d\gamma \right) = \sup_{x \in X/G} \left( \int_G \left| c_1(\gamma^{-1}x) \right|^2 \, d\gamma \right) < \infty.
\]

Remind that \( C \) depends only on \( K_X \), not on \( s \). We conclude that

\[
\left\| \int_G \gamma[P] \, d\gamma \right\|^2_{L^2(Y;\mathbb{W})} = \left\| \int_G F_s(\gamma) \, d\gamma \right\|^2_{L^2(Y;\mathbb{W})} \leq |Z| \left\| F_s \right\|^2_{L^2(G)} \leq |Z|C \cdot \left\| P \right\|^2_{op} \left\| s \right\|^2_{L^2(X;\mathbb{V})}.
\]

\[ \square \]

**Lemma 3.6** [Kas16, Lemma 5.3] Let \( P \) be a bounded positive operator on \( L^2(X;\mathbb{V}) \) with a compactly supported distributional kernel. Then the scalar product

\[
(s_1, s_2) \mapsto \left< s_1, \left( \int_G \gamma[P] \, d\gamma \right) s_2 \right>_{\mathcal{E}(\mathbb{V})} \in C^*(G)
\]

is well defined and positive for any \( s_1 = s_2 \in C_c(X;\mathbb{V}) \).
Proof. Note that

\[ \langle \gamma[s], P(\gamma[s]) \rangle_{L^2(X;V)} = \left\langle \sqrt{P}(\gamma[s]), \sqrt{P}(\gamma[s]) \right\rangle_{L^2(X;V)} \]

for \( \gamma \in G \) and \( s \in C_c(X;V) \). Regarding the each side of the above equation as a function in \( \gamma \in G \), it is clear that the left hand side vanishes outside some compact subset in \( G \) depending on the support of \( s \) and \( P \). This implies \( \sqrt{P}(\gamma[s]) \) has a compact support in \( G \). Take any unitary representation space \( \mathcal{H} \) of \( G \) and \( h \in \mathcal{H} \). By the above observation of the compact support,

\[ v := \int_G \Delta(\gamma)^{-\frac{1}{2}} \sqrt{P}(\gamma[s]) \otimes \gamma[h] \, d\gamma \in L^2(X;V) \otimes \mathcal{H} \]

is well-defined. Then we obtain that

\[
0 \leq \|v\|^2 = \int_G \int_G \Delta(\gamma)^{-\frac{1}{2}} \Delta(\eta)^{-\frac{1}{2}} \left\langle \sqrt{P}(\gamma[s]), \sqrt{P}(\eta[s]) \right\rangle_{L^2(X;V)} \langle \gamma[h], \eta[h] \rangle_{\mathcal{H}} \, d\gamma \, d\eta \\
= \int_G \int_G \Delta(\gamma)^{-\frac{1}{2}} \Delta(\eta)^{-\frac{1}{2}} \left\langle s, \gamma^{-1}[P(\eta[s])] \right\rangle_{L^2(X;V)} \langle h, \gamma^{-1}[\eta[h]] \rangle_{\mathcal{H}} \, d\gamma \, d\eta \\
= \int_G \int_G \Delta(\gamma)^{-1} \Delta(\eta)^{-\frac{1}{2}} \left\langle s, \gamma^{-1}[P(\gamma^{-1}[\eta[s]])] \right\rangle_{L^2(X;V)} \langle h, \gamma^{-1}[\eta[h]] \rangle_{\mathcal{H}} \, d\gamma \, d(\gamma^{-1}[\eta]) \\
= \int_G \int_G \Delta(\gamma)^{-\frac{1}{2}} \left\langle s, \left( \int_G \gamma[P] \, d\gamma \right)(\gamma[s]) \right\rangle_{L^2(X;V)} \langle h, \gamma[h] \rangle_{\mathcal{H}} \, d\gamma \, d\zeta \\
= \int_G \left\langle s, \left( \int_G \gamma[P] \, d\gamma \right)(\gamma[s]) \right\rangle_{E(V)} \langle \zeta, h, \gamma[h] \rangle_{\mathcal{H}} \, d\zeta
\]

Recall that the action of \( f := \left\langle s, \left( \int_G \gamma[P] \, d\gamma \right)(\gamma[s]) \right\rangle_{E(V)} \in C_c(G) \) on \( \mathcal{H} \) is given by \( f[h] = \int_G f(\zeta) \zeta[h] \, d\zeta \) for \( h \in \mathcal{H} \). Thus, by rewriting the above inequality, we have \( \langle h, f[h] \rangle_{\mathcal{H}} \geq 0 \) for any \( h \), which means that this \( f \) is a positive operator on any unitary representation space \( \mathcal{H} \). To conclude, \( f \) is positive in \( C^*(G) \) for any \( s \in C_c(E(V)) \).

Proof of Proposition 3.4. Let \( T_1 := \frac{1}{2} (cT^*T + T^*Tc) \), which is bounded self-adjoint operator \( L^2(X;V) \to L^2(X;V) \). Moreover the distributional kernel of \( T_1 \) is contained in \( K \times K \) for some compact subset \( K \subset X \). By Lemma 3.5, \( \int_G \gamma[T_1] \) is well-defined in \( L(L^2(X;V)) \) and

\[
\int_G \gamma[T_1] = \int_G \frac{1}{2} \left( \gamma[cT^*T + T^*Tc] \right) = T^*T.
\]

Consider a compactly supported continuous function \( f \in C_c(X;[0,1]) \) satisfying that \( c_1 = 1 \) on \( K \) so that \( c_1T_1c_1 = T_1 \) holds. Consider the following self-adjoint operator;

\[ P := c_1 \left( \|T\|^2 \|c\| - T_1 \right) c_1 = c_1^2 \|T\|^2 \|c\| - T_1 \in L(L^2(X;V)). \]

Obviously \( P \) is compactly supported and since \( T_1 \leq \|T\| \leq \|T\|^2 \|c\| \), \( P \) is positive. Using Lemma 3.6, for any \( s \in C_c(V) \), the following value is positive;

\[
0 \leq \left\langle s, \left( \int_G \gamma[P] \, d\gamma \right)(\gamma) \right\rangle_{E(V)} \\
\leq C \|T\|^2 \|c\| \left\langle s, s \right\rangle_{E(V)} - \left\langle s, \left( \int_G \gamma[T_1] \right)(\gamma) \right\rangle_{E(V)} \in C^*(G),
\]

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where $C$ is the maximum of a $G$-invariant bounded function $\int_X \gamma[c_i^2]$, which is independent of $s$. To conclude,

$$\langle T(s), T(s) \rangle_{\mathcal{E}(\mathcal{W})} = \langle s, T^*T(s) \rangle_{\mathcal{E}(\mathcal{V})} = \left\langle s, \left(\int_G \gamma[T_1] s \right) \right\rangle_{\mathcal{E}(\mathcal{V})} \leq C \|T\|^2 \|c\| \langle s, s \rangle_{\mathcal{E}(\mathcal{V})}.$$ 

\[\square\]

### 3.2 Proof of Theorem A

The theorem we will discuss is the following:

**Theorem A** Let $X$ and $Y$ be oriented even-dimensional complete Riemannian manifolds and let a locally compact Hausdorff group $G$ acts on $X$ and $Y$ isometrically, properly and co-compactly. $f: Y \to X$ be a $G$-equivariant orientation preserving homotopy equivalent map. Let $\partial_X$ and $\partial_Y$ be the signature operators. Then $\text{ind}_G(\partial_X) = \text{ind}_G(\partial_Y) \in K_0(C^*(G)).$

From now on we will slightly change the notation for simplicity. We will only consider $\mathcal{V}$ for the cotangent bundle $\Lambda^* T^*X \otimes \mathbb{C}$. Let us use $\mathcal{E}_X$ for $\mathcal{E}(\Lambda^* T^*X \otimes \mathbb{C})$. Let $\Omega^k_c(X)$ be the space consisting of compactly supported smooth differential forms on $X$, namely, $C^\infty_c(X; \mathcal{V})$. We will prove Theorem A using Lemma 2.10.

**Definition 3.7** Let us introduce the following data $(\mathcal{E}, Q, \delta)$ to present the $G$-index of the signature operator:

- Let $C^*(G)$-valued quadratic form $Q_X$ be defined by the formula;

$$Q_X(\nu, \xi)(\gamma) := i^{k(n-k)} \Delta(\gamma)^{-\frac{1}{2}} \int_X \bar{\nu} \wedge \gamma[\xi] \quad \text{for} \quad \nu \in \Omega^k_c(X), \quad \nu \in \Omega^{n-k}_c(X), \quad \gamma \in G, \quad (3.3)$$

where $\bar{\nu}$ denotes the complex conjugate. If $\text{deg}(\nu) + \text{deg}(\xi) \neq \text{dim}(X)$ then $Q_X(\nu, \xi) := 0$. This deg means the degree of the differential form.

- The grading $U_X$ determined by $Q_X$ is given by

$$U_X(\xi) = i^{-k(n-k)} \ast \xi \quad \text{for} \quad \xi \in \Omega^k_c(X), \quad (3.4)$$

where $\ast$ denotes the Hodge $\ast$-operation.

Clearly, $U_X^2 = 1$ and $Q_X(\nu, U_X(\xi)) = \langle \nu, \xi \rangle_{\mathcal{E}_X}$ hold.

- $\delta_X(\xi) := i^k d_X \xi$ for $\xi \in \Omega^k_c(X)$, where $d_X$ denotes the exterior derivative on $X$.

We will also use the similar notations for $Y$.

**Lemma 3.8** $(\mathcal{E}_X, Q_X, \delta_X) \in \mathbb{J}(C^*(G))$ and $\Psi(\mathcal{E}_X, Q_X, \delta_X) = \text{ind}_G(\partial_X)$, where $\partial_X$ is the signature operator of $X$.

**Proof.** First, obviously $\delta^2 = 0$. Applying Theorem 3.2 to the signature operator on $X$, it follows that $\delta_X - U_X \delta_X U_X: \Omega^k_c(X) \to \mathcal{E}_X$ is closable and its closure is self-adjoint. Let us use $\delta_X - U_X \delta_X U_X$ for also its closure. Since $\text{Im}(\delta_X)$ and $\text{Im}(-U_X \delta_X U_X)$ are orthogonal to each other with respect to the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{E}_X}$, it follows that $\delta_X$ itself is a closed operator on $\mathcal{E}$. Moreover, set $\sigma = \tau := \frac{\delta_X^*}{1 + (\delta_X + \delta_X)^*}$. 

They belong to $\mathbb{K}(\mathcal{E}_X)$ since $\frac{\delta_X}{\delta_X + \delta_X i} \in L(\mathcal{E}_X)$ and $\frac{1}{\delta_X + \delta_X i} \in \mathbb{K}(\mathcal{E}_X)$. Then from Theorem 3.2, we obtain

$$\sigma \delta_X + \delta_X \tau - 1 = \frac{-1}{1 + (\delta_X + \delta_X)^2} \in \mathbb{K}(\mathcal{E}_X).$$

Therefore, $(\mathcal{E}_X, Q_X, \delta_X) \in \mathbb{J}(C^*(G))$ and $\Psi(\mathcal{E}_X, Q_X, \delta_X) = \text{ind}_G(\partial_X)$ by the definition of $\Psi$.

