Nonperturbative equation of state of quark-gluon plasma.

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Abstract

The paper is devoted to the systematic derivation of nonperturbative thermodynamics of quark-gluon plasma, on the basis of the background perturbation theory. Vacuum background fields enter only in the form of field correlators, which are known from lattice and analytic calculations. In the lowest order in $\alpha_s$ the purely nonperturbative sQGP thermodynamics is expressed through single quark and gluon lines (single line approximation) which are interacting nonperturbatively with vacuum fields and with other lines. Nonperturbative EOS is obtained in terms (of the absolute value) of fundamental (adjoint) Polyakov loop $L_{\text{fund}(\text{adj})}$ and spatial string tension $\sigma_s(T)$. In the lowest approximation the pressure for quarks (gluons) has a simple factorized form $P_{q(g)} = P_{SB}L_{\text{fund}(\text{adj})}$, where $L_i$ describe the action of vacuum colorelectric fields on particle trajectory.

1 Introduction

The study of the deconfined phase of QCD has a long history and was based mainly on the idea of weakly interacting gas of quarks and gluons, where perturbative and some nonperturbative methods can be used, (see a book in [1] and reviews in [2]).
Recently, however, it was realized that there is a strong interaction in the QCD vacuum above the deconfinement temperature $T_c$, as witnessed by the RHIC experiments, (see [3] for reviews) and lattice calculations (see e.g. [4] for a review and further references).

Independently of these developments, another approach was suggested in QCD almost two decades ago ([5], see [6, 7] for reviews), where the Nonperturbative (NP) dynamics, instead of the perturbative one, was at the basis of QCD, and systematic formalism was developed both for $T = 0$ ([8, 9]) and for $T > 0$ ([10]-[12]). According to this approach, and in agreement with existing lattice data, the new picture of the deconfined phase of QCD was suggested, where NP dynamics plays the dominant role. The latter is described by simplest (Gaussian) Field Correlators (FC) of color electric and color magnetic fields, $D^E(x), D^H(x), D^E_1(x)$ and $D^H_1(x)$ which can be found from lattice measurements [13, 14] or from analytic calculations [15]. Here correlators $D^E(x), D^H(x)$ are of pure NP origin, while $D^E_1, D^H_1$ contain all perturbative diagrams [16]. The phase transition was predicted to be due to the disappearance of $D^E(x)$, which ensures colorelectric confinement, and the transition temperature $T_c$ was calculated (in terms of gluonic condensate) in good agreement with lattice data [17]. As a result above $T_c$ the NP dynamics was predicted with colormagnetic confinement due to $D^H(x)$ and this prediction was supported later by detailed lattice measurements [14, 18].

This NP dynamics of (color)magnetic confinement plus the NP component of $D^E_1$ (which survives the phase transition) suggest a picture which is much different from the popular perturbative approach. In particular, the strong interaction in systems of quarks and gluons, leading to the formation of e.g. $Q\bar{Q}$ bound state was predicted 15 years ago [19], and the role of the spatial string tension $\sigma_s(T)$ for $T \geq T_c$ was stressed in [10, 17], and supported by lattice data [13, 14, 20].

Recently, as a new challenge to the perturbative picture of quark-gluon plasma, the Debye masses have been computed through $\sigma_s(T)$ [21], in good agreement with lattice data [22].

An important dynamical role in the formalism of field correlators is played by Polyakov loops, and the latter have been computed in terms of correlators and the property of Casimir scaling for them was theoretically established in [23]. Recently the Casimir scaling for five lowest representations SU(3) was accurately checked on the lattice [24], thus supporting the quantitative accuracy of the method also for $T > T_c$. At this point it is clear that a systematic NP formalism for all phenomena in $T > 0$ QCD is needed, which
should be a basis for calculation of EOS, Polyakov loops, susceptibilities etc. In the present paper, we undertake this task, giving a detailed derivation of the expressions for generating functions, pressure and internal energies of quarks and gluons, where perturbative contributions are considered as a series of Background Perturbation Theory (BPTh) and NP effects due to NP background can be taken fully in each term of $O(\alpha_s^n)$.

The present paper is concerned with the lowest order $O(\alpha_s^0)$ of BPTh and gives the pressure of quarks and gluons in the form, where the leading contribution is due to single quark and gluon line interacting with the NP background (Single Line Approximation (SLA)), while line-line NP interaction is a correction. In SLA the pressure can be written in a simple factorized form Eqs. (66), (67), where NP factors are of the absolute value Polyakov loops, while colormagnetic background fields enter in the line-line correction.

Our main region of interest is the low temperature region, $T_c < T < 2T_c$, where nonperturbative effects are most important, while at higher temperatures one expects possible connections of this formalism with the well developed perturbation theory now known to $O(g^6)$ (see [25] and refs. therein).

Two limitations are present in this paper: we do not consider chiral effects, and possible accumulation of quark and gluon bound states near $T_c$.

The plan of the paper is as follows. In section 2 the basics of BPTh and path integrals is given and free quark-gluon gas result is derived as an illustration. In section 3 NP interaction of quarks and gluons with background is expressed in the form of Polyakov loops (for colorelectric fields). Polyakov loops are studied in section 4 and EOS is summarized in section 5. Section 6 is devoted to line-line interaction. Results and discussion are given in the concluding section.

## 2 Background Perturbation Theory at $T > 0$

### 2.1 Basic equations

We start with standard formulae of the background field formalism [8] generalized to the case of nonzero temperature in [10]-[12]. We assume that the gluonic field $A_\mu$ can be split into the background field $B_\mu$ and the (valence gluon) quantum field $a_\mu$

$$A_\mu = B_\mu + a_\mu, \quad (1)$$
both satisfying the periodic boundary conditions (PBC)

\[ B_\mu(z_4, z_i) = B_\mu(z_4 + n\beta, z_i); \quad a_\mu(z_4, z_i) = a_\mu(z_4 + n\beta, z_i), \]

(2)

where \( n \) is an integer and \( \beta = 1/T \). The partition function can be written as

\[ Z(V, T) = \langle Z(B) \rangle_B \]

(3)

with

\[ Z(B) = N \int D\phi \exp \left( -\int_0^\beta d\tau \int d^3 x L_{\text{tot}}(x, \tau) \right) \]

(4)

and where \( \phi \) denotes all set of fields \( a_\mu, \Psi, \Psi^+ \); \( L_{\text{tot}} \) is defined in [10]-[12] and for the convenience of the reader in Appendix 1. \( N \) is a normalization constant. Furthermore, in Eq. (3) \( \langle \rangle_B \) means averaging over (nonperturbative) background fields \( B_\mu \). The precise form of this averaging is not needed for our purpose.

Integration over the ghost and gluon degrees of freedom in Eq. (3) yields the same answer as for the case \( T = 0 \) [19], but where now all fields are subject to the periodic boundary conditions (2). Disregarding for simplicity quark terms and source terms in \( L_{\text{tot}} \) in (4), one obtains

\[ Z(B) = N' (\det W(B))^{-1/2} [\det(-D_\mu(B)D_\mu(B + a))]_{a=\delta J} \times \left\{ 1 + \sum_{l=1}^{\infty} S_{\text{int}} \left( a = \frac{\delta J}{\delta \beta} \right)^l \right\} \exp \left( -\frac{1}{2} J^G J \right) \]

(5)

where the valence gluon Green’s function \( G \equiv W^{-1} \) is \( G_{\mu\nu} = (\tilde{D}_\lambda^2 \cdot \hat{1} + 2ig\tilde{F}_{\mu\nu})^{-1} \), and the tilde sign here and below refers to the operators in the adjoint representation, e.g. \( \tilde{F}_{\mu\nu}^{bc} \equiv iF_{\mu\nu}^a f^{abc} \). We can consider strong background fields, so that \( gB_\mu \) is large (as compared to \( \Lambda_{\text{QCD}}^2 \)), while \( \alpha_s = g^2/4\pi \) in that strong background is small at all distances. Moreover, it was shown that \( \alpha_s \) is frozen at large distances [9]. In this case Eq. (5) is a perturbative sum in powers of \( g^n \), arising from the expansion in \( (ga_\mu)^n \).

In what follows we shall discuss the Feynman graphs for the thermodynamic potential \( F(T, \mu) \), connected to \( Z(B) \) via

\[ F(T, \mu) = -T \ln \langle Z(B) \rangle_B. \]

(6)

It is clear that (6) contains all possible interactions, perturbative and non-perturbative (NP) between quarks and gluons, and in particular creation or
dissociation of bound states. It is impossible to take into account all possible subsystems and interaction between them, and it is imperative to choose the strategy of approximations for the quark-gluon plasma as a deconfined state of quarks and gluons.

We assume that the whole system of $N_g$ gluons and $N_q, N_{\bar{q}}$ quarks and antiquarks for $T > T_c$ stays gauge-invariant, as it was for $T < T_c$, however in case of deconfinement and neglecting in the first approximation all perturbative and NP interaction, any white system will have the same energy depending only on the number and type of constituents. In this case $Z$ factorizes into a product of one-gluon or one-quark (antiquark) contributions and we calculate the corresponding thermodynamic potential.

This first step is called the Single Line Approximation (SLA) and in the next two subsections we calculate the known results for free gluon and quark gas in our path-integral formalism following [10].

However already in SLA there exists a strong interaction of a gluon (or $q, \bar{q}$) with the NP vacuum fields. It consists of colorelectric (CE) and colormagnetic (CM) parts, as shown in section 3. The CE part in the deconfinement case creates the individual NP self-energy contribution for every gluon or quark, and this is a factor of the corresponding Polyakov loop discussed in section 4. Note, that the quark (gluon) Polyakov loop factor is computed from the gauge invariant $q\bar{q}(gg)$ Wilson loop, which for the NP $D_1^E$ interaction separates into individual quark (gluon) contributions equal to the modulus of the corresponding Polyakov loop. Therefore our Polyakov loop factors are expressed through color singlet 9heavy-quark) free energy $F_s(\infty, T)$ and the popular $Z(N_c)$ symmetry is irrelevant for them.

With the CM part the situation is more subtle. Strong CM fields in the deconfined QCD vacuum introduced in [10], [17] and measured on lattice [13, 14] create bound states of white combinations of quarks and gluons, and this is formally beyond SLA. To take these bound states into account one needs to use source terms, introduced in Appendix 1, and do calculations as shown in Appendix 2.

Therefore the CM fields are taken into account as two-body correlations, which are discussed in section 5 and Appendix 2.

The SLA result for the pressure with the CE is given in section 6. The effects of possible $q\bar{q}, 3q, gg, 3g$, etc bound states which might be important nearby $T \geq T_c$, as it is discussed in [23], are deferred to future publications.
2.2 The lowest order gluon contribution

To lowest order in \( g_a^\mu \) (keeping all dependence on \( g B^\mu \) explicit) we have

\[
Z_0 = e^{-F_0(T)/T} = N' \langle \exp(-F_0(B)/T) \rangle_B, \tag{7}
\]

where using Eq. (5) \( F_0(B) \) can be written as

\[
\frac{1}{T} F_0^g(B) = \frac{1}{2} \ln \det G^{-1} - \ln \det(-D^2(B)) = \text{Sp} \left\{ -\frac{1}{2} \int_0^\infty \xi(s) \frac{ds}{s} e^{-sG^{-1}} + \int_0^\infty \xi(s) \frac{ds}{s} e^{sD^2(B)} \right\}. \tag{8}
\]

In Eq. (8) \( \text{Sp} \) implies summing over all variables (Lorentz and color indices and coordinates) and \( \xi(t) \) is a regularization factor. Graphically, the first term on the r.h.s. of Eq. (8) is a gluon loop in the background field, while the second term is a ghost loop.\(^1\)

Let us turn now to the averaging procedure in Eq. (7). With the notation \( f = -F_0(B)/T \), we can exploit in Eq. (7) the cluster expansion \[26\]

\[
\langle \exp f \rangle_B = \exp \left( \sum_{n=1}^\infty \frac{\langle f^n \rangle_B}{n!} \right) = \exp \{ \langle f \rangle_B + \frac{1}{2} [\langle f^2 \rangle_B - \langle f \rangle_B^2] + O(f^3) \}. \tag{9}
\]

Below in this paper we consider mostly the leading term in the cluster expansion, namely \( \langle f \rangle_B \), leaving the quadratic and higher terms for the section 7.

