Infinitely degenerate exact vacuum solutions in $f(R)$ gravity

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Abstract

We obtain an infinite number of exact static spherically symmetric vacuum solutions for a class of $f(R)$ theories of gravity. We analytically derive two exact vacuum black-hole solutions for the same class of $f(R)$ theories. The two black-hole solutions have the event-horizon at the same point; however, their asymptotic features are different. Our results point that the Birkhoff theorem is not valid for all modified gravity theories. We discuss the implications of our work to distinguish modified gravity theories from general relativity in gravitational wave detections.

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I. INTRODUCTION

General Relativity (GR) is a hugely successful description of gravity. Both theory and observations suggest that GR might have significant classical and quantum corrections in strong gravity regime \([1, 2]\). The direct detection of gravitational waves has provided a possibility to look for modifications to GR in the strong gravity regime \([3, 4]\). Current constraints on deviations from GR rely on partial, parametrized waveforms or propagation effects \([5]\).

The real potential to test GR in the merger of two black-holes is constrained due to two reasons: First, there is no unique way to modify general relativity \([5]\). Each of these modifications to GR has different features. To circumvent this, recently, strong gravity diagnostic parameters are proposed to distinguish GR and modified gravity theories \([6]\). However, these parameters work only if the black-hole solutions in modified gravity theories are identical to GR. Second, we do not have waveforms for the GW emissions in modified gravity theories \([7]\). While the waveforms are model-specific, waveforms in any modified theory will serve as a test-bed to obtain constraints on deviations from GR.

In particular, the merger of two black-holes is a cataclysmic event, and it is unclear whether GR is an accurate description during the merger \([3, 7]\). Even if the initial black-holes are described by GR, due to different dynamics, the final black-hole after the merger \textit{may not} be described by GR. Thus, if the final black-hole in the modified gravity theory is different from the Kerr black-hole, then the ring-down phase can help to distinguish modified gravity and GR. However, one of the crucial assumptions is that the final black-hole in a given modified gravity theory is unique. In this work, we show that this assumption \textit{may not} hold for modified gravity theories.

Before we proceed, let us look at the uniqueness of the final black-hole in GR. As a consequence of no-hair theorem, GR predicts that the Kerr metric describes all astrophysical black holes \([8]\). This is because the isolated black-holes do not radiate and are axisymmetric \([8]\). In the case of spherical symmetry, Birkhoff’s theorem guarantees that the most general spherically symmetric, electrovac solution of the Einstein-Maxwell equations is the static Reissner-Nordstrom solution \([9]\). The result is linked to the absence of spin-0 modes in the linearized field equations. In other words, since a spherically symmetric system cannot couple to higher spin excitations when spin-0 is absent, no emission or absorption of radia-
tion is possible, forcing the solution to be static [10]. However, there is no Birkhoff theorem for the Kerr metric. Outside a rotating star, the metric is not described by Kerr. A generic rotating star can have gravitational multipoles that are not the same as Kerr. Mass ($M$) and angular momentum ($a$) describe the monopole and magnetic dipole moments of Kerr. Kerr does have higher multipole moments, but they are all expressible in terms of $M$, and $a$ [8]. Thus, in GR, the Kerr solution does not describe the space-time outside a rotating star, while the Schwarzschild solution describes the space-time outside a non-rotating star.

In this work, we investigate whether there exists a unique solution analogous to Schwarzschild in modified gravity theories. We show explicitly that this is not the case for $f(R)$ theories of gravity. More specifically, we show that there exists an infinite number of spherically symmetric vacuum solutions in 4-D $f(R)$ theories of gravity without transforming to a conformal frame.

$f(R)$ theories of gravity are the most straightforward modifications to GR [2]. The higher-order Ricci scalar terms encapsulate high energy modifications to GR. Although the equations of motion are higher-order, they do not suffer from Oströgradsky instability [2]. Thus, $f(R)$ theories provide a natural arena for understanding many exhaustive features of gravity. Unlike GR, $f(R)$ theories have an extra field equation and have a longitudinal mode [6, 11].

