MOTIVIC DECOMPOSITION AND
INTERSECTION CHOW GROUPS I

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Abstract. For an arbitrary quasiprojective variety \( S \), defined over a field \( k \), assumed to be finitely generated over its prime field, we define a category \( CHM(S) \) of pure Chow motives over \( S \). Assuming conjectures of Grothendieck and Murre, we prove that the decomposition theorem holds in \( CHM(S) \). As a consequence, the intersection complex \( IS \) of \( S \) makes sense as an object of \( CHM(S) \). Part II will give an unconditional definition of “intersection” Chow groups and study some of their properties.

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References

1. Introduction

We will begin explaining our original motivation for writing this paper, then move on to summarise our main results and outline the contents of each section.

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Motivation.

[GoMa] introduced the intersection cohomology groups

\[ IH^i(X(\mathbb{C}), \mathbb{Q}) \]

of a (singular) algebraic variety \( X \) defined over the field of complex numbers (the \étaile version of this theory was constructed in [BBD] for varieties defined over fields finitely generated over the prime field). We recall the construction of these groups in the beginning of §4. We list some of their properties:

1. There is a factorisation

\[ H^i(X, \mathbb{Q}) \to IH^i(X, \mathbb{Q}) \to H_{2 \dim X - i}^{BM}(X, \mathbb{Q}) \]

of the Poincaré map \( H^i(X, \mathbb{Q}) \to H_{2 \dim X - i}^{BM}(X, \mathbb{Q}) \).

2. There is an intersection product

\[ IH^i(X, \mathbb{Q}) \times IH^j(X, \mathbb{Q}) \to H_{2 \dim X - i - j}(X, \mathbb{Q}) \]

which is nondegenerate for proper varieties and intersection cycles of complementary dimensions.

3. Cohomology acts on intersection cohomology

\[ H^i(X, \mathbb{Q}) \times IH^j(X, \mathbb{Q}) \to IH^{i+j}(X, \mathbb{Q}) \]

This research started out as a program (carried out—to an extent—in part II) to define a Chow theoretic analogue \( ICH^r(X, \mathbb{Q}) \) of the intersection cohomology groups, satisfying corresponding properties, namely:

1’ There should be a factorisation

\[ CHC^r(X, \mathbb{Q}) \to ICH^r(X, \mathbb{Q}) \to CH^r(X, \mathbb{Q}) \]

of the natural map \( CHC^r(X, \mathbb{Q}) \to CH^r(X, \mathbb{Q}) \). Here \( CHC^\bullet \) means “Chow cohomology”, not the operational theory of [FM], which does not have a cycle class map [To], but the theory developed in [Ha2,5].

2’ There should be an intersection product

\[ ICH^r(X, \mathbb{Q}) \times ICH^s(X, \mathbb{Q}) \to CH^{r+s}(X, \mathbb{Q}) \]

3’ Chow cohomology should act on intersection Chow groups

\[ CHC^r(X, \mathbb{Q}) \times ICH^s(X, \mathbb{Q}) \to ICH^{r+s}(X, \mathbb{Q}) \]
Moreover, there should be a cycle class map $cl : ICH^r(X, \mathbb{Q}) \to IH^{2r}(X, \mathbb{Q})$ and $(1' - 3')$ should be compatible, via the cycle class maps, with $(1 - 3)$.

This seemed to us an important and interesting question. Since their discovery, $IH^\bullet(X(\mathbb{C}), \mathbb{Q})$, resp. $IH^\bullet(X \times \overline{k}, \mathbb{Q}_l)$ have been shown to carry a natural pure Hodge structure, resp. pure Galois module structure [SaM], resp. [BBD]. These structures are only known to arise in nature as the cohomology groups of a pure Chow motive (direct summands of cohomology groups of a smooth algebraic variety. We summarise Grothendieck’s definition of pure motives in the beginning of §2). If this “intersection motive” can be identified and constructed, one can then take its Chow groups.

Where to look for such a motive? Let us assume for simplicity that $X$ is a variety with a single isolated singularity, having a resolution $f : Z \to X$ introducing a single smooth exceptional divisor $E$. According to the decomposition theorem [BBD, SaM]

$$Rf_*\mathbb{Q}_Z = IC_X \oplus V$$

where $IC_X$—in Borel’s notation [Bo]—is the intersection complex of $X$ (its hypercohomology groups are the $IH^i(X, \mathbb{Q})$) and $V$ is a sheaf supported on the singular point. For the reader’s convenience, we will recall the definition of intersection complexes and the statement of the decomposition theorem early on in §4. If the decomposition theorem is to hold for motives, we expect the “intersection motive” $IX$ of $X$ to be a direct summand of the motive $hZ$ of the desingularisation. If the decomposition theorem is to hold for motivic sheaves on $X$, we expect $IX$ to be of the form $(Z, i_*P)$ where $P \in CH_{\text{dim } X}(E \times E)$ is a projector in the correspondence ring of $E$ and $i : E \times E \to Z \times Z$ the inclusion. It is relatively easy to figure out what the cohomology class of $P$ must be: $E$ is naturally polarised by the dual of its normal bundle, and $P$ must induce the Hodge $\Lambda$ operator (see the beginning of §3 if you don’t remember this) relative to this polarisation. In other words, in this particular case, the motivic decomposition theorem is equivalent to the standard conjecture of Lefschetz type for $E$ (actually, there is a further quite subtle problem which will not be discussed here in the introduction, to justify that the Chow motive $(Z, i_*P)$ is independent on the choice of $P$). On the one hand this is quite disappointing: there seems no way to have a reasonable theory without proving the standard conjecture. On the other hand, we are at least able now to place the original question in its proper framework.

Main results.

This paper is not actually concerned with intersection Chow groups, we plan to do those in part II. Our first result is

**Theorem 1.** (See §2 for precise statements) let $S$ be an arbitrary quasiprojective variety, defined over a field $k$, assumed to be finitely generated over its prime field.
There is a category $\text{CHM}(S)$ of pure Chow motives over $S$, with realisation in $\mathcal{D}^b_{\text{cc}}(S)$, which is the relative analogue of the category $\text{CHM}$ defined by Grothendieck. As $\text{CHM}$, $\text{CHM}(S)$ arises from a correspondence category $\text{CHC}(S)$ whose objects are smooth varieties $X$, together with a projective morphism $X \to S$, and morphisms are defined as

$$\text{Hom}_{\text{CHC}}(X,Y) = \bigoplus \text{CH}_{\dim Y_\alpha}(X \times_S Y_\alpha)$$

the sum being taken over all irreducible components $Y_\alpha$ of $Y$. The composition of morphisms uses Fulton and Mac Pherson’s refined Gysin maps.

The construction of $\text{CHM}(S)$ is very easy and generalises an earlier idea by [DM]. We are convinced that $\text{CHM}(S)$ is a useful language, see for instance [DM], [Sc1].

Our second main result is

**Theorem 2.** (See §6 for the precise statement) Consider, as before, a quasiprojective variety $S$ over $k$, finitely generated over its prime field. Assume that resolutions of singularities exist for varieties over $k$. Then, assuming Grothendieck’s standard conjectures and Murre’s conjecture, a decomposition theorem holds in $\text{CHM}(S)$ which realises to the (topological) decomposition theorem in $\mathcal{D}^b_{\text{cc}}(S)$ of [BBD].

The assumption on resolutions of singularities is probably unnecessary: the modifications of [DJ1–2] are possibly sufficient for our purposes. We haven’t pursued this, since it is hardly the point of the paper.

The result is part of a larger program, due to M. H., to construct a triangulated category $\mathcal{D}(S)$ of “mixed motivic sheaves” on $S$, and show that, assuming the standard conjectures and Murre’s conjecture (and, in addition, the vanishing conjecture for $K$-groups of Soulé and Beilinson), it possesses the expected $t$-structure.

The decomposition “theorem” allows to make sense of intersection complexes and intersection Chow groups. In part II we shall use the ideas and result of part I to propose an unconditional definition of intersection Chow groups, and study some of their properties, sometimes with the aid of the conjectures.

We are sure that the reader will perceive our liberal use of various long standing conjectures to be a significant weakness of our study. As a partial answer to this possible objection, we would like to make 2 remarks. First, we are making the point in this paper, following an insight of M. H., that the standard conjectures, which were designed primarily to deal with motives over the point, are indeed enough to determine the first order (i.e., pure) behaviour of motivic sheaves. Second, there are interesting concrete contexts where the conjectural assumptions are satisfied, or could conceivably be shown. These include families of curves, surfaces, abelian varieties, toric morphisms. We believe that our theory will prove to be useful in these situations.
Before giving a quick detailed description of the contents of each section, we would like to say that we invested an inordinate amount of time to make this paper as self contained as possible. We assume a basic knowledge of algebraic geometry and intersection theory, as can be accessed through the 1st 6 chapters of [Fu], and a working knowledge of Verdier duality as can be obtained e.g. by looking into [KS]. We shall recall or summarise everything else we need, and that includes the bivariant theory of [FM], the standard conjectures, Murre’s conjecture on the natural filtration on the Chow groups of smooth and projective varieties, perverse sheaves, the decomposition theorem, etc. We especially hope that this paper will be accessible to nonexperts and graduate students seeking an introduction to the field.

Summary of contents.

§2 is devoted to the construction of $CHM(S)$ and the study of its first properties, especially the realisation in $D_{cc}^b(S)$. In §3 we recall the standard conjectures of Grothendieck, Murre’s conjecture on the canonical filtration of the Chow groups of smooth and projective varieties, and S. Saito’s proposed unconditional definition of this filtration. In §4 we define a category $\mathcal{M}(S)$ of “Grothendieck” motives over $S$ and, assuming the standard conjectures, we show that it is abelian and semisimple. This is an intermediate step in the direction of the decomposition theorem in $CHM(S)$, which we finally prove in §6, after extending Saito’s filtration to Chow groups of quasiprojective varieties in §5. For more information on the material covered in the various sections and the logical dependencies between them, the reader is invited to consult the short summary that we provide at the beginning of each.

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2. Pure motives over a base

The main object of this section is to generalise Grothendieck’s construction of pure motives to the relative situation over an arbitrary quasiprojective variety $S$. In short, we will define a category $CHM(S)$ of Chow motives over $S$; a way to think of it is the category of pure motivic sheaves over $S$. After reviewing our conventions for cohomology theories, we recall Grothendieck’s construction of the category $CHM$ of Chow motives, then move on to define $CHM(S)$ along very similar lines: the new element is the definition of composition of morphisms in $CHM(S)$ using Fulton
and Mac Pherson’s refined Gysin maps. We close the section with the construction of a natural realisation functor \( CH_M(S) \rightarrow D^b_{cc}(S) \). This construction is quite technical and we will complete it after a short summary of the topological bivariant theory of [FM], which is an essential ingredient in the proof: we advise the reader to skip it on first reading.

**Cohomology theories.**

In this subsection, we explain our notation and conventions for cohomology theories.

We will need some properties which are not shared by all Weil cohomology theories (in particular De Rham theory), especially the bivariant formalism, but later on, as we progress in the study, we will also need perverse sheaves and reasonable specialisation properties. For this reason we will work either with Betti cohomology or with étale cohomology.

1. We fix a field \( k \), finitely generated over the prime field, and consider quasiprojective varieties defined over \( k \).

   The notation \( H^i X \) means either:
   - \( H^i(X(\mathbb{C}), \mathbb{Q}) \), if \( k \) has characteristic 0, where we always assume to have chosen an embedding \( \sigma : k \hookrightarrow \mathbb{C} \), or
   - \( H^i(X, \mathbb{Q}_l) \), if \( \text{char } k \neq l \), where \( \overline{X} = X \otimes \bar{k} \).

   These are vector spaces over \( \mathbb{Q} \) or \( \mathbb{Q}_l \), depending on the context. The mixed Hodge structure or mixed Galois module structure on these spaces will be unimportant for us and, for this reason, we do not keep track of Tate twists in our notation for cohomology groups. We denote \( H^{BM}_i X \) the Borel-Moore homology theory companion to \( H^i X \). Homology and cohomology are part of the more general bivariant formalism alluded to above, which we shall summarise below when needed. There is a natural cycle class map \( \text{cl} : CH_r X \rightarrow H^{BM}_{2r} X \).

2. If \( S \) is a quasiprojective variety defined over \( k \), \( D^b_{cc} S \) denotes either \( D^b_{cc}(S(\mathbb{C}), \mathbb{Q}) \), the derived category of cohomologically constructible (for the euclidian topology) sheaves, or \( D^b_{cc}(\overline{S}, \mathbb{Q}_l) \), the category constructed in [BBD]. This has the 6 operations of Grothendieck, Verdier duality etc. \( Q_S \), resp. \( D_S \) will denote the constant sheaf, resp. dualising sheaf, i.e., \( Q_S \) is either \( \mathbb{Q}_S(\mathbb{C}) \) or \( \mathbb{Q}_{\overline{S}, l} \). Cohomology, Borel-Moore homology and the bivariant formalism alluded to above arise from \( D^b_{cc} \) and the 6 operations in a familiar way [FM], which is also briefly recalled below.

3. In §3 we will use the following specialisation property of \( H^i X \). Let \( T \) be the spectrum of a discrete valuation ring with residue field \( k \) and quotient field \( K \) (both finitely generated over their prime field), \( 0, \eta \in T \) the central and generic point, \( X \rightarrow T \) a smooth and proper morphism. There is then a specialisation isomorphism

\[
\text{hsp} : H^i X_\eta \xrightarrow{\cong} H^i X_0
\]
compatible via the cycle class with the specialisation homomorphism for Chow groups

\[
\begin{align*}
CH^r X_\eta \xrightarrow{csp} CH^r X_0 \\
\downarrow cl \quad \quad \quad \downarrow cl
\end{align*}
\]

\[
H^{2r} X_\eta \xrightarrow{hsp} H^{2r} X_0
\]

**Reminder of Grothendieck motives.**

In this subsection, we give a quick reminder of Grothendieck’s classical construction of motives (over the point), while also fixing our notation. This construction is in 3 steps: first the construction of a correspondence category, followed by pseudoabelianisation and the introduction of Tate objects and twists by them.

The standard references for this material are [De], [Ma], [Sc].

**Correspondences.**

We fix a field \(k\), finitely generated over its prime field. We consider smooth and projective varieties \(X\) over \(k\), and denote \(C^i X\) the group of algebraic cycles of codimension \(i\) on \(X\) modulo a suitable equivalence relation. The examples are:

1. \(C^i X = CH^i X\), the Chow group of cycles modulo rational equivalence.
2. \(C^i X = A^i X = H^{2i}X_{alg} = \text{Im}(CH^i X \to H^{2i} X)\) is the group of cycles modulo homological equivalence.
3. \(C^i X = \) cycles on \(X\) modulo numerical equivalence.

We will now construct the categories \(CC\) of \(C\)-correspondences.

**2.1 Definition.** An object of \(CC\) is a smooth and projective, not necessarily connected, variety \(X\).

Morphisms in \(CC\) are correspondences:

\[
\text{Hom}_{CC}(X, Y) = \bigoplus C^{\dim X_\alpha} X_\alpha \times Y
\]

where \(X = \bigsqcup X_\alpha\) is the decomposition of \(X\) into its connected components \(X_\alpha\).

Let \(u : X_1 \to X_2\) and \(v : X_2 \to X_3\) be correspondences, let \(p_{ij} : X_1 \times X_2 \times X_3 \to X_i \times X_j\) be the projection. The composition is defined as follows:

\[
v \circ u = p_{13*}(p_{23}^* v \cdot p_{12}^* u)
\]

It is easy to see that, with the above definitions, \(CC\) is an additive category, with the disjoint union of varieties being the categorical direct sum.

Since the intersection product for Chow groups is compatible with the cup product for cohomology classes, we have a forgetful functor \(CHC \to AC\) from the category of Chow correspondences to the category of homological correspondences.
Chow and Grothendieck motives.

We first recall the construction of the pseudoabelianisation of an additive category. Let \( \mathcal{A} \) be an additive category, \( A \) an object of \( \mathcal{A} \). A projector is an arrow \( P : A \to A \) such that \( P^2 = P \). It is possible to give a categorical definition of the image of a projector:

2.2 Definition. The image of a projector \( P : A \to A \) is an object \( \text{Im} P \) of \( \mathcal{A} \), together with a factorisation of \( P : A \to A \) (commutative diagram)

\[
\begin{array}{c}
A \xrightarrow{P} A \\
\downarrow \quad \downarrow \\
\text{Im} P
\end{array}
\]

satisfying the 2 identities

\[
\text{Hom}(\cdot, \text{Im} P) = P \circ \text{Hom}(\cdot, A)
\]

\[
\text{Hom}(\text{Im} P, \cdot) = \text{Hom}(A, \cdot) \circ P
\]

\( \mathcal{A} \) is pseudoabelian if every projector has an image.

