MULTIPLICATIVITY IN MANDELL’S INVERSE $K$-THEORY

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ABSTRACT. We show that Mandell’s inverse $K$-theory functor from $\Gamma$-categories to permutative categories preserves multiplicative structure. This is a first step towards an equivariant generalization that would be inverse to the construction of Bohmann and Osorno.

1. Introduction

Segal showed that symmetric monoidal categories give rise to spectra in $[7]$, and May gave a simpler construction in the case of permutative categories in $[6]$. Both constructions deserve to be called algebraic $K$-theory constructions. Mandell and the author showed in $[2]$ that a modification of May’s construction actually preserves multiplicative structure, and Bohmann and Osorno $[1]$ used this multiplicativity to construct an equivariant version, making crucial use of the work of Guillou and May $[4]$ characterizing equivariant spectra as presheaves of spectra over a spectral version of the Burnside category.

All the spectra arising from the Segal-May construction are connective, and Thomason showed in $[9]$ that all connective spectra arise in this fashion. His construction was quite obscure, however, and Mandell gave a much more comprehensible one in $[5]$. The aim of this paper is to show that the main step in Mandell’s construction preserves multiplicative structure, and therefore can serve as a first step towards giving an inverse construction to the equivariant generalization of Bohmann and Osorno. The conjecture is that all connective equivariant spectra, that is, those all of whose fixed-point spectra are connective, arise from the Bohmann-Osorno construction.

The first part of Mandell’s construction is simply to use Thomason’s equivalence between spaces and categories from $[8]$ to construct a $\Gamma$-category from a $\Gamma$-space, by using levelwise double subdivision and categorification. The substantial portion of Mandell’s construction is the passage from $\Gamma$-categories to permutative categories, and it is this part of his construction that we show preserves multiplicative structure. The proof relies on a factorization of Mandell’s construction as a composite of three
functors, all of which preserve multiplicative structure. The first starts with a Γ-category and produces a multifunctor with source a multicategory derived from the natural numbers, and with target a multicategory of categories. The second step is a wreath product construction that starts from such a multifunctor and produces a single multicategory. The third step is the left adjoint to the forgetful functor from permutative categories to multicategories; it was somewhat surprising to the author to find that this also preserves multiplicative structure, but the proof is fairly simple once one thinks to look for it.

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2. Outline and statement of results

We begin with a Γ-category $X$, and wish to end with a permutative category using a construction that preserves multiplicative structure, as captured by multicategory structure. To set terminology and notation, let $\Gamma^{\text{op}}$ be the category with objects the based sets $\underline{n} = \{0, 1, \ldots, n\}$ with basepoint 0 for $n \geq 0$, and morphisms the based functions. Let $\text{Cat}^*$ be the category of small based categories, that is, small categories with a select base object. Then a Γ-category is a functor $X : \Gamma^{\text{op}} \to \text{Cat}^*$ for which $X(0) = \ast$, a category with one object and one morphism.

Next, let $N$ temporarily denote the permutative category whose objects are the unbased sets $[n] = \{1, 2, \ldots, n\}$ for $n \geq 0$, morphisms all functions, and monoidal product given by $[m] \oplus [n] := [m+n]$, where we use the canonical bijection $[m] \amalg [n] \cong [m+n]$ to make this into a bifunctor. Being a permutative category, so is its opposite category, and our permanent use of $N$ will be as the underlying multicategory of the permutative category given by this opposite category. Explicitly, then, an $r$-morphism in $N$ from $([n_1], \ldots, [n_r])$ to $[m]$ consists of an ordinary function $[m] \to [n_1] \amalg \cdots \amalg [n_r]$. It is important to note that $N$ is actually a based multicategory: there is a canonical multifunctor from the terminal multicategory $\ast$ with one object and one morphism of each arity to our multicategory $N$, picking out the single object $[0] = \emptyset$ and making it into a commutative monoid in the multicategorical sense: this just means that the basepoint-defining multifunctor $\ast \to N$ lands on the object $[0]$. The terminology about commutative monoids arises from the fact that $\ast$ parametrizes commutative monoids in any symmetric monoidal category.

We now define a multicategory structure on $\text{Cat}^*$, or more precisely its opposite category. Note that a based category is the same thing as a category with a specified functor $\ast \to C$ from any terminal category with one object and one morphism. Since $\text{Cat}^*$ is a symmetric monoidal category using its categorical product, which is the cartesian product of categories, so is its opposite category. We will use $\text{Cat}^{\ast, \text{op}}$ to
denote “the” underlying multicategory of this opposite category, where by “the” underlying multicategory we mean any choice of underlying multicategory: all of them are canonically isomorphic. Explicitly, an $r$-morphism in $\text{Cat}_*^{op}$ from $(D_1, \ldots, D_r)$ to $C$ consists of a based functor

$$C \to D_1 \times \cdots \times D_r.$$ 

We note that this multicategory is itself based, with base object any one-point, one-morphism category.

Since both $\mathbb{N}$ and $\text{Cat}_*^{op}$ are based multicategories, we can look at

$$\text{Mult}_*(\mathbb{N}, \text{Cat}_*^{op}),$$

the collection of based multifunctors from $\mathbb{N}$ to $\text{Cat}_*^{op}$. The first step in our construction takes a $\Gamma$-category $X$ and produces a multifunctor $AX \in \text{Mult}_*(\mathbb{N}, \text{Cat}_*^{op})$; we use the notation $A$ to follow Mandell’s notation to some extent, and the construction is given in Section 4.

**Theorem 1.** There are multicategory structures on $\Gamma\text{-Cat}_*$ and $\text{Mult}_*(\mathbb{N}, \text{Cat}_*^{op})$ for which the construction

$$A : \Gamma\text{-Cat}_* \to \text{Mult}_*(\mathbb{N}, \text{Cat}_*^{op})$$

is a multifunctor.

In fact, the multicategory structure on $\Gamma\text{-Cat}_*$ arises from the symmetric monoidal structure given by the Day convolution induced by the smash product of based categories and the smash product $\Gamma^{op} \times \Gamma^{op} \to \Gamma^{op}$. However, the multicategory structure on $\text{Mult}_*(\mathbb{N}, \text{Cat}_*^{op})$ is not the one given by the enrichment of $\text{Mult}_*$ over itself; we explain in Section 4.

The next step is a wreath product construction that produces a based multicategory from a (based) multifunctor into $\text{Cat}_*^{op}$. The construction will be given in Section 5. We will show that

**Theorem 2.** The wreath product construction gives a multifunctor

$$\text{Wr} : \text{Mult}_*(\mathbb{N}, \text{Cat}_*^{op}) \to \text{Mult}_*.$$ 

Here the multicategory structure on the target $\text{Mult}_*$ is the one underlying the symmetric monoidal structure from $\Gamma$.

