Kinetic Energy Plus Penalty Functions for Sparse Estimation

Zhihua Zhang and Shibo Zhao
Department of Computer Science & Engineering
Shanghai Jiao Tong University
800 Dong Chuan Road, Shanghai, China 200240
zhzhang@gmail.com

Zebang Shen
College of Computer Science & Technology
Zhejiang University
38 Zheda Road, Hangzhou, China 310028
shenzebang@gmail.com

Shuchang Zhou
Key Laboratory of Computer System and Architecture
Institute of Computing Technology
Chinese Academy of Sciences, Beijing, China
shuchang.zhou@gmail.com

July 8, 2014

Abstract
Motivated by iteratively reweighted $\ell_q$ methods, we propose and study a family of sparsity-inducing penalty functions. Since the penalty functions are related to the kinetic energy in special relativity, we call them kinetic energy plus (KEP) functions. We construct the KEP function by using the concave conjugate of a $\chi^2$-distance function and present several novel insights into the KEP function with $q = 1$. In particular, we derive a thresholding operator based on the KEP function, and prove its mathematical properties and asymptotic properties in sparsity modeling. Moreover, we show that a coordinate descent algorithm is especially appropriate for the KEP function. Additionally, we discuss the relationship of KEP with the penalty functions $\ell_{1/2}$ and MCP. The theoretical and empirical analysis validates that the KEP function is effective and efficient in high-dimensional data modeling.

Keywords: iteratively reweighted minimization methods, kinetic energy plus penalties, nonconvex penalization, stability, concave conjugate

1. Introduction
Sparsity is an important attribute in statistical modeling for high-dimensional data sets, especially when the underlying model has a sparse representation. Typically, the penalty theory has been used for capturing sparsity. A principled approach is to employ the $\ell_1$-norm penalty as a convex relaxation of the $\ell_0$-norm (Tibshirani, 1996). Additionally, some non-convex alternatives, such as the bridge penalty $\ell_q$ ($q \in (0,1)$), the log-penalty (Mazumder et al., 2011), the nonconvex EXP (Bradley and Mangasarian, 1998, Gao et al., 2011), the
smoothly clipped absolute deviation (SCAD) penalty (Fan and Li, 2001) and the minimax concave plus (MCP) penalty (Zhang, 2010a), have attracted wide attention.

On one hand, nonconvex penalties usually have nice consistency properties (Fan and Li, 2001, Zhang and Zhang, 2012). On the other hand, they would yield computational challenges due to their nonconvexity and nondifferentiability. In order to address this challenge, Fan and Li (2001) proposed a local quadratic approximation (LQA), while Zou and Li (2008) then devised a local linear approximation (LLA). These methods enjoy a so-called majorization-minimization (MM) procedure (Lange et al., 2000, Hunter and Li, 2005). In the same spirit, iteratively reweighted $\ell_q$ ($q = 2$ or 1) methods have been also developed to find sparse solutions (Chartrand and Yin, 2008, Candès et al., 2008, Wipf and Nagarajan, 2008, Daubechies et al., 2010, Wipf and Nagarajan, 2010). Additionally, Mazumder et al. (2011) developed a SparseNet algorithm based on coordinate descent for the MCP penalty.

Our work is mainly motivated by the iteratively reweighted $\ell_q$ method of Daubechies et al. (2010) and by the coordinate descent algorithm of Mazumder et al. (2011). Daubechies et al. (2010) demonstrated the elegant performance of their method theoretically and empirically. However, there are still several issues that deserve to be further studied. First, the penalty function corresponding to the method is not explicitly available. This results in that the corresponding thresholding operator is also unknown. Second, it is unclear whether the estimator has some properties such as unbiasedness, continuity and asymptotic consistency.

Within and beyond these issues, we develop a family of novel penalty functions. First, we derive the expression of the penalty function by using the concave conjugate of a $\chi^2$-distance function. Interestingly, when $q = 2$, the expression is mathematically the same with the kinetic energy in special relativity. We thus refer to them as kinetic energy plus (KEP) functions. We explore the connection of the KEP penalty with the $\ell_q$-norm and $\ell_{q/2}$-norm. The constructive method encourages us to rederive the iteratively reweighted $\ell_q$ method of Daubechies et al. (2010) via an augmented Lagrangian methodology.

In this paper we are especially concerned with the case of $q = 1$, because the corresponding KEP penalty is nonconvex. Theoretically, we give mathematical properties and asymptotic behaviors of the resulting estimators built on the work of Fan and Li (2001), Knight and Fu (2000), Zhao and Yu (2006), Zou and Li (2008). Specifically, the asymptotic behaviors are studied both in the conventional fixed $p$ (the number of features) setting and in the large $p$ setting as $n$ (the training sample size) increases.

Computationally, we develop the corresponding thresholding operator. We show that the thresholding operator bridges the soft thresholding operator based on the lasso and the half thresholding operator based on the $\ell_{1/2}$ penalty (Xu et al., 2012). However, compared with the soft thresholding operator, our thresholding operator has unbiasedness and oracle properties. Compared with the half thresholding operator, our thresholding operator is continuous, which makes it stable in model prediction. These properties assure that the KEP function is suitable for coordinate descent algorithms. Moreover, the convergence property of the coordinate descent algorithm can be ensured (Mazumder et al., 2011).

