HIGHER ORDER MASSEY PRODUCTS AND APPLICATIONS

IVAN LIMONCHENKO AND DMITRY MILLIONSCHIKOV

Abstract. In this survey, we discuss two research areas related to Massey’s higher operations. The first direction is connected with the cohomology of Lie algebras and the theory of representations. The second main theme is at the intersection of toric topology, homotopy theory of polyhedral products, and the homology theory of local rings, Stanley–Reisner rings of simplicial complexes.

Introduction

Higher order Massey operations proved to be a very effective algebraic tool for describing various obstructions to the existence or continuation of the diverse topological, geometric or algebraic structures and their deformations. In this short review, we attempted to systematize a series of results obtained over the past decade concerned with two important topics related to applications of the Massey higher operations. The starting point for the first topic is the Duady’s work [34] of the 1960, where the relation of Massey products to the theory of deformations was observed. This topic has been actively developed in the following decades, we especially note the works by Palamodov [70], Retakh [73, 74], in which the connection between the higher Massey products and the Kodaira-Spencer theory of deformations was studied. In 1975, Deligne, Griffiths, Morgan and Sullivan proved that simply connected compact Kähler manifolds are formal [30]. In particular it means that the existence of non-trivial Massey products in the cohomology $H^*(M, \mathbb{R})$ is an obstruction for a manifold $M$ to be Kähler [30]. A blow-up of a symplectic manifold $M$ along its submanifold $N$ inherits non-trivial Massey products [31, 8, 9]. For the first time this idea was used by McDuff [65] in her construction of a simply connected symplectic manifold with no Kähler structures. Later Babenko and Taimanov applied the symplectic blow-up procedure for the construction of simply connected non-formal symplectic manifolds in dimensions $\geq 10$ [8, 9]. However, in the papers of Babenko and Taimanov [9, 10] we were attracted primarily by their approach to the definition of Massey products in the language of formal connections and the Maurer-Cartan equation. The idea of such an approach has been encountered in literature before, first May’s [62, 63], then in Palamodov’s article [70], but it...
was precisely in the articles of Babenko and Taimanov that this approach was comprehensively developed [9, 10].

In the Section 1 we recall the elements of the Babenko-Taimanov approach to the definition of Massey products [9, 10]. The analogy with the classical Maurer-Cartan equation, which is especially transparent in the case of Massey products of 1-dimensional cohomology classes \( \langle \omega_1, \ldots, \omega_n \rangle \), is discussed in Section 2. The relation of this special case to representation theory was discovered in [35, 37].

The related material, which is presented in the form of a publication for the first time, is contained in Section 3. We are talking about variations of Massey products, which we called \( k \)-step Massey products. It is well known that the classical Massey products are multi-valued and partially defined operations. We propose to consider successive obstructions \( \langle \omega_1, \ldots, \omega_n \rangle_k, k = 1, \ldots, n-1 \), arising in constructing a formal connection (defining system) \( A \) for the classical Massey product \( \langle \omega_1, \ldots, \omega_n \rangle \) as \( k \)-step Massey products (see Definition 3.6).

The main feature uniting the results that we relate to the first topic is the Massey products in the cohomology of Lie algebras. Particular attention in this part is given to non-trivial Massey products, it was motivated by applications. Two very important and interesting positively graded Lie algebras were chosen as the main examples of this article. Firstly, this is the positive part \( W^+ \) of the Witt algebra, and secondly, its associated graded algebra with respect to the filtration by the ideals of the lower central series \( m_0 \). Sometimes \( m_0 \) is called the infinite dimensional filiform Lie algebra. We discuss in Section 4 the proof of Buchstaber’s conjecture that the cohomology \( H^*(W^+) \) are generated by means of non-trivial Massey products by the one-cohomology \( H^1(W^+) \). The corresponding Theorem 4.4 was proved by the second author in [67]. We consider in Section 4 also the structure results on the Massey products in the cohomology \( H^*(m_0) \).

The second central theme of our review originates also in the early 1960s from pioneering paper of Golod [40], who showed that Poincaré series of a local ring \( A \) achieve its (coefficientwise) upper bound, previously identified by Serre, precisely in the case when multiplication and all Massey products, triple and higher, vanish in Koszul homology of \( A \). Thanks to toric topology, this result acquired a topological interpretation by means of the theorem due to Buchstaber and Panov [24, 25], who proved that cohomology algebra of a moment-angle-complex \( Z_K \) over a commutative ring with unit \( k \) is isomorphic to Koszul homology of the corresponding Stanley-Reisner ring \( k[K] \). This gives us a tool to identify a class of simplicial complexes with formal moment-angle complexes and it also enables one to construct non-formal (and therefore, non-Kähler) moment-angle manifolds, having non-trivial Massey products in their cohomology. First examples of non-trivial triple Massey products in cohomology of moment-angle-complexes were found by Baskakov [16]; the first author [58, 59] proved the existence of polyhedral products (moment-angle manifolds) having non-trivial higher Massey products of any prescribed order in cohomology.

In the last two sections we give a survey on the construction of Massey products in the algebraic context of Koszul homology of local rings and discuss the Golod property for Stanley-Reisner rings of simplicial complexes. We deal with the results on non-trivial triple and higher
Massey products in cohomology of moment-angle-complexes and moment-angle manifolds, emphasizing the case of strictly defined (i.e., containing a single element) non-trivial higher Massey products.

1. Massey products in cohomology

Let $A = \bigoplus_{l \geq 0} A^l$ be a differential graded algebra over a field $K$. It means that the following operations are defined: an associative multiplication

$$\wedge : A^l \times A^m \to A^{l+m}, \quad l, m \geq 0, \quad l, n \in \mathbb{Z}.$$ 

such that $a \wedge b = (-1)^{lm} b \wedge a$ for $a \in A^l$, $b \in A^m$, and a differential $d$, $d^2 = 0$

$$d : A^l \to A^{l+1}, \quad l \geq 0,$$

satisfying the Leibniz rule

$$d(a \wedge b) = da \wedge b + (-1)^l a \wedge db$$

for $a \in A^l$.

Of course, the most natural example of differential graded algebra $A$ is the de Rham complex $A = \Lambda^*(M)$, $K = \mathbb{R}$, of smooth forms of a smooth manifold $M$. However, in this article we will pay special attention to the following two examples.

Example 1.1. $A = \Lambda^*(g)$ is the cochain complex of a Lie algebra.

Example 1.2. A Koszul complex $K_A = \Lambda^m A$ of a commutative Noetherian local ring $(A, m, k)$ (see the section 5 for details).

For a given differential graded algebra $(A, d)$ we denote by $T_n(A)$ a space of all upper triangular $(n+1) \times (n+1)$-matrices with entries from $A$, vanishing at the main diagonal. The standard matrix multiplication turns the vector space $T_n(A)$ into an algebra, we assume that matrix entries are multiplying as elements of $A$. One can define the differential $d$ on $T_n(A)$ by

$$(1) \quad dA = (da_{ij})_{1 \leq i, j \leq n+1}.$$ 

We extend the involution $a \to \bar{a} = (-1)^{k+1} a, a \in A^k$ of $A$ to the involution of $T_n(A)$ by the rule

$$\bar{A} = (\bar{a}_{ij})_{1 \leq i, j \leq n+1}.$$ 

It satisfies the following properties

$$\bar{A} = A, \quad \bar{AB} = -\bar{A}B, \quad d\bar{A} = -dA.$$ 

Also we have the generalized Leibniz rule for the differential $d$

$$d(AB) = (dA)B - A(dB).$$

Consider a two-sided ideal $I_n(A)$ of matrices of the following form

$$\begin{pmatrix}
0 & \ldots & 0 & \tau \\
0 & \ldots & 0 & 0 \\
\vdots & & & \\
0 & \ldots & 0 & 0
\end{pmatrix}, \quad \tau \in A$$

Obviously, the ideal $I_n(A)$ belongs to the center $Z(T_n(A))$ of the algebra $T_n(A)$. 

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Definition 1.3 ([10]). A matrix \( A \in T_n(\mathcal{A}) \) is called the matrix of a formal connection if it satisfies the Maurer-Cartan equation
\[
\mu(A) = dA - \bar{A} \cdot A \in I_n(\mathcal{A}).
\]

Proposition 1.4 ([10]). The generalized Bianchi identity for the Maurer-Cartan operator \( \mu(A) = dA - \bar{A} \cdot A \) holds
\[
d\mu(A) = \bar{\mu}(A) \cdot A + A \cdot \mu(A).
\]

Proof. Indeed it’s easy to verify the following equalities
\[
d\mu(A) = -d(\bar{A} \cdot A) = -d\bar{A} \cdot A + dA = \bar{\mu}(A) \cdot A + A \cdot \mu(A).
\]

\( \square \)

Corollary 1.5 ([10]). Let \( A \) be the matrix of a formal connection, then the entry \( \tau \in \mathcal{A} \) of the matrix \( \mu(A) \in I_n(\mathcal{A}) \) in the definition (2) is closed.

Now let \( A \) be the matrix of a formal connection, then the matrix \( \mu(A) \) belongs to the ideal \( I_n(\mathcal{A}) \) and hence \( d\mu(A) = 0 \). In a formal sense \( \mu(A) \) plays the role of the curvature matrix of a formal connection \( A \).

Let \( A \) be an upper triangular matrix from \( T_n(\mathcal{A}) \).

\[
A = \begin{pmatrix}
0 & a(1,1) & a(1,2) & \cdots & a(1,n-1) & a(1,n) \\
0 & 0 & a(2,2) & \cdots & a(2,n-1) & a(2,n) \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & a(n-1,n-1) & a(n-1,n) \\
0 & 0 & 0 & \cdots & 0 & a(n,n) \\
0 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}
\]

Proposition 1.6. A matrix \( A \in T_n(\mathcal{A}) \) is the matrix of a formal connection if and only if the following equalities hold
\[
a(i,i) = a_i \in \mathcal{A}^{p_i}, \quad i = 1, \ldots, n;
\]
\[
a(i,j) \in \mathcal{A}^{p(i,j)+1}, \quad p(i,j) = \sum_{r=i}^{j} (p_r - 1);
\]
\[
da(i,j) = \sum_{r=i}^{j-1} \bar{a}(i,r) \wedge a(r+1,j), \quad (i,j) \neq (1,n).
\]

The system (4) is just the Maurer-Cartan equation rewritten in terms of the entries of the matrix \( A \) and it is a part of the classical definition [56] of the defining system for a Massey product.

Definition 1.7 ([56]). A collection of elements, \( A = (a(i,j)) \), for \( 1 \leq i \leq j \leq n \) and \( (i,j) \neq (1,n) \) is said to be a defining system for the product \( \langle a_1, \ldots, a_n \rangle \) if it satisfies (4).
Under these conditions the \((p(1, n) + 2)\)-dimensional cocycle

\[
c(A) = \sum_{r=1}^{n-1} \bar{a}(1, r) \wedge a(r + 1, n)
\]

is called the related cocycle of the defining system \(A\).

One can verify that the notion of the defining system is equivalent to the notion of the formal connection. We have only to remark that an entry \(a(1, n)\) of a formal connection \(A\) does not belong to the corresponding defining system. It can be taken as an arbitrary element from \(A\) and for the only one nonzero (possibly) entry \(\tau \in \mu(A)\) we have

\[
\tau = -c(A) + da(1, n).
\]

**Definition 1.8** (\[56\]). The \(n\)-fold product \(\langle a_1, \ldots, a_n \rangle\) is defined if there exists at least one defining system for it (a formal connection \(A\) with entries \(a_1, \ldots, a_n\) at the second diagonal). If it is defined, then the value \(\langle a_1, \ldots, a_n \rangle\) is the set of all cohomology classes \(\alpha \in H^{p(1, n) + 2}(A)\) for which there exists a defining system \(A\) such that the cocycle \(c(A)\) (or equivalently \(-\tau\)) represents \(\alpha\).

**Theorem 1.9** (see \[56, 10\]). The product \(\langle a_1, \ldots, a_n \rangle\) depends only on the cohomology classes of the elements \(a_1, \ldots, a_n\).

**Definition 1.10** (\[56\]). A set of closed elements \(a_i, i = 1, \ldots, n\) from \(A\) representing some cohomology classes \(\alpha_i \in H^{p_i}(A), i = 1, \ldots, n\) is said to be a defining system for the Massey \(n\)-fold product \(\langle \alpha_1, \ldots, \alpha_n \rangle\) if it is one for \(\langle a_1, \ldots, a_n \rangle\). The Massey \(n\)-fold product \(\langle \alpha_1, \ldots, \alpha_n \rangle\) is defined if \(\langle a_1, \ldots, a_n \rangle\) is defined, in which case \(\langle \alpha_1, \ldots, \alpha_n \rangle = \langle a_1, \ldots, a_n \rangle\) as subsets in \(H^{p(1, n) + 2}(A)\).