Let $f: Y \to X$ be a $G$-equivariant proper orientation preserving homotopy equivalent map between $n$-dimensional proper co-compact Riemannian $G$-manifolds. In order to construct a map $T \in L(\mathcal{E}_X, \mathcal{E}_Y)$ satisfying the hypothesis of Lemma 2.10, it is sufficient to construct an $L^2$-bounded $G$-invariant operator $T: \Omega^*_c(X) \to \Omega^*_c(Y)$ due to Proposition 3.4.

**Remark 3.9** Note that $f^*: \Omega^*_c(X) \to \Omega^*_c(Y)$ may not be $L^2$-bounded unless $f: Y \to X$ is submersion. For instance, let $Y = X = [-1, 1]$ and $f(y) = y^3$. Consider an $L^2$-form $\omega$ on $X$ given by $\omega(x) = \frac{1}{|x|^{1/2}}$. Actually $\|\omega\|^2_{L^2(X)} = \int_{-1}^1 \frac{1}{|y|^{1/2}} dy = 2$, however, $\|f^*\omega\|^2_{L^2(Y)} = \int_{-1}^1 \frac{1}{|y|^{1/2}} dy = +\infty$. So we need to replace $f^*$ by a suitable operator.

Let us construct operator $T$ that we need and investigate its properties in a slightly more general condition.

- $X$ and $Y$ are Riemannian manifold and $G$ acts on them isometrically and properly. For a while, $X$ and $Y$ may have boundary and the action may not be co-compact if not mentioned.
- Let $W$ be an oriented $G$-invariant fiber bundle over $Y$ whose typical fiber is an even dimensional unit open disk $B^k \subset \mathbb{R}^k$. Let $q: W \to Y$ denote the canonical projection map and $q_I: \Omega^*_{c+k}(W) \to \Omega^*_c(Y)$ be the integration along the fiber.
- Let us fix $\omega \in \Omega^k(W)$ be a $G$-invariant closed $k$-form with fiber-wisely compact support such that the integral along the fiber is always equal to 1; $q_I(\omega)(y) = \int_{W_y} \omega = 1$ for any $y \in Y$. Let $e_\omega$ denote the operator given by $e_\omega(\zeta) = \zeta \wedge \omega$ for $\zeta \in \Omega^*(W)$.

We can construct a $G$-invariant $\omega$ as follows: Let $\tau \in \Omega^k(W)$ be a $k$-form inducing a Thom class of $W$. We may assume that $\int_{W_y} \tau = 1$ for any $y \in Y$. Then $\omega := \int_G \gamma [\tau] \gamma^* d\gamma$ is a desired $G$-invariant form.

- Suppose that we have a $G$-equivariant submersion $p: W \to X$ whose restriction on $\text{supp}(\omega) \subset W$ is proper.

**Definition 3.10** For the above data, let us set $T_{p, \omega} := q_I e_\omega p^*: \Omega^*_c(X) \to \Omega^*_c(Y)$. We may write just $T_p$ for simplicity.

$$\begin{array}{ccc}
W & \xrightarrow{q} & Y \\
\downarrow p & & \downarrow X \\
\Omega^*_c(X) & \xrightarrow{p^*} & \Omega^*_c(W) \\
\xrightarrow{e_\omega} & & \xrightarrow{q_I} \\
\Omega^*_c(W) & \xrightarrow{e_\omega p^*} & \Omega^*_c(Y).
\end{array}$$

**Lemma 3.11** If the actions of $G$ are co-compact, then $T_{p, \omega}$ determines an operator in $L(\mathcal{E}_X, \mathcal{E}_Y)$.

**Proof.** By Proposition 3.4, it is sufficient to check that $T_{p, \omega}$ is $L^2$-bounded.

Since $q_I$ is obviously $L^2$-bounded, only the boundedness of $e_\omega p^*: \Omega^*_c(X) \to \Omega^*_c(W)$ is non-trivial. Note that our proper submersion $p$ restricted on $\text{supp}(\omega) \subset W$ is locally trivial $G$-invariant fibration. Let $p_I$ denotes the integration along this fibration. Then

$$\int_W \zeta = \int_X p_I \zeta$$
holds for any compactly supported differential form \( \zeta \in \Omega^*(W) \) satisfying \( \text{supp}(\zeta) \subset \text{supp}(\omega) \), in particular, \( \zeta = ((p^* \xi) \wedge \omega)^2 \text{vol}_W \in \Omega^{2+k}(W) \) for \( \xi \in \Omega^*_c(X) \). Let \( C_\omega \) be the maximum of the norm of bounded \( G \)-invariant form \( p_I ((\omega)^2 \text{vol}_W) \in \Omega^n(X) \).

\[
\|e_\omega p^*(\xi)\|_{L^2(W)}^2 = \int_W |(p^* \xi) \wedge \omega|^2 \text{vol}_W \equiv \int_X |\xi|^2 \text{vol}_X = C_\omega \|\xi\|_{L^2(X)}^2 \quad \text{for} \quad \xi \in \Omega^*_c(X).
\]

The equation (†) holds because the function \( p^* |\xi|^2 \) is constant along the fiber \( p^{-1}(x) \).

**Lemma 3.12** Let us consider proper co-compact \( G \)-manifold \( X, Y, Z \) and \( q_1 : W \to Y \) and \( q_2 : V \to Z \) be \( G \)-invariant oriented disk bundles over \( Y \) and \( Z \) with typical fiber \( B^{k_1} \) and \( B^{k_2} \). Fix \( G \)-invariant closed forms \( \omega_1 \in \Omega^1(W) \) and \( \omega_2 \in \Omega^{k_2}(V) \) with fiber-wisely compact support satisfying \( (q_j)_I(\omega_j) = 1 \). Let \( p_1 : W \to X \) \( p_2 : V \to Y \) be \( G \)-equivariant submersions whose restriction on \( \text{supp}(\omega_j) \) are proper.

On the other hand, as in the diagram below, let us consider the pull-back bundle \( p_2^* W = \{(v, w) \in V \times W | p_2(v) = w \} \) over \( V \) and let us regard it as a fiber bundle over \( Z \) with projection denoted by \( q_{21} \). Let us set \( \omega_{21} : = p_2^* \omega_1 \wedge \tilde{q}_1 \omega_2 \in \Omega^*(p_2^* W) \), \( p_{21} : = p_1 \tilde{p}_2 \), where \( \tilde{q}_1 : p_2^* W \to V \) denotes the projection and \( \tilde{p}_2 : p_2^* W \to W \) denotes the map induced by \( p_2 \).

Then \( T_{p_2} T_{p_1} = T_{p_{21}} : \mathcal{E}_X \to \mathcal{E}_Z \).

**Proof.** First we can see that for \( \xi \in \Omega^*_c(X) \),

\[
T_{p_{21}}(\xi) = (q_{21})_I \circ e_{\omega_2} p_{21}^* (\xi)
= (q_2)_I (\tilde{q}_1)_I \{ \tilde{p}_2^* p_1^* \xi \wedge (p_2^* \omega_1 \wedge \tilde{q}_1^* \omega_2) \}
= (q_2)_I (\tilde{q}_1)_I \{ \tilde{p}_2^* (p_1^* \xi \wedge \omega_1) \wedge \tilde{q}_1^* \omega_2 \}
= (q_2)_I (\tilde{q}_1)_I \{ (\tilde{q}_1)_I \circ (\tilde{p}_2^* (p_1^* \xi \wedge \omega_1)) \wedge \omega_2 \}
= (q_2)_I (\tilde{q}_1)_I e_{\omega_2} (\tilde{q}_1)_I e_{\omega_2} p_1^* (\xi).
\]

\[
T_{p_2} T_{p_1}(\xi) = (q_2)_I e_{\omega_2} (q_1)_I e_{\omega_1} p_1^* (\xi).
\]

Note that \( (\tilde{q}_1)_I \) in the second bottom row is well defined because the differential form \( \tilde{p}_2^* e_{\omega_1} p_1^*(\xi) \) is compactly supported along each fiber of \( \tilde{q}_1 : p^* W \to V \). We need to prove the commutativity of the following diagram:

\[
\begin{array}{ccc}
\Omega^*(V) & \xrightarrow{p_2^*} & \Omega^*(W) \\
\xrightarrow{(q_1)_I} & \downarrow \tilde{p}_2^* & \\
\Omega^*(W) & \xrightarrow{p_1^*} & \Omega^*(Y)
\end{array}
\]
It is easy to check this using local trivializations. Suppose that \( W \to Y \) is trivialized on \( U \subset Y \). Then \( p_2^*W \) is trivialized on \( p_2^{-1}U \subset V \). We write these trivialization as \( W|_U \cong U \times B^k \) and \( p_2^*W|_U \cong p_2^{-1}U \times B^k \). Then for \( \zeta(y,w) = f(y,w)dy \wedge dw \in \Omega^*|_U \),

\[
((\tilde{q}_1)_I \tilde{p}_2^* \zeta)(v) = \int_{B^k} (f(p_2(v),w)p_2^*(dy)) \, dw = (p_2^*(q_1)_I \zeta)(v) \quad \text{for} \quad v \in p^{-1}U \subset V.
\]

\[\square\]

We will use the following proposition repeatedly.

**Proposition 3.13** Let \( W_1 \) and \( W_2 \) be oriented \( G \)-invariant disk bundles over \( Y \) with typical fiber \( B^{k_1} \) and \( B^{k_2} \), and let \( q_j : W_j \to Y \) be the projection. Let \( \omega_j \in \Omega^{k_j} \) be closed forms with fiber-wisely compact support satisfying \( (p_j)_* \omega_j = 1 \).

Suppose that there exist \( G \)-equivariant submersions \( p_j : W_j \to X \) whose restriction on the 0-sections \( p_j(\cdot,0) : Y \to X \) are \( G \)-equivariant homotopic to each other.

Then, there exists a properly supported \( G \)-equivariant \( L^2 \)-bounded operator \( \psi : \Omega^*_c(X) \to \Omega^*_c(Y) \) satisfying that \( T_{p_2\omega_2} - T_{p_1\omega_1} = d_X \psi + \psi d_Y \).

First, let us prove the following lemma;

**Lemma 3.14** Let \( Q : \tilde{W} \to Y \times [0,3] \) be a \( G \)-invariant disk bundle over \( Y \times [0,3] \) and let \( \omega \in \Omega^k(\tilde{W}) \) be a closed form with fiber-wisely compact support satisfying \( Q_1(\omega) = 1 \). Suppose that there exists a \( G \)-equivariant submersion \( P : \tilde{W} \to X \) whose restriction on \( \text{supp}(\omega) \) is proper. Then there exists a properly supported \( G \)-equivariant \( L^2 \)-bounded operator \( \psi : \Omega^*_c(X) \to \Omega^*_c(Y) \) satisfying that \( T_{P(\cdot,3),\omega(3)} - T_{P(\cdot,0),\omega(0)} = d_X \psi + \psi d_Y \).