At this point in the construction we forget about the based structure of the objects of $\text{Mult}_*$, that is, we apply the forgetful functor $\text{Mult}_* \to \text{Mult}$. This is a multifunctor since based multilinear maps of based multicategories are in particular multilinear maps of their underlying unbased multicategories. Now the third and final step in our factorization of Mandell’s construction is the left adjoint to the forgetful functor from permutative categories to multicategories, which we denote by $F$ and
describe in Section 6. Let \( \text{Perm} \) denote the multicategory of permutative categories. Our third multiplicativity theorem is then

**Theorem 3.** The functor \( F : \text{Mult} \to \text{Perm} \) extends to a multifunctor; i.e., it preserves multiplicative structure.

The following observation is our factorization of Mandell’s construction.

**Observation 4.** Given a \( \Gamma \)-category \( X \), Mandell’s construction \( PX \) in [5] of a permutative category coincides with the construction \( F(\mathbb{N} \wr AX) \).

Since all three steps in this construction preserve multiplicative structure, we may conclude

**Corollary 5.** Mandell’s construction \( P \) defines a multifunctor from \( \Gamma\text{-Cat} \) to \( \text{Perm} \) inverse to the \( K \)-theory construction.

The fact that \( P \) is inverse to the \( K \)-theory construction is one of Mandell’s results from [5].

3. MUltIFUNCTORS FROM SYMMETRIC MONOIDAL CATEGORIES

This section is devoted to a lemma that we will exploit in the proofs of the above theorems. We refer to the unit \( e \) in a symmetric monoidal category as strong if the natural map \( e \otimes x \to x \) is an isomorphism.

**Lemma 6.** Let \((C, \otimes, e)\) be a symmetric monoidal category with strong unit \( e \), and let \( M \) be a multicategory. Let \( UC \) be “the” underlying multicategory of \( C \) (there are many choices, but they’re all canonically isomorphic.) Then a multifunctor \( F : UC \to M \) determines and is determined by:

1. a functor on underlying categories,
2. a 2-morphism \( \lambda : (Fx, Fy) \to F(x \otimes y) \) for each pair of objects \( x \) and \( y \), and
3. a 0-morphism \( \eta : () \to Fe \),

all subject to the four diagrams listed below.
The diagrams that we require $\lambda$ and $\eta$ to satisfy are as follows. First, we require $\lambda$ to be coherently associative, in the sense that

\[
(F(x \otimes y), Fz) \xrightarrow{\lambda} F((x \otimes y) \otimes z)
\]

commutes, where $\alpha$ is the associativity isomorphism in $C$. We also require it to be consistent with the interchange isomorphism, in the sense that

\[
(Fx, Fy) \xrightarrow{\tau} (Fy, Fx)
\]

\[
\lambda \downarrow \downarrow \lambda
\]

\[
Fx \otimes Fy \xrightarrow{F\tau} Fy \otimes Fx
\]

commutes. In addition, we require a naturality condition for $\lambda$: given morphisms $f_1 : x_1 \to y_1$ and $f_2 : x_2 \to y_2$ in $C$, we require the following diagram to commute:

\[
(Fx_1, Fx_2) \xrightarrow{(Ff_1, Ff_2)} (Fy_1, Fy_2)
\]

\[
\lambda \downarrow \downarrow \lambda
\]

\[
Fx_1 \otimes x_2 \xrightarrow{F(f_1 \otimes f_2)} F(y_1 \otimes y_2).
\]

Finally, we require $\lambda$ to be coherent with $\eta$ in the sense that the following diagram commutes:

\[
Fx \xrightarrow{(\eta, 1)} (Fe, Fx) \xrightarrow{\lambda} F(e \otimes x)
\]

\[
\approx \downarrow \downarrow \approx \quad Fx.
\]

(The corresponding diagram for $x \otimes e$ now follows from diagram 2.)

**Proof.** Suppose first given a multifunctor $F : UC \to M$. Then certainly $F$ descends to a functor on the underlying categories. We also have a canonical and natural 2-morphism

\[
(x, y) \to x \otimes y
\]

in $UC$ given by the identity morphism on $x \otimes y$, and applying $F$ gives us our 2-morphism

\[
\lambda : (Fx, Fy) \to F(x \otimes y)
\]
in $M$.

Since 0-morphisms in $U\mathcal{C}$ are given by morphisms from $e_{\mathcal{C}}$, we have a canonical 0-morphism $() \to e_{\mathcal{C}}$ given by the identity on $e_{\mathcal{C}}$, and applying $F$ gives us

$$\eta : () \to F(e_{\mathcal{C}}).$$

The required diagrams for $\lambda$ and $\eta$ now follow from the properties of a symmetric monoidal category.

Now suppose given just a functor $F : U\mathcal{C} \to M$ on underlying categories, together with a 2-morphism $\lambda$ and 0-morphism $\eta$ subject to the given diagrams. We wish to extend $F$ to a multifunctor.

Let $(x_1, \ldots, x_r)$ be an $r$-tuple of objects in $\mathcal{C}$. We have by induction an $r$-morphism in $M$

$$\lambda_r : (Fx_1, \ldots, Fx_r) \to F(x_1 \otimes \cdots \otimes x_r).$$

Explicitly, we take $\lambda_0 = \eta$, and

$$\lambda_{r+1} := \lambda \circ (\lambda_r, \text{id}).$$

Note in particular that $\lambda_1 = \text{id}$, by diagram [1]. Now given an arbitrary $r$-morphism $f : (x_1, \ldots, x_r) \to y$ in $U\mathcal{C}$, there is a specified 1-morphism $f : x_1 \otimes \cdots \otimes x_r \to y$ in $\mathcal{C}$, and we agree to apply the ordinary functor $F$ to this morphism and define the $r$-morphism $Ff$ in $M$ by the composite

$$(Fx_1, \ldots, Fx_r) \xrightarrow{\lambda_r} F(x_1 \otimes \cdots \otimes x_r) \xrightarrow{Ff} Fy.$$

Note that if $f$ happens to be a 0-morphism, we have an empty tensor product which we agree to be $e_{\mathcal{C}}$, and the definition still makes sense. We claim that these assignments make $F$ into a multifunctor.

