A GENERALIZATION OF RANDOM MATRIX ENSEMBLE II: CONCRETE EXAMPLES AND INTEGRATION FORMULAE

JINPENG AN, ZHENGDONG WANG, AND KUIHUA YAN

Abstract. According to the classification scheme of the generalized random matrix ensembles, we present various kinds of concrete examples of the generalized ensemble, and derive their joint density functions in an unified way by one simple formula which was proved in [2]. Particular cases of these examples include Gaussian ensemble, chiral ensemble, new transfer matrix ensembles, circular ensemble, Jacobi ensembles, and so on. The associated integration formulae are also given, which are just many classical integration formulae or their variation forms.

1. Introduction

Guided by Dyson’s idea in [5], many authors have investigated the method of deriving the joint density functions of various kinds of random matrix ensembles in terms of Riemannian symmetric spaces. One of the most important works in this direction was made by Dueñez [4], in which the joint density functions for the circular ensemble and various kinds of Jacobi ensembles was obtained using an integration formula associated with the $KAK$ decomposition of compact Lie group, according to Cartan’s classification of compact irreducible Riemannian symmetric spaces. The achievement of this direction was summarized by the excellent review article of Caselle and Magnnea [3].

This is a sequel paper of [2], in which a generalization of the random matrix ensemble was defined. First we give a sketch of the content of [2]. Suppose a Lie group $G$ acts on an $n$-dimensional Riemannian manifold $X$ by $\sigma : G \times X \to X$, and suppose the induced Riemannian measure $dx$ is $G$-invariant. Let $Y$ be a closed submanifold of $X$ with the induced Riemannian measure $dy$, and let $K = \{g \in G : \sigma_g(y) = y$, $\forall y \in Y\}$. Define the map $\varphi : G/K \times Y \to X$ by $\varphi([g], y) = \sigma_g(y)$.

Let $X_z \subset X$, $Y_z \subset Y$ be closed subsets of measure zero in $X$ and $Y$, respectively. Denote $X' = X \setminus X_z$, $Y' = Y \setminus Y_z$. Suppose the following conditions hold.

(a) (invariance condition) $X' = \bigcup_{y \in Y'} O_y$.

(b) (transversality condition) $T_yX = T_yO_y \oplus T_yY$, $\forall y \in Y'$.

(c) (dimension condition) $\dim G_y = \dim K$, $\forall y \in Y'$.

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(d) (orthogonality condition) \( T_y Y \perp T_y O_y, \quad \forall y \in Y'. \)

Suppose \( d\mu \) is a \( G \)-invariant smooth measure on \( G/K \), and suppose \( p(x) \) is a \( G \)-invariant smooth function on \( X \). Then the system \((G, \sigma, X, p(x)dx, Y, dy)\) is called a generalized random matrix ensemble. \( X \) and \( Y \) are called the integration manifold and the eigenvalue manifold, respectively. It is proved that there is a quasi-smooth measure \( dv \) on \( Y \), which is called the generalized eigenvalue distribution, such that \( \varphi^*(p(x)dx) = dp\alpha dv \). We write \( dv \) as the form \( dv(y) = P(y)dy = p(y)J(y)dy \), where \( P(y) = p(y)J(y) \) is called the generalized joint density function. For \( y \in Y' \) define the map \( \Psi_y : l \rightarrow T_y O_y \) by \( \Psi_y(\xi) = \frac{1}{d\pi l}e^{t\xi}p(x)dx, \quad \forall \xi \in l \), where \( l \) is a linear subspace of \( g \) such that \( g = \mathfrak{t} \oplus l \). The main result in [2] is

\[
J(y) = C|\det \Psi_y|, \quad C = |\det((d\pi)_{\mathfrak{t}})|^{-1}.
\]

If in addition the following covering condition holds:

e (covering condition) The map \( \varphi : G/K \times Y' \rightarrow X' \) is a \( d \)-sheeted covering map, with \( d < +\infty \).

Then it is also proved in [2] that

\[
\int_X f(x)p(x)dx = \frac{1}{d} \int_Y \left( \int_{C/K} f(\sigma(y))d\mu(y) \right) dv(y)
\]

for all \( f \in C^\infty(X) \) with \( f \geq 0 \) or with \( f \in L^1(X, p(x)dx) \). [2] also give a classification scheme of the generalized ensembles, that is, the linear ensemble, the nonlinear noncompact ensemble, the compact ensemble, the group ensemble, the algebra ensemble, the pseudo-group ensemble, and the pseudo-algebra ensemble.

We should point out that though the proof of Formula [1] is not difficult, it provides a direct and unified way to compute the joint density functions for various kinds of random matrix ensembles. In this paper we will show that all the classical ensembles are particular cases of the generalized ensemble, and the corresponding density functions can be derived directly from [1]. The density functions for some new examples of the generalized ensemble can also derived form [1] explicitly.

According to the classification scheme of the generalized ensembles which was given in [2], we will present various kinds of concrete examples of generalized ensemble, and derive their joint density functions explicitly. The associated integration formulae will also be given. In §2 we will consider the linear ensemble, and present examples associated with \( GL(n, \mathbb{K}), O(m, n)_0, U(m, n), \) and \( Sp(m, n) \). Particular cases associated with \( GL(n, \mathbb{K}) \) is the Gaussian ensemble, particular cases associated with \( O(m, n)_0, U(m, n), \) and \( Sp(m, n) \) are the chiral ensembles (which are also called Laguerre ensembles). The four classes of the BdG ensemble and the two classes of the p-wave ensemble are also particular cases of the linear ensemble. The associated integration formula is just the integration formula for the Cartan decomposition of reductive Lie algebra in [6]. In §3 examples associated with \( GL(n, \mathbb{K}), O(m, n)_0, U(m, n), \) and \( Sp(m, n) \) of the nonlinear noncompact ensemble will be presented. Particular cases associated with \( GL(n, \mathbb{K}) \) are the so-called new transfer matrix ensembles. The three cases of the transfer matrix ensemble are also particular cases of the nonlinear noncompact ensemble. The associated integration formula is a variation form of the integration formula for Riemannian symmetric
we choose an orthogonal basis \( \mathfrak{g} \). In \( \S 4 \) we present examples of the compact ensemble associated with \( G_\ast = GL(n, \mathbb{K}) \) and \( G = SO(m + n), U(m + n), Sp(m + n) \). Particular case associated with \( GL(n, \mathbb{K}) \) is the circular ensemble, and particular cases associated with \( SO(m + n), U(m + n), \) and \( Sp(m + n) \) are Jacobi ensembles. The associated integration formula is a variation form of the integration formula for Riemannian symmetric space of compact type in [9]. \( \S 5 \) and \( \S 6 \) will be devoted to various examples of the group ensemble and the algebra ensemble. Examples associated with \( U(n), SO(2n + 1), Sp(n), SO(2n), SL(n, \mathbb{C}), Sp(n, \mathbb{C}), SO(2n, \mathbb{C}), \) and \( SO(2n + 1, \mathbb{C}) \) will be presented. We note that in some literatures, the group \( Sp(n) \) is denoted by \( USp(2n) \). Here we follow the notation in Knapp [8]. The associated integration formulae will be the Weyl integration formula for compact groups and its Lie algebra version. As a corollary of the associated integration formula, we will recover Harish-Chandra’s integration formula for real reductive group and its Lie algebra version.

2. Linear Ensembles

In this section we consider the linear ensemble. Let \( G \) be a real reductive Lie group with Lie algebra \( \mathfrak{g} \). Then \( G \) admits a global Cartan involution \( \Theta \), which induces a Cartan involution \( \theta \) of \( \mathfrak{g} \) with the associated Cartan decomposition \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \). Let \( K = \{ g \in G : \Theta(g) = g \} \), which is a maximal compact subgroup of \( G \) with Lie algebra \( \mathfrak{k} \). Let \( \mathfrak{a} \) be a maximal abelian subspace of \( \mathfrak{p} \), and let \( A = \exp(\mathfrak{a}) \) be the connected subgroup of \( G \) with Lie algebra \( \mathfrak{a} \). Then \( \mathfrak{p} = \bigcup_{k \in K} A_k(\mathfrak{a}), P = \bigcup_{k \in K} \sigma_k(\mathfrak{a}) \). Let \( M = \{ k \in K : A_k(\eta) = \eta, \forall \eta \in \mathfrak{a} \} = \{ k \in K : \sigma_k(\mathfrak{a}) = \mathfrak{a}, \forall \mathfrak{a} \in \mathfrak{A} \} \), \( m = \{ \xi \in \mathfrak{t} : [\xi, \eta] = 0, \forall \eta \in \mathfrak{a} \} \), then \( M \) is a closed subgroup of \( K \) with Lie algebra \( \mathfrak{m} \). Let \( \Sigma \) be the restricted root system associated with \( \mathfrak{a} \) with the Weyl group \( W = W(\Sigma) \). For \( \lambda \in \Sigma \), let \( \mathfrak{g}_\lambda \) be the corresponding root space. We choose a notion of positivity in \( \Sigma \) and denote by \( \Sigma^+ \) the set of positive restricted roots. There is a nondegenerate symmetric bilinear form \( B \) on \( \mathfrak{g} \) which is invariant under \( \theta \) and \( \text{Ad}(g) \) for all \( g \in G \), and satisfies that \( \mathfrak{t} \) and \( \mathfrak{p} \) are orthogonal under \( B \), \( B|\mathfrak{t} \) is negative definite, and \( B|\mathfrak{p} \) is positive definite. So \( (\xi, \eta) = -B(\xi, \Theta(\eta)) \) defines an inner product on \( \mathfrak{g} \). We write \( \mathfrak{b} = \mathfrak{a}^\perp \) in \( \mathfrak{p} \) and \( \mathfrak{l} = \mathfrak{m}^\perp \) in \( \mathfrak{t} \), then \( \mathfrak{b} \oplus \mathfrak{l} = \bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda \). For each \( \lambda \in \Sigma^+ \) we choose an orthogonal basis \( \{ \gamma_{\lambda, 1}, \cdots, \gamma_{\lambda, \beta_\lambda} \} \) of \( \mathfrak{g}_\lambda \) such that \( |\gamma_{\lambda, j}| = \frac{\lambda^2}{2} \), where \( \beta_\lambda = \dim \mathfrak{g}_\lambda \). For each \( \gamma_{\lambda, j} \), denotes \( \xi_{\lambda, j} = \gamma_{\lambda, j} + \theta_{\lambda_{\lambda, j}}, \zeta_{\lambda, j} = \gamma_{\lambda, j} - \theta_{\lambda_{\lambda, j}} \), then \( |\xi_{\lambda, j}| = |\zeta_{\lambda, j}| = 1 \) and we have \( \theta_{\xi_{\lambda, j}} = \xi_{\lambda, j}, \theta_{\zeta_{\lambda, j}} = -\zeta_{\lambda, j} \). So \( \xi_{\lambda, j} \in \mathfrak{t} \cap \bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda = \mathfrak{l}, \zeta_{\lambda, j} \in \mathfrak{p} \cap \bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda = \mathfrak{b} \). But the set \( \{ \xi_{\lambda, j}, \zeta_{\lambda, j} : \lambda \in \Sigma^+, j = 1, \cdots, \beta_\lambda \} \) is linearly independent, so

\[
\{ \xi_{\lambda, j} : \lambda \in \Sigma^+, j = 1, \cdots, \beta_\lambda \} \subset \mathfrak{l}
\]

is an orthonormal basis for \( \mathfrak{l} \), and

\[
\{ \zeta_{\lambda, j} : \lambda \in \Sigma^+, j = 1, \cdots, \beta_\lambda \} \subset \mathfrak{b}
\]

is an orthonormal basis for \( \mathfrak{b} \). And then we have \( \dim \mathfrak{l} = \dim \mathfrak{b} = \sum_{\lambda \in \Sigma^+} \beta_\lambda \). Let \( P = \exp(\mathfrak{p}) \), which is a closed submanifold of \( G \) satisfies \( T_e P = \mathfrak{p} \). In fact, \( P \) is the identity component of the set \( \{ g \in G : \Theta(g) = g^{-1} \} \) (see [11]). The exponential map

\[
e^{\theta a} = \exp(\theta a)
\]
exp : \mathfrak{p} \to P is a diffeomorphism, so we can define its inverse map \log : P \to \mathfrak{p}. We also have the global Cartan decomposition \( G = K \times P \). It is known that \( \mathfrak{p} \) is an invariant subspace of the adjoint action \( \text{Ad}|_K \), and \( P \) is also invariant under the conjugate action of \( K \). We denote \( A_k = \text{Ad}(k)|_{\mathfrak{p}} \) and \( \sigma_k(p) = kp_k^{-1} \) for \( k \in K \) and \( p \in P \).

In this section we consider the action \( A_k \) of \( K \) on \( \mathfrak{p} \). The inner product \( \langle \cdot, \cdot \rangle \) induces a linear Riemannian structure on \( \mathfrak{p} \), which is \( K \)-invariant under the action \( A_k \). So it induces a \( K \)-invariant Riemannian measure \( d\lambda \) on \( \mathfrak{p} \), which is just the Lebesgue measure on \( \mathfrak{p} \). Let \( p(\xi) \) be a \( K \)-invariant positive smooth function on \( \mathfrak{p} \), then \( p(\xi)d\lambda(\xi) \) is a \( K \)-invariant smooth measure. The Riemannian structure on \( \mathfrak{p} \) also induces a \( K \)-invariant Riemannian measure \( dY \) on \( \mathfrak{a} \), which is also the Lebesgue measure. Define the map \( \varphi : K/M \times \mathfrak{a} \to \mathfrak{p} \) by \( \varphi([k], \eta) = A_k(\eta) \). It is easy to prove that \( \text{Ad}_M(l) \subset \mathfrak{l} \), so under the natural identification \( (d\pi)_\mathfrak{l} : \mathfrak{l} \to T_{[e]}(K/M) \), the \( \text{Ad}_M \)-invariant inner product \( \langle \cdot, \cdot \rangle|_\mathfrak{l} \) on \( \mathfrak{l} \) induces a \( K \)-invariant Riemannian structure on \( K/M \), and then induces a \( K \)-invariant smooth measure \( d\mu \) on \( K/M \). These are sufficient for us to form a concrete example of the generalized random matrix ensemble with integration manifold \( \mathfrak{p} \) and eigenvalue manifold \( \mathfrak{a} \).

**Theorem 2.1.** The system \( (K, \mathfrak{A}, \mathfrak{p}, p(\xi)d\lambda(\xi), \mathfrak{a}, dY) \) is a generalized random matrix ensemble. Its generalized joint density function \( P(\eta) = p(\eta)J(\eta) \) is given by

\[
J(\eta) = \prod_{\lambda \in \Sigma^+} |\lambda(\eta)|^{\beta\lambda}.
\]

**Proof.** For \( \eta \in \mathfrak{a} \), we consider the map \( \Psi_\eta : \mathfrak{l} \to T_\eta O_\eta \) defined by \( \Psi_\eta(\xi) = \frac{d}{dt}|_{t=0} A_{\exp t\xi}(\eta) \). We have

\[
\Psi_\eta(\xi_{\lambda,j}) = \frac{d}{dt}|_{t=0} A_{\exp t\xi_{\lambda,j}}(\eta) = [\xi_{\lambda,j}, \eta] = -\lambda(\eta)(\xi_{\lambda,j}).
\]

So for \( \eta \in \mathfrak{a} \setminus \bigcup_{\lambda \in \Sigma^+} \ker \lambda \), \( \Psi_\eta \) is an isomorphism, hence \( T_\eta O_\eta = \text{Im}(\Psi_\eta) = \mathfrak{b} \). Let \( \mathfrak{p}_\lambda = \varphi(K/M, \bigcup_{\lambda \in \Sigma^+} \ker \lambda) \), then \( \mathfrak{p}_\lambda \) and \( \mathfrak{a}_\lambda = \mathfrak{a}_\lambda \cap \mathfrak{p}_\lambda = \mathfrak{a} \setminus \bigcup_{\lambda \in \Sigma^+} \ker \lambda \) are lower-dimensional sets in \( \mathfrak{p} \) and \( \mathfrak{a} \), respectively (in the sense of [8], Section 8.1), thus they have measures zero in the corresponding spaces.

Now we check the conditions (a), (b), (c), and (d). Let \( \mathfrak{p}' = \mathfrak{p} \setminus \mathfrak{p}_\lambda \) and \( \mathfrak{a}' = \mathfrak{a} \cap \mathfrak{p}' = \mathfrak{a} \setminus \bigcup_{\lambda \in \Sigma^+} \ker \lambda \). We have shown the condition (a) holds. For \( \eta \in \mathfrak{a}' \), by the definition of the Riemannian structure on \( \mathfrak{p} \), \( T_\eta \mathfrak{p} = \mathfrak{p} = \mathfrak{a} \oplus \mathfrak{b} = T_\eta \mathfrak{a} \oplus T_\eta O_\eta \) orthogonally, so the conditions (b) and (d) hold. For \( \eta \in \mathfrak{a}' \), suppose the isotropic subgroup associated with \( \eta \) is \( K_\eta \), then

\[
\dim K_\eta = \dim K - \dim O_\eta = \dim K - \dim \mathfrak{b} = \dim K - \dim \mathfrak{l} = \dim M.
\]

So the condition (c) also holds. This proves the system \( (K, \mathfrak{A}, \mathfrak{p}, p(\xi)d\lambda(\xi), \mathfrak{a}, dY) \) is a generalized random matrix ensemble.
We have seen above that \( \Psi_\eta(\xi_{\lambda,j}) = -\lambda(\eta)(\xi_{\lambda,j}) \) for each \( \lambda \in \Sigma^+ \) and \( j = 1, \cdots, \beta_\lambda \). By Formula (1.1),

\[
J(\eta) = C \prod_{\lambda \in \Sigma^+} |\lambda(\eta)|^{\beta_\lambda},
\]

where \( C = |\det((d\pi)_e)|^{-1} \). But \( (d\pi)_e \) is isometric, so \( C = 1 \). This complete the proof of the theorem. \( \square \)

From Theorem 2.1 we know that the generalized eigenvalue distribution \( d\nu \) is given by

\[
d\nu(\eta) = p(\eta) \prod_{\lambda \in \Sigma^+} |\lambda(\eta)|^{\beta_\lambda} dY(\eta).
\]

The generalized random matrix ensemble in Theorem 2.1 is called linear ensemble.

