Application of a triangular finite element with Lagrange correction factors in calculations of thin shells of ellipsoidal type

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Abstract. The study of the stress-strain state of a thin shell of an ellipsoidal type was carried out. The finite element method was used as a research tool using the triangular fragment of the middle surface as the discretization element. The nodal variable parameters of the triangular element of discretization selected the components of the displacement vector, their first and second derivatives with respect to global curvilinear coordinates. Fifth-degree polynomials were used as polynomial functions. To improve the consistency of the triangular element of discretization on the boundaries of adjacent elements, the Lagrange correction factors in the nodes entered in the midpoints of the sides of the triangular element were additionally used as unknowns. The analysis of the results of the calculation of the thin shell of the ellipsoidal type showed high efficiency of using the modified triangular finite element.

1. Introduction
Thin-walled elements of many building and engineering constructions have the shape of a triaxial ellipsoid or its fragments. The study of the stress-strain state (SSS) of a thin shell of ellipsoidal type requires the preliminary solution of the problem of parametrization of its middle surface. In monograph [1], it is proposed to set the surface of a triaxial ellipsoid using parametric dependencies. However, when using numerical calculation methods [2-8], it is preferable to use parameters that have convenient geometric interpretation for fixing, for example, the coordinates of the discrete grid points on the ellipsoid surface. To this end, this article proposes to use the formula of the radius vector, which is a function of two global curvilinear coordinates: the axial coordinate and the angle, measured from the vertical axis parallel to the axis in a plane perpendicular to the axis. As a tool for investigating the SSS of a triaxial ellipsoid, it is proposed to apply a numerical finite element method (FEM) using the triangular fragment of the middle surface as a discretization element with high-order interpolation polynomials. To improve the consistency of the triangular discretization element at the boundaries of adjacent elements, it is proposed to additionally use Lagrange multipliers at the nodes entered in the middle of the sides of the triangular element as unknown.

2. Materials and methods
2.1. The geometry of a triaxial ellipsoid
The radius vector defining the surface of a triaxial ellipsoid in a cartesian coordinate system is determined by the expression

\[ \mathbf{R}^0 = x\mathbf{i} + y\mathbf{j} + c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\mathbf{k}, \]

where \(a, b, c\) - the lengths of the semi-axes of the triaxial ellipsoid.
Vectors of the local basis tangent to the surface of the ellipsoid can be obtained by differentiating (1) with respect to $x$, and $y$:

$$\vec{a}_1^0 = \vec{R}_x^0 = \vec{i} - \frac{cx}{a^2\sqrt{1-(x/a)^2} - (y/b)^2} \vec{k};$$

$$\vec{a}_2^0 = \vec{R}_y^0 = \vec{j} - \frac{cy}{b^2\sqrt{1-(x/a)^2} - (y/b)^2} \vec{k}. $$

(2)

It is not quite convenient to use formulas (2), since it is necessary to observe the condition

$$1 - (x/a)^2 - (y/b)^2 > 0. $$

(3)

Therefore, this paper proposes the use of the following expression for the radius vector of a triaxial ellipsoid

$$\vec{R}^0 = \vec{x}^0 + r(\theta)\sin \theta \vec{y} + r(\theta)\cos \theta \vec{k}, $$

(4)

where $\theta$ - is angle measured from the vertical counterclockwise in a plane perpendicular to the axis $Ox$.

The function $r(x, \theta)$ included in (4) is determined by the following formula

$$r(x, \theta) = \frac{bc\sqrt{1-(x/a)^2}}{\sqrt{b^2\cos^2 \theta + c^2\sin^2 \theta}}. $$

(5)

The tangential vectors of the local basis of an arbitrary point on the surface of the ellipsoid can be determined by differentiating (4) with respect to $x$ and $\theta$:

$$\vec{a}_1 = \vec{R}_x = \vec{i} + r_\theta \sin \theta \vec{j} + r_\theta \cos \theta \vec{k};$$

$$\vec{a}_2 = (r_\theta \sin \theta + r \cos \theta) \vec{j} + (r_\theta \cos \theta - r \sin \theta) \vec{k}, $$

