Isotropic Markov semigroups on ultra-metric spaces

A. D. Bendikov, A. A. Grigor’yan, Ch. Pittet, and W. Woess

Abstract. Let $(X, d)$ be a separable ultra-metric space with compact balls. Given a reference measure $\mu$ on $X$ and a distance distribution function $\sigma$ on $[0, \infty)$, a symmetric Markov semigroup $\{P^t\}_{t \geq 0}$ acting in $L^2(X, \mu)$ is constructed. Let $\{\mathcal{X}_t\}$ be the corresponding Markov process. The authors obtain upper and lower bounds for its transition density and its Green function, give a transience criterion, estimate its moments, and describe the Markov generator $\mathcal{L}$ and its spectrum, which is pure point. In the particular case when $X = Q_p^n$, where $Q_p$ is the field of $p$-adic numbers, the construction recovers the Taibleson Laplacian (spectral multiplier), and one can also apply the theory to the study of the Vladimirov Laplacian. Even in this well-established setting, several of the results are new. The paper also describes the relation between the processes involved and Kigami’s jump processes on the boundary of a tree which are induced by a random walk. In conclusion, examples illustrating the interplay between the fractional derivatives and random walks are provided.

Bibliography: 66 titles.

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Contents

1. Introduction 590
2. Heat semigroup and heat kernel 596
   2.1. Averaging operator 596
   2.2. Basic properties of heat semigroups 597
   2.3. Spectral distribution function 600

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1. Introduction

In the past three decades there has been an increasing interest in various constructions of Markov chains on ultra-metric spaces such as the Cantor set or the field of $p$-adic numbers. In this paper we introduce and study a class of symmetric Markov semigroups and their generators on ultra-metric spaces. Our construction is very transparent, and it leads to a number of new results as well as to a better understanding of previously known results.

Let $(X, d)$ be a metric space. The metric $d$ is called an ultra-metric if it satisfies the ultra-metric inequality

$$d(x, y) \leq \max\{d(x, z), d(z, y)\},$$

which is obviously stronger than the usual triangle inequality. In this case $(X, d)$ is called an ultra-metric space.
We will always assume in addition that the ultra-metric space \((X, d)\) in question is separable, and that every closed ball
\[
B_r(x) = \{ y \in X : d(x, y) \leq r \}
\]
is compact. The latter implies that \((X, d)\) is complete.

The ultra-metric property (1.1) implies that balls in an ultra-metric space \((X, d)\) look very different from familiar Euclidean balls. In particular, any two ultra-metric balls of the same radius are either disjoint or identical. Consequently, the collection of all distinct balls of the same radius \(r\) forms a partition of \(X\).

One of the best-known examples of an ultra-metric space is the field \(\mathbb{Q}_p\) of \(p\)-adic numbers endowed with the \(p\)-adic norm \(\|x\|_p\) and the \(p\)-adic ultra-metric \(d(x, y) = \|x - y\|_p\). Moreover, for any integer \(n \geq 1\), the \(p\)-adic \(n\)-space \(\mathbb{Q}_p^n = \mathbb{Q}_p \times \cdots \times \mathbb{Q}_p\) is also an ultra-metric space with the ultra-metric \(d_n(x, y)\) defined as
\[
d_n(x, y) = \max\{d(x_1, y_1), \ldots, d(x_n, y_n)\}.
\]

If the group of isometries of an ultra-metric space \((X, d)\) acts transitively on \(X\), then \((X, d)\) is in fact a locally compact Abelian group, which in particular is the case for \(\mathbb{Q}_p^n\).

In the literature one distinguishes the following two subclasses of ultra-metric spaces:

(i) \((X, d)\) is discrete and infinite;
(ii) \((X, d)\) is perfect (that is, \(X\) contains no isolated points).

Various constructions of Markov processes on non-compact perfect locally compact Abelian groups have been developed by Evans [22], Haran [29], [30], Ismagilov [33], Kochubei [38], [39], Albeverio and Karwowski [1], [2], Albeverio and Zhao [3], Del Muto and Figà-Talamanca [42], [43], and Rodríguez-Vega and Zúñiga-Galindo [50], [66]. They studied \(X\)-valued infinitely divisible random variables and processes by using tools of Fourier analysis; for general references, see Hewitt and Ross [31], Taibleson [55], and Kochubei [39]. Indeed, Taibleson’s spectral multipliers on \(\mathbb{Q}_p^n\) were early forerunners of the Laplacians which we consider here.

Pearson and Bellissard [45] and Kigami [36], [37] considered random walks on the Cantor set and the Cantor set minus one point, respectively. In [36], [37] the main focus is on the interplay between random walks on trees and jump process on their boundaries. In this context, we also mention Aldous and Evans [4] and Chen, Fukushima, and Ying [15]. We shall come back to Kigami’s work in the last three sections of this paper.

An entirely different approach was developed by Vladimirov, Volovich, and Zelenov [57], [59]. They were concerned with \(p\)-adic analysis (Bruhat distributions, Fourier transform, and so on) related to the concept of \(p\)-adic quantum mechanics, and they introduced a class of pseudodifferential operators on \(\mathbb{Q}_p\) and on \(\mathbb{Q}_p^n\). In particular, they considered the \(p\)-adic Laplacian defined on \(\mathbb{Q}_p^3\) and studied the corresponding \(p\)-adic Schrödinger equation. Among other results, they explicitly computed (as series expansions) certain heat kernels as well as the Green function of the \(p\)-adic Laplacian. In connection with the theory of pseudodifferential operators on general totally disconnected groups we mention here the pioneering work of Saloff-Coste [51].
Discrete ultra-metric spaces \((X, d)\) were treated by Bendikov, Grigor’yan, and Pittet in [7], the direct forerunner of the present work. Among the examples of such spaces we mention the class of locally finite groups: a countable group \(G\) is locally finite if any of its finite subsets generates a finite subgroup. Every locally finite group \(G\) is the union of an increasing sequence of finite subgroups \(\{G_n\}\). An ultra-metric \(d\) in \(G\) can be defined as follows: \(d(x, y)\) is the minimal value of \(n\) such that \(x\) and \(y\) belong to a common coset of \(G_n\).

Since locally finite groups are not finitely generated, the basic notions of geometric group theory such as the word metric, volume growth, isoperimetric inequalities, and so on (cf., for instance, [16], [28], [46]–[48], [52], [56], [61]) do not apply in this setting. The notion of an ultra-metric can be used instead of the word metric in this setting (see [5]–[7]).

Selecting a set of generators for each subgroup \(G_n\) of a locally finite group \(G\), one thereby defines a random walk, that is, a Markov kernel on \(G_n\). Taking a convex combination of the Markov kernels with respect to all the \(G_n\), one obtains a Markov kernel on \(G\) which determines a random walk on \(G\). Such random walks have been studied by Darling and Erdős [17], Kesten and Spitzer [35], Flatto and Pitt [26], Fereig and Molchanov [25], Kasymdzhanova [34], Cartwright [13], Lawler [40], and Brofferio and Woess [11] (see also Bendikov and Saloff-Coste [9]). In particular, [40] has a remarkable general criterion of recurrence for such random walks. Further results on Markov processes on ultra-metric spaces can be found in [18], [19], [23], [24], [41], [49].

Many of the results in the above-mentioned literature are subsumed by our approach via ultra-metrics. We develop tools to analyze a class of very natural Markov processes on ultra-metric spaces without assuming any group structure. In particular, the nature of our argument allows us to bring into consideration an arbitrary Radon measure \(\mu\) on \(X\) (instead of the Haar measure in the case of groups), which is used as a speed measure for a Markov process.

So, given an ultra-metric space \((X, d)\), fix a Radon measure \(\mu\) on \(X\) with full support and define the family \(\{Q_r\}_{r>0}\) of averaging operators acting on non-negative or bounded Borel functions \(f: X \to \mathbb{R}\) by

\[
Q_r f(x) = \frac{1}{\mu(B_r(x))} \int_{B_r(x)} f \, d\mu.
\]  

(1.3)

Note that \(0 < \mu(B_r(x)) < \infty\) for all \(x \in X\) and \(r > 0\). The operator \(Q_r\) has the kernel

\[
K_r(x, y) = \frac{1}{\mu(B_r(x))} \mathbf{1}_{B_r(x)}(y).
\]  

(1.4)

It is symmetric in \(x, y\) because \(B_r(x) = B_r(y)\) for any \(y \in B_r(x)\). Clearly, \(Q_r\) is a Markov operator on the space \(\mathcal{B}_b(X)\) of bounded Borel functions on \(X\), that is, \(Q_r f \geq 0\) if \(f \geq 0\) and \(Q_r 1 = 1\). Hence, \(Q_r\) extends to a bounded self-adjoint operator on \(L^2(X, \mu)\).

Let us choose a function \(\sigma\) which satisfies the following assumptions:

\[
\sigma: [0, \infty) \to [0, 1] \text{ is a strictly monotone increasing left-continuous function such that } \sigma(0+) = 0 \text{ and } \sigma(\infty) = 1.
\]  

(1.5)
Then the operator
\[ P f = \int_0^\infty Q_r f \, d\sigma(r), \]  
(1.6)
is also a Markov operator on \( B_b(X) \) as well as a bounded self-adjoint operator on \( L^2(X, \mu) \).

The operator \( P \) determines a discrete-time Markov chain \( \{X_n\}_{n \in \mathbb{N}} \) on \( X \) with the following transition rule: \( X_{n+1} \) is \( \mu \)-uniformly distributed in \( B_r(X_n) \), where the radius \( r \) is chosen at random according to the probability distribution \( \sigma \). For that reason we refer to \( \sigma \) as the distance distribution function.

Note that \( P \) is determined by the triple \( (d, \mu, \sigma) \). We refer to \( P \) as an isotropic Markov operator associated with \( (d, \mu, \sigma) \). The isotropic Markov operator \( P \) has some unique features arising from the ultra-metric property. First, there is the simple identity
\[ Q_r Q_s = Q_s Q_r = Q_{\max\{r,s\}}. \]
(1.7)
Indeed, for any ball \( B \) of radius \( r \), any point \( x \in B \) is a centre of \( B \). Since the value \( Q_r f(x) \) is the average of \( f \) in \( B \), we see that \( Q_r f(x) \) does not depend on \( x \in B \); that is, \( Q_r f = \text{const} \) on \( B \). Now if \( s \leq r \), then the application of \( Q_s \) to \( Q_r f \) does not change this constant, whence we get that \( Q_r Q_s f = Q_r f \). On the other hand, if \( s > r \), then any ball \( B \) of radius \( s \) is the disjoint union of finitely many balls \( B_j \) of radius \( r \). Then \( Q_r f = \text{const}_j \) on \( B_j \), and by a one-line computation, \( Q_r Q_s f = Q_s f \).

Since by (1.7) we get that \( Q_r^2 = Q_r \), \( Q_r \) is an orthoproyection\(^1\) on \( L^2 \). In particular, \( \text{spec} \, Q_r \subset [0, 1] \).

It follows from (1.6) that the spectral projections in the spectral decomposition of \( P \) are the averaging operators \( Q_r \), up to a change of variables (cf. (2.6)). The fact that the spectral projections are themselves Markov operators brings up a new insight, new technical possibilities, and a new type of results which have no analogue in other commonly used settings.

In particular, the Markov operator \( P \) is non-negative-definite, which allows us to define the powers \( P^t \) for all \( t \geq 0 \). Then \( \{P^t\}_{t \geq 0} \) is a symmetric strongly continuous Markov semigroup. It follows from (1.6) that \( P^t \) admits for \( t > 0 \) the following representation:
\[ P^t f(x) = \int_0^\infty Q_r f(x) \, d\sigma^t(r). \]
(1.8)
In a more elementary way, one can also define \( P^t \) by (1.8) and use the formula (1.7) to deduce that \( P^t P^s = P^{s+t} \).

The semigroup \( \{P^t\}_{t \geq 0} \) determines a continuous-time Markov process \( \{X_t\}_{t \geq 0} \). Since the \( n \)-step transition operator of the discrete-time Markov chain \( \{X_n\}_{n \in \mathbb{N}} \) is \( P^n \), we see that the discrete-time Markov chain coincides with the restriction of the continuous-time Markov process \( \{X_t\} \) to integer values of \( t \). This allows us to concentrate on the study of the continuous-time process \( \{X_t\}_{t \geq 0} \).

\(^1\)We note for comparison that the analogous averaging operator in \( \mathbb{R}^n \) is also bounded and self-adjoint, but it has a non-empty negative part of the spectrum. In particular, it is not an orthoproyection.
We refer to the Markov semigroup \( \{ P^t \}_{t \geq 0} \) defined by (1.3)–(1.8) as an isotropic semigroup, and to the process \( \{ \mathcal{X}_t \}_{t \geq 0} \) as an isotropic jump process associated with the triple \((d, \mu, \sigma)\).

Let us briefly describe the content of the present paper, which is devoted to the study of isotropic semigroups.

In §2 we construct the isotropic semigroup as mentioned above and provide explicit formulae for its heat kernel \( p(t, x, y) \) (that is, the transition density of the process \( \{ \mathcal{X}_t \} \)). As indicated above, our approach is based upon the observation that the building blocks of the operator \( P \), namely, the averaging operators \( Q_r \) of (1.3), are orthogonal projections on \( L^2(X, \mu) \), which enables us to engage the methods of spectral theory and functional calculus at an early stage.

We establish some basic properties of the heat kernel, for example, its continuity away from the diagonal, and prove upper and lower bounds in terms of \( t \) and \( d(x, y) \).

For example, in \( \mathbb{Q}_p \) with the \( p \)-adic ultra-metric \( \| x - y \|_p \) and the Haar measure \( \mu \), the most natural choice of the distance distribution function is

\[
\sigma(r) = \exp\left(-\left(\frac{p}{r}\right)^\alpha\right), \quad \alpha > 0, \tag{1.9}
\]

when the associated heat kernel admits the estimate

\[
p_t(x, y) \approx \frac{t}{(t^{1/\alpha} + \| x - y \|_p)^{1+\alpha}} \tag{1.10}
\]

for all \( t > 0 \) and \( x, y \in \mathbb{Q}_p \). Note that the estimate (1.10) is similar to the heat kernel bound for a symmetric stable process in \( \mathbb{R} \) of index \( \alpha \).

We also obtain an explicit expression for the Green function of the isotropic semigroup and provide a transience criterion in terms of the volume growth. Unlike the previously known transience criteria (cf. [40]), ours does not assume any group structure.

In §3 we are concerned with the spectral properties of the isotropic Laplacian \( \mathcal{L} \) which is the (positive-definite) generator of the isotropic semigroup. We provide a full description of the spectrum, in particular, we show that the spectrum is pure point, and list explicitly all the eigenfunctions by means of ultra-metric balls. Also, we show that the spectra of the extensions of \( \mathcal{L} \) acting in the spaces \( L^p, 1 \leq p < \infty \), do not depend on \( p \).

A striking property of the isotropic Laplacian \( \mathcal{L} \) is that, for any increasing bijection \( \psi: [0, \infty) \to [0, \infty) \), the operator \( \psi(\mathcal{L}) \) is also an isotropic Laplacian (for another distance distribution function). In particular, \( \mathcal{L}^\alpha \) is an isotropic Laplacian for any \( \alpha > 0 \). Recall for comparison that, for a general symmetric Markov generator \( \mathcal{L} \), the operator \( \mathcal{L}^\alpha \) generates a Markov semigroup only for \( 0 < \alpha \leq 1 \).

In §4 we obtain two-sided estimates of moments of the Markov process \( \{ \mathcal{X}_t \} \).

In the case when \( X \) is a locally compact group, our results apply with an arbitrary Radon measure \( \mu \) instead of the Haar measure. Some of the aforementioned questions are particularly sensitive to the choice of the measure \( \mu \), for example, the heat kernel and Green function estimates. On the other hand, the spectrum of the Laplacian and escape rate bounds do not depend on \( \mu \). These quantities depend strongly on the choice of the ultra-metric \( d \), whereas the eigenfunctions depend on both \( d \) and \( \mu \).
In §5 we compare our isotropic Laplacian with other previously known ‘differential’ operators in $\mathbb{Q}_p$ and $\mathbb{Q}_p^n$. The notion of fractional derivative $D^\alpha$ on functions on $\mathbb{Q}_p$ was introduced by Vladimirov [57] by means of the Fourier transform in $\mathbb{Q}_p$, which coincides with the operator of Taibleson [55], introduced in the quite different context of Riesz multipliers on $\mathbb{Q}_p^n$. We show that $D^\alpha$ coincides with our isotropic Laplacian $L_\alpha$ associated with the distance distribution function (1.9). In particular, this implies that the heat kernel of $D^\alpha$ satisfies the estimate (1.10). Note that previously only an upper bound for the heat kernel of $D^\alpha$ was known (cf. Kochubei [39], Chap. 4.1, Lemma 4.1). We also give a simple proof for a previously known explicit formula for the fundamental solution of $D^\alpha$.

Using the functional calculus of the operator $D^1$, we give a full description of the class of all rotation-invariant Markov generators on $\mathbb{Q}_p$, which includes but is not restricted to the isotropic Laplacians. As a consequence, we find that the class of all rotation-invariant Markov processes in $\mathbb{Q}_p$ coincides with the class of Markov processes constructed by Albeverio and Karwowski [2] using much more involved technical tools.

Next we consider ‘differential’ operators on $\mathbb{Q}_p^n$. The $p$-adic Laplace operator of Vladimirov on $\mathbb{Q}_p^n$ is defined as the direct sum of the operators $D^\alpha$ acting separately on each coordinate. Although this operator is not an isotropic Laplacian, it can be studied within our setting, which gives simple direct proofs of many results of [59], without using Fourier analysis and the theory of Bruhat distributions.

Another multidimensional generalization of $D^\alpha$ is the Taibleson operator $T^\alpha$ in $\mathbb{Q}_p^n$ which is defined by means of the Fourier transform in $\mathbb{Q}_p^n$. We show that the operator $T^\alpha$ is an isotropic Laplacian, which enables us to obtain very detailed analytic results.

In §6 we use the fact that every locally compact ultra-metric space arises as the boundary of a locally finite tree. Using that, we relate random walks\(^2\) on the tree with isotropic jump processes on its boundary. In a recent work, Kigami [36] starts with a transient nearest-neighbour random walk on a tree and constructs a naturally associated jump process on the boundary of the tree: given the Dirichlet form of the random walk on the vertex set of the tree, the boundary process is induced by the Dirichlet form which reproduces the energy (‘power’) of a harmonic function on the tree via its boundary values. This is analogous to the well-known Douglas integral [21] on the unit disk. Using this approach, [36] undertakes a detailed analysis of the process on the boundary.

Restricting attention at first to the compact case, in §7 we answer the obvious question as to how the approach of Kigami and that of the present paper are related. The relation is basically one-to-one: every boundary process induced by a random walk is an isotropic process of our setting. Conversely, we show that up to a unique linear time change, every isotropic jump process on the boundary of a tree arises from a uniquely determined random walk as the process of [36]. In addition, we explain how a boundary process on a tree transforms into an isotropic jump process on the non-compact ultra-metric space given by a punctured boundary of the tree. This should be compared with the very recent work [37].

\(^2\)Discrete-time random walks of nearest-neighbour type on a tree are very well understood (see the book [63] by Woess, Chap. 9).
Finally, in §8 we construct specific examples of the \( p \)-adic fractional derivative on the (compact) group of \( p \)-adic integers and the corresponding random walk on the associated rooted tree, as well as the random walk corresponding to the fractional derivative on the whole of \( \mathbb{Q}_p \).

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2. Heat semigroup and heat kernel

Throughout this paper, \((X,d)\) is an ultra-metric space which is separable and such that all \( d \)-balls \( B_r(x) \) are compact.

2.1. Averaging operator. Recall that for any \( r > 0 \)

\[
Q_r f(x) = \frac{1}{\mu(B_r(x))} \int_{B_r(x)} f \, d\mu
\]

is an orthoprojection on \( L^2 \equiv L^2(X, \mu) \) (cf. (1.3)), and the image of \( Q_r \) is the subspace \( \mathcal{V}_r \) of \( L^2 \) which consists of all functions taking constant values on each ball of radius \( r \).

Clearly, the family \( \{\mathcal{V}_r\}_{r>0} \) is monotone decreasing with respect to set inclusion. It follows that the limit

\[
Q_\infty := \lim_{r \to \infty} Q_r
\]

exists in the strong operator topology, and it is an orthoprojection onto \( \mathcal{V}_\infty = \bigcap_{r>0} \mathcal{V}_r \). It follows that \( \mathcal{V}_\infty \) consists of constant functions. If \( \mu(X) = \infty \), then \( \mathcal{V}_\infty = \{0\} \) and \( Q_\infty = 0 \), while in the case \( \mu(X) < \infty \) we have \( \dim \mathcal{V}_\infty = 1 \) and

\[
Q_\infty f = \frac{1}{\mu(X)} \int_X f \, d\mu. \tag{2.1}
\]

Also, let \( Q_0 := \text{id} \).

Lemma 2.1. The family \( \{Q_r\}_{r \in [0, \infty)} \) of orthoprojections is strongly right continuous with respect to \( r \).

Proof. Let us first show that \( r \mapsto Q_r \) is strongly continuous at \( r = 0 \), that is,

\[
\lim_{s \to 0^+} Q_s = \text{id}. \tag{2.2}
\]

Let \( f \) be a continuous function on \( X \) with compact support. Then for any \( x \in X \)

\[
Q_s f(x) \to f(x) \quad \text{as } s \to 0.
\]

Since the family \( \{Q_s f\}_{s \in (0,1)} \) is uniformly bounded by \( \sup |f| \) and is uniformly compactly supported, it follows by the dominated convergence theorem that

\[
\|Q_s f - f\|_{L^2} \to 0 \quad \text{as } s \to 0. \tag{2.3}
\]

Since the space of continuous functions with compact support is dense in \( L^2 \), (2.3) extends to all \( f \in L^2 \) by a standard approximation argument, whence (2.2) follows.
Next, let us prove that \( r \mapsto Q_r \) is strongly right continuous at any \( r > 0 \), that is,
\[
    s\lim_{s \to r^+} Q_s = Q_r. \tag{2.4}
\]

It suffices to show that, for any continuous function \( f \) with compact support,
\[
    \|Q_s f - Q_r f\|_{L^2} \to 0 \quad \text{as } s \to r^+. \tag{2.5}
\]

Indeed, for any \( x \in X \) the function \( r \mapsto Q_r f(x) \) is right continuous by (1.3) since balls are closed, and thus (2.5) follows by the dominated convergence theorem. \( \square \)

For any \( \lambda \in \mathbb{R} \) let
\[
    E_\lambda = \begin{cases} Q_{1/\lambda}, & \lambda > 0, \\ 0, & \lambda \leq 0. \end{cases} \tag{2.6}
\]

Note that \( E_{0^+} = Q_{\infty} \). It follows from the above properties of \( Q_r \) that the family \( \{E_\lambda\} \) of orthoprojections on \( L^2 \) is a left-continuous spectral resolution. Consequently, for any Borel function \( \varphi : [0, \infty) \to \mathbb{R} \) the integral
\[
    \int_{[0, \infty)} \varphi(\lambda) \, dE_\lambda
\]
determines a self-adjoint non-negative-definite operator, which is bounded if and only if \( \varphi \) is bounded.

2.2. Basic properties of heat semigroups. Consider now the operator \( P \) defined by (1.6) with a function \( \sigma \) as in (1.5). Observe that the integral in (1.6) converges in the strong operator topology since, for any \( f \in L^2 \),
\[
    \int_0^\infty \|Q_r f\|_{L^2} \, d\sigma(r) < \infty.
\]

On the other hand, for any \( f \in \mathcal{B}_b(X) \) the integral (1.6) converges pointwise. Moreover, in this case the function \( P f \) is continuous, because for any \( \varepsilon > 0 \) the function
\[
    x \mapsto \int_{\varepsilon}^\infty Q_r f(x) \, d\sigma(r)
\]
is locally constant and hence continuous, and it converges uniformly to \( P f(x) \) as \( \varepsilon \to 0 \).

As already observed, \( P \) is a self-adjoint operator in \( L^2 \) and \( \text{spec } P \subset [0, 1] \). In particular, for any \( t > 0 \) the power \( P^t \) is well defined. Also, let \( P^0 := \text{id} \). In the next statement we collect basic properties of \( P^t \).

**Theorem 2.2.** (a) The family \( \{P^t\}_{t \geq 0} \) is a strongly continuous symmetric Markov semigroup on \( L^2(X, \mu) \).

(b) For any \( t > 0 \) the operator \( P^t \) has the representation (1.8), that is,
\[
    P^t f = \int_{[0, \infty)} Q_r f \, d\sigma^t(r). 
\]
(c) For any \( t > 0 \) the operator \( P^t \) admits an integral kernel \( p(t, x, y) \), that is, for all \( f \in \mathcal{B}_b \) and \( f \in L^2 \)

\[
P^t f(x) = \int_X p(t, x, y) f(y) \, d\mu(y),
\]

where \( p(t, x, y) \) is given by

\[
p(t, x, y) = \int_{d(x,y) \leq \infty} \frac{d\sigma(r)}{\mu(B_r(x))}.
\]

The function \( p(t, x, y) \) is called the heat kernel of the semigroup \( \{P^t\} \). It is clear from (2.8) that \( p(t, x, y) < \infty \) for all \( t > 0 \) and \( x \neq y \), whereas under certain conditions \( p(t, x, x) \) can be equal to \( \infty \).

For \( f \in \mathcal{B}_b \) the identity (2.7) holds pointwise, that is, for all \( x \in X \), whereas for \( f \in L^2 \) it is an identity of two \( L^2 \)-functions, that is, it holds for \( \mu \)-almost all \( x \).

**Proof.** It follows from (1.6) by integrations by parts that for any \( f \in L^2 \)

\[
P f = \int_{[0, \infty)} Q_r f \, d\sigma(r) = Q_\infty f - \int_{(0, \infty)} \sigma(r) \, dQ_r f.
\]

Changing \( \lambda = 1/r \) and using (2.6), we get that

\[
P f = E_{0+} f + \int_{(0, \infty)} \sigma \left( \frac{1}{\lambda} \right) \, dE_\lambda f = \int_{[0, \infty)} \sigma \left( \frac{1}{\lambda} \right) \, dE_\lambda f,
\]

using the convention \( \sigma(\infty) = 1 \). Hence, we obtain the spectral resolution of \( P \) in the form

\[
P = \int_{[0, \infty)} \sigma \left( \frac{1}{\lambda} \right) \, dE_\lambda.
\]

It follows that

\[
P^t = \int_{[0, \infty)} \sigma^t \left( \frac{1}{\lambda} \right) \, dE_\lambda.
\]

(a) The semigroup identity \( P^t P^s = P^{t+s} \) is a straightforward consequence of (2.11), as observed in the Introduction. It remains to show that

\[
s- \lim_{t \to 0^+} P^t = \text{id},
\]

which easily follows from (2.11) because \( \sigma(1/\lambda) > 0 \) for \( \lambda \in [0, \infty) \), and therefore \( \sigma^t(1/\lambda) \to 1 \) as \( t \to 0^+ \).

(b) Reversing the argument in the derivation of (2.11) from (2.9), we get from (2.11) that

\[
P^t f = \int_{[0, \infty)} Q_r f \, d\sigma^t(r).
\]
Corollary 2.5. For all $s, t > 0$ and $\sigma$ is strictly monotone increasing (cf. (1.5)). For that theorem, it suffices to assume that $\sigma$ is monotone increasing and $\sigma(r) > 0$ for $r > 0$.

Remark 2.3. In the proof of Theorem 2.2 we have not used at full strength the fact that $\sigma$ is strictly monotone increasing (cf. (1.5)). For that theorem, it suffices to assume that $\sigma$ is monotone increasing and $\sigma(r) > 0$ for $r > 0$.

Remark 2.4. If one takes (1.8) as the definition of the operator $P^t$, then one can prove the semigroup identity $P^t P^s = P^{t+s}$ by means of (1.7). Indeed, for any given $s, t > 0$ and $f \in L^2$, we have

$$P^s P^t f(x) = \int_0^\infty d\sigma^s(r) \int_0^\infty d\sigma^t(r') Q_r Q_{r'} f(x)$$

$$= \int_0^\infty d\sigma^s(r) \int_0^\infty d\sigma^t(r') Q_{\max\{r, r'\}} f(x).$$

Let $\xi_1$ and $\xi_2$ be two independent random variables with distributions $\sigma^s$ and $\sigma^t$, respectively. Then the distribution of the random variable $\xi = \max\{\xi_1, \xi_2\}$ is $\sigma^{t+s}$. It follows that

$$P^s P^t f(x) = E(Q_{\max\{\xi_1, \xi_2\}} f(x)) = \int_0^\infty Q_r f(x) d\sigma^{t+s}(r) = P^{t+s} f(x).$$

**Corollary 2.5.** For all $x, y \in X$ and all $t > 0$, $p(t, x, y) > 0$, $p(t, x, y) = p(t, y, x)$, and

$$p(t, x, y) \leq \min\{p(t, x, x), p(t, y, y)\}. \quad (2.12)$$

**Proof.** The strict positivity of $p(t, x, y)$ follows from (2.8) and the strict monotonicity of $\sigma$.

In the integral in (2.8) we have $r \geq d(x, y)$, whence it follows that $B_r(x) = B_r(y)$ and $p(t, x, y) = p(t, y, x)$. Alternatively, the symmetry of the heat kernel also follows from the fact that $P^t$ is self-adjoint.

By (2.8)

$$p(t, x, y) = \int_{[d(x,y), \infty)} \frac{d\sigma^t(r)}{\mu(B_r(x))} \leq \int_{[0, \infty)} \frac{d\sigma^t(r)}{\mu(B_r(x))} = p(t, x, x),$$

and thus (2.12) follows. □

Note that in general, heat kernels only satisfy the estimate

$$p(t, x, y) \leq \sqrt{p(t, x, x)p(t, y, y)}.$$

The estimate (2.12) is clearly stronger, reflecting a special feature of ultra-metricity.
Corollary 2.6. For any \( t > 0 \), the function
\[
x, y \mapsto \begin{cases} 
  \frac{1}{p(t, x, y)}, & x \neq y, \\
  0, & x = y,
\end{cases}
\]
is an ultra-metric.

Proof. Let
\[
F(x, r) = \left( \int_{[r, +\infty)} \frac{d\sigma^t(s)}{\mu(B_s(x))} \right)^{-1}
\]
for \( r > 0 \), \( F(x, 0) = 0 \), and observe the following two properties of \( F \):
(a) \( r \mapsto F(x, r) \) is monotone increasing with respect to \( r \);
(b) \( F(x, r) = F(y, r) \) if \( r \geq d(x, y) \), since in this case \( B_s(x) = B_s(y) \) for all \( s \geq r \).

For any function \( F \) with these properties, \( \rho(x, y) := F(x, d(x, y)) \) is an ultra-metric, because the symmetry follows from (b), while the ultra-metric inequality (1.1) follows from (a) and (b): if \( d(x, y) \leq d(x, z) \) then
\[
\rho(x, y) = F(x, d(x, y)) \leq F(x, d(x, z)) = \rho(x, z),
\]
and if \( d(x, y) \leq d(y, z) \) then
\[
\rho(x, y) = F(y, d(x, y)) \leq F(y, d(y, z)) = \rho(y, z).
\]
\( \square \)

2.3. Spectral distribution function. For the Markov semigroup \( P \) associated with the triple \((d, \mu, \sigma)\), define the intrinsic ultra-metric \( d_* \) by
\[
\frac{1}{d_*(x, y)} = \log \frac{1}{\sigma(d(x, y))}.
\]
(2.14)
Since \( d_* \) is expressed as a strictly monotone increasing function of \( d \) which vanishes at 0, it follows that \( d_* \) is an ultra-metric on \( X \). Denote by \( B_r^*(x) \) the metric balls of \( d_* \).

Lemma 2.7. For any \( r \geq 0 \) let
\[
s = \frac{1}{\log(1/\sigma(r))}.
\]
Then the following identity holds for all \( x \in X \):
\[
B_s^*(x) = B_r(x).
\]
Consequently, the metrics \( d \) and \( d_* \) determine the same set of balls and the same topology.

Proof. We have
\[
B_s^*(x) = \{ y \in X : d_*(x, y) \leq s \}
= \{ y \in X : \sigma(d(x, y)) \leq \sigma(r) \}
= \{ y \in X : d(x, y) \leq r \}
= B_r(x),
\]
where we have used that \( \sigma \) is strictly monotone increasing. \( \square \)
Definition 2.8. For any $x \in X$ we define the spectral distribution function

$$N(x, \cdot) : [0, \infty) \to [0, \infty)$$

as

$$N(x, \tau) = \frac{1}{\mu(B_{1/\tau}^*(x))}$$  \hspace{1cm} (2.15)

(See Figs. 1, 2, and 3).

Figure 1. The graph of the function $\tau \mapsto N(x, \tau)$ in the case when $\mu(X) < \infty$.

Figure 2. The graph of the function $\tau \mapsto N(x, \tau)$ in the case when $\mu(x) > 0$.

Let us define $\sigma_*(r)$ as the distribution function of the ‘inverse exponential distribution’, that is, let

$$\sigma_*(r) = \exp\left(-\frac{1}{r}\right), \quad r > 0.$$  \hspace{1cm} (2.16)

As a distance distribution function, $\sigma_*$ will play an important role in what follows.
Figure 3. The graph of the function $\tau \mapsto N(x, \tau)$ in the case when $\mu(x) = 0$ and $\mu(X) = \infty$.

**Definition 2.9.** The isotropic Markov operator $P$ associated with a triple $(d, \mu, \sigma_*)$ will be referred to as the *standard* Markov operator associated with $(d, \mu)$.

