Reconstruction of solutions to a generalized Moisil-Teodorescu system in Jordan domains with rectifiable boundary

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Abstract

In this paper we consider the problem of reconstructing solutions to a generalized Moisil-Teodorescu system in Jordan domains of $\mathbb{R}^3$ with rectifiable boundary. In order to determine conditions for existence of solutions to the problem we embed the system in an appropriate generalized quaternionic setting.

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1 Introduction

Quaternionic analysis is a function theory that offers a natural and elegant extension of classic complex analysis in the plane to the quaternion skew-field (denoted by $\mathbb{H}$) generated by a real unit $1$ and the imaginary units $i, j, k$. This analysis relies heavily on results on quaternion-valued functions that are defined in domains of $\mathbb{R}^s, s = 3, 4$ and that are null solutions of generalized Cauchy-Riemann or Dirac operator [25, 30].

Nowadays, quaternionic analysis has proven to be a good tool to study numerous mathematical models of spatial physical phenomena related to many different extensions of the Moisil-Teodorescu system, see [21, 13, 11, 24, 17, 6, 31].

The use of a general orthonormal basis $\psi := \{\psi^1, \psi^2, \psi^3\} \in \mathbb{H}^3$, called structural set, instead of the standard basis in $\mathbb{R}^3$, and introducing a generalized Moisil-Teodorescu system associated to $\psi$ are the cornerstone of a generalized quaternionic analysis.

The pioneers works on this theory, centered around the concept of $\psi$-hyperholomorphic functions defined on domains in $\mathbb{R}^3$ (or $\mathbb{R}^4$), were proposed independently by Naser [26], Nono [19, 20], Shapiro and Vasilevsky [32, 33]. More information about the importance of the structural sets can be found in [14, 23, 12, 8, 1, 2, 7].

The structural set $\psi^\theta := \{i, ie^{i\theta}j, e^{i\theta}j\}$, for $0 \leq \theta \leq 2\pi$ fixed and its associated operator

$$\psi^\theta D := \frac{\partial}{\partial x_1}i + \frac{\partial}{\partial x_2}ie^{i\theta}j + \frac{\partial}{\partial x_3}e^{i\theta}j$$
are used in [9] to study solvability conditions for a non-homogeneous generalized Moisil-
Teodorescu system. To achieve our goals, we consider the homogeneous case of such
Moisil-Teodorescu system.

Let $\Omega \subset \mathbb{R}^3$ be a Jordan domain (see [18]) with rectifiable boundary $\Gamma$ (see [3]), i.e., $\Gamma$ is
the image of some bounded subset of $\mathbb{R}^2$ under a Lipschitz mapping, and let $0 \leq \theta \leq 2\pi$.

\begin{equation}
\begin{aligned}
\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} - \frac{\partial f_3}{\partial x_3} &= 0, \\
\frac{\partial f_2}{\partial x_3} - \frac{\partial f_3}{\partial x_1} &= 0, \\
\frac{\partial f_3}{\partial x_2} - \frac{\partial f_1}{\partial x_1} - \frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_2} - \frac{\partial f_2}{\partial x_3} - \frac{\partial f_3}{\partial x_3} &= 0.
\end{aligned}
\end{equation}

(1.1)

The system (1.2) is better known in the following form:

\begin{equation}
\begin{aligned}
\text{div} \vec{f} &= 0, \\
\text{rot} \vec{f} &= 0,
\end{aligned}
\end{equation}

(1.3)

where $\vec{f} = f_1 \mathbf{i} + f_2 \mathbf{j} + f_3 \mathbf{k}$. Whenever $f_1 \mathbf{i} + f_2 \mathbf{j} + f_3 \mathbf{k}$ is a solution of (1.1), then $f_1 \mathbf{i} + f_3 \mathbf{j} + f_2 \mathbf{k}$ is so of (1.3) when $\theta = 0$ is taken.

- **The homogeneous Cimmino system** (see [10]).

A particular case of the homogeneous Cimmino system is given by:

\begin{equation}
\begin{aligned}
\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} - \frac{\partial f_3}{\partial x_3} &= 0, \\
\frac{\partial f_2}{\partial x_3} - \frac{\partial f_3}{\partial x_1} &= 0, \\
\frac{\partial f_3}{\partial x_2} - \frac{\partial f_1}{\partial x_1} - \frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_2} - \frac{\partial f_2}{\partial x_3} - \frac{\partial f_3}{\partial x_3} &= 0.
\end{aligned}
\end{equation}

(1.4)

where $\vec{f} = f_1 \mathbf{i} + f_2 \mathbf{j} + f_3 \mathbf{k}$ and each function $f_m$, depends only of $(x_1, x_2, x_3)$. This case is obtained from (1.1) for $\theta = \frac{\pi}{2}$.
\* The Riesz system (see [16, 14, 28]).

Let \( f : (x_0, x_1, x_2) \in \mathbb{R}^3 \rightarrow \text{span}_{\mathbb{R}^3}\{1, i, j\} \)

\[
\begin{align*}
\frac{\partial f_0}{\partial x_0} - \frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_2} &= 0, \\
\frac{\partial f_0}{\partial x_1} + \frac{\partial f_1}{\partial x_0} &= 0, \\
\frac{\partial f_0}{\partial x_2} + \frac{\partial f_2}{\partial x_0} &= 0, \\
\frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_1} &= 0,
\end{align*}
\]

which is equivalent with the so-called Riesz system

\[
\begin{align*}
\text{div} \bar{f} &= 0, \\
\text{rot} \bar{f} &= 0,
\end{align*}
\]

where \( \bar{f} := f_0 - f_1i - f_2j \). If \( f_1i + f_2j + f_3k \) is a solution of (1.1), then \( f_1 + f_3i + f_2j \) is so of the inhomogeneous system (1.6) for \( \theta = \pi \).