To get a closer look at \( \langle \varphi \rangle_B \) we first should discuss the thermal propagators of the gluon and ghost in the background field. We start with the thermal ghost propagator and write the Fock-Feynman-Schwinger Representation (FFSR) for it \[10\]

\[
(-\tilde{D}^2)^{-1}_{xy} = \langle x | \int_0^\infty ds e^{sD^2(B)} | y \rangle = \int_0^\infty ds (Dz)_{xy} e^{-K \tilde{\Phi}(x, y)}. \tag{10}
\]

Here \( \tilde{\Phi} \) is the parallel transporter in the adjoint representation along the trajectory of the ghost:

\[
\tilde{\Phi}(x, y) = P \exp(i \int \tilde{B}_\mu(z) dz_\mu) \tag{11}
\]

\(^1\)In what follows we shall call the contribution of gluon (ghost) Green’s functions \( G(D^{-2}(B)) \) the gluon(ghost) line contribution to distinguish it from the loop expansion in standard perturbative analysis.
and \((D\zeta)^w_{xy}\) is a path integration with boundary conditions imbedded (denoted by the subscript \((xy)\)) and with all possible windings in the Euclidean temporal direction (denoted by the superscript \(w\)). We can write it explicitly as

\[
(D\zeta)^w_{xy} = \lim_{N \to \infty} \prod_{m=1}^{N} \frac{d^4\zeta(m)}{(4\pi\varepsilon)^2} \sum_{n=0,\pm,\ldots} \frac{d^4p}{(2\pi)^4} \exp \left[ ip_\mu \left( \sum_{m=1}^{N} \zeta_\mu(m) - (x-y)_\mu - n\beta \delta_\mu 4 \right) \right]. \tag{12}
\]

Here, \(\zeta(k) = z(k) - z(k-1),\ N\varepsilon = s\). We can readily verify that in the free case, \(\hat{B}_\mu = 0\), Eq. (10) reduces to the well-known form of the free propagator

\[
(-\partial^2)^{-1}_{xy} = \int_{0}^{\infty} ds \exp \left[ -\sum_{m=1}^{N} \frac{\zeta^2(m)}{4\varepsilon} \right] \prod_{m=1}^{N} \frac{d\zeta(m)}{(2\pi)^4} \sum_{n=0,\pm,\ldots} \frac{d^4p}{(2\pi)^4} \exp \left[ ip_\mu \left( \sum_{m=1}^{N} \zeta_\mu(m) - (x-y)_\mu - n\beta \delta_\mu 4 \right) \right]. \tag{13}
\]

with

\[
d\zeta(m) \equiv \frac{d\zeta(m)}{(4\pi\varepsilon)^2}.
\]

Using the Poisson summation formula

\[
\frac{1}{2\pi} \sum_{n=0,\pm1,\pm2,\ldots} \exp(ip_4n\beta) = \sum_{k=0,\pm1,\ldots} \delta(p_4\beta - 2\pi k) \tag{14}
\]

we finally obtain the standard form

\[
(-\partial^2)^{-1}_{xy} = \sum_{k=0,\pm1,\ldots} \int T \frac{d^3p}{(2\pi)^3} \frac{\exp[-ip_i(x-y)_i - i2\pi k T(x_4 - y_4)]}{p_i^2 + (2\pi k T)^2}. \tag{15}
\]

Note that, as expected, the propagators (10) and (15) correspond to a sum of ghost paths with all possible windings around the torus. The momentum integration in Eq. (12) asserts that the sum of all infinitesimal "walks" \(\zeta(m)\) should be equal to the distance \((x-y)\) modulo \(N\) windings in the compactified fourth coordinate. For the gluon propagator in the background gauge we obtain similarly to Eq. (10)

\[
G_{xy} = \int_{0}^{\infty} ds (Dz)^w_{xy} e^{-K\tilde{\Phi}(x,y)}, \tag{16}
\]
\[ \tilde{\Phi}_F(x, y) = P_F P \exp \left( 2ig \int_0^t \tilde{F}(z(\tau)) d\tau \right) \exp \left( ig \int_y^x \tilde{B}_\mu d\tau \right). \quad (17) \]

The operators \( P_F P \) are used to order insertions of \( \hat{F} \) on the trajectory of the gluon.

Now we come back to the first term in Eq. (9), \( \langle \phi \rangle_B \), which can be represented with the help of Eqs. (10) and (16) as

\[ \langle F_{gl0}(B) \rangle_B = -T \int ds \xi(s) d^4x (Dz)_x e^{-K} \left[ \frac{1}{2} tr \langle \bar{\Phi}_F(x, x) \rangle_B - \langle tr \tilde{\Phi}(x, x) \rangle_B \right], \quad (18) \]

where \( tr \) implies summation over Lorentz and color indices. One can easily show \[10\] that Eq. (18) yields for \( B_\mu = 0 \) the usual result for the free gluon gas:

\[ F_{gl0}(B = 0) = \frac{1}{2} \ln det(m_q^2 - \hat{D}^2(B')) = -\frac{1}{2} Sp \int_0^\infty \xi(s) \frac{ds}{s} e^{-s m_q^2 + s \hat{D}^2(B')}, \quad (21) \]

where \( Sp \) has the same meaning as in Eq. (8) and

\[ \hat{D}^2 = (D_\mu \gamma_\mu)^2 = D_\mu^2(B') - g F_{\mu\nu} \sigma_{\mu\nu} \equiv D^2 - g \sigma F; \]

\[ \sigma_{\mu\nu} = \frac{1}{4i} (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu). \quad (22) \]

Our aim now is to exploit the FSR to represent Eq. (21) in a form of the path integral, as was done for gluons in Eq. (10). The equivalent form for
quarks must implement the antisymmetric boundary conditions pertinent to fermions. We find \cite{10}

\[
\frac{1}{T} F_0^q(B') = -\frac{1}{2} \text{tr} \int_0^\infty \frac{ds}{s} d^4x \overline{x(Dz)^w} e^{-K} x \quad \text{W}_\sigma(C_n), \tag{23}
\]

where \( tr \) implies summation over spin and color indices,

\[
W_\sigma(C_n) = P_F P_A \exp \left( ig \int_{C_n} B'_\mu dz_\mu \right) \exp g \int_0^s (\sigma_{\mu\nu} F_{\mu\nu}) d\tau, \tag{24}
\]

and

\[
(Dz)^w_{xy} = \prod_{m=1}^N \frac{d^4\zeta(m)}{(4\pi^2)^2} \times \sum_{n=0,\pm 1, \pm 2, \ldots} \frac{(-1)^n}{(2\pi)^4} \exp \left[ ip \left( \sum_{m=1}^N \zeta(m) - (x - y) - n\beta \delta_{\mu4} \right) \right]. \tag{24}
\]

It can readily be checked that in the case \( B_\mu = 0, \ \mu \equiv 0 \) the well known expression for the free quark gas is recovered, i.e. for \( m_q = 0 \)

\[
F_0^q(\text{free quark}) = -\frac{7\pi^2}{180} N_c V_3 T^4 \cdot n_f, \tag{25}
\]

where \( n_f \) is the number of flavors. The derivation of Eq. \eqref{25} starting from the path-integral form \eqref{23} is done similarly to the gluon case given in the Appendix of the last reference in Ref. \cite{10}.

The path \( C_n \) in Eq. \eqref{23} corresponds to \( n \) windings in the fourth direction. Above the deconfinement transition temperature \( T_c \) one sees in Eq. \eqref{23} the appearance of the factor

\[
\Omega_{\text{fund}} = P \exp \left[ ig \int_0^\beta B_4(z) dz_4 \right].
\]

In the next section it is shown that it will generate a gauge-invariant quantity

\[
L_{\text{fund}} = \frac{1}{N_c} \langle tr \Omega_{\text{fund}} \rangle. \tag{26}
\]

where \( L_{\text{fund}} \) is the Polyakov loop in the fundamental representation.

For a nonzero \( \mu \) and \( n_f \) massless flavors with \( B_\mu \equiv 0 \) one has

\[
\langle F \rangle = -\frac{4N_c V_3 T^4}{\pi^2} n_f \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} \cosh \left( \frac{\mu n}{T} \right). \tag{27}
\]

In what follows, we consider arbitrary fields \( B_\mu(x) \) and express \( \langle F \rangle \) through field correlators.
3 Nonperturbative dynamics of single quark and gluon lines

As was shown in the previous section, Eq. (18), the pressure for gluons can be written as (the $T$-independent term with $n = 0$ is subtracted, and the relation $P_{gl}V_3 = -(F_0(B))_B$ is used),

$$P_{gl} = (N_c^2 - 1) \int_0^\infty \frac{ds}{s} \sum_{n \neq 0} G^{(n)}(s)$$  \hspace{1cm} (28)

and for quark one can write using (23)

$$P_q = 2N_c \int_0^\infty \frac{ds}{s} e^{-m_q^2 s} \sum_{n=1}^{\infty} (-1)^{n+1} [S^{(n)}(s) + S^{(-n)}(s)].$$  \hspace{1cm} (29)

Here $G^{(n)}$ and $S^{(n)}$ are the proper-time kernels of gluon and quark Green’s functions at nonzero $T$, which are defined as follows

$$G^{(n)}(s) = \int (Dz)^w_{on} e^{-K \hat{tr}_a \langle W_\Sigma(C_n) \rangle}$$  \hspace{1cm} (30)

$$S^{(n)}(s) = \int (Dz)^w_{on} e^{-K \hat{tr}_f \langle W_\sigma(C_n) \rangle}$$  \hspace{1cm} (31)

where

$$\hat{tr}_f W_\sigma(C_n) = \frac{1}{N_c} tr_f W_\sigma(C_n)$$  \hspace{1cm} (32)

and

$$\hat{tr}_a \langle W_\Sigma(C_n) \rangle = \frac{tr_a}{(N_c^2 - 1)} \left( \frac{1}{2} \hat{\Phi}_F(C_n) - \Phi(C_n) \right)$$  \hspace{1cm} (33)

In what follows we shall neglect the first exponent in (17) both in $W_\Sigma$ and in $W_\sigma$, since this gives spin-dependent interaction, which can be treated as a correction (we relegate this treatment to the Appendix 2). Therefore the calculation of $G^{(n)}(s)$ and $S^{(n)}(s)$ needs first the evaluation of the Wilson loop $\langle W(C_n) \rangle$.

Our purpose is to calculate $G^{(n)}(s)$, $S^{(n)}(n)$ using for $\langle W(C_n) \rangle$ the field correlators.

To proceed one should look more carefully into the topology of the Wilson loop $W_{\Sigma,\sigma}(C_n)$, which is to be a closed loop in $4d$. At this point one should
again emphasize that the basic states $|k>$ with the quark-gluon numbers $N^k_g, N^k_q, N^k_{\bar{q}}$ entering into the partition function $Z$ are gauge invariant, $Z = \sum_k |k> e^{-H/T} |k>$ and one write for $|k>$ a generic decomposition of the type, $|k> = \{(gg)(gg)(gg)(q\bar{q})...\}$ where particles in parentheses form white combinations.

Correspondingly (and neglecting interaction between white subsystems) for each white subsystem one has contribution proportional to the product, e.g. for the $(gg)$ or $(q\bar{q})$ system

$$Z_{(12)} = \int \int d\Gamma_1 d\Gamma_2 \langle trW_{\Sigma,\sigma}(C_n^{(1)}, C_n^{(2)})\rangle,$$

with

$$\int d\Gamma_i = \int ds_i e^{-K_i(Dz^{(i)})}.$$

Here $\langle trW_{\Sigma,\sigma}(C_n^{(1)}, C_n^{(2)})\rangle$ is the closed Wilson loop made of paths $(C_n^{(1)}, C_n^{(2)})$ of two gluons (or $q\bar{q}$) and parallel transporters (Schwinger lines) in the initial and final states, necessary to make them gauge invariant.

