Bäcklund transformations between the AKNS and DNLS hierarchies.

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Abstract. Starting from the functional representation of the Ablowitz-Kaup-Newell-Segur (AKNS) and derivative nonlinear Schrödinger (DNLS) hierarchies and using the chains of the Miura-like transformations we derive a set of Bäcklund transformations that link solutions of these systems. It is shown that the extended AKNS and DNLS hierarchies possess common set of tau-functions and their connection with the Ablowitz-Ladik hierarchy is established. These results are another manifestation of the already known fact that the AKNS and DNLS hierarchies are closely related and can be viewed as particular cases of a more general system.

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1. Introduction.

This paper is devoted to the Bäcklund transformations (BTs) between the Ablowitz-Kaup-Newell-Segur (AKNS) hierarchy and the derivative nonlinear Schrödinger (DNLS) hierarchy. The relationships between these integrable systems have been studied by different authors (see, e.g., book [1] and references therein). Moreover, the nonlinear Schrödinger (NLS) equation, which is the simplest equation of the AKNS hierarchy, and DNLS equation can be viewed as particular cases of a more general system. So, the authors of [2,3,4] have shown that the generalized (or mixed, or hybrid) NLS equation, which has both usual cubic nonlinear term (as in the NLS equation) and a derivative cubic term (as in the DNLS equation), is also an integrable system and elaborated the correspondent versions of the inverse scattering transform. Later Flaschka, Newell and Ratiu demonstrated in [5] that both hierarchies (AKNS and DNLS) are parts of the greater hierarchy connected with $\tilde{sl}(2,C)$. Also we would like to mention the results obtained by Dimakis and Müller-Hoissen [6] who recovered the DNLS hierarchy as AKNS-type one using the deformations of associative products. However, usually these equations/hierarchies are treated as different and a natural question is to study the transformations between solutions of these closely related systems. This problem has been solved for the NLS and DNLS equations by Ishimori [7], Wadati and Sogo [8] (see also [9] for the multicomponent case), who constructed the gauge transformations between the matrices which form the zero-curvature representation of the corresponding equations.

In this work the results of [7,8] are extended to the level of hierarchies. In doing this we will not use the approach based on inverse scattering technique and will not develop the gauge transformations between the scattering problems of the hierarchies. Instead all the relationships are formulated directly in terms of solutions starting from the functional representation of the AKNS and DNLS hierarchies (instead of the zero-curvature one).

After presenting in section 2 some facts related to the functional representation of the AKNS and DNLS hierarchies and their Miura-like transformations, we introduce and prove the Bäcklund relations (sections 3, 6) which then are simplified in section 4. Finally, we rewrite in section 5 the obtained BTs in the bilinear form and demonstrate their relations with the Ablowitz-Ladik hierarchy (ALH).

2. The AKNS and DNLSE hierarchies.

The key feature of the presented work is the so-called functional representation of the hierarchies when an infinite number of differential equations is replaced by a few functional equations generated by the Miwa’s shifts applied to the functions of an infinite number of arguments:

$$E_ξQ(t) = Q(t + i[ξ])$$

(2.1)
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where

\[ Q(t) = Q(t_1, t_2, ...) = Q(t_k) \] (2.2)

and

\[ Q(t + i[\xi]) = Q(t_1 + i\xi, t_2 + i\xi^2/2, ...) = Q(t_k + i\xi^k/k) \] (2.3)

(see, e.g., [10, 11]). In what follows we use the functional representations of the AKNS and DNLS hierarchies that have been derived in [10, 12].

2.1. AKNS hierarchy.

The AKNS hierarchy was introduced in 1974 in [13]. In that work the authors generalized the inverse scattering approach of Zakharov and Shabat that was developed in [14, 15] for solution of the NLS equation. This hierarchy consists of an infinite family of integrable equations which, in the framework of the inverse scattering method, can be associated with the Zakharov-Shabat eigenvalue problem used in [14, 15].

The AKNS hierarchy can be written as the system of two-parametric equations

\[
\begin{align*}
\mathfrak{A}^Q(\xi, \eta) &= 0 \\
\mathfrak{A}^R(\xi, \eta) &= 0
\end{align*}
\] (2.4)

where the generating relations \( \mathfrak{A}^Q(\xi, \eta) \) and \( \mathfrak{A}^R(\xi, \eta) \) are defined by

\[
\begin{align*}
\mathfrak{A}^Q(\xi, \eta) &= (\xi - \eta) \hat{Q} - \left( 1 + \xi \eta \hat{Q} R \right) \left( \xi \hat{Q} - \eta \bar{Q} \right) \\
\mathfrak{A}^R(\xi, \eta) &= (\xi - \eta) R - \left( 1 + \xi \eta \hat{Q} R \right) \left( \xi \bar{R} - \eta \hat{R} \right)
\end{align*}
\] (2.5)

with the shortcuts that will be repeatedly used throughout the paper,

\[
\hat{Q} = \mathbb{E}_\xi Q, \quad \bar{Q} = \mathbb{E}_\eta Q. \quad (2.6)
\]

Equations (2.4) can be simplified by different limiting procedures. For example, sending \( \eta \) to zero one comes to a system that was used in [12]:

\[
\begin{align*}
Q - \hat{Q} + i\xi \partial_1 \hat{Q} - \xi^2 \hat{Q}^2 R &= 0 \\
\bar{R} - R - i\xi \partial_1 R - \xi^2 \hat{Q} R^2 &= 0
\end{align*}
\] (2.7)

where \( \partial_j = \partial/\partial t_j \). Expanding these functional equations in the power series in \( \xi \) one obtains an infinite set of differential ones, the first of which are the NLS equation,

\[
\begin{align*}
i\partial_2 Q + \partial_1 Q + 2Q^2 R &= 0 \\
-i\partial_2 R + \partial_1 R + 2QR R^2 &= 0
\end{align*}
\] (2.8)

and the complex mKdV equation,

\[
\begin{align*}
\partial_3 Q + \partial_{11} Q + 6QR \partial_1 Q &= 0 \\
\partial_3 R + \partial_{11} R + 6QR \partial_1 R &= 0
\end{align*}
\] (2.9)

