SUMS AND DIFFERENCES OF FOUR $k$-TH POWERS

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Abstract. We prove an upper bound for the number of representations of a positive integer $N$ as the sum of four $k$-th powers of integers of size at most $B$, using a new version of the determinant method developed by Heath-Brown, along with recent results by Salberger on the density of integral points on affine surfaces.

More generally we consider representations by any integral diagonal form. The upper bound has the form $O_B(N^{c/k})$, whereas earlier versions of the determinant method would produce an exponent for $B$ of order $k^{-1/3}$ (uniformly in $N$) in this case.

Furthermore, we prove that the number of representations of a positive integer $N$ as a sum of four $k$-th powers of non-negative integers is at most $O_\epsilon(N^{1/k+2/k^3/2+\epsilon})$ for $k \geq 3$, improving upon bounds by Wisdom.

1. Introduction

In this paper, we shall study the number of representations of a positive integer $N$ using four $k$-th powers. We consider two different versions of this problem. The main part of the paper concerns solutions to the equation

$$x_1^k \pm x_2^k \pm x_3^k \pm x_4^k = N$$

in integers $x_i$, positive or negative. Our treatment of this problem is inspired by a recent paper of Heath-Brown where he studies the equation

$$x_1^k \pm x_2^k \pm x_3^k = N.$$  

More precisely, he estimates the number of integral solutions to (2), with $\max |x_i| \leq B$, that are not trivial in the sense that $\pm x_i^k = N$ for some $i$. Assuming that $N \ll B$, Heath-Brown proves that there are $O_k(B^{10/k})$ such solutions for $k \geq 3$.

The method used by Heath-Brown is a new approach to the determinant method of Bombieri and Pila. Rather than counting integral points on the affine surface defined by (2), an approach that would yield an exponent of order $1/\sqrt{k}$ (using the version of the determinant method developed in [6] and [7]), he studies rational points near the projective curve given by $x_1^k \pm x_2^k \pm x_3^k = 0$.
Our aim in this paper is to study the corresponding problem in four variables, using the approach of [8]. The method works for arbitrary non-singular diagonal forms, so we state our main result in that generality. Let \(a = (a_1, a_2, a_3, a_4)\) be a quadruple of non-zero integers, \(k \geq 3\) an integer, and \(N\) a positive integer. Let \(R(N, B)\) be the number of quadruples \((x_1, x_2, x_3, x_4)\) \(\in \mathbb{Z}^4\) satisfying
\[
a_1x_1^k + a_2x_2^k + a_3x_3^k + a_4x_4^k = N
\]
and \(\max |x_i| \leq B\). We note that the trivial estimate \(R(N, B) = O(B^{2+\varepsilon})\) can be deduced easily using known results for Thue equations (see Proposition 5.2 below).

We call a solution \(x\) to (3) special if either \(a_ix_i^k = N\) for some index \(i\) or \(a_ix_i^k + a_jx_j^k = N\) for some pair of indices \(i, j\). If \(X \subset \mathbb{A}^4\) denotes the hypersurface defined by (3), then these solutions are all contained in a proper subvariety of \(X\) (see Section 4). We shall see (Proposition 5.3) that the special solutions contribute at most \(O(k, \varepsilon)\) to \(R(N, B)\). Thus, let \(R_0(N, B)\) be the number of non-special solutions to (3) satisfying \(\max |x_i| \leq B\). Then we shall prove the following estimate.

**Theorem 1.1.** For any \(\varepsilon > 0\) we have
\[
R_0(N, B) \ll a, N, \varepsilon B^{16/(3\sqrt{3}k)} B^{2/(\sqrt{k}) + 1/2}+6/(k+3)).
\]
In particular, \(R(N, B) \ll a, N, B\) for \(k \geq 27\).

**Remark.** The exponent \(16/(3\sqrt{3}k)\) in Theorem 1.1 is to be compared with the exponent \(3/k^{1/3}\) that could be obtained (uniformly in \(N\)) by applying the “ordinary” determinant method of Heath-Brown [7, Thm. 15] in this case. Furthermore, we remark that the bound (4) is non-trivial for \(k \geq 8\).

The estimate in Theorem 1.1 is proven by combining the ideas from [8] with recent results by Salberger [13] about the density of integral points on affine surfaces.

In Sections 2 and 3 we adapt Heath-Brown’s arguments to the four-variable case. As with other instances of the determinant method, the output is a number of auxiliary forms, allowing us to estimate \(R(N, B)\) through counting integral points of bounded height on a number of affine algebraic surfaces. In doing this, we use results by Salberger, discussed in Section 4 concerning the geometry of Fermat hypersurfaces. The proof of Theorem 1.1 is finished in Section 5.

It is implicit in Theorem 1.1 that \(N\) is fixed and small. If \(N\) is allowed to grow as \(B \to \infty\), we have the following more precise estimate.

**Theorem 1.2.** Suppose that \(N = O(B^{k-\tau})\), where \(4/3 < \tau < k\). Then we have
\[
R_0(N, B) \ll a, \tau, \varepsilon B^{16/(3\sqrt{3}k)+\varepsilon} N^{-24/(3\tau)} B^{2/(\sqrt{k}) - 16/(3k)} (B^{2/(\sqrt{k}) + 1/(k+3)} + B^{1/(\sqrt{k})+6/(k+3)}).
\]
for any $\varepsilon > 0$.

Note that, as in \cite{S}, the determinant method discussed in Sections 2 and 3 applies to any non-singular form. It is only in the later steps of the proof of Theorem 1.1 that we specialize to the case of a diagonal form.

The second result of the paper concerns the number $R_k(N)$ of representations of a positive integer $N$ as a sum of four $k$-th powers

$$ x_1^k + x_2^k + x_3^k + x_4^k = N, $$

where $x_i$ are non-negative integers and $k \geq 3$. One easily proves, for example using Proposition 5.2 below, that $R_k(N) = O(\varepsilon N^{2k+\varepsilon})$.

Hooley \cite{H} has studied sums of four cubes, and proved the remarkable estimate $R_3(N) = O(\varepsilon N^{11/18+\varepsilon})$. Wisdom \cite{W1, W2} extended Hooley’s methods to prove that $R_k(N) = O(\varepsilon N^{11/(6k)+\varepsilon})$ for odd integers $k \geq 3$.

Our result is the following:

**Theorem 1.3.**

$$ R_k(N) \ll_{k, \varepsilon} N^{1/k+2/k^{3/2}+\varepsilon} $$

for any $\varepsilon > 0$.

