Geometry of almost-product Lorentzian manifolds and relativistic observer

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Abstract

The notion of relativistic observer is confronted with Naveira’s classification of (pseudo-)Riemannian almost-product structures on spacetime manifolds. Some physical properties and their geometrical counterparts are shortly discussed.
1 Introduction

In Einstein’s General Relativity, a gravitational interaction is represented by a metric with Lorentzian signature $(-, +, +, +)$ living on a (curved) four-dimensional spacetime manifold and satisfying Einstein’s field equations. An observer is an independent notion and, according to a nowadays point of view, can be identified with an arrow of time. More precisely, the observer is determined by a timelike normalized (local) vector field on spacetime. We can also think of it as the collection of its integral curves, considered as world lines (also known as the congruence of world lines of point observers) of some continuous material object (e.g. relativistic fluid). From a mathematical perspective, it provides a one-dimensional (timelike) foliation. It appears that a pair, the metric and the vector field, determines a differential-geometric structure which is called an almost-product structure. From a physical perspective, a relativistic observer is tautologically defined as a field of his own four-velocities. Having chosen an observer, one can define relativistic observables, i.e. relative measurable quantities. They include the relative (three-)velocity of another observer or test particles (see e.g. [1], [2], [3]), as well as Noether conserved currents in diffeomorphism covariant field theories [4]. The well-known splitting of the electromagnetic field into measurable electric and magnetic components is also relative to the observer. In the more traditional approach to General Relativity, the measurable quantities are related to coordinates. In fact, given a coordinate system, one can associate to it a (local) observer, indicated by a time variable. However the notion adopted here is more general, coordinate-free and can be globalized.

In the presented note we provide the correspondence between Naveira’s classes of a pseudo-Riemannian manifold [5] implemented by the observer and its physical characteristics as introduced in [6].

The paper is organized as follows. In section 2 we introduce the notation and basic notions. In section 3 we shortly recall Gil-Medrano’s theorem [7], which provides a differential geometric interpretation for Naveira’s classes. The advantages of the almost-product structures in physics are discussed in section 4 (see also [8] in this context). They extend the possible characteristics for a given observer on a Lorentzian manifold. Finally, we provide a few illustrative examples in section 5.

2 Preliminaries and definitions

Let $M$ and $TM$ denote respectively an $n$-dimensional smooth manifold and its tangent bundle. A $k$-dimensional ($k < n$) tangent distribution ($k$-distribution in short) is a map $D$ which associates a $k$-dimensional subspace $D_p \subset T_p M$ to the point $p \in M$:

$$D : p \rightarrow D_p \subset T_p M.$$  
(1)
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$D$ can be also considered as a subbundle of $TM$. Locally, one can say that a $k$-distribution is generated by a set of $k$ linearly independent vector fields iff in every point $p$ their values span the $k$-dimensional subspace $D_p$, i.e. $D_p = \text{span}\{X_1(p), \ldots, X_k(p)\}$. In this case we shall write $X_i \in \Gamma(D)$, where $\Gamma(D)$ stands for a submodule of cross sections of the subbundle $D \subset TM$.

An embedded submanifold $N \subset M$ is called an integral manifold of the distribution $D$ if $T_pN = D_p$ in every point $p \in N$. We say that $D$ is involutive if, for each pair of local vector fields $(X, Y)$ belonging to $D$, their Lie bracket $[X, Y]$ is also a vector field from $D$.

The distribution $D$ is completely integrable if for each point $p \in M$ there exists an integral manifold $N$ of the distribution $D$ passing through $p$ such that the dimension of $N$ is equal to the dimension of $D$. It turns out that every involutive distribution is completely integrable (local Frobenius theorem).

Every smooth 1-dimensional distribution is integrable.

The integrability of a distribution is closely related to the notion of foliation. We have the following (global) Frobenius theorem:

**Theorem 1.** Let $D$ be an involutive $k$-dimensional tangent distribution on a smooth manifold $M$. The collection of all maximal connected integral manifolds of $D$ forms a foliation of $M$.