**Proof.** Let \( \xi \in \Omega^*_c(X) \) and \( \theta := Q_1(P^*\xi \wedge \omega) \in \Omega^*_c(Y \times [0,3]) \). Then it is easy to see that

\[
\int_{[0,3]} d\theta = -d \left( \int_{[0,3]} \theta \right) + (i_3^*\theta - i_0^*\theta),
\]

where \( i_\cdot : Y \times \{\cdot\} \hookrightarrow Y \times [0,3] \) denotes the inclusion map. Note that \( i_0^*\theta = T_{P(\cdot,0),\omega(\cdot,0)} \xi \).

Now, set \( \psi : \Omega^*_c(X) \to \Omega^*_c(Y) \) by the formula; \( \psi(\xi) := \int_{[0,3]} Q_1(P^*\xi \wedge \omega) \) for \( \xi \in \Omega^*_c(X) \). Note that the identity map \( L^1([0,3]) \to L^2([0,3]) \) is a continuous inclusion due to the finiteness of \( \text{vol}([0,3]) \) hence, the map \( \int_{[0,3]} : \Omega^*_c(Y \times [0,3]) \to \Omega^*_c(Y) \) is \( L^2 \)-bounded. Moreover, since \( P^*\xi \wedge \omega \) vanishes at the boundary of each fiber of \( \tilde{W} \), the integration along the fiber commutes with taking exterior derivative, in particular,

\[
d\theta = dQ_1(P^*\xi \wedge \omega) = Q_1d(P^*\xi \wedge \omega) = Q_1(P^*(d\xi) \wedge \omega).
\]

To conclude, we obtain

\[
\psi(d\xi) = \int_{[0,3]} dQ_1(P^*\xi \wedge \omega) = -d\psi(\xi) + T_{P(\cdot,3),\omega(\cdot,3)} \xi - T_{P(\cdot,0),\omega(\cdot,0)} \xi,
\]

\[\square\]

**Proof of Proposition 3.13.** We need to construct \( \tilde{W} \) and \( P \) as above satisfying \( T_{P(\cdot,0),\omega(\cdot,0)} = T_{P_1\omega_1} \) and \( T_{P(\cdot,3),\omega(\cdot,3)} = T_{p_2\omega_2} \).

Let \( h : Y \times [0,3] \to X \) be a re-parametrized \( G \)-homotopy between \( p_1(\cdot,0) \) and \( p_2(\cdot,0) \), that is, \( h \) is a \( G \)-equivariant smooth map satisfying

\[
h(y,t) = p_1(y,0) \quad \text{for} \quad t \in [0,1]
\]

\[
\text{and} \quad h(y,t) = p_2(y,0) \quad \text{for} \quad t \in [2,3].
\]
here $G$ acts on $[0,3]$ trivially. Moreover, consider the following fiber product $W_1 \times_Y W_2 = \{(y_1, w_1), (y_2, w_2) \in W_1 \times W_2 \}$. Let us introduce a smooth map $\chi: [0,3] \to [0,1]$ satisfying that

$$\chi(t) = 0 \quad \text{for} \quad t \in [0, \frac{11}{10}] \cup \left(\frac{29}{10}, 3\right]$$

and

$$\chi(t) = 1 \quad \text{for} \quad t \in \left(\frac{9}{10}, \frac{21}{10}\right).$$

Then

$$\tilde{h}: (W_1 \times_Y W_2) \times [0,3] \to X,$$

$$((y,t), w_1, w_2, v) \mapsto \begin{cases} p_1(y, (1 - \chi(t))w_1) & \text{for} \quad t \in [0,1], \\ h(y,t) & \text{for} \quad t \in [1,2], \\ p_2(y, (1 - \chi(t))w_2) & \text{for} \quad t \in [2,3]. \end{cases}$$

This $\tilde{h}$ is submersion as long as $\chi(t) \neq 1$ due to the submergence of $p_1$ and $p_2$. Let $BX := \{ v \in TX \mid \|v\| < 1 \}$ be the unit disk tangent bundle and consider the pull-back bundle $\tilde{W} := h^*BX$ and let us regard it as a bundle over $Y \times [0,3]$ and set

$$P: \tilde{W} \to X,$$

$$((y,t), w_1, w_2, v) \mapsto \exp_{\tilde{h}(y,t), w_1, w_2}(\chi(t)v).$$

Due to the $(\chi(t)v)$-component, $P$ is submersion also when $\chi(t) \neq 0$ not only when $\chi(t) \neq 1$.

Moreover, define $\omega \in \Omega^*(W)$ as $\omega := \pi^*_1 \omega_1 \wedge \pi^*_2 \omega_2 \wedge \tilde{h}^* \omega_{BX}$, where $\pi_j: \tilde{W} \to W_j$ for $j = 1,2$ and $\omega_{BX} \in \Omega^*(BX)$ is a $G$-invariant differential with fiber-wisely compact support satisfying $\int_{BX} \omega_{BX} = 1$. These $\tilde{W}$, $P$ and $\omega$ satisfy the assumption of Lemma 3.14.

It is easy to see that $T_{P(\cdot,0),\omega(\cdot,0)} = T_{p_1,\omega_1}$ and $T_{P(\cdot,3),\omega(\cdot,3)} = T_{p_2,\omega_2}$ as follows. For the simplicity, let

$\pi: \tilde{W}_{Y \times \{0\}} \to W_1$ denote the projection. Note that $P(y,0) = p_1 \pi$ and we can write $\omega(\cdot,0) = \pi^* \omega_1 \wedge \tilde{\omega}$, using some $\tilde{\omega} \in \Omega^*(\tilde{W}_{Y \times \{0\}})$ satisfying $\pi_1 \tilde{\omega} = 1$. Then we obtain that

$$T_{P(\cdot,0),\omega(\cdot,0)}(\xi) = (q_1)_f \pi(\pi^* p_1^* \xi \wedge \pi^* \omega_1 \wedge \tilde{\omega})$$

$$= (q_1)_f \pi(\pi^* (p_1^* \xi \wedge \omega_1) \wedge \tilde{\omega})$$

$$= (q_1)_f ((p_1^* \xi \wedge \omega_1) \wedge \pi_1 \tilde{\omega})$$

$$= (q_1)_f (p_1^* \xi \wedge \omega_1) = T_{p_1,\omega_1}(\xi),$$

and similarly, $T_{P(\cdot,3),\omega(\cdot,3)} = T_{p_2,\omega_2}$.}

Now let us define a map $T \in \mathbb{L}(\mathcal{E}_X, \mathcal{E}_Y)$ which satisfies the assumption of Lemma 2.10. First, remark that our map $f: Y \to X$ is a proper map by Lemma 1.5.

**Definition 3.15** Let $BX := \{ v \in TX \mid \|v\| < 1 \}$ be the unit disk tangent bundle and let $W := f^*BX$ be the pull-back on $Y$, that is, $W = \{(y,v) \in Y \times BX \mid v \in BX \}$. Let $\tilde{f}: W \to BX$ be a map given by $\tilde{f}(x,v) := (f(x),v)$. Since the action of $G$ on $X$ is isometric and $f$ is $G$-equivariant, $G$ acts on $BX$ and also on $W$. Consider a $G$-equivariant submersion given by the formula;

$$p: \quad W \to X,$$

$$(y,v) \mapsto \exp_{f(y),(v)}. \tag{3.6}$$

Let us fix a $G$-invariant $\mathbb{R}$-valued closed $n$-form $\omega_0 \in \Omega^n(BX)$ with fiber-wisely compact support whose integral along the fiber is always equal to 1, and let $\omega := \tilde{f}^* \omega_0 \in \Omega^n(W)$ For these $W$, $p$ and $\omega$, let us set $T := T_{p,\omega}$. 

\]
Lemma 3.16 The adjoint with respect to quadratic forms $Q_X$ and $Q_Y$ is given by $T' = p_f e_\omega q^*$. Proof. Note that $\text{deg}(\omega) = \text{dim}(X)$ is even, hence, $\omega$ commutes with other differential forms. For $\nu \in \Omega^k_c(Y)$ and $\xi \in \Omega^{n-k}_c(X)$,

\[
\int_X p_f e_\omega q^*(\nu) \wedge \xi = \int_X p_f (q^* \nu \wedge \omega) \wedge \xi = \int_X p_f (q^* \nu \wedge \omega \wedge p^* \xi) = \int_BX q^* \nu \wedge \omega \wedge p^* \xi
\]

\[
= \int_Y q_f (q^* \nu \wedge p^* \xi \wedge \omega) = \int_Y \nu \wedge q_f (p^* \xi \wedge \omega) = \int_Y \nu \wedge T(\xi).
\]

Since $Q_X(\nu, \xi)(\gamma) := \frac{1}{k} \Delta(\gamma) \int_X \nu \wedge \gamma(\xi)$, the proof complete replacing $\nu$ and $\xi$ by $\nu$ and $\gamma(\xi)$ respectively and using the $G$-invariance of $T$. \hfill \Box

Proposition 3.17 There exists $\phi \in \mathcal{L}(E_X)$ such that $1 - T'\theta = d_X \phi + \phi d_X.$

Proof. Consider the fiber product $W \times_Y W$ and let $q_1$ and $q_2$: $W \times_Y W \rightarrow W$ denote the projections given by $q_j(y, v_1, v_2) := (y, v_j)$. Take $\zeta \in \Omega^2_c(W)$, here $W$ is regarded as the first component of $W \times_Y W$. Using the commutativity of the diagram (3.5), we obtain that

\[
e_\omega q^* q_f(\zeta) = e_\omega (q_2)_f q_f(\zeta) = (q_2)_f (q_f(\zeta) \wedge \omega) = (q_2)_f (q_f(\zeta) \wedge q_f \omega)
\]

and hence,

\[
T'\theta = p_f e_\omega q^* q_f e_\omega p^* = p_f (q_2)_f e_\omega q_f^* e_\omega p^*.
\]

On the other hand, since $q_1(y, 0) = q_2(y, 0)$, by Proposition 3.13, there exists a properly supported $G$-equivariant $L^2$-bounded operator $\psi_W: \Omega^*_c(W) \rightarrow \Omega^*_c(W)$ satisfying

\[
(q_2)_f e_\omega q_f^* - (q_2)_f e_\omega q_f^* = d\psi_W + \psi_W d.
\]

Moreover, it is obvious that $(q_2)_f e_\omega q_f^* = \text{id}_{\Omega^*_c(W)}$, so we obtain

\[
p_f e_\omega p^* - T'\theta = p_f \left( \text{id}_{\Omega^*_c(W)} - (q_2)_f e_\omega q_f^* \right) e_\omega p^*
\]

\[
= p_f \left( d\psi_W + \psi_W d \right) e_\omega p^*
\]

\[
= d \circ p_f \psi_W e_\omega p^* + p_f \psi_W e_\omega p^* \circ d.
\]

Remark that $p_f \circ d = d \circ p_f$ because the act on differential forms with compact support, and $e_\omega \circ d = d \circ e_\omega$ because $\omega$ is a closed form.

