Approximation of deterministic and stochastic Navier-Stokes equations in vorticity-velocity formulation

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Abstract

We consider a time discretization of incompressible Navier-Stokes equations with spatial periodic boundary conditions in the vorticity-velocity formulation. The approximation is based on freezing the velocity on time subintervals resulting in linear parabolic equations for vorticity. Probabilistic representations for solutions of these linear equations are given. At each time step, the velocity is expressed via vorticity using a formula corresponding to the Biot–Savart-type law. We show that the approximation is divergent free and of first order. The results are extended to two-dimensional stochastic Navier-Stokes equations with additive noise, where, in particular, we prove the first mean-square convergence order of the vorticity approximation.

Keywords. Navier-Stokes equations, vorticity, numerical method, stochastic partial differential equations, mean-square convergence.

AMS 2000 subject classification. 65C30, 60H15, 60H35, 35Q30

1 Introduction

Navier-Stokes equations (NSE), both deterministic and stochastic, are important for a number of applications and, consequently, development and analysis of numerical methods for simulation of NSE are of significant interest. The theory and applications of deterministic NSE can be found, e.g. in [6, 8, 15, 10, 28, 29] and for stochastic NSE – e.g. in [9, 16, 18]. The literature on numerics for deterministic NSE is extensive [11, 27, 30] (see also references therein) while the literature on numerics for stochastic NSE is still rather sparse, let us mention [4, 8, 2, 7, 26].

In this paper we consider incompressible NSE in the vorticity-velocity formulation with periodic boundary conditions (see, e.g. [15] for the deterministic case and [12] for the stochastic one). In the deterministic case we deal with both two-dimensional and three-dimensional NSE, in the stochastic case we are interested in two-dimensional NSE with additive noise. We study their time discretization which is based on freezing the
velocity at every time step. Consequently, at every step we just need to solve a system of linear parabolic PDEs. To compute the velocity, we express it via the vorticity field, i.e. we derive a periodic version of Biot-Savart’s law (see e.g. [15, p. 50 and p. 71]). The constructed approximations of both vorticity and velocity are divergent free. We prove convergence theorems for the suggested approximation. The second part of the paper deals with NSE with additive noise.

The paper is organised as follows. We introduce function spaces required, recall the Helmholtz-Hodge decomposition, and derive the periodic version of Biot-Savart’s law in Section 2. In the deterministic case we suggest to use probabilistic representations together with ideas of weak-sense numerical integration of SDEs for solving the system of linear parabolic PDEs at every step of the time discretization. In Section 3 we present probabilistic representations appropriate for this task which are based on [19] (see also [22, 23]). The numerical method is proposed and analysed in Section 4 where in particular its first order convergence in $L^2$-norm is proved. Then the ideas of Section 4 are transferred over to the case of NSE with additive noise in Section 5 including a proof of first-order mean-square convergence of the time-discretization of the stochastic NSE in the vorticity-velocity formulation. Ideas used in the proof are of potential interest for convergence analysis of numerical methods for a wider class of semilinear SPDEs.

## 2 Preliminaries

In the first part of this paper (Sections 3 and 4), we consider the two- and three-dimensional deterministic incompressible NSE for velocity $v$ and pressure $q$ with space periodic conditions:

\[
\begin{align*}
\frac{\partial v}{\partial s} + (v, \nabla)v + \nabla q - \frac{\sigma^2}{2} \Delta v &= F, \quad (2.1) \\
\text{div } v &= 0. \quad (2.2)
\end{align*}
\]

In (2.1)-(2.2) we have $0 < s \leq T, x \in \mathbb{R}^n, v \in \mathbb{R}^n, F \in \mathbb{R}^n, n = 2, 3, q$ is a scalar. The velocity vector $v = (v^1, \ldots, v^n)^\top$ satisfies initial conditions

\[v(0, x) = \varphi(x)\] (2.3)

and spatial periodic conditions

\[v(s, x + Le_i) = v(s, x), \quad i = 1, \ldots, n, \quad 0 \leq s \leq T.\] (2.4)

Here $\text{div } \varphi = 0, \{e_i\}$ is the canonical basis in $\mathbb{R}^n$, and $L > 0$ is the period. For simplicity in writing, the periods in all the directions are taken to be equal. The function $F = F(s, x)$ and pressure $q = q(s, x)$ are assumed to be spatial periodic as well.

In what follows we will consider the deterministic NSE with negative direction of time which is convenient for probabilistic representations considered in Section 3. By an appropriate change of the time variable and functions, the NSE (2.1)-(2.2) with positive direction of time can be rewritten in the form:

\[
\begin{align*}
\frac{\partial u}{\partial t} + \frac{\sigma^2}{2} \Delta u - (u, \nabla)u - \nabla p + f &= 0, \quad (2.5) \\
\text{div } u &= 0, \quad (2.6)
\end{align*}
\]
where \(0 \leq t < T, \; x \in \mathbb{R}^n, \; u \in \mathbb{R}^n, \; f \in \mathbb{R}^n, \; n = 2, 3,\) the pressure \(p\) is a scalar. The velocity vector \(u = (u^1, \ldots, u^n)^\top\) satisfies the terminal condition
\[
u(T, x) = \varphi(x)
\]
and spatial periodic conditions
\[
u(t, x + L e_i) = \nu(t, x), \; i = 1, \ldots, n, \; 0 \leq t \leq T.
\]

Throughout the paper we will assume that this problem has a unique, sufficiently smooth classical solution. In the two-dimensional case \((n = 2)\) the corresponding theory is available, e.g. in \([10, 15]\). In the remaining part of this preliminary section we recall the required function spaces \([8, 28, 29]\) and write the NSE in vorticity formulation.

### 2.1 Function spaces and the Helmholtz-Hodge decomposition

Let \(\{e_i\}\) be the canonical basis in \(\mathbb{R}^n\). We shall consider spatial periodic \(n\)-vector functions \(u(x) = (u^1(x), \ldots, u^n(x))^\top\) in \(\mathbb{R}^n: \; u(x + L e_i) = u(x), \; i = 1, \ldots, n,\) where \(L > 0\) is the period in \(i\)th direction. Denote by \(Q = (0, L]^n\) the cube of the period. We denote by \(L^2(Q)\) the Hilbert space of functions on \(Q\) with the scalar product and the norm
\[
(u, v) = \int_Q \sum_{i=1}^n u^i(x)v^i(x)dx, \; \|u\| = \langle u, u \rangle^{1/2}.
\]

We keep the notation \(| \cdot |\) for the absolute value of numbers and for the length of \(n\)-dimensional vectors, for example,
\[
|u(x)| = [(u^1(x))^2 + \cdots + (u^n(x))^2]^{1/2}.
\]

We denote by \(H^m_p(Q), \; m = 0, 1, \ldots,\) the Sobolev space of functions which are in \(L^2(Q)\), together with all their derivatives of order less than or equal to \(m,\) and which are periodic functions with the period \(Q.\) The space \(H^m_p(Q)\) is a Hilbert space with the scalar product and the norm
\[
(u, v)_m = \int_Q \sum_{i=1}^n \sum_{\alpha \leq m} D^{\alpha_i} u^i(x) D^{\alpha_i} v^i(x)dx, \; \|u\|_m = [(u, u)_m]^{1/2},
\]
where \(\alpha = (\alpha_1, \ldots, \alpha_n), \; \alpha_i \in \{0, \ldots, m\}, \; [\alpha^i] = \alpha_1^i + \cdots + \alpha_n^i,\) and
\[
D^{\alpha_i} = D^{\alpha_1}_1 \cdots D^{\alpha_n}_n = \frac{\partial^{[\alpha^i]}}{\partial(x^1)^{\alpha_1} \cdots \partial(x^n)^{\alpha_n}}, \; i = 1, \ldots, n.
\]

Note that \(H^0_p(Q) = L^2(Q).\)

Introduce the Hilbert subspaces of \(H^m_p(Q):\)
\[
V^m_p = \{v : \; v \in H^m_p(Q), \; \text{div} v = 0\}, \; m > 0,
\]
\[
V^0_p = \text{the closure of } V^m_p, \; m > 0 \text{ in } L^2(Q).
\]
Denote by $P$ the orthogonal projection in $\mathbf{H}_p^m(Q)$ onto $\mathbf{V}_p^m$ (we omit $m$ in the notation $P$ here). The operator $P$ is often called the Leray projection. Due to the Helmholtz-Hodge decomposition, any function $u \in \mathbf{H}_p^m(Q)$ can be represented as

$$u = Pu + \nabla g, \ \nabla Pu = 0,$$

where $g = g(x)$ is a scalar $Q$-periodic function such that $\nabla g \in \mathbf{H}_p^m(Q)$. It is natural to introduce the notation $P^\perp u := \nabla g$ and hence write

$$u = Pu + P^\perp u$$

with

$$P^\perp u \in (\mathbf{V}_p^m)^\perp = \{v: v \in \mathbf{H}_p^m(Q), \ v = \nabla g\}.$$

Let

$$u(x) = \sum_{n \in \mathbb{Z}^n} u_n e^{i(2\pi/L)(n,x)}, \ g(x) = \sum_{n \in \mathbb{Z}^n} g_n e^{i(2\pi/L)(n,x)}, \ g_0 = 0, \ (2.9)$$

$$Pu(x) = \sum_{n \in \mathbb{Z}^n} (Pu)_n e^{i(2\pi/L)(n,x)}, \ P^\perp u(x) = \nabla g(x) = \sum_{n \in \mathbb{Z}^n} (P^\perp u)_n e^{i(2\pi/L)(n,x)}$$

be the Fourier expansions of $u$, $g$, $Pu$, and $P^\perp u = \nabla g$. Here $u_n$, $(Pu)_n$, and $(P^\perp u)_n = (\nabla g)_n$ are $n$-dimensional vectors and $g_n$ are scalars. We note that $g_0$ can be any real number but for definiteness we set $g_0 = 0$ without loss of generality \[10\]. The coefficients $(Pu)_n$, $(P^\perp u)_n$, and $g_n$ can be easily expressed in terms of $u_n$:

$$(Pu)_n = u_n - \frac{u_n^\top n}{|n|^2} n, \ (P^\perp u)_n = i\frac{2\pi}{L} g_n n = \frac{u_n^\top n}{|n|^2} n, \ (2.10)$$

$$g_n = -i\frac{L}{2\pi} \frac{u_n^\top n}{|n|^2}, \ n \neq 0, \ g_0 = 0.$$

We have

$$\nabla e^{i(2\pi/L)(n,x)} = n e^{i(2\pi/L)(n,x)} \cdot \frac{2\pi}{iL},$$

hence $u_n e^{i(2\pi/L)(n,x)} \in \mathbf{V}_p^m$ if and only if $(u_n, n) = 0$. We obtain from here that the orthogonal basis of the subspace $(\mathbf{V}_p^m)^\perp$ consists of $n e^{i(2\pi/L)(n,x)}$, $n \in \mathbb{Z}^n$, $n \neq 0$; and an orthogonal basis of $\mathbf{V}_p^m$ consists of $k u_n e^{i(2\pi/L)(n,x)}$, $k = 1, \ldots, n - 1$, $n \in \mathbb{Z}^n$, where under $n \neq 0$ the vectors $k u_n$ are orthogonal to $n$: $(k u_n, n) = 0$, $k = 1, \ldots, n - 1$, and they are orthogonal among themselves: $(k u_n, \ m u_n) = 0$, $k, m = 1, \ldots, n - 1$, $m \neq k$, and finally, for $n = 0$, the vectors $k u_0$, $k = 1, \ldots, n$, are orthogonal. In particular, in the two-dimensional case ($n = 2$), these bases are, correspondingly (for $n \neq 0$):

$$\begin{bmatrix} n_1 \\ n_2 \end{bmatrix} e^{i(2\pi/L)(n,x)} \quad \text{and} \quad \begin{bmatrix} -n_2 \\ n_1 \end{bmatrix} e^{i(2\pi/L)(n,x)}, \ n = (n_1, n_2)^\top. \ (2.11)$$

We shall consider the case of zero space average (see e.g. \[10\]), i.e. when

$$\int_Q u(x) = 0. \ (2.12)$$

In this case the Fourier series expansion for $u(x)$ does not contain the constant term and $\sum_{n \in \mathbb{Z}^n}$ in (2.9) can be replaced by $\sum_{n \in \mathbb{Z}^n, n \neq 0}$, which in what follows we will write as simply $\sum$. 

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We recall Parseval’s identity
\[ \|u\|^2 = \int_Q |u(x)|^2 \, dx = L^n \sum |u_n|^2, \quad n = 2, 3. \] (2.13)
We also note the following two relationships. Since the vector field \( u = u(x) \) is real valued, we have
\[ u_{-n} = \bar{u}_n, \quad n \in \mathbb{Z}^n, \quad n \neq 0, \]
where \( \bar{u}_n \) denotes the complex conjugate of \( u_n \). The divergence-free condition reads
\[ u_n^\top n = (u_n, n) = \sum_{k=1}^n u_n^k n^k = 0, \quad n = 2, 3. \]

We will need the following estimate for the tri-linear form (see [6, p. 50, eq. (6.10)] or [29, p. 12, eq. (2.29)]):
\[ |((v, \nabla)u, g)| \leq K\|v\|_{m_1}\|u\|_{m_2+1}\|g\|_{m_3}, \] (2.14)
where \( K > 0 \) is a constant, \( m_1, m_2 \) and \( m_3 \) are such that \( m_1 + m_2 + m_3 \geq n/2 \) and \((m_1, m_2, m_3) \neq (0, 0, n/2), (0, n/2, 0), (n/2, 0, 0)\), and \( u, v, g \) are arbitrary functions from the corresponding spaces. Further, we recall the standard interpolation inequality for Sobolev spaces (see e.g. [29, p. 11]):
\[ \|u\|^m \leq \|u\|^{1-l}\|u\|^l_{m_2}, \] (2.15)
where \( m = (1-l)m_1 + lm_2 \), \( m_1, m_2 \geq 0 \), \( l \in (0,1) \), and \( u \in H_p^{\text{max}(m_1,m_2)}(Q) \). For any \( c > 0 \), we get from (2.15) and Young’s inequality:
\[ \|u\|^m_m \leq \|u\|^{2-2l}_{m_1}\|u\|_{m_2}^{2l} \leq (1-l)c\|u\|^{2}_{m_1} + lc^{1-l}\|u\|^2_{m_2}. \] (2.16)
Let us take \( m_1 = 1 \), \( m_2 = 0 \) and \( m_3 = 1/2 \) in (2.14), then, using (2.16), we get (see also [12, p. 1028, eq. (A6)]) for any \( c_1 > 0 \) and \( c_2 > 0 \):
\[ |((v, \nabla)u, g)| \leq K\|v\|_{1}\|u\|_{1}\|g\|_{1/2} \leq K^{2}\|g\|_{1/2}^{2} + c_1\|v\|_{1}^{2}\|u\|_{1}^{2} \] (2.17)
\[ \leq \frac{K^{2}}{4c_1}\|g\|_{1/2}^{2} + c_1\|v\|_{1}^{2}\|u\|_{1}^{2} \]
\[ \leq c_2\|g\|_{1}^{2} + \frac{K^{4}}{6c_1^2c_2}\|g\|_{1}^{2} + c_1\|v\|_{1}^{2}\|u\|_{1}^{2}, \]
where we used (2.16) with \( m = 1/2, m_1 = 1, m_2 = 0 \) and \( l = 1/2 \) (fractional \( H_p^m(Q) \) spaces are defined in the usual way via the Fourier series expansions, see e.g. [29, pp. 7-8]).

We also recall (see e.g. [10, p. 20, eq. (4.14)]) Poincare’s inequality for functions \( u \in H_p^1(Q) \) satisfying (2.12):
\[ \|u\| \leq \alpha\|\nabla u\| \] (2.18)
for some constant \( \alpha > 0 \) which depends only on the shape of \( Q \) and on the period \( L \). We note that here and in what follows: when \( u(x) \) is a vector, \( \nabla u(x) \) means the matrix with elements \( \partial u^i/\partial x^j \) and \( \|\nabla u\| \) means \( L^2\)-norm of the Frobenius norm of the matrix \( \nabla u(x) \).
2.2 Equation for vorticity

Introduce the vorticity \( \omega \):

\[
\omega = \begin{bmatrix}
\omega^1 \\
\omega^2 \\
\omega^3
\end{bmatrix} := \text{curl } u = \text{curl } \begin{bmatrix}
u^1 \\
u^2 \\
u^3
\end{bmatrix} = \begin{bmatrix}
i \frac{\partial}{\partial x^1} - j \frac{\partial}{\partial x^2} + k \frac{\partial}{\partial x^3}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial u^3}{\partial x^2} - \frac{\partial u^2}{\partial x^3} \\
\frac{\partial u^1}{\partial x^3} - \frac{\partial u^3}{\partial x^1} \\
\frac{\partial u^2}{\partial x^1} - \frac{\partial u^1}{\partial x^2}
\end{bmatrix}.
\tag{2.19}
\]

We note that (2.19) implies \( \text{div } \omega = 0 \).

