Parameter inversion of cantilever beam based on polynomial model

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Abstract. Inverse problem is a kind of problem that “effects” are used to get the “causes”. It has broad application prospects in the field of applied mathematics and physics. The paper makes an inversion analysis based on a cantilever beam via polynomial model. An iterative formula is deduced based on Gauss-Newton method to tackle inherent parameter of cantilever beam. In the process of inversing, direct problem is solved for many times. The polynomial model is constructed and taken as a direct problem solver. The method proposed in this paper can make parameter inversion of cantilever beam with variable Young’s modulus. The result shows that the method has good stability. It can give some guidance for engineers to solve other inversion problem in engineering.

1. Introduction
Parameter inversion, which is a kind of inverse problem, is the opposite of solving a direct problem. Generally speaking, a direct problem is that the model parameter of the system is known. In other words, the input of the system is known. And then, output of the system is calculated. However, engineers may encounter many inverse problems in engineering practice. That is to say, output of the system is known, but the parameter or input of the system is unknown. The process of solving this kind of problem is called as solving the inverse problem or the inversion process [1]. In fact, many physical problems can be described by the mathematical model of partial differential equation, whose general form can be expressed in the formula below.

\[ \phi_1(p,s) = \phi_2(f) \]  

(1)

In the formula (1), \( p \) is the model parameter and the internal cause of the system. \( s \) is the state variable of the system, which can be measured. \( f \) is the input of the system, usually representing external actions such as external force and control. \( \phi_1 \) and \( \phi_2 \) are partial differential operators. Given \( \phi_1 \) and \( \phi_2 \); when you know \( p, f \), all the boundary conditions and the initial conditions; \( s \) can be solved. This is a cause-and-effect process, which is known as solving a direct problem. However, in many practical engineering problems, it is difficult to obtain complete information of \( p, f \), boundary conditions or initial conditions even though \( \phi_1 \) and \( \phi_2 \) are known. To get this information is to solve an inverse problem [2].

When solving the inverse problem, it is necessary to construct an appropriate iterative algorithm. Moreover, in the process of solving inverse problem, direct problem should be solved for several times.
Simulation software is usually used as the solver of the direct problem. But it is a time consuming process [3]. In this paper, the polynomial model is used as the solver of the direct problem to improve the solving efficiency [4].

Solving inverse problems has a positive effect on engineering practice. For example, for a given structure, some parameters of the structure can be obtained by observing the displacement change under the load case, which is helpful to know whether there is damage in the structure [5]. This makes it easier for people to monitor the health of the structure [6]. The cantilever beam has simple structural style, but it is widely used in mechanical engineering, civil engineering and aerospace engineering. The paper makes parameter inversion of cantilever beam. The significance of this process is to test the feasibility of monitoring the structure health.

2. Derivation of the iterative formula for solving the inverse problem

2.1. General idea of inverse problem solving

To some extent, inverse problem of solid structure can be described as an optimization with constraint conditions.

\[
\begin{align*}
\text{Min} & \quad G(p) \\
\text{s.t.} & \quad K(p, u) = F(f)
\end{align*}
\]

Formula (2) is an objective function. Formula (3) is constraint condition, which is also a description of a physical problem. A detailed introduction is made about the parameters in the formulas above. \(p\) is the inherent parameter of the physical model, which needs solving. \(u\) is the state variable. \(p\) and \(u\) are vectors in this paper, that is to say there are several elements in \(p\) and \(u\) respectively. \(f\) is the external force. \(K\) and \(F\) are the partial differential operators. The objective function \(G(p)\) is a measure that represents the deviation between the new state variable under new input and the real state variable.

The process of inversion can be described in the Figure 1. \(p_0\) represents the real inherent parameter of the structure. \(p\) represents the inherent parameter that is generated by iterative formula in the inversion process except for the first iteration step. \(u_0\) is state variable in the real physical model which can be measured by equipment. \(u\) is state variable that is calculated in the inversion process. When the \(p\) is put into the physical model, \(u\) can be obtained quickly. Then the deviation between \(u\) and \(u_0\) can be expressed in an objective function. This function can be further modified into a function with \(p\) as a variable. Then, the function is operated in a special way to get new \(p\). This is an iterative process. At last, \(p\) is found, which is close or equal to \(p_0\).

