Monotonicity Formula On Cigar Soliton

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Abstract. In this paper, we prove a monotonicity formula on cigar soliton. Using the monotonicity, we obtain a three-ball type theorem. There might be some people who have proved the three-ball theorem on Cigar Soliton before, but we use a new method. We add a weight function in the norm of u. By doing so, we make the calculation process much easier because when using integral by part, the term that integrates at the surface of ball would disappear as the weight function we add would be zero.

1. Introduction
This Monotonicity formula on cigar soliton contains profound geometrical significance and would help prove the three-ball theorem on it [1-4]. Many people studied similar type inequalities, such as [5-10]. In the paper [5], Tobias H. Colding, Camillo De Lellis, and William P. Minicozzi II use monotonicity formula to prove the three-ball type theorem for a manifold with finitely many cylindrical ends. In the paper [6], Tobias H. Colding and William P. Minicozzi II prove monotonicity of a parabolic frequency on manifolds. In the paper [7], Tobias H. Colding and William P. Minicozzi II discuss some monotonicity formula for both parabola and elliptic operators and their geometrical meaning. In the paper [8], Tobias Holck Colding proves three monotonicity for manifolds with a lower Ricci curvature bound. In the paper [10], Jiuyi Zhu uses the monotonicity of a frequency function to prove the three-ball theorem for Schrödinger equations and higher order elliptic equations. In the paper [9], Jianyu Ou uses the same method to prove the three-ball type theorem on gradient shrinking Ricci soliton with constant scalar curvature.

Ricci solitons are generalization of Einstein metrics, special solutions to Hamilton’s Ricci flow and they are important in the singularity study of the Ricci flow [11]. A complete Riemannian manifold \((M, g)\) is called a Ricci soliton if

\[ \text{Ric} + \nabla \nabla f = \lambda g \quad (1) \]

Cigar soliton is a simple example of steady Ricci soliton which is discovered by Hamilton [12]. Its base manifold is \(\mathbb{R}^2\), and its metric is

\[ g = \frac{dx_1^2 + dx_2^2}{1 + x_1^2 + x_2^2} \quad (2) \]

By integration, we can also get the distance of any point from the origin [13],

\[ \rho = \ln \left( \sqrt{x_1^2 + x_2^2} + \sqrt{x_1^2 + x_2^2 + 1} \right) \quad (3) \]

For some eigenfunction

\[ \Delta u = -\lambda u \quad (4) \]

we can find the monotonicity formula on cigar soliton using similar method as Jiuyi Zhu uses in [10],

Theorem 1.1.

\[ (e^{20r} + 7 \ln r N(r) + \lambda e^{30r})' \geq 0 \quad (5) \]
where \( N(\rho) = \frac{I(r)}{H(\rho)} = \frac{(\alpha+1) \int_{B(r)} u \cdot \nabla \rho^2 > (r^2 - \rho^2)^\alpha dx}{\int_{B(r)} u^2 (r^2 - \rho^2)^\alpha dx} \) for arbitrary \( \alpha \geq 2 \).

Furthermore, by the monotonicity formula we find, we can prove the three-ball type theorem on cigar soliton [14].

**Theorem 1.2.**

\[
 h(r_2)C_1 + C_3 \leq \left( \frac{4^\alpha}{3^\alpha} C_1 + C_3 \right) \frac{e^{C_2} r_2^{2,\alpha} C_1 + C_3}{r_2^{2,\alpha} C_1 + 2^\alpha C_3} h(r_1)C_1 h(r_3)C_3
\]

where \( C_1, C_2, C_3, C_4 \) are constants depend only on \( r_1, r_2, r_3, r_1 < r_2 < 2r_2 < r_3 \), and \( h(r) = \int_{B(r)} u^2 dx \).

**2. Monotonicity formula**

First, we define

\[
 H(\rho) = \int_{B(r)} u^2 (r^2 - \rho^2)^\alpha dx
\]

where \( \alpha \) is any integer bigger than 1.

To find the monotonicity formula, we need to find the maximum value of \( H'(r) \) [15][16]. To prove the three-ball type theorem, we also need the minimum value of \( H'(r) \).