(here \( \partial_{jk} \) stand for \( \partial^2/\partial t_j \partial t_k \), etc) that are the first equations of the AKNS hierarchy. However, in what follows it is more convenient to work with the ‘algebraic’ equations (2.4) than with the ‘differential’ ones (2.7).
Another key ingredient of this work is to consider, instead of a single solution of the hierarchy, \((Q, R)\), an infinite set of solutions \((Q_n, R_n)\), \(n \in (-\infty, \infty)\) related by the Miura-like transformations:

\[
\begin{align*}
\mathcal{M}_Q^n(\xi) &= 0 \\
\mathcal{M}_R^n(\xi) &= 0
\end{align*}
\] (2.10)

where

\[
\begin{align*}
\mathcal{M}_Q^n(\xi) &= \hat{Q}_n - \Gamma^{QR}_n(\xi) \left( Q_n + \xi \hat{Q}_{n+1} \right) \\
\mathcal{M}_R^n(\xi) &= R_{n+1} - \Gamma^{QR}_n(\xi) \left( \hat{R}_{n+1} + \xi R_n \right)
\end{align*}
\] (2.11)

with

\[
\Gamma^{QR}_n(\xi) = 1 - \xi \hat{Q}_n R_{n+1}.
\] (2.12)

It can be shown that if a pair \((Q_{n_0}, R_{n_0})\) solves equations (2.4), then, by virtue of (2.10), so do the pairs \((Q_{n_0 \pm 1}, R_{n_0 \pm 1})\) and hence the pairs \((Q_n, R_n)\) for all values of \(n\) (see Appendix A).

### 2.2. DNLS hierarchy.

There are different forms to write the DNLS equation [16, 17, 18]. In this work, we use the one that was presented by Chen, Lee and Liu in [17].

By analogy with the AKNS case, discussed in the previous section, the DNLS hierarchy can be written as

\[
\begin{align*}
\mathcal{A}^U(\xi, \eta) &= 0 \\
\mathcal{A}^V(\xi, \eta) &= 0
\end{align*}
\] (2.13)

where

\[
\begin{align*}
\mathcal{A}^U(\xi, \eta) &= \left[ \xi - \eta + \xi \eta \left( \hat{U} - \hat{U} \right) V \right] \hat{U} - \xi \hat{U} + \eta \hat{U} \\
\mathcal{A}^V(\xi, \eta) &= \left[ \xi - \eta + \xi \eta \hat{U} \left( \hat{V} - \hat{V} \right) \right] V + \eta \hat{V} - \xi \hat{V}
\end{align*}
\] (2.14)

(see [10]). These functional equations in the \(\eta \to 0\) limit become

\[
\begin{align*}
i \xi \partial_1 \hat{U} + \left( 1 + \xi \hat{U} \right) \left( U - \hat{U} \right) &= 0 \\
i \xi \partial_1 \hat{V} + \left( 1 + \xi \hat{U} \right) \left( \hat{V} - V \right) &= 0
\end{align*}
\] (2.15)

and lead to an infinite set of differential equations, the simplest of which is the DNLS equation:

\[
\begin{align*}
(i \partial_2 + \partial_{11}) U + 2i UV \partial_1 U &= 0 \\
(-i \partial_2 + \partial_{11}) V - 2i UV \partial_1 V &= 0
\end{align*}
\] (2.16)

The Miura-like transformations for the DNLS hierarchy are defined by

\[
\begin{align*}
\mathcal{M}_U^\xi(\xi) &= 0 \\
\mathcal{M}_V^\xi(\xi) &= 0
\end{align*}
\] (2.17)
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with
\[ M_n(\xi) = \Gamma_{n+1}(\xi) \dot{U}_n - \left( U_n + \xi \dot{U}_{n+1} \right) \]
\[ M_n(\xi) = \Gamma_n(\xi) V_{n+1} - \left( \dot{V}_{n+1} + \xi V_n \right) \]  
\eqno(2.18)

and
\[ \Gamma_n(\xi) = 1 + \xi \dot{U}_n V_n \]  
\eqno(2.19)

and can be shown to be compatible with (2.13) (see Appendix B).

3. Bäcklund transformations between the AKNS and DNLS hierarchies.

The generating relations for the BTs between the AKNS and DNLS hierarchies can be presented as follows:
\[
\begin{cases}
\mathcal{G}_n(\xi) = 0 \\
\mathcal{h}_n = 0
\end{cases}
\]  
\eqno(3.1)

where
\[ \mathcal{G}_n(\xi) = \Gamma_{n-1}(\xi) \Gamma_n(\xi) - 1 \]
\[ \mathcal{h}_n = 1 - (1 + Q_n R_{n+1}) (1 - U_n V_{n+1}) \]  
\eqno(3.2)

The proof of the fact that (3.1) indeed relate solutions of (2.4) with ones of (2.13) is based on expressing the quantities \( A_{Q,R} \) and \( A_{U,V} \), that are given by (2.5) and (2.14) with \( Q, R, U, \) and \( V \) being replaced by \( Q_n, R_n, U_n, \) and \( V_n \), as linear combinations of \( \mathcal{G}_n \) and \( \mathcal{h}_n \) (and their \( E \)-shifted values). These calculations are, on the one hand, rather simple but, on the other hand, are rather cumbersome and fatiguing. Here we present only the main steps verifying some of statements in Appendix C and leaving the simplest ones without a proof.

Consider the quantities
\[ \tilde{\mathcal{A}}_n^{QU} = \dot{U}_{n+1} \left( V_{n+1} - V_n \right) + \xi Q_n R_n \]  
\eqno(3.3)
\[ \tilde{\mathcal{A}}_n^{RV} = \left( U_n - \dot{U}_n \right) V_n + \xi Q_n R_n \]  
\eqno(3.4)

In the case when (3.1) hold they are related to \( \mathcal{A}_{Q,R} \) and \( \mathcal{A}_{U,V} \) by
\[ \frac{2 \dot{U}_{n+1} n}{\dot{U}_{n+1} - \dot{Q}_n} = \xi \eta \left[ \tilde{\mathcal{A}}_n^{QU}(\xi) - \tilde{\mathcal{A}}_n^{QU}(\eta) \right] \]  
\eqno(3.5)
\[ \frac{2 \dot{V}_n}{V_n - \dot{R}_n} = \xi \eta \left[ \tilde{\mathcal{A}}_n^{RV}(\xi) - \tilde{\mathcal{A}}_n^{RV}(\eta) \right] \]  
\eqno(3.6)