The proof of the theorem and the precise definition of a foliation can be found in [9]. Roughly speaking, a foliation is a collection of submanifolds $N_i$ such that each submanifold proceeds smoothly into another one. They do not cross each other. Particularly, a class of globally hyperbolic spacetimes $M = T \times \Sigma$, where $T$ is an open interval in the real line $\mathbb{R}$ and $\Sigma$ is a three-manifold, serve as a typical example of global foliation [10].

Let us recall [11, 12] that an almost-product structure on $M$ is determined by a field of endomorphisms of $TM$, i.e. a $(1, 1)$ tensor field $P$ on $M$, such that $P^2 = I$ ($I =$Identity). In this case, at any point $p \in M$, one can consider two subspaces of $T_PM$ corresponding respectively to two eigenvalues $\pm 1$ of $P$.

It defines two complementary distributions on $M$, i.e. $TM = D^+ \oplus D^-$. Moreover, if $M$ is equipped with a (pseudo-)Riemannian metric $g$ such that

$$g(PX, PY) = g(X, Y); \quad X, Y \in \Gamma(TM),$$

then both distributions are mutually orthogonal. In this case, $P$ is called a (pseudo-) Riemannian almost-product structure. It is to be noticed that some modified gravity models admit almost-product structures as solutions [13].
3 Geometric characterization of distributions on (pseudo-)Riemannian manifolds

Let $D$ be a distribution on $(M, g)$ and $D^\perp$ the distribution orthogonal to $D$. At every point $p \in M$, we have then $T_pM = D_p \oplus D_p^\perp$. Thus we can uniquely define a $(1,1)$ tensor field $P$ such that $P^2 = I$, $P|_D = 1$, $P|_{D^\perp} = -1$. It is clear that $P$ becomes automatically a (pseudo-)Riemannian almost-product structure. One has (see [7]):

**Definition 1.** The distribution $D$ is called geodesic, minimal or umbilical if and only if $D$ has property $D_1$, $D_2$ or $D_3$ respectively, where:

- $D_1 \iff (\nabla_A P)A = 0$,
- $D_2 \iff \alpha(X) = 0$,
- $D_3 \iff g((\nabla_A P)B, X) + g((\nabla_B P)A, X) = \frac{k}{2} g(A, B) \alpha(X)$,

where $X \in \Gamma(D)^\perp$; $A, B \in \Gamma(D)$. Here $\{e_a\}_{a=1}^k (k = \dim D)$ is a local orthonormal frame of $D$ and $\alpha(X) = \sum_{a=1}^k g((\nabla_{e_a} P)e_a, X)$.

It implies that a distribution has the property $D_1$ if and only if it has the properties $D_2$ and $D_3$. Their meanings in the case of integrability are explained below.

**Theorem 2.** (O. Gil-Medrano) A foliation $D$ is called totally geodesic, minimal or totally umbilical if and only if $D$ has the property $F_1$, $F_2$ or $F_3$ respectively, where

$$F_i \iff F + D_i, \quad i = 1, 2, 3$$  \hspace{1cm} (3)

and

$$F \iff (\nabla_A P)B = (\nabla_B P)A \quad \forall A, B \in \Gamma(D).$$  \hspace{1cm} (4)

The proof of this theorem can be found in [7]. It is easy to see that the property $F$ is equivalent to Frobenius’ theorem, i.e. a distribution $D$ with this property is a maximal foliation. The theorem says that, in principle, one deals with three special types of foliations:

- $(F_1)$ Totally geodesic foliation: it means that every geodesic of an arbitrary integral submanifold $N$ (the leaf of foliation), if considered together with the induced metric (the first fundamental form), is at the same time geodesic of the total manifold $M$. Moreover, it is equivalent to the statement that the second fundamental form of $N$ (i.e. extrinsic curvature) vanishes. In other words, the extrinsic curvature measures the failure of a geodesic of the manifold $N$ to be a geodesic of $M$.
- $(F_2)$ Minimal foliation: If there is a surface with the smallest possible value of the area bounded by a certain curve, that surface is called a minimal surface. The condition for a distribution to be a

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1 The case of null distributions is more complicated and should be discussed separately, see e.g. [14].
minimal distribution is that the trace of the second fundamental form vanishes. The trace of the extrinsic curvature is also called mean curvature, that is, the average of the principal curvatures. Examples of minimal surfaces in $\mathbb{R}^3$ are the catenoid and the helicoid.