The important result of this paper, shown below, is that colorelectric correlator $D_E$ yields factorized contribution, i.e. $Z_{(12)} \rightarrow Z_1 Z_2$, where each of the factors $Z_i$ contains only a part of the common loop in the form of the Polyakov loop factor (with the singlet free energy in the exponent).

The contribution of colormagnetic fields does not factorize, however, as shown in Appendix 2, this contribution can be considered as a correction therefore in this section it will be neglected, leaving the topic to section 5.

As we shall see, the 4d path integral will be decomposed into a product, $D^4z \rightarrow D^3z Dz_4$, and the interaction kernel also factorizes in the Gaussian approximation $\langle W(C_n)\rangle = \langle W_1^{(1)} \rangle \langle W_3^{(2)} \rangle$, which leads finally to the relation (42). To proceed we shall use as in [23] the Field Correlator Method, where for any closed loop $C_n$ one can apply the nonabelian Stokes theorem and Gaussian approximation and write (see [6, 7] for discussion and refs)

$$\hat{tr}_f \langle W(C_n^{(1)} C_n^{(2)})\rangle = \frac{tr_c}{N_c} \langle P \exp ig \int_{C_n} B_\mu dz_\mu \rangle =$$

$$= \frac{tr_c}{N_c} \exp \left( -\frac{g^2}{2} \int_{S_n} \int_{S_n} d\sigma_{\mu\nu}(u) d\sigma_{\lambda\sigma}(v) \langle F_{\mu\nu}(u)F_{\lambda\sigma}(v) \rangle \right). \quad (34)$$

Here one should specify the surface $S_n$ with the surface elements $d\sigma_{\mu\nu}(u)$ in the integral on the r.h.s. of (34). For this analysis it is important to consider
different terms in the sum over \( \mu, \nu \) in (34) separately. For the field correlator in (34) one can use the decomposition (second ref. in [5]) which for nonzero \( T \) should be written separately for colorelectric field \( E_i \) and colormagnetic field \( H_i \) [10, 17]. With the definition \( H_k \equiv \frac{1}{2} \varepsilon_{ijk} F_{ij} \) one has

\[
g^2 \langle \hat{r} f [E_i(x) \Phi(x, y) E_k(y) \Phi(y, x)] \rangle = \delta_{ik} \left[ D^E + D_1^E + u_4 \frac{\partial D_1^E}{\partial u_4^2} \right] + u_i u_k \frac{\partial D_1^E}{\partial u_4^2},
\]

\[
g^2 \langle \hat{r} f [H_i(x) \Phi(x, y) H_k(y) \Phi(y, x)] \rangle = \delta_{ik} \left[ D^H + D_1^H + u^2 \frac{\partial D_1^H}{\partial u^2} \right] - u_i u_k \frac{\partial D_1^H}{\partial u^2},
\]

\[
g^2 \langle \hat{r} f [E_i(x) \Phi(x, y) H_k(y) \Phi(y, x)] \rangle = -\frac{1}{2} \varepsilon_{ikn} u_n \frac{\partial D_1^{HE}}{\partial u_4}. \tag{35}
\]

We take first the term with \( \nu = \sigma = 4 \) (note that by definition \( \mu < \nu \) and \( \lambda < \sigma \), see [6, 7]).

\[
J_n^E \equiv \frac{1}{2} \int D_{4,4k4} (u, v) d\sigma_{44} (u) d\sigma_{k4} (v), \tag{36}
\]

where we have defined \( D_{\mu\nu,\lambda\sigma} \equiv g^2 \langle \hat{r} f F_{\mu\nu}(u) \Phi(u, v) F_{\lambda\sigma}(v) \Phi(v, u) \rangle \). At this point one takes into account that the surface \( S_n \) is inside the winding Wilson loop for the gauge-invariant \( q \bar{q} \) system, as it is done in [21] and Appendix 2. The contribution of \( D_1 \) leads to the sum of terms for \( q \) and \( \bar{q} \) separately of the form (38), which fact justifies the use of singlet \( V_1(\infty) \) for single line quark (or gluon) term in \( P_q \) (or \( P_{gl} \)). For a single quark one obtains in this way

\[
J_n^E = \frac{1}{2} \int_0^{n\beta} du_4 \int_0^{n\beta} dv_4 \int_0^\infty \xi d\xi \frac{D_1^E}{D_4^E}(\sqrt{(u_4 - v_4)^2 + \xi^2}) = \frac{1}{2} n\beta \int_0^{n\beta} dv \left(1 - \frac{\nu}{n\beta}\right) \int_0^\infty \xi d\xi \frac{D_1^E}{D_4^E}(\sqrt{\nu^2 + \xi^2}). \tag{37}
\]

Note that for \( n = 1 \) one recovers the expression for the Polyakov loop obtained in [23], namely

\[
L_{fund} = \exp(-\frac{1}{2T} V_1(\infty)), \quad \frac{1}{2T} V_1(\infty) = J_1^E \tag{38}
\]
At this point an important simplification occurs. Namely, originally the path $C_n$ is a complicated path $z_\mu(\tau)$ in 4d, however the integral (37) over $d\xi$ is always from some point $z_\mu(\tau)$ on $C_n$ to infinity and does not depend on the exact form of $C_n$, and is the same as for the straight-line Polyakov loop. This is true, however, only for the $D_{i4,k4}$ and not for $D_{ik,lm}$.

Another important observation is that correlators $D_{\mu\nu,\lambda\sigma}(u,\nu)$ are not periodic function of $(u_4 - v_4)$, in contrast to fields $F_{\mu\nu}(u)$ and $F_{\lambda\sigma}(v)$. This will be true also for path integrals from any point $x$ to arbitrary point $y$, (temperature Green’s function) and is a consequence of the vacuum average of parallel transporter $\Phi(u, v)$ present in $D_{\mu\nu,\lambda\sigma}(u, v)$.

We turn now to other contributions in (34),

$$J_n^{EH} \equiv \frac{1}{2} \int D_{i4,kl}(u, v) d\sigma_{i4}(u) d\sigma_{kl}(v).$$

(39)

This term is treated in Appendix 2, and is shown to be $T$ independent, we shall neglect it in what follows.

We finally turn to the spatial term, which will play in what follows a special role. Here the main contribution comes from the colormagnetic correlator $D^H(u - v)$, which provides the area law for the (closed) spatial projection $A_3$ of the surface $S_n$. Correspondingly we shall denote this term as (for $A_3 \gg \lambda^2$, where $\lambda$ is the gluon correlation length, $D^H(x) \sim \exp(-x/\lambda)$)

$$\langle W_3(C_n) \rangle = \exp(-\sigma_s A_3) \quad (40)$$

where

$$\sigma_s = \frac{1}{2} \int D^H(x) d^2x$$

(41)

and $A_3$ is the minimal area of the spacial projection of the surface $S_n$. As in the $J_n^{EH}$ case, one should start with white states of $q\bar{q}, gg$ or $3q$. As it is shown in Appendix 2, the colormagnetic vacuum ($D^H$) acts only in the $L \neq 0$ states of these systems, and the resulting contribution of $D^H$ does not separate into single-line terms, but rather acts pairwise (triple-wise for $3q$). Therefore one can account for color magnetic interaction in the higher (2-line or 3-line) terms. One can take it approximately averaging single line in the gas of quarks and gluons, and denoting the corresponding term as Magnetic Loop Factor (MLF) $G_3(s)$ and $S_3(s)$ for gluons and quarks respectively. In what follows we shall neglect colormagnetic interaction in the first approximation. As a result the 4d dynamics in (30), (31) separates into 3d and 1d, and one can write
and similarly $S^{(n)}(s) = S_4^{(n)}(s)S_3(s)$, with the free quark (gluon) factors

$$S_3^{(0)}(s) = G_3^{(0)}(s) = \int (D^3z)_{00} e^{-K_3} = \frac{1}{(4\pi s)^{3/2}}. \tag{43}$$

One should notice at this point, that $G^{(n)}(s)$ and $S^{(n)}(s)$, Eqs. (30) and (31) differ in spin-factors (17) and (23), and in color group representation. Dropping spin-factors in the lowest approximation, one obtains $S^{(n)}$ from $G^{(n)}$ replacing adjoint quantities (like $\sigma_s(\text{adj})$) by fundamental ones ($\sigma_s(\text{fund})$). At this point one can use Casimir scaling [23, 24] to write $\sigma_s(\text{adj}) = \frac{9}{4}\sigma_s(\text{fund})$.

To compute $G_4^{(n)}(s)$ one can notice, that $J^E_n$ does not depend on $z_4$ and can be taken out of the integral, while the integral over $z_4$ can be taken exactly, namely splitting the proper-time interval $N\varepsilon = s, N \rightarrow \infty$ one can write

$$\int (Dz_4)_{00} e^{-K_4} = \prod_{k=1}^{N} \left( \frac{d\Delta z_4(k)}{\sqrt{4\pi \varepsilon}} e^{-\frac{(\Delta z_4(k))^2}{4\varepsilon}} e^{ip_4\Delta z_4(k)} \right) e^{-ip_4 n \beta} \frac{dp_4}{2\pi} =$$

$$= \int \frac{dp_4}{2\pi} e^{ip_4 n \beta - \frac{s^2}{4s}} = \frac{1}{2\sqrt{\pi s}} e^{-n^2 s^2} \tag{44}$$

As a result one has for $G^{(n)}(s)$

$$G^{(n)}(s) = \frac{1}{2\sqrt{\pi s}} e^{-\frac{n^2 s^2}{4s} - \hat{J}^E_n} G_3^{(0)}(s) \tag{45}$$

$$S^{(n)}(s) = \frac{1}{2\sqrt{\pi s}} e^{-\frac{n^2 s^2}{4s} - \hat{J}^E_n} S_3^{(0)}(s); \quad \hat{J}^E_n = \frac{9}{4} J^E_n$$

In the next section we consider the Polyakov loop factor $J^E_n$

4 Properties of Polyakov loops

As it was discussed in the previous section, Eq. (28), the lowest order contribution to the free energy of the quark and gluon contains the path integral
$S^{(n)}(s)$ and $G^{(n)}(s)$, Eqs. (30) and (31), which is expressed through the Polyakov loop with $n$ windings, namely (cf Eq. (45)).

$$G^{(n)}(s) = \frac{1}{2\sqrt{\pi} s} e^{-\frac{s^2}{2\beta}} G_3^{(0)}(s) L_{adj}^{(n)}$$

(46)

where

$$L_{adj}^{(n)} \equiv \exp(-\bar{J}_n^E), \quad \bar{J}_n^E = \frac{9}{4} J_n^E$$

(47)

and $J_n^E$ is given in (37),

$$J_n^E = \begin{array}{c} \frac{n\beta}{2} \int_0^{\beta_n} d\nu \left(1 - \frac{\nu}{n\beta}\right) \int_0^\infty \xi d\xi D_1^E(\sqrt{\nu^2 + \xi^2}). 
\end{array}$$

(48)

In what follows we shall discuss the following properties of Polyakov loops:

a) Dependence of $L^{(n)}$ on $n$. (in what follows $L$ stands for $L_i$, $i = \text{fund, adj.}$)

b) Renormalization of $L^{(n)}$.

c) Calculation of $L_{adj}$ and $L_{fund}$ through the field correlators.

a) We start with $n = 1$ and write as in (38)

$$L_{fund} = \exp\left(-\frac{1}{2T} V_1(\infty)\right), \quad \frac{1}{2T} V_1(\infty) \equiv J_1^E$$

(49)

where $V_1(r)$ was introduced in [23] as a potential between heavy quark $Q$ and antiquark $\bar{Q}$ at a distance $r$, obtained from the correlator of the Polyakov loops. As it is seen in [38] $V_1(\infty)$ (as well as $V_1(r)$) is calculated directly through the correlator $D_1^E$ (for $T \geq T_c$). The latter has perturbative and nonperturbative contributions which we write as (we neglect possible interference terms, which contribute near $r = 0$ [27])

$$D_1^E(x) = D_{1\text{pert}}^E(x) + D_{1\text{nonp.}}^E(x)$$

(50)

and it is clear that $D_{1\text{pert}}^E$ contributes to the color-Coulomb law between $Q$ and $\bar{Q}$ and to the perturbative self-energy correction, which should be properly subtracted and renormalized, as discussed below in the point b). The NP ingredient to $L$ is given by $D_{1\text{nonp}}^E$ and this is responsible for the
NP thermodynamics of the $qq$ plasma, which is the leading contribution considered in this paper.