**Corollary 2.2.** Let \( f \in C^\infty(p) \) satisfies \( f \geq 0 \) or \( f \in L^1(p, p(\xi)dX(\xi)) \). Then we have the following integration formula

\[
\int_p f(\xi)p(\xi)dX(\xi) = \frac{1}{|W|} \int_a \left( \int_{K/M} f(A_k(\eta))d\mu([k]) \right) d\nu(\eta),
\]

where \( |W| \) is order of the Weyl group \( W \).

**Proof.** By Formula (1.2), it is sufficient to show that the covering condition (e) holds, and the covering sheet is \( |W| \). For each \( \eta \in \mathfrak{a}' \), suppose \( A_k(\eta) = \text{Ad}(k)\eta = \eta \) for some \( k \in K \). Since \( \text{Ad}(k) \) is an automorphism of \( \mathfrak{g} \), \( \text{Ad}(k) \) must fix \( \mathfrak{z}_\eta = \mathfrak{g}_\eta \), but \( \mathfrak{g} \) is also fixed by \( \text{Ad}(k) \), so \( \text{Ad}(k) \) fix \( \mathfrak{g}_\eta \cap \mathfrak{a} = \mathfrak{a} \), that is \( k \) is in the normalizer \( N_K(\mathfrak{a}) = \{k \in K : \text{Ad}(k)(\mathfrak{a}) = \mathfrak{a}\} \) of \( \mathfrak{a} \). But it is known that the analytic Weyl group \( W(G, A) = N_K(\mathfrak{a})/M \) is coincides with \( W(\Sigma) \) (see [3], Proposition 7.32), so the action \( \text{Ad}(k) \) on \( \mathfrak{a} \) coincides with some \( w \in W \). But \( \eta \) is a regular element and \( w(\eta) = \eta \), this force that \( w = 1 \), that is \( k \in Z_K(\mathfrak{a}) \). This proves that for each \( \eta \in Y' \), the isotropic subgroup \( K_\eta = M \). Next, also by the relation \( W(G, A) = W(\Sigma) \), it follows that for each \( \eta \in \mathfrak{a}' \), \( O_\eta \cap Y' \) has \( |W| \) points. By Corollary 3.6 in [2], \( \varphi : K/M \times Y' \to X' \) is a \( |W| \)-sheeted covering map. This proves the corollary. \( \square \)

**Remark 2.1.** Formula (2.3) has appeared in Helgason [9] (Chapter 1, Theorem 5.17). Here we recover it from the viewpoint of generalized random matrices.

**Example 2.1.** Let \( G = GL(n, \mathbb{K}) \), where \( \mathbb{K} \) is \( \mathbb{R} \), \( \mathbb{C} \), or \( \mathbb{H} \). Then \( G \) is real reductive, when we view \( GL(n, \mathbb{C}) \) and \( GL(n, \mathbb{H}) \) as real Lie groups. The Cartan involution of the Lie algebra \( \mathfrak{g} = \mathfrak{gl}(n, \mathbb{K}) \) can be chosen as \( \theta(\xi) = -\xi^* \), where the symbol “\( \xi^* \)” means the transpose of \( \xi \) when \( \mathbb{K} = \mathbb{R} \), and the conjugate transpose when \( \mathbb{K} = \mathbb{C} \) or \( \mathbb{H} \). The corresponding Cartan decomposition is \( \mathfrak{h} = \{\xi \in \mathfrak{gl}(n, \mathbb{K}) : \xi^* = -\xi\} \), \( \mathfrak{p} = \{\xi \in \mathfrak{gl}(n, \mathbb{K}) : \xi^* = \xi\} \). The space \( \mathfrak{a} = \{\eta = \text{diag}(x_1, \cdots, x_n) : x_k \in \mathbb{R}\} \) is a maximal abelian subspace of \( \mathfrak{p} \) in each of the three cases. The corresponding global Cartan involution of \( GL(n, \mathbb{K}) \) is \( \Theta(\xi) = (\xi^*)^{-1} \), and the maximal compact subgroup \( K = \{g \in G : \Theta(\xi) = g\} \) is \( O(n) \), \( U(n) \), or \( Sp(n) \) when \( \mathbb{K} = \mathbb{R}, \mathbb{C}, \) or \( \mathbb{H} \), respectively. Let \( \mathfrak{e}_r \in \mathfrak{a}^* \) denotes \( \mathfrak{e}_r(\text{diag}(x_1, \cdots, x_n)) = x_r \) for each \( 1 \leq r \leq n \), then one can choose the positive restricted root system as \( \Sigma^+ = \{\mathfrak{e}_r - \mathfrak{e}_s : 1 \leq r < s \leq n\} \), and \( \beta_{\mathfrak{e}_r - \mathfrak{e}_s} = \dim \mathfrak{g}_{\mathfrak{e}_r - \mathfrak{e}_s} = \beta \), where \( \beta = 1, 2, \) or 4 when \( \mathbb{K} = \mathbb{R}, \mathbb{C}, \) or \( \mathbb{H} \), respectively. Let \( p(\xi) \) be a \( K \)-invariant positive smooth function on \( \mathfrak{p} \). Then
by Theorem 2.1, the density function $P(\eta) = \frac{p(\eta)J(\eta)}{J(\eta)}$ for the linear ensemble $(K, A, p, p(\xi)dX(\xi), a, dY)$ is determined by

\begin{equation}
J(\eta) = \prod_{1 \leq r < s \leq n} |x_r - x_s|^\beta.
\end{equation}

If the function $p(\xi)$ is of the particular form $p(\xi) = \exp(-\alpha tr\xi^2 + b tr\xi + c)$ such that $p(\xi)dX(\xi)$ is a probability measure, the linear ensemble $(K, A, p, p(\xi)dX(\xi), a, dY)$ is just the Gaussian orthogonal, unitary, and symplectic ensembles. Thus we recover the joint density functions for the three cases of Gaussian ensemble from the viewpoint of generalized random matrix ensemble.

\[ \square \]

**Example 2.2.** Let $G = O(m, n)_0, U(m, n),$ or $Sp(m, n)$, which are all real reductive. These groups are defined to be the connected component of

\[ \{g \in GL(m + n, \mathbb{K}) : g^*I_{m,n}g = I_{m,n}\}, \]

where $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, respectively, where $I_{m,n} = \begin{pmatrix} I_m & 0 \\ 0 & -I_n \end{pmatrix}$. Without loss of generality, we may assume $m \geq n$. The Lie algebras $\mathfrak{o}(m, n), \mathfrak{u}(m, n),$ and $\mathfrak{sp}(m, n)$ of the three groups are

\[ \mathfrak{g} = \{ \xi \in \mathfrak{gl}(m + n, \mathbb{K}) : \xi^*I_{m,n} + I_{m,n}\xi = 0 \} \]

\[ = \left\{ \begin{pmatrix} A & B \\ -B^* & D \end{pmatrix} : A + A^* = 0, D + D^* = 0 \right\}. \]

The Cartan involution of $\mathfrak{g}$ can be chosen as $\Theta(\xi) = -\xi^*$, and the corresponding Cartan decompositions is $\mathfrak{k} = \begin{pmatrix} A_0 & 0 \\ 0 & D_0 \end{pmatrix}$, $\mathfrak{p} = \begin{pmatrix} 0 & B_0 \\ -B_0^* & 0 \end{pmatrix}$. Let $E_{rs}$ denotes the $(m + n)$-by-$(m + n)$ matrix with 1 at the $(r, s)$-th entry and 0 elsewhere. Then one can easily checked the space

\begin{equation}
\mathfrak{a} = \left\{ \eta = \sum_{k=1}^{n} x_k(E_{m-k+1,m+k} + E_{m+k,m-k+1}) : x_k \in \mathbb{R} \right\}
\end{equation}

is a maximal abelian subspace of $\mathfrak{p}$ in each of the three cases. The corresponding global Cartan involution is $\Theta(g) = (g^*)^{-1}$, and the maximal compact subgroup $K = \{ g \in G : \Theta(g) = g \} = S(O(m) \times O(n))$ (which means the subgroup of $O(m) \times O(n)$ consists of elements with determinant 1), $U(m) \times U(n)$, or $Sp(m) \times Sp(n)$ when $G = O(m, n)_0, U(m, n),$ or $Sp(m, n)$, respectively. Let $e_r, e_s \in \mathfrak{a}^*$ denotes $e_r(\eta) = x_r$ for each $1 \leq r \leq n$, then one can choose the positive restricted root system as

\begin{equation}
\Sigma^+ = \{ e_r \pm e_s : 1 \leq r < s \leq n \} \cup \{ e_r, 2e_r : 1 \leq r \leq n \},
\end{equation}

and it can be shown that $\beta_{e_r \pm e_s} = \beta, \beta_{e_r} = \beta(m - n),$ and $\beta_{e_s} = \beta - 1$, where $\beta = 1, 2,$ or 4 when $G = O(m, n)_0, U(m, n),$ or $Sp(m, n)$ (if $\beta = 0$ for some $\lambda \in \Sigma^+$, the root $\lambda$ should be omitted). By Theorem 2.1 we can compute the factor $J(\eta)$ as

\begin{equation}
J(\eta) = 2^{(\beta-1)n} \prod_{1 \leq r < s \leq n} |x_r - x_s|^\beta \prod_{r=1}^{n} |x_r|^{\beta(m-n+1)-1}.
\end{equation}

Let $p(\xi)$ be a $K$-invariant positive smooth function on $\mathfrak{p}$, then the density function $P(\eta) = \frac{p(\eta)J(\eta)}{J(\eta)}$ for the three cases of the linear ensemble $(K, A, p, p(\xi)dX(\xi), a, dY)$ is determined by Formula (2.7). If the function $p(\xi)$ is chosen of the particular form $p(\xi) = \exp(-\alpha tr(\xi^*))$ such that $p(\xi)dX(\xi)$ is a probability measure, the linear ensemble $(K, A, p, p(\xi)dX(\xi), a, dY)$ is just the chiral orthogonal, unitary, and symplectic ensembles (see [3]).

\[ \square \]
The four classes of the BdG ensemble and the two classes of the \( p \) wave ensemble are also particular cases of the linear ensemble. For the lack of space, we only point out what the corresponding groups \( G \) and \( K \) are. The reader can easily obtain the other objects and derive their joint density functions from Theorem 2.1 For the BdG ensembles, \( G = SO(4n, \mathbb{C}), Sp(n, \mathbb{C}), SO^*(4n) \), and \( Sp(n, \mathbb{R}) \), the corresponding \( K = SO(4n), Sp(n), U(2n) \), and \( U(n) \), respectively. For the \( p \)-wave ensembles, \( G = SO(2n + 1, \mathbb{C}) \) and \( SO^*(4n + 2) \), the the corresponding \( K = SO(2n + 1) \) and \( U(2n + 1) \).

3. Nonlinear noncompact ensembles

In some sense, the nonlinear noncompact ensemble is the nonlinear version of the linear ensemble. But something will be different. Let \( G \) be a reductive Lie group, and keep the notations at the beginning of §2. Recall that the group \( K \) acts on \( P \) by \( \sigma_k(p) = kp^{k-1} \). The inner product \( \langle \cdot, \cdot \rangle \) induces a \( G \)-left invariant and \( K \)-bi-invariant Riemannian structure on \( G \), and then induces Riemannian structures on \( P \) and \( A \) as well as the Riemannian measures \( dx \) and \( da \) on \( P \) and \( A \), respectively. Since the induced Riemannian structure on \( P \) is \( K \)-invariant, the measure \( dx \) on \( P \) is also \( K \)-invariant. As in the previous section, the inner product \( -B_{ij} \) induces a \( K \)-invariant Riemannian structure on \( K/M \), then induces a \( K \)-invariant Riemannian measure \( d\mu \). Let \( p(x) \) be a \( K \)-invariant positive smooth function on \( P \). Define the map \( \varphi : K/M \times A \to P \) by \( \varphi([k], a) = \sigma_k(a) \). Then we can construct the nonlinear noncompact ensemble with integration manifold \( P \) and eigenvalue manifold \( A \) as follows.

**Theorem 3.1.** Let the objects be as above. Then the system \( (K, \sigma, P, p(x)dx, A, da) \) is a generalized random matrix ensemble. Its generalized joint density function \( P(a) = p(a)J(a) \) is given by

\[
J(a) = 2^{\dim \mathfrak{l}} \prod_{\lambda \in \Sigma^+} \left( \frac{\sinh \frac{\lambda(\eta)}{2}}{\sqrt{\cosh \lambda(\eta)}} \right)^{\beta_{\lambda}},
\]

where \( \eta = \log a \).

**Proof.** For \( a \in A \), consider the map \( \Psi_a : I \to T_a O_a, \Psi_a(\xi) = \frac{d}{dt}_{|t=0} \sigma \exp_t \xi(a) \). Then we have

\[
\Psi_a(\xi_{\lambda,j}) = \frac{d}{dt}_{|t=0} e^{t \xi_{\lambda,j}} ae^{-t \xi_{\lambda,j}} = (da) \left( e^{-\ad(a^{-1}) \xi_{\lambda,j}} e^{-t \xi_{\lambda,j}} \right) = (da) \left( \ad(a^{-1}) \xi_{\lambda,j} - \xi_{\lambda,j} \right) = (da) \left( e^{-\ad(a^{-1}) \xi_{\lambda,j} - \xi_{\lambda,j}} \right) = (da) \left( e^{-\lambda(\eta) \gamma_{\lambda,j} + e^{\lambda(\eta)} \theta \gamma_{\lambda,j} - \xi_{\lambda,j}} \right) = (da) \left( -\sinh \lambda(\eta) \zeta_{\lambda,j} + (\cosh \lambda(\eta) - 1) \xi_{\lambda,j} \right).
\]
Since $dl_a$ is isometric,
\[
\left| \Psi_a(\xi_{\lambda,j}) \right| = |(-\sinh\lambda(\eta))\xi_{\lambda,j} + (\cosh\lambda(\eta) - 1)\xi_{\lambda,j}| = \sqrt{\sinh^2\lambda(\eta) + (\cosh\lambda(\eta) - 1)^2} = 2 \left| \sinh\frac{\lambda(\eta)}{2} \right| \sqrt{\cosh\lambda(\eta)}.
\]

So if $\lambda(\eta) \neq 0$, that is $a \notin \exp(\ker \lambda)$, then $|\Psi_a(\xi_{\lambda,j})| \neq 0$. Let $A_\lambda = \bigcup_{\lambda \in \Sigma^+} \exp(\ker \lambda)$, then for $a \in A' = A' \setminus A_\lambda$, $\Psi_a$ is an isomorphism. Let $P'_z = \varphi(K/M, A_\lambda)$ and $P' = P \setminus P'_z$. Then $A_\lambda$ and $P'_z$ are lower-dimensional sets in $A$ and $P$, respectively, and it is obvious that condition (a) holds. By the computation above, $\Psi_a(\xi_{\lambda,j}) \perp T_a A$, so $Im(\Psi_a) = T_a O_a \perp T_a A$. But for $a \in A' \dim T_a A + \dim T_a O_a = \dim a + \dim l = \dim p = \dim T_a P$, so $T_a P = T_a A \perp T_a O_a$ orthogonally. This means conditions (b) and (d) hold. Similar to the proof of Theorem 2.1 the dimension condition (c) also holds. This proves that the system $(K, \sigma, p, p(x)dx, A, da)$ is a generalized random matrix ensemble. Since $|\det((d\pi)e_1)| = 1$, by Formula (1.1), we have
\[
J(a) = |\det \Psi_a| = \prod_{\lambda \in \Sigma^+} \prod_{j=1}^{\beta_\lambda} |\Psi_a(\xi_{\lambda,j})| = 2^{\dim A} \prod_{\lambda \in \Sigma^+} \left( \left| \sinh\frac{\lambda(\eta)}{2} \right| \sqrt{\cosh\lambda(\eta)} \right)^{\beta_\lambda}.
\]

The above theorem tells us that the generalized eigenvalue distribution $d\nu$ is given by
\[
d\nu(a) = 2^{\dim A} p(a) \prod_{\lambda \in \Sigma^+} \left( \left| \sinh\frac{\lambda(\eta)}{2} \right| \sqrt{\cosh\lambda(\eta)} \right)^{\beta_\lambda} da,
\]
where $\eta = \log a$.

