TWISTED SIMPLICIAL GROUPS AND TWISTED HOMOLOGY OF CATEGORIES

J. Y. LI, V. V. VERSHININ, AND J. WU

Abstract. Let $A$ be either a simplicial complex $K$ or a small category $C$ with $V(A)$ as its set of vertices or objects. We define a twisted structure on $A$ with coefficients in a simplicial group $G$ as a function

$$
\delta: V(A) \to \text{End}(G), \quad v \mapsto \delta_v
$$

such that $\delta_v \circ \delta_w = \delta_w \circ \delta_v$ if there exists an edge in $A$ joining $v$ with $w$ or an arrow either from $v$ to $w$ or from $w$ to $v$. We give a canonical construction of twisted simplicial group as well as twisted homology for $A$ with a given twisted structure. Also we determine the homotopy type of of this simplicial group as the loop space over certain twisted smash product.

1. Introduction

Motivated by the applications of algebraic topology to dynamic processes in the network and other areas of mathematics, some new versions of homology (cohomology) theory and combinatorial homotopy theory of graphs and simplexes were introduced recently [1, 2, 6, 8, 9, 10, 11, 16, 17]. In particular, the classical simplicial homology was varied by considering homology with local coefficient system for colored posets [8, 17] or by inserting $\delta$-factors as numerical data on vertices in the differentials of simplicial chain complexes [9, 10]. The purpose of this paper is to give a canonical twisted construction of simplicial groups as well as twisted homology for a simplicial complex and a small category with a given twisted structure. In particular, our twisted homology (twisted cohomology) is a generalization of the $\delta$-homology ($\delta$-cohomology) introduced in [9, 10]. Categories considered in the paper

2000 Mathematics Subject Classification. Primary 55U10; Secondary 18G30.

The first author is supported by NSFC (11201314, 11302136) of China and the Excellent Young Scientist Fund of Shijiazhuang Tiedao University. The second author is partially supported by the Laboratory of Quantum Topology of Chelyabinsk State University (Russian Federation government grant 14.Z50.31.0020) and RFBR grants 14-01-00014 and 13-01-92697-IND. The last author is partially supported by the Singapore Ministry of Education research grant (AcRF Tier 1 WBS No. R-146-000-190-112) and a grant (No. 11329101) of NSFC of China.
are small and we omit mentioning it. All necessary definitions will be
given later in the text of the paper.

Let $A$ be either a simplicial complex $K$ or a category $\mathcal{C}$. We are
considering category from geometrical point of view, using the termin-
ology of underlying quiver, for example we are using word "vertices"
instead of objects. However composition of morphisms and identity
morphisms are involved in our constructions. Let $V(A)$ be the vertex
set of $A$, $G$ be a simplicial group. We denote by $\text{End}(G)$ the monoid
of simplicial endomorphisms of $G$, namely $\text{End}(G)$ consists of graded
endomorphisms $f = \{f_n : G_n \to G_n\}_{n \geq 0}$ such that $d_i \circ f = f \circ d_i$ and
$s_i \circ f = f \circ s_i$ for the face homomorphisms $d_i : G_n \to G_{n-1}$ and the
degeneracy homomorphisms $s_i : G_n \to G_{n+1}$ for $0 \leq i \leq n$. A twisted
structure on $A$ with coefficients in a simplicial group $G$ is a function

$$\delta : V(A) \to \text{End}(G), \quad v \mapsto \delta_v$$

satisfying the following commuting rule:

$$(1.1) \quad \delta_w \circ \delta_v = \delta_{w \circ v}$$

if there exists an edge in $A$ (if $A$ is a simplicial complex) joining $v$ with
$w$ or an arrow either from $v$ to $w$ or from $w$ to $v$ (if $A$ is a category).
A twisted structure on $A$ with coefficients in $G$ is called non-singular
if $\delta_v : G \to G$ is a simplicial automorphism for each $v \in V(A)$.

Now let us suppose that a simplicial complex $K$ has a total order.
There is a canonical associated simplicial set $S(K)$ by allowing the
repeating of vertices in the sequences of paths. For a category $\mathcal{C}$ take
$S = S(\mathcal{C})$ to be a nerve of $\mathcal{C}$. The simplicial set $S = S(A)$ has a nature
that the set $S_n$ of $n$-paths ($n$-simplices) admits a form of the sequences

$$v_0 \alpha_1 v_1 \cdots \alpha_n v_n$$

with $v_i \in V(\mathcal{C})$ and $\alpha_i : v_{i-1} \to v_i$ is a morphism or the sequences
$(v_0, v_1, \ldots, v_n)$ for $v_i \in V(K)$ with $v_i \leq v_j$ for $i \leq j$ under a total order
on $V(K)$ and $\{v_0, \ldots, v_n\}$ being a simplex in $K$. The face operation
d_i : S_n \to S_{n-1}, \quad 0 \leq i \leq n,$

is given by removing the vertex $v_i$ followed
the composition $\alpha_i \alpha_{i+1} : v_{i-1} \to v_{i+1},$ and the degeneracy operation
s_i : S_n \to S_{n+1}, \quad 0 \leq i \leq n,$

is given by doubling the vertex $v_i$ with
inserting the identity arrow $e_{v_i} : v_i \to v_i$ if it is the case of category.

We construct a $\Delta$-group and a simplicial group $F^G[A]$ depending
on the twisted structure $\delta$. This $\Delta$-group seems to be interesting. If
the $\delta$-structure is given by $\delta_v \equiv \text{id}_G$ for $v \in V(A)$, then $F^G[A]$ is
Carlsson’s $J_G(X)$-construction [5] or Quillen’s tensor product [15] with
its geometric realization having the homotopy type of $\Omega(B[G] \wedge |A|)$,

where $|X|$ is the geometric realization of a simplicial set $X$. 


The main result in this article is to determine the homotopy type of $F^G_δ[A]$ as the loop space of the twisted smash product $B|G| \land_δ |A|$. See section 4 for the definition of twisted smash product.

The article is organized as follows. In Section 2, we give a review on the path complexes of categories. The twisted construction of the $Δ$-groups and simplicial groups is given in section 3 by examining the $Δ$-identity and simplicial identities. The homotopy type of the twisted simplicial group $F^G_δ[A]$ is studied in section 4, where the main result is Theorem 4.5. In section 5, we explore the twisted homology of simpli-
cial complexes as well as categories.

2. Path complexes of categories and simplicial set extension of simplicial complexes

2.1. Path complexes of categories. Recall that a quiver is a quadruple $Q = (V(Q), E(Q), s, t)$, where the set $V$ is called the set of vertices of $Q$, the set $E$ is called the set of edges of $Q$, $s$ and $t$ are two maps $s : E \to V$, giving the start or source of the edge, and $t : E \to V$, giving the target of the edge. The elements in $E(Q)$ we denote by arrows $α : v \to w$, or $vαw$.