![Figure 1. Idea diagram of inverse problem solving.](image)

2.2. Iterative formula based on Gauss-Newton method

The objective function is defined as formula (4). \(y\) is state variable that can be measured. State variable is displacement of beam in this paper. \(y\) is considered to be accurate and is not disturbed by measurement noise. \(u\) is the state variable that is generated by iterative formula. \(S\) is select matrix. The purpose of \(S\) is to choose the proper \(u\) to take part in the calculation.

\[
G(p) = \left(y - Su\right)^T \left(y - Su\right)
\]

Gauss-Newton method is expressed as follow [7].

\[
p_k = p_{k-1} + \Delta p_{k-1}
\]
In formula (5), \( k \) and \( k-1 \) represent the \( k \) iteration step and \( k-1 \) iteration step respectively. The key point in the formula is to get \( \Delta p^{k-1} \). In order to get \( \Delta p^{k-1} \), \( G(p^k) \) is expanded by Taylor expansion at point \( p^{k-1} \), which can be seen in formula (6).

\[
G(p^k) \approx q \equiv G(p^{k-1}) + \frac{\partial G(p^{k-1})}{\partial p} \Delta p^{k-1} + \frac{1}{2} (\Delta p^{k-1})^T \frac{\partial^2 G(p^{k-1})}{\partial p^2} \Delta p^{k-1}
\]  

(6)

\( R \) is defined as a residual vector. \( J \) is defined as a sensitivity matrix.

\[
R = y - Su
\]  

(7)

\[
J = \frac{\partial R}{\partial p} = S \frac{\partial u}{\partial p}
\]  

(8)

Therefore, the first-order partial derivative and second-order partial derivative in formula (6) can be simplified into the following formulas.

\[
\frac{\partial G(p)}{\partial p} = 2J^T R
\]  

(9)

\[
\frac{\partial^2 G(p)}{\partial p^2} = 2 \frac{\partial^2 R^c}{\partial p^2} - R + 2J^T J
\]  

(10)

In order to get the minimum value of objective function, the partial derivative of \( q \) to \( p \) is calculated. And then, make the partial derivative equal to zero.

\[
\frac{\partial q}{\partial \Delta p^{k-1}} = \frac{\partial G(p^{k-1})}{\partial p} + \frac{\partial^2 G(p^{k-1})}{\partial p^2} \Delta p^{k-1} = 0
\]  

(11)

Put formula (9) and formula (10) into formula (11), formula (12) is obtained.

\[
2(J^{k-1})^T R^{k-1} + 2(J^{k-1})^T J^{k-1} \Delta p^{k-1} = 0
\]  

(12)

\( \Delta p^{k-1} \) can be solved easily by formula (12).

\[
\Delta p^{k-1} = -[(J^{k-1})^T J^{k-1}]^{-1} (J^{k-1})^T R^{k-1}
\]  

(13)

Put formula (13) into formula (5), the final iterative formula can be obtained as formula (14).

\[
p^k = p^{k-1} - [(J^{k-1})^T J^{k-1}]^{-1} (J^{k-1})^T R^{k-1}
\]  

(14)

Formula (14) is used in the inversion process in subsequent sections. If the solution meets the formula (15), the solution is considered to be convergent.

\[
\max \left[ \left| \frac{\Delta u_i^k}{u_i^k} \right|, \left| \frac{\Delta p_j^k}{p_j^k} \right| \right] < \epsilon ; \quad i = 1, 2, \ldots, N_u; \quad j = 1, 2, \ldots, N_p
\]  

(15)

In formula (15), \( u_i^k \) is the \( i \) component in \( u \) at the \( k \) iteration step; \( \Delta u_i^k \) is the increment of the \( i \) component in \( u \) at the \( k \) iteration step; \( p_j^k \) is the \( j \) component in \( p \) at the \( k \) iteration step; \( \Delta p_j^k \) is the increment of the \( j \) component in \( p \) at the \( k \) iteration step; \( N_u \) and \( N_p \) are the numbers of components in \( u \) and \( p \), respectively.