This estimate is non-trivial for $k \geq 5$, and sharper than Wisdom’s for $k > 5$. Theorem 1.3 is proven in Section 6 as an easy corollary of the next result. The estimate in Theorem 1.4 was mentioned already in \cite{S}, and is in principle contained in Salberger’s work \cite{S3}, but we shall give a proof in Section 6 for the sake of completeness.

**Theorem 1.4.** Let $a_1, a_2, a_3, M$ be non-zero integers. Let $r_0(M, B)$ be the number of solutions $(x_1, x_2, x_3) \in \mathbb{Z}^3$ to the equation

$$ a_1 x_1^k + a_2 x_2^k + a_3 x_3^k = M $$

satisfying $|x_i| \leq B$ and $a_i x_i^k \neq M$ for $i = 1, 2, 3$. Then

$$ r_0(M, B) = O(\varepsilon B^{2/\sqrt{k}+\varepsilon}). $$

In fact, with no extra work, the proof of Theorem 1.3 yields the following more general result.

**Theorem 1.5.** Let $k, \ell \geq 3$. Let $R_{k, \ell}(N)$ be the number of solutions to the equation

$$ x_1^k + x_2^k + x_3^k + x_4^\ell = N $$

in non-negative integers $x_i$. Then we have the estimate

$$ R_{k, \ell}(N) \ll_{k, \varepsilon} N^{1/\ell+2/k^{3/2}+\varepsilon} $$

for any $\varepsilon > 0$. 
The corresponding trivial estimate is $N^{1/\ell+1/k+\varepsilon}$. We also note that Wisdom [17] has proved that

$$R_{3,3}(N) = O_\varepsilon(N^{5/9+\varepsilon}) \text{ and } R_{3,5}(N) = O_\varepsilon(N^{47/90+\varepsilon}),$$

bounds which are sharper than the ones given by Theorem 1.5.

**Notation.** The following notation shall be used. If $U \subseteq \mathbb{A}^n$ is a locally closed subset, let $U(\mathbb{Z})$ be the set of integral points in $U$. Then we define

$$U(\mathbb{Z}, B) = U(\mathbb{Z}) \cap [-B, B]^n$$

for any positive real number $B$, and

$$N(U, B) = \#U(\mathbb{Z}, B).$$

We shall also use the notation

$$N_+(U, B) = \#(U(\mathbb{Z}) \cap [0, B]^n).$$

Finally, we adopt the following convention for the $O$- and $\ll$-notation. The implied constants are allowed to depend upon the coefficients of the polynomial $F$ under consideration (that is, on the $a_i$, in the case of a diagonal form) unless we indicate uniformity through the subscript $k$.

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2. **Parameterization of points near projective surfaces**

In this section we generalize, in a completely straightforward fashion, some preparatory results in Heath-Brown’s paper [8]. The proof of Lemma 1 in [8] generalizes readily to $\mathbb{R}^4$, to yield our first lemma.

**Lemma 2.1.** Let $F(x_1, x_2, x_3, x_4)$ be a non-singular homogeneous polynomial of degree $k$. There is a natural number $M_0$, depending only on $F$, with the following property: if the unit cube $[-1, 1]^3$ is partitioned into smaller cubes

$$[a_1, a_1 + (M_0 M)^{-1}] \times [a_2, a_2 + (M_0 M)^{-1}] \times [a_3, a_3 + (M_0 M)^{-1}],$$

for some positive integer $M$, then the number of such cubes containing a solution $(t_1, t_2, t_3) \in \mathbb{R}^3$ to the inequality

$$|F(t_1, t_2, t_3, 1)| \leq \frac{1}{M_0 M}$$

(8)
is at most $O(M^2)$. Moreover, if $S$ is such a cube containing a solution to $(8)$, then for some index $i$ we have
\[
\left| \frac{\partial F}{\partial x_i}(a_1, a_2, a_3, 1) \right| \gg 1.
\]

Following Heath-Brown, let us call such a cube $S$ a “good” cube. Let us also call a solution $t = (t_1, t_2, t_3)$ to $(8)$ a “good” point. For a good cube, we can prove the following result. Again, the proof is an easy generalization of that of [8, Lemma 2].

**Lemma 2.2.** Retaining the notation of the previous lemma, suppose that
\[
S = [a_1, a_1 + (M_0 M)^{-1}] \times [a_2, a_2 + (M_0 M)^{-1}] \times [a_3, a_3 + (M_0 M)^{-1}]
\]
is a good cube, and that
\[
\left| \frac{\partial F}{\partial x_3}(a_1, a_2, a_3, 1) \right| \gg 1.
\]

If $(t_1, t_2, t_3) = (a_1 + u_1, a_2 + u_2, a_3 + u_3) \in S$, put
\[
w = F(t_1, t_2, t_3, 1) - F(a_1, a_2, a_3, 1).
\]

Then there exist, for each $m \in \mathbb{N}$, two polynomials
\[
\Phi_m(u_1, u_2, w), \quad \Psi_m(u_1, u_2, u_3, w),
\]
such that $\Phi_m$ has no constant term and $\Psi_m$ has no term of degree less than $m$, and such that the relation
\[
u_3 = \Phi_m(u_1, u_2, w) + u_3\Psi_m(u_1, u_2, u_3, w)
\]
holds throughout $S$. Moreover, $\Phi_m$ and $\Psi_m$ have degree $O_m(1)$ and coefficients of size $O_m(1)$.

In other words, the lemma states that the relation
\[
F(a_1 + u_1, a_2 + u_2, a_3 + u_3) = F(a_1, a_2, a_3) + w
\]
defines $u_3$, approximately, as a function of $u_1, u_2$ and $w$. It may thus be viewed as a form of the Implicit function theorem.

3. **Application of the determinant method**

Let $F \in \mathbb{Z}[x_1, x_2, x_3, x_4]$ be a non-singular form of degree $k \geq 3$, $N$ a positive integer, and $B \geq 1$ a real number. Our aim in this section is to exhibit a set $\mathcal{C}$ of homogeneous polynomials $A_i \in \mathbb{Z}[x_1, x_2, x_3, x_4]$ of the same degree $\delta$, such that every solution $x \in \mathbb{Z}^4 \cap [-B, B]^4$ to the inequality
\[
(9) \quad |F(x_1, x_2, x_3, x_4)| \leq N
\]
satisfies at least one of the equations $A_i(x) = 0$. These polynomials shall be called auxiliary forms. Let $\mathcal{A}(F, N, B, \delta)$ be the smallest possible cardinality of such a collection $\mathcal{C}$ of auxiliary forms. This is a well-defined quantity since there are only finitely many solutions to $(9)$. Our arguments will conclude in two different estimates for
\( A(F, N, B, \delta) \). However, we begin with some considerations that apply to both situations, following closely the arguments in [8].

