ROOT DATA WITH GROUP ACTIONS

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Abstract. Suppose $k$ is a field, $G$ is a connected reductive algebraic
$k$-group, $T$ is a maximal $k$-torus in $G$, and $\Gamma$ is a finite group that acts on
$(G,T)$. From the above, one obtains a root datum $\Psi$ on which $\text{Gal}(k) \times \Gamma$
acts. Provided that $\Gamma$ preserves a positive system in $\Psi$, not necessarily
invariant under $\text{Gal}(k)$, we construct an inverse to this process. That is,
given a root datum on which $\text{Gal}(k) \times \Gamma$ acts, we show how to construct
a pair $(G,T)$, on which $\Gamma$ acts as above.

Although the pair $(G,T)$ and the action of $\Gamma$ are canonical only up
to an equivalence relation, we construct a particular pair for which $G$
is $k$-quasisplit and $\Gamma$ fixes a $\text{Gal}(k)$-stable pinning of $G$. Using these
choices, we can define a notion of taking “$\Gamma$-fixed points” at the level
of equivalence classes, and this process is compatible with a general
“restriction” process for root data with $\Gamma$-action.

1. Introduction

Let $k$ be a field with separable closure $k^{\text{sep}}$. Let $\Gamma$ be a finite group.

Suppose $\Psi$ is a (reduced) based root datum on which the absolute Galois
group $\text{Gal}(k)$ acts. Then it is well known ([7, Theorem 6.2.7]) that there
exists a connected, reductive, $k$-quasisplit $k$-group $G$, uniquely determined
up to $k$-isomorphism, such that the root datum of $G$ (with respect to a
maximal $k$-torus contained in a Borel $k$-subgroup) is isomorphic to $\Psi$ and
carries the same action of $\text{Gal}(k)$. We generalize this result in two directions.

(A) Suppose $G$ is a connected reductive $k$-group, and $T$ is an arbitrary
maximal torus. Then the root datum $\Psi(G,T)$ carries an action of
$\text{Gal}(k)$. We show that one can reverse this process. That is, given a
root datum $\Psi$ with an action of $\text{Gal}(k)$, one can obtain a pair $(G,T)$
that gives rise to $\Psi$. In general, the pair $(G,T)$ need not be uniquely
determined up to $k$-isomorphism. However, we can always choose $G$ to
be $k$-quasisplit, and all possibilities for $G$ must be $k$-inner forms of each
other.

(B) Now suppose that $\Gamma$ acts on a pair $(G,T)$ as above via $k$-automorphisms.
Then $\Gamma$ acts on the root datum $\Psi(G,T)$, and the actions of $\Gamma$ and
$\text{Gal}(k)$ commute. We show that one can reverse this process under mild

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conditions. That is, suppose that $Ψ$ is a root datum with an action of $\text{Gal}(k) \times Γ$. Assume that $Γ$ (but not necessarily $\text{Gal}(k)$) preserves a base. Then one can obtain a pair $(G, T)$ as above, carrying an action of $Γ$. That is, under appropriate conditions, we can lift an action of $Γ$ from a root datum to a reductive group. Moreover, one can choose $G$ to be $k$-quasisplit, and can choose the action of $Γ$ to preserve a pinning.

The above are all contained in our main result, Theorem 1. In order to state it more precisely, let us consider the collection of abstract root data $Ψ$ that carry an action of $\text{Gal}(k) \times Γ$ such that the action of $Γ$ stabilizes a base for $Ψ$. We consider two data $Ψ$ and $Ψ'$ with such actions to be equivalent if there is a $\text{Gal}(k) \times Γ$-equivariant isomorphism $Ψ \to Ψ'$. Let $R_Γ$ denote the set of equivalence classes of reduced data with such actions.

Let $G$ be a connected reductive $k$-group and $T \subseteq G$ a maximal $k$-torus. Suppose there exists some Borel subgroup $B \subseteq G$ (not necessarily defined over $k$) containing $T$, and a homomorphism $ϕ$ from $Γ$ to the group $\text{Aut}_k(G, B, T)$ of $k$-automorphisms of $G$ stabilizing $T$ and $B$. Suppose $G'$, $T'$, and $ϕ'$ are defined similarly. We say that the triples $(G, T, ϕ)$ and $(G', T', ϕ')$ are equivalent if there exists an isomorphism $ν: G \to G'$ whose restriction gives a $Γ$-equivariant $k$-isomorphism $T \to T'$. (In this situation, $ν$ must be an inner twisting by [5, §3.2].) Let $T_Γ$ be the set of equivalence classes of such triples $(G, T, ϕ)$.

A triple $(G, T, ϕ)$ as above naturally determines a root datum with appropriate actions of $\text{Gal}(k)$ and $Γ$, hence an element of $R_Γ$. It is easily seen that if $(G', T', ϕ')$ and $(G, T, ϕ)$ are equivalent, then they determine the same class in $R_Γ$. Hence we have a natural map $r_Γ: T_Γ \to R_Γ$.

Our main result is the following:

**Theorem 1.** The map $r_Γ: T_Γ \to R_Γ$ is a bijection.

We also prove a generalization of a second well-known result. Suppose $\text{char} k = 0$, and $Γ$ is a finite, cyclic group acting algebraically on a connected reductive group $G$. Then $Γ$ must fix a Borel-torus pair $(B, T)$ in $G$. If $Γ$ fixes a pinning, then the root system of the connected part $\bar{G} := (G^{Γ})^\circ$ of the group of fixed points is obtained as follows. The set of restrictions of roots of $G$ from $T$ to $\bar{T} := (T^{Γ})^\circ$ is a root system, not necessarily reduced, but there is a preferred way to choose a maximal reduced subsystem.

We generalize the above result in several directions.

(C) Instead of assuming $\text{char} k = 0$, we impose the weaker condition that $Γ$ fix a Borel-torus pair.

(D) The group $Γ$ need not be cyclic.

(E) We describe the root datum, not just the root system, of $\bar{G}$ with respect to $\bar{T}$.

The above are all contained in Theorem 2. To state this result more precisely, suppose that the triple $(G, T, ϕ)$ represents an element of $T_Γ$. Then we know [1, Proposition 3.5] that $G$ is a connected reductive $k$-group,
and $T$ is a maximal $k$-torus in $\hat{G}$. Thus, if we let “1” represent the map from the trivial group 1 to $\text{Aut}(G)$, then the triple $(G, \hat{T}, 1)$ represents an element of $\mathcal{R}_1$. The equivalence class of $(G, T, \varphi)$ does not determine that of $(G, \hat{T}, 1)$, or even the $k_{\text{sep}}$-isomorphism class of $\hat{G}$. Nonetheless, we can obtain a well-defined map $\mathcal{T}_\Gamma \to \mathcal{T}_1$ as follows: From Remark 25 we will see that every class in $\mathcal{T}_\Gamma$ contains a triple $(G, T, \varphi)$ such that $G$ is $k$-quasisplit and $\varphi$ fixes a $\text{Gal}(k)$-invariant pinning. Use this choice of triple to define $\hat{G}$ and $\hat{T}$, and it is straightforward to show that our choices determine $\hat{G}$ and $\hat{T}$ up to $k$-isomorphism.

