Second-order effective energy-momentum tensor of gravitational scalar perturbations with perfect fluid

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ABSTRACT: We investigate the second-order gravitational scalar perturbations for a barotropic fluid. We derive the effective energy-momentum tensor described by the quadratic terms of the gravitational and the matter perturbations. We show that the second-order effective energy-momentum tensor is gauge dependent. We impose three gauge conditions (longitudinal, spatially-flat, and comoving gauges) for dust and radiation. The resulting energy-momentum tensor is described only by a gauge invariant variable, but the functional form depends on the gauge choice. In the matter-dominated epoch with dust-like fluid background, the second-order effective energy density and pressure of the perturbations evolve as $1/a^2$ in all three gauge choices, like the curvature density of the Universe, but they do not provide the correct equation of state. The value of this parameter depends also on the gauge choice. In the radiation-dominated epoch, the perturbations in the short-wave limit behave in the same way as the radiation-like fluid in the longitudinal and the spatially-flat gauges. However, they behave in a different way in the comoving gauge. As a whole, we conclude that the second-order effective energy-momentum tensor of the scalar perturbation is strictly gauge dependent.

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1 Introduction

The theory of cosmological perturbations is a great success in explaining the structure formation in the Universe. The density perturbation produced during inflation evolves in the subsequent Friedman universe, and produces the inhomogeneities in the matter distribution which can be observed in the cosmic microwave background and in the distribution of galaxies. The observation techniques have developed very rapidly, which allows us to investigate cosmology with a very high precision. Thus, now we are able to enter the regime of non-linearity with precise cosmological observations. This will reveal the new physics relevant for structure formation only accessible by non-linear cosmological perturbations.

The second-order cosmological perturbation with the scalar field in the inflation period was investigated in [1, 2]. The authors investigated the second-order effective energy-momentum tensor (2EMT) of cosmological perturbations and its gauge invariance. Regarding the back-reaction effect of 2EMT in the Universe, for the long-wavelength perturbations, they found that the effective equation of state is given by $p_s \approx -\rho_s$ with $\rho_s < 0$ for scalar perturbations, and $p_{gw} \approx -\rho_{gw}/3$ for tensor perturbations. More works have investigated the gauge invariance of the second-order perturbations in [3–7], for example. However, the gauge invariance of 2EMT is questionable [8, 9] and thus its physical relevance, under any approximation, is doubtful.

In this work, we will study the second-order cosmological scalar perturbations with fluid matter. From the Einstein’s equation, we obtain 2EMT constructed by the quadratic combinations of scalar perturbations, and investigate its gauge invariance. Using the first-order equations, the matter perturbations can be expressed by the gravitational ones, in terms of which we can write 2EMT completely. The results show that 2EMT cannot be written only in terms of the gauge invariant variables such as the so-called Bardeen variable [10], but there remains gauge dependence. That is, for the cosmological scalar perturbations with fluid, the 2EMT is not gauge invariant.

More specifically, we investigate 2EMT in the Friedmann universe described by a barotropic fluid-like radiation and dust. Once we impose a specific gauge condition, e.g. longitudinal, spatially-flat, and comoving gauges, the resulting 2EMT can be expressed by the Bardeen variable only. However, the functional form of 2EMT varies depending on the gauge choice. Further we investigate some limits of this 2EMT depending on the wavelengths to see if the effect of 2EMT converges in all gauge choices. In particular, we investigate the second-order effective equation of state evaluated from 2EMT. The result shows that the values of the equation-of-state parameter are different for different gauge choices. This means that this effect of 2EMT is strictly gauge dependent, so is not physically meaningful in any wavelength limit.

This paper is composed as following. In Section 2, we introduce the gravitational scalar and the matter perturbations, derive 2EMT from the Einstein’s equation, and investigate its gauge invariance. In Section 3, we investigate 2EMT in the Friedmann universe after imposing gauge conditions. In Section 4, we conclude.
2 Einstein’s equation and 2EMT

In this section, we derive the zeroth- and the first-order Einstein’s equations. We also introduce gauge invariant variables, and write the first-order equation in a gauge invariant form. Then we construct 2EMT.

2.1 Einstein Tensor

Let us consider the cosmological perturbations in the Friedmann universe. The most general metric perturbations are written as

\[ ds^2 = a^2(\eta) \left[ - (1 + 2A) d\eta^2 - 2B_i d\eta dx^i + \left( \delta_{ij} + 2C_{ij} \right) dx^i dx^j \right], \]

(2.1)

where the background metric represents a flat Friedmann universe, and the non-linear metric perturbations A, B, and C, are both time- and space-dependent functions and are expanded in all orders, e.g. \( A = A^{(1)} + A^{(2)} + A^{(3)} + \cdots \). With \( H \equiv a'/a = da/d\eta/a \), we can write each component of the Einstein tensor \( G_{\mu\nu} \) up to second order in perturbations as follows [11]:

\[
G_{00} = 3\dot{H}^2 + 2HB_{k,k} + 2H'C_{kk} - \Delta C_{kk} + C_{kl,kl} - (2\dot{H}^2 + \dot{H}^2)B_kB_k + \frac{1}{2}B_{k,k}B_{l,l} \\
- \frac{1}{4}B_{k,l}B_{k,l} - \frac{1}{4}B_{k,l}B_{l,k} + B_kB_{l,l,k} - B_k\Delta B_k - \frac{1}{2}C_{ik}C_{jl} + \frac{1}{2}C_{ik}C_{jl}' \Delta C_{kl}C_{kl}' \\
+ \frac{1}{2}C_{ik}C_{jl}C_{kl,m} - C_{kl,m}C_{km,l} - \frac{1}{2}(C_{kk,l} - 2C_{lk,k})(C_{mm,l} - 2C_{lm,m}) \\
+ 2C_{kl}(C_{mm,k} - \Delta C_{kl} - 2C_{mk,m}) - 4HA_kB_k + 2B_k(C_{mm,k} - C_{km,m}) \\
+ 2HB_k(C_{mm,k} - 2C_{km,m}) - B_{k,l}(C_{kl} + 4HC_{kl}) + B_{k,l}C_{lk} - 2A(\Delta C_{kk} - C_{kl,kl}), \tag{2.2}
\]

\[
G_{0i} = 2HA_i + B_{[i,k]} + (2\dot{H}' + \dot{H}^2)B_i + C_{ik} - C_{ik}' - 4HAA_i - B_k(B_{[i,k]}') + 2HB_{[i,k]} \\
+ B_{i}(B_{k,k} + 2HB_{k,k}) + (C_{mm,k} - 2C_{km,m})C_{ik}' + 2C_{kl}(C_{kl,i} - C_{ik,l}') + C_{ik}'C_{kl,i} \\
- 2H\dot{A}'B_i + A_{i,k}B_k - A_{i,k}B_{l,k} - A_{i,k}B_{[i,k]} - 2(2H' + \dot{H}^2)AB_i \\
- (\Delta A)B_i + 2HB_kC_{ik}' + 2HB_kC_{kk}' - 2B_{[i,k]}C_{kl} + 2B_{[i,k]}(C_{lk,k} - 2C_{kl,l}) \\
- 2B_{[i,k]}C_{i[l,j]} - B_{i}(\Delta C_{kk} - C_{kl,kl}) + A_iC_{kk}' - A_{i,k}C_{ik}', \tag{2.3}
\]

