Some remarks on port-based teleportation

Satoshi Ishizaka

Graduate School of Integrated Arts and Sciences, Hiroshima University,
1-7-1 Kagamiyama, Higashi-Hiroshima, 739-8521, Japan
(Dated: June 8, 2015)

Port-based teleportation (PBT) is a teleportation scheme such that the teleported state appears in one of receiver’s multiple output ports without any correcting operation on the output port. In this paper, we make some remarks on PBT. Those include the possibility of recoverable PBT (a hybrid protocol between PBT and the standard teleportation scheme), the possibility of port-based superdense coding (a dual protocol to PBT), and the fidelity upper bound expected from the entanglement monogamy relation in asymmetric universal cloning.

PACS numbers: 03.67.Hk, 03.67.Ac, 03.67.Bg, 03.67.Lx

I. INTRODUCTION

Quantum teleportation [1] is the most fundamental protocol in quantum information science, and indeed has always played a crucial role in the progress of quantum information theory and technology (for a good review, see [2]). The standard teleportation scheme (STS) [1] transfers an unknown quantum state from Alice to Bob as follows: Alice performs a joint measurement on the state to be teleported and half of the previously shared entangled state, tells the outcome of the measurement to Bob, and Bob applies a unitary transformation, depending on the outcome, to the remaining half of the entangled state. To ensure no-signaling (no faster than light communication), every teleportation scheme must accompany some sort of communication and Bob’s operation depending on the communication content.

In so-called port-based teleportation (PBT) [3, 4], Bob’s operation is quite simple: he regards his half of (large) entangled state as a collection of N output ports, and he only picks up an output port (and discards all the other ports). The output port contains the teleported state as it is, without any correcting operation on the output port. The absence of the correcting operation leads to an application of PBT as a universal programmable quantum processor [3, 5], which is a device to play back the record of the (past) experiences of a quantum object. The universality of the device is so powerful that arbitrary experiences can be recoded and played back (just like a machine in science fiction to relive your childhood), including not only those described as unitary evolution but also measurements (working as a quantum multimeter [6] in this case). The drawback is that huge amount of entanglement is necessary to increase the fidelity or success probability. It has been shown, however, that the most of the huge amount of entanglement can be recycled for subsequent PBT [7].

Moreover, it has been shown in [8] that a combined protocol of PBT and STS works as if it could break the barrier of spacetime as follows: Suppose that Alice and Bob, separated in spacetime, each has a quantum system $A_0$ and $B_0$, respectively. Alice can then consider, without waiting for communication, that a port of her half of the shared entangled state contains the state of $A_0B_0$, i.e. she already has the non-local state of $A_0B_0$ somewhere in her hand (though she can know which port contains the state only after the communication from Bob) [8]. This technique is used for attacking position-based cryptography and for instantaneous non-local quantum computation [8]. Moreover, PBT has been used as a tool to investigate the relation of quantum communication complexity and the Bell non-locality [9].

However, the properties of PBT have not been completely clarified yet, in particular, for teleporting a high dimensional quantum state. This is because, in contrast to STS, the simple multiple use of PBT for a qubit (quantum bit) does not result in PBT for higher dimension. For teleporting a state of a qudit (d-dimensional system), only a lower bound of the teleportation fidelity of deterministic PBT [4, 8] and an upper bound of the success probability of probabilistic PBT [10] have been obtained so far. More studies will be necessary to clarify the properties of PBT.

In this paper, we make some remarks on PBT. After recalling the formulation of PBT in Sec. III in Sec. IV we pay attention to the fact that, in most cases of $d = 2$, the optimal measurements of Alice agree with each other. In Sec. V we propose a hybrid protocol between PBT and STS (say recoverable PBT), where Bob has another choice (in addition to adopt usual PBT) to adopt a faithful teleportation by utilizing all the N output ports. In Sec. VI we consider the setting of the port-based superdense coding, a dual protocol to PBT, and rederive the upper bound of success probability of probabilistic PBT. This bound is tight even for $d = 3$ and $N = 2$. In Sec. VII we obtain an upper bound of the teleportation fidelity by using the entanglement monogamy relation in asymmetric $1 \rightarrow N$ universal cloning. In Sec. VIII we finally remark that the superdense coding capacity can be asymptotically achieved in a limit different from the fidelity, and hence port-based superdense coding is possible. A summary is given in Sec. VIII.
II. FORMULATION OF PBT

To begin with, let us recall the formulation of (deterministic) PBT, where Bob has \( N \) output ports and a teleported state appears in one of the \( N \) ports without any correcting operation on each port. As a preparation of PBT, Bob has \( N \) qudits: \( B_1, B_2, \ldots, B_N \), where each corresponds to the output port of PBT. In this paper, \( B_1, \ldots, B_N \) are denoted by \( B \) as a whole. Alice also has \( N \) qudits: \( A_1, A_2, \ldots, A_N \), which are denoted by \( A \) as a whole. Let us then describe an entangled state between \( A \) and \( B \) for PBT as

\[
|\Psi\rangle = (O_A \otimes \mathbb{1})|\psi^-\rangle A_1 B_1 |\psi^-\rangle A_2 B_2 \cdots |\psi^-\rangle A_N B_N.
\]

