Extremal Black Holes as Bound States

J. Rahmfeld†

Center for Theoretical Physics
Physics Department
Texas A & M University
College Station, Texas 77843

ABSTRACT

We consider a simple static extremal multi-black hole solution with constituents charged under different $U(1)$ fields. Each of the constituents by itself is an extremal dilatonic black hole of coupling $a = \sqrt{3}$. For a special case with two electrically and two magnetically charged black holes the multi-black hole solution interpolates between the familiar $a = \sqrt{3}, 1, \frac{1}{\sqrt{3}}$ and 0 solutions, depending on how many black holes are placed at infinity. This proves the hypothesis that black holes with the above dilaton couplings arise in string theory as bound states of fundamental $a = \sqrt{3}$ states with zero binding energy. We also generalize the result to states where the action does not admit a single scalar truncation and show that a wide class of dyonic black holes in toroidally compactified string theory can be viewed as bound states of fundamental $a = \sqrt{3}$ black holes.

† e-mail: joachim@tam2000.tamu.edu
1 Introduction

The role of black holes in string theory is a topic of active research. In one of the earliest papers discussing the relation between extremal black holes and massive string states, the observation was made that the charge and mass quantum numbers of special extremal black holes are consistent with a bound state interpretation. Those solutions could each be described by an effective action

\[ I_a = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[ R - \frac{1}{2}(\phi')^2 - \frac{1}{4}e^{-a\phi}F^2 \right], \]  

(1.1)

where \( a \) is the dilaton coupling parameter, \( \phi \) is an effective dilaton, and \( F \) is an effective field strength. \( \phi \) and \( F \) are typically linear combinations of a variety of fields in the underlying theory. The couplings consistent with supersymmetry are \( a = \sqrt{3}, 1, \frac{1}{\sqrt{3}}, \frac{1}{3} \) [1, 13, 8, 14, 15]. Incidentally, the extremal black hole solutions with precisely these values of \( a \) have regular null surface [16].

The conjecture was that two, three or four fundamental \( a = \sqrt{3} \) black holes (with appropriate charges) can combine with neutral binding energy to create \( a = 1, \frac{1}{\sqrt{3}}, \frac{1}{3} \) states. For example, the extreme Reissner-Nordström black hole is interpreted as a bound state of four Kaluza-Klein black holes! Our goal is to prove this hypothesis and see whether we can extend it to a wider class of solutions.

The starting point of every discussion is the bosonic action of heterotic string theory toroidally compactified to \( D = 4 \) and broken to \( U(1)^{28} \). In canonical coordinates it is given by

\[ I_H = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[ R - \frac{1}{2}(\eta')^2 - \frac{e^{-2\eta}}{12}F^2 + \frac{1}{8\pi G} \text{Tr}(\partial M L \partial M L) - \frac{1}{4}e^{-\eta}F^T(LML)F \right], \]  

(1.2)

where \( L \) is the metric of \( O(6,22) \). \( M = M^T \in O(6,22)/O(6) \times O(22) \) parametrizes the scalars in the sigma model, the 28 \( F_{\mu\nu} \)'s are the \( U(1) \) fields strengths, and \( \eta \) is the four dimensional dilaton. Let us work on a special point in the moduli space and set the asymptotic value of \( M \) to \( M^{(0)} = I \). All other backgrounds are equivalent to this one by an \( O(6,22) \) T-duality transformation. T-duality also allows further simplifications: for the solutions we are interested in, charged black holes, we can use \( O(6) \times O(22) \) transformations to truncate all field strengths but four, \( F_1, F_2, F_3 \) and \( F_4 \). The first two are Kaluza-Klein fields, the other two are winding modes. It is noteworthy that T-duality ensures that the only relevant degrees of freedom arise from the compactification from six to four dimensions on a torus.

