Superspace approach to anomalous dimensions 
in $\mathcal{N} = 4$ SYM

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ABSTRACT

In a $\mathcal{N} = 1$ superspace setup and using dimensional regularization, we give a general and simple prescription to compute anomalous dimensions of composite operators in $\mathcal{N} = 4$, $SU(N)$ supersymmetric Yang-Mills theory, perturbatively in the coupling constant $g$. We show in general that anomalous dimensions are responsible for the appearance of higher order poles in the perturbative expansion of the two–point function and that their lowest contribution can be read directly from the coefficient of the $1/\epsilon^2$ pole. As a check of our procedure we rederive the anomalous dimension of the Konishi superfield at order $g^2$. We then apply this procedure to the case of the double trace, dimension 4, superfield in the $20$ of $SU(4)$ recently considered in the literature. We find that its anomalous dimension vanishes for all $N$ in agreement with previous results.

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1 Introduction

Recently there has been a renewal of interest in four dimensional conformal field theories. In particular, new results are now available in the context of $\mathcal{N} = 4$ $SU(N)$ supersymmetric Yang-Mills theory, mainly because of its connection with the AdS/CFT correspondence [1]. The existence of non-renormalization theorems for the class of the so-called Chiral Primary Operators (CPO) has been suggested on the basis of the correspondence [2] and subsequently proven at perturbative and nonperturbative level: these operators have no anomalous dimension and their two and three point functions are protected from getting quantum corrections [3, 4, 5].

The operators of the theory have been recently classified according to the behavior of their anomalous dimensions in the strong coupling regime and in the large-$N$ limit [6]. In this contest it is important to investigate anomalous dimensions also in the perturbative regime and for any value of $N$.

The anomalous dimension of a composite operator $\mathcal{O}$ can be determined by the renormalization of the source term $\mathcal{J}\mathcal{O}$ appearing in the generating functional of correlation functions with insertions of the composite operator. Performing a quantum–background splitting, one then needs evaluate order by order Feynman diagrams with the external background structure $\mathcal{J}\mathcal{O}$. However, in the case of theories with a rich spectrum of composite operators, this procedure is often not very convenient from a technical point of view. This is certainly the case of $\mathcal{N} = 4$ $SU(N)$ supersymmetric Yang–Mills theory when considering operators that are not flavor or colour singlets. In these cases it can be more efficient to infer anomalous dimensions from the divergences of Green’s functions [7, 6, 8].

In this paper we develop a simple and direct method to compute perturbatively anomalous dimensions of composite operators from their two–point function. The method is general and can be applied to any superconformal field theory in four dimensions. Given a composite operator, we compute its two–point function order by order in perturbation theory. We use dimensional regularization to deal with UV divergences which then appear as poles in the regularization parameter $\epsilon$. The conformal invariance of the theory constrains the result to have at most simple pole divergences which are associated to the short distance singularity of the correlation function. However, we show that in general a nontrivial renormalization for the operator at hand (i.e. the presence of nonvanishing anomalous dimensions) is responsible for the appearance of higher order poles in the perturbative expansion of the two-point function, much like it produces logarithmic contributions in a point–splitting regularization [8, 9]. The search for higher order poles in the two–point correlators represents a simple but strong criterium to select operators which are subject to renormalization. Moreover, we show that the anomalous dimension can be read from the coefficient of the $1/\epsilon^2$ pole.

We apply our procedure to composite operators of $\mathcal{N} = 4$ $SU(N)$ supersymmetric Yang–Mills theory in a $\mathcal{N} = 1$ superspace description. As a check of our method we rederive the anomalous dimension of the $N = 1$ Konishi superfield at order $g^2$ in perturbation theory, obtaining a result in agreement with the known value [7, 10].
We then determine the anomalous dimension of the double trace, dimension 4 operator, belonging to the \(20\) of the R-symmetry group \(SU(4)\) appearing in the product of two dimension 2 CPO’s. This operator has been found to be protected in the large-\(N\) limit \([3, 11]\), and its protection for \(N\) finite has also been conjectured \([11, 12]\), despite the fact that the group unitarity arguments of \([13]\) do not prevent it from acquiring anomalous dimensions (see however \([14]\)). This unexpected result has been found in a rather indirect way, studying the partial non-renormalization of the four point functions of CPO’s. In this paper we check in a direct and simple manner the protection of this operator. We find that the anomalous dimension of the double trace operator written in terms of \(N = 1\) superfields vanishes for all values of \(N\) in agreement with the previous results.

The paper is organized as follows. In Section 2 we describe the general procedure that allows to compute the anomalous dimensions of composite operators in \(\mathcal{N} = 4\) \(SU(N)\) supersymmetric Yang-Mills theory, by using a description in terms of \(\mathcal{N} = 1\) superfields. In Section 3 we make use of this procedure to reproduce the known value of the anomalous dimension of the Konishi operator at order \(g^2\). In Section 4 we compute the anomalous dimension of the double trace operator \(O^{(20)}_{ij}\) at order \(g^2\) and we find zero. The meaning of our result and its implications are then discussed in the concluding Section 5. We have found convenient to add two Appendices. The first one collects our conventions, basic rules and useful identities to perform pertubative calculations. Appendix B describes in details the construction of the double trace \(O^{(20)}_{ij}\) operator in terms of \(\mathcal{N} = 1\) superfields.

## 2 The general prescription to compute anomalous dimensions

\(\mathcal{N} = 4\) supersymmetric Yang-Mills theory describes the dynamics of a spin-1 Yang-Mills vector, four spin-\(\frac{1}{2}\) Majorana spinors and six spin-0 particles in the \(6\) of the R–symmetry group \(SU(4)\), all the fields being in the adjoint representation of the \(SU(N)\) gauge group.

A description convenient for perturbative calculations is the \(\mathcal{N} = 1\) superspace description where the field content of the theory is given in terms of one real vector superfield \(V\) and three chiral superfields \(\Phi^i\) containing the six scalars organized into the \(3 \times \bar{3}\) of \(SU(3) \subset SU(4)\). The classical action is (we use notations of \([15]\))

\[
S = \int d^8 z \, \text{Tr} \left( e^{-gV} \Phi_i e^{gV} \Phi^i \right) + \frac{1}{2g^2} \int d^6 z \, \text{Tr} W^\alpha W_\alpha \\
+ \frac{ig}{3!} \int d^6 z \, \epsilon_{ijk} \Phi^i [\Phi^j, \Phi^k] + \frac{ig}{3!} \int d^6 \bar{z} \, \epsilon_{ijk} \Phi^i [\bar{\Phi}^j, \bar{\Phi}^k]
\]

where \(W_\alpha = i \bar{D}^2 (e^{-gV} D_\alpha e^{gV})\), and \(V = V^\alpha T^\alpha\), \(\Phi_i = \Phi^a T^a\), \(T^\alpha\) being \(SU(N)\) matrices in the fundamental representation. The theory is known to be finite and then sits at its conformal fixed point. Possible divergences can arise only in correlators of composite operators.
In Euclidean space, we introduce the generating functional

$$W[J, \bar{J}] = \int D\Phi \ D\bar{\Phi} \ DV \ e^{S + \int d^4z (\bar{J}O + J\bar{O})} \quad (2.2)$$

