Likelihood-based Spacings Goodness-of-Fit Statistics for Univariate Shape-constrained Densities

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Abstract

A variety of statistics based on sample spacings has been studied in the literature for testing goodness-of-fit to parametric distributions. To test the goodness-of-fit to a nonparametric class of univariate shape-constrained densities, including widely studied classes such as k-monotone and log-concave densities, a likelihood ratio test with a working alternative density estimate based on the spacings of the observations is considered, and is shown to be asymptotically normal and distribution-free under the null, consistent under fixed alternatives, and admits bootstrap calibration. The distribution-freeness under the null comes from the fact that the asymptotic dominant term depends only on a function of the spacings of transformed outcomes that are uniformly distributed. Applications and extensions of theoretical results in the literature of shape-constrained estimation are required to show that the average log-density ratio converges to zero at a faster rate than the sample spacing term under the null, and diverges under the alternatives. Numerical studies are conducted to demonstrate that the test is applicable to various classes of shape-constrained densities and has a good balance between type-I error control under the null and power under alternative distributions.

1 Introduction

Scientific knowledge and theoretical understanding of a problem can often provide hypotheses about the underlying data-generating mechanism, which could be in the form of a completely known parametric distribution, a parametric family of distributions with unknown finite-dimensional parameters, or flexible nonparametric constraints on the shape of the density, e.g. monotonicity or log-concavity, which motivated many recent theoretical and methodological development.

Goodness-of-fit tests of parametric distributions using sums of functions of sample spacings, which are the lengths of the intervals between adjacent order statistics from a sample, have been studied since the 1940s. Early developments include [1, 2, 3, 4]. Since one can
use the distribution function to transform data with a known distribution to the uniform
distribution on the unit interval, it suffices to study the theoretical properties of testing
the goodness-of-fit to the uniform distribution. Asymptotic power of tests based on sample
spacing was studied in [5, 6], who showed a theoretical limitation but motivated subsequent
developments of goodness-of-fit tests based on higher-order spacings, or \( \nu \)-spacings \((\nu > 1)\),
which are the lengths of the intervals between two order statistics for which the order differs
by \( \nu \) \[7, 8\], and have a power advantage over sample spacings \((\nu = 1)\). [9, 10] studied the
power of \( \nu \)-spacings goodness-of-fit statistics when the order \( \nu \) diverges with the sample size
\( n \).

To test whether the data-generating mechanism fits a parametric model with unknown
parameters, which is typically a composite null hypothesis, [11, 12, 13] studied functions of
sample spacings using an estimated distribution function transformed with a plug-in esti-
mator of the unknown parameters, whose null distribution was shown to be asymptotically
equivalent to the case where the parameters are indeed known. However, it is unclear whether
similar procedures can be used for testing the goodness-of-fit to a nonparametric class with
an estimated density.

Nonparametric density estimation under shape constraints has been receiving increasing
attention in recent years. See, e.g., [14] for various examples. A particularly valuable feature
of estimation under shape constraints is that nonparametric maximum likelihood estimators
are often fully automatic, requiring no tuning parameters, in comparison to kernel smoothing
methods.

For univariate distributions, two general classes, \( k \)-monotonicity and log-concavity, have
recently been studied in detail. A density function \( f \) on \( \mathbb{R}^+ \) is 1-monotone if it is nonin-
creasing. [15] first showed that the nonparametric maximum likelihood estimator (NPMLE)
under the nonincreasing (1-monotone) density assumption can be characterized as the left
derivative of the least concave majorant of the empirical distribution function. As a result,
the NPMLE in this case is often referred to as the Grenander estimator. Its asymptotic
distribution at a fixed interior point at which the derivative is strictly negative was obtained
by [16] and [17].

A density function is 2-monotone if it is nonincreasing and convex. The corresponding
NPMLE was studied in [18]. Furthermore, a density function is \( k \)-monotone for \( k \geq 3 \) if
\((-1)^j f^{(j)}\) is non-negative, nonincreasing and convex for \( j = 0, \ldots, k - 2 \), with NPMLE being
studied in [19] and [20]. A density on \( \mathbb{R}^+ \) is called completely monotone if \((-1)^j f^{(j)}(t) \geq 0\)
for all non-negative integers \( j \) and all \( t > 0 \), which can be seen as the limit of a \( k \)-monotone
density as \( k \) goes to infinity and can be represented as a scale mixture of exponential dis-
tributions. [21] established the unique existence of the NPMLE for a completely monotone
density and the consistency of the mixing distribution function. To the best of our knowledge,
there is no pointwise limit distribution theory or global rates of convergence for continuous
completely monotone densities.

A density function on \( \mathbb{R} \) is log-concave if its logarithm is concave. This class is a subset
of unimodal densities and contains many of the commonly used parametric distributions and
can be regarded as an infinite-dimensional generalization of the class of Gaussian densities.
Estimation and theoretical properties have been extensively studied in recent years; see, for
eexample, [22, 23, 24, 25, 26, 27].

Apart from estimation, various hypothesis testing problems related to shape restrictions
on monotone functions have been considered in the literature. Many of these tests make use of distances between two estimators, where one is only valid under the null hypothesis and the other is valid in general. For example, [28] tests the null hypothesis of a constant hazard rate against the alternative of an increasing hazard rate; [29] tests the null hypothesis that a hazard rate is monotone nondecreasing; [30] tests for local monotonicity of a hazard function; [31] tests a parametric null hypothesis that respects a monotonicity constraint; and [32] tests for monotone density. These tests typically have complicated null distributions, and additional regularity conditions on the true underlying distribution are often imposed. For testing multivariate log-concave densities, see [33] and [34].

In this article, we study a unified nonparametric likelihood ratio test (NPLRT) for testing whether the underlying univariate density belongs to a particular hypothesis class of functions, focusing on $k$-monotone, completely monotone and log-concave densities. An attractive feature of our test is that the limiting null distribution remains the same across different hypothesis classes and does not depend on the unknown density or its derivatives. In a conventional likelihood ratio test, we have to maximize the likelihood over the union of the null and alternative hypotheses. In such cases, the union is the collection of all densities on $\mathbb{R}$, where it is well-known that the corresponding maximum likelihood is infinite and resulting in an ill-posed likelihood ratio statistic. Instead, we use a histogram-type estimator that depends on the $\nu$-spacings.

Our proposed likelihood ratio test statistic depends only on the NPMLE under the corresponding shape constraint and the spacings of the observations, and can be computed easily. Moreover, the asymptotic null distribution of the test statistic is distribution-free, which follows from the convergence of a sum of functions of uniform $\nu$-spacings. An average log-likelihood ratio is required to converge at a faster rate than the spacings term under the null, and to diverge under fixed alternatives. We verify these properties for $k$-monotone, completely monotone, and log-concave hypotheses classes, by applying and extending existing results in the shape-constrained estimation literature.

We also consider bootstrapping for both theoretical and practical reasons. Although it is known that the nonparametric bootstrap and bootstrapping from the NPMLE for 1-monotone densities do not work for finding the pointwise limiting distribution of the NPMLE at an interior point [35] [36], we show that bootstrapping from the NPMLE is valid for approximating the distribution of the NPLRT. The main reason bootstrapping from the NPMLE works in this context is that the statistic is a global measure instead of a local one, and the bootstrap distribution only needs to be continuous without further requirements on differentiability. However, the nonparametric bootstrap remains inapplicable because the lack of continuity of the bootstrap distribution. For practical reasons, given the relatively slow rate of convergence of the log-likelihood ratio and the remainder term, bootstrap calibration of the critical value can provide better control of the type-I error and improve the power under alternatives.

The following sections are organized as follows. In Section 2 we introduce our NPLRT and establish the asymptotic distribution of the NPLRT statistic under the null hypothesis, as well as the consistency of the test under the alternative hypothesis, given general conditions. In Section 3 we specifically discuss the average log-likelihood ratio for $k$-monotone (including 1-monotone and completely monotone) and log-concave densities under the null and alternative hypotheses, where we also establish a rate of convergence of the maximum
likelihood estimator for the $k$-monotone densities under relaxed conditions. Section 4 discusses a valid bootstrap procedure. In Section 5, simulation studies are performed to evaluate the performance of the test in various situations. Section 6 presents several real data applications of our tests. Conclusion and additional remarks are discussed in Section 7. All the proofs for the theoretical results are given in the Supplementary Material.

2 Nonparametric Goodness-of-Fit Test and Main Results

2.1 Definition and Main Results

Let $X_1, \ldots, X_n$ be a random sample from a univariate distribution $F_0$ with density $f_0$. The likelihood function is

$$L_n(f) := \prod_{i=1}^{n} f(X_i).$$

Our aim is to propose a nonparametric likelihood ratio test (NPLRT) for testing the null hypothesis $H_0 : f_0 \in \mathcal{F}$, where $\mathcal{F}$ is a nonparametric class of densities for which the NPMLE, $\hat{f}_n := \arg\max_{f \in \mathcal{F}} L_n(f)$, exists, versus $H_1 : f_0 \notin \mathcal{F}$. In particular, we focus on the hypothesis classes of shape-constrained densities, including the classes of (i) decreasing densities, (ii) $k$-monotone densities ($k \geq 2$), (iii) completely monotone densities, and (iv) log-concave densities, where all the NPMLEs exist and are unique.

To define a likelihood ratio test, we need an estimator that works under both the null and alternative hypotheses. To this end, we consider a histogram-type estimator, which leads to the desirable properties that the test statistic is asymptotically distribution-free under the null hypothesis. Additionally, it does not require strong assumptions such as bounded support or upper-boundedness of the underlying density to establish the asymptotic null distribution and the consistency of the test. Let $Z_1 < \ldots < Z_n$ be the order statistics of $X_1, \ldots, X_n$, and let $\nu \in \mathbb{N}$.

Without loss of generality, and for simplicity of presentation, we assume throughout the article that $\frac{n-1}{\nu}$ is an integer, and we denote by $n_\nu$. We define a piecewise constant density function with $n_\nu$ steps as follows: for $j = 0, \ldots, n_\nu - 1$, $x \in (Z_{j+1}, Z_{(j+1)\nu+1}]$, let

$$f_{n,\nu}(x) := \frac{1}{n_\nu(Z_{(j+1)\nu+1} - Z_{j\nu+1})}. $$

Define $f_{n,\nu}(Z_1) := f_{n,\nu}(Z_1^+)$ and $f_{n,\nu}(x) = 0$ if $x \notin [Z_1, Z_n]$. Thus, $f_{n,\nu}$ is a function of $\nu$-spacings. Modifications can be made accordingly when $n_\nu$ is not an integer by dividing the data into $\lfloor n_\nu \rfloor$ groups as evenly as possible, where for $x \in \mathbb{R}_+$, $\lfloor x \rfloor$ denotes the smallest integer less than or equal to $x$.

Our proposed likelihood ratio test statistic is

$$T_n := -\frac{1}{n} \log \frac{\prod_{i=1}^{n} \hat{f}_n(X_i)}{\prod_{i=1}^{n} f_{n,\nu}(X_i)}. $$

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To study its asymptotic properties, we can write

$$T_n = -\frac{1}{n} \sum_{i=1}^{n} \log \frac{\hat{f}_n(X_i)}{f_0(X_i)} - \frac{1}{n} \sum_{i=1}^{n} \log \frac{f_0(X_i)}{f_{H,n,\nu}(X_i)}. \quad (1)$$

Denote

$$S_n := -\frac{1}{n} \sum_{i=1}^{n} \log \frac{\hat{f}_n(X_i)}{f_0(X_i)}$$

which is the first term in the right-hand-side of (1). We further decompose the second term into two terms,

$$-\frac{1}{n} \sum_{i=1}^{n} \log \frac{f_0(X_i)}{f_{H,n,\nu}(X_i)} = M_n + R_n, \quad (2)$$

where

$$M_n := -\frac{\nu}{n} \sum_{j=0}^{n_\nu-1} \log \frac{F_0(Z_{(j+1)\nu+1}) - F_0(Z_{j\nu+1})}{\nu/(n-1)},$$

$$R_n := -\frac{1}{n} \sum_{j=0}^{n_\nu-1} \sum_{l=1}^{\nu} \log \frac{f_0(Z_{(j+1)\nu+l+1})(Z_{(j+1)\nu+1} - Z_{j\nu+1})}{F_0(Z_{(j+1)\nu+1}) - F_0(Z_{j\nu+1})}$$

$$-\frac{1}{n} \log \frac{f_0(Z_1)}{f_{H,n,\nu}(Z_1)}.$$

We first provide a high-level description of how the decomposition $T_n = S_n + M_n + R_n$ will be used in the theoretical development. The term $S_n$ is an average log-likelihood ratio between the NPMLE and the true density, $M_n$ is an average log-spacings of transformed data, and $R_n$ is an asymptotically negligible remainder term under both the null and the alternative hypotheses. The NPMLE under a specific hypothesis class appears only in the first, $S_n$, but not in the other two terms. The key observation is that the terms, $S_n$ and $M_n$, contribute differently to the behavior of the test statistic under the null and the alternative hypotheses. Under the null, $M_n$ asymptotically dominates $S_n$, leading to an asymptotic normal and pivotal null distribution. Under fixed alternatives, $S_n$ diverges, ensuring the test is consistent.

However, the NPMLE for each hypothesis class, and therefore $S_n$, differs substantially. Thus, the above argument needs to be established on a case-by-case basis, as discussed in Section 3. The convergence of $M_n$ is discussed in in Section 2.2. Sufficient conditions for $R_n$ being asymptotically negligible are provided in Section 2.3.

Let $\Gamma(\cdot), \psi(\cdot)$ and $\psi_1(\cdot)$ denote the gamma, digamma and trigamma functions, respectively. Specifically, $\Gamma(y) = \int_{0}^{\infty} t^{y-1}e^{-t}dt$ for $y > 0$, $\psi(y) = \frac{d}{dy} \log \Gamma(y)$, and $\psi_1(y) = \frac{d}{dy} \psi(y)$.

The following result provides the asymptotic null distribution of our goodness-of-fit test statistics.

**Theorem 2.1 (Asymptotic Null Distribution).** Suppose that $\nu = O(n^{1/3} \log n)^{-1}$, $\sqrt{n\nu} S_n = o_p(1)$ and $R_n = O_p(\frac{\nu \log n}{n})$. Then,

$$\sqrt{\frac{n\nu}{\nu^2 \psi_1(\nu)} - \nu} (T_n - \log \nu + \psi(\nu)) \xrightarrow{d} N(0, 1). \quad (3)$$
The following theorem states the consistency of our test under fixed alternatives, where the key condition is the existence of some sequence $L_n \uparrow \infty$ such that $\sqrt{n\nu S_n} > L_n$ with probability approaching 1. In Section 3, we show this will be satisfied when the minimum Hellinger distance between the true underlying density and any densities in the hypothesis class is strictly greater than 0, where the Hellinger distance between two densities $f$ and $g$ is defined as

$$h(f, g) = \frac{1}{\sqrt{2}} \left[ \int_0^\infty \left\{ \sqrt{f(t)} - \sqrt{g(t)} \right\}^2 dt \right]^{1/2}.$$ 

**Theorem 2.2** (Consistency). Suppose that $\nu = O(n^{1/3}\log n^{-1})$. Under $H_1$, suppose that there exists some sequence $L_n \uparrow \infty$ such that $\lim_{n \to \infty} P(\sqrt{n\nu S_n} > L_n) = 1$ and $R_n = O_p(\frac{\nu \log n}{n})$. Then, the NPLRT is consistent. That is, for any $0 \leq c < \infty$,

$$\lim_{n \to \infty} P \left( \sqrt{\frac{n\nu}{\nu^2 \psi_1(\nu)} - \nu} \left( T_n - \log \nu + \psi(\nu) \right) > c \right) = 1.$$ 

Note that Theorems 2.1 and 2.2 holds for both fixed $\nu$ and when $\nu$ diverges to infinity. When $\nu = 1$, $\psi(1) = -\gamma$, where $\gamma = 0.57721...$ is the Euler-Mascheroni constant, and $\psi_1(1) = \pi^2/6$. As $\nu \to \infty$,

$$\psi_1(\nu) = \frac{1}{\nu} + \frac{1}{2\nu^2} + O(\nu^{-3}),$$

therefore $\nu^2 \psi_1(\nu) - \nu \to 1/2$ as $\nu \to \infty$. Thus, in Theorems 2.1 and 2.2 $\nu^2 \psi_1(\nu) - \nu$ may be replaced by 1/2 for diverging $\nu$.

### 2.2 Asymptotic distribution of $M_n$

Since $F_0$ is assumed to be continuous, $(F_0(Z_1), \ldots, F_0(Z_n))$ and $(U_1, \ldots, U_n)$ have the same distribution, where $0 < U_1 < \ldots < U_n < 1$ are the order statistics from a random sample of size $n$ from a Uniform(0, 1) distribution. Therefore, $M_n$, as a function of $(F_0(Z_1), \ldots, F_0(Z_n))$, is distribution-free for any finite sample size $n$.

It is well known that

$$(U_1, U_2 - U_1, \ldots, U_n - U_{n-1}) \overset{d}{=} \left( \frac{E_1}{\sum_{h=1}^{n+1} E_h}, \ldots, \frac{E_n}{\sum_{h=1}^{n+1} E_h} \right),$$

where $E_h, h = 1, \ldots, n + 1$, are independent random variables, each following the standard exponential distribution with mean 1; see, for example, Theorem 2.2 in [37]. Therefore,

$$(U_{(\nu+1)}, U_{(2\nu+1)} - U_{(\nu+1)}, \ldots, U_{(n)} - U_{(n-\nu)}) \overset{d}{=} \left( \frac{\sum_{l=2}^{\nu+1} E_l}{\sum_{h=1}^{n+1} E_h}, \frac{\sum_{l=\nu+2}^{2\nu+1} E_l}{\sum_{h=1}^{n+1} E_h}, \ldots, \frac{\sum_{l=n-\nu+1}^{n+1} E_l}{\sum_{h=1}^{n+1} E_h} \right).$$

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The numerators are sum of \( \nu \) independent standard exponential random variables, each of which follows a gamma distribution with shape parameter \( \nu \) and scale parameter 1. Let \( \tilde{E}_j, j = 0, \ldots, n_\nu - 1 \) be independent and identically distributed gamma random variables with shape parameter \( \nu \) and scale parameter 1. We will show in the supplementary materials that if \( \nu = o(n) \), \( M_n - \log \nu + \psi(\nu) \) is asymptotically equivalent to

\[
\frac{1}{n} \sum_{j=0}^{n_\nu-1} (\tilde{E}_j - \nu \log(\tilde{E}_j) - \nu + \nu \psi(\nu)).
\]

An interesting fact is that \( \text{Var}(\tilde{E}_1) = O(\nu) \) and \( \text{Var}(\nu \log(\tilde{E}_1)) = O(\nu) \), but \( \text{Var}(\tilde{E}_1 - \nu \log(\tilde{E}_1)) = O(1) \). We have the following result.

**Theorem 2.3.** If \( \nu = o(n) \), then

\[
\sqrt{n} \frac{\nu \psi_1(\nu) - \nu}{\nu^2 \psi_1(\nu) - \nu} (M_n - \log \nu + \psi(\nu)) \xrightarrow{d} N(0,1).
\]

Theorem 2.3 holds for both fixed \( \nu \) and diverging \( \nu \). For fixed \( \nu \), the result (in a slightly different form) was given in [38]. For diverging \( \nu \), the result is given in [10], with \( \nu^2 \psi_1(\nu) - \nu \) replaced by its limit \( 1/2 \), but that result does not hold for fixed \( \nu \).

### 2.3 Sufficient Conditions for \( R_n \) to be asymptotically negligible

In this subsection, we provide some sufficient conditions for \( R_n = O_p(\nu \log n) \) in Theorem 2.4.

Here, \( f_0 \) can belong to either \( H_0 \) or \( H_1 \). These conditions relate only to the true underlying density and not to the NPMLE. Intuitively, it is expected that \( \log f_0(Z_{j+1})/(Z_{j+1} - Z_j) \approx \log f_0(Z_{j+1}) - f_0(Z_j) \), so that \( R_n \) approaches 0 at a certain rate. Given a density \( f \), let \( \tau_f \) and \( \sigma_f \) denote its left endpoint and right endpoint of the support, respectively.

**Conditions:**

(A) There exists \( x_0 > 0 \) such that for all \( |x| \geq x_0 \), \( f_0(x) \leq |x|^{-\gamma} \) for some \( \gamma > 1 \).

(B) For \( a \) such that \( f(a+) = \infty \) or \( f(a-) = \infty \), there exists \( \delta > 0 \) such that for \( x \) with \( |x-a| < \delta \), \( f(x) \leq (x-a)^{\gamma_2-1} \) for some \( \gamma_2 \in (0,1) \).

(C) (i) \( f_0 \) is monotone (decreasing or increasing); or:

(ii) There exist \( K_1 > \tau_f \) and \( K_2 < \sigma_f \), such that \( f_0 \) is monotone on \( (\tau_f, K_1] \) and \( [K_2, \sigma_f) \), and \( \log f_0 \) is Lipschitz continuous on \( [K_1, K_2] \) with a Lipschitz constant \( L \).

Condition (A) essentially requires the tail of \( f_0 \) to decay at a rate that is not too slow if its support is unbounded. In particular, a regularly varying random variable with any tail index \( \alpha > 1 \) whose density is ultimately monotone satisfies Condition (A) by the

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1 A random variable is regularly varying with tail index \( \alpha > 0 \) if its survival function \( \mathbb{P}(X > x) = x^{-\alpha} l(x) \) for some slowly varying function \( l \).

2 A function \( f \) is said to be ultimately monotone if \( f \) is monotone on \( (x, \infty) \) for some \( x > 0 \). Thus, \( f_0 \) satisfying (C)(i) or (C)(ii) is ultimately monotone.

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monotone density theorem (see Theorem 1.2.9 in [39]). When \( \alpha \in (0, 1) \), such a regularly varying random variable has an infinite mean.

Condition (B) essentially requires that \( f_0 \) does not grow to infinity too quickly if it is unbounded. We also allow for the possibility that the density grows to infinity at multiple points.

**Theorem 2.4.** Suppose that Conditions (A), (B), and one of Conditions (C) hold. Then, we have

\[
R_n = O_p \left( \frac{\nu \log n}{n} \right).
\]

For log-concave densities, we have the following corollary of Theorem 2.4 as they satisfy Conditions (A), (B) and (C) (ii).

**Corollary 2.5.** For any univariate log-concave densities, \( R_n = O_p \left( \frac{\nu \log n}{n} \right) \).

## 3 Average log density ratio for Shape-Constrained Densities

In this section, we study the behavior of the average log density ratio \( S_n \) under both the null and alternative hypotheses. Specifically, Section 3.1 will focus on \( k \)-monotone densities; Section 3.2 on completely monotone densities; and Section 3.3 on log-concave densities. In each subsection, we will develop the rate of convergence of \( S_n \) under the null, which in general is faster than that of \( M_n \). To show the divergence of the log-likelihood ratio under alternatives, one approach is to study the behavior of the NPMLE under misspecification. However, this problem has only been studied for 1-monotone densities [40] and log-concave densities [25]. By using a probability inequality for the likelihood ratio from [41], we are able to show the consistency of the test under the alternative without first investigating the limit of the NPMLE under misspecification of the hypothesis classes under general regularity conditions.

### 3.1 \( k \)-monotone densities

In this subsection, we study the behavior of the average log density ratio for the class of \( k \)-monotone densities. Denote \( \mathcal{F}_k \) as the class of all \( k \)-monotone densities with support being \([0, \infty)\) or its subsets, for any \( k \in \mathbb{N} \). Let \( \hat{f}_{n,k} \) be the NPMLE over the whole class \( \mathcal{F}_k \). Define \( S_{n,k} \) by

\[
S_{n,k} := -\frac{1}{n} \sum_{i=1}^{n} \log \frac{\hat{f}_{n,k}(X_i)}{f_0(X_i)}.
\]

The main result in this section is to establish the rate of convergence of \( S_{n,k} \) to 0 under \( H_0 \), and the divergence of \( S_{n,k} \) under \( H_1 \), when \( f_0 \) is allowed to be unbounded at 0 and may have an unbounded support \([0, \infty)\).

We first review existing results for bounded density with bounded support. Denote \( \mathcal{F}_k([0, A]) \) as the class of \( k \)-monotone densities with support contained in \([0, A]\), and \( \mathcal{F}_k^B([0, A]) \)
as the subclass of densities in $\mathcal{F}_k([0, A])$ that are all bounded above by $B$. When $f_0 \in \mathcal{F}_k^B([0, A])$, [42] obtained an upper bound on the bracketing entropy of the class $\mathcal{F}_k^B([0, A])$ under the Hellinger distance $h$:

$$\log N_{\varepsilon}(\varepsilon, \mathcal{F}_k^B([0, A]), h) \leq C \log AB + \frac{1}{2} \log \varepsilon^{-\frac{1}{k}}, \quad (5)$$

where $C$ is a constant that depends only on $k$, and $N_{\varepsilon}(\varepsilon, \mathcal{G}, \rho)$ is the bracketing number, which is the minimum number of $\varepsilon$-brackets, defined using the distance metric $\rho$, needed to cover a function class $\mathcal{G}$. Note that [42] considered $\mathcal{F}_k^B([0, A])$ instead of $\mathcal{F}_k$ because the latter is not totally bounded. Using (5), for $f_0 \in \mathcal{F}_k^B([0, A])$, one can derive, for example, by following the argument in Corollary 7.5 in [43], that

$$S_{n,k} = O_p\left(n^{-\frac{2k}{2k+1}}\right); \quad (6)$$

see also Theorem [3.4](i).