**Corollary 3.2.** Let $f \in C^\infty(P)$ satisfies $f \geq 0$ or $f \in L^1(P, p(x)dx)$. Then we have the following integration formula
\[
\int_P f(x)p(x)dx = \frac{1}{|W|} \int_A \left( \int_{K/M} f(\sigma_k(a))d\mu(|k|) \right) d\nu(a).
\]

**Proof.** Similar to the proof of Corollary 2.2 it is sufficient to check the covering condition (e) hold with covering sheet $|W|$. But we notice that $\exp|_P : p \to P$ is a diffeomorphism, and $\exp|_P(p_z) = P_z$, $\exp|_P(p'_z) = P'$. So the proof reduces to that of Corollary 2.2. \qed

Note that the space $G/K$ is a Riemannian symmetric space of noncompact type, and the map $\phi : G/K \to P$ defined by $\phi([g]) = g\Theta(g)^{-1}$ is a diffeomorphism (see [1]). So Corollary 3.2 can be viewed as an integration formula for symmetric space of noncompact type. Now we make it precisely. Under the identification $p \cong T_{[g]}(G/K)$, the inner product $B_{[g]}$ induces a $G$-invariant Riemannian structure on $G/K$, and then induces a $G$-invariant measure $d\mu_1$ on $G/K$. Then we have
Corollary 3.3. Let $f \in C^\infty(G/K)$ satisfies $f \geq 0$ or $f \in L^1(G/K, d\mu_1)$. Then

$$\int_{G/K} f([g])d\mu_1([g]) = \frac{1}{|W|} \int_A \left( \int_{K/M} f([ka])d\mu([k]) \right) \delta(a)da,$$

where

$$\delta(a) = \prod_{\lambda \in \Sigma^+} |\sinh \lambda(\eta)|^{\beta}\lambda,$$

here $\eta = \log a$.

Proof. First we compute the expression $|\det(d\phi)_{[a]}|$ for $a \in A$. Choose an orthonormal basis $\eta_1, \cdots, \eta_{\dim a}$ of $a$. Then the set

$$\left\{ \frac{d}{dt}_{|t=0} [ae^{t\eta_j}] : 1 \leq j \leq \dim a \right\} \bigcup \left\{ \frac{d}{dt}_{|t=0} [ae^{t\lambda,j}] : \lambda \in \Sigma^+, 1 \leq j \leq \beta_\lambda \right\}$$

is an orthonormal basis of $T_{[a]}(G/K)$. It is easy to show that

$$(d\phi)_{[a]} \left( \frac{d}{dt}_{|t=0} [ae^{t\eta_j}] \right) = (dl_{a^2})2\eta_j,

(d\phi)_{[a]} \left( \frac{d}{dt}_{|t=0} [ae^{t\lambda,j}] \right) = (dl_{a^2})2(e^{-\lambda(\eta)}\gamma_{\lambda,j} - e^{\lambda(\eta)}\theta\gamma_{\lambda,j}),$$

where $\eta = \log a$. Then

$$|\det(d\phi)_{[a]}| = \prod_{j=1}^{\dim a} |2\eta_j| \prod_{\lambda \in \Sigma^+} \prod_{j=1}^{\beta_\lambda} |2(e^{-\lambda(\eta)}\gamma_{\lambda,j} - e^{\lambda(\eta)}\theta\gamma_{\lambda,j})|$$

$$= 2^{\dim a} \prod_{\lambda \in \Sigma^+} \left( \cosh 2\lambda(\eta) \right)^{\beta_\lambda}.$$

To fulfill the proof of the corollary, we define some auxiliary maps. Let $\psi : K/M \times A \to P$ be $\psi([k], a) = ka^2k^{-1}$, $\rho : K/M \times A \to G/K$ be $\rho([k], a) = [ka]$, and $sq : A \to A$ be $sq(a) = a^2$. Then one can easily check that $\psi = \phi \circ \rho = \phi \circ (id \times sq)$, form which we can easily get

$$\rho^*(d\mu_1) = |\det(d\phi)_{[a]}|^{-1} 2^{\dim a} J(a^2)d\mu da

= \prod_{\lambda \in \Sigma^+} |\sinh \lambda(\eta)|^{\beta_\lambda} d\mu da.$$

Since $\rho = (\phi)^{-1} \circ \phi \circ (id \times sq)$ is a $|W|$ sheeted covering map, by Proposition 3.1 in [2], we get the desired integration formula (3.4). \qed

Remark 3.1. Formula (3.4) has appeared in Helgason [3] (Chapter 1, Theorem 5.8).

The following two examples are nonlinear versions of Example 2.1 and 2.2 in the previous section.

Example 3.1. Let $G = GL(n, \mathbb{K})$, where $\mathbb{K}$ is $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Then $G$ is real reductive. We choose the Cartan involution of $g = \mathfrak{gl}(n, \mathbb{K})$ as $\theta(\xi) = -\xi^*$, then the corresponding global Cartan involution of $GL(n, \mathbb{K})$ is $\Theta(g) = (g^*)^{-1}$ (see Example 2.1). Recall that The corresponding Cartan decomposition of $\mathfrak{gl}(n, \mathbb{K})$ is $\mathfrak{k} = \{ \xi \in \mathfrak{gl}(n, \mathbb{K}) : \xi^* = -\xi \}$, $\mathfrak{p} = \{ \xi \in \mathfrak{gl}(n, \mathbb{K}) : \xi^* = \xi \}$. The space $\mathfrak{a} = \{ \eta = \text{diag}(x_1, \cdots, x_n) : x_k \in \mathbb{R} \}$ is a maximal abelian subspace of $\mathfrak{p}$ for each of the three cases, and the subgroup $A = \exp(\mathfrak{a}) = \{ a = \text{diag}(a_1, \cdots, a_n) : a_k > 0 \}$. For $\mathbb{K} = \mathbb{R}$, the maximal compact subgroup $K = \{ g \in GL(n, \mathbb{R}) : (g^t)^{-1} = g \}$ is $O(n)$. The closed submanifold $P = \exp(\mathfrak{p})$, which is the identity component of $\{ g \in GL(n, \mathbb{R}) : g^t = g \}$, is the set of all real symmetric positive-definite matrices.
For $K = \mathbb{C}$, the maximal compact subgroup $K = \{ g \in GL(n, \mathbb{C}) : (g^*)^{-1} = g \}$ is $U(n)$. Now the closed submanifold $P$ is the set of all complex Hermitian positive-definite matrices. For the case that $K = \mathbb{H}$, the maximal compact subgroup $K = \{ g \in GL(n, \mathbb{H}) : (g^*)^{-1} = g \}$ is $Sp(n)$. Now the closed submanifold $P$, which is the identity component of $\{ g \in GL(n, \mathbb{H}) : g^* = g \}$, is the set of all quaternion self-adjoint positive-definite matrices. Recall that we can choose the positive restricted root system as $\Sigma^+ = \{ e_r - e_s : 1 \leq r < s \leq n \}$ for each case, and $\beta_{e_r - e_s} = \beta$, where $\beta = 1, 2$, or 4 when $K = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. For $a = \text{diag}(a_1, \cdots, a_n) \in A$, let $\eta = \log a = \text{diag}(x_1, \cdots, x_n) \in a$, where $x_k = \log a_k$. Then by Theorem 3.1

$$J(a) = 2^{\frac{\beta n(n-1)}{2}} \prod_{1 \leq r < s \leq n} \left| \frac{x_r - x_s}{2} \right| \left( \cosh(x_r - x_s) \right)^{\frac{\beta}{2}}$$

$$= 2^{\frac{\beta n(n-1)}{2}} \prod_{1 \leq r < s \leq n} |a_r - a_s|^\beta (a_r^2 + a_s^2)^{\frac{\beta}{2}}.$$  

Let $p(x)$ be a $K$-invariant positive smooth function on $P$, then the density function $P(a) = p(a)J(a)$ for the nonlinear noncompact ensemble $(K, \sigma, P, p(x)dx, A, da)$ is determined by Formula (3.5). For some particular choice of $p(x)$, these kinds of ensembles are called the new transfer matrix ensembles in [3]. But their density functions were not derived there. \hfill \square

**Example 3.2.** Let $G = O(m, n)_0, U(m, n), \text{or} Sp(m, n)$. Without loss of generality, we assume $m \geq n$. The Lie algebra for each of the three groups has been given in Example 2.2. We choose the Cartan involution of the Lie algebra as $\Theta(\xi) = -\xi^*$ with the corresponding global Cartan involution of the group as $\Theta(g) = (g^*)^{-1}$. Then the corresponding Cartan decomposition is $\mathfrak{k} = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \right\}, \mathfrak{p} = \left\{ \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix} \right\}$, and the corresponding maximal compact subgroup $K = S(O(m) \times O(n)), U(m) \times U(n)$, or $Sp(m) \times Sp(n)$, respectively. It is easy to show that the subgroup $A = \exp(\mathfrak{a})$ corresponding to the maximal abelian subspace $\langle \mathfrak{a} \rangle$ of $\mathfrak{p}$ is

$$A = \left\{ a = \sum_{k=1}^n a_k (E_{m-k+1,m-k} + E_{m-k,m+k}) + \sum_{k=1}^n \pm \sqrt{a_k^2 - 1}(E_{m-k+1,m+k} + E_{m+k,m-k+1}) : a_k \geq 1 \right\}.$$  

In fact, under the exponential map $\exp : \mathfrak{a} \to A$, $a_k = \cosh x_k$. The closed submanifold $P = \exp(\mathfrak{p})$ is the identity component of the set $\{ g \in G : g^* = g \}$ for each case. Recall that we can choose the positive restricted root system as $\Sigma^+$, and we have $\beta_{e_r + e_s} = \beta$, $\beta_{e_r - e_s} = \beta(\beta - n)$, $\beta_{e_r} = \beta - 1$, where $\beta = 1, 2$, or 4 when $G = O(m, n)_0, U(m, n), \text{or} Sp(m, n)$. By Theorem 3.1 the factor $J(a)$ is given by

$$J(a) = 2^{\frac{\beta n(n-1)}{2}} \prod_{1 \leq r < s \leq n} \left| \frac{x_r + x_s}{2} \right| \left( \cosh(x_r + x_s) \right)^{\frac{\beta}{2}}$$

$$\prod_{r=1}^n \left| \frac{x_r}{2} \right| \left( \cosh x_r \right)^{\frac{\beta(n-1)}{2}} \left| \sinh x_r \right|^{\beta - 1} \left( \cosh 2x_r \right)^{\frac{\beta - 1}{2}}$$

$$= 2^{\frac{\beta n(n-1)}{2}} \prod_{1 \leq r < s \leq n} |a_r - a_s|^\beta (a_r^2 + a_s^2 - 1)^{\frac{\beta}{2}}$$

$$\prod_{r=1}^n ((a_r^2 - 1)(2a_r^2 - 1))^{\frac{\beta(n-1)}{2}} (a_r(a_r - 1))^{\frac{\beta(n-1)}{2}}.$$
Let $p(x)$ be a $K$-invariant positive smooth function on $P$, then the density function 
$\mathcal{P}(a) = p(a)J(a)$ for the nonlinear noncompact ensemble $(K, \sigma, P, p(x)dx, A, da)$ is given by (4). This may be viewed as nonlinear noncompact orthogonal, unitary, and symplectic ensembles with parameter $(m, n)$.

The three classes of the transfer matrix ensemble are also particular cases of the nonlinear noncompact ensemble. The corresponding group $G = Sp(n, \mathbb{R}), U(n, n)$, and $SO^*(4n)$, and the corresponding $K = U(n), U(n) \times U(n)$, and $U(2n)$. The reader can easily obtain the other objects and derive their joint density functions from Theorem 3.1.

4. Compact ensembles

Consider a connected compact Lie group $G$ with Lie algebra $\mathfrak{g}$. Suppose $\Theta$ is a global involution automorphism of $G$ with the induced involution $\theta = d\Theta$ of $\mathfrak{g}$. Let $K = \{g \in G : \Theta(g) = g^{-1}\}$, and let $\mathfrak{k}$ and $\mathfrak{p}$ be the eigenspaces of $\theta$ with eigenvalue 1 and $-1$, respectively. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, and we have $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. Let $P = \exp(\mathfrak{p})$, then $P$ is invariant under the conjugate action of $K$. It was proved in [1] that $P$ is a closed submanifold of $G$ satisfies $T_xP = \mathfrak{p}$, which is just the identity component of the set $\{g \in G : \Theta(g) = g^{-1}\}$, and we have $G = KP$.

Let $G_C$ be the complexification of $G$ with Lie algebra $\mathfrak{g}_C$, then the real Lie algebra $\mathfrak{g}_* = \mathfrak{k} \oplus \mathfrak{p}_*$ is a real form of $\mathfrak{g}_C$, where $\mathfrak{p}_* = i\mathfrak{p}$. Let $G_*$ be the subgroup of $G_C$ with Lie algebra $\mathfrak{g}_*$ such that $G \cap G_* = K$, then $G_*$ is reductive, and the direct sum $\mathfrak{g}_* = \mathfrak{k} \oplus \mathfrak{p}_*$ is just the Cartan decomposition of $\mathfrak{g}_*$. Note that given $G_*$, we can recover the groups $G_C$ and $G$, since $G_C$ is a (connected) complexification of $G_*$, and $G$ is a maximal compact group of $G_C$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$, $A$ be the connected subgroup with Lie algebra $\mathfrak{a}$, which is a torus of $G$. Since $K$ is a maximal compact subgroup of $G_*$ with Lie algebra $\mathfrak{k}$ and $\mathfrak{a}$ is a maximal abelian subspace of $i\mathfrak{p}$, we have $i\mathfrak{p} = \bigcup_{k \in K} \text{Ad}(k)\mathfrak{a}$. So $\mathfrak{p} = \bigcup_{k \in K} \text{Ad}(k)\mathfrak{a}$, and then $P = \bigcup_{k \in K} k\mathfrak{ak}^{-1}$. We denote $\sigma_k(p) = kp_k^{-1}$ for $k \in K$ and $p \in P$.

Let $M = \{k \in K : \sigma_k(a) = a, \forall a \in A\}$. Then $M$ is a closed subgroup of $K$ with Lie algebra $\mathfrak{m} = \{\xi \in \mathfrak{k} : [\xi, \eta] = 0, \forall \eta \in \mathfrak{a}\}$. The Lie algebra $\mathfrak{g}_* = \mathfrak{k} \oplus i\mathfrak{p}_*$ has the restricted root space decomposition $\mathfrak{g}_* = (\mathfrak{g}_*)_0 \oplus \bigoplus_{\lambda \in \Sigma_*} (\mathfrak{g}_*)_\lambda$, where $\Sigma_* \subset i\mathfrak{a}^*$ is the restricted root system of $\mathfrak{g}_*$. Define $\lambda = i\lambda_\mathfrak{a}$, and $\Sigma = i\Sigma_*$, then $\Sigma = \{\lambda : \lambda_\mathfrak{a} \in \Sigma_*\} \subset \mathfrak{a}^*$. Let $\Sigma^+ \subset \Sigma$ be the set of positive restricted roots.

As in §2, we can write $\mathfrak{t} = \mathfrak{m} \oplus \mathfrak{i}$ and $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{b}$ orthogonally. We also choose an orthogonal basis $\{\gamma_{\lambda,1}, \cdots, \gamma_{\lambda,b}\}$ of $(\mathfrak{g}_*)_\lambda$ for each $\lambda \in \Sigma^+$ with $|\gamma_{\lambda,j}| = \frac{1}{2}$, and let $\xi_{\lambda,j} = \gamma_{\lambda,j} + \theta\gamma_{\lambda,j}$, $\xi_{\lambda,j} = i(\gamma_{\lambda,j} - \theta\gamma_{\lambda,j})$. Then

$$\{\xi_{\lambda,j} : \lambda \in \Sigma^+, j = 1, \cdots, b\} \subset \mathfrak{i}$$

is an orthonormal basis for $\mathfrak{i}$, and

$$\{\xi_{\lambda,j} : \lambda \in \Sigma^+, j = 1, \cdots, \beta\} \subset \mathfrak{t}$$

is an orthonormal basis for $\mathfrak{b}$. And we have $\text{dim}\mathfrak{i} = \text{dim}\mathfrak{b} = \sum_{\lambda \in \Sigma^+} \beta$. Let $\mathfrak{d}$ be a maximal abelian subalgebra of $\mathfrak{m}$, then $\mathfrak{t} = \mathfrak{a} \oplus \mathfrak{d}$ is a maximal abelian subalgebra of $\mathfrak{g}$. Let $T$ be the maximal torus with Lie algebra $\mathfrak{t}$, and let $\Delta \subset \mathfrak{t}^*$ be the corresponding root system. Then $\Sigma = \{\alpha_\mathfrak{a} : \alpha \in \Delta, \alpha_\mathfrak{a} \neq 0\}$. Hence for each $\lambda \in \Sigma$, there exists a character $\vartheta_\lambda$ of $A$ satisfies $\vartheta_\lambda(e^n) = e^{i\lambda(\eta)}, \forall \eta \in \mathfrak{a}$. 


Furthermore, we have \( t \oplus \left( g \cap \bigoplus_{\alpha \neq 0} g_{\alpha} \right) = a \oplus m \), where \( g_{\alpha} \) is the corresponding root space in \( g_{\mathbb{C}} \) (see [3], Formula (6.48c)).

**Remark 4.1.** The global involution \( \Theta \) exists if and only if \( G/K \) can be endowed with a Riemannian structure such that it is a Riemannian symmetric space.