for the $n$–point functions of the generically complex composite operator $O$

$$\langle O(z_1) \cdots \bar{O}(z_n) \rangle = \frac{\delta^n W}{\delta J(z_1) \cdots \delta \bar{J}(z_n)} \bigg|_{J=\bar{J}=0} \quad (2.3)$$

where $z \equiv (x, \theta, \bar{\theta})$. The perturbative evaluation of the $n$–point function is equivalent to computing the contributions to $W[J, \bar{J}]$ at order $n$ in the sources. In particular, for the two–point function we need evaluate the quadratic terms

$$W[J, \bar{J}] \rightarrow \int d^4x_1 \ d^4x_2 \ d^4\theta \ J(x_1, \theta, \bar{\theta}) \ F(g^2, N) \ \mathcal{P}(D_\alpha, D_{\bar{\alpha}}) J(x_2, \theta, \bar{\theta}) \quad (2.4)$$

where we have defined $\gamma_0$ to be the naive dimensions of the operator, while $\gamma(g)$ is the anomalous dimension ($\gamma(g = 0) = 0$). The particular dependence on the coordinates is fixed by the conformal invariance of the theory, $\mathcal{P}(D_\alpha, D_{\bar{\alpha}})$ is an operatorial expression built up with spinorial derivatives and $F(g^2, N)$ is a function of the coupling constant which can be determined perturbatively.

To deal with possible divergences in the correlation functions we perform dimensional regularization by analytically continuing the theory to $n$ dimensions, $n = 4 - 2\epsilon$. We find convenient to work in momentum space where a generic power factor $(x^2)^{-\nu}$ is given by

$$\frac{1}{(x^2)^\nu} = 2^{n-2\nu} \pi^{\frac{n}{2}} \frac{\Gamma(\frac{n}{2} - \nu)}{\Gamma(\nu)} \int \frac{d^np}{(2\pi)^n} \frac{e^{-ipx}}{(p^2)^{\frac{n}{2}-\nu}} \quad \nu \neq 0, -1, -2, \cdots$$

$$\nu \neq \frac{n}{2}, \frac{n}{2} + 1, \cdots \quad (2.5)$$

For any power $\nu \geq 2$ the factor $\Gamma(\frac{n}{2} - \nu) = \Gamma(-\nu + 2 - \epsilon)$, when analytically continued to the region of positive arguments, develops a $1/\epsilon$ singularity which encodes the divergent behavior of $(x^2)^{-\nu}$ at short distances. In performing perturbative calculations in momentum space this is the leading behavior which is sufficient to look for when computing two–point correlation functions for CPO’s \[4\]. As extensively discussed in \[5\] finite contributions in momentum space would correspond in $x$–space to terms proportional to $\epsilon$ which give rise only to contact terms.

In this framework, we now concentrate on the perturbative evaluation of the anomalous dimensions $\gamma$. According to the standard definition in quantum field theory, a composite operator acquires anomalous dimension as a consequence of a renormalization which has to be implemented in order to cancel divergences of correlation functions containing insertions of that operator. The simplest way to compute these divergences is to evaluate perturbatively divergent contributions to the source term $J\bar{O}$ in the generating functional \[2.2\]. In dimensional regularization, divergences show up as poles $1/\epsilon^k$ and they can be
cancelled by adding suitable counterterms to the original action in (2.2). Performing minimal subtraction scheme, one ends up with the general structure for the source term given by

\[ \int d^8 z \ J \mathcal{O} \to \int d^8 z \ J \mathcal{O} \left( 1 + \sum_{k=0}^{\infty} \frac{a_k(g)}{\epsilon^k} \right) \]  

(2.6)

Defining the bare operator

\[ \mathcal{O}_B \equiv \mathcal{O} \left( 1 + \sum_{k=0}^{\infty} \frac{a_k(g)}{\epsilon^k} \right) \equiv \mathcal{O} Z \]  

(2.7)

the anomalous dimension is given by

\[ \gamma \equiv -\mu \frac{\partial}{\partial \mu} \log Z = \alpha d_{a_1} (\alpha) \]

(2.8)

where \( \mu \) is the mass scale of dimensional regularization and \( \alpha \equiv \frac{g^2}{4\pi} \) the effective coupling of the theory. It is important to notice that, being the theory finite, no renormalization of the fundamental fields is present, so that the renormalization function \( Z \) contains only divergences arising from Feynman diagrams with insertions of the composite operator.

In order to find the anomalous dimension of \( \mathcal{O} \) it is therefore sufficient to determine order by order in \( \alpha \) the simple pole divergences of Feynman diagrams with \( J \) and \( \mathcal{O} \) on the external legs. However, this approach can be sometimes difficult to apply, especially when dealing with higher dimensional composite operators which are not flavor or colour singlets. In this cases it is more convenient to evaluate the anomalous dimension by computing perturbative contributions to singlet structures constructed from the operator at hand. The simplest singlet we can compute is the two–point function of the operator with itself (or its hermitian conjugate if it complex). We are then going to discuss in details how anomalous dimensions can be easily read from the higher order poles in the two–point function.

We will be mainly concerned with the class of nonderivative operators which are functions of the fundamentals superfields \( \Phi, \bar{\Phi} \) and \( V \). For composite operators which are homogeneous polynomials of the scalar superfields the naive dimension \( \gamma_0 \) is simply the number of powers of scalars appearing in its expression (remember that in 4d the fundamental scalar superfields have dimension one, while \( V \) is dimensionless). In \( n \) dimensions, \( n = 4 - 2\epsilon \), the naive dimension of the operator is continued to the value \( \gamma_0 (1 - \epsilon) \). Therefore, the power factor appearing in (2.4), when using (2.5) is given by

\[ \frac{1}{(x^2)^{\gamma_0(1-\epsilon)+\gamma}} = 2^{4-\gamma_0-2\gamma+2\gamma_{(\gamma_0-1)}} \pi^{2-\epsilon} \frac{\Gamma(2-\gamma_0 - \gamma + \epsilon(\gamma_0 - 1))}{\Gamma(\gamma_0 (1 - \epsilon) + \gamma)} \int \frac{d^n p}{(2\pi)^n} \frac{e^{-ipx}}{(p^2)^{2-\gamma_0-\gamma+\epsilon(\gamma_0 - 1)}} \]  

(2.9)

For \( \gamma \) small, the gamma function \( \Gamma(2 - \gamma_0 - \gamma + \epsilon(\gamma_0 - 1)) \) is singular for \( \epsilon \to 0 \), whenever \( \gamma_0 \) is an integer greater or equal to 2. To extract this singularity we perform analytic
continuation by using the standard identity \(x \Gamma(x) = \Gamma(x + 1)\). We write

\[
\Gamma(2 - \gamma_0 - \gamma + \epsilon(\gamma_0 - 1)) = \frac{1}{(2 - \gamma_0 - \gamma + \epsilon(\gamma_0 - 1))(3 - \gamma_0 - \gamma + \epsilon(\gamma_0 - 1)) \cdots (-1 - \gamma + \epsilon(\gamma_0 - 1))}
\times \frac{1}{-\gamma + \epsilon(\gamma_0 - 1)} \Gamma(1 - \gamma + \epsilon(\gamma_0 - 1))
\]

(2.10)

