A novel improved action for SU(3) lattice gauge theory

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SU(3) lattice gauge theory is studied by means of an improved action where a $2 \times 2$ Wilson loop is supplemented to the standard plaquette term. By contrast to earlier studies using a tree level improvement, the prefactor of the $2 \times 2$ Wilson term is determined by minimizing the breaking of rotational symmetry detected from the static quark-antiquark potential. On coarse lattices, the novel action is superior to the Iwasaki action and comparable with DBW2 action. The scaling behavior of the novel action is studied by using the static quark potential and the ratio of the deconfinement temperature and the string tension.

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A systematic approach to the non-perturbative regime of Yang-Mills theory is provided by computer simulations of lattice gauge theory. Regularization is introduced by replacing continuous spacetime by a hypercubic lattice with lattice spacing $a$. Improved actions for those lattice gauge simulations have attracted much interest in the recent past, since they allow for simulations on coarse lattices without an overwhelming impact of discretization errors. For example if the infra red regime of QCD Green functions is addressed (see e.g. [1, 2]), the low lying lattices without an overwhelming impact of discretization errors.

In lattice gauge theory, dynamical degrees of freedom are unitary matrices $U_\mu(x) \in SU(3)$ associated with the links of the lattice. The partition function is defined by

$$Z = \int DU_\mu \exp\{-S_{\text{latt}}[U]\}.$$  

In order to recover continuum Yang-Mills theory, one demands in the limit $a \to 0$ (adopting minimal Landau gauge)

$$U_\mu(x) = \exp\{it^b A_\mu^b(x)a\},$$  

$$S_{\text{latt}} = \sum_{x,\mu,\nu} \left[ \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a a^4 + O_1 a^6 + \ldots \right],$$

where $t^a$ are the generators of the SU(3) group, $A_\mu^b(x)$ are the gauge fields and $F_{\mu\nu}^b$ is the usual non-Abelian field strength tensor. When one designs a realization of $S_{\text{latt}}$ in terms of the matrices $U_\mu(x)$, it is essential that $S_{\text{latt}}$ is gauge invariant for any choice of the lattice spacing $a$, thereby enforcing gauge invariance in the continuum limit. The most simple choice for such an action is the Wilson action:

$$S_{\text{latt}}^{\text{wil}} = \beta \sum_{x,\mu,\nu} \left[ \frac{1}{3} \text{Re} \text{tr} \{1 - W_{\mu\nu}^{1 \times 1}(x)\} \right],$$

where $W_{\mu\nu}^{n \times m}(x)$ is the $n \times m$ rectangular Wilson loop, located at $x$, with extension $n$ in $\mu$ and extension $m$ in $\nu$ direction. $\beta$ is related to the bare gauge coupling $g_0$ by $\beta = 6/g_0^2$. Universality implies that any other lattice action which meets these requirements also serves the purpose in the limit $a \to 0$. The crucial point, however, is that computer simulations are carried out on finite lattices consisting of $N^4$ lattice points. The corresponding extension of the lattice in one particular direction is given by $L = Na$. If we insist on $L > 1.2$ fm, the lattice spacing must not be smaller than $a = 0.075$ fm for a $16^4$ grid. The central idea is to construct a lattice action which minimizes the contribution of the irrelevant terms such as $O_1$ in [2]. At the expense of a more complicated lattice action, simulations might be performed on coarse lattices and might produce results with little corrections even for sizable values $a$. Several proposals have been made for such actions: tadpole improved actions [4, 5], improved actions obtained by lattice perturbation theory [3, 4], improved actions based on renormalization group studies [4, 5], and “perfect” actions operating close to the fixed point of the theory [3, 5].

