Optimization of traffic control in $\text{MMAP}[2]/\text{PH}[2]/S$ priority queueing model with $\text{PH}$ retrial times and the preemptive repeat policy

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Abstract

The presented study elaborates a multi-server priority queueing model considering the preemptive repeat policy and phase-type distribution ($\text{PH}$) for retrial process. The incoming heterogeneous calls are categorized as handoff calls and new calls. The arrival and service processes of both types of calls follow marked Markovian arrival process ($\text{MMAP}$) and $\text{PH}$ distribution with distinct parameters, respectively. An arriving new call will be blocked when all the channels are occupied, and consequently will join the orbit (virtual space) of infinite capacity to retry following $\text{PH}$ distribution. When all the channels are occupied and a handoff call arrives at the system, out of the following two scenarios, one might take place. In the first scenario, if all the channels are occupied with handoff calls, the arriving handoff call will be lost from the system. While in the second one, if all the channels are occupied and at least one of them is serving a new call, the arriving handoff call will be provided service by using preemptive priority over that new call and the preempted new call will join the orbit. Behaviour of the proposed system is modelled by the level dependent quasi-birth-death ($\text{LDQBD}$) process. The expressions of various performance measures have been derived for the numerical illustration. An optimization problem for optimal channel allocation and traffic control has been formulated and dealt by employing appropriate heuristic approaches.

Keywords — Channel Failure, LDQBD Process, Markovian Arrival Process, Preemptive Repeat Priority Policy, Phase-Type Distribution, Particle Swarm Optimization, Retrial Queue, Simulated Annealing.

1 Introduction

In modern wireless cellular networks, priority queueing models are effectively used in various applications where the prioritization of traffic is essential, e.g., priority of voice traffic is ensured for voice and data transmission in multiprocessor switching. Such prioritization of traffic can be classified depending on the various features of the
system, e.g., arrival discipline, service discipline, types of services, etc. In the literature, two types of priority queueing models (PQMs) based on the service discipline, named as the non-preemptive priority queueing model (NPPQM) and the preemptive priority queueing model (PPQM), have been discussed (refer, [3, 6, 22, 24], etc.). In the NPPQM, the service of a lower priority traffic is not interrupted by the arrival of a higher priority traffic. Whereas, in the PPQM, the service of an ongoing lower priority traffic is terminated by the arrival of a higher priority traffic. This higher priority traffic gets the service in place of the preempted lower priority traffic. In this work, the incoming traffic is categorized into handoff calls and new calls where preemptive priority is assigned to handoff calls over new calls. The preempted new call joins the orbit and retries for the service. The preemptive policy may be distinguished as the preemptive repeat priority policy or as the preemptive resume priority policy.

In this study, the preemptive repeat priority policy has been adopted for the prioritization of handoff calls over new calls. An overview of research works on the preemptive repeat PQM can be found in the articles (refer, [1, 6, 9, 13], etc.). In these studies, challenges arising due to the consideration of the preemptive repeat policy have been emphasized. However, the applicability of these models has been diminished in the present scenario, since the incoming call arrival follows Poisson process and service times is considered exponentially distributed.

In the cutting edge wireless technologies, the input flow of calls possesses burstiness and correlation properties rather than the memory-less property of stationary Poisson flow. Despite the enormous number of single/multi server PQMs, the literature devoted to the analysis of such models with the consideration of more general arrival and service processes, e.g., Markovian arrival process (MAP), marked Markovian arrival process (MMAP) and phase-type (PH) distribution, appears rather moderate. Some of the relevant studies are as follows. He and Alfa [16] derived stationary distribution for a single-server queueing system following MMAP for the arrival process and different PH distributions for the service of distinct classes of customers. He and Li [17] obtained stability conditions for a single-server preemptive repeat PQM by assuming MMAP arrival and general distributed service times for distinct classes of customers. Sun et al. [27] presented a MAP + MAP/M2/S queueing model with infinite buffer considering the preemptive repeat priority policy and reservation policy. The consideration of exponential distribution for service times restricts the applicability of the model in real life scenario. Recently, Klimenok et al. [21] proposed a MMAP[2]/PH[2]/S priority queueing model and analyzed the presented system without considering the retrial behavior of the customers.

Though, there exist a vast literature over the PPQM with single/multi server, yet a handful studies have shown the impact of retrial phenomenon over the PPQM with generalized arrival and service processes. For a detailed survey of studies over retrial queueing models, readers may refer [19] and references cited herein. Some of the relevant literature for the PPQM with retrial phenomenon is discussed here. Dudin et al. [10] proposed a multi-server retrial queueing system following MMAP for the arrival of customers with the preemptive repeat priority policy under a random environment. They confined the model considering exponentially distributed service and retrial times. Dudin et al. [11] constructed a retrial PPQM considering MMAP and PH distribution for the arrival and service processes, respectively. Their objective was to analyze the model by considering the channel sharing scheme in cognitive radio network. Though, the above mentioned studies considered generalized arrival and service processes yet retrial process is explained by exponential distribution only. In wireless cellular networks, the inter-retrial times are notably brief in comparison to the service times. Since, the retrial attempt is just a matter of pushing one button, these retrial customers will make numerous attempts during any given
service interval. Therefore, the consideration of exponential retrial times in place of non-exponential ones could lead to under or over estimating the system parameters as shown by various studies in the literature (refer, \[5, 8, 26\], etc.). Therefore, to obtain realistic performance measures for retrial phenomenon, a more generalized approach, i.e., \(PH\) distribution, has been applied in the proposed model.