2.3 Remark. The definition of image of \( P \) simply means that, for all objects \( X \), \( \text{Hom}(X, I) \) is the image, in the category of abelian groups of \( P \circ \cdot \) by means of the following diagram

\[
\begin{array}{c}
\text{Hom}(X, A) \xrightarrow{P \circ} \text{Hom}(X, A) \\
\downarrow \quad \downarrow \\
\text{Hom}(X, I)
\end{array}
\]

and, at the same time, \( \text{Hom}(I, X) \) is the image of \( \cdot \circ P \) by means of the following diagram

\[
\begin{array}{c}
\text{Hom}(A, X) \xrightarrow{\cdot \circ P} \text{Hom}(A, X) \\
\downarrow \quad \downarrow \\
\text{Hom}(I, X)
\end{array}
\]
2.4 Definition. The pseudoabelianisation of $\mathcal{A}$ is the category $\tilde{\mathcal{A}}$ defined as follows. Objects of $\tilde{\mathcal{A}}$ are pairs $(A, P)$ of an object $A$ of $\mathcal{A}$ and a projector $P : A \to A$. Morphisms in $\tilde{\mathcal{A}}$ are defined as follows

$$\text{Hom}_{\tilde{\mathcal{A}}}(A, P, (B, Q)) = Q \circ \text{Hom}_\mathcal{A}(A, B) \circ P$$

It is a simple observation that this is the same as morphisms $f : A \to B$ in $\mathcal{A}$ such that $f = Q \circ f \circ P$.

The following result is a formal exercise:

2.5 Theorem. The category $\tilde{\mathcal{A}}$ is pseudoabelian. There is a natural functor $F : \mathcal{A} \to \tilde{\mathcal{A}}$. Let $\mathcal{B}$ be a pseudoabelian category and $G : \mathcal{A} \to \mathcal{B}$ a functor. Then there exist a unique functor $H : \tilde{\mathcal{A}} \to \mathcal{B}$ such that $G = H \circ F$.

We now define the category $\mathcal{CM}$ of $C$-motives. This is made by taking the pseudoabelianisation of $\mathcal{CC}$ and then inserting Tate objects and twists by them:

2.6 Definition. An object of $\mathcal{CM}$ is a triple $(X, P, r)$ also denoted $(X, P)(r)$, where $X$ is a smooth projective, not necessarily connected variety, $P \in \text{End}_{\mathcal{CC}}(X, X)$ a projector, and $r \in \mathbb{Z}$ is an integer.

Morphisms in $\mathcal{CM}$ are defined as

$$\text{Hom}_{\mathcal{CM}}((X, P, r), (Y, Q, s)) = Q \circ (\bigoplus_{i} C^\text{dim } X_\alpha + s - r (X_\alpha \times Y)) \circ P$$

where $X = \bigsqcup X_\alpha$ is the decomposition of $X$ into its connected components $X_\alpha$. Composition is by means of the same formula used for composing correspondences.

2.7 Remarks, terminology, notation, etc..

(1) For $C = CH$, the category $\mathcal{CHM}$ is called the category of Chow motives. If $C$ is cycles modulo numerical equivalence, the corresponding category of motives is denoted simply $\mathcal{M}$ and called the category of Grothendieck motives. As noted below in 3.5, one of the consequences of the standard conjectures is that $\mathcal{AM} = \mathcal{M}$.

(2) Denoting $\mathcal{V}$ the category of smooth and projective varieties over $k$, there are natural contravariant cohomological $h : \mathcal{V} \to \mathcal{CHM}$ and covariant homological $h^\vee : \mathcal{V} \to \mathcal{CHM}$ functors. As an object, $hX = X$ regarded as a Chow motive, and for a morphism $f : X \to Y$, $h(f) = c\Gamma_f^t$ is the cycle class of the transpose $\Gamma_f^t \subset Y \times X$ of the graph $\Gamma_f$ of $f$. Similarly, $h^\vee X = \bigoplus_{\alpha} X_\alpha(\text{dim } X_\alpha)$, where $X = \bigsqcup X_\alpha$ is the decomposition in connected components, and $h^\vee(f) = c\Gamma_f$. 

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Similarly, there are natural contravariant cohomological \( H : V \to \mathcal{M} \) and covariant homological \( H^\vee : V \to \mathcal{M} \) functors. These are defined in a similar way to \( h \) and \( h^\vee \).

(4) If \( Vec_Q \) is the category of vector spaces over \( Q \), where the cohomology theory \( H^iX \) is taking values, there are also realisation functors \( H^* : \mathcal{M} \to Vec_Q \) sending a motive to its cohomology.

**Pure motives over a base.**

We extend Grothendieck’s construction to the case of varieties over an arbitrary quasiprojective base variety \( S \). In doing so, we generalise [DM]. Realisation functors need more work and we treat them in the next subsection.

**Correspondences over \( S \).**

We fix a field \( k \), finitely generated over its prime field. We consider quasiprojective varieties \( Z \) over \( k \), and denote \( C_iZ \) the group of \( i \)-dimensional algebraic cycles on \( Z \) modulo a suitable equivalence relation. The examples are:

1. \( C_iZ = CH_iZ \), the Chow group of cycles modulo rational equivalence.
2. \( C_iZ = A_iZ = H_{BM}^{2i}X_{alg} = \text{Im}(CH_iX \to H_{BM}^{2i}X) \) is the group of cycles modulo homological equivalence.

Let us fix an arbitrary quasiprojective variety \( S \). We will now construct the categories \( CC(S) \) of \( C \)-correspondences over \( S \).

**2.8 Definition.** An object of \( CC(S) \) is a smooth, not necessarily connected, variety \( X \), together with a projective morphism \( f : X \to S \).

Morphisms in \( CC(S) \) are correspondences:

\[
\text{Hom}_{CC(S)}(X, Y) = \bigoplus C_{\text{dim } Y} \alpha (X \times_s Y \alpha)
\]

where \( Y = \bigsqcup Y_\alpha \) is the decomposition of \( Y \) into its connected components \( Y_\alpha \).

The composition of morphisms is realized with the help of the following fibre square diagram

\[
\begin{array}{ccc}
X \times_s Y \times_s Z & \longrightarrow & (Y \times_s Z) \times (X \times_s Y) \\
\downarrow & & \downarrow \\
Y & \longrightarrow & Y \times Y
\end{array}
\]

For \( u : X \to Y \), \( v : Y \to Z \) we define the composition \( v \bullet u : X \to Z \) by

\[
v \bullet u = p_{XZ} \circ \delta^!(v \times u)
\]

where \( p_{XZ} \) is the projection on the first and third factor

\[
p_{XZ} : X \times_s Y \times_s Z \to X \times_s Z
\]
and $\delta'$ is Fulton’s refined Gysin map for local complete intersection (lci) morphisms [Fu, BFM]. We are assuming that $Y$ is smooth, therefore the diagonal embedding $Y \to Y \times Y$ is lci.

It is easy to see that, with the above definitions, $CC(S)$ is an additive category, with the disjoint union of varieties being the categorical direct sum.

Since the intersection product for Chow groups is compatible with the cup product for Borel-Moore homology classes [BFM], we have a forgetful functor $CHC(S) \to AC(S)$.

**Chow and homological motives over $S$.**

We now define the category $CM(S)$ of pure $C$-motives over $S$. This is made by taking the pseudoabelianisation of $CC(S)$ and then inserting Tate objects and twists by them:

**2.9 Definition.** An object of $CM(S)$ is a triple

$$(X, P, r)$$

also denoted $(X, P)(r)$, where $X$ is a smooth, not necessarily connected, variety, together with a projective morphism $f : X \to S$, $P \in \text{End}_{CCS}(X, X)$ a projector, and $r \in \mathbb{Z}$ is an integer.

Morphisms in $CM(S)$ are defined as

$$\text{Hom}_{CM}(((X, P, r), (Y, Q, s)) = Q \circ (\oplus C_{\dim Y \alpha + r - s}(X \times S Y \alpha)) \circ P$$

where $Y = \coprod Y \alpha$ is the decomposition of $Y$ into its connected components $Y \alpha$. The composition of morphisms is by means of the same formula used for composing correspondences.

**2.10 Remarks, terminology, notation, etc.**

(1) For $C = CH$, the category $CHM(S)$ is called the category of Chow motives over $S$. If $C = A$ is cycles modulo homological equivalence, $AM(S)$ is the category of homological motives over $S$. We will only need $AM(S)$ very briefly in §4 and §5. At this time, we are not in a position of constructing the analogue of Grothendieck motives $M$: this will be done in §4.

(2) Denoting $V(S)$ the category of smooth varieties $X$, projective over $S$, there are natural contravariant cohomological $h_S : V(S) \to CHM(S)$ and covariant homological $h^\vee_S : V(S) \to CHM(S)$ functors. As an object, $h_S X = X$ regarded as a Chow motive, and for a morphism $f : X \to Y$ covering the identity of $S$, $h_S(f) = cl \Gamma^f_\text{t}$ is the cycle class of the transpose $\Gamma^f_\text{t} \subset Y \times S X$ of the graph $\Gamma_f$ of $f$. Similarly, $h^\vee_S X = \oplus X_\alpha(\dim X_\alpha)$, where $X = \coprod X_\alpha$ is the decomposition in connected components, and $h^\vee_S(f) = cl \prod X_\alpha$. 

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If \( f : X \to S \) is the morphism to \( S \), we will sometimes use the following alternative notation for the objects \( h_S X \) and \( h_Y X \):

\[
\begin{align*}
h_S X &= \mathcal{C}Rf_* Q_X \\
h_Y X &= \mathcal{C}Rf_* D_X
\end{align*}
\]

In the coming subsection, we will construct a realisation functor \( CHM(S) \to D^{bc}_c(S) \). The notation is meant to suggest, for instance, that \( h_S X = \mathcal{C}Rf_* Q_X \) realises to \( Rf_* Q_X \), and is therefore a Chow theoretic “\( Rf_* \)”. This notation will be particularly useful in the statement of the decomposition theorem in §6. Similar remarks apply to the dualising sheaf \( D_X \).

(4) As already said, realisations will be constructed in the coming subsection.

**Realisations.**

This subsection is devoted to the proof of the following basic result:

**2.11 Theorem.** There is a natural (faithful) realisation functor

\[
AM(S) \to D^{bc}_c(S)
\]

Therefore, composing with the forgetful functor \( CHM(S) \to AM(S) \), there is also a realisation functor

\[
CHM(S) \to D^{bc}_c(S)
\]

**2.12 Remark.** We have no special notation for the realisation functors. In the instances where we will need to emphasise it, we will simply denote it \( \text{real} \). For instance \( \text{real} : CHM(S) \to D^{bc}_c(S) \) is the realisation functor.

**Proof.** The idea is to send the object \( f : X \to S \) to the sheaf \( Rf_* Q_X \). The actual proof is made of the following ingredients:

(1) By lemma 2.14 below, the category \( D^{bc}_c(S) \) is pseudoabelian.

(2) Let \( X, Y \) be smooth and \( p : X \to S, q : Y \to S \) be projective morphisms. By lemma 2.15 below, there is a natural isomorphism

\[
\varphi : \text{Hom}_S(Rp_* Q_X[2r], Rq_* Q_Y[2s]) \cong H_{2 \dim Y + 2r - 2s}^{BM} X \times S Y
\]

(3) By lemma 2.17, the morphism \( \varphi \) is compatible with composition, i.e. \( \varphi(v \circ u) = \varphi(v) \cdot \varphi(u) \). As for the Chow theory, the last quantity is defined as \( p_{XZ} \cdot \delta^l (v \times u) \): \( \delta \) carries a natural orientation class in Borel-Moore homology, compatible with the Chow theoretic orientation class, see [BFM].

This allows to define realisations by sending

\[
(X, P, r) \to \text{Im}(\varphi^{-1} cl P : Rf_* Q_X \to Rf_* Q_X)[2r]
\]
where $\text{cl} P \in H_*^{BM} X \times_S X$ is the Borel-Moore class of $P$. □

The construction of the realisation functors relies on 3 lemmas which we shall now state and prove. These are technical routines in $\mathcal{D}^b_{cc} S$, but basically easy and we advise to skip them on first reading: after all, the statement is rather plausible. Of these, the hardest is 2.17, stating that composition in $\text{CH.M}(S)$ is compatible with composition in $\mathcal{D}^b_{cc} S$. After some standard yoga, this is seen to be equivalent to the statement that, on a smooth variety $Y$, cup product of cohomology classes is compatible, via the Poincaré duality isomorphism, with intersection of Borel-Moore homology classes. The key point is finally dealt with in 2.18, which will use the topological bivariant theory of [FM] in an essential way, and will be shown after a quick reminder of [FM].

2.13 Warning. Working in $\mathcal{D}^b_{cc} (S)$, we make no attempts to get the signs right: in what follows all formulas are to be understood up to a $\pm$ sign. This policy is well established in the literature [KS] [SaM].

2.14 Lemma. The category $\mathcal{D}^b_{cc} (S)$ is pseudoabelian.

Proof. We will give a brief outline of a proof of this statement, for which we could find no reference in the literature. In fact the proof works for the full subcategory $D^b$ of cohomologically bounded objects in any triangulated category $D$ with $t$-structure.

Step 1. Let $p^2 = p : M \to M$ be a projector, we wish to construct the kernel and image $K$ and $I$ of $p$. The proof is by induction on the cohomological amplitude of $M$. For a suitable $i, M' = \tau_{\leq i} M$ and $M'' = \tau_{>i} M$ both have smaller cohomological amplitude then $M$, so we may assume by induction that $p' = \tau_{\leq i} p$, resp. $p'' = \tau_{>i} p$, have kernel and image $K'$ and $I'$, resp. $K''$ and $I''$. Note that we have a morphism of exact triangles

$$
\begin{array}{c}
M' \rightarrow M \rightarrow M'' [1] \\
\downarrow p' \quad \quad \quad \downarrow p \quad \quad \quad \downarrow p'' \\
M' \rightarrow M \rightarrow M'' [1]
\end{array}
$$

Step 2. With the identification $M' = K' \oplus I'$, $p' : M' \to M'$ is the projection to the second factor, and similarly for $M''$. Denoting $\delta : M'' \to M'[1]$ the map of degree 1, we have a commutative diagram

$$
\begin{array}{c}
K'' \oplus I'' \rightarrow \delta \rightarrow K'[1] \oplus I'[1] \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
K'' \oplus I'' \rightarrow \delta \rightarrow K'[1] \oplus I'[1]
\end{array}
$$
where the vertical arrows are projection on the second factor. We deduce that \( \delta(K'') \subset K'[1] \) (this has an obvious meaning in any additive category). Similarly, arguing with \( 1 - p \) instead of \( p \), we also have that \( \delta(I'') \subset I'[1] \).

**Step 3.** Choose now a triangle

\[
K' \rightarrow K \rightarrow K'' \rightarrow K'[1]
\]

there is then a morphism \( \varepsilon : K \rightarrow M \) so that the following is a morphism of triangles

\[
\begin{array}{cccc}
K' & \rightarrow & K & \rightarrow & K'' & \rightarrow & K'[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
M' & \rightarrow & M & \rightarrow & M'' & \rightarrow & M'[1]
\end{array}
\]

Replacing \( \varepsilon \) with \( \varepsilon - p \circ \varepsilon \), we may assume that \( p \circ \varepsilon = 0 \).

**Step 4.** Let now \( F : D \rightarrow \text{abelian groups} \) be any cohomological functor. Then the following sequence is exact

\[
0 \rightarrow FK \overset{\varepsilon}{\rightarrow} FM \overset{p}{\rightarrow} FM
\]

Using \( p \circ \varepsilon = 0 \) and step 2, the claim follows from a never ending diagram chase along the paths and trails of the diagram

\[
\begin{array}{cccc}
0 & \rightarrow & F^{-1}K'' & \rightarrow & FK' & \rightarrow & FK & \rightarrow & FK'' & \rightarrow & F^1K' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
F^{-1}K'' \oplus F^{-1}I'' & \rightarrow & FK' \oplus FI' & \rightarrow & FM & \rightarrow & FK'' \oplus FI'' & \rightarrow & F^1K' \oplus F^1I' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
F^{-1}K'' \oplus F^{-1}I'' & \rightarrow & FK' \oplus FI' & \rightarrow & FM & \rightarrow & FK'' \oplus FI'' & \rightarrow & F^1K' \oplus F^1I'
\end{array}
\]

**Step 5.** Apply step 4 to \( F = \text{Hom}(U, -) \) where \( U \) is an arbitrary object. This shows that \( K = \text{Ker}(p) \). Then \( I = \text{Ker}(1 - p) \). \( \square \)

**2.15 Lemma.** Let \( p : X \rightarrow S, q : Y \rightarrow S \) be morphisms of varieties, and consider the following fibre square diagram
Then:

(1) For sheaves $F \in \mathcal{D}^b_{cc} X$, $G \in \mathcal{D}^b_{cc} Y$, there is a natural isomorphism

$$Rf_*R\text{Hom}_{X \times S Y}(q'^* F, p'^! G) = R\text{Hom}_S(Rp! F, Rq_* G)$$

(2) In particular, if $p$ is proper and $Y$ is smooth, there is a natural isomorphism

$$\varphi : \text{Hom}_S(Rp_* Q_X[i], Rq_* Q_Y[j]) \xrightarrow{\cong} H_{2 \dim Y + i - j}^{BM} X \times S Y$$

**Proof.** (1) follows from Verdier duality and proper base change

$$Rf_*R\text{Hom}_{X \times S Y}(q'^* F, p'^! G) = (\text{standard duality}) \quad Rp_* R\text{Hom}_X(F, Rq'_* p'^! G)$$

$$= (\text{proper base change}) \quad Rp_* R\text{Hom}_X(F, p'_! Rq_* G)$$

$$= (\text{Verdier duality}) \quad R\text{Hom}_S(Rp! F, Rq_* G)$$

If $p$ is proper and $Y$ is smooth, $Rp! = Rp_*$ and $D_Y = Q_Y[2 \dim Y]$. Setting $F = Q_X[i]$, $G = Q_Y[j]$ in (1), and taking $H^0$, we obtain

$$\text{Hom}_S(Rp_* Q_X[i], Rq_* Q_Y[j]) = \text{Hom}_{X \times S Y}(Q_{X \times S Y}[i], p'^! Q_Y[j])$$

$$= \text{Hom}_{X \times S Y}(Q_{X \times S Y}[i], p'^! D_Y[-2 \dim Y + j])$$

$$= \text{Hom}_{x \times S Y}(Q_{X \times S Y}, D_{X \times S Y}[-2 \dim Y - i + j])$$

$$= H_{2 \dim Y + i - j}^{BM} X \times S Y$$

that is, (2). \qed

2.16 Remark. In the proof of (1) we went from $X \times S Y$ to $S$ passing through $X$. We could have gone there passing through $Y$, getting the same isomorphism.
2.17 Lemma. Let $p_1 : X \rightarrow S$, $p_2 : Y \rightarrow S$, $p_3 : Z \rightarrow S$ be morphisms of varieties with $p_1$, $p_2$ proper and $Y$, $Z$ smooth. Let $u : Rp_1^*Q_X[i] \rightarrow Rp_2^*Q_Y[j]$, $v : Rp_2^*Q_Y[j] \rightarrow Rp_3^*Q_Z[k]$ be morphisms, and $\varphi$ be the isomorphism of 2.15(2) above. Then

$$\varphi(v \circ u) = \varphi(v) \bullet \varphi(u)$$

The proof of 2.17 uses the following statement, with $Y$, $X \times_SY$, $Y \times_SZ$ in place of $T$, $U$, $V$, and we postpone it until the end of the subsection.

