\textbf{$p$-ADIC MARKOV PROCESS AND THE PROBLEM OF THE FIRST RETURN OVER BALLS}

O. F. CASAS-SÁNCHEZ, J. GALEANO-PÉNALOZA, AND J. J. RODRÍGUEZ-VEGA

\textbf{Abstract.} Let $\langle x \rangle^\alpha = (\max\{|x|_p, p^\alpha\})^\alpha$ and $H^\alpha \varphi = \mathcal{F}^{-1}[\langle \xi \rangle^\alpha - p^\alpha \mathcal{F} \varphi]$, in this article we study the Markov process associated to this operator and the first passage time problem associated to $H^\alpha$.

\textbf{Keywords:} Random walks, ultradiffusion, $p$-adic numbers, non-archimedean analysis.

\textbf{MSC2010:} 82B41, 82C44, 26E30.

1. \textbf{Introduction}

Avetisov et al. have constructed a wide variety of models of ultrametric diffusion constrained by hierarchical energy landscapes (see \cite{2}, \cite{3}). From a mathematical point of view, in these models the time-evolution of a complex system is described by a $p$-adic master equation (a parabolic-type pseudodifferential equation) which controls the time evolution of a transition function of a random walk on an ultrametric space, and the random walk describes the dynamics of the system in the space of configurational states which is approximated by an ultrametric space ($\mathbb{Q}_p$).

The problem of the first return in dimension 1 was studied in \cite{4}, and in arbitrary dimension in \cite{6} and \cite{10}. In these articles, pseudodifferential operators with radial symbols were considered. More recently, Chacón-Cortés \cite{7} considers pseudodifferential operators over $\mathbb{Q}_4^+$ with non-radial symbol; he studies the problem of first return for a random walk $X(t, w)$ whose density distribution satisfies certain diffusion equation.

In this paper we define the operator

$$H^\alpha \varphi = \mathcal{F}^{-1}[\langle \xi \rangle^\alpha - p^\alpha \mathcal{F} \varphi],$$

for $\varphi \in \mathcal{S}(\mathbb{Q}_p)$, where $\langle \xi \rangle = \max\{|\xi|_p, p^\alpha\}$. We also define the heat-kernel $Z_r$ as

$$Z_r(x, t) := \int_{\mathbb{Q}_p} \chi(-x \xi) e^{-t(\langle \xi \rangle^\alpha - p^\alpha)} \, d\xi,$$

heat kernels of this type have been studied in \cite{5}, we show that function

$$u(x, t) = Z_r(x, t) * \Omega(|x|_p) = \int_{\mathbb{Q}_p} \chi(-x \xi) e^{-t(\langle \xi \rangle^\alpha - p^\alpha)} \Omega(|\xi|_p) \, d\xi$$

is a solution of Cauchy problem

\begin{equation}
\begin{cases}
\frac{\partial u}{\partial t}(x, t) + (H^\alpha u)(x, t) = 0, & x \in \mathbb{Q}_p, \quad t \in (0, T], \quad \alpha > 0 \\
u(x, 0) = \Omega(|\xi|_p),
\end{cases}
\end{equation}

1
and we show that \( Z_r(x, t) \) is the transition density of a time and space homogeneous Markov process, which is bounded, right-continuous and has no discontinuities other than jumps.

Finally, we study the first passage time problem associated to the operator \( H^\alpha \).

## 2. Preliminaries

In this section we fix the notation and collect some basic results on \( p \)-adic analysis that we will use through the article. For a detailed exposition on \( p \)-adic analysis the reader may consult [11] [10] [11]

### 2.1. The field of \( p \)-adic numbers

Along this article \( p \) will denote a prime number. The field of \( p \)-adic numbers \( \mathbb{Q}_p \) is defined as the completion of the field of rational numbers \( \mathbb{Q} \) with respect to the \( p \)-adic norm \( | \cdot |_p \), which is defined as

\[
|x|_p = \begin{cases} 
0 & \text{if } x = 0, \\
p^{-\gamma} & \text{if } x = p^\gamma a/b,
\end{cases}
\]

where \( a \) and \( b \) are integers coprime with \( p \). The integer \( \gamma := ord(x) \), with \( ord(0) := +\infty \), is called the \( p \)-adic order of \( x \).

