Far field imaging by a planar lens: diffraction versus superresolution

Nicholas A. Kuhta and Viktor A. Podolskiy
Department of Physics, 301 Weniger Hall, Oregon State University, Corvallis OR 97331 USA
Alexei L. Efros
Department of Physics, University of Utah, Salt Lake City UT, 84112 USA

We resolve the long standing controversy regarding the imaging by a planar lens made of left-handed media and demonstrate theoretically that its far field image has a fundamentally different origin depending on the relationship between losses inside the lens and the wavelength of the light \( \lambda \). At small enough \( \lambda \) the image is always governed by diffraction theory, and the resolution is independent of the absorption if both \( \text{Im} \varepsilon \ll 1 \) and \( \text{Im} \mu \ll 1 \). For any finite \( \lambda \), however, a critical absorption exists below which the superresolution regime takes place, though this absorption is extremely low and can hardly be achieved. We demonstrate that the transition between diffraction limited and superresolution regimes is governed by the universal parameter combining absorption, wavelength, and lens thickness. Finally, we show that this parameter is related to the resonant excitation of the surface plasma waves.

PACS numbers: 42.30.-d,73.20.Mf,42.25.Fx

The left-handed medium (LHM), introduced by Veselago\textsuperscript{1} in 1967 as a medium with simultaneously negative and real \( \mu \) and \( \varepsilon \) provides a negative refraction at its interface with a regular medium (RM). This effect allows creation of a unique imaging device, sometimes called the “Veselago lens”, formed by a planar slab of LHM, with its refractive index and impedance ideally matched to the surrounding RM (Fig.1). Interest in the planar lens significantly increased after the work by Pendry\textsuperscript{2,3}, who suggested that the planar lens can in principle focus all Fourier components of a 2D image, introducing a term “the perfect lens”. However, taken literally, this statement contradicts Electrodynamics: since there is no source at the focal point the field of the source cannot be exactly reproduced in the vicinity of the focal point. As it was pointed out in Refs.\textsuperscript{4,6}, in the absorption-less limit (\( \text{Im} \varepsilon = \text{Im} \mu = 0 \)), the solution proposed in\textsuperscript{2} exponentially diverges inside a 3D domain between two foci (see Fig.1), and therefore it cannot be a solution of Maxwell’s equations. Pokrovsky and Efros\textsuperscript{5,6} proposed a diffraction theory that should be valid at large \( k_0 = \omega/c \) and that does not contain any superresolution. Haldane\textsuperscript{7} explained that the superresolution should be connected to the resonance of the plasma waves on both sides of the slab, but he added that the theory should be regularized. The regularization due to a very small absorption inside the LHM, and its implication for resolution were analyzed by many authors\textsuperscript{7,8,9,10,11,12}. Near field behavior has been studied since 1994\textsuperscript{13,14}, both theoretically and experimentally\textsuperscript{15,16,17,18}. The goal of this work is to develop a quantitative criterion of applicability of diffraction theories for planar lens systems.

It has been shown that the far-field imaging of the planar lens has the following properties:

(a) The stationary solution of the Maxwell equations does not exist when \( \text{Im} \varepsilon = \text{Im} \mu = 0 \). When absorption is small, but non-zero, there are two "resonant" regions shown in Fig. 1; the fields inside resonant regions diverge as absorption tends to zero. Both foci are at the boundaries of the resonant regions. Therefore the focus outside the lens is quasi virtual: an observer to the right of the focal point can see subwavelength focus while an observer to the left of the focus can see only large and highly oscillating fields – an indication of resonantly excited surface plasma waves propagating at the back interface of the lens. The existence of such a quasi-focus does not contradict general theorems.

(b) Milton et al\textsuperscript{2} showed that total absorption of electric energy inside the slab \( \propto (\text{Im} \varepsilon)^{(2a/d-1)} \), where \( a \) is the distance from the source to the lens and \( d \) is the width of the lens. If \( a < d/2 \), the resonant regions overlap. In this case the total absorption inside the lens tends to infinity as \( \text{Im} \varepsilon \) tends to zero. Moreover, the electrostatic lens cloaks two dimensional dipoles rather than imaging them “perfectly” (See also Ref.\textsuperscript{14}).

