FORMALIZING RELATIONS IN TYPE THEORY

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Abstract. Type theory plays an important role in foundations of mathematics as a framework for formalizing mathematics and a base for proof assistants providing semi-automatic proof checking and construction. Derivation of each theorem in type theory results in a formal term encapsulating the whole proof process. In this paper we use a variant of type theory, namely the Calculus of Constructions with Definitions, to formalize the standard theory of binary relations. This includes basic operations on relations, criteria for special properties of relations, invariance of these properties under the basic operations, equivalence relation, well-ordering, and transfinite induction. Definitions and proofs are presented as flag-style derivations.

1. Introduction

First type theories were proposed by B. Russell [10] as a foundation of mathematics. Other important type theories are typed $\lambda$-calculus introduced by A. Church [2] and intuitionistic type theory introduced by P. Martin-Löf [7]. A higher-order typed $\lambda$-calculus known as Calculus of Constructions (CoC) was created by T. Coquand [3]. Variants of CoC make formal bases of proof assistants, which are computer tools for formalizing and developing mathematics. In particular, the well-known proof assistant Coq is based on the strong variant of CoC called the Calculus of Inductive Constructions (CIC).

Here we use the variant $\lambda D$ of CoC developed in [8]; $\lambda D$ is called the Calculus of Constructions with Definitions. We choose $\lambda D$ because of its following useful properties.

– In $\lambda D$, as in other variants of CoC, proofs are expressed as formal terms and thus are incorporated in the system.
– In $\lambda D$ type checking is decidable and therefore proof checking is decidable. So the correctness of a proof can be checked by an algorithm.
– $\lambda D$ is strongly normalizing, which implies the logical consistency of this theory, even with classical logic (when no extra axioms are added) - see [1].

The theory $\lambda D$ is weaker than CIC because $\lambda D$ does not have inductive types. This does not limit its capability for formalizing mathematics because in $\lambda D$ we can use axiomatic approach and higher-order logic to express the objects that CIC defines with inductive types.

In Section 2 we briefly describe the theory $\lambda D$, derived rules of intuitionistic logic in $\lambda D$, and the classical axiom of excluded third that can be added to $\lambda D$ if necessary; we also briefly explain the flag format derivation. In Section 3 we describe the equality in $\lambda D$ and its derived properties.

In Section 4 we study binary relations in $\lambda D$, operations on relations, and their properties. In Section 5 we formally prove criteria of reflexivity, symmetry, antisymmetry and transitivity, and study the invariance of these properties under some basic operations. In Section 6 we formally define partitions in $\lambda D$ and provide a proof of their correspondence with equivalence relations. In Section 6 we also provide an example of partial order with a formal proof, definition of well-ordering in $\lambda D$ and a formal proof of the principle of transfinite induction.

In our formalizations we aim to keep the language and theorems as close as possible to the ones of standard mathematics. In definitions and proofs we use the flag-style derivation described in [8]. Long formal derivations are moved from the main text to Appendices for better readability.

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2. Type Theory \( \lambda D \)

In [8] Nederpelt and Geuvers developed a formal theory \( \lambda D \) and formalized some parts of logic and mathematics in it. Here we briefly describe main features of \( \lambda D \).

2.1. Type Theory \( \lambda D \)

The language of \( \lambda D \) described in [8] has an infinite set of variables, \( V \), and an infinite set of constants, \( C \); these two sets are disjoint. There are also special symbols \( \boxdot \) and \( * \).

**Definition 2.1.** Expressions of the language are defined recursively as follows.

1. Each variable is an expression.
2. Each constant is an expression.
3. Constant \( * \) is an expression.
4. Constant \( \boxdot \) is an expression.
5. (Application) If \( A \) and \( B \) are expressions, then \( AB \) is an expression.
6. (Abstraction) If \( A \) and \( B \) are expressions and \( x \) is a variable, then \( \lambda x : A.B \) is an expression.
7. (Dependent Product) If \( A \) and \( B \) are expressions and \( x \) is a variable, then \( \Pi x : A.B \) is an expression.
8. If \( A_1, A_2, \ldots, A_n \) are expressions and \( c \) is a constant, then \( c(A_1, A_2, \ldots, A_n) \) is an expression.

An expression \( A \rightarrow B \) is introduced as a particular type of Dependent Product from (7) when \( x \) is not a free variable in \( B \).

**Definition 2.2.**

1. A statement is of the form \( M : N \), where \( M \) and \( N \) are expressions.
2. A declaration is of the form \( x : N \), where \( x \) is a variable and \( N \) is an expression.
3. A descriptive definition is of the form:
   \[
   \bar{x} : \bar{A} \vdash c(\bar{x}) := M : N,
   \]
   where \( \bar{x} \) is a list \( x_1, x_2, \ldots, x_n \) of variables, \( \bar{A} \) is a list \( A_1, A_2, \ldots, A_n \) of expressions, \( c \) is a constant, and \( M \) and \( N \) are expressions.
4. A primitive definition is of the form:
   \[
   \bar{x} : \bar{A} \vdash c(\bar{x}) := \perp : N,
   \]
   where \( \bar{x} \), \( \bar{A} \), and \( c \) are described the same way as in (3), and \( N \) is an expression. The symbol \( \perp \) denotes the non-existing definiens. Primitive definitions are used for introducing axioms where no proof terms are needed.
5. A definition is a descriptive definition or a primitive definition.
6. A judgement is of the form:
   \[
   \Delta; \Gamma \vdash M : N,
   \]
   where \( M \) and \( N \) are expressions of the language, \( \Delta \) is an environment (a properly constructed sequence of definitions) and \( \Gamma \) is a context (a properly constructed sequence of declarations).

For brevity we often use implicit variables in definitions, that is we omit the previously declared variables \( \bar{x} \) in \( c(\bar{x}) \) in (3) and (4).

The following informally explains the meaning of expressions.

1. If an expression \( M \) appears in a derived statement of the form \( M : * \), then \( M \) is interpreted as a type, which represents a set or a proposition.
   
   Note: There is only one type \( * \) in \( \lambda D \). But informally we often use \( *_p \) for propositions and \( *_s \) for sets to make proofs more readable.

2. If an expression \( M \) appears in a derived statement of the form \( M : N \), where \( N \) is a type, then \( M \) is interpreted as an object at the lowest level.

   When \( N \) is interpreted as a set, then \( M \) is regarded as an element of this set.
When \( N \) is interpreted as a proposition, then \( M \) is regarded as a proof (or a proof term) of this proposition.

3. The symbol \( \Box \) represents the highest level.

4. **Sort** is * or \( \Box \). Letters \( s, s_1, s_2, \ldots \) are used as variables for sorts.

5. If an expression \( M \) appears in a statement of the form \( M : \Box \), then \( M \) is called a **kind**. \( \lambda D \) contains the derivation rule:

\[
\emptyset ; \emptyset \vdash * : \Box,
\]

which is its (only) axiom because it has an empty environment and an empty context.

Further details of the language and derivation rules of the theory \( \lambda D \) can be found in [8]. Judgments are formally derived in \( \lambda D \) using the derivation rules.

### 2.2. Flag Format of Derivations

The flag-style deduction was introduced by Jaśkowski [5] and Fitch [4]. A derivation in the flag format is a linear deduction. Each "flag" (a rectangular box) contains a declaration that introduces a variable or an assumption; a collection of already introduced variables and assumptions makes the current context. The scope of the variable or assumption is established by the "flag pole". In the scope we construct definitions and proof terms for proving statements/theorems in \( \lambda D \). Each new flag extends the context and at the end of each flag pole the context is reduced by the corresponding declaration. For brevity we can combine several declarations in one flag.

More details on the flag-style deduction can be found in [9] and [8].

### 2.3. Logic in \( \lambda D \)

The rules of intuitionistic logic are derived in the theory \( \lambda D \) as shown in [8]. We briefly describe it here by showing the introduction and elimination rules for logical connectives and quantifiers.

#### 2.3.1. Implication

The logical implication \( A \Rightarrow B \) is identified with the arrow type \( A \rightarrow B \). The rules for implication follow from the following general rules for the arrow type (we write them in the flag format):

| var \( A : s_1 \mid B : s_2 \) |
|---|
| \( A \rightarrow B : s_2 \) |
| \( u : A \rightarrow B \mid v : A \) |
| \( uv : B \) |
| var \( x : A \) |
| ... |
| \( M : B \) |
| \( \lambda x : A.M : A \rightarrow B \) |

Here \( x \) is not a free variable in \( B \).

In \( \lambda D \) arrows are right associative, that is \( A \rightarrow B \rightarrow C \) is a shorthand for \( A \rightarrow (B \rightarrow C) \).

#### 2.3.2. Falsity and Negation

Falsity \( \bot \) is introduced in \( \lambda D \) by:

\[
\bot := \Pi A : * A : *.
\]

From this definition we get a rule for falsity:
The rule states that falsity implies any proposition.
As usual, negation is defined by: \( \neg A := A \rightarrow \bot \).
Other logical connectives and quantifiers are also defined using second order encoding. Here we only list their derived rules and names of the corresponding terms, without details of their construction. The exact values of the terms can be found in [8].
Some of our flag derivations contain the proof terms that will be re-used in other proofs; such proof terms are written in bold font, e.g. \( \wedge\)-in in the first derived rule for conjunction as follows.