When taking the limit \(\epsilon \to 0\) potential divergences can arise only from the factor \(1/(-\gamma + \epsilon(\gamma_0 - 1))\) for \(\gamma\) small. Keeping \(\epsilon\) finite, we expand this factor in powers of the anomalous dimension

\[
\frac{1}{-\gamma + \epsilon(\gamma_0 - 1)} = \frac{1}{\epsilon(\gamma_0 - 1)} \left[ 1 + \frac{\gamma}{\epsilon(\gamma_0 - 1)} + O \left( \left( \frac{\gamma}{\epsilon(\gamma_0 - 1)} \right)^2 \right) \right]
\]

(2.11)

Inserting back in (2.9) we can write

\[
\frac{1}{(x^2)^{\gamma_0(1-\epsilon)+\gamma}} \sim \pi^{2-\epsilon} \frac{\Gamma(1 - \gamma + \epsilon(\gamma_0 - 1))}{\Gamma(\gamma_0(1-\epsilon) + \gamma)} \times \frac{1}{\epsilon^{\gamma}} \frac{1}{\epsilon^{\gamma_0}} \frac{1}{\Gamma(\gamma_0(1-\epsilon))} \times \int \frac{d^4 p}{(2\pi)^4} \frac{e^{-ipx}}{\sin(\gamma(\gamma_0 - 1)(\gamma_0 - 1))}
\]

(2.12)

This relation immediately gives a general criteria to establish whether a composite operator has anomalous dimensions: The existence of a nonvanishing anomalous dimension is signaled by the presence of higher order poles in the \(\epsilon\)-expansion of the two-point function. On the contrary, whenever the operator has \(\gamma = 0\) the two-point function has at the most \(1/\epsilon\) poles at every order of perturbation theory. This is in agreement with the results found \([4, 5]\) in the perturbative evaluation of two-point functions for CPO’s beyond leading order, and proves the more general no-go theorem according to which the two-point correlator of any protected operator cannot diverge faster than \(1/\epsilon\), for \(\epsilon\) small.

In the case of nonvanishing anomalous dimensions, the identity (2.12) can be used to actually compute \(\gamma\). If we write the anomalous dimension as a perturbative expansion in the coupling constant

\[
\gamma = \sum_{n=1}^{\infty} a_n(g^2)^n
\]

(2.13)

the expression (2.12) can be organized as a double expansion in powers of \(g^2\) and \(\epsilon\). At zero order in the coupling constant the identity (2.12) gives the expected \(1/\epsilon\) divergence which accounts for the singular behavior of the correlation function at coincident points

\[
\frac{1}{(x^2)^{\gamma_0(1-\epsilon)}} \sim \frac{\pi^2}{\epsilon \Gamma(\gamma_0)(2 - \gamma_0)(3 - \gamma_0) \cdots (-1)(\gamma_0 - 1)} \times \int \frac{d^4 p}{(2\pi)^4} \frac{e^{-ipx}}{(p^2)^{2-\gamma_0}}
\]

(2.14)
At order $g^2$ we keep only linear terms in $\gamma$ ($\gamma = a_1 g^2$) and taking the limit $\epsilon \to 0$, the leading order behavior of the correlation function is given by

$$\frac{1}{(x^2)^{\gamma_0(1-\epsilon)+\gamma}} \sim \frac{\gamma}{\epsilon^2} \frac{\pi^2}{\Gamma(\gamma_0)} (2-\gamma_0)(3-\gamma_0) \cdots (-1)(\gamma_0-1)^2 \times \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ipx}}{(p^2)^{2-\gamma_0}} \quad (2.15)$$

The anomalous dimension is then given by $(\gamma_0 - 1)$ times the ratio of the coefficients of the $g^2$ and $g^0$ leading divergences.

Practically, we compute all the two–point diagrams contributing to $W[J, \bar{J}]$ with external $J$, $\bar{J}$ sources up to $g^2$–order, directly in momentum space. We obtain in general simple pole divergent contributions at tree level and $1/\epsilon^2$ poles at order $g^2$. We extract the coefficients of these two divergent terms and from their ratio we read the anomalous dimensions. Possible subleading divergences arising at order $g^2$ give nontrivial quantum corrections to the coefficient $F$ in (2.4).

The extension of our procedure to higher orders in $g^2$ is in principle trivial: At every order it is sufficient to determine the coefficient of the double pole divergence. However, since from the $g^4$–order on, this is expected to be a subleading divergence, one has to deal with possible scheme dependent contributions.

### 3 Anomalous dimension of the Konishi scalar superfield

As a check of our procedure we rederive in a manifestly supersymmetric way the known result [7, 10] for the lowest order contribution to the anomalous dimension of the Konishi operator. In $\mathcal{N} = 1$ superspace the gauge invariant Konishi superfield for $\mathcal{N} = 4$ SYM theory is

$$K = \text{Tr} \{ e^{-gV} \bar{\Phi}^i e^{gV} \Phi^i \} \quad (3.1)$$

where a sum over the $SU(3)$ flavor index is meant. We determine its anomalous dimension using both the standard approach which amounts to compute quantum corrections directly to the source term in (2.2) and our procedure which requires the evaluation of $1/\epsilon^2$ terms in the two–point function. In both cases the general strategy we follow is: we select diagrams contributing at order $g^2$ and compute all factors coming from combinatorics, colour and flavor structures. We then perform the superspace $D$-algebra following standard techniques so reducing the various contributions to multi-loop momentum integrals. We evaluate the divergent part of these integrals by using massless dimensional regularization and minimal subtraction scheme $\mathbb{3}$. We observe that gauge-fixing the classical action requires the introduction of corresponding Yang-Mills ghosts. However they only couple to the vector multiplet and do not enter our specific calculation.

Useful formulae for dealing with the colour and flavor structures and to compute the momentum integrals are collected in appendix A.

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3 Notice that, even if the theory is finite, multiproof diagrams containing insertions of anomalous composite operators do possess UV divergent subdiagrams.
To compute one–loop contributions to the source term $JK$ in the generating functional we first perform quantum–background splitting and concentrate on contributions with external background structure $J \text{Tr}\{\bar{\Phi}\Phi\}$. In fact, in order to compute the anomalous dimension of $K$ it is sufficient to consider contributions at zero order in the background $V$ since higher order contributions in $V$ will necessarily give the same answer due to gauge invariance.

One loop contributions are given by diagrams in Fig. 1 where $J$ is a classical real source. Performing the $D$–algebra on the diagrams (1a) and (1b) and keeping only potentially divergent contributions reduces them to self–energy type integrals

$$\int \frac{d^n k}{k^2(p - k)^2} \sim \frac{1}{\epsilon}$$

where the identity (A.3) has been used to select the leading behavior for $\epsilon \to 0$. No $D$–algebra must be performed on diagrams (c) and (d) which again give self–energy type contributions.

Figure 1: One–loop contributions to $J \text{Tr}\{\bar{\Phi}\Phi\}$. Dashed lines correspond to vector propagators

Computing the overall coefficient from combinatorics, coefficients from vertices and propagators and colour and flavor structures, we eventually have

$$(1a) \to \frac{2g^2N}{(4\pi)^2} \frac{1}{\epsilon}$$

$$(1b) \to -\frac{4g^2N}{(4\pi)^2} \frac{1}{\epsilon}$$
\[(1c) = (1d) \rightarrow -\frac{2g^2N}{(4\pi)^2} \frac{1}{\epsilon} \quad (3.3)\]

These divergent contributions can be cancelled by adding a suitable counterterm to the source term in the generating functional

\[
\int d^8z JK \rightarrow \int d^8z JK \left(1 + \frac{3g^2N}{8\pi^2} \frac{1}{\epsilon}\right) \quad (3.4)
\]

Comparing with eq. (2.6) and using the identity (2.8) we finally obtain \(\gamma = \frac{3g^2N}{8\pi^2}\), as expected.