\* Another particular case.

\[
\begin{align*}
\frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} &= 0, \\
\frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_1} + \frac{\partial f_3}{\partial x_3} &= 0, \\
\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3} &= 0, \\
\frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3} &= 0.
\end{align*}
\]

To get the system (1.7), take \( \theta = \frac{3\pi}{2} \) in (1.1). This example is related to the time-harmonic relativistic Dirac bispinor theory (see [21, 27]).

The purpose of this work is to study the problem of reconstruction of solutions to the generalized Moisil-Teodorescu system (1.1), which is formulated as follows: Given a continuous three-dimensional vector field \( f : \Gamma \rightarrow \mathbb{C}^3 \), under which conditions can \( f \) be decomposed on \( \Gamma \) as a sum:

\[
f(t) = f^+(t) + f^-(t), \quad \forall t \in \Gamma,
\]

where \( f^\pm \) are extendable to vector fields \( F^\pm \) that satisfy the generalized Moisil-Teodorescu system (1.1) in respectively \( \Omega_+ := \Omega \) and \( \Omega_- := \mathbb{R}^3 \setminus \{\Omega \cup \Gamma\} \), with \( f^-(\infty) = 0 \).

In [3], is considered the problem of reconstruction of solutions to the Div-rot system (1.2) by using quaternionic analysis tools. Our results extend the achievements of [3] to the aforementioned variety of systems.

The integral

\[
\psi \mathcal{K}_\Gamma[f](x) := \int_\Gamma \mathcal{K}_\phi(x - \xi)\langle \nu_\phi(\xi), f(\xi) \rangle dS_\xi + \int_\Gamma \mathcal{K}_\phi(x - \xi), |\nu_\phi(\xi), f(\xi)| dS_\xi,
\]

with \( x \in \Omega_+ \), plays the role of the Cauchy transform in the theory of three dimensional continuous vector fields \( f \), where \( dS_\xi \) is the two-dimensional surface area element on \( \Gamma \) and \( \nu_\phi(\xi) := \sum_{k=1}^3 \psi^k \nu_k(\xi) \) is outward pointing normal vector to \( \Gamma \). The function \( \mathcal{K}_\phi \) plays a similar role in the \( \psi \)-hyperholomorphic function theory as the Cauchy kernel does in complex analysis.

Similarly, the singular Cauchy transform is defined for \( t \in \Gamma \) as

\[
\psi \mathcal{S}_\Gamma[f](t) := \int_\Gamma \mathcal{K}_\phi(t - \tau)\langle \nu_\phi(\tau), (f(\tau) - f(t)) \rangle dS_\tau + \int_\Gamma \mathcal{K}_\phi(t - \tau), |\nu_\phi(\tau), (f(\tau) - f(t))| dS_\tau + f(t),
\]

with \( x \in \Omega_+ \), plays the role of the Cauchy transform in the theory of three dimensional continuous vector fields \( f \), where \( dS_\xi \) is the two-dimensional surface area element on \( \Gamma \) and \( \nu_\phi(\xi) := \sum_{k=1}^3 \psi^k \nu_k(\xi) \) is outward pointing normal vector to \( \Gamma \). The function \( \mathcal{K}_\phi \) plays a similar role in the \( \psi \)-hyperholomorphic function theory as the Cauchy kernel does in complex analysis.
where the integral is being understood in the sense of the Cauchy principal value.
We shall use the following notation:

\[
\mathcal{M}_\psi := \left\{ f : \int_\Gamma (\mathcal{K}_\psi (x - \xi), [\nu_\psi (\xi), f(\xi)]) dS_\xi = 0, x \not\in \Gamma \right\}, \tag{1.11}
\]

\[
\mathcal{M}_\psi^* := \left\{ f : \int_\Gamma (\mathcal{K}_\psi (x - \xi), [\nu_\psi (\xi), f(\xi)]) dS_\xi = 0, x \in \Gamma \right\}. \tag{1.12}
\]

The set \( \mathcal{M}_\psi^* \) can be described in purely physical terms (see [34]).

After this brief introduction let us give a description of the structure of the paper. Section 2 presents some preliminaries on the \( \psi \)-hyperholomorphic function theory. Section 3 is devoted to the study the Cauchy transform \( \psi^\ast \mathcal{K}_G[f] \). Section 4 contains a pair of generalizations of the results presented in [3, Theorem 3.3 and 3.4]. In Section 5 our main results are stated and proved.

2 Preliminaries

2.1 Basics of \( \psi \)-hyperholomorphic functions theory

Let \( \mathbb{H} := \mathbb{H}(\mathbb{R}) \) and \( \mathbb{H}(\mathbb{C}) \) denote the sets of real and complex quaternions respectively. If \( a \in \mathbb{H} \) or \( a \in \mathbb{H}(\mathbb{C}) \), then \( a = a_0 + a_1 i + a_2 j + a_3 k \), where the coefficients \( a_k \in \mathbb{R} \) if \( a \in \mathbb{H} \) and \( a_k \in \mathbb{C} \) if \( a \in \mathbb{H}(\mathbb{C}) \). The symbols \( i, j, \) and \( k \) denote different imaginary units, i.e. \( i^2 = j^2 = k^2 = -1 \) and they satisfy the following multiplication rules \( ij = -ji = k; \ jk = -kj = i; \ ki = -ik = j \). The (complex) imaginary unit \( i \in \mathbb{C} \) commutes with every quaternionic unit imaginary.