\[
G_{ij} = G_{ij} \delta_{ij} - A_{i,j} + B_{[i,j]} + 2HB_{[i,j]} + C_{ij}' + 2HC_{ij}' - \Delta C_{ij} - 2(2H' + \dot{H}^2)C_{ij} + 2C_{(i,j)k} \\
- C_{kk,i}A_{i} + A_{i}A_{j} + 2AA_{ij} + B_{k,k}B_{i,j} + B_kB_{[i,j]k} - \frac{1}{2}B_{i,k}B_{j,k} - \frac{1}{2}B_{k,i}B_{k,j} \\
- B_{k,k}B_{i,j} - 2C_{kk}'C_{ij} + C_{ik}C_{jl}' - 2C_{ik}'C_{jkl}' - 4HC_{kk}'C_{ij} + 2(\Delta C_{kk} - C_{kl,kl})C_{ij} \\
+ 2C_{kl}(C_{ij,kl} + C_{kl,ij} - 2C_{ij(k,l)}) - (C_{kk,l} - 2C_{lk,k})(C_{ij,l} - 2C_{ij(l,k)}) + C_{ik,l}C_{kl,j} \\
+ 2C_{ik,l}(C_{jkl} - C_{j,l,k}) - A'B_{i,j} - 2A(B_{[i,j]}' + 2HB_{[i,j]}) - 2B_{k}(C_{(i,j)k}' - C_{ij,k}') \\
- (B_{k}' + 2HB_{k})(2C_{k,(i,j)} - C_{(i,j)k} + B_{k}(C_{ij}' - 4HC_{ij}) - 2B_{k,k}'C_{ij}' - B_{[i,k]}C_{ik}' + B_{[i,j]}'C_{kk}) \\
- B_{i,k}'C_{ij} - B_{j,k}'C_{ik}' - 2A(C_{ij}' + 2HC_{ij}') - A'(C_{ij}' - 4HC_{ij}) + 4(2H' + \dot{H}^2)AC_{ij} \\
+ A_{k}(2C_{h(i,j)} - C_{i,j,k}) + (\Delta A)C_{ij}, \tag{2.4}
\]
where $G_D$ multiplied by the Kronecker delta $\delta_{ij}$ in $G_{ij}$ is given by

$$G_D = - (2\mathcal{H}' + \mathcal{H}^2) + 2\mathcal{H}'A + 2(2\mathcal{H}' + \mathcal{H}^2)A + \Delta A - B_k, k - 2\mathcal{H}B_k, k - C_{kk}'' - 2\mathcal{H}C_{kk}'$$

$$+ \Delta C_{kk} - C_{kl,kl} - 8\mathcal{H}AA' - (\nabla A)^2 - 2\Delta A - 4(2\mathcal{H}' + \mathcal{H}^2)A^2 + 2\mathcal{H}B_k B_k'$$

$$+ (2\mathcal{H}' + \mathcal{H}^2)B_k B_k + 3\frac{B_k, B_k, B_k, B_k - \frac{1}{4} B_{k, B_k, B_k, B_k} - \frac{1}{2} B_{k, B_k, B_k, B_k} - B_{k, B_k, B_k, B_k} + B_k \Delta B_k}{2}$$

$$+ 2C_{kl}C_{kl}' + \frac{3}{2} C_{kl} C_{kl}' - \frac{1}{2} C_{kk} C_{ll}' + 4\mathcal{H}C_{kl} C_{kl}' + \frac{1}{2} (C_{kk, l} - 2C_{lk, k})(C_{mm, l} - 2C_{lm, m})$$

$$- \frac{3}{2} C_{lm, k} C_{lm, k} + C_{lm, k} C_{lk, m} - 2C_{kl}(C_{mm, kl} + \Delta C_{kl} - 2C_{mk, ml} + A'B_{k, k} + 2AB_{k, k}'$$

$$+ 2\mathcal{H}A_{k} B_{k} + 4\mathcal{H}A_{k} B_{k} + 2B_{k, l} C_{kl} - B_{k, l}(C_{ll, k} - 2C_{lk, l}) - 2B_{k}(C_{ll, k} - C_{kl, l})$$

$$- B_{k, k} C_{ll} + B_{k, l} C_{ll} + 4\mathcal{H}B_{k, l} C_{kl} - 2\mathcal{H}B_{l}(C_{kk, l} - 2C_{lk, l}) + 2AC_{kk}'' + A' C_{kk}'$$

$$+ 4\mathcal{H}A_{k} C_{kk} - 2A_{k} C_{kl} + A_{k}(C_{ll, k} - 2C_{kl, l}).$$

(2.5)

### 2.2 Matter Energy-Momentum Tensor

Now we consider a perfect fluid matter of which the energy-momentum tensor is given by

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu},$$

(2.6)

where $\rho$ is the energy density, $p$ is the pressure, and $u^\mu$ is the four-velocity. The perturbations of fluid are written as

$$\rho = \rho_0 + \delta \rho \quad \text{and} \quad p = p_0 + \delta p,$$

(2.7)

where the subscript 0 denotes the background unperturbed quantity. The perturbations are, as those in the metric, expanded in all orders, for example, $\delta \rho = \delta \rho^{(1)} + \delta \rho^{(2)} + \delta \rho^{(3)} + \cdots$. If we introduce the peculiar velocity $v^i$ defined by [12]

$$v^i = \frac{dx^i}{dx^0} = \frac{u^i}{u^0},$$

(2.8)

we can find up to second order in perturbations, with the normalization condition $u^\mu u_\mu = -1$,

$$u^0 = \frac{1}{a} \left(1 - A + \frac{3}{2} A^2 - B_k v_k + \frac{1}{2} v_k v_k\right),$$

(2.9)

$$u^i = \frac{1}{a} (1 - A) v^i.$$

(2.10)

Then each component of $T_{\mu\nu}$ is obtained as, up to second order in perturbations,

$$T_{00} = a^2 \left[ \rho_0 + \delta \rho + 2A\rho_0 + 2A\delta \rho + (\rho_0 + p_0)v_k v_k \right],$$

(2.11)

$$T_{0i} = a^2 \left[ \rho_0(B_i - v_i) - p_0 v_i + \delta \rho (B_i - v_i) - \delta \rho v_i - 2(\rho_0 + p_0)C_{ik} v_j \right],$$

(2.12)

$$T_{ij} = a^2 \left[ p_0 \delta_{ij} + 2p_0 C_{ij} + \delta \rho \delta_{ij} + 2C_{ij} \delta p + (\rho_0 + p_0)(B_i - v_i)(B_j - v_j) \right].$$

(2.13)

In this work, we shall consider only a perfect fluid matter.
2.3 Einstein’s Equation

The Einstein’s equation order by order can be constructed by equating the Einstein tensor in Eqs. (2.2)-(2.4) and the corresponding matter energy-momentum tensor in Eqs. (2.11)-(2.13).

2.3.1 Zeroth-order equations

The zeroth-order, background equations are given by the 00 and ij components:

\[ 3H^2 = 8\pi G a^2 \rho_0, \]  
\[ 2H' + H^2 = -8\pi G a^2 p_0. \]  

(2.14)  
(2.15)

Note that combining these two equations gives the conservation equation for the background energy density \( \rho_0 \), which can be also derived from the conservation of energy-momentum tensor:

\[ \rho'_0 + 3H (\rho_0 + p_0) = 0. \]  

(2.16)

2.3.2 First-order equations and gauge invariant variables

Using the background equations, we can cast the first-order Einstein’s equation into the following:

\[ 6H^2 A - 2HB_{k,k} - 2HC'_{kk} + \Delta C_{kk} - C_{kl,kl} = -8\pi G a^2 \delta \rho, \]  
\[ 2HA_i + B_{[i,k]} + 2(H' - H^2)B_i + C'_{ik,k} - C'_{k,k,i} = 2(H' - H^2)v_i, \]  
\[ \delta_{ij} \left[ 2HA' + 2(2H' + H^2)A + \Delta A - B'_{i,k} - 2HB_{k,k} - C''_{kk} - 2HC'_{kk} + \Delta C_{kk} - C_{kl,kl} \right] - A_{,ij} + B'_{(i,j)} + 2HB_{(i,j)} + C''_{ij} + 2HC'_{ij} + \Delta C_{ij} + 2C_{k(i,j)k} - C_{kk,ij} = 8\pi G a^2 \delta \rho \delta_{ij}. \]  

(2.17)  
(2.18)  
(2.19)

The first-order conservation equations read

\[ \delta \rho' + 3H(\delta \rho + \delta p) + (C'_{kk} + v_{k,k})(\rho_0 + p_0) = 0, \]  
\[ \delta p_{,i} - (B_i - v_i)p'_0 + [A_{,i} - (B'_i - v'_i) - H(B_i - v_i)](\rho_0 + p_0) = 0. \]  

(2.20)  
(2.21)

Now let us consider the gauge transformation of cosmological perturbations. As is well known, not all degrees of freedom in cosmological perturbations are physical. This is because a generic coordinate transformation should not modify the physics of perturbations. But in general cosmological perturbations are subject to the change of coordinates: Let a function \( Q \) be any scalar, vector or tensor quantity. Then, under the infinitesimal coordinate transformation \( x^\mu \rightarrow x^\mu + \xi^\mu \), the corresponding gauge transformation for \( Q \) can be written as

\[ Q \rightarrow Q - L_\xi Q, \]  