We now use the alternative procedure to extract the anomalous dimension from the two–point correlation function. At tree level the contribution to the two–point function \(\langle K(z_1)K(z_2) \rangle\) is given in Fig. 2.

![Figure 2: Tree level contribution to \(\langle K(z_1)K(z_2) \rangle\)](image)

Performing the \(D\)-algebra and using eq. (A.4) we end up with two different contributions, both associated to self–energy type integrals (3.2) but with \(\bar{D}^2J\tilde{D}^2J\) and \(\bar{D}^aJi\partial_{\alpha\dot{\alpha}}D^aJ\) as external source terms. The colour and flavor structures which emerge after contraction are

\[\text{Tr}(T_aT_b)\text{Tr}(T_aT_b) \delta^i_j \delta^i_j = 3(N^2 - 1) \quad (3.5)\]

where the identity (A.3) have been used. Putting toghether with the combinatorics, the coefficients from vertices and propagators and the result (3.2) we finally obtain

\[W[J]_{tree} \rightarrow \frac{1}{\epsilon} \frac{3(N^2 - 1)}{2(4\pi)^2} \int \frac{d^4p}{(2\pi)^4} d^4\theta J(-p, \theta, \bar{\theta}) \left[\bar{D}^2D^2 + \frac{1}{2}p_{\alpha\dot{\alpha}}\bar{D}^\alpha\tilde{D}^\dot{\alpha}\right] J(p, \theta, \bar{\theta}) \quad (3.6)\]

The lowest order perturbative contribution is given by diagrams in Fig. 3. To draw diagrams (3a) and (3b) we only need the interaction vertices (A.2), whereas (3c) and (3d) also contain the external vertex \(gJ\text{Tr}([\bar{\Phi}_i, V]\Phi^i)\) from the expansion of the Konishi operator in powers of \(gV\).

\[\text{Note that there is a mismatch of a factor 2 compared to the result in [5, 10] due to the fact that our coupling constant is } \sqrt{2} \text{ times the one used there.}\]

\[\quad\]

8
We perform the $D$–algebra on the diagrams (3a), (3b) and (3c) (for the diagram (3d) no $D$–algebra is necessary) and select only contributions which potentially diverge as $1/\epsilon^2$. The effective diagrams after completion of $D$–algebra are shown in Fig. 4 where, by suitable integrations by parts on the external sources, diagrams on the first line are proportional to $\bar{D}^2 J D^2 J$ and the ones on the second line to $\bar{D}^\alpha J D^\alpha J$.

The first three diagrams reduce to the same momentum integral whose leading divergence for $\epsilon \to 0$ can be easily inferred by repeated use of identity (A.3)

$$I \equiv \int \frac{d^nk d^nq}{k^2(p-k)^2 q^2(q-k)^2} \sim \frac{1}{2 \epsilon^2} \quad (3.7)$$
These diagrams contain a divergent subdiagram associated with the $q$–loop. The subtraction of this subdivergence gives a contribution $-\frac{1}{\epsilon^2}$. Therefore, the actual leading divergence from these diagrams is

$$I - I_{\text{sub}} \sim -\frac{1}{2\epsilon^2} \quad (3.8)$$

The diagrams on the second line of Fig. 4 give rise to the momentum integral

$$\int \frac{d^nkd^nq}{k^2(p-k)^2q^2(q-k)^2} \quad (3.9)$$

Using identities (A.3, A.4), for $\epsilon \to 0$ its leading behavior is given by $\frac{1}{2\epsilon^2}p_{\alpha\dot{\alpha}}(I - I_{\text{sub}})$.

Evaluating the momentum integral associated with diagram (3d) gives a simple pole divergence $1/\epsilon$ (see (A.5)). Therefore, this diagram does not contribute to the anomalous dimension and we will neglect it in what follows.

At this point we need only compute the overall factor from combinatorics and group structures. Using identities in Appendix A, we find

$$(3a) : \quad 6g^2N(N^2 - 1)$$

$$(3b) : \quad -12g^2N(N^2 - 1)$$

$$(3c) : \quad -12g^2N(N^2 - 1) \quad (3.10)$$

Restoring a factor $1/(4\pi)^2$ for each loop (see Appendix A) the final leading contribution at order $g^2$ to the generating functional is

$$W[J]_{1-\text{loop}} \to \frac{1}{\epsilon^2} \cdot 9 \frac{g^2N}{(4\pi)^4}(N^2-1) \int \frac{d^4p}{(2\pi)^4}d^4\theta J(-p, \theta, \bar{\theta}) \left[ \bar{D}^2D^2 + \frac{1}{2}p_{\alpha\dot{\alpha}}ar{D}^{\dot{\alpha}}D^\alpha \right] J(p, \theta, \bar{\theta}) \quad (3.11)$$

According to the general prescription given in Section 2, the anomalous dimension at order $g^2$ is then given by the ratio of the coefficients in (3.11) and (3.6), times $(\gamma_0 - 1)$. Taking into account that for the Konishi operator $\gamma_0 = 2$, we finally obtain the correct result $\gamma = \frac{3g^2N}{8\pi^2}$.

Before closing this Section, we would like to make some comments on the connection between higher order poles in $\epsilon$ and the appearance of a nonvanishing anomalous dimension. From our calculation it is clear that $1/\epsilon^2$ poles can arise only from diagrams with self–energy subdiagrams containing only one external vertex (self–energy subdiagrams containing both, when subtracted, give rise to tadpole–like integrals which vanish in dimensional regularization). On the other hand, these subdiagrams are exactly the ones responsible for the renormalization of the composite operator (in our particular example one can easily convince by comparing the topologies of divergent subdiagrams in Fig. 4 with the ones in Fig 1 with $D$ algebra solved). This observation, together with the fact that multiple self–energy diagrams have only simple pole divergences (see eq. (A.5)), gives an intuitive explanation of why in the evaluation of two–point functions higher order poles emerge only in the presence of anomalous dimensions.
4 Anomalous dimension of the $O_{ij}^{(20)}$ scalar superfield

The gauge invariant $O_{ij}^{(20)}$ superfield built in Appendix B (see eq. (B.10))

\[
O_{ij}^{(20)} \equiv \text{Tr}(\Phi_k \Phi_i)\text{Tr}(e^{gV} \Phi_j)e^{-gV}\Phi_k - \frac{1}{3}\delta_{ij}e^{gV} \Phi_i e^{-gV} \Phi_j
\]  

contains, as lowest component, a double trace scalar operator of dimension 4 in the 20 of SU(4) which is expected not to renormalize perturbatively [6, 11, 12]. We apply our method to evaluate its anomalous dimension pertubatively in superspace with the purpose to give an independent check of the previous results.

