Orlicz property of operator spaces and eigenvalue estimates

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Abstract
As is well known absolute convergence and unconditional convergence for series are equivalent only in finite dimensional Banach spaces. Replacing the classical notion of absolutely summing operators by the notion of 1 summing operators

\[ \sum_k \|Tx_k\| \leq c \sum_k e_k \otimes x_k \]

in the category of operator spaces, it turns out that there are quite different interesting examples of 1 summing operator spaces. Moreover, the eigenvalues of a composition \( TS \) decreases of order \( n^{1/4} \) for all operators \( S \) factorizing completely through a commutative \( C^* \)-algebra if and only if the 1 summing norm of the operator \( T \) restricted to a \( n \)-dimensional subspace is not larger than \( cn^{1-1/4} \), provided \( q > 2 \). This notion of 1 summing operators is closely connected to the notion of minimal and maximal operator spaces.

Introduction
In Banach space theory the Orlicz property and its connection to unconditional and absolute convergence is well understood. For instance, unconditional convergence and absolute convergence only coincide in finite dimensional spaces, but unconditional converging series are at least square summable in the spaces \( L_q, 1 \leq q \leq 2 \). This was discovered by Orlicz, hence the name Orlicz property. Furthermore, this is best possible in arbitrary infinite dimensional Banach space by Dvoretzky’s theorem. In the category of operator spaces there are several possibilities to generalize the classical Orlicz property. We have choose a definition where only sequences are involved and which is motivated by the theory of absolutely summing operators introduced by Grothendieck. To be more precise, we recall that an unconditional converging series \( (x_k)_k \subset E \) in a Banach space \( E \) corresponds to an operator defined on \( c_0 \) with values in \( E \). This is a consequence of the contraction principle.

\[ \left\| \sum_k e_k \otimes x_k \right\|_{\ell_1 \otimes \varepsilon E} = \sup_{|\alpha_k| \leq 1} \left\| \sum_k \alpha_k x_k \right\| \leq 4 \sup_{\varepsilon_k = \pm 1} \left\| \sum_k \varepsilon_k x_k \right\| . \]

In order to involve the operator space structure of an operator space \( E \subset B(H) \) we define an operator \( T : E \rightarrow F \) to be 1-summing if there exits a constant \( c > 0 \) such that

\[ \sum_k \|Tx_k\| \leq c \left\| \sum_k e_k \otimes x_k \right\|_{\ell_1 \otimes_{\min} E} . \]

The best possible constant will be denoted by \( \pi_{1,cb}(T) \). Here \( \min \) denotes the minimal or spatial tensor product and \( \ell_1 \) is considered as an operator space (for example by identification of the unit vectors with the generators of a free group). Anyhow, in the definition of absolutely

\[ \sum_k \|Tx_k\| \leq c \left\| \sum_k e_k \otimes x_k \right\|_{\ell_1 \otimes_{\min} E} . \]
summing operators we simple replace the norm by the cb-norm of the corresponding operator. Obviously, in this definition only the Banach space structure of $F$ is involved. That’s why this notion lives from the interplay of operator space and Banach space theory. For a more complete notion which is entirely defined in the category of operator spaces and where matrices instead of sequences are considered we refer to the work of Pisier about completely $p$-summing operators and factorization problems. With this background 1-summing operators turns to be the weakest possible notion. The classical notion of absolutely summing operators is included, by defining an operator spaces structure of a Banach space via the embedding of $E$ in the commutative $C^*$ algebra $C(B_{E^*})$. Following Paulsen we will denote this operator space by $\min(E)$. In the first chapter we collect basic properties of 1 summing operators and study the relation between 1-summing operators and $(1,C^*)$-summing operators defined early by Pisier for $C^*$-algebras. This is connected with Haagerup’s characterization of injective von Neumann algebras.

In this paper the framework of eigenvalue estimates for operators factorizing completely through a commutative $C^*$-algebra is used to distinguish different operator spaces. The first part is based on a generalization of Maurey’s inequality:

**Theorem 1** Let $2 < q < \infty$. For an operator space $E$ a Banach space $F$ and $T : E \to F$ the following assertions are equivalent.

i) There exists a constant $c_1$ such that for all $n$ dimensional subspaces $G \subset E$ and $(x_k)_k \subset G$

$$\sum_{k} \|Tx_k\| \leq c_1 n^{1-\frac{1}{q}} \left\| \sum_{k} e_k \otimes x_k \right\|_{\ell_1 \otimes \min E}.$$ 

ii) There exists a constant $c_2 > 0$ such that for all $n \in \mathbb{N}$ and $x_1, .., x_n \in E$ one has

$$\sum_{1}^{n} \|Tx_k\| \leq c_2 n^{1-\frac{1}{q}} \left\| \sum_{1}^{n} e_k \otimes x_k \right\|_{\ell_1^q \otimes \min E}.$$ 

iii) There exists a constant $c_3$ such that for all operators $S : F \to E$ which factor completely through a $C(K)$ space, i.e. $S = PR$, $R : F \to C(K)$ bounded and $P : C(K) \to E$ completely bounded one has

$$\sup_{n \in \mathbb{N}} n^{\frac{1}{q}} |\lambda_n(ST)| \leq c_3 \|P\|_{cb} \|R : F \to C(K)\|_{op}.$$ 

iv) There exists a constant $c_4$ such that for all operators $S : F \to E$ which factors completely through $B(H)$, i.e. $S = PR$, $R : F \to B(H)$ and $P : B(H) \to E$ completely bounded one has

$$\sup_{n \in \mathbb{N}} n^{\frac{1}{q}} |\lambda_n(ST)| \leq c_4 \|P\|_{cb} \|R : \min(F) \to B(H)\|_{cb}.$$ 

where $(\lambda_n(ST))_{n \in \mathbb{N}}$ denotes the sequence of eigenvalues in non-increasing order according to their multiplicity.

As an application for identities we see that projection constant of an $n$-dimensional subspace in a $(q, 1)$-summing Banach space is at most $n^{\frac{1}{q}}$. This can already be deduced from a corresponding theorem for identities on Banach spaces which was proved in [12]. In the operator spaces setting we see that estimates for the growth rate of the 1-summing norm are useful to measure the ‘distance’ of an operator space and its subspaces to $\ell_\infty$ spaces.
Let us note that the theorem is not valid for values \( q < 2 \). For identities on Banach space this is not relevant, all the properties are only satisfied for finite dimensional spaces. In contrast to this operators spaces with 1-summing identity are interesting spaces. For example the generators of the Clifford algebra spans such a space \( \text{CL} \). This is probably not so surprising, since this example has proved to be relevant also for the closely connected notion of \((2, oh)-\text{summing operators}\), introduced by Pisier. Starting with \( CL \) we construct a scale of operator spaces with different growth rates for the 1-summing norm. Further examples with small 1-summing norm were given by randomly chosen \( n \)-dimensional subspaces of the matrix algebra \( M_N \), provided \( n \leq N \). This was the starting point to discover, independently of Paulsen and Pisier, the fact there are only few completely bounded operators between minimal and maximal operator spaces. Paulsen studied all possible operator space structures on a given Banach space \( E \) and realized that there is a minimal and maximal one. The minimal one is given by the commutative structure already defined and the maximal by the embedding \( E \leftrightarrow (\min(E^*))^* \), where * denotes the operators space dual discovered by Effros/Ruan and Blecher/Paulsen. This is called the maximal operator space \( \max(E) \). Our approach is contained in the following proposition which is a refinement of Paulsen/Pisier’s result, unfortunately with a worse constant.

**Proposition 2** Let \( E \) be a maximal and \( F \) be a minimal operator space. For an operator \( T : F \to E \) of rank at most \( n \) one has

\[
\|T\|_{cb} \leq \gamma_2^*(T) \leq 170 \|T \otimes \text{Id}_{M_n} : M_n(F) \to M_n(E)\|,
\]

where \( \gamma_2^* \) is defined by trace duality with respect to Hilbert space factorizing norm \( \gamma_2 \). In particular,

\[
\frac{\sqrt{n}}{170} \leq \|\text{Id} : \min(E) \to \max(E)\|_{cb},
\]

for all \( n \)-dimensional Banach space \( E \).

This is contained in the second part of this paper where the study of 1-summing operators is continued. This turns out to be quite fruitful in the context of dual operator spaces. For instance, maximal operator spaces are 1-summing if and only if they are isomorphic to Hilbert spaces. Moreover, the 1-summing norm of a \( n \)-dimensional subspace of \( \max(\ell_r) \), \( \max(\ell_{r'}) \), \( \max(S_r) \) \( \max(S_{r'}) \) is less then \( 4n^{1-r} \) for all \( 2 \leq r \leq \infty \). Most of the techniques for maximal operator spaces carry over to duals of exact operator spaces by the key inequality of [JP]. The lack of local reflexivity in operator spaces leads to the notion of exactness defined by Pisier and motivated by Kirchberg’s work. In this context we will say that an operator space is exact if all its finite dimensional subspaces are uniformly cb-isomorphic to subspaces of the matrix algebra’s \( M_N \). In the next theorem the connection between 1-summing operators and factorization properties is established for duals of exact operator spaces.

**Theorem 3** Let \( 1 < p < 2 \), \( G \) an exact operator space, \( E \subset G^* \) and \( F \) a minimal operator space. For an operator \( T : E \to F \) the following are equivalent.

i) There exists a constant \( c_1 > 0 \) such that

\[
\sum_{1}^{n} \|Tx_k\| \leq c_1 n^{1-\frac{1}{p}} \left\| \sum_{1}^{n} e_i \otimes x_k \right\|_{e_1^p \otimes \min E}.
\]
ii) There is a constant $c_2$ such that for all completely bounded operators $S : F \to E$ one has

$$
\sup_k k^{\frac{1}{n}} |\lambda_k(TS)| \leq c_3 \|S : \min(F) \to E\|_{cb}.
$$

In the limit cases $p = 1$ the eigenvalues are summable if and only if the operator is 1-summing. In this case there exists a 1-summing extension $\hat{T} : G^* \to \min(F^{**})$ which factors completely bounded through $R \cap C$. Furthermore, every completely bounded $S : \min F \to E$ is absolutely 2-summing and hence the eigenvalues of a composition $TS$ are in $\ell_2$.

Let us note an application for an exact space $E \subset B(H)$ with quotient map $q : B(H)^* \to E^*$. The Banach space $E^*$ is of cotype 2 and $B(\ell_\infty, E^*) \subset CB(\ell_\infty, E^*)$ if and only if it is a cotype 2 space satisfying Grothendieck’s theorem. A non trivial example is the dual $A(D)^*$ of the disk algebra.