For \(n = 2\) the matrix \(A\) of a formal connection is

\[
A = \begin{pmatrix}
0 & a & c \\
0 & 0 & b \\
0 & 0 & 0
\end{pmatrix}
\]

and the generalized Maurer-Cartan equation is equivalent to the system \(da = 0, \ db = 0\). Hence a 2-fold Massey product \(\langle \alpha, \beta \rangle\) is always defined and \(\langle \alpha, \beta \rangle = \bar{\alpha} \wedge \bar{\beta}\).

Let \(\alpha, \beta, \) and \(\gamma\) be the cohomology classes of closed elements \(a \in A^p, b \in A^q\), and \(c \in A^r\). The Maurer-Cartan equation for

\[
A = \begin{pmatrix}
0 & a & f & h \\
0 & 0 & b & g \\
0 & 0 & 0 & c \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

is equivalent to the system

\[
d f = (-1)^{p+1} a \wedge b, \quad d g = (-1)^{q+1} b \wedge c.
\]
Hence the triple Massey product $\langle \alpha, \beta, \gamma \rangle$ is defined if and only if the following conditions hold
$$\alpha \cdot \beta = \beta \cdot \gamma = 0 \text{ in } H^*(A).$$

The triple Massey product $\langle \alpha, \beta, \gamma \rangle$ is defined as a subspace $H^{p+q+r-1}(A)$ of elements
$$\langle \alpha, \beta, \gamma \rangle = \left\{ \left[ (\alpha \beta + (-1)^{p+q} f \cdot \gamma) \right] \right\}.$$

Since $f$ and $g$ are defined by (5) up to closed elements from $A_{p+q-1}$ and $A_{q+r-1}$ respectively, the triple Massey product $\langle \alpha, \beta, \gamma \rangle$ is an affine subspace of $H^{p+q+r-1}(A)$ parallel to $\alpha \cdot H^{q+r-1}(A) + H^{p+q-1}(A) \cdot \gamma$.

Sometimes the triple Massey product $\langle \alpha, \beta, \gamma \rangle$ is defined as a quotient $\langle \alpha, \beta, \gamma \rangle/\alpha \cdot H^{q+r-1}(A) + H^{p+q-1}(A) \cdot \gamma$.

**Definition 1.11.** Let an $n$-fold Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle$ be defined. It is called trivial if it contains the trivial cohomology class: $0 \in \langle \alpha_1, \ldots, \alpha_n \rangle$.

**Proposition 1.12.** Let a Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle$ is defined. Then all Massey products $\langle \alpha_l, \ldots, \alpha_q \rangle, 1 \leq l < q \leq n, q - l < n - 1$ are defined and trivial.

The triviality of all Massey products $\langle \alpha_l, \ldots, \alpha_q \rangle, 1 \leq l < q \leq n, q - l < n - 1$ is only a necessary condition for a Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle$ to be defined. It is sufficient only in the case $n = 3$.

Let us denote by $GT_n(K)$ a group of non-degenerate upper triangular $(n+1, n+1)$-matrices of the form:

$$C = \begin{pmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,n} & c_{1,n+1} \\ 0 & c_{2,2} & \cdots & c_{2,n} & c_{2,n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & c_{n,n} & c_{n,n+1} \\ 0 & 0 & \cdots & 0 & c_{n+1,n+1} \end{pmatrix}.$$  \hspace{1cm} (6)

**Proposition 1.13.** Let $A \in T_n(A)$ be the matrix of a formal connection and $C$ an arbitrary matrix from $GT_n(K)$. Then the matrix $C^{-1}AC \in T_n(A)$ and satisfies the Maurer-Cartan equation, i.e. it is again the matrix of a formal connection.

**Proof.**
$$d(C^{-1}AC) - \bar{C}^{-1} \bar{A} \bar{C} \wedge C^{-1}AC = C^{-1} (dA - \bar{A} \wedge A) C \in I_n(A).$$

It follows also that the associated classes $[c(A)]$ and $[c(C^{-1}AC)]$ with $C$ from (6) are related
$$[c(C^{-1}AC)] = \frac{c_{n+1,n+1}[c(A)]}{c_{1,1}}.$$ 
\(\square\)
Example 1.14. Let \( A \in T_n(A) \) be the matrix of a formal connection (defining system) for a Massey product \( \langle \alpha_1, \ldots, \alpha_n \rangle \). Then a matrix \( C^{-1}AC \) with
\[
C = \begin{pmatrix}
1 & 0 & \ldots & 0 & 0 \\
0 & x_1 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & x_1 \cdots x_{n-1} & 0 \\
0 & 0 & \cdots & 0 & x_1 \cdots x_{n-1}x_n
\end{pmatrix}
\]
is a defining system for \( \langle x_1\alpha_1, \ldots, x_n\alpha_n \rangle = x_1 \cdots x_n \langle \alpha_1, \ldots, \alpha_n \rangle \).

Definition 1.15. Two matrices \( A \) and \( A' \) of formal connections are equivalent if there exists a non-degenerate scalar matrix \( C \in GT_n(\mathbb{K}) \) such that \( A' = C^{-1}AC \).

It is obvious that we can consider only the subgroup of non-degenerate diagonal matrices instead of the whole \( GT_n(\mathbb{K}) \) in the last definition.

Following the original Massey’s paper [61], some higher order cohomological operations that we call now Massey products were introduced in the 1960s in [56] and [62]. May briefly noticed in [62] that there is a relation between the definition of the Massey products and Maurer-Cartan equation. However, this analogy was not developed till the Babenko-Taimanov paper [10].

In the present article we deal only with Massey products of non-trivial cohomology classes. In general situation it is more natural to consider so-called matric Massey products that were first introduced by May in [62] and developed in [10].

2. Massey products and Lie algebras representations

Consider the cochain complex with trivial coefficients \( \mathbb{K} \) of an \( n \)-dimensional Lie algebra \( g \)
\[
\mathbb{K} \xrightarrow{d_0 = 0} \mathfrak{g}^* \xrightarrow{d_1} \Lambda^2(\mathfrak{g}^*) \xrightarrow{d_2} \ldots \xrightarrow{d_{n-1}} \Lambda^n(\mathfrak{g}^*) \xrightarrow{d_n} 0.
\]
where \( d_1 : \mathfrak{g}^* \rightarrow \Lambda^2(\mathfrak{g}^*) \) is a dual mapping to the Lie bracket \([.,.] : \Lambda^2\mathfrak{g} \rightarrow \mathfrak{g}\). The differential \( d \) (the whole collection of \( d_p \)) is the derivation of the exterior algebra \( \Lambda^*(\mathfrak{g}^*) \) that continues \( d_1 \)
\[
d(\rho \wedge \eta) = d\rho \wedge \eta + (-1)^{\deg \rho} \rho \wedge d\eta, \forall \rho, \eta \in \Lambda^*(\mathfrak{g}^*).
\]
It is easy to see that the condition \( d^2 = 0 \) is equivalent to the Jacobi identity of \( \mathfrak{g} \).

Continue to use dual language and write with its help the definition of the representation of a Lie algebra by square \( (n+1, n+1) \)-matrices.

Proposition 2.1. A \((n+1, n+1)\)-matrix \( A \) with entries from \( \mathfrak{g}^* \) defines a representation \( \rho : \mathfrak{g} \rightarrow \mathfrak{gl}_n(\mathbb{K}) \) if and only if \( A \) satisfies the strong Maurer-Cartan equation
\[
dA - \bar{A} \wedge A = 0.
\]

Proof. \((dA - \bar{A} \wedge A)(x, y) = A([x, y]) - [A(x), A(y)], \forall x, y \in \mathfrak{g}\). \(\square\)

In the Section [1] the involution of a graded \( \mathcal{A} \) was defined as \( \bar{a} = (-1)^{k+1}a, a \in \mathcal{A}^k \). Thus, for a matrix \( A \) with entries in \( \mathfrak{g}^* \) we have \( \bar{A} = A \). One has to remark that \( \bar{a} \) differs by the sign from the definition of the involution \( \bar{a} \) in [56], however, in [63] there is the same sign rule.
From now on, we will consider only representations in upper triangular matrices. So, we denote by $T_n(K) \subset gl_n(K)$ the Lie subalgebra of upper triangular $(n + 1, n + 1)$-matrices and consider representations $\rho : g \to T_n(K)$ of a Lie algebra $g$ for some fixed value $n$.

Consider $n = 1$ and a linear map

$$\rho : x \in g \to A(x) = \begin{pmatrix} 0 & a(x) \\ 0 & 0 \end{pmatrix}.$$ 

It is evident that $\rho$ is a Lie algebra homomorphism if and only if the linear form $a \in g^*$ is closed. Or equivalently, $A$ satisfies the strong Maurer-Cartan equation $dA - \bar{A} \wedge A = 0$.

The Lie algebra $T_n(K)$ has a one-dimensional center $I_n(K)$ spanned by the matrix

$$\begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix}.$$ 

One can consider an one-dimensional central extension

$$0 \longrightarrow K \cong I_n(K) \longrightarrow T_n(K) \longrightarrow T_n(K) \longrightarrow 0.$$ 

**Proposition 2.2** ([37], [35]). Fixing a Lie algebra homomorphism $\bar{\varphi} : g \to \tilde{T}_n(K)$ is equivalent to fixing a defining system $A$ with elements from $g^* = \Lambda^1(g)$. The related cocycle $c(A)$ is cohomologous to zero if and only if $\bar{\varphi}$ can be lifted to a homomorphism $\varphi : g \to T_n(K)$, $\bar{\varphi} = \pi \varphi$.

There is a standard definition.

**Definition 2.3.** Two representations $\varphi : g \to T_n(K)$ and $\varphi' : g \to T_n(K)$ are called equivalent if there exists a non-degenerate matrix $C \in GL(n+1, K)$ such that

$$\varphi'(g) = C^{-1} \varphi(g) C, \ \forall g \in g.$$ 

It is evident that this definition is equivalent to the Definition [1.15] for Massey products of 1-cohomology classes $\omega_1, \ldots, \omega_n$. Moreover, these linear forms are on the second (top) main diagonal of the matrix $A$ of the formal connection.

**Proposition 2.4.** Let a Massey product $\langle \omega_1, \omega_2, \ldots, \omega_n \rangle$ be defined and trivial in $H^2(g)$ for some 1-cohomology classes $\omega_i \in H^1(g)$ of a Lie algebra $g$. Then $\langle x_1 \omega_1, x_2 \omega_2, \ldots, x_n \omega_n \rangle$ is also defined and trivial for any choice of non-zero constants $x_1, x_2, \ldots, x_n$.

3. **k-step Massey products in Lie algebra cohomology**

This paragraph is written largely under the influence of an article by Ido Efrat [32]. His main observation was the following theorem.
Theorem 3.1 (Efrat [32]). Suppose that for every set \( \omega_1, \ldots, \omega_n \) of 1-cocycles from \( H^1(g) \) there is a matrix \( A \) of formal connection. Then the Massey product
\[
\langle \cdot, \ldots, \cdot \rangle : H^1(g) \times \cdots \times H^1(g) \to H^2(g)
\]
is a well-defined single-valued map.

However, in this section we want to discuss other ideas of [32] in a revised form, in particular, to define higher order Massey products \( \langle a_1, a_2, \ldots, a_n \rangle \) as successive obstructions \( \langle a_1, a_2, \ldots, a_n \rangle_k, k = 1, 2, \ldots, n - 1 \), to the existence of a formal connection \( A \) for a given set of one-dimensional cocycles \( a_1, a_2, \ldots, a_k \) from the differential algebra \( A \).