We must first show that $F$ preserves the composition in $U\mathcal{C}$. So suppose that for $1 \leq i \leq s$, we have a $t_i$-morphism $g_i : \langle w_{ij} \rangle_{j=1}^{t_i} \to x_i$, and an $s$-morphism $f : \langle x_i \rangle_{i=1}^{s} \to y$. We need to show that

$$F(f \circ \langle g_i \rangle_{i=1}^{s}) = Ff \circ \langle F(g_i) \rangle_{i=1}^{s}.$$

Let $t = \sum_{i=1}^{s} t_i$; both sides of the proposed equality are $t$-morphisms. We have the following diagram, which we claim commutes, where $\odot$ denotes concatenation of lists:

$$
\begin{array}{c}
\langle (Fw_{ij})_{j=1}^{t_i} \rangle_{i=1}^{s} \\
\odot \\
\langle (\langle Fw_{ij} \rangle_{j=1}^{t_i} \rangle_{i=1}^{s} \rangle_{i=1}^{s} \rangle_{i=1}^{s} \\
\odot \\
\langle (Fg_i) \rangle_{i=1}^{s} \\
\xrightarrow{\lambda_s} \\
\langle F(\odot_{j=1}^{t_i} w_{ij}) \rangle_{i=1}^{s} \\
\xrightarrow{\lambda_s} \\
F(\odot_{i=1}^{s} \otimes_{j=1}^{t_i} w_{ij}) \\
\xrightarrow{F(\odot_{i=1}^{s} g_i)} \\
\langle Fx_i \rangle_{i=1}^{s} \\
\xrightarrow{\lambda_s} \\
F(\odot_{i=1}^{s} x_i) \\
\xrightarrow{Ff} \\
Fy.
\end{array}
$$
Tracing counterclockwise gives the right hand side, and clockwise gives the left hand side. The top square commutes up to an explicit associativity isomorphism by induction using the associativity of $\lambda$, or the unit diagram in case any of the $g_i$ are 0-morphisms, and the bottom square by induction using the naturality of $\lambda$. The extended functor therefore preserves composition in the sense of multicategories.

We must also show that $F$ preserves the permutation actions on sets of morphisms. Since permutations are generated by transpositions, it suffices to show preservation of transpositions. Suppose given a 2-morphism $f : (x, y) \to z$ in $\mathcal{U}C$, which amounts to a 1-morphism $f : x \otimes y \to z$, abusively denoted by the same letter. Then $f \tau$ is given by the composite

$$y \otimes x \xrightarrow{\tau} x \otimes y \xrightarrow{f} z.$$

Now the commutative diagram

$$
\begin{array}{ccc}
(Fy, Fx) & \xrightarrow{\tau} & (Fx, Fy) \\
\downarrow{\lambda} & & \downarrow{\lambda} \\
F(y \otimes x) & \xrightarrow{F\tau} & F(x \otimes y) \xrightarrow{Ff} z
\end{array}
$$

shows that $F(f \tau) = (Ff)\tau$. We have therefore shown that $F$ extends to a multifunctor. It is straightforward to check that the two constructions are inverse to each other. \qed

4. FROM $\Gamma$-CATEGORIES TO MULTIFUNCTORS

This section is devoted to proving Theorem [11] that the construction $A : \Gamma\text{-Cat}_* \to \text{Mult}_* (\mathbb{N}, \text{Cat}_{\text{op}})$, given as follows, extends to a multifunctor.

Here is the construction of $AX$ given a $\Gamma$-category $X$. We need to first assign a based category $AX[n]$ to any object $[n] \in \mathbb{N}$, and we just use the category $X(n)$. Next, given a morphism $f : ([n_1], \ldots, [n_r]) \to [m]$ in $\mathbb{N}$, we need an associated morphism $(AX[n_1], \ldots, AX[n_r]) \to AX[m]$ in $\text{Cat}_*^{\text{op}}$. But the morphism $f$ consists just of a function

$$f : [m] \to [n_1] \amalg \cdots \amalg [n_r],$$

to which we can attach a disjoint basepoint on each side, considered as new elements 0, and obtain a map of based sets

$$f_+ : m \to n_1 \vee \cdots \vee n_r.$$

For any index $i$ with $1 \leq i \leq r$, we can then collapse all the wedge summands except $n_i$ to the basepoint, producing a map $f_i : m \to n_i$ in $\Gamma^{\text{op}}$. Since $X$ is a $\Gamma$-category,
this induces a functor \( X(f_i) : X(m) \to X(n_i) \) which we use as the \( i \)'th coordinate map to the product of the \( X(n_i) \)'s, giving us a functor

\[
AX(f) : X(m) \to X(n_1) \times \cdots \times X(n_r);
\]

this functor is based since all the components are induced by maps in a \( \Gamma \)-category, which must be based functors. This concludes the description of the functor \( A \); we now turn to showing it extends to a multifunctor structure, for which we need multicategory structures on both source and target.

For the multicategory structure on \( \Gamma\text{-Cat}_* \), we exploit the Day convolution construction. There is a permutative category structure on \( \Gamma^{\text{op}} \) in which we use lexicographic order to identify \( m \land n \) with \( m \cdot n \). Now given two \( \Gamma \)-categories \( X \) and \( Y \), noting that for all \( n \) both \( X(n) \) and \( Y(n) \) are based categories, we form the smash product \( X \land Y \) as the left Kan extension

\[
\Gamma^{\text{op}} \times \Gamma^{\text{op}} \xrightarrow{X \times Y} \text{Cat}_* \times \text{Cat}_* \xrightarrow{\land} \text{Cat}_*.
\]

This gives us a symmetric monoidal structure on \( \Gamma\text{-Cat}_* \); the unit object is the \( \Gamma \)-category \( B \) given by

\[
B(n) = \begin{cases} S^0 & \text{if } n = 1, \\ * & \text{if } n \neq 1. \end{cases}
\]

Here \( S^0 \) is a two-object discrete category with one object the base; it is the unit for the smash product of based categories. We therefore have an underlying multicategory structure on \( \Gamma\text{-Cat}_* \).

To give the multicategory structure on \( \text{Mult}_* (\mathbb{N}, \text{Cat}_*^{\text{op}}) \), we need extra structure on both \( \mathbb{N} \) and \( \text{Cat}_*^{\text{op}} \), which we call ring structure.

**Definition 7.** Let \( M \) be a based multicategory. A ring structure on \( M \) consists of a based bilinear map of multicategories

\[
\lambda : (M, M) \to M
\]

together with a unit multifunctor \( \eta : u \to M \), where \( u \) is the unit for the smash product of based multicategories (see [3] for further details; such a multifunctor amounts simply to a choice of object of \( M \).) These are then subject to the same associativity and symmetry conditions one imposes on a symmetric monoidal category.

The ring structures we have in mind are given by the following two lemmas.

**Lemma 8.** The multicategory \( \mathbb{N} \) has a ring structure given by

\[
\lambda([m], [n]) := [m \cdot n],
\]
using lexicographic order to identify \([m \cdot n] \text{ with } [m] \times [n]\) in order to extend to morphisms. The unit object is \([1] = \{1\}\).

The proof is straightforward and left to the reader: bilinearity of \(\lambda\) is an aspect of the distributive law.

The next lemma gives the ring structure on \(\text{Cat}^{\text{op}}\).

