A tropical atmosphere model with moisture: global well-posedness and relaxation limit

Jinkai Li\textsuperscript{1} and Edriss S Titi\textsuperscript{1,2}

\textsuperscript{1} Department of Computer Science and Applied Mathematics, Weizmann Institute of Science, Rehovot 76100, Israel
\textsuperscript{2} Department of Mathematics, Texas A&M University, 3368 TAMU, College Station, TX 77843-3368, USA

E-mail: jklmath@gmail.com, titi@math.tamu.edu and edriss.titi@weizmann.ac.il

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Abstract

In this paper, we consider a nonlinear interaction system between the barotropic mode and the first baroclinic mode of the tropical atmosphere with moisture, which was derived in Frierson \textit{et al} (2004 \textit{Comm. Math. Sci.} \textbf{2} 591–626). We establish the global existence and uniqueness of strong solutions to this system, with initial data in $H^1$, for each fixed convective adjustment relaxation time parameter $\varepsilon > 0$. Moreover, if the initial data possess slightly more regularity than $H^1$, then the unique strong solution depends continuously on the initial data. Furthermore, by establishing several appropriate $\varepsilon$-independent estimates, we prove that the system converges to a limiting system as the relaxation time parameter $\varepsilon$ tends to zero, with a convergence rate of the order $O(\sqrt{\varepsilon})$. Moreover, the limiting system has a unique global strong solution for any initial data in $H^1$ and such a unique strong solution depends continuously on the initial data if the initial data possess slightly more regularity than $H^1$. Notably, this solves the \textsc{viscous version} of an open problem proposed in the above mentioned paper of Frierson, Majda and Pauluis.

Keywords: tropical–extratropical interactions, atmosphere with moisture, primitive equations, relaxation limit, variational inequality

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1. Introduction

1.1. The primitive equations for planetary atmospheric dynamics

In the context of the large-scale atmosphere, the ratio of the vertical scale to the horizontal scale is very small which, by scale analysis (see, e.g. [40, 44]), leads to the hydrostatic approximation in the vertical momentum equation. This small aspect ratio limit can be rigorously justified, see [1, 30]. Taking into account the Boussinesq approximation and the hydrostatic approximation to the Navier–Stokes equations, one obtains the primitive equations, which model the large-scale atmospheric dynamics.

The primitive equations read (see, e.g. [18, 28, 34, 40, 44, 45, 47])

\[
\begin{align*}
\partial_t \mathbf{V} + (\mathbf{V} \cdot \nabla_h) \mathbf{V} + W \partial_z \mathbf{V} - \mu \Delta_h \mathbf{V} - \nu \partial_z^2 \mathbf{V} + \nabla_h \Phi &= 0, \\
\partial_t \Phi &= \frac{S \Theta}{\theta_0}, \\
\partial_t \Theta + \mathbf{V} \cdot \nabla_h \Theta + W \partial_z \Theta + \frac{N^2 \theta_0}{g} W &= S \Theta, \\
\nabla \cdot \mathbf{V} + \partial_z W &= 0,
\end{align*}
\]

where the unknowns \( \mathbf{V} = (V_1, V_2)^T \), \( W, \Phi \) and \( \Theta \) are the horizontal velocity field, vertical velocity, pressure and potential temperature, respectively, \( S \Theta \) is the temperature source, which generally combines three kinds of effects (the radiative cooling, the sensible heat flux and the precipitation), while the non-negative constants \( \mu \) and \( \nu \) are the horizontal and vertical viscosity coefficients, respectively. The total potential temperature is given by

\[
\Theta_{\text{total}}(x, y, z, t) = \theta_0 + \bar{\theta}(z) + \Theta(x, y, z, t),
\]

where \( \theta_0 \) is a positive reference constant temperature and \( \bar{\theta} \) defines the vertical profile background stratification, satisfying \( N^2 = (g/\theta_0) \partial_z \bar{\theta} > 0 \), where \( N \) is the Brunt–Väisälä buoyancy frequency. Here we use \( \nabla_h \) to denote the horizontal gradient and \( \mathbf{V}^\perp = (-V_2, V_1)^T \).

During the last two decades, a lot of effort has been devoted to the mathematical study of the primitive equations. To date, it has been known that the primitive equations, with full viscosity and full diffusivity, have global weak solutions (but the uniqueness is still unclear; see [31–33]) and have a unique global strong solution (see [10, 23, 25, 26], and also see [5, 6, 11, 29] for some recent developments in the direction of partial dissipation cases). Moreover, recent works [7–9] show that the horizontal viscosity turns out to be more crucial than the vertical one for the global well-posedness, because the results there show that the vertical viscosity is not required for the global well-posedness of strong solutions to the primitive equations. Notably, the inviscid primitive equations may develop finite time singularities, see [4, 46]. Combining the results of [7–9] and those of [4, 46], one can conclude that the horizontal viscosity is necessary for the global well-posedness of the primitive equations and, if ignoring the temperature effect, the horizontal viscosity is also sufficient for the global well-posedness.

1.2. The barotropic and the first baroclinic modes’ interaction system

In the tropics, the wind in the lower troposphere is of equal magnitude but with opposite sign to that in the upper troposphere, in other words, the primary effect is captured in the first baroclinic mode, in particular in areas of deep convection, where latent heating in the mid-troposphere creates the baroclinic structure. First baroclinic mode models have been used in many studies of tropical atmospheric dynamics dating back to [16, 38]. However, for the
study of tropical–extratropical interactions, where the transport of momentum between the barotropic and baroclinic modes plays an important role, it is necessary to retain both the barotropic and baroclinic modes of the velocity.

Consider the primitive equations (1.1) in the layer \( \mathbb{R}^2 \times (0, H) \), for a positive constant \( H \). Since we consider the tropical atmosphere and take into consideration the tropical–extratropical interactions, we can impose an ansatz of the form

\[
\begin{align*}
V(x, y, z, t) &= (u(x, y, t), v(x, y, t), p(x, y, t)) + \sqrt{\frac{2}{\pi}} \cos(\pi z/H), \\
W(x, y, z, t) &= (w(x, y, t), \theta(x, y, t)) + \sqrt{\frac{2}{\pi}} \sin(\pi z/H),
\end{align*}
\]

which carry the barotropic and first baroclinic modes of the unknowns.

By performing the Galerkin projection of the primitive equations in the vertical direction onto the barotropic mode and the first baroclinic mode, one derives the following dimensionless interaction between the barotropic mode and the first baroclinic mode, the system for the tropical atmosphere (see [34] and [15, 20, 36, 43] for details):

\[
\begin{align*}
\phi + \nabla \cdot \nabla u - \Delta u + \nabla p + \nabla \cdot (v \otimes v) &= 0, \\
\nabla \cdot u &= 0, \\
\phi + \nabla \cdot \nabla v - \Delta v + (v \cdot \nabla)u &= \nabla \theta, \\
\partial_t \theta + u \cdot \nabla \theta - \nabla \cdot v &= S_\theta,
\end{align*}
\]

where \( u = (u_1, u_2) \) is the barotropic velocity, and \( v = (v_1, v_2) \), \( p \) and \( \theta \), respectively, are the first baroclinic modes of the velocity, pressure and the temperature. The system is now defined on \( \mathbb{R}^2 \), and the operators \( \nabla \) and \( \Delta \) are therefore those for the variables \( x \) and \( y \).

We would like to point out that the systems derived in [15, 20, 34, 36, 43] are inviscid, because the authors’ starting point was the inviscid version of the primitive equations, to which the Galerkin projections on the relevant modes were performed. In other words, the Laplacian terms \( \Delta u \) and \( \Delta v \) in system (1.2) were not involved in the systems derived in the above-mentioned papers. However, in this paper we consider the corresponding viscous version of the systems derived in [15, 20, 34, 36, 43], i.e. system (1.2), which is derived following exactly the same procedure as in the above mentioned papers, from the viscous primitive equations (with full viscosity or with only horizontal viscosity). Physically, and due to the strong horizontal turbulent currents and mixing, it is natural to add a horizontal turbulence eddy viscosity. In view of this it is natural to add the horizontal viscous terms, which are understood as the eddy viscous ones, in the primitive equations for the large-scale atmosphere. As a result, recalling that we perform the Galerkin projection on the primitive equations in the vertical direction only, the horizontal viscous terms are kept in the deduction of system (1.2). Moreover, recalling the finite time blow up results [4, 46] for the inviscid primitive equations and the global well-posedness results [7–9] for the primitive equations with only horizontal viscosity, no matter how weak the horizontal viscosity is. Therefore, one cannot ignore the inherent dissipation mechanism caused by the horizontal viscosity of the atmosphere in the analytical study of the atmospheric dynamics. Because of the above, in this paper we focus on the viscous system, i.e. system (1.2). However, the interesting question of the global regularity, or possible finite time blow up, of the solutions of the corresponding inviscid system of (1.2) is a subject for future study.
1.3. The moisture equation

An important ingredient in the tropical atmospheric circulation is water vapour. Water vapour is the most abundant greenhouse gas in the atmosphere and it is responsible for amplifying the long-term warming or cooling cycles. Therefore, one should also consider the coupling with an equation modelling moisture in the atmosphere.

Following [15], we couple system (1.2) with the following large-scale moisture equation

\[ \partial_t q + u \cdot \nabla q + \bar{Q} \nabla \cdot v = -P, \]  

(1.3)

where \( \bar{Q} \) is the prescribed gross moisture stratification. The precipitation rate \( P \) is parameterized, according to [15, 21, 39, 43], as

\[ P = \frac{1}{\varepsilon} (q - \alpha \theta - \hat{q})^+, \]  

(1.4)

where \( f^+ = \max\{f, 0\} \) denotes the positive part of \( f \), \( \varepsilon \) is a convective adjustment time scale parameter, and \( \alpha \) and \( \hat{q} \) are constants, with \( \hat{q} > 0 \).

In order to close system (1.2)–(1.3), one still needs to parameterize the source term \( \theta_S \) in the temperature equation. Generally, the temperature source \( \theta_S \) combines three kinds of effects: the radiative cooling, the sensible heat flux and the precipitation \( P \). However, for simplicity, and as in [15, 37], we only consider in this paper the precipitation source term, i.e. we set

\[ \theta_S = P, \]

with \( P \) given by (1.4).

As in [15, 37], by introducing the equivalent temperature \( T_e \) and the equivalent moisture \( q_e \) as

\[ T_e = q + \theta, \quad q_e = q - \alpha \theta - \hat{q}, \]

system (1.2)–(1.3) can be rewritten as

\[ \partial_t u + (u \cdot \nabla) u - \Delta u + \nabla p + \nabla \cdot (v \otimes v) = 0, \]  

(1.5)

\[ \nabla \cdot u = 0, \]  

(1.6)

\[ \partial_t v + (u \cdot \nabla) v - \Delta v + (v \cdot \nabla) u = \frac{1}{1 + \alpha} \nabla(T_e - q_e), \]  

(1.7)

\[ \partial_t T_e + u \cdot \nabla T_e - (1 - \bar{Q}) \nabla \cdot v = 0, \]  

(1.8)

\[ \partial_t q_e + u \cdot \nabla q_e + (\bar{Q} + \alpha) \nabla \cdot v = -\frac{1 + \alpha}{\varepsilon} q_e^+, \]  

(1.9)

in \( \mathbb{R}^2 \times (0, \infty) \), where the constants \( \alpha \) and \( \bar{Q} \) are required to satisfy (see [15])

\[ 0 < \bar{Q} < 1, \quad \alpha + \bar{Q} > 0. \]  

(1.10)
1.4. Main results

We will work in the framework of strong solutions, which are defined below.

**Definition 1.1.** Given a positive time \( T \) and the initial data \((u_0, v_0, T_0, q_0)\), a function \((u, v, T, q)\) is called a strong solution to system (1.5)–(1.9), on \( \mathbb{R}^2 \times (0, T) \), with initial data \((u_0, v_0, T_0, q_0)\), if it enjoys the following regularities

\[
(u, v) \in C([0, T]; H^4(\mathbb{R}^2)) \cap L^2(0, T; H^2(\mathbb{R}^2)),
\]

\[
(\partial_t u, \partial_t v, \partial_t T, \partial_t q) \in L^2(0, T; L^2(\mathbb{R}^2)),
\]

\[
(T, q) \in C([0, T]; L^2(\mathbb{R}^2)) \cap L^\infty(0, T; H^2(\mathbb{R}^2)),
\]

and satisfies equations (1.5)–(1.9), a.e. on \( \mathbb{R}^2 \times (0, T) \), and has the initial value

\[
(u, v, T, q)|_{t=0} = (u_0, v_0, T_0, q_0).
\]

**Definition 1.2.** A function \((u, v, T, q)\) is called a global strong solution to system (1.5)–(1.9) if it is a strong solution to system (1.5)–(1.9) on \( \mathbb{R}^2 \times (0, T) \), for any positive time \( T \).

Throughout this paper, for positive integer \( k \) and positive \( q \in [1, \infty] \), we use \( L^q(\mathbb{R}^2) \) and \( W^{k,q}(\mathbb{R}^2) \) to denote the standard Lebesgue and Sobolev spaces, respectively, and when \( q = 2 \) we use \( H^k(\mathbb{R}^2) \) instead of \( W^{k,2}(\mathbb{R}^2) \). For simplicity, we usually use \( \|f\|_{L^q(\mathbb{R}^2)} \) to denote \( \|f\|_{L^q(\mathbb{R}^2)} \).

The first main result of this paper is on the global existence, uniqueness and well-posedness of the strong solutions to the Cauchy problem of system (1.5)–(1.9).

**Theorem 1.1.** Suppose that (1.10) holds, and the initial data

\[
(u_0, v_0, T_0, q_0) \in H^1(\mathbb{R}^2), \quad \text{with} \quad \nabla \cdot u_0 = 0.
\]

Then, we have the following:

(i) There is a unique global strong solution \((u, v, T, q)\) to system (1.5)–(1.9) with initial data \((u_0, v_0, T_0, q_0)\) such that

\[
\sup_{0 \leq t \leq T} \left( \|u, v, T, q(t)\|_{H^2}^2 + \int_0^T \left( \|\nabla^2 u\|_{H^2}^2 + \|\nabla^2 v\|_{H^2}^2 + \|\nabla u\|_{H^2} \right) dt \right) + \int_0^T \|\nabla (u, v, T, q)\|_{H^2}^2 dt \leq C(\alpha, \tilde{Q}, T, \|(u_0, v_0, T_0, q_0)\|_{H^2}),
\]

for any positive time \( T \), here and in what follows, we use \( C(\cdots) \) to denote a general positive constant depending only on the quantities in the parenthesis.

(ii) Suppose, in addition to (1.11), that \( q_{e,0} \leq 0 \), a.e. on \( \mathbb{R}^2 \), then

\[
\sup_{0 \leq t \leq T} \left( \|q_{e,0}(t)\|_{L^1}^2 + \int_0^T \|\partial_t q_{e,0}\|_{L^2}^2 dt \right) \leq C(\alpha, \tilde{Q}, T, \|(u_0, v_0, T_0, q_{e,0})\|_{H^2}),
\]

for any positive time \( T \).
(iii) Suppose, in addition to (1.11), that $(\nabla T_{e,0}, \nabla q_{e,0}) \in L^m(\mathbb{R}^2)$ for some $m \in (2, \infty)$, then the following estimate holds:

$$
\sup_{0 \leq t \leq T} \|(\nabla T, \nabla q_{c})(t)\|_{L^2}^2 \leq C(\alpha, \bar{Q}, T, \|(u_{0,0}, v_{0,0}, T_{e,0}, q_{e,0})\|_{H^p}, \|(\nabla T_{e,0}, \nabla q_{e,0})\|_{L^m}),
$$

for any positive time $T$, and the unique strong solution $(u, v, T, q_{c})$ depends continuously on the initial data on any finite interval of time.

Formally, by taking the relaxation limit as $\varepsilon \to 0^+$, system (1.5)–(1.9) will converge to the following limiting system:

$$
\begin{align*}
\partial_t u + (u \cdot \nabla) u - \mu \Delta u + \nabla p + \nabla \cdot (v \otimes v) &= 0, & (1.12) \\
\nabla \cdot u &= 0, & (1.13) \\
\partial_t v + (u \cdot \nabla) v - \mu \Delta v + (v \cdot \nabla) u &= \frac{1}{1 + \alpha} \nabla (T - q_e), & (1.14) \\
\partial_t T_e + u \cdot \nabla T_e - (1 - \bar{Q}) \nabla \cdot v &= 0, & (1.15) \\
\partial_t q_e + u \cdot \nabla q_e + \bar{Q} \nabla \cdot v &\leq 0, & (1.16) \\
q_e &\leq 0, & (1.17) \\
\partial_t q_e + u \cdot \nabla q_e + (\bar{Q} + \alpha) \nabla \cdot v &= 0, & \text{a.e. on } \{q_e < 0\}. & (1.18)
\end{align*}
$$

Note that equation (1.9) is now replaced by two inequalities (1.16)–(1.17) and one equality (1.18).

Inequality (1.16) comes from equation (1.9) by noticing the negativity of the term $\frac{1 - \bar{Q}}{1 + \alpha} q_e$; while inequality (1.17) is derived by multiplying both sides of equation (1.9) by $\varepsilon$ and taking the formal limit $\varepsilon \to 0^+$. Inequality (1.18) can be derived by the following heuristic argument. Let $(u_\varepsilon, v_\varepsilon, T_{e,\varepsilon}, q_{e,\varepsilon})$ be a solution to system (1.5)–(1.9) and suppose that $(u_\varepsilon, v_\varepsilon, T_{e,\varepsilon}, q_{e,\varepsilon})$ converges to $(u, v, T_e, q_e)$, with $q_e \leq 0$, for any compact subset $K$ of the set $\{(x, y, t) \in \mathbb{R}^2 \times (0, \infty) : q_e(x, y, t) < 0\}$, since $q_{e,\varepsilon}$ converges to $q_e$, one may have $q_{e,\varepsilon} > 0$ on $K$ for sufficiently small positive $\varepsilon$. Therefore, by equation (1.9), it follows that $\partial_t q_{e,\varepsilon} + u_\varepsilon \cdot \nabla q_{e,\varepsilon} + (\bar{Q} + \alpha) \nabla \cdot v = 0$ a.e. on $K$ from which, by taking $\varepsilon \to 0^+$, one can see that (1.18) is satisfied a.e. on $K$ and furthermore a.e. on $\{q_e < 0\}$.

