BALANCE LAWS IN MICROMORPHIC ELASTICITY

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ABSTRACT: We derive the Eshelby stress tensor, the angular momentum tensor and the dilatational vector flux for micromorphic elasticity. We give the corresponding balance laws and the $J$, $L$, and $M$ integrals. Also we discuss when the balance laws become conservation laws.

Keywords: Micromorphic elasticity, balance laws, Eshelby stress tensor

1 INTRODUCTION

Conservation and balance laws are a very important research field in theories of elasticity, generalized elasticity, material science and engineering science [1]. Each infinitesimal symmetry group of the strain energy is associated with a conservation law [2].

In this paper we consider the microcontinuum field theory of micromorphic elasticity. A micromorphic theory can be used to model materials with microstructure. A micromorphic continuum, as we consider, is built up from particles which possess an inherent orientation. It possesses twelve degrees of freedom: three translational ones and nine ones of the microstructure (rotation, shear, dilatation).

It is the purpose of the present paper to derive balance and conservation laws for micromorphic elasticity. We take account for material inhomogeneity, anisotropy, body forces and body couples. Also we will discuss the corresponding $J$, $L$ and $M$ integrals.

2 MICROMORPHIC ELASTICITY

We consider the theory of micromorphic elasticity [3,4,5]. Micromorphic elasticity views a material as a continuous collection of deformable particles. Each particle is attached with a microstructure of finite size. In addition to macro-deformations, we have to consider micro-deformations in order to describe microstructural effects (micro-rotation, micro-dilatation and micro-shear of the microstructure). The basic macro-field is the displacement vector $u(x)$ and the basic micro-field is the so-called micro-distortion tensor $\phi(x)$.
Let the strain energy density of micromorphic elasticity be of the form
\[ W = W(x, u, \phi_{\alpha\beta}, u_{\alpha,i}, \phi_{\alpha\beta,i}). \] (1)

The Euler-Lagrange equations are obtained according to
\[ E_{\alpha}(W) = \frac{\partial W}{\partial u_{\alpha}} - D_i \frac{\partial W}{\partial u_{\alpha,i}} = 0, \] (2)
\[ E_{\alpha\beta}(W) = \frac{\partial W}{\partial \phi_{\alpha\beta}} - D_i \frac{\partial W}{\partial \phi_{\alpha\beta,i}} = 0, \] (3)
where \( D_i \) is the so-called total derivative
\[ D_i = \frac{\partial}{\partial x_i} + u_{\alpha,i} \frac{\partial}{\partial u_{\alpha}} + u_{\alpha,ij} \frac{\partial}{\partial u_{\alpha,j}} + \phi_{\alpha\beta,i} \frac{\partial}{\partial \phi_{\alpha\beta}} + \phi_{\alpha\beta,ij} \frac{\partial}{\partial \phi_{\alpha\beta,j}} + \ldots. \] (4)

In linear micromorphic elasticity, the strains are related to a displacement vector and a micro-distortion tensor according
\[ \gamma_{kl} = u_{k,l} - \phi_{kl}, \quad 2e_{kl} = \phi_{kl} + \phi_{lk}, \quad \kappa_{klm} = \phi_{kl,m}. \] (5)

Here \( \gamma_{kl}, e_{kl} \) and \( \kappa_{klm} \) are the relative distortion, the micro-strain and the wryness tensors.

Using the strain tensors, the strain energy density with external sources is of the form
\[ W = \frac{1}{2} t_{kl} \gamma_{kl} + \frac{1}{2} s_{kl} e_{kl} + \frac{1}{2} m_{ijk} \kappa_{ijk} - u_{\alpha} F_{\alpha} - \phi_{\alpha\beta} L_{\alpha\beta}. \] (6)