Consider the descending series \( \{ T_n^{-k}(A), k = 0, \ldots, n - 1 \} \) of two-sided ideals
\[
T_n^{-1}(A) = T_n(A) \subset T_n^{-2}(A) \subset \cdots \subset T_n^{-k}(A) \subset \cdots \subset T_n^{0}(A) = I_n(A) \subset \{0\},
\]
where \( T_n^{-k}(A) \) denotes the subspace of upper triangular \((n+1,n+1)\)-matrices \( B \) with entries from \( A \) of the following form
\[
B = \begin{pmatrix}
0 & 0 & \ldots & 0 & b(1, k) & \ldots & b(1, n) \\
0 & \ddots & \ddots & \ddots & 0 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & \ldots & \ldots & \ldots & 0 & 0
\end{pmatrix}, (i, j) \in A.
\]

Obviously for non-negative integers \( k, m, k + m \leq n \), we have inclusions
\[
(7) \quad T_n^{-k}(A) \cdot T_n^{-m}(A) \subset T_n^{-k-m}(A).
\]

Definition 3.2. Consider a non-negative integer \( k, 0 \leq k \leq n - 1 \). A matrix \( A_k \in T_n(A) \) is called the matrix of a \( k \)-step formal connection if it satisfies the \( k \)-step Maurer-Cartan equation
\[
(8) \quad \mu(A_k) = dA_k - \bar{A}_k \cdot A_k \in T_n^{-k}(A).
\]

Proposition 3.3. Let \( A_k \) be the matrix of a \( k \)-step formal connection. Then
\[
d\mu(A_k) \in T_n^{n-k-1}(A).
\]

Proof. It follows from (7). Indeed the generalized Bianci identity [3] means
\[
d\mu(A_k) = -d(\bar{A}_k \cdot A_k) = -\mu(A_k) \cdot A_k + A_k \cdot \mu(A_k) \in T_n^{-k}(A) \cdot T_n^{n-1}(A).
\]

Corollary 3.4. Let \( A_k \) be the matrix of a \( k \)-step formal connection, \( 0 \leq k \leq n - 1 \). Then all elements on the \( k \)-th parallel main diagonal of the matrix \( \mu(A_k) = (\tau(i, j)) \) are closed
\[
d\tau(1, k) = 0, \; d\tau(2, k+1) = 0, \; \ldots, \; d\tau(n-k, n) = 0.
\]
From now on, we will consider matrices of formal connections with elements \( \omega \) from the dual space \( g^* \) to the Lie algebra \( g \). It means that they are all 1-forms and \( \bar{\omega} = \omega \) for every \( \omega \in g \).

**Definition 3.5.** Let \( A_k \) be the matrix of a \( k \)-step formal connection, \( 0 \leq k \leq n-1 \). The set of \( n-k \)-two-dimensional cocycles
\[
c(A_k) = \left( \sum_{r=1}^{n-1} \bar{a}(1,r) \wedge a(r+1,k), \ldots, \sum_{r=1}^{n-1} \bar{a}(n-k,r) \wedge a(r+1,n) \right).
\]
is called the related set \( c(A_k) \) of cocycles of the formal \( k \)-step connection \( A_k \).

**Definition 3.6.** The \( k \)-step \( n \)-fold product \( \langle a_1, \ldots, a_n \rangle_k \) is defined if there exists at least one \( k \)-step formal connection \( A_k \) for it with entries \( a_1, \ldots, a_n \) at the second diagonal. If it is defined, then the value \( \langle a_1, \ldots, a_n \rangle_k \) is the set of all \( (n-k) \)-tuples of cohomology classes \( (\alpha_1, \ldots, \alpha_k+1) \in H^2(A) \times \ldots H^2(A) \) for which there exists a \( k \)-step formal connection \( A_k \) such that the sequence \( c(A_k) \) (or equivalently \( (-\tau_1, \ldots, -\tau_{n-k}) \)) represents \( (\alpha_1, \ldots, \alpha_{n-k}) \).

\[
\mu(A_k) = \begin{pmatrix}
0 & \ldots & 0 & \tau_1 & \ast & \ldots & \ast \\
0 & \ldots & \tau_2 & \ddots & \vdots & \\
\vdots & \ldots & \ddots & \ast & \\
\vdots & \ldots & \ast & \tau_{n-k} & \\
0 & \ldots & \ldots & \ldots & \vdots & \\
0 & \ldots & \ldots & \ldots & \ldots & \\
\end{pmatrix}, \tau_i \in g^*, d\tau_i = 0, i = 1, \ldots, n-k.
\]

Using the standard arguments from [56][10] one can prove that the product \( \langle a_1, \ldots, a_n \rangle_k \) depends only on the cohomology classes of the elements \( a_1, \ldots, a_n \) as it holds in the classic case.

**Example 3.7.** The 1-step product \( \langle a_1, \ldots, a_n \rangle_1 \) is defined, single-valued and equal to the following \( (n-1) \)-tuple of double products
\[
\langle a_1, \ldots, a_n \rangle = (a_1 \wedge a_2, a_2 \wedge a_3, \ldots, a_{n-2} \wedge a_{n-1}, a_{n-1} \wedge a_n) \in H^2(g) \times \ldots H^2(g).
\]

**Remark.** If we do not pay attention to the subscript \( k \), the \( n \)-fold Massey product \( \langle a_1, \ldots, a_n \rangle \) is now always defined, although we a priori do not know in which space its value will be located.

By analogy with the classical case, we give the following definition

**Definition 3.8.** Let a \( k \)-step \( n \)-fold Massey product \( \langle \alpha_1, \ldots, \alpha_n \rangle_k \) in the Lie algebra cohomology be defined. It is called trivial if it contains the trivial cohomology class of the vector space
\[
\left( H^2(g) \times \ldots H^2(g) \right)_{n-k}
\]
\( (0, \ldots, 0) \in \langle \alpha_1, \ldots, \alpha_n \rangle_k. \)
Proposition 3.9. Let a $k$-step Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle_k$ be defined. Then all Massey products $\langle \alpha_l, \ldots, \alpha_q \rangle_s$, $1 \leq l < q \leq n$, $q - l < n - 1$, $s \leq k$, are defined and trivial.

Proposition 3.10. Let a $k$-step Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle_k$ be defined and trivial. Then the $(k+1)$-step Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle_{k+1}$ is defined.

Returning to Example 3.7, we see that a two-step formal connection $A_2$ for the classes $a_1, \ldots, a_n$, exists if and only if all the cocycle products $a_i \wedge a_{i+1}$, $i = 1, \ldots, n - 1$, are trivial in the cohomology $H^*(A)$. Moreover, if we find 1-forms $a(i, i+1)$ solving the equations

$$a(i, i) \wedge a(i+1, i+1) = da(i, i+1), i = 1, \ldots, n - 1,$$

we can explicitly write down the formal connection matrix $A_2$ with their help and (4). New elements $a(i, i+1)$ make up the second diagonal of the matrix $A_2$.

Corollary 3.11. Let a standard $n$-fold Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle$ be defined. Then all $k$-step Massey products $\langle \alpha_1, \ldots, \alpha_n \rangle_k$ with $k < n$ are defined and trivial.

Proof. It means that the $(n - 1)$-step Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle_{n-1}$ is defined. Hence the statement follows from Proposition 3.9. \square

To conclude this section, we want to note that the technique of $k$-step Massey products constructed in it is not something completely unknown. Anyone who searched for (constructed) the defining systems of Massey products came across this recursive procedure at one level or another. But it seems however useful to formalize it in this specific way thinking on the applications to representations theory of nilpotent Lie algebras.

4. Non-trivial Massey products in Lie algebra cohomology

Buchstaber and Shokurov discovered [26] that the tensor product $S \otimes \mathbb{R}$ of the Landweber-Novikov algebra $S$ (the complex cobordism theory) by real numbers $S \otimes \mathbb{R}$ is isomorphic to the universal enveloping algebra $U(W^+)$ of the Lie algebra $W^+$ of polynomial vector fields on the real line $\mathbb{R}^1$ with vanishing non-positive Fourier coefficients. $W^+$ is a maximal pro-nilpotent subalgebra of the Witt algebra.

The Witt algebra $W$ is spanned by differential operators on the real line $\mathbb{R}^1$ with a fixed coordinate $x$

$$e_i = x^{i+1} \frac{d}{dx}, i \in \mathbb{Z}, \quad [e_i, e_j] = (j - i)e_{i+j}, \quad \forall i, j \in \mathbb{Z}.$$ 

We denote by $W^+$ the so-called positive part of the Witt algebra, i.e. the subalgebra of $W$ spanned by all $e_k$ with positive indices $k$. Obviously $W^+$ is a $\mathbb{N}$-graded Lie algebra and its cohomology is the bigraded associative commutative algebra $H^*(W^+) = \oplus_{q,k} H^q_k(W^+)$.

It is easy to see that $H^1(W^+)$ is spanned by $e^1$ and $e^2$, the duals of two generators $e_1$ and $e_2$ of $w^+$.

Theorem 4.1 (Goncharova [41]). The Betti numbers $\dim H^q(W^+) = 2$, for every $q \geq 1$, more precisely

$$\dim H^q_k(W^+) = \begin{cases} 
1, & \text{if } k = \frac{3q^2 + q}{2} \\
0, & \text{otherwise}.
\end{cases}$$
We will denote by $g^q_\pm$ a basis in the spaces $H^q_{3q^2+q}(W^+)$. The numbers $\frac{3q^2+q}{2} = e_\pm(q)$ are so-called Euler pentagonal numbers. It is easy to verify that the sum

$$e_\pm(q) + e_\pm(p) \neq e_\pm(p+q), p, q \in \mathbb{N}.$$ 

Hence the cohomology algebra $H^*(W^+)$ has a trivial multiplication. Buchstaber conjectured that the algebra $H^*(W^+)$ is generated with respect to some non-trivial Massey products by its first cohomology $H^1(W^+)$. Feigin, Fuchs and Retakh [37] represented the basic homogeneous cohomology classes from $H^*(W^+)$ as Massey products [37].

**Theorem 4.2 (Feigin, Fuchs, Retakh [37]).** For any $q \geq 2$ we have inclusions

$$g^q_- \in \langle g^q_- e^1, \ldots, e^1 \rangle_{2q-1}, \quad g^q_+ \in \langle g^q_+ e^1, \ldots, e^1 \rangle_{3q-1}.$$ 

Feigin, Fuchs and Retakh proposed to consider the following matrix of formal connection

$$A = \begin{pmatrix}
0 & g^k_+ & \Omega_1 & \Omega_2 & \cdots & \Omega_{n-1} & * \\
0 & 0 & e^1 & \alpha e^2 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & e^1 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix},$$

with homogeneous forms $\Omega_i \in \Lambda^{k+i-1}_{\frac{1}{2}(3(k-1)^2+(k-1))+i}(W^+)$ and parameter $\alpha \in \mathbb{K}$. The corresponding cocycle $c(A) \in \Lambda_{\frac{1}{2}(3k^2+2k-1)+n-1}^k(W^+)$ can be non-trivial only if $n = 2k$ or $n = 3k$ that corresponds to $H^k_{\frac{1}{2}(3k^2+2k)}(W^+)$. Feigin, Fuchs and Retakh have shown that the triviality of the cocycle $c(A)$ is equivalent to the triviality of the differential $d_n$ of some spectral sequence $E^p_{i,j}$ converging to the cohomology $H^*(W^+, V)$ with coefficients in the graded $W^+$-module $V$. The infinite dimensional module $V$ depends on the parameter $\alpha$ and can be defined by its basis $f_1, f_2, \ldots, f_n, \ldots$ and relations

$$e_1 f_j = f_{j+1}, \quad e_2 f_j = \alpha f_{j+2}, j \geq 1; \quad e_k f_j = 0, k, j \in \mathbb{N}.$$ 

Feigin, Fuchs and Retakh established that

1) the differential $d_2k$ is trivial if and only if $\alpha \in \left\{ \frac{1}{6}, \frac{1}{24}, \ldots, \frac{1}{6(k-1)^2} \right\}$;

2) the differential $d_{3k}$ is defined and trivial if and only if:

a) $\alpha \in \left\{ \frac{1}{6}, \frac{1}{6}, \ldots, \frac{1}{6(k-3)^2}, \frac{1}{6(k-1)^2} \right\}$ in the case of even $k$;

b) $\alpha \in \left\{ \frac{1}{24}, \frac{1}{96}, \ldots, \frac{1}{6(k-3)^2}, \frac{1}{6(k-1)^2} \right\}$ if $k$ is odd.

**Corollary 4.3.** All Massey products from the Theorem 4.2 are trivial.
The main technical problem in the proof of Theorem 4.2 is to find explicit formulas for the entries $\Omega_i, i = 1, \ldots, n - 1$. One can verify directly that the cocycles $g^2_2 = e^2 \wedge e^3$ and $g^2_+ = e^2 \wedge e^5 - e^3 \wedge e^4$ span the homogeneous subspaces $H^2_2(W^+)$ and $H^2_2(W^+)$ respectively. But explicit formulas for all Goncharova’s cocycles $g^k_\pm$ in terms of exterior forms from $\Lambda^*(W^+)$ are still unknown. Fuchs, Feigin and Retakh proposed [37] an elegant way how to establish non-triviality of differentials for some values of the parameter $\alpha$ of the spectral sequence $E^p_2$ that converge to the cohomology $H^*(W^+, V)$.

Artel’nykh [3] represented a part of basic cocycles in $H^*(W^+)$ by means of non-trivial Massey products, but his very brief article does not contain any sketch of the proof.