Taking the curl of equation (2.5) gives the evolution equation for the vorticity \( \omega = \text{curl } u \):

\[
\frac{\partial \omega}{\partial t} - (u, \nabla)\omega + (\omega, \nabla)u + \frac{\sigma^2}{2} \Delta \omega + g = 0,
\tag{2.20}
\]

where \( g = \text{curl } f \). From (2.7)-(2.8), we get

\[
\omega(T, x) = \text{curl } \phi(x) := \phi(x)
\tag{2.21}
\]

and spatial periodic conditions

\[
\omega(t, x + L e_i) = \omega(t, x), \quad i = 1, \ldots, n, \quad 0 \leq t \leq T.
\tag{2.22}
\]

Analogously to (2.9), we write the Fourier expansion for \( \omega \):

\[
\omega(t, x) = \sum \omega_n(t) e^{i(2\pi/L)(n, x)},
\tag{2.23}
\]

where

\[
\omega_n(t) = \frac{1}{L^n} \left( \omega(t, \cdot), e^{i(2\pi/L)(n, \cdot)} \right)
\]

\[
= \frac{1}{L^n} \int_Q \omega(t, x) e^{-i(2\pi/L)(n, x)} dx, \quad n \in \mathbb{Z}^n, \quad n \neq 0.
\]

Substituting the Fourier expansions for \( \omega \) and \( u \) in (2.19), we obtain

\[
\omega = \begin{bmatrix}
\sum \omega_n^1(t) e^{i(2\pi/L)(n, x)} \\
\sum \omega_n^2(t) e^{i(2\pi/L)(n, x)} \\
\sum \omega_n^3(t) e^{i(2\pi/L)(n, x)}
\end{bmatrix} = \frac{2\pi}{i L} \begin{bmatrix}
\sum (u_n^1(t)n^2 - u_n^2(t)n^3) e^{i(2\pi/L)(n, x)} \\
\sum (u_n^2(t)n^3 - u_n^3(t)n^1) e^{i(2\pi/L)(n, x)} \\
\sum (u_n^3(t)n^1 - u_n^1(t)n^2) e^{i(2\pi/L)(n, x)}
\end{bmatrix}.
\tag{2.24}
\]

The equality (2.24) gives for any \( n \neq 0 \) the equations with respect to \( u_n^k, \quad k = 1, 2, 3 \):

\[
n^2 u_n^3 - n^3 u_n^2 = -\frac{iL}{2\pi} \omega_n^1
\tag{2.25}
\]

\[
n^3 u_n^1 - n^1 u_n^3 = -\frac{iL}{2\pi} \omega_n^2
\]

\[
n^1 u_n^2 - n^2 u_n^1 = -\frac{iL}{2\pi} \omega_n^3.
\]

Thanks to \( \text{div } u = 0 \) and \( \text{div } \omega = 0 \), we also have for any \( n \):

\[
n^1 u_n^1 + n^2 u_n^2 + n^3 u_n^3 = 0
\tag{2.26}
\]

\[
n_n^1 \omega_n^1 + n_n^2 \omega_n^2 + n_n^3 \omega_n^3 = 0.
\tag{2.27}
\]
Due to the property (2.27), the system (2.25)-(2.26) with respect to $u^k$ is compatible. It is not difficult to prove directly that the solution of this system is unique. This also follows from the two observations that the vector field $u_ne^{i(2\pi/L)(n,x)}$ is solenoidal because it is divergence-free and that in the case of $\omega^k_n = 0$, $k = 1, 2, 3$, the field is irrotational, i.e. potential. But if a vector field is simultaneously solenoidal and potential, it is trivial, i.e. $u_n = 0$. Thus, the homogeneous system corresponding to (2.25)-(2.26) has the trivial solution only, and hence the solution to the system (2.25)-(2.26) exists and it is unique.

Our nearest goal consists in solving this system, i.e., in expressing $u$ via $\omega$. We observe that

$$\text{curl } u = \omega, \quad \text{div } u = 0. \quad (2.28)$$

$$\text{div } u = 0. \quad (2.29)$$

**Proposition 2.1** For a sufficiently smooth $\psi$, let $\text{div } \psi = 0$. Then

$$\text{curl } [\text{curl } \psi] = \Delta \psi. \quad (2.30)$$

**Proof.** The first component of the vector $\text{curl } [\text{curl } \psi]$ is equal to

$$- \frac{\partial^2 \psi^2}{\partial x^1 \partial x^2} + \frac{\partial^2 \psi^1}{(\partial x^2)^2} + \frac{\partial^2 \psi^1}{(\partial x^3)^2} - \frac{\partial^2 \psi^3}{\partial x^1 \partial x^3}$$

$$= \frac{\partial}{\partial x^1} \left( - \frac{\partial \psi^2}{\partial x^2} - \frac{\partial \psi^3}{\partial x^3} \right) + \frac{\partial^2 \psi^1}{(\partial x^2)^2} + \frac{\partial^2 \psi^1}{(\partial x^3)^2}.$$ 

Because of the condition $\text{div } \psi = 0$, this component is equal to $\Delta \psi^1$. Analogously, the second and third components are equal to $\Delta \psi^2$ and $\Delta \psi^3$, correspondingly. The proposition is proved.

Let us look for $u$ in the form

$$u = - \text{curl } \psi, \quad (2.31)$$

where $\text{div } \psi = 0$. Due to (2.28)-(2.30), we now have to solve

$$\Delta \psi = \omega, \quad \omega = \sum \omega_n e^{i(2\pi/L)(n,x)}. \quad (2.32)$$

Equation (2.32) is solvable (uniquely, if we assume $\psi_0 = 0$):

$$\psi = \sum \psi_n e^{i(2\pi/L)(n,x)}, \quad \psi_n = \frac{\omega_n L^2}{4\pi^2 |n|^2},$$

i.e.,

$$\psi^j_n = - \frac{\omega^j_n L^2}{4\pi^2 |n|^2}, \quad j = 1, 2, 3.$$ 

Hence, using (2.31), we have

$$u = - \text{curl } \psi = \frac{L}{2\pi} \left[ \sum \frac{1}{|n|^2} e^{i(2\pi/L)(n,x)} (\omega_n^3 n^2 - \omega_n^2 n^3) \right] = U \omega, \quad (2.33)$$
where $U$ is a linear operator. It is not difficult to verify that the equality $\text{div} \ u = 0$ (see (2.29)) holds for $u$ from (2.33) under arbitrary $\omega$. However, the equality (2.28) is not fulfilled by $u$ from (2.33) for arbitrary $\omega$. But the considered $\omega$ is not arbitrary, it is divergent free and we show below that for a divergent-free $\omega$ the equality (2.28) is satisfied by $u$ from (2.33).

For $u$ from (2.33), we get

$$\text{curl} \ u = -\left[ \sum_{n=1}^{3} \frac{1}{n!} e^{i(2\pi/L)(n,x)} \left[ -\omega_1^{n} (n^2)^2 + (n^3)^2 + n^1(n^2 \omega_n + n^3 \omega_3) \right] \right].$$

Recall that (2.27) holds for a divergent-free $\omega$. Then, using (2.27), we have

$$n^1(n^2 \omega_n + n^3 \omega_3) = -(n^1)^2 \omega_1,$$

$$n^2(n^1 \omega_n + n^3 \omega_3) = -(n^2)^2 \omega_2,$$

$$n^3(n^1 \omega_n + n^2 \omega_2) = -(n^3)^2 \omega_3,$$

and therefore

$$\text{curl} \ u = -\left[ \sum_{n=1}^{3} \frac{1}{n!} e^{i(2\pi/L)(n,x)} \omega_n \left| n \right|^2 \right],$$

i.e., $\text{curl} \ u = \omega$. Thus, the following theorem is proved.

**Theorem 2.1** The velocity field $u$ (with $u_0 = 0$) is determined explicitly through vorticity field $\omega = \text{curl} \ v$ by formula (2.33). The closed-form equation for $\omega$ is given by

$$\frac{\partial \omega}{\partial t} - (U \omega, \nabla) \omega + (\omega, \nabla) U \omega + \frac{\sigma^2}{2} \Delta \omega + g = 0,$$  \hspace{1cm} (2.34)

where $U$ is from (2.33).

This theorem is related to analogous results for periodic 2D flows (see [15, p. 50]) and for flows in the whole space (see [15, p. 71]).

Let us mention the corresponding formulas in the 2D case, for which we have

$$u(x) = \begin{bmatrix} u^1(x^1, x^2) \\ u^2(x^1, x^2) \\ 0 \end{bmatrix} = \sum_n \begin{bmatrix} u_n^1 \\ u_n^2 \\ 0 \end{bmatrix} e^{i(2\pi/L)(n^1 x^1 + n^2 x^2)},$$

i.e., $u^1(x)$ and $u^2(x)$ are independent of $x^3$ and $u^3 = 0$. Hence

$$\omega = \begin{bmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{\partial u^2}{\partial x^1} - \frac{\partial u^1}{\partial x^2} \end{bmatrix}.$$  

We shall denote the scalar $\omega^3(x)$ as $\omega(x) = \frac{\partial u^2}{\partial x^1}(x) - \frac{\partial u^1}{\partial x^2}(x)$ and the two dimensional vector $(u^1(x^1, x^2), u^2(x^1, x^2))^T$ as $u(x)$. This does not lead to confusion. We have $\text{div} \ u(x) = \frac{\partial u^1}{\partial x^1} + \frac{\partial u^2}{\partial x^2} = 0$, i.e., $u_n^1 n^1 + u_n^2 n^2 = 0$ for any $n = (n^1, n^2)$, and

$$\omega(x) = \sum_n \omega_n e^{i(2\pi/L)(n^1 x^1 + n^2 x^2)},$$
where
\[ \omega_n = i \frac{2\pi}{L} (u_n^2 n_1 - u_n^1 n_2) \]

Due to (2.33), \( u \) is expressed through \( \omega \):
\[ u(x) = \frac{L}{2\pi} \left[ \sum_{|n|} \frac{1}{|n|^2} \left( e^{i(2\pi/L)(n,x)} \omega_n n_2 \right) - \sum_{|n|} \frac{1}{|n|^2} \left( e^{i(2\pi/L)(n,x)} \omega_n n_1 \right) \right] := U_\omega. \tag{2.35} \]

Clearly, \( (\omega, \nabla) u = 0 \) in the 2D case. Hence (2.20) takes the form
\[ \frac{\partial \omega}{\partial t} - u^1(t,x) \frac{\partial \omega}{\partial x^1}(t,x) - u^2(t,x) \frac{\partial \omega}{\partial x^2}(t,x) + \frac{\sigma^2}{2} \Delta \omega(t,x) + g(t,x^1,x^2) = 0. \tag{2.36} \]

Example 2.1 Consider the Stokes equation with negative direction of time:
\[ \frac{\partial u}{\partial t} + \frac{\sigma^2}{2} \Delta u - \nabla p + f = 0 \tag{2.37} \]

with the conditions (2.6)–(2.8). In this case the vorticity \( \omega(t,x) \) satisfies the equation
\[ \frac{\partial \omega}{\partial t} + \frac{\sigma^2}{2} \Delta \omega + g = 0, \tag{2.38} \]

where \( g(t,x) = \sum g_n e^{i(2\pi/L)(n,x)} \). Substituting the Fourier expansions for \( \omega, g \) in (2.38), we get
\[ \sum \omega_n'(t) e^{i(2\pi/L)(n,x)} + \frac{\sigma^2}{2} \sum \omega_n(t) \left( -\frac{4\pi^2}{L^2} \right) |n|^2 e^{i(2\pi/L)(n,x)} + \sum g_n e^{i(2\pi/L)(n,x)} = 0, \]

whence
\[ \frac{d\omega_n^k}{dt} - \frac{2\pi^2 \sigma^2}{L^2} |n|^2 \omega_n^k + g_n^k = 0, \quad \omega_n^k(T) = \phi_n^k, \]

where \( \phi_n \) are the Fourier coefficients for \( \phi := \text{curl} \varphi \). Hence
\[ \omega_n^k(t) = \phi_n^k \exp \left( \frac{2\pi^2 \sigma^2}{L^2} |n|^2 (t-T) \right) + \int_T^t \exp \left( \frac{2\pi^2 \sigma^2}{L^2} |n|^2 (t-s) \right) g_n^k(s) ds. \]

In future we will need the following estimates. One can obtain from (2.33) that for \( m \geq 1 \)
\[ \|u\|_m = \|U_\omega\|_m \leq K \|\omega\|_{m-1} \tag{2.39} \]
for some \( K > 0 \). Further, we note that
\[ \|\omega\|^2_1 = \|\omega\|^2 + \|\nabla \omega\|^2 \tag{2.40} \]
and then by (2.18)
\[ \|\omega\|^2_1 \leq K \|\nabla \omega\|^2 \tag{2.41} \]
for some \( K > 0 \). Using (2.17), (2.39), (2.40) and (2.41), we get for \( \omega, v, g \) from appropriate spaces and arbitrary \( c_1 > 0 \) and \( c_2 > 0 \):
\[ \|((U_\omega, \nabla)v, g)\| \leq c_2\|g\|^2_1 + \frac{K^4}{64c_1c_2}\|g\|^2 + c_1\|U_\omega\|^2_1\|v\|^2_1 \tag{2.42} \]
\[
\leq c_2 \|\nabla g\|^2 + (c_2 + \frac{K^4}{64c_1^2c_2}) \|g\|^2 + Kc_1 \|\omega\|^2 \|\nabla v\|^2 \\
= c_2 \|\nabla g\|^2 + K \|g\|^2 + c_3 \|\omega\|^2 \|\nabla v\|^2,
\]
where in the third line \(c_3 > 0\) is an arbitrary constant and \(K > 0\) is some constant dependent on \(c_2\) and \(c_3\) (it differs from \(K > 0\) in the first and second line but this should not cause any confusion).

3 Probabilistic representations of solutions to linear systems of parabolic equations with application to vorticity equations

In this section we derive probabilistic representations for systems of parabolic equations based on the approach developed in [19]. They can be used for constructing probabilistic methods for NSE in vorticity-velocity formulation (2.20) (see probabilistic numerical methods for semilinear PDEs in e.g. [21, 23] and for NSE in velocity formulation in e.g. [25]). For this purpose, it is useful to have a wide class of such probabilistic representations, and, in addition to [19], we also exploit ideas from [22, 23]. Note that we obtain more general probabilistic representations than in [4].

3.1 The basic probabilistic representation

We consider the following Cauchy problem for system of parabolic equations

\[
\frac{\partial u^k}{\partial t} + \frac{1}{2} \sum_{i,j=1}^{n} \sum_{r=1}^{l} \sigma_r \sigma_r^\top \frac{\partial^2 u^k}{\partial x^i \partial x^j} + \sum_{i=1}^{n} \sum_{j=1}^{m} \left( \sum_{r=1}^{l} \sigma_r^i \sigma_r^j \right) \frac{\partial u^j}{\partial x^i} + \sum_{i=1}^{n} a^i \frac{\partial u^k}{\partial x^i} + [B^\top u]^k + f^k = 0, \quad k = 1, \ldots, m,
\]

\[u(T, x) = \varphi(x).\] (3.2)

Introduce the system of SDEs

\[
dX = a(s, X) ds + \sum_{r=1}^{l} \sigma_r(s, X) dw_r(s), \quad X(t) = x, \] (3.3)

\[
dY = B(s, X) Y ds + \sum_{r=1}^{l} \vartheta_r(s, X) Y dw_r(s), \quad Y(t) = y. \] (3.4)

In (3.1)-(3.4), \(0 \leq t \leq s \leq T; x\) and \(X\) are column-vectors of dimension \(n\); \(y\) and \(Y\) are column-vectors of dimension \(m\); \(w_r, r = 1, \ldots, l\), are independent standard Wiener processes; \(a(s, x)\) and \(\sigma_r(s, x)\) are column-vectors of dimension \(n\); \(B(s, x)\) and \(\vartheta_r(s, x)\) are \(m \times m\) - matrices; \(u(s, x), f(s, x)\), and \(\varphi(x)\) are column-vector of dimension \(m\) with components \(u^k, f^k, \varphi^k, k = 1, \ldots, m\). We assume that there exist a sufficiently smooth solution of the problem (3.1)-(3.2) and a unique solution of the problem (3.3)-(3.4).
Introduce the process
\[ \xi_{t,x,y}(s) = \int_t^s f^T(s', X_{t,x}(s'))Y_{t,x,y}(s')ds' + u^T(s, X_{t,x}(s))Y_{t,x,y}(s). \] (3.5)

Using Ito’s formula, we get
\[ d\xi = \sum_{k=1}^m f^k Y^k ds + d(u^TY), \] (3.6)

\[ d(u^TY) = \sum_{k=1}^m d[u^k Y^k] = \sum_{k=1}^m \frac{\partial u^k}{\partial s} Y^k ds + \sum_{k=1}^m \sum_{i=1}^n \frac{\partial u^k}{\partial x^i} [a^i ds + \sum_{r=1}^l \sigma^i_r dw_r(s)]Y^k \]
\[ + \frac{1}{2} \sum_{k=1}^m \sum_{i,j=1}^n \frac{\partial^2 u^k}{\partial x^i \partial x^j} \sum_{r=1}^l \sigma^i_r \sigma^j_r ds Y^k + \sum_{k=1}^m u^k [BY]^k ds \]
\[ + \sum_{k=1}^m u^k \left[ \sum_{r=1}^l \partial_r Y dw_r(s) \right] + \sum_{k=1}^m \sum_{i=1}^n \frac{\partial u^k}{\partial x^i} dX^i Y^k. \]

Further,
\[ \sum_{k=1}^m u^k [BY]^k ds = \sum_{k=1}^m [B^T u]^k Y^k, \] (3.7)

\[ \sum_{k=1}^m \sum_{i=1}^n \frac{\partial u^k}{\partial x^i} dX^i Y^k = \sum_{k=1}^m \sum_{i=1}^n \frac{\partial u^k}{\partial x^i} \sum_{r=1}^l \sigma^i_r dw_r(s) \sum_{r'=1}^l \sum_{j=1}^m \frac{\partial^k j_r Y^j r'}{\partial x^{r'}} dw_r(s) \]
\[ = \sum_{k=1}^m \sum_{i=1}^n \frac{\partial u^k}{\partial x^i} \sum_{r=1}^l \sum_{r'=1}^l \sigma^i_r \sigma^j_{r'} Y^j ds = \sum_{k=1}^m \sum_{i=1}^n \sum_{j=1}^m \sigma^i_r \sigma^j_{r'} \frac{\partial u^k}{\partial x^i} Y^j ds. \] (3.8)

In (3.6)–(3.9) all the coefficients and functions have \( s, X_{t,x}(s) \) as their arguments.

Substituting (3.7)–(3.9) in (3.6) and taking into account that \( u \) is a solution of (3.1), we get
\[ d\xi = \sum_{r=1}^l \sum_{k=1}^m (u^k(\partial_r Y)^k + n \frac{\partial u^k}{\partial x^i} \sigma^i_r Y^k) dw_r(s). \] (3.10)

It is known that if
\[ E \int_t^T \left[ \sum_{k=1}^m (u^k(\partial_r Y)^k + n \frac{\partial u^k}{\partial x^i} \sigma^i_r Y^k) \right]^2 ds < \infty \] (3.11)
then
\[ E \int_t^T \sum_{r=1}^l \sum_{k=1}^m (u^k(\partial_r Y)^k + n \frac{\partial u^k}{\partial x^i} \sigma^i_r Y^k) dw_r(s) = E[\xi_{t,x,y}(T) - \xi_{t,x,y}(t)] = 0. \] (3.12)
At the same time (see (3.5))

$$\xi_{t,x,y}(T) - \xi_{t,x,y}(t) = \int_t^T f^\top(s', X_{t,x}(s')) Y_{t,x,y}(s') ds' + \varphi^\top(X_{t,x}(T)) Y_{t,x,y}(T) - u^\top(t, x)y.$$  \hspace{1cm} (3.13)

Hence

$$u^\top(t, x)y = E \left[ \int_t^T f^\top(s', X_{t,x}(s')) Y_{t,x,y}(s') ds' + \varphi^\top(X_{t,x}(T)) Y_{t,x,y}(T) \right].$$ \hspace{1cm} (3.14)

So, we have obtained that under certain conditions ensuring existence of a sufficiently smooth solution of (3.1)-(3.2), existence and uniqueness of solution of (3.3)-(3.4), and boundedness (3.11), the probabilistic representation of the solution to the problem (3.1)-(3.2) is given by formula (3.14).