Since we are only interested in the order of growth of \( A(F, N, B, \delta) \) as a function of \( B \), it is clearly enough to consider solutions to (9) for which

\[
|x_4| \geq \max(|x_1|, |x_2|, |x_3|).
\]

We may even restrict ourselves to solutions satisfying

\[
B/2 < \max_i(|x_i|) = |x_4| \leq B,
\]
deducting the final estimate from this case by dyadic summation.

If we assume that

\[
M \leq (B/2)^k M_0^{-1} N^{-1},
\]
then every solution \( x \) to (9) and (10), and such that \( t \) which by Lemma 2.1 lies in some good cube. Thus, let \( S \) be a good cube, and let \( R \) be as in Lemma 2.2. Furthermore, let \( \delta \) be any positive integer, and let \( s = (\delta+3) \) be the number of different monomials in \( x_1, x_2, x_3, x_4 \) of degree \( \delta \). Consider the \( s \times J \)-matrix

\[
A = (f_i(x^{(j)})),
\]
where \( f_i \) runs over all monomials of degree \( \delta \). We shall prove that it is possible to choose \( M \) in such a way that \( \text{rank } A < s \). This implies the existence of a homogeneous polynomial \( A(x_1, x_2, x_3, x_4) \) of degree \( \delta \) vanishing at all the \( x^{(j)} \).

If \( J < s \), we are done. Otherwise, we proceed by choosing a subset of \( R \) of cardinality \( s \), without loss of generality \( \{x^{(1)}, \ldots, x^{(s)}\} \), and evaluating the corresponding \( s \times s \)-subdeterminant

\[
\Delta_1 = \det \left( f_i(x^{(j)}) \right)_{1 \leq i,j \leq s}.
\]

Our aim is to prove that \( |\Delta_1| < 1 \). In that case, being an integer, \( \Delta_1 \) has to vanish. We have

\[
\Delta_1 = \prod_{j=1}^{s} (x_4^{(j)})^\delta \Delta_2 \ll B^{s\delta}|\Delta_2|,
\]
where \( \Delta_2 = \det (f_i(t^{(j)}_1, t^{(j)}_2, t^{(j)}_3, 1)) \).

At this point, we make the “variable change” suggested by Lemma 2.2. Suppose, without loss of generality, that \( |\partial F/\partial x_3(a_1, a_2, a_3, 1)| \gg 1 \). For \( (t_1, t_2, t_3) \in S \), let \( u_1, u_2, u_3, w \) be as in Lemma 2.2. Furthermore, let \( \xi = F(t_1, t_2, t_3, 1) = w + F(a_1, a_2, a_3, 1) \). For any monomial \( f_i \), we have

\[
f_i(t_1, t_2, t_3, 1) = G_i(u_1, u_2, u_3),
\]
where $G_i = G_{i,S}$ is a polynomial of degree $\delta$. For any $m \in \mathbb{N}$, an application of Lemma 2.2 now yields

$$f_i(t_1, t_2, t_3, 1) = G_i(u_1, u_2, \Phi_m(u_1, u_2, w) + u_3 \Psi_m(u_1, u_2, u_3, w)) = Q_i(u_1, u_2, w) + u_3 H_i(u_1, u_2, u_3, w),$$

where $Q_i = Q_{i,S}$ and $H_i = H_{i,S}$ are polynomials of degree $O_m(1)$ and all terms of $H_i$ have degree at least $m$.

Now, if $t$ is a good point, we have $u_i \ll M^{-1}$ and $\xi \ll M^{-1}$, and thus also $w \ll M^{-1}$, so we get

$$f_i(t_1, t_2, t_3, 1) = g_i(u_1, u_2, \xi) + O_m(M^{-(m+1)}),$$

for some polynomials $g_i$ of degree $O_m(1)$. Here, $g_i$ depends on the chosen cube $S$, but the size of the coefficients of $g_i$ is bounded in terms of $m$. We conclude that

$$\Delta_2 = \Delta_3 + O_m(M^{-(m+1)}),$$

where $\Delta_3 = \det (g_i(u_1^{(j)}, u_2^{(j)}, \xi^{(j)}))$.

To estimate the determinant $\Delta_3$, we shall use a variant of the argument in [1] where we take into account the fact that one of the variables, $\xi$, takes only small values. Let us recall the notation from [2]: let $n, D, H$ be positive integers. Given real numbers $0 \leq X_1, \ldots, X_n \leq 1$, we define the size $\|m_i\|$ of a monomial $m_i(x_1, \ldots, x_n) = x_1^{a_1} \cdots x_n^{a_n}$ by

$$\|m_i\| = X_1^{a_1} \cdots X_n^{a_n}.$$ 

Furthermore, we enumerate the monomials $m_1, m_2, \ldots$ in $x_1, \ldots, x_n$ in such a way that $\|m_1\| \geq \|m_2\| \geq \cdots$. Finally, by abuse of notation, by the height $\|f\|$ of a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ we mean the maximum modulus of its coefficients. Heath-Brown proves the following result.

**Lemma 3.1** ([2] Lemma 3). Let $f_1, \ldots, f_H \in \mathbb{C}[x_1, \ldots, x_n]$ be polynomials of degree at most $D$. Let $x^{(1)}, \ldots, x^{(H)} \in \mathbb{C}^n$ satisfy $|x_i^{(j)}| \leq X_i$ for all $i$ and $j$. Then we have the estimate

$$\det((f_i(x^{(j)}))_{1 \leq i,j \leq H}) \ll_{H,D} (\max_i \|f_j\|)^H \prod_{j=1}^H \|m_i\|.$$

In the application of Lemma 3.1 we take $X_1 = X_2 = (M_0 M)^{-1}$ and $X_3 = N(B/2)^{-k}$, according to our a priori bounds for $|u_1|$, $|u_2|$ and $|\xi|$, respectively.