(2.22)

where \( L_\xi \) denotes the Lie derivative in the direction of \( \xi^\mu \). Then the metric perturbations in (2.1) transform as

\[ A \rightarrow A - \xi^0, \]  
\[ B_i \rightarrow B_i - \xi^0, i + \xi'_i, \]  
\[ C_{ij} \rightarrow C_{ij} - H\xi^0 \delta_{ij} - \xi_{(i,j)}. \]  

(2.23)  
(2.24)  
(2.25)

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If we consider only scalar perturbations such that
\[ A = \alpha, \quad B_i = \beta_i, \quad C_{ij} = -\psi \delta_{ij} + E_{ij}, \]
they transform as follows [13]:
\[ \alpha \rightarrow \alpha - \xi^0'H - H\xi^0, \]
\[ \beta \rightarrow \beta - \xi^0' + \xi', \]
\[ \psi \rightarrow \psi + H\xi^0, \]
\[ E \rightarrow E - \xi, \]
where \( \xi = \xi_i = 0 \). Then, as in Ref. [10], one choice of gauge invariant variables is the following:
\[ \Phi = \alpha - Q' - HQ, \]
\[ \Psi = \psi + HQ, \]
\[ \delta \rho = \delta \rho - \rho'_0 Q, \]
\[ \delta p = \delta p - p'_0 Q, \]
\[ v_i = v_i + E'_i, \]
where
\[ Q = \beta + E'. \]
From the off-diagonal components of Eq. (2.19), we have \( \Phi = \Psi \). This means that the off-diagonal component of \( T_{ij} \) vanishes. Then the first-order equations (2.17)-(2.19) can be written in a gauge invariant form as
\[ 3H\Psi' + 3H^2\Psi - \Delta\Psi = -4\pi Ga^2\delta \rho, \]
\[ (\Psi' + HQ)_i = (H' - H^2) v_i, \]
\[ \Psi'' + 3H\Psi' + (2H' + H^2)\Psi = 4\pi Ga^2\delta p. \]

2.3.3 Second-order effective energy-momentum tensor

The second-order effective energy-momentum tensor (2EMT) is constructed in the following way [1, 2, 9]. Let us consider the Einstein’s equation at second-order in perturbations:
\[ G^{(2)}_{\mu\nu} = 8\pi GT^{(2)}_{\mu\nu}. \]
The second-order Einstein tensor consists of two parts,
\[ G^{(2)}_{\mu\nu} = G^{(1)}_{\mu\nu} [g^{(2)}] + G^{(2)}_{\mu\nu} [g^{(1)}], \]
where \( G^{(1)}_{\mu\nu} [g^{(2)}] \) represents the linear terms in genuine second-order perturbations, and \( G^{(2)}_{\mu\nu} [g^{(1)}] \) represents the quadratic terms in first-order perturbations. The former describes the second-order part of the geometry, while the latter contributes as a part of the second-order energy-momentum tensor. \( T^{(2)}_{\mu\nu} \) can also be decomposed into two parts as (2.41), and the second-order effective energy-momentum tensor is constructed as
\[ G^{(1)}_{\mu\nu} [g^{(2)}] = 8\pi GT^{(1)}_{\mu\nu} [g^{(2)}] + G^{(2)}_{\mu\nu} [g^{(1)}] + 8\pi GT^{(2)}_{\mu\nu} [g^{(1)}], \]
where
\[ G^{(2)}_{\mu\nu} [g^{(1)}] = G^{(2)}_{\mu\nu} [g^{(1)}] \equiv 8\pi GT^{(2, \text{eff})}_{\mu\nu}. \]
Using the first-order Eqs. (2.17)-(2.19) [or (2.37)-(2.39)], we can replace the matter perturbations with the metric ones in $T_{\mu\nu}^{(2)}$. Then finally we can obtain $T_{\mu\nu}^{(2,\text{eff})}$ in terms of the Bardeen variable $\Psi$ and the gauge dependent variables $Q = \beta + E'$ and $E$. After integrating over several wavelengths, which is denoted by braket notations, the total derivative terms are integrated out, and finally we find $\tau_{\mu\nu} \equiv \left\langle T_{\mu\nu}^{(2,\text{eff})} \right\rangle$ as

\[
\tau_{00} = \frac{1}{8\pi G} \left[ -\frac{2}{\mathcal{H}' - \mathcal{H}^2} \left( (\nabla \Psi')^2 \right) - \frac{4\mathcal{H}}{\mathcal{H}' - \mathcal{H}^2} \langle \nabla \Psi' \cdot \nabla \Psi \rangle + \frac{5\mathcal{H}' - 7\mathcal{H}^2}{\mathcal{H}' - \mathcal{H}^2} \langle (\nabla \Psi)^2 \rangle \\
- 3\langle (\Psi')^2 \rangle - 12\mathcal{H}^2 \langle \Psi^2 \rangle + 6\langle (\mathcal{H}'Q - \mathcal{H}Q' - 4\mathcal{H}^2Q)\Psi' \rangle + 2\langle \nabla Q \cdot \nabla \Psi \rangle \\
- 4\mathcal{H} \langle \nabla Q \cdot \nabla \Psi \rangle - 24\mathcal{H}^2 \langle (Q' + \mathcal{H}Q)\Psi \rangle - 2\mathcal{H} \langle \nabla Q' \cdot \nabla Q \rangle - 3\langle (\mathcal{H}'Q - \mathcal{H}Q')^2 \rangle \\
+ 12\mathcal{H}^2 \langle 2\mathcal{H}' - \mathcal{H}^2 \rangle \langle Q^2 \rangle + \langle \Delta E' \left( 2\mathcal{H}' + 4\mathcal{H}^2Q - 3\mathcal{H}^2E' + 2\mathcal{H}\Delta E \right) \rangle \\
- 2\langle \Delta E \left( 2\mathcal{H}'\Psi - \Delta \Psi - 2\mathcal{H}'Q - 2\mathcal{H}H'Q \right) \rangle \right],
\]

(2.43)

\[
\tau_{0i} = 0,
\]

(2.44)

\[
\tau_{ij} = \frac{1}{8\pi G} \delta_{ij} \left\{ -\frac{2}{3(\mathcal{H}' - \mathcal{H}^2)} \left( (\nabla \Psi')^2 \right) - \frac{4\mathcal{H}}{3(\mathcal{H}' - \mathcal{H}^2)} \langle \nabla \Psi' \cdot \nabla \Psi \rangle - \frac{\mathcal{H}' + \mathcal{H}^2}{3(\mathcal{H}' - \mathcal{H}^2)} \langle (\nabla \Psi)^2 \rangle \\
+ \left( (\Psi')^2 \right) + 8\mathcal{H} \langle \Psi^2 \rangle + 4\langle (2\mathcal{H}' + \mathcal{H}^2)\Psi^2 \rangle + 4\langle (Q' + 2\mathcal{H}Q)\Psi'' \rangle \\
+ 2\langle (Q'' + 9\mathcal{H}Q' + (\mathcal{H}' + 12\mathcal{H}^2Q)\Psi) \rangle \\
+ 4\langle 2\mathcal{H}Q' + 4(\mathcal{H}' + \mathcal{H}^2)Q' + 2\mathcal{H}(3\mathcal{H}' + \mathcal{H}^2)Q \rangle \Psi \rangle + \frac{2}{3} \langle \nabla Q'' \cdot \nabla Q \rangle \\
+ \frac{2}{3} \left( (\nabla Q')^2 \right) + \frac{4}{3} \mathcal{H} \langle \nabla Q' \cdot \nabla Q \rangle + 2\mathcal{H} \langle Q''Q' \rangle - 2\mathcal{H}' \langle Q' Q \rangle + \mathcal{H}^2 \langle Q^2 \rangle \\
- 2\langle 2\mathcal{H}'' + 3\mathcal{H}\mathcal{H}' \rangle \langle Q' Q \rangle - (8\mathcal{H}\mathcal{H}'' + 3\mathcal{H}'^2 - 4\mathcal{H}^4) \langle Q^2 \rangle \\
- \frac{2}{3} \langle \Delta E'' \langle \Psi + 2\mathcal{H}Q - 3\mathcal{H}E' + 2\Delta E \rangle \rangle \\
- \frac{1}{3} \langle \Delta E' \left[ 4\Psi' + 10\mathcal{H} \Psi + 8\mathcal{H}Q' + 8(\mathcal{H}' + \mathcal{H}^2)Q - 3(2\mathcal{H}' + \mathcal{H}^2)E' \right. \\
+ 2\Delta E' + 8\mathcal{H}\Delta E] \rangle \\
+ \frac{2}{3} \left. \langle \Delta E \left[ 2\Psi'' + 4\mathcal{H}' \Psi - 2\mathcal{H}Q'' - 4(\mathcal{H}' + \mathcal{H}^2)Q' - 2(\mathcal{H}'' + 2\mathcal{H}'H)Q \right. \\
+ \Delta (E'' + 2\mathcal{H}E') \rangle \right\} \right].
\]

(2.45)

This energy-momentum tensor $\tau_{\mu\nu}$ is the explicit expression of the 2EMT of the scalar perturbations for perfect fluid. As we can see, the explicit form of $\tau_{\mu\nu}$ is expressed not only by the gauge invariant variable $\Psi$, but also by the gauge dependent variables $Q$ and $E$. Therefore, we can conclude that 2EMT is definitely gauge dependent.