For the case where the density is unbounded and/or has an unbounded support, to the best of our knowledge, the corresponding literature on the rate of convergence of $S_{n,k}$ is lacking, since this class does not have a finite bracketing entropy in terms of Hellinger distance [42]. Novel extensions are needed to address this problem. The approach we use here relies on the following facts at the NPMLE: (i) it is bounded by $k$ times the inverse of the minimum order statistic, and (ii) the right endpoint of its support is bounded by $k$ times the maximum order statistic; see Lemma 3.1 and Lemma 3.3, respectively.

Lemma 3.1. If $f_0 \in \mathcal{F}_k$, then $\hat{f}_{n,k}(0+) \leq kZ_1^{-1}$. If $f_0$ is also bounded from above, then $\hat{f}_{n,k}(0+) = O_p(1)$.

Remark 3.2. In [44], the authors studied the asymptotic behavior of the Grenander estimator for a 1-monotone density near zero. Specifically, they considered the situation when the true density is unbounded at zero and established the rate at which $\hat{f}_n(0+)$ diverges to infinity under certain regularity conditions. For example, if $f_0(x) = \gamma x^{\gamma-1}I(0 < x \leq 1)$ with $0 < \gamma < 1$, then Theorem 1.1 in [44] implies that $\hat{f}_{n,k}(0+) = O_p(n^{1/\gamma-1})$. On the other hand, the bound $\hat{f}_{n,k}(0+) \leq kZ_1^{-1}$ implies that $\hat{f}_{n,k}(0+) = O_p(n^{1/\gamma})$. While our bound is weaker, it does not require any additional assumptions made in [44] and is sufficient for establishing that $\sqrt{n\nu}S_{n,k} = o_p(1)$; see Theorem 3.4 and Corollary 3.6. Furthermore, to the best of our knowledge, there is no known limit theory of the NPMLE of a $k$-monotone density near zero when $k > 1$.

Lemma 3.3. If $f_0 \in \mathcal{F}_k$, then $\sigma_{\hat{f}_{n,k}} \leq kZ_n$.

In general, to obtain a rate of convergence of the NPMLE, it suffices to consider a subspace where the NPMLE lies with probability approaching 1. In view of Lemmas 3.1 and 3.3, we know that $\hat{f}_{n,k} \in \mathcal{F}_k^{kZ_1^{-1}}([0, kZ_n])$. Thus, while $f_0$ may not be bounded from above or may not have a bounded support, its NPMLE is bounded from above and has a bounded support. Then, the rate of convergence of $S_{n,k}$ to 0 depends on how fast $Z_1$ approaches 0 and/or how fast $Z_n$ approaches $\infty$. In the following theorem, we establish that as long as $f_0$
satisfies Conditions (A) and (B), there is only a difference in logarithm order compared to the case where \( f_0 \in \mathcal{F}_B^k([0, A]) \).\(^3\)

**Theorem 3.4.** Suppose that \( f_0 \in \mathcal{F}_k \).

(i) If \( f_0 \in \mathcal{F}_B^k([0, A]) \), then \( S_{n,k} = O_p \left( n^{-\frac{2k}{2k+1}} \right) \), as stated in (6).

(ii) If \( f_0 \) has an unbounded support but satisfies Condition (A) and/or \( f_0 \) is not bounded from above but satisfies Condition (B), then

\[
S_{n,k} = O_p \left( n^{-\frac{2k}{2k+1}} (\log(n))^{\frac{1}{2(2k+1)}} \right).
\]

**Remark 3.5.** Similar to Theorem 3.4, if \( f_0 \) satisfies Conditions (A) and (B), we can also obtain

\[
h(\hat{f}_{n,k}, f_0) = O_p \left( n^{-\frac{2k}{2k+1}} (\log(n))^{\frac{1}{2(2k+1)}} \right).
\]

Following Corollary 2.1, Theorems 2.4 and 3.4, we obtain the following Corollary 3.6 establishing the asymptotic null distribution of the NPLRT for the class of \( k \)-monotone densities.

**Corollary 3.6.** Under Conditions in Theorem 3.4 and suppose that \( \nu = O(n^{1/3}(\log n)^{-1}) \), then (3) holds.

Now, we turn to the study of the NPLRT under \( H_1 \). Theorem 3.7, where we make use of a large deviation inequality for likelihood ratios studied in [11] in its proof.

**Theorem 3.7.** If \( \inf_{f \in \mathcal{F}_k} h(f, f_0) > 0 \) and \( f_0 \) satisfies Conditions (A) and (B), then there exists some constant \( c > 0 \) such that

\[
\lim_{n \to \infty} P(S_{n,k} > c) = 1.
\]

**Corollary 3.8** (Consistency of the NPLRT for \( k \)-monotone Densities). Consider \( H_0 : f_0 \in \mathcal{F}_k \) and \( H_1 : f_0 \notin \mathcal{F}_k \). Let \( \nu = O(n^{1/3}(\log n)^{-1}) \). Under \( H_1 \), suppose that \( \inf_{f \in \mathcal{F}_k} h(f, f_0) > 0 \), \( f_0 \) satisfies Conditions (A), (B), and either (C)(i) or (C)(ii). Then, the NPLRT is consistent.

### 3.2 Completely Monotone Densities

For completely monotone densities, where global rates of convergence have not been developed, we obtain a rough rate of convergence that is sufficient for the application of the proposed test.

---

\(^3\)For simplicity, we state the results that cover most of the common practical cases of interest. A more general statement is given in the supplementary material, where \( f_0 \) satisfying Conditions (A) and (B) is a special case.
Denote $\mathcal{F}_\infty$ as the class of all completely monotone densities on $(0, \infty)$. By the Bernstein’s theorem \[45\],

$$
\mathcal{F}_\infty = \left\{ f : (0, \infty) \to (0, \infty) : f(t) = \int_0^\infty \lambda e^{-\lambda t} dM(\lambda), \right\}
$$

where $M$ is a distribution function.

Note that a completely monotone density can be unbounded at 0. For example, the gamma distribution with shape parameter in $(0, 1)$ is unbounded at 0 and completely monotone. A completely monotone density can also have a heavy tail. For example, the density $f(t) = \beta(1 + t)^{-\beta - 1}$ for $\beta > 0$ has tail that decays polynomially and is completely monotone.

Similar to the class of $k$-monotone densities with unbounded support, we know $\mathcal{F}_\infty$ is not totally bounded from \[42\]. Moreover, the NPMLE $\hat{f}_{n,\infty}$ for a completely monotone density always has an unbounded support, as $\hat{f}_{n,\infty}(t) = \int_0^\infty \lambda e^{-\lambda t} d\hat{M}_n(\lambda)$ for $t > 0$, where $\hat{M}_n$ is the NPMLE of the corresponding mixing distribution. Although completely monotone densities are often viewed as $k$-monotone with $k \in \infty$, the approach in Subsection 3.1 is not applicable and we will derive a rate of convergence of the log-likelihood ratio using a different method.

Here, we do not aim to derive a tight bound for the convergence rate of the log-likelihood ratio in the completely monotone case, as this is not the primary focus of this paper. Instead, we provide a loose bound in Lemma 3.9 which is sufficient for showing that $\sqrt{n} \nu S_{n,\infty} = o_p(1)$ for our NPLRT under the null. The slow rate is merely a consequence of our method of proof.

The key idea is that we can still obtain an upper bound of the NPMLE. Then, we view the NPMLE as a sum of two decreasing functions with supports of the form $[0, c_{2n}]$ and $(c_{2n}, \infty)$ for some increasing sequence $c_{2n}$. Similar to the 1-monotone case, we can obtain a finite bracketing entropy for the class of decreasing functions that are bounded from above and have the bounded support $[0, c_{2n}]$. The sequence $c_{2n}$ is chosen such that the tail of the NPMLE is bounded by $1/t^2$. Using the method in Lemma 7.10 in \[43\], we can then obtain a finite bracketing entropy for the class of the decreasing functions with unbounded support that are bounded by $1/t^2$.

**Lemma 3.9.** Suppose that $f_0 \in \mathcal{F}_\infty$ and satisfies Conditions (A) and (B). Then,

$$
S_{n,\infty} = O_p \left( n^{-\frac{2}{3}} \log n \right).
$$

**Corollary 3.10.** Under the conditions in Lemma 3.9. Suppose that $\nu = O(n^{1/3}(\log n)^{-1})$, then (3) holds.

Under $H_1$, Theorem 3.11 and Corollary 3.12 are similar to the corresponding results for $k$-monotone densities.

**Theorem 3.11.** If $\inf_{f \in \mathcal{F}_\infty} h(f, f_0) > 0$, $f_0$ satisfies Conditions (A) and (B), then there exists some constant $c > 0$ such that

$$
\lim_{n \to \infty} \mathbb{P}(S_{n,\infty} > c) = 1.
$$

**Corollary 3.12** (Consistency of the NPLRT for Complete Monotone Densities). Consider $H_0 : f_0 \in \mathcal{F}_\infty$ and $H_1 : f_0 \notin \mathcal{F}_\infty$. Let $\nu = O(n^{1/3}(\log n)^{-1})$. Under $H_1$, suppose that $\inf_{f \in \mathcal{F}_\infty} h(f, f_0) > 0$, $f_0$ satisfies Conditions (A), (B), and either (C)(i) or (C)(ii). Then, the NPLRT is consistent.
3.3 Log-concave Densities

In this subsection, let $\mathcal{F}_{lc}$ denote the class of log-concave densities on $\mathbb{R}$, and let $\hat{f}_{n,lc}$ be the NPMLE over $\mathcal{F}_{lc}$. Define

$$S_{n,lc} := -\frac{1}{n} \sum_{i=1}^{n} \log \frac{\hat{f}_{n,lc}(X_i)}{f_0(X_i)}.$$ 

Corollary 3.2 in [46] shows that $S_{n,lc} = O_p(n^{-4/5})$. Additionally, $R_n = O_p(\nu \log n)$, as stated in Corollary 2.5. Therefore, we can establish the asymptotic null distribution for the class of log-concave densities in the absence of any additional regularity conditions in Corollary 3.13, following Corollary 2.1.

**Corollary 3.13.** Suppose that $f_0$ is a log-concave density on $\mathbb{R}$ and $\nu = O(n^{1/3}(\log n)^{-1})$, then (A) holds.

In [46], a bracketing entropy bound is derived for the following subset of $\mathcal{F}_{lc}$:

$$\mathcal{F}_{lc}^M := \left\{ f \in \mathcal{F}_{lc} : \sup_{x \in \mathbb{R}} f(x) \leq M, \frac{1}{M} \leq \inf_{x \in [-1,1]} f(x) \right\},$$

where $0 < M < \infty$. Using this result, along with the deviation inequality in Theorem 1 in [41], we establish the following theorem.

**Theorem 3.14.** If $\inf_{f \in \mathcal{F}_{lc}} h(f, f_0) > 0$ and

$$\int_{\mathbb{R}} |x| f_0(x) dx < \infty,$$

then there exists some finite constant $c > 0$ such that

$$\lim_{n \to \infty} \mathbb{P}(S_{n,lc} > c) = 1.$$ 

**Corollary 3.15** (Consistency of NPLRT for Log-concave Densities). Consider $H_0 : f_0 \in \mathcal{F}_{lc}$ and $H_1 : f_0 \notin \mathcal{F}_{lc}$. Let $\nu = O(n^{1/3}(\log n)^{-1})$. Under $H_1$, suppose that $\inf_{f \in \mathcal{F}_{lc}} h(f, f_0) > 0$, $f_0$ satisfies Conditions (A), (B), and either (C)(i) or (C)(ii). Then, the NPLRT is consistent.

4 Bootstrap Calibration of the Test

While the asymptotic distribution under the null hypothesis is distribution-free, and thus can be used to determine the critical value of the test, the smaller order terms, particularly $\sqrt{n\nu} S_n$, may converge to 0 slowly. This can make the asymptotic approximation inaccurate in finite samples. One possible way to improve the approximation is to analyze the leading term in $S_n$ and debias accordingly. We will first explore this idea for 1-monotonicity which shows its limitations, and discuss a bootstrap method which can be broadly applied.

With additional regularity conditions, we obtain the following Theorem [4.1] for 1-monotonicity and $\nu = 1$. Its proof makes use of a generalization of Theorem 1.1 in [47] and Theorem 2.1 in [48]. The former establishes the asymptotic normality of a weighted $L_2$-error between $\hat{f}_n$ and $f_0$, while the latter establishes the asymptotic normality of a weighted $L_1$-norm between $\hat{F}_n$ and $\mathbb{F}_n$, where $\mathbb{F}_n$ is the empirical distribution function and $\hat{F}_n$ is the distribution function corresponding to the MLE, which is the least concave majorant of $\mathbb{F}_n$ in this case.
Theorem 4.1. Suppose that $f_0$ is a twice continuously differentiable and strictly decreasing density function with support contained in $[0, 1]$ and that it satisfies the following conditions:

(i) $0 < f_0(1) \leq f_0(0) < \infty$;

(ii) $0 < \inf_{t \in (0, 1)} |f_0'(x)|$.

Then,

$$
\sqrt{n}S_{n, 1} = -\left(\mu_{2, f_0}^2 + \kappa_{f_0}\right)n^{-1/6} + O_p(n^{-\frac{1}{3} + \delta}),
$$

for any $\delta > 0$, where

$$
\mu_{2, f_0} := \left[\mathbb{E}[V(0)]^2 \int_0^1 \{2|f_0'(t)|^2 f_0(t)^{-1}\}^{\frac{1}{2}} \, dt\right]^{1/2},
$$

$$
\kappa_{f_0} := \mathbb{E}[\zeta(0)] \int_0^1 \{2|f_0'(t)|^2 f_0(t)^{-1}\}^{\frac{1}{2}} \, dt,
$$

with $V(c) = \arg\max_{t \in \mathbb{R}} \{\mathbb{W}(t) - (t - c)^2\}$,

$\zeta(c) = [\text{CM}_R Z](c) - Z(c)$ and $Z(c) = \mathbb{W}(c) - c^2$, for $c \in \mathbb{R}$. Here, $\mathbb{W}$ is a standard two-sided Brownian motion with $\mathbb{W}(0) = 0$ and $\text{CM}_R Z$ denotes the least concave majorant of $Z$ on $\mathbb{R}$.

However, the result in Theorem 4.1 cannot be readily applied to improve the accuracy of the test, as $\mu_{2, f_0}$ and $\kappa_{f_0}$ are unknown, and their estimation is not straightforward as they depend on the derivative of $f_0$.

A more practical and generally applicable way to improve the test accuracy is to use a bootstrap procedure to determine appropriate critical values. Let $\{\hat{f}_n\}$ be a sequence of estimated densities, which may be $k$-monotone, completely monotone, or log-concave, where $\hat{f}_n$ depends on $X_1, \ldots, X_n$, and let $\hat{F}_n$ be the corresponding distribution function. Let $X_{n, 1}^*, \ldots, X_{n,n}^*$ be independent random variables simulated from the density $\hat{f}_n$, and denote $Z_{n, 1}^* < \ldots < Z_{n,n}^*$ as their order statistics. Let $\hat{f}_n^* \ast$ be the NPMLE based on $X_{n, 1}^*, \ldots, X_{n,n}^*$ under the relevant hypothesis class (either $k$-monotonicity, complete monotonicity, or log-concavity) and let $f_{n, \nu}^H \ast$ be the corresponding histogram-type estimator. Define

$$
T_n^* := -\frac{1}{n} \log \prod_{i=1}^n \frac{\hat{f}_n^*(X_{n,i}^*)}{f_{n, \nu}^H(X_{n,i}^*)}.
$$

Let $\Omega$ denote the sample space, and let $\mathbb{P}^*$ represent the conditional probability given the entire sequence $(X_1, X_2, \ldots)$. In this section, we also emphasize the probability measure in
the $o_p$ notation by using $o_{P^*_\omega}$ for $\omega \in \Omega$ when applicable. Let

$$S^*_n := -\frac{1}{n} \sum_{i=1}^{n} \log \frac{\hat{f}_n^*(X^*_ni)}{\hat{f}_n(X^*_ni)},$$

$$M^*_n := -\nu \frac{\nu}{n} \sum_{j=0}^{\nu-1} \log \frac{\tilde{F}_n(Z^*_{n,(j+1)\nu+1}) - \tilde{F}_n(Z^*_{n,j\nu+1})}{\nu/(n-1)},$$

$$R^*_n := -\frac{1}{n} \sum_{j=0}^{\nu-1} \sum_{l=1}^{\nu} \left[ \log \frac{\bar{Z}^*_{n,(j+1)\nu+1} - Z^*_{n,j\nu+1}}{\bar{F}_n(Z^*_{n,(j+1)\nu+1}) - \bar{F}_n(Z^*_{n,j\nu+1})} \right] \log \frac{\hat{f}_n(Z^*_n)}{\hat{f}_{H,\nu}(Z^*_n)}.$$

**Theorem 4.2.** Suppose that conditional on $X_1, X_2, \ldots$, every subsequence $\{n_k\}$ of $\{n\}$ has a further subsequence $\{n_{k_l}\}$ along which for almost all $\omega \in \Omega$,

(i) $\sqrt{n_k}\nu S^*_{n_{k_l}} = o_{P^*_\omega}(1)$; and

(ii) $\sqrt{n_k}R^*_{n_{k_l}} = o_{P^*_\omega}(1)$.

Then,

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P} \left( \sqrt{\nu^2 \psi_1(\nu)} \left( T^*_n - \log \nu + \psi(\nu) \right) \leq x \right) - \mathbb{P} \left( Z \leq x \right) \right| \overset{P}{\rightarrow} 0,$$

where $Z \sim N(0, 1)$.

In the supplementary materials, we give details on the conditions in Theorem 4.2 are satisfied for $k$-monotone, completely monotone, and log-concave densities. Specifically, for testing $k$-monotonicity and complete monotonicity, these conditions hold if the true underlying density satisfies Conditions (A) and (B); for testing log-concavity, the true density having a finite first absolute moment is a sufficient condition. As a result of Theorem 4.2, we have the following bootstrap procedure for finding the critical value of the test.

**Bootstrap Procedure.**

1. Simulate $X^{(b)}_{n1}, \ldots, X^{(b)}_{nn}$ as independent random variables from the NPMLE $\hat{f}_n$ of $X_1, \ldots, X_n$.

2. Compute the NPMLE $\hat{f}_n^{(b)}$ and the histogram-type estimator $\hat{f}^{H,(b)}_{n,\nu}$ from $X^{(b)}_{n1}, \ldots, X^{(b)}_{nn}$.

3. Set

$$T^{(b)}_n := -\frac{1}{n} \log \prod_{i=1}^{n} \frac{\hat{f}^{(b)}_{n,\nu}(X^{(b)}_{ni})}{\hat{f}^{H,(b)}_{n,\nu}(X^{(b)}_{ni})}. $$
4. Repeat Steps 1 to 3 \( B \) times and use the quantiles of \( T_n^{(1)}, \ldots, T_n^{(B)} \) as the critical values of the test.

In the simulation studies, we observe that this procedure controls the size of the test well and provides good power for the tests across various settings.

**Remark 4.3.** Recall that an important fact used in establishing the asymptotic null distribution is that \((F_0(Z_1), \ldots, F_0(Z_n)) \overset{d}{=} (U(1), \ldots, U(n))\). If we bootstrap from the distribution function \( \hat{F}_n \) corresponding to the NPMLE \( \hat{f}_n \), then we still have \((\hat{F}_n(Z_{n1}^*), \ldots, \hat{F}_n(Z_{nn}^*)) \overset{d}{=} (U(1), \ldots, U(n))\) because \( \hat{F}_n \) is continuous. However, if we bootstrap from the empirical distribution, this property is lost because the empirical distribution is not continuous. Therefore, to find the null distribution of the NPLRT, we cannot bootstrap from the empirical distribution.

**Remark 4.4.** Although it is known that for 1-monotone densities, bootstrapping from the empirical distribution or the NPMLE does not provide a consistent estimator of the distribution of \( n^{1/3} \{ \hat{f}_{n,1}(t_0) - f_0(t_0) \} \), where \( t_0 \in (0, \infty) \) is an interior point; see \[35\] and \[30\]; our result does not contradict the above literature because the nature of the statistics of interest is different. Our statistic is a global measure between the NPMLE and the true density, whereas the statistic considered in the above two papers concerns a local property of the NPMLE.

On a more technical level, bootstrapping from the NPMLE fails to capture the pointwise limiting distribution of the NPMLE at an interior point because of the lack of smoothness in the Grenander estimator, from which the bootstrap samples are generated, while the true underlying distribution function is assumed to have a differentiable density with a nonzero derivative at that interior point. In contrast, we only require the true underlying distribution function to be continuous, and we do not need to assume the true underlying density has a nonzero derivative at any point when deriving the null asymptotic distribution.

**Remark 4.5.** Regardless of whether \( f_0 \) belongs to the hypothesis class, the NPMLE of the density is always within the class, Therefore, bootstrap simulations from the NPMLE would always approximate the null distribution.

5 Simulation Studies

In this section, we conduct simulation studies to evaluate the finite-sample performance of the proposed tests and compare them with some other tests. For our proposed NPLRT, we have to determine the value of \( \nu \) in the histogram-type estimator. Following the standard way of choosing a tuning parameter in density estimation problems; see \[19\], we choose \( \nu \) by minimizing the cross-validation error, defined as

\[
CV(\nu) := \int f_{n,\nu}^H(x)dx - \frac{2}{n} \sum_{i=1}^{n} f_{n-1,\nu}^{H,-i}(X_i).
\]

Here \( f_{n-1,\nu}^{H,-i} \) denotes the histogram-type estimator without the \( i \)th observation. The critical values are determined using the bootstrap procedure as discussed in Section \[4\]. Specifically, after generating the samples from the NPMLE, we use the same value of \( \nu \) to compute the
bootstrapped test statistic. To estimate the sizes and empirical powers, we conduct 1,000 independent repetitions. All significance levels are set to be 0.05.

The Grenander estimator, the MLE for the 1-monotone case, can be computed by posing it as an isotonic regression problem. See, for instance, [50] for the pool adjacent violators algorithm for isotonic regression. Here, we make use of the R package fdrtool [51]. Using the nonparametric mixture representation, the computation of $k$-monotone and completely monotone MLE is done using the R package nspmix; see [52]. For the computation of the log-concave MLE, we use the R package logcondens; see [53].

5.1 Decreasing Densities

We first consider testing whether a density is decreasing. The distributions considered are given in Table 1. Under $H_0$, we consider decreasing densities with an unbounded support, with a bounded support, and densities that are not bounded from above. Specifically, Exp(1) denotes the exponential distribution with density $f(x) = e^{-x}I(x > 0)$, which has an unbounded support. Beta$(a, b)$ denotes the beta distribution with density $f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}x^{a-1}(1-x)^{b-1}I(0 < x < 1)$, which has a bounded support. In particular, Beta$(1, 4)$ has a decreasing density. “Unbounded” refers to the decreasing density $f(x) = \frac{1}{2x}I(0 < x < 1)$, which is not bounded from above. Under $H_1$, we consider different scenarios where a density deviates from a decreasing shape. Let $f_1(d, M)$ denote the density

$$f_1(x; d, M) \propto x^{-0.1}I(x < d) + MxI(d \leq x \leq 1)$$

for $d \in (0, 1)$ and $M > 0$. The density $f_1(x; d, M)$ is decreasing on $(0, d)$ and increasing on $(d, 1)$. Smaller $d$ and/or larger $M$ leads to more deviation from a decreasing density. The densities of Beta$(1.2, 1.5)$ and Beta$(1.5, 3)$ are both initially increasing and then decreasing. "Mixture" refers to the density $0.7e^{-2x} + 0.3I(x \in [1, 2])$, which is a mixture of truncated exponential and uniform distribution on $[1, 2]$, having two distinct decreasing regions. Lastly, $f_2(c)$, for $c > 0$, denotes the density

$$f_2(x; c) \propto xI(0 < x < c) + e^{-x}I(x \geq c),$$

where $f_2(x; c)$ is increasing on $(0, c)$ and decreasing on $(c, \infty)$. Larger values of $c$ result in greater deviation from a decreasing density. Figure 1 shows the plots of these densities.

Other natural test statistics for testing monotonicity are to based on some distance measures between the empirical distribution function $F_n$ and its least concave majorant $\hat{F}_n$. In particular, [32] studied the $L_k$-distance between $F_n$ and $\hat{F}_n$ and derived the asymptotic distribution of the corresponding test statistic. However, their establishment of the asymptotic distribution requires additional assumptions about the underlying $f$. Specifically, they assumed $f$ is twice continuous differentiable, bounded away from 0 and bounded from above, and that its negative derivative is also bounded away from 0. In addition, their method also requires the estimation of the derivative $f'$, for example, by using kernel estimators.

The simulation reported in [32] shows that using the critical value determined by their asymptotic distribution has inflated type I errors, making power comparison not meaningful. Because of these issues, we consider the $L_2$-test when the critical value is determined by a
Figure 1: The density functions considered under $H_1$. The density of $f_2(0.2)$ is not shown as it is similar to that of $f_2(0.05)$.

bootstrap procedure where we draw samples from the Grenander estimator. Specifically, the $L_2$ test statistics is

$$L_2 := \int (\hat{F}_n(t) - F_n(t))^2 dt.$$  

In [32], two other test statistics were compared:

$$L'_2 := \int (\hat{F}_n(t) - F_n(t))^2 dF_n(t),$$

$$L_\infty := \sup_t (\hat{F}_n(t) - F_n(t)).$$

They showed that the uniform distribution is least favorable among all decreasing densities on $[0, 1]$, ensuring that the type I error can be bounded by $\alpha$. In our comparison, the critical values of these two tests are also determined using the same bootstrap procedure as in the $L_2$-test.