To compute the two–point correlation function $\langle O_{ij}^{(20)}(z_1)\bar{O}^{(20)}_{i'j'}(z_2) \rangle$ we find convenient to separate the flavor contributions from the rest of the calculation. To this end we write the operator as

\[
O_{ij}^{(20)} = D_{(ij)}^{klpq} \text{Tr}(\Phi_k \Phi_l)\text{Tr}(e^{gV} \Phi_p e^{-gV} \Phi_q) \equiv D_{(ij)}^{klpq} O_{klpq}
\]  

with the flavor tensor $D_{(ij)}^{klpq}$ given by

\[
D_{(ij)}^{klpq} = \frac{1}{2}\delta^q(k \delta^l(i \delta^j)p) - \frac{1}{3}\delta^l(i \delta^j)(i \delta^j)k \delta^l
\]  

It is symmetric in $(kl)$ and satisfies the property

\[
D_{(ij)}^{klpq} \delta_{pq} = 0
\]  

We then compute $\langle O_{klpq}(z_1)\bar{O}^{k't'p'q'}_{i'j'}(z_2) \rangle$ both at tree level and one–loop and leave the contructions with the flavor tensors $D_{(ij)}^{klpq}$ and $D_{(i'j')}^{k't'p'q'}$ at the end.

The calculation follows the general recipe described in the previous Sections. At tree level the two–point function is given by the diagram in Fig. 5

![Diagram of tree level contribution to $\langle O_{ij}^{(20)}(z_1)\bar{O}^{(20)}_{i'j'}(z_2) \rangle$](image)

The D–algebra performed on this diagram is similar to the one for the Konishi superfield and eventually produces two independent superspace structures, $\bar{D}^2 J^{ij} D^2 J^{i'j'}$ associated to a multiple self–energy integral

\[
I \equiv \int \frac{d^n q_1 d^n q_2 d^n q_3}{q_1^2(q_2 - q_1)^2(q_3 - q_2)^2(p - q_3)^2}
\]
\[ I_{\dot{\alpha} \dot{\alpha}} \equiv \int \frac{d^n q_1 d^n q_2 d^n q_3 (q_1)_{\dot{\alpha} \dot{\alpha}}}{q_1^2 (q_2 - q_1)^2 (q_3 - q_2)^2 (p - q_3)^2} \]  

The momentum integral \((4.5)\) can be easily evaluated by making use of \((A.5)\) for \(k = 4\) and \(a = 0\). Its leading term is \(\frac{1}{3!} \frac{1}{(p^2)^2} \). For the second integral, we use identity \((A.6)\) to obtain \(I_{\dot{\alpha} \dot{\alpha}} \sim \frac{1}{4} p_{\dot{\alpha} \dot{\alpha}} I\) for \(\epsilon \to 0\).

Performing the flavor and colour combinatorics and using the normalization \((A.9)\) one finds

\[ \delta_{bb'} \sum_{\sigma} \delta_{aa'}^{\sigma} \delta_{kk'}^{\sigma} \equiv \mathcal{T} \]  

where \(a, b, a', b'\) are colour indices and the short notation \(\delta_{aa'}^{\sigma} \delta_{kk'}^{\sigma} \equiv \delta_{aa'}^{\sigma} \delta_{kk'}^{\sigma}\) has been used. A sum over all possible permutations of flavor and colour indices is explicitly indicated. The colour combinatorics can be easily evaluated. Using the labellings \((A.17)\) for the different flavor structures the previous result can be written as

\[ \mathcal{T} = (N^2 - 1)^2 (\mathcal{F}_1 + \mathcal{F}_4) + (N^2 - 1) (\mathcal{F}_2 + \mathcal{F}_3 + \mathcal{F}_5 + \mathcal{F}_6) \]  

As a last step we have to contract the previous expression with the external tensors \(D^{kpq}_{(ij)}\) and \(D^{k'p'q'}_{(i'j')}\). Making use of the identities \((A.19)\) we find

\[ \mathcal{D} \mathcal{T} \mathcal{D}' = \frac{20}{9} (N^2 - 1) (3N^2 - 2) (\delta_{ii'} \delta_{jj'} + \delta_{ij'} \delta_{i'j}) \]  

Therefore, reinserting a factor \(\frac{1}{(4\pi)^2}\) for each loop according to our conventions, we can finally write

\[ W[J, \bar{J}]_{tree} \to \frac{1}{\epsilon} \frac{1}{(3!)^2 (4\pi)^6} \frac{20}{9} (N^2 - 1) (3N^2 - 2) \int \frac{d^4 p}{(2\pi)^4} d^4 \theta (p^2)^2 J^{ij} (-p, \theta, \bar{\theta}) \left[ \bar{D}^2 D^2 + \frac{1}{4} p_{\dot{\alpha} \dot{\alpha}} D^\alpha D^\alpha \right] \bar{J}^{ij} (p, \theta, \bar{\theta}) \]  

We now concentrate on the one–loop contribution. All possible diagrams contributing at this order to the quadratic terms in \(W[J, \bar{J}]\) are listed in Fig. 6. Performing the \(D–\) algebra produces a huge amount of potentially divergent diagrams with different structures of spinorial derivatives acting on the external sources. However, using the results in \((A.3, A.6, A.7)\) allows to conclude that most of the diagrams have only simple pole divergences, so they can be neglected when computing anomalous dimensions. Moreover diagram 6g gives zero after summing over colour and flavor indices. The only nonvanishing diagrams which manifest a \(1/\epsilon^2\) pole are the diagrams which reduce, after \(D–\) algebra, to the topological structures in Fig. 7. Diagrams 7a, 7b and 7c are produced by 6a, 6c, 6d and 6e while 7d and 7e come from 6h.  

\[ \text{footnote} \]

\[ \text{We are very grateful to G. Rossi and Y. Stanev for having pointed us the nonvanishing contribution from diagram 6h.} \]
By suitable integration by parts the external field content of diagrams of type 7a and 7d can be written in the form $\bar{D}^2 J D^2 \bar{J}$, whereas diagrams in Figs. 7b, 7c and 7e are proportional to $\bar{D}^\alpha J D^\alpha \bar{J}$. The corresponding integrals, with momentum lines as given in
Fig. 8, are

\[ I_{7a} \equiv \int \frac{d^n q_1 \, d^n q_2 \, d^n q_3}{q_1^2 (q_2 - q_1)^2 (q_3 - q_2)^2 (p - q_3)^2} \int \frac{d^n k}{k^2 (q_1 - k)^2} \] (4.11)

\[ I_{7b} \equiv \int \frac{d^n q_1 \, d^n q_2 \, d^n q_3 \, (p - q_3)_{\alpha\dot{\alpha}}}{q_1^2 (q_2 - q_1)^2 (q_3 - q_2)^2 (p - q_3)^2} \int \frac{d^n k}{k^2 (q_1 - k)^2} \]

\[ I_{7c} \equiv \int \frac{d^n q_1 \, d^n q_2 \, d^n q_3 \, (q_1)_{\alpha\dot{\alpha}}}{q_1^2 (q_2 - q_1)^2 (q_3 - q_2)^2 (p - q_3)^2} \int \frac{d^n k}{k^2 (q_1 - k)^2} \]

and

\[ I_{7d} \equiv \int \frac{d^n q_2 \, d^n q_3}{(q_3 - q_2)^2 (p - q_3)^2} \int \frac{d^n k}{k^2 (q_2 - k)^2} \int \frac{d^n r}{r^2 (q_2 - r)^2} \] (4.12)

\[ I_{7e} \equiv \int \frac{d^n q_2 \, d^n q_3 \, (p - q_3)_{\alpha\dot{\alpha}}}{(q_3 - q_2)^2 (p - q_3)^2} \int \frac{d^n k}{k^2 (q_2 - k)^2} \int \frac{d^n r}{r^2 (q_2 - r)^2} \]

Figure 8: Momentum structures of leading divergent diagrams

The \( k \) and \( r \) self-energy integrals are given in (A.3), while the rest of the integrations can be easily performed with the help of identities (A.5,A.6).