(1)

Hereafter, the qudits are regarded as \( s \)-spins \((d = 2s+1)\), and \( d \) and \( 2s+1 \) will be used interchangeably. The spin basis is denoted by \(|s,m\rangle \) \((m = -s, \ldots, s)\). Then,

\[
|\psi^-\rangle = \frac{1}{\sqrt{2s+1}} \sum_{m=-s}^{s} (-1)^{s-m} |s,m\rangle |s,m\rangle
\]

(2)

is a state of spin 0 in two \( s \)-spins, which is maximally entangled between the two. The operator \( O \) specifies the actual form of \(|\Psi\rangle\), and \( \text{tr} O A = d^N \) so that \(|\Psi\rangle\) is normalized. Note that, in PBT, the teleportation fidelity is maximized when \( O \neq \mathbb{1} \) in general, i.e. when \(|\Psi\rangle\) is not maximally entangled.

To teleport the state of the \( C \) qudit, Alice performs a joint measurement with \( N \) possible outcomes \((1, 2, \ldots, N)\) on the \( A \) and \( C \) qudits. Let us denote the positive operator valued measure (POVM) of her measurement by \( \{\Pi_i\} \) and hence \( \sum_{i=1}^{N} \Pi_i = \mathbb{1}_{AC} \). When Alice obtains the outcome \( i \), the state of \( B_i \) qudit is close to the state of the \( C \) qudit as it is. It is then found that the entanglement fidelity of PBT is given by

\[
F = \frac{1}{d^2} \sum_{i=1}^{N} \text{tr} \Pi_i \mathbb{1}_{AC} \rho_{AC} \sigma_{AC}^{(i)} O_{A}^{(i)} \mathbb{1}_{A},
\]

(3)

where

\[
\sigma_{AC}^{(i)} = \frac{1}{d^{N-1}} P_{AC}^{-} \otimes \mathbb{1}_{A_i},
\]

(4)

\( P^{-} \equiv |\psi^-\rangle \langle \psi^-| \), and \( A_i \) denotes the \( A \) qudits except for \( A_i \) (i.e. \( A_1A_2\cdots A_{i-1}A_{i+1}\cdots A_N \)).

In analyzing the properties of PBT, the following operator:

\[
\rho \equiv \sum_{i=1}^{N} \sigma^{(i)} = \frac{1}{d^{N-1}} \sum_{i=1}^{N} P_{AC}^{-} \otimes \mathbb{1}_{A_i}
\]

(5)

frequently plays a crucial role. Indeed, Alice’s optimal measurement in the case of \( d = 2 \) and \( O = \mathbb{1} \) is the square-root measurement (SRM) \([3, 4]\) [also known as a pretty good measurement (PGM) or least-squares measurement (LSM) \([11, 16]\)] for distinguishing the quantum signals \( \{\sigma^{(i)}\} \), and the corresponding entanglement fidelity is given by

\[
F = \frac{1}{d^2} \sum_{i=1}^{N} \rho^{-1/2} \sigma^{(i)} \rho^{-1/2} \sigma^{(i)}.
\]

(6)

To investigate the general properties of \( \rho \), let us decompose \( \Pi_i \) into the spin components:

\[
\Pi_i = \sum_{j=j_{\min}}^{(N-1)s} \Pi \rho_{AC} \sigma_{AC}^{(i)} O_{A}^{(i)} \otimes \mathbb{1}_{A_i},
\]

(7)

where \( \Pi \) is an identity on the subspace where the total spin angular momentum of \( A_i \) is \( j \) \((j_{\min} \text{ is the possible minimum value})\). Then, \( \rho \) is also decomposed into

\[
\rho \equiv \frac{1}{d^{N-1}} \sum_{i=1}^{N} P_{AC}^{-} \otimes \mathbb{1}_{A_i}.
\]

(8)

Since the addition of spin 0 and spin \( j \) results in spin \( j \) only, each term \( P_{AC}^{-} \otimes \mathbb{1}_{A_i} \) is an operator on the subspace of total spin \( j \) \((\text{though the spin function constructed by the addition is different, depending on } i)\), and hence \( \rho \) is also an operator on the subspace of total spin \( j \). Therefore, \( \text{tr} \rho(j) \rho(j') = 0 \) for \( j \neq j' \), i.e. \( \rho \) is block diagonal with respect to the total spin angular momentum (and clearly its \( z \)-component also) of \( N+1 \) spins \((AC)\), but the maximum momentum is limited to \((N-1)s\).

As far as we know, the eigenvalues and eigenstates of \( \rho \) for \( d > 2 \) have not been obtained yet, which leads to the difficulty in analyzing PBT in higher dimension. The only exception is the maximum eigenvalue, which is proved in Appendix A to be

\[
\lambda_{\max} = \frac{N + d - 1}{d^N}.
\]

(9)

This leads to a known monogamy relation of singlet fraction \([17]\) for a multipartite state \( \Omega \) such that

\[
\frac{1}{N} \sum_{i=1}^{N} F_i \leq \frac{1}{N} \sum_{i=1}^{N} \text{tr} P_{AC}^{-} \otimes \mathbb{1}_{A_i} \Omega AC \leq \frac{d^{N-1}}{N} \text{tr} \rho \Omega \leq \frac{N + d - 1}{N^d},
\]

(10)

which explains the fidelity limit of symmetric \( 1 \rightarrow N \) universal cloning \([18, 20]\) in a simpler way.