Although Schwarzschild black-hole is a solution to vacuum $f(R)$ theories of gravity [12, 13], it is unclear whether Schwarzschild is a unique vacuum solution for these theories. The reason for such a possibility to arise is due to the extra field equation satisfied by $R$. Unlike GR, $f(R)$ gravity has 11 dynamical variables — 10 metric variables ($g_{\mu\nu}$) and the Ricci scalar ($R$). In other words, in $f(R)$ theories, the scalar curvature $R$, plays a non-trivial role in the determination of the metric itself. Recently, the effects on the extra mode are used to obtain diagnostic parameters to distinguish GR and modified gravity theories using the Quasi-normal mode spectrum of the identical black-hole solutions in the two theories [6]. Here, the focus is to test the validity of this assumption. In other words, to verify whether the black-hole solutions in the two theories are indeed identical. We show that $f(R)$ admits multiple space-time geometries with the horizon for the same stress-tensor configuration (vacuum in this case). The concept of the horizon is, in general, observer-dependent. However, for spherically symmetric static space-times, by horizon, we refer to a horizon associated with static observers.
In Sec. (II), we introduce the $f(R)$ model. In Sec. (III), we obtain an infinite number of exact static spherically symmetric vacuum solutions for the model and discuss the important features of the same. In Sec. (IV), we derive two exact black-hole solutions and discuss their properties. Finally, in Sec. (V), we summarize the results and discuss the implications. We use $(-, +, +, +)$ signature for the 4-D space-time metric \cite{10}, Greek alphabets for the 4-D space-time, and $\kappa^2 = 8\pi G/c^4$ where $G$ is the Newton’s constant. We shall denote the derivative of any function with $r$ by an overprime, and overdot denotes the partial derivative w.r.t Ricci scalar ($R$).

II. $f(R)$ THEORY AND THE MODEL

The action for $f(R)$ gravity with no external matter fields (vacuum) is given by \cite{2}:

$$ S[g_{\mu\nu}] = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \, f(R) \quad (1) $$

where $f(R)$ is an arbitrary, smooth function of the Ricci scalar $R$. The modified Einstein tensor ($G_{\mu\nu}$) vanishes, i.e.,

$$ G_{\mu\nu} \equiv \ddot{f}(R) R_{\mu\nu} - \nabla_{\mu} \nabla_{\nu} \dot{f}(R) + g_{\mu\nu} \Box \dot{f}(R) - \frac{\ddot{f}(R)}{2} g_{\mu\nu} = 0, \quad (2) $$

where $\dot{f}(R) \equiv F(R) = \partial f/\partial R$ and $\Box = \nabla^\mu \nabla_\mu$. The generalized Bianchi identity leads to \cite{14}:

$$ \ddot{f}(R) (R_{\mu\nu} \nabla^\mu R) = 0. \quad (3) $$

For GR, $f(R) = R$. Hence, $\ddot{f}(R)$ vanishes and the above equation is trivially satisfied. However, $\ddot{f}(R)$ is non-zero for modified gravity theories, hence, the generalized Bianchi identity (3) leads to four constraints on the Ricci tensor. While, GR and $f(R)$ have four constraints on the field variables, the number of dynamical variables are different. For the above $f(R)$ action (1), the trace of the field equation (2) is dynamical:

$$ R \, \dot{f}(R) + 3 \Box \dot{f}(R) - 2 \, f(R) = 0 \quad (4) $$

As a result, $f(R)$ gravity has 11 dynamical variables — 10 metric variables ($g_{\mu\nu}$) and Ricci scalar ($R$). However, General Relativity has only 10 metric variables ($g_{\mu\nu}$). In other words, in $f(R)$, the scalar curvature $R$, plays a non-trivial role in the determination of the metric itself.
One may still find the trivial solution where field equations reduce to the Einstein field equations with an effective cosmological constant and an effective gravitational constant [15]. This includes the case where \( R = 0 \). Thus, all known black-hole solutions in GR also exist in this \( f(R) \) model. However, our interest in this work is to look for non-trivial solutions that take into account the dynamical aspect of Ricci scalar through the trace equation (4).