When \( r > r_0 \),

\[
 H'(r) = \lim_{r_0 \to r} \frac{H(r) - H(r_0)}{r - r_0} = \lim_{r_0 \to r} \int_{B(r) \setminus B(r_0)} u^2 (r^2 - \rho^2)^\alpha dx + \int_{B(r_0)} u^2 [(r^2 - \rho^2)^\alpha - (r^2 - \rho^2)^\alpha] dx
\]

\[
 \leq \lim_{r_0 \to r} \int_{B(r) \setminus B(r_0)} u^2 (r^2 - \rho^2)^\alpha dx + 2\alpha \int_{B(r)} u^2 (r^2 - \rho^2)^\alpha dx + 2\alpha r \int_{B(r)} u^2 (r^2 - \rho^2)^\alpha dx - 2\alpha r \int_{B(r)} u^2 (r^2 - \rho^2)^\alpha dx
\]

\[
 = 2\alpha \int_{B(r)} u^2 (r^2 - \rho^2)^\alpha dx
\]

On the other hand, using the same method, we can prove that \( H'(r) \geq 2\alpha r \int_{B(r)} u^2 (r^2 - \rho^2)^\alpha dx \) when \( r < r_0 \).

Therefore,

\[
 H'(r) = H(r) + \frac{1}{r} \int_{B(r)} u < \nabla u \nabla \rho^2 > (r^2 - \rho^2)^\alpha dx + \frac{1}{2r} \int_{B(r)} u^2 \Delta \rho^2 (r^2 - \rho^2)^\alpha dx
\]

\[
 = \frac{2\alpha}{r} H(r) + \frac{1}{r} \int_{B(r)} u < \nabla u \nabla \rho^2 > (r^2 - \rho^2)^\alpha dx + \frac{1}{2r} \int_{B(r)} u^2 \Delta \rho^2 (r^2 - \rho^2)^\alpha dx
\]

The second term in the right side would later be used in the frequency function. Therefore, we only need to find the upper bound and lower bound of the third term in the right hand side.

\[
 0 \leq \frac{1}{r} \int_{B(r)} u^2 \frac{\rho}{\sqrt{x_1^2 + x_2^2}} (r^2 - \rho^2)^\alpha dx
\]

\[
 = \frac{1}{r} \int_{B(r)} u^2 e^{2\rho_0 - e^{-2\rho_0}} (r^2 - \rho^2)^\alpha dx
\]

\[
 \leq \frac{1}{r} \int_{B(r)} u^2 e^{2\rho_0 - e^{-2\rho_0}} (r^2 - \rho^2)^\alpha dx \leq \frac{1}{r} H(r)
\]

where \( \rho_0 \) is one of the solutions of \( \frac{1}{e^{2\rho_0 - e^{-2\rho_0}}} \) and where \( \frac{4\rho}{e^{2\rho_0 - e^{-2\rho_0}}} \) has its maximum value.

So we find the upper bound and lower bound of \( H'(r) \) now.
\[
\frac{2\alpha + 1}{r} H(r) + \frac{1}{r} \int_{B(r)} u < \nabla u \nabla \rho^2 > (r^2 - \rho^2)^{\alpha} dx \\
\leq H'(r) \leq \frac{2\alpha + 2}{r} H(r) + \frac{1}{r} \int_{B(r)} u < \nabla u \nabla \rho^2 > (r^2 - \rho^2)^{\alpha} dx
\]

Let
\[
I(r) = (\alpha + 1) \int_{B(r)} u < \nabla u \nabla \rho^2 > (r^2 - \rho^2)^{\alpha} dx
\]
and we get
\[
\frac{2\alpha + 1}{r} H(r) + \frac{1}{(\alpha + 1)r} I(r) \leq H'(r) \leq \frac{2\alpha + 2}{r} H(r) + \frac{1}{(\alpha + 1)r} I(r)
\]

Also, we let \(N(r) = \frac{I(r)}{H(r)}\). Later we will try to prove the monotonicity of the transformation of \(N(r)\).