It is known that \( \mathbb{H} \) is a skew-field and \( \mathbb{H}(\mathbb{C}) \) is an associative, non-commutative complex algebra with zero divisors.

If \( a \in \mathbb{H} \) or \( a \in \mathbb{H}(\mathbb{C}) \), \( a \) can be represented as \( a = a_0 + \tilde{a} \), with \( \tilde{a} = a_1 i + a_2 j + a_3 k \), \( Sc(a) := a_0 \) is called the scalar part and \( Vec(a) := \tilde{a} \) is called the vector part of the quaternion \( a \). Also, if \( a \in \mathbb{H}(\mathbb{C}) \), \( a \) can be represented as \( a = \alpha_1 + i\alpha_2 \) with \( \alpha_1, \alpha_2 \in \mathbb{H} \).

Let \( a, b \in \mathbb{H}(\mathbb{C}) \), the product between these quaternions can be calculated by the formula:

\[
ab = a_0b_0 - \langle \tilde{a}, \tilde{b} \rangle + a_0\tilde{b} + b_0\tilde{a} + [\tilde{a}, \tilde{b}], \tag{2.1}
\]

where

\[
\langle \tilde{a}, \tilde{b} \rangle := \sum_{k=1}^{3} a_kb_k, \quad \langle \tilde{a}, \tilde{b} \rangle := \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}. \tag{2.2}
\]

We define the conjugate of \( a = a_0 + \tilde{a} \in \mathbb{H}(\mathbb{C}) \) by \( \overline{a} := a_0 - \tilde{a} \).

The Euclidean norm of a quaternion \( a \in \mathbb{H} \) is the number \( |a| \) given by:

\[
|a| = \sqrt{a\overline{a}} = \sqrt{\overline{a}a}. \tag{2.3}
\]

We define the quaternionic norm of \( a \in \mathbb{H}(\mathbb{C}) \) by:

\[
|a|_c := \sqrt{|a_0|_c^2 + |a_1|_c^2 + |a_2|_c^2 + |a_3|_c^2}, \tag{2.4}
\]

where \( |a_k|_c \) denotes the complex norm of each component of the quaternion \( a \). The norm of a complex quaternion \( a = \alpha_1 + i\alpha_2 \) with \( \alpha_1, \alpha_2 \in \mathbb{H} \) can be rewritten in the form

\[
|a|_c = \sqrt{|\alpha_1|^2 + |\alpha_2|^2}. \tag{2.5}
\]

If \( a \in \mathbb{H} \), \( b \in \mathbb{H}(\mathbb{C}) \), then

\[
|ab|_c = |a||b|_c. \tag{2.6}
\]

If \( a \in \mathbb{H}(\mathbb{C}) \) is not a zero divisor then \( a^{-1} := \frac{a}{a\overline{a}} \) is the inverse of \( a \).
2.2 Notations

- A complex quaternionic valued function \( f : \Omega \to \mathbb{H}(\mathbb{C}) \) will be expressed as

\[
    f := f_0 + f_1 i + f_2 j + f_3 k.
\]

If \( f_0 \equiv 0 \) in \( \Omega \), \( f \) will be called a vector field (written \( f = f \)).

- We say that \( f \) has properties in \( \Omega \) such as continuity and real differentiability of order \( p \) whenever all \( f_j \) have these properties. These spaces are usually denoted by \( C^p(\Omega, \mathbb{H}(\mathbb{C})) \) with \( p \in (\mathbb{N} \cup \{0\}) \).

- Throughout this work, \( Lip_\mu(\Omega, \mathbb{H}(\mathbb{C})), 0 < \mu \leq 1 \), denotes the set of Hölder continuous functions defined on \( \Omega \) with values on \( \mathbb{H}(\mathbb{C}) \) and Hölder exponent \( \mu \).

As defined in [21], consider on \( C^1(\Omega, \mathbb{H}(\mathbb{C})) \) an operator \( \psi D \) by the formula

\[
    \psi D[f] := \sum_{k=1}^{3} \psi^k \frac{\partial f}{\partial x_k},
\]

where \( \psi := \{\psi^1, \psi^2, \psi^3\}, \psi^k \in \mathbb{H} \). Denote \( \overline{\psi} := \{\overline{\psi^1}, \overline{\psi^2}, \overline{\psi^3}\} \). Then, the equality

\[
    \psi D \overline{\psi} = \overline{\psi} D\psi = \Delta_3,
\]

is true if and only if

\[
    \psi^j \psi^k + \psi^k \psi^j = 2 \delta_{jk},
\]

for \( j, k \in \{1, 2, 3\} \).

Any set of real quaternions \( \psi := \{\psi^1, \psi^2, \psi^3\}, \psi^k \in \mathbb{H} \) with the property (2.10) is called structural set.

Similarly (see [21]), on \( C^1(\Omega, \mathbb{H}(\mathbb{C})) \) is defined an operator \( D\psi \)

\[
    D\psi[f] := \sum_{k=1}^{3} \frac{\partial f}{\partial x_k} \psi^k.
\]

**Definition 2.1.** [21]. We say that \( f \in C^1(\Omega, \mathbb{H}(\mathbb{C})) \) is left (right)-\( \psi \)-hyperholomorphic in \( \Omega \) if \( \psi D[f](x) = 0 \) \((D\psi[f](x) = 0)\) for all \( x \in \Omega \).