The form of the $D_{\text{inonp}}^E$ has been found previously at $T = 0$ analytically \cite{15, 23, 18} and on the lattice \cite{13, 14} and it was shown that $D_{\text{inonp}}^E$ can be connected to the gluelump Green’s function, the gluelump mass $M_1$, found in \cite{28}, defining the range of $D_{\text{inonp}}^E$.

$$D_{\text{inonp}}^E(x) \approx \text{const} \exp(-M_1|x|), |x| \gtrsim 1/M_1$$  \hspace{1cm} (51)

In \cite{13, 14} $D_1^E$ and the value of $M_1$ was computed on the lattice, and in \cite{23} this calculation was compared to the lattice data \cite{29} for Polyakov loop correlator showing a reasonable agreement. Therefore one can use as an acceptable the value of $M_1 \approx 0.7 \div 1$ GeV, which is comparable to the inverse of gluon correlation length $\lambda$ at $T = 0$; $1/\lambda \approx 1$ GeV.

The asymptotic behaviour (51) is also kept at $T > T_c$, as shown by lattice data \cite{13, 14}.

At this point one should use the behaviour (51) in (48). It is important to stress, that $D_1^E$ is not periodic in $\nu \equiv x_4 - y_4$, we remind that

$$D_1^E(\nu, \xi) \text{ is decreasing for large } |\nu| \text{ exponentially as in (51).}$$

One can define in the integral (48) two possible regimes: 1) $\beta M_1 \gg 1$, and 2) $\beta M_1 \ll 1$.

In the first case one can always neglect in (48) all dependence on $n$ except for the first factor, writing

$$J_n^E \approx \langle tr E_i(x_4, x) \tilde{\Phi}(x, y) E_i(y_4, y) \rangle$$  \hspace{1cm} (52)

in contrast to $E_i(x)$, which satisfies PBC $E_i(x_4 + n\beta, x) = E_i(x_4, x)$, and this is a consequence of the presence of the factor $\Phi(x, y)$ which is not periodic in $x_4 - y_4$. Hence $D_1^E(\nu, \xi)$ is decreasing for large $|\nu|$ exponentially as in (51).

One can define in the integral (48) two possible regimes: 1) $\beta M_1 \gg 1$, and 2) $\beta M_1 \ll 1$.

In the first case one can always neglect in (48) all dependence on $n$ except for the first factor, writing

$$J_n^E \approx \frac{n\beta}{2} \int_0^\infty d\nu \int_0^\infty \xi d\xi D_1^E(\sqrt{\xi^2 + \nu^2}) \approx nJ_1^E.$$  \hspace{1cm} (53)

This results in a simple relation

$$L_i^{(n)} \approx (L_i)^n, \quad i = \text{adj, fund.}$$  \hspace{1cm} (54)

In the second case $D_1^E$ extends over the region of several periods $n$, and for such $n$, that $n\beta M_1 \lesssim 1$, $J_n^E$ behaves quadratically in $n$. Since $D_1^E$ is positive, this means that $\langle L^n \rangle$ decreases at large $T, T \gg M_1$, faster than is given by the law (54). Hence the general conclusion is that for $T \ll M_1$ the
law (54) holds, while for \( T \gg M_1 \) (54) is replaced by a stronger dependence of the type \( L^{(n)} \approx (L)^{n^2} \) for the first \( n \) terms satisfying \( n < n^* = \frac{T}{M_1} \).

b) We now turn to the point b) and discuss renormalization of Polyakov loop, which was introduced on the lattice in [29] and was considered analytically in [23].

In [23] it was found that the Polyakov loop expressed through the correlator \( D_1 \), can be written as

\[
L_{\text{fund}} = \exp\left( -\frac{V_1^{(\text{np})(\infty)} + V_1^{(\text{pert})(\infty)}}{2T} \right)
\]

(55)

where \( V_1^{(\text{np})(\infty)} \) is expressed through the NP (and regular at \( x = 0 \) part \( D_{1\text{np}}(x) \)), while \( V_1^{(\text{pert})(\infty)} \) is expressed through \( D_{1\text{pert}}(x) \) (cf. (50)) and for the lattice-type minimal distance \( a \) in the lowest order in \( g^2 \) is equal to

\[
V_1^{(\text{pert})(\infty)} = \frac{2C_2 \alpha_s}{\pi} \left( \frac{1}{a} - T \ln a \right)
\]

(56)

Neglecting the next order and possible perturbative-nonperturbative contributions, one can define the renormalized Polyakov loop as

\[
L_{\text{fund}}^{\text{ren}} \equiv \exp\left( -\frac{V_1^{(\text{np})(\infty)}}{2T} \right)
\]

(57)

This definition can be compared to the renormalization of color singlet free energy \( F_s(r) \) used in [29], where it was noticed that \( F_s(r) \) does not depend on temperature (at least for \( T < 2T_c \)) for distances \( r \leq 0.15 \text{ fm} \). Thus normalization of the data at short distances allows to have a smooth limit to the \( T = 0 \) case.

Since \( V_1^{(\text{np})(r)} \sim r^2 \) [23, 5] at small distances, \( F_s(r) \) actually coincides there with \( V_1^{(\text{pert})(r)} \), and the divergent part of \( V_1^{(\text{pert})(r)} \) is in \( V_1^{(\text{pert})(\infty)} \). Hence one can deduce, that normalization to the small distance part of \( V_1^{(\text{np})(r)} \) at \( T = 0 \), with

\[
F_s^{\text{ren}}(r) \propto V_1(r) - V_1^{(\text{pert})(\infty)} \equiv V_1^{(\text{ren})(r)}
\]

(58)

is a justifiable renormalization procedure equivalent to dropping the divergent part, \( V_1^{(\text{pert})(\infty)} \) in the definition of \( V_1^{(\text{ren})} \). In what follows we shall define \( L^{\text{ren}} \) as in (57).

One might ask how at all the (unphysical) \( \frac{1}{a} \) divergence can occur in the expression for the free energy (pressure), where physical quark and gluon
Green’s function enter. Indeed, we have used the path-integral form (30), (31), which is exact and corresponds to the standard perturbation theory at small distances. Therefore self-energy parts due to gluon exchange have a normal logarithmic divergence which is eliminated by a standard renormalization procedure, setting the renormalized self-energy equal to the pole mass for quarks or zero for gluons. Therefore the divergent part is effectively eliminated from the Green’s function when general form as in (28) is used. Note, that the factorized form (42) already used the fact that $J_E^E$ is a constant, not depending on $z_4, z$, which implies that the self-energy divergence is eliminated and only nonperturbative contribution to $L$ is kept. Therefore $J_E^E$ in (45) is already $V(r^\infty)/2T$ and corresponds to the renormalized Polyakov loop (57).

At this point one should be careful to distinguish between the (theoretical) definition of the Polyakov loop (57) and lattice values $L_{lat}$, since the latter, as discussed in [23], contain all possible excitations of the heavy-light system, not present in the light quark or gluon loop. For more discussion see last section of this paper.

c) Here we discuss calculation of $L_{fund}$ and $L_{adj}$ through $D$ and $D_1$ both below and above $T_c$. As found in [23], $L_{adj}$ is

$$L_{adj} = \exp\left\{ -\left( \frac{9}{4}V_D(r^*, T) + \frac{9}{8}V_{1np}(\infty, T) \right) / T \right\}$$

(59)

where $V_D$ is given in Appendix A of [23], $T < T_c$

$$V_D(r, T) = 2 \int_0^\beta d\nu (1 - \nu T) \int_0^r (r - \xi) d\xi D_E(\sqrt{\xi^2 + \nu^2})$$

(60)

and $r^*$ is an average distance between the adjoint line (static adjoint charge) and a gluon of vacuum, forming together a bound state (a gluelump). In a rough approximation one can replace $\frac{9}{4}V_D(r^*, T) \approx M_{glp}$, with $M_{glp}$-the gluelump mass of the order of $1.1 \div 1.4$ GeV [28]. In the similar way one has for a quark Polyakov loop

$$L_{fund} = \exp\left\{ -(V_D(r^*_1, T) + \frac{1}{2}V_{1np}(\infty, T))/T \right\}, \quad L_{adj} = (L_{fund})^{9/4}$$

(61)

where $r^*_1$ corresponds to the average size of the heavy-light meson, $r^*_1 \sim 1.2 \div 1.4$ fm, and $V_D(r^*, T) \approx \varepsilon_{HL} \approx 0.6 \div 0.7$ GeV.

Therefore both $L_{adj}$ and $L_{fund}$ for $n_f > 0$ are nonzero also below $T_c$, however small in this region. Above $T_c$ one has a Casimir scaling relation,
derived first in [23] as a consequence of the fact that Gaussian correlators $D, D_1$ proportional to the charge squared.

Recently the Casimir scaling as in (61) was accurately checked and confirmed on the lattice [24] for five lowest representations of SU(3). This fact shows that also for Polyakov loops as well as for Wilson loops [30] the lowest (quadratic) field correlators $D, D_1$ are dominant with accuracy of few percent. Hence the QCD vacuum is Gaussian dominant at all temperatures, (unless a fine-tuned cancellation of higher correlators takes place [30]).

5 Nonperturbative Equation of State (Single Line Approximation)

We are now in position to summarize our derivations of the nonperturbative free energies for quarks and gluons in the Single Line Approximation (SLA) as follows: for gluons from (28), (45),

$$P_{gSLA} = \frac{(N_c^2 - 1)}{2 \sqrt{\pi}} \int_0^\infty \frac{ds}{s^{3/2}} G_3^{(0)}(s) \sum_{n \neq 0} e^{-\frac{a^2}{4T^2 s}} L_0^{(n)} =$$

$$= \frac{(N_c^2 - 1)}{16 \pi^2} \int_0^\infty \frac{ds}{s^3} \sum_{n \neq 0} e^{-\frac{a^2}{4T^2 s}} e^{-J_n^E} \tag{62}$$

where $J_n^E$ is given in (47), (48), (53).

For quarks of $n_f$ flavours from (29), (45),

$$P_{qSLA} = \frac{2N_c}{\sqrt{\pi}} n_f \int_0^\infty \frac{ds}{s^{3/2}} e^{-m_q^2 s} S_3^{(0)}(s) \sum_{n=1}^\infty (-1)^{n+1} e^{-\frac{a^2}{4T^2 s}} L_n^{(n)} e^{-J_n^E} \cosh \frac{\mu n}{T} =$$

$$= n_f \frac{N_c}{4 \pi^2} \int_0^\infty \frac{ds}{s^3} e^{-m_q^2 s} \sum_{n=1}^\infty (-1)^{n+1} e^{-\frac{a^2}{4T^2 s}} e^{-J_n^E} \cosh \frac{\mu n}{T}, \tag{63}$$

The Polyakov loop exponents, $J_n^E$, are expressed through the function $D_1^E$ in (37) and can be either computed analytically, as in [23, 18], or taken from the lattice data (see e.g. [31]).

As a first check we consider the limit of free gluons and quarks, i.e. $J_n^E \to 1$. 