### 3.2 A family of probabilistic representations

Now we restrict ourselves to the case $\vartheta_r = 0, \ r = 1, \ldots, l$, i.e. (see (3.1)-(3.4)):

$$\frac{\partial u^k}{\partial t} + \frac{1}{2} \sum_{i,j=1}^n \left[ \sum_{r=1}^l \sigma_r \sigma_r^\top \right]^{ij} \frac{\partial^2 u^k}{\partial x^i \partial x^j} + \sum_{i=1}^n a^i \frac{\partial u^k}{\partial x^i} + [B^\top u]^k + f^k = 0, \ k = 1, \ldots, m,$$ \hspace{1cm} (3.15)

$$u(T, x) = \varphi(x),$$ \hspace{1cm} (3.16)

and

$$dX = a(s, X) ds + \sum_{r=1}^l \sigma_r(s, X) dW_r(s), \ X(t) = x,$$ \hspace{1cm} (3.17)

$$dY = B(s, X) Y ds, \ Y(t) = y.$$ \hspace{1cm} (3.18)

Introduce the system

$$dX = a(s, X) ds - \sum_{r=1}^l \mu_r(s, X) \sigma_r(s, X) ds + \sum_{r=1}^l \sigma_r(s, X) dW_r(s), \ X(t) = x,$$ \hspace{1cm} (3.19)

$$dY = B(s, X) Y ds, \ Y(t) = y,$$ \hspace{1cm} (3.20)

$$dQ = \sum_{r=1}^l \mu_r(s, X) Q dW_r(s), \ Q(t) = 1,$$ \hspace{1cm} (3.21)

$$dZ = Q f^\top(s, X) Y ds + \sum_{r=1}^l F_r^\top(s, X) Y dW_r(s), \ Z(t) = 0.$$ \hspace{1cm} (3.22)

In (3.19)-(3.22) $\mu_r, Q,$ and $Z$ are scalars; $F_r, \ r = 1, \ldots, l,$ are column-vectors of dimension $m; \ \mu_r$ and $F_r$ are arbitrary functions, however, with good analytical properties.

Introduce the process

$$\eta_{t,x,y}(s) = Q_{t,x,y,1}(s) u^\top(s, X_{t,x}(s)) Y_{t,x,y}(s) + Z_{t,x,y,1,0}(s).$$ \hspace{1cm} (3.23)
Using Ito’s formula and taking into account that $u$ is a solution of (3.15), we get that under arbitrary $\mu_r$ and $F_r$

$$d\eta = \sum_{r=1}^{l} Q \left( \sum_{i=1}^{n} \frac{\partial u^T}{\partial x^i} \sigma^i_r + \mu_r u^T + F^T_r \right) Y dw_r(s). \quad (3.24)$$

If

$$E \int_t^T \sum_{r=1}^{l} \left[ Q \left( \sum_{i=1}^{n} \frac{\partial u^T}{\partial x^i} \sigma^i_r + \mu_r u^T + F^T_r \right) Y \right] ds < \infty \quad (3.25)$$

then

$$E \int_t^T \sum_{r=1}^{l} Q \left( \sum_{i=1}^{n} \frac{\partial u^T}{\partial x^i} \sigma^i_r + \mu_r u^T + F^T_r \right) Y dw_r(s) = E(\eta_{t,x,y}(T) - \eta_{t,x,y}(t)) = 0. \quad (3.26)$$

We have

$$\eta_{t,x,y}(T) - \eta_{t,x,y}(t) = Q(T)\varphi^T(X_{t,x}(T))Y(T) - u(t,x)y$$

$$+ \int_t^T Q_{t,x,y,1}(s') f^T(s', X_{t,x}(s'))Y_{t,x,y}(s')ds'$$

$$+ \int_t^T Q_{t,x,y,1}(s') \sum_{r=1}^{l} F^T_r(s', X_{t,x}(s'))Y_{t,x,y}(s')dw_r(s'). \quad (3.27)$$

Under the natural assumption

$$E \left[ \int_t^T Q_{t,x,y,1}(s') \sum_{r=1}^{l} F^T_r(s', X_{t,x}(s'))Y_{t,x,y}(s')dw_r(s') \right] = 0,$$

using (3.26) and (3.27), we obtain the family of probabilistic representations for the solution of (3.15)-(3.16):

$$u^T(t,x)y = E \left[ Q_{t,x,y,1}(T)\varphi^T(X_{t,x}(T))Y_{t,x,y}(T) + \int_t^T Q_{t,x,y,1}(s') f^T(s', X_{t,x}(s'))Y_{t,x,y}(s')ds' \right], \quad (3.28)$$

where the expressions under sign $E$ depend on a choice of $\mu_r$ and $F_r$. We see that the expectation of $\eta_{t,x,y}(T)$ in the right hand side of (3.28) is equal to $u(t,x)y$ and it is independent of a choice of $\mu_r$ and $F_r$. At the same time, the variance $Var[\eta_{t,x,y}(T)]$ does depend on $\mu_r$ and $F_r$.

### 3.3 Probabilistic representations for the vorticity

System (2.20) has the form of (3.15) with $m = 3$, $n = 3$,

$$\sigma_1 = \begin{bmatrix} \sigma & 0 \\ 0 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & \sigma \\ 0 & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$a(t,x) = - \begin{bmatrix} u^1(t,x) \\ u^2(t,x) \\ u^3(t,x) \end{bmatrix}, \quad B^T = \begin{bmatrix} \partial u^1/\partial x^1 & \partial u^1/\partial x^2 & \partial u^1/\partial x^3 \\ \partial u^2/\partial x^1 & \partial u^2/\partial x^2 & \partial u^2/\partial x^3 \\ \partial u^3/\partial x^1 & \partial u^3/\partial x^2 & \partial u^3/\partial x^3 \end{bmatrix},$$

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with \( \omega \) instead of \( u \) and \( g \) instead of \( f \), \( \sigma \) is a positive constant, i.e., the more detailed writing of (2.20) has the form:

\[
\frac{\partial \omega^k}{\partial t} + \frac{1}{2} \sigma^2 \Delta \omega^k - \sum_{i=1}^{3} u^i(t, x) \frac{\partial \omega^k}{\partial x^i} + \sum_{i=1}^{3} \frac{\partial u^k}{\partial x^i}(t, x) \omega^i + g^k(t, x) = 0, \quad (3.29)
\]

\[
\omega^k(T, x) = \phi^k(x), \ k = 1, 2, 3. \quad (3.30)
\]

Let us put \( \mu_r(s, x) = -a^r(s, x), \ F^r(s, x) = 0 \) in the family of representations (3.19)-(3.22), (3.28) for the problem (3.29)-(3.30). We get the components \( \omega^1, \omega^2, \omega^3 \) of \( \omega \) are obtained from (3.35) under \( y \) equal subsequently to \( (1, 0, 0)^T, (0, 1, 0)^T, (0, 0, 1)^T \).

**Example 3.1** *(The Monte Carlo calculation of the Fourier coefficients)* Due to (2.23), we have

\[
dX^i = \sigma dw_i(s), \quad X^i(t) = x^i, \ i = 1, 2, 3, \quad (3.31)
\]

\[
dY^i = \sum_{j=1}^{3} \frac{\partial u^j}{\partial x^i}(s, X)Y^j ds, \quad Y^i(t) = y^i, \ i = 1, 2, 3, \quad (3.32)
\]

\[
dQ = -Q \sum_{j=1}^{3} u^j(s, X)dw_j(s), \quad Q(t) = 1, \quad (3.33)
\]

\[
dZ = Q \sum_{j=1}^{3} g^j(s, X)Y^j ds, \quad Z(t) = 0, \quad (3.34)
\]

\[
\omega^\top(t, x)y = E \left[ Q_{t,x,y,1}(T)\phi^\top(X_t,x(T))Y_{t,x,y}(T) + Z_{t,x,y,10}(T) \right]. \quad (3.35)
\]

The components \( \omega^1, \omega^2, \omega^3 \) of \( \omega \) can be computed subsequently by applying the Monte Carlo technique and weak-sense approximation of SDEs to the representation (3.28), (3.31)-(3.35).

**Example 3.1** *(The Monte Carlo calculation of the Fourier coefficients)* Due to (2.23), we have

\[
\omega^j_n(t) = \frac{1}{L^3} \left( \omega^j(t, x), e^{i(2\pi/L)(n,x)} \right) = \frac{1}{L^3} \int_Q \omega^j(t, x)e^{-i(2\pi/L)(n,x)} dx, \quad n \in \mathbb{Z}^3, \ n \neq 0, \ j = 1, 2, 3. \quad (3.36)
\]

Let \( \xi \) be a random variable uniformly distributed on \( Q \). Then (3.30) can be written as

\[
\omega^j_n(t) = E \left[ \omega^j(t, \xi)e^{-i(2\pi/L)(n,\xi)} \right],
\]

where the expectation can be approximated using the Monte Carlo technique and hence

\[
\omega^j_n(t) \approx \frac{1}{M} \sum_{m=1}^{M} \omega^j(t, \xi^{(m)})e^{-i(2\pi/L)(n,\xi^{(m)})}
\]

with \( \xi^{(m)} \) being independent realizations of \( \xi \). In turn, every \( \omega(t, \xi^{(m)}) \) can be computed by applying the Monte Carlo technique and weak-sense approximation of SDEs to the representation (3.28), (3.31)-(3.35).

### 4 Approximation method based on vorticity

Let us introduce a uniform partition of the time interval \([0, T]\): \( 0 = t_0 < t_1 < \cdots < t_N = T \) and the time step \( h = T/N \) (we restrict ourselves to the uniform time discretization for simplicity only). In this section we derive an approximation for the vorticity (Section 4.1) and study its properties (divergence free property in Section 4.2, one-step error in Section 4.3, and global convergence in Section 4.4).
4.1 Construction of the method

Let \( \omega(t_{k+1}, x) \), \( k = 0, \ldots, N - 1 \), be known exactly. Then \( u(t_{k+1}, x) \) can be calculated exactly due to (2.33): \( u(t_{k+1}, x) = U \omega(t_{k+1}, x) \). The formula (3.33),

\[
\omega^T(t_k, x) y = E \left[ Q_{t_k,x,y}(t_{k+1}) \omega^T(t_{k+1}, X(t_{k+1})) Y_{t_k,x,y}(t_{k+1}) + Z_{t_k,x,y,1,0}(t_{k+1}) \right],
\]

(4.1)
gives the value of the solution of (2.20) at \( t_k \) assuming that \( u(t, x) \), \( t_k \leq t < t_{k+1} \), is known exactly. We note that knowing this \( u(t, x) \) is necessary for equations (3.32)-(3.33).

Let us replace the unknown \( u(t, x) \) in (3.29) by the function

\[
\hat{u}(t, x) := u(t_{k+1}, x) := \check{u}(x), \quad t_k \leq t < t_{k+1}.
\]

(4.2)

As an approximation of \( \omega(t, x) \) on \( t_k \leq t \leq t_{k+1} \), we take \( \check{\omega}(t, x) \) satisfying the system

\[
\frac{\partial \check{\omega}^i}{\partial t} + \frac{\sigma^2}{2} \Delta \check{\omega}^i - \sum_{j=1}^{3} \check{u}^j(x) \frac{\partial \check{\omega}^i}{\partial x^j} + \sum_{i=1}^{3} \frac{\partial \check{u}^i}{\partial x^j}(x) \check{\omega}^j + g(t, x) = 0, \quad t_k \leq t < t_{k+1},
\]

\[
\check{\omega}^i(t_{k+1}, x) = \omega^i(t_{k+1}, x), \quad \check{\omega}^i(t_{k+1}, x + Le_j) = \check{\omega}^i(t_{k+1}, x), \quad j = 1, 2, 3.
\]

(4.3)

We observe that (4.3)-(4.4) can be also obtained from (3.29) by freezing the velocity \( u(t, x) \) on every time step according to (4.2).

Now we propose the method for solving the problem (2.5)-(2.8) with negative direction of time. On the first step of the method we set

\[
\check{\omega}(t_N, x) = \text{curl } u(t_N, x) = \phi(x) = \text{curl } \varphi(x)
\]

and

\[
\check{u}(x) = \check{u}(t, x) = u(t_N, x) = \varphi(x), \quad t_{N-1} \leq t \leq t_N.
\]

Then we solve the system (4.3)-(4.4) on \([t_{N-1}, t_N]\) to obtain \( \check{\omega}(t, x) \) and to construct

\[
\check{u}(t_{N-1}, x) = U \check{\omega}(t_{N-1}, x).
\]

(4.4)

On the second step we solve (4.3)-(4.4) on \([t_{N-2}, t_{N-1}]\) having \( \check{\omega}(t_{N-1}, x) \) and setting \( \check{u}(t, x) = \check{u}(x) = \check{u}(t_{N-1}, x) \) for \( t_{N-2} \leq t < t_{N-1} \). As a result, we obtain \( \check{\omega}(t, x) \) on \([t_{N-2}, t_{N-1}]\) and \( \check{u}(t_{N-2}, x) = U \check{\omega}(t_{N-2}, x) \), and so on. Proceeding in this way, we obtain on the \( N \)-th step the approximation \( \check{\omega}(t, x) \) on \([t_0, t_1]\) for \( \omega(t, x) \) having \( \check{\omega}(t_1, x) \) and

\[
\check{u}(t, x) = \check{u}(t_1, x) = U \check{\omega}(t_1, x)
\]

and setting \( \check{u}(t, x) = \check{u}(x) = \check{u}(t_1, x) \) for \( t_0 \leq t < t_1 \). Finally, \( \check{u}(t_0, x) = U \check{\omega}(t_0, x) \).

It is also useful to introduce

\[
\check{u}(t, x) := U \check{\omega}(t, x), \quad t_0 \leq t \leq t_N.
\]

(4.5)

In contrast to \( \check{u} \), the function \( \check{u} \) is continuous in \( t \). These functions coincide at \( t = t_k \), \( k = 0, \ldots, N \).

At each step of this method one has to solve the system (4.3)-(4.4). In contrast to the system (2.5)-(2.8), the system (4.3)-(4.4) does not have the divergence-free condition and it is linear. Then the solution of (4.3)-(4.4) can be found using probabilistic representations.
We pay attention to the fact that in the vorticity formulation of the NSE the pressure term disappears.

In order to realise the approximation process described above, it is sufficient that on every time interval \([t_k, t_{k+1}]\), \(k = N - 1, N - 2, \ldots, 1, 0\), there exists a solution of the linear parabolic system \((4.3)-(4.4)\) (we denote such a solution \(\tilde{\omega}_k(t, x)\)) which satisfies the condition

\[
\tilde{\omega}_k(t_{k+1}, x) = \left\{
\begin{array}{ll}
\text{curl } \varphi(x), & k = N - 1, \\
\tilde{\omega}_{k+1}(t_{k+1}, x), & k = N - 2, \ldots, 0,
\end{array}
\right.
\]

and has the time-independent \(\hat{u}(x)\) within each interval \([t_k, t_{k+1}]\) defined as

\[
\hat{u}(x) := \hat{u}_k(x) = U\tilde{\omega}_k(t_{k+1}, x), \quad t_k \leq t < t_{k+1}.
\]

Clearly, \(\hat{u}(x)\) used in \((4.3)\) are different on the time intervals \([t_k, t_{k+1}]\).

### 4.2 The divergence-free property of the method

The evolution equation \((2.20)\) for vorticity has the form

\[
\frac{\partial \omega}{\partial t} = \text{curl}[\ldots].
\]

Due to this fact, any solution of \((2.20)\) with \(\text{div } \omega(t_k, x) = 0\) is divergence free for \(t \leq t_k : \text{div } \omega(t, x) = 0, t \leq t_k\). Indeed, this property can be seen after applying the operator \(\text{div}\) to \((2.20)\) and taking into account the equality \(\text{div } \text{curl}[\ldots] = 0\).

A very important property of the proposed method is that the constructed approximation \(\tilde{\omega}_k(t, x)\) is also divergence free.

**Theorem 4.1** The solution \(\tilde{\omega}(t, x), \quad t_k \leq t \leq t_{k+1},\) of \((4.3)-(4.4)\) is divergent free.