2.18 Lemma. Let $T$ be a smooth variety, and let $p : U \rightarrow T$, $q : V \rightarrow T$ be morphisms, with $p$ proper. There are natural isomorphisms

$$\lambda' : \text{Hom}_T(Rp_*Q_U, QT[i]) \xrightarrow{\cong} H_{2 \dim T-i}BM_U$$

$$\mu' : \text{Hom}_T(Q_T, Rq_*q'_*Q_T[j]) \xrightarrow{\cong} H_{2 \dim T-j}BM_V$$

$$\nu' : \text{Hom}_T(Rp_*Q_U, Rq_*q'_*Q_T[i+j]) \xrightarrow{\cong} H_{2 \dim T-i-j}BM_{U \times_T V}$$

satisfying the identity

$$\nu'(v \circ u) = \delta'((\mu'(v) \times \lambda'(u)))$$

The proof of 2.18 will keep us busy for some time. Let us explain the idea, which is pretty basic. In the simplest case where both $p : U \rightarrow T$ and $q : V \rightarrow T$ are the identity map $T = T$, the lemma just says that cup product in cohomology is compatible, via the Poincaré duality isomorphism, with intersection product in homology. Indeed $\text{Hom}_T(Q_T, Q_T[i]) = H^iT$, $\text{Hom}_T(Q_T[i], Q_T[i+j]) = H^jT$, and $v \circ u \in \text{Hom}_T(Q_T, Q_T[i+j])$ is the cup product $v \cup u$. Here we let $\lambda' = \mu' = \nu' = P : H^*T \rightarrow H_{2 \dim T-i}BM_T$ be the Poincaré duality isomorphism. The lemma then says $P(v \cup u) = \delta'(Pv \times Pu)$, but this is fine because $\delta'(PV \times PU) = PV \cdot PU$ is the intersection product. The actual proof of 2.18 uses the topological bivariant theory in an essential way. We will now give a quick reminder of [FM], followed by a generalisation of cup and intersection products, and finally the proof of 2.18. In closing this subsection, we shall prove 2.17 using 2.18.

**Topological bivariant theory.**

This is a very quick summary of the relevant bits of [FM]. For more information, the reader is invited to consult the original source.

(1) The topological bivariant theory associates to a morphism $f : X \rightarrow Y$ of algebraic varieties $Q$-vector spaces $H^i(X \rightarrow Y)$. Particular cases are $H^i(X = X) = H^iX$ is ordinary cohomology, $H^i(X \rightarrow pt) = H_{-i}BMX$ is Borel-Moore homology, and, for the inclusion $X \hookrightarrow Y$ of a locally closed subvariety, $H^i(X \hookrightarrow Y) = H^i_XY$ is local cohomology.

(2) The natural operations are *products, proper push forward and pull back*. 

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If $\alpha \in H^i(X \to Y)$ and $\beta \in H^j(Y \to Z)$, there is a product $\alpha \cdot \beta \in H^{i+j}(X \to Z)$.

If $f : X \to Y$ is proper and $Y \to Z$ is arbitrary, we get a push forward homomorphism

$$f_* : H^i(X \to Z) \to H^i(Y \to Z)$$

If

$$\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow & & \downarrow \\
Y' & \longrightarrow & Y
\end{array}$$

is a fibre square, we get a pull back

$$f^* : H^i(X \to Y) \to H^i(X' \to Y')$$

(3) Proper push forward and pull back are functorial and satisfy a number of natural compatibility axioms with products, like the projection formula, which we will not bother writing down.

(4) A strong orientation for $f : X \to Y$ is a class $\omega \in H^j(X \to Y)$ such that

$$H^i(U \to X) \ni \alpha \to \alpha \cdot \omega \in H^{i+j}(U \to Y)$$

is an isomorphism for all $U \to X$. With the aid of a strong orientation we can define unexpected proper push forward $f_! : H^\bullet X \to H^\bullet Y$ and pull back $f^! : H^BM_Y \to H^BM_X$.

A class of maps closed under composition possesses canonical orientations if all maps in the class are oriented in a way that products of orientations are compatible with compositions in the class. The key example of such a class is morphisms of smooth varieties (see below), but also local complete intersection (lci) morphisms.

(5) One way to construct the topological bivariant theory is to do so on top of the derived category $D^b_{cc}$ as follows. For a morphism $f : X \to Y$ one defines

$$H^i(X \to Y) = \text{Hom}_Y(Rf_!Q_X, Q_Y[i]) = \text{(Verdier duality)} \text{Hom}_X(Q_X, f^!Q_Y[i])$$

The product is essentially given by composition. Indeed let $f : X \to Y$, $g : Y \to Z$ and $\alpha : Rf_!Q_X \to Q_Y[i]$, $\beta : Rg_!Q_Y \to Q_Z[j]$ be bivariant classes. The product $\alpha \cdot \beta$ is the composition

$$R(g \circ f)_!Q_X \xrightarrow{Rg_!\alpha} Rg_!Q_Y[i] \xrightarrow{\beta} Q_Z[i+j]$$

Note that we might as well have done the composition on $Y$ or on $X$, with the same output.
If $f : X \to Y$ is proper, $g : Y \to Z$ is arbitrary, and $\alpha : R(g \circ f)_!Q_X \to Q_Z[i]$ a bivariant class, the **proper push forward** $f_*\alpha : Rg_!Q_Y \to Q_Z[i]$ uses the canonical trace map $Q_Y \to Rf_*Q_X$ and is defined to be the composition

$$Rg_!Q_Y \xrightarrow{Rg_!tr} Rg_!Rf_*Q_X = R(g \circ f)_!Q_X \xrightarrow{\alpha} Q_Z[i]$$

Finally, given a fibre square

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & X \\
\downarrow{g'} & & \downarrow{g} \\
Y' & \xrightarrow{f} & Y
\end{array}
\]

and a class $\alpha : Rg_!Q_X \to Q_Y[i]$, the **pull back** $f^*\alpha$ uses the base change isomorphism $Rg'_!f'^* = f^*Rg_!$:

$$Rg'_!Q_{X'} = Rg'_!f'^*Q_X = f^*Rg_!Q_X \xrightarrow{f^*\alpha} f^*Q_Y[i] = Q_{Y'}[i]$$

If $X$ and $Y$ are smooth (and, for simplicity, equidimensional), $f : X \to Y$ possesses a **canonical orientation**. Indeed, the duality homomorphism $Rf_!D_X \to D_Y$ is here nothing but a morphism $Rf_!Q_X[2 \dim X] \to Q_Y[2 \dim Y]$, i.e. a class in $H^{2 \dim Y - 2 \dim X}(X \to Y)$.

**Cup and intersection products.**

2.19 **Definition.** Given $p : U \to T$ and $q : V \to T$ we define a cup product

$$\cup : H^j(V \to T) \times H^i(U \to T) \to H^{i+j}(U \times_T V \to T)$$

by the formula

$$\alpha \cup \beta = q^*(\beta) \cdot \alpha$$

via the diagram

\[
\begin{array}{ccc}
U \times_T V & \xrightarrow[]{\text{q}} & V \\
\downarrow{p} & & \downarrow{q} \\
U & \xrightarrow[]{p} & T
\end{array}
\]

(it is the same as $(-1)^{i+j}p^*\alpha \cdot \beta$).
If $T$ is smooth, then it has a canonical orientation class $\omega \in H^{2\dim T}(T \to pt)$. For any $W \to T$ we denote $P_\omega$ the associated “Poincaré duality” isomorphism

$$H^i(W \to T) \ni \alpha \xrightarrow{P_\omega} \alpha \cdot \omega \in H^{i-2\dim T}(W \to pt)$$

The diagonal morphism $\delta : T \to T \times T$ also has a canonical orientation

$$or_\delta = P_{\omega \times \omega}^{-1}(\omega) \in H^{2\dim T}(T \to T \times T)$$

this allows 2 a priori different, but in the end equal, ways to define intersection products in Borel-Moore homology.

**2.20 Definition-Proposition.** Given $U \to T$ and $q : V \to T$ we define an intersection product

$$\bullet : H^{BM}_i U \times H^{BM}_j V \to H^{BM}_{2\dim T-i-j} U \times_T V$$

in any of the 2 equivalent ways

$$a \bullet b = P_\omega(P_{\omega}^{-1}a \cup P_{\omega}^{-1}b) = \delta^! (a \times b)$$

**Proof.** To prove that

$$P_\omega(P_{\omega}^{-1}a \cup P_{\omega}^{-1}b) = \delta^! (a \times b)$$

the reader is invited to stare at the following diagram, where every square is a fibre square

```
  U \quad U \times_T V
  |          |
  |          |
U \times T\quad U \times V
  |          |
  |          |
  T \quad \delta \quad V
  |          |
  |          |
T \times T\quad T \times V
  |          |
  |          |
  T \quad V
  |          |
  |          |
  pt \quad T \quad V
```

\[ \square \]
Proof of 2.18.

Step 1. We begin constructing natural isomorphisms

\[ \lambda : \text{Hom}_T(Rp_*Q_U, Q_T[i]) \xrightarrow{\cong} H^i(U \to T) \]

\[ \mu : \text{Hom}_T(Q_T, Rq_*q^!Q_T[j]) \xrightarrow{\cong} H^j(V \to T) \]

\[ \nu : \text{Hom}_T(Rp_*Q_U, Rq_*q^!Q_T[i + j]) \xrightarrow{\cong} H^{i+j}(U \times_T V \to T) \]

satisfying the identity

\[ \nu(v \circ u) = \mu(v) \cup \lambda(u) \]

The three isomorphisms are defined as follows. \( \lambda \) is the identity since \( (p \) is proper) by definition

\[ \text{Hom}_T(Rp!Q_U, Q_T[i]) = H^i(U \to T) \]

\( \mu \) is defined with a single application of standard duality

\[ \text{Hom}_T(Q_T, Rq_*q^!Q_T[j]) = \text{Hom}_V(Q_V, q^!Q_T[j]) = H^j(V \to T) \]

For the rest of the proof, we fix the notation in the following diagram

To define \( \nu \), let first

\[ \chi : \text{Hom}_V(q^*Rp_*Q_U, q^!Q_T[k]) \xrightarrow{\cong} H^k(U \times_T V \to T) \]

be the isomorphism obtained composing the following natural identifications

\[ \text{Hom}_V(q^*Rp_*Q_U, q^!Q_T[k]) = (\text{base change, } p \text{ proper}) \text{Hom}_V(Rp'_*q'^*Q_U, q'^!Q_T[k]) \]

\[ = \text{Hom}_V(Rp'_*Q_U \times_T V, q'^!Q_T[k]) = (p \text{ proper}) \text{Hom}_{U \times_T V}(Q_U \times_T V, p'^!q'^!Q_T[k]) \]

\[ = H^k(U \times_T V \to T) \]

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As a small digression, let $u : Rp_*Q_U \to Q_T[i]$, $v : Q_T \to Rq_*q^!Q_T[j]$, and let $v' : q^*Q_T = Q_V \to q'^*Q_T[j]$ correspond to $v$ under Verdier duality. We like to observe, at this point, that

$$
\mu(v) \cup \lambda(u) = q^*(\lambda(u)) \cdot \mu(v) = \chi(v' \circ q^*(u))
$$

This ends the digression. Now, to come back to the definition of $\nu$, we just compose $\chi$ with a standard duality isomorphism

$$
\text{Hom}_T(Rp_*Q_U, Rq_*q^!Q_T[k]) = \text{Hom}_V(q^*Rp_*Q_U, q'^*Q_T[k])
\xrightarrow{\Delta} H^k(U \times_T V \to T)
$$

Look now at the commutative diagram

$$
\begin{array}{ccc}
Rp_*Q_U & \xrightarrow{tr} & Rq_*q^*Rp_*Q_U \\
\downarrow{u} & & \downarrow{Rq_*q^*\lambda(u)} \\
Q_T[i] & \xrightarrow{tr} & Rq_*q^*Q_T[i] \\
\downarrow{v} & & \downarrow{Rq_*v'} \\
Rq_*q^!Q_T[i+j] & \xrightarrow{\sim} & Rq_*q^!Q_T[i+j]
\end{array}
$$

The diagram shows that $v' \circ q^*u$ corresponds to $v \circ u$ under the standard duality isomorphism

$$
\text{Hom}_T(Rp_*Q_U, Rq_*q^!Q_T[i + j]) = \text{Hom}_V(q^*Rp_*Q_U, q'^*Q_T[i + j])
$$

which was used in the definition of $\nu$. Therefore

$$
\nu(v \circ u) = \chi(v' \circ q^*u)
$$

On the other hand, as we have seen in the digression

$$
\mu(v) \cup \lambda(u) = \chi(v' \circ q^*u)
$$

and combining the last 2 displayed formulas concludes step 1.

**Step 2.** We now define

$$
\lambda' = P_\omega \circ \lambda : \text{Hom}_T(Rp_*Q_U, Q_T[i]) \xrightarrow{\cong} H_{2 \dim T - i}^{BM} U
$$
\[ \mu' = P_\omega \circ \mu : \text{Hom}_T(Q_T, Rq_*q^T[j]) \xrightarrow{\cong} H_{2 \dim T - j}^{BM} \]

\[ \nu' = P_\omega \circ \nu : \text{Hom}_T(Rp_*Q_U, Rq_*q^T[i + j]) \xrightarrow{\cong} H_{2 \dim T - i - j}^{BM} \]

The statement now follows from step 1, 2.20, and a simple calculation

\[ \nu'(v \circ u) = P_\omega (\nu(v \circ u)) = P_\omega (\mu(v) \cup \lambda(u)) =
\]

\[ = P_\omega (P_\omega^{-1} \mu'(v) \cup P_\omega^{-1} \lambda'(u)) = \delta^i (\mu'(v) \times \lambda'(u)) \]

This finishes the proof of 2.18.

**Proof of 2.17.**

We summarise the notation for the various spaces and maps in the following commutative diagram

\[ \begin{array}{c}
X \times S Z \\
| \\
p_{xz} \\
| \\
X \times S Y \times S Z \\
p_1, p_2, p_3, p_1', p_2', p_3' \\
| \\
X \times S Y \\
p_2' \\
| \\
X \\
p_1 \\
| \\
S \\
\end{array} \]

The proof of the lemma results from contemplating the following commutative diagram, which is commented upon below

\[ \begin{array}{c}
Rp_{1*}Q_X[i] \xrightarrow{u} Rp_{2*}p_2^*Rp_{1*}Q_X[i] \xrightarrow{Rp_{2*}u'} Rp_{2*}Rp_{1*}'Q_{X \times S Y}[i] \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
Rp_{2*}Q_Y[j] \xrightarrow{v} Rp_{2*}Q_Y[j] \xrightarrow{Rp_{2*}v'} Rp_{2*}Q_Y[j] \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
Rp_{3*}Q_Z[k] \xrightarrow{Rp_{2*}p_2^*Rp_{3*}Q_Z[k]} \xrightarrow{Rp_{2*}Rp_{3*}'P_3'Q_Y[2 \dim Y - 2 \dim Z + k]} \end{array} \]
Here \( u' : p_2^*R_1^*Q_X[i] \to Q_Y[j] \) corresponds to \( u \) via the standard duality

\[
\text{Hom}_Y(p_2^*R_1^*Q_X[i], Q_Y[j]) = \text{Hom}_S(R_1^*Q_X[i], R_2^*Q_Y[j])
\]

and \( tr : R_1^*Q_X[i] \to R_2^*p_2^*R_1^*Q_X[i] \) is the trace map giving rise to (or arising from, depending on the reader’s preference) the duality. Similar comments apply to \( v' \).