(F3) Umbilical foliation: We recall that an umbilical manifold is a manifold for which all points are umbilical points. Umbilical points, in turn, are locally spherical: every tangent vector at such point is a principal direction and all principal curvatures are equal $[15]$. For example, a sphere is an umbilical manifold. In the case of integral submanifolds, the second fundamental form has to be proportional to the induced metric.

4 Almost-product structure related to a spacetime observer

In the present section we are going to apply the formalism presented above to the special case of a relativistic observer on a spacetime manifold. These new tools will be used at the end of the section for a final classification.

From now on $(M, g)$ denotes a four-dimensional manifold (spacetime) equipped with Lorentzian signature metric $g_{\alpha\beta}$. An observer is represented by a timelike vector field $u^\alpha$ which, according to our sign convention $(-, +, +, +)$, is normalized to

$$u^\alpha u_\alpha = -1. \quad (5)$$

Strictly speaking, the normalization condition (5) prevents the existence of critical points and one can deal with a one-dimensional (timelike) distribution instead. Such a distribution is always integrable and provides a foliation with world-lines as leaves. Each leaf can then be parameterized by arc length (proper time), making $u^\alpha$ a four-velocity field. This implies that the only nontrivial question one can ask about a one-dimensional distribution is whether it is geodesic or not (see below the tables).

Because of this, one should concentrate on its orthogonal (transverse) completion $D$. This is a spacelike three-dimensional distribution with Euclidean signature. These two distributions provide a $3+1$ (orthogonal) decomposition of the tangent bundle $TM = D \oplus D^\perp$, with the one-dimensional timelike distribution denoted as $D^\perp$. It is easy to find out that the corresponding three-dimensional projection tensor has the form:

$$h_\alpha^\beta = \delta_\alpha^\beta + u^\alpha u_\beta, \quad (6)$$

which, due to (5), implies $h^\rho_\alpha h^\rho_\beta = h^\rho_\beta$. We would like to stress that in what follows we shall always use the original metric $g_{\alpha\beta}$ for lowering and rising indices. Thus covariant and contravariant components of tensors can be used exchangeably. For example, the second-rank symmetric tensor

$$h_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta, \quad (7)$$
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plays the role of induced Euclidean metric on the distribution \( D \). When \( D \) is integrable, then \( \text{induced Euclidean metric on the distribution } D \). When \( D \) is integrable, then (7) is the first fundamental form (i.e. induced metric) on each leaf. The corresponding foliation by spacelike hypersurfaces has the physical meaning of clock synchronization and divides the spacetime into equal-time pieces identified as three-dimensional spaces. One should mention that the integrability of \( D \) is always required in the case of 3 + 1 splitting which is necessary for the Hamiltonian formalism of General Relativity (see e.g. [16]).

More generally, to any tensor \( A^\alpha_{\beta\cdots} \) living in the spacetime one can assign its projected three-dimensional counterpart

\[
\tilde{A}^\alpha_{\beta\cdots} = h^\alpha_\mu h^\nu_\beta \cdots A^\mu_{\nu\cdots}. \tag{8}
\]

According to widely spread ideas (see e.g. [6, 10, 17]), only projected three-dimensional tensors are good candidates for measurable relativistic observables. Obviously, such quantities are relative, i.e. observer dependent. For example, for an anti-symmetric covariant two-tensor \( F_{\alpha\beta} = -F_{\beta\alpha} \) (two-form), which under the closedness condition \( (dF = 0) \) can be interpreted as an electromagnetic field, one gets

\[
F_{\alpha\beta} = H_{\alpha\beta} + u_\alpha E_\beta - u_\beta E_\alpha, \tag{9}
\]

where \( H_{\alpha\beta} = \tilde{F}_{\alpha\beta} = h^\mu_\alpha h^\nu_\beta F_{\mu\nu} \) and \( E_\alpha = u^\mu h^\nu_\alpha F_{\mu\nu} \) are measurable electric and magnetic components.