To model modified gravity in the strong-gravity regime, we consider \( f(R) \) to be a polynomial in \( R \), i.e.,

\[
 f(R) = \beta_0 + \beta_1 R + \beta_2 R^2 + \cdots ,
\]

where \( \beta_i \)'s \((i = 0, 1, 2 \cdots)\) are constants with appropriate dimensions. To keep the calculations tractable, we assume that the above form of \( f(R) \) can be written in a binomial form, i.e.,

\[
 f(R) = (\alpha_0 + \alpha_1 R)^p ,
\]

where, \( p \) is the power index, and \( \alpha_0 \) and \( \alpha_1 \) being positive constants. Thus, all the \( \beta_i \)'s in (5) are related to the two constants \( \alpha_0 \) and \( \alpha_1 \). For \( p = 1 \), the above action reduces to:

\[
 f(R) = \alpha_0 + \alpha_1 R .
\]

Thus, \( \alpha_0 \) acts like the cosmological constant and \( \alpha_1 \) is a dimensionless constant which modifies the Newton’s constant. [Note that \( \alpha_1 \) is dimensionless and \( \alpha_0 \) has dimensions of \([L]^{-2} \).]

Since, we are interested in the strong-gravity corrections to GR, we take \( p > 1 \). In principle, \( p \) need not be an integer. We aim to look for a generic, static spherically symmetric solutions for the above \( f(R) \) model without transforming to conformal frame [15].

III. A CLASS OF EXACT SOLUTIONS FOR \( f(R) \) THEORY

The static, spherically symmetric metric in 4-D can be written in the following form:

\[
 ds^2 = -A(r)e^\delta(r)dt^2 + \frac{dr^2}{A(r)} + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) 
\]

where \( A(r) \) and \( \delta(r) \) are unknown functions of the Schwarzschild radial coordinate \( r \). Substituting the above line-element in the modified Einstein’s equations (2) for the model \( f(R) \)
As expected, the equations of motion contain up to fourth-order derivatives in \( A(r) \) and \( \delta(r) \), and their derivatives. More specifically, (i) \( T_3 \) and \( T_5 \) are non-linear, and contain up to 4th order derivatives of \( A(r) \) and \( \delta(r) \), and (ii) \( T_4 \) is non-linear and contain up to 3rd order derivatives of \( A(r) \) and \( \delta(r) \). (iii) Even in the special case of \( \delta(r) = 0 \), \( G_i^\prime \neq G_r^r \). Hence, we do not expect to get identical solutions as in GR.

The exact forms of \( T_3, T_4 \) and \( T_5 \) are not relevant for the rest of the calculations, hence, they are not reported here. They can be seen in the MAPLE code available in the Dropbox folder. As expected, the equations of motion contain up to fourth-order derivatives in \( A(r) \) and \( \delta(r) \). Thus, an exact solution to these equations will contain up to four independent constants.

Fig. (1) contains the procedure we have adopted to reduce these highly non-trivial equations into a product of two second-order non-linear differentials in \( A(r) \) and \( \delta(r) \). Interestingly, both the procedures lead to the following equation:

\[
\frac{2}{r} \frac{(p - \frac{1}{2})}{p (p - 1)} \frac{\phi}{\rho} = 0
\]

where \( \Phi(r) = r \left( \delta'(r) + \ln A(r) \right) \),

\[
T_1[A(r), \delta(r)] = \left( \Phi + \frac{p + 1}{p - \frac{1}{2}} \right) \cdot \left( \frac{\Phi}{r} \right) - \frac{3r}{2} \left( \delta'(r)^3 + \left( \ln A(r) \right)'^3 \right)
\]

\[
+ (2 \Phi + 1) \delta'(r)^2 + \frac{5 \Phi}{2} - 1 \left( \ln A(r)' \right)^2 + \frac{1}{r} \left( \ln A(r)' \right)
\]