The two equal signs in the right portion of the diagram are obtained from the proper base change isomorphism, for instance the lower one (which is the hardest) is derived as follows

\[
p_2^1R_3^*Q_Z[k] = (\text{base change}) \ R_3'^1p_2'^*Q_Z[k] \\
= (Z \text{ is smooth}) \ R_3'^1p_2'^*D_Z[-2 \dim Z + k] \\
= R_3'^1D_{Y \times S Z}[-2 \dim Z + k] \\
= (Y \text{ is smooth}) \ R_3'^1p_3'^*Q_Y[2 \dim Y - 2 \dim Z + k]
\]

Contemplating the diagram in the light of how \( \phi \) is defined (lemma 2.15 and remark 2.16), and using 2.18, with \( X \times_S Y \), \( Y \times_S Z \) and \( Y \) in place of \( U \), \( V \) and \( T \) respectively, we evince the following

\[
\phi(v \circ u) = p_{XZ}^*u'(v' \circ u') \\
\phi(u) = \lambda'(u') \\
\phi(v) = \mu'(v')
\]

The result then follows immediately from 2.18. This finishes the proof of 2.17.

### 3 Standard conjectures and canonical filtrations

In this section, which is intended mainly for reference, we begin recalling Grothendieck’s standard conjectures, which were introduced, among other things, to determine the behaviour of the category \( \mathcal{M} \) of Grothendieck motives. We will mainly need them in §4, when we will define the relative analogue \( \mathcal{M}(S) \) of \( \mathcal{M} \) and show, assuming the conjectures, that it is an abelian semisimple category and the decomposition theorem holds in \( \mathcal{M}(S) \). Then we recall Murre’s conjecture, which we only need later on in §5 and §6, implying the existence of a natural filtration \( F^\bullet \) on the Chow groups of smooth and projective varieties, and explain how this conjecture can be used to fill in part of the gap between \( \mathcal{M} \) and \( CHM \), making it possible to define a noncanonical decomposition of a Chow motive into its cohomology groups. Finally, we recall S. Saito’s unconditional definition of a
filtration, having all the expected categorical properties, except that it is not known to be separated. If it is separated, it coincides with Murre’s, and Murre’s conjecture holds.

It is Saito’s filtration that will be extended, in §5, to the Chow groups of arbitrary quasiprojective varieties. This will be used in the final §6 to prove the decomposition theorem in $CHM(S)$, and in the forthcoming part II, when we will propose an unconditional definition of the intersection Chow groups $ICH^r X$.

**Standard conjectures.**

Before stating the conjectures, we introduce some notation and recall some well known facts on the cohomology of algebraic varieties.

Let $X$ be a smooth projective variety of complex dimension $d$, with a fixed ample divisor class $L \in H^2X$. The Lefschetz theorem asserts that, for $i \leq d$, the $d - i$-th iterated cup product with $L$ is an isomorphism of $H^iX$ to $H^{2d-i}X$

$$L^{d-i} : H^iX \xrightarrow{\cong} H^{2d-i}X$$

For $i \leq d$ we then define the primitive cohomology of $X$ to be

$$P^iX = \ker L^{d-i+1} \subset H^iX$$

We have the hard Lefschetz decomposition of the cohomology of $X$

$$H^iX = \bigoplus_{j \geq 0} L^j P^{i-2j}X$$

if $i \leq d$ and

$$H^iX = \bigoplus_{j \geq i-d} L^j P^{i-2j}X$$

if $i > d$.

**3.1 Definition.** The Lefschetz operator $\Lambda : H^iX \to H^{i-2}X$ relative to the ample class $L$ is defined as follows. Let $\alpha \in H^iX$ and write, using the Lefschetz decomposition

$$\alpha = \sum_j L^j \alpha^{i-2j}$$

with $\alpha^{i-2j} \in P^{i-2j}X$. Then by definition

$$\Lambda \alpha = \sum_j L^{j-1} \alpha^{i-2j}$$

(i.e., $\Lambda$ removes one $L$).

Grothendieck [Gr] proposed the following 2 standard conjectures:
3.2 Standard conjecture of Lefschetz type. The $\Lambda$ operator is algebraic.

3.3 Standard conjecture of Hodge type. The rational quadratic form

$$(\alpha, \beta) \rightarrow (-1)^i \text{tr}(\alpha \cup \beta \cup L^{d-2i})$$

is positive definite on $P^{2i} \cap H^{2i}X_{algebraic}$.

Progress is occasionally made on these conjectures [Ja1] [Sm].

In the proof of the decomposition theorem in $M(S)$, §4, we will need the following simple consequence of the conjecture of Lefschetz type:

3.4 Proposition. Assume the standard conjecture of Lefschetz type. Let $S$ be a smooth quasiprojective variety, and $f : X \to S$ be a smooth projective morphism, with relatively ample divisor class $L \in H^2X$. There exists a cycle $Z \in CH_{\text{dim}X+1}(X \times_S X)$ such that, for every $s \in S$ and fibre $X_s$, $Z|_{X_s \times X_s}$ induces the $\Lambda_s$ operator (relative to the class $L_s = L|_{X_s}$) of that fibre.

Proof. The proof uses a standard “spreading out” argument followed by specialisation. By conjecture 1, there is a cycle $Z_\eta$ on $X_\eta \times X_\eta$ inducing $\Lambda_\eta$. Let $U \subset S$ be a neighbourhood of $\eta$ and $Z_U$ a cycle on $X_U \times_U X_U$ such that $Z_U|_{\eta} = Z_\eta$, and let $Z$ on $X \times_S X$ be its Zariski closure. I claim that for all scheme theoretic points $s \in S$, $Z|_{X_s \times X_s}$ induces $\Lambda_s$. By considering a chain of points

$s \in \overline{s_1} \in \overline{s_2} \in \cdots \in \overline{\eta}$

with

$$\text{cod}_{s_i} s_{i+1} = 1$$

we are reduced to the case of the spectrum $T$ of a discrete valuation ring with central point 0 and generic point $s$, and a morphism $T \to S$. Assuming that $Z|_{X_s \times X_s}$ induces $\Lambda_s$, we need to prove that $Z|_{X_0 \times X_0}$ induces $\Lambda_0$. In this situation, letting $k(s)$ be the function field of $T$, there are well defined specialisation maps

$$CH^iX_s \times X_s \xrightarrow{\text{csp}} CH^iX_0 \times X_0$$

$$H^{2i}X_s \times X_s \xrightarrow{\text{hsp}} H^{2i}X_0 \times X_0$$
Let us recall the construction of the Chow theoretic specialisation homomorphism. We have a diagram

\[
\begin{array}{ccc}
CH^{i-1}X_0 \times X_0 & \xrightarrow{i_*} & CH^i X \times_T X \xrightarrow{j^*} CH^i X_s \times X_s & \rightarrow 0 \\
\downarrow & & \downarrow \ i^! & \\
CH^i X_0 \times X_0 & & \\
\end{array}
\]

where the row is exact. Because \( X_0 \sim 0 \), we can define \( \text{csp}(\alpha) = i^! \alpha' \) where \( \alpha' \in CH^i X \times_T X \) is anything such that \( \alpha'|_{X_s \times X_s} = \alpha \), and the result does not depend on \( \alpha' \). By construction of \( \text{csp} \) then

\[
Z|_{X_0 \times X_0} = \text{csp}(Z|_{X_s \times X_s})
\]

On the other hand clearly \( \Lambda_0 = hsp(\Lambda_s) \), indeed by what we just said about specialisation \( L_0 = \text{csp}(L_s) \), so “removing one \( L \)” specialises to “removing one \( L \)”. Therefore

\[
cl(Z|_{X_0 \times X_0}) = \Lambda_0
\]

\[\square\]

The most important consequence of the standard conjectures is the following [Kl1, Kl2]:

3.5 Theorem. Assuming the standard conjectures, then

- (1) \( AM = M \), in other words homological and numerical equivalence of algebraic cycles are the same.
- (2) The category \( M \) of Grothendieck motives is abelian and semisimple.

Proof. See [Kl1]. \[\square\]

Murre’s conjecture.

The standard conjectures are perfectly adequate in determining the behaviour of Grothendieck motives. There is a large gap between Grothendieck motives and Chow motives, which one begins to appreciate when trying to decompose a Chow motive \( h_X \) into its pieces \( h^i X[-i] \) in an unambiguous way. To address this issue, Murre [Mu] proposed the following:

3.6 Murre’s conjecture. Let \( X \) be a smooth variety of complex dimension \( d \), and \( \pi^i \in H^i X \otimes H^{2d-i} X \subset H^{2d} X \times X \) be the Künneth components of the diagonal. Then:

- (A) The \( \pi^i \) lift to an orthogonal set of projectors \( \Pi^i \in CH^{\dim X}(X \times X) \) such that \( \Delta = \sum \Pi^i \).
(B) The correspondences $\Pi^{2r+1}, \ldots, \Pi^{\dim X}$ act as zero on $CH^r X$.

(C) For each $\nu$, $F^\nu CH^r X = \text{Ker} \Pi^{2r} \cap \cdots \cap \text{Ker} \Pi^{2r-\nu+1}$ is independent of the choice of the $\Pi^r$.

(D) $F^1 CH^r X = CH^r X_{\text{hom}}$ is the group of cycles homologically equivalent to zero.

Jannsen proved [Ja2, pg. 294–296 and 259]:

3.7 Theorem. Assume 3.6. The filtration $F^*$ satisfies the following properties:
(a) $F^0 CH^r X = CH^r X$, $F^1 CH^r X = CH^r X_{\text{hom}}$.
(b) $F^\nu CH^r X \cdot F^{\nu} CH^s X \subset F^{\nu+s} CH^r X$.
(c) If $f : X \to Y$ is a morphism of smooth projective varieties, $f_*$ and $f^*$ respect the filtration (no shifts involved).
(d) Let $\Gamma \in CH_{\dim X} X \times X$ be a correspondence. Assume that $\Gamma$ acts trivially on $H^{2r-\nu} X$. Then
$$\Gamma_* : F^\nu CH^r X \to F^{\nu+1} CH^r X$$

(e) Assuming moreover the standard conjecture of Lefschetz type, $F^{r+1} CH^r X = (0)$. □

We will show momentarily that, assuming the standard conjecture of Lefschetz type, Murre’s $A + B + C + D$ is equivalent to $A + B' + D$ below:

3.8 Conjecture B' (vanishing). Let $X$ be a smooth projective variety and $P \in CH_{\dim X} X \times X$ a projector. Assume $P_* H^i X = 0$ for $i \leq 2r$. Then $P_* CH^r X = (0)$.

3.9 Proposition. Assuming the standard conjecture of Lefschetz type, Murre’s $A + B + C + D$ implies $B'$.

Proof. Assume $A + B + C + D$. By 3.7(d), $P_* F^\nu CH^r X \subset F^{\nu+1} CH^r X$ for all $\nu \leq 2r$. But $P$ is a projector, so $P_* CH^r X = P^2_* CH^r X \subset P_* F^1 CH^r X = P^2_* F^1 CH^r X \subset P_* F^2 CH^r X \ldots \subset F^{r+1} CH^r X = 0$ by 3.7(e). □

To prove that $A + B' + D$ implies $A + B + C + D$ we now make a small digression to discuss the decomposition of Chow motives. The decomposition theorem in $CHM(S)$ in §6 will follow the same basic strategy:

3.10 Definition. A Chow motive $M$ has cohomological degree $\leq m$, resp. $\geq m$ if the cohomology groups
$$H^i M = 0$$
vanish for $i > m$, resp. $i < m$. $M$ has degree exactly $m$ if it has degree $\geq m$ and $\leq m$. 27
3.11 Proposition. Conjecture $B'$ implies

(1) If $M$ has cohomological degree $\leq m$ and $N$ has cohomological degree $> m$, then

$$\text{Hom}_{CH^M}(M, N) = 0$$

(2) If $M$ and $N$ have cohomological degree exactly $m$, then the natural homomorphism

$$\text{Hom}_{CH^M}(M, N) \to \text{Hom}_M(M, N)$$

is an isomorphism.

Proof. Assume for simplicity that $M = (X, P)$ and $N = (Y, Q)$. Let $Z = X \times Y$ and consider the projector

$$CH^i Z \ni \alpha \to \Psi \alpha = Q \circ \alpha \circ P \in CH^i Z$$

To prove both statements, it is enough to show that $\Psi = 0$ if $cl\Psi = 0$, but this is $B'$. □

We draw 2 consequences

3.12 Corollary (decomposition of Chow motives). Assume $A + B'$, then for all smooth projective varieties $X$

(1) There is a noncanonical direct sum decomposition

$$hX = \sum (X, \Pi^i)$$

(2) The monomorphisms

$$\tau_{\leq i} hX = \sum_{m \leq i} (X, \Pi^m) \to hX$$

(where the 1st equality is a definition of $\tau_{\leq i} hX$) are specified up to canonical isomorphism. In particular so are the “subquotients”

$$h^i X[-i] = (X, \Pi^i)$$

(this is a definition of $h^i X[-i]$) specified up to canonical isomorphism. □

3.13 Corollary. $A + B' + D$ implies $A + B + C + D$.

Proof. We can see that

$$F^\nu CH^i X = \text{Hom}_{CH^M}(pt, (\tau_{\leq 2i-\nu} hX)(i))$$

is independent on the $\Pi^i$s, by 3.12. □
Saito’s filtration.

S. Saito [SaS] gave an unconditional definition of a filtration on the Chow groups of smooth projective algebraic varieties over $k$ and proved that, assuming the standard conjectures, it coincides with Murre’s filtration. We now recall Saito’s definition and his results:

3.14 Definition. [SaS] For a smooth projective variety $X$ we define a filtration

$$CH^r X = F^0CH^r X \supset F^1CH^r X \supset \cdots \supset F^\nu CH^r X \supset \cdots$$

in the following inductive way:

1. $F^0CH^r X = CH^r X$.
2. Assume $F^\nu CH^r X$ defined for all $X$ and all $r$. Then we set:

$$F^{\nu+1} CH^r X = \sum_{Y,q,\Gamma} \Gamma_* F^\nu CH^{r-q} Y$$

where $Y$, $q$ and $\Gamma$ range over the following data:

- (2.1) $Y$ is smooth and projective,
- (2.2) $q$ is an integer (the operation yields nothing unless $r - \dim Y \leq q \leq r$, since otherwise $CH^{r-q} Y = 0$),
- (2.3) $\Gamma \in CH^{\dim Y + q}(Y \times X) = Hom_{CM}(Y, X(q))$ is a correspondence such that

$$\Gamma_* H^{2r-2q-\nu} Y \subset N^{r-\nu+1} H^{2r-\nu} X$$

where $N^\bullet$ is Grothendieck’s coniveau filtration (see 3.15 below).

3.15 Reminder. Recall that the coniveau filtration on the cohomology of a smooth projective algebraic variety $X$ is defined as

$$N^p H^i X = \sum_{Y,f} f_* H^{i-2q} Y$$

where the sum ranges over all smooth projective $Y$ with $q = \dim X - \dim Y \geq p$ and morphisms $f : Y \to X$.

3.16 Theorem. [SaS]

1. The filtration defined in 3.14 satisfies the properties b, c, d in 3.7 and $F^1CH^r X = CH^r X_{hom}$.
2. If the filtration is separated, i.e. $F^\nu CH^1 X = 0$ for $\nu$ large, then Murre’s conjecture is true and the filtrations are the same.
3. Assuming the standard and Murre’s conjectures, the filtrations are the same (in particular they are separated). □
3.17 Remark. It would have been possible to define $F^\bullet$ just as in 3.14, but replacing 3.14(2.3) with the easier condition

$$0 = \Gamma_* H^{2r-2q-\nu}Y \to H^{2r-\nu}X$$

3.16 would still be true (in fact, slightly easier to prove) for this filtration.

The following property is not stated explicitly in [SaS] and will be used in §5.

3.18 Proposition. For $X$, $Y$ smooth projective and $f : X \to Y$ a morphism,

$$f_* : CH_s X \to CH_s Y$$

is strictly compatible with the $F^\bullet$ filtration as defined in 3.14.

Proof. Choose a diagram

$$\begin{array}{ccc}
Z & \rightarrow & X \\
\downarrow & \pi & \downarrow f \\
& \searrow & \searrow \\
& & Y
\end{array}$$

where $Z$ is smooth projective, $i : Z \hookrightarrow X$ is a closed embedding, and $\pi : Z \to Y$ a generically finite morphism of degree $d$. Let $\alpha \in F^\nu CH^r X$. Then

$$\alpha = \frac{1}{d} \pi_* \pi^* \alpha = \frac{1}{d} f_* i_* \pi^* \alpha$$

It is clear that

$$i_* \pi^* \alpha \in F^\nu CH^r X$$

□

4. Grothendieck motives over a base, semisimplicity and decomposition

This section is divided into 3 subsections. In the first, for the convenience of the reader and to fix the notation, we recall the notion of perverse sheaves and the statement of the topological decomposition theorem. The standard references are [Bo], [BBD]. For the expert, we say right away that we found it convenient to use Deligne’s convention for perverse sheaves, because better suited for taking direct images under a closed embedding, and Borel’s convention for intersection complexes. With our conventions, therefore, intersection complexes are not perverse (but a suitable shift is). In the second subsection we define a category $\mathcal{M}(S)$, which we call the category of Grothendieck motives over a variety $S$: this is the correct analogue of the category of Grothendieck motives over the point and is built precisely in order to have a faithful realisation in the graded category of perverse sheaves. In the third subsection, assuming the standard conjectures, we prove that $\mathcal{M}(S)$ is abelian and semisimple and, as a consequence, we derive a decomposition theorem in $\mathcal{M}(S)$ which realizes to the topological decomposition theorem.
Perverse sheaves and the topological decomposition theorem.