In this work, a \(\text{MMAP}[2]/\text{PH}[2]/S\) model with \(PH\) distributed retrial times is introduced. To the best of authors’ knowledge, the proposed model is the first one that deals with such complex system considering the preemptive repeat priority policy. The arrival and service processes for both types of calls are described by applying \(\text{MMAP}\) and \(PH\) distributions with different parameters, respectively. The new call, which finds all the channels busy upon its arrival will join the orbit (virtual space) of infinite capacity and will be referred as a retrial call \([13]\). The retrial call following \(PH\) distribution can either retry for service or exit the system without obtaining the service. If all the channels are occupied at the arrival epoch of a handoff call, out of the following two cases, one might occur. In the first case, the arriving handoff call will be lost from the system when all the channels are occupied with the handoff calls. In the second case, the handoff call will be provided preemptive priority over the ongoing new call when at least one of the channel is occupied with that new call. The handoff call commenced its service in place of the preempted new call and this preempted new call joins the orbit. The underlying process of the system is modelled by level dependent quasi-birth-death \((LDQBD)\) process \([4]\). Further, a matrix analytic algorithm, proposed by Baumann and Sandmann \([2]\), is applied for the analysis of the proposed model. The detailed study over \(LDQBD\) process can be found in \([15]\) and \([23]\). Due to the consideration of the preemptive repeat priority policy, the dropping probability for handoff calls in the system decreases and simultaneously the frequent termination of services for new calls increases the probability of preemption. Both types of probabilities (say, loss probabilities) are majorly affected by the arrival of handoff calls. Thus, an optimal value of handoff call arrival rate is estimated in order to obtain a minimum value of the total number of channels in such a way that the dropping probability and preemption probability should not exceed some pre-defined values. On the basis of the above mentioned assumptions, an optimization problem has been proposed and solved by applying appropriate heuristic algorithms.

The layout of this work is arranged in seven sections. In Section 2, a \(\text{MMAP}[2]/\text{PH}[2]/S\) model with \(PH\) distributed retrial times is described. In Section 3, the infinitesimal generator matrix for the proposed \(LDQBD\) process has been derived and steady-state probabilities has been computed through matrix analytic algorithm. In Section 4, formulas of key performance measures to analyse the system efficiency are derived explicitly. Numerical illustrations to point out the impact of various intensities over the system performance are presented in Section 5. An optimization problem has been formulated to evaluate the behaviour of the system in Section 6. Finally, the underlying model is concluded with the insight for the future works in Section 7.

## 2 Model Description

This work considers a multi-server \(PQM\) with the preemptive repeat policy and \(PH\) distributed retrial process. All the other assumptions are described as follows.

- **Arrival Process:**

  The arrival of a handoff call/new call follows \(\text{MMAP}\). The arrival in \(\text{MMAP}\) is directed by the underlying process \(\{\nu_t, t \geq 0\}\), which is an irreducible continuous time Markov chain with the state space of dimension
Fig. 1: A multi-server MMAP[2]/PH[2]/S model with PH retrial times and the preemptive repeat priority policy.

The MMAP is defined by the matrices $C_0, C_N, C_H$; where $C_0$ represents the rates of transitions due to the occurrence of no arrival. The rates of transitions accompanied by arrival of a new call and a handoff call are described by $C_N$ and $C_H$, respectively. The steady-state vector $\pi$ is the unique solution to the system $\pi C = 0$, $\pi e = 1$, where $C = C_0 + C_N + C_H$. Here 0 is a row vector consisting of 0’s of appropriate dimension and $e$ is a column vector consisting of 1’s. The fundamental arrival rates of handoff and new call are given by $\lambda_H = \pi C_H e$ and $\lambda_N = \pi C_N e$, respectively. The total fundamental arrival rate is $\lambda = \lambda_H + \lambda_N$. For more details over MMAP, authors suggest to refer [13].

- Service Process:

  The system provides different types of services to handoff calls and new calls which follow PH distributions with distinct parameters. The service times of a handoff call following PH distribution has $(\beta_H, A_H)$ representation with dimension $M_H$, i.e., $A_H e + A_0^H = 0$. The average service rate of a handoff call is given by $1/\mu_H = -\beta_H(A_H)^{-1} e$. Similarly, the service times of a new call follows PH distribution with representation $(\beta_N, A_N)$ of dimension $M_N$, i.e., $A_N e + A_0^N = 0$. The average service rate of a new call is given by $1/\mu_N = -\beta_N(A_N)^{-1} e$.

- Retrial Process:

  The retrial times of a retrial call is PH distributed with representation $(\gamma, \Gamma)$ and dimension $N$, i.e., $\Gamma e + \Gamma^0(1) + \Gamma^0(2) = 0$. Here $\Gamma^0(1)$ shows the absorption due to the departure from the cell and $\Gamma^0(2)$ denotes the absorption due to the retrial attempt. The average retrial rate is given by $1/\theta = -\gamma(\Gamma)^{-1} e$. 
3 Mathematical Analysis

The underlying process \( \{ \Xi(t), t \geq 0 \} \) for a cell is defined by the following state space:

\[
\Omega = \{(l, \kappa, j, v, s_3, s_2, r); l \geq 0, \ 0 \leq \kappa \leq S, \ 0 \leq j \leq S, \ 1 \leq v \leq L \},
\]

where,

- \( l \) is the number of retrial calls,
- \( \kappa \) is the number of busy channels,
- \( j \) is the number of handoff calls in the system receiving service,
- \( v \) is the current phase of MMAP,
- \( s_N = \{(s_3^N, s_2^N, \ldots, s_j^N); 1 \leq s_j^N \leq M_N; 1 \leq \nu_1 \leq i \} \), where the \( \nu_1^{th} \) handoff call, out of \( j \) number of handoff calls, is being served in phase \( s_j^N \),
- \( s_N = \{(s_3^N, s_2^N, \ldots, s_j^{N-1}); 1 \leq s_j^{N-1} \leq M_N; 1 \leq \nu_2 \leq \kappa - j \} \), where the \( \nu_2^{th} \) new call, out of \( \kappa - j \) number of new calls, is being served in phase \( s_j^{N-1} \),
- \( r = \{(r^1, r^2, \ldots, r^\nu); 1 \leq r^h \leq N; 1 \leq h \leq \ell \} \), where the \( h^{th} \) retrial call in the orbit is in phase \( r^h \).