Before proceeding further, let us answer the question of when the one-dimensional foliation spanned by \( u \) is totally geodesic. This can be easily done by studying the auto-parallel (geodesic) equation

\[
u^\beta u_\alpha;\beta = 0, \tag{10}
\]

where \( u_\alpha;\beta = \nabla_\beta u_\alpha \) denotes the Levi-Civita covariant derivative of \( u \). Thus introducing the acceleration vector \( \dot{u}_\alpha = u^\beta u_\alpha;\beta \) one can conclude that the vanishing of \( \dot{u}^\alpha \) is equivalent to the geodesic equation \( \text{(10)} \). One should notice that \( \dot{u}_\alpha \) is, in fact, a three-vector, since \( u^\alpha \dot{u}_\alpha = 0 \).

In general, one can decompose the space components of the two-tensor \( u_{\alpha;\sigma} \) into irreducible parts with respect to the three-dimensional orthogonal group:

\[
\tilde{u}_{\alpha;\beta} = h^\sigma_\alpha u_{\alpha;\sigma} = \omega_{\alpha\beta} + \sigma_{\alpha\beta} + \frac{1}{3} \Theta h_{\alpha\beta}, \tag{11}
\]

where \( \omega_{\alpha\beta} \) denotes its antisymmetric part, \( \sigma_{\alpha\beta} \) is the traceless symmetric component and finally \( \Theta \) stands for the trace. This is a kinematical decomposition. Using \( \text{(8)} \) we shall obtain \( \text{(10)} \) and \( \text{(11)} \):

\[
u^\beta u_{\alpha;\beta} = \omega_{\alpha\beta} + \sigma_{\alpha\beta} + \frac{1}{3} \Theta h_{\alpha\beta}, \tag{11}
\]

There is a well-known interpretation of the observer in terms of relativistic hydrodynamics, treating it as a flow of material points constituting a (perfect) fluid (continuous medium), with the world lines

\[2\text{The dynamical equation is known as Raychaudhuri equation (see e.g. [17, 18]).}\]
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being particle trajectories: one line of the flow passes through every point \( x^\alpha \) of a certain spacelike (possibly bounded) region in spacetime. Accordingly, the tensor \( u_{\alpha\beta} \) determines the rate of change in the position of one point with respect to the other one in the material \[20\].

Keeping in mind this fluid analogy, each irreducible component of the projected tensor \( u_{\alpha\beta} \) admits a physical interpretation which is contained in self-explanatory and intuitive names (for more detailed explanations see e.g. \[6, 20, 21\]). In a more explicit form, one has to take into account the following three-dimensional quantities:

\[
\omega_{\alpha\beta} = u_{[\alpha;\beta]} + \dot{u}_{[\alpha} u_{\beta]} \quad \text{is the rotation tensor,} \tag{13}
\]

\[
\sigma_{\alpha\beta} = u_{(\alpha;\beta)} + \dot{u}_{(\alpha} u_{\beta)} - \frac{1}{3} \Theta h_{\alpha\beta} \quad \text{is the shear tensor,} \tag{14}
\]

\[
\Theta = u^\alpha_{;\alpha} \quad \text{is the expansion scalar,} \tag{15}
\]

\[
\dot{u}^\alpha = u^\beta_{;\beta} \quad \text{is the acceleration vector.} \tag{16}
\]

It is more convenient to use the scalars

\[
\dot{u} \equiv \left( \dot{u}_\alpha u^\alpha \right)^{\frac{1}{2}}, \quad \omega \equiv \left( \frac{1}{2} \omega_{\alpha\beta} \omega^{\alpha\beta} \right)^{\frac{1}{2}}, \quad \sigma \equiv \left( \frac{1}{2} \sigma_{\alpha\beta} \sigma^{\alpha\beta} \right)^{\frac{1}{2}}. \tag{17}
\]

These are non-negative and vanish at the same time as their corresponding tensors. An observer is rotation-free, shear-free or expansion-free when \( \omega = 0, \sigma = 0 \) or \( \Theta = 0 \) respectively. If all quantities vanish, then the observer is called rigid.

