Analysis of propagating wave structures of the cold bosonic atoms in a zig-zag optical lattice via comparison with two different analytical techniques

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Abstract
The wave propagation has the significant role in the field of coastal engineering and ocean. In the geographical fields, waves are primary source of environmental process owed to energy conveyance on floating structure. This study aims to investigate the system of cold bosonic atoms in zig-zag optics lattices. The solitonic patterns of the considered model successfully surveyed by using two integrated analytical techniques new extended direct algebraic and \( \frac{G'}{G} \) expansion method. The exact solutions are presented by rational, trigonometric, hyperbolic and exponential functions. On the basis of solitons, we need to show that which one is more integrated and robust scheme. These solutions will help to understood the dynamics of cold bosonic atoms in zig-zag optical lattices and the several other systems. Three dimensional as well as two dimensional comparison presented for a cold bosonic atoms model solutions which are revealed diagrammatically for appropriate parameters by using Mathematica. This study will help physicists to predict some new hypothesis and theories in the field of mathematical physics.

Keywords New direct extended algebraic method · \( \frac{G'}{G} \) expansion method · The cold bosonic atoms in a zig-zag optical lattice · Traveling wave solutions

1 Introduction
Nonlinear partial differential equation is one of the significant tools to examine the features of nonlinear physical phenomenons rigorously. The Schrödinger type governing equation is a remarkable mechanism to interpret the complex physical nonlinear model more accurately and has vital applications in the fields of plasma, fiber-optic, mathematical physics, telecommunication engineering, and optic. The extraction of analytical exact solutions for the Schrödinger equations is a glaring research area because exact solutions have a consequential role to express the physical aspects of non-linear systems in applied mathematics (Gao et al. 2020, 2019; Ali ...
et al. 2020). Since the last decade, the study of non-linear PDE’s have become more and more significant in pure and applied mathematics. For mathematicians, the use of computer technology has opened up new frontiers in the field of applied sciences. The nonlinear models which are frequently practiced in engineering sciences and mathematical physics, are getting prominence.

The partial differential equations (PDEs) are mainly used in developing mathematical models of substantial tangible phenomena of several areas of non-linear sciences and engineering. The partial differential equations containing non-linearity have a lot of tangible applications in heat and mass conveyance, continuum mechanics, hydrodynamics, acoustics, wave theory, polyphase streams, chemical technology and plasma physics (Cheema et al. 2019; He et al. 2019) statistical demographic mechanics Barkai et al. (2000), biology Barfield et al. (2018), population ecology Lin and Gao (2019), economics Scalas et al. (2000), plasma waves Vallejos et al. (2017), civil engineering Baudouin et al. (2018), quantum mechanics Laskin (2019), and soil mechanics Shao et al. (2017) etc.

The exact solitonic solutions of the fractional partial differential equations provide vital and innovative ideas to the researcher about the sensitive phenomenons, because of this curiosity, at present, it is a hot research discipline in the scientific fields for scientists. Schrödinger equation has several diligence’s in fibre optics with artificial optic media such as photonic crystallization and plasmon Thomas et al. (2012) and the traveling wave theory Carles et al. (2010). It is also applied to describe the Bose-Einstein condensates in the theory of quantum field in the sort of solitons Kartashov et al. (2011). Atangana and Araz has opened a new window of research by introducing piece-wise differential and integral operators Atangana et al. (2021). Mekkaoui et al. (2021) developed an efficient fractional numerical scheme. Atangana and Araz established a new mathematical model for Covid-19 to forecast by using piece-wise operator Atangana et al. (2021). Zhang et al. (2022) applied extended algebraic scheme on fractional couple Drinfeld–Sokolov–Wilson system to secure fractional propagating waves. Jhangeer et al. (2022) has established fractional solitonic structures by an analytical approach. Faridi et al. (2021) has discussed the fusion and fission process in sort of fractional theory. Asjad et al. (2021) found interesting fractional soliton wave profiles with the aid of computational algorithm.

The recent work related to this field is presenting. Seadawy et al. (2022) developed the new solitary waves for higher order nonlinear Schrödinger equation by using the an extended modified auxiliary equation mapping method. Rizvi et al. (2022) obtained the chirped periodic waves of the coupled nonlinear Schrödinger equation by using the Jacobi elliptic functions. Rizvi et al. (2022) discussed the M-shaped rational soliton, double exponential form, periodic cross-kink (PCK), periodic cross-rational (PCR), multi-waves and homoclinic breather approach to perturbed nonlinear Schrödinger equation. Seadawy et al. (2022) applied ansatz functions technique to the generalized Fitzhugh–Nagumo (gFN)-equation and established the novel wave structures and multiple lump-soliton solutions. Aziz et al. (2022) has depicted the chirped solitons (CS) solutions to the nonlinear Schrödinger’s equation (NLSE) under the polynomial nonlinearity. Seadawy et al. (2022a) investigated the double chain deoxyribonucleic acid (DNA) system and discussed distinct types of solitons. Seadawy et al. (2022b) used a newly developed sub-ODE approach to disclose Weierstrass elliptic, Jacobi elliptic, Bell type, kink type, brilliant soliton, periodic, and some other soliton solutions for the (1 + 1)–dimensional dissipative nonlinear Schrödinger equation. Ali et al. (2022) examined lump, lump one-strip, lump two-strip, rogue wave, manifold periodic type accurate solutions are created by suitable transformation scheme by taking the Kerr Law features of the model describing nematic liquid crystals into consideration. The auxiliary equation approach was used by Tariq et al. (2022) to investigate some novel precise solutions to the (2+1)-dimensional nonlinear Schrödinger equation (NLSE) with Kerr
media. Rizvi et al. (2022) analyzed chirped pulses (CP) in the context of a cubic-quintic nonlinear nonparaxial pulse propagation (CQ-NNP-PP) model. Seadawy et al. (2022) addressed the many kinds of lump: lump one stripe, lump two stripe, generalised breathers, Akhmediev breather, multiwave, M-shaped rational, and rogue wave solutions for the complex cubic quintic Ginzburg Landau (CQGL) problem with intrapulse Raman scattering (IRS) via suitable transformations technique.

The energy spectrum and intrinsic localized modes related to modulations instability of boson chains has been accomplished by employing Bose–Hubbard model. On atomic chains a lot of other work have been done (Tang et al. 2014, 2017). In Heisenberg ferromagnetic spin chains, the author has controlled the quantum breathers by executing an oblique magnetic field Tang et al. (2017). The cold bosonic atoms in a zig-zag optical lattice phenomenon is governed by following equation Tala-Tebue et al. (2021),

\[ u \text{ is the boson-boson interaction.} \]

The boson-boson interaction is representing by \( u \). The interaction is repulsive and attractive when \( u > 0 \) and \( u < 0 \) respectively. The first nearest neighbor hopping and the second nearest neighbor hopping are presented by \( \mathcal{X}_1 > 0 \) and \( \mathcal{X} > 0 \) respectively. Experiments with ultracold atoms are important in order to understand quantum phase transitions and exploration into bosonic super fluidity, Bose-Einstein condensation, quantum magnetism, many-body spin dynamics, and other topics Tala-Tebue et al. (2021).

To the our best knowledge, no study of the considered model has been discussed in the existing literature so far independently with the new extended direct algebraic and V-expansion method. Further, comparative analysis for this model has not been found yet. Therefore, our main contribution is to analysis he cold bosonic atoms in a zig-zag optical lattice comparatively with the help of two different analytical techniques. Such kind of approach for any model can be helpful to predict of soliton solutions and the accuracy nature of analytical solutions.

