RANK OF A CO-DOUBLY COMMUTING SUBMODULE IS 2

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Dedicated to the memory of our friend and colleague Sudipta Dutta

Abstract. We prove that the rank of a non-trivial co-doubly commuting submodule is 2. More precisely, let \( \varphi, \psi \in H^\infty(D) \) be two inner functions. If \( Q_\varphi = H^2(D)/\varphi H^2(D) \) and \( Q_\psi = H^2(D)/\psi H^2(D) \), then

\[
\text{rank} (Q_\varphi \otimes Q_\psi)^\perp = 2.
\]

An immediate consequence is the following: Let \( S \) be a co-doubly commuting submodule of \( H^2(D^2) \). Then \( \text{rank} S = 1 \) if and only if \( S = \Phi H^2(D^2) \) for some one variable inner function \( \Phi \in H^\infty(D^2) \). This answers a question posed by R. G. Douglas and R. Yang [4].

1. Introduction

Let \( T = (T_1, \ldots, T_n) \) be an \( n \)-tuple of commuting bounded linear operators on a Hilbert space \( \mathcal{H} \). For a subset \( E \subseteq \mathcal{H} \) we denote \( [E]_T \) by the close subspace \( \text{span}\{T_1^{k_1} \cdots T_n^{k_n} E : k_j \in \mathbb{N}, j = 1, \ldots, n\} \) of \( \mathcal{H} \). Then the rank of \( T \) [3] is the unique number

\[
\text{rank}(T) = \min \{ \#E : [E]_T = \mathcal{H}, E \subseteq \mathcal{H} \}.
\]

A closed subspace \( S \) of \( H^2(D^n) \), the Hardy space over the unit polydisc \( D^n \), is said to be shift invariant if \( M_z(S) \subseteq S \) for \( i = 1, 2, \ldots, n \), where \( M_z \) is the co-ordinate multiplication operator on \( H^2(D^n) \). The rank of a shift invariant subspace \( S \) of \( H^2(D^n) \) is the rank of the corresponding \( n \)-tuple of restricted co-ordinate shift operators, that is

\[
\text{rank} S = \text{rank} (M_{z_1}|S, \ldots, M_{z_n}|S).
\]

The rank of a bounded linear operator (or, of a commuting tuple of bounded linear operators) on a Hilbert space is an important numerical invariant. Very briefly, the rank of a bounded linear operator is the cardinality of a minimal generating set (see the definition below). One of the most intriguing and important problems in operator theory and function theory is the existence of a finite generating set for a commuting tuple of operators. Alternatively, one may ask when the rank of a commuting tuple of operators is finite.

Prototype examples of rank one operators are the co-ordinate multiplication operator tuple \( (M_{z_1}, \ldots, M_{z_n}) \) on the Hardy space, the (weighted) Bergman space over the unit ball and the polydisc in \( \mathbb{C}^n \), \( n \geq 1 \), and the Drury-Arveson space over the unit ball in \( \mathbb{C}^n \). Moreover, a particular version of the celebrated invariant subspace theorem of Beurling says: A shift invariant (or, shift co-invariant) subspace of the one variable Hardy space is of rank one.

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Computation of ranks of shift invariant as well as shift co-invariant subspaces beyond the case of the one variable Hardy space is an excruciatingly difficult problem, even if one considers only shift invariant (as well as co-invariant) subspaces of the Hardy space over the unit polydisc in $\mathbb{C}^n$, $n > 1$ (see however [2, 6, 7, 8, 14]).

The purpose of this paper is to compute the rank of a tractable class of shift invariant subspaces of the two variables Hardy space, $H^2(D^2)$, over the bidisc $D^2$ in $\mathbb{C}^2$. In order to state the precise contribution of this paper, we need to introduce first some definitions and notations.

We denote the open unit disc of $\mathbb{C}$ by $D$, and the unit circle by $T$. The Hardy space over the unit disc $D$ (bidisc $D^2$), denoted by $H^2(D)$ ($H^2(D^2)$), is the Hilbert space of all square summable holomorphic functions on $D$ (on $D^2$). Also we will denote by $M_z$ and $M_w$ the multiplication operators on $H^2(D^2)$ by the coordinate functions $z$ and $w$, respectively. It is easy to see that $(M_z, M_w)$ is a pair of commuting isometries, that is,

$$M_z M_w = M_w M_z, \quad M_z^* M_z = M_w^* M_w = I_{H^2(D^2)}.$$ 

Identifying $H^2(D^2)$ with the 2-fold Hilbert space tensor product $H^2(D) \otimes H^2(D)$, one can represent $(M_z, M_w)$ as $(M_z \otimes I_{H^2(D)}, I_{H^2(D)} \otimes M_w)$.