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Preliminaries

In what follows $c_0, c_1, \ldots$ always denote universal constants. We use standard Banach space notation. In particular, the classical spaces $\ell_q$ and $\ell_q^*$, $1 \leq q \leq \infty$, $n \in \mathbb{N}$, are defined in the usual way. We will also use the Lorentz spaces $\ell_p(\infty)$. This space consists of all sequences $\sigma \in \ell_\infty$ such that

$$
\|\sigma\|_{p(\infty)} := \sup_{n \in \mathbb{N}} n^{\frac{1}{p}} \sigma_n^* < \infty.
$$

Here $\sigma^* = (\sigma_n^*)_{n \in \mathbb{N}}$ denotes the non-increasing rearrangement of $\sigma$. The standard reference on operator ideals is the monograph of Pietsch [PIE]. The ideals of linear bounded operators, finite rank operators, integral operators are denoted by $\mathcal{B}$, $\mathcal{F}$, $\mathcal{I}$. Given an operator ideal $(A, \alpha)$ the adjoint operator ideal $(A^*, \alpha^*)$ is defined by the set of bounded operators $T : Y \to X$ such that

$$
\alpha^*(T) := \sup \left\{ \|\text{tr}(ST)\| : S \in \mathcal{F}(X, Y), \alpha(S) \leq 1 \right\}
$$

is finite. In particular, the ideal of integral operator is adjoint to bounded operators with

$$
t_1(T) := \|\cdot\|^* (T).
$$

We recall that an operator $T \in B(X, Y)$ factors through a Hilbert space $(T \in \Gamma_2(X, Y))$ if there are a Hilbert spaces $H$ and operators $S : X \to H$, $R : H \to Y^{**}$ such that $\iota_Y \cdot T = RS$, where $\iota_Y : Y \to Y^{**}$ is the canonical embedding of $Y$ into its bidual. The corresponding norm $\gamma_2(T)$ is defined as $\inf \{\|S\| \|R\|\}$, where the infimum is taken over such factorizations.
Let $1 \leq q \leq p \leq \infty$ and $n \in \mathbb{N}$. For an operator $T \in \mathcal{B}(X, Y)$ the pq-summing norm of $T$ with respect to $n$ vectors is defined by
\[
\pi^n_{pq}(T) := \sup\left\{ \left( \sum_{i=1}^{n} \|Tx_i\|^p \right)^{1/p} \left| \sup_{\|x\|_{X^*} \leq 1} \left( \sum_{i=1}^{n} \langle x_i, x^* \rangle \right)^{q} \right|^{1/q} \leq 1 \right\}.
\]
An operator is said to be absolutely pq-summing $(T \in \Pi_{pq}(X, Y))$ if
\[
\pi_{pq}(T) := \sup_n \pi^n_{pq}(T) < \infty.
\]
Then $(\Pi_{pq}, \pi_{pq})$ is a maximal and injective Banach ideal (in the sense of Pietsch). As usual we abbreviate $(\Pi_q, \pi_q) := (\Pi_{pq}, \pi_{pq})$. For further information about absolutely pq-summing operators we refer to the monograph of Tomczak-Jaegermann [TOJ].

The definition of some s-numbers of an operator $T \in \mathcal{B}(E, F)$ is needed. The $n$-th approximation number is defined by
\[
a_n(T) := \inf\{ \|T - S\| \mid \text{rank}(S) < n \},
\]
whereas the $n$-th Weyl number is given by
\[
x_n(T) := \sup\{ a_n(Tu) \mid u \in \mathcal{B}(\ell_2, E) \text{ with } \|u\| \leq 1 \}.
\]

Let $s \in \{a, x\}$. By $\mathcal{L}^{(s)}_{pq}$ we denote the ideal of operators $T$ such that $(s_n(T))_{n \in \mathbb{N}} \in \ell_{pq}$ with the associated quasi-norm $\ell_{pq}^{(s)}(T) := \|(s_n(T))_{n \in \mathbb{N}}\|_{\ell_{pq}}$. If $H$ is a Hilbert space the spaces $S_{pq}(H) = \mathcal{L}^{(a)}_{pq}$ are normable. Indeed all s-numbers coincide for operators on Hilbert spaces. If $p = q$ we will briefly write $S_p(H)$. This includes $S_2(H)$ the set of Hilbert-Schmidt operators.

By Ruan characterization theorem there are two possibilities to introduce operator spaces. Either as subspaces of $\mathcal{B}(H)$, where $H$ is a Hilbert space or as a Banach space $E$ together with a sequence of norms on the spaces of $n \times n$ matrices $M_n(E)$ with values in $E$. To guarantee that such a sequence of norms is induced by an embedding into some $\mathcal{B}(H)$ the following axioms are required.

i) If $O = (O_{ij})$, $P = (P_{ij})$ are scalar $n \times n$ matrices and $x = (x_{ij})$ in $M_n(E)$ one has
\[
\left\| \left( \sum_{kl} O_{ikl}P_{kl} \right)_{ij} \right\|_{M_n(E)} \leq \|O\| \|x\|_{M_n(E)} \|P\|.
\]

ii) If a matrix $B = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ consists of two disjoint blocs one has
\[
\|B\| = \max\{\|x\|, \|y\|\}.
\]

A major step for the development of operator space theory is the right definition of an operator space dual. Indeed, the norm of a matrix $(x^*_{ij}) \subset E^*$ is given by
\[
x^*_{ij} \left\|_{M_n(E^*)} = \left\| (x^*_{ij}) : E \rightarrow M_n \right\|_{cb} = \sup \left\{ \left\| \langle x^*_{ij}, x_{kl} \rangle \right\|_{M_{n^2}} \right\| x_{ij} \left\|_{M_n(E)} \leq 1 \right\} .
\]

For further information on this and operator space theory we refer to the paper of Blecher and Paulsen, [BPT].
1 The notion of 1-summing operators on operator spaces

Given two Banach spaces $X$ and $Y$ a matrix structure corresponding to operator spaces is defined on $\mathcal{B}(X, Y)$ in the following way. The norm of a matrix $(T_{ij}) \subset \mathcal{B}(X, Y)$ is induced by considering this matrix as element in $\mathcal{B}(\ell_2^n(X), \ell_2^n(Y))$

$$\|T_{ij}\|_n := \sup \left\{ \left( \sum_{i=1}^n \left\| \sum_{j=1}^n T_{ij}(x_j) \right\|^2 \right)^{\frac{1}{2}} \left| \sum_{i=1}^n \|x_j\|^2 \leq 1 \right\} .$$

Following \cite{PSC} an operator $u \in \mathcal{B}(E, F)$, where $E \subset \mathcal{B}(X_1, Y_1)$ and $F \subset \mathcal{B}(X_2, Y_2)$ is said to be completely bounded if there is a constant $c > 0$ such that for $(T_{ij}) \subset E$

$$\|u(T_{ij})\|_n \leq c \|T_{ij}\|_n .$$

The infimum over all such constants is denoted by $\|u\|_{\text{cb}}$. As usual $\ell_\infty^n$ will be considered as a subspace of $\mathcal{B}(\ell_2^n)$. The matrix norm induced by this embedding corresponds to the $\varepsilon$ tensor product. In analogy to the classical theory of absolutely r1-summing operators we define the r1-summing norm (with n vectors) for an operator $T \in \mathcal{B}(E, F)$, where $F$ is a Banach space and $E \subset \mathcal{B}(X, Y)$ as follows

$$\pi_{r_1, cb}^n(T) := \sup \left\{ \left( \sum_{k=1}^n \|Tu(e_k)\|^r \right)^{\frac{1}{r}} \left| \|u : \ell_\infty^n \rightarrow E\|_{\text{cb}} \leq 1 \right\}$$

and

$$\pi_{r_1, cb}(T) := \sup_{n \in \mathbb{N}} \pi_{r_1, cb}^n(T) .$$

An operator $T$ is said to be r1-\textit{summing} if $\pi_{r_1, cb}(T)$ is finite. The notion of absolutely r1-summing operators is included in this definition if we consider $E$ to be embedded into $C(B_{E^*}) \subset \mathcal{B}(\ell_2(B_{E^*}), \ell_2(B_{E^*}^*))$. A basic tool for the notion of r1-summing operators is a description of the cb norm for operators acting on $\ell_\infty$. This is well-known but since it is crucial for the following we give a proof.

\textbf{Lemma 1.1} Let $E \subset \mathcal{B}(X, Y)$ and $u \in \mathcal{B}(\ell_\infty^n, E)$ with $x_k = u(e_k)$. Then we have

$$\|u\|_{\text{cb}} = \sup \left\{ \sum_{k=1}^n \sigma_1(vx_kw) \left| v \in \mathcal{B}(Y, \ell_2), w \in \mathcal{B}(\ell_2, X) and \pi_2(v), \pi_2(w^*) \leq 1 \right\} ,$$

where $\sigma_1$ denotes the trace class norm.

\textbf{Proof:} Clearly, the supremum on the right hand remains unchanged if we replace all operators $v \in \mathcal{B}(Y, \ell_2), w \in \mathcal{B}(\ell_2, X)$ by the supremum over $m \in \mathbb{N}$ and $v \in \mathcal{B}(Y, \ell_2^m), u \in \mathcal{B}(\ell_2^m, X)$. By a well known characterization of 2-summing operators, see \cite{PSL}, every operator $v \in \mathcal{B}(Y, \ell_2^m)$ can be written in the form $v = Oz$ with

$$\left( \sum_{k=1}^n \|z^*(e_k)\|^2 \right)^{\frac{1}{2}} \left\| O : \ell_2^N \rightarrow \ell_2^m \right\| \leq (1 + \varepsilon) \pi_2(v) ,$$

where
for $\varepsilon > 0$ arbitrary. Hence we get

$$\sup \left\{ \sum_{i=1}^{n} \sigma_1(vx_kw) \mid v \in B(Y, \ell_2), w \in B(\ell_2, X) \text{ and } \pi_2(v), \pi_2(w^*) \leq 1 \right\} =$$

$$= \sup_{N \in \mathbb{N}} \sup \left\{ \left( \sum_{k=1}^{n} \sum_{i=1}^{N} \left| v^*(e_i) \right| \sum_{j=1}^{N} \left| A^k_{ji}x_k(w(e_j)) \right| \left( A^k : \ell_2 \rightarrow \ell_2^N \right) \leq 1, \sum_{i=1}^{N} \left| w(e_i) \right|^2 = 1 \right\}$$

$$= \sup_{N \in \mathbb{N}} \sup \left\{ \left\| \left( \sum_{k=1}^{n} e_k \otimes A^k_{ji} \right) \right\|_N \sup_{k} \left\| A^k \right\| \leq 1 \right\}$$

$$= \left\| u \right\|_{cb}.$$

\[ \square \]

**Remark 1.2** If $E \subset X^* \cong B(X, \mathcal{C})$ or $E \subset Y \cong B(\mathcal{C}, Y)$ the formula above reduces to

$$\|u : \ell_\infty^m \rightarrow E\|_{cb} = \pi_2(u).$$

Therefore the 1-summing norm of an operator $T \in B(E, F)$ coincides with the absolutely 2-summing norm

$$\pi_{1, cb}(T) = \pi_2(T).$$

If the space $E$ has Cotype 2 (or is $(2, 1)$ mixing, see [H]) every absolutely 2-summing operator is absolutely 1-summing and therefore all these notions coincide. The most canonical examples are given by the row space $R = B(\mathcal{C}, \ell_2)$ and the column space $C = B(\ell_2, \mathcal{C})$. In this cases it is a consequence of the "little Grothendieck inequality", see [OJ],

$$\pi_1(T) \leq \frac{2}{\sqrt{\pi}} \pi_2(T) = \frac{2}{\sqrt{\pi}} \pi_{1, cb}(T).$$

By interpolation the same remains true for the operator Hilbert space $OH$.

