Deterministic Parameterized Algorithms for Matching and Packing Problems

Meirav Zehavi
Department of Computer Science, Technion - Israel Institute of Technology, Haifa 32000, Israel
meizeh@cs.technion.ac.il

Abstract. We present three deterministic parameterized algorithms for well-studied packing and matching problems, namely, Weighted q-Dimensional p-Matching ((q, p)-WDM) and Weighted q-Set p-Packing ((q, p)-WSP). More specifically, we present an $O^*(2.85043^{(q−1)p})$ time deterministic algorithm for (q, p)-WDM, an $O^*(8.04143^p)$ time deterministic algorithm for the unweighted version of (3, p)-WDM, and an $O^*((0.56201\cdot 2.85043^p)p)$ time deterministic algorithm for (q, p)-WSP. Our algorithms significantly improve the previously best known $O^*$ running times in solving (q, p)-WDM and (q, p)-WSP, and the previously best known deterministic $O^*$ running times in solving the unweighted versions of these problems. Moreover, we present kernels of size $O(c^q(p−1)^q)$ for (q, p)-WDM and (q, p)-WSP, improving the previously best known kernels of size $O(q^q(p−1)^q)$ for these problems.

1 Introduction

We consider the following well-studied matching and packing problems.

Weighted q-Dimensional p-Matching ((q, p)-WDM)

- Input: Pairwise disjoint universes $U_1, \ldots, U_q$, a set $S \subseteq U_1 \times \ldots \times U_q$, a weight function $w : S \to \mathbb{R}$, and a parameter $p$.
- Output: A subset $S' \subseteq S$ of $p$ disjoint tuples, which maximizes $\sum_{S \in S'} w(S)$.

Weighted q-Set p-Packing ((q, p)-WSP)

- Input: A universe $U$, a set $S$ of subsets of size $q$ of $U$, a weight function $w : S \to \mathbb{R}$, and a parameter $p$.
- Output: A subset $S' \subseteq S$ of $p$ disjoint sets, which maximizes $\sum_{S \in S'} w(S)$.

The q-Dimensional p-Matching ((q, p)-DM) problem is the special case of (q, p)-WDM in which all of the tuples in $S$ have the same weight. Similarly, the q-Set p-Packing ((q, p)-SP) problem is the special case of (q, p)-WSP in which all of the tuples in $S$ have the same weight. Note that (q, p)-WDM is a special case of (q, p)-WSP.

As noted by Chen et al. [2], matching and packing problems form an important class of NP-hard problems. In particular, the six “basic” NP-complete problems include 3-Dimensional Matching [10].

A parameterized algorithm solves an NP-hard problem by confining the combinatorial explosion to a parameter $k$. More precisely, a problem is fixed-parameter tractable (FPT) with respect to a parameter $k$ if an instance of size $n$ can be solved in time $O^*(f(k))$ for some function $f(k)$ [17]. A kernelization algorithm for a problem $P$ is a polynomial-time algorithm that, given an instance $x$ of $P$ and a parameter $k$, returns an instance $x'$ of $P$ whose size is bounded by some function $f(k)$, such that there is a solution to $x$ iff there is a solution to $x'$. We then say that $P$ has a kernel of size $f(k)$.

In this paper we present three deterministic parameterized algorithms and deterministic kernelization algorithms for (q, p)-WDM and (q, p)-WSP, where the parameter is $(p + q)$.

Prior Work and Our Contribution: A lot of attention has been paid to (q, p)-WDM and (q, p)-WSP. Tables 1 and 2 present a summary of parameterized algorithms for these problems. In particular, Chen et al. [2] gave a deterministic algorithm for (q, p)-WDM that runs in time $O^*(4^{(q−1)p+o(qp)})$. This algorithm has the previously best known $O^*$ running time for (q, p)-WDM (for any $q$), and the previously best known deterministic $O^*$ running time for (q, p)-DM (for any $q$). Our first result is a deterministic algorithm for (q, p)-WDM that runs in time $O^*(2.851^{(q−1)p})$. We thus achieve a significant improvement over the previously best known $O^*$ running time for (q, p)-WDM (for any $q$), and the previously best known deterministic $O^*$ running time for (q, p)-DM (for any $q$). Our second result is a deterministic

\footnote{O^* hides factors polynomial in the input size.}
algorithm for \((3, p)\)-DM, which further reduces the \(O^*\) running time of our first algorithm, when applied to \((3, p)\)-DM, from \(O^*(8.125p)\) to \(O^*(8.042p)\).

Chen et al. \cite{2} gave a randomized algorithm for \((q, p)\)-WSP that runs in time \(O^*(4^{(q-0.1)p+o(qp)})\), and Chen et al. \cite{3} gave a deterministic algorithm for \((q, p)\)-WSP that runs in time \(O^*(4^{(q-0.5)p+o(qp)})\). These algorithms have the previously best known \(O^*\) running time for \((q, p)\)-WSP (for any \(q\)) and the previously best known deterministic \(O^*\) running time for \((q, p)\)-SP (for any \(q\)). Our third result is a deterministic algorithm for \((q, p)\)-WSP that runs in time \(O^*((0.563 \cdot 2.851^q p)\)) where the special case of \((3, p)\)-WSP runs in time \(O^*(12.155^p)\). We thus achieve a significant improvement over the previously best known deterministic \(O^*\) running time for \((q, p)\)-WSP (for any \(q\)) and the previously best known deterministic \(O^*\) running time for \((q, p)\)-SP (for any \(q\)).

Assuming that \(q = O(1)\), Chen et al. \cite{3} gave kernels of size \(O(q^2 qp^3)\) for \((q, p)\)-WDM and \((q, p)\)-WSP. Fellows et al. \cite{5} gave kernels of size \(O(q^2 q(p-1)^q)\) for \((q, p)\)-DM and \((q, p)\)-SP, which can be extended to kernels of the same size for \((q, p)\)-WDM and \((q, p)\)-WSP. Dell et al. \cite{6} proved that \((q, p)\)-DM is unlikely to admit a kernel of size \(O(f(q)p^q\epsilon^{-r})\) for any function \(f(q)\) and \(\epsilon > 0\) (improving upon a result by Hermelin et al. \cite{11}). Our fourth result presents kernels of size \(O(e^q q(p-1)^q)\) for \((q, p)\)-WDM and \((q, p)\)-WSP.

**Organization:** Section 2 gives some background about representative sets and two related results by Fomin et al. \cite{9}. Sections 3, 4, and 5 present deterministic algorithms for \((q, p)\)-WDM, \((3, p)\)-DM and \((q, p)\)-WSP, respectively. Finally, Section 6 gives kernels for \((q, p)\)-WDM and \((q, p)\)-WSP, and uses them to improve the running times of the algorithms presented in the previous three sections.

## 2 Representative Sets

Recently, Fomin et al. \cite{9} presented two new efficient computations of representative sets, which they then used to design improved deterministic parameterized algorithms for “graph connectivity” problems such as \(k\)-Path (i.e., finding a path of length at least \(k\) in a given graph). Our algorithms rely on these results, which we present in this section.

**Definition 1.** Let \(U\) be a universe, \(s, r \in \mathbb{Z}\), and \(A\) be a set of triples \((X, S', W)\) s.t. \(X \subseteq U\), \(|X| = s\) and \(W \subseteq \mathbb{R}\).