We uncover an inherent connection between the KEP and MCP functions. Specifically, the MCP function can be also defined as the concave conjugate of the $\chi^2$-distance function. The difference between KEP and MCP is then due to asymmetricity of the $\chi^2$-distance function. This difference makes the KEP outperform MCP in that KEP enjoys a nesting property—a desirable property stated by Mazumder et al. (2011).
It is worth noting that Palmer et al. (2006) and Wipf and Nagarajan (2008) considered the application of concave conjugates for non-Gaussian latent variable models. The notion of concave conjugates has been also used by (Zhang, 2010b, Zhang and Tu, 2012, Zhang et al., 2013) in construction of nonconvex penalty functions. For example, Zhang et al. (2013) employed the concave conjugate of the squared Euclidean distance function for defining the MCP function. Zhang and Tu (2012) then showed that the nonconvex LOG and EXP functions can be defined as the concave conjugate of the Kullback-Leibler (KL) divergence. Interestingly, asymmetricity of the KL divergence implies the connection between LOG and EXP, which stands in parallel with the connection between KEP and MCP.

The remainder of the paper is organized as follows. Section 2 reviews the iteratively reweighted \( \ell_q \) method of Daubechies et al. (2010). We propose the KEP penalty in Section 3, and study sparse estimation based on the KEP function in Section 4. In Section 5 we explore the relationship between MCP and KEP. In Section 6 we give asymptotic consistent results of sparse estimators. In Section 7 we conduct our experimental evaluations. Finally, we conclude our work in Section 8. Some proofs are given in Appendix.

2. Problem Formulations

Typically, supervised learning can be formulated as an optimization problem under the regularization framework or penalty theory:

\[
\min_\theta \left\{ L(\theta; \mathcal{X}) + P(\theta; \lambda) \right\},
\]

where \( \mathcal{X} = \{(x_i, y_i); i = 1, \ldots, n\} \) is a training dataset, \( \theta \) the model parameter vector, \( L(\cdot) \) the loss function penalizing data misfit, \( P(\cdot) \) the regularization term penalizing model complexity, and \( \lambda (> 0) \) the tuning parameter of balancing the relative significance of the loss function and the penalty.

The choice of the loss function depends very much on the supervised learning problem at hand. Our presentation is mainly based on the linear regression problem

\[
L(b; \mathcal{X}) = \frac{1}{2} \sum_{i=1}^{n} (y_i - x_i^T b)^2 = \frac{1}{2} \| y - Xb \|_2^2,
\]

where \( y = (y_1, \ldots, y_n)^T \in \mathbb{R}^n \), \( X = [x_1, \ldots, x_n]^T \in \mathbb{R}^{n \times p} \), and \( b = (b_1, \ldots, b_p)^T \in \mathbb{R}^p \). We can also consider extensions involving other exponential family models.

A widely used setting for penalty is \( P(b; \lambda) = \lambda \sum_{j=1}^{p} P_j(b_j) \), which implies that the penalty function consists of \( p \) separable subpenalties and all subpenalties share a common tuning parameter \( \eta \). In order to find a sparse solution of \( b \), one imposes the \( \ell_0 \)-norm penalty to \( b \). However, the resulting optimization problem is usually NP-hard. Thus, the \( \ell_1 \)-norm penalty \( P(b; \lambda) = \lambda \| b \|_1 = \lambda \sum_{j=1}^{p} |b_j| \) is an effective convex alternative. Additionally, some nonconvex alternatives, such as the bridge penalty \( \ell_q \) \((q \in (0, 1))\), SCAD and MCP, have been employed. Meanwhile, iteratively reweighted \( \ell_q \) \((q = 1 \text{ or } 2)\) minimization methods were developed for finding sparse solutions.

Specifically, Daubechies et al. (2010) proposed an iteratively reweighted least-squares (or \( \ell_2 \)) minimization method. This method can be slightly modified as an iteratively reweighted
\(\ell_1\) minimization version. Thus, we here consider a general \(\ell_q\) version. In particular, the method introduces a set of auxiliary variables, including a real number \(\epsilon > 0\) and a weight vector \(\omega = (\omega_1, \ldots, \omega_p)^T \in \mathbb{R}^p\) with \(\omega_j > 0\) for \(j = 1, \ldots, p\). Subsequently, the iteratively reweighted \(\ell_q\) method solves the following optimization problem:

\[
\min \left\{ J(b, \omega, \epsilon) := \frac{1}{2} \|y - Xb\|^2_2 + \frac{\lambda}{2} \sum_{j=1}^{p} |b_j|^q \omega_j + \left( \frac{\epsilon^2}{2} \omega_j + \omega_j^{-1} \right) \right\},
\]

where \(\lambda > 0\). Furthermore, given the \(t\)th estimates \((b^{(t)}, \omega^{(t)}, \epsilon^{(t)})\), one recursively defines

\[
b^{(t+1)} = \arg\min_b J(b, \omega^{(t)}, \epsilon^{(t)})
\]

\[
= \arg\min_b \frac{1}{2} \|y - Xb\|^2_2 + \frac{\lambda}{2} \sum_{j=1}^{p} |b_j|^q \omega^{(t)}_j,
\]

and

\[
\omega^{(t+1)} = \arg\min_{\omega > 0} J(b^{(t+1)}, \omega, \epsilon^{(t+1)})
\]

\[
= \arg\min_{\omega > 0} \sum_{j=1}^{p} \left[ |b_j^{(t+1)}|^q \omega_j + \frac{(\epsilon^{(t+1)})^2 \omega_j^2 + 1}{\omega_j} \right],
\]

where \(\epsilon^{(t+1)} = \min \left( \epsilon^{(t)}, r_{k+1}(b^{(t)}/p) \right)\). Here \(k < p\) is a prespecified positive integer and \(r_i(b)\) is the \(i\)th largest element of the vector \((|b_1|, \ldots, |b_p|)^T\). It is directly obtained that

\[
\omega_j^{(t+1)} = \frac{1}{\sqrt{|b_j^{(t+1)}|^q + (\epsilon^{(t+1)})^2}}, \quad j = 1, \ldots, p.
\]

Daubechies et al. (2010) demonstrated the performance of the iteratively reweighted \(\ell_q\) method theoretically and empirically. However, there are still several questions that would be interesting. For example,

- (1) What are the explicit expressions of the penalty function and its corresponding thresholding operator for the penalized regression problem in (1)?
- (2) Does the estimator resulted from the problem in (1) have properties such as “unbiasedness,” “sparsity,” and “continuity,” and “asymptotic consistency?”

