Symmetry breaking differential operators, the source operator and Rodrigues formulæ

Jean-Louis Clerc

Abstract

A Rodrigues type formula for the symbols of symmetry breaking differential operators is obtained in three situations: for the Rankin-Cohen operators, for the Juhl operators and for the conformally covariant bi-differential operators.

2010 Mathematics Subject Classification: 43A85. Secondary 58J70
Key words: symmetry breaking differential operators, transvectants, Rankin-Cohen brackets, conformally covariant differential operators, Rodrigues formula

Introduction

Symmetry breaking differential operators (SBDO for short) are a classical notion in physics, and they have interested many authors in mathematics during the recent years. T. Kobayashi has designed a program to study the existence, the uniqueness and the construction of such operators and proposes the F-method as a tool to solve such questions (see [3]). We will use a restricted version of the notion of SBDO, adapted to the present article and we will be concerned mostly by the constructive part of the program, even more precisely in an effort to give explicit expressions for these operators. Here is a (non exhaustive) list of recent papers on the subject [1, 2, 3, 4, 5, 7, 9, 10, 13].

Let $M$ be a manifold, $G$ (called the big group) a Lie group action on $M$, $N$ a submanifold of $M$ and $H$ (called the small group) a closed Lie subgroup of $G$ which preserves $N$. Let $\pi$ be a smooth representation of $G$ on $C^\infty(M)$ and $\rho$ a smooth representation of $H$ on $C^\infty(N)$. Let $D$ be a differential operator from $C^\infty(M)$ into $C^\infty(N)$. Then $D$ is said to be a symmetry breaking differential operator (SBDO for short) if $D$ intertwines $\pi|_H$ and $\rho$, i.e.

$$\forall h \in H, \quad D \circ \pi(h) = \rho(h) \circ D.$$
In order to do explicit computations (typically when using a local chart), it is useful to have a slightly weaker version of the previous definition, adapted to the context of a rational action of $G$ on a vector space $E$, $H$ being a subgroup of $G$ preserving a vector subspace $F$ of $E$. We omit details and present the main results of the paper, which concern three situations.

1. **The generalized transvectants/Rankin-Cohen brackets**

Let $E = \mathbb{R} \times \mathbb{R}$ and $F = \text{diag}(E) \simeq \mathbb{R}$. Let $G_1 = SL(2, \mathbb{R})$, $G = G_1 \times G_1$ and $H = \text{diag}(G) \simeq G_1$. $G_1$ acts rationally on $\mathbb{R}$ by

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}), \quad x \in \mathbb{R} \quad g(x) = \frac{ax + b}{cx + d}.$$ 

For $\lambda \in \mathbb{C}$ and $\epsilon = \pm$, let $\pi_{\lambda, \epsilon}$ be the principal series representation of $G_1$, realized in the noncompact picture, acting on $C^\infty(\mathbb{R})$. Then set

$$\pi = \pi_{\lambda, \epsilon} \otimes \pi_{\mu, \eta}, \quad \rho = \pi_{\lambda + \mu + 2k, \epsilon \eta},$$

where $(\lambda, \epsilon), (\mu, \eta) \in \mathbb{C} \times \{\pm\}$ and $k \in \mathbb{N}$. Let $\text{res} : C^\infty(E) \longrightarrow C^\infty(F)$ be the restriction map to the diagonal.

**Theorem A1.** Let $\alpha, \beta \in \mathbb{C}$ and for $k \in \mathbb{N}$, let $Q^{\alpha, \beta}_k$ the polynomial of two variables defined by

$$Q^{\alpha, \beta}_k(\xi, \eta) = \xi^{-\alpha} \eta^{-\beta} \left( \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \right)^k \xi^{\alpha + k} \eta^{\beta + k}. \quad (1)$$

Then the differential operator

$$RC^{(k)}_{\lambda, \mu} = \text{res} \circ Q^{\lambda - 1, \mu - 1}_k \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) \quad C^\infty(\mathbb{R} \times \mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$$

satisfies for $g \in SL(2, \mathbb{R})$

$$RC^{(k)}_{\lambda, \mu} \circ \left( \pi_{\lambda, \epsilon}(g) \otimes \pi_{\mu, \eta}(g) \right) = \pi_{\lambda + \mu + 2k, \epsilon \eta}(g) \circ RC^{(k)}_{\lambda, \mu}.$$

2. **Juhl operators**

Let $E = \mathbb{R}^n$ and let $F$ be the hyperplane defined by the equation $x_n = 0$. Let $G = SO_0(1, n + 1)$ acting conformally on $E$ and let $H \simeq SO_0(1, n)$ be the subgroup of $G$ preserving $F$. For $\lambda \in \mathbb{C}$ let $\pi_{\lambda}$ be the scalar principal series representation of $G$ in the noncompact picture acting on $C^\infty(E)$. Similarly, for $\mu \in \mathbb{C}$, let $\pi'_{\mu}$ be the scalar
principal series representation of $H$ in the noncompact picture acting on $C^\infty(F)$. For $\lambda \in \mathbb{C}$ and $k \in \mathbb{N}$, set

$$\pi = \pi_\lambda, \quad \rho = \pi_{\lambda+k}.$$  

Let $\text{res} : C^\infty(E) \to C^\infty(F)$ be the restriction map to the hyperplane $F$.

**Theorem A2.** Let $\gamma \in \mathbb{C}$ and for $k \in \mathbb{N}$, let $B^\gamma_k$ be the polynomial on $\mathbb{R}^n$ defined by

$$B^\gamma_k(\xi) = |\xi|^{-2\gamma} \left( \frac{\partial}{\partial \xi_n} \right)^k |\xi|^{2(\gamma+k)}.$$  

(2)

Let $\lambda \in \mathbb{C}$. Then the differential operator defined by

$$J^{(k)}_\lambda = \text{res} \circ B^{\lambda-2\frac{n}{2}}_k \left( \frac{\partial}{\partial x} \right) \quad C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^{n-1})$$

satisfies for $g \in SO_0(1,n)$

$$J^{(k)}_\lambda \circ \pi_\lambda(g) = \pi'_{\lambda+k}(g) \circ J^{(k)}_\lambda.$$  

3. The conformally covariant bi-differential operators

Let $E = \mathbb{R}^n \times \mathbb{R}^n$ and $F = \text{diag}(E) \simeq \mathbb{R}^n$. Let $G_1 = SO_0(1,n+1)$ acting conformally on $\mathbb{R}^n$. Let $G = G_1 \times G_1$ acting on $E$ and $H = \text{diag}(G) \simeq G_1$ preserving $F$. For $\lambda \in \mathbb{C}$, let $\pi_\lambda$ be the scalar principal series representation of $G_1$ in the noncompact picture acting on $C^\infty(\mathbb{R}^n)$. For $\lambda, \mu \in \mathbb{C}$ and $k \in \mathbb{N}$, let

$$\pi = \pi_\lambda \otimes \pi_\mu, \quad \rho = \pi_{\lambda+\mu+2k}.$$

Let $q$ the quadratic form on $\mathbb{R}^n$ given by $q(\xi) = |\xi|^2 = \xi_1^2 + \xi_2^2 + \cdots + \xi_n^2$.

Finally, let $\text{res} : C^\infty(E) \to C^\infty(F)$ be the restriction map to the diagonal.

**Theorem A3.** Let $\alpha, \beta \in \mathbb{C}$ and for $k \in \mathbb{N}$, let $R^{\alpha,\beta}_k$ be the polynomial on $\mathbb{R}^n \times \mathbb{R}^n$ defined by

$$R^{\alpha,\beta}_k(\xi, \eta) = q(\xi)^{-\alpha} q(\eta)^{-\beta} q \left( \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \right)^k q(\xi)^{(\alpha+k)} q(\eta)^{(\beta+k)}.$$  

(3)

Let $\lambda, \mu \in \mathbb{C}$. Then the differential operator defined by

$$D^{(k)}_{\lambda,\mu} = \text{res} \circ R^{\lambda-\frac{n}{2},\mu-\frac{n}{2}}_k \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) \quad C^\infty(\mathbb{R}^n \times \mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$$

satisfies for $g \in SO_0(1,n+1)$

$$D^{(k)}_{\lambda,\mu} \circ (\pi_\lambda(g) \otimes \pi_\mu(g)) = \pi_{\lambda+\mu+2k}(g) \circ D^{(k)}_{\lambda,\mu}.$$
The scheme of the proof goes in each example through five steps, which can be presented as follows

CONSTRUCTION OF THE SOURCE OPERATOR

CONSTRUCTION OF SBDO FROM THE SOURCE OPERATOR

RECURSION RELATION FOR THE SYMBOLS OF THE SBDO

RODRIGUES FORMULA FOR THE SYMBOLS OF THE SBDO

LINK WITH FAMILIES OF ORTHOGONAL POLYNOMIALS

The first step was inspired by the $\Omega$ process, used long ago to construct the transvectants (see [11] for more on this subject and the rôle of the $\Omega$-process in classical invariant theory). Because of lack of reference, the $\Omega$ process, somewhat generalized, is presented for the benefit of the reader with full details in Section 2, and is used later to treat the first example. The construction of the source operator for the second and third example is more sophisticated and only the final result is recalled. For Juhl operators, the source operator was constructed in [4]. For the conformally covariant bi-differential operators, this is done in [1], done again and more systematically in [2], Section 10. Let us mention that the source operator method has been used in many other situations. In [2] it is used for the conformal group of a real Jordan algebra. The method also works for vector valued situations: see [5], [3] for the case of differential forms and/or spinors. In any case, the source operator is a differential operator with polynomial coefficients on $E$, which satisfies a covariance property w.r.t. the little group $H$ (but not for the big group $G$).