With axion-like fields set to zero we arrive at

\[ I_t = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left\{ R \right. \left. - \frac{1}{2} \left[ (\partial \eta)^2 + (\partial \sigma)^2 + (\partial \rho)^2 \right] - \frac{e^{-\eta}}{4} \left[ e^{-\sigma+\rho}F_1^2 - e^{-\sigma+\rho}F_2^2 - e^{\sigma+\rho}F_3^2 - e^{\sigma-\rho}F_4^2 \right] \right\}. \]  

(1.3)

e\^\sigma and \( e^{-\rho} \) belong to the Kähler form and complex structure of the torus. This action was thoroughly analyzed in [8]. Especially, the triality of the three \( SL(2) \) duality groups was emphasized.
The extremal black holes we have in mind to illustrate the point, namely those that allow for a description by an action of type (1.1) with $a = \sqrt{3}, 1, \frac{1}{\sqrt{3}}, 0$, have the following electric/magnetic charge quantum numbers and masses [1, 8]:

(i) $a = \sqrt{3}$: $q = (1, 0, 0, 0)$, $p = (0, 0, 0, 0)$, $m = 1$,
(ii) $a = 1$: $q = (1, 0, 1, 0)$, $p = (0, 0, 0, 0)$, $m = 2$,
(iii) $a = \frac{1}{\sqrt{3}}$: $q = (1, 0, 1, 0)$, $p = (0, 1, 0, 0)$, $m = 3$,
(iv) $a = 0$: $q = (1, 0, 1, 0)$, $p = (0, 1, 0, 1)$, $m = 4$,

where $q_A$ and $p_A$ are the electric and magnetic charges of field strength $F_A$. These charges and masses are certainly consistent with the bound state interpretation. However, to prove the hypothesis we have to show that there is a no-force condition between the relevant fundamental black holes, since a look at the masses shows that the binding energy vanishes. The best way to guarantee vanishing total force is to find a static multi-black hole solution (with four single black holes) for arbitrary distances between the constituents. This solution should have the correct limits if one, two or three black holes are pushed out to infinity. Recent papers discussed cases of that kind by making use of the chiral null model [18]. In [19, 20, 21] solutions were constructed which are relevant for the $a = 1$ case. In this letter, we will extend the idea to include dyonic charges to account for the other two dilaton couplings. Our results are consistent with [22] where it was argued that only dilatonic black holes with $a > 1$ can behave like elementary particles.

Most black holes in the string spectrum cannot be described by a single scalar action. The natural question arises whether the bound state interpretation can be generalized to these solutions as well. The answer is yes, at least for a certain class, as will be discussed in section 3.

2 The Basic Multi-Black Hole Solution

Let us look for solutions of (1.3) (one has to keep in mind that axion-like fields are set to zero which imposes some constraints). The equations of motion of are:

$$\nabla_{\mu} \left( e^{-\eta - \sigma - \rho} F_{1}^{\mu \nu} \right) = 0,$$

$$\nabla_{\mu} \nabla^{\mu} \eta = -\frac{1}{4} e^{-\eta} \left( e^{-\sigma - \rho} F_{1}^{2} + e^{\sigma + \rho} F_{2}^{2} + e^{-\sigma + \rho} F_{3}^{2} + e^{\sigma - \rho} F_{4}^{2} \right),$$

$$R_{\mu \nu} = \frac{1}{2} \left( \partial_{\mu} \eta \partial_{\nu} \eta + \partial_{\mu} \sigma \partial_{\nu} \sigma + \partial_{\mu} \rho \partial_{\nu} \rho \right) +$$

$$+ \frac{e^{-\eta - \sigma}}{2} \left[ e^{-\rho} \left( F_{1 \mu \lambda} F_{1 \nu} \lambda - \frac{1}{4} g_{\mu \nu} F_{1}^{2} \right) + e^{\rho} \left( F_{2 \mu \lambda} F_{2 \nu} \lambda - \frac{1}{4} g_{\mu \nu} F_{2}^{2} \right) \right] +$$

$$+ \frac{e^{-\eta + \sigma}}{2} \left[ e^{\rho} \left( F_{3 \mu \lambda} F_{3 \nu} \lambda - \frac{1}{4} g_{\mu \nu} F_{3}^{2} \right) + e^{-\rho} \left( F_{4 \mu \lambda} F_{4 \nu} \lambda - \frac{1}{4} g_{\mu \nu} F_{4}^{2} \right) \right] \quad (2.1)$$

plus three additional Maxwell and two scalar equations.
To demonstrate the pattern, let us briefly review the solutions that can alternatively be described by (1.1). With
\[ \Delta = \left(1 + \frac{Q}{r}\right)^\frac{1}{2}, \quad r = \sqrt{x^2 + y^2 + z^2}. \] (2.2)
the solutions are:

(i) \(a = \sqrt{3} :\)

\[
\begin{align*}
    ds^2 &= -\Delta^{-1}dt^2 + \Delta(dx^m dx^n \delta_{mn}) \\
    e^{-\eta} &= \Delta, \quad e^{-\sigma} = \Delta, \quad e^{-\rho} = \Delta \\
    F_{1tm} &= \frac{Q x^m}{r^3 \Delta^4}
\end{align*}
\] (2.3)

(ii) \(a = 1 :\)

\[
\begin{align*}
    ds^2 &= -\Delta^{-2}dt^2 + \Delta^2(dx^m dx^n \delta_{mn}) \\
    e^{-\eta} &= \Delta^2, \quad e^{-\sigma} = 1, \quad e^{-\rho} = 1 \\
    F_{1tm} &= F_{3tm} = \frac{Q x^m}{r^3 \Delta^4}
\end{align*}
\] (2.4)

(iii) \(a = \frac{1}{\sqrt{3}} :\)

\[
\begin{align*}
    ds^2 &= -\Delta^{-3}dt^2 + \Delta^3(dx^m dx^n \delta_{mn}) \\
    e^{-\eta} &= \Delta, \quad e^{-\sigma} = \Delta^{-1}, \quad e^{-\rho} = \Delta \\
    F_{1tm} &= F_{3tm} = \tilde{F}_{2tm} = \frac{Q x^m}{r^3 \Delta^4}
\end{align*}
\] (2.5)

(iv) \(a = 0 :\)

\[
\begin{align*}
    ds^2 &= -\Delta^{-4}dt^2 + \Delta^4(dx^m dx^n \delta_{mn}) \\
    e^{-\eta} &= 1, \quad e^{-\sigma} = 1, \quad e^{-\rho} = 1 \\
    F_{1tm} &= F_{3tm} = \tilde{F}_{2tm} = \tilde{F}_{4tm} = \frac{Q x^m}{r^3 \Delta^4}
\end{align*}
\] (2.6)

with \(\tilde{F}_{2/4} = e^{-\eta \pm (-\sigma + \rho)} F_{2/4}^*\), and where * denotes the Hodge dual. One should add that the solutions of type \(a = \sqrt{3}\) with other gauge fields (and possibly magnetic charges) are obtained from (2.3) by the obvious changes in \(F\) and in the scalar fields, as indicated by (1.3). \(Q\) is understood to be quantized, since the electric and magnetic charge vectors of string states belong to an even self-dual lattice \([23, 17]\).

The behavior of the metric is noteworthy: \(g_{tt} = -\Delta^n\) and \(g_{mm} = \Delta^n\) where \(n\) is the number of conjectured constituents. Also, the scalar fields seem to behave rather linearly in the contributions of the individual black holes. This suggests a very simple solution for the four-black hole configuration. It seems that the scalar and Maxwell fields should just be added and the metric components multiplied. For the case of two extremal \(a = \sqrt{3}\) black holes, electrically charged under \(F_1\) and \(F_3\), (2.4) coincides with a limit of the chiral null model, as discussed in \([19, 20]\).
Let four black holes of type $a = \sqrt{3}$ and electric charges in $F_1, \tilde{F}_2, F_3, \tilde{F}_4$, respectively, be placed at $y^m_A$ with $A = 1, 2, 3, 4$. The straightforward generalizations of $\Delta$ and $r$ are

$$\Delta_A = (1 + \frac{Q}{r_A})^{\frac{1}{2}}, \quad r_A = \sqrt{(x^1 - y^1_A)^2 + (x^2 - y^2_A)^2 + (x^3 - y^3_A)^2}.$$  \tag{2.7}$$