In this paper we introduce penalty functions that we call \textit{kinetic energy plus} (KEP) functions to address these questions. In Section 3 we derive the KEP function by using the concave conjugate of a \(\chi^2\)-distance function. In Section 4 we develop a sparse estimation approach based on the KEP penalty with \(q = 1\) and present some important properties for assisting our approach. In Section 6 we present asymptotic consistent results about the sparse estimator. Thus, our work not only deals with the questions mentioned above but also provides new insights into sparse estimation problems well.

1. Daubechies et al. (2010) originally considered the iteratively reweighted \(\ell_q\) method for a sparse recovery problem with equality constraints. The method also applies to the problem in the presence of noise, that is, the problem in (1).
3. Kinetic Energy Plus (KEP) Penalty Functions

Before presenting our work, we first give some notations. We denote \( \mathbb{R}^p_+ = \{ u = (u_1, \ldots, u_p)^T \in \mathbb{R}^p : u_j \geq 0 \text{ for } j = 1, \ldots, p \} \) and \( \mathbb{R}^p_{++} = \{ u = (u_1, \ldots, u_p)^T \in \mathbb{R}^p : u_j > 0 \text{ for } j = 1, \ldots, p \} \). Furthermore, if \( u \in \mathbb{R}^p_+ \) (or \( u \in \mathbb{R}^p_{++} \)), we also write \( u \geq 0 \) (or \( u > 0 \)). Additionally, we denote \( |u|^q = (|u_1|^q, \ldots, |u_p|^q)^T \) and \( \|u\|_q^q = \sum_{j=1}^p |u_j|^q \).

We observe that the minimization problem in (3) is equivalent to the following problem

\[
\min_{\omega > 0} \sum_{j=1}^p \frac{1}{2} \left[ |b_j|^q \omega_j + \frac{(\omega_j \epsilon - 1)^2}{\omega_j} \right].
\]

By direct calculations, the corresponding minimum is given by

\[
\sum_{j=1}^p \left( \sqrt{|b_j|^q + \epsilon^2} - \epsilon \right).
\] (4)

We are now able to answer the first question given in Section 2. That is, \( \sqrt{|b_j|^q + \epsilon^2} - \epsilon \) is the penalty associated with the iteratively reweighted \( \ell_q \) minimization method of Daubechies et al. (2010). In other words, the method is used to solve the following penalization problem:

\[
\min_{b} \left\{ L(b; X) + \lambda \sum_{j=1}^p \left( \sqrt{|b_j|^q + \epsilon^2} - \epsilon \right) \right\},
\] (5)

which can in turn be formulated into the optimization problem in (1).

We now present an alternative derivation of the above penalty function and establish its connection with the kinetic energy in special relativity. Note that \( \frac{(\omega-\epsilon)^2}{\omega} \) is related to the \( \chi^2 \)-distance. For nonnegative \( \omega \) and \( \eta \), the \( \chi^2 \)-distance between them is \( \frac{(\omega-\eta)^2}{\eta} \). This motivates us to develop a new approach for the construction of KEP penalty functions.

We first study a nonseparable version. In this case, given \( \alpha > 0 \) and \( \eta > 0 \), we consider the following optimization problem

\[
\min_{\omega > 0} \omega \|b\|_q^q + \frac{1}{2\alpha} \frac{(\omega - \eta)^2}{\omega}.
\]

It is immediate that the corresponding minimum is given by

\[
\frac{\eta}{\alpha} \left( \sqrt{2\alpha \|b\|_q^q + 1} - 1 \right) \quad \text{(denoted Ke}(|b|^q; \eta, \alpha)).
\]

Interestingly, if \( q = 2, p = 3, 2\alpha = 1/(mc^2) \) and \( 2\eta = 1/m \) where \( m > 0 \) is the mass at rest and \( c > 0 \) is the velocity of light, we can obtain that

\[
\text{Ke} = \sqrt{m^2c^4 + c^2 \|b\|_2^2 - mc^2},
\]

which is the kinetic energy in relativity theory.
We next study a separable version. Alternatively, we are concerned with the following optimization problem

$$
\min_{\omega > 0} Q(\omega | b, \eta) := \omega^T |b|^q + \frac{1}{2\alpha} \sum_{j=1}^p \frac{(\omega_j - \eta)^2}{\omega_j}.
$$

Let $C(|b|^q)$ denote the minimum of the above problem, which is the concave conjugate of $-\frac{1}{2\alpha} \sum_{j=1}^p \frac{(\omega_j - \eta)^2}{\omega_j}$ with respect to (w.r.t.) $|b|^q$. It is easily computed that

$$
C(|b|^q) = \sum_{j=1}^p \frac{\eta}{\alpha}(\sqrt{2\alpha|b_j|^q + 1} - 1)
$$

at $\hat{\omega}_j = \eta(2\alpha|b_j|^q + 1)^{-1/2}$. With $C(|b|^q)$ as the penalty, the corresponding iteratively reweighted $\ell_q$ minimization method is then used to solve the following penalization problem:

$$
\min_b \left\{ J(b) := L(b; \mathcal{X}) + \sum_{j=1}^p \frac{\eta}{\alpha}(\sqrt{2\alpha|b_j|^q + 1} - 1) \right\},
$$

which in turn be formulated as the optimization problem:

$$
\min_b \min_{\omega > 0} \left\{ L(b; \mathcal{X}) + \omega^T |b|^q + \frac{1}{2\alpha} \sum_{j=1}^p \frac{(\omega_j - \eta)^2}{\omega_j} \right\}.
$$

Clearly, when we set $2\eta = \lambda/\epsilon$ and $2\alpha = 1/\epsilon^2$, the problems (6) and (7) are respectively equivalent to (5) and (1). In this case, we further see that $\epsilon = mc$ and $\lambda = c$. Thus, we have very interesting physical meanings of the hyperparameters $\lambda$ and $\epsilon$ in the iteratively reweighted least squares method of Daubechies et al. (2010).