The second step is easy and is the same for the three examples. For each integer $k$, by composing appropriately $k$ source operators followed by composition with the restriction map from $E$ to $F$, a family of symmetry breaking differential operators depending on $k$ is constructed.

In the third step, the construction process of the SBDO is analyzed and, using some calculation within the Weyl algebra of $E$, translated into a recursive relation on the symbols of the SBDO.

In the fourth step, a solution to this recursive relation is obtained through a Rodrigues type formula. This leads to the three theorems quoted in the introduction.

The fifth step is an effort to relate the Rodrigues type formula with known Rodrigues formulæ for orthogonal polynomials. In the first example, this links the symbols of the Rankin-Cohen operators to Jacobi polynomials, and in the second example, it links the symbols of the Juhl operators with the Gegenbauer polynomials. This had already
been observed for the first example (see [11]) and for the second example (see [7, 9]). For the third example, the Rodrigues type formula (cf. (3)) is new and no connection to a family of orthogonal polynomials or special functions is known (see more comments at the end of Section 4).

In each example, the Rodrigues type formula for the symbols of the symmetry breaking differential operators is spectacularly simple, and this suggests possible extension to other situations. The construction of a source operator and associated SBDO in the context of the conformal group of a real simple Jordan algebra was done in [2], but did not lead to explicit formulæ. There is however a clear candidate for a Rodrigues type formula for the symbols of the SBDO, in the spirit of (3), based on the determinant of the Jordan algebra in place of the quadratic form. We hope to come back to these questions in the next future.

1 The Weyl algebra

This section is meant to introduce notation for later computations in the Weyl algebra. All results are given without proofs, as they are elementary.

Let $E$ be a real vector space, of dimension $n$. After a choice of a basis, the space $E$ will be identified with $\mathbb{R}^n$. A typical element of $E$ is $x = (x_1, x_2, \ldots, x_n)$. The dual space $E^*$ of $E$ is also identified with $\mathbb{R}^n$, but to distinguish $E$ from $E^*$, elements of $E^*$ will be denoted by greek letters, i.e. $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$.

Let $P(E) \cong P(\mathbb{R}^n)$ be the algebra of polynomial complex valued functions on $E$. To a $n$-tuple $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ associate

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}.$$

The polynomials $(x^\alpha)_{\alpha \in \mathbb{N}_n}$ form a basis of $P(E)$. To each $p \in P(E)$ is associated the multiplication operator by $p$.

Dually the algebra of constant coefficients differential operators on $E$ is related to the algebra $P(E^*) \cong P(\mathbb{R}^n)$ by the symbol map. To a $n$-tuple $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ associate the differential operator $(\frac{\partial}{\partial x})^\alpha$ given by

$$\left( \frac{\partial}{\partial x} \right)^\alpha = \left( \frac{\partial}{\partial x_1} \right)^{\alpha_1} \left( \frac{\partial}{\partial x_2} \right)^{\alpha_2} \ldots \left( \frac{\partial}{\partial x_n} \right)^{\alpha_n}.$$

To any $d \in P(E^*)$ given by $d(\xi) = \sum_{\alpha} a_{\alpha} \xi^\alpha$ associate the differential operator

$$D = d \left( \frac{\partial}{\partial x} \right) = \sum_{\alpha} a_{\alpha} \left( \frac{\partial}{\partial x} \right)^\alpha.$$
Conversely, if $D$ is a constant coefficients differential operator on $E$, there exists a unique $d \in \mathcal{P}(E^*)$, called the *symbol of* $D$, such that $D = d \left( \frac{\partial}{\partial x} \right)$.

The *Weyl algebra* $\mathcal{W} = \mathcal{W}(E \times E^*)$ is the algebra of differential operators on $E$ with polynomial coefficients. The polynomial algebra $\mathcal{P}(E)$ is viewed as a subalgebra of $\mathcal{W}(E \times E^*)$ (multiplications by polynomial functions), and similarly $\mathcal{P}(E^*)$ is viewed as a subalgebra of $\mathcal{W}(E \times E^*)$ (constant coefficients differential operators). The Weyl algebra is not commutative. The multiplication of a polynomial function $p$ and a constant coefficients differential operator $D = d \left( \frac{\partial}{\partial x} \right)$ is denoted following the *normal rule* : if differentiation comes first and then multiplication, we simply use $p D$ or $p(x) d \left( \frac{\partial}{\partial x} \right)$, whereas if multiplication comes first and then differentiation, we use the notation $D \circ p$ or $d \left( \frac{\partial}{\partial x} \right) \circ p(x)$. There is a linear homomorphism between $\mathcal{P}(E \times E^*)$ and $\mathcal{W}(E \times E^*)$, given by

$$p(x, \xi) \mapsto p \left( x, \frac{\partial}{\partial x} \right)$$

obtained by extending linearly the map

$$x^\alpha \xi^\beta \mapsto x^\alpha \left( \frac{\partial}{\partial x} \right)^\beta .$$

This homomorphism is easily seen to be a (linear) isomorphism. Given an element $D$ of $\mathcal{W}(E \times E^*)$, there exists a unique polynomial $d \in \mathcal{P}(E \times E^*)$ such that $D = d \left( x, \frac{\partial}{\partial x} \right)$ and $d$ is called the *symbol* of $D$.  

There is an important formula giving the symbol of the composition of two elements of the Weyl algebra. Let $D = d \left( x, \frac{\partial}{\partial x} \right)$ and $E = e \left( x, \frac{\partial}{\partial x} \right)$ and consider $F = D \circ E$. Let $f$ be its symbol. Then $f = d \# e$, where

$$(d \# e)(x, \xi) = \sum_\alpha \frac{1}{\alpha!} \left( \frac{\partial}{\partial \xi} \right)^\alpha d(x, \xi) \left( \frac{\partial}{\partial x} \right)^\alpha e(x, \xi) , \quad (4)$$

with the usual convention that for $\alpha = (\alpha_1, \alpha_2, \alpha_n)$ an $n$-tuple,

$$\alpha! = \alpha_1! \alpha_2! \ldots \alpha_n! .$$

\[1\text{The usual convention used by analysts is different, essentially } \frac{1}{!} \frac{\partial}{\partial \xi} \text{ is used instead of } \frac{\partial}{\partial \xi} \text{ because of the relation with the Fourier transform.}\]
Let $F$ be a vector subspace of $E$ of dimension $p$. We may always (and hence do) assume that $F$ is the subspace

$$F = \{ x \in E, x_{p+1} = x_{p+2} = \cdots = x_{n} = 0 \} ,$$

and use the notation

$$x = (x_1, x_2, \ldots, x_p, x_{p+1}, \ldots, x_n) = (x', x'')$$

where $x = (x_1, x_2, \ldots, x_p) \in \mathbb{R}^p$ and $x'' = (x_{p+1}, x_{p+2}, \ldots, x_n) \in \mathbb{R}^{n-p}$. Then $F$ is the subspace defined by the equation $x'' = 0$ and can be identified with $\mathbb{R}^p$ through the map $x' \mapsto (x', 0)$.

Given $p$ in $\mathcal{P}(E)$, its restriction $\text{res}(p)$ to $F$ is given by

$$\text{(res}(p)) (x') = p(x', 0) .$$

The map $\text{res}$ is a surjective homomorphism form $\mathcal{P}(E)$ onto $\mathcal{P}(F)$, and its kernel consists in polynomials $p \in \mathcal{P}(E)$ which can be written as

$$p(x) = \sum_{j=p+1}^{n} x_j p_j(x)$$

for some $p_j \in \mathcal{P}(E)$. As for notation, let

$$\ker(\text{res}) = \sum_{j=p+1}^{n} x_j \mathcal{P}(E) = x'' \mathcal{P}(E) .$$

Extend this notation by letting

$$x'' \mathcal{P}(E \times E^*) = \sum_{j=p+1}^{n} x_j \mathcal{P}(E \times E^*) , \quad x'' \mathcal{W}(E \times E^*) = \sum_{j=p+1}^{n} x_j \mathcal{W}(E \times E^*) .$$

Generalizing the classical notion, an operator from $C^\infty(E)$ into $C^\infty(F)$ is called a differential operator, if

$$Df(x') = \sum_{\alpha \in \mathbb{N}^n} a_\alpha(x') \left( \left( \frac{\partial}{\partial x} \right)^\alpha f \right) (x', 0) .$$

where for any $n$-tuple $\alpha$, $a_\alpha$ is a $C^\infty$ function on $E$, the family being identically 0 except for finitely many $\alpha$’s. Notice that the coefficients are uniquely determined by the operator $D$. In the present context, further assume that the coefficients $a_\alpha$ are polynomials functions on $F$ and denote by $\mathcal{W}(F \times E^*)$ the corresponding family of operators. The polynomial function $d \in \mathcal{P}(F \times E^*)$ given by

$$d(x', \xi) = \sum_{\alpha} a_\alpha(x') \xi^\alpha$$
is called the *symbol* of the operator $D \in \mathcal{W}(F \times E^*)$.