Next let us consider submersion $p_X : BX \rightarrow X$ given by $(x, v) \mapsto \exp_x(v)$. Note that $p = p_X \tilde{f}$.  

\[
f^*BX \xrightarrow{\tilde{f}} BX \\
p \downarrow \downarrow p_X \\
\tilde{f} \downarrow \\
X
\]
Now we want to check that $p_I e_\omega p^* = (p_X)_I e_\omega_X p^*_X$. For any $\nu \in \Omega^*_c(X)$ and $\zeta \in \Omega^*_c(BX)$,
\[
\int_X \nu \wedge p_I (\tilde{f}^* \zeta) = \int_W p^* \nu \wedge \tilde{f} \zeta = \int_W \tilde{f}^* (p^*_X \nu \wedge \zeta) = \deg(\tilde{f}) \int_{BX} p^*_X \nu \wedge \zeta = \int_{BX} p^*_X \nu \wedge \zeta = \int_X \nu \wedge (p_X)_I(\zeta),
\]
since $f$ is an orientation preserving proper homotopy equivalent. In particular, we obtain
\[
p_I (\tilde{f}^* \zeta) = (p_X)_I(\zeta) \tag{3.8}
\]
Put $\zeta := p^*_X \xi \wedge \omega_0$ for $\xi \in \Omega^*_c(X)$ to obtain
\[
p_I e_\omega p^* (\xi) = p_I (\tilde{f}^* p^*_X \xi \wedge \tilde{f} \omega_0) = p_I (\tilde{f}^* (p^*_X \xi \wedge \omega_0)) = (p_X)_I (p^*_X \xi \wedge \omega_0) = (p_X)_I e_\omega_X p^*_X (\xi). \tag{3.9}
\]
Let $\pi: BX \to X$ be the natural projection. Since $p^*_X (x, 0) = \pi(x, 0)$, by Proposition 3.13, there exists a properly supported $G$-equivariant $L^2$-bounded operator $\psi_X: \Omega^*_c(X) \to \Omega^*_c(X)$ satisfying
\[
\pi_I e_\omega_X \pi^* - (p_X)_I e_\omega_X p^*_X = d\psi_X + \psi_X d. \tag{3.9}
\]
On the other hand, it is obvious that $\pi_I e_\omega_X \pi^* = \text{id}_{\Omega_c(X)}$. Therefore, combining (3.7), (3.8) and (3.9), we conclude
\[
\text{id}_{\Omega_c(X)} - T'T = d\phi + \phi d,
\]
where $\phi = p_I \psi_W e_\omega p^* + \psi_X$. Since $\phi$ is properly supported $G$-invariant $L^2$-bounded operator, it defines an element in $L(\mathcal{E}_X)$.

\textbf{Proof of Theorem A.} First, let us check that $T$ satisfies the assumption (1) of Lemma 2.10. Since $\omega$ is a closed form and has fiber-wisely compact support, it follows that $T\delta_X = \delta_Y T$. Let $g: X \to Y$ be the $G$-equivariant homotopy inverse of $f$ and consider a map $S \in L(\mathcal{E}_Y, \mathcal{E}_X)$ constructed in the same method as $T$ from $g$ instead of $f$ in Definition 3.15. By 3.12, the composition $ST$ is equal to the map $T_p \in L(\mathcal{E}_X)$ for $p$ satisfying that $p(\cdot, 0)$ is $G$-equivariant homotopic to $\text{id}_X$. Then by Proposition 3.13, there exists $\phi_X \in L(\mathcal{E}_X)$ satisfying that $ST - (\delta_X \phi_X + \phi_X \delta_X) = T_{\text{id}_X} = \text{id}_{\mathcal{E}_X}$. Thus, $ST$ induces the identity map on $\text{Ker}(\delta_X)/\text{Im}(\delta_X)$. Similarly $TS$ induces the identity map on $\text{Ker}(\delta_Y)/\text{Im}(\delta_Y)$, and hence, $T$ induces an isomorphism $\text{Ker}(\delta_X)/\text{Im}(\delta_X) \to \text{Ker}(\delta_Y)/\text{Im}(\delta_Y)$.

The assumption (2) of Lemma 2.10 is obtained from 3.17.

Finally, let $\varepsilon(\xi) := (-1)^k \xi$ for $\xi \in \Omega^*_c(X)$. Clearly, $\varepsilon$ determines an operator $\varepsilon \in L(\mathcal{E}_X)$, $\varepsilon^2 = 1$ and satisfies $\varepsilon f = f, \varepsilon(\text{dom}(\delta_X)) \subset \text{dom}(\delta_X)$ and $\varepsilon \delta_X = -\delta_X \varepsilon$. Moreover since neither $T$ nor $T'$ changes the order of the differential forms, $\varepsilon$ commutes with $1 - T'T$. Thus $\varepsilon$ satisfies the assumption (3) of Lemma 2.10. To conclude, we obtain $\text{ind}_G(\partial_X) = \Psi(\mathcal{E}_X, Q_X, \delta_X) = \Psi(\mathcal{E}_Y, Q_Y, \delta_Y) = \text{ind}_G(\partial_Y)$. \hfill \Box

### 3.3 On proof of Corollary B

To prove Corollary B, we will combine [F, Theorem A] with Theorem A. Suppose, in addition, that $G$ is unimodular and $H_1(X; \mathbb{R}) = H_1(Y; \mathbb{R}) = \{0\}$. Let $f: Y \to X$ be a $G$-equivariant orientation preserving homotopy invariant map and consider a $G$-manifold $Z := X \sqcup (-Y)$, the disjoint union of $X$ and orientation reversed $Y$. Let $\partial_Z$ be the signature operator, then we have that $\text{ind}_G(\partial_Z) = \text{ind}_G(\partial_X) - \text{ind}_G(\partial_Y) = 0 \in K_0(C^*(G))$. Although the $G$-manifold should be connected in [F, Theorem A], however in this case, we can apply it to $Z$ after replacing some arguments in [F] as follows.
Lemma 4.2 Using a cut-off function given line bundle, we just use a line bundle $L$ the formula; $f$ bundles of line bundles $\{L_t\}$ over $X$ in the same way and pull back on $Y$ to obtain a family $\{f^*L_t\}$. To be specific, $f^*L_t$ is a trivial bundle $Y \times \mathbb{C}$, equipped with the connection given by $\nabla^t = d + itf^*\eta$ and the action of $G_{\alpha_t}$ is given by

$$(\gamma, u)(y, z) = (\gamma y, \exp[-itf^*\psi(x)]uz) \quad \text{for} \quad (\gamma, u) \in G_{\alpha_t}, \ y \in Y, \ z \in \mathbb{C} = (L_t)_x.$$ 

Then consider a family of $G_{\alpha_t}$-line bundles $\{L_t \sqcup f^*L_t\}$ over $Z$. We also need the similar replacement in [F, Definition 7.19] to obtain the global section on $L_t \sqcup f^*L_t$. Then the rest parts proceed similarly.

4 Index of Dirac operators twisted by almost flat bundles

Now we will discuss on the Dirac operators twisted by a family of Hilbert module bundles $\{E^k\}$ whose curvature tend to zero and prove Theorem C. Such an family is called a family of almost flat bundles. In this section, it is convenient to formulate the index map using $KK$-theory.

4.1 $G$-index map in $KK$-theory

Lemma 4.1 [Kas88, Theorem 3.11] Let $G$ be a second countable locally compact Hausdorff group. For any $G$-algebras $A$ and $B$ there exists a natural homomorphism

$$j^G: KK^G(A, B) \to KK(C^*(G; A), C^*(G; B)).$$

Furthermore if $x \in KK^G(A, B)$ and $y \in KK^G(B, D)$, then $j^G(x \otimes_B y) = j^G(x) \hat{\otimes}_{C^*(G; B)} j^G(y).$ \hfill $\square$

Lemma 4.2 Using a cut-off function $c \in C_c(X)$, one can define an idempotent $p \in C_c(G; C_0(X))$ by the formula;

$$\hat{c}(\gamma)(x) = \sqrt{c(x)c(\gamma^{-1}x)}\Delta(\gamma)^{-1}.$$ 

In particular it defines an element of $K$-homology denoted by $[c] \in K_0(C^*(G; C_0(X)))$. Moreover the element of $K$-homology $[c] \in K_0(C^*(G; C_0(X)))$ does not depend on the choice of cut-off functions. \hfill $\square$

Definition 4.3 (G-Index) [Kas16, Theorem 5.6.] Define

$$\mu^G: KK^G(C_0(X), C) \to K_0(C^*(G))$$

as the composition of

- $j^G: KK^G(C_0(X), C) \to KK(C^*(G; C_0(X)), C^*(G))$ and
- $[c] \tilde{\otimes}: KK(C^*(G; C_0(X)), C^*(G)) \to KK(C, C^*(G)) \simeq K_0(C^*(G))$, i.e.,

$$\mu^G(-) := [c] \tilde{\otimes}_{C^*(G; C_0(X))} j^G(-) \in K_0(C^*(G)).$$

Remark 4.4 As in [Kas16, Remark 4.4.] or [F, Subsection 5.2], it is sufficient to consider only in the case of Dirac type operators for calculating the index.

Let $B$ be a unital $C^*$-algebra. Following the definition 4.3, we define the index maps with coefficients;

Definition 4.5 For unital $C^*$-algebras $B$, define the index map

$$\text{ind}_G: KK^G(C_0(X), B) \to K_0(C^*(G; B))$$

as the composition of
\( j^G: KK^G(C_0(X), B) \to KK(C^*(G; C_0(X)), C^*(G; B)) \) and

\( [c] \hat{\otimes}: KK(C^*(G; C_0(X)), C^*(G; B)) \to K_0(C^*(G; B)), \) i.e.,

\[
\text{ind}_G(\cdot) := [c] \hat{\otimes}_{C^*(G; C_0(X))} j^G(\cdot) \in K_0(C^*(G; B)).
\]

The crossed product \( C^*(G; B) \) is either maximal or reduced one. In this paper, we assume that \( G \) acts on \( B \) trivially. Then \( C^*_\text{Max}(G; B) \) and \( C^*_\text{red}(G; B) \) will be naturally identified with \( C^*_\text{Max}(G) \otimes_{\text{Max}} B \) and \( C^*_\text{red}(G) \otimes_{\text{min}} B \) respectively. Moreover if \( B \) is nuclear, \( \otimes_{\text{Max}} B \) and \( \otimes_{\text{min}} B \) are identified.

**Definition 4.6** Let \( E \) be a finitely generated projective \((\mathbb{Z}/2\mathbb{Z})\)-graded Hilbert \( B \)-module \( G \)-bundle. Define \( C_0(X; E) \) as a space consisting of sections \( s: X \to E \) vanishing at infinity. It is considered as a \((\mathbb{Z}/2\mathbb{Z})\)-graded Hilbert \( C_0(X; B) \)-module with the right action given by point-wise multiplications and the scalar product given by

\[
\langle s_1, s_2 \rangle(x) := \langle s_1(x), s_2(x) \rangle_{E_x} \in C_0(X; B).
\]

**Remark 4.7** The \( C^* \)-algebra \( C_0(X; B) \) consisting of \( B \)-valued function vanishing at infinity is naturally identified with \( C_0(X) \hat{\otimes} B \) by [We, 6.4.17. Theorem]. Similarly, if \( E = X \times E_0 \) is a trivial Hilbert \( B \)-module bundle over \( X \), then \( C_0(X; E) \) is naturally identified with \( C_0(X) \hat{\otimes} E_0 \) as Hilbert \((C_0(X; B) \cong C_0(X) \hat{\otimes} B)\)-modules.