3. Structure parameter inversion

3.1. Cantilever beam model

There are some important inherent parameters of cantilever beam; such as Young’s modulus, poisson’s ratio, density, size of beam section and so on. In this paper, the parameter needs to be inverses is Young’s modulus of the beam. A cantilever beam with variable Young’s modulus is selected. The Young’s modulus is written into a function of position coordinates, which can be expressed as
E(x) = 150cos(x × π/3)(GPa) (currently unknown). There is a concentrated load on the free end of beam. The state variable is the deflection (displacement in Y direction) of the beam at different position coordinates. It is necessary to make a hypothesis that the measured displacement is accurate and free from noise. The displacement measurement points are on the upper surface of the beam along the X direction of the beam. The detail information of the beam model can be seen in the Figure 2 below.

Figure 2. Cantilever beam model.

3.2. The flow chart of inversion

The flow chart of the parameter inversion is shown in Figure 3 below. The key parts of the inversion process are: constructing the polynomial model to solve the direct problem, giving a proper initial solution of parameters and calculating parameters by iterative formula correctly. The next subsection makes an introduction about constructing the polynomial model.

Figure 3. Flow chart of inversion.

3.3. Construct the polynomial model

Instead of numerical simulation software, the polynomial model is taken as the direct problem solver. The cantilever beam is divided into 5 parts evenly. The length of each part is 0.2m. Assuming that each part has constant Young’s modulus. 5 measurement points are distributed at each end of these parts. According to the basic theory in mechanics of materials, the deflection of cantilever beam with a concentrated load at its free end is inversely proportional to Young’s modulus [8]. Therefore, the relationship between Young’s modulus E (GPa) and displacement v (mm) is assumed as following
formula. $E_i$ represents Young’s modulus in 5 different parts; where $i = 1/2/\ldots/5$. $v_i$ represents displacement at the end of five parts, where $i = 1/2/\ldots/5$.

\[
\begin{pmatrix}
  v_1 \\
  v_2 \\
  v_3 \\
  v_4 \\
  v_5 \\
\end{pmatrix} =
\begin{bmatrix}
  k_{11} & k_{12} & k_{13} & k_{14} & k_{15} \\
  k_{21} & k_{22} & k_{23} & k_{24} & k_{25} \\
  k_{31} & k_{32} & k_{33} & k_{34} & k_{35} \\
  k_{41} & k_{42} & k_{43} & k_{44} & k_{45} \\
  k_{51} & k_{52} & k_{53} & k_{54} & k_{55} \\
\end{bmatrix}
\begin{pmatrix}
  1 \\
  1 \\
  1 \\
  1 \\
  1 \\
\end{pmatrix}
\]

(16)

Formula (16) is a polynomial model. $k_{ij}$ in the formula needs to be solved. Simulation software, which is based on finite element method, is used to determine the $k_{ij}$ in the formula above. Finite element model is constructed in the software with real geometry parameter. Different displacements at measurement points are obtained with the change of Young’s modulus. And then, $k_{ij}$ is solved, as shown in Table 1.

**Table 1.** $k_{ij}$ in polynomial model.

| $i$ | $j=1$ | $j=2$ | $j=3$ | $j=4$ | $j=5$ |
|-----|-------|-------|-------|-------|-------|
| 1   | 0.07468 | 0 | 0 | 0 | 0 |
| 2   | 0.21871 | 0.05867 | 0 | 0 | 0 |
| 3   | 0.36273 | 0.17070 | 0.04267 | 0 | 0 |
| 4   | 0.50676 | 0.28272 | 0.12269 | 0.02667 | 0 |
| 5   | 0.65079 | 0.39474 | 0.20270 | 0.07468 | 0.01066 |

3.4. The process and result of inversion

Comsol Multiphysics software is taken to get the displacement of the beam at 5 measurement points instead of measuring by equipment. The beam, which is under load, and the displacement of measurement points are shown in the Figure 4 below.

**Figure 4.** Schematic diagram of beam and measurement point displacement.

A set of appropriate initial solution is given as follows: $E_1 = E_2 = E_3 = E_4 = E_5 = 100$ GPa. The inversion process is shown in the Figure 5 below.

**Figure 5.** The process of iteration for the parameter inversion.
The inversed Young’s modulus of cantilever beam is shown in Table 2.

