Kappa-Deformed Phase Space and Uncertainty Relations

Anatol NOWICKI∗
Institute of Physics, Pedagogical University
Plac Słowiański 6, 65-029 Zielona Góra, POLAND

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Abstract

We consider two realizations of the $\kappa$-deformed phase space obtained as a cross product algebra extension of $k$-Poincaré algebra. Two kinds of the $\kappa$-deformed uncertainty relations are briefly discussed.

1 Introduction

Deformations of space-time symmetries are extensively investigated in last years. In this approach the notion of symmetries is generalized to quantum groups i.e. Hopf algebras [1]. In this way the Poincaré symmetry can be extended to deformed one. Further, we shall consider the special deformation of space-time symmetry associated to $k$-deformed Poincaré algebra with a Hopf algebra structure [2]. In this framework, using the concept of duality, for a given momentum space one can define configuration space which turns out to be noncommuting space-time [3]. It appears that both momentum and configuration spaces form a pair of dual Hopf algebras.

It is known that from such pair of dual algebras one can construct three different double algebras: Drinfeld double and Drinfeld codouble, both with Hopf algebra structure, and Heisenberg double (or more generally, cross product algebra), which is not a Hopf algebra. Only cross product algebra can be regarded as a generalization of the standard quantum mechanical phase space, because in this case we obtain in nondeformed limit the usual commutation relations between momentum and position operators.

Therefore, deformed phase space cannot be a Hopf algebra.

∗Supported by KBN grant 2PO3B130.12; e-mail: anowicki@omega.wsp.zgora.pl
In chapter 2 we recall the construction of a cross product algebra [1], which in our case it is equivalent to the notion of a Heisenberg double or smash product algebra.

In chapter 3 we show that deformed phase space which is an extension of the $\kappa$-deformed Minkowski space, dual to the momentum sector of $\kappa$-Poincaré algebra [4] can be defined ambiguously. Roughly speaking, one can define the commutators between position and momentum in different ways. In particular, we show that for the $\kappa$-Minkowski space we can construct two different phase spaces using the notion of the cross product algebra. In this way we obtain two different sets of the position and momentum commutation relations, which generalize in $\kappa$-deformed way the usual canonical commutation relations. One type of commutation relations was considered in [4,5] and the physical implications were discussed in [6,7]. The second form of the commutation relations give us the uncertainty relations

$$\Delta x \Delta p \geq \hbar + \alpha G (\Delta p)^2$$

(where $G$ - gravitational coupling, $\alpha$ - constant related with the string tension) on the algebraic level as a consequence of the $\kappa$-deformed phase space. Recently, this kind of uncertainty relations are investigated in context of quantum gravity and string theories [8,9].

2 Cross Product Algebra

Let $\mathcal{P}$ be an algebra, $\mathcal{X}$ a vector space.

- A **left action (representation)** of $\mathcal{P}$ on $\mathcal{X}$ is a linear map

$$\triangleright : \mathcal{P} \otimes \mathcal{X} \rightarrow \mathcal{X} : p \otimes x \rightarrow p \triangleright x$$

such that

$$(pp) \triangleright x = p \triangleright (\tilde{p} \triangleright x) \quad 1 \triangleright x = x$$

- We say that $\mathcal{X}$ is a **left $\mathcal{P}$-module**.

In the case where $\mathcal{X}$ is an algebra and $\mathcal{P}$ a bialgebra (**or Hopf algebra**)

- We say that $\mathcal{X}$ is a **left $\mathcal{P}$-module algebra** if

$$p \triangleright (x\tilde{x}) = (p_{(1)} \triangleright x)(p_{(2)} \triangleright \tilde{y}) \quad p \triangleright 1 = \epsilon(p) 1$$

where $\epsilon$ denotes counit and we use the Sweedler’s notation

$$\Delta(p) = \sum p_{(1)} \otimes p_{(2)}$$
Let \( P \) be a Hopf algebra and \( \mathcal{X} \) a left \( P \)-module algebra.

- **Left cross product algebra** (*smash product*) \( \mathcal{X} \rtimes P \) is a vector space \( \mathcal{X} \otimes P \) with product \([1]\)

\[
(x \otimes p)(\tilde{x} \otimes \tilde{p}) = x(p(1) \triangleright \tilde{x}) \otimes p(2)\tilde{p} \quad (\text{left cross product})
\]

with unit element \( 1 \otimes 1 \), where \( x, \tilde{x} \in \mathcal{X} \) and \( p, \tilde{p} \in P \).