2 Description of the methods

Assume a general non-linear partial differential equation:

\[ P(\Psi, \Psi', \Psi_x, \Psi_y, \Psi_{tt}, \Psi_{xx}, \ldots) = 0. \]

It has susceptibility to attain the form of non-linear ordinary differential equation on the execution of complex traveling wave transformation such as:

\[ \Psi(x, y, t) = \Phi(\tau)e^{i\psi}, \]

where \( \tau = \kappa_1 t + \kappa_2 x + \kappa_3 y \) and \( \psi = \kappa_4 t + \kappa_5 x + \kappa_6 y + \theta \), we get:

\[ f(\Phi, \Phi', \Phi'', \ldots) = 0, \]

here, the notation of differentiation depicted by the prime in Eq. (4).
2.1 New extended direct algebraic technique

Assume that, the solution of Eq. (4) is:

\[ f(\tau) = \sum_{i=0}^{m} a_i(P(\tau))^i \]  

along with,

\[ P'(\tau) = \ln(\Psi)(\mathfrak{M} + \mathfrak{B}P(\tau) + \mathfrak{Z}P^2(\tau)), \quad \Psi \neq 0, 1, \]  

The solutions concerning to real parameters \(\mathfrak{M}, \mathfrak{B}\) and \(\mathfrak{Z}\) of Eq. (4) are:

**Category** (1) If \(\mathfrak{Z}^2 - 4\mathfrak{M}\mathfrak{Z} < 0\), and \(\mathfrak{Z} \neq 0\), then,

\[ P_1(\tau) = -\frac{\mathfrak{B}}{23} + \frac{\sqrt{-\Theta}}{23} \tan_{\Psi}\left(\frac{\sqrt{-\Theta}}{2} \tau\right). \]

\[ P_2(\tau) = -\frac{\mathfrak{B}}{23} - \frac{\sqrt{-\Theta}}{23} \cot_{\Psi}\left(\frac{\sqrt{-\Theta}}{2} \tau\right). \]

\[ P_3(\tau) = -\frac{\mathfrak{B}}{23} + \frac{\sqrt{-\Theta}}{23} \left(\tan_{\Psi}\left(\sqrt{-\Theta} \tau\right) \pm \sqrt{mn} \sec_{\Psi}\left(\sqrt{-\Theta} \tau\right)\right). \]

\[ P_4(\tau) = -\frac{\mathfrak{B}}{23} + \frac{\sqrt{-\Theta}}{23} \left(\cot_{\Psi}\left(\sqrt{-\Theta} \tau\right) \pm \sqrt{mn} \csc_{\Psi}\left(\sqrt{-\Theta} \tau\right)\right). \]

\[ P_5(\tau) = -\frac{\mathfrak{B}}{23} + \frac{\sqrt{-\Theta}}{43} \left(\tan_{\Psi}\left(\frac{\sqrt{-\Theta}}{4} \tau\right) - \cot_{\Psi}\left(\frac{\sqrt{-\Theta}}{4} \tau\right)\right). \]

**Category** (2) If \(\mathfrak{Z}^2 - 4\mathfrak{M}\mathfrak{Z} > 0\), and \(\mathfrak{Z} \neq 0\), then,

\[ P_6(\tau) = -\frac{\mathfrak{B}}{23} - \frac{\sqrt{\Theta}}{23} \tanh_{\Psi}\left(\frac{\sqrt{\Theta}}{2} \tau\right). \]

\[ P_7(\tau) = -\frac{\mathfrak{B}}{23} - \frac{\sqrt{\Theta}}{23} \coth_{\Psi}\left(\frac{\sqrt{\Theta}}{2} \tau\right). \]

\[ P_8(\tau) = -\frac{\mathfrak{B}}{23} + \frac{\sqrt{\Theta}}{23} \left(-\tanh_{\Psi}\left(\sqrt{\Theta} \tau\right) \pm i\sqrt{mn} \sech_{\Psi}\left(\sqrt{\Theta} \tau\right)\right). \]

\[ P_9(\tau) = -\frac{\mathfrak{B}}{23} + \frac{\sqrt{\Theta}}{23} \left(-\coth_{\Psi}\left(\sqrt{\Theta} \tau\right) \pm \sqrt{mn} \csch_{\Psi}\left(\sqrt{\Theta} \tau\right)\right). \]
\[ P_{10}(\tau) = -\frac{m}{2} \sqrt{\Theta} \left( \tanh_{\psi} \left( \frac{\sqrt{\Theta}}{4} \tau \right) + \coth_{\psi} \left( \frac{\sqrt{\Theta}}{4} \tau \right) \right). \] (16)

**Category** (3) If \( \mathcal{M} > 0 \), and \( \mathcal{W} = 0 \), then,

\[ P_{11}(\tau) = \sqrt{\frac{m}{3}} \tan_{\psi} \left( \sqrt{\mathcal{M}} \tau \right). \] (17)

\[ P_{12}(\tau) = -\sqrt{\frac{m}{3}} \cot_{\psi} \left( \sqrt{\mathcal{M}} \tau \right). \] (18)

\[ P_{13}(\tau) = \sqrt{\frac{m}{3}} \left( \tan_{\psi} \left( 2\sqrt{\mathcal{M}} \tau \right) \pm \sqrt{mn} \sec_{\psi} \left( 2\sqrt{\mathcal{M}} \tau \right) \right). \] (19)

\[ P_{14}(\tau) = \sqrt{\frac{m}{3}} \left( -\cot_{\psi} \left( 2\sqrt{\mathcal{M}} \tau \right) \pm \sqrt{mn} \csc_{\psi} \left( 2\sqrt{\mathcal{M}} \tau \right) \right). \] (20)

\[ P_{15}(\tau) = \frac{1}{2} \sqrt{\frac{m}{3}} \left( \tan_{\psi} \left( \frac{\sqrt{\mathcal{M}}}{2} \tau \right) - \cot_{\psi} \left( \frac{\sqrt{\mathcal{M}}}{2} \tau \right) \right). \] (21)

**Category** (4) If \( \mathcal{M} < 0 \), and \( \mathcal{W} = 0 \), then,

\[ P_{16}(\tau) = -\sqrt{\frac{m}{3}} \tanh_{\psi} \left( \sqrt{-\mathcal{M}} \tau \right), \] (22)

\[ P_{17}(\tau) = -\sqrt{\frac{m}{3}} \coth_{\psi} \left( \sqrt{-\mathcal{M}} \tau \right), \] (23)

\[ P_{18}(\tau) = \sqrt{\frac{m}{3}} \left( -\tan_{\psi} \left( 2\sqrt{-\mathcal{M}} \tau \right) \pm i \sqrt{mn} \sech_{\psi} \left( 2\sqrt{-\mathcal{M}} \tau \right) \right). \] (24)

\[ P_{19}(\tau) = \sqrt{\frac{m}{3}} \left( -\coth_{\psi} \left( 2\sqrt{-\mathcal{M}} \tau \right) \pm \sqrt{mn} \csch_{\psi} \left( 2\sqrt{-\mathcal{M}} \tau \right) \right). \] (25)

\[ P_{20}(\tau) = -\frac{1}{2} \sqrt{\frac{m}{3}} \left( \tanh_{\psi} \left( \frac{\sqrt{-\mathcal{M}}}{2} \tau \right) + \coth_{\psi} \left( \frac{\sqrt{-\mathcal{M}}}{2} \tau \right) \right). \] (26)