Let the monomials $m_i(u_1, u_2, \xi)$ be defined as above. The strategy of our method is to ensure that for small $i$, $m_i$ does not contain a positive power of $\xi$. In that way our determinant will behave almost as if we were considering points on a projective surface instead of points on an affine threefold. Our approach now differs from that in [2] in that
we allow $\delta$ to grow as large as required to obtain an optimal bound, whereas Heath-Brown only considers $\delta < k$. Thus, suppose that

$$ (M_0M)^{\alpha} = N^{-1}(B/2)^{k}, $$

where $\alpha$ is to be chosen properly. Then Lemma 3.1 yields an estimate

$$ \Delta_3 \ll_{\delta,m} \prod_{i=1}^{s} \|m_i\| \ll_{\delta,m} M^{-f}, $$

where $f = \sum(n_1 + n_2 + \alpha n_3)$, the sum being taken over all non-negative integers $n_1, n_2, n_3$ such that $u_1^{n_1} u_2^{n_2} \xi^{n_3}$ occurs among the monomials $m_i$, $1 \leq i \leq s$.

To estimate $f$, we need to determine the first monomials $m_1, \ldots, m_s$. Thus, suppose that $m_s(u_1, u_2, \xi) = u_1^{e_1} u_2^{e_2} \xi^{e_3}$, and let

$$ \nu = e_1 + e_2 + \alpha e_3, $$

so that $\|m_s\| = (M_0M)^{\nu}$. To determine the relationship between $\delta$ and $\nu$, we note that

$$ \sum_{n_1 + n_2 + \alpha n_3 \leq \nu - 1} 1 < s \leq \sum_{n_1 + n_2 + \alpha n_3 \leq \nu} 1. $$

The left sum in (15), i.e. the number of integer points inside the tetrahedron $T_1 \subset \mathbb{R}^3$ defined by the inequalities

$$ x \geq 0, \ y \geq 0, \ z \geq 0, \ x + y + \alpha z \leq \nu - 1, $$

can be interpreted as the volume of the three-dimensional body

$$ S_1 = \bigcup_{n_1, n_2, n_3 \geq 0 \atop n_1 + n_2 + \alpha n_3 \leq \nu - 1} [n_1, n_1 + 1] \times [n_2, n_2 + 1] \times [n_3, n_3 + 1]. $$

Since $T_1 \subset S_1 \subset T_2$, where $T_2$ is the tetrahedron

$$ x \geq 0, \ y \geq 0, \ z \geq 0, \ x - 1 + y - 1 + \alpha(z - 1) \leq \nu - 1, $$

we get the estimate

$$ \frac{(\nu - 1)^3}{6\alpha} = \text{vol}(T_1) < \#(T_1 \cap \mathbb{Z}^3) < \text{vol}(T_2) = \frac{(\nu + 1 + \alpha)^3}{6\alpha}. $$

Similarly, the right sum in (15) is the number of integer points inside the tetrahedron $T_3$ defined by the inequalities

$$ x \geq 0, \ y \geq 0, \ z \geq 0, \ x - 1 + y - 1 + \alpha(z - 1) \leq \nu, $$

so that,

$$ \sum_{n_1, n_2, n_3 \geq 0 \atop n_1 + n_2 + \alpha n_3 \leq \nu} 1 \leq \text{vol}(T_3) = \frac{(\nu + 2 + \alpha)^3}{6\alpha}. $$

By the definition of $s$ we conclude that

$$ \delta = \frac{\nu}{\alpha^{1/3}} + O_\alpha(\nu^{2/3}). $$
A lower bound for $f$ is given by the sum
\[ \tilde{f} = \sum_{(n_1, n_2, n_3) \in T_1 \cap \mathbb{Z}^3} (n_1 + n_2 + \alpha n_3). \]

We can estimate $\tilde{f}$ from below by considering the integral
\[ I = \int_{T_1} (x + y + \alpha z) \, dx \, dy \, dz = \frac{(\nu - 1)^4}{8\alpha}. \]

Since $T_1 \subset S_1$, we have
\[ I < \sum_{(n_1, n_2, n_3) \in T_1 \cap \mathbb{Z}^3} (n_1 + 1 + n_2 + 1 + \alpha(n_3 + 1)) = \tilde{f} + (2 + \alpha)\#(T_1 \cap \mathbb{Z}^3). \]

By (16) we conclude that
\[ f > \frac{(\nu - 1)^4}{8\alpha} - \frac{(2 + \alpha)(\nu + 1 + \alpha)^3}{6\alpha} = \frac{\nu^4}{8\alpha} + O_{a,\nu}(\nu^3). \]

Tracing our steps back to the estimates (12) and (13), and choosing $m = f = O_{a,\nu}(1)$, we get the estimate
\[ \Delta_1 \ll_{F,\delta,\alpha} B^{s\delta}(M_0 M)^{-f} \ll_{F,\delta,\alpha} B^\beta, \]
where, upon recalling the relation (14), we have
\[ \beta = s\delta - \frac{f}{\alpha} \left( k - \frac{\log N}{\log B} \right). \]

Using (15), (17) and (18) this implies that
\[ \beta = \left( \frac{1}{6} - \left( k - \frac{\log N}{\log B} \right) \frac{1}{8\alpha^{2/3}} \right) \delta^4 + O_{a}(\delta^3) \]

Let us first consider the case where $N$ remains fixed as $B \to \infty$. Given $\epsilon > 0$, choose $\lambda = \lambda(k, \epsilon) > 0$ small enough that
\[ \frac{16}{3\sqrt{3}k} (1 - \lambda)^{-1} \leq \frac{16}{3\sqrt{3}k} + \epsilon \quad \text{and} \quad (1 - \lambda) \left( \frac{3k}{4} \right)^{3/2} > 1, \]
and put
\[ \alpha = (1 - \lambda) \left( \frac{3k}{4} \right)^{3/2}, \]
so that
\[ \frac{2k}{\alpha} = \frac{16}{3\sqrt{3}k} (1 - \lambda)^{-1} \leq \frac{16}{3\sqrt{3}k} + \epsilon. \]

Then there is a positive constant $c_1 = c_1(k, \epsilon)$ such that
\[ \frac{1}{6} - \left( k - \frac{\log N}{\log B} \right) \frac{1}{8\alpha^{2/3}} < -c_1, \]
that is, the leading coefficient in $\beta$ is negative, as soon as $B \geq B_0 = B_0(N, k, \epsilon)$. Thus, we have $\beta < -1$, say, as soon as $\delta \geq \delta_0 = \delta_0(k, \epsilon)$. 