The stochastic process \( \{ \Xi(t), t \geq 0 \} \) can be modelled as \( LDEQD \) process with the tri-diagonal infinitesimal generator matrix provided as follows:

\[
\mathcal{L} = \begin{pmatrix}
\mathcal{D}_{0,0} & \mathcal{D}_{0,1} & 0 & 0 & 0 \\
\mathcal{D}_{1,0} & \mathcal{D}_{1,1} & \mathcal{D}_{1,2} & 0 & 0 \\
0 & \mathcal{D}_{2,1} & \mathcal{D}_{2,2} & \mathcal{D}_{2,3} & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
\]

We define the following notations in order to carry out the analysis:

- \( I_N \) is an identity matrix of dimension \( u \).
- \( O \) is a zero matrix of appropriate dimension.
- \( O(A) \) denotes the order of matrix \( A \).
- \( \Psi_N(\kappa - j) = A_N \oplus A_N \oplus \ldots \oplus A_N \) represents that \( \kappa - j \) number of new calls are receiving services.
- \( \Psi_N(j) = A_N \oplus A_N \oplus \ldots \oplus A_N \) represents that \( j \) number of handoff calls are receiving services.
- \( \Psi_{orbit}(l) = \Gamma \oplus \Gamma \oplus \ldots \oplus \Gamma \) represents that \( l \) number of new calls are retrying.
- \( \hat{\Psi}_{orbit}(l) = (I^0(2) \gamma) \oplus (I^0(2) \gamma) \oplus \ldots \oplus (I^0(2) \gamma) \) represents that \( l \) number of new calls are having unsuccessful retrial attempts.
- \( \Phi_N(\kappa - j) = \sum_{y=0}^{\kappa - 1} I_{M_N^y} \oplus A_N \oplus I_{M_N^{\kappa - 1 - y} - y} \) represents that any one out of the \( \kappa - j \) number of new calls has completed the service.
- \( \Phi_N(j) = \sum_{y=0}^{j - 1} I_{M_N^y} \oplus A_N \oplus I_{M_N^{j - 1} - y} \) represents that any one out of the \( j \) number of handoff calls has completed the service.
• $\Phi_{\text{orbit}}(l + 1) = \sum_{y=0}^{l} I_{N^y} \otimes I^{\text{0}}(1) \otimes I_{N^l-y}$ represents that any one out of the $l$ number of new calls leaves the orbit as well as the cell without getting connected.

• $\hat{\Phi}_{\text{orbit}}(l + 1) = \sum_{y=0}^{l} I_{N^y} \otimes (I^{\text{0}}(2) \otimes \beta_N) \otimes I_{N^l-y}$ represents that any one out of the $l$ number of new calls is getting service after its successful retrial attempt.

Note that notations $\oplus$ and $\otimes$ are used for the Kronecker sum and the Kronecker product of two matrices, respectively. For more description over Kronecker sum and Kronecker product, refer [7].

The intensities of the upper diagonal of the $\mathcal{D}$ matrix represent the scenario when one new call joins the orbit due to the non availability of idle channels. The intensities of the lower diagonal show the loss of one retrial call either due to the successful retrial or due to the departure from the orbit without obtaining the service. The main diagonal represents transitions due to the arrival or service of handoff calls and new calls or the transitions of a retrial call from one phase to another phase. The number of retrial calls is not changed during these transitions. The block matrices are defined as follows.

**Upper Diagonal :**

$$ \mathcal{D}_{l, l+1} = \text{diag}\{X_i(0), X_i(1), \ldots, X_i(S)\}; \quad l \geq 0,$$

$$X_i(\kappa) = 0; \quad \kappa = 0, S - 1,$$

$$X_i(S) = \begin{pmatrix} X_i(S, 0) & \tilde{X}_i(S, 0) & 0 & \cdots & 0 & 0 \\ 0 & X_i(S, 1) & \tilde{X}_i(S, 1) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & X_i(S, S-1) & \tilde{X}_i(S, S-1) \\ 0 & 0 & 0 & \cdots & 0 & X_i(S, S) \end{pmatrix},$$

$$X_i(S, j) = (C_N \otimes I_{M_N^1 M_N^{S-1-1}}) \otimes \gamma_j; \quad \forall l \geq 0, \quad j = 0, S,$$

$$\tilde{X}_i(S, j) = (C_N \otimes I_{M_N^1}) \otimes \beta_R \otimes I_{M_N^{S-1-1}} \otimes \delta(M_N) \otimes I_{N^l} \otimes \gamma_j; \quad \forall l \geq 0, \quad j = 0, S - 1.$$