On the other hand, 2D and 3D diffraction theories demonstrate a real focus with different widths in the lateral and in the longitudinal directions. According to these theories, fields near the focus are the universal function of \( r/\lambda \), where \( r \) is the radius-vector with an ori-
gin at the focal point. Although diffraction theory uses spherical waves rather than plane waves, results of both 2D and 3D diffraction theories coincide with solutions in the plane wave representation with evanescent waves omitted. The diffraction theory is known to be the first approximation at small deviations from geometrical optics. However, in the case of the planar lens we have a paradoxical situation. If Im $\epsilon \ll 1$ and Im $\mu \ll 1$, the width of the focus, as obtained by the diffraction theory, is independent of absorption, while the exact solution has a singularity at zero absorption even at small wavelengths. Therefore, the criterion of applicability of the diffraction theory cannot be related to a trivial ratio of wavelength to width of the slab, and must also depend on absorption.

Podolsky and Narimanov studied the width of the focus of the 2D planar lens. They have found that it is close to the diffraction limit almost at all cases where the wavelength is smaller than the width $d$ of the LHM slab. They also made an important observation that the boundaries of the diffraction regions depend on the absorption.

First we describe our results. We consider TM polarization with a 2D Green’s function as a source. The magnetic field with an amplitude $h$ near the source at the origin has a form $G_R = (i/4)\hbar H_0^{(1)}(\rho k_0)$, with $H$ being the Hankel function $H_0^{(1)} = J_0 + iN_0$, and $\rho = \sqrt{x^2 + y^2}$. This function can be represented in the form $G_R = H_p + H_{ev}$, where

$$H_p = \frac{i\hbar}{\pi} \int_{-k_0}^{k_0} \exp \left( iky + z\sqrt{k^2 - k_0^2 - \omega t} \right) \frac{dk}{\sqrt{k^2 - k_0^2}}$$  \hspace{1cm} (1)$$
contains only propagating waves while

$$H_{ev} = \frac{\hbar}{|k|} \int_{|k|>k_0} \exp \left( iky - z\sqrt{k^2 - k_0^2 - \omega t} \right) \frac{dk}{\sqrt{k^2 - k_0^2}}$$  \hspace{1cm} (2)$$
contains only evanescent waves (EW).

The 2-dimensional diffraction theory shows that the field near the focus contains only propagating modes. If the slab is at $a < z < d + a$ and Im $\epsilon \ll 1$ and Im $\mu \ll 1$, the field has a form

$$H_p^f(y, z') = \frac{i\hbar}{\pi} \int_{-k_0}^{k_0} \exp \left( ik'y + z'\sqrt{k^2 - k_0^2 - \omega t} \right) \frac{dk}{\sqrt{k^2 - k_0^2}}$$  \hspace{1cm} (3)$$
where $z' = z - (2d - a)$ is a distance from the focus in $z$-direction. At $z' = 0$ one gets $H_p^f(y, 0) = i\hbar J_0(2\pi y/\lambda)$. Note that the halfwidth of the first maximum of the square of the Bessel function is $\Delta y \approx 0.3\lambda$. This small value of the width in the lateral direction has been misinterpreted as superlensing.

As a criterion of transition from the supersolution regime to the diffraction regime we choose the ratio $P = H_{ev}/H_p^f(0, 0) = H_{ev}/i\hbar$, where both magnetic fields of EW’s and magnetic fields given by the diffraction theory are taken at the focal point $z = 2d - a$. Then if $P$ is small, we have the diffraction regime, while at large $P$ we have the regime of supersolution.

One should keep in mind that the creation of the LHM is a difficult and controversial problem. Usually LHM’s are metamaterials with periodic or quasi-periodic structure and their magnetic permeability $\mu$ may exhibit strong spatial dispersion $[\mu = \mu(\omega, \vec{k})]$. All these problems are outside the scope of this paper. Here we consider the basic model – a homogeneous and isotropic “hypothetical” slab that has

$$\epsilon = \mu = -1 + i\delta,$$  \hspace{1cm} (4)$$
where $\delta < 1$; $\epsilon = \mu = 1$ outside the slab. Our goal is to show what one can expect in the best case scenario.