(see equations (C.32), (C.33) of Appendix C). On the other hand, one can show that (3.1) imply that \( \tilde{\mathcal{A}}_n^{QU} \) and \( \tilde{\mathcal{A}}_n^{RV} \) satisfy
\[ \tilde{\mathcal{A}}_n^{QU} = - \frac{\dot{Q}_n}{\Gamma_{n+1}} M_n - \frac{U_{n+1}}{\Gamma_n} M_n \]  
\eqno(3.7)
\[ \tilde{\mathcal{A}}_n^{RV} = - \frac{\dot{R}_n}{\Gamma_{n-1}} M_{n-1} - \frac{V_n}{\Gamma_{n+1}} M_n \]  
\eqno(3.8)
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(see equations (C.28)–(C.31) of Appendix C). It is easy to conclude from the above formulae that

\[
\begin{align*}
\mathcal{G}_n = h_n &= 0 \\
\mathcal{A}_n^Q &= \mathcal{A}_n^R = 0
\end{align*}
\Rightarrow \ {\mathcal{A}_n^U = \mathcal{A}_n^V = 0}_{n=-\infty, \ldots, \infty} \tag{3.9}
\]

and

\[
\begin{align*}
\mathcal{G}_n = h_n &= 0 \\
\mathcal{A}_n^U &= \mathcal{A}_n^V = 0
\end{align*}
\Rightarrow \ {\mathcal{A}_n^Q = \mathcal{A}_n^R = 0}_{n=-\infty, \ldots, \infty} \tag{3.10}
\]

which means that (3.1), (3.2) are indeed transformations that link solutions of the AKNS and DNLS hierarchies.

Let us prove, for example, the first of these statements. The conditions from the left-hand side of (3.9) together with the definitions of \(Q_{n+1}\) and \(R_{n+1}\), \(\mathcal{M}_n^Q = \mathcal{M}_n^R = 0\), and identities (3.7), (3.8) imply that \(\mathcal{F}_n^{QR} = \mathcal{F}_n^{RV} = 0\) for all \(n\). Then, it follows from (3.5), (3.6) that \(\mathcal{A}_n^U = \mathcal{A}_n^V = 0 \ (n = -\infty, \ldots, \infty)\) which proves (3.9). The second statement, (3.10), can be demonstrated in a similar way.

4. Solution of Bäcklund equations.

The aim of this section is to look at BTs (3.1), (3.2) from the practical viewpoint and to express solutions of the AKNS hierarchy in terms of solutions of the DNLS hierarchy and vice versa. To this end consider the quantities

\[
\mathcal{R}_n^{QR}(\xi) = \frac{1 + Q_{n-1}R_{n+1}}{\Gamma_n^{QR}} - \frac{1 + \hat{Q}_{n-1}\hat{R}_{n+1}}{\Gamma_{n-1}^{QR}} \tag{4.1}
\]

\[
\mathcal{R}_n^{UV}(\xi) = \frac{1 - U_nV_{n+1}}{\Gamma_n^{UV}} - \frac{1 - \hat{U}_n\hat{V}_{n+1}}{\Gamma_{n+1}^{UV}} \tag{4.2}
\]

together with

\[
\mathcal{L}_n^Q = \frac{Q_n}{Q_{n-1}} - \frac{U_{n+1}}{U_n} (1 - U_nV_{n+1}) \tag{4.3}
\]

\[
\mathcal{L}_n^R = \frac{R_n}{R_{n+1}} - \frac{V_n}{V_{n+1}} (1 - U_nV_{n+1}) \tag{4.4}
\]

and

\[
\mathcal{L}_n^U = \frac{U_{n+1}}{U_n} - \frac{Q_n}{Q_{n-1}} (1 + Q_{n-1}R_{n+1}) \tag{4.5}
\]

\[
\mathcal{L}_n^V = \frac{V_n}{V_{n+1}} - \frac{R_n}{R_{n+1}} (1 + Q_{n-1}R_{n+1}). \tag{4.6}
\]

As is shown in Appendix C these functions vanish together with Bäcklund generating relations:

\[
\mathcal{G}_n = h_n = 0 \Rightarrow \ \mathcal{R}_n^{QR} = \mathcal{R}_n^{UV} = 0, \ \mathcal{L}_n^x = 0 \ (x = q, r, u, v). \tag{4.7}
\]
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Using this fact and introducing the new functions, $F_{QR}^n$ and $F_{UV}^n$, related to $Q_n$, $R_n$ and $U_n$, $V_n$ non-locally,

\begin{equation}
1 + Q_{n-1} R_{n+1} = \frac{F_{n+1}^{QR}}{F_{n}^{QR}}
\end{equation}

and

\begin{equation}
1 - U_n V_{n+1} = \frac{F_{n}^{UV}}{F_{n+1}^{UV}},
\end{equation}

one can perform easy calculations leading to the solution of our problem. Noting that

\begin{equation}
h_n = \frac{F_{n+1}^{QR}}{F_{n}^{QR}} \frac{F_{n}^{UV}}{F_{n+1}^{UV}} - 1
\end{equation}

one can conclude that equation $h_n = 0$ implies

\begin{equation}
\frac{F_{n}^{UV}}{F_{n}^{QR}} = \alpha
\end{equation}

where $\alpha$ does not depend on $n$. Rewriting $\mathcal{R}_{QR}^n$ and $\mathcal{R}_{UV}^n$ as

\begin{equation}
\mathcal{R}_{QR}^n = \frac{\hat{F}_{n+1}^{QR}}{F_{n}^{QR}} \left( \frac{F_{n+1}^{QR}}{F_{n+1}^{QR} \Gamma_{QR}^n} - \frac{F_{n}^{QR}}{F_{n}^{QR} \Gamma_{QR}^{n-1}} \right)
\end{equation}

\begin{equation}
\mathcal{R}_{UV}^n = \frac{F_{n}^{UV}}{F_{n+1}^{UV}} \left( \frac{\hat{F}_{n+1}^{UV}}{F_{n+1}^{UV} \Gamma_{UV}^n} - \frac{\hat{F}_{n}^{UV}}{F_{n}^{UV} \Gamma_{UV}^{n-1}} \right)
\end{equation}