2.3.3. Conjunction. These are derived rules for conjunction \( \wedge \):

\[
\begin{align*}
\text{var } A, B : *_p \\
\ldots \\
u : \bot \\
u : \Pi A : *_p . A \\
uB : B
\end{align*}
\]

\( \wedge\)-in is defined as follows:

\[
\wedge\text{-in}(A, B, u, v) : A \wedge B
\]

\( \wedge\text{-el}_1(A, B, w) : A \)

\( \wedge\text{-el}_2(A, B, w) : B \)

2.3.4. Disjunction. These are derived rules for disjunction \( \vee \):

\[
\begin{align*}
\text{var } A, B : *_p \\
u : A | v : B
\end{align*}
\]

\( \vee\text{-in}_1(A, B, u) : A \vee B \)

\( \vee\text{-in}_2(A, B, u) : A \vee B \)

\( C : *_p \\
u : A \vee B | v : A \Rightarrow C | w : B \Rightarrow C \\
\vee\text{-el}(A, B, C, u, v, w) : C
\)

2.3.5. Bi-implication. Bi-implication \( \Leftrightarrow \) has the standard definition:

\( (A \Leftrightarrow B) := (A \Rightarrow B) \land (B \Rightarrow A) \).

Lemma 2.3. We will often use this lemma to prove bi-implication \( A \Leftrightarrow B \):

\[
\begin{align*}
\text{var } A, B : *_p \\
u : A \Rightarrow B | v : B \Rightarrow A
\end{align*}
\]

\( \text{bi-impl}(A, B, u, v) := \wedge\text{-in}(A \Rightarrow B, B \Rightarrow A, u, v) : A \Leftrightarrow B \)
2.3.6. **Universal Quantifier.** The universal quantifier $\forall$ is defined through the dependent product:

\[
\begin{align*}
\text{var } & S : *_s | P : S \to *_p \\
\text{Definition } & \forall(S, P) := \Pi x : S.Px : *_p \\
\text{Notation: } & (\forall x : S.Px) \text{ for } \forall(S, P)
\end{align*}
\]

2.3.7. **Existential Quantifier.** These are derived rules for the existential quantifier $\exists$.

\[
\begin{align*}
\text{var } & S : *_s | P : S \to *_p \\
\text{var } & y : S | u : Py \\
\exists\text{-in}(S, P, y, u) : (\exists x : S.Px) \\
C : & *_p \\
u : (\exists x : S.Px) | v : (\forall x : S.(Px \Rightarrow C)) \\
\exists\text{-el}(S, P, u, C, v) : C
\end{align*}
\]

Here $x$ is not a free variable in $C$.

2.3.8. **Classical Logic.** We use mostly intuitionistic logic. But sometimes classical logic is needed; in these cases we add the following **Axiom of Excluded Third**:

\[
\begin{align*}
\text{var } & A : *_p \\
\text{exc-thrd}(A) := \perp : A \lor \neg A
\end{align*}
\]

This axiom implies the **Double Negation** theorem:

\[
\begin{align*}
\text{var } & A : *_p \\
doub-neg(A) : (\neg\neg A \Rightarrow A)
\end{align*}
\]

2.4. **Sets in $\lambda D$**

Here we briefly repeat some definitions from $\llbracket S \rrbracket$ relating to sets, in particular, subsets of type $S$.

\[
\begin{align*}
\text{var } & S : *_s \\
ps(S) := & S \to *_p \quad \text{Power set of } S \\
\text{var } & V : ps(S) \\
\text{Notation: } & \{ x : S | x \in V \} \text{ for } \lambda x : S.Vx \\
\text{var } & x : S \\
el(S, x, V) := & Vx : *_p \\
\text{Notation: } & x \in S \text{ or } x \in V \text{ for } element(S, x, V)
\end{align*}
\]

Thus, a subset $V$ of $S$ is regarded as a predicate on $S$ and $x \in V$ means $x$ satisfies the predicate $V$.

### 3. Intensional Equality in $\lambda D$

Here we introduce intensional equality for elements of any type; we will call it just equality. In the next section we will introduce extensional equality and the axiom of extensionality relating the two types of equality.
3.1. Properties of Equality

3.1.1. Reflexivity. The following diagram proves the reflexivity property of equality in $\lambda D$.

\[
\begin{align*}
\text{eq-refl}(S, x) &= \lambda P : S \to *_p. (\Pi P : S \to *_p. (P x \Rightarrow P y)) \\
\text{eq-refl}(S, x) &= x =_S x
\end{align*}
\]

Proof terms are constructed similarly for the following properties of Substitutivity, Congruence, Symmetry, and Transitivity (see [8]).

3.1.2. Substitutivity. Substitutivity means that equality is consistent with predicates of corresponding types.

\[
\begin{align*}
\text{eq-subs}(S, P, x, y, u, v) &= P y \\
\end{align*}
\]

3.1.3. Congruence. Congruence means that equality is consistent with functions of corresponding types.

\[
\begin{align*}
\text{eq-cong}(Q, S, f, x, y, u) &= f x =_S f y \\
\end{align*}
\]

3.1.4. Symmetry. The following diagram expresses the symmetry property of equality in $\lambda D$.

\[
\begin{align*}
\text{eq-sym}(S, x, y, u) &= y =_S x \\
\end{align*}
\]

3.1.5. Transitivity. The following diagram expresses the transitivity property of equality in $\lambda D$.

\[
\begin{align*}
\text{eq-trans}(S, x, y, z, u) &= x =_S z \\
\end{align*}
\]

Intensional equality
4. Relations in Type Theory

4.1. Sets in $\lambda D$

Here we briefly repeat some definitions from [8] relating to sets, in particular, subsets of type $S$.

| var $S$ : *s |
|---------|
| $ps(S) := S \to *p$ | Power set of $S$ |
| var $V$ : $ps(S)$ |
| Notation: $\{x : S \mid x \epsilon V\}$ for $\lambda x : S.Vx$ |
| var $x : S$ |
| $element(S, x, V) := Vx : *p$ |
| Notation: $x \epsilon S V$ or $x \epsilon V$ for $element(S, x, V)$ |

Thus, a subset $V$ of $S$ is regarded as a predicate on $S$ and $x \epsilon V$ means $x$ satisfies the predicate $V$.

4.2. Defining Binary Relations in $\lambda D$

Binary relations are introduced in [8], together with the properties of reflexivity, symmetry, antisymmetry, and transitivity, and definitions of equivalence relation and partial order. We use them as a starting point for formalizing the theory of binary relations in $\lambda D$.

A relation on $S$ is a binary predicate on $S$, which is regarded in $\lambda D$ as a composition of unary predicates. For brevity we introduce the type $br(S)$ of all binary relations on $S$:

| var $S$ : *s |
|---------|
| Definition $br(S) := S \to S \to *p$ : $\square$ |

In the rest of the article we call binary relations just relations. The equality of relations and operations on relations are defined similarly to the set equality and set operations.

Next we define the extensional equality of relations vs the intentional equality introduced in the previous section.

| var $S$ : *s |
|---------|
| var $R, Q : br(S)$ |
| Definition $\subseteq (S, R, Q) := (\forall x, y : S.(Rxy \Rightarrow Qxy)) : *p$ |
| Notation: $R \subseteq Q$ for $\subseteq (S, R, Q)$ |
| Definition $Ex-eq(S, R, Q) := R \subseteq Q \land Q \subseteq R : *p$ |
| Notation: $R = Q$ for $Ex-eq(S, R, Q)$ | Extensional equality |

We add to the theory $\lambda D$ the following axiom of extensionality for relations.

| var $S$ : *s |
|---------|
| var $R, Q : br(S)$ |
| $u : R = Q$ |
| $ext$-axiom$(S, R, Q, u) := \perp : R =_{br(S)} Q$ | Extensionality Axiom |
The axiom is introduced in the last line by a primitive definition with the symbol \( \perp \) replacing a non-existing proof term. The Extensionality Axiom states that the two types of equality are the same for binary relations. So we will use the symbol = for both and we will not elaborate on details of applying the axiom of extensionality when converting one type of equality to the other.

4.3. Operations on Binary Relations

Using the flag format, we introduce the identity relation \( id_S \) on type \( S \) and converse \( R^{-1} \) of a relation \( R \).

\[
\text{var } S : * \\
\text{Definition } id_S := \lambda x, y : S. (x = S y) : \text{br}(S) \\
\text{Identity relation}
\]

\[
\text{var } R : \text{br}(S) \\
\text{Definition } \text{conv}(S, R) := \lambda x, y : S. (Ryx) : \text{br}(S) \\
\text{Notation: } R^{-1} \text{ for } \text{conv}(S, R) \\
\text{Converse relation}
\]

Next we introduce the operations of union \( \cup \), intersection \( \cap \), and composition \( \circ \) of relations.

\[
\text{var } S : * \\
\text{var } R, Q : \text{br}(S) \\
\text{Definition } \cup (S, R, Q) := \lambda x, y : S. (Rxy \lor Qxy) : \text{br}(S) \\
\text{Notation: } R \cup Q \text{ for } \cup (S, R, Q) \\
\text{Union}
\]

\[
\text{Definition } \cap (S, R, Q) := \lambda x, y : S. (Rxy \land Qxy) : \text{br}(S) \\
\text{Notation: } R \cap Q \text{ for } \cap (S, R, Q) \\
\text{Intersection}
\]

\[
\text{Definition } \circ (S, R, Q) := \lambda x, y : S. (\exists z : S. (Rxz \land Qzy)) : \text{br}(S) \\
\text{Notation: } R \circ Q \text{ for } \circ (S, R, Q) \\
\text{Composition}
\]

4.4. Properties of Operations

The following two technical lemmas will be used in some future proofs.