In this paper we define the following penalty function:

$$
\Psi(|b|^q; \eta, \alpha) = \frac{\eta}{\alpha}(\sqrt{2\alpha|b|^q + 1} - 1).
$$

We refer to it as the kinetic energy plus (KEP) function of $b$, due to the relationship with the kinetic energy in relativity theory. To explore the relationship of $\Psi(|b|^q; \eta, \alpha)$ with the $\ell_q$-norm, we let $\eta = \frac{\lambda}{\sqrt{2\alpha} + 1}$ for some $\lambda > 0$. Accordingly, we define

$$
\Phi(|b|^q; \alpha) = \frac{\sqrt{2\alpha|b|^q + 1} - 1}{\sqrt{2\alpha} + 1 - 1} = \frac{(\sqrt{2\alpha+1}+1)|b|^q}{\sqrt{2\alpha|b|^q + 1} + 1},
$$

which goes through the points $(0, 0)$ and $(1, 1)$ like the $\ell_q$-norm. The derivative of $\Phi(|b|^q; \alpha)$ w.r.t. $|b|^q$ is

$$
\Phi'(|b|^q; \alpha) := \frac{\partial \Phi(|b|^q; \alpha)}{\partial |b|^q} = \frac{\sqrt{2\alpha+1}+1}{2\sqrt{2\alpha|b|^q + 1}}.
$$

We now present the following proposition.

**Proposition 1** Let $\Phi(|b|^q; \alpha)$ be defined in (9). Then,
(i) $\Phi(|b|^q; \alpha)$ is a nonnegative, nondecreasing and concave function of $|b|^q$.

(ii) $\lim_{\alpha \to \infty} \Phi(|b|^q; \alpha) = |b|^q$ and $\lim_{\alpha \to \infty} \Phi'(|b|^q; \alpha) = \frac{1}{2|b|^q}$.

(iii) $\lim_{\alpha \to 0^+} \Phi(|b|^q; \alpha) = |b|^q$ and $\lim_{\alpha \to 0^+} \Phi'(|b|^q; \alpha) = 1$.

The proof is immediately. This proposition shows that $\Phi(|b|^q; \alpha)$ can be regarded as a penalty for $b$. Specifically, $\Phi(|b|^2; \alpha)$ (i.e., $q = 2$) defines a convex penalty of $b$, while $\Phi(|b|; \alpha)$ (i.e., $q = 1$) defines a nonconvex penalty of $b$. Moreover, Proposition 1 says that $\Phi(|b|^q; \alpha)$ bridges the $\ell_q$-norm and the $\ell_{q/2}$-norm. Figure 1 illustrates $\Phi(|b|^q; \alpha)$ when $p = 1$ and $p = 2$.

Figure 1: The KEP functions $\Phi(|b|; \alpha)$. 
4. Sparse Estimation Based on the KEP Penalty

When \( q = 1 \) the KEP function defines a nonconvex penalty for \( b \) and is singular at the origin. Thus, such a penalty is able to induce sparsity. We now study the mathematical properties of the sparse estimator in the settings \( q = 1 \). These properties show that the KEP penalty is suitable for a coordinate descent algorithm (Mazumder et al., 2011).

4.1 Threshold Operators

Following Fan and Li (2001), we define the penalized least squares problem

\[
J_1(b) := \frac{1}{2}(z - b)^2 + \Psi(|b|; \eta, \alpha),
\]

where \( z = x^T y \). Fan and Li (2001) stated that a good penalty should result in an estimator with three properties. (1) Unbiasedness: it is nearly unbiased when the true unknown parameter is large; (2) Sparsity: it is a thresholding rule, which automatically sets small estimated coefficients to zero; (3) Continuity: it is continuous in data \( z \) to avoid instability in model prediction.

According to the discussion in Fan and Li (2001), the resulting estimator is nearly unbiased due to that \( \Psi'(|b|) = \frac{\eta}{\sqrt{2\alpha|b| + 1}} \to 0 \) as \( |b| \to \infty \). Note that

\[
\lim_{|b| \to \infty} \frac{\eta}{\sqrt{2\alpha|b| + 1}} \left/ \frac{1}{2|b|^{1/2}} \right. = \frac{2\eta}{\sqrt{2\alpha}}.
\]

Thus, for the KEP penalty \( \Psi(|b|) \) and the \( \ell_{1/2} \)-norm penalty \( |b|^{1/2} \), the convergence rates of their derivatives to zero are same.

As also stated in Fan and Li (2001), it suffices for the resulting estimator to be a thresholding rule that the minimum of the function \( |b| + \Psi'(|b|) \) is positive. Moreover, a sufficient and necessary condition for “continuity” is the minimum of \( |b| + \Psi'(|b|) \) is attained at 0. In fact, we have the following theorem.