Let $D$ be an element of $\mathcal{W}(E \times E^*)$. Then $\text{res}(D) := \text{res} \circ D$ belongs to $\mathcal{W}(F \times E^*)$. Moreover, if $D$ has symbol $d$, then the symbol $\text{res}(d)$ of $\text{res}(D)$ is given by

$$\text{res}(d)(x', \xi) = d((x', 0), \xi).$$

(5)

**Proposition 1.1.** Let $D$ an element of $\mathcal{W}(E \times E^*)$, and let $d$ be its symbol. Then the following are equivalent:

i) $\text{res}(D) = 0$

ii) $\text{res}(d) = 0$

iii) $D$ belongs to $\mathcal{W}(E \times E^*)$

iv) $d$ belongs to $\mathcal{P}(E \times E^*)$.

The proof is elementary and left to the reader.

### 2 The Rankin-Cohen operators

#### 2.1 The $\Omega$-process and the Rankin-Cohen operators

The group $G = SL(2, \mathbb{R})$ acts on $\mathbb{R}^2 = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, x_1, x_2 \in \mathbb{R} \right\}$. Consider the diagonal action of $G$ on $\mathbb{R}^2 \times \mathbb{R}^2 = \left\{ (x, y), x, y \in \mathbb{R}^2 \right\}$, given by

$$g(x, y) = (gx, gy), \quad \text{for } g \in G.$$

This action coincides with the action of $G$ by left multiplication on the space $\text{Mat}(2, \mathbb{R}) = \left\{ \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}, x_1, x_2, y_1, y_2 \in \mathbb{R} \right\}$. The group $G$ acts on the space $C^\infty(\mathbb{R}^2)$ by

$$F \mapsto F \circ g^{-1},$$

and by diagonal extension on the space $C^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$. Introduce the Cayley operator

$$\Omega = \frac{\partial^2}{\partial x_1 \partial y_2} - \frac{\partial^2}{\partial x_2 \partial y_1}.$$

Its symbol is given by

$$\sigma_\Omega \left( \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \right) = \det \left( \begin{pmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \end{pmatrix} \right),$$

which is invariant by the (contragredient) action of $G$ and hence the operator $\Omega$ commutes with the action of $G$ on $C^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$.

---

2Notation departs from the general notation presented in the introduction. The big group is $SL(2\mathbb{R}) \times SL(2, \mathbb{R})$ and the small group is its diagonal, identified with $SL(2, \mathbb{R})$. 8
For $t \in \mathbb{R}^\times$, let $\delta_t$ be the dilation $\delta_t : x \mapsto tx$. The dilations form a group, isomorphic to $\mathbb{R}^\times$, which commutes with the action of $G$ on $\mathbb{R}^2$.

For $(\lambda, \epsilon) \in \mathbb{C} \times \{\pm\}$, $t \in \mathbb{R}^\times$, let

$$t^{\lambda, \epsilon} = \begin{cases} |t|^\lambda & \epsilon = + \\
\text{sgn}(t)|t|^\lambda & \epsilon = - \end{cases}$$

For convenience, let $\mathbb{R}^{2^\times} = \{x \in \mathbb{R}^2, x \neq (0,0)\}$. Let

$$\mathcal{H}_{\lambda, \epsilon} = \left\{ F \in C^\infty(\mathbb{R}^{2^\times}), \quad F(tx) = t^{-\lambda, \epsilon} F(x) \right\},$$

equipped with its standard topology. Clearly the space $\mathcal{H}_{\lambda, \epsilon}$ is invariant by the action of $G$ and in fact defines a smooth representation of $G$ on $\mathcal{H}_{\lambda, \epsilon}$, denoted by $\pi_{\lambda, \epsilon}$. The representations $(\pi_{\lambda, \epsilon})_{\lambda \in \mathbb{C}, \epsilon \in \{\pm\}}$ form the (nonunitary) principal series of $G$.

Another realization of these representations is obtained in the so-called noncompact picture. To any $F \in \mathcal{H}_{\lambda, \epsilon}$, associate the function $f$ on $\mathbb{R}$ given by

$$f(x) = F(x, 1).$$

The function $F$ can be recovered by the formula

$$F(x_1, x_2) = x_2^{-\lambda, \epsilon} f \left( \frac{x_1}{x_2} \right). \quad (6)$$

The function $f$ cannot be an arbitrary smooth function on $\mathbb{R}$, as the function $F$ defined by (6) is not defined when $x_2 = 0$ and it might be impossible to continuously extend $F$ to yield a smooth function on $\mathbb{R}^2$.

In this model, the representation $\pi_{\lambda, \epsilon}$ is given by

$$\pi_{\lambda, \epsilon}(g)f(x) = (cx + d)^{-\lambda, \epsilon} f \left( \frac{ax + b}{cx + d} \right), \quad (7)$$

for $g \in G$, $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

As the action of $G$ on $\mathbb{R}$ is rational (not everywhere defined), (7) does not make sense for an arbitrary function $f \in C^\infty(\mathbb{R})$. Let us observe however that if $f$ has (say) compact support, then for $g$ in a sufficiently small neighborhood of the identity in $G$ (depending on the support of $f$), (7) defines a $C^\infty$ function on $\mathbb{R}$ with compact support. This shows that $\pi_{\lambda, \epsilon}$ is well-defined as a local representation of $G$ on $C^\infty(\mathbb{R})$ and in particular this allows to define the associated derived representation of the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$. For the purpose of the present paper, this will cause no serious difficulty, it will be enough to consider these substitutes (either the local representation or the derived
representation) and we will speak somewhat loosely of the noncompact realization of $\pi_{\lambda, \epsilon}$.

For $(\lambda, \epsilon), (\mu, \eta) \in \mathbb{C} \times \{\pm\}$ let

$$\mathcal{H}_{(\lambda, \epsilon), (\mu, \eta)} = \left\{ F \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^2), \quad F(sx, ty) = s^{-\lambda \epsilon - \mu \eta} F(x, y) \right\}.$$ 

Equipped with its standard topology, it can be viewed as a realization of the standard completion of the tensor product $\mathcal{H}_{\lambda, \epsilon} \otimes \mathcal{H}_{\mu, \eta}$. The diagonal action of $G$ on $\mathbb{R}^2 \times \mathbb{R}^2$ leads to a smooth representation of $G$ on $\mathcal{H}_{(\lambda, \epsilon), (\mu, \eta)}$, which is a realization of the tensor product representation $\pi_{\lambda, \epsilon} \otimes \pi_{\mu, \eta}$.

The corresponding operator denoted by $\Omega_{(\lambda, \epsilon), (\mu, \eta)}$ intertwines the representations $\pi_{\lambda, \epsilon} \otimes \pi_{\mu, \eta}$ and we will speak somewhat loosely of the noncompact realization of $\Omega_{(\lambda, \epsilon), (\mu, \eta)}$. Denote by $\Omega_{(\lambda, \epsilon), (\mu, \eta)}$ the operator obtained by restriction. The operator $\Omega_{(\lambda, \epsilon), (\mu, \eta)}$ intertwines the representations $\pi_{\lambda, \epsilon} \otimes \pi_{\mu, \eta}$ and $\pi_{\lambda + 1, -\epsilon} \otimes \pi_{\mu + 1, -\eta}$. More generally, for $k \in \mathbb{N}$, $\Omega^k$ maps $\mathcal{H}_{(\lambda, \epsilon), (\mu, \eta)}$ into $\mathcal{H}_{(\lambda + k, (-1)^k \epsilon), (\mu + k, (-1)^k \eta)}$.

The corresponding operator denoted by $\Omega_{(\lambda, \epsilon), (\mu, \eta)}^{(k)}$ intertwines the representations $\pi_{\lambda, \epsilon} \otimes \pi_{\mu, \eta}$ and $\pi_{\lambda + k, (-1)^k \epsilon} \otimes \pi_{\mu + k, (-1)^k \eta}$.

There is a corresponding noncompact realization of $\pi_{\lambda, \epsilon} \otimes \pi_{\mu, \eta}$. An elementary computation, using (6) shows that operator $\Omega_{(\lambda, \epsilon), (\mu, \eta)}^{(k)}$ is now realized as a differential operator $E_{\lambda, \mu}$ on $\mathbb{R}^2$ given by

$$E_{\lambda, \mu} = (x - y) \frac{\partial^2}{\partial x \partial y} - \mu \frac{\partial}{\partial x} + \lambda \frac{\partial}{\partial y} .$$

Notice that the expression does not depend on $\epsilon$ nor on $\eta$ and this remark is reflected in the notation.

The operator $\Omega^k$ also has a realization in the noncompact model as a differential operator $E_{\lambda, \mu}^{(k)}$. Observe that

$$E_{\lambda, \mu}^{(k)} = E_{\lambda + k - 1, \mu + k - 1} \circ \cdots \circ E_{\lambda, \mu} .$$

The differential operator $E_{\lambda, \mu}^{(k)}$ is of degree $2k$ and has polynomial coefficients, depending polynomially on the parameters $(\lambda, \mu)$.