**Lemma 9.** The multicategory \(\text{Cat}^{\text{op}}\) has a ring structure given by

\[
\lambda(C, D) := C \wedge D.
\]

Given an \(r\)-morphism \(f : (C_1, \ldots, C_r) \to C\), i.e., a functor \(f : C \to C_1 \times \cdots \times C_r\), the induced \(r\)-morphism \(\lambda(f, D) : (C_1 \wedge D, \ldots, C_r \wedge D) \to C \wedge D\) is given by the functor

\[
\begin{array}{ccc}
C \wedge D & \longrightarrow & (C_1 \wedge D) \times (C_2 \wedge D) \times \cdots \times (C_r \wedge D) \\
\lambda(f, D) & & \\
\lambda(C, g) & & \lambda(f, D)
\end{array}
\]

whose \(i\)th coordinate map is given by \(f_i \wedge D : C \wedge D \to C_i \wedge D\), and similarly in the variable \(D\). The unit category is the unit \(S^0\) for the smash product of based categories: it is discrete with two objects, one of which is the base object.

**Proof.** The main issue is to show that the multifunctorialities in \(C\) and \(D\) interact correctly. For this, suppose given an \(r\)-map \(f : C \to C_1 \times \cdots \times C_r\) and an \(s\)-map \(g : D \to D_1 \times \cdots \times D_s\). Then we have

\[
\begin{array}{ccc}
\prod_{i=1}^r (C_i \wedge D) & \longrightarrow & \prod_{i=1}^r (C_i \wedge D_j) \\
\prod_{j=1}^s (C \wedge D_j) & \longrightarrow & \prod_{j=1}^s (C_i \wedge D_j)
\end{array}
\]

which commutes since the projection to the factor \(C_i \wedge D_j\) is always just \(f_i \wedge g_j\). This establishes the coherence necessary to have a bilinear map of multicategories. The other verifications are straightforward. \(\square\)

Given a ring structure \(\lambda : (M, M) \to M\), we can iterate and obtain a canonical \(r\)-linear map \(\lambda_r : (M, \ldots, M) \to M\), where there are \(r\) copies of \(M\) in the source. We think of the unit map as being \(\lambda_0\). We can now define the multicategory structure on \(\text{Mult}_*(\mathbb{N}, \text{Cat}^{\text{op}}_*)\) of interest to us; as mentioned above, this is not the multicategory structure given by the enrichment of \(\text{Mult}_*\) over itself given by its symmetric monoidal structure, which makes no use of the ring structures. Instead, this is a multicategory structure that twists by means of the ring structures.
Definition 10. Let $F_1, \ldots, F_r$ and $G$ be elements of $\text{Mult}_*(\mathbb{N}, \text{Cat}_*^{\text{op}})$. We define an $r$-map $(F_1, \ldots, F_r) \to G$ to be a based $r$-linear transformation $\phi$ as in the following diagram:

$$
\begin{array}{c}
(N, \ldots, N) \xrightarrow{(F_1, \ldots, F_r)} (\text{Cat}_*^{\text{op}}, \ldots, \text{Cat}_*^{\text{op}}) \\
\xrightarrow{\lambda_r} \xrightarrow{\phi} \xrightarrow{\lambda_r} \text{Cat}_*^{\text{op}}.
\end{array}
$$

Unpacking this a bit, for each $r$-tuple $([m_1], \ldots, [m_r])$ of objects of $\mathbb{N}$, we require a based functor

$$
\phi_{m_1, \ldots, m_r} : F_1m_1 \land \cdots \land F_rm_r \to G(m_1 \cdots m_r)
$$

that is multifunctorial in each variable $m_i$ separately, basepoint preserving, and based bilinear (in the sense of [3], Definition 2.8) in each pair of variables. To illustrate in just the case of a bilinear transformation $(F_1, F_2) \to G$, we require based functors

$$
\phi_{m,n} : F_1m \land F_2n \to G(mn)
$$

for all $m, n \in \mathbb{N}$, and given an $r$-morphism $f : (m_1, \ldots, m_r) \to m'$ in $\mathbb{N}$, that is, a function

$$
f : [m'] \to [m_1] \amalg \cdots \amalg [m_r],
$$

we require the following diagram to commute:

$$
\begin{array}{c}
F_1m' \land F_2n \xrightarrow{\phi_{m',n}} G(m'n) \\
\xrightarrow{\lambda_2(F_1f,F_2n)} \xrightarrow{\prod_{i=1}^r(F_1m_i \land F_2n)} \xrightarrow{\prod_{i=1}^r\phi_{m_i,n_i}} G(m'in),
\end{array}
$$

along with an analogous diagram in the variable $n$. Given in addition an $s$-morphism

$$
g : [n'] \to [n_1] \amalg \cdots \amalg [n_s],
$$

we also require the bilinearity diagram

$$
\begin{array}{c}
\prod_{i=1}^r(F_1m_i \land F_2n') \xrightarrow{\lambda_2(F_1f,F_2n')} \xrightarrow{\prod_{i=1}^r\phi_{m_i,n'}} \xrightarrow{\prod_{i=1}^rG(m'in')} \xrightarrow{\prod_{j=1}^sG(m'n_j)} \xrightarrow{\prod_{j=1}^sG(f \cdot n_j)} \xrightarrow{=\sim} \prod_{i=1}^r\prod_{j=1}^sG(m_in_j)
\end{array}
$$
to commute. These diagrams are then modified as appropriate for larger numbers of variables, but with no significant differences.

We can now begin the proof of Theorem 1. Since $\Gamma\text{-Cat}_s$ forms a symmetric monoidal category, Lemma 6 tells us that we need to check that $A$ gives us a functor, specify a natural 2-morphism $\lambda : (AX, AY) \to A(X \wedge Y)$ in $\text{Mult}_*(\mathbb{N}, \text{Cat}_s^{\text{op}})$ for $\Gamma$-categories $X$ and $Y$, specify a 0-morphism $\eta : () \to A(B)$, and show that these choices satisfy the four diagrams listed after Lemma 6.