**Lemma 4.1.** This lemma gives a shortcut for constructing an element of a composite relation.

\[
\text{var } S : * | R, Q : \text{br}(S) | x, y, z : S \\
u : Rxy | v : Qyz \\
a := \land-in (Rxy, Qyz, u, v) : Rxy \land Qyz \\
\text{prod-term} (S, R, Q, x, y, z, u, v) := \exists-in (S, \lambda t. Rtx \land Qtz, y, a) : (R \circ Q)xz
\]

**Lemma 4.2.** This lemma gives a shortcut for proving equality of two relations.

\[
\text{var } S : * | R, Q : \text{br}(S) \\
u : R \subseteq Q | v : Q \subseteq R \\
\text{rel-equal}(S, R, Q, u, v) := \land-in (R \subseteq Q, Q \subseteq R, u, v) : R = Q
\]

**Theorem 4.3.** For relations \( R, P \) and \( Q \) on \( S \) the following hold.

1) \( (R^{-1})^{-1} = R \).
2) \((R \circ Q)^{-1} = Q^{-1} \circ R^{-1}\).
3) \((R \cap Q)^{-1} = R^{-1} \cap Q^{-1}\).
4) \((R \cup Q)^{-1} = R^{-1} \cup Q^{-1}\).
5) \(R \circ (P \cup Q) = R \circ P \cup R \circ Q\).
6) \((P \cup Q) \circ R = P \circ R \cup Q \circ R\).
7) \(R \circ (P \cap Q) \subseteq R \circ P \cap R \circ Q\).
8) \((P \cap Q) \circ R \subseteq P \circ R \cap Q \circ R\).
9) \((R \circ P) \circ Q = R \circ (P \circ Q)\).

The formal proof is in part A of Appendix. The proof of part 2) has the form:

\[
\text{var } S : * \mid R, Q : \mathit{br}(S)
\]

\[
\ldots
\]

\[
\mathit{conv-prod}(S, R, Q) := \ldots : (R \circ Q)^{-1} = Q^{-1} \circ R^{-1}
\]

Its proof term \(\mathit{conv-prod}(S, R, Q)\) will be re-used later in the paper.

5. Properties of Binary Relations

The properties of reflexivity, symmetry, antisymmetry, transitivity, and the relations of equivalence and partial order are defined in \([8]\) as follows.

\[
\text{var } S : * \mid R : \mathit{br}(S)
\]

\[
\text{Definition } \mathit{refl}(S, R) := \forall x : S. (Rxx) : *p
\]

\[
\text{Definition } \mathit{sym}(S, R) := \forall x, y : S. (Rxy \Rightarrow Ryx) : *p
\]

\[
\text{Definition } \mathit{antisym}(S, R) := \forall x, y : S. (Rxy \Rightarrow Ryx \Rightarrow x =_S y) : *p
\]

\[
\text{Definition } \mathit{trans}(S, R) := \forall x, y, z : S. (Rxy \Rightarrow Ryz \Rightarrow Rxz) : *p
\]

\[
\text{Definition } \mathit{equiv-relation}(S, R) := \mathit{refl}(S, R) \wedge \mathit{sym}(S, R) \wedge \mathit{trans}(S, R) : *p
\]

\[
\text{Definition } \mathit{part-ord}(S, R) := \mathit{refl}(S, R) \wedge \mathit{antisym}(S, R) \wedge \mathit{trans}(S, R) : *p
\]

**Theorem 5.1.** Suppose \(R\) is a relation on type \(S\). Then the following hold.

1) **Criterion of reflexivity.** \(R\) is reflexive \(\Leftrightarrow \mathit{id}_S \subseteq R\).

2) **First criterion of symmetry.** \(R\) is symmetric \(\Leftrightarrow R^{-1} \subseteq R\).

3) **Second criterion of symmetry.** \(R\) is symmetric \(\Leftrightarrow R^{-1} = R\).

4) **Criterion of antisymmetry.** \(R\) is antisymmetric \(\Leftrightarrow R^{-1} \cap R \subseteq \mathit{id}_S\).

5) **Criterion of transitivity.** \(R\) is transitive \(\Leftrightarrow R \circ R \subseteq R\).

The formal proof is in part B of Appendix. The proof of part 3) has the form:

\[
\text{var } S : * \mid R : \mathit{br}(S)
\]

\[
\ldots
\]

\[
\mathit{sym-criterion}(S, R) := \ldots : \mathit{sym}(S, R) \Leftrightarrow R^{-1} = R
\]
Its proof term $\text{sym-criterion}(S, R)$ will be re-used later in the paper.

**Theorem 5.2.** Relation $R$ on $S$ is reflexive, symmetric and antisymmetric $\Rightarrow R = \text{id}_S$.

**Proof.** The formal proof is in the following flag diagram.

| var $S : \ast_s | R : \text{br}(S)$ |
|-----------------------------------|
| $u_1 : \text{refl}(S, R) | u_2 : \text{sym}(S, R) | u_3 : \text{antisym}(S, R)$ |
| $\var x, y : S | v : Rxy$ |
| $a_1 = u_2xyv : Ryx$ |
| $a_2 = u_3xyva_1 : x =_S y$ |
| $a_3 := \lambda x, y : S.\lambda v : Rxy.a_2 : (R \subseteq \text{id}_S)$ |

| var $x, y : S | v : \text{id}_Sxy$ |
|-------------|
| $v : x =_S y$ |
| Notation $P := \lambda z : S.Rxz : S \rightarrow \ast_p$ |
| $a_4 = u_1x : Rxx$ |
| $a_4 : P x$ |
| $a_5 := \text{eq-subs}(S, P, x, y, v, a_4) : Py$ |
| $a_5 : Rxy$ |
| $a_6 := \lambda x, y : S.\lambda v : \text{id}_Sxy.a_5 : (\text{id}_S \subseteq R)$ |
| $a_7 := \text{rel-equal}(S, R, \text{id}_S, a_3, a_6) : R = \text{id}_S$ |

**Theorem 5.3.** Invariance under converse operation. Suppose $R$ is a relation on type $S$. Then the following hold.

1) $R$ is reflexive $\Rightarrow R^{-1}$ is reflexive.

2) $R$ is symmetric $\Rightarrow R^{-1}$ is symmetric.

3) $R$ is antisymmetric $\Rightarrow R^{-1}$ is antisymmetric.

4) $R$ is transitive $\Rightarrow R^{-1}$ is transitive.

**Proof.** 1)
Theorem 5.4. **Invariance under intersection.** Suppose $R$ and $Q$ are relations on type $S$. Then the following hold.

1) $R$ and $Q$ are reflexive $\Rightarrow$ $R \cap Q$ is reflexive.
2) $R$ and $Q$ are symmetric $\Rightarrow$ $R \cap Q$ is symmetric.
3) $R$ or $Q$ is antisymmetric $\Rightarrow$ $R \cap Q$ is antisymmetric.
4) $R$ and $Q$ are transitive $\Rightarrow$ $R \cap Q$ is transitive.
Proof. 1)

\[ \text{var } S : * \mid R, Q : \text{br}(S) \]
\[ u : \text{refl}(S, R) \mid v : \text{refl}(S, Q) \]
\[ \text{var } x : S \]
\[ a_1 := ux : Rxx \]
\[ a_2 := vx : Qxx \]
\[ a_3 := \land \text{-in } (Rxx, Qxx, a_1, a_2) : (R \cap Q)xx \]
\[ a_4 := \lambda x : S. a_3 : \text{refl}(S, R \cap Q) \]

2)

\[ \text{var } S : * \mid R, Q : \text{br}(S) \]
\[ u : \text{sym}(S, R) \mid v : \text{sym}(S, Q) \]
\[ \text{var } x, y : S \mid w : (R \cap Q)xy \]
\[ w : Rxy \land Qxy \]
\[ a_1 := \land \text{-el}_1(Rxy, Qxy, w) : Rxy \]
\[ a_2 := \land \text{-el}_2(Rxy, Qxy, w) : Qxy \]
\[ a_3 := uxya_1 : Ryx \]
\[ a_4 := vxya_2 : Qyx \]
\[ a_5 := \land \text{-in } (Ryx, Qyx, a_3, a_4) : (R \cap Q)yx \]
\[ a_6 := \lambda x, y : S. \lambda w : (R \cap Q)xy. a_5 : \text{sym}(S, R \cap Q) \]

3)

\[ \text{var } S : * \mid R, Q : \text{br}(S) \]

Notation \( A := \text{antisym}(S, R) : *_p \)
Notation \( B := \text{antisym}(S, Q) : *_p \)
Notation \( C := \text{antisym}(S, R \cap Q) : *_p \)
\[ u : A \lor B \]
\[ v : A \]
\[ \text{var } x, y : S \mid w_1 : (R \cap Q)xy \mid w_2 : (R \cap Q)yx \]
\[ w_1 : Rxy \land Qxy \]
\[ a_1 := \land \text{-el}_1(Rxy, Qxy, w_1) : Rxy \]
\[ w_2 : Ryx \land Qyx \]
\[ a_2 := \land \text{-el}_1(Ryx, Qyx, w_2) : Ryx \]
\[ vxy : (Rxy \Rightarrow Ryx \Rightarrow x = y) \]
\[ a_3 := vxya_1a_2 : x = y \]
\[ a_4 := \lambda v : A. \lambda x, y : S. \lambda w_1 : (R \cap Q)xy. \lambda w_2 : (R \cap Q)yx. a_3 : (A \Rightarrow C) \]
Theorem 5.5. Invariance under union. Suppose $R$ and $Q$ are relations on type $S$. Then the following hold.