We say that a subset \(\tilde{A} \subseteq A\) (max) \(r\)-represents \(A\) if for every \(Y \subseteq U\) s.t. \(|Y| \leq r\) the following holds: if there is \((X, S', W) \in A\) s.t. \(X \cap Y = \emptyset\), then there is \((X^*, S^*, W^*) \in \tilde{A}\) s.t. \(X^* \cap Y = \emptyset\) and \(W^* \geq W\).

By Section 4.2 in \cite{9}, we have a deterministic algorithm, that we call \(R\text{-Alg}(U, s, r, A)\), whose input, output and running time are as follows.

- **Input:** A universe \(U\), \(s, r \in \mathbb{Z}\), and a set \(A\) of triples \((X, S', W)\) s.t. \(X \subseteq U\), \(|X| = s\) and \(W \subseteq \mathbb{R}\).
- **Output:** A subset \(\tilde{A} \subseteq A\) s.t. \(|\tilde{A}| \leq \left(\frac{s+r}{s}\right)^{2s(s+r)} \log |U|\), which \(r\)-represents \(A\).
- **Running time:** \(O(|A|\left(\frac{s+r}{s}\right)^s \log |U|)\).

By Section 4.1 in \cite{9}, we have a deterministic algorithm, that we call \(K\text{-Alg}(U, s, r, A)\), whose input, output and running time are as follows.
– Input: A universe $U$, $s,r \in \mathbb{Z}$, and a set $A$ of tuples $(X,S',W)$ s.t. $X \subseteq U$, $|X|=s$ and $W \in \mathbb{R}$.
– Output: A subset $\hat{A} \subseteq A$ s.t. $|\hat{A}| \leq \binom{|s+r|}{s}$, which $r$-represents $A$.
– Running time: $O(|A|\binom{|s+r|}{s} \tilde{w}^{-1}\log(s||U||^2))$, where $\tilde{w}<2.373$ is the matrix multiplication exponent [20].

We also need the following observation from [9].

**Observation 1.** Let $U$ be a universe, $s,r \in \mathbb{Z}$, and $A, \hat{A}$ and $\tilde{A}$ be sets of triples $(X,S',W)$ s.t. $X \subseteq U$, $|X|=s$ and $W \in \mathbb{R}$. If $\hat{A}$ r-represents $A$ and $\tilde{A}$ r-represents $A$, then $\tilde{A}$ r-represents $A$.

### 3 An Algorithm for $(q,p)$-WDM

Let $< \leq$ be an order on $U_1$. Roughly speaking, the idea of the algorithm is to iterate over $U_1$ in an ascending order, such that when we reach an element $u \in U_1$, we have already computed representative sets of $r$-sets of "partial solutions" that include only tuples whose first elements are smaller than $u$. Then, we try to extend the "partial solutions" by adding tuples whose first element is $u$ and computing new representative sets accordingly. Note that the elements in $U_1$ that appear in the "partial solutions" do not appear in any tuple whose first element is at least $u$, and that any tuple whose first element is at least $u$ does not contain elements in $U_1$ that appear in the "partial solutions". This allows us to use "better" representative sets, which improves the running time of the algorithm.

We next give the notation used in this section. Then we describe the algorithm and give its pseudocode. Finally, we prove its correctness and running time.

**Notation:** Denote $U = U_1 \cup \ldots \cup U_q$. Let $u_s$ (resp. $u_q$) be the smallest (resp. greatest) element in $U_1$. Given $u \in U_1$, denote $S_u = \{S \in S : S$ includes $u\}$. Given a tuple $S$, let $\text{set}(S)$ be the set of elements in $S$, excluding its first element. Given a set of tuples $S'$, denote $\text{tri}(S') = (\bigcup_{S \in S'} \text{set}(S), S', \sum_{S' \in S'} w(S))$. Given a set of sets of tuples $S$, denote $\text{tri}(S) = \{\text{tri}(S') : S' \in S\}$. Given $S \subseteq S$ and $1 \leq j \leq q$, let $S_j$ denote the tuple including the first $j$ elements in $S$, and define $w(S_j) = w(S)$.

Given $u \in U_1$ and $1 \leq i \leq p$, let $SOL_{u,i}$ be the set of all sets of $i$ disjoint tuples in $S$ whose first elements are at most $u$ (i.e., $SOL_{u,i} = \{S' \subseteq \bigcup_{u' \in U_1} \text{s.t. } u' \leq u : |S'| = i, \text{the tuples in } S' \text{ are disjoint}\})$. Note that for all $(X,S',W) \in \text{tri}(SOL_{u,i})$, we have that $|X| = (q-1)i$. Given also $S \subseteq S_u$ and $1 \leq j \leq q$, let $SOL_{u,i,S,j}$ be the set of all sets of disjoint tuples that include $S_j$ and $i-1$ tuples in $S$ whose first elements are smaller than $u$ (i.e., $SOL_{u,i,S,j} = \{S' \subseteq \{S_j\} \cup \bigcup_{u' \in U_1} \text{s.t. } u' < u : S_j \in S'\}$).

| Reference     | Randomized | Deterministic | Variation | Running Time |
|---------------|------------|--------------|-----------|--------------|
| Chen et al. [3] | D          | (3, p)-SP    | $O^*(p^{O(p)})$ |
| Downey et al. [7] | D          | (3, p)-WSP   | $O^*(p^{O(p)})$ |
| Fellows et al. [8] | D          | (3, p)-WSP   | $O^*(2^{O(p)})$ |
| Liu et al. [15] | D          | (3, p)-WSP   | $O^*(2^{O(p)})$ |
| Koutis [12]   | D          | (3, p)-SP    | $O^*(2^{O(p)})$ |
| R            | (3, p)-SP  | $O^*(1.285.475^p)$ |
| Wang et al. [18] | D          | (3, p)-WSP   | $O^*(43.028^p)$ |
| Chen et al. [9] | D          | (3, p)-DM    | $O^*(21.907^p)$ |
| Liu et al. [16] | D          | (3, p)-SP    | $O^*(97.973^p)$ |
| D          | (3, p)-DM  | $O^*(21.254^p)$ |
| R            | (3, p)-DM  | $O^*(1.248^p)$ |
| Wang et al. [19] | D          | (3, p)-SP    | $O^*(97.973^p)$ |
| Chen et al. [2] | D          | (3, p)-WSP   | $O^*(32^{O(p)})$ |
| R            | (3, p)-WDM | $O^*(16^{O(p)})$ |
| R            | (3, p)-DM  | $O^*(2^{O(p)})$ |
| Koutis [13]  | D          | (3, p)-SP    | $O^*(8^p)$ |
| R            | (3, p)-SP  | $O^*(4^p)$ |
| Björklund et al. [1] | R  | (3, p)-SP    | $O^*(3.344^p)$ |
| R            | (3, p)-DM  | $O^*(2^p)$ |
| This paper   | D          | (3, p)-WSP   | $O^*(12.155^p)$ |
| D          | (3, p)-WDM | $O^*(8.128^p)$ |
| D          | (3, p)-DM  | $O^*(8.042^p)$ |

Table 2. Known parameterized algorithms for $(3,p)$-WDM and $(3,p)$-WSP.
We start by proving the following lemma regarding WDM-Add.