Any \( p \)-adic number \( x \neq 0 \) has a unique expansion \( x = p^{ord(x)} \sum_{j=0}^{\infty} x_j p^j \), where \( x_j \in \{0, 1, 2, \ldots, p-1\} \) and \( x_0 \neq 0 \). By using this expansion, we define the fractional part of \( x \in \mathbb{Q}_p \), denoted \( \{ x \}_p \), as the rational number

\[
\{ x \}_p = \begin{cases} 
0 & \text{if } x = 0 \text{ or } ord(x) \geq 0, \\
p^{ord(x)} \sum_{j=0}^{-ord(x)-1} x_j p^j & \text{if } ord(x) < 0.
\end{cases}
\]

For \( r \in \mathbb{Z} \), denote by \( B_r(a) = \{ x \in \mathbb{Q}_p : |x-a|_p \leq p^r \} \) the ball of radius \( p^r \) with center at \( a \in \mathbb{Q}_p \), and take \( B_r(0) := B_r \).

### 2.2. The Bruhat-Schwartz space

A complex-valued function \( \varphi \) defined on \( \mathbb{Q}_p \) is called locally constant if for any \( x \in \mathbb{Q}_p \) there exists an integer \( l(x) \in \mathbb{Z} \) such that

\[
(2.1) \quad \varphi(x + x') = \varphi(x) \text{ for } x' \in B_{l(x)}.
\]

The space of locally constant functions is denoted by \( \mathcal{E}(\mathbb{Q}_p) \). A function \( \varphi : \mathbb{Q}_p \to \mathbb{C} \) is called a Bruhat-Schwartz function (or a test function) if it is locally constant with compact support. The \( \mathbb{C} \)-vector space of Bruhat-Schwartz functions is denoted by \( \mathbf{S}(\mathbb{Q}_p) \). For \( \varphi \in \mathbf{S}(\mathbb{Q}_p) \), the largest of such number \( l = l(\varphi) \) satisfying (2.1) is called the exponent of local constancy of \( \varphi \).

Let \( \mathbf{S}'(\mathbb{Q}_p) \) denote the set of all functionals (distributions) on \( \mathbf{S}(\mathbb{Q}_p) \). All functionals on \( \mathbf{S}(\mathbb{Q}_p) \) are continuous.

Set \( \chi(y) = \exp(2\pi i \{ y \}_p) \) for \( y \in \mathbb{Q}_p \). The map \( \chi(\cdot) \) is an additive character on \( \mathbb{Q}_p \), i.e. a continuous map from \( \mathbb{Q}_p \) into \( S \) (the unit circle) satisfying \( \chi(y_0 + y_1) = \chi(y_0)\chi(y_1) \), \( y_0, y_1 \in \mathbb{Q}_p \).

### 2.3. Fourier transform

Given \( \xi \) and \( x \in \mathbb{Q}_p \), the Fourier transform of \( \varphi \in \mathbf{S}(\mathbb{Q}_p) \) is defined as

\[
(\mathcal{F} \varphi)(\xi) = \int_{\mathbb{Q}_p} \chi(\xi x)\varphi(x)dx \quad \text{for } \xi \in \mathbb{Q}_p,
\]

where \( dx \) is the Haar measure on \( \mathbb{Q}_p \) normalized by the condition \( vol(B_0) = 1 \). The Fourier transform is a linear isomorphism from \( \mathbf{S}(\mathbb{Q}_p) \) onto itself satisfying
Lemma 1. For all $\alpha \in \mathbb{C}$ we define the following pseudodifferential operator
\[
H^\alpha \varphi = \mathcal{F}^{-1}[(\langle \xi \rangle^\alpha - p^\alpha) \mathcal{F} \varphi], \quad \varphi \in S(Q_p).
\]
Where $\langle \xi \rangle^\alpha = \max\{\langle \xi \rangle_p, p^\alpha\}$.

It is clear that the map $H^\alpha : S(Q_p) \to S(Q_p)$ is continuous. Also it is possible to show that the pseudodifferential operator $H^\alpha$ has the following representation integral
\[
(H^\alpha \varphi)(x) = \frac{1 - p^\alpha}{1 - p^{-\alpha + 1}} \int_{|y|_p \leq p^{-r}} \varphi(x - y) \, dy - \frac{p^{\alpha+1}}{|y|_p^{\alpha+1}} \int_{|y|_p \leq p^{-r}} \frac{\varphi(x - y) - \varphi(x)}{|y|_p^{\alpha+1}} \, dy.
\]

Definition 2. Set $\alpha_k := \frac{2k\pi i}{\ln p}$, $k \in \mathbb{Z}$,
\[
K_{\alpha}(x) := \begin{cases} \frac{1 - p^\alpha}{1 - p^{-\alpha - 1} + p^{\alpha+1}} & \frac{1 - p^\alpha}{1 - p^{-\alpha + 1}} \
(1 - p^{-1})\Omega_p(|x|_p)((1 - r) - \log_p |x|_p) 
\end{cases}
\]
for $\alpha \neq -1 + \alpha_k$ and for $\alpha = -1 + \alpha_k$.