Suppose that the root datum $\Psi$ represents an element of $\mathcal{R}_\Gamma$. We will see in §2 that the action of $\Gamma$ on $\Psi$ allows us to construct a “restricted” root datum $\bar{\Psi}$ that has a preferred choice of reduced subdatum. We thus obtain a map $\mathcal{R}_\Gamma \to \mathcal{R}_1$.

**Theorem 2.** Our maps $\mathcal{T}_\Gamma \to \mathcal{T}_1$ and $\mathcal{R}_\Gamma \to \mathcal{R}_1$ above are compatible with the maps of Theorem 1, in the sense that the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{T}_\Gamma & \xrightarrow{\mathcal{r}} & \mathcal{T}_1 \\
\downarrow & & \downarrow \\
\mathcal{R}_\Gamma & \xrightarrow{\mathcal{r}_1} & \mathcal{R}_1
\end{array}
\]

We prove both theorems in §4.

2. Restrictions of root data

Let $\Psi = (X^*, \Phi, X_*, \Phi^\vee)$ be a root datum. (We do not assume that $\Phi$ is reduced.) Let $\Gamma$ denote a finite group of automorphisms of $\Psi$. We assume that there exists a $\Gamma$-stable set $\Delta$ of simple roots in $\Phi$. Let $V^* = X^* \otimes \mathbb{Q}$ and $V_* = X_* \otimes \mathbb{Q}$. Let $i^*$ denote the quotient map from $V^*$ to its space $\bar{V}^* := V^*_{\Gamma}$ of $\Gamma$-coinvariants. From [1, §2], there is an embedding $\iota: \bar{V}^* \to V^*$ with image $V^*_{\Gamma}$ given by

\[\iota(\bar{v}) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma v,\]

where $v$ is any preimage in $V^*$ of $\bar{v} \in \bar{V}^*$. Let $\bar{X}^*$ and $\bar{\Phi}$ denote the images of $X^*$ and $\Phi$ under $i^*$. Then $\bar{X}^*$ is the torsion-free part of the module $X^*_{\Gamma}$ of $\Gamma$-coinvariants of $X^*$. It is straightforward to see that $\bar{X}^*$ and $\bar{X}_* := X^*_{\Gamma}$, and thus $\bar{V}^*$ and $\bar{V}_* := V^*_{\Gamma}$, are in duality via the pairing given by $\langle \bar{x}, \bar{\lambda} \rangle := \langle i_* \bar{x}, i_* \bar{\lambda} \rangle$, where $i_*: \bar{X}_* \to X_*$ is the inclusion map. With respect to these pairings, $i^*$ is the transpose of $i_*$. For each $\beta \in \Phi$, let $w_\beta$ denote the automorphism of $X^*$ defined by

\[w_\beta(x) = x - \langle x, \beta^\vee \rangle \beta.\]

Let $W$ denote the Weyl group of $\Psi$, i.e., the (finite) subgroup of $\text{Aut}(X^*)$ generated by the $w_\beta$. Then $\Gamma$ acts naturally on $W$, and $W$ acts on $X^*$. The group $W^\Gamma$ of $\Gamma$-fixed elements of $W$ acts on on both $\bar{V}^*$ and $\bar{X}^*$ via the rule
Lemma 3 (cf. [3 §1.32(a)]). The natural action of $W^\Gamma$ on $X^\ast$ is faithful.

Proof. Let $w$ be a nontrivial element of $W^\Gamma$. Then there exists a positive root $\beta \in \Phi$ such that $w(\beta)$ is negative. Since $\Gamma$ stabilizes $\Delta$, it follows that $w(\gamma \cdot \beta) = \gamma \cdot (w\beta)$ is also negative for every $\gamma \in \Gamma$. Thus $i(w(i^*\beta))$ is a linear combination of roots in $\Delta$ in which all of the coefficients are nonpositive, so $w(i^*\beta) \neq i^*\beta$. \hfill \Box

Notation 4. For each root $\beta \in \Phi$, define a $\Gamma$-orbit $\Xi_\beta$ in $\Phi$ as in [1 §5]. That is, let $\Xi_\beta = \Gamma \cdot \beta$ if this is an orthogonal set. Otherwise, for each $\theta \in \Gamma \cdot \beta$, there exists a unique root $\theta' \neq \theta$ in $\Gamma \cdot \beta$ such that $\theta$ and $\theta'$ are not orthogonal. Moreover, $\theta + \theta'$ is a root in $\Phi$ and does not belong to $\Gamma \cdot \beta$. In this case, let $\Xi_\beta = \{\theta + \theta' | \theta \in \Gamma \cdot \beta\}$.

Remark 5. Thus, in all cases, $\Xi_\beta$ is an orthogonal $\Gamma$-orbit of roots.

Lemma 6. If $\alpha \in \Phi$, then $i^*^{-1}(\alpha)$ is a $\Gamma$-orbit of roots in $\Phi$.

Proof. This argument is similar to but more general than that given in the proof of [3 Lemma 10.3.2(ii)]. Suppose $\beta \in \Phi$ and $i^*(\beta) = \alpha$. Then clearly $i^*\theta = \alpha$ for any $\theta \in \Gamma \cdot \beta$.

Now suppose $\beta' \in \Phi$, $\beta' \neq \beta$, and $i^*\beta' = \alpha$. Since $i(i^*(\beta' - \beta)) = 0$ and since $\Gamma$ preserves $\Delta$, when $\beta' - \beta$ is written as a linear combination of simple roots, the coefficients must sum to 0. In particular, $\beta' - \beta \notin \Phi$. Since $\beta'$ cannot be multiple of $\beta$, we have that $\langle \beta', \beta' \rangle \leq 0$ by standard results about root systems. Similarly, $\langle \beta', \theta' \rangle \leq 0$ for all $\theta \neq \beta'$ in $\Gamma \cdot \beta$. Therefore,

$$\langle \beta' - \beta, \sum_{\theta \in \Gamma \cdot \beta} \theta' \rangle = \langle \beta' - \beta, i_\ast \left( \sum_{\theta \in \Gamma \cdot \beta} \theta' \right) \rangle = \langle i^*(\beta' - \beta), \sum_{\theta \in \Gamma \cdot \beta} \theta' \rangle,$$

and since $i^*(\beta' - \beta) = 0$, this pairing vanishes. Thus $\sum_{\theta \in \Gamma \cdot \beta} \langle \beta', \theta' \rangle = 2$ or 1, depending on whether or not $\Gamma \cdot \beta$ is orthogonal. (This follows from the properties of root orbits discussed in [1 §5].) Since $\langle \beta', \theta' \rangle \leq 0$ for all $\theta \neq \beta'$ in $\Gamma \cdot \beta$, it follows that $\beta' \in \Gamma \cdot \beta$. \hfill \Box

For each $\alpha \in \Phi$, define

$$\alpha^\vee \mid_\Gamma \beta \mid \sum_{\xi \in \Xi_\beta} \xi^\vee \in \tilde{X}_\ast,$$

where $\beta$ is any element of $\Phi$ such that $i^*\beta = \alpha$. The element of $\tilde{X}_\ast$ defined by the above formula is independent of the particular choice of $\beta$ by Lemma [3]. Note that $\alpha^\vee$ does indeed lie in $\tilde{X}_\ast$ since $|\Gamma \cdot \beta|/|\Xi_\beta| = 1$ or 2. Let $\tilde{\Phi}^\vee = \{\alpha^\vee | \alpha \in \tilde{\Phi}\}$.