**Algorithm 1 WDM-Add**(U1, ..., Uq, S, w, p)

1: let M be a matrix that has a cell [u, i] for all u ∈ U1 and 1 ≤ i ≤ p, which is initialized to {}.
2: for all u ∈ U1 ascending do
3:    \( A \leftarrow \text{R-Alg}(U, q - 1, (q - 1)(p - 1), \text{tri}(\{S : S \in \bigcup_{u' \in U_1 \text{ s.t. } u' \leq u} S_{u'}\})) \).
4:    M[u, 1] \leftarrow \{S' : \exists X, W \text{ s.t. } (X, S', W) \in A\}.
5:    if u = u_s then skip the iteration. else let u' be the element preceding u in U1.
6:    for i = 2, ..., p do
7:        A \leftarrow \text{tri}(M[u', i]) \cup \bigcup_{S \in S_u} \text{WDM-Add}(i, S, M[u', i - 1])
8:        A \leftarrow \text{R-Alg}(U, (q - 1)i, (q - 1)(p - i), A).
9:    M[u, i] \leftarrow \{S' : \exists X, W \text{ s.t. } (X, S', W) \in A\}.
10: end for
11: end for
12: if M[u_q, p] = \emptyset then reject. else return S' ∈ M[u_q, p] that maximizes \( \sum_{S \in S'} w(S) \).

**Algorithm 1 WDM-Add**(i, S, S)

1: \( B_1 \leftarrow \{(X, S' \cup \{S_i\}, W + w(S)) : (X, S', W) \in \text{tri}(S)\} \).
2: for j = 2, ..., q do
3:    \( B_j \leftarrow \{(X \cup \{u_j\}, (S' \setminus \{S_{j - 1}\}) \cup \{S_j\}, W) : (X, S', W) \in B_{j - 1}, u_j \text{ is the } j^{th} \text{ element in } S, u_j \notin X\} \).
4:    \( B_j \leftarrow \text{R-Alg}(U, (q - 1)(i - 1) + (j - 1), (q - 1)(p - i) + (q - j), B_j) \).
5: end for
6: return \( B_q \).

**Correctness and Running Time:** We start by proving the following lemma regarding WDM-Add.

This approach results in a running time better than that achieved by adding all the elements of the tuple "at once" and only then using R-Alg.
**Lemma 1.** Given $2 \leq i \leq p$, $S \in S_u$ for some $u \in U_1$, and $S$ s.t. $\text{tri}(S)$ $(q-1)(p-i)-$represents $\text{tri}(\text{SOL}_u,i-1)$ where $u'$ is the element preceding $u$ in $U_1$, $\text{WDM-Add}$ returns a set that $(q-1)(p-i)-$represents $\text{tri}(\text{SOL}_u,i,S)$.

**Proof.** By using induction on $j$, we prove that for all $1 \leq j \leq q$, $\mathcal{B}_j ((q-1)(p-i) + (q-j))$-represents $\text{tri}(\text{SOL}_u,i,j)$. By Step 4 since $\text{tri}(S) (q-1)(p-i)-$represents $\text{tri}(\text{SOL}_u,i-1)$, we have that $\mathcal{B}_1 ((q-1)(p-i) + (q-1))$-represents $\text{tri}(\text{SOL}_u,i,1)$.

Next consider some $2 \leq j \leq q$, and assume that the claim holds for all $1 \leq j' < j$. By the definition of $R$-Alg, Observation 2, and Step 4, it is enough to prove that $\mathcal{B}_j ((q-1)(p-i) + (q-j))$-represents $\text{tri}(\text{SOL}_u,i,j)$. By the induction hypothesis and Step 3 we get that $\mathcal{B}_j \subseteq \text{tri}(\text{SOL}_u,i,j)$. Assume that there are $(X, S', W) \in \text{tri}(\text{SOL}_u,i,j)$ and $Y \subseteq U \setminus X$ s.t. $|Y| \leq ((q-1)(p-i) + (q-j))$, since otherwise the claim clearly holds. Let $u_j$ be the $j$th element in $S$. Note that $(X \setminus \{u_j\}, (S' \setminus \{S_j\}) \cup \{S_j-1\}, W) \in \text{tri}(\text{SOL}_u,i,j-1)$. Thus, by the induction hypothesis, there is $(X^*, S^*, W^*) \in \mathcal{B}_{i-1}$ s.t. $X^* \cap (Y \cup \{u_j\}) = \emptyset$ and $W^* \subseteq W$. We get that $(X^* \cup \{u_j\}, (S^* \setminus \{S_j-1\}) \cup \{S_j\}, W^*) \in \mathcal{B}_j$. Since $(X^* \cup \{u_j\}) \cap Y = \emptyset$ and $W^* \subseteq W$, we get that the claim holds. □

**Theorem 1.** $\text{WDM-Alg}$ solves $(q, p)$-$\text{WDM}$ in $O(2.85043^{(q-1)p}|S||U|\log^2 |U|)$ deterministic time. In particular, it solves $(3, p)$-$\text{WDM}$ in $O^*(8.12492^p)$ deterministic time.

**Proof.** The following lemma clearly implies the correctness of the algorithm.

**Lemma 2.** For all $u \in U_1$ and $1 \leq i \leq p$, $\text{tri}(M[u,i]) (q-1)(p-i)-$represents $\text{tri}(\text{SOL}_u,i)$.

**Proof.** We prove the lemma by using induction on the order of the computation of $M$. For all $u \in U_1$, $\text{SOL}_u = \{\{\cdot\} : S \in \bigcup_{u' \in U_1} \text{tri}(M[u',i])\}$, and thus, by the definition of $R$-Alg and Steps 3 and 4, $\text{tri}(M[u,i]) (q-1)(p-i)-$represents $\text{tri}(\text{SOL}_u,i)$. For all $2 \leq i \leq p$, $\text{SOL}_u,i = \{\}\$, and thus, by the initialization of $M$, $\text{tri}(M[u,i]) (q-1)(p-i)-$represents $\text{tri}(\text{SOL}_u,i)$.

Next consider an iteration of Step 4 that corresponds to some $u \in U_1 \setminus \{u_1\}$ and $2 \leq i \leq p$, and assume that the lemma holds for the element $u'$ preceding $u$ in $U_1$ and all $1 \leq i' < i$. By the definition of $R$-Alg, Observation 2, and Step 4, it is enough to prove that $A (q-1)(p-i)-$represents $\text{tri}(\text{SOL}_u,i)$. By the induction hypothesis, Step 4 and Lemma 2 we have that $A \subseteq \text{tri}(\text{SOL}_u,i)$. Assume that there are $(X, S', W) \in \text{tri}(\text{SOL}_u,i)$ and $Y \subseteq U \setminus X$ s.t. $|Y| \leq (q-1)(p-i)$, since otherwise the lemma clearly holds. We have two possible cases as follows.

1. $(X, S', W) \in A$ s.t. $X' \cap Y = \emptyset$ and $W^* \subseteq W$.

2. $(X, S', W) \in A$ s.t. $X' \cap Y = \emptyset$ and $W^* \subseteq W$.