**Theorem 2.10.** Let $d_*$ and $\sigma_*$ be defined by (2.14) and (2.16).

(a) The triples $(d, \mu, \sigma)$ and $(d_*, \mu, \sigma_*)$ induce the same isotropic Markov operators.

(b) The heat kernel $p(t, x, y)$ associated with the triple $(d, \mu, \sigma)$ satisfies for all $x, y \in X$ and $t > 0$ the following identities:

$$p(t, x, y) = \int_0^{t/d_*(x, y)} N\left(x, \frac{s}{t}\right) e^{-s} ds \quad (2.17)$$

and

$$p(t, x, y) = t \int_0^{1/d_*(x, y)} N(x, \tau) \exp(-\tau t) d\tau. \quad (2.18)$$

Consequently, $p(t, x, y)$ is a finite continuous function of $t, x, y$ for all $t > 0$ and $x \neq y$.

As follows from (a), any isotropic Markov operator is at the same time the standard Markov operator associated with $(d_*, \mu)$.

**Proof.** (a) It suffices to show that

$$p(t, x, y) = \int_{[d_*(x, y), \infty)} \frac{d\sigma_*^t(u)}{\mu(B_u^*(x))}, \quad (2.19)$$

where by Theorem 2.2 the right-hand side represents the heat kernel associated with the triple $(d_*, \mu, \sigma_*)$. Consider the function

$$u(r) = \frac{1}{\log(1/\sigma(r))}, \quad r \in [0, \infty),$$

and observe that
1) \( u(d(x, y)) = d_*(x, y) \), \( u(\infty) = \infty \);
2) \( \sigma_*(u(r)) = \exp(-1/u(r)) = \sigma(r) \);
3) \( B^*_u(r) = B_r(x) \) by Lemma 2.7.

Making the change \( u = u(r) \) in the integral in (2.19), we find that

\[
\int_{[d_*(x, y), \infty)} \frac{d\sigma^t_*(u)}{\mu(B^*_u(x))} = \int_{[d(x, y), \infty)} \frac{d\sigma^t(r)}{\mu(B_r(x))},
\]

which together with (2.8) implies (2.19). Clearly, (2.18) follows from (2.17) as \( d_*(x, x) = 0 \).

(b) The change \( s = t/u \) in (2.19) yields

\[
p(t, x, y) = \int_{[d_*(x, y), \infty)} \frac{d \exp(-t/u)}{\mu(B^*_u(x))}
= \int_0^{t/d_*(x, y)} \frac{de^{-s}}{\mu(B^*_u/s(x))}
= \int_0^{t/d_*(x, y)} N(x, s/t) e^{-s} ds,
\]

which proves (2.17). Another change \( s = t\tau \) transforms (2.17) into (2.18). □

In the case \( x = y \) we obtain from (2.17) and (2.18) that

\[
p(t, x, x) = \int_0^\infty N(x, s/t) e^{-s} ds = t \int_0^\infty N(x, \tau) \exp(-\tau t) d\tau.
\]

Depending on the function \( N(x, \tau) \), the on-diagonal value \( p(t, x, x) \) can be equal to \( \infty \). For any \( x \in X \) let

\[
T(x) := \limsup_{\tau \to \infty} \frac{\log N(x, \tau)}{\tau}.
\]

**Corollary 2.11.** The function \( t \mapsto p(t, x, x) \) is monotone decreasing and \( p(t, x, x) < \infty \) for all \( t > T(x) \).

**Proof.** The monotonicity of \( p(t, x, x) \) follows from the first identity in (2.20), while the second claim follows from the second identity. Also, if \( \lim_{\tau \to \infty} \frac{\log N(x, \tau)}{\tau} \) exists and hence is equal to \( T(x) \), then \( p(t, x, x) = \infty \) for \( t < T(x) \). □

**Proposition 2.12.** Assume that \( T(x) < \infty \) for some \( x \in X \).

(a) For all \( y \in X \)

\[
\lim_{t \to \infty} p(t, x, y) = \frac{1}{\mu(X)},
\]

where the convergence is locally uniform with respect to \( y \in X \).

(b) For all \( y \in X \)

\[
\lim_{t \to \infty} \frac{p(t, x, y)}{p(t, x, x)} = 1,
\]

where the convergence is locally uniform with respect to \( y \in X \).
Proof. (a) As $t \to \infty$ we have

$$N\left(x, \frac{s}{t}\right) \to N(x, 0) = \frac{1}{\mu(X)}$$

and $t/d_*(x, y) \to \infty$. Hence, we get from (2.17) that

$$\lim_{t \to \infty} p(t, x, y) = \int_0^\infty \frac{1}{\mu(X)} e^{-s} \, ds = \frac{1}{\mu(X)},$$

provided we justify that the integral and limit are interchangeable. The latter follows from the dominated convergence theorem, because the hypothesis $T(x) < \infty$ implies that for some $A, a > 0$ and all $\tau > 0$

$$N(x, \tau) \leq A \exp(a \tau),$$

whence

$$N\left(x, \frac{s}{t}\right) e^{-s} \leq A \exp\left(\frac{a}{t} - 1\right)s \leq A \exp\left(-\frac{1}{2}s\right)$$

for $t > 2a$, so that the domination condition is satisfied.

(b) Let $r = d_*(x, y)$. It follows from (2.17) and (2.20) that

$$p(t, x, x) - p(t, x, y) = \int_{t/r}^\infty N\left(x, \frac{s}{t}\right) e^{-s} \, ds.$$

Assuming that $t > 2a$ and applying (2.23), we have

$$p(t, x, x) - p(t, x, y) \leq A \int_{t/r}^\infty e^{-s/2} \, ds = 2A \exp\left(-\frac{t}{2r}\right),$$

whereas

$$p(t, x, x) \geq \int_{t/(4r)}^\infty N\left(x, \frac{s}{t}\right) e^{-s} \, ds \geq N\left(x, \frac{1}{4r}\right) \exp\left(-\frac{t}{4r}\right).$$

It follows that

$$\frac{p(t, x, x) - p(t, x, y)}{p(t, x, x)} \leq \frac{2A \exp(-t/(4r))}{N(x, 1/(4r))} \to 0 \quad \text{as } t \to \infty. \quad \square$$

2.4. Estimates of the heat kernel. The purpose of this section is to provide some estimates of the isotropic heat kernel. Recall that by Theorem 2.10

$$p(t, x, y) = \int_0^{t/d_*(x, y)} N\left(x, \frac{s}{t}\right) e^{-s} \, ds. \quad (2.24)$$

Definition 2.13. A monotone increasing function $\Phi: \mathbb{R}_+ \to \mathbb{R}_+$ is said to satisfy the doubling property if there is a constant $D > 0$ such that

$$\Phi(2s) \leq D \Phi(s) \quad \text{for all } s > 0.$$
It is known (Potter’s theorem) that if \( \Phi \) is doubling, then
\[
\Phi(s_2) \leq D \left( \frac{s_2}{s_1} \right)^\gamma \Phi(s_1) \quad \text{for all } 0 < s_1 < s_2, \quad \text{where } \gamma = \log_2 D. \tag{2.25}
\]

**Theorem 2.14.** Suppose that, for some \( x \in X \), the function \( \tau \mapsto N(x, \tau) \) is doubling. Then
\[
\frac{ct}{t + d_*(x, y)} N \left( x, \frac{1}{t + d_*(x, y)} \right) \leq p(t, x, y) \leq \frac{Ct}{t + d_*(x, y)} N \left( x, \frac{1}{t + d_*(x, y)} \right)
\tag{2.26}
\]
for all \( t > 0 \) and \( y \in X \) and some constants \( C, c > 0 \) depending on the doubling constant.

In what follows we will use the relation \( f \simeq g \) between two positive functions \( f, g \), which means that the ratio \( f/g \) is bounded from above and below by positive constants, for a specified range of the variables. In particular, we can write (2.26) in the shortened form
\[
p(t, x, y) \simeq \frac{t}{t + d_*(x, y)} N \left( x, \frac{1}{t + d_*(x, y)} \right) \tag{2.27}
\]
for a fixed \( x \) and all \( y \in X \) and \( t > 0 \).

**Example 2.15.** Assume that, for some \( x \in X \) and \( \alpha > 0 \),
\[
N(x, \tau) \simeq \tau^\alpha \quad \text{for all } \tau > 0.
\]

Then by (2.27)
\[
p(t, x, y) \simeq \frac{t}{(t + d_*(x, y))^{1+\alpha}} \simeq \frac{t}{(t^2 + d_*(x, y)^2)^{(1+\alpha)/2}},
\]
that is, \( p(t, x, y) \) behaves like the Cauchy distribution in ‘\( \alpha \)-dimensional’ space.

**Example 2.16.** More generally, assume that for some \( \alpha, \beta > 0 \)
\[
N(x, \tau) \simeq \begin{cases} \tau^\alpha, & 0 < \tau \leq 1, \\ \tau^\beta, & \tau > 1. \end{cases} \tag{2.28}
\]

Then from (2.27) we get that
\[
p(t, x, y) \simeq \begin{cases} \frac{t}{(t + d_*(x, y))^{1+\beta}}, & t + d_*(x, y) \leq 1, \\ \frac{t}{(t + d_*(x, y))^{1+\alpha}}, & t + d_*(x, y) > 1. \end{cases} \tag{2.29}
\]

For example, let \( X \) be a discrete locally finite group, like \( X = \bigoplus_{k=1}^{\infty} \mathbb{Z}(n_k) \), and let \( \mu \) be the Haar measure, normalized to \( \mu(x) = 1 \). With the discrete ultra-metric we get by (2.15) that \( N(x, \tau) \simeq 1 \) for sufficiently large \( \tau \). Assuming additionally that
\[
N(x, \tau) \simeq \tau^\alpha \quad \text{for small } \tau,
\]
we see that (2.28) and hence (2.29) hold with \( \beta = 0 \) (cf. [13]).
Example 2.17. Assume that \( \tau \mapsto N(x, \tau) \) is doubling and, for some \( \alpha > 0 \),

\[
N(x, \tau) \simeq \left( \log \frac{1}{\tau} \right)^{-\alpha} \quad \text{for } \tau < \frac{1}{2}.
\]

Then by (2.27)

\[
p(t, x, y) \simeq \frac{t}{(t + d_*(x, y)) \log(t + d_*(x, y))},
\]

provided \( t + d_*(x, y) > 2 \).

Example 2.18. Assume that, for some \( \alpha > 0 \),

\[
N(x, \tau) \simeq \exp(-\tau - \alpha).
\]

In this case Theorem 2.14 does not apply. An ad hoc method of estimating the integral in (2.24) yields in this case

\[
p(t, x, y) \leq \frac{C_3 t}{t + d_*(x, y)} \exp(-c_3(t^{\alpha/(\alpha+1)} + d_*(x, y)\alpha))
\]

and

\[
p(t, x, y) \geq \frac{C_4 t}{t + d_*(x, y)} \exp(-c_4(t^{\alpha/(\alpha+1)} + d_*(x, y)\alpha))
\]

for all \( x, y \in X \) and \( t > 0 \) and some positive constants \( C_3, C_4, c_3, c_4 \).

For the proof of Theorem 2.14 we need a sequence of lemmas.

Lemma 2.19. For all \( x, y \in X \) and \( t > 0 \) the following estimates hold.

(a) \( p(t, x, y) \leq \frac{t}{d_*(x, y)} N\left(x, \frac{1}{d_*(x, y)}\right) \); \hspace{1cm} (2.30)

(b) \( p(t, x, y) \geq \frac{1}{2e} \begin{cases} \frac{t}{d_*(x, y)} \frac{N\left(x, \frac{1}{2d_*(x, y)}\right)}{N\left(x, \frac{1}{2t}\right)}, & t \leq d_*(x, y), \\ N\left(x, \frac{1}{2t}\right), & t > d_*(x, y); \end{cases} \) \hspace{1cm} (2.31)

(c) \( p(t, x, x) \geq \frac{1}{e} N\left(x, \frac{1}{t}\right) \). \hspace{1cm} (2.32)

Proof. (a) The inequality (2.30) follows from (2.24) using the monotonicity of \( \tau \mapsto N(x, \tau) \), which implies that

\[
N\left(x, \frac{s}{t}\right) e^{-s} \leq N\left(x, \frac{1}{d_*(x, y)}\right).
\]

(b) Let \( a = \min\left(1, \frac{t}{d_*(x, y)}\right) \). It follows from (2.24) that

\[
p(t, x, y) \geq \int_{a/2}^{a} N\left(x, \frac{s}{t}\right) e^{-s} ds \geq N\left(x, \frac{a}{2t}\right) \frac{a}{2e},
\]

which is equivalent to (2.31).
(c) By (2.20)
\[ p(t, x, x) \geq \int_1^\infty N\left(x, \frac{s}{t}\right) e^{-s} ds \geq N\left(x, \frac{1}{t}\right) \int_1^\infty e^{-s} ds, \]
whence (2.32) follows. □

**Lemma 2.20.** The following inequalities hold for all \( x, y \in X \) and \( t > 0 \):
\[
p(t, x, y) \geq \frac{1}{2e} \frac{t}{t + d_*(x, y)} N\left(x, \frac{1}{2(t + d_*(x, y))}\right) \tag{2.33}
\]
and
\[
p(t, x, y) \leq 2e \frac{t}{t + d_*(x, y)} p\left(\frac{t + d_*(x, y)}{2}, x, x\right). \tag{2.34}
\]

**Proof.** The lower bound (2.33) follows immediately from (2.31). To prove (2.34), observe that by (2.30) and (2.32)
\[
p(t, x, y) \leq e \frac{t}{d_*(x, y)} p(d_*(x, y), x, x),
\]
which yields (2.34) in the case \( t \leq d_*(x, y) \), since the function \( p(\cdot, x, x) \) is monotone decreasing. In the case \( t > d_*(x, y) \), (2.34) follows trivially from (2.12), that is, from
\[
p(t, x, y) \leq p(t, x, x),
\]
again using the monotonicity of \( p(\cdot, x, x) \). □

**Lemma 2.21.** For any given \( x \in X \), the following two properties are equivalent.

(i) For some constant \( C \) and all \( t > 0 \),
\[
p(t, x, x) \leq CN\left(x, \frac{1}{t}\right). \tag{2.35}
\]

(ii) The function \( \tau \mapsto N(x, \tau) \) is doubling, that is, for some constant \( D \),
\[
N(x, 2\tau) \leq DN(x, \tau).
\]

**Proof.** (ii) ⇒ (i). The estimate (2.35) follows from (2.20) and (2.25) as follows:
\[
p(t, x, x) = N\left(x, \frac{1}{t}\right) \int_0^\infty N\left(x, \frac{s}{t}\right) \frac{e^{-s}}{N(x, 1/t)} ds
\]
\[
\leq DN\left(x, \frac{1}{t}\right) \int_0^\infty \max\{1, s^\gamma\} e^{-s} ds
\]
\[
= CN\left(x, \frac{1}{t}\right).
\]

(i) ⇒ (ii). The upper bound (2.35) implies that for any \( t > 0 \)
\[
CN\left(x, \frac{1}{t}\right) \geq p(t, x, x) \geq \int_2^\infty N\left(x, \frac{s}{t}\right) e^{-s} ds
\]
\[
\geq e^{-2} N\left(x, \frac{2}{t}\right),
\]
whence the doubling property of \( N(x, \cdot) \) follows. □
Proof of Theorem 2.14. The lower bound in (2.26) follows from (2.33), and the upper bound follows from (2.34) and (2.35). □

In conclusion to this section we provide sufficient conditions for the validity of the doubling property of $N(x, \cdot)$.

**Definition 2.22.** A monotone increasing function $\Psi: \mathbb{R}_+ \to \mathbb{R}_+$ is said to have the reverse doubling property if there is a constant $\delta \in (0, 1)$ such that for all $r > 0$

$$\Psi(r) \geq 2\Psi(\delta r).$$

**Proposition 2.23.** Fix some $x \in X$. The function $\tau \mapsto N(x, \tau)$ is doubling provided the following two conditions hold:

(i) the function $\Psi(r) = -1/\log \sigma(r)$ has the reverse doubling property;

(ii) the volume function $r \mapsto \mu(B_r(x))$ has the doubling property.

**Proof.** We use the following short notation for the balls centred at $x$: $B_r = B_r(x)$ and $B_r^* = B_r^*(x)$. It follows from Definition 2.8 of the spectral distribution function that $\tau \mapsto N(x, \tau)$ is doubling if and only if the function $s \mapsto \mu(B_s^*)$ has the doubling property. Let $\Phi = \Psi^{-1}$ and observe that the reverse doubling property for $\Psi$ is equivalent to the doubling property for $\Phi$. By Lemma 2.7 we have $B^*_r = B^*_{\Psi(r)}$, which implies that $B^*_s = B^*_{\Phi(s)}$. Using the hypotheses (ii) and (2.25) for the function $\mu(B_r)$, we have

$$\mu(B^*_s) = D\left(\frac{\Phi(2s)}{\Phi(s)}\right)^{\gamma} \mu(B^*_{\Phi(s)}) \leq \text{const} \mu(B^*_s),$$

which was to be proved. □

2.5. Heat kernels in $\mathbb{Q}_p$. Given a prime $p$, the $p$-adic norm on $\mathbb{Q}$ is defined as follows: if $x = p^na/b$, where $a$, $b$ are integers not divisible by $p$, then

$$\|x\|_p := p^{-n}. $$

If $x = 0$, then $\|x\|_p := 0$. The $p$-adic norm on $\mathbb{Q}$ satisfies the ultra-metric inequality. Indeed, if $y = p^mc/d$ and $m \leq n$, then

$$x + y = p^n\left(\frac{p^{n-m}a}{b} + \frac{c}{d}\right),$$

and therefore

$$\|x + y\|_p \leq p^{-m} = \max\{\|x\|_p, \|y\|_p\}. $$

Hence, $\mathbb{Q}$ with the metric $d(x, y) = \|x - y\|_p$ is an ultra-metric space, and so is its completion $\mathbb{Q}_p$, the field of $p$-adic numbers.

Every $p$-adic number $x$ has a representation

$$x = \sum_{k=-N}^{\infty} a_kp^k = \ldots a_k \ldots a_2a_1a_0.a_{-1}a_{-2} \ldots a_{-N},$$

(2.36)

where $N \in \mathbb{N}$ and the $a_k \in \{0, \ldots, p - 1\}$ are $p$-adic digits. The rational number $0.a_{-1} \ldots a_{-N} = \sum_{k=-N}^{-1} a_kp^k$ is called the fractional part of $x$ and the rest, $\sum_{k=0}^{\infty} a_kp^k$, is the integer part of $x$. 

For any $n \in \mathbb{Z}$, the $d$-ball $B_{p^{-n}}(x)$ consists of all numbers

$$y = \sum_{k=-N}^{\infty} b_k p^k = \ldots b_k \ldots b_2 b_1 b_0. b_{-1} b_{-2} \ldots b_{-N}$$

such that the $b_k$ are arbitrary for $k \geq n$ and $b_k = a_k$ for $k < n$. It follows that $B_{p^{-n}}(x)$ decomposes into a disjoint union of $p$ balls of radii $p^{-(n+1)}$ depending on the choice of $b_n$.

For example, $B_1(0)$ coincides with the set $\mathbb{Z}_p$ of all $p$-adic integers, that is, any $y \in B_1(0)$ has the form

$$y = \ldots b_k \ldots b_2 b_1 b_0$$

with arbitrary $p$-adic digits $b_k$. For any fixed $c \in \{0, 1, \ldots, p-1\}$, the additional restriction $b_0 = c$ determines a ball of radius $1/p$ centred at $c$, so that $B_1(0)$ is a disjoint union of $p$ such balls, as in the following diagram, where every cell represents one of the balls $B_{1/p}(c)$:

\[
\begin{array}{ccc}
\ldots b_k \ldots b_2 b_1 0 & \ldots b_k \ldots b_2 b_1 1 & \ldots \\
\end{array}
\]

Let $\mu$ be the additive Haar measure on $\mathbb{Q}_p$ normalized so that $\mu(B_1(0)) = 1$. Since

$$B_r(x) = x + B_r(0)$$

and $\mu$ is translation invariant, $\mu(B_r(x))$ does not depend on $x$. The above argument with the decomposition of the ball $B_{p^{-n}}(x)$ implies that

$$\mu(B_{p^{-n}}(x)) = p \mu(B_{p^{-(n+1)}}(x)),$$

whence it follows that

$$\mu(B_{p^{-n}}(x)) = p^{-n}. \quad (2.37)$$

For any $r > 0$, the ball $B_r(x)$ coincides with $B_{p^{-n}}(x)$, where $n \in \mathbb{Z}$ is such that $p^{-n} \leq r < p^{-(n-1)}$, which implies that, for all $r > 0$,

$$\frac{r}{p} < \mu(B_r(x)) \leq r. \quad (2.38)$$

**Example 2.24.** Let $(X, d, \mu)$ be $\mathbb{Q}_p$ with the $p$-adic distance and the Haar measure $\mu$. Consider the distance distribution function

$$\sigma(r) = \exp\left(-\left(\frac{b}{r}\right)^\alpha\right),$$

where $\alpha, b > 0$. Since

$$\Psi(r) := \frac{1}{\log(1/\sigma(r))} = \left(\frac{r}{b}\right)^\alpha,$$

we get by (2.14) that

$$d_*(x, y) = \Psi(d(x, y)) = \left(\frac{||x - y||_p}{b}\right)^\alpha. \quad (2.39)$$
By Lemma 2.7, we have
\[ B_s^* (x) = B_{\Psi^{-1}(s)} (x), \]
which together with (2.38) implies that
\[ \mu(B_s^* (x)) \simeq s^{1/\alpha}. \]
(2.40)
Consequently, we have
\[ N(x, \tau) \simeq \tau^{1/\alpha}. \]
Since this function is doubling, Theorem 2.14 (cf. also Example 2.15) yields the estimate
\[ p(t, x, y) \simeq \frac{t}{(t + d_*(x, y))^{1+1/\alpha}} \simeq \frac{t}{(t^{1/\alpha} + \|x - y\|_p)^{1+\alpha}}. \]
In particular, for all \( t > 0 \) and \( x \in X \)
\[ p(t, x, x) \simeq t^{1/\alpha}. \]

Example 2.25. Let \( X = \mathbb{Z}_p \), that is, \( X \) is the unit ball \( B_1(0) \) in \( \mathbb{Q}_p \) with the \( p \)-adic distance and the Haar measure \( \mu \). Consider the distance distribution function
\[ \sigma(r) = \exp(1 - \exp(r^\alpha)) \]
for some \( \alpha > 0 \). Since for \( r \leq 1 \)
\[ \Psi(r) := \frac{1}{\log(1/\sigma(r))} = \frac{1}{\exp(r^\alpha) - 1} \simeq \exp(-r^{-\alpha}), \]
we get that
\[ d_*(x, y) = \Psi(d(x, y)) \simeq \exp(-\|x - y\|_p^{-\alpha}). \]
By Lemma 2.7 and (2.38) we have, for all \( s \leq 1/2 \),
\[ \mu(B_s^* (x)) = \mu(B_{\Psi^{-1}(s)}(x)) \simeq \Psi^{-1}(s) \simeq \frac{1}{\log^{1/\alpha}(1/s)}, \]
whereas for \( s > 1/2 \) we have \( \mu(B_s^* (x)) \simeq 1 \). Therefore, for all \( \tau > 0 \)
\[ N(x, \tau) = \frac{1}{\mu(B_{1/\tau}^* (x))} \simeq \log^{1/\alpha}(2 + \tau). \]
Hence, the function \( N(x, \tau) \) is doubling, and we obtain by (2.27) that
\[ p(t, x, y) \simeq \frac{t}{t + \exp(-\|x - y\|_p^{-\alpha})} \log^{1/\alpha} \left(2 + \frac{1}{t + \exp(-\|x - y\|_p^{-\alpha})}\right). \]
Example 2.26. Let $X$ be the subset of $\mathbb{Q}_p$ consisting of all $p$-adic fractions, that is, numbers of the form $x = 0.a_{-1} \ldots a_{-N}$. Then the $p$-adic distance $d$ on $X$ takes only integer values, so that $(X, d)$ is a discrete space. Let $\mu$ be the counting measure on $X$, that is, $\mu(x) = 1$ for any $x \in X$. Consider the following distance distribution function:

$$\sigma(r) = \exp\left(-\frac{1}{\log^2(2r)}\right) \text{ for } r \geq 1,$$

arbitrarily extended to $r < 1$ to be strictly monotone increasing and to have $\sigma(0) = 0$. Since

$$\Psi(r) := \frac{1}{\log(1/\sigma(r))} = \log^2(2r) \text{ for } r \geq 1,$$

we find that for $x \neq y$

$$d_s(x, y) = \Psi(d(x, y)) = \log^2(2\|x - y\|_p).$$

For $s \geq s_0 := \log^2 2$ we have

$$\mu(B^*_s(x)) = \mu(B_{\Psi^{-1}(s)}(x)) \simeq \Psi^{-1}(s) = \frac{1}{2} \exp(s^{1/2}),$$

whereas for $s < s_0$ we have $\mu(B^*_s(x)) \simeq \mu(x) = 1$. We see that (2.43) holds for all $s > 0$. It follows that, for all $\tau > 0$,

$$N(x, \tau) = \frac{1}{\mu(B^*_1/\tau(x))} \simeq \exp(-\tau^{-1/2}).$$

By Example 2.18,

$$p(t, x, y) \leq \frac{Ct}{t + \log^2_+(2\|x - y\|_p)} \exp\left(-c\left(t^{1/(\alpha+1)} + \log_+(2\|x - y\|_p)\right)\right),$$

and a similar lower bound.

2.6. Green function and transience. Given an isotropic heat semigroup $\{P^t\}$, define the Green operator $G$ on non-negative Borel functions $f$ on $X$ by

$$Gf(x) = \int_0^\infty P^t f(x) \, dt.$$

Of course, the value of $Gf(x)$ could be $\infty$. By Fubini’s theorem,

$$Gf(x) = \int_X g(x, y) f(y) \, d\mu(y),$$

where

$$g(x, y) = \int_0^\infty p(t, x, y) \, dt.$$

Substituting the heat kernel from (2.18) and again using Fubini’s theorem, we get that

$$g(x, y) = \int_0^{1/d_*(x, y)} N(x, \tau) \, d\tau = \int_{d_*(x, y)}^\infty \frac{ds}{\mu(B^*_s(x))},$$

(2.45)
where the second identity follows from (2.15). The function \( g(x, y) \) is called the Green function of the semigroup \( \{P^t\} \). Note that the Green function may be identically equal to \( \infty \). For example, this is the case when \( \mu(x) < \infty \) (cf. Fig. 1) and the second integral (2.45) diverges at \( \infty \).

**Definition 2.27.** The process \( \{\mathcal{X}_t\} \) and the semigroup \( \{P^t\} \) are said to be transient if \( Gf \) is a bounded function whenever \( f \) is bounded and has compact support, and recurrent otherwise.

**Theorem 2.28.** The following statements are equivalent:

(i) the semigroup \( \{P^t\} \) is transient;

(ii) \( g(x, y) < \infty \) for some/all distinct \( x, y \in X \);

(iii) for some/all \( x \in X \)

\[
\int_0^\infty \frac{ds}{\mu(B^*_s(x))} < \infty. \tag{2.46}
\]

The inequality (2.46) is equivalent to

\[
\int_0^\infty \frac{N(x, \tau) d\tau}{\tau^2} < \infty. \tag{2.47}
\]

Observe that, in the transient case, the function \( x, y \mapsto 1/g(x, y) \) determines an ultra-metric on \( X \) (proved like Corollary 2.6).

**Proof.** The validity of the condition (2.46) is independent of the choice of \( x \), because for any two \( x, x' \in X \) the balls \( B^*_s(x) \) and \( B^*_s(x') \) are identical if \( s \geq d(x, x') \). The finiteness of the second integral in (2.45) for \( x \neq y \) is clearly equivalent to (2.46), and thus the equivalence (ii) \( \iff \) (iii) follows, with all combinations of some/all options.

The finiteness of \( Gf \) for any bounded function \( f \) with compact support clearly implies that \( g(x, y) \neq \infty \), that is, (i) \( \Rightarrow \) (ii). So, it remains to prove (iii) \( \Rightarrow \) (i). It suffices to show that \( Gf \) is bounded for \( f = 1_A \), where \( A \) is a bounded Borel subset of \( X \). Let \( R \) be the diameter of \( A \) with respect to the distance \( d^* \). Then we have \( A \subset B^*_R(x) \) for any \( x \in A \), and therefore by (2.45)

\[
Gf(x) = \int_A g(x, y) \, d\mu(y) \leq \int_{B^*_R(x)} g(x, y) \, d\mu(y)
= \int_{B^*_R(x)} \int_0^{\infty} 1_{[d_*(x, y), \infty)}(s) \frac{ds}{\mu(B^*_s(x))} \, d\mu(y)
= \int_0^\infty \frac{1}{\mu(B^*_s(x))} \left( \int_{B^*_R(x)} 1_{[0, s]}(d_*(x, y)) \, d\mu(y) \right) ds
= \int_0^\infty \frac{1}{\mu(B^*_s(x))} \mu(B^*_R(x) \cap B^*_s(x)) \, ds.
\]

For \( s \geq R \) the integrand is equal to \( \frac{1}{\mu(B^*_s(x))} \mu(B^*_R(x)) \), so that the convergence at \( \infty \) follows from (2.46). The convergence is clearly uniform with respect to \( x \in A \).
because $\mu(B^*_R(x))$ and $\mu(B^*_s(x))$ are independent of $x \in A$ for $s \geq R$. For $s \leq R$ the integrand is equal to

$$\frac{1}{\mu(B^*_s(x))} \mu(B^*_s(x)) = 1,$$

whence the uniform convergence at 0 follows. Hence, $\sup_A Gf(x) < \infty$. The fact that $\sup_X Gf(x) < \infty$ follows from the decay of $g(x,y)$ with respect to $d_*(x,y)$. □

Let us note that if $X$ is a locally finite group with the Haar measure $\mu$ then the transience criterion (iii) of Theorem 2.28 coincides with the general sufficient condition of transience of [40].

Now let us provide some estimates of the Green function. Let

$$V(x,r) = \mu(B^*_r(x)).$$

(2.48)

**Theorem 2.29.** Assume that there are constants $1 < c < c' < c''$ such that for all $r > r_0 > 0$ and some $x \in X$

$$c' \leq \frac{V(x,cr)}{V(x,r)} \leq c''.$$  

(2.49)

Then the semigroup $\{P^t\}$ is transient, and for all $y \in X$ such that $r := d_*(x,y) > r_0$

$$g(x,y) \approx \frac{r}{V(x,r)}.$$

Note that the condition $\frac{V(x,cr)}{V(x,r)} \leq c''$ is equivalent to the doubling property of $r \mapsto V(x,r)$ (cf. Definition 2.13), whereas the condition $\frac{V(x,cr)}{V(x,r)} \geq c'$ with $c' > c$ is somewhat stronger than the reverse doubling property (cf. Definition 2.22). For example, (2.49) holds for $V(x,r) \simeq r^\alpha$ if and only if $\alpha > 1$.

**Proof of Theorem 2.29.** For simplicity of notation let $V(s) := V(x,s)$. For $r > r_0$ we have

$$g(x,y) = \int_r^\infty \frac{ds}{V(s)} = \sum_{k=0}^\infty \int_{c^k r}^{c^{k+1} r} \frac{ds}{V(s)} = \sum_{k=0}^\infty c^k \int_r^{cr} \frac{ds}{V(c^k s)}.$$

Using the lower bound in (2.49), we see that

$$\int_r^\infty \frac{ds}{V(s)} \leq \sum_{k=0}^\infty c^k \int_r^{cr} \frac{(c')^{-k} ds}{V(s)} \leq \sum_{k=0}^\infty \left( \frac{c}{c'} \right)^k \frac{cr}{V(r)} \leq \text{const} \frac{r}{V(r)},$$

where the series converges because $c' > c$. Similarly, using the upper bound in (2.49), we have

$$\int_r^\infty \frac{ds}{V(s)} \geq \int_r^{cr} \frac{ds}{V(s)} \geq \frac{(c-1)r}{V(cr)} \geq \text{const} \frac{r}{V(r)},$$

which finishes the proof. □
Example 2.30. Let \((X,d,\mu)\) and \(\sigma\) be as in Example 2.24, that is, \(X = \mathbb{Q}_p\) is the field of \(p\)-adic numbers with ultra-metric \(d(x,y) = \|x-y\|_p\), and \(\sigma(r) = \exp(-(b/r)^\alpha)\). Then by (2.39)
\[d_*(x,y) = \text{const} \|x-y\|_p^\alpha,\]
and by (2.40)
\[V(x,r) \simeq r^{1/\alpha}.
\]
Therefore, by Theorem 2.28 the semigroup \(\{P_t\}\) is transient if and only if \(\alpha < 1\). Moreover, the condition (2.49) is fulfilled also if and only if \(\alpha < 1\), and in this case we obtain by Theorem 2.29 that for all \(x\) and \(y\)
\[g(x,y) \simeq d_*(x,y)^{1-1/\alpha} \simeq \|x-y\|_p^{\alpha-1}.
\]

Example 2.31. Let \((X,d,\mu)\) and \(\sigma\) be as in Example 2.26, that is, \(X\) is the set of fractional \(p\)-adic numbers and \(\sigma\) is given by (2.41). By (2.42) we have for \(x \neq y\)
\[d_*(x,y) = \log^\alpha(2\|x-y\|_p),\]
and by (2.43)
\[V(x,r) \simeq \exp(r^{1/\alpha}).\]
By Theorem 2.28 we conclude that the semigroup \(\{P_t\}\) is transient. Theorem 2.29 does not apply in this case, but a direct estimate of the integral in (2.45) yields
\[g(x,y) = \int_r^\infty \frac{ds}{V(x,s)} \simeq \int_r^\infty \exp(-s^{1/\alpha}) ds \simeq r^{1-1/\alpha} \exp(-r^{1/\alpha})
\]
for \(r := d^*(x,y)\), and thus
\[g(x,y) \simeq \|x-y\|_p^{-1} \log^{\alpha-1}(2\|x-y\|_p)
\]
for \(x \neq y\).

