Weyl Consistency Conditions and $\gamma_5$

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Abstract

The treatment of $\gamma_5$ in Dimensional Regularization leads to ambiguities in field-theoretic calculations, of which one example is the coefficient of a particular term in the four-loop gauge $\beta$-functions of the Standard Model. Using Weyl Consistency Conditions, we present a scheme-independent relation between the coefficient of this term and a corresponding term in the three-loop Yukawa $\beta$-functions, where a semi-naïve treatment of $\gamma_5$ is sufficient, thereby fixing this ambiguity. We briefly outline an argument by which the same method fixes similar ambiguities at higher orders.

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1 Introduction

The treatment of $\gamma_5$ in Dimensional Regularization is a well-known theoretical issue \cite{1}, and can be summarized in the following statement: given a four-dimensional, Poincaré-invariant quantum field theory, there is no gauge-invariant regularization method that preserves chiral symmetry \cite{2}. The precise connection between the two is most easily demonstrated using the ABJ anomaly, the derivation of which requires

$$\text{tr} \left[ \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_5 \right] = 4i\epsilon^{\mu\nu\rho\sigma}, \quad \epsilon_{0123} = -\epsilon^{0123} = 1 \quad (1.1)$$

in four dimensions, whereas the $d$-dimensional $\gamma$-matrix algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}1, \quad g^{\mu\nu}g_{\mu\nu} = d, \quad \{\gamma^\mu, \gamma^5\} = 0 \quad (1.2)$$

plus trace-cyclicity directly implies

$$\text{tr} \left[ \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_5 \right] = 0 \quad (1.3)$$

even when $d \to 4$. Thus, if one wishes to renormalize a gauge theory with chiral fermions, one must sacrifice either cyclicity of the trace over Dirac matrices involving $\gamma_5$, or break gauge invariance at intermediate stages of a calculation in perturbation theory. The former option is preferable for the purpose of calculating higher-order perturbative corrections, but will inevitably give rise to ambiguities in loop integrals stemming from the precise location of $\gamma_5$ in the Dirac traces. Such ambiguities may appear for the first time at three loops, however the $\beta$-functions of the gauge \cite{3} and scalar \cite{4} couplings in the Standard Model are spared, due to the cancellation of the ABJ anomaly. Furthermore, for the Yukawa couplings, one can use a “semi-naïve” treatment of $\gamma_5$,

$$\text{tr} \left[ \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_5 \right] = 4i\tilde{\epsilon}^{\mu\nu\rho\sigma} + O(\epsilon), \quad \tilde{\epsilon}^{\mu\nu\rho\sigma} \epsilon_{\alpha\beta\gamma\delta} = g^{\mu[a}g^{\nu\beta}g^{\sigma\rho]}g^{\gamma\delta]}, \quad \tilde{\epsilon}^{\mu\nu\rho\sigma} \xrightarrow{d \to 4} \epsilon^{\mu\nu\rho\sigma} \quad (1.4)$$

in order to show that the resulting ambiguity in the relevant Feynman integral is $O(\epsilon)$, and hence cannot affect the Yukawa $\beta$-function \cite{5}. Unfortunately, such minor miracles no longer hold at four loops; by parametrizing the integrals according to the position of $\gamma_5$, the resulting ambiguity in the four-loop strong-coupling $\beta$-function, $\beta_{\alpha_s}^{(4)}$, has been explicitly calculated as \cite{7,8}

$$\beta_{\alpha_s}^{(4)} \supset R \left( \frac{16}{3} + 32\zeta_3 \right) T_F^2\alpha_s^3\alpha_t^2, \quad R = 1, 2, \text{ or } 3. \quad (1.5)$$

While the pursuit of higher-order loop calculations has motivated many significant computational developments, there have also been notable advances in our understanding of renormalization itself, which are not yet as well-known in the phenomenological community. One such development is the notion of Weyl Consistency Conditions \cite{9}: if one extend
a theory to curved spacetime and local couplings, then the Wess-Zumino consistency conditions for the trace anomaly imply a plethora of relations between various RG quantities, amongst them Osborn’s equation\(^1\)

\[
\partial_I \tilde{A} \equiv \frac{\partial \tilde{A}}{\partial g^I} = T_{IJ} \beta^J, \quad T_{IJ} = G_{IJ} + 2\partial_{[I} W_{J]} + 2\tilde{\rho}_{[I} Q_{J]} \tag{1.6}
\]

where \(g^I\) labels the marginal couplings of the theory. For the purpose of calculation, it is easier to work with an equivalent equation, obtained by multiplying (1.6) by \(dg^I\):

\[
d\tilde{A} \equiv dg^I \partial_I \tilde{A} = dg^I T_{IJ} \beta^J \tag{1.7}
\]