### 3 2EMT in Friedmann universe: gauge choices

In the previous section, we have obtained the explicit form of 2EMT $\tau_{\mu\nu}$ and have noticed that it is gauge dependent. But, nevertheless, we may expect that under different gauge conditions we still obtain similar behavior of the equation of state in certain wavelength limits
so that the explicit gauge dependence does not matter practically. In this section, we investigate the 2EMT in the Friedmann universe, in particular, during the radiation-dominated epoch (RDE) and the matter-dominated epoch (MDE). The corresponding background equation of state \( w \equiv p_0/\rho_0 \) is 1/3 and 0 respectively. We evaluate the 2EMT in three gauge choices – longitudinal, spatially-flat, and comoving gauges.

For isentropic process \((\delta S = 0)\), using the relation
\[
\delta p = \left( \frac{\partial p}{\partial \rho} \right)_S \delta \rho + \left( \frac{\partial p}{\partial S} \right)_\rho \delta S \equiv c_s^2 \delta \rho,
\]
where \( c_s^2 = w \) for a barotropic fluid, the second-order differential equation for the Bardeen variable \( \Psi \) is obtained from Eqs. (2.37) and (2.39):
\[
\Psi'' + 3(1 + w)H\Psi' - w\Delta \Psi + [2H' + (1 + 3w)H^2]\Psi = 0.
\]
With the Fourier-mode expansion
\[
\Psi(\eta, x) = \sum_k \Psi_k(\eta)e^{ik \cdot x},
\]
we find the solution for MDE \((w = 0)\) and RDE \((w = 1/3)\) as \[\operatorname{[14]}\]
\[
\Psi_k(\eta) = \begin{cases} 
        c_1(k) + c_2(k) \frac{\eta}{\eta^3} & \text{(MDE)} \\
        \frac{d_1(k)}{\eta^3} \left[ k\eta \sqrt{3} \cos \left( \frac{k\eta}{\sqrt{3}} \right) - \sin \left( \frac{k\eta}{\sqrt{3}} \right) \right] + \frac{d_2(k)}{\eta^3} \left[ k\eta \sqrt{3} \sin \left( \frac{k\eta}{\sqrt{3}} \right) + \cos \left( \frac{k\eta}{\sqrt{3}} \right) \right] & \text{(RDE)}
    \end{cases}
\]
where \( c_1(k), c_2(k), d_1(k) \) and \( d_2(k) \) are momentum-dependent functions constant in time, and \( 1/\sqrt{3} \) is the sound speed during RDE. Now we fix the gauge condition, obtain \( \tau_{\mu \nu} \), and plug in the above solutions to see the behaviour of 2EMT in the Friedmann universe.

We interpret the components of 2EMT as the second-order effective energy density and pressure, \( \tau_{\mu \nu} = \text{diag}(-\rho, p, p, p) \). We present only the dominant terms in \( \tau_{\mu \nu} \) here, with more complete results being given in Appendix A.

### 3.1 Longitudinal gauge

Let us take the longitudinal gauge by imposing the conditions,
\[
\beta = E = 0,
\]
which gives \( Q = 0 \). As the gauge variables \( E \) and \( Q \) in Eqs. (2.43)-(2.45) vanish, \( \tau_{\mu \nu} \) is expressed only by the Bardeen variable \( \Psi \) as
\[
\tau_{00} = \frac{1}{8\pi G} \left[ -\frac{2}{H' - H^2} \langle (\nabla \Psi')^2 \rangle - \frac{4H}{H' - H^2} \langle \nabla \Psi' \cdot \nabla \Psi \rangle + \frac{5H' - 7H^2}{3(H' - H^2)} \langle \nabla \Psi \rangle^2 \right.
\]
\[
\left. - 3\langle (\Psi')^2 \rangle - 12H^2\langle \Psi^2 \rangle \right],
\]
\[
\tau_{ij} = \frac{1}{8\pi G} \delta_{ij} \left[ -\frac{2}{3(H' - H^2)} \langle (\nabla \Psi')^2 \rangle - \frac{4H}{3(H' - H^2)} \langle \nabla \Psi' \cdot \nabla \Psi \rangle - \frac{H' + H^2}{3(H' - H^2)} \langle (\nabla \Psi)^2 \rangle \right.
\]
\[
\left. + 8H \langle \Psi \Psi' \rangle + \langle (\Psi')^2 \rangle + 4(H' + H^2)\langle \Psi^2 \rangle \right].
\]
3.1.1 MDE

Plugging the solution (3.4) in Eqs. (3.6) and (3.7), we find the dominant terms as

\[ \tau_{00} = \frac{1}{8\pi G} \left[ \frac{19k^2}{3} |c_1|^2 + O(\eta^{-2}) \right], \tag{3.8} \]

\[ \tau_{ij} = \frac{1}{8\pi G} \delta_{ij} \left[ \frac{k^2}{9} |c_1|^2 + O(\eta^{-5}) \right]. \tag{3.9} \]

Recalling that \( a(\eta) \propto \eta^2 \) during MDE, for the most dominant terms, we have

\[ \varrho = \frac{\tau_{00}}{a^2} \propto \frac{1}{a^2} \quad \text{and} \quad p = \frac{\tau_{ij}}{a^2} \propto \frac{1}{a^2}, \tag{3.10} \]

which individually exhibit the same \( a \)-dependence as the curvature density. But their relation, \( p \approx 57\varrho \), is not that of the curvature density.

3.1.2 RDE

Using the solution (3.4) for (3.6) and (3.7), we find very complicated results containing a lot of sine and cosine functions. To simplify our discussions, we present the results only in two limits.

(i) Long-wavelength limit (\( k\eta/\sqrt{3} \ll 1 \)): In this case, we find

\[ \tau_{00} = \frac{1}{8\pi G\eta^8} \left\{ -39|d_2|^2 + O \left[ \left( \frac{k\eta}{\sqrt{3}} \right)^2 \right] \right\}, \tag{3.11} \]

\[ \tau_{ij} = \frac{1}{8\pi G\eta^8} \delta_{ij} \left\{ -19|d_2|^2 + O \left[ \left( \frac{k\eta}{\sqrt{3}} \right)^2 \right] \right\}. \tag{3.12} \]

Recalling that \( a(\eta) \propto \eta \) in RDE, for the most dominant terms, we have

\[ \varrho \propto \frac{1}{a^{10}} \quad \text{and} \quad p \propto \frac{1}{a^{10}}, \tag{3.13} \]

which decay quickly as the Universe expands.