**Lower Diagonal :**

$$\mathcal{D}_{l+1, l} = \begin{pmatrix} Z_i(0) & \hat{Z}_i(0) & 0 & \cdots & 0 & 0 \\ 0 & Z_i(1) & \hat{Z}_i(1) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & Z_i(S - 1) & \hat{Z}_i(S - 1) \\ 0 & 0 & 0 & \cdots & 0 & Z_i(S) \end{pmatrix}; \quad l \geq 0,$$

$$Z_i(\kappa) = \text{diag}\{Z_i(\kappa, 0), Z_i(\kappa, 1), \ldots, Z_i(\kappa, \kappa)\}; \quad \kappa = 0, S, \quad O(Z_i(\kappa)) = (\kappa + 1) \times (\kappa + 1),$$

$$\hat{Z}_i(\kappa) = I_{L_{M_N^1 S_N}} \otimes \Phi_{\text{orbit}}(l + 1); \quad \forall l \geq 0, \quad \kappa = 0, S, \quad j = 0, \kappa,$$

$$\hat{Z}_i(\kappa) = \text{diag}\{\hat{Z}_i(\kappa, 0), \hat{Z}_i(\kappa, 1), \ldots, \hat{Z}_i(\kappa, \kappa - 1)\}; \quad \kappa = 0, S - 1, \quad O(\hat{Z}_i(\kappa)) = (\kappa + 1) \times (\kappa + 2),$$

$$\hat{Z}_i(\kappa) = I_{L_{M_N^1 S_N}} \otimes \Phi_{\text{orbit}}(l + 1); \quad \forall l \geq 0, \quad \kappa = 0, S - 1, \quad j = 0, \kappa.$$

**Main Diagonal :**
Since the structure of the generator matrix is complex, a compact analytical form of the steady-state distributions is strenuous to achieve. Therefore, an algorithmic procedure is adopted to solve the proposed matrix of \( 1 \times \) form.

Let \( \Psi(s) = \sum_{j=0}^{\infty} \Psi(s,j) = \Psi(s) \Xi(s) \Xi(s)^{-1} \), called rate matrices, be the steady-state probability vector of generator matrix \( \Psi(s) = \Psi(s) \Xi(s) \Xi(s)^{-1} \). Here, element \( \Psi(s,j) \) contains

\[
O(\Psi(s,j)) = (k + 1) \times (k + 2),
\]

\[
\Psi(s,j) = (C_N \otimes I_{M_t} \otimes \beta_N) \otimes I_{N_t}; \quad \forall \ell \geq 0, \quad \kappa = 0, S - 1, \quad j = 0, \kappa,
\]

\[
\Psi(s,j) = (C_N \otimes I_{M_t} \otimes \beta_N) \otimes I_{N_t}; \quad \forall \ell \geq 0, \quad \kappa = 0, S - 1, \quad j = 0, \kappa,
\]

\[
Y(l) = \text{diag} \{Y(l,0), Y(l,1), \ldots, Y(l,k)\}; \quad \kappa = 0, S, \quad O(Y(l)) = (k + 1) \times (k + 1),
\]

\[
Y(l,j) = C_0 \otimes \Psi_N(j) \otimes \Psi_N(S-j) \otimes \Psi_{arb}(l) + I_{LM_t} \otimes \hat{\Psi}_{arb}(l); \quad \forall \ell \geq 0, \quad \kappa = 0, S - 1, \quad j = 0, S - 1,
\]

\[
Y(l,j) = C_0 \otimes \Psi_N(j) \otimes \Psi_N(S-j) \otimes \Psi_{arb}(l) + I_{LM_t} \otimes \hat{\Psi}_{arb}(l); \quad \forall \ell \geq 0, \quad \kappa = 0, S - 1, \quad j = 0, S - 1,
\]

Since the structure of the generator matrix is complex, a compact analytical form of the steady-state distributions is strenuous to achieve. Therefore, an algorithmic procedure is adopted to solve the proposed LDQBD process.

### 3.1 Steady-State Analysis

Let \( z_s = \{z_s(0), z_s(1), z_s(2), \ldots, z_s(M - 1), z_s(M), \ldots \} \) be the steady-state probability vector of generator matrix \( \Psi(s) = \Psi(s) \Xi(s) \Xi(s)^{-1} \). Here, element \( z_s(j) \) contains

\[
1 \times \sum_{k=0}^{S} \sum_{j=0}^{\infty} L_{M_t} \otimes M_{N_t}^{-1} N_t \text{ vector components, } z_s(j) = \{z_s(j,0), z_s(j,1), z_s(j,2), \ldots, z_s(j,k)\}; \quad z_s(j,k) \text{ consists of } 1 \times \sum_{k=0}^{S} \sum_{j=0}^{\infty} L_{M_t} \otimes M_{N_t}^{-1} N_t \text{ vector components, } z_s(j,k) = \{z_s(j,k,0), z_s(j,k,1), z_s(j,k,2), \ldots, z_s(j,k,k)\}; \quad z_s(j,k,i) \text{ is } 1 \times \sum_{k=0}^{S} \sum_{j=0}^{\infty} L_{M_t} \otimes M_{N_t}^{-1} N_t \text{ vector; } \ell \geq 0, 0 \leq \kappa \leq S, 0 \leq j \leq k.
\]

According to Neuts [23], \( z_s \) satisfies the matrix-geometric relationship \( z_s(j+1) = z_s(j)R^{(j)}; \quad \ell \geq 0, \) where the family of the matrices \( \{R^{(j)}; \ell \geq 0\} \), called rate matrices, are the minimal non-negative solutions to the following
system of equations

\[ \mathcal{D}_{l,l+1} + \mathcal{R}^{(l)} \mathcal{D}_{l,l} + \mathcal{R}^{(l)} (\mathcal{D}_{l,l+1}^{(l+1)}) = 0; \quad l \geq 0, \]

(1)

and \( z_s(0) \) is the solution of the equation \( z_s(0)(\mathcal{D}_{0,0} + \mathcal{R}^{(0)} \mathcal{D}_{1,0}) = 0 \). For the numerical computation, the infinite generator matrix \( \mathcal{D} \) is truncated up to a large finite number, say \( M \). Once the truncation level \( M \) is computed, the original system with infinite orbit size is approximated by the system with orbit size \( M \). Therefore, the system will have a unique stationary distribution. Further, a matrix analytic algorithm, provided by Baumann and Sandmann [2], has been applied to compute the steady-state probability vector of the proposed LDQBD system.