The results are given in Table 1. We observe that our LRT with bootstrap-calibrated critical values, controls the sizes of the test reasonably well, with the other 3 tests tends to be over conservative using the same type of bootstrapping procedure. Under the alternative hypothesis, the proposed LRT has higher power than the rest of the three tests in most of the settings considered.

5.2 2-monotone

In this subsection, we consider testing whether a density is 2-monotone using the proposed NPLRT. Table 2 lists the distributions considered under $H_0$ and $H_1$. The definitions of “Unbounded”, $f_1$, “Mixture” and $f_2$ are the same as in Subsection 5.1. In Table 2, we observe
| Method     | $L_2$ | $L'_2$ | $L_\infty$ | Method     | $L_2$ | $L'_2$ | $L_\infty$ |
|------------|-------|--------|------------|------------|-------|--------|------------|
| $H_0$      |       |        |            | $H_1$      |       |        |            |
| Exp(1)     | 0.031 | 0.018  | 0.016      | $f_1(0.7,1)$ | 0.146 | 0.041  | 0.041      |
| Beta(1,4)  | 0.056 | 0.013  | 0.021      | $f_1(0.9,2)$ | 0.408 | 0.252  | 0.222      |
| Unbound    | 0.021 | 0.019  | 0.010      | Beta(1,4)   | 0.170 | 0.125  | 0.132      |
| Beta(1.2,1.5) | 0.220 | 0.084  | 0.069      | Beta(1.2,1.5) | 0.265 | 0.151  | 0.157      |
| Beta(1.5,3) | 0.500 | 0.233  | 0.222      | Beta(1.5,3) | 0.494 | 0.370  | 0.353      |
| Mixture    | 0.500 | 0.233  | 0.222      | Mixture     | 0.756 | 0.485  | 0.475      |
| $f_2(0.05)$ | 0.304 | 0.021  | 0.039      | $f_2(0.05)$ | 0.614 | 0.056  | 0.118      |
| $f_2(0.2)$ | 0.500 | 0.233  | 0.393      | $f_2(0.2)$ | 0.358 | 0.110  | 0.281      |

Table 1: Test for 1-monotonicity: Comparison of the proposed LRT with tests based on some distance measures between the empirical distribution function and its least concave majorant at sample sizes $n = 100, 250, 500, 1000$ under $H_0$ and $H_1$. 

$n = 100$  

| Method     | $L_2$ | $L'_2$ | $L_\infty$ | Method     | $L_2$ | $L'_2$ | $L_\infty$ |
|------------|-------|--------|------------|------------|-------|--------|------------|
| $H_0$      |       |        |            | $H_1$      |       |        |            |
| Exp(1)     | 0.036 | 0.005  | 0.011      | $f_1(0.7,1)$ | 0.295 | 0.062  | 0.051      |
| Beta(1,4)  | 0.052 | 0.000  | 0.015      | $f_1(0.9,2)$ | 0.935 | 0.790  | 0.763      |
| Unbound    | 0.035 | 0.011  | 0.011      | Beta(1,4)   | 0.500 | 0.208  | 0.172      |
| Beta(1.2,1.5) | 0.450 | 0.190  | 0.172      | Beta(1.2,1.5) | 0.698 | 0.737  | 0.701      |
| Beta(1.5,3) | 0.612 | 0.381  | 0.351      | Beta(1.5,3) | 0.885 | 0.971  | 0.974      |
| Mixture    | 0.733 | 0.210  | 0.393      | Mixture     | 0.816 | 0.680  | 0.640      |
| $f_2(0.05)$ | 0.991 | 1.000  | 1.000      | $f_2(0.05)$ | 0.999 | 0.999  | 0.878      |
| $f_2(0.2)$ | 1.000 | 1.000  | 1.000      | $f_2(0.2)$ | 1.000 | 1.000  | 1.000      |
that the empirical rejection proportions are close to 0.05 under $H_0$, and the power increases as the sample size grows. Compared to table 1, the power under the null hypothesis of 2-monotone is higher than that of 1-monotone when the density is neither 1 nor 2-monotone.

### 5.3 Completely Monotone Densities

Next, we consider testing whether a density is completely monotone using the proposed NPLRT. Table 3 lists the distributions considered under $H_0$ and $H_1$. Consider the mixture density $\sum_{k=1}^{K} p_k \lambda_k e^{-\lambda_k x}, x > 0$. “MixExp1” corresponds to the case when $K = 2$, $p = (0.3, 0.7), \lambda = (1, 5)$, while “MixExp2” corresponds to the case when $K = 3$, $p = (0.3, 0.3, 0.4), \lambda = (1, 5, 10)$. The results are also presented in Table 3, where we observe that the empirical rejection proportions are close to 0.05 under $H_0$, and the power increases as the sample size grows.

### 5.4 Log-concave Densities

Next, we consider testing log-concave densities. The distributions considered are shown in Table 4. Figure 2 shows the log-density plots of the alternatives considered for the log-
Figure 2: Log density of the distributions considered in the alternative for log-concave case.

concave case. The density of Beta(α, β) is $x^{α-1}(1-x)^{β-1}B(α,β)I(x \in (0,1))$, where $B(·, ·)$ is the beta function. LogNormal(µ, σ) denotes the distribution when the logarithm of the random variable is normally distributed with mean µ and standard deviation σ. Let $ϕ(·)$ denote the density of the standard normal distribution. MixNormal(µ) denotes the mixture of two normal distributions with density $\frac{1}{2}ϕ(x) + \frac{1}{2}ϕ(x-µ)$, which is log-concave only when µ ≤ 2.

We compare our test with the trace test proposed in [33] and the split LRT proposed in [34]. The specification of the split LRT requires an additional density estimator and we follow [34] to use a kernel density estimator (KDE). In particular, we follow [34] to use the kde1d function from the kde1d package in R, which allows user to restrict the support of the KDE. We consider three cases for specifying the support in kde1d: (i) the true support; (ii) without any restriction; and (iii) [min $X_i$, max $X_i$], an estimated support. The resulting tests are labelled Split1, Split2, Split3, respectively, in Tables 4. Note that for the mixtures of normal distributions, the support is the entire real line, so Split1 and Split2 are identical in that case.

The results are given in Tables 4. We found that the trace test can have inflated type I errors ranging from 0.2 to 0.3 for exponential and Laplace distributions, when it should be close to 0.05. This may explain why it has high powers under the alternatives. On the other hand, while the split LRT controls the type I error level in finite samples through sample splitting, it tends to be overly conservative. Additionally, the performance of the split LRT with KDE relies on a good specification of the support. When the support is not provided in kde1d, the powers drop significantly, possibly because of the well-known boundary issue of the KDE. Our proposed test is more powerful than the split LRT in most of the settings considered and does not suffer from boundary problems as in KDE, as its uses a histogram-type estimator for the alternatives. It also controls the type-I error around 0.05.
| Method         | LRT | Trace | Split1 | Split2 | Split3 | LRT | Trace | Split1 | Split2 | Split3 |
|---------------|-----|-------|-------|-------|-------|-----|-------|-------|-------|-------|
| $n = 100$     |     |       |       |       |       |     |       |       |       |       |
| $H_0$ Exp(1)  | 0.071 | 0.223 | 0.000 | 0.000 | 0.000 | 0.065 | 0.155 | 0.000 | 0.000 | 0.000 |
| Laplace(1)    | 0.053 | 0.361 | 0.000 | 0.000 | 0.000 | 0.045 | 0.313 | 0.000 | 0.000 | 0.000 |
| Normal(0, 1)  | 0.023 | 0.015 | 0.000 | 0.000 | 0.000 | 0.018 | 0.011 | 0.000 | 0.000 | 0.000 |
| MixNormal(2)  | 0.023 | 0.010 | 0.000 | 0.000 | 0.000 | 0.023 | 0.013 | 0.000 | 0.000 | 0.000 |
| $H_1$ Beta(2, 0.5) | 0.869 | 0.870 | 0.678 | 0.000 | 0.514 | 0.996 | 0.992 | 0.999 | 0.000 | 0.996 |
| Beta(0.5, 10) | 0.945 | 0.973 | 0.784 | 0.000 | 0.867 | 0.996 | 0.999 | 0.999 | 0.000 | 0.999 |
| Beta(0.6, 10) | 0.719 | 0.791 | 0.277 | 0.000 | 0.365 | 0.959 | 0.971 | 0.862 | 0.000 | 0.938 |
| Beta(0.7, 10) | 0.389 | 0.551 | 0.036 | 0.000 | 0.043 | 0.631 | 0.712 | 0.279 | 0.000 | 0.401 |
| LogNormal(0, 1) | 0.212 | 0.865 | 0.001 | 0.000 | 0.000 | 0.371 | 0.984 | 0.000 | 0.000 | 0.025 |
| LogNormal(0, 1.1) | 0.344 | 0.951 | 0.029 | 0.002 | 0.010 | 0.591 | 0.997 | 0.403 | 0.002 | 0.223 |
| LogNormal(0, 1.2) | 0.485 | 0.981 | 0.125 | 0.002 | 0.072 | 0.792 | 1.000 | 0.754 | 0.007 | 0.607 |
| MixNormal(3.5) | 0.211 | 0.406 | 0.000 | 0.000 | 0.000 | 0.433 | 0.864 | 0.001 | 0.001 | 0.000 |
| MixNormal(4)  | 0.453 | 0.723 | 0.000 | 0.000 | 0.000 | 0.863 | 0.995 | 0.085 | 0.085 | 0.000 |
| MixNormal(4.5) | 0.756 | 0.905 | 0.002 | 0.002 | 0.000 | 0.991 | 1.000 | 0.688 | 0.688 | 0.055 |
| $n = 500$     |     |       |       |       |       |     |       |       |       |       |
| $H_0$ Exp(1)  | 0.054 | 0.129 | 0.000 | 0.000 | 0.000 | 0.036 | 0.115 | 0.000 | 0.000 | 0.000 |
| Laplace(1)    | 0.029 | 0.264 | 0.000 | 0.000 | 0.000 | 0.025 | 0.241 | 0.000 | 0.000 | 0.000 |
| Normal(0, 1)  | 0.018 | 0.011 | 0.000 | 0.000 | 0.000 | 0.007 | 0.005 | 0.000 | 0.000 | 0.000 |
| MixNormal(2)  | 0.014 | 0.015 | 0.000 | 0.000 | 0.000 | 0.009 | 0.010 | 0.000 | 0.000 | 0.000 |
| $H_1$ Beta(2, 0.5) | 1.000 | 1.000 | 1.000 | 0.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| Beta(0.5, 10) | 1.000 | 1.000 | 1.000 | 0.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| Beta(0.6, 10) | 0.997 | 0.998 | 0.996 | 0.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| Beta(0.7, 10) | 0.845 | 0.915 | 0.744 | 0.000 | 0.877 | 0.985 | 0.995 | 0.992 | 0.000 | 0.999 |
| LogNormal(0, 1) | 0.579 | 0.999 | 0.524 | 0.010 | 0.270 | 0.829 | 1.000 | 0.968 | 0.014 | 0.845 |
| LogNormal(0, 1.1) | 0.794 | 1.000 | 0.899 | 0.011 | 0.766 | 0.973 | 1.000 | 0.999 | 0.047 | 0.997 |
| LogNormal(0, 1.2) | 0.943 | 1.000 | 0.997 | 0.019 | 0.986 | 0.998 | 1.000 | 1.000 | 0.054 | 1.000 |
| MixNormal(3.5) | 0.644 | 0.997 | 0.059 | 0.059 | 0.000 | 0.851 | 1.000 | 0.713 | 0.713 | 0.100 |
| MixNormal(4)  | 0.984 | 1.000 | 0.860 | 0.860 | 0.141 | 0.999 | 1.000 | 1.000 | 1.000 | 0.824 |
| MixNormal(4.5) | 1.000 | 1.000 | 1.000 | 1.000 | 0.772 | 1.000 | 1.000 | 1.000 | 1.000 | 0.847 |

Table 4: Comparison of the proposed NPLRT testing for log-concavity with the trace test and split LRT at sample sizes $n = 100, 250, 500, 1000$ under both $H_0$ and $H_1$. 


6 Real Data Examples

6.1 Coal-mining accidents

[54] considered the time intervals between explosions in British mines that resulted in the loss of ten lives or more from 1875 to 1951. The dataset, comprising 109 observations, is also presented in the same paper. A histogram of the data suggests the density should be decreasing; see Figure 3. In particular, [54] fitted an exponential distribution to the data. More recently, [55] proposed a method called the $\alpha$-power transformation to introduce an extra parameter to a family of parametric distributions and applied their proposed method to this coal-mining accidents data.

We tested whether the underlying density is decreasing by applying our proposed LRT along with $L_2, L'_2, L_\infty$ tests for comparisons. The resulting $p$-values were 0.16, 0.64, 0.61, and 0.70, respectively. Since these $p$-values are not small, a model with decreasing density is considered plausible. Thus, if a nonparametric method without any tuning parameter is desired, the Grenander estimator may be used. If one wishes to obtain a smoother nonparametric estimate and/or needs to account for the inconsistency of the Grenander estimator at 0, a smoothed Grenander estimator can be utilized; see [14].

6.2 Hospital length of stay

Understanding and modeling hospital length of stay, which refers to the number of days that a patient will remain in hospital, is important as it is a useful indicator of resource utilization and cost-efficiency. [56] empirically found that the length of stay of patients in departments of geriatric medicine fits extremely well with a mixture of two exponential distributions and proposed a compartmental model to explain this observation. This implies that the density is completely monotone. More generally, the length of stay depends on patient diagnoses and characteristics [57, 58].

Here, we consider the hospital length of stay data from Microsoft Machine Learning Server\footnote{https://github.com/Microsoft/ML-Server}. We consider the two subgroups of patients that were flagged for renal disease and
substance dependence during encounter. Since there are a large number of tied data, we break the ties by assuming the length of stay is uniformly distributed within the day so that a standard uniform random variable is added to each data. Histograms for these two subgroups are shown in Figures 4 and 5, respectively.

From Figure 4, the density of the LoS for the renal group is unlikely to be decreasing. The $p$-values of the NPLRT test and the other 3 tests for decreasing density, as discussed in Section 5, are all 0. Based on the figure, one might suspect a log-concave density may fit. However, the $p$-value of our NPLRT test for log-concavity is 0.026, suggesting a log-concave density may not fit well.

For the substance dependence group, the $p$-value for testing whether the density is decreasing is 0.170, and the $p$-values of the other 3 tests are also greater than 0.05. Thus, we do not reject the null hypothesis that it is decreasing. From Figure 5 the density does not appear to be convex, and our NPLRT confirms this ($p$-value is 0).

### 6.3 Reliability

[22] considered modeling the reliability of a certain device. The data, available in the R package logcondens, exhibited a skewed and non-Gaussian distribution. A kernel density
estimator with a small bandwidth revealed a multimodal distribution curve, while a large bandwidth tended to overestimate the variance and place excessive emphasis on the tails. [22] employed a slightly smoothed version of the log-concave MLE, which is also log-concave. We applied our LRT to this data to test for log-concavity. The resulting $p$-value is 0.995, indicating that a log-concave density provides a good fit to the data, consistent with the findings in [22].

7 Discussion

In this paper, we studied a nonparametric likelihood ratio test for univariate shape-constrained densities, focusing on the classes of $k$-monotonicity, complete monotonicity, and log-concavity. A universal asymptotic null distribution is derived, and a bootstrap simulation procedure for determining the critical values of the null distribution is shown to be valid. Consistency of the test is also established. Our empirical findings suggest that the proposed test is competitive when compared to other methods. One of the key advantages of our method is that it does not require strong assumptions about the underlying density in order to establish the asymptotic null distribution, the consistency of test, and the validity of a bootstrap procedure. In contrast, $L_p$-distance-based tests often require additional assumptions on the underlying density in order to establish, for example, the asymptotic null distribution. In the following, we discuss some additional remarks and related problems.

First, the NPLRT proposed in this article is not restricted to testing the classes of $k$-monotone, completely monotone, and log-concave densities. It is also valid for testing parametric classes of densities where we usually have $\frac{1}{n} \sum_{i=1}^{n} \log \frac{\hat{f}_n(X_i)}{f_0(X_i)} = O_p(n^{-1})$. It can also be applied to other nonparametric class of densities that may not be shape-constrained, for instance, the class of $\alpha$-Holder densities:

$$\mathcal{P}_\alpha := \{ f : [0, 1] \to \mathbb{R}, |f(x) - f(y)| \leq M|x - y|^{\alpha}, x, y \in [0, 1] \},$$

where $M > 0$ and $\alpha > 0$. If $f_0$ is bounded away from 0 and $\alpha > 1/2$, one can obtain the rate $h(\hat{f}_n, f_0) = O_p(n^{-\alpha/(2\alpha+1)})$ and $\frac{1}{n} \sum_{i=1}^{n} \log \frac{\hat{f}_n(X_i)}{f_0(X_i)} = O_p(n^{-2\alpha/(2\alpha+1)})$; see Example 7.4.6 in [43]. Thus, $\sqrt{n}S_n = o_p(1)$ under the null hypothesis when $\nu$ does not grow too fast, and the asymptotic null distribution would be valid. On the other hand, when $\alpha < 1/2$, [59] showed that the rate of convergence of MLE in Hellinger distance is not better than $(n(\log n))^{-\alpha/2}$. In this case, Theorem 2.1 does not apply because $S_n$ becomes the dominating term under $H_0$.

While our proposed likelihood ratio test makes use of a histogram-type estimator because the MLE for the class of all densities do not exist, such a modification is not necessary when a smaller class of alternatives is used in which the MLE exists. For instance, suppose we want to test the hypothesis that $H_0 : f_0$ is 2-monotone versus $H_1 : f_0$ is 1-monotone but not 2-monotone. Denote $\hat{f}_{n,k}$ as the $k$-monotone MLE. Then, the log-likelihood ratio

$$\Lambda_n := -\frac{1}{n} \log \frac{\prod_{i=1}^{n} \hat{f}_{n,2}(X_i)}{\prod_{i=1}^{n} \hat{f}_{n,1}(X_i)}$$
is well-defined. We can expand $\Lambda_n$ as

$$
\Lambda_n = \frac{1}{n} \sum_{i=1}^{n} \log \frac{\hat{f}_{n,1}(X_i)}{f_0(X_i)} - \frac{1}{n} \sum_{i=1}^{n} \log \frac{\hat{f}_{n,2}(X_i)}{f_0(X_i)}.
$$

In Theorem 3.4 we already know $\frac{1}{n} \sum_{i=1}^{n} \log \frac{\hat{f}_{n,1}(X_i)}{f_0(X_i)} = O_P(n^{-2/3})$ and $\frac{1}{n} \sum_{i=1}^{n} \log \frac{\hat{f}_{n,2}(X_i)}{f_0(X_i)} = O_P(n^{-4/5})$ under some mild regularity conditions. Therefore, to determine the asymptotic distribution of $\Lambda_n$, it suffices to study that of $\frac{1}{n} \sum_{i=1}^{n} \log \frac{\hat{f}_{n,1}(X_i)}{f_0(X_i)}$. In Theorem 4.1 we have shown that

$$
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \log \frac{\hat{f}_{n,1}(X_i)}{f_0(X_i)} = (\mu_{2,f_0} + \kappa_{f_0})n^{-1/6} + O_P(n^{-1/3+\delta}).
$$

A more detailed analysis is needed to establish its asymptotic distribution and to investigate whether bootstrap procedure is valid.

Some multivariate extensions of spacings has been considered in the literature, see for example [60, 61, 62]. Goodness-of-fit test for parametric multivariate distribution using multivariate spacings is an under-explored area, and extensions to multivariate shape-constraint distributions remain to be studied in the future.

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8 Proofs for Section 2

8.1 Proofs for Sections 2.1 and 2.2

Proof of (2). Using the definition of $f_{n,\nu}^H$,

\[ -\frac{1}{n} \sum_{i=1}^{n} \log \frac{f_0(X_i)}{f_{n,\nu}^H(X_i)} \]

\[ = \frac{1}{n} \sum_{j=0}^{\frac{n}{\nu}-1} \left( \sum_{l=1}^{\nu} \log \frac{f_0(Z_{j\nu+l+1})}{f_{n,\nu}^H(Z_{j\nu+l+1})} \right) - \frac{1}{n} \log \frac{f_0(Z_1)}{f_{n,\nu}^H(Z_1)} \]

\[ = -\frac{1}{n} \sum_{j=0}^{\frac{n}{\nu}-1} \left( \sum_{l=1}^{\nu} \log \frac{f_0(Z_{j\nu+l+1})(Z_{(j+1)\nu+1} - Z_{j\nu+1})}{\nu/(n-1)} \right) - \frac{1}{n} \log \frac{f_0(Z_1)}{f_{n,\nu}^H(Z_1)} \]

\[ = -\frac{1}{n} \sum_{j=0}^{\frac{n}{\nu}-1} \left( \sum_{l=1}^{\nu} \log \frac{f_0(Z_{j\nu+l+1})(Z_{(j+1)\nu+1} - Z_{j\nu+1})}{f_0(Z_{(j+1)\nu+1}) - f_0(Z_{j\nu+1})} + \nu \log \frac{f_0(Z_{(j+1)\nu+1}) - f_0(Z_{j\nu+1})}{\nu/(n-1)} \right) \]

\[ - \frac{1}{n} \log \frac{f_0(Z_1)}{f_{n,\nu}^H(Z_1)} \]

\[ = M_n + R_n, \]

where the last equality follows from the definitions of $M_n$ and $R_n$. \qed

Before we prove Theorems 2.1, 2.2, and 2.3, we first define some additional notations and establish several lemmas.

Let $E_1, \ldots, E_{n+1}$ be i.i.d. standard exponential random variables. Denote $\tilde{E}_j := \sum_{l=j\nu+2}^{(j+1)\nu+1} E_l$ for $j = 0, \ldots, \frac{n}{\nu} - 1$. Then, $\tilde{E}_j$’s are i.i.d. random variables following the Gamma distribution with shape parameter $\nu$ and scale parameter 1. Define

\[ Y_n := \frac{1}{n} \sum_{j=0}^{\frac{n}{\nu}-1} (\tilde{E}_j - \nu) + \frac{1}{n-1} (E_1 + E_{n+1}). \]

The following Lemma 8.1 decomposes $M_n$ into a deterministic part, a term involving independent and identically distributed random variables ($M_{1n}$ below), and a remainder term, which is shown to be asymptotically negligible in Lemma 8.2. With the help of Lemma 8.3, the asymptotic distribution of $M_{1n}$ is established in Lemma 8.4.

Lemma 8.1. We have

\[ M_n - \frac{n-1}{n} \log \nu \overset{d}{=} M_{1n} + M_{2n}, \]

where

\[ M_{1n} := \frac{1}{n} \sum_{j=0}^{\frac{n}{\nu}-1} \left( \tilde{E}_j - \nu - \nu \log \tilde{E}_j \right), \]

\[ M_{2n} := \frac{1}{n-1} (E_1 + E_{n+1}) - \frac{Y_n}{n} - (1 - n^{-1}) \frac{Y_n Y_n^*}{1 + Y_n^*}, \]

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and \( Y_n^* \) lies between 0 and \( Y_n \).

**Proof of Lemma 8.1** Note that \((F_0(Z_1), \ldots, F_0(Z_n)) \) and \((U_1, \ldots, U(n)) \) have the same distribution, where \( U(i)'s \) are the order statistics from a random sample from Uniform(0, 1) with sample size \( n \). Also,

\[
(U(1), U(2) - U(1), \ldots, U(n) - U(n-1)) \overset{d}{=} \left( \frac{E_1}{\sum_{h=1}^{n+1} E_h}, \ldots, \frac{E_n}{\sum_{h=1}^{n+1} E_h} \right),
\]

where \( E_h \)'s are independent random variables each following the standard exponential distribution with mean 1; see, for example, Theorem 2.2 in [37]. Therefore,

\[
(F_0(Z_{\nu+1}) - F_0(Z_1), F_0(Z_{\nu+1}) - F_0(Z_{\nu+1}), \ldots, F_0(Z_n) - F_0(Z_{n-\nu})) \overset{d}{=} (U(\nu+1) - U(1), U(\nu+1) - U(\nu+1), \ldots, U(n) - U(n-\nu))
\]

\[
= \left( \frac{\sum_{j=\nu+1}^{\nu+1} E_j}{\sum_{h=1}^{\nu+1} E_h}, \frac{\sum_{j=\nu+2}^{\nu+1} E_j}{\sum_{h=1}^{\nu+2} E_h}, \ldots, \frac{\sum_{j=n-\nu+1}^{n} E_j}{\sum_{h=1}^{n-\nu+1} E_h} \right).
\]

Then, by the definition of \( \tilde{E}_j \),

\[
M_n = -\frac{\nu}{n} \sum_{j=0, \ldots, \frac{n-1}{\nu}-1} \log \left( \frac{\sum_{j=\nu+1}^{\nu+1} E_j}{\sum_{h=1}^{\nu+1} E_h} \right)
\]

\[
= -\frac{\nu}{n} \sum_{j=0, \ldots, \frac{n-1}{\nu}-1} \log \left( \sum_{j=\nu+1}^{\nu+1} E_j \right) + \frac{n-1}{n} \log \left( \frac{n-1}{\nu} \sum_{h=1}^{\nu} E_h \right)
\]

\[
= -\frac{\nu}{n} \sum_{j=0, \ldots, \frac{n-1}{\nu}-1} \log \tilde{E}_j + \frac{n-1}{n} \log \left( \frac{\nu}{n-1} \sum_{j=0, \ldots, \frac{n-1}{\nu}-1} \tilde{E}_j \right) + \frac{\nu}{n-1} \left( E_1 + E_{n+1} \right)
\]

As \( \mathbb{E}(\tilde{E}_j) = \nu \), we write

\[
M_n = -\frac{\nu}{n} \sum_{j=0, \ldots, \frac{n-1}{\nu}-1} \log \tilde{E}_j + \frac{n-1}{n} \log \left( \nu + \frac{\nu}{n-1} \sum_{j=0, \ldots, \frac{n-1}{\nu}-1} \left( \tilde{E}_j - \nu \right) + \frac{\nu}{n-1} \left( E_1 + E_{n+1} \right) \right)
\]

\[
= -\frac{\nu}{n} \sum_{j=0, \ldots, \frac{n-1}{\nu}-1} \log \tilde{E}_j + \frac{n-1}{n} \log \nu + \frac{n-1}{n} \log (1 + Y_n),
\]

where the last equality follows from the definition of \( Y_n^* \). By Taylor’s theorem,

\[
\log (1 + Y_n) = \frac{Y_n}{1 + Y_n} = Y_n - \frac{Y_n Y_n^*}{1 + Y_n^*},
\]

where \( Y_n^* \) lies between 0 and \( Y_n \). Thus,

\[
M_n = -\frac{\nu}{n} \sum_{j=0, \ldots, \frac{n-1}{\nu}-1} \log \tilde{E}_j + \frac{n-1}{n} \log \nu + Y_n - \frac{Y_n}{n} - (1 - n^{-1}) \frac{Y_n Y_n^*}{1 + Y_n^*}.
\]

The claim in the lemma follows from the definitions of \( Y_n, M_{1n} \) and \( M_{2n} \). \( \square \)
Lemma 8.2. Suppose that $\nu = o(n)$. Then, $\sqrt{n}Y_n \xrightarrow{d} N(0, 1)$ and $M_{2n} = O_p(n^{-1})$.