**Category** (5) If \( \mathcal{W} = 0 \), and \( \mathcal{M} = \mathcal{M} \), then,

\[ P_{21}(\tau) = \tan_{\psi} (\mathcal{M} \tau), \] (27)

\[ P_{22}(\tau) = -\cot_{\psi}(\mathcal{M} \tau), \] (28)
\[ P_{23}(\tau) = \tan_{\mathfrak{p}}(2\mathfrak{M}\tau) \pm \sqrt{mn} \sec_{\mathfrak{p}}(2\mathfrak{M}\tau), \] (29)

\[ P_{24}(\tau) = -\cot \mathfrak{z}_{\mathfrak{p}}(2\mathfrak{M}\tau) \pm \sqrt{mn} \csc_{\mathfrak{p}}(2\mathfrak{M}\tau), \] (30)

\[ P_{25}(\tau) = \frac{1}{2} \left( \tan_{\mathfrak{p}} \left( \frac{2\mathfrak{M}}{2} \tau \right) - \cot \mathfrak{z}_{\mathfrak{p}} \left( \frac{2\mathfrak{M}}{2} \tau \right) \right). \] (31)

**Category** (6) If \( \mathfrak{B} = 0 \), and \( \mathfrak{J} = -\mathfrak{M} \), then,

\[ P_{26}(\tau) = -\tanh_{\mathfrak{p}}(\mathfrak{M}\tau), \] (32)

\[ P_{27}(\tau) = -\coth_{\mathfrak{p}}(\mathfrak{M}\tau), \] (33)

\[ P_{28}(\tau) = -\tanh_{\mathfrak{p}}(2\mathfrak{M}\tau) \pm i\sqrt{mn} \sech_{\mathfrak{p}}(2\mathfrak{M}\tau), \] (34)

\[ P_{29}(\tau) = -\cot \mathfrak{z}_{\mathfrak{p}}(2\mathfrak{M}\tau) \pm \sqrt{mn} \csch_{\mathfrak{p}}(2\mathfrak{M}\tau), \] (35)

\[ P_{30}(\tau) = -\frac{1}{2} \left( \tanh_{\mathfrak{p}} \left( \frac{2\mathfrak{M}}{2} \tau \right) + \cot \mathfrak{z}_{\mathfrak{p}} \left( \frac{2\mathfrak{M}}{2} \tau \right) \right). \] (36)

**Category** (7) If \( \mathfrak{B}^2 = 4\mathfrak{M}\mathfrak{J} \), then,

\[ P_{31}(\tau) = \frac{-2\mathfrak{M}(\mathfrak{B}\tau \ln \mathfrak{B} + 2)}{\mathfrak{B}^2 \tau \ln \mathfrak{B}}. \] (37)

**Category** (8) If \( \mathfrak{B} = p, \mathfrak{M} = pq, (q \neq 0) \), and \( \mathfrak{J} = 0 \), then,

\[ P_{32}(\tau) = \mathfrak{p}^{\mu r} - q. \] (38)

**Category** (9) If \( \mathfrak{B} = \mathfrak{J} = 0 \), then,

\[ P_{33}(\tau) = \mathfrak{M}\tau \ln(\mathfrak{B}). \] (39)

**Category** (10) If \( \mathfrak{B} = \mathfrak{M} = 0 \), then,

\[ P_{34}(\tau) = \frac{-1}{3\tau \ln \mathfrak{B}}. \] (40)

**Category** (11) If \( \mathfrak{M} = 0 \), and \( \mathfrak{B} \neq 0 \), then,

\[ P_{35}(\tau) = -\frac{m\mathfrak{B}}{3(\cosh_{\mathfrak{p}}(\mathfrak{B}\tau) - \sinh_{\mathfrak{p}}(\mathfrak{B}\tau) + m)}. \] (41)

\[ P_{36}(\tau) = -\frac{\mathfrak{B}(\sinh_{\mathfrak{p}}(\mathfrak{B}\tau) + \cosh_{\mathfrak{p}}(\mathfrak{B}\tau))}{3(\sinh_{\mathfrak{p}}(\mathfrak{B}\tau) + \cosh_{\mathfrak{p}}(\mathfrak{B}\tau) + n)}. \] (42)

**Category** (12) If \( \mathfrak{B} = p, \mathfrak{J} = pq, (q \neq 0 \) and \( \mathfrak{M} = 0 \), then,
\[ P_{37}(\tau) = -\frac{m\psi^r}{m - q\Phi^r} \]  

\[ \sinh_{\psi}(\tau) = \frac{m\psi^r - n\Phi^r}{2} \quad \cosh_{\psi}(\tau) = \frac{m\psi^r + n\Phi^r}{2} \]
\[ \tanh_{\psi}(\tau) = \frac{2\psi^r - n\Phi^r}{m\psi^r + n\Phi^r} \quad \coth_{\psi}(\tau) = \frac{2\psi^r + n\Phi^r}{m\psi^r - n\Phi^r} \]
\[ \sin_{\psi}(\tau) = \frac{1}{2i} \quad \cos_{\psi}(\tau) = \frac{1}{2} \]
\[ \tan_{\psi}(\tau) = -\frac{i\psi^r - n\Phi^r}{m\psi^r + n\Phi^r} \quad \cot_{\psi}(\tau) = \frac{i\psi^r + n\Phi^r}{m\psi^r - n\Phi^r} \]
\[ \sec_{\psi}(\tau) = \frac{2}{m\psi^r + n\Phi^r} \quad \csc_{\psi}(\tau) = \frac{2i}{m\psi^r - n\Phi^r} \]

where, \( m, n > 0 \).

### 2.2 Description of \( \mathcal{G}^r/\mathcal{G}^2 \)-expansion method

Consider the solution of Eq. (4) is:

\[ U(\Gamma) = a_0 + \sum_{i=1}^{m} \left( a_i V^i(\Gamma) + b_i V^{-i}(\Gamma) \right) \]  

(44)

Which satisfies the following Riccati equation,

\[ \left( \frac{G'}{G^2} \right)' = \mathcal{M} + \lambda V^2 \]  

(45)

where \( \mathcal{M}, \lambda \in \mathbb{R} \), are constants. We gain below the solution to Eq. (45) due to the different conditions of \( \lambda \) and \( \mathcal{M} \). The solution of Eq. (45) with respect to parameters \( \mathcal{M}, \lambda \) is given as:

1. \( \mathcal{M}\lambda > 0 \)

\[ \frac{G'}{G^2} = \frac{\sqrt{\mathcal{M}}}{\lambda} \left( \frac{r_1 \cos(\sqrt{\mathcal{M}\lambda}\xi) + r_2 \sin(\sqrt{\mathcal{M}\lambda}\xi)}{r_2 \cos(\sqrt{\mathcal{M}\lambda}\xi) - r_1 \sin(\sqrt{\mathcal{M}\lambda}\xi)} \right) \]  

(46)

2. \( \mathcal{M}\lambda < 0 \)

\[ \frac{G'}{G^2} = -\frac{\sqrt{|\mathcal{M}\lambda|}}{\lambda} \left( \frac{r_1 \sinh(\sqrt{|\mathcal{M}\lambda|}\xi) + r_2 \cosh(\sqrt{|\mathcal{M}\lambda|}\xi) + r_2}{r_1 \sinh(\sqrt{|\mathcal{M}\lambda|}\xi) + r_1 \cosh(\sqrt{|\mathcal{M}\lambda|}\xi) - r_2} \right) \]  

(47)

3. \( \mathcal{M} = 0, \lambda \neq 0 \),
\[
\frac{G'}{G^2} = -\left( \frac{r_1}{\lambda(r_1^2 + r_2)} \right)
\]  
(48)

### 3 Application of the methods

In this section we will find soliton solutions of partial differential equation. We have the following equation:

\[
l\frac{d\Omega}{dt} = (\epsilon_0 - 2\mathfrak{T}_1 - 2\mathfrak{T}_2)\Omega - \delta^2(\mathfrak{T}_1 + 4\mathfrak{T}_2)\frac{d^2\Omega}{dx^2} + u(n - 1)|\Omega|^2\Omega,
\]  
(49)

where delta is the space between two adjacent sections. We set

\[
\phi(x, t) = f(\tau)e^{i(hx+\omega t)},
\]  
(50)

where \(\tau = \beta x + \eta\).