(6)

where $r_\theta = \partial r(x, \theta) / \partial \theta x = \frac{-x}{\sqrt{1-(x/a)^2}} \cdot \frac{bc}{\sqrt{b^2\cos^2 \theta + c^2\sin^2 \theta}}$;

$$r_\theta = \partial r(x, \theta) / \partial \theta \theta = \frac{-bc\sqrt{1-(x/a)^2} \sin 2\theta(c^2 - b^2)}{2(b^2\cos^2 \theta + c^2\sin^2 \theta)^{3/2}}, \quad r \equiv r(x, \theta).$$

Orth normal to the surface of the ellipsoid is determined by the vector product

$$\vec{a}^0 = \vec{a}_1^0 \times \vec{a}_2^0 \sqrt{a^0} = \left[ (-r_\theta \cdot r) \vec{i} + (r \sin \theta - r_\theta \cos \theta) \vec{j} + (r_\theta \cos \theta - r \sin \theta) \vec{k} \right] \sqrt{a^0}, $$

(7)

where $a^0 = (\vec{a}_1^0 \cdot \vec{a}_2^0) - (\vec{a}_1^0 \cdot \vec{a}_2^0)^2$ - determinant of the metric tensor.

Based on (6) and (7), we can form a direct and inverse matrix dependencies

$$\left\{ \vec{a}_i^0 \right\}_{3 \times 1} = \left[ \vec{a}_i^0 \right]_{3 \times 3} \left\{ \vec{a}_j^0 \right\}_{3 \times 1}, \quad \left\{ \vec{i} \right\}_{3 \times 1} = \left[ \vec{a}_i^0 \right]^{-1} \left\{ \vec{a}_i^0 \right\}_{3 \times 1}, $$

(8)

where $\left\{ \vec{a}_i^0 \right\}_{3 \times 1} = \left\{ \vec{a}_1^0, \vec{a}_2^0, \vec{a}_3^0 \right\}_{3 \times 1}$; $\left\{ \vec{i} \right\}_{3 \times 1} = \left\{ \vec{i}, \vec{j}, \vec{k} \right\}_{3 \times 1}$.

Differentiation (6) and (7) by $x$ and $\theta$ it is possible to obtain derivatives of the vectors of the local basis of an arbitrary point on the surface of the ellipsoid.
\[ \ddot{a}_0^{0,\text{x}} = r_{,\text{x}} \sin \theta \dot{a}_1 + r_{,\text{x}} \cos \theta \ddot{k}; \quad \ddot{a}_0^{0,\text{y}} = (r_{,\theta} \sin \theta + r_{,\text{y}} \cos \theta) \dot{a}_1 + (r_{,\theta} \cos \theta - r_{,\text{y}} \sin \theta) \ddot{k}; \]

\[ \ddot{a}_0^{0,\text{z}} = \hat{i} \left( -r_{,\text{z}} \cdot r - (r_{,\text{z}})^2 \right) \frac{1}{\sqrt{a^0}} - (r_{,\text{z}} \cdot r) \left( \frac{1}{\sqrt{a^0}} \right)_{,\text{z}} + \]

\[ + \hat{j} \left( r_{,\text{z}} \sin \theta - r_{,\theta} \cos \theta \right) \frac{1}{\sqrt{a^0}} + (r \sin \theta - r_{,\beta} \cos \theta) \left( \frac{1}{\sqrt{a^0}} \right)_{,\beta} + \]

\[ + \hat{k} \left( r_{,\beta} \sin \theta + r_{,\text{y}} \cos \theta \right) \frac{1}{\sqrt{a^0}} + (r_{,\beta} \sin \theta + r \cos \theta) \left( \frac{1}{\sqrt{a^0}} \right)_{,\beta}; \]

\[ \ddot{a}_1^{0,\theta} = (r_{,\theta} \sin \theta + r_{,\text{y}} \cos \theta) \dot{a}_1 + (r_{,\theta} \cos \theta - r_{,\text{y}} \sin \theta) \ddot{k}; \]

\[ \ddot{a}_1^{0,\beta} = (r_{,\beta} \sin \theta + 2r_{,\beta} \cos \theta - r \sin \theta) \dot{a}_1 + (r_{,\beta} \cos \theta - 2r_{,\beta} \sin \theta - r \cos \theta) \ddot{k}; \]

\[ \ddot{a}_1^{0,\text{z}} = \hat{i} \left( -r_{,\text{z}} \cdot r - (r_{,\text{z}})^2 \right) \frac{1}{\sqrt{a^0}} - (r_{,\text{z}} \cdot r) \left( \frac{1}{\sqrt{a^0}} \right)_{,\text{z}} + \]