**Definition 4.8** \( E \) define an element in \( KK \)-theory

\[
[E] = (C_0(X; E), 0) \in KK^G(C_0(X), C_0(X) \hat{\otimes} B).
\]

The action of \( C_0(X) \) on \( C_0(X; E) \) is the point-wise multiplication.

**Definition 4.9** Let \( E \) be a finitely generated Hilbert \( B \)-module bundle over \( X \) equipped with a Hermitian connection \( \nabla^E \). Let \( R^E \in C^\infty \left( X; \text{End}(E) \otimes \wedge^2(T^*(X)) \right) \) denote its curvature. Then define its norm as follows: First, define the point-wise norm as the operator norm given by

\[
\|R^E\|_x := \sup \left\{ \|R^E(u \wedge v)\|_{L(E)} \mid u, v \in T_x X, \|u \wedge v\| = 1 \right\} \quad \text{for} \quad x \in X.
\]

Then define the global norm as the supremum in \( x \in X \) of the point-wise norm: \( \|R^E\| := \sup_{x \in X} \|R^E\|_x \)

To describe the Theorem which we will prove,

**Theorem C** Let \( X \) be a complete oriented Riemannian manifold and let \( G \) be a locally compact Hausdorff group acting on \( X \) isometrically, properly and co-compactly. Moreover we assume that \( X \) is simply connected. Let \( D \) be a \( G \)-invariant properly supported elliptic operator of order 0 on \( G \)-Hermitian vector bundle over \( X \).

Then there exists \( \varepsilon > 0 \) satisfying the following: for any finitely generated projective Hilbert \( B \)-module \( E \) over \( X \) equipped with a \( G \)-invariant Hermitian connection such that \( \|R^E\| < \varepsilon \), we have

\[
\text{ind}_G \left( [E] \hat{\otimes}_{C_0(X)} [D] \right) = 0 \in K_0(C^*_\text{Max}(G) \otimes_{\text{Max}} B)
\]

if \( \text{ind}_G([D]) = 0 \in K_0(C^*_\text{Max}(G)) \). If we only consider commutative \( C^* \)-algebras for \( B \), then the same conclusion is also valid for \( C^*_\text{red}(G) \).
4.2 Infinite product of $C^*$-algebras

Definition 4.10 Let $B_k$ be a sequence of $C^*$-algebras.

- Define $\prod B_k$ as the $C^*$-algebra consisting of norm-bounded sequences
  \[
  \prod B_k := \left\{ \{b_1, b_2, \ldots\} \mid b_k \in B_k, \sup_k \|b_k\|_{B_k} < \infty \right\}.
  \]
  The norm of $B_k$ is given by $\|\{b_1, b_2, \ldots\}\|_{\prod B_k} := \sup_k \|b_k\|_{B_k}$.

- Let $\bigoplus B_k$ a closed two-sided ideal in $\prod B_k$ consisting of sequences vanishing at infinity
  \[
  \bigoplus B_k := \left\{ \{b_1, b_2, \ldots\} \mid b_k \in B_k, \lim_{k \to \infty} \|b_k\| = 0 \right\}.
  \]
  In other words, $\bigoplus_{k \in \mathbb{N}} B_k$ is a closure of the sub-space in $\prod_{k \in \mathbb{N}} B_k$ consisting of sequences $\{b_1, b_2, \ldots, 0, 0, \ldots\}$ whose entries are zero except for finitely many of them.

- Define $Q B_k$ as the quotient algebra given by
  \[
  Q B_k := \left( \prod B_k \right) / \left( \bigoplus B_k \right).
  \]
  The norm of $QB_k$ is given by $\|\{b_1, b_2, \ldots\}\|_{QB_k} := \limsup_{k \to \infty} \|b_k\|_{B_k}$.

- If $\mathcal{E}_k$ are Hilbert $B_k$-modules, one can similarly define $\prod \mathcal{E}_k$ as a Hilbert $\prod B_k$-module consisting of bounded sequences
  \[
  \prod \mathcal{E}_k := \left\{ \{s_1, s_2, \ldots\} \mid s_k \in \mathcal{E}_k, \sup_k \|s_k\|_{\mathcal{E}_k} < \infty \right\}.
  \]
  The action of $\prod B_k$ and $\prod B_k$-valued scalar product are defined as follows;
  \[
  \{s_k\} \cdot \{b_k\} := \{s_k \cdot b_k\} \in \prod \mathcal{E}_k \quad \text{for} \quad \{s_k\} \in \prod \mathcal{E}_k, \{b_k\} \in \prod B_k,
  \]
  \[
  \langle \{s^1_k\}, \{s^2_k\} \rangle_{\prod \mathcal{E}_k} := \left\{ \langle s^1_k, s^2_k \rangle \right\} \in \prod B_k \quad \text{for} \quad \{s^1_k\}, \{s^2_k\} \in \prod \mathcal{E}_k.
  \]
  One can define similarly
  \[
  \bigoplus \mathcal{E}_k := \left\{ \{s_1, s_2, \ldots\} \mid s_k \in \mathcal{E}_k, \lim_{k \to \infty} \|s_k\|_{\mathcal{E}_k} = 0 \right\}
  \]
  as a Hilbert $\prod B_k$-module, and define
  \[
  Q \mathcal{E}_k := \left( \prod \mathcal{E}_k \right) \hat{\otimes}_\pi (QB_k) = \left( \prod \mathcal{E}_k \right) / \left( \bigoplus \mathcal{E}_k \right)
  \]
  as a Hilbert $QB_k$-module, where $\pi : \prod B_k \to QB_k$ denotes the projection.

Example 4.11 If all of $B_k$ are $\mathbb{C}$, then, $\prod \mathbb{C} = \ell^\infty(\mathbb{N})$ and $\bigoplus \mathbb{C} = C_0(\mathbb{N})$.

Following [Ha12, Section 3.], we will construct “infinite product bundle $\prod E_k$” over $X$ which has a structure of finite generated projective $\prod B_k$-module.
Definition 4.12 Let us fix some notations about the holonomy.

- Two paths $p_0$ and $p_1$ from $x$ to $y$ in $X$ are thin homotopic to each other if there exists an end points preserving homotopy $h: [0, 1] \times [0, 1] \to X$ with $h(\cdot, j) = p_j$ that factors through a finite tree $T$,

$$h: [0, 1] \times [0, 1] \to T \to X$$

such that both restrictions of the first map $[0, 1] \times \{j\} \to T$ are piecewise-linear for $j = 0, 1$.

- The path groupoid $\mathcal{P}_1(X)$ is a groupoid consisting of all the points in $X$ as objects. The morphism from $x$ to $y$ are the equivalence class of piece-wise smooth paths connecting given two points

$$\mathcal{P}_1(X)[x, y] := \{p: [0, 1] \to X | p(0) = x, p(1) = y\} / \sim .$$

The equivalent relationship is given by re-parametrization and thin homotopy.

- If a Hilbert $B$-module $G$-bundle $E$ over $X$ is given, the transport groupoid $\mathcal{T}(X; E)$ is a groupoid with the same objects as $\mathcal{P}_1(X)$. The morphism from $x$ to $y$ are the unitary isomorphisms between the fibers $\mathcal{T}(X; E)[x, y] := \text{Iso}_B(E_x, E_y)$.

Definition 4.13 A parallel transport of $E$ is a continuous functor $\Phi^E: \mathcal{P}_1(X) \to \mathcal{T}(X; E)$. $\Phi^E$ is called $\varepsilon$-close to the identity if for each $x \in X$ and contractible loop $p \in \mathcal{P}_1(X)[x, x]$, it follows that

$$\|\Phi^E_p - \text{id}_{E_x}\| < \varepsilon \cdot \text{area}(D)$$

for any two dimensional disk $D \subset X$ spanning $p$. $D$ may be degenerated partially or completely.

Remark 4.14 Let $E$ be a Hermitian vector bundle, in other words, a finitely generated Hilbert $C^*$-module bundle, equipped with a compatible connection $\nabla$. Let $\Phi^E$ be the parallel transport with respect to $\nabla$ in the usual sense. If its curvature $R^E \in C^\infty \left(X; \text{End}(E) \otimes \Lambda^2(T^*(X))\right)$ has uniformly bounded operator norm $\|R^E\| < C$, then for any loop $p \in \mathcal{P}_1(X)[x, x]$ and any two dimensional disk $D \subset X$ spanning $p$, it follows that $\|\Phi^E_p - \text{id}_{E_x}\| < \int_D \|R^E\| < C \cdot \text{area}(D)$ so it is $C$-closed to identity.

Proposition 4.15 Let $\{E^k\}$ be a sequence of Hilbert $B_k$-module $G$-bundles over $X$ with $B_k$ unital $C^*$-algebras. Assume that each parallel transport $\Phi^k$ for $E^k$ is $\varepsilon$-close to the identity uniformly, that is, $\varepsilon$ is independent of $k$.

Then there exists a finitely generated Hilbert $(\prod_k B_k)$-module $G$-bundle $V$ over $X$ with Lipschitz continuous transition functions in diagonal form and so that the $k$-th component of this bundle is isomorphic to the original $E^k$.

Moreover, if the parallel transport $\Phi^k$ for each of $E^k$ comes from the $G$-invariant connection $\nabla^k$ on $E^k$, $V$ is equipped with a continuous $G$-invariant connection induced by $E^k$.

Proof. We will essentially follow the proof of [Ha12, Proposition 3.12]. For each $x \in X$ take a open ball $U_x \subset X$ of radius $\ll 1$ whose center is $x$. Assume that each $U_x$ is geodesically convex. Due to the corollary 1.4 of the slice theorem, there exists a sub-family of finitely many open subsets $\{U_{x_1}, \ldots U_{x_N}\}$ such that $X = \bigcup_{\gamma \in G} \bigcup_{i=1}^N \gamma(U_{x_i})$.

Fix $k$. In order to simplify the notation, let $U_i := U_{x_i}$ and $\Phi_{y,x}^k: E^k_y \to E^k_x$ denote the parallel transport of $E^k$ along the minimal geodesic from $y$ to $x$ for $x$ and $y$ in the same neighborhood $\gamma(U_i)$. Trivialize $E^k$ via $\Phi_{y,x}^k: E^k_y \to E^k_x$ on each $U_i$. Similarly trivialize $E^k$ on each $\gamma(U_i)$ for $\gamma \in G$ via $\Phi_{y,\gamma x}^k: E^k_{\gamma y} \to E^k_{\gamma x}$. Note that since parallel transport commute with the action of $G$, it follows that $\Phi_{y,\gamma x}^k = \gamma \circ \Phi_{y,x}^k \circ \gamma^{-1}$.