Applying Eqs. (49–50)) get imaginary and real parts respectively,

\[
[-\eta + 2b\beta\delta^2(\mathfrak{T}_1 + 4\mathfrak{T}_2)]f' = 0,
\]  
(51)

\[
[b^2\delta^2(\mathfrak{T}_1 + 4\mathfrak{T}_2) + \omega + \epsilon_0 - 2(\mathfrak{T}_1 + \mathfrak{T}_2)]f + u(n - 1)f^3 + \beta^2\delta^2(\mathfrak{T}_1 + 4\mathfrak{T}_2)f'' = 0,
\]  
(52)

from Eq. (51), we have,

\[
\eta = 2b\beta\delta^2(\mathfrak{T}_1 + 4\mathfrak{T}_2).
\]  
(53)

#### 3.1 New extended direct algebraic method

The homogeneous balancing principal in Eq. (52), in which highest order derivative is two and highest power of the function is three yield \(N = 1\) so, by Eq. (5), the solution becomes:

\[
\Omega(x, t) = a_0 + a_1P(\tau),
\]  
(54)

along with,

\[
P'(\tau) = \ln(\mathfrak{M})(\mathfrak{M} + \mathfrak{B}P(\tau) + \mathfrak{Z}P^2(\tau)), \quad \mathfrak{B} \neq 0, 1.
\]  
(55)

The Eq. (54) is plugging into Eq. (52) and get the algebraic system as:

\[
P^0 : a_0b^2\delta^2\mathfrak{T}_1 + 4a_0b^2\delta^2\mathfrak{T}_2 + uma_0 + a_0\epsilon_0 - 2a_0\mathfrak{T}_1 - 2a_0\mathfrak{T}_2 - uma_0 - \delta^2\beta^2a_1\ln(\mathfrak{M})^2\mathfrak{M}\mathfrak{M} - 4\delta^2\beta^2\mathfrak{T}_2a_1\ln(\mathfrak{M})^2\mathfrak{M}\mathfrak{M} = 0,
\]  
(56)

\[
P^1 : -3uma_0a_1 + a_1\omega + a_1\epsilon - 2a_1\mathfrak{T}_1 - 2a_1\mathfrak{T}_2 - \delta^2\beta^2\mathfrak{T}_1a_1\ln(\mathfrak{M})^2\mathfrak{M}\mathfrak{M} - 4\delta^2\beta^2\mathfrak{T}_2a_1\ln(\mathfrak{M})^2\mathfrak{M}\mathfrak{M}
- 2\delta^2\beta^2\mathfrak{T}_1a_1\ln(\mathfrak{M})^2\mathfrak{M}\mathfrak{M} - 8\delta^2\beta^2\mathfrak{T}_2a_1\ln(\mathfrak{M})^2\mathfrak{M}\mathfrak{M} + a_1b^2\delta^2\mathfrak{T}_1 + 4a_1b^2\delta^2\mathfrak{T}_2 + 3uma_0a_1 = 0,
\]  
(56)

\[
P^2 : -3uma_0a_1^2 - 12\delta^2\beta^2\mathfrak{T}_2a_1(\mathfrak{M})^2\mathfrak{M}\mathfrak{M} - 3\delta^2\beta^2\mathfrak{T}_1a_1(\mathfrak{M})^2\mathfrak{M}\mathfrak{M} + 3uma_0a_1 = 0,
\]  
(56)

\[
P^3 : uma_1 - uma_1^3 - 2\delta^2\beta^2\mathfrak{T}_1a_1\ln(\mathfrak{M})^2\mathfrak{M}\mathfrak{M} - 8\delta^2\beta^2\mathfrak{T}_2a_1\ln(\mathfrak{M})^2\mathfrak{M}\mathfrak{M} = 0.
\]  
(56)
The Maple software package is used to solve the above mentioned system (56) and get the solution set,

\[
\begin{align*}
\alpha_0 &= 2\Xi, \quad \alpha_1 = 2\Xi, \quad \text{and} \quad \beta = \frac{\sqrt{\left(\frac{\Xi}{\theta(n+1)}\right)}}{\ln(\Psi)}.
\end{align*}
\] (57)

We obtained the general solution by plugging Eq. (57) in Eq. (54). It should be noticed that, we can construct many solutions by taking \(P_i\) from Eqs. (7–43).

\[
\Omega(t, x) = \Psi \Xi + 2\Xi(P_i(\kappa)),
\] (58)

where,

\[
\Xi = \pm \sqrt{\frac{b^2 \delta^2 \Xi_1 + 4b^2 \delta^2 \Xi_2 + \varepsilon + \omega - 2\Xi_1 - 2\Xi_2}{\Theta u(n-1)}}, \quad \text{and} \quad \Theta = \Psi^2 - 4\mathcal{M}. \tag{59}
\]

**Category (1)** When \(\Psi^2 - 4\mathcal{M}\Xi < 0\) and \(\Xi \neq 0\),

\[
\Omega_{E,1}(t, x) = \Xi \sqrt{-\Theta} \tan \Psi \left(\frac{\sqrt{-\Theta}}{2} \tau\right) e^{j(-kx+wt)},
\] (60)

\[
\Omega_{E,2}(t, x) = -\Xi \sqrt{-\Theta} \cot \frac{\sqrt{-\Theta}}{2} \tau e^{j(-kx+wt)},
\] (61)

\[
\Omega_{E,3}(t, x) = \Xi \sqrt{-\Theta} \tan \Psi \left(\frac{\sqrt{-\Theta}}{2} \tau\right) \pm \sqrt{mn} \sec \Psi \left(\frac{\sqrt{-\Theta}}{2} \tau\right) e^{j(-kx+wt)}, \tag{62}
\]

\[
\Omega_{E,4}(t, x) = \Xi \sqrt{-\Theta} \cot \Psi \left(\frac{\sqrt{-\Theta}}{2} \tau\right) \pm \sqrt{mn} \csc \Psi \left(\frac{\sqrt{-\Theta}}{2} \tau\right) e^{j(-kx+wt)}, \tag{63}
\]

\[
\Omega_{E,5}(t, x) = \Xi \sqrt{-\Theta} \tan \Psi \left(\frac{\sqrt{-\Theta}}{4} \tau\right) - \cot \Psi \left(\frac{\sqrt{-\Theta}}{4} \tau\right) e^{j(-kx+wt)}. \tag{64}
\]