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**TABLE I:** Coupling constants of the “1×2” term contributing to the improved action.

| Wilson | Tadpole improved | Iwasaki | DBW2 |
|--------|------------------|---------|------|
| $f$    | 0                | 1/20$a_0^2$ | 0.0907 | 0.1148 |

2 Wilson term is determined by minimizing the breaking of rotational symmetry detected from the static quark-antiquark potential. On coarse lattices, the novel action is superior to the Iwasaki action and comparable with DBW2 action. The scaling behavior of the novel action is studied by using the static quark potential and the ratio of the deconfinement temperature and the string tension.
Let us focus on improved actions of the type

\[ S_{\text{latt}}^{\text{imp}} = \beta \sum_{x, \mu > \nu} \left[ \frac{1}{3} \text{Re} \left\{ 1 - W_{\mu \nu}^{1 \times 1}(x) \right\} - f \frac{1}{3} \text{Re} \left\{ 2 - W_{\mu \nu}^{1 \times 2}(x) - W_{\mu \nu}^{2 \times 1}(x) \right\} \right]. \]

Different ideas on improvement result in different proposals for \( f \). More than ten years ago, Iwasaki used perturbation theory for a renormalization group analysis and suggested \( f = 0.0907 \). The so-called DBW2 action, derived from “double blocked Wilson” configurations [7], chooses \( f = 0.1148 \). The “tadpole improved tree level” action [3] (see also [10] for details) favors a \( \beta \) dependent factor \( f = 1/20 u_0(\beta) \), where

\[ u_0(\beta) = \left[ \frac{1}{3} \text{Re} \left\langle \text{tr} W_{\mu \nu}^{1 \times 1}(x) \right\rangle \right]^{1/4} \]

must be self-consistently determined. See Table II for a summary. A thorough study of the scaling behavior of several “1 \( \times \) 2” actions was recently performed by Necco [11]. It was found at least for small \( a \) that the DBW2 action shows larger lattice artefacts than the Iwasaki action.

| \( \beta \) | 6.0 | 6.2 | 6.4 |
|----------|-----|-----|-----|
| \( \sigma a^2 \) | 0.343(3) | 0.247(1) | 0.178(10) |

Here, we propose a new type of improvement which is based upon the lattice action of type

\[ S_{\text{latt}}^{\text{new}} = \beta \sum_{x, \mu > \nu} \left[ \frac{1}{3} \text{Re} \left\{ 1 - W_{\mu \nu}^{1 \times 1}(x) \right\} - f \frac{1}{3} \text{Re} \left\{ 1 - W_{\mu \nu}^{2 \times 2}(x) \right\} \right]. \]

Since \( W_{\mu \nu}^{2 \times 2}(x) \) possesses the same (lattice) symmetry structure than the plaquette, the same symmetry breaking terms show up upon an expansion with respect to small lattice spacings. One therefore can hope that for a suitable choice of \( f \) almost all of the breaking terms can be removed. Improved actions containing a \( W_{\mu \nu}^{2 \times 2} \) term were considered in [12, 13] where the prefactor \( f \) was determined at tree level, i.e., \( f_{\text{tree}} = 1/64 \). It turned out that already the tree level improved action substantially progressed the analysis of the hot gluon plasma. Rather than using the tree level improvement, we here pursue a non-perturbative improvement: \( f \) will be determined by minimizing the breaking of rotational symmetry by the underlying lattice. Since \( O_1 \) in (2) explicitly contains terms which break rotational symmetry, the absence of rotational symmetry breaking is directly a measure for the suppression of irrelevant terms and, therefore, for improvement.

For an analysis of rotational symmetry breaking, let us consider the potential of a static quark-antiquark pair separated by a distance vector \( \vec{r} \). Choosing one of the main axis of the cubic lattice, i.e., \( \vec{r} = r \hat{e}_i, \ i = 1 \ldots 3\), yields the “on-axis” potential \( V_{\text{on}}(r) \). Alternatively, one might choose \( \vec{r} = r/\sqrt{2} (1,1,0)^T \) (and permutations) and \( \vec{r} = r/\sqrt{3} (1,1,1)^T \). Contributions from these directions to the potential results in the “off-axis” potential. The
The key idea of non-perturbative improvement is to free the parameter $f = 1/64$ of the $2 \times 2$ action, and to find a choice for $f$ which is optimal with respect to the symmetry breaking effects. One possibility for this program is to perform a renormalization group study employing lattice perturbation theory, see [4], or using blocked configurations, see [9]. Here, we adopt a practical point of view: For a given $\beta$, we calculate the quantity $\delta_v$ for several values $f$. Starting with the tree level value $f = 1/64$, we systematically increase $f$ in steps of $10^{-3}$, and we are monitoring $\delta_v$. Once the order $a^2$-terms, the terms generating the linear slope of $\delta_v$ in figure [1] are eliminated, we stop the procedure. For this aim, 100 configurations were generated using a $10^9$ lattice. Configurations were obtained by virtue of the Cabibbo Marinari algorithm [10] supplemented with micro-canonical reflections in order to reduce autocorrelations (details can be found in [11]). The above procedure for determining $f$ is repeated for several values $\beta$ implying that $f$ could in principle be a function of $\beta$. However, it turned out that the choice