**Proof.** Let us take \(\text{div}\) of the equation \((4.3)\). In \((4.3)\) we have that \(\hat{u}(t, x) = \hat{u}(x) = u(t_{k+1}, x), \quad t_k \leq t < t_{k+1},\) and \(\hat{u}(x)\) is divergent free: \(\text{div } \hat{u} = 0\). Besides, \(\text{div } g = 0\). We have

\[
-\langle \hat{u}, \nabla \hat{\omega} \rangle = -\left(\hat{u}^1 \frac{\partial }{\partial x^1} + \hat{u}^2 \frac{\partial }{\partial x^2} + \hat{u}^3 \frac{\partial }{\partial x^3}\right)\hat{\omega} = -\begin{bmatrix}
\hat{u}^1 \frac{\partial \hat{\omega}^1}{\partial x^1} + \hat{u}^2 \frac{\partial \hat{\omega}^1}{\partial x^2} + \hat{u}^3 \frac{\partial \hat{\omega}^1}{\partial x^3} \\
\hat{u}^1 \frac{\partial \hat{\omega}^2}{\partial x^1} + \hat{u}^2 \frac{\partial \hat{\omega}^2}{\partial x^2} + \hat{u}^3 \frac{\partial \hat{\omega}^2}{\partial x^3} \\
\hat{u}^1 \frac{\partial \hat{\omega}^3}{\partial x^1} + \hat{u}^2 \frac{\partial \hat{\omega}^3}{\partial x^2} + \hat{u}^3 \frac{\partial \hat{\omega}^3}{\partial x^3}
\end{bmatrix},
\]

\[
\text{div } -\langle \hat{u}, \nabla \hat{\omega} \rangle = -\left(\frac{\partial \hat{u}^1}{\partial x^1} \frac{\partial \hat{\omega}^1}{\partial x^1} + \frac{\partial \hat{u}^2}{\partial x^1} \frac{\partial \hat{\omega}^1}{\partial x^2} + \frac{\partial \hat{u}^3}{\partial x^1} \frac{\partial \hat{\omega}^1}{\partial x^3}
\right)
\]

Analogously,

\[
\text{div } [\langle \hat{\omega}, \nabla \hat{u} \rangle] = \begin{bmatrix}
\frac{\partial \hat{\omega}^1}{\partial x^1} \frac{\partial \hat{u}^1}{\partial x^1} + \frac{\partial \hat{\omega}^2}{\partial x^1} \frac{\partial \hat{u}^1}{\partial x^2} + \frac{\partial \hat{\omega}^3}{\partial x^1} \frac{\partial \hat{u}^1}{\partial x^3} \\
\frac{\partial \hat{\omega}^1}{\partial x^2} \frac{\partial \hat{u}^2}{\partial x^1} + \frac{\partial \hat{\omega}^2}{\partial x^2} \frac{\partial \hat{u}^2}{\partial x^2} + \frac{\partial \hat{\omega}^3}{\partial x^2} \frac{\partial \hat{u}^2}{\partial x^3} \\
\frac{\partial \hat{\omega}^1}{\partial x^3} \frac{\partial \hat{u}^3}{\partial x^1} + \frac{\partial \hat{\omega}^2}{\partial x^3} \frac{\partial \hat{u}^3}{\partial x^2} + \frac{\partial \hat{\omega}^3}{\partial x^3} \frac{\partial \hat{u}^3}{\partial x^3}
\end{bmatrix}.
\]
\[ (+\tilde{\omega}^1 \frac{\partial}{\partial x^1} \text{div} \, \hat{u} + \tilde{\omega}^2 \frac{\partial}{\partial x^2} \text{div} \, \hat{u} + \tilde{\omega}^3 \frac{\partial}{\partial x^3} \text{div} \, \hat{u}). \]

Since \( \text{div} \, \hat{u} = 0 \), we get

\[ \text{div} [-(\hat{u}, \nabla) \tilde{\omega}] + \text{div} [(\tilde{\omega}, \nabla) \hat{u}] = -(\hat{u}^1 \frac{\partial}{\partial x^1} \text{div} \, \tilde{\omega} + \hat{u}^2 \frac{\partial}{\partial x^2} \text{div} \, \tilde{\omega} + \hat{u}^3 \frac{\partial}{\partial x^3} \text{div} \, \tilde{\omega}). \]

Hence, taking \( \text{div} \) of (4.3) gives the following equation for \( \text{div} \, \tilde{\omega} \):

\[ \frac{\partial \text{div} \, \tilde{\omega}}{\partial t} - (\hat{u}^1 \frac{\partial}{\partial x^1} \text{div} \, \tilde{\omega} + \hat{u}^2 \frac{\partial}{\partial x^2} \text{div} \, \tilde{\omega} + \hat{u}^3 \frac{\partial}{\partial x^3} \text{div} \, \tilde{\omega}) + \frac{\sigma^2}{2} \Delta \text{div} \, \tilde{\omega} = 0, \tag{4.8} \]

\[ t_k \leq t < t_{k+1}, \quad \text{div} \, \tilde{\omega}(t_{k+1}, x) = 0. \tag{4.9} \]

From here, due to uniqueness of solution to the problem (4.8)-(4.9), we obtain

\[ \text{div} \, \tilde{\omega}(t, x) = 0, \quad t_k \leq t \leq t_{k+1}, \quad x \in \mathbb{R}^3. \]

Theorem 4.1 is proved.

### 4.3 The one-step error of the method

For estimating the local error (the one-step error) in the 2D case, together with the solution \( \omega(t, x), \ t_k \leq t \leq t_{k+1}, \) of (2.36), we consider the approximation \( \tilde{\omega}(t, x) \), which satisfies the equation

\[ \frac{\partial \tilde{\omega}}{\partial t} - \hat{u}^1(x) \frac{\partial \tilde{\omega}}{\partial x^1}(t, x) - \hat{u}^2(x) \frac{\partial \tilde{\omega}}{\partial x^2}(t, x) + \frac{\sigma^2}{2} \Delta \tilde{\omega}(t, x) + g(t, x^1, x^2) = 0 \tag{4.10} \]

and the Cauchy condition

\[ \tilde{\omega}(t_{k+1}, x) = \omega(t_{k+1}, x). \tag{4.11} \]

The difference

\[ \delta \omega(t, x) := \omega(t, x) - \tilde{\omega}(t, x), \]

which is the one step error, is a solution to the problem

\[ \frac{\partial \delta \omega}{\partial t} + \frac{\sigma^2}{2} \Delta \delta \omega - u^1 \frac{\partial \delta \omega}{\partial x^1} - u^2 \frac{\partial \delta \omega}{\partial x^2} - (u^1 - \hat{u}^1) \frac{\partial \tilde{\omega}}{\partial x^1} - (u^2 - \hat{u}^2) \frac{\partial \tilde{\omega}}{\partial x^2} = 0, \tag{4.12} \]

\[ \delta \omega(t_{k+1}, x) = 0. \tag{4.13} \]

**Theorem 4.2** The one-step error of \( \tilde{\omega}(t, x), \ t_k \leq t \leq t_{k+1}, \) which solves (4.3)-(4.4) is of second order with respect to \( h \):

\[ |\delta \omega(t, x)| \leq Kh^2, \ t_k \leq t \leq t_{k+1}, \quad x \in \mathbb{R}^2. \tag{4.14} \]

**Proof.** Let us write the probabilistic representation of the form (3.31)-(3.35) for the solution to problem (4.12)-(4.13):

\[ dX^i = \sigma dw_i(s), \quad X^i(t) = x^i, \ i = 1, 2, \tag{4.15} \]

\[ dQ = -Q(u^1 dw_1 + u^2 dw_2), \quad Q(t) = 1, \tag{4.16} \]
\[ dZ = -Q((u^1 - \hat{u}^1) \frac{\partial \tilde{\omega}}{\partial x^1} + (u^2 - \hat{u}^2) \frac{\partial \tilde{\omega}}{\partial x^2})ds, \quad Z(t) = 0, \quad (4.17) \]

\[ \delta \omega(t, x) = -E \int_t^{t_{k+1}} Q((u^1 - \hat{u}^1) \frac{\partial \tilde{\omega}}{\partial x^1} + (u^2 - \hat{u}^2) \frac{\partial \tilde{\omega}}{\partial x^2})ds. \quad (4.18) \]

Using boundedness of \( \frac{\partial \tilde{\omega}}{\partial x^i}, \quad i = 1, 2 \), and the inequalities

\[ |u^i(s, X_{t,x}(s)) - \hat{u}^i(X_{t,x}(s))| = |u^i(s, X_{t,x}(s)) - u^i(t_{k+1}, X_{t,x}(s))| \leq Ch, \]

for \( t_k \leq s \leq t_{k+1} \), we get

\[ |\delta \omega(t, x)| \leq \int_t^{t_{k+1}} E|Q|ds \cdot Kh. \]

But \( Q > 0 \) and \( E|Q| = EQ = 1 \), whence \((4.14)\) follows. Theorem 4.2 is proved.

Introduce the one-step error for \( \tilde{u}(t, x) \) from \((4.5)\):

\[ \delta u(t, x) := u(t, x) - U \tilde{\omega}(t, x) = U \delta \omega(t, x), \quad (4.19) \]

where \( \tilde{\omega}(t, x), \ t_k \leq t \leq t_{k+1}, \) is the solution of \((4.3)-(4.4)\).

**Corollary 4.1** The one-step error of \( \tilde{u}(t, x) \) from \((4.5)\) is of second order with respect to \( h \) in \( L^2 \)-norm:

\[ ||\delta u(t, x)||_{L^2} \leq Kh^2, \ t_k \leq t \leq t_{k+1}. \quad (4.20) \]

**Proof.** Let the Fourier coefficients for \( \delta \omega(t, x) \) be \((\delta \omega(t, \cdot))_n\), i.e.

\[ \delta \omega(t, x) = \sum(\delta \omega(t, \cdot))_n e^{i(2\pi/L)(n,x)}. \]

Hence (cf. \((2.35)\))

\[ \delta u(t, x) = \frac{iL}{2\pi} \sum \frac{1}{|n|^2} e^{i(2\pi/L)(n,x)}(\delta \omega(t, \cdot))_n \left[ \begin{array}{c} n^2 \\ -n^1 \end{array} \right], \]

i.e., the Fourier coefficients for \( \delta u(t, x) \) are

\[ (\delta u(t, \cdot))_n = \frac{iL}{2\pi} \frac{1}{|n|^2} (\delta \omega(t, \cdot))_n \left[ \begin{array}{c} n^2 \\ -n^1 \end{array} \right]. \]

Then, by Parseval’s identity \((2.13)\), we have

\[ ||\delta u(t, \cdot)||_{L^2}^2 = \int_Q |\delta u(t, x)|^2 dx = L^2 \sum ||(\delta u(t, \cdot))_n||^2 \]

\[ = \frac{L^4}{4\pi^2} \sum \frac{|(\delta \omega(t, \cdot))_n|^2 (n^1)^2 + (n^2)^2}{|n|^4} \]

\[ = \frac{L^4}{4\pi^2} \sum \frac{|(\delta \omega(t, \cdot))_n|^2}{|n|^2} \leq \frac{L^4}{4\pi^2} \sum ||(\delta \omega(t, \cdot))_n||^2 \]


\[
\frac{L^2}{4\pi^2} \int_Q |\delta_\omega(t, x)|^2 dx \leq \frac{L^2}{4\pi^2} \max_x |\delta_\omega(t, x)|^2,
\]

which together with (4.14) implies (4.20). Corollary 4.1 is proved.

The result of Theorem 4.2 is carried over to the 3D case without any substantial changes in the proof. In the 3D case the difference \( \delta_\omega(t, x) := \omega(t, x) - \tilde{\omega}(t, x) \) is a solution to the problem

\[
\frac{\partial \delta_\omega}{\partial t} + \frac{\sigma^2}{2} \Delta \delta_\omega - \sum_{i=1}^3 u^i \frac{\partial \delta_\omega}{\partial x^i} + \sum_{i=1}^3 \frac{\partial u^i}{\partial x^i} \delta_\omega - \sum_{i=1}^3 (u^i - \hat{u}^i) \frac{\partial \tilde{\omega}}{\partial x^i} + \sum_{i=1}^3 (\frac{\partial u^i}{\partial x^i} - \frac{\partial \hat{u}^i}{\partial x^i}) \omega^i = 0,
\]

(4.22)

\[
\delta_\omega(t_{k+1}, x) = 0.
\]

(4.23)

**Theorem 4.3** The one-step error of \( \tilde{\omega}(t, x) \), \( t_k \leq t \leq t_{k+1} \), which solves (4.3)-(4.4), is of second order with respect to \( h \):

\[
|\delta_\omega(t, x)| \leq Kh^2, \; t_k \leq t \leq t_{k+1}, \; x \in \mathbb{R}^3.
\]

(4.24)

**Proof.** We apply the probabilistic representation (3.31)-(3.35) to the solution of (4.22)-(4.23):

\[
dX^i = \sigma dw_i(s), \; X^i(t) = x^i, \; i = 1, 2, 3, \quad (4.25)
\]

\[
dY^i = \sum_{j=1}^3 \frac{\partial u^j}{\partial x^i} Y^j ds, \; Y^i(t) = y^i, \; i = 1, 2, 3, \quad (4.26)
\]

\[
dQ = -Q \sum_{j=1}^3 u^j(s, X) dw_j(s), \; Q(t) = 1, \quad (4.27)
\]

\[
dZ = Q \left[ \sum_{j=1}^3 \sum_{i=1}^3 \left( \frac{\partial u^j}{\partial x^i} - \frac{\partial \hat{u}^j}{\partial x^i} \right) \omega^i Y^j - \sum_{j=1}^3 \sum_{i=1}^3 (u^i - \hat{u}^i) \frac{\partial \tilde{\omega}^j}{\partial x^i} Y^j \right] ds, \; Z(t) = 0, \quad (4.28)
\]

\[
E \int_t^{t_{k+1}} \delta_\omega^\top(t, y) = EZ_{t,x,y,10}(t_{k+1}) \quad (4.29)
\]

Using boundedness of \( \omega^i, \partial \omega^i / \partial x^i, Y^i(s), \; i, j = 1, 2, 3 \), the inequalities

\[
|u^i(s, X_{t,x}(s)) - \hat{u}^i(X_{t,x}(s))| = |u^i(s, X_{t,x}(s)) - u^i(t_{k+1}, X_{t,x}(s))| \leq Ch,
\]

\[
\left| \frac{\partial u^j}{\partial x^i}(s, X_{t,x}(s)) - \frac{\partial \hat{u}^j}{\partial x^i}(X_{t,x}(s)) \right| = \left| \frac{\partial u^j}{\partial x^i}(s, X_{t,x}(s)) - \frac{\partial \hat{u}^j}{\partial x^i}(t_{k+1}, X_{t,x}(s)) \right| \leq Ch,
\]

for \( t_k \leq s < t_{k+1} \), and the properties \( Q > 0, \; E|Q| = EQ = 1 \), we get (4.24). Theorem 4.3 is proved.

We note that the one-step error estimate (4.20) for \( \tilde{u} \) from Corollary 4.1 is also valid in the three-dimensional case.
4.4 Convergence theorems

In this section we first consider the global error for the approximation $\tilde{u}(t, x)$ from (4.5), i.e., we are interested in estimating the difference

$$D_{\tilde{u}} := u(t_0, x) - \tilde{u}(t_0, x),$$

where $u(t_0, x)$ is the solution of the NSE (2.5)-(2.8).

Let us introduce the auxiliary functions $k u(t, x)$ on the time intervals $[t_0, t_k]$, $k = 1, \ldots, N$:

$$k u(t, x) := u(t, t_k, \tilde{u}(t_k, \cdot)), \quad t_0 \leq t \leq t_k, \quad (4.30)$$

where $u(t; t_k, \tilde{u}(t_k, \cdot))$ denotes the solution of the NSE (2.5)-(2.8) with the terminal condition $\varphi(\cdot) = \tilde{u}(t_k, \cdot)$ prescribed at $T = t_k$. To prove the convergence theorem, we assume that all the functions $k u(t, x)$ are bounded together with their derivatives up to some order.

Since $\tilde{u}(t_N, x) = u(t_N, x)$, we have $N u(t, x) = u(t, x)$, $t_0 \leq t \leq t_N$. Also, note that $\tilde{u}(t_0, x) = u(t_0, x)$. Then we can re-write the global error as

$$D_{\tilde{u}} = \sum_{k=0}^{N-1} (k+1 u(t_0, x) - k u(t_0, x)). \quad (4.31)$$

We have

$$k+1 u(t_0, x) = u(t_0, x; t_{k+1}, \tilde{u}(t_{k+1}, \cdot)) = u(t_0, x; t_k, u(t_k, \cdot; t_{k+1}, \tilde{u}(t_{k+1}, \cdot))), \quad (4.32)$$

$$k u(t_0, x) = u(t_0, x; t_k, \tilde{u}(t_k, \cdot)).$$

Note that the difference

$$k \delta_u(t_k, x) = u(t_k, x; t_{k+1}, \tilde{u}(t_{k+1}, \cdot)) - \tilde{u}(t_k, x)$$

is a one-step error (see (4.19)), which $L^2$-estimate is of order $h^2$ according to Corollary 4.1. We remark that $k+1 u_0 - k u_0$ is the propagation error which is due to the error in the terminal condition propagated along the trajectory of the NSE solution.

To estimate the propagation error, we are making use of the basic energy estimate from [15, p. 89], where it is proved in the whole space, but it can be derived for the periodic case as well. In our case this energy estimate takes the form

$$\sup_{t_0 \leq t \leq t_k} \| k+1 u(t_0, \cdot) - k u(t_0, \cdot) \|_{L^2} \leq C \| k+1 u(t_k, \cdot) - k u(t_k, \cdot) \|_{L^2}, \quad (4.33)$$

where the constant $C > 0$ depends on the function $k u(t, x)$.

Due to (4.20) and (4.33), we obtain

$$\| k+1 u(t_0, \cdot) - k u(t_0, \cdot) \|_{L^2} \leq K h^2, \quad (4.34)$$

where $K > 0$ combines the constant $K$ from (4.20) and $C$ from (4.33). From (4.34) and (4.31), we get

$$\| D_{\tilde{u}} \|_{L^2} \leq Kh.$$

Thus, we have proved the following theorem.
Theorem 4.4  The approximation $\tilde{u}(t, x)$ from (4.5) for the solution of the NSE (2.3)-(2.8) is of first order in $h$.

We note that the proof of Theorem 4.4 tacitly used an assumption of existence, uniqueness and regularity of solutions of the NSE problems involved in the error estimates. Such an assumption is natural to make in the work aimed at deriving approximations and we do not consider here how one can prove such properties of $k u(t, x)$ from (4.30).

Now we analyse the global error of $\tilde{\omega}_k(t, x)$.

Theorem 4.5  The approximation $\tilde{\omega}_k(t, x)$ (see (1.5), (1.6)) for the solution of the NSE (2.3)-(2.8) converges with order 1 in $L^2$-norm.