The other aim of this paper is to prove the global existence and uniqueness of strong solutions to the limiting system (1.12)–(1.18) and rigorously justify the above formal convergences as $\varepsilon \to 0^+$. Strong solutions to system (1.12)–(1.18) are defined in a similar way as those to system (1.5)–(1.9).

**Theorem 1.2.** Suppose that (1.10) holds and the initial data

$$(u_0, v_0, T_{e,0}, q_{e,0}) \in H^1(\mathbb{R}^2), \quad \nabla \cdot u_0 = 0, \quad q_{e,0} \leq 0, \quad \text{a.e. on } \mathbb{R}^2.$$
Then, there is a unique global strong solution \((u, v, T, q)\) to system (1.12)–(1.18), with initial data \((u_0, v_0, T_0, q_0)\), such that
\[
\begin{align*}
\sup_{\varepsilon \in [0, T]} \|u, v, T, q\|_{H^1}^2 + \int_0^T \left( \|u, v\|_{H^1}^2 + \|\nabla u\|_\infty + \|\partial u, \partial v, \partial T, \partial q\|_2^2 \right) dt \\
\leq C(\alpha, \bar{Q}, T_0, \|(u_0, v_0, T_0, q_0)\|_{H^1}),
\end{align*}
\]
for any positive time \(T\).

If we assume, in addition, that \((\nabla T, \nabla q) \in L^m(\mathbb{R}^2)\) for some \(m \in (2, \infty)\), then we have further that
\[
\sup_{\varepsilon \in [0, T]} \|\nabla T, \nabla q\|_{L^m}^2 \leq C(\alpha, \bar{Q}, T, \|(u_0, v_0, T_0, q_0)\|_{H^1}, \|\nabla T, \nabla q\|_{L^m}),
\]
for any positive time \(T\), and the unique strong solution \((u, v, T, q)\) depends continuously on the initial data.

**Theorem 1.3.** Suppose that (1.10) holds and the initial data
\[
\begin{align*}
(u_0, v_0, T_0, q_0) &\in H^1(\mathbb{R}^2), \quad \nabla \cdot u_0 = 0, \\
(\nabla T_0, \nabla q_0) &\in L^m(\mathbb{R}^2), \quad q_{e,0} \leq 0, \text{ a.e. on } \mathbb{R}^2,
\end{align*}
\]
for some \(m \in (2, \infty)\). Denote by \((u_\varepsilon, v_\varepsilon, T_\varepsilon, q_\varepsilon)\) and \((u, v, T, q)\) the unique global strong solutions to systems (1.5)–(1.9) and (1.12)–(1.18), respectively, with the same initial data \((u_0, v_0, T_0, q_0)\).

Then, we have the estimate
\[
\begin{align*}
\sup_{\varepsilon \in [0, T]} \|u_\varepsilon - u, v_\varepsilon - v, T_\varepsilon - T, q_\varepsilon - q\|_{L^2}^2 \\
+ \int_0^T \left( \|\nabla(u_\varepsilon - u, \nabla(v_\varepsilon - v))\|_2^2 + \frac{\|q_\varepsilon^+\|_2^2}{\varepsilon} \right) dt \leq C\varepsilon,
\end{align*}
\]
for any finite positive time \(T\), where \(C\) is a positive constant depending only on \(\alpha, \bar{Q}, m, T\), and the initial norm \(\|(u_0, v_0, T_0, q_0)\|_{H^1} + \|\nabla T_0, \nabla q_0\|_{L^m}\).

Therefore, in particular, we have the convergences
\[
\begin{align*}
(u_\varepsilon, v_\varepsilon) &\to (u, v) \quad \text{in } L^\infty(0, T; L^2(\mathbb{R}^2)) \cap L^2(0, T; H^1(\mathbb{R}^2)), \\
(T_\varepsilon, q_\varepsilon) &\to (T, q) \quad \text{in } L^\infty(0, T; L^2(\mathbb{R}^2)), \quad q_\varepsilon^+ \to 0 \quad \text{in } L^2(0, T; L^2(\mathbb{R}^2)),
\end{align*}
\]
for any positive time \(T\), and the convergence rate is of order \(O(\sqrt{\varepsilon})\).

**Remark 1.1.**

(i) In the absence of the barotropic mode, the global existence and uniqueness of strong solutions to the inviscid limiting system were proved in [37] and the relaxation limit, as \(\varepsilon \to 0^+\), was also studied there, but the convergence rate was not achieved. Note that in
the absence of the barotropic mode the limiting system is linear, while in the presence of the barotropic mode the limiting system is nonlinear.

(ii) The existence and uniqueness of solutions to the limiting system (1.12)–(1.18), without viscosity, was proposed as an open problem in [15] and also in [22, 35, 37]. Notably, theorem 1.2 settles this open problem for the viscous version of (1.12)–(1.18). Note that we only add viscosity to the velocity equations and we do not use any diffusivity in the temperature and moisture equations.

**Remark 1.2.** The global well-posedness of strong solutions to a coupled system of the primitive equations with moisture (therefore, it is a different system from those considered in this paper) was recently addressed in [12], where the system under consideration has full dissipation in all dynamical equations and, in particular, has diffusivity in the temperature and moisture equations. Note that we do not need any diffusivity in the temperature and moisture equations in order to establish global regularity of the systems considered in this paper. It is worth mentioning that the global regularity of the coupled three-dimensional primitive equations with moisture and with partial dissipation is the subject of a forthcoming paper.

The rest of this paper is organized as follows: in section 2, we state and prove several preliminary lemmas, while the proofs of theorems 1.1–1.3 are given in section 3–5, respectively. The last section is an appendix in which we prove some parabolic estimates that are used in this paper and which are of general interest on their own.

### 2. Preliminaries

We will frequently use the following Ladyzhenskaya inequality (see, e.g. [27])

\[
\|f\|_{L^r(\mathbb{R}^2)} \leq C \|f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|\nabla f\|_{L^4(\mathbb{R}^2)}^{\frac{1}{2}}, \quad \forall f \in H^1(\mathbb{R}^2).
\]

The following lemma on the Gronwall type inequality will be used to establish the global in time a priori estimates to the strong solutions to system (1.5)–(1.9) later.

**Lemma 2.1.** Given a positive time \( T \), a positive integer \( n \) and positive numbers \( r_i \in [1, \infty), 1 \leq i \leq n \). Let \( a_0, a_i \ and \ b_i, 1 \leq i \leq n \), be non-negative functions, such that \( a_0, a_i \in L^\infty((0, T)) \) and \( b_i \in L^1((0, T)) \). Suppose that the non-negative measurable function \( f \) satisfies

\[
f(t) \leq a_0(t) + \sum_{i=1}^{n} a_i(t) \left( \int_0^t b_i(s) f^r(s) ds \right)^{\frac{1}{r}},
\]

for any \( t \in [0, T] \). Then, the following holds

\[
\|f\|_{L^\infty(0, T)} \leq (n + 1) \|a_0\|_{\infty} \exp \left\{ (n + 1)^{-1} \sum_{i=1}^{n} \|a_i\|_{\infty} (1 + \|b_i\|)^{r+1} \right\},
\]

where \( r = \max_{1 \leq i \leq n} r_i \) and \( \|\cdot\|_1 \) and \( \|\cdot\|_\infty \) denote the \( L^1((0, T)) \) and \( L^\infty((0, T)) \) norms, respectively.
Proof. By the Hölder and Young inequalities, we deduce

\[
\left( \int_0^t b(s) f''(s) \, ds \right)^{\frac{1}{r}} \leq \left[ \int_0^t b_0^{\frac{r-1}{r}} \left( \frac{1}{r} b_1^r \right) f(s) \, ds \right]^\frac{1}{r} \\
\leq \left( \int_0^t b(s) \, ds \right)^{\frac{r-1}{r}} \left( \int_0^t b_1(s) f''(s) \, ds \right)^\frac{1}{r} \\
\leq (1 + \|b_1\|) \left( \int_0^t b_1(s) f''(s) \, ds \right)^\frac{1}{r},
\]

for \( 1 \leq i \leq n \). Therefore, by assumption, we have

\[
f(t) \leq \|a_0\|_{\infty} + \sum_{i=1}^n \|a_i\|_{\infty} (1 + \|b_i\|) \left( \int_0^t b_1(s) f''(s) \, ds \right)^{\frac{1}{r}},
\]

from which, taking the \( r \)th powers to both sides of the above inequality and using the elementary inequality \( \left( \sum_{i=0}^n c_i \right)^r \leq (n + 1)^{-1} \sum_{i=0}^n c_i^r \), where \( c_i \) are positive numbers, we arrive at

\[
f(t) \leq (n + 1)^{-1} \|a_0\|_{\infty}^r + (n + 1)^{-1} \sum_{i=1}^n \|a_i\|_{\infty}^r (1 + \|b_i\|)^r \left( \int_0^t b_1(s) f''(s) \, ds \right).
\]

Applying the Gronwall inequality to the above inequality, we have

\[
f'(t) \leq (n + 1)^{-1} \|a_0\|_{\infty}^r \exp \left\{ (n + 1)^{-1} \sum_{i=1}^n \|a_i\|_{\infty}^r (1 + \|b_i\|)^r \int_0^t b_1(s) \, ds \right\} \\
\leq (n + 1)^{-1} \|a_0\|_{\infty}^r \exp \left\{ (n + 1)^{-1} \sum_{i=1}^n \|a_i\|_{\infty}^r (1 + \|b_i\|)^{1 + r} \right\},
\]

from which, taking the \( r \)th power root to both sides of the above inequality, and taking the supremum with respective to \( t \) over \([0, T]\), one obtains the proof. \( \square \)

The next lemma will be employed to prove the uniqueness of strong solutions.

Lemma 2.2. Given a positive time \( T \), and let \( m_1, m_2 \) and \( S \) be non-negative functions on \((0, T)\), such that

\[
m_1, S \in L^1((0, T)), \quad m_2 \in L^2((0, T)), \quad \text{and} \quad S > 0, \text{a.e. on } (0, T).
\]

Suppose that \( f \) and \( G \) are two non-negative functions on \((0, T)\), with \( f \) being absolutely continuous on \([0, T]\), that satisfy

\[
\begin{aligned}
f'(t) + G(t) &\leq m_1(t) f(t) + m_2(t) \left[ f(t) G(t) \log \left( \frac{S(t)}{G(t)} \right) \right]^\frac{1}{2}, \quad \text{a.e. on } (0, T), \\
f(0) &= 0,
\end{aligned}
\]

where \( \log^+ z = \max \{0, \log z\} \) for \( z \in (0, \infty) \), and when \( G(t) = 0 \), at some time \( t \in [0, T] \), we adopt the following natural convention.
\[ G(t) \log \left( \frac{S(t)}{G(t)} \right) = \lim_{z \to 0^+} z \log^z \left( \frac{S(t)}{z} \right) = 0. \]

Then, we have \( f \equiv 0 \) on \([0, T]\).

**Proof.** Suppose, by contradiction, that there is some time \( t_0 \in (0, T) \), such that \( f(t_0) > 0 \). Recalling that \( f \) is absolutely continuous on \([0, T]\), by the property of continuous functions, there must be a time \( t_0 \in (0, T) \), such that \( f(t_0) = 0 \) and \( f(t) > 0 \), for any \( t \in (t_0, t] \). In the rest of the proof, we will focus on the time interval \([t_0, T]\).

For any \( \sigma \in (0, \infty) \), one can easily check that

\[ \log^\sigma z \leq \frac{z^\sigma}{\sigma e}, \quad \text{for } z \in (0, \infty). \]

Recall the Young inequality of the form \( ab \leq \frac{a^p}{p} + \frac{b^q}{q} \), for any non-negative numbers \( a, b \) and for any \( p, q \in (1, \infty) \), with \( \frac{1}{p} + \frac{1}{q} = 1 \). Thanks to the above inequality, and choosing \( \sigma \in (0, 1) \), it follows from the assumption and the Young inequality that

\[ f' + G \leq m_1 f + m_2 \left( f \frac{1}{e \sigma} \left( \frac{S(t)}{G(t)} \right)^{\sigma} \right)^{\frac{1}{\sigma}} \]

\[ \leq m_1 f + \frac{1 - \sigma}{2} G + \frac{1 + \sigma}{2} m_2 \left( \frac{f}{e \sigma} \right)^{\frac{2}{1+\sigma}} \]

\[ = m_1 f + \frac{1 - \sigma}{2} G + \frac{1 + \sigma}{2} m_2 S^{\frac{\sigma}{1+\sigma}} \left( \frac{f}{e \sigma} \right)^{\frac{1}{1+\sigma}} \]

\[ \leq m_1 f + G + m_2^{\frac{2}{1+\sigma}} S^{\frac{\sigma}{1+\sigma}} \left( \frac{f}{\sigma} \right)^{\frac{1}{1+\sigma}}, \quad \text{a.e. on } (0, T). \]

Note that the arguments used in the above inequality are for the time when \( G(t) > 0 \). However, for the time when \( G(t) = 0 \), recalling that we understood the term involving \( G \) as zero, the above inequality result holds trivially. Therefore, we obtain

\[ f' \leq m_1 f + m_2^{\frac{2}{1+\sigma}} S^{\frac{\sigma}{1+\sigma}} \left( \frac{f}{\sigma} \right)^{\frac{1}{1+\sigma}}, \]

for any \( \sigma \in (0, 1) \), and for a.e. \( t \in [t_0, t] \). Recall that \( f(t) > 0 \), for \( t \in (t_0, t] \). Dividing both sides of the above inequality by \( f^{\frac{1}{1+\sigma}} \), one can deduce

\[ \left( f^{\frac{\sigma}{1+\sigma}} \right) \leq \frac{\sigma}{1 + \sigma} m_1 f^{\frac{\sigma}{1+\sigma}} + \frac{\sigma^{\frac{1}{1+\sigma}}}{1 + \sigma} m_2^{\frac{2}{1+\sigma}} S^{\frac{\sigma}{1+\sigma}} \]

\[ \leq \frac{\sigma}{1 + \sigma} m_1 f^{\frac{\sigma}{1+\sigma}} + \sigma^{\frac{1}{1+\sigma}} m_2^{\frac{2}{1+\sigma}} S^{\frac{\sigma}{1+\sigma}}, \]

for a.e. \( t \in (t_0, t] \). Applying the Gronwall inequality to the above inequality, and recalling that \( f(t_0) = 0 \), it follows from the Hölder inequality that

\[ \ldots \]
\[
\int_0^t f(s) ds \leq \int_0^t \int_0^s (m_2(s) - m_3(s)) ds ds
\]

from which, taking the \(1+\sigma\)th power to both sides of the above inequality, one obtains

\[
f(t) \leq \sigma \left( \int_0^t m_2(s) ds \right)^{\frac{1+\sigma}{\sigma}} \int_0^t S(s) ds,
\]

for any \(t \in [t_0, t_1]\) and for any \(\sigma \in (0, 1)\). By taking \(\sigma \to 0^+\), this implies that \(f \equiv 0\), for any \(t \in [t_0, t_1]\), which contradicts the assumption that \(f(t) > 0\), for any \(t \in (t_0, t_\ast)\). This contradiction implies that there is no such \(t_\ast \in (0, T)\) that \(f(t_\ast) > 0\), in other words, recalling that \(f\) is a non-negative function, we have \(f \equiv 0\) on \([0, T)\). This completes the proof. □

We also will use the following elementary lemma.

**Lemma 2.3.** Let \(\Omega \subseteq \mathbb{R}^d\) be a measurable set of positive measure and \(f\) be a measurable function defined on \(\Omega\). Suppose that for any positive number \(\eta \leq t_\ast - t_0\), such that \(\int_{t_0}^{t_\ast} m_2(s) ds \leq 1\) for any \(t \in [t_0, t_1 + \eta]\). Therefore, the above inequality implies

\[
f(t) \leq \sigma \left( \int_0^t m_2(s) ds \right)^{\frac{1}{\sigma}} \int_0^t S(s) ds,
\]

for any \(t \in [t_0, t_1 + \eta]\) and for any \(\sigma \in (0, 1)\). By taking \(\sigma \to 0^+\), this implies that \(f \equiv 0\), for any \(t \in [t_0, t_1 + \eta]\), which contradicts the assumption that \(f(t) > 0\), for any \(t \in (t_0, t_\ast)\). This contradiction implies that there is no such \(t_\ast \in (0, T)\) that \(f(t_\ast) > 0\), in other words, recalling that \(f\) is a non-negative function, we have \(f \equiv 0\) on \([0, T)\). This completes the proof. □

3. **Global existence and uniqueness of the system with positive \(\varepsilon\)**

In this section, we will prove the global existence and uniqueness of strong solutions to the Cauchy problem of system (1.5)–(1.9) for any positive \(\varepsilon\). Several \(\varepsilon\)-independent *a priori* estimates will also be obtained.

Let us start with the following result on the local existence and uniqueness of strong solutions to the Cauchy problem to system (1.5)–(1.9).
Proposition 3.1. Suppose that (1.10) holds. Then, for any initial data
\[(u_0, v_0, T_e, q_e, 0) \in H^1(\mathbb{R}^3), \text{ with } \nabla \cdot u_0 = 0,\]
there is a unique local strong solution \((u, v, T_e, q_e)\) to system (1.5)–(1.9) on \(\mathbb{R}^2 \times (0, T)\), with initial data \((u_0, v_0, T_e, q_e, 0)\), where the existence time \(T\) depends on \(\alpha, Q, \varepsilon\) and the initial norm \(||(u_0, v_0, T_e, q_e, 0)||_{H^1}\).

Proof.