**Proof:** Let $E \subset Y \cong B(\mathcal{C}, Y)$ and $u \in B(\ell_\infty^m, E)$. Trace duality for the absolutely 2 summing operators implies

$$\|u\|_{cb} = \sup_{\pi_2(v), \pi_2(w^*) \leq 1} \sum_{i=1}^{n} \sigma_1(v(e_i \otimes y_i))w \leq \sup_{\pi_2(v), \pi_2(w^*) \leq 1} \sum_{i=1}^{n} \|v(y_i)\| \|w\|$$

$$= \sup_{\pi_2(v) \leq 1} \|u_v\| \leq \sup_{\pi_2(v), \|w\| \leq 1} \|tr(vuw)\| \leq \sup_{\|w\| \leq 1} \pi_2(wu) = \pi_2(u).$$

The argument for $E \subset B(X, \mathcal{C})$ is similar. For $T \in B(E, F)$ we use Pietsch’ factorization theorem, again trace duality and the fact that absolutely 1-summing operators on $\ell_\infty$ are integral

$$\pi_{1, cb}(T) = \sup \left\{ \pi_1(Tu) \mid \pi_2(u : \ell_\infty^m \rightarrow E) \leq 1 \right\}$$

$$= \sup \left\{ \|u\|_{cb} \mid \pi_2(u : F \rightarrow E) \leq 1 \right\} = \pi_2(T).$$

\[ \square \]
Nowadays it can be considered as a standard application of the Hahn-Banach separation theorem to deduce a factorization theorem for 1-summing operators. We refer to [PSP] for the required modification in the infinite dimensional case.

**Proposition 1.3** Let $X$, $Y$, $F$ be Banach spaces, $E \subset \mathcal{B}(X,Y)$ and $u \in \mathcal{B}(E,F)$.

1. Let us assume that $X$ and $Y$ finite dimensional, of dimension $n$ and $m$ in $\mathbb{N}$, say. The operator $T$ is 1-summing if and only if there exists a constant $C > 0$ and a probability measure $\mu$ on the compact space $K := B_{\Pi_2^d(\ell_2^n, X)} \times B_{\Pi_2^d(\ell_2^m, Y)}$ such that

$$\|Tx\| \leq C \int_K \sigma_1(vxu) \, d\mu(u,v).$$

2. $T$ is 1-summing if and only if there exists a constant $C > 0$ and an ultrafilter $\mathcal{U}$ over an index set $A$ together with finite sequences $(\lambda_i^a)_{i \in I}$, $(u_i^a, v_i^a)_{i \in I}$ such that

$$\|Tx\| \leq C \lim_{\alpha \in \mathcal{U}} \sum_{i \in I} \sigma_1(u_i^a x u_i^a) .$$

In both cases $C$ can be chosen to be $\pi_{1,cb}(T)$. In particular if $E \subset \mathcal{B}(H)$ is an operator space and $F$ carries its minimal (commutative) operator spaces structure then every 1-summing operator is completely 1-summing in the sense of Pisier, [PSP].

In the next proposition we list the relations between the notion of r1-summing operators and $(r1,C^*)$-summing operators defined on $C^*$-algebra’s by Pisier. More generally, let us recall that an element $z \in \mathcal{B}(X,\overline{X}^*)$, $\overline{X}^*$ the anti dual, is said to be positive if $\langle z(x), x \rangle \geq 0$ for all $x \in X$. An operator $u : \ell_1^\infty \rightarrow \mathcal{B}(X,\overline{X}^*)$ is positive, if $u$ maps positive sequences into positive elements.

**Proposition 1.4** Let $X$ be a Banach space.

1. An operator $u : \ell_1^\infty \rightarrow \mathcal{B}(X,\overline{X}^*)$ is completely bounded if and only if $u$ is decomposable into positive operators and

$$\|u\|_{cb} \leq \inf \left\{ \sum_j |\lambda_j| \|u_j\| \mid u = \sum_j \lambda_j u_j , \text{ u_j positive} \right\} \leq 4 \|u\|_{cb} .$$

Therefore an operator $T : \mathcal{B}(X,\overline{X}^*) \rightarrow F$ is $r1$-summing if and only if

$$\left( \sum_1^n \|Tz_k\| \right)^{\frac{1}{r}} \leq C \left\| \sum_1^n z_k \right\|$$

for all finite sequences of positive elements $(z_k)_1^n \subset \mathcal{B}(X,\overline{X}^*)$. The corresponding constants are equivalent up to a factor 4. Given an operator $v : X^* \rightarrow G$ then operator $T := v \otimes \bar{v} : \mathcal{B}(X,\overline{X}^*) \rightarrow G \otimes_\varepsilon \overline{G}$ is 1-summing if and only if $v$ is absolutely 2-summing.

2. If $E$ is a subspace of a $C^*$-algebra and $T \in \mathcal{B}(E,F)$ is a r1-summing operator then it is $(r1,C^*)$-summing, i. e. for all $(x_k)_k \subset C^*$

$$\left( \sum_1^n \|u(x_k)\| \right)^{\frac{1}{r}} \leq 4 \pi_{r1,cb}(T) \left\| \sum_1^n \left( \frac{x_k^* x_k + x_k x_k^*}{2} \right)^{\frac{1}{2}} \right\|_{C^*} .$$
Conversely, if $E$ is a von Neumann algebra, $E$ is injective if and only if every $(1,C^*)$-
summing operator is 1-summing and satisfies

$$\pi_{1,cb}(T) \leq c \pi_{1,C^*}(T),$$

where $c$ is a constant depending on $E$ ($\pi_{1,C^*}$ denotes the best constant in the inequality
above for $r = 1$). In this case also $\pi_{r,cb}(T) \leq c\pi_{r,C^*}(T)$ for all $1 \leq r < \infty$.

3. If $T : E \rightarrow F$ is a 1-summing operator defined on an operator space $E \subset B(H)$ it is $(2,oh)$,
$(2,R)$ and $(2,C)$-summing. This means

$$\left( \sum_{1}^{n} \|T(x_k)\|^2 \right)^{\frac{1}{2}} \leq \pi_{1,cb}(T) \left\| \sum_{1}^{n} x_k \otimes x_k^* \right\|_{E \otimes_{min} F},$$

and

$$\left( \sum_{1}^{n} \|T(x_k)\|^2 \right)^{\frac{1}{2}} \leq \pi_{1,cb}(T) \left\| \left( \sum_{1}^{n} x_k x_k^* \right) \right\|_{B(H)},$$

$$\left( \sum_{1}^{n} \|T(x_k)\|^2 \right)^{\frac{1}{2}} \leq \pi_{1,cb}(T) \left\| \left( \sum_{1}^{n} x_k^* x_k \right) \right\|_{B(H)}.$$  

**Proof:** For the following let us denote by $\pi_{r1}^+(T)$ the best constant $C$ satisfying

$$\left( \sum_{1}^{n} \|T(z_k)\|^r \right)^{\frac{1}{r}} \leq C \left\| \sum_{1}^{n} z_k \right\|_{B(X,X^*)}$$

for all positive elements $(z_k)_1^n$. Then we have trivially

$$\left( \sum_{1}^{n} \|T(u(e_k))\|^r \right)^{\frac{1}{r}} \leq \pi_{r1}^+(T) \left\| u : \ell^n \rightarrow B(X,X^*) \right\|_{dec},$$

where

$$\left\| u \right\|_{dec} := \inf \left\{ \sum_{j} |\lambda_j| \left\| u_j \right\|_{op} \right\} = \sum_{j} \lambda_j u_j \text{ positive}.$$  

We will first show that for a positive operator $u$

$$\left\| u \right\|_{cb} = \left\| \sum_{1}^{n} u(e_k) \right\| = \left\| u \right\|_{op}.$$  

For this we can assume that $z_k = u(e_k)$ are positive elements in $B(X,X^*)$. Let us note that
positive elements are automatically $\Gamma_2$ operators. On the tensor product $\ell_2 \otimes X$ we use the norm
induced by the absolutely 2 summing norm of the corresponding operator from $X^*$ with values in $\ell_2$. With this norm each element $x_k$ defines a positive, possibly degenerated, scalar product

$$\phi_k : (\ell_2 \otimes X) \times (\ell_2 \otimes X) \rightarrow \mathbb{C} \text{ with } \phi_k(v,w) := tr(v^* z_k w).$$
From Lemma [1.1], Hölder’s and the Cauchy-Schwartz inequality we deduce

\[ \|u\|_{cb} = \sup \left\{ \sum_{1}^{n} tr(A^k v^* z_k w) \mid \pi_2(v^*), \pi_2(w^*), \|A^k\| \leq 1 \right\} \]

\[ = \sup \left\{ \sum_{1}^{n} \phi_k(vA^k, w) \mid \pi_2(v^*), \pi_2(w^*), \|A^k\| \leq 1 \right\} \]

\[ \leq \sup \left\{ \left( \sum_{1}^{n} \phi_k(vA^k, vA^k) \right)^{\frac{1}{2}} \left( \sum_{1}^{n} \phi_k(w, w) \right)^{\frac{1}{2}} \mid \pi_2(v^*), \pi_2(w^*), \|A^k\| \leq 1 \right\} \]

\[ \leq \sup \left\{ \sum_{1}^{n} \sigma_1(v^* z_k v) \mid \pi_2(v^*) \leq 1 \right\} = \sup \left\{ \sum_{1}^{n} tr(v^* z_k v) \mid \pi_2(v^*) \leq 1 \right\} \]

\[ \leq \gamma_2(\sum_{1}^{n} z_k) \leq \left\| \sum_{1}^{n} z_k \right\| = \|u(1, ..., 1)\| \leq \|u\|_{op}. \]

Where we used that for a positive element \( z_k \) the composition \( v^* z_k v \) actually defines a positive operator on \( \ell_2 \) and that for the positive element \( z_k \) the \( \gamma_2 \)-norm and the operator norm coincide. (If \( X = H \) the whole statement can be deduced from [PAU theorem 2.4., proposition 3.5.]) In particular we obtain

\[ \|u\|_{cb} \leq \|u\|_{dec} \quad \text{and} \quad \pi_{r_1}^+(T) \leq \pi_{r_1,cb}(T). \]

1: Let \( u : \ell_\infty^3 \to B(X, X^*) \) be a completely bounded operator. By Pisier’s version [PSC] of the Haagerup/Wittstock factorization theorem, there exists a *-representation \( \pi : B(\ell_2^3) \to B(H) \) and operators \( V, W : H \to X \) such that

\[ u(\alpha) = W^* \pi(D_\alpha) V \quad \text{and} \quad \|V\| = \|W\| \leq \sqrt{\|u\|_{cb}}, \]

where \( D_\alpha \) denotes the diagonal operator with entries \( \alpha \). It is standard to see that the operators

\[ u^k(\alpha) := \frac{1}{4} (V + i^k W) \pi(D_\alpha) (V + i^k W) \quad k = 0, ..., 3 \]

are positive and of norm less than \( \|u\|_{cb} \). But \( u = u^0 - u^2 + i(u^1 - u^3) \) implies \( \|u\|_{dec} \leq 4 \|u\|_{cb} \). The second statement about operators \( T \) of the form \( v \otimes \overline{w} \) is a simple consequence of the observation that elementary tensors \( z_i = x^*_i \otimes x^*_j \) are clearly positive. For the reverse implication one simply uses Pietsch factorization theorem for absolutely 2 summing operators.