Theorem 7. With the above notation, $\tilde{\Psi} := (\tilde{X}_\ast, \tilde{\Phi}, \tilde{X}_\ast, \tilde{\Phi}^\vee)$ is a root datum.
Remark 8. If $\Psi$ comes equipped with an action of $\text{Gal}(k)$, and the action of $\Gamma$ commutes with that of $\text{Gal}(k)$, then it is clear that the action of $\text{Gal}(k)$ preserves $\bar{\Psi}$.

Proof of Theorem 7. According to [5, §1.1], it suffices to show that

- $\tilde{X}^*$ and $\tilde{X}_s$ are in duality (which we have already observed),
- $\langle \alpha, \alpha^\vee \rangle = 2$ for all $\alpha \in \Phi$, and
- The automorphisms $w_\alpha$ of $X^*$ of the form

$$w_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha \quad \text{(for } \alpha \in \bar{\Phi} \text{)}$$

stabilize $\Phi$ and generate a finite subgroup of $\text{Aut}(\tilde{X}^*)$.

Let $\alpha \in \bar{\Phi}$. Choose $\beta \in \Phi$ such that $i^*\beta = \alpha$, and choose $\xi_0 \in \Xi_\beta$. Then we have

$$\langle \alpha, \alpha^\vee \rangle = \langle i\alpha, i*\alpha^\vee \rangle$$

$$= \left\langle \frac{1}{|\Gamma \cdot \beta|} \sum_{\theta \in \Gamma \cdot \beta} \theta, \frac{|\Gamma \cdot \beta|}{|\Xi_\beta|} \sum_{\xi \in \Xi_\beta} \xi^\vee \right\rangle$$

$$= \frac{1}{|\Xi_\beta|} \left\langle \sum_{\theta \in \Gamma \cdot \beta} \theta, \sum_{\xi \in \Xi_\beta} \xi^\vee \right\rangle$$

$$= \frac{1}{|\Xi_\beta|} \left\langle \sum_{\xi' \in \Xi_\beta} \xi', \sum_{\theta \in \Xi_\beta} \xi^\vee \right\rangle \quad \text{(by the definition of } \Xi_\beta \text{)}$$

$$= \langle \xi_0, \xi_0^\vee \rangle \quad \text{(by Remark [5])}$$

$$= 2,$$

as desired.

Now let $\bar{x} \in \bar{X}^*$, and choose $x \in X^*$ such that $i^*x = \bar{x}$. Then

$$\langle \bar{x}, \alpha^\vee \rangle = \langle x, i*\alpha^\vee \rangle$$

$$= \left\langle x, \frac{|\Gamma \cdot \beta|}{|\Xi_\beta|} \sum_{\xi \in \Xi_\beta} \xi^\vee \right\rangle \quad \text{(by Remark [5])}$$

(9)

$$= \frac{|\Gamma \cdot \beta|}{|\Xi_\beta|} \sum_{\xi \in \Xi_\beta} \langle x, \xi^\vee \rangle.$$
It follows that
\[
\begin{align*}
    w_\alpha(x) &= \bar{x} - \langle \bar{x}, \alpha^\vee \rangle \alpha \\
    &= i^*x - \langle \bar{x}, \alpha^\vee \rangle i^*\beta \\
    &= i^*x - \frac{\langle \bar{x}, \alpha^\vee \rangle}{|\Gamma : \beta|} i^* \left( \sum_{\theta \in \Gamma : \beta} \theta \right) \quad \text{(by Lemma 6)} \\
    &= i^*x - \frac{1}{|\Xi_\beta|} \sum_{\xi' \in \Xi_\beta} \langle x, \xi'^\vee \rangle i^* \left( \sum_{\xi' \in \Xi_\beta} \xi' \right) \quad \text{(by the definition of } \Xi_\beta) \\
    &= i^*x - \sum_{\xi \in \Xi_\beta} \langle x, \xi'^\vee \rangle i^* \left( \frac{1}{|\Xi_\beta|} \sum_{\xi' \in \Xi_\beta} \xi' \right).
\end{align*}
\]

But by Remark 5, for any \( \xi \in \Xi_\beta \),
\[
i^* \left( \frac{1}{|\Xi_\beta|} \sum_{\xi' \in \Xi_\beta} \xi' \right) = i^* \xi,
\]
so we have
\[
\begin{equation}
    (10) \quad w_\alpha(x) = i^* \left( x - \sum_{\xi \in \Xi_\beta} \langle x, \xi'^\vee \rangle \xi \right).
\end{equation}
\]

Also by Remark 5, the reflections \( w_\xi \in W \) for \( \xi \in \Xi_\beta \) all commute with one another. If \( w \) denotes their product, then
\[
x - \sum_{\xi \in \Xi_\beta} \langle x, \xi'^\vee \rangle \xi = w(x),
\]
so by (10), we have
\[
\begin{equation}
    (11) \quad w_\alpha(x) = i^*\left( w(x) \right).
\end{equation}
\]

In particular, if \( \alpha' \in \bar{\Phi} \), and \( \beta' \in \Phi \) satisfies \( i^* \beta' = \alpha' \), then
\[
w_\alpha(\alpha') = i^*\left( w(\beta') \right) \in i^*(\Phi) = \bar{\Phi},
\]
so \( \Phi \) is stable under the action of \( w_\alpha \), as desired.