For the first integral the result is

\[ I_{7a} \sim \frac{1}{\epsilon^2} \frac{1}{48} (p^2)^2 \quad , \quad \epsilon \to 0 \] (4.13)

The diagram contains a divergent subdiagram associated to the self-energy \( k \) integral. Its subtraction corresponds to an extra contribution \(-\frac{1}{\epsilon^2} \frac{1}{36} (p^2)^2\). The final leading term of the momentum integral (4.11) is then

\[ I_{7a} - I_{7a}_{\text{sub}} \sim \frac{1}{\epsilon^2} \frac{1}{4(3!)^2} (p^2)^2 \] (4.14)

whereas the leading behavior of the integrals \( I_{7b} \) and \( I_{7c} \) is simply given by \( \frac{1}{4} p_{\alpha\dot{\alpha}}(I_{7a} - I_{7a}_{\text{sub}}) \), so that the tree-level structure \( \bar{D}^2 D^2 + \frac{1}{4} p_{\alpha\dot{\alpha}} \bar{D}^\alpha D^\alpha \) is reproduced (see (1.10)).

Similarly, the integral (4.12) is given by

\[ I_{7d} \sim \frac{1}{\epsilon^2} \frac{1}{24} (p^2)^2 \quad , \quad \epsilon \to 0 \] (4.15)
and after the subtraction of the two $k$ and $r$ divergent subdiagrams

\[ I^{7d} - I^{7d}_{\text{sub}} \sim -\frac{1}{c^2} \frac{1}{2(3!)^2} (p^2)^2 \quad (4.16) \]

whereas the leading behavior of the integral $I^{7e}$ is given by $\frac{1}{4} p_{\alpha\dot{\alpha}} (I^{7d} - I^{7d}_{\text{sub}})$, so again the tree-level structure $\bar{D}^2 D^2 + \frac{1}{4} p_{\alpha\dot{\alpha}} \bar{D}^\alpha D^\alpha$ is reproduced.

We are now left with the evaluation of the combinatorial factors from each diagram. The combinatorics from diagrams 6a, 6d and 6e simply reduce to the tree level structure. Keeping also coefficients from vertices, propagators and combinatorics from $D$–algebra we have

\[
\begin{align*}
(6a) : & \quad 4g^2 N \mathcal{T} \\
(6d) : & \quad -4g^2 N \mathcal{T} \\
(6e) : & \quad -4g^2 N \mathcal{T} \quad (4.17)
\end{align*}
\]

where the tensor $\mathcal{T}$ has been defined in (4.11). The evaluation of the combinatorics for diagrams 6c and 6h is more involved. Indeed, in these cases the flavor indices from the internal vertices enter the expression through the product of two Levi–Civita tensors (see eq. (A.2)) and the use of the standard identity

\[ \epsilon_{ijk} \epsilon_{imn} = \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km} \quad (4.18) \]

produces new flavor structures which do not appear at tree level. A long but straightforward calculation gives

\[
\begin{align*}
(6c) : & \quad 4g^2 N \mathcal{T} \\
& \quad -4g^2 N (N^2 - 1) \left[ 2\mathcal{F}_1 - \mathcal{F}_2 - \mathcal{F}_3 - \mathcal{G}_1 - \mathcal{G}_2 + k \leftrightarrow l \right] \\
& \quad -4g^2 N (N^2 - 1)^2 \delta_{pq} \delta_{p'q'} (\delta_{kk'} + \delta_{kk'} \delta_{k'k'}) \\
& \quad -4g^2 N (N^2 - 1)^2 \delta_{pq} \delta_{l'l'} (\delta_{l'l'} + \delta_{l'l'} \delta_{k'k'}) \\
& \quad -4g^2 N (N^2 - 1)^2 \delta_{pq} \delta_{k'k'} (\delta_{l'l'} + \delta_{l'l'} \delta_{k'k'}) \\
& \quad -4g^2 N (N^2 - 1)^2 \delta_{pq} \delta_{k'k'} (\delta_{l'l'} + \delta_{l'l'} \delta_{k'k'}) \\
& \quad -4g^2 N (N^2 - 1)^2 \delta_{pq} \delta_{k'k'} (\delta_{l'l'} + \delta_{l'l'} \delta_{k'k'}) \\
& \quad -4g^2 N (N^2 - 1)^2 \delta_{pq} \delta_{k'k'} (\delta_{l'l'} + \delta_{l'l'} \delta_{k'k'}) \quad (4.19)
\end{align*}
\]

\[
\begin{align*}
(6h) : & \quad -2g^2 N (N^2 - 1) \left[ 2\mathcal{F}_1 - \mathcal{F}_2 - \mathcal{F}_3 + k \leftrightarrow l \right] \quad (4.20)
\end{align*}
\]

At this point it is easy to perform the contractions with the flavor tensors $\mathcal{D}$ and $\mathcal{D}'$. By exploiting the identity (4.4) we immediately realize that the last three terms in (4.19)
give vanishing contributions. By using the results in (A.19, A.20) we finally have

\[(6a) : \quad 4g^2N \, \mathcal{DTD}'\]
\[(6c) : \quad 4g^2N \, \mathcal{DTD}' + 4g^2N(N^2 - 1) \frac{100}{9} (\delta_{ii'}\delta_{jj'} + \delta_{ij'}\delta_{i'j})\]
\[(6d) : \quad -4g^2N \, \mathcal{DTD}'\]
\[(6e) : \quad -4g^2N \, \mathcal{DTD}'\]
\[(6h) : \quad -2g^2N(N^2 - 1) \frac{100}{9} (\delta_{ii'}\delta_{jj'} + \delta_{ij'}\delta_{i'j})\]

where \(\mathcal{DTD}'\) is given in (1.9).

At this point we take into account also the factors from the integrals, in particular we note that the integral (4.16) coming from diagram 6h has an extra factor of 2 with respect to the integral (4.14) coming from all the other diagrams. If we sum all the contributions we then have complete cancellation of the leading \(\mathcal{T}\)–terms, and also of the subleading terms produced by diagrams 6c and 6h, so we can conclude

\[W[J, \bar{J}]_{one-loop} = 0 \quad (4.22)\]

proving that at order \(g^2\) the anomalous dimension of the double trace operator \(O_{ij}^{(20)}\) vanishes, in agreement with previous results [6, 11, 12].

Our result gives further support to the belief that the double trace superfield (4.1) is the protected operator whose existence is expected on the basis of general arguments [11, 12, 14].

5 Conclusions

In this paper we have developed a rather simple procedure to compute anomalous dimensions directly from the two–point correlation functions. Our prescription works in any superconformal field theory in four dimensions and allows to read perturbative contributions to unprotected operators directly from few higher order poles divergent diagrams. We have applied this procedure to the particular case of \(\mathcal{N} = 4\) supersymmetric Yang–Mills theory with gauge group \(SU(N)\), written in \(\mathcal{N} = 1\) superspace. Our result represents the first direct superspace calculation of anomalous dimensions of operators other than Konishi [7, 10].