3. The isotropic Laplacian and its spectrum

In this section we are concerned with the properties of the generator of the isotropic semigroup \(\{P_t\}\). By definition, the generator \(\mathcal{L}\) of a strongly continuous semigroup \(\{P_t\}_{t \geq 0}\) in a Banach space is defined by
\[\mathcal{L}f = s\lim_{t \to 0} \frac{f - P_t f}{t},\]
and the domain \(\text{dom}\mathcal{L}\) consists of those \(f\) for which the above limit exists. Since the isotropic semigroup \(\{P_t\}\) is symmetric and acts in a Hilbert space \(L^2(X,\mu)\), the above definition is equivalent to the following: \(\mathcal{L}\) is a self-adjoint (unbounded) operator acting in \(L^2(X,\mu)\) such that
\[P_t = \exp(-t\mathcal{L}) \quad \text{for all } t > 0.
\]
Obviously, this is equivalent to \(P = \exp(-\mathcal{L})\), which leads to the identity
\[\mathcal{L} = \log \frac{1}{P},\]
where the right-hand side is understood in the sense of functional calculus of self-adjoint operators. We refer to \(\mathcal{L}\) as an isotropic Laplace operator associated with \((d,\mu,\sigma)\).
3.1. Subordination. Using the spectral decomposition (2.10) of $P$, we get that

$$L = \int_{[0, +\infty)} \log \frac{1}{\sigma(1/\lambda)} dE_\lambda,$$

where $\{E_\lambda\}$ is the spectral resolution defined by (2.6). For simplicity let

$$\varphi(\lambda) := \log \frac{1}{\sigma(1/\lambda)}, \quad (3.1)$$

so that

$$L = \int_{(0, +\infty)} \varphi(\lambda) dE_\lambda. \quad (3.2)$$

The domain $\text{dom}_L$ is then given by

$$\text{dom}_L = \left\{ f \in L^2 : \int_{0}^{\infty} \varphi(\lambda)^2 d(E_\lambda f, f) < \infty \right\}.$$

Observe that the function $\varphi$ has the following properties as a consequence of the assumptions (1.5) about $\sigma$:

$$\varphi : [0, \infty] \to [0, \infty] \text{ is a strictly monotone increasing}$$

$$\text{right-continuous function such that } \varphi(0) = 0 \text{ and } \varphi(\infty - ) = \infty. \quad (3.3)$$

Conversely, any function $\varphi$ satisfying (3.3) determines the function

$$\sigma(\lambda) = \exp\left(-\varphi\left(\frac{1}{\lambda}\right)\right),$$

which satisfies (1.5). This observation leads us to the following interesting subordination property of isotropic Laplacians.

**Theorem 3.1.** Let $L$ be an isotropic Laplacian associated with $(d, \mu, \sigma)$. Let $\psi$ be any function satisfying (3.3). Then $\psi(L)$ is also an isotropic Laplacian associated with $(d, \mu, \tilde{\sigma})$ for some other distance distribution function $\tilde{\sigma}$.

**Proof.** It follows from (3.2) that

$$\psi(L) = \int_{[0, +\infty)} \psi \circ \varphi(\lambda) dE_\lambda.$$

Since the composition $\psi \circ \varphi$ also satisfies (3.3), $\psi(L)$ is an isotropic Laplacian. Moreover, using (3.1), we obtain the following formula for $\tilde{\sigma}$:

$$\tilde{\sigma}(r) = \exp\left(-\psi\left(\log \frac{1}{\sigma(r)}\right)\right). \quad \square$$
Remark 3.2. Any non-negative-definite, self-adjoint operator \( L \) acting in \( L^2 \) generates a semigroup \( \{e^{-tL}\}_{t \geq 0} \). We refer to \( L \) as a Laplacian if the semigroup \( \{e^{-tL}\} \) is Markovian. In general, by Bochner’s theorem, for any Laplacian \( L \), the operator \( \psi(L) \) is again a Laplacian, provided \( \psi \) is a Bernstein function (see, for example, Schilling, Song and Vondraček [53]). It is known that \( \psi(\lambda) = \lambda^\alpha \) is a Bernstein function if and only if \( 0 < \alpha \leq 1 \). Thus, for a general Laplacian \( L \), the power \( L^\alpha \) is guaranteed to be a Laplacian only for \( \alpha \leq 1 \).

For example, for the classical Laplace operator \( L = -\Delta \) in \( \mathbb{R}^n \), the power \( (-\Delta)^\alpha \) with \( \alpha > 1 \) is not a Laplacian. This is in striking contrast to the fact that, by Theorem 3.1, the powers \( L^\alpha \) of the isotropic Laplacian are again Laplacians for all \( \alpha > 0 \).

3.2. The \( L^2 \)-spectrum of the Laplacian. Our next goal in this section is to give an explicit expression for \( Lf \) and to describe the spectrum of \( L \). Recall that by Theorem 2.10 the triples \( (d, \mu, \sigma) \) and \( (d^*, \mu, \sigma^*) \) induce the same Markov operator \( P \) and, hence, the same Laplace operator \( L \), where \( d^* \) is the intrinsic ultra-metric defined by (2.14) and

\[
\sigma^*(r) = \exp\left(-\frac{1}{r}\right).
\]

From now on we will use only the metrics \( d^* \) and \( \sigma^* \). Let the spectral resolution \( \{E_\lambda\} \) also be defined using the metric \( d^* \), which means that in the definition (2.6) of \( E_\lambda \) we now use the averaging operator \( Q_r \) with respect to the metric \( d^* \). The function \( \varphi^* \) associated with \( \sigma^* \) by (3.1) has an especially simple form:

\[
\varphi^*(\lambda) = \lambda.
\]

Therefore, we obtain from (3.2) the spectral decomposition of \( L \) in the classical form

\[
L = \int_{[0, +\infty)} \lambda dE_\lambda = \int_{(0, \infty)} \lambda dE_\lambda.
\]  

(3.4)

The change \( s = 1/\lambda \) gives

\[
L = -\int_{(0, \infty)} \frac{1}{s} dQ_s.
\]

For any \( x \in X \), denote by \( \Lambda(x) \) the set of values of \( d^*(x, y) \) for all \( y \in X, y \neq x \), that is,

\[
\Lambda(x) = \{d(x, y) : y \in X \setminus \{x\}\}.
\]  

(3.5)

Lemma 3.3. The set \( \Lambda(x) \) has no accumulation point in \( (0, \infty) \). Consequently, \( \Lambda(x) \) is at most countable.

Proof. Let \( r \in (0, \infty) \) be an accumulation point of \( \Lambda(x) \), that is, there is a sequence \( \{r_k\} \) of points in \( \Lambda(x) \setminus \{r\} \) such that \( r_k \to r \) as \( k \to \infty \). Then \( r_k = d^*(x, y_k) \) for some \( y_k \in X \). Since the sequence \( \{y_k\} \) is bounded, by the compactness of all balls in \( X \) it has a convergent subsequence. Without loss of generality, we can then assume that \( \{y_k\} \) converges, say to \( y \in X \). Then we have \( r = d(x, y) \). Since \( r > 0 \), we have for sufficiently large \( k \) that \( r_k > r/2 \) and \( d(y, y_k) < r/2 \). Then we get by the ultra-metric inequality that

\[
r_k \leq \max(r, d(y, y_k)) = r
\]
and analogously
\[ r \leq \max(r_k, d(y, y_k)) = r_k, \]
whence \( r_k = r \), which contradicts the assumptions. \( \square \)

**Definition 3.4.** For any ball \( B \) in \( X \) denote by \( \rho(B) \) the minimal \( d_\ast \)-radius of \( B \).

Note that \( \rho(B) \) exists, because all balls are defined as closed balls.

**Lemma 3.5.** If \( \rho(B) > 0 \), then \( \rho(B) \in \Lambda(x) \) for any \( x \in B \). Conversely, any number in \( \Lambda(x) \) is equal to \( \rho(B) \) for some ball \( B \) containing \( x \).

**Proof.** Let \( r = \rho(B) \), so that \( B = B_r^\ast(x) \). For any \( y \in B \) we have \( d_\ast(x, y) \leq r \), and we must show that \( d_\ast(x, y) = r \) for some \( y \). Assume that \( d_\ast(x, y) < r \) for all \( y \in B \). Then the set
\[ \{ d_\ast(x, y) : y \in B \setminus \{x\} \} \]
is a subset of \((0, r) \cap \Lambda(x)\). By Lemma 3.3 the latter set has a maximal element, say \( r' \). Then \( B \subset B_{r'}^\ast(x) \), which contradicts the minimality of the radius \( r \). Conversely, if \( r \in \Lambda(x) \) then the ball \( B = B_r(x) \) has \( \rho(B) = r \) since there is a \( y \in X \) with \( d(y, x) = r \). \( \square \)

**Definition 3.6.** Let \( B, C \) be two balls in \( X \) such that \( C \subset B \). We say that \( C \) is a **child** or **successor** of \( B \) (and \( B \) is a **parent** or **predecessor** of \( C \)) if \( C \neq B \) and, for any ball \( A \) such that \( C \subset A \subset B \) we have \( A = C \) or \( A = B \). In other words, \( B \) is a minimal ball containing \( C \) as a proper subset. If \( C \) is a child of \( B \), then we write \( C \prec B \).

Denote by \( \mathcal{K} \) the family of all balls \( C \) in \( X \) with positive radii. If \( C = B_r^\ast(x) \) is a ball in \( \mathcal{K} \) with \( r > 0 \), then for the minimal radius \( \rho(C) \) we have two possibilities:
1) \( \rho(C) > 0 \),
2) \( \rho(C) = 0 \) and the centre of \( C \) is an isolated point of \( X \).

**Lemma 3.7.** Any ball \( C \in \mathcal{K} \) such that \( C \neq X \) has a unique parent ball \( B \). For any ball \( B \) with \( \rho(B) > 0 \) the number \( \deg(B) \) of its children satisfies \( 2 \leq \deg(B) < \infty \). Moreover, all the children of \( B \) are disjoint and their union is equal to \( B \).

**Proof.** Fix some \( x \in C \). It follows from Lemma 3.3 and the definition of \( \mathcal{K} \) that the set \((\rho(C), \infty) \cap \Lambda(x)\) has a minimum which we denote by \( r \). Then the ball \( B_r^\ast(x) \) is a parent of \( C \). The uniqueness of the parent follows from the definition.

If \( C_1 \) and \( C_2 \) are two distinct children of \( B \), then \( C_1 \) and \( C_2 \) are disjoint. Indeed, if they intersect then one of them contains the other, say \( C_1 \subset C_2 \). By the definition of a parent/child, we must then have \( C_2 = C_1 \) or \( C_2 = B \), from which \( C_1 = C_2 \) follows.

Let us show that for any \( x \in B \) there is a ball \( C \) such that \( x \in C \prec B \). Indeed, if the set \((0, \rho(B)) \cap \Lambda(x)\) is empty, then \( C = B_0^\ast(x) = \{x\} \) is the child of \( B \). If the set \((0, \rho(B)) \cap \Lambda(x)\) is non-empty, then by Lemma 3.3 it has a maximum, say \( r \). Then \( C = B_r^\ast(x) \) is a child of \( B \). Hence, the set of all children of \( B \) is a covering of \( B \).

Each child \( C \) of \( B \) is an open set (being also a closed ball) because \( C \) coincides with an open ball of radius \( \rho(B) \). Since \( B \) is compact, it follows that the set of its
children is finite, that is, $\deg(B) < \infty$. Finally, $\deg(B)$ cannot be equal to 1 since then $B$ would coincide with its only child. Hence, $\deg(B) \geq 2$. □

For any $C \in \mathcal{X}$ define the function $f_C$ on $X$ as follows. If $C$ is a proper subset of $X$, then, denoting by $B$ the parent of $C$, we set

$$f_C = \frac{1}{\mu(C)} 1_C - \frac{1}{\mu(B)} 1_B \quad (3.6)$$

(note that always $\mu(C) > 0$). Also, let $\lambda(C) := 1/\rho(B)$. If $C = X$ (which can only be the case when $X$ is compact), then let $f_C \equiv 1$ and $\lambda(C) = 0$.

**Theorem 3.8.** For any $C \in \mathcal{X}$ the function $f_C$ is an eigenfunction of $\mathcal{L}$ with the eigenvalue $\lambda(C)$. The family $\{f_C : C \in \mathcal{X}\}$ is complete (its linear span is dense) in $L^2(X, \mu)$. Consequently, the operator $\mathcal{L}$ has a complete system of compactly supported eigenfunctions.

**Proof.** Fix a ball $C \in \mathcal{X}$ of radius $r = \rho(C)$, and let $B$ be the parent of radius $r' = \rho(B)$. Any ball of radius $s < r'$ is either disjoint with $C$ or contained in $C$, which implies that $1_C$ is constant in any such ball. It follows that for any $s < r'$ we have $Q_s 1_C = 1_C$ and similarly $Q_s 1_B = 1_B$, so that

$$Q_s f_C = f_C.$$

For $s \geq r'$ any ball of radius $s$ either contains both $C$ and $B$ or is disjoint from $B$. Since the averages of the two functions $\frac{1}{\mu(C)} 1_C$ and $\frac{1}{\mu(B)} 1_B$ over any ball containing $C$ and $B$ are equal, we find that in this case $Q_s f_C = 0$. It follows that

$$\mathcal{L} f_C = -\int_{(0, \infty)} \frac{1}{s} Q_s f_C \, ds = \frac{1}{r'} f_C = \lambda(C) f_C,$$

which proves that $f_C$ is an eigenfunction of $\mathcal{L}$ with the eigenvalue $\lambda(C)$. In the case of compact $X$ we have $Q_s f_X = f_X$ for all $s > 0$, whence $\mathcal{L} f_X = 0 = \lambda(X)$.

Let us show that the system $\{f_C : C \in \mathcal{X}\}$ is complete. We assume that some function $f \in L^2$ is orthogonal to all the functions $f_C$ and prove that $f \equiv \text{const}$. We have for any $r > 0$

$$(Q_r f, f_C)_{L^2} = (f, Q_r f_C)_{L^2} = \text{const}(f, f_C)_{L^2} = 0,$$

where we have used the fact that any eigenfunction of $\mathcal{L}$ is also an eigenfunction of $Q_r$ with an eigenvalue which we denoted by const. Hence, $Q_r f$ is also orthogonal to all $f_C$. We will prove below that $Q_r f = 0$, which will imply by (2.2) that $f = 0$.

Since $Q_r f$ is constant in any ball of radius $r$, by renaming $Q_r f$ back to $f$ we can assume from now on that $f$ is constant in any ball of radius $r$. Fix some ball $C \in \mathcal{X}$ and its parent $B$. It follows from (3.6) that the equality $(f, f_C)_{L^2} = 0$ is equivalent to

$$\frac{1}{\mu(C)} \int_C f \, d\mu = \frac{1}{\mu(B)} \int_B f \, d\mu,$$

that is, the average value of $f$ over a ball is preserved when switching to its parent. Starting with two balls $C_1$ and $C_2$ of radii $r$, we can build a sequence of their
predecessors which ends up with the same (sufficiently large) ball. This implies that the averages of $f$ in $C_1$ and $C_2$ are the same. Since $f$ is constant in $C_1$ and $C_2$, it follows that the values of these constants are the same. It follows that $f \equiv \text{const}$ on $X$. If $\mu(X) = \infty$, then $f \equiv 0$. If $\mu(X) < \infty$, then using the orthogonality of $f$ to $f_X \equiv 1$, we again get that $f \equiv 0$. \hfill \Box

For any ball $B$ with $\rho(B) > 0$ define the subspace $\mathcal{H}_B$ of $L^2$ as

$$\mathcal{H}_B = \text{span}\{f_C : C \prec B\}. \quad (3.7)$$

By Theorem 3.8, all non-zero functions in $\mathcal{H}_B$ are eigenfunctions of $\mathcal{L}$ with eigenvalue $1/\rho(B)$.

It follows from Lemma 3.7 that the functions $\{1_C : C \prec B\}$ are linearly independent and

$$\sum_{C \prec B} 1_C = 1_B. \quad (3.8)$$

This implies that

$$\sum_{C \prec B} \mu(C)f_C = 0 \quad (3.8)$$

and that this is the only dependence among the functions $f_C$. Hence,

$$\dim \mathcal{H}_B = \deg(B) - 1. \quad (3.9)$$

Clearly, the spaces $\mathcal{H}_B$ and $\mathcal{H}_B'$ are orthogonal provided the balls $B, B'$ are disjoint.

Let

$$\Lambda := \{d_*(x, y) : x, y \in X, \; x \neq y\} = \bigcup_{x \in X} \Lambda(x). \quad (3.10)$$

Theorem 3.8 implies the following.

**Corollary 3.9.** The spectrum $\text{spec} \mathcal{L}$ of the Laplacian $\mathcal{L}$ is pure point and

$$\text{spec} \mathcal{L} = \left\{\frac{1}{r} : r \in \Lambda \right\} \cup \{0\}.$$

The space $L^2(X, \mu)$ decomposes into an orthogonal sum of finite-dimensional eigenspaces as follows: if $\mu(X) = \infty$, then

$$L^2(X, \mu) = \bigoplus_{\rho(B) > 0} \mathcal{H}_B,$$

and if $\mu(X) < \infty$, then

$$L^2(X, \mu) = \{\text{const}\} \oplus \bigoplus_{\rho(B) > 0} \mathcal{H}_B.$$

**Example 3.10.** Let $(X, d, \mu)$ be as in Example 2.24, that is, $X = \mathbb{Q}_p$, $d(x, y) = \|x - y\|_p$ is the $p$-adic distance, and $\mu$ is the Haar measure. For some $\alpha > 0$ let

$$\sigma(r) = \exp\left(-\left(\frac{p}{r}\right)^\alpha\right),$$
so that by (2.39)
\[ d_*(x, y) = \left( \frac{\|x - y\|_p}{p} \right)^\alpha. \]
Since the set of non-zero values of \( \|x - y\|_p \) is \( \{ p^k \}_{k \in \mathbb{Z}} \), it follows that the set \( \Lambda \) of all non-zero values of \( d_*(x, y) \) is
\[ \Lambda = \{ p^{\alpha k} : k \in \mathbb{Z} \}. \]
Hence,
\[ \text{spec} \mathcal{L} = \{ p^{\alpha k} : k \in \mathbb{Z} \} \cup \{ 0 \}. \]

Corollary 3.11. Let \((X, d)\) be a non-compact, proper ultra-metric space. Let \( M \subset [0, \infty) \) be any closed set (unbounded if \( X \) contains at least one non-isolated point) which accumulates at 0. Then the following are true.

(a) There is a proper ultra-metric \( d' \) on \( X \) which generates the same topology as \( d \) and for which the isotropic Laplacian \( \mathcal{L}' \) of the triple \((d', \mu, \sigma_*)\) has the spectrum \( \text{spec} \mathcal{L}' = M \).

(b) Suppose in addition that there is a partition of \( X \) into \( d \)-balls which consists of infinitely many non-singletons. Then the ultra-metric \( d' \) of part (a) can be chosen so that the collections of \( d \)-balls and \( d' \)-balls coincide.

Proof. The set
\[ D = \{ x \in (0, \infty) : x^{-1} \in M \} \cup \{ 0 \} \]
is a closed, unbounded subset of \([0, \infty)\) containing 0. Then the statement (a) is equivalent to the existence of a proper ultra-metric \( d' \) on \( X \) which generates the same topology as \( d \) and is such that the closure of the value set \( \{ d'(x, y) \}_{x, y \in X} \) of \( d' \) coincides with \( D \). This metric property was proved by Bendikov and Krupski in [8], §2. Given \( \mu \), the Laplacian associated with the triple \((d', \mu, \sigma_*)\) has the required property by Corollary 3.9. The proof of (b) follows in the same way from a result in [8], §2. \( \square \)

3.3. The Dirichlet form and jump kernel. We construct a Dirichlet form \((\mathcal{E}, \text{dom}_\mathcal{E})\) associated with the isotropic semigroup \( \{ P^t \} \). It is well known that if \( P^t 1 = 1 \), which is the case here, then
\[ \mathcal{E}(f, f) = \lim_{t \to 0} \frac{1}{2t} \int_X \int_X p(t, x, y) (f(x) - f(y))^2 \, d\mu(x) \, d\mu(y) \]
and
\[ \text{dom}_\mathcal{E} = \{ f \in L^2 : \mathcal{E}(f, f) < \infty \} \]
(see [27]). Using the identity (2.18), we get that
\[ \frac{p(t, x, y)}{t} \to \int_0^{1/d_*(x,y)} N(x, \tau) \, d\tau \text{ as } t \searrow 0. \]
Setting
\[ J(x, y) := \int_0^{1/d_*(x,y)} N(x, \tau) \, d\tau = \int_{d_*(x,y)}^\infty \frac{1}{V(x, s)} \frac{ds}{s^2}, \quad (3.11) \]
we get by the monotone convergence theorem that for all \( f \in L^2 \)

\[
\mathcal{E}(f, f) = \frac{1}{2} \int_X \int_X (f(x) - f(y))^2 J(x, y) \, d\mu(x) \, d\mu(y).
\]

Note that \( 0 < J(x, y) = J(y, x) < \infty \) for all \( x \neq y \), while \( J(x, x) = \infty \).

The polarization identity then implies that for all \( f, g \in \text{dom} \mathcal{E} \)

\[
\mathcal{E}(f, g) = \frac{1}{2} \int_X \int_X (f(x) - f(y)) (g(x) - g(y)) J(x, y) \, d\mu(x) \, d\mu(y).
\]  

(3.12)

The function \( J \) is called the **jump kernel** of the Dirichlet form \( \mathcal{E} \). We show here that it can also be used to describe the generator \( L \) of \( \{P_t\} \). Recall that by the theory of Dirichlet forms, the generator \( L \) has the following equivalent definition: it is the self-adjoint operator with \( \text{dom} L \subset \text{dom} \mathcal{E} \) in \( L^2 \) such that

\[
(Lf, g) = \mathcal{E}(f, g)
\]

for all \( f \in \text{dom} \mathcal{E} \) and \( g \in \text{dom} \mathcal{E} \).

Denote by \( \mathcal{V}_r \) the image of the operator \( Q_r \) (defined with respect to \( d_\ast \)), that is, the space of all \( L^2 \)-functions which are constant on each ball of radius \( r \). Also, let

\[
\mathcal{V} := \bigcup_{r > 0} \mathcal{V}_r
\]

and observe that \( \mathcal{V} \) is a linear subspace of \( L^2 \). Observe also that the space \( \mathcal{V}_c \) of all locally constant functions with compact support is contained in \( \mathcal{V} \).

**Theorem 3.12.** The space \( \mathcal{V} \) is dense in \( L^2 \), it is a subset of \( \text{dom} \mathcal{E} \), and for any \( f \in \mathcal{V} \)

\[
L f(x) = \int_X (f(x) - f(y)) J(x, y) \, d\mu(y).
\]  

(3.13)

**Proof.** That \( \mathcal{V} \) is dense in \( L^2 \) follows from (2.2). In fact, \( \mathcal{V}_c \) is also dense in \( L^2 \), which follows from the fact that all the eigenfunctions of \( \mathcal{L} \) lie in \( \mathcal{V}_c \).

By (2.6) and (3.4) we have \( Q_r = 1_{[0, 1/r)}(\mathcal{L}) \). Therefore, \( \mathcal{L} Q_r \) is a bounded operator, which implies that \( \text{dom} \mathcal{E} \supset \mathcal{V}_r \) and, hence, \( \text{dom} \mathcal{E} \supset \mathcal{V} \).

Fix a function \( f \in \mathcal{V}_r \) with \( r > 0 \) and let

\[
u(x) = \int_X |f(x) - f(y)| J(x, y) \, d\mu(y).
\]

We show that \( u \in L^2 \). Observe that \( f(x) = f(y) \) whenever \( d_\ast(x, y) \leq r \). Hence, we can restrict the integration to the domain \( \{d_\ast(x, y) > r\} \). By the Cauchy–Schwarz inequality we have

\[
u^2(x) \leq \left( \int_X |f(x) - f(y)|^2 J(x, y) \, d\mu(y) \right) \left( \int_{\{y: d_\ast(x, y) > r\}} J(x, y) \, d\mu(y) \right).
\]  

(3.14)

Let us show that

\[
\int_{\{y: d_\ast(x, y) > r\}} J(x, y) \, d\mu(y) \leq \frac{1}{r}.
\]
Indeed, by (3.11) and Fubini’s theorem, the latter integral is equal to

\[
\int_{\{y: d_*(x,y) > r\}} \int_{\{s: s \geq d_*(x,y)\}} \frac{1}{V(x,s)} \frac{ds}{s^2} d\mu(y)
\]

\[
= \int_r^\infty \frac{ds}{s^2 V(x,s)} \int_{\{y: r < d_*(x,y) \leq s\}} d\mu(y)
\]

\[
= \int_r^\infty \frac{V(x,s) - V(x,r)}{s^2 V(x,s)} ds
\]

\[
\leq \int_r^\infty \frac{ds}{s^2} = \frac{1}{r} .
\]

It follows from (3.14) that

\[
\int_X u^2 d\mu \leq \frac{1}{r} \mathcal{E}(f,f).
\]

Since \( f \in \text{dom} \mathcal{L} \subset \text{dom} \mathcal{E} \), we find that \( u \in L^2 \). In particular, \( u(x) < \infty \) for almost all \( x \in X \). Consequently, for almost all \( x \in X \) the function

\[
y \mapsto (f(x) - f(y))J(x,y)
\]

is in \( L^1 \), and its integral

\[
v(x) = \int_X (f(x) - f(y))J(x,y) d\mu(y)
\]

is an \( L^2 \) function. We need to verify that \( \mathcal{L} f = v \). For that purpose it suffices to verify that, for any \( g \in \text{dom} \mathcal{E} \),

\[
(v,g)_{L^2} = \mathcal{E}(f,g).
\]

Indeed, using Fubini’s theorem, we have

\[
(v,g)_{L^2} = \int_X \int_X (f(x) - f(y))g(x)J(x,y) d\mu(y) d\mu(x)
\]

\[
= \int_X \int_X (f(y) - f(x))g(y)J(y,x) d\mu(x) d\mu(y)
\]

\[
= \frac{1}{2} \int_X \int_X (f(x) - f(y))(g(x) - g(y))J(x,y) d\mu(x) d\mu(y)
\]

\[
= \mathcal{E}(f,g),
\]

which was to be proved. \( \square \)
3.4. The $L^p$-spectrum of the Laplacian. It is known that any continuous symmetric Markov semigroup can be extended to all the spaces $L^p$, $1 \leq p < \infty$, as a continuous contraction semigroup. In particular, this is true for the semigroup $\{P^t\}$. We use the same notation for the extended semigroup, while we denote by $\mathcal{L}_p$ its infinitesimal generator and by $\text{dom}\mathcal{L}_p$ its domain in $L^p$.

**Theorem 3.13.** For all $1 \leq p < \infty$

$$\text{spec}\mathcal{L}_p = \text{spec}\mathcal{L}_2.$$ 

**Proof.** Since by Theorem 3.8 all the eigenfunctions of $\mathcal{L}_2$ are compactly supported, they also belong to $L^p$, which implies that

$$\text{spec}\mathcal{L}_2 \subset \text{spec}\mathcal{L}_p.$$ 

To prove the opposite inclusion, we choose $\lambda_0 \notin \text{spec}\mathcal{L}_2$ and show that $\lambda_0 \notin \text{spec}\mathcal{L}_p$. For that purpose it suffices to show that the resolvent operator

$$R := (\mathcal{L}_2 - \lambda_0 \text{id})^{-1},$$ 

being a bounded operator on $L^2$, extends to a bounded operator on $L^p$. The latter amounts to showing that, for any functions $f \in L^2 \cap L^p$ and $g \in L^2 \cap L^q$, where $q = p/(p-1)$ is the Hölder conjugate of $p$, the following inequality holds:

$$|\langle Rf, g \rangle_{L^2}| \leq C \|f\|_{L^p} \|g\|_{L^q}$$ 

with a constant $C$ which does not depend on $f$, $g$.

Let us restrict ourselves to the case $\lambda_0 > 0$ (the case when $\lambda_0 < 0$ is simpler). Choose $a, b > 0$ such that $a < \lambda_0 < b$ and $[a, b]$ is disjoint from $\text{spec}\mathcal{L}_2$. Using the spectral decomposition (3.4), we obtain

$$R = \int_{\text{spec}\mathcal{L}_2} \frac{dE_\lambda}{\lambda - \lambda_0} = \int_{[0,a)} \frac{dE_\lambda}{\lambda - \lambda_0} + \int_{[b,\infty)} \frac{dE_\lambda}{\lambda - \lambda_0},$$ 

whence

$$\langle Rf, g \rangle = \int_{[0,a)} \frac{d(E_\lambda f, g)}{\lambda - \lambda_0} + \int_{[b,\infty)} \frac{d(E_\lambda f, g)}{\lambda - \lambda_0}.$$

Integration by parts gives

$$\langle Rf, g \rangle = \left(\frac{E_a f, g}{a - \lambda_0} + \int_{[0,a)} \frac{(E_\lambda f, g)}{(\lambda - \lambda_0)^2} d\lambda \right) - \left(\frac{E_b f, g}{b - \lambda_0} + \int_{[b,\infty)} \frac{(E_\lambda f, g)}{(\lambda - \lambda_0)^2} d\lambda \right).$$

Since $E_\lambda = Q_{1/\lambda}$ is a Markov operator, it extends in the standard way to a bounded operator on $L^p$ with the norm bound 1, so that

$$|E_\lambda f, g| \leq \|f\|_{L^p} \|g\|_{L^q}.$$

It follows that

$$|\langle Rf, g \rangle| \leq \|f\|_{L^p} \|g\|_{L^q} \left(\frac{1}{\lambda_0 - a} + \frac{1}{b - \lambda_0} + \int_{[0,a) \cup [b,\infty)} \frac{d\lambda}{(\lambda - \lambda_0)^2} \right),$$

which finishes the proof since the quantity in the large parentheses is finite. $\square$
The last theorem of this section concerns a Liouville property. Note that the semigroup $\{P^t\}$ defined by (1.6) acts on the space $\mathcal{B}_b$ of bounded Borel functions as a contraction semigroup, but it is not continuous unless $X$ is discrete. Define convergence of sequences in $\mathcal{B}_b$ as bounded pointwise convergence, that is, a sequence $\{f_k\} \subset \mathcal{B}_b$ converges in $\mathcal{B}_b$ to a function $f$ if the whole sequence $\{f_k\}$ is uniformly bounded and $f_k(x) \to f(x)$ as $k \to \infty$ for all $x \in X$. We define a weak infinitesimal generator $L_\infty$ of the semigroup $\{P^t\}$ on $\mathcal{B}_b$ as follows: the domain $\text{dom } L_\infty$ consists of the functions $f \in \mathcal{B}_b$ such that the limit $L_\infty f := \lim_{t \to 0} f - P^t f$ exists in the sense of convergence in $\mathcal{B}_b$. This implies that $L_\infty f \in \mathcal{B}_b$ for any $f \in \text{dom } L_\infty$.

**Theorem 3.14** (Strong Liouville property). Any Borel function $f: X \to [0, \infty)$ which satisfies $P f = f$ must be constant.

Consequently, 0 is an eigenvalue of $L_\infty$ of multiplicity 1.

**Proof.** Since $P$ and $Q_r$ commute, we get from $f = P f$ and

$$P f = \int_0^\infty Q_s f \, d\sigma_*(s)$$

that for all $r \geq 0$

$$Q_r f = PQ_r f = \int_{[0,\infty)} Q_s Q_r f \, d\sigma_*(s).$$

Observing that

$$Q_s Q_r = Q_{\text{max}(r,s)},$$

we have

$$Q_r f = \int_{[0,r)} Q_r f \, d\sigma_*(s) + \int_{[r,\infty)} Q_s f \, d\sigma_*(s).$$

The first integral here is equal to $\sigma_*(r)Q_r f$, which implies that

$$(1 - \sigma_*(r))Q_r f = \int_{[r,\infty)} Q_s f \, d\sigma_*(s).$$

(3.16)

Fix some $x \in X$. By Lemma 3.3, the set $\Lambda(x)$ of all values $d_*(x,y)$ for $y \neq x$ has no accumulation point in $(0, +\infty)$. Choose $r_0$ as follows: if $\Lambda(x)$ does not accumulate at 0, then $r_0 = 0$, and if $\Lambda(x)$ accumulates at 0, then $r_0$ is any value in $\Lambda(x)$. In both cases, the set $\Lambda(x) \cap (r, \infty)$ consists of a (finite or infinite) sequence $r_1 < r_2 < \cdots$ which converges to $\infty$ in the case when it is infinite. Applying (3.16) to $r = r_k$ and $r = r_{k+1}$ instead of $r$, where $k \geq 0$, we see that

$$(1 - \sigma_*(r_k))Q_{r_k} f(x) - (1 - \sigma_*(r_{k+1}))Q_{r_{k+1}} f(x) = \int_{[r_k, r_{k+1})} Q_s f(x) \, d\sigma_*(s)$$

$$= Q_{r_k} f(x) (\sigma_*(r_{k+1}) - \sigma_*(r_k)).$$
from which it follows that
\[(1 - \sigma_*(r_{k+1})) Q_{r_k} f(x) = (1 - \sigma_*(r_{k+1})) Q_{r_{k+1}} f(x),\]
and hence
\[Q_{r_k} f(x) = Q_{r_{k+1}} f(x).\]
Consequently,
\[Q_{r_k} f(x) = Q_{r_0} f(x) \quad \text{for all} \quad k \geq 1.\]
Since \(r_0\) can be chosen arbitrarily close to \(0\), \(Q_r f(x)\) does not depend on \(r\). For any two points \(x, y \in X\) we have \(Q_r f(x) = Q_r f(y)\) for \(r \geq d_*(x, y)\). Therefore, the function \(Q_r f(x)\) is constant with respect to both \(r\) and \(x\). It follows from (3.15) that \(f = P f\) is also a constant.