These provide a local trivializations for $E^k$ whose transition functions have uniformly bounded Lipschitz constants. More precisely we have to fix unitary isomorphisms $\phi_{\gamma x}^k: E^k_{\gamma x} \to E^k_x$ between
the fiber on $\gamma x_i$ and the typical fiber $E^k$. Our local trivialization is $\phi_{\gamma x_i} \Phi_{y;\gamma x_i} : E^k_y \to E^k$. If $y, z \in \gamma(U_i) \cap \eta(U_j) \neq \emptyset$, we can consider the transition function

$$y \mapsto \psi_{\gamma(U_i), \eta(U_j)}(y) := (\phi_{y;\eta x_j} \circ \Phi_{y;\gamma x_i}) (\phi_{\gamma x_i} \circ \Phi_{y;\gamma x_i})^{-1} \in \text{End}_{B_k}(E^k).$$

Now we will estimate its Lipschitz constant as follows;

$$\psi_{\gamma(U_i), \eta(U_j)}(y) - \psi_{\gamma(U_i), \eta(U_j)}(z) = (\phi_{y;\eta x_j} \Phi_{y;\gamma x_i}) (\phi_{y;\gamma x_i} \Phi_{y;\gamma x_i})^{-1} - (\phi_{z;\eta x_j} \Phi_{y;\gamma x_i}) (\phi_{\gamma x_i} \Phi_{y;\gamma x_i})^{-1}$$

$$= \phi_{y;\eta x_j} \left( (\Phi_{y;\eta x_j} \Phi_{y;\gamma x_i})^{-1} - (\Phi_{z;\eta x_j} \Phi_{y;\gamma x_i})^{-1} \right) \phi_{\gamma x_i}^{-1}$$

$$= \phi_{y;\eta x_j} \left( (\Phi_{y;\eta x_j} \Phi_{y;\gamma x_i})^{-1} - (\Phi_{z;\eta x_j} \Phi_{y;\gamma x_i})^{-1} \right) \phi_{\gamma x_i}^{-1}.$$

Since $\phi$'s and $\Phi$'s are isometry, it follows that

$$\left| \psi_{\gamma(U_i), \eta(U_j)}(y) - \psi_{\gamma(U_i), \eta(U_j)}(z) \right| \leq \left| \Phi_{y;\eta x_j} - \Phi_{z;\eta x_j} \right| + \left| \Phi_{y;\gamma x_i}^{-1} - \Phi_{z;\gamma x_i}^{-1} \right|$$

$$= \left| \Phi_{y;\eta x_j} \Phi_{z;\gamma x_i} - \Phi_{z;\eta x_j} \Phi_{y;\gamma x_i} \Phi_{y;\gamma x_i} \right| + \left| \Phi_{y;\gamma x_i}^{-1} - \Phi_{z;\gamma x_i}^{-1} \right|$$

$$\leq \varepsilon \cdot (\text{area}(D_1) + \text{area}(D_2)). \quad (4.1)$$

Here $D_1 \subset \eta(U_j)$ is a two dimensional disk spanning the piece-wise geodesic loops connecting $\eta x_j$, $y$, $z$, and $\eta x_j$ and $D_2 \subset \gamma(U_j)$ is a two dimensional disk spanning the piece-wise geodesic loop connecting $\gamma x_i$, $y$, $z$, and $\gamma x_i$.

We claim that there exists a constant $C$ depending only on $X$ such that

$$\text{area}(D_1), \text{area}(D_2) \leq C \cdot \text{dist}(y, z) \quad (4.2)$$

if we choose suitable disks $D_1$ and $D_2$.

We verify this using the geodesic coordinate $\exp_{\eta x_j}^{-1} : \eta(U_j) \to T_{\eta x_j}X$ centered at $\eta x_j \mapsto 0$. More precisely, let $p$ denote the minimal geodesic from $y = p(0)$ to $z = p(\text{dist}(y, z))$ with unit speed. Consider

$$D_0 := \{ (r \cos \theta, r \sin \theta) \in \mathbb{R}^2 \mid 0 \leq r, 0 \leq \theta \leq \text{dist}(y, z) \} \subset \mathbb{R}^2$$

and $F : D_0 \to \eta(U_j) \subset X$ given by

$$F(r \cos \theta, r \sin \theta) := \exp_{\eta x_j} \left( r \exp_{\eta x_j}^{-1}(p(\theta)) \right).$$

Set $D_1 := F(D_0)$. $F$ is injective if $\exp_{\eta x_j}^{-1}(y)$ and $\pm \exp_{\eta x_j}^{-1}(z)$ are on different radial directions, in which case $F$ is a homeomorphism onto its image, and hence $F(D_0)$ is a two dimensional disk spanning the target loop. The Lipschitz constant of $F$ is bounded by a constant depending on the curvature on $\eta(U_j)$, so there exists a constant $C_{\eta, j}$ depending on the Riemannian curvature on $\eta(U_j)$ satisfying

$$\text{area}(D_1) \leq C_{\eta, j} \cdot \text{area}(D_0) \leq C_{\eta, j} \cdot \text{dist}(y, z).$$

However, the constant $C_{\eta, j}$ can be taken independent of $\eta(U_j)$ due to the bounded geometry of $X$ implied by the slice theorem (Corollary 1.4). In the case of $\exp_{\eta x_j}^{-1}(y)$ and $\pm \exp_{\eta x_j}^{-1}(z)$ are on the same radial direction, $D_1$ is completely degenerated and $\text{area}(D_1) = 0$. We can construct $D_2$ in the same manner so the claim $(4.2)$ has been verified.

Therefore combining $(4.1)$ and $(4.2)$, we conclude that the Lipschitz constants of the transition functions of these local trivialization are less than $2C\varepsilon$, which are independent of $E^k$, $U_i$ and $\gamma \in G$, in particular, the product of them

$$\Psi_{\gamma(U_i), \eta(U_j)} := \left\{ \psi^k_{\gamma(U_i), \eta(U_j)} \right\}_{k \in \mathbb{N}} : \gamma(U_i) \cap \eta(U_j) \to \mathbb{L}(\prod B_k) \left( \prod B_k^k \right)$$
are Lipschitz continuous. So it is allowed to use them to define the Hilbert $\prod_k B_k$-module bundle $V$ as required. Precisely $V$ can be constructed as follows:

$$V := \bigsqcup_{\gamma, i} \left( \gamma(U_i) \times \prod_k E^k \right) / \sim.$$ 

Here, $(x, v) \in \gamma(U_i) \times \prod_k E^k$ and $(y, w) \in \eta(U_j) \times \prod_k E^k$ are equivalent if and only if $x = y \in \gamma(U_i) \cap \eta(U_j)$ and $\Phi_{\gamma(U_i), \eta(U_j)}(v) = w$. By the construction of $V$, if $p_n : \prod_k B_k \to B_n$ denotes the projection onto the $n$-th component, $V \otimes_{p_n} B_n$ is isomorphic to the original $n$-th component $E^n$.

In order to verify the continuity of the induced connection, let $\{e_i\}$ be an orthonormal local frame on $U_i$ for an arbitrarily fixed $E^k$ obtained by the parallel transport along the minimal geodesic from the center $x_i \in U_i$, namely, $e_i(x_i) = \Phi_{x_i, y_i} e_i(x_i)$. It is sufficient to verify that $\|\nabla^k e_i\| \leq C$. Let $v \in T_y X$ be a unit tangent vector and $p(t) := \exp_y(tv)$ be the geodesic of unit speed with direction $v$.

$$\nabla^k e_i(y) = \lim_{t \to 0} \frac{1}{t} \left( \Phi_{p(t); p(0)} e_i(p(t)) - e_i(p(0)) \right)$$

$$= \lim_{t \to 0} \frac{1}{t} \left( \Phi_{p(t); p(0)} \Phi_{x_i; p(t)} - \Phi_{x_i; p(0)} \right) e_i(x_i),$$

$$\|\nabla^k e_i(y)\| \leq \lim_{t \to 0} \frac{1}{|t|} \| \Phi_{p(t); p(0)} \Phi_{x_i; p(t)} - \Phi_{x_i; p(0)} \|$$

$$\leq \lim_{t \to 0} \frac{1}{|t|} \epsilon \cdot \text{area}(D(t)),$$

where $D(t)$ is a 2-dimensional disk in $U_i$ spanning the piece-wise geodesic connecting $x_i, p(0) = y, p(t)$ and $x_i$. As above, we can find a constant $C > 0$ and disks $D(t)$ satisfying

$$\text{area}(D(t)) \leq C \cdot \text{dist}(p(0), p(t)) = C|t|$$

for $|t| \ll 1$. Hence, we obtain $\|\nabla^k e_i(y)\| \leq C\epsilon$.

**Definition 4.16** Let us define a Hilbert $(Q B_k)$-module bundle

$$W := V \otimes_{\pi} (Q B_k),$$

where $\pi : \prod B_k \to Q B_k$ denotes the projection.

The family of parallel transport of $E^k$ induces the parallel transport $\Phi^W$ of $W$ which commutes with the action of $G$.

**Proposition 4.17** If the parallel transport of $E^k$ is $C_k$-close to the identity with $C_k \searrow 0$, then the $G$-bundle $W$ constructed above is a flat bundle. More precisely the parallel transport $\Phi^W(p) \in \text{Hom}(W_x, W_y)$ depends only on the ends-fixing homotopy class of $p \in P_1(X)[x, y]$.