It is known that a quaternionic valued function can be left-\( \psi \)-hyperholomorphic but no right-\( \psi \)-hyperholomorphic and vice-versa; or can be left-right-\( \psi \)-hyperholomorphic.

The function

\[
    \mathcal{K}_\psi(x) := \frac{1}{4\pi} \frac{(x)_\psi}{|x|^3}, \quad x \in \mathbb{R}^3 \setminus \{0\},
\]

where

\[
    (x)_\psi := \sum_{k=1}^{3} x_k \psi^k,
\]

is a left-right-\( \psi \)-hyperholomorphic fundamental solution of \( \psi D \) in \( x \in \mathbb{R}^3 \setminus \{0\} \). Observe that \( |(x)_\psi| = |x| \) for all \( x \in \mathbb{R}^3 \).

Consider the special case of the structural set \( \psi := \psi^\theta := \{i, ie^{i\theta} j, e^{i\theta} j\} \), for \( 0 \leq \theta \leq 2\pi \) fixed, then the operator \( \psi^\theta D \) takes the form

\[
    \psi^\theta D := \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} ie^{i\theta} j + \frac{\partial}{\partial x_3} e^{i\theta} j.
\]

We define the following partial operators for \( f \in C^1(\Omega, \mathbb{H}(\mathbb{C})) \):

\[
    \psi^\theta \text{div}[\vec{f}] := \frac{\partial f_1}{\partial x_1} + \left( \frac{\partial f_2}{\partial x_2} - \frac{\partial f_3}{\partial x_3} \right) ie^{i\theta},
\]

(2.15)
\[
\psi^0 \text{grad}[f_0] := \frac{\partial f_0}{\partial x_1} i + \frac{\partial f_0}{\partial x_2} i e^{i\theta} j + \frac{\partial f_0}{\partial x_3} e^{i\theta} j, 
\]
\[
\psi^0 \text{rot}[\vec{f}] := \left( -\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3} \right) e^{i\theta} + \left( -\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1} \right) j + \left( \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} i e^{i\theta} \right) k.
\]

If we apply \( \psi^D \) to \( f \in C^1(\Omega, \mathbb{H}(\mathbb{C})) \) we obtain
\[
\psi^D[f] = -\psi^0 \text{div}[\vec{f}] + \psi^0 \text{grad}[f_0] + \psi^0 \text{rot}[\vec{f}],
\]
which implies that \( \psi^D[f] = 0 \) is equivalent to
\[
-\psi^0 \text{div}[\vec{f}] + \psi^0 \text{grad}[f_0] + \psi^0 \text{rot}[\vec{f}] = 0.
\]
For \( f_0 = 0 \), (2.19) is reduced to
\[
-\psi^0 \text{div}[\vec{f}] + \psi^0 \text{rot}[\vec{f}] = 0.
\]

The following Lemma will be used in the proof of our main result.

**Lemma 2.3.** Let \( f = f_0 + \vec{f} \in C^1(\Omega, \mathbb{H}(\mathbb{C})) \). Define \( f^w := X E_0(\vec{f}) \), where \( X \) is the characteristic function in \( \Omega \cup \Gamma \) and \( E_0(\vec{f}) \) is the Whitney operator (see [29]) applied to \( \vec{f} \).

**Proof.** The proof is based on the fact that (2.20) and (2.25) are equivalent to (1.1).

**Definition 2.2.** A continuously differentiable vector field \( f \) defined in \( \Omega \) is said to be a \( \psi^\theta \)-Laplacian vector field if \( f \) is left-right-\( \psi^\theta \)-hyperholomorphic in \( \Omega \).

2.3 Whitney’s extension theorem

The following results are generalizations of those presented in [4, Theorem 4.1, Proposition 4.1], which is due to the fact that the proofs make no appeal to which structural set is assumed.

**Theorem 2.4.** Let \( f \in \text{Lip}_\theta(\Gamma, \mathbb{H}(\mathbb{R})) \). Define \( f^w := X E_0(\vec{f}) \), where \( X \) is the characteristic function in \( \Omega \cup \Gamma \) and \( E_0(\vec{f}) \) is the Whitney operator (see [29]) applied to \( f \). Then
The singular Cauchy transform is defined by
\[ \psi_2.4 \text{ Integral operators for } \psi^\theta \text{-hyperholomorphic function theory} \]

a) \( f^w \in \text{Lip}_p(\Omega \cup \Gamma, \mathbb{H}(\mathbb{R})) \).
b) \( |\frac{\partial f^w}{\partial x_i}(x)| \leq c(\text{dist}(x, \Gamma))^\mu-1 \).
c) \( |\psi^\theta Df^w(x)| \leq c_1(\text{dist}(x, \Gamma))^\mu-1 \),

where \( c, c_1 \) are constants.

Proposition 2.5. Let \( f \in \text{Lip}_p(\Gamma, \mathbb{H}(\mathbb{R})) \). Then \( \psi^\theta D[f^w] \in L_p(\mathbb{R}^3, \mathbb{H}(\mathbb{R})) \) for \( p < \frac{1}{1-\mu} \).