(i) The existence. The existence of strong solutions to system (1.5)–(1.9), with initial data \((u_0, v_0, T_e, q_e, 0)\) can be proven by the standard regularization argument as follows.

(a) Adding the diffusivity terms \(-\eta \Delta T_e\) and \(-\eta \Delta q_e\) to the left-hand sides of equations (1.8) and (1.9), respectively. In other words, we consider the following regularized system
\[
\begin{aligned}
\partial_t u + (u \cdot \nabla) u - \Delta u + \nabla p + \nabla \cdot (v \otimes v) &= 0, \\
\nabla \cdot u &= 0, \\
\partial_t v + (u \cdot \nabla) v - \Delta v + (v \cdot \nabla) u &= \frac{1}{1 + \alpha} \nabla (T_e - q_e), \\
\partial_t T_e + u \cdot \nabla T_e - (1 - \tilde{Q}) \nabla \cdot v - \eta \Delta T_e &= 0, \\
\partial_t q_e + u \cdot \nabla q_e + (\tilde{Q} + \alpha) \nabla \cdot v - \eta \Delta q_e &= -\frac{1 + \alpha}{\varepsilon} q_e^+.
\end{aligned}
\]
(b) For each \(\eta > 0\), the Cauchy problem of the regularized system (3.1) with initial data \((u_0, v_0, T_{e,0}, q_{e,0})\) has a unique short time strong solution \((\tilde{u}^{(\eta)}, \tilde{v}^{(\eta)}, \tilde{T}_{e, \eta}, \tilde{q}_{e, \eta})\), which satisfies some \(\eta\)-independent \textit{a priori} estimates on some \(\eta\)-independent time interval \((0, T)\) for a positive time \(T\) depending only on \(\alpha, Q, \varepsilon\) and the initial norm \(||(u_0, v_0, T_{e,0}, q_{e,0})||_{H^1}||\).
(c) Thanks to these \(\eta\)-independent estimates, by adopting the Cantor diagonal argument, one can apply the Aubin–Lions lemma and take the limit \(\eta \to 0^+\) to show the local existence of strong solutions to the Cauchy problem of system (1.5)–(1.9) with initial data \((u_0, v_0, T_{e,0}, q_{e,0})\). Since the proof is standard we omit it here. However, the key part of the proof, i.e. the relevant \textit{a priori} estimates, are essentially contained in the ‘formal’ proofs of propositions 3.2–3.5, below. As was mentioned above, these formal estimates can be rigorously justified by establishing them first to be \(\eta\)-independent for the regularized system (3.1) and then passing with the limit as \(\eta \to 0^+\).

(ii) The uniqueness. Let \((u, v, T_e, q_e)\) and \((\tilde{u}, \tilde{v}, \tilde{T}_e, \tilde{q}_e)\) be two strong solutions to system (1.5)–(1.9), with the same initial data \((u_0, v_0, T_{e,0}, q_{e,0})\) on the time interval \((0, T)\). Define the new functions
\[
(\delta u, \delta v, \delta T_e, \delta q_e) = (u, v, T_e, q_e) - (\tilde{u}, \tilde{v}, \tilde{T}_e, \tilde{q}_e).
\]
Then, one can easily check that
\[
\partial_t \delta u + (u \cdot \nabla) \delta u + (\delta u \cdot \nabla) \tilde{u} - \Delta \delta u + \nabla \delta p + \nabla \cdot (v \otimes \delta v + \delta v \otimes \tilde{v}) = 0,
\]
\[
\nabla \cdot \delta u = 0,
\]

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\[\partial_t \delta v + (u \cdot \nabla) \delta v + (\delta u \cdot \nabla) \delta v - \Delta \delta v + (v \cdot \nabla) \delta u = 0, \tag{3.4}\]

\[\partial_t \delta T_e + u \cdot \nabla \delta T_e + \delta u \cdot \nabla \delta T_e - (1 - \tilde{Q}) \nabla \cdot \delta v = 0, \tag{3.5}\]

\[\partial_t \delta q_e + u \cdot \nabla \delta q_e + \delta u \cdot \nabla \delta q_e + (\tilde{Q} + \alpha) \nabla \cdot \delta v = -\frac{1}{\varepsilon} \frac{1}{1 + \alpha} (q_e^+ - q_e^-). \tag{3.6}\]

Since equations (3.2)–(3.5) hold in \(L^2(0, T; L^2(\mathbb{R}^2))\), we multiply equations (3.2), (3.4) and (3.5) by \(\delta u, \delta v\) and \(\delta T_e\), respectively, and integrate over \(\mathbb{R}^2\). Then it follows from integration by parts that

\[
\frac{1}{2} \frac{d}{dt} ([\|\delta u\|_2^2 + \|\delta v\|_2^2 + \|\delta T_e\|_2^2]) + \|\nabla \delta u\|_2^2 + \|\nabla \delta v\|_2^2
\]

\[
= -\int_{\mathbb{R}^2} \left\{[(\delta u \cdot \nabla) \delta v + (v \cdot \nabla) \delta u + (\delta u \cdot \nabla) \delta u] \cdot \delta v + \frac{\delta T_e - \delta q_e}{1 + \alpha} \nabla \cdot \delta v \right\} dx dy
\]

\[
= -\int_{\mathbb{R}^2} [\delta v \cdot \nabla T_e - (1 - \tilde{Q}) \nabla \cdot \delta v] \delta T_e dx dy =: I_1 + I_2 + I_3. \tag{3.7}\]

By the Young inequality, we can estimate \(I_1, I_2\) and \(I_3\) as follows:

\[
I_1 \leq \int_{\mathbb{R}^2} \left[|\delta u||\nabla \delta u| + (|v| + |\tilde{v}|)|\nabla \delta v| + (|\nabla v| + |\nabla \tilde{v}|)|\delta v|\right]|\delta u| dx dy
\]

\[
\leq \frac{1}{6} \int_{\mathbb{R}^2} |\nabla \delta v|^2 dx dy + C \int_{\mathbb{R}^2} (|\nabla v| + |\nabla \tilde{v}|)(|\delta u|^2 + |\delta v|^2) dx dy
\]

\[
+ C \int_{\mathbb{R}^2} (|v|^2 + |\tilde{v}|^2)(|\delta u|^2 + |\delta v|^2) dx dy,
\]

\[
I_2 \leq \int_{\mathbb{R}^2} \left[|\delta u||\nabla \tilde{v}| + |v||\nabla \delta u| + |\delta u||\nabla \delta u|\right]|\delta v| + \frac{|\delta T_e| + |\delta q_e|}{1 + \alpha} |\nabla \delta v| dx dy
\]

\[
\leq \frac{1}{6} \int_{\mathbb{R}^2} |\nabla \delta v|^2 dx dy + C \int_{\mathbb{R}^2} (|\nabla v| + |\nabla \tilde{v}|)(|\delta u|^2 + |\delta v|^2) dx dy
\]

\[
+ C \int_{\mathbb{R}^2} (|\delta T_e|^2 + |\delta q_e|^2) dx dy,
\]

and

\[
I_3 \leq \int_{\mathbb{R}^2} \left[|\delta u||\nabla T_e||\delta T_e| + (1 - \tilde{Q}) |\nabla \delta v||\delta T_e|\right] dx dy
\]

\[
\leq \frac{1}{6} \int_{\mathbb{R}^2} |\nabla \delta v|^2 dx dy + C \int_{\mathbb{R}^2} (|\delta T_e|^2 + |\nabla T_e||\delta u||\delta T_e|) dx dy.
\]

Substituting the above estimates on \(I_1, I_2\) and \(I_3\) into (3.7) we obtain
\[
\frac{d}{dt} \left[ \| (\delta u, \delta v, \delta T_e) \|^2 + \| \nabla \delta u \|^2 + \| \delta T_e \|^2 \right] \\
\leq C \int_{\mathbb{R}^2} \left[ |\nabla \delta u| + |\nabla \delta v| + |\nabla \delta T_e| \right] \delta u \delta T_e \, dx \, dy,
\]
(3.8)

Multiplying equation (3.6) by \( \delta q_e \), integrating the resultant over \( \mathbb{R}^2 \). Then it follows from integration by parts and the Young inequality that

\[
\frac{1}{2} \frac{d}{dt} \| \delta q_e \|^2 + \frac{1 + \alpha}{\varepsilon} \int_{\mathbb{R}^2} (q_e^+ - \tilde{q}_e^-)(q_e - \tilde{q}_e) \, dx \, dy
\]

\[
= - \int_{\mathbb{R}^2} (\delta u \cdot \nabla \tilde{q}_e + (\alpha + \tilde{Q}) \nabla \cdot \delta q_e \delta q_e \, dx \, dy
\]

\[
\leq \frac{1}{4} \| \nabla \delta v \|^2 + C \int_{\mathbb{R}^2} (|\delta q_e|^2 + |\nabla \delta q_e| |\delta q_e|) \, dx \, dy.
\]

from which, noticing that the function \( z^+ \) is non-decreasing in \( z \), \((q_e^+ - \tilde{q}_e^-)(q_e - \tilde{q}_e) \geq 0 \) and one obtains

\[
\frac{d}{dt} \| \delta q_e \|^2 \leq \frac{1}{2} \| \nabla \delta v \|^2 + C \int_{\mathbb{R}^2} (|\delta q_e|^2 + |\nabla \delta q_e| |\delta q_e|) \, dx \, dy.
\]

Summing the above inequality with (3.8) yields

\[
\frac{d}{dt} \left[ \| (\delta u, \delta v, \delta T_e, \delta q_e) \|^2 + \frac{1}{2} (\| \nabla \delta u \|^2 + \| \nabla \delta v \|^2) \right]
\]

\[
\leq C \int_{\mathbb{R}^2} \left[ |\nabla \delta u| + |\nabla \delta v| + |\nabla \delta T_e| \right] \delta u \delta T_e \, dx \, dy,
\]
(3.9)

from which, by the Hölder, Ladyzhenskay and Young inequalities, we deduce

\[
\frac{d}{dt} \left[ \| (\delta u, \delta v, \delta T_e, \delta q_e) \|^2 + \frac{1}{2} (\| \nabla \delta u \|^2 + \| \nabla \delta v \|^2) \right]
\]

\[
\leq C \left( \| \nabla \delta u \|^2 + \| \nabla \delta v \|^2 \right) \| \delta u \| \| \delta v \| \| \delta T_e \| \| \delta q_e \|
\]

\[
+ C \left( \| \delta T_e \| \| \delta q_e \| \right) \| \delta u \| \| \delta v \| \| \delta T_e \| \| \delta q_e \|
\]

\[
\leq \frac{1}{4} \left( \| \nabla \delta u \|^2 + C \left( \| \nabla \delta u \|^2 + \| \delta u \| \| \delta v \| \right) \right) \| \delta T_e \| \| \delta q_e \|
\]

\[
+ C \left( \| \delta T_e \| \| \delta q_e \| \right) \| \delta u \| \| \delta v \| \| \delta T_e \| \| \delta q_e \|
\]

Therefore, one has

\[
\frac{d}{dt} \left[ \| (\delta u, \delta v, \delta T_e, \delta q_e) \|^2 + \frac{1}{4} (\| \delta u \| \| \delta v \|) \right] \\
\leq C \left( 1 + \| (\delta u, \delta v) \|^2 + \| \nabla \delta u \|^2 + \| \nabla \delta v \|^2 \right) \| \delta u \| \| \delta v \| \| \delta T_e \| \| \delta q_e \|
\]

\[
+ C \left( \| \nabla \delta T_e \| \| \nabla \delta q_e \| \right) \| \delta u \| \| \delta v \| \| \delta T_e \| \| \delta q_e \|.
\]
(3.10)
Recalling the following Brezis–Gallouet–Wainger inequality (see \([2, 3]\) )

\[
\|f\|_{L^\infty(\mathbb{R}^2)} \leq C\|f\|_{H^1(\mathbb{R}^2)} \log^\frac{1}{\epsilon} \left( \frac{\|f\|_{L^2(\mathbb{R}^2)}}{\|f\|_{H^1(\mathbb{R}^2)}} + e \right),
\]

and denoting \( U = (u, v) \), \( \tilde{U} = (\tilde{u}, \tilde{v}) \) and \( \delta U = (\delta u, \delta v) \), we have

\[
\|\delta U\|_{L^\infty} \leq C\|\delta U\|_{H^1} \log^\frac{1}{\epsilon} \left( \frac{\|\delta U\|_{H^1}^2}{\|\delta U\|_{H^1}^2 + e} + 1 \right) \leq C\|\delta U\|_{H^1} \log^\frac{1}{\epsilon} \left( \frac{S(t)}{\|\delta U\|_{H^1}^2} \right)
\]

\[
= C \left( \|\delta U\|_{H^1}^2 \log^\frac{1}{\epsilon} \left( \frac{S(t)}{\|\delta U\|_{H^1}^2} \right) \right)^{\frac{1}{2}},
\]

(3.11)

where

\[
S(t) = \|U\|_{H^1}^2 + \|	ilde{U}\|_{H^1}^2 + e(\|U\|_{H^1}^2 + \|	ilde{U}\|_{H^1}^2).
\]

Note that when \( \delta U \equiv 0 \) (3.11) still holds, as long as we understand the quantity on the right-hand side as zero, in the natural way as in lemma 2.2.

Denoting

\[
f = \|\delta u, \delta v, \delta T_c, \delta q_c\|_{L^1}^2,
\]

\[
G = \frac{1}{4} \|\delta u, \delta v\|_{H^1}^2,
\]

\[
m_1 = C(1 + \|(\tilde{u}, \tilde{v})\|_{L^6}^2 + \|\nabla \tilde{u}, \nabla \tilde{v}\|_{L^6}^2),
\]

\[
m_2 = C(\|\nabla \tilde{T}_c, \nabla \tilde{q}_c\|_{L^6})
\]

then it follows from (3.10) and (3.11) that

\[
f' + G \leq m_1 f + m_2 \left[ fG \log \left( \frac{S/4}{G} \right) \right]^{\frac{1}{2}}.
\]

Here, at the time when \( G(t) = 0 \), the term involving \( G(t) \) on the right-hand side of the above inequality is understood as zero, as it was in lemma 2.2. Recalling the regularities of \((u, v, T_c, q_c)\) and \((\tilde{u}, \tilde{v}, \tilde{T}_c, \tilde{q}_c)\), one can easily check, thanks to the Ladyzhanskaya inequality, that \( m_1, S \in L^1((0, T)) \) and \( m_2 \in L^2((0, T)) \). Therefore, we can apply lemma 2.2 to conclude that \( f \equiv 0 \), which proves the uniqueness.
Navier–Stokes equations. Observe, however, that for the rest of the proof of the main result it is sufficient to have energy inequality.

**Proposition 3.2.** We have the following estimate

\[
\frac{1}{2} \frac{d}{dt} \left( \|u\|_{L^2}^2 + \|v\|_{L^2}^2 + \frac{\|T_q\|_{L^2}^2}{(1 + \alpha)(1 - Q)} + \frac{\|q\|_{L^2}^2}{(1 + \alpha)(Q + \alpha)} \right) + \|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \frac{\|q\|_{L^2}^2}{\epsilon(Q + \alpha)} = 0,
\]

for any \( t \in (0, T) \).

**Proof.** Multiplying equations (1.5) and (1.7) by \( u \) and \( v \), respectively, summing the resultants up and integrating over \( \mathbb{R}^2 \), then it follows from integration by parts that

\[
\frac{1}{2} \frac{d}{dt} \left( \|u\|_{L^2}^2 + \|v\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 \right) = \frac{1}{1 + \alpha} \int_{\mathbb{R}^2} (q - T_q) \nabla \cdot v \, dx \, dy,
\]

where we have used the following fact

\[
\int_{\mathbb{R}^2} \nabla \cdot (v \otimes v) \cdot u + (v \cdot \nabla) u \cdot v \, dx \, dy = \int_{\mathbb{R}^2} [(v \cdot \nabla) u] \cdot v - (v \otimes v) : \nabla u \, dx \, dy = 0.
\]

Multiplying equation (1.8) by \((1 + \alpha)^{-1}(1 - Q)^{-1} T_q\) and integrating over \( \mathbb{R}^2 \), follows from integration by parts that

\[
\frac{1}{2(1 + \alpha)(1 - Q)} \frac{d}{dt} \|T_q\|_{L^2}^2 = \frac{1}{1 + \alpha} \int_{\mathbb{R}^2} T_q \nabla \cdot v \, dx \, dy = 0.
\]

Multiplying equation (1.9) by \((1 + \alpha)^{-1}(Q + \alpha)^{-1} q\) and integrating over \( \mathbb{R}^2 \), it follows from integration by parts that

\[
\frac{1}{2(1 + \alpha)(Q + \alpha)} \frac{d}{dt} \|q\|_{L^2}^2 = \frac{1}{1 + \alpha} \int_{\mathbb{R}^2} q \nabla \cdot v \, dx \, dy = -\frac{1}{\epsilon(Q + \alpha)} \int_{\mathbb{R}^2} |q|^2 \, dx \, dy.
\]

Summing (3.12)–(3.14) up concludes the proof.

As an intermediate step to obtain the \( L^\infty(0, T; L^2(\mathbb{R}^2)) \) estimate for \((u, v, T, q)\), we prove the \( L^\infty(0, T; L^4(\mathbb{R}^2)) \) estimate in the next proposition.

**Proposition 3.3.** Denote \( U = (u, v) \). Then, we have the estimate

\[
\sup_{0 \leq t < T} \|\left( U, T, q \right)\|_{L^4}^4 + \int_0^T \left( \|U\|_{L^4}^4 + \|\nabla v\|_{L^4}^4 \right) dt \leq C,
\]

for a positive constant \( C \) depending only on the parameters \( \alpha, Q, T \) and the initial norm \( \|\left( U_0, T_0, q_0 \right)\|_{L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)} \) and, in particular, \( C \) is independent of \( \epsilon \).