Assume now that $B \geq B_0$ and, in addition, that
\[ M = M_0^{-1}N^{-1/\alpha}(B/2)^{k/\alpha} \]
is an integer (we may clearly restrict ourselves to such values of $B$). As $\alpha > 1$, the requirement (11) is fulfilled, so the above arguments are indeed valid. In particular, for $\delta \geq \delta_0$ we get
\[ \Delta_1 \ll_{F, \varepsilon} B^{-1}, \]
so that $|\Delta_1| < 1$ as soon as $B \gg_{F, \varepsilon} 1$. In this situation, as already explained, we obtain an auxiliary form of degree $\delta$ for each good cube $S$. By Lemma 2.1, the total number of good cubes is
\[ O_{F}(M^2) = O_{F,N}(B^{2k/\alpha}) = O_{F,N}(B^{16/3\sqrt{3k}+\varepsilon}), \]
and so this constitutes an upper estimate for $A(F, N, B, \delta)$. Thus we can summarize our findings so far in the following result:

**Proposition 3.2.** Let $F \in \mathbb{Z}[x_1, x_2, x_3, x_4]$ be a non-singular homogeneous polynomial of degree $k \geq 3$. Let $N \in \mathbb{N}$ be given. Then for any $\varepsilon > 0$, there is an integer $\delta$, depending only on $k$ and $\varepsilon$, such that
\[ A(F, N, B, \delta) = O_{N, \varepsilon}(B^{16/3\sqrt{3k}+\varepsilon}). \]

Next, let us allow $N$ to vary as $B$ grows. In this situation, we shall prove the following estimate.

**Proposition 3.3.** Let $F \in \mathbb{Z}[x_1, x_2, x_3, x_4]$ be a non-singular homogeneous polynomial of degree $k \geq 3$. Let $N \in \mathbb{N}$ be given such that $N = O(B^{k-\tau})$, where $4/3 < \tau < k$. Then for any $\varepsilon > 0$, there is an integer $\delta$, depending only on $k, \tau$ and $\varepsilon$, such that
\[ A(F, N, B, \delta) = O_{\tau, \varepsilon}(B^{16/(3\sqrt{3k})+\varepsilon}). \]

To prove Proposition 3.3 let $t$ be an arbitrary real number in the interval $\tau \leq t < k$, and put
\[ \alpha = (1 - \lambda) \left( \frac{3t}{4} \right)^{3/2}, \]
where $\lambda > 0$ is to be chosen properly in terms of $k$, $\varepsilon$ and $\tau$. Put $\gamma = k - t$. Suppose now that $\frac{1}{2}B^\gamma < N \leq B^\gamma$, say, and that $M = M_0^{-1}N^{-1/\alpha}(B/2)^{k/\alpha}$ is an integer. It clearly suffices to prove our estimate for such $N$ and $B$, as $t$ runs over the interval $[\tau, k)$, allowing for an implicit constant $O_{k, \varepsilon, \tau}(1)$ in our final estimate. Now we have
\[ \frac{1}{6} - \left( k - \frac{\log N}{\log B} \right) \frac{1}{80^{2/3}} \leq \frac{1}{6} \left( 1 - (1 - \lambda)^{-2/3} \right) \leq -c_2, \]
where $c_2 = c_2(\lambda) > 0$, so that
\[ \beta \leq -c_2 \delta^4 + O_{k}(\delta^3). \]
As above, this implies that $|\Delta_1| < 1$ as soon as $\delta \gg_{k, \lambda} 1$ and $B \gg_{F, \lambda} 1$, in which case
\begin{equation}
A(F, N, B, \delta) \ll_F M^2.
\end{equation}

We shall now see how to choose $\lambda$ to obtain the optimal estimate. From (20) we have
\[
\frac{1}{\alpha} = \left(\frac{4}{3k}\right)^{3/2} (1 - \lambda)^{-1} = \left(\frac{4}{3k}\right)^{3/2} (1 - \lambda)^{-1} \left(1 - \frac{\gamma}{k}\right)^{-3/2}.
\]
By Taylor expansion of the function $x \mapsto (1 + x/t)^{3/2}$ at $x = 0$, one sees that
\[
\left(1 - \frac{\gamma}{k}\right)^{-3/2} = \left(1 + \frac{\gamma}{t}\right)^{3/2} \leq 1 + \frac{3k^{1/2}}{2t^{3/2}} \gamma.
\]
Furthermore, we may assume that $\lambda < 1/2$, so that $(1 - \lambda)^{-1} < 1 + 2\lambda$. Thus, for any $\varepsilon > 0$, we have
\[
\frac{8}{3^{3/2}k^{3/2}} < \frac{1}{\alpha} < \frac{8}{3^{3/2}k^{3/2}} + \frac{\varepsilon}{2k} + \frac{\gamma}{31/2} \frac{4}{3^{3/2}k^{3/2}}
\]
upon choosing $\lambda \ll_{k, \varepsilon, \tau} 1$. (Note that $\lambda$ does not depend on $\gamma$.) Using this estimate and the fact that $\gamma \leq \log(2N)/\log B$, we get
\[
B^{2k/\alpha} \ll B^{\frac{16}{(3\varepsilon)^{3/2}}} N^{\frac{24}{(3\varepsilon)^{3/2}}},
\]
and hence
\[
M^2 < B^{2k/\alpha} N^{-2/\alpha} \ll B^{\frac{16}{(3\varepsilon)^{3/2}}} N^{\frac{24}{(3\varepsilon)^{3/2}}} \ll B^{\frac{16}{(3\varepsilon)^{3/2}}} N^{\frac{24}{(3\varepsilon)^{3/2}}}.
\]
In view of (21), this establishes the estimate in Proposition 3.3.