After some calculations it is possible to show the following result.

Theorem 1. The Fourier transform of $K_{\alpha}$ is given by $\langle \xi \rangle^\alpha$ for all $\alpha \in \mathbb{C}$.

Definition 3. For $x \in Q_p$, $t \in \mathbb{R}$ the heat kernel is defined as
\[
Z_r(x, t) := \int_{Q_p} \chi(-x, t) e^{-t[(\langle \xi \rangle^\alpha - p^\alpha)]} \, d\xi.
\]

The following properties are proved in [5].

Lemma 1. For $\alpha > 0$, $t > 0$, the following assertions hold.
1. $Z_r(x, t) \in C(Q_p, \mathbb{R}) \cap L^1(Q_p) \cap L^2(Q_p)$, for $t > 0$.
2. $Z_r(x, t) \geq 0$ for all $x \in Q_p$.
3. $\int_{Q_p} Z_r(x, t) \, dx = \int_{|x|_p \leq p^{-r}} Z_r(x, t) \, dx = 1$.
4. $\lim_{t \to 0^+} Z_r(x, t) \ast \varphi(x) = \varphi(x)$, for $\varphi \in S(Q_p)$.
5. $Z_r(x, t) \ast Z_r(x, t') = Z(x, t + t')$, for $t, t' > 0$.
6. $Z_r(x, t) \leq Ct|x|_p^{-1} \left(\frac{p}{px} \right)^{\alpha}$. 

3. Pseudodifferential operators
If we set for \( \varphi \in S(\mathbb{Q}_p) \)

\[
(3.4) \quad u(x, t) := \begin{cases} 
Z_r(x, t) * \varphi(x), & \text{if } t > 0 \\
\varphi(x), & \text{if } t = 0,
\end{cases}
\]

then it is easy to see that \( u(x, t) \in S(\mathbb{Q}_p) \) for \( t \geq 0 \), and also it is possible to show that for \( t \geq 0 \), \( \alpha > 0 \)

\[
H^\alpha(u(x, t)) = F^{-1}_{\xi \to x} \left[ (\langle \xi \rangle^\alpha - p^{\alpha})e^{-t(\langle \xi \rangle^\alpha - p^{\alpha})} \hat{\varphi}(\xi) \right].
\]

**Theorem 2.** Consider the following Cauchy problem

\[
(3.5) \quad \begin{cases} 
\frac{\partial u}{\partial t}(x, t) + (H^\alpha u)(x, t) = 0, & x \in \mathbb{Q}_p, \ t \in (0, T], \ \alpha > 0 \\
u(x, 0) = \varphi(x), & \varphi \in S(\mathbb{Q}_p),
\end{cases}
\]

then the function \( u(x, t) \) defined in (3.4) is a solution.

**Proof.** See Theorem 3.14 in [5]. \( \square \)

4. \( p \)-adic Markov process over balls

The space \( (\mathbb{Q}_p, |\cdot|_p) \) is a complete non-Archimedean metric space. Let \( \mathcal{B} \) the Borel \( \sigma \)-algebra of \( \mathbb{Q}_p \); thus \( (\mathbb{Q}_p, \mathcal{B}, dx) \) is a measure space. By using the terminology and results of [8, Chapters 2, 3], we set

\[
p(t, x, y) := Z_r(x - y, t), \quad t > 0, \ x, y \in \mathbb{Q}_p
\]

and

\[
P(t, x, B) = \begin{cases} 
\int_B p(t, x, y) \, dy & \text{for } t > 0, \ x \in \mathbb{Q}_p, \ B \in \mathcal{B} \\
1_B(x) & \text{for } t = 0.
\end{cases}
\]

**Lemma 2.** With the above notation the following assertions hold:

1. \( p(t, x, y) \) is a normal transition density.
2. \( P(t, x, B) \) is a normal transition function.

