CHARACTER RINGS AND FUSION ALGEBRAS

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Abstract. We present an overview of the close analogies between the character rings of finite groups and the fusion rings of rational conformal models, which follow from general principles related to orbifold deconstruction.

1. Introduction

Analogies between group representation theory and 2D conformal field theory have been noticed by several authors over the years. Some of these have a natural interpretation because of the group theoretic origin of the relevant conformal models (e.g. WZNW models based on affine Lie algebras [4, 14], or holomorphic orbifolds based on finite groups [7]), but in other cases the relation is less obvious. A new approach to the subject is provided by recent advances in orbifold deconstruction, and the aim of the present note is to give a sketchy overview of the relevant results.

Orbifold deconstruction [1, 3] is a procedure aimed at recognizing whether a given 2D conformal model is an orbifold [8, 11] of another one, and if so, to identify (up to isomorphism) the relevant twist group and the original model. The basic ideas have been described in [1, 3], focusing on the conceptual issues, while (part of) the relevant mathematical details have been discussed in [2]. The basic observation is that every orbifold has a distinguished set of primaries, the so-called vacuum block, consisting of the descendants of the vacuum, and that this vacuum block has quite special properties: it is closed under the fusion product, and all its elements have integral conformal weight and quantum dimension. Such sets of primaries, called twisters because of their relation to twisted boundary conditions, provide the input for the deconstruction procedure: each twister corresponds to a different deconstruction, possibly with a different twist group and/or deconstructed model.

Most of the basic notions related to twisters may be formulated in the much more general setting of sets of primaries closed under the fusion product [2], called FC sets for short. As it turns out, these behave in many ways as character rings of finite groups, especially those – the integral ones – all of whose elements have integral quantum dimension. In particular, one may show that the collection of all FC sets of a conformal model form a modular lattice (with the integral ones forming

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a sublattice), allowing the generalization to FC sets of such group theoretic notions as nilpotency, solubility, being Abelian, and so on. Moreover, there is a well-defined notion of center and of central extensions, which fit perfectly in the above mentioned analogy with group theory. Of course, all this is completely natural for twisters, which are nothing but character rings of twist groups according to the general principles of orbifold deconstruction, but their meaning for general FC sets still needs to be clarified. In the sequel, we shall try to sketch the highlights of this circle of questions.

2. FC sets and their classes

Let’s consider a rational unitary conformal model \([9, 12]\): we’ll denote by \(d_p\) and \(h_p\) the quantum dimension and conformal weight of a primary \(p\), and by \(\mathcal{N}(p)\) the associated fusion matrix, whose matrix elements are given by the fusion rules

\[ [\mathcal{N}(p)]^r_q = N^r_{pq} \]  

The fusion matrices generate a commutative matrix algebra over \(\mathbb{C}\), the Verlinde algebra \(\mathcal{V}\), whose irreducible representations \(\rho_p\), all of dimension 1, are in one-to-one correspondence with the primaries of the model.

An FC set is a set \(g\) of primaries containing the vacuum that is fusion closed, which means that

\[ \sum_{\gamma \in g} N^-_{\alpha \beta} d_\gamma = d_\alpha d_\beta \]  

for all \(\alpha, \beta \in g\). Taking into account that quantum dimensions are positive numbers, this is tantamount to the requirement that \(N^-_{\alpha \beta} > 0\) and \(\alpha, \beta \in g\) implies \(\gamma \in g\).

The fusion matrices \(\mathcal{N}(\alpha)\) for \(\alpha \in g\) generate a subalgebra \(\mathcal{V}_g\) of the Verlinde algebra. Because \(\mathcal{V}\) is commutative, the irreducible representations of this subalgebra coincide with the different restrictions of the representations \(\rho_p\) of \(\mathcal{V}\). A \(g\)-class is defined to be the set \(\mathcal{C}\) of all those primaries \(p\) whose associated representations \(\rho_p\) coincide when restricted to the subalgebra \(\mathcal{V}_g\); we shall denote by \(\rho_{\mathcal{C}}\) this common restriction. Clearly, the collection \(\mathcal{C}(g)\) of \(g\)-classes (whose cardinality equals that of \(g\)) provides a partition of the set of all primaries.