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Integrating over $ds$ in (62), (63) one obtains for the free gas with $m_q = 0$, $\mu = 0$

$$P_{SB}^{gl}(\sigma \rightarrow 0) = \frac{\pi^2 N_c T^4}{45}, \quad P_{SB}^{q}(\sigma \rightarrow 0) = \frac{7\pi^2 N_c T^4}{180} n_f,$$

(64)

where we have used the standard sum values

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} = \frac{7\pi^4}{720}.$$  \hspace{1cm} (65)

Another simple approximation consists of neglecting all terms in the sum over $n$, except the term with $n = 1$. The accuracy of this approximation can be estimated from the sum (65), where all terms with $n > 1$ contribute less than 8%, and for nonzero $J_E$ (for $T < 1.5T_c$, $L_{fund}$ is less than 0.7 for $n_f = 2$ [32]) higher terms in $n$ are suppressed even more.

Therefore with accuracy better than 10% (for $\frac{\mu}{T} < 1$) one can suggest the following expressions for $P_{SLA}^{gl}$, $P_{SLA}^{q}$

$$P_{SLA}^{gl} = \frac{2(N_c^2 - 1)}{\pi^2} L_{adj} T^4$$

(66)

$$P_{SLA}^{q} = \frac{4N_c n_f}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} (L_{fund})^n \varphi_n^{(n)}(T) \cosh \frac{\mu n}{T} \approx \frac{4N_c}{\pi^2} n_f L_{fund} T^4 \varphi_1^{(1)}(T) \cosh \frac{\mu}{T},$$

(67)

where we have defined

$$\varphi_n^{(n)}(T) = \frac{n^4}{16 T^4} \int_0^{\infty} ds e^{-m_q^2 s} e^{-\frac{s^2}{4T^2}} = \frac{n^2 m_q^2}{2T^2} K_2 \left( \frac{m_q n}{T} \right).$$

(68)

A representation similar to (66) was suggested a decade ago in [12], for gluons where instead of $\frac{\pi^2}{45}$, as in (66), the Stefan-Boltzmann factor $\frac{\pi^2}{45}$ was used. With the lattice data for $L_{adj}$ known at that time, the resulting agreement of $P_{SLA}^{gl}$ was quite satisfactory.

In the next paper [33] we exploit new accurate data for the renormalized Polyakov loops, fundamental with $n_f = 2$ [31] and $n_f = 0$ [29] and recent data for the adjoint loop [24], showing a reasonable agreement with (66), (67).

Summing up the series (67) for $m_q = 0$ one easily obtains another form for $P_{SLA}^{q}$.
\[ P_{q}^{SLA}(m_q = 0) = \frac{n_f T^4}{\pi^2} \left\{ \int_{0}^{\infty} \frac{z^3 dz}{e^{z-(\frac{\mu}{T}+J^k_T)} + 1} + \int_{0}^{\infty} \frac{z^3 dz}{e^{z+\frac{\mu}{T}+J^k_T} + 1} \right\} \]  

and for \( m_q \neq 0 \) (see [33] for details) one has instead
\[ P_{q}^{SLA}(m_q) = \frac{n_f T^4}{\pi^2} \int_{0}^{\infty} \frac{z^4 dz}{\sqrt{z^2 + \nu^2}} \left\{ \frac{1}{e^{\sqrt{z^2 + \nu^2}-(\frac{\mu}{T}+J^k_T)} + 1} + \frac{1}{e^{\sqrt{z^2 + \nu^2}+\frac{\mu}{T}+J^k_T} + 1} \right\} \]  

where \( \nu = \frac{m_q}{T} \).

Another useful quantity to compare with lattice data is the internal energy density \( \varepsilon \),
\[ \varepsilon = T^2 \frac{\partial}{\partial T} \left( \frac{P}{T} \right)_V = \varepsilon_{gl} + \varepsilon_q \]

and from (66), (67) we obtain (for \( \mu = 0 \))
\[ \varepsilon_{SLA}^{gl} = \frac{2}{\pi^2} T^2 (N_c^2 - 1) \frac{d}{dT} (T^3 L_{adj}) \]
\[ \varepsilon_{SLA}^{q} = \frac{4N_c}{\pi^2} n_f T^2 \frac{d}{dT} (T^3 L_{fund} \varphi_q^{(1)}(T)) \]

and finally for the “nonideality” of the quark-gluon plasma one obtains
\[ \frac{\varepsilon_{SLA}^{q} - 3P_{SLA}^{q}}{T^4} = 2(N_c^2 - 1) T \frac{d}{dT} (L_{adj}) + \]  
\[ + \frac{4N_cT}{\pi^2} T \frac{d}{dT} (L_{fund} \varphi_q^{(1)}(T)). \]  

It is interesting to study the limit of large \( T \) for \( \frac{P_{SLA}}{T^4} \) given in Eqs. (69), (70), and compare it to the Stefan-Boltzmann value \( \frac{P_{SB}}{T^4}(T = \infty) \). For that purpose one needs the large \( T \) limit of \( L_{i}, \varphi_q, i = gl, q \).

The situation is more transparent with the large limit of \( \varphi_q^{(1)}(T) \).

From (47) one can conclude that \( \varphi_q^{(1)}(T) \) tends asymptotically to 1 with the (small) correction proportional to \( \frac{\tilde{m}_q^2}{T^2} \). As to \( L_i \), from (47), (48), (49) it is clear that the problem with \( L_i \) boils down to the large \( T \) behaviour of \( V_1(\infty, T)/T \). In this paper we take \( L_i \) as given by lattice [27, 29, 31] and analytic calculations, [15, 18, 24] from which \( L_i \) grows with \( T \) and reaches the value of 1 around \( T = 2T_c \), and may become larger than 1 beyond.
This is connected to the negative value of the lattice free energy of static quark in this range of temperature $T_{c}$. The actual limit of $L_i$ at infinite $T$ is connected to the fact, how the magnetic confinement, viz. $\sigma_s(T)$ enters in $L_i$, and this will be discussed in one of the next papers. In this way one expects that all corrections to $L_i - 1$ should be proportional to some power of $\sqrt{\sigma_s(T)} \sim O\left(\frac{1}{\ln T/\Lambda_{\sigma}}\right)$, since $\sigma_s(T)$ is the only nonperturbative dimensionful quantity which defines the dynamics of high temperature QCD – colormagnetic interaction [17, 19].

6 Beyond Single Line Approximation

In previous sections one-particle contributions were considered, i.e. SLA was used, however it was stressed, that some interactions like the colormagnetic one, acts in white ensembles like $q\bar{q}, gg$ etc. These latter contributions generated by the source terms listed in Appendix 1 were neglected except for contributions of $D^E_i$, which acts in $q\bar{q}(gg)$ but produced effectively single-line term $V_1(\infty)$. In addition $D^E_i$ as well as $D^H_i$ produce $q\bar{q}, gg, 3q$ etc bound systems [23, 18, 34] which may affect strongly dynamics at $T \geq T_c$, as will be discussed below.

In general one can write the contribution of the pair (triple) interaction, starting from the pair (triple) Green’s function, e.g. for the $q\bar{q}$ or $gg$ system as

$$G_{12}^{(n)} = \int d\Gamma_1 d\Gamma_2 \langle tr W_{\Sigma,\sigma}(C_1^{(n)}, C_2^{(n)}) \rangle$$

with $d\Gamma_i = \int_0^\infty d\tau_i(D\tau_i)^s_{x_1, y_1} e^{-K_i}$.

Here the closed Wilson loop is formed in the same way, as discussed at the beginning of section 3. From the $(q\bar{q})$ or $(gg)$ Green’s function one can proceed to the derivation of the two-body Hamiltonian $H_{q\bar{q}}$ or $H_{gg}$ in the same way, as it was done for $T = 0$ in [35],

$$G_{12}(x_1 x_2 | y_1 y_2) = \langle x_1 x_2 | e^{-H_{12}/T} | y_1 y_2 \rangle$$

Our purpose in this section is to discuss the properties of the resulting $H_{q\bar{q}}, H_{gg}$ for $T > T_c$, and possible consequences for the quark-gluon thermodynamics. Writing $H_i = K_i + V_i$, $i = (q\bar{q}), (gg), (3q), ...$, one can consider two distinct dynamics, the Colorelectric (CE) one due $D^E_i$, and colormagnetic (CM) due to $D^H_i, D^H_1$. 

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In the CE case the interaction can be written as (see [23, 18] for more discussion)

\[ V_{q\bar{q}}^{(CE)}(r, T) \equiv v_1(r, T) \equiv V_1(r, T) - V_1(\infty, T) \]  

(77)

and [23]

\[ V_1(r, T) = \int_0^{1/T} (1 - \nu T) d\nu \int_0^r \xi d\xi D_1^E(\sqrt{\nu^2 + \xi^2}) \]  

(78)

In (77) we have subtracted \( V_1(\infty, T) \), which is already accounted for in SLA in the form of Polyakov loops, see Eq. (55).

Bound states of \( q\bar{q}, gg, \) and \( gq \) systems have been studied in the field correlator formalism in [23, 18] and on the lattice (see [29] and refs. in [23, 18]).

Characteristic feature of \( v_1(r, T) \) is that it is short-ranged (\( r_{\text{eff}} \sim 0.3 \text{ fm} \)) and can support bound \( S \)-states in \((c\bar{c})_1, (gg)_1 (cg)_3, (gg)_8 \) of weak binding, \(|\varepsilon| \lesssim 0.14 \text{ GeV} \) for \( T \lesssim 1.5T_c \).

On the lattice in addition to \( S \)-states of \((c\bar{c})_1, (b\bar{b})_1 \) and light \((q\bar{q}) \) also lowest \( P \) state is claimed to exist (see [36] for a review).

It is clear that weakly bound states should fast dissociate in the dense \( qg \) plasma and therefore hardly produce any significant effect on the production rate, however can be important for the kinetic coefficients.

We now turn to the CM case. Here dynamics of \( q\bar{q} \) system was carefully studied in [34] using the formalism of [35]. The resulting interaction has several interesting features.

1. Interaction of quark and gluon systems with the CM vacuum occurs only in the states with the orbital momentum \( L > 0 \).

2. Centrifugal barrier is effectively killed for \( r^2 \gg \frac{L(L+1)}{\sigma} \) and no spin-independent interaction is found in this region.

3. Long-range spin-orbit forces are predominant at large \( r \),

\[ V_{so}(r) \approx -\mathbf{LS}\sigma_s/\mu_{eff}^2 r, \]  

where \( \mu_{eff}^2 \) is the effective mass squared to be determined from the extremum of the relativistic Hamiltonian (we disregard in this section the large \( r \) damping of \( V_{so}(r) \) due to string inertia in \( \mu \)).

The latter can be written as (the \( q\bar{q} \) system with equal masses is treated here for simplicity)

\[ H_{II} = \frac{p_r^2 + m^2}{\mu} + \mu + V_{SI}^{(II)}(r) + V_{so}(r) \]  

(79)
where the sum of potentials can be effectively rewritten as (see [34] for exact expressions) for $J = L + S$

$$V_{CM}(r) = \frac{L(L+1)}{(\mu + \sigma_r r)r^2} - \frac{\sigma_s L}{2\mu(\mu + \sigma r)r}.$$  (80)

Note that $V_{CM}(r) \equiv 0$ for $L = 0$, as was noted before. For large $r$ the first term decreases as $O\left(\frac{1}{r^4}\right)$.

Numerical analysis shows that the lowest bound states appear in the situation when $r \lesssim \mu/\sigma$, when $V_{CM}(r)$ can be rewritten as

$$V_{CM}(r) \approx \frac{L(L+1)}{\mu r^2} - \frac{\sigma_s L}{2\mu^2 r} = \frac{L(L+1)}{\mu r^2} - \frac{\alpha_{eff}}{r}$$  (81)

with $\alpha_{eff} = \frac{\sigma_s L}{2\mu^2 r}$.