2.4 Integral operators for \( \psi^\theta \)-hyperholomorphic function theory

Now we enunciate the most essential integral formulas for the theory of \( \psi^\theta \)-hyperholomorphic functions, starting with the quaternionic Stokes’ formula (see [21]), which is a consequence of the Stokes’ theorem in real analysis.

Let \( f, g \in C^1(\Omega, \mathbb{H}(\mathbb{C})) \cap C(\Omega \cup \Gamma, \mathbb{H}(\mathbb{C})) \). Then

\[ \int_\Gamma g(\xi)\nu^{\psi^\theta}(\xi)f(\xi)dS_\xi = \int_\Omega [D^{\psi^\theta}[g](\xi)f(\xi) + g(\xi)f^\psi Df(\xi)]d\xi. \] (2.26)

Definition 2.6. [21]. Let \( f \) be a continuous function defined on \( \Omega \cup \Gamma \). The quaternionic Cauchy and Teodorescu transforms of \( f \) are defined by

\[ \psi^\theta T[f](x) := \int_\Omega \mathcal{K}_{\psi^\theta}(x - \xi)f(\xi)d\xi, \quad x \in \mathbb{R}^3, \] (2.27)

\[ \psi^\theta K_\Gamma[f](x) := -\int_\Gamma \mathcal{K}_{\psi^\theta}(x - \xi)\nu^{\psi^\theta}(\xi)f(\xi)dS_\xi, \quad x \in \mathbb{R}^3 \setminus \Gamma. \] (2.28)

The singular Cauchy transform is defined by

\[ \psi^\theta \mathcal{S}_\Gamma[f](t) := 2\psi^\theta \Phi[f](t) + f(t), \quad t \in \Gamma, \] (2.29)

where

\[ \psi^\theta \Phi[f](t) := \lim_{r \to 0} \int_{\Gamma \setminus \Gamma_{t,r}} \mathcal{K}_{\psi^\theta}(t - \tau)\nu^{\psi^\theta}(\tau)(f(\tau) - f(t))dS_\tau, \quad t \in \Gamma, \] (2.30)

with

\[ \Gamma_{t,r} := \{ \xi \in \Gamma : |t - \xi| \leq r \}. \] (2.31)

In a similar way, the operators \([f]^{\psi^\theta} K_\Gamma\) and \([f]^{\psi^\theta} \mathcal{S}_\Gamma\) are defined (the quaternionic Cauchy kernel appears on the right side of the integral and the normal vector is placed between the function \( f \) and the kernel \( \mathcal{K}_{\psi^\theta} \)). It is worth pointing out that meanwhile \( \psi^\theta K_\Gamma[f] \) is left-\( \psi^\theta \)-hyperholomorphic, \([f]^{\psi^\theta} K_\Gamma \) is right-\( \psi^\theta \)-hyperholomorphic.

If \( f \in C^1(\Omega, \mathbb{H}(\mathbb{C})) \cap C(\Omega \cup \Gamma, \mathbb{H}(\mathbb{C})) \), the Borel Pompieu formula is true

\[ \psi^\theta K_\Gamma[f](x) + \psi^\theta T[\psi^\theta D[f]](x) = \begin{cases} f(x) & \text{if } x \in \Omega_+, \\ 0 & \text{if } x \in \Omega_. \end{cases} \] (2.32)

Under the above assumptions, if moreover \( f \) is left-\( \psi^\theta \)-hyperholomorphic in \( \Omega \), from \( (2.32) \) we get the Cauchy integral formula

\[ \psi^\theta K_\Gamma[f](x) = f(x), \quad x \in \Omega. \] (2.33)

Theorem 2.7. Let \( f \in C^1(\Omega, \mathbb{H}(\mathbb{C})) \cap C(\Omega \cup \Gamma, \mathbb{H}(\mathbb{C})) \). Then

\[ \psi^\theta D[\psi^\theta T[f]](x) := \begin{cases} f(x) & \text{if } x \in \Omega_+, \\ 0 & \text{if } x \in \Omega_. \end{cases} \] (2.34)
**Theorem 2.8.** Let \( f \in L_p(\Omega, \mathbb{H}(\mathbb{R})) \), for \( p > 3 \). Then \( \psi^\theta T[f] \) satisfy the following inequalities
\[
|\psi^\theta T[f](x)| \leq C_1(\Omega, p)||f||_{L_p},
\]
\[
|\psi^\theta T[f](x) - \psi^\theta T[f](x')| \leq C_2(\Omega, p)||f||_{L_p}|x - x'|^{\frac{\nu-3}{\nu}}, \quad x \neq x',
\]
where
\[
||f||_{L_p} := \left( \int_{\Omega} |f|^p(\xi)d\mu(\xi) \right)^{\frac{1}{p}}.
\]

**Proof.** Applying a analogous reasoning to that used in [3, Theorem 2.3.2] and the fact that \(|(x)_{\psi^\theta}| = |x|\), the result is obtained. 

**Theorem 2.9.** Let \( f \in C(\Gamma, \mathbb{H}(\mathbb{C})) \) and suppose that
\[
\Psi_\Gamma[f](t) = \frac{1}{4\pi} \lim_{\delta \to 0} \int_{\Gamma \setminus \Gamma, \delta} \frac{|f(\xi) - f(t)|_{2\nu}dS_\xi}{|\xi - t|^2},
\]
exists uniformly with respect to \( t \in \Gamma \). Then there exist the singular integral \( \psi^\theta \Phi[f] \) and the Cauchy transform \( \psi^\theta K[f] \) satisfies the Sokhotski-Plemelj formulas
\[
\psi^\theta K^\pm[f](t) := \lim_{\Omega_{\pm, \delta} \ni t} \psi^\theta K_\Gamma[f](x) = \frac{1}{2}(\psi^\theta \mathcal{H}_\Gamma[f](t) \pm f(t)), \quad t \in \Gamma.
\]

**Proof.** Reasoning as in [3, Theorem 3.1] and applying that \(|(x)_{\psi^\theta}| = |x|\) for all \( x \in \mathbb{R}^3 \) we get the proof.