After a check of the method on the anomalous dimension of the Konishi superfield, we have evaluated at lowest order the anomalous dimension of the double trace superfield \(O_{ij}^{(20)}\) in the 20 of \(SU(4)\). The result is that at one–loop \(O_{ij}^{(20)}\) does not renormalize. This is in agreement with previous calculations done in components, but has the advantage to keep \(\mathcal{N} = 1\) supersymmetry manifest.

The way we carried on the calculation for \(O_{ij}^{(20)}\) allows for a quite immediate extension to other superfield representations, in particular in the decomposition (B.4). This subject will be extensively studied elsewhere [17].
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A Conventions and basic rules

In this Appendix we list the main ingredients we need to compute perturbative divergent contributions to two–point correlators as described in the main text.

The classical action for $\mathcal{N} = 4$ super Yang–Mills theory is given in (2.1). Its quantization requires the introduction of a gauge fixing and the corresponding ghosts. The ghost superfields only couple to the vector multiplet and do not enter our calculation at order $\hat{g}^2$. Working in Feynman gauge and momentum space the superfield propagators are

$$< V^a V^b > = - \frac{\delta^{ab}}{p^2} \quad < \Phi_i^a \bar{\Phi}_j^b > = \frac{\delta_{ij} \delta^{ab}}{p^2}$$ (A.1)

The vertices can be obtained from the interaction terms in (2.1). The ones that we need are the following

$$V_1 = ig f_{abc} \delta^{ij} \bar{\Phi}_i^a V^b_j \Phi^c \quad V_2 = - \frac{g}{3!} \epsilon^{ijk} f_{abc} \Phi^a_i \Phi^b_j \Phi^c_k$$

with additional $\bar{D}^2$, $D^2$ factors for chiral, antichiral lines respectively.

All the calculations are performed in $n$ dimensions, with $n = 4 - 2\epsilon$ and in momentum space. The momentum integrals can be computed by making repeated use of the one loop results

$$\int \frac{d^n k}{(k^2)^a [(p-k)^2]^b} = \frac{\Gamma(n-a-b) \Gamma(n-a-b) \Gamma(n-a-b)}{\Gamma(n-a-b) \Gamma(n-a-b) (p^2)^{a+b-\frac{4}{2}}}$$

(A.3)

$$\int \frac{d^n k}{(k^2)^a [(p-k)^2]^b} = \frac{1}{2} \frac{n-2a}{n-a-b} b^{\alpha \beta} \int \frac{d^n k}{(k^2)^a [(p-k)^2]^b}$$

(A.4)

In particular, the basic formulae are

$$I \equiv \int \frac{d^n q_1 \cdots d^n q_{k-1}}{(q_1^2)^{1+\alpha}(q_2 - q_1)^2(q_3 - q_2)^2 \cdots (p - q_{k-1})^2} =$$

$$= \frac{1}{(a+1)\epsilon} \int \frac{d^n q_2 \cdots d^n q_{k-1}}{(q_2^2)^{(a+1)}(q_3 - q_2)^2 \cdots (p - q_{k-1})^2} + \mathcal{O}(\epsilon^0) =$$

$$= \frac{1}{\epsilon} \frac{(-1)^k (k-1)}{[(k-1)!]^2 (k-1+a)} (p^2)^{k-2-(k-1+\alpha)} + \mathcal{O}(\epsilon^0)$$

(A.5)

and

$$\int \frac{d^n q_1 \cdots d^n q_{k-1} q_1^{\alpha \bar{\alpha}}}{(q_1^2)^{1+\alpha}(q_2 - q_1)^2(q_3 - q_2)^2 \cdots (p - q_{k-1})^2} \sim \frac{1}{k} p^{\alpha \bar{\alpha}} I$$

$$\int \frac{d^n q_1 \cdots d^n q_{k-1} (p - q_{k-1})^{\alpha \bar{\alpha}}}{(q_1^2)^{1+\alpha}(q_2 - q_1)^2(q_3 - q_2)^2 \cdots (p - q_{k-1})^2} =$$

$$= \frac{1}{(a+1)\epsilon} \int \frac{d^n q_2 \cdots d^n q_{k-1} (p - q_{k-1})^{\alpha \bar{\alpha}}}{(q_2^2)^{(a+1)}(q_3 - q_2)^2 \cdots (p - q_{k-1})^2} + \mathcal{O}(\epsilon^0) \sim \frac{1}{k} p^{\alpha \bar{\alpha}} I$$ (A.6)
where in the last two equations the leading behavior for $\epsilon \to 0$ has been extracted. Moreover, it is useful to remind the finiteness of the following two-loop integral

$$
\int \frac{d^n k \ d^n l}{k^2 l^2 (k - l)^2 (p - k)^2 (p - l)^2} = \frac{1}{(p^2)^{1+2\epsilon}} [6 \zeta(3) + \mathcal{O}(\epsilon)]
$$

where $T_a$ are the generators and $f_{abc}$ the structure constants. The matrices $T_a$'s are normalized as

$$
\text{Tr}(T_a T_b) = \delta_{ab}
$$

We specialize to the case of $SU(N)$ Lie algebra whose generators $T_a$, $a = 1, \cdots, N^2 - 1$ are taken in the fundamental representation, i.e. they are $N \times N$ traceless matrices. The basic relation which allows to deal with products of $T_a$'s is the following

$$
T_{ij}^a T_{kl}^a = \left( \delta_{il} \delta_{jk} - \frac{1}{N} \delta_{ij} \delta_{kl} \right).
$$

From this identity, together with (A.8), we can easily obtain all the identities used to compute the colour structures associated to the Feynman diagrams relevant for the two-point correlation functions. They are

$$
f_{acd} f_{bcd} = 2N \delta_{ab}
$$

$$
\text{Tr}(T_c T_b T_a T_d) \text{Tr}(T_d T_c T_a T_b) = \frac{1}{N^2} (N^2 - 1)(N^2 + 3)
$$

$$
\text{Tr}(T_c T_b T_a T_d) \text{Tr}(T_a T_b T_c T_d) = \frac{1}{N^2} (N^2 - 1)(N^2 - 3)
$$

$$
\text{Tr}(T_c T_b T_a T_d) \text{Tr}(T_a T_b T_c T_d) = \frac{1}{N^2} (N^2 - 1)(N^4 - 3N^2 + 3)
$$

and

$$
\text{Tr}(T_c T_a T_b T_c T_a T_b) f_{c1mb1} f_{c2mb2} = - (\delta_{b1a1} \delta_{b2a2} + \delta_{b2a1} \delta_{b1a2})
$$

$$
\text{Tr}(T_c T_a T_b T_c T_a T_b) f_{c1mb1} f_{c2mb2} = \delta_{b1b2} \delta_{a1a2} + N \text{Tr}(T_b T_c T_b T_c)
$$