Recall also that there is a forgetful functor from small categories to quivers

\[ Cat \to Quiv. \]

By an $n$-path for a category $C$ we mean the sequence $v_0α_1 \cdots v_{n-1}α_nv_n$ with $n \geq 1$. There exists a trivial path (the identity arrow) $e_v : v \to v$ for each vertex $a \in V(C)$. Let $S_n(C)$ be the set of all $n$-paths with $n \geq 1$ and $S_0(C) = V(C)$.

The face operations in $S(C)$ are given by removing a vertex followed by the composition of arrows in the following sense:

\[ d_i(v_0α_1v_1α_2v_2 \cdots α_nv_n) = v_0α_1v_1 \cdots v_{i-1}(α_{i+1} \circ α_i)v_{i+1} \cdots α_nv_n \]

for $0 \leq i \leq n$, where $d_0$ removes the first vertex together with the first arrow and the last face $d_n$ removes the last vertex together with the last arrow. The degeneracy operations

\[ s_i : S_n(C) \to S_{n+1}(C) \]

for $0 \leq i \leq n$ are defined by doubling the $i$-vertex with inserting the trivial path. Then $S(C) = \{S_n(C)\}_{n \geq 0}$ is the classifying simplicial set (or nerve) of the category $C$.

Choose a vertex $a_0 \in V(C)$ as a basepoint. Consider the constant $n$-path $a_0^n = a_0ε_{a_0}a_0 \cdots ε_{a_0}a_0$ as a basepoint for the set $S_n(C)$. Then $S(C)$ is a pointed simplicial set [18, Chapter 2].
2.2. Simplicial set extension of simplicial complexes. Let $K$ be a simplicial complex with its vertex set $V(K)$ which has a total order. Recall that the simplicial set $S(K)$ induced by $K$ is given as follows:

1. The set $S(K)_n$ consists of sequences $(v_0, v_1, \ldots, v_n)$ of vertices of $K$ such that $v_0 \leq v_1 \leq \cdots \leq v_n$ and $\{v_0, \ldots, v_n\}$ forms a simplex in $K$.

2. The face operation $d_i : S(K)_n \to S(K)_{n-1}, 0 \leq i \leq n$, is given by

\[
d_i(v_0, v_1, \ldots, v_n) = (v_0, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n).
\]

3. The degeneracy operation $s_i : S(K)_n \to S(K)_{n+1}, 0 \leq i \leq n$, is given by

\[
s_i(v_0, v_1, \ldots, v_n) = (v_0, v_1, \ldots, v_{i-1}, v_i, v_i, v_i, v_{i+1}, \ldots, v_n).
\]

The geometric realization of the simplicial set $S(K)$ is homeomorphic to the polyhedron $|K|$ (see, for example, [7]). We choose a basepoint in $S(K)_n$ by $a_0^n = (a_0, a_0, \ldots, a_0)$ for a vertex $a_0 \in V(K)$ so that $S(K)$ is a pointed simplicial set.

3. THE TWISTED CONSTRUCTION OF $\Delta$-GROUPS AND SIMPLICIAL GROUPS

A $\Delta$-set is a sequence of sets $X = \{X_n, n \geq 0\}$ with maps called faces

\[
d_i : X_n \to X_{n-1}, 0 \leq i \leq n,
\]

such that the following condition is hold

\[
d_id_j = d_jd_{i+1}
\]

for $i \geq j$. It is called the $\Delta$-identity [18]. Roughly speaking $\Delta$-set is a simplicial set without degeneracies.

A $\Delta$-set $G = \{G_n\}_{n \geq 0}$ is called a $\Delta$-group if each $G_n$ is a group, and each face $d_i$ is a group homomorphism [18].

We proceed the following steps for constructing a $\Delta$-group and a simplicial group depending on the twisted structure $\delta$ on $A$:

1) Choose a vertex $a_0 \in V(A)$ as a basepoint. Consider the constant $n$-path $a_0^n = a_0a_0a_0 \cdots a_0a_0$ as a basepoint for the set $S_n$. Define $F^G[A]_n$ to be the free product of the group $G_n$ with indices running over all non-basepoint elements in $S_n(A)$. More precisely, let $(G_n)_x$ be a copy of $G_n$ labelled by an element $x \in S_n(A)$. Then $F^G[A]_n$ is the quotient group of the free product $\ast_{x \in S_n(G_n)_x}$ subject to the relations that $(G_n)a_0^n = \{1\}$. 
2) Let \( g_x \) denote the element \( g \in G_n \) in its copy group \((G_n)_x\) labelled by \( x \). Define a (twisted) face operation \( d^\delta_i : F^G[A]_n \to F^G[A]_{n-1} \), \( 0 \leq i \leq n \) to be the (unique) group homomorphism such that \( d^\delta_i \) restricted to each copy \((G_n)_{v_0a_1v_1 \cdots a_nv_n}\) is given by the twisted formula

\[
d^\delta_i|((G_n)_{v_0a_1v_1 \cdots a_nv_n}) = (\delta v_i(d_i(g)))d_i(v_0a_1v_1 \cdots a_nv_n)\).
\]

Namely the element \( d_i g \in (G_n-1)_{d_i(v_0a_1v_1 \cdots a_nv_n)} \) is twisted by the endomorphism \( \delta v_i : G_{n-1} \to G_{n-1} \). By Proposition 3.1 below, the sequence of groups \( F^G[A]_\delta = \{F^G[A]_n\}_{n \geq 0} \) with the twisted face operations forms a \( \Delta \)-group, where the commuting rule \( (\mathbb{I \! I \! I}) \) assures the \( \Delta \)-identity.

3) Suppose that the \( \delta \)-structure is non-singular. We define a twisted degeneracy operation \( s^\delta_i : F^G[A]_n \to F^G[A]_{n+1} \), \( 0 \leq i \leq n \), to be the (unique) group homomorphism such that \( s^\delta_i \) restricted to each copy \((G_n)_{v_0a_1v_1 \cdots a_nv_n}\) is given by the twisted formula

\[
s^\delta_i|((G_n)_{v_0a_1v_1 \cdots a_nv_n}) = ((\delta v_i)^{-1}(s_i(g)))s_i(v_0a_1v_1 \cdots a_nv_n).
\]

By Proposition 3.1 the sequence of groups \( F^G[A]_\delta = \{F^G[A]_n\}_{n \geq 0} \) with the twisted face operations \( d^\delta_i \) in step 2 and the above twisted degeneracy operations \( s^\delta_i \) forms a simplicial group \([18, \text{Chapter } 2]\), where the commuting rule \( (\mathbb{I \! I \! I}) \) and invertibility of \( \delta_v \) are essential for assuring the simplicial identities.