Proof.  Let $D_\omega(t, x; k)$ be the global error for $\tilde{\omega}$ on the interval $[t_k, t_{k+1}]$, i.e.

$$D_\omega(t, x; k) := \omega(t, x) - \tilde{\omega}_k(t, x),$$

and $D_\tilde{u}(t, x; k)$ be the global error for $\tilde{u}_k$ on the interval $[t_k, t_{k+1}]$, i.e.

$$D_\tilde{u}(t, x; k) := u(t, x) - \tilde{u}_k(x).$$

We have analogously to (1.12)-(1.13):

$$-\frac{\partial D_\omega(t, x; k)}{\partial t} = \frac{\sigma^2}{2} \Delta D_\omega(t, x; k) - (u(t, x), \nabla) D_\omega(t, x; k)$$

$$(D_\tilde{u}(t, x; k), \nabla) \tilde{\omega}_k(t, x), \quad t_k \leq t < t_{k+1}, \quad k = N - 1, \ldots, 0, \quad D_\omega(t_N, x; N - 1) = 0,$$

$$D_\omega(t_{k+1}, x; k) = D_\omega(t_{k+1}, x; k + 1), \quad k = N - 2, \ldots, 0. \quad (4.37)$$

Then

$$-\frac{1}{2} \frac{d}{dt} ||D_\omega(t, \cdot; k)||^2 = -\frac{\sigma^2}{2} ||\nabla D_\omega||^2 - ((u, \nabla) D_\omega, D_\omega) - ((D_\tilde{u}(t, \cdot; k), \nabla) \tilde{\omega}_k, D_\omega), \quad (4.38)$$

$$t_k \leq t < t_{k+1}.$$  

Since $u$ is divergence free, we get

$$((u, \nabla) D_\omega, D_\omega) = 0. \quad (4.39)$$

Note that (see (4.7)):

$$D_\tilde{u}(t, x; k) = u(t, x) - \tilde{u}_k(x) = U \omega(t, x) - U \tilde{\omega}_k(t_{k+1}, x)$$

$$= UD_\omega(t, x; k) + U \tilde{\omega}_k(t, x) - U \tilde{\omega}_k(t_{k+1}, x).$$

Then, using (2.42) with $c_2 = \sigma^2/2$, we obtain for some $K > 0$:

$$||(D_\tilde{u}(t, \cdot; k), \nabla) \tilde{\omega}_k, D_\omega)|| \leq \frac{\sigma^2}{2} ||\nabla D_\omega||^2 + K ||D_\omega||^2$$

$$+ K ||\nabla \tilde{\omega}_k||^2 ||D_\omega + \tilde{\omega}_k(t, \cdot) - \tilde{\omega}_k(t_{k+1}, \cdot)||^2$$

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Using boundedness of \( \|\frac{d}{dt}\bar{\omega}_k(t, \cdot)\|^2 \), we get
\[
\|\bar{\omega}_k(t, \cdot) - \bar{\omega}_k(t_{k+1}, \cdot)\|^2 \leq Kh^2,
\]
which together with boundedness of \( \|\nabla\bar{\omega}_k\|^2 \) implies
\[
\|((D_k(t, \cdot; k), \nabla)\bar{\omega}_k, D_\omega)| \leq \frac{\sigma^2}{2}\|\nabla D_\omega\|^2 + K\|D_\omega\|^2 + Kh^2. \tag{4.41}
\]
It follows from (4.38), (4.39) and (4.41) that
\[
\frac{-d (\|D_\omega(t, \cdot; k)\|^2 + h^2)}{\|D_\omega(t, \cdot; k)\|^2 + h^2} \leq 2K dt, \ t_k \leq t < t_{k+1}.
\]
Then
\[
\|D_\omega(t_k, \cdot; k)\|^2 + h^2 \leq e^{2Kh}(\|D_\omega(t_{k+1}, \cdot; k)\|^2 + h^2).
\]
From here and due to (4.37), we get
\[
\|D_\omega(t_k, \cdot; k-1)\|^2 \leq e^{2Kkh}\|D_\omega(t_{k+1}, \cdot; k)\|^2 + (e^{2Kk} - 1) h^2, \ k = N - 1, \ldots, 1.
\]
Denoting \( R_k := \|D_\omega(t_{k+1}, \cdot; k)\|^2, \ k = N - 1, \ldots, 0 \), we obtain (see (4.36)):
\[
R_{k-1} = e^{2Kh} R_k + (e^{2Kh} - 1) h^2, \ k = N - 1, \ldots, 1,
\]
\[
R_N = 0,
\]
and using the discrete Gronwall lemma (see e.g. [23, p. 7]), we arrive at \( R_0 = \|D_\omega(t_1, \cdot; 0)\|^2 \leq Kh^2 \). Theorem 4.3 is proved.

5 Stochastic Navier-Stokes equations

In this section we carry over the results of Section 4 for the deterministic NSE to two-dimensional NSE with additive noise. After introducing the stochastic NSE in velocity-vorticity formulation, we prove two auxiliary lemmas (Section 5.1) about its solution; we consider a one-step approximation of vorticity and its properties (Section 5.2); we introduce the numerical method for vorticity and prove boundedness of its moments in Section 5.3 and, finally, we prove first-order mean-square convergence of the method in Section 5.4. The global convergence proof contains ideas, which can potentially be exploited in analysis of numerical methods for a wider class of semilinear SPDEs.

Let \((\Omega, \mathcal{F}, P)\) be a probability space and \((w(t), \mathcal{F}^w_t) = ((w_1(t), \ldots, w_q(t))^\top, \mathcal{F}_t)\) be a \(q\)-dimensional standard Wiener process, where \( \mathcal{F}_t, 0 \leq t \leq T \), is an increasing family of \(\sigma\)-algebras of \( \mathcal{F} \) induced by \(w(t)\). We consider the system of stochastic Navier-Stokes equations (SNSE) with additive noise for velocity \(v\) and pressure \(p\) in a viscous incompressible flow:
\[
dv(t) = \left[\frac{\sigma^2}{2}\Delta v - (v, \nabla)v - \nabla p + f(t, x)\right] dt + \sum_{r=1}^q \gamma_r(t, x) dw_r(t), \tag{5.1}
\]
with spatial periodic conditions
\[ v(t, x + Le_i) = v(t, x), \quad p(t, x + Le_i) = p(t, x), \]
\[ 0 \leq t \leq T, \quad i = 1, 2, \tag{5.3} \]
and the initial condition
\[ v(0, x) = \varphi(x). \tag{5.4} \]

In (5.1)-(5.3), \(v, f, \) and \(\gamma_r\) are two-dimensional functions; \(p\) is a scalar; \(\{e_i\}\) is the canonical basis in \(\mathbb{R}^2\) and \(L > 0\) is the period. The functions \(f = f(t, x)\) and \(\gamma_r(t, x)\) are assumed to be spatial periodic as well. Further, we require that \(\gamma_r(t, x)\) are divergence free:
\[ \text{div} \, \gamma_r(t, x) = 0, \quad r = 1, \ldots, q. \tag{5.5} \]

For simplicity of proofs, we assume that the number of noises \(q\) is finite but it can be shown that the theoretical results of this section are also valid when \(q\) is infinite if \(\|\gamma_r(t, x)\|_m\) for some \(m \geq 0\) decay exponentially fast with increase of \(r\).

**Assumption 5.1.** We assume that the coefficients \(f(t, x)\) and \(\gamma_r(s, x), \quad r = 1, \ldots, q,\) belong to \(H^{m+1}_p(Q)\) and the initial condition \(\varphi(x)\) belongs to \(H^{m+2}_p(Q)\) for some \(m \geq 0\).

Under this assumption the problem (5.1)-(5.4) has a unique solution \(v(t, x), \quad p(t, x), \) \((t, x) \in [0, T] \times \mathbb{R}^2, \) so that for some \(m \geq 0\) and \(l \geq 2\) [16] [17]:
\[ E\|v(t, \cdot)\|_{m+2} \leq K, \tag{5.6} \]
where \(K > 0\) may depend on \(l, m, T, f(t, x), \gamma_r(t, x), \) and \(\varphi(x)\). The solution \(v(t, x), \quad p(t, x), \) \((t, x) \in [0, T] \times \mathbb{R}^2,\) to (5.1)-(5.4) is \(\mathcal{F}_r\)-adaptive, \(v(t, \cdot) \in V^{m+2}_p\) and \(\nabla p(t, \cdot) \in (V^{m+2}_p)^\perp\) for every \(t \in [0, T]\) and \(\omega \in \Omega\). We note that if we were interested in variational solutions of (5.1)-(5.4) then it is more natural to put \(m \geq -1\) in Assumption 5.1; but here our focus is on the vorticity formulation and then it is natural to require more, \(m \geq 0\).

The vorticity formulation of the problem (5.1)-(5.4) has the form
\[ d\omega = \left[ \frac{\sigma^2}{2} \Delta \omega - (v, \nabla)\omega + g(t, x) \right] dt + \sum_{r=1}^q \mu_r(t, x) dw_r(t), \tag{5.7} \]
where \(g = \text{curl} \, f\) and \(\mu_r = \text{curl} \, \gamma_r\). The vorticity satisfies the initial and periodic boundary conditions
\[ \omega(0, x) = \text{curl} \, \varphi(x) := \phi(x) \tag{5.8} \]
and spatial periodic conditions
\[ \omega(t, x + Le_i) = \omega(t, x), \quad i = 1, 2, \quad 0 \leq t \leq T. \tag{5.9} \]

We note that \(\omega(t, x)\) is a one-dimensional function here. Using the linear operator \(U\) from (2.35), we can re-write (5.7) as
\[ d\omega = \left[ \frac{\sigma^2}{2} \Delta \omega - (U\omega, \nabla)\omega + g(t, x) \right] dt + \sum_{r=1}^q \mu_r(t, x) dw_r(t). \tag{5.10} \]
Similarly to the solution \( v(t, x) \) of (5.1)-(5.4), the solution \( \omega(t, x) \) to the vorticity problem (5.7)-(5.9) under Assumption 5.1 is so that for some \( m \geq 0 \) and \( p \geq 2 \):
\[
E \| \omega(t, \cdot) \|_{m+1}^p \leq K,
\]
where \( K > 0 \) depends on \( p, m, g, \mu_r \), and \( \phi \). Note that under Assumption 5.1 the coefficients \( g(t, x) \) and \( \mu_r(s, x) \), \( r = 1, \ldots, q \), belong to \( H^m_p(Q) \) and the initial condition \( \phi(x) \) belongs to \( H^{m+1}_p(Q) \). As it is clear from the context, we are dealing here with solutions understood in the strong sense probabilistically and PDE-wise in the variational sense.

### 5.1 Two technical lemmas

For proving convergence of the numerical method in Section 5.4, we need two further properties of the solution \( \omega(t, x) \) which are formulated in the next two lemmas.

It is convenient to introduce the notation for the solution \( \omega(t, x) \) of the problem (5.7)-(5.9) which reflects its dependence on the initial condition \( \phi(x) \) prescribed at time \( s \leq t \):
\[
\omega(t, x) = \omega(t, x; s, \phi).
\]

Let us prove a technical lemma which is related to Lemmas 4.10(1) and A.1 from [12].

**Lemma 5.1** Let Assumption 5.1 hold with \( m = 0 \). There exist constants \( \beta_0 > 0 \) and \( \alpha > 0 \) such that for any \( \beta \in (0, \beta_0] \) and \( 0 \leq t \leq t + h \leq T \):
\[
E \exp \left( \beta \left[ ||\omega(t + h, \cdot, \phi)||^2 - ||\phi||^2 \right] + \frac{\beta \sigma^2}{4} \int_t^{t+h} ||\nabla \omega(s, \cdot, \phi)||^2 ds \right) \leq \exp \left( \beta \int_t^{t+h} \left( \frac{2}{\alpha \sigma^2} ||g(s, \cdot)||^2 + \sum_{r=1}^q ||\mu_r(s, \cdot)||^2 \right) ds \right).
\]

**Proof.** By the Ito formula, integration by parts and using \( \text{div} \ v = 0 \), we obtain
\[
\frac{1}{2} d||\omega(s, \cdot)||^2 = \left[ -\frac{\sigma^2}{2} ||\nabla \omega(s, \cdot)||^2 + g(s, \cdot, \omega(s, \cdot)) + \frac{1}{2} \sum_{r=1}^q ||\mu_r(s, \cdot)||^2 \right] ds
\]
\[
+ \sum_{r=1}^q (\mu_r(s, \cdot), \omega(s, \cdot)) dw_r(s), \quad t < s \leq t + h,
\]
\[
||\omega(t, \cdot)||^2 = ||\phi||^2.
\]

Using the elementary inequality, we get for any \( \alpha > 0 \):
\[
\frac{1}{2} d||\omega(s, \cdot)||^2 \leq \left[ -\frac{\sigma^2}{2} ||\nabla \omega(s, \cdot)||^2 + \frac{1}{\alpha \sigma^2} ||g(s, \cdot)||^2 + \frac{\sigma^2}{4} \alpha ||\omega(s, \cdot)||^2 + \frac{1}{2} \sum_{r=1}^q ||\mu_r(s, \cdot)||^2 \right] ds
\]
\[
+ \sum_{r=1}^q (\mu_r(s, \cdot), \omega(s, \cdot)) dw_r(s).
\]
By Poincare's inequality (2.18), for some $\alpha > 0$, we have
\[ ||\nabla \omega(t, \cdot)||^2 \geq \alpha ||\omega(t, \cdot)||^2. \] (5.15)

By (5.15), we obtain
\[
\begin{align*}
\frac{d}{ds} ||\omega(s, \cdot)||^2 &\leq \left[ -\frac{\sigma^2}{4} ||\nabla \omega(s, \cdot)||^2 - \frac{\sigma^2}{4} \alpha ||\omega(s, \cdot)||^2 + \frac{2}{\alpha \sigma^2} ||g(s, \cdot)||^2 \\
&\quad + \sum_{r=1}^{q} ||\mu_r(s, \cdot)||^2 \right] ds + 2 \sum_{r=1}^{q} (\mu_r(s, \cdot), \omega(s, \cdot)) dw_r(s),
\end{align*}
\] (5.16)
then for any $c > 0$
\[
\begin{align*}
&c ||\omega(t + h, \cdot)||^2 - c ||\phi||^2 + \frac{\sigma^2}{4} \int_{t}^{t+h} ||\nabla \omega(s, \cdot)||^2 ds \\
&\quad - c \int_{t}^{t+h} \left( \frac{2}{\alpha \sigma^2} ||g(s, \cdot)||^2 + \sum_{r=1}^{q} ||\mu_r(s, \cdot)||^2 \right) ds \\
&\quad \leq 2c \int_{t}^{t+h} \sum_{r=1}^{q} (\mu_r(s, \cdot), \omega(s, \cdot)) dw_r(s) - \alpha \sigma^2 c \int_{t}^{t+h} ||\omega(s, \cdot)||^2 ds.
\end{align*}
\] (5.17)

Let
\[ M(t, t') := 2c \int_{t}^{t'} \sum_{r=1}^{q} (\mu_r(s, \cdot), \omega(s, \cdot)) dw_r(s) \]
which is a continuous $L^2$-martingale with quadratic variation
\[ < M > (t, t') := 4c^2 \int_{t}^{t'} \sum_{r=1}^{q} (\mu_r(s, \cdot), \omega(s, \cdot))^2 ds. \]

There exists a constant $\beta_0 > 0$ (independent of $h$ and $c$) so that for all $\beta \in (0, \beta_0]$
\[ \alpha \sigma^2 c \int_{t}^{t'} ||\omega(s, \cdot)||^2 ds \geq \frac{\beta}{2c} < M > (t, t'). \]
Hence
\[
\begin{align*}
&c ||\omega(t + h, \cdot)||^2 - c ||\phi||^2 + \frac{\sigma^2}{4} \int_{t}^{t+h} ||\nabla \omega(s, \cdot)||^2 ds \\
&\quad - c \int_{t}^{t+h} \left( \frac{2}{\alpha \sigma^2} ||g(s, \cdot)||^2 + \sum_{r=1}^{q} ||\mu_r(s, \cdot)||^2 \right) ds \\
&\quad \leq M(t, t + h) - \frac{\beta}{2c} < M > (t, t + h).
\end{align*}
\] (5.18)
For $c = \beta$, the right-hand side of (5.18) is logarithm of a local exponential martingale and therefore
\[
E \exp \left[ \beta ||\omega(t + h \wedge \tau_n, \cdot)||^2 - \beta ||\phi||^2 + \frac{\sigma^2}{4} \int_{t}^{t+h \wedge \tau_n} ||\nabla \omega(s, \cdot)||^2 ds \right]
\]
where \( \tau_n = \inf \{ s > t : M > (t, s) \geq n \} \) for a natural number \( n \). Tending \( n \) to infinity, we arrive at (5.12). Lemma 5.1 is proved.

Note that it follows from (5.12) that

\[
E \exp \left( \frac{\beta \sigma^2}{4} \int_t^{t+h} ||\nabla \omega(s, \cdot; t, \phi)||^2 ds \right) \leq \exp \left( \beta ||\phi||^2 + \beta \int_t^{t+h} \left( \frac{2}{\alpha \sigma^2} ||g(s, \cdot)||^2 + \sum_{r=1}^q ||\mu_r(s, \cdot)||^2 \right) ds \right).
\]

We also pay attention that the prove of Lemma 5.1 is not relying on smallness of the time step \( h \) and, after replacing \( t \) with 0 and \( t + h \) with \( T \), the result remains valid:

\[
E \exp \left( \frac{\beta \sigma^2}{4} \int_0^T ||\nabla \omega(s, \cdot; 0, \phi)||^2 ds \right) \leq \exp \left( \beta ||\phi||^2 + \beta \int_0^T \left( \frac{2}{\alpha \sigma^2} ||g(s, \cdot)||^2 + \sum_{r=1}^q ||\mu_r(s, \cdot)||^2 \right) ds \right).
\]

We now prove the next lemma which gives us dependence of the solution \( \omega(s, x; t, \phi) \) on the initial data.

**Lemma 5.2** Let Assumption 5.1 hold with \( m = 2 \) and \( \phi_i(t, x), i = 1, 2 \), be \( \mathcal{F}_t \)-measurable processes satisfying (5.17) with \( m = 2 \). There exists a constant \( c_0 > 0 \) such that for every \( c \in (0, c_0) \) there is a sufficiently small \( h > 0 \) so that we have for \( t \leq s \leq t + h \):

\[
\omega(s, x; t, \phi_1(t, \cdot)) - \omega(s, x; t, \phi_2(t, \cdot)) = \phi_1(t, x) - \phi_2(t, x) + \eta(s, x)
\]

for which

\[
||\omega(s, \cdot; t, \phi_1) - \omega(s, \cdot; t, \phi_2)||^2 \leq ||\phi_1(t, \cdot) - \phi_2(t, \cdot)||^2 \exp \left( K(s - t) + c \int_t^s ||\nabla \omega(s', \cdot; t, \phi_1(t, \cdot))||^2 ds' \right),
\]

where \( K > 0 \) is a constant.

The process \( \eta(s) \) satisfies the following estimate

\[
||\eta(s, \cdot)||^2 \leq (s - t)||\phi_1(t, \cdot) - \phi_2(t, \cdot)||^2 + C(s, \omega)(s - t)^3,
\]

where \( C(s, \omega) > 0 \) is an \( \mathcal{F}_s \)-adapted process with bounded moments of a sufficiently high order.

**Proof.** Let

\[
\theta(s, x) := \omega(s, x; t, \phi_1) - \omega(s, x; t, \phi_2)
\]

We have

\[
d\theta(s, x) = \left[ \frac{\sigma^2}{2} \Delta \theta - (U \theta, \nabla)\omega(s, \cdot; t, \phi_1) - (U \omega(s, \cdot; t, \phi_2), \nabla)\theta \right] ds, \ t < s \leq t + h,
\]

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\[ \theta(t,x) = \phi_1(t,x) - \phi_2(t,x). \]

Then
\[ \frac{1}{2} \frac{d}{dt} ||\theta(s,\cdot)||^2 = \left[ -\frac{\sigma^2}{2} ||\nabla \theta(s,\cdot)||^2 - ((U\theta(s,\cdot), \nabla)\omega(s,\cdot; t, \phi_1), \theta(s,\cdot)) \right] ds, \ t < s \leq t + h, \]

\[ (5.24) \]

\[ ||\theta(t,\cdot)||^2 = ||\phi_1(t,\cdot) - \phi_2(t,\cdot)||^2. \]

Using the inequality (2,42) with \( c_2 = \sigma^2 / 4 \), we have that there exists \( K > 0 \) such that for any \( c > 0 \)
\[ 2 ||((U\theta(s,\cdot), \nabla)\omega(s,\cdot; t, \phi_1), \theta(s,\cdot))|| \leq \frac{\sigma^2}{2} ||\nabla \theta(s,\cdot)||^2 + K ||\theta(s,\cdot)||^2 (5.25) \]
and hence
\[ d||\theta(s,\cdot)||^2 \leq [K + c||\nabla \omega(s,\cdot; t, \phi_1)||^2] ||\theta(s,\cdot)||^2 ds, \ t < s \leq t + h, \]
which implies
\[ ||\theta(s,\cdot)||^2 \leq ||\phi_1(t,\cdot) - \phi_2(t,\cdot)||^2 \exp \left( K(s - t) + c \int_t^s ||\nabla \omega(s',\cdot; t, \phi_1(t,\cdot))||^2 ds' \right). \]  \( (5.26) \)

Thus we have proved the inequality (5.22).