**Proof.** It is sufficient to prove that for any contractive loop $p \in P_1(X)[x, x]$, it satisfies $\Phi^W(p) = \text{id}_{W_x}$. Fix a two dimensional disk $D \subset X$ spanning the loop $p$. For arbitrary $\epsilon > 0$ there exists $n_0$ such that every $k \geq n_0$ satisfies that $\Phi^{E^k}$ is $\frac{\epsilon}{1 + \text{area}(D)}$-close to the identity.

$$\|\Phi^W(p) - \text{id}_{W_x}\| = \limsup_{k \to \infty} \left\| \Phi^{E^k}(p) - \text{id} \right\| \leq \sup_{k \geq n_0} \left\| \Phi^{E^k}(p) - \text{id} \right\| \leq \frac{\epsilon}{1 + \text{area}(D)} \cdot \text{area}(D) \leq \epsilon$$

This implies $\Phi^W(p) = \text{id}_{W_x}$. \qed
4.3 Index of the product bundle

**Proposition 4.18**

(1) Let \( p_n : \prod B_k \to B_n \) denote the projection onto the \( n \)-th component and consider

\[
(1 \otimes p_n)_* : K_0 \left( C^* (G) \hat{\otimes} (\prod B_k) \right) \to K_0 \left( C^* (G) \hat{\otimes} B_n \right).
\]

Then

\[
(1 \otimes p_n)_* \text{ind}_G \left( \prod E^k \right) \hat{\otimes} [D] = \text{ind}_G \left( [E^n] \hat{\otimes} [D] \right).
\]

(2) Let \( \pi : \prod B_k \to Q B_k \) denote the quotient map and consider

\[
(1 \otimes \pi)_* : K_0 \left( C^* (G) \hat{\otimes} (\prod B_k) \right) \to K_0 \left( C^* (G) \hat{\otimes} (Q B_k) \right).
\]

Then

\[
(1 \otimes \pi)_* \text{ind}_G \left( \prod E^k \right) \hat{\otimes} [D] = \text{ind}_G ([W] \hat{\otimes} [D]).
\]

**Proof.** As for the first part, \( [E^n] = (p_n)_* \prod E^k \in \text{KK}^G (C_0 (X), C_0 (X) \hat{\otimes} B_n) \) by the construction of \( \prod E^k \). Then it follows that

\[
\text{ind}_G \left( [E^n] \hat{\otimes} [D] \right) = \text{ind}_G \left( (p_n)_* \left[ \prod E^k \right] \hat{\otimes} [D] \right)
\]

\[
= \text{ind}_G \left( [\prod E^k] \hat{\otimes} [D] \hat{\otimes} \mathbb{P}_n \right)
\]

\[
= \text{ind}_G \left( [\prod E^k] \hat{\otimes} [D] \hat{\otimes} j^G (\mathbb{P}_n) \right)
\]

\[
\text{ind}_G \left( \left[ \prod E^k \right] \hat{\otimes} [D] \right) \hat{\otimes} j^G (\mathbb{P}_n)
\]

\[
= \text{ind}_G \left( \left[ \prod E^k \right] \hat{\otimes} [D] \right) \hat{\otimes} j^G (\mathbb{P}_n)
\]

\[
= (1 \otimes p_n)_* \text{ind}_G \left( \left[ \prod E^k \right] \hat{\otimes} [D] \right),
\]

where \( \mathbb{P}_n = (B_n, p_n, 0) \in \text{KK} (\prod_{k \in \mathbb{N}} B_k, B_n) \). Then note that \( j^G (\mathbb{P}_n) = (C^* (G) \hat{\otimes} B_n, 1 \otimes p_n, 0) \in \text{KK} \left( C^* (G) \hat{\otimes} (\prod_{k \in \mathbb{N}} B_k), C^* (G) \hat{\otimes} B_n \right) \). Since \( \pi_* \left[ \prod E^k \right] = [W] \in \text{KK}^G \left( C_0 (X), C_0 (X) \hat{\otimes} (Q B_k) \right) \) by the construction of \( W \), the second part can be proved in the similar way.

**Proposition 4.19** Let \( [D] \) be a \( K \)-homology element in \( \text{KK}^G (C_0 (X), \mathbb{C}) \) determined by a Dirac operator on a \( G \)-Hermitian vector bundle \( V \) over \( X \). Suppose that \( W \) is a finitely generated flat \( B \)-module \( G \)-bundle. Assume that \( X \) is simply connected.

Then \( \text{ind}_G ([W] \hat{\otimes} [D]) = 0 \) if \( \text{ind}_G ([D]) = 0 \).

In order to prove this, we introduce an element \( [W]_{\text{rpn}} \in \text{KK}^G (\mathbb{C}, B) \) using the holonomy representation.

**Definition 4.20**

- Let \( \Phi_{x,y} \) denote the parallel transport of \( W \) along an arbitrary path from \( x \in X \) to \( y \in X \). Since \( X \) is simply connected and \( W \) is flat, it depends only on the ends of the path.

- Let us fix a base point \( x_0 \in X \) and \( W_{x_0} \) be the fiber on \( x_0 \). Define \( [W]_{\text{rpn}} \) as

\[
[W]_{\text{rpn}} := (W_{x_0}, 0) \in \text{KK}^G (\mathbb{C}, B)
\]

The action of \( G \) on \( W_{x_0} \) is given by the holonomy \( \rho : G \to \text{End}_Q (W_{x_0}) \)

\[
\rho [\gamma] (w) = (\Phi_{x_0, x_0})^{-1} x_0) \gamma (w) \quad \text{for} \quad \gamma \in G, \ w \in W_{x_0}
\]
Lemma 4.21

\[ [W] \hat{\otimes}_{C_0(X)} [D] = [D] \hat{\otimes}_{C} [W]_{\text{rpn}} \in KK^G(C_0(X), B). \]

Proof. Recall that \([D] \in KK^G(C_0(X), \mathbb{C})\) is given by \((L^2(X; V), F_D)\), where \(F_D\) denotes the operator \(\frac{D}{\sqrt{1+D^2}}\), and that

\[ [W] \hat{\otimes}_{C_0(X)} [D] = \left( C_0(X; W) \hat{\otimes}_{C_0(X)} L^2(X; V), F_{DW} \right), \]

where \(D^W\) is the Dirac operator twisted by \(W\) acting on \(L^2(X; W \otimes V) \simeq C_0(X; W) \hat{\otimes}_{C_0(X)} L^2(X; V)\), that is,

\[ D^W = \sum_j (\text{id}_W \otimes c(e_j)) \left( \nabla^{W}_{e_j} \otimes \text{id}_V + \text{id}_W \otimes \nabla^{V}_{e_j} \right), \]

where \(\{e_j\}\) denotes an orthogonal basis for \(TX\) and \(c(\cdot)\) denotes the Clifford multiplication by \(\text{Cliff}(TX)\) on \(V\). The action of \(C_0(X)\) on \(C_0(X; W)\) and \(L^2(X; V)\) are the point-wise multiplications. On the other hand,

\[ [D] \hat{\otimes}_{C} [W]_{\text{rpn}} = (L^2(X; V) \hat{\otimes}_{C} W_{x_0}, F_{D} \hat{\otimes} \text{id}) \]

The action of \(C_0(X)\) is the point-wise multiplications. Note that the action of \(G\) on \(W_{x_0}\) is given by the holonomy representation \(\rho\). It is sufficient to give a \(G\)-equivariant isomorphism

\[ \varphi : L^2(X; V) \hat{\otimes}_{C} W_{x_0} \to C_0(X; W) \hat{\otimes}_{C_0(X)} L^2(X; V), \]

which is compatible with \(D^W\) and \(D \hat{\otimes} \text{id}\). Set a section for \(W\) given by

\[ \overline{w} : x \mapsto \Phi_{x_0; x} w \in W_x \quad (4.3) \]

and define \(\varphi\) on a dense sub space \(C_c(X; V) \hat{\otimes} W_{x_0}\) as

\[ \varphi(s \otimes w) := \overline{w} \cdot \chi \otimes s \quad \text{for } s \in C_c(X; V) \text{ and } w \in W_{x_0}, \]

where \(\chi \in C_0(X)\) is an arbitrary compactly supported function on \(X\) with values in \([0, 1]\) satisfying that \(\chi(x) = 1\) for all \(x \in \text{supp}(s)\).

\(\varphi\) is independent of the choice of \(\chi\) and hence well-defined. Indeed, Let \(\chi' \in C_c(X)\) be another such function, and let \(\rho \in C_c(X)\) be a compactly supported function on \(X\) with values in \([0, 1]\) satisfying that \(\rho(x) = 1\) for all \(x \in \text{supp}(\chi) \cup \text{supp}(\chi')\). Then in \(C_0(X; W) \hat{\otimes}_{C_0(X)} C_c(X; V)\),

\[ \overline{w} \cdot \chi \otimes s - \overline{w} \cdot \chi' \otimes s = \overline{w} \cdot (\chi - \chi') \otimes s = \overline{w} \cdot \rho \cdot (\chi - \chi') \otimes s = \overline{w} \cdot \rho \otimes (\chi - \chi') s = 0. \]

Now we obtain that

\[ D^W \circ \varphi(s \otimes w) = D^W(\overline{w} \otimes s) = \overline{w} \otimes D(s) = \varphi \circ (D \hat{\otimes} \text{id})(s \otimes w) \]

for \(s \in C_c(V)\) and \(w \in W_{x_0}\). This is because \(\nabla^W \overline{w} = 0\) by its construction.

Compatibility with the action of \(G\) is verified as follows;

\[ \varphi(\gamma(s \otimes w))(x) = \Phi_{x_0; x}(\rho[\gamma](w)) \otimes \gamma(s(\gamma^{-1}x)) = \Phi_{x_0; x}(\Phi_{x_0; \gamma x_0})^{-1} \gamma(w) \otimes \gamma(s(\gamma^{-1}x)) = \Phi_{\gamma x_0; x}(\gamma(b)) \otimes \gamma(s(\gamma^{-1}x)), \]

\[ \gamma(\varphi(s \otimes w))(x) = \gamma((\Phi_{x_0; \gamma^{-1}x}(w) \otimes s(\gamma^{-1}x)) = \Phi_{\gamma x_0; x}(\gamma(w)) \otimes \gamma(s(\gamma^{-1}x)), \]
Let us check that \( \varphi \) induces an isomorphism. For \( s_1 \otimes w_1,\ s_2 \otimes w_2 \in C_c(X;\mathbb{V}) \otimes_{C} W_{x_0} \), it follows that

\[
\begin{align*}
\left\langle \varphi(s_1 \otimes w_1), \varphi(s_2 \otimes w_2) \right\rangle_{C_0(X;W) \otimes_{C_0(X;X)} L^2(X;\mathbb{V})} &= \left\langle s_1, \left\langle \overline{w}_1 \chi, \overline{w}_2 \chi \right\rangle_{C_0(X;W)} s_2 \right\rangle_{L^2(X;\mathbb{V})} \\
&= \int_X \left\langle s_1(x), \left\langle (\Phi_{x_0;x} w_1) \chi(x), (\Phi_{x_0;x} w_2) \chi(x) \right\rangle_{W_x} s_2(x) \right\rangle_{\mathbb{V}_x} \, d\text{vol}(x) \\
&= \int_X \left\langle w_1, w_2 \right\rangle_{W_x} \chi(x)^2 \left\langle s_1(x), s_2(x) \right\rangle_{\mathbb{V}_x} \, d\text{vol}(x) \\
&= \left\langle w_1, w_2 \right\rangle_{W_0} \left\langle s_1, s_2 \right\rangle_{L^2(X;\mathbb{V})} \\
&= \left\langle s_1 \otimes w_1, s_2 \otimes w_2 \right\rangle_{L^2(X;\mathbb{V})} \otimes_{\mathbb{V}} W_{x_0},
\end{align*}
\]

where \( \chi \in C_0(X) \) is a compactly supported function on \( X \) satisfying that \( \chi(x) = 1 \) for all \( x \in \text{supp}(s_1) \cup \text{supp}(s_2) \). This implies that \( \varphi \) is continuous and injective.