For functoriality, given a morphism $q : X \to Y$ of $\Gamma$-categories, we get a 1-morphism $Aq : AX \to AY$ of multifunctors in $\text{Mult}_*(\mathbb{N}, \text{Cat}_s^{\text{op}})$ as follows. For each $m \in \mathbb{N}$, we have $q[m] : X(m) \to Y(m)$ as the component of the natural map $q$ at the object $[m]$ of $\Gamma^{\text{op}}$; this gives us the required morphisms $AX[m] = X(m) \to Y(m) = AY[m]$. To see that these maps give us a multinatural transformation, suppose given an $r$-morphism $f : [m'] \to [m_1] \amalg \cdots \amalg [m_r]$ in $\mathbb{N}$. Then we get the induced maps $f_i : m' \to m_i$ for each $i$ with $1 \leq i \leq r$, and naturality of $q$ now tells us that

\[
\begin{array}{ccc}
X(m') & \xrightarrow{q(m')} & Y(m') \\
\downarrow_{X(f_i)} & & \downarrow_{Y(f_i)} \\
X(m_i) & \xrightarrow{q(m_i)} & Y(m_i)
\end{array}
\]

commutes. Since each $X(f_i)$ and $Y(f_i)$ are the coordinate maps for the induced map in $\text{Cat}_s^{\text{op}}$, we find that

\[
\begin{array}{ccc}
X(m') & \xrightarrow{q(m')} & Y(m') \\
\downarrow_{AX(f)} & & \downarrow_{AY(f)} \\
\prod_{i=1}^r X(m_i) & \xrightarrow{\prod_{i=1}^r q(m_i)} & \prod_{i=1}^r Y(m_i)
\end{array}
\]

commutes, showing that $Aq$ is in fact multinatural. It is easy to see that composition of 1-morphisms is preserved.

For the construction of $\lambda : (AX, AY) \to A(X \wedge Y)$, we use the natural transformations giving $X \wedge Y$ as a left Kan extension: just pasting on the functor $\mathbb{N} \to \Gamma^{\text{op}}$ that attaches the disjoint basepoint 0, we get the necessary transformations from the pasting diagram

\[
\begin{array}{ccc}
\mathbb{N} \times \mathbb{N} & \xrightarrow{\chi_2} & \Gamma^{\text{op}} \times \Gamma^{\text{op}} \\
\downarrow_{\chi_2} & & \downarrow_{\wedge} \\
\mathbb{N} & \xrightarrow{\wedge} & \text{Cat}_s^{\text{op}} \times \text{Cat}_s^{\text{op}}
\end{array}
\]

\[
\begin{array}{ccc}
\chi_2 & \wedge & \chi_2 \\
\downarrow & & \downarrow \\
\chi_2 & \wedge & \chi_2 \\
\downarrow & & \downarrow \\
\chi_2 & \wedge & \chi_2
\end{array}
\]
The first three coherence diagrams now follow from the universal property of the left Kan extension defining \( X \land Y \).

To define the 0-morphism \( \eta : () \to A(B) \), we note that the smash product of an empty list of based categories must be the unit \( S^0 \) for the smash product, and the product of an empty list of integers is 1. We therefore just need to specify a map of based categories from \( S^0 \) to \( B(1) \), but since \( B(1) = S^0 \), we can just use the identity functor. The fourth coherence diagram now follows. This concludes the proof of Theorem 1.

5. The Multicategorical Wreath Product

This section is devoted to proving Theorem 2, which states that the multicategorical wreath product provides a multifunctor from \( \mathbf{Mult}_*(\mathbb{N}, \mathbf{Cat}_*)^\mathbf{op} \) to \( \mathbf{Mult}_* \). We begin with the description of the construction.

Suppose given a based multicategory \( M \) and a based multifunctor \( F : M \to \mathbf{Cat}_*^\mathbf{op} \). We define a based multicategory \( M \wr F \) as follows. The objects of \( M \wr F \) are given as

\[
\text{Ob}(M \wr F) := \bigsqcup_{a \in M} \text{Ob}(Fa),
\]

with the base object given by the unique object of the terminal category to which the base object of \( M \) gets mapped. Given a source string \((x_1, \ldots, x_r)\) with \( x_i \in Fa_i \) and a target object \( y \in Fb \), a morphism \((x_1, \ldots, x_r) \to y\) in \( M \wr F \) consists of

1. an \( r \)-morphism \( f : (a_1, \ldots, a_r) \to b \) in \( M \), which induces an \( r \)-morphism \( Ff : (Fa_1, \ldots, Fa_r) \to Fb \) in \( \mathbf{Cat}_*^\mathbf{op} \), in other words, a functor \( Ff : Fb \to Fa_1 \times \cdots \times Fa_r \),

2. an \( r \)-tuple of morphisms \( \psi_i \in Fa_i \) assembling to

\[
\prod_{i=1}^r \psi_i : (x_1, \ldots, x_r) \to (Ff)(y)
\]

as a morphism in \( \prod_{i=1}^r Fa_i \).

Composition is now straightforward to construct. This completes the description of the multicategorical wreath product we wish to use (there are other variants.)

Now given an \( r \)-morphism \((F_1, \ldots, F_r) \to G\) in \( \mathbf{Mult}_*(\mathbb{N}, \mathbf{Cat}_*)^\mathbf{op} \), we need to produce an \( r \)-morphism \((\mathbb{N} \wr F_1, \ldots, \mathbb{N} \wr F_r) \to \mathbb{N} \wr G\) in \( \mathbf{Mult}_* \). We start with just the case \( r = 2 \), so suppose given a 2-morphism \( \phi : (F_1, F_2) \to G \), and we wish to produce a based bilinear map of based multicategories \( \mathbb{N} \wr \phi : (\mathbb{N} \wr F_1, \mathbb{N} \wr F_2) \to \mathbb{N} \wr G \).

On objects, given a pair of objects \((x, m)\) and \((y, n)\) of \( \mathbb{N} \wr F_1 \) and \( \mathbb{N} \wr F_2 \) respectively, so \( x \in F_1[m] \) and \( y \in F_2[n] \), the 2-morphism \( \phi \) provides us with an object \( \phi_{m,n}(x, y) \in \)
$G[m \cdot n]$, so we can define the map on objects by

$$(\mathbb{N} \cdot \phi)((x, m), (y, n)) := (\phi_{m, n}(x, y), m \cdot n) \in \mathbb{N} \cdot G.$$ 

We need to see that this assignment can be made based multifunctorial in each variable, and based bilinear.

Suppose we have an $r$-morphism $((x_1, m_1), \ldots, (x_r, m_r)) \rightarrow (x', m')$ in $\mathbb{N} \cdot F_1$, so this unpacks as an $r$-morphism

$$f : [m'] \rightarrow [m_1] \prod R \cdots R [m_r]$$

in $\mathbb{N}$, which induces

$$F_1f : F_1m' \rightarrow F_1m_1 \times \cdots \times F_1m_r,$$

an $r$-morphism in $\mathbf{Cat}_\mathbf{op}$; we then also require an $r$-tuple of morphisms $\langle \psi_i \rangle_{i=1}^r$ giving a morphism in $\prod_{i=1}^r F_1m_i$ from $(x_1, \ldots, x_r)$ to $(F_1f)(x')$. We wish to construct from these data an $r$-morphism $\langle (\phi_{m_i, n}(x_i, y), m_i \cdot n) \rangle \rightarrow (\phi_{m', n}(x', y), m' \cdot n)$ in $\mathbb{N} \cdot G$.