Let us now prove (5.23). We have
\[ \eta(s,x) = \int_t^s \left[ \frac{\sigma^2}{2} \Delta \theta - (U\theta, \nabla)\omega(s',\cdot; t, \phi_1) - (U\omega(s',\cdot; t, \phi_2), \nabla)\theta \right] ds', \ t < s \leq t + h, \]
which together with (5.6) and (5.11) implies that
\[ ||\eta(s,x)|| \leq C(s,\omega)(s - t), \]
where \( C(s,\omega) > 0 \) is an \( \mathcal{F}_s \)-adapted process with bounded moments of a sufficiently high order. It is not difficult to see that the inequality (5.27) is also valid for \( ||\nabla \eta(s,x)|| \) and \( ||\Delta \eta(s,x)|| \):
\[ ||\nabla \eta(s,x)|| \leq C(s,\omega)(s - t), \ ||\Delta \eta(s,x)|| \leq C(s,\omega)(s - t). \]  \( (5.28) \)

We have
\[ d||\eta(s',x)||^2 = [(\sigma^2 \Delta \eta, \eta) - 2((U\theta, \nabla)\omega(s',\cdot; t, \phi_1), \eta) - 2((U\omega(s',\cdot; t, \phi_2), \nabla)\theta, \eta)] ds'. \]

Using integration by parts, (5.22), and (5.28) (we also recall that \( s' - t \leq h \) which is sufficiently small), we get
\[ |(\sigma^2 \Delta \theta, \eta)| = \sigma^2 |(\theta, \Delta \eta)| \leq \sigma^2 ||\theta||||\Delta \eta|| \leq C(s',\omega)(s' - t)||\phi_1(t,\cdot) - \phi_2(t,\cdot)||. \]

By (2,14) with \( m_1 = 1, m_2 = 0, \) and \( m_3 = 1, (2,39), (5,11), (5,22), (5,27) \) and (5.28), we obtain
\[ |2((U\theta, \nabla)\omega(s,\cdot; t, \phi_1), \eta)| \leq K||U\theta||||\omega|| \leq K||\theta||||\omega||. \]
\[ \leq C(s', \omega)(s' - t)||\phi_1(t, \cdot) - \phi_2(t, \cdot)||. \]

And by (2.14) with \( m_1 = 1, m_2 = 1, \) and \( m_3 = 0, \) we arrive at
\[ |2((U\omega(s', \cdot; t, \phi_2), \nabla)\theta, \eta)| = 2|((U\omega(s', \cdot; t, \phi_2), \nabla)\eta, \theta)| \leq K||\omega||||\eta||||\theta|| \]
\[ \leq C(s', \omega)(s' - t)||\phi_1(t, \cdot) - \phi_2(t, \cdot)||. \]

Then we have
\[ d||\eta(s', x)||^2 \leq C(s', \omega)(s' - t)||\phi_1(t, \cdot) - \phi_2(t, \cdot)|| ds' \]
\[ \leq ||\phi_1(t, \cdot) - \phi_2(t, \cdot)||^2 ds' + \frac{C^2(s', \omega)}{4}(s' - t)^2 ds' \]
from which (5.23) follows. Lemma 5.2 is proved.

### 5.2 One-step approximation

Similarly to derivation of the approximation for the deterministic NSE in Section 4, we can approximate the stochastic NSE (5.7)-(5.9) by freezing the velocity as in (4.2):

\[ v(t, x) \approx \hat{v}(t, x) := v(t_k, x) := \hat{v}(x), \quad t_k < t \leq t_{k+1}, \quad (5.29) \]

and obtain an approximation \( \tilde{\omega}(t, x) \) of \( \omega(t, x) \) on \( t_k \leq t \leq t_{k+1}, \) as follows

\[ d\tilde{\omega} = \left[ \frac{\sigma^2}{2} \Delta \tilde{\omega} - (\hat{v}, \nabla)\tilde{\omega} + g(t, x) \right] dt + \sum_{r=1}^{q} \mu_r(t, x) dw_r(t), \quad t_k < t \leq t_{k+1}, \quad (5.30) \]
\[ \tilde{\omega}(t_k, x) = \omega(t_k, x), \quad \tilde{\omega}(t_k + Le_j) = \tilde{\omega}(t_k, x), \quad j = 1, 2. \quad (5.31) \]

It is not difficult to see that the local error \( \delta_{\omega}(t, x) = \tilde{\omega}(t, x) - \omega(t, x), \) \( t_k \leq t \leq t_{k+1}, \) for the approximation \( \tilde{\omega}(t, x) \) of the solution \( \omega(t, x) \) of the stochastic NSE (5.7)-(5.9) satisfies the problem of the same form as (4.12)-(4.13) but with positive direction of time:

\[ d\delta_{\omega} = \left[ \frac{\sigma^2}{2} \Delta \delta_{\omega} - (v, \nabla)\delta_{\omega} - ((v - \hat{v}), \nabla)\tilde{\omega} \right] dt, \quad (5.32) \]
\[ \delta_{\omega}(t_k, x) = 0. \quad (5.33) \]

We note that the main difference of (5.32)-(5.33) with (4.12)-(4.13) is that the functions in (5.32) are random and non-smooth in time, they have the same regularity in time as Wiener processes.

Moments of \( ||\tilde{\omega}||_3 \) (and hence of \( ||\delta_{\omega}||_3 \)) up to a sufficiently high order are bounded under Assumption 5.1 with \( m = 2: \) for \( t_k < t \leq t_{k+1} \) and \( p \geq 1: \)

\[ E||\tilde{\omega}(t, \cdot)||_3^{2p} \leq K, \quad (5.34) \]

where \( K > 0 \) is a constant, which can be proved by arguments similar to boundedness of the global approximation (see Theorems 5.2 and 5.3) but not considered here.

To obtain bounds for the one-step error \( \delta_{\omega}, \) we first prove the following lemma.
Lemma 5.3 Let Assumption 5.1 hold with \( m = 1 \). For \( v(t, x) \) from (5.11)-(5.14), \( \hat{v}(x) \) from (5.29), and \( \tilde{\omega}(t, x) \) from (5.30)-(5.31), we have for \( t_k < t \leq t_{k+1} \) and sufficiently small \( h > 0 \):\[
\|E[((v - \hat{v}), \nabla)\tilde{\omega}|\mathcal{F}_{t_k}]\| \leq C(t_k, \omega)h, \tag{5.35}
\]
\[
(E\|v - \hat{v}\|^2_p)^{1/2} \leq Kh^{1/2}, \quad p \geq 1, \tag{5.36}
\]
where \( C(t_k, \omega) > 0 \) is an \( \mathcal{F}_{t_k} \)-measurable random variable with moments of a sufficiently high order bounded by a constant independent of \( h \) and \( K > 0 \) is a constant independent of \( h \).

Proof. From (5.1) and (5.29), we have for \( t_k < t \leq t_{k+1} \):
\[
v(t, x) - \hat{v}(x) = \int_{t_k}^{t} \left[ \frac{\sigma^2}{2} \Delta v - (v, \nabla)v - \nabla p + f(s, x) \right] ds + \int_{t_k}^{t} \sum_{r=1}^{q} \gamma_r(s, x)dw_r(s). \tag{5.37}
\]
Then it is not difficult to obtain the estimate (5.36) using (5.6) and the assumptions on \( f \) and \( \gamma_r \).

From (5.37) and (5.31), we have
\[
((v - \hat{v}), \nabla)\tilde{\omega} = \left( \int_{t_k}^{t} \left[ \frac{\sigma^2}{2} \Delta v - (v, \nabla)v - \nabla p + f(s, x) \right] ds, \nabla \right) \tilde{\omega}(t, x)
\]
\[
+ \left( \int_{t_k}^{t} \sum_{r=1}^{q} \gamma_r(s, x)dw_r(s), \nabla \right)
\]
\[
\left\{ \tilde{\omega}(t_k, x) + \int_{t_k}^{t} \left[ \frac{\sigma^2}{2} \Delta \tilde{\omega} - (\hat{v}, \nabla)\tilde{\omega} + g(s, x) \right] ds + \int_{t_k}^{t} \sum_{r=1}^{q} \mu_r(s, x)dw_r(s) \right\}
\]
from which it is not difficult to see that the inequality (5.35) holds. Lemma 5.3 is proved.

Now we proceed to proving estimates for the one-step error of \( \tilde{\omega}(t, x) \).

Theorem 5.1 Let Assumption 5.1 hold with \( m = 2 \). The one-step error of \( \tilde{\omega}(t, x) \), \( t_k \leq t \leq t_{k+1} \), which solves (5.30)-(5.31), has the following bounds for \( t_k \leq t \leq t_{k+1} \) and sufficiently small \( h > 0 \):
\[
\|E[\delta_\omega(t, \cdot)|\mathcal{F}_{t_k}]\| \leq C(t_k, \omega)h^2, \tag{5.38}
\]
\[
(E\|\delta_\omega(t, \cdot)\|^2)^{1/2} \leq Kh^{3/2}, \tag{5.39}
\]
where \( C(t_k, \omega) > 0 \) is an \( \mathcal{F}_{t_k} \)-measurable random variable with moments of a sufficiently high order bounded by a constant independent of \( h \) and \( K > 0 \) is a constant independent of \( h \).

Proof. Taking scalar product of (5.32) and \( \delta_\omega(t, x) \), using integration by parts and the property \( \text{div } v(t, x) = 0 \), we get
\[
\frac{1}{2}d\|\delta_\omega(t, \cdot)\|^2 = \frac{\sigma^2}{2} (\Delta \delta_\omega(t, \cdot), \delta_\omega(t, \cdot))dt - ([(v(t, \cdot), \nabla)\delta_\omega(t, \cdot), \delta_\omega(t, \cdot)]dt) \tag{5.40}
\]
\[-((v(t, \cdot) - \hat{v}(\cdot), \nabla)\tilde{\omega}(t, \cdot), \delta_\omega(t, \cdot))dt\]
\[= -\frac{\sigma^2}{2}||\nabla\delta_\omega(t, \cdot)||^2 dt - ((v(t, \cdot) - \hat{v}(\cdot), \nabla)\tilde{\omega}(t, \cdot), \delta_\omega(t, \cdot))dt.\]

Then
\[\frac{1}{2} dE||\delta_\omega(t, \cdot)||^2 = -\frac{\sigma^2}{2} E||\nabla\delta_\omega(t, \cdot)||^2 dt - E(((v(t, \cdot) - \hat{v}(\cdot), \nabla)\tilde{\omega}(t, \cdot), \delta_\omega(t, \cdot))dt. \tag{5.41}\]

For the last term in (5.41), we get
\[|E(((v(t, \cdot) - \hat{v}(\cdot), \nabla)\tilde{\omega}(t, \cdot), \delta_\omega(t, \cdot))| \leq KE||v(t, \cdot) - \hat{v}(\cdot)|| \cdot ||\tilde{\omega}(t, \cdot)|| \cdot ||\delta_\omega(t, \cdot))|| \leq K \left(E||v(t, \cdot) - \hat{v}(\cdot)||^2 \cdot ||\tilde{\omega}(t, \cdot)||^2 \right)^{1/2} \left(E||\delta_\omega(t, \cdot)||^2 \right)^{1/2} \leq K \left(E||v(t, \cdot) - \hat{v}(\cdot)||^4 \cdot \left(E||\tilde{\omega}(t, \cdot)||^4 \right)^{1/4} \left(E||\delta_\omega(t, \cdot)||^2 \right)^{1/2} \right.

where for the first line we used the inequality (2.14) with \(m_1 = 0, m_2 = 2, m_3 = 0\); we applied the Cauchy-Bunyakovsky inequality twice to arrive at the pre-last line; and we used the error estimate (5.36) with \(p = 2\) and boundedness of the moment \(E||\tilde{\omega}(t, \cdot)||^4\) (see (5.34)) to obtain the last line.

Thus
\[\frac{1}{2} dE||\delta_\omega(t, \cdot)||^2 \leq K h^{1/2} \left(E||\delta_\omega(t, \cdot)||^2 \right)^{1/2} dt,\]
and since \(\delta_\omega(t_k, x) = 0\), we arrive at
\[\int_{t_k}^t \frac{1}{2} \left(E||\delta_\omega(s, \cdot)||^2 \right)^{1/2} = \left(E||\delta_\omega(t, \cdot)||^2 \right)^{1/2} \leq K h^{3/2} \]

confirming (5.39).

Now we are to prove (5.38). Using (5.32), we write the equation for \(dE[\delta_\omega(t, x)|\mathcal{F}_{t_k}]\) and, after taking scalar product of the components of this equation and \(E[\delta_\omega(t, x)|\mathcal{F}_{t_k}]\) and doing integration by parts, we arrive
\[\frac{1}{2} dE[\delta_\omega(t, \cdot)|\mathcal{F}_{t_k}]^2 \]
\[= -\frac{\sigma^2}{2} ||\nabla\delta_\omega(t, \cdot)|\mathcal{F}_{t_k}||^2 dt - (E[((v(t, \cdot) - \hat{v}(\cdot), \nabla)\tilde{\omega}(t, \cdot)|\mathcal{F}_{t_k}], E[\delta_\omega(t, \cdot)|\mathcal{F}_{t_k}])dt

By (5.35), we get for the third term in (5.43):
\[|E[((v(t, \cdot) - \hat{v}(\cdot), \nabla)\tilde{\omega}(t, \cdot)|\mathcal{F}_{t_k}], E[\delta_\omega(t, \cdot)|\mathcal{F}_{t_k}])| \leq \|E[((v(t, \cdot) - \hat{v}(\cdot), \nabla)\tilde{\omega}(t, \cdot)|\mathcal{F}_{t_k}]\| \cdot \|E[\delta_\omega(t, \cdot)|\mathcal{F}_{t_k}]\| \leq C(t_k, \omega) h \|E[\delta_\omega(t, \cdot)|\mathcal{F}_{t_k}]\|,\]

where \(C(t_k, \omega) > 0\) is an \(\mathcal{F}_{t_k}\)-measurable random variable which has moments up to a sufficiently high order and does not depend on \(h\).
By simple manipulations and using (2.14) \( m_1 = 2, m_2 = 0, m_3 = 0 \) as well as (2.41), the Cauchy-Bunyakovski inequality, (5.11), a conditional version of (5.39), and (5.34), we obtain for the second term in (5.43):

\[
\begin{align*}
\| (E[(v(t,\cdot),\nabla)\delta_\omega(t,\cdot)|F_{t_k}], E[\delta_\omega(t,\cdot)|F_{t_k}]) \| \\
= \| E[((v(t,\cdot),\nabla)\delta_\omega(t,\cdot), E[\delta_\omega(t,\cdot)|F_{t_k}]) | F_{t_k}) \| \\
\leq E [((v(t,\cdot),\nabla)\delta_\omega(t,\cdot), E[\delta_\omega(t,\cdot)|F_{t_k}]) | F_{t_k}) \\
= E [((v(t,\cdot),\nabla)E[\delta_\omega(t,\cdot)|F_{t_k}], \delta_\omega(t,\cdot)) | F_{t_k}) \\
\leq KE [||v(t,\cdot)||^2 \cdot |E[\delta_\omega(t,\cdot)|F_{t_k}]| \cdot |\delta_\omega(t,\cdot)| |F_{t_k}) \\
= K||E[\delta_\omega(t,\cdot)|F_{t_k}]||_1 E [||v(t,\cdot)||^2 \cdot |\delta_\omega(t,\cdot)| |F_{t_k}) \\
\leq K ||\nabla E[\delta_\omega(t,\cdot)|F_{t_k}]|| E [||\omega(t,\cdot)||^2 \cdot |\delta_\omega(t,\cdot)| |F_{t_k}) \\
\leq K ||\nabla E[\delta_\omega(t,\cdot)|F_{t_k}]|| (E [||\omega(t,\cdot)||^2 |F_{t_k})^{1/2} (E [||\delta_\omega(t,\cdot)||^2 |F_{t_k})^{1/2}
\end{align*}
\]

where \( C(t_k,\omega) > 0 \) is a \( F_{t_k} \)-measurable random variable which has moments up to a sufficiently high order and does not depend on \( h \).

Combining (5.43)-(5.45), we arrive at

\[
\begin{align*}
\frac{1}{2} d||E[\delta_\omega(t,\cdot)|F_{t_k}]||^2 & \leq -\frac{\sigma^2}{2} ||\nabla E[\delta_\omega(t,\cdot)|F_{t_k}]||^2 dt + \\
& \quad + C(t_k,\omega) h^{3/2} ||\nabla E[\delta_\omega(t,\cdot)|F_{t_k}]|| dt + C(t_k,\omega) h ||E[\delta_\omega(t,\cdot)|F_{t_k}]|| dt \\
& = -\frac{1}{2} \left( \frac{\sigma||\nabla E[\delta_\omega(t,\cdot)|F_{t_k}]|| - C(t_k,\omega) h^{3/2}}{\sigma} \right)^2 dt + C(t_k,\omega) h ||E[\delta_\omega(t,\cdot)|F_{t_k}]|| dt \\
& \quad + C(t_k,\omega) h ||E[\delta_\omega(t,\cdot)|F_{t_k}]|| dt \\
& \leq \frac{C^2(t_k,\omega)}{2\sigma^2} h^3 dt + C(t_k,\omega) h ||E[\delta_\omega(t,\cdot)|F_{t_k}]|| dt.
\end{align*}
\]

Then, for some \( F_{t_k} \)-measurable independent of \( h \) random variable \( C(t_k,\omega) > 0 \), we have

\[
d||E[\delta_\omega(t,\cdot)|F_{t_k}]||^2 \leq C(t_k,\omega) h^3 dt + \frac{1}{h} ||E[\delta_\omega(t,\cdot)|F_{t_k}]||^2 dt,
\]

from which (5.38) follows taking into account that \( ||E[\delta_\omega(t_k,\cdot)|F_{t_k}]|| = 0 \). Theorem 5.1 is proved.