2: Clearly we have \( \pi_{r_1}^+(T) \leq \pi_{r_1,C^*}(T) \). For the converse we only have to note that every element \( x \) in a \( C^* \) algebra admits a decomposition \( x = x^1 - x^2 + i(x^3 - x^4) \) in positive elements such that

\[ x^k \leq \left( \frac{x^* x + xx^*}{2} \right)^{\frac{1}{2}}. \]

Hence we get \( \pi_{r_1,C^*}(T) \leq 4 \pi_{r_1}^+(T) \). If \( E \) is a von Neumann algebra we see that the existence of a constant \( c_1 > 0 \)

\[ \pi_{1,cb}(T) \leq c_1 \pi_{1,C^*}(T), \]

for all operators \( T : E \to \ell_\infty^3 \) is equivalent with the existence of a constant \( c_2 \)

\[ t^0(T) = \pi_{1,cb}(T) \leq c_2 \pi_{1}^+(T), \]

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where \( \ell^0 \) is the operator integral norm. Hence trace duality implies that the condition above is equivalent to 
\[
\|u\|_{dec} \leq c_2 \|u\|_{cb}
\]
for all \( u : \ell^0_{\infty} \to E \). By Haagerup’s theorem, see [HA], this holds if and only if \( E \) is injective. Together with the proof of 1., we see that for an injective von Neumann algebra the notion of \( r_1 \)-summing and \( (r_1, C^*) \)-summing coincide.

3: This is an easy variant of Kwapien’s argument. By the remark 1.2, we deduce that for all diagonal operator \( D_\sigma : \ell^0_{\infty} \to \ell^2_2 \) and \( G_n \in \{ R_n, C_n, OH_n \} \)
\[
\|D_\sigma : \ell^0_{\infty} \to G_n\|_{cb} = \pi_2(D_\sigma) = \|\sigma\|_2.
\]
Let us denote by \((e_k)^n\) the sequence of unit vectors of \( G_n \). Then we get for all \( w \in B(G_n, E) \)
\[
\left( \sum_1^n \|Tw(e_k)\|^2 \right)^{\frac{1}{2}} = \sup_{\|\sigma\|_2 \leq 1} \sum_1^n \|TwD_\sigma(e_k)\| \leq \pi_1(cb) \sup_{\|\sigma\|_2 \leq 1} \|wD_\sigma\|_{cb} \leq \pi_1(cb)(T) \|w\|_{cb}.
\]
The assertion is proved by identifying the complete bounded norm of \( w \) with the corresponding expressions on the right hand side in 2.. For \( G_n = OH_n \) this was done in [PSO]. For the two other cases we refer to [BPT].

**Remark 1.5** For an operator space \( E \subset B(H) \) which is of operator cotype 2 the a priori different notions of summability coincide. Indeed, using the same arguments as in the commutative theory, see [PSL], one can deduce that every operator \( S \in B(\ell^0_{\infty}, E) \) factors through \( OH_n \) with \( \gamma_{oh}(S) \leq c(E) \|S\| \). For notation and information see [PSO]. A use of “little Grothendieck” inequality implies
\[
\pi_1(T) \leq c_0 \ c(E) \ \pi_2_{,oh}(T).
\]
For all \( (2,oh) \)-summing operator \( T \in B(E, \ell^2_2) \). Finally the factorization properties of \( (2,oh) \)-summing operators imply for all operators \( T \in B(E, F) \)
\[
\frac{1}{c_0c(E)} \pi_1(T) \leq \pi_2_{,oh}(T) \leq \pi_1_{,cb}(T) \leq \pi_1(T).
\]
The proof of the first theorem in the introduction is based on a similar statement for the absolutely-summing norm of operators defined on \( C(K) \) spaces.

**Proposition 1.6** Let \( 2 < r < \infty \), \( K \) a compact Haussdorf space, \( F \) a Banach space and \( T : C(K) \to F \). If there exists a constant \( C > 0 \) such that
\[
\sum_1^n \|Tx_k\| \leq C n^{1-\frac{1}{r}} \sup_{t \in K} \sum_1^n |x_k(t)|
\]
for all elements \((x_k)^n\) \( \subset C(K) \), then we have
\[
\ell^{(x)}_{r,\lambda_k}(T) \leq c_0 \left( \frac{1}{2} - \frac{1}{r} \right)^{-1} C,
\]
where \( c_0 \) is an absolute constant. If \( F \) and \( C(K) \) are complex Banach spaces one has for every \( S : F \to C(K) \)
\[
\sup_{k \in \mathbb{N}} k^{1/r} \lambda_k(TS) \leq c_0^2 \left( \frac{1}{2} - \frac{1}{r} \right)^{-1} C \|S\|.
\]
Proof: First we show
\[ \| (\| Tx_k \|_F)_r \|_{r,\infty} \leq C \sup_{t \in K} \sum_k |x_k(t)| \]
for all \((x_k)_n \subset C(K)\). Indeed we can assume \(\| Tx_j \|\) non increasing. For fixed \(1 \leq k \leq n\) we get
\[ k \| Tx_k \| \leq \sum_1^k \| Tx_l \| \leq C k^{1-\frac{1}{r}} \sup_{t \in K} \sum_1^k |x_j(t)| \]
Dividing by \(k^{1-\frac{1}{r}}\) and taking the supremum over all \(1 \leq k \leq n\) yields the estimate. Now we choose \(2 < q < r\) with \(\frac{1}{2} + \frac{1}{r} = \frac{2}{q}\). For \((x_k)_n \subset C(K)\) we obtain
\[
\left( \sum_1^n \| Tx_k \|^q \right)^{1/q} \leq \left( \sum_1^n k^{-q/r} \right)^{1/q} (\| Tx_k \|)_{r,\infty} \\
\leq \left( \frac{1}{q} - \frac{1}{r} \right)^{-1/q} n^{1/q-1/r} c_2 \sup_{t \in K} \sum_1^n |x_k(t)| .
\]
Therefore we have
\[ \pi_{q1}^n(T) \leq C \left( \frac{1}{q} - \frac{1}{r} \right)^{-1/q} n^{1/q-1/r} . \]
Using Maurey’s theorem, see [TOJ, theorem 21.7], this implies with our choice of \(q\)
\[ \pi_{q2}^n(T) \leq C c_0 \left( \frac{1}{2} - \frac{1}{q} \right)^{1/q-1} \left( \frac{1}{q} - \frac{1}{r} \right)^{-1/q} n^{1/q-1/r} \\
\leq C 2 c_0 \left( \frac{1}{2} - \frac{1}{r} \right)^{-1} n^{1/q-1/r} . \]
Now let \(u \in B(\ell_2, C(K))\). By a Lemma, probably due to Lewis, see [PIE, Lemma 2.7.1], one can find for all \(n \in \mathbb{N}\) an orthonormal family \((o_k)_n \subset \ell_2\) with
\[ a_k(Tu) \leq 2 \| Tu(o_k) \| \quad \text{for all} \quad k = 1, \ldots, n . \]
Hence we deduce
\[ n^{1/q} a_n(Tu) \leq 2 \left( \sum_1^n \| Tu(o_k) \|^q \right)^{1/q} \leq 2 \pi_{q2}^n(T) \sup_{t \in K} \left( \sum_1^n |u(o_k(t))|^2 \right)^{1/2} \\
\leq 4 C c_0 \left( \frac{1}{2} - \frac{1}{r} \right)^{-1} n^{1/q-1/r} \sup_{\|o\|_2 \leq 1} \left\| u \left( \sum_1^n a_k o_k \right) \right\|_{C(K)} \\
\leq 4 C c_0 \left( \frac{1}{2} - \frac{1}{r} \right)^{-1} n^{1/q-1/r} \| u \| . \]
Dividing by the factor \(n^{1/q-1/r}\) and taking the supremum over \(n \in \mathbb{N}\) yields
\[ \sup_{n \in \mathbb{N}} n^{1/r} a_n(Tu) \leq 4 c_0 \left( \frac{1}{2} - \frac{1}{r} \right)^{-1} C \| u \| . \]
Now taking the supremum over all \(u\) with norm less than 1 the desired estimate for the Weyl numbers is proved. For the estimates of the eigenvalues we use the fact that the ideal \(L^{(\infty)}_r\) is of eigenvalue type \(\ell_{r,\infty}\). [PIE 3.6.5].
\[ \Box \]
**Remark 1.7** In fact all these conditions are equivalent as far as $2 < r < \infty$. If $1 < r < 2$ let us consider the embedding $I : \ell_1 \to C[0, 2\pi]$ given by the Rademacher functions $r_j(t) = \text{sign} \sin(2^jt)$ and the corresponding projection $P : C[0, 2\pi] \to \ell_2$. By Kintchine's inequality $P$ is $r_1$-summing for all $r > 1$. On the other hand if we compose with a continuous diagonal operator $D_r : \ell_2 \to \ell_1$ we see that the best possible eigenvalue behaviour for $r_1$-summing operators is actually $(\lambda_k(PD_r))_{k \in \mathbb{N}} \in \ell_2$. For $r = 2$ a more complicated example was constructed by [KOE]. This shows that the assumption $r > 2$ is really necessary.

**Remark 1.8** Since for an operator $A \in \mathcal{B}(\ell_1^n, \ell_1^m)$ the operator norm coincides with completely bounded norm we have for $1 \leq r \leq \infty$

$$
\pi_{r_1, cb}^n(u) = \sup \left\{ \pi_{r_1}^n(uw) \left| \|w : \ell_1^m \to \ell_1^c\| \leq 1 \right. \right\} .
$$

Therefore the results of [DJ2] can be applied to deduce for each operator $u$ of rank at most $n$

$$
\pi_{r_1, cb}^n(u) \leq c_0 \begin{cases} 
\left( \frac{1}{r} - \frac{1}{2} \right)^{\frac{n}{r'}} \pi_{r_1, cb}^{[n'/2]}(u) & \text{for } 1 < r < 2 \\
\pi_{[n/1+\ln n]}(u) & \text{for } r = 2 \\
\left( \frac{1}{2} - \frac{1}{r} \right)^{\frac{n}{r'}} \pi_{r_1, cb}^n(u) & \text{for } 2 < r < \infty ,
\end{cases}
$$

where $r'$ is the conjugate index to $r$.

An operator $u \in \mathcal{B}(F, E), E \subset \mathcal{B}(X, Y)$ is said to be completely $\infty$-factorable ($u \in \Gamma_{\infty}(F, E)$) if there is a factorization $u = SR$, where $R \in CB(F, B(H)), S \in CB(B(H), E)$, $H$ a Hilbert space. The $\gamma_{\infty}$-norm of $u$ is defined as $\inf\{\|S\|_{cb} \|R\|_{cb}\}$ where the infimum is taken over all such factorizations. As in the commutative case this turns out to be a norm. Now we can prove the first theorem of the introduction.