1) $R$ or $Q$ is reflexive $\Rightarrow R \cup Q$ is reflexive.
2) $R$ and $Q$ are symmetric $\Rightarrow R \cup Q$ is symmetric.

Proof. 1)

\[
\begin{array}{l}
\text{var } S : \ast_s \mid R, Q : \text{br}(S) \\
u : \text{refl}(S, R) \mid x : S \\
u x : Rx x \\
a_1 := \lor-\text{in}_1(R x x, Q x x, u x) : (R \cup Q)x x
\end{array}
\]
\[ a_2 := \lor\text{-}in_2(Rxx, Qxx, ux) : (Q \cup R)xx \]
\[ a_3 := \lambda u : refl(S, R).\lambda x : S.a_1 : \text{refl}(S, R) \Rightarrow \text{refl}(S, R \cup Q) \]
\[ a_4(R, Q) := \lambda u : refl(S, R).\lambda x : S.a_2 : \text{refl}(S, R) \Rightarrow \text{refl}(S, Q \cup R) \]
\[ a_5 := a_4(Q, R) : \text{refl}(S, Q) \Rightarrow \text{refl}(S, R \cup Q) \]
\[ u : \text{refl}(S, R \cup \text{refl}(S, Q)) \]
\[ a_7 := \lor\text{-}el(\text{refl}(S, R), \text{refl}(S, Q), \text{refl}(S, R \cup Q), u, a_3, a_5) : \text{refl}(S, R \cup Q) \]

\[ 2) \]
\[ \forall S : \ast_s | R, Q : \text{br}(S) \]
\[ u_1 : \text{sym}(S, R) \mid u_2 : \text{sym}(S, Q) \]
\[ \forall x, y : S | v : (R \cup Q)xy \]
\[ v : Rxy \lor Qxy \]
\[ w : Rxy \]
\[ a_1 := u_1xyw : Ryx \]
\[ a_2 := \lor\text{-}in_1(Ryx, Qyx, a_1) : (R \cup Q)yx \]
\[ a_3 := \lambda w : Rxy.a_2 : (Rxy \Rightarrow (R \cup Q)yx) \]
\[ w : Qxy \]
\[ a_4 := u_2xyw : Qyx \]
\[ a_5 := \lor\text{-}in_2(Ryx, Qyx, a_4) : (R \cup Q)yx \]
\[ a_6 := \lambda w : Qxy.a_5 : (Qxy \Rightarrow (R \cup Q)yx) \]
\[ a_7 := \lor\text{-}el(Rxy, Qxy, (R \cup Q)yx, v, a_3, a_6) : (R \cup Q)yx \]
\[ a_8 := \lambda x, y : S.\lambda v : (R \cup Q)xy.a_7 : \text{sym}(S, R \cup Q) \]

\[ \square \]

**Theorem 5.6. Invariance under composition.** Suppose \( R \) and \( Q \) are relations on type \( S \). Then the following hold.

1) \( R \circ R^{-1} \) is always symmetric.
2) \( R \) and \( Q \) are reflexive \( \Rightarrow R \circ Q \) is reflexive.
3) Suppose \( R \) and \( Q \) are symmetric. Then \( R \circ Q \) is symmetric \( \Leftrightarrow R \circ Q = Q \circ R \).

**Proof.** 1)
and the proof term \( \text{conv} \)

Here we use the proof term \( \text{conv} \)

\[
\begin{align*}
\var S & : \ast, | R, Q : \text{br}(S) \\
u & : \text{refl}(S, R) | v : \text{refl}(S, Q) \\
\var x & : S \\
u x & : R x x \\
v x & : Q x x \\
a_1 & := \text{prod-term} (S, R, Q, x, x, u x, v x) : (R \circ Q) x x \\
a_2 & := \lambda x : S. \lambda v : P z. a_3 : (\forall z : S. (P z \Rightarrow (R \circ R^{-1}) y x)) \\
a_3 & := \exists \text{-el} (S, P, u, (R \circ R^{-1}) y x, a_4) : (R \circ R^{-1}) y x \\
a_4 & := \lambda z : S. \lambda v : P z. a_3 : (\forall z : S. (P z \Rightarrow (R \circ R^{-1}) y x)) \\
a_5 & := \exists \text{-el} (S, P, u, (R \circ R^{-1}) y x, a_4) : (R \circ R^{-1}) y x \\
a_6 & := \lambda x, y : S. \lambda u : (R \circ R^{-1}) y x. a_5 : \text{sym}(S, R \circ R^{-1})
\end{align*}
\]

2) \( \var S : \ast, | R, Q : \text{br}(S) \)

\[
\begin{align*}
\var x & : S \\
u x & : R x x \\
v x & : Q x x \\
a_1 & := \text{prod-term} (S, R, Q, x, x, u x, v x) : (R \circ Q) x x \\
a_2 & := \lambda x : S. a_1 : \text{refl}(S, R \circ Q)
\end{align*}
\]

3) Here we use the proof term \( \text{sym-criterion}(S, R) \) from Theorem 5.1.3) for the second criterion of symmetry and the proof term \( \text{conv-prod} \) from Theorem 5.3.2).
Notation $A := \text{sym}(S, R \circ Q) : *_p$
Notation $B := (R \circ Q = Q \circ R) : *_p$

$w : A$

- $a_9 := a_2(R \circ Q)w : (R \circ Q)^{-1} = R \circ Q$
- $a_{10} := \text{eq-sym}(br(S), (R \circ Q)^{-1}, R \circ Q, a_9) : R \circ Q = (R \circ Q)^{-1}$
- $a_{11} := \text{eq-trans}(br(S), R \circ Q, (R \circ Q)^{-1}, Q \circ R, a_{10}, a_8) : R \circ Q = Q \circ R$
- $a_{12} := \lambda w : A. a_{11} : A \Rightarrow B$

$w : B$

- $a_{13} := \text{eq-sym}(br(S), R \circ Q, Q \circ R, w) : Q \circ R = R \circ Q$
- $a_{14} := \text{eq-trans}(br(S), (R \circ Q)^{-1}, Q \circ R, R \circ Q, a_8, a_{13}) : (R \circ Q)^{-1} = R \circ Q$
- $a_{15} := a_3(R \circ Q)a_{14} : \text{sym}(S, R \circ Q)$
- $a_{16} := \lambda w : B. a_{15} : B \Rightarrow A$
- $a_{17} := \text{bi-impl}(A, B, a_{12}, a_{16}) : (\text{sym}(S, R \circ Q) \Leftrightarrow R \circ Q = Q \circ R)$

6. Special Binary Relations

6.1. Equivalence Relation and Partition

Theorem 6.1. Invariance of equivalence relation under converse operation and intersection. Suppose $R$ and $Q$ are equivalence relations on type $S$. Then the following hold.

1) $R^{-1}$ is an equivalence relation on $S$.
2) $R \cap Q$ is an equivalence relation on $S$.

Proof. 1) can easily be derived from Theorem 5.3.1), 2), 4) using intuitionistic logic.
2) can easily be derived from Theorem 5.4.1), 2), 4) using intuitionistic logic.

We skip the formal proofs.

Next we formalize the fact that there is a correspondence between equivalence relations on $S$ and partitions of $S$. Equivalence classes are introduced in [8] as follows.

```
var S : *_s | R : br(S) | u : equiv-rel(S, R)
```

```
var x : S
class(S, R, u, x) := \{ y : S | Rxy \} : ps(S)
```

Notation $[x]_R$ for $\text{class}(S, R, u, x)$

Next we define a partition of type $S$:

```
var S : *_s | R : S \rightarrow ps(S)
```

```
partition(S, R) := (\forall x : S. x \in Rx) \land (\forall x, y, z : S. (z \in Rx \Rightarrow z \in Ry \Rightarrow Rx = Ry))
```

As usual, we can regard a partition $R$ as a collection $Rx (x \in S)$ of subsets of $S$. From this point of view, the above diagram expresses the standard two facts for a partition:
(1) any element of $S$ belongs to one of subsets from the collection (namely $Rx$); 
(2) if the intersection of two subsets $Rx$ and $Ry$ is non-empty, then they coincide. 

(1) implies that each subset from the collection is non-empty and that the union of all subsets from the collection is $S$.