Now we consider the action of \( K \) on \( P \) by \( \sigma_k(p) = kp^{-1} \). The nondegenerate bilinear form \( B_* \) on \( g_* \) induces a bi-invariant Riemannian structure on \( G \), such that the linear subspaces \( a, m, \mathbb{R} \xi_{\lambda,j}, \mathbb{R} \zeta_{\lambda,j} \) of \( g \) are mutually orthogonal. Similar to the previous section, it induces the Riemannian measures \( dx \) and \( da \) on \( P \) and \( A \), and \( dx \) is \( K \)-invariant. Since \( P \) is compact, we can normalize the Riemannian structure on \( G \) such that \( dx \) is a probability measure. Choose a \( K \)-invariant smooth function \( p(x) \) on \( P \). As before, the inner product \( -B_* \) induce a \( K \)-invariant Riemannian structure and a \( K \)-invariant Riemannian measure \( d\mu \) on \( K/M \). Similar to the previous section, we define the map \( \varphi : K/M \times A \to P \) by \( \varphi([k], a) = \sigma_k(a) \). Then we can construct the compact ensemble with integration manifold \( P \) and eigenvalue manifold \( A \) as follows.

**Theorem 4.1.** Let the objects be as above. Then the system \((K, P, p(x)dx, A, da)\) is a generalized random matrix ensemble. Its generalized joint density function \( P(a) = p(a)J(a) \) is given by

\[
J(a) = 2^{\dim \mathfrak{g}} \prod_{\lambda \in \Sigma^+} \left| \sin \frac{\lambda(\eta)}{2} \right|^{\beta_{\lambda}},
\]

where \( \eta \in \mathfrak{a} \) such that \( e^\eta = a \).

**Proof.** Similar to the proof of Theorem 3.1 for \( a \in A \), we have

\[
\Psi_a(\xi_{\lambda,j}) = \frac{d}{dt} \bigg|_{t=0} e^{it\lambda(\eta)} a e^{-it\lambda(\eta)}
\]

\[
= (da) \left( \frac{d}{dt} \bigg|_{t=0} e^{it\lambda(\eta) + \theta(\gamma_{\lambda,j} - \xi_{\lambda,j})} \right)
\]

\[
= (da) \left( \frac{d}{dt} \bigg|_{t=0} e^{it\lambda(\eta)} \right) \left( \gamma_{\lambda,j} + \theta(\gamma_{\lambda,j} - \xi_{\lambda,j}) \right)
\]

\[
= (da) \left( \frac{d}{dt} \bigg|_{t=0} e^{it\lambda(\eta)} \right) \left( \gamma_{\lambda,j} + \theta(\gamma_{\lambda,j} - \xi_{\lambda,j}) \right)
\]

\[
= (da) \left( \sin \lambda(\eta) \xi_{\lambda,j} + (\cos \lambda(\eta) - 1) \xi_{\lambda,j} \right).
\]

So

\[
|\Psi_a(\xi_{\lambda,j})| = |\sin \lambda(\eta) \xi_{\lambda,j} + (\cos \lambda(\eta) - 1) \xi_{\lambda,j}|
\]

\[
= \sqrt{\sin^2 \lambda(\eta) + (\cos \lambda(\eta) - 1)^2}
\]

\[
= 2 \left| \sin \frac{\lambda(\eta)}{2} \right|.
\]

Let \( A' = \{a \in A : \varphi_a(\eta) \neq 1, \forall \lambda \in \Sigma \} \). For \( a = e^\eta \in A' \), \( e^{i\lambda(\eta)} = \varphi_a(e^\eta) \neq 1 \). This implies \( \lambda(\eta) \neq 2k\pi \), and hence \(|\Psi_a(\xi_{\lambda,j})| \neq 0 \). This means that for \( a \in A' \), \( \Psi_a \) is an isomorphism. Let \( A_x = A' \setminus A, P_x = \varphi(K/M, A_x) \), and \( P' = P \setminus P_x \). Then similar to the proof of Theorem 3.1 one can easily check that the conditions (a), (b), (c), and (d) hold. So the system \((K, P, p(x)dx, A, da)\) is a generalized random matrix ensemble. And then

\[
J(a) = 2^{\dim \mathfrak{g}} \prod_{\lambda \in \Sigma^+} \left| \sin \frac{\lambda(\eta)}{2} \right|^{\beta_{\lambda}}.
\]
Proof. We will prove that for each \( |\cdot| \) and the covering sheet is \( |\cdot| \) directly.

\[
|\alpha|^\dim |\alpha| = 2\begin{pmatrix} \sin \frac{\lambda(g)}{2} \end{pmatrix} |\beta| da,
\]

where \( \eta \in \mathfrak{a} \) satisfies \( e^\eta = a \). Formula (4.2) has been obtained, when \( p \equiv 1 \) and omitting the constant \( 2^{2\dim |\cdot|} \), by Dueñez [4] using an integration formula associated with the \( KAK \) decomposition of compact groups. Here we recover it from Formula (1.1) directly.

**Corollary 4.2.** Let \( f \in C^\infty(P) \) satisfies \( f \geq 0 \) or \( f \in L^1(P,p(x)dx) \). Then we have the following integration formula

\[
\int_P f(x)p(x)dx = \frac{1}{|W|} \int_A \left( \int_{K/M} f(\sigma_k(a))d\mu([k]) \right) d\nu(a),
\]

where \( W \) is the Weyl group of the restricted root system \( \Sigma \).

**Proof.** We will prove that for each \( a \in A' \), the isotropic subgroup \( K_a = M \) and \( |O_a \cap A'| = |W| \). Then by Corollary 3.6 in [2], the covering condition (e) holds and the covering sheet is \( |W| \). By Formula (1.1), we get the desired integration formulae.

Now we prove \( K_a = M \) for \( a \in A' \). First we consider the group \( Z_G(a) \). It is obvious that \( MA \subseteq Z_G(a) \). If \( \xi \in \mathfrak{g} \) lies in the Lie algebra of \( Z_G(a) \), then \( \text{Ad}_a(\xi) = \xi \), that is, \( \xi \in t \oplus \left( \mathfrak{g} \cap \bigoplus_{\alpha|a=1} \mathfrak{g}_\alpha \right) \). But by the definition of \( A' \), \( \alpha|a \neq 0 \) implies \( \vartheta_\alpha(a) \neq 1 \), so in fact \( \bigoplus_{\alpha|a=1} \mathfrak{g}_\alpha \). Hence \( \xi \in t \oplus \left( \mathfrak{g} \cap \bigoplus_{\alpha|a=0} \mathfrak{g}_\alpha \right) = \mathfrak{a} \oplus \mathfrak{m} \). This implies that the Lie algebra of \( Z_G(a) \) is \( \mathfrak{a} \oplus \mathfrak{m} \). We claim that \( Z_G(a) \) is connected. In fact, let \( g \in Z_G(a) \), then the closed subgroup generated by \( a \) and \( g \) is a closed abelian subgroup of \( G \), hence has a generator \( h \). Let \( T_1 \) be a maximal torus of \( G \) containing \( h \), then \( a, g \in T_1 \). This implies \( g \in T_1 \subseteq Z_G(a) \), thus there is a continuous path in \( T_1 \subseteq Z_G(a) \) connecting \( g \) and \( e \). In a word, \( Z_G(a) \) is the connected subgroup of \( G \) with Lie algebra \( \mathfrak{a} \oplus \mathfrak{m} \). But \( MA \subseteq Z_G(a) \), so in fact \( Z_G(a) = MA \). (This also shows \( MA \) is connected.) Hence \( K_a = Z_K(a) = Z_G(a) \cap K = MA \cap K = M \). Here the equality \( MA \cap K = M \) can be shown by an easy argument. Next we show that \( |O_a \cap A'| = |W| \) for each \( a \in A' \).

By the definition of \( A' \), \( Z_p(a) := \{ \xi \in \mathfrak{p} : \text{Ad}_a(\xi) = \xi \} = \mathfrak{a}, \forall a \in A' \). If some \( k \in K \) such that \( \sigma_k(a) = kak^{-1} \in A' \), then \( \text{Ad}_k(a) = \text{Ad}_k(Z_p(a)) = Z_p(\sigma_k(a)) = a \), that is, \( k \in N_K(a) \). Hence \( |O_a \cap A'| = |N_K(a) : Z_K(a)| = |N_K(a) : M| = |W| \). This complete the proof of the corollary.

As we have pointed out in Remark 4.1 the space \( G/K \) is a Riemannian symmetric space of compact type. The map \( \phi : G/K \to P \) defined by \( \phi([g]) = g\Theta(g)^{-1} \) is a diffeomorphism (see [1]). So similar to Corollary 3.3 we can derive an integration formula for symmetric space of compact type. Let \( \Gamma = A \cap K \), which is a finite group. We define the \( G \)-invariant measure \( d\mu_1 \) on \( G/K \) as in Corollary 3.3.

**Corollary 4.3.** Under the above conditions, we have

\[
\int_{G/K} f([g])d\mu_1([g]) = \frac{1}{|W|} \int_A \left( \int_{K/M} f([ka])d\mu([k]) \right) \delta(a)da,
\]
where
\[ \delta(a) = \prod_{\lambda \in \Sigma^+} |\sin \lambda(\eta)|^{\beta_\lambda}, \]
here \( \eta \in a \) is chosen such that \( e^{\eta} = a \).

Proof. If we define the twisted conjugate action of \( G \) on \( P \) by \( \tau_g(p) = gp\Theta(g)^{-1} \) (note that \( \tau_k = \sigma_k \) for \( k \in K \)), then it is easy to show that the measure \( dx \) is \( G \)-invariant, and \( (\phi^{-1})^* (d\mu_1) = 2^{d-\dim P} dx \). As in Corollary 3.3, we define the maps
\[ \psi : K/M \times A \to P \] by \( \psi([k], a) = ka^2k^{-1}, \rho : K/M \times A \to G/K \) by \( \rho([k], a) = [ka] \), and \( sq : A \to A \) by \( sq(a) = a^2 \). We also have \( \psi \circ \rho = \varphi \circ (id \times sq) \), form which one can easily get
\[ \rho^*(d\mu_1) = \prod_{\lambda \in \Sigma^+} |\sin \lambda(\eta)|^{\beta_\lambda} d\mu_a. \]
Since \( \Gamma = \ker(sq) \), \( \rho = (\phi)^{-1} \circ \varphi \circ (id \times sq) \) is a \( |\Gamma||W| \)-sheeted covering map. By Proposition 3.1 in [2], the desired integration formula (4.12) is proved. \( \square \)

Remark 4.2. Formula (4.11) has appeared in Helgason [6] (Chapter 1, Theorem 5.10).

Example 4.1. Let \( G_* = GL(n, \mathbb{K}) \), where \( \mathbb{K} = \mathbb{R}, \mathbb{C}, \text{ or } \mathbb{H} \). We recover \( G_C \) and \( G \) below, and see what the corresponding compact ensemble is.

First consider the case \( G_* = GL(n, \mathbb{R}) \). Its Lie algebra \( g_* = gl(n, \mathbb{R}) \). \((G_*)_C = GL(n, \mathbb{C})\) is a connected complexification of \( GL(n, \mathbb{R}) \), and \( G = U(n) \) is a maximal compact subgroup of \( G_* \). Now \( \mathfrak{t} = \mathfrak{g} \cap \mathfrak{g}_* = u(n) \cap gl(n, \mathbb{R}) = so(n) \), and \( K = G \cap G_* = U(n) \cap GL(n, \mathbb{R}) = O(n) \). In the associated Cartan decomposition of \( \mathfrak{g}_* = \mathfrak{t} \oplus \mathfrak{p}_* \), the space \( \mathfrak{p}_* = \{ \xi \in \mathfrak{g}(n, \mathbb{R}) : \xi^t = \xi \} \), so \( \mathfrak{p} = i\mathfrak{p}_* = \{ i\xi : \xi \in \mathfrak{g}(n, \mathbb{R}), \xi^t = \xi \} \), and \( \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p} \). In fact, the global involution \( \Theta(g) = (g^t)^{-1} \) of \( G = U(n) \) is compatible with the above scheme. One can prove that the set \( \{ g \in U(n) : \Theta(g) = g^{-1} \} \) of symmetric unitary matrices is connected, so we have \( \mathfrak{p} \), and the corresponding eigenvalue manifold \( A = \exp(\mathfrak{a}) = \{ a = \text{diag}(a_1, \ldots, a_n) : a_k = \exp(i\lambda \xi) \} \).

Next we let \( G_* = GL(n, \mathbb{C}) \). Since \( g_* = gl(n, \mathbb{C}) \) has a complex structure itself, \( g_C \cong gl(n, \mathbb{C}) \cong gl(n, \mathbb{C}) \) as complex Lie algebras (see Theorem 6.94 in [3]). So \( GL(n, \mathbb{C}) \times GL(n, \mathbb{C}) \) is a complexification of \( GL(n, \mathbb{C}) \), if we identify \( G_* = GL(n, \mathbb{C}) \) with the subgroup \( G'_* = \{ (g, \overline{g}) : g \in GL(n, \mathbb{C}) \} \) of \( GL(n, \mathbb{C}) \times GL(n, \mathbb{C}) \). The group \( G = U(n) \times U(n) \) is a maximal compact subgroup of \( G_* \). Now \( K = G \cap G'_* = U(n) \times U(n) \cap \{ (g, \overline{g}) : g \in GL(n, \mathbb{C}) \} = \{ (g, \overline{g}) : g \in U(n) \} \cong U(n) \), and \( \mathfrak{t} = \mathfrak{g} \cap \mathfrak{g}'_* = \{ (\xi, \overline{\xi}) : \xi \in u(n) \} \cong u(n) \). So in the associated Cartan decomposition of \( \mathfrak{g}_* = \mathfrak{t} \oplus \mathfrak{p}_* \), the space \( \mathfrak{p}_* \cong \{ (\xi, \overline{\xi}) : \xi = \xi^* \} \), so \( \mathfrak{p} \cong \{ (\xi, \overline{\xi}) : \xi \in u(n) \} \), and \( \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p} \). Hence \( P = \exp(\mathfrak{p}) = \{ (p, p^t) : p \in U(n) \} \), which is isomorphic to \( U(n) \). In fact, the global involution \( \Theta(g_1, g_2) = (\overline{g_2}, \overline{g_1}) \) of \( G = U(n) \times U(n) \) is compatible with the above scheme. The group \( K = \{ (g, \overline{g}) : g \in U(n) \} \) acts on \( P = \{ (p, p^t) : p \in U(n) \} \) by \( \sigma_k(p) = kpk^{-1} \), that is, \( \sigma_k(p, p^t) = (gkp^{-1}, \overline{g}^t\overline{g}p^t) \). So under the identification of \( G_* \) with \( G'_* \), \( K = P \cong U(n) \), and the action \( \sigma \) is just the conjugate action of \( U(n) \). The space \( \mathfrak{a} = \{ \text{diag}(ix_1, \ldots, ix_n), \text{diag}(ix_1, \ldots, ix_n) : x_k \in \mathbb{R} \} \) is a maximal abelian subspace of \( \mathfrak{p} \), so under the identification of \( \mathfrak{g}_* \) with \( \mathfrak{g}'_* \), \( \mathfrak{a} \cong \{ \text{diag}(ix_1, \ldots, ix_n), \text{diag}(ix_1, \ldots, ix_n) : x_k \in \mathbb{R} \} \).
\[ \{ \eta = \text{diag}(ix_1, \cdots, ix_n) : x_k \in \mathbb{R} \}. \] Then the corresponding eigenvalue manifold \( A = \exp(a) = \{ a = \text{diag}(a_1, \cdots, a_n) : a_k = e^{ix_k} \}. \]

Now we let \( G_* = GL(n, \mathbb{H}) \). To see what the complexification \( GL(n, \mathbb{H})_\mathbb{C} \) is, we expand the definition of the quaternions. Recall that an quaternion in \( \mathbb{H} \) is an element of the form \( z_0 + iz_1 + jz_2 + kz_3 \), where \( z_i \in \mathbb{R} \). The multiplication in \( \mathbb{H} \) is defined by the linear expansion of the relation

\[ i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j. \]