We get that there is $(X^*, S^*, W^*) \in A$ s.t. $X^* \cap Y = \emptyset$ and $W^* \subseteq W$. □

By the definition of $R$-Alg and the pseudocode, the algorithm runs in time

$$O\left(\sum_{u \in U_1} \sum_{i=1}^{p} \sum_{j=1}^{q} \binom{(q-1)p}{(q-1)(i-1)+j-1} 2^{(q-1)p}|S||U|^{(q-1)(p-i)+q-j} \log^2 |U|\right) =$$

$$O\left(2^{(q-1)p}|S||U| \log^2 |U| \cdot \max_{t=0}^{(q-1)p} \left\{ \left(\frac{(q-1)p}{t}\right)^{(q-1)p-t} \right\} \right)$$

The maximum is achieved at $i = \alpha(q-1)p$, where $\alpha = 1 + \frac{1}{\sqrt{4e}}$. Thus, the running time of the algorithm is $O(2.85043^{(q-1)p}|S||U| \log^2 |U|)$. □

**4 An Algorithm for $(3, p)$-DM**

Roughly speaking, the algorithm is based on combining the following lemma from 3 with the algorithm presented in Section 3 as we next describe in more detail.
Lemma 3. If there is a solution to the input, then for any set $\mathcal{P} \subseteq \mathcal{S}$ of $p-1$ disjoint tuples, there is a solution to the input whose tuples contain at least $2(p-1)$ elements of tuples in $\mathcal{P}$.

Denote $U = U_1 \cup U_2 \cup U_3$, and let $< \in \mathcal{U}$ be an order on $U$. The algorithm first computes a set $\mathcal{P} \subseteq \mathcal{S}$ of $p-1$ disjoint tuples (by using recursion). By Lemma 3 there is $t \in \{1, 2, 3\}$ such that if there is a solution to the input, then there is a solution to the input whose tuples contain at least $\lceil 4(p-1)/3 \rceil$ elements in $U \setminus U_t$ that appear in (the tuples of) $\mathcal{P}$.

For each $t \in \{1, 2, 3\}$, the algorithm iterates over $U_t$ in an ascending order and over subsets of the set of elements in $U \setminus U_t$ that appear in $\mathcal{P}$, such that when we reach an element $u \in U_t$ and a subset $P$, we have already computed representative sets of sets of "partial solutions" that include only tuples whose $t^{th}$ elements are smaller than $u$ and whose set of elements in $U \setminus U_t$ that appear in $\mathcal{P}$ is a subset of $P$. Then, we try to extend the "partial solutions" by adding tuples whose $t^{th}$ element is $u$ and computing new representative sets accordingly. The representative sets do not need to hold information on elements in $U \setminus U_t$ that appear in $\mathcal{P}$ (we store the necessary information on such elements separately). Moreover, the elements in $U_t$ that appear in the "partial solutions" do not appear in any tuple whose $t^{th}$ element is at least $u$, and any tuple whose $t^{th}$ element is at least $u$ does not contain elements in $U_t$ that appear in the "partial solutions". We can thus use "better" representative sets, which improves the running time of the algorithm.

We next give the notation used in this section. We then describe the algorithm and give its pseudocode. Finally, we prove its correctness and running time.

Notation: Let $t \in \{1, 2, 3\}$ and $P_t \subseteq U \setminus U_t$. Let $u_t^1$ (resp. $u_t^2$) be the smallest (resp. greatest) element in $U_t$. Given $u \in U_t$ and $P \subseteq P_t$, denote $\mathcal{S}_{u,P,P_t} = \{S \subseteq \mathcal{S} : u \in P, S$ is the set of elements in $S$ that appear in $P_t \}$. Given $S \subseteq \mathcal{S}$, let $\text{set}_{t,P}(S)$ be the set of elements in $S$, excluding its $t^{th}$ element and elements that belong to $P_t$. Given $S' \subseteq S$, denote $\text{tri}_{t,P}(S') = (\cup_{S'' \subseteq S, \text{set}_{t,P}(S'')} \mathcal{S}_{u,P,P_t})$. The notation $\text{tri}_{t,P}(S)$ will not hold "too many" sets from $\mathcal{S}_{u,P,P_t}$. If there is a solution containing exactly $2$ tuples, then by using representative sets, $\text{DM-Alg}$ guarantees that each cell $M[u, i, P]$ will hold "enough" sets from $\mathcal{S}_{u,P,P_t}$, such that when the computation of $M$ is finished, $\cup_{P \subseteq P_t} M[u, i, P]$ will hold some $S' \subseteq \bigcup_{P \subseteq P_t} \mathcal{S}_{u,P,P_t}$ (clearly, such a set $S'$ is a solution). Moreover, by using representative sets, $\text{DM-Alg}$ guarantees that each cell $M[u, i, P]$ will not hold "too many" sets from $\mathcal{S}_{u,P,P_t}$, since then we will not get an improved running time.

The Algorithm: We now describe our algorithm for $(3, p)$-DM, that we call $\text{DM-Alg}$ (see the pseudocode below). In Step 2, $\text{DM-Alg}$ computes a set $\mathcal{P} \subseteq \mathcal{S}$ of $p-1$ disjoint tuples. Then, in Step 3, it iterates over each $t \in \{1, 2, 3\}$ and $r \in \{0, \ldots, \lceil (2p+4)/3 \rceil \}$, where $r$ notes the number of elements in $U \setminus U_t$ that do not appear in $\mathcal{P}$ and should appear in the currently desired solution. Next consider an iteration corresponding to such $t$ and $r$.

$\text{DM-Alg}$ introduces a matrix $M$, where each cell $M[u, i, P]$ will hold a subset of $\mathcal{S}_{u,P,P_t}$. It then iterates over $U_t$ in an ascending order and over every subset $P$ of $P_t$ s.t. $2 - r \leq |P| \leq 2p - r$. In each iteration, corresponding to such $u$ and $P$, $\text{DM-Alg}$ computes any cell of the form $M[u, i, P]$ s.t. $1 \leq i \leq p$ by using $M[u', i, P]$ and $M[u', i - 1, P]$ for all $P' \subseteq P$ (where $u'$ is the element preceding $u$ in $U_t$). In other words, for any $1 \leq i \leq p$, $\text{DM-Alg}$ computes a subset of $\mathcal{S}_{u,P,P_t}$ by using subsets of $\mathcal{S}_{u',P',P_t}$ and $\cup_{P' \subseteq P} \mathcal{S}_{u',P',i-1,P,P'}$. If there is a solution containing exactly $2p-r$ elements from $U \setminus U_t$ that appear in $P_t$, then by using representative sets, $\text{DM-Alg}$ guarantees that each cell $M[u, i, P]$ will hold "enough" sets from $\mathcal{S}_{u,P,P_t}$, such that when the computation of $M$ is finished, $\cup_{P \subseteq P_t} M[u, i, P]$ will hold some $S' \subseteq \cup_{P \subseteq P_t} \mathcal{S}_{u,P,P_t}$ (clearly, such a set $S'$ is a solution). Moreover, by using representative sets, $\text{DM-Alg}$ guarantees that each cell $M[u, i, P]$ will not hold "too many" sets from $\mathcal{S}_{u,P,P_t}$, since then we will not get an improved running time.