**Proof.** The result follows from Theorem 4.1 (see [8, Section 2.1], for further details). \( \square \)

**Lemma 3.** The transition function \( P(t, x, B) \) satisfies the following two conditions:

(i) \( L(B) \) For each \( u \geq 0 \) and compact \( B \),

\[
\lim_{|x|_p \to \infty} \sup_{t \leq u} P(t, x, B) = 0.
\]

(ii) \( M(B) \) For each \( \epsilon > 0 \) and compact \( B \),

\[
\lim_{t \to 0^+} \sup_{x \in B} P(t, x, \mathbb{Q}_p \setminus B_\epsilon(x)) = 0.
\]
Proof. (i) By Lemma 1 (6), we have

\[ P(t, x, B) = \int_B Z_r(x - y, t) dy \]

\[ \leq Ct \int_B |x - y|^{-1}_p \left( (p(x - y)^{-1})^\alpha - p^\alpha \right) dy \]

for \( x \in \mathbb{Q}_p \setminus B \), we have \( |x|_p = |x - y|_p \)

\[ = Ct|x|^{-1}_p \left( (px^{-1})^\alpha - p^\alpha \right) \int_B dy. \]

Therefore, \( \lim_{|x|_p \to \infty} \sup_{t \leq u} P(t, x, B) = 0 \).

(ii) By using Lemma 1 (6), \( \alpha > 0 \), we have

\[ P(t, x, \mathbb{Q}_p \setminus B_\epsilon(x)) \leq Ct \int_{|x - y| > \epsilon} |x - y|^{-1}_p \left( (p(x - y)^{-1})^\alpha - p^\alpha \right) dy \]

\[ = Ct \int_{|z| > \epsilon} |z|^{-1}_p \left( (pz^{-1})^\alpha - p^\alpha \right) dz \]

if \( p^{-\epsilon - 1} \leq |z|_p \) or \( \epsilon < p^{-\epsilon - 1} \leq |z|_p \), then \( (pz^{-1})^\alpha = p^\alpha \) and

\[ \int_{|z| > \epsilon} |z|^{-1}_p \left( (pz^{-1})^\alpha - p^\alpha \right) dz = 0. \]

Therefore,

\[ P(t, x, \mathbb{Q}_p \setminus B_\epsilon(x)) \leq Ct \int_{|x - y| > \epsilon} |x - y|^{-1}_p \left( (p(x - y)^{-1})^\alpha - p^\alpha \right) dy \]

\[ = Ct \int_{p^{-\epsilon - 1} > |z| > \epsilon} |z|^{-1}_p \left( |z|^{-1}_p \alpha - p^\alpha \right) dz \]

\[ \leq Ctp^{-1} \int_{p^{-\epsilon - 1} > |z| > \epsilon} |z|^{-1 - \alpha}_p dz \]

\[ = Ctp^{-1}C_1 \]

Therefore, \( \lim_{t \to 0^+} \sup_{x \in B} P(t, x, \mathbb{Q}_p \setminus B_\epsilon(x)) = 0. \)

\[ \Box \]

**Theorem 3.** \( Z_r(x, t) \) is the transition density of a time and space homogeneous Markov process, called \( \mathcal{S}(t, \omega) \), which is bounded, right-continuous and has no discontinuities other than jumps.

**Proof.** The result follows from [3] Theorem 3.6 by using that \((\mathbb{Q}_p, |x|_p)\) is a semi-compact space, i.e., a locally compact Hausdorff space with a countable base, and \( P(t, x, B) \) is a normal transition function satisfying conditions \( L(B) \) and \( M(B) \). \[ \Box \]
5. The First Passage Time

By Proposition 2, the function

\[ u(x, t) = \int_{Q_p} \chi(-x\xi)e^{-t(\langle \xi \rangle^\alpha - p^{-\alpha})}\Omega(|\xi|_p) \, d\xi \]

is a solution of

\[
\begin{cases}
\frac{\partial u}{\partial t}(x, t) + (H^\alpha u)(x, t) = 0, & x \in Q_p, \ t > 0, \\
u(x, 0) = \Omega(|x|_p).
\end{cases}
\]