**Theorem 4.4** (Millionshchikov [67]). The cohomology $H^*(W^+)$ is generated by two elements $e^1, e^2 \in H^1(W^+)$ by means of two series of non-trivial Massey products. More precisely the recurrent procedure is organized as follows

1) elements $e^1$ and $e^2$ span $H^1(W^+)$;
2) the triple Massey product $\langle e^1, e^2, e^2 \rangle$ is single-valued and determines non-trivial cohomology class $g^2_2 = (e^1, e^2, e^2) \in H^2_2(W^+)$;
3) the 5-fold product $\langle e^1, e^2, e^1, e^1, e^2 \rangle$ is non-trivial and it is an affine line $\{g^2_2 + t g^2_-, t \in \mathbb{K}\}$ on the plane $H^2(W^+)$, where $g^2_2$ denotes some generator from $H^2_2(W^+)$. Denote by $g^2_+$ an arbitrary element in $(e^1, e^2, e^1, e^1, e^2)$.

Let us suppose that we have already constructed some basis $g^k_\pm, \tilde{g}^k_\pm$ of $H^k(W^+), k \geq 2$, such that the cohomology class $g^k_\pm$ spans the subspace $H^k_{\frac{3k-2}{2}}(W^+)$. Then

4) the $(2k + 1)$-fold Massey product

$$\langle e^1, \ldots, e^1, e^2, e^1, \ldots, e^1, \tilde{g}^k_+ \rangle = g^{k+1}_-, m + n = 2k - 1,$$

is single-valued and spans the subspace $H^k_{\frac{3k-1}{2}}(3(k+1)^2-(k+1))(W^+)$. 

5) the $(3k + 2)$-fold product

$$\langle e^1, \ldots, e^1, e^2, e^1, \ldots, e^1, \tilde{g}^k_+ \rangle$$

is non-trivial and it is an affine line on the two-dimensional plane $H^{k+1}(W^+)$ parallel to the one-dimensional subspace $H^k_{\frac{3k+1}{2}}(3(k+1)^2-(k+1))(W^+)$. One can take an arbitrary element in $\langle e^1, \ldots, e^1, e^2, e^1, \ldots, e^1, \tilde{g}^k_+ \rangle$ as the second basic element $\tilde{g}^{k+1}_+$ of the subspace $H^{k+1}(W^+)$. 

The proof of Theorem 4.4 is rather complicated [67]. It is based on the technique developed in [37], to which a set of fundamental additions was proposed. The first was the construction of the special graded thread $W^+$-module $V_{gr}$, which is uniquely determined by its main properites. The non-triviality of the corresponding Massey products largely follows from such a rigidity. To calculate the differentials of the spectral sequence associated with the constructed Massey products, explicit formulas for the special vectors of Verma modules over the Virasoro algebra were applied. Finally the nontriviality of the corresponding differentials $d_k$ was established by
explicit calculus of the cohomology $H^*(W^+, V_{gr})$ using the so-called Feigin-Fuchs-Rocha-Caridi-Wallach resolution \cite{67}.

As an example, we will reproduce the simplest part of the proof \cite{67} related to the second cohomology $H^*(W^+)$, but which nevertheless well illustrates the basics of the proof technique.

For an arbitrary formal connection $A$ that corresponds to the product $\langle e^2, e^2, e^1 \rangle$ the cocycle $c(A) = -e^2 \wedge e^3 + \alpha d(e^3)$ for some scalar $\alpha$. Hence the single-valued triple product $\langle e^1, e^2, e^2 \rangle = -[e^2 \wedge e^3] \not= 0$ spans the subspace $H^2_0(W^+)$. 

Consider $\langle e^1, e^2, -e^1, -2e^1, -e^2 \rangle$ instead of $\langle e^1, e^2, e^1, e^1, e^2 \rangle$ and a formal connection $A$

$$A = \begin{pmatrix}
0 & e^1 & e^3 & e^4 & e^5 & * \\
0 & 0 & e^2 & e^3 & e^4 & 0 \\
0 & 0 & 0 & -e^1 & e^2 & -e^4 - te^2 \\
0 & 0 & 0 & 0 & -2e^1 & 2e^3 \\
0 & 0 & 0 & 0 & 0 & -e^2 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$ 

The corresponding cocycle will be $c(A) = (e^2 \wedge e^3 - 3e^2 \wedge e^4) + te^2 \wedge e^3$. On the another hand for an arbitrary defining system $A'$ the corresponding cocycle will have the form $c(A') = (e^2 \wedge e^3 - 3e^2 \wedge e^4) + \ldots$, where dots stand for the summands with the second grading strictly less than 7.

**Example 4.5.** The Lie algebra $m_0$ is defined by its infinite basis $e_1, e_2, \ldots, e_n, \ldots$ with commutator relations:

$$[e_1, e_i] = e_{i+1}, \forall i \geq 2; [e_i, e_j] = 0, i, j \not= 1.$$

Let $g$ be a pro-nilpotent or $\mathbb{N}$-graded Lie algebra. The ideals $g^k$ of the descending central sequence define a decreasing filtration of the Lie algebra $g$

$$g = g^1 \supset g^2 \supset \ldots \supset g^k = [g^1, g^{k-1}] \supset \ldots; [g^k, g^l] \subset g^{k+l}, k, l \in \mathbb{N}.$$

Consider the associated graded Lie algebra

$$\text{gr } g = \bigoplus_{k=1}^{+\infty} (\text{gr } g)_k, \ (\text{gr } g)_k = g^k / g^{k+1}, k \in \mathbb{N}.$$

**Proposition 4.6.** We have the following isomorphisms:

$$\text{gr } W_+ \cong \text{gr } m_0 \cong m_0.$$

The corresponding natural grading of $m_0$ is defined by

$$(\text{gr } m_0)_1 = \text{Span}(e_1, e_2), \ (\text{gr } m_0)_i = \text{Span}(e_{i+1}), \ i \geq 2.$$

Let $g = \bigoplus_{n} g_n$ be a $\mathbb{N}$-graded (pro-nilpotent) Lie algebra and $V$ is a finite-dimensional nilpotent $g$-module. There is a decreasing filtration of the $g$-module module $V$

$$V^1 = V \subset V^2 = gV \subset \ldots V^k = gV^{k-1} \subset \ldots$$

One can define the associated graded module $\text{gr } V$ over the associated graded Lie algebra $\text{gr } g$

$$\text{gr } V = \bigoplus_{i=1}^{+\infty} (\text{gr } V)_i, \ \text{gr } V_i = V^i / V^{i+1}, (\text{gr } g)_i (\text{gr } V)_j \subset (\text{gr } V)_{i+j}, i, j \in \mathbb{N}.$$
Thus, we came to the problem of description of Massey products in the cohomology $H^*(m_0)$. As we saw earlier, it is useful to describe trivial Massey products $\langle \omega_1, \ldots, \omega_n \rangle$ of 1-cohomology classes $\omega_1, \ldots, \omega_n$. The purpose of this interest is to consider Massey products of the form $\langle \omega_1, \ldots, \omega_n, \Omega \rangle$, where $\Omega$ is a cocycle from $H^p(g)$ for $p > 1$.

Consider $p = 2$. It was found out in [38] that $H^2(m_0)$ is spanned by the cohomology classes of the following set of cocycles [38]

$$\omega(e^k \wedge e^{k+1}) = \sum_{l=0}^{k-2} (-1)^l e^{k-l} \wedge e^{k+1+l}, k \geq 2.$$

All of the cocycles (9) can be represented as Massey products. Namely let us consider the following matrix of a formal connection

$$A = \begin{pmatrix}
0 & e^2 & -e^3 & \ldots & (-1)^k e^k & (-1)^{k+1} e^{k+1} & 0 \\
0 & 0 & e^1 & 0 & \ldots & 0 & e^{k+1} \\
0 & 0 & 0 & e^1 & 0 & \ldots & e^k \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & \ldots & 0 & e^3 \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & 0 & 0 & \ldots & 0
\end{pmatrix}.$$

For the related cocycle $c(A)$ we have

$$c(A) = \sum_{l=2}^{k+1} (-1)^l e^l \wedge e^{k+3-l} = 2\omega(e^k \wedge e^{k+1}).$$

So we proved that

$$2\omega(e^k \wedge e^{k+1}) \in \langle e^2, e^1, \ldots, e^1, e^2 \rangle, \quad k \geq 2.$$

The space $H^1(m_0)$ is spanned by $e^1$ and $e^2$ and therefore an arbitrary $n$-fold Massey product of elements from $H^1(m_0)$ has a form

$$\langle \alpha_1 e^1 + \beta_1 e^2, \alpha_2 e^1 + \beta_2 e^2, \ldots, \alpha_n e^1 + \beta_n e^2 \rangle.$$

It follows from $e^1 \wedge e^2 = de^3$ that a triple product

$$\langle \omega_1, \omega_2, \omega_3 \rangle = \langle \alpha_1 e^1 + \beta_1 e^2, \alpha_2 e^1 + \beta_2 e^2, \alpha_3 e^1 + \beta_3 e^2 \rangle$$

is defined for all values $\alpha_i, \beta_i \in \mathbb{K}, i = 1, 2, 3$.

$$A = \begin{pmatrix}
0 & \omega_1 & \gamma_1 e^3 & 0 \\
0 & 0 & \omega_2 & \gamma_2 e^3 \\
0 & 0 & 0 & \omega_3 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \gamma_1 = \alpha_1 \beta_2 - \alpha_2 \beta_1, \gamma_2 = \alpha_2 \beta_3 - \alpha_3 \beta_2,$$

The related cocycle $c(A) = \gamma_2 \omega_1 \wedge e^3 - \gamma_1 \omega_3 \wedge e^3$ is trivial if and only if

$$\beta_1 (\alpha_2 \beta_3 - \alpha_3 \beta_2) - \beta_3 (\alpha_1 \beta_2 - \alpha_2 \beta_1) = 0.$$
Let us consider the operator $D_1 = ad^*e_1 : \Lambda^*(e_2, e_3, \ldots) \to \Lambda^*(e_2, e_3, \ldots)$ acting on the chain subcomplex $\Lambda^*(e_2, e_3, \ldots) \subset \Lambda^*(e_1, e_2, e_3, \ldots)$ of $m_0$.

(11) \[ D_1(e^2) = 0, \quad D_1(e^i) = e^{i-1}, \quad \forall i \geq 3, \]

The operator $D_1$ has the right inverse operator $D_{-1} : \Lambda^*(e^2, e^3, \ldots) \to \Lambda^*(e^2, e^3, \ldots)$, defined by the formulas

(12) \[ D_{-1}e^i = e^{i+1}, \quad D_{-1}(\xi \wedge e^i) = \sum_{l \geq 0} (-1)^l D_1^l(\xi) \wedge e^{i+1+l}, \]

where $i \geq 2$ and $\xi$ stands for an arbitrary form in $\Lambda^*(e^2, e^3, \ldots)$. One can verify that $D_1 D_{-1} = Id$ on $\Lambda^*(e^2, e^3, \ldots)$.

The sum in the definition (12) of $D_{-1}$ is always finite because $D_1^l$ strictly decreases the second grading by $l$. For instance, $D_{-1}(e^i \wedge e^k) = \sum_{l=0}^{i-2} (-1)^l e^{i-l} \wedge e^{k+l+1}$.

There is an explicit formula for $D_{-1}(e^{i_1} \wedge \ldots \wedge e^{i_2} \wedge e^{i_3}) = D_{-1}(0)$

\[ \omega(e^{i_1} \wedge \ldots \wedge e^{i_2} \wedge e^{i_3+1}) = \sum_{l \geq 0} (-1)^l D_1^l(e^{i_1} \wedge \ldots \wedge e^{i_2}) \wedge e^{i_3+1+l}, \]

this sum is also always finite and determines a homogeneous closed $(q+1)$-form of the second grading $i_1 + \ldots + i_{q-1} + 2i_q + 1$.

**Theorem 4.7** (Fialowski, Millionshchikov [38]). The bigraded cohomology algebra $H^*(m_0) = \bigoplus_{k,q} H_k^q(m_0)$ is spanned by the cohomology classes of the following homogeneous cocycles:

(13) \[ e^1, \quad e^2, \quad \omega(e^{i_1} \wedge \ldots \wedge e^{i_2} \wedge e^{i_3+1}) = \sum_{l \geq 0} (-1)^l (ad^*e_1)^l (e^{i_1} \wedge \ldots \wedge e^{i_2} \wedge e^{i_3}) \wedge e^{i_3+1+l}, \]

where $q \geq 1$, $2 \leq i_1 < i_2 < \ldots < i_q$.