\[ + \hat{j} \left( 2r_{,\beta} \sin \theta + r \cos \theta - r_{,\theta} \cos \theta \right) \frac{1}{\sqrt{a^0}} + (r \sin \theta - r_{,\beta} \cos \theta) \left( \frac{1}{\sqrt{a^0}} \right)_{,\beta} + \]

\[ + \hat{k} \left( r_{,\beta} \sin \theta + 2r_{,\beta} \cos \theta - r \sin \theta \right) \frac{1}{\sqrt{a^0}} + (r_{,\beta} \sin \theta + r \cos \theta) \left( \frac{1}{\sqrt{a^0}} \right)_{,\beta}. \]

Derivatives of the vectors of the local basis (9) can be represented in a matrix form

\[ \left\{ \begin{array}{c} \ddot{a}_0^{0} \\ \ddot{a}_0^{0} \end{array} \right\} = \left[ \begin{array}{c} d_0^{0} \\ d_0^{0} \end{array} \right] \left[ \begin{array}{c} \ddot{r} \\ \ddot{r} \end{array} \right]; \quad \left\{ \begin{array}{c} \ddot{a}_0^{0} \\ \ddot{a}_0^{0} \end{array} \right\} = \left[ \begin{array}{c} d_0^{0} \\ d_0^{0} \end{array} \right] \left[ \begin{array}{c} \ddot{r} \\ \ddot{r} \end{array} \right]. \]

where \( \left\{ \ddot{a}_0^{0} \right\} = \left\{ \ddot{a}_0^{0} \right\} \left\{ \ddot{a}_0^{0} \right\}; \quad \left\{ \ddot{a}_0^{0} \right\} = \left\{ \ddot{a}_0^{0} \right\} \left\{ \ddot{a}_0^{0} \right\} \)

Taking into account (8) from (10), it is possible to obtain matrix relations that allow us to represent the derivatives of the vectors of the local basis in the local basis of the same point on the surface of the ellipsoid

\[ \left\{ \begin{array}{c} \ddot{a}_0^{0} \\ \ddot{a}_0^{0} \end{array} \right\} = \left[ \begin{array}{c} d_0^{0} \\ d_0^{0} \end{array} \right] \left[ \begin{array}{c} \ddot{r} \\ \ddot{r} \end{array} \right]; \quad \left\{ \begin{array}{c} \ddot{a}_0^{0} \\ \ddot{a}_0^{0} \end{array} \right\} = \left[ \begin{array}{c} d_0^{0} \\ d_0^{0} \end{array} \right] \left[ \begin{array}{c} \ddot{r} \\ \ddot{r} \end{array} \right]. \]

In the process of deformation point \( M^0 \) the middle surface of the ellipsoidal shell will move to a points \( M \), a point \( M^0 \), distance from the point \( M^0 \) along the distance normal \( \zeta \), take a new position \( M^\zeta \). Point moves \( M^0 \) and \( M^0 \) in new positions \( M \) and \( M^\zeta \) will be characterized by displacement vectors \( \tilde{v} \) and \( \tilde{V} \) respectively. Vector of the displacement of points \( M^0 \) and its derivatives by \( x \) and \( \theta \) determined by the expressions

\[ \tilde{v} = \dot{a}_1^{0} + \dot{v} a_1^{0} + \dot{w} a_1^{0}; \quad \tilde{v}_z = t_z^1 a_1^{0} + t_z^2 a_2^{0} + t_z a_3^{0}; \quad \tilde{v}_0 = t_{\theta} a_1^{0} + t_{\theta} a_2^{0} + t_{\theta} a_3^{0}, \]
where \( u, v \) - tangential, \( w \) - normal components of the displacement vector; \( t_1', t_1^2, t_1, t_2', t_2 \) - polynomials containing the first derived components of the displacement vector.

Point displacement vector \( M^{0c} \) based on the hypothesis of a direct normal, can be determined by the formula

\[
\vec{v} = \vec{v} + \vec{c} (\vec{a} - \vec{a}^0)
\]  

(13)

where \( \vec{a} \) - ort normal to the surface of the ellipsoid in the deformed state.