Steps of the algorithm for computing \( z_s \) will be as follows:

1. Define \( \mathcal{R}^{(M)} = 0 \). Choose \( M \) a large finite number such that \( ||\mathcal{R}^{(M)} - \mathcal{R}^{(M-1)}||_\infty \leq \epsilon \), where \( \epsilon \) is a pre-defined positive value.
2. For \( l = M, M-1, \ldots, 1, \) compute and store \( \mathcal{R}^{(l-1)} = -\mathcal{D}_{l-1,l} (\mathcal{D}_{l,l} + \mathcal{R}^{(l)} \mathcal{D}_{l,l+1})^{-1} \).
3. Determin a solution of \( x_s(0)(\mathcal{D}_{0,0} + \mathcal{R}^{(0)} \mathcal{D}_{1,0}) = 0 \).
4. For \( l = M-1, M-2, \ldots, 1, 0 \), compute \( x_s(l+1) = x_s(l) \mathcal{R}^{(l)} \).
5. By normalizing \( x_s = (x_s(0), x_s(1), \ldots, x_s(M)) \), determine \( z_s \), i.e.,

\[ z_s = \frac{x_s}{c}, \quad \text{where} \quad c = \sum_{l=0}^{M} ||x_s||, \]

where \( ||.|| \) is the row sum norm.

### 4 Performance Measures

The following relevant performance measures for the proposed system are calculated, after computing the steady-state distribution \( z_s \).

1. The dropping probability of a handoff call:

\[ P_d = \frac{1}{\lambda_N} \left( \sum_{l=0}^{M-1} z_s(l, S, S) \times C_{N\epsilon} \right). \]

2. The blocking probability of a new call:

\[ P_b = \frac{1}{\lambda_N} \left( \sum_{l=0}^{S} z_s(M-1, S, l) \times C_{N\epsilon} \right). \]

3. The probability that there are \( j \) number of handoff calls receiving service:

\[ P_{c}(j) = \sum_{l=0}^{M-1} \sum_{\kappa=1}^{S} z_s(l, \kappa, i) \epsilon. \]

4. The probability that there are \( j' \) number of new calls receiving service:

\[ P_{n}(j') = \sum_{l=0}^{M-1} \sum_{\kappa=1}^{S} z_s(l, \kappa, \kappa - j') \epsilon. \]
5. Expected number of handoff calls receiving service:

\[ E_H = \sum_{j=1}^{S} jP_{H}(j)e. \]

6. Expected number of new calls receiving service:

\[ E_N = \sum_{j'=1}^{S} j'P_{N}(j')e. \]

7. The probability that there are \( l \) number of retrial calls:

\[ P_{\text{orbit}}(l) = \sum_{\kappa=0}^{S} \sum_{j=0}^{\kappa} \mu_{z}(l, \kappa, j)e. \]

8. Expected number of retrial calls:

\[ E_{\text{orbit}} = \sum_{l=0}^{M-1} lP_{\text{orbit}}(l)e. \]

9. The intensity at which both types of calls are served successfully:

\[ T_{P} = \sum_{l=0}^{M-1} \sum_{\kappa=1}^{S} \sum_{j=0}^{\kappa-1} \theta_{z}(l, \kappa, j)e. \]

10. The probability that an arriving handoff call preempts the service of an ongoing new call:

\[ P_{\text{preempt}} = \frac{1}{\lambda_{H}} \sum_{l=0}^{M-1} \sum_{\kappa=0}^{S-1} \sum_{j=0}^{\kappa-1} \theta_{z}(l, S, j) \times C_{N}e. \]

11. The intensity by which a retrial call is successfully connected to an available channel:

\[ \theta_{\text{succ}} = \sum_{l=1}^{M-1} \sum_{\kappa=0}^{S-1} \sum_{j=0}^{\kappa-1} \theta_{z}(l, \kappa, j)(e(\lambda_{N})m_{N}^{-1}) \otimes (I^{0}(2) \otimes \beta_{N})e. \]

12. The probability that a retrial call will exit the system without obtaining the service:

\[ P_{\text{no-service,orbit}} = \frac{1}{\theta_{\text{succ}}} \left( \sum_{l=1}^{M-1} \sum_{\kappa=0}^{S-1} \sum_{j=0}^{\kappa-1} \theta_{z}(l, \kappa, j)(e(\lambda_{N})m_{N}^{-1}) \otimes (I^{0}(1))e. \right) \]

13. The probability that an arriving new call will directly join the orbit:

\[ P_{\text{join,orbit}} = \frac{1}{\lambda_{N}} \left( \sum_{l=0}^{M-1} \sum_{\kappa=0}^{S} \sum_{j=0}^{\kappa} \theta_{z}(l, S, j) \times C_{N}e. \right) \]

In the next section, the behaviour of the key performance measures will be explored with respect to the several rates for the presented model.