We now describe an iteration of Step 3 corresponding to some $u$ and $P$, in more detail. By using $\text{R-Alg}$, $\text{DM-Alg}$ first computes a set that $(r - (2 - |P|))$-represents $\text{tri}(\mathcal{S}_{u,1,P,P_t})$ (Step 4), and assigns its corresponding set of sets of tuples to $M[u, 1, P]$ (Step 7). If $u = u_s$, then $\mathcal{S}_{u_s,P,P_t}$ is empty for all $2 \leq i \leq p$, and $\text{DM-Alg}$ skips the rest of the iteration accordingly (thus $M[u_s, P]$ stays empty, as it is initialized, for all $2 \leq i \leq p$). Next assume that $u > u_s$, and consider an iteration of Step 10 corresponding to some $2 \leq i \leq p$. First, in Step 11 $\text{DM-Alg}$ computes a set that $(r - (2i - |P|))$-represents $\text{tri}(\mathcal{S}_{u,i,P,P_t})$ by using the sets in $M[u', i, P]$ and adding tuples in $\mathcal{S}_{u,P,P_t}$ to sets in $\mathcal{S}_{u',i-1,P,P_t}$ for all $P' \subseteq P$. Then, in Step 12 $\text{DM-Alg}$ uses $\text{R-Alg}$ to compute a representative set of the representative set it has just computed in Step 11 in order to reduce its size. Finally, in Step 13 $\text{DM-Alg}$ assigns the corresponding set of sets of tuples to $M[u, i, P]$.

Correctness and Running Time: We summarize in the following theorem.

Theorem 2. $\text{DM-Alg}$ solves $(3, p)$-DM in $O((8.04143p|S||U| \log^2 |U|)$ deterministic time.
Thus, there are 1
\[R-Alg\]
S
Thus, the following lemma implies the correctness of the algorithm.

We get that there is \((X, S', 1) \in A\), s.t. \(2 \leq |P| \leq 2p - r\) and \((X', S, 1) \in A\).

Next consider an iteration of Step 10 that corresponds to some \(u, i \in P\). Note that the definition of R-Alg and the pseudocode, the algorithm runs in time

\[O(3 \sum_{t=1}^{2p/3} \sum_{r=0}^{2p/3} \sum_{u \in U_i \cap P \subseteq P_t \text{ s.t. } 2 - r \leq |P| \leq 2p - r} \sum_{i=1}^{\left\lfloor \frac{|P|+r}{2r} \right\rfloor} \left(\left(\frac{r}{2i - |P|}\right)2^{p(r)}|S|\left(\frac{r}{r - (2i - |P|)}\right)^{r-(2i-|P|)}\log^2 |U|\right) = \]

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\[ O(4^p \cdot |S| |U| \log^2 |U|) \cdot \max_{t=0}^{[2p/3]} \left( \frac{[2p/3]}{t} \right)^{\left\lfloor \frac{2p/3}{t} \right\rfloor - [2p/3]} \cdot |U|^{\left\lfloor \frac{2p/3}{t} \right\rfloor - [2p/3]} - t) \]

The maximum is achieved at \( t = \alpha \cdot [2p/3] \), where \( \alpha = 1 + \frac{1}{2p/3} \). Thus, the running time of the algorithm is \( O(4^p \cdot 2.850423^{2p/3} |S| |U| \log^2 |U|) = O(8.04143^p \cdot |S| |U| \log^2 |U|) \).

5 An Algorithm for \((q,p)\)-WSP

Let \( \prec \) be an order on \( U \). Roughly speaking, the algorithm is based on combining the following lemma from [5] with the algorithm presented in Section 3, as we next describe in more detail.

**Lemma 5.** Let \( S' \subseteq S \), and denote \( S_{\min} = \{ u : \exists S \in S', \text{ s.t. } u \text{ is the smallest element in } S \} \). Then, any \( S \in S \) whose smallest element is greater than \( \max(S_{\min}) \) does not contain any element from \( S_{\min} \).

The algorithm iterates over \( U \) in an ascending order, such that when we reach an element \( u \in U \), we have already computed representative sets of sets of "partial solutions" that include only sets whose smallest elements are smaller than \( u \). Then, we try to extend the "partial solutions" by adding sets whose smallest element is \( u \) and computing new representative sets accordingly. By Lemma 5 the elements in \( U \) that are the smallest elements of sets in the "partial solutions" do not appear in any set whose smallest element is at least \( u \). This allows us to use "better" representative sets, which improves the running time of the algorithm. We note that the sets in the "partial solutions" can contain \( u \) (and elements greater than \( u \)); thus the running time of WDM-Alg (see Section 3) is better than the running time of the algorithm presented in this section.

We next give the notation used in this section. Since the algorithm is similar to WDM-Alg (see Section 3), we only give its pseudocode. Finally, we prove its correctness and running time.

**Notation:** Let \( u_u \) (resp. \( u_u \)) be the smallest (resp. greatest) element in \( U \). Given \( u \in U \), denote \( S_u = \{ S \in S : u \text{ is the smallest element in } S \} \). Given a set \( S \), denote \( w(S) \) be the set of elements in \( S \), excluding its smallest element. Given a set of sets \( S' \), denote \( \text{tri}(S') = \bigcup_{S \in S'} \text{set}(S), S' \), \( \sum S \in S \cdot w(S) \). Given a set of sets \( S \), denote \( \text{tri}(S) = \{ \text{tri}(S') : S' \in S \} \). Given \( S \in S \) and \( 1 \leq j \leq q \), let \( S_j \) denote the set including the \( j \) smallest elements in \( S \), and define \( w(S_j) \).

Given \( u \in U \) and \( 1 \leq i \leq p \), define \( \text{SOL}_{u,i} = \{ S' \subseteq \bigcup_{u \in U \cdot u \leq u} S' : |S'| = i, \text{ the sets in } S' \text{ are disjoint} \} \). Note that for all \( (X, S', W) \in \text{tri}(\text{SOL}_{u,i}) \), we have that \( |X| = (q-1)i \). Given also \( S_u \) and \( 1 \leq j \leq q \), define \( \text{SOL}_{u,i,S,j} = \{ S' \subseteq S_j \cup \bigcup_{u \in U \cdot u \leq u} S_u : S_j \in S', |S'| = i, \text{ the sets in } S' \text{ are disjoint} \} \). Note that for all \( (X, S', W) \in \text{tri}(\text{SOL}_{u,i,S,j}) \), we have that \( |X| = (q-1)(i-1) + j - 1 \).