**Proof.** Multiplying equations (1.5) and (1.7) by \( |U|^2 u \) and \( |U|^2 v \), respectively, summing the resultants up and integrating over \( \mathbb{R}^2 \), it follows from integration by parts and the Hölder inequality that
\[
\frac{1}{4} \frac{d}{dr} \| U \|^4_r + \int_{\mathbb{R}^2} \left( |U|^2 |\nabla U|^2 + \frac{1}{2} |\nabla (|U|^2)|^2 \right) dxdy \\
= \int_{\mathbb{R}^2} \left[ p \nabla \cdot (|U|^2 u) - (\nabla \cdot (v \otimes v)) |U|^2 u \\
- (v \cdot \nabla) u |U|^2 v + \frac{q_\alpha - T_\alpha}{1 + \alpha} \nabla \cdot (|U|^2 v) \right] dxdy \\
\leq 3 \int_{\mathbb{R}^2} \left[ |p| |U|^2 |\nabla U| + |U|^4 |\nabla U| + (|q_\alpha| + |T_\alpha|) |U|^2 |\nabla U| \right] dxdy \\
\leq 3 \left( \| p \|_4 + \left\| \left\| U \right\|_4^2 + \left\| T_\alpha \right\|_4 + \left\| q_\alpha \right\|_4 \right) \| U \|_4 \| U |\nabla U| \|_2 \right), \quad (3.15)
\]

Applying the divergence operator to equation (1.5), in view of (1.6), one can see that
\[- \Delta p = \nabla \cdot (u \otimes u + v \otimes v).\]

Note that \( p \) is uniquely determined by the above elliptic equation by assuming that \( p \to 0 \), as \((x, y) \to \infty\). Thus, by the elliptic estimates, one has
\[\| p \|_4 \leq C \| u \otimes u + v \otimes v \|_4 \leq C \| U \|_4^2.\]

Substituting this estimate into (3.15), and using the Ladyzhenskaya and Young inequalities, one deduces
\[\frac{1}{4} \frac{d}{dr} \| U \|^4_r + \| U |\nabla U|_2^2 + \frac{1}{2} \| \nabla |U|^2 \|_2^2 \leq C \left( \| U \|_4^2 + \| T_\alpha \|_4 + \| q_\alpha \|_4 \right) \| U \|_4 \| U |\nabla U| \|_2 \]
\[\leq C \left( \| U \|_4^2 + \| T_\alpha \|_4 + \| q_\alpha \|_4 \right) \| U \|_4 \| U |\nabla U| \|_2 \]
\[\leq \frac{1}{2} \left( \| U |\nabla U|_2^2 + \| \nabla |U|^2 \|_2^2 \right) + C \left( \| U \|_4 \| \left\| T_\alpha \right\|_2^2 + \left\| q_\alpha \right\|_2 \right) \| U \|_4 \| U |\nabla U| \|_2 \]
\[\leq C \left( 1 + \| U \|_4^2 \right) \left( \| T_\alpha \|_2^2 + \| q_\alpha \|_2^2 + \| U \|_4^2 \right) \]
\[+ \frac{1}{2} \left( \| U |\nabla U|_2^2 + \| \nabla |U|^2 \|_2^2 \right) \]
and thus
\[\frac{d}{dr} \| U \|^4_r + 2 \| U |\nabla U|_2^2 \leq C \left( 1 + \| U \|_4^2 \right) \left( \| T_\alpha \|_2^2 + \| q_\alpha \|_2^2 + \| U \|_4^2 \right). \quad (3.16)\]

Multiplying equation (1.8) by \(|T_\alpha|^2 T_\alpha\), and integrating over \(\mathbb{R}^2\), then it follows from integration by parts and the Hölder inequality that
\[\frac{1}{4} \frac{d}{dr} \| T_\alpha \|^2_r = (1 - \bar{Q}) \int_{\mathbb{R}^2} \nabla \cdot (v \otimes \nabla) T_\alpha \| U \|^2_r dxdy \leq (1 - \bar{Q}) \| \nabla v \|_4 \| T_\alpha \|_4^2,\]
which implies
\[
\frac{d}{dt} \|\mathcal{T}_t\|_{L^2}^2 \leq 2(1 - \tilde{Q}) \|\nabla v\|_4 \|\mathcal{T}_t\|_4.
\]  
(3.17)

Similar manipulation to equation (1.9) yields
\[
\frac{d}{dt} \|q_t\|_{L^2}^2 \leq 2(\tilde{Q} + \alpha) \|\nabla v\|_4 \|q_t\|_4.
\]  
(3.18)

Summing (3.16)–(3.18) up, and integrating the resultant in \(t\) yields
\[
\begin{align*}
(\|U\|_{L^4}^4 + \|\mathcal{T}_t\|_{L^2}^2 + \|q_t\|_{L^2}^2)(t) &+ 2 \int_0^t \left\| U |\nabla U| \right\|_{L^2}^2 ds \\
&\leq \|U_0\|_{L^4}^4 + \|\mathcal{T}_0\|_{L^2}^2 + \|q_0\|_{L^2}^2 + 2(1 + \alpha) \int_0^t \|\nabla v\|_4 (\|T_t\|_4 + \|q_t\|_4) ds \\
&\quad + C \int_0^t (1 + \|U\|_{L^4}^4)(\|U\|_{L^2}^2 + \|\mathcal{T}_t\|_{L^2}^2 + \|q_t\|_{L^2}^2) ds,
\end{align*}
\]  
(3.19)

for \(t \in [0, T]\).

We need to estimate the term \(\int_0^t \|\nabla v\|_4 (\|\mathcal{T}_t\|_4 + \|q_t\|_4) ds\) on the right-hand side of (3.19). To this end, applying lemma A.2 (in the appendix) to equation (1.7) yields
\[
\int_0^t \|\nabla v\|_4^2 ds \leq C \left[ \|\nabla v_0\|_4^2 + \left( \int_0^t \left\| U |\nabla U| \right\|_{L^2}^2 ds \right)^2 + \int_0^t (\|T_t\|_{L^2}^2 + \|q_t\|_{L^2}^2) ds \right],
\]  
(3.20)

for all \(t \in [0, T]\), where \(C\) is a positive constant independent of \(t\). Thanks to this estimate, it follows from the Hölder and Young inequalities that
\[
\begin{align*}
2(1 + \alpha) \int_0^t \|\nabla v\|_4 (\|T_t\|_4 + \|q_t\|_4) ds &\leq C T^2 \left( \int_0^t \|\nabla v\|_4^2 ds \right)^{\frac{1}{2}} \left( \int_0^t (\|T_t\|_{L^2}^2 + \|q_t\|_{L^2}^2) ds \right)^{\frac{1}{2}} \\
&\quad + C T^2 \left( \int_0^t (\|T_t\|_{L^2}^2 + \|q_t\|_{L^2}^2) ds \right)^{\frac{1}{2}} \\
&\quad + C \int_0^t \left( \|T_t\|_{L^2}^2 + \|q_t\|_{L^2}^2 \right) ds + C \left( \int_0^t (\|T_t\|_{L^2}^2 + \|q_t\|_{L^2}^2) ds \right)^{\frac{1}{2}} + C \|\nabla v_0\|_{L^2}^2.
\end{align*}
\]  
(3.21)

Substituting (3.21) into (3.19) and denoting
\[
f(t) = (\|U\|_{L^4}^4 + \|\mathcal{T}_t\|_{L^2}^2 + \|q_t\|_{L^2}^2)(t) + \int_0^t \left\| U |\nabla U| \right\|_{L^2}^2 ds,
\]
we have
\[ f(t) \leq f(0) + C\|\nabla v_0\|_2^2 + C\left(\int_0^t f^2(s)ds\right)^{1/2} + C\int_0^t (1 + \|U\|_2^2) f(s)ds, \]
for all \( t \in [0, T) \). By proposition 3.2 and using the Ladyzhenskaya inequality, one can easily check that \( \int_0^T (1 + \|U\|_2^2) dt \leq C \) for a positive constant \( C \) depending only on \( \alpha, \tilde{Q}, T \) and the initial norm \( \|u_0, v_0, T_0, T, q_0\|_{L^2} \). Therefore, applying lemma 2.1 to the above inequality, one obtains
\[ \sup_{0 \leq t < T} \|(U_0, T, q_0)(t)\|_2^2 + \int_0^T \|U\|_2^2 \leq C, \]
and further recalling (3.20) concludes the proof.

Thanks to the a priori estimate stated in the above proposition, one can immediately obtain the \( L^\infty(0, T, H^1(\mathbb{R}^2)) \) estimate on \( u \) as stated in the following proposition.

**Proposition 3.4.** We have the following estimates
\[ \sup_{0 \leq t < T} \|\nabla u(t)\|_2^2 + \int_0^T \|\Delta u\|_2^2 ds \leq C, \]
for a positive constant \( C \) depending only on the parameters \( \alpha, \tilde{Q}, T \) and the initial norm \( \|u_0\|_{H^1(\mathbb{R}^2)} + \|(v_0, T_0, q_0)\|_{L^2(\mathbb{R}^2) \rightarrow H^1(\mathbb{R}^2)} \) and in particular is independent of \( \varepsilon \).

**Proof.** Multiplying equation (1.5) by \(-\Delta u\) and integrating over \(\mathbb{R}^2\), it follows from integration by parts that
\[ \frac{1}{2} \frac{d}{dt} \|\nabla u\|_2^2 + \|\Delta u\|_2^2 = \int_{\mathbb{R}^2} [(u \cdot \Delta u) + \nabla (v \otimes v)] \cdot \Delta u dx dy \leq 3 \int_{\mathbb{R}^2} |U| |\nabla U| |\Delta u| dx dy \leq \frac{1}{2} \|\Delta u\|_2^2 + C \|U|\|_2^2 \|\nabla U\|_2^2, \]
where, again, \( U = (u, v) \) and thus
\[ \frac{d}{dt} \|\nabla u\|_2^2 + \|\Delta u\|_2^2 \leq C \|U|\|_2^2 \|\nabla U\|_2^2, \]
for all \( t \in [0, T) \). From which, in view of proposition 3.3, the proof is concluded.

Finally, we are ready to prove the \( L^\infty(0, T, H^1(\mathbb{R}^2)) \) estimate on \((v, T, q_e)\), using is the following proposition.
Proposition 3.5. The following estimate holds

\[ \sup_{0 \leq t < T} \left( \| \nabla v, \nabla T_e, \nabla q_e(t) \|_2^2 + \int_0^T \left( \| \Delta v \|_2^2 + \frac{\| \nabla q_e^{-1/2} \|}{\varepsilon} + \| \nabla u \|_\infty \right) \, dt \right) \leq C, \]

where \( C \) is a positive constant depending only on \( \tilde{\alpha}, \bar{T} \) and the initial norms \( \|(u_0, v_0, T_e(0), q_e(0))\|_H^2 \) and in particular is independent of \( \varepsilon \).

Proof. Multiplying equation (1.8) by \(-\Delta T_e\) and integrating over \(\mathbb{R}^2\), it follows from integration by parts and the Hölder inequality that

\[ \frac{1}{2} \frac{d}{dt} \| \nabla T_e \|_2^2 = (1 - \bar{Q}) \int_{\mathbb{R}^2} \nabla T_e \cdot \nabla (\nabla \cdot v) \, dx \, dy - \int_{\mathbb{R}^2} \partial_t u \cdot \nabla T_e \partial T_e \, dx \, dy \leq (1 - \bar{Q}) \| \Delta v \|_2 \| \nabla T_e \|_2 + \| \nabla u \|_\infty \| \nabla q_e \|_2. \]

Similarly, one can derive from equation (1.9) that

\[ \frac{1}{2} \frac{d}{dt} \| \nabla q_e \|_2^2 + \frac{1 + \alpha}{\varepsilon} \| \nabla q_e^{-1/2} \|_2^2 \leq (\alpha + \bar{Q}) \| \Delta v \|_2 \| \nabla q_e \|_2 + \| \nabla u \|_\infty \| \nabla q_e \|_2. \]

Summing the previous two inequalities up yields

\[ \frac{1}{2} \frac{d}{dt} \left( \| \nabla T_e \|_2^2 + \| \nabla q_e \|_2^2 \right) + \frac{1 + \alpha}{\varepsilon} \| \nabla q_e^{-1/2} \|_2^2 \leq (1 + \alpha) \| \Delta v \|_2 \left( \| \nabla T_e \|_2 + \| \nabla q_e \|_2 \right) + \| \nabla u \|_\infty \| \nabla q_e \|_2 \]

\[ \leq \frac{1}{4} \| \Delta v \|_2^2 + \| \nabla u \|_\infty + 2(\alpha + 1)^2 \left( \| \nabla T_e \|_2^2 + \| \nabla q_e \|_2^2 \right), \]

and thus

\[ \frac{d}{dt} \left( \| \nabla T_e \|_2^2 + \| \nabla q_e \|_2^2 \right) + \frac{1 + \alpha}{\varepsilon} \| \nabla q_e^{-1/2} \|_2^2 \]

\[ \leq 2 \| \nabla u \|_\infty + 2(\alpha + 1)^2 \left( \| \nabla T_e \|_2^2 + \| \nabla q_e \|_2^2 \right) + \frac{1}{2} \| \Delta v \|_2^2. \quad (3.22) \]

Multiplying equation (1.7) by \(-\Delta v\) and integrating over \(\mathbb{R}^2\), it follows from integration by parts and the Young inequality that

\[ \frac{1}{2} \frac{d}{dt} \| \nabla v \|_2^2 + \| \Delta v \|_2^2 = \int_{\mathbb{R}^2} \left[ \frac{1}{1 + \alpha} \Delta (T_e - q_e) - (\alpha \cdot \nabla)v - (v \cdot \nabla)u \right] \, \Delta v \, dx \, dy \]

\[ \leq \frac{1}{4} \| \Delta v \|_2^2 + C \left( \| \nabla T_e \|_2^2 + \| \nabla q_e \|_2^2 + \| U | \nabla U \|_2^2 \right), \]

and thus

\[ \frac{d}{dt} \| \nabla v \|_2^2 + \frac{3}{2} \| \Delta v \|_2^2 \leq C \left( \| \nabla T_e \|_2^2 + \| \nabla q_e \|_2^2 + \| U | \nabla U \|_2^2 \right). \quad (3.23) \]
Summing (3.22) with (3.23) up yields
\[
\frac{d}{dt} \left( \|\nabla v, \nabla T, \nabla q_e\|_2^2 + \|\Delta v\|_2^2 + \frac{1}{\varepsilon} \|\nabla q_e\|_2^2 \right) \\
\leq C \left\| U \right\|_{L^2}^2 + C(\|\nabla u\|_\infty + 1)(\|\nabla T\|_2^2 + \|\nabla q_e\|_2^2),
\]
from which, by the Gronwall inequality and using proposition 3.3, one obtains
\[
\sup_{0 \leq t < T} (\|\nabla v, \nabla T, \nabla q_e(t)\|_2^2 + \int_0^T (\|\Delta v\|_2^2 + \frac{1}{\varepsilon} \|\nabla q_e\|_2^2) dt) \\
\leq C \int_0^T \left( \|\nabla u\|_\infty + 1 \right) dt \\
\leq C(\alpha, \bar{Q}, T, \|(U_0, T_{e,0}, q_{e,0})\|_{H'}) \exp \left\{ C \int_0^T (\|\nabla u\|_\infty + 1) dt \right\}.
\]

(3.24)

To complete the proof, one still needs to estimate \(\int_0^T \|\nabla u\|_\infty dt\). It follows from propositions 3.3–3.4 and the Ladyzhenskaya inequality that
\[
\int_0^T (\|\nabla u\|_2^2 + \|\nabla v\|_2^2) dt \leq C \int_0^T (\|\nabla u\|_2^2 + \|\Delta u\|_2^2 + \|\nabla v\|_2^2) dt \\
\leq C(\alpha, \bar{Q}, T, \|(U_0, v_0, T_{e,0}, q_{e,0})\|_{H'})
\]

(3.25)

We decompose \(u\) as \(u = \bar{u} + \hat{u}\), where \(\bar{u}\) and \(\hat{u}\), respectively, are the unique solutions to the following two systems
\[
\begin{align*}
\partial_t \bar{u} - \Delta \bar{u} + \nabla \bar{p} &= -(u \cdot \nabla)u - \nabla \cdot (v \otimes v), \\
\nabla \cdot \bar{u} &= 0, \\
\bar{u}\big|_{t=0} &= 0,
\end{align*}
\]

(3.26)

and
\[
\begin{align*}
\partial_t \hat{u} - \Delta \hat{u} + \nabla \hat{p} &= 0, \\
\nabla \cdot \hat{u} &= 0, \\
\hat{u}\big|_{t=0} &= u_0.
\end{align*}
\]

(3.27)

We are going to estimate \(\bar{u}\) and \(\hat{u}\). Let us first estimate \(\bar{u}\). By the \(L^q(0, T; W^2,q)\) type estimates for the Stokes equations (see, e.g. Solonnikov [41, 42]), we have
\[
\|\partial_t \bar{u}, \Delta \bar{u}\|_{L^q(\mathbb{R}^3 \times (0, T))} \leq C \|U\|_{L^q(\mathbb{R}^3 \times (0, T))},
\]

for any \(q \in (1, \infty)\) and thus it follows from the Hölder inequality and Gagliardo–Nirenberg inequality, \(\|\varphi\|_2^3 \leq C \|\varphi\|_2^2 \|\nabla \varphi\|_4\) (3.25) and proposition 3.3 that
\[
\int_0^T \|\Delta u\|_2^2 dt \leq C \int_0^T \|U\|_{L^2}^2 \|\nabla U\|_2^3 dt \leq C \int_0^T \|\nabla U\|_2^3 \|U\|_2^3 dt \\
\leq C \int_0^T \|\nabla U\|_{L^2}^2 \|U\|_{L^2}^3 dt \leq C(\alpha, \bar{Q}, T, \|(u_0, v_0, T_{e,0}, q_{e,0})\|_{H'})
\]
One can deduce easily from equation (3.26) by using proposition 3.3 that
\[
\sup_{0 \leq r < T} \left\| \nabla \bar{u}(r) \right\|_2^2 + \int_0^T \left\| \Delta \bar{u} \right\|_2^2 dr \leq C \int_0^T \left\| U \right\|_2^2 dr
\leq C(\alpha, \bar{Q}, T, \left\| (u_0, v_0, T_0, q_{e,0}) \right\|_{H^s}).
\]