## 5 Numerical Illustration

In this section, the qualitative behaviour of the proposed model is explored through a few experiments. It can be noticed that in the proposed model computational complexities inevitably arise for a large value of \( M \) due to the consideration of \( PH \) distributed inter-retrial times. Therefore, it is needed that the proposed model should be truncated up to a finite number \( M \). For this purpose, consider \( \epsilon = 10^{-5} \). For the numerical computation, the
matrices for the $M/MAP$ are computed as follows,

$$C_0 = \begin{pmatrix} -1.3431 & 0.0230 \\ 0 & -17.183 \end{pmatrix}, \quad C_H = \begin{pmatrix} 0.6600 & 0 \\ 0.2567 & 8.3351 \end{pmatrix}, \quad C_N = \begin{pmatrix} 0.6600 & 0 \\ 0.2567 & 8.3351 \end{pmatrix}.$$

The correlation coefficients for both types of calls are $C_T^{(1)} = C_T^{(2)} = 0.2211$ and the variation coefficients for both types of calls are $C_V^{(1)} = C_V^{(2)} = 0.2211$. The arrival rates $\lambda_H = \lambda_N = 1$. Let $PH$ distributions parameters for the service rates of a handoff and a new call be

$$\beta_H = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad A_H = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, \quad \beta_N = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad A_N = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

The fundamental service rates are $\mu_H = \mu_N = 0.5$. The retrial rate of a retrial call, following $PH$ distribution, is given by the parameters

$$\gamma = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} -2 & 2 \\ 0 & -2 \end{pmatrix}, \quad \theta = 1.33.$$

To demonstrate the feasibility of the developed model, some interesting observations of the proposed system are described through the following numerical experiments. These experiments will present the behaviour of performance measures with respect to arrival, service and retrial rates.

**Experiment 1:** The objective here is to analyze the impact of arrival rate of handoff call ($\lambda_H$) and arrival rate of new call ($\lambda_N$) over the loss probabilities, i.e., dropping probability ($P_d$) and preemption probability ($P_{\text{preempt}}$).

Figures 2(a) and 2(b) represent $P_d$ and $P_{\text{preempt}}$ as functions of total number of channels $S$ and $\lambda_H$. It can be observed from Figure 2(a) that $P_d$ increases with respect to $\lambda_H$ for a fixed value of $S$. Moreover, under the same value of $\lambda_N$, $P_d$ decreases with respect to $S$. When handoff calls arrive frequently in the system, all the channels are most likely to be occupied by handoff calls and consequently extra handoff calls will be dropped. Therefore, the value of $P_d$ increases with increasing $\lambda_N$. In this scenario, if the total channels are increased in the system, more handoff call will be able to obtain the service, as a result, $P_d$ decreases. Figure 2(b) exhibits the impact of $\lambda_N$ and $S$ over $P_{\text{preempt}}$. It is seen from the graph that the values of $P_{\text{preempt}}$ for different $S$, first increase, and then decrease. The cause for this behavior of $P_{\text{preempt}}$ lies in the following explanation. When $\lambda_N$ is relatively small, an arriving handoff call often finds at least one channel available, and consequently the ongoing service of a new call is not preempted by the arriving handoff call. As $\lambda_N$ increases, the number of handoff calls also increase in the system. If an arriving handoff call finds all the channels occupied and at least one of them is serving a new call, the service of that new call will be preempted by the arriving handoff call. Hence, $P_{\text{preempt}}$ increases and reaches maximum at some value of $\lambda_N$. Further, the decreasing behaviour of $P_{\text{preempt}}$ is explained by the fact that, with the increment in $\lambda_N$, all the channels are occupied with handoff calls. Thus, the number of new calls in the service decreases and the probability that an arriving handoff call preempts the service of a new call decreases. These figures provide some advantageous results for the proposed model and these results are opted to formulate an optimization problem later on.

Figures 3(a) and 3(b) describe the behaviour of $P_d$ and $P_{\text{preempt}}$ corresponding to $\lambda_N$ and $S$. The plot 3(a) exhibits that $P_d$ is less affected by an increment in $\lambda_N$ but $P_d$ decreases with the increasing value of $S$. Whereas, an obvious increasing behaviour of $P_{\text{preempt}}$ is observed from Figure 3(b) when $\lambda_N$ increases. More interestingly, $P_{\text{preempt}}$ decreases when $S$ increases in the system. Such behaviour of $P_{\text{preempt}}$ can be explained as follows.

When the number of new calls receiving service in the system is very less, the probability of preemption for new
calls also decreases and as \( \lambda_N \) increases, \( P_{\text{preempt}} \) also increases. Though, this probability can be reduced by increasing \( S \) in the system.

**Experiment 2:** In this experiment, the impact of the service rate of handoff calls (\( \mu_H \)) is shown over \( P_d \) and \( P_{\text{preempt}} \).

Figures 4(a) and 4(b) show the decreasing behaviour of \( P_d \) and \( P_{\text{preempt}} \) as \( \mu_H \) increases in the system. Further, it can also be observed from Figures 4(a) and 4(b) that, for the fixed value of \( \mu_H \), \( P_d \) and \( P_{\text{preempt}} \) decrease as \( S \) increases. An intuitive explanation for this finding can easily be given as follows. As \( \mu_H \) increases, the calls are served with increasing rate, consequently the loss probabilities for both types of calls decrease. If \( S \) is increased in the system, the chances for both types of calls to obtain the service also increase and consequently \( P_d \) and \( P_{\text{preempt}} \) decrease.

**Experiment 3:** The main purpose of this experiment is to observe the behaviour of intensity by which a retrial call is successfully connected to an available channel (\( \theta_{\text{succ}} \)) and the probability that a retrial call will exit the system without obtaining the service (\( P_{\text{no-service leave}} \)) with respect to the retrial rate (\( \theta \)).