Radius vectors defining points \( M^{0c}, M \) and \( M^\xi \) determined by the appropriate expressions

\[
\vec{R}^{0c} = \vec{R}^0 + \zeta \vec{a}^0; \quad \vec{R} = \vec{R}^0 + \vec{v}; \quad \vec{R}^\xi = \vec{R}^{0c} + \vec{V}.
\]  

(14)

Differentiating (14) by \( x \) and \( \theta \) can get the corresponding basis vectors

\[
\vec{g}_x^0 = \vec{g}_x^{0c}; \quad \vec{g}_\theta^0 = \vec{g}_\theta^{0c}; \quad \vec{g}_x = \vec{g}_x^{\xi}; \quad \vec{g}_\theta = \vec{g}_\theta^{\xi}.
\]  

(15)

scalar products which determine the components of the metric tensor of the original and deformed states

\[
g_{xx}^0 = \vec{g}_x^0 \cdot \vec{g}_x^0; \quad g_{\theta\theta}^0 = \vec{g}_\theta^0 \cdot \vec{g}_\theta^0; \quad g_{xx} = \vec{g}_x \cdot \vec{g}_x; \quad g_{\theta\theta} = \vec{g}_\theta \cdot \vec{g}_\theta; \quad g_{x\theta} = \vec{g}_x \cdot \vec{g}_\theta; \quad g_{\theta x} = \vec{g}_\theta \cdot \vec{g}_x.
\]  

(16)

Deformations at the point of the ellipsoidal shell spaced at a distance from the middle surface \( \zeta \), calculated using continuum mechanics formulas

\[
e_{xx}^\xi = \frac{(g_{xx} - g_{xx}^0)}{2}; \quad e_{\theta\theta}^\xi = \frac{(g_{\theta\theta} - g_{\theta\theta}^0)}{2}; \quad e_{x\theta}^\xi = \frac{(g_{x\theta} - g_{x\theta}^0)}{2}.
\]  

(17)

2.2. Triangle sampling element

The median surface of the ellipsoidal shell is approximated by an ensemble of triangular discretization elements with nodes \( i, j, k \). For convenience of numerical integration over the area, a triangular fragment of the median surface of the ellipsoid is mapped onto a right-angled triangle with unit lengths in the local coordinate system \( 0 \leq \xi, \eta \leq 1 \).

Considering that the deformations (17) are functions of the components of the displacement vector, as well as their first and second derivatives by \( x \) and \( \theta \) in addition to the displacement vector components, the first and second derivatives with respect to the global \( x, \theta \) or local \( \xi, \eta \) coordinates

\[
\begin{align*}
\{ lG_y \}_{1x54} & = \{ lG_y \}_{1x18} \{ lG_y \}_{1x18} \\
\{ lL_y \}_{1x54} & = \{ lL_y \}_{1x18} \{ lL_y \}_{1x18}
\end{align*}
\]  

(18)

\[
\begin{align*}
\{ lG_y \}_{1x54} & = \{ lG_y \}_{1x18} \{ lG_y \}_{1x18} \\
\{ lL_y \}_{1x54} & = \{ lL_y \}_{1x18} \{ lL_y \}_{1x18}
\end{align*}
\]  

(19)

where

\[
\begin{align*}
\{ q_j \}_{1x18} & = \{ q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_j q_
\[ q = \begin{bmatrix} \phi_j \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} + \begin{bmatrix} \varphi_i \end{bmatrix} \begin{bmatrix} \zeta \end{bmatrix} \begin{bmatrix} \ell \end{bmatrix} \]

(20)

Differentiating (20) by \( x \) and \( \theta \) you can get the first and second derivatives of the displacement vector components