**Theorem 2** Consider the penalized least squares problem in (10).

(i) If \( \eta \geq \frac{1}{\alpha} \), then the resulting estimator is a thresholding rule; that is,

\[
\hat{b} = S_{\alpha}(z, \eta) := \begin{cases} 
\text{sgn}(z) \frac{\eta}{\sqrt{2\alpha|b| + 1}} \kappa(|z|) & \text{if } |z| > \frac{3}{2\alpha}(\alpha\eta)^{3/2} - \frac{1}{2\alpha}, \\
0 & \text{if } |z| \leq \frac{3}{2\alpha}(\alpha\eta)^{3/2} - \frac{1}{2\alpha},
\end{cases}
\]

where

\[
\kappa(|z|) = \frac{4(2\alpha|z| + 1)}{3} \cos^2 \left[ \frac{1}{3} \arccos \left( -\alpha\eta \left( \frac{3}{2\alpha|z| + 1} \right)^{3/2} \right) \right] - 1.
\]

(ii) If \( \eta < \frac{1}{\alpha} \), then the resulting estimator is defined as

\[
\hat{b} = S_{\alpha}(z, \eta) := \begin{cases} 
\text{sgn}(z) \frac{\eta}{\sqrt{2\alpha}} \kappa(|z|) & \text{if } |z| > \eta, \\
0 & \text{if } |z| \leq \eta,
\end{cases}
\]

which is continuous in \( z \).
Remarks} In both the cases, we always have $|\hat{b}| \leq |z|$. The objective function $J_1(b)$ in (10) is strictly convex in $b$ whenever $\eta \leq \frac{1}{\alpha}$. Moreover, according to Lemma 7 in Appendix A, the estimator $\hat{b}$ in both the cases is strictly increasing w.r.t. $|z|$, and $\hat{b}$ is Lipschitz continuous when $\eta < \frac{1}{\alpha}$ (also see Lemma 7).

We now explore connection of the thresholding operator (function) based on the KEP penalty with the soft thresholding operator based on Lasso and the half thresholding operator based on the $\ell_{1/2}$-norm penalty (Xu et al., 2012). For this purpose, in terms of Proposition 1 we let $\eta = \frac{\lambda_\alpha}{\sqrt{2\alpha+1-1}}$ where $\lambda > 0$ does not rely on $\alpha$. Obviously, $\frac{\alpha}{\sqrt{2\alpha+1-1}} = \sqrt{\frac{2\alpha+1-1}{2}} \geq 1$. Hence, $|z| \geq \frac{\lambda_\alpha}{\sqrt{2\alpha+1-1}}$ implies $|z| \geq \lambda$. Moreover, $\frac{\sqrt{2\alpha+1-1}}{2}$ is increasing but $\frac{1}{\sqrt{2\alpha+1-1}}$ is decreasing in $\alpha$. This implies that the KEP penalty ($q = 1$) to some extent satisfies the nesting property (see Figure 2-(a)), a desirable property for thresholding functions pointed out by Mazumder et al. (2011).

Furthermore, we have $\lim_{\alpha \to 0} \frac{1}{\alpha} = \infty$ and $\lim_{\alpha \to 0} \frac{\lambda_\alpha}{\sqrt{2\alpha+1-1}} = \lambda$. In this limiting case, it is clear that our thresholding function approaches the soft thresholding function:

$$
\lim_{\alpha \to 0^+} S_\alpha(z, \eta) = S(z, \lambda) := \text{sgn}(z)(|z| - \lambda)_+ = \begin{cases} 0 & \text{if } |z| \leq \lambda, \\ \text{sgn}(z)(|z| - \eta) & \text{if } |z| > \lambda. 
\end{cases}
$$

Next, we take the limits that $\lim_{\alpha \to \infty} \frac{1}{\alpha} = 0$, $\lim_{\alpha \to \infty} \frac{3(\frac{\alpha^2}{2\alpha+1-1})^2}{2\alpha} = 3(\frac{4}{4})^{2/3}$, and $\lim_{\alpha \to \infty} \frac{\lambda^2 \left(\frac{3}{2|z|+1}\right)^2}{\sqrt{2\alpha+1-1}} = \lambda^2 (\frac{3}{|z|})^2$, $\frac{4}{4}$ in this limiting case, $\eta > \frac{1}{\alpha}$ is always met. Thus, the resulting estimator in Theorem 2-(i) degenerates to

$$
\hat{b} = S_{\frac{1}{2}}(z, \lambda) := \begin{cases} \text{sgn}(z) \frac{4|z|}{3} \cos^2 \left[ \frac{1}{4} \arccos \left( -\frac{\lambda}{\frac{4}{4}} \left(\frac{3}{|z|}\right)^2 \right) \right] & \text{if } |z| > 3(\frac{4}{4})^{2/3}, \\ 0 & \text{if } |z| \leq 3(\frac{4}{4})^{2/3}, 
\end{cases}
$$

which is well established by Xu et al. (2012). Obviously, the above thresholding function is not continuous at $|z| = 3(\frac{4}{4})^{2/3}$. However, the KEP penalty ($q = 1$) can make the resulting estimators have “unbiasedness,” “sparsity” and “continuity” by assuming $\eta \leq \frac{1}{\alpha}$. Moreover, the KEP penalty satisfies the nesting property.

The previous analysis implies that $\alpha > 0$ plays a role of “temperature” in statistical physics. When $\alpha \to \infty$, the thresholding function becomes discontinuous from continuous status, yielding a “phase transition” phenomenon.

In Figure 2-(b) we compare the thresholding rules for the hard ($\ell_0$), soft ($\ell_1$ or Lasso), half ($\ell_{1/2}$) and the KEP penalty. In Section 5 we explore the relationship between KEP and MCP as well as the relationship between the thresholding functions based on KEP and MCP.

### 4.2 The Coordinate Descent Algorithm

Given the training dataset $(y, X)$, we consider the following minimization problem

$$
J(b) = \frac{1}{2}\|y - Xb\|_2^2 + \sum_{j=1}^{p} \Psi(|b_j|; \eta, \alpha).
$$
Based on the discussion in the previous subsection, the KEP penalty with \( q = 1 \) is suitable for the coordinate descent algorithm. Particularly, the coordinate descent procedure of solving the above minimization problem is given Algorithm 1.