It is worth mentioning that the observer (four-velocity) field \( u^\alpha \) can be used to construct the energy momentum tensor of an ideal (incompressible) fluid,

\[
T_{\alpha\beta} = (p + \rho) u_\alpha u_\beta + p g_{\alpha\beta}, \tag{18}
\]

where the matter density \( \rho \) and the pressure \( p \) are internal fluid parameters determining its thermodynamical behavior. The same energy momentum tensor treated on the right-hand side of Einstein’s equations as the source of the gravitational field influences the metric. This suggests possible relationships between metric and fluid observer, which are an interesting subject for future research (see e.g. \[18, 22\]).

Now we are ready to classify all almost-product structures related to relativistic observers in gravitational spacetimes. As we have already mentioned, the tensor \( h_\beta^\alpha \) projects on a three-dimensional subspace while \( -u^\alpha u_\beta = \delta_\beta^\alpha - h_\beta^\alpha \) projects on the one-dimensional complementary distribution spanned by \( u^\alpha \). It turns out that the difference:

\[
P_\beta^\alpha = h_\beta^\alpha - (-u^\alpha u_\beta) = \delta_\beta^\alpha + 2u^\alpha u_\beta. \tag{19}
\]
represents an almost-product structure compatible with the metric $g$. Now the almost-product structure (19) can be used to encode the observer $u^\alpha$. Since the issue of one-dimensional distributions have already been solved, we should concentrate on the three-dimensional one. One has 4 conditions to be imposed on $P$ (see Definition 1 and Theorem 2). The umbilical case $D_3$, after some manipulations, produces

$$u_{(\alpha;\beta)} + \dot{u}_{(\alpha} u_{\beta)} = \frac{1}{3} \sum_{i=1}^{3} \epsilon_i^\beta u_{\alpha;\beta} e_i^\alpha. \quad (20)$$

Here $\{e_i\}_{i=1}^3$ denotes a local orthonormal frame of $D$. We notice that the sum of the right hand side is a three-dimensional trace of the tensor $u_{\alpha;\beta}$, thus $D_3$ is equivalent to the vanishing of the shear tensor. The condition $D_2$ (a minimal distribution) leads to

$$\sum_{i=1}^{3} \epsilon_i^\beta u_{\alpha;\beta} e_i^\alpha = 0, \quad (21)$$

which is equivalent to the vanishing a scalar of expansion. For a geodesic distribution ($D_1$), one obtains

$$u_{(\alpha;\beta)} + \dot{u}_{(\alpha} u_{\beta)} = 0, \quad (22)$$

which is equivalent to the vanishing of both characteristics: shear and expansion. In the free falling case ($\dot{u} = 0$) the condition (22) denotes that the normalized timelike vector $u^\alpha$ is a Killing vector for the metric $g_{\mu\nu}$. Similarly, one can show that the integrability condition ($F$) reduces to

$$u_{[\alpha;\beta]} + \dot{u}_{[\alpha} u_{\beta]} = 0, \quad (23)$$

which means vanishing of a rotation. The final results are presented in the Table 1 and Table 2. The first one concerns accelerated $\dot{u} \neq 0$ observers with all possibilities for the three-dimensional distribution taken into account. Similarly, the second table concerns free-falling observers ($\dot{u} = 0$). The bracket $(1,3)$ in the first column indicates that the first symbol is for the one-dimensional distribution and the second for the three-dimensional one. The tables also contain the physical interpretations for each class of almost-product Lorentzian manifolds.