### 3 The Cauchy integral formula: vector fields case

In this section we analyze the quaternionic Cauchy transform and the Cauchy integral formula restricted to the vector fields case.

Let \( f \) be a vector field defined in \( \Gamma \), then the Cauchy transform of \( f \) is given by:
\[
\psi^\theta K_\Gamma[f](x) = \int_\Gamma \mathcal{H}_{\psi^\theta}(x - \xi)\langle \nu_{\psi^\theta}(\xi), f(\xi) \rangle dS_\xi + \int_\Gamma \langle \mathcal{H}_{\psi^\theta}(x - \xi), [\nu_{\psi^\theta}(\xi), f(\xi)] \rangle dS_\xi +
- \int_\Gamma [\mathcal{H}_{\psi^\theta}(x - \xi), [\nu_{\psi^\theta}(\xi), f(\xi)]]dS_\xi.
\]

(3.1)

We can rewrite \( \psi^\theta K_\Gamma[f] \) as
\[
\psi^\theta K_\Gamma[f](x) = Sc(\psi^\theta K[f](x)) + Vcc(\psi^\theta K[f](x)),
\]
where
\[
Sc(\psi^\theta K_\Gamma[f](x)) = \int_\Gamma [\mathcal{H}_{\psi^\theta}(x - \xi), [\nu_{\psi^\theta}(\xi), f(\xi)]]dS_\xi,
\]
(3.3)

\[
Vcc(\psi^\theta K_\Gamma[f](x)) = \int_\Gamma \mathcal{H}_{\psi^\theta}(x - \xi)\langle \nu_{\psi^\theta}(\xi), f(\xi) \rangle dS_\xi +
- \int_\Gamma [\mathcal{H}_{\psi^\theta}(x - \xi), [\nu_{\psi^\theta}(\xi), f(\xi)]]dS_\xi.
\]

(3.4)

From the previous observation we can see that in general for a vector field \( f \) the Cauchy transform is not a purely vectorial complex quaternion.

**Remark 3.1.** If \( f \in \mathcal{M}_{\psi^\theta} \) the Cauchy transform is expressed as follows:
\[
\psi^\theta K_\Gamma[f](x) = \int_\Gamma \mathcal{H}_{\psi^\theta}(x - \xi)\langle \nu_{\psi^\theta}(\xi), f(\xi) \rangle dS_\xi - \int_\Gamma [\mathcal{H}_{\psi^\theta}(x - \xi), [\nu_{\psi^\theta}(\xi), f(\xi)]]dS_\xi.
\]

(3.5)
In the next corollary the Cauchy integral formula is presented for the case of left-$\psi^\theta$-hyperholomorphic vector fields.

**Corollary 3.2.** Let $f$ be a $\psi^\theta$-Laplacian vector field in $\Omega \cup \Gamma$. Then

$$
\int_\Gamma \mathcal{K}_{\psi^\theta}(x - \xi)(\nu_{\psi^\theta}(\xi), f(\xi))dS_\xi - \int_\Gamma [\mathcal{K}_{\psi^\theta}(x - \xi), [\nu_{\psi^\theta}(\xi), f(\xi)]]dS_\xi = f(x),
$$

(3.6)

$x \in \Omega$.

**Proof.** The proof of this theorem is obtained from the Cauchy integral formula. $\Box$

**Theorem 3.3.** Let $f \in \mathcal{M}_{\psi^\theta}$ be a continuous vector field and let the integral

$$
\Psi_\Gamma[f](t) := \frac{1}{4\pi \delta} \lim_{\delta \to 0} \int_{\Gamma \setminus \Gamma_\delta} \frac{|f(\xi) - f(t)|^2}{|\xi - t|^2} dS_\xi.
$$

(3.7)

exists uniformly with respect to $t \in \Gamma$. Then, for every $t \in \Gamma$ there exists the Cauchy singular integral $\psi^\theta \mathcal{S}_\Gamma[f]$, the Cauchy transform $\psi^\theta K_\Gamma[f]$ has continuous limit values on $\Gamma$ and the following analogues of Sokhotski-Plemelj formulas hold:

$$
\psi^\theta K_\Gamma[f](t) := \lim_{\delta \to 0} \psi^\theta K_\Gamma[f](x) = \frac{1}{2} (\psi^\theta \mathcal{S}_\Gamma[f](t) \pm f(t)), \ t \in \Gamma.
$$

(3.8)

The class of continuous vector fields that satisfy (3.7) is denoted by $\mathcal{D}(\Gamma, \mathbb{H}(\mathbb{C}))$.

**Proof.** Let $f \in \mathcal{M}_{\psi^\theta} \cap \mathcal{D}(\Gamma, \mathbb{H}(\mathbb{C}))$, then the Cauchy transform $\psi^\theta K_\Gamma[f] := \psi^\theta K_\Gamma[f]$ is a complex quaternionic function with null scalar part. Now, we only apply the Theorem 2.39 to the function $f := f$ to get the result. $\Box$

### 4 Boundary values of the Cauchy transform

The following theorems provide two sets of assertions equivalent with the decomposition given by (1.8).