It is useful to introduce shifted coordinates $x^m_m = x^m - y^m_A$. The solution of (2.1) is

$$ds^2 = -(\Delta_1 \Delta_2 \Delta_3 \Delta_4)^{-1}dt^2 + (\Delta_1 \Delta_2 \Delta_3 \Delta_4)(dx^m dx^n \delta_{mn}),$$

$$e^{-\eta} = \frac{\Delta_1 \Delta_3}{\Delta_2 \Delta_4}, \quad e^{-\sigma} = \frac{\Delta_1 \Delta_4}{\Delta_2 \Delta_3}, \quad e^{-\rho} = \frac{\Delta_1 \Delta_2}{\Delta_3 \Delta_4},$$  \tag{2.8}$$

$$F_{1/3\; tm} = \frac{Q x^m_{1/3}}{r_{1/3} \Delta_{1/3}}$$

$$\tilde{F}_{2/4\; tm} = \frac{Q x^m_{2/4}}{r_{2/4} \Delta_{2/4}}.$$

It interpolates between the extremal black holes we are targeting. In the limit of $n$ constituents at the origin and $4 - n$ at infinity, we recover the standard $a = \sqrt{\frac{4}{n} - 1}$ solutions.

Note that in this simple case $a$ reveals a new interpretation: $a^2$ is the ratio of black holes at infinity and black holes at the origin.

The supersymmetry breaking is determined by the central charge $s$. Since mass and charge are evaluated in the asymptotic region, the multi-black hole solution preserves as many supersymmetries as the appropriate single dilatonic black hole.

Another feature is also interesting: from the supersymmetry point of view, the simple bound state interpretation seems to break down after we reach $n = 4$ constituents with distinct charges. This finds its correspondence in the solutions of (1.1), the power of $\Delta^{-1}$ in the metric is limited by 4, since

$$g_{tt} = -\Delta^{-\frac{4}{1+a^2}}! $$  \tag{2.9}$$

It seems very surprising to find such a simple multi-black hole solution, since the Einstein equations are non-linear. After all, with the ansatz

$$ds^2 = -f^{-1}dt^2 + f dx^m dx^n \delta_{mn},$$  \tag{2.10}$$

the non-vanishing components of the Ricci tensor become

$$R_{tt} = -\frac{1}{2f^2} \Sigma_k \partial_k \partial_k \ln f,$$

$$R_{ii} = -\frac{1}{2} \Sigma_k \partial_k \partial_k \ln f - \frac{1}{2} \partial_i \ln f \partial_i \ln f,$$

$$R_{ij} = -\frac{1}{2} \partial_i \ln f \partial_j \ln f,$$  \tag{2.11}$$

which is partially non-linear in the contributions of the individual constituents. Also, the scalar terms on the right hand side of the Einstein equations are non-linear. However, for the solution (2.8) the non-linearities on both sides cancel! Essentially, this can be seen by noting that in

$$(\ln f)^2 + \eta^2 + \sigma^2 + \rho^2 = 4 \left[ (\ln \Delta_1)^2 + (\ln \Delta_2)^2 + (\ln \Delta_3)^2 + (\ln \Delta_4)^2 \right]$$  \tag{2.12}$$
the non-linearities on the left-hand side vanish. This cancellation translates directly to the Einstein equations.

Since (2.8) is static it implies a no-force condition between all four fundamental $a = \sqrt{3}$ extremal black holes. One might wonder what cancels the gravitational attraction, after all the charges do not provide the repulsion. It turns out that the scalars (or some of them) are repulsive for our type of solution. This behavior is analogous to the case considered recently in [11].

The new multi-black hole solution provides us with the proof for the bound state interpretation of dilatonic black holes with the standard couplings. However, we can push the idea even further, which is what we will do in the next chapter.

3 More General Dyonic Black Holes As Bound States

It is well-known that, in general, black hole solutions in string theory cannot be described by an effective single scalar, single gauge field action [1]. The procedure outlined above shows that with ansatz (2.8) the equations decompose into independent equations of motion for states charged under a single gauge field. Therefore, we can use the familiar multi-monopole ansatz [24, 25], of which (2.7) was a special case, for each charge sector. In the most general case, each individual $\Delta$ becomes

$$\Delta = \left(1 + \frac{Q_i}{|\vec{r} - \vec{r}_i|}\right)^{\frac{1}{2}} \quad (3.1)$$

where $Q_i$ and $\vec{r}_i$ denote the charges and positions of the black holes with common gauge field. The gauge field itself is modified in the well-known way. Overall, we obtain a static multi-black hole solution of mixed type: a combination of linear superpositions of identical black holes and a product combination of black hole sectors with different charges.