\[
q_x = \begin{bmatrix} \phi_j \end{bmatrix} \begin{bmatrix} \zeta_x \end{bmatrix} + \begin{bmatrix} \varphi_i \end{bmatrix} \begin{bmatrix} \eta_x \end{bmatrix} \begin{bmatrix} \ell \end{bmatrix};
q_\theta = \begin{bmatrix} \phi_j \end{bmatrix} \begin{bmatrix} \zeta_\theta \end{bmatrix} + \begin{bmatrix} \varphi_i \end{bmatrix} \begin{bmatrix} \eta_\theta \end{bmatrix} \begin{bmatrix} \ell \end{bmatrix};
q_{xx} = \begin{bmatrix} \phi_j \end{bmatrix} \begin{bmatrix} \zeta_{xx} \end{bmatrix} + 2\begin{bmatrix} \varphi_i \end{bmatrix} \begin{bmatrix} \zeta_x \eta_x \end{bmatrix} + \begin{bmatrix} \varphi_i \end{bmatrix} \begin{bmatrix} \eta_x \end{bmatrix}^2 \begin{bmatrix} \ell \end{bmatrix};
q_{\theta\theta} = \begin{bmatrix} \phi_j \end{bmatrix} \begin{bmatrix} \zeta_{\theta\theta} \end{bmatrix} + 2\begin{bmatrix} \varphi_i \end{bmatrix} \begin{bmatrix} \zeta_\theta \eta_\theta \end{bmatrix} + \begin{bmatrix} \varphi_i \end{bmatrix} \begin{bmatrix} \eta_\theta \end{bmatrix}^2 \begin{bmatrix} \ell \end{bmatrix};
q_{x\theta} = \begin{bmatrix} \phi_j \end{bmatrix} \begin{bmatrix} \zeta_x \zeta_\theta \end{bmatrix} + \begin{bmatrix} \varphi_i \end{bmatrix} \begin{bmatrix} \eta_x \eta_\theta \end{bmatrix} + \begin{bmatrix} \varphi_i \end{bmatrix} \begin{bmatrix} \zeta_x \eta_\theta \end{bmatrix} + \begin{bmatrix} \varphi_i \end{bmatrix} \begin{bmatrix} \zeta_\theta \eta_x \end{bmatrix} \begin{bmatrix} \ell \end{bmatrix}. \tag{21}
\]

3. Results

To improve the compatibility of the considered element of discretization, it is proposed to introduce three additional nodes 1, 2 and 3, one in each of the midpoints. If in additional nodes we calculate the derivatives of the normal component of the displacement vector along the normals to the sides of the triangular elements, then in the values of these derivatives, calculated from the side of the neighboring elements, there will be a jump, which is caused by the incompatibility of this element of discretization. To eliminate the noted deficiency, you can enter Lagrange multipliers as nodal unknown additional nodes \( \lambda_i \) \((i = 1, 2, 3)\). Then the following equality holds

\[
\left( \frac{\partial w_1}{\partial S_{m1}} + \frac{\partial w_2}{\partial S_{m2}} \right) \lambda_1 = 0,
\]

(22)

where the bar indicates the values related to the adjacent sample element.

For a single triangular finite element, you can write the following expression

\[
\frac{\partial w_1}{\partial S_{m1}} \lambda_1 + \frac{\partial w_2}{\partial S_{m2}} \lambda_2 + \frac{\partial w_3}{\partial S_{m3}} \lambda_3 = 0.
\]

(23)

Derivatives \( \frac{\partial w_i}{\partial S_{m1}} \) can be expressed in terms of derivatives of the normal component of the displacement vector in global coordinates \( x \) and \( \theta \)

\[
\frac{\partial w_i}{\partial S_{m1}} = \frac{\partial w_i}{\partial x} \cos \alpha_i + \frac{\partial w_i}{\partial \theta} \cos \beta_i,
\]

(24)

where \( \alpha_i \) and \( \beta_i \) - angles between the normals to the side of the triangular discretization element and the tangential vectors of the local basis \( \alpha_i^0, \alpha_i^2 \) in additional node \( i \).

Cosines of corners \( \alpha_i \) and \( \beta_i \) can be calculated by the formulas

\[
\cos \alpha_i = \frac{\alpha_i^0 \cdot \vec{n}_i}{|\alpha_i^0| |\vec{n}_i|}; \quad \cos \beta_i = \frac{\alpha_i^2 \cdot \vec{n}_i}{|\alpha_i^2| |\vec{n}_i|}.
\]

(25)

Included in (25) the normal vectors to the sides of the triangular element in the additional nodes can be determined by the vector product

\[
\vec{n}_i = \alpha_i^0 \times \vec{r}_i.
\]

(26)
where $\vec{a}_l$ - ort normal to the surface of the ellipsoid in the additional node $l$; $\vec{t}_i$ - vector tangent to the side of a triangular discretization element, defined by the formula

$$\vec{t}_i = (\vec{R}_i^a) + (\vec{R}_i^\theta)(\partial \theta/\partial x)_i. \quad (27)$$