one can easily see that

\begin{equation}
\Gamma_{QR}^n(\xi) = \frac{1}{\gamma_{QR}^n} \frac{F_{n+1}^{QR}}{F_{n}^{QR}},
\end{equation}

and

\begin{equation}
\Gamma_{UV}^n(\xi) = \frac{1}{\gamma_{UV}^n} \frac{\hat{F}_{n}^{UV}}{F_{n}^{UV}}
\end{equation}

where, again, $\gamma_{QR}^n$ and $\gamma_{UV}^n$ are constants with respect to $n$. Further, the functions $\mathcal{L}_n^x$ can be presented as

\begin{equation}
\mathcal{L}_n^Q = \frac{U_{n+1}}{Q_{n-1} F_{n+1}^{UV}} \left( \frac{Q_n F_{n+1}^{UV}}{U_{n+1}} - \frac{Q_{n-1} F_{n}^{UV}}{U_{n}} \right)
\end{equation}

\begin{equation}
\mathcal{L}_n^R = \frac{V_n F_{n}^{UV}}{R_{n+1}} \left( \frac{R_n}{V_{n+1} F_{n}^{UV}} - \frac{R_{n+1}}{V_{n} F_{n}^{UV}} \right)
\end{equation}

and

\begin{equation}
\mathcal{L}_n^U = \frac{Q_n F_{n+1}^{QR}}{U_{n+1}} \left( \frac{Q_{n+1} F_{n}^{QR}}{Q_n F_{n+1}^{QR}} - \frac{U_{n+1}}{Q_{n+1} F_{n+1}^{QR}} \right)
\end{equation}

\begin{equation}
\mathcal{L}_n^V = \frac{R_n}{V_{n+1} F_{n}^{QR}} \left( \frac{V_n}{R_{n+1} F_{n}^{QR}} - \frac{V_{n+1} F_{n}^{QR}}{R_{n+1}} \right).
\end{equation}
Hence, equations $\mathcal{L}_n^X = 0$ for $x = q, r, u, v$ lead to

$$Q_n = \beta^q U_{n+1}/F_{n+1}^{UV}, \quad R_n = \beta^r V_n F_n^{UV}$$

(4.20)

and

$$U_n = \beta^u Q_{n-1} F_{n}^{QR}, \quad V_n = \beta^v R_n / F_n^{QR}.$$  

(4.21)

At last, analysis of other consequences of equations (3.1)–(3.2) impose some restrictions on the constants $\alpha^X, \beta^X$ and $\gamma^X$ (which are presented without derivation):

$$\beta^q(t)\beta^r(t) = \beta^u(t)\beta^v(t) = 1$$

(4.22)

$$\gamma^{UV}(t, \xi) = \frac{E_{\xi} \beta^q(t)}{\beta^q(t)} = \frac{\beta^r(t)}{E_{\xi} \beta^r(t)}$$

(4.23)

$$\gamma^{QR}(t, \xi) = \frac{E_{\xi} \beta^u(t)}{\beta^v(t)} = \frac{\beta^v(t)}{E_{\xi} \beta^v(t)}$$

(4.24)

Choosing of all the functions $\alpha^X, \beta^X$ and $\gamma^X$ two independent ones, say, $\alpha(t)$ and $\beta(t) = \beta^q(t)$ one comes to the final formulæ:

$$Q_n = \beta U_{n+1}/F_{n+1}^{UV}$$

(4.26)

$$R_n = \frac{1}{\beta} V_n F_n^{UV}$$

(4.27)

$$F_n^{QR} = \frac{1}{\alpha} F_n^{UV}$$

(4.28)

and

$$U_n = \frac{\alpha}{\beta} Q_{n-1} F_n^{QR}$$

(4.29)

$$V_n = \frac{\beta}{\alpha} R_n / F_n^{QR}$$

(4.30)

$$F_n^{UV} = \alpha F_n^{QR}$$

(4.31)

that complete ‘solution’ of the Bäcklund equations (3.1), (3.2).

5. Bäcklund transformations, bilinearization and the ALH.

In this section, we present some bilinear equations related to the problem we are dealing with. However, they are not the traditional bilinear BTs which are bilinear equations relating the $\tau$-functions of two hierarchies. Instead, it is shown that the extended AKNS and DNLS hierarchies (‘extended’ means considered for an infinite set of solutions related by the Miura-like transformations) admit common set of $\tau$-functions. In terms of these $\tau$-functions the action of BTs obtained above is reduced to simple changes of the index $n$. 

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Let us start with the final formulae of the previous section. The structure of (4.26)–(4.31) suggests the following parametrization:

\[ Q_n = \frac{k^Q \sigma_{n+1}}{\tau_n}, \quad R_n = \frac{1}{k^Q} \frac{\rho_{n-1}}{\tau_n} \]  

and

\[ U_n = \frac{k^U \sigma_n}{\tau_n}, \quad V_n = \frac{1}{k^U} \frac{\rho_{n-1}}{\tau_{n-1}}. \]  

Rewriting definitions (4.8), (4.9) as

\[ \frac{F_{n+1}^{QR}}{F_{n}^{QR}} = 1 + Q_{n-1} R_{n+1} = \frac{1}{\tau_{n-1} \tau_{n+1}} \left( \tau_n^2 + \Delta^\tau_n \right) \]  

\[ \frac{F_{n+1}^{UV}}{F_{n}^{UV}} = 1 - U_n V_{n+1} = \frac{1}{\tau_n^2} \left( \tau_{n-1} \tau_{n+1} - \Delta^\tau_n \right) \]

with

\[ \Delta^\tau_n = \tau_{n-1} \tau_{n+1} + \rho_n \sigma_n - \tau_n^2 \]

one can see that they are compatible with (4.28) and (4.31) and lead to

\[ F_{n}^{QR} = \hat{k}^Q \frac{\tau_{n-1}}{\tau_n} \]

\[ F_{n}^{UV} = \alpha \hat{k}^U \frac{\tau_{n-1}}{\tau_n} \]

provided \( \Delta^\tau_n = 0 \). Calculating \( \Gamma_n^{QR}(\xi) \) and \( \Gamma_n^{UV}(\xi) \),

\[ \Gamma_n^{QR} = \hat{k}^Q \left[ \frac{k^U \tau_n \hat{\tau}_{n+1}}{k^U \tau_{n+1} \hat{\tau}_n} + \frac{\Delta^\sigma_{n+1}}{\tau_{n+1} \hat{\tau}_n} \right] \]