III. REMARKS FOR \( d = 2 \)

Fortunately, all the eigenvalues and eigenstates of \( \rho \) for \( d = 2 \) can be analytically obtained as we showed in \([3, 4]\), but the results are not simple. Therefore, it may be worth to summarize it again in a tractable way. A special property held for \( d = 2 \) is that the spin angular
momentum of the A spins (denoted by $S_A$ hereafter) is a good quantum number, i.e., $\rho$ is block diagonal with respect to $S_A$ also. Indeed, $\rho(j)$ for $d = 2$ is written as

$$\rho(j) = \frac{N - 2j + 1}{2N+1} \mathbb{I}(j)_{AC}^{\pm \frac{1}{2}} + \frac{N + 2j + 3}{2N+1} \mathbb{I}(j)_{AC}^{\pm \frac{1}{2}},$$

(11)

where $\mathbb{I}(j)_{AC}^{\pm \frac{1}{2}}$ is an identity on the subspace where total spin angular momentum of $AC$ is $j$ and $S_A = J \pm \frac{1}{2}$. Note that, since the total spin of $j$ is the result of the addition of $S_A$ and $\frac{1}{2}$ spin of $C$, $\mathbb{I}(j) = \mathbb{I}(j)^{-\frac{1}{2}} + \mathbb{I}(j)^{\frac{1}{2}}$ holds.

When $|\Psi\rangle$ used for PBT is fixed to a maximally entangled state ($O = \mathbb{I}$), the optimal measurement of Alice to provide the maximum entanglement fidelity is SRM, i.e. the POVM elements are given by

$$\Pi_i = \rho^{-1/2}\sigma(i)\rho^{-1/2}.$$  

(12)

Here, it is implicitly assumed that the excess term $(1/N)\Pi_0$ is added to every $\Pi_i$ so that $\sum_i \Pi_i = \mathbb{I}$. Since $j$ in $\rho(j)$ takes a value only for $j \leq (N - 1)/2$ as mentioned before, we have

$$\Pi_0 = \mathbb{I} - \mathbb{I}_p = \mathbb{I}(\mathbb{N}^{N/2})$$

(13)

where $\mathbb{I}_p$ is an identity on the support of $\rho$.

The corresponding entanglement fidelity can be more increased by optimizing $O$. The optimization result is

$$O = \frac{N}{\sqrt{\sum_j h[N]^2(2j+1)^2}} \sum_j h[N]^2(2j+1)^2 \Pi(j)_A.$$  

(14)

where $h[N]^2(j)$ is the number of states with total spin $j$ in $N \frac{1}{2}$-spins and hence $h[N]^2(j) = (2j + 1)^2\left(\binom{N}{2j+1}\right)$. We showed in [4] the corresponding optimal measurement of Alice in the form of $\hat{\Pi}_i \equiv O^i\Pi_i O$. However, when the actual POVM elements $\Pi_i$ are derived from $\hat{\Pi}_i$, it will be found that those agree with Eq. (12). Note that, since $\rho$ is block diagonal with respect to $S_A$ and the optimal $O$ is an identity on each subspace, we have $\langle\rho, O\rangle = 0$ and as a result the measurement can also be considered as SRM for distinguishing $\{\sigma(i)\}$.

In the probabilistic version of PBT, the optimal $O$ which provides the maximum success probability of faithful teleportation $p = N/(N + 3)$ is given by

$$O = \sqrt{\frac{2N}{\sum_j (2j+1)^2} \sum_j h[N](2j+1)^2 \Pi(j)_A}.$$  

(15)

We showed in [4] the corresponding optimal measurement in the form of $\Pi_i = P^A_{AC} \otimes \hat{\Pi}_i A$, but it will be found that the actual POVM elements again agree with Eq. (12) without the implicit excess term of $(1/N)\Pi_0$. In this probabilistic case, $\Pi_0$ of Eq. (13) by itself constitutes a POVM element, such that $\Pi_0$ indicates the failure of faithful teleportation.

In this way, the optimal measurement of Alice for $d = 2$ is given by Eq. (12) in many cases: both $O = \mathbb{I}$ and optimal $O$ in the deterministic version, and optimal $O$ in the probabilistic version. The only exception is the case of $O = \mathbb{I}$ in the probabilistic version. This seems to rely on the property that $\rho$ is block diagonal with respect to $S_A$. Unfortunately, this property does not hold for general $d$ as shown in Appendix [3] where the result of $d = 3$ and $N = 2$ is explicitly shown. It is quite interesting that, even in this case, the optimal measurement of probabilistic PBT again agrees with SRM for distinguishing $\{\sigma(i)\}$.