For the second statement of the theorem, \(0\) is an eigenvalue of \(L_\infty\) because \(L_\infty 1 = 0\). We assume that \(L_\infty f = 0\) and prove that \(f \equiv \text{const}\), which will imply that the multiplicity of \(0\) is \(1\). By assumption we have \(f \in \mathcal{B}_b\) and
\[
\frac{f - P_t f}{t} \xrightarrow{\mathcal{B}_b} 0 \quad \text{as} \quad t \to 0.
\]
Since the family \(\left\{ \frac{f - P_t f}{t} \right\}_{t > 0}\) is uniformly bounded, we get from the dominated convergence theorem that for any \(r \geq 0\)
\[Q_r \left( \frac{f - P_t f}{t} \right) \xrightarrow{\mathcal{B}_b} 0 \quad \text{as} \quad t \to 0,
\]
which in turn implies that for all \(s \geq 0\)
\[
\frac{P^s f - P^{s+t} f}{t} = P^s \left( \frac{f - P_t f}{t} \right) \xrightarrow{\mathcal{B}_b} 0 \quad \text{as} \quad t \to 0.
\]
It follows that for any \(x \in X\) the function \(s \mapsto P^s f(x)\) has derivative \(0\) and hence is constant. It follows that \(f = P f\), and by the first statement of the theorem, we conclude that \(f = \text{const}\).

4. Moments of the Markov process

Let \(\mathcal{X}_t\) be the Markov process associated with the semigroup \(\{P^t\}\). For any \(\gamma > 0\), the moment of order \(\gamma\) of the process is defined as
\[M_\gamma(x, t) = E_x (d_*(x, \mathcal{X}_t)^\gamma),\]
where \(E_x\) is the expectation with respect to the probability measure on the trajectory space of \(\mathcal{X}_t\) that governs the process starting at \(x\). In terms of the heat kernel \(p(t, x, y)\) the moment is given by
\[M_\gamma(x, t) = \int_X d_*(x, y)^\gamma p(t, x, y) \, d\mu(y). \quad (4.1)\]
The aim of this section is to estimate \(M_\gamma(x, t)\) as a function of \(t\) and \(\gamma\).
Let us start with two lemmas. We use the intrinsic volume function (2.48), that is,

\[ V(x, r) = \mu(B^*_r(x)) \]

and its average moment function of order \( \gamma \), that is,

\[ R_\gamma(x, \tau) = \frac{1}{V(x, \tau)} \int_{(0, \tau]} r^\gamma dV(x, r). \]

**Lemma 4.1.** For all \( x \in X \), \( t > 0 \), and \( \gamma > 0 \),

\[ M_\gamma(x, t) = t \int_0^\infty R_\gamma(x, \frac{1}{t}) e^{-\tau t} d\tau = \int_0^\infty R_\gamma(x, \frac{t}{s}) e^{-s} ds. \]

**Proof.** Using the equations (4.1) and (2.24), as well as Definition 2.8 of the spectral distribution function in terms of the volume function, we get that

\[ M_\gamma(x, t) = \int_X d_*(x, y)^\gamma p(t, x, y) d\mu(y) \]

\[ = \int_{(0, \infty)} r^\gamma \left( t \int_0^{1/r} N(x, \tau) e^{-\tau t} d\tau \right) dV(x, r) \]

\[ = \int_0^\infty \left( \int_{(0, 1/\tau)} r^\gamma dV(x, r) \right) te^{-\tau t} d\tau = \int_0^\infty R_\gamma(x, \frac{1}{\tau}) te^{-\tau t} d\tau. \]

In the third identity, we have used Fubini’s theorem. \( \square \)

The volume function \( r \mapsto V(x, r) \) is non-decreasing and varies from 0 to \( \mu(X) \).

In the compact case, \( V(x, r) = \mu(X) \) for all \( r \geq r^*_\text{max} = r^*_\text{max}(x) \), the largest value in \( \Lambda(x) \) (see (3.5)). When \( x \) is isolated, \( V(x, r) = \mu\{x\} \) for all \( 0 \leq r < r^*_0 = r^*_0(x) \), the smallest positive value in \( \Lambda(x) \).

**Lemma 4.2.** For any given \( x \in X \) and \( \gamma > 0 \), the following properties hold.

(a) The function \( \tau \mapsto R_\gamma(x, \tau) \) is non-decreasing.

- If \( X \) is compact, then \( R_\gamma(x, \tau) = R_\gamma(x, r^*_\text{max}(x)) \) for all \( \tau \geq r^*_\text{max}(x) \).

- If \( X \) is discrete and infinite, then \( R_\gamma(x, \tau) = R_\gamma(x, r^*_0(x)) \) for all \( 0 \leq \tau \leq r^*_0(x) \).

(b) For all \( \tau > 0 \)

\[ R_\gamma(x, \tau) \leq \tau^\gamma, \]

and if the volume function \( r \mapsto V(x, r) \) has the reverse doubling property, then there is a constant \( c > 0 \) such that

\[ R_\gamma(x, \tau) \geq c\tau^\gamma \] (4.2)

for all \( \tau > 0 \). In the non-discrete compact case, if the volume function only has the reverse doubling property at zero, then (4.2) holds for all \( 0 < \tau < r^*_\text{max}(x) \). In the discrete infinite case, if the volume function only has the reverse doubling property at infinity, then (4.2) holds for all \( \tau > r^*_0(x) \).
Proof. For the first part of (a), we integrate by parts:

\[
R_\gamma(x, \tau) = \frac{1}{V(x, \tau)} \left( \tau^\gamma V(x, \tau) - \int_{(0, \tau]} V(x, s) \, ds^\gamma \right)
\]

\[
= \int_{(0, \tau]} \left( 1 - \frac{V(x, s)}{V(x, \tau)} \right) ds^\gamma,
\]

whence \( \tau \mapsto R_\gamma(x, \tau) \) is non-decreasing.

The second part of (a) is straightforward.

Regarding (b), the general upper bound on \( R_\gamma(x, \tau) \) is obvious. If the volume function has the reverse doubling property, then in the corresponding range,

\[
R_\gamma(x, \tau) \geq \frac{1}{V(x, \tau)} (\delta \tau)^\gamma (V(x, \tau) - V(x, \delta \tau))
\]

\[
= (\delta \tau)^\gamma \left( 1 - \frac{V(x, \delta \tau)}{V(x, \tau)} \right) \geq \delta^\gamma (1 - \kappa) \tau^\gamma = c \tau^\gamma
\]

for suitable constants \( 0 < \kappa, c < 1 \).

Now in order to estimate the moment function \( t \mapsto M_\gamma(x, t) \), we need to estimate a Laplace-type integral as given by the formula of Lemma 4.1. We will treat such estimates in the two technical Propositions 4.6 and 4.7 at the end of this section. Before that, in the next three theorems we anticipate the statements of the results regarding the moment function.

**Theorem 4.3.** Assume that \((X, d)\) is non-compact and has no isolated points. Then the following properties hold.

1. For all \( x \in X \), \( t > 0 \), and \( 0 < \gamma < 1 \)

\[
M_\gamma(x, t) \leq \frac{t^\gamma}{1 - \gamma}.
\]

2. If for some \( x \in X \) the volume function has the reverse doubling property, then for any \( 0 < \gamma < 1 \)

\[
M_\gamma(x, t) \geq \frac{c}{1 - \gamma} t^\gamma
\]

for all \( x \), all \( t > 0 \), and some \( c > 0 \). Moreover,

\[
M_\gamma(z, t) = \infty
\]

for all \( z \), all \( t > 0 \), and all \( \gamma \geq 1 \).

**Theorem 4.4.** Assume that \((X, d)\) is discrete and infinite. Then the following properties hold.

(a) For all \( x \), all \( t > 0 \), and \( 0 < \gamma < 1 \)

\[
M_\gamma(x, t) \leq \frac{C}{1 - \gamma} \min\{t, t^\gamma\}
\]

for some \( C > 0 \).
(b) If for some (equivalently, all) \( x \in X \) the volume function has the reverse doubling property at infinity, then for any \( 0 < \gamma < 1 \)

\[
M_\gamma(z, t) \geq \frac{c}{1 - \gamma} \min\{t, t^\gamma\}
\]

for all \( z \), all \( t > 0 \), and some \( c > 0 \). Moreover,

\[
M_\gamma(z, t) = \infty
\]

for all \( z \), all \( t > 0 \), and all \( \gamma \geq 1 \).

Assume now that \( (X, d) \) is compact and let \( D \) be its \( d_\ast \)-diameter. By Lemmas 4.1 and 4.2, for all \( x \in X, \gamma > 0 \), and \( t > 0 \),

\[
M_\gamma(x, t) \leq R_\gamma(x, D) \leq D^\gamma,
\]

and from this we obtain information about the behaviour of the moment function \( t \mapsto M_\gamma(x, t) \) at zero.

**Theorem 4.5.** Assume that \( (X, d) \) is non-discrete and compact. Then the following properties hold.

1. There is a constant \( C > 0 \) such that

\[
M_\gamma(x, t) \leq C \begin{cases} t & \text{if } \gamma > 1, \\ t \left( \log \frac{1}{t} + 1 \right) & \text{if } \gamma = 1, \\ t^\gamma & \text{if } \gamma < 1 \end{cases}
\]

holds for all \( x \) and all \( 0 < t \leq 1 \).

2. If for some \( x \in X \) the volume function has the reverse doubling property at zero, then there is a constant \( c > 0 \) such that

\[
M_\gamma(z, t) \geq c \begin{cases} t & \text{if } \gamma > 1, \\ t \left( \log \frac{1}{t} + 1 \right) & \text{if } \gamma = 1, \\ t^\gamma & \text{if } \gamma < 1 \end{cases}
\]

holds for all \( z \) and all \( 0 < t \leq 1 \).

We now provide the technical details regarding the Laplace-type estimates which imply Theorems 4.3, 4.4, and 4.5. In the following two propositions, \( M \) and \( R \) will always be two non-negative, non-decreasing functions related by the Laplace-type integral

\[
M(t) = \int_0^\infty R\left( \frac{t}{\tau} \right) e^{-\tau} d\tau.
\]

**Proposition 4.6.** Let \( \gamma > 0 \) be given.

1. Assume that

\[
As^\gamma \geq R(s) \quad \text{(or that, respectively, } R(s) \geq Bs^\gamma) \tag{4.3}
\]
for some $A > 0$ (respectively, $B > 0$) and all $s > 0$. Then the inequality
\[
\frac{A t^\gamma}{1 - \gamma} \geq M(t) \quad \left( \text{respectively, } M(t) \geq \frac{B t^\gamma}{(1 - \gamma)e} \right)
\]
holds for all $0 < \gamma < 1$ and all $t > 0$.

(2) Assume that there is a $t_0 > 0$ such that $R(s) = 0$ for all $0 < s < t_0$. Assume also that the first (respectively, the second) inequality in (4.3) holds for all $s > t_0$. Then
\[
M(t) \leq \frac{c}{1 - \gamma} \min \left\{ \frac{t}{t_0}, \left( \frac{t}{t_0} \right)^\gamma \right\}
\]
(respectively $M(t) \geq \frac{c'}{1 - \gamma} \min \left\{ \frac{t}{t_0}, \left( \frac{t}{t_0} \right)^\gamma \right\}$)
for all $0 < \gamma < 1$, all $t > 0$, and some constants $c, c' > 0$.

(3) The assumption $\gamma \geq 1$ and the lower bound $R(s) \geq B s^\gamma$ imply that $M(t) = \infty$ for all $t > 0$.

Proof. It is known that for $0 < \gamma < 1$ the gamma function satisfies
\[
\frac{1}{(1 - \gamma)e} < \Gamma(1 - \gamma) < \frac{1}{1 - \gamma},
\]
so the first claim follows by the monotonicity of the Laplace-type integral.

To prove the second statement, we write
\[
M(t) = \int_{\{t/s \geq t_0\}} R \left( \frac{t}{s} \right) e^{-s} ds.
\]
First assume that $R(s) \leq A s^\gamma$ for all $0 < s < \infty$. Then
\[
M(t) \leq A \int_{\{t/s \geq t_0\}} \left( \frac{t}{s} \right)^\gamma e^{-s} ds = At^\gamma \int_{\{s \leq t/t_0\}} s^{-\gamma} e^{-s} ds
\]
\[
\leq At^\gamma \int_0^{t/t_0} s^{-\gamma} ds = \left( \frac{t}{t_0} \right) \frac{At_0^{-\gamma}}{1 - \gamma}
\]
and
\[
M(t) \leq At^\gamma \int_0^\infty s^{-\gamma} e^{-s} ds \leq \frac{At^\gamma}{1 - \gamma} = \left( \frac{t}{t_0} \right)^\gamma \frac{At_0^{-\gamma}}{1 - \gamma}.
\]
It follows that
\[
M(t) \leq \frac{A \max\{t_0, t_0^{-1}\}}{1 - \gamma} \min \left\{ \frac{t}{t_0}, \left( \frac{t}{t_0} \right)^\gamma \right\}.
\]
Second, assume that $R(s) \geq B s^\gamma$ for all $s \geq t_0$. Then for $t/t_0 \geq 1$
\[
M(t) \geq B t^\gamma \int_0^{t/t_0} s^{-\gamma} e^{-s} ds \geq \frac{B t^\gamma}{e} \int_0^1 s^{-\gamma} ds = \frac{B t^\gamma}{(1 - \gamma)e} = \frac{B t^\gamma}{(1 - \gamma)e} \left( \frac{t}{t_0} \right)^\gamma.
\]
When \( t/t_0 \leq 1 \) we have

\[
M(t) \geq Bt^\gamma \int_0^{t/t_0} s^{-\gamma}e^{-s\gamma} \, ds \\
= \frac{Bt^\gamma}{(1-\gamma)e} \left( \frac{t}{t_0} \right)^{1-\gamma} = \frac{Bt_0^\gamma}{(1-\gamma)e} \left( \frac{t}{t_0} \right).
\]

It follows that

\[
M(t) \geq \frac{Bt_0^\gamma}{(1-\gamma)e} \min \left\{ \frac{t}{t_0}, \left( \frac{t}{t_0} \right)^\gamma \right\} \geq B \min \{t_0, 1\} \min \left\{ \frac{t}{t_0}, \left( \frac{t}{t_0} \right)^\gamma \right\}.
\]

This proves the second claim.

For the third claim observe that if \( R(s) \geq Bs^\gamma \) for all \( s \geq t_0 \) and \( \gamma \geq 1 \), then

\[
M(t) \geq Bt^\gamma \int_0^{t/t_0} s^{-\gamma}e^{-s\gamma} \, ds = \infty
\]

for all \( t > 0 \). 

**Proposition 4.7.** Assume that there is a \( t_0 > 0 \) such that \( R(s) = R(t_0) \) for all \( s \geq t_0 \). Assume also that the first (respectively, the second) inequality in (4.3) holds for all \( 0 < s \leq t_0 \). Then

\[
M(t) \leq \begin{cases} 
  c_1 \frac{t}{t_0} & \text{if } \gamma > 1, \\
  c_2 t \left( \log \frac{t_0}{t} + 1 \right) & \text{if } \gamma = 1, \\
  c_3 \left( \frac{t}{t_0} \right)^\gamma & \text{if } \gamma < 1,
\end{cases}
\]

respectively,

\[
M(t) \geq \begin{cases} 
  c'_1 \frac{t}{t_0} & \text{if } \gamma > 1, \\
  c'_2 t \left( \log \frac{t_0}{t} + 1 \right) & \text{if } \gamma = 1, \\
  c'_3 \left( \frac{t}{t_0} \right)^\gamma & \text{if } \gamma < 1,
\end{cases}
\]

for all \( 0 < t \leq t_0 \) and some positive constants \( c_1, c'_1, c_2, c'_2, c_3, c'_3 \).

**Proof.** Let \( \gamma > 1 \) and \( 0 < t < t_0 \). According to our assumption

\[
M(t) = \int_{\{t/s \leq t_0\}} R \left( \frac{t}{s} \right) e^{-s\gamma} \, ds + R(t_0)(1 - e^{-t/t_0}).
\]

Observe that for \( 0 < t < t_0 \)

\[
\frac{t}{2t_0} \leq 1 - e^{-t/t_0} \leq \frac{t}{t_0}.
\]
First, if $R(s) \leq As^\gamma$ for all $0 < s < t_0$, then
\[
M(t) \leq At^\gamma \int_{t/t_0}^{\infty} s^{-\gamma} e^{-s} \, ds + \frac{R(t_0)t}{t_0} \leq As^\gamma \int_{t/t_0}^{\infty} s^{-\gamma} \, ds + \frac{R(t_0)t}{t_0}
\]
\[
\leq \frac{At^\gamma}{\gamma - 1} \left( \frac{t}{t_0} \right)^{1-\gamma} + \frac{R(t_0)t}{t_0} = \frac{t}{t_0} \left( R(t_0) + \frac{At_0^\gamma}{\gamma - 1} \right).
\]
Second, if $R(s) \geq Bs^\gamma$ for all $0 < s < t_0$, then
\[
M(t) \geq \frac{R(t_0)}{2} \frac{t}{t_0}.
\]
Assume that $0 < \gamma < 1$ and $0 < t < t_0$. Again, first, if $R(s) \leq As^\gamma$ for all $0 < s < t_0$, then
\[
M(t) \leq At^\gamma \int_{t/t_0}^{\infty} s^{-\gamma} e^{-s} \, ds + \frac{R(t_0)t}{t_0} \leq At^\gamma \Gamma(1 - \gamma) + R(t_0) \frac{t}{t_0}
\]
\[
\leq \frac{At^\gamma}{1-\gamma} + R(t_0) \frac{t}{t_0} = \frac{At_0^\gamma}{1-\gamma} \left( \frac{t}{t_0} \right)^\gamma + R(t_0) \frac{t}{t_0}
\]
\[
\leq \left( \frac{t}{t_0} \right)^\gamma \left( \frac{At_0^\gamma}{1-\gamma} + R(t_0) \right).
\]
Second, once more, when $R(s) \geq Bs^\gamma$ for all $0 < s < t_0$, then
\[
M(t) \geq Bt^\gamma \int_{t/t_0}^{\infty} s^{-\gamma} e^{-s} \, ds \geq Bt^\gamma \int_{1}^{\infty} s^{-\gamma} e^{-s} \, ds \geq \left( \frac{t}{t_0} \right)^\gamma B \min\{t_0, 1\} \frac{e}{e^2}.
\]
Finally, assume that $\gamma = 1$ and $0 < t < t_0$. First, if $R(s) \leq As^\gamma$ for all $0 < s < t_0$, then
\[
M(t) \leq At \int_{t/t_0}^{\infty} s^{-1} e^{-s} \, ds + \frac{R(t_0)t}{t_0}
\]
\[
= At \left( \int_{1}^{\infty} s^{-1} e^{-s} \, ds + \int_{t/t_0}^{1} s^{-1} e^{-s} \, ds \right) + \frac{R(t_0)t}{t_0}
\]
\[
\leq At \left( \int_{1}^{\infty} \frac{ds}{s^2} + \int_{t/t_0}^{1} \frac{ds}{s} \right) + \frac{R(t_0)t}{t_0} = \left( A + \frac{R(t_0)}{t_0} \right) t \left( \log \frac{t_0}{t} + 1 \right).
\]
And finally, if $R(s) \geq Bs^\gamma$ for all $0 < s < t_0$, then
\[
M(t) \geq Bt \int_{t/t_0}^{\infty} s^{-1} e^{-s} \, ds + \frac{R(t_0)t}{2t_0}
\]
\[
\geq \frac{Bt}{e} \int_{t/t_0}^{1} \frac{ds}{s} + \frac{R(t_0)t}{2t_0} = \frac{Bt}{e} \log \frac{t_0}{t} + \frac{R(t_0)t}{2t_0}
\]
\[
= \frac{Bt}{e} \left( \log \frac{t_0}{t} + \frac{R(t_0)e}{2Bt_0} \right) \geq \min \left\{ \frac{R(t_0)}{2t_0}, \frac{B}{e} \right\} t \left( \log \frac{t_0}{t} + 1 \right). \quad \square
\]

Theorems 4.3, 4.4, and 4.5 follow.
5. Analysis in $\mathbb{Q}_p$ and $\mathbb{Q}_p^n$

5.1. The $p$-adic fractional derivative. Consider the field $\mathbb{Q}_p$ of $p$-adic numbers endowed with the $p$-adic norm $\|x\|_p$ and the $p$-adic ultra-metric $d_p(x, y) = \|x - y\|_p$. Let $\mu_p$ be the Haar measure on $\mathbb{Q}_p$, normalized so that $\mu_p(\mathbb{Z}_p) = 1$. Let $\mathcal{V}_c$ be the space of locally constant functions on $\mathbb{Q}_p$ with compact support, which will be considered as test functions on $\mathbb{Q}_p$.

The notion of $p$-adic fractional derivative, closely related to the concept of $p$-adic quantum mechanics, was introduced in several papers by Vladimirov [57], Vladimirov and Volovich [58], and Vladimirov, Volovich and Zelenov [59]. In particular, a one-parameter family $\{D_\alpha\}_{\alpha > 0}$ of operators called fractional derivative operators of order $\alpha$ was introduced in [57].

Recall that the Fourier transform $\mathcal{F}: f \mapsto \hat{f}$ of a function $f$ on the self-dual locally compact Abelian group $\mathbb{Q}_p$ is defined by

$$\hat{f}(\theta) = \int_{\mathbb{Q}_p} \langle x, \theta \rangle f(x) \, d\mu_p(x),$$

where $x, \theta \in \mathbb{Q}_p$,

$$\langle x, \theta \rangle = \exp(2\pi \sqrt{-1} \{x\theta\})$$

and $\{x\theta\}$ is the fractional part of the $p$-adic number $x\theta$ (cf. (2.36)). It is known that $\mathcal{F}$ is a linear isomorphism of $\mathcal{V}_c$ onto itself.

**Definition 5.1.** The operator $(\mathfrak{D}^\alpha, \mathcal{V}_c), \alpha > 0$, is defined via the Fourier transform on the locally compact Abelian group $\mathbb{Q}_p$ by

$$\mathfrak{D}^\alpha f(\xi) = \|\xi\|_p^\alpha \hat{f}(\xi), \quad \xi \in \mathbb{Q}_p.$$

It was shown by the above-named authors that each operator $(\mathfrak{D}^\alpha, \mathcal{V}_c)$ can be written as a Riemann–Liouville-type singular integral operator

$$\mathfrak{D}^\alpha f(x) = \frac{p^\alpha - 1}{1 - p^{-\alpha - 1}} \int_{\mathbb{Q}_p} \frac{f(x) - f(y)}{\|x - y\|_p^{1+\alpha}} \, d\mu_p(y). \quad (5.1)$$

The aim of this section is in particular to show that the operator $(\mathfrak{D}^\alpha, \mathcal{V}_c)$ is in fact the restriction to $\mathcal{V}_c$ of an appropriate isotropic Laplacian. We use the following distance distribution function:

$$\sigma_\alpha(r) = \exp\left(-\left(\frac{p}{r}\right)^\alpha\right).$$

Denote by $\{P^t_\alpha\}$ the isotropic semigroup associated with the triple $(d_p, \mu_p, \sigma_\alpha)$, and let $\mathcal{L}_\alpha$ be the corresponding Laplacian.

**Theorem 5.2.** For any $\alpha > 0$

$$(\mathcal{L}_\alpha, \mathcal{V}_c) = (\mathfrak{D}^\alpha, \mathcal{V}_c). \quad (5.2)$$

**Proof.** By Theorem 3.12, we have, for any $f \in \mathcal{V}_c$,

$$\mathcal{L}_\alpha f(x) = \int_{\mathbb{Q}_p} (f(x) - f(y)) J_\alpha(x, y) \, d\mu_p(y),$$

where

$$J_\alpha(x, y) = \int_{\mathbb{Q}_p} \frac{1}{\|x - y\|_p^{1+\alpha}} d\mu_p(z).$$

Since $\sigma_\alpha(r)$ is a distance distribution function, the statement of the theorem follows.
where
\[ J_\alpha(x, y) = \int_0^\infty s^{-2} \frac{ds}{\mu_p(B_s^*(x))}. \]

As in Example 2.24,
\[ d_*(x, y) = \left(\frac{\|x - y\|_p}{p}\right)^\alpha, \quad (5.3) \]

whence
\[ B_s^*(x) = B_{ps^{1/\alpha}}(x). \]

The change \( r = ps^{1/\alpha} \) yields
\[ J_\alpha(x, y) = p^\alpha \int_0^\infty \frac{\alpha r^{-\alpha - 1} dr}{\mu_p(B_r(x))}. \]

Since the value set of the metric \( \|x - y\|_p \) is \( \{p^n\}_{n \in \mathbb{Z}} \), we see from (2.37) that
\[ \mu_p(B_r(x)) = p^n \text{ if } p^n \leq r < p^{n+1}, \quad (5.4) \]

which implies for \( \|x - y\|_p = p^k \) that
\[
\int_{p^k}^\infty \frac{\alpha r^{-\alpha - 1} dr}{\mu_p(B_r(x))} = \sum_{n \geq k} \left( \int_{p^n}^{p^{n+1}} \frac{\alpha r^{-\alpha - 1} dr}{\mu_p(B_r(x))} \right).
\]
\[
= \sum_{n \geq k} \frac{p^{n+1}}{p^n} \frac{1}{p^n} = \sum_{n \geq k} \frac{1}{p^n} \left( \frac{1}{p^{n\alpha}} - \frac{1}{p^{(n+1)\alpha}} \right)
\]
\[
= \left( 1 - \frac{1}{p^{\alpha k}} \right) \sum_{n \geq k} \frac{1}{p^{n(\alpha + 1)}} = \left( 1 - \frac{1}{p^{\alpha}} \right) \frac{p^{-k(\alpha + 1)}}{1 - p^{-(\alpha + 1)}}
\]
\[
= \frac{1 - p^{-\alpha}}{1 - p^{-(\alpha + 1)}} \left( \frac{1}{p^k} \right)^{\alpha + 1} = \frac{1 - p^{-\alpha}}{1 - p^{-(\alpha + 1)}} \left( \frac{1}{\|x - y\|_p} \right)^{\alpha + 1}.
\]

Hence, we obtain the identity
\[ J_\alpha(x, y) = \frac{p^\alpha - 1}{1 - p^{-\alpha - 1}} \frac{1}{\|x - y\|_p^{\alpha + 1}}, \quad (5.5) \]

which in view of (5.1) finishes the proof. □

The heat kernel for the semigroup \( \{P^t_\alpha\} \) was estimated in Example 2.24. We restate this estimate here as a theorem.

**Theorem 5.3.** The semigroup \( \{P^t_\alpha\} \) has a continuous transition density \( p_\alpha(t, x, y) \) with respect to Haar measure \( \mu_p \), and for all \( t > 0 \) and \( x, y \in \mathbb{Q}_p \) it satisfies the estimate
\[ p_\alpha(t, x, y) \asymp \frac{t}{(t^{1/\alpha} + \|x - y\|_p)^{1+\alpha}}. \quad (5.6) \]

The upper bound in (5.6) was also obtained by a different method by Kochubei ([39], Chap. 4.1, Lemma 4.1).
Theorem 5.4. The semigroup \( \{P^t_\alpha\} \) is transient if and only if \( \alpha < 1 \). In the transient case its Green function \( g_\alpha \) is given explicitly by

\[
g_\alpha(x, y) = \frac{1 - p^{-\alpha}}{1 - p^{-\alpha - 1}} \|x - y\|_p^{-1}. \tag{5.7}
\]

This formula for a fundamental solution of \( \mathfrak{D}_\alpha \) acting in the Bruhat–Schwartz space of distributions \( \gamma'_\alpha \) was obtained by Vladimirov ([57], Theorem 1) and Kochubei ([39], Chap. 2.2).

Proof. That \( \alpha < 1 \) is equivalent to transience was shown in Example 2.30. Assuming that \( \alpha < 1 \), we get by (2.45) that

\[
g_\alpha(x, y) = \int_{d_\alpha(x, y)}^\infty ds \frac{\mu_p(B^*_s(x))}{\mu_p(B_r(x))} = \frac{1}{p^\alpha} \int_{\|x - y\|_p}^\infty \alpha r^{\alpha - 1} dr \mu_p(B_r(x)).
\]

Setting \( \|x - y\|_p = p^k \) and using (5.4), we have

\[
g_\alpha(x, y) = \frac{1}{p^\alpha} \sum_{n \geq k} \int_{p^n}^{p^{n+1}} \frac{dr}{p^n} = \frac{1}{p^\alpha} \sum_{n \geq k} \frac{1}{p^n} (p^{(n+1)\alpha} - p^{n\alpha}) = \frac{1 - p^{-\alpha}}{1 - p^{\alpha - 1}} \lambda^{(\alpha - 1)k}.
\]

Denote by \( \mathcal{L}_{\alpha,q} \) the generator of the semigroup \( \{P^t_\alpha\} \) acting in \( L^q(\mu_p) \), \( 1 \leq q < \infty \). Applying Corollary 3.9 and Theorem 3.13, we obtain the following.

Theorem 5.5. For any \( \alpha > 0 \) and \( 1 \leq q < \infty \)

\[
\text{spec } \mathcal{L}_{\alpha,q} = \{p^{\alpha k} : k \in \mathbb{Z}\} \cup \{0\}.
\]

Each \( \lambda_k = p^{\alpha k} \) is an eigenvalue with infinite multiplicity.

Proof. We only need to show that the multiplicity of \( \lambda_k \) is infinite. In the general setting of Theorem 3.8 and Corollary 3.9, some eigenvalues may well have finite multiplicity and some not. Indeed, each ball \( B \) with the minimal positive \( d_* \)-radius \( \rho \) generates a finite-dimensional eigenspace \( \mathcal{H}_B \) which consists of eigenfunctions with the eigenvalue \( 1/\rho \). It follows that the eigenvalue \( 1/\rho \) has finite multiplicity if and only if there are only finitely many distinct balls of \( d_* \)-radius \( \rho \).

In the present setting in \( \mathbb{Q}_p \) there are infinitely many disjoint balls of the same radius \( \rho \), since they can all be obtained by translations of one such ball. Thus, all the eigenvalues have infinite multiplicity. \( \square \)

Let \( \{X_t\} \) be the Markov process on \( \mathbb{Q}_p \) driven by the Markov semigroup \( \{P^t_\alpha\}_{t>0} \). The semigroup is translation invariant, and thus the process has independent and stationary increments. For any given \( \gamma > 0 \) and \( t > 0 \), consider the moment of order \( \gamma \) of \( \mathcal{X}_t \) defined in terms of the \( p \)-adic distance \( d_p(x, y) \):

\[
\mathcal{M}_\gamma(t) = E(\|\mathcal{X}_t\|_p^\gamma),
\]

where \( E \) is expectation with respect to the probability measure on the trajectory space of the process starting at 0. Applying Theorem 4.3 and using the relation (5.3) between \( d_* \) and \( \|\cdot\|_p \), we obtain the following estimates.
Theorem 5.6. The moment $M_{\gamma}(t)$ is finite if and only if $\gamma < \alpha$. In that case there is a constant $\kappa = \kappa(\alpha) > 0$ such that

$$\frac{\kappa t^{\gamma/\alpha}}{\alpha - \gamma} \leq M_{\gamma}(t) \leq \frac{\alpha t^{\gamma/\alpha}}{\alpha - \gamma}.$$ 

5.2. Rotation-invariant Markov semigroups. Let $\{P_t\}_{t \geq 0}$ be a symmetric translation-invariant Markov semigroup on the additive Abelian group $Q_p$. This semigroup acts in $C_0(Q_p)$, the Banach space of continuous functions vanishing at $\infty$. It follows that there is a weakly continuous convolution semigroup $\{p_t\}_{t > 0}$ of symmetric probability measures on $Q_p$ such that

$$P_t f(x) = p_t \ast f(x). \quad (5.8)$$

Since the probability measures $p_t$ are symmetric, the following identity holds, and it is basic in the theory of infinite divisible distributions:

$$\tilde{p}_t(\zeta) = \exp(-t\Psi(\zeta)),$$

where $\Psi: Q_p \to \mathbb{R}_+$ is a non-negative-definite symmetric function on $Q_p$. By the Lévy–Khinchin formula,

$$\Psi(\zeta) = \int_{Q_p \setminus \{0\}} (1 - \Re\langle x, \zeta \rangle) \, d\mathcal{J}(x),$$

where $\mathcal{J}$ is a symmetric Radon measure on $Q_p \setminus \{0\}$, the Lévy measure associated with the non-negative-definite function $\Psi$ (see for the details the book of Berg and Forst [10]).