Figures 5(a) and 5(b) represent the behaviour of \( \theta_{\text{succ}} \) and \( P_{\text{no-service leave}} \) with respect to \( \theta \), respectively. It can be seen from the graphs when the value of \( \theta \) increases, \( \theta_{\text{succ}} \) increases. This impact can easily be explained as follows. When \( \theta \) increases, the probability of a retrial call getting a connection also increases. For the fixed value of \( \theta \), this probability is substantially greater for the large value of \( S \). The opposite impact of \( \theta \) over \( P_{\text{no-service leave}} \) can be observed from Figure 5(b). When \( \theta \) increases, the probability that a retrial call leaves the system without obtaining the service decreases. Moreover, decreasing behaviour of \( P_{\text{no-service leave}} \) is observed with respect to \( S \) for the fixed value of \( \theta \). When the number of available channels increases in the system, the probability for retrial call to obtain the service also increases. From 2(a) and 2(b) it can be observed that \( \lambda_H \) has vital impact over \( P_d \) and \( P_{\text{preempt}} \) when all other parameters of the system are fixed according to the system requirement. This observation is the main motivation for the formulation of the optimization problem illustrated in Section 6.
Fig. 3: Dependence of the dropping probability $P_d$ and preemption probability $P_{preempt}$ over arrival rate of a new call $\lambda_N$.

Fig. 4: Dependence of the dropping probability $P_d$ and preemption probability $P_{preempt}$ over service rate of a handoff call $\mu_H$. 
Fig. 5: Dependence of the intensity by which a retrial call is successfully connected to an available channel $\theta_{\text{succ}}$ and the probability that a retrial call will exit the system without obtaining the service $P_{\text{leave}}$ over retrial rate $\theta$.

6 Optimization Problem for Traffic Control

The loss probabilities are considered as performance determining factors for cellular networks. Therefore, $P_d$ and $P_{\text{preempt}}$ are considered crucial factors in the proposed model. An increment in $P_d$ as well as in $P_{\text{preempt}}$ indicates unsatisfactory level of service for the customers. Thus, it is required to find the optimal values of parameters in such a way that the loss probabilities should not exceed some pre-defined values. In Section 5, a detailed analysis of loss probabilities with respect to several parameters has been provided. It can be observed from the results that $P_d$ and $P_{\text{preempt}}$ are mostly affected by the arrival rate of handoff calls $\lambda_H$ and the total number of channels $S$. The service provider certainly cannot determine $\lambda_H$, yet an approximated value of $\lambda_H$ can be estimated in order to keep sufficient channels to provide service. Thus, a non-trivial optimization problem is proposed with the decision variables $S$ and $\lambda_H$ given as:

$$\min_S \quad S$$

subject to,  
$$P_d(S, \lambda_H) \leq \epsilon_1,$$
$$P_{\text{preempt}}(S, \lambda_H) \leq \epsilon_2,$$
$$S, \lambda_H \geq 0.$$  

Here, $\epsilon_1$ and $\epsilon_2$ are pre-defined values depending on the tolerance of the system for dropping probability and preemption probability, respectively. The above defined constraints are non-linear and highly complex. Thus heuristic approaches have been used to obtain the optimal solution such as Direct Search method ($DS$), Particle Swarm Optimization method ($PSO$) and Simulated Annealing method ($SA$). Let the objective function be denoted as $F$ and the optimal solution as $\hat{x} = (S^*, \lambda_H^*)$ for the algorithms. Consider $\epsilon_1 = 10^{-4}$ and $\epsilon_2 = 10^{-4}$ for the numerical computation of the proposed optimization problem. The detailed sensitivity analysis of the optimization problem is provided as follows.
Table 1: Optimal values of $S^*$ and $\lambda^*_N$ for different values of $\mu_N$ and $\lambda_N$ by applying DS Method

### 6.1 Direct Search Method

The DS method can be applied to non-linear, convex or non-convex optimization problems. It does not need to compute the gradient of the objective function like gradient based optimization methods \cite{25}. Due to the easy implementation and computation, this method is used widely to obtain the optimal solution for any type of optimization problem. The algorithm for this method works as follows:

**Algorithm 1:**

- **Step 1:** Fix $\lambda_N$, $\mu_N$, $\mu_S$, $\theta$, $\epsilon_1$ and $\epsilon_2$. Initialize $S = 2$.
- **Step 2:** Find the values of $\lambda_1^*_N$ and $\lambda_2^*_N$ which satisfy $P_d(S, \lambda_1^*_N) \leq \epsilon_1$ and $P_{preempt}(S, \lambda_2^*_N) \leq \epsilon_2$, respectively.
- **Step 3:** If $\lambda_3^*_N = \lambda_4^*_N = \lambda_5^*_N$, declare $S$ and $\lambda_3^*_N$ as optimal values. Otherwise put $S = S + 1$ and repeat Step 2.

Table 1 represents the optimal values of $S^*$ and $\lambda^*_N$ obtained by applying DS method for different values of $\mu_N$ and $\lambda_N$.

### 6.2 Particle Swarm Optimization Method

The algorithm for PSO method was developed by \cite{12}. In this method, a population of particles are explored and exploited by applying search techniques in different dimensions. Each particle has velocity and position which defines a particle’s best known local and global positions in the feasible space. The PSO method is suitable for
non-differentiable constrained and unconstrained optimization problems as it does not require the computation of the gradient. This technique can search discrete and continuous decision variables at the same time. Since, the proposed optimization problem is constrained optimization problem, penalty function approach has been applied to convert the same into an unconstrained optimization problem. The algorithm for PSO method works as follows:

Algorithm 2:

- Step 1: Set parameters of PSO, i.e., $w_{\text{min}}$, $w_{\text{max}}$, $c_1$ and $c_2$; where $w \in [w_{\text{min}}, w_{\text{max}}]$ is an inertia factor and $c_1, c_2$ are acceleration factors. Fix the parameters, $\lambda_N$, $\mu_N$, $\mu_H$, $\theta$, $\epsilon_1$ and $\epsilon_2$ for the computation of the proposed optimization problem as per the system requirement.
- Step 2: Initialize population of particles having positions $x$, velocities $V$ and maximum iterations $\text{maxite}$.
- Step 3: Convert a constrained optimization problem to an unconstrained optimization problem by using penalty function approach. Compute fitness function of particles $F_i = F(x_i), \forall i$; where $i$ is population of particles and $x = \{x^{(1)}, x^{(2)}\} = \{S, \lambda_H\}$.
- Step 4: Initialize the partial best solution $P_B = x$ and the global best solution $G_B = \arg\min\{F(x); x \in P_B\}$.
- Step 5: Set $\text{Iteration} = 1$. Generate two random numbers $U_1 = U(0,1)$ and $U_2 = U(0,1)$.
- Step 6: Update the particle’s positions and velocity as $x = x + V$ and $V = wV + c_1U_1(P_B - x) + c_2U_2(G_B - x)$.
- Step 7: Update the partial best solution $P_B$ and $G_B$.
- Step 8: Repeat Steps 5-7 until $\text{Iteration} < \text{maxite}$.
- Step 9: Obtain the optimal solution $\hat{x} = G_B$ and $F^* = F(\hat{x})$.

For various values of $\mu_N$ and $\lambda_H$, the PSO method is executed with initial 60 random generated particles, $w_{\text{max}} = 0.9$, $w_{\text{min}} = 0.4$, $c_1 = c_2 = 2$, and $\text{maxite} = 200$. Table 2 exhibits the optimal values of $S^*$ and $\lambda_H^*$ obtained by applying PSO for different combinations of $\mu_N$ and $\lambda_N$.

6.3 Simulated Annealing Method

SA is a meta-heuristic approach which has the convergence rate comparatively slower than other heuristic methods, but this method does not stuck in the pool of local optimal solutions. Thus, this method guarantees to obtain the global optimal solution successfully. This method is applicable for complex non-differentiable constrained and unconstrained optimization problems. The algorithm for SA method works as follows:

Algorithm 3:

- Step 1: Fix the parameters $\lambda_N$, $\mu_N$, $\mu_H$, $\theta$, $\epsilon_1$ and $\epsilon_2$ for the computation of the objective function. Convert a constrained optimization problem to an unconstrained optimization problem by using penalty function approach.
- Step 2: Initialize $x$ as the initial state. Generate $x' = x + \Delta x$ a neighbor state of $x$, where $x = \{x^{(1)}, x^{(2)}\} = \{S, \lambda_H\}$.
- Step 3: Compute $\Delta F(x) = F(x) - F(x')$. 
\( \text{Table 2: Optimal values of } S^* \) and \( \lambda^*_M \) for different values of \( \mu_M \) and \( \lambda_N \) by applying PSO method

- Step 4: Generate \( P = e^{-\Delta F(x)/T} \), the acceptance probability of \( x \), where \( T \), a temperature parameter, is evaluated randomly by computing the mean of different values of given objective function.
- Step 5: Generate \( R = U(0, 1) \), the acceptance probability of \( x' \).
- Step 6: If \( \Delta F(x) < 0; x \) is the actual state else if \( R > P; x' \) is the actual state.
- Step 7: Repeat Steps 2-6 until \( \Delta F(x) < 10^{-6} \).
- Step 8: Obtain the optimal solution \( \hat{x} = (S^*, \lambda^*_M) \).

Table 2 exhibits the optimal values of \( S^* \) and \( \lambda^*_M \) obtained by employing SA method for different combinations of \( \mu_S \) and \( \lambda_N \).

All the results were obtained by MATLAB software, which were run on a computer with Intel Core i7-6700 3.40GHz CPU and 16 GB of RAM. It can be observed from Table 1, Table 2 and Table 3 that PSO method and SA method are more efficient. These methods provide optimal solution for each possible combination of \( \mu_S \) and \( \lambda_N \). Though, the number of iterations in SA method are more than the number of iterations in PSO method, SA method always provide the global optimal solution. For this optimization problem both the methods are providing identical results which determines the reliability of the optimal solutions.
In the leading edge wireless technologies, traffic of different classes, e.g., video, voice, images, data, etc., are assigned different categories of importance, and consequently their services are effectuated in accordance with an appropriate priority policy. In cellular networks, the kinds of systems where a higher priority traffic has an advantage in access to service compared to less important ones, are explored through priority queueing models. Therefore, in this study a $\text{MMAP}[2]/\text{PH}[2]/S$ preemptive repeat priority queueing model with $\text{PH}$ distributed retrial times is investigated. The incoming traffic is classified into handoff calls and new calls where preemptive repeat priority is assigned to handoff calls over new calls. Due to the brief span of inter-retrial times in comparison to service times, a more generalized approach, $\text{PH}$ distributed retrial times is used so that the performance of the system is not over or under estimated. The analysis of the proposed model is implemented via investigation of steady-state behaviour of $\text{LDQBD}$ process. The expressions for the important performance measures have been derived and successfully implemented to demonstrate the influence of various rates on the performance measures of the system. Further, to prevent frequent termination of services for new calls due to the consideration of the preemptive repeat priority policy, a traffic control optimization problem has been formulated to estimate the optimal values of parameters $\lambda_S$ and $S$ such that the loss probabilities $P_d$ and $P_{\text{preempt}}$ must not exceed some

7 Conclusions
pre-defined threshold values. The proposed optimization problem has been investigated by employing DS, PSO and SA methods. The obtained identical numerical results exhibit the reliability of its optimal solutions. The presented results of the optimization problems can be used in modern wireless cellular networks where the flows of traffic might be essentially heterogeneous with respect to arrival and service processes. In the future, authors propose to extend this model by using the preemptive resume priority policy for various classes of traffic.

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