Among other properties, the function \( u(x, t) = \int_{Q_p} \chi(-x\xi)(\langle \xi \rangle^\alpha - p^{-\alpha})e^{-t(\langle \xi \rangle^\alpha - p^{-\alpha})}\Omega(|\xi|_p) \, d\xi \), \( t \geq 0 \), is pointwise differentiable in \( t \) and, by using the Dominated Convergence Theorem, we can show that its derivative is given by the formula

\[ \frac{\partial u}{\partial t}(x, t) = \int_{Q_p} \chi_p(-x\xi)(\langle \xi \rangle^\alpha - p^{-\alpha})e^{-t(\langle \xi \rangle^\alpha - p^{-\alpha})}\Omega(|\xi|_p) \, d\xi. \]

**Lemma 4.** If \( \alpha > 0 \) and \( r < 0 \), then

\[ 0 < -\int_{1 <|y|_p \leq p^{-r}} K_\alpha(y)dy < 1. \]

**Proof.**

\[
-\int_{1 <|y|_p \leq p^{-r}} K_\alpha(y)dy = \frac{1 - p^{\alpha}}{1 - p^{\alpha+1}} \left[ p^{\alpha+1} \int_{1 <|y|_p \leq p^{-r}} \frac{1}{|y|_p^{\alpha+1}}dy - p^{\alpha(\alpha+1)} \int_{1 <|y|_p \leq p^{-r}} dy \right]
\]

\[
< \frac{1 - p^{\alpha}}{1 - p^{\alpha+1}} \left[ p^{\alpha+1} \int_{1 <|y|_p \leq p^{-r}} \frac{1}{|y|_p^{\alpha+1}}dy - p^{\alpha(\alpha+1)}(1 - p^{-r} - 1) \right]
\]

\[
= \frac{1 - p^{-1}}{1 - p^{-\alpha+1}} - \frac{1 - p^{\alpha}}{1 - p^{\alpha+1}} p^{\alpha(1 - p^{-r})}
\]

\[
= 1 - \frac{1 - p^{\alpha}}{1 - p^{\alpha+1}} (1 + p^{\alpha}(1 - p^{-r}))
\]

\[
< 1.
\]

Now

\[
-\int_{1 <|y|_p \leq p^{-r}} K_\alpha(y)dy = \frac{1 - p^{\alpha}}{1 - p^{\alpha+1}} \left[ p^{\alpha+1} \int_{1 <|y|_p \leq p^{-r}} \frac{1}{|y|_p^{\alpha+1}}dy - p^{\alpha(\alpha+1)} \int_{1 <|y|_p \leq p^{-r}} dy \right]
\]

\[
> \frac{p^{\alpha}(p - 1)(1 - p^{\alpha})}{p^{\alpha+1} - 1} + \frac{p^{\alpha}(1 - p^{\alpha})}{p^{\alpha+1} - 1}
\]

\[
> 0.
\]

\( \square \)

The rest of this section is dedicated to the study of the following random variable.
Definition 4. The random variable $\tau_{\Omega(|x|_p)}(\omega) : \mathcal{Y} \to \mathbb{R}_+$ defined by

$$\inf\{t > 0 \mid T(t, \omega) \in \Omega(|x|_p) \text{ and there exists } t' \text{ such that } 0 < t' < t \text{ and } T(t', \omega) \notin \Omega(|x|_p)\}$$

is called the first passage time of a path of the random process $T(t, \omega)$ entering the domain $\Omega(|x|_p)$.

Lemma 5. The probability density function for a path of $T(t, \omega)$ to enter into $\Omega(|x|_p)$ at the instant of time $t$, with the condition that $T(0, \omega) \in \Omega(|x|_p)$ is given by

$$g(t) = \int_{1 < |y|_p \leq p^{-r}} K_\alpha(y) u(y, t) dy.$$

Proof. We first note that, for $x, y \in \Omega(|z|_p)$, we have

$$u(x - y, t) = \int_{\Omega(|\xi|_p)} \chi_p(-(x - y) \cdot \xi) e^{-t((\xi)_n - p^{r\alpha})} d\xi$$

$$= \int_{\Omega(|\xi|_p)} e^{-t((\xi)_n - p^{r\alpha})} d\xi = \int_{\Omega(|\xi|_p)} \chi_p(-x \cdot \xi) e^{-t((\xi)_n - p^{r\alpha})} d\xi$$

$$= u(x, t).$$

i.e. $u(x - y, t) - u(x, t) \equiv 0$ for $x, y \in \Omega(|z|_p)$.