Let $S$ and $Q$ be closed subspaces of $H^2(D^2)$. Then $S$ is said to be a submodule if $M_z(S) \subseteq S$ and $M_w(S) \subseteq S$. We say that $Q$ is a quotient module if $Q^\perp$ is a submodule.

A well-known result due to Beurling states that if $S$ is a submodule of $H^2(D)$ (that is, $S$ is a closed subspace of $H^2(D)$ and $M_z S \subseteq S$), then $S$ can be represented as

$$S = S_\varphi := \varphi H^2(D),$$

where $\varphi \in H^\infty(D)$ is an inner function (that is, $\varphi$ is a bounded holomorphic function on $D$ and $|\varphi| = 1$ a.e. on $T$). Consequently, a quotient module $Q$ (that is, $Q$ is a closed subspace of $H^2(D)$ and $M_z Q \subseteq Q$) of $H^2(D)$ can be represented as

$$Q = Q_\varphi := (S_\varphi)^\perp = H^2(D)/\varphi H^2(D).$$

It readily follows that

$$\text{rank } (M_z|_{S_\varphi}) = \text{rank } (P_{Q_\varphi} M_z|_{Q_\varphi}) = 1.$$ 

Rudin [10], however, pointed out that there exists a submodule $S$ of $H^2(D^2)$ such that the rank of $S$ is not finite (see also [7, 12] and [13]).

A quotient module $Q$ of $H^2(D^2)$ is doubly commuting if $C_z C_w^* = C_w^* C_z$, where $C_z = P_Q M_z|_Q$ and $C_w = P_Q M_w|_Q$. A submodule $S$ of $H^2(D^2)$ is co-doubly commuting if the quotient module $S^\perp (\cong H^2(D^2)/S)$ is doubly commuting.

The following useful characterization of co-doubly commuting submodules is essential for our study (see [1, 14]): If $Q$ is a quotient module of $H^2(D^2)$, then $Q$ is a doubly commuting quotient module if and only if

$$Q = Q_1 \otimes Q_2,$$

for some quotient modules $Q_1$ and $Q_2$ of $H^2(D)$.

Let $S = (Q_1 \otimes Q_2)^\perp$ be a non-zero co-doubly commuting submodule. If $Q_j = H^2(D)$, for some $j = 1, 2$, then it is easy to see that

$$\text{rank } S = 1.$$
Now let both $Q_1$ and $Q_2$ be non-trivial quotient modules of $H^2(\mathbb{D})$, that is, $Q_j \neq \{0\}, H^2(\mathbb{D})$, $j = 1, 2$. Then there exist inner functions $\varphi, \psi \in H^\infty(\mathbb{D})$ such that $Q_1 = Q_\varphi$ and $Q_2 = Q_\psi$. The main purpose of the present paper is to prove that (see Theorem 2.1)

$$\text{rank } (Q_\varphi \otimes Q_\psi)^\perp = 2.$$ 

As a consequence of this, we give a complete and affirmative answer to a conjecture of Douglas and Yang (see page 220 [1]): If $S$ is a rank one co-doubly commuting submodule, then $S = \Phi H^2(\mathbb{D}^2)$ for some one variable inner function $\Phi \in H^\infty(\mathbb{D})$.

2. Proof of the main result

We begin with a simple but crucial observation on the rank of a joint semi-invariant subspace of a commuting tuple of operators.