It remains to show that the group \( \bar{W} := \langle w_\alpha \mid \alpha \in \bar{\Phi} \rangle \subset \text{Aut}(\bar{X}^*) \) is finite. To accomplish this, we show that \( \bar{W} \) embeds naturally in the finite group \( W^\Gamma \). By Lemma 3, there is a natural injection
\[
W^\Gamma \longrightarrow \text{Aut}(\bar{X}^*).
\]

To construct an embedding \( \bar{W} \longrightarrow W^\Gamma \), it is therefore enough to show that the image of this injection contains \( \bar{W} \). Thus, given \( \bar{w} \in \bar{W} \), we will show that there exists \( w \in W^\Gamma \) whose action on \( \bar{X}^* \) coincides with that of \( \bar{w} \). It suffices to prove the existence of \( w \) only in the case in which \( \bar{w} \) is a reflection \( w_\alpha \) through a root \( \alpha \in \bar{\Phi} \). In this case, let \( w = \prod_{\xi \in \Xi_\beta} w_\xi \), where \( \beta \in \Phi \) is
such that \( i^*\beta = \alpha \). It follows from Remark 5 that \( w \in W^\Gamma \), and it follows from \((\Pi)\) that for any \( x \in X^* \),
\[
w_\alpha(i^*x) = i^*(w(x)) = w(i^*x).
\]
This establishes the existence of the desired embedding. \( \square \)

Remark 12. If \( \Phi \) is reduced, then so is the root system \( \bar{\Phi} \) constructed above, unless \( \Phi \) has an irreducible factor of type \( A_{2n} \) whose stabilizer in \( \Gamma \) acts upon it nontrivially. To see this, it is easy to reduce to the case where \( \Phi \) is irreducible and \( \Gamma \) is cyclic (see [1, Proposition 3.5]). The result then follows from [3, §1.3].

Remark 13. There is a way to choose a maximal reduced subsystem that we will later see is preferred. Specifically, take only the nondivisible (resp. nonmultipliable) roots of \( \bar{\Phi} \) according as \( \text{char } k \neq 2 \) (resp. \( \text{two} \)).

**Lemma 14.** The map \( i^* \) induces a bijection between the set of \( \Gamma \)-invariant positive systems in \( \Phi \) and the set of positive systems in \( \bar{\Phi} \).

**Proof.** Let \( \Pi \subseteq \Phi \) be a \( \Gamma \)-invariant positive system. Let \( \bar{\Pi} = i^*(\Pi) \subseteq \bar{\Phi} \).

Then there is some vector \( v \in V_\alpha \) such that for every root \( \beta \in \Phi \), we have that \( \langle \beta, v \rangle \neq 0 \), and \( \langle \beta, v \rangle > 0 \) if and only if \( \beta \in \Pi \). Since \( \Pi \) is \( \Gamma \)-invariant, we may replace \( v \) by \( \sum_{\gamma \in \Gamma} \gamma v \), and thus assume that \( v \) is \( \Gamma \)-invariant, and so lies in \( V^* \). Suppose \( \alpha \in \bar{\Phi} \). Then \( \alpha = i^*\beta \) for some \( \beta \in \Phi \), so \( \langle \alpha, v \rangle = \langle \beta, v \rangle \).

Thus, \( \langle \alpha, v \rangle \neq 0 \), and \( \langle \alpha, v \rangle > 0 \) if and only if \( \alpha \in \bar{\Pi} \). This shows that \( \bar{\Pi} \) is a positive system in \( \bar{\Phi} \).

Conversely, suppose that \( \bar{\Pi} \subseteq \bar{\Phi} \) is a positive system, and let \( \Pi = i^*^{-1}\bar{\Pi} \).

Then there is some vector \( \bar{v} \in V^* \) such that for every root \( \alpha \in \Phi \), we have that \( \langle \alpha, \bar{v} \rangle \neq 0 \), and \( \langle \alpha, \bar{v} \rangle > 0 \) if and only if \( \alpha \in \bar{\Pi} \). For every root \( \beta \in \Phi \), we have \( \langle \beta, i_*v \rangle = \langle i^*\beta, v \rangle \), which is never zero, and is positive if and only if \( \beta \in \bar{\Pi} \). Thus, \( \Pi \subseteq \Phi \) is a positive system. Since \( i_*v \) is \( \Gamma \)-invariant, so is \( \Pi \). \( \square \)

**Corollary 15.** Let \( \bar{W} \) be the Weyl group of \( \bar{\Psi} \). Then the embedding of \( \bar{W} \) into \( W^\Gamma \) in the proof of Theorem 2 is an isomorphism.

**Proof.** Since \( \bar{W} \) acts simply transitively on the set of positive systems in \( \bar{\Phi} \), and \( W^\Gamma \) acts simply transitively on the set of \( \Gamma \)-invariant positive systems in \( \Phi \), the result follows from Lemma 14. \( \square \)

### 3. FROM AUTOMORPHISMS OF ROOT DATA TO AUTOMORPHISMS OF REDUCTIVE GROUPS

Let \( \Psi = (X^*, \Phi, X^*, \Phi^\vee) \) be a root datum on which a group \( \Lambda \) acts via automorphisms. Choosing a root basis \( \Delta \) of \( \Phi \), we obtain a corresponding based root datum \( \bar{\Psi} \). Then \( \bar{\Psi} \) also carries an action of \( \Lambda \). Namely, for \( \sigma \in \Lambda \), there exists a unique element \( c(\sigma) \) in the Weyl group \( W(\Psi) \) of \( \Psi \) such that \( \sigma(\Delta) = c(\sigma)(\Delta) \). If we define \( \sigma^* \) to be the automorphism of \( \bar{\Psi} \) given by

\[
\sigma^*\chi = c(\sigma)^{-1}(\sigma\chi)
\]
for $\chi \in X^*$, then the action of $\Lambda$ on $\dot{\Psi}$ is given by $\sigma \mapsto \sigma^\ast$.

Since $\Lambda$ acts on $\Psi$ and $\dot{\Psi}$, it acts naturally on $\text{Aut}(\Psi)$ and $\text{Aut}(\dot{\Psi})$, as well as on the Weyl groups $W(\Psi) \subset \text{Aut}(\Psi)$ and $W(\dot{\Psi}) \subset \text{Aut}(\dot{\Psi})$. Just as the actions of $\Lambda$ on $\dot{\Psi}$ and on $\Psi$ differ, so the actions of $\Lambda$ on $W(\dot{\Psi})$ and on $W(\Psi)$ differ, even though the Weyl groups themselves are equal. For $\sigma \in \Lambda$ and $w \in W(\dot{\Psi})$, let $(\sigma, w) \mapsto \sigma^\ast(w)$ denote the action of $\Lambda$ on $W(\dot{\Psi})$. Then we have

$$\sigma w = c(\sigma)(\sigma^\ast w)c(\sigma)^{-1}.$$

One can check readily that map $c : \Lambda \rightarrow W(\dot{\Psi})$ is a cocycle in $Z^1(k, W(\dot{\Psi}))$.

We now turn our attention to based root data arising from reductive algebraic groups. Let $G$ be a connected reductive $k$-group, $B$ a Borel subgroup of $G$, and $T \subseteq B$ a maximal torus of $G$. Let $\dot{\Psi}(G, B, T)$ denote the corresponding based root datum.