We start with the $r$-morphism in $\mathbb{N}$ given by

$$f \times [n] : [m' \cdot n] \rightarrow [m_1 \cdot n] \prod R \cdots R [m_r \cdot n]$$

obtained from the lexicographic identification $[m_i] \times [n] \cong [m_i \cdot n]$. This induces the functor $G(f \times [n]) : G(m' \cdot n) \rightarrow \prod_{i=1}^r G(m_i n)$, and we also desire an $r$-tuple of morphisms from $\langle \phi_{m_i, n}(x_i, y) \rangle$ to $G(f \times [n])(\phi_{m', n}(x', y))$ in $\prod_{i=1}^r G(m_i, n)$. We know that the square

$$\begin{array}{ccc}
F_1m' \times F_2n & \xrightarrow{\phi_{m', n}} & G(m'n) \\
\downarrow \lambda_2(F_1f, F_2n) & & \downarrow G(f \times [n]) \\
\prod_{i=1}^r (F_1m_i \times F_2n) & \xrightarrow{\prod_{i=1}^r \phi_{m_i, n}} & \prod_{i=1}^r G(m_i n),
\end{array}$$

commutes, so

$$G(f \times [n])(\phi_{m', n}(x', y)) = \left( \prod_{i=1}^r \phi_{m_i, n} \right)((F_1f)(x'), y) = (\phi_{m_i, n}((F_1f)(x'_i), y))_{i=1}^r,$$

where we write $(F_1)(x'_i)$ for the component of $(F_1f)(x')$ in $F_1m_i$. But now we can exploit the bifunctoriality of all the $\phi_{m_i, n}$'s to insert the $\langle \psi_i \rangle$'s into the first slot, giving us an $r$-tuple of maps

$$\langle \phi_{m_i, n}(\psi_i, y) \rangle_{i=1}^r : \langle \phi_{m_i, n}(x_i, y) \rangle_{i=1}^r \rightarrow \langle \phi_{m_i, n}((F_1f)(x'_i), y) \rangle_{i=1}^r$$

which assemble to a single morphism in $\prod_{i=1}^r G(m_i n)$. We have therefore produced the desired $r$-morphism in $\mathbb{N} \cdot G$. We proceed similarly given an $s$-morphism in $\mathbb{N} \cdot F_2$. If there are a larger number of $F$'s, the appropriate modifications to the construction are straightforward. Preservation of composition requires several large diagrams that the reader is encouraged to construct for herself.
For the bilinearity diagram, suppose we have an $r$-morphism
$$(f, \langle \psi_i \rangle_{i=1}^r) : \langle (x_i, m_i) \rangle_{i=1}^r \to (x', m')$$
in $\mathbb{N} \wr F_1$, and an $s$-morphism
$$(g, \langle \xi_j \rangle_{j=1}^s) : \langle (y_j, n_j) \rangle_{j=1}^s \to (y', n')$$
in $\mathbb{N} \wr F_2$. We wish the following diagram to commute in $\mathbb{N} \wr G$:
$$
\begin{aligned}
\langle \langle (\mathbb{N} \wr \phi)((x_i, m_i), (y_j, n_j)) \rangle_{i=1}^r \rangle_{j=1}^s \xrightarrow{(f, \langle \psi_i \rangle)_*} \langle \langle (\mathbb{N} \wr \phi)((x', m'), (y_j, n_j)) \rangle_{j=1}^s \rangle_{i=1}^r \\
\end{aligned}
\begin{aligned}
\langle \langle (\mathbb{N} \wr \phi)((x_i, m_i), (y_j, n_j)) \rangle_{i=1}^r \rangle_{j=1}^s \xrightarrow{(g, \langle \xi_j \rangle)_*} \langle \langle (\mathbb{N} \wr \phi)((x', m'), (y_j, n_j)) \rangle_{j=1}^s \rangle_{i=1}^r \\
\end{aligned}
\begin{aligned}
\langle \langle (\mathbb{N} \wr \phi)((x_i, m_i), (y_j, n_j)) \rangle_{i=1}^r \rangle_{j=1}^s \xrightarrow{(f, \langle \psi_i \rangle)_*} \langle \langle (\mathbb{N} \wr \phi)((x', m'), (y_J, n_J)) \rangle_{j=1}^s \rangle_{i=1}^r \\
\end{aligned}
\begin{aligned}
\langle \langle (\mathbb{N} \wr \phi)((x_i, m_i), (y_J, n_J)) \rangle_{i=1}^r \rangle_{j=1}^s \xrightarrow{(g, \langle \xi_j \rangle)_*} \langle \langle (\mathbb{N} \wr \phi)((x', m'), (y', n')) \rangle_{j=1}^s \rangle_{i=1}^r \\
\end{aligned}
\begin{aligned}
\langle \langle (\mathbb{N} \wr \phi)((x_i, m_i), (y_J, n_J)) \rangle_{i=1}^r \rangle_{j=1}^s \xrightarrow{(f, \langle \psi_i \rangle)_*} \langle \langle (\mathbb{N} \wr \phi)((x', m'), (y_J, n_J)) \rangle_{j=1}^s \rangle_{i=1}^r \\
\end{aligned}
\begin{aligned}
\langle \langle (\mathbb{N} \wr \phi)((x_i, m_i), (y_J, n_J)) \rangle_{i=1}^r \rangle_{j=1}^s \xrightarrow{(g, \langle \xi_j \rangle)_*} \langle \langle (\mathbb{N} \wr \phi)((x', m'), (y', n')) \rangle_{j=1}^s \rangle_{i=1}^r \\
\end{aligned}
$$

Tracing clockwise, we have the $r$-tupel of morphisms
$$\langle \psi_i \rangle_{i=1}^r : \langle x_i \rangle_{i=1}^r \to (F_1 f)(x'),$$
or again writing $F_1 f)(x')_i$ for the component of $F_1 f)(x')$ in $F_1 m_i$, we have
$$\psi_i : x_i \to (F_1 f)(x')_i$$
as a morphism in $F_1 m_i$ for $1 \leq i \leq r$. Exploiting the bifunctoriality of $\phi$ as required, we obtain an $r$-tupel of morphisms for all $j$ as follows:
$$\phi_{m_i, n_j}(\psi_i, y_j) : \phi_{m_i, n_j}(x_i, y_j) \to \phi_{m_i, n_j}((F_1 f)(x'), y_j).$$

Since we have
$$G(f \times [n])((\phi m', n_j')(x', y_j)) = \langle (\phi m_i, n_j)((F_1 f)(x'), y_j) \rangle_{i=1}^r,$$
these assemble to give us the required morphism defining an $r$-map in $\mathbb{N} \wr G$ for each $1 \leq j \leq s$.