Proof of Lemma 8.2. First, note that $E_1$ and $E_{n+1}$ are $O_p(1)$. Thus,

$$Y_n = \frac{1}{n} \sum_{j=0, \ldots, \frac{n-1}{\nu}} (\tilde{E}_j - \nu) + O_p(n^{-1}).$$

Note that

$$\sum_{j=0, \ldots, \frac{n-1}{\nu}} \text{Var}(\tilde{E}_j) = \frac{n-1}{\nu} \cdot \nu = n - 1.$$

By the Lindeberg’s central limit theorem, if for all $\varepsilon > 0$, we have

$$\lim_{n \to \infty} \frac{1}{n-1} \sum_{j=0, \ldots, \frac{n-1}{\nu}} \mathbb{E}((\tilde{E}_j - \nu)^2 \mathbb{1}(|\tilde{E}_j - \nu| > \varepsilon \sqrt{n-1})) = 0,$$

then

$$\frac{1}{\sqrt{n-1}} \sum_{j=0, \ldots, \frac{n-1}{\nu}} (\tilde{E}_j - \nu) \xrightarrow{d} N(0, 1)$$

so that $\sqrt{n}Y_n \xrightarrow{d} N(0, 1)$ by Slutsky’s theorem. Now, we verify the Lindeberg’s condition \[(10)\]. Fix $\varepsilon > 0$. Since $\tilde{E}_j$’s have the same distribution,

$$\frac{1}{n-1} \sum_{j=0, \ldots, \frac{n-1}{\nu}} \mathbb{E}((\tilde{E}_j - \nu)^2 \mathbb{1}(|\tilde{E}_j - \nu| > \varepsilon \sqrt{n-1})) = \frac{1}{\nu} \mathbb{E}((\tilde{E}_1 - \nu)^2 \mathbb{1}(|\tilde{E}_1 - \nu| > \varepsilon \sqrt{n-1})).$$

Note that

$$\mathbb{E} \left( \frac{\tilde{E}_1 - \nu}{\sqrt{n-1}} \right)^2 = \frac{\nu}{n-1} \to 0$$

as $n \to \infty$ since $\nu = o(n)$. Thus, $|\tilde{E}_1 - \nu|/\sqrt{n-1} \xrightarrow{p} 0$. For any $\varepsilon' > 0$,

$$\mathbb{P}((\tilde{E}_j - \nu)^2 \mathbb{1}(|\tilde{E}_j - \nu| > \varepsilon \sqrt{n-1})) > \varepsilon') \leq \mathbb{P}(|\tilde{E}_1 - \nu|/\sqrt{n-1} > \varepsilon) \to 0,$$

as $n \to \infty$. Thus, $(\tilde{E}_j - \nu)^2 \mathbb{1}(|\tilde{E}_j - \nu| > \varepsilon \sqrt{n-1}) \xrightarrow{p} 0$. As

$$\frac{1}{\nu} (\tilde{E}_j - \nu)^2 \mathbb{1}(|\tilde{E}_j - \nu| > \varepsilon \sqrt{n-1}) \leq \frac{(\tilde{E}_j - \nu)^2}{\nu}$$

and

$$\mathbb{E} \left( \frac{(\tilde{E}_j - \nu)^2}{\nu} \right) = 1,$$

we have, by the dominated convergence theorem,

$$\frac{1}{\nu} \mathbb{E}((\tilde{E}_1 - \nu)^2 \mathbb{1}(|\tilde{E}_1 - \nu| > \varepsilon \sqrt{n-1})) \to 0$$

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as \( n \to \infty \), verifying (10). For the claim that \( M_{2n} = O_p(n^{-1}) \), observe that \( |Y_n^*| \leq |Y_n| = O_p(n^{-1/2}) \). Therefore,

\[
M_{2n} = \frac{O_p(1)}{n} + \frac{O_p(n^{-1/2})}{n} + \frac{O_p(n^{-1/2})O_p(n^{-1/2})}{1 + O_p(n^{-1/2})} = O_p(n^{-1}).
\]

Recall that \( \Gamma(\cdot) \), \( \psi(\cdot) \) and \( \psi_1(\cdot) \) denote the gamma, diagamma, and trigamma functions, respectively. It is well known that the log-Gamma random variable \( \log \tilde{E}_j \) has an expected value of \( \psi(\nu) \) and variance of \( \psi_1(\nu) \). For completeness, these results are provided in Lemma 8.3 (a) and (b).

**Lemma 8.3.** Let \( Y \sim \text{Gamma}(\nu, 1) \). We have

(a) For \( m \in \mathbb{N} \), \( \mathbb{E}((\log(Y))^m) = \Gamma^{(m)}(\nu)/\Gamma(\nu) \), where \( \Gamma^{(m)} \) is the \( m \)th derivative of \( \Gamma \). In particular, \( \mathbb{E}(\log(Y)) = \psi(\nu) \);

(b) \( \text{Var}(\log(Y)) = \psi_1(\nu) \);

(c) \( \text{Cov}(Y, \log(Y)) = 1 \).

**Proof of Lemma 8.3.** (a) First, the density of \( \log Y \) is given by

\[
f_{\log Y}(y) = \frac{1}{\Gamma(\nu)} e^{\nu y - e^y}, \quad y \in \mathbb{R}.
\]

Thus, for any \( y \in \mathbb{R} \) and \( m \in \mathbb{N} \),

\[
\frac{d^m}{d\nu^m} e^{\nu y} e^{-e^y} = y^m e^{\nu y} e^{-e^y} = \Gamma(\nu) y^m f_{\log Y}(y).
\]

Integrating both sides of the above equation with respect to \( y \),

\[
\Gamma(\nu) \mathbb{E}((\log Y)^m) = \int_{-\infty}^{\infty} \frac{d^m}{d\nu^m} e^{\nu y} e^{-e^y} dy = \frac{d^m}{d\nu^m} \int_{-\infty}^{\infty} e^{\nu y} e^{-e^y} dy = \Gamma^{(m)}(\nu).
\]

In particular, \( \mathbb{E}(\log(Y)) = \Gamma'(\nu)/\Gamma(\nu) = \frac{d}{d\nu} \log \Gamma(\nu) = \psi(\nu) \).

(b) \( \text{Var}(\log(Y)) = \frac{\Gamma''(\nu)}{\Gamma(\nu)} - \psi^2(\nu) = \frac{d}{d\nu} \psi(\nu) = \psi_1(\nu) \).

(c) First,

\[
\mathbb{E}(Y \log(Y)) = \int_{-\infty}^{\infty} \frac{1}{\Gamma(\nu)} e^{\nu y} e^{\nu y - e^y} dy = \frac{1}{\Gamma(\nu)} \int_{-\infty}^{\infty} \frac{d}{d\nu} e^{\nu(\nu+1)} e^{-e^y} dy
\]

\[
= \frac{1}{\Gamma(\nu)} \frac{d}{d\nu} \int_{-\infty}^{\infty} e^{\nu(\nu+1)} e^{-e^y} dy = \frac{\Gamma'(\nu + 1)}{\Gamma(\nu)} \cdot \frac{\Gamma(\nu + 1)}{\Gamma(\nu)} = \frac{\Gamma'(\nu + 1)}{\Gamma(\nu + 1)}
\]

\[
= \nu \psi(\nu + 1) + 1/\nu,
\]

Using the recurrence relation \( \psi(\nu + 1) = \psi(\nu) + 1/\nu \), we obtain

\[
\text{Cov}(Y, \log(Y)) = \nu \psi(\nu + 1) - \nu \psi(\nu) = \nu(\psi(\nu + 1) - \psi(\nu)) = \nu \cdot \frac{1}{\nu} = 1.
\]

\[\square\]
Lemma 8.4. If $\nu = o(n)$, then
\[
\frac{1}{\sqrt{n^{-1}(\nu^2 \psi_1(\nu) - \nu)}} \sum_{j=0, \ldots, n^{-1}-1} (\tilde{E}_j - \nu - \nu \log \tilde{E}_j + \nu \psi(\nu)) \overset{d}{\to} N(0,1). \tag{11}
\]

Proof of Lemma 8.4. We shall apply the Lindeberg’s central limit theorem. By Lemma 8.3 (a),
\[
\mathbb{E}(\tilde{E}_j - \nu - \nu \log \tilde{E}_j + \nu \psi(\nu)) = 0.
\]
In addition, by Lemma 8.3 (b) and (c),
\[
\text{Var}(\tilde{E}_j - \nu - \nu \log \tilde{E}_j) = \text{Var}(\tilde{E}_j) + \nu^2 \text{Var}(\log \tilde{E}_j) - 2\nu \text{Cov}(\tilde{E}_j, \log \tilde{E}_j) = \nu + \nu^2 \psi_1(\nu) - 2\nu = \nu^2 \psi_1(\nu) - \nu.
\]
If the Lindeberg’s condition holds, we have (11). Thus, it remains to verify the Lindeberg’s condition that for all $\varepsilon > 0$,
\[
\lim_{n \to \infty} \frac{1}{n^{-1}(\nu^2 \psi_1(\nu) - \nu)} \sum_{j=0, \ldots, n^{-1}-1} \mathbb{E}\left\{ (\tilde{E}_j - \nu - \nu \log \tilde{E}_j + \nu \psi(\nu))^2 \cdot 1 \left( |\tilde{E}_j - \nu - \nu \log \tilde{E}_j + \nu \psi(\nu)| > \varepsilon \sqrt{n^{-1}(\nu^2 \psi_1(\nu) - \nu)} \right) \right\} = 0.
\]
Denote $F_\nu := \tilde{E}_1 - \nu - \nu \log \tilde{E}_1 + \nu \psi(\nu)$. The above condition is equivalent to
\[
\lim_{n \to \infty} \frac{1}{\nu^2 \psi_1(\nu) - \nu} \mathbb{E}\left( F_\nu^2 \cdot 1 \left( |F_\nu| > \varepsilon \sqrt{n^{-1}(\nu^2 \psi_1(\nu) - \nu)} \right) \right) = 0. \tag{12}
\]
As $\nu = o(n)$, we have
\[
\mathbb{E}\left( \frac{\nu}{n-1} \cdot \frac{F_\nu^2}{\nu^2 \psi_1(\nu) - \nu} \right) = \frac{\nu}{n-1} \to 0
\]
as $n \to \infty$. This implies that $\sqrt{\frac{\nu}{n-1}} |F_\nu| / \sqrt{\nu^2 \psi_1(\nu) - \nu} \overset{p}{\to} 0$ which in turn implies that
\[
F_\nu^2 \cdot 1 \left( |F_\nu| > \varepsilon \sqrt{n^{-1}(\nu^2 \psi_1(\nu) - \nu)} \right) \overset{p}{\to} 0.
\]
As
\[
F_\nu^2 \cdot 1 \left( |F_\nu| > \varepsilon \sqrt{n^{-1}(\nu^2 \psi_1(\nu) - \nu)} \right) \leq F_\nu^2,
\]
$\mathbb{E}(F_\nu^2) = \nu^2 \psi_1(\nu) - \nu$ and $\lim_{n \to \infty}(\nu^2 \psi_1(\nu) - \nu) = 1/2$ if $\nu \to \infty$, (12) holds by the dominated convergence theorem.

\[\square\]
Proof of Theorem 2.3. By Lemma 8.1,

\[ M_n - \frac{n-1}{n} \log \nu \overset{d}{=} M_{1n} + M_{2n}. \]  

(13)

As \( \nu = o(n) \), by Lemma 8.2

\[ M_{2n} = O_p(n^{-1}). \]  

(14)

Note that

\[ \sqrt{n\nu}(T_n - \log \nu + \psi(\nu)) \]

\[ = \sqrt{\frac{n\nu}{\nu^2\psi_1(\nu) - \nu}} (M_n - \log \nu + \psi(\nu)) + \sqrt{\frac{n\nu}{\nu^2\psi_1(\nu) - \nu}} (S_n + R_n) \]

\[ = \sqrt{\frac{n\nu}{\nu^2\psi_1(\nu) - \nu}} (M_n - \log \nu + \psi(\nu)) + o_p(1) + O(n^{2/3}(\log n)^{-1/2})O_p(n^{1/3}/n) \]

\[ \overset{d}{\to} N(0, 1). \]  

(15)

where the last convergence follows from Lemma 8.4 as \( \nu = o(n) \). In view of (13), (14), (15) and Slutsky’s theorem, the proof is completed.

Proof of Theorem 2.2. Fix \( c \geq 0, \varepsilon > 0 \) and \( \delta > 0 \). From the condition \( \nu = O(n^{1/3}/\log n) \) and \( R_n = O_p(\nu \log n / n) \), we have as in the proof of Theorem 2.1 that \( \sqrt{n\nu R_n} = o_p(1) \). For all sufficiently large \( n \), we have

\[ P_{H_1}(|\sqrt{n\nu R_n}| < \delta) > 1 - \frac{\varepsilon}{3}. \]  

(16)

From \( \lim_{n \to \infty} P(\sqrt{n\nu S_n} > L_n) = 1 \), we have for all sufficiently large \( n \) that

\[ P_{H_1}(\sqrt{n\nu} > L_n) > 1 - \frac{\varepsilon}{3}. \]  

(17)

By Theorem 2.3 there exists \( K > 0 \) such that for all sufficiently large \( n \),

\[ P_{H_1}\left(\left|\sqrt{\frac{n\nu}{\nu^2\psi_1(\nu) - \nu}} (M_n - \log \nu + \psi(\nu))\right| < K\right) > 1 - \frac{\varepsilon}{3}. \]  

(18)
Hence, in view of (16)-(18) and the fact that $L_n \uparrow \infty$, for all sufficiently large $n$,

\[
\mathbb{P}_{H_1} \left( \sqrt{\frac{n\nu}{\nu^2 \psi_1(\nu) - \nu}} (T_n - \log \nu + \psi(\nu)) > c \right) \\
\geq \mathbb{P}_{H_1} \left( \sqrt{\frac{n\nu}{\nu^2 \psi_1(\nu) - \nu}} (T_n - \log \nu + \psi(\nu)) > -K - \delta + L_n \right) > 1 - \varepsilon,
\]

and the claim follows as desired.

8.2 Proofs for Section 2.3

In this subsection, we will prove Theorem 2.4 and Corollary 2.5. We will first establish a few lemmas.

Lemma 8.5. Suppose one of Conditions (C) (i) or (ii) holds. Then, we have

\[
|R_n| \lesssim \frac{\nu}{n} \left( C + |\log f_0(Z_1)| + |\log f_0(Z_n)| \right) + O_p \left( \frac{\log n}{n} \right),
\]

for some constant $C$ that depends on $K_1, K_2, L$ if Condition (ii) holds and is a universal constant if Condition (i) holds.

Proof of Lemma 8.5. Write

\[
R_n = R_{1n} + R_{2n},
\]

where

\[
R_{1n} := -\frac{1}{n} \sum_{j=0,\ldots,\frac{n-1}{\nu}} \sum_{l=1}^{\nu} \log \frac{f_0(Z_{j\nu+l+1})(Z_{(j+1)\nu+1} - Z_{j\nu+1})}{f_0(Z_{(j+1)\nu+1}) - f_0(Z_{j\nu+1})},
\]

\[
R_{2n} := -\frac{1}{n} \log \frac{f_0(Z_1)}{f_{n,\nu}^H(Z_1)}.
\]

For $R_{1n}$, by the mean value theorem, there exists $Z_j^*$ lying between $Z_{j\nu+1}$ and $Z_{(j+1)\nu+1}$ such that $f_0(Z_{(j+1)\nu+1}) - f_0(Z_{j\nu+1}) = f_0(Z_j^*)(Z_{(j+1)\nu+1} - Z_{j\nu+1})$ and so

\[
R_{1n} = -\frac{1}{n} \sum_{j=0,\ldots,\frac{n-1}{\nu}} \sum_{l=1}^{\nu} \log \frac{f_0(Z_{j\nu+l+1})}{f_0(Z_j^*)}.
\]

If Condition (i) holds and $f_0$ is monotone increasing, then

\[
|nR_{1n}| \leq \left| \sum_{j=0,\ldots,\frac{n-1}{\nu}} \sum_{l=1}^{\nu} \log \frac{f_0(Z_{j\nu+1})}{f_0(Z_{(j+1)\nu+1})} \right| = \nu |\log f_0(Z_1) - \log f_0(Z_n)|
\]

\[
\leq \nu (|\log f_0(Z_1)| + |\log f_0(Z_n)|).
\]

(19)
We have the same inequality when \( f_0 \) is monotone decreasing. If Condition (ii) holds, let

\[
R_{1n}^{(m)} := \sum_{j=0,\ldots,n-1} \sum_{l=1}^{\nu} \nu_{ijkl} \log \frac{f_0(Z_{jl+1})}{f_0(Z_j^*)},
\]

where

\[
\begin{align*}
I_{n,jl}^{(1)} &:= 1(Z_{jl+1} < Z_j^* \in [K_1, K_2]), \\
I_{n,jl}^{(2)} &:= 1(Z_j^* < K_2 < Z_{jl+1}), \\
I_{n,jl}^{(3)} &:= 1(Z_{jl+1} < K_2 < Z_j^*), \\
I_{n,jl}^{(4)} &:= 1(Z_j^* < K_1 < Z_{jl+1}), \\
I_{n,jl}^{(5)} &:= 1(Z_{jl+1} < K_1 < Z_j^*), \\
I_{n,jl}^{(6)} &:= 1(Z_{jl+1}, Z_j^* > K_2), \\
I_{n,jl}^{(7)} &:= 1(Z_{jl+1}, Z_j^* < K_1).
\end{align*}
\]

Then,

\[
|nR_{1n}| \leq \sum_{m=1}^{7} |R_{1n}^{(m)}|.
\]

For \( R_{1n}^{(1)} \), by the Lipschitz continuity of \( \log f_0 \) on \([K_1, K_2]\),

\[
|R_{1n}^{(1)}| \leq \sum_{j=0,\ldots,n-1} \sum_{l=1}^{\nu} \nu_{ijkl} |Z_{jl+1} - Z_j^*|
\]

\[
\leq \sum_{j=0,\ldots,n-1} \sum_{l=1}^{\nu} \nu_{ijkl} (Z_{(j+1)l+1} - Z_j^*)
\]

\[
\leq \nu L(K_2 - K_1),
\]

where the second and third inequalities follow from the fact that \( Z_j^* \) is between \( Z_{jl+1} \) and \( Z_{(j+1)l+1} \).

For \( R_{1n}^{(2)} \), note that there will be at most one \( j, \) say, \( j_0 \), such that \( 1(Z_{j_l}^* < K_2 < Z_{jl+1}) \) for some \( l \). If \( Z_j^* > K_1 \), by telescoping and the triangle inequality,

\[
|R_{1n}^{(2)}| \leq \sum_{l=1}^{\nu} \left( |\log f_0(Z_{jl+1}) - \log f_0(K_2)| + |\log f_0(K_2) - \log f_0(Z_j^*)| \right)
\]

\[
\leq \nu (|\log f_0(Z_j)| + |\log f_0(K_2)| + L(K_2 - K_1)),
\]

where the last inequality follows from the monotonicity of \( f_0 \) beyond \( K_2 \) and the Lipschitz continuity of \( \log f_0 \) on \([K_1, K_2]\). If \( Z_j^* < K_1 \), we have

\[
|R_{1n}^{(2)}| \leq \sum_{l=1}^{\nu} \left( |\log f_0(Z_{jl+1}) - \log f_0(K_2)| + |\log f_0(K_2) - \log f_0(K_1)|
\]

\[
+ |\log f_0(K_1) - \log f_0(Z_j^*)| \right)
\]

\[
\leq \nu \left( |\log f_0(Z_j) + |\log f_0(K_2)| + L(K_2 - K_1) + |\log f_0(K_1)| + |\log f_0(Z_j)| \right).
\]
Thus, we always have

\[ |R_{1n}^{(2)}| \leq \nu(C + |\log f_0(Z_1)| + |\log f_0(Z_n)|), \]

where \( C := L(K_2 - K_1) + |\log f_0(K_2)| + |\log f_0(K_1)|. \) It is straightforward to see We have the same bound \( |R_{1n}^{(3)}|, |R_{1n}^{(4)}|, |R_{1n}^{(5)}| \). The derivation of these bounds are similar to that for \( |R_{1n}^{(2)}| \) and is therefore omitted.

Now, we consider \( R_{n}^{(6)} \). Let \( j^* \) be the first \( j \) such that \( Z_j^* > K_2 \). If \( f_0 \) is decreasing after \( K_2 \),

\[
|R_{1n}^{(6)}| \leq \sum_{l=1}^{\nu} (\log f_0(K_2) - \log f_0(Z_{(j^*+1)\nu+1})) + \sum_{j=j^*+1, \ldots, n-1}^{\nu} \sum_{l=1}^{\nu} (\log f_0(Z_{j\nu+1}) - \log f_0(Z_{(j+1)\nu+1})) \\
\leq \nu(\log f_0(K_2) - \log f_0(Z_n)).
\]

If \( f_0 \) is increasing after \( K_2 \), the bound becomes \( \nu(\log f_0(Z_n) - \log f_0(K_2)) \). In general, we have

\[
|R_{1n}^{(6)}| \leq \nu (|\log f_0(K_2)| + |\log f_0(Z_n)|).
\]

Similarly, we also have

\[
|R_{1n}^{(7)}| \leq \nu (|\log f_0(K_1)| + |\log f_0(Z_1)|).
\]

Combining the bounds for \( |R_{1n}^{(1)}|, \ldots, |R_{1n}^{(7)}| \), we have

\[
|nR_{1n}| \leq \nu(C_1 + 5|\log f_0(Z_1)| + 5|\log f_0(Z_n)|),
\]

where \( C_1 := 5L(K_2 - K_1) + 5|\log f_0(K_1)| + 5|\log f_0(K_2)|. \) In view of \((55)\) and \((20)\), we have

\[
|R_{1n}| \lesssim \frac{\nu}{n}(C + |\log f_0(Z_1)| + |\log f_0(Z_n)|).
\]

For \( R_{2n} \). Note that

\[ R_{2n} = -\frac{1}{n} \log \frac{f_0(Z_1)(Z_{\nu+1} - Z_1)}{F_0(Z_{\nu+1}) - F_0(Z_1)} - \frac{1}{n} \log \frac{F_0(Z_{\nu+1}) - F_0(Z_1)}{\nu/(n - 1)}. \]

By the mean value theorem, there exists \( Z_1^* \) lying between \( Z_{\nu+1} \) and \( Z_1 \) such that \( F_0(Z_{\nu+1}) - F_0(Z_1) = f_0(Z_1^*)(Z_{\nu+1} - Z_1) \). Thus, using \( (F_0(Z_1), F_0(Z_{\nu+1})) \overset{d}{=} (U_1, U_{\nu+1}) \) as in the proof of Lemma 8.1 we have

\[
R_{2n} \overset{d}{=} -\frac{1}{n} \log \frac{f_0(Z_1)}{f_0(Z_1^*)} - \frac{1}{n} \log (U_{\nu+1} - U_1) + \frac{1}{n} \log \frac{n-1}{\nu}.
\]

Now, if Condition (i) holds, then

\[
\left| \log \frac{f_0(Z_1)}{f_0(Z_1^*)} \right| \leq \left| \log \frac{f_0(Z_1)}{f_0(Z_n)} \right| \leq |\log f_0(Z_1)| + |\log f_0(Z_n)|.
\]
If Condition (ii) holds, we have

\[
\begin{align*}
|\log f_0(Z_1^*)| \\
= & \left| (\log f_0(Z_1^*))1(Z_1^* < K_1) + (\log f_0(Z_1^*))1(K_1 < Z_1^* < K_2) + (\log f_0(Z_1^*))1(Z_1^* > K_2) \right| \\
\leq & \left( |\log f_0(Z_1)| + |\log f_0(K_1)| \right) + (L(K_2 - K_1) + |\log f_0(K_1)|) \\
& + (L(K_2 - K_1) + |\log f_0(Z_n)|) \\
= & C_2 + |\log f_0(Z_1)| + |\log f_0(Z_n)|, \quad (24)
\end{align*}
\]

where

\[ C_2 := 2|\log f_0(K_1)| + L(K_2 - K_1) + |\log f_0(K_2)|. \]

Let \( E_h \)'s be i.i.d. standard exponential random variables the proof of Lemma 8.1, then

\[
\begin{align*}
|\log(U_{\nu+1}) - U(1))| & \leq |\log(U(2) - U(1))| \leq |\log E_2| + \left| \log \left( \sum_{h=1}^{n+1} E_h \right) \right| \\
& \leq |\log E_2| + \left| \log \left( \frac{1}{n+1} \sum_{h=1}^{n+1} E_h \right) \right| + \log(n+1) = O_p(\log n). \quad (25)
\end{align*}
\]

In view of (22)-(25), we have

\[
|R_{2n}| \leq \frac{2}{n} |\log f_0(Z_1)| + \frac{1}{n} |\log f_0(Z_n)| + O_p \left( \frac{\log n}{n} \right). \quad (26)
\]

The claim in the lemma follows in view of (21) and (26). \( \Box \)

The following lemma provides sufficient conditions for deriving the orders of \( \log f_0(Z_1) \) and \( \log f_0(Z_n) \).