The first analogy with character rings of finite groups comes from the existence of the orthogonality relations

\[ \sum_{\mathcal{C} \in \mathcal{C}(g)} \alpha(\mathcal{C}) \overline{\beta(\mathcal{C})} = \begin{cases} 1 & \text{if } \alpha = \beta; \\ 0 & \text{otherwise} \end{cases} \]  

for \(\alpha, \beta \in g\), and

\[ \sum_{\alpha \in g} \alpha(\mathcal{C}_1) \overline{\alpha(\mathcal{C}_2)} = \begin{cases} [\mathcal{C}_1] & \text{if } \mathcal{C}_1 = \mathcal{C}_2; \\ 0 & \text{otherwise} \end{cases} \]
for \( C_1, C_2 \in \mathcal{C}(g) \), where
\[
[C] = \frac{\sum_{p \in C} d_p^2}{\sum_{p \in C} d_p^2} \tag{2.5}
\]
is the *extent* of the class \( C \in \mathcal{C}(g) \), and \( \alpha(C) = p_C(\alpha) \).

The class containing the vacuum primary is of special importance: we shall denote it by \( g^\perp \) and call it the *trivial class*. Note that \( \alpha(g^\perp) = d_\alpha \) for \( \alpha \in g \), and
\[
[g^\perp] = \sum_{\alpha \in g} d_\alpha^2 \tag{2.6}
\]

One can show that the trivial class maximizes both extent and the product of size and extent, i.e. \( [C] \leq [g^\perp] \) and \( |C|[C] \leq |g^\perp|[g^\perp] \) for every class \( C \in \mathcal{C}(g) \).

A most important property of the trivial class is the *product rule*: if \( N_{pq}^r > 0 \) for some \( p \in g^\perp \), then the primaries \( q \) and \( r \) belong to the same \( g \)-class. It follows at once that the trivial class \( g^\perp \) is itself an FC set, the *dual* of \( g \), hence all previous notions and results go over verbatim to it. In particular, the set of all primaries is partitioned into \( g^\perp \)-classes, which we shall call \( g \)-blocks (or simply blocks) to avoid confusion with \( g \)-classes. It follows from what has been said for classes that the set \( B\ell(g) \) of \( g \)-blocks provides a partition of the set of all primaries. A simple argument shows that the primaries \( p \) and \( q \) belong to the same \( g \)-block iff \( N_{qp}^r > 0 \) for some \( \alpha \in g \), and in particular, the \( g \)-block containing the vacuum is \( g \) itself, that is \( (g^\perp)^+ = g \). This illustrates the inherent duality of FC sets: \( g \) and its dual \( g^\perp \) determine each other, while their extents are, roughly speaking, reciprocal, since the product \( [g][g^\perp] \) can be shown to be the same for every FC set \( g \). This duality implies that any result proven about classes gives a corresponding result about blocks, and *vice versa*, a seemingly trivial observation that turns out to be quite useful.

The inclusion relation makes the collection \( \mathcal{L} \) of FC sets partially ordered, with maximal element the FC set containing all primaries, and minimal element the trivial FC set consisting of the vacuum primary solely. Because the intersection of two FC sets is obviously an FC set again, \( \mathcal{L} \) is actually a finite lattice, and one may show that the join \( g \vee h \) of the FC sets \( g \) and \( h \) (the smallest FC set that contains both of them) is the dual of the intersection of their duals, i.e.
\[
g \vee h = (g^\perp \cap h^\perp)^\perp \tag{2.7}
\]
If \( g \) and \( h \) are FC sets such that \( h \subseteq g \), then every \( h \)-class is a union of \( g \)-classes, in particular \( g^\perp \subseteq h^\perp \), and every \( g \)-block is a union of \( h \)-blocks; moreover, the number of \( g \)-classes contained in \( h^\perp \) equals the number of \( h \)-blocks contained in \( g \). It follows that the map sending each FC set to its dual is an isomorphism between the lattice \( \mathcal{L} \) and its dual. An important consequence of the above results, crucial from...
the viewpoint of orbifold deconstruction, is that \( \mathcal{L} \) is a modular (even Arguesian) lattice, but usually neither distributive nor complemented. A better understanding of the lattice theoretic properties of \( \mathcal{L} \) would be highly desirable.