- **commutation relations in a cross product algebra.** The obvious isomorphism \( \mathcal{X} \sim \mathcal{X} \otimes 1 \), \( P \sim 1 \otimes P \) gives us the following cross relations between the algebras \( \mathcal{X} \) and \( P \)

\[
[x, p] = x \circ p - p \circ x \quad \text{where} \quad x \circ p = x \otimes p \quad p \circ x = (p(1) \triangleright x) \otimes p(2)
\]

3 Kappa-Deformed Phase Space as Cross Product Algebra

We consider two cases depending on the choice of the momentum sector basis which give us two different commutation relations between position and momentum and in consequence yield two kinds of the uncertainty relations.

3.1 \( \kappa \)-Poincaré Algebra in the Bicrossproduct Basis

The \( \kappa \)-deformed Poincaré algebra in the bicrossproduct basis [10] is given by \((g_{\mu \nu} = (-1, 1, 1, 1))\)

- nondeformed (classical) Lorentz algebra

\[
[M_{\mu}M_{\nu}, M_{\rho\tau}] = i(g_{\mu\rho}M_{\nu\tau} + g_{\nu\tau}M_{\mu\rho} - g_{\mu\tau}M_{\nu\rho} - g_{\nu\rho}M_{\mu\tau})
\]

- deformed covariance relations \((M_i = \frac{1}{2}\epsilon_{ijk}M_{jk}, N_i = M_{i0})\)

\[
[M_i, P_j] = \imath\epsilon_{ijk}P_k \\
[M_i, P_0] = 0 \\
[N_i, P_j] = \imath\delta_{ij} \left[ \kappa c \sinh \left( \frac{P_0}{\kappa c} \right) e^{-\frac{P_0}{\kappa c}} + \frac{1}{2\kappa c}(P^2) \right] - \frac{\imath}{\kappa c}P_iP_j \\
[N_i, P_0] = \imath P_i \\
[P_\mu, P_\nu] = 0
\]

where \( \kappa \) - massive deformation parameter and \( c \) - light velocity and \( \kappa \)-deformed mass Casimir is given by

\[
C_2 = (2\kappa \sinh \frac{P_0}{2\kappa c})^2 - \frac{1}{c^2}P^2 e^{\frac{P_0}{\kappa c}} = M^2
\]

defining coproduct \( \Delta \), antipode \( S \) and counit \( \epsilon \) as follows
\[
\begin{align*}
\Delta(M_i) &= M_i \otimes 1 + 1 \otimes M_i \\
\Delta(N_i) &= N_i \otimes 1 + e^{-\frac{\hbar}{\kappa c}} N_i + \frac{1}{\kappa c} \epsilon_{ijk} P_j \otimes M_k \\
\Delta(P_0) &= P_0 \otimes 1 + 1 \otimes P_0 \\
\Delta(P_i) &= P_i \otimes 1 + e^{-\frac{\hbar}{\kappa c}} \otimes P_i \\
S(M_i) &= -M_i \\
S(N_i) &= -e^{\frac{\hbar}{\kappa c}} N_i + \frac{1}{\kappa c} \epsilon_{ijk} e^{\frac{\hbar}{\kappa c}} P_j M_k \\
S(P_0) &= -P_0 \\
S(P_i) &= -P_i e^{\frac{\hbar}{\kappa c}} \\
\epsilon(X) &= 0 \quad \text{for} \quad X = M_i, N_i, P_\mu 
\end{align*}
\]

Using the duality relations with second fundamental constant \( \hbar \) (Planck’s constant)
\[
< x_\mu, P_\nu > = -i\hbar g_{\mu \nu} \quad g_{\mu \nu} = (-1, 1, 1, 1)
\]
we obtain the noncommutative \( \kappa \)-deformed configuration space \( X_\kappa \) as a Hopf algebra with the following algebra and coalgebra structure
\[
\begin{align*}
[x_0, x_k] &= -\frac{i\hbar}{\kappa c} x_k, \\
\Delta(x_\mu) &= x_\mu \otimes 1 + 1 \otimes x_\mu \\
S(x_\mu) &= -x_\mu \\
\epsilon(x_\mu) &= 0
\end{align*}
\]

We consider \( \kappa \)-deformed phase space \( \Pi_\kappa \sim X_\kappa \otimes P_\kappa \) as a left cross product algebra with a product
\[
(x \otimes p)(\tilde{x} \otimes \tilde{p}) = x(p(1) \triangleright \tilde{x}) \otimes p(2)\tilde{p}
\]
and left action
\[
p \triangleleft x = < p, x(2) > x(1)
\]
using the isomorphism \( x \sim x \otimes 1, \ p \sim 1 \otimes P \) we obtain the commutation relations for \( \Pi_\kappa \) in the form
Introducing the dispersion of the observable $a$ in quantum mechanical sense by

$$\Delta(a) = \sqrt{<a^2> - <a>}^2 \quad \Delta(a)\Delta(b) \geq \frac{1}{2} |c|$$