The multiplicative structure is defined by the rules

(14) \[ [e^1] \wedge \omega(\xi \wedge e^i \wedge e^{i+1}) = 0, \quad [e^2] \wedge \omega(\xi \wedge e^i \wedge e^{i+1}) = \omega(e^2 \wedge \xi \wedge e^i \wedge e^{i+1}), \]

\[ \omega(\xi \wedge e^i \wedge e^{i+1}) \wedge \omega(\eta \wedge e^j \wedge e^{j+1}) = \sum_{l=0}^{j-i-2} (-1)^l \omega((ad^*e_1)^l (\xi \wedge e^i) \wedge e^{i+1+l} \wedge \eta \wedge e^j \wedge e^{j+1}) + \]

\[ + (-1)^{i+j+\deg \eta} \sum_{s \geq 1} \omega((ad^*e_1)^{i-j-1+s} (\xi \wedge e^i) \wedge (ad^*e_1)^s (\eta \wedge e^j) \wedge e^{i+s} \wedge e^{j+s+1}), \]

where $i < j$, $\xi$ and $\eta$ are arbitrary homogeneous forms in $\Lambda^*(e^2, \ldots, e^{i-1})$ and $\Lambda^*(e^2, \ldots, e^{j-1})$, respectively.

It can be seen that the cohomology of the two algebras $m_0$ and $W^+$ are very different from each other. The cohomology $H^p(m_0)$ is infinite-dimensional for $p \geq 2$, while ring multiplication in cohomology $H^*(m_0)$ is not trivial, although there is a sufficient number of trivial products, in particular, the product mapping $H^1(m_0) \wedge H^1(m_0) \to H^2(m_0)$ vanishes.
We recall that if an $n$-fold Massey product $\langle \omega_1, \omega_2, \ldots, \omega_n \rangle$ is defined than all $(p+1)$-fold Massey products $\langle \omega_i, \omega_{i+1}, \ldots, \omega_{i+p} \rangle$ for $1 \leq i \leq n-1, 1 \leq p \leq n-2, i+p \leq n$ are defined and trivial.

The following theorem shows that cohomology $H^*(m_0)$, as well as $H^*(W^+)$ is generated by non-trivial Massey products, if we include in their number ordinary wedge products, as double Massey products.

**Theorem 4.8** (Millionshchikov [68]). The cohomology algebra $H^*(m_0)$ is generated with respect to the non-trivial Massey products by $H^1(m_0)$, namely

$$\omega(e^2 \wedge e^2 \wedge \ldots \wedge e^q \wedge e^q + 1) = e^2 \wedge \omega(e^2 \wedge \ldots \wedge e^q \wedge e^q + 1),$$

$$2\omega(e^k \wedge e^{k+1}) \in \langle e^2, e^1, \ldots, e^1, e^2 \rangle,$$

$$(15)$$

$$(-1)^{i_1} \omega(e^{i_1} \wedge e^{i_2} \wedge \ldots \wedge e^{i_q} \wedge e^{i_q + 1}) \in \langle e^2, e^1, \ldots, e^1, \omega(e^{i_2} \wedge \ldots \wedge e^{i_q} \wedge e^{i_q + 1}) \rangle,$$

First of all we present a graded defining system (a graded formal connection) $A$ for a Massey product $\langle e^2, e^1, \ldots, e^1, \omega(e^2 \wedge \ldots \wedge e^q \wedge e^q + 1) \rangle$. To simplify the formulas we will write $\omega$ instead of $\omega(e^2 \wedge \ldots \wedge e^q \wedge e^q + 1)$.

One can verify that the following matrix $A$ with non-zero entries at the second diagonal, first line and first row gives us an answer.

$$A = \begin{pmatrix}
0 & e^2 & -e^3 & e^4 & \ldots & (-1)^{i_1} e^{i_1} & 0 \\
0 & 0 & e^1 & 0 & \ldots & 0 & D^{i_1-2}_1 \omega \\
0 & 0 & 0 & e^1 & \ldots & 0 & D^{i_1-3}_1 \omega \\
0 & 0 & 0 & 0 & \ldots & e^1 & D_{-1} \omega \\
0 & 0 & 0 & 0 & \ldots & 0 & \omega \\
0 & 0 & 0 & 0 & \ldots & 0 & 0
\end{pmatrix}.$$}

The proof follows from

$$d(D^{-1}_- \omega) = e^1 \wedge D^{-1}_- \omega, \quad d((-1)^k e^k) = (-1)^{k-1} e^{k-1} \wedge e^1.$$}

The related cocycle $c(A)$ is equal to

$$c(A) = \sum_{l \geq 2} (-1)^l e^l \wedge D^{i_1-l}_1 \omega = (-1)^{i_1} \sum_{k \geq 0} (-1)^k D^k e_{i_1} \wedge D^k D_{-1} \omega$$

**Example 4.9.** We take $\langle e^2, e^1, \omega(e^4 \wedge e^5) \rangle$.

$$A = \begin{pmatrix}
0 & e^2 & -e^3 & 0 \\
0 & 0 & e^1 & D_{-1} \omega(e^4 \wedge e^5) \\
0 & 0 & 0 & \omega(e^4 \wedge e^5) \\
0 & 0 & 0 & 0
\end{pmatrix}.$$
Compute the related cocycle $c(A)$

$$c(A) = \epsilon^2 \wedge D_1 \omega(\epsilon^4 \wedge \epsilon^5) - \epsilon^3 \wedge \omega(\epsilon^4 \wedge \epsilon^5) =$$

$$= \epsilon^2 \wedge (\epsilon^4 \wedge \epsilon^6 - 2\epsilon^3 \wedge \epsilon^7 + 3\epsilon^2 \wedge \epsilon^8) - \epsilon^3 \wedge (\epsilon^4 \wedge \epsilon^5 - \epsilon^3 \wedge \epsilon^6 + \epsilon^2 \wedge \epsilon^7) =$$

$$= -\epsilon^2 \wedge \epsilon^4 \wedge \epsilon^5 + \epsilon^2 \wedge \epsilon^4 \wedge \epsilon^6 - \epsilon^2 \wedge \epsilon^3 \wedge \epsilon^7 = -\omega(\epsilon^3 \wedge \epsilon^4 \wedge \epsilon^5).$$

Let us return to Theorem 3.1. Consider the question: for which sets $a_1, \ldots, a_n$ of one-dimensional cocycles from $H^*(m_0)$, all $k$-step Massey products $\langle a_1, \ldots, a_n \rangle_k$ are defined and trivial? The following theorem answers this question.

**Theorem 4.10** (Millionshchikov [68]). Up to an equivalence the following trivial $n$-fold Massey products of non-zero cohomology classes from $H^1(m_0)$ are defined:

| name | Massey product | parameters |
|------|----------------|------------|
| $A^{n+1}_\lambda$ | $\langle \alpha e^1 + \beta e^2, \alpha e^1 + \beta e^2, \ldots, \alpha e^1 + \beta e^2 \rangle_n$ | $n \geq 3, \lambda = (\alpha, \beta) \in \mathbb{K}P^1$ |
| $B^{n+1}_{\alpha, \beta}$ | $\langle \lambda_1 e^1 + e^2, \lambda_2 e^1 + e^2, \ldots, \lambda_n e^1 + e^2 \rangle_n$ | $n \geq 3, \lambda_i = i\alpha + \beta, \alpha, \beta \in \mathbb{K}, \alpha \neq 0$ |
| $C^{n+1}_{l, \alpha}$ | $\langle e^1, \ldots, e^1, e^2 + \alpha e^1, e^1, \ldots, e^1 \rangle_{n-l-1}$ | $\alpha \in \mathbb{K}, n \geq 3, 0 \leq l \leq n-1$ |
| $D^{2k+3}_{\alpha, \beta}$ | $\langle e^2 + \alpha e^1, e^1, \ldots, e^1, e^2 + \beta e^1 \rangle_{2k}$ | $k \geq 1, \alpha, \beta \in \mathbb{K}$ |

The statement of the present theorem is equivalent to Benoist’s classification [18] of indecomposable graded thread modules over $m_0$.

**5. Massey Products in Koszul Homology of Local Rings**

Starting with this section, we turn to a discussion of (higher) Massey products in Koszul homology of local rings and their applications in toric topology.

**Definition 5.1.** Let $(A, m, k)$ be a (commutative Noetherian) local ring, its unique maximal ideal $m$ having minimal set of generators $(x_1, \ldots, x_m)$ and its residue field being $k = A/m$. A *Koszul complex* of $(A, m, k)$ is defined to be an exterior algebra $K_A = \Lambda A^m$, where $A^m$ denotes the free $A$-module generated by a set $\{e_1, \ldots, e_m\}$, which is a differential graded algebra with a differential $d$ acting as follows:

$$d(e_{i_1} \wedge \ldots \wedge e_{i_k}) = \sum_{r=1}^{k} (-1)^{r-1} x_{i_r} e_{i_1} \wedge \ldots \wedge \widehat{e_{i_r}} \wedge \ldots \wedge e_{i_k}.$$
Definition 5.2. Let \((A, m, k)\) be a local ring. Then for an \(A\)-module \(M\) we define its Poincaré series to be a formal power series of the type

\[
P_A(M; t) = \sum_{i=0}^{\infty} \dim_k \text{Tor}^A_i(M, k)t^i.
\]

By definition, the \(k\)-module \(\text{Tor}^A_i(M, k)\) is the \(i\)th homology of a projective resolution for \(k\) (the latter viewed as an \(A\)-module via the quotient map \(A \to A/m = k\)) tensored by \(M\).

We call \(P_A := P_A(k; t)\) Poincaré series of the local ring \(A\).

A classical problem of homological algebra was to prove a conjecture by Serre and Kaplansky, which asserted that \(P_A\) is a rational function. Anick [2] (1982) found a counterexample being a quotient ring of a polynomial ring over an ideal generated by a set of monomials alongside with one binomial.

On the other hand, in 1982 Backelin [11] proved the conjecture is true for monomial rings, that is, a quotient ring of a polynomial ring by an ideal generated by monomials. More precisely, a Poincaré series of a monomial ring

\[
A = k[x_1, \ldots, x_n]/I
\]

has a form

\[
P_A = \frac{(1 + t)^m}{Q_{A,k}(t)},
\]

where \(m\) is a number of elements in a minimal set of generators of \(I\). It remains to be an open problem to determine effectively the coefficients of the denominator polynomial \(Q_{A,k}(t)\) for various classes of local rings.

The properties of the Poincaré series of a loop space \(\Omega X\) on finite simply connected CW complex \(X\) were related to the homotopy properties of \(X\), in particular, to the generating function for ranks of the homotopy groups of \(X\), in a series of influential works by Babenko [4, 6, 7]. An extensive survey on the problem of rationality and growth for Poincaré series in the context of homological algebra and homotopy theory can be found in [7].

Using a spectral sequence associated with a presentation of a local ring as a quotient ring of a regular local ring, Serre showed that for any local ring \(A\) the following coefficient-wise inequality for its Poincaré series holds:

\[
P_A \leq \frac{(1 + t)^m}{1 - \sum b_it^{i+1}},
\]

where \(m = \dim_k m/m^2\) and \(b_i = \dim_k H_i(K_A)\).

In 1962 Golod obtained a key result linking an important class of local rings with rational Poincaré series with vanishing of multiplication and all Massey products in its Koszul homology.

Theorem 5.3 ([40]). Let \(A\) be a local ring. Then Serre’s inequality (17) turns into equality if and only if multiplication and all Massey products in \(H_+(K_A)\) are vanishing.
In their monograph [48] Gulliksen and Levin proposed to refer to the local rings with the above property as Golod rings.

**Example 5.4** ([40]). Let $A$ be a free reduced nilpotent algebra, that is, a quotient ring $A_{n,r} = \frac{k[x_1, \ldots, x_n]}{(x_1, \ldots, x_n)^r}$. Golod [40] observed that multiplication and all Massey products are trivial in Koszul homology of $A_{n,r}$ and, furthermore, its Betti numbers are equal to $b_i = \binom{i+r-2}{r-1} \binom{n+r-1}{i+r-1}$.

Therefore, by Theorem 5.3, $A_{n,r}$ is a Golod ring and

$$P_{A_{n,r}} = \frac{(1+t)^n}{1 - \sum_{i=1}^{n} \binom{i+r-2}{r-1} \binom{n+r-1}{i+r-1} t^{i+1}},$$

which generalizes computation of Poincaré series given by Kostrikin and Shafarevich [55].

From now on we concentrate mainly on the special class of monomial rings called face rings, or Stanley–Reisner rings of simplicial complexes, see their definition below. Note that application to a monomial ring of a procedure called ‘polarization of a monomial ideal’ [49, 50] results in a Stanley–Reisner ring with the same homological properties as the ones of the initial ring. The Stanley–Reisner ring is a quotient ring of a polynomial algebra over a square-free monomial ideal.

**Definition 5.5.** By an (abstract) simplicial complex on a vertex set $[m] = \{1, 2, \ldots, m\}$ we mean a subset $K$ of $2^m$ such that if $\sigma \in K$ and $\tau \in \sigma$, then $\tau \in K$. The singleton elements of $K$ are called its vertices and the dimension of $K$ is defined to be one less than the maximal number of vertices in an element of $K$.