**Theorem 4.1.** Let $f \in \mathcal{D}(\Gamma, \mathbb{H}(\mathbb{C})) \cap \mathcal{M}_{\psi^\theta}$ a vector field. Then the following statements are equivalent:

1. The vector field $f$ admits on $\Gamma$ a decomposition of the form (1.8).
2. The left-$\psi^\theta$-hyperholomorphic transform $\psi^\theta K_\Gamma[f]$ is right-$\psi^\theta$-hyperholomorphic in $\mathbb{R}^3 \setminus \Gamma$.
3. The vector field $Vcc(\psi^\theta K_\Gamma[f])$ is a $\psi^\theta$-Laplacian vector field.
4. $Sc(\psi^\theta K_\Gamma[f]) = 0$ in $\mathbb{R}^3$.

**Proof.** We give only the proof of 3 $\Rightarrow$ 4. The remaining implications can be proved using similar arguments as in [3, Theorem 3.3]. To this end, let us consider $\psi^\theta K_\Gamma[f] = Sc(\psi^\theta K_\Gamma[f]) + Vcc(\psi^\theta K_\Gamma[f])$. By hypothesis, $Vcc(\psi^\theta K_\Gamma[f])$ is a $\psi^\theta$-Laplacian vector field on $\mathbb{R}^3 \setminus \Gamma$. Acting $\psi^\theta D$ on $Sc(\psi^\theta K_\Gamma[f])$ gives $\psi^\theta \text{grad} \left[ Sc(\psi^\theta K_\Gamma[f]) \right](x) = 0$ for all $x \in \mathbb{R}^3 \setminus \Gamma$.

For abbreviation, we write $L$ instead of $Sc(\psi^\theta K_\Gamma[f])$. It is a simple matter to show that

$$
\psi^\theta \text{grad}[L](x) = \frac{\partial L}{\partial x_1}(x)i + \frac{\partial L}{\partial x_2}(x)ke^\theta j + \frac{\partial L}{\partial x_3}(x)e^\theta j =
$$

$$
= \frac{\partial L}{\partial x_1}(x)i + \left( \frac{\partial L}{\partial x_3}(x) \cos(\theta) - \frac{\partial L}{\partial x_2}(x) \sin(\theta) \right) j + \left( \frac{\partial L}{\partial x_3}(x) \sin(\theta) + \frac{\partial L}{\partial x_2}(x) \cos(\theta) \right) k = 0,
$$

(3.9)
is equivalent to the system
\[
\begin{align*}
\frac{\partial L}{\partial x_1}(x) &= 0, \\
\frac{\partial L}{\partial x_3}(x)\cos(\theta) - \frac{\partial L}{\partial x_2}(x)\sin(\theta) &= 0, \\
\frac{\partial L}{\partial x_3}(x)\sin(\theta) + \frac{\partial L}{\partial x_2}(x)\cos(\theta) &= 0.
\end{align*}
\]
whose solution is such that \( \frac{\partial L}{\partial x_3} = 0 \) in \( \mathbb{R}^3 \), \( s \in \{1, 2, 3\} \). This implies that \( \text{Sc}(\psi^\theta K_\Gamma[f]) \) is constant in \( \Omega_+ \). Moreover, as \( f \in \mathcal{D}(\Gamma, \mathbb{H}(\mathbb{C})) \), in view of the Sokhotski-Plemelj formulas
\[
\lim_{\Omega_+ \ni x \to t} \text{Sc}(\psi^\theta K_\Gamma[f])(x) = \lim_{\Omega_- \ni x \to t} \text{Sc}(\psi^\theta K_\Gamma[f])(x),
\]
which implies that \( \text{Sc}(\psi^\theta K_\Gamma[f]) \) is continuous in \( \mathbb{R}^3 \) and since \( \psi^\theta K_\Gamma[f] \) vanish at infinity \( 4 \) holds.\[\square\]

**Theorem 4.2.** Let \( f \in \mathcal{D}(\Gamma, \mathbb{H}(\mathbb{C})) \), then the following assertions are equivalent:

1. The Cauchy transform \( \psi^\theta K_\Gamma[f] \) is right-\( \psi^\theta \)-hyperholomorphic in \( \mathbb{R}^3 \setminus \Gamma \).
2. \( f \in \mathcal{M}^\psi \).
3. \( \psi^\theta \mathcal{A}[f] = [f] \psi^\theta \mathcal{A} \).

**Proof.** The proof is obtained reasoning as in \[3\] Theorem 3.4. \[\square\]

## 5 Main result

Before beginning the proof of our main result we give some important remarks.

Let \( f \) be a \( \psi^\theta \)-Laplacian vector field. Applying \( \psi^\theta D \) and \( D\psi^\theta \) to \( f \) we obtain that equations
\[
\begin{align*}
\psi^\theta D[f] &= -\psi^\theta \text{div}[f] + \psi^\theta \text{rot}[f] = 0, \\
D\psi^\theta[f] &= -\psi^\theta \text{div}[f] + \psi^\theta \text{rot}[f] = 0,
\end{align*}
\]
are equivalent. In other words, the class of \( \psi^\theta \)-Laplacian vector fields is identified with the set of purely vectorial \( \psi^\theta \)-hyperholomorphic functions.