For linear and anisotropic micromorphic elasticity, the constitutive relations for the stresses are
\[ t_{ij} = \frac{\partial W}{\partial \gamma_{ij}} = A_{ijkl} \gamma_{kl} + E_{ijkl} e_{kl} + F_{ijklm} \kappa_{klm}, \] (7)
\[ s_{ij} = \frac{\partial W}{\partial e_{ij}} = E_{klj} \gamma_{kl} + B_{ijkl} e_{kl} + G_{ijklm} \kappa_{klm}, \] (8)
\[ m_{ijk} = \frac{\partial W}{\partial \kappa_{ijk}} = F_{lmijk} \gamma_{lm} + G_{lmijk} e_{lm} + C_{ijkmn} \kappa_{lmn}, \] (9)
with \( s_{ij} = s_{ji} \). Here \( t_{ij} \) is the force stress tensor, \( s_{kl} \) is the micro-stress tensor and \( m_{klm} \) is the stress moment tensor. The tensors \( A_{ijkl}, B_{ijkl}, C_{ijklmn}, E_{ijkl}, F_{ijklm} \) and \( G_{ijklm} \) are constitutive tensors. They fulfill the symmetry relations:
\[ A_{ijkl} = A_{klji}, \quad B_{ijkl} = B_{klij} = B_{jikl} = B_{ijkl}, \quad C_{ijklmn} = C_{lminj}, \quad E_{ijkl} = E_{jikl}, \quad G_{ijklm} = G_{jiklm}. \] (10)

The external body forces \( F_{\alpha} \) and body couples \( L_{\alpha\beta} \) are defined by
\[ F_{\alpha} := -\frac{\partial W}{\partial u_{\alpha}}, \quad L_{\alpha\beta} := -\frac{\partial W}{\partial \phi_{\alpha\beta}}. \] (11)

Therefore, the forces and couples are conservative. In addition, the Lagrangian may depend explicitly on \( x_i \). In this case, the material force (or inhomogeneity force) is defined by
\[ f_{\alpha \text{inh}} := -\frac{\partial W}{\partial x_i}, \] (12)
which is caused by material inhomogeneities.

Using eqs. (2) and (3), the Euler-Lagrange equations read in terms of the canonical conjugate quantities:
\[ D_it_{\alpha\beta} + F_{\alpha} = 0, \] (13)
\[ D_im_{\alpha\beta} + (t_{\alpha\beta} - s_{\alpha\beta}) + L_{\alpha\beta} = 0. \] (14)

1The usual notations \( A_i = \frac{\partial A}{\partial (x_i)} \) and \( \dot{A} = \frac{\partial A}{\partial t} \) are used.
TRANSLATIONAL, ROTATIONAL AND DILATATIONAL FLUXES

In micromorphic elasticity the flux is given by \[6\]

\[
A_i = U_\alpha \frac{\partial W}{\partial u_{\alpha,i}} + \Phi_{\alpha\beta} \frac{\partial W}{\partial \phi_{\alpha\beta,i}} + X_i W - X_j \left( u_{\alpha,j} \frac{\partial W}{\partial u_{\alpha,i}} + \phi_{\alpha\beta,j} \frac{\partial W}{\partial \phi_{\alpha\beta,i}} \right).
\]

(15)

Here the infinitesimal generators are defined by

\[
X_i(x, u, \phi) := \left. \frac{\partial x_i'}{\partial \varepsilon} \right|_{\varepsilon=0}, \quad U_\alpha(x, u, \phi) := \left. \frac{\partial u'_\alpha}{\partial \varepsilon} \right|_{\varepsilon=0}, \quad \Phi_{\alpha\beta}(x, u, \phi) := \left. \frac{\partial \phi'_{\alpha\beta}}{\partial \varepsilon} \right|_{\varepsilon=0}
\]

(16)

and the group action is of the form

\[
x'_i = x_i + \varepsilon X_i(x, u, \phi) + \cdots,
\]

(17)

\[
u'_\alpha = u_\alpha + \varepsilon U_\alpha(x, u, \phi) + \cdots,
\]

(18)

\[
\phi'_{\alpha\beta} = \phi_{\alpha\beta} + \varepsilon \Phi_{\alpha\beta}(x, u, \phi) + \cdots,
\]

(19)

where \(\varepsilon\) is a group parameter.

3.1 Translations

The translation in space is given by the formulae:

\[
x'_i = x_i + \varepsilon \delta_{ki}, \quad u'_\alpha = u_\alpha, \quad \phi'_{\alpha\beta} = \phi_{\alpha\beta}.
\]

(20)

The corresponding generators of infinitesimal transformations are

\[
X_{ki} = \delta_{ki}, \quad U_\alpha = 0, \quad \Phi_{\alpha\beta} = 0.
\]

(21)

Using eqs. (15) and (21), the translational flux is given by

\[
P_{ki} = W \delta_{ki} - u_{\alpha,k} \frac{\partial W}{\partial u_{\alpha,i}} - \phi_{\alpha\beta,k} \frac{\partial W}{\partial \phi_{\alpha\beta,i}}.
\]

(22)

In terms of stresses it reads

\[
P_{ki} = W \delta_{ki} - u_{\alpha,k} t_{\alpha,i} - \phi_{\alpha\beta,k} m_{\alpha\beta,i}.
\]

(23)

Eq. (23) is nothing but the Eshelby stress tensor for micromorphic elasticity [6, 7, 8].