**Theorem 1.9** Let $2 < r < \infty$, $X, Y, F$ Banach spaces and $E \subset \mathcal{B}(X, Y)$. For an operator $T : E \to F$ the following assertions are equivalent.

i) There is a constant $c_1$ such that for all $n \in \mathbb{N}$

$$
\pi_{1, cb}^n(T) \leq c_1 n^{1-\frac{1}{r}} .
$$

ii) There is a constant $c_2$ such that for all operators $R \in \mathcal{B}(F, C(K)), S \in CB(C(K), E), K$ a compact Hausdorff space

$$
\sup_{k \in \mathbb{N}} k^{1/r} \|\lambda_k(TSR)\| \leq c_2 \|R\| \|S\|_{cb} .
$$

iii) There is a constant $c_3$ such that for all $n$-dimensional subspaces $E_1 \subset E$ one has

$$
\pi_{1, cb}(T_{I_{E_1}}) \leq c_3 n^{1-\frac{1}{r}} .
$$

Moreover the best constants satisfy

$$
c_1 \leq c_3 \leq c_0 c_2 \leq c_0^2 \left( \frac{1}{2} - \frac{1}{r} \right)^{-1} c_1 .
$$
If $E \subset B(H)$ is an operator space and $F = \min F$ carries its minimal operator space structure these conditions are equivalent to

$$\sup_{k \in \mathbb{N}} k^{1/r} |\lambda_k(TS)| \leq c_4 \gamma_\infty^0(S)$$

for all completely $\infty$-factorable operators $S$.

Proof: $i) \Rightarrow ii)$ By the remark above we have for all $S \in CB(C(K),E)$

$$\pi_1(uS) \leq c_1 \|S\|_{cb} n^{1-1/r}.$$ 

By Proposition 1.6 this implies for all $R \in B(F,C(K))$

$$\sup_{k \in \mathbb{N}} k^{1/r} |\lambda_k(uSR)| \leq c_0 \ell_\infty^0(uSR) \leq c_0^2 \left( \frac{1}{2} - \frac{1}{r} \right)^{-1} c_1 \|S\|_{cb} \|R\|.$$ 

For the implication $ii) \Rightarrow iii)$ let $u : \ell_\infty^m \rightarrow E_1$ be a completely bounded map and $(y_k^*)_1^m \subset B_Y$, such that

$$\|Tu(e_k)\| = \langle Tu(e_k), y_k \rangle.$$ 

We define the operator $S : Y \rightarrow \ell_\infty^m; S(y) = ((y, y_k^*))_1^m$ which is of norm at most 1 and get

$$\sum_1^m \|Tu(e_k)\| = tr(STu) \leq 2 n^{1-\frac{1}{r}} \sup_k k^{\frac{1}{2}} |\lambda_k(STu)| \leq 2 n^{1-\frac{1}{r}} c_2 \|S\| \|u\|_{cb}.$$ 

The implication $iii) \Rightarrow ii)$ is obvious. Since $\ell_\infty^m$ is a completely complemented subspace of $M_n$ we only have to show the eigenvalue estimate. In fact, let $S = PR, R : \min(F) \rightarrow B(H), P : B(H) \rightarrow E$ completely bounded. Since $F$ is considered as a subspace of $C(K)$ for some compact Hausdorff space $K$, there is a completely bounded extension $\tilde{R} : C(K) \rightarrow B(H)$ of the same cb-norm by Wittstock’s extension theorem, see [PAU]. If we apply $ii)$ to $S = (P\tilde{R})_{\ell_F}, \iota_F$ the inclusion map we obtain the assertion. 

$$\square$$

2 1-summing operators in connection with minimal and exact operator spaces

In contrast to Banach space theory there are infinite dimensional operator spaces such that the identity is 1-summing. This is possible because this notion does not respect the whole operator space structure. In fact we will see that these examples appear in different contexts. We will start with a probabilistic approach.

Lemma 2.1 Let $n, N \in \mathbb{N}$. Then there exists a biorthogonal sequence $(x_j)_1^n \subset M_N$, i.e. $tr(x_j^*x_i) = \delta_{ij}$ with

$$\left\| \sum_1^n e_j \otimes x_j : \ell_2^n \rightarrow M_N \right\|_{op} \leq \pi(1 + \sqrt{2}) \left( \frac{1}{\sqrt{N}} + \frac{\sqrt{n}}{\sqrt{2N}} \right).$$

In fact a random frame for $n$-dimensional subspaces of $M_N$ satisfies this inequality up to a constant.
Proof: Let $J$ be a subset of cardinality $n$ in $I = \{(i, j) \mid i, j = 1, \ldots, N\}$. We set $y_s := e_i \otimes e_j \in M_N$, but $x_s := e_i \otimes e_j$ only for $s \in J$ and 0 else. For $(s, t) \in I \times I$ let $h_{s,t} = \frac{1}{\sqrt{2}} (g_{st} + i g_{st}')$ be a sequence of independent, normalized, complex gaussian variables (Clearly $(g_{st})$ and $(g_{st}')$ are assumed to be independent.) Applying Chevet’s inequality twice we obtain

$$
\mathbb{E} \left\| \sum_{s \in J, t \in I} h_{s,t} x_s \otimes y_t \right\|_{op} = \mathbb{E} \left\| \sum_{s \in J, t \in I} g_{s,t} x_s \otimes \frac{y_t}{\sqrt{2}} + \sum_{s \in J, t \in I} g_{s,t}'(ix_s) \otimes \frac{y_t}{\sqrt{2}} \right\|
\leq \left( \omega_2 \{x_s, ix_s\} \mathbb{E} \left\| \sum_{t \in I} \frac{g_t + g_t'}{\sqrt{2}} y_t \right\|_{M_N} + \frac{1}{\sqrt{2}} \omega_2 \{y_t, y_t\} \mathbb{E} \left\| \sum_{s \in J} g_s x_s + g_s' ix_s \right\|_{(S_2^N)^*} \right)
\leq \left( 2\sqrt{N} + \sqrt{2n} \right),
$$

where $\omega_2 \{y_t, y_t\}$ corresponds to the operator norm of the corresponding real linear operator.

Using the comparison principle between random unitary matrices in $U_{N^2}$ and gaussian $N \times N$ matrices, see [MAP], we get

$$
\mathbb{E} \left\| \sum_{s \in J} x_s \otimes U(x_s) \right\|_{op} = \mathbb{E} \left\| \sum_{s \in J} \langle y_t, U(x_s) \rangle x_s \otimes y_t \right\|
\leq \pi (1 + \sqrt{2}) \frac{1}{2 \sqrt{N^2}} \mathbb{E} \left\| \sum_{s, t} h_{s,t} x_s \otimes y_t \right\|
\leq \pi (1 + \sqrt{2}) \left( \frac{1}{\sqrt{N}} + \frac{\sqrt{n}}{\sqrt{2N}} \right).
$$

For every $\varepsilon > 0$ we can find a unitary $U$ such that the norm estimate is satisfied up to $(1 + \varepsilon)$ by Chebychev’s inequality. By passing to a limit we can even find a unitary $U$ satisfying the norm estimate for $\varepsilon = 0$. Since $U$ is a unitary in $\ell_2^{N^2}$ we use the usual identification between trace and scalar product to see that the elements $U(x_s)$ are biorthogonal. An application of the concentration phenomenon [MIS] gives the assertion for random frames of $n$-dimensional spaces subspaces of $M_N$.

The notion of random subspaces of a given $N$-dimensional Banach space $F$ is always defined by a "natural" scalar product and the group of unitaries of the associated Hilbert space. A property of random subspace, means that this property is satisfied with "high probability" for subspaces of a fixed dimension $n$. In this case the probability measure is taken from the surjection $U \mapsto \text{span}\{U(e_1), \ldots, U(e_k)\}$ with respect to the normalized Haar measure on the group of unitaries. Implicitly, it is understood that the constant may depend on how close to 1 the probability is chosen. However, if the expected value can be estimated the concentration phenomenon on the group of unitaries yields reasonable estimates. For further and more precise information of this concept see the book of Milman/Schechtman [MIS]. In this sense we formulate the following.
Corollary 2.2 Let \( n \leq N \) and \( E \) a random subspace of \( M_N \), then \( E \) is 1-summing with
\[
\pi_{1,cb}(id_E) \leq C ,
\]
where \( C \) depends on the probability not on the dimension.

Proof: We keep the notation from the proof above. A random \( n \)-dimensional subspace of \( M_N \) is of the form \( E = \text{span}\{U(x_s) \mid s \in J\} \). By lemma 2.1 we can assume that with high probability the operator
\[
v := \sum_{s \in J} x_s \otimes U(x_s)
\]
is of norm less than \( \frac{C}{\sqrt{N}} \). The operator \( vv^* \) acts as a projection \( E \) and therefore we have the following factorization
\[
Id_E = (\sqrt{N}v)(\sqrt{N}v)^* \left( \frac{1}{N} Id : M_N \rightarrow S_N^1 \right) \iota_E ,
\]
where \( \iota_E \) is the canonical embedding and \( (\sqrt{N}v)(\sqrt{N}v)^* \) should be considered as an operator from \( S_N^1 \) to \( M_N \). As such it is of norm at most \( C^2 \). By the trivial part of the factorization theorem for 1-summing operators 1.3 we get the assertion. \( \square \)

Paulsen, [PAU], proved that a unique operator space structure for a given Banach space is only possible in small dimensional spaces. This is based on the study of cb maps between minimal and maximal operator spaces. In this setting the author discovered lemma 2.1 above in a preliminary version of this paper, noticing that this implies an estimate for the operator integral norm for the identity \( \max(\mathbb{L}_2^n) \rightarrow \min(\mathbb{L}_2^n) \). Indeed, such a factorization has just been constructed with the help of the random spaces \( E \) above. However, the constant which can be deduced from this approach is worse than that obtained by Paulsen/Pisier. Before we indicate our proof of Paulsen/Pisier result let us recall an easy lemma which is merely the definition of the dual space, see also [JP]. For this we will use the following notation \( \|T\|_n := \|Id_{M_n} \otimes T : M_n(E) \rightarrow M_n(F)\| \) for an operator \( T \) between to operator spaces \( E, F \).

Lemma 2.3 Let \( E, F \) operator spaces and \( T : E \rightarrow F \) then we have
\[
\|T\|_n = \sup \left\{ \sum_{i,j=1}^n \langle y_{ij}, a_{ik}T(x_{kl})b_{lj} \rangle \bigg| hs(a) \leq 1, \|x_{ij}\|_{M_n(E)}, \|y_{ij}\|_{M_n(F^*)} \leq 1 \right\} .
\]

With the probabilistic approach we can prove that it suffices to consider \( n \times n \) matrices for rank \( n \) operators between minimal and maximal spaces, improving Paulsen/Pisier’s result.