\[
+ \frac{\alpha_0 \alpha_1}{\alpha_1 A(r) \left( p - \frac{1}{2} \right) \rho} \left( \frac{\Phi}{r} - 1 \right) + \frac{1}{A(r) r^2} \left( \frac{A(r) + 1}{p - \frac{1}{2}} \right) \frac{\Phi}{r} - 4(A(r) - 1)
\]

\[
T_2[A(r), \delta(r)] = r^2 A(r) \left[ \left( \frac{\Phi}{r} \right)' + \left( \frac{4 + r}{2 r} \right) \left( \ln A(r) \right)' + \frac{3}{2} \left( \ln A(r) \right)' \left( \frac{\Phi}{r} \right) \right]
\]

\[
- \frac{r^2 A(r)}{2} \left( \ln A(r) \right)^2 + \left( \frac{4 + r}{2 r} \right) \left( \ln A(r) \right)' - \frac{4}{r} \left( \frac{1}{r + 2} \right) - 2 \left( 1 + \frac{\alpha_0 r^2}{2 \alpha_1} \right)
\]

and prime denotes derivative w.r.t \( r \).
Two procedures adopted to reduce Eqs. (9)

(A) Eliminate fourth-order derivatives of $A(r)$ in Eqs. (9a), (9c) leading to third order derivatives of $A(r)$

(B) Obtain third-order differential equation for $A(r)$ from Eq. (9b)

(C) Eliminate third-order derivatives of $A(r)$ from (A) & (B)

(D) Leads to Eq. (10)

(I) Eliminate fourth-order derivatives of $\delta(r)$ in Eqs. (9a), (9c) leading to third order derivatives of $\delta(r)$

(II) Obtain third-order differential equation for $A(r)$ from Eq. (9b)

(III) Eliminate third-order derivatives of $\delta(r)$ from (I) & (II)

(IV) Leads to Eq. (10)

FIG. 1. Flow-chart of the two procedures leading to Eq. (10).

This is the first important result regarding which we would like to stress the following points: First, as mentioned above, we have obtained the same equation (10) using two different approaches. This implies that Eq. (10) is a unique differential equation for this $f(R)$ model for the static spherically symmetric space-time (8). Second, the above simplified equation is a product of two second-order non-linear differentials of $A(r)$ and $\delta(r)$. Thus, the above equation drastically simplifies the procedure to obtain the exact black-hole solutions for any value of $p$. Third, the immediate consequence of the above equation are the conditions it imposes on $p$, $A(r)$ and $\delta(r)$. More specifically, if we demand a non-trival solution to be satisfied for any finite value of $r$, we get,

$$ p \neq 0, \frac{1}{2}, 1; \phi(r) \neq -4 \text{ or } 2. $$

(12)

Since $p = 1$ is not allowed, the non-trivial exact solutions we are looking is only valid for
modified theories of gravity. While $p = 0$ and 1 will lead to divergence, $p = 1/2$ will lead to trivial solutions. Fourth, the above condition of $\Phi(r)$ implies that

$$\delta(r) + \ln A(r) \neq \ln(r^2) \text{ or } \ln(r^{-4}) \quad (13)$$

If we assume $\delta(r) = $ constant, then $A(r) \neq c_0 r^2 + c_1 r^{-4}$, where $c_0, c_1$ are constants. Lastly, non-trivial solutions for the above equation (10) are possible if $T_1$ or $T_2$ vanish, i.e.,

$$T_1[A(r), \delta(r)] = 0 \quad \text{or} \quad T_2[A(r), \delta(r)] = 0 \quad (14)$$

In principle, for a given $A(r)$, we can have two forms of $\delta(r)$ that satisfy either $T_1 = 0$ or $T_2 = 0$. This leads to the immediate question: how can we obtain $A(r)$? Using the condition (12) on $\Phi(r)$, we get

$$\delta'(r) + (\ln[A(r)])' = \mu(r) \quad \mu(r) \neq \frac{-4}{r} \text{ or } \frac{2}{r} \quad (15)$$

Thus, for a given $\mu(r)$, we have a functional relation between $A(r)$ and $\delta(r)$. Substituting this relation in the constraint relation (14), we obtain a differential equation in terms of $A(r)$ or $\delta(r)$. Note that $\delta(r) = 0$ trivially satisfies Eqs. (11).