Integrate by part to transform \(I(r)\)
\[
I(r) = (\alpha + 1) \int_{B(r)} u < \nabla u \nabla \rho^2 > (r^2 - \rho^2)^{\alpha} dx
\]
\[
= \int_{B(r)} |\nabla u|^2 (r^2 - \rho^2)^{\alpha+1} dx + \int_{B(r)} u \Delta u (r^2 - \rho^2)^{\alpha+1} dx
\]
\[
= \int_{B(r)} |\nabla u|^2 (r^2 - \rho^2)^{\alpha+1} dx - \lambda \int_{B(r)} u^2 (r^2 - \rho^2)^{\alpha+1} dx
\]

To find the monotonicity formula, we need to find the lower bound of \(I'(r)\)
\[
I'(r) = 2(\alpha + 1) r \int_{B(r)} |\nabla u|^2 (r^2 - \rho^2)^{\alpha+1} dx - 2(\alpha + 1) r \lambda \int_{B(r)} u^2 (r^2 - \rho^2)^{\alpha} dx
\]
\[
= \frac{2(\alpha + 1)}{r} \int_{B(r)} |\nabla u|^2 (r^2 - \rho^2)^{\alpha+1} dx
\]
\[
+ \frac{2(\alpha + 1)}{r} \int_{B(r)} \rho^2 |\nabla \rho|^2 |\nabla u|^2 (r^2 - \rho^2)^{\alpha} dx
\]
\[
- 2(\alpha + 1) r \lambda \int_{B(r)} u^2 (r^2 - \rho^2)^{\alpha} dx
\]
\[
= \frac{2(\alpha + 1)}{r} \int_{B(r)} |\nabla u|^2 (r^2 - \rho^2)^{\alpha+1} dx
\]
\[
- \frac{1}{2r} \int_{B(r)} |\nabla u|^2 < \nabla \rho^2 \nabla (r^2 - \rho^2)^{\alpha+1} > dx
\]
\[
- 2(\alpha + 1) r \lambda \int_{B(r)} u^2 (r^2 - \rho^2)^{\alpha} dx
\]

Since \(|\nabla \rho|^2 = 1\), we can add it to the second term in the right hand side
\[
I'(r) = \frac{2(\alpha + 1)}{r} \int_{B(r)} |\nabla u|^2 (r^2 - \rho^2)^{\alpha+1} dx
\]
\[
+ \frac{2(\alpha + 1)}{r} \int_{B(r)} \rho^2 |\nabla \rho|^2 |\nabla u|^2 (r^2 - \rho^2)^{\alpha} dx
\]
\[
- 2(\alpha + 1) r \lambda \int_{B(r)} u^2 (r^2 - \rho^2)^{\alpha} dx
\]
\[
= \frac{2(\alpha + 1)}{r} \int_{B(r)} |\nabla u|^2 (r^2 - \rho^2)^{\alpha+1} dx
\]
\[
- \frac{1}{2r} \int_{B(r)} |\nabla u|^2 < \nabla \rho^2 \nabla (r^2 - \rho^2)^{\alpha+1} > dx
\]
\[
- 2(\alpha + 1) r \lambda \int_{B(r)} u^2 (r^2 - \rho^2)^{\alpha} dx
\]

Use integral by part for the second term in the right hand side, we get
\[
I'(r) = \frac{2(\alpha + 1)}{r} \int_{B(r)} |\nabla u|^2 (r^2 - \rho^2)^{\alpha+1} dx + \frac{1}{2r} \int_{B(r)} < \nabla |\nabla u|^2 \nabla \rho^2 > (r^2 - \rho^2)^{\alpha+1} dx
\]
\[
+ \frac{1}{2r} \int_{B(r)} |\nabla u|^2 (r^2 - \rho^2)^{\alpha+1} \Delta \rho^2 dx - 2(\alpha + 1) r \lambda \int_{B(r)} u^2 (r^2 - \rho^2)^{\alpha} dx
\]

Since \(\Delta \rho^2 = 2 \rho \Delta \rho + 2|\nabla \rho|^2 \geq 2 \)
\[
I'(r) \geq \frac{2(\alpha + 3)}{r} \int_{B(r)} |\nabla u|^2 (r^2 - \rho^2)^{\alpha+1} dx + \frac{1}{2r} \int_{B(r)} < \nabla |\nabla u|^2 \nabla \rho^2 > (r^2 - \rho^2)^{\alpha+1} dx
\]
\[
- 2(\alpha + 1) r \lambda \int_{B(r)} u^2 (r^2 - \rho^2)^{\alpha} dx
\]