In this subsection, we recall the theory of perverse sheaves and the topological decomposition theorem. The standard reference for this material is [BBD]. For ease of notation and terminology, we will assume that $k = \mathbb{C}$ and refer the reader to the original source for the language suitable to the étale situation.

4.1 Definition. Let $\mathcal{D}$ be a triangulated category. A $t$-structure on $\mathcal{D}$ is a pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ of full subcategories of $\mathcal{D}$, satisfying the following axioms:

1. $\mathcal{D}^{\leq 0}[1] \subset \mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 0} \subset \mathcal{D}^{\geq 0}[1]$.
2. $\text{Hom}(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0}) = 0$.
3. For every object $K$ of $\mathcal{D}$, there is a (necessarily unique up to canonical isomorphism) triangle

$$K' \to K \to K'' \to$$

with $K' \in \mathcal{D}^{\leq 0}$, $K'' \in \mathcal{D}^{\geq 0}$.

The assignment $K$ to $K' = \tau_{\leq 0}K$ is functorial and the corresponding functor is called the truncation functor relative to the $t$-structure. $\tau_{\leq m}K$ is defined to be $(\tau_{\leq 0}(K[m]))[-m]$, similarly $\tau_{\geq m}$, and $\mathcal{H}^m(\cdot) = (\tau_{\leq m}\tau_{\geq m}(\cdot))[m]$ is the $m$-th cohomology functor relative to the $t$-structure.

The main theorem [BBD] about $t$-structures asserts that the heart $\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ is an abelian category.

We now come to the most important example of $t$-structure, the perverse $t$-structure on $\mathcal{D}_{cc}(S)$, but first:

4.2 Definition. Let $S$ be a quasi projective variety over a field $k$. A good stratification of $S$ is a stratification

$$S = \coprod T_k$$

where $T_k$ is a Zariski locally closed subset of complex dimension $k$, satisfying the following axioms:

1. each stratum $T_k$ is smooth,
2. the stratification is topologically normally locally trivial.

From now on, we will assume that all varieties $S$ are equipped with a good stratification.

4.3 Notation. If $S = \coprod_k T_k$ is a good stratification, we denote $i_{T_k} : T_k \to S$ the inclusion and $S_k = \coprod_{h \leq k} T_h$ the Zariski closure of $S_k$.

4.4 Definition.

1. Let $\mathcal{T} = \{T_k\}$ be a good stratification

$$\mathcal{D}_\mathcal{T}(S) = \{K \in \mathcal{D}^b(S) \mid K|_{T_k} \text{ is cohomologically locally constant } \forall k\}$$
(2) The bounded derived category of cohomologically constructible sheaves is defined as
\[ D_{bc}^b(S) = \bigcup D_T(S) \]
the union being taken over all good stratifications of \( S \).

From now on, when dealing with a sheaf \( K \in D_{bc}^b(S) \), in connection with a preexisting good stratification \( T = \{ T_k \} \), we will assume that \( K \) is cohomologically locally constant along all strata \( T_k \) of \( T \).

4.5 Definition. The perverse \( t \)-structure on \( D_{bc}^b(S) \) is defined as follows
\[
\begin{align*}
pD_{\leq 0} &= \bigcup pD_T^{\leq 0} \\
pD_{\geq 0} &= \bigcup pD_T^{\geq 0}
\end{align*}
\]
the union being taken over all good stratifications \( T \), where
\[
\begin{align*}
pD_T^{\leq 0} &= \{ K \in D_T \mid \mathcal{H}^i i_T^* K = 0, \ i > -k \} \\
pD_T^{\geq 0} &= \{ K \in D_T \mid \mathcal{H}^i i_T^! K = 0, \ i < -k \}
\end{align*}
\]

It is well known that the above data define a \( t \)-structure, whose heart = \( Perv(S) \) is the category of perverse sheaves on \( S \). We will denote the truncation, resp. cohomology functors of the perverse \( t \)-structure with the symbol \( p\tau_{\leq 0} \), resp. \( p\mathcal{H}^m \).

The most important construction in the theory of perverse sheaves is that of the intersection complexes:

4.6 Intersection complexes. Let \( T \) be a good stratification and \( V \) a rational local system on the largest stratum \( T_d \). The intersection complex \( ICV \) is characterised by the properties
\[
\begin{align*}
(0) \quad & ICV|_{T_d} = V \\
(-) \quad & \mathcal{H}^i i_{T_k}^* ICV = 0, \ i \geq d - k \ (k < d) \\
(+) \quad & \mathcal{H}^i i_{T_k}^! ICV = 0, \ i \leq d - k \ (k < d)
\end{align*}
\]
It follows immediately from the characterisation just given that the shift \( ICV[d] \) is a perverse sheaf on \( S \).

It is important to understand that \( ICV[d] \) is not characterised by being a perverse sheaf and restricting to \( V[d] \) on the largest stratum. In fact, there are lots and lots of perverse sheaves which restrict to \( V[d] \) on \( T_d \), and \( ICV[d] \) is built to be as much in the middle of \( Perv(S) \) as possible.
The intersection cohomology of $S$ is defined as

$$IH^m S = H^m \mathcal{IC}Q_{T_d}$$

Similarly, if $V_k$ is a local system on the stratum $T_k$ of dimension $k$, the intersection complex $\mathcal{IC}V_k$ is a complex supported on the Zariski closure $S_k$ characterised by the properties

\begin{align*}
(0) & \quad \mathcal{IC}V_k|T_k = V_k \\
(-) & \quad \mathcal{H}^i i^*_T \mathcal{IC}V = 0, \quad i \geq k - h \quad (h < k) \\
(+) & \quad \mathcal{H}^i i^!_T \mathcal{IC}V = 0, \quad i \leq k - h \quad (h < k)
\end{align*}

It follows immediately from the characterisation just given that $\mathcal{IC}V_k[k]$ is a perverse sheaf on $S$.

4.7 Topological decomposition theorem. Let $X$ be a smooth variety and $f : X \to S$ be a projective morphism.

(1) There is a noncanonical direct sum decomposition

$$Rf_* Q_X \cong \sum p R^m f_* Q_X[-m]$$

in $\mathcal{D}^b_{cc}(S)$, where $p R^m f_* Q_X$ denotes the $m$th perverse cohomology of $Rf_* Q_X$. The decomposition itself is not unique, but the subobjects

$$p \tau_{\leq m} Rf_* Q_X = \sum_{i \leq m} p R^i f_* Q_X[-i]$$

are uniquely specified.

(2) Let $\mathcal{T} = \{T_k\}$ be a good stratification with the property that $p R^m f_* Q_X \in \mathcal{D}_{\mathcal{T}}(S)$. There are local systems $V_k^m$ on $T_k$ and a canonical isomorphism

$$p R^m f_* Q_X = \sum_k \mathcal{IC}V_k^m[k]$$

4.8 Remark. The uniqueness of the subobjects $p \tau_{\leq m} Rf_* Q_X = \sum_{i \leq m} p R^i f_* Q_X[-i]$ is an immediate consequence of the axiom $\text{Hom}(\mathcal{D}^{\leq 0}, \mathcal{D}^{> 0}) = 0$ for $t$-structures.
Grothendieck motives over $S$.

Let $\overline{M} \in AM(S)$ and $M = \text{real} \overline{M} \in D^b_{cc}(S)$ be its realisation. Recall that $AM(S)$ is the category of homological motives over $S$, constructed in §2. As a consequence of the topological decomposition theorem, $M$ is naturally equipped with an increasing filtration (perverse Leray filtration)

$$\ldots \subset L_m M \subset L_{m+1} M \subset \ldots \subset M$$

defined as

$$L_m M = \tau_{\leq m} M$$

with $gr^L_m M = p^H^m(M)[−m]$. Let $u : M \rightarrow N$ be a morphism in $AM(S)$ and $\overline{u} = \text{real} u : M \rightarrow N$ be its realisation in $D^b_{cc}(S)$. Because real and $p^H^m$ are functors, we get a system of compatible morphisms

$$L_m u : L_m M \rightarrow L_m N$$

To elaborate more on this point, choose decompositions

$$M = \sum p^H^m M[−m]$$

$$N = \sum p^H^m N[−m]$$

With respect to these decompositions, $u = \sum u^l_m$, where $u^l_m : p^H^m M[−m] \rightarrow p^H^l N[−l]$. Here $u^l_m$ can be regarded as an extension of perverse sheaves

$$u^l_m \in \text{Hom}_{D^b_{cc}}(p^H^m M[−m], p^H^l N[−l]) = \text{Ext}^{m−l}_{Perv_S}(p^H^m H, p^H^l K)$$

hence $u^l_m = 0$ if $m < l$, corresponding to the fact that we have morphisms $L_m u : L_m M \rightarrow L_m N$, and $u$ can be therefore visualised as an upper triangular matrix

$$u = \begin{pmatrix} \vdots & \vdots & \vdots \\ \vdots & u^1_1 & u^1_2 & \ldots \\ \vdots & 0 & u^2_2 & \ldots \\ \vdots & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The above considerations imply that, passing to the corresponding graded objects, we have a perverse realisation functor

$$p \text{ real} : AM(S) \rightarrow \text{grPerv}(S)$$
4.9 Definition. The category $\mathcal{M}(S)$ of Grothendieck motives over $S$ has the same objects as $A\mathcal{M}(S)$ and morphisms

$$\text{Hom}_{\mathcal{M}(S)}(M, N) = \text{Im}(\text{Hom}_{A\mathcal{M}(S)}(M, N) \rightarrow \text{Hom}_{gr\mathcal{Perv}(S)}(p\text{ real } M, p\text{ real } N))$$

Semisimplicity and decomposition in $\mathcal{M}(S)$.
In this subsection we assume that desingularisations of varieties over $k$ exist.

4.10 Notation.
(1) In this subsection only, underlined capital letters $M$ denote objects in $\mathcal{M}(S)$, while non underlined letters $M$ denote the corresponding realisation in $gr\mathcal{Perv}(S)$. The same letter will denote a morphism in $\mathcal{M}(S)$ or its realisation in $gr\mathcal{Perv}(S)$: the context will always make it clear which is meant. When we want to specifically emphasise the realisation functor, we call it $p\text{ real } : \mathcal{M}(S) \rightarrow gr\mathcal{Perv}(S)$.

(2) If $X$ is a smooth variety and $f : X \rightarrow S$ a projective morphism, we denote

$$pRf_*Q_X$$

the corresponding object in $\mathcal{M}(S)$.

We will prove the following results:

4.11 Theorem. Assuming the standard conjectures, the category $\mathcal{M}(S)$ is abelian and semisimple.

4.12 Decomposition theorem in $\mathcal{M}(S)$. Assume the standard conjectures. Let $X$ be a smooth variety and $f : X \rightarrow S$ be a projective morphism.

(1) There is a canonical direct sum decomposition in $\mathcal{M}(S)$

$$pRf_*Q_X \cong \sum pR^m f_*Q_X[-m]$$

where $pR^m f_*Q_X[-m]$ denotes an object in $\mathcal{M}(S)$, together with a given isomorphism in $gr\mathcal{Perv}(S)$

$$p\text{ real } pR^m f_*Q_X[-m] \cong pR^m f_*Q_X[-m]$$

(2) There is a canonical direct sum decomposition:

$$pR^m f_*Q_X[-m] = \sum_k IC^{\cdot}_V^m[k - m]$$

where $V_k$ is a local system on a Zariski locally closed subvariety $T_k \subset S$ and $IC^{\cdot}_V^m[k - m]$ denotes an object in $\mathcal{M}(S)$, together with a given isomorphism in $gr\mathcal{Perv}(S)$

$$p\text{ real } IC^{\cdot}_V^m[k - m] \cong IC^{\cdot}_V^m[k - m]$$
4.13 Remark. The decomposition is unique (contrary to 4.7) because of the way \( \mathcal{M}(S) \) is built as a faithful subcategory of \( \text{grPerv}(S) \).

4.11 is an immediate consequence of the following (as is the case for motives over a point, compare [Kl1]) proposition 4.14 which will be shown, together with theorem 4.12, at the very end of this subsection.

4.14 Proposition. Let \( X \) be a smooth variety and \( f : X \to S \) a projective morphism. Assuming the standard conjectures, \( \text{End}_{\mathcal{M}_S}^p Rf_*Q_X \) is a semisimple ring, finite dimensional over \( \mathbb{Q} \).

The proof of the proposition will depend on the following:

4.15 Lemma (decomposition mechanism). Let \( \mathcal{A} \) be an abelian semisimple category, \( \mathcal{B} \) an additive category and \( \mathcal{A} \subset \mathcal{B} \) a fully faithful embedding. Assume given objects \( A, A' \) of \( \mathcal{A} \) and \( B \) of \( \mathcal{B} \), and morphisms

\[
A \xrightarrow{i} B \xrightarrow{i^*} A'
\]

denote \( a = i^*i_* : A \to A' \): by assumption this is a morphism in \( \mathcal{A} \).

There is a non unique projector \( \beta : B \to B \), with image in \( B \), and a natural isomorphism

\[
\varphi(\beta) : \text{Im} \beta \xrightarrow{\cong} \text{Im} a
\]

Proof. Let \( V = \text{Im} a \), \( t : A \to V \) and \( t' : V \to A' \) the natural maps. Choose \( s : V \to A, s' : A' \to V \) such that

\[
ts s = \text{Id}_V, \quad s' t' = \text{Id}_V
\]

and let \( \alpha = ss' : A' \to A \). It is immediate to verify that \( \alpha a \alpha = \alpha \) and \( a \alpha a = a \).

Let now \( \beta = i_* \alpha i^* \), it is immediate that \( \beta^2 = \beta \).

Claim. With the diagram

\[
\begin{array}{ccc}
B & \xrightarrow{\beta} & B \\
\downarrow{s^*i_*} & & \downarrow{i_*s} \\
V & & \\
\end{array}
\]

\( V = \text{Im} \beta \). Let \( X \) be any object of \( \mathcal{B} \). We wish to check that \( \text{Hom}(X, B) \), resp. \( \text{Hom}(B, X) \) is an image of \( \circ \beta \), resp. \( \beta \circ \_ \), in the category of abelian groups, via the diagram

\[
\begin{array}{ccc}
\text{Hom}(X, B) & \xrightarrow{} & \text{Hom}(X, B) \\
\downarrow & & \downarrow \\
\text{Hom}(X, V) & & \\
\end{array}
\]
resp. the diagram

\[
\begin{array}{ccc}
\text{Hom}(B, X) & \rightarrow & \text{Hom}(B, X) \\
\downarrow & & \downarrow \\
\text{Hom}(V, X) & & \\
\end{array}
\]

In other words, we may assume that \( \mathcal{B} \) is an abelian category. The claim then follows from the observation that \( s'i^* \) is surjective (indeed \( s'i^*i_* = t \) is surjective) and \( i_*s \) is injective (indeed \( i^*i_*s = t' \) is injective). \( \square \)

4.16 Example (conic bundles). As an example, we use the decomposition mechanism to very briefly outline the calculation of the Chow motive of a conic bundle, following [Be].

Let \( X \) be a smooth 3-fold, \( f : X \to S \) a conic bundle structure. We assume, for simplicity, that the discriminant \( \Delta \subset S \) is a smooth divisor, denote \( Y = f^{-1}\Delta \) and \( i : Y' \to X \) the normalisation of \( Y \). In an obvious way \( Y' \to \Delta \) factors through a \( \mathbb{P}^1 \)-bundle \( p : Y' \to D \), with a distinguished section (the conductor) \( s : D \to Y' \), by which we may think \( D \subset Y' \), and an étale double cover \( D \to \Delta \). Let \( \tau : D \to D \) be the involution associated to this double cover.

**Step 1.** Let

\[
a = i^*i_* : h_S Y'(-1) \to h_S Y'
\]

in \( CHM(S) \), then

\[
a = c_1(N\Delta S) - s_*(s^* - \tau_*s^*)
\]

where, abusing notation slightly, \( N\Delta S \) is the pull back to \( Y \) of the normal bundle \( N\Delta S \) of \( \Delta \) in \( S \). Indeed, let \( \Gamma = \Gamma_i \subset Y' \times X \) be the graph of \( i \), and \( \gamma \in CH_2(Y' \times X) \) its class. Then \( i_* = \gamma \) and \( i^* = t' \gamma \), while by definition \( a = p_{13*}(p_{23}^*t' \gamma \cdot p_{12}^*\gamma) \) (definition 2.1). We can calculate the intersection product \( p_{23}^*t' \gamma \cdot p_{12}^*\gamma \) with the help of the following fibre square diagram

\[
\begin{array}{ccc}
D \coprod Y' & \rightarrow & Y' \times Y' \\
\downarrow_{(s,s\tau)} \coprod \Delta_Y' & & \downarrow_{(1,i) \times 1} \\
Y' \times Y' & \rightarrow & Y' \times X \times Y'
\end{array}
\]

By the excess intersection formula then

\[
a = c_1(E) + s_*\tau_*s^*
\]

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where $E$ is the excess bundle on $Y'$, defined by the exact sequence

$$0 \to T_{Y'} \to i^*T_X \to E \to 0$$

Finally, it is easy to convince oneself that $E = N_\Delta S(-D)$, giving $c_1(E) = c_1(N_\Delta S) - D = c_1(N_\Delta S) - s_*s^*$, which proves our formula.