4. Curves of low degree on Fermat threefolds

The following result was proven by Salberger:

**Theorem 4.1** ([13, Thm. 8.1]). Let $K$ be an algebraically closed field of characteristic 0, and let $(a_0, \ldots, a_n)$ be an $(n+1)$-tuple of non-zero elements of $K$. Let $X \subset \mathbb{P}^n_K$ be the Fermat hypersurface given by $a_0x_0^k + \cdots + a_nx_n^k = 0$. Suppose that $C \subset X$ is an integral curve of degree $e$ that does not lie on any other hypersurface defined by a diagonal form $b_0x_0^k + \cdots b_nx_n^k$. Then the following holds:
\[
(n + 1)(k - (n - 1)) \leq nd + \frac{n(n - 1)(e - 3)}{2}.
\]

From this, Salberger draws the following conclusion:

**Theorem 4.2** ([13, Thm. 8.4]). Let $K$ be an algebraically closed field of characteristic 0. Let $(a_0, a_1, a_2, a_3)$ be a quadruple of non-zero elements of $K$ and $X \subset \mathbb{P}^3_K$ the Fermat hypersurface given by $a_0x_0^k + a_1x_1^k + a_2x_2^k + a_3x_3^k = 0$, where $k \geq 3$. Let $C \subset X$ be an integral curve. If
\[
\deg C < (k + 1)/3,
\]
then $C$ is one of the $3k^2$ lines defined on $X$ by the equation $a_0x_0^k + a_jx_j^k = 0$, where $j = 1, 2$ or $3$.

We shall now derive an analogous statement for Fermat threefolds in $\mathbb{P}^4$. Let $K$ be an algebraically closed field of characteristic 0. Let $(a_0, \ldots, a_4)$ be a quintuple of non-zero elements of $K$ and $X \subset \mathbb{P}^4_K$ the Fermat hypersurface given by $a_0x_0^k + \cdots + a_4x_4^k = 0$, where $k \geq 4$. For any partition
\[
\{0, 1, 2, 3, 4\} = \{i_0, i_1\} \cup \{i_2, i_3, i_4\},
\]
the subvariety of $X$ defined by
\[
a_{i_0}x_{i_0}^k + a_{i_1}x_{i_1}^k = a_{i_2}x_{i_2}^k + a_{i_3}x_{i_3}^k + a_{i_4}x_{i_4}^k = 0
\]
is covered by a one-dimensional family of lines, called standard lines $\mathbb{P}^3$. It is well-known \cite[Ex. 2.5.3]{3} that all lines contained in $X$ are standard if $k \geq 4$. The following result strengthens that statement to say that all curves on $X$ of sufficiently low degree are in fact standard lines.

**Proposition 4.3.** Let $C \subset X$ be an integral curve. If $\deg C < (k + 3)/6$, then $C$ is a standard line.

**Proof.** We shall first prove the following statement

(I) There exists a three-element subset $\{i_0, i_1, i_2\}$ of $\{0, 1, 2, 3, 4\}$ and a diagonal form $c_0x_{i_0}^k + c_1x_{i_1}^k + c_2x_{i_2}^k$ that vanishes on $C$, with all $c_i \neq 0$.

By Theorem 4.2 there is a diagonal form $b_0x_0^k + \cdots + b_4x_4^k$, linearly independent from $a_0x_0^k + \cdots + a_4x_4^k$, that vanishes on $C$. Choosing a suitable linear combination of the two forms, we can assume that either one or two of the coefficients $b_i$ vanish. If there are exactly three non-zero coefficients we are done, so let us assume that there are four. By permuting the variables, we assume for the sake of simplicity that $b_4 = 0$ and $b_i \neq 0$ for $i < 4$.

Next, let $\pi : \mathbb{P}^4 \rightarrow \mathbb{P}^3$ be the rational map given by projection onto the first four coordinates. Let $Y \subset \mathbb{P}^3$ be the hypersurface given by $b_0x_0^k + \cdots + b_3x_3^k = 0$. Then the image $\pi(C)$ is an irreducible curve $C' \subset Y$. Indeed, the image is either a point or a curve, but the first alternative would imply that $C$ were a line containing the point $(0 : 0 : 0 : 1)$, which would contradict the fact that $a_4 \neq 0$. Furthermore we have $\deg C' \leq (k+3)/6 \leq (k+1)/3$, so by Theorem 4.2, $C'$ is a standard line. In other words, there is a partition $\{0, 1, 2, 3\} = \{j_0, j_1\} \cup \{j_2, j_3\}$ such that $b_{j_0}x_{j_0}^k + b_{j_1}x_{j_1}^k = b_{j_2}x_{j_2}^k + b_{j_3}x_{j_3}^k = 0$ on $C$. Choosing a suitable linear combination of the forms $a_0x_0^k + \cdots + a_4x_4^k$, $b_{j_0}x_{j_0}^k + b_{j_1}x_{j_1}^k$ and $b_{j_2}x_{j_2}^k + b_{j_3}x_{j_3}^k$, we get (I).

Having proven (I), we may assume, by permuting the variables, that the form $c_0x_0^k + c_1x_1^k + c_2x_2^k$ vanishes on $C$. We shall prove that $C$ is a
line. To this end, let \( \pi_1 : \mathbb{P}^4 \to \mathbb{P}^2 \) be the projection onto the first three coordinates. Let \( Z \subseteq \mathbb{P}^2 \) be the subvariety given by \( c_0 x_0^k + c_1 x_1^k + c_2 x_2^k = 0 \). As above, \( \pi_1(C) \) is either a point or an irreducible curve contained in \( Z \). But this curve would have degree less than \((k + 3)/6\), which would contradict the fact that \( Z \) is an irreducible curve of degree \( k \). Therefore \( \pi_1(C) \) is a single point, say \((y_0 : y_1 : 1)\) without loss of generality.

This means that \( C \) is contained in the plane \( \Pi_1 \subset \mathbb{P}^4 \) given by the equations \( x_0 - y_0 x_2 = x_1 - y_1 x_2 = 0 \). Inserting this into the equation for \( X \), we infer that

\[
a_2' x_2^k + a_3 x_3^k + a_4 x_4^k = 0
\]
on \( C \), where \( a_2' = a_0 y_0^k + a_1 y_1^k + a_2 \). If \( a_2' = 0 \), we infer that \( C \) is one of the \( k \) lines given by the equations

\[
a_3 x_3^k + a_4 x_4^k = x_0 - y_0 x_2 = x_1 - y_1 x_2 = 0.
\]

If \( a_2' \neq 0 \), then by the same argument as above, \( C \) is mapped to a point by the projection \( \pi_2 : \mathbb{P}^4 \to \mathbb{P}^2 \) onto the last three coordinates, which implies that \( C \) is contained in some plane \( \Pi_2 \subset \mathbb{P}^4 \), necessarily distinct from \( \Pi_1 \). \( C \) is then the line \( \Pi_1 \cap \Pi_2 \). It would now be easy to proceed by showing that \( C \) is one of the standard lines, but as remarked above, this is a known result. \( \square \)