(ii) Short-wavelength limit (\( k\eta/\sqrt{3} \gg 1 \)): In this case, we find

\[ \tau_{00} = \frac{1}{4\pi G\eta^8} \left\{ 3(|d_1|^2 + |d_2|^2) \left( \frac{k\eta}{\sqrt{3}} \right)^6 + O \left[ \left( \frac{k\eta}{\sqrt{3}} \right)^5 \right] \right\}, \tag{3.14} \]

\[ \tau_{ij} = \frac{1}{4\pi G\eta^8} \delta_{ij} \left\{ (|d_1|^2 + |d_2|^2) \left( \frac{k\eta}{\sqrt{3}} \right)^6 + O \left[ \left( \frac{k\eta}{\sqrt{3}} \right)^5 \right] \right\}. \tag{3.15} \]

We then have

\[ p \approx \frac{1}{3} \varrho \propto \frac{1}{a^4}, \tag{3.16} \]

which behaves as radiation in this limit.
3.2 Spatially-flat gauge

Let us take the spatially-flat gauge by imposing the conditions

$$\psi = E = 0.$$  \tag{3.17}

From Eqs. (2.31) and (2.32), we have

$$\Psi = H Q.$$  \tag{3.18}

As the gauge variables become $E = 0$ and $Q = \Psi/H$ in Eqs. (2.43)-(2.45), $\tau_{\mu\nu}$ is expressed only by the Bardeen variable $\Psi$ as

$$\tau_{00} = \frac{1}{8\pi G} \left[ -\frac{2}{H' - H^2} \left( \nabla' \Psi \cdot \nabla\Psi \right)^2 - \frac{4H}{H^2} \langle \nabla\Psi' \cdot \nabla\Psi \rangle + \frac{(2H' - 3H^2)(H' + H^2)}{H^2(H' - H^2)} \langle (\nabla\Psi)^2 \rangle ight. \\
- 12\langle (\Psi')^2 \rangle + \frac{24(H' - 2H^2)}{H} \langle \Psi' \Psi \rangle - \frac{12(H' - 2H^2)}{H^2} \langle \Psi^2 \rangle \right], \tag{3.19}
$$

$$\tau_{ij} = \frac{1}{8\pi G} \delta_{ij} \left[ \frac{2}{3H^2} \langle \nabla\Psi'' \cdot \nabla\Psi \rangle + \frac{2(H' - 2H^2)}{3H^2(H' - H^2)} \langle (\nabla\Psi')^2 \rangle \\
- \frac{4(2H^2 - 3H'H^2 + 2H^4)}{3H^2(H' - H^2)} \langle \nabla\Psi' \cdot \nabla\Psi \rangle + \frac{8}{H}\langle \Psi'' \Psi' \rangle - \frac{8(H' - 2H^2)}{H^2} \langle \Psi'' \Psi \rangle \\
- \frac{2H(H' - H^2)H'' - 6H^3 + 10H^2H^2 - 3H'H^4 + H^6}{3H^2(H' - H^2)} \langle (\nabla\Psi)^2 \rangle \\
- \frac{4(2H' - 5H^2)}{H^2} \langle (\Psi')^2 \rangle - \frac{8(HH'' - 2H'^2 + 3H'H^2 - 6H^4)}{H^3} \langle \Psi' \Psi \rangle \\
+ \frac{4(H' - 2H^2)(2HH'' - 2H'^2 - 3H'H^2 - 2H^4)}{H^4} \langle \Psi^2 \rangle \right]. \tag{3.20}
$$

3.2.1 MDE

Using the solution (3.4) for Eqs. (3.19) and (3.20), then we find

$$\tau_{00} = \frac{1}{8\pi G} \left[ \frac{4k^2}{3} |c_1|^2 + O(\eta^{-2}) \right], \tag{3.21}
$$

$$\tau_{ij} = \frac{1}{8\pi G} \delta_{ij} \left[ \left( 4 + \frac{17k^2}{18} \right) |c_1|^2 + O(\eta^{-2}) \right]. \tag{3.22}
$$

For the most dominant term, we have the same $a$-dependence (3.10) as in the longitudinal gauge

$$\varrho \propto \frac{1}{a^2}, \quad \text{and} \quad p \propto \frac{1}{a^2}, \tag{3.23}
$$

but with $p \approx (4 + 17k^2/18)/(4k^2/3)\varrho$.

3.2.2 RDE

Again, we consider two limits in RDE.
(i) Long-wavelength limit ($k\eta/\sqrt{3} \ll 1$): We find in this limit

$$
\tau_{00} = \frac{1}{8\pi G\eta^2} \Bigg\{ 9|d_2|^2 \left( \frac{k\eta}{\sqrt{3}} \right)^2 + O \left[ \left( \frac{k\eta}{\sqrt{3}} \right)^4 \right] \Bigg\}, \quad (3.24)
$$

$$
\tau_{ij} = \frac{1}{8\pi G\eta^2} \delta_{ij} \Bigg\{ 4|d_2|^2 + O \left[ \left( \frac{k\eta}{\sqrt{3}} \right)^2 \right] \Bigg\}. \quad (3.25)
$$

Considering the most dominant term, the pressure $p \propto 1/a^8$ is more significant than the energy density $\rho \propto k^2/a^8$.

(ii) Short-wavelength limit ($k\eta/\sqrt{3} \gg 1$): We obtain, in this limit, the most dominant terms are the same as Eqs. (3.14) and (3.15) in the longitudinal gauge, and the equation of state is also the same with Eq. (3.16), which behaves as radiation.

### 3.3 Comoving gauge

Let us take the comoving gauge by imposing the conditions

$$
\beta_{i} = v_{i} \quad \text{and} \quad E = 0. \quad (3.26)
$$

Using $v_i$ in Eq. (2.38), we have

$$
Q = \beta = \Psi' + \frac{H\Psi}{H' - H^2}. \quad (3.27)
$$

Similar to the previous gauge conditions, $\tau_{\mu\nu}$ is expressed only by the Bardeen variable $\Psi$ as

$$
\tau_{00} = \frac{1}{8\pi G} \left[ -\frac{2H}{(H' - H^2)^2} \langle \nabla\Psi'' \cdot \nabla\Psi' \rangle - \frac{2H^2}{(H' - H^2)^2} \langle \nabla\Psi'' \cdot \nabla\Psi \rangle + \frac{2H(H'' - 3H'H + H^3)}{(H' - H^2)^3} \langle (\nabla\Psi')^2 \rangle + \frac{4H(H'H'' - 2H'^2 + H^2H' - H^4)}{(H' - H^2)^3} \langle \nabla\Psi' \cdot \nabla\Psi \rangle + 5H'^3 + 2H^3H'' - 23H^2H'^2 + 25H^4H' - 11H^6 \langle (\nabla\Psi)^2 \rangle - \frac{3H^2}{(H' - H^2)^2} \langle \Psi''^2 \rangle + \frac{6H^2(H'' - 2H'H)}{(H' - H^2)^3} \langle \Psi''\Psi' \rangle + \frac{6H^2(HH'' - 4H'^2 + 6H^2H' - 4H^4)}{(H' - H^2)^3} \langle \Psi''\Psi \rangle - \frac{3H^2(H'' - 4H'H + 2H^3)(H'' - 2H^3)}{(H' - H^2)^3} \langle (\Psi')^2 \rangle - \frac{6H^2(HH'^2 - 4H''(H^2 - H^2H' + H^4) + 12H^3H^2 - 28H^3H'^2 + 28H^5H' - 8H^7)}{(H' - H^2)^4} \langle \Psi'\Psi \rangle - \frac{3H^2(HH'' - 6H'^2 + 12H^2H' - 8H^4)(HH'' - 2H'^2)}{(H' - H^2)^4} \langle \Psi^2 \rangle \right]. \quad (3.28)
$$
\[
\tau_{ij} = \frac{1}{8\pi G} \delta_{ij} \left[ \frac{2}{3(H' - H^2)^2} \langle \nabla \Psi'' \cdot \nabla \Psi' \rangle + \frac{2H}{3(H' - H^2)^2} \langle \nabla \Psi'' \cdot \nabla \Psi \rangle + \frac{2F_1}{3(H' - H^2)^3} \langle \nabla \Psi'' \rangle^2 - \frac{2F_1}{3(H' - H^2)^3} \langle \nabla \Psi' \cdot \nabla \Psi \rangle - \frac{2F_2}{3(H' - H^2)^3} \langle \nabla \Psi'' \cdot \nabla \Psi \rangle \right. \\
+ \frac{2F_3}{3(H' - H^2)^3} \langle \nabla \Psi'' \rangle^2 - \frac{2F_4}{3(H' - H^2)^3} \langle \nabla \Psi' \cdot \nabla \Psi \rangle - \frac{2F_5}{3(H' - H^2)^3} \langle \nabla \Psi'' \cdot \nabla \Psi \rangle \\
+ \frac{2H}{(H' - H^2)^2} \langle \Psi'' \Psi'' \rangle - \frac{2F_6}{(H' - H^2)^3} \langle \Psi'' \Psi' \rangle - \frac{2F_7}{(H' - H^2)^3} \langle \Psi'' \Psi' \rangle - \frac{F_8}{(H' - H^2)^3} \langle \Psi'' \rangle^2 \\
+ \frac{2F_9}{(H' - H^2)^3} \langle \Psi'' \Psi' \rangle - \frac{2F_{10}}{(H' - H^2)^3} \langle \Psi'' \Psi' \rangle + \frac{F_{11}}{(H' - H^2)^3} \langle \Psi' \rangle^2 - \frac{2F_{12}}{(H' - H^2)^3} \langle \Psi' \rangle \\
+ \frac{F_{13}}{(H' - H^2)^3} \langle \Psi' \rangle^2 \right], \\
(3.29)
\]