The operator $E_{\lambda, \mu}$ is called the source operator. The choice of this name is motivated by the fact that this operator can be used to construct a family of bi-differential operators (acting from $C^\infty(\mathbb{R} \times \mathbb{R})$ into $C^\infty(\mathbb{R})$), depending on $\lambda, \mu$ and $k \in \mathbb{N}$, and which are covariant under the action of $G$. This family includes as special cases both the transvectants and the Rankin-Cohen brackets as will be explained later.

Let $\text{res}$ be the restriction from $\mathbb{R}^2 \times \mathbb{R}^2$ to the diagonal $\{(x, x), x \in \mathbb{R}^2\} \simeq \mathbb{R}^2$ defined by

$$\text{res} : C^\infty(\mathbb{R}^2 \times \mathbb{R}^2) \longrightarrow C^\infty(\mathbb{R}^2), \quad \text{res}(f)(x) = f(x, x) .$$

It is easily verified that $\text{res}$ maps $\mathcal{H}_{(\lambda, \epsilon), (\mu, \eta)}$ into $\mathcal{H}_{\lambda + \mu, \epsilon \eta}$. Consequently, the composition map

$$\text{res} \circ \Omega_{(\lambda, \epsilon), (\mu, \eta)}^k$$
yields a differential operator from $H_{(\lambda, \epsilon)}$ into $H_{\lambda+\mu+2k, \epsilon\eta}$, intertwining the representations $\pi_{\lambda, \epsilon} \otimes \pi_{\mu, \eta}$ and $\pi_{\lambda+\mu+2k, \epsilon\eta}$.

The same constructions can be made in the noncompact picture. Using the same notation as before, let $\text{res} : C^\infty(\mathbb{R} \times \mathbb{R}) \rightarrow C^\infty(\mathbb{R})$ be the restriction map from $\mathbb{R} \times \mathbb{R}$ to the diagonal $\{(x, x) : x \in \mathbb{R}\}$ given by

$$C^\infty(\mathbb{R} \times \mathbb{R}) \ni f \mapsto \text{res}(f)(x) = f(x, x).$$

By composition, the operator

$$RC_{\lambda, \mu}^{(k)} = \text{res} \circ E_{\lambda, \mu}^{(k)} = \text{res} \circ E_{\lambda+k-1, \mu+k-1} \circ \cdots \circ E_{\lambda, \mu}$$

is a bi-differential operator mapping $C^\infty(\mathbb{R} \times \mathbb{R})$ into $C^\infty(\mathbb{R})$ and satisfies the covariance property, for any $g \in G$

$$RC_{\lambda, \mu}^{(k)} \circ (\pi_{\lambda, \epsilon}(g) \otimes \pi_{\mu, \eta}(g)) = \pi_{\lambda+\mu+2k, \epsilon\eta}(g) \circ RC_{\lambda, \mu}^{(k)}. \quad (9)$$

The operators $RC_{\lambda, \mu}^{(k)}$ are called the Rankin-Cohen operators. The first values are easily computed from (8)

$$R_{\lambda, \mu}^{(0)} = \text{Id}, \quad R_{\lambda, \mu}^{(1)} = -\mu \frac{\partial}{\partial x} + \lambda \frac{\partial}{\partial y},$$

$$R_{\lambda, \mu}^{(2)} = \mu(\mu + 1) \frac{\partial^2}{\partial x^2} - 2(\mu + 1)(\lambda + 1) \frac{\partial^2}{\partial x \partial y} + \lambda(\lambda + 1) \frac{\partial^2}{\partial y^2}.$$ 

As announced earlier, the family contains two special subfamilies, discovered much before the general case, and corresponding to cases where the representations $\pi_{\lambda, \epsilon}$ and $\pi_{\mu, \eta}$ are reducible.

### 2.2 The transvectants

Let $\lambda = -l$ for some $l \in \mathbb{N}$, and let

$$\epsilon_l = +1 \text{ if } l \text{ is even}, \quad \epsilon_l = -1 \text{ if } l \text{ is odd}.$$ 

Then the space $\mathcal{H}_{l, \epsilon}$ contains the space $\mathcal{P}_l$ of polynomials on $\mathbb{R}^2$ which are homogeneous of degree $l$. This space is invariant under the representation $\pi_{l, \epsilon}$ and hence induces by restriction a representation $\rho_l$ of $G$ on $\mathcal{P}_l$. In the noncompact model, $\rho_l$ corresponds to the classical action of $\text{SL}(2, \mathbb{R})$ on the space $\tilde{\mathcal{P}}_l$ of polynomials on $\mathbb{R}$ of degree less that or equal to $l$, given by

$$(\rho_l(g)p)(x) = (cx + d)^l p \left( \frac{ax + b}{cx + d} \right).$$

for $g \in \text{SL}(2, \mathbb{R}), g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. 

11
Let also $\mu = -m$ for some $m$ in $\mathbb{N}$. The space $\mathcal{P}_m$ is an invariant subspace of $\mathcal{H}_{m,\epsilon_m}$ and $\mathcal{P}_l \otimes \mathcal{P}_m$ is an invariant subspace of $\mathcal{H}_{(l,\epsilon_l), (m,\epsilon_m)}$.

For $k \in \mathbb{N}$ such that $l + m \geq 2k$, the space $\mathcal{H}_{l-m+2k, \epsilon_{l-m}}$ contains the invariant subspace $\mathcal{P}_{m+l-2k}$ and the operator $\Omega^k$ maps $\mathcal{P}_l \otimes \mathcal{P}_m$ into $\mathcal{P}_{l+m-2k}$. In the non compact picture the operator $\Omega^k$ is expressed by $RC_{l,m}^{(k)}$ and

$$RC_{l,m}^{(k)} : \tilde{P}_l \otimes \tilde{P}_m \rightarrow \tilde{P}_{l+m-2k}$$

satisfies the covariance property, for $g \in G$

$$RC_{l-m}^{(k)} \circ (\rho_l(g) \otimes \rho_m(g)) = \rho_{l+m+2k}(g) \circ RC_{l-m}^{(k)}$$

It is related to the classical notion of transvectant: more precisely, for $p \in \tilde{P}_l$ and $q \in \tilde{P}_m$, $RC_{l-m}^{(k)}(p,q)$ is (up to a normalization constant depending on $l, m$ and $k$) the $k$-th transvectant of $p$ and $q$.

### 2.3 The Rankin-Cohen brackets

Now assume that $\lambda = l \in \mathbb{N}$, and let

$$\epsilon = + \quad \text{if } l \text{ is even,} \quad \epsilon_l = - \quad \text{if } l \text{ is odd.}$$

Let $f$ be $C^\infty$ function on $\mathbb{R}$ which is the boundary value of a holomorphic functions $F$ in the complex upper half-plane $\Im(z) > 0$. For $g \in G$ such that $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the function

$$(cz + d)^{-l} F \left( \frac{az + b}{cz + d} \right)$$

as $cz + d \neq 0$ on $\Im(z) > 0$, is a well-defined holomorphic function in $\Im(z) > 0$ and admits the boundary value

$$(cx + d)^{-l} f \left( \frac{ax + b}{cx + d} \right) = (\pi_{l,\epsilon_l}(g)f)(x).$$

Hence the subspace $\mathcal{H}_l^{\text{hol}}$ of such functions is an invariant subspace under $\pi_{l,\epsilon_l}$.

Assume also that $\mu = m \in \mathbb{N}$. Then $\mathcal{H}_m^{\text{hol}}$ is an invariant space under $\pi_{m,\epsilon_m}$, and $\mathcal{H}_l^{\text{hol}} \otimes \mathcal{H}_m^{\text{hol}}$ is an invariant subspace under the representation $\pi_{l,\epsilon_l} \otimes \pi_{m,\epsilon_m}$. Finally for $k \in \mathbb{N}$ the space $\mathcal{H}_{l+m+2k}^{\text{hol}}$ is an invariant space under the representation $\pi_{l+m+2k, \epsilon_{l+m+2k}}$, and notice that $\epsilon_{l+m+2k} = \epsilon_l \epsilon_m$. The differential operator $RC_{l,m}^{(k)}$ is covariant with respect to the representations $\pi_{l,\epsilon_l} \otimes \pi_{m,\epsilon_m}$ and $\pi_{l+m+2k, \epsilon_{l+m+2k}}$.

Hence, the holomorphic extension of the differential operator $RC_{l,m}^{(k)}$ maps $\mathcal{H}_l^{\text{hol}} \otimes \mathcal{H}_m^{\text{hol}}$ into $\mathcal{H}_{l+m+2k}^{\text{hol}}$. These operators coincide (up to scalars) with the classical Rankin-Cohen brackets, used in the theory of automorphic forms.
2.4 Rodrigues formula for the symbols of the Rankin-Cohen operators

The last part of this section is devoted to finding a more explicit formula for the operators $RC_{\lambda,\mu}^{(k)}$.

The operator $E_{\lambda,\mu}$ has polynomial coefficients, in other words belongs to the Weyl algebra $W = W(\mathbb{R}^2 \times (\mathbb{R}^2^*))$. Its symbol is given by

$$e_{\lambda,\mu}(x, y, \xi, \eta) = (x - y)\xi\eta - \mu\xi + \lambda\eta.$$  \hspace{1cm} (10)

The operator $E_{\lambda,\mu}^{(k)}$ also belongs to $W$ and denote its symbol by $e_{\lambda,\mu}^{(k)}$.