\[ \Gamma_n^{UV} = \hat{k}^U \left[ \frac{k^Q \tau_n \hat{\tau}_{n-1}}{k^Q \tau_{n-1} \hat{\tau}_n} - \frac{\Delta^\sigma_n}{\tau_{n-1} \hat{\tau}_n} \right] \]

where

\[ \Delta^\sigma_n(\xi) = \frac{k^Q}{k^Q} \frac{k^U}{k^U} \left( \hat{\tau}_{n-1} - \xi \rho_{n-1} \hat{\sigma}_n \right) \]

and substituting the result in (3.2) one can obtain

\[ \mathcal{G}_n(\xi) = \frac{k^Q k^U}{\tau_{n-1} \tau_n \hat{\tau}_{n-1} \hat{\tau}_n} \Delta^\sigma_n(\xi) \]

and

\[ h_n = \frac{\rho_n \sigma_n}{\tau_{n-1} \tau_n \hat{\tau}_{n+1}} \Delta^\tau_n. \]

Thus, the the relationship between solutions of the AKNS and DNLS hierarchies can be re-formulated as presentation (5.1), (5.2) together with the conditions

\[ \Delta^\tau_n = \Delta^\sigma_n(\xi) = 0. \]  

Rewriting the functions generating Miura-like transformations, \( \mathcal{M}_n^X(\xi) \), and then \( \mathcal{A}_n^X(\xi, \eta) \) one comes to a set of bilinear functional equations, similar to (5.13). These equations, which are not presented here, are nothing but equations forming the
functional representation of the ALH [10]. Thus, the simplest way to summarize the above results is to say that solutions of the AKNS and DNLS hierarchies can be constructed of the same set of $\tau$-functions (the ALH $\tau$-functions) by the rule given by (5.1) and (5.2).

6. One-point transformations.

The question discussed in this section is the ‘traditional’ BTs that do not use the lattice representation (based on the Miura-like transformations) of hierarchies, but relate one solution of the AKNS hierarchy, $(Q,R)$, with one solution of the DNLS hierarchy, $(U,V)$. We will not re-derive these transformations from scratch, but use the results presented in the previous sections. The main idea can be explained as follows. Among many generating relations determining the BTs $(\mathfrak{g}_n, h_n, \tilde{\mathfrak{g}}_n^X, \tilde{\mathfrak{h}}_n^X, \tilde{\mathfrak{g}}_n^X)$, one can select a subset that links $(U_n, V_n)$ with $(Q_{n'}, R_{n'})$ only, without invoking $(Q_{n\pm 1}, R_{n\pm 1})$. In doing this one can find the two possibilities: (i) transformations that connect $(U_n, V_n)$ with $(Q_n, R_n)$ and (ii) transformations that connect $(U_n, V_n)$ with $(Q_{n-1}, R_{n-1})$.

6.1. $(Q_n, R_n) \leftrightarrow (U_n, V_n)$ transformations.

The first kind of transformations follows from the identities

$$
\frac{\mathfrak{A}_n^{V}(\xi, \eta)}{V_n} - \frac{\mathfrak{A}_n^{R}(\xi, \eta)}{R_n} = \xi \eta \left[ E_{\eta} \tilde{\mathfrak{G}}_n^{RV} (\xi) - E_{\xi} \tilde{\mathfrak{G}}_n^{RV} (\eta) \right]
+ \xi \tilde{\mathfrak{G}}_n^{R}(\eta) - \eta \tilde{\mathfrak{G}}_n^{R}(\xi)
$$

(6.1)

where

$$
\mathfrak{D}_n^{RV}(\xi, \eta) - \mathfrak{D}_n^{QV}(\xi, \eta) = \left( E_{\eta} - 1 \right) \tilde{\mathfrak{G}}_n^{RV}(\xi) + (1 - E_{\xi}) \tilde{\mathfrak{G}}_n^{RV}(\eta)
$$

(6.2)

$$
\mathfrak{D}_n^{QR}(\xi, \eta) = \hat{Q}_n \left( \eta \hat{R}_n - \xi \hat{R}_n \right) + \left( \xi \hat{Q}_n - \eta \hat{Q}_n \right) R_n,
$$

(6.3)

$$
\mathfrak{D}_n^{UV}(\xi, \eta) = \left( E_{\xi} - 1 \right) (\hat{U}_n - U_n) V_n + (1 - E_{\eta}) \left( \hat{U}_n - U_n \right) V_n,
$$

(6.4)

and the quantities $\tilde{\mathfrak{G}}_n^{R}(\xi)$ are defined by

$$
\tilde{\mathfrak{G}}_n^{R} = \frac{\hat{R}_n}{R_n} - \Gamma_{n}^{UV} \frac{\hat{V}_n}{V_n}.
$$

(6.5)

Both $\tilde{\mathfrak{G}}_n^{RV}(\xi)$ and $\tilde{\mathfrak{H}}_n^{UV}(\xi)$ do not invoke $Q_{n\pm 1}$, $R_{n\pm 1}$ and vanish when (3.1) hold. At the same time the above equations (6.1), (6.2) demonstrate that if $\tilde{\mathfrak{G}}_n^{RV}(\xi) = \tilde{\mathfrak{G}}_n^{R}(\xi) = 0$ and $\mathfrak{A}_n^{Q} = \mathfrak{A}_n^{R} = 0$, then $\mathfrak{A}_n^{U} = \mathfrak{A}_n^{V} = 0$ and vice versa (3.4).
Bäcklund transformations between the AKNS and DNLS hierarchies.

ensure $\mathfrak{A}_n^Q = \mathfrak{A}_n^R = 0$ for any $n$. To summarize, rewriting $\mathfrak{F}_n^{QV}(\xi)$ and $\mathfrak{F}_n^R(\xi)$ with the index $n$ being omitted one can state that transformations defined by

$$
\begin{aligned}
\begin{cases}
\xi (\mathbb{E}_\xi Q) R = [(\mathbb{E}_\xi U) - U] V \\
\frac{\mathbb{E}_\xi R}{R} = [1 + \xi (\mathbb{E}_\xi U)] \frac{\mathbb{E}_\xi V}{V}
\end{cases}
\end{aligned}
$$

(6.6)

are the BTs between the AKNS and DNLS hierarchies. In the $\xi \to 0$ limit one comes to the transformations between solutions of the NLS and DNLS equations similar to ones derived by Wadati and Sogo in [8].