Note that $\mu$ is to be found from the minimum of the Hamiltonian eigenvalues, as it is always done in the einbein Hamiltonian formalism [35]. Moreover $V_{CM}(r)$ and especially $V_{so}(r)$ is not treated as a correction: in contrast to the standard $\frac{1}{M}$ expansion, the spin-dependent interaction in the Field-Correlator Method is obtained in the framework of the Gaussian (quadratic) approximation, which is proved to be accurate, see [30], [24] for more details, and [37] for the derivation of spin-dependent interaction.

For the fixed $\mu$, one obtains the bound state mass from $H_{ll}$ as

$$M_{n_r,L}(\mu) = \frac{m^2}{\mu} + \mu - \frac{\mu \alpha_{eff}}{2(n_r + L + 1)^2}.$$  (82)

The physical mass of CM bound ($q\bar{q}$) is obtained after an extremum in $\mu$ is found for some $\mu = \mu_0$. As was found in [34], this extremum exists for $m \gtrsim 0.25$ GeV, i.e. for $(b\bar{b}), (c\bar{c})$ and possibly $(s\bar{s})$ systems with the binding energies 0.007, 0.19 and 90 MeV for $\sigma_s = 0.2$ GeV$^2$.

There is no extremum for $m < 0.2$ GeV, which means that relativistic system of light $q\bar{q}$ is not stable (no lower bound for the mass). The situation is the same as for the famous $Z > 137$ problem in QED, where relativistic treatment of bound states for $\alpha_{eff} > 1$ leads to inconsistencies (see [38] for reviews). Physically the problem should be treated taking into account pair creation, i.e. in the one-body (heavy-light) Dirac formalism, as in [38], considering also lower continuum of negative energy states.

In the framework of the relativistic (einbein) Hamiltonians [35] this be studied using the matrix Hamiltonian, suggested in [39].
Thus light quarks in the \((q\bar{q})\) system or together with heavy antiquarks are unstable in the CM vacuum with respect to multiple light pair creation. A similar, and even more pronounced situation should occur for gluonic systems, like \((gg)\) or a gluon circling around a heavy quark. Here Hamiltonian is the same as in \((79)\) with the replacement \(\sigma_s \rightarrow \frac{9}{4}\sigma_s\) and quark spin by the gluon spin, so that

\[
\alpha_{e_{\text{eff}}}(\text{gluon}, \mu) = \frac{9}{2}\alpha_{e_{\text{eff}}}(\text{quark}, \mu).
\]

Again the gluon systems are unstable and should be stabilized by creation of multiple \((gg)\) pairs, which means that the CM vacuum stimulates numerous \((q\bar{q})\) and \((gg)\) pair creation.

Numerical estimates of these processes and spectra of produced quarks and gluons are of immediate interest for the ion-ion collisions and shall be treated elsewhere.

At this point one should consider the effect of quark-gluon medium on the existence of CM bound pairs. It is clear, that the size of the pairs is (from the minimum of \(V_{CM}\) and \(\alpha_{e_{\text{eff}}} \approx 1\)) \(r_{\text{eff}} \sim L^{3/2} / \sqrt{\sigma_s} \sim 0.5\) fm \(L^{3/2}\), and in the medium of density \(n\) with closest neighbor distance \(r_0 \sim n^{-1/3}\), one expects strong screening for \(r_{\text{eff}} > r_0\). Hence for \(T > 1.1T_c\) when the energy density exceeds 2 GeV/fm\(^3\), \(r_0 \sim 0.5\) fm the high-\(L\) levels are already screened. This might explain a fast drop of entropy \(S_\infty(T)\) beyond \(1.05T_c\) found on the lattice \([40]\), as well as large value of \(S_\infty(T)\) near \(T_c\), since large -\(L\), large-\(J\) bound states might bring about a large entropy.

## 7 Summary and conclusions

We have used BPTH and field correlators to find the NP contribution to the quark-gluon EOS. As a result the free energy (pressure) is represented as a double sum in powers of \(\alpha_s^n(T)\) and in the number of interacting single-particle trajectories. In the lowest order in \(\alpha_s^n\) one obtains the purely NP corrections to the free (Stefan-Boltzmann) result, which contains as a leading term the single particle trajectories interacting only with the NP vacuum (background) fields – Single Line Approximation discussed in chapters 1-6.

In the next terms of this series also NP interaction between quark (gluon) trajectories is taken into account in chapter 6.
The most interesting result of the investigation above is that in the leading approximation the influence of the NP background can be represented by two factors multiplying the Stephan-Boltzmann result: one is the Polyakov loop for quark and another for gluon taking into account colorelectric vacuum fields.

Thus the main contribution to the pressure $P(T)$ and all other thermodynamic quantities is given in our approach by the NP impact on the free otherwise trajectories of quarks and gluons. Perturbative interaction is included in the next orders of $\alpha_s^n, n \geq 1$, and yields additional correlation between quark and gluon trajectories (line-line interaction), which together with the NP correlation leads to a possible formation of quark and gluon bound states in the region $T_c \leq T \leq 1.2T_c$ and should be considered as a next step, discussed in chapter 6.

As discussed in section 5 and Appendix 2, the CM interaction produces a rather strong potential with the dominant spin-orbit term $V_{so} \sim -\frac{L\sigma_s}{\mu^2}$, which for quarks with momentum $p \sim \mu$ and $L \sim p$ yields the plasma interaction ratio $\Gamma^q = \frac{|V_{so}|}{E_{kin}} \sim \frac{2}{\mu^2}$ and $\Gamma^g = \frac{3}{2} \Gamma^q$. Both $\Gamma^q$ and $\Gamma^g$ are large and presume a strong interacting plasma in the temperature region, where screening due to high pressure is not operating. At large $T$ the Polyakov and magnetic loop factors tend to unity, and the pressure tends to the Stefan-Boltzmann limit, as it is in the standard perturbation theory. Comparison of our results, Eqs. (62), (63) and (66), (67) to the lattice data is given in detail in the next paper [33], where it will be shown, that a reasonable agreement is achieved already in the situation when line-line interaction is neglected, as in (66), (67) in the region at $T > 1.2T_c$, while in the narrow region $T_c \leq T \leq 1.2T_c$ one possibly needs to take into account line-line corrections. This result is in agreement with an earlier investigation (see the first reference of [12]), where also a reasonable agreement was found for the gluonic part of the spectrum.

At this point it is interesting to compare two distinct pictures of the sQGP. In the first one the main dynamics of sQGP is due to interaction between the quarks and gluons, which should be taken into account to high orders, as in [25], with possible partial resummation of the perturbative series, see e.g. [41]. It is important, that vacuum fields do not enter directly into the picture, however can be taken into account by adjusting to the dimensionally reduced theory at larger $T$.

From this prospective the theoretical method suggested in the present and next papers of this series, is approaching sQGP from the opposite direction.
as compared to the perturbation theory. Indeed, the main contribution is obtained from the vacuum (NP background) and perturbative corrections are treated as small at all temperatures. One important result of the proposed method is the calculation \[21\] of the Debye mass \(m_D\) which is expressed through the spatial string tension \(\sigma_s\): \(m_D = c_D \sqrt{\sigma_s}\) with calculable coefficient \(c_D\). A good agreement with the lattice data demonstrated in \[21\] gives a strong support to the idea, that the NP vacuum is after all the main source of dynamics both in the confined and in the deconfined phase. The main outcome is the picture of almost independent quasiparticles.

This is in agreement with the lattice measurements of the correlation between strangeness and baryon number or electric charge, demonstrating almost zero effect for \(T > T_c\) \[42\].

The method suggested in the present paper can be easily extended to find the phase diagram in the \((\mu, T)\) plane, which is done in the next paper of this series. N.O.Agasian actively participated in the first stage of this work, and the author is indebted to him for suggestions and discussions. The author is grateful for useful discussions to K.G.Boreskov, S.N.Fedorov, A.B.Kaidalov, O.V.Kancheli, B.O.Kerbikov and M.B.Voloshin. This work is supported by the grants RFBR 06-02-17012, grant for scientific schools NSh-843.2006.2 and the State Contract 02.445.11.7424.

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\section*{Appendix 1. Background Perturbation Theory for \(T > 0\)}

We start with standard formulas of the background field formalism \[8, 9\] generalized to the case of nonzero temperature. We assume that the gluonic field \(A_\mu\) can be split into the background field \(B_\mu\) and the quantum field \(a_\mu\)

\[A_\mu = B_\mu + a_\mu,\]

(A1.1)

both satisfying periodic boundary conditions

\[B_\mu(z_4, z_i) = B_\mu(z_4 + n\beta, z_i), a_\mu(z_4, z_i) = a_\mu(z_4 + n\beta, z_i),\]

(A1.2)

where \(n\) is an integer and \(\beta = 1/T\). It will be convenient in what follows
to use as in [8] the background Lorenz gauge

\[ D_\mu(B)a_\mu = 0, \quad D_\mu^a(B) \equiv \partial_\mu \delta_{ca} - igT^b_{ca}B^b_\mu, \]  

(A1.3)

and to split the gauge transformation as follows (in fundamental representation)

\[ A_\mu(x) = U^+(x)(A_\mu(x) + \frac{i}{g} \partial_\mu)U(x), \]
\[ B_\mu(x) = U^+(x)(B_\mu(x) + \frac{i}{g} \partial_\mu)U(x) \]
\[ a_\mu(x) = U^+(x)a_\mu(x)U(x) \]

(A1.4)

One can see, that the form of gauge condition (A1.3) is invariant under gauge transformations (A1.4). The partition function can be written as

\[ Z(V, T, \mu) = \left< Z(B) \right>_B, \]

\[ Z(B) = N \int D\phi \exp(-\int_0^{\beta} d\tau \int d^3x L(x, \tau)) \]  

(A1.5)

where \( \phi \) denotes all set of fields \( a_\mu, \Psi, \Psi^+, N \) is a normalization constant, and the sign \( <>_B \) means some averaging over (nonperturbative) background fields \( B_\mu \), the exact form of this averaging is not needed for our purposes. Furthermore, we have

\[ L_{\text{tot}}(x, \tau) = \sum_{i=1}^{8} L_i + L(j^{(a)}, a_\mu, \Psi, \Psi^+), \]  

(A1.6)

where

\[ L_1 = \frac{1}{4}(F^a_{\mu\nu}(B))^2; \quad L_2 = \frac{1}{2}a^a_\mu W^{ab}_\mu a^b_\nu, \]
\[ L_3 = \Theta^a(D^2(B))_{ab}\Theta^b; \quad L_4 = -ig\Theta^a(D_\mu, a_\mu)_{ab}\Theta^b \]
\[ L_5 = \frac{1}{2}\alpha(D_\mu(B)a_\mu)^2; \quad L_6 = L_{\text{int}}(a^3, a^4) \]
\[ L_7 = -a_\mu D_\mu(B)F_{\mu\nu}(B); \quad L_8 = \Psi^+(m + \mu\gamma_4 + \hat{D}(B + a))\Psi \]

Here \( \Theta, \emptyset \) are ghost fields, \( \alpha \) - gauge–fixing constant, \( L_6 \) contains 3–gluon– and 4–gluon vertices, and we keep the most general background field \( B_\mu \), not satisfying classical equations, hence the the appearance of \( L_7 \).
An additional term in (A1.6), \( L(j^{(n)}(...)) \) serves as a generating functional for creation of white multiquark or multigluon systems, which are singled out at low temperatures. One can write

\[
L(j^{(n)}, a_{\mu}, \Psi, \Psi^+) = \int dx_1 dx_2 \{ j^{(qq)}_{\mu\nu}(x, x_1, x_2) tr(a_{\mu}(x_1) \tilde{\Phi}(x_1, x_2) a_{\nu}(x_2)) + j^{(q\bar{q})}_{\lambda}(x, x_1, x_2) tr(\Psi^{+}(x_1) \gamma_{\lambda} \Phi(x_1, x_2) \Psi(x_2)) \} + ... \quad (A1.8)
\]

Here \( \tilde{\Phi}(x_1, x_2) \) is the parallel transporter in adjoint representation (marked by tilde) made of field \( B_{\mu} \) only

\[
\tilde{\Phi}(x_1, x_2) = P \exp(ig \int_{x_1}^{x_2} \tilde{B}_{\mu}(z) dz_{\mu}) \quad (A1.9)
\]

\( \Phi(x_1, x_2) \) is the same but in the fundamental representation. One can see that \( L \) and \( j^{(n)} \) are gauge invariant. Differentiating in \( j^{(n)} \) one generates white states of quark and gluons which enter in the total sum over \( n \)

\[
Z(V, T) = \sum_n \langle n | \exp(-\beta H) | n \rangle \quad (A1.10)
\]

The inverse gluon propagator in the background gauge is

\[
W_{\mu\nu}^{ab} = -D^2(B)_{ab} \cdot \delta_{\mu\nu} - 2g F_{\mu\nu}^{c}(B) f^{acb} \quad (A1.11)
\]

where

\[
(D_{\lambda})_{ca} = \partial_{\lambda} \delta_{ca} - ig T_{ca}^{b} B_{\lambda}^{b} \equiv \partial_{\lambda} \delta_{ca} - g f_{eca} B_{\lambda}^{b} \quad (A1.12)
\]

We consider first the case of pure gluodynamics, \( L_8 \equiv 0 \), and omit the source term, \( L \) in (A1.6).