### IV. Recoverable PBT

According to the no-go theorem for the faithful and deterministic universal programmable processor [5], a deterministic PBT protocol is inevitably forced to be an approximate one for finite $N$. Therefore, it may be convenient if, for the same measurement of Alice, Bob can lately choose between two choices: (1) usual PBT (with non-unit fidelity) by selecting one of the $N$ output ports or (2) faithful teleportation (with unit fidelity) by utilizing all the $N$ output ports. This protocol, say recoverable PBT, is indeed possible as shown below.

To this end, let us consider the optimal probabilistic PBT for $d = 2$ to teleport the $C$ qubit of $P_{CD}$. When Alice-obtains the outcome $\Pi_i$ with $i \neq 0$ in her measurement, the state of the $C$ qubit is faithfully teleported to the $B_i$ qubit, and hence the resulting state is $P_{DB_i}$. When Alice-obtains $\Pi_0$ that indicates the failure of faithful teleportation, the state of $BD$ is given by

$$\text{tr}_{AC} P_{0AC}O_A(P_{CD}^\perp \otimes P_{AB_1}^\perp \otimes \cdots \otimes P_{ANB_N}^\perp)O_A$$

$$= \frac{1}{2N+1} O_B P_{BD} O_B = \frac{1}{2N+1} O_B \mathbb{I}(\mathbb{N}^{N+1})_{BD} O_B$$

$$= \frac{6 \cdot 2^N}{(N + 2)(N + 3)} \sum_m |\frac{N+1}{2}, m\rangle \langle \frac{N+1}{2}, m|,$$

(16)

where we used Eq. (13) and (15), and

$$|\frac{N+1}{2}, m\rangle = \sqrt{\frac{N + m + \frac{1}{2}}{N + 1}} |\uparrow\rangle |D\rangle |\frac{N+1}{2}, m - \frac{1}{2}\rangle_B$$

$$+ \sqrt{\frac{N - m + \frac{1}{2}}{N + 1}} |\downarrow\rangle |D\rangle |\frac{N+1}{2}, m + \frac{1}{2}\rangle_B.$$  

(17)

Now, suppose that, after Alice obtains $\Pi_0$, she further measures the $z$-component of the total spin of $AC$ to determine $m$ in Eq. (18). When she obtains $m$, the state of $BD$ becomes proportional to $|\frac{N+1}{2}, m\rangle$, which is not maximally entangled between $D$ and $B$ unless $m = 0$ from Eq. (17). In this case, the initial entanglement of $C$ is not transferred to $B$ and Bob cannot recover the lost entanglement anymore. Instead of this measurement, therefore, suppose that Alice performs the measurement in the basis of

$$\{|e_m\rangle \equiv \sqrt{\frac{N + m + \frac{1}{2}}{N + 1}} |\frac{N+1}{2}, m\rangle + \sqrt{\frac{N - m + \frac{1}{2}}{N + 1}} |\frac{N+1}{2}, -m\rangle, \quad \text{for } m > 0,\$$

$$|e_0\rangle \equiv |\frac{N+1}{2}, 0\rangle, \quad \text{for } m = 0.$$  

(18)
It is not difficult to see that, when \( N \) is odd and thus \( m \) is an integer, all the above states are maximally entangled between \( D \) and \( B \), and hence the entanglement of \( C \) is completely transferred to \( B \) in this case. This implies that, if Bob knows the outcome of the measurement (denoted by \( (m, \pm) \)) and he applies an appropriate unitary transformation on \( B \) according to the outcome, he can completely recover the state of \( C \) in his hand. Note that, this does not work well for even \( N \), because \( |e_m^\pm\rangle \) is not maximally entangled for \( m = 1/2 \).

To summarize, the explicit protocol of recoverable PBT is as follows: Alice performs a measurement \( \{\Pi_0, \Pi_1, \cdots, \Pi_i, \cdots, \Pi_N\} \) on \( AC \) as in the case of optimal probabilistic PBT and obtains the outcome \( i \). When she obtains \( i = 0 \), she further performs the measurement in the basis of Eq. (18) and obtains \( (m, \pm) \). She then sends the outcome \( i \) and \( (m, \pm) \) to Bob. For \( i \neq 0 \), the state of the \( C \) qubit is faithfully teleported to the \( B_i \) qubit. For \( i = 0 \), Bob has two choices. If he ignores \( (m, \pm) \) and randomly picks up one of the \( B \) qubits as an output port, the protocol works as deterministic PBT. The entanglement fidelity is equal to the probability of obtaining \( i \neq 0 \), because \( \text{tr}(\Pi_0 \sigma^{(i)}) = 0 \), and hence \( F = N/(N + 3) \). If Bob utilizes the information of \( (m, \pm) \) to apply an appropriate unitary transformation to the whole of the \( B \) qubits, he can obtain the state of the \( C \) qubit faithfully. The recoverable PBT is considered to be a hybrid of PBT and STS. Indeed, the protocol completely agrees with STS for \( N = 1 \), where \( |e_1^\pm\rangle = |\phi^\pm\rangle \) and \( |e_0\rangle = |\psi^\pm\rangle \) in the standard notation of the Bell basis.