**Algorithm:** The pseudocode of our algorithm for \((q,p)\)-WSP, called WSP-Alg, is given below.

**Algorithm 3** WSP-Alg \((U, S, w, p)\)

1. \( \text{let M be a matrix that has a cell } [u,i] \text{ for all } u \in U \text{ and } 1 \leq i \leq p, \text{ which is initialized to } \{ \} \).
2. for all \( u \in U \) ascending do
3. \( \hat{A} \leftarrow \text{R-Alg}(U, q-1, q(p-1)-\text{tri}([S] : S \in \bigcup_{u \in U \cdot u \leq u} S_u)) \).
4. \( M[u, 1] \leftarrow \{ S' : \exists X, W \text{ s.t. } (X, S', W) \in \hat{A} \} \).
5. if \( u = u_u \) then skip the iteration. else let \( u' \) be the element preceding \( u \) in \( U \).
6. for \( i = 2, \ldots, p \) do
7. \( \hat{A} \leftarrow \text{tri}(M[u', i]) \cup \bigcup_{S \in S_u} \text{WSP-Add}(i, S, M[u', i-1]) \).
8. \( \hat{A} \leftarrow \text{R-Alg}(U, q-1, q(p-1)-\hat{A}) \).
9. \( M[u, i] \leftarrow \{ S' : \exists X, W \text{ s.t. } (X, S', W) \in \hat{A} \} \).
10. end for
11. end for
12. if \( M[u_u, p] = \emptyset \) then reject. else return \( S' \in M[u_u, p] \) that maximizes \( \sum_{S \in S_u} w(S) \).

**Correctness and Running Time:** By using the new definitions of set() and tri(), the next lemma can be proved similarly to Lemma 3 (see Appendix A).

**Lemma 6.** Given \( 2 \leq i \leq p, S \in S_u \) for some \( u \in U \), and \( S \) s.t. \( \text{tri}(S) \cdot q(p-(i-1))-\text{represents} \), \( \text{tri}(\text{SOL}_{u,i-1}) \) where \( u' \) is the element preceding \( u \) in \( U \), WSP-Add returns a set that \( q(p-i)-\text{represents} \), \( \text{tri}(\text{SOL}_{u,i,S,q}) \).


Algorithm 3 WSP-Add(i, S, S)
1: \( B_1 \leftarrow \{(X, S' \cup \{S_1\}, W + w(S)) : (X, S', W) \in \text{tri}(S)\}, \) no set in \( S' \) includes the element in \( S_1 \).
2: for \( j = 2, \ldots, q \) do
3: \( B_j \leftarrow \{(X \cup \{u_j\}, (S' \setminus \{S_{j-1}\}) \cup \{S_j\}, W) : (X, S', W) \in B_{j-1}, u_j \) is the \( j^{\text{th}} \) smallest element in \( S, u_j \notin X \). \)
4: \( B_q \leftarrow \text{R-Alg}(U, (q-1)(i-1) + (j-1), q(p-i) + (q-j), B_j) \).
5: end for
6: return \( B_q \).

Theorem 3. WSP-Alg solves \((q, p)\)-WSP in \( O((0.56201 \cdot 2.85043^p)\|S\|U\|U_2 \log^2 |U|) \) deterministic time. In particular, it solves \((3, p)\)-WSP in \( O^*(12.15493^p) \) deterministic time.

Proof. By using the new definitions of set() and tri(), the next lemma, which clearly implies the correctness of the algorithm, can be proved similarly to Lemma 2 (see Appendix A).

Lemma 7. For all \( u \in U \) and \( 1 \leq i \leq p \), \( \text{tri}(M[u,i]) \) \( q(p-i) \)-represents \( \text{tri}(SOL_{u,i}) \).

Denote \( x = 2^{o(qp)}\|S\|U\log^2 |U| \). By the definition of R-Alg and the pseudocode, the algorithm runs in time

\[
O \left( \sum_{u \in U} \sum_{i=1}^{p} \sum_{j=1}^{q} \left( \frac{qp - i}{(q-1)(i-1) + j - 1} \right) \right) 2^{o(qp)}\|S\|U \left( \frac{qp - i}{qp - qi + q - j} \right)^{q - qi + q - j} \log^2 |U| =
\]

\[
O(x \cdot \max_{i=1}^{p} \max_{j=1}^{q} \left\{ \left( \frac{qp - i}{qi - i - q + j} \right) \left( \frac{qp - i}{qp - qi + q - j} \right)^{q - qi + q - j} \right\}) =
\]

\[
O(x \cdot \max_{i=1}^{q} \left\{ \left( \frac{qp - \left( \frac{t}{t/q} \right)}{t - \left( \frac{t}{t/q} \right)} \right) \left( \frac{qp - \left( \frac{t}{t/q} \right)}{qp - t} \right)^{qp - t} \right\}) = O(x \cdot \max_{i=1}^{q} \left\{ \left( \frac{(qp - \left( \frac{t}{t/q} \right))^{2q - t - \left( \frac{t}{t/q} \right)} \right) \left( \frac{(t - \left( \frac{t}{t/q} \right))^{t - \left( \frac{t}{t/q} \right)}(qp - t)^{2qp - 2t} \right) \right\}) =
\]

\[
O(x \cdot \max_{0 < \alpha < 1} \left\{ \left( \frac{qp - \alpha}{(qp - \alpha)^{2q - \alpha} - \alpha} \right) \left( \frac{qp - \alpha}{(qp - \alpha)^{2q - \alpha} - \alpha} \right)^{2qp - 2q} \right\}) =
\]

\[
O(x \cdot \max_{0 < \alpha < 1} \left\{ \left( \frac{(q - \alpha)^{2q - \alpha} - \alpha}{(q - \alpha)^{2q - \alpha} - \alpha} \right) \left( \frac{(q - \alpha)^{2q - \alpha} - \alpha}{(q - \alpha)^{2q - \alpha} - \alpha} \right)^{2q - \alpha} \right\}) =
\]

\[
O(x \cdot \max_{0 < \alpha < 1} \left\{ \left( \frac{\alpha^\alpha \left( \frac{1}{\alpha^\alpha (1 - \alpha)^{2q - \alpha}} \right)}{\alpha^\alpha \left( \frac{1}{\alpha^\alpha (1 - \alpha)^{2q - \alpha}} \right)} \right)^p \right\} = (*)
\]

When \( q = 3 \), the maximum of (*) is achieved at \( \alpha \approx 0.58226 \). Thus, WSP-Alg solves \((3, p)\)-WSP in \( O^*(12.15493^p) \) deterministic time. Now, note that

\[
(*) = O(x \cdot \max_{0 < \alpha < 1} \left\{ \left( \frac{\alpha^\alpha \left( \frac{1}{\alpha^\alpha (1 - \alpha)^{2q - \alpha}} \right)}{\alpha^\alpha \left( \frac{1}{\alpha^\alpha (1 - \alpha)^{2q - \alpha}} \right)} \right)^p \right\}
\]

As we increase \( q \), the \( \alpha \) for which we get the maximum decreases, staying greater than \( \alpha^* = 1 + \frac{1 - \sqrt{1 + 4e}}{2e} \) (since this \( \alpha^* \) maximizes \( \left( \frac{1}{\alpha^\alpha (1 - \alpha)^{2q - \alpha}} \right)^p \)). When \( q = 1, 500 \), the maximum of (*) is achieved at \( \alpha' < 0.550148 \), and thus when \( q \geq 1, 500 \), we get that WSP-Alg runs in time \( O(x \cdot \alpha^\alpha \left( \frac{1}{\alpha^\alpha (1 - \alpha)^{2q - \alpha}} \right)^p) = O(x \cdot (0.56201 \cdot 2.85043^p)) \). Since this expression bounds (*) for smaller values for \( q \), we get the desired running time. \( \square \)
6 Kernels for $(q,p)$-WDM and $(q,p)$-WSP

We first give the notation used in this section. Then we present our kernel for $(q,p)$-WDM, followed by our kernel for $(q,p)$-WSP. Finally, by using these kernels, we improve the running times (though not the $O^*$ running times) of the algorithms presented in the previous three sections. In this section, given an input to $(q,p)$-WDM or $(q,p)$-WSP, assume that any element in the universe(s) appears in some tuple∈set in $S$, since otherwise we can delete it.