3.2 Rotations

The three-dimensional group of rotations acts in the space of the independent \(x\) and dependent variables \(u\) and \(\phi\). Its infinitesimal action is given by:

\[
x'_i = x_i + \varepsilon_{kji} x_j \varepsilon_k, \quad u'_\alpha = u_\alpha + \varepsilon_{k\beta\alpha} u_\beta \varepsilon_k, \quad \phi'_{\alpha\beta} = \phi_{\alpha\beta} + \varepsilon_{k\beta\alpha} \phi_{\beta\beta} \varepsilon_k + \varepsilon_{k\beta\beta} \phi_{\alpha\beta} \varepsilon_k.
\]

(24)

Eventually, the infinitesimal generators are easily obtained

\[
X_{ik} = \varepsilon_{ikj} x_j, \quad U_{\alpha k} = \varepsilon_{\alpha k\beta} u_\beta, \quad \Phi_{\alpha \beta k} = \varepsilon_{\alpha k j} \phi_{\beta j} + \varepsilon_{\beta k j} \phi_{\alpha j}.
\]

(25)
If we use eqs. (15) and (25), the rotational flux is given by

\[
M_{ki} = \epsilon_{kji} x_j W + \epsilon_{kja} \left( u_j \frac{\partial W}{\partial u_{\alpha,i}} + \phi_{jl} \frac{\partial W}{\partial \phi_{\alpha,l,i}} + \phi_{lj} \frac{\partial W}{\partial \phi_{\alpha,j,i}} \right)
- \epsilon_{kjn} x_j \left( u_{\alpha,n} \frac{\partial W}{\partial u_{\alpha,i}} + \phi_{\alpha\beta,n} \frac{\partial W}{\partial \phi_{\alpha\beta,i}} + \phi_{\beta\alpha,n} \frac{\partial W}{\partial \phi_{\alpha\beta,i}} \right),
\]

(26)

In terms of the Eshelby tensor (23) and the stress tensors we obtain

\[
M_{ki} = \epsilon_{kjn} x_j P_{ni} + u_j t_{ni} + \phi_{lj} m_{ni} + \phi_{jl} m_{ni},
\]

(27)

It is the total angular momentum tensor. It can be decomposed into the orbital and intrinsic (spin) angular momentum tensors

\[
M_{ki} = M_{ki}^{(o)} + M_{ki}^{(i)}.
\]

(28)

The orbital angular momentum tensor reads

\[
M_{ki}^{(o)} = \epsilon_{kjn} x_j P_{ni}.
\]

(29)

The intrinsic (spin) angular momentum tensor is given by

\[
M_{ki}^{(i)} = \epsilon_{kjn} \left( u_j t_{ni} + \phi_{lj} m_{ni} + \phi_{jl} m_{ni} \right).
\]

(30)

3.3 Scaling

The scaling group acts in infinitesimal form on the independent and dependent variables

\[
x'_i = (1 + \varepsilon) x_i, \quad u'_\alpha = (1 + \varepsilon d_u) u_\alpha, \quad \phi'_{\alpha\beta} = (1 + \varepsilon d_\phi) \phi_{\alpha\beta},
\]

(31)

where \(d_u\) and \(d_\phi\) denote the (scaling) dimensions of the displacement vector \(u\) and the micro-deformation tensor \(\phi\). The infinitesimal generators are given by

\[
X_i = x_i, \quad U_\alpha = d_u u_\alpha, \quad \Phi_{\alpha\beta} = d_\phi \phi_{\alpha\beta},
\]

(32)

where

\[
d_u = -\frac{n - 2}{2}, \quad d_\phi = -\frac{n}{2},
\]

(33)

and \(\delta_{ll} = n\). If we substitute (32) into (15), we obtain for the scaling flux

\[
Y_i = x_i W + \left( d_u u_\alpha - x_k u_{\alpha,k} \right) \frac{\partial W}{\partial u_{\alpha,i}} + \left( d_\phi \phi_{\alpha\beta} - x_k \phi_{\alpha\beta,k} \right) \frac{\partial W}{\partial \phi_{\alpha\beta,i}}.
\]