6.2. $(Q_{n-1}, R_{n-1}) \leftrightarrow (U_n, V_n)$ transformations.

In a similar way, one can restrict himself to the functions $\tilde{\mathfrak{F}}_{n-1}^{QU}(\xi)$ and $\tilde{\mathfrak{F}}_{n-1}^Q(\xi)$, defined by

$$
\tilde{\mathfrak{F}}_{n-1}^Q = \frac{Q_n}{Q_{n-1}} - \frac{\Gamma_{n+1}^{UV} U_{n+1}}{U_{n+1}}.
$$

(6.7)

They vanish when (3.1) hold and ensure the linear relations between $\mathfrak{A}_{n-1}^{U,R}$ and $\mathfrak{A}_n^{Q,R}$:

$$
\begin{aligned}
\frac{\mathfrak{A}_n^{U}(\xi, \eta)}{U_n} - \frac{\mathfrak{A}_{n-1}^{Q}(\xi, \eta)}{Q_{n-1}} &= \xi \eta \left[ \mathfrak{F}^{Q}_{n-1}(\xi) - \mathfrak{F}^{Q}_{n-1}(\eta) \right] \\
&+ \xi \mathbb{E}_\xi \tilde{\mathfrak{F}}_{n-1}^Q(\eta) - \eta \mathbb{E}_\eta \tilde{\mathfrak{F}}_{n-1}^Q(\xi)
\end{aligned}
$$

(6.8)

Thus, rewriting the definitions of $\mathfrak{F}^{Q}_{n-1}(\xi)$ and $\tilde{\mathfrak{F}}_{n-1}^Q(\xi)$ in terms of $Q = Q_{n-1}$, $R = R_{n-1}$ and $U = U_n$, $V = V_n$ as

$$
\begin{aligned}
\begin{cases}
\xi (\mathbb{E}_\xi Q) R = (\mathbb{E}_\xi U) [V - (\mathbb{E}_\xi V)] \\
\frac{\mathbb{E}_\xi Q}{Q} = \frac{1}{1 + \xi (\mathbb{E}_\xi U)} \frac{\mathbb{E}_\xi U}{U}
\end{cases}
\end{aligned}
$$

(6.10)

one comes to the second BT between the DNLS and AKNS hierarchies.

7. Conclusion.

To conclude we would like to mention some important questions that have not been discussed in this paper. The first problem is related to the generalized NLS equation. We know the relations between solutions of the NLS and DNLS equations. Also we know that both equations are particular cases of the generalized NLS equation. So it is interesting to obtain the functional representation of the generalized NLS hierarchy together with the reductions to the AKNS and DNLS cases. In other words, this problem can be re-formulated as to find the ‘interpolation’ between (5.1) and (5.2), the task which seems to be not trivial.

Another very important question is related to the involution, or complex conjugation. Usually in physical applications both NLS and DNLS equations appear in
the form when \( R = \pm \overline{Q} \) and \( V = \pm \overline{U} \) (here overline stands for the complex conjugation). This reduction changes many of the properties of the hierarchies. Most noticeable manifestation of this fact one can get considering the Miura-like transformations: apparently they destroy the involution. So one has to make additional efforts in order to apply results of this paper to the physical situations. An example of how that can be done can be found in [19] where the author was dealing with the equation closely related to the DNLS hierarchy. More general approach to the problem of the involution is based on the incorporation of the so-called ‘negative flows’. This question that is outside the scope of this paper surely deserves separate studies.

**Appendix A. Lattice representation of the AKNS hierarchy.**

In this appendix, one can find the identities related to the functional representation of the AKNS hierarchy and its Miura-like transformations that provide proof of the implication \( \mathfrak{A}^{Q,R}_n \Rightarrow \mathfrak{A}^{Q,R}_{n+1} \) as well as of some other statements of this paper.

Consider the following linear combinations of the functions \( \mathfrak{M}^{Q,R}_n \):

\[
\tilde{\mathfrak{M}}^{Q}_n (\xi, \eta) = \xi \overline{E}_\xi \mathfrak{M}^{Q}_n (\eta) - \eta \overline{E}_\eta \mathfrak{M}^{Q}_n (\xi) \tag{A.1}
\]

\[
\tilde{\mathfrak{M}}^{R}_n (\xi, \eta) = \xi \overline{E}_\xi \mathfrak{M}^{R}_n (\eta) - \eta \overline{E}_\eta \mathfrak{M}^{R}_n (\xi) \tag{A.2}
\]

and

\[
\hat{\mathfrak{M}}^{QR}_n (\xi) = \frac{1}{\Gamma^{QR}_n (\xi)} \left[ R_{n+1} \mathfrak{M}^{Q}_n (\xi) - \hat{Q}_n \mathfrak{M}^{R}_n (\xi) \right]. \tag{A.3}
\]

By straightforward calculations, one can demonstrate the identities

\[
R_{n+1} \mathfrak{A}^{Q}_n (\xi, \eta) + \hat{Q}_n \mathfrak{A}^{R}_{n+1} (\xi, \eta) = R_{n+1} \tilde{\mathfrak{M}}^{Q}_n (\xi, \eta) + \hat{Q}_n \tilde{\mathfrak{M}}^{R}_n (\xi, \eta) \tag{A.4}
\]

and

\[
\mathfrak{D}^{QR}_n (\xi, \eta) = \mathfrak{D}^{QR}_{n+1} (\xi, \eta) = (E_\xi - 1) \hat{\mathfrak{M}}^{QR}_n (\eta) + (1 - E_\eta) \hat{\mathfrak{M}}^{QR}_n (\xi) \tag{A.5}
\]

where \( \mathfrak{D}^{QR}_n \) is defined by (6.3). One can easily see that

\[
\mathfrak{M}^{Q,R}_n = 0 \quad \Rightarrow \quad \mathfrak{M}^{Q,R}_n = \mathfrak{M}^{QR}_n = 0. \tag{A.6}
\]