5. Counting integral points on affine surfaces

From now on we consider the case of a diagonal form. Thus, let

\[
F(x_1, x_2, x_3, x_4) = a_1 x_1^k + a_2 x_2^k + a_3 x_3^k + a_4 x_4^k,
\]
where \( a_i \) are non-zero integers, let \( N \) be a positive integer and \( B \geq 1 \) a real number. Furthermore, let \( X \subseteq \mathbb{A}^4 \) be the hypersurface defined by

\[
F(x_1, x_2, x_3, x_4) = N.
\]

Let \( V_i \subseteq \mathbb{A}^4 \), for \( 1 \leq i \leq 4 \), be the closed subvariety defined by

\[
a_i x_i^k = N, \quad \sum_{j \neq i} a_j x_j^k = 0
\]
and \( W_{i,j} \), for \( 1 \leq i < j \leq 4 \), be defined by

\[
a_i x_i^k + a_j x_j^k = N, \quad \sum_{\ell \neq i,j} a_{\ell} x_{\ell}^k = 0.
\]

As is shown in Section 4, the algebraic set

\[
V = \left( \bigcup_{1 \leq i \leq 4} V_i \right) \cup \left( \bigcup_{1 \leq i < j \leq 4} W_{i,j} \right)
\]
is precisely the union of all lines on \( X \). The quantity we wish to estimate is then \( \mathcal{R}(N, B) = N(X_0, B) \), where \( X_0 := X \setminus V \).

By Proposition 3.2 we know that every \( x \in X(\mathbb{Z}, B) \) satisfies

\[
F(x_1, x_2, x_3, x_4) = N, \quad A_i(x_1, x_2, x_3, x_4) = 0,
\]
for one of $O_{N,\varepsilon}(B^{16/(3\sqrt{3k})+\varepsilon})$ forms $A_i$ of degree $O_\varepsilon(1)$.

Remark. We shall only write out the proof of Theorem 1.1. If we would supply the more precise estimate of Proposition 3.3 at this point, we would obviously get a proof of Theorem 1.2.

Let $\tilde{Y} \subset A^4$ be any one of the varieties defined by (22). Since $A_i$ is homogeneous, it cannot vanish entirely on $X$, so the dimension of $\tilde{Y}$ is 2. Let $Y$ be an irreducible component of $\tilde{Y}$. As we shall see shortly, we may assume that $Y$ is in fact geometrically irreducible. Then, as $\tilde{Y}$ is a closed subvariety of the non-singular hypersurface $\tilde{X}$, where $\tilde{X}, \tilde{Y} \subset P^4$ denote the respective projective closures, it follows from the Noether-Lefschetz theorem [5, pp. 180-1] that the degree $d$ of $\tilde{Y}$ is divisible by $k$.

It is then possible [11, Prop. 6.2] to find an affine projection $\pi : Y \to A^3$ that is birational onto its image, and such that integral points of height at most $B$ are mapped onto integral points of height at most $cB$ for some constant $c \ll_k 1$. Then $W = \pi(Y) \subset A^3$ is an irreducible closed subvariety of dimension 2 and degree $d$, and $\pi^{-1}(x)$ consists of at most $d$ points for any $x \in W$.

Now we use the new version of the determinant method developed by Salberger. For the sake of convenience, we recall the following result from [13].

**Theorem 5.1** ([13, Thm. 7.2]). Let $X \subset A^3_Q$ be a geometrically integral surface of degree $d$. Then there is a collection of $O_{d,\varepsilon}(B^{1/\sqrt{d}+\varepsilon})$ geometrically integral curves $D_\lambda \subset X, \lambda \in \Lambda$, of degree $O_d(1)$, such that

$$N(X \setminus \bigcup_{\lambda \in \Lambda} D_\lambda, B) = O_{d,\varepsilon}(B^{2/\sqrt{d}+\varepsilon}).$$

From Theorem [5.1] we infer that there is a collection of $O_{k,\varepsilon}(B^{1/\sqrt{k}+\varepsilon})$ irreducible curves on $W$ of degree $O_k(1)$ such that all but $O_{k,\varepsilon}(B^{2/\sqrt{k}+\varepsilon})$ points of $W(Z, B)$ lie on one of these curves. Pulling these curves and points back by $\pi$, we get $O_{k,\varepsilon}(B^{1/\sqrt{k}+\varepsilon})$ irreducible curves of bounded degree on $Y$, the union of which contains all but $O_{k,\varepsilon}(B^{2/\sqrt{k}+\varepsilon})$ points of $Y(Z, B)$.

Concerning the case where $Y$ is integral but not geometrically integral, we can say more. Indeed, one can argue as in [14, Proof of Thm. 2.1] to conclude that all rational points on $Y$ lie on a single curve, the sum of the degrees of the irreducible components of which is bounded in terms of $k$. Thus these irreducible components can be absorbed in the collection of curves and points of the previous paragraph.

To investigate the nature of such a curve, we shall use Proposition [1.3] on the hypersurface

$$\tilde{X} = \{ -Nx_0^k + a_1x_1^k + a_2x_2^k + a_3x_3^k + a_4x_4^k = 0 \} \subset P^4.$$
Any irreducible curve on $X$ of degree less than $(k + 3)/6$ gives rise to an irreducible curve of the same degree on $\bar{X}$, and must therefore in fact be one of the lines in $V$.

Since the number of irreducible components of a surface $\bar{Y}$ as above is bounded in terms of $k$, we conclude that

$\quad (23) \quad X_0(\mathbb{Z}, B) \subseteq \left( \bigcup C(\mathbb{Z}, B) \right) \cup \left( \bigcup \{ y \} \right),$

where $C$ runs over a collection of

$O(B^{16/(3\sqrt{3k})+1/\sqrt{k}+\varepsilon})$

irreducible curves of degree at least $(k + 3)/6$, and $y$ runs over a collection of

$O(B^{16/(3\sqrt{3k})+2/\sqrt{k}+\varepsilon})$

points.

To obtain the estimate (24), we now apply Pila’s estimate [12]. If $C \subset \mathbb{A}^4$ is an irreducible curve of degree $d$, then we have

$\quad (24) \quad N(C, B) \ll d, \varepsilon \cdot B^{1/d+\varepsilon}.$

Thus we conclude that

$\quad R_0(N, B) \ll \varepsilon \cdot B^{16/(3\sqrt{3k})+1/\sqrt{k}+6/(k+3)+\varepsilon} + B^{16/(3\sqrt{3k})+2/\sqrt{k}+\varepsilon},$

which establishes the main estimate in Theorem 1.1.