Thanks to the above two estimates, it follows from the Gagliardo–Nirenberg, \( \| \varphi \|_\infty \leq C \| \varphi \|_2^{\frac{1}{2}} \| \Delta \varphi \|_2^{\frac{3}{2}} \) and the Hölder inequalities that
\[
\int_0^T \left\| \nabla \bar{u} \right\|_\infty dr \leq C \int_0^T \left\| \nabla \bar{u} \right\|_2^{\frac{1}{2}} \| \Delta \bar{u} \|_2^{\frac{3}{2}} dr
\leq C\left( \int_0^T \left\| \nabla \bar{u} \right\|_2^2 \right)^{\frac{1}{2}} \left( \int_0^T \| \Delta \bar{u} \|_2^3 \right)^{\frac{1}{3}} T^{\frac{2}{3}}
\leq C(\alpha, \bar{Q}, T, \left\| (u_0, v_0, T_0, q_{e,0}) \right\|_{H^s}).
\] (3.28)

Next, we estimate \( \hat{u} \). Multiplying equation (3.27) by \( (\hat{t} \Delta - \Delta) \hat{u} \), and integrating the resultant over \( \mathbb{R}^2 \), it follows from integration by parts that
\[
\frac{1}{2} \frac{d}{dr} \left( \| \nabla \hat{u} \|_2^2 + \| \sqrt{\hat{t}} \Delta \hat{u} \|_2^2 \right) + \frac{1}{2} \| \Delta \hat{u} \|_2^2 + \| \sqrt{\hat{t}} \nabla \Delta \hat{u} \|_2^2 = 0.
\]
Therefore, we have
\[
\sup_{0 \leq r < T} \left( \| \nabla \hat{u} \|_2^2 + \| \sqrt{\hat{t}} \Delta \hat{u} \|_2^2 \right) + \int_0^T \left( \| \Delta \hat{u} \|_2^2 + \| \sqrt{\hat{t}} \nabla \Delta \hat{u} \|_2^2 \right) dr \leq \| \nabla u_0 \|_2^2.
\]

Thanks to this estimate, it follows from the Gagliardo–Nirenberg (Agmon), \( \| \varphi \|_\infty \leq C \| \varphi \|_2^{\frac{1}{2}} \| \Delta \varphi \|_2^{\frac{3}{2}} \) and Hölder inequalities that
\[
\int_0^T \left\| \nabla \hat{u} \right\|_\infty dr \leq C\left( \int_0^T \left\| \nabla \hat{u} \right\|_2^2 \left\| \nabla \Delta \hat{u} \right\|_2^2 \right)^{\frac{1}{2}} \left( \int_0^T \| \sqrt{\hat{t}} \nabla \Delta \hat{u} \|_2^2 \right)^{\frac{1}{2}} dr
\leq C\left( \int_0^T \left\| \nabla \hat{u} \right\|_2^2 \right)^{\frac{1}{2}} \left( \int_0^T T^{-\frac{1}{2}} \right)^{\frac{3}{2}}
\leq C T^{\frac{1}{4}} \| \nabla u_0 \|_2^{\frac{1}{2}} \| \nabla u_0 \|_2^{\frac{1}{2}} T^{\frac{1}{2}} = C T^{\frac{1}{4}} \| \nabla u_0 \|_2.
\]

Combining the above estimate with (3.28), one has
\[
\int_0^T \left\| \nabla u \right\|_\infty dr \leq \int_0^T \left( \| \nabla u \|_\infty + \| \nabla \hat{u} \|_\infty \right) dr \leq C(\alpha, \bar{Q}, T, \left\| (u_0, v_0, T_0, q_{e,0}) \right\|_{H^s}).
\]
which, when substituted into (3.24), concludes the proof.

As a corollary of propositions 3.2–3.5 we have the \textit{a priori} estimate to \( (u, v, T, q) \) as stated in the following.
Corollary 3.1. Suppose that (1.10) holds and the initial data
\[ (u_0, v_0, T_{e,0}, q_{e,0}) \in H^4(\mathbb{R}^2), \quad \nabla \cdot u_0 = 0. \] (3.29)

Let \((u, v, T, q)\) be the unique strong solution to system (1.5)–(1.9), on \(\mathbb{R}^2 \times (0, T)\), \(0 < T < \infty\), with initial data \((u_0, v_0, T_{e,0}, q_{e,0})\). Then, the following hold.

(i) We have the estimate
\[
\sup_{0 \leq t < T} \left\| \left( u, T, q_e \right)(t) \right\|^2_{\mu_T} + \int_0^T \left( \frac{\|q_e^+\|^2_{\mu_T}}{\varepsilon} + \|\left( u, v \right)\|^2_{H^2} + \|\nabla u\|_{\infty} \right) \, dt
\]
\[
+ \int_0^T \| (\partial_t u, \partial_t v, \partial_t T_e) \|^2_{L^2} \, dt \leq C(\alpha, \hat{Q}, T, \|(u_0, v_0, T_{e,0}, q_{e,0})\|_{\mu_T}).
\]

(ii) Suppose in addition to (3.29) that \(q_{e,0}^+ = 0\), a.e. on \(\mathbb{R}^2\), then we have
\[
\sup_{0 \leq t < T} \frac{\|q_e^+\|^2}{\varepsilon} + \int_0^T \| \partial_t q_e \|^2 \, dt \leq C(\alpha, \hat{Q}, T, \|(u_0, v_0, T_{e,0}, q_{e,0})\|_{\mu_T}).
\]

(iii) Assume in addition to (3.29) that \((\nabla T_{e,0}, \nabla_q e, 0) \in L^m(\mathbb{R}^2)\) for some \(m \in (2, \infty)\), then we have the estimate
\[
\sup_{0 \leq t < T} \| (\nabla T_e, \nabla q_e) \|^2_m \leq C(\alpha, \hat{Q}, T_m, \|(u_0, v_0, T_{e,0}, q_{e,0})\|_{\mu_T}, \| (\nabla T_{e,0}, \nabla_q e, 0)\|_m).
\]

Proof.

(i) The estimate on all the terms, except those involving the time derivatives, follow directly from propositions 3.2–3.5. The desired estimate for \((\partial_t u, \partial_t v)\) follows directly from the \textit{a priori} estimate in propositions 3.3 and 3.5, by using the \(L^2(0, T; H^2)\) type estimates to the Stokes and heat equations. By propositions 3.2–3.5, it follows from equation (1.8) and the Sobolev embedding inequalities that
\[
\int_0^T \|\partial_t T_e\|^2 \, dt \leq \int_0^T [(1 - \hat{Q})\|\nabla v\|^2_{L^2} + \|u\|^2_{H^2} \|\nabla T_e\|^2] \, dt
\]
\[
\leq C + C \int_0^T \|u\|^2_{H^2} \, dt \leq C + C \int_0^T \|u\|^2_{H^2} \, dt \leq C.
\]

(ii) Multiplying equation (1.9) by \(\partial_t q_e\) and integrating over \(\mathbb{R}^2\), it follows from the Young and Sobolev embedding inequalities and proposition 3.5 that
\[
\frac{1 + \alpha}{2\varepsilon} \frac{d}{dt} \|q_e^+\|^2_{L^2} + \|\partial_t q_e\|^2_{L^2} = -\int_{\mathbb{R}^2} \left[ u \cdot \nabla q_e + (\hat{Q} + \alpha) \nabla \cdot v \right] \partial_t q_e \, dx \, dy
\]
\[
\leq \frac{1}{2} \|\partial_t q_e\|^2_{L^2} + C(\|u\|^2_{L^2} \|\nabla q_e\|^2_{L^2} + \|\nabla v\|^2_{L^2})
\]
\[
\leq \frac{1}{2} \|\partial_t q_e\|^2_{L^2} + C(\|u\|^2_{H^2} + 1),
\]
from which, by (i), the conclusion in (ii) follows.
(iii) Applying the operator $\nabla$ to equation (1.8), multiplying the resultant by $|\nabla T_2|^{m-2} \nabla T_2$ and integrating over $\mathbb{R}^2$, it follows from integration by parts and the Hölder inequality that
\[
\frac{1}{m} \frac{d}{dt} \|\nabla T_2\|_m^m = (1 - \bar{Q}) \int_{\mathbb{R}^2} |\nabla T_2|^{m-2} \nabla T_2 \cdot \nabla (\nabla \cdot \nu) \, dx \, dy - \int_{\mathbb{R}^2} \partial_t u \cdot \nabla T_2 |\nabla T_2|^{m-2} \partial_t T \, dx \, dy \leq (1 - \bar{Q}) \|\nabla^2 v\|_m \|\nabla T_2|^{m-1} + \|\nabla u\|_\infty \|\nabla T_2\|_m^m.
\]
Thus
\[
\frac{d}{dt} \|\nabla T_2\|_m \leq (1 - \bar{Q}) \|\nabla^2 v\|_m + \|\nabla u\|_\infty \|\nabla T_2\|_m.
\]
Similarly, one can derive from equation (1.9) that
\[
\frac{d}{dt} \|\nabla q_e\|_m \leq (\alpha + \bar{Q}) \|\nabla^2 v\|_m + \|\nabla u\|_\infty \|\nabla q_e\|_m.
\]
Summing the above two inequalities one obtains
\[
\frac{d}{dt} (\|\nabla T_2\|_m + \|\nabla q_e\|_m) \leq (1 + \alpha) \|\nabla^2 v\|_m + \|\nabla u\|_\infty (\|\nabla T_2\|_m + \|\nabla q_e\|_m),
\]
from which, integrating with respect to $t$, we have
\[
\|(\nabla T_2, \nabla q_e)\|_m(t) \leq C \int_0^t \|\nabla^2 v\|_m \, ds + C \int_0^t \|\nabla u\|_\infty \|(\nabla T_2, \nabla q_e)\|_m \, ds, \quad (3.30)
\]
for all $t \in [0, T]$. Applying lemma A.3 (see the appendix) to equation (1.7) and using the Sobolev embedding inequality, one deduces
\[
\int_0^t \|\nabla^2 v\|_m \, ds \leq C \left[ \|\nabla v_0\|_2 + \left( \int_0^t \|(\nabla T_2, \nabla q_e, |u| \nabla v, |v| \nabla u)\|_m^m \, ds \right)^{\frac{1}{2}} \right],
\]
for any $t \in [0, T)$, where $C$ is a positive constant depending only on $m$ and $T$ and is, in particular, independent of $t \in [0, T)$. By (i) the above inequality implies
\[
\int_0^t \|\nabla^2 v\|_m \, ds \leq C + C \left( \int_0^t \|(\nabla T_2, \nabla q_e)\|_m^m \, ds \right)^{\frac{1}{2}},
\]
for any $t \in [0, T)$ and for a positive constant $C$ independent of $t \in [0, T)$. Substituting the above estimate into (3.30) and setting $f(t) = \|(\nabla T_2, \nabla q_e)\|_m(t)$ yield
for any $t \in [0, T)$ where $C$ is a positive constant independent of $t \in [0, T)$. Recalling (i) and applying lemma 2.1, the conclusion stated in (iii) follows. □

Now, we are ready to prove the global existence, uniqueness and well-posedness of strong solutions to the Cauchy problem of system (1.5)–(1.9).

**Proof of theorem 1.1.** The uniqueness of the strong solutions follows from proposition 3.1 directly, while the a priori estimates in (i)–(iii) follow from (i)–(iii) of corollary 3.1, respectively. Therefore, we still need to prove the global existence of strong solutions as stated in (i) and the continuous dependence of the strong solutions on the initial date as stated in (iii).

To prove the global existence of strong solutions, it suffices to extend the local solution established in proposition 3.1 to be a global one. By repeating proposition 3.1, one can extend the local solution $(u_0, v_0, T_0, q_0)$ to the maximal interval of existence $[T_0, T_*]$. Then, we need to show that $T_* = \infty$. Suppose, by contradiction, that $T_* < \infty$, then we must have

\[
\lim_{t \to T_*} \|(u, v, T, q)\|_{H^2} = \infty.
\]

However, by corollary 3.1, which holds since $T_* < \infty$, the quantity $\|(u, v, T, q)\|_{H^2}$ is bounded on $[0, T_*]$ which is a contradiction, and thus $T_* = \infty$.

We now prove the continuous dependence of the unique strong solutions on the initial data as stated in (iii) on any finite interval $[0, T]$. Therefore, we choose arbitrary $T \in (0, \infty)$ and focus on the interval $[0, T]$. Let $(u^{(1)}, v^{(1)}, T^{(1)}_e, q^{(1)}_e)$ and $(u^{(2)}, v^{(2)}, T^{(2)}_e, q^{(2)}_e)$ be the unique solutions to system (1.5)–(1.9), respectively, with initial data $(u^{(1)}_0, v^{(1)}_0, T^{(1)}_{e,0}, q^{(1)}_{e,0})$ and $(u^{(2)}_0, v^{(2)}_0, T^{(2)}_{e,0}, q^{(2)}_{e,0})$.

Denote by

\[
\delta \equiv (\delta u, \delta v, \delta T_e, \delta q_e) = (u^{(1)}, v^{(1)}, T^{(1)}_e, q^{(1)}_e) - (u^{(2)}, v^{(2)}, T^{(2)}_e, q^{(2)}_e),
\]

and

\[
\delta u_0, \delta v_0, \delta T_{e,0}, \delta q_{e,0}) = (u^{(1)}_0, v^{(1)}_0, T^{(1)}_{e,0}, q^{(1)}_{e,0}) - (u^{(2)}_0, v^{(2)}_0, T^{(2)}_{e,0}, q^{(2)}_{e,0}).
\]

Then, similar to (3.9), we have

\[
\frac{d}{dt} \|\delta u, \delta v, \delta T_e, \delta q_e\|^2 + \frac{1}{2} \left(\|\nabla \delta u\|^2 + \|\nabla \delta v\|^2\right) \\
\leq C \int_{\mathbb{R}^2} \left( |\nabla u^{(2)}| + |\nabla v^{(2)}| + |\nabla v^{(1)}| + |v^{(1)}|^2 + |v^{(2)}|^2 \right) \delta u^2 + |\delta v|^2 + |\delta T_e|^2 + |\delta q_e|^2 + \|\delta T_e\| \|\delta u\| \|\delta u\| \|\delta u\| \|\delta q_e\| \|\delta q_e\| \|\delta q_e\| \|dx, dy, \right)
\]

for all $t \in (0, T]$. All the integrals on the right-hand side of the above inequality, except for the last two terms, can be dealt with as before in (3.10), while we estimate the last two terms by the Hölder, Sobolev embedding and Young inequalities as follows:
\begin{align*}
C & \int_{\mathbb{R}^2} \left( |\nabla T_{\varepsilon}^{(2)}| |\delta u| |\delta T_{\varepsilon}| + |\nabla q_{\varepsilon}^2| |\delta u| |\delta q_{\varepsilon}| \right) dx dy \\
& \leq C \| \nabla T_{\varepsilon}^{(2)} \|_{\infty} \| \delta u \|_{L^2} \sum_{m=2}^{2m} \| \delta T_{\varepsilon}^m \|_2 + C \| \nabla q_{\varepsilon}^2 \|_{\infty} \| \delta u \|_{L^2} \sum_{m=2}^{2m} \| \delta q_{\varepsilon}^m \|_2 \\
& \leq C \| \nabla T_{\varepsilon}^{(2)} \|_{\infty} \| \delta u \|_{H^1} \| \delta T_{\varepsilon} \|_2 + C \| \nabla q_{\varepsilon}^2 \|_{\infty} \| \delta u \|_{H^1} \| \delta q_{\varepsilon} \|_2 \\
& \leq \frac{1}{8} \| \delta u \|_{H^1}^2 + C (\| \nabla T_{\varepsilon}^{(2)} \|_{\infty}^2 + \| \nabla q_{\varepsilon}^2 \|_{\infty}^2)(\| \delta T_{\varepsilon} \|_2^2 + \| \delta q_{\varepsilon} \|_2^2).
\end{align*}

for all \( t \in (0, T] \). Therefore, we deduce from (3.31) that
\[
\frac{d}{dt} \left( \| (\delta u, \delta v, \delta T_{\varepsilon}, \delta q_{\varepsilon}) \|_{H^1}^2 + \frac{1}{8} \| (\delta u, \delta v) \|_{H^1}^2 \right)
\leq C (1 + \| (u^{(2)}, v^{(2)}) \|_{L^4}^4 + \| (\nabla u^{(2)}, \nabla v^{(2)}, \nabla v^{(11)}) \|_{L^4}^2) \| (\delta u, \delta v, \delta T_{\varepsilon}, \delta q_{\varepsilon}) \|_{H^1}^2
\]
\[
+ C (\| \nabla T_{\varepsilon}^{(2)} \|_{\infty}^2 + \| \nabla q_{\varepsilon}^2 \|_{\infty}^2)(\| \delta T_{\varepsilon} \|_2^2 + \| \delta q_{\varepsilon} \|_2^2),
\]

for all \( t \in (0, T] \). Applying the Gronwall inequality to the above inequality yields
\[
\sup_{0 \leq s \leq t} \left( \| (\delta u, \delta v, \delta T_{\varepsilon}, \delta q_{\varepsilon})(s) \|_{H^1}^2 + \frac{1}{8} \int_0^t \| (\delta u, \delta v) \|_{H^1}^2 ds \right)
\leq C \int_0^t \left( 1 + \| (u^{(2)}, v^{(2)}) \|_{L^4}^4 + \| (\nabla u^{(2)}, \nabla v^{(2)}, \nabla v^{(11)}) \|_{L^4}^2 + \| (\nabla T_{\varepsilon}^{(2)}, \nabla q_{\varepsilon}^2) \|_{L^4}^2 \right) ds
\]
\[
\times \left( \| (\delta u_0, \delta v_0, \delta T_{\varepsilon, 0}, \delta q_{\varepsilon, 0}) \|_{H^1}^2 \right),
\]

for all \( t \in (0, T] \). Recalling the regularities in (i) and (iii), the above inequality implies the continuous dependence of the strong solution on the initial data on \([0, T]\), for any arbitrary \( T \in (0, \infty) \). This completes the proof. \( \square \)

4. Global existence and uniqueness of the limiting system

In this section we prove the global existence and uniqueness of strong solutions to the Cauchy problem of the limiting system (1.12)–(1.18).