The survival probability, by definition

$$S(t) := S_{\Omega(|x|_p)}(t) = \int_{\Omega(|x|_p)} u(x, t) d^n x,$$

is the probability that a path of $T(t, \omega)$ remains in $\Omega(|x|_p)$ at the time $t$. Because there are no external or internal sources,

$$S'(t) = \text{Probability that a path of } T(t, \omega) \text{ goes back to } \Omega(|x|_p) \text{ at the time } t$$

$$- \text{Probability that a path of } T(t, \omega) \text{ exits } \Omega(|x|_p) \text{ at the time } t$$

$$= g(t) - C \cdot S(t) \text{ with } 0 < C \leq 1.$$
by using the derivative (5.3)
\[ S'(t) = \int_{\Omega(|x|_p)} \frac{\partial u(x,t)}{\partial t} \, dx \]
\[ = -\frac{1 - p^{\alpha}}{1 - p^{\alpha+1}} \left[ p^{\alpha+1} \int_{|x|_p \leq 1} \int_{1<|y|_p \leq p^{-r}} u(x - y, t) - u(x, t) \, dy \, dx \right. \]
\[ \left. - p^{\alpha+1} \int_{|x|_p \leq 1} \int_{1<|y|_p \leq p^{-r}} \frac{u(x - y, t) - u(x, t)}{|y|_p^{\alpha+1}} \, dy \, dx \right] \]
\[ = -\frac{1 - p^{\alpha}}{1 - p^{\alpha+1}} \left[ p^{\alpha+1} \int_{|x|_p \leq 1} \int_{1<|y|_p \leq p^{-r}} u(x - y, t) \, dy \, dx \right. \]
\[ \left. - p^{\alpha+1} \int_{|x|_p \leq 1} \int_{1<|y|_p \leq p^{-r}} \frac{u(x, t)}{|y|_p^{\alpha+1}} \, dy \, dx \right] \]
\[ + \frac{1 - p^{\alpha}}{1 - p^{\alpha+1}} \left[ p^{\alpha+1} \int_{|x|_p \leq 1} \int_{1<|y|_p \leq p^{-r}} u(x, t) \, dy \, dx \right. \]
\[ \left. - p^{\alpha+1} \int_{|x|_p \leq 1} \int_{1<|y|_p \leq p^{-r}} \frac{u(x, t)}{|y|_p^{\alpha+1}} \, dy \, dx \right]. \]

Now if \( y \in \Omega(p^{r}|y|_p \setminus \Omega(|y|_p) \) and \( x \in \Omega(|x|_p), \) then \( u(x - y, t) = u(y, t), \) consequently
\[ S'(t) = \int_{1<|y|_p \leq p^{-r}} K_\alpha(y)u(y, t) dy + \int_{|x|_p \leq 1} K_\alpha(y) dy \int_{1<|y|_p \leq p^{-r}} u(x, t) \, dx \]
\[ = \frac{\partial}{\partial t} \int_{1<|y|_p \leq p^{-r}} K_\alpha(y) dy + \int_{|x|_p \leq 1} K_\alpha(y) dy \int_{1<|y|_p \leq p^{-r}} u(x, t) \, dx \]
\[ = g(t) - CS(t), \]
where \( C = -\int_{1<|y|_p \leq p^{-r}} K_\alpha(y) dy. \] □

**Proposition 1.** The probability density function \( f(t) \) of the random variable \( \tau_{\Omega(|x|_p)}(\omega) \) satisfies the non-homogeneous Volterra equation of second kind
\[ g(t) = \int_{0}^{\infty} g(t - \tau) f(\tau) d\tau + f(t). \]  

**Proof.** The result follows from Lemma 5 by using the argument given in the proof of Theorem 1 in [4]. □

**Proposition 2.** The Laplace transform \( G_r(s) \) of \( g(t) \) is given by
\[ G_r(s) = \int_{1<|y|_p \leq p^{-r}} K_\alpha(y) \int_{|x|_p \leq 1} \frac{\chi_p(-\xi \cdot y)}{s + ((\xi \cdot y)^{r \alpha})} \, d\xi \, dy. \]
Proof. We first note that $e^{-st}K_\alpha(y)e^{-t((\zeta)^\alpha - p^{-r})}\Omega([\xi]_p) \in L^1((0, \infty) \times \Omega([\eta]_p) \Omega([\xi]_p) \times \mathbb{Q}_p, dt dy d\zeta$ for $s \in \mathbb{C}$ with $Re(s) > 0$. The announced formula follows now from (5.4) and (5.1) by using Fubini’s Theorem. □

Definition 5. We say that $\Xi(t, \omega)$ is recurrent with respect to $\Omega([x]_p)$ if

\[(5.6) \quad P\{\{\omega \in \Omega : \tau_{\Omega([x]_p)}(\omega) < \infty\}\} = 1.\]

Otherwise, we say that $\Xi(t, \omega)$ is transient with respect to $\Omega([x]_p)$.