Now we divide the second term in the right side to find its lower bound
\[
\frac{1}{2r} \int_{B(r)} < \nabla |\nabla u|^2 \nabla \rho^2 > (r^2 - \rho^2)^{\alpha+1} dx
\]
\[
= \frac{1}{2r} \int_{B(r)} (1 + x_1^2 + x_2^2) \partial_j [(1 + x_1^2 + x_2^2) \partial_i u \partial_i u \partial_j \rho^2 (r^2 - \rho^2)^{\alpha+1} dx
\]
\[
= \frac{1}{2r} \int_{B(r)} (1 + x_1^2 + x_2^2) \partial_j (1 + x_1^2 + x_2^2) \partial_i u \partial_i u \partial_j \rho^2 (r^2 - \rho^2)^{\alpha+1} dx
\]
\[
+ \frac{1}{r} \int_{B(r)} (1 + x_1^2 + x_2^2) \partial_i u \partial_i u \partial_j \rho^2 (r^2 - \rho^2)^{\alpha+1} dx
\]

Because
\[ (1 + x_1^2 + x_2^2) \geq 0, \partial_j u \partial_i u \geq 0, \partial_j (1 + x_1^2 + x_2^2) \partial_j \rho^2 \geq 0, (r^2 - \rho^2)^{\alpha + 1} \geq 0 \] 

Therefore,

\[ \frac{1}{2r} \int_{B(r)} (1 + x_1^2 + x_2^2) \partial_j (1 + x_1^2 + x_2^2) \partial_i u \partial_j u \partial_i \rho^2 (r^2 - \rho^2)^{\alpha + 1} dx \geq 0 \] 

and

\[ \frac{1}{2r} \int_{B(r)} < \nabla |\nabla u|^2 \nabla \rho^2 > (r^2 - \rho^2)^{\alpha + 1} dx \]

\[ \geq \frac{1}{r} \int_{B(r)} (1 + x_1^2 + x_2^2)^2 \partial_i u \partial_i u \partial_i \rho^2 (r^2 - \rho^2)^{\alpha + 1} dx \]

Use integral by part, we get

\[ \frac{1}{2r} \int_{B(r)} < \nabla |\nabla u|^2 \nabla \rho^2 > (r^2 - \rho^2)^{\alpha + 1} dx \]

\[ \geq \frac{1}{r} \int_{B(r)} u < \nabla u \nabla \rho^2 > (r^2 - \rho^2)^{\alpha + 1} dx \]

\[ - \frac{1}{r} \int_{B(r)} (1 + x_1^2 + x_2^2)^2 \partial_j u \partial_i u \partial_j \rho^2 (r^2 - \rho^2)^{\alpha + 1} dx \]

\[ + \frac{\alpha + 1}{r} \int_{B(r)} < \nabla u \nabla \rho^2 >^2 (r^2 - \rho^2)^{\alpha} dx \]

\[ - \frac{2}{r} \int_{B(r)} < \nabla u \nabla \rho^2 > (r^2 - \rho^2)^{\alpha} dx \]

Put it back into \( I'(r) \), and we get

\[ I'(r) \geq \frac{2(\alpha + 1)}{r} I(r) + \frac{1}{r} \int_{B(r)} |\nabla u|^2 (r^2 - \rho^2)^{\alpha + 1} dx \]

\[ - \frac{\lambda}{2r} \int_{B(r)} u^2 \Delta \rho^2 (r^2 - \rho^2)^{\alpha + 1} dx \]

\[ - \frac{1}{r} \int_{B(r)} (1 + x_1^2 + x_2^2)^2 \partial_j u \partial_i u \partial_j \rho^2 (r^2 - \rho^2)^{\alpha + 1} dx \]

\[ + \frac{\alpha + 1}{r} \int_{B(r)} < \nabla u \nabla \rho^2 >^2 (r^2 - \rho^2)^{\alpha} dx \]

\[ - \frac{2}{r} \int_{B(r)} < \nabla u \nabla \rho^2 > (r^2 - \rho^2)^{\alpha} dx \]

Since the third term in the right side has a negative coefficient, we need to find its upper bound

\[ \frac{\lambda}{2r} \int_{B(r)} u^2 \Delta \rho^2 (r^2 - \rho^2)^{\alpha + 1} dx \leq \frac{2\lambda}{r} \int_{B(r)} u^2 (r^2 - \rho^2)^{\alpha + 1} \leq 2r \lambda \int_{B(r)} u^2 (r^2 - \rho^2)^{\alpha} \] 