By a (convex) $n$-dimensional simple polytope $P$ with $m$ facets we mean a bounded intersection of $m$ halfspaces in $\mathbb{R}^n$ such that the supporting hyperplanes of those halfspaces are in general position. The latter condition is equivalent to saying that each vertex of $P$ is an intersection of exactly $n$ of its facets (i.e. faces of codimension one).

**Example 5.6.** Let $P^*$ be the polytope combinatorially dual to $P$. Then $P^*$ is a convex hull of $m$ vertices, which are in general position in the ambient Euclidean space $\mathbb{R}^n$. Therefore, all proper faces of $P^*$ are simplices (we say, that $P^*$ is a simplicial polytope) and its boundary $K_P = \partial P^*$ is a simplicial complex of dimension $n - 1$ with $m$ vertices.

**Definition 5.7.** Let $k$ be a commutative ring with unit, $\deg(v_i) = 2, 1 \leq i \leq m$, and $K$ be a simplicial complex on $[m]$. Then we call a graded ring

$$k[K] = \frac{k[v_1, \ldots, v_m]}{(v_{i_1} \cdots v_{i_k} | \{i_1, \ldots, i_k\} \notin K)}$$

the face ring, or the Stanley–Reisner ring of $K$.

By definition, $k[K]$ is a finite $k$-algebra and also a finitely generated $k[m] = k[v_1, \ldots, v_m]$-module via the natural projection. Thus, a graded version of definition 5.1 can be stated as follows.
Definition 5.8. Let \( \text{mdeg}(u_i) = (-1; 0, \ldots, 2, \ldots, 0) \), \( \text{mdeg}(v_i) = (0; 0, \ldots, 2, \ldots, 0) \) for \( 1 \leq i \leq m \), and \( a \in \mathbb{Z}_2^n \) (this grading will acquire its topological interpretation below). Then a \textit{multigraded Tor-module} of \( k[K] \) is a direct sum of \( k \)-modules

\[
\text{Tor}_{k[m]}^{-i,2a}(k[K], k) = H^{-i,2a}(k[K] \otimes_k \Lambda[u_1, \ldots, u_m], \partial),
\]

where in the latter differential graded algebra its differential \( \partial \) acts as \( \partial(u_i) = v_i \) and \( \partial(v_i) = 0 \) for all \( 1 \leq i \leq m \).

In the above notation, the next result due to Hochster [50] describes the \( k \)-module structure of \( \text{Tor}_{k[m]}^{-i,2a}(k[K], k) \) in terms of induced subcomplexes in \( K \).

Theorem 5.9 ([50]). Denote by \( K_I = K \cap 2^I \) the induced subcomplex of \( K \) on the vertex set \( I \subset [m] \). Then

\[
\text{Tor}_{k[m]}^{-i,2a}(k[K], k) \cong \tilde{H}^{-|I|-i-1}(K_I),
\]

where the \( k \)-th component of \( I \) is either 0 or 1, depending on whether or not \( k \in [m] \) is an element of \( I \subset [m] \), and \( |I| \) denotes the cardinality of \( I \).

The next class of spaces with a compact torus action is the main object of study in toric topology. Let \( (X, A) = \{(X_i, A_i)\}_{i=1}^m \) be an ordered set of topological pairs. The case \( X_i = X, A_i = A \) was firstly introduced in [24] as a \( K \)-\textit{power} and was then intensively studied and generalized in a series of more recent works, among which are [12] [46] [51].

Definition 5.10. A \textit{polyhedral product} over a simplicial complex \( K \) on the vertex set \( [m] \) is a topological space

\[
(X, A)^K = \bigcup_{I \in K} (X, A)^I \subseteq \prod_{i=1}^m X_i,
\]

where \( (X, A)^I = \prod_{i=1}^m Y_i \) and \( Y_i = X_i \) if \( i \in I \), and \( Y_i = A_i \) if \( i \notin I \).

The term ‘polyhedral product’ was suggested by Browder (cf. [12]).

Example 5.11. Suppose \( X_i = X \) and \( A_i = A \) for all \( 1 \leq i \leq m \). Then the next spaces are particular cases of the above construction of a polyhedral product \( (X, A)^K = (X, A)^K \).

1. Moment-angle-complex \( Z_K = (\mathbb{D}^2, S^1)^K \);
2. Real moment-angle-complex \( R_K = ([-1; 1], \{-1, 1\})^K \);
3. Davis–Januszkiewicz space \( DJ(K) := E\mathbb{T}^m \times_{\mathbb{T}^m} Z_K \simeq (\mathbb{C}P^\infty, \ast)^K \);
4. Cubical complex \( \text{cc}(K) = (I^1, 1)^K \) in the \( m \)-dimensional cube \( I^m = [0, 1]^m \), which is PL-homeomorphic to a cone over a barycentric subdivision of \( K \).

As shown by Buchstaber and Panov [24], one has a commutative diagram

\[
\begin{array}{ccc}
Z_K & \longrightarrow & (\mathbb{D}^2)^m \\
\downarrow r & & \downarrow \rho \\
\text{cc}(K) & \underset{i_v}{\longrightarrow} & I^m
\end{array}
\]
where \( i_c: \text{cc}(K) \hookrightarrow I^m = (I^1, I^1)^{[m]} \) is an embedding of a cubical subcomplex, induced by an embedding of pairs: \((I^1,1) \subset (I^1, I^1)\), the maps \( r \) and \( \rho \) are projection maps onto the orbit spaces of a \( \mathbb{T}^m \)-action, induced by coordinatewise action of \( \mathbb{T}^m \) on the complex unitary polydisk \( (D^2)^m \) in \( \mathbb{C}^m \).

Buchstaber and Panov [24] gave the following construction for the moment-angle manifold \( Z_P \) over a simple \( n \)-dimensional polytope \( P \) with \( m \) facets. This space was first introduced by Davis and Januszkiewicz in [29].

**Definition 5.12.** Let a moment-angle manifold \( Z_P \) of a polytope \( P \) be a pullback from the following commutative diagram

\[
\begin{array}{ccc}
Z_P & \xrightarrow{i_Z} & \mathbb{C}^m \\
\downarrow & & \downarrow \mu \\
P & \xrightarrow{i_P} & \mathbb{R}^m_\geq
\end{array}
\]

where \( i_P \) is an affine embedding of \( P \) into \( \mathbb{R}^m \) and \( \mu(z_1, \ldots, z_m) = (|z_1|^2, \ldots, |z_m|^2) \). The projection map \( Z_P \to P \) in the above diagram is a quotient map of the canonical \( \mathbb{T}^m \)-action on \( Z_P \), induced by the standard coordinatewise action of \( \mathbb{T}^m = \{ z \in \mathbb{C}^m : |z_i| = 1 \text{ for } i = 1, \ldots, m \} \) on \( \mathbb{C}^m \). Therefore, \( \mathbb{T}^m \) acts on \( Z_P \) with an orbit space \( P \) and \( i_Z \) is a \( \mathbb{T}^m \)-equivariant embedding.

It follows that \( Z_P \) is a nonsingular intersection of Hermitian quadrics embedded into \( \mathbb{C}^m \) with a trivial normal bundle. Furthermore, it was also shown in [24] to be a smooth closed \( 2 \)-connected \( (m + n) \)-dimensional manifold, which is \( \mathbb{T}^m \)-equivariantly homeomorphic to \( Z_K \).

It was proved in [24] that there exists a homotopy fibration of polyhedral products

\[
Z_K \to (\mathbb{C}P^\infty, *)^K \to BT^m,
\]

where the first map is induced by a natural map of pairs \((D^2, S^1) \to (\mathbb{C}P^1, *)\) and the second is induced by inclusion \((\mathbb{C}P^\infty, *) \hookrightarrow (\mathbb{C}P^\infty, \mathbb{C}P^\infty)\).

Applying Eilenberg–Moore spectral sequence argument and analyzing topology of the polyhedral products involved, Baskakov, Buchstaber and Panov obtained the next fundamental result, which links toric topology with combinatorial commutative algebra and, in particular, with homology theory of face rings.

**Theorem 5.13 ([25, Theorem 4.5.4]).** Cohomology algebra of \( Z_K \) over a commutative ring with unit \( k \) is given by isomorphisms

\[
H^*(Z_K; k) \cong \text{Tor}_{k[v_1, \ldots, v_m]}^*(k[K], k) \cong H^*[R(K), d] \cong \bigoplus_{I \subseteq [m]} \tilde{H}^*(K_I; k),
\]

where the differential (multi)graded algebra \( R(K) := k[K] \otimes_k A[u_1, \ldots, u_m]/(u_i^2 = u_i v_i = 0, 1 \leq i \leq m) \) and \( d \) acts as above. Here we denote by \( \tilde{H}^*(K_I; k) \) reduced simplicial cohomology of
a simplicial complex $K_I$ and set $\tilde{H}^{-1}(\emptyset; k) \cong k$. The last isomorphism above is a sum of isomorphisms

$$H^p(\mathbb{Z}_K; k) \cong \sum_{I \subset [m]} \tilde{H}^{p-|I|-1}(K_I; k).$$

In order to determine a product of two cohomology classes $\alpha = [a] \in \tilde{H}^p(K_I_1; k)$ and $\beta = [b] \in \tilde{H}^q(K_I_2; k)$, consider an embedding of subsets $i : K_{I_1 \sqcup I_2} \to K_{I_1} * K_{I_2}$ and a canonical $k$-module isomorphism of cochains:

$$j : \tilde{C}^p(K_{I_1}; k) \otimes \tilde{C}^q(K_{I_2}; k) \to \tilde{C}^{p+q+1}(K_{I_1 \sqcup I_2}; k).$$

Then a product of the classes $\alpha$ and $\beta$ is given by:

$$\alpha \cdot \beta = \begin{cases} 0, & \text{if } I_1 \cap I_2 \neq \emptyset; \\
   i^*[j(a \otimes b)] \in \tilde{H}^{p+q+1}(K_{I_1 \sqcup I_2}; k), & \text{if } I_1 \cap I_2 = \emptyset. \end{cases}$$

It turned out that Golodness of a face ring $k[K]$ is closely related to the case when $\mathbb{Z}_K$ is homotopy equivalent to a wedge of spheres; by Theorem 5.13, the latter implies the former. However, the opposite statement is not true, see [44, Example], [57], and [53]. Using methods of stable homotopy theory and toric topology such as the stable homotopy decomposition of polyhedral products by Bahri, Benderski, Cohen, and Gitler [12, 13, 14] and the fat wedge filtration method due to Iriye and Kishimoto [52], several authors, among which are Grbić and Theriault [45], Iriye and Kishimoto [51], Grbić, Panov, Theriault, and Wu [44], were able to identify several important classes of simplicial complexes for which $k[K]$ is a Golod ring. For a detailed exposition on this problem we refer the reader to a survey article by Grbić and Theriault [47].

By definition, Golodness of $k[K]$ implies triviality of multiplication in its Koszul homology $H_*(K_{k[K]})$, or equivalently, $\cup(\mathbb{Z}_K) = 1$. Another related area of research, which also attracts much attention from the scholars these days, emerged from a false claim made in [17], namely, that Golodness of $K$ is equivalent to $\cup(\mathbb{Z}_K) = 1$. The first counterexample was constructed by Katthän in 2015. More precisely, he proved the next result.

**Theorem 5.14** ([54]). The following statements hold.

1. If $\dim K \leq 3$, then $\cup(\mathbb{Z}_K) = 1 \iff K$ is a Golod complex, that is, $k[K]$ is Golod of any field $k$;
2. There exists a 4-dimensional simplicial complex $K$ such that
   a. $\cup(\mathbb{Z}_K) = 1$;
   b. There is a nontrivial triple Massey product $\langle \alpha_1, \alpha_2, \alpha_3 \rangle \subset H^*(\mathbb{Z}_K)$; therefore, $K$ is not a Golod complex.

Since that time there have already appeared a number of works devoted to identification of the class of simplicial complexes $K$ for which Golodness of $K$ is equivalent to $\cup(\mathbb{Z}_K) = 1$. Such simplicial complexes and face rings are referred to as quasi-Golod, that is the case when there are no non-trivial triple or higher order Massey products in their Koszul homology. The
The next statement is probably the most general one so far, characterizing a class of Golod (and quasi-Golod) simplicial complexes and face rings.