By definition of \( \mathcal{M}^\psi \) and \( \mathcal{M}^\psi_{\psi^\theta} \), we have
\[
\mathcal{D}(\Gamma, \mathbb{H}(\mathbb{C})) \cap \mathcal{M}^\psi := \left\{ f \in \mathcal{D}(\Gamma, \mathbb{H}(\mathbb{C})) : \int_\Gamma \langle \mathcal{K}^\psi(x - \xi), [\nu^\psi(\xi), f(\xi)] \rangle dS_\xi = 0, x \notin \Gamma \right\},
\]
\[
\mathcal{D}(\Gamma, \mathbb{H}(\mathbb{C})) \cap \mathcal{M}^\psi_{\psi^\theta} := \left\{ f \in \mathcal{D}(\Gamma, \mathbb{H}(\mathbb{C})) : \int_\Gamma \langle \mathcal{K}^\psi(x - \xi), [\nu^\psi(\xi), f(\xi)] \rangle dS_\xi = 0, x \in \Gamma \right\}.
\]

**Theorem 3.3** now shows that
\[
2 \lim_{\Omega_+ \ni x \to t} \text{Sc}(\psi^\theta K_\Gamma[f])(x) = 2 \text{Sc}(\psi^\theta K_\Gamma^{+\Delta}[f])(t) = 2 \text{Sc}(\psi^\theta \mathcal{A}[f])(t),
\]
which implies that
\[
\text{if } f \in \mathcal{D}(\Gamma, \mathbb{H}(\mathbb{C})) \cap \mathcal{M}^\psi, \text{ then } f \in \mathcal{D}(\Gamma, \mathbb{H}(\mathbb{C})) \cap \mathcal{M}^\psi_{\psi^\theta}.
\]

Now we are in conditions to state and proof our main result.
**Theorem 5.1.** Let \( \Omega \) be a Jordan domain in \( \mathbb{R}^3 \) with rectifiable boundary \( \Gamma \). Let \( f \in Lip_p(\Gamma, \mathbb{H}(\mathbb{C})) \cap \mathcal{M}_{\psi} \). Then, for \( \frac{2}{3} < \mu \leq 1 \) the vector field \( f \) admits on \( \Gamma \) a unique decomposition of the form \( (1.8) \).

**Proof.** The proof will be divided into two steps.

**Existence:** Let \( f \in Lip_p(\Gamma, \mathbb{H}(\mathbb{C})) \cap \mathcal{M}_{\psi} \). Inspired by the Borel Pompieu formula, we consider a new kind of Cauchy transform given by:

\[
\psi^\theta K_{\hat{T}}^\psi f(x) := -\psi^\theta T[\psi^\theta D[f^w]](x) + f^w(x), \quad x \in \mathbb{R}^3 \setminus \Gamma.
\]

We can write \( f^w = f_{1}^w + if_{2}^w \), by Proposition 2.5 it follows that \( \psi^\theta D[f_{1,2}^w] \in L_p(\Omega, \mathbb{H}) \) for \( p < \frac{1}{1 - \mu} \). The condition \( \frac{2}{3} < \mu \) implies that \( \psi^\theta D[f_{1,2}^w] \in L_p(\Omega, \mathbb{H}) \) for some \( p > 3 \). Due the fact that

\[
-\psi^\theta T[\psi^\theta D[f^w]](x) = -\psi^\theta T[\psi^\theta D[f_1^w]](x) - i\psi^\theta T[\psi^\theta D[f_2^w]](x), \quad x \in \mathbb{R}^3 \setminus \Gamma.
\]

by Theorem 2.8 the integrals on the right side of (5.2) exist and their sum represent a continuous function in \( \mathbb{R}^3 \). Hence, \( \psi^\theta K_{\hat{T}}^\psi f \) exists and possesses continuous extensions to the closures of the domains \( \Omega_{\pm} \). Consequently, the problem (1.8) has a solution given by

\[
F^+(x) := -\psi^\theta T[\psi^\theta D[f^w]](x) + f^w(x),
\]

\[
F^-(x) := \psi^\theta T[\psi^\theta D[f^w]](x).
\]

The difference between the limit values of \( \psi^\theta K_{\hat{T}}^\psi f(x) \) in \( \Gamma \) approaching \( x \) from \( \Omega_{\pm} \) is equal to

\[
F^+(t) + F^-(t) = f^w(t) = f(t), \quad \forall t \in \Gamma.
\]

The vector fields \( F^\pm \) are left-\( \psi^\theta \)-hyperholomorphics in \( \Omega_{\pm} \) and each has null scalar part, therefore \( F^\pm \) are right-\( \psi^\theta \)-hyperholomorphics in \( \Omega_{\pm} \).

**Uniqueness:** Since \( F^\pm = F^\pm_1 + iF^\pm_2 \), it follows that \( f = f_1 + if_2 \), where

\[
f_1(t) = F^+_1(t) + F^-_1(t), \quad \forall t \in \Gamma,
\]

\[
f_2(t) = F^+_2(t) + F^-_2(t), \quad \forall t \in \Gamma,
\]

and \( F^\pm_{1,2} \) are left-right-\( \psi^\theta \)-hyperholomorphic in \( \Omega_{\pm} \), hence it is enough to prove that \( F^\pm_{1,2} \) are unique. Indeed, the uniqueness of such functions follow from a sort of Painlevé Theorem (see [3 Corollary 5.3]) taking into account that the considered structural set plays no role in the proof.

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