**Proof of theorem 1.2.**

(i) The global existence and regularities. By theorem 1.1, for any positive \( \varepsilon \) there is a unique global strong solution \((u_{\varepsilon}, v_{\varepsilon}, T_{\varepsilon, \varepsilon}, q_{\varepsilon, \varepsilon})\) to system (1.5)–(1.9), with initial data \((u_0, v_0, T_{\varepsilon, 0}, q_{\varepsilon, 0})\), such that
\[
\sup_{0 \leq t \leq T} \left( \frac{\| q_{\varepsilon, \varepsilon}^+ (t) \|_{L^2}^2}{\varepsilon} + \| (u_{\varepsilon}, v_{\varepsilon}, T_{\varepsilon, \varepsilon}, q_{\varepsilon, \varepsilon}) (t) \|_{H^1}^2 \right)
\leq \int_0^T \frac{\| \nabla q_{\varepsilon, \varepsilon}^+ \|_{L^2}^2}{\varepsilon} + \| (u_{\varepsilon}, v_{\varepsilon}) \|_{H^1}^2 dt
\]
\[
+ \int_0^T (\| (\partial u_{\varepsilon}, \partial v_{\varepsilon}, \partial T_{\varepsilon, \varepsilon}, \partial q_{\varepsilon, \varepsilon}) \|_{L^2}^2 + \| \nabla u_{\varepsilon} \|_{L^2}) dt \leq C,
\]

for any positive finite time \( T \), where \( C \) is a constant depending only on \( \alpha, Q, T \) and initial norms \( \| (u_0, v_0, T_{\varepsilon, 0}, q_{\varepsilon, 0}) \|_{H^1} \) and, in particular, is independent of \( \varepsilon \). Moreover, if in addition \((\nabla T_{\varepsilon, 0}, \nabla q_{\varepsilon, 0}) \in L^m (\mathbb{R}^2)\) for some \( m \in (2, \infty) \), then we have further that
for any positive finite time \( T \) and, again, the estimate is independent of \( \varepsilon \).

Thanks to the above \( \varepsilon \)-independent estimates there is a subsequence, still denoted by 
\((u_{\varepsilon}, v_{\varepsilon}, T_{\varepsilon}, q_{\varepsilon})\) and \((u, v, T_{q}, q)\), such that

\[
\begin{align*}
(u_{\varepsilon}, v_{\varepsilon}) & \to (u, v), \quad \text{in } \mathcal{L}^{\infty}(0, T; H^{1}(\mathbb{R}^{2})), \\
(u_{\varepsilon}, v_{\varepsilon}) & \to (u, v), \quad \text{in } \mathcal{L}^{2}(0, T; H^{2}(\mathbb{R}^{2})), \\
(\partial_{t}u_{\varepsilon}, \partial_{t}v_{\varepsilon}) & \to (\partial_{t}u, \partial_{t}v), \quad \text{in } \mathcal{L}^{2}(0, T; \mathcal{L}^{2}(\mathbb{R}^{2})), \\
(T_{\varepsilon}, q_{\varepsilon}) & \to (T_{q}, q), \quad \text{in } \mathcal{L}^{\infty}(0, T; H^{1}(\mathbb{R}^{2})), \\
(T_{\varepsilon}, q_{\varepsilon}) & \to (T_{q}, q), \quad \text{in } \mathcal{L}^{2}(0, T; \mathcal{L}^{2}(\mathbb{R}^{2})), \\
q_{\varepsilon}^+ & \to 0, \quad \text{in } \mathcal{L}^{\infty}(0, T; \mathcal{L}^{2}(\mathbb{R}^{2})) \cap \mathcal{L}^{2}(0, T; H^{1}(\mathbb{R}^{2})),
\end{align*}
\]

for any positive finite time \( T \), where \( \to \) and \( \to^{*} \) are the weak and weak-* convergences, respectively. The last convergence in the above implies that

\[
q_{e}^+ = 0, \text{ or equivalently } q_{e} \leq 0, \quad \text{a.e. in } \mathbb{R}^{2} \times (0, T).
\]

Moreover, by the Aubin–Lions lemma and using the Cantor diagonal argument, we have a subsequence, still denoted by 
\((u_{\varepsilon}, v_{\varepsilon}, T_{\varepsilon}, q_{\varepsilon})\), such that

\[
\begin{align*}
(u_{\varepsilon}, v_{\varepsilon}) & \to (u, v), \quad \text{in } \mathcal{C}([0, T]; \mathcal{L}^{2}(B_{R})) \cap \mathcal{L}^{2}(0, T; H^{1}(B_{R})), \\
(T_{\varepsilon}, q_{\varepsilon}) & \to (T_{q}, q), \quad \text{in } \mathcal{C}([0, T]; \mathcal{L}^{2}(B_{R})),
\end{align*}
\]

for any positive finite time \( T \) and disc \( B_{R} \subset \mathbb{R}^{2} \) of arbitrary radius \( R > 0 \).

Thanks to the previous convergences, one can take the limit \( \varepsilon \to 0^{+} \) in the equations (1.5)–(1.8) for \((u_{\varepsilon}, v_{\varepsilon}, T_{\varepsilon}, q_{\varepsilon})\) to deduce that \((u, v, T_{q}, q_{e})\) satisfies equations (1.5)–(1.8) a.e. in \( \mathbb{R}^{2} \times (0, \infty) \), since \( R \) in the previous strong convergences is arbitrary. Moreover, by the lower semi-continuity of the norms the \textit{a priori} estimates stated in theorem 1.2 hold. In order to complete the proof of existence, we still need to prove that \(q_{e}\) satisfies inequalities (1.16)–(1.18). Inequality (1.17) has already been verified previously. While for (1.16), note that equation (1.9) for \( q_{e} \) implies that

\[
\partial_{t}q_{e} + u_{e} \cdot \nabla q_{e} + (\bar{Q} + \alpha) \nabla \cdot v_{e} \leq 0, \quad \text{a.e. in } \mathbb{R}^{2} \times (0, \infty),
\]

from which, recalling the previous convergences, one can take the limit \( \varepsilon \to 0^{+} \) to see that

\[
\partial_{t}q_{e} + u \cdot \nabla q_{e} + (\bar{Q} + \alpha) \nabla \cdot v \leq 0, \quad \text{a.e. in } \mathbb{R}^{2} \times (0, \infty),
\]

which is (1.16).

It remains to verify (1.18). To this end, let us define the set

\[
O^{-} = \{ (x, t) | q_{e}(x, t) < 0, \ x \in \mathbb{R}^{2}, \ t \in (0, \infty) \},
\]
and for any positive integers \( j, k, l \), we define
\[
\mathcal{O}_{jk}^- = \left\{ (x, t) \left| q_e(x, t) < -\frac{1}{j}, x \in B_k, t \in (0, l) \right. \right\},
\]
where \( B_k \subset \mathbb{R}^2 \) is a disc of radius \( k \) and \( j, k, l \in \mathbb{N} \). Noting that
\[
\mathcal{O}^- = \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} \mathcal{O}_{jk}^-,
\]
to prove that (1.18) holds a.e. on \( \mathcal{O}^- \), it suffices to show that it holds a.e. on \( \mathcal{O}_{jk}^- \) for any positive integers \( j, k, l \). Now, let us fix the positive integers \( j, k, l \). Recalling that \( q_{ee} \to q_e \) in \( C([0, T ]; L^2(B_k)) \), for any positive time \( T \) and positive radius \( R \) it is straightforward that \( q_{ee} \to q_e \) in \( L^2(\Omega_{jk}) \). Therefore, there is a subsequence, still denoted by \( q_{ee} \), such that \( q_{ee} \to q_e \) a.e. on \( \mathcal{O}_{jk}^- \). By the Egoroff theorem, for any positive number \( \eta > 0 \) there is a subset \( E_\eta \) of \( \mathcal{O}_{jk}^- \) with \( |E_\eta| < \eta \), such that
\[
q_{ee} \to q_e \quad \text{uniformly on } \mathcal{O}_{jk}^- \setminus E_\eta.
\]
Recalling the definition of \( \mathcal{O}_{jk}^- \), this implies that for sufficiently small positive \( \varepsilon \) it holds that
\[
q_{ee} \leq q_e + \frac{1}{2j} \leq -\frac{1}{2j} < 0 \quad \text{on } \mathcal{O}_{jk}^- \setminus E_\eta.
\]
As a result, by equation (1.9) for \( q_{ee} \) we have, for any sufficiently small positive \( \varepsilon \), that
\[
\mathcal{G}_\varepsilon := \partial_t q_{ee} + u \cdot \nabla q_{ee} + (Q + \alpha) \nabla \cdot v = 0, \quad \text{a.e. on } \mathcal{O}_{jk}^- \setminus E_\eta.
\]
Noting that
\[
\mathcal{G}_\varepsilon \to \partial_t q_e + u \cdot \nabla q_e + (Q + \alpha) \nabla \cdot v =: \mathcal{G}, \quad \text{in } L^2(0, T; L^2(\mathbb{R}^2)),
\]
for any positive finite time \( T \), which in particular implies \( \mathcal{G}_\varepsilon \to \mathcal{G} \), in \( L^2(\Omega_{jk}^- \setminus E_\eta) \). Since \( \mathcal{G}_\varepsilon = 0 \) a.e. on \( \mathcal{O}_{jk}^- \setminus E_\eta \), we have \( \mathcal{G} = 0 \) a.e. on \( \mathcal{O}_{jk}^- \setminus E_\eta \) that is
\[
\partial_t q_e + u \cdot \nabla q_e + (Q + \alpha) \nabla \cdot v = 0, \quad \text{a.e. on } \Omega_{jk}^- \setminus E_\eta.
\]
By lemma 2.3, this implies that the above equation holds a.e. on \( \mathcal{O}_{jk}^- \), and further on \( \mathcal{O}^- \), in other words, (1.18) holds. Therefore, \((u, v, T_e, q_e)\) is a global strong solution to system (1.12)–(1.18), with initial data \((u_0, v_0, T_{e,0}, q_{e,0})\), satisfying the regularities stated in the theorem.

(ii) The uniqueness. Let \((u, v, T_e, q_e)\) and \((\tilde{u}, \tilde{v}, \tilde{T}_e, \tilde{q}_e)\) be two strong solutions to system (1.12)–(1.18), with the same initial data \((u_0, v_0, T_{e,0}, q_{e,0})\). Define the new functions
\[
(\delta u, \delta v, \delta T_e, \delta q_e) = (u, v, T_e, q_e) - (\tilde{u}, \tilde{v}, \tilde{T}_e, \tilde{q}_e).
\]
Then, one can easily check that \((\delta u, \delta v, \delta T_e, \delta q_e)\) satisfies equations (3.2)–(3.5) and the same argument as that for (3.8) yields

\[
\frac{d}{dt} \left[ \|(\delta u, \delta v, \delta T_e)\|_2^2 + \|\nabla \delta u\|_2^2 + \|\nabla \delta v\|_2^2 \right]
\leq C \int_{\Omega} \left[ (|\nabla \bar{u}| + |\nabla \bar{v}| + |\nabla v|^2 + |\bar{v}|^2) \delta u_2 + |\delta v|^2 \right]
+ |\delta T_e|^2 + |\delta q_e|^2 + |\nabla \delta u||\delta T_e| \, dx \, dy. \tag{4.1}
\]

We need to estimate \(\delta q_e\). To this end, we first derive the equation for \(\delta q_e\). We divide the domain \(\Omega := \mathbb{R}^2 \times (0, \infty)\) as follows

\[
\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4,
\]

where

\[
\begin{align*}
\Omega_1 &= \{ q_e < 0 \} \cap \{ \tilde{q}_e < 0 \}, \\
\Omega_2 &= \{ q_e < 0 \} \cap \{ \tilde{q}_e = 0 \}, \\
\Omega_3 &= \{ q_e = 0 \} \cap \{ \tilde{q}_e < 0 \}, \\
\Omega_4 &= \{ q_e = 0 \} \cap \{ \tilde{q}_e = 0 \}.
\end{align*}
\]

On the set \(\Omega_1\), \(q_e\) and \(\tilde{q}_e\) satisfy, respectively,

\[
\begin{align*}
\partial_t \delta q_e + u \cdot \nabla \delta q_e + (\bar{Q} + \alpha) \nabla \cdot v &= 0, \\
\partial_t \tilde{q}_e + \bar{u} \cdot \nabla \tilde{q}_e + (\bar{Q} + \alpha) \nabla \cdot \bar{v} &= 0.
\end{align*}
\]

Subtracting the above two equations yields

\[
\partial_t \delta q_e + u \cdot \nabla \delta q_e + \delta u \cdot \nabla \tilde{q}_e + (\bar{Q} + \alpha) \nabla \cdot \delta v = 0, \quad \text{on } \Omega_1. \tag{4.2}
\]

On the set \(\Omega_2\), \(q_e\) satisfies

\[
\partial_t \delta q_e + u \cdot \nabla \delta q_e + (\bar{Q} + \alpha) \nabla \cdot v = 0,
\]

while for \(\tilde{q}_e\), since \(\tilde{q}_e \equiv 0\) on \(\Omega_2\), one has \(\partial_t \tilde{q}_e, \nabla \tilde{q}_e = 0\) a.e. on \(\Omega_2\) and thus \(\partial_t \tilde{q}_e \equiv \bar{u} \cdot \nabla \tilde{q}_e = 0\) a.e. on \(\Omega_2\). Here, we have used the well-known fact that the derivatives of a function \(f \in W_0^{1,1}(\Omega)\) vanish a.e. on any level set \(\{ (x, y, t) \in \Omega \mid f(x, y, t) = c \}\), see, e.g. [14] or page 297 of [17]. We will use, without any further mention, this fact several times in the proof of this part. Therefore, one has

\[
\partial_t \delta q_e + u \cdot \nabla \delta q_e + \delta u \cdot \nabla \tilde{q}_e + (\bar{Q} + \alpha) \nabla \cdot \delta v = 0 \quad \text{a.e. on } \Omega_2. \tag{4.3}
\]

Similar to (4.3), on the domain \(\Omega_3\) one has

\[
\partial_t \delta q_e + u \cdot \nabla \delta q_e + \delta u \cdot \nabla \tilde{q}_e - (\bar{Q} + \alpha) \nabla \cdot \bar{v} = 0 \quad \text{a.e. on } \Omega_3. \tag{4.4}
\]

Finally, since \(\tilde{q}_e = q_e = 0\), on \(\Omega_4\) one has

\[
\partial_t \delta q_e + u \cdot \nabla \delta q_e + \delta u \cdot \nabla \tilde{q}_e = 0 \quad \text{a.e. on } \Omega_4.
\]

Thanks to the last equation, as well as (4.2)–(4.4), we obtain the equation for \(\delta q_e\) as
\[
\partial_t \delta q_e + u \cdot \nabla \delta q_e + \delta u \cdot \nabla \delta q_e = -(\bar{Q} + \alpha)[\nabla \cdot (\delta v \chi_{\Omega_e}) + \delta v \cdot v \chi_{\Omega_e}] - \nabla \cdot \bar{v} \chi_{\Omega_e},
\]
\[
= -(\bar{Q} + \alpha)[\nabla \cdot \delta v - \nabla \cdot (\delta v \chi_{\Omega_e}) + \delta v \cdot \bar{v} \chi_{\Omega_e} - \nabla \cdot v \chi_{\Omega_e}],
\tag{4.5}
\]
a.e. on \( \Omega = \mathbb{R}^2 \times (0, \infty) \). Moreover, equation (4.5) holds in \( L^2_{\text{loc}}((0, \infty); L^2(\mathbb{R}^2)) \).

Multiplying equation (4.5) by \( \delta q_e \) and integrating over \( \mathbb{R}^2 \), it follows from integration by parts that
\[
\frac{1}{2} \frac{d}{dt} \|\delta q_e\|^2 \leq -\int_{\mathbb{R}^2} [(\delta u \cdot \nabla \delta q_e + (\bar{Q} + \alpha)\nabla \cdot \delta v)\delta q_e] \, dx \, dy
\]
\[
- (\bar{Q} + \alpha) \int_{\mathbb{R}^2} (\nabla \cdot \bar{v} \chi_{\Omega_e} - \nabla \cdot v \chi_{\Omega_e})(\delta q_e - \bar{q}_e) \, dx \, dy
\]
\[
\leq \frac{1}{4} \int_{\mathbb{R}^2} |\nabla \delta v|^2 \, dx \, dy + C \int_{\mathbb{R}^2} (\|\delta q_e\|^2 + |\nabla \bar{q}_e|)|\delta u||\delta q_e| \, dx \, dy
\]
\[
- (\bar{Q} + \alpha) \int_{\mathbb{R}^2} (\nabla \cdot \bar{v} \chi_{\Omega_e} - \nabla \cdot v \chi_{\Omega_e})(\delta q_e - \bar{q}_e) \, dx \, dy.
\tag{4.6}
\]

Recalling that \( \bar{q}_e = 0 \) on \( \Omega_e \), we have \( \partial_t \bar{q}_e + \bar{a} \cdot \nabla \bar{q}_e = 0 \) a.e. on \( \Omega_e \) and thus it follows from (1.16) for \((\bar{a}, \bar{v}, \bar{T}_e, \bar{q}_e)\) that \( \nabla \cdot \bar{v} \leq 0 \) a.e. on \( \Omega_e \). Similarly, one has \( \nabla \cdot v \leq 0 \) a.e. on \( \Omega_e \). Thanks to these facts we deduce
\[
\nabla \cdot \bar{v} \chi_{\Omega_e}(q_e - \bar{q}_e) = \nabla \cdot \bar{v} \chi_{\Omega_e} q_e > 0,
\]
\[
- \nabla \cdot v \chi_{\Omega_e}(q_e - \bar{q}_e) = \nabla \cdot v \chi_{\Omega_e} \bar{q}_e > 0.
\]

Therefore, it follows from (4.6) that
\[
\frac{d}{dt} \|\delta q_e\|^2 \leq \frac{1}{2} \|\nabla \delta v\|^2 + C \int_{\mathbb{R}^2} (|\delta q_e|^2 + |\nabla \bar{q}_e|)|\delta u||\delta q_e| \, dx \, dy.
\]

Summing the above inequality with (4.1) yields
\[
\frac{d}{dt} \|(\delta u, \delta v, \delta T_e, \delta q_e)\|^2 \leq \frac{1}{2}(\|\nabla \delta u\|^2 + \|\nabla \delta v\|^2)
\]
\[
\leq C \int_{\mathbb{R}^2} (|\nabla \bar{u}| + |\nabla \bar{v}| + |\nabla v| + |v|^2 + |\bar{v}|^2)(|\delta u|^2 + |\delta v|^2)
\]
\[
+ |\delta T_e|^2 + |\delta q_e|^2 + |\nabla \bar{T}_e||\delta u||\delta T_e| + |\nabla \bar{q}_e||\delta u||\delta q_e| \, dx \, dy,
\tag{4.7}
\]
which is exactly the same as inequality (3.9) from which, by the same argument as that in the proof of the uniqueness part of proposition 3.1, one obtains
\[
\|(\delta u, \delta v, \delta T_e, \delta q_e)\|^2 \equiv 0.
\]
This proves the uniqueness.