**Step 2.** Now $Y' \to D$ is a $\mathbb{P}^1$-bundle, therefore we have an isomorphism

$$(p^*, s_*) : h_SD \oplus h_SD(-1) \to h_SY'$$

Using this isomorphism, it is quite easy to see that

$$a : h_SD(-1) \oplus h_SD(-2) \to h_SD \oplus h_SD(-1)$$

can be written in matrix form as

$$a = \begin{pmatrix} c_1(N_\Delta S) & 1 - \tau^* \\ 0 & c_1(N_\Delta S \otimes N^\vee_D Y' \otimes \tau^* N_D Y') \end{pmatrix}$$

**Step 3.** From the previous step and the decomposition mechanism we can see, for instance, the classical result stating that the Prym motive $(h^1D, 1 - \tau^*)$ is a direct summand of the intermediate motive $h^3X(1)$.

Before we embark in the proof of proposition 4.14 and Theorem 4.12, we need a definition and a lemma.

**4.17 Definition.** Let $X$ be a smooth variety, $f : X \to S$ a projective morphism. An **equisingular stratification** of $f$ is a pair $(\mathcal{T}, \mathcal{Y}^1 \rightrightarrows \mathcal{Y}^0 \to \mathcal{Y})$ where:

1. $\mathcal{T} = \{T_k\}$ is a good stratification of $S$;
2. $\mathcal{Y} = \{Y_k\}$ and $Y_k$ is defined by the fibre square

$$\begin{array}{ccc} Y_k & \longrightarrow & X \\ \downarrow & & \downarrow \\ T_k & \longrightarrow & S \end{array}$$

3. $\mathcal{Y}^1 = \{Y^1_k\}$, $\mathcal{Y}^0 = \{Y^0_k\}$ and $Y_k^1 \rightrightarrows Y_k^0 \to Y_k$ is a truncated simplicial resolution.

The above data are subjected to the following condition:

4. The compositions $Y_k^1 \to T_k$, $Y_k^0 \to T_k$ are all smooth (not necessarily equidimensional). In particular, for all $t \in T_k$

$$Y_k^1 \rightrightarrows Y_k^0 \to Y_{k,t}$$

is a truncated simplicial resolution.

It is a consequence of our assumption on the existence of resolutions of singularities, that equisingular stratifications of $f : X \to S$ exist. We will need the following:
4.18 Key lemma. Let $X$ be a smooth variety, $f : X \to S$ a projective morphism. Fix an equisingular stratification $(\mathcal{T}, \mathcal{Y}_1 \supseteq \mathcal{Y}_0 \to \mathcal{Y})$ of the morphism $f$. Let $T_0$ be the smallest stratum of $\mathcal{T}$. We have a natural isomorphism (cf. the notation in the statement of the topological decomposition theorem 4.7)

$$i_{T_0*}V_0^m = \text{Im}(R^m f_* i_{Y_0*} i_{Y_0!} Q_X \to R^m f_* Q_X)$$

Proof. Step 1. First of all, if $h < k$ the natural maps

$$\mathcal{H}^j i_{T_h*} i_{T_h!} IC V^m_k \to \mathcal{H}^j IC V^m_k$$

are zero for all $j$.

**Warning:** this is not saying that $i_{T_h*} i_{T_h!} IC V^m_k \to IC V^m_k$ is the zero map in $\mathcal{D}^{bc}_c(S)$.

Indeed: $\mathcal{H}^j i_{T_h!} IC V^m_k = 0$ for $j \leq k - h$, and $\mathcal{H}^j i_{T_h*} IC V^m_k = 0$ for $j \geq k - h$.

Step 2. We have a fibre square

$$
\begin{array}{ccc}
Y_0 & \xrightarrow{i_{Y_0}} & X \\
\downarrow & & \downarrow f \\
T_0 & \xrightarrow{i_{T_0}} & S
\end{array}
$$

From it, using the proper base change theorem and the topological decomposition theorem, we derive the following commutative diagram

$$
\begin{array}{ccc}
Rf_* i_{Y_0*} i_{Y_0!} Q_X & \to & Rf_* Q_X \\
\downarrow & & \downarrow \\
i_{T_0*} i_{T_0!} Rf_* Q_X & \to & Rf_* Q_X \\
\downarrow & & \downarrow \\
\sum_{k>0} \sum_m i_{T_0*} i_{T_0!} IC V^m_k [k-m] & \oplus & \sum_{k>0} \sum_m IC V^m_k [k-m] \\
\sum_m V_0^m [-m] & \oplus & \sum_m V_0^m [-m]
\end{array}
$$

The result then follows from step 1, upon taking $\mathcal{H}^m$ of both sides of the bottom portion of the diagram. □
Proof of proposition 4.14 and theorem 4.12.

Fix an equisingular stratification \((\mathcal{T}, \mathcal{Y}^1 \Rightarrow \mathcal{Y}^0 \Rightarrow \mathcal{Y})\) of the morphism \(f : X \to S\). The proof is by induction on \(\dim S\) and the number of strata in \(\mathcal{T}\). The basis for the induction is solid because:

(a) If \(\dim S = 0\), \(\mathcal{M}(S) = \mathcal{M}\) is semisimple by 3.5, proven in [Kl1]. The decomposition theorem 4.12 in this case can be proven as follows. Again in [Kl1] it is shown that, if the standard conjecture of Lefschetz type holds, then there are cycles \(\Pi^i\) representing the Künneth components \(\pi^i\) of the diagonal \(\Delta \subset X \times X\), and the sought for decomposition is then

\[ X = \sum (X, \Pi^i) \]

(b) If \(\mathcal{T}\) has only one stratum, note that, by definition of equisingular stratification, this happens if and only if \(f : X \to S\) is a smooth morphism. By 3.4, there is a cycle \(Z\) on \(X \times_S X\) inducing the \(\Lambda\) operator on each fibre. The same proof as in [Kl1] will then show that \(\text{End}_{\mathcal{M}(S)} X\) is a semisimple ring, finite dimensional over \(\mathbb{Q}\). Then again, as in [Kl1], there are classes \(\Pi^i \in CH_{\dim X} X \times_S X\) inducing on each fibre the Künneth components of the diagonal of that fibre, and one can get the decomposition as above

\[ X = \sum (X, \Pi^i) \]

Let now \(T_0\) be the smallest stratum. We apply the decomposition mechanism 4.15 with the following setup. \(A = \mathcal{M}(T_0)\), which is by inductive assumption abelian and semisimple, \(B = \mathcal{M}(S)\). We take

\[ A = \text{Cok}(^pRf_*D_{\mathcal{Y}^1}(\dim X) \to ^pRf_*D_{\mathcal{Y}^0}(\dim X)) \]
\[ A' = \text{Ker}(^pRf_*Q_{\mathcal{Y}^0} \to ^pRf_*Q_{\mathcal{Y}^1}) \]

and

\[ B = ^pRf_*Q_X \]

\(A\) and \(A'\) are objects of \(\mathcal{M}(T_0)\), but if we like we can think of them as being in \(\mathcal{M}(S)\) via the obvious inclusion \(\mathcal{M}(T_0) \subset \mathcal{M}(S)\). There are obvious maps \(i_* : A \to B\) and \(i^* : B \to A'\). We will not need this, but we still like to say that the assignment \(X\) to \(A\), resp. \(A'\) is functorial, in other words it does not depend on the choice of the equisingular stratification. As we anticipated, we will denote \(A, A'\) and \(B\) the realisations in \(gr\mathcal{Perv}(S)\).

Claim 1. Let \(R^m = \mathcal{H}^m A, R^{m,*} = \mathcal{H}^m A'\). Then there is a natural isomorphism

\[ V_0^m = \text{Im}(\mathcal{H}^m \alpha : R^m \to R^{m,*}) \]
In fact we know from the key lemma that

$$V_0^m = \text{Im}(\mathcal{H}^m i_{T_0}^! Rf_*Q_X \to \mathcal{H}^m i_{T_0}^* Rf_*Q_X)$$

Since $\mathcal{H}^m i_{T_0}^! Rf_*Q_X$ (resp. $\mathcal{H}^m i_{T_0}^* Rf_*Q_X$) has weights $\geq m$ (resp. $\leq m$), we have

$$i_{T_0*}V_0^m = \text{Im}(gr_m^W \mathcal{H}^m i_{T_0}^! Rf_*Q_X \to gr_m^W \mathcal{H}^m i_{T_0}^* Rf_*Q_X)$$

The claim now follows from the identifications

$$gr_m^W \mathcal{H}^m i_{T_0}^! Rf_*Q_X \to gr_m^W \mathcal{H}^m i_{T_0}^* Rf_*Q_X$$

The decomposition mechanism, together with the claim, provides a projector $\beta \in \text{End}_M S_p Rf_*Q_X$ s.t., upon setting $\underline{M} = \text{Ker} \beta$, $\underline{V} = \text{Im} \beta$, we have

$$p Rf_*Q_X = \underline{M} \oplus \underline{V}$$

whose realisations in $gr \mathcal{Perv}(S)$ are

$$V = \sum_m i_{T_0*} V_0^m [-m]$$

$$M = \sum_{k>0} \sum_m IC V_k^m [k - m]$$

Let us now prove 4.14, i.e., let us show that $\text{End}_M S_p Rf_*Q_X$ is a semisimple ring, finite dimensional over $\mathbb{Q}$. According to the above decomposition

$$\text{End}_M S_p Rf_*Q_X = \text{End}_M \underline{M} \oplus \text{End}_M \underline{V}$$

Now $\text{End}_M \underline{V}$ is semisimple finite dimensional over $\mathbb{Q}$ by inductive assumption on $\text{dim} S$. The same is true of $\text{End}_M \underline{M}$ by inductive assumption on the number of strata, by the

Claim 2.

$$\text{End}_M \underline{M} = \text{End}_{M(S \setminus T_0)}(\underline{M}|_{S \setminus T_0})$$

Indeed we have a diagram

$$\text{End}_M(\underline{M}) \xrightarrow{\text{inj}} \text{End}_{gr \mathcal{Perv} S}(\sum_{m,k>0} IC V_k^m)$$

$$\text{End}_{M(S \setminus T_0)}(\underline{M}|_{S \setminus T_0}) \xrightarrow{\text{inj}} \text{End}_{gr \mathcal{Perv}(S \setminus T_0)}(\sum_{m,k>0} IC V_k^m|_{S \setminus T_0})$$
where surjectivity of the left vertical arrow follows from the fact that cycles can be closed:

$CH \times_S X \xrightarrow{\text{surj}} H^{BM}_X \times_S X$

This finishes the proof of 4.14, because $\text{End}_{\mathcal{M}} X$ is direct sum of 2 rings, each of which is semisimple and finite dimensional over $\mathbb{Q}$.

Finally, we shall now prove the decomposition theorem 4.12. By induction on the number of strata, we may assume that the decomposition theorem holds over $S \setminus T_0$:

$pRf_\ast Q_X|_{S \setminus T_0} = \sum_{k>0} \sum_{m} \mathcal{IC}V^m_k [k-m]$

To give such a decomposition is equivalent to giving the projectors $\Pi^m_k \in \text{End}_{\mathcal{M}}(S \setminus T_0)$ down to $\mathcal{IC}V^m_k$. Since by claim 2 $\text{End}_{\mathcal{M}} \mathcal{M} = \text{End}_{\mathcal{M}}(S \setminus T_0)(\mathcal{M}|_{S \setminus T_0})$, this also decomposes $pRf_\ast Q_X$ over $S$ into the desired pieces.

5. Filtrations for quasiprojective varieties

Throughout this section we assume that desingularisations of varieties over $k$ exist. For a quasiprojective variety $X$ we will put a canonical filtration on its rational Chow group $CH_s X$ so that some functorial properties are satisfied, theorem 5.1. The filtration is shown to satisfy additional properties if one assumes the conjectures of Grothendieck and Murre, 5.2.

As an application, assuming these conjectures, we show that the projectors in §4 can be lifted to an orthogonal set of projectors in the Chow group of relative self correspondences, 5.10. This is done by showing that the map

$CH_{\dim X} \times_S X \to \text{End}_{\mathcal{M}} pRf_\ast Q_X$

is a surjective homomorphism with nilpotent kernel.

5.1 Theorem. For a quasi-projective variety $X$, there is a decreasing finite filtration (the canonical filtration) on its Chow group $CH_s X$

$CH_s X = F^0 CH_s X \supset F^1 CH_s X \supset F^2 CH_s X \supset \cdots$

subject to the following conditions:

(i) If $f : X \to Y$ is a proper map of quasi-projective varieties, then the induced map $f_\ast : CH_s X \to CH_s Y$ respects the filtrations, i.e.

$f_\ast F^\nu CH_s X \subset F^\nu CH_s Y$
for each \( \nu \). If \( f \) is proper and surjective, then \( f_* F^\nu CH_s X = F^\nu CH_s Y \) (in other words, the surjection \( f_* \) is strictly compatible with \( F^\bullet \)).

(ii) If \( j : U \to X \) is an open immersion of quasi-projective varieties, then the restriction \( j^* : CH_s X \to CH_s U \) is strictly compatible with \( F^\bullet : j^* F^\nu CH_s X = F^\nu CH_s U \).

(iii) For a smooth projective \( X \)

\[
F^1 CH_s X = CH_s(X)_{hom} = \text{Ker}(cl : CH_s X \to H_{2s} X)
\]

where \( cl \) is the cycle class map.

(iv) The external product map \( CH_s X \otimes CH_t Y \to CH_{s+t}(X \times Y) \) respects \( F^\bullet \): if \( z \in F^\nu CH_s X \) and \( w \in F^\mu CH_t Y \) then \( z \times w \in F^{\nu+\mu} CH_{s+t}(X \times Y) \).

(v) The internal product respects \( F^\bullet \): if \( X \) smooth quasi-projective equidimensional, \( z \in F^\nu CH_s X \) and \( w \in F^\mu CH_t X \) then \( z \cdot w \in F^{\nu+\mu} CH_{s+t-dim X} \).

(vi) Refined Gysin maps respect \( F^\bullet \). Let \( i : X \to Y \) be a regular embedding of codimension \( d \), and

\[
\begin{array}{ccc}
X' & \longrightarrow & Y' \\
\downarrow & & \downarrow \\
X & \rightarrow & Y \\
\end{array}
\]

be a Cartesian square where \( Y' \) is an arbitrary quasiprojective variety and \( Y' \to Y \) is an arbitrary map. Then the refined Gysin map [Fu, Chap. 6]

\[
i^! : CH_s Y' \to CH_{s-d} X'
\]

respects \( F^\bullet \).

(vii) If \( X, Y \) are quasi-projective varieties, \( X \) equidimensional, and \( p_Y : X \times Y \to Y \) is the projection, then the map \( p_Y^* : CH_s Y \to CH_{s+dim X}(X \times Y) \) respects \( F^\bullet \).

(viii) Let \( W, X \) be smooth projective equidimensional, \( \Gamma \in CH_{dim X - i}(W \times X)_{hom} \), and

\[
\Gamma_* : CH_{s+i} W \to CH_s X
\]

the induced map (by (i), (v) and (vii), \( \Gamma_* \) respects \( F^\bullet \)). Then the map induces zero on the \( F \)-graded pieces:

\[
Gr_F \Gamma_* = 0 : Gr_F CH_{s+i} W \to Gr_F CH_s X
\]
5.2 Remarks.

(1) As already noted in 3.14 and 3.18, S. Saito defined a filtration $F^\bullet$ on $CH_s(X)$ for $X$ smooth projective satisfying the conditions (i), (iii), (v), (vii) (where $X, Y$ are smooth projective) and (viii).

If we further assume Murre’s conjecture, it follows that Saito’s filtration is separated, i.e. for any $X$ and $s$, one has $F^\nu CH_sX = 0$ for $\nu$ large.

In the proof of Theorem 5.1, we take Saito’s filtration and show that it uniquely extends to a filtration for $X$ quasiprojective, so that the conditions (i)-(viii) are satisfied.

(2) (iv) and (v) follow from (vi) and (vii).

(3) We will use the following properties of refined Gysin maps [Fu, Chap. 6]: compatibility with proper push forwards, compatibility with flat pull-backs, the excess intersection formula, and the fact: if $i$ is of codimension one, namely if $X \subset Y$ is a Cartier divisor, then $i^!$ coincides with intersection with $X$.

(4) There is an interpretation of the filtrations in terms of mixed motives; we will not need this.

5.3 Theorem. Assuming Grothendieck’s and Murre’s conjectures, the filtration in theorem 5.1 satisfies in addition the following properties:

(ix) For any quasiprojective variety

$$F^1 CH_sX = CH_s(X)_{hom} = \text{Ker} (cl : CH_sX \to H^{BM}_{2s} X)$$

where $cl : CH_sX \to H^{BM}_{2s} X$ is the cycle map into Borel-Moore homology.