Also, we need to find the upper bound for the fourth term in the right side for its negative coefficient.

\[ \frac{1}{r} \int_{B(r)} (1 + x_1^2 + x_2^2)^2 \partial_i u \partial_i u \partial_i \rho^2 (r^2 - \rho^2)^{\alpha + 1} dx \]

\[ = \frac{1}{r} \int_{B(r)} (1 + x_1^2 + x_2^2)^2 \partial_i u \partial_i u \partial_{i1} \rho^2 (r^2 - \rho^2)^{\alpha + 1} dx \]

\[ + \frac{1}{r} \int_{B(r)} (1 + x_1^2 + x_2^2)^2 \partial_j u \partial_j u \partial_{j2} \rho^2 (r^2 - \rho^2)^{\alpha + 1} dx \]

\[ + \frac{\lambda}{r} \int_{B(r)} (1 + x_1^2 + x_2^2)^2 \partial_i u \partial_i u \partial_{i2} \rho^2 (r^2 - \rho^2)^{\alpha + 1} dx \]

\[ \partial_{11} \rho^2 = 2(\partial_1 \rho)^2 + 2\rho \partial_{11} \rho = \frac{2\rho}{(x_1^2 + x_2^2)(1 + x_1^2 + x_2^2)} \]
\[ +2 \ln \left( \sqrt{x_1^2 + x_2^2} + \sqrt{1 + x_1^2 + x_2^2} \right) - \frac{-x_1^2 + x_2^2}{x_1^2 + x_2^2} \leq \frac{2r + 4}{1 + x_1^2 + x_2^2} \]  

We can also prove that

\[ \partial_2 \rho \leq \frac{2r + 4}{1 + x_1^2 + x_2^2}, \quad \partial_1 \rho \leq \frac{2r + 4}{1 + x_1^2 + x_2^2} \]

using the same method.

Therefore,

\[ \frac{1}{r} \int_{B(r)} (1 + x_1^2 + x_2^2)^2 \partial_i u \partial_j u \partial_1 \rho^2 (r^2 - \rho^2)^{a+1} dx \leq \frac{4r + 8}{r} \int_{B(r)} |\nabla u|^2 (r^2 - \rho^2)^{a+1} dx \]  

Now to find the upper bound for the last term in the right side

\[ \frac{2}{r} \int_{B(r)} \nabla u \nabla \rho^2 > \partial_i u \partial_i (1 + x_1^2 + x_2^2) (r^2 - \rho^2)^{a+1} dx \]

\[ = \frac{1}{r} \int_{B(r)} \partial_i u \partial_j \rho^2 \partial_i u \partial_i (1 + x_1^2 + x_2^2)^2 (r^2 - \rho^2)^{a+1} dx \]

\[ = \frac{1}{r} \int_{B(r)} \partial_1 u \partial_1 \rho^2 \partial_1 u \partial_1 (1 + x_1^2 + x_2^2)^2 (r^2 - \rho^2)^{a+1} dx \]

\[ + \frac{1}{r} \int_{B(r)} \partial_2 u \partial_2 \rho^2 \partial_2 u \partial_2 (1 + x_1^2 + x_2^2)^2 (r^2 - \rho^2)^{a+1} dx \]

\[ + \frac{1}{r} \int_{B(r)} \partial_1 u \partial_1 \rho^2 \partial_1 u \partial_1 (1 + x_1^2 + x_2^2)^2 (r^2 - \rho^2)^{a+1} dx \]  

\[ \partial_1 \rho^2 \partial_1 (1 + x_1^2 + x_2^2)^2 = 2 \rho \frac{1}{\sqrt{x_1^2 + x_2^2}} \frac{2(1 + x_1^2 + x_2^2)}{2x_1} \leq 8r (1 + x_1^2 + x_2^2) \]

Similarly, we can prove that

\[ \partial_2 \rho \leq \frac{2r + 4}{1 + x_1^2 + x_2^2} \]

\[ \partial_1 \rho^2 \partial_1 (1 + x_1^2 + x_2^2)^2 \leq 8r (1 + x_1^2 + x_2^2) \]

Using the same method.