This equation therefore demonstrates the existence of a function, \(\tilde{A}\), of the couplings in a general renormalizable theory, which places constraints on the corresponding \(\beta\)-functions. Central to these constraints is the “3-2-1” phenomenon, where the gauge \(\beta\)-function is related to the Yukawa \(\beta\)-function one loop below, and the scalar \(\beta\)-function two loops below. The reason for this ordering is topological, and is thus manifestly preserved to all orders; consequently, given enough information at lower orders, one can use (1.6) to predict coefficients of terms at higher orders. Most importantly, the \(\beta\)-functions in (1.6) are precisely the four-dimensional functions that one should obtain after taking the \(\epsilon \to 0\) limit of Dimensional Regularization. This is the crux of our approach: if there exists a consistency condition relating the ambiguous term in \(\beta^{(4)}_{\phi}\) to lower-order \(\beta\)-function coefficients, and if the consistency condition is simple enough, then it may be possible to fix the ambiguity inherent in the treatment of \(\gamma_5\).

2 Constraints from Weyl Consistency Conditions

In order to derive constraints on the four-loop gauge \(\beta\)-function, one must construct \(\tilde{A}\) at five loops. This is already a somewhat awkward task, but there is a further complication: in order to isolate particular contributions to the \(\beta\)-function, such as those stemming from the integrals involving \(\gamma_5\), one must work with a completely general theory, described in terms of tensor couplings between arbitrary multiplets of matter fields\(^2\). Expressing the matter content as \(n_\phi\) real scalars \(\phi_a\) and \(n_\psi\) Weyl fermions \(\psi_j\), the Lagrangian density of a general theory with a semi-simple gauge symmetry group \(G = G_1 \times \ldots \times G_n\), containing

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\(^1\)This equation is not generally known by any set name, but one of us (CP) is fed up of using phrases such as the technically-incorrect “gradient-flow equation”, the correct-but-cumbersome “gradient-flow-like equation”, and the frankly horrific “equation defining the four-dimensional perturbative \(A\)-function”. As the power of this equation is only now being realized, we feel it appropriate that its author be suitably recognised.

\(^2\)This is related to well-known questions regarding which terms actually contribute to the \(\beta\)-functions in a particular scheme.
at most one U(1) factor\(^3\), is given by

\[
\mathcal{L} = \sum_{\alpha=1}^{n} \left( -\frac{1}{4} (G_\alpha)^{\mu\nu}_{a\alpha}(G_\alpha)^{a\mu}_{a\alpha} \right) + i\bar{\psi}_j \sigma^\mu D_\mu \psi^j + \frac{1}{2} D_\mu \phi_a D^\mu \phi_a \\
- \frac{1}{2} (Y_{a0j} \phi_a \bar{\psi}_i \psi^j + \bar{Y}_{a0j} \phi_a \bar{\psi}_i \psi^j) - \frac{1}{4!} \lambda_{abcd} \phi_a \bar{\psi}_b \psi^c \phi_d \\
+ \text{mass terms} + \text{gauge-fixing} + \text{ghost terms} \tag{2.1}
\]

with covariant derivatives

\[
D_\mu \psi_j = \partial_\mu \psi_j - iq_j (A_1)_\mu \psi_j - i \sum_{\alpha=2}^{n} \sum_{a=1}^{[G_\alpha]} g_\alpha (R_\alpha)_{a\mu} (A_\alpha)_{a\mu} \psi_k \\
D_\mu \phi_a = \partial_\mu \phi_a - iq_a (A_1)_{\mu} \phi_a - i \sum_{\alpha=2}^{n} \sum_{a=1}^{[G_\alpha]} g_\alpha (S_\alpha)_{a\mu} (A_\alpha)_{a\mu} \phi_b
\tag{2.2}
\]

where we have assumed that \( G_1 = U(1) \). The fermions transform under a representation \( R_\alpha \) of the corresponding gauge group \( G_\alpha \), with Hermitian generators \((R_\alpha)_{a\mu} \rightarrow (R_\alpha)^{\mu}_{a\nu}\) and the scalars likewise transform under a representation \( S_\alpha \) with antisymmetric, Hermitian generators \((S_\alpha)_{a\mu} \rightarrow -(S_\alpha)^{a\mu}_{ab}\). When constructing \( \bar{A} \), it proves convenient to assemble the Yukawa couplings and fermion generators into matrices,