The interesting aspect of this more general ansatz is that it extends the bound state interpretation to solutions with arbitrary (quantized) left- and right-handed charges. For simplicity, let us consider an example with black holes electrically charged under $F_1$ or $F_3$ and take the limit in which all black holes are located at the origin. The multi-black hole solution then reduces to:

$$ds^2 = -\left(1 + \frac{Q_1}{r}\right)^{-\frac{1}{2}} \left(1 + \frac{Q_3}{r}\right)^{-\frac{1}{2}} dt^2 + \left(1 + \frac{Q_1}{r}\right)^{\frac{1}{2}} \left(1 + \frac{Q_3}{r}\right)^{\frac{1}{2}} dx^m dx^n \delta_{mn},$$

$$e^{-\eta} = \left(1 + \frac{Q_1}{r}\right)^{\frac{1}{2}} \left(1 + \frac{Q_3}{r}\right)^{\frac{1}{2}},$$

$$e^{-\rho} = \left(1 + \frac{Q_1}{r} + \frac{Q_3}{r}\right)^{\frac{1}{2}},$$

$$F_{1 \; tm} = \frac{Q_1 x^m}{r^3 \left(1 + \frac{Q_1}{r}\right)^2},$$

$$F_{3 \; tm} = \frac{Q_3 x^m}{r^3 \left(1 + \frac{Q_3}{r}\right)^2},$$

(3.2)
where $Q_1$ and $Q_3$ are the total charges of all black holes charged under $F_1$ and $F_3$, respectively. With

\[
Q_R = \frac{1}{2}(Q_1 + Q_3) \\
Q_L = \frac{1}{2}(Q_1 - Q_3)
\]

it is clear that (3.2) can have arbitrary half-integer left- and right-handed electric charges. A simple calculation reveals that (3.2) agrees with Sen’s extreme non-rotating black hole solutions [3] for states electrically charged under $F_1$ and $F_3$.

The above example can be easily generalized to include magnetic charges, in which case the multi-monopole ansatz (3.1) applies to $\Delta_2, \Delta_4, \tilde{F}_2$ and $\tilde{F}_4$ also. Again, the right- and left-handed magnetic charges can take any half-integer value.

Overall, we find that the bound state interpretation can be extended to dyonic states with charge and mass quantum numbers

\[
q = (Q_1, 0, Q_3, 0), \quad p = (0, Q_2, 0, Q_4), \quad m = Q_1 + Q_2 + Q_3 + Q_4, \quad (3.4)
\]

which, in general, do not admit a single scalar truncation. Note that the mass/charge relation in (3.4) does not agree with the Schwarz-Sen mass formula [23], since generically, dyonic black holes break more than one half of the supersymmetries. However, $m$ is perfectly consistent with [3, 8].

4 Conclusion

In this letter, we discussed an extended multi-monopole solution which established a no-force condition between four $a = \sqrt{3}$ black holes, with electric/magnetic charges in different gauge fields. The gravitational attraction is balanced by scalar repulsion. This proved that black holes of dilaton coupling $a = 1, \frac{1}{\sqrt{3}}, 0$ can be viewed as bound states of two, three and four $a = \sqrt{3}$ black holes with zero binding energy, as conjectured in [1]. The new interpretation of these dilatonic black holes explains very nicely the behavior of the metric. So far, there was no physical understanding of the relative powers of $\Delta$ in $g_{tt}$ and $g_{mm}$ of the extremal solutions with the discussed couplings. The multi-black hole solution fills this gap, it gives a physical interpretation of the “quantization” of powers for the crucial $a$ values. Also, the coupling $a$ becomes a new physical meaning: $a^2$ is the ratio of black holes at infinity and black holes at the origin.

It turned out that the bound state interpretation could be generalized to a wide class of dyonic black holes in string theory as well.