The \( \Delta \)-group \( F^G[A]_\delta \) is a generalization of \( \delta \)-homology \([11, 10]\) in the following sense. Let \( G \) be any abelian group. Let \( \delta : V(A) \to \End(G) \) be a function satisfying the commuting rule \( (\mathbb{I \! I \! I}) \). We consider \( G \) as a discrete simplicial group with \( G_n = G \) and the faces and degeneracies given by the identity. Then we have the \( \delta \)-structure on \( A \). By taking the abelianization of the (non-commutative) \( \Delta \)-group \( F^G[A]_\delta \) in step 2, we have an abelian \( \Delta \)-group with the twisted faces given in \((3.2)\) on chains

\[
(F^G[A]_n)^{ab} = \left( \bigoplus_{x \in S_n} G_x \right) / (G)_{a_{0}}
\]

having the differentials given by

\[
\partial_n(g_{v_0a_1v_1 \cdots a_nv_n}) = \sum_{i=0}^{n} (-1)^i(\delta v_i(d_i(g)))d_i(v_0a_1v_1 \cdots a_nv_n).
\]

In the case where \( G \) is a commutative ring \( R \) and \( \delta_v \) is the translation given by an element \( \delta_v \in R \), the above formula on the differentials is exactly given by inserting the factor \( \delta_v \) as described in \([9, 10]\).
Proposition 3.1. Let $S$ be $S(A)$ where $A$ is either a simplicial complex $K$ or a category $C$ with twisted structure in a simplicial group $G$. Then

1. The sequence of groups $F_{\delta}^{\Delta}[A] = \{F^G[A]_n\}_{n\geq 0}$ with the twisted face operations defined by formula 3.2 forms a $\Delta$-group.
2. Suppose that the twisted structure $\delta$ is nonsingular. Then the sequence of groups $F_{\delta}^{G_i}[A] = \{F^{G_i}[A]_n\}_{n\geq 0}$ with the twisted face operations $d_{\delta}^i$ given by formula 3.2 and the twisted degeneracy operations $s_{\delta}^i$ given by formula 3.3 forms a simplicial group.

Proof. The proof is given by examining the $\Delta$-identity for assertion (1), and simplicial identities for assertion (2). Since the examination on the $\Delta$-identity for assertion (1) is part of the simplicial identities for assertion (2), we only need to prove assertion (2) with paying attention that the invertibility of $\delta_v$ is not required in the $\Delta$-identity for face operations.

For the basepoint $a_0^n$ in $S_n$, we see that each $d_{\delta}^i$ sends $(G_n)_0^n$ into $(G_{n-1})_{0}^{n-1}$, so

$$d_{\delta}^i : F_{\delta}^{G_i}[A]_n \rightarrow F_{\delta}^{G_i}[A]_{n-1}$$

is a well-defined homomorphism. The same is true for

$$s_{\delta}^i : F_{\delta}^{G_i}[A]_n \rightarrow F_{\delta}^{G_i}[A]_{n+1}.$$ 

Now we show that the simplicial-identity holds for the face and degeneracy operations. For simplicity of notations, we only consider the case $S = S(K)$. For the case $S(C)$ of a category $C$, the proof follows from the same lines with keeping in mind the following rules:

1. When a vertex $v_i$ is removed from the sequence $(v_0, \ldots, v_n)$, the arrow from $v_{i-1} \rightarrow v_{i+1}$ in the sequence $(v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$ is given by the composition $v_{i-1} \rightarrow v_i \rightarrow v_{i+1}$.
2. When a vertex $v_i$ is doubled from the sequence $(v_0, \ldots, v_n)$, we insert the trivial path $e_{v_i} : v_i \rightarrow v_i$ in the sequence

$$(v_0, \ldots, v_{i-1}, v_i, v_i, v_{i+1}, \ldots, v_n).$$
Let \( g_{(v_0, \ldots, v_n)} \in (G_n)_{(v_0, \ldots, v_n)} \), and let \( i \geq j \) with \( 0 \leq j \leq n - 1 \) and \( 0 \leq i \leq n \). Then

\[
d^i_j \circ d^j_0(g_{(v_0, \ldots, v_n)}) = d^i_j(( \delta^n_{v_{j+1}} - 1(d_j g))(v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n))
\]

\[
= d^i_j(\delta^n(v_{j+1})(g))(v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n)
\]

\[
= (\delta^n_{v_{j+1}} - 1(d_j g))(v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n)
\]

\[
= (\delta^n_{v_{j+1}} - 1(d_j g))(v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n)
\]

\[
= (\delta^n_{v_{j+1}} - 1(d_j g))(v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n)
\]

\[
= (\delta^n_{v_{j+1}} - 1(d_j g))(v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n)
\]

\[
= (\delta^n_{v_{j+1}} - 1(d_j g))(v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n)
\]

where \( \delta^n_{v_{j+1}} \circ \delta^n_{v_j} = \delta^n_{v_j} \circ \delta^n_{v_{j+1}} \) because \( v_j v_{j+1} \) forms a 1-simplex in \( K \). In the case of \( S = S(C) \), \( \delta^n_{v_{j+1}} \circ \delta^n_{v_j} = \delta^n_{v_j} \circ \delta^n_{v_{j+1}} \) because there is an arrow given by a composition from \( v_j \) to \( v_{j+1} \).

Let \( i \geq j \), and let \( g_{(v_0, \ldots, v_n)} \in (G_n)_{(v_0, \ldots, v_n)} \). We shorten the notation \( s^i_j \) as \( s_i \). Then

\[
s_j s_i(g_{(v_0, \ldots, v_n)}) = s_j((\delta^n_{v_{j+1}} - 1(s_i g))(v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n))
\]

\[
= s_j((\delta^n_{v_{j+1}} - 1(s_i g))(v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n))
\]

\[
= (\delta^n_{v_{j+1}} - 1(s_j s_i((\delta^n_{v_j} - 1(g)))(v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n))
\]

\[
= (\delta^n_{v_{j+1}} - 1(s_j s_i((\delta^n_{v_j} - 1(g)))(v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n))
\]

\[
= (\delta^n_{v_{j+1}} - 1(s_j s_i((\delta^n_{v_j} - 1(g)))(v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n))
\]

\[
= s_i s_j(g_{(v_0, \ldots, v_n)}).
\]

We shorten the notation \( d^i_j \) as \( d_i \). Let \( i < j \), and let \( g_{(v_0, \ldots, v_n)} \in (G_n)_{(v_0, \ldots, v_n)} \). Then

\[
d_i s_j(g_{(v_0, \ldots, v_n)}) = d_i((\delta^n_{v_{j+1}} - 1(s_j g))(v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n))
\]

\[
= d_i((\delta^n_{v_{j+1}} - 1(s_j g))(v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n))
\]

\[
= (\delta^n_{v_{j+1}} - 1(s_j s_i(g))(v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n))
\]

\[
= (\delta^n_{v_{j+1}} - 1(s_j s_i(g))(v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n))
\]

\[
= (\delta^n_{v_{j+1}} - 1(s_j s_i(g))(v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n))
\]

\[
= s_i s_j(g_{(v_0, \ldots, v_n)}).
\]