Apply to the present situation the considerations developed in the last part of Section 1. So $E = \mathbb{R}^2$ and $F = \text{diag}(\mathbb{R}^2) = \{(x, y) \in \mathbb{R}^2, x = y\}$. The quantity denoted by $x''$ in the general context now is equal to $\frac{1}{2}(x - y)$, so that $x''W$ is denoted by $(x - y)W$.

The operators $RC_{\lambda,\mu}^{(k)} = \text{res} \circ E_{\lambda,\mu}^{(k)}$ are bi-differential operators from $\mathbb{R}^2$ into $\mathbb{R}$, so belong to $W(\mathbb{R} \times (\mathbb{R}^2^*))$. The covariance property \[(9)\] under the action of translations imply that they have constant coefficients, and moreover, the covariance property under the action of the dilations imply that they are homogeneous of degree $k$. In other words, the symbols $rc_{\lambda,\mu}^{(k)}$ can be written as

$$rc_{\lambda,\mu}^{(k)}(x, \xi, \eta) = q_{k}^{\lambda,\mu}(\xi, \eta)$$

where $q_{k}^{\lambda,\mu}$ is a homogeneous polynomials of degree $k$ in the variables $(\xi, \eta)$. Consequently (cf Proposition [10]),

$$e_{\lambda,\mu}^{(k)} = q_{k}^{\lambda,\mu} \mod (x - y)W \hspace{1cm} (11)$$

**Proposition 2.1.** For $\lambda, \mu \in \mathbb{C}$ and $k \geq 1$

$$q_{\lambda,\mu}^{(k)} = (-\mu\xi + \lambda\eta)q_{\lambda+1,\mu+1}^{(k-1)} + \xi\eta \left(\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\eta}\right)q_{\lambda+1,\mu+1}^{(k-1)}$$  \hspace{1cm} (12)

**Proof.** By definition

$$E_{\lambda,\mu}^{(k)} = (E_{\lambda+k-1,\mu+k-1} \circ \cdots \circ E_{\lambda+1,\mu+1}) \circ E_{\lambda,\mu} = E_{\lambda+1,\mu+1}^{(k-1)} \circ E_{\lambda,\mu}$$

Hence

$$e_{\lambda,\mu} = e_{\lambda+1,\mu+1}^{(k-1)} \# e_{\lambda,\mu}$$

and consequently

$$q_{k}^{\lambda,\mu} = q_{k+1}^{\lambda,\mu} \# e_{\lambda,\mu} \mod (x - y)P$$

Now use \[10\] and the composition formula \[11\] to get
\[ q_k^{\lambda,\mu}(\xi, \eta) = q_{k+1,\mu+1}(\xi, \eta)(-\mu\xi + \lambda\eta) + \xi\eta \left( \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \right) q_{k+1,\mu+1}(\xi, \eta). \]

Together with the initial condition \( q_0^{\lambda,\mu} = 1 \), the recursion relation (12) determines \( q_k^{\lambda,\mu} \) by induction on \( k \). The next proposition introduces a family of polynomials which will be shown to solve the recursion relation.

**Proposition 2.2.** Let \( \alpha, \beta \in \mathbb{C} \). For any \( l \in \mathbb{N} \), there exists a (unique) polynomial \( Q^{\alpha,\beta}_l(\xi, \eta) \), homogeneous of degree \( l \) such that

\[ \left( \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \right)^l \xi^{\alpha+l-1} \eta^{\beta+l-1} = \xi^{\alpha} \eta^\beta Q^{\alpha,\beta}_l(\xi, \eta). \]  

(13)

For \( l = 0, 1, 2 \), the explicit expression is given by

\[ Q^{\alpha,\beta}_0(\xi, \eta) = 1, \quad Q^{\alpha,\beta}_1(\xi, \eta) = -(\beta + 1)\xi + (\alpha + 1)\eta, \quad Q^{\alpha,\beta}_2(\xi, \eta) = (\beta + 2)(\beta + 1)\xi^2 - 2(\alpha + 2)(\beta + 2)\xi\eta + (\alpha + 2)(\alpha + 1)\eta^2. \]

**Proposition 2.3.** The polynomials \( Q^{\alpha,\beta}_k \) satisfy the following relation

\[ Q^{\alpha,\beta}_k(\xi, \eta) = \xi\eta \left( \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \right) Q^{\alpha+1,\beta+1}_{k-1}(\xi, \eta) + (- (\beta + 1)\xi + (\alpha + 1)\eta) \right) Q^{\alpha+1,\beta+1}_{k-1}(\xi, \eta). \]  

(14)

**Proof.** Observe that

\[ \left( \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \right)^l (\xi^{\alpha+l-1} \eta^{\beta+l-1}) = \left( \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \right) \left( \left( \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \right)^{l-1} \xi^{\alpha+l-1} \eta^{\beta+l-1} \right) \]

and use Leibniz rule to conclude.

**Theorem 2.1.** For \( \lambda, \mu \in \mathbb{C} \), and \( k \in \mathbb{N} \)

\[ q_k^{\lambda,\mu} = Q_k^{\lambda-1,\mu-1}. \]  

(15)

**Proof.** As \( q_0^{\lambda,\mu} = Q_k^{\lambda-1,\mu-1} = 1 \), compare (12) and (13) (for \( l = k \) and \( \alpha = \lambda - 1, \beta = \mu - 1 \)) to conclude by induction on \( k \).
Notice that Theorem A1 in the introduction is merely a reformulation of Theorem A1.

Finally, the polynomials $Q_{k}^{\alpha,\beta}$ are closely related to the Jacobi polynomials. First recall the Rodrigues formula which can be taken as a definition of the Jacobi polynomials (see [6] 8.960.1 p. 998).

\[(1 - t)^{\alpha}(1 + t)^{\beta} P_{k}^{\alpha,\beta}(t) = \frac{(-1)^{k}}{2^{k}k!} \left( \frac{d}{dt} \right)^{k} ((1 - t)^{k+\alpha}(1 + t)^{k+\beta}) \] (16)

In [9], the authors define a family of homogeneous polynomials $\tilde{P}_{k}^{\alpha,\beta}$ of two variables by the formula

\[\tilde{P}_{k}^{\alpha,\beta}(\xi, \eta) = (-1)^{k}(\xi + \eta)^{k} P_{k}^{\alpha,\beta} \left( \frac{\eta - \xi}{\xi + \eta} \right) \] (17)

**Proposition 2.4.** Let $\alpha, \beta \in \mathbb{C}$. Then for all $l \in \mathbb{N}$

\[Q_{k}^{\alpha,\beta} = (-1)^{k} k! \tilde{P}_{k}^{\alpha,\beta} \] (18)

**Proof.** For $F$ a function of two variables $(\xi, \eta)$, associate the function $f$ of one variable given by

\[f(t) = F(1 - t, 1 + t)\]

Then,

\[\left( \frac{d}{dt} \right)^{k} f(t) = (-1)^{k} \left( \left( \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \right)^{k} F \right) (1 - t, 1 + t)\]

Apply this result to the function $F(\xi, \eta) = \xi^{k+\alpha}\eta^{k+\beta}$, which corresponds to $f(t) = (1 - t)^{k+\alpha}(1 + t)^{k+\beta}$. On one hand, by Rodrigues formula

\[\left( \frac{d}{dt} \right)^{k} (1 - t)^{\alpha}(1 - t)^{\beta} = (-1)^{k} 2^{k} k! (1 - t)^{\alpha}(1 + t)^{\beta} P_{k}^{\alpha,\beta}(t)\]

whereas on the other hand,

\[\left( \left( \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \right)^{k} F \right) (1 - t, 1 + t) = (1 - t)^{\alpha}(1 + t)^{\beta} Q_{k}^{\alpha,\beta}(1 - t, 1 + t)\]

Hence

\[2^{k} k! P_{k}^{\alpha,\beta}(t) = Q_{k}^{\alpha,\beta}(1 - t, 1 + t)\]

Now let $(\xi, \eta) \in \mathbb{R}^{2}$ such that $\xi + \eta = 2$. Write $\xi = 1 - t$ and $\eta = 1 + t$, so that $t = \frac{\eta - \xi}{\xi + \eta}$.

\[k! (\xi + \eta)^{k} P_{k}^{\alpha,\beta} \left( \frac{\eta - \xi}{\xi + \eta} \right) = Q_{k}^{\alpha,\beta}(\xi, \eta)\]
or equivalently, using (17)

\[-1 \rightleftharpoons k! P_{k}^{\alpha, \beta}(\xi, \eta) = Q_{k}^{\alpha, \beta}(\xi, \eta)\]

whenever \(\xi + \eta = 2\). As both sides are homogeneous polynomials of degree \(k\), the conclusion follows.

As the Jacobi polynomials have explicit formulæ for their coefficients, this last result yields an explicit formula for the Rankin-Cohen operators.

3 The Juhl operators

Some years ago, A. Juhl (see [7]) introduced a family of differential operators from \(\mathbb{R}^n\) to \(\mathbb{R}^{n-1}\) which are covariant for the subgroup of the conformal group of \(\mathbb{R}^n\) which preserves the hyperplane \(\mathbb{R}^{n-1}\). Recently, I presented a new approach to these operators (see [4]), based on the source operator method.