(34)

In terms of the stress tensors the scaling flux reads

\[
Y_i = x_j P_{ji} + d_u u_j t_{ji} + d_\phi \phi_{jl} m_{ji}.
\]

(35)

4 BALANCE AND CONSERVATION LAWS

We now turn to the discussion of the properties of the fluxes. The information of a symmetry defined by the transformation law of the fields lies in the properties of the divergence of the corresponding currents. If the divergence is zero, we speak of a conservation law. If it is not zero, we have a balance law.
For micromorphic elasticity we obtain with the corresponding Eshelby stress tensor (23), total angular momentum tensor (27) and scaling flux vector (35) the following balance laws:

\[ D_i P_{ki} = -f_{k}^{\text{inh}}, \quad (36) \]
\[ D_i M_{ki} = \epsilon_{kjn} \left( \gamma_{ij} t_{ni} + \gamma_{ji} t_{ni} + 2 \epsilon_{ij} s_{in} + \kappa_{ijl} m_{inl} + \kappa_{ji} m_{nl} + \kappa_{lj} m_{in} \right) \]
\[ - \epsilon_{kjn} \left( x_j f_{n}^{\text{inh}} + u_j F_{n} + \phi_{ji} L_{ni} + \phi_{ij} L_{ni} \right), \quad (37) \]
\[ D_i Y_i = -\kappa_{\alpha\beta\gamma} m_{\alpha\beta\gamma} - x_i f_{i}^{\text{inh}} - \frac{n + 2}{2} u_{\alpha} F_{\alpha} - \frac{n}{2} \phi_{\alpha\beta} L_{\alpha\beta}. \quad (38) \]

Eq. (36) follows essentially from the lack of translational invariance and it can be called the canonical momentum balance law. In eq. (37) the terms in the first parentheses on the right hand side vanish if the micromorphic material is isotropic [6]. The other terms on the right hand side are resulting vector moments caused by inhomogeneities, external body forces and couples. Eq. (37) may be called the canonical angular momentum balance law. The source terms in eq. (38) are ‘scalar moments’ breaking the scaling symmetry. The first source term breaks the dilatational symmetry due to the stress moment tensor \( m_{ijk} \). Additional source terms appear which account for material inhomogeneities, external forces and couples. Eq. (38) can be called the scalar moment of momentum balance law.

In integral form we obtain the \( J, L \) and \( M \) integrals of micromorphic elasticity:

\[ J_k = \int_S P_{ki} n_i \, dS = -\int_V f_{k}^{\text{inh}} \, dV, \quad (39) \]
\[ L_k = \int_S M_{ki} n_i \, dS = \int_V \epsilon_{kjn} \left[ \gamma_{ij} t_{ni} + \gamma_{ji} t_{ni} + 2 \epsilon_{ij} s_{in} + \kappa_{ijl} m_{inl} + \kappa_{ji} m_{nl} + \kappa_{lj} m_{in} \right] \]
\[ - \epsilon_{kjn} \left( x_j f_{n}^{\text{inh}} + u_j F_{n} + \phi_{ji} L_{ni} + \phi_{ij} L_{ni} \right) \, dV, \quad (40) \]
\[ M = \int_S Y_i n_i \, dS = -\int_V \left( \kappa_{\alpha\beta\gamma} m_{\alpha\beta\gamma} + x_i f_{i}^{\text{inh}} + \frac{n + 2}{2} u_{\alpha} F_{\alpha} + \frac{n}{2} \phi_{\alpha\beta} L_{\alpha\beta} \right) \, dV. \quad (41) \]

For an isotropic material with vanishing body forces, body moments and material forces we obtain:

\[ J_k = \int_S P_{ki} n_i \, dS = 0, \quad (42) \]
\[ L_k = \int_S M_{ki} n_i \, dS = 0, \quad (43) \]
\[ M = \int_S Y_i n_i \, dS = -\int_V \kappa_{\alpha\beta\gamma} m_{\alpha\beta\gamma} \, dV. \quad (44) \]

Thus, in this case the \( J \) and \( L \) integrals are zero and the Eshelby stress and total angular momentum tensors give conservation laws because there are divergence-less. On the other hand, the \( M \) integral is non-zero because the term \( m_{ijk} \) breaking the scaling symmetry survives. Therefore, it is not a conservation integral.

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