Definition 5.7. For any $a \in Q_p$ with $\|a\|_p = 1$ we define the rotation operator $\theta_a: Q_p \to Q_p$ by $\theta_a(x) = ax$. We say that the Markov semigroup $\{P_t\}$ is rotation invariant if

$$\theta_a(p_t) = p_t \quad \text{for all } a \in Q_p \text{ with } \|a\|_p = 1. \quad (5.9)$$

Let $\mathcal{L}$ be the generator of $\{P_t\}$, that is, $P_t = \exp(-t\mathcal{L})$. It is easy to see that (5.9) is equivalent to $\theta_a \circ \mathcal{L} = \mathcal{L} \circ \theta_a$. In this case we also say that $\mathcal{L}$ is rotation invariant. By construction, any isotropic Markov semigroup $\{P^t\}$ defined on the ultra-metric measure space $(Q_p, d_p, \mu_p)$ is rotation invariant. As we will see, the class of all isotropic Markov semigroups is actually a proper subset of the class of rotation-invariant Markov semigroups.

Assume that the semigroup $\{P_t\}$ is rotation invariant. Then for all $a$ such that $\|a\|_p = 1$ we have

$$\Psi(a\zeta) = \Psi(\zeta) \quad \text{and} \quad \theta_a(\mathcal{J}) = \mathcal{J}. \quad (5.10)$$

Since the Haar measure $\mu_p$ of each sphere is strictly positive, (5.9) and (5.10) imply that the measures $p_t$ and $\mathcal{J}$ are absolutely continuous with respect to $\mu_p$ and have densities $p_t(x)$ and $J(x)$ which depend only on $\|x\|_p$. The same is true for the function $\Psi$, so that

$$J(x) = j(\|x\|_p) \quad \text{and} \quad \Psi(\zeta) = \psi(\|\zeta\|_p).$$
All the above shows that, for the generator \( L \) of \( \{P_t\} \), we have \( V_c \subset \text{dom}_L \) and

\[
L u = \psi(\mathcal{D}) u, \quad u \in V_c,
\]

where \( \mathcal{D} = \mathcal{D}^1 \) is the fractional derivative operator of order \( \alpha = 1 \), which we identify with the isotropic Laplacian \( L_1 \) by Theorem 5.2.

It follows from (5.11) and (5.2) that the eigenfunctions \( \{f_C : C \in \mathcal{K}\} \) of the operator \((L, V_c)\) acting in \( L^2 \) form a complete system of eigenfunctions as described in Theorem 3.8. Associated with each ball \( B \) of radius \( p^m \) there is the \((p - 1)\)-dimensional eigenspace \( \mathcal{H}_B \) spanned by all the functions \( f_C \), where \( C \) runs through all the balls that are children of \( B \), and the corresponding eigenvalue is

\[
\lambda(m) = \psi(p^{-m+1}).
\]

Let \( \{a(m)\}_{m \in \mathbb{Z}} \) be a sequence of real numbers satisfying

\[
a(m) \geq a(m + 1), \quad a(+\infty) = 0 \quad \text{and} \quad 0 < a(-\infty) = W \leq +\infty.
\]

Define the sequence \( \{\lambda(m)\}_{m \in \mathbb{Z}} \) by

\[
\lambda(m) = a(m) - (p - 1)^{-1} \{a(m + 1) - a(m)\}.
\]

**Theorem 5.8.** A sequence \( \{\lambda(m)\}_{m \in \mathbb{Z}} \) of real numbers represents the spectrum \( \text{spec} L \) of a rotation-invariant Laplacian \( L \) on \( \mathbb{Q}_p \) if and only if it is given by (5.13) with a sequence \( a(m) \) which satisfies (5.12).

**Proof.** Consider a rotation-invariant Laplacian \( L = \psi(\mathcal{D}) \). Let us compute the non-negative-definite function \( \Psi(\zeta) = \psi(\|\zeta\|_p) \) associated with \( L \). We have

\[
\psi(\|\zeta\|_p) = \int_{\mathbb{Q}_p \setminus \{0\}} (1 - \text{Re}(\langle x, \zeta \rangle)) j(\|x\|_p) \, d\mu_p(x)
\]

\[
= \sum_{k \in \mathbb{Z}} j(p^k) \int_{\{x : \|x\|_p = p^k\}} (1 - \text{Re}(\langle x, \zeta \rangle)) \, d\mu_p(x).
\]

According to Vladimirov ([57], Example 4),

\[
\int_{\{x : \|x\|_p = p^k\}} \langle x, \zeta \rangle \, d\mu_p(x) = \begin{cases} p^k - p^{k-1} & \text{if } \|\zeta\|_p \leq p^{-k}, \\ -p^{k-1} & \text{if } \|\zeta\|_p = p^{-k+1}, \\ 0 & \text{if } \|\zeta\|_p \geq p^{-k+2}. \end{cases}
\]

In particular, we have

\[
\int_{\{x : \|x\|_p = p^k\}} d\mu_p(x) = p^k - p^{k-1}.
\]

Let \( \|\zeta\|_p = p^{-m+1} \); then the above computations give us that

\[
\psi(p^{-m+1}) = j(p^m)p^m + (1 - p^{-1}) \sum_{k \geq m+1} j(p^k)p^k.
\]

(5.14)
Define the non-increasing sequence \( \{a(m)\}_{m \in \mathbb{Z}} \) by
\[
a(m) = (1 - p^{-1}) \sum_{k \geq m} j(p^k)p^k = (1 - p^{-1}) \int_{\{x: \|x\|_p \geq p^m\}} j(\|x\|_p) \, d\mu_p(x).
\] (5.15)

By (5.15), the equalities (5.14) will acquire the following form:
\[
\psi(p^{-m+1}) = \frac{p}{p-1} (a(m) - a(m+1)) + a(m+1) = a(m) - (p^{-1})^{-1}(a(m+1) - a(m)).
\] (5.16)

Let \( \lambda(m) \) be the eigenvalue of the Laplacian \( (\psi(\mathcal{D}), \mathcal{Y}_c) \) corresponding to the ball \( B \) of radius \( p^m \). Then \( \lambda(m) = \psi(p^{-m+1}) \), and the identity (5.16) gives the desired result, namely, the equation (5.13).

Conversely, given a sequence \( \{a(m)\} \) as in (5.12), we define the sequence \( \{\lambda(m)\} \) by (5.13) and we set
\[
\Psi(\xi) = \psi(\|\xi\|_p), \quad \text{where } \psi(p^m) = \lambda(-m+1),
\]
and \( J(x) = j(\|x\|_p) \), where \( j(p^m) = \frac{a(m) - a(m+1)}{p^m - p^{m-1}} \). (5.17)

It is straightforward to show that
\[
\Psi(\zeta) = \int_{\mathbb{Q}_p \setminus \{0\}} \left(1 - \text{Re}\langle x, \zeta \rangle\right) J(x) \, d\mu_p(x),
\]
whence \( \Psi \) is a non-negative-definite function. It follows that the function \( \exp(-t \Psi) \) is positive-definite, so it is the Fourier transform of a probability measure \( p_t \). Clearly, \( \{p_t\}_{t>0} \) is a weakly continuous convolution semigroup of probability measures. By construction, each measure \( p_t \) is rotation invariant. Finally, we can define the translation-invariant Markov semigroup by \( P_t f = f * p_t \).

**Corollary 5.9.** In the above notation the following statements are equivalent:

1. the sequence \( \lambda(m) \) is non-increasing;
2. the sequence \( \psi(p^m) \) is non-decreasing;
3. the sequence \( j(p^m) \) is non-increasing.

In particular, if the sequence \( a(m) \) is convex, then each of the equivalent properties (1)–(3) holds.

**Proof.** The equivalence (1) \( \Leftrightarrow \) (2) follows from the relation \( \lambda(m) = \psi(p^{-m+1}) \). To prove that (1) \( \Leftrightarrow \) (3), we apply (5.17) and find that
\[
\lambda(m) - \lambda(m+1) = (p^m - p^{m-1})(j(p^m) - j(p^{m+1})).
\]
The equivalence (1) \( \Leftrightarrow \) (2) \( \Leftrightarrow \) (3) follows. Finally, (5.13) and the convexity of \( a(m) \) imply (1). □

Next, we consider strict monotonicity.
Corollary 5.10. The following statements are equivalent:

(i) the sequence \( \lambda(m) \) is strictly decreasing and \( \lambda(-\infty) = +\infty \);

(ii) the sequence \( \psi(p^m) \) is strictly increasing and \( \psi(+\infty) = +\infty \);

(iii) the sequence \( j(p^m) \) is strictly increasing and

\[
\int j(\|x\|_p) \, d\mu_p(x) = +\infty;
\]

(iv) the associated rotation-invariant Markov semigroup \( \{P_t\} \) is isotropic.

In particular, if the sequence \( a(m) \) is strictly convex and \( a(-\infty) = +\infty \), then each of the equivalent properties (i)–(iv) holds.

Proof. The equivalence (i) \( \iff \) (ii) \( \iff \) (iii) follows by the same arguments as in the proof of Corollary 5.9. The convexity of \( a(m) \) together with \( a(-\infty) = +\infty \) imply (i) following the same argument. We are left to show that (iv) \( \iff \) (ii).

Assume that \( \{P_t\} \) is an isotropic Markov semigroup as constructed in (1.3)–(1.8). The semigroup admits a continuous transition density \( p(t, x, y) = p_t(x - y) \) with respect to the Haar measure \( \mu_p \); the function \( p_t \) is given by

\[
p_t(y) = \int_0^\infty q_s(y) \, d\sigma_t(s), \quad \text{where } q_s(y) = \frac{1}{\mu_p(B_s(0))} 1_{B_s(0)}(y). \quad (5.18)
\]

To find the Fourier transform \( \hat{p}_t(\xi) \), we argue as follows. The ball \( B_s(0), p^k \leq s < p^{k+1} \), is the compact subgroup \( p^{-k}Z_p \) of \( \mathbb{Q}_p \), and therefore the measure \( \omega_s = q_s \mu_p \) coincides with the normalized Haar measure of that compact subgroup. Since for any locally compact Abelian group, the Fourier transform of the normalized Haar measure of any compact subgroup is the indicator of its annihilator group, and in our particular case the annihilator of the group \( p^{-k}Z_p \) is the group \( p^kZ_p \), we obtain

\[
\widehat{\omega}_s(\xi) = 1_{p^kZ_p}(\xi) = 1_{[0, p^{-k}]}(\|\xi\|_p), \quad \text{where } p^k \leq s < p^{k+1}.
\]

It follows that when \( \|\xi\|_p = p^{-l} \),

\[
\hat{p}_t(\xi) = \sum_{k: k \leq l} (\sigma^t(p^{k+1}) - \sigma^t(p^k)) = \sigma^t(p^{l+1}) = \exp(-t\psi(\|\xi\|_p)),
\]

whence

\[
\psi(p^{-l}) = \log \frac{1}{\sigma(p^{l+1})}.
\]

According to (1.5), the sequence \( \sigma(p^l) \) is assumed to be strictly increasing and to tend to zero as \( l \to -\infty \). Thus, \( \psi(p^m) \) is as claimed in (ii). Conversely, if a strictly increasing sequence \( \psi(p^m) \) as in (ii) is given, we define the strictly increasing sequence

\[
\sigma(p^m) = \exp(-\psi(p^{-m+1})).
\]

Let \( \sigma: [0, \infty) \to [0, 1) \) be any increasing bijection which takes the values \( \sigma(p^m) \) at the points \( p^m \). We define the function \( p_t(y) \) by the equation (5.18). Since \( \sigma(+\infty) = 1 \), this is a probability density with respect to \( \mu_p \). It is straightforward that \( \{p_t\}_{t \geq 0} \) gives rise to a weakly continuous convolution semigroup of probability measures on \( \mathbb{Q}_p \). Moreover, each \( p_t \) is rotation invariant by construction. Thus, the semigroup \( P_t: f \mapsto f \ast p_t \) is isotropic. \( \Box \)
Remark 5.11. In [2] Albeverio and Karwowski began with a sequence \(\{a(m)\}_{m}^{\infty}\) as in (5.12) and used the classical approach of backward and forward Kolmogorov equations to construct a Markov semigroup \(\{P_{t}\}\) on the ultra-metric measure space \((Q_{p}, d_{p}, \mu_{p})\). In particular, they showed in [2], Theorem 3.2 that the Laplacian \(L\) of that semigroup has a pure point spectrum \(\{\lambda(m)\}\) as in (5.13), and the \(\lambda(m)\)-eigenspace is spanned by the functions \(f_{B}\), where \(B\) runs over all balls of radius \(p^{m-1}\). Our Theorem 5.8 shows that in fact the class of Markov semigroups constructed in [2] coincides with the class of rotation-invariant Markov semigroups.

5.3. Product spaces. Let \(\{(X_{i}, d_{i})\}_{i=1}^{n}\) denote a finite sequence of ultra-metric spaces; we assume that all the \((X_{i}, d_{i})\) are separable and that all balls are compact. Let \((X, d)\) be their Cartesian product, \(X = X_{1} \times \cdots \times X_{n}\), and for \(x = (x_{i}) \in X\) and \(y = (y_{i}) \in X\) let

\[d(x, y) = \max \{d_{i}(x_{i}, y_{i}) : i = 1, 2, \ldots, n\}.\]

Thus, \((X, d)\) is a separable ultra-metric space, all balls in \((X, d)\) are compact, and moreover, each \(d\)-ball \(B_{r}(a)\) in \(X\) is a product of \(d_{i}\)-balls \(B_{i}^{r}(a_{i})\) in \(X_{i}\) of the same radius.

Given a Radon measure \(\mu_{i}\) on each \((X_{i}, d_{i})\), we define \(\mu = \otimes \mu_{i}\) on \((X, d)\). Let \(\mathcal{V}_{c}\) be the set of all compactly supported locally constant functions on \((X, d)\).

Consider the ultra-metric measure space \((X, d, \mu)\). According to the previous sections, there is a rich class of isotropic Markov semigroups and corresponding Laplacians on \((X, d, \mu)\) as constructed in (1.3)–(1.8). Thanks to the product structure of \((X, d, \mu)\) one can define in a natural way a non-trivial and interesting class of Markov semigroups and Laplacians which are not isotropic. Namely, choosing on each \((X_{i}, d_{i}, \mu_{i})\) an isotropic Markov semigroup \(\{P_{t}^{i}\}\), we define a Markov semigroup \(\{P_{t}\}\) on \((X, d, \mu)\) as the tensor product of the \(\{P_{t}^{i}\}\),

\[P_{t} = \bigotimes_{i=1}^{n} P_{t}^{i}.\]

The semigroup \(\{P_{t}\}\) has the following heat kernel:

\[p(t, x, y) = \prod_{i=1}^{n} p_{i}(t, x_{i}, y_{i}),\]

where \(p_{i}\) is the heat kernel of \(\{P_{t}^{i}\}\).

The generator \(\mathcal{L}\) of \(P_{t}\) can be described as follows: \(\mathcal{V}_{c} \subset \text{dom}\mathcal{L}\) and for any \(f \in \mathcal{V}_{c}\) we have

\[\mathcal{L} f(x) = \sum_{i=1}^{n} \mathcal{L}_{i} f(x),\] (5.19)

where \(x = (x_{1}, \ldots, x_{n})\) and \(\mathcal{L}_{i}\) acts on \(x_{i}\). It follows that

\[\mathcal{L} f(x) = \int_{X} (f(x) - f(y)) J(x, dy),\]
where
\[ J(x, dy) = \sum_{i=1}^{n} J_i(x_i, y_i) d\mu_i(y_i) \]
and \( J_i(x_i, y_i) \) is the jump kernel of \( \mathcal{L}_i \).

In particular, we see that for each \( x \in X \) the measures \( J(x, dy) \) and \( \mu(dy) \) are not necessarily mutually absolutely continuous (in the case when at least one of \( X_t \) is perfect, \( J(x, dy) \) is singular with respect to \( \mu \)), which implies that the semigroup \( \{P_t\} \) is not necessarily an isotropic Markov semigroup.

In this paper we do not intend to develop a general theory on product spaces. Our aim is to study in detail two specific examples related to \( p \)-adic analysis.

In the first example we consider the Vladimirov Laplacian, which matches the above general construction well. In the second example we consider the Taibleson Laplacian defined in terms of the multidimensional Riesz kernels (see Taibleson [55] and Rodríguez-Vega and Zúñiga-Galindo [50]). We show that the Taibleson Laplacian is isotropic. This will allow us to improve the heat kernel bounds from [50] and to obtain some new results (transience/recurrence, independence of \( \frac{1}{16} < p < \infty \) for the \( L^p \)-spectrum, sharp bounds for the moments of the corresponding Markov process, and so on).

Consider the linear space \( \mathbb{Q}_p^n = \mathbb{Q}_p \times \cdots \times \mathbb{Q}_p \) over the field \( \mathbb{Q}_p \) and define in \( \mathbb{Q}_p^n \) a norm
\[ \|z\|_p = \max\{\|z_i\|_p : i = 1, 2, \ldots, n\}. \]
(5.20)
It clearly satisfies the ultra-metric triangle inequality (1.1) and is homogeneous in the following sense:
\[ \|az\|_p = \|a\|_p \|z\|_p \quad \text{for all} \ a \in \mathbb{Q}_p, \ z \in \mathbb{Q}_p^n. \]
Let
\[ d_p(x, y) = \|x - y\|_p, \]
so that \( (\mathbb{Q}_p^n, d_p) \) is an ultra-metric space.

Let \( \mu_p = \bigotimes \mu_{p,i} \) be the additive Haar measure on the Abelian group \( \mathbb{Q}_p^n \). As before, let \( \mathcal{V}_c \) be the set of all compactly supported locally constant functions on the ultra-metric space \( (\mathbb{Q}_p^n, d_p) \). Recall that \( \mathcal{V}_c \) is a dense subset of \( L^2 = L^2(\mathbb{Q}_p^n, \mu_p) \).

5.3.1. The Vladimirov Laplacian. For any given \( n \)-tuple \( \alpha = (\alpha_1, \ldots, \alpha_n) \) with entries \( \alpha_i > 0 \) we define the ultra-metric
\[ d_{p,\alpha}(x, y) = \max\{\|x_i - y_i\|_p^{\alpha_i} : i = 1, 2, \ldots, n\}, \]
also denoted by \( \|x - y\|_{p,\alpha} \). In particular, the ultra-metric \( d_{p}(x, y) \) defined above corresponds to the case \( \alpha = (1, \ldots, 1) \). The identity map
\[ (\mathbb{Q}_p^n, d_{p,\alpha}) \to (\mathbb{Q}_p^n, d_p) \]
is a homeomorphism, but is not bi-Lipschitz, unless \( \alpha_i = 1 \) for all \( i \). This fact plays an essential role in the study of the class of Laplacians introduced next as a special instance of (5.19).
Definition 5.12. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$. For any function $f \in V_c$ we define the operator

$$V^\alpha f(x) = \sum_{i=1}^{n} D_{x_i}^{\alpha_i} f(x),$$

where $x = (x_1, \ldots, x_n)$ and $D_{x_i}^{\alpha_i}$ is the $p$-adic fractional derivative of order $\alpha_i$ acting on $x_i$.

The operator $V^\alpha$ on $Q^3_p$ with $\alpha = (2, 2, 2)$ was introduced by Vladimirov [57] as an analogue of the classical Laplace operator in $\mathbb{R}^3$. This operator, which we denote for short by $V^2$, is translation invariant and homogeneous, that is,

$$V^2 \tau_y(f) = \tau_y(V^2 f), \text{ where } \tau_y f(x) = f(x + y),$$

and

$$V^2 \theta_a(f) = \|a\|^2_p \theta_a(V^2 f), \text{ where } \theta_a f(x) = f(ax_1, ax_2, ax_3).$$

It follows that the Green function $g(x, y)$ of the operator $V^2$ on $Q^3_p$ is also translation invariant and homogeneous:

$$g(x, y) = g(x - z, y - z), \quad g(ax, ay) = \frac{g(x, y)}{\|a\|^p}, \quad a \in Q_p.$$

In particular, setting $E(x) = g(x, 0)$, we obtain for all non-zero $a \in Q_p$ the identity

$$E(a, a, a) = \frac{E(1, 1, 1)}{\|a\|^p}. \tag{5.21}$$

This identity was observed in [57]. It gives an idea of how the Green function of the operator $V^2$ (in Vladimirov's terminology; the fundamental solution of the equation $V^2 E = \delta$) behaves at infinity/at zero. Below, in Proposition 5.15, we will prove that, for all non-zero $a = (a_1, a_2, a_3) \in Q^3_p$,

$$E(a_1, a_2, a_3) \sim \frac{1}{\|a\|^p}. \tag{5.21}$$

In fact, we shall prove similar estimates for more general operators $V^\alpha$ without the homogeneity property. We start by listing some properties of the operator $(V^\alpha, V_c)$ in Definition 5.12 which follow directly from the corresponding properties of the ‘one-dimensional Laplacians’ $D^\alpha_i$.

1. $(V^\alpha, V_c)$ is a non-negative-definite symmetric operator.
2. $(V^\alpha, V_c)$ admits a complete system of compactly supported eigenfunctions. In particular, the operator $(V^\alpha, V_c)$ is essentially self-adjoint.
3. The semigroup $\exp(-tV^\alpha)$ is symmetric and Markovian. It has a heat kernel $p_\alpha(t, x, y)$ of the form

$$p_\alpha(t, x, y) = \prod_{i=1}^{n} p_{\alpha_i}(t, x_i, y_i).$$

4. The semigroup $\exp(-tV^\alpha)$ is transient if and only if

$$A := \sum_{i=1}^{n} \frac{1}{\alpha_i} > 1.$$
5. For all $f \in \mathcal{V}_c$

$$\mathfrak{W}^\alpha f(x) = \int_{\mathbb{Q}_p^n} (f(x) - f(y)) \, J_\alpha(x, dy),$$

where

$$J_\alpha(x, dy) = \sum_{i=1}^{n} J_{\alpha_i}(x_i - y_i) \, d\mu_{p,i}(y_i)$$

and

$$J_{\alpha_i}(x_i - y_i) = \frac{p^{\alpha_i} - 1}{1 - p^{-\alpha_i - 1}} \frac{1}{\|x_i - y_i\|_p^{1 + \alpha_i}}.$$

In particular, the semigroup $\exp(-t \mathfrak{W}^\alpha)$ is in general not an isotropic Markov semigroup.

Observe that due to the group structure of $\mathbb{Q}_p^n$, the functions $(x, y) \mapsto p_\alpha(t, x, y)$ and $(x, y) \mapsto g_\alpha(x, y)$ are translation invariant. Hence, setting

$$p_\alpha(t, z) = p_\alpha(t, z, 0) \quad \text{and} \quad g_\alpha(z) = g_\alpha(z, 0),$$

we get that

$$p_\alpha(t, x, y) = p_\alpha(t, x - y) \quad \text{and} \quad g_\alpha(x, y) = g_\alpha(x - y).$$

**Proposition 5.13.** Let

$$A = \sum_{i=1}^{n} \frac{1}{\alpha_i}.$$  

Then the heat kernel satisfies the estimate

$$p_\alpha(t, z) \simeq t^{-A} \prod_{i=1}^{n} \min \left\{ 1, \frac{t^{1+1/\alpha_i}}{\|z_i\|_p^{1+\alpha_i}} \right\} \quad (5.22)$$

uniformly for all $t > 0$ and $z \in \mathbb{Q}_p^n$. In particular, for all $t > \|z\|_{p,\alpha}$

$$p_\alpha(t, z) \simeq t^{-A}. \quad (5.23)$$

**Proof.** By Theorem 5.3 we have

$$p_{\alpha_i}(t, z_i) \simeq \frac{t}{(t^{1/\alpha_i} + \|z_i\|_p)^{1+\alpha_i}} \simeq \frac{1}{t^{1/\alpha_i}} \frac{1}{\|z_i\|_p^{1+\alpha_i}} \min \left\{ 1, \frac{t^{1+1/\alpha_i}}{\|z_i\|_p^{1+\alpha_i}} \right\},$$

and the claim follows. $\Box$

**Proposition 5.14.** The semigroup $\exp(-t \mathfrak{W}^\alpha)$ is transient if and only if $A > 1$. If $A > 1$, then for all $z \in \mathbb{Q}_p^n$ and some $C_1 > 0$

$$g_\alpha(z) \geq C_1 \left( \frac{1}{\|z\|_{p,\alpha}} \right)^{A-1}.$$
For any \( \kappa > 0 \) let
\[
\Omega(\kappa) = \left\{ x \in \mathbb{Q}_+^n : \max_i \{ \|x_i\|^{\alpha_i} \} \leq \kappa \min_i \{ \|x_i\|^{\alpha_i} \} \right\}.
\]

Then, for all \( z \in \Omega(\kappa) \) and some constant \( C_2 > 0 \) depending on \( \kappa \),
\[
g_\alpha(z) \leq C_2 \left( \frac{1}{\|z\|_{p,\alpha}} \right)^{A-1}.
\]

**Proof.** The transience criterion \( A > 1 \) follows from \( p_\alpha(t, x, x) \simeq t^{-A} \). To prove the lower bound, we use (5.23) and write
\[
g_\alpha(z) = \int_0^\infty p_\alpha(t, z) \, dt \geq \int_{\|z\|_{p,\alpha}}^\infty p_\alpha(t, z) \, dt \geq C_1 \int_{\|z\|_{p,\alpha}}^\infty t^{-A} \, dt = c_1 \left( \frac{1}{\|z\|_{p,\alpha}} \right)^{A-1}.
\]

On the other hand,
\[
g_\alpha(z) = \left( \int_0^\infty p_\alpha(t, z) \, dt \right) + \left( \int_{\|z\|_{p,\alpha}}^\infty p_\alpha(t, z) \, dt \right) =: I + II.
\]

To estimate the second term \( II \) we again use (5.23):
\[
II \simeq \int_{\|z\|_{p,\alpha}}^\infty t^{-A} \, dt \simeq \left( \frac{1}{\|z\|_{p,\alpha}} \right)^{A-1}.
\]

To estimate the first term \( I \) we use (5.22):
\[
I \leq c \int_{\|z\|_{p,\alpha}}^\infty t^{-A} \prod_{i=1}^n \frac{t^{1+\alpha_i}}{\|z_i\|^{1+\alpha_i}} \, dt = c \int_{\|z\|_{p,\alpha}}^\infty \prod_{i=1}^n \frac{1}{\|z_i\|^{1+\alpha_i}} t^n \, dt = c' \prod_{i=1}^n \frac{1}{\|z_i\|^{1+\alpha_i}} \|z\|_{p,\alpha}^{n+1}.
\]

When \( z \in \Omega(\kappa) \), we have
\[
I \leq c'' \prod_{i=1}^n \frac{1}{\|z_i\|^{1+\alpha_i}} \left( \min_i \{ \|z_i\|_{p}^{\alpha_i} \} \right)^{n+1} \leq c'' \min_i \{ \|z_i\|_{p}^{\alpha_i} \} \prod_{i=1}^n \frac{1}{\|z_i\|_{p}}
\]
\[
= c'' \min_i \{ \|z_i\|_{p}^{\alpha_i} \} \prod_{i=1}^n \frac{1}{(\|z_i\|_{p}^{\alpha_i})^{1/\alpha_i}}.
\]

Next,
\[
\prod_{i=1}^n \frac{1}{(\|z_i\|_{p}^{\alpha_i})^{1/\alpha_i}} \leq \prod_{i=1}^n \frac{1}{(\min_j \{ \|z_j\|_{p}^{\alpha_j} \})^{1/\alpha_i}} = \left( \frac{1}{\min_j \{ \|z_j\|_{p}^{\alpha_j} \}} \right)^A,
\]
whence
\[
I \leq c'' \left( \frac{1}{\min_j \{ \|z_j\|_{p}^{\alpha_j} \}} \right)^{A-1}.
\]
Again using the fact that \( z \in \Omega(\kappa) \), we write

\[
\left( \frac{1}{\min_j \{ \| z_j \|_{p,\alpha} \}^\kappa} \right)^{A-1} \leq \left( \frac{\kappa}{\max_j \{ \| z_j \|_{p,\alpha} \}^\kappa} \right)^{A-1} = c(\kappa) \left( \frac{1}{\| z \|_{p,\alpha}} \right)^{A-1}.
\]

The upper bounds obtained for the integrals I and II imply the required upper bound for \( g_\alpha(z) \). 

**Proposition 5.15.** Let \( \alpha = (\alpha_1, \ldots, \alpha_n) = (\beta, \ldots, \beta) \) be an n-tuple having all entries equal to \( \beta \). Assume that \((n-1)/2 < \beta < n\). Then the semigroup \( \exp(-t \mathfrak{V}^\alpha) \) is transient and the Green function \( g_\alpha(z) \) satisfies the estimates

\[
g_\alpha(z) \simeq \left( \frac{1}{\| z \|_{p,\alpha}} \right)^{A-1}
\]

for all \( z \in \mathbb{Q}_p^n \) and some \( c_1, c_2 > 0 \).

Since \( A = n/\beta \) and \( \| z \|_{p,\alpha} = \| z \|_{p,\beta} \), the estimate (5.24) is equivalent to

\[
g_\alpha(z) \simeq \left( \frac{1}{\| z \|_p} \right)^{n-\beta}.
\]

**Proof.** Transience follows from Proposition 5.14 because \( A = n/\beta > 1 \). The same proposition yields the desired lower bound of the Green function. To prove the upper bound, we observe that the Laplacian \( \mathfrak{V}^\alpha \) is homogeneous, that is,

\[
\mathfrak{V}^\alpha \circ \theta_a = \| a \|_{\beta} \cdot \theta_a \circ \mathfrak{V}^\alpha
\]

for all \( a \in \mathbb{Q}_p \). This implies that the Green function \( g_\alpha(z) \) is also homogeneous, that is,

\[
g_\alpha(az) = \| a \|_{p}^{n-\beta} g_\alpha(z)
\]

for all \( a \in \mathbb{Q}_p \) and \( z \in \mathbb{Q}_p^n \).

Without loss of generality assume that \( \| z \|_{p,\alpha} = \| z_1 \|_{p,\beta} > 0 \). Then

\[
g_\alpha(z) = g_\alpha \left( z_1 \left( 1, \frac{z_2}{z_1}, \ldots, \frac{z_n}{z_1} \right) \right) = \| z_1 \|_{p,\beta}^{n-\beta} g_\alpha \left( 1, \frac{z_2}{z_1}, \ldots, \frac{z_n}{z_1} \right)
\]

\[
= \left( \frac{1}{\| z \|_{p,\alpha}} \right)^{A-1} g_\alpha \left( 1, \frac{z_2}{z_1}, \ldots, \frac{z_n}{z_1} \right)
\]

\[
\leq \left( \frac{1}{\| z \|_{p,\alpha}} \right)^{A-1} \sup \{ g_\alpha(1, x_2, \ldots, x_n) : x_i \in \mathbb{Z}_p \}.
\]

Next we apply our assumption \( \beta > (n-1)/2 \) and get from (5.22) that

\[
g_\alpha(1, x_2, \ldots, x_n) = \int_0^\infty p_\alpha(t, (1, x_2, \ldots, x_n)) \, dt
\]

\[
= \left( \int_0^1 + \int_1^\infty \right) p_\alpha(t, (1, x_2, \ldots, x_n)) \, dt
\]

\[
\leq c \int_0^1 t^{-n/\beta} t^{1+1/\beta} \, dt + c' \int_1^\infty t^{-n/\beta} \, dt = c_2 < \infty,
\]

which implies the desired upper bound. \( \square \)
5.3.2. The Taibleson Laplacian. The Fourier transform \( \mathcal{F} : f \mapsto \hat{f} \) of a function \( f \) on the locally compact Abelian group \( \mathbb{Q}_p^n \) is defined by

\[
\hat{f}(\theta) = \int_{\mathbb{Q}_p^n} \langle x, \theta \rangle f(x) \, d\mu^n_p(x),
\]

where \( x = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n, \theta = (\theta_1, \ldots, \theta_n) \in (\mathbb{Q}_p^n)^* = \mathbb{Q}_p^n, \)

\[
\langle x, \theta \rangle = \prod_{k=1}^n \langle x_k, \theta_k \rangle
\]

and \( d\mu^n_p(x) = d\mu_p(x_1) \cdots d\mu_p(x_n) \) is the Haar measure on \( \mathbb{Q}_p^n \). It is known that \( \mathcal{F} \) is a linear isomorphism from \( \mathcal{V}_c \) onto itself, which justifies the following definition (compare with Definition 5.1).

**Definition 5.16.** The Taibleson operator \( \mathcal{T}^\alpha \) for \( \alpha > 0 \) is defined on functions \( f \in \mathcal{V}_c \) by

\[
\mathcal{T}^\alpha f(\zeta) = \|\zeta\|_p^\alpha \hat{f}(\zeta), \quad \zeta \in \mathbb{Q}_p^n.
\]

It follows that \( (\mathcal{T}^\alpha, \mathcal{V}_c) \) is an essentially self-adjoint and non-negative-definite operator acting in \( L^2 \). This operator was introduced by Taibleson [55], and the associated semigroup \( \exp(-t \mathcal{T}^\alpha) \) was studied by Rodríguez-Vega and Zúñiga-Galindo [50]. In particular, it was shown that

\[
\mathcal{T}^\alpha f(x) = \frac{p^\alpha - 1}{1 - p^{-\alpha-n}} \int_{\mathbb{Q}_p^n} \frac{f(x) - f(y)}{\|x-y\|_p^{\alpha+n}} \, d\mu^n_p(y). \tag{5.26}
\]

The equation (5.26) implies that the operator \( (-\mathcal{T}^\alpha, \mathcal{V}_c) \) satisfies the maximum principle, therefore, its semigroup is Markovian. Our aim is to show that \( \exp(-t \mathcal{T}^\alpha) \) is an isotropic Markov semigroup on the ultra-metric measure space \( (\mathbb{Q}_p^n, d_p, \mu^n_p) \).