Assume \((y, X)\) lies on a compact set and no column of \(X\) is degenerate. It then follows from Theorem 4 of Mazumder et al. (2011) that the univariate maps \( b \mapsto J_1(b) \) are strictly convex and that the sequence \( \{b(t)\}; t = 1, 2, \cdots \) generated via Algorithm 1 converges to a (local) minimum of the objective function \( J(b) \).

It is worth pointing out that the iteratively reweighted \( \ell_1 \) method of Daubechies et al. (2010) is essentially equivalent to the multi-state LLA procedure of Zhang (2010b). The multi-state LLA for the minimization problem in (10) gives the following update

\[ b^{(t+1)} = S(z, w_0^{(t)}) = \arg\min_b \frac{1}{2}(b - z)^2 + w_0^{(t)}|b| \]

where \( w_0^{(t)} = \eta / \sqrt{1 + 2\alpha|b^{(t)}|} \), i.e., the derivative of \( \Psi(|b|; \eta, \alpha) \) at \( |b| = |b^{(t)}| \). Since \( J_1(b) \) is strictly convex when \( \eta\alpha < 1 \), it is also reasonable to let \( \eta\alpha < 1 \) when applying the multi-state LLA method.

Figure 2: (a) Threshold rules for KEP with \( \eta = \frac{\lambda}{2}(\sqrt{1 + 2\alpha + 1}) \) where \( \lambda \) is fixed and \( \alpha \) varies. (b) Threshold rules for the Hard (\( \ell_0 \)) (\( zI(|z| \geq \sqrt{2\lambda}) \)), Soft (\( \ell_1 \)) (\( \text{sgn}(z)(|z| - \lambda)_+ \)), Half (\( \ell_{1/2} \)) and KEP with \( \eta = \frac{\lambda}{2}(\sqrt{1 + 2\alpha + 1}) \).
Algorithm 1 The coordinate descent algorithm

Input: \{\mathbf{x}_i, y_i\}_{i=1}^n \text{ where each column of } \mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_n]^T \text{ is standardized to have mean 0 and length 1, a grid of increasing values } \Lambda = \{\lambda_1, \ldots, \lambda_L\}, \text{ a grid of decreasing values } \Gamma = \{\alpha_1, \ldots, \alpha_K\} \text{ where } \alpha_K \text{ indexes the Lasso penalty.}

Set \hat{b}_{\alpha_K, \lambda L + 1} = 0.

for each value of \( l \in \{L, L - 1, \ldots, 1\} \) do

Initialize \( \tilde{b} = \hat{b}_{\alpha_K, \lambda L + 1} \);

for each value of \( k \in \{K, K - 1, \ldots, 1\} \) do

Compute \( \eta_{lk} = \frac{\lambda_l}{2}(1 + \sqrt{1 + 2\alpha_k}) \)

if \( \eta_{lk} \alpha_k < 1 \) then

Cycle through the following one-at-a-time updates

\[ \tilde{b}_j = S_{\alpha_k} \left( \sum_{i=1}^n (y_i - z_i^j) x_{ij}, \eta_{lk} \right), \quad j = 1, \ldots, p \]

where \( z_i^j = \sum_{k \neq j} x_{ik} \tilde{b}_k \), until the updates converge to \( b^* \);

\( \hat{b}_{\alpha_K, \lambda l} \leftarrow b^* \).
end if

Increment \( k \);

end for

Increment \( l \);

Decrement \( l \);

Output: Return the two-dimensional solution \( \hat{b}_{\alpha, \lambda} \) for \((\alpha, \lambda) \in \Gamma \times \Lambda\).

For the sake of simplicity, we assume that \( z \geq 0 \). If \( z \geq \frac{\eta}{\sqrt{1 + 2\alpha|b^{(t)}|}} \) for any \( 1 \leq i \leq k \), we obtain \( b^{(t)} \geq 0 \). Using the fact that \( 1/\sqrt{1 + 2as} \) is convex in \( s \geq 0 \), we have

\[ b^{(t+1)} = z - \eta/\sqrt{1 + 2\alpha|b^{(t)}|} \leq z - \eta + \eta \alpha b^{(t)} \leq (z - \eta) \sum_{i=0}^t (\eta \alpha)^i + (\eta \alpha)^{(t+1)} b^{(1)}, \]

which implies that the multi-state LLA procedure converges to the minimum of \( J_1(b) (\eta \alpha < 1) \) at rate \( O((\eta \alpha)^t) \) in the worst case. This result agrees with that of Mazumder et al. (2011) about the univariate MCP penalized squares problem. As a result, the number of iterations required for the multi-state LLA procedure to converge with an \( \epsilon \) tolerance of the minimizer of \( J_1(b) \) is of order \( -\frac{\log(\epsilon)}{\log(\eta \alpha)} \). Thus, the multi-state LLA based coordinate-wise method is less efficient than Algorithm 1.

5. Relationships Between KEP and MCP

In Sections 3 and 4 we discuss the relationship of KEP with the \( \ell_1/2 \) and \( \ell_1 \) norms. In this section we explore the relationship between KEP and MCP.

Note that \( \chi^2 \) distance \( \frac{(w-\eta)^2}{w} \) between \( w \) and \( \eta \) is not symmetric. Thus, it is also interesting to consider the concave conjugate of \( \frac{(w-\eta)^2}{\eta} \). In this regard, the corresponding
concave conjugate is given by

$$\min_{w \geq 0} \left\{ ws + \frac{1}{2\alpha} \frac{(w - \eta)^2}{\eta} \right\}.$$

We denote the minimum as $\eta M(s)$ where

$$M(s) = \begin{cases} \frac{1}{2\alpha} & \text{if } s \geq \frac{1}{\alpha}, \\ s - \frac{\alpha s^2}{2} & \text{if } s < \frac{1}{\alpha}, \end{cases}$$

which is in fact the MCP function of Mazumder et al. (2011) when setting $\frac{1}{\alpha} = \lambda \gamma$ and $\eta = \lambda$ therein. This recovers an important connection between KEP and MCP; that is, both are based on the $\chi^2$-distance. Note that Zhang et al. (2013) constructed the MCP function using the concave conjugate of the squared Euclidean distance function. Their construction approach is essentially equivalent to the previous construction, because $|w - \eta|^2$ is the squared Euclidean distance and $\alpha \eta$ can be treated as a new single parameter.