**Lemma 8.6.** Let \( X_1, X_2, \ldots \) be a sequence of random variables that are identically distributed with a common density \( f \) but not necessarily independent of each other.

(i) Suppose that \( \int_{\mathbb{R}} f(x)^{-\alpha} \, dx < \infty \) for some \( \alpha > 0 \). Then

\[
P \left( \limsup_{n \to \infty} \max_{i=1,\ldots,n} \left[ -\{\log f(X_i)\}(\log n)^{-1} \right] \leq \frac{2}{\alpha} \right) = 1. \quad (27)
\]

(ii) Suppose that \( \int_{\mathbb{R}} f(x)^{1+\alpha} \, dx < \infty \) for some \( \alpha > 0 \). Then

\[
P \left( \limsup_{n \to \infty} \max_{i=1,\ldots,n} \left[ \{\log f(X_i)\}(\log n)^{-1} \right] \leq \frac{2}{\alpha} \right) = 1. \quad (28)
\]

**Proof of Lemma 8.6.** We only prove (27) while (28) could be proven similarly and we omit it. First, observe that

\[
\mathbb{E}[f(X_1)^{-\alpha}] = \int_{\mathbb{R}} f(x)^{-\alpha} \, dx < \infty.
\]
By Markov’s inequality, for any $m > 1$,
\[
\mathbb{P}(f(X_n)^{-\alpha} > n^m) \leq \frac{\mathbb{E}(f(X_n)^{-\alpha})}{n^m} = \frac{\mathbb{E}(f(X_1)^{-\alpha})}{n^m}.
\]
Hence,
\[
\sum_{n=1}^{\infty} \mathbb{P}(f(X_n)^{-\alpha} > n^m) \leq \mathbb{E}(f(X_1)^{-\alpha}) \sum_{n=1}^{\infty} \frac{1}{n^m} < \infty.
\]
By the first Borel-Cantelli lemma, with probability one, there exists an integer $N_1$ such that for all $n \geq N_1$,
\[
f(X_n)^{-\alpha} \leq n^m.
\]
There also exists another integer $N_2 \geq N_1$ such that for all $i = 1, \ldots, N_2$, $f(X_i)^{-\alpha} \leq n^m$. In other words, for all $n \geq N_2$ and $i \leq n$, $f(X_i)^{-\alpha} \leq n^m$. Hence, with probability one,
\[
\limsup_{n \to \infty} \max_{i=1,\ldots,n}[-\{\log f(X_i)}(\log n)^{-1}] \leq \frac{m}{\alpha}
\]
and (27) follows by taking $m = 2$.

\[\Box\]

**Lemma 8.7.** (a) Suppose that Condition (A) holds, we have $\int_{\mathbb{R}} f_0(x)^{1-\alpha} dx < \infty$ for some $\alpha \in (0, 1)$.

(b) Suppose that Condition (B) holds, we have $\int_{\mathbb{R}} f_0(x)^{1+\alpha} dx < \infty$ for some $\alpha \in (0, 1)$.

**Proof of Lemma 8.7.** (a) Since $\gamma > 1$, there exists $\alpha \in (0, 1)$ such that $\gamma(1-\alpha) > 1$. Then,
\[
\int_{\mathbb{R}} f_0(x)^{1-\alpha} dx \leq \int_{|x| \leq x_0} f_0(x)^{1-\alpha} dx + \int_{|x| > x_0} |x|^{-\gamma(1-\alpha)} dx.
\]
(29)
Since $\gamma(1-\alpha) > 1$, $\int_{|x| > x_0} |x|^{-\gamma(1-\alpha)} dx < \infty$. For the first term on the RHS of (29), we have
\[
\int_{|x| \leq x_0} f_0(x)^{1-\alpha} dx = \int_{|x| \leq x_0, f_0(x) \leq 1} f_0(x)^{1-\alpha} dx + \int_{|x| \leq x_0, f_0(x) > 1} f_0(x)^{1-\alpha} dx
\leq \int_{|x| \leq x_0} 1 dx + \int_{|x| \leq x_0} f_0(x) dx < \infty.
\]
(b) Since $\gamma_2 > 0$, there exists $\alpha \in (0, 1)$ such that $(1+\alpha)(\gamma_2-1) > -1$. Then
\[
\int_{|x-a| < \delta} f_0(x)^{1+\alpha} dx \leq \int_{|x-a| < \delta} |x-a|^{(1+\alpha)(\gamma_2-1)} dx < \infty.
\]
Now, $\sup_{|x-a| \geq \delta, f_0(x) \geq 1} f_0(x) < \infty$ and as $f_0$ is a density, the set such that $f_0(x) \geq 1$ must have finite Lebesgue measure. Thus,
\[
\int_{|x-a| \geq \delta} f_0(x)^{1+\alpha} dx = \int_{|x-a| \geq \delta, f_0(x) \geq 1} f_0(x)^{1+\alpha} dx + \int_{|x-a| \geq \delta, f_0(x) < 1} f_0(x)^{1+\alpha} dx
\leq \sup_{|x-a| \geq \delta, f_0(x) \geq 1} f_0(x)^{1+\alpha} \int_{f_0(x) \geq 1} dx + \int_{|x-a| \geq \delta, f_0(x) < 1} f_0(x) dx < \infty.
\]
\[\Box\]
Proof of Theorem 2.4. Since one of Conditions (C) (i) or (ii) holds, by Lemma 8.5 we have
\[ |R_n| \lesssim \frac{\nu}{n} (C + |\log f_0(Z_1)| + |\log f_0(Z_n)|) + O_p \left( \frac{\log n}{n} \right). \]

It remains to show that \( \log f_0(Z_1) = O_p(\log n) \) and \( \log f_0(Z_n) = O_p(\log n) \). Note that
\[ \log f_0(Z_1) \leq \max_{i=1,\ldots,n} f_0(X_i) \quad \text{and} \quad -\log f_0(Z_1) \leq \max_{i=1,\ldots,n} \{-\log f_0(X_i)\}. \]

By Lemma 8.7, since Conditions (A) and (B) hold, we have \( R f \pm \alpha_0 < \infty \) for some \( \alpha > 0 \).

Then, by Lemma 8.6, we have with probability one,
\[ \limsup_{n \to \infty} |\log f_0(Z_1)| (\log n)^{-1} \leq \frac{2}{\alpha}. \]

This implies that \( \log f_0(Z_1) = O_p(\log n) \). The same argument also leads to \( \log f_0(Z_n) = O_p(\log n) \).

Proof of Corollary 2.5. We shall show that the conditions in Theorem 2.4 are satisfied so that the result follows. First, a log-concave density must be unimodal. It is either monotone over the support, or there exists \( K_1 > \tau_f \) and \( K_2 < \sigma_f \) such that \( f_0 \) is monotone on \((\tau_f, K_1]\) and \([K_2, \sigma_f)\). Furthermore, because \( \log f_0 \) is concave, \( \log f_0 \) is Lipschitz continuous on \([K_1, K_2]\).

Note that any log-concave density has an exponential tail (see Lemma 1 in [25]), that is, there exist \( a > 0 \) and \( b \in \mathbb{R} \) such that \( f_0(x) \leq e^{-a|x|+b} \) for all \( x \). Thus, Conditions (A) and (B) are satisfied.

9 Appendix for Section 3.1: \( k \)-monotone Densities

9.1 Upper bound and support of \( k \)-monotone MLE

To prove Lemmas 3.1 and 3.3, we first recall that the MLE \( \hat{f}_{n,k} \) is of the form:
\[
\hat{f}_{n,k}(x) = \sum_{j=1}^{m} \hat{w}_j \frac{k(\hat{a}_j - x)_+^{k-1}}{\hat{a}_j^k},
\]
where \( m \in \mathbb{Z}^+ \), and \( \{\hat{w}_1, \ldots, \hat{w}_m\} \) and \( \{\hat{a}_1, \ldots, \hat{a}_m\} \) are respectively the family of weights and support points of the maximizing mixing distribution corresponding to \( \hat{f}_{n,k} \). Here, \( a_+ \) denotes \( \max(a, 0) \) for any \( a \in \mathbb{R} \); see Lemma 2 in [20]. Without loss of generality, we can assume that \( \hat{a}_1 \leq \ldots \leq \hat{a}_m \). The likelihood function at \( \hat{f}_{n,k} \) is:
\[
L_n(\hat{f}_{n,k}) = \prod_{i=1}^{n} \left\{ \sum_{j=1}^{m} \hat{w}_j \frac{k(\hat{a}_j - X_i)_+^{k-1}}{\hat{a}_j^k} \right\}.
\]

Proof of Lemma 3.1. We first show that \( \hat{f}_{n,k}(0+) = O_p(1) \) if \( f_0 \) is bounded from above. Proposition 6 in [42] demonstrates that when \( f_0 \in \mathcal{F}_k^B([0, A]) \), we have \( \hat{f}_{n,k}(0+) = O_p(1) \).
A closer inspection of their proof reveals that the main step is to make use of the characterization of the NPMLE, which does not depend on the boundedness of the support of \( f_0 \). Therefore, the same proof remains valid in this case.

Now, we shall show that, in general, \( \hat{f}_{n,k}(0+) \leq kZ_1^{-1} \). For \( k = 1 \), \( \hat{f}_{n,1}(0)Z_1 = \int_0^{Z_1} \hat{f}_{n,k}(x) \, dx \leq \int_0^{\infty} \hat{f}_n(x) \, dx = 1 \). Therefore, \( \hat{f}_n(0) \leq Z_1^{-1} \). For \( k \geq 2 \), we use the form in (30) and claim that \( \hat{a}_1 > Z_1 \). To see that, fix the values of \( \hat{a}_2, \ldots, \hat{a}_m \) and \( \hat{w}_1, \ldots, \hat{w}_m \). If \( \hat{a}_1 \leq Z_1 \), then \( (\hat{a}_1 - X_i)^{k-1}_+ = 0 \) for all \( i \) while if \( \hat{a}_1 > Z_1 \), \( (\hat{a} - X_i)^{k-1}_+ > 0 \) for some \( i \). Hence, the maximum value of \( L_n \) must be obtained when \( \hat{a}_1 > Z_1 \). Finally, from the proof of Proposition 6 in [42], we have:

\[
\hat{f}_{n,k}(0) \leq \frac{kP_n(\hat{a}_1)}{\hat{a}_1} \leq \frac{k}{Z_1}.
\]

\( \square \)

**Proof of Lemma 3.3.** For \( k = 1 \), it is well-known that \( \sigma_{\hat{f}_{n,1}} = Z_n \). In the rest of this proof, we assume that \( k \geq 2 \). In view of (30), \( \hat{a}_m = \sigma_{\hat{f}_{n,k}} \). For fixed \( \hat{a}_1, \ldots, \hat{a}_{m-1} \) and \( \hat{w}_1, \ldots, \hat{w}_m \), we investigate how the change of \( \hat{a}_m \) affects the value of the likelihood. By the definition of the MLE, we know that \( \hat{a}_m > Z_n \), since \( k \geq 2 \) (see also Remark 2 in [20]). Therefore, for each \( i \), we shall consider the term

\[
\hat{w}_m \frac{k(\hat{a}_m - X_i)^{k-1}_+}{\hat{a}_m^k}.
\]

Consider the function \( g_i(y) = \frac{(y - X_i)^{k-1}_+}{y^k} \) for \( y > X_i \). Then,

\[
g_i'(y) = \frac{(y - X_i)^{k-2}_+}{y^{k+1}}(kX_i - y) \text{ for } y > X_i,
\]

with which we can see that \( g_i \) increases and then decreases in \( y \), and attains its maximum at \( kX_i \). Therefore, if \( \hat{a}_m > kZ_n \), the term in (31) becomes smaller for all \( i = 1, \ldots, n \) if \( \hat{a}_m \) is set at a larger value. This shows that the optimality can be ensured only if \( \hat{a}_m \leq kZ_n \).

\( \square \)

## 9.2 General Results on \( S_{n,k} \) under \( H_0 \) and \( H_1 \)

In this subsection, we consider a more general version of Theorems 3.4 and 3.7; see Theorems 9.1 and 9.2 respectively.

Now, suppose that \( \{a_n\} \) and \( \{b_n\} \) are two sequences of real numbers such that \( \log(a_nb_n) = o(n^{2k}) \),

\[
\lim_{n \to \infty} P(kZ_n^{-1} > a_n) = 0 \quad \text{and} \quad \lim_{n \to \infty} P(kZ_n > b_n) = 0.
\]

Then, with a probability approaching 1, we have \( \hat{f}_n \in \mathcal{F}_k^{\omega,\varepsilon}([0, b_n]) \), which has a finite bracketing entropy for each \( n \) (recall that [42] showed that \( N_{\varepsilon}(\omega, \mathcal{F}_k^{B}([0, A]), h) < \infty \) for each small enough \( \varepsilon \) and any \( A, B > 0 \)). The price of this relaxation is an extra factor, up to a difference in logarithm order, to be shown in Theorem 9.1 that appears in the rate of convergence. If \( f_0 \) has a bounded support, we can simply take \( b_n \) as \( k\sigma_{f_0} \), which is finite; otherwise, given a sequence \( d_n \) that converges to 0, \( b_n \) can be chosen to be \( b_n = kF_0^{-1}((1 - d_n)^{1/n}) \). If \( f_0 \) is bounded from above, the condition \( \lim_{n \to \infty} P(kZ_n^{-1} > a_n) = 0 \) is not needed as we in that case have \( \hat{f}_{n,k}(0+) = O_p(1) \); otherwise \( a_n \) can be chosen to be \( k/F_0^{-1}(d_n^{1/n}) \).
Theorem 9.1. Suppose that $f_0 \in \mathcal{F}_k$. Let $\{a_n\}$ and $\{b_n\}$ be two sequences that satisfy (32).

(i) If $f_0$ is unbounded at 0 and has an unbounded support,
$$S_{n,k} = O_p\left(n^{-\frac{2k}{2k+1}} |\log(a_n b_n)|^{\frac{1}{2k+1}}\right).$$

(ii) If $f_0$ is bounded from above and has an unbounded support,
$$S_{n,k} = O_p\left(n^{-\frac{2k}{2k+1}} |\log b_n|^{\frac{1}{2k+1}}\right).$$

(iii) If $f_0$ is unbounded at 0 and has a bounded support,
$$S_{n,k} = O_p\left(n^{-\frac{2k}{2k+1}} |\log a_n|^{\frac{1}{2k+1}}\right).$$

(iv) If $f_0$ is bounded from above and has a bounded support, then $S_{n,k} = O_p\left(n^{-\frac{2k}{2k+1}}\right)$, as stated in (3).

Now, we turn to the study of the NPLRT under $H_1$. Theorem 9.2 below provides mild sufficient conditions for the existence of $L_n \uparrow \infty$ such that $\lim_{n \to \infty} P(\sqrt{n}\nu S_{n,k} > L_n) = 1$ for the $k$-monotone density case, where we make use of a large deviation inequality for likelihood ratios studied in [41] in its proof.

Theorem 9.2. If $\inf_{f \in \mathcal{F}_k} h(f, f_0) > 0$ and $\log(a_n b_n) = o(n^{2k})$, then there exists some constant $c > 0$ such that
$$\lim_{n \to \infty} P(S_{n,k} > c) = 1.$$

9.3 Proofs for Subsection 9.2

First, we shall introduce some notations. Given two functions $l$ and $u$, the bracket $[l, u]$ is the set of all functions $g$ with $l \leq g \leq u$. Given a distance $\rho$, an $\varepsilon$-bracket is a bracket $[l, u]$ for some $l$ and $u$ so that $\rho(l, u) < \varepsilon$. For a class of functions $\mathcal{G}$, the bracketing number $N_{[\cdot]}(\varepsilon, \mathcal{G}, \rho)$ is the minimum number of $\varepsilon$-brackets needed to cover $\mathcal{G}$. The entropy with bracketing is the logarithm of the bracketing number $N_{[\cdot]}(\varepsilon, \mathcal{G}, \rho)$. The Hellinger distance between two densities $f$ and $g$ is defined as:
$$h(f, g) = \frac{1}{\sqrt{2}} \left[ \int_{0}^{\infty} \left\{ \sqrt{f(t)} - \sqrt{g(t)} \right\}^2 dt \right]^{1/2}.$$

Let $\tilde{J}_{[\cdot]}(\delta, \mathcal{G}, h) := \int_{0}^{\delta} \sqrt{1 + \log N_{[\cdot]}(\varepsilon, \mathcal{G}, h)} d\varepsilon$. For more details of these definitions, see [63]. The notation $\lesssim$ is used to indicate the left side is bounded by the right side but up to a (universal) constant. Finally, denote $\mathbb{F}_n$ to be the empirical distribution function induced by the random sample $X_1, \ldots, X_n$, and denote the empirical process by $\mathbb{G}_n(f) := \sqrt{n} \int f d(\mathbb{F}_n - F_0)$.

The method of proof of Theorem 9.1 follows the empirical process theory as described in, for example, [43].
Proof of Theorem 9.1. We shall only prove (i). The other cases can be proven similarly. Fix $k \in \mathbb{N}$. For simplicity, we write $\hat{f}_n$ instead of $f_{n,k}$. From the proof of Lemma 4.1 and Lemma 4.2 in [H3], we have

$$0 \leq -\frac{1}{4} S_{n,k} \leq \frac{1}{2} \int \frac{\hat{f}_n + f_0}{2f_0} d(\mathbb{F}_n - F_0) - h^2 \left( \frac{\hat{f}_n + f_0}{2} , f_0 \right)$$

$$\leq \frac{1}{2} \int \log \frac{\hat{f}_n + f_0}{2f_0} d(\mathbb{F}_n - F_0) - \frac{1}{16} h^2 \left( \hat{f}_n , f_0 \right).$$

Let $m_f(t) := \log \frac{f(t) + f_0(t)}{2}$. Fix $M > 0$. Let $\delta_n := n^{-\frac{k}{2k+1}} \log(a_n b_n)^{\frac{1}{2(2k+1)}}$. Then, we have

$$\mathbb{P} \left( -\frac{\sqrt{n}}{2} S_{n,k} \geq \sqrt{n} 2M \delta_n^2 \right) \leq \mathbb{P} \left( \mathcal{G}_n(m_f) - \frac{\sqrt{n}}{8} h^2(\hat{f}_n, f_0) \geq \sqrt{n} 2M \delta_n^2, kZ_1 \leq a_n, kZ_n \leq b_n \right) + \mathbb{P}(kZ^{-1}_1 > a_n) + \mathbb{P}(kZ_n > b_n),$$

where the last two terms in the last display go to 0 as $n$ goes to infinity by assumptions on $a_n$ and $b_n$. Denote

$$A_n := \mathbb{P} \left( \mathcal{G}_n(m_f) - \frac{\sqrt{n}}{8} h^2(\hat{f}_n, f_0) \geq \sqrt{n} 2M \delta_n^2, kZ_1 \leq a_n, kZ_n \leq b_n \right).$$

Using the peeling device, we obtain

$$A_n \leq \sum_{s=0}^{S_n} \mathbb{P} \left( \sup_{f \in \mathcal{F}^a_k([0, b_n]): 2^{M+s} \delta_n \leq h(f, f_0) < 2^{M+s+1} \delta_n} \left( \mathcal{G}_n(m_f) - \frac{\sqrt{n}}{8} h^2(f, f_0) \right) \geq \sqrt{n} 2^M \delta_n^2 \right)$$

$$+ \mathbb{P} \left( \sup_{f \in \mathcal{F}^a_k([0, b_n]): h(f, f_0) < 2M \delta_n} \left( \mathcal{G}_n(m_f) - \frac{\sqrt{n}}{8} h^2(f, f_0) \right) \geq \sqrt{n} 2^M \delta_n^2 \right)$$

$$\leq \sum_{s=0}^{S_n} \mathbb{P} \left( \sup_{f \in \mathcal{F}^a_k([0, b_n]): 2^{M+s} \delta_n \leq h(f, f_0) < 2^{M+s+1} \delta_n} \mathcal{G}_n(m_f) \geq \sqrt{n} 2^{M}(2^{2s} + 1) \delta_n^2 \right)$$

$$+ \mathbb{P} \left( \sup_{f \in \mathcal{F}^a_k([0, b_n]): h(f, f_0) < 2M \delta_n} \mathcal{G}_n(m_f) \geq \sqrt{n} 2^M \delta_n^2 \right)$$

$$\leq \sum_{s=0}^{S_n} \mathbb{P} \left( \sup_{f \in \mathcal{F}^a_k([0, b_n]): h(f, f_0) < 2^M \delta_n} \mathcal{G}_n(m_f) \geq \sqrt{n} 2^{M} \delta_n^2 \right)$$

$$+ \mathbb{P} \left( \sup_{f \in \mathcal{F}^a_k([0, b_n]): h(f, f_0) < 2^M \delta_n} \mathcal{G}_n(m_f) \geq \sqrt{n} 2^M \delta_n^2 \right)$$

where $S_n := \min \{ s : 2^{M+s+1} \delta_n \geq 1 \}$. Define

$$\mathcal{M}_{n, \delta} := \{ m_f : f \in \mathcal{F}^a_k([0, b_n]), h(f, f_0) < \delta \}.$$
By Markov’s inequality,
\[ A_n \leq \sum_{s=0}^{S_n} \frac{\mathbb{E}[|G_n||\mathcal{M}_{n,2^M+1}\delta_n]}{\sqrt{n}2^M2^s\delta_n^2} + \frac{\mathbb{E}[|G_n||\mathcal{M}_{n,2^M}\delta_n]}{\sqrt{n}2^M\delta_n^2}, \]
where \( \mathbb{E}[|G_n||\mathcal{M}_{n,\delta}] := \sup_{f \in \mathcal{M}_{n,\delta}} \mathbb{E}[|G_n(f)|] \). By Theorem 3.4.4 in [41],
\[ \mathbb{E}[|G_n||\mathcal{M}_{n,\delta}] \lesssim J_2(\delta, \mathcal{F}_k^{an}([0, b_n]), h) \left( 1 + \frac{\tilde{J}_2(\delta, \mathcal{F}_k^{an}([0, b_n]), h)}{\delta^2} \right). \]
From [4], we have
\[ \log N_{\mathcal{E}}(\delta, \mathcal{F}_k^{an}([0, b_n]), h) \leq C_k \log(a_n b_n)^{1/3} \frac{\delta}{\pi} \delta^{-1/3}. \]
A direct calculation gives
\[ \tilde{J}_2(\delta, \mathcal{F}_k^{an}([0, b_n]), h) \lesssim C_k^{1/2} |\log(a_n b_n)|^{1/3} \delta^{-1/3} \lesssim |\log(a_n b_n)|^{1/3} \delta^{-1/3} \]
for all large enough \( n \) as \( C_k \) only depends on \( k \). Define
\[ \xi_n(\delta) := |\log(a_n b_n)|^{1/3} \delta^{-1/3} \left( 1 + |\log(a_n b_n)|^{1/3} \delta^{-1/3} \right). \]
Note that \( \xi_n(\delta)/\delta \) is decreasing in \( \delta \) for any fixed \( n \). Thus, for any \( j > 0 \),
\[ \frac{\xi_n(2^j \delta_n)}{\delta_n^2} \leq \frac{2^j \xi_n(\delta_n)}{\delta_n^2} = 2^{j+1} \sqrt{n}. \]
Hence,
\[ A_n \leq \sum_{s=0}^{S_n} \frac{\xi_n(2^M+1)\delta_n}{\sqrt{n}2^M2^s\delta_n^2} + \frac{\xi_n(2^M)\delta_n}{\sqrt{n}2^M\delta_n^2} \leq \sum_{s=0}^{\infty} \frac{2^M+2}{\sqrt{n}2^M2^s} + \frac{2^M+1}{\sqrt{n}2^M} = 10 \frac{2^M}{2^M}, \]
which goes to 0 as \( M \) goes to \( \infty \). This implies that
\[ S_{n,k} = O_p \left( n^{-\frac{2k}{2k+1}} |\log(a_n b_n)|^{1/2k+1} \right). \]

**Proof of Theorem 9.2** Let \( \varepsilon := \inf_{f \in \mathcal{F}_k} h(f, f_0) > 0 \). Also let \( c_j, j = 1, \ldots, 4 \), be the positive constants in Theorem 1 in [41]. First, note that
\[ \mathbb{P} \left( \prod_{i=1}^{n} \frac{\hat{f}_{n,k}(X_i)}{f_0(X_i)} \geq e^{-c_1 n \varepsilon^2} \right) \]
\[ \leq \mathbb{P} \left( \prod_{i=1}^{n} \frac{\hat{f}_{n,k}(X_i)}{f_0(X_i)} \geq e^{-c_1 n \varepsilon^2}, \hat{f}_{n,k} \in \mathcal{F}_k^{an}[0, b_n] \right) + \mathbb{P}(\hat{f}_{n,k} \notin \mathcal{F}_k^{an}[0, b_n]) \]
\[ \leq \mathbb{P} \left( \sup_{f \in \mathcal{F}_k^{an}[0, b_n]} \prod_{i=1}^{n} \frac{f(X_i)}{f_0(X_i)} \geq e^{-c_1 n \varepsilon^2} \right) + \mathbb{P}(\hat{f}_{n,k} \notin \mathcal{F}_k^{an}[0, b_n]). \]

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To verify (3.1) in [41], we have, using Theorem 3 in [42],
\[
\int_{\frac{\epsilon}{\sqrt{2}\pi}}^2 \sqrt{\log N_\epsilon}\left( \frac{u}{c_3}, \mathcal{F}_k^a([0, b_n]), h \right) du \leq \int_0^{\frac{\epsilon}{\sqrt{2}\pi}} C_k^{\frac{1}{2}} |\log(a_n b_n)| \frac{h}{\sqrt{\log N_\epsilon}} du \\
\leq C_k^{\frac{1}{2}} C_3^{\frac{1}{2}} 2^{\frac{1}{4}} \epsilon^{1/4} |\log(a_n b_n)| \frac{1}{\sqrt{\log N_\epsilon}}.
\]

As \( \log(a_n b_n) = o(n^{2k}) \), the right hand side of the last inequality is smaller than \( c_4 n^{1/2} \epsilon^2 \) for all large enough \( n \). Therefore, by Theorem 1 in [41], for all large enough \( n \),
\[
P\left( \sup_{f \in \mathcal{F}_k^a([0, b_n])} \prod_{i=1}^n \frac{f(X_i)}{f_0(X_i)} \geq e^{-c_1 \epsilon n^2} \right) \leq 4 \exp(-c_2 \epsilon n^2).
\]
Hence,
\[
\limsup_{n \to \infty} P\left( \prod_{i=1}^n \frac{\hat{f}_{n,k}(X_i)}{f_0(X_i)} \geq e^{-c_1 \epsilon n^2} \right) \leq \limsup_{n \to \infty} 4 \exp(-c_2 \epsilon n^2) + \limsup_{n \to \infty} P(\hat{f}_{n,k} \notin \mathcal{F}_k^a([0, b_n])) = 0,
\]
as \( \hat{f}_{n,k} \in \mathcal{F}_k^a([0, b_n]) \) with a probability going to 1. Finally, this implies that
\[
\lim_{n \to \infty} P\left( \frac{1}{n} \sum_{i=1}^n \log \frac{f_0(X_i)}{\hat{f}_{n,k}(X_i)} > c_1 \epsilon^2 \right) = \lim_{n \to \infty} P\left( \prod_{i=1}^n \frac{\hat{f}_{n,k}(X_i)}{f_0(X_i)} < e^{-c_1 \epsilon n^2} \right) = 1.
\]

\[\square\]

### 9.4 Proofs of Theorems 3.4, 3.7, Corollaries 3.6, 3.8

**Lemma 9.3.** Under Conditions (A) and (B), there exist \( \{a_n\} \) and \( \{b_n\} \) satisfying (32) and \( \log(a_n b_n) = O(\log n) \).