Under this multiplication, \( \mathbb{H} \) is a division algebra over \( \mathbb{R} \). Our expanded quaternion is the element of the form \( z_0 + iz_1 + jz_2 + kz_3 \), where \( z_i \in \mathbb{C} \). We denote the set of all such elements by \( \mathbb{H}_\mathbb{C} \). The multiplication in \( \mathbb{H}_\mathbb{C} \) is defined by the complex linear expansion of the relation \[ H \] in \( n \). This makes \( \mathbb{H}_\mathbb{C} \) an algebra over \( \mathbb{C} \), and \( \mathbb{H}_\mathbb{C} \) is the complexification of \( \mathbb{H} \). But \( \mathbb{H}_\mathbb{C} \) is not divisible. We denote the set of all \( n \)-by-\( n \) matrices with entries in \( \mathbb{H}_\mathbb{C} \) by \( gl(n, \mathbb{H}_\mathbb{C}) \), and the set of all invertible elements in \( gl(n, \mathbb{H}_\mathbb{C}) \) by \( GL(n, \mathbb{H}_\mathbb{C}) \). \( GL(n, \mathbb{H}_\mathbb{C}) \) is a Lie group with Lie algebra \( gl(n, \mathbb{H}_\mathbb{C}) \). Then it is easy to see that \( GL(n, \mathbb{H}_\mathbb{C}) = gl(n, \mathbb{H}_\mathbb{C}) \), \( GL(n, \mathbb{H}_\mathbb{C}) = GL(n, \mathbb{H}_\mathbb{C}) \). In fact, \( gl(n, \mathbb{H}_\mathbb{C}) \cong gl(2n, \mathbb{C}) \) as Lie algebras over \( \mathbb{C} \). The isomorphism can be defined as follows. For \( \xi \in gl(n, \mathbb{H}_\mathbb{C}) \), let \( \xi = \xi_0 + i\xi_1 + j\xi_2 + k\xi_3 \), where \( \xi_i \in gl(n, \mathbb{C}) \). Define

\[
\Phi(\xi) = \left( \begin{array}{cc}
\xi_0 + i\xi_1 & \xi_2 + i\xi_3 \\
-\xi_2 + i\xi_3 & \xi_0 - i\xi_1
\end{array} \right) \in gl(2n, \mathbb{C}),
\]

then \( \Phi \) is an isomorphism. In particular, we have \( \mathbb{H}_\mathbb{C} \cong gl(2, \mathbb{C}) \). For \( \xi = \xi_0 + i\xi_1 + j\xi_2 + k\xi_3 \in gl(n, \mathbb{H}_\mathbb{C}) \), define the conjugation \( \bar{\xi} \) of \( \xi \) by \( \bar{\xi} = \xi_0 + i\xi_1 + j\xi_2 + k\xi_3 \), and the dual \( \xi^R \) of \( \xi \) by \( \xi^R = \xi_0^R - i\xi_1^R - j\xi_2^R - k\xi_3^R \). Define \( \xi^* = (\bar{\xi})^R \). Denote \( U(n, \mathbb{H}_\mathbb{C}) = \{ g \in GL(n, \mathbb{H}_\mathbb{C}) : gg^* = I_n \} \), \( u(n, \mathbb{H}_\mathbb{C}) = \{ \xi \in gl(n, \mathbb{H}_\mathbb{C}) : \xi + \xi^* = 0 \} \), then \( G = U(n, \mathbb{H}_\mathbb{C}) \) is a maximal compact subgroup of \( GL(n, \mathbb{H}_\mathbb{C}) \) with Lie algebra \( u(n, \mathbb{H}_\mathbb{C}) \), and \( K = G_* \cap G = Sp(n) \). Note that under the isomorphism \( \Phi \) above, \( U(n, \mathbb{H}_\mathbb{C}) \cong U(2n) \). It is easy to show that \( p = \{ \xi \in u(n, \mathbb{H}_\mathbb{C}) : \xi^R = \xi \} \), and \( P = \{ p \in U(n, \mathbb{H}_\mathbb{C}) : p^R = p \} \), which is the set of self-dual unitary matrices in \( GL(n, \mathbb{H}_\mathbb{C}) \). In fact, the global involution \( \Theta(g) = \bar{g} \) of \( G = U(n, \mathbb{H}_\mathbb{C}) \) is compatible with the above scheme. The group \( Sp(n) \) acts on \( P \) by \( \sigma_k(p) = kp^{-1} \). The space \( a = \{ \eta = \text{diag}(ix_1, \cdots, ix_n) : x_k \in \mathbb{R} \} \) is a maximal abelian subspace of \( p \), and the corresponding eigenvalue manifold \( A = \exp(a) = \{ a = \text{diag}(a_1, \cdots, a_n) : a_k = e^{ix_k} \}. \]

We can choose the set of positive restricted roots as \( \Sigma^+ = \{ e_r - e_s : 1 \leq r < s \leq n \} \) for each case, and \( \beta_{e_r - e_s} = \beta \), where \( \beta = 1, 2 \) or 4 when \( G_* \) is \( GL(n, \mathbb{R}) \), \( GL(n, \mathbb{C}) \), or \( GL(n, \mathbb{H}) \), respectively. Let \( p(x) \) be a \( K \)-invariant positive smooth function on \( P \). By Theorem 12, the density function \( \mathcal{P}(a) = p(a)J(a) \) for the compact ensemble \( (G, \sigma, P, p(x)dx, A, da) \) is determined by

\[
J(a) = 2^{\frac{\beta(a-1)}{2}} \prod_{1 \leq r < s \leq n} \left| \sin \frac{x_r - x_s}{2} \right|^\beta = \prod_{1 \leq r < s \leq n} |a_r - a_s|^\beta.
\]

In the particular case that \( p = 1 \), the corresponding ensembles is just the three cases of the circular ensemble.

**Example 4.2.** Let \( G = SO(m + n), U(m + n) \), or \( Sp(m + n) \). We choose the global involution \( \Theta \) of \( G \) as \( \Theta(g) = I_{m,n} g I_{m,n} \), then \( K = \{ g \in G : \Theta(g) = g \} \).
\( g \) = \( S(O(m) \times O(n)), U(m) \times U(n), \text{ or } Sp(m) \times Sp(n) \), respectively, and \( P = \{ g \in G : \Theta(g) = g^{-1} \} \). The induced involution \( \theta = d\Theta \) of \( g \) is \( \theta(\xi) = I_{m,n}\xi I_{m,n} \) for \( \xi \in g \), and the corresponding \( \mathfrak{t} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A + A^* = 0, D + D^* = 0 \right\}, \mathfrak{p} = \left\{ \begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix} \right\} \). The space
\[
a = \left\{ \eta = \sum_{k=1}^n x_k(E_{m-k+1,m+k} - E_{m+k,m-k+1}) : x_k \in \mathbb{R} \right\}
\]
is a maximal abelian subspace of \( \mathfrak{p} \), and the corresponding
\[
A = \left\{ a = \sum_{k=1}^n a_k(E_{m-k+1,m+k} + E_{m+k,m-k}) \right\}
\]
In fact, under the exponential map, \( a_k = \cos x_k \). It is easy to show that \( G_\ast \) is isomorphic to \( O(m,n) \), \( U(m,n) \), or \( Sp(m,n) \) when \( G = SO(m+n), U(m+n) \), or \( Sp(m+n) \), and the positive restricted root system \( \Sigma^+ \) and the associated \( \beta_\lambda \) for \( \lambda \in \Sigma^+ \) are the same as in Example 2.2 for each of the three cases. A computation similar to that of in Example 2.2 shows that
\[
J(a) = 2^{\frac{\beta(m+1)}{2}} \prod_{1 \leq r < s \leq n} |a_r - a_s|^\beta \prod_{r=1}^n |1 + a_r|^{\frac{\beta + 1}{2}} |1 - a_r|^{\frac{\beta(m-n+1)}{2}},
\]
where \( \beta = 1, 2 \), or 4 when \( G = SO(m+n), U(m+n) \), or \( Sp(m+n) \), respectively. Let \( p(x) \) be a \( K \)-invariant positive smooth function on \( P \), then the density function \( P(a) = p(a)J(a) \) for the compact ensemble \( (K, \sigma, P, p(x)dx, A, da) \) is determined by (4.7). In the particular case that \( p \equiv 1 \), These are just three cases of the Jacobi ensembles in Dueñez [4].

5. GROUP AND ALGEBRA ENSEMBLES ASSOCIATED WITH COMPACT GROUPS

In this section we examine the group ensemble and algebra ensemble associated with connected compact Lie group. First we give some general arguments.

Suppose \( G \) is a Lie group with Lie algebra \( g \). Consider the conjugate action \( \sigma_g(h) = ghg^{-1} \) of \( G \) on itself and the adjoint action \( \text{Ad}_g = d\sigma_g \) of \( G \) on \( g \). To get the group and algebra ensembles, we need a \( \sigma \)-invariant smooth measure \( p(g)dg \) on \( G \) and an \( \text{Ad} \)-invariant smooth measure \( p(\xi)d\mathcal{X}(\xi) \) on \( g \), where \( dg \) is the Haar measure on \( G \) and \( d\mathcal{X} \) is the Lebesgue measure on \( g \). One can easily show that such measures exist if and only if \( G \) is unimodular. In this case, we can always endow Riemannian structures on \( G \) and \( g \) inducing the measure \( dg \) and \( d\mathcal{X} \), respectively. To choose the zero measure subsets \( X_\mathcal{Z} \) and \( Y_\mathcal{Z} \), we need to consider the set of singular elements in Lie groups and Lie algebras. We denote the sets of regular elements and singular elements in a Lie group \( G \) by \( G_r \) and \( G_s \), and denote the sets of regular elements and singular elements in a Lie algebra \( g \) by \( g_r \) and \( g_s \).

**Lemma 5.1.** Let \( M \) be a real or complex analytic manifold, \( f \) an analytic function on \( M \) which is not identically zero. Then the set \( \{ x \in M : f(x) = 0 \} \) has measure zero.
Proof. Because a complex manifold is automatically real analytic, we need only to prove the real case. In the following we always let \( f \) be an analytic function on \( M \) which is not identically zero. We denote the zero set of \( f \) by \( Z \). First we suppose \( M = (-1,1)^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_j \in (-1,1), j = 1, \ldots, n\} \). We prove by induction that the zero set \( Z \) of \( f \) has measure zero. For \( n = 1 \) the conclusion is obvious true. Suppose the conclusion is true for \( n - 1 \). Then for the case of \( n \), since \( f \) is not identically zero, the set

\[
A = \{x_1 \in (-1,1) : f(x_1, \ldots, x_n) = 0, \forall x_j \in (-1,1), j = 2, \ldots, n\}
\]

is discrete, which must have measure zero in \((-1,1)\). For \( x_1 \in (-1,1) \setminus A \), by the induction hypothesis, \( Z \cap (\{x_1\} \times (-1,1)^{n-1}) \) has measure zero in \( \{x_1\} \times (-1,1)^{n-1} \). So by Fubini’s Theorem,

\[
\int_{(-1,1)^n} \chi_Z dx_1 \cdots dx_n
\]

\[
= \int_{x_1 \in (-1,1) \setminus A} \left( \int_{\{x_1\} \times (-1,1)^{n-1}} \chi_Z(x_1, \ldots, x_n) dx_2 \cdots dx_n \right) dx_1
\]

\[
= 0,
\]

where \( \chi_Z \) is the characteristic function of \( Z \). Hence \( Z \) has measure zero. For the general \( M \), we can choose countable may coordinate charts \( \{U_j\}_{j \in \mathbb{N}} \) covering \( M \) such that \( U_j \) is diffeomorphic to \((-1,1)^n, \forall j \in \mathbb{N} \). Then \( f \) is not identically zero on each \( U_j \). Let \( \nu \) be a smooth measure on \( M \), then \( \nu(Z) \leq \sum_{j=1}^\infty \nu(Z \cap U_j) = 0 \). \( \square \)

**Proposition 5.2.** The set of singular elements in a Lie group or a Lie algebra always has measure zero.

**Proof.** Since the set of singular elements is defined to be the zero locus of some analytic function, the proposition is obvious from the above Lemma. \( \square \)

Suppose \( G \) is a connected compact group. Choose an Ad-invariant inner product \( \langle \cdot, \cdot \rangle \) on \( g \). Then it induces a bi-invariant Riemannian structure on \( G \) and an Ad-invariant linear Riemannian structure on \( g \), and then induces a Haar measure \( dg \) on \( G \) and an Ad-invariant Lebesgue measure \( dX \) on \( g \). Without loss of generality, we may assume \( dg \) is a probability measure. Let \( p_{\text{grp}}(g) \) and \( p_{\text{alg}}(\xi) \) be \( \sigma \)-invariant smooth function on \( G \) and Ad-invariant smooth function on \( g \), respectively. Let \( T \) be a maximal torus of \( G \) with Lie algebra \( t \). Then the Riemannian structure on \( G \) also induces a Haar measure \( dt \) on \( T \), and the Riemannian structure on \( g \) also induces a Lebesgue measure \( dY \) on \( t \). Under the identification \( t^\perp \cong T[\mathfrak{g}](G/T) \), the inner product \( \langle \cdot, \cdot \rangle \) induces a \( G \)-invariant Riemannian structure on \( G/T \), and then induces a \( G \)-invariant measure \( d\mu \) on \( G/T \). Since \( T = \{g \in G : \sigma_g(t) = t, \forall t \in T\} = \{g \in G : \text{Ad}_g(\eta) = \eta, \forall \eta \in t\} \), we can form the maps \( \varphi_{\text{grp}} : G/T \times T \to G \) and \( \varphi_{\text{alg}} : G/T \times t \to g \) by \( \varphi_{\text{grp}}([g], t) = \sigma_g(t), \varphi_{\text{alg}}([g], \eta) = \text{Ad}_g(\eta) \), respectively. Let \( \Delta \subset t^* \) be the root system. For \( \alpha \in \Delta \), let \( \vartheta_\alpha \) be the character of \( T \) defined by \( \vartheta_\alpha(e^\eta) = e^{i\alpha(\eta)}, \forall \eta \in t \).

**Theorem 5.3.** Let the objects be as above. Then

(1) \( (G, \sigma, G, p_{\text{grp}}(g)dg, T, dt) \) is a generalized random matrix ensemble. Its generalized joint density function \( P(t) = p_{\text{grp}}(t)J_{\text{grp}}(t) \) is given by

\[
J_{\text{grp}}(t) = \prod_{\alpha \in \Delta} |1 - \vartheta_\alpha(t^{-1})|.
\]
(2) \((G, \text{Ad}, \mathfrak{g}, p_{\text{alg}}(\xi)dX(\xi), t, dY)\) is a generalized random matrix ensemble. Its generalized joint density function \(P(\eta) = p_{\text{alg}}(\eta)J_{\text{alg}}(\eta)\) is given by

\[
J_{\text{alg}}(\eta) = \prod_{\alpha \in \Delta} |\alpha(\eta)|.
\]

**Proof.** Let \(G_{z} = G_{s}T_{z} = T \cap G_{z}\). By Proposition 5.2, \(G_{z}\) has measure zero in \(G\). Since \(T_{z} = \bigcup_{\alpha \in \Delta} \ker \vartheta_{\alpha}\), \(T_{z}\) has measure zero in \(T\). Let \(G' = G \setminus G_{s} = G_{r}\) and \(T' = T \setminus T_{z}\). By the theory of compact Lie groups, one can easily show that the conditions (a), (b), (c), and (d) hold, and then uses Formula (5.1) to prove formula (5.1). This proves (1). (2) can be proved similarly. 

**Corollary 5.4.** Let \(G\) be a compact group with a maximal torus \(T\), and let \(g\) and \(t\) be their Lie algebras. Then we have

\[
\int_{G} f(g)dg = \frac{1}{|W|} \int_{T} \left( \int_{G/T} f(\sigma_{t}(g))d\mu([g]) \right) J_{\text{grp}}(t)dt,
\]

\[
\int_{g} f(\xi)dX(\xi) = \frac{1}{|W|} \int_{t} \left( \int_{G/T} f(\text{Ad}_{g}(\eta))d\mu([g]) \right) J_{\text{alg}}(\eta)dY(\eta).
\]

**Proof.** Using Corollary 3.6 in \([2]\), one can easily shows that for both cases in Theorem 5.3 the covering condition (e) holds, and the covering sheet is \(|W|\). So the corollary directly from Formula (5.2).

**Remark 5.1.** Formula (5.3) is just the Weyl integration formula for compact Lie groups. Formula (5.4) can be viewed as the linear version of the the Weyl integration formula. Here we recover them from the viewpoint of generalized random matrices.

**Example 5.1.** Let \(G = U(n), SO(2n + 1), Sp(n),\) or \(SO(2n)\). We derive the joint density functions for the corresponding group ensemble and algebra ensemble by deriving the factor \(J_{\text{alg}}(\eta) \) and \(J_{\text{grp}}(t)\) for each case.

First we let \(G = U(n)\). Then \(T = \{ t = \text{diag}(t_{1}, \ldots, t_{n}) : |t_{k}| = 1 \}\) is a maximal torus of \(U(n)\) with Lie algebra \(\mathfrak{t} = \{ \eta = \text{diag}(ix_{1}, \ldots, ix_{n}) : x_{k} \in \mathbb{R} \}\). The associated root system \(\Delta = \{ \pm (e_{r} - e_{s}) : 1 \leq r < s \leq n \}\), where \(e_{r} \in \mathfrak{t}^{*}\) is defined by \(e_{r}(\text{diag}(ix_{1}, \ldots, ix_{n})) = x_{r}\). Then by Theorem 5.3 for the algebra ensemble \((U(n), \text{Ad}, u(n), p(\xi)dX(\xi), t, dY)\), the factor

\[
J_{\text{alg}}(\eta) = \prod_{1 \leq r < s \leq n} |x_{r} - x_{s}|^{2}.
\]

And for the group ensemble \((U(n), \sigma, U(n), dg, T, dt)\), the density function

\[
J_{\text{grp}}(t) = \prod_{1 \leq r \leq s \leq n} |1 - e^{i(x_{r}, -x_{s})}|^{2} = \prod_{1 \leq r < s \leq n} |t_{r} - t_{s}|^{2},
\]

where we have chosen \(\eta = \text{diag}(ix_{1}, \ldots, ix_{n}) \in \mathfrak{t}\) such that \(t = e^{\eta}\).