where
\[
F_1 = 4H'' - 13H'H' + 5H^3, \\
F_2 = 4H'H'' - 2H^2 - 9H^2H' + 3H^4, \\
F_3 = (H' - H^2)H'' - 3H^2 - 3H^4 + 4H(4H' - H^2)H'' - 24H^2H^2 + 19H^4H' - 4H^6, \\
F_4 = 2H(H' - H^2)H'' - (6H'H'' - 3H^2 - 26H^2H' + 5H^4)H'' + 4H(4H'^3 + 6H^2H^2 \nonumber \\
- 5H^4H' + H^6), \\
F_5 = 2H^2(H' - H^2)H'' + 2H(3H^2 + 10H^2H' - H^4)H'' - 6H^2H'^2 - H^4 - 22H^2H^3 \nonumber \\
- 6H^4H^2 + 6H^6H' - H^8, \\
F_6 = H(H'' - 2H'H'), \\
F_7 = H(H'H'' - 4H^2 + 6H^2H' - 4H^4), \\
F_8 = 4H'H'' - 4H^2 - 3H^2H' - H^4, \\
F_9 = 4H'^2H'' - H(H' - H^2)H'' + 20H^5H'^2 - 2(2H^2 + 5H^2H' + H^4)H'' - 28H^3H^2 \nonumber \\
+ 38H^5H' - 14H^7, \\
F_{10} = H^2 (H' - H^2)H'' - 4H^2H'^2 + H(13H^2 - 8H^2H' + 11H^4)H'' - 10H^4 - 10H^2H^3 \nonumber \\
- 52H^2H' + 40H^4H^2 + 16H^8, \\
F_{11} = 2H(H' - H^2)(H'' - 2H'H')H'' - 4H'H'^3 + (4H'^2 + 17H^2H' + 3H^4)H'^2 \nonumber \\
- 4H(10H'^3 - 11H^2H'^2 + 20H^4H' - 7H^6)H'' + 4H^2(18H^4 - 46H^2H^2 + 70H^4H^2 \nonumber \\
- 43H^6H' + 9H^8), \\
F_{12} = 4H^2H'^3 - 2H(H' - H^2)(H'' - 2H'H'^2 + 2H^2H' - 2H^4)H'' \nonumber \\
- H(13H^2 - 12H^2H' + 12H^4)H'^2 + 2(5H'^4 + 22H^2H^3 - 40H^4H^2 + 50H^6H' \nonumber \\
- 13H^8)H'' - 4H(15H^3 - 28H^2H^4 + 16H^4H'^3 + 19H^6H^2 - 18H^8H' + 4H^10), \\
F_{13} = 2H^2(H' - H^2)(H'' - 4H^2 + 6H^2H' - 4H^4)H'' - 4H^3H'^3 + (22H^2H'^2 - 19H^4H' \nonumber \\
+ 21H^2H'' - 4H(7H^4 + 4H^2H^3 - 17H^4H^2 + 22H^6H' - 4H^3)H'' \nonumber \\
+ 4(6H^4 + 3H^2H^3 - 25H^4H^2 + 32H^6H' - 8H^8)H'^2. \\
(3.42)
3.3.1 MDE

Plugging the solution (3.4) in Eqs. (3.28) and (3.29), then we obtain

\[ \tau_{00} = \frac{1}{8\pi G} \left[ \frac{77k^2}{9} |c_1|^2 + \mathcal{O}(\eta^{-5}) \right], \quad (3.43) \]

\[ \tau_{ij} = \frac{1}{8\pi G} \delta_{ij} \left[ \left( \frac{16}{9} + \frac{13k^2}{27} \right) |c_1|^2 + \mathcal{O}(\eta^{-2}) \right], \quad (3.44) \]

which give the same \( a \)-dependence as in the other gauges, Eqs. (3.10) and (3.23), but the equation of state becomes \( p \approx (16 + 13k^2/3)/(77k^2)\varrho \).

3.3.2 RDE

Again, we consider two limits in RDE.

(i) Long-wavelength limit \( (k\eta/\sqrt{3} \ll 1) \): The most dominant terms are given by

\[ \tau_{00} = \frac{1}{8\pi G\eta^8} \left\{ \frac{45}{2} |d_2|^2 \left( \frac{k\eta}{\sqrt{3}} \right)^2 + \mathcal{O} \left( \frac{k\eta}{\sqrt{3}} \right)^4 \right\}, \quad (3.45) \]

\[ \tau_{ij} = \frac{1}{8\pi G\eta^8} \delta_{ij} \left\{ |d_2|^2 + \mathcal{O} \left( \frac{k\eta}{\sqrt{3}} \right)^2 \right\}. \quad (3.46) \]

Similar to the spatially-flat gauge case, the pressure \( p \propto 1/a^8 \) is more significant than the energy density \( \varrho \propto k^2/a^8 \).

(ii) Short-wavelength limit \( (k\eta/\sqrt{3} \gg 1) \): The dominant terms are given by

\[ \tau_{00} = \frac{1}{4\pi G\eta^8} \left\{ \frac{45}{2} (|d_1|^2 + |d_2|^2) \left( \frac{k\eta}{\sqrt{3}} \right)^4 + \mathcal{O} \left( \frac{k\eta}{\sqrt{3}} \right)^3 \right\}, \quad (3.47) \]

\[ \tau_{ij} = \frac{1}{4\pi G\eta^8} \delta_{ij} \left\{ \frac{3}{2} (|d_1|^2 + |d_2|^2) \left( \frac{k\eta}{\sqrt{3}} \right)^4 + \mathcal{O} \left( \frac{k\eta}{\sqrt{3}} \right)^3 \right\}. \quad (3.48) \]

For the most dominant terms, \( \varrho \propto 1/a^6 \) and \( p \propto 1/a^6 \) and the equation of state is \( p \approx \varrho/15 \).

4 Conclusions

In this work, we have investigated the gauge invariance of 2EMT in the Friedmann universe. Introducing the gravitational scalar perturbations as well as the matter ones of fluid, we have kept the contributions up to second order in perturbations. The second-order terms consist of two parts: (i) the linear terms in second-order perturbations, and (ii) the quadratic combinations of first-order perturbations. In the Einstein tensor, (i) \( G^{(1)}_{\mu\nu}[g^{(1)}] \) is regarded as a pure second-order geometric contribution, and (ii) \( G^{(2)}_{\mu\nu}[g^{(1)}] \) is regarded as a contribution to the effective energy-momentum tensor along with \( T^{(2)}_{\mu\nu}[g^{(1)}, \delta \rho^{(1)}, \delta p^{(1)}] \). As a result, 2EMT is given by \( T^{(2, \text{eff})}_{\mu\nu} = T^{(2)}_{\mu\nu}[g^{(1)}, \delta \rho^{(1)}, \delta p^{(1)}] - G^{(2)}_{\mu\nu}[g^{(1)}] / 8\pi G \). Finally we have integrated over several wavelengths to obtain \( \tau_{\mu\nu} = \left\langle T^{(2, \text{eff})}_{\mu\nu} \right\rangle \).
Using the first-order equations, we could evaluate $\tau_{\mu\nu}$ in terms of the gravitational perturbations in Section 2.3.3. The result shows that $\tau_{\mu\nu}$ depends not only on the gauge-invariant Bardeen variable $\Psi$, but also the gauge variables, $Q = \beta + E$ and $E$. This indicates that $\tau_{\mu\nu}$ is definitely gauge dependent. The fact that $\tau_{\mu\nu}$ is not gauge invariant is not unreasonable: In general, the tensor components change after gauge (or infinitesimal coordinate) transformations. Even the rank-zero tensor (scalar) changes if it is a local quantity, e.g. the energy density. Therefore, the gauge dependence of 2EMT of cosmological scalar perturbations is in fact not exceptional.

Any truly observable quantity is supposed to be gauge invariant from the beginning. In our case, $\tau_{\mu\nu}$ is not in this category. However, if the value of the quantity converges in a certain limit after imposing gauge conditions, we may hope that it may have some (in)direct connection to observables. In this sense, we have examined $\tau_{\mu\nu}$ in three gauge choices – longitudinal, spatially-flat, and comoving gauges.