Thus, we have established that two branches of solutions exist by setting $T_1 = 0$ or $T_2 = 0$. Using the arbitrary function $\mu(r)$, we can obtain an infinite solutions for the spherically symmetric metric (8). This leads to the following question: Whether any arbitrary function $\mu(r)$ satisfying $T_1 = 0$ or $T_2 = 0$ is indeed a solution to the vacuum field equations (9)?

To address this, for any $p$, we write a formal solution to the above equation Eq. (15) as

$$A(r) = e^{-\delta(r)} \gamma(r) \quad \text{where} \quad \gamma(r) = \exp \left( \int \mu(r) dr \right) \quad (16)$$

Substituting $A(r)$ in-terms of $\delta(r)$ in $T_2[A(r), \delta(r)] = 0$, we obtain a differential equation in $\delta''(r)$. Substituting these in Eq. (9), we get,

$$T_3[\delta(r), \gamma(r)] = T_5[\delta(r), \gamma(r)] = 4\alpha_1 p(p-1)\gamma(r) = 4 \gamma(r) \gamma(r) \gamma(r) e^{-\delta(r)} [\delta'(r)r - 8]^2 T_2[\delta(r), \gamma(r)] \quad (17a)$$

$$T_4[\delta(r), \gamma(r)] = -2\alpha_1 p(p-1)\gamma(r) \gamma(r) \gamma(r) e^{-\delta(r)} [\delta'(r)r - 8] \gamma(r) \gamma(r) \gamma(r) \gamma(r) \gamma(r) + 4) T_2[\delta(r), \gamma(r)] \quad (17b)$$

Since, we have obtained the above expressions using the condition $T_2[A(r), \delta(r)] = 0$, we have $G_1^r = G_2^r = G_3^r = G_4^r = 0$. This implies that $A(r)$ given by Eq. (16) satisfying $T_2[A(r), \delta(r)] = 0$ is an exact solution for the $f(R)$ model (6). Since $A(r)$ depends on the arbitrary function
\( \mu(r) \), for the same observer with Schwarzschild time \( t \), there are \textit{infinite number of exact static, spherically symmetric solutions} for this model.

This is the key result of this work. As mentioned earlier, the Birkhoff theorem in GR guarantees that the most general spherically symmetric vacuum solution is the static Schwarzschild solution [9]. However, the trace equation (4) provides a non-trivial structure for the Ricci scalar as a function of \( r \), which is not possible for GR. This provides an infinite set of static solutions for \( f(R) \) theories of gravity. To our knowledge, such an explicit calculation is new for any modified theories of gravity. All the earlier analyses, either restrictive or use a conformal frame to confirm/infirm Birkhoff theorem [15, 16]. Here, we have not made any approximation or performed a conformal transformation to obtain a class of exact spherically symmetric solutions. Our results point that the Birkhoff theorem is not valid for all modified gravity theories.

As noted in Eq. (14), we can obtain non-trivial solutions if \( T_1 \) or \( T_2 \) vanish. Until now, we have shown that \( T_2 = 0 \) yields an infinitely many vacuum spherically symmetric solutions. Unlike \( T_2 \), \( T_1 \) is not a common factor of the field equations (17); hence, \( T_1 = 0 \) alone can not provide valid solutions. Thus, beside \( T_1 = 0 \), we must use either of the three equations (9) to obtain a unique solution.

In the next section, we obtain two particular vacuum black-hole solutions where \( A(r) \) is the same, with different \( \delta(r) \).

**IV. TWO VACUUM BLACK-HOLE SOLUTIONS**

In the earlier section, we showed that there exists an infinite number of spherically symmetric vacuum solutions for the \( f(R) \) model (6). In this section, we obtain two black-hole solutions which satisfy the condition \( T_2[A(r), \delta(r)] = 0 \).