Our first observation is that the spectrum of the symmetric operator \( (\mathcal{T}^\alpha, \mathcal{V}_c) \) coincides with the range of the function \( \zeta \mapsto \|\zeta\|_p^\alpha \),

\[
\text{spec} \mathcal{T}^\alpha = \{p^{k\alpha} : k \in \mathbb{Z}\} \cup \{0\}.
\]

The eigenspace \( \mathcal{H}(\lambda) \) of the operator \( (\mathcal{T}^\alpha, \mathcal{V}_c) \) corresponding to the eigenvalue \( \lambda = p^{k\alpha} \) is spanned by the function

\[
f_k = \frac{1}{\mu^n_p(p^k \mathbb{Z}_p^n)} 1_{p^k \mathbb{Z}_p^n} - \frac{1}{\mu^n_p(p^{k-1} \mathbb{Z}_p^n)} 1_{p^{k-1} \mathbb{Z}_p^n}
\]

and all its shifts \( f_k(\cdot + a) \) with \( a \in \mathbb{Q}_p^n/p^k \mathbb{Z}_p^n \). Indeed, computing the Fourier transform of the function \( f_k \),

\[
\hat{f}_k(\zeta) = 1_{\|\zeta\|_p \leq p^k} - 1_{\|\zeta\|_p \leq p^{k-1}} = 1_{\|\zeta\|_p = p^k},
\]

we find that

\[
\hat{\mathcal{T}^\alpha f_k}(\zeta) = \|\zeta\|_p^\alpha \hat{f}_k(\zeta) = p^{k\alpha} \hat{f}_k(\zeta).
\]
All the above shows that the operator $\mathcal{T}^\alpha$ coincides with the isotropic Laplacian $\mathcal{L}_\alpha$ on $(\mathbb{Q}_p^n, d_p, \mu_p^n)$ associated with the distance distribution function

$$\sigma_\alpha(r) = \exp\left(-\left(\frac{p}{r}\right)^\alpha\right),$$

and the semigroup $\exp(-t\mathcal{T}^\alpha)$ coincides with the isotropic semigroup $\{P^t\}$.

Observe that the associated intrinsic ultra-metric is

$$d_p^*(x, y) = \left(\frac{\|x - y\|_p}{p}\right)^\alpha.$$

The spectral distribution function $N_\alpha(x, \tau) = N_\alpha(\tau)$ is the non-decreasing left-continuous staircase function which has jumps at the points $\tau_k = p^{k\alpha}$, $k \in \mathbb{Z}$, and takes values $N_\alpha(\tau_k) = p^{(k-1)n}$ at these points. It follows that

$$N_\alpha(\tau) \simeq \tau^{n/\alpha}.$$

In particular, $\tau \mapsto N_\alpha(\tau)$ is a doubling function, and Theorem 2.14 implies the following result.

**Theorem 5.17.** The semigroup $\exp(-t\mathcal{T}^\alpha)$ on $\mathbb{Q}_p^n$ admits a continuous heat kernel $p_\alpha(t, x, y)$ which satisfies the estimate

$$p_\alpha(t, x, y) \simeq \frac{t}{(t^{1/\alpha} + \|x - y\|_p)^{n+\alpha}}. \quad (5.27)$$

In particular, the semigroup $\exp(-t\mathcal{T}^\alpha)$ is transient if and only if $\alpha < n$. In the transient case the Green function (=Taibleson’s Riesz kernel) satisfies the identity

$$g_\alpha(x, y) = \frac{1 - p^{-\alpha}}{1 - p^{\alpha-n}} \frac{1}{\|x - y\|_p^{\alpha-n}}.$$

The upper bound in (5.27) was proved in [50].

Definition 5.7 of a rotation-invariant Laplacian on $\mathbb{Q}_p$ can be carried over to $\mathbb{Q}_p^n$. The Taibleson operator $\mathcal{T}^\alpha$ is an example of a rotation-invariant Laplacian. Theorem 5.8, Corollary 5.9, and Corollary 5.10 and their proofs remain valid also for $\mathbb{Q}_p^n$. Here we provide a short proof of a slightly weaker result which is of significance for us. Let $\mathcal{Y} = \mathcal{T}^1$.

**Theorem 5.18.** The equation $(\mathcal{L}, \mathcal{Y}_c) = (\psi(\mathcal{Y}), \mathcal{Y}_c)$, where $\psi$ is an arbitrary increasing bijection $[0, \infty) \to [0, \infty)$, gives a complete description of the class of isotropic Laplacians on the ultra-metric measure space $(\mathbb{Q}_p^n, d_p, \mu_p^n)$.

**Proof.** Let $\psi: [0, \infty) \to [0, \infty)$ be an increasing bijection. By Theorem 3.1, the operator $(\psi(\mathcal{Y}), \mathcal{Y}_c)$ is an isotropic Laplacian.

Conversely, let $(\mathcal{L}, \mathcal{Y}_c)$ be an isotropic Laplacian on $(\mathbb{Q}_p^n, d_p, \mu_p^n)$. Let $d_{p*}$ be the intrinsic distance associated with $\mathcal{L}$. By construction, $d_{p*}$ is an increasing function of $d_p$, see (2.14). Since the range of $d_p$ is the set $\{p^k: k \in \mathbb{Z}\} \cup \{0\}$, one can choose an increasing bijection $\phi: [0, \infty) \to [0, \infty)$ such that $d_{p*} = \phi(d_p)$. Let $\lambda(B)$ and
\( \tau(B) \) be the eigenvalues of \((L, \mathcal{V}_c)\) and \((\bar{T}, \mathcal{V}_c)\), respectively, corresponding to the ball \( B \subset \mathbb{Q}_p^n \). Since the intrinsic distance associated with \( \bar{T} \) is \( p^{-1}d_p \), we get that

\[
\lambda(B) = \frac{1}{\text{diam}_{p^*}(B)} = \frac{1}{\varphi(\text{diam}_p(B))} = \frac{1}{\varphi(p/\tau(B))} =: \psi(\tau(B)),
\]

where \( \psi(s) = 1/\varphi(p/s) \) is an increasing bijection of \([0, \infty)\) onto itself.

Since both \((L, \mathcal{V}_c)\) and \((\psi(\bar{T}), \mathcal{V}_c)\) are isotropic Laplacians defined on the ultra-metric measure space \((\mathbb{Q}_p^n, d_p, \mu_p^n)\) and their sets of eigenvalues coincide, we have

\[
(L, \mathcal{V}_c) = (\psi(\bar{T}), \mathcal{V}_c),
\]

or equivalently, in terms of the Fourier transform,

\[
\hat{L}\hat{f}(\zeta) = \psi(\|\zeta\|_p)\hat{f}(\zeta)
\]

for all \( f \in \mathcal{V}_c \) and \( \zeta \in \mathbb{Q}_p^n \). \( \square \)

6. Random walks on a tree and jump processes on its boundary

6.1. Rooted trees and their boundaries. A tree is a connected graph \( T \) without cycles (closed paths of length \( \geq 3 \)). We tacitly identify \( T \) with its vertex set, which is assumed to be infinite. We write \( u \sim v \) if \( u, v \in T \) are neighbours. For any pair of vertices \( u, v \in T \), there is a unique shortest path, called a geodesic segment

\[
\pi(u, v) = [u = v_0, v_1, \ldots, v_k = v],
\]

such that \( v_{i-1} \sim v_i \) and all the \( v_i \) are distinct. If \( u = v \), then this is the empty or trivial path. The number \( k \) is the length of the path (the graph distance between \( u \) and \( v \)). In \( T \) we choose and fix a root vertex \( o \). We write \( |v| \) for the length of \( \pi(o, v) \).

The choice of the root induces a partial order on \( T \), where \( u \leq v \) when \( u \in \pi(o, v) \). Every \( v \in T \setminus \{o\} \) has a unique predecessor \( v^- = v^-_o \) with respect to \( o \), which is the unique neighbour of \( v \) on \( \pi(o, v) \). Thus, the set of all (unoriented) edges of \( T \) is

\[
E(T) = \{[v^-, v] : v \in T, v \neq o\}.
\]

For \( u \in T \), the elements of the set

\[
\{v \in T : v^- = u\}
\]

are the successors of \( u \), and its cardinality \( \deg_+^+(u) \) is the forward degree of \( u \).

In this and the next section, we assume that

\[
2 \leq \deg_+^+(u) < \infty \quad \text{for every } u \in T. \quad (6.1)
\]

A (geodesic) ray in \( T \) is a one-sided infinite path \( \pi = [v_0, v_1, v_2, \ldots] \) such that \( v_{i-1} \sim v_i \) and all the \( v_i \) are distinct. Two rays are equivalent if their symmetric difference (as sets of vertices) is finite. An end of \( T \) is an equivalence class of rays.
We shall typically use letters $x$, $y$, $z$ to denote ends (and letters $u$, $v$, $w$ for vertices). The set of all ends of $T$ is denoted $\partial T$. This is the boundary at infinity of the tree. For any $u \in T$ and $x \in \partial T$, there is a unique ray $\pi(u, x)$ representing the end (equivalence class) $x$ and starting at $u$. We write

$$\hat{T} = T \cup \partial T.$$

For $u \in T$ the branch of $T$ rooted at $u$ is the subtree $T_u$, which we identify with its set of vertices

$$T_u = \{ v \in T : u \leq v \}, \quad (6.2)$$

so that $T_o = T$. We write $\partial T_u$ for the set of all ends of $T$ which have a representative path contained in $T_u$, and we set $\hat{T}_u = T_u \cup \partial T_u$.

For $w, z \in \hat{T}$ we define their confluent $w \wedge z = w \wedge_o z$ with respect to the root $o$ by the relation

$$\pi(o, w \wedge z) = \pi(o, w) \cap \pi(o, z).$$

This is the last common element on the geodesics $\pi(o, w)$ and $\pi(o, z)$; it is a vertex of $T$ unless $w = z \in \partial T$. See Fig. 4.

![Figure 4](image)

One of the most common ways of defining an ultra-metric on $\hat{T}$ is

$$d_e(z, w) = \begin{cases} 
0 & \text{if } z = w, \\
 e^{-|z \wedge w|} & \text{if } z \neq w. 
\end{cases} \quad (6.3)$$

Then $\hat{T}$ is compact, and $T$ is open and dense. We are mostly interested in the compact ultra-metric space $\partial T$. In the metric $d_e$ of (6.3), each $d_e$-ball with centre $x \in \partial T$ is of the form $\partial T_u$ for some $u \in \pi(o, x)$. Indeed,

$$\partial T_u = B_{e^{-|u|}}(x) \quad \text{for every } u \in \pi(o, x)$$

and

$$\Lambda_{d_e}(x) = \{ e^{-|u|} : u \in \pi(o, x) \}.$$
Conversely, we can start with a compact ultra-metric space \((X, d)\) which does not possess isolated points, and construct a tree \(T\) as follows: the vertex set of \(T\) is the collection 
\[
\mathcal{B} = \{ B_r(x) : x \in X, \ r > 0 \}
\]
of all closed balls in \((X, d)\), already encountered in §3. Here, we may assume (if we wish) that \(r \in \Lambda_d(x)\).

We now consider any ball \(v = B \in \mathcal{B}\) as a vertex of a tree \(T\). We choose our root vertex as \(o = X\), which belongs to \(\mathcal{B}\) by compactness. The notion of a neighbourhood is determined by the predecessor relation of balls, as given by Definition 3.6. That is, if \(v = B\), then \(u = B'\) is the predecessor vertex \(v^-\) of \(v\) in the tree \(T\). By compactness, each \(x\) has only finitely many successors, and since there are no isolated points in \(X\), every vertex has at least two successors, so that (6.1) holds.

This defines the tree structure. For any \(x \in X\), the collection of all balls \(B_r(x), \ r \in \Lambda_d(x)\), ordered decreasingly, forms the set of vertices of a ray in \(T\) which starts at \(o\). Via a straightforward exercise, the mapping which associates \(x\) with the end of \(T\) represented by that ray is a homeomorphism from \(X\) onto \(\partial T\). Thus, we can identify \(X\) and \(\partial T\) as ultra-metric spaces.

In this identification, if a vertex \(u\) was originally interpreted as a ball \(B_r(x), \ r \in \Lambda_d(x)\), then the set \(\partial T_u\) of ends of the branch \(T_u\) simply coincides with the ball \(B_r(x)\). That is, we are identifying each vertex \(u\) of \(T\) with the set \(\partial T_u\).

If we start with an arbitrary locally finite tree and take its space of ends as the ultra-metric space \(X\), then the above construction does not recover vertices with forward degree \(1\), so that in general we do not get back the tree we started with. However, via the above construction, the correspondence between compact ultra-metric spaces without isolated points (perfect ultra-metric spaces) and locally finite rooted trees with forward degrees \(\geq 2\) is bijective (cf. [32]).

It is well known that any ultra-metric space \(X\) which is both compact and perfect is homeomorphic to the ternary Cantor set \(C \subset [0,1]\). When \(X\) is not compact but still perfect, we have a homeomorphism \(X \simeq C \setminus \{p\}\), where \(p \in C\) is any fixed point.

For the rest of this section and in the next section, we shall abandon the notation \(X\) for a compact and perfect ultra-metric space.

We consider \(X\) as the boundary \(\partial T\) of a locally finite, rooted tree with forward degrees \(\geq 2\).

At the end, we shall comment on how one can handle the presence of vertices with forward degree 1, as well as the non-compact case.

There are many ways to equip \(\partial T\) with an ultra-metric which has the same topology and the same compact-open balls \(\partial T_x, \ x \in T\), (possibly with different radiii) as in the standard metric (6.3). The following is an ultra-metric analogue of a length element.

**Definition 6.1.** Let \(T\) be a locally finite, rooted tree \(T\) with \(\deg^+(x) \geq 2\) for all \(x.\) An ultra-metric element is a function \(\phi: T \to (0, \infty)\) with

(i) \(\phi(v^-) > \phi(v)\) for every \(v \in T \setminus \{o\}\);
(ii) \(\lim \phi(v_n) = 0\) along every geodesic ray \(\pi = [v_0, v_1, v_2, \ldots]\).
It induces the ultra-metric $d_\phi$ on $\partial T$ given by
\[
d_\phi(x, y) = \begin{cases} 
0 & \text{if } x = y, \\
\phi(x \wedge y) & \text{if } x \neq y.
\end{cases}
\]

The balls in this ultra-metric are again the sets
\[
\partial T_u = B_{\phi(u)}(x), \quad x \in \partial T_u.
\]

Note that condition (ii) in the definition is needed to ensure that each end of $T$ is non-isolated in the metric $d_\phi$. The metric $d_e$ of (6.3) is of course induced by $\phi(x) = e^{-|x|}$.

**Lemma 6.2.** For a tree as in Definition 6.1, every ultra-metric on $\partial T$ whose closed balls are the sets $\partial T_u$, $u \in T$, is induced by an ultra-metric element on $T$.

**Proof.** Given an ultra-metric $d$ as stated, we set $\phi(v) = \text{diam}(\partial T_v)$, the diameter with respect to the metric $d$. Since $\deg^+(v^-) \geq 2$ for any $v \in T \setminus \{o\}$, the ball $\partial T_{v^-}$ is the disjoint union of at least two balls $\partial T_u$ with $u^- = v^-$. Therefore, we must have $\text{diam}(\partial T_v) < \text{diam}(\partial T_{v^-})$, and property (i) holds. Since no end is isolated, $\phi$ satisfies (ii). It is now straightforward that $d_\phi = d$. \(\square\)

In view of this correspondence, in the sequel we shall replace the subscript $d$ referring to the metric $d = d_\phi$ by the subscript $\phi$ referring to the ultra-metric element. We note that
\[
diam_\phi(\partial T) = \phi(o), \quad A_\phi(x) = \{\phi(u) : u \in \pi(o, x)\}, \quad \text{and} \quad A_\phi = \{\phi(v) : v \in T\}.
\]

We also note here that for any $x \in \partial T$ and $v \in \pi(o, x)$, the balls with respect to $d_\phi$ are
\[
B_r(x) = B^\phi_r(x) = \begin{cases} 
\partial T_v & \text{for } \phi(v) \leq r < \phi(v^-) \text{ if } v \neq o; \\
\partial T & \text{for } r \geq \phi(o) \text{ if } v = o.
\end{cases}
\]

**6.2. Isotropic jump processes on the boundary of a tree.** In view of the explanations given above, we can consider the isotropic jump processes of (1.3)–(1.8) on $X = \partial T$. Since this space is compact, we may assume that the reference measure $\mu$ is a probability measure on $\partial T$. Given $\mu$, a distance distribution $\sigma$ with properties (1.5), and an ultra-metric element $\phi$ on $T$, we can now refer to the $(d_\phi, \mu, \sigma)$-process simply as the $(\phi, \mu, \sigma)$-process on $\partial T$. We can write the semigroup and its transition probabilities in detail as follows. By (6.5), for $x \in \partial T$ and $\pi(0, x) = [o = v_0, v_1, v_2, \ldots]$,
\[
P^t f(x) = \sum_{n=0}^{\infty} c^t_n Q_{\phi(v_n)} f(x),
\]

where
\[
c^t_0 = 1 - \sigma^t(\phi(v_0)), \quad c^t_n = \sigma^t(\phi(v_{n-1})) - \sigma^t(\phi(v_n)), \quad n \geq 1.
\]
Thus, for arbitrary $u \in T$ and $x \in \partial T$ as above,

$$
P[X_t \in \partial T_u \mid X_0 = x] = \sum_{n=0}^{\infty} c_n \frac{\mu(\partial T_{v_n} \cap \partial T_u)}{\mu(\partial T_{v_n})}.
$$

(6.6)

Recall Definition 2.9 of the standard $(d, \mu)$-process, now to be re-named the standard $(\phi, \mu)$-process.

6.3. Nearest-neighbour random walks on a tree. On a tree as a discrete structure, there are other, very well studied stochastic processes, namely, random walks. Our aim is to analyze how they are related to isotropic jump processes on the boundary of the tree. A good part of the material outlined next is taken from the book of Woess [63]. An older recommended reference is the seminal paper of Cartier [12].

A nearest-neighbour random walk on the locally finite, infinite tree $T$ is induced by its stochastic transition matrix $\mathcal{P} = (p(u, v))_{u, v \in T}$ with the property that $p(u, v) > 0$ if and only if $u \sim v$. The resulting discrete-time Markov chain (random walk) is written as $(Z_n)_{n \geq 0}$. Its $n$-step transition probabilities

$$p^{(n)}(u, v) = P_u[Z_n = v], \quad u, v \in T,$$

are the elements of the $n$th power of the matrix $\mathcal{P}$. The notation $P_u$ refers to the probability measure on the trajectory space which governs the random walk starting at $u$. We assume that the random walk is transient, that is, with probability 1 it visits any finite set only finitely often. Thus, $0 < G(u, v) < \infty$ for all $u, v \in T$, where

$$G(u, v) = \sum_{n=0}^{\infty} p^{(n)}(u, v)$$

is the Green kernel of the random walk. In addition, we shall also make crucial use of the quantities

$$F(u, v) = P_u[Z_n = v \text{ for some } n \geq 0],$$

$$U(v, v) = P_v[Z_n = v \text{ for some } n \geq 1].$$

We shall need several identities connecting these, and we start with some that are valid for all $u, v \in T$:

$$G(u, v) = F(u, v)G(v, v);$$

(6.7)

$$G(v, v) = \frac{1}{1 - U(v, v)};$$

(6.8)

$$U(v, v) = \sum_{u} p(v, u)F(u, v);$$

(6.9)

$$F(u, v) = F(u, w)F(w, v) \quad \text{whenever } w \in \pi(u, v).$$

(6.10)

The first three hold for arbitrary denumerable Markov chains, while (6.10) is specific for trees. The identities show that those quantities are already completely
determined by all the $F(u, v)$ with $u \sim v$. More identities, as can be found in [63], Chap. 9, will be displayed and used later on.

By transience, the random walk $Z_n$ must converge to a random end, a simple and well-known fact (see, for instance, [12] or [63], Theorem 9.18).

**Lemma 6.3.** There is a $\partial T$-valued random variable $Z_\infty$ such that for every starting point $u \in T$,

$$P_u[Z_n \to Z_\infty \text{ in the topology of } \hat{T}] = 1.$$  

In brief, the argument is as follows: by transience, random-walk trajectories must accumulate at $\partial T$ almost surely. If such a trajectory had two distinct accumulation points, say $x$ and $y$, then by the nearest-neighbour property, the trajectory would visit the vertex $x \land y$ infinitely often, which can only occur with probability 0.

For each $u \in T$ we can consider the limit distribution $\nu_u$, which is a Borel measure on $\partial T$ defined on Borel sets $B \subset \partial T$ by

$$\nu_u(B) = P_u[Z_\infty \in B].$$  

The sets $\partial T_u$, $u \in T$ (plus the empty set), form a semi-algebra which generates the Borel $\sigma$-algebra of $\partial T$. Thus, each $\nu_u$ is determined by the values on those sets. There is an explicit formula (see [12] or [63], Proposition 9.23). For $v \neq o$,

$$\nu_u(\partial T_v) = \begin{cases} F(u, v) \frac{1 - F(v, v^-)}{1 - F(v^-, v)F(v, v^-)} & \text{if } u \in \{v\} \cup (T \setminus T_v), \\ 1 - F(u, v) \frac{F(v, v^-) - F(v^-, v)F(v, v^-)}{1 - F(v^-, v)F(v, v^-)} & \text{if } u \in T_v. \end{cases}$$  

(6.11)

A harmonic function is a function $h : T \to \mathbb{R}$ with $\mathcal{P}h = h$, where

$$\mathcal{P}h(u) = \sum_v p(u, v)h(v).$$

For any Borel set $B \subset \partial T$, the function $u \mapsto \nu_u(B)$ is a bounded harmonic function. One deduces that all the $\nu_u$ are comparable: $p^{(k)}(u, v)\nu_u \leq \nu_v$, where $k$ is the length of $\pi(u, v)$. Thus, for any function $\varphi \in L^1(\partial T, \nu_o)$ the function $h_\varphi$ defined by

$$h_\varphi(u) = \int_{\partial T} \varphi \, d\nu_u,$$

is finite and harmonic on $T$. It is often called the Poisson transform of $\varphi$.

We next define a measure $m$ on $T$ via its atoms:

$$m(o) = 1,$$

and for $v \in T \setminus \{o\}$ with $\pi(o, v) = [o = v_0, v_1, \ldots, v_k = v]$,

$$m(v) = \frac{p(v_0, v_1)p(v_1, v_2)\cdots p(v_{k-1}, v_k)}{p(v_1, v_0)p(v_2, v_1)\cdots p(v_k, v_{k-1})}. \quad (6.12)$$

Then for all $u, v \in T$

$$m(u)p(u, v) = m(v)p(v, u), \quad \text{and consequently} \quad m(u)G(u, v) = m(v)G(v, u); \quad (6.13)$$
the random walk is reversible. This would allow us to use the electrical network interpretation of \((T, \mathcal{P}, m)\), for which there are various references: see, for instance, Yamasaki [65], Soardi [54], or—with the notation used here—[63], Chap. 4. Each edge \(e = [v^-, v] \in E(T)\) is thought of as an electric conductor with a conductance 
\[
a(v^-, v) = m(v)p(v, v^-).
\]

We obtain the Dirichlet form \(E_T = E_T, \mathcal{P}\) for functions \(f, g : T \to \mathbb{R}\), defined by
\[
E_T(f, g) = \sum_{[v^-, v] \in E(T)} (f(v) - f(v^-))(g(v) - g(v^-))a(v^-, v). \tag{6.14}
\]

It is well defined for \(f, g\) in the space
\[
\mathcal{D}(T) = \mathcal{D}(T, \mathcal{P}) = \{f : T \to \mathbb{R} \mid E_T(f, f) < \infty\}. \tag{6.15}
\]

**6.4. Harmonic functions of finite energy and their boundary values.** We are interested in the subspace 
\[
\mathcal{H}D(T) = \mathcal{H}D(T, \mathcal{P}) = \{h \in \mathcal{D}(T, \mathcal{P}) : \mathcal{P}h = h\}
\]
of harmonic functions with finite energy. The terminology comes from the interpretation of such a function as the potential of an electric flow (or current), and then \(E_T(h, h)\) is the energy of that flow.\(^3\)

Every function in \(\mathcal{H}D(T, \mathcal{P})\) is the Poisson transform of some function \(\varphi \in L^2(\partial T, \nu_o)\). This is valid not only for trees, but also for general finite-range reversible Markov chains, and it is a consequence of the following facts.

1. Every function in \(\mathcal{H}D\) is the difference of two non-negative functions in \(\mathcal{H}D\).
2. Every non-negative function in \(\mathcal{H}D\) can be approximated, monotonically from below, by a sequence of non-negative bounded functions in \(\mathcal{H}D\).
3. Every bounded harmonic function (not necessarily with finite energy) is the Poisson transform of a bounded function on the boundary \(\partial T\).

The boundary \(\partial T\) is the (active part of) the Martin boundary, with \(\nu_u\) being the limit distribution of the Markov chain, starting from \(u\), on that boundary. The facts (1) and (2) are contained in [65] and [54], while (3) is part of general Martin boundary theory (see, for instance, [63], Theorem 7.61).

Thus, we can introduce a form \(\mathcal{E}_{\mathcal{H}D}\) on \(\partial T\) by setting
\[
\mathcal{D}(\partial T, \mathcal{P}) = \{\varphi \in L^1(\partial T, \nu_o) : \mathcal{E}_T(h_\varphi, h_\varphi) < \infty\},
\]
\[
\mathcal{E}_{\mathcal{H}D}(\varphi, \psi) = \mathcal{E}_T(h_\varphi, h_\psi) \quad \text{for} \quad \varphi, \psi \in \mathcal{D}(\partial T, \mathcal{P}). \tag{6.16}
\]

**6.5. Jump processes on the boundary of a tree.** Kigami [36] develops an expression for the form \(\mathcal{E}_{\mathcal{H}D}(\varphi, \psi)\) of (6.16) by dint of considerable effort, demonstrates its regularity properties, and then studies the jump process on \(\partial T\) induced by this Dirichlet form. We call this the boundary process associated with the random walk on \(T\).

\(^3\)In the mathematical literature, the expression ‘energy’ is commonly used for \(E_T(h, h)\), but it seems that ‘power’ would be the more appropriate terminology from physics.
Now there is a rather simple expression for $\mathcal{E}_{\mathcal{D}}$. We define the \textit{Naïm kernel} on $\partial T \times \partial T$ by

$$\Theta_o(x, y) = \begin{cases} \frac{m(o)}{G(o, o)F(o, x \wedge y)F(x \wedge y, o)} & \text{if } x \neq y, \\ +\infty & \text{if } x = y. \end{cases} \quad (6.17)$$

In our case, $m(o) = 1$, but we might want to change the base point, or normalize the measure $m$ in a different way.

**Theorem 6.4.** For any transient nearest-neighbour random walk on the tree $T$ with root $o$, and for all functions $\varphi, \psi$ in $\mathcal{D}(\partial T, \mathcal{P})$,

$$\mathcal{E}_{\mathcal{D}}(\varphi, \psi) = \frac{1}{2} \int_{\partial T} \int_{\partial T} (\varphi(x) - \varphi(y))(\psi(x) - \psi(y))\Theta_o(x, y) \, dv_o(x) \, dv_o(y).$$

A proof of Theorem 6.4 is given in [20] in a setting of abstract potential theory on Green spaces, which are locally Euclidean. The definition of the Naïm kernel in [44] refers to the same type of setting. However, infinite networks, even when seen as metric graphs, are not locally Euclidean. In this sense, until now the definition of the Naïm kernel and a proof of Theorem 6.4 for transient reversible random walks have not been well enough reflected in the literature. In a forthcoming paper, Georgakopoulos and Kaimanovich will provide those ‘missing links’. Here we give a direct and simple proof of Theorem 6.4 for the specific case of trees, starting with the following observation.

**Lemma 6.5.** The measure $\Theta_o(x, y) \, dv_o(x) \, dv_o(y)$ on $\partial T \times \partial T$ is invariant with respect to changing the base point (root) $o$.

**Proof.** We want to replace $o$ with some other $u \in T$. We may assume that $u \sim o$. Indeed, then we may replace the current base point step by step by one of its neighbours to obtain the result for arbitrary $u$.

Recall that the confluent which appears in the definition (6.17) of $\Theta_o$ depends on the root $o$, while for $\Theta_x$ it becomes the confluent with respect to $x$ as the new root. It is a well-known fact that

$$\frac{dv_u}{dv_o}(x) = K(u, x) := \frac{G(u, u \wedge o \, x)}{G(o, u \wedge o \, x)},$$

where $K$ is the Martin kernel. Thus, we have to show that for all $x, y \in \partial T$ ($x \neq y$),

$$\frac{m(o)}{G(o, o)F(o, x \wedge y)F(x \wedge y, o)} = \frac{m(u)K(u, x)K(u, y)}{G(u, u)F(u, x \wedge u \, y)F(x \wedge u \, y, u)}. \quad \text{Case 1: } x, y \in \partial T_u. \text{ Then } x \wedge_o y = x \wedge u y =: v \in T_u \text{ and } u \wedge_o x = u \wedge_o y = u. \text{ Thus, using (6.7), (6.10) and the fact that by (6.13)}$$

$$\frac{m(u)}{G(u, u)} = \frac{m(o)}{G(o, o)},$$
we see that
\[
\frac{m(u)K(u,x)K(u,y)}{G(u,u)F(u,x \wedge_y)F(x \wedge_y, u)} = \frac{m(u)}{G(u,u)F(v,u)F(v,u)} \left( \frac{G(u,u)}{G(o,u)} \right)^2
\]
\[
= \frac{m(o)G(u,u)}{F(u,v)F(v,u)G(o,u)G(u,o)}
\]
\[
= \frac{m(o)G(o,u)G(u,o)}{F(u,v)F(v,u)F(o,u)F(u,o)G(o,o)}
\]
\[
= \frac{m(o)G(o,o)G(u,o)}{F(o,v)F(v,o)G(o,o)},
\]
as required.

There are 3 more cases.

Case 2: \(x, y \in \partial T \setminus \partial T_u\). Then
\[
x \wedge_o y = x \wedge_u y =: w \in T \setminus T_u, \quad u \wedge_o x = u \wedge_o y = o.
\]

Case 3: \(x \in \partial T_u, y \in \partial T \setminus \partial T_u\). Then
\[
x \wedge_o y = o, \quad x \wedge_u y = u, \quad u \wedge_o x = u, \quad u \wedge_o y = o.
\]

Case 4: \(x \in \partial T \setminus \partial T_u, y \in \partial T_u\). This is like Case 3, with the roles of \(x\) and \(y\) exchanged.

In all Cases 2–4, the computation is very much like Case 1, a straightforward exercise. □

To prove Theorem 6.4, we need a few more facts related to the network setting; compare, for example, [63], § 4.D.

The space \(\mathcal{D}(T)\) in (6.15) is a Hilbert space when equipped with the inner product
\[
(f, g) = \mathcal{E}_T(f, g) + f(o)g(o).
\]

The subspace \(\mathcal{D}_0(T)\) is defined as the closure of the space of finitely supported functions in \(\mathcal{D}(T)\). It is a proper subspace if and only if the random walk is transient, and then the function \(G_v(u) = G(u, v)\) is in \(\mathcal{D}_0(T)\) for any \(v \in T\) [65], [54]. We need the formula
\[
\mathcal{E}_T(f, G_v) = m(v)f(v) \text{ for every } f \in \mathcal{D}_0(T).
\] (6.18)

Given a branch \(T_w\) of \(T, (w \in T \setminus \{o\})\), we can consider it as a subnetwork equipped with the same conductances \(a(u, v)\) for \([u, v] \in E(T_w)\). The associated measure on \(T_w\) is
\[
m_{T_w}(u) = \sum_{v \in T_w : v \sim u} a(u, v) = \begin{cases} m(u) & \text{if } u \in T_w \setminus \{w\}, \\ m(w) - a(w, w^-) & \text{if } u = w. \end{cases}
\]

The resulting random walk on \(T_w\) has transition probabilities
\[
\rho_{T_w}(u, v) = \frac{a(v, w)}{m_{T_w}(u)} = \begin{cases} p(u, v) & \text{if } u \in T_w \setminus \{w\}, v \sim u, \\ p(w, v) & \text{if } u = w, v \sim u. \end{cases}
\]
We have \( F_{T_w}(u, u^-) = F(u, u^-) \) and thus also \( F_{T_w}(u, w) = F(u, w) \) for every \( u \in T_w \setminus \{w\} \), because before its first visit to \( w \), the random walk on \( T_w \) obeys the same transition probabilities as the original random walk on \( T \). It is then easy to see ([63], p. 241) that the random walk on \( T_w \) is transient if and only if for the original random walk \( F(w, w^-) < 1 \), which in turn holds if and only if \( \nu_v(\partial T_w) > 0 \). (In other parts of this and the preceding two sections, this is always assumed, but for the proof of Theorem 6.4, we just assume the random walk on the whole of \( T \) to be transient.) Conversely, if \( F(w, w^-) = 1 \), then \( F(u, w) = 1 \) for all \( u \in T_w \).

Below we shall need the following formula for the limit distributions.