Once we select a gauge condition, $\tau_{\mu\nu}$ can be expressed only by the gauge invariant variable $\Psi$. However, the functional form of $\tau_{\mu\nu}$ is dependent on the gauge choice. In order to investigate $\tau_{\mu\nu}$ in certain limits, we have performed the Fourier-mode expansion for $\Psi$, and have solved the equation for $\Psi$ in the matter- and radiation-dominated epoch. Plugging the solution for $\Psi$ in $\tau_{\mu\nu}$, the results show that $\tau_{\mu\nu}$ never converges in any wavelength limit. In addition, interpreting the components of $\tau_{\mu\nu}$ as the effective second-order energy density and pressure, $\tau^{\mu\nu} = \text{diag}(-\varrho, p, p, p)$, the effective equation-of-state parameter $p/\varrho$ does not converge to a single value in any limit of three gauge choices. As a conclusion, $\tau_{\mu\nu}$ and its effects are strictly gauge dependent.

We may also well quantize $\Psi$ as

$$
\Psi(\eta, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3/2} \left[ \hat{\Psi}_k(\eta)e^{i\mathbf{k} \cdot \mathbf{x}}\hat{a}_k + \hat{\Psi}_k^*(\eta)e^{-i\mathbf{k} \cdot \mathbf{x}}\hat{a}_k^\dagger \right],
$$

where the annihilation and the creation operators satisfy the usual commutator relations with proper normalizations. Solving the field equation (3.2) for $\Psi$ with this quantization, we find the same solutions for $\hat{\Psi}_k(\eta)$ as Eq. (3.4). Using the ordering $\Psi^{(m)}(\eta)\Psi^{(n)} = (\Psi^{(m)}(\eta)\Psi^{(n)} + \Psi^{(n)}(\eta)\Psi^{(m)})/2$, and applying the operator on the ground state, we find the same $\tau_{\mu\nu}$ as obtained in Section 3. Therefore, $\tau_{\mu\nu}$ has no quantum effect, or at least any quantum contributions are indistinguishable from classical ones (see also Refs. [15–22] for the quantum effects of the backreaction).

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In this section, we present more complete functional forms of 2EMT during MDE and RDE in the three gauge conditions presented in the main text. In this section, for notational simplicity we define $k\eta/\sqrt{3} \equiv T$, $\sin(k\eta/\sqrt{3}) \equiv S$, $\cos(k\eta/\sqrt{3}) \equiv C$ and $(c_1, c_2) \equiv c_1^*c_2 + c_1c_2^* = 2\Re(c_1^*c_2)$.

A.1 Longitudinal gauge

A.1.1 MDE

$$
\begin{align*}
\tau_{00} &= \frac{1}{8\pi G} \left[ \frac{19k^2}{3} |c_1|^2 - \frac{48}{\eta^2} |c_1|^2 + \frac{3k^2}{\eta^3} (c_1, c_2) - \frac{48}{\eta^4} |c_1|^2 + \frac{8k^2}{\eta^5} |c_2|^2 - \frac{123}{\eta^7} |c_2|^2 \right], \quad (A.1) \\
\tau_{ij} &= \frac{1}{8\pi G} \delta_{ij} \left[ \frac{k^2}{9} |c_1|^2 - \frac{k^2}{\eta^3} (c_1, c_2) - \frac{40}{\eta^4} |c_1|^2 + \frac{2k^2}{3\eta^6} |c_2|^2 - \frac{55}{\eta^7} |c_2|^2 \right]. \quad (A.2)
\end{align*}
$$

A.1.2 RDE in long-wavelength limit

$$
\begin{align*}
\tau_{00} &= \frac{1}{8\pi G \eta^8} \left[ -39|d_2|^2 + 6|d_2|^2 T^2 + 4(d_1, d_2) T^3 + 12|d_2|^2 T^4 - \frac{4(d_1, d_2)}{5} T^5 \\
&\quad + \frac{13|d_2|^2 - 4|d_1|^2}{3} T^6 - \frac{113(d_1, d_2)}{35} T^7 + \frac{34(|d_1|^2 - |d_2|^2)}{15} T^8 + \mathcal{O}(T^9) \right], \quad (A.3) \\
\tau_{ij} &= \frac{1}{8\pi G \eta^8} \delta_{ij} \left[ -19|d_2|^2 - 14|d_2|^2 T^2 + \frac{16(d_1, d_2)}{3} T^3 + \frac{28(d_1, d_2)}{15} T^5 \\
&\quad + \frac{13|d_2|^2 - 4|d_1|^2}{9} T^6 - \frac{3(d_1, d_2)}{35} T^7 - \frac{4(|d_1|^2 - |d_2|^2)}{45} T^8 + \mathcal{O}(T^9) \right]. \quad (A.4)
\end{align*}
$$
A.1.3 RDE in short-wavelength limit

\[ \tau_{00} = \frac{1}{8\pi G\eta^8} \left[ 3 \left( S^2|d_1|^2 + C^2|d_2|^2 - SC(d_1, d_2) \right) T^6 
+ \left( 12SC(|d_1|^2 - |d_2|^2) - 6(C^2 - S^2)(d_1, d_2) \right) T^5 
+ \left( (27C^2 - 15S^2)|d_1|^2 + (27S^2 - 15C^2)|d_2|^2 + 42SC(d_1, d_2) \right) T^4 
+ \left( 72SC(|d_2|^2 - |d_1|^2) + 36(C^2 - S^2)(d_1, d_2) \right) T^3 
+ \left( - 84SC(d_1, d_2) - 39(C^2|d_1|^2 + S^2|d_2|^2) + 45(S^2|d_1|^2 + C^2|d_2|^2) \right) T^2 
+ \left( - 78SC(|d_1|^2 - |d_2|^2) + 39(S^2 - C^2)(d_1, d_2) \right) T 
+ \left( - 39(S^2|d_1|^2 + C^2|d_2|^2) + 39SC(d_1, d_2) \right) \right], \tag{A.5} \]

\[ \tau_{ij} = \frac{1}{8\pi G\eta^8} \delta_{ij} \left[ \left( S^2|d_1|^2 + C^2|d_2|^2 - SC(d_1, d_2) \right) T^6 
+ \left( 4SC(|d_1|^2 - |d_2|^2) - 2(C^2 - S^2)(d_1, d_2) \right) T^5 
+ \left( 3(C^2 - S^2)(|d_1|^2 - |d_2|^2) + 6SC(d_1, d_2) \right) T^4 
+ \left( 8SC(|d_2|^2 - |d_1|^2) + 4(C^2 - S^2)(d_1, d_2) \right) T^3 
+ \left( - 24SC(d_1, d_2) - 19(C^2|d_1|^2 + S^2|d_2|^2) + 5(S^2|d_1|^2 + C^2|d_2|^2) \right) T^2 
+ \left( 38SC(|d_1|^2 - |d_2|^2) + 19(S^2 - C^2)(d_1, d_2) \right) T 
+ \left( - 19(S^2|d_1|^2 + C^2|d_2|^2) + 19SC(d_1, d_2) \right) \right]. \tag{A.6} \]