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3It is easy to see that $P$ satisfies the conditions $P^2 = I$, as well as (2).
### Table 1: Accelerated observers

| Class \((1,3)\) | Physical meaning | 3-distribution |
|-----------------|------------------|----------------|
| \((F,-)\)       | \(u_{\alpha\beta} = \sigma_{\alpha\beta} + \omega_{\alpha\beta} + \frac{1}{3}\Theta h_{\alpha\beta} - \dot{u}_\alpha u_\beta\) | non-integrable distribution |
| \((F,D_2)\)     | \(\dot{\Theta} = 0 \Rightarrow u_{\alpha\beta} = \sigma_{\alpha\beta} + \omega_{\alpha\beta} - \dot{u}_\alpha u_\beta\) | expansion-free minimal |
| \((F,D_3)\)     | \(\sigma = 0 \Rightarrow u_{\alpha\beta} = \omega_{\alpha\beta} + \frac{1}{3}\Theta h_{\alpha\beta} - \dot{u}_\alpha u_\beta\) | shear-free umbilical |
| \((F,F)\)       | \(\omega = 0 \Rightarrow u_{\alpha\beta} = \sigma_{\alpha\beta} + \frac{1}{3}\Theta h_{\alpha\beta} - \dot{u}_\alpha u_\beta\) | rotation-free foliation |
| \((F,F_2)\)     | \(\omega = \dot{\Theta} = 0 \Rightarrow u_{\alpha\beta} = \sigma_{\alpha\beta} - \dot{u}_\alpha u_\beta\) | rotation-free & expansion-free minimal |
| \((F,F_3)\)     | \(\omega = \sigma = 0 \Rightarrow u_{\alpha\beta} = \frac{1}{3}\Theta h_{\alpha\beta} - \dot{u}_\alpha u_\beta\) | rotation-free shear-free totally umbilical |
| \((F,F_1)\)     | \(u_{\alpha\beta} = -\dot{u}_\alpha u_\beta\) | rigid totally geodesic |

### Table 2: Free-falling observers

| Class \((1,3)\) | Geodesic (free falling) observers \(\dot{u} = 0\) | Physical meaning | 3-distribution |
|-----------------|---------------------------------|-------------------|----------------|
| \((F_1,-)\)    | \(u_{\alpha\beta} = u_{[\alpha;\beta]} + \sigma_{\alpha\beta} + \frac{1}{3}\Theta h_{\alpha\beta}\) | geodesic non-integrable distribution |
| \((F_1,D_2)\)  | \(\Theta = 0 \Rightarrow u_{\alpha\beta} = u_{[\alpha;\beta]} + \sigma_{\alpha\beta}\) | geodesic minimal |
| \((F_1,D_3)\)  | \(\sigma = 0 \Rightarrow u_{\alpha\beta} = u_{[\alpha;\beta]} + \frac{1}{3}\Theta h_{\alpha\beta}\) | geodesic shear-free umbilical |
| \((F_1,D_1)\)  | \(\sigma = \dot{\Theta} = 0 \Rightarrow u_{\alpha\beta} = u_{[\alpha;\beta]}\) | geodesic shear-free & geodesic |
| \((F_1,F)\)    | \(\omega = 0 \Rightarrow u_{\alpha\beta} = \sigma_{\alpha\beta} + \frac{1}{3}\Theta h_{\alpha\beta}\) | geodesic rotation-free foliation |
| \((F_1,F_2)\)  | \(\omega = \dot{\Theta} = 0 \Rightarrow u_{\alpha\beta} = \sigma_{\alpha\beta}\) | geodesic rotation-free & expansion-free minimal |
| \((F_1,F_3)\)  | \(\omega = \sigma = 0 \Rightarrow u_{\alpha\beta} = \frac{1}{3}\Theta h_{\alpha\beta}\) | geodesic rotation-free & shear-free totally umbilical |
| \((F_1,F_1)\)  | \(u_{\alpha\beta} = 0\) | geodesic rigid totally geodesic |
5 Illustrative examples

5.1 Minkowski spacetime

The most extreme case in Naveira’s classification is \((F_1, F_1)\) class, i.e. both distributions are totally geodesic foliations. In Minkowski spacetime the metric is flat in the Cartesian coordinate system \(\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)\), so one can replace the covariant derivatives with partial ones. Then the \((F_1, F_1)\) case becomes just \(u_{\alpha,\beta} = 0\). There exists a solution in the form of a constant vector. In fact, any constant timelike vector field can be changed by a linear transformation of coordinates (i.e. Lorentz transformation) into

\[
    u^\alpha = [1, 0, 0, 0].
\]  