Let \( i = j \), and let \( g_{(v_0, \ldots, v_n)} \in (G_n)_{(v_0, \ldots, v_n)} \). Then

\[
d_j s_j(g_{(v_0, \ldots, v_n)}) = d_j((\delta^n_{v_{j+1}} - 1(s_j g))(v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n))
\]

\[
= d_j((\delta^n_{v_{j+1}} - 1(s_j g))(v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n))
\]

\[
= (\delta^n_{v_{j+1}} - 1(s_j s_i(g))(v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n))
\]

\[
= (\delta^n_{v_{j+1}} - 1(s_j s_i(g))(v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n))
\]

\[
= g_{(v_0, \ldots, v_n)}.
\]
Let $i = j + 1$, and let $g_{(v_0,\ldots,v_n)} \in (G_n)_{(v_0,\ldots,v_n)}$. We remind that the rule is such that we add the twisting of the vertex we are removing. When we apply $d_{j+1}$, we should add twisting $\delta_{v_j}$ because the second $v_j$ is located in the position $(j + 1)$ under removing. Then we have

$$d_{j+1}s_j(g_{(v_0,\ldots,v_n)}) = d_{j+1}(\delta_{v_j}^{n+1}(s_j(g)))_{(v_0,\ldots,v_j,v_j,\ldots,v_n)}$$

$$= d_{j+1}(s_j\delta_{v_j}^n)(g)_{(v_0,\ldots,v_j,v_j,\ldots,v_n)}$$

$$= \delta_{v_{j+1}}^n(d_{j+1}s_j\delta_{v_j}^n)(g)_{(v_0,\ldots,v_j,v_j,\ldots,v_n)}$$

$$= (d_{j+1}s_j\delta_{v_j}^n)(d_{v_{j+1}}^n)(g)_{(v_0,\ldots,v_j,v_j,\ldots,v_n)}$$

$$= s_jd_{v_{j+1}}^n(g_{(v_0,\ldots,v_n)}).$$

Let $i > j + 1$, and let $g_{(v_0,\ldots,v_n)} \in (G_n)_{(v_0,\ldots,v_n)}$. Then

$$d_is_j(g_{(v_0,\ldots,v_n)}) = d_i(s_j\delta_{v_j}^n)(g)_{(v_0,\ldots,v_j,v_j,\ldots,v_n)}$$

$$= d_i(s_j\delta_{v_j}^n)(g)_{(v_0,\ldots,v_j,v_j,\ldots,v_n)}$$

$$= \delta_{v_{i-1}}^n(d_is_j\delta_{v_j}^n)(g)_{(v_0,\ldots,v_j,v_j,\ldots,v_n)}$$

$$= (d_is_j\delta_{v_j}^n)(d_{v_{i-1}}^n)(g)_{(v_0,\ldots,v_j,v_j,\ldots,v_n)}$$

$$= (\delta_{v_{i-1}}^n)(s_jd_{v_{i-1}}^n(d_is_j\delta_{v_j}^n)(g))_{(v_0,\ldots,v_j,v_j,\ldots,v_n)}$$

$$= s_jd_{v_{i-1}}^n(g_{(v_0,\ldots,v_n)}).$$

\[\square\]

4. Homotopy Type of $F^S[G][S]$\]

4.1. The $\delta$-twisted Cartesian products. We generalize now the notion of twisted structure for a simplicial set, not necessary a simplicial group $G$. Let $Y$ be a simplicial set. We denote by $\text{End}(Y)$ the monoid of self-simplicial maps of $Y$. Let $A$ be a simplicial complex or a category with vertex set $V(A)$. A twisted structure on $A$ in a simplicial set $Y$ is a function

$$\delta: V(A) \to \text{End}(Y), \quad v \mapsto \delta_v$$

such that the commuting rule

$$\delta_w \circ \delta_v = \delta_v \circ \delta_w$$

holds if there is an edge joint $v$ and $w$ in the case of simplicial complexes and arrow between $v$ and $w$ in the case of category. A twisted structure on $A$ is called nonsingular if $\delta_v: Y \to Y$ is a simplicial isomorphism for each $v \in V(A)$.

Let $S = S(A)$ be a simplicial set defined in section 2. Given any twisted structure $\delta$ on $A$ in a simplicial set $Y$, we can proceed the following construction of $\delta$-twisted Cartesian product:

1. The sequence of sets is given by $Y_n \times S_n$ for $n \geq 0$. 


Proposition 4.2. Let theory \([3, 7]\) different from that the classical twisted Cartesian product in simplicial
Remark 4.1. The definition of our \(\delta\) structure \(A\) homology (cohomology) with twisted coefficients. Let
simplicial set \(Y\) proposition 3.1 for examining the \(\Delta\)-identity and the simplicial identities.
Proof. The assertions follow from the lines in the proof of Proposition (4.2) for having
homology (cohomology) with twisted coefficients. Let \(A\) have a twisted structure \(\delta\) in a simplicial set \(Y\). Let \(G\) be an abelian group. The twisted homology (twisted cohomology) of \(A\) with coefficients in \(G\) is defined by \(H^\text{twist}_n(A; G) = H_n(G(Y \times \Delta S(A)))\), \(H^\text{twist}_n(A; G) = H^n(G(Y \times \Delta S(A)))\) for \(n \geq 0\), where \(G(X)\) for a simplicial set \(X\) is given by \(G(X) = \oplus_{x \in X} (G)_x\) with \((G)_x\) a copy of \(G\) labelled by \(x\). A consequence of (1) of Proposition (4.2) is the following corollary.
Corollary 4.3. Let \(A\) have a twisted structure \(\delta\) in a simplicial set \(Y\). Then the twisted homology \(H^\text{twist}_n(A; G)\) (twisted cohomology \(H^\text{twist}_n(A; G)\)) is the homology (cohomology) of the chain complex \(C^\text{twist}_n(A)\) with coefficients in \(G\), where
\[
C^\text{twist}_n(A) = \mathbb{Z} \cdot Y_n \oplus \mathbb{Z} \cdot S(A)\]
with the differential \(d^\text{twist}_n\) given by
\[
d^\text{twist}_n(y \otimes v_0 \alpha_1 v_1 \cdots \alpha_n v_n) = \sum_{i=0}^{n} (-1)^i \delta_{v_i} (d_i y) \otimes d_i (v_0 \alpha_1 v_1 \cdots \alpha_n v_n)\]
for \( n \geq 0 \). □

From the second assertion of Proposition 4.2, there is a connection between \( \delta \)-twisted Cartesian products and simplicial fibre bundles which is described as follows. Let \( S = S(A) \) be with a nonsingular twisted structure \( \delta \) in a simplicial set \( Y \). Observe that the coordinate projection

\[
p: Y \times_\delta S \to S, \quad (g, v_0 \alpha_1 v_1 \cdots \alpha_n v_n) \mapsto v_0 \alpha_1 v_1 \cdots \alpha_n v_n
\]

is a simplicial map. Recall from [7, page 155] that a simplicial map \( p: E \to B \) is called a simplicial fibre bundle with fibre \( F \) if for each \( b \in B_n \), there is a commutative diagram

\[
\begin{array}{ccc}
\Delta[n] \times F & \xrightarrow{\alpha_b} & E' \\
\downarrow \text{proj.} & & \downarrow \text{pull-back} \\
\Delta[n] & \xrightarrow{f_b} & B,
\end{array}
\]

where \( \Delta[n] \) is the standard simplicial \( n \)-simplex, \( f_b: \Delta[n] \to B \) is the representing map of \( b \in B_n \) and \( \alpha_b \) is a simplicial isomorphism.