Moreover, choose arbitrary \( F \in C_c(X;W) \otimes_{C} W_{x_0} \) and \( s \in C_c(X;\mathbb{V}) \). Since \( \Phi_{x_0;x}^{-1} \) provides a trivialization of \( W \simeq X \times W_{x_0} \), we have an isomorphism \( C_c(X;W) \simeq C_c(X;\mathbb{V}) \otimes_{C} W_{x_0} \). Remark that, however, this is not a \( G \)-equivariant isomorphism, just as pre-Hilbert \( (C_0(X;B) \cong C_0(X) \otimes B) \)-modules. Then there exist countable subsets \( \{f_1, f_2, \ldots\} \subset C_c(X) \) and \( \{w_1, w_2, \ldots\} \subset W_{x_0} \) satisfying that \( \sum_{j \in \mathbb{N}} f_j w_j = F \) in \( C_0(X;W) \). Now it follows that

\[
\varphi \left( \sum_{j \in \mathbb{N}} f_j s \otimes w_j \right) = \sum_{j \in \mathbb{N}} \left( \overline{w}_j \chi \otimes f_j s \right) = \sum_{j \in \mathbb{N}} (\overline{w}_j \chi f_j \otimes s) = \left( \sum_{j \in \mathbb{N}} \overline{w}_j f_j \right) \cdot \chi \otimes s = F \otimes s,
\]

where \( \chi \in C_0(X) \) is a compactly supported function on \( X \) satisfying that \( \chi(x) = 1 \) for all \( x \in \text{supp}(F) \cup \text{supp}(s) \). This implies that the image of \( \varphi \) is dense in \( C_0(X;W) \otimes L^2(X;\mathbb{V}) \). Therefore \( \varphi \) induces an isomorphism. \( \square \)

**Proof of the Proposition 4.19.** Due to the previous lemma, it follows that

\[
\text{ind}_G([W] \otimes [D]) = \text{ind}_G([D] \otimes C_0(X) \otimes L^2(X;\mathbb{V})) = \text{ind}_G([D] \otimes C_0(X) \otimes L^2(X;\mathbb{V})).
\]

Thus the assumption \( \text{ind}_G[D] = 0 \) implies \( \text{ind}_G([W] \otimes [D]) = 0 \). \( \square \)

### 4.4 Proof of Theorem C

**Proof of Theorem C.** As Remark 4.4, we may assume that \( D \) is a Dirac type operator. Assume that \( \text{ind}_G[D] = 0 \) and we assume the converse. that is, for each \( k \in \mathbb{N} \) there exits a Hilbert \( B_k \)-module \( G \)-bundle \( E^k \) over \( X \) whose curvature norm is less than \( \frac{1}{k} \) satisfying that

\[
\text{ind}_G([E^k] \otimes [D]) \neq 0 \quad \in K_0(C^*(G) \otimes B_k).
\]

To begin with, we have an exact sequence;

\[
0 \rightarrow \bigoplus B_k \rightarrow \prod B_k \rightarrow \mathcal{Q} B_k \rightarrow 0,
\]
where $\iota$ and $\pi$ are natural inclusion and projection. We also have the following exact sequence [We, Theorem T.6.26]:

$$0 \to C^*_\text{Max}(G) \otimes_{\text{Max}} \left( \bigoplus B_k \right) \xrightarrow{\mathbf{1} \otimes \iota} C^*_\text{Max}(G) \otimes_{\text{Max}} \left( \prod B_k \right) \xrightarrow{\mathbf{1} \otimes \pi} C^*_\text{Max}(G) \otimes_{\text{Max}} (Q B_k) \to 0.$$  

We have the exact sequence of $K$-groups

$$K_0 \left( C^*_\text{Max}(G) \otimes_{\text{Max}} \left( \bigoplus B_k \right) \right) \to K_0 \left( C^*_\text{Max}(G) \otimes_{\text{Max}} \left( \prod B_k \right) \right) \to K_0 \left( C^*_\text{Max}(G) \otimes_{\text{Max}} (Q B_k) \right).$$

If all of $B_k$ are commutative, then $QB_k$ is also commutative and hence nuclear. In that case, we also have the same exact sequences in which $C^*_\text{Max}(G)$ and $\otimes_{\text{Max}}$ are replaced by $C^*_\text{red}(G)$ and $\otimes_{\text{min}}$ respectively.

Let us start with $\text{ind}_G (\prod E^k \otimes [D]) \in K_0 (C^*(G) \otimes (\prod B_k)).$

Due to the flatness of $W$ (Proposition 4.17) and Proposition 4.19 and 4.18, we have

$$(1 \otimes \pi)_* \text{id}_G \left( \prod E^k \otimes [D] \right) = \text{id}_G \left( [W] \otimes [D] \right) = 0.$$  

It follows from the exactness that there exists $\zeta \in K_0 (C^*(G) \otimes (\bigoplus B_k))$ such that

$$(1 \otimes \iota)_* (\zeta) = (1 \otimes \iota)_* \text{id}_G \left( \prod E^k \otimes [D] \right).$$

**Lemma 4.22** $A \otimes (\bigoplus_{k \in N} B_k)$ is naturally isomorphic to $\bigoplus_{k \in N} (A \otimes B_k) = \lim_n \bigoplus_{k=1}^n (A \otimes B_k).$

**Proof.** Let $C$ denote the direct product $\lim_n \bigoplus_{k=1}^n (A \otimes B_k)$. Note that for the finite direct product, we have the natural isomorphism $\bigoplus_{k=1}^n (A \otimes B_k) \cong A \otimes (\bigoplus_{k=1}^n B_k)$. For each $n \in N$, we have the following commutative diagram:

\[
\begin{array}{ccc}
A \otimes (\bigoplus_{k=1}^n B_k) & \xrightarrow{id_A \otimes \iota^+_{n+1}} & A \otimes (\bigoplus_{k=1}^{n+1} B_k) \\
\downarrow{id_A \otimes \iota^+_{n}} & & \downarrow{id_A \otimes \iota^+_{n+1}} \\
A \otimes (\bigoplus_{k \in N} B_k) & & \\
\end{array}
\]

Now by using the universal property of the direct limit, we obtain a map $\phi$:

\[
\begin{array}{ccc}
A \otimes (\bigoplus_{k=1}^n B_k) & \xrightarrow{\lim_n} & A \otimes (\bigoplus_{k \in N} B_k) \\
\downarrow & & \downarrow \phi \\
A \otimes (\bigoplus_{k \in N} B_k) & & \\
\end{array}
\]

Since $id_A \otimes \iota_n$ are isometric and injective, $\phi$ is isometric and injective on each sub-space $A \otimes (\bigoplus_{k=1}^n B_k) \subset \lim_n A \otimes (\bigoplus_{k=1}^n B_k)$. Since the union of such sub-spaces is dense in $\lim_n A \otimes (\bigoplus_{k=1}^n B_k)$, it follows that $\phi$ itself is isometric and injective.

As for the surjectivity of $\phi$, take any $a \otimes \{b_k\} \in A \otimes (\bigoplus_{k \in N} B_k)$. For any $\varepsilon > 0$, there exists $n \in N$ such that $\|b_k\| < \frac{\varepsilon}{1 + \|a\|}$ for $k \geq n$. Then replace $b_k$ by 0 for all $k \geq n$ to obtain an element $\beta := \{b_1, b_2, \ldots, 0, 0, \ldots\} \in \bigoplus_{k \in N} B_k$. Now we have that

$$a \otimes \beta = (id_A \otimes \iota_n)(a \otimes \{b_1, b_2, \ldots, b_{n-1}\}) = \phi(a \otimes \{b_1, b_2, \ldots, b_{n-1}\}) \in \text{Im}(\phi)$$

and $\|a \otimes \{b_k\} - a \otimes \beta\| \leq \|a\| \|\{b_k\} - \beta\| \leq \varepsilon$. These imply that $\text{Im}(\phi)$ is dense in $A \otimes (\bigoplus_{k \in N} B_k)$ and hence, $\phi$ is surjective since it has closed range. 

\[\square\]
By this lemma, $C^*(G)\hat{\otimes} (\bigoplus B_k)$ is naturally isomorphic to $\bigoplus (C^*(G)\hat{\otimes} B_k)$. Besides, we have the natural isomorphism $K_0 (\bigoplus (C^*(G)\hat{\otimes} B_k)) \simeq \bigoplus K_0 (C^*(G)\hat{\otimes} B_k)$ [HR, 4.15 Proposition, 4.2.3 Remark], with the last $\bigoplus$ meaning the algebraic direct sum. Thus we can consider the following diagram:

$$
\begin{array}{ccc}
K_0 (C^*(G)\hat{\otimes} (\bigoplus B_k)) & \xrightarrow{(1\otimes p_k)_*} & K_0 (C^*(G) \otimes (\prod B_k)) \\
\{\{1\otimes p_k\}_*\} & \cong & \{\{1\otimes p_k\}_*\} \\
\bigoplus K_0 (C^*(G)\hat{\otimes} B_k) & \xrightarrow{\text{inclusion}} & \prod K_0 (C^*(G)\hat{\otimes} B_k)
\end{array}
$$

Since $p_k = \iota p_k$, this diagram commutes. Note that both $\bigoplus$ and $\prod$ in the bottom row are in the algebraic sense. Again due to Proposition 4.18,

$$
\left\{ \text{ind}_G \left( [E^n] \hat{\otimes} [D] \right) \right\}_{k \in \mathbb{N}} = \left\{ (1 \otimes p_k)_* \left( \text{ind}_G \left( \prod E^k \hat{\otimes} [D] \right) \right) \right\} = \left\{ (1 \otimes p_k)_* (1 \otimes \iota)_* (\xi) \right\} = \left\{ (1 \otimes p_k)_* (\xi) \right\} \in \bigoplus K_0 (C^*(G)\hat{\otimes} B_k).
$$

This implies that all of $\text{ind}_G \left( [E^n] \hat{\otimes} [D] \right) \in K_0 (C^*(G)\hat{\otimes} B_n)$ are equal to zero except for finitely many $n \in \mathbb{N}$, which contradicts to our assumption. \qed

4.5 On proof of Corollary D

To prove Corollary D, we will combine Theorem C with Theorem A. Consider the same conditions as Theorem A on $X$, $Y$ and $G$ and assume additionally that $X$ and $Y$ are simply connected. Let $f: Y \to X$ be a $G$-equivariant orientation preserving homotopy invariant map. Assume that for each $k \in \mathbb{N}$ there exits a Hilbert $B_k$-module $G$-bundle $E^k$ over $X$ whose curvature norm is less than $\frac{1}{k}$ satisfying that

$$
\text{ind}_G ([E^k] \hat{\otimes} [\partial X]) \neq \text{ind}_G ([f^* E^k] \hat{\otimes} [\partial Y]) \in K_0 (C^*(G)\hat{\otimes} B_k).
$$

as in the proof of Theorem C. Consider a $G$-manifold $Z := X \sqcup (-Y)$, the disjoint union of $X$ and orientation reversed $Y$ and the signature operator $\partial_Z$ on it. Although $Z$ is not connected, however, we may apply Theorem C to $\partial_Z$, after replacing some argument in the proof as follows. Consider a family of Hilbert $B_k$-module bundles $\{ E^k \sqcup f^* E^k \}$ over $Z$ and obtain a flat bundle $W \sqcup f^* W$ as in subsection 4.2. In order to obtain a global section $\overline{w}$ as in (4.3) in the proof of Lemma 4.21, we have used the connectedness of the base space. In this case, construct a section $\overline{w}: X \to W$ on $X$ in the same way and pull back it on $Y$ by $f$ to obtain a global section on $Z$. The other parts are the same as above.

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