A Fixed Point Theorems in L-Fuzzy Quasi-Metric Spaces

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Abstract: At first we considered the L-fuzzy metric space notation which is useful in modeling some phenomena where it is necessary to study the relationship between two probability functions as well observed in Gregori et al. [A note on intuitionistic fuzzy metric spaces. Chaos, Solitons and Fractals 2006; 28: 902-905]. Then we introduced the concept of fixed point theorem in L-fuzzy metric space and finally, showed that every contractive mapping on an L-fuzzy metric space has a unique fixed point.

Key words: Fixed-point theorem, fuzzy sets

INTRODUCTION

The concept of fuzzy sets was introduced initially by Zadeh[1] in 1965. Various concepts of fuzzy metric spaces were considered in George and Veeramani[2] and Mihet[3,4].

In this research, at first we shall adopt the usual terminology, notation and conventions of L-fuzzy metric spaces introduced by Saadati et al.[5] which are a generalization of fuzzy metric spaces[2] and intuitionistic fuzzy metric spaces[6,7]. Then we consider the fixed point theorem on such spaces and show that every contractive mapping on non-Archimedean L-fuzzy metric space has a unique fixed point.

Definitions 1.1: Goguen[8] let $L = (L, \leq)$ be a complete lattice and $U$ a non-empty set called universe. An L-fuzzy set $A$ on $U$ is defined as a mapping $A: U \to L$. For each $u$ in $U$, $A(u)$ represents the degree (in $L$) to which $u$ satisfies $A$.

Classically, a triangular norm $T$ on $([0,1], \leq)$ is defined as an increasing, commutative, associative mapping $T: [0,1]^2 \to [0,1]$ satisfying $1, x) = x$ for all $x \in [0,1]$. These definitions can be straightforwardly extended to any lattice $L = (L, \leq)$.

Definition 1.2: A triangular norm (t-norm) on $L$ is a mapping $\tau: L^2 \to L$ satisfying the following conditions:

- $(\forall x \in L)(\tau(x,1_L) = x)$ (boundary condition)
- $(\forall (x,y) \in L^2)(\tau(x,y) = \tau(y,x))$ (commutativity)
- $(\forall (x,y,z) \in L^3)(\tau(x,\tau(y,z)) = \tau(\tau(x,y),z))$ (associativity)
- $(\forall (x,x',y,y') \in L^4)$
- $(x \leq_L x' \land y \leq_L y' \Rightarrow \tau(x,y) \leq_L \tau(x',y'))$ (monotonicity)

The t-norm $\tau$ is Hadzic type if $\tau(x,y) \geq_L \land(x,y)$ for every $x, y \in L$ where

$$\land(x,y) = \begin{cases} x, & \text{if } x \leq_L y; \\ y, & \text{if } y \leq_L x. \end{cases}$$

Triangle norms are recursively defined by $\tau^1 = \tau$ and

$$\tau^n(x_{(i)},...,x_{(a+i)}) = \tau^{n-1}(x_{(i)},...,x_{(a+i)}).$$

for $n \geq 2$, $x_{(i)} \in L$ and $i \in \{1,2,...,n+1\}$.

Definition 1.3: Deschrijver et al.[9] A negator on $L$ is any decreasing mapping $N: L \to L$ satisfying $N(0_L) = 1_L$ and $N(1_L) = 0_L$. If $N(N(x)) = x$ for all $x \in L$, then $N$ is called an involutive negator.

In this research the negator $N: L \to L$ is fixed. The negator $N$, on $([0,1], \leq)$ defined as $N(x) = 1-x$, for all $x \in [0,1]$, is called the standard negator on $([0,1], \leq)$.

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Definition 1.4: The triple \((X, M, \tau)\) is said to be an L-fuzzy quasi-metric space if \(X\) is an arbitrary (non-empty) set, \(\tau\) is a continuous \(t\)-norm on \(L\) and \(M\) is an \(L\)-fuzzy set on \(X \times [0, +\infty[^{\tau}\) satisfying the following conditions for every \(x, y, z \in X\) and \(t, s \in \]0, +\infty[^{\tau}\):

- \(M(x, y, t) > l_1\)
- \(M(x, y, t) = M(y, x, t) = l_1\) for all \(t > 0\) if and only if \(x = y\)
- \(\tau(M(x, y, t), M(y, z, t+s)) \leq l_1 M(x, t+s)
- \(M(x, y, t) : 0, +\infty[^{\tau}\) is continuous
- \(\lim_{t \to +\infty[^{\tau}} M(x, y, t) = l_1\).