Next we consider the case \(G = SO(2n+1)\). Then the maximal torus of \(SO(2n+1)\) can be chosen as

\[
T = \{ t = \text{diag} \left( \left( \begin{array}{c} t_{1}^{1} & -t_{1}^{1} \\ t_{1}^{1} & t_{1}^{1} \end{array} \right), \ldots, \left( \begin{array}{c} t_{n}^{n} & -t_{n}^{n} \\ t_{n}^{n} & -t_{n}^{n} \end{array} \right), 1 \right) : t_{k}, t_{k}' \in [-1, 1], t_{k}^{2} + t_{k}'^{2} = 1 \},
\]

whose Lie algebra is

\[
t = \{ \eta = \text{diag} \left( \left( \begin{array}{c} 0 & -x_{1} \\ x_{1} & 0 \end{array} \right), \ldots, \left( \begin{array}{c} 0 & -x_{n} \\ x_{n} & 0 \end{array} \right), 0 \right) : x_{k} \in \mathbb{R} \}.\]
The root system $\Delta = \{ \pm (e_r + e_s), \pm (e_r - e_s) : 1 \leq r < s \leq n \} \cup \{ \pm e_r : 1 \leq r \leq n \}$, where $e_r(\eta) = x_r$. By Theorem 5.3 for the algebra ensemble $(SO(2n + 1), \text{Ad}, \text{so}(2n + 1), p(\xi) dX(\xi), t, dY)$, the factor

(5.7) \[ J^{\text{alg}}(\eta) = \prod_{1 \leq r < s \leq n} |x_r + x_s|^2 |x_r - x_s|^2 \prod_{r=1}^n |x_r|^2 \]

If for $t \in T$ we choose $\eta \in t$ such that $t = e^{\eta}$, that is, $t_k = \cos x_k, t'_k = \sin x_k$, then the density function for the group ensemble $(SO(2n + 1), \sigma, SO(2n + 1), dg, T, dt)$ is

(5.8) \[ J^{\text{grp}}(t) = \prod_{1 \leq r < s \leq n} |1 - e^{i(x_r + x_s)}|^2 |1 - e^{i(x_r - x_s)}|^2 \prod_{r=1}^n |1 - e^{ix_r}|^2 \]

\[ = 2^{n^2} \prod_{1 \leq r < s \leq n} (t_r - t_s)^2 \prod_{r=1}^n (1 - t_r). \]

Now we let $G = Sp(n)$. Then $T = \{ t = \text{diag}(t_1, \cdots, t_n, \overline{t}_1, \cdots, \overline{t}_n) : |t| = 1 \}$ is a maximal torus of $U(n)$ with Lie algebra $\mathfrak{t} = \{ \eta = \text{diag}(x_1, \cdots, x_n, -x_1, \cdots, -x_n) : x_k \in \mathbb{R} \}$. The root system $\Delta = \{ \pm \{ e_r + e_s \}, \pm \{ e_r - e_s \} : 1 \leq r < s \leq n \} \cup \{ \pm 2e_r : 1 \leq r \leq n \}$, where $e_r(\eta) = x_r$. So by Theorem 5.3 for the algebra ensemble $(Sp(n), \text{Ad}, \text{sp}(n), p(\xi) dX(\xi), t, dY)$, the factor

(5.9) \[ J^{\text{alg}}(\eta) = \prod_{1 \leq r < s \leq n} |x_r + x_s|^2 |x_r - x_s|^2 \prod_{r=1}^n |2x_r|^2 = 2^{2n} \prod_{1 \leq r < s \leq n} |x_r^2 - x_s^2|^2 \prod_{r=1}^n |x_r|^2. \]

For $t \in T$, choose $\eta \in t$ such that $e^{\eta} = t$, then the density function for the group ensemble $(Sp(n), \sigma, Sp(n), dg, T, dt)$ is

(5.10) \[ J^{\text{grp}}(t) = \prod_{1 \leq r < s \leq n} |1 - e^{i(x_r + x_s)}|^2 |1 - e^{i(x_r - x_s)}|^2 \prod_{r=1}^n |1 - e^{2ix_r}|^2 \]

\[ = \prod_{1 \leq r < s \leq n} |t_r - t_s|^2 |1 - t_r t_s|^2 \prod_{r=1}^n |1 - t_r|^2. \]

For the last group $G = SO(2n)$, the maximal torus of $SO(2n + 1)$ can be chosen as

\[ T = \{ t = \text{diag} \left( \left( \begin{array}{cc} t_{i_1} & -t_{i_1} \\ t_{i_1} & t_{i_1} \end{array} \right), \cdots, \left( \begin{array}{cc} t_{i_n} & -t_{i_n} \\ t_{i_n} & t_{i_n} \end{array} \right) \right) : t_k, t'_k \in [-1, 1], t_k^2 + t'_k^2 = 1 \}, \]

whose Lie algebra is

\[ \mathfrak{t} = \{ \eta = \text{diag} \left( \left( \begin{array}{cc} 0 & -x_1 \\ x_1 & 0 \end{array} \right), \cdots, \left( \begin{array}{cc} 0 & -x_n \\ x_n & 0 \end{array} \right) \right) : x_k \in \mathbb{R} \}. \]

The root system $\Delta = \{ \pm (e_r + e_s), \pm (e_r - e_s) : 1 \leq r < s \leq n \}$, where $e_r(\eta) = x_r$. Then by Theorem 5.3 for the algebra ensemble $(SO(2n), \text{Ad}, \text{so}(2n), p(\xi) dX(\xi), t, dY)$, the factor

(5.11) \[ J^{\text{alg}}(\eta) = \prod_{1 \leq r < s \leq n} |x_r + x_s|^2 |x_r - x_s|^2 = \prod_{1 \leq r < s \leq n} |x_r^2 - x_s^2|^2. \]
If for \( t \in T \) we choose \( \eta \in \mathfrak{t} \) such that \( t = e^\eta \), then the density function for the group ensemble \((SO(2n), \sigma, SO(2n), dg, T, dt)\) is

\[
J^{\text{grp}}(t) = \prod_{1 \leq r < s \leq n} |1 - e^{(t_r + t_s)}|^2 |1 - e^{(t_r - t_s)}|^2 \\
= 2^{n(n-1)} \prod_{1 \leq r < s \leq n} (t_r - t_s)^2.
\]

Let \( p_{\text{grp}}(g) \) and \( p_{\text{alg}}(\xi) \) be \( \sigma \)-invariant smooth function on \( G \) and \( \text{Ad} \)-invariant smooth function on \( \mathfrak{g} \), then the density functions \( P(t) = p_{\text{grp}}(t)J_{\text{grp}}(t) \) and \( P(\eta) = p_{\text{alg}}(\eta)J_{\text{alg}}(\eta) \) for the group ensemble \((G, \sigma, G, p_{\text{grp}}(g)dg, T, dt)\) and the algebra ensemble \((G, \text{Ad}, \mathfrak{g}, p_{\text{alg}}(\xi)d\mathfrak{X}, t, dY)\) are determined by the above formulae. In the particular case that \( p_{\text{grp}} \equiv 1 \), these four classes of group ensembles were particularly interesting for number theorist, since they have close relation with the distribution of the Riemann zeta function and \( L \)-functions (see [7]). Note that when \( p_{\text{grp}} \equiv 1 \), the group ensemble associated with \( U(n) \) is just the circular unitary ensemble, and for suitable choice of \( p_{\text{alg}} \), the algebra ensemble associated with \( U(n) \) is just the Gaussian unitary ensemble up to multiplication by \( i \).

6. Group and algebra ensembles associated with complex semisimple Lie groups

Now we consider the group ensemble and the algebra ensemble associated with a connected complex semisimple Lie group \( G \) with Lie algebra \( \mathfrak{g} \). Let \( \mathfrak{h} \) be a Cartan subalgebra of \( \mathfrak{g} \). \( \mathfrak{h} \) be the connected Lie subgroup of \( G \) with Lie algebra \( \mathfrak{h} \), which is called a Cartan subgroup of \( G \). Then \( H = \{ g \in G : \sigma_g(h) = h, \forall h \in H \} = \{ g \in G : \text{Ad}_g(\eta) = \eta, \forall \eta \in \mathfrak{h} \} \). Similarly, we can define the map \( \varphi_{\text{grp}} : G/H \times \mathfrak{h} \to G \) and \( \varphi_{\text{alg}} : G/H \times \mathfrak{h} \to \mathfrak{g} \) by \( \varphi_{\text{grp}}([g], h) = \sigma_g(h) \), \( \varphi_{\text{alg}}([g], \eta) = \text{Ad}_g(\eta) \). Note that unlike the case for compact Lie groups, the maps \( \varphi_{\text{grp}} \) and \( \varphi_{\text{alg}} \) are not surjective in general, but every regular element of \( G \) or \( \mathfrak{g} \) lies in the image of \( \varphi_{\text{grp}} \) or \( \varphi_{\text{alg}} \). Let \( \Delta = \Delta(\mathfrak{g}, \mathfrak{h}) \) be the root system. For each \( \alpha \in \Delta \), let \( \vartheta_\alpha \) be the restriction of the adjoint representation of \( H \) on the root space \( \mathfrak{g}_\alpha \). Note that \( \vartheta_\alpha(e^\eta) = e^{\alpha(\eta)} \) for \( \eta \in \mathfrak{h} \). Choose a left invariant Riemannian structure on \( G \) such that \( \mathfrak{h} \) and the root spaces \( \mathfrak{g}_\alpha \) are mutually orthogonal. It induces Haar measures \( dg, dh \) on \( G \) and \( H \), and the associated linear Riemannian structure on \( \mathfrak{g} \) induces Lebesgue measures \( dX, dY \) on \( \mathfrak{g} \) and \( \mathfrak{h} \). We choose a Riemannian structure on \( G/H \) which induces a \( G \)-invariant Riemannian measure \( d\mu \) on \( G/H \), such that the identification \( \mathfrak{h}^\perp = \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \cong T[e](G/H) \) is isometric. To simplify to notations, we let the functions \( p_{\text{grp}} \equiv 1 \) and \( p_{\text{alg}} \equiv 1 \).

**Theorem 6.1.** Let the objects be as above. Then

(1) \((G, \sigma, G, dg, H, dh)\) is a generalized random matrix ensemble. Its generalized joint density function \( P(h) = J^{\text{grp}}(h) \) is given by

\[
J^{\text{grp}}(h) = \prod_{\alpha \in \Delta} |1 - \vartheta_\alpha(h^{-1})|^2.
\]
(2) $(G, \text{Ad}, g, dX, h, dY)$ is a generalized random matrix ensemble. Its generalized joint density function $\mathcal{P}(\eta) = J^{\text{alg}}(\eta)$ is given by

$$J^{\text{alg}}(\eta) = \prod_{\alpha \in \Delta} |\alpha(\eta)|^2.$$  

**Proof.** We first prove (1). Let $G_z = G_s, H_z = H \cap G_z = \bigcup_{\alpha \in \Delta} \ker \vartheta_\alpha$, then $G_z$ and $H_z$ have measure zero in $G$ and $H$, respectively. Let $G' = G \setminus G_z = G_r$ and $H' = H \setminus H_z$. We prove the conditions (a), (b), (c), and (d) hold. For every $g \in G'$, there is some $g' \in G$ such that $\sigma_{g'}(g) \in H'$, so every orbit in $G'$ intersects $H'$. Then the invariance condition (a) holds. We denote $g_1 = \bigoplus_{\alpha \in \Delta} g_\alpha$. For $h \in H$, consider the map $\Psi_h : g_1 \to T_hO_h$, $\Psi_h(\xi) = \frac{d}{dt} \big|_{t=0} \sigma_{\exp t\xi}(h)$. If we identify $T_hG$ with $g = T_oG$ by left translation, we have, for $\xi \in g_\alpha$,

$$\Psi_h(\xi) = (\text{d}h_{\lambda_\alpha}) \left. \frac{d}{dt} \right|_{t=0} \exp(t\xi) \exp(-t\xi)$$

$$= \frac{d}{dt} \big|_{t=0} (h^{-1} \exp(t\xi)h) \exp(-t\xi)$$

$$= \text{Ad}(h^{-1})(\xi) - \xi$$

$$= (\vartheta_\alpha(h^{-1}) - 1) \xi.$$  

If $h \in H'$, $\vartheta_\alpha(h^{-1}) - 1 \neq 0, \forall \alpha \in \Delta$, so $\Psi_h$ is an isomorphism from $g_1$ onto $T_hO_h = g_1$. Then we have $T_hG = h \oplus T_hO_h$ orthogonally, that is, the conditions (b) and (d) hold. By Corollary 7.106 in [8], the identity component of $G_h$ is $H$, $\forall h \in H'$, so the dimension condition (c) also holds. We have shown that $\Psi_h$ acts on $g_1 = \bigoplus_{\alpha \in \Delta} g_\alpha$ diagonally with eigenvalues $\{\lambda_\alpha = \vartheta_\alpha(h^{-1}) - 1 : \alpha \in \Delta\}$. Each eigenspace has complex dimension 1. But what we are looking for is the norm of the “determinant” $|\det \Psi_h|$ of $\Psi_h$, which was regarded as a real linear map. Note that if we view $C$ as a 2-dimensional real vector space with a basis $(1, i)$, then multiplication by $\lambda_\alpha$ induces a linear transformation with matrix $\begin{pmatrix} \text{Re}\lambda_\alpha & -\text{Im}\lambda_\alpha \\ \text{Im}\lambda_\alpha & \text{Re}\lambda_\alpha \end{pmatrix}$, whose determinant is $|\lambda_\alpha|^2$. Note that the identification $g_1 \cong T_{[\text{c}]}(G/H)$ is isometric, we have

$$J^{\text{grp}}(h) = |\det \Psi_h|$$

$$= \prod_{\alpha \in \Delta} |\lambda_\alpha|^2$$

$$= \prod_{\alpha \in \Delta} |\vartheta_\alpha(h^{-1}) - 1|^2.$$  

This proves (1). The proof of (2) is similar but more easy. We omit it here. \(\square\)

**Corollary 6.2.** Let $G$ be a complex semisimple Lie group with a Cartan subgroup $H$, and let $\mathfrak{g}$ and $\mathfrak{h}$ be their Lie algebras. Then we have

$$\int_G f(g)dg = \frac{1}{|W|} \int_H \left( \int_{G/H} f(\sigma_\eta(h)) d\mu([g]) \right) J^{\text{grp}}(h)dh,$$

$$\int_{\mathfrak{g}} f(\xi)d\xi = \frac{1}{|W|} \int_{\mathfrak{h}} \left( \int_{G/H} f(\text{Ad}_\eta(h)) d\mu([g]) \right) J^{\text{alg}}(\eta)d\eta,$$

where $W = W(\Delta)$ is the Weyl group.

**Proof.** By Formula (1.2), it is sufficient to check the covering condition (c) for both cases and show the covering sheet is $|W|$. For the Lie algebra case, by the structure theory of complex semisimple Lie group, for every $\eta \in \mathfrak{h}'$, the isotropic
subgroup $G_\eta$ of $G$ associated with $\eta$ equals to $H$. It is also known that for every $\xi \in \mathfrak{g}'$, there exists some $g \in G$ such that $\text{Ad}_g(\xi) \in \mathfrak{h}'$. Such $\text{Ad}_g$ are labelled by $N_G(\mathfrak{h}) = \{ \tau \in \text{Int}(\mathfrak{g}) : \tau(\mathfrak{h}) = \mathfrak{h} \}$, that is, if $\text{Ad}_g$ and $\text{Ad}_{g'}$ both send $\xi$ into $\mathfrak{h}'$, then $\text{Ad}_g = \tau \circ \text{Ad}_{g'}$ for some $\tau \in N_G(\mathfrak{h})$, and such $\tau$ is unique. But it is known that $W \cong N_G(\mathfrak{h})/H$. So every orbit in $\mathfrak{g}'$ intersects $\mathfrak{h}'$ at $|W|$ points. By Corollary 3.6 in [2], the covering condition (e) holds for the Lie algebra case, and the covering sheet is $|W|$.  

Now we prove the Lie group case. We want to show that $\forall g \in G'$, $(\varphi^{\text{grp}})^{-1}(g)$ has $|W|$ points. By the same reason as in the proof of Corollary 3.6 in [2], we need only to show the case $g \in H'$, and the general case can be reduced to it. Thus we let $h \in H'$, and let $g_1, \ldots, g_{|W|} \in N_G(\mathfrak{h})$, one in each component. Then $\{(g_i, g_i^{-1}h) : i = 1, \ldots, |W|\} \subset (\varphi^{\text{grp}})^{-1}(h)$. Let $(g, h') \in (\varphi^{\text{grp}})^{-1}(h)$, then $gh'g^{-1} = h$. But $h$ and $h'$ are regular, their centralizer in $\mathfrak{g}$ must be the Cartan subalgebra $\mathfrak{h}$. So $\text{Ad}(g)$ fixes $\mathfrak{h}$, and then $g \in N_G(\mathfrak{h})$, so $\{(g, h') = (g_i, g_i^{-1}h)\}$ for some $i_0 \in \{1, \ldots, |W|\}$. Thus in fact we have $(\varphi^{\text{grp}})^{-1}(h) = \{(g_i, g_i^{-1}h) : i = 1, \ldots, |W|\}$, which has $|W|$ point. By Proposition 3.5 in [2], $\varphi^{\text{grp}}$ is a $|W|$ sheeted covering map.  

Remark 6.1. Notice that when we prove $\varphi^{\text{grp}}$ is a covering map, we make use of Proposition 3.5 in [2] directly, ignoring Corollary 3.6 in [2]. In fact, the conditions of Corollary 3.6 in [2] are not satisfied in general, that is, here the phenomena of sudden variation of the isotropic subgroups may happen (see Remark 3.2 in [2]). $G = \text{SL}(2, \mathbb{C})/\{\pm 1\}$ is such an example. For details see ([8], Section 7.8).