**Notation:** Given a tuple or a set $S$, let set$(S)$ be the set of elements in $S$. Given a set of tuples or sets $S'$, denote tri$(S') = \{(\text{set}(S), \text{set}(w(S)) : S \in S')\}$.

**A Kernel for $(q,p)$-WDM:** We now present a kernelization algorithm, that we call WDM-Ker, for $(q,p)$-WDM (see the pseudocode below).

**Theorem 4.** Given an input $(U_1, \ldots, U_q, S,w,p)$ for $(q,p)$-WDM, WDM-Ker returns an input $(\hat{U}_1, \ldots, \hat{U}_q, \hat{S},w,p)$ for $(q,p)$-WDM, s.t. $\sum_{i=1}^{q}{|U_i|^q} \leq |\hat{S}|^q$, $|\hat{S}| = O(e^q(p-1)^q)$, and a set $S'$ solves $(U_1, \ldots, U_q, S,w,p)$ iff there is a solution $S''$ to $(\hat{U}_1, \ldots, \hat{U}_q, \hat{S},w,p)$ s.t. $\sum_{S'\subseteq S''}{w(S)} = \sum_{S'\subseteq S''}{w(S)}$. WDM-Ker runs in time $O(\min(|S|, e^q(p-1)^q))^{\frac{q}{q^2+1}} |S|^q \log |U|).

**Proof.** If $|S| \leq e^q(p-1)^q$, then by Step 1 the algorithm is clearly correct and runs in the desired time; thus next assume that $|S| > e^q(p-1)^q$.

By the definition of K-Alg and Steps 2-4 we get that $\sum_{i=1}^{q}{|U_i|^q} \leq |\hat{S}|^q$ and $|\hat{S}| \leq (\frac{q}{q^2+1})^q \leq O(e^q(p-1)^q)$. Moreover, we get that the algorithm runs in time bounded by

$$O(|S|^{-q^2+1} \log(q! \prod_{i=1}^{q}{|U|^q})) = O(|S|(e^q(p-1)^q)^{\frac{q}{q^2+1}} q^2 \log |U|).$$

By the definition of K-Alg and Steps 2-4 we get that $(\forall i \in \{1, \ldots, q\}): U_i \subseteq U_i$ and $\hat{S} \subseteq \hat{S}$. Thus, if $(U_1, \ldots, U_q,S,w,p)$ does not have a solution, then $(\hat{U}_1, \ldots, \hat{U}_q, \hat{S},w,p)$ does not have a solution, and if a set $\hat{S}$ is a solution to $(U_1, \ldots, U_q,S,w,p)$, then $(\hat{U}_1, \ldots, \hat{U}_q, \hat{S},w,p)$ does not have a solution $\hat{S}'$ s.t. $\sum_{S'\subseteq S''}{w(S)} < \sum_{S'\subseteq S''}{w(S)}$.

It is now enough to prove that given a solution $\hat{S}'$ to $(U_1, \ldots, U_q,S,w,p)$, there is a set of disjoint tuples $S'' \subseteq S'$ s.t. $\sum_{S'\subseteq S}{w(S)} \leq \sum_{S'}{w(S)}$. Consider the following lemma.

**Lemma 8.** Let $S' = \{S_1', \ldots, S_p'\}$ be a solution to $(U_1, \ldots, U_q,S,w,p)$. For all $i \in \{0, \ldots, p\}$, there is a set of disjoint tuples $S_i' = \{S_1', \ldots, S_i'\} \subseteq S'$ s.t. $\sum_{j=1}^{i}{w(S_j') \leq \sum_{j=1}^{i}{w(S_j')}$, whose tuples are disjoint from those in $\{S_1', \ldots, S_p'\}$.

**Proof.** We prove the lemma by using induction on $i$. The claim clearly holds for $i = 0$, since then we can choose $S_0' = \{\}$. Next consider some $i \in \{1, \ldots, p\}$ and assume that the claim holds for $i - 1$. By the induction hypothesis, there is a set of disjoint tuples $S'_{i-1} = \{S_1', \ldots, S_{i-1}'\} \subseteq S'$ s.t. $\sum_{j=1}^{i-1}{w(S_j') \leq \sum_{j=1}^{i-1}{w(S_j')}$, whose tuples are disjoint from those in $\{S_1', \ldots, S_p'\}$. By the definition of K-Alg and Steps 2 and 3 there is a tuple $S_i' \subseteq S'$ s.t. $w(S_i') \leq w(S_i')$, which is disjoint from the tuples in $\{S_1', \ldots, S_{i-1}', S_{i+1}', \ldots, S_p'\}$. Thus, by defining $S_i' = \{S_1', \ldots, S_i'\} \subseteq S'$, we conclude the lemma.

**Lemma 8** implying the existence of the required set, concludes the theorem.

**A Kernel for $(q,p)$-WSP:** By trivial modifications of WDM-Ker (see Appendix B), we get a kernelization algorithm, that we call WSP-Ker, which satisfies the following result.

**Theorem 5.** Given an input $(U,S,w,p)$ for $(q,p)$-WSP, WSP-Ker returns an input $(\hat{U}, S', \hat{S},w,p)$ for $(q,p)$-WSP, s.t. $|\hat{U}|^q \leq |\hat{S}|^q$, $|\hat{S}| = O(e^q(p-1)^q)$, and a set $S'$ solves $(U,S,w,p)$ iff there is a solution $S''$ to $(\hat{U}, S', \hat{S},w,p)$ s.t. $\sum_{S'\subseteq S''}{w(S)} = \sum_{S'\subseteq S''}{w(S)}$. WSP-Ker runs in time $O(\min(|S|, e^q(p-1)^q))^{\frac{q}{q^2+1}} |S|^q \log |U|).$
Improving the Running Times of WDM-Alg, DM-Alg and WSP-Alg: Since $e^q(p-1)^q = O(2^{O(q \log p)}) = O(2^{(q \log p)})$, Theorems 1 and 2 imply the following results.

- $(q, p)$-WDM can be solved in $O(2^{(q \log p)} |S| \log |U| + 2.85043^{(q-1)p})$ deterministic time. In particular, $(3, p)$-WDM can be solved in $O(2^{(q \log p)} |S| \log |U| + 8.12492^p)$ deterministic time.
- $(3, p)$-DM can be solved in $O(2^{(q \log p)} |S| \log |U| + 8.04143^p)$ deterministic time.
- $(q, p)$-WSP can be solved in $O(2^{(q \log p)} |S| \log |U| + (0.56201 \cdot 2.85043^p)^p)$ deterministic time. In particular, $(3, p)$-WSP can be solved in $O(2^{(q \log p)} |S| \log |U| + 12.15493^p)$ deterministic time.