**Proof of Lemma 9.3.** Under Condition (A), for some \( \gamma > 1, x_0 > 0 \), we have for all \( x \geq x_0 \),
\[
P(X > x) \leq \frac{x^{-\gamma+1}}{\gamma - 1}.
\]
Let \( b_n = n^{2/(\gamma-1)} \). Then for any \( \epsilon > 0 \),
\[
n P(X > \epsilon b_n) \leq \frac{n \epsilon^{-\gamma+1}}{n^{2(\gamma-1)}} \to 0
\]
as \( n \to \infty \). By Theorem 3 in [64], we have \( Z_n/b_n \to 0 \). This implies that
\[
\lim_{n \to \infty} P(k Z_n > b_n) = 0.
\]
Under Condition (B), for some $\delta > 0$, for all $x < \delta$, $F_0(x) \leq x^{\gamma_2}$ for some $\gamma_2 > 0$. Let $a_n = n^{2/\gamma_2}$.

$$n\mathbb{P}(X^{-1} > \varepsilon a_n) = n\mathbb{P}(X < (\varepsilon a_n)^{-1}) \leq \frac{n}{\varepsilon^{\gamma_2} n^{2}} \to 0.$$  

Note that $Z_1$ is the maximum of $X_1^{-1}, \ldots, X_n^{-1}$. By Theorem 3 in [64], we have $Z_1/a_n \overset{P}{\to} 0$. This implies that

$$\lim_{n \to \infty} \mathbb{P}(kZ_1^{-1} > a_n) = 0.$$  

Finally, we clearly have $\log(a_n b_n) = O(\log n)$.

Proof of Theorem 3.4. This is a consequence of Lemma 9.3 and Theorem 9.1.

Proof of Theorem 3.7. This is a consequence of Lemma 9.3 and Theorem 9.2.

Proof of Corollary 3.6. We shall apply Theorem 2.1 by verifying $\sqrt{n \nu S_{n,k}} = o_p(1)$ and $R_n = O_p(\nu^{\log n} n)$. First, as $\nu = O(n^{1/3}/\log n)$, by Theorem 3.4, for any $k \in \mathbb{N}$,

$$\sqrt{n \nu S_{n,k}} = O\left(n^{\frac{2}{3}}(\log n)^{-\frac{1}{2}}\right) O_p \left(n^{-\frac{2k}{3\nu \log n}} \left(\log n\right)^{\frac{1}{3\nu \log n}}\right) = o_p(1).$$  

As $f_0$ is decreasing, by Theorem 2.4

$$R_n = O_p(\nu^{\log n} n).$$  

Proof of Corollary 3.8. By Theorem 3.7, $\lim_{n \to \infty} \mathbb{P}(\sqrt{n \nu S_n} > \sqrt{n \nu c}) = 1$ for some $c > 0$. By Theorem 2.4

$$R_n = O_p(\nu^{\log n} n).$$  

The result then follows from Theorem 2.2.

10 Appendix for Section 3.2: Completely Monotone Densities

10.1 General Results on $S_{n,\infty}$ under $H_0$ and $H_1$

In this subsection, we consider a more general version of Lemma 3.9 and Theorem 3.11; see Lemma 10.1 and Theorem 10.2, respectively.

Let $\{c_{1n}\}$ and $\{c_{2n}\}$ be sequences such that $\log(c_{1n} c_{2n}) = o(n^2)$,

$$\lim_{n \to \infty} \mathbb{P}(Z_1^{-1} > c_{1n}) = 0 \quad \text{and} \quad \lim_{n \to \infty} \mathbb{P}(Z_n^{1+\beta} > c_{2n}) = 0,$$

for some $\beta > 0$. Denote

$$S_{n,\infty} := -\frac{1}{n} \sum_{i=1}^{n} \log \frac{\hat{f}_{n,\infty}(X_i)}{f_0(X_i)}.$$  

Lemma 10.1. Suppose that $f_0 \in \mathcal{F}_\infty$, $\{c_{1n}\}$ and $\{c_{2n}\}$ satisfy (34). Then,

$$S_{n,\infty} = O_p \left(n^{-\frac{2}{3}} |\log(c_{1n} c_{2n})|^{\frac{1}{3}}\right).$$

The slow rate in Lemma 10.1 is a consequence of our method of proof but is sufficient for the application of the NPLRT under mild conditions on $c_{1n}$ and $c_{2n}$.

Theorem 10.2. If $\inf_{f \in \mathcal{F}_\infty} h(f, f_0) > 0$ and $\log(c_{1n} c_{2n}) = o(n^2)$, then there exists some constant $c > 0$ such that

$$\lim_{n \to \infty} \mathbb{P}(S_{n,\infty} > c) = 1.$$  

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10.2 Proofs for Subsection 10.1

Proof of Lemma 10.1. For simplicity, we write \( \hat{f}_n \) instead of \( \hat{f}_{n,∞} \). First, from [21], we know that the MLE is of the form

\[
\hat{f}_n(t) = \sum_{i=1}^{r_n} \hat{\lambda}_i \hat{p}_i e^{-\hat{\lambda}_i t},
\]

for some \( r_n \leq n \), with \( \sum_{i=1}^{r_n} \hat{p}_i = 1 \) and \( \hat{\lambda}_i \in [Z_n^{-1}, Z_1^{-1}] \) for each \( i = 1, \ldots, r_n \). Without loss of generality, we assume that \( \hat{\lambda}_1 \leq \ldots \leq \hat{\lambda}_{r_n} \). We shall first obtain an upper bound for \( \hat{f}_n \) for a large \( t \). Note that

\[
\hat{f}_n(t) \leq \sum_{i=1}^{r_n} \hat{p}_i \max_{j=1,\ldots,r_n} \hat{\lambda}_j e^{-\hat{\lambda}_j t} = \max_{j=1,\ldots,r_n} \hat{\lambda}_j e^{-\hat{\lambda}_j t},
\]

since \( \sum_{i=1}^{r_n} \hat{p}_i = 1 \). For any fixed \( t \), the function \( \lambda \mapsto \lambda e^{-\lambda t} \) is unimodal and achieves its maximum when \( \lambda = \frac{1}{t} \). Thus, if \( t > Z_n \) so that \( \frac{1}{t} < \frac{1}{Z_n} \leq \hat{\lambda}_1 \), we have

\[
\hat{f}_n(t) \leq \hat{\lambda}_1 e^{-\hat{\lambda}_1 t} \leq \frac{1}{Z_n} e^{-\frac{t}{Z_n}}.
\]

Since \( f_0 \) has an unbounded support, \( Z_n \overset{a.s.}{\rightarrow} \infty \). Thus, with a probability approaching 1, we have

\[
\hat{f}_n(t) \leq \frac{1}{Z_n} e^{-\frac{t}{Z_n}} \leq \frac{1}{t^2} \quad \text{as } n \to \infty. \tag{37}
\]

We shall also obtain a rough upper bound for \( \hat{f}_n(0) \). Observe that

\[
\hat{f}_n(0) = \sum_{i=1}^{r_n} \hat{\lambda}_i \hat{p}_i \leq \sum_{i=1}^{r_n} \hat{p}_i \max_{j=1,\ldots,r_n} \hat{\lambda}_j \leq \max_{j=1,\ldots,r_n} \hat{\lambda}_j \leq Z_1^{-1}.
\]

To obtain a rate of convergence of \( \hat{f}_n \), we shall find an upper bound for the bracketing entropy of a class of functions where the MLE will be in with a probability approaching 1. To this end, write

\[
\hat{f}_n(t) = \hat{f}_n(t) I(t \leq c_{2n}) + \hat{f}_n(t) I(t > c_{2n}).
\]

Define

\[
\mathcal{F}_n := \mathcal{F}_1^{c_{2n}}([0, c_{2n}]) + \mathcal{G}_n,
\]

where

\[
\mathcal{F}_1^{c_{2n}}([0, c_{2n}]) := \left\{ f : [0, c_{2n}] \to \mathbb{R}^+ : f \text{ is decreasing, } \int_0^{c_{2n}} f(x) dx \leq 1, f(0) \leq c_{1n} \right\}
\]

and

\[
\mathcal{G}_n := \left\{ f : (c_{2n}, \infty) \to [0, 1] : f \text{ is decreasing, } f(t) \leq 1/t^2 \right\}.
\]
Because of (34), \( \lim_{n \to \infty} \mathbb{P}(\hat{f}_n(t) I(t \leq c_{2n}) \in \tilde{F}_{1n}^c([0, c_{2n}])) = 1 \). In view of (37), we have \( \lim_{n \to \infty} \mathbb{P}(\hat{f}_n(t) I(t > c_{2n}) \in G_n) = 1 \). Therefore,

\[
\lim_{n \to \infty} \mathbb{P}(\hat{f}_n \in \tilde{F}_n) = 1. \tag{38}
\]

We shall now show that \( \tilde{F}_n \) has a finite bracketing entropy with respect to the Hellinger distance. First, from the proof of Theorem 3 in [42], we know the result of that theorem is also valid for \( \tilde{F}_{1n}^c([0, c_{2n}]) \), where for any function \( f \) in this class such that \( \int_0^{c_{2n}} f(x) \, dx \leq 1 \) rather than just \( \int_0^{c_{2n}} f(x) \, dx = 1 \). Thus,

\[
\log N[\varepsilon, \tilde{F}_{1n}^c([0, c_{2n}]), h] \leq C \log(c_{1n} c_{2n})^{1/2} \varepsilon^{-1}, \tag{39}
\]

where \( C \) is a universal constant. Then, note that if \( f \in G_n \), then \( f^{1/2} \in G_n^{1/2} \), where

\[
G_n^{1/2} := \{ f : (c_{2n}, \infty) \to [0, 1] : f \text{ is decreasing}, f(t) \leq 1/t \}.
\]

As \( \int_{c_{2n}}^\infty 1/x^{2(1-\gamma)} \, dx < \infty \) for some \( \gamma \in (0, 1) \), we can apply the idea of Lemma 7.10 in [43] to obtain that

\[
\log N[\varepsilon, G_n^{1/2}, || \cdot ||_2] \leq A \varepsilon^{-1}, \tag{40}
\]

where \( A \) is a universal constant and \( || \cdot ||_2 \) is the \( L^2 \)-norm with respect to the Lebesgue measure on \([0, \infty)\). Now, simply note that

\[
\log N[\varepsilon, G_n^{1/2}, || \cdot ||_2] = \log N[\varepsilon, G_n, h]. \tag{41}
\]

Combining (39) to (41), we obtain that

\[
\log N[\varepsilon, \tilde{F}_n, h] \leq \{ C \log(c_{1n} c_{2n})^{1/2} + A \} \varepsilon^{-1} \lesssim \log(c_{1n} c_{2n})^{1/2} \varepsilon^{-1}, \tag{42}
\]

for all large \( n \). With (38) and (42), we can obtain (36) as in the proof of Theorem 9.1 and we omit the details.

The proof of Theorem 10.2 is similar to that of Theorem 9.2 and is therefore omitted.

### 10.3 Proofs for Subsection 3.2

The proof of the following lemma is essentially the same as that of Lemma 9.3 and is therefore omitted.

**Lemma 10.3.** Under Conditions (A) and (B), there exist \( \{c_{n1}\} \) and \( \{c_{n2}\} \) satisfying (34) and \( \log(c_{n1} c_{n2}) = O(\log n) \).

With Lemma 10.3, we can establish Lemma 3.9 in view of Lemma 10.1. The proof of Corollary 3.10 is similar to that of Corollary 3.6 for \( k \)-monotone densities and is therefore omitted. The proofs of Theorem 3.11 and Corollary 3.12 are similar to those of Theorem 3.7 and Corollary 3.8 for \( k \)-monotone densities and are therefore omitted.
11 Appendix for Section 3.3: Log-concave Densities

Note that the derivation of Corollary 3.13 is explained in the paragraph preceding it.

**Proof of Theorem 3.14.** The proof is similar to the one in Theorem 9.2, where we apply the large deviation inequality in Theorem 1 in [41]. To this end, it suffices to verify (3.1) in [41] and our claim then follows as a result. Without loss of generality, assume that the interval $[-1, 1]$ is strictly contained in the support of $f_0$. Because $\int_{\mathbb{R}} |x| f_0(x) \, dx < \infty$, by Lemma 3 in [65], $\hat{f}_{n,lc} \in F_{lc}^M$ for some $M > 0$ with a probability approaching 1 as $n \to \infty$. By Theorem 3.1 in [46], $\log N_1(u/c_3, F_{lc}^M, h) \lesssim u^{-1/2}$. Hence,

$$\int_{c^2 \varepsilon^2}^{\sqrt{2} \varepsilon} \sqrt{\log N_1(u/c_3, F_{lc}^M, h)} \, du \lesssim \int_0^{\sqrt{2} \varepsilon} u^{-1/4} \, du \lesssim \varepsilon^{3/4}.$$ 

Thus, the right hand side of the last inequality is smaller than $c_4 n^{1/2} \varepsilon^2$ for all large enough $n$, and so (3.1) in [41] is satisfied.

**Proof of Corollary 3.15.** By Theorem 3.14 there exists $c > 0$ such that $\mathbb{P}(\sqrt{n} \nu S_{n,lc} > \sqrt{n} \nu c) = 1$. Under Conditions (A), (B), and either (C)(i) or (C)(ii), $R_n = O_p(\nu \log n)$ by Theorem 2.4. The result then follows from Corollary 2.2.

12 Proofs for Section 4

12.1 Proof of Theorem 4.1

For simplicity, we write $\hat{f}_n$ for $\hat{f}_{n,1}$ in this section. Recall that $\hat{F}_n$ denotes the least concave majorant of the empirical distribution function $F_n$. To prove Theorem 4.1, we write

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \log \frac{\hat{f}_n(X_i)}{f_0(X_i)} = \sqrt{n} \int_0^1 \log \frac{\hat{f}_n(t)}{f_0(t)} \, d\hat{F}_n(t) + \sqrt{n} \int_0^1 \log \frac{\hat{f}_n(t)}{f_0(t)} \, d\{F_n(t) - \hat{F}_n(t)\} =: A_n + B_n.$$ 

Here, $A_n$ can be viewed as the Kullback–Leibler divergence of $f_0$ from $\hat{f}_n$, and thus $A_n$ is nonnegative by a simple application of Jensen's inequality. The following Lemma 12.1 will demonstrate that $A_n$ is asymptotically equivalent to a weighted $L_2$-error between $f_n$ and $f_0$, which is asymptotically normally distributed by a generalization of Theorem 1.1 in [47]; see Lemma 12.1. Lemma 12.2 will show that $B_n$ is also nonnegative and asymptotically equivalent to a weighted $L_1$-norm between $\hat{F}_n$ and $F_n$, which is also asymptotically normally distributed by Theorem 2.1 in [43].

**Lemma 12.1.** Under the conditions in Theorem 4.1 we have

$$\sqrt{n} \int_0^1 \log \frac{\hat{f}_n(t)}{f_0(t)} \, d\hat{F}_n(t) = \sqrt{n} \int_0^1 \log \frac{f_n(t) - f_0(t)}{2f_0(t)} \, dt + O_p(n^{-1/3+\delta}),$$

for any $\delta > 0$. 

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Lemma 12.2. Under the conditions in Theorem 4.1, we have

\[
\sqrt{n} \int_0^1 \log \frac{\hat{f}_n(t)}{f_0(t)} d\{F_n(t) - \hat{F}_n(t)\} = \sqrt{n} \int_0^1 \{\hat{F}_n(t) - F_n(t)\} \frac{|f'_0(t)|}{f_0(t)} dt.
\]

In addition,

\[
\sqrt{n} \int_0^1 \{\hat{F}_n(t) - F_n(t)\} \frac{|f'_0(t)|}{f_0(t)} dt = \kappa_n n^{-1/6} + O_p(n^{-1/3}).
\]

Remark 12.3. Since \(\hat{F}_n\) is the least concave majorant of \(F_n, \hat{F}_n \geq F_n\). Thus, \(\hat{F}_n(t) - F_n(t) = |\hat{F}_n(t) - F_n(t)|\) and so the term \(\int_0^1 \{\hat{F}_n(t) - F_n(t)\} |f'_0(t)|/f_0(t) dt\) is a weighted \(L_1\)-norm of the difference between \(\hat{F}_n\) and \(F_n\).

We first provide Lemmas 12.4 - 12.6 that are used in the proof of Lemma 12.1.

Lemma 12.4. Suppose that \(f_0\) is a decreasing density with support on \([0, 1]\) and \(0 < f_0(1) \leq f_0(0) < \infty\). Then, \(1/\hat{f}_n(\hat{Z}_n) = O_p(1)\). In other words, for any \(\epsilon > 0\), there exists \(\delta > 0\) such that for all sufficiently large \(n\),

\[\mathbb{P}(\hat{f}_n(\hat{Z}_n) > \delta) > 1 - \epsilon.\]

Proof of Lemma 12.4. From page 326-328 of [50], we know that

\[\hat{f}_n(Z_n) = \min_{0 \leq i \leq n-1} \frac{n - i}{n(Z_n - Z_i)}.\]

Then

\[1/\hat{f}_n(Z_n) = \max_{0 \leq i \leq n-1} \frac{n(Z_n - Z_i)}{n - i} = \max_{h=1,...,n} \frac{n(Z_n - Z_{n-h})}{h} = O_p(1),\]

by Corollary 5.2 (i) in [66].

Lemma 12.5. Under the conditions in Theorem 4.1, we have

\[
\sqrt{n} \int_0^1 |\hat{f}_n(t) - f_0(t)|^3 dt = O_p(n^{-1/3+\delta})
\]

for any \(\delta > 0\).

Proof of Lemma 12.5. For any \(0 < \delta' < 2.5\), we can bound this third-degree moment term by

\[
\int_0^1 |\hat{f}_n(t) - f_0(t)|^3 dt = \int_0^1 \left|\hat{f}_n(t) - f_0(t)\right|^{2.5-\delta'} \left|\hat{f}_n(t) - f_0(t)\right|^{0.5+\delta'} dt \\
\leq [2\max\{f_0(0), \hat{f}_n(0+))\}]^{0.5+\delta'} \int_0^1 \left|\hat{f}_n(t) - f_0(t)\right|^{2.5-\delta'} dt,
\]

where \([2\max\{f_0(0), \hat{f}_n(0+))\}]^{0.5+\delta'} = O_p(1). Theorem 1.1 in [47] with \(k = 2.5 - \delta'\) implies that

\[
n^{1/6} \left[n^{1/3} \left\{ \int_0^1 \left|\hat{f}_n(t) - f_0(t)\right|^{2.5-\delta'} dt \right\}^{1/(2.5-\delta')} - \mu_{2.5-\delta'} \right] = O_p(1),
\]

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for some finite constant $\mu_{2.5-\delta'}$. By straightforward algebra, we have

$$\left\{ \int_0^1 \left| \hat{f}_n(t) - f_0(t) \right|^{2.5-\delta'} dt \right\}^{1/(2.5-\delta')} = O_p(n^{-1/2}) + \mu_{2.5-\delta'} n^{-1/3} = O_p(n^{-1/3})$$

and so

$$\int_0^1 \left| \hat{f}_n(t) - f_0(t) \right|^{2.5-\delta'} = O_p(n^{-5/6+\delta}),$$

where $\delta = \delta'/3$ and the result in the lemma follows.

It is indicated in Remark 1.1 in [47] that one may obtain the asymptotic normality of the following weighted version of the $L_{\mu}$-error of $\hat{f}_n$:

$$n^{1/6} \left\{ n^{k/3} \int_0^1 \frac{\left| \hat{f}_n(t) - f_0(t) \right|^k}{2f_0(t)} dt - \mu_{k,f_0} \right\}$$

for some $\mu_{k,f_0} > 0$. We state such claim formally in Lemma 12.6 and omit the proof.

**Lemma 12.6** (Modified version of Theorem 1.1 in [47]). Under the conditions in Theorem 4.1, we have, for $1 \leq k < 2.5$,

$$n^{1/6} \left\{ n^{k/3} \int_0^1 \frac{\left| \hat{f}_n(t) - f_0(t) \right|^k}{2f_0(t)} dt - \mu_{k,f_0} \right\}$$

converges in distribution to a zero-mean normal random variable with a finite variance as $n \to \infty$, where

$$\mu_{k,f_0} := \left\{ \mathbb{E}|V(0)|^k \int_0^1 2^{\frac{k}{2}-1} f_0(t)^{\frac{k}{2}-1} |f_0'(t)|^{\frac{k}{3}} dt \right\}^{1/k},$$

with $V(\cdot)$ defined in Theorem 4.1.

**Proof of Lemma 12.6.** Using Taylor’s expansion, for some $f_n^*(t)$ lying between $f_0(t)$ and $\hat{f}_n(t)$, we have

$$\sqrt{n} \int_0^1 \log \frac{\hat{f}_n(t)}{f_0(t)} \hat{f}_n(t) dt = \sqrt{n} \int_0^1 \frac{\hat{f}_n(t) - f_0(t)}{f_0(t)} \hat{f}_n(t) dt - \sqrt{n} \int_0^1 \frac{\{\hat{f}_n(t) - f_0(t)\}^2}{2f_0^2(t)} \hat{f}_n(t) dt$$

$$+ \sqrt{n} \int_0^1 \frac{\{\hat{f}_n(t) - f_0(t)\}^3}{3\{f_n^*(t)\}^3} \hat{f}_n(t) dt,$$

$$=: A_{n1} + A_{n2} + A_{n3}.$$

Interestingly, at the first glance, $A_{n1}$ and $A_{n2}$ do not share the same rate of convergence. However, they are equally important for the Kullback–Leibler divergence from $\hat{f}_n$ to $f_0$. In the following, we shall show that both $A_{n1}$ and $A_{n2}$ are related to a weighted $L_2$-error between $\hat{f}_n$ and $f_0$ asymptotically. For $A_{n3}$, it is relatively small and comparatively negligible.
(i) For $A_{n1}$, because $\hat{f}_n$ and $f_0$ are both densities, we have $\int_0^1 \{\hat{f}_n(t) - f_0(t)\} \, dt = 0$. Thus,

$$A_{n1} = \sqrt{n} \int_0^1 \frac{\hat{f}_n(t) - f_0(t)}{f_0(t)} \hat{f}_n(t) \, dt - \sqrt{n} \int_0^1 \frac{\hat{f}_n(t) - f_0(t)}{f_0(t)} f_0(t) \, dt$$

$$= \sqrt{n} \int_0^1 \left\{ \frac{\hat{f}_n(t) - f_0(t)}{f_0(t)} \right\} \hat{f}_n(t) \, dt.$$

(ii) For $A_{n2}$, we have

$$A_{n2} = - \sqrt{n} \int_0^1 \left\{ \frac{\hat{f}_n(t) - f_0(t)}{2f_0(t)} \right\} \frac{\hat{f}_n(t)}{f_0(t)} \, dt$$

$$= - \sqrt{n} \int_0^1 \left\{ \frac{\hat{f}_n(t) - f_0(t)}{2f_0(t)} \left( \frac{f_0(t)}{f_0(t)} + \frac{\hat{f}_n(t) - f_0(t)}{f_0(t)} \right) \right\} \, dt$$

$$= - \sqrt{n} \int_0^1 \left\{ \frac{\hat{f}_n(t) - f_0(t)}{f_0(t)} \right\} \, dt - \sqrt{n} \int_0^1 \left\{ \frac{\hat{f}_n(t) - f_0(t)}{f_0(t)} \right\} \frac{f_0(t)}{2f_0(t)} \, dt.$$

Under the conditions in Theorem 4.1, we can apply Lemma 12.5 to obtain that the second term in the last displayed equation is of the order $O_p(n^{-1/3+\delta})$ because $f_0(1) > 0$.