**Theorem 4.4.** Let \( S = S(A) \) be with a non-singular twisted structure \( \delta \) in a simplicial set \( Y \). Then the coordinate projection map

\[
p: Y \times_\delta S \to S
\]

is a simplicial fibre bundle with fibre \( Y \).

**Proof.** Let \( \sigma = (0, 1, \ldots, n) \in \Delta[n] \) be the standard non-degenerate \( n \)-simplex. For each \( b = v_0 \alpha_1 v_1 \cdots \alpha_n v_n \in S_n \), we have \( f_b(\sigma) = b \) by definition. By taking iterated faces, we have \( f_b(i) = v_i \). The twisted structure on \( A \) induces a function

\[
\delta: V(\Delta[n]) = \{(0), (1), \ldots, (n)\} \to \text{End}(Y)
\]

given by

\[
\delta(i) = \delta_{v_i}
\]

for \( 0 \leq i \leq n \). For \( 0 \leq i < j \leq n \), since there is an edge between \( v_i \) and \( v_j \) or an arrow \( v_i \to v_j \), we have

\[
\delta_{(i)} \circ \delta_{(j)} = \delta_{(j)} \circ \delta_{(i)}
\]

by the commuting rule \( \delta_{v_i} \circ \delta_{v_j} = \delta_{v_j} \circ \delta_{v_i} \) and so a twisted structure on \( \Delta[n] \) in \( Y \). From the definition of \( \delta \)-twisted Cartesian product, we have the simplicial map

\[
(id, f_b): Y \times_\delta \Delta[n] \to E = Y \times_\delta S
\]
inducing a fibrewise simplicial isomorphism

\[
\begin{array}{ccc}
Y \times \delta \Delta[n] & \longrightarrow & E' \\
\downarrow p & & \downarrow p \\
\Delta[n] & = & \Delta[n],
\end{array}
\]

where \(E'\) is given by the pull-back in diagram (4.5) with \(E = Y \times \delta S\) and \(B = S\). Now we show that there is a fibrewise simplicial isomorphism

\[
Y \times \Delta[n] \overset{\alpha_b}{\longrightarrow} Y \times \delta \Delta[n]
\]

\[
\begin{array}{ccc}
\Delta[n] & = & \Delta[n].
\end{array}
\]

The simplicial isomorphism \(\alpha_b\) is constructed by untwisting the twisted faces and degeneracies. More precisely, observe that, each element \(w\) in \(\Delta[n]_q\) can be uniquely expressed as a monotone sequence

\[
w = (j_1(w), \ldots, j_l(w), \ldots, j_{t_w}(w), \ldots, j_{t_w}(w))
\]

with \(l_i(w) \geq 1\) for \(1 \leq i \leq t\), \(0 \leq j_1(w) < j_2(w) < \cdots < j_{t_w}(w) \leq n\) and \(\sum_{i=1}^{t_w} l_i(w) = q + 1\). Let

\[
\{i_1(w), \ldots, i_{s_w}(w)\} = \{0, 1, \ldots, n\} \setminus \{j_1(w), j_2(w), \ldots, j_{t_w}(w)\}
\]

with \(i_1(w) < i_2(w) < \cdots < i_{s_w}(w)\) be the set of missing vertices in \(w\). Define the function

\[
\alpha_b : Y_q \times \Delta[n]_q \longrightarrow Y_q \times \Delta[n]_q
\]

by setting

\[
\alpha_b(y, w) = (\delta_{j_{t_w}(w)}^{l_{t_w}(w)+1} \cdots \delta_{j_1(w)}^{l_1(w)+1} \delta_{i_1(w)} \cdots \delta_{i_{s_w}(w)})(y, w).
\]

Clearly \(\alpha_b : Y_q \times \Delta[n]_q \longrightarrow Y_q \times \Delta[n]_q\) is bijective for \(q \geq 0\). It suffices to show that \(\alpha_b\) is a simplicial map. We shorten the notation \(l_i(w)\) as \(l_i\), and similarly for other functions on \(w\). Note that, for \(0 \leq i \leq q\),

\[
d_i^q(\alpha_b(y, w)) = d_i^q(\delta_{j_1}^{l_1+1} \cdots \delta_{j_{l_1}}^{l_1+1} \delta_{i_1} \cdots \delta_{i_{s_w}}(y, w)) = (d_i(\delta_{j_1}^{l_1+1} \cdots \delta_{j_{l_1}}^{l_1+1} \delta_{i_1} \cdots \delta_{i_{s_w}})(y, w), d_i(w)),
\]
where \((i')\) is the \(i\)-th vertex of the \(q\)-simplex \(w\). Let \(k\) be the minimal integer such that
\[
l_1 + \cdots + l_k - 1 \geq i.
\]
Then \(i' = j_k\) and
\[
d^\delta_i (\alpha_b(y, w)) = (\delta_{(j_1)}^{-l_1+1} \cdots \delta_{(j_i)}^{-l_i+1} \delta_{(j_1)} \cdots \delta_{(i_1)}(d_i(y)), d_i(w)).
\]
Note that
\[
d_i w = (j_1(w), \ldots, j_1(w), \ldots, j_k(w), \ldots, j_k(w), \ldots, j_{l_1}(w), \ldots, j_{l_1}(w)).
\]
From the definition of \(\alpha_b\), we have
\[
\alpha_b(d_i(y), d_i(w)) = d^\delta_i (\alpha_b(y, w)).
\]
Consider the degeneracy operation \(s_i, 0 \leq i \leq q\). We have
\[
s_i w = (j_1(w), \ldots, j_1(w), \ldots, j_k(w), \ldots, j_k(w), \ldots, j_{l_1}(w), \ldots, j_{l_1}(w)).
\]
Then
\[
s^\delta_i (\alpha_b(y, w)) = s^\delta_i (\alpha_b(y, w)) = (\delta_{(j_1)}^{-l_1+1} \cdots \delta_{(j_i)}^{-l_i+1} \delta_{(j_1)} \cdots \delta_{(i_1)}(s_i(y)), s_i(w)) = (\delta_{(j_1)}^{-l_1+1} \cdots \delta_{(j_i)}^{-l_i+1} \delta_{(j_1)} \cdots \delta_{(i_1)}(s_i(y)), s_i(w)) = \alpha_b(s_i(y), s_i(w)).
\]

4.2. The \(\delta\)-Twisted Smash Products. Now we consider the pointed constructions. Let \(Y\) be a pointed simplicial set with the base-point
\[
* = \{s^n_0\}_{n \geq 0}.
\]
A reduced twisted structure on \(A\) in \(Y\) is a function
\[
\delta: V(K) \rightarrow \text{End}_*(Y), \quad v \mapsto \delta_v
\]
such that the commuting rule holds, where \(\text{End}_*(Y)\) is the monoid of base-point-preserving self-simplicial maps of \(Y\). A reduced twisted structure on \(A\) is called non-singular if \(\delta_v: Y \rightarrow Y\) is a simplicial isomorphism for each \(v \in V(A)\).