We now wish to compose this $s$-tupel of morphisms in $\mathbb{N} \wr G$ with the single $s$-morphism induced by $(g, \langle \xi_j \rangle_{j=1}^s)$. Explicitly, we have
$$g : [n'] \to \prod_{j=1}^s [n_j]$$
which induces
$$F_2 g : F_2 n' \to \prod_{j=1}^s F_2 n_j,$$
and we also have
$$\xi_j : y_j \to (F_2 g)(y')_j.$$
as a morphism in $F_2n_j$ for all $j$. In the same manner as before, we have
\[ G([m'] \times g)(\phi_{m',n_j}(x', y')) = \langle \phi_{m',n_j}(x', (F_2g)(y')) \rangle_{j=1}^{s}, \]
and so bifunctoriality of $\phi_{m',n_j}$ gives us maps
\[ \phi_{m',n_j}(x', \xi_j) : \phi_{m',n_j}(x', y_j) \to \phi_{m',n_j}(x', (F_2g)(y')) \]
for all $j$, giving us the required $s$-morphism in $N \triangleright G$.

Now composing these data, we first have the composite in $N$ given by
\[
\begin{array}{c}
\begin{array}{c}
 [m'n'] \\
\end{array}
\end{array}
\xrightarrow{[m'] \times g}
\prod_{j=1}^{s} [m'n_j]
\xrightarrow{\prod(f \times [n_j])}
\prod_{j=1}^{s} \prod_{i=1}^{r} [m_in_j]
\]
inducing the functor composite
\[
G(m'n') \xrightarrow{G([m'] \times g)} \prod_{j=1}^{s} G(m'n_j) \xrightarrow{\prod[G(f \times [n_j])]} \prod_{j=1}^{s} \prod_{i=1}^{r} G(m_in_j).
\]
But because $G$ is a multifunctor, and therefore commutes with permutations in the “source,” (actually the target of the functors, since we’re working in $\mathbf{Cat}_{\ast, \text{op}}$), this is all actually equivalent up to reordering the factors in the products to
\[ G(f \times g) : G(m'n') \to \prod_{(i,j) \in \mathcal{P} \times \mathcal{Q}} G(m_in_j). \]

Next, we need to compose the morphisms
\[ G(f \times [n_j])(\phi_{m',n_j}(x', \xi_j)) = \langle \phi_{m,n_j}((F_1f)(x'), \xi_j) \rangle_{i=1}^{r}, \]
whose components we can identify and rewrite as
\[
\begin{align*}
G(f \times [n_j])_i(\phi_{m',n_j}(x', \xi_j)) & = G(f \times [n_j])_i(\phi_{m',n_j}(x', y_j)) \\
& = G(f \times [n_j])_i(\phi_{m,n_j}(\psi_i, (F_2g)(y'))_j) \\
& = \phi_{m,n_j}(\psi_i, (F_2g)(y'))_j,
\end{align*}
\]
with the morphisms
\[ \phi_{m,n_j}(\psi_i, y_j) : \phi_{m,n_j}(x_i, y_j) \to G(f \times [n_j])_i(\phi_{m',n_j}(x', y_j)). \]
But as before, we have
\[ G(f \times [n_j])_i(\phi_{m',n_j}(x', \xi_j)) = \phi_{m,n_j}((F_1f)(x')_i, \xi_j), \]
so the composites we need to form end up being
\[ \phi_{m,n_j}((F_1f)(x')_i, \xi_j) \circ \phi_{m,n_j}(\psi_i, y_j) = \phi_{m,n_j}(\psi_i, \xi_j), \]
which is independent of the priority order of the indices. We therefore get the same result on composing in the other order, establishing the bilinearity in $N \triangleright G$. 


6. The Free Permutative Category on a Multicategory

In this section we prove Theorem 3: the left adjoint to the forgetful functor from permutative categories to multicategories is actually a multifunctor.

We begin by describing the forgetful functor and its left adjoint $F$. Given a permutative category $C$, its underlying multicategory has the same objects, and an $r$-morphism $(a_1, \ldots, a_r) \to b$ consists of a morphism in $C$

$$f : a_1 \oplus \cdots \oplus a_r \to b.$$ 

If $r = 0$, we consider an empty sum to be given by the identity object of the permutative category. Composition is given by taking sums of sums, and composing within $C$. The $\Sigma_n$-actions are induced from the transposition isomorphism in the permutative structure.

The left adjoint to this construction is as follows. Given a multicategory $M$, we construct a permutative category $FM$ by first specifying its objects to be

$$\text{Ob}(FM) := \prod_{n=0}^{\infty} (\text{Ob}M)^n,$$

so the objects of $FM$ consists of lists of objects of $M$, including an empty list, which gives the identity object. Given a source string $\langle x_i \rangle_{i=1}^r = (x_1, \ldots, x_r)$ and a target string $\langle y_j \rangle_{j=1}^s = (y_1, \ldots, y_s)$, we define a morphism $\langle x_i \rangle_{i=1}^r \to \langle y_j \rangle_{j=1}^s$ to consist of a function $\phi : \{1, \ldots, r\} \to \{1, \ldots, s\}$ and, for each $j$ with $1 \leq j \leq s$, a morphism $\psi_j : \langle x_i \rangle_{\phi(i)=j} \to y_j$ in $M$, where $\langle x_i \rangle_{\phi(i)=j}$ is the tuple of entries in $\langle x_i \rangle_{i=1}^r$ whose indices get mapped to $j$. If there are no such $i$, then $\langle x_i \rangle_{\phi(i)=j}$ is the empty list, and $\psi_j$ is a 0-morphism in $M$. The permutative structure is given by concatenation of lists. We remark that multifunctors are sent to strict maps of permutative categories by this construction.

Exploiting Lemma 6 we begin by describing a bilinear map $\lambda : (FM, FN) \to F(M \otimes N)$ for multicategories $M$ and $N$, where $M \otimes N$ is the tensor product of multicategories originally due to Boardman and Vogt, and described in detail in [3]. In particular, the objects of $M \otimes N$ consist of $\text{Ob}(M) \times \text{Ob}(N)$, with a typical object written $x \otimes y$, and the morphisms of $M \otimes N$ are generated by those of the form $x \otimes \psi$ or $\phi \otimes y$, where $x$ is an object of $M$, $\psi$ is a morphism of $N$, $\phi$ is a morphism of $M$, and $y$ is an object of $N$; these are the induced morphisms from the universal bilinear map $(M, N) \to M \otimes N$ of multicategories. See [3], Construction 4.10 and Proposition 4.16.