Thus,

\[
\begin{cases}
\mathfrak{A}^{Q}_n = \mathfrak{A}^{R}_n = 0 \\
\mathfrak{M}^{Q}_n = \mathfrak{M}^{R}_n = 0
\end{cases}
\Rightarrow
\begin{cases}
\mathfrak{A}^{R}_{n+1} = 0 \\
\mathfrak{D}^{QR}_{n+1} = 0
\end{cases}
\Rightarrow
\begin{cases}
\mathfrak{A}^{Q}_{n+1} = 0 \\
\mathfrak{A}^{R}_{n+1} = 0
\end{cases} \tag{A.7}
\]

by virtue of (A.4) and (A.5). This proves the fact that if a pair \((Q_n, R_n)\) solves equations (2.4) and the pair \((Q_{n+1}, R_{n+1})\) is defined by \(\mathfrak{M}^{Q}_n = \mathfrak{M}^{R}_n = 0\), then it is another solution of (2.4). In a similar way, one can prove that the pair \((Q_{n-1}, R_{n-1})\) defined by \(\mathfrak{M}^{Q}_{n-1} = \mathfrak{M}^{R}_{n-1} = 0\) also solves (2.4).
Bäcklund transformations between the AKNS and DNLS hierarchies.

Appendix B. Lattice representation of the DNLS hierarchy.

Along the lines of the previous appendix consider the quantities

\[ \tilde{M}_n^U (\xi, \eta) = \xi \Xi \tilde{M}_n^U (\eta) - \eta \Xi \tilde{M}_n^U (\xi) \]  
\[ \tilde{M}_n^V (\xi, \eta) = \xi \Xi \tilde{M}_n^V (\eta) - \eta \Xi \tilde{M}_n^V (\xi) \]  

and

\[ \tilde{M}_n^{UV} (\xi) = V_n \tilde{M}_n^U (\xi) - \hat{U}_{n+1} \tilde{M}_n^V (\xi) \]  

which are related to \( \mathcal{A}_{n}^{UV} \) by

\[ V_{n+1} \mathcal{A}_n^U (\xi, \eta) + \hat{U}_n \mathcal{A}_{n+1}^U (\xi, \eta) = V_{n+1} \tilde{M}_n^U (\xi, \eta) + \hat{U}_n \tilde{M}_n^V (\xi, \eta) \]  

and

\[ \mathcal{D}_n^{UV} (\xi, \eta) - \mathcal{D}_{n+1}^{UV} (\xi, \eta) = (\Xi - 1) \tilde{M}_n^{UV} (\eta) + (1 - \Xi) \tilde{M}_n^{UV} (\xi) \]  

where \( \mathcal{D}_n^{UV} \) is defined by (6.4). These identities can be used to prove the fact that if a pair \( (U_n, V_n) \) solves equations (2.13) and the pairs \( (U_{n \pm 1}, V_{n \pm 1}) \) are defined by \( \tilde{M}_n^U = \tilde{M}_n^V = 0 \) and \( \tilde{M}_{n-1}^U = \tilde{M}_{n-1}^V = 0 \), then they also solve (2.13).

Appendix C. Generating relations for Bäcklund transformations.

Expanding the definition of the generating function \( \mathcal{G}_n (\xi) \),

\[ \xi^{-1} \mathcal{G}_n (\xi) = \hat{U}_n V_n - \hat{Q}_{n-1} R_n - \xi \hat{Q}_{n-1} R_n \hat{U}_n V_n \]  
\[ = \hat{U}_n V_n - \hat{Q}_{n-1} R_n \Gamma^{UV}_n (\xi) \]  
\[ = \hat{U}_n V_n \Gamma^{QR}_{n-1} (\xi) - \hat{Q}_{n-1} R_n \]  

and taking the limit \( \xi \to 0 \),

\[ g_n = \lim_{\xi \to 0} \xi^{-1} \mathcal{G}_n (\xi) = U_n V_n - Q_{n-1} R_n \]  

one can conclude than the functions

\[ \mathcal{J}_n^Q = \hat{Q}_n / Q_n - \hat{U}_{n+1} / U_{n+1} \Gamma^{UV}_{n+1} \]  
\[ \mathcal{J}_n^R = R_n / R_n - V_n / V_n \Gamma^{UV}_n \]  

and

\[ \mathcal{J}_n^U = \hat{U}_n / U_n - \hat{Q}_{n-1} / Q_{n-1} \Gamma^{QR}_{n-1} \]  
\[ \mathcal{J}_n^V = V_n / V_n - R_n / R_n \Gamma^{QR}_{n-1} \]  

vanish together with \( \mathcal{G}_n (\xi) \):

\[ \mathcal{J}_n^X = 0 \mod \mathcal{G}_n \quad (x = Q, R, U, V). \]  

The same is true,

\[ \tilde{\mathcal{J}}_n^X = 0 \mod \mathcal{G}_n \quad (x = Q, R, U, V), \]
for the quantities
\[
\tilde{\gamma}_n^Q = Q_n/R_n - \Gamma_{n+1}^{UV} U_{n+1}/\hat{U}_{n+1} \tag{C.9}
\]
and
\[
\tilde{\gamma}_n^R = \dot{R}_n/R_n - \Gamma_{n+1}^{UV} \hat{V}_n/V_n \tag{C.10}
\]
and
\[
\begin{align*}
\tilde{\gamma}_n^U &= U_n/\dot{U}_n - \Gamma_{n+1}^{QR} Q_{n+1}^{-1}/\dot{Q}_{n+1} \tag{C.11} \\
\tilde{\gamma}_n^V &= \dot{V}_n/V_n - \Gamma_{n+1}^{QR} \hat{R}_n/R_n.
\end{align*}
\]