(24)

Such a vector field is a canonical inertial observer in Minkowski spacetime. A less restrictive class is \((F, F_1)\), which implies that the observer should accelerate. An example of such an observer is Rindler’s:

\[
    u^\alpha = \left[ \frac{x}{\sqrt{x^2 - t^2}}, \frac{t}{\sqrt{x^2 - t^2}}, 0, 0 \right],
\]

(25)

for whom only a part of Minkowski space is available. The only non-vanishing characteristic is the acceleration

\[
    \dot{u} = (x^2 - t^2)^{-1/2},
\]

(26)

which is constant along each trajectory. Again, by introducing adapted (Rindler’s) coordinates one can simplify expressions.

Let us consider the rotating observer in the \((x, y)\) plane,

\[
    u^\alpha = \left[ \sqrt{2}, \frac{-y}{\sqrt{x^2 + y^2}}, \frac{x}{\sqrt{x^2 + y^2}}, 0 \right],
\]

(27)

belonging to the class \((F, D_2)\) with the following characteristics:

\[
    \Theta = 0,
\]

(28)

\[
    \dot{u} = (x^2 + y^2)^{-1/2},
\]

(29)

\[
    \omega = \frac{\sqrt{2}}{2} (x^2 + y^2)^{-1/2},
\]

(30)

\[
    \sigma = \frac{\sqrt{2}}{2} (x^2 + y^2)^{-1/2},
\]

(31)

constant along particles trajectories. The last example for Minkowski spacetime (in the spherical coordinates \(g_{\mu\nu} = \text{diag}(-1, 1, r^2, r^2 \sin^2 \theta)\)) is the observer

\[
    u^\alpha = \left[ \frac{r}{\sqrt{r^2 - t^2}}, \frac{t}{\sqrt{r^2 - t^2}}, 0, 0 \right].
\]

(32)
It turns out that observer (32) has the following characteristics:

\[ \dot{u} = (r^2 - t^2)^{-1/2}, \]
\[ \omega = 0, \]
\[ \Theta = \frac{2t}{r}(r^2 - t^2)^{-1/2}, \]
\[ \sigma = \frac{\sqrt{3}t}{3r}(r^2 - t^2)^{-1/2}, \]

and belongs to the class \((F, F)\).

### 5.2 Schwarzschild spacetime

There is no observer belonging to the class \((F_1, F_1)\) in Schwarzschild spacetime. It should satisfy the sixteen equations \(u_{\alpha;\beta} = 0\), which turn out to be inconsistent.

Let us consider the observer

\[ u^\alpha = \left[ (1 - \frac{2M}{r})^{-1/2}, 0, 0, 0 \right]. \] (37)

The only non-vanishing characteristic is the acceleration \(\dot{u} = \frac{M}{r^2}(1 - \frac{2M}{r})^{-1/2}\), which implies that observer (37) belongs to the class \((F, F_1)\).

The geodesic observer \([17]\) is of the form

\[ u^\alpha = \left[ (1 - \frac{3M}{r})^{-1/2}, 0, \sqrt{\frac{M}{r^2(r - 3M)}}, 0 \right]. \] (38)

It shows a singular expansion at the north and south poles:

\[ \Theta = \sqrt{\frac{M}{r^2(r - 3M)}} \cot \theta. \] (39)

The rotation and shear scalars are

\[ \omega = \frac{1}{4} \sqrt{\frac{M}{r^3}} \left( \frac{1 - 6M/r}{1 - 3M/r} \right), \] (40)
\[ \sigma = \sqrt{\frac{27M \sin^2 \theta (r - 2M)^2 + 16Mr(r - 3M) \cos^2 \theta}{48r^3(r - 3M)^2 \sin^2 \theta}}. \] (41)

It belongs to the \((F, -)\) class, so for this observer, as well as for \([27]\) of Minkowski spacetime, there is no three-dimensional orthogonal distribution providing a foliation of the spacetime manifold \((M, g)\) i.e. there are no three-dimensional equal-time subspaces relative to these observers.
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