**Lemma 6.6.** For \( u \in T \setminus \{o\} \),

\[
\nu_u(\partial T_u) = 1 - p(u, u^-)(G(u, u) - G(u^-, u)).
\]

**Proof.** By (7.7),

\[
G(u, u)p(u, u^-) = \frac{F(u, u^-)}{1 - F(u, u^-)F(u^-, u)}
\]

Thus,

\[
p(u, u^-)(G(u, u) - G(u^-, u)) = (1 - F(u^-, u))G(u, u)p(u, u^-) = 1 - \nu_u(\partial T_u)
\]

after a short computation using (6.11). \( \square \)

**Proof of Theorem 6.4.** We first prove the Doob–Naïm formula (for short, the D-N formula) for the case when \( \varphi = 1_{\partial T_v} \) and \( \psi = 1_{\partial T_w} \) for two proper branches \( T_v \) and \( T_w \) of \( T \). They are either disjoint, or one of them contains the other.

**Case 1:** \( T_w \subset T_v \). (The case \( T_v \subset T_w \) is analogous by symmetry.) This means that \( w \in T_v \). For \( x, y \in \partial T \) we have

\[
(\varphi(x) - \varphi(y))(\psi(x) - \psi(y)) = 1
\]

if \( x \in \partial T_w \) and \( y \in \partial T \setminus \partial T_v \) or conversely, and

\[
(\varphi(x) - \varphi(y))(\psi(x) - \psi(y)) = 0
\]

otherwise. By Lemma 6.5 we may choose \( v \) as the base point. Thus, the right-hand side of the identity is

\[
\int_{\partial T \setminus \partial T_v} \int_{\partial T_w} \Theta_v(x, y) d\nu_v(x) d\nu_v(y) = \frac{m(v)}{G(v, v)} \nu_v(\partial T \setminus \partial T_v)\nu_v(\partial T_w),
\]

since \( x \wedge_v y = v \) and \( F(v, v) = 1 \).

Let us now turn to the left-hand side of the D-N formula. The Poisson transforms of \( \varphi \) and \( \psi \) are

\[
h_\varphi(u) = \nu_u(\partial T_v) \quad \text{and} \quad h_\psi(u) = \nu_u(\partial T_w).
\]

By (6.11),

\[
h_\varphi(u) = F(u, v)\nu_v(\partial T_v), \quad u \in \{v\} \cup (T \setminus T_v);
\]

\[
1 - h_\varphi(u) = F(u, v)\nu_v(\partial T \setminus \partial T_v), \quad u \in T_v.
\]
We set $F_v(u) = F(u, v)$ and write
\[ h_\varphi(u) - h_\varphi(u^-) = (1 - h_\varphi(u^-)) - (1 - h_\varphi(u)) \]
whenever this is convenient, and analogously for $h_\psi$. Then
\[
\mathcal{E}_T(h_\varphi, h_\psi) = \sum_{[u,u^-] \in E(T) \setminus E(T_u)} a(u, u^-)(F(u, v) - F(u^-, v))\nu_v(\partial T_v)
\times (F(u, v) - F(u^-, v))\nu_v(\partial T_w)
- \sum_{[u,u^-] \in E(T_w) \setminus E(T_u)} a(u, u^-)(F(u, v) - F(u^-, v))\nu_v(\partial T \setminus \partial T_v)
\times (F(u, v) - F(u^-, v))\nu_v(\partial T_w)
+ \sum_{[u,u^-] \in E(T_w)} a(u, u^-)(F(u, v) - F(u^-, v))\nu_v(\partial T \setminus \partial T_v)
\times (F(u, v) - F(u^-, v))\nu_v(\partial T \setminus \partial T_w)
= \mathcal{E}_T(F_v, F_w)\nu_v(\partial T_v)\mathcal{E}_w(F_v, F_w)\nu_v(\partial T_w) - \mathcal{E}_T(F_v, F_w)\nu_v(\partial T_w)
+ \mathcal{E}_w(F_v, F_w)\nu_v(\partial T \setminus \partial T_v),
\]
where, of course, $\mathcal{E}_T$ is the Dirichlet form of the random walk on the branch $T_v$, as discussed above, and analogously for $\mathcal{E}_w$. Now $F_v = G_v/G(v, v)$ by (6.7), and thus by (6.18)
\[
\mathcal{E}_T(F_v, F_w) = \frac{\mathcal{E}_T(G_v, F_w)}{G(v, v)} = \frac{m(v) F(v, w)}{G(v, v)}.
\]
Recall that for the random walk on $T_v$ we have $F_{T_v}(u, v) = F(u, v)$ for every $u \in T_v$. Also,
\[
m_{T_v}(v) = m(v) - a(v, v^-) = m(v)(1 - p(v, v^-)).
\]
We apply (6.19) to that random walk and obtain
\[
\mathcal{E}_{T_v}(F_v, F_w) = \frac{m(v)(1 - p(v, v^-))F(v, w)}{G_{T_v}(v, v)}.
\]
We now apply (6.8) and (6.9), recalling in addition that
\[
p_{T_v}(v, u) = \frac{p(v, u)}{1 - p(v, v^-)}
\]
for $u \in T_v$, and we have
\[
\frac{1 - p(v, v^-)}{G_{T_v}(v, v)} = 1 - p(v, v^-) - (1 - p(v, v^-))U_{T_v}(v, v)
= 1 - p(v, v^-) - \sum_{u: u = v} p(v, u)F(u, v)
= 1 - p(v, v^-) - (U(v, v) - p(v, v^-)F(v^-, v))
= \frac{1}{G(v, v)} - p(v, v^-)(1 - F(v^-, v))
= \frac{\nu_v(\partial T_v)}{G(v, v)},
\]
where in the last step we have used Lemma 6.6. We have shown that
\[
\mathcal{E}_{T_v}(F_v, F_{w}) = \frac{m(v)F(v, w)}{G(v, v)}\nu_v(\partial T_v).
\]
In the same way, exchanging the roles of $T_w$ and $T_v$ and using the reversibility (6.13), we see that
\[
\mathcal{E}_{T_w}(F_v, F_{w}) = \frac{m(w)F(w, v)}{G(w, w)}\nu_w(\partial T_w) = \frac{m(v)}{G(v, v)}\nu_v(\partial T_v).
\]
Putting things together, we get that
\[
\mathcal{E}_{T}(h_\varphi, h_\psi) = \mathcal{E}_{T_w}(F_v, F_{w})\nu_v(\partial T \setminus \partial T_v) = \frac{m(v)}{G(v, v)}\nu_v(\partial T \setminus \partial T_v),
\]
as proposed.

Case 2: $T_w \cap T_v = \emptyset$. In view of Lemma 6.5, both sides of the D-N formula are independent of the root $o$. Thus, we may declare our root to be one of the neighbours of $v$ which is not on $\pi(v, w)$. Also, let $\bar{v}$ be the neighbour of $v$ on $\pi(w, v)$. Then, with our chosen new root, the complement of the ‘old’ $T_v$ is $T_{\bar{v}}$, which contains $T_w$ (the latter remains the same with respect to the new root).

Thus, we can apply the result of Case 1 to $T_{\bar{v}}$ and $T_w$. This means that we have to replace the functions $\varphi$ and $h_\varphi$ with $1 - \varphi$ and $1 - h_\varphi$, respectively, which just means that we change the sign on both sides of the identity. We are re-conducted to Case 1 without further computations.

From what we have done so far, and from the linearity of the Poisson transform and the bilinearity of the forms on both sides of the D-N formula we deduce that this formula holds for linear combinations of the indicator functions of the sets $\partial T_v$. These indicator functions are dense in the space $C(\partial T)$ with respect to the max-norm. Thus, the D-N formula holds for all continuous functions on $\partial T$. The extension to all of $\mathcal{D}(\partial T, \mathcal{P})$ is by the standard approximation. $\square$

7. The duality of random walks on trees and isotropic processes on their boundaries

When looking at our isotropic processes and at the boundary process of Kigami [36], it is natural to ask the following two questions.

**Question I.** Given a transient random walk on $T$ associated with the Dirichlet form $\mathcal{E}_T$ in (6.14), does the boundary process on $\partial T$ induced by the form $\mathcal{E}_{\mathcal{H}\mathcal{P}}$ in (6.16) arise as one of the isotropic processes (1.8) on $\partial T$ with transition probabilities (6.6) with respect to the measure $\mu = \nu_o$ on $\partial T$, some ultra-metric element $\phi$ on $T$, and a suitable distance distribution function $\sigma$ on $[0, \infty)$?

**Question II.** Conversely, given data $\mu$, $\phi$, and $\sigma$, is there a random walk on $T$ with limit distribution $\nu_o = \mu$ such that the isotropic process induced by $\mu$, $\phi$, and $\sigma$ is the boundary process with Dirichlet form $\mathcal{E}_{\mathcal{H}\mathcal{P}}$?
Before answering these two questions, we need to specify the assumptions more precisely. When starting with \((\phi, \mu, \sigma)\), we always assume as before that \(\mu\) is supported by the whole of \(\partial T\).

Thus, as far as the random walk is concerned, we also require that \(\text{supp}(\nu_0) = \partial T\). This is equivalent to the requirement that \(\nu_0(\partial T_v) > 0\) for every \(v \in T\). By (6.11) this is in turn equivalent to

\[
F(v, v^-) < 1 \quad \text{for every } v \in T \setminus \{o\}.
\]

(7.1)

Indeed, we shall see that we need a little more, namely, that

\[
\lim_{v \to \infty} G(v, o) = 0,
\]

(7.2)

that is, for every \(\varepsilon > 0\) there is a finite set \(A \subset T\) such that \(G(v, o) < \varepsilon\) for all \(v \in T \setminus A\). This condition is necessary and sufficient for the solubility of the Dirichlet problem: for any \(\varphi \in C(\partial X)\), its Poisson transform \(h_\varphi\) provides the unique continuous extension of \(\varphi\) to \(\hat{T}\) which is harmonic in \(T\). See, for instance, [63], Corollary 9.44.

We shall restrict our attention to random walks with the properties (7.1) and (7.2) on a rooted tree with forward degrees \(\geq 2\).

### 7.1. Answer to Question I.

We start with a random walk which fulfills the above requirements. We know from §1 that each \((\mu, \phi, \sigma)\)-process arises as the standard process of Definition 2.9 with respect to the intrinsic metric (cf. Theorem 2.10); given \(\phi\) and \(\sigma\), the intrinsic metric is induced by the ultra-metric element

\[
\phi_\ast(u) = -\frac{1}{\log \sigma(\phi(u))}.
\]

(7.3)

Thus, we can eliminate \(\sigma\) from our considerations by looking for an ultra-metric element \(\phi\) such that the boundary process is the standard process on \(\partial T\) associated with \((\phi, \nu_0)\).

Since symmetric processes are determined by their Dirichlet forms, we infer from Theorems 3.12 and 6.4 that we are looking for \(\phi\) such that \(J(x, y) = \Theta_o(x, y)\) for all \(x, y \in \partial T\) with \(x \neq y\), where \(J(x, y)\) is given by (3.11). By rewriting \(J(x, y)\) in terms of \(\phi, \nu_o\), and the tree structure, this becomes

\[
\frac{1}{\phi(o)} + \int_{1/\phi(o)}^{1/\phi(x \wedge y)} \frac{dt}{\nu_o(B_{1/t}^\phi(x))} = \frac{m(o)}{G(o, o)F(o, x \wedge y)F(x \wedge y, o)}.
\]

(7.4)

In our case, \(m(o) = 1\), but we keep track of what happens when one changes the root or the normalization of \(m\). First, since \(\text{deg}^+(o) \geq 2\), there are \(x, y \in \partial T\) such that \(x \wedge y = o\). We insert these two boundary points in (7.4). Since \(F(o, o) = 1\), we must have

\[
\phi(o) = \frac{G(o, o)}{m(o)}.
\]

Now take \(v \in T \setminus \{o\}\). Since forward degrees are greater than or equal to 2, there are \(x, y, y' \in \partial T\) such that \(x \wedge y = v\) and \(x \wedge y' = v^-\). We write (7.4) first for \((x, y')\)
and then for \((x, y)\) and then take the difference, leading to the equation
\[
\int_{1/\phi(v)}^{1/\phi(v^-)} \frac{dt}{\nu_o(B_{1/t}^\phi(x))} = \frac{m(o)}{G(o, o)F(o, v)F(v, o)} - \frac{m(o)}{G(o, o)F(o, v^-)F(v^-, o)}.
\] (7.5)

By (6.5), within the range of the last integral we must have \(B_{1/t}^\phi(v) = \partial T_v\), hence that integral reduces to
\[
\left(\frac{1}{\phi(v)} - \frac{1}{\phi(v^-)}\right) \frac{1}{\nu_o(\partial T_v)}.
\]

We multiply equation (7.5) by \(\nu_o(\partial T_v)\) and simplify the resulting right-hand side
\[
\left(\frac{m(o)}{G(o, o)F(o, v)F(v, o)} - \frac{m(o)}{G(o, o)F(o, v^-)F(v^-, o)}\right)\nu_o(\partial T_v),
\]
by use of the identities (6.7)–(6.10) and the first of the two formulae in (6.11) (for \(\nu_o\)). We get that the ultra-metric element which we are looking for should satisfy
\[
\frac{1}{\phi(v)} - \frac{1}{\phi(v^-)} = \frac{m(o)}{G(v, o)} - \frac{m(o)}{G(v^-, o)} \quad \text{for every } v \in T \setminus \{o\}.
\] (7.6)

This determines \(1/\phi(v)\) recursively, and with \(m(o) = 1\) we see that
\[
\phi(v) = G(v, o).
\]

Since by (6.7) and (6.10)
\[
G(v, o) = F(v, v^-)G(v^-, o),
\]
the assumptions (7.1) and (7.2) imply that \(\phi\) is an ultra-metric element. Tracing back the last computations, we find that with this choice of \(\phi\), we have indeed that \(J(x, y) = \Theta_o(x, y)\) for all \(x, y \in \partial T\) with \(x \neq y\). We have proved the following.

**Theorem 7.1.** Let \(T\) be a locally finite rooted tree with \(\deg^+(o) \geq 2\). Consider a transient nearest-neighbour random walk on \(T\) which satisfies (6.7) and (6.10). Then the boundary process on \(\partial T\) induced by the Dirichlet form (6.16) coincides with the standard process associated with the ultra-metric element \(\phi = G(\cdot, o)\) and the limit distribution \(\nu_o\) of the random walk.

Let \(\mathcal{L}\) be the Laplacian associated with the boundary process of Theorem 7.1. \(\mathcal{L}\) acts on locally constant functions \(f\) by
\[
\mathcal{L}f(x) = \int_{\partial T} (f(x) - f(y))\Theta_o(x, y)\,d\nu_o(y).
\]

In view of the identification of balls in \(\partial T\) with vertices of \(T\), the eigenfunctions in (3.6) now become
\[
f_v = \frac{1_{\partial T_v}}{\nu_o(\partial T_v)} - \frac{1_{\partial T_v^-}}{\nu_o(\partial T_v^-)}, \quad v \in T \setminus \{o\}.
\]

In addition, we set \(f_o = 1\) and note that it is an eigenfunction of \(\mathcal{L}\) with eigenvalue 0. Applying Theorem 3.8, we obtain the following result.
Corollary 7.2. \( \mathcal{L} f_v = G(v^-,o)^{-1} f_v \) for \( v \in T \setminus \{ o \} \), and the set of eigenfunctions \( \{ f_v \}_{v \in T} \) is complete. In particular,

\[
\text{spec } \mathcal{L} = \{ G(v,o)^{-1} : v \in T \} \cup \{ 0 \}.
\]

Remark 7.3. For any two vertices \( v \) and \( w \) in \( T \setminus \{ o \} \) such that \( v^- = w^- = u \) the functions \( f_v \) and \( f_w \) are eigenfunctions of \( \mathcal{L} \) corresponding to the eigenvalue \( \lambda = 1/G(u,o) \). Hence, the eigenspace \( \mathcal{H}(u) \) corresponding to the vertex \( u \) is spanned by the functions \( \{ f_v : v^- = u \} \). Since the rank of the system \( \{ f_v : v^- = u \} \) is \( \text{deg}^+(u) - 1 \), where \( \text{deg}^+(u) \geq 2 \) is the forward degree of the vertex \( u \), we obtain

\[
\dim \mathcal{H}(u) = \text{deg}^+(u) - 1
\]

(cf. (3.9)).

Remark 7.4. Given a random walk on \( T \) and the associated boundary process on \( \partial T \), we might want to realize it as the \( (\nu_o, \phi, \sigma) \)-process for an ultra-metric element \( \phi \) different from \( G(\cdot,o) \). This means that we have to look for a suitable distance distribution \( \sigma \) on \([0, \infty)\) different from the inverse exponential distribution (2.16). In view of 7.3, we are looking for \( \sigma \) such that for our given generic \( \phi \)

\[
\sigma(\phi(v)) = e^{-1/G(v,o)}.
\]

For this it is necessary that \( \phi(u) = \phi(v) \) whenever \( G(u,o) = G(v,o) \): we need \( \phi \) to be constant on equipotential sets. In that case, the distribution function \( \sigma(r) \) is determined by the above equation for \( r \) in the value set \( \Lambda_\phi \) of the ultra-metric \( d_\phi \). We can ‘interpolate’ that function in an arbitrary way (monotone increasing, left continuous) and obtain a feasible measure \( \sigma \).

7.2. Answer to Question II. Answering Question II means that we start with \( \phi \) and \( \mu \) and then look for a random walk with limit distribution \( \nu_o = \mu \) such that the standard \( (\phi, \mu) \)-process is the boundary process associated with the random walk. We know from Theorem 7.1 that in this case we should have \( \phi(v) = G(v,o) \), whence in particular, \( \phi(o) > 1 \). Thus, we cannot expect that every \( \phi \) is suitable. The most natural choice is to replace \( \phi \) by \( C \cdot \phi \) for some constant \( C > 0 \). For the standard processes associated with \( \phi \) and \( C \cdot \phi \), respectively, this simply gives rise to a linear time change: if the old process is \( \{ X_t \}_{t>0} \), then the new one is \( \{ X_{t/C} \}_{t>0} \).

Theorem 7.5. Let \( T \) be a locally finite, rooted tree with \( \text{deg}^+(o) \geq 2 \). Consider an ultra-metric element \( \phi \) on \( T \) and a fully supported probability measure \( \mu \) on \( \partial T \). Then there exist a unique constant \( C > 0 \) and a unique transient nearest-neighbour random walk on \( T \) which satisfies (6.7) and (6.10) with the following properties:

1) \( \mu = \nu_o \) is the limit distribution of the random walk;

2) The associated boundary process coincides with the standard process on \( \partial T \) induced by the ultra-metric element \( C \cdot \phi \) and the given measure \( \mu \).
For the proof, we shall need three more formulae. The first two are taken from [63], Lemma 9.35, while the third is immediate from (6.10) and (6.11):

\[
G(u, u)p(u, v) = \frac{F(u, v)}{1 - F(u, v)F(v, u)} \quad \text{if } u \sim v, \tag{7.7}
\]

\[
G(u, u) = 1 + \sum_{v : v \sim u} \frac{F(u, v)F(v, u)}{1 - F(u, v)F(v, u)}, \tag{7.8}
\]

\[
F(v^-, v) = \frac{\nu_o(\partial T_v)/F(o, v^-)}{1 - F(v, v^-) + F(v, v^-)\nu_o(\partial T_v)/F(o, v^-)}. \tag{7.9}
\]

**Proof of Theorem 7.5.** We proceed as follows: we start with \( \phi \) and \( \mu \) and replace \( \phi \) by a new ultra-metric element \( C \cdot \phi \), with \( C \) to be determined, and \( \mu \) will be a candidate for the limit distribution of the random walk we are looking for. Using the various formulae at our disposal, we first construct in the only possible way the quantities \( F(u, v) \) for \( u, v, \in T \), in particular when \( u \sim v \). In turn, they lead to the Green kernel \( G(u, v) \). Up to here, these will be only ‘candidate’ quantities whose feasibility will have to be verified. Until that verification, we shall denote them by \( \tilde{F}(u, v) \) and \( \tilde{G}(u, v) \). Via (7.7), they will lead to definitions of the transition probabilities \( p(u, v) \). The stochasticity of the resulting transition matrix \( \mathcal{P} \) will also have to be verified.

Only then will we use a potential-theoretic argument to show that \( \tilde{G}(u, v) \) really is the Green kernel associated with \( \mathcal{P} \), so that the question mark which is implicit in the ‘\( \sim \)’ can be removed.

First, in view of Theorem 7.1 we must have

\[
C \cdot \phi(v) = \tilde{G}(v, o),
\]

and thus by (6.7) and (6.10)

\[
\tilde{F}(v, v^-) = \frac{\phi(v)}{\phi(v^-)} \quad \text{for } v \in T \setminus \{o\}, \tag{7.10}
\]

and more generally

\[
\tilde{F}(v, u) = \frac{\phi(v)}{\phi(u)} \quad \text{when } u \leq v.
\]

We note immediately that \( 0 < \tilde{F}(v, u) < 1 \) when \( u < v \) and that \( \tilde{F}(u, u) = 1 \).

Next, we use (7.9) to construct \( \tilde{F}(v^-, v) \) and \( \tilde{F}(o, v) \) recursively. We start with \( \tilde{F}(o, o) = 1 \). If \( v \neq o \) and \( \tilde{F}(o, v^-) \) is already given, with

\[
\mu(\partial T_{v^-}) \leq \tilde{F}(o, v^-) \leq 1
\]

(the lower bound is required by (6.11)), then we have to set

\[
\tilde{F}(v^-, v) = \frac{\mu(\partial T_v)/\tilde{F}(o, v^-)}{1 - \tilde{F}(v, v^-) + \tilde{F}(v, v^-)\mu(\partial T_v)/\tilde{F}(o, v^-)} \tag{7.11}
\]

and

\[
\tilde{F}(o, v) = \tilde{F}(o, v^-)\tilde{F}(v^-, v).
\]
Since
\[ \tilde{F}(o, v^-) \geq \mu(\partial T_v^-) \geq \mu(\partial T_v), \]
we see that
\[ 0 < \tilde{F}(v^-, v) \leq 1. \]
As imposed by (6.10), we set
\[ \tilde{F}(o, v) = \tilde{F}(o, v^-)\tilde{F}(v^-, v). \]
Formula (7.11) (re-)transforms into
\[ \begin{align*}
\mu(\partial T_v) &= \tilde{F}(o, v^-)\tilde{F}(v^-, v) \frac{1 - \tilde{F}(v^-)}{1 - \tilde{F}(v^-, v)} \leq \tilde{F}(o, v) \leq 1,
\end{align*} \]
as needed for our recursive construction. At this point, we have all the values \( \tilde{F}(u, v) \), initially for \( u \sim v \), and then for all \( u, v \) by taking products along geodesic paths.

We can now compute the constant \( C \): (7.8), combined with (7.10) and (7.12) for \( u \sim o \), implies that
\[ C \Phi(o) = \tilde{G}(o, o) = 1 + \sum_{u : u \sim o} \frac{\tilde{F}(o, u)\tilde{F}(u, o)}{1 - \tilde{F}(o, u)\tilde{F}(u, o)} \]
\[ = 1 + \sum_{u : u \sim o} \frac{\tilde{F}(u, o)}{1 - \tilde{F}(u, o)} \mu(\partial T_u) \]
\[ = 1 + \sum_{u : u \sim o} \frac{\phi(u)/\phi(o)}{1 - \phi(u)/\phi(o)} \mu(\partial T_u). \]

Therefore,
\[ C = \frac{1}{\phi(o)} + \sum_{u : u \sim o} \frac{\phi(u)/\phi(o)}{\phi(o) - \phi(u)} \mu(\partial T_u). \]

We now construct \( \tilde{G}(u, u) \) via (7.8):
\[ \tilde{G}(u, u) = 1 + \sum_{v : v \sim u} \frac{\tilde{F}(u, v)\tilde{F}(v, u)}{1 - \tilde{F}(u, v)\tilde{F}(v, u)}. \]

For \( u = o \), we know that this is compatible with our choice of \( C \). Lastly, our only choice for the Green kernel is
\[ \tilde{G}(u, v) = \tilde{F}(u, v)\tilde{G}(v, v), \quad u, v \in T. \]

Now we finally arrive at the only way of defining the transition probabilities, via (7.7):
\[ p(u, v) = \frac{1}{\tilde{G}(u, u)} \frac{\tilde{F}(u, v)}{1 - \tilde{F}(u, v)\tilde{F}(v, u)}. \]

\textbf{Claim 7.1.} \( \mathcal{P} \) is stochastic.
Proof. Combining (7.15) with (7.14), we deduce that we have to verify that for every $u \in T$,
\[
\sum_{v : v \sim u} \frac{\tilde{F}(u, v)(1 - \tilde{F}(v, u))}{1 - \tilde{F}(u, v)\tilde{F}(v, u)} = 1. \tag{7.16}
\]
If $u = o$, then by (7.12) this is just
\[
\sum_{v : v \sim o} \mu(\partial T_v) = 1.
\]
If $u \neq o$, then, again by (7.12), the left-hand side of (7.16) is
\[
\sum_{v : v \sim u} \frac{\tilde{F}(u, v)(1 - \tilde{F}(v, u)) + \tilde{F}(u, u^-)(1 - \tilde{F}(u^-, u))}{1 - \tilde{F}(u, v)\tilde{F}(v, u)} = \sum_{v : v \sim u} \frac{\mu(\partial T_v)}{\tilde{F}(o, u)} + 1 - \frac{1 - \tilde{F}(u^-, u)}{1 - \tilde{F}(u, u^-)\tilde{F}(u^-, u)} = 1. \quad \square
\]

Claim 7.2. For any $u_0 \in T$ the function $\tilde{g}_{u_0}(u) = \tilde{G}(u, u_0)$ satisfies $\partial \tilde{g}_{u_0} = \tilde{g}_{u_0} - 1_{u_0}$.

Proof. First, we combine (7.14) with (7.15) to obtain
\[
P\tilde{g}_{u_0}(u_0) = \sum_{v : v \sim u_0} p(u_0, v)\tilde{F}(v, u_0)\tilde{G}(u_0, u_0)
= \sum_{v : v \sim u_0} \frac{\tilde{F}(u_0, v)\tilde{F}(v, u_0)}{1 - \tilde{F}(u_0, v)\tilde{F}(v, u_0)} = \tilde{g}_{u_0}(u_0) - 1,
\]
and Claim 7.2 is true at $u = u_0$. Second, for $u \neq u_0$, let $w$ be the neighbour of $u$ on $\pi(u, u_0)$. Then
\[
P\tilde{g}_{u_0}(u) = \sum_{v : v \sim u, v \neq w} p(u, v)\tilde{F}(v, u)\tilde{G}(u_0, u_0) + p(u, w)\tilde{G}(w, u_0)
= \sum_{v : v \sim u} \frac{\tilde{F}(u, v)\tilde{F}(v, u)}{1 - \tilde{F}(u, v)\tilde{F}(v, u)} \tilde{G}(u_0, u_0)
\quad \tilde{G}(u, u) - 1
- p(u, w)\tilde{F}(w, u)\tilde{G}(u, u_0) + p(u, w)\tilde{G}(w, u_0)
= G(u, u_0) \left( 1 - \frac{1}{\tilde{G}(u, u)} - p(u, w)\tilde{F}(w, u) + p(u, w) \frac{1}{\tilde{F}(u, w)} \right)
= \tilde{g}_{u_0}(u),
\]
since
\[
\frac{p(u, w)}{\tilde{F}(u, w)} - p(u, w)\tilde{F}(w, u) = \frac{1}{\tilde{G}(u, u)}
\]
by (7.15). This completes the proof of Claim 7.2.
Now we can conclude that the function $\tilde{g}_{u_0}$ is non-constant, positive, and super-harmonic. Therefore, the random walk with transition matrix $P$ given by (7.15) is transient and does possess a Green function $G(u, v)$. Furthermore, by the Riesz decomposition theorem, we have

$$\tilde{g}_{u_0} = Gf + h,$$

where $h$ is a non-negative harmonic function and the charge $f$ of the potential

$$Gf(u) = \sum_v G(u, v)f(v)$$

is $f = \tilde{g}_{u_0} - P\tilde{g}_{u_0} = 1_{u_0}$, That is,

$$\tilde{G}(u, u_0) = G(u, u_0) + h(x) \quad \text{for all } u \in T.$$

Now let $x \in \partial T$ and $v = u_0 \land x$. If $u \in T_v$, then by our construction

$$\tilde{G}(u, u_0) = \tilde{G}(u, o)\frac{\tilde{G}(v, u_0)}{\phi(v)}\phi(u) \to 0 \quad \text{as } u \to x.$$

Therefore, $\tilde{G}(\cdot, u_0)$ vanishes at infinity, and the same must hold for $h$. By the maximum principle, $h \equiv 0$.

We conclude that $\tilde{G}(u, v) = G(u, v)$ for all $u, v \in T$. But then $\tilde{F}(u, v) = F(u, v)$ also holds by our construction. Comparing (7.12) with (6.11), we see that $\mu = \nu_o$.

\[\square\]

7.3. The non-compact case. Our general approach in the present work is not restricted to compact spaces. In case of a non-compact, locally compact ultra-metric space without isolated points, one constructs the tree in the same way: the vertex set corresponds to the collection of all closed balls, and the notion of a neighbourhood in the resulting tree is defined as above: if a vertex $v$ corresponds to a ball $B$, then the predecessor $v^-$ is the vertex corresponding to the ball $B'$ (see Definition 3.6), and one has the edge $[v^-, v]$. Now every vertex has a predecessor (while in the compact case, the root vertex has none), and the tree has its root at infinity, that is, the ultra-metric space becomes $\partial^*T = \partial T \setminus \{\varpi\}$, where $\varpi$ is a fixed reference end of $T$. See Fig. 5 below.

We now start with this situation: given a tree $T$ and a reference end $\varpi \in \partial T$, the predecessor $v^- = v^-_{\varpi}$ of a vertex $v$ with respect to $\varpi$ is the neighbour of $v$ on the geodesic $\pi(v, \varpi)$. Given two elements $w, z \in \hat{T} \setminus \{\varpi\}$, their confluent $w \land z$ with respect to $\varpi$ is again defined as the last common element on the geodesics $\pi(\varpi, w)$ and $\pi(\varpi, z)$; it is a vertex, unless $v = w \in \partial^*T$ (Fig. 5). Again, it is natural to assume that each vertex has at least two forward neighbours.

In this situation, for Definition 6.1 of an ultra-metric element $\phi: T \to (0, \infty)$ we need in addition to monotonicity $[\phi(v) < \phi(v^-)]$ that $\phi$ tends to $\infty$ along $\pi(o, \varpi)$, while it has to tend to 0 along any geodesic going to $\partial^*T$. The associated ultra-metric on $\partial^*T$ is then given in the same way as before:

$$d_\phi(x, y) = \begin{cases} 0 & \text{if } x = y, \\ \phi(x \land y) & \text{if } x \neq y. \end{cases}$$
Let us note here that when $\phi$ does not tend to $\infty$ along $\pi(x, \varpi)$, this also defines an ultra-metric, but then $(\partial^* T, d_\phi)$ will not be complete. Also, if the inequality $\phi(v) \leq \phi(v^-)$ is not strict, then one obtains an ultra-metric, but the above construction of the tree of closed balls does not recover the original tree from $(\partial^* T, d_\phi)$. Finally, if $\phi$ does not tend to 0 along some geodesic $\pi(o, x)$, $x \in \partial^* T$, then $x$ will be an isolated point in $(\partial^* T, d_\phi)$. (The last two observations are also true in the compact case, for a tree with a root vertex.)

Returning to our setting, the reference measure $\mu$ of a $(\phi, \mu, \sigma)$-process may have infinite mass: this is the case of a Radon measure supported on the whole of $\partial^* T$. Again, we know that it is sufficient to study the standard $(\phi, \mu)$-process. We give a brief outline of the duality of such processes with random walks on $T$. This should be compared with the final part of Kigami’s second paper [37] (the preprint of which became available when the bulk of this work had been done, and in particular, the preliminary version [64], which contains §§6 and 7 of the present paper, had already been circulated).

With respect to $\varpi$, the branch of $T$ rooted at $u \in T$ is now

$$T_u = T_{\varpi, u} = \{v \in T: u \in \pi(v, \varpi)\}.$$  

Then $\partial T_u$ is a compact subset of $\partial^* T$, being a ball with $d_\phi$-diameter $\phi(u)$. Here, it will be convenient to write $T_{o, u}$ for the branch with respect to a root vertex $o \in T$, as defined in (6.2). We note that $T_{\varpi, u} = T_{o, u}$ if and only if $u \notin \pi(o, \varpi)$. In addition to the reference end $\varpi$, we choose such a root $o$ and write $o_n$ for its $n$th predecessor, that is, the vertex on $\pi(o, \varpi)$ at graph distance $n$ from $o$.

Now let $\mathcal{P} = (p(u, v))_{u,v \in T}$ be the transition matrix of a transient nearest-neighbour random walk on $T$. We assume once more that (7.1) holds: $F(v, v^-) < 1$ for every $v \in T$, but now predecessors refer to $\varpi$. (Indeed, this implies (7.1) with respect to any choice of the root vertex.) We now consider the Dirichlet form $\mathcal{E}_H^D$ and look at the formula of Theorem 6.4. We would like to move $o$ to $\varpi$ in that formula. We know from Lemma 6.5 that the measures $\Theta_{o_n}(x, y) d\nu_{o_n}(x) d\nu_{o_n}(y)$
are the same for all $n$. However, the measures $\nu_{o_n}$ restricted to $\partial^* T$ will typically converge weakly to 0. Thus, we normalize by defining

$$\mu_n = \frac{1}{\nu_{o_n}(\partial T_o)} \nu_{o_n} \quad \text{and} \quad J_n(x,y) = \Theta_{o_n}(x,y)(\nu_{o_n}(\partial T_o))^2.$$ 

For the following lemma, recall that $T_u = T_{\omega,u}$, and note that $u \land o = o_k$ for some $k \geq 0$.