Remark 6.2. Formula (6.4) is just Harish-Chandra’s integration formula for complex semisimple Lie groups. (6.4) is the linear version of (6.3). Here we recover them form the viewpoint of generalized random matrices.

Example 6.1. Let $G = \text{SL}(n, \mathbb{C})(n \geq 2)$, which is a complex simple Lie group.

$$\mathfrak{h} = \{ \eta = \text{diag}(x_1, \ldots, x_n) : x_k \in \mathbb{C}, \sum_{k=1}^n x_k = 0 \}$$

is a Cartan subalgebra of the Lie algebra $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ of $G$. The corresponding Cartan subgroup is

$$H = \{ h = \text{diag}(h_1, \ldots, h_n) : h_k \in \mathbb{C}, \prod_{k=1}^n h_k = 1 \}.$$  

The root system $\Delta = \{ \pm (\mathbf{e}_r - \mathbf{e}_s) : 1 \leq r < s \leq n \}$, where $\mathbf{e}_r \in \mathfrak{h}^*$ is defined by $\mathbf{e}_r(\text{diag}(x_1, \ldots, x_n)) = x_r$. So by Theorem 6.1 the generalized joint density function $p(\eta) = J^{\text{alg}}(\eta)$ for the algebra ensemble $(\text{SL}(n, \mathbb{C}), \text{Ad}, \mathfrak{sl}(n, \mathbb{C}), dX, \mathfrak{h}, dY)$ is

(6.5)  

$$J^{\text{alg}}(\eta) = \prod_{1 \leq r < s \leq n} |x_r - x_s|^4.$$  

For $h = \text{diag}(h_1, \ldots, h_n) \in H$, choose some $\eta = \text{diag}(x_1, \ldots, x_n) \in \mathfrak{h}$ such that $h = e^\eta$, that is, $h_i = e^{x_i}$ for each $r$. Since $\partial_\alpha(h) = \partial_\alpha(e^\eta) = e^{\alpha(\eta)}$ for each $\alpha \in \Delta$, by Theorem 6.1 we get the density function $p(h) = J^{\text{grp}}(h)$ for the group ensemble.
(SL(n, C), σ, SL(n, C), dg, H, dh) as

\[
J^{grp}(h) = \prod_{1 \leq r < s \leq n} |1 - e^{x_r - x_s}|^2 |1 - e^{x_s - x_r}|^2
\]

(6.6)

\[
= \prod_{1 \leq r < s \leq n} |e^{x_r} - e^{x_s}|^4 \prod_{r=1}^n |e^{-2(n-1)x_r}|
\]

\[
= \prod_{1 \leq r < s \leq n} |h_r - h_s|^4.
\]

Note that in this example the “eigenvalue manifolds” \( H \) and \( \mathfrak{h} \) are really consist of eigenvalues of the matrices in the corresponding integration manifolds.

**Example 6.2.** Let \( G = Sp(n, C) = \{ g \in SL(2n, C) : g^tJ_{n,n}g = J_{n,n}\} \) which is a complex simple Lie group, where \( J_{n,n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \). Its Lie algebra \( \mathfrak{g} = \mathfrak{sp}(n, C) = \{ \xi \in sl(2n, C) : \xi^tJ_{n,n} + J_{n,n}\xi = 0\} \).

\( \mathfrak{h} = \{ \eta = \text{diag}(x_1, \ldots, x_n, -x_1, \ldots, -x_n) : x_k \in C \} \)

is a Cartan subalgebra of \( \mathfrak{sp}(n, C) \). The corresponding Cartan subgroup is

\( H = \{ h = \text{diag}(h_1, \ldots, h_n, h_1^{-1}, \ldots, h_n^{-1}) : h_k \in C, h_k \neq 0\} \).

The root system \( \Delta = \{ \pm(e_r + e_s), \pm(e_r - e_s) : 1 \leq r < s \leq n\} \cup \{ \pm 2e_r : 1 \leq r \leq n\} \).

By Theorem 6.1, the density function \( \mathcal{P}(\eta) = J^{\text{alg}}(\eta) \) for the algebra ensemble \( (Sp(n, C), \text{Ad}, \mathfrak{sp}(n, C), dX, \mathfrak{h}, d\mathcal{Y}) \) is

\[
J^{\text{alg}}(\eta) = 2^{4n} \prod_{1 \leq r < s \leq n} |x_r^2 - x_s^2|^4 \prod_{r=1}^n |x_r|^4.
\]

(6.7)

For \( h = \text{diag}(h_1, \ldots, h_n, h_1^{-1}, \ldots, h_n^{-1}) \in H \), choose some \( \eta = \text{diag}(x_1, \ldots, x_n, -x_1, \ldots, -x_n) \in \mathfrak{h} \) such that \( h = e^\eta \), that is, \( h_r = e^{x_r} \) for each \( r \). Then we have \( \partial_\alpha(h) = e^{a(\eta)} \) for each \( \alpha \in \Delta \). By Theorem 6.1, the density function \( \mathcal{P}(h) = J^{grp}(h) \) for the group ensemble \( (Sp(n, C), \sigma, \mathfrak{sp}(n, C), dg, H, dh) \) is

\[
J^{grp}(h) = \prod_{1 \leq r < s \leq n} |1 - e^{x_r + x_s}|^2 |1 - e^{-(x_r + x_s)}|^2 |1 - e^{x_s - x_r}|^2 |1 - e^{x_r - x_s}|^2
\]

\[
= \prod_{1 \leq r < s \leq n} |h_r - h_s|^4 |1 - h_r h_s|^4 \prod_{r=1}^n |1 - h_r^2|^4 |h_r|^{-2n(n+1)}.
\]

(6.8)

Similar to Example 6.1, here the “eigenvalue manifolds” \( H \) and \( \mathfrak{h} \) are also consist of eigenvalues of the matrices in the corresponding integration manifolds.

**Example 6.3.** Let \( G = SO(2n, C) \). Then

\( \mathfrak{h} = \{ \eta = \text{diag} \left( \left( \begin{array}{cc} 0 & -x_1 \\ x_1 & 0 \end{array} \right), \ldots, \left( \begin{array}{cc} 0 & -x_n \\ x_n & 0 \end{array} \right) \right) : x_k \in C \} \)

is a Cartan subalgebra of \( \mathfrak{so}(2n, C) \), the corresponding Cartan subgroup is

\( H = \{ h = \text{diag} \left( \begin{array}{cc} h_1 & -h_1' \\ h_1' & h_1 \end{array} \right), \ldots, \begin{array}{cc} h_n & -h_n' \\ h_n' & h_n \end{array} \} : h_k, h_k' \in C, h_k h_k' = 1 \} \).

A routine computation similar to that of in Example 6.1 and 6.2 shows that

\[
J^{\text{alg}}(\eta) = \prod_{1 \leq r < s \leq n} |x_r^2 - x_s^2|^4,
\]

(6.9)
Example 6.4. Let $G = SO(2n + 1, \mathbb{C})$. Then
\[ h = \{ \eta = \text{diag} \left( \begin{pmatrix} 0 & -x_1 & \cdots & 0 \\ x_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix} \right) : x_k \in \mathbb{C} \} \]
is a Cartan subalgebra of $\mathfrak{so}(2n + 1, \mathbb{C})$, and the corresponding Cartan subgroup is
\[ H = \left\{ h = \text{diag} \left( \begin{pmatrix} h_{11} & -h_{12} & \cdots & 0 \\ h_{12} & h_{22} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & h_{nn} \end{pmatrix} \right), 1 : h_{kk}, h_{kk}' \in \mathbb{C}, h_{kk}^2 + h_{kk}'^2 = 1 \right\}. \]
Then we can derive that
\[ J^{\text{alg}}(\eta) = \prod_{1 \leq r < s \leq n} |x_r^2 - x_s^2|^4 \prod_{r=1}^n |x_r|^4, \]
\[ J^{\text{grp}}(h) = 2^{2n^2} \prod_{1 \leq r < s \leq n} |h_r - h_s|^4 \prod_{r=1}^n |1 - h_r|^2. \]

7. Pseudo-group and pseudo-algebra ensembles

In this section we consider the pseudo-group ensemble and the pseudo-algebra ensemble associated with a real reductive group. Strictly speaking, they are not generalized ensembles, since the integration manifolds may have singularities. But this doesn’t matter, since integration manifold is the closure of an open submanifold of a real reductive group or a real reductive Lie algebra, whose boundary has measure zero. Let $G$ be a real reductive group with Lie algebra $\mathfrak{g}$. Let $\theta$ be a Cartan involution of $\mathfrak{g}$, and let $\mathfrak{h}_1, \ldots, \mathfrak{h}_m$ be a maximal set of mutually nonconjugate $\theta$ stable Cartan subalgebras of $\mathfrak{g}$. The corresponding Cartan subgroups of $G$ are $H_1 = Z_G(\mathfrak{h}_1), \ldots, H_m = Z_G(\mathfrak{h}_m)$. Let $G' = G_{r-s}H'_j \subset G'$, and let $\mathfrak{g}' = \mathfrak{g}_r, \mathfrak{h}_j' = \mathfrak{h}_j \cap \mathfrak{g}'$. Then it is known that $G' = \bigsqcup_{j=1}^m \bigcup_{g \in G} gH'_j g^{-1}, \mathfrak{g}' = \bigsqcup_{j=1}^m \bigcup_{g \in G} \text{Ad}(g)(\mathfrak{h}_j')$. Denote $G' = \bigcup_{g \in G} gH'_j g^{-1}, \mathfrak{g}' = \bigcup_{g \in G} \text{Ad}(g)(\mathfrak{h}_j)$. Then each $G_j'$ is an open set in $\mathfrak{g}$, and each $G'_j$ is an open set in $G$. Let $G_j = \overline{G_j'}, \mathfrak{g}_j = \overline{\mathfrak{g}_j'}$. It is easy to show that $\{ g \in G : \sigma_g(h) = h, \forall h \in H_j \} = Z(H_j)$, whose Lie algebra is $\mathfrak{h}_j$. So we can form the maps $\varphi_j^{\text{grp}} : G/Z(H_j) \times H_j \rightarrow G_j$ and $\varphi_j^{\text{alg}} : G/H_j \times H_j \rightarrow G_j$ by $\varphi_j^{\text{grp}}([g], h) = \sigma_g(h)$ and $\varphi_j^{\text{alg}}([g], \eta) = \text{Ad}_g(\eta)$, respectively. The maps $\varphi_j^{\text{grp}}$ and $\varphi_j^{\text{alg}}$ may not be surjective in general. But since $G'_j \subset \text{Im}(\varphi_j^{\text{grp}}) \subset G_j$ and $\mathfrak{g}_j' \subset \text{Im}(\varphi_j^{\text{alg}}) \subset \mathfrak{g}_j$, the sets $G_j \setminus \text{Im}(\varphi_j^{\text{grp}})$ and $\mathfrak{g}_j \setminus \text{Im}(\varphi_j^{\text{alg}})$ have measure zero. Choose a Hermitian product on the complexification $\mathfrak{g}_C$ of $\mathfrak{g}$ such that $(\mathfrak{h}_j)_C$ and the associated root spaces are mutually orthogonal. It induces a left invariant Riemannian structure on $G$, then induces a $G$-invariant measure on $G_j$ and a Haar measure on $H_j$. Note that the measure $d\mu_j$ is the restriction of a Haar measure on $G$ for each $j$. Similarly, it induces a $G$-invariant measure on $\mathfrak{g}_j$ (which is the restriction of a Lebesgue measure on $\mathfrak{g}$ for each $j$) and a Lebesgue measure on $\mathfrak{h}_j$. Let $d\mu'_j, d\mu_j$ be $G$-invariant measures on $G/Z(H_j)$.
and \( G/H_j \), which are induced by Riemannian structures on \( G/Z(H_j) \) and \( G/H_j \)
such that the identifications \( h_j \cong T [ \varrho | (G/Z(H_j)) \cong T [ \varrho | (G/H_j) \) are isometric.
For each \( \alpha \in \Delta_j = \Delta(\mathfrak{g}_C, (h_j)_C) \), let \( \vartheta_\alpha \) be the restriction of the adjoint representation of \( H_j \) on the root space \( \mathfrak{g}_\alpha \), which satisfies \( \vartheta_\alpha (e^\varrho) = e^{\alpha(\eta)} \) for \( \eta \in h_j \). Then we have

**Theorem 7.1.** Let the objects be as above. Then for each \( 1 \leq j \leq m \),
(1) \((G, \sigma, G_j, d\varrho_j, H_j, d\varrho_j)\) is a generalized random matrix ensemble. Its generalized joint density function \( \mathcal{P}_j(h) = J_j^{gX}(h) \) is given by

\[
J_j^{gX}(h) = \prod_{\alpha \in \Delta_j} |1 - \vartheta_\alpha(h^{-1})|.
\]

(2) \((G, \text{Ad}, g_j, dX_j, h_j, dY_j)\) is a generalized random matrix ensemble. Its generalized joint density function \( \mathcal{P}_j(\eta) = J_j^{\text{alg}}(\eta) \) is given by

\[
J_j^{\text{alg}}(\eta) = \prod_{\alpha \in \Delta_j} |\alpha(\eta)|.
\]

*Proof.* First we prove (1). We let \((G_j)_\varrho = G_j \setminus G'_j, (H_j)_\varrho = H_j \setminus H'_j \). Then by the discussions above, the condition (a) holds automatically. For \( h \in H'_j \), since \( h \) is regular, the Lie algebra of \( G'_h = \{ g \in G : ghg^{-1} = h \} \) is \( h_j \), so the dimension condition (c) holds. For \( h \in H_j \) and \( \xi \in h_j^\perp \), under the identification of \( T_hG \) with \( g = T_eG \) by left multiplication, it is easy to show that

\[
\Psi_h(\xi) = \frac{d}{dt}|_{t=0} \sigma_{\exp t\xi}(h) = (\text{Ad}(h^{-1}) - I) \xi.
\]

Let \( \mathfrak{g}_C = (h_j)_C \oplus \bigoplus_{\alpha \in \Delta_j} \mathfrak{g}_\alpha \) be the root space decomposition of \( \mathfrak{g}_C \), then \((h_j^\perp)_C = \bigoplus_{\alpha \in \Delta_j} \mathfrak{g}_\alpha \). Let \( \Delta_j^1 = \{ \alpha \in \Delta_j : h_j^\perp \cap \mathfrak{g}_\alpha \neq 0 \} \), and let \( \Delta_j^2 = \Delta_j \setminus \Delta_j^1 \). For each \( \alpha \in \Delta_j \), choose a \( \xi_\alpha \in \mathfrak{g}_\alpha \) such that \( \xi_\alpha \in h_j^\perp \) for \( \alpha \in \Delta_j^1 \). Then for \( \alpha \in \Delta_j^1 \),

\[
\Psi_h(\xi_\alpha) = (\vartheta_\alpha(h^{-1}) - 1) \xi_\alpha.
\]

Now let \( \alpha \in \Delta_j^2 \). For each \( h \in H_j \), since \( \text{Ad}(h) \xi_\alpha = \vartheta_\alpha(h) \xi_\alpha \), we have \( \text{Ad}(h) \overline{\xi_\alpha} = \overline{\vartheta_\alpha(h)} \overline{\xi_\alpha} \), where \( \overline{\xi_\alpha} \) is the conjugation \( \xi_\alpha \) of with respect to \( h_j \). This means that \( \overline{\xi_\alpha} \) belongs to some root space \( \mathfrak{g}_\alpha^* \). Denote \( \alpha' = \tau(\alpha) \), then \( \tau \) is a permutation of \( \Delta_j^2 \) without fixed point, and \( \tau^2 = 1 \). So \( \Delta_j^2 \) has a partition \( \Delta_j^2 = \Delta_j^2 \setminus \tau(\Delta_j^2) \).

Modifying the Hermitian product on \( \mathfrak{g}_C \) if necessary, we may assume \( |\xi_\alpha| = |\overline{\xi_\alpha}| \).

Then it is easy to show that \( \{ \xi_\alpha : \alpha \in \Delta_j^1 \} \cup \{ \xi_\alpha + \overline{\xi_\alpha}, i(\xi_\alpha - \overline{\xi_\alpha}) : \alpha \in \Delta_j^2 \} \) is an orthogonal basis of \( h_j^\perp \). Now for \( \alpha \in \Delta_j^2 \), we have

\[
\Psi_h(\xi_\alpha + \overline{\xi_\alpha}) = (\vartheta_\alpha(h^{-1}) - 1) \xi_\alpha + (\overline{\vartheta_\alpha(h^{-1})} - 1) \overline{\xi_\alpha},
\]

and

\[
\Psi_h(i(\xi_\alpha - \overline{\xi_\alpha})) = i(\vartheta_\alpha(h^{-1}) - 1) \xi_\alpha - i(\overline{\vartheta_\alpha(h^{-1})} - 1) \overline{\xi_\alpha}.
\]

If \( h \in H'_j \), then \( \vartheta_\alpha(h^{-1}) - 1 \neq 0, \forall \alpha \in \Delta_j \). This means that \( \Psi_h : h_j^\perp \to T_hG_h \) is an isomorphism, and \( T_hG'_h = h_j \oplus T_hO_h \) orthogonally. So the conditions (b) and (d) hold. Combining (7.2), (7.3), and (7.4), we get

\[
J_j^{gX}(h) = |\det \Psi_h| = \prod_{\alpha \in \Delta_j} |1 - \vartheta_\alpha(h^{-1})|.
\]

This proves (1). The proof of (2) is similar, which was omitted here. \qed
Corollary 7.2. Let the objects be as above. Then for each $1 \leq j \leq m$, we have

\begin{equation}
\int_{G_j} f(g)dg_j = \frac{1}{|W_j|} \int_{H_j} \left( \int_{G/H_j} f(\sigma_g(h))d\mu_j([g]) \right) J^{\text{grp}}_j(h)dh_j,
\end{equation}

\begin{equation}
\int_{g_j} f(\xi)dX_j(\xi) = \frac{1}{|W_j|} \int_{H_j} \left( \int_{G/H_j} f(\text{Ad}_g(\eta))d\mu_j([\eta]) \right) J^{\text{alg}}_j(\eta)dY_j(\eta),
\end{equation}

where $W_j$ is the analytic Weyl group $W_j = N_G(h_j)/H_j$ associated with $H_j$.