Suppose that \(S = S(A)\) has non-singular reduced twisted structure \(\delta\) in a pointed simplicial set \(Y\). Since \(\delta_v(*) = *\) for \(v \in V(A)\) there is a canonical inclusion
\[
* \times S = * \times_\delta S \longrightarrow Y \times_\delta S.
\]
Let $a_0$ be a vertex of $A$ treated as the base-point with the induced $\delta$-structure by the restriction. Then there is a canonical simplicial inclusion

$$Y \times_\delta a_0 \rightarrow Y \times_\delta S.$$ 

The $\delta$-twisted smash product $Y \wedge_\delta S$ is then defined as the simplicial quotient set

$$(4.6) \quad Y \wedge_\delta S := (Y \times_\delta S) / ((Y \times_\delta a_0) \cup (* \times_\delta S)).$$

4.3. The classifying spaces of the twisted simplicial groups. Recall that there is a functor $\bar{W}$ from simplicial groups to simplicial sets, which plays the role of classifying space functor in simplicial theory. The construction of the functor $\bar{W}$ is briefly reviewed using the terminology of categorical digraphs as follows.

Let $G$ be a group (without assuming simplicial structure). Let $\mathcal{G}$ be the category with a single vertex $a$ and a collection of arrows labeled by the elements $g \in G$, where the identity $1 = e_a : a \rightarrow a$ is considered as the trivial path of the vertex $a$. Then $\mathcal{G}$ is a categorical digraph with the composition operation induced by the multiplication of the group $G$. Let $W(G) = S(\mathcal{G})$ defined in section 2.

Now let $G = \{G_n\}_{n \geq 0}$ be a simplicial group. The face homomorphisms $d_i : G_n \rightarrow G_{n-1}$ and the degeneracy homomorphisms $s_i : G_n \rightarrow G_{n+1}$ induce simplicial face maps $d_i = W(d_i) : W(G_n) \rightarrow W(G_{n-1})$ and simplicial degeneracy maps $s_i = W(s_i) : W(G_n) \rightarrow W(G_{n+1})$ for $0 \leq i \leq n$ so that $\{W(G_n)\}_{n \geq 0}$ is a bi-simplicial set. The simplicial set $W(G)$ is then defined as the diagonal simplicial set of the bi-simplicial set $W(G_s)_s$.

**Theorem 4.5.** Let $S = S(A)$ has a non-singular twisted structure in a simplicial group $G$. Then there is a natural simplicial map

$$\theta : W(G) \wedge_\delta S \rightarrow W(F_\delta^G[S]),$$

which is a homotopy equivalence after geometric realization, where the twisted structure of $A$ in the simplicial set $W(G)$ is induced from its twisted structure in $G$ through the functor $W$.

**Proof.** Let $v$ be a vertex of $A$. By applying the functor $\bar{W}$ to the simplicial isomorphism $\delta_v : G \rightarrow G$, we have a sequence of isomorphisms of pointed simplicial sets

$$\bar{W}(\delta_v^n) : \bar{W}(G_n) \rightarrow \bar{W}(G_n)$$
and the commutative diagram

\[
\begin{array}{ccc}
\widetilde{W}(G_{n-1}) & \xrightarrow{\widetilde{W}(d_i)} & \widetilde{W}(G_n) \\
\downarrow & & \downarrow \\
\widetilde{W}(\delta_{n}^{i-1}) & \xrightarrow{\widetilde{W}(\delta_{n}^{i})} & \widetilde{W}(\delta_{n}^{i+1}) \\
\end{array}
\]

for \(0 \leq i \leq n\) and \(v \in V(A)\).

Recall from the definition that the group \((F_v^{G}[S])_n = F_v^{G}[S]_n\) is the quotient group of the free product \(*_{x \in S_n}(G_n)_x\) subject to the relations \((G_n)_{a_0^n} = \{1\}\). For \(x \in S_n\), the inclusion \((G_n)_x \hookrightarrow F_v^{G}[S]_n\) induces a simplicial map

\[ j_x : \widetilde{W}((G_n)_x) \longrightarrow \widetilde{W}((F_v^{G}[S])_n) \]

and so a simplicial map

\[ \tilde{\theta}_n : \widetilde{W}(G_n) \times S_n \longrightarrow \widetilde{W}((F_v^{G}[S])_n) \quad (y, x) \mapsto j_x(y), \]

where we consider \(S_n\) as a discrete simplicial set with \((S_n)_q = S_n\) for \(q \geq 0\) and faces and degeneracies being given by the identity map. Since \((G_n)_{a_0^n} = \{1\}\), we have

\[ \tilde{\theta}_n(\widetilde{W}(G_n) \times \{a_0^n\}) = * . \]

Since \(j_x(*) = *\) for any \(x \in S_n\), we have

\[ \tilde{\theta}_n(* \times S_n) = * \]

and so \(\tilde{\theta}\) factors through the smash product. Let

\[ \theta_n : \widetilde{W}(G_n) \wedge S_n \longrightarrow \widetilde{W}((F_v^{G}[S])_n) \]

be the resulting simplicial map. Now we let \(n\) be varied. Let

\[ x = v_0a_1v_1 \cdots a_nv_n \in S_n = (S_n)_q \quad \text{and} \quad y = a_{g_1}a_{g_2}a \cdots a_{g_q}a \in \widetilde{W}(G_n)_q \]

with \(g_1, \ldots, g_q \in G_n\). Then

\[ \theta_n(y \wedge x) = a_{g_1}a_{g_2}a \cdots a_{g_q}a. \]