To deal with the flavor structure we find convenient to introduce the following tensors (to shorten the notation we avoid to write explicitly the flavor indices on the tensors)

$$
\mathcal{F}_1 \equiv \delta_{kk'} \delta_{ll'} \delta_{pp'} \delta_{qq'}
$$

$$
\mathcal{F}_2 \equiv \delta_{kk'} \delta_{ll'} \delta_{pl'} \delta_{qq'}
$$

$$
\mathcal{F}_3 \equiv \delta_{kk'} \delta_{ll'} \delta_{pq'} \delta_{qq'}
$$

$$
\mathcal{F}_4 \equiv \delta_{kk'} \delta_{ll'} \delta_{pp'} \delta_{qq'}
$$

$$
\mathcal{F}_5 \equiv \delta_{kk'} \delta_{ll'} \delta_{pl'} \delta_{qq'}
$$

$$
\mathcal{F}_6 \equiv \delta_{kk'} \delta_{ll'} \delta_{pq'} \delta_{qq'}
$$
and
\[ G_1 \equiv \delta_{kq} \delta_{l'}^{l''} \delta_{pp'} \delta_{p''} \quad G_2 \equiv \delta_{kq} \delta_{l'q'} \delta_{lk'} \delta_{pp'} \] (A.18)

The contractions with the tensors \( D^{kipq}_{(ij)} \equiv D \) and \( D^{k'j'p'q'}_{(i'j')} \equiv D' \) as defined in the main text, eq. (4.3), give the following results
\[
\begin{align*}
DF_1D' &= DF_4D' = \frac{10}{3} (\delta_{ii'} \delta_{jj'} + \delta_{ij'} \delta_{i'j}) \\
DF_2D' &= DF_3D' = DF_5D' = DF_6D' = \frac{5}{9} (\delta_{ii'} \delta_{jj'} + \delta_{ij'} \delta_{i'j})
\end{align*}
\] (A.19)

and
\[
\begin{align*}
DG_1D' &= DG_2D' = \frac{50}{9} (\delta_{ii'} \delta_{jj'} + \delta_{ij'} \delta_{i'j})
\end{align*}
\] (A.20)

**B Construction of the \( O_{ij}^{(20)} \) superfield**

In this Appendix we describe in detail the procedure to construct in terms of \( N = 1 \) superfields the dimension 4 operator in the \( 20 \) of \( SU(4) \) appearing in the product of two dimension 2 CPO's, also belonging to the \( 20 \).

The \( N = 1 \) fundamental superfields of \( \mathcal{N} = 4 \) supersymmetric Yang–Mills theory are chiral \( \Phi_i \) and antichiral \( \bar{\Phi}^i \) superfields \( (i = 1, \ldots, 3) \) in the \( 3 \) and \( \bar{3} \) of \( SU(3) \), respectively. The six scalars of the theory which belong to the (real) \( 6 \) of the \( R \)-symmetry group \( SO(6) \sim SU(4) \) can be expressed in terms of the lowest components of these superfields according to the following decomposition under \( SU(4) \to SU(3) \times U(1) \)
\[ 6 \to 3\left(\frac{2}{3}\right) + 3\left(-\frac{2}{3}\right) \] (B.1)

where the \( 3 \) and \( \bar{3} \) representations contain, respectively, the highest and the lowest weight of the \( 6 \). The small indices \( \left(\frac{2}{3}\right) \) and \( \left(-\frac{2}{3}\right) \) are (conventional) \( U(1) \) charges. The decomposition \( (B.1) \) can be deduced from the one of the fundamental \( 4 \) representation of \( SU(4) \)
\[ 4 \to 3\left(-\frac{2}{3}\right) + 1(1) \] (B.2)

Analogously, the decomposition of the (real) \( 20 \) can be found to be
\[ 20 \to 6\left(\frac{4}{3}\right) + 8(0) + 6\left(-\frac{4}{3}\right) \] (B.3)

where the \( 6 \) and \( \bar{6} \) representations contain, respectively, the highest and the lowest weight of the \( 20 \).

We want to build the superfield expression corresponding to the highest weight \( 6 \) (or to its complex conjugate lowest weight \( \bar{6} \)) that appear in the decomposition under \( (B.3) \) of our \( 20 \) operator on the right hand side of
\[ (20 \times 20)_{S} = 105 + 84 + 20 + 1 \] (B.4)
To this purpose we use (B.3) on the left hand side of (B.4) and we produce all the following terms

\[
\begin{align*}
(6_{\frac{4}{3}} \times 6_{\frac{4}{3}})_S &= \bar{6}_{\frac{8}{3}} + 15_{\frac{8}{3}} \\
(8_{(0)} \times 8_{(0)})_S &= 1_{(0)} + 8_{(0)} + 27_{(0)} \\
(\bar{6}_{\frac{4}{3}} \times \bar{6}_{\frac{4}{3}})_S &= 6_{\frac{8}{3}} + 15_{\frac{8}{3}} \\
6_{\frac{4}{3}} \times 8_{(0)} &= \bar{3}_{\frac{4}{3}} + 6_{\frac{4}{3}} + 15_{\frac{4}{3}} + 24_{\frac{4}{3}} \\
6_{\frac{4}{3}} \times \bar{6}_{\frac{4}{3}} &= 1_{(0)} + 8_{(0)} + 27_{(0)} \\
\bar{6}_{\frac{4}{3}} \times 8_{(0)} &= 3_{\frac{4}{3}} + \bar{6}_{\frac{4}{3}} + 15_{\frac{4}{3}} + 24_{\frac{4}{3}}
\end{align*}
\]  

(B.5)

We see that the 6 can be obtained in two ways, from \((\bar{6} \times \bar{6})_S\) and from \(6 \times 8\). However, only the last one has the correct \(U(1)\) charge +\(\frac{4}{3}\) (see eq. (B.3)).

In terms of \(\mathcal{N} = 1\) superfields, objects in the 6 and 8 are obtained as bilinear forms (being \(\gamma_0 = 2\))

\[
\text{Tr}(\Phi_i \Phi_j) \equiv A_{ij}
\]

and

\[
\text{Tr}(\Phi_i \bar{\Phi}^j - \frac{1}{3} \delta_i^j \Phi_k \bar{\Phi}^k) \equiv B_{ij}^k
\]

respectively. The operator in the \(6 \times 8\) is then \(A_{ij}B_{ij}^k\). The contraction of one of the chiral indices of the operator in the 6 with the antichiral index of the operator in the 8 gives the two irreducible representations

\[
A_{k[i}B_{j]}^k \quad A_{k(i}B_{j)}^k
\]

where the first one is the \(\bar{3}\) and the second one is the 6 appearing in the product \(6 \times 8\).

The explicit form of the operator is then given by

\[
\text{Tr}(\Phi_k \Phi_{(i)} \text{Tr}(\Phi_{j)\bar{\Phi}^k - \frac{1}{3} \delta_{j}^{k} \Phi_{l} \bar{\Phi}^l))
\]

(B.9)

From the point of view of the \(R\)-symmetry group representations this is the correct superfield structure we were looking for. However, in order to guarantee also its gauge invariance, we need introduce exponentials of the gauge superfield \(V\), obtaining

\[
\text{Tr}(\Phi_k \Phi_{(i)} \text{Tr}(e^{gV} \Phi_{j)e^{-gV}} \bar{\Phi}^k - \frac{1}{3} \delta_{j}^{k} e^{gV} \Phi_{l} e^{-gV} \bar{\Phi}^l))
\]

(B.10)

This is the form of the superfield \(O_{ij}^{(20)}\) operator we have used in our calculation.
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