Proposition 2.4 Let \( E \) be a minimal, \( F \) be a maximal operator space and \( T : E \rightarrow F \) an operator of rank at most \( n \) then we have
\[
\gamma_2^*(T) \leq 170 \|Id_{M_n} \otimes T : M_n(E) \rightarrow M_n(F)\| .
\]
Furthermore, for every \( n \) dimensional space we have
\[
\sqrt{n} \leq (\pi(1 + \sqrt{2}))^2 \|Id : \min(E) \rightarrow \max(E)\| ,
\]
where \( \min(E), \max(E) \) means \( E \) equipped with its minimal, maximal operator space structure, respectively.
Proof: First we will prove an estimate for operators \( T : \ell_2^n \to \ell_2^n \)

\[
|\text{tr}(T)| \leq 170 \|T : \min(\ell_2^n) \to \max(\ell_2^n)\|_n.
\]

Indeed, we use \( N = n \) in lemma 2.1 and consider the elements

\[
z_{kl} = \sum_i \langle x_i(e_k), e_l \rangle \otimes e_i \in M_n(\min(\ell_2^n))
\]

which are of norm at most \( \frac{\pi^{(2+\frac{1}{2})}}{\sqrt{n}} \). In lemma 2.3 we use \( a = b = \frac{1}{\sqrt{n}} I_{\ell_2^n} \) to deduce

\[
|\text{tr}(T)| = \left| \sum_1^n \langle T(e_i), e_j \rangle \right| = \left| \sum_{i,j} \text{tr}(x_i x_j^*) \langle T(e_i), e_j \rangle \right| \leq \left| \sum_{kl} \langle z_{kl}^*, T(z_{kl}) \rangle \right|
\]

\[
\leq \|T\|_n \text{hs}(id)^2 \|z\|_{M_n(\min(\ell_2^n))} \|z^*\|_{M_n(\min(\ell_2^n))}
\]

\[
\leq \|T\|_n \frac{\pi^2(2 + \frac{3}{\sqrt{2}})^2}{n} \leq 170 \|T\|_n.
\]

For an arbitrary operator \( T : \min(E) \to \max(F) \) we use trace duality. Indeed, let \( S : F \to E \) an operator which factors through a Hilbert space, i.e. \( S = uv, v : F \to H, u : H \to E \). In order to estimate \( S \) we can modify \( S \) by inserting the orthogonal projection on \( v(\text{Im}(T)) \).

Therefore there is no loss of generality to assume \( H = \ell_2^n \). Hence we get

\[
|\text{tr}(TS)| = |\text{tr}(vTu)| \leq 170 \|vTu\|_n \leq 170 \|v\|_n \|T\|_n \|u\|_n \leq 170 \|T\|_n \|v\| \|u\|.
\]

We used that by the definition of the minimal operator space every operator with values in \( \min(E) \) is automatically completely bounded. Taking the infimum over all factorizations we get the first assertion. The second one follows from duality by applying the estimate for the identity operator and recalling that John’s theorem \( \gamma_2(Id_E) \leq \sqrt{n} \). The better constant is obtained by letting \( N \) tend to infinity in lemma 2.7 and the corresponding modification in the proof above.

As a consequence one obtains that the identity on \( \max(\ell_2) \) is indeed a 1-summing operator. More general results hold in the context of duals of exact operator spaces using the key inequality of [JP]. We will need some notation. Given a Hilbert space \( H \) there are at least two natural ways to associate an operator spaces with \( H \), the column space

\[
C_H := \{ x \otimes y \in B(H) \mid x \in H \}
\]

and the row space \( R_H := \{ y \otimes x \in B(H) \mid x \in H \} \),

where \( y \) is a fixed, normalized element in \( H \). It is quite easy to check that the corresponding norm of a matrix \( (x_{ij}) \subset H \) is given by

\[
\|x_{ij}\|_{M_n(C_H)} = \left( \sum_k (x_{ki}, x_{kj}) \right)_{ij} \quad \text{and} \quad \|x_{ij}\|_{M_n(R_H)} = \left( \sum_k (x_{jk}, x_{ik}) \right)_{ij}.
\]

where we assume the scalar product \( \langle \cdot, \cdot \rangle \) to be antilinear in the first component. It turns out that \( R_H^* = C_H \) in the category of operator spaces. The space \( R_H \cap C_H \) is \( H \) equipped with matrix norm given by the supremum of \( R_H \) and \( C_H \). The dual space \( (R_H \cap C_H)^* = C_H + R_H \) carries a natural operator spaces structure and was intensively studied by Lust-Picard, Haagerup
and Pisier, see [LP,HP]. In connection with this row and column spaces it is very useful to consider the following notion. Let $E \subset B(K)$ an operator space and $F$ Banach space. An operator $T : E \to F$ is $(2,RC)$-summing if there exists a constant $c > 0$ such that

$$\sum_k \|T(x_k)\| \leq c \max \left\{ \left\| \sum_k x_k^* x_k \right\|_{B(K)}, \left\| \sum_k x_k x_k^* \right\|_{B(K)} \right\}.$$ 

The best possible constant is denoted by $\pi_{2,RC}(T)$. Let us note that the right hand side is a weight in the sense of [PSI]. We start with a description of $(2,RC)$ summing operators with values in a Hilbert space, which was suggested by C. le Merdy.

**Proposition 2.5** Let $E \subset B(K)$ an operator space, $H$ a Hilbert space and $T : E \to H$ a bounded linear operator. $T$ is $(2,RC)$ summing if and only if there is a bounded extension $\hat{T} : B(K) \to H$ of $T$ if and only if there is a completely bounded extension $\hat{T} : B(K) \to R_H + C_H$ of $T$.

**Proof:** Let us observe that by the non-commutative Grothendieck inequality every bounded $S : B(K) \to H$ is $(2,RC)$ summing, see e.g. [PSI]. Therefore, we are left to prove the existence of a cb extension for $(2,RC)$-summing operators $T : E \to H$. Using a variant of Pietch’s factorization theorem,(for more precise information see [PSI]), there are states $\phi, \psi$ on $B(K)$ and $0 \leq \theta \leq 1$ such that

$$\|v(x)\| \leq \pi_{2,RC}(T) \left( \theta \phi(x^* x) + (1 - \theta) \psi(x^* x) \right)^{1/2}.$$

We define the sesquilinearforms $\langle x,y \rangle_\phi := \phi(y x^*)$ and $\langle x,y \rangle_\psi := \psi(x^* y)$. Furthermore, we denote by $C_\phi, R_\psi$ the column, row Hilbert space which is induced by the corresponding scalar product. It is easy to check that the identities $I_\phi : B(K) \to C_\phi, I_\psi : B(K) \to R_\psi$ are in fact completely bounded of norm 1. We denote by

$$M := \text{cl}\{ (\sqrt{\theta} x, \sqrt{1-\theta} x) \mid x \in E \} \subset C_\phi \oplus_2 R_\psi$$

the closure of the image of $J := \sqrt{\theta} I_\phi \oplus \sqrt{1 - \theta} I_\psi$ restricted to $E$. $P_M$ denotes the orthogonal projection of $C_\phi \oplus_2 R_\psi$ onto $M$. Then we get an extension $\hat{T} = \tilde{v} J$ of $T$, where $\tilde{v}$ acts as $v$ but considered as an operator on $M$. By the first inequality $\tilde{v}$ is of norm at most $\pi_{2,RC}(T)$ and by definition of $R_H + C_H$ we get $\|\tilde{v} : R_M + C_M \to R_H + C_H\|_{cb} \leq \pi_{2,RC}(T)$. By duality it is easy to see that $P_M : C_\phi \oplus_1 R_\psi \to R_M + C_M$ is completely bounded of norm 1. On the other hand the cb norm of

$$\sqrt{\theta} I_{C_\phi} \oplus \sqrt{1-\theta} I_{R_\psi} : C_\phi \oplus_\infty R_\psi \to C_\phi \oplus_1 R_\psi$$

is at most $\sqrt{2}$. \qed

Now we will give a description of completely bounded operators between the class of exact operator spaces and maximal operator spaces. Pisier’s notion of exact operator spaces, [PSE], is motivated by Kirchberg’s work. One possible definition says that an operator space is exact if its finite dimensional subspaces are uniformly cb isomorphic to subspaces of the spaces of compact operators.
Proposition 2.6 Let $E \subset B(K)$ be either an exact operator space and $F$ a maximal operator space, i.e., a quotient of $\ell_1(I)$ for some index set $I$, or $E$ a $C^*$ algebra and $F = \ell_1(I)$. For an operator $T : E \to F$ the following are equivalent.

i) $T$ is completely bounded.

ii) There is a Hilbert space $H$ and operators $v : E \to H$, $u : H \to F$ such that $v$ is $(2,RC)$ summing and $u^*$ is absolutely 2 summing.

iii) There is a completely bounded extension $\hat{T} : B(K) \to \ell_1(I)$ of $T$.

iv) $T$ factors completely through $R_H + C_H$ for some Hilbert space $H$.

Moreover, the corresponding constants are equivalent.

Proof: The implication $i) \Rightarrow ii)$ is either the non-commutative Grothendieck inequality, see [PSL], or the key inequality in [JP]. The implications $ii) \Rightarrow iii), iv)$ are direct consequences of proposition 2.5 and the extension properties of absolutely 2 summing operators. We only have to note that an absolutely 2 summing operator $u^* : \ell_\infty(I) \to R_H \cap C_H$ is completely bounded. The rest is trivial. ◻

For the proof of theorem 3 we will need some more notation. Let $1 < p < \infty$, $E$ be an operator space and $F$ a Banach space. An operator $T : E \to F$ belongs to $\Gamma_p,RC$ if

$$\gamma_p,RC(T) := \sup \{ \sigma_{p,\infty}(vTu) \mid v \in \Pi_2(F,\ell_2), u \in CB(R + C, E) \pi_2(v), \|u\|_{cb} \leq 1 \} < \infty.$$ 

For $p = 1$ we will use $\Gamma_{1,RC}$, $\gamma_{1,RC}$ for the corresponding expressions with $\sigma_{p,\infty}$ replaced by $\sigma_1$. This notion is modeled close to the notion of Hilbert space factoring operators and forms a 'graduation' of $\Gamma_{1,RC}$ in the cases $p > 1$. This has already been proved to be useful for eigenvalue estimates.

Theorem 2.7 Let $1 < p < 2$, $G$ an exact operator space, $E \subset G^*$ and $F$ a minimal operator space. For an operator $T : E \to F$ the following are equivalent.

i) There exists a constant $c_1 > 0$ such that

$$\sum_1^n \|Tx_k\| \leq c_1 n^{1-\frac{p}{2}} \left\| \sum_1^n e_i \otimes x_i \right\|_{\ell_1^n \otimes \min E}.$$ 

ii) $T$ is in $\Gamma_p,RC$

iii) There is a constant $c_3$ such that for all completely bounded operator $S : F \to E$ one has

$$\sup_k k^{\frac{p}{2}} |\lambda_k(TS)| \leq c_3 \|S : \min(F) \to E\|_{cb}.$$ 

In the limit cases $p = 1$ the same remains true if we replace the $\ell_{p,\infty}$ norm of the eigenvalues by the $\ell_1$ norm. Furthermore, every completely bounded $S : \min F \to E$ is absolutely 2-summing and hence the eigenvalues of a composition $TS$ are in $\ell_2$. 