The meaning of (5.6) is that every path of $\Xi(t, \omega)$ is sure to return to $\Omega([x]_p)$. If (5.6) does not hold, then there exist paths of $\Xi(t, \omega)$ that abandon $\Omega([x]_p)$ and never go back.

Theorem 4. For all $\alpha > 0$ the processes $\Xi(t, \omega)$ is recurrent with respect to $\Omega([x]_p)$.

Proof. By Proposition 4 the Laplace transform $F(s)$ of $f(t)$ equals $G_r(s) / (1 + G_r(s))$, where $G_r(s)$ is the Laplace transform of $g(t)$, and thus

\[F(0) = \int_0^\infty f(t) \, dt = 1 - \frac{1}{1 + G_r(0)}.\]

Hence in order to prove that $\Xi(t, \omega)$ is recurrent is sufficient to show that

\[G_r(0) = \lim_{s \to 0} G_r(s) = \infty,\]

and to prove that it is transient that

\[G_r(0) = \lim_{s \to 0} G_r(s) = \infty.\]

Therefore $\lim_{s \to 0} G_r(s) = \infty$ and the process $\Xi(t, \omega)$ is recurrent. □

References

[1] S. Albeverio, A. Yu. Khrennikov and V. M. Shelkovich, Theory of $p$-adic Distributions. Linear and Nonlinear Models, London Mathematical Society Lecture Note Series 370, Cambridge University Press (2010).

[2] V.A. Avetisov, A.Kh. Bikulov, S.V. Kozyrev and V.A. Osipov, $p$-Adic models of ultrametric diffusion constrained by hierarchical energy landscapes, J. Phys. A.Math. Gen. 35, 177-189 (2002).
[3] V.A.Avetisov and A.Kh. Bikulov, *Protein ultrametricity and spectral diffusion in deeply frozen proteins*, Rev. Lett. 3(3), 2008.
[4] Avetisov V. A., Bikulov A. Kh., Zubarev, A. P. First passage time distribution and the number of returns for ultrametric random walks, J. Phys. A 42 (2009), no. 8, 085003, 18 pp.
[5] O. F. Casas-Sánchez, J. J. Rodríguez-Vega, *Parabolic type equations on p-adic balls*, Boletín de Matemáticas, 22 (1) (2015), 97-106.
[6] L. F. Chacón-Cortes, and W. A. Zúñiga-Galindo, *Nonlocal operators, parabolic-type equations, and ultrametric random walks*, J. Math. Phys., 55 (2014), no. 10
[7] L. F. Chacón-Cortes, The Problem of the First Passage Time for Some Elliptic Pseudodifferential Operators Over the p-adics, Rev. Colombiana Mat. 48 (2014), No. 2, 191?209.
[8] E. B. Dynkin *Markov processes*, Vol. I. Springer-Verlag, 1965.
[9] M. H. Taibleson, Fourier analysis on local fields, Princeton University Press, 1975.
[10] A. Torresblanca-Badillo, W.A. Zúñiga-Galindo, *Ultrametric Diffusion, Exponential Landscapes, and the First Passage Time Problem*, Acta Appl Math (2018), 1-24.
[11] V. S. Vladimirov, I. V. Volovich and E. I. Zelenov, *p-Adic Analysis and Mathematical Physics*, Series on Soviet and East European Mathematics 1, World Scientific, River Edge, NJ, 1994.

**Universidad Pedagógica y Tecnológica de Colombia, Escuela de Matemáticas y Estadística, Tunja, Colombia**

*E-mail address: oscar.casas01@uptc.edu.co*

**Universidad Nacional de Colombia, Departamento de Matemáticas, Ciudad Universitaria, Bogotá D.C., Colombia**

*E-mail address: jgaleanop@unal.edu.co*

**Universidad Nacional de Colombia, Departamento de Matemáticas, Ciudad Universitaria, Bogotá D.C., Colombia**

*E-mail address: jjrodriguezv@unal.edu.co*