Therefore,

\[ \frac{2}{r} \int_{B(r)} \nabla u \nabla \rho^2 > (r^2 - \rho^2)^{a+1} \partial_i u \partial_i (1 + x_1^2 + x_2^2) dx \]

\[ \leq 16 \int_{B(r)} |\nabla u|^2 (r^2 - \rho^2)^{a+1} dx \]  

Put them back to \( I' (r) \), we obtain

\[ I' (r) \geq \frac{2(a + 1)}{r} \int_{B(r)} |\nabla u|^2 (r^2 - \rho^2)^{a+1} dx - 2r \lambda \int_{B(r)} u^2 (r^2 - \rho^2)^a dx \]

\[ - \frac{20 \rho + 8}{r} \int_{B(r)} |\nabla u|^2 (r^2 - \rho^2)^{a+1} dx + \frac{r + 1}{r} \int_{B(r)} \nabla u \nabla \rho^2 > (r^2 - \rho^2)^a dx \]

\[ \geq \frac{2(a + 1)}{r} \int_{B(r)} |\nabla u|^2 (r^2 - \rho^2)^{a+1} dx - 2r \lambda \int_{B(r)} u^2 (r^2 - \rho^2)^a dx - \frac{20 \rho + 7}{r} I (r) dx \]

\[ - \frac{20 \rho + 8}{r} \int_{B(r)} u^2 (r^2 - \rho^2)^{a+1} dx + \frac{a + 1}{r} \int_{B(r)} \nabla u \nabla \rho^2 > (r^2 - \rho^2)^a dx \]

\[ \geq \frac{2(a + 1)}{r} \int_{B(r)} |\nabla u|^2 (r^2 - \rho^2)^{a+1} dx - 2r \lambda \int_{B(r)} u^2 (r^2 - \rho^2)^a dx - \frac{20 \rho + 7}{r} I (r) dx \]

\[ + \frac{a + 1}{r} \int_{B(r)} \nabla u \nabla \rho^2 > (r^2 - \rho^2)^a dx \]  

To find the monotonicity of some type of \( N (r) \), we need to prove that its derivative is always non-negative.
\[ N'(r) = \frac{H(r)I'(r) - I(r)H'(r)}{H^2(r)} \]
\[ \geq \frac{1}{H^2(r)} \left[ \frac{2(\alpha + 1)}{r} I(r) - \frac{20r^7 + 7}{r} I(r) - (20r^2 + 9r)\lambda H(r) \right] \]
\[ + \frac{\alpha + 1}{r} \int_{B(r)} < \nabla u \nabla p^2 > ^2 (r^2 - \rho^2)^a dx \]
\[ - I(r) \left[ \frac{2(\alpha + 1)}{r} H(r) + \frac{1}{(\alpha + 1)r} I(r) \right] \]
\[ \geq \frac{1}{H^2(r)} \left[ \frac{20r^7 + 7}{r} I(r) - (20r^2 + 9r)\lambda H(r) \right] \]
\[ + \frac{\alpha + 1}{r} \int_{B(r)} < \nabla u \nabla p^2 > ^2 (r^2 - \rho^2)^a dx - \frac{1}{(\alpha + 1)r} I^2(r) \] (38)

According to Cauchy-Schwarz inequality,
\[ \int_{B(r)} < \nabla u \nabla p^2 > ^2 (r^2 - \rho^2)^a dx \int_{B(r)} u^2 (r^2 - \rho^2)^a dx \]
\[ \geq (\int_{B(r)} u < \nabla u \nabla p^2 > (r^2 - \rho^2)^a dx)^2 \]

Thus,
\[ N'(r) \geq \frac{1}{H^2(r)} \left[ H(r) \left[ - \frac{20r + 7}{r} I(r) - (20r^2 + 9r)\lambda H(r) \right] \right] \]
\[ \geq \frac{1}{20r + 7} - N(r) - (20r^2 + 9r)\lambda \] (40)

That means
\[ 0 \leq N'(r) + \frac{20r + 7}{r} N(r) + (20r^2 + 9r)\lambda \]
\[ = \frac{1}{e^{20r + 7lnr}} \left[ \left( e^{20r + 7lnr} N(r) \right)' + (20r^2 + 9r)\lambda e^{20r + 7lnr} \right] \]
\[ \leq \frac{1}{e^{20r + 7lnr}} \left[ \left( e^{20r + 7lnr} N(r) \right)' + 29\lambda e^{29r} \right] \]
\[ \leq \frac{1}{e^{20r + 7lnr}} \left[ e^{20r + 7lnr} N(r) + \lambda e^{30r} \right]' \] (41)

Therefore,
\[ (e^{20r + 7lnr} N(r) + \lambda e^{30r})' \geq 0 \] (42)

Theorem 1.1 is proved.