**Table 2.** The result of parameter inversion.

|   | \( E_1 \) (GPa) | \( E_2 \) (GPa) | \( E_3 \) (GPa) | \( E_4 \) (GPa) | \( E_5 \) (GPa) |
|---|----------------|----------------|----------------|----------------|----------------|
|   | 148.99775      | 142.925        | 129.994        | 110.982        | 88.073         |

The stability of the inversion method should be tested with different cases of initial solution. Three cases of initial solution (unit is GPa) are given as follow:

(a) \( E_1=1, E_2=2, E_3=3, E_4=4, E_5=5 \);
(b) \( E_1=290, E_2=280, E_3=270, E_4=260, E_5=250 \);
(c) \( E_1=1, E_2=20, E_3=70, E_4=150, E_5=250 \).

Three different cases are all convergent at last. It indicates that the method has good stability. The process of iteration can be seen in the Figure 6 below.

![Figure 6](image-url)

*Figure 6.* The process of iteration for different initial solutions.

The convergent results with three different cases of initial solution are consistent with those in Table 2. Inversion result and the real parameter of cantilever beam are plotted in one figure, as shown in Figure 7 below.

![Figure 7](image-url)

*Figure 7.* Inversed parameter (beam is divided into 5 parts) and the real parameter.

It can be seen that the result is rough because the beam is divided only in 5 parts. However, the inversion method is successful. If the cantilever beam is divided into more parts, better result will be obtained. Therefore, parameter inversion is done again with the beam which is evenly divided into 20 parts along the X direction. 20 measurement points are arranged to get the displacement in the Y direction. And 20 parameters need to be inversed. That is to say, \( E_1, E_2, \ldots, E_{19}, E_{20} \) should be solved. The process of inversion is the same as the that of beam divided into 5 parts. A polynomial model is
constructed firstly, and then iteration is done with a given initial solution. The case of initial solution is that $E_1 = E_2 = \ldots = E_{19} = E_{20} = 100$ GPa. The result of parameter inversion is shown in the Figure 8 below, which is better than that in Figure 7. It can be expected that with the increase of the number of divided parts, the inversion result will be closer to the true value of parameters.

**Figure 8.** Inversed parameter (beam is divided into 20 parts) and the real parameter.

4. Conclusions
The paper makes parameter inversion on a kind of cantilever beam with variable Young’s modulus. An iterative formula is deduced particularly. A polynomial model is constructed as the direct problem solver. The paper uses iterative formula and polynomial model to do the parameter inversion. The result shows that the method which is proposed in the paper is effective and feasible. The method in this paper can also be applied in other solid structure for parameter inversion, which has a positive effect on structural parameter identification and structure health monitoring.

References
[1] Vatul Yan A O 2010 The theory of inverse problems in the linear mechanics of a deformable solid *Journal of Applied Mathematics & Mechanics* 74(6) 648-653
[2] Caseiro J F and Campos A A 2011 An evolutionary-inspired optimisation algorithm suitable for solid mechanics engineering inverse problems *International Journal of Mechatronics & Manufacturing Systems* 4(5) 415
[3] Goksel O Eskandari H and Salcudean S E 2013 Mesh adaptation for improving elasticity reconstruction using the fem inverse problem *IEEE Transactions on Medical Imaging* 32(2) 408-418
[4] Muller J, Shoemaker C A and Piche R 2013 So-mi: a surrogate model algorithm for computationally expensive nonlinear mixed-integer black-box global optimization problems *Computers & Operations Research* 40(5) 1383-1400
[5] Bureerat S and Pholdee N 2018 Inverse problem based differential evolution for efficient structural health monitoring of trusses *Applied Soft Computing* 66 21
[6] Roux P, Gueguen P, Baillet L and Hamze A 2014 Structural-change localization and monitoring through a perturbation-based inverse problem *Journal of the Acoustical Society of America* 136(5) 86-97
[7] Burke J V and Ferris M C 1995 A gauss-newton method for convex composite optimization *Mathematical Programming* 71(2) 179-194
[8] Sun X F, Fang X S and Guan L T 2009 *Mechanics of Materials (I)* (Beijing: Higher Education Press) p 372