\[
y_a = \begin{pmatrix} Y^a & 0 \\ 0 & \bar{Y}^a \end{pmatrix}, \quad \dot{y}_a = \begin{pmatrix} \bar{Y}^a & 0 \\ 0 & Y^a \end{pmatrix} = \sigma_1 y_a \sigma_1, \\
T^a = \begin{pmatrix} R^a & 0 \\ 0 & -(R^a)^* \end{pmatrix}, \quad \dot{T}^a = \sigma_1 T^a \sigma_1 = -(T^a)^T
\tag{2.3}
\]

so that there is a single Yukawa interaction involving Weyl fermions assembled into Majorana spinors \( \Psi^T = (\psi_i^T, (\bar{\psi}^j)^T) \) \([10]\). Yet another complication is the identities that the gauge generators must satisfy, in order for a theory with scalar and Yukawa interactions to be gauge-invariant:

\[
0 = (T^p)^T_{ik} y_{akj} + y_{aik} T^p_{kj} + S^p_{ba} y_{bij} \\
0 = S^p_{ae} \lambda_{ebcd} + S^p_{be} \lambda_{acde} + S^p_{ce} \lambda_{abed} + S^p_{de} \lambda_{abcde} \tag{2.4}
\]

These identities relate various gauge-dependent tensor structures, leading to redundancies; one must therefore reduce the set of tensors in each \( \beta \)-function to a basis. Taking all this into account, the construction of \( \bar{A} \) for a general four-dimensional theory with a single gauge group has been done at four loops, using a diagrammatic representation of the tensor couplings to express the \( \beta \)-functions as a sum over tensor structures, each multiplied

\(^3\)Extension to multiple U(1)s requires an extended treatment due to kinetic mixing \([13]\); such treatment may be implemented by uniting the U(1)s into a single object with a matrix coupling \( G \equiv G_{\alpha\beta} \) \([14]\).
Figure 1: Tensor structures related to the ambiguous treatment of $\gamma_5$ in the Standard Model.

by a coefficient$^4$ [10]. The authors extracted scheme-independent relations between the coefficients of $\beta^{(1)}_\lambda \equiv \beta_{abcd}$, $\beta^{(2)}_Y \equiv \beta_{aij}$, and $\beta^{(3)}_g$, and used $\overline{\text{MS}}$ results to show how one could deduce many of the coefficients in $\beta^{(3)}_g$ without explicit calculation. By expressing the gauge couplings as entries in a diagonal matrix $G_k^{\alpha\beta} \equiv \text{diag}(g^k_1, \ldots, g^k_n)$, representing $\beta_{G}$ as a two-point tensor, and adopting the convention that contracted gauge lines sum over all gauge couplings and associated generators, we have extended the notation of [10] to a general semi-simple gauge group, whereby a dot on a gauge line with label $k$ represents the new coupling matrix $G_k^{\alpha\beta}$, and have written a bespoke Mathematica procedure to automate the generation of consistency conditions in the same manner [15].

The diagram of interest in $\beta^{(4)}_G$ is given in Fig. 1a, and indeed such a diagram appears in the basis of terms generated by our program. In line with our extended notation, it is easy to see that $\tilde{A}$ will receive a contribution to the diagram in Fig. 1b by contracting $\beta^{(4)}_G$ with the leading-order tensor $T^{(1)}_{GG} = T^{(1)}_G G^{-2} \delta_{\gamma_5}$, at which point certain special features become obvious. Fig. 1b is in fact topologically equivalent to a cube, hence has no subdiagrams in the form of a subdivergence, and therefore has no other contributions from higher-order terms in $T_{GJ}$. Similarly, the basis of terms in the general three-loop Yukawa $\beta$-function $\beta^{(3)}_Y$ contains the tensor shown in Fig. 1c, so $\tilde{A}$ receives the same contribution by contracting $\beta^{(3)}_Y$ with the leading-order tensor $T^{(2)}_{YY} = T^{(2)}_Y \delta_{YY}$, and there are again no possible higher-order contributions from $T_{YJ}$. Consequently, (1.6) and (1.7) imply that, if $b^{(4)}$ is the coefficient of Fig. 1a and $y^{(3)}$ the coefficient of Fig. 1c, then the coefficient $a^{(5)}$ of Fig. 1b must satisfy

$$4a^{(5)} = T^{(1)}_G b^{(4)} = T^{(2)}_Y y^{(3)} \quad (2.5)$$

Using the leading-order calculations of $T_{IJ}$ in [11], we obtain the desired scheme-independent consistency condition

$$y^{(3)} = 12b^{(4)} \quad (2.6)$$

$^4$Scheme-dependence of the $\beta$-functions then simply corresponds to changes in these coefficients.