The group \(G = SO_0(1, n + 1)\) acts conformally on \(\mathbb{R}^n\) by a rational action. For \(g \in G\) defined at \(x \in \mathbb{R}^n\), let \(\kappa(g, x)\) be the conformal factor of \(g\) at \(x\), so that for every \(v \in \mathbb{R}^n\)

\[\forall v \in \mathbb{R}^n, \quad |Dg(x)v| = \kappa(g, x)|v| .\]

For \(\lambda \in \mathbb{C}\), the principal series representation \(\pi_\lambda\) of \(G\) (in the noncompact picture) is given by

\[\pi_\lambda(g) f(x) = \kappa(g^{-1}, x)^\lambda f(g^{-1}(x))\]

where \(f \in C^\infty(\mathbb{R}^n)\).

Let identify the hyperplane \(\{ x \in \mathbb{R}^n, x_n = 0 \}\) with \(\mathbb{R}^{n-1}\) and write \(x = (x', x_n)\) where \(x' \in \mathbb{R}^{n-1}\). The subgroup \(H\) of \(G\) which stabilizes this hyperplane can be identified with \(SO_0(1, n)\). For \(\mu \in \mathbb{C}\), the scalar principal series representation \(\pi'_\mu\) of \(H\) is realized on \(C^\infty(\mathbb{R}^{n-1})\) and given by

\[\pi'_\mu(h) f(x') = \kappa(h^{-1}, x')^\mu f(h^{-1}(x')) , \quad h \in H\]

where \(f \in C^\infty(\mathbb{R}^{n-1})\).

For \(\lambda \in \mathbb{C}\), let \(E_\lambda\) be the differential operator on \(\mathbb{R}^n\) given by

\[E_\lambda = x_n \Delta + (2\lambda - n + 2) \frac{\partial}{\partial x_n} . \quad (19)\]

where \(\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}\) is the usual Laplacian on \(\mathbb{R}^n\). The operator \(E_\lambda\) belongs to the Weyl algebra \(\mathcal{W} = \mathcal{W}(\mathbb{R}^n \times (\mathbb{R}^n)^*)\) For further reference, notice that its symbol is given by

\[e_\lambda(x, \xi) = x_n |\xi|^2 + (2\lambda - n + 2)\xi_n . \quad (20)\]
Proposition 3.1. For any $h \in H$

$$E_\lambda \circ \pi_\lambda(h) = \pi_{\lambda+1}(h) \circ E_\lambda.$$  \hfill (21)

See [4] for the proof. The operator $E_\lambda$ is called the source operator
and plays the same rôle for the construction of Juhl operators as the source operator $E_{\lambda,\mu}$ (see [5]) did for the construction of the Rankin-Cohen operators. For $k \geq 1$ define

$$E_\lambda^{(k)} = E_{\lambda+k-1} \circ \cdots \circ E_\lambda.$$  \hfill (22)

The operator $E_\lambda^{(k)}$ also belongs to $W$ and its symbol is denoted by $e_\lambda^{(k)}$. It satisfies the covariance relation

$$\forall h \in H, \quad E_\lambda^{(k)} \circ \pi_\lambda(h) = \pi_{\lambda+k}(h) \circ E_\lambda^{(k)}.$$  \hfill (23)

Define the restriction map $\text{res}$ by

$$C^\infty(\mathbb{R}^n) \ni f \mapsto \text{res}(f) \in C^\infty(\mathbb{R}^{n-1}), \quad \text{res}(f)(x') = f(x',0).$$

Notice that for any $\lambda \in \mathbb{C}$

$$\forall h \in H, \quad \text{res} \circ \pi_\lambda(h) = \pi'_\lambda(h) \circ \text{res}$$

For $k \in \mathbb{N}$, define

$$J_\lambda^{(k)} = \text{res} \circ E_\lambda^{(k)} = \text{res} \circ E_{\lambda+k-1} \circ \cdots \circ E_\lambda.$$  \hfill (24)

Proposition 3.2. For any $h \in H$,

$$J_\lambda^{(k)} \circ \pi_\lambda(h) = \pi'_{\lambda+k}(h) \circ J_\lambda^{(k)}.$$  \hfill (25)

The operator $J_\lambda^{(k)}$ is a differential operator from $\mathbb{R}^n$ into $\mathbb{R}^{n-1}$ which belongs to $W(\mathbb{R}^{n-1} \times (\mathbb{R}^n)^*)$. Moreover, $J_\lambda^{(k)}$ has constant coefficients, as a consequence of the covariance property (25) for $h$ a translation along a vector in $\mathbb{R}^{n-1}$. Hence the symbol $j_\lambda^{(k)}$ of $J_\lambda^{(k)}$ can be written as

$$j_\lambda^{(k)}(x',\xi) = p^k_\lambda(\xi)$$

where $p^k_\lambda \in \mathcal{P}(\mathbb{R}^{n*})$.

The definition of the operators $J_\lambda^{(k)}$ implies a recurrence relation for their symbols.

Proposition 3.3. The polynomials $p^\lambda_k$ satisfy the following relation

$$p^\lambda_k = (2\lambda - n + 2) \xi_n p^{\lambda+1}_{k-1} + |\xi|^2 \frac{\partial}{\partial \xi_n} p^{\lambda+1}_{k-1}.$$  \hfill (26)
Proof. First, as \( \lambda^k = \text{res} \circ E^k \),
\[
\varepsilon^k(x, \xi) = p^k(\xi) \mod x_n \mathcal{P}(\mathbb{R}^n \times (\mathbb{R}^n)^*) .
\] (27)
Next
\[
E^{(k)} = (E_{\lambda_{k-1}} \circ \cdots \circ E_{\lambda_{1+1}}) \circ E_{\lambda} = \circ E^{(k-1)}_{\lambda_{k-1}} \circ E_{\lambda}
\] (28)
so that
\[
\varepsilon^k = \varepsilon^{(k-1)}_{\lambda_{k-1}} \mod e_{\lambda}
\]
and hence
\[
p^k(\xi) = p^{k+1}_{k-1} \# e_{\lambda} \mod x_n \mathcal{P}(\mathbb{R}^n \times (\mathbb{R}^n)^*)
\]
Use (4) and (20) to get
\[
p^k(\xi) = p^{k+1}_{k-1}(\xi)(2\lambda - n + 2) \xi_n + \left( \frac{\partial}{\partial \xi_n} p^{k+1}_{k-1} \right) \xi|\xi|^2 .
\]
Together with the initial condition \( p^0_0 = 1 \), the recurrence relation (26) determines the polynomials \( p^k_0 \) by induction over \( k \). To solve this recurrence relation, define for \( \gamma \in \mathbb{C} \) the sequence of polynomials \( B^\gamma_k \) on \( \mathbb{R}^n \) by
\[
B^\gamma_0 = 1, \quad \left( \frac{\partial}{\partial \xi_n} \right)^k |\xi|^{2(\gamma+k)} = B^\gamma_k(\xi)|\xi|^{2\gamma}
\] (29)
Lemma 3.1. Let \( \gamma \in \mathbb{C} \). For any \( k \geq 1 \),
\[
B^\gamma_k = 2(\gamma + 1) \xi_n B^{\gamma+1}_{k-1} + |\xi|^2 \frac{\partial}{\partial \xi_n} B^{\gamma+1}_{k-1} .
\] (30)
Proof.
\[
\frac{\partial}{\partial \xi_n} |\xi|^{2(\gamma+k)} = \frac{\partial}{\partial \xi_n} \left( \left( \frac{\partial}{\partial \xi_n} \right)^k -1 |\xi|^{2(\gamma+1)+(k-1)} \right) = \frac{\partial}{\partial \xi_n} \left( |\xi|^{2(\gamma+1)} B^{\gamma+1}_{k-1}(t) \right)
\]
\[
= |\xi|^{2(\gamma+1)} \frac{\partial}{\partial \xi_n} B^{\gamma+1}_{k-1}(\xi) + 2(\gamma + 1) \xi_n |\xi|^{2\gamma} B^{\gamma+1}_{k-1}(\xi) ,
\]
and the conclusion follows.

Theorem 3.1. For any \( \lambda \in \mathbb{C} \) and \( k \in \mathbb{N} \)
\[
p^k_\lambda = B^\lambda_{k-\frac{n}{2}} .
\] (31)
Proof. Notice that \( p^0_\lambda = B^\lambda_{0-\frac{n}{2}} = 1 \), and compare (20) and (30) for \( \gamma = \lambda - \frac{n}{2} \).
Notice that Theorem A2 in the introduction is merely a reformulation of Theorem 3.1.