Any map $\vartheta \in \text{Aut}(G)$ determines an obvious isomorphism

$$\vartheta^* : \dot{\Psi}(G, B, T) \rightarrow \dot{\Psi}(G, \vartheta(B), \vartheta(T)).$$

There is a natural homomorphism

$$\pi : \text{Aut}(G) \rightarrow \text{Aut}(\dot{\Psi}(G, B, T))$$

defined as follows. For $\vartheta \in \text{Aut}(G)$, choose $g_\vartheta \in G(k_{\text{sep}})$ such that $\text{Int}(g_\vartheta)$ takes $\vartheta(B)$ to $B$, and $\vartheta(T)$ to $T$. Then $\text{Int}(g_\vartheta) \circ \vartheta$ stabilizes $B$ and $T$, and we let $\pi(\vartheta)$ be the automorphism $(\text{Int}(g_\vartheta) \circ \vartheta)^\ast$ of $\dot{\Psi}(G, B, T)$ (which is, in fact, independent of the choice of $g_\vartheta$).

Now suppose that $T$ is defined over $k$. Then an element $\sigma \in \text{Gal}(k)$ naturally determines an automorphism of $\dot{\Psi}(G, T)$ hence an automorphism $\sigma^\ast$ of $\dot{\Psi}(G, B, T)$ as defined in [16]. We thus obtain an action of $\text{Gal}(k)$ on $\dot{\Psi}(G, B, T)$, hence one on $\text{Aut}(\dot{\Psi}(G, B, T))$ as above. These actions are independent of the particular choice of $B$ and $T$ in the sense that if $g \in G(k_{\text{sep}})$ and $\sigma \in \text{Gal}(k)$, then we have

$$\sigma^\ast \circ \text{Int}(g)^\ast = \text{Int}(g)^\ast \circ \sigma^\ast,$$

where we use the notation $\sigma^\ast$ to denote both the action of $\sigma$ on $\dot{\Psi}(G, B, T)$ and on $\dot{\Psi}(G, gB, gT)$.

There is a well-known exact sequence

$$1 \rightarrow \text{Inn}(G) \rightarrow \text{Aut}(G) \xrightarrow{\pi} \text{Aut}(\dot{\Psi}(G, B, T)) \rightarrow 1.$$

We note that the homomorphisms in [18] are $\text{Gal}(k)$-equivariant.

Remark 19. Let $\Delta$ be the set of simple roots for $(G, B, T)$. Let $\{X_\alpha\}_{\alpha \in \Delta} \subset \text{Lie}(G)(k_{\text{sep}})$ be a pinning. It is well known [5, Cor. 2.14] that $\{X_\alpha\}$ determines a unique splitting $\psi$ of [18]. Namely, if $f \in \text{Aut}(\dot{\Psi}(G, B, T))$, define $\psi(f)$ to be the automorphism of $G$ such that

- $\psi(f)$ stabilizes $B$ and $T$,
- the restriction of $\psi(f)$ to $T$ is determined by the automorphism of $X^\ast(T)$ given by $f$, and
Our result is trivial, so assume that $G$ is simple. Then $\bar{G}$ and $\bar{T}$ are defined over $k$, and $\{X_\alpha\}$ is Gal($k$)-stable, it follows from [2] §3.10 that $\psi$ is Gal($k$)-equivariant.

Lemma 20. Retain the notation of the previous remark, and assume that $B$ and $T$ are defined over $k$, and $\{X_\alpha\}$ is Gal($k$)-stable. Suppose a group $\Gamma$ acts on $G$ via $k$-automorphisms, preserving $B$, $T$, and $\{X_\alpha\}$. Then $G = (G^\Gamma)^\circ$ is a reductive $k$-group, $B = (B^\Gamma)^\circ$ is a Borel $k$-subgroup of $G$, $\bar{T} = (T^\Gamma)^\circ$ is a maximal $k$-torus in $B$, and $W(G,T)^\Gamma = W(G,\bar{T})$.

We prove this result by reducing to the well-known case where $\Gamma$ is cyclic.

Proof. The statements about $\bar{G}$ and $\bar{T}$ follow from [1] Proposition 3.5. The lemma follows for $G$ if it holds for a central quotient of $G$. Therefore, we may assume that (over $k^{\text{sep}}$) $G$ is a direct product of almost simple groups. We can also reduce to the case where $\Gamma$ acts transitively on the factors of $G$. As in the proof loc. cit., we may identify the factors of $G$ with each other, and replace $\Gamma$ by a group $S \times \Gamma_1$ such that $\Gamma_1$ acts by permuting the factors in our product decomposition of $G$, and $\Gamma_1$ preserves each factor and acts in the same way on each. It is clear from the construction that $S \times \Gamma_1$ preserves $\{X_\alpha\}$.

Working in stages, we may assume that $\Gamma$ is simple. Thus, either $\Gamma$ acts by permutation of the factors of $G$, or $G$ is simple. In the former case, our result is trivial, so assume that $G$ is simple. Then $G$ has a connected Dynkin diagram, whose automorphism group is solvable. Since $\Gamma$ embeds in this automorphism group, it must be cyclic.

We let $\Psi = (X^*(T),\Phi(G,T),X_*(T),\Phi^*(G,T))$ and will freely use the notation of §2. We may identify $\bar{X}^*$ with $X^*(\bar{T})$. Under this identification, the restriction $\beta_{\text{res}}$ of a root $\beta \in \Phi(G,T)$ to $\bar{T}$ corresponds to $i^*\beta$. It follows from [3] §8.2(2)] that since $\Gamma$ fixes a pinning (i.e., $c_\beta = 1$ for each $\beta \in \Delta$, in the terminology loc. cit.), then for each $\beta \in \Phi(G,T)$, there exists a root $\alpha \in \Phi(G,\bar{T})$ proportional to $\beta_{\text{res}}$. Meanwhile, it follows from [1] Proposition 3.5(iv)] that every root in $\Phi(\bar{G},\bar{T})$ is the restriction of a root in $\Phi(G,T)$. It follows that the Weyl group $\bar{W}$ of $\Psi$ is equal to $W(G,T)$. But $\bar{W}$ is canonically isomorphic to $W(G,T)^\Gamma$ by Corollary 15.

4. PROOFS OF THEOREMS

Proof of Theorem 1. Consider an abstract root datum $\Psi = (X^*,\Phi,X_*,\Phi^*)$ with an action of Gal($k$) $\times \Gamma$. Suppose that $\Delta$ is a $\Gamma$-stable base for $\Psi$. Let $\hat{\Psi}$ be the corresponding based root datum. As discussed in [3] the action of Gal($k$) $\times \Gamma$ on $\Psi$ determines one of Gal($k$) $\times \Gamma$ on $\hat{\Psi}$. Since $\Delta$ is $\Gamma$-stable, the actions of $\Gamma$ on $\Psi$ and $\hat{\Psi}$ coincide. In the notation of [11] with $\Lambda = \text{Gal}(k)$, the elements $c(\sigma) \in W(\Psi)$ that arise from the action of Gal($k$) on $\Psi$ must lie in $W(\hat{\Psi})^\Gamma$ since this action commutes with that of $\Gamma$. Therefore, the
map $c : \text{Gal}(k) \longrightarrow W(\Psi)^\Gamma$ is a cocycle in $Z^1(k, W(\Psi)^\Gamma)$. We note that the \text{Gal}(k) \times \Gamma$-isomorphism class of $\Psi$ depends only on that of $\Psi$.