Our main results are:

(i) The parameter responsible for the far-field ($k_0 d \gg 1$) transition between diffraction limited and superresolution regimes is not described by the wavelength alone, but also depends on absorption:

$$S = k_0 d \sqrt{\text{Im}(\epsilon + \mu)}/2 = k_0 d \sqrt{\delta}.$$  \hspace{1cm} (5)$$

(ii) At $S \gg 1$ the system is always described by diffraction theory, and

$$P(S) = \frac{1}{k_0 d} \frac{4(i \sin S - \cos S) \exp(-S)}{S(i-1)} \ll 1.$$  \hspace{1cm} (6)$$

(iii) At $S < 1$ and $k_0 d > 1$ the system may exhibit superresolution, with

$$P(S) = \frac{1}{i\pi} \sinh^{-1}(y_f/(2k_0 d)).$$  \hspace{1cm} (7)$$
where $y_f$ is the root of equation

$$4S^4 \exp y_f y_f^4 = 1. \quad (8)$$

(iv) Finally, when $-\ln(\delta/4) \gg k_0 d$ ($S \ll 1$) the system exhibits superresolution with quasi-static-like behavior

$$P(S) \approx \frac{2}{i\pi} \ln \left[ -\frac{2\ln(\delta/4)}{k_0 d} \right] \propto \ln \ln(1/S). \quad (9)$$

$P(S) \to \infty$ as $S \to 0$. However, divergence of $P(S)$ is extremely slow, and in practice $|P(S)| \gtrsim 1$ is unachievable in realistic far-field structures.

Note that the limit $\lambda \to 0$ ($k_0 \to \infty$) at a fixed $\delta$ and the limit $\delta \to 0$ at fixed $k_0(\lambda)$ do not commute. Indeed, at small absorption the diffraction regime exists at large enough $k$ and that at large $k$ the superresolution regime exists at small enough absorption: as described above, the solution at $\delta = S = 0$ does not exist. Below we provide the derivation of Eqs. (15-19), and demonstrate the connection between the superresolution and interaction of plasmonic surface waves, which disappears at $S > 1$.

To derive Eqs. (15-19) we use transmission coefficients for EW’s as calculated by Podolskij and Narimanov and take into account that $H_y^i(0,0) = i\hbar$. Then

$$P = \frac{2}{i\pi} \int_{k_0}^{\infty} \sqrt{k^2 - k_0^2 k_0^2} d(k) \exp(-d\sqrt{k^2 - k_0^2} D \sinh(\sqrt{k^2 - k_0^2} \mu d) + \cosh(\sqrt{k^2 - k_0^2} \mu d)) \quad (10)$$

$$D = \frac{k_0^2 \epsilon(\epsilon + \mu) - k^2(1 + \epsilon^2)}{2\epsilon\sqrt{k^2 - k_0^2} \sqrt{k^2 - k_0^2} \mu} \quad (11)$$

At $k_0 d \gg 1$ the values of $k$, that are important in this integral are very close to $k_0$. Namely,

$$k^2 - k_0^2 \approx 1/d^2. \quad (12)$$

Then $(k - k_0)/k_0 \approx 1/d^2 k_0^3 \ll 1$, $k^2 - k_0^2 \approx 2k_0(k - k_0)$, and $\epsilon^2 = \epsilon_\mu \approx 1 - 2i\delta$ [see Eq. (1)]. Thus,

$$P = \frac{2}{\pi k_0 d} \int_0^{\infty} \sqrt{1 + iS^2/2y^2} \frac{\sinh(y) \cosh(\sqrt{y^2 + iS^2/2} + \cosh(\sqrt{y^2 + iS^2/2}}) \quad (13)$$

As it can be explicitly verified, in the regime $S \gg 1$, the denominator of this integral can be further simplified:

$$\sqrt{1 + iS^2/2y^2} \sinh(y) + \cosh(y^2 + iS^2/2) \approx \sqrt{1 + iS^2/2y^2} \sinh(y) + iS^2/2, \text{leading to Eq. (10).}$$

If $S = 0$ the integral in Eq. (12) diverges which means that there is no solution without absorption. To consider the case of small $S$ it is more convenient to start with the transmission given by Eq. (2) of Ref. 12, that is written for the case of small absorption. Introducing a new variable of integration $y = 2\sqrt{k^2 - k_0^2} d$ one gets

$$P = \frac{1}{\pi k_0 d} \int_0^{\infty} \frac{dy}{\sqrt{\sqrt{y^2/(2k_0 d)^2 + 1}(1 + \phi^2 \exp(y))}}} \quad (14)$$

where

$$\phi(y) = \frac{1}{2} \left( \Im \epsilon + \frac{4\delta(k_0 d)^2}{y^2} \right) \quad (15)$$

In the limit of extremely small absorption, this integral is dominated by the first term in Eq. (14), yielding a quasi-electrostatic-like regime [Eq. (19)], predicted by Nicorovici et al., where the results are independent of $\mu$. Note that this quasi-electrostatic regime may exist even at $k_0 d \gg 1$ if absorption is small enough.