(iii) We aim to show that $A_{n3} = O_p(n^{-1/3+\delta})$. Because $\hat{f}_n(t) = 0$ when $t > Z_n$ and, for all $t$, $\hat{f}_n(t) \leq \hat{f}_n(0+) = O_p(1)$, the absolute value of $A_{n3}$ is bounded above by

$$\sqrt{n} \int_0^Z \left| \frac{\hat{f}_n(t) - f_0(t)}{3f_n^*(t)^3} \hat{f}_n(t) \right| \, dt \leq O_p(1) \sqrt{n} \int_0^Z \frac{\left| \hat{f}_n(t) - f_0(t) \right|}{3f_n^*(t)^3} \, dt.$$

Although $f_n^*(t)$ is lying between and $\hat{f}_n$ and $f_0$, where for all $t$, $f_0(t) > f_0(1) > 0$, it is not guaranteed that $\hat{f}_n$ is bounded away from zero as well. By Lemma 12.4, we show that $1/\hat{f}_n(1) = O_p(1)$. As a result, we have

$$\sqrt{n} \int_0^Z \frac{\left| \hat{f}_n(t) - f_0(t) \right|}{3f_n^*(t)^3} \, dt \leq \sqrt{n} \int_0^Z \frac{\left| \hat{f}_n(t) - f_0(t) \right|}{3\min\{f_n^*(t)^3, f_0(t)^3\}} \, dt$$

$$\leq O_p(1) \sqrt{n} \int_0^1 \left| \hat{f}_n(t) - f_0(t) \right|^3 \, dt = O_p(n^{-1/3+\delta}),$$

by Lemma 12.5

Combining (i)-(iii), the claim in the lemma follows.

\[\square\]

**Proof of Lemma 12.2** First, from the proof of Lemma 4.2 in [67], we know

$$\int_0^1 \log \hat{f}_n(t) \, d\mathbb{F}_n(t) = \int_0^1 \log \hat{f}_n(t) \, d\hat{\mathbb{F}}_n(t).$$

Therefore,

$$\int_0^1 \log \frac{\hat{f}_n(t)}{f_0(t)} \, d\{\mathbb{F}_n(t) - \hat{\mathbb{F}}_n(t)\} = - \int_0^1 \log f_0(t) \, d\{\mathbb{F}_n(t) - \hat{\mathbb{F}}_n(t)\}. \quad (43)$$
Also, because $F_n$ and $\hat{F}_n$ are both distribution functions over $[0, 1]$, we have

$$\sqrt{n} \int_0^1 \log f_0(0) \, d\{F_n(t) - \hat{F}_n(t)\} = 0.$$ 

Thus,

$$- \int_0^1 \log f_0(t) \, d\{F_n(t) - \hat{F}_n(t)\} = - \int_0^1 \{\log f_0(t) - \log f_0(0)\} \, d\{F_n(t) - \hat{F}_n(t)\}$$

$$= \int_0^1 \int_0^t \frac{|f_0'(w)|}{f_0(w)} \, dw \, d\{F_n(t) - \hat{F}_n(t)\}$$

$$= \int_0^1 \left[ \int_w^1 \{f_0(w) - F_n(t)\} \right] \frac{|f_0'(w)|}{f_0(w)} \, dw$$

$$= \int_0^1 \left\{ \hat{F}_n(w) - F_n(w) \right\} \frac{|f_0'(w)|}{f_0(w)} \, dw, \quad (44)$$

where the third equality holds by Fubini’s theorem. Under the conditions in Theorem 4.1, we can apply Theorem 2.1 in [48] to obtain that

$$n^{1/3} \left[ n^{2/3} \int_0^1 \{\hat{F}_n(w) - F_n(w)\} \frac{|f_0'(w)|}{f_0(w)} \, dw - \kappa_{f_0} \right]$$

converges to a normal distribution with zero mean and a finite variance. The above convergence implies that

$$\int_0^1 \left\{ \hat{F}_n(w) - F_n(w) \right\} \frac{|f_0'(w)|}{f_0(w)} \, dw = \kappa_{f_0} n^{-1/6} + O_p(n^{-1/3}) \quad \text{(45)}$$

where $\kappa_{f_0}$ is defined in Theorem 4.1. The claim in the lemma follows in view of (43), (44) and (45). 

Proof of Theorem 4.1. By Lemmas 12.1 and 12.2

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n \log \frac{\hat{f}_n(X_i)}{f_0(X_i)} = -\sqrt{n} \int_0^1 \frac{|\hat{f}_n(t) - f_0(t)|^2}{2 f_0(t)} \, dt - \kappa_{f_0} n^{-1/6} + O_p(n^{-1/3+\delta}).$$

It remains to show that

$$\sqrt{n} \int_0^1 \frac{|\hat{f}_n(t) - f_0(t)|^2}{2 f_0(t)} \, dt = n^{-1/6} \mu_{2, f_0}^2 + O_p(n^{-1/3+\delta}),$$

which is implied by Lemma 12.6.
12.2 Proof of Theorem 4.2

Proof of Theorem 4.2: Let $X_n = \{X_1, \ldots, X_n\}$. As before, we suppress the dependence of $n$ in $\nu$ in the notation. To prove (9) is equivalent to show that every subsequence $\{n_k\}$ of $\{n\}$ has a further subsequence $\{n_{k_l}\}$ along which (9) holds almost surely instead of in probability. Similar to $T_n$, we can write $T_n^*$ as

$$T_n^* = S_n^* + M_n^* + R_n^*.$$  

Clearly, $M_n^*$ is distribution-free because $(\hat{F}_n(Z_{n1}), \ldots, \hat{F}_n(Z_{nm})) \overset{d}{=} (U_{(1)}, \ldots, U_{(nm)})$, where $U_{ni}$’s are a random sample from the standard uniform distribution. Thus, for the entire sequence $\{n\}$, for almost all $\omega \in \Omega$, $\sqrt{n} \nu^2 \psi_1(\nu)^{-\nu} (M_n^* - \log \nu + \psi(\nu)) \overset{d}{\to} N(0, 1)$ conditional on $X_n(\omega)$ as in the proof of Theorem 2.3.

By conditions (i) and (ii), every subsequence $\{n_k\}$ of $\{n\}$ has a further subsequence $\{n_{k_l}\}$ along which $\sqrt{n_{k_l}} \nu^2 S_{n_{k_l}} = o_{\mathbb{P}^*}(\omega)$ (1) and $\sqrt{n_{k_l}} \nu^2 R_{n_{k_l}}^* = o_{\mathbb{P}^*}(\omega)$ (1) for almost all $\omega \in \Omega$. Hence, along that subsequence $\{n_{k_l}\}$, for almost all $\omega \in \Omega$, we have $\sqrt{n_{k_l}} \nu^2 (T_n^* - \log \nu + \psi(\nu)) \overset{d}{\to} N(0, 1)$ conditional on $X_{n_{k_l}}(\omega)$. The result then follows as $N(0, 1)$ is a continuous distribution. □

13 Bootstrap consistency for particular classes

13.1 Convergence under a sequence of underlying distributions

To establish bootstrap consistency in Section 4, we have to consider the MLE under the setting that the samples are generated from a sequence of underlying distributions defined on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$. To this end, let $X_{n1}, \ldots, X_{nn}$ be a random sample from $f_n$, a deterministic sequence of densities. Let $F_n$ be the distribution function of $f_n$ and $\mathbb{P}_n$ the empirical distribution from $X_{ni}$’s. Denote the corresponding order statistics of $X_{ni}$’s by $Z_{n1} \leq \ldots \leq Z_{nn}$. The $k$-monotone MLE, completely monotone MLE, and log-concave MLE for $f_n$ from the sample $X_{ni}$’s are denoted by $f_{nk}, \hat{f}_{nn,\infty}$ and $\hat{f}_{nn,lc}$, respectively. To prepare for the proofs of the bootstrap consistency in Theorems 13.5 - 13.7, we first provide upper bounds for $-\log f_n(Z_{n1})$ and $-\log f_n(Z_{nn})$ in the following Lemma 13.1 and establish the rates of convergence to 0 of the respective $\hat{f}_{nk}, \hat{f}_{nn,\infty}$ and $\hat{f}_{nn,lc}$ in terms of the log-likelihood ratio to $f_n$ in Lemmas 13.2 to 13.4 when $f_n$ is $k$-monotone, completely monotone and log-concave, respectively. To indicate the dependence of a sample point $\omega$ in the sample space $\Omega$ in different functions, notations such as $Z_n(\omega)$ and $\hat{f}_{nn,k}(\cdot; \omega)$ will also be used.

Lemma 13.1. (a) Let $f_n$ be a sequence of decreasing densities satisfying $\int_{-\infty}^{\infty} f_n^{1+\alpha}(x) dx \leq n^{m_0}$ for some $\alpha, m_0 > 0$ for all sufficiently large $n$. Then,

$$| \log f_n(Z_{n1}) | + | \log f_n(Z_{nn}) | = O_p(\log n).$$

(b) Let $f_n$ be a sequence of log-concave densities satisfying $\int_{-\infty}^{\infty} f_n^{1-\alpha}(x) dx \leq n^{m_0}$ for some $\alpha, m_0 > 0$ and $f_n \leq C$ for all sufficiently large $n$. Then,

$$| \log f_n(Z_{n1}) | + | \log f_n(Z_{nn}) | = O_p(\log n).$$
Proof of Lemma 13.1. (a) Let \( m > m_0 + 1 \). Since \( f_n \) is decreasing,

\[
\mathbb{P}((f_n(Z_{nn}))^{- \alpha} > n^m) = 1 - \mathbb{P}((f_n(Z_{nn}))^{- \alpha} \leq n^m) \\
= 1 - \mathbb{P}(f_n(X_{n1})^{- \alpha} \leq n^m) \\
= 1 - \left[ 1 - \mathbb{P}(f_n(X_{n1})^{- \alpha} > n^m) \right]^n \\
= 1 - \left[ 1 - \mathbb{E}(f_n(X_{n1})^{- \alpha}) \right]^n \\
\leq 1 - \left( 1 - \frac{n^{m_0}}{n^m} \right)^n \\
\leq 1 - \frac{1}{n^2}.
\]

Hence, \( \sum_{n=1}^{\infty} \mathbb{P}((f_n(Z_{nn}))^{- \alpha} > n^m) < \infty \). By the first Borel-Cantelli Lemma, with probability one, for all sufficiently large \( n \), \( f_n(Z_{nn})^{- \alpha} \leq n^m \) and thus \( -\log f_n(Z_{nn}) \leq \frac{m}{\alpha} \log n \). Similar calculation gives

\[
\mathbb{P}((f_n(Z_{nn}))^{\alpha} > n^m) \leq \frac{1}{n^2}.
\]

The same argument gives with probability one, \( \log f_n(Z_{nn}) \leq \frac{m}{\alpha} \log n \) for all sufficiently large \( n \). Therefore, \( \log f_n(Z_{nn}) = O_p(\log n) \). For \( f_n(Z_{n1}) \), we have

\[
\mathbb{P}((f_n(Z_{n1}))^{- \alpha} > n^m) = \mathbb{P}((f_n(X_{n1}))^{- \alpha} > n^m)^n \leq \left[ \mathbb{E}(f_n(X_{n1})^{- \alpha}) \right]^n \\
\leq \left( \frac{n^{m_0}}{n^m} \right)^n \leq \frac{1}{n^2}.
\]

Similarly,

\[
\mathbb{P}((f_n(Z_{n1}))^{\alpha} > n^m) \leq \frac{1}{n^2}.
\]

Using the same argument as above, we obtain \( \log f_n(Z_{n1}) = O_p(\log n) \).

(b) Since \( f_n \leq C \) for all sufficiently large \( n \),

\[
|\log f_n(Z_{n1})| + |\log f_n(Z_{nn})| \leq 2 \max\{ |\log C|, -\log \min(f_n(Z_{n1}), f_n(Z_{nn})) \}. \quad (46)
\]

Fix \( m > m_0 + 1 \). Then

\[
\mathbb{P}(\min\{f_n(Z_{n1}), f_n(Z_{nn})\}^{- \alpha} > n^m) = 1 - \mathbb{P}(\min\{f_n(Z_{n1}), f_n(Z_{nn})\}^{- \alpha} \leq n^m) \\
= 1 - \{ \mathbb{P}(f_n^{- \alpha}(X_{n1}) \leq n^m) \}^n \\
= 1 - \{ 1 - \mathbb{P}(f_n^{- \alpha}(X_{n1}) > n^m) \}^n \\
\leq 1 - \left( 1 - \frac{\mathbb{E}(f_n^{- \alpha}(X_{n1}))}{n^m} \right)^n \\
\leq 1 - \left( 1 - \frac{n^{m_0}}{n^m} \right)^n \\
\leq \frac{1}{n^2}.
\]
where the second equality follows because \( f_n \), being log-concave, is unimodal, and the first inequality follows from Markov’s inequality. Hence, \( \sum_{n=1}^{\infty} \mathbb{P}(\min\{f_n(Z_n), f_n(Z_{nn})\} > n^m) < \infty \). By the first Borel-Cantelli Lemma, with probability one, for all sufficiently large \( n \), \( \min\{f_n(Z_n), f_n(Z_{nn})\} \leq n^m \) and thus \( - \log \min\{f_n(Z_n), f_n(Z_{nn})\} \leq \frac{m}{a} \log n \). Together with (46) the claim of the lemma follows.

\[ \square \]

**Lemma 13.2 (for \( k \)-monotone densities).** Let \( \{M_n\} \) and \( \{b_n\} \) be sequences that can possibly go to \( \infty \). Suppose that \( f_n \in \mathcal{F}_k^{M_n}([0, b_n]) \) for all sufficiently large \( n \). Then,

\[
\frac{1}{n} \sum_{i=1}^{n} \log \frac{\hat{f}_{nn,k}(X_{ni})}{f_n(X_{ni})} = O_p \left( n^{-\frac{2k}{2k+1}} \log(M_n b_n)^{\frac{1}{2k+1}} \right),
\]

where \( \tilde{M}_n \) satisfies \( \tilde{M}_n/M_n \to \infty \).

**Proof of Theorem 13.2** First, note that if \( f_n \in \mathcal{F}_k^{M_n}([0, b_n]) \), then \( \hat{f}_{nn,k}(0+) = O_p(M_n) \). To see this, following the proof of Proposition 6 in [42], we know that

\[
\hat{f}_{nn,k}(0+) \leq k \sup_{t > 0} \frac{F_{nn}(t)}{F_n(t)} f_n(0+) = O_p(M_n),
\]

since

\[
\sup_{t > 0} \frac{F_{nn}(t)}{F_n(t)} = \sup_{0 < t \leq 1} \frac{G_{nn}(t)}{t} = O_p(1)
\]

by Daniels’ theorem (see Theorem 2 on p. 345 in [68]), where

\[
G_{nn}(t) := \frac{1}{n} \sum_{i=1}^{n} I(F_n(X_{ni}) \leq t).
\]

Since \( f_n \in \mathcal{F}_k^{M_n}([0, b_n]) \), it follows that \( Z_{nn} \leq b_n \). By Lemma 3.3 we also know \( \sigma_{f_{nn}} \leq k Z_{nn} \). Hence, \( \sigma_{f_{nn}} \leq k b_n \). Since \( \tilde{M}_n/M_n \to \infty \) by assumption, we then have

\[
\lim_{n \to \infty} \mathbb{P}(\hat{f}_{nn,k} \notin \mathcal{F}_k^{\tilde{M}_n}([0, k b_n])) = \lim_{n \to \infty} \mathbb{P}(\hat{f}_{nn,k}(0+) > \tilde{M}_n) = 0. \tag{47}
\]

Denote

\[
K_{nn,k} := -\frac{1}{n} \sum_{i=1}^{n} \log \frac{\hat{f}_{nn,k}(X_{ni})}{f_n(X_{ni})}.
\]

Let \( \delta_n := n^{-\frac{k}{2k+1}} \log(M_n b_n)^{\frac{1}{2k+1}} \) and \( m_{f_n}(t) := \log \frac{f(t) + f_n(t)}{2} \). Similar to the proof of Theorem 9.1, we have

\[
\mathbb{P}\left(-\frac{\sqrt{n}}{2} K_{nn,k} \geq \sqrt{n} 2^M \delta_n^2 \right) \leq \mathbb{P}\left(G_{nn}(m_{f_{nn}}, n) - \frac{\sqrt{n}}{8} k^2(\hat{f}_{nn}, f_n) \geq \sqrt{n} 2^M \delta_n^2, \hat{f}_{nn,k} \in \mathcal{F}_k^{\tilde{M}_n}([0, k b_n]) \right) + \mathbb{P}(\hat{f}_{nn,k} \notin \mathcal{F}_k^{\tilde{M}_n}([0, k b_n])),
\]

where the last term on the above display goes to 0 as \( n \) goes to \( \infty \) by (47). The rest of the proof is similar to that of Theorem 9.1 and is therefore omitted. \( \square \)
Lemma 13.3 (for completely monotone densities). Let $f_n \in F_\infty$ such that $\lim_{n \to \infty} \mathbb{P}(Z_{n1}^{-1} > \tilde{c}_{1n}) = 0$ and $\lim_{n \to \infty} \mathbb{P}(Z_{n1}^{1+\beta} > \tilde{c}_{2n}) = 0$ for some $\beta > 0$. Then

$$\frac{1}{n} \sum_{i=1}^{n} \log \frac{\hat{f}_{nn,\infty}(X_{ni})}{f_n(X_{ni})} = O_p \left(n^{-\frac{2}{3}} |\log(\tilde{c}_{1n}\tilde{c}_{2n})|^{\frac{1}{3}}\right),$$

Proof of Lemma 13.3. The proof is essentially the same as that of Lemma 10.1 and is therefore omitted.

Lemma 13.4 (for log-concave densities). Let $f_n \in F_{lc}$ with support $[l_n, u_n]$, where $l_n$ and $u_n$ are sequences of real numbers that are decreasing and increasing, respectively. Suppose that there exists $M > 0$ such that $\sup_n \sup_{x \in \mathbb{R}} f_n(x) \leq M$ and for any compact set $S_1 \subset (\lim l_n, \lim u_n)$, there exists $m = m(S_1) > 0$ such that $\liminf_n \inf_{x \in S_1} f_n(x) \geq m$. Furthermore, suppose that there exists $f^* \in F_{lc}$ such that $f^*(x) \leq e^{-a_0|x|+b_0}$, for some $a_0 > 0$ and $b_0 \in \mathbb{R}$, and for any $a \in (0, a_0)$,

$$\lim_{n \to \infty} \int_{\mathbb{R}} e^{a|x|} |f_n(x) - f^*(x)| \, dx = 0. \quad (48)$$

Then:

(a) There exists a constant $C > 0$ such that

$$\mathbb{P} \left( \limsup_{n \to \infty} \sup_{x \in \mathbb{R}} \hat{f}_{nn,lc}(x) \leq C \right) = 1.$$

(b) For any compact set $S_2 \subset (\lim l_n, \lim u_n)$, there exists a constant $c = c(S_2) > 0$ such that

$$\mathbb{P} \left( \liminf_{n \to \infty} \inf_{x \in S_2} \hat{f}_{nn,lc}(x) \geq c \right) = 1.$$

(c)

$$\frac{1}{n} \sum_{i=1}^{n} \log \frac{\hat{f}_{nn,lc}(X_{ni})}{f_n(X_{ni})} = O_p(n^{-4/5}),$$

which is the same as the rate when $f_n = f_0$ for all $n$, where $f_0$ is a fixed log-concave density.

The main idea of the proof of Lemma 13.4 follows from the proof for Lemma 3 in [25]. In our case, we need to consider the lower and upper bounds of the log-concave MLE from a random sample from a sequence of log-concave densities instead from a fixed density.

Proof of Lemma 13.4. (a) Let $g(x) = \exp(-|x|+b)$ for $x \in \mathbb{R}$, where $b$ is the normalization constant such that $g$ is a density. First, note that for any $a > 0$, $\lim_{x \to \infty} |x|/e^{a|x|} = 0.$
Therefore, \( \{ x : |x| > e^{|x|} \} \) is a bounded subset of \( \mathbb{R} \) and \( \sup \{ |x| : |x| > e^{|x|} \} < \infty \). Now, for some \( a \in (0, a_0) \), we have
\[
\int_{\mathbb{R}} |x| f_n(x) \, dx = \int_{\{x : |x| < e^{|x|}\}} |x| f_n(x) \, dx + \int_{\{x : |x| > e^{|x|}\}} |x| f_n(x) \, dx
\leq \int_{\mathbb{R}} e^{|x|} f_n(x) \, dx + \int_{\{x : |x| > e^{|x|}\}} |x| f_n(x) \, dx
\leq \int_{\mathbb{R}} e^{|x|} f_n(x) \, dx + C_1,
\]
where \( C_1 := M \sup \{ |x| : (\log |x|)/|x| > a \} \lambda(\{ x : (\log |x|)/|x| > a \}) \) and \( \lambda \) is the Lebesgue measure. Thus,
\[
\int_{\mathbb{R}} f_n(x) \log g(x) \, dx = -\int_{\mathbb{R}} |x| f_n(x) \, dx + b
\geq -\int_{\mathbb{R}} e^{|x|} f_n(x) \, dx - C_1 + b.
\]
As a consequence of (48), we have
\[
\liminf_{n \to \infty} \int_{\mathbb{R}} f_n \log g \geq -\int_{\mathbb{R}} e^{|x|} f^\ast(x) \, dx - C_1 + b =: q + 1,
\]
say, with \( f^\ast \) as specified in the statement of the lemma and \( \int_{\mathbb{R}} e^{|x|} f^\ast(x) \, dx < \infty \) because \( a \in (0, a_0) \). Now, let \( \tilde{M} := e^{M_2} \) where \( M_2 \) is large enough such that \( M_2 > q + 1 \) and such that \( \int_D f_n \leq 1/4 \) whenever \( D \subset \mathbb{R} \) such that \( \lambda(D) \leq 2^4(M_2 - q) e^{-M_2} \). This is possible because \( f_n \leq M \) for all \( n \). Let \( f \) be any log-concave density with \( \sup_{x \in \mathbb{R}} f(x) = \tilde{M} \). We claim that, for all sufficiently large \( n \), the log-concave density \( g \) has larger in value of the log-likelihood. Equivalently, we shall show that
\[
\mathbb{P}\left( \frac{1}{n} \sum_{i=1}^{n} \log f(X_{ni}) > \frac{1}{n} \sum_{i=1}^{n} \log g(X_{ni}) \ i.o. \right) = 0,
\]
where i.o. stands for infinitely often. Observe that
\[
\mathbb{P}\left( \frac{1}{n} \sum_{i=1}^{n} \log f(X_{ni}) > \frac{1}{n} \sum_{i=1}^{n} \log g(X_{ni}) \ i.o. \right)
\leq \mathbb{P}\left( \frac{1}{n} \sum_{i=1}^{n} \log g(X_{ni}) < q \ i.o. \right) + \mathbb{P}\left( \frac{1}{n} \sum_{i=1}^{n} \log f(X_{ni}) > q \ i.o. \right).
\]
Because of (50),
\[
\mathbb{P}\left( \frac{1}{n} \sum_{i=1}^{n} \log g(X_{ni}) < q \ i.o. \right) \leq \mathbb{P}\left( \frac{1}{n} \sum_{i=1}^{n} \log g(X_{ni}) - \int_{\mathbb{R}} f_n \log g < -1 \ i.o. \right)
\]
We now claim that the right-hand side of (53) is 0. First, using similar bounds as in (49), it is easy to see that \( \sup_{n,i} \mathbb{E}\{\log g(X_{ni}) - \int f_n \log g\}^4 < \infty \). For example, to show that \( \mathbb{E}\{\log g(X_{ni})\}^4 \leq \infty \), we have for some \( a \in (0,a_0) \),

\[
\mathbb{E}\{\log g(X_{ni})\}^4 \leq 4 \int_{\mathbb{R}} |x|^4 f_n(x) \, dx + 4|b|^4 \\
\leq 4 \int_{\mathbb{R}} e^{a|x|} f_n(x) \, dx + 4M \sup \{ |x| : |x|^4 > e^{a|x|} \} \lambda(\{ x : |x|^4 > e^{a|x|} \}) + 4|b|^4.
\]

To see the bound of the last inequality is finite, simply note that by (48), \( \lim_n \int_{\mathbb{R}} e^{a|x|} f_n(x) \, dx = \int_{\mathbb{R}} e^{a|x|} f^*(x) \, dx < \infty \) for \( a \in (0,a_0) \). Let \( \{Y_{ni}\}_{i=1,\ldots,n,n \in \mathbb{N}} \) be a triangular array of scalar random variables \( Y_{ni} := \log g(X_{ni}) - \int f_n \log g \). Then, the row \( Y_{n1}, \ldots, Y_{nn} \) is a collection of independent random variables with the mean 0 and \( \sup_{i=1,\ldots,n,n} \mathbb{E}(Y_{ni}^4) < \infty \). Hence, by a strong law of large numbers for triangular arrays,

\[
\frac{1}{n} \sum_{i=1}^n \log g(X_{ni}) - \int f_n \log g = \frac{1}{n} \sum_{i=1}^n Y_{ni} \overset{a.s.}{\longrightarrow} 0
\]

and so the right-hand side of (53) is 0. The proof that shows the second term on the right-hand side of (52) equals 0 is similar to the corresponding proof in Lemma 3(a) in (65) (p.262) with \( X_i \) replaced by \( X_{ni} \), \( f_0 \) replaced by \( f_n \) and the fact that Hoeffding’s inequality is valid for each \( n \), and is therefore omitted.