Integration over ghost and gluon degrees of freedom in (A1.5) yields

\[
Z(B) = N'(\det W(B))^{-1/2} \det(-D_{\mu}(B)D_{\mu}(B + a)) \mid_{a=0} \times \times \{ 1 + \sum_{l=1}^{\infty} \frac{S^l_{int}(a = \delta)}{l!} \} \exp\left( -\frac{1}{2} JW^{-1}J \right) J_{\mu} = D_{\mu}(B)F_{\mu\nu}(B) \quad (A1.13)
\]

**Appendix 2. Calculation of the Color Magnetic Interaction**
Taking into account any interaction between gluons or quarks, and in particular when the colormagnetic interaction is considered, one should recur to gauge invariant initial and final states $\Psi$ in the Matsubara expansion for $P_g, P_q$, and define $\Psi_{gg}, \Psi_{q\bar{q}}$ as follows

$$\Psi_{gg}^{(0)}(x, y) = \text{tr}(\hat{a}_\mu(x)\hat{\Phi}(x, y)\hat{a}_\nu(y)) \quad (A2.1)$$

$$\Psi_{gg}^{(n)}(x', y') = \Psi_{gg}(x + n\beta e_4, y + n\beta e_4). \quad (A2.2)$$

For the double-line term in (93) one obtains

$$\langle Z(B) \rangle_{\text{two line}} = \int d\Gamma d\Gamma' \langle \text{tr} W_{\Gamma\Gamma'} \rangle \quad (A2.3)$$

where $W_{\Gamma\Gamma'}$ is a rectangular Wilson loop, formed by parallel transporters $\hat{\Phi}(x, y), \hat{\Phi}(x', y')$ and gluon paths $(x, x')$ and $(y, y')$.

First of all one can calculate $\langle \text{tr} W_{\Gamma\Gamma'} \rangle$ using Gaussian approximation and cluster expansion [5, 6]. For fundamental charges ($q\bar{q}$ system) one has

$$\langle \text{tr} c_N W_{\Gamma\Gamma'} \rangle = \exp(-g^2/2 \int d\pi_{\mu\nu}(u)d\pi_{\lambda\sigma}(v)\langle F_{\mu\nu}(u)F_{\lambda\sigma}(v) \rangle) \quad (A2.4)$$

where we have omitted $\Phi(u, v), \Phi(v, u)$ in field correlator for brevity and defined

$$d\pi_{\mu\nu}(u) = ds_{\mu\nu}(u) - i\sigma_{\mu\nu}d\tau \quad (A2.5)$$

and $ds_{\mu\nu}(u)$ is the surface element, and $d\tau$ is proper time integration present in $W_{\sigma}$, Eq. (23).

We start with the terms coming from the product $ds_{\mu\nu}(u)d\pi_{\lambda\sigma}(v)$ in (A2.4), while all other terms have been treated in [37] and we shall write the resulting equation at the end of this Appendix.

In what follows we shall derive as in [35] the effective Hamiltonian for the $q\bar{q}(gg)$ system, but taking into account that colorelectric string tension $\sigma_E$ and the colormagnetic one $\sigma_H \equiv \sigma_s$ can be different and one of them ($\sigma_E$) may vanish. In what follows we are closely following the paper [34].

With the notations of (35), (36) one can write

$$\langle \text{tr} W_{\Gamma\Gamma'}^{(n)} \rangle = \exp\left(-\frac{1}{2} \int \int ds_{\mu\nu}(u)d\pi_{\lambda\sigma}(v)D_{\mu\nu,\lambda\sigma}(u, v)\right) =$$

$$= \exp(-J_n(D_1^E) - J_n^{EH} - J_n^H - J_n(D^E)) \quad (A2.6)$$

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where \( J_n(D^E) \equiv 2J_n^E \) and \( J_n^E \) is given in (36), while \( J_n^{EH} \) is

\[
J_n^{EH} \equiv \frac{1}{2} \left( \int D_{i4,k4}(u, v) d\sigma_{i4}(u) d\sigma_{k4}(v) = \right.
\]

\[
= \frac{1}{4} \int \int (\delta_{ik} w_k - \delta_{il} w_l) \frac{\partial D_{i4,k4}^E(w)}{\partial w_4} ds_{i4}(u) ds_{k4}(v)
\]

(A2.7)

with \( w_\mu = u_\mu - v_\mu \). The integral \( du_4 \) in (A2.7) reduces to \( D_{i4,k4}^E(n\beta - v_4) - D_{i4,k4}^E(-v_4) \) and is small for \( T \ll \frac{n}{\lambda_{EH}} = nM_{EH} \), moreover this integral vanishes for any flat surface.

We shall not discuss this term in what follows and now turn to the term \( J_n(D^E) \), which can be written as

\[
J_n(D^E) = \frac{1}{2} \int \int ds_{i4}(u) ds_{k4}(v) D_{i4,k4}(u, v) = \]

\[
= \frac{1}{2} \int \int dv_4 dv_4 dv_4 D^E \left( \sqrt{(u_4 - v_4)^2 + (u_i - v_i)^2} \right)
\]

(A2.8)

Introducing variables \( r, \rho \)

\[
r = x - y, \quad \rho = [r \times \dot{x}], \quad w = \beta x + (1 - \beta)y,
\]

(A2.9)

where we have parametrized the Wilson surface \( W_{1\Gamma} \) with \( 0 \leq t \leq n/T; 0 \leq \beta \leq 1 \), so that the surface elements are

\[
ds_{i4}(u) ds_{i4}(v) = r(t)r(t') dt dt' d\beta d\beta'
\]

(A2.10)

while

\[
ds_{ik}(u) ds_{ik}(v) = 2\rho(t)\rho(t') dt dt' d\beta d\beta'.
\]

(A2.11)

One can also write using induced metrics \( g_{ab} \)

\[
(u - v)^2 = (u(t, \beta) - v(t', \beta'))^2 = g^{ab} \xi_a \xi_b, \quad \xi_1 = t - t', \quad \xi_2 = \beta - \beta'
\]

(A2.12)

and \( \det g = r^2 + \rho^2 \). In this way one finally obtains

\[
J_n(D^E) = \sigma_E \int_0^{n/T} dt \int_0^1 d\beta \frac{r^2}{\sqrt{r^2 + \rho^2}}
\]

(A2.13)

\[
J_n(D^H) = \sigma_H \int_0^{n/T} dt \int_0^1 d\beta \frac{\rho^2}{\sqrt{\rho^2 + r^2}}
\]

(A2.14)
In case when $T = 0$, one has $D^E = D^H$, $\sigma_E = \sigma_H$ and one obtains

$$J_n(D^E) + J_n(D^H) = \sigma \int_0^{n/T} dt \int_0^1 d\beta \sqrt{r^2 + \rho^2}$$  \hspace{1cm} (A2.15)

where the standard Nambu-Goto action used in [35] is $\sqrt{\dot{w}^2(w')^2 - (\dot{w}_\mu w'_\mu)^2} = \sqrt{r^2 + \rho^2}$.

In the action for equal mass quarks, $m_q = q = m$, taking into account that [35]

$$K_q + K_{\bar{q}} = \int_0^{T_0} \mu (x^2 + y^2) dt + \int_0^{T_0} \mu dt \hspace{1cm} (A2.16)$$

one finally obtains the Hamiltonian for $T = 0$ [35]

$$H = \frac{p_r^2 + m^2}{\mu} + \mu + \frac{L^2/r^2}{2(\frac{\mu}{2} + \int (\beta - \frac{1}{2})^2 \nu d\beta)} + \int \frac{\sigma_1^2 \nu^2}{2\nu} + \nu + \sigma_2 r \hspace{1cm} (A2.17)$$

where the einbein parameter $\nu$ is to be found from the extremum condition.

For the case of electric deconfinement, when $\sigma_E = 0$ and $\sigma_H \neq 0$, one obtains instead [34]

$$H = \frac{p_r^2 + m^2}{\mu} + \mu + \int \frac{L^2/r^2}{2(\frac{\mu}{2} + \int (\beta - \frac{1}{2})^2 \nu d\beta)} + \int \sigma_1^2 \nu^2 + \nu + \sigma_2 r \hspace{1cm} \equiv H_{kin} + h_{HL} \hspace{1cm} (A2.18)$$

and in the case when one mass is much larger than $\sqrt{\sigma_H}, \sqrt{\sigma_E}$ one obtains for the heavy-light system with both $\sigma_H, \sigma_E$ nonzero [34],

$$H_{HL} = \frac{p_r^2 + m^2}{2\mu} + \mu + \int d\beta \left( \frac{\sigma_1^2 \nu^2}{2\nu} + \nu + \sigma_2 r \right) + \frac{L(L+1)}{2r^2 \left[ \frac{\mu}{2} + \int d\beta \left( \beta - \frac{1}{2} \right)^2 \nu \right]} \equiv H_{kin} + h_{HL} \hspace{1cm} (A2.19)$$

where we have defined $H_{kin} \equiv \frac{p_r^2 + m^2}{2\mu} + \frac{\mu}{2}$, and

$$\sigma_1 = \sigma_H + \eta^2 (\sigma_H - \sigma_E), \quad \sigma_2 = 2\eta (\sigma_E - \sigma_H).$$  \hspace{1cm} (A2.20)

Here $\eta$ is another einbein parameter (cf [35]). Consider first the case when $L = 0$. Taking extremum of (A2.18), (A2.19) with respect to $\nu$ one has

$$H(L = 0) = H_{kin} + r \int_0^1 d\beta \left[ \sigma_E + (\sigma_H - \sigma_E)(1 - \eta)^2 \right] \hspace{1cm} (A2.21)$$

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and taking extremum in $\eta$, which yields $\eta_0 = 1$ one has for both Hamiltonians (A2.18), (A2.19)

$$H(L = 0) = H_{\text{kin}} + \sigma Er.$$  \hfill (A2.22)

Hence for $\sigma_E = 0, \sigma_H \neq 0$ and $L = 0$ magnetic “confining” vacuum does not actually confine quarks (or gluons). Hence quotation marks for magnetic “confinement”.