As we have mentioned earlier, \( \delta(r) = 0 \) is a trivial solution. Hence, setting \( \delta(r) = 0 \), we obtain the following solution for \( A(r) \)

\[
A(r) = 1 + C_2 r^2 - \frac{C_3}{r^2} \quad \text{where} \quad C_2 = \frac{\alpha_0}{12\alpha_1} \tag{18}
\]

which satisfies the null-energy condition [10]. \( C_3 \) is a constant of integration and can take any real value. We like to list the following important points regarding the above solution: First, it is easy to verify that the above solution satisfies the modified Einstein’s equations
Second, $C_2$ is a positive constant, since, $\alpha_0$ and $\alpha_1$ are positive constants. Physically, $C_2$ acts like an effective cosmological constant. For $C_3 > 0$, the metric (8) has a horizon at

$$r_h = \sqrt{\frac{\sqrt{1 + 4C_2C_3} - 1}{2C_2}}$$  \hspace{1cm} (19)$$

In the limit of $\alpha_0 \to 0$, $C_2 \to 0$, the metric (8) has a horizon at $r = \sqrt{C_3}$. Thus, $\alpha_0 \to 0$ is a smooth limit. Third, the term $c_3/r^2$ is a reminiscence of the charge in the Reissner-Nordström solution in GR [10]. In GR, $C_3/r^2$ term can not exist without the mass term. However, in this case, the metric coefficients $g_{tt}$ and $g_{rr}$ do not contain $1/r$ term. In our case, the $1/r^2$ term is present in the absence of $1/r$ term. This result is similar to the one obtained sometime back in the context of black-holes on the brane [17]. Physically, $C_3$ corresponds to the mass of the black-hole. Lastly, it is easy to verify that the above solution satisfies the modified Einstein’s equations (9). The Kretschmann scalar for the above form of $A(r)$ is

$$R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = \frac{\alpha_0^2}{6\alpha_1^4} + \frac{56C_2^2}{r^8}.$$  \hspace{1cm} (20)$$

Thus, the metric has a singularity at $r = 0$ and is finite everywhere else. For finite $\alpha_0$, the Kretschmann scalar is a positive constant at asymptotic infinity which corresponds to asymptotic de Sitter space-times in GR [10].

In the above case, we have assumed $\delta(r) = 0$. Let us now substitute the above form of $A(r)$ in $T_2[A(r), \delta(r)] = 0$, this leads to the following differential equation for $\delta(r)$:

$$\delta''(r) + \frac{1}{2}\delta'(r)^2 + \frac{5C_2r^4 + 2r^2 + C_3}{(2r^4 + r^2 - C_3)} \frac{\delta'(r)}{r} = 0 \hspace{1cm} (21)$$

where $C_2$ is defined in Eq. (18). This differential equation is highly non-linear, however, it has the following exact solution:

$$e^{\delta(r)/2} = \frac{C_4}{2} - \frac{C_5}{2(12\alpha_1)^{3/2}} \frac{(2C_2r^2 + 1)}{(4C_2C_3 + 1)} \frac{1}{(2r^4 + r^2 - C_3)^{1/2}}$$  \hspace{1cm} (22)$$

where $C_4$ and $C_5$ are arbitrary constants. [$C_4$ is dimensionless while $C_5$ has dimension of [L].] The horizon for this new solution is again given by (19). The Kretschmann scalar again can be evaluated and near origin is singular, i. e.

$$R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} \sim \frac{\Gamma(C_2, C_3, C_4, C_5)}{r^8} \hspace{1cm} (23)$$

and is finite everywhere else. Thus, the constants $C_4$ and $C_5$ determine the behaviour of the second black-hole solution. We have reconfirmed the results in the earlier section.
and shown that there are at least two black-hole solutions corresponding to the same matter configuration (in this case, vacuum). To our knowledge, this is a novel result for any modified theories of gravity and confirms that the Birkhoff’s theorem is not valid for all modified gravity theories.