$^5$In [12], the topologically-equivalent case of constructing $\tilde{A}$ for six-dimensional $\phi^3$ theory demonstrated the exact same behaviour, whereby the only contribution to $A^{(5)}_s$ came from the tensor $g^{(3k)}_{(3d)}$ contracted with the leading-order $T^{(2)}_{2g}$. 


We have, of course, used our program to generate the full set of consistency conditions for a completely general theory, and found precisely this condition \[ (2.6) \].

### 3 Standard Model \( \beta \)-functions and \( \gamma_5 \)

The consistency condition \((2.6)\) relates two tensor structures that may receive non-trivial contributions from integrals involving \( \gamma_5 \), and holds for a completely general renormalizable theory with a semi-simple gauge group. The Standard Model is, of course, precisely such a theory, and so by inserting the SM matter content we may extract relations between various terms in the SM \( \beta \)-functions. As indicated in [3–8], the SM matter content is such that integrals involving \( \gamma_5 \) only contribute to these tensors, and so \((2.6)\) directly relates the ambiguous treatment in \( \beta^{(4)}_{\alpha_S} \) to the semi-naïve treatment in \( \beta^{(3)}_Y \). By considering the set of tensor structures in the general \( \beta^{(3)}_{a_{i,j}} \) that contain four generators, a trace over two Yukawa tensors, and an additional untraced Yukawa tensor, we can extract the \( \overline{\text{MS}} \) coefficient \( y^{(3)} \) by using the results in appendix D of [3] and matching with the SM calculations in [6]:

\[
y^{(3)} = 12 + 72 \zeta_3 \tag{3.1}
\]

Equation \((2.6)\) then requires that

\[
b^{(4)} = 1 + 6 \zeta_3 \tag{3.2}
\]

so expanding out the tensor structure in Fig. 1a and multiplying by \((3.2)\) gives

\[
\beta^{(4)}_{\alpha_S} \supset (16 + 96 \zeta_3) T_T^2 \alpha_S^3 \alpha_I^2 \tag{3.3}
\]

By comparing with \((1.5)\), we are therefore forced to take

\[
R = 3 \tag{3.4}
\]

in the \( \beta^{(4)}_{\alpha_S} \) calculation of [7, 8], corresponding to a reading of the traces whereby one insert \( \gamma_5 \) at any of the internal vertices. While [7] gave some theoretical justifications for preferring this value of \( R \), we believe this constitutes the first proof that it must be so. We stress that there is no wiggle-room in the conclusion: \((2.6)\) relates the final \( \beta \)-function coefficients after removal of the regulator, and holds for all perturbative renormalization schemes, thus the four-loop integral involving \( \gamma_5 \) must be treated in this manner.

The topological argument guaranteeing that no higher-order \( T_{1,1} \) contributions influence the consistency condition can easily be extended to higher loops: if the tensor structure in \( \tilde{A}^{(n)} \) is topologically equivalent to a connected symmetric graph\(^6\), and the associated

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\(^6\)A symmetric graph generally refers to a graph with a set number of edges connected to each vertex, such that the automorphism group acts transitively on both the associated vertex- and edge-graph; a connected symmetric graph is then a symmetric graph with no disconnected vertices or subgraphs. Due to the multiple interaction types, the graph topologies that contribute to the \( A \)-function and lead to a simple consistency condition like \((2.6)\) are more general - we are unaware of a classification scheme for all such topologies, but the connected symmetric graphs form a well-defined subset.

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primitive tensors in $\beta^{(n-1)}_G$, $\beta^{(n-2)}_{aij}$ and/or $\beta^{(n-3)}_{abcd}$ contain non-trivial contributions from $\gamma_5$, then one can quickly derive an analogous consistency condition to fix the potential ambiguity, as parametrized by the same trace-cutting procedure used at four loops. It may of course be possible that, for a particular theory, $\gamma_5$ does not contribute to the terms in these simple conditions. If this is so, it is still possible to use the full set of consistency conditions to infer a consistent treatment, although the amount of work required will be dramatically increased.

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