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**Theorem:** There is a factorization of $S$ completely bounded and $u : R_H \cap C_H \to G^\ast$ is completely bounded. Using an orthogonal projection $P$ on $u^{-1}(E)$ together with the homogeneity of the spaces $R_H$ and $C_H$, we can assume that $(u(H) \subset E$ and therefore $S = uv$. Using the well-known eigenvalue estimate of the class $S_{p,\infty}$ and the principle of related operators, [PIE], we get

$$\sup_{k} k^{\frac{1}{p}} |\lambda_k(TS)| = \sup_{k} k^{\frac{1}{p}} |\lambda_k(vTu)| \leq c_0 \sup_{k} k^{\frac{1}{p}} a_k(vTu) \leq c_0 \gamma_{p,RC}(T) \|u\|_{cb} \varepsilon_2(v) \leq c_0 b_0 \|S\|_{cb} \gamma_{p,RC}(T),$$

where $b_0 \leq 4\sqrt{2}$ is the constant from proposition 2.6. In order to prove $i) \Rightarrow ii)$ we will use the notion of Grothendieck numbers for an operator $R : X \to Y$ introduced by S. Geiss.

$$\Gamma_n(R) := \sup \left\{ |\text{det}((R(x_i), y_j))_{ij}|^{\frac{1}{n}} : (x_i)_{i}^n \subset B_X, (y_j)_{j}^n \subset B_Y \right\}.$$  

Using an inequality of [DJ1], we have to show that

$$\sup_{n} n^{\frac{1}{p} - \frac{1}{2}} \Gamma_n(Tu) \leq c_2 \|u\|_{cb}$$

for all operator $u : R + C \to E$. By the definition of the Grothendieck numbers we have to consider elements $(y_k^n)_{1}^n \subset B_{F^n}$ and $v := \sum_{1}^{n} y_k^n \otimes e_k : F \to \ell_1^n$ which is of norm 1. If $\iota_{2,\infty} : \ell_1^n \to \ell_2^n$ denotes the canonical inclusion map we have to show

$$\Gamma_n(\iota_{2,\infty} vTu) \leq c_2 n^{\frac{1}{p} - \frac{1}{2}} \|u\|_{cb}.$$  

Now let $w : \ell_2^n \to H$ such that

$$\sum_{1}^{n} a_j(\iota_{2,\infty} vTu) = tr(\iota_{2,\infty} vTu)w.$$  

Using basic properties of Grothendieck numbers, see [GE1] and the geometric/arithmetic mean inequality we get for $S := uvw \iota_{2,\infty} : \ell_1^n \to E$

$$\Gamma_n(\iota_{2,\infty} vTu) \leq \left( \prod_{1}^{n} a_j(\iota_{2,\infty} vTu) \right)^{\frac{1}{n}} \leq \frac{1}{n} \sum_{1}^{n} a_j(\iota_{2,\infty} vTu) = \frac{1}{n} |tr(vTS)| \leq \frac{1}{n} \sum_{1}^{n} \|TS(e_k)\| \sup_{i} \|y_k^n\| \leq c_1 n^{-\frac{1}{p}} \|S\|_{cb} \leq c_1 n^{-\frac{1}{p}} \varepsilon_2(\iota_{2,\infty}) \|wu : R_n \cap C_n \to E\|_{cb} \leq c_1 n^{\frac{1}{p} - \frac{1}{2}} \|u\|_{cb},$$

where we have used the homogeneity of the space $R_H \cap C_H$ and remark 1.2 to estimate the cb norm of $\iota_{2,\infty} : \ell_1^n \to R_n \cap C_n$. In the case $p = 1$ we have to estimate $\sigma_1(vTu)$ for a 1-summing $T$, absolutely 2 summing $v$ and completely bounded $u : R_n \cap C_n \to E$ By Pietsch’s factorization theorem, see [PIE], there is a factorization of $v = SR$, $S : \ell_2^n \to \ell_2$, with absolutely 2 summing $S$. Since $uvwS$ is completely bounded for all bounded $w$ we can use the definition to see that $TuwS$ is integral in the Banach space sense and hence the trace of $uvwTu = SRw$ can be estimated by the 1-summing norm. This gives the right estimate for trace class norm, and hence the eigenvalues of $ST$, provided $w$ is chosen by polar decomposition as above.

\[\square\]
Indeed, this duality concept was studied in the more general framework of $T_{\Psi}$. We want to indicate the connection to 1-summing operators in this context. Given an equivalent to $S$ It was shown by Pisier and Lust-Picard [LPP] that $\alpha(S)$ for all operators $S : F^{**} \to G^{*}$ which admit a factorization $S = vu$, $\pi_2(v) \leq 1$ and $\pi_2,R_C(u^{*}) \leq 1$. Indeed, this duality concept was studied in the more general framework of $\gamma$-norms by Pisier [PS]. We want to indicate the connection to 1-summing operators in this context. Given a 1-summing operator $T : E \subset G^{*} \to F$ we observe that $T$ corresponds by trace duality to a linear functional on $F^{*} \otimes_{\alpha} E$ where $\alpha(S) := \inf\{\pi_2(v) \parallel u : R \cap C \to E\parallel_{cb}\}$ and the infimum is taken over all factorizations $S = vu$. Since $F^{*} \otimes_{\alpha} E$ embeds isometrically into $F^{*} \otimes_{\alpha} G^{*}$ an application of Hahn-Banach yields a norm preserving functional on the whole tensor product, i.e. an extension $\hat{T} : G^{*} \to F^{**}$ of $T$, which is also 1-summing by theorem 2.7. As a consequence of the key inequality in [JP] and proposition 2.5 we deduce that for all $u : R_H \cap C_H \to G^{*}$ the cb-norm is equivalent to $\pi_{2,R_C}(u^{*})$. Therefore, we can apply the modification of Kwapien’s argument, see also [PS], to obtain a completely bounded factorization of $\hat{T} : G^{*} \to \min(F^{**})$ through $R_H \cap C_H$ for some Hilbert space $H$. Clearly, if $\hat{T}$ admits such a factorization it must be 1-summing and all these properties coincide due to the fact that $G$ is exact.

**Remark 2.8** A variant of Kwapien theorem for Hilbert space factorizing operators shows that an operator $T : G^{*} \to \min(F^{**})$ factors completely through $R_H \cap C_H$ if and only if $|tr(TS)| \leq C$ for all operators $S : F^{**} \to G^{*}$ which admit a factorization $S = vu$, $\pi_2(v) \leq 1$ and $\pi_2,R_C(u^{*}) \leq 1$. In particular, a maximal operator space satisfies one of the conditions above if and only if it is an extension $\hat{\pi}(\hat{T})$ of the key inequality in [JP] and proposition 2.5 we deduce that for all $u : R_H \cap C_H \to G^{*}$ the cb-norm is equivalent to $\pi_{2,R_C}(u^{*})$. Therefore, we can apply the modification of Kwapien’s argument, see also [PS], to obtain a completely bounded factorization of $\hat{T} : G^{*} \to \min(F^{**})$ through $R_H \cap C_H$ for some Hilbert space $H$. Clearly, if $\hat{T}$ admits such a factorization it must be 1-summing and all these properties coincide due to the fact that $G$ is exact.

**Corollary 2.9** Let $G$ be an exact operator space, $q : B(H)^{*} \to G^{*}$ the quotient map and $E \subset G^{*}$. The following conditions are equivalent

1. The Banach space $E$ is of cotype 2 and every bounded operator $u : c_0 \to E$ is completely bounded.

2. The Banach space $E$ is of cotype 2 and every operator $v : E \to R \cap C$ which admits a completely bounded extension $\hat{\pi} : G^{*} \to R \cap C$ is absolutely 1-summing.

3. There exists a constant $c > 0$, such that for every sequence $(x_{ik})_{i} \subset E$ there is a sequence $(\tilde{x}_{ik})_{i} \subset B(H)^{*}$ such that $q(\tilde{x}_{ik}) = x_{ik}$ and

$$\mathbb{E} \left\| \sum_{1}^{n} \tilde{x}_{ik} \varepsilon_{k} \right\|_{B(H)^{*}} \leq c \mathbb{E} \left\| \sum_{1}^{n} x_{ik} \varepsilon_{k} \right\|_{E}.$$ 

In particular, a maximal operator space satisfies one of the conditions above if and only if it is of operator cotype 2 if and only if it is a G.T. space of cotype 2, see [PS].

**Proof:** Let $X$ be a Banach space we define $Rad(X) \subset L_{2}(\mathbb{D}, X)$ to be the span of $\{\varepsilon_{i} \otimes x_{i}\}$, where $\mathbb{D} = \{-1, 1\}^{\mathbb{N}}$ is the group of signs with its Haar measure $\mu$ and $\varepsilon_{i}$ the i-th coordinate. For a sequence $(x_{i})_{i}$ the norm in $Rad(X)$ is given by

$$\|(x_{i})_{i}\| := \left( \int_{\mathbb{D}} \left\| \sum_{1}^{n} \varepsilon_{i} x_{i} \right\|_{X}^{2} \, d\mu \right)^{\frac{1}{2}}.$$

It was shown by Pisier and Lust-Picard [LPP] that $Rad(B(H)^{*})$ and $(R + C)(B(H)^{*})$ have equivalent norms. Since the map $Id_{R+C} \otimes q$ is a complete quotient map, condition $iii)$ is equivalent to

$$\|Id \otimes \iota_{E} : Rad(E) \to (R + C)(G^{*})\| < \infty,$$

where $\iota_{E} : E \to G^{*}$ is the inclusion map. We deduce from theorem 2.7 and remark 2.8 that condition $i)$ and $ii)$ are equivalent by trace duality. Moreover, all conditions imply that $E$ is of
cotype 2, since $\mathcal{B}(H)^*$ is of cotype 2, [104]. Now let $v := \sum_i x_i^* \otimes e_i$ be an operator from $E$ to $R \cap C$. We deduce from [PSL, 5.16]

$$\frac{1}{C_1(E)} \pi_1(v) \leq \pi_2(v) \leq \|x_i^*\|_{(\text{Rad}(E))^*} \leq C_2(E) \pi_2(v) \leq C_2(E) \pi_1(v),$$

where $C_2(E)$ is the cotype 2 constant of $E$ and $C_1(G)$ only depends on $C_2(E)$. Finally we note that $CB(G^*, R \cap C) = (R \cap C)(G^{**})$. But this means that the set of operators admitting a cb extension can be identified with the dual space of $(R + C)^{inj}(E) := (\text{Id} \otimes \iota_E)^{-1}(R + C)(G^*)$. Therefore condition $\text{i}i)$ is equivalent to

$$\left\| \text{Id} \otimes \text{Id}_G : (R + C)^{inj}(E) \to (\text{Rad}(E))^* \right\| < \infty.$$ 

Duality implies the assertion. In the situation of maximal operator spaces we deduce from remark [L5] that a maximal operator space $X = \ell_1(I)/S$ with operator cotype 2 satisfies condition $\text{i}$ whereas $\text{iii}$) implies operator cotype 2 since $\ell_1(I)$ has operator cotype 2. (This seems not to be the case for $S_1(H)$.)

In the last part we will study the operator spaces associated to the Clifford algebra. Recalling that the generators of the Clifford algebra have already been useful to find an example of a $(2, oh)$-summing space, see [PSO], it is probably not surprising that this space is also 1-summing. More precisely, let $(u_i)_{i \in \mathbb{N}} \subseteq \bigoplus_{n \in \mathbb{N}} M_2$ be the generators of the Clifford algebra, i.e.

$$u_i = u_i^* \quad \text{and} \quad u_i^2 = \text{Id} \quad \text{for } i \in \mathbb{N},$$

$$u_i u_j + u_j u_i = 0 \quad \text{if } i \neq j.$$

By $\text{CL}$ we denote of the span of these generators. The next proposition collects some facts about this space. ($\text{OH}$ is the operator Hilbert space introduced and studied by Pisier [PSO]).