As in the deterministic case, we define

\[
\tilde{v}(t,x) := U\tilde{\omega}(t,x), \quad t_k \leq t \leq t_{k+1},
\]

where the operator \( U \) is from (2.35).

Using the idea of the proof of Corollary 4.1, it is not difficult to prove the following corollary to Theorem 5.1.

**Corollary 5.1** The one-step error \( \delta_\omega(t,x) := v(t,x) - \tilde{v}(t,x) = U\delta_\omega(t,x) \) of \( v(t,x) \), \( t_k \leq t \leq t_{k+1} \), has the following bounds for \( t_k \leq t \leq t_{k+1} \):

\[
\begin{align*}
||E[\delta_\omega(t,\cdot)|| & \leq Kh^2, \\
\left(E[||\delta_\omega(t,\cdot)||^2]\right)^{1/2} & \leq Kh^{3/2},
\end{align*}
\]

where \( K > 0 \) is independent of \( h \).
5.3 The method

Analogously, how it was done in the deterministic case (see Section 4.1), we construct the global approximation for the stochastic NSE \((5.7)-(5.9)\) based on the on-step approximation \((5.30)-(5.31)\). On the first step of the method we set

\[
\tilde{\omega}(t_0, x) = \text{curl} v(t_0, x) = \phi(x) = \text{curl} \varphi(x)
\]

and

\[
\hat{v}(x) = \hat{v}(t, x) = u(t_0, x) = \varphi(x), \quad 0 = t_0 \leq t \leq t_1.
\]

Then we solve the linear SPDE \((5.30)-(5.31)\) on \([0, t_1]\) to obtain \(\tilde{\omega}(t, x)\) and to construct

\[
\hat{v}(t_1, x) = U\tilde{\omega}(t_1, x).
\]

On the second step we solve \((5.30)-(5.31)\) on \([t_1, t_2]\) having \(\tilde{\omega}(t_1, x)\) and setting \(\hat{v}(t, x) = \hat{v}(t_1, x)\) for \(t_1 < t \leq t_2\). As a result, we obtain \(\tilde{\omega}(t, x)\) on \([t_1, t_2]\) and \(\hat{v}(t_2, x) = U\tilde{\omega}(t_2, x)\), and so on. Proceeding in this way, we obtain on the \(N\)-th step the approximation \(\tilde{\omega}(t, x)\) on \([t_{N-1}, t_N]\) for \(\omega(t, x)\) having \(\tilde{\omega}(t_{N-1}, x)\) and \(\hat{v}(x) = \hat{v}(t_{N-1}, x) = U\tilde{\omega}(t_{N-1}, x)\) and setting \(\hat{v}(t, x) = \hat{v}(t_{N-1}, x)\) for \(t_{N-1} < t \leq t_N\). Finally, \(\hat{v}(T, x) = U\tilde{\omega}(T, x)\).

In order to realise the approximation process described above, it is sufficient that on every time interval \([t_k, t_{k+1}]\), \(k = 0, \ldots, N - 1\), there exists a solution of the linear SPDE \((5.30)-(5.31)\), we denote such a solution \(\tilde{\omega}(t, x)\) which satisfies the condition

\[
\tilde{\omega}_k(t_k, x) = \begin{cases} 
\text{curl} \varphi(x), & k = 0, \\
\tilde{\omega}_{k-1}(t_k, x), & k = 1, \ldots, N,
\end{cases}
\]

and has the time-independent \(\hat{u}(x)\) within each interval \((t_k, t_{k+1}]\) defined as

\[
\hat{v}(x) := \hat{v}_k(x) = U\tilde{\omega}_k(t_k, x), \quad t_k < t \leq t_{k+1}.
\]

Clearly, \(\hat{v}(x)\) used in \((5.30)\) are different on the time intervals \((t_k, t_{k+1}]\).

Before considering global errors of the approximation in Section 5.4, we now prove boundedness of the approximation’s moments.

**Theorem 5.2** Let Assumption 5.1 hold with \(m = 0\). The moments of the global approximation \(\tilde{\omega}_k(t_k, x)\) and \(\hat{v}_k(x)\) are uniformly bounded in \(h\) and \(k\):

\[
E||\tilde{\omega}_k(t_{k+1}, \cdot)||^{2p} \leq ||\phi(\cdot)||^{2p} + K \int_0^{t_{k+1}} \left(||g(s, \cdot)||^{2p} + \sum_{r=1}^q ||\mu_r(s, \cdot)||^{2p}\right) ds, \quad (5.51)
\]

\[
E||\hat{v}_k(\cdot)||^{2p} \leq KE||\tilde{\omega}_k(t_k, \cdot)||^{2p}, \quad (5.52)
\]

where \(K > 0\) is independent of \(h\) and \(t_k\) but depends on \(p\).

**Proof.** For every sufficiently large integer \(n\), define the stopping time

\[
\tau_n = \inf\{0 < t \leq T : ||\tilde{\omega}(t, \cdot)||^2 \geq n\}.
\]

Using the Ito formula, doing integration by parts and taking into account that \(\hat{v}_k(x)\) is divergence free, we obtain

\[
d||\tilde{\omega}_k(t, \cdot)||^{2p} = 2p||\tilde{\omega}_k(t, \cdot)||^{2(p-1)}
\]

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\[
\cdot \left[ -\frac{\sigma^2}{2} ||\nabla \bar{w}_k(t, \cdot)||^2 + (g(t, \cdot), \bar{w}_k(t, \cdot)) + \frac{2p-1}{2} \sum_{r=1}^{q} ||\mu_{r}(t, \cdot)||^2 \right] dt \\
+ 2p ||\bar{w}_k(t, \cdot)||^{2(p-1)} \sum_{r=1}^{q} (\mu_{r}(t, \cdot), \bar{w}_k(t, \cdot)) dw_r(t), \ t_k \land \tau_n \leq t \leq t_{k+1} \land \tau_n, \\
||\bar{w}_k(t_k, \cdot)||^{2p} = E||\bar{w}_{k-1}(t_k, \cdot)||^{2p}.
\]

We have
\[
dE||\bar{w}_k(t, \cdot)||^{2p} \\
= 2p \left[ -\frac{\sigma^2}{2} E(||\bar{w}_k(t, \cdot)||^{2(p-1)}||\nabla \bar{w}_k(t, \cdot)||^2) + E||\bar{w}_k(t, \cdot)||^{2(p-1)}(g(t, \cdot), \bar{w}_k(t, \cdot)) \\
+ \frac{2p-1}{2} E||\bar{w}_k(t, \cdot)||^{2(p-1)} \sum_{r=1}^{q} ||\mu_{r}(t, \cdot)||^2 \right] dt, \ t_k \land \tau_n \leq t \leq t_{k+1} \land \tau_n, \\
E||\bar{w}_k(t_k, \cdot)||^2 = E||\bar{w}_{k-1}(t_k, \cdot)||^2.
\]

By Poincare’s inequality (2.18) and doing simple re-arrangements, we arrive at
\[
dE||\bar{w}_k(t, \cdot)||^{2p} \\
\leq 2p \left[ -\frac{\alpha \sigma^2}{2} E||\bar{w}_k(t, \cdot)||^{2p} + \alpha \frac{\sigma^2}{4} E||\bar{w}_k(t, \cdot)||^{2p} + \frac{1}{\alpha \sigma^2} ||g(t, \cdot)||^2 \\
+ \frac{\alpha^2}{4} E||\bar{w}_k(t, \cdot)||^{2p} + 2 \frac{((2p-1)(p-1))^{p}}{(\alpha \sigma^2)^{p-1}(p-1)} \left[ \sum_{r=1}^{q} ||\mu_{r}(t, \cdot)||^2 \right]^{p} \right] dt.
\]

We note that the constant \( \alpha > 0 \) in the above expression is due to Poincare’s inequality (2.18) and it is, of course, independent of \( h \) and \( k \). Hence
\[
dE||\bar{w}_k(t, \cdot)||^{2p} \leq K \left[ ||g(t, \cdot)||^2 + \left[ \sum_{r=1}^{q} ||\mu_{r}(t, \cdot)||^2 \right]^{p} \right] dt,
\]
where the constant \( K > 0 \) depends on \( p \) but independent of \( h \) and \( k \). The previous inequality implies
\[
E||\bar{w}_k(t_{k+1} \land \tau_n, \cdot)||^{2p} \leq E||\bar{w}_{k-1}(t_k, \cdot)||^{2p} \\
+ KE \int_{t_k \land \tau_n}^{t_{k+1} \land \tau_n} \left( ||g(s, \cdot)||^2 + \left[ \sum_{r=1}^{q} ||\mu_{r}(s, \cdot)||^2 \right]^{p} \right) ds \\
\leq E||\phi(\cdot)||^{2p} + E \int_{0}^{t_{k+1} \land \tau_n} \left( K ||g(s, \cdot)||^2 + \sum_{r=1}^{q} ||\mu_{r}(s, \cdot)||^2 \right) ds,
\]
and letting \( n \to \infty \) we arrive at (5.51). The estimate (5.52) is evident (see e.g. (2.35)). Theorem [5.2] is proved.

**Remark 5.1** It is not difficult to see that repeating the proof of Lemma 5.1 word by word, we immediately get that the exponential moment for \( ||\bar{w}_k(t_{k+1}, \cdot)||^2 \) is bounded, more precisely the estimate of the form (5.12) holds for \( ||\bar{w}_k(t_{k+1}, \cdot)||^2 \) under Assumption 5.1 with \( m = 0 \).
Now we consider uniform bounds for moments of higher Sobolev norms of $\tilde{\omega}_k$.

**Theorem 5.3** Let Assumption 5.1 hold with $m > 0$. Then

$$E||\tilde{\omega}_k(t_{k+1}, \cdot)||_{m}^{2p} \leq ||\phi(\cdot)||_{m}^{2p} + K t_{k+1},$$

where $K > 0$ is independent of $h$ and $t_k$.

**Proof.** The proof is by induction. To this end, we assume that moments $E||\tilde{\omega}_k(t\cdot)||_{m-1}^{2p}$ are bounded (uniformly in $k$ and $h$) and for sufficiently large $p \geq 1$ (note that Theorem 5.2 guarantees their boundedness for $m = 0$).

We will be adapting recipes from [29, Section 4.1]. Let the operator $\Lambda$ be such that $\Lambda^2 = -\Delta$. We have for an integer $m \geq 1$ (cf. [29, p. 29] and also [16, Section 3.4]):

$$d||\tilde{\omega}_k(t\cdot)||_{m}^{2p} = 2p||\tilde{\omega}_k(t\cdot)||_{m}^{2(p-1)} \left[-\frac{\sigma^2}{2}||\tilde{\omega}_k(t\cdot)||_{m+1}^{2} + (\tilde{\omega}_k(t\cdot), g(t\cdot))_m \right] dt$$

$$-((\dot{v}_k(\cdot), \nabla)\tilde{\omega}_k(t\cdot), \Lambda^{2m}\tilde{\omega}_k(t\cdot)) + \frac{2p - 1}{2} \sum_{r=1}^{q} ||\mu_r(t\cdot)||_{m}^{2} dt$$

$$+ 2p||\tilde{\omega}_k(t\cdot)||_{m}^{2(p-1)} \sum_{r=1}^{q} (\mu_r(t\cdot), \tilde{\omega}_k(t\cdot))_m dw_r(t),$$

Here $\tau_n$ is as in Theorem 5.2.

Let us analyze terms in the right-hand side of (5.55). We have (e.g. see [29, Eq. (4.4)]):

$$||\tilde{\omega}_k(t\cdot, g(t\cdot))_m || \leq ||g(t\cdot)||_{m-1} ||\tilde{\omega}_k(t\cdot)||_{m+1}$$

$$\leq \frac{4}{\sigma^2}||g(t\cdot)||_{m-1}^{2} + \frac{\sigma^2}{16}||\tilde{\omega}_k(t\cdot)||_{m+1}^{2},$$

and

$$K||\tilde{\omega}_k(t\cdot)||_{m}^{2(p-1)}||g(t\cdot)||_{m-1}^{2} \leq \frac{K^p}{p} \left( \frac{16p}{\alpha \sigma^2 (p - 1)} \right)^{p-1} ||g(t\cdot)||_{m-1}^{2p} + \alpha \frac{\sigma^2}{16}||\tilde{\omega}_k(t\cdot)||_{m}^{2p},$$

where as before the constant $\alpha > 0$ is due to Poincare’s inequality [2.18]. Also, for some $K > 0$ dependent on $p$ :

$$p(2p - 1)||\tilde{\omega}_k(t\cdot)||_{m}^{2(p-1)} \sum_{r=1}^{q} ||\mu_r(t\cdot)||_{m}^{2} \leq K \left( \sum_{r=1}^{q} ||\mu_r(t\cdot)||_{m}^{2} \right)^{p} + \frac{\sigma^2}{8}||\tilde{\omega}_k(t\cdot)||_{m}^{2p}.$$
for the second line we used (2.39) and that
where for the first line we used the recipe from \[29, \text{pp. 29-30}\] and the inequality (2.14);
Thus, for some
\(K > 0\) that the constants
\(K > 0\) and
\(t_k \land \tau_n \leq t \leq t_{k+1} \land \tau_n,\)
\[||\tilde{\omega}_k(t_k, \cdot)||^2_m = ||\tilde{\omega}_{k-1}(t_k, \cdot)||^2_m.\]
Let us now estimate the trilinear-form:
\(\left|\left((v_k(\cdot), \nabla)\tilde{\omega}_k(t, \cdot), \Lambda^{2m}\tilde{\omega}_k(t, \cdot)\right)\right| \leq K \sum_{l=1}^{m} ||\tilde{v}_k(\cdot)||_l ||\tilde{\omega}_k(t, \cdot)||_{m-l+3/2} ||\tilde{\omega}_k(t, \cdot)||_{m+1} \]
\(\leq K ||\tilde{\omega}_k(t, \cdot)||_{m-1} ||\tilde{\omega}_k(t, \cdot)||_{m+1} ||\tilde{\omega}_k(t, \cdot)||_{m+1} \]
\(\leq K ||\tilde{\omega}_k(t, \cdot)||_{m-1}^{5/4} ||\tilde{\omega}_k(t, \cdot)||_{m+1}^{7/4} \]
\(\leq \frac{K^8}{8} \left(\frac{56}{\sigma^2}\right)^7 ||\tilde{\omega}_k(t_k, \cdot)||_{m-1}^{10} + \frac{\sigma^2}{8} ||\tilde{\omega}_k(t, \cdot)||_{m+1}^{2p}.\)
where for the first line we used the recipe from \[29, \text{pp. 29-30}\] and the inequality (2.14); for the second line we used (2.39) and that \(||u(\cdot)||_{m_1} \leq ||u(\cdot)||_{m_2}\) for \(m_2 \geq m_1\); the third line is obtained using (2.15); and the fourth line follows from Young’s inequality. Note that the constants \(K > 0\) in the first and second lines are different.
Further,
\[K ||\tilde{\omega}_k(t, \cdot)||_{m-1}^{10} ||\tilde{\omega}_k(t, \cdot)||_{m+1}^{2p-1} \]
\[\leq \frac{K^p}{p} \left(\frac{8p}{\alpha \sigma^2 (p-1)}\right)^{p-1} ||\tilde{\omega}_k(t_k, \cdot)||_{m-1}^{10p} \]
Thus, for some \(K > 0\)
\[dE||\tilde{\omega}_k(t, \cdot)||^2_m \leq \]
\[+K \left[||g(t, \cdot)||_{m-1}^{2p} + E||\tilde{\omega}_k(t_k, \cdot)||_{m-1}^{10p} + \left(\sum_{r=1}^{q} ||\mu_r(t, \cdot)||_m^2\right)^p\right] dt,\]
\(t_k \land \tau_n \leq t \leq t_{k+1} \land \tau_n,\)
\[E||\tilde{\omega}_k(t_k, \cdot)||^2_m = E||\tilde{\omega}_{k-1}(t_k, \cdot)||^2_m.\]
By the Cauchy-Bunyakovskii inequality and the induction assumption at the start of the proof, we get
\[E||\tilde{\omega}_k(t_k, \cdot)||_{m-1}^{10p} \leq K\]
with a constant \(K > 0\) independent of \(h\) and \(k\). Hence
\[dE||\tilde{\omega}_k(t, \cdot)||^2_m \leq K dt,\]
\(t_k \land \tau_n \leq t \leq t_{k+1} \land \tau_n,\)
\[E||\tilde{\omega}_k(t_k, \cdot)||^2_m = E||\tilde{\omega}_{k-1}(t_k, \cdot)||^2_m\]
and
\[E||\tilde{\omega}_k(t_k+1 \land \tau_n, \cdot)||^2_m \leq E||\tilde{\omega}_{k-1}(t_k, \cdot)||^2_m + Kh,\]
from which (5.54) follows by the standard arguments. Theorem (5.3) is proved.
5.4 Mean-square convergence theorem

To prove the global convergence of \( \hat{\omega}_k(t, \cdot) \), we use the idea of the proof of the fundamental theorem of mean-square convergence for SDEs [20] (see also [23, Section 1.1]).

**Theorem 5.4** Let Assumption 5.1 hold with \( m = 2 \). The global approximation \( \hat{\omega}_k(t+1, x) \) for the problem (5.7)–(5.9) has the first mean-square order accuracy.

**Proof.** We note that in the proof we shall again use letters \( K \) and \( C(\cdot, \omega) \) to denote various deterministic constants and random variables, respectively, which are independent of \( h \) and \( k \), and \( K \) is also independent of \( h \) and \( k \); their values may change from line to line.

We have

\[
R(t_{k+1}, x) := \omega(t_{k+1}, x; 0, \phi) - \omega_k(t_{k+1}, x; 0, \phi) = \omega(t_{k+1}, x; t_k, \omega(t_k, \cdot)) - \omega(t_{k+1}, x; t_k, \omega_k(t_k, \cdot)) = \mathbf{K} + \mathbf{W}_k,
\]

\[
K := \delta \omega(t_{k+1}, x) := \omega(t_{k+1}, x; t_k, \omega_k(t_k, \cdot)) - \omega(t_{k+1}, x; t_k, \omega_k(t_k, \cdot)),
\]

where \( \cdot \) reflects function dependence of solutions on the initial conditions. The first difference in the right-hand side of (5.56) is the error of the solution arising due to the error in the initial data at time \( t_k \), accumulated at the \( k \)-th step. The second difference is the one-step error at the \( (k+1) \)-step:

\[
\delta \omega(t_{k+1}, x) := \omega(t_{k+1}, x; t_k, \omega_k(t_k, \cdot)) - \omega(t_{k+1}, x; t_k, \omega_k(t_k, \cdot))
\]

for which estimates are given in Theorem 5.1 taking into account that Theorems 5.2 and 5.3 guarantees boundedness of moments of \( ||\hat{\omega}_k(t, \cdot)||_3 \) under the conditions of this theorem. Taking the \( L^2 \)-norm of both sides of (5.56), we obtain

\[
||R(t_{k+1}, \cdot)||^2 = ||\omega(t_{k+1}, \cdot; t_k, \omega(t_k, \cdot)) - \omega(t_{k+1}, \cdot; t_k, \omega_k(t_k, \cdot))||^2 + ||\delta \omega(t_{k+1}, \cdot)||^2 + 2 \langle \omega(t_{k+1}, \cdot; t_k, \omega(t_k, \cdot)) - \omega(t_{k+1}, \cdot; t_k, \omega_k(t_k, \cdot)), \delta \omega(t_{k+1}, \cdot) \rangle,
\]

where the first \( \cdot \) in each \( \omega \) or \( \omega_k \) reflects that we took \( L^2 \)-norm.