It remains to estimate the number of special solutions, using known bounds for Thue equations.

**Proposition 5.2.** Let $a, b, h \in \mathbb{Z} \setminus \{0\}$, and let $k \geq 3$ be an integer. Then the number of integer solutions $(x, y)$ to the equation $ax^k + by^k = h$ is $O(h^{\varepsilon})$ for any $\varepsilon > 0$, where the implied constant depends only on $k$ and $\varepsilon$.

**Proof.** More precisely, the number of solutions is at most $C^{1+\omega(h)}$, where $C$ is a constant depending only on $k$. This follows from Evertse’s estimate [4, Cor. 2] for Thue-Mahler equations. Thus the proposition follows from the observation that $\omega(h) \ll \log h / \log \log h$. \qed

This result immediately implies the trivial bound

$\quad \mathcal{R}(N, B) = O(B^{2+\varepsilon}).$

We shall now estimate $N(V_1, B)$ and $N(W_{i,j}, B)$, where clearly it suffices to handle the case $(i, j) = (1, 2)$. Beginning with $N(V_1, B)$, we have at most two choices for the value of $x_1$. The number of integer triples $(x_2, x_3, x_4)$ satisfying

$\quad (25) \quad a_2x_2^k + a_3x_3^k + a_4x_4^k = 0, \quad -B \leq x_2, x_3, x_4 \leq B$

is $O_k(B)$. Indeed, choose $\varepsilon > 0$ so that $\theta := 2/k + \varepsilon < 1$. Then the number of primitive solutions to (25) is $O_k(B^\theta)$ by Heath-Brown’s estimate [6, Thm. 3]. Employing Möbius inversion, as in [9, Ex. F.16],...
one sees that the total number of solutions is $O_k(B)$. Thus we conclude that $N(V_1, B) = O_k(B)$.

Next we consider $N(W_{1,2}, B)$. Here we have $O(B)$ choices for $(x_3, x_4)$, and by Proposition 5.2 there are $O_{k,\varepsilon}(N^\varepsilon)$ possibilities for $(x_1, x_2)$, so $N(W_{1,2}, B) = O_{k,\varepsilon}(BN^\varepsilon)$. Thus we have established the following result.

**Proposition 5.3.** Let $R_1(N, B)$ be the number of special solutions to (3) satisfying $\max_i |x_i| \leq B$. Then we have

$$R_1(N, B) = O_{k,\varepsilon}(BN^\varepsilon).$$

6. The sum of three $k$-th powers and an $\ell$-th power

We begin by proving Theorem 1.4. Let $Y \subset A_3$ be the closed subvariety defined by the equation

$$a_1x_1^k + a_2x_2^k + a_3x_3^k = M,$$

and $Y_0 \subset Y$ the open subset defined by $a_i x_i^k \neq M$ for $i = 1, 2, 3$. The estimate we seek to establish is then

$$N(Y_0, B) = O_{k,\varepsilon}(B^{2/\sqrt{k}+\varepsilon}).$$

By Theorem 5.1 there is a collection $C$ of (geometrically integral) curves $C \subset Y$ of degree $O_k(1)$, such that all but $O_{k,\varepsilon}(B^{2/\sqrt{k}+\varepsilon})$ points in $Y(\mathbb{Z}, B)$ belong to one of the curves $C \in C$, where $\#C = O_{k,\varepsilon}(B^{1/\sqrt{k}+\varepsilon})$. In other words, we have

$$N(Y_0, B) \leq \sum_{C \in C} N(C \cap Y_0, B) + O_{k,\varepsilon}(B^{2/\sqrt{k}+\varepsilon}).$$

Let $\bar{Y} \subset \mathbb{P}^3$ be the projective closure of $Y$, that is the Fermat hypersurface given by the equation

$$-Mx_0^k + a_1x_1^k + a_2x_2^k + a_3x_3^k = 0.$$

Since $\bar{Y}$ is smooth, it follows from a theorem of Colliot-Thélène [2] that the number of geometrically integral curves on $\bar{Y}$ that have degree at most $k-2$ is $O_k(1)$.

Using the results of Salberger in Section 4, we can say more about the degrees of these curves. Indeed, unless $C \in C$ is one of the standard lines, in which case $C \cap Y_0 = \emptyset$, it has degree at least $(k+1)/3$.

Again we use Pila’s estimate (24). The bounded number of curves $C \in C$ with deg $C \leq k-2$ thus contribute $O_{k,\varepsilon}(B^{3/(k+1)+\varepsilon})$ to (23), while the curves with degree at least $k-1$ contribute $O_{k,\varepsilon}(B^{1/\sqrt{k+1}(k-1)+\varepsilon})$. In sum, as $k \geq 3$, we get

$$N(Y_0, B) \ll_{k,\varepsilon} B^{2/\sqrt{k}+\varepsilon} + B^{1/\sqrt{k+1}(k-1)+\varepsilon} + B^{3/(k+1)+\varepsilon} \ll B^{2/\sqrt{k}+\varepsilon},$$

as desired.

Now we turn to the proof of Theorem 1.5. Let $X \subset A_4$ be the hypersurface defined by the equation (7). We shall count integral points
on hyperplane sections of $X$. Thus, for each integer $a \in [0, N^{1/\ell})$, let $X_a$ be the intersection of $X$ with the hyperplane given by $x_4 = a$. Viewed as a subvariety of $\mathbb{A}^3$, $X_a$ is given by the equation

$$x_1^k + x_2^k + x_3^k = N - a^\ell. \tag{27}$$

Let $B = N^{1/k}$. It is then obvious that we have

$$R_{k,\ell}(N) \leq \sum_{0 \leq a < N^{1/\ell}} N_+(X_a, B) + 1. \tag{28}$$

As we are now only considering non-negative solutions to (27), Theorem 1.4 implies that

$$N_+(X_a, B) = O_{k,\varepsilon}(B^{2/\sqrt{k}+\varepsilon}) = O_{k,\varepsilon}(N^{2/(3\sqrt{k})+\varepsilon}).$$

Inserting this into (28), we get

$$R_{k,\ell}(N) \ll_{k,\varepsilon} N^{1/\ell+2/(3\sqrt{k})+\varepsilon},$$

which proves Theorem 1.5.

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