3. Three-ball Type Theorem

Here, by using the monotonicity formula we find in section 2, we aim to prove the three-ball type theorem on cigar soliton.

First, we define \( r_1 < r_2 < 2r_2 < r_3 \).

By (14)
\[ \frac{H'(r)}{H(r)} \leq \frac{2\alpha + 2}{r} + \frac{1}{(\alpha + 1)r} N(r) \] (43)
\[ (\ln H(r))' = (2\alpha + 2)(\ln r)' + \frac{1}{(\alpha + 1)r} N(r) \] (44)

That means integral from \( r_1 \) to \( 2r_2 \)
\[ \int_{r_1}^{2r_2} (\ln H(r))' dr \leq \frac{2\alpha + 2}{(\alpha + 1)r_1} \int_{r_1}^{2r_2} (2\alpha + 2)(\ln r)' dr + \int_{r_1}^{2r_2} \frac{1}{(\alpha + 1)r} N(r) dr \] (45)

So
\[ \ln \frac{H(2r_2)}{H(r_1)} \leq (2\alpha + 2) \ln \frac{2r_2}{r_1} + \frac{1}{(\alpha + 1)r_1} \int_{r_1}^{2r_2} \frac{1}{r} N(r) dr \] (46)

Now we use the monotonicity formula
\[ \ln \frac{H(2r_2)}{H(r_1)} \leq (2\alpha + 2) \ln \frac{2r_2}{r_1} + \frac{1}{(\alpha + 1)r_1} \int_{r_1}^{2r_2} \frac{e^{20r + 7lnr} N(r) + \lambda e^{30r}}{e^{20r + 7lnr} r} dr \]
\[ \leq (2\alpha + 2) \ln \frac{2r_2}{r_1} + \frac{e^{40r + 7lnr} N(2r_2) + \lambda e^{60r}}{(\alpha + 1)2r_1} \int_{r_1}^{2r_2} \frac{1}{e^{20r + 7lnr} r} dr - \frac{\lambda e^{10r_1}}{256(\alpha + 1)2^2} (2r_2 - r_1) \] (47)

Similarly, because
we can derive that
\[
\frac{H'(r)}{H(r)} \geq \frac{2a+1}{r} + \frac{1}{(a+1)}N(r)
\]  
(48)

Therefore,
\[
N(2r_2) \geq \frac{(\alpha + 1)r_1^{2\beta}}{128e^{40r_2^2}(2r_2 - r_1)} \ln H(2r_2) - \frac{(\alpha + 1)2r_1^{2\beta}\ln r_2^{r_2}}{r_1} - \frac{\lambda e^{20r_2}r_2^{128e}}{215e^{40r_2^2}r_2^{15}}
\]  
(49)

On the other hand,
\[
N(2r_2) \leq \frac{(\alpha + 1)e^{2\beta r_3}}{128e^{40r_2^2}(r_3 - 2r_2)} \ln H(r_3) - \frac{(2\alpha^2 + 3\alpha + 1)e^{2\beta r_3}r_3^{r_3}}{2r_2} - \frac{\lambda e^{20r_2}r_2^{128e}}{215e^{40r_2^2}r_2^{15}}
\]  
(50)

Let \(C_1 = \frac{(\alpha + 1)r_1^{2\beta}}{128e^{40r_2^2}(2r_2 - r_1)}\), \(C_2 = -\frac{(\alpha + 1)r_1^{2\beta}\ln r_2^{r_2}}{r_1} - \frac{\lambda e^{20r_2}r_2^{128e}}{215e^{40r_2^2}r_2^{15}}\), \(C_3 = \frac{(\alpha + 1)e^{2\beta r_3}}{128e^{40r_2^2}(r_3 - 2r_2)}\), \(C_4 = \frac{\lambda e^{2\beta r_3}}{215e^{40r_2^2}r_2^{15}}\).