**Theorem 6.2.** Any equivalence relation $R$ on type $S$ is a partition of $S$ and vice versa.

**Proof.** The type of partitions of $S$ is $S \rightarrow ps(S)$, which is $S \rightarrow S \rightarrow \ast_p$, and it is the same as the type $br(S)$ of relations on $S$. The proof consists of two steps.

**Step 1.** Any equivalence relation is a partition.

```plaintext
var S : \ast_s | R : S \rightarrow S \rightarrow \ast_p

u : equiv-rel(S, R)

a_1 := \land-\text{el}_1(\text{refl}(S, R), \text{sym}(S, R), \land-\text{el}_1(\text{refl}(S, R) \land \text{sym}(S, R), \text{trans}(S, R), u)) : \text{refl}(S, R)

var x : S

a_2 := a_1x : Rxx

a_2 : (x \in Rx)

a_3 := \lambda x : S.a_2 : (\forall x : S.x \in X)
```

This proves the first part of the definition of $\text{partition}(S, R)$ and the second part was proven in \cite{3}, pg. 291.

**Step 2.** Any partition is an equivalence relation.

```plaintext
var S : \ast_s | R : S \rightarrow S \rightarrow \ast_p

u : \text{partition}(S, R)

Notation $A := \forall x : S.(x \in Rx)$

Notation $B := \forall x, y, z : S.(z \in Rx \Rightarrow z \in Ry \Rightarrow Rx = Ry)$

u : A \land B

a_1 := \land-\text{el}_1(A, B, u) : A

a_2 := \land-\text{el}_2(A, B, u) : B

var x : S

a_3 := a_1x : x \in Rx

a_3 : Rxx

a_4 := \lambda x : S.a_3 : \text{refl}(S, R)

var x, y : S | v : Rxy

a_5 := a_1y : (y \in Ry)

v : (y \in Rx)

a_6 := a_2xyva_5 : Rx = Ry

a_7 := a_1x : (x \in Rx)

a_8 := \text{eq-sub}(ps(S), \lambda Z : ps(S).x \in Z, Rx, Ry, a_6, a_7) : (x \in Ry)

a_8 : Ryx
```
\begin{align*}
a_9 & := \lambda x, y : S. \lambda v : Rx. a_8 : \text{sym}(S, R) \\
\text{var } x, y, z : S \mid v : Rx \mid w : Ryz \\
v & := y \in Rx \\
a_{10} & := a_9 yzw : Rzy \\
a_{10} & := (y \in Rz) \\
a_{11} & := a_2 zxy a_{10} v : Rz = Rx \\
a_{12} & := a_1 z : (z \in Rz) \\
a_{13} & := \text{eq-sub}(ps(S), \lambda Z : ps(S). z \in Z, Rz, Rx, a_{11}, a_{12}) : \text{eq}(Rz, Rx) \\
a_{13} & := \text{trans}(S, R) \\
a_{14} & := \lambda x, y, z : S. \lambda v : Rxy. \lambda w : Ryz. a_{13} : \text{trans}(S, R) \\
a_{15} & := \land \text{-in}(\text{refl}(S, R) \land \text{sym}(S, R), \text{trans}(S, R), \land \text{-in}(\text{refl}(S, R), \text{sym}(S, R), a_4, a_9), a_{14}) : \text{equiv-rel}(S, R)
\end{align*}

\[\square\]

6.2. Partial Order

\textbf{Theorem 6.3. Invariance of partial order under converse operation and intersection.} Suppose \(R\) and \(Q\) are partial orders on type \(S\). Then the following hold.

1) \(R^{-1}\) is a partial order on \(S\).
2) \(R \cap Q\) is a partial order on \(S\).

\textit{Proof.} 1) can easily be derived from Theorem 5.3.1), 3), 4) using intuitionistic logic. 
2) can easily be derived from Theorem 5.4.1), 3), 4) using intuitionistic logic. We skip the formal proofs. \[\square\]

\textbf{Example 6.4.} \(\subseteq\) is a partial order on the power set \(ps(S)\) of type \(S\).

\textit{Proof.} This is the formal proof.

\begin{align*}
\text{var } S : \ast_s \\
\text{Notation } R := \lambda X, Y : ps(S). X \subseteq Y : \text{br}(ps(S)) \\
\text{Notation } A := \text{refl}(ps(S), R) \\
\text{Notation } B := \text{antisym}(ps(S), R) \\
\text{Notation } C := \text{trans}(ps(S), R) \\
\text{var } X : ps(S) \\
a_1 := \lambda x : S. \lambda u : (x \in X). u : X \subseteq X \\
a_2 := \lambda X : ps(S). a_1 \ : A \\
\text{var } X, Y : ps(S) \mid u : X \subseteq Y \mid v : Y \subseteq X \\
a_3 := \land \text{-in}(X \subseteq Y, Y \subseteq X, u, v) \ : X = Y \\
a_4 := \lambda X, Y : ps(S). \lambda u : X \subseteq Y, \lambda v : Y \subseteq X. a_3 \ : B \\
\text{var } X, Y, Z : ps(S) \mid u : X \subseteq Y \mid v : Y \subseteq Z \\
\text{var } x : S \mid w : x \in X \\
a_5 := u x w \ : (x \in Y)
\end{align*}
6.3. Well-Ordering and Transfinite Induction

We will use the notation \( S \leq \) for partial order. In the following diagram we define the strict order \(<\), the least element of a partially ordered set, and well-ordering of type \( S \).

\[
\text{var } S : \ast_s \mid \leq: \text{br}(S) \mid u : \text{part-ord}(S, \leq) \\
\text{Definition } < := \lambda x, y : S. (x \leq y \land \neg (x = y)) \\
\text{var } X : \text{ps}(S) \mid x : S \\
\text{Definition } \text{least}(S, \leq, X, x) := x \leq X \land \forall y : S. (y \leq X \Rightarrow x \leq y) \\
\text{Definition } \text{well-ord}(S, \leq) := \text{part-ord}(S, \leq) \land \forall X : \text{ps}(S), \exists x : S.x \leq X \Rightarrow \exists x : S. \text{least}(S, \leq, X, x) \\
\]

**Theorem 6.5. Transfinite Induction.** Suppose \( \leq \) is a well-ordering of type \( S \). Then for any predicate \( P \) on \( S \):

\[
\forall x : S. [ (\forall y : S. (y < x \Rightarrow Py) \Rightarrow Px) \Rightarrow \forall x : S. Px ]
\]

**Proof.** Here is the formal proof.

\[
\begin{align*}
\text{var } S : \ast_s \mid \leq: \text{br}(S) \mid u_1 : \text{well-ord}(S, \leq) \mid P : S \to \ast_p \\
u_2 : \forall x : S, [ \forall y : S. (y < x \Rightarrow Py) \Rightarrow Px ] \\
\text{Notation } A := \text{part-ord}(S, \leq) \\
\text{Notation } B := \forall X : \text{ps}(S), \exists x : S.x \leq X \Rightarrow \exists x : S. \text{least}(S, \leq, X, x)) \\
u_1 : A \land B \\
a_1 := \land \text{-el}_1(A, B, u_1) : A \\
a_2 := \land \text{-el}_2(A, B, u_1) : B \\
a_3 := \land \text{-el}_1(\text{refl}(S, \leq) \land \text{antisym}(S, \leq), \text{trans}(S, \leq), a_1) : \text{refl}(S, \leq) \land \text{antisym}(S, \leq) \\
a_4 := \land \text{-el}_2(\text{refl}(S, \leq), \text{antisym}(S, \leq), a_3) : \text{antisym}(S, \leq) \\
\text{Notation } X := \lambda x : S. \neg Px : \text{ps}(S) \\
v_1 : (\exists x : S.x \leq X) \\
a_5 := a_2 X v_1 : [ \exists x : S. \text{least}(S, \leq, X, x) ] \\
\text{var } x : S \mid v_2 : \text{least}(S, \leq, X, x) \\
a_6 := \land \text{-el}_1(x \leq X, \forall y : S. (y \leq X \Rightarrow x \leq y), v_2) : x \leq X \\
a_6 : \neg Px \\
a_7 := \land \text{-el}_2(x \leq X, \forall y : S. (y \leq X \Rightarrow x \leq y), v_2) : [ \forall y : S. (y \leq X \Rightarrow x \leq y) ]
\end{align*}
\]
Here we used (twice) the Double Negation theorem with the proof term \textit{doub-neg}. This is the only place in this paper where we use the classical (not intuitionistic) logic. □

7. Conclusion

Starting with the definitions from [8] of binary relations and properties of reflexivity, symmetry, antisymmetry, and transitivity, we formalize in the theory $\lambda D$ (the Calculus of Constructions with Definitions) criteria for these properties and prove their invariance under operations of union, intersection, composition, and taking converse. We provide a formal definition of partition and formally prove correspondence between equivalence relations and partitions. We derive a formal proof that $\subseteq$ is a partial order on power set. Finally we formally prove the principle of transfinite inductions for a type with well-ordering.