Lemma 2.1. Let $T = (T_1, \ldots, T_n)$ be an $n$-tuple of commuting operators on a Hilbert space $\mathcal{H}$. Let $S_1$ and $S_2$ be two joint $T$-invariant subspaces of $\mathcal{H}$ and $S_2 \subseteq S_1$. If $S = S_1 \ominus S_2$, then

$$\text{rank } (P_S T_1|_S, \ldots, P_S T_n|_S) \leq \text{rank } (T_1|_{S_1}, \ldots, T_n|_{S_1}).$$

Proof. Let $m \in \mathbb{N}$ be the right side of the above inequality. Let $\{f_j\}_{j=1}^m \subseteq S_1$ be a generating set for $(T_1|_{S_1}, \ldots, T_n|_{S_1})$. Clearly, $P_S T_j P_S = P_S T_j|_{S_1}$ for all $j = 1, \ldots, n$. This yields

$$\text{rank } (P_S T_i|_S)(P_S T_j P_S) = P_S (T_i T_j)|_{S_1} \quad (i, j = 1, \ldots, n).$$

It hence follows that $\{P_S f_j\}_{j=1}^m$ is a generating set for $(P_S T_1|_S, \ldots, P_S T_n|_S)$. This completes the proof. \hfill $\square$

We now prove the main result of this paper.

Theorem 2.1. Let $\varphi, \psi \in H^\infty(\mathbb{D})$ be two inner functions. If

$$S = (Q_\varphi \otimes Q_\psi)^\perp,$$

then $\text{rank } S = 2$.

Proof. Let $X = I_{H^2(\mathbb{D}^2)} - (I_{H^2(\mathbb{D}^2)} - M_\varphi M_\varphi^* \otimes I_{H^2(\mathbb{D}^2)})(I_{H^2(\mathbb{D}^2)} - I_{H^2(\mathbb{D})} \otimes M_\psi M_\psi^*)$. Since $S = \text{ran } X$,

and

$$X = ((M_\varphi M_\varphi^*) \otimes (I_{H^2(\mathbb{D}^2)} - M_\psi M_\psi^*)) \oplus (I_{H^2(\mathbb{D})} \otimes M_\psi M_\psi^*),$$

it follows that

$$S = (S_\varphi \otimes Q_\psi) \oplus (H^2(\mathbb{D}) \otimes S_\psi).$$

Since by Theorem 6.2 of [1], $\text{rank } S \leq 2$, we only need to show that $\text{rank } S \geq 2$. Set

$$E = S \ominus (S_\varphi \otimes S_\psi).$$

It follows that

$$E = (S_\varphi \otimes Q_\psi) \oplus (Q_\varphi \otimes S_\psi).$$

Since $S_\varphi \otimes S_\psi \subseteq S$ is a submodule of $H^2(\mathbb{D}^2)$, by Lemma 2.1 it follows that

(2.1) \hfill \text{rank}(P_E M_z|_E, P_E M_w|_E) \leq \text{rank}(M_z|_S, M_w|_S) = \text{rank}(S).
Note that

$$P_\mathcal{E} = (P_{S_\varphi} \otimes P_{Q_\psi}) \oplus (P_{Q_\varphi} \otimes P_{S_\psi}).$$

and hence, an easy calculation yields

$$P_\mathcal{E} M_z|_\mathcal{E} = (M_z|_{S_\varphi} \otimes P_{Q_\psi}) \oplus (P_{Q_\varphi} M_z|_{S_\psi} \otimes P_{S_\psi}),$$

and

$$P_\mathcal{E} M_w|_\mathcal{E} = (P_{S_\varphi} \otimes P_{Q_\psi} M_w|_{Q_\psi}) \oplus (P_{Q_\varphi} \otimes M_w|_{S_\psi}).$$

Therefore it follows from the above equalities that $(S_{\varphi^2} \otimes Q_\psi) \oplus (Q_\varphi \otimes S_{\psi^2})$ is a joint $(P_\mathcal{E} M_z|_\mathcal{E}, P_\mathcal{E} M_w|_\mathcal{E})$ invariant subspace of $\mathcal{E}$. Set

$$\tilde{\mathcal{E}} = \mathcal{E} \oplus ((S_{\varphi^2} \otimes Q_\psi) \oplus (Q_\varphi \otimes S_{\psi^2})).$$