Derivatives of the normal components of the displacement vector along the normals to the sides of the triangular element in the additional nodes, taking into account (24), can be expressed in terms of a column of nodal unknowns

$$\frac{\partial w_l}{\partial S_{nj}} = \left(\left[\phi_{x,0} \cdot \xi_x + \left[\phi_{y,0}\right] \cdot \eta_y\right] \cos \alpha_i + \left[\phi_{x,0} \cdot \xi_{x,0} + \left[\phi_{y,0}\right] \cdot \eta_{y,0}\right] \cos \beta_i \right) \mu_{i,j}^T \text{ or in matrix form}
$$

$$\frac{\partial W_l}{\partial S_{nj}} = \left[\chi_{3,ij}^T \left[T^T \chi_i^G \right] \right], \quad i=1 \ldots 54, \quad j=54 \ldots 54+1. \quad (28)$$

where $[T]$ - transition matrix from the nodal unknowns column in the local coordinate system $\{U_j^L\}$ to the column of unknowns in the global coordinate system $\{U_j^G\}$.

Relation (23) with regard to (28) can be represented in the matrix form

$$\left[\chi_{1,i}^T \left[T^T \chi_i^G \right] \right] = \left[\chi_{2,ij}^T \left[T^T \chi_i^G \right] \right] = \left[\chi_{3,ij}^T \left[T^T \chi_i^G \right] \right] = \left[\chi_{4,ij}^T \left[T^T \chi_i^G \right] \right] = \left[\chi_{5,ij}^T \left[T^T \chi_i^G \right] \right] = \ldots = \left[\chi_{54,ij}^T \left[T^T \chi_i^G \right] \right] = 0. \quad (29)$$

The Lagrange functional for the triangular element of discretization, taking into account (29), will be written as

$$\Phi_L = \int_{V} \left[\phi_{x,0} \cdot \xi_x + \left[\phi_{y,0}\right] \cdot \eta_y\right] \cdot \sigma_{x,0} \cdot \sigma_{x,0}^T \cdot dV - \int_{F} \left[\{U_j^L\} \cdot \{P\} \cdot \{\lambda\} \cdot \left[B\right] \cdot \{U_j^G\}\right] \cdot dF. \quad (30)$$

where $\{\phi_{x,0}\} = \left[\phi_{x,0} \cdot \xi_x + \left[\phi_{y,0}\right] \cdot \eta_y\right] -$ matrix rows of deformations and stresses; at a point at a distance from the middle surface $\zeta$; $\{U_j^L\} = \{u \cdot v \cdot w\}$ - line component of the displacement vector of the middle surface point; $\{P\} = \left[p_{1,1} \cdot p_{2,2} \cdot p_{1,2}\right]$ - external surface load string.

Minimizing the functionality (30) by $\{U_j^G\}$ and $\{\lambda\}$ can get a system of equations, from which you then assemble the stiffness matrix $[K]$ and nodal force column $\{R\}$ triangular bin

$$[K] = \left[\begin{array}{c|c} \left[K'\right] & \left[B\right]^T \\ \hline \left[0\right] & \left[0\right]\end{array}\right], \quad \{R\} = \left[\begin{array}{c|c} \left[R'\right] \end{array}\right], \quad (31)$$

where $[K']$ and $\{R'\}$ - stiffness matrix and nodal force column of a standard triangular element without Lagrange multipliers $\dot{\lambda}_i$.

Further construction of the stiffness matrix and the column of nodal forces for the calculation of the ellipsoidal shell is performed in the standard way for FEM [9–19].

**4. Discussion**

As an example, the SSS of a triaxial ellipsoid loaded with an internal intensity pressure was investigated $q$. Due to the presence of symmetry planes, 1/8 of the shell was calculated.
The calculations were carried out in two variants: in the first variant, a triangular element without Lagrange multipliers with a stiffness matrix was used as a discretization element \(54 \times 54\); in the second variant, a triangular discretization element with Lagrange multipliers with a stiffness matrix \(57 \times 57\). The results of the variant calculation showed significant advantages of the second version of the calculation in the accuracy of finite-element solutions.

5. Conclusion
Based on the analysis of the calculation results, we can conclude that the triangular discretization element with Lagrange multipliers \(57 \times 57\) has the advantage over the standard triangular finite element with the size of the stiffness matrix \(54 \times 54\) in the problems of analyzing the SSS of thin shells of ellipsoidal type.

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