The results can be transferred to the proof assistants that are based on the Calculus of Constructions. Since binary relations are the abstract concepts used in many areas of mathematics, the results can be useful for further formalizations of mathematics in $\lambda D$. Our next direction of research is formalization of parts of probability theory in $\lambda D$ that we outlined in [6].
APPENDIX

Appendix A. Proof of Theorem 4.3

Proof. 1)

\begin{verbatim}
var S : * | R : br(S)

var x, y : S
\hline
u : (R^{-1})^{-1}xy
u : R^{-1}yx
u : Rxy

a_1 := \lambda u : (R^{-1})^{-1}xy. u : (R^{-1})^{-1}xy \Rightarrow Rxy
u : Rxy
u : R^{-1}yx
u : (R^{-1})^{-1}xy

a_2 := \lambda u : Rxy. u : Rxy \Rightarrow (R^{-1})^{-1}xy

a_3 := \lambda x, y : S. a_1 : (R^{-1})^{-1} \subseteq R
a_4 := \lambda x, y : S. a_2 : R \subseteq (R^{-1})^{-1}

a_5 := \text{rel-equal}(R^{-1})^{-1}, R, a_3, a_4) : (R^{-1})^{-1} = R
\end{verbatim}

2)

\begin{verbatim}
var S : * | R, Q : br(S)

Notation A := (R \circ Q)^{-1} : br(S)
Notation B := Q^{-1} \circ R^{-1} : br(S)
var x, y : S
\hline
Notation P_1 := \lambda z : S.Ryz \land Qzx : S \rightarrow *
Notation P_2 := \lambda z : S.Q^{-1}xz \land R^{-1}zy : S \rightarrow *

u : Axy
\hline
u : (R \circ Q)yx
u : (\exists z : S.P_1 z)
\hline
z : S | v : P_1 z
\hline
v : Ryz \land Qzx
\hline
a_1 := \land-e_1(Ryz, Qzx, v) : Ryz
a_2 := \land-e_2(Ryz, Qzx, v) : Qzx
a_3 : R^{-1}zy
a_2 : Q^{-1}xz
a_3 := \text{prod-term}(S, Q^{-1}, R^{-1}, x, z, y, a_2, a_1) : (Q^{-1} \circ R^{-1})xy
a_3 : Bxy
\end{verbatim}
\[ a_4 := \lambda z : S . \lambda v : P_1 z . a_3 : (\forall z : S . (P_1 z \Rightarrow Bxy)) \]
\[ a_5 := \exists -el (S, P_1, u, Bxy, a_4) : Bxy \]
\[ a_6 := \lambda x, y : S . \lambda u : Axy. a_5 : A \subseteq B \]

\[ \text{var } x, y : S | u : Bxy \]
\[ u : (\exists z : S . P_2 z) \]
\[ z : S | v : P_2 z \]
\[ v : Q^{-1}xz \land R^{-1}zy \]
\[ a_7 := \land -el_1 (Q^{-1}xz, R^{-1}zy, v) : Q^{-1}xz \]
\[ a_8 := \land -el_2 (Q^{-1}xz, R^{-1}zy, v) : R^{-1}zy \]
\[ a_7 := Qzx \]
\[ a_8 := Ryz \]
\[ a_9 := \text{prod-term} (S, R, Q, y, z, a_8, a_7) : (R \circ Q)y \]
\[ a_9 := (R \circ Q)^{-1}xy \]
\[ a_9 := Axy \]
\[ a_{10} := \lambda z : S . \lambda v : P_2 z . a_9 : (\forall z : S . (P_2 z \Rightarrow Axy)) \]
\[ a_{11} := \exists -el (S, P_2, u, Axy, a_{10}) : Axy \]
\[ a_{12} := \lambda x, y : S . \lambda u : Bxy. a_{11} : B \subseteq A \]
\[ \text{conv-prod} (S, R, Q) := \text{rel-equal} (A, B, a_6, a_{12}) : (R \circ Q)^{-1} = Q^{-1} \circ R^{-1} \]

3)

\[ \text{var } S : * | R, Q : \text{br}(S) \]

Notation \( A := (R \cap Q)^{-1} : \text{br}(S) \)
Notation \( B := R^{-1} \cap Q^{-1} : \text{br}(S) \)

\[ \text{var } x, y : S | u : Axy \]
\[ u : (R \cap Q)^{-1}xy \]
\[ u : (R \cap Q)y \]
\[ u : Ry \land Qyx \]
\[ a_1 := \land -el_1 (Ryx, Qyx, v) : Ry \]
\[ a_2 := \land -el_2 (Ryx, Qyx, v) : Qyx \]
\[ a_1 := R^{-1}xy \]
\[ a_2 := Q^{-1}xy \]
\[ a_3 := \land -in (R^{-1}xy, Q^{-1}xy, a_1, a_2) : Bxy \]
\[ a_4 := \lambda x, y : S . \lambda u : Axy. a_3 : A \subseteq B \]

\[ \text{var } x, y : S | u : Bxy \]
\[ u : R^{-1}xy \land Q^{-1}xy \]
\[ a_5 := \land -el_1 (R^{-1}xy, Q^{-1}xy, v) : R^{-1}xy \]
\[ a_6 := \land-cl_2(R^{-1}xy, Q^{-1}xy, v) : Q^{-1}xy \]
\[ a_5 : Ryx \]
\[ a_6 : Qyx \]
\[ a_7 := \land-in(Ryx, Qyx, a_5, a_6) : (R \cap Q)yx \]
\[ a_7 : (R \cap Q)^{-1}xy \]
\[ a_7 : Axy \]
\[ a_8 := \lambda x, y : S, \lambda u : Bxy, a_7 : Bxy, a_7 : Axy \]
\[ a_9 := rel-equal(A, B, a_4, a_8) : (R \cap Q)^{-1} = R^{-1} \cap Q^{-1} \]

4)

\[ \text{var } S : *_{s} | R, Q : br(S) \]

Notation \( A := (R \cup Q)^{-1} : br(S) \)

Notation \( B := R^{-1} \cup Q^{-1} : br(S) \)

\[ \text{var } x, y : S | u : Axy \]

\[ u : (R \cup Q)yx \]
\[ u : Ryx \lor Qyx \]
\[ v : Ryx \]
\[ v : R^{-1}xy \]
\[ a_1 := \lor-in_1(R^{-1}xy, Q^{-1}xy, v) : Bxy \]
\[ a_2 := \lambda v : Ryx. a_1 := Ryx \Rightarrow Bxy \]
\[ v : Qyx \]
\[ v : Q^{-1}xy \]
\[ a_3 := \lor-in_2(R^{-1}xy, Q^{-1}xy, v) : Bxy \]
\[ a_4 := \lambda v : Qyx. a_3 := Qyx \Rightarrow Bxy \]
\[ a_5 := \lor-el (Ryx, Qyx, Bxy, u, a_2, a_4) : Bxy \]
\[ a_6 := \lambda x, y : S, \lambda u : Axy, a_5 : Axy \subseteq B \]

\[ \text{var } x, y : S | u : Bxy \]

\[ u : R^{-1}xy \lor Q^{-1}xy \]
\[ v : R^{-1}xy \]
\[ v : Ryx \]
\[ a_7 := \lor-in_1(Ryx, Qyx, v) : Ryx \lor Qyx \]
\[ a_7 := (R \cup Q)^{-1}xy \]
\[ a_7 := Axy \]
\[ a_8 := \lambda v : R^{-1}xy. a_7 := R^{-1}xy \Rightarrow Axy \]
\[ v : Q^{-1}xy \]
\[ v : Qyx \]
5) \(\text{var } S : *s \mid R, P, Q : \text{br}(S)\)

Notation \(A := R \circ (P \cup Q) : \text{br}(S)\)
Notation \(B := R \circ P \cup R \circ Q : \text{br}(S)\)

\var x, y : S

\text{Notation } P_0 := \lambda z : S.Rxz \land (P \cup Q)zy : S \rightarrow *p

\text{u : Axy}

\text{u : } (\exists z : S.P_0 z)
\text{z : S | v : P_0 z}

\text{v : Rxz \land (P \cup Q)zy}
\text{a_1 := } \land \text{-el}(Rxz, (P \cup Q)zy, v, v) : Rxz
\text{a_2 := } \land \text{-el}(Rxz, (P \cup Q)zy, v) : (P \cup Q)zy
\text{a_3 := prod-term (S, R, P, x, z, y, a_1, w) : (R \circ P)xy}
\text{a_4 := } \lor \text{-in}_1((R \circ P)xy, (R \circ Q)xy, a_3) : Bxy
\text{a_5 := } \lambda w : Pzy.a_4 : Pzy \Rightarrow Bxy
\text{w : Pzy}

\text{w : Qzy}
\text{a_6 := prod-term (S, R, Q, x, z, y, a_1, w) : (R \circ Q)xy}
\text{a_7 := } \lor \text{-in}_2((R \circ P)xy, (R \circ Q)xy, a_5) : Bxy
\text{a_8 := } \lambda w : Qzy.a_7 : Qzy \Rightarrow Bxy
\text{a_9 := } \lor \text{-el}(Pzy, Qzy, Bxy, a_2, a_5, a_8) : Bxy
\text{a_{10} := } \lambda z : S.\lambda v : P_0 z.a_9 : (\forall z : S.(P_0 z \Rightarrow Bxy))
\text{a_{11} := } \exists \text{-el}(S, P_0, u, Bxy, a_{10}) : Bxy
\text{a_{12} := } \lambda x, y : S.\lambda u : Axy.a_{11} : A \subseteq B