(x) For each $X$, one has $F^\nu CH_sX = 0$ for $\nu$ large.

To show Theorem 5.1, we take Saito’s filtration $CH_sX$ for $X$ smooth projective and attempt to extend it to $X$ general.

First consider the case $X$ is smooth quasiprojective. Take a smooth projective variety $\overline{X}$ and an open immersion $j : X \hookrightarrow \overline{X}$. It induces the surjective map

$$j^* : CH_s\overline{X} \to CH_sX$$

and $CH_sX$ is given the induced filtration: $F^\nu CH_sX = j^* F^\nu CH_s\overline{X}$. This filtration is independent of the choice of a compactification. In fact, let $j' : X \to \overline{X}'$ be another smooth compactification. Since $\overline{X}$ and $\overline{X}'$ are dominated by a third compactification, one may assume that there is a map $f : \overline{X}' \to \overline{X}$ such that $f \circ j' = j$. 

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The diagram

```
CH_sX'  ↓
f_* ↓         ↓
CH_sX  ←         ←

CH_sX
```

commutes. By the strictness of \( f_* \) (5.1 (i) for surjective maps of smooth projective varieties) one has

\[ f_* F^\nu CH_sX' = F^\nu CH_sX \]

so \( j^* \) and \( j'^* \) induce the same filtrations.

5.4 Proposition.

(1) If \( X \) and \( Y \) are smooth quasiprojective varieties and \( f : X \to Y \) is proper (resp. proper surjective) then the map \( f_* : CH_sX \to CH_sY \) respects \( F^\bullet \) (resp. strictly compatible with \( F^\bullet \)).

(2) If \( j : U \hookrightarrow X \) is an open immersion of smooth varieties, \( j^* : CH_sX \to CH_sU \) is strictly compatible with \( F^\bullet \).

Proof. For (1) we take smooth compactifications \( \overline{X}, \overline{Y} \) of \( X, Y \), respectively, so that \( f \) extends to a map \( \overline{f} : \overline{X} \to \overline{Y} \).

Consider the commutative diagram

```
CH_sX  \xrightarrow{f_*}  CH_sY  \\
\uparrow  \quad \uparrow  \\
CH_s\overline{X}  \xrightarrow{\overline{f}_*}  CH_s\overline{Y}
```

where the vertical arrows are the pull backs by open immersions. Since \( \overline{f}_* \) respects \( F^\bullet \) (resp. strictly compatible with \( F^\bullet \) if \( f \) is surjective) \( \overline{f}_* F^\nu CH_sX \subset F^\nu CH_sY \) (resp. equal). On the other hand, by definition \( F^\nu CH_sX \) surjects to \( F^\nu CH_sX \), and \( F^\nu CH_sY \) surjects to \( F^\nu CH_sY \). Hence \( f_* F^\nu CH_sX \subset F^\nu CH_sY \) (resp. equal).

For (2) take a compactification \( j : X \to \overline{X} \) and consider the commutative diagram

```
CH_sX  \xrightarrow{j^*}  CH_sU  \\
\downarrow  \quad \downarrow  \\
CH_s\overline{X}
```

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where all the arrow are restrictions by open immersions. Since $F^\nu CH_sX$ surjects to both $F^\nu CH_sX$ and $F^\nu CH_sU$, one has $j^*F^\nu CH_sX = F^\nu CH_sU$. □

For an arbitrary quasiprojective variety $X$, take a desingularization $\pi : \tilde{X} \to X$ and equip $CH_sX$ with the filtration induced by the surjective map $\pi_* : CH_s\tilde{X} \to CH_sX$. By an argument using 5.4(1), one sees that the filtration is well defined independent of the choice of $\tilde{X}$.

5.5 Proposition.

(1) If $X, Y$ are quasi-projective varieties and $f : X \to Y$ is proper (resp. proper surjective), then $f_* : CH_sX \to CH_sY$ respects $F^\bullet$ (resp. strictly compatible with $F^\bullet$).

(2) If $j : U \to X$ is an open immersion of quasi-projective varieties, $j^* : CH_sX \to CH_sU$ is strictly compatible with $F^\bullet$.

Proof. To prove (1), take desingularizations $\pi : \tilde{X} \to X$, $\pi' : \tilde{Y} \to Y$ so that $f$ extends to a map $\tilde{f} : \tilde{X} \to \tilde{Y}$. Then $\pi' \circ \tilde{f} = f \circ \pi$ and one has a commutative diagram

$$
\begin{array}{ccc}
CH_sX & \xrightarrow{f_*} & CH_sY \\
\pi_* \downarrow & & \downarrow \pi'_* \\
CH_s\tilde{X} & \xrightarrow{\tilde{f}_*} & CH_s\tilde{Y}
\end{array}
$$

where the arrows are proper push forwards. Since the map $\tilde{f}_*$ respects $F^\bullet$ (resp. strictly compatible with $F^\bullet$) by 5.4(1), and so are the vertical surjective maps by definition, $f_*$ respects $F^\bullet$ (resp. is strictly compatible with $F^\bullet$). The proof of (2) is similar. □

Proof of Theorem 5.1. The properties (i) and (ii) have been verified, and we started with the filtration satisfying (iii) and (viii).

(vii) This is verified by reducing first to the case where $X$ and $Y$ are both smooth, and then to the case they are smooth projective.

(vi) Follows from strictness of the filtration under proper maps and open immersions (i) and (ii), lemma 5.6 below and compatibility of the filtration under action of correspondences on smooth projective varieties (i), (v), (vii). □

5.6 Lemma. Let $i : T \to S$ be a regular embedding of codimension $d$

$$
\begin{array}{ccc}
Y & \xrightarrow{g} & X \\
\downarrow & & \downarrow f \\
T & \xrightarrow{i} & S
\end{array}
$$

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a fibre square, and
\[ i^! : CH_r X \to CH_{r-d} Y \]
the associated refined Gysin map [Fu, Ch. 6]. There are:

1. smooth varieties \( U, V \), proper surjective maps \( p : U \to X, q : V \to Y \),
2. a commutative diagram

\[
\begin{array}{ccc}
V & \xrightarrow{q} & U \\
\downarrow & & \downarrow p \\
Y & \xrightarrow{f} & X
\end{array}
\]

3. smooth compactifications \( U \subset \overline{U}, V \subset \overline{V} \) and a correspondence \( \Gamma \in CH_r V \times \overline{U} \), such that for a cycle \( \alpha \in CH_r U \)

\[ i^! p_*(\alpha|U) = q_*(\langle \Gamma^* \alpha \rangle|V) \]

**Proof.**

**Step 1.** In this step we reduce the problem to the case where \( Y = Y_1 \coprod Y_2 \hookrightarrow X \), \( X \) is smooth, \( Y_1 \hookrightarrow X \) is a normal crossing divisor, and \( Y_2 \hookrightarrow X \) is the inclusion of a bunch of connected components.

Let indeed \( \delta : X' \to X \) be a resolution of singularities such that \( Y' = Y \times_X X' = Y_1' \coprod Y_2' \hookrightarrow X' \) is as above. We have a commutative diagram of fibre squares

\[
\begin{array}{ccc}
Y' & \xrightarrow{\varepsilon} & X' \\
\downarrow & & \downarrow \delta \\
Y & \xrightarrow{g} & X \\
\downarrow & & \downarrow f \\
T & \xrightarrow{i} & S
\end{array}
\]

By compatibility of refined Gysin maps with proper push forward [Fu, Ch 6] we have

\[ i^! \delta_* = \varepsilon_* i^! \]

Therefore, it is enough to prove the result for \( Y' \to X' \).

**Step 2.** We now assume that \( j : Y \hookrightarrow X \) is a normal crossing divisor inside a smooth quasiprojective variety. Let \( E = g^* N_T S/N_Y X \) be the excess bundle (by assumption, it has rank \( d - 1 \)). The excess intersection formula reads

\[ i^! \alpha = c_{d-1} E \cap j^! \alpha \]
If now \( U = X, U \subset \overline{U} \) is a smooth projective compactification of \( U \) such that the closure \( \overline{Y} \) of \( Y \) in \( \overline{U} \) is a normal crossing divisor, then denoting \( c_{d-1}E \) any extension of \( c_{d-1}E \) to \( \overline{U}, \nu : V \to \overline{Y} \) the normalization, \( V = \nu^{-1}Y \) the normalization of \( Y \), and \( h : V \to \overline{U} \) the composition \( V \to \overline{Y} \to \overline{U} \), we have for a class \( \alpha \in \text{CH}_r(U) \)

\[
j^! (\alpha|U) = (\nu|V)_* ((h^* \alpha)|V)
\]

and

\[
i^! (\alpha|U) = (\nu|V)_* ((c_{d-1}E \cap h^* \alpha)|V)
\]

To conclude the proof now just take a correspondence

\[
\Gamma \in \text{CH}_* \overline{V} \times \overline{U}
\]

such that

\[
c_{d-1}E \cap h^* \cdot = \Gamma^* \cdot
\]

**Step 3.** Finally, we treat the case where \( j : Y \hookrightarrow X \) is the inclusion of a bunch of connected components. This is quite a bit easier than step 2: here

\[
i^! \alpha = c_d(E) \cup j^* \alpha
\]

We let \( V = Y, U = X, U \subset \overline{U} \) a smooth compactification and \( c_dE \) any extension of \( c_dE \) to \( \overline{U}, \overline{j} : V \to \overline{U} \) the corresponding compactification of \( V \hookrightarrow U \). \( \Gamma \) works if \( \Gamma^* \cdot = c_dE \cup \overline{j}^* \cdot \). □

The following proposition will be needed in the proof of Theorem 5.3.

**5.7 Proposition.** *(Assume Grothendieck’s standard conjectures.)* For a quasiprojective variety \( X \), let

\[
H_{2s}^{BM}(X)_{\text{alg}} = \text{Im} \left( cl : \text{CH}_sX \to H_{2s}^{BM}X \right)
\]

which is a \( \mathbb{Q} \)-vector space.

(1) If \( j : U \hookrightarrow X \) is an open immersion of quasiprojective varieties and \( i : Z = X - U \hookrightarrow X \) is the closed immersion of the complement of \( U \), the exact sequence

\[
H_{2s}^{BM}Z \xrightarrow{i_*} H_{2s}^{BM}X \xrightarrow{j^*} H_{2s}^{BM}U
\]

induces the following exact sequence on algebraic parts:

\[
H_{2s}^{BM}(Z)_{\text{alg}} \xrightarrow{i_*} H_{2s}^{BM}(X)_{\text{alg}} \xrightarrow{j^*} H_{2s}^{BM}(U)_{\text{alg}} \to 0
\]
(2) Let

\[
\begin{array}{c}
Z' \xrightarrow{i'} X' \\
\downarrow q \quad \downarrow p \\
Z \xrightarrow{i} X
\end{array}
\]

be a Cartesian square of quasiprojective varieties such that the horizontal maps are closed immersions, the vertical maps are proper surjective and \( p \) induces an isomorphism \( X' \to Z' \xrightarrow{\cong} X - Z \). Then the exact sequence

\[
H^{BM}_{2s} Z' \xrightarrow{i_*^* q_*} H^{BM}_{2s} X' \oplus H^{BM}_{2s} Z \xrightarrow{p_* - i_*} H^{BM}_{2s} X
\]

induces an exact sequence

\[
H^{BM}_{2s} (Z')_{alg} \xrightarrow{i_*^* q_*} H^{BM}_{2s} (X')_{alg} \oplus H^{BM}_{2s} (Z)_{alg} \xrightarrow{p_* - i_*} H^{BM}_{2s} (X)_{alg} \to 0
\]

Proof. (1) We recall that \( H^{BM}_i X \) for \( X \) quasiprojective has a weight filtration \( W_\bullet \), the weights are \( \geq -i \), and the maps \( i_* \), \( j^* \) are strictly compatible with the weight filtrations.

The first exact sequence induces, upon taking \( \text{Gr}^{W}_{2s} \), the exact sequence

\[
W_{-2s} H^{BM}_{2s} Z \to W_{-2s} H^{BM}_{2s} X \to W_{-2s} H^{BM}_{2s} U
\]

This may be viewed as an exact sequence in the category of Grothendieck motives \( \mathcal{M} \). More specifically in the weight spectral sequence

\[
W E_1^{p q} \Rightarrow H^{BM}_{-p - q} X
\]

which induces the weight filtration of \( H^{BM}_{-p - q} X \), each \( W E_1^{p q} \) is the cohomology of a smooth projective variety and can be regarded as a Grothendieck motive denoted \( W E_1^{p q} \); the differentials \( d_1^{p q} \) are morphisms of Grothendieck motives. Define

\[
\text{Gr}_q^W H_{-p - q}^{BM} X = W Z_{1}^{p q} \big/ W B_{1}^{p q}
\]

Taking the cohomological realisation \( H^* \) we have

\[
H^*(\text{Gr}_q^W H_{-p - q}^{BM} X) = \text{Gr}_q^W H_{-p - q}^{BM} X
\]

because \( H^* \) is exact (by the standard conjectures, \( \mathcal{M} \) is a semisimple abelian category) and the weight spectral sequence above degenerates at \( E_2 \).
Since $\mathcal{M}$ is semisimple, the functor $\text{Hom}(pt(-s),-) : \mathcal{M} \to \text{Vec}_Q$ is also exact, and the above exact sequence induces an exact sequence under this functor. It is the desired exact sequence, except possibly at the end, because: for a quasiprojective variety $X$

$$\text{Hom}(pt(-s), W_{-2s} H_{2s}^{BM} X) = H_{2s}^{BM} (X)_{alg}$$

The surjectivity at the end is obvious since $j^* : CH_s X \to CH_s U$ is surjective.

(2) One has a commutative diagram

\[
\begin{array}{c}
H_{2s}^{BM} Z' \xrightarrow{i'_*} H_{2s}^{BM} X' \xrightarrow{p_*} H_{2s}^{BM} (X' \setminus Z') \\
\downarrow q_* \quad \downarrow p_* \quad \downarrow p_* \\
H_{2s}^{BM} Z \xrightarrow{i_*} H_{2s}^{BM} X \xrightarrow{} H_{2s}^{BM} (X \setminus Z)
\end{array}
\]

where the rows are exact and the third vertical map is an isomorphism. Hence the exact sequence of Borel-Moore homology.

Applying (1) to the closed immersions $i'$ and $i$ respectively one obtains a similar commutative diagram with exact rows

\[
\begin{array}{c}
H_{2s}^{BM} (Z')_{alg} \xrightarrow{i'_*} H_{2s}^{BM} (X')_{alg} \xrightarrow{p_*} H_{2s}^{BM} (X' \setminus Z')_{alg} \\
\downarrow q_* \quad \downarrow p_* \quad \downarrow p_* \\
H_{2s}^{BM} (Z)_{alg} \xrightarrow{i_*} H_{2s}^{BM} (X)_{alg} \xrightarrow{} H_{2s}^{BM} (X \setminus Z)_{alg}
\end{array}
\]

(the third vertical map is an isomorphism); the last exact sequence follows from this. □

**Proof of Theorem 5.3.** (x) is verified by reducing to the smooth projective case where it holds true, see 5.2(1).

To show (ix) first assume $X$ is smooth quasiprojective. Take a compactification $j : X \to \overline{X}$, let $Z = \overline{X} - X$, and consider the commutative diagram with exact rows 5.7(1)

\[
\begin{array}{c}
H_{2s}^{MB} (Z)_{alg} \xrightarrow{i_*} H_{2s}^{MB} (\overline{X})_{alg} \xrightarrow{j^*} H_{2s}^{MB} (X)_{alg} \xrightarrow{} 0 \\
\downarrow \quad \downarrow \quad \downarrow \\
CH_s Z \xrightarrow{i_*} CH_s \overline{X} \xrightarrow{j^*} CH_s X \xrightarrow{} 0
\end{array}
\]
where the vertical arrows are the cycle class maps. Since $F^1CH_sX$ surjects to $F^1CH_sX$ the inclusion $F^1CH_sX \subset CH_s(X)_{hom}$ is obvious. The other inclusion follows from the above diagram.

The proof in the general case is reduced to the smooth case by taking desingularisations and using 5.7(2). □

As an application, let $S$ be a quasiprojective variety, $X$ a smooth quasiprojective equidimensional variety, and $f : X \to S$ a projective map. We recall that the group $CH_{\dim X}X \times_S X$ is a ring where the multiplication is defined by

$$v \cdot u = \delta^i(v \times u)$$

$\delta^i$ being the refined Gysin map related to the diagonal embedding $X \to X \times X$. Hence we have:

**5.8 Proposition.** The multiplication of the ring $CH_{\dim X}X \times_S X$ respects $F^\bullet$. Namely if $u \in F^\nu CH_{\dim X}X \times_S X$ and $v \in F^\mu CH_{\dim X}X \times_S X$, then

$$v \cdot u \in F^{\nu + \mu} CH_{\dim X}X \times_S X$$

In particular, if we assume the conjectures of Murre and of Grothendieck, the ideal $F^1CH_{\dim X}X \times_S X = CH_{\dim X}(X \times_S X)_{hom}$ is a nilpotent ideal. □

**5.9 Theorem.** (Assume Grothendieck’s and Murre’s conjectures) Let $f : X \to S$ be as above. Then the surjective map

$$\rho : CH_{\dim X}X \times_S X \to End_{M_{S}}^p Rf_*Q_X$$

has a nilpotent kernel.