Under the twisted faces in the simplicial group \(F_v^{G}[S]\), we have

\[
\begin{align*}
\widetilde{W}(d_v^i)(\theta_n(y \wedge x)) &= \widetilde{W}(d_v^i)(a_{g_1}a_{g_2}a \cdots a_{g_q}a) \\
&= a(d_v^i((g_1)_x))a(d_v^i((g_2)_x))a \cdots a(d_v^i((g_q)_x))a \\
&= a((\delta_{v_1}^{n-1}(d_i(g_1)))_{d,x})a \cdots a((\delta_{v_1}^{n-1}(d_i(g_q)))_{d,x})a \\
&= \theta_{n-1}(\widetilde{W}(\delta_{v_1}^{n-1}(d_i(y))) \wedge d_i(x))
\end{align*}
\]
for $0 \leq i \leq n$. Similarly, under the twisted faces in the simplicial group $F^G_\delta[S]$, we have
\[
\tilde{W}(s_i^\delta(\theta_n(y \wedge x))) = \theta_{n+1}((\tilde{W}((\delta_{v_i}^{n+1})^{-1})(s_i(y))) \wedge s_i(x)).
\]

We define a $\delta$-twisted structure on the sequence of simplicial sets
\[
\{\tilde{W}(G_n) \wedge S_n\}_{n \geq 0}
\]
by the formulae
\[
d^\delta_i(y \wedge x) = (\tilde{W}((\delta_{v_i}^{n+1})^{-1})(d_i(y))) \wedge d_i(x),
\]
\[
s^\delta_i(y \wedge x) = (\tilde{W}((\delta_{v_i}^{n+1})^{-1})(s_i(y))) \wedge s_i(x)
\]
for $0 \leq i \leq n$, $x = v_0a_1v_1 \cdots a_nv_n \in S_n = (S_n)_q$ and $y \in \tilde{W}(G_n)_q$.

Since each $\theta_n$ is injective and $\tilde{W}((F^G_\delta[S])_*)$ is a bi-simplicial set, the simplicial identities hold for $d^\delta_i$ and $s^\delta_i$ in $T_{*,*} = \{\tilde{W}(G_n) \wedge S_n\}_{n \geq 0}$ so that $T_{*,*}$ is a bi-simplicial set with
\[
\theta_* : T_{*,*} \longrightarrow \tilde{W}((F^G_\delta[S])_*),
\]

being a morphism of bi-simplicial sets.

Recall that, by the Whitehead Theorem [13, Proposition 4.3], the canonical inclusion
\[
\tilde{W}(G') \vee \tilde{W}(G'') \rightarrow \tilde{W}(G' \ast G'')
\]
is a homotopy equivalence after geometric realization for any simplicial groups $G'$ and $G''$. It follows that
\[
\theta_n : \tilde{W}(G_n) \wedge S_n \longrightarrow \tilde{W}((F^G_\delta[S])_n)
\]
is a homotopy equivalence after geometric realization for $n \geq 0$. According to [4], the geometric realization
\[
|\theta_*| : |T_{*,*}| \longrightarrow |\tilde{W}((F^G_\delta[S])_*)|
\]
is a homotopy equivalence. Moreover, by [4], the geometric realization of the diagonal simplicial simplicial associated to a bi-simplicial set $X_{*,*}$ is homotopy equivalent to $|X_{*,*}|$. From the definition of the functor $\tilde{W}$, the simplicial set $\tilde{W}(F^G_\delta[S])$ is given by the diagonal simplicial set of the bi-simplicial set $\tilde{W}((F^G_\delta[S])_*).$ By the definition of $\delta$-twisted smash product, it is evident that the diagonal simplicial set of $T_{*,*}$ is $\tilde{W}(G) \wedge_\delta S$. 

\[\square\]
5. Twisted homology of simplicial complexes and categories

In this section, we give some remarks on the twisted $\Delta$-groups $F^G,\Delta_\delta[S]$, where $S = S(A)$ with a twisted structure $\delta$ in a simplicial group $G$. The construction $F^G,\Delta_\delta[S]$ seems to be general, which only requires the commuting rule (1.1), and so one has the homology groups from the chain complexes given by $\mathbb{Z}(F^G,\Delta_\delta[S])$ and $(F^G,\Delta_\delta[S])^{ab}$ (as well as $\mathbb{Z}((W(G) \times_{\delta,\Delta} S)$ in section 11) for any twisted structure. On the other hand, the $\Delta$-groups $F^G,\Delta_\delta[S]$ could be very wild in general. For instance, if $\delta_v$ is the trivial endomorphism of $G$ for any vertex $v$, all twisted face operations in $F^G,\Delta_\delta[S]$ become trivial, which concludes that the differentials in $\mathbb{Z}(F^G,\Delta_\delta[S])$, $(F^G,\Delta_\delta[S])^{ab}$ and $\mathbb{Z}((W(G) \times_{\delta,\Delta} S)$ are zero maps and so their homology are given by the chains themselves. This indicates that the homology groups from the chain complexes given by $\mathbb{Z}(F^G,\Delta_\delta[S])$, $(F^G,\Delta_\delta[S])^{ab}$ and $\mathbb{Z}((W(G) \times_{\delta,\Delta} S)$ may not be homotopy invariants for general twisted structures $\delta$. If the twisted structure is nonsingular, the homology groups from the chain complexes given by $\mathbb{Z}(F^G,\Delta_\delta[S])$, $(F^G,\Delta_\delta[S])^{ab}$ and $\mathbb{Z}((W(G) \times_{\delta,\Delta} S)$ coincide with the homology groups from the chain complexes given by the simplicial abelian groups $\mathbb{Z}(F^G[S])$, $(F^G[S])^{ab}$ and $\mathbb{Z}((W(G) \times_{\delta,\Delta} S)$ by [7, Section 5] and so Theorems 4.4 and 4.5 assure that these homology groups only depend on the homotopy type of the corresponding fibre bundles.

Some simplicial techniques can be used for understanding the twisted $\Delta$-groups $F^G,\Delta_\delta[S]$.

Proposition 5.1. Let $S = S(K)$ for a simplicial complex $K$ with a twisted structure in a simplicial group $G$. Let $K_1$ and $K_2$ be simplicial sub-complexes of $K$ such that $K_1 \cup K_2 = K$ and $K_1 \cap K_2 \neq \emptyset$. Let $\delta_{K_i}$ be the restriction of the twisted structure $\delta_K$ on $K_i$. We choose the basepoint in $K_1 \cap K_2$. Then

1. The $\Delta$-group

$$F^G,\Delta_\delta[S(K)] = F^G,\Delta_\delta[S(K_1)] \ast_{F^G,\Delta_\delta[S(K_1 \cap K_2)]} F^G,\Delta_\delta[S(K_2)].$$

is the free product with amalgamation.