**Category (2)** When \(\Psi^2 - 4\mathcal{M}\Xi > 0\) and \(\Xi \neq 0\),

\[
\Omega_{E,6}(t, x) = -\Xi \sqrt{-\Theta} \tanh \Psi \left(\frac{\sqrt{-\Theta}}{2} \tau\right) e^{j(-kx+wt)},
\] (65)

\[
\Omega_{E,7}(t, x) = -\Xi \sqrt{-\Theta} \coth \Psi \left(\frac{\sqrt{-\Theta}}{2} \tau\right) e^{j(-kx+wt)},
\] (66)

\[
\Omega_{E,8}(t, x) = \Xi \sqrt{-\Theta} \left(\tan \Psi \left(\frac{\sqrt{-\Theta}}{4} \tau\right) \pm i\sqrt{mn} \sech \Psi \left(\frac{\sqrt{-\Theta}}{4} \tau\right)\right) e^{j(-kx+wt)}. \tag{67}
\]
\[
\Omega_{E,9}(t, x) = \Xi \sqrt{\Theta} \left( \coth \varphi \left( \sqrt{\Theta} \right) \pm \sqrt{mn} \csc \varphi \left( \sqrt{\Theta} \right) \right) e^{i(-kx + wt)},
\]

(68)

\[
\Omega_{E,10}(t, x) = -\Xi \frac{\sqrt{\Theta}}{2} \left( \tanh \varphi \left( \frac{\sqrt{\Theta}}{4} \right) + \coth \varphi \left( \frac{\sqrt{\Theta}}{4} \right) \right) e^{i(-kx + wt)}.
\]

(69)

**Category (3)** When \( \mathfrak{M} \not> 0 \) and \( \mathfrak{B} = 0 \),

\[
\Omega_{E,11}(t, x) = \pm \sqrt{\frac{b^2 \delta^2 \mathcal{I}_1 + 4b^2 \delta^2 \mathcal{I}_2 + \varepsilon + \omega - 2\mathcal{I}_1 - 2\mathcal{I}_2}{u(1 - n)}} \left( \tanh \varphi \left( \sqrt{\mathfrak{M} \mathfrak{Z}} \right) \right) e^{i(-kx + wt)}.
\]

(70)

\[
\Omega_{E,12}(t, x) = \mp \sqrt{\frac{b^2 \delta^2 \mathcal{I}_1 + 4b^2 \delta^2 \mathcal{I}_2 + \varepsilon + \omega - 2\mathcal{I}_1 - 2\mathcal{I}_2}{u(1 - n)}} \left( \coth \varphi \left( \sqrt{\mathfrak{M} \mathfrak{Z}} \right) \right) e^{i(-kx + wt)}.
\]

(71)

\[
\Omega_{E,13}(t, x) = \pm \sqrt{\frac{b^2 \delta^2 \mathcal{I}_1 + 4b^2 \delta^2 \mathcal{I}_2 + \varepsilon + \omega - 2\mathcal{I}_1 - 2\mathcal{I}_2}{u(1 - n)}} \left( \tanh \varphi \left( \sqrt{\mathfrak{M} \mathfrak{Z}} \right) \pm \sqrt{mn} \sec \varphi \left( \sqrt{\mathfrak{M} \mathfrak{Z}} \right) \right) e^{i(-kx + wt)},
\]

(72)

\[
\Omega_{E,14}(t, x) = \pm \sqrt{\frac{b^2 \delta^2 \mathcal{I}_1 + 4b^2 \delta^2 \mathcal{I}_2 + \varepsilon + \omega - 2\mathcal{I}_1 - 2\mathcal{I}_2}{u(1 - n)}} \left( - \coth \varphi \left( \sqrt{\mathfrak{M} \mathfrak{Z}} \right) \pm \sqrt{mn} \csc \varphi \left( \sqrt{\mathfrak{M} \mathfrak{Z}} \right) \right) e^{i(-kx + wt)},
\]

(73)

\[
\Omega_{E,15}(t, x) = \pm \sqrt{\frac{b^2 \delta^2 \mathcal{I}_1 + 4b^2 \delta^2 \mathcal{I}_2 + \varepsilon + \omega - 2\mathcal{I}_1 - 2\mathcal{I}_2}{u(1 - n)}} \left( \tanh \varphi \left( \frac{\sqrt{\mathfrak{M} \mathfrak{Z}}}{2} \right) \right) e^{i(-kx + wt)}.
\]

(74)

**Category (4)** When \( \mathfrak{M} \mathfrak{Z} < 0 \) and \( \mathfrak{B} = 0 \),

\[
\Omega_{E,16}(t, x) = \mp \sqrt{\frac{b^2 \delta^2 \mathcal{I}_1 + 4b^2 \delta^2 \mathcal{I}_2 + \varepsilon + \omega - 2\mathcal{I}_1 - 2\mathcal{I}_2}{u(n - 1)}} \left( \tanh \varphi \left( \sqrt{-\mathfrak{M} \mathfrak{Z}} \right) \right) e^{i(-kx + wt)},
\]

(75)

\[
\Omega_{E,17}(t, x) = \mp \sqrt{\frac{b^2 \delta^2 \mathcal{I}_1 + 4b^2 \delta^2 \mathcal{I}_2 + \varepsilon + \omega - 2\mathcal{I}_1 - 2\mathcal{I}_2}{u(n - 1)}} \left( \coth \varphi \left( \sqrt{-\mathfrak{M} \mathfrak{Z}} \right) \right) e^{i(-kx + wt)},
\]

(76)
\[ \Omega_{E,18}(t,x) = \mp \sqrt{\frac{b^2 \delta^2 \mathcal{X}_1 + 4b^2 \delta^2 \mathcal{X}_2 + \epsilon + \omega - 2 \mathcal{X}_1 - 2 \mathcal{X}_2}{u(n-1)}} \left( - \tanh_{\wp} \left( 2 \sqrt{-\mathcal{M}} \mathcal{Z} \tau \right) \pm i \sqrt{mn} \text{sech}_{\wp} \left( 2 \sqrt{-\mathcal{M}} \mathcal{Z} \tau \right) \right) e^{i(-k_x \omega t)}, \] 

\[ \Omega_{E,19}(t,x) = \mp \sqrt{\frac{b^2 \delta^2 \mathcal{X}_1 + 4b^2 \delta^2 \mathcal{X}_2 + \epsilon + \omega - 2 \mathcal{X}_1 - 2 \mathcal{X}_2}{u(n-1)}} \left( - \coth_{\wp} \left( 2 \sqrt{-\mathcal{M}} \mathcal{Z} \tau \right) \pm \sqrt{mn} \text{csch}_{\wp} \left( 2 \sqrt{-\mathcal{M}} \mathcal{Z} \tau \right) \right) e^{i(-k_x \omega t)}, \] 

\[ \Omega_{E,20}(t,x) = \mp \frac{b^2 \delta^2 \mathcal{X}_1 + 4b^2 \delta^2 \mathcal{X}_2 + \epsilon + \omega - 2 \mathcal{X}_1 - 2 \mathcal{X}_2}{2u(n-1)} \left( \tanh_{\wp} \left( \frac{\sqrt{-\mathcal{M}} \mathcal{Z}}{2} \tau \right) + \coth_{\wp} \left( \frac{\sqrt{-\mathcal{M}} \mathcal{Z}}{2} \tau \right) \right) e^{i(-k_x \omega t)}. \]