For an FC set \( g \), the of restriction of the indices of the fusion matrices \( N(\alpha) \) to the primaries belonging to a given block \( b \in \mathcal{B}(g) \) results in non-negative integer matrices \( N_b(\alpha) \) that form a representation \( \Delta_b \) of the subalgebra \( \mathcal{V}_g \). This representation decomposes into a direct sum of the irreducible representations \( \rho_c \), and the overlap \( \langle b, c \rangle \) of \( b \) with the class \( c \in \mathcal{C}(g) \) is defined as the multiplicity of \( \rho_c \) in the irreducible decomposition of \( \Delta_b \). The overlap \( \langle b, c \rangle \) may be shown to equal the rank of the minor of the modular \( S \)-matrix obtained by restricting the row indices to \( b \in \mathcal{B}(g) \) and the column indices to \( c \in \mathcal{C}(g) \). Alternatively, one has the expression

\[
\langle b, c \rangle = \sum_{p \in b} \sum_{q \in c} |S_{pq}|^2 \tag{2.8}
\]

In particular, this means that \( \langle b, c \rangle = 0 \) implies \( S_{pq} = 0 \) for all \( p \in b \) and \( q \in c \), explaining the appearance of large blocks of zeroes in the modular \( S \)-matrix of many conformal models. Let’s note that for twistors (to be introduced in Section 4) the overlap has another, more profound group theoretic interpretation \([1, 3]\).

3. Central extensions

As mentioned previously, the extent \( [c] \) of a class \( c \in \mathcal{C}(g) \) is bounded from above by the extent of the trivial class: \( [c] \leq [g^+]. \) Those classes that saturate this bound – the central ones – form the center \( Z(g) = \{ z \in \mathcal{C}(g) \mid [z] = [g^+] \} \) of the FC set \( g \in \mathcal{L}. \) Since \( g^+ \in Z(g) \), the center is never empty, and a straightforward argument shows that \( z \in Z(g) \) iff \( |\alpha(z)| = d_\alpha \) for all \( \alpha \in g. \) One may show that \( Z(g) = g \), i.e. all classes are central precisely when \( d_\alpha = 1 \) for all \( \alpha \in g. \) Such FC sets are called (for obvious reasons) Abelian; in the language of 2D CFT, they correspond to groups of simple currents.

The following generalization of the product rule holds: if the primary \( p \) belongs to the class \( c \in \mathcal{C}(g) \) and \( q \) belongs to the central class \( z \in Z(g) \), then all primaries \( r \) for which \( N_{pq}^r > 0 \) belong to the same \( g \)-class, denoted \( zc. \) What is more, if \( z_1, z_2 \in Z(g) \) are central classes, then \( z_1 z_2 \) is also central, and \( z_1 z_2 = z_2 z_1 \), hence the center \( Z(g) \) of an FC set \( g \in \mathcal{L} \) is an Abelian group, and since \( (z_1 z_2) c = z_1 (z_2 c) \) for any class \( c \in \mathcal{C}(g) \), the center permutes the \( g \)-classes.

For a subgroup \( Z < Z(g) \) of the center, \( g/Z = \{ \alpha \in g \mid \alpha(z) = d_\alpha \) for all \( z \in Z \} \) is again an FC set, the central quotient of \( g \) by \( Z. \) The usefulness of this notion rests on the explicit knowledge of its structure, for one has complete control over its
classes and blocks in terms of those of \( g \), in particular its dual is given by

\[
(g/Z)^\perp = \bigcup_{z \in Z} z
\]  

(3.1)

Moreover, if \( h \in \mathcal{L} \) is such that \( g/Z \subseteq h \subseteq g \), then \( h = g/H \) for a suitable subgroup \( H < Z \). It follows that there is an order reversing one-to-one correspondence between central quotients of \( g \in \mathcal{L} \) and subgroups of its center \( Z(g) \).