**Theorem 5.15** ([44, 69]). If $K$ is a flag simplicial complex, that is, all its minimal non-faces have two vertices, then the following statements are equivalent:

1. $\text{sk}^1(K)$ is a chordal graph, that is, it does not have any induced cycles of length greater than three;
2. $\bigcup(Z_K) = 1$, i.e. multiplication in $H^+(Z_K; k)$ is trivial;
3. $Z_K$ is homotopy equivalent to a wedge of spheres;
4. $K$ is a Golod complex;
5. Commutator subgroup $\pi_1(R_K) = RC'_K$ of the right-angled Coxeter group $RC_K$ is a free group;
6. Associated graded Lie algebra $\text{gr}(RC'_K)$ is free.

Polyhedral products of the type $(X, *)^K$ are rationally formal, provided the space $X$ is rationally formal, see [25, Chapter 8]. In particular, there are no non-trivial triple, or higher order Massey products in their (singular) cohomology (over $\mathbb{Q}$).

The next section will be devoted to the opposite case, in which there exist non-trivial Massey products in cohomology of moment-angle-complexes and moment-angle manifolds.

6. **Massey products in Toric Topology and nonformality of polyhedral products**

Since for any simplicial complex $K$ its moment-angle-complex $Z_K$ is 2-connected, the lowest degree cohomology classes in $H^*(Z_K)$ having a non-trivial Massey product may arise only in degree three. For triple Massey products the latter case was first analyzed by Denham and Suciu [33] (strictly defined products) and recently by Grbić and Linton [42] (products with non-zero indeterminacy).

The first example of a non-trivial triple Massey product for polyhedral products was constructed by Baskakov [16]. His construction was later generalized by the first author [58, 59, 60], by Buchstaber and Limonchenko [23], who developed a theory of direct families of polytopes with non-trivial Massey products, and recently by Grbić and Linton [43] to the case of a non-trivial Massey product of any prescribed order. The current section is devoted to a discussion of the above mentioned results.

At first, let us consider the case of a non-trivial triple Massey product of three dimensional classes in $H^*(Z_K)$. By Theorem 5.13 a 3-dimensional element in $H^*(Z_K)$ corresponds to a pair of vertices of $K$ not connected by an edge.

In fact, the next criterion shows that non-trivial triple Massey products $\langle \alpha_1, \alpha_2, \alpha_3 \rangle \subset \tilde{H}^*(Z_K)$ for $\dim \alpha_i = 3, 1 \leq i \leq 3$ are determined by the graph $\text{sk}^1(K)$ of the simplicial complex $K$.

**Theorem 6.1** ([33, 42]). Let $\langle \alpha_1, \alpha_2, \alpha_3 \rangle \subset \tilde{H}^*(Z_K)$ for $\dim \alpha_i = 3, 1 \leq i \leq 3$ be a defined Massey product. Then
Figure 1. Nontrivial triple Massey products of 3-dimensional classes.

(a) $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is non-trivial and strictly defined if and only if $\alpha_i$ generates $\tilde{H}^0(K_{i,i+1})$ for $1 \leq i \leq 3$ in Figure 1(a);
(b) $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is non-trivial and has non-zero indeterminacy if and only if $\alpha_i$ generates $\tilde{H}^0(K_{i,i+1})$ for $1 \leq i \leq 3$ in Figure 1(b).

Remark. Statement (a) was proved firstly by Denham and Suciu as [33, Theorem 6.1.1]. However, it was asserted there that any non-trivial triple Massey product of 3-dimensional classes in $H^*(Z_K)$ has such a form; the additional two graphs of Figure 1(b) were found and the statement (b) above was proved by Grbić and Linton [42]. This also provided us with a first example of a non-strictly defined non-trivial Massey product in cohomology of a moment-angle-complex.

Now we turn to a discussion of higher order non-trivial Massey products in Koszul homology of a Stanley–Reisner ring $H_*(K_{K[K]}) \cong H^*(Z_K;k)$. Following [60], we fix the notation as indicated below.

Let us consider a set of induced subcomplexes $K_{I_j}$ on pairwisely disjoint subsets of vertices $\{I_j\}$ for $1 \leq j \leq k$ and their cohomology classes $\alpha_j \in \tilde{H}^{d(j)}(K_{I_j})$, $1 \leq j \leq k$. If an $s$-fold Massey product $(s \leq k)$ of consecutive classes $\langle \alpha_{i+1}, \ldots, \alpha_{i+s} \rangle$ for $1 \leq i + 1 < i + s \leq k$ is defined, then $\langle \alpha_{i+1}, \ldots, \alpha_{i+s} \rangle$ is a subset of $\tilde{H}^{d(i+1,i+s)}(K_{I_{i+1} \sqcup \ldots \sqcup I_{i+s}})$, where $d(i+1,i+s) = d(i+1) + \ldots + d(i+s) + 1$.

Now our goal is to determine the conditions sufficient for a Massey product $\langle \alpha_1, \ldots, \alpha_k \rangle$ of cohomology classes introduced above to be defined and, furthermore, to be strictly defined.

We are going to give a complete proof of the following theorem, in which statements (1) and (2) were originally proved by the first author as [60] Lemma 3.3. Alongside with statement (3), this result provides an effective tool to determine non-trivial $k$-fold Massey products for $k \geq 3$ in $H^*(Z_K)$ for a given simplicial complex $K$, when one knows multigraded (or, algebraic) Betti numbers of $K$ and the combinatorial structure of the corresponding full subcomplex $K_{I_1 \sqcup \ldots \sqcup I_k}$.

Theorem 6.2. Let $k \geq 3$. Then

(1) If $\tilde{H}^{d(s,r+s)}(K_{I_1 \sqcup \ldots \sqcup I_{r+s}}) = 0$, $1 \leq s \leq k-r$, $1 \leq r \leq k-2$, then the $k$-fold Massey product $\langle \alpha_1, \ldots, \alpha_k \rangle$ is defined;
(2) If $\tilde{H}^{d(s,r+s)-1}(K_{I_1 \sqcup \ldots \sqcup I_{r+s}}) = 0$, $1 \leq s \leq k-r$, $1 \leq r \leq k-2$ and the $k$-fold Massey product $\langle \alpha_1, \ldots, \alpha_k \rangle$ is defined, then the $k$-fold Massey product $\langle \alpha_1, \ldots, \alpha_k \rangle$ is strictly defined.
(3) In the latter case, there exists a defining system $C = (c_{i,j})_{i,j=1}^{k+1}$ for $\langle \alpha_1, \ldots, \alpha_k \rangle$ such that $c_{s,r+s+1} \in C^{d(s,r+s) - 1}(K_{I_1 \sqcup \ldots \sqcup I_{r+s}})$, $1 \leq s \leq k - r$, $1 \leq r \leq k - 2$

and

$$\langle \alpha_1, \ldots, \alpha_k \rangle = \{[a(C)]\} \in H^{d(1,k)}(K_{I_1 \sqcup \ldots \sqcup I_k}), \quad a(C) = -\sum_{j=1}^{k-1} \bar{c}_{1,1+j} \wedge c_{1+j,k+1},$$

in the differential graded algebra $\oplus_{I \subseteq \{n\}} \mathbb{C} \mathbb{T}(K_I) \cong \mathbb{C}^*(\mathbb{Z}_K)$.

Proof. The proof goes by induction on $k \geq 3$. First, we prove statement (1).

For $k = 3$ the condition (1) implies that the 2-fold products $\langle \alpha_1, \alpha_2 \rangle$ and $\langle \alpha_2, \alpha_3 \rangle$ vanish and the triple Massey product $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is defined. By the condition (2), indeterminacy in $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is trivial, and therefore, $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is strictly defined.

The inductive hypothesis follows, since all the Massey products of consecutive elements of orders $2, \ldots, k - 1$ for $k \geq 4$ are defined by the inductive assumption and contain only zero elements by Theorem 5.13. Therefore, a defining system $C$ exists and the corresponding cocycle $a(C)$ represents an element in the Massey product $\langle \alpha_1, \ldots, \alpha_k \rangle$.

Now we prove statement (2).

We need to show that $[a(C)] = [a(C')]$ for any two defining systems $C$ and $C'$. By the inductive assumption, suppose that the statement holds for defined higher Massey products of orders less than $k \geq 4$. The induction step can be divided into two parts, and in both of them we also act by induction.

I. Let us construct a sequence of defining systems $C(1), \ldots, C(k - 1)$ for the defined Massey product $\langle \alpha_1, \ldots, \alpha_k \rangle$ such that the following properties:

1. $C(1) = C$;
2. $c_{ij}(r) = c'_{ij}$, if $j - i \leq r$;
3. $[a(C(r))] = [a(C(r + 1))]$, for all $1 \leq r \leq k - 2$.

Observe that (2) implies $c_{i,i+1}'(r) = a_i = a_i'$ for all $1 \leq i \leq k$, and $C(k - 1) = C'$. We apply induction on $r \geq 1$ to determine the defining systems $C(r)$. Since $C(1) = C$ by (1), we need to prove the induction step assuming that $C(r)$ is already defined.

For any $1 \leq s \leq k - r$ let us consider a cochain $b_s = c'_{s,r+s+1} - c_{s,r+s+1}(r)$.

By definition of a Massey product,

$$d(b_s) = d(c'_{s,r+s+1}) - d(c_{s,r+s+1}(r)) = \sum_{p=s+1}^{r+s} \frac{c'_{s,p}}{c'_{p,r+s+1}} - \sum_{p=s+1}^{r+s} \bar{c}_{s,p}(r) \wedge c_{p,r+s+1}(r) = 0$$

by property (2) of $C(r)$ above. It follows that $b_s$ is a cocycle, and therefore,

$$[b_s] \in \bar{H}^{d(s,r+s) - 1}(K_{I_1 \sqcup \ldots \sqcup I_{r+s}}) = 0,$$

for all $1 \leq s \leq k - r$, by condition (2) of our statement, since here we have $1 \leq r \leq k - 2$ by property (3) above.
II. Observe that the construction of the defining system $C(r + 1)$ will be finished if one is able to determine a sequence of defining systems $C(r, s)$ ($0 \leq s \leq k - r$) for $(\alpha_1, \ldots, \alpha_k)$ with the following properties:

\begin{enumerate}
  \item[(1')] $C(r, 0) = C(r)$;
  \item[(2')] $c_{ij}(r, s) = c_{ij}(r)$, if $j - i \leq r$, and
  \begin{align*}
  c_{ij}(r, s) &= \begin{cases} 
  c_{ij}(r), & \text{if } i > s; \quad (*) \\
  c_{ij}(r) + b_i, & \text{if } i \leq s \quad (**) 
  \end{cases}
  \end{align*}
  \end{enumerate}

when $j - i = r + 1 \geq 2$;

\begin{enumerate}
  \item[(3')] $[a(C(r, s))] = [a(C(r, s + 1))]$, for all $0 \leq s \leq k - r - 1$.
\end{enumerate}

It is easy to see that property (2')(**) for $j = i + r + 1$ implies that $c_{ij}(r, k - r) = c_{ij}(r) + (c'_{i,r+1+i} - c_{i,r+1+i}(r)) = c'_{ij}$, the latter being equal to $c_{ij}(r + 1)$ by property (2) above, and therefore, one can take $C(r + 1) = C(r, k - r)$ and the proof will be completed by induction.

So, to finish the proof, it suffices to construct a sequence of defining systems $C(r, s)$. Now we shall do it acting by induction on $s \geq 0$. The base of induction $s = 0$ follows by property (1').

Now assume that we have already constructed $C(r, s)$ and let us determine the defining system $C(r, s + 1)$.

If $i > s + 1$, then one can take $c_{ij}(r, s + 1) = c_{ij}(r, s)$, see (2')(*). Similarly, one can also take $c_{ij}(r, s + 1) = c_{ij}(r, s)$ if $j < s + r + 2$, see (2')(**). Suppose $1 \leq i \leq s + 1 < s + 2 + r \leq j \leq k + 1$.