The polynomials \( B_k^\gamma \) are connected with the classical Gegenbauer polynomials. In fact, the latter may be defined through the Rodrigues formula (see [6] page 993)

\[
C_k^\lambda(t) = c_k(\lambda) (1 - t^2)^{-(\lambda - \frac{1}{2})} \left( \frac{d}{dt} \right)^k (1 - t^2)^{k+\lambda - \frac{1}{2}} \quad (32)
\]

where

\[
c_k(\lambda) = \frac{(-1)^k \Gamma(\lambda + \frac{1}{2}) \Gamma(k + 2\lambda)}{2^k k! \Gamma(2\lambda) \Gamma(k + \lambda + \frac{1}{2})}.
\]

Observe first that \( B_k^\gamma \) is a homogeneous polynomial of degree \( k \). Next, as \(|\xi|^2 = |x'|^2 + \xi_n^2\), \( B_k^\lambda(\xi', \xi_n) \) can be written as a polynomial in \( \xi_n \) and \(|\xi'|^2\), so set

\[
B_k^\gamma(\xi', \xi_n) = A_k^\gamma(|\xi'|, \xi_n)
\]

where \( A_k^\gamma \) is a polynomial of two variables, homogeneous of degree \( k \) and even in the first variable. With this notation, (29) implies

\[
\left( \frac{\partial}{\partial t} \right)^k (s^2 + t^2)^{\gamma + k} = A_k^\gamma(s, t)(s^2 + t^2)^{\gamma} \quad (33)
\]

Proposition 3.4.

\[
A_k^\gamma(s, t) = c_k \left( \gamma + \frac{1}{2} \right)^{-1} (-i)^{k} s^k C_k^{\gamma + \frac{1}{2}} \left( \frac{t}{is} \right). \quad (34)
\]

Proof. Let \( f \) be a function of one variable, and associate the function of two variables given by \( F(s, t) = f \left( \frac{t}{is} \right) \). Then

\[
\left( \frac{\partial}{\partial t} \right)^k F(s, t) = (-i)^{k} s^{-k} f^{(k)} \left( \frac{t}{is} \right)
\]

Apply this relation to

\[
f(t) = (1 - t^2)^{k+\lambda - \frac{1}{2}}, \quad F(s, t) = s^{-2k-2\lambda+1}(s^2 + t^2)^{k+\lambda - \frac{1}{2}}.
\]

Now, by (32)

\[
f^{(k)}(t) = c(k, \lambda)^{-1} (1 - t^2)^{\lambda - \frac{1}{2}} C_k^\lambda(t)
\]

whereas by (33) and letting \( \gamma = \lambda - \frac{1}{2} \)

\[
\left( \frac{\partial}{\partial t} \right)^k F(s, t) = s^{-2k-2\lambda+1}(s^2 + t^2)^{\lambda - \frac{1}{2}} A_k^{\lambda - \frac{1}{2}}(s, t).
\]

(32) follows. \( \square \)
4 Conformally covariant bi-differential operators

We now turn to the conformally covariant bi-differential operators. Using notation of the previous section, let now $G = SO_0(1, n + 1)$ act conformally on $\mathbb{R}^n \times \mathbb{R}^n$ by the diagonal action. Let $\lambda, \mu \in \mathbb{C}$ and consider the tensor product representation $\pi_\lambda \otimes \pi_\mu$ realized on $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. In [12] the authors studied a family of bi-differential operators, mapping $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ into $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, depending on $\lambda, \mu$ and $k \in \mathbb{N}$, covariant w.r.t. $(\pi_\lambda \otimes \pi_\mu, \pi_{\lambda+\mu+2k})$. In [1] a new construction of these operators was presented, implicitly based on the source operator method. Later, in [2] Section 10, this was redone in a more systematic (and likely more readable) presentation of the source operator method, as a particular case among a much larger context.

Denote by $\Delta_x$ (resp. $\Delta_y$) the Laplacian on $\mathbb{R}^n$ with respect to the variable $x$ (resp. $y$) and let

$$\nabla_{xy} = \sum_{j=1}^n \frac{\partial^2}{\partial x_j \partial y_j}$$

The source operator in this case is the differential operator $E_{\lambda,\mu}$ on $\mathbb{R}^n \times \mathbb{R}^n$ given by

$$E_{\lambda,\mu} = |x - y|^2 \Delta_x \Delta_y + 2(2\lambda - n + 2) \sum_{j=1}^n (x_j - y_j) \frac{\partial}{\partial x_j} \Delta_y + 2(2\mu - n + 2) \sum_{j=1}^n (y_j - x_j) \frac{\partial}{\partial y_j} \Delta_x + 2\mu(2\mu - n + 2) \Delta_x - 2(2\lambda - n + 2)(2\mu - n + 2) \nabla_{x,y} + 2\lambda(2\lambda - n + 2) \Delta_y .$$

(35)

It satisfies

$$\forall g \in G, \quad E_{\lambda,\mu}(g) \circ (\pi_\lambda(g) \otimes \pi_\mu(g)) = (\pi_{\lambda+1}(g) \otimes \pi_{\mu+1}(g)) \circ E_{\lambda,\mu} .$$

For further reference, the symbol $e_{\lambda,\mu}$ of the source operator $E_{\lambda,\mu}$ is given by

$$e_{\lambda,\mu}(x, y, \xi, \eta) = |x - y|^2 |\xi|^2 |\eta|^2 + 2(2\lambda - n + 2) \sum_{j=1}^n (x_j - y_j) \xi_j |\eta|^2 + 2(2\mu - n + 2) \sum_{j=1}^n (y_j - x_j) \eta_j |\xi|^2 + 2\mu(2\mu - n + 2) |\xi|^2 - 2(2\lambda - n + 2)(2\mu - n + 2) \xi_\eta + 2\lambda(2\lambda - n + 2) |\eta|^2 .$$

(36)

For $k \in \mathbb{N}$, define the operator $E^{(k)}_{\lambda,\mu}$ by

$$E^{(k)}_{\lambda,\mu} = E_{\lambda+k-1,\mu+k-1} \circ \cdots \circ E_{\lambda,\mu} .$$

(37)
The operator $E_{\lambda,\mu}^{(k)}$ satisfies for any $g \in G$
\[
\forall g \in G, \quad E_{\lambda,\mu}^{(k)} \circ (\pi_\lambda(g) \otimes \pi_\mu(g)) = (\pi_{\lambda+k}(g) \otimes \pi_{\mu+k}(g)) \circ E_{\lambda,\mu}^{(k)} . \quad (38)
\]

Further, define the restriction map $\text{res} : C^\infty(\mathbb{R}^n \times \mathbb{R}^n) \rightarrow C^\infty(\mathbb{R}^n)$ by
\[
\text{res}(f)(x) = f(x,x) .
\]
Then, for any $g \in G$,
\[
\forall g \in G, \quad \text{res} \circ \pi_\lambda(g) \otimes \pi_\mu(g) = \pi_{\lambda+\mu}(g) \circ \text{res} . \quad (39)
\]
Now define the operator $D_{\lambda,\mu}^{(k)}$ by
\[
D_{\lambda,\mu}^{(k)} = \text{res} \circ E_{\lambda+k-1,\mu+k-1} \circ \cdots \circ E_{\lambda,\mu} . \quad (40)
\]
Combining (37) and (39), the operator $D_{\lambda,\mu}^{(k)}$ satisfies for any $g \in G$
\[
D_{\lambda,\mu}^{(k)} \circ (\pi_\lambda(g) \otimes \pi_\mu(g)) = \pi_{\lambda+\mu+2k}(g) \circ D_{\lambda,\mu}^{(k)} . \quad (41)
\]
These operators are called \textit{conformally covariant bi-differential operators} and may be regarded as generalized Rankin-Cohen operators, as $SL(2,\mathbb{R})$ is isogeneous to $SO_0(1,2)$ and acts conformally on $\mathbb{R}$.

The rest of this section is devoted to finding a more explicit expression for the operators $D_{\lambda,\mu}^{(k)}$.

The differential operator $E_{\lambda,\mu}^{(k)}$ has polynomials coefficients, hence belongs to $\mathcal{W} = \mathcal{W}((\mathbb{R}^n \times \mathbb{R}^n) \times (\mathbb{R}^n \times \mathbb{R}^n)^*)$. Denote its symbol by $e_{\lambda,\mu}^{(k)}$.

The diagonal of $\mathbb{R}^n \times \mathbb{R}^n$ is viewed as the subspace of $\mathbb{R}^n \times \mathbb{R}^n$ defined by $x - y = 0$, and is identified with $\mathbb{R}^n$. The operator $D_{\lambda,\mu}^{(k)}$ belongs to $\mathcal{W}(\mathbb{R}^n \times (\mathbb{R}^n \times \mathbb{R}^n)^*)$. The covariance property 111 used for translations (along diagonal vectors) implies that $D_{\lambda,\mu}^{(k)}$ has constant coefficients and hence its symbol $d_{\lambda,\mu}^{(k)} \in \mathcal{P}(\mathbb{R}^n \times (\mathbb{R}^n \times \mathbb{R}^n)^*)$ does not depend on the first variable $x \in \mathbb{R}^n$, so can be written as $d_{\lambda,\mu}^{(k)}(\xi, \eta)$ where $d_{\lambda,\mu}^{(k)}$ belongs to $\mathcal{P}((\mathbb{R}^n \times \mathbb{R}^n)^*)$. Moreover, using 111 for the dilations, it follows that $d_{\lambda,\mu}^{(k)}$ is a homogeneous polynomial of degree $2k$.
Proposition 4.1. $d_{k}^{\lambda,\mu}$ satisfies the following recurrence relation

$$d_{k}^{\lambda,\mu}(\xi, \eta) =$$

$$\left(2\mu(2\mu - n + 2)|\xi|^2 - 2(2\lambda - n + 2)(2\mu - n + 2)|\xi|\eta + 2\lambda(2\lambda - n + 2)|\eta|^2\right)d_{k-1}^{\lambda+1,\mu+1}(\xi, \eta)$$

$$+ 2(2\lambda - n + 2)|\eta|^2 \sum_{j=1}^{n} \xi_j \left(\frac{\partial}{\partial \xi_j} - \frac{\partial}{\partial \eta_j}\right) d_{k-1}^{\lambda+1,\mu+1}(\xi, \eta)$$

$$+ 2(2\mu - n + 2)|\xi|^2 \sum_{j=1}^{n} \eta_j \left(\frac{\partial}{\partial \eta_j} - \frac{\partial}{\partial \xi_j}\right) d_{k-1}^{\lambda+1,\mu+1}(\xi, \eta)$$

$$+ |\xi|^2|\eta|^2 q \left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}\right) d_{k-1}^{\lambda+1,\mu+1}(\xi, \eta).$$

(42)

Proof. As $D_{\lambda,\mu}^{(k)} = \text{res} \circ E_{\lambda,\mu}^{(k)}$, $E_{\lambda,\mu}^{(k)} = d_{k}^{\lambda,\mu} \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \mod (x - y)W$.