V. RE Derivation of Probability Bound

It has been shown that the success probability of probabilistic PBT for any \( d \) is upper bounded by [10]

\[
p \leq \frac{N}{N + d^2 - 1}. \tag{19}
\]

It seems very plausible that this bound is indeed reachable, because the bound agrees with the optimal probability for \( d = 2 \) with any \( N \), and even for the case of \( d = 3 \) with \( N = 2 \), where \( \rho \) is not block diagonal with respect to \( S_A \), as shown in Appendix B. Here, we rederive the bound in a way different from [10], which is convenient for the later discussions.

To this end, let us consider the setting of port-based superdense coding as shown in Fig. 1 where Alice and Bob previously share \( |\Psi\rangle \), Bob sends the \( B_k \) qubit to Alice, and Alice performs a measurement on \( B_k A \) to know the actual value of \( k \). Note that the roles of Alice and Bob are opposite to the usual setting of superdense coding, and note that \( |\Psi\rangle \) is not necessarily a maximally entangled state. Suppose that, to know \( k \), Alice performs the same measurement as probabilistic PBT, whose POVM elements are \( \{\Pi_0, \Pi_1, \cdots, \Pi_N\} \), and let \( q_{ik} \) be the probability that Alice obtains outcome \( i (\neq 0) \) when Bob sent \( B_k \) to Alice. Since the state that Alice measures is obtained by projecting \( P_{DB_k}^− \) to \( P_{CD} \otimes |\Psi\rangle \langle \Psi| \), we have

\[
q_{ik} = d^2 p_i \text{tr} (P_{DB_k}^− \otimes \chi_{B_k}) P_{DB_k}^− = \begin{cases}
  p_i & \text{for } i \neq 0, \\
  d^2 p_k & \text{for } i = 0.
\end{cases}
\tag{21}
\]

The success probability of PBT is given by \( p = \sum_{i \neq 0} p_i \). Since \( \sum_k \sum_{i \neq 0} q_{ik} \leq N \) and

\[
\sum_k \sum_{i \neq 0} q_{ik} = \sum_k \left( \sum_{i \neq 0} p_i - p_k + d^2 p_k \right) = (N + d^2 - 1) p,
\tag{22}
\]

we obtain the bound of Eq. (19). Note that, in this derivation, we only used the fact that the state of \( C \) is faithfully teleported to \( B \) in probabilistic PBT. Note further that \( q_{0k} = 0 \) must hold so that the bound of Eq. (19) is tight.

VI. FIDELITY BOUND DUE TO MONOGAMY

In the same setting as Fig. 1, let us now suppose that Alice performs the same measurement as deterministic PBT. The post-measurement state, denoted by \( \chi_{DB} \) hereafter, is close to but not equal to \( P_{DB}^- \). Then, we have

\[
q_{ik} = d^2 p_i \text{tr}(\chi_{DB}) P_{DB_k}^− = d^2 p_i F_{ik},
\tag{23}
\]

where \( F_{ik} \) is the entanglement fidelity with respect to \( P_{DB_k}^- \) when the state of \( C \) is teleported to \( B_k \). For the sake of simplicity, let us consider the symmetric case such that \( p_i = 1/N \), \( F_{ik} = F \) (irrespective of \( k \)), and \( F_{ik} = F' \) for \( i \neq k \), as this permutation symmetry generally holds in PBT. Namely, \( F \) stands for the (usual)
entanglement fidelity of the correct output port, and $F'$ stands for the fidelity of the other output port. Then, from the condition $\sum_{i \neq 0} q_{i|k} = 1$ in this deterministic case, we have

$$F + (N - 1)F' = \frac{N}{d^2}. \quad (24)$$

This equality already implies that faithful and deterministic PBT is impossible for finite $N$. Indeed, when $F = 1$ for the output port $B_i$, the reduced post-measurement state for the other port $B_k$ must have the form of $\chi_{DB_i} = (1/d)I_0 \otimes \chi_{B_k}$, and hence $F' = 1/d^2$, but those $F$ and $F'$ cannot satisfy Eq. (24) for finite $N$. In this way, Eq. (24) is a constraint on entanglement monogamy from above by $1/N$.

The fully entangled fraction is obtained by maximizing the following twirling operation is applied to the post-measurement state $\chi_{DB}$:

$$\int dU(U_D^* \otimes U_{B_1} \cdots U_{B_N}) \chi_{DB}(U_D^* \otimes \epsilon U_{B_1} \cdots U_{B_N})^1, \quad (27)$$

the resulting reduced states $\chi_{DB_i}$ are all isotropic states, whose fully entangled fraction has been obtained in [22]. We then have $F = F = 1 - \epsilon$ for the output port $B_i$ and $F' = (1 - F')/(d^2 - 1) = N/[d^2(N - 1)] + O(\epsilon/N)$ for the other output port $B_k$ because $F' < 1/d^2$. Namely, the fully entangled fraction of $\chi_{AB_i}$ can take this value, at least. Putting $F$ and $F'$ into Eq. (26), we obtain

$$F = 1 - \epsilon \leq 1 - \frac{1}{4(d - 1)N^2} + O(\frac{1}{N^2}). \quad (28)$$

In this way, the monogamy relation in asymmetric universal cloning bounds the entanglement fidelity of PBT from above by $1 - O(N^{-2})$. Note that this bound is tight (leaving for the coefficient) for $d = 2$, where $F = \cos^2 \pi/(N + 2) \to 1 - \pi^2/(2N^2)$ [4].