Now, by induction on $j - i \geq r + 1$ we shall determine a set of cochains $\{b_{ij}\}$ such that

\begin{align*}
  c_{ij}(r, s + 1) &= c_{ij}(r, s) + b_{ij} \\
  \text{(***)}.
\end{align*}

Equality (**) implies that $c_{s+1,r+2+s}(r, s + 1) = c_{s+1,r+2+s}(r, s) + b_{s+1}$ and the latter is equal to $c_{s+1,r+2+s}(r, s) + b_{s+1}$ by equality (*). Therefore, one can set $b_{s+1,r+2+s} = b_{s+1}$. By inductive assumption, assume that for all $r + 1 \leq j - i < w$ the cochains $b_{ij}$ have already been defined. Then equality (***)) implies that

\begin{align*}
  d(b_{ij}) &= d(c_{ij}(r, s + 1)) - d(c_{ij}(r, s)) = \sum_{p=i+1}^{j-1} (\bar{\alpha}_{i,p}(r, s) + \bar{b}_{i,p}) \wedge (c_{p,j}(r, s) + b_{p,j}) - \\
  &- \sum_{p=i+1}^{j-1} \bar{c}_{i,p}(r, s) \wedge c_{p,j}(r, s) = \sum_{p=i+1}^{s+1} \bar{c}_{i,p}(r, s) \wedge b_{p,j} + \sum_{p=r+s+2}^{j-1} \bar{b}_{i,p} \wedge c_{p,j}(r, s),
\end{align*}

where the last equality holds, since $\sum_{p=i+1}^{j-1} \bar{b}_{i,p} \wedge b_{p,j} = 0$, because $b_{p,j} = 0$ when $p > s + 1$, and $b_{i,p} = 0$ when $p < r + 2 + s$, and one gets the following equality for any $r + 1 \leq j - i < w$:

\begin{align*}
  d(b_{ij}) &= \sum_{p=i+1}^{s+1} \bar{c}_{i,p}(r, s) \wedge b_{p,j} + \sum_{p=r+s+2}^{j-1} \bar{b}_{i,p} \wedge c_{p,j}(r, s) \quad (1.1)
\end{align*}
Observe that for \( j - i = w \) the right hand side of the equality (1.1) is a cocycle \( a \) representing an element \( \alpha = [a] \) in

\[- \langle \alpha_i, \ldots, \alpha_s, [b_{s+1}], \alpha_{s+r+2}, \ldots, \alpha_{j-1} \rangle\]  

(1.2)

This can be shown acting by induction on \( j - i \geq r + 2 \). Indeed, for \( j - i = r + 2 \) and \( 1 \leq i \leq s + 1 < s + 2 + r \leq j \leq k + 1 \) we have only two possible cases: (1) \( i = s + 1, j = s + r + 3 \) and the right hand side of (1.1) has the form \( \bar{b}_{s+1,r+s+2} \wedge c_{s+r+2,s+r+3} = \bar{b}_{s+1}a_{s+r+2} \). The latter cocycle represents \(-\langle [b_{s+1}], \alpha_{s+r+2} \rangle\); (2) \( i = s, j = s + r + 2 \) and the right hand side of (1.1) has the form \( \bar{c}_{s,s+1}b_{s+1,s+r+2} = \bar{a}_s \wedge b_{s+1} \). The latter cocycle represents \(-\langle \alpha_s, [b_{s+1}] \rangle\). The induction step follows from the equality (1.1), definition of a (higher) Massey product, and the inductive assumption.

Since \([b_{s+1}] = 0\), one concludes that the Massey product given by the formula (1.2) above is trivial. Furthermore, as \( r \geq 1 \) it follows that the order of this Massey product is less than \((j - 1) - i + 1 = j - i \leq k\). Therefore, we can apply the inductive assumption on \( k \) to this Massey product, since \([b_{s+1}] \in \bar{H}^\beta(K_{I_1, \ldots, I_s, I_{s+1}})\) for \( \beta = d(s + 1, r + s + 1) - 1 = d(s + 1) + \ldots + d(r + s + 1) \). Thus, by the inductive assumption on \( k \), we obtain that the Massey product

\[0 \in \langle \alpha_i, \ldots, \alpha_s, [b_{s+1}], \alpha_{s+r+2}, \ldots, \alpha_{j-1} \rangle\]

is strictly defined, that is, it contains only zero. It follows that equality (1.1) above has a solution for \( j - i = w \). Therefore, for all \( 1 \leq i < j \leq k + 1 \) equality (1.1) implies

\[d(b_{ij}) = \sum_{p=i+1}^{j-1} \bar{c}_{i,p}(r, s + 1) \wedge c_{p,j}(r, s + 1) - \sum_{p=i+1}^{j-1} \bar{c}_{i,p}(r, s) \wedge c_{p,j}(r, s)\]

The above equality means that: (i) \( C(r, s + 1) \) is also a defining system for the Massey product \( \langle \alpha_1, \ldots, \alpha_k \rangle \) and (ii) \([a(C(r, s + 1))] = [a(C(r, s))]\) (when in the above formula \( j - i = k \)). The whole proof of the statement (2) is now completed by induction on the order \( k \) of a Massey product.

Finally we prove statement (3).

We proceed by induction on \( k \) again, using the fact that the right hand side in Theorem 5.13 admits a multigraded refinement, that is, the differential \( d \) respects the multigraded structure on the Tor-module, given by Definition 5.8 and Theorem 5.9. Therefore, induction on \( j - i = r \geq 1 \) in the defining system \( C \), see condition (3) above, gives us simplicial cochains \( c_{s,r+s+1} \in C^d(s,r+s+1)(K_{I_1, \ldots, I_r}) \) for all \( 1 \leq s \leq k - r, 1 \leq r \leq k - 2 \), satisfying the relations for elements of a defining system themselves and giving the unique element \([a(C)]\) of the \( k \)-fold Massey product \( \langle \alpha_1, \ldots, \alpha_k \rangle \). The latter class is an element of the group \( H^d(1,k)(K_{I_1, \ldots, I_k}) \) by definition of multiplication in \( H^*(Z_K) \), see Theorem 5.13. This finishes the proof of the theorem.

Remark. It is easy to see that Theorem 6.2 implies the triple Massey products in \([16] \) and \([83] \) are all strictly defined. On the other hand, in the example of a trivial triple Massey product in \( H^*(Z_P) \) when \( P \) is a hexagon, see \([60] \) Example 3.4.1] as well as in the case of Theorem 6.1 (b) the condition (2) of Theorem 6.2 is not satisfied and those Massey products are not strictly defined.
First examples of non-trivial higher Massey products of any order in $H^*(\mathbb{Z}_K)$ were constructed by Limonchenko in a short note [58]. A complete proof of nontriviality and an example of computation of a non-trivial 4-fold Massey product were given by the first author in [59]. We describe this construction below, following [23].

**Definition 6.3.** [58, 23] Let $Q^0$ be a point and $Q^1 \subset \mathbb{R}^1$ be a segment $[0,1]$. Denote by $I^n = [0,1]^n, n \geq 2$ an $n$-dimensional cube with facets $F_1, \ldots, F_{2n}$ in such a way that $F_i, 1 \leq i \leq n$ contains the origin $0$, $F_i$ and $F_{n+i}$ are parallel for all $1 \leq i \leq n$. Then the face ring of the cube $I^n$ has the form:

$$k[I^n] = k[v_1, \ldots, v_n, v_{n+1}, \ldots, v_{2n}] / I^n,$$

where $I^n = (v_1v_{n+1}, \ldots, v_nv_{2n})$.

Consider a polynomial ring

$$k[v_1, \ldots, v_{2n}, v_{k',n+k'+i'}, v_{k',n+k'+i''}, \ldots, v_{k',n+k'+i''}]_{1 \leq i' \leq n-2, 1 \leq k' \leq n-i'},$$

and its monomial ideal, generated by square free monomials:

$$I = \left\langle v_kv_{n+k+i}, v_{k',n+k'+i'}, v_{k',n+k'+i''}, v_{k',n+k'+i''}, v_{k',n+k'+i''} \right\rangle,$$

where $v_j$ corresponds to $F_j$ for all $1 \leq j \leq 2n$, and

$$0 \leq i \leq n-2, 1 \leq k \leq n-i, 1 \leq i', i'' \leq n-2, 1 \leq k' \leq n-i',$$

$$1 \leq k'' \leq n-i'' \leq n-2, 1 \leq p \neq k' \leq k' + i', 0 \leq l \neq i' \leq n-2,$$

$$k' + i' = k'' \text{ or } k'' + i'' = k'.$$

Let us define $Q^n \subset \mathbb{R}^n$ to be a simple polytope such that $I_{Q^n} = I$. Note that $Q^n$ has a natural realization as a 2-truncated cube, and moreover, its combinatorial type does not depend on the order in which the faces of the cube $I^n$ are truncated (the generators $v_{i,j}$ correspond to the truncated faces $F_i \cap F_j$ of $I^n$).

Then by [60, Theorem 3.6], for any $n \geq 2$ there exists a strictly defined non-trivial $n$-fold Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle \subset H^*(\mathbb{Z}_{Q^n})$, where $\alpha_i$ is a generator of $H^0(F_i \cup F_{n+i})$.

Finally, using the above construction, Theorem 6.2, and the simplicial multiwedge operation (or, $J$-construction), introduced in the framework of toric topology by Bahri, Bendersky, Cohen, and Gitler [12], the first author proved the following result.

**Theorem 6.4 (60).** For any $n \geq 2$ there exists a strictly defined non-trivial Massey product of order $k$ in $H^*(\mathbb{Z}_{Q^n})$ for all $k$, $2 \leq k \leq n$. Furthermore, there exists a family of moment-angle manifolds $\mathcal{F}$ such that for any given $l, r \geq 2$ there is an $l$-connected manifold $M \in \mathcal{F}$ with a strictly defined non-trivial $r$-fold Massey product in $H^*(M)$.

Applying the theory of nestohedra [36, 72] and the theory of the differential ring of polytopes, introduced by Buchstaber [20], the previous result was generalized by Buchstaber and Limonchenko as follows.

**Theorem 6.5 (23).** For any given $\ell \geq 2$ and $n_1, \ldots, n_r \geq 2, r \geq 1$ there exists a polyhedral product $\mathcal{Z}(P_B; (D_2^n, S^{2n-1})) = \mathcal{Z}_{P_B}(J), J = J(\ell, n_1, \ldots, n_r) := (j_1, \ldots, j_m)$ for $m = f_0(P_B)$, over a flag nestohedron $P_B$ on a connected building set $B$ such that
• The moment-angle manifold $Z_{P_b(J)}$ is $\ell$-connected;
• There exist strictly defined non-trivial Massey products of orders $n_1, \ldots, n_r$ in $H^*(Z_{P_b(J)})$.

Another way to generalize Theorem 6.4 has been recently suggested by Grbić and Linton. Their approach is based on a careful investigation, on the level of cochains, of cup and Massey products of the cohomology classes occurred in Theorem 6.2. Note that below the non-trivial Massey products, which are stated to exist, are no longer strictly defined, in general.

**Theorem 6.6** ([43]). The following statements hold.

(1) For any given simplicial complexes $K_1, \ldots, K_n$ there exists a sequence of stellar subdivisions of their join $K_1 \ast \ldots \ast K_n$ resulting in a simplicial complex $K$ such that there exists a non-trivial $n$-fold Massey product in $H^*(Z_K)$;

(2) If a simplicial complex $K'$ is obtained from a simplicial complex $K$ by a sequence of a special type edge truncations and there exists a non-trivial $n$-fold Massey product in $H^*(Z_{K'})$, then the same property holds in $H^*(Z_K)$.

**Remark.** The case of $n = 3$ in Theorem 6.6 (1) coincides with the construction due to Baskakov [16].

Using Theorem 6.6 (2) and Theorem 6.1 (1), Grbić and Linton obtain a result due to Zhuravleva [75], who proved that for any Pogorelov polytope $P$ (see [71, 1, 21, 22]) there exists a non-trivial triple Massey product in $H^*(Z_P)$.

Moreover, using Theorem 6.6 (2) and Theorem 6.4, Grbić and Linton showed that for any $n \geq 2$ there exists a non-trivial Massey product in $H^*(Z_{P_{e^n}})$ of any order $k$, $2 \leq k \leq n$, where $P_{e^n}$ is an $n$-dimensional permutohedron.

It was earlier proved by the first author [60, Lemma 4.9, Theorem 4.10] that if $P$ is a graph-associahedron, in particular, an $n$-dimensional permutohedron, see [27], then the following conditions are equivalent: 1) $Z_P$ is rationally formal; 2) there exist no non-trivial strictly defined triple Massey products in $H^*(Z_P)$; 3) $P$ is a product of segments, pentagons, and hexagons.

Finally, it should be mentioned that there exists a simple polytope $P$ such that there are no non-trivial triple Massey products of 3-dimensional classes in $H^*(Z_P)$, but there exists a non-trivial strictly defined 4-fold Massey product in $H^*(Z_P)$. This was proved by Baralić, Grbić, Limonchenko, and Vučić [15] for $P$ being the dodecahedron using Theorem 6.2.

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Faculty of Computer Science, National Research University Higher School of Economics, Moscow, Russia
E-mail address: ilimonchenko@yandex.ru

The Fields Institute for Research in Mathematical Sciences, Department of Mathematics, University of Toronto, Toronto, Canada
E-mail address: ilimonch@fields.utoronto.ca

Department of Mechanics and Mathematics, Moscow State University, 119992 Moscow, Russia
E-mail address: mitiam@hotmail.com

Gubkin Russian State University of Oil and Gas (National Research University), 65 Leninsky Prospekt, 119991 Moscow, Russia

Department of Mechanics and Mathematics, Moscow State University, Leninskie Gory, 119992 Moscow, Russia, E-mail: mitiam@hotmail.com