(b) Let \( S \) be a compact subset of the interval \( (\lim_n l_n, \lim_n u_n) \) and \( \delta > 0 \) be small enough that \( S^\delta := \{ x \in \mathbb{R} : \text{dist}(x,S) \leq \delta \} \subset (\lim_n l_n, \lim_n u_n) \). Let \( f \) be any log-concave density on \( \mathbb{R} \). We claim that if \( c := 2 \inf_{x \in S} f(x) \) is sufficiently small but positive, then \( f \) cannot be the NPMLE for any large \( n \) with probability one, that is, (51) holds. Indeed, by Lemma 13.4 (a), we can assume that \( \sup_{x \in \mathbb{R}} f(x) \leq C \). If \( B \subset \mathbb{R} \) contains a ball of radius \( \delta/2 \) centered at a point in \( S^{\delta/2} \), say \( B_{\delta/2}(x_0) \), where \( B_{\delta/2}(x_0) \) is the closed ball of radius \( \delta \) centered at \( x_0 \), for some \( x_0 \in S^{\delta/2} \), then

\[
\lim inf_n \int_B f_n \geq \lim inf_n \int_{B_{\delta/2}(x_0)} f_n \geq \lim inf_n \inf_{x \in S^{\delta/2}} \int_{B_{\delta/2}(x)} f_n \geq \delta \lim inf \inf_{x \in S^{\delta}} f_n := p > 0.
\]

Recall the density \( g \) defined in (a) and the constant \( q \) defined in (50); suppose that \( c \in (0,C] \) is small enough such that

\[
\frac{p}{2} \log c + \left(1 - \frac{p}{2}\right) \log C \leq q.
\]

Denote \( B := \{ x \in S^\delta : f(x) \leq c \} \). Then, \( B \) contains a ball of radius \( \delta/2 \) centered at a point in \( S^{\delta/2} \), so \( \lim inf_n \int_B f_n \geq p \). Then

\[
\mathbb{P}\left( \frac{1}{n} \sum_{i=1}^n \log f(X_{ni}) > q \right) \leq \mathbb{P}\left( \frac{1}{n} \sum_{i=1}^n I\{X_{ni} \in B\} \leq \frac{p}{2} \right) \leq e^{-np^2/2},
\]

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by (54) and Hoeffding’s inequality. By the first Borel-Cantelli lemma,
\[ \mathbb{P}\left( \frac{1}{n} \sum_{i=1}^{n} \log f(X_{ni}) > q \right) = 0. \]

Finally, arguing as in the proof of Lemma 13.4 (a) above (see (52) and (53)), we have (51).

(c) Without loss of generality, we can assume that \([-1, 1]\) is strictly inside \((\lim l_n, \lim u_n)\). By parts (a) and (b) in this lemma, we have, with a probability going to 1 that \(\hat{f}_{nn,lc} \in \mathcal{F}_{lc}^{M}\) for some finite \(M > 0\), where \(\mathcal{F}_{lc}^{M}\) is defined in (8). With the above result and the result from Theorem 3.1 in [46] that \(\log N_{[\varepsilon, \mathcal{F}_{lc}^{M}, h]} \lesssim \varepsilon^{-1/2}\), the proof of the claim regarding the rate of convergence to 0 in this part is similar to that for the \(k\)-monotone case and is therefore omitted.

\[ \square \]

### 13.2 Bootstrap Consistency

In this subsection, we establish the bootstrap consistency of the NPLRT for \(k\)-monotone densities, completely monotone densities, and log-concave densities under mild regularity conditions.

**Theorem 13.5** (Bootstrap Consistency for \(k\)-monotone Densities). Fix \(k \in \mathbb{N}\). Suppose that \(f_0\) is a density that may not be bounded from above or have a bounded support. Without loss of generality, assume that \(\tau_{f_0} = 0\). Suppose that \(f_0\) satisfies Conditions (A) and (B). Let \(\nu = O(n^{1/3}(\log n)^{-1})\). Regardless of whether \(f_0\) is \(k\)-monotone or not,

\[
\sup_{x \in \mathbb{R}} \left| \mathbb{P}^* \left( \frac{n \nu}{\sqrt{2 \psi_1(\nu)}} \left( T_n - \log \nu + \psi(\nu) \right) \leq x \right) - \mathbb{P}(Z \leq x) \right| \to 0,
\]

where \(Z \sim N(0, 1)\) and \(\hat{f}_n\) in the bootstrap procedure is the \(k\)-monotone MLE \(\hat{f}_{n,k}\).

**Proof of Theorem 13.5.** First, from the proof of Lemma 9.3 we have \(Z^{-1} = o_p(n^{m_0})\) and \(Z_n = o_p(n^{m_0})\) for some \(m_0 > 0\).

In view of Theorem 4.2 it suffices to verify Conditions (i) and (ii) in Theorem 4.2.

Let \(M_n = n^{m_0}, \tilde{b}_n = n^{m_0}, \text{ and } \tilde{M}_n = n^{2m_0}\). Recall that by Lemma 3.1 and 3.3 we have \(\hat{f}_{n,k}(0+) \leq kZ^{-1}n\) and \(\sigma_{\hat{f}_n} \leq kZ_n\). Thus,

\[
\mathbb{P}(\hat{f}_{n,k} \notin \mathcal{F}_{k}^{M_n}([0, \tilde{b}_{n}])) \leq \mathbb{P}(\hat{f}_{n,k}(0+) > M_n) + \mathbb{P}(\hat{f}_{n}(< \tilde{b}_{n})) \leq \mathbb{P}(kZ^{-1} > M_n) + \mathbb{P}(kZ > \tilde{b}_n) \to 0,
\]

as \(n\) goes to \(\infty\) by assumptions. This implies that \(I(\hat{f}_{n,k} \notin \mathcal{F}_{k}^{M_n}([0, \tilde{b}_{n}])) \to 0\). The last result together with the assumption that \(Z_n = o_p(n^{m_0})\) imply that for any subsequence \(\{n_k\}\) of \(\{n\}\), there exists a further subsequence \(\{n_{k_l}\}\) and \(\Omega_0\) such that \(\mathbb{P}(\Omega_0) = 1\) and for all \(\omega \in \Omega_0\), \(\hat{f}_{n_{k_l}, k}(\cdot; \omega) \in \mathcal{F}_{k}^{M_{k_l}}([0, \tilde{b}_{n_{k_l}}])\) and \(Z_{n_{k_l}}(\omega)/n_{k_l}^{m_0} \leq 1\) for all large enough \(l\). Therefore, Lemma 13.2 implies that Condition (i) is satisfied as

\[
\sqrt{n\nu n^{-\frac{2k}{2k+1}}} \log(M_n \tilde{b}_n)^{\frac{1}{2k+1}} = o(1),
\]

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for all \( k \in \mathbb{N} \) under \( \nu = O(\nu = O(n^{1/3}/\log n)) \).

In the following, the dependence on \( \omega \) is not always written explicitly for the sake of notational simplicity. We now verify Condition (ii) in Theorem 4.2. Write

\[
R_{nk_l}^* = R_{1nk_l}^* + R_{2nk_l}^*,
\]

where

\[
R_{1nk_l}^* = -\frac{1}{n_{k_l}} \sum_{j=0, \ldots, n_{k_l}^{-1} - 1} \sum_{l=1}^{\nu} \log \frac{\hat{f}_{nk_l,k}(Z_{j\nu+l+1})}{\hat{F}_{nk_l,k}(Z_{(j+1)\nu+1})},
\]

\[
R_{2nk_l}^* = -\frac{1}{n} \log \frac{\hat{f}_{n,k}(Z_1)}{\hat{f}_{n,k}^{R,*}(Z_1)}.
\]

For \( R_{1nk_l}^* \), by the mean value theorem, there exists \( \hat{Z}_j^* \) lying between \( Z_{j\nu+1}^* \) and \( Z_{(j+1)\nu+1}^* \) such that

\[
\hat{F}_{nk_l,k}(Z_{j\nu+1}^*) - \hat{F}_{nk_l,k}(Z_{(j+1)\nu+1}) = \hat{f}_{nk_l,k}(\hat{Z}_j^*)(Z_{(j+1)\nu+1}^* - Z_{j\nu+1}^*)
\]

and so

\[
R_{1nk_l}^* = -\frac{1}{n_{k_l}} \sum_{j=0, \ldots, n_{k_l}^{-1} - 1} \sum_{l=1}^{\nu} \log \frac{\hat{f}_{nk_l,k}(Z_{j\nu+l+1}^*)}{\hat{f}_{nk_l,k}(Z_{j}^*)}.
\]

By the monotonicity of \( \hat{f}_{nk_l,k} \),

\[
|n_{k_l}R_{1nk_l}^*| \leq \left| \sum_{j=0, \ldots, n_{k_l}^{-1} - 1} \sum_{l=1}^{\nu} \log \frac{\hat{f}_{nk_l,k}(Z_{j\nu+l+1})}{\hat{f}_{nk_l,k}(Z_{(j+1)\nu+1})} \right| = \nu \left| \log \hat{f}_{nk_l,k}(Z_1^*) - \log \hat{f}_{nk_l,k}(Z_{n_{k_l}}^*) \right|
\]

\[
\leq \nu (| \log \hat{f}_{nk_l,k}(Z_1^*)| + | \log \hat{f}_{nk_l,k}(Z_{n_{k_l}}^*)|).
\]

We now apply Lemma 13.1 (a) with \( f_{nk_l} = \hat{f}_{nk_l,k} \). To verify the condition in that lemma, note that \( \int_0^\infty \hat{f}_{nk_l,k}^{1+\alpha}(x) dx \leq k \int_{nk_l}^{1+\alpha}(0) Z_{nk_l} \leq k M_{nk_l}^{1+\alpha} b_{nk_l} = kn_{k_l}^{m_1(1+\alpha)+m_0} \leq n_{k_l}^{m_0(1+\alpha)+m_0+1} \) for some \( \alpha \in (0, 1) \) and for all large \( l \). By Lemma 13.1 (a), we have \( | \log \hat{f}_{nk_l,k}(Z_1^*)| + | \log \hat{f}_{nk_l,k}(Z_{n_{k_l}}^*)| = O_{\mathbb{F}_2}(\log n_{k_l}) \). Therefore,

\[
|\sqrt{n_{k_l}}\nu R_{1nk_l}^*| \leq \frac{\nu^{3/2}}{\sqrt{n_{k_l}}} O_{\mathbb{F}_2}(\log n_{k_l}) = o_{\mathbb{F}_2}(1)
\]

when \( \nu = O(n^{1/3}/\log n) \). Following the proof of Lemma 8.5, we have

\[
|n_{k_l}R_{2nk_l}^*| \leq | \log \hat{f}_{nk_l,k}(Z_1^*)| + | \log \hat{f}_{nk_l,k}(Z_{n_{k_l}}^*)| + | \log (\hat{F}_{nk_l,k}(Z_{(j+1)\nu+1}) - \hat{F}_{nk_l,k}(Z_{j\nu+1}))| + \log \frac{n_{k_l} - 1}{\nu},
\]

which is \( O_{\mathbb{F}_2}(\log n_{k_l}) \) as in (25). Thus, Condition (ii) in Theorem 4.2 is satisfied. 

\[\square\]
Theorem 13.6 (Bootstrap Consistency for Completely Monotone Densities). Suppose that $f_0$ is a density that may not be bounded from above or have a bounded support. Without loss of generality, assume that $\tau_{f_0} = 0$. Suppose that $f_0$ satisfies Conditions (A) and (B). Let $\nu = O(n^{1/3}(\log n)^{-1})$. Regardless of whether $f_0$ is completely monotone or not,

$$
\sup_{x \in \mathbb{R}} \left[ \mathbb{P}^* \left( \sqrt{\frac{n\nu}{\nu^2 \psi_1(\nu)}} (T_n^* - \log \nu + \psi(\nu)) \leq x \right) - \mathbb{P}(Z \leq x) \right] \xrightarrow{P} 0,
$$

where $Z \sim N(0, 1)$ and $\tilde{f}_n$ in the bootstrap procedure is the completely monotone MLE $\hat{f}_{n, \infty}$.

Proof of Theorem 13.6. First, from the proof of Lemma 9.3 we have $Z_t^{-1} = o_p(n^{m_0})$ and $Z_n = o_p(n^{m_0})$ for some $m_0 > 0$.

Since $Z_t^{-1} = o_p(n^{m_0})$ and $Z_n = o_p(n^{m_0})$, for any subsequence $\{n_k\}$ of $\{n\}$, there exists a further subsequence $\{n_{k_l}\}$ and $\Omega_0$ such that $\mathbb{P}(\Omega_0) = 1$ and for all $\omega \in \Omega_0$, $Z_t^{-1}(\omega)/n_{k_l} \to 0$ and $Z_{n_{k_l}}(\omega)/n_{k_l} \to 0$. Now, note that the distribution function of the completely monotone MLE $\hat{F}_{n, \infty}(t)$ is given by $1 - \sum_{i=1}^{r_n} \hat{p}_i e^{-\lambda_i t}$. Thus,

$$
1 - e^{-\frac{t}{n}} \leq 1 - \max_{i=1,\ldots,r_n} e^{-\lambda_i t} \leq \hat{F}_n(t) \leq 1 - \min_{i=1,\ldots,r_n} e^{-\lambda_i t} \leq 1 - e^{-\frac{t}{n}}.
$$

Hence, we have

$$
\mathbb{P}^*_\omega (\{Z_{n_{k_l}, 1}^* > t\}^{-1}) = 1 - \mathbb{P}^*_\omega (\{Z_{n_{k_l}, 1}^* \leq t\})^{-1} = 1 - \mathbb{P}^*_\omega (\{X_{n_{k_l}, 1}^* \leq t\})^{n_{k_l}} = 1 - \mathbb{P}^*_\omega \left( X_{n_{k_l}, 1}^* > \frac{1}{t} \right)^{n_{k_l}} = 1 - \{1 - \hat{F}_{n_{k_l}}(1/t; \omega)\}^{n_{k_l}} \leq 1 - e^{-\frac{t}{Z_{n_{k_l}}(\omega)}}
$$

and

$$
\mathbb{P}^*_\omega (Z_{n_{k_l}, n_{k_l}}^* > t) = 1 - \mathbb{P}^*_\omega (X_{n_{k_l}, 1}^* \leq t)^{n_{k_l}} \leq 1 - \left(1 - e^{-\frac{t}{Z_{n_{k_l}}(\omega)}}\right)^{n_{k_l}}.
$$

Therefore, as $l$ goes to infinity,

$$
\mathbb{P}^*_\omega (\{Z_{n_{k_l}, 1}^* > n_{k_l}^{m_0+1}\}) \leq 1 - e^{-\frac{Z_{n_{k_l}}^{-1}(\omega)}{n_{k_l}}} \to 0
$$

and

$$
\mathbb{P}^*_\omega (Z_{n_{k_l}, n_{k_l}}^* > n_{k_l}^{m_0+1}) \leq 1 - \left(1 - \exp \left(-\frac{n_{k_l}^{m_0}}{Z_{n_{k_l}}(\omega)} n_{k_l}\right)\right)^{n_{k_l}} \to 1 - e^0 = 0.
$$

We now apply Lemma 13.3 with $f_{n_{k_l}} = \hat{f}_{n_{k_l}, \infty}(\cdot; \omega)$, $\bar{c}_{1n_{k_l}} = n_{k_l}^{m_0+1}$ and $\bar{c}_{2n_{k_l}} = n_{k_l}^{m_0+1/\gamma+\beta}$. Therefore, Condition (i) in Theorem 4.2 is satisfied.

To verify Condition (ii) in Theorem 4.2, similar to the $k$-monotone case, it suffices to show that

$$
|\log \hat{f}_{n_{k_l}, \infty}(Z_1^*)| + |\log \hat{f}_{n_{k_l}, \infty}(Z_{n_{k_l}, n_{k_l}}^*)| = O_{\mathbb{P}^*_\omega}(\log n).
$$
To apply Lemma \[13.1\] (a), note that \( \hat{f}_{n_k} \approx (0) \leq Z_1^{-1} \) and for all large \( l \), \( \hat{f}_{n_k} \approx x^2 \) for all \( x \geq Z_{n_k}^{1+\beta} (\hat{\omega}) \) (see the proof of Lemma \[10.1\]). Hence, for all large \( l \),
\[
\int_0^\infty \hat{f}_{n_k}^{1+\alpha} (x; \hat{\omega}) dx = \int_0^\infty \hat{f}_{n_k}^{1+\alpha} (x; \hat{\omega}) dx + \int_0^\infty \hat{f}_{n_k}^{1+\alpha} (x; \hat{\omega}) dx \leq Z_1^{-(1+\alpha)} (\hat{\omega}) Z_{n_k}^{1+\beta} (\hat{\omega}) + \int_0^\infty \frac{1}{\nu(1+2\alpha)} dx \leq n_k^{-m_0(1+\alpha)} + m_0(1+\beta) + \frac{1}{Z_{n_k}^{1+\beta}(1+2\alpha)} (\hat{\omega}) \leq n_k^{-m_0(1+\alpha)} + m_0(1+\beta) + 1,
\]
as \( Z_{1}^{(1+\beta)(1+2\alpha)} (\hat{\omega}) > 1 \) for all large \( l \). Therefore, we can apply Lemma \[13.1\] (a) and (56) is satisfied.

**Theorem 13.7** (Bootstrap Consistency for Log-concave Densities). **Regardless of whether** \( f_0 \) **is log-concave or not, suppose that** \( \int_{\mathbb{R}} x |f_0(x) dx < \infty \) **and** \( \nu = O(n^{1/3} (\log n)^{-1}) \). **We have**
\[
\sup x \in \mathbb{R} \left[ \mathbb{P}^* \left( \nu^2 \psi_1 (\nu) - \nu \left(T_n^* - \log \nu + \psi(\nu) \right) \leq x \right) - \mathbb{P}(Z \leq x) \right] \xrightarrow{p} 0,
\]
where \( Z \sim N(0, 1) \) and \( \hat{f}_n \) in the bootstrap procedure is the log-concave MLE \( \hat{f}_n \).

**Proof of Theorem 13.7** In view of Theorem \[4.2\], it suffices to verify Conditions (i) and (ii) in Theorem \[4.2\]. First, note that the support of \( f_{n,lc} \) is \([Z_1, Z_n]\). We now claim that \( Z_n - Z_1 = o_p(n^2) \). Indeed, note that \( |Z_n - Z_1| \leq |Z_n| + |Z_1| \leq 2 \max_{i=1,\ldots,n} |X_i| \) and for any \( \delta > 0 \),
\[
\mathbb{P} \left( \max_{i=1,\ldots,n} |X_i| > \delta n^2 \right) = 1 - \left( 1 - \mathbb{P}(|X_i| > \delta n^2) \right)^n \leq 1 - \left( 1 - \frac{\mathbb{E}(|X_i|)}{\delta n^2} \right)^n \rightarrow 0,
\]
as \( n \rightarrow \infty \), where the inequality follows from Markov’s inequality. Hence, \( Z_n - Z_1 = o_p(n^2) \), which then implies that for any subsequence \( \{n_k\} \) of \( \{n\} \), there exists a further subsequence \( \{n_k\} \) and \( \Omega_0 \) such that \( \mathbb{P}(\Omega_0) = 1 \) and for any \( \omega \in \Omega_0 \), \( \{Z_{n_k}(\omega) - Z_1(\omega)\} n_k^2 = o(n_k^2) \). Now, fix a \( \omega \in \Omega_0 \). Because \( \int_{\mathbb{R}} x |f_0(x) dx < \infty \), Lemma 3 and Theorem 4 in \[65\] imply that \( \hat{f}_{n,lc}(\cdot; \omega) \) satisfies all the conditions in Lemma \[13.4\] for all large enough \( n \) (where without loss of generality, we assume that the sets with probability one from Lemma 3 and Theorem 4 in \[65\] contain \( \Omega_0 \)). Hence, Lemma \[13.4\] (c) implies that Condition (i) is satisfied.

In the following, the dependence on \( \omega \) is not always written explicitly for the sake of notational simplicity. We now verify Condition (ii) in Theorem \[4.2\]. Write
\[
R_{n_k}^* = R_{1n_k}^* + R_{2n_k}^*,
\]
where
\[
R_{1n_k}^* := -\frac{1}{n_k} \sum_{j=0,\ldots,n_k^{-1}} \sum_{l=1}^\nu \log \frac{\hat{f}_{n_k,lc}(Z_{j+l+1}^*) (Z_{j+l+1}^* - Z_{j+l}^*)}{\hat{F}_{n_k,lc}(Z_{j+l+1}^*) - \hat{F}_{n_k,lc}(Z_{j+l}^*)},
\]
\[
R_{2n}^* := -\frac{1}{n} \log \frac{\hat{f}_{n,lc}(Z_1)}{f_{n,lc}(Z_1)}.
\]

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For $R^*_{1n_{ki}}$, by the mean value theorem, there exists $\tilde{Z}_j^*$ lying between $Z_{j+1}^*$ and $Z_j^*$ such that $\hat{F}_{n_{ki},lc}(Z_{(j+1)\nu+1}) - \hat{F}_{n_{ki},lc}(Z_{j\nu+1}) = \hat{f}_{n_{ki},lc}(\tilde{Z}_j^*)(Z_{(j+1)\nu+1} - Z_j^*)$ and so

$$R^*_{1n_{ki}} = -\frac{1}{n_{ki}} \sum_{j=0}^{n_{ki}-1} \sum_{\nu=1}^{\nu} \log \frac{\hat{f}_{n_{ki},lc}(Z_{j\nu+1})}{\hat{f}_{n_{ki},lc}(Z_j^*)}.$$ 

Since $\hat{f}_{n_{ki},lc}$ is log-concave, it is unimodal. Let $s_{n_{ki}}$ be its mode. Let $j^*$ be such that $s_{n_{ki}} \in [Z_{j^*\nu+1}, Z_{(j^*+1)\nu+1}]$. We have

$$|n_{ki}R^*_{1n_{ki}}| \leq \left| \sum_{j=0}^{n_{ki}-1} \sum_{\nu=1}^{\nu} \log \frac{\hat{f}_{n_{ki},lc}(Z_{(j+1)\nu+1})}{\hat{f}_{n_{ki},lc}(Z_{j\nu+1})} \right| + \left| \sum_{j=j^*+1}^{n_{ki}-1} \sum_{\nu=1}^{\nu} \log \frac{\hat{f}_{n_{ki},lc}(Z_{(j+1)\nu+1})}{\hat{f}_{n_{ki},lc}(Z_{j\nu+1})} \right|$$

$$+ \left| \sum_{\nu=1}^{\nu} \log \frac{\hat{f}_{n_{ki},lc}(s_{n_{ki}})}{\hat{f}_{n_{ki},lc}(Z_{j\nu+1})} \right| + \left| \sum_{\nu=1}^{\nu} \log \frac{\hat{f}_{n_{ki},lc}(Z_{j^*\nu+1})}{\hat{f}_{n_{ki},lc}(s_{n_{ki}})} \right|$$

$$= \nu \left| \log \hat{f}_{n_{ki},lc}(s_{n_{ki}}) - \log \hat{f}_{n_{ki},lc}(Z_{j^*}) \right| + \nu \left| \log \hat{f}_{n_{ki},lc}(s_{n_{ki}}) - \log \hat{f}_{n_{ki},lc}(Z_{n}) \right|$$

$$\leq 2\nu \left| \log \hat{f}_{n_{ki},lc}(s_{n_{ki}}) \right| + \nu \left( \left| \log \hat{f}_{n_{ki},lc}(Z_{j^*}) \right| + \left| \log \hat{f}_{n_{ki},lc}(Z_{n}) \right| \right).$$

Note that for all large $n$, we know $\hat{f}_{n_{ki},lc} \leq C$ for some constant $C$; see [25]. We now verify the condition in Lemma [13.1] (b),

$$\int_{-\infty}^{\infty} \hat{f}_{n_{ki},lc}^{1-\alpha}(x) \, dx \leq C^{1-\alpha}(\sigma_{f_{n_{ki}}} - \tau_{f_{n_{ki}}}) = C^{1-\alpha}(Z_{n_{ki}}(\omega) - Z_1(\omega)) = o(n_{ki}^2).$$

Thus, by Lemma [13.1] (b), we have

$$|n_{ki}R^*_{1n_{ki}}| = O_{\tilde{\mathcal{P}}}(\log n_{ki}).$$

The proof of $|n_{ki}R^*_{2n_{ki}}| = O_{\tilde{\mathcal{P}}}(\log n_{ki})$ is similar to that in the proof of Theorem [13.5] and is therefore omitted. Thus, Condition (ii) in Theorem [4.2] is satisfied. 

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