For somewhat larger absorptions, the far field regime is described by the second term in Eq. (14), leading to

$$P = \frac{1}{\pi k_0 d} \int_0^{\infty} \frac{dy}{\sqrt{(y/(2k_0 d)^2 + 1)(1 + 4S^2\exp(y)/y^4)}}. \quad (15)$$

At $S \ll 1$ one gets an adequate approximation for the integral Eq. (15) assuming that $1/[1 + 4S^2 \exp(y)/y^4] = \theta(y - y)$, where $\theta(x)$ is the Heaviside Step function and $y_f$ is given by Eq. (10). This leads to Eq. (17).

Fig. 2 shows the comparison of our analytical results for $P(S)$ for $S \ll 1$ and $S \gg 1$ to the results of numerical integration of Eq. (10).

We now explain the physical meaning of parameter $S$. The condition $k_0 d \gg 1$ is not sufficient for the applicability of diffraction theory in the planar lens because of the resonance interaction of the surface plasmon waves. These waves exist in TM polarization under the condition $k^2_0 < k_0^2$. Thus, the ratio $k_0/\epsilon$ changes sign at the interface of the vacuum and the LHM. At $\delta \ll 1$ it reads

$$\sqrt{k^2 - k_0^2} = \sqrt{k^2 - k_0^2 + k_0^2 \frac{2i\delta}{\epsilon}}. \quad (16)$$

The mismatch responsible for the breakdown of resonant excitation of surface modes can be related to the term
At $S \gg 1$ one can use Eq. (11) to find that the absolute value of the ratio of the mismatch term to $k^2 - k_0^2$ is $S^2 \gg 1$. Thus, at $S \gg 1$ there are no traces of the resonance and regular diffraction theory should be applicable. At $S \ll 1$ it follows from Eq. (13) that $k^2 - k_0^2 \sim y^2/d^2$. Ignoring in this estimate the logarithmic factor in $yf$ and assuming $yf \sim 1$ we resolve that the relative mismatch in Eq. (15) is also of the order of $S^2$. Therefore, the violation due to the absorption is small at $S \ll 1$ and the main features of the resonance should be preserved.

Finally, we use the developed formalism to analyze the range of parameters where one could expect superresolution (and quasi virtual focus with subwavelength thickness). These results are summarized in Fig. 3, which shows the dependence of normalized lens thickness $d/\lambda$ as a function of absorption $\delta$ for a set of fixed values of the parameter $|P|$, calculated using direct numerical calculation of the integral in Eq. (10). One can see that the contribution of the evanescent waves at the focal point is practically negligible at $d/\lambda > 2$ at any reasonable value of absorption.

In conclusion, we have developed an approach to calculate a quantitative measure of superlensing in a planar hypothetical LHM-based lens, and used the developed formalism to separate the regions of the superresolution and diffraction in the far field regime. We demonstrated that the limits of absorption $\delta \rightarrow 0$ and wavelength $\lambda \rightarrow 0$ do not commute; the former limit yields superlensing, while the latter leads to diffraction limited behavior, which typically dominates the far-field image of realistic planar lenses. We demonstrated that if $\lambda$ and $\delta$ are both finite, the behavior of the planar lens is described by a universal parameter $S$ which depends on both geometrical sizes and absorption, and found analytically asymptotical behaviors for $S \rightarrow 0$ and $S \rightarrow \infty$. A connection between the value of $S$ and the existence of resonant excitation of plasmonic waves has also been demonstrated. Understanding the onset of the diffraction limit presented in our work is important for the further development and design of imaging systems with negative refraction.

FIG. 3: (color online) The normalized lens thickness as a function of absorption for constant values of parameter $|P|$: filled triangles, squares, starts, diamonds, and empty triangles correspond to $|P| = 1$, $|P| = 1/2$, $|P| = 1/4$, $|P| = 1/8$, and $|P| = 1/16$ respectively. In the part of the plane above the curve with a given $|P|$, the relative contribution of evanescent waves is less than $|P|$.