Given an FC set \( g \), it is natural to ask whether it is a central quotient of another FC set. Given an Abelian group \( A \), an \( A \)-extension of \( g \) is an FC set \( h \in \mathcal{L} \) such that \( h/Z = g \) for some central subgroup \( Z < Z(h) \) isomorphic to \( A \). One may show that the different \( A \)-extensions of \( g \) are in one-to-one correspondence with subgroups of \( Z(g^\perp) \) isomorphic to \( A \), and in particular, every \( g \in \mathcal{L} \) has a maximal central extension, namely \( (g^\perp/Z(g^\perp))^\perp \), the dual of the maximal central quotient of \( g^\perp \).

Note that, while central quotients are related to groups of central classes, central extensions have a similar relation to groups of central blocks, i.e. blocks \( b \in \mathcal{B}(g) \) that satisfy \([b] = [g]\), which form the center \( Z(g^\perp) \) of the dual of \( g \). An interesting observation is that any class (hence any block) that contains a simple current (a primary of dimension 1) is automatically central, but the converse need not be true.

4. Local FC sets

It follows from the results discussed in Section 2 that every \( g \)-class is a union of \( g \)-blocks precisely when \( g \subseteq g^\perp \). It turns out that such FC sets play a basic role in orbifold deconstruction [1, 3], hence they deserve a special name: we’ll call them local FC sets. The point is that the vacuum block of an orbifold model (the set of primaries that originate from the vacuum) is an FC set whose classes correspond to the different twisted sectors, i.e. collections of twisted modules whose twist element belong to the same conjugacy class, while its blocks correspond to orbits of twisted modules. Since the conjugacy class of a twist element is the same for all twisted modules on the same orbit, every block should be included in a well-defined class, hence the vacuum block, considered as an FC set, should be local by the above. Note that, while the intersection of local FC sets is clearly local, this is not necessarily the case for their join, hence they do not form a lattice.

Actually, the vacuum block of an orbifold belongs to a special class of local FC sets, termed twisters because of their relation to twisted boundary conditions, characterized by all of their elements having integral conformal weight. Indeed, since elements of the vacuum block descend from the vacuum primary, they all have integral conformal weight, hence the vacuum block is necessarily a twister.

One may show that a local FC set is either itself a twister, or a \( \mathbb{Z}_2 \)-extension of a twister. More precisely, every local FC set that is not a twister has a distinguished
central class $R$, the so-called Ramond class, such that the corresponding central quotient is a twister. The rationale for this appellation is that, in a suitable fermionic generalization of orbifold deconstruction, the blocks contained in the trivial class account for the Neveu-Schwarz (bosonic) sector of the deconstructed model, while those contained in the Ramond class describe the fermionic (Ramond) sector. This is substantiated by the observation that a block is contained in the Ramond class precisely when the conformal weights of its elements differ by integers, and that the number of blocks contained in the trivial and Ramond classes (the number of bosonic and fermionic degrees of freedom) are equal.

Local FC sets have many striking properties. For example, one may show that all of their elements have (rational) integer dimension, and either integer or half-integer conformal weight\(^1\). Ultimately, all this follows from the observation that, if the primaries $\alpha, \beta$ belong to a local FC set and $\mathcal{N}^\gamma_{\alpha\beta} > 0$, then the conformal weight $h_\gamma$ differs by an integer from the sum $h_\alpha + h_\beta$. From a categorical point of view this means that the elements of a local FC set are the simple objects of a Tannakian subcategory of the modular tensor category associated to the model. According to a result of Deligne [6], such a category is (tensor-)equivalent to the representation category of some finite group, hence the associated subalgebra may be identified with the character ring of that group. This has a natural interpretation in terms of orbifold deconstruction: the elements of the vacuum block correspond to irreducible representations of the twist group, hence their fusion rules describe the decomposition of tensor products of the latter, implying that the subalgebra $\mathcal{V}_g$ of the Verlinde algebra is nothing but the character ring of the twist group.