A.2 Spatially-flat gauge

A.2.1 MDE

\[ \tau_{00} = \frac{1}{8\pi G} \left[ \frac{4k^2}{3} |c_1|^2 - \frac{300}{\eta^2} |c_1|^2 - \frac{2k^2}{\eta^6} (c_1, c_2) + \frac{3k^2}{\eta^{10}} |c_2|^2 \right], \tag{A.7} \]

\[ \tau_{ij} = \frac{1}{8\pi G} \delta_{ij} \left[ 4|c_1|^2 + \frac{17k^2}{18} |c_1|^2 - \frac{16}{\eta^2} |c_1|^2 + \frac{4 - k^2}{\eta^5} (c_1, c_2) - \frac{16}{\eta^7} (c_1, c_2) + \frac{4(k^2 + 1)}{\eta^{10}} |c_2|^2 
- \frac{16}{\eta^{12}} |c_2|^2 \right]. \tag{A.8} \]
A.2.2 RDE in long-wavelength limit

\[
\tau_{00} = \frac{1}{8\pi G} \left[ \frac{9}{\eta^8} |d_2|^2 \tau^2 - \frac{15}{\eta^8} |d_2|^2 \tau^4 + \frac{15(d_1, d_2)}{\eta^8} \tau^5 + \frac{15|d_2|^2 - 12|d_1|^2}{\eta^8} \tau^6 \\
- \frac{41(d_1, d_2)}{5\eta^8} \tau^7 + \frac{4(|d_1|^2 - |d_2|^2)}{\eta^8} \tau^8 + O(\tau^9) \right]. \tag{A.9}
\]

\[
\tau_{ij} = \frac{1}{8\pi G} \delta_{ij} \left[ 4\left( \frac{1}{\eta^6} + \frac{1}{\eta^8} \right) |d_2|^2 + \left( \frac{4}{\eta^8} + \frac{11}{\eta^8} \right) |d_2|^2 \tau^2 - \frac{4}{3} \left( \frac{1}{\eta^6} - \frac{1}{\eta^8} \right) (d_1, d_2) \tau^3 \\
- \frac{1}{9\eta^8} |d_2|^2 \tau^4 - \frac{1}{15} \left( \frac{8}{\eta^6} - \frac{143}{\eta^8} \right) (d_1, d_2) \tau^5 + \left( \frac{4}{9\eta^6} - \frac{59}{3\eta^8} \right) (d_1, d_2) \tau^7 \\
- \frac{1}{45} \left( \frac{4}{\eta^6} + \frac{26}{\eta^8} \right) (|d_1|^2 - |d_2|^2) \tau^8 + O(\tau^9) \right]. \tag{A.10}
\]

A.2.3 RDE in short-wavelength limit

\[
\tau_{00} = \frac{1}{8\pi G \eta^8} \left[ 3 \left( S^2 |d_1|^2 + C^2 |d_2|^2 - SC(d_1, d_2) \right) \tau^6 \\
+ \left( 12SC(|d_1|^2 - |d_2|^2) - 6(C^2 - S^2)(d_1, d_2) \right) \tau^5 \\
+ \left( (9C^2 - 24S^2)|d_1|^2 + (9S^2 - 24C^2)|d_2|^2 + 3SC(d_1, d_2) \right) \tau^4 \\
+ \left( 18SC(|d_2|^2 - |d_1|^2) + 9(C^2 - S^2)(d_1, d_2) \right) \tau^3 \\
+ \left( - 9SC(d_1, d_2) + 9(S^2 |d_1|^2 + C^2 |d_2|^2) \right) \tau^2 \right], \tag{A.11}
\]

\[
\tau_{ij} = \frac{1}{8\pi G \eta^8} \delta_{ij} \left[ \left( 3S^2 - 2C^2 \right) |d_1|^2 + \left( 3C^2 - 2S^2 \right) |d_2|^2 - 5SC(d_1, d_2) \right) \tau^6 \\
+ \left( 24SC(|d_1|^2 - |d_2|^2) - 12(C^2 - S^2)(d_1, d_2) \right) \tau^5 \\
+ \left( (15C^2 - 26S^2)|d_1|^2 + (26C^2 - 15S^2)|d_2|^2 + 41SC(d_1, d_2) \right) \tau^4 \\
+ \left( 30SC(|d_2|^2 - |d_1|^2) + 15(C^2 - S^2)(d_1, d_2) \right) \tau^3 \\
+ \left( - 19SC(d_1, d_2) - 4(C^2 |d_1|^2 + S^2 |d_2|^2) + 15(S^2 |d_1|^2 + C^2 |d_2|^2) \right) \tau^2 \\
+ \left( 4\eta^2 \left( SC(d_1, d_2) + (C^2 |d_1|^2 + S^2 |d_2|^2) \right) \right) \tau \right] \\
+ 4\eta^2 \left( \left( C^2 - S^2 \right)(d_1, d_2) - 2SC(|d_1|^2 - |d_2|^2) \right) \tau \\
+ 4(1 - \eta^2) \left( - (S^2 |d_1|^2 + C^2 |d_2|^2) + SC(d_1, d_2) \right]. \tag{A.12}
\]
A.3 Comoving gauge

A.3.1 MDE

\[
\tau_{00} = \frac{1}{8\pi G} \left[ \frac{77k^2}{9} c_1^2 - \frac{77k^2}{9\eta^2} (c_1, c_2) + \frac{77k^2}{9\eta} (c_1, c_2)^2 \right],
\]

(A.13)

\[
\tau_{ij} = \frac{1}{8\pi G} \delta_{ij} \left[ \frac{16}{9} c_1^2 + \frac{13k^2}{27} c_1^2 |c_1|^2 - \frac{64}{9\eta^2} |c_1|^2 (c_1, c_2) + \frac{48 + 13k^2}{27\eta^3} (c_1, c_2) - \frac{64}{9\eta^2} (c_1, c_2) 
+ \frac{48 + 13k^2}{27\eta^3} |c_2|^2 - \frac{64}{9\eta^2} |c_2|^2 \right].
\]

(A.14)

A.3.2 RDE in long-wavelength limit

\[
\tau_{00} = \frac{1}{8\pi G \eta^6} \left[ \frac{45}{2} |d_2|^2 T^2 + \frac{45}{2} |d_2|^2 T^4 - \frac{15}{2} (|d_1|, |d_2|) T^7 + \frac{5}{2} (|d_1|^2 - |d_2|^2) T^8 
+ \mathcal{O} \left( T^9 \right) \right],
\]

(A.15)

\[
\tau_{ij} = \frac{1}{8\pi G} \delta_{ij} \left[ \left( \frac{1}{\eta^6} - \frac{1}{\eta^6} \right) |d_2|^2 + \left( \frac{1}{\eta^6} + \frac{1}{2\eta^8} \right) |d_2|^2 T^2 - \frac{1}{3} \left( \frac{1}{\eta^6} - \frac{1}{\eta^8} \right) (d_1, d_2) T^3 
+ \frac{3}{2\eta^8} |d_2|^2 T^4 - \frac{1}{15} \left( \frac{2}{\eta^6} + \frac{11}{2\eta^8} \right) (d_1, d_2) T^5 + \frac{1}{9} \left( \frac{1}{\eta^6} - \frac{1}{\eta^8} \right) (|d_1|^2 - |d_2|^2) T^6 
+ \frac{1}{35} \left( \frac{2}{\eta^6} - \frac{9}{\eta^8} \right) (d_1, d_2) T^7 - \frac{1}{45} \left( \frac{1}{\eta^6} - \frac{17}{2\eta^8} \right) (|d_1|^2 - |d_2|^2) T^8 + \mathcal{O} \left( T^9 \right) \right].
\]

(A.16)

A.3.3 RDE in short-wavelength limit

\[
\tau_{00} = \frac{1}{8\pi G} \times \frac{45}{2\eta^8} \left[ \left( C^2 |d_1|^2 + S^2 |d_2|^2 + SC(d_1, d_2) \right) T^4 
+ \left( 2SC(|d_2|^2 - |d_1|^2) + (C^2 - S^2)(d_1, d_2) \right) T^3 
+ \left( S^2 |d_1|^2 + C^2 |d_2|^2 - SC(d_1, d_2) \right) T^2 \right],
\]

(A.17)

\[
\tau_{ij} = \frac{1}{8\pi G} \delta_{ij} \left[ \frac{3}{2\eta^8} \left( C^2 |d_1|^2 + S^2 |d_2|^2 + SC(d_1, d_2) \right) T^4 
+ \frac{3}{2\eta^8} \left( 2SC(|d_2|^2 - |d_1|^2) + (C^2 - S^2)(d_1, d_2) \right) T^3 
+ \frac{1}{2\eta^8} \left\{ - 5SC(d_1, d_2) - 2(C^2 |d_1|^2 + S^2 |d_2|^2) + 3(S^2 |d_1|^2 + C^2 |d_2|^2) \right\} 
+ \frac{1}{\eta^6} \left\{ SC(d_1, d_2) + (C^2 |d_1|^2 + S^2 |d_2|^2) \right\} T^2 
+ \frac{1}{\eta^8} \left\{ 2SC(|d_1|^2 - |d_2|^2) + (S^2 - C^2)(d_1, d_2) \right\} 
+ \frac{1}{\eta^8} \left\{ (C^2 - S^2)(d_1, d_2) - 2SC(|d_1|^2 - |d_2|^2) \right\} T 
+ \left( \frac{1}{\eta^6} - \frac{1}{\eta^8} \right) \left( - (S^2 |d_1|^2 + C^2 |d_2|^2) + SC(d_1, d_2) \right) \right].
\]

(A.18)
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