Then,
\[
(C_1 + C_2) \ln H(2r_2) + C_3 \leq C_1 \ln H(r_3) + C_2 \ln H(r_3) + (C_4 - C_2)
\]  
(52)

Here we define
\[
h(r) = \int B(r) u^2 \, dx
\]  
(54)

It’s clear that:
\[
H(r) \leq \int B(r) u^2 r^{2\alpha} \, dx = r^{2\alpha} h(r)
\]  
(55)

Let \(0 < t < r\),
\[
h(t) = \int B(r) u^2 \, dx \leq \int B(r) u^2 \left(\frac{r^2 - t^2}{r^2 - t^2}\right)^{\alpha} \, dx \leq \frac{H(r)}{(r^2 - t^2)^{\alpha}}
\]  
(56)

Let \(t = \frac{r}{2}\), then
\[
\frac{3}{4^a} r^{2\alpha} h(\frac{r}{2}) \leq H(r) \leq r^{2\alpha} h(r)
\]  
(57)

Thus,
\[
(C_1 + C_3) \ln \frac{3}{4^a} r^{2\alpha} + (C_1 + C_3) \ln h(r_2)
\]  
\[\leq C_1 \ln r_3^{2\alpha} + C_3 \ln r_3^{2\alpha} + C_1 \ln h(r_1) + C_3 \ln h(r_3) + (C_4 - C_2)
\]  
(58)

We obtain
\[
(C_1 + C_3) \ln h(r_2) \leq C_1 \ln h(r_1) + C_3 \ln h(r_3) + C_1 \ln r_1^{2\alpha} + C_3 \ln r_3^{2\alpha} - (C_1 + C_3) \ln \frac{3}{4^a} r_2^{2\alpha} + (C_4 - C_2)
\]  
(59)

which implies
\[
h(r_2)^{C_1 + C_3} \leq \frac{3}{4^a} C_1 + C_3 e^{C_1 - C_2} r_2^{2\alpha} C_1^{2\alpha} C_3 h(r_1)^{C_1} h(r_3)^{C_3}
\]  
(60)
where \( C_1 = \frac{(\alpha+1)r_1^2}{128e^{40\pi r_2^2}(2r_2-r_1)} \), \( C_2 = -\frac{(\alpha+1)r_1^2ln(2r_2)}{64e^{40\pi r_2^2}(2r_2-r_1)} - \frac{Le^{30\pi r_2}}{128r_2^2} \), \( C_3 = \frac{(\alpha+1)e^{20\pi r_3}}{128e^{40\pi r_2^2}(r_3-2r_2)} \), \( C_4 = \frac{Le^{30\pi r_3}}{2 \cdot 128e^{40\pi r_2^2}r_2^2} \).

Thus, Theorem 1.2 is proved.

4. Conclusion
By using the monotonicity formula we derive in part 2
\[
e^{20r+7\ln r}N(r) + \lambda e^{30r}
\]
we finally derive the three-ball theorem on Cigar Soliton in part 3
\[
h(r_2)^{C_1+C_3} \leq \left( \frac{4e^{20r+7\ln r}N(r)}{30r} \right)^{C_1+C_3} \left( \frac{e^{20r}+e^{20r}}{2ac_1} \right)^{2ac_3} h(r_1)^{C_1} h(r_3)^{C_3}
\]
(62)

As we use the method Jiuyi Zhu uses in [8], the second third part of this paper, which is to deduce the three-ball theorem, is quite easy with the help of the monotonicity of the frequency function, and the most difficult part of this paper is the process of finding the monotonicity formula for that frequency function, in which the most challenging part is to deduce the lower bound of \( I'(r) \), which should only contain \( H(r) \) with an arbitrary constant coefficient, \( H'(r) \) with a certain coefficient, and a certain term that can be used in Cauchy-Schwarz inequality. Since the author is not well acquainted with other technique, nearly all the calculation is about integral by part, and there is no any complicated formula used in this paper.

With the three-ball theorem on Cigar Soliton, we might be able to derive some other meaningful geometrical properties such as its double property and the boundary of its vanishing order.

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