where $c = [a, b]$, we obtain $\kappa$-deformed uncertainty relations in cross product basis $(x_0 = ct, E = cp_0) \ [4,5,6]$

$$\Delta(t)\Delta(x_k) \geq \frac{\hbar}{2\kappa c} |x_k| \quad \Delta(p_k)\Delta(x_l) \geq \frac{\hbar}{2\kappa c} \delta_{kl}$$

$$\Delta(E)\Delta(t) \geq \frac{\hbar}{2\kappa c}$$

$$\Delta(p_k)\Delta(t) \geq \frac{\hbar}{2\kappa c} |p_k| \quad \frac{1}{2} |c|$$

where $l_\kappa = \frac{\hbar}{\kappa c}$ describes the fundamental length at which the time variable should already be considered noncommutative. In the recent estimates $\kappa > 10^{12} GeV$ therefore $l_\kappa < 10^{-26} cm$; in particular one can put $\kappa$ equal to the Planck mass what implies that $l_\kappa = l_p \simeq 10^{-33} cm$ [see discussion in [6] ].

### 3.2 $\kappa$-Poincaré Algebra in the Standard Basis

In the standard basis of $\kappa$-Poincaré algebra [2] the commutation relations are given by

- nondeformed (classical) $O(3)$-rotation algebra ($M_i = \frac{1}{2}\epsilon_{ijk}M_{jk}$)

$$[M_{ij}, M_{kl}] = i(\delta_{ik}M_{jl} + \delta_{jl}M_{ik} - \delta_{il}M_{jk} - \delta_{jk}M_{il})$$

- deformed covariance relations

$$[M_i, N_j] = i\epsilon_{ijk}N_k \quad [P_\mu, P_\nu] = 0$$

$$[M_i, P_j] = i\epsilon_{ijk}P_k \quad [M_i, P_0] = 0$$

$$[N_i, N_j] = -i\epsilon_{ijk} \left(M_k \cosh\left(\frac{P_0}{\kappa c}\right) - \frac{1}{4(\kappa c)^2}P_kP_lM_l\right)$$

$$[N_i, P_j] = i\delta_{ijk}\kappa c \sinh\left(\frac{P_0}{\kappa c}\right) \quad [N_i, P_0] = iP_i$$

with $\kappa$-deformed mass Casimir

$$C_2 = (2\kappa \sinh\left(\frac{P_0}{2\kappa c}\right))^2 - \frac{1}{c^2} F^2 = M^2$$
In this basis the $\kappa$-Poincaré algebra becomes a Hopf algebra if we define coproduct $\Delta$, antipode $S$ and counit $\epsilon$ in the following way

\[
\Delta(M_i) = M_i \otimes 1 + 1 \otimes M_i \\
\Delta(N_i) = N_i \otimes e^{\frac{p_0}{2\kappa c}} + e^{-\frac{p_0}{2\kappa c}} \otimes N_i + \frac{1}{2\kappa c} \epsilon_{ijk} \left( P_j \otimes M_k e^{\frac{p_0}{2\kappa c}} + e^{-\frac{p_0}{2\kappa c}} M_j \otimes P_k \right) \\
\Delta(P_0) = P_0 \otimes 1 + 1 \otimes P_0 \\
\Delta(P_i) = P_i \otimes e^{-\frac{p_0}{2\kappa c}} + e^{\frac{p_0}{2\kappa c}} \otimes P_i
\]

(25)

\[
S(M_i) = -M_i \quad S(N_i) = -N_i + \frac{3i}{2\kappa c} P_i \\
S(P_\mu) = -P_\mu \\
\epsilon(X) = 0 \quad \text{for} \quad X = M_i, N_i, P_\mu
\]

(26)

(27)

As in the previous case we see, that $\kappa$-Poincaré algebra contains the Hopf subalgebra of $\kappa$-deformed fourmomentum $P_\kappa$

\[
[P_\mu, P_\nu] = 0 \\
\Delta(P_0) = P_0 \otimes 1 + 1 \otimes P_0 \\
\Delta(P_i) = P_i \otimes e^{-\frac{p_0}{2\kappa c}} + e^{\frac{p_0}{2\kappa c}} \otimes P_i \\
S(P_0) = -P_0 \quad S(P_i) = -P_i \quad \epsilon(P_\mu) = 0
\]

(28)

(29)

Now, applying the duality relations we obtain the noncommutative $\kappa$-deformed configuration space $X_\kappa$ as a Hopf algebra with the following algebra and coalgebra structure

\[
[x_0, x_k] = -\frac{i\hbar}{\kappa c} x_k, \quad [x_k, x_l] = 0 \quad [x_k, x_\mu] = 0 \\
\Delta(x_\mu) = x_\mu \otimes 1 + 1 \otimes x_\mu \\
S(x_\mu) = -x_\mu \quad \epsilon(x_\mu) = 0
\]