Acknowledgments:

It is a pleasure to thank Michael Duff, Ramzi Khuri and Sudipta Mukherji for inspiring conversations.
Note added:

After completion of this work we became aware of the work by Cvetič and Tseytlin [26]. They discuss extreme dyonic single-centered black hole solutions with the same charges as those in the present work, also characterized by four harmonic functions.

References

[1] M. J. Duff and J. Rahmfeld, Massive string states as extreme black holes, Phys. Lett. B 345 (1995) 441.

[2] M. J. Duff, R. R. Khuri, R. Minasian and J. Rahmfeld, New black hole, string and membrane solutions of the four dimensional heterotic string, Nucl. Phys. B 418 (1994) 195.

[3] A. Sen, Black hole solutions in heterotic string theory on a torus, Nucl. Phys. B 440 (1995) 421.

[4] A. Sen, Extremal black holes and elementary string states, Mod. Phys. Lett. A 10 (1995) 2081.

[5] B. R. Greene, D. R. Morrison and A. Strominger, Black hole condensation and the unification of string vacua, Nucl. Phys. B 451 (1995) 109.

[6] M. Cvetič and D. Youm, Dyonic BPS saturated black holes of heterotic string on a six-torus, UPR-672-T, hep-th/9507091.

[7] R. R. Khuri and R. C. Myers, Dynamics of extreme black holes and massive string states, Phys. Rev. D 52 (1995) 6988.

[8] M. J. Duff, J. T. Liu and J. Rahmfeld, Four-dimensional string/string/string triality, CTP-TAMU-27-95, hep-th/9508094.

[9] C. G. Callan, Jr., J. M. Maldacena and A. W. Peet, Extremal black holes as fundamental strings, PUPT-1565, hep-th/9510134.

[10] A. Dabholkar, J. P. Gauntlett, J. A. Harvey and D. Waldram, Strings as solitons & black holes as strings, CALT-68-2028, hep-th/9511053.

[11] R. Kallosh, and A. Linde, Supersymmetric balance of forces and condensation of BPS states, SU-ITP-95-26, hep-th/9511115.

[12] L. Susskind and J. Uglum, String Physics and Black Holes, SU-ITP-95-31, hep-th/9511227.

[13] G. W. Gibbons, G. T. Horowitz and P. K. Townsend, Higher dimensional resolution of dilatonic black hole singularities, Class. Quant. Grav. 12 (1995) 297.
[14] H. Lü and C. N. Pope, *P-brane solitons in maximal supergravities*, CTP-TAMU-47-95, hep-th/9512012.

[15] R. R. Khuri and T. Ortin, *Supersymmetric black holes in N=8 supergravity*, CERN-TH-95-348, hep-th/9512177.

[16] C. M. Hull and P. K. Townsend, *Unity of superstring dualities*, Nucl. Phys. B 438 (1995) 109.

[17] A. Sen, *Strong-weak coupling duality in four-dimensional string theory*, Int. J. Mod. Phys. A 9 (1994) 3707.

[18] G. T. Horowitz and A. A. Tseytlin, *A new class of exact solutions in string theory*, Phys. Rev. D 51 (1995) 2896.

[19] K. Behrndt, *The 10-d chiral null model and the relation to 4-d string solutions*, Phys. Lett. B 348 (1995) 495.

[20] K. Behrndt, *About a class of exact string backgrounds*, Nucl. Phys. B 455 (1995) 188.

[21] K. Behrndt and R. Kallosh, *O(6,22) BPS configurations of the heterotic string*, SU-ITP-95-19, hep-th/9509102.

[22] C. F. E. Holzhey and F. Wilczek, *Black holes as elementary particles*, Nucl. Phys. B 380(1992) 447.

[23] J. H. Schwarz and A. Sen, *Duality Symmetries of 4-d heterotic strings*, Phys. Lett. B 312 (1993) 105.

[24] R. R. Khuri, *A multimonopole solution in string theory*, Phys. Lett. B294 (1992) 325.

[25] J. P. Gauntlett, J. A. Harvey and J. T. Liu, *Magnetic Monopoles in String Theory*, Nucl. Phys. B409 (1993) 363-381.

[26] M. Cvetić and A. A. Tseytlin, *Solitonic Strings and BPS saturated dyonic black holes*, ASSNS-HEP-95-102, hep-th/9512031.