$$f = 0.056(1)$$

yields satisfactory results for all $\beta$ values under investigations. Note that this value is roughly a factor 3.5 larger than the tree level choice, i.e., $f_{tree} = 1/64 = 0.015625$. We find at the end that the “2 $\times$ 2” action with $f = 0.056$ discussed in the present paper yields results of the same quality than the DBW2 action, in particular, in the regime of coarse lattices. In addition, the novel action seems to be comparable with the Iwasaki action at small $a$.

Let us finally explore the scaling properties of the novel action for a $16^3$ lattice. For each $\beta \in \{6.0, 6.2, 6.4\}$, 136 independent configurations have been generated. The static quark-antiquark potential is fitted with the ansatz

$$V_{on}(r) = a + b/r + cr.$$
The Polyakov line, i.e., parameter of the transition is the expectation value of the Wilson action. The order parameter of rotational symmetry for lattice spacings larger than $\sigma^2 > 0.25$, the present action yields results in this regime which are even comparable with those of the DBW2 action. On the other hand, scaling violations, detected from $T_c/\sqrt{\sigma}$, are of the same order as those from the Iwasaki action.

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FIG. 4: Scaling analysis for $T_c/\sqrt{\sigma}$ for the Wilson action and the novel action.

The results for the string tension are listed in table III. Using these values for $\sigma a^2$, we finally express the static potential (as well as $r$) in physical units ($\sigma = (440 \text{fm})^2$ is set for the reference scale). The result is shown in figure 2. We observe almost perfect scaling: the data points obtained from different $\beta$ values fall on top of a single curve. In addition, effects from the breaking of the rotational symmetry by the underlying lattice are invisible.

As a further application, let us study the deconfinement phase transition at finite temperatures. The order parameter of the transition is the expectation value of the Polyakov line, i.e.,

$$y := \frac{1}{L^3} \left\langle \sum_{\vec{x}} \frac{1}{3} \text{tr} \mathcal{P}(\vec{x}) \right\rangle, \quad (9)$$

where $L$ is the spatial extent of the lattice, and

$$\mathcal{P}(\vec{x}) = \prod_{n=1}^{N_t} U_0(\vec{x}, n)$$

with $N_t$ being the number of lattice points in Euclidean time direction. The temperature $T$ is given by $T = 1/N_t a(\beta)$. Since we are only interested in the scaling properties of the novel action, we have chosen a physical situation which is accessible by a reasonable amount of computer time: Rather than to aspire the infinite volume limit $L^3 \to \infty$, we studied the finite volume case $L = 3/T_c$. Thereby, $T_c$ is defined as follows: Let $P(y)$ denote the probability distribution for the variable $y$ in (9). At the deconfinement phase transition, the probability distribution shows a characteristic double peak structure reflecting the first order nature of the transition (see figure 3). In the finite volume system, $T_c$ is defined as the temperature at which the two peaks of $P(y)$ are of equal height. For a given $N_t$, simulations extracted the critical coupling $\beta_c$ for the novel action and for the Wilson action for comparison. The findings for the novel action are summarized in table III. For the largest lattice and $\beta = 6.91$, the auto correlation length was estimated to $\tau \approx 12$, whereas 1300 configurations were used to estimate $P(y)$. Our final results for $T_c/\sqrt{\sigma}$ are summarized in figure 4.

In conclusion, a thorough study of the non-perturbatively improved $2 \times 2$ action was performed. While the Iwasaki action shows a significant breaking of rotational symmetry for lattice spacings larger than $\sigma a^2 > 0.25$, the present action yields results in this regime which are even comparable with those of the DBW2 action. On the other hand, scaling violations, detected from $T_c/\sqrt{\sigma}$, are of the same order as those from the Iwasaki action.
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