In this case, \(M\) is called an \(L\)-fuzzy quasi-metric.

If, in the above definition, the triangular inequality (c) is replaced by

\[
\text{(NA) } \tau(M(x, y, t), M(y, z, t)) \leq l_1 M(x, z, \max\{t, s\}) \quad \forall x, y, z \in X, \quad \forall t, s > 0
\]

or, equivalently,

\[
\tau(M(x, y, t), M(y, z, t)) \leq l_1 M(x, z, t) \quad \forall x, y, z \in X, \quad t > 0.
\]

Then the triple \((X, M, \tau)\) is called a non-Archemidean \(L\)-fuzzy quasi-metric space\(^{[4]}\).

For \(t \in \]0, +\infty[^{\tau}\), we define the closed ball \(B[x, r, t]\) with center \(x \in X\) and radius \(r \in L\{0, l_1, 1\}\), as

\[
B[x, r, t] = \{y \in X : M(x, y, t) \geq l_1 N(r)\}.
\]

Definition 1.5: A sequence \(\{x_n\}_{n \in \mathbb{N}}\) in an \(L\)-fuzzy quasi-metric space \((X, M, \tau)\) is called a right (left) Cauchy sequence if, for each \(\varepsilon \in L\{0, l_1\}\) and \(t > 0\), there exists \(n_0 \in \mathbb{N}\) such that, for all \(m, n \geq n_0\) (\(m, n \geq m_0\)),

\[
M(x_n, x_m, t) \geq l_1 N(\varepsilon).
\]

The sequence \(\{x_n\}_{n \in \mathbb{N}}\) is said to be convergent to \(x \in X\) in the \(L\)-fuzzy quasi-metric space \((X, M, \tau)\) (denoted by \(x_n \xrightarrow{M} x\)) if \(M(x_n, x, t) = M(x, x_n, t) \to l_1\), whenever \(n \to +\infty[^{\tau}\) for every \(t > 0\). An \(L\)-fuzzy quasi-metric space is said to be right (left) complete if and only if every right (left) Cauchy sequence is convergent.

Definition 1.6: Let \((X, M, \tau)\) be an \(L\)-fuzzy metric space and let \(N\) be a negator on \(L\). Let \(A\) be a subset of \(X\), then the \(L\)-fuzzy diameter of the set \(A\) is the function defined as:

\[
\delta_A(s) = \sup_{x \in A} \inf_{y \in A} M(x, y, t).
\]

A sequence \(\{A_n\}_{n \in \mathbb{N}}\) of subsets of an \(L\)-fuzzy quasi-metric space is called decreasing sequence if \(A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots\).

The following lemma gives conditions under which the intersection of such sequences is nonempty.

Lemma 1.7: Let \((X, M, \tau)\) be a left complete \(L\)-fuzzy metric space and let \(\{A_n\}_{n \in \mathbb{N}}\) be a decreasing sequence of nonempty closed subsets of \(X\) such that \(\delta_{A_n}(t) \to 1_L\) as \(n \to +\infty[^{\tau}\). Then \(A = \bigcap_{n=1}^{+\infty[^{\tau}} A_n\) contains exactly one point.

Proof: From the assumption \(\delta_{A_n}(t) \to 1_L\), it is evident that the set \(A\) can’t contain more than one element. So it is enough to show that \(A\) is nonempty. Let \(x_n\) be a point in \(A_n\). Since \(\delta_{A_n}(t) \to 1_L\), by definition of \(5F\)-diameter, \(\{x_n\}_{n \in \mathbb{N}}\) is a left Cauchy sequence in \(X\). Since \((X, M, \tau)\), is left complete, \(\{x_n\}_{n \in \mathbb{N}}\), has a limit \(x\). We show that \(x\) is in \(A\) and for this it suffices to show that \(x\) is in \(A_{n_0}\) for some fixed \(n_0\). If \(\{x_n\}_{n \in \mathbb{N}}\), has only finitely many distinct points, then \(x\) is that point infinitely repeated and is therefore in \(A_{n_0}\). If \(\{x_n\}_{n \in \mathbb{N}}\), has infinitely many distinct points, then \(x\) is a limit point of the set of points of the sequence, so it is a limit point of the subset \(\{x_n\}_{n \in \mathbb{N}_0}\) of the set of points of the sequence which implies it is a limit point of \(A_{n_0}\) and since \(A_{n_0}\) is closed, it is in \(A_{n_0}\).