VII. PORT-BASED SUPERDENSE CODING

Superdense coding is a protocol dual to quantum teleportation, where the classical information capacity of $2 \log_2 d$ bits is achieved per qudit sent from Bob to Alice. In this section, we remark that the capacity $2 \log_2 d$ bits can be asymptotically achieved, i.e. port-based superdense coding is possible in the setting of Fig. 1.

When Bob sends $B_k$ qudit to Alice, the probability that Alice can obtain the outcome $i$ by the same measurement as deterministic PBT is given by Eq. (23). The entanglement fidelity employing SRM and maximally entangled $|\Psi\rangle$ is lower bounded by $F \geq 1 - (d^2 - 1)/N$ [3], but this bound has been slightly improved in [8] as

$$F \geq \frac{N}{N + d^2 - 1}. \quad (29)$$

The derivation of this bound using a convenient property of $\rho$, instead of using $\text{tr} \rho^2$, is given in Appendix C. We then have

$$q_{k|i} = \frac{d^2F}{N} \geq \frac{d^2}{N + d^2 - 1}. \quad (30)$$

Using this no-error probability, the mutual information between Bob and Alice, which takes maximum for Bob’s equal prior probability $1/N$, is

$$I(B : A) = \log_2 \frac{N}{N + d^2 - 1} + \frac{d^2}{N + d^2 - 1} \log_2 d^2. \quad (31)$$

At first glance, port-based superdense coding seems impossible because $I(B : A) \to 0$ in the limit of $d^2 \ll N \to \infty$, in quite contrast to $F \to 1$ in the same limit. However, $I(B : A)$ takes the maximum at $N = (d^2 - 1)^2/[(\log e \, d^2 - 1)[d^2 + 1]]$, and therefore with keeping $N = d^2/\log e \, d^2$ in the limit of $N \to \infty$, we have

$$I(B : A) \to 2 \log_2 d - \log_2 \log e \, d^2. \quad (32)$$

In this way, the mutual information asymptotically approaches to the superdense coding capacity, in the limit different from the fidelity of PBT. Although the application of port-based superdense coding is unknown, this may provide an intriguing example to investigate the duality between teleportation and superdense coding.

VIII. SUMMARY

In this paper, we first recalled the optimal protocols of PBT for $d = 2$ and paid attention to the fact that, in most cases of $d = 2$, the optimal measurements of Alice agree with SRM for distinguishing $\{\sigma^{(i)}\}$. We showed that, even in the higher dimension of $d = 3$, the optimal measurement of probabilistic PBT for $N = 2$ is SRM. It might be conjectured that this holds for any $d$ and $N$.

Next, we proposed a hybrid protocol between PBT and STS. In this protocol of recoverable PBT, Bob has two choices, to adopt PBT with an approximate fidelity by selecting one of $N$ output ports, or to adopt faithful teleportation by applying a unitary transformation to all the $N$ output ports as STS. We showed that recoverable PBT is possible at least for $d = 2$ and odd $N$. 

Moreover, we considered the setting of the port-based superdense coding as shown in Fig. 1 and rederived the upper bound of success probability of probabilistic PBT.\textsuperscript{10} In the same setting, we obtained a constraint between the entanglement fidelities of the output ports in PBT. We then regarded PBT as asymmetric $1 \to N$ universal cloning, and derived the upper bound of the fidelity expected from the entanglement monogamy relation in the asymmetric cloning. The obtained bound can explain why the entanglement fidelity of PBT is limited in the asymmetric cloning. The obtained bound can explain why the entanglement fidelity of PBT is limited.

Finally, we remarked that port-based superdense coding is possible. Indeed, the capacity of $2 \log_2 d$ bits per qudit sent is asymptotically achieved in the limit of $N, d^2 \to \infty$ with keeping $N = d^2 / \log_e d^2$, while $F \to 1$ in the limit of $d^2 \ll N \to \infty$ in PBT. Namely, in spite that port-based superdense coding and PBT are dual to each other, the perfect transmission of classical and quantum information, respectively, is achieved in the different limiting conditions. This will be a good example to deepen our understanding about the duality between superdense coding and teleportation.

Acknowledgments

This work was supported by JSPS KAKENHI Grants No. 23246071 and No.24540405.