Next,

$$E_{\lambda,\mu}^{(k)} = E_{\lambda+1,\mu+1}^{(k-1)} \circ E_{\lambda,\mu} = d_{k-1}^{\lambda+1,\mu+1} \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \circ E_{\lambda,\mu} \mod (x - y)W.$$

Hence

$$d_{k}^{\lambda,\mu} = d_{k-1}^{\lambda+1,\mu+1} \# e_{\lambda,\mu} \mod (x - y)\mathcal{P}.$$

Now, using the composition formula (4) for the symbols

$$d_{k}^{\lambda,\mu}(\xi, \eta) =$$

$$d_{k-1}^{\lambda+1,\mu+1}(\xi, \eta) \left(2\mu(2\mu - n + 2)|\xi|^2 - 2(2\lambda - n + 2)(2\mu - n + 2)|\xi|\eta + 2\lambda(2\lambda - n + 2)|\eta|^2\right)$$

$$+ 2(2\lambda - n + 2)\sum_{j=1}^{n} \left(\frac{\partial}{\partial \xi_j} - \frac{\partial}{\partial \eta_j}\right) d_{k-1}^{\lambda+1,\mu+1}(\xi, \eta) \xi_j |\xi|^2$$

$$+ 2(2\mu - n + 2)\sum_{j=1}^{n} \left(\frac{\partial}{\partial \eta_j} - \frac{\partial}{\partial \xi_j}\right) d_{k-1}^{\lambda+1,\mu+1}(\xi, \eta) \eta_j |\xi|^2$$

$$+ q \left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}\right) d_{k-1}^{\lambda+1,\mu+1}(\xi, \eta) |\xi|^2|\eta|^2$$

and the proposition follows.

Together with the condition $d_{0}^{\lambda,\mu} = 1$, the recurrence relation (42) determines $d_{k}^{\lambda,\mu}$ by induction on $k$. To solve this recurrence relation, let
us introduce for $\alpha, \beta \in \mathbb{C}$ the family of polynomials $p_{k}^{\alpha,\beta}$ on $(\mathbb{R}^{n} \times \mathbb{R}^{n})^{*}$ defined by

$$q \left( \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \right)^{k} \left( |\xi|^{2(\alpha+k)} |\eta|^{2(\beta+k)} \right) = p_{k}^{\alpha,\beta}(\xi, \eta) |\xi|^{2\alpha} |\eta|^{2\beta} .$$

(43)

It is easily seen that $p_{k}^{\alpha,\beta}$ thus defined is a homogeneous polynomial of degree $2k$.

**Proposition 4.2.** The polynomials $p_{k}^{\alpha,\beta}$ satisfy the recurrence relation

$$p_{k}^{\alpha,\beta}(\xi, \eta) = \left| \xi \right|^{2} \left| \eta \right|^{2} q \left( \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \right)^{k} p_{k-1}^{\alpha+1,\beta+1}$$

$$+ 2(2\alpha + 2) |\eta|^{2} \sum_{j=1}^{n} \xi_{j} \left( \frac{\partial}{\partial \xi_{j}} - \frac{\partial}{\partial \eta_{j}} \right) p_{k-1}^{\alpha+1,\beta+1}$$

$$+ 2(2\beta + 2) |\xi|^{2} \sum_{j=1}^{n} \eta_{j} \left( \frac{\partial}{\partial \eta_{j}} - \frac{\partial}{\partial \xi_{j}} \right) p_{k-1}^{\alpha+1,\beta+1}$$

$$+ \left( (2\alpha + 2)(2\alpha + n) |\eta|^{2} - 2(2\alpha + 2)(2\beta + 2) \langle \xi, \eta \rangle + (2\beta + 2)(2\beta + n) |\xi|^{2} \right) p_{k-1}^{\lambda+1,\mu+1}.$$  

(44)

**Proof.** Observe that

$$q \left( \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \right)^{k} \left| \xi \right|^{2(\alpha+k)} \left| \eta \right|^{2(\beta+k)}$$

$$= q \left( \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \right)^{k-1} \left| \xi \right|^{2(\alpha+(k-1)}) \left| \eta \right|^{2(\beta+(k-1))}$$

$$= q \left( \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \right) \left( p_{k-1}^{\alpha+1,\beta+1} \right) \left| \xi \right|^{2(\alpha+1)} \left| \eta \right|^{2(\beta+1)} .$$

A straightforward calculation done *asinus trottans* yields the result. 

□

**Theorem 4.1.** For $\lambda, \mu \in \mathbb{C}$ and for $k \in \mathbb{N}$,

$$d_{k}^{\lambda,\mu}(\xi, \eta) = p_{k}^{\lambda-\frac{n}{2},\mu-\frac{n}{2}}(\xi, \eta) .$$

(45)

**Proof.** As $d_{0}^{\lambda,\mu} = 1$ and $p_{0}^{\alpha,\beta} = 1$, the conclusion follows by comparing (42) and (44) after setting $\alpha = \lambda - \frac{n}{2}, \beta = \mu - \frac{n}{2}$. 

□

23
Notice that Theorem A3 in the introduction is merely a reformulation of Theorem 4.1.

In this last example, the fifth step is missing. An inspection of the symmetries of the polynomials \( p_k^{\alpha,\beta} \) shows that they are invariant under the (diagonal action) of \( SO(n) \) and hence, by the first fundamental theorem of invariant theory, can be written as a polynomial in the variables \( |\xi|^2, \xi.\eta \) and \( |\eta|^2 \), homogenous of degree \( k \), i.e.

\[
p_k^{\alpha,\beta}(\xi,\eta) = \sum_{r,s,t,r+s+t=k} c_{r,s,t}^{k,\alpha,\beta} |\xi|^{2r} (\xi.\eta)^s |\eta|^{2t}.
\]

Linking the coefficients \( c_{r,s,t}^{k,\alpha,\beta} \) with the coefficients of some family of orthogonal polynomials would require polynomials of several variables.

Let us finally mention two other older approaches relevant for the third example. In [12] the authors were the first to consider these conformally covariant bi-differential operators, and in [13] the author uses the F-method for the construction of these operators.

References

[1] R. Beckmann, J.-L. Clerc, Singular invariant trilinear forms and covariant (bi-)differential operators under the conformal group, J. Funct. Anal. 262 (2012), 4341–4376
[2] S. Ben Saïd, J.-L. Clerc, K. Koufany, Conformally covariant bi-differential operators on a simple Jordan algebra, Int. Math. Res. Notices, on line
[3] S. Ben Saïd, J.-L. Clerc, K. Koufany, Conformally covariant bi-differential operators for differential forms, arXiv : 1819.06290
[4] J.-L. Clerc, Another approach to Juhl’s conformally covariant differential operators from \( S^n \) to \( S^{n-1} \), SIGMA 13 (2017), Paper No. 28, 18 pp.
[5] M. Fischmann, B. Ørsted, P. Somberg Bernstein-Sato identities and conformal symmetry breaking operators, arXiv:1711.01546
[6] I.S. Gradshteyn, I. Ryzhik, Table of integrals, series and products, 7th edition edition, Elsevier/Academic Press, Amsterdam (2007)
[7] A. Juhl, Families of conformally covariant differential operators, Q-curvature and holography, Progress in Mathematics 275 (2009), Birkhäuser
[8] T. Kobayashi, F-method for symmetry breaking operators, Diff. Geometry and its Appl. 33 (2014), 272–289
[9] T. Kobayashi, M. Pevzner, Differential symmetry breaking operators II. Rankin-Cohen operators for symmetric pairs, Selecta Math. 22 (2016), 847–911
[10] T. Kobayashi, B. Speh, *Symmetry breaking for representations of rank one orthogonal groups*, Mem. Amer. Math. Soc. 238, Amer. Math. Soc., Providence, RI (2015)

[11] P. Olver, *Classical invariant theory*, London Mathematical Society Student Texts, 44, Cambridge University Press (1999).

[12] V. Ovsienko and P. Redou, *Generalized transvectants, Rankin-Cohen brackets*, Lett. Math. Phys. 63 (2003), 19–28

[13] P. Somberg, *Rankin-Cohen brackets for orthogonal Lie algebras and bilinear conformally invariant differential operators*, preprint, [arXiv:1301.2687](http://arxiv.org/abs/1301.2687)

Institut Elie Cartan de Lorraine, Université de Lorraine et CNRS
jean-louis.clerc@univ-lorraine@univ-lorraine.fr