\var x, y : S

\text{Notation } P_1 := \lambda z : S.Rxz \land Pzy : S \rightarrow *p
\text{Notation } P_2 := \lambda z : S.Rxz \land Qzy : S \rightarrow *p
\( u : B_{xy} \)
\[
\begin{align*}
u &: ( R \circ P )_{xy} \lor ( R \circ Q )_{xy} \\
v &: ( R \circ P )_{xy} \end{align*}
\]
\( v : ( \exists z : S.P_1 z ) \)
\[
\begin{align*}z &: S \mid w : P_1 z \end{align*}
\]
\( w &: Rxz \land Pzy \)
\[
\begin{align*}a_{13} &: \land \text{-el}_1(Rxz, Pzy, w) : Rxz \\
a_{14} &: \land \text{-el}_2(Rxz, Pzy, w) : Pzy \\
a_{15} &: \lor \text{-in}_1(Pzy, Qzy, a_{14}) : (P \cup Q)zy \end{align*}
\]
\( a_{16} &: \text{prod-term}(S, R, (P \cup Q), x, z, y, a_{13}, a_{15}) : Axy \)
\[
\begin{align*}a_{17} &: \lambda z : S.\lambda w : P_1 z. a_{16} : (\forall z : S.(P_1 z \Rightarrow Axy)) \\
a_{18} &: \exists \text{-el}(S, P_1, v, Axy, a_{17}) : Axy \\
a_{19} &: \lambda v : ( R \circ P )_{xy}. a_{18} : ((R \circ P)_{xy} \Rightarrow Axy) \end{align*}
\]
\( v : ( R \circ Q )_{xy} \)
\[
\begin{align*}v &: ( \exists z : S.P_2 z ) \end{align*}
\]
\( z &: S \mid w : P_2 z \)
\[
\begin{align*}a_{20} &: \land \text{-el}_1(Rxz, Qzy, w) : Rxz \\
a_{21} &: \land \text{-el}_2(Rxz, Qzy, w) : Qzy \\
a_{22} &: \lor \text{-in}_2(Pzy, Qzy, a_{21}) : (P \cup Q)zy \end{align*}
\]
\( a_{23} &: \text{prod-term}(S, R, (P \cup Q), x, z, y, a_{20}, a_{22}) : Axy \\
a_{24} &: \lambda z : S.\lambda w : P_2 z. a_{23} : (\forall z : S.(P_2 z \Rightarrow Axy)) \\
a_{25} &: \exists \text{-el}(S, P_2, v, Axy, a_{24}) : Axy \\
a_{26} &: \lambda v : ( R \circ Q )_{xy}. a_{25} : ((R \circ Q)_{xy} \Rightarrow Axy) \\
a_{27} &: \lor \text{-el}((R \circ P)_{xy}, (R \circ Q)_{xy}, Axy, u, a_{19}, a_{26}) : Axy \\
a_{28} &: \lambda x, y : S.\lambda u : B_{xy}. a_{27} : B \subseteq A \\
a_{29} &: \text{rel-equal}(A, B, a_{12}, a_{28}) : R \circ (P \cup Q) = R \circ P \cup R \circ Q \end{align*}
\]
6) is proven similarly to 5).
\[ \text{Notation } A := R \circ (P \cap Q) : \text{br}(S) \]
\[ \text{Notation } B := R \circ P \cap R \circ Q : \text{br}(S) \]

7) \[ \text{var } S : \ast_s | R, P, Q : \text{br}(S) \]
\[ \text{Notation } A := R \circ (P \cap Q) : \text{br}(S) \]
\[ \text{Notation } B := R \circ P \cap R \circ Q : \text{br}(S) \]
\[ \text{var } x, y : S \]
\[ \text{Notation } P := \lambda z : S.Rxz \land (P \cap Q)zy : \ast_p \]
\[ \text{u : Axy} \]
\[ \text{u : } (\exists z : S.Pz) \]
\[ \text{var } z : S \mid v : Pz \]
\[ v : Rxz \land (P \cap Q)zy \]
\[ a_1 := \land \text{-el}_1(Rxz, (P \cap Q)zy, v) : Rxz \]
\[ a_2 := \land \text{-el}_2(Rxz, (P \cap Q)zy, v) : (P \cap Q)zy \]
\[ a_2 := Pzy \land Qzy \]
\[ a_3 := \land \text{-el}_1(Pzy, Qzy, a_2) : Pzy \]
\[ a_4 := \land \text{-el}_2(Pzy, Qzy, a_2) : Qzy \]
\[ a_5 := \text{prod-term } (S, R, P, x, z, y, a_3) : (R \circ P)xy \]
\[ a_6 := \text{prod-term } (S, R, Q, x, z, y, a_4) : (R \circ Q)xy \]
\[ a_7 := \land \text{-in } ((R \circ P)xy, (R \circ Q)xy, a_5, a_6) : Bxy \]
\[ a_8 := \lambda z : S.\lambda v : Pz.a_7 : (\forall z : S.(Pz \Rightarrow Bxy)) \]
\[ a_9 := \exists \text{-el } (S, P, u, Bxy, a_8) : Bxy \]
\[ a_{10} := \lambda x, y : S.\lambda u : Axy.a_9 : R \circ (P \cap Q) \subseteq R \circ P \cap R \circ Q \]

8) is proven similarly to 7).

9) \[ \text{var } S : \ast_s | R, P, Q : \text{br}(S) \]
\[ \text{Notation } A := (R \circ P) \circ Q : \text{br}(S) \]
\[ \text{Notation } B := R \circ (P \circ Q) : \text{br}(S) \]
\[ \text{var } x, y : S \]
\[ \text{Notation } P_1(x, y) := \lambda z : S.(R \circ P)xz \land Qzy : S \rightarrow \ast_p \]
\[ \text{Notation } P_2(x, y) := \lambda z : S.Rxz \land (P \circ Q)zy : S \rightarrow \ast_p \]
\[ \text{Notation } P_3(x, y) := \lambda z : S.Rxz \land Pzy : S \rightarrow \ast_p \]
\[ \text{Notation } P_4(x, y) := \lambda z : S.Pxz \land Qzy : S \rightarrow \ast_p \]
\[ \text{var } x, y : S \mid u : Axy \]
\[ \text{u : } (\exists z : S.P_1(x, y)z) \]
\[ \text{var } z : S \mid v : P_1(x, y)z \]
\[ a_1 := \land \text{-el}_1((R \circ P)xz, Qzy, v) : (R \circ P)xz \]
\[ a_2 := \land \text{-el}_2((R \circ P)xz,Qzy,v) : Qzy \]
\[ a_1 : (\exists z_1 : S.P_3(x,z)z_1) \]
\textbf{var} \[ z_1 : S \mid w : P_3(x,z)z_1 \]
\[ w : Rxz_1 \land Pz_1z \]
\[ a_3 := \land \text{-el}_1(Rxz_1,Pz_1z,w) : Rxz_1 \]
\[ a_4 := \land \text{-el}_2(Rxz_1,Pz_1z,w) : Pz_1z \]
\[ a_5 := \text{prod-term} (S,P,Q,z_1,z,y,a_4,a_2) : (P \circ Q)z_1y \]
\[ a_6 := \text{prod-term} (S,R,(P \circ Q),x,z_1,y,a_3,a_5) : Bxy \]
\[ a_7 := \lambda z_1 : S.\lambda w : P_3(x,z)z_1.a_6 : (\forall z_1 : S.(P_3(x,z)z_1 \Rightarrow Bxy)) \]
\[ a_8 := \exists \text{-el} (S,P_3(x,z),a_1,Bxy,a_7) : Bxy \]
\[ a_9 := \lambda z : S.\lambda v : P_1(x,y)z.a_8 : (\forall z : S.(P_1(x,y)z \Rightarrow Bxy)) \]
\[ a_{10} := \exists \text{-el} (S,P_1(x,y),u,Bxy,a_9) : Bxy \]
\[ a_{11} := \lambda x,y : S.\lambda u : Axy.a_{10} : A \subseteq B \]
\textbf{var} \[ x,y : S \mid u : Bxy \]
\[ u : (\exists z : S.P_2(x,y)z) \]
\textbf{var} \[ z : S \mid v : P_2(x,y)z \]
\[ a_{12} := \land \text{-el}_1(Rxz,(P \circ Q)zy,v) : Rxz \]
\[ a_{13} := \land \text{-el}_2(Rxz,(P \circ Q)zy,v) : (P \circ Q)zy \]
\[ a_{13} := (\exists z_1 : S.P_4(z,y)z_1) \]
\textbf{var} \[ z_1 : S \mid w : P_4(z,y)z_1 \]
\[ w : Pz_1z \land Qz_1y \]
\[ a_{14} := \land \text{-el}_1(Pz_1z,Qz_1y,w) : Pz_1z \]
\[ a_{15} := \land \text{-el}_2(Pz_1z,Qz_1y,w) : Qz_1y \]
\[ a_{16} := \text{prod-term} (S,R,P,x,z,z_1,a_{12},a_{14}) : (R \circ P)xz_1 \]
\[ a_{17} := \text{prod-term} (S,R \circ P,Q,x,z_1,y,a_{16},a_{15}) : Axy \]
\[ a_{18} := \lambda z_1 : S.\lambda w : P_4(z,y)z_1.a_{17} : (\forall z_1 : S.(P_4(z,y)z_1 \Rightarrow Axy)) \]
\[ a_{19} := \exists \text{-el} (S,P_4(z,y),a_{13},Axy,a_{18}) : Axy \]
\[ a_{20} := \lambda z : S.\lambda v : P_2(x,y)z.a_{19} : (\forall z : S.(P_2(x,y)z \Rightarrow Axy)) \]
\[ a_{21} := \exists \text{-el} (S,P_2(x,y),u,Axy,a_{20}) : Axy \]
\[ a_{22} := \lambda x,y : S.\lambda u : Bxy.a_{21} : B \subseteq A \]
\[ a_{23} := \text{rel-equal}(A,B,a_{11},a_{22}) : (R \circ P) \circ Q = R \circ (P \circ Q) \]
\[ \square \]
Appendix B. Proof of Theorem 5.1

Proof. Each statement here is a bi-implication, so we use the proof term \textit{bi-impl} from Lemma 2.3.