Appendix A: Maximum eigenvalue of $\rho$

Let $|j, m, \alpha\rangle$ be a spin state of $\bar{A}_i$ where $\alpha$ distinguishes the permutation degeneracy, and

$$|\xi^{(i)}(j, m, \alpha)\rangle \equiv |\psi_-\rangle_{A_i} |j, m, \alpha\rangle_{\bar{A}_i}. \quad (A1)$$

By this, the block submatrix of $\rho$ with total spin angular momentum $j$ and its $z$-component $m$ is written as

$$\rho(j, m) = \frac{1}{d^{N-1}} \sum_{i=1}^{N} \sum_{\alpha} |\xi^{(i)}(j, m, \alpha)\rangle \langle \xi^{(i)}(j, m, \alpha)|. \quad (A2)$$

Let us then define the Gram matrix $\Gamma$ using $|\xi^{(i)}(j, m, \alpha)\rangle$, i.e. the matrix elements of $\Gamma$ are given by

$$\Gamma_{i\alpha,k\beta} = \langle \xi^{(i)}(j, m, \alpha)| \xi^{(k)}(j, m, \beta)\rangle. \quad (A3)$$

When the spin state $|j, m, \alpha\rangle$ of $\bar{A}_i$ for $i \neq 1$ is defined such that

$$|j, m, \alpha\rangle_{\bar{A}_i} \equiv |j, m, \alpha\rangle_{\bar{A}_i} \rangle_{A_i \to \bar{A}_i}, \quad (A4)$$

and hence $|\xi^{(i)}(j, m, \alpha)\rangle \equiv V_{A_i \bar{A}_i} |\xi^{(i)}(j, m, \alpha)\rangle$ with $V$ begin a swap operator, it is found that the matrix elements of $\Gamma$ are given by

$$(2s+1)\Gamma_{i\alpha,k\beta} = \begin{cases} (2s+1)\delta_{\alpha\beta} & \text{for } i = k, \\ \delta_{\alpha\beta} & \text{for } i \neq k \text{ but } i = 1 \text{ or } k = 1, \\ (j, m, \alpha) |V_{A_i \bar{A}_i}| |j, m, \beta\rangle & \text{otherwise}. \quad (A5)$$

Then, $\Gamma$ is a real symmetric matrix because the Clebsch-Gordan (CG) coefficients are all real. When $\gamma$ is an eigenvalue of $\Gamma$ and the corresponding normalized eigenvector is $\bar{\psi} = (\cdots, c_{i\alpha}, \cdots)^t$, it is not difficult to see that $|\psi\rangle = \sum c_{i\alpha} |\xi^{(i)}(j, m, \alpha)\rangle$ is an eigenstate of $\rho(j, m)$ and the eigenvalue is $\gamma/d^{N-1}$. Moreover, $|\psi\rangle$ is not normalized and $\langle \psi | \psi \rangle = \gamma$. Now, let us rewrite $|\psi\rangle$ as

$$|\psi\rangle = \sum_i d_i |\psi_{-i\alpha} \rangle_{A_i} \sum_{\alpha} c_{i\alpha} |j, m, \alpha\rangle_{\bar{A}_i}, \quad (A6)$$

where $d_i^2 = \sum_{\alpha} c_{i\alpha}^2$, and hence $\sum_i d_i^2 = 1$ and every $|f^{(i)}\rangle$ is normalized. Then,

$$\langle \psi | \psi \rangle = \sum_{i,k} d_i d_k \langle f^{(i)} | \bar{\psi}_{-i\alpha} \rangle_{A_i} \langle \psi_{-j\beta} | f^{(k)} \rangle_{A_k}$$

$$= \sum_i d_i^2 + \frac{1}{2s+1} \sum_{i \neq k} d_i d_k \langle f^{(i)} | [f^{(k)}] \rangle_{A_i \to A_k}$$

$$\leq \sum_i d_i^2 + \frac{1}{2s+1} \sum_{i \neq k} |d_i d_k|$$

$$= \frac{2s}{2s+1} \frac{1}{N} \frac{d-1}{d} \frac{N}{2s+1} + \frac{N}{2s+1}, \quad (A7)$$

where the Cauchy-Schwarz inequality was used in the second inequality. As a result, it is found that the maximum eigenvalue of $\rho$ is upper bounded by $(N+d-1)/d^N$, which is indeed achieved when $d_i = 1/\sqrt{N}$ and every $|f^{(i)}\rangle$ is the same symmetric function, e.g. when $j$ takes the maximum spin angular momentum $j = (N-1)s$.

Appendix B: Optimal probability for $d = 3$ and $N = 2$

In this case, $\rho$ has only one spin component $j = 1$. Let us denote the spin state on $CA_1A_2$ by $|1, m^{J}\rangle$, which is constructed by the addition of $S_A = J (J = 0, 1, 2)$ and 1-spin of C. Using the standard relation of the CG coefficients\textsuperscript{24, 25}, we have

$$\langle 1m^{J}| P_{BA_1}^{-1}| 1, m^{J}\rangle = \frac{1}{9} (-1)^{j+j'} \sqrt{(2J+1)(2J'+1)}$$

$$\langle 1m^{J}| P_{BA_2}^{-1}| 1, m^{J}\rangle = \frac{1}{9} \sqrt{(2J+1)(2J'+1)}$$

and therefore the matrix elements of $\rho(1, m)$ in the basis of $|1, m^{J}\rangle$ is

$$\rho(1, m) = \frac{2}{27} \begin{pmatrix} 1 & 0 & \sqrt{5} \\ 0 & 3 & 0 \\ \sqrt{5} & 0 & 5 \end{pmatrix} \ . \quad (B2)$$
where the rows and columns are indexed by \( J \). Now, by choosing \( O \) as