By [4, Theorem 6.2.7], there exists a triple $(G, B_0, T_0)$, unique up to \text{k}-isomorphism, consisting of a \text{k}-quasisplit connected reductive group $G$, a Borel $\text{k}$-subgroup $B_0$ of $G$, and a maximal $\text{k}$-torus $T_0$ of $B_0$, such that the associated based root datum $\dot{\Psi}(G, B_0, T_0)$ is Gal($\text{k}$)-isomorphic to $\Psi$. We will identify $\dot{\Psi}$ and $\dot{\Psi}(G, B_0, T_0)$ via such an isomorphism.

Let $\{X_\alpha\}$ be a Gal($\text{k}$)-stable pinning for $G$ relative to $B_0$ and $T_0$. The action of $\Gamma$ on $\dot{\Psi}$ determines a homomorphism $\phi : \Gamma \longrightarrow \text{Aut}(\dot{\Psi})$. Let $\varphi$ be the composition

$$\varphi : \Gamma \overset{\phi}{\longrightarrow} \text{Aut}(\dot{\Psi}) = \text{Aut}(\dot{\Psi}(G, B_0, T_0)) \overset{\psi}{\longrightarrow} \text{Aut}(G, B_0, T_0, \{X_\alpha\}),$$

where $\psi : \text{Aut}(\dot{\Psi}(G, B_0, T_0)) \longrightarrow \text{Aut}(G, B_0, T_0, \{X_\alpha\})$ is the homomorphism from Remark 19.

Let $\dot{G} = (G^\varphi(\Gamma))^\circ$ and $\dot{T}_0 = (T_0^\varphi(\Gamma))^\circ$. By Lemma 20, $\dot{G}$ is a $\text{k}$-quasisplit reductive group, $\dot{T}_0$ a maximal $\text{k}$-torus of $\dot{G}$, and

$$W(\Psi)^\Gamma = W(G, T_0)^\varphi(\Gamma) = W(\dot{G}, \dot{T}_0).$$

Thus we may view $c$ as a cocycle in $Z^1(k, W(\dot{G}, \dot{T}_0))$.

By [4, Theorem 1.1], there is some $g \in \dot{G}(\text{k}^{\text{sep}})$ such that for all $\sigma \in \text{Gal}(\text{k})$, $g^{-1}\sigma(g)$ lies in the normalizer $N_{\dot{G}}(\dot{T}_0)(\text{k}^{\text{sep}})$, and the image of $g^{-1}\sigma(g)$ in $W(G, T_0)$ is equal to $c(\sigma)$. Let $T = gT_0$ and $B = gB_0$. Since $g$ is $\varphi(\Gamma)$-fixed, $T$ is a $\varphi(\Gamma)$-stable maximal $\text{k}$-torus of $G$, and $B$ is a $\varphi(\Gamma)$-stable Borel subgroup of $G$ containing $T$. We have therefore associated to $\Psi$ a triple $(G, T, \varphi)$ of the required type.

Suppose we vary the arbitrary choices made in the above construction of $(G, T, \varphi)$. That is, suppose we choose

- another based root datum $\dot{\Psi}'$ with underlying datum $\Psi$, and hence a cocycle $c'$ in $Z^1(k, W(\dot{\Psi}')^\Gamma)$;
- another triple of $\text{k}$-groups $(G', B_0', T_0')$ $\text{k}$-isomorphic to $(G, B_0, T_0)$ and an identification of $\dot{\Psi}'(G', B_0', T_0')$ with $\dot{\Psi}$; and
- a Gal($\text{k}$)-stable pinning $\{X'_\alpha\}$ of $G'$ relative to $B_0'$ and $T_0'$, along with the associated map $\psi' : \text{Aut}(\dot{\Psi}'(G', B_0', T_0')) \longrightarrow \text{Aut}(G', B_0', T_0', \{X'_\alpha\})$ from Remark 19.

We will show that these choices lead to a triple $(G', T', \varphi')$ that is equivalent to $(G, T, \varphi)$. We note that replacing $\Psi$ by another datum in its Gal($\text{k}$) \times $\Gamma$-isomorphism class has no additional effect on the triple arising from the construction.

Use a particular $\text{k}$-isomorphism between $(G', B_0', T_0')$ and $(G, B_0, T_0)$ to identify these triples. Following the above construction, we obtain a homomorphism $\varphi' : \Gamma \longrightarrow \text{Aut}(G, B_0, T_0, \{X'_\alpha\})$, as well as an element $g' \in (G'^\varphi(\Gamma))^\circ$ and a $\text{k}$-torus $T' = g'T_0$, analogous to $g$ and $T$, respectively.

There is a unique element $w \in W(\Psi)$ mapping $\Psi$ to $\dot{\Psi}'$, and by uniqueness, $w$ must in fact lie in $W(\Psi)^\Gamma$, and the mapping it induces is equivariant with
respect to the actions of \( \text{Gal}(k) \) on these based root data. Via conjugation, the element \( w \) induces \( \Gamma \)-equivariant isomorphisms \( W(\Psi) \rightarrow W(\Psi') \) and \( \tau: Z^1(k, W(\Psi)) \rightarrow Z^1(k, W(\Psi')) \).

We have a unique element \( \kappa \) of \( \text{Aut}_k(\Psi(G, B_0, T_0)) \) that produces a commutative square

\[
\begin{array}{ccc}
\Psi & \rightarrow & \Psi(G, B_0, T_0) \\
\downarrow w & & \downarrow \kappa \\
\Psi' & \rightarrow & \Psi(G, B_0, T_0)
\end{array}
\]

Here the horizontal arrows are the identifications chosen in the respective constructions of \( \varphi \) and \( \varphi' \). (That \( \kappa \) is \( \text{Gal}(k) \)-equivariant follows from the equivariance of the other three maps in the square.) We therefore obtain a diagram

\[
\begin{array}{ccc}
\text{Aut}(\Psi) & \rightarrow & \text{Aut}(\Psi(G, B_0, T_0)) \\
\downarrow & & \downarrow \\
\Gamma & \rightarrow & \text{Aut}(\Psi) \cap \text{Aut}(\Psi') \\
\downarrow & & \downarrow \\
\text{Aut}(\Psi') & \rightarrow & \text{Aut}(\Psi(G, B_0, T_0))
\end{array}
\]

in which the square on the right is induced by (21) (and hence commutes), the vertical maps are given respectively by conjugation by \( w \) and \( \kappa \), the diagonal maps are given by inclusion, and the map out of \( \Gamma \) is given by the action of \( \Gamma \) on \( \Psi \).