Let us return to the KEP function $\Psi(s; \eta, \alpha)$ defined in (8) where $q = 1$ and $s = |b|$. Furthermore, we define $\Psi(s; \eta, \alpha) = \eta K(s)$ where $K(s) = \frac{1}{\alpha}(\sqrt{2\alpha s + 1} - 1)$. For a fixed $\alpha > 0$, it is easily verified that $M(s) \leq K(s) \leq s$, with equality only if $s = 0$ (also see Figure 3(a)). Additionally, $K(s)$ is infinitely differentiable on $[0, \infty)$. However, $M(s)$ is only first-order differentiable on $[0, \infty)$. The second-order derivative of $M(s)$ at $s = 1/\alpha$ does not exist (see Figure 3(b)). However, the convergence result of Mazumder et al. (2011) is built on the assumption that the second-order derivative exists (see Theorem 4 therein).

![Figure 3](image-url)

**Figure 3:** (a) The functions: KEP $K(|b|)$, MCP $M(|b|)$ and $\ell_1$-norm $|b|$ w.r.t. $b \in (-\infty, \infty)$. (b) The derivatives of $K(s)$, $M(s)$ and $s$ w.r.t $s \geq 0$. 

12
To obtain the thresholding function w.r.t. MCP, we also need to consider the two cases that \( \eta \alpha < 1 \) and \( \eta \alpha \geq 1 \). In the first case that \( \eta \alpha < 1 \), the thresholding function is given as

\[
S_{\alpha}(z, \eta) = \begin{cases} 
0 & \text{if } |z| \leq \eta \\
\text{sgn}(z) \frac{|z| - \eta}{1 - \alpha \eta} & \text{if } \eta < |z| \leq \frac{1}{\alpha} \\
z & \text{if } |z| > \frac{1}{\alpha}, 
\end{cases}
\]

which is identical to the one of Mazumder et al. (2011) when setting \( \frac{1}{\alpha} = \lambda \gamma \) and \( \eta = \lambda \). The resulting rule \( S_{\alpha}(z, \eta) \) is obviously continuous. However, \( S_{\alpha}(z, \eta) \) is not smooth for \( |z| > \eta \). Specifically, \( S_{\alpha}(z, \eta) \) is not differentiable at \( |z| = \frac{1}{\alpha} \). Recall that the thresholding function w.r.t. KEP is always smooth for \( |z| > \eta \) in the case that \( \eta \alpha < 1 \) (see Theorem 2). In Figure 4-(a), we illustrate comparison of MCP with the \( \ell_1 \)-norm and KEP. As we see, KEP can be treated as a trade-off of the \( \ell_1 \)-norm and MCP in unbiasedness and differentiability.

In the second case that \( \eta \alpha \geq 1 \), the thresholding function w.r.t. MCP is

\[
H_{\alpha}(z, \eta) = \begin{cases} 
0 & \text{if } |z| \leq \frac{1}{\alpha} \\
z & \text{if } |z| > \frac{1}{\alpha}, 
\end{cases}
\]

The derivation is based on some direct computations, so we omit it. Clearly, \( H_{\alpha}(z, \eta) \) is not continuous at \( |z| = \frac{1}{\alpha} \) in this case (see Figure 4-(b)). Especially, when \( \alpha \eta = 1 \), the thresholding function is also not continuous at \( |z| = \frac{1}{\alpha} \). However, it is obtained from Theorem 2 that the thresholding function w.r.t. KEP is still continuous when \( \alpha \eta = 1 \) (see Figure 4-(c)).

Figure 4: (a) Threshold rules for the Soft (Lasso) \( \text{sgn}(z)(|z| - \eta)_+ \), KEP and MCP under the condition \( \eta \alpha > 1 \); (b) KEP and MCP under the condition \( \eta \alpha > 1 \); (c) KEP and MCP under the condition \( \eta \alpha = 1 \).

We now take behaviours as \( \alpha \) approaches to limiting cases. First, we immediately have that \( \lim_{\alpha \to 0^+} M(s) = s \) and

\[
\lim_{\alpha \to 0^+} S_{\alpha}(z, \eta) = S(z, \eta) := \begin{cases} 
0 & \text{if } |z| \leq \eta \\
\text{sgn}(z)(|z| - \eta) & \text{if } |z| > \eta. 
\end{cases}
\]
Second, let $\eta = \frac{\lambda}{M(1)}$ where $\lambda > 0$ is a constant that independents on $\alpha$. We have that

$$
\lim_{\alpha \to 0^+} \frac{M(s)}{M(1)} = s \quad \text{and} \quad \lim_{\alpha \to \infty} \frac{M(s)}{M(1)} = \begin{cases} 
0 & \text{if } s = 0 \\
1 & \text{if } s \neq 0.
\end{cases}
$$

This shows that $\frac{M(s)}{M(1)}$ get the entire continuum from the $\ell_1$-norm to the $\ell_0$-norm, as varying from $\alpha \to 0^+$ to $\alpha \to \infty$. However, it is not tractable to derive the thresholding function corresponding to the penalty function $\frac{M(s)}{M(1)}$ because $M(1)$ as a function of $\alpha$ is not smooth. We feel that this would be an important reason that MCP does not hold the nesting property (Mazumder et al., 2011). In contrast, KEP can keep this property by setting $\eta = \frac{\lambda \alpha}{\sqrt{1+2\alpha-1}}$.