**Category (5)** When \( \mathcal{M} = 0 \) and \( \mathcal{Z} = \mathcal{M} = \mathcal{N} \),

\[ \Omega_{E,21}(t,x) = \mp \sqrt{\frac{b^2 \delta^2 \mathcal{X}_1 + 4b^2 \delta^2 \mathcal{X}_2 + \epsilon + \omega - 2 \mathcal{X}_1 - 2 \mathcal{X}_2}{u(1-n)}} \left( \tanh_{\wp} (\mathcal{M} \tau) \right) e^{i(-k_x \omega t)}, \] 

\[ \Omega_{E,22}(t,x) = \mp \sqrt{\frac{b^2 \delta^2 \mathcal{X}_1 + 4b^2 \delta^2 \mathcal{X}_2 + \epsilon + \omega - 2 \mathcal{X}_1 - 2 \mathcal{X}_2}{u(1-n)}} \left( \coth_{\wp} (\mathcal{M} \tau) \right) e^{i(-k_x \omega t)}, \] 

\[ \Omega_{E,23}(t,x) = \pm \sqrt{\frac{b^2 \delta^2 \mathcal{X}_1 + 4b^2 \delta^2 \mathcal{X}_2 + \epsilon + \omega - 2 \mathcal{X}_1 - 2 \mathcal{X}_2}{u(n-1)}} \left( \tanh_{\wp} (2\mathcal{M} \tau) \pm \sqrt{mn} \sec_{\wp} (2\mathcal{M} \tau) \right) e^{i(-k_x \omega t)}, \] 

\[ \Omega_{E,24}(t,x) = \mp \sqrt{\frac{b^2 \delta^2 \mathcal{X}_1 + 4b^2 \delta^2 \mathcal{X}_2 + \epsilon + \omega - 2 \mathcal{X}_1 - 2 \mathcal{X}_2}{u(n-1)}} \left( \coth_{\wp} (2\mathcal{M} \tau) \pm \sqrt{mn} \csc_{\wp} (2\mathcal{M} \tau) \right) e^{i(-k_x \omega t)}, \] 

\[ \Omega_{E,25}(t,x) = \mp \sqrt{\frac{b^2 \delta^2 \mathcal{X}_1 + 4b^2 \delta^2 \mathcal{X}_2 + \epsilon + \omega - 2 \mathcal{X}_1 - 2 \mathcal{X}_2}{u(n-1)}} \left( \tanh_{\wp} \left( \frac{\mathcal{M}}{2} \tau \right) - \coth_{\wp} \left( \frac{\mathcal{M}}{2} \tau \right) \right) e^{i(-k_x \omega t)}. \]

**Category (6)** When \( \mathcal{M} = 0 \) and \( \mathcal{Z} = -\mathcal{M} \),
\[\Omega_{E,26}(t,x) = \mp \sqrt{\frac{b^2 \delta^2 \mathcal{I}_1 + 4b^2 \delta^2 \mathcal{I}_2 + \varepsilon + \omega - 2 \mathcal{I}_1 - 2 \mathcal{I}_2}{u(n-1)}} \left( \tanh_q \left( \mathcal{M} \tau \right) \right) e^{i(-kx+wt)}, \tag{85}\]

\[\Omega_{E,27}(t,x) = \mp \sqrt{\frac{b^2 \delta^2 \mathcal{I}_1 + 4b^2 \delta^2 \mathcal{I}_2 + \varepsilon + \omega - 2 \mathcal{I}_1 - 2 \mathcal{I}_2}{u(n-1)}} \left( \coth_q \left( \mathcal{M} \tau \right) \right) e^{i(-kx+wt)}, \tag{86}\]

\[\Omega_{E,28}(t,x) = \mp \sqrt{\frac{b^2 \delta^2 \mathcal{I}_1 + 4b^2 \delta^2 \mathcal{I}_2 + \varepsilon + \omega - 2 \mathcal{I}_1 - 2 \mathcal{I}_2}{u(n-1)}} \left( \tanh_q \left( 2 \mathcal{M} \tau \right) \pm i \sqrt{\frac{mn}{\mathcal{I}_1}} \text{sech}_q \left( 2 \mathcal{M} \tau \right) \right) e^{(-kx+wt)}, \tag{87}\]

\[\Omega_{E,29}(t,x) = \mp \sqrt{\frac{b^2 \delta^2 \mathcal{I}_1 + 4b^2 \delta^2 \mathcal{I}_2 + \varepsilon + \omega - 2 \mathcal{I}_1 - 2 \mathcal{I}_2}{u(n-1)}} \left( \coth_q \left( 2 \mathcal{M} \tau \right) \pm \sqrt{\frac{mn}{\mathcal{I}_1}} \text{csch}_q \left( 2 \mathcal{M} \tau \right) \right) e^{i(-kx+wt)}, \tag{88}\]

\[\Omega_{E,30}(t,x) = \mp \sqrt{\frac{b^2 \delta^2 \mathcal{I}_1 + 4b^2 \delta^2 \mathcal{I}_2 + \varepsilon + \omega - 2 \mathcal{I}_1 - 2 \mathcal{I}_2}{2u(n-1)}} \left( \tanh_q \left( \frac{\mathcal{M}}{2} \tau \right) + \coth_q \left( \frac{\mathcal{M}}{2} \tau \right) \right) e^{i(-kx+wt)}. \tag{89}\]

**Category (7)** When \( \mathfrak{B}^2 = 4 \mathfrak{M} \mathfrak{I} \),

\[\Omega_{E,31}(t,x) = \text{Undefined} \tag{90}\]

**Category (8)** When \( \mathfrak{B} = p, \mathfrak{M} = pq, (q \neq 0) \) and \( \mathfrak{I} = 0 \),

\[\Omega_{E,32}(t,x) = \pm \sqrt{\frac{b^2 \delta^2 \mathcal{I}_1 + 4b^2 \delta^2 \mathcal{I}_2 + \varepsilon + \omega - 2 \mathcal{I}_1 - 2 \mathcal{I}_2}{2u(n-1)}} e^{i(-kx+wt)} \tag{91}\]

**Category (9)** When \( \mathfrak{B} = \mathfrak{I} = 0 \),

\[\Omega_{E,33}(t,x) = 0, \tag{92}\]

**Category (10)** When \( \mathfrak{B} = \mathfrak{I} = 0 \),

\[\Omega_{E,34}(t,x) = \text{Undefined}. \tag{93}\]

**Category (11)** When \( \mathfrak{M} = 0 \) and \( \mathfrak{B} \neq 0 \),
\[ \Omega_{E,35}(t,x) = \pm \sqrt{\frac{b^2 \delta^2 T_1 + 4b^2 \delta^2 T_2 + \epsilon + \omega - 2T_1 - 2T_2}{u(n-1)}} \left( 1 - \frac{m}{(\cosh \beta (\mathbf{G}) - \sinh \beta (\mathbf{G}) + m)} \right) e^{i(-kx + wt)}, \]  
(94)

\[ \Omega_{E,36}(t,x) = \pm \sqrt{\frac{b^2 \delta^2 T_1 + 4b^2 \delta^2 T_2 + \epsilon + \omega - 2T_1 - 2T_2}{u(n-1)}} \left( 1 - \frac{2(\sinh \beta (\mathbf{G}) + \cosh \beta (\mathbf{G}))}{(\sinh \beta (\mathbf{G}) + \cosh \beta (\mathbf{G}) + n)} \right) e^{i(-kx + wt)}, \]  
(95)

Category (12) When \( \mathfrak{B} = p, \mathfrak{Z} = pq, \) \( q \neq 0 \) and \( \mathfrak{M} = 0, \)

\[ \Omega_{E,37}(t,x) = \pm \sqrt{\frac{b^2 \delta^2 T_1 + 4b^2 \delta^2 T_2 + \epsilon + \omega - 2T_1 - 2T_2}{u(n-1)}} \left( 1 - \frac{2qm \mathfrak{M}^p}{m - qn \mathfrak{M}^p} \right) e^{i(-kx + wt)}. \]  
(96)