The map \( \tau(c): \sigma \mapsto wc(\sigma)w^{-1} \) is a cocycle in \( Z^1(k, W(\Psi')^{\Gamma}) \), cohomologous to \( c' \); more precisely, for \( \sigma \in \text{Gal}(k) \),

\[
c'(\sigma) = w^{-1}wc(\sigma)w^{-1}\sigma^{\star}(w) = w^{-1}(\tau(c)(\sigma))\sigma^{\star}(w),
\]

where \( \sigma^{\star} \) denotes the result of the action of \( \sigma \) on \( w \), viewed as an element of \( W(\Psi') \). Identifying \( c \) and \( c' \) respectively with cocycles in \( Z^1(k, W(G, T_0)^{\varphi(\Gamma)}) \) and \( Z^1(k, W(G, T_0)^{\varphi'(\Gamma)}) \) as in the above construction, it follows from (22) that

\[
c'(\sigma) = w^{-1}(\kappa \circ c(\sigma) \circ \kappa^{-1})\sigma(w),
\]

where \( \sigma(w) \) here denotes the result of \( \sigma \in \text{Gal}(k) \) acting on the element \( w \in W(\Psi') \) via the identification of this group with the concrete Weyl group \( W(G, T_0) \) in (22).

Let \( n \in N_G(T_0)(k_{\text{sep}}) \) be a representative for \( w \) and set \( \mu = \psi(\kappa) \in \text{Aut}_k(G, B_0, T_0) \). Then by (23), \( g'^{-1}\sigma(g') \) and \( n^{-1}\mu(g^{-1}\sigma(g))\sigma(n) \) have the same image in \( W(G, T_0) \). Rearranging terms and letting \( h = g'n^{-1}\mu(g)^{-1} \),
we obtain that $\sigma(h)$ and $h$ have the same image modulo
\begin{equation}
\sigma(\mu(g)n)T_0 = \sigma(\mu(g)nT_0) = \sigma(\mu(g)T_0) = \sigma(\mu(T)) = \mu(T).
\end{equation}

Let $\nu = \text{Int}(h) \circ \mu$. Since
\begin{equation*}
\nu(T) = \text{Int}(h)(\mu(T)) = \text{Int}(g'n^{-1}\mu(g)^{-1})(\mu(T)) = \text{Int}(g'n^{-1})(\mu^{-1}T)
\end{equation*}
\begin{equation*}
= \text{Int}(g'n^{-1})(\mu(T_0)) = \text{Int}(g'n^{-1})(T_0) = g'T_0 = T',
\end{equation*}
it follows from (24) that $\nu$ gives a $k$-isomorphism $T \rightarrow T'$. 

To show that $(G', T', \varphi')$ is equivalent to $(G, T, \varphi)$, it remains to show that $\nu$ is $\Gamma$-equivariant. It follows from the construction of $\varphi$ that $\pi \circ \varphi$ is equal to the composition $\Gamma \rightarrow \text{Aut}(\dot{\Psi}) \rightarrow \text{Aut}((\dot{\Psi}(G, B_0, T_0)))$ appearing in (22).

Similarly, $\pi \circ \varphi'$ is equal to the analogous composition $\Gamma \rightarrow \text{Aut}(\dot{\Psi}) \rightarrow \text{Aut}((\dot{\Psi}(G, B_0, T_0)))$. Thus for any $\gamma \in \Gamma$,
\begin{equation*}
\pi(\varphi'(\gamma)) = \kappa \circ \pi(\varphi(\gamma)) \circ \kappa^{-1}.
\end{equation*}
Applying $\psi$ to this equality and noting that $\psi \circ \pi \circ \varphi = \varphi$ by construction, we obtain
\begin{equation*}
\psi(\pi(\varphi'(\gamma))) = \mu \circ \varphi(\gamma) \circ \mu^{-1}.
\end{equation*}

Note that by definition, $\psi(f)$ and $\psi'(f)$ agree on $T_0$ for any $f \in \text{Aut}(\dot{\Psi}(G, B_0, T_0))$. Therefore, as automorphisms of $T_0$, we have
\begin{equation*}
\varphi'(\gamma) = \psi'(\pi(\varphi'(\gamma))) = \psi(\pi(\varphi(\gamma))) = \mu \circ \varphi(\gamma) \circ \mu^{-1}.
\end{equation*}

It follows that, as maps on $T$,
\begin{equation*}
\varphi'(\gamma) \circ \nu
= \varphi'(\gamma) \circ \text{Int}(h) \circ \mu
= \varphi'(\gamma) \circ \text{Int}(g'n^{-1}\mu(g)^{-1}) \circ \mu
= \text{Int}(g') \circ \varphi'(\gamma) \circ \text{Int}(\mu(g)n)^{-1} \circ \mu
= \text{Int}(g') \circ \mu \circ \varphi(\gamma) \circ \mu^{-1} \circ \text{Int}(\mu(g)n)^{-1} \circ \mu
= \text{Int}(g') \circ \mu \circ \text{Int}(\mu^{-1}(n))^{-1}
= \text{Int}(g') \circ \mu \circ \text{Int}(\mu^{-1}(n))^{-1} \circ \varphi(\gamma),
\end{equation*}
where the last equality above comes from the fact that $g \in \bar{G}(k^{\text{sep}})$ and $\text{Int}(\mu^{-1}(n)) \in W(G, T_0)_{\varphi'(\Gamma)}$. Thus $\varphi'(\gamma) \circ \nu$ is equal to
\begin{equation*}
\text{Int}(g'n^{-1}\mu(g)^{-1}) \circ \mu \circ \varphi(\gamma) = \nu \circ \varphi(\gamma),
\end{equation*}
showing that $\nu$ is $\Gamma$-equivariant. Therefore, $(G', T', \varphi')$ is equivalent to $(G, T, \varphi)$, and our construction induces a well-defined map $s_{\Gamma} : \mathcal{R}_{\Gamma} \rightarrow \mathcal{R}_{\Gamma}$.