It follows from the above considerations that results from character theory [13, 15, 16] should go over to local FC sets, and this observation allows the generalization of many group theoretic concepts to arbitrary FC sets, providing a host of non-trivial conjectural results that seem to hold in greater generality. In this vein one can generalize [2] to FC sets such concepts as nilpotency, (super)solubility, and so on. For example, an FC set $g$ is nilpotent if it can be obtained from the trivial FC set by a sequence of central extensions. The rationale for this terminology is that if $g$ is local, hence the algebra $\mathcal{V}_g$ is isomorphic to the character ring of some finite group $G$, then $g$ is nilpotent precisely when the corresponding group $G$ is. One may show that if the FC set $g$ is nilpotent, then $[g^+] \in \mathbb{Z}$, and this in turn implies that the quantum dimension $d_\alpha$ of any element $\alpha \in g$ is either an integer

\(^1\)Note that the converse is not true: there are many FC sets in which all conformal weights belong to $\frac{1}{2}\mathbb{Z}$, but are nevertheless not local; on the other hand, the integrality of conformal weights implies locality, hence every twister is automatically local.
or the square root of an integer\(^2\). We conjecture that many results about (finite) nilpotent groups would carry over to nilpotent FC sets, e.g. if \( \mathfrak{g} \) is nilpotent and \( d \) is an integer dividing \( [\mathfrak{g}]^+ \), then there would exist an FC set \( \mathfrak{h} \subseteq \mathfrak{g} \) such that \( [\mathfrak{h}]^+ = d \).

As explained above, for a local FC set \( \mathfrak{g} \) the algebra \( \mathcal{V}_\mathfrak{g} \) is isomorphic to the character ring of some finite group, hence usual properties of character rings \([13, 15, 16]\) should apply to it. This means in particular that

1. the extent \( [C] \) of any \( \mathfrak{g} \)-class \( C \in \mathcal{E}(\mathfrak{g}) \) is a rational integer dividing \( [\mathfrak{g}]^+ \);
2. the dimension \( d_\alpha \) of every \( \alpha \in \mathfrak{g} \) is a rational integer dividing \( \frac{[\mathfrak{g}]^+}{|\mathcal{Z}(\mathfrak{g})|} \);
3. if \( \alpha \in \mathfrak{g} \) has dimension \( d_\alpha > 1 \), then \( \alpha(C) = 0 \) for some class \( C \in \mathcal{E}(\mathfrak{g}) \);
4. if the class \( C \in \mathcal{E}(\mathfrak{g}) \) is such that \( \frac{[\mathfrak{g}]^+}{[C]} \) is coprime to \( d_\alpha \) for some \( \alpha \in \mathfrak{g} \), then either \( |\alpha(C)| = d_\alpha \) or \( \alpha(C) = 0 \).

All these assertions are well-known properties of character rings, e.g. the first one just states that the size of a conjugacy class is an integer dividing the order of the group, while the second one is Ito’s celebrated theorem \([13]\). What is really amazing is that, as suggested by extensive computational evidence, these properties (and many similar ones) seem to hold for a much larger class of FC sets \([2]\), the so-called integral ones characterized by the property that all of their elements have integer dimension\(^3\). This is truly surprising, as one can show explicit examples of integral FC sets which are not the character ring of any finite group. From this point of view, one may consider integral FC sets as ‘character rings’ of some suitable generalization of the group concept.

5. Summary

As we have seen, fusion closed sets of primaries of a conformal model (or modular tensor category) have a fairly deep structure, generalizing many aspects of the character theory of finite groups. Of course, this is no accident, since the vacuum block of an orbifold model, which corresponds on general grounds to the character ring of the twist group, is a special variety of FC set, a so-called twister. But it turns out that the parallel with character theory goes much further, even for FC sets that have no group theoretic interpretation. Many notions from group theory (like nilpotency, solubility, etc.) may be generalized to arbitrary FC sets, and the corresponding properties seem to go over almost verbatim in this more general setting. But it should be stressed that FC sets are more than just some fancy

\(^2\)That the latter possibility can occur is exemplified by the maximal FC set of the Ising model (the minimal Virasoro model of central charge \( \frac{1}{2} \)), which is nilpotent while having a primary of dimension \( \sqrt{\pi} \).

\(^3\)Actually, they seem to hold (after suitable amendments) in the more general case of FC sets whose elements have integer squared dimension.
generalization of the group concept, since they possess genuinely new structures and properties, which await careful study.

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