The key ingredient in the proof of the Birkhoff theorem in GR is the absence of spin-0 modes in the linearized field equations. The spherically symmetric space-time cannot couple to higher-spin excitations when spin-0 is absent [10, 18]. In the case of \( f(R) \) theories, the differential equation satisfied by the Ricci scalar \( R \) plays a non-trivial role in the determination of the metric itself. Thus, a non-trivial dependence between the metric and the Ricci scalar \( R \) leads to the breaking of the Birkhoff theorem in \( f(R) \).

V. CONCLUSIONS AND DISCUSSION

We have obtained an infinite number of exact static spherically symmetric vacuum solutions for \( f(R) \) gravity. To emphasize this unique feature, we obtained two exact vacuum black-hole solutions to the \( f(R) \) model. We showed that two solutions have the event-horizon at the same point; however, their asymptotic features are different. Our results point that the Birkhoff theorem is not valid for all modified gravity theories. The two black-hole solutions have a space-time singularity at the origin.

Unlike in the literature, we have obtained the exact solutions \textit{without} transforming to a conformal frame. It is then natural to ask what the infinite solutions correspond to in the conformal frame? Under conformal transformations, the \( f(R) \) action (1) transforms to [2]:

\[
S^E = \int d^4x \sqrt{-\tilde{g}} \left[ \frac{1}{2\kappa^2} \tilde{R} - \frac{1}{2} \tilde{g}^{\alpha\beta} \partial_{\alpha} \varphi \partial_{\beta} \varphi - U(\varphi) \right] \tag{24}
\]

where \( \tilde{g}_{\mu\nu} = F(R) g_{\mu\nu} \), and \( R \), and \( f(R) \) are expressed in terms of \( \varphi \), i. e.,

\[
\varphi = \sqrt{\frac{3}{2\kappa^2}} \ln F, \quad U(\varphi) = \frac{RF - f}{2\kappa^2 F^2}. \tag{25}
\]

Interestingly, for our model, \( \varphi \) can be written as

\[
\varphi = \sqrt{\frac{3}{2\kappa^2}}(p + 1) \ln \left( \frac{T_2[A(r), \delta(r)]}{r^2} \right) + \sqrt{\frac{3}{2\kappa^2}} \ln(\alpha_1 p) \tag{26}
\]

The solutions we have obtained in the original frame is for \( T_2 = 0 \). In the limit of \( T_2 \to 0 \), the scalar field \( \varphi \) takes infinite values. Thus, in this limit, we have an infinite number of
degenerate scalar field states. This also provides a physical meaning for the variable $T_2$ as it related to $F(R)$. Due to this deep relation, between $T_2$ and $F(R)$, our results are valid for any value of $p \neq 1$.

Our analysis shows the deficiency of finding solutions in the conformally transformed frame. The conformal transformations are not well-defined near $T_2 \to 0$ and, hence, the conformal frame will not be able to pick off the solutions we have obtained. However, the solution corresponding to $T_1[A(r), \delta(r)] = 0$ will be well-defined in the conformal frame. Since these equations are highly non-linear, we plan to use the publically available NeuroDiffEq package to obtain new non-trivial solutions in $f(R)$ models [19].

To keep the calculations tractable, we have used a binomial form for $f(R)$. However, the solutions we have derived should be true for any $f(R)$ model. The condition that $F(R)$ vanishes ensures that all the field equations are satisfied when $R$ takes a constant value. Hence, we can build infinitely many interesting $f(R)$ models for the same metric, which yield a constant $R$.

One of the prospects of the gravitational wave observations is to find signatures for the modified gravity theories. We have shown that if the modified theories belong one of the degenerate classes with $T_2 = 0$, then our analysis shows that the prospect of detection needs different methodologies than the one that is currently used [5].

The analysis in this work has been restricted to spherically symmetric space-times. It is natural to ask whether the same feature will be seen for axially rotating space-times. We plan to report this elsewhere.

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