Notice that for any inner function $\theta \in H^\infty(\mathbb{D})$, we have

$$S_\theta \ominus S_{\theta^2} = \theta Q_\theta.$$

From this and the representation of $\mathcal{E} = (S_\varphi \otimes Q_\psi) \oplus (Q_\varphi \otimes S_\psi)$ it follows that

$$\tilde{\mathcal{E}} = ((S_\varphi \otimes Q_\psi) \oplus (Q_\varphi \otimes S_\psi)) \oplus ((S_{\varphi^2} \otimes Q_\psi) \oplus (Q_\varphi \otimes S_{\psi^2}))$$

$$= (\varphi Q_\varphi \otimes Q_\psi) \oplus (Q_\varphi \otimes \psi Q_\psi).$$

Then Lemma 2.1 and 2.1 implies that

$$\text{rank}(P_\mathcal{E} M_z|_\mathcal{E}, P_\mathcal{E} M_w|_\mathcal{E}) \leq \text{rank}(P_\mathcal{E} M_z|_\mathcal{E}, P_\mathcal{E} M_w|_\mathcal{E}) \leq \text{rank}(\mathcal{S}) \leq 2.$$

To finish the proof of the theorem it is now enough to prove the following:

$$\text{rank}(P_\mathcal{E} M_z|_\mathcal{E}, P_\mathcal{E} M_w|_\mathcal{E}) > 1.$$

Equivalently, it is enough to prove that the set $\{\xi\}$, for any $\xi \in \tilde{\mathcal{E}}$, is not a generating set corresponding to $(P_\mathcal{E} M_z|_\mathcal{E}, P_\mathcal{E} M_w|_\mathcal{E})$. Equivalently, given $\xi \in \tilde{\mathcal{E}}$, we show that there exists $\eta_\xi(\neq 0) \in \tilde{\mathcal{E}}$ such that

$$\langle (z^p \otimes w^q)\xi, \eta_\xi \rangle = 0 \quad (p, q \in \mathbb{N}).$$

To this end, let $\{f_i\}$ and $\{g_j\}$ be orthonormal bases of $Q_\varphi$ and $Q_\psi$, respectively, and let $\xi \in \tilde{\mathcal{E}}$ where

$$\xi = (\sum_{k,l} a_{kl} \varphi f_k \otimes g_l) \oplus (\sum_{k,l} b_{kl} f_k \otimes \psi g_l),$$

$\{a_{kl}\}, \{b_{kl}\} \subseteq \mathbb{C}$, and

$$\sum_{k,l} |a_{kl}|^2, \sum_{k,l} |b_{kl}|^2 < \infty.$$

Again we observe that for any inner function $\theta \in H^\infty(\mathbb{D})$ and $f = \sum_{m \geq 0} c_m z^m \in Q_\theta$ we have

$$M_\varphi^*(\theta \bar{f}) \in Q_\theta,$$

where $\bar{f} = \sum_{m \geq 0} \bar{c}_m e^{-imt} \in L^2(\mathbb{T})$. This follows from the fact that $\theta$ is a bounded holomorphic function on $\mathbb{D}$ and $M_\varphi^*(\theta \bar{f}) \perp z^m$ for all $m < 0$ (which gives that $M_\varphi^*(\theta \bar{f}) \in H^2(\mathbb{D})$), and then $M_\varphi^*(\theta \bar{f}) \perp z^m$ in $L^2(\mathbb{T})$ for all $m \geq 0$ (which gives that $M_\varphi^*(\theta \bar{f}) \in Q_\theta$). It should be noted that $M_\varphi^*(\theta \bar{f}) = \theta z^n \bar{f} = C_\theta(f)$, where the conjugation map $C_\theta : Q_\theta \rightarrow Q_\theta$, $f \mapsto M_\varphi^*(\theta \bar{f})$, is
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called a C-symmetry and it is used extensively in the study of Toeplitz operators on model spaces (for more details see [5]).

Coming back to our context, this immediately yields that

\[ M^*_z(\varphi f_k) \otimes M^*_w(\psi g_l) \in \mathcal{Q}_\varphi \otimes \mathcal{Q}_\psi \quad (k, l \geq 0), \]

and hence \( s_0 \otimes s_1, t_0 \otimes t_1 \in \mathcal{Q}_\varphi \otimes \mathcal{Q}_\psi \), where

\[ s_0 \otimes s_1 := - \sum_{k,l} a_{kl} M^*_z(\varphi f_k) \otimes M^*_w(\psi g_l) = -(M^*_z \otimes M^*_w)(\varphi \otimes \psi)(\sum_{k,l} a_{kl} f_k \otimes g_l) \]

and

\[ t_0 \otimes t_1 := \sum_{k,l} b_{kl} M^*_z(\varphi f_k) \otimes M^*_w(\psi g_l) = (M^*_z \otimes M^*_w)(\varphi \otimes \psi)(\sum_{k,l} b_{kl} f_k \otimes g_l). \]