(iii) **Continuous dependence.** Let \((u^{(i)}, v^{(i)}, T_e^{(i)}, q_e^{(i)})\) be the unique solutions to system (1.12)–(1.18), with initial data \((u_0^{(i)}, v_0^{(i)}, T_e^{(0), i}, q_e^{(0), i})\), \( i = 1, 2 \). Suppose in addition that
\[
(\nabla T_e^{(i)}, \nabla q_e^{(i)}) \in L^m(\mathbb{R}^2), \text{ for some } m \in (2, \infty).\]
Then, recalling what we have proven in (i), \((u^{(i)}, v^{(i)}, T_e^{(i)}, q_e^{(i)})\) has the additional regularity that \((T_e^{(i)}, q_e^{(i)}) \in L^\infty(0, T; L^m(\mathbb{R}^2))\) for any positive time \( T \).
Denote by

$$(\delta u, \delta v, \delta T_e, \delta q_e) = (u^{(1)}, v^{(1)}, T_e^{(1)}, q_e^{(1)}) - (u^{(2)}, v^{(2)}, T_e^{(2)}, q_e^{(2)}),$$

and

$$(\delta u_0, \delta v_0, \delta T_{e,0}, \delta q_{e,0}) = (u^{(1)}_0, v^{(1)}_0, T_{e,0}^{(1)}, q_{e,0}^{(1)}) - (u^{(2)}_0, v^{(2)}_0, T_{e,0}^{(2)}, q_{e,0}^{(2)}).$$

Then, similar to (4.7), we have

$$\frac{d}{dt} \|\delta u, \delta v, \delta T_e, \delta q_e\|^2 + \frac{1}{2} (\|\nabla \delta u\|^2 + \|\nabla \delta v\|^2) \leq C \int_{\mathbb{R}^2} \left[ |\nabla u^{(2)}| + |\nabla v^{(2)}| + |\nabla v^{(1)}| + |v^{(1)}|^2 + |v^{(2)}|^2 \right] (|\delta u|^2 + |\delta v|^2)

+ |\delta T_e|^2 + |\delta q_e|^2 + \|\nabla T_e^{(2)}\| |\delta T_e| + \|\nabla q_e^{(2)}\| |\delta q_e| \, dx \, dy,$$

which is exactly of the same form as (3.31). Therefore, by the same argument as that in the proof of the continuous dependence part of (iii) of theorem 1.1, we obtain

$$\sup_{0 \leq s \leq T} \|\delta u(s), \delta v(s), \delta T_e(s), \delta q_e(s)\|^2 + \frac{1}{8} \int_0^T \|\delta u, \delta v\|^2 \, ds \leq C \left[ (1 + \|u^{(2)}, v^{(2)}\|^2) \|\nabla u^{(2)}, \nabla v^{(2)}\|^2 + \|\nabla T_e^{(2)}\|^2 \right] \|\delta u_0, \delta v_0, \delta T_{e,0}, \delta q_{e,0}\|^2,$$

Recalling the regularities of $(u^{(i)}, v^{(i)}, T_e^{(i)}, q_e^{(i)})$, $i = 1, 2$, the above inequality implies the continuous dependence of strong solutions on the initial data. This completes the proof of theorem 1.2.

□

5. Strong convergence of the relaxation limit

In this section, we prove the strong convergence of the relaxation limit, as $\varepsilon \to 0^+$, of system (1.5)--(1.9) to the limiting system (1.12)--(1.18).

**Proof of theorem 1.3.** Define the difference function $(\delta u_\varepsilon, \delta v_\varepsilon, \delta T_{e,\varepsilon}, \delta q_{e,\varepsilon})$ as

$$(\delta u_\varepsilon, \delta v_\varepsilon, \delta T_{e,\varepsilon}, \delta q_{e,\varepsilon}) = (u_\varepsilon, v_\varepsilon, T_{e,\varepsilon}, q_{e,\varepsilon}) - (u, v, T_e, q_e).$$

Taking the subtraction between equations (1.5)--(1.8), for $(u_\varepsilon, v_\varepsilon, T_{e,\varepsilon}, q_{e,\varepsilon})$ and equations (1.12)--(1.15), for $(u, v, T_e, q_e)$, one can easily check that

$$\begin{align*}
\partial_t \delta u_\varepsilon + (\delta u_\varepsilon \cdot \nabla) \delta u_\varepsilon + (\delta u_\varepsilon \cdot \nabla) u + (u \cdot \nabla) \delta u_\varepsilon - \Delta \delta u_\varepsilon \\
+ \nabla \delta p_\varepsilon + (\delta u_\varepsilon \cdot \nabla) \delta u_\varepsilon + (\delta v_\varepsilon \geq 0, \\
\nabla \cdot \delta u_\varepsilon = 0, \\
\partial_t \delta v_0 + (\delta u_\varepsilon \cdot \nabla) \delta v_\varepsilon + (\delta u_\varepsilon \cdot \nabla) v + (u \cdot \nabla) \delta v_\varepsilon - \Delta \delta v_\varepsilon + (\delta v_\varepsilon \cdot \nabla) \delta u_\varepsilon \\
+ (\delta v_\varepsilon \cdot \nabla) u + (v \cdot \nabla) \delta u_\varepsilon = \frac{1}{1 + \alpha} \nabla (\delta T_{e,\varepsilon} - \delta q_{e,\varepsilon}),
\end{align*}$$

(5.1)
\[ \partial_t \delta T_{ee} + \delta u_e \cdot \nabla \delta T_{ee} + \delta u_e \cdot \nabla T_e + u \cdot \nabla \delta T_{ee} - (1 - \bar{Q}) \nabla \cdot \delta v = 0, \quad (5.4) \]

where (5.1)–(5.4) hold a.e. on \( \mathbb{R}^2 \times (0, \infty) \) and in \( L^2_{\text{loc}}(\mathbb{R}^2) \).

Multiplying equations (5.1), (5.3) and (5.4) by \( \delta \epsilon u_e \), \( \delta \epsilon v_e \) and \( \delta \epsilon T_e \), respectively, summing the resultants, integrating over \( \mathbb{R}^2 \) and noting that

\[ \int_{\mathbb{R}^2} [\nabla \cdot (\delta v_e \otimes \delta v_e) \cdot \delta u_e + (\delta v_e \cdot \nabla) \delta u_e \cdot \delta v_e] \, dx \, dy = 0, \]

it follows from integration by parts that

\[
\frac{1}{2} \frac{d}{dt} \left\| (\delta u_e, \delta v_e, \delta T_e) \right\|_2^2 + \left\| (\nabla \delta u_e, \nabla \delta v_e) \right\|_2^2
\]

\[ = - \int_{\mathbb{R}^2} [(\delta u_e \cdot \nabla) u + \nabla \cdot (\delta v_e \otimes v + v \otimes \delta v_e)] \cdot \delta u_e \, dx \, dy
\]

\[ - \int_{\mathbb{R}^2} [(\delta u_e \cdot \nabla) v + (\delta v_e \cdot \nabla) u + (v \cdot \nabla) \delta u_e] \cdot \delta v_e \, dx \, dy
\]

\[ - \frac{1}{1 + \alpha} \int_{\mathbb{R}^2} (\nabla \cdot \delta v_e) (\delta T_e - \delta q_e) \, dx \, dy
\]

\[ - \int_{\mathbb{R}^2} [\delta u_e \cdot \nabla T_e - (1 - \bar{Q}) \nabla \cdot \delta v_e] \delta T_e \, dx \, dy, \]

from which, by the Young inequality, we deduce

\[
\frac{1}{2} \frac{d}{dt} \left\| (\delta u_e, \delta v_e, \delta T_e) \right\|_2^2 + \left\| (\nabla \delta u_e, \nabla \delta v_e) \right\|_2^2
\]

\[ \leq \int_{\mathbb{R}^2} [\nabla u_c \cdot \delta u_e + 2 |\nabla v| |\delta v_e| + 2 |v| |\nabla \delta v_e|] \, dx \, dy + \int_{\mathbb{R}^2} |\nabla v_e| (|\delta T_e| + |\delta q_e|) \, dx \, dy
\]

\[ + \int_{\mathbb{R}^2} (1 - \bar{Q}) |\nabla \delta v_e| |\delta T_e| + |\nabla T_e| |\delta u_e| |\delta T_e| \, dx \, dy
\]

\[ \leq \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u_e|^2 + |\nabla \delta v_e|^2) \, dx \, dy + C \int_{\mathbb{R}^2} [\nabla u_c \cdot \delta T_e + |\delta T_e|^2 + |\delta q_e|^2 + |\nabla T_e| |\delta u_e| |\delta T_e|] \, dx \, dy. \]

Therefore, we obtain

\[
\frac{d}{dt} \left\| (\delta u_e, \delta v_e, \delta T_e) \right\|_2^2 + \left\| (\nabla \delta u_e, \nabla \delta v_e) \right\|_2^2
\]

\[ \leq C \int_{\mathbb{R}^2} [\nabla u_c \cdot \delta T_e + |\delta T_e|^2 + |\delta q_e|^2 + |\nabla T_e| |\delta u_e| |\delta T_e|] \, dx \, dy. \quad (5.5) \]

We still need to estimate \( \| \delta q_e \|_2^2 \). To this end, we first derive the equation for \( \delta q_e \). On the set \( \{ (x, y, t) \in \mathbb{R}^2 \times (0, \infty) \mid q_e(x, y, t) < 0 \} \), \( q_e \) and \( q e \) satisfy equations (1.9) and (1.18), respectively, and thus \( \delta q_e \) satisfies
\[ \partial_t \delta q_{\epsilon} + \delta u_e \cdot \nabla \delta q_{\epsilon} + \delta u_e \cdot \nabla q_e + u \cdot \nabla \delta q_{\epsilon} + (\bar{Q} + \alpha) \nabla \cdot \delta \nu = -\frac{1 + \alpha}{\epsilon} q_{\epsilon}^+, \]

a.e. on \( \{(x,y,t) \in \mathbb{R}^2 \times (0,\infty) | \delta q_e(x,y,t) < 0\} \). On the set \( \mathcal{O} := \{(x,t) \in \mathbb{R}^2 \times (0,\infty) | q_e(x,t) = 0\} \), recalling again the well-known fact that the derivatives of a function \( f \in W^{1,1}_{loc}(\mathbb{R}^2 \times (0,\infty)) \) vanish a.e. on any level set \( \{(x,y,t) \in \mathbb{R}^2 \times (0,\infty) | f(x,y,t) = c\} \), we have \( \partial_t q_e + u \cdot \nabla q_e = 0 \) a.e. on \( \mathcal{O} \) and \( q_e \) satisfies (1.9). Consequently, \( \delta q_{\epsilon} \) satisfies

\[ \partial_t \delta q_{\epsilon} + \delta u_e \cdot \nabla \delta q_{\epsilon} + \delta u_e \cdot \nabla q_e + u \cdot \nabla \delta q_{\epsilon} + (\bar{Q} + \alpha) \nabla \cdot \nu = -\frac{1 + \alpha}{\epsilon} q_{\epsilon}^+, \]

a.e. on \( \mathcal{O} \). Combining the above two equations, one can see that \( \delta q_{\epsilon} \) satisfies

\[ \partial_t \delta q_{\epsilon} + \delta u_e \cdot \nabla \delta q_{\epsilon} + \delta u_e \cdot \nabla q_e + u \cdot \nabla \delta q_{\epsilon} + (\bar{Q} + \alpha) \nabla \cdot \nu = -\frac{1 + \alpha}{\epsilon} q_{\epsilon}^+, \tag{5.6} \]

a.e. on \( \mathbb{R}^2 \times (0,\infty) \) and in \( L^2_{loc}(\mathbb{R}^2) \).

Multiplying equation (5.6) by \( \delta q_{\epsilon} \) and integrating over \( \mathbb{R}^2 \), it follows from integration by parts that

\[ \frac{1}{2} \frac{d}{dt} \|\delta q_{\epsilon}\|^2 + \frac{1 + \alpha}{\epsilon} \int_{\mathbb{R}^2} q_{\epsilon}^+ \delta q_{\epsilon} \, dx \, dy = -\int_{\mathbb{R}^2} [\delta u_e \cdot \nabla q_e + (\bar{Q} + \alpha)(\nabla \cdot \delta \nu + \nabla \cdot v_{\mathcal{O}}(x,y,t))] \delta q_{\epsilon} \, dx \, dy, \tag{5.7} \]

a.e. \( t \in (0,\infty) \). Recalling that \( \delta q_{\epsilon} \leq 0 \) we have

\[ \int_{\mathbb{R}^2} q_{\epsilon}^+ \delta q_{\epsilon} \, dx \, dy = \int_{\mathbb{R}^2} q_{\epsilon}^+(q_{\epsilon} - q_e) \, dx \, dy \geq \int_{\mathbb{R}^2} q_{\epsilon}^+ \, dx \, dy = \|q_{\epsilon}^+\|^2 \tag{5.8} \]

Note that \( \partial_t q_e + u \cdot \nabla q_e = 0 \) a.e. on \( \mathcal{O} \), it follows from (1.16) that \( \nabla \cdot \nu \leq 0 \) a.e. on \( \mathcal{O} \), and thus

\[ -\nabla \cdot v_{\mathcal{O}}(x,y,t) \delta q_{\epsilon} = -\nabla \cdot v_{\mathcal{O}}(x,y,t) q_e \leq -\nabla \cdot v_{\mathcal{O}}(x,y,t) q_{\epsilon}^-. \]

Thanks to the above inequality, it follows from (5.7), (5.8) and the Young inequality that

\[ \frac{1}{2} \frac{d}{dt} \|\delta q_{\epsilon}\|^2 + \frac{1 + \alpha}{\epsilon} \|q_{\epsilon}^+\|^2 \leq -\int_{\mathbb{R}^2} [\delta u_e \cdot \nabla q_e + (\bar{Q} + \alpha)(\nabla \cdot \delta \nu + \nabla \cdot v_{\mathcal{O}}(x,y,t))] \delta q_{\epsilon} \, dx \, dy \]

\[ \leq \int_{\mathbb{R}^2} |\delta q_e||\delta q_{\epsilon}| \, dx \, dy + \frac{1}{4} \|\nabla \delta \nu\|^2 + (\bar{Q} + \alpha)^2 \|\delta q_{\epsilon}\|^2 \]

\[ + \frac{1 + \alpha}{2\epsilon} \|q_{\epsilon}^+\|^2 + \frac{(\bar{Q} + \alpha)^2}{2(1 + \alpha)} \epsilon \|\nabla \nu\|^2, \]
and thus
\[
\frac{d}{dt} \| \delta q_c \|_2^2 + \frac{1 + \alpha}{\varepsilon} \| \delta q_c \|_2^2 \leq \frac{1}{2} \| \nabla \delta v \|_2^2 + 2(Q + \alpha)^2 \| \delta q_c \|_2^2 + \frac{(Q + \alpha)^2}{1 + \alpha} \varepsilon \| \nabla v \|_2^2 + 2 \int_{\mathbb{R}^3} | \nabla q_c | | \delta u_c | | \delta q_c | \, dx dy.
\] (5.9)

Summing (5.5) with (5.9) yields
\[
\frac{d}{dt} \| (\delta u_c, \delta v, \delta T_c, \delta q_c) \|_2^2 + \frac{1 + \alpha}{\varepsilon} \| \delta q_c \|_2^2 \leq C \int_{\mathbb{R}^3} [(| \nabla u | + | \nabla v | + | v |^3)(| \delta u_c |^2 + | \delta v |^2) + | \delta T_c |^2 + | \delta q_c |^2
+ | \nabla T_c | | \delta u_c | | \delta T_c | + | \nabla q_c | | \delta u_c | | \delta q_c | ] \, dx dy + C \varepsilon \| \nabla v \|_2^2 = J.
\] (5.10)

We are going to estimate \( J \). First, by the H"older, Ladyzhenskaya and Young inequalities, we deduce
\[
J_1 = C \int_{\mathbb{R}^3} (| \nabla u | + | \nabla v | + | v |^2)(| \delta u_c |^2 + | \delta v |^2) \, dx dy
\leq C (|| \nabla u ||_2 + || \nabla v ||_2 + || v ||_2^2)(|| \delta u_c ||_2^2 + || \delta v ||_2^2)
\leq C (|| \nabla u ||_2 + || \nabla v ||_2 + || v ||_2^2)(|| \delta u_c ||_2^2 + || \nabla \delta u_c ||_2 + || \delta v ||_2 || \nabla \delta v ||_2)
\leq \frac{1}{8} (|| \nabla \delta u_c ||_2^2 + || \nabla \delta v ||_2^2) + C (|| \nabla u ||_2^2 + || \nabla v ||_2^2
+ || v ||_2^2 || \nabla v ||_2^2)(|| \delta u_c ||_2^2 + || \delta v ||_2^2).
\]