2. Suppose that $G$ is a simplicial abelian group. Then there is a short exact sequence of the chain complexes

$$F^G,\Delta_\delta[S(K_1 \cap K_2)]^{ab} \longrightarrow F^G,\Delta_\delta[S(K_1)]^{ab} \oplus F^G,\Delta_\delta[S(K_2)]^{ab} \longrightarrow F^G,\Delta_\delta[S(K)]^{ab}$$

Proof. Assertion (2) follows from assertion (1) by taking abelianization. For proving assertion (1), we check that $S(K) = S(K_1) \cup_{S(K_1 \cap K_2)} S(K_2)$. By the definition, the elements in $S(K)_n$ are given by $(v_0, \ldots, v_n)$ with
$v_0 \leq v_1 \leq \cdots \leq v_n$ such that $\{v_0, \ldots, v_n\}$ form a simplex in $K$. Since $K = K_1 \cup K_2$, the simplex $\{v_0, \ldots, v_n\}$ is either in $K_1$ or $K_2$. This shows that $S(K) = S(K_1) \cup S(K \cap K_2) S(K_2)$. Assertion (1) then follows from our construction. □

The second assertion in the above proposition assures that the Mayer-Vietoris sequence can be applied in the homology theory given by $H_\ast(F^G_\delta, \Delta [S(K)])^{ab}$ for twisted simplicial complexes. Similar results hold for the case $S(C)$ for category $C$ with twisted structure.

Now let us consider the special case of the twisted $\Delta$-groups of the cones of simplicial complexes and categories. For a simplicial complex $K$, the cone $CK = a \ast K$ is defined in the usual way. For a category $C$, the cone $CC = a \ast C$ is the category obtained by adding an initial vertex $a$ with assigning a unique arrow $e_{a,v}: a \to v$ for each $v \in V(C)$ and the identical arrow $id_a : a \to a$. Suppose that $\delta: V(K) \to \text{End}(G)$ or $V(C) \to \text{End}(G)$ is a twisted structure in a simplicial group $G$. A twisted structure $\delta_{CK}$ (or $\delta_{CC}$) is called a regular extension of the twisted structure $\delta_{K}$ (or $\delta_{C}$) if

1. $\delta_{CK}|V(K) = \delta_{K}$ (or $\delta_{CC}|V(C) = \delta_{C}$) and
2. $\delta(a) \in \text{Aut}(G)$.

We recall that, since there is an edge or arrow between $a$ and any other vertex, the commuting rule [11] of the $\delta$-structure forces that $\delta_a \circ \delta_v = \delta_v \circ \delta_a$ for any vertex $v$.

Recall also that any group $G$ can be considered as a discrete simplicial group with $G_n = G$ and faces and degeneracies being the identity map.

**Proposition 5.2.** Let $S = S(a \ast A)$ where $A$ is a simplicial complex $K$ or a category $C$ with a twisted structure given as a regular extension of a twisted structure on $A$ in a group $G$. Then the following chain complexes

$$\mathbb{Z}(F^G_\delta, \Delta [S]), \quad F^G_\delta, \Delta [S]^{ab}$$

are contractible.

**Proof.** Let

$$\Phi: (F^G_\delta, \Delta [S])_n = F^G[S]_n \longrightarrow (F^G_\delta, \Delta [S])_{n+1} = F^G[S]_{n+1}$$

be the (unique) group homomorphism such that

$$\Phi|_{(G_n)_{v_0 v_1 \cdots v_n} (g_{v_0 v_1 \cdots v_n})} = (\delta_a)^{-1}(g)_{v_0 v_1 v_2 \cdots v_n}.$$
Let us compute \( d^i \circ \Phi \). We shorten the notation \( d^i \) as \( d_i \).

\[
d_0 \Phi(g_{v_0 \alpha_1 v_1 \cdots \alpha_n v_n}) = d_0((\delta_a)^{-1}(g))_{ae_{v_0 v_0 v_{\alpha_1 v_1 \cdots \alpha_n v_n}}}
\]
\[
= (\delta_a(d_0((\delta_a)^{-1}(g))))_{v_{\alpha_1 v_1 \cdots \alpha_n v_n}}
\]
\[
= (\delta_a((\delta_a)^{-1}(g)))_{v_{\alpha_1 v_1 \cdots \alpha_n v_n}} \text{ because } G \text{ is discrete}
\]
\[
= g_{v_0 \alpha_1 v_1 \cdots \alpha_n v_n}.
\]

Thus

\[
d_0 \Phi = \text{id}.
\]

Now let \( i > 0 \). Then

\[
d_i \Phi(g_{v_0 \alpha_1 v_1 \cdots \alpha_n v_n}) = d_i((\delta_a)^{-1}(g))_{ae_{v_0 v_0 v_{\alpha_1 v_1 \cdots \alpha_n v_n}}}
\]
\[
= (\delta_{v_{i-1}}(d_i((\delta_a)^{-1}(g))))_{ae_{v_0 v_0 v_{\alpha_1 v_1 \cdots \alpha_{i-2} \alpha_{i-1} \alpha_i v_i \cdots \alpha_n v_n}}
\]
\[
= ((\delta_a)^{-1}\delta_{v_{i-1}} g)_{ae_{v_0 v_0 v_{\alpha_1 v_1 \cdots \alpha_{i-2} \alpha_{i-1} \alpha_i v_i \cdots \alpha_n v_n}} \text{ because } G \text{ is discrete}
\]
\[
= \Phi d_{i-1}(g_{v_0 \alpha_1 v_1 \cdots \alpha_n v_n}).
\]

Thus we have the following identity:

\[
d_i \Phi = \Phi d_{i-1}.
\]

It follows that, in the chain complex \( \mathbb{Z}(F^G, \Delta[S]) \),

\[
\partial_{n+1} \circ \Phi = \sum_{i=0}^{n+1} (-1)^i d_i \Phi = \text{id} - \sum_{i=1}^{n+1} (-1)^{i-1} \Phi d_{i-1}
\]
\[
= \text{id} - \Phi \circ \partial_n.
\]

and so \( \mathbb{Z}(F^G, \Delta[S]) \) is contractible. By taking the abelianization, the homomorphisms \( \Phi_{ab} \) defines a null homotopy for the chain complex \( F^{G, \Delta[S]}_{ab} \).

\[\square\]

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DEPARTMENT OF MATHEMATICS AND PHYSICS, SHIJIAZHUANG TIEDAO UNIVERSITY 050000, CHINA
E-mail address: yanjinglee@163.com

DÉPARTEMENT DES SCIENCES MATHÉMATIQUES, UNIVERSITÉ DE MONTPELLIER, PLACE EUGÈNE BATAILLON, 34095 MONTPELLIER CEDEX 5, FRANCE
E-mail address: vladimir.verchinine@univ-montp2.fr

SOBOLEV INSTITUTE OF MATHEMATICS, NOVOSIBIRSK 630090, RUSSIA
E-mail address: versh@math.nsc.ru

LABORATORY OF QUANTUM TOPOLOGY, CHELYABINSK STATE UNIVERSITY, BRAT’EV KASHIRINYYKH STREET 129, CHELYABINSK 454001, RUSSIA

DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF SINGAPORE, 2 SCIENCE DRIVE 2 SINGAPORE 117542
E-mail address: matwj@nus.edu.sg
URL: www.math.nus.edu.sg/˜matwjie