\[
X = OO^\dagger = \frac{6}{5} \mathbb{1}(0)_A + \frac{3}{5} \mathbb{1}(1)_A + \frac{6}{5} \mathbb{1}(2)_A, \tag{B3}
\]

the matrix elements of \( I_C \otimes X_A \) in the basis of \( |1, m^J \) are

\[
\mathbb{1}_C \otimes X_A = \frac{1}{5} \begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}, \tag{B4}
\]

because \( \mathbb{1}_C \otimes \mathbb{1}(J)_A = \sum_{j=|J|-s}^{J+s} \mathbb{1}(j)_{AC} \). Then, the POVM elements of

\[
\tilde{\Pi}_i = O \Pi_i O^\dagger = \frac{9}{10} P_{CA_i} \otimes \mathbb{1}_{\tilde{A}_i}, \tag{B5}
\]

satisfy the constraints of probabilistic PBT \[4\] evas \( \text{tr} X = 0 \) and \( \sum_i \tilde{\Pi}_i = \frac{27}{10} / \mathbb{1}_C \otimes X_A \). The corresponding success probability is \( p = (1/3^3) \sum_i \text{tr} \tilde{\Pi}_i = 1/5 \), which agrees with the upper bound of Eq. (19). It is intriguing that, even in this case, we have from Eq. (B1), \[12\] and \[B3\],

\[
\Pi_i = (\mathbb{1}_C \otimes X_A)^{-1/2} \left( \frac{10}{9} P_{BA_i} \right) (\mathbb{1}_C \otimes X_A)^{-1/2} = \rho^{-1/2} (\frac{1}{3} P_{BA_i} \rho^{-1/2} = \rho^{-1/2} \sigma(i) \rho^{-1/2}, \tag{B6}
\]

and hence the optimal measurement is SRM. Note that \( [\rho, O] = 0 \) also holds, and

\[
\Pi_0 = \mathbb{1} - \mathbb{1}_{\rho} = \mathbb{1}(N + 1) + \sum_m |\kappa_m \rangle \langle \kappa_m |, \tag{B7}
\]

where \( |\kappa_m \rangle = (5|1, m^J=0 \rangle - |1, m^J=2 \rangle) / \sqrt{6} \) is the eigenstate with a zero eigenvalue of \( \rho(1, m) \)

### Appendix C: Derivation of fidelity lower bound

For the operator \( \rho \), the following convenient relations hold:

\[
\langle \psi_{A_i}^\dagger | \rho | \psi_{A_i}^\dagger \rangle = \frac{N + d^2 - 1}{d^{N+1}} \mathbb{1}_{\tilde{A}_i},
\]

\[
\langle \psi_{A_i}^\dagger | \mathbb{1}_{\rho} | \psi_{A_i}^\dagger \rangle = \mathbb{1}_{\tilde{A}_i}. \tag{C1}
\]

By using \( X^{-1/2} \geq (3/2) \mathbb{1} - (1/2) X \) for \( X \geq 0 \), and by using \( |\xi^{(i)}(j, m, \alpha)\rangle \equiv |\psi_{A_i} \rangle_{ij} |m, \alpha \rangle_{A_i} \) defined in Appendix A, we have

\[
F = \frac{1}{d^2} \text{tr} \sum_{i=1}^N \rho^{-1/2} \sigma(i) \rho^{-1/2} \sigma(i) \geq \frac{N}{d^2 N} \sum_{j,m,\alpha} \left| \langle \xi^{(i)}(j, m, \alpha) | (\mathbb{1}_\rho - a \mathbb{1}) | \xi^{(i)}(j, m, \alpha) \rangle \right|^2
\]

\[
= \frac{N}{d^2 N} \sum_{j,m,\alpha} \left| \langle \xi^{(i)}(j, m, \alpha) | (\mathbb{1}_\rho - a \mathbb{1}) | \xi^{(i)}(j, m, \alpha) \rangle \right|^2
\]

\[
= \frac{N}{d^2 N} \left( \frac{3}{2} - \frac{a(N + d^2 - 1)}{2d^{N+1}} \right)^2, \tag{C2}
\]

where see Eq. \[A5\] for the second equality. This lower bound is maximized when \( a = d^{N+1} / (N + d^2 - 1) \), and hence we obtain Eq. \[B4\].
[20] H. Fan, Y.-N. Wang, L. Jing, J.-D. Yue, H.-D. Shi, Y.-L. Zhang, and L.-Z. Mu, Phys. Rep. 544, 241 (2014).
[21] A. Kay, R. Ramanathan, and D. Kaszlikowski, Quant. Inf. Comput. 13, 880 (2013).
[22] M. J. Donald, M. Horodecki, and O. Rudolph, J. Phys. A: Math. Theor. 43, 275203 (2010).
[23] R. F. Werner, J. Phys. A: Math. Gen. 34, 7081 (2001).
[24] A. Messiah, Quantum Mechanics (Dover Publications, Inc., Mineola, New York, 1999).
[25] D. A. Varshalovich, A. N. Moskalev, and V. K. Khe- sonskii, Quantum Theory of Angular Momentum (World Scientific, 1988).