Second, it follow from the H"older, Gagliardo--Nirenberg, \( \| \varphi \| \frac{2m}{m-2} \leq C \| \varphi \|_2^2 \| \nabla \varphi \|_2^2 \) and Young inequalities that
\[
J_2 = C \int_{\mathbb{R}^3} (| \nabla T_c | | \delta u_c | | \delta T_c | + | \nabla q_c | | \delta u_c | | \delta q_c | ) \, dx dy
\leq C || \delta u_c ||_2 \frac{2m}{m-2} (|| \nabla T_c ||_{m-1} || \delta T_c ||_2 + || \nabla q_c ||_m || \delta q_c ||_2)
\leq C || \delta u_c ||_2 \frac{m-2}{m-2} (|| \nabla T_c ||_{m-1} || \delta T_c ||_2 + || \nabla q_c ||_m || \delta q_c ||_2)
\leq \frac{1}{8} || \nabla \delta u_c ||_2^2 + C (|| \delta u_c ||_2^\frac{m-2}{m-1} (|| \nabla T_c ||_{m-1} || \delta T_c ||_2^\frac{m}{m-1} + || \nabla q_c ||_m^\frac{m}{m-1} || \delta q_c ||_2^m
\leq \frac{1}{8} || \nabla \delta u_c ||_2^2 + C (|| \nabla T_c ||_{m-1} || \nabla q_c ||_m^\frac{m}{m-1} (|| \delta u_c ||_2^2 + || \delta T_c ||_2^2 + || \delta q_c ||_2^2).
\]

Then, by the aid of the above estimates on \( J_1 \) and \( J_2 \) we can estimate \( J \) as
\[
J = J_1 + J_2 + C (|| \delta T_c ||_2^2 + || \delta q_c ||_2^2 + \varepsilon || \nabla v ||_2^2)
\leq \frac{1}{4} (|| \nabla \delta u_c ||_2^2 + || \nabla \delta v ||_2^2) + C (|| \nabla u ||_2^2 + || \nabla v ||_2^2 + || v ||_2^2 || \nabla v ||_2^2 + || \nabla T_c ||_{m-1}^\frac{m}{m-1}
+ || \nabla q_c ||_{m-1}^\frac{m}{m-1} + 1)(|| \delta u_c ||_2^2 + || \delta v ||_2^2 + || \delta T_c ||_2^2 + || \delta q_c ||_2^2) + C \varepsilon \| \nabla v \|_2^2.
\]
Therefore, substituting the above estimate into (5.10) yields
\[
\frac{d}{dt} \left\| (\delta u, \delta v, \delta T, \delta q) \right\|_2^2 + \frac{1}{4} \left\| (\nabla \delta u, \nabla \delta v) \right\|_2^2 + \frac{1 + \alpha}{\varepsilon} \left\| q^+ \right\|_2^2 \\
\leq C \left( 1 + \left\| (\nabla u, \nabla v) \right\|_2^2 + \left\| v \right\|_2^2 \left\| \nabla v \right\|_2^2 + \left\| (\nabla T, \nabla q) \right\|_m^{m-1} \right) \\
\times \left\| (\delta u, \delta v, \delta T, \delta q) \right\|_2^2 + C \varepsilon \left\| \nabla v \right\|_2^2.
\]

Applying the Gronwall inequality to the above inequality and recalling the regularities of \((u, v, T, q)\) yield
\[
\sup_{0 \leq t \leq T} \left\| (\delta u, \delta v, \delta T, \delta q) (t) \right\|_2^2 + \int_0^T \left( \left\| (\nabla \delta u, \nabla \delta v) \right\|_2^2 + \frac{\left\| q^+ \right\|_2^2}{\varepsilon} \right) dt \leq C \varepsilon,
\]
for a positive constant \(C\) depending only on \(\alpha, Q, T, m, \left\| (u_0, v_0, T_{0,0}, q_{0,0}) \right\|_H\) and \(\left\| (\nabla T_{0,0}, \nabla q_{0,0}) \right\|_m\). This proves the desired estimate in the theorem, while the strong convergences are direct consequences of this estimate. \(\square\)

6. Conclusion

We addressed the rigorous mathematical analyses on a nonlinear interaction system between the barotropic mode and the first baroclinic mode of the tropical atmosphere with moisture, which is derived from the viscous primitive equations (with full or only horizontal viscosity). The system considered in this paper is the viscous version of the system derived in [15], which is a generalization of the classic first baroclinic mode model used in many tropical atmospheric dynamics models. In view of the presence of the horizontal eddy viscosity, created by the horizontal strong turbulence mixing of the atmosphere, it is natural for us to consider such a viscous version of the system.

The following results on the global well-posedness and strong convergence are established in this paper:

1. The global existence and uniqueness of strong solutions to the finite-time relaxation system, for any \(H^1\) initial data, and the continuous dependence on the initial data when the initial data enjoy slightly more regularity than \(H^1\).
2. The global existence and uniqueness of strong solutions to the instantaneous-relaxation limiting system for any \(H^1\) initial data, and the continuous dependence on the initial data when the initial data enjoy slightly more regularity than \(H^1\).
3. Strong convergence, with a convergence rate of the order \(O(\sqrt{\varepsilon})\), from the finite-time relaxation system to the instantaneous-relaxation limiting system, as the convective adjustment time \(\varepsilon\) tends to zero.

In particular, result (3) justifies the appropriateness of the instantaneous-relaxation model in the viscous case. Moreover, our results, combined with the finite time blow up results [4, 46] for the inviscid primitive equations, as well as the global well-posedness results [7–9] for the
primitive equations with only horizontal viscosity, indicate that one may not ignore the hori-
zontal viscous mechanism created by the horizontal turbulence mixing in the atmosphere, no
matter how weak it is, because the horizontal eddy viscosity plays an essential role in stabiliz-
ing the flow.

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Appendix

In this appendix, we state and prove several parabolic estimates which have been used in the
previous sections.

Lemma A.1. Given a time \( T \in (0, \infty) \) and a function \( g \in L^\alpha(0, T; L^\beta(\mathbb{R}^2)) \), with \( 1 < \alpha, \beta < \infty \),
let \( U \) be the unique solution to

\[
\begin{cases}
\partial_t U - \Delta U = g, & \text{in } \mathbb{R}^2 \times (0, T), \\
U|_{t=0} = 0, & \text{in } \mathbb{R}^2.
\end{cases}
\]

Then, we have the estimate

\[
\|\partial_t U\|_{L^\alpha(0,T;L^\beta(\mathbb{R}^2))} + \|\Delta U\|_{L^\alpha(0,T;L^\beta(\mathbb{R}^2))} \leq C_{\alpha,\beta} \|g\|_{L^\alpha(0,T;L^\beta(\mathbb{R}^2))},
\]

where \( C_{\alpha,\beta} \) is a positive constant depending only on \( \alpha, \beta \) and, in particular, is independent of
\( T \) and \( g \).

Proof. Introducing the scaled functions \( U_T \) and \( g_T \) as

\[
U_T(x,t) = U(\sqrt{T}x, Tt), \quad g_T(x,t) = g(\sqrt{T}x, Tt), \quad x \in \mathbb{R}^2, t \in (0, 1),
\]

then one can easily verify that \( U_T \) and \( g_T \) satisfy

\[
\begin{cases}
\partial_t U_T - \Delta U_T = T g_T, & \text{in } \mathbb{R}^2 \times (0, 1), \\
U_T|_{t=0} = 0, & \text{in } \mathbb{R}^2.
\end{cases}
\]

Applying the maximal regularity theory for parabolic equations to the above system (see, e.g.
[13], [19] and [24]), one has

\[
\|\partial_t U_T\|_{L^\alpha(0,1;L^\beta(\mathbb{R}^2))} + \|\Delta U_T\|_{L^\alpha(0,1;L^\beta(\mathbb{R}^2))} \leq C_{\alpha,\beta} T \|g_T\|_{L^\alpha(0,T;L^\beta(\mathbb{R}^2))}.
\]

From which, and after observing that
\[ \| \partial_t U \|_{L^2(0,1;L^2(\mathbb{R}^2))} = T^{\frac{1}{2}} \| \partial_t U \|_{L^2(0,T;L^2(\mathbb{R}^2))}, \]
\[ \| \Delta U \|_{L^2(0,1;L^2(\mathbb{R}^2))} = T^{\frac{1}{2}} \| \Delta U \|_{L^2(0,T;L^2(\mathbb{R}^2))}, \]
\[ \| u \|_{L^2(0,1;L^2(\mathbb{R}^2))} = T^{\frac{1}{2}} \| u \|_{L^2(0,T;L^2(\mathbb{R}^2))}, \]

one concludes the proof. □

**Lemma A.2.** Given a time \( T \in (0, \infty) \), let \( f \) and \( g \) be two functions, such that \( f \in L^2(\mathbb{R}^2 \times (0, T)) \) and \( g \in L^2(\mathbb{R}^2 \times (0, T)) \). Let \( v \) be the unique solution to

\[
\begin{aligned}
\partial_t v - \Delta v &= f + \nabla g, & \text{in} \ \mathbb{R}^2 \times (0, T), \\
v_{|t=0} &= v_0 \in H^1(\mathbb{R}^2), & \text{in} \ \mathbb{R}^2.
\end{aligned}
\]

Then we have the following estimate

\[
\int_0^T \| \nabla v(t) \|_{L^2}^2 \, dt \leq C \left( \| \nabla v_0 \|_{L^2}^2 + \left( \int_0^T \| f \|_{L^2}^2 \, dt \right)^2 + \int_0^T \| g \|_{L^2}^2 \, dt \right),
\]

where \( C \) is an absolute constant and in particular is independent of \( T, v_0, f \) and \( g \).

**Proof.** Decompose \( v \) as \( v = \bar{v} + \hat{v} \), where \( \bar{v} \) and \( \hat{v} \) are the unique solutions to systems

\[
\begin{aligned}
\partial_t \bar{v} - \Delta \bar{v} &= f, & \text{in} \ \mathbb{R}^2 \times (0, T), \\
\bar{v}_{|t=0} &= v_0 \in H^1(\mathbb{R}^2), & \text{in} \ \mathbb{R}^2,
\end{aligned}
\]

and

\[
\begin{aligned}
\partial_t \hat{v} - \Delta \hat{v} &= \nabla g, & \text{in} \ \mathbb{R}^2 \times (0, T), \\
\hat{v}_{|t=0} &= 0, & \text{in} \ \mathbb{R}^2,
\end{aligned}
\]

respectively. The standard energy approach (multiplying the equation for \( \bar{v} \) by \( -\Delta \bar{v} \), integrating over \( \mathbb{R}^2 \), integration by parts, using the Young inequality and integrating with respect to \( t \) over \( (0, T) \)) to the system for \( \bar{v} \) leads to

\[
\sup_{0 \leq t \leq T} \| \nabla \bar{v}(t) \|_{L^2}^2 + \int_0^T \| \Delta \bar{v} \|_{L^2}^2 \, dt \leq \| \nabla v_0 \|_{L^2}^2 + \int_0^T \| f \|_{L^2}^2 \, dt.
\]

Defining \( U \) to be the unique solution to the system

\[
\begin{aligned}
\partial_t U - \Delta U &= g, & \text{in} \ \mathbb{R}^2 \times (0, T), \\
U_{|t=0} &= 0, & \text{in} \ \mathbb{R}^2.
\end{aligned}
\]

Then \( \nabla U \) satisfies the same system as that for \( \bar{v} \), and therefore, by the uniqueness of the solutions to system \((A.1)\), we have \( \bar{v} = \nabla U \). Thanks to this fact, and applying lemma A.1, it follows from the elliptic estimates that

\[
\| \nabla \bar{v} \|_{L^2(0,T;L^2(\mathbb{R}^2))} \leq C \| \nabla^2 U \|_{L^2(0,T;L^2(\mathbb{R}^2))} \leq C \| \Delta U \|_{L^2(0,T;L^2(\mathbb{R}^2))} \leq C \| \nabla^2 U \|_{L^2(0,T;L^2(\mathbb{R}^2))}.
\]
for an absolute positive constant $C$.
Combining the estimates for $\bar{v}$ and $\hat{v}$, we deduce from the Ladyzhenskaya inequality that
\[
\int_0^T \|\nabla v\|^2_m \, dt \leq C \int_0^T \|\nabla v\|^2_m \, dt + C \int_0^T \|\nabla \bar{v}\|^2_m \, dt \\
\leq C \left( \sup_{0 \leq t \leq T} \|\nabla v(t)\|^2_m \int_0^T \|\nabla \bar{v}\|^2_m \, dt + C \|g\|^2_m \|L^2_{0, T; L^m(\mathbb{R}^2)}\| \right) \\
\leq C \left( \|\nabla v_0\|^2_m + \left( \int_0^T \|f\|^2_m \, dt \right)^\frac{1}{2} + \int_0^T \|g\|^2_m \, dt \right),
\]
for an absolute positive constant $C$. This completes the proof. □

**Lemma A.3.** Given a time $T \in (0, \infty)$ and a number $m \in (2, \infty)$, let $f \in L^2(0, T; L^m(\mathbb{R}^2))$ and $v$ be the unique solution to
\[
\begin{cases}
\partial_t v - \Delta v = f, & \text{in } \mathbb{R}^2 \times (0, T), \\
v_{|t=0} = v_0 \in H^1(\mathbb{R}^2).
\end{cases}
\]
Then, we have the following estimate
\[
\int_0^T \|\Delta v\|^2_m \, dt \leq C_m (1 + \sqrt{T}) \left( \|\nabla v_0\|^2_m + \left( \int_0^T \|f\|^2_m \, dt \right)^\frac{1}{2} \right),
\]
where $C_m$ is a positive constant depending only on $m$, and in particular is independent of $T$, $f$ and $v_0$.

**Proof.** Decompose $v$ as $v = \bar{v} + \hat{v}$, where $\bar{v}$ and $\hat{v}$ are the unique solutions to systems
\[
\begin{cases}
\partial_t \bar{v} - \Delta \bar{v} = f, & \text{in } \mathbb{R}^2 \times (0, T), \\
\bar{v}_{|t=0} = 0,
\end{cases}
\]
and
\[
\begin{cases}
\partial_t \hat{v} - \Delta \hat{v} = 0, & \text{in } \mathbb{R}^2 \times (0, T), \\
\hat{v}_{|t=0} = v_0 \in H^1(\mathbb{R}^2)
\end{cases}
\]
respectively.
By lemma A.1 and using the Hölder inequality for $\bar{v}$, we have the estimate
\[
\int_0^T \|\Delta \bar{v}\|^2_m \, dt \leq C_m T \left( \int_0^T \|\Delta \bar{v}\|^2_m \, dt \right)^\frac{1}{2} \leq C_m T \left( \int_0^T \|f\|^2_m \, dt \right)^\frac{1}{2}.
\]
To estimate $\hat{v}$, we multiplying equation (A.2) by $t \Delta^2 \hat{v} - \Delta \hat{v}$, integrating the resultant over $\mathbb{R}^2$, then it follows from integration by parts that
\[
\frac{1}{2} \frac{d}{dt} (\|\nabla \hat{v}\|^2_2 + \|\sqrt{T} \Delta \hat{v}\|^2_2) + \frac{1}{2} \|\Delta \hat{v}\|^2_2 + \|\sqrt{T} \nabla \Delta \hat{v}\|^2_2 = 0,
\]
from which integrating with respect to $t$ yields

$$\sup_{0 \leq t \leq T} (\|\nabla \hat{v}(t)\|_{L^2}^2 + \|\sqrt{t}\nabla \Delta \hat{v}(t)\|_{L^2}^2) + \int_0^T (\|\Delta \hat{v}\|_{L^2}^2 + \|\sqrt{t}\nabla \Delta \hat{v}\|_{L^2}^2) dt \leq \|\nabla v_0\|_{L^2}^2.$$

Thanks to this estimate, by the Gagliardo–Nirenberg, $\|\varphi\|_{m} \leq C\|\varphi\|_{L^2}^{2m} \|\nabla \varphi\|_{L^2}^{1-\frac{2m}{m}}$ and Hölder inequalities, we deduce

$$\int_0^T \|\Delta \hat{v}\|_{m} dt \leq C \int_0^T \|\Delta \hat{v}\|_{L^2}^{2m} \|\nabla \Delta \hat{v}\|_{L^2}^{1-\frac{2m}{m}} t^{\frac{1}{2m}} \|\nabla \Delta \hat{v}\|_{L^2}^{1-\frac{2m}{m}} dt \leq C \left( \int_0^T \|\Delta \hat{v}\|_{L^2}^{2m} dt \right)^{\frac{1}{2m}} \left( \int_0^T \|\nabla \Delta \hat{v}\|_{L^2}^{1-\frac{2m}{m}} dt \right)^{\frac{1}{1-\frac{2m}{m}}} \leq C \sqrt{m} T^{\frac{1}{2m}} \|\nabla v_0\|_{L^2}.$$

Combining the estimates for $\hat{v}$ and $\hat{\varphi}$, we then deduce from the Young inequality (recalling $m > 2$) that

$$\int_0^T \|\Delta \hat{v}\|_{m} dt \leq \int_0^T (\|\Delta \hat{v}\|_{m} + \|\Delta \hat{\varphi}\|_{m}) dt \leq C_n T^{\frac{1}{2}} \left( \int_0^T \|\hat{f}\|_{m}^2 dt \right)^{\frac{1}{2}} + C \sqrt{m} T^{\frac{1}{2m}} \|\nabla v_0\|_{L^2} \leq C_n (1 + \sqrt{T}) \left[ \|\nabla v_0\|_{L^2} + \left( \int_0^T \|\hat{f}\|_{m}^2 dt \right)^{\frac{1}{2}} \right],$$

concluding the proof. \qed

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