Corollary 1.8: Let \((X, M, \tau)\) be a left complete \(L\)-fuzzy metric space and let \(\{A_i\}_{i \in I}\) be a family of closed subsets of \(X\), which has the finite intersection property and for each \(\varepsilon > 0\), contains a set of \(L\)-fuzzy diameter less than \(\varepsilon\), then \(\bigcap_{i \in I} A_i \neq \emptyset\).

Proof: For each \(n = 1, 2, \ldots\) let \(i_n \in I\) denote an index such that

\[
\delta_{A_{i_n}}(t) = M(x, y, l_n^{\tau})
\]

for every \(x \neq y\). The set \(A_i = \bigcap_{n=1}^{+\infty[^{\tau}} A_{i_n}\) satisfy the assumption of the last lemma. Therefore \(\bigcap_{i \in I} A_i\),
contains exactly one point say $x_0$. Then $x_0 \in A_i$, for $i \in I$. Indeed define $A'_n = A_i \cap A_{a_i}$ for $n = 1, 2, \ldots$. Now

$$\emptyset \neq \bigcap_{a \in A_i} A'_{a} = A_i \cap \bigcap_{a \in A_i} A_{a} = A_i \cap \{x_0\}.$$  

**Definition 1.9:** Let $(X, M, \tau)$ be an L-fuzzy metric space. A mapping $\Delta: X \to X$ is said to be contractive if whenever $x$ and $y$ are distinct point in $X$, we have

$$M(\Delta x, \Delta y, t) > \delta M(x, y, t).$$

**MAIN RESULT**

**Theorem 2.1:** Let $(X, M, \tau)$ be non-Archimedean L-fuzzy metric space, in which $\tau$ is Hadzic type. If $\Delta: X \to X$ is a contractive mapping then $\Delta$ has a unique fixed point.

**Proof:** Let $B_x = B[x, \eta(t), t]$ with $\eta(x, t) = N(m(x, \Delta x, t))$ and $t > 0$. Let $A$ be the collection of all these balls for all $x \in X$. The relation $B_x \subseteq B_y$, if and only if $B_y \subseteq B_x$ is a partial order in $A$. Consider a totally ordered subfamily $A_i$ of $A$. From Corollary 1.8, we have

$$\bigcap_{B_{x_i}} B_x = B \neq \emptyset.$$  

Let $y \in B$ and $B_x \in A_i$, then

$$M(x, y, t) \geq \delta N(N(M(x, Ax, t))) = M(x, Ax, t) \quad (1)$$

Now, if $x_0 \in B_y$, then

$$M(x_0, y, t) \geq \delta N(N(M(y, Ay, t)))$$

$$\geq \delta^2 M(y, Ax, t),$$

Thus

$$M(x_0, y, t) \geq \delta M(x, Ax, t) \quad (2)$$

Now, by using (1) and (2) we obtain

$$M(x_0, x, t) \geq \delta M(x_0, y, t) \geq \delta \tau M(x, Ax, t) \geq \delta M(x, Ax, t).$$

Therefore $x_0 \in B_x$ and $B_x \subseteq B_y$ implies that $B_z \subseteq B_y$ for all $B_z \in A_i$. Thus $B_y$ is an upper bound in $A$ for family $A_i$. Hence by Zorn’s Lemma, $A$ has a maximal element, say, $B_z$, for some $z \in X$. We claim that $z = \Delta z$.

Suppose that $z \neq \Delta z$. Since $\Delta$ is contractive, therefore

$$M(\Delta z, \Delta^2 z, t) > \delta M(z, \Delta z, t),$$

where $\Delta^2 = \Delta \Delta$ and

$$\Delta z \in B[z, \eta(\Delta z, t), t] \cap B[z, \eta(z, t), t]$$

Therefore $B_z \subseteq B_x$ and $z$ is not in $B_z$. Thus $B_x \subseteq B_z$, which contradicts the maximality of $B_z$. Hence $\Delta$ has a fixed point.

Uniqueness easily follows from contractive condition.

**CONCLUSION**

In this research we introduce the concept of fixed point theorem in L-fuzzy metric spaces and present some results.

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