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A Some Proofs

A.1 Proof of Lemma 6

By using induction on $j$, we prove that for all $1 \leq j \leq q$, $\hat{B}_j (q(p-i) + (q-j))$-represents tri$(S_{u,i,S,j})$. By Step 1, since tri$(S) q(p-(i-1))$-represents tri$(S_{u,i-1,1})$, we have that $\hat{B}_1 (q(p-i)+(q-1))$-represents tri$(S_{u,i,S,1})$.

Next consider some $2 \leq j \leq q$, and assume that the lemma holds for all $1 \leq j' < j$. By the definition of R-Alg, Observation 1 and Step 2 it is enough to prove that $B_j (q(p-i) + (q-j))$-represents tri$(S_{u,i-1,S,j})$.

By the induction hypothesis and Step 3 we get that $B_j \subseteq$ tri$(S_{u,i,S,j})$. Assume that there are $(X, S', W) \in$ tri$(S_{u,i,S,j})$ and $Y \subseteq U \setminus X$ s.t. $|Y| \leq (q(p-i) + (q-j))$, since otherwise the claim clearly holds. Let $u_j$ be the $j^i$th smallest element in $S$. Note that $(X \setminus \{u_j\}, \{S' \setminus \{S_j\} \cup \{S_{j-1}\}, W) \in$ tri$(S_{u,i,S,j-1})$. Thus, by the induction hypothesis, there is $(X^*, S^*, W^*) \in \hat{B}_{j-1}$ s.t. $X^* \cap (Y \cup \{u_j\}) = \emptyset$ and $W^* \geq W$. We get that $(X^* \cup \{u_j\}, \{S' \setminus \{S_{j-1}\} \cup \{S_j\}, W^*) \in B_j$. Since $(X^* \cup \{u_j\}) \cap Y = \emptyset$ and $W^* \geq W$, we get that the claim holds. \qed
A.2 Proof of Lemma 7

We prove the lemma by using induction on the order of the computation of M. For all $u \in U$, $SOL_{u,1} = \{S : S \in \bigcup_{u' \in U} \text{ s.t. } u' \lesssim u \}$, and thus, by the definition of R-Alg and Steps 3 and 4, tri($M[u,1]$) $q(p-1)$-represents tri($SOL_{u,1}$). For all $2 \leq i \leq p$, $SOL_{u,i} = \emptyset$, and thus, by the initialization of M, tri($M[u,i]$) $q(p-i)$-represents tri($SOL_{u,i}$).

Next consider an iteration of Step 6 that corresponds to some $u \in U \{u_0\}$ and $2 \leq i \leq p$, and assume that the lemma holds for the element $u'$ preceding $u$ in $U$ and all $1 \leq i' \leq i$. By the definition of R-Alg, Observation 1, and Steps 8 and 9, it is enough to prove that $A \leq \{S : \sum S \}$, the induction hypothesis, there is a set of disjoint sets $A$.

Observation 1 and Steps 8 and 9, it is enough to prove that $A \leq \{S : \sum S \}$, then (

We have two possible cases as follows.

1. $S' \cap \bigcup_{u} = \emptyset$. Note that $S' \in R_{u'}$, and thus, by the induction hypothesis, there is a set of disjoint sets $A$.

2. $S' \cap \bigcup_{u} = \{S\}$ for some $S$. Note that $S' \in R_{u,i,S,q}$, thus, by the induction hypothesis and Lemma 7, WSP-Add($i, S, M[u',i-1]$) returns a set that includes a triple $(X, S, W, p)$ s.t. $X \cap Y = \emptyset$ and $W' \geq W$; and therefore $(X, S, W') \in A$.

We get that there is $(X, S, W^*) \in A$ s.t. $X \cap Y = \emptyset$ and $W^* \geq W$.

B A Kernel for $(q,p)$-WSP

We now present a kernelization algorithm, that we call WSP-Ker, for $(q,p)$-WSP (see the pseudocode below).

Algorithm 4 WSP-Ker($U, S, w, p$)

1: if $|S| \leq e^q(p-1)^q$ then return $\{U, S, w, p\}$.
2: $A \Leftarrow K-Alg(U, q, p-1, \text{tri}(|S|))$.
3: $U^* \Leftarrow \{u \in U : \exists (X, S', W) \in A \text{ s.t. } u \in X\}$.
4: return $(U^*, \{S : \exists X, W \text{ s.t. } (X, S, W) \in A\}, w, p)$.

Proof (Theorem 2). If $|S| \leq e^q(p-1)^q$, then by Step 1, the algorithm is clearly correct and runs in the desired time; thus next assume that $|S| > e^q(p-1)^q$.

By the definition of K-Alg and Steps 2, we get that $\sum |U^*| \leq q|S^*|$ and $|S^*| \leq (\frac{e^q}{q}) = O(e^q(p-1)^q)$. Moreover, we get that the algorithm runs in time bounded by

$$O(|S| (\frac{e^q}{q})^{\frac{1}{q}} \log(q(|U|^{\frac{1}{q}}))) = O(|S| e^q(p-1)^q)$$

By the definition of K-Alg and Steps 2, we get that $|U^*| \leq U$ and $S^* \subseteq S$. Thus, if $(U, S, w, p)$ does not have a solution, then $(U^*, S^*, w, p)$ does not have a solution, and if a set $S'$ is a solution to $(U, S, w, p)$, then $(U^*, S^*, w, p)$ does not have a solution.

It is now enough to prove that given a solution $S'$ to $(U, S, w, p)$, there is a set of disjoint sets $S'' \subseteq S^*$ s.t. $\sum_{S \in S''} w(S) < \sum_{S \in S'} w(S)$.

Consider the following lemma.

Lemma 9. Let $S' = \{S'_1, \ldots, S'_p\}$ be a solution to $(U, S, w, p)$. For all $i \in \{0, \ldots, p\}$, there is a set of disjoint sets $S^i = \{S'_1, \ldots, S'_i\} \subseteq S^*$ s.t. $\sum_{j=1}^{i} w(S'_j) \leq \sum_{j=1}^{i} w(S'_j)$, whose sets are disjoint from those in $\{S'_i, \ldots, S'_p\}$.

Proof. We prove the lemma by using induction on $i$. The claim clearly holds for $i = 0$, since then we can choose $S'_0 = \emptyset$. Next consider some $i \in \{1, \ldots, p\}$ and assume that the claim holds for $i - 1$. By the induction hypothesis, there is a set of disjoint sets $S'_{i-1} = \{S'_1, \ldots, S'_{i-1}\} \subseteq S^*$ s.t. $\sum_{j=1}^{i-1} w(S'_j) \leq \sum_{j=1}^{i-1} w(S'_j)$, whose sets are disjoint from those in $\{S'_i, \ldots, S'_p\}$. By the definition of K-Alg and Steps 2, there is a set $S'_i \subseteq S^*$ s.t. $w(S'_i) \leq w(S'_i)$, which is disjoint from the sets in $\{S'_1, \ldots, S'_{i-1}, S'_{i+1}, \ldots, S'_p\}$. Thus, by defining $S'_i = \{S'_1, \ldots, S'_i\} \subseteq S^*$, we conclude the lemma.

Lemma 9 implying the existence of the required set, concludes the theorem. □