**Proof.** We claim first that the map of rings

$$H^2BM_{\dim X}X \times_S X \to End_{grPerVs(S)} \sum_p R^i f_*Q_X[-i]$$

has a nilpotent kernel. If one chooses a decomposition

$$Rf_*Q_X = \sum_p R^i f_*Q_X[-i]$$

in $D^b_{cc}(S)$, an endomorphism $u$ of $Rf_*Q_X$ may be represented by a matrix according to the decomposition. The matrix is upper triangular since the maps $pR^i f_*Q_X[-i] \to pR^j f_*Q_X[-j]$ are zero if $i < j$.

Consider now an element

$$u \in H^2BM_{\dim X}X \times_S X = End_{D^b_{cc}S} Rf_*Q_X$$
which maps to zero in $\text{End}_{P\text{erv}S} \sum p^i R^i f_* Q_X[-i]$. Then the matrix representing $u$ as above is strictly upper triangular. Hence there exists $N$ such that $u^N = 0$ as an endomorphism of $Rf_* Q_X$.

The kernel of the homomorphism

$$CH_{\dim X} X \times_S X \to H^{BM}_{2 \dim X} X \times_S X$$

equals $CH_{\dim X}(X \times_S X)_{\text{hom}}$. Under the conjectures, it is a nilpotent ideal, 5.9. Hence the kernel of $\rho$ is also nilpotent. □

5.10 Corollary. Any set of orthogonal projectors $\{\pi^i\}$ of $\text{End}_{\mathcal{M}S} p Rf_* Q_X$ such that $\Delta_X = \sum \pi^i$ can be lifted to a set of orthogonal projectors $\{\Pi^i\}$ of $CH_{\dim X}(X \times_S X)$ such that $\Delta_X = \sum \Pi^i$.

Proof. More generally the following holds (cf. [Ja1, Lemma 5.4]). □

5.11 Proposition. Let $\phi: A \to B$ be a surjective homomorphism of not necessarily commutative rings with nilpotent kernel. Then any orthogonal set $\{p_1, \ldots, p_m\}$ of idempotents of $B$ (i.e. $p_i p_j = \delta_{i,j} p_i$) adding up to $1_B$ can be lifted to an orthogonal set of idempotents of $A$ adding up to $1_A$. □

6. Decomposition in $CH\mathcal{M}(S)$

Throughout this section we assume that desingularisations of varieties over $k$ exist. The aim is to prove the following:

6.1 Decomposition theorem in $CH\mathcal{M}(S)$. Assume the standard conjectures and Murre’s conjecture. Let $X$ be a smooth variety and $f: X \to S$ be a projective morphism.

(1) There is a noncanonical direct sum decomposition in $CH\mathcal{M}(S)$

$$CRf_* Q_X \cong \sum_{m,k} CIC V^m_{k}[k-m]$$

where $V_k$ is a local system on a Zariski locally closed subvariety $T_k \subset S$ and $CIC V^m_{k}[k-m]$ denotes an object in $CH\mathcal{M}(S)$, together with a given isomorphism in $D^b_{cc}(S)$

$$\text{real} CIC V^m_{k}[k-m] \cong IC V^m_{k}[k-m]$$

(2) The monomorphisms

$$p_{\tau \leq m} CRf_* Q_X = \sum_{i \leq m} \sum_k CIC V^i_{k}[k-i] \to CRf_* Q_X$$
(where the 1st equality is the definition of $p^{\tau \leq m} CR f_* Q_X$) are specified up to canonical isomorphism. In particular so are the “subquotients”

$$p^{\tau \leq m} CR f_* Q_X[-m] = \sum_k CIC V^m_k [k-m]$$

(\text{where the 1st equality is the definition of } p^{\tau \leq m} CR f_* Q_X[-m]) specified up to canonical isomorphism.

(3) The decompositions

$$p^{\tau \leq m} CR f_* Q_X[-m] = \sum_k CIC V^m_k [k-m]$$

are uniquely specified.

\textbf{Proof.} Using 4.12, choose a decomposition in $\mathcal{M}(S)$

$$p^R f_* Q_X \cong \sum_{m,k} V^m_k [k-m]$$

and let

$$\pi^m_k \in \text{End}_{\mathcal{M}S} X$$

be the projector onto $V^m_k [k-m]$. By 5.10, the $\pi^m_k$ lift to an orthogonal set of projectors

$$\Pi^m_k \in \text{End}_{C \mathcal{H}M S} X$$

Now set

$$CIC V^m_k [k-m] = (X, \Pi^m_k)$$

This proves the existence of the sought for decomposition.

The uniqueness statement (2) is more subtle, and will be deduced from the corresponding uniqueness statements for the decomposition in $\mathcal{M}(S)$. According to definition 6.2 below, $p^{\tau \leq m} CR f_* Q_X$ (resp. $p^{\tau > m} CR f_* Q_X$) has cohomological degree $\leq m$ (resp. $> m$). Then, by 6.3(2) below

$$\text{Hom}_{C \mathcal{H}M S} (p^{\tau \leq m} CR f_* Q_X, p^{\tau > m} CR f_* Q_X) = 0$$

independently of the choice of the liftings $\Pi^m_k$. This implies (2).

Finally, to prove the uniqueness statement in (3), note that by construction $p^{\tau \leq m} CR f_* Q_X[-m]$ has cohomological degree exactly $m$, hence the statement follows from the decomposition theorem in $\mathcal{M}(S)$ 4.12, and 6.3(2). \[\square\]

The rest of this section is devoted to the proof of 6.3, which was used in the proof of 6.1.
6.2 Definition. A Chow motive $(X, P)$ over $S$ has cohomological degree $\leq m$ (resp. $> m$) if
\[ p^i \text{real}(X, P) = 0 \]
for all $i > m$ (resp. $i \leq m$). Finally, $(X, P)$ has degree exactly $m$ if it has degree $\geq m$ and $\leq m$.

6.3 Theorem. (Assuming the standard and Murre’s conjectures)
(1) If $(X, P)$ has cohomological degree $\leq m$ and $(Y, Q)$ has cohomological degree $> m$, then
\[ \text{Hom}_{CHMS}((X, P), (Y, Q)) = 0 \]
(2) If $(X, P)$ and $(Y, Q)$ have degrees exactly $m$, then
\[ \text{Hom}_{CHMS}((X, P), (Y, Q)) = \text{Hom}_{MS}((X, P), (Y, Q)) \]

Proof. Let $Z = X \times_S Y$. We consider the operators
\[ CH_s Z \ni \alpha \rightarrow \Psi \alpha = Q \cdot \alpha \cdot P \in CH_s Z \]
\[ H_i^{BM} Z \ni a \rightarrow \psi a = [Q] \cdot a \cdot [P] \in H_i^{BM} Z \]
Note that $\Psi^2 = \Psi$ and $\psi^2 = \psi$, i.e. both operators are projectors.

To prove (1) and (2), it is enough to show that if $\psi = 0$, then $\Psi = 0$. Most of the argument will be spent showing that, because $\psi = 0$, $\Psi F^\nu \subset F^{\nu+1}$ for all $\nu$.

Step 1. Let $T \subset Z$ be the singular set. Make a diagram

\[
\begin{array}{c}
\overline{R} \\
\uparrow \\
R \\
\downarrow \\
T \\
\rightarrow \\
\rightarrow \\
W \\
\uparrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
Z
\end{array}
\]

where:
(a) $\overline{W}$ is smooth and projective and $\overline{R} \subset \overline{W}$ is a smooth normal crossing divisor,
(b) $(W, R) = (\overline{W} - B, \overline{R} - B)$ for some divisor $B \subset \overline{R}$,
(c) $p : W \rightarrow Z$ is a resolution of singularities and $R = p^{-1} T \subset W$.

By 6.8 below, $\Psi$ is a class $C$ operator, hence by 6.5 there is a correspondence
\[ \Gamma \in CH_*(\overline{W} \times \overline{W}) \]
(\overline{W} \text{ is not necessarily equidimensional}) such that
\[
\Psi(p_*(\alpha|W)) = p_*((\Gamma_*\alpha)|W) \tag{*}
\]
for all \(\alpha \in CH_*Z\), resp.
\[
\psi(p_*(a|W)) = p_*((\Gamma_*a)|W) \tag{**}
\]
for all \(a \in H_*^{BM}Z\).

**Step 2.** We have an exact sequence
\[
H_*^{BM}\overline{R} \to H_*^{BM}R \oplus H_*^{BM}\overline{W} \to H_*^{BM}W
\]
If \(j : W \hookrightarrow \overline{W}\), \(j' : R \hookrightarrow \overline{R}\) are the natural inclusions, the exact sequence arises, by means of a familiar construction, from the following morphism of distinguished triangles
\[
\begin{array}{ccc}
D_B & \to & D_{\overline{R}} \\
\downarrow & & \downarrow \\
D_B & \to & D_{\overline{W}}
\end{array} \quad \begin{array}{ccc}
Rj'_*D_R & \to & Rj_*D_R \\
\to & & \to \\
Rj'_*D_R & \to & Rj_*D_W
\end{array}^{[1]}
\]

**Step 3.** This is the crucial step. We show that
\[
\gamma = cl\Gamma \in N^1H_*^{BM}\overline{W} \times \overline{W}
\]
lies in the codimension 1 piece of the coniveau filtration.

Let \(i : \overline{R} \hookrightarrow \overline{W}\) and \(i' : R \hookrightarrow W\) be the inclusions. First of all, from the standard exact sequence
\[
H_*^{BM}R \to H_*^{BM}T \oplus H_*^{BM}W \to H_*^{BM}Z
\]
we deduce that if
\[
p_*(a|W) = 0
\]
for some \(a \in H_*^{BM}\overline{W}\), then
\[
a|W = i'_*a'
\]
for some \(a' \in H_*^{BM}R\). Then, from the exact sequence in Step 2,
\[
a = i_*a''
\]
for some \(a'' \in H_*^{BM}\overline{R}\). Now, for any \(a \in H_*^{BM}\overline{W}\)
\[
p_*(((\Gamma_*a)|W) = \psi(p_*(a|W)) = 0
\]
by assumption, so, because of what has been said
\[ \Gamma_* a \in \text{Im} H^B_{BM} \]
which finishes step 3.

**Step 4.** Let \( \Gamma' = \Gamma \circ \Gamma \circ \ldots \) (many times). Since \( \Psi^2 = \Psi \) and \( \psi^2 = \psi \), equations (\( \ast \)) and (\( \ast \ast \)) are still satisfied with \( \Gamma' \) in place of \( \Gamma \). In addition we have
\[ \gamma' = c \Gamma' \in N^{\text{many times}} H_{BM} \times \overline{W} = 0 \]
Then \( \Gamma'_\nu H_{BM} \overline{W} \subset F^{\nu+1} H_{BM} \overline{W} \) for all \( \nu \) and, by the strictness properties of the \( F \)-filtration
\[ \Psi F^{\nu} CH_Z \subset F^{\nu+1} CH_Z \]
for all \( \nu \).

**Step 5.** We are assuming the standard conjectures. Therefore \( F^{\nu} CH_{BM} \overline{Z} = 0 \) for \( \nu \) large. This implies that \( \Psi = 0 \), which concludes the proof. \( \square \)

The rest of the paper is devoted to finishing the proof of 6.3.

**6.4 Definition.**

(1) Let \( X \) be a quasiprojective variety. For the purpose of the following discussion, a **smooth cover** of \( X \) is a diagram

\[
\begin{array}{ccc}
U & \longrightarrow & \overline{U} \\
\downarrow & & \downarrow \\
X & \quad \quad & \overline{U} \\
\end{array}
\]

sometimes simply denoted \( \overline{U} \supset U \xrightarrow{p} X \), where \( \overline{U} \) is a smooth projective variety, \( U \subset \overline{U} \) an open subvariety with smooth normal crossing boundary divisor \( \overline{U} \setminus U \), and \( p : U \rightarrow X \) a projective morphism. It is a consequence of our assumptions on existence of resolution of singularities, that smooth covers always exist.

(2) Let \( X, Y \) be quasiprojective varieties. We say that an operator
\[ \Psi : CH_{BM} X \rightarrow CH_{BM} Y \]
is of **class C** (\( \mathcal{C} \) stands for “correspondence”) if there are smooth covers \( \overline{U} \supset U \xrightarrow{p} X \) of \( X \) and \( \overline{V} \supset V \xrightarrow{q} Y \) of \( Y \), and a correspondence \( \Gamma \in CH_{BM} \overline{U} \times \overline{V} \) such that
\[ \Psi(p_*(\alpha|U)) = q_*(\Gamma_*\alpha|V) \]
for all \( \alpha \in CH_{BM} \overline{U} \). In this case, we say that \( \Gamma \) **induces** \( \Psi \).

The following is the basic point
6.5 Lemma. Let $\Psi : CH_\bullet X \to CH_\bullet_{-c} Y$ be of class $C$, and let $U \supset U \xrightarrow{p_1} X$, $V \supset V \xrightarrow{q_1} Y$ be arbitrary smooth covers of $X$, $Y$. Then, there exists a correspondence $\Gamma \in CH_\bullet U \times V$ inducing $\Psi$

Proof. By assumption, there are some smooth covers $U_1 \supset U_1 \xrightarrow{p_1} X$ of $X$, $V_1 \supset V_1 \xrightarrow{q_1} Y$ of $Y$, and some correspondence $\Gamma_1 \in CH_\bullet U_1 \times V_1$ such that

$$\Psi(p_1^*(\alpha|U_1)) = q_1^*((\Gamma_1^*\alpha)|V_1)$$

for all $\alpha \in CH_\bullet U_1$.

Now, any 2 smooth covers can be housed under a third

$$U_2 \subset U_2 \xrightarrow{} U_1 \subset U_1 \xrightarrow{} U \subset U$$

so, in the end, we may assume that there is either a morphism $U_1 \subset U_1 \xrightarrow{\pi} U \subset U$, or the other way around, and similarly for $V$.

Case 1. Assume that there is a morphism $\pi : U \subset U \xrightarrow{} U_1 \subset U_1$ and let

$$\Gamma = \Gamma_1 \circ \Gamma \pi \in CH_\bullet (U \times V_1)$$

Then $\Gamma$ induces $\Psi$, since

$$\Psi(p^*(\alpha|U)) = \Psi(p_1^*\pi^*(\alpha|U)) = \Psi(p_1^*((\pi^*\alpha)|U_1)) = q_1^*((\Gamma_1^*\pi^*\alpha)|V_1) = q_1^*((\Gamma^*\alpha)|V_1)$$

Case 2. Assume now that there is a morphism $\pi : U_1 \subset U_1 \xrightarrow{} U \subset U$. Let $i : \overline{W} \subset \overline{U_1}$ be a smooth projective subvariety generically finite over $U$

$$\overline{W} \xrightarrow{i} \overline{U_1} \xrightarrow{\pi} \overline{U}$$

and let us agree that $d$ be the generic degree of $\pi \circ i$. Let us fix ourselves a correspondence $\Gamma \in CH_\bullet U \times V_1$ with the property that

$$\Gamma = d \Gamma_1 \circ i^* \pi^*$$
Then $\Gamma$ induces $\Psi$, since

$$
\Psi(p_*(\alpha|U)) = \Psi(p_*(\frac{1}{d}\pi_*i^*\pi^*\alpha|U)) = \\
= \Psi(p_*(\frac{1}{d}i_*i^*\pi^*\alpha|U)) = \\
= q_1*((\frac{1}{d}\Gamma_1i_*i^*\pi^*\alpha)|V_1) = q_1*((\Gamma_\star \alpha)|V_1)
$$

To summarise, in both cases we were able to find $\Gamma \in CH_\bullet U \times V_1$ inducing $\Psi$. Working now the $V$s in a similar fashion, we can also find $\Gamma \in U \times V$ inducing $\Psi$, i.e., prove the lemma. □

6.6 Lemma. Let $\Psi : CH_\bullet X \to CH_\bullet -cY$ and $\Phi : CH_\bullet Y \to CH_\bullet -eZ$ be of class $C$. Then, the composition $\Phi \circ \Psi : CH_\bullet X \to CH_\bullet -c-eZ$ is also of class $C$.

Proof. Obvious. □

6.7 Lemma.

(1) Let $f : X \to Y$ be a proper map. Then $f_\star : CH_iX \to CH_iY$ is of class $C$.

(2) Let $i : Y \hookrightarrow X$ be a regular embedding of codimension $c$ and

$\begin{array}{c}
\begin{array}{c}
Y' \\
Y
\end{array}
\end{array} \longrightarrow \begin{array}{c}
\begin{array}{c}
X' \\
X
\end{array}
\end{array}$

be a fibre square. The refined Gysin map

$$
i^! : CH_\bullet X' \to CH_\bullet -eY'
$$

is of class $C$

Proof. (1) is obvious and (2) is 5.6. □

6.8 Corollary. The operator $\Psi$ in the proof of 6.3 is of class $C$.

Proof. Indeed, $\Psi$ is a composition of proper push forward and refined Gysin maps. All these are of class $C$ 6.7, and so their composition is 6.6. □

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