Let us now turn to the case of $L \neq 0, \sigma_E = 0$ and consider large $L$ limit, when one expects as in \cite{35} that $\nu_0 \gg \mu_0$, where the subscript refers to the extremum values of $\nu, \mu$. In this case taking extremum in $\nu$ in (A2.19) and neglecting $\mu$ in the last term on the r.h.s. of (A2.19) one obtains

$$\nu_0 = \left[\frac{\sigma_H^2 (1 + \eta^2)^2 r^2 + 3L(L+1)}{r^2}\right]^{1/2} \hfill (A2.23)$$

and for the $\eta_0$ - extremum value of $\eta$, one has an equation

$$\left(\eta_0^4 - 1\right)(\eta_0^2 + 1) = \frac{3L(L+1)}{\sigma_H^4 r^4} \equiv c \hfill (A2.24)$$

with the solution

$$\eta_0^2 = -\frac{1}{3} + \left\{\frac{1}{2} \left(\frac{16}{27} + c\right) + \sqrt{\frac{c(c+32)}{54}}\right\}^{1/3} +$$

$$+ \left\{\frac{1}{2} \left(\frac{16}{27} + c\right) - \sqrt{\frac{c(c+32)}{54}}\right\}^{1/3} \hfill (A2.25)$$

The resulting Hamiltonian is

$$h_{HL} = \sqrt{\frac{\sigma_H^2 (1 + \eta_0^2)^2 r^2 + 3L(L+1)}{r^2}} - 2\eta_0 \sigma_H r \hfill (A2.26)$$

where $\eta_0$ is defined in (A2.25). It is easy to check that for $L = c = 0$ one obtains $\eta_0 = 1$ and hence $h_{HL}(l = 0) = 0$, in agreement with (A2.22) for $\sigma_E = 0$.

For small $c$ (A2.26) and (A2.25) yield

$$h_{HL} = \frac{3L(L+1)}{4\sigma_H^3 r^4}, \quad 3L(L+1) \ll \sigma_H^2 r^4. \hfill (A2.27)$$
In the opposite limit, $c \gg 1(3L(L+1) \gg \sigma_H^{-2})$ one should take into account also $\mu$ in the denominator of $h_{HL}$ in (A2.19) and one obtains the standard centrifugal term

$$h_{HL} = \frac{L(L+1)}{2\mu r^2}. \quad (A2.28)$$

One can see, that the centrifugal barrier of light particle is essentially destroyed by the magnetic vacuum for $r > r_0$, $r_0 = \left(\frac{3L(L+1)}{\sigma_H^2}\right)^{1/4}$ while for $r < r_0$ the barrier is essential.

The case of equal mass particles (light-light or heavy-heavy) is done in the same way as above, the only difference being that instead of $3L(L+1)$ one has $12L(L+1)$ which should be substituted in all Eqs. (A2.23-A2.28). Thus the conclusion for the spin-independent part of the Hamiltonian, Eqs. (A2.18), (A2.19), is that in magnetic vacuum the centrifugal barrier is effectively destroyed for $r > r_0$, which fact stresses the importance of large distance spin-dependent interaction – to be discussed now.

The SDI is obtained from (A2.4), when one considers products of $ds_{ik} \cdot \sigma_{i\lambda} d\tau$ (for spin-orbit forces) and $\sigma_{i\mu} d\tau \sigma_{i\lambda} d\tau'$ (for spin-spin and tensor forces), (see [37] and [43] for details of derivation).

The resulting SDI can be represented in the Eichten-Feinberg form [44]:

$$V_{SD}^{(diag)}(R) = \left(\frac{\bar{\sigma}_1 \bar{L}_1}{4\mu_1^2} - \frac{\bar{\sigma}_2 \bar{L}_2}{4\mu_2^2}\right)\left(\frac{1}{R} \frac{d \varepsilon}{dR} + \frac{2dV_1(R)}{R dR}\right) + \frac{\bar{\sigma}_2 \bar{L}_1 - \bar{\sigma}_1 \bar{L}_2}{2\mu_1\mu_2} \frac{1}{R dR} \frac{dV_2}{R dR} + \frac{\bar{\sigma}_1 \bar{\sigma}_2 V_4(R)}{12\mu_1\mu_2} + \frac{(3\bar{\sigma}_1 \bar{R} \bar{\sigma}_2 \bar{R} - \bar{\sigma}_1 \bar{\sigma}_2 R^2)V_3}{12\mu_1\mu_2 R^2}. \quad (A2.29)$$

where SDI potentials $V_i$ are expressed via field correlators as in [37], see also definition of correlators in (35).

$$\frac{1}{R} \frac{dV_1}{dR} = -\int_{-\infty}^{\infty} d\nu \int_{0}^{R} d\lambda \frac{1}{R} \left(1 - \frac{\lambda}{R}\right) D^H(\lambda, \nu) \quad (A2.30)$$

\[\text{\textsuperscript{2}}\text{Here one should take into account that each factor } ds_{ik}\text{ contains } \rho \text{ and hence } \bar{w} \text{ and each } \bar{z}_i \text{ for particle } i \text{ yields as in (A2.19) the factor } \bar{\mu}_i = \mu_i + \int_{0}^{1} \nu(\beta) \beta^2 d\beta \text{ in the denominator instead of } \mu_i, \text{ while each } \sigma_i \text{ brings about a factor } \frac{1}{\bar{\mu}_i}. \text{ This additional } \nu(\beta) \text{ term in } \bar{\mu}_i \text{ was never taken into account before, and is inserted in [34] and below in (A2.40). (A2.41).} \]
\[
\frac{1}{R} \frac{dV_2}{dR} = \int_{-\infty}^{\infty} d\nu \int_{0}^{R} \frac{\lambda d\lambda}{R^2} \left[ D^H(\lambda, \nu) + D^H_1(\lambda, \nu) + \lambda^2 \frac{\partial D^H_1}{\partial \lambda^2} \right] \quad (A2.31)
\]

\[
V_3 = -\int_{-\infty}^{\infty} d\nu R^2 \frac{\partial D^H_1(R, \nu)}{\partial R^2} \quad (A2.32)
\]

\[
V_4 = \int_{-\infty}^{\infty} d\nu \left( 3D^H(R, \nu) + 3D^H_1(R, \nu) + 2R^2 \frac{\partial D^H_1}{\partial R^2} \right) \quad (A2.33)
\]

In the deconfinement phase \( D^E \) vanishes, while \( D^H, D^E_1, D^H_1 \) are nonzero, and the main part of \( D^E, D^H_1 \) is perturbative, yielding the known from of SDI essential at small distances. One can assume (in agreement with lattice data \([13, 14]\)) that \( D^E_1, D^H_1 \) do not change with temperature, which yields \( \left( D^E_1^{\text{pert}} = D^H_1^{\text{pert}} = \frac{16\alpha_s}{3\pi^2} \right) \)

\[
\frac{1}{R} V_{2\text{pert}}(R) = \frac{4\alpha_s}{3R^3}, \quad \frac{1}{R} \varepsilon'_{\text{pert}}(R) = + \frac{4\alpha_s}{3R^3}, \quad V_{3\text{pert}}(R) = \frac{4\alpha_s}{R^3} \quad (A2.34)
\]

Since NP colormagnetic fields at \( T > T_c \) act only on states with nonzero \( L \), one can neglect \( V_{4np} \). Writing \( D^{E,H}_1 = D^{E,H}_{1\text{pert}} + D^{E,H}_{1\text{np}} \), one has from \( (A2.34) \) (for \( T > T_c, \beta \equiv 1/T \))

\[
\frac{1}{R} \varepsilon'_{np} = \int_{0}^{\beta} (1 - \frac{y}{\beta}) dy D^E_{1\text{np}}(\sqrt{y^2 + R^2}),
\]

\[
\frac{1}{R} V'_{2\text{np}} = \int_{0}^{\beta} (1 - \frac{y}{\beta}) dy D^H_{1\text{np}}(\sqrt{y^2 + R^2}) + 2 \int_{0}^{\beta} d\nu (1 - \frac{\nu}{\beta}) \int_{0}^{R} \frac{\lambda d\lambda}{R^2} D^H(\lambda, \nu) \quad (A2.36)
\]

Finally for \( V_1' = V'_{1\text{np}} \) one has from \([37]\)

\[
\frac{1}{R} V_1' = -2 \int_{0}^{\beta} (1 - \frac{\nu}{\beta}) d\nu \int_{0}^{R} \frac{d\lambda}{R} \left( 1 - \frac{\lambda}{R} \right) D^H(\sqrt{\lambda^2 + \nu^2}). \quad (A2.37)
\]

We have corrected in \( (A2.35),(A2.37) \) equations from \([37]\), taking into account the finiteness of the Euclidean time integration in the temporal Green's
functions at $T > 0$. As one can see, the presence of upper limit becomes essential only when $\frac{1}{T} \lesssim t_g$, where $t_g$ is the correlation length of the vacuum, $D^H(x) \sim e^{-|x|/t_g}$, for $|x| \gg t_g$, and for $T = 0$ one has $t_g \approx 1$ GeV$^{-1}$. Thus one expects, that for $T < 1$ GeV$\approx 6 T_c$ the upper limit $\beta$ is inessential and one can use the same integrals as in [37], i.e. replacing $\beta$ by $\infty$.

At large $R$ one has from (A2.36), (A2.37)

$$\frac{1}{R} V'(R) = -\frac{\sigma_s}{R}, \quad \frac{1}{R} V'_{2np}(R) = \frac{\gamma_H}{R^2}, \quad R \to \infty \quad \text{(A2.38)}$$

where $\gamma_H$ is

$$\gamma_H = \int_{-\infty}^{\infty} d\nu \int_{0}^{\infty} \lambda d\lambda D^H(\sqrt{\lambda^2 + \nu^2}) \approx O(t_g \cdot \sigma_s). \quad \text{(A2.39)}$$

Hence the main term at large $R$ in $V_{SD}(A3.29)$ is

$$V_{SD}(R \to \infty) \approx -\left(\frac{\sigma_1 L_1}{2\mu_1 \hat{\mu}_1} - \frac{\sigma_2 L_2}{2\mu_2 \hat{\mu}_2}\right) \frac{\sigma_s}{R}. \quad \text{(A2.40)}$$

For equal mass particles $\mu_1 = \mu_2 = \mu$, $L_2 = -L_1 = -L$, $S = \frac{1}{2}(\sigma_1 + \sigma_2)$ and one obtains

$$V_{SD}(r \to \infty) \approx -\frac{\sigma_s S L}{2\mu \hat{\mu} R} \left(1 + O\left(\frac{t_g}{R}\right)\right) + O\left(\frac{\alpha_s}{\mu^2 R^3}\right), \quad \text{(A2.41)}$$

where $\hat{\mu} = \mu + 2 \int_0^1 d\beta (\beta - \frac{1}{2})^2 \nu(\beta) \approx \mu + \frac{\sigma_R}{6}$.

It is clear that interaction (A2.41) can produce infinite number of bound states for $S L > 0$, i.e. for $S = 1$ and $J = L + 1$, independently of the dynamical mass $\mu$. In this way (A2.41) embodies the suggestion made in [19].

One can also estimate the interaction (A2.41) for the free quark with the momentum $p$, when $\mu \approx p$ and $L = [r \times p] = [\rho \times p]$, where $\rho$ is the impact parameter to the neighboring antiquark.

One has $|V_{SD}| \approx \frac{\sigma_s \rho}{p^2 R} \sim \frac{\sigma_s}{p}$, and the standard plasma interaction ratio $\Gamma$ for average momentum $\langle p \rangle \sim T$

$$\Gamma_g = \frac{|V_{SD}|}{T_{kin}} \approx \frac{\sigma_s}{p^2} \approx \frac{\sigma_s}{T^2}; \quad \Gamma_g = 2 \cdot \frac{9}{4} \frac{\sigma_s}{T^2} \quad \text{(A2.42)}$$

where in $\Gamma_g$ the first factor (2) is due to gluon spin which is twice the quark spin, and the second is Casimir scaling factor.
One can see that for $T_c \leq T \leq 2T_c$ this ratio is larger than unity, implying strong interaction due to magnetic vacuum, especially among gluons. Therefore gluons (and quarks) cannot be considered as a gas of weakly interacting objects, e.g. for $T \approx T_c \approx 0.2$ GeV one has $\Gamma_q \approx 5$, $\Gamma_g \approx 20$ and this is closer to a liquid rather than a gas.

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