The deformation parameters \( m, n > 0. \)

### 3.2 \( \frac{G'}{G^2} \)-expansion method

By homogeneous balancing principal in Eq. (52) highest order derivative is two and highest power of the function is three so, \( 3N = N + 2 \) simplify to get \( N = 1 \) so that we will take the solution:

\[ \Omega(x, t) = a_0 + a_1 V(t) + b_1 V(t)^{-1}, \]  
(97)

where,

\[ \left( \frac{G'}{G^2} \right)' = \mathfrak{M} + \lambda V^2, \]  
(98)

where \( \mathfrak{M}, \lambda \in \mathbb{R}, \) putting Eq. (97) in Eq. (52) and equating the coefficients of the distinct power of \( \frac{G'}{G^2}, \) we get the following system of equations:

\[ \left( \frac{G'}{G^2} \right)^0 = a_0 \delta^2 k^2 T_1 + 4a_0 \delta^2 k^2 T_2 + a_0^3 nu + 6a_0 a_1 b_1 nu - a_0^3 u - 6a_0 a_1 b_1 u + a_0 c + a_0 \omega - 2a_0 T_1 - 2a_0 T_2 = 0. \]  
(99)

\[ \left( \frac{G'}{G^2} \right)^1 = -2a_1 \beta^2 \delta^2 \lambda \mathfrak{M} T_1 - 8a_1 \beta^2 \delta^2 \lambda \mathfrak{M} T_2 + a_1 \delta^2 k^2 T_1 + 4a_1 \delta^2 k^2 T_2 + 3a_2 a_1 nu + 3a_1 b_1 nu - 3a_2 a_1 u - 3a_1 b_1 u + a_1 c + a_1 \omega - 2a_1 T_1 - 2a_1 T_2 = 0. \]  
(100)
\[
\left( \frac{G'}{G^2} \right)^2 : 3a_0a_1^2\nu - 3a_0a_1^3u = 0. \tag{101}
\]

\[
\left( \frac{G'}{G^2} \right)^3 : -2a_1\beta^2\delta^2\lambda^2 \mathcal{I}_1 - 8a_1\beta^2\delta^2\lambda^2 \mathcal{I}_2 + a_1^2\nu - a_1^3u = 0. \tag{102}
\]

\[
\left( \frac{G'}{G^2} \right)^{-3} : -2b_1\beta^2\delta^2\mathcal{M}^2 \mathcal{I}_1 - 8b_1\beta^2\delta^2\mathcal{M}^2 \mathcal{I}_2 + b_1^3\nu - b_1^3u = 0. \tag{103}
\]

\[
\left( \frac{G'}{G^2} \right)^{-2} : 3a_0b_1^2\nu - 3a_0b_1^2u = 0. \tag{104}
\]

\[
\left( \frac{G'}{G^2} \right)^{-1} : -2b_1\beta^2\delta^2\lambda \mathcal{I}_1 - 8b_1\beta^2\delta^2\lambda \mathcal{I}_2 + b_1\delta^2k^2 \mathcal{I}_1 + 4b_1\delta^2k^2 \mathcal{I}_2 + 3a_0^2b_1\nu + 3a_1b_1^2\nu - 3a_0^2\beta_1u - 3a_1b_1^2u + b_1\nu - 2b_1 \mathcal{I}_1 - 2b_1 \mathcal{I}_2 = 0. \tag{105}
\]

Solve the above system of equations from Eqs. (99–105) by using Maple software we get:

**Set-1**

\[
\begin{align*}
    a_0 &= 0, a_1 = 0, b_1 = \pm \Lambda \mathcal{M} \beta \delta, b &= \pm \sqrt{\frac{2\beta^2\delta^2\lambda \mathcal{M} \mathcal{I}_1 + 8\beta^2\delta^2\lambda \mathcal{M} \mathcal{I}_2 - \nu - \omega + 2 \mathcal{I}_1 + 2 \mathcal{I}_2}{\delta^2(\mathcal{I}_1 + 4 \mathcal{I}_2)}}
\end{align*} \tag{106}
\]

**Set-2**

\[
\begin{align*}
    a_0 &= 0, a_1 = \pm \Lambda \beta \delta, b_1 = 0, b &= \pm \sqrt{\frac{2\beta^2\delta^2\lambda \mathcal{M} \mathcal{I}_1 + 8\beta^2\delta^2\lambda \mathcal{M} \mathcal{I}_2 - \nu - \omega + 2 \mathcal{I}_1 + 2 \mathcal{I}_2}{\delta^2(\mathcal{I}_1 + 4 \mathcal{I}_2)}}
\end{align*} \tag{107}
\]

**Set-3**

\[
\begin{align*}
    a_0 &= 0, a_1 = \pm \Lambda \beta \delta, b_1 = \pm \Lambda \mathcal{M} \beta \delta, b &= \pm \sqrt{\frac{2\beta^2\delta^2\lambda \mathcal{M} \mathcal{I}_1 + 8\beta^2\delta^2\lambda \mathcal{M} \mathcal{I}_2 - \nu - \omega + 2 \mathcal{I}_1 + 2 \mathcal{I}_2}{\delta^2(\mathcal{I}_1 + 4 \mathcal{I}_2)}}
\end{align*} \tag{108}
\]

### 3.2.1 Set-1

**Case-1** When \( \mathcal{M} \lambda > 0 \), the periodic soliton solution is as follows:

\[
\Omega_{V,1,1}(x,t) = \pm \sqrt{\mathcal{M} \lambda} \Lambda \beta \delta \left( \frac{r_1 \cos(\sqrt{\mathcal{M} \lambda} \tau) - r_2 \sin(\sqrt{\mathcal{M} \lambda} \tau)}{r_1 \cos(\sqrt{\mathcal{M} \lambda} \tau) - r_2 \sin(\sqrt{\mathcal{M} \lambda} \tau)} \right) e^{(-kx + \omega t)}. \tag{109}
\]

**Case-2** When \( \lambda \mathcal{M} < 0 \), then the singular soliton solution is obtained as:
\[ \Omega_{x,1,2}(x, t) = \pm \sqrt{\| \mathcal{M} \| \Lambda \delta \beta} \left( \frac{2r_1 \sinh(\sqrt{\| \mathcal{M} \| \Lambda} \tau) \cosh(\sqrt{\| \mathcal{M} \| \Lambda} \tau) + 2r_1 \cosh^2(\sqrt{\| \mathcal{M} \| \Lambda} \tau) - r_1 - r_2) \right) e^{i(-kx + wt)}. \]  

(110)

**Case-3** When \( \lambda \neq 0 \), \( \mathcal{M} = 0 \), then the solution is obtained as:

\[ \Omega_{V,1,3}(x, t) = \pm \Lambda \left( \frac{r_1 \tau + r_2}{r_1 \tau + r_2} \right) e^{i(-kx + wt)}. \]  

(111)

### 3.2.2 Set-2

**Case-4** When \( \mathcal{M} \lambda > 0 \), the periodic soliton solution is as follows:

\[ \Omega_{V,2,1}(x, t) = \pm \sqrt{\| \mathcal{M} \| \Lambda \delta \beta} \left( \frac{r_1 \cos(\sqrt{\| \mathcal{M} \| \Lambda} \tau) + r_2 \sin(\sqrt{\| \mathcal{M} \| \Lambda} \tau)}{r_1 \cos(\sqrt{\| \mathcal{M} \| \Lambda} \tau) - r_2 \sin(\sqrt{\| \mathcal{M} \| \Lambda} \tau)} \right) e^{i(-kx + wt)}. \]  