**Lemma 7.6.** Let $A \subset \partial^* T$ be compact, so that there is a vertex $u$ such that $A \subset \partial T_u$. If $u \land o = o_k$, then for all $n \geq k$ and for all $x, y \in \partial T_{o_k}$

$$\mu_n(A) = \mu_k(A) =: \mu(A) \quad \text{and} \quad J_n(x,y) = J_k(x,y) =: J(x,y).$$

It follows that

$$J(x,y) = j(x \land y) \quad \text{for} \quad j(v) = \frac{\vartheta^2}{K(v,\omega)^2} \frac{G(v,v)}{m(v)}, \quad v \in T,$$

where

$$\vartheta = \frac{m(o)\nu_o(\partial T_o)}{G(o,o)},$$

and

$$K(v,\omega) = \frac{F(v,v \land o)}{F(o,v \land o)} = \frac{F(v,v \land o \omega)}{F(o,v \land o \omega)}$$

is the Martin kernel at $\omega$.

**Proof.** Since $\partial T_{o_k}$ contains both $\partial T_o$ and $A$, we have for $n \geq k$,

$$\mu_n(A) = \frac{\nu_{o_n}(A)}{\nu_{o_n}(\partial T_o)} = \frac{F(o_n,o_k)\nu_{o_k}(A)}{F(o_n,o_k)\nu_{o_k}(\partial T_o)} = \mu_k(A).$$

Analogously, let $x, y \in \partial T_{o_k}$ and $x \land y = v$, an element of $T_{o_k}$. We use the identity

$$m(v)G(v,w) = m(w)G(w,v),$$

which implies that

$$m(o_n)F(o_n,o) = m(o_n)G(o_n,o) = m(o_n)G(o,o).$$

and compute for $n \geq k$

$$J_n(x,y) = \frac{\nu_{o_n}(\partial T_o)^2 m(o_n)}{F(o_n,v)G(v,o_n)} = \nu_o(\partial T_o)^2 \frac{m(o_n)^2 F(o_n,o)^2}{m(o_n) F(o_n,v) G(v,o_n)}$$

$$= \frac{\nu_o(\partial T_o)^2 m(o)^2}{G(o,o)^2} \frac{G(o,o)^2}{G(v,o_n)^2} \frac{G(v,v)}{m(v)} ,$$

which implies the proposed formula since

$$G(o,o_n) = F(o,o \land v)G(o \land v,o_n) \quad \text{and} \quad G(v,o_n) = F(v,o \land v)G(o \land v,o_n). \quad \square$$
Now it is not hard to deduce the following.

**Theorem 7.7.** Let $T$ and its reference end $v$ be as outlined above. Consider a nearest-neighbour random walk on $T$ which satisfies the condition $F(v, v^-) < 1$ for every $v \in T$. Let $\mu$ and $J$ be as in Lemma 7.6. Then for all compactly supported continuous functions $\varphi, \psi$ on $\partial^* T$ the Dirichlet form $(6.16)$ can be written as

$$\mathcal{E}_{\partial^* T}(\varphi, \psi) = \mathcal{E}_J(\varphi, \psi) + \partial \cdot \nu_o(\{v\}) \int_{\partial^* T} \varphi(x) \psi(x) d\mu(x)$$

where

$$\mathcal{E}_J(\varphi, \psi) = \frac{1}{2} \int_{\partial^* T} \int_{\partial^* T} (\varphi(x) - \varphi(y))(\varphi(x) - \varphi(y)) J(x, y) d\mu(x) d\mu(y).$$

When the random walk is Dirichlet regular (in which case $\nu_o(\{v\}) = 0$), the form $\mathcal{E}_J = \mathcal{E}_{\partial^* T}$ induces the standard $(\mu, \phi)$-process, where the ultra-metric element $\phi$ with respect to $v$ is given by

$$\phi(v) = \frac{1}{\partial} K(v, v),$$

and $\partial$ and the Martin kernel $K(v, v)$ are as defined in Lemma 7.6.

In particular, the $(\mu, \phi)$-process is the boundary process with a time change.

**Proof.** There is a $k$ such that the compact supports of $\varphi$ and $\psi$ are contained in $\partial T_{o_k}$. Let $n \geq k$. By Lemmas 6.5 and 7.6,

$$\mathcal{E}_{\partial^* T}(\varphi, \psi) = \frac{1}{2} \int_{\partial^* T} \int_{\partial^* T} (\varphi(x) - \varphi(y))(\varphi(x) - \varphi(y)) J_n(x, y) d\mu_n(x) d\mu_n(y)$$

$$= \frac{1}{2} \int_{\partial T_{o_n}} \int_{\partial T_{o_n}} (\varphi(x) - \varphi(y))(\varphi(x) - \varphi(y)) J(x, y) d\mu(x) d\mu(y)$$

$$+ \int_{\partial T_{o_n}} \varphi(x) \psi(x) \int_{\partial^* T \setminus \partial T_{o_n}} J_n(x, y) d\mu_n(y) d\mu(x).$$

As $n \to \infty$,

$$\frac{1}{2} \int_{\partial T_{o_n}} \int_{\partial T_{o_n}} (\varphi(x) - \varphi(y))(\varphi(x) - \varphi(y)) J(x, y) d\mu(x) d\mu(y) \to \mathcal{E}_J(\varphi, \psi).$$

Let us look at the second term. We have

$$f_n(x) = \int_{\partial^* T \setminus \partial T_{o_n}} \Theta_{o_n}(x, y) \nu_{o_n}(\partial T_o) d\nu_{o_n}(y).$$

For $x \in \partial T_{o_n}$ and $y \in \partial^* T \setminus \partial T_{o_n}$, their confluent with respect to $o_n$ is $o_n$ itself. Therefore, by $(6.17)$ and $(6.11)$

$$\Theta_{o_n}(x, y) \nu_{o_n}(\partial T_o) = \frac{m(o)}{G(o, o)} F(o, o) \nu_o(\partial T_o) = \frac{m(o)}{G(o, o)} G(o, o) \nu_o(\partial T_o)$$

$$= \frac{m(o)}{G(o, o)} G(o, o) \nu_o(\partial T_o) F(o, o) = \partial F(o, o).$$
Now note that for \( y \in \partial T \setminus \partial T_{o_n} \) we have \( F(o, o_n) \, d\nu_{o_n}(y) = d\nu_o(y) \). Therefore,

\[
f_n(x) = \vartheta \int_{\partial T \setminus \partial T_{o_n}} F(o, o_n) \, d\nu_{o_n}(y) = \vartheta \cdot \nu_o(\partial T \setminus \partial T_{o_n}) \rightarrow \vartheta \cdot \nu_o(\{\varpi\}),
\]

and for \( n \to \infty \) we can use dominated convergence to get that

\[
\int_{\partial T_{o_n}} \varphi(x) \psi(x) \int_{\partial T \setminus \partial T_{o_n}} J_n(x, y) \, d\mu_n(y) \, d\mu(x) = \int_{\partial^* T} \varphi(x) \psi(x) f_n(x) \, d\mu(x)
\]

\[
\rightarrow \vartheta \cdot \nu_o(\{\varpi\}) \int_{\partial^* T} \varphi(x) \psi(x) \, d\mu(x),
\]

as stated. To prove the formula for the corresponding ultra-metric element, we proceed as in the proof of Theorem 7.1 (see (7.4) and the subsequent lines). We find that the ultra-metric element must satisfy

\[
\frac{1}{\Phi(v)} - \frac{1}{\Phi(v^-)} = (j(v) - j(v^-)) \mu(\partial T_v).
\]

The right-hand side of this equation can be computed: we have \( v^- \land o = o_k \) for some \( k \geq 0 \), and combining the arguments after (7.4) with those in the proof of Lemma 7.6, we have

\[
(j(v) - j(v^-)) \mu(\partial T_v) = \left( \frac{m(o_k)}{F(o_k, v)G(v, o_k)} - \frac{m(o_k)}{F(o_k, v^-)G(v^-, o_k)} \right) \nu_{o_k}(\partial T_v) \nu_{o_k}(\partial T_0)
\]

\[
= \left( \frac{G(o, o_k)}{G(v, o_k)} - \frac{G(o, o_k)}{G(v^-, o_k)} \right) F(o_k, o) \nu_o(\partial T_0)
\]

\[
= \frac{G(o, o_k)}{G(v, o_k)} \frac{m(o) \nu_o(\partial T_0)}{G(o, o)}
\]

\[
= \frac{\vartheta}{K(v, \varpi)} - \frac{\vartheta}{K(v^-, \varpi)}.
\]

We infer that \( 1/\Phi(\cdot) - \vartheta/K(\cdot, \varpi) \) must be constant. By Dirichlet regularity of the random walk, \( K(v, \varpi) \to \infty \) as \( v \to \varpi \). On the other hand, \( \phi(o_n) \) must also tend to infinity. Thus, the constant is 0, and \( \phi \) has the proposed form. \( \square \)

Lemma 7.6 and Theorem 7.7 lead to clearer insight and simpler proofs concerning the material on random walks in [37] (§§10, 11, and in particular, Theorem 11.3). Namely, our limit measure \( \mu \) coincides with the \( \nu_* \) in [37]. We also note here that there are examples where \( \mu(\partial^* T) = \infty \), as well as examples where \( \mu(\partial^* T) < \infty \), even though the ultra-metric space is non-compact.

Remark 7.8. In §§6 and 7 we have always assumed that the ultra-metric space has no isolated points, which for the tree means that \( \deg^+ \geq 2 \). The theme of [7] is the opposite situation, where all points are isolated, that is, the space is discrete. In that case the ultra-metric space is also the boundary of a tree and does not consist of ends but of terminal vertices, that is, vertices with only one neighbour.
From the point of view of the present section, the mixed situation works equally well. If we start with a locally compact ultra-metric space having both isolated and non-isolated points, then we can construct the tree in the same way. The vertex set is the collection of all closed balls. The isolated points will then become terminal vertices of the tree, which have no neighbours other than the predecessor, as for example the vertices $x$ and $y$ in Fig. 6. All interior (non-terminal) vertices will have forward degree at least two.

In the compact case, the boundary $\partial T$ of that tree consists of the terminal vertices together with the space of ends. In the non-compact case, we will again have a reference end $\varpi$ as before, and $\partial^* T$ consists of all ends except $\varpi$, plus the terminal vertices. The definition of an ultra-metric element remains the same, but we only need to define it on interior vertices. In this general setting, the construction of $(\phi, \mu, \sigma)$-processes remains unchanged.

Even in presence of isolated points, the duality between $(\phi, \mu, \sigma)$-processes and random walks on the associated tree remains as explained earlier. The random walk should then be such that the terminal vertices are absorbing, and that the Green kernel tends to 0 at infinity. The Doob–Naim formula extends readily to that setting.

Remark 7.9. Let us again consider the general situation when we start with a transient random walk on a locally finite rooted tree $T$.

The limit distribution $\nu_\circ$ will in general not be supported by the whole of $\partial T$. The boundary process can of course still be constructed (see [36]), but will evolve naturally on $\text{supp}(\nu_\circ)$ only. Thus, we can consider our ultra-metric space to be just $\text{supp}(\nu_\circ)$. The tree associated with this ultra-metric space will in general not be the tree we started with, nor its transient skeleton as defined in [63], (9.27) (the subtree induced by $o$ and all $v \in T \setminus \{o\}$ with $F(v, v^-) < 1$, where $v^- = v^o$).

The reasons are twofold. First, the construction of the tree associated with $\text{supp}(\nu_\circ)$ will never return vertices with forward degree 1. Second, some end contained in $\text{supp}(\nu_\circ)$ may be isolated within that set, while not being isolated in $\partial T$. But then this element will become a terminal vertex in the tree associated with the
ultra-metric (sub)space \( \text{supp}(\nu_o) \). This occurs precisely when the transient skeleton has isolated ends.

Thus, one should work with a modified ‘reduced’ tree plus a random walk in order to maintain the duality between random walks and isotropic jump processes. The same observations apply to the non-compact case, with a reference end in the place of the root and the measure \( \mu \) of Lemma 7.6 in the place of \( \nu_o \).

**Remark 7.10.** For a transient random walk on the rooted tree \( T \), [36] also recovers an intrinsic metric of the boundary process on \( \partial T \) (compact case!) in terms of what is called an ultra-metric element in the present paper. This is of course \( \phi(x) = G(x, o) \), denoted \( D_x \) in [36], where it is shown that, for \( \nu_o \)-almost every \( \xi \in \partial T \), \( D_x \to 0 \) along the geodesic ray \( \pi(o, \xi) \). This has the following potential-theoretic interpretation.

A point \( x \in \partial T \) is called regular for the Dirichlet problem if for every \( \varphi \in C(\partial T) \), its Poisson transform \( h_\varphi \) satisfies

\[
\lim_{v \to x} h_\varphi(v) = \varphi(x).
\]

It is known from Cartwright, Soardi, and Woess [14] (Remark 2) that \( x \) is regular if and only if \( \lim_{u \to x} G(u, o) = 0 \) (provided \( T \) has at least two ends); see also [63], Theorem 9.43. By the latter theorem, the set of regular points has \( \nu_o \)-measure 1. That is, the Green kernel vanishes at \( \nu_o \)-almost every boundary point.

**Remark 7.11.** In the proof of Theorem 7.5, we have reconstructed random-walk transition probabilities from \( C \cdot \phi(u) = G(u, o) \) and \( \mu = \nu_o \).

A similar (somewhat simpler) question was addressed by Vondraček [60]: how to reconstruct the transition probabilities from all limit distributions \( \nu_u, u \in T \), on the boundary. This, as well as our method, basically come from (6.11) and (7.7)–(7.8), which can be traced back to Cartier [12].

## 8. Random walk associated with \( p \)-adic fractional derivative

In this section we consider a doubly specific example which unites the approaches of §5 and §§6, 7. We start with the compact case.

### 8.1. The \( p \)-adic fractional derivative on \( \mathbb{Z}_p \)

Let \( \mathbb{Z}_p \subset \mathbb{Q}_p \) be the group of \( p \)-adic integers. As a counterpart of the operator \( \mathcal{D}^\alpha \) we introduce the fractional derivative operator \( \mathbb{D}^\alpha \) on \( \mathbb{Z}_p \). We show that it is the Laplacian of an appropriate isotropic Markov semigroup. Then we construct a random walk associated with \( \mathbb{D}^\alpha \) in the sense of §§6, 7.

Since \( \mathbb{Z}_p \) is a compact Abelian group, its dual \( \hat{\mathbb{Z}}_p \) is a discrete Abelian group. It is known that \( \hat{\mathbb{Z}}_p \) can be identified with the group

\[
Z(p^\infty) = \{ p^{-n}m : 0 \leq m < p^n, \ n = 1, 2, \ldots \},
\]

equipped with addition (of real numbers) modulo 1 as the group operation. As sets (but not as groups) \( Z(p^\infty) \subset \mathbb{Q}_p \), whence the function \( \xi \mapsto \|\xi\|_p \) is well defined on the group \( Z(p^\infty) \).
Definition 8.1. The operator \((D^\alpha, \mathcal{V}_\alpha), \alpha > 0\), is defined via the Fourier transform on the compact Abelian group \(\mathbb{Z}_p\) by
\[
\widehat{D^\alpha f}(\xi) = \|\xi\|_p^\alpha \hat{f}(\xi), \quad \xi \in Z(p^\infty),
\]
where \(\mathcal{V}_\alpha\) is the space of locally constant functions on \(\mathbb{Z}_p\) (cf. Definition 5.1 of the operator \(D^\alpha\)).

An immediate consequence is that \(D^\alpha\) is a non-negative-definite self-adjoint operator whose spectrum coincides with the range of the function
\[
\xi \mapsto \|\xi\|_p^\alpha : Z(p^\infty) \to \mathbb{R}^+,
\]
that is,
\[
\text{spec } D^\alpha = \{0, p^\alpha, p^{2\alpha}, \ldots \}.
\]
The eigenspace \(\mathcal{H}(\lambda)\) of \(D^\alpha\) corresponding to the eigenvalue \(\lambda = p^{k\alpha}, k \geq 1\), is spanned by the function
\[
f_k = \frac{1}{\mu_p(p^k\mathbb{Z}_p)} 1_{p^k\mathbb{Z}_p} - \frac{1}{\mu_p(p^{k-1}\mathbb{Z}_p)} 1_{p^{k-1}\mathbb{Z}_p}
\]
and its shifts \(f_k(\cdot + a), a \in \mathbb{Z}_p/p^k\mathbb{Z}_p\).

Indeed, computing the Fourier transform of the function \(f_k\)
\[
\widehat{f_k}(\xi) = 1_{\{\|\xi\|_p \leq p^k\}} - 1_{\{\|\xi\|_p \leq p^{k-1}\}} = 1_{\{\|\xi\|_p = p^k\}},
\]
we find that
\[
\widehat{D^\alpha f_k}(\xi) = \|\xi\|_p^\alpha \hat{f_k}(\xi) = p^{k\alpha} \hat{f_k}(\xi).
\]
The maximal number of linearly independent functions in the set \(\{f_k(\cdot + a) : a \in \mathbb{Z}_p/p^k\mathbb{Z}_p\}\) is \(p^{k-1}(p-1)\), and therefore
\[
\dim \mathcal{H}(\lambda) = p^{k-1}(p-1).
\]
All the above shows that \(D^\alpha\) coincides with the Laplacian of some isotropic Markov semigroup \(\left(\mathbb{P}_t^\alpha\right)_{t>0}\) on the ultra-metric measure space \((\mathbb{Z}_p, d_P, \mu_p)\). In particular, using the complete description of \(\text{spec } D^\alpha\), we compute the intrinsic distance, call it \(d_{p,\alpha}(x, y)\):
\[
d_{p,\alpha}(x, y) = \left(\frac{\|x - y\|_p}{p}\right)^\alpha.
\]
It is now straightforward to compute the spectral distribution function \(N_\alpha(x, \tau) \equiv N_\alpha(\tau)\) and then the jump kernel \(J_\alpha(x, y) \equiv J_\alpha(x - y)\) of the operator \(D^\alpha\). We claim that
\[
J_\alpha(x, y) = \frac{p^\alpha - 1}{1 - p^{-\alpha - 1}} \left(\frac{p^{-\alpha} - p^{-\alpha - 1}}{1 - p^{-\alpha}} + \frac{1}{\|x - y\|_p^{1+\alpha}}\right).
\]
(8.1)
Recall for comparison that according to (5.5) the jump kernel \(J_\alpha(x, y)\) of the operator \(D^\alpha\) is given by
\[
J_\alpha(x, y) = \frac{p^\alpha - 1}{1 - p^{-\alpha - 1}} \frac{1}{\|x - y\|_p^{1+\alpha}}.
\]
To prove (8.1), we compute \( J_\alpha(z) \). Let \( \|z\|_p = p^{-l} \). Then \( d_{p,\alpha}(0,z) = p^{-(l+1)\alpha} \) and

\[
J_\alpha(z) = \int_0^{1/d_{p,\alpha}(0,z)} N_\alpha(\tau)\,d\tau = \int_0^{p^{(l+1)\alpha}} N_\alpha(\tau)\,d\tau.
\]

The function \( N_\alpha(\tau) \) is a non-decreasing left-continuous staircase function having jumps at the points \( \tau_k = p^{k\alpha}, k = 1, 2, \ldots, \) and taking the values \( N_\alpha(\tau_k) = p^{k-1} \) at these points, whence

\[
J_\alpha(z) = 1 \cdot p^\alpha + p(2^{2\alpha} - p^\alpha) + p^2(3^{3\alpha} - p^{2\alpha}) + \cdots + p^l(p^{(l+1)\alpha} - p^{l\alpha}) \\
= \frac{1 - p^{-1}}{1 - p^{-\alpha - 1}} + \frac{p^\alpha - 1}{1 - p^{-\alpha - 1}}p^{(\alpha+1)} \\
= \frac{p^\alpha - 1}{1 - p^{-\alpha - 1}}\left(\frac{p^{-\alpha} - p^{-\alpha - 1}}{1 - p^{-\alpha}} + \frac{1}{\|z\|_p^{1+\alpha}}\right),
\]

as required. Next, we apply Theorem 3.12 and get that

\[
\mathcal{D}^\alpha f(x) = \int_{\mathbb{Z}_p} (f(x) - f(y))J_\alpha(x-y)\,d\mu_p(y). \tag{8.2}
\]

The equations (8.1), (8.2) and (5.1) now yield the following result.

**Corollary 8.2.** For any function \( f \) defined on \( \mathbb{Z}_p \subset \mathbb{Q}_p \) let \( \tilde{f} = f \) on \( \mathbb{Z}_p \) and \( \tilde{f} = 0 \) otherwise. Then

\[
f \in \text{dom}(\mathcal{D}^\alpha) \implies \tilde{f} \in \text{dom}(\mathcal{D}^\alpha)
\]

and

\[
\mathcal{D}^\alpha f(x) = \mathcal{D}^\alpha \tilde{f}(x), \quad (\mathcal{D}^\alpha f, f) = (\mathcal{D}^\alpha \tilde{f}, \tilde{f}) \tag{8.3}
\]

for \( x \in \mathbb{Z}_p, \ f \in \text{dom}(\mathcal{D}^\alpha), \) and \( (1, f) = 0. \)

### 8.2. Nearest-neighbour random walk on the rooted tree \( \mathbb{T}_p^\circ \)

As an illustration of Theorem 7.5 we construct a random walk on the rooted tree associated with \( \mathbb{Z}_p \) whose boundary process coincides with the isotropic process driven by the operator \( \mathcal{C} \cdot \mathcal{D}^\alpha, \) where \( \mathcal{C} = p^{-\alpha}(1 - p^{-\alpha}). \)

The Abelian group \( \mathbb{Z}_p \) can be identified with the boundary of the tree \( \mathbb{T}_p^\circ \) with root \( o, \) where every vertex \( v \) has \( p \) forward neighbours. In our identification, this is the tree of balls of the ultra-metric space \( (\mathbb{Z}_p, d_p) \) with root \( o \) corresponding to the whole of \( \mathbb{Z}_p \) and the ultra-metric \( d_p(x,y) = \|x-y\|_p. \) See Fig. 4 above, where \( p = 2. \) We fix a constant \( c \in (0,1) \) and consider the nearest-neighbour random walk on \( \mathbb{T}_p^\circ \) with

\[
p(v,v^-) = 1 - c \quad \text{and} \quad p(v^-,v) = \begin{cases} 1/p & \text{if } v^- = o, \\ c/p & \text{otherwise.} \end{cases} \tag{8.4}
\]

Using [63] (Theorem 1.38 and Proposition 9.3), one can compute precisely the Green function \( G(v,o) \), the hitting probability \( F(v,o) \), and other quantities associated with our random walk. In particular, choosing \( c = (1 + p^{-\alpha})^{-1} \), we get that

\[
F(v,o) = p^{-\alpha|v|} \quad \text{and} \quad G(v,o) = \frac{p^{-\alpha|v|}}{1 - p^{-\alpha}}, \tag{8.5}
\]
where $|v|$ is the graph distance from $v$ to $o$. We see that the Green function vanishes at infinity, so that the random walk is Dirichlet regular.

The transition probabilities are invariant with respect to all the automorphisms of the tree. Every such automorphism must fix $o$ and every level of the tree. Let $\nu = \nu_o$ be the limit distribution on $\partial T^o_p$ of the random walk starting at $o$. Then $\nu$ is also invariant under the automorphism group of the tree (whose action extends to the boundary). In particular, it is invariant under the action of $\mathbb{Z}_p$. Thus, under the identification of $\partial T^o_p$ with $\mathbb{Z}_p$, we have that $\nu = \mu_p$, the normalized Haar measure of $\mathbb{Z}_p$.

We now look at the boundary process induced by our random walk as a jump process on $\mathbb{Z}_p$. By Theorem 7.1, the boundary process arises as an isotropic jump process with the reference measure $\mu_p$. Let $L$ be its Laplacian. By Corollary 7.2, the set $\text{spec } L$ coincides with the range of the function $v \mapsto 1/G(o,v), v \in T^o_p$, together with $\{0\}$. In view of the above formula for $G(o,v)$, this means that

$$\text{spec } L = \{0, (1 - p^{-\alpha}), p^\alpha(1 - p^{-\alpha}), p^{2\alpha}(1 - p^{-\alpha}), \ldots \}.$$  

Recall that

$$\text{spec } D^\alpha = \{0, p^\alpha, p^{2\alpha}, \ldots \} = \frac{p^\alpha}{1 - p^{-\alpha}} \text{spec } L.$$  

Since both $D^\alpha$ and $L$ have the same orthonormal basis of eigenfunctions, we conclude that they are proportional, that is,

$$D^\alpha = \frac{p^\alpha}{1 - p^{-\alpha}} L. \quad (8.6)$$

Thus, we finally come to the following conclusion.

**Proposition 8.3.** The boundary process $\{X_t\}_{t>0}$ associated with the random walk defined in (8.4) with parameter $c = (1 + p^{-\alpha})^{-1}$ and the isotropic jump process $\{X^\alpha_t\}_{t>0}$ driven by the operator $D^\alpha$ are related by the linear time change $X_t/C = X^\alpha_t$, where $C = p^{-\alpha}(1 - p^{-\alpha})$.

The equation (8.6) implies that the jump kernels $J^\alpha(x,y)$ and $\Theta_o(x,y)$ of the operators $D^\alpha$ and $L$, respectively, are related by

$$J^\alpha(x,y) = \frac{p^\alpha}{1 - p^{-\alpha}} \Theta_o(x,y). \quad (8.7)$$

We now show how to directly compute the Na"im kernel

$$\Theta_o(x,y) = \frac{1}{G(o,o) F(o,v) F(v,o)}, \quad \text{where } v = x \land y,$n

using the data in (8.5). We do not yet have $F(o,v)$. We shall compute

$$N(v) = \frac{1}{F(o,v) F(v,o)}.$$  

Since this depends only on the level $k$ of $v$, we consider an arbitrary geodesic ray $[o = v_0, v_1, \ldots]$ and set up a linear recursion for $N(v_k)$. Denoting by $w_1$ an arbitrary
Isotropic Markov semigroups on ultra-metric spaces
675

neighbour of \( o \) different from \( v_1 \) and using [63] (Proposition 9.3(b)) and (8.5), we see that

\[
F(o, v_1) = \frac{1}{p} + \frac{p-1}{p} F(w_1, o) F(o, v_1) = \frac{1}{p} + \frac{(p-1)p^{-\alpha}}{p} F(o, v_1),
\]

and thus

\[
F(o, v_1) = \frac{p^\alpha}{p^{\alpha+1} - p + 1}.
\]

Hence, we obtain the initial values

\[
N(v_0) = 1 \quad \text{and} \quad N(v_1) = p^{\alpha+1} - p + 1.
\]

Next, for \( k \geq 1 \), we let \( w_{k+1} \) be a forward neighbour of \( v_k \) different from \( v_{k+1} \). Again using [63] (Proposition 9.3(b)) and (8.5)), we get that

\[
F(v_k, v_{k+1}) = \frac{p^\alpha}{p(p^\alpha + 1)} + \frac{(p-1)p^\alpha}{p(p^\alpha + 1)} F(w_{k+1}, v_k) F(v_k, v_{k+1})
\]

\[
+ \frac{1}{p^{\alpha+1} + 1} F(v_{k-1}, v_k) F(v_k, v_{k+1}).
\]

We insert the value \( F(w_{k+1}, v_k) = p^{-\alpha} \) and divide by

\[
F(o, v_{k+1}) = F(o, v_k) F(v_k, v_{k+1}) = F(o, v_{k-1}) F(v_{k-1}, v_k) F(v_k, v_{k+1}).
\]

Then

\[
\frac{1}{F(o, v_k)} = \frac{p^\alpha}{p(p^\alpha + 1)} \frac{1}{F(o, v_{k+1})} + \frac{p-1}{p(p^\alpha + 1)} \frac{1}{F(o, v_k)} + \frac{1}{p^{\alpha+1} + 1} \frac{1}{F(o, v_{k-1})}.
\]

Now we multiply both sides by \( 1/F(v_k, o) = p^{\alpha k} \) and obtain

\[
N(v_k) = \frac{1}{p(p^\alpha + 1)} N(v_{k+1}) + \frac{p-1}{p(p^\alpha + 1)} N(v_k) + \frac{p^\alpha}{p^{\alpha+1} + 1} N(v_{k-1}).
\]

This is a homogeneous second-order linear recursion with constant coefficients. Its characteristic polynomial has roots 1 and \( p^{\alpha+1} \). Therefore,

\[
N(v_k) = A + B p^{(\alpha+1)k}.
\]

Inserting the initial values, we easily find the values of \( A \) and \( B \). In order to obtain the Naïm kernel, we have to multiply by \( 1/G(o, o) = 1 - p^{-\alpha} \). Thus,

\[
\Theta_o(x, y) = \frac{(1 - p^{-\alpha})(p-1)}{p^{\alpha+1} - 1} + \frac{(1 - p^{-\alpha})(p^{\alpha+1} - p)}{p^{\alpha+1} - 1} p^{(\alpha+1)k} = \frac{1 - p^{-\alpha}}{p^{\alpha}} J_\alpha(x, y),
\]

as required.
8.3. The random walk corresponding to $\mathcal{D}^\alpha$ on $\mathbb{Q}_p$. We can combine the preceding considerations with the material of Lemma 7.6 and Theorem 7.7 concerning the duality between isotropic processes and random walks in the non-compact case. It is now easy to understand the random walk corresponding to the fractional derivative on the whole of $\mathbb{Q}_p$.

The tree associated with $\mathbb{Q}_p$ is the homogenous tree $T = T_p$ with degree $p + 1$. We have to choose a reference end $\varpi$. Then we can identify its lower boundary $\partial T_p^\varpi$ with the field of $p$-adic numbers. With respect to $\varpi$, every vertex $v$ has its predecessor $v^-$ and $p$ successors. Every subtree $T_v = T_{\varpi, v}$ is isomorphic to the rooted tree $T_o$ considered above in the compact case of the $p$-adic integers. In particular, we choose the root vertex $o$ such that $\partial T_o = \mathbb{Z}_p$. See Fig. 5 above, where $p = 2$.

We now define the random walk on $T_p$ as in (8.4), but with predecessors referring to $\varpi$:

$$p(v, v^-) = 1 - c \quad \text{and} \quad p(v^-, v) = \frac{c}{p}, \quad \text{where} \quad c = (1 + p^{-\alpha})^{-1}. \quad (8.8)$$

For the following quantities, see, for instance, [62], pp. 423, 424. For all $v \in T_p$,

$$F(v, v^-) = p^{-\alpha}, \quad F(v^-, v) = p^{-1}, \quad G(v, v) = \frac{1 + p^{-\alpha}}{1 - p^{-\alpha - 1}}$$

and

$$\nu_v(\partial T_v) = \frac{1 - p^{-\alpha}}{1 - p^{-\alpha - 1}}.$$

This implies that the reference measure $\mu$ of the boundary process with respect to $\varpi$, as given by Lemma 7.6, is the standard Haar measure on $\mathbb{Q}_p$.

We compute $\vartheta = (1 - p^{-\alpha})/(1 + p^{-\alpha})$. Furthermore, let $h(v) = d(v, v \land o) - d(o, v \land o)$ (where $d$ is the graph metric). This is the horocycle number of $v$. That is, the vertices with $h(v) = k$, $k \in \mathbb{Z}$, are the elements in the $k$th generation $H_k$ of the tree (see Fig. 5), and $\partial T_v$ corresponds to a ball with radius $p^{-k}$ in the standard ultra-metric on $\mathbb{Q}_p$. Then

$$K(v, \varpi) = p^{\alpha h(v)} \quad \text{and} \quad m(v) = p^{(\alpha - 1)h(v)}.$$

Putting things together, we have

$$\Phi(v) = \frac{1 + p^{-\alpha}}{1 - p^{-\alpha}} p^{-\alpha h(v)} \quad \text{and} \quad j(v) = \frac{(1 - p^{-\alpha})^2}{(1 + p^{-\alpha})(1 - p^{-\alpha - 1})} p^{(\alpha + 1)h(v)}.$$

Retranslating this into $p$-adic notation, we conclude that the intrinsic metric and jump kernel of the boundary process with respect to $\varpi$ are given by

$$d_\varphi(x, y) = \frac{1 - p^{-\alpha}}{1 + p^{-\alpha}} \|x - y\|_p^{\alpha},$$

$$J(x, y) = \frac{(1 - p^{-\alpha})^2}{(1 + p^{-\alpha})(1 - p^{-\alpha - 1})} \frac{1}{\|x - y\|_p^{\alpha + 1}} = \frac{1 - p^{-\alpha}}{p^\alpha + 1} J_\alpha(x, y),$$

where $J_\alpha$ is the jump kernel associated with $\mathcal{D}^\alpha$. So at last we obtain the following.
Proposition 8.4. The boundary process \( \{ X_t \}_{t>0} \) with respect to the reference end \( \varpi \) associated with the random walk (8.8) on \( \mathbb{T}_p \) and the isotropic jump process \( \{ X^\alpha_t \}_{t>0} \) driven by the operator \( \mathcal{D}^\alpha \) on \( \mathbb{Q}_p \) are related by the linear time change \( X_t/C^\ast = X^\alpha_t \), where \( C^\ast = (1 - p^{-\alpha})/(p^\alpha + 1) \).

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