Using (5.39) from Theorem 5.1 together with Theorems 5.2 and 5.3 we obtain for the second term in (5.58):

\[
||\delta \omega(t_{k+1}, \cdot)||^2 \leq C(t_{k+1}, \omega)h^2
\]

where \( C(t_{k+1}, \omega) > 0 \) is an \( \mathcal{F}_{t_{k+1}} \)-measurable with bounded second moment. By (5.22) from Lemma 5.2 together with Theorems 5.2 and 5.3 we get for the first term in (5.58):

\[
||\omega(t_{k+1}, x; t_k, \omega(t_k, \cdot)) - \omega(t_{k+1}, x; t_k, \omega_k(t_k, \cdot))||^2 \leq ||R(t_{k}, \cdot)||^2 \exp \left( K h + c \int_{t_k}^{t_{k+1}} ||\nabla \omega(s', \cdot; t_k, \omega(t_k, \cdot))||^2 ds' \right).
\]

The difference \( \omega(t_{k+1}, x; t_k, \omega(t_k, \cdot)) - \omega(t_{k+1}, x; t_k, \omega_k(t_k, \cdot)) \) in the last summand in (5.58) can be treated using (5.21) from Lemma 5.2.

\[
\omega(t_{k+1}, x; t_k, \omega(t_k, \cdot)) - \omega(t_{k+1}, x; t_k, \omega_k(t_k, \cdot)) = R(t_{k}, x) + \eta(t_{k}, x).
\]
Using a conditional version of (5.39) from Theorem 5.1 and (5.23) from Lemma 5.2, together with Theorems 5.2 and 5.3, we get

\[ \|(\eta(t_k, \cdot), \delta(t_{k+1} ,\cdot))\| \leq \|\eta(t_k, \cdot)\| \|\delta(t_{k+1}, \cdot)\| \leq \|\eta(t_k, \cdot)\|^2 + \frac{1}{4} \|\delta(t_{k+1}, \cdot)\|^2 \]  

(5.61)

\[ \leq h \|R(t_k, \cdot)\|^2 + C(t_{k+1}, \omega) h^3, \]

where \( C(t_{k+1}, \omega) > 0 \) is an \( \mathcal{F}_{k+1} \)-measurable with bounded second moment.

Combining the above, we arrive at

\[ \|R(t_{k+1}, \cdot)\|^2 \leq \|R(t_k, \cdot)\|^2 \exp \left( Kh + c \int_{t_k}^{t_{k+1}} \|\nabla \omega(s', \cdot; t_k, \omega(t_k, \cdot))\|^2 ds' \right) + h \|R(t_k, \cdot)\|^2 + (R(t_k, \cdot), \delta(t_{k+1}, \cdot)) + C(t_{k+1}, \omega) h^3. \]  

(5.62)

Since \( \|R(0, \cdot)\| = 0 \), summing (5.62) from \( k = 0 \) to \( n \), we get

\[ \|R(t_{n+1}, \cdot)\|^2 \leq \sum_{k=1}^{n} \|R(t_k, \cdot)\|^2 \left[ \exp \left( Kh + c \int_{t_k}^{t_{k+1}} \|\nabla \omega(s', \cdot; t_k, \omega(t_k, \cdot))\|^2 ds' \right) - 1 + h \right] 

+ h^3 \sum_{k=0}^{n} C(t_{k+1}, \omega) + \sum_{k=1}^{n} (R(t_k, \cdot), \delta(t_{k+1}, \cdot)) \]

\[ \leq \sum_{k=1}^{n} \|R(t_k, \cdot)\|^2 \left[ \exp \left( Kh + c \int_{t_k}^{t_{k+1}} \|\nabla \omega(s', \cdot; t_k, \omega(t_k, \cdot))\|^2 ds' \right) - 1 \right] 

+ h^3 \sum_{k=0}^{n} C(t_{k+1}, \omega) + \sum_{k=1}^{n} (R(t_k, \cdot), \delta(t_{k+1}, \cdot)) \]

From which, by a version of Gronwall’s lemma (see, e.g. [1]), we obtain

\[ \|R(t_{n+1}, \cdot)\|^2 \leq F_n + \sum_{k=1}^{n} F_{k-1} \]

\[ \cdot \left[ \exp \left( Kh + c \int_{t_k}^{t_{k+1}} \|\nabla \omega(s', \cdot; t_k, \omega(t_k, \cdot))\|^2 ds' \right) - 1 \right] \]

\[ \cdot \prod_{j=k+1}^{n} \exp \left( Kh + c \int_{t_j}^{t_{j+1}} \|\nabla \omega(s', \cdot; t_k, \omega(t_k, \cdot))\|^2 ds' \right), \]

where

\[ F_k := h^3 \sum_{j=0}^{k} C(t_{j+1}, \omega) + \sum_{j=1}^{k} (R(t_j, \cdot), \delta(t_{j+1}, \cdot)). \]

We have

\[ \|R(t_{n+1}, \cdot)\|^2 \leq F_n \]

\[ + \sum_{k=1}^{n} F_{k-1} \cdot \left[ \exp \left( Kh + c \int_{t_k}^{t_{k+1}} \|\nabla \omega(s', \cdot; t_k, \omega(t_k, \cdot))\|^2 ds' \right) - 1 \right], \]  

(5.64)

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\begin{align*}
&\cdot \exp \left( K(t_{n+1} - t_{k+1}) + c \int_{t_{k+1}}^{t_{n+1}} \| \nabla \omega(s', \cdot; t_{k+1}, \omega(t_{k+1}, \cdot)) \|^2 ds' \right) \\
&= F_n + \sum_{k=1}^n F_{k-1} \cdot \left[ \exp \left( K(t_{n+1} - t_k) + c \int_{t_k}^{t_{n+1}} \| \nabla \omega(s', \cdot; t_k, \omega(t_k, \cdot)) \|^2 ds' \right) \\
&\quad - \exp \left( K(t_{n+1} - t_k) + c \int_{t_k}^{t_{n+1}} \| \nabla \omega(s', \cdot; t_k+1, \omega(t_k+1, \cdot)) \|^2 ds' \right) \right] \\
&= \sum_{k=1}^n (F_k - F_{k-1}) \exp \left( K(t_{n+1} - t_k) + c \int_{t_k}^{t_{n+1}} \| \nabla \omega(s', \cdot; t_k+1, \omega(t_k+1, \cdot)) \|^2 ds' \right) \\
&\quad + h^3 C(t_1, \omega) \exp \left( K(t_{n+1} - t_1) + c \int_{t_1}^{t_{n+1}} \| \nabla \omega(s', \cdot; t_1, \omega(t_1, \cdot)) \|^2 ds' \right) \\
&\quad + \lambda(s)dw_r(s),
\end{align*}

For the last term in the right-hand side of (5.64), we obtain using the Cauchy-Bunyakovsky inequality and Lemma 5.1,

\[ E \left\{ h^3 C(t_1, \omega) \exp \left( K(t_{n+1} - t_1) + c \int_{t_1}^{t_{n+1}} \| \nabla \omega(s', \cdot; 0, \phi(\cdot)) \|^2 ds' \right) \right\} \leq Kh^3. \quad (5.65) \]

Consider now the first term in the right-hand side of (5.64). We have

\[ (F_k - F_{k-1}) \exp \left( K(t_{n+1} - t_k) + c \int_{t_k}^{t_{n+1}} \| \nabla \omega(s', \cdot; t_k+1, \omega(t_k+1, \cdot)) \|^2 ds' \right) \quad (5.66) \]

\[ = \exp \left( K(t_{n+1} - t_k) + c \int_{t_k}^{t_{n+1}} \| \nabla \omega(s', \cdot; t_k+1, \omega(t_k+1, \cdot)) \|^2 ds' \right) \times \left[ h^3 C(t_{k+1}, \omega) + \left( R(t_k, \cdot), \delta \omega(t_{k+1}, \cdot) \right) \right]. \]

Expectation of the first term from the right-hand side of (5.66) is estimated by \( Kh^3 \) as in (5.65). Let us now consider the second term.

By the martingale representation theorem and Lemma 5.1, we can obtain

\[ E \left[ \exp \left( K(t_{n+1} - t_k) + c \int_{t_k}^{t_{n+1}} \| \nabla \omega(s', \cdot; t_k+1, \omega(t_k+1, \cdot)) \|^2 ds' \right) \bigg| \mathcal{F}_{t_k+1} \right] \quad (5.67) \]

\[ = E \left[ \exp \left( K(t_{n+1} - t_k) + c \int_{t_k}^{t_{n+1}} \| \nabla \omega(s', \cdot; t_k+1, \omega(t_k+1, \cdot)) \|^2 ds' \right) \right] + \sum_{r=1}^q \int_{t_k+1}^{t_{n+1}} \lambda_r(s)dw_r(s), \]

where \( \lambda_r(s) \) are \( \mathcal{F}_r \)-adapted square-integrable stochastic processes.

Using (5.38) and a conditional version of (5.39) from Theorem 5.1 together with Theorems 5.2 and 5.3 and also using Lemma 5.1 and (5.67), we arrive at

\[ \left| E \left\{ \exp \left( K(t_{n+1} - t_k) + c \int_{t_k}^{t_{n+1}} \| \nabla \omega(s', \cdot; t_k+1, \omega(t_k+1, \cdot)) \|^2 ds' \right) \right\} \right| \]

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Therefore,

\[
\begin{align*}
&\left| (R(t_k, \cdot), \delta\omega(t_{k}+1, \cdot)) \right|_{F_{tk}} \\
= & \left| \left( R(t_k, \cdot), E \left\{ \exp \left( K(t_{n+1} - t_{k+1}) + c \int_{t_k+1}^{t_{n+1}} \| \nabla \omega(s', \cdot; t_{k+1}, \omega(t_{k+1}, \cdot)) \|^2 ds' \right) \right\} \right|_{F_{tk}} \\
\leq & \left| R(t_k, \cdot) \right| \left| E \left\{ \exp \left( K(t_{n+1} - t_{k+1}) + c \int_{t_k+1}^{t_{n+1}} \| \nabla \omega(s', \cdot; t_{k+1}, \omega(t_{k+1}, \cdot)) \|^2 ds' \right) \right\} \right|_{F_{tk}} \\
\leq & \left| R(t_k, \cdot) \right| E \left\{ \left( E \left[ \exp \left( K(t_{n+1} - t_{k+1}) + c \int_{t_k+1}^{t_{n+1}} \| \nabla \omega(s', \cdot; t_{k+1}, \omega(t_{k+1}, \cdot)) \|^2 ds' \right) \right] \right) \right\}_{F_{tk}} \\
\leq & \left| R(t_k, \cdot) \right| C(t_k, \omega) h^2 + \left| R(t_k, \cdot) \right| \left| \sum_{r=1}^{q} \int_{0}^{t_k} \lambda_r(s) dw_r \delta\omega(t_{k}+1, \cdot) \right|_{F_{tk}} \\
+ & \left| R(t_k, \cdot) \right| \left( E \left[ \left[ \delta\omega(t_{k}+1, \cdot) \right]^2 \right] \right)^{1/2} \\
\cdot & \left( \sum_{r=1}^{q} \left( E \left[ \left( \int_{t_k}^{t_{k+1}} \lambda_r(s) dw_r \right)^2 \right] \right)^{1/2} \right)^{1/2} \\
\leq & \left| R(t_k, \cdot) \right| C(t_k, \omega) h^2 \leq h \| R(t_k, \cdot) \|^2 + \frac{h^4}{4} C^2(t_k, \omega).
\end{align*}
\]

Therefore,

\[
E \left\{ (F_k - F_{k-1}) \exp \left( K(t_{n+1} - t_{k+1}) + c \int_{t_k+1}^{t_{n+1}} \| \nabla \omega(s', \cdot; t_{k+1}, \omega(t_{k+1}, \cdot)) \|^2 ds' \right) \right\} \leq h E \| R(t_k, \cdot) \|^2 + Kh^3.
\]

From (5.64), (5.65), and (5.68), we obtain

\[
E \| R(t_{n+1}, \cdot) \|^2 \leq h \sum_{k=1}^{n} E \| R(t_k, \cdot) \|^2 + Kh^2,
\]

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from which it follows by a version of Gronwall’s lemma that

$$E\|R(t_{n+1}, \cdot)\|^2 \leq Kh^2$$

as required. Theorem 5.4 is proved.

**Remark 5.2** Various approaches can be used to turn the method, introduced at the start of Section 5.3 for the problem (5.7)-(5.9), into a numerical algorithm. To obtain a constructive numerical algorithm, we need to approximate the linear SPDE (5.30)-(5.31) at every step. To this end, for instance, we can discretize this SPDE in space using the spectral method based on the Fourier expansion and use a finite difference for time discretization (see such an algorithm in the deterministic setting in e.g. [27]). Alternatively, we can apply the method based on averaging characteristics to (5.30)-(5.31) [24]. We leave construction, analysis and testing of such algorithms for a future work.

**References**

[1] P.R. Beesack. More generalised discrete Gronwall inequalities. *ZAMM Z. Angew. Math. Mech.* 65 (1985), 589–595.

[2] H. Bessaih, Z. Brzeźniak, A. Millet. Splitting up method for the 2D stochastic Navier–Stokes equations. *SPDEs: An. Comp.* 2 (2014), 433–470.

[3] Z. Brzeźniak, E. Carelli, A. Prohl. Finite element based discretizations of the incompressible Navier-Stokes equations with multiplicative random forcing. *IMA J. Num. Anal.* 34 (2014), 502–549.

[4] B. Busnello, F. Flandoli, M. Romito. A probabilistic representation for the vorticity of a 3D viscous fluid and for general systems of parabolic equations. *Proc. Edinb. Math. Soc.* 48 (2005), 295–336.

[5] E. Carelli, A. Prohl. Rates of convergence for discretizations of the stochastic incompressible Navier-Stokes equations. *SIAM J. Num. Anal.* 50 (2012), 2467–2496.

[6] P. Constantin, C. Foias. *Navier-Stokes Equations.* University of Chicago Press, 1988.

[7] P. Dörsek. Semigroup splitting and cubature approximations for the stochastic Navier-Stokes equations. *SIAM J. Num. Anal.* 50 (2012), 729-746.

[8] T. Dubois, F. Jauberteau, R. Temam. *Dynamic Multilevel Methods and the Numerical Simulation of Turbulence.* Cambridge Univ. Press, 1999.

[9] F. Flandoli. An introduction to 3D stochastic fluid dynamics. In: SPDE in Hydrodynamic: Recent Progress and Prospects, *Lecture Notes in Mathematics* 1942, Springer, 2008, 51–150.

[10] C. Foias, O. Manley, R. Rosa, R. Temam. *Navier-Stokes Equations and Turbulence.* Cambridge University Press, 2001.

[11] V. Girault, P.A. Raviart. *Finite Element Methods for Navier-Stokes Equations.* Springer, 1986.
[12] M. Hairer, J.C. Mattingly. Ergodicity of the 2D Navier-Stokes equations with degenerate stochastic forcing. *Ann. Math.* **164** (2006), 993–1032.

[13] R. Kruse, M. Scheutzow. A discrete stochastic Gronwall lemma. *Math. Comp. Simul.* **143** (2018), 149–157.

[14] J.L. Lions, E. Magenes. *Nonhomogeneous Boundary Value Problems and Applications*. Springer, 1972.

[15] A.J. Majda, A.L. Bertozzi. *Vorticity and Incompressible Flow*. Cambridge Univ. Press, 2003.

[16] J.C. Mattingly. *The Stochastic Navier-Stokes Equation: Energy Estimates and Phase Space Contraction*. Ph.D. Thesis, Princeton University, Princeton, 1998.

[17] J.C. Mattingly. The dissipative scale of the stochastics Navier–Stokes equation: regularization and analyticity. *J. Stat. Phys.* **108** (2002), 1157–1179.

[18] R. Mikulevicius, B. Rozovskii. Global L2-solutions of stochastic Navier-Stokes equations. *Ann. Prob.* **33** (2005), 137–176.

[19] G.N. Milstein. Probabilistic solution of linear systems of elliptic and parabolic equations. *Theor. Prob. Appl.* **23** (1978), 851–855.

[20] G.N. Milstein. A theorem on the order of convergence of mean-square approximations of solutions of systems of stochastic differential equations. *Teor. Veroyat. Primenen.*, **32** (1987), 809–811.

[21] G.N. Milstein. The probability approach to numerical solution of nonlinear parabolic equations. *Num. Meth. PDE* **18** (2002), 490–522.

[22] G.N. Milstein, J.G.M. Schoenmakers, V. Spokoiny. Transition density estimation for stochastic differential equations via forward-reverse representations. *Bernoulli* **10** (2004), 281–312.

[23] G.N. Milstein, M.V. Tretyakov. *Stochastic Numerics for Mathematical Physics*. Springer, 2004.

[24] G.N. Milstein, M.V. Tretyakov. Solving parabolic stochastic partial differential equations via averaging over characteristics. *Math. Comp.* **78** (2009), 2075–2106.

[25] G.N. Milstein, M.V. Tretyakov. Layer methods for the incompressible Navier-Stokes equations with space periodic conditions. *Adv. App. Prob.* **45** (2013), 742–772.

[26] G.N. Milstein, M.V. Tretyakov. Layer methods for Navier-Stokes equations with additive noise using simplest characteristics. *J. Comp Appl. Maths.* **302** (2016), 1–23.

[27] R. Peyret. *Spectral Methods for Incompressible Viscous Flow*. Springer, 2002.

[28] R. Temam. *Navier-Stokes Equations, Theory and Numerical Analysis*. AMS Chelsea Publishing, 2001.

[29] R. Temam. *Navier-Stokes Equations and Nonlinear Functional Analysis*. SIAM, 1995.
[30] P. Wesseling. *Principles of Computational Fluid Dynamics*. Springer, 2001.