The quantities \(\mathcal{R}_n^{QR}(\xi)\) and \(\mathcal{R}_n^{UV}(\xi)\) defined by (4.1) and (4.2) can be presented as
\[
\begin{align*}
\mathcal{R}_n^{QR}(\xi) &= \frac{-U_{n+1} V_n \mathcal{J}_n^{U} + \hat{U}_{n+1} \hat{V}_n \mathcal{J}_n^{V}}{Q_n R_n} - \frac{\mathcal{J}_n^{QR}}{Q_n R_n \Gamma_{n+1}^{QR}} + \frac{\tilde{\gamma}_n^{QR}}{Q_n R_n \Gamma_{n+1}^{QR}} \tag{C.12} \\
\mathcal{R}_n^{UV}(\xi) &= \frac{-Q_n R_n \mathcal{J}_n^{Q} + \hat{Q}_n \hat{R}_n \mathcal{J}_n^{R}}{U_{n+1} V_n} - \frac{\mathcal{J}_n^{QR}}{U_{n+1} V_n \Gamma_{n+1}^{QR}} + \frac{\tilde{\gamma}_n^{QR}}{U_{n+1} V_n \Gamma_{n+1}^{QR}} \tag{C.13}
\end{align*}
\]
where
\[
\begin{align*}
\mathcal{J}_n^{QR} &= Q_n R_n - U_{n+1} V_n (1 - U_n V_{n+1}) \tag{C.14} \\
\mathcal{J}_n^{UV} &= U_{n+1} V_n - Q_n R_n (1 + Q_{n+1} R_{n+1}) \tag{C.15}
\end{align*}
\]

It is easy to show that
\[
\begin{align*}
Q_{n-1} R_{n+1} \mathcal{J}_n^{QR} &= U_{n+1} V_n \mathcal{J}_n^{QR} \tag{C.16} \\
U_n V_{n+1} \mathcal{J}_n^{UV} &= -Q_n R_n \mathcal{J}_n^{QR} + \mathcal{J}_n^{QR} \tag{C.17}
\end{align*}
\]

with
\[
\begin{align*}
\mathcal{J}_n^{QR} &= U_n U_{n+1} V_n - Q_{n-1} Q_n R_n R_{n+1} \tag{C.18} \\
&= U_{n+1} V_n \mathcal{J}_n^{QR} + Q_{n-1} R_n \mathcal{J}_n^{QR} + Q_n R_n \mathcal{J}_n^{QR} \tag{C.19} \\
&= Q_n R_{n+1} \mathcal{J}_n^{QR} + U_n V_n \mathcal{J}_n^{QR} \tag{C.20}
\end{align*}
\]
which implies
\[
\mathcal{J}_n^{QR} = 0 \mod \mathcal{J}_n^{QR} \quad (xy = QR, UV) \tag{C.21}
\]
and hence
\[
\mathcal{R}_n^{QR}(\xi) = 0 \mod \mathcal{J}_n^{QR} \quad (xy = QR, UV). \tag{C.22}
\]

In a similar way one can treat functions (4.3) – (4.6):
\[
\begin{align*}
\mathcal{L}_n^{Q} &= \frac{1}{U_n V_n} \left( \mathcal{J}_n^{QR} + \frac{Q_n}{Q_{n-1}} \mathcal{J}_n^{QR} \right) \tag{C.23} \\
\mathcal{L}_n^{R} &= \frac{1}{U_{n+1} V_{n+1}} \left( \mathcal{J}_n^{QR} + \frac{R_n}{R_{n+1}} \mathcal{J}_n^{QR} \right) \tag{C.24} \\
\mathcal{L}_n^{U} &= \frac{1}{Q_{n-1} R_n} \left( \mathcal{J}_n^{UV} - \frac{U_{n+1}}{U_n} \mathcal{J}_n^{QR} \right) \tag{C.25} \\
\mathcal{L}_n^{V} &= \frac{1}{Q_n R_{n+1}} \left( \mathcal{J}_n^{UV} - \frac{V_{n+1}}{V_n} \mathcal{J}_n^{QR} \right). \tag{C.26}
\end{align*}
\]
Hence,
\[
\mathcal{L}_n^x = 0 \mod \mathfrak{g}_n, \mathfrak{h}_n \quad (x = q, r, u, v).
\] (C.27)

Finally, the properties of the functions \( \tilde{\mathfrak{F}}^{QU}_n \) and \( \tilde{\mathfrak{F}}^{RV}_n \) that are used in the body of the paper (in particular, equations (3.5)–(3.8)) follow from the identities
\[
\tilde{\mathfrak{F}}^{QU}_n (\xi) = - \frac{\dot{Q}_n}{\Gamma^{QR}_n} m^{U}_n - Q_n R_{n+1} \mathcal{I}_n^{U} - \frac{\dot{U}_{n+1}}{U_{n+1}} g_{n+1} + \dot{h}_{n+1} \quad (C.28)
\]
\[
= - \frac{\dot{U}_{n+1}}{U_{n+1}} m^{U}_n + \frac{\xi V_n}{V_n \Gamma^{UV}_n} \dot{h}_n - \xi \dot{Q}_n R_n \mathcal{I}_n^{R} \quad (C.29)
\]
\[
\tilde{\mathfrak{F}}^{RV}_n (\xi) = - \frac{R_n}{\Gamma^{QR}_n} m^{Q}_{n-1} - \dot{Q}_{n-1} \dot{R}_n \mathcal{I}_n^{V} + g_n - \frac{V_n}{V_n} \dot{g}_n \quad (C.30)
\]
\[
= - \frac{V_n}{U_{n+1} \Gamma^{UV}_n} m^{V}_n + \frac{\xi \dot{U}_{n+1}}{U_{n+1}} \dot{h}_n^{QR} + \xi Q_n R_n \mathcal{I}_n^{Q} \quad (C.31)
\]

and
\[
\frac{\mathfrak{A}^{U+1}_n}{\dot{U}_{n+1}} - \frac{\mathfrak{A}^{Q}_n}{\dot{Q}_n} = \xi \eta [\tilde{\mathfrak{F}}^{QU}_n (\xi) - \tilde{\mathfrak{F}}^{QU}_n (\eta)] + \xi \mathcal{E}_\xi \tilde{\mathfrak{F}}^{Q}_n (\eta) - \eta \mathcal{E}_\eta \tilde{\mathfrak{F}}^{Q}_n (\xi) \quad (C.32)
\]
\[
\frac{\mathfrak{A}^{V}_n}{V_n} - \frac{\mathfrak{A}^{R}_n}{R_n} = \xi \eta [\mathcal{E}_\eta \tilde{\mathfrak{F}}^{RV}_n (\xi) - \mathcal{E}_\xi \tilde{\mathfrak{F}}^{RV}_n (\eta)] + \xi \tilde{\mathfrak{F}}^{R}_n (\eta) - \eta \tilde{\mathfrak{F}}^{R}_n (\xi). \quad (C.33)
\]

that can be verified directly.

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