To give the bilinear map $\lambda : (FM, FN) \to F(M \otimes N)$ of permutative categories, we must first give a map on objects $\text{Ob}(FM) \times \text{Ob}(FN) \to \text{Ob}(F(M \otimes N))$. We do so by assigning

$$\lambda(\langle x_i \rangle_{i=1}^r, \langle y_j \rangle_{j=1}^s) := \langle \langle x_i \otimes y_j \rangle_{i=1}^r \rangle_{j=1}^s.$$
with the indices prioritized as written. Notice that if \( r = 0 \) or \( s = 0 \), then the result is the empty list in \( F(M \otimes N) \), which is the unit object, as required for a bilinear map of permutative categories.

This assignment must give us a bifunctor on the underlying categories, so suppose given a morphism \( (f, \langle \psi_k \rangle_{k=1}^t) : \langle x_i \rangle_{i=1}^r \to \langle z_k \rangle_{k=1}^t \) in \( FM \), so \( f : [r] = \{1, \ldots, r\} \to \{1, \ldots, t\} = [t] \) and for each \( k \in [t] \), we have \( \psi_k : \langle x_i \rangle_{f(i)=k} \to \langle z_k \rangle \in M \), and similarly suppose given \( (g, \langle \xi_q \rangle_{q=1}^p) : \langle y_j \rangle_{j=1}^s \to \langle w_q \rangle_{q=1}^p \) in \( FN \); we must produce an induced morphism

\[
\langle \langle x_i \otimes y_j \rangle_{i=1}^r \rangle_{j=1}^s \rightarrow \langle \langle z_k \otimes w_q \rangle_{k=1}^t \rangle_{q=1}^p.
\]

In order to do so, we first use the product map

\[
f \times g : [rs] \cong [r] \times [s] \to [t] \times [p] \cong [tp],
\]

where the bijections are given by lexicographic order. We then observe that for \( (k, q) \in [t] \times [p] \), we have \( (f \times g)^{-1}(k, q) = f^{-1}(k) \times g^{-1}(q) \), so

\[
\langle x_i \otimes y_j \rangle_{(f \times g)(i,j)=(k,q)} = \langle \langle x_i \otimes y_j \rangle_{f(i)=k} \rangle_{g(j)=q}.
\]

The required map \( \langle x_i \otimes y_j \rangle_{(f \times g)(i,j)=(k,q)} \to \langle z_k \otimes w_q \rangle \) is then given by either way of traversing the bilinearity rectangle

\[
\begin{array}{ccc}
\langle x_i \otimes y_j \rangle_{f(i)=k} & \xrightarrow{\phi_k \otimes y_j} & \langle z_k \otimes y_j \rangle_{g(j)=q} \\
\downarrow & \cong & \downarrow \\
\langle z_k \otimes y_j \rangle_{g(j)=q} & \xrightarrow{\phi_k \otimes w_q} & \langle z_k \otimes w_q \rangle \\
\langle x_i \otimes w_q \rangle_{f(i)=k} & \xrightarrow{\phi_k \otimes w_q} & \langle z_k \otimes w_q \rangle.
\end{array}
\]

This gives us the required bifunctor underlying our bilinear map.

We also require distributivity maps

\[
\delta_1 : \lambda(\langle x_i \rangle_{i=1}^r, \langle y_j \rangle_{j=1}^s) \otimes \lambda(\langle z_k \rangle_{k=1}^t, \langle y_j \rangle_{j=1}^s) \rightarrow \lambda(\langle x_i \rangle_{i=1}^r \otimes \langle z_k \rangle_{k=1}^t, \langle y_j \rangle_{j=1}^s)
\]

and

\[
\delta_2 : \lambda(\langle x_i \rangle_{i=1}^r, \langle y_j \rangle_{j=1}^s) \otimes \lambda(\langle y_j \rangle_{j=1}^s, \langle w_q \rangle_{q=1}^p) \rightarrow \lambda(\langle x_i \rangle_{i=1}^r, \langle y_j \rangle_{j=1}^s \otimes \langle w_q \rangle_{q=1}^p)
\]

subject to the coherence conditions of [2], Definition 3.2. For \( \delta_1 \), we expand the source and obtain

\[
\lambda(\langle x_i \rangle_{i=1}^r, \langle y_j \rangle_{j=1}^s) \otimes \lambda(\langle z_k \rangle_{k=1}^t, \langle y_j \rangle_{j=1}^s)
= \langle \langle x_i \otimes y_j \rangle_{i=1}^r \rangle_{j=1}^s \otimes \langle \langle z_k \otimes y_j \rangle_{k=1}^t \rangle_{j=1}^s.
\]
while expanding the target gives us
\[ \lambda(\langle x_i \rangle_{i=1}^s \otimes \langle z_k \rangle_{k=1}^t \otimes \langle y_j \rangle_{j=1}^s) = \langle \langle x_i \otimes y_j \rangle_{i=1}^s \otimes \langle z_k \otimes y_j \rangle_{k=1}^t \rangle_{j=1}^s. \]

Shuffling from one side to the other gives us a well-defined element of $\Sigma_{s(r+t)}$, which we adopt as our definition of $\delta_1$.

For $\delta_2$, expanding the source gives us
\[ \lambda(\langle x_i \rangle_{i=1}^r \otimes \langle y_j \rangle_{j=1}^s) \circ \lambda(\langle x_i \rangle_{i=1}^r \otimes \langle w_q \rangle_{q=1}^p) = \langle \langle x_i \otimes y_j \rangle_{i=1}^s \otimes \langle x_i \otimes w_q \rangle_{i=1}^r \rangle_{q=1}^p, \]

while expanding the target gives us
\[ \lambda(\langle x_i \rangle_{i=1}^r \otimes \langle y_j \rangle_{j=1}^s \otimes \langle w_q \rangle_{q=1}^p) = \langle \langle x_i \otimes y_j \rangle_{i=1}^r \otimes \langle x_i \otimes w_q \rangle_{i=1}^r \rangle_{q=1}^p, \]

which is exactly the same thing. We therefore use the identity for $\delta_2$, and the coherence relations for just $\delta_2$ follow immediately. The other coherence relations involving $\delta_1$ and both $\delta_1$ and $\delta_2$ follow from the fact that they all involve a well-defined shuffling of terms from one side to the other. We have therefore constructed the desired bilinear $\lambda : (FM, FN) \to F(M \otimes N)$.

Now the objects of both $((M \otimes N) \otimes P)$ and $(M \otimes (N \otimes P))$ take the form of $x \otimes y \otimes z$, with the only difference being the insertion of parentheses, and similarly with the generating morphisms, which are of the form $\phi \otimes y \otimes z$, $x \otimes \psi \otimes z$, or $x \otimes y \otimes \xi$. The associativity coherence diagram for $\lambda$ now follows by inspection. So do the consistency with transposition, and the naturality diagram. Theorem 3 therefore follows, and this concludes the proof that Mandell’s construction is multiplicative.

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