Set

\[ \eta_\xi = (\varphi t_0 \otimes t_1) \oplus (s_0 \otimes \psi s_1) \in \mathcal{E}. \]

Then \( \eta_\xi \neq 0 \) and for every \( p, q \in \mathbb{N} \) we have

\[
\langle (z^p \otimes w^q)\xi, \eta_\xi \rangle = \langle (z^p \otimes w^q)(\sum_{k,l} a_{kl} \varphi f_k \otimes g_l) \oplus (\sum_{k,l} b_{kl} f_k \otimes \psi g_l), (\varphi t_0 \otimes t_1) \oplus (s_0 \otimes \psi s_1) \rangle \\
= \langle (z^p \otimes w^q)(\sum_{k,l} a_{kl} \varphi f_k \otimes g_l), \varphi t_0 \otimes t_1 \rangle \\
+ \langle (z^p \otimes w^q)(\sum_{k,l} b_{kl} f_k \otimes \psi g_l), s_0 \otimes \psi s_1 \rangle \\
= \langle (z^p \otimes w^q)(\sum_{k,l} a_{kl} f_k \otimes g_l), t_0 \otimes t_1 \rangle + \langle (z^p \otimes w^q)(\sum_{k,l} b_{kl} f_k \otimes g_l), s_0 \otimes s_1 \rangle \\
= \langle (z^{p+1} \otimes w^{q+1})(\sum_{k,l} a_{kl} f_k \otimes g_l), (\varphi \otimes \psi)(\sum_{k,l=1}^\infty b_{kl} f_k \otimes g_l) \rangle \\
- \langle (z^{p+1} \otimes w^{q+1})(\sum_{k,l} b_{kl} f_k \otimes g_l), (\varphi \otimes \psi)(\sum_{k,l} a_{kl} f_k \otimes g_l) \rangle \\
= 0.
\]

We have thus shown that \( \{\xi\} \) is not a minimal generating subset of \( \mathcal{E} \) with respect to \( (P_\xi M_z|_\xi, P_\xi M_w|_\xi) \) as desired. \( \square \)

As a consequence of the above theorem we have the following corollary which provides an affirmative answer of the question raised by Douglas and Yang [4].

**Corollary 2.2.** Let \( \mathcal{S} \) be a co-doubly commuting submodule of \( H^2(\mathbb{D}^2) \). Then \( \text{rank} \ (\mathcal{S}) = 1 \) if and only if \( \mathcal{S} = \Theta H^2(\mathbb{D}^2) \) for some one variable inner function \( \Theta \in H^\infty(\mathbb{D}) \).

**Proof.** If \( \mathcal{S} = \Theta H^2(\mathbb{D}^2) \) for some one variable inner function \( \Theta \in H^\infty(\mathbb{D}) \), then \( \mathcal{S} \cong H^2(\mathbb{D}^2) \) and hence \( \text{rank} \ \mathcal{S} = 1 \). To prove the the sufficient part let \( \mathcal{S} \) be a rank one co-doubly
commuting submodule of $H^2(D^2)$. Then there exist quotient modules $Q_1$ and $Q_2$ of $H^2(D^2)$ such that (see [9, 11])

$$S = (Q_1 \otimes Q_2)\perp.$$ 

Since rank $(S) = 1$, it follows from Theorem 2.4 that $Q_j = H^2(D)$, for some $j = 1, 2$. This shows that

$$S = S_\varphi \otimes H^2(D), \quad \text{or} \quad S = H^2(D) \otimes S_\psi,$$

for some inner functions $\varphi, \psi \in H^\infty(D)$. This concludes the proof of the corollary. □

There is now the following interesting and natural question: Let $m \geq 2$ and let $\{\varphi_j\}_{j=1}^m \subseteq H^\infty(D)$ be inner functions. Is then

$$\text{rank} \left( Q_{\varphi_1} \otimes \ldots \otimes Q_{\varphi_m} \right) \perp = m?$$

Our present approach does not seem to work for $m > 2$ case.

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