We now show that $\tau_{\Gamma} \circ s_{\Gamma}$ is the identity map on $\mathcal{R}_{\Gamma}$. Let $\Psi$ be a root datum representing some class in $\mathcal{R}_{\Gamma}$, and let $(G, T, \varphi)$ be a triple representing the image of the class of $\Psi$ under $s_{\Gamma}$. We need to show that $\Psi(G, T)$ is $\text{Gal}(k) \times \Gamma$-isomorphic to $\Psi$. We will make free use of the notation developed in the construction of $s_{\Gamma}$.
The $\text{Gal}(k)$-equivariant isomorphism of based root data $\Psi \to \hat{\Psi}(G, B_0, T_0)$ chosen in the definition of $s_\Gamma$ is $\Gamma$-equivariant by construction (where the action of $\Gamma$ on $\hat{\Psi}(G, B_0, T_0)$ is induced by $\varphi$). We may therefore identify $\Psi$ and $\hat{\Psi}(G, B_0, T_0)$ as based root data with $\text{Gal}(k) \times \Gamma$-action via this isomorphism. This allows us to identify $\Psi$ and $\hat{\Psi}(G, T_0)$ as root data with $\Gamma$-action (but not necessarily with $\text{Gal}(k)$-action since the actions of $\text{Gal}(k)$ on $\Psi$ and $\hat{\Psi}$ differ in general).

Recall the element $g \in \hat{G}(k^{\text{sep}})$ chosen in the definition of $s_\Gamma$. The map $\text{Int}(g)^*: \Psi = \Psi(G, T_0) \to \hat{\Psi}(G, T)$ is $\Gamma$-equivariant since $g$ is $\varphi(\Gamma)$-fixed. Furthermore, $\text{Int}(g)^*$ is $\text{Gal}(k)$-equivariant since for $\sigma \in \text{Gal}(k)$ and $\chi \in X^*(T_0)$,

$$\text{Int}(g)^*(\sigma \chi) = \text{Int}(g)^*(c(\sigma)(\sigma^* \chi))$$

$$= \sigma^{-1}g^{-1}(\sigma(\sigma^* \chi))$$

$$= \sigma(\sigma^* \chi)$$

$$= \sigma^g \chi$$

$$= \sigma(\text{Int}(g)^*(\chi)).$$

Thus $\Psi(G, T)$ is $\text{Gal}(k) \times \Gamma$-isomorphic to $\Psi$, as desired.

Finally, we show that $s_\Gamma \circ r_\Gamma$ is the identity map on $\mathcal{T}_\Gamma$. Let $(G', T', \varphi')$ represent a class in $\mathcal{T}_\Gamma$, and let $(G, T, \varphi)$ represent the image of this class under $s_\Gamma \circ r_\Gamma$. Since $r_\Gamma \circ (s_\Gamma \circ r_\Gamma) = (r_\Gamma \circ s_\Gamma) \circ r_\Gamma = r_\Gamma$, it follows that there is a $\text{Gal}(k) \times \Gamma$ isomorphism $\Psi(G, T) \to \Psi(G', T')$. By [5, Theorem 2.9], this isomorphism is induced by an isomorphism $\nu: G \to G'$ that restricts to a $\Gamma$-equivariant $k$-isomorphism $T \to T'$. Thus $(G, T, \varphi)$ and $(G', T', \varphi')$ are equivalent.

Remark 25. Observe that in the definition of the map $s_\Gamma$ above, the triple $(G, T, \varphi)$ is constructed in such a way that $G$ is $k$-quasisplit and $\varphi$ fixes a $\text{Gal}(k)$-invariant pinning of $G$. Thus, since $s_\Gamma \circ r_\Gamma$ is the identity map on $\mathcal{T}_\Gamma$, we see that every equivalence class in $\mathcal{T}_\Gamma$ contains such a triple.

Moreover, suppose that $(G, T, \varphi)$ is a triple of this kind. Applying the construction of $s_\Gamma \circ r_\Gamma$ to this triple, we see that the triple we obtain is precisely $(G, T, \varphi)$, provided that we make appropriate choices.

Remark 26. Recall that in the proof, it is shown that if $(G, T, \varphi)$ and $(G', T', \varphi')$ are two triples that arise by applying the $s_\Gamma$ construction to a root datum $\Psi$, then $(G, T, \varphi)$ and $(G', T', \varphi')$ are equivalent. We note that the equivalence $\nu$ constructed in this case is of a special kind. Namely, $\nu$ is of the form $\text{Int}(h) \circ \mu$, where $h \in G(k^{\text{sep}})$ and $\mu \in \text{Aut}_k(G, B_0, T_0)$.

Now suppose that $(G, T, \varphi)$ and $(G', T', \varphi')$ are arbitrary equivalent triples with the properties that $G$ and $G'$ are $k$-quasisplit and $\varphi$ and $\varphi'$ fix $\text{Gal}(k)$-invariant pinnings for $G$ and $G'$, respectively. Then combining the first part of this remark with Remark 25, it follows that there is an equivalence $\nu$ between $(G, T, \varphi)$ and $(G', T', \varphi')$ of the above special form.
Remark 27. Suppose that $G'$ is $k$-quasisplit and $T'$ is a maximal $k$-torus of $G'$. Suppose that the finite group $\Gamma$ acts via $\text{Gal}(k)$-equivariant automorphisms on $\Psi(G', T')$ preserving a base. Then the equivalence class of $\Psi(G', T')$ lies in $\mathcal{R}_\Gamma$. Applying the construction in the definition of $s_\Gamma$ to $\Psi(G', T')$, we obtain a triple $(G, T, \varphi)$ where $G$ is $k$-quasisplit. Since $\Psi(G', T')$ and $\Psi(G, T)$ are $\text{Gal}(k)$-isomorphic, $G'$ can be taken to equal $G$. Moreover, if $g \in G(k^{\text{sep}})$ is chosen such that $T' = g T_0$, then the cocycle $c$ used to define $T$ can be taken to be the image of $\sigma \mapsto g^{-1} \sigma(g)$ in $Z^1(k, W(G, T_0))$. In particular, it follows from [1, Proposition 6.1] that $T'$ is stably conjugate to $T$.

Proof of Theorem 2. Consider a class in $\mathcal{R}_\Gamma$. From Remark 25, we can represent this class by a triple $(G, T, \varphi)$, where $G$ is $k$-quasisplit and the action $\varphi$ of $\Gamma$ on $G$ fixes a $\text{Gal}(k)$-invariant pinning. Let $\bar{G} = (G^{\varphi(\Gamma)})^\circ$ and $\bar{T} = (T^{\varphi(\Gamma)})^\circ$. Then $\bar{\Psi} = \Psi(G, T)$ comes equipped with an action of $\Gamma$. Consider the restricted root datum $\bar{\Psi}$ of Theorem 7. Construct a new root datum $\bar{\Psi}'$ by replacing the root system $\bar{\Phi}$ of $\bar{\Psi}$ by a maximal reduced subsystem $\bar{\Phi}'$ as in Remark 13 and do likewise with the coroot system. It is clear from the constructions that the root data $\Psi'$ and $\bar{\Psi}(\bar{G}, \bar{T})$ are equivalent, provided that their root systems are equivalent.

As in the proof of Lemma 20, one may reduce to the case where $\Gamma$ is cyclic. From the proof of [8, §8.2(2′′′′)], the root system $\Phi'$ must contain the root system $\Phi(\bar{G}, \bar{T})$, with equality since $\Gamma$ fixes a pinning.

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