Proof. The proof of (7.7) is essentially same to the proof of (6.4) in Corollary 6.2. Now we prove formula (7.6). Similar to the proof of (6.3) in Corollary 6.2, we can get an integration formula

\begin{equation}
\int_{G_j} f(g)dg_j = \frac{1}{|N_G(H_j)/Z_G(H_j)|} \int_{H_j} \left( \int_{G/H_j} f(\sigma_g(h))d\mu_j([g]) \right) J^{\text{grp}}_j(h)dh_j.
\end{equation}

But $Z_G(H_j) = Z(H_j)$, and it is easily to prove $N_G(H_j) = N_G(h_j)$. So $|N_G(H_j)/Z_G(H_j)| = |N_G(h_j)/Z(H_j)| = |N_G(h_j)/H_j|\cdot |H_j/Z(H_j)| = |W_j|\cdot |H_j/Z(H_j)|$. Hence to prove (7.6), by (7.8), it is sufficient to show that

\begin{equation}
\int_{G/Z(H_j)} f(\sigma_g(h))d\mu_j([g]) = |H_j/Z(H_j)| \int_{G/H_j} f(\sigma_g(h))d\mu_j([g]).
\end{equation}

But the natural projection $\psi : G/Z(H_j) \to G/H_j$ is a $|H_j/Z(H_j)|$-sheeted covering map, and $\psi^*(d\mu_j) = d\mu_j$. Hence (7.6) follows directly from Proposition 3.1 in [2]. This proof the complete proof of the corollary.

Corollary 7.3. Let the objects be as above. Then we have

\begin{equation}
\int_{G} f(g)dg = \sum_{j=1}^m \frac{1}{|W_j|} \int_{H_j} \left( \int_{G/H_j} f(\sigma_g(h))d\mu_j([g]) \right) J^{\text{grp}}_j(h)dh_j,
\end{equation}

\begin{equation}
\int_{g} f(\xi)dX(\xi) = \sum_{j=1}^m \frac{1}{|W_j|} \int_{H_j} \left( \int_{G/H_j} f(\text{Ad}_g(\eta))d\mu_j([\eta]) \right) J^{\text{alg}}_j(\eta)dY_j(\eta).
\end{equation}

Proof. Since $G' = \bigsqcup_{j=1}^m G_j$, $g' = \bigsqcup_{j=1}^m g_j$, and the sets of singular elements $G_s = G\setminus G'$ and $g_s = g\setminus g'$ have measure zero, and also notice that $H_j\setminus H'_j$ and $h_j\setminus h'_j$ have measure zero in the corresponding spaces, the proof follows directly from Corollary 7.2.

Remark 7.1. Formula (7.10) is just the Harish-Chandra’s integration formula for real reductive groups (see [8], Theorem 8.64), and formula (7.11) is its linear version. Here we recover them from the viewpoint of generalized random matrices.

Example 7.1. Let $G = SL(2, \mathbb{R})$. Its Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) = \left\{ \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \right\}$, where $x, y, z \in \mathbb{R}$. $\theta \begin{pmatrix} x & y \\ z & -x \end{pmatrix} = -\begin{pmatrix} x & y \\ z & -x \end{pmatrix}^t = -\begin{pmatrix} -x & -z \\ -y & x \end{pmatrix}$ is a Cartan involution of $\mathfrak{sl}(2, \mathbb{R})$. There are exactly 2 mutually nonconjugate stable Cartan subalgebras $h_1 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$, $h_2 = \left\{ \begin{pmatrix} 0 & y \\ -y & 0 \end{pmatrix} \right\}$. The corresponding Cartan subgroups are $H_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{R}, a \neq 0 \right\}$, $H_2 = SO(2)$ (see [8], page 487). Note that $H_2$ is connected, but $H_1$ has two connected components. It is easy to show that $\mathfrak{g}' = \mathfrak{g}_r = \{ \xi \in \mathfrak{sl}(2, \mathbb{R}) : \det \xi \neq 0 \}$,
\( g_1 = \{ \xi \in \mathfrak{sl}(2, \mathbb{R}) : \det \xi \leq 0 \} \),
\( g_2 = \{ \xi \in \mathfrak{sl}(2, \mathbb{R}) : \det \xi \geq 0 \} \).

Similarly,
\[
G' = G_r = \{ g \in SL(2, \mathbb{R}) : |\text{tr} g| \neq 2 \},
\]
\( G_1 = \{ g \in SL(2, \mathbb{R}) : |\text{tr} g| \geq 2 \}, \)
\( G_2 = \{ g \in SL(2, \mathbb{R}) : |\text{tr} g| \leq 2 \} \).

The corresponding root systems are \( \Delta_1 = \{ \pm \alpha_1 \} \) with \( \pm \alpha_1 \left[ \begin{array}{cc} x & 0 \\ 0 & -x \end{array} \right] = \pm 2x \), and
\( \Delta_2 = \{ \pm \alpha_2 \} \) with \( \pm \alpha_2 \left[ \begin{array}{cc} 0 & y \\ -y & 0 \end{array} \right] = \pm 2iy \). By Theorem 4.1, the density function \( P_1 = J_1^{\text{alg}} \) for the pseudo-algebra ensemble \( (SL(2, \mathbb{R}), \text{Ad}, g_1, dX_1, h_1, dY_1) \) is
\[
J_1^{\text{alg}} \left[ \begin{array}{cc} x & 0 \\ 0 & -x \end{array} \right] = 4x^2,
\]
and the density function \( P_2 = J_2^{\text{alg}} \) for the pseudo-algebra ensemble \( (SL(2, \mathbb{R}), \text{Ad}, g_2, dX_2, h_2, dY_2) \) is
\[
J_2^{\text{alg}} \left[ \begin{array}{cc} 0 & y \\ -y & 0 \end{array} \right] = 4y^2.
\]

For the group ensembles, it is easy to show that \( \vartheta_{\pm \alpha_1} \left[ \begin{array}{cc} a & 0 \\ 0 & a^{-1} \end{array} \right] = a^{\pm 2} \), and
\( \vartheta_{\pm \alpha_2} \left[ \begin{array}{cc} \cos y & \sin y \\ -\sin y & \cos y \end{array} \right] = e^{\pm 2iy} \). So by Theorem 4.1, the density function \( P_1 = J_1^{\text{grp}} \) for the pseudo-group ensemble \( (SL(2, \mathbb{R}), \sigma, G_1, dg_1, H_1, dh_1) \) is
\[
J_1^{\text{grp}} \left[ \begin{array}{cc} a & 0 \\ 0 & a^{-1} \end{array} \right] = (a - a^{-1})^2,
\]
and the density function \( P_2 = J_2^{\text{grp}} \) for the pseudo-group ensemble \( (SL(2, \mathbb{R}), \sigma, G_2, dg_2, H_2, dh_2) \) is
\[
J_2^{\text{grp}} \left[ \begin{array}{cc} \cos y & \sin y \\ -\sin y & \cos y \end{array} \right] = 4 \sin^2 y.
\]

\[\square\]

Example 7.2. Let \( G = GL(n, \mathbb{R}) \), which is a reductive group. Then \( \theta(\xi) = -\xi^t \) is a Cartan involution of \( g = \mathfrak{gl}(n, \mathbb{R}) \). \( g \) has exactly \( m = \left[ \frac{n}{2} \right] + 1 \) mutually nonconjugate \( \theta \) stable Cartan subalgebras, which can be write explicitly as
\( h_j = \{ \eta = D_j(x_1 \cdots x_n) : x_k \in \mathbb{R} \} \),
\( j = 0, \cdots, \left[ \frac{n}{2} \right] \), where we denote
\[
D_j(x_1 \cdots x_n) = \left( \begin{array}{ccc}
\text{diag}(x_1 \cdots x_j) & \text{diag}(x_{j+1} \cdots x_{2j}) & 0 \\
-\text{diag}(x_{j+1} \cdots x_{2j}) & \text{diag}(x_1 \cdots x_j) & 0 \\
0 & 0 & \text{diag}(x_{2j+1} \cdots x_n)
\end{array} \right)
\]
(see [9], page 95). Using the explicit form of the Cartan subalgebra \( h_j \), one can easily prove that an \( n \)-by-\( n \) real matrix commutes with all elements in \( h_j \) if and only if it is of the form \( D_j(a_1 \cdots a_n) \), whose determinant is \( \prod_{r=1}^j (a_r^2 + a_{j+r}^2) \prod_{r=2j+1}^n a_r \). So by definition, the Cartan subgroup
\[
H_j = \{ h = D_j(a_1 \cdots a_n) : a_k \in \mathbb{R}, \prod_{r=1}^j (a_r^2 + a_{j+r}^2) \prod_{r=2j+1}^n a_r \neq 0 \}.
\]
It is easily seen that $H_j$ has $2^{n-2j}$ components. The integration manifolds $\mathfrak{g}_j = \bigcup_{g \in G} \text{Ad}_g (\mathfrak{h}_j)$, $G_j = \bigcup_{g \in G} gH_j g^{-1}$. More precisely, one can prove that

$$
\mathfrak{g}_j' = \{ \xi \in \mathfrak{g}' : \xi \text{ has exactly } n - 2j \text{ real eigenvalues}\},
$$

$$
G_j' = \{ g \in G' : g \text{ has exactly } n - 2j \text{ real eigenvalues}\}.
$$

So

$$
\mathfrak{g}_j = \mathfrak{g}_j' = \{ \xi \in \mathfrak{g} : \text{for some suitable permutation } \lambda_1, \ldots, \lambda_n \text{ of the eigenvalues of } \xi, \lambda_1 = \overline{\lambda}_{j+1}, \ldots, \lambda_j = \overline{\lambda}_{2j}, \ldots, \lambda_n \text{ are real}\},
$$

$$
G_j = G_j' = \{ g \in G : \text{for some suitable permutation } \lambda_1, \ldots, \lambda_n \text{ of the eigenvalues of } g, \lambda_1 = \overline{\lambda}_{j+1}, \ldots, \lambda_j = \overline{\lambda}_{2j}, \ldots, \lambda_n \text{ are real}\}.
$$

The root system associated with the Cartan subalgebra $\mathfrak{h}_0 = \{ \text{diag}(x_1, \ldots, x_n) \}$ is $\Delta_0 = \{ (\pm (e_r - e_s)) : 1 \leq r < s \leq n \}$, where $e_r \in \mathfrak{h}_0$ is defined by $e_r(\text{diag}(x_1, \ldots, x_n)) = x_r$. Denote the matrix $L = \begin{pmatrix} i_1 & -i_1 & 0 \\ i_1 & 0 & i_{n-2j} \end{pmatrix}$, then we have $L^{-1}D_j(x_1 \cdots x_n)L = \text{diag}(y_1, \ldots, y_n)$, where

$$
y_r = \begin{cases} 
  x_r + ix_{j+r}, & 1 \leq r \leq j; \\
  x_{r-j} - ix_r, & j + 1 \leq r \leq 2j; \\
  x_r, & 2j + 1 \leq r \leq n.
\end{cases}
$$

By Theorem 7.1, the density function $P_j(\eta) = J_j^{\text{alg}}(\eta)$ for the pseudo-algebra ensemble $(GL(n, \mathbb{R}), \text{Ad}, \mathfrak{g}_j, dX_j, \mathfrak{h}_j, dY_j)$ is

$$
J_j^{\text{alg}}(\eta) = \prod_{\alpha \in \Delta_0} |\alpha(\text{diag}(y_1, \ldots, y_n))|
$$

$$
= \prod_{1 \leq r < s \leq n} |y_r - y_s|^2
$$

$$
= \prod_{1 \leq r < s \leq n} \left| [(x_r + ix_{j+r}) - (x_s + ix_{j+s})][(x_r - ix_{j+r}) - (x_s - ix_{j+s})] \right|^2
$$

$$
\prod_{2j+1 \leq r < s \leq n} |(x_r - x_s)|^2 \prod_{1 \leq r, s \leq j} |(x_r + ix_{j+r}) - (x_s - ix_{j+s})|^2
$$

$$
\prod_{1 \leq r, s \leq j \leq 2j+1 \leq s \leq n} |[(x_r + ix_{j+r}) - x_s][x_r - ix_{j+r}) - x_s]|^2
$$

$$
= 4^{j} \prod_{r=1}^{j} x_{j+r}^2 \prod_{2j+1 \leq r < s \leq n} |x_r - x_s|^2 \prod_{1 \leq r, s \leq j \leq 2j+1 \leq s \leq n} ((x_r - x_s)^2 + x_{j+r}^2)^2
$$

$$
\prod_{1 \leq r < s \leq j} ((x_r - x_s)^2 + (x_{j+r} - x_{j+s})^2)((x_r - x_s)^2 + (x_{j+r} + x_{j+s})^2)^2.
$$

Now we come to the groups ensembles associated with $G = GL(n, \mathbb{R})$. A direct computation shows that the root spaces $\mathfrak{g}_\alpha (\alpha \in \Delta_j)$ associated with $(\mathfrak{g}_\mathbb{C}, (\mathfrak{h}_j)_\mathbb{C})$ are of the form

$$
\{ \mathfrak{g}_\alpha : \alpha \in \Delta_j \} = \{ \mathbb{C}(LE_{rs}L^{-1}) : r \neq s \},
$$

where $E_{rs}$ denotes the $n$-by-$n$ matrix with $1$ at the $(r, s)$ position and $0$ elsewhere. We denote the root $\alpha \in \Delta_j$ corresponding to $E_{rs}$ by $\alpha_{rs}$. One can also easily
computes that for each \( h \in H_j \) and \( \xi_{rs} \in \mathfrak{g}_{\alpha_{rs}} \), \( h\xi_{rs}h^{-1} = \frac{\lambda}{l_s} \xi_{rs} \), where
\[
  l_r = \begin{cases}
    h_r + ih_{j+r}, & 1 \leq r \leq j; \\
    h_{r-j} - ih_r, & j + 1 \leq r \leq 2j; \\
    h_r, & 2j + 1 \leq r \leq n.
  \end{cases}
\]
So \( \vartheta_{\alpha_{rs}}(h) = \frac{l_r}{l_s} \), and then by Theorem \[7\]
\[
  J^{\text{up}}_j(h) = \prod_{1 \leq r,s \leq n, r \neq s} \left| 1 - \vartheta_{\alpha_{rs}}(h^{-1}) \right| = \prod_{1 \leq r < s \leq n} \left| 1 - \frac{l_r}{l_s} \right| \left| 1 - \frac{l_s}{l_r} \right| = \prod_{1 \leq r < s \leq n} \frac{|l_r - l_s|^2}{|l_r l_s|}.
\]
Note that the expression \( \prod_{1 \leq r<s \leq n} |l_r - l_s|^2 \) has been computed in Formula \[16\], if we replace \( y_r \) by \( l_r \). On the other hand,
\[
  \prod_{1 \leq r < s \leq n} |l_r l_s| = \prod_{r=1}^n |l_r|^{n-1} = \prod_{r=1}^j (h_r^2 + h_{j+r}^2)^{n-1} \prod_{r=2j+1}^n |h_r|^{n-1}.
\]
Combining these two results, we get
\[
  J^{\text{up}}_j(h) = 4^j \prod_{r=1}^j h_r^2 (h_{j+r}^2 + h_{j+r}^2)^{-(n-1)} \prod_{r=2j+1}^n |h_r|^{-(n-1)} \prod_{2j+1 \leq s \leq n} |h_r - h_s|^2 \\
(7.17) \prod_{1 \leq r < s \leq j} \left( (h_r - h_s)^2 + (h_{j+r} - h_{j+s})^2 \right)^2 \left( (h_r - h_s)^2 + (h_{j+r} + h_{j+s})^2 \right)^2 \\
\prod_{1 \leq r < s, 2j+1 \leq s \leq n} \left( (h_r - h_s)^2 + h_{j+r}^2 \right)^2.
\]

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SCHOOL OF MATHEMATICAL SCIENCE, PEKING UNIVERSITY, BEIJING, 100871, P. R. CHINA
E-mail address: anjinpeng@math.pku.edu.cn

SCHOOL OF MATHEMATICAL SCIENCE, PEKING UNIVERSITY, BEIJING, 100871, P. R. CHINA
E-mail address: zdwang@pku.edu.cn

SCHOOL OF MATHEMATICS AND PHYSICS, ZHEJIANG NORMAL UNIVERSITY, ZHEJIANG JINHUA, 321004, P. R. CHINA
E-mail address: yankh@zjnu.cn