(112)

**Case-5** When \( \lambda \mathcal{M} < 0 \), then the singular soliton solution is obtained as:

\[ \Omega_{V,2,2}(x, t) = \pm \sqrt{\| \mathcal{M} \| \Lambda \delta \beta} \left( \frac{2r_1 \sinh(\sqrt{\| \mathcal{M} \| \Lambda} \tau) \cosh(\sqrt{\| \mathcal{M} \| \Lambda} \tau) + 2r_1 \cosh^2(\sqrt{\| \mathcal{M} \| \Lambda} \tau) - r_1 + r_2)}{2r_1 \sinh(\sqrt{\| \mathcal{M} \| \Lambda} \tau) \cosh(\sqrt{\| \mathcal{M} \| \Lambda} \tau) + 2r_1 \cosh^2(\sqrt{\| \mathcal{M} \| \Lambda} \tau) - r_1 - r_2) \right) e^{i(-kx + wt)}. \]  

(113)

**Case-6** When \( \lambda \neq 0 \), \( \mathcal{M} = 0 \), then the solution is obtained as:

\[ \Omega_{V,2,3}(x, t) = \pm \Lambda \left( \frac{\delta r_1}{r_1 \tau + r_2} \right) e^{i(-kx + wt)}. \]  

(114)

### 3.2.3 Set-3

**Case-7** When \( \mathcal{M} \lambda > 0 \), the periodic soliton solution is as follows:

\[ \Omega_{V,3,1}(x, t) = \pm \sqrt{\| \mathcal{M} \| \Lambda} \left( \frac{3 \cos^2(\sqrt{\| \mathcal{M} \| \Lambda} \tau) r_2^2 - 3 \cos^2(\sqrt{\| \mathcal{M} \| \Lambda} \tau) r_2^2 - 2 \cos(\sqrt{\| \mathcal{M} \| \Lambda} \tau) r_1 - r_2^2}{\cos^2(\sqrt{\| \mathcal{M} \| \Lambda} \tau) r_2^2 + \cos^2(\sqrt{\| \mathcal{M} \| \Lambda} \tau) r_2^2 - r_2^2} \right) e^{i(-kx + wt)}. \]  

(115)

**Case-8** When \( \lambda \mathcal{M} < 0 \), then the singular soliton solution is obtained as:

\[ \Omega_{V,3,2}(x, t) = \pm \sqrt{\| \mathcal{M} \| \Lambda} \left( \frac{r_1 \sinh(2\sqrt{\| \mathcal{M} \| \Lambda} \tau) + r_1 \cosh(2\sqrt{\| \mathcal{M} \| \Lambda} \tau) + r_2}{r_1 \sinh(2\sqrt{\| \mathcal{M} \| \Lambda} \tau) + r_1 \cosh(2\sqrt{\| \mathcal{M} \| \Lambda} \tau) - r_2} \right) e^{i(-kx + wt)}. \]  

(116)

**Case-9** When \( \lambda \neq 0 \), \( \mathcal{M} = 0 \), then the solution is obtained as:

\[ \Omega_{V,3,3}(x, t) = \pm \Lambda \left( \frac{2(r_1 \tau + r_2)^2 \mathcal{M} + r_1^2}{(r_1 \tau + r_2)^2} \right) e^{i(-kx + wt)}. \]  

(117)
Fig. 1 Three dimensional and contour of solution \( (\Omega_1(t, x)) \) is presented with \( a = 10, n = 31, b = 1, \delta = 1, \Sigma_1 = 0.4, \Sigma_2 = 0.8, \epsilon = 1, \eta = 0.05, u = 7, \Theta = 5, \) and \( \omega = 1 \)
4 Graphical visualization and explanation

In this section, the graphical comparision and description will present to show the novelty and effectiveness of the utilized analytical approaches.

Figures 1 and 2 are presenting the 2-D, 3-D, and Contour graphic view of solution \((\Omega_{E,1}(t,x))\). Figure 1a, b are 3-D and Contour dark soliton visualization at \(\eta = 0.05\). Figure 1c, d are 3-D and Contour dark soliton visualization at \(\eta = 0.25\). Figure 1e, f are 3-D and Contour dark-singular soliton visualization at \(\eta = 0.05\). Figure 2a, c, e, are depicting the impact of \(\eta\) on solution at \(t = 0, t = 10,\) and \(t = 20\), respectively. Figure 2b, d, f, are depicting the impact of time on solution at \(\eta = 0.05, \eta = 0.25,\) and \(\eta = 0.35\), respectively.

Remark One can observe that, solution is displaying the dark soliton to dark-singular soliton on increasing the \(\eta\). The considered soliton is obtained by new direct extended algebraic method but the \(\frac{G'}{G}\)-expansion method did not generate the dark soliton. It
means that the new extended algebraic method is generating more solitons as compare to $G^2$-expansion algorithm.

Figure 3 is demonstrating 2-D, 3-D, and Contour comparision of rational solutions obtained by NEDAE and $G^2$-expansion technique at $\eta = 0.05$. 

Fig. 3 2-D, 3-D and Contour representation of solutions $n = 31$, $b = 0.1$, $r_1 = 2$, $r_2 = 0.4$, $\mathcal{W} = 1$, $\delta = 1$, $\mathcal{S}_1 = 0.4$, $\mathcal{S}_2 = 0.8$, $\eta = 0.3$, $\alpha = 7$, $\omega = 0.3$, and $\eta = 0.05$.
Figure 4 is demonstrating 2-D, 3-D, and Contour comparision of rational solutions obtained by NEDAE and $G^2$-expansion technique at $\eta = 0.25$.

Figure 5 is demonstrating 2-D, 3-D, and Contour comparision of rational solutions obtained by NEDAE and $G^2$-expansion technique at $\eta = 0.35$. 
Remark One can observe that the solution $\Omega_{E,37}$ is explaining the more generalized soliton behaviour as compare to $\Omega_{G,3,3}$. It is an evidence that the new extended direct algebraic scheme is more effective and generalized analytical approach as compare to $\frac{G'}{G}$-expansion method.

Fig. 5 2-D, 3-D and Contour representation of solutions for $n = 31, b = 0.1, r_1 = 2, r_2 = 0.4, \eta = 1, \delta = 1, \xi_1 = 0.4, \xi_2 = 0.8, \eta = 0.3, \alpha = 7, \omega = 0.3$, and $\eta = 0.35$
Figure 6 is describing the 2-D comparison and influence of $\eta$ and time on the solution.

5 Conclusion

A cold bosonic atom in a zig-zag optical lattices have been investigated in this article. Numerous solutions have been suggest by using two different methods. The obtained results are bright solutions, dark solutions, trigonometric solution, hyperbolic, exponential, logarithmic, and pulse solution. The new extended direct algebraic method has provided twelve categories and thirty seven solutions containing various families of functions while the $G'/G$ expansion approach provided three families and nine solutions containing rational and trigonometric function. To notice their shape in the space and in time, we plotted 2-D, 3-D and contour graphs and observed the impact of wave frequency parameter...
and time. The graphical comparison for utilized techniques is displayed and assured that new extended direct algebraic method is more reliable and generalized analytical approach then $G'/G$-expansion scheme. It is observed that, many types of solutions like, dark, bright, and singular solitons do not generated by $G'/G$-expansion while other method has generated such types of solitons for considered model and can be useful to develop various types of solitons for other significant models and systems. The established results will help to well understand the nonlinear phenomena in a cold bosonic atoms.

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