1) \textbf{var} \ S : \ast_s | \ R : \text{br}(S)

Notation \( A := \text{refl}(S, R) : \ast_p \)

Notation \( B := \text{id}_s \subseteq R : \ast_p \)

\( u : A \)

\textbf{var} \( x, y : S | v : (id_S)xy \)

\( v : x =_S y \)

Notation \( P := \lambda z : S. Rxz : S \rightarrow \ast_p \)

\( ux : Px \)

\( a_1 := \text{eqsubs}(S, P, x, y, v, ux) : Py \)

\( a_1 : Rxy \)

\( a_2 := \lambda x, y : S. \lambda v : (id_S)xy. a_1 : (id_S \subseteq R) \)

\( a_2 : B \)

\( a_3 := \lambda u : A.a_2 : (A \Rightarrow B) \)

\( u : B \)

\textbf{var} \( x : S \)

\( a_4 := \text{eq-refl}(S, x) : x =_S x \)

\( a_4 : (id_S)xx \)

\( uxx : (id_S)xx \Rightarrow Rxx \)

\( a_5 := uxxa_4 : Rxx \)

\( a_6 := \lambda x : S.a_5 : (\forall x : S.Rxx) \)

\( a_6 : A \)

\( a_7 := \lambda u : B.a_6 : (B \Rightarrow A) \)

\( a_8 := \text{bi-impl}(A, B, a_3, a_7) : \text{refl}(S, R) \iff \text{id}_s \subseteq R \)

2) and 3) are proven together as follows.

\textbf{var} \ S : \ast_s | \ R : \text{br}(S)

Notation \( A := \text{sym}(S, R) : \ast_p \)

Notation \( B := R^{-1} \subseteq R : \ast_p \)

Notation \( C := R^{-1} = R : \ast_p \)

\( u : A \)

\textbf{var} \( x, y : S | v : R^{-1}xy \)

\( v : Ryx \)

\( uyx : (Ryx \Rightarrow Rxy) \)
\[ a_1 := uyxv : Rxy \]
\[ a_2 := \lambda x, y : S. \lambda u : R^{-1}xy.a_1 : (R^{-1} \subseteq R) \]
\[
\text{var} x, y : S \mid v : Rxy
\]
\[ uxy : (Rxy \Rightarrow Ryx) \]
\[ a_3 := uxyv : Ryx \]
\[ a_3 : R^{-1}xy \]
\[ a_4 := \lambda x, y : S. \lambda u : Rxy.a_3 : (R \subseteq R^{-1}) \]
\[ a_5 := \text{rel}-\text{equal}(S, R^{-1}, R, a_2, a_4) : R^{-1} = R \]
\[ a_6 := \lambda u : A. a_2 : A \Rightarrow B \]
\[ a_7 := \lambda u : A. a_5 : A \Rightarrow C \]

\[
\text{u} : B
\]
\[
\text{var} x, y : S \mid v : Rxy
\]
\[ v : R^{-1}yx \]
\[ uyx : (R^{-1}yx \Rightarrow Ryx) \]
\[ a_8 := uyxv : Ryx \]
\[ a_9 := \lambda x, y : S. \lambda v : Rxy.a_8 : \text{sym}(S, R) \]
\[ a_{10} := \lambda u : B. a_8 : (B \Rightarrow A) \]

\[
\text{u} : C
\]
\[ u : R^{-1} \subseteq R \land R \subseteq R^{-1} \]
\[ a_{11} := \land-\text{el}_1(R^{-1} \subseteq R, R \subseteq R^{-1}, u) : R^{-1} \subseteq R \]
\[ a_{11} : B \]
\[ a_{12} := a_{10}a_{11} : A \]
\[ a_{13} := \lambda u : C. a_{12} : (C \Rightarrow A) \]
\[ a_{14} := \text{bi}-\text{impl}(A, B, a_6, a_{10}) : \text{sym}(S, R) \Leftrightarrow R^{-1} \subseteq R \]

\text{sym-criterion}(S, R) := \text{bi}-\text{impl}(A, C, a_7, a_{13}) : \text{sym}(S, R) \Leftrightarrow R^{-1} = R

\]

4)
\[
\text{var} S : * \mid R : \text{br}(S)
\]
Notation \( A := \text{antisym}(S, R) : *_p \)
Notation \( B := R \cap R^{-1} \subseteq \text{id}_S : *_p \)
\[
\text{u} : A
\]
\[
\text{var} x, y : S \mid v : (R \cap R^{-1})xy
\]
\[ v : R^{-1}xy \land Rx \]
\[ a_1 := \land-\text{el}_1(R^{-1}xy, Rx, v) : R^{-1}xy \]
\[ a_2 := \land-\text{el}_2(R^{-1}xy, Rx, v) : Rx \]
\[ a_1 : Ryx \]
uxy : Rxy \Rightarrow Ryx \Rightarrow x = y \\
a_3 := uxya_2a_1 : (x = y) \\
a_3 : (ids)xy \\
a_4 := \lambda x, y : S.\lambda v : (R \cap R^{-1})xy.a_3 : (R \cap R^{-1} \subseteq ids) \\
a_4 : B \\
a_5 := \lambda u : A.a_4 : (A \Rightarrow B) \\
u : B \\
\text{var } x, y : S | v : Rxy | w : Ryx \\
w : R^{-1}xy \\
a_6 := \land \text{-in}_1(R^{-1}xy, Rxy, w, v) : (R^{-1} \cap R)xy \\
a_7 := uxya_6 : (idS)xy \\
a_7 : x = y \\
a_8 := \lambda x, y : S.\lambda v : Rxv.a_7 : \text{antisym}(S, R) \\
a_8 : A \\
a_9 := \lambda u : B.a_8 : (B \Rightarrow A) \\
a_{10} := bi-impl(A, B, a_5, a_9) : (\text{antisym}(S, R) \Leftrightarrow (R \cap R^{-1} \subseteq ids)) \\
5) \\
\text{var } S : * \mid R : br(S) \\
\text{Notation } A := trans(S, R) : *_p \\
\text{Notation } B := R \circ R \subseteq R : *_p \\
u : A \\
\text{var } x, y : S \\
\text{Notation } P := \lambda z : S.Rxz \land Rzy : S \rightarrow *_p \\
v : (R \circ R)xy \\
v : (\exists z : S.Pz) \\
\text{var } z : S | w : Pz \\
w : Rzx \land Rzy \\
a_1 := \land \text{-el}_{1}(Rzx, Rzy, w) : Rzx \\
a_2 := \land \text{-el}_{2}(Rzx, Rzy, w) : Rzy \\
a_3 := uxyzy_a_2 : Rxy \\
a_4 := \lambda z : S.\lambda w : Pz.a_3 : (\forall z : S.(Pz \Rightarrow Rxy)) \\
a_5 := \exists \text{-el}(S, P, v, Rxv, a_4) : Rxy \\
a_6 := \lambda x, y : S.\lambda v : (R \circ R)xy.a_5 : (R \circ R \subseteq R) \\
a_6 : B \\
a_7 := \lambda u : A.a_6 : (A \Rightarrow B)
\[ \begin{align*}
\text{\textbf{u : B}} \\
\text{\textbf{\texttt{var } x, y, z : S | v : Rxy | w : Ryz}} \\
\text{\texttt{a_8} := \texttt{prod-term}(S, R, R, x, y, z, v, w) : (R \circ R)xz} \\
\text{\texttt{a_9} := \texttt{uxz} : ((R \circ R)xz} \Rightarrow Rxz) \\
\text{\texttt{a_{10}} := a_9a_8} : Rxz \\
\text{\texttt{a_{11}} := \lambda x, y, z : S. \lambda v : Rxy. \lambda w : Ryz.a_{10}} : \texttt{trans(S, R)} \\
\text{\texttt{a_{11}} := a_{10}a_8} : Rxz \\
\text{\texttt{a_{12}} := \lambda u : B. a_{11}} : (B \Rightarrow A) \\
\text{\texttt{a_{13}} := \texttt{bi-impl}(A, B, a_7, a_{12})} : (\texttt{trans(S, R)} \Leftrightarrow (R \circ R \subseteq R))
\end{align*} \]

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