(30)

It is worthwhile to stress that for both realizations of bases of fourmomentum sector i.e. bicrossproduct and standard, we obtain the same commutation relations for configuration space $X_\kappa$. Both bases are related by nonlinear transformation

\[
P_i \rightarrow P_i e^{\frac{p_0}{2\kappa c}}
\]

Using the left cross product algebra construction we get commutation relations for $\kappa$-deformed phase space $\Pi_\kappa$ in the standard basis (see also [5])

\[
[x_0, x_k] = -\frac{i\hbar}{\kappa c} x_k, \quad [x_k, x_l] = 0 \quad [x_k, x_\mu] = 0 \\
[p_\mu, p_\nu] = 0 \\
[x_k, p_l] = i\hbar \delta_{kl} e^{\frac{p_0}{2\kappa c}} \quad [x_k, p_0] = 0 \\
[x_0, p_k] = \frac{i\hbar}{2\kappa c} p_k \quad [x_0, p_0] = -i\hbar
\]

(31)

Therefore, the $\kappa$-deformed uncertainty relations read $(x_0 = ct, E = cp_0)$

\[
\Delta(t) \Delta(x_k) \geq \frac{\hbar}{2 \kappa c} |< x_k >| = \frac{1}{2} \frac{\hbar}{c} |< x_k >| \\
\Delta(p_k) \Delta(x_l) \geq \frac{1}{2} \hbar \delta_{kl} |< x_l >| \\
\Delta(E) \Delta(t) \geq \frac{1}{2} \frac{\hbar}{c} \\
\Delta(p_k) \Delta(t) \geq \frac{\hbar}{4 \kappa c} |< p_k >| = \frac{1}{4} \frac{\hbar}{c} |< p_k >|
\]

(32)
and the mass-shell condition

\[(2\kappa \sinh \frac{P_0}{2\kappa c})^2 - \frac{1}{c^2} \vec{P}^2 = M^2\]  \hspace{1cm} (33)

This relation one can rewrite as follows

\[\exp\left(\frac{P_0}{2\kappa c}\right) = \sqrt{\frac{\vec{P}^2 + M^2}{4\kappa^2 c^2}} + \sqrt{1 + \frac{\vec{P}^2 + M^2}{4\kappa^2 c^2}}\]  \hspace{1cm} (34)

This formula allows us to consider the uncertainty relations (32) between momentum and position in the nonrelativistic limit \(c \to \infty\)

\[\Delta(p_k)\Delta(x_l) > \frac{\hbar}{4} \delta_{kl} \left[ 1 + \left(1 + \frac{M}{2\kappa}\right)^2 \right] > \frac{\hbar}{2} \delta_{kl} \left(1 + \frac{M}{2\kappa}\right)\]  \hspace{1cm} (35)

Therefore, we see that it is mass dependent relation, however for \(\kappa \gg M\) we get the standard quantum mechanical uncertainty relation.

On the other hand, neglecting the first term on r.h.s in (34) we have an estimation

\[\Delta(p_k)\Delta(x_l) > \frac{\hbar}{2} \delta_{kl} \left(1 + \frac{(\Delta p)^2}{8\kappa^2 c^2}\right)\]  \hspace{1cm} (36)

using the relation \(<\vec{P}^2> = <\vec{P}^2> + (\Delta p)^2\) in the regime \(<\vec{P}^2> + M^2 c^2 \ll \kappa^2 c^2, \Delta p \leq \kappa c\) one gets

\[\Delta(p_k)\Delta(x_l) > \frac{\hbar}{2} \delta_{kl} \left(1 + \frac{(\Delta p)^2}{8\kappa^2 c^2}\right)\]  \hspace{1cm} (37)

a modified uncertainty relation which follows from the analysis of string collisions at Planckian energies (see [4,5]).

4 Final Remarks

We considered two realizations of the fourmomentum sector of \(\kappa\)-Poincaré algebra - the bicrossproduct and standard bases. In both cases we discussed the \(\kappa\)-deformed phase space obtained by the cross product algebra construction (Heisenberg double) and related uncertainty relations. We see that noncommutativity of space-time coordinates is the same for both realizations of the phase space, however the commutation relations between position and momenta operators are different which give us two kinds of the uncertainty relations.

In the case of the standard basis we showed that the \(\kappa\)-deformed uncertainty relations in appropriate limit give us the modified uncertainty relations postulated in string theory, now derived on the algebraic level.
It is worthwhile to mention here, that the choice of the bases in the momentum space physically means the choice of generators which one interpretes as describing physical momenta. It appears that the choice of the physical momenta changes radically the uncertainty relations.

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