**Proposition 2.10** 1. $\text{CL}$ is $\sqrt{2}$ isomorphic to a Hilbert space.

2. The identity $id_{\text{CL}}$ is 1-summing with $\pi_1, cb(T) \leq 2$ and for every operator $T : \text{CL} \to \text{CL}$ we have

$$\sum_k |\lambda_k(T)| \leq c_0 \gamma^c_\infty(T).$$

3. Let $G \in \{\text{OH}, C, R, C + R, R \cap C\}$ and $u : G \to \text{CL}$ then one has

$$\|u\|_{cb} \sim_c \pi_2(u).$$

**Proof:** By approximation it is sufficient to consider the finite dimensional case. Therefore we fix $n \in \mathbb{N}$ and $u_1, ..., u_n \in \bigotimes_{k=1}^n M_2 \simeq M_{2^n}$. For an element $x = \sum_j \alpha_j u_j$ we have

$$x^* x + xx^* = \sum_{kj} \alpha_k \alpha_j u_k u_j + \sum_{jk} \alpha_j \alpha_k u_j u_k$$

$$= 2 \sum_1^n |\alpha_k|^2 u_k^2 + \sum_{k<n} \alpha_k \alpha_j (u_k u_j + u_j u_k) + \sum_{k>j} \alpha_k \alpha_j (u_k u_j + u_j u_k)$$

$$= 2 \|\alpha\|^2 Id.$$
In particular, we get
\[ \|\alpha\|_2 = \left\| \frac{x^* x + xx^*}{2} \right\|_{M_{2^n}}^{\frac{1}{2}} \leq \|x\|_{M_{2^n}} \leq \sqrt{2} \left\| \frac{x^* x + xx^*}{2} \right\|^{\frac{1}{2}} = \sqrt{2} \|\alpha\|_2 \, . \]

This is the first assertion. In order to estimate the 1-summing norm we define \( \hat{x} = \begin{pmatrix} x & 0 \end{pmatrix} \) in \( S_1^{2n+1} \). With the triangle inequality in \( S_1^{2n+1} \) we get
\[ 2^n \|\alpha\|_2 = \left\| \frac{x^* x + xx^*}{2} \right\|_1^{\frac{1}{2}} = \frac{1}{\sqrt{2}} \left\| \hat{x} \right\|_{S_1^{2n+1}} \leq \sqrt{2} \|x\|_{S_1^n} \, . \]

Combining these estimates we have found a factorization of the identity on \( CL^n \) through the restriction of \( 2^{-n}Id : M_{2^n} \to S_1^{2n} \) on \( E \). By proposition [13] the 1-summing norm of identity on \( CL^n \) is at most 2. As a consequence every operator \( T : CL \to CL \) which factors completely through a \( C(K) \) space is integral and since \( CL \) is isomorphic to a Hilbert space the eigenvalues are absolutely summing. To prove 3. let \( u : R \cap C \to CL \). In order to show that this operator is absolutely 2-summing we use trace duality. For this let \( v : CL \to R \cap C \) which is absolutely 2-summing. By Pietsch factorization theorem \( v \) factors through a 2-summing operator \( S : C(K) \to R \cap C \), which is completely bounded, see [12]. Since \( CL \) is 1-summing the composition \( Su \) is integral we get the right estimate for the trace. Vice versa, we consider an absolutely 2 summing operator \( u : R + C \to CL \). All the underlying Banach spaces are isomorphic to Hilbert spaces and therefore \( u \) admits a factorization \( u = vv^*, v^* \) absolutely 2-summing and \( w : \ell_1 \to CL \). This operator \( w \) is automatically completely bounded, whereas \( w \) is completely bounded in view of [12] and duality.

Now we will construct operator spaces \( E_r, E_r^n \) which are isomorphic to \( \ell_2, \ell_2^n \), respectively, but the 1 summing norm has a certain growth rate. For \( 1 < r < 2 \) we define a matrix structure on \( \ell_2, (\ell_2^n) \) as follows
\[ \|(x_{ij})\|_r := \sup_{k \in \mathbb{N}} k^{\frac{1}{r} - 1} \sup \left\{ \|(P_H x_{ij})\|_{CL} : \right. \]
\[ \left. H \subset \ell_2, (H \subset \ell_2^n) \dim H \leq k \right\} , \]
where we identify \( CL \) and \( \ell_2 \) via the isomorphism from proposition [2.10] and \( P_H : \ell_2 \to H \) denotes the orthogonal projection on \( H \). The next proposition states the properties of this operator spaces.

**Proposition 2.11** Let \( 1 < r < 2 < p < \infty \) with \( \frac{1}{r} = \frac{1}{2} + \frac{1}{p} \).

i) \( E_r \) is an operator space which is 2 isomorphic to \( \ell_2 \).

ii) For all \( n \in \mathbb{N} \) one has \( \pi^n_{1,cb}(id_{E_r}) \sim_2 n^{\frac{1}{r}} \).

iii) For all completely \( \infty \)-factorable operators \( T \in \Gamma_\infty(E_r, E_r) \) one has
\[ \sup_{n \in \mathbb{N}} n^{\frac{1}{r}} |\lambda_n(T)| \leq c_r \gamma^{\infty}_\infty(T : \min(E_r) \to E_r) \, . \]

iv) For the completely bounded operators with values in \( E_r \) and defined on \( \ell_\infty \) or \( G \in \{R, C, R+C, R \cap C, OH\} \) one has
\[ CB(\ell_\infty, E_r) = L^{(a)}_{p,\infty}(\ell_\infty, E_r) \quad \text{and} \quad CB(G, E_r) = L^{(a)}_{p,\infty}(OH, E_r) \, . \]

A similar statement holds uniformly in \( n \) for the spaces \( E_r^n \).
Remark 2.12 An easy modification of the spaces above allows us to construct an operator with $1$-summing. For $i$ we note that by definition and the fact that $CL$ is $1$ summing we have

$$
\|T : \ell_\infty \to CL\|_{cb} \sim 2 \sup_{k \in \mathbb{N}, \dim H \leq k} k^\frac{1}{p} \sigma_1(P_H T).
$$

For an operator $u : \ell_2 \to \ell_\infty$ we deduce by Schmidt decomposition

$$
\sup_k k^\frac{1}{p} a_k(Tu) \leq \sup_{k \in \mathbb{N}, \dim H \leq k} k^\frac{1}{p} - 1 \sigma_1(P_H Tu) \leq 2 \|T\|\|u\|.
$$

For the converse implication we use an interpolation argument. Indeed, by standard relations between different $s$-numbers, $[\text{PIE}]$, one has

$$
\mathcal{L}_{r,\infty}^{(x)} \subset \mathcal{L}_{p,\infty}^{(a)} \subset (\mathcal{L}_{2,1}^{(a)}, \mathcal{L}_{\infty,\infty}^{(a)})_{\theta,\infty},
$$

with $\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{\infty}$ and $\ell_{2,1}^{(a)}(T)$ is the norm of the approximation numbers in the Lorentz spaces $\ell_{2,1}$. By definition of the $K_t$ functional for $t = \sqrt{k}$ we can find a decomposition $T = T_1 + T_2$ such that $\ell_{2,1}^{(a)}(T_1) + \sqrt{k} \|T_2\| \leq c_p k^{\frac{1}{2} - \frac{1}{p}} \ell_{r,\infty}^{(a)}(T)$. An application of "little Grothendieck’s inequality", $[\text{FSI}]$, gives $\iota(T_1) \leq c_1 \ell_{2,1}^{(a)}(T_1)$. Hence we get for every $k$ dimensional subspace $H$

$$
\iota(P_H T) \leq \iota(P_H T_1) + \iota(P_H T_2) \leq \iota(T) + \sqrt{k} \pi_2(T_2)
\leq (c_1 + \frac{2}{\sqrt{\pi}}) \left( \ell_{2,1}^{(a)}(T_1) + \sqrt{k} \|P_H T_2\| \right) \leq c_p (c_1 + \frac{2}{\sqrt{\pi}}) k^{1 - \frac{1}{p}} \ell_{r,\infty}^{(a)}(T).
$$

The second formula is proved along the same line, although Grothendieck’s inequality is not used in this argument. The key point here is the following formula which we deduce from proposition 2.10

$$
\|T : G \to E_r\|_{cb} = \sup_{k \in \mathbb{N}} k^{\frac{1}{p}} \sup_{H, \dim(H) \leq k} \|P_H T : G \to CL\|_{cb}
\sim_c \sup_{k \in \mathbb{N}} k^{\frac{1}{p} - 1} \sup_{H, \dim(H) \leq k} \pi_2(P_H T : G \to CL).
$$

Remark 2.12 An easy modification of the spaces above allows us to construct an operator space $E_1$ such that the identity is not $1$-summing, but

$$
\sup_{n \in \mathbb{N}} n |\lambda_n(T)| \leq c_0 \gamma_{\infty}^c(T)
$$

for $T \in \Gamma_\infty^c(E_1, E_1)$. In fact, we define the matrix norm on $E_1$ by

$$
\|(x_{ij})\| := \sup_{k \in \mathbb{N}} \sup_{H \subset \ell_2, \dim(H) \leq k} \|P_H x_{ij}\|_{CL}
\text{ where } H_k := \text{span} \{e_j | k \leq j \}.
$$

Using similar arguments as above we can prove

$$
CB(\ell_\infty, E_1) \subset \mathcal{L}_{1,\infty}^{(x)}(\ell_\infty, E_1),
$$

and the diagonal operator $D_\sigma \in \mathcal{B}(\ell_\infty, E_1)$ defined by $\sigma_k = \frac{1}{k}$ is completely bounded, but not $1$-summing.
Example 2.13 At this point we want to give a review of infinite dimensional operator spaces such that \( \pi_{1,cb}^{\nabla_1} \leq n^{1-\frac{1}{1-r}} \) for some \( 1 < r < 2 \). By theorem 2.7 it is easy to see that this holds for max(\( X \)) if and only if \( X \) is a so called weak \( r \) Hilbertian Banach spaces, see [PI3], [GEI] and [DJ1]. Standard examples are obtained by interpolation \( X = (H,Y)_\theta = (1-\theta)^{-1}H + \theta^{-1}Y \), where \( H \) is a Hilbert space and \( Y \) an arbitrary Banach space. Therefore, max(\( \ell_\theta \)) and max(\( \ell_\theta' \)) are typical examples, but also max(\( \mathcal{S}_r \)) and max(\( \mathcal{S}_r' \)). Moreover, we see that \( \pi_{1,cb}^{\nabla_1}(Id_{\text{max}(E)}) \leq n^{1-\frac{1}{1-r}} \) if and only if the same holds for max(\( E' \)). In the limit case \( r = 1 \) the identity of a maximal operator space is 1-summing if and only if the associated Banach space is isomorphic to a Hilbert space, whereas a subspace of max(\( E \)) is 1-summing if and only if it is a complemented Hilbert space in \( E \).

It is easy to see that the operator space \( CL \) spanned by the generators of the Clifford algebra is an exact operator space. Moreover, the exactness constant [PSE] is uniformly in \( n \) bounded for the spaces \( E^n_r \). Using theorem 2.7 and the last proposition it is quite standard to deduce that the operator space dual \( CL^* \) is not 1-summing, but \( \pi_{1,cb}^k(id_{\text{max}(E^*)}) \sim_{cr} k^{\frac{1}{2}} \) for \( k \leq n \).

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