When we let $s = |b|^2$, $K(|b|^2)$ is convex in $|b|$ (see Section 3). In fact, $K(|b|^2)$ is used by Daubechies et al. (2010) in devising the iterative reweighted $\ell_2$ method (see Section 3). However, $M(|b|^2)$ is neither convex nor concave in $|b|$.

6. Asymptotic Properties

We discuss asymptotic properties of sparse estimators. Following the setup of Zou and Li (2008), we assume two conditions: (1) $y_i = x_i^T \mathbf{b}^* + \epsilon_i$ where $\epsilon_1, \ldots, \epsilon_n$ are iid errors with mean 0 and variance $\sigma^2$; (2) $X^T X/n \to C$ where $C$ is a positive definite matrix. Let $\mathcal{A} = \{ j : b^*_j \neq 0 \}$. Without loss of generality, we assume that $\mathcal{A} = \{ 1, 2, \ldots, r \}$ with $r < p$. Thus, partition $C$ as

$$
\begin{bmatrix}
C_{11} & C_{12} \\
C_{21} & C_{22}
\end{bmatrix},
$$

where $C_{11}$ is $r \times r$. Additionally, let $b_1^* = \{ b^*_j : j \in \mathcal{A} \}$ and $b_2^* = \{ b^*_j : j \notin \mathcal{A} \}$.

Recall that the iteratively reweighted $\ell_1$ method of Daubechies et al. (2010) can be regarded as a multi-stage LLA estimator (Zhang, 2010b). Specifically, we study the oracle property of the one-step LLA suggested by Zou and Li (2008). Based on the KEP with $q = 1$, we consider the following one-step sparse estimator:

$$
b_n^{(1)} = \arg\min_{\mathbf{b}} ||y - X\mathbf{b}||_2^2 + \sum_{j=1}^p \eta_n \sqrt{1+2\alpha_n|b_{0j}^*|} |b_j|,
$$

where $\mathbf{b}^{(0)} = (b_{10}^{(0)}, \ldots, b_{p0}^{(0)})^T$ is a root-$n$-consistent estimator to $\mathbf{b}^*$. The following theorem shows that this estimator has the oracle property. That is,

**Theorem 3** Let $b_n^{(1)} = \{ b_n^{(1)} : j \in \mathcal{A} \}$ and $A_n^{(1)} = \{ j : b_n^{(1)} \neq 0 \}$. Suppose that $n^{-3/4} \eta_n \to 0$, $\eta_n/\sqrt{n} \to \infty$, and $\alpha_n/\sqrt{n} \to c_1$ where $c_1 \in (0, \infty)$. Then $b_n^{(1)}$ satisfies the following properties:

1. **Consistency in variable selection:**

   $$
   \lim_{n \to \infty} P(A_n^{(1)} = \mathcal{A}) = 1.
   $$

14
(2) **Asymptotic normality:**

\[
\sqrt{n}(\hat{b}_{n1}^{(1)} - b_1^*) \xrightarrow{d} N(0, \sigma^2C_{11}^{-1}).
\]

As we mentioned earlier, \(2\alpha = 1/e^2\) and \(2\eta = \lambda/\epsilon\) in the iteratively reweighted \(\ell_1\) method of Daubechies et al. (2010). In this case, we make the assumption that \(\lambda_n/n^{1/4} \to \infty\), \(\lambda_n/\sqrt{n} \to 0\), \(\epsilon^2 = O(n^{-1/2})\) (or \(\epsilon^{-2}n^{-1/2} \to c_1 \in (0,\infty)\)). Then the resulting estimators have the oracle properties. Recall that Daubechies et al. (2010) set \(\epsilon^{(t+1)} = \min(\epsilon^{(t)}, r_k(b^{(t)})/p)\). This makes it sense that \(\epsilon^{-2}n^{-1/2} \to c_1\).

Let us return to the sparse estimator based on the penalty function \(\Psi(|b|; \eta, \alpha)\) itself. That is,

\[
\hat{b}_n = \text{argmin}_b \|y - Xb\|^2 + \frac{\eta_n}{\alpha_n} \sum_{j=1}^p \left[ \sqrt{1+2\alpha_n|b_j|} - 1 \right].
\]

**Theorem 4** Let \(\tilde{b}_n = \{\tilde{b}_{nj} : j \in \mathcal{A}\}\) and \(\tilde{A}_n = \{j : \tilde{b}_{nj} \neq 0\}\). Suppose that \(\eta_n/n^{3/4} \to 0\), \(\eta_n/\sqrt{n} \to \infty\), and \(\alpha_n/\sqrt{n} \to c_3 \in (0,\infty)\). Then \(\tilde{b}_n\) satisfies the following properties:

1. **Consistency in variable selection:**

\[
\lim_{n \to \infty} P(\tilde{A}_n = \mathcal{A}) = 1.
\]

2. **Asymptotic normality:**

\[
\sqrt{n}(\hat{b}_{n1} - b_1^*) \xrightarrow{d} N(0, \sigma^2C_{11}^{-1}).
\]

It is worth noting that we set \(\eta_n = \frac{\lambda \alpha_n}{\sqrt{2\alpha_n+1}}\) to define \(\Phi(|b|; \alpha)\) in (9). In this setting, if we assume that \(\lambda_n/n^{1/4} \to \infty\), \(\lambda_n/n^{1/2} \to 0\), and \(\alpha_n/\sqrt{n} \to c_3 \in (0,\infty)\), then we can obtain that \(\eta_n/n^{1/2} \to \infty\), \(\lambda_n/n^{3/4} \to 0\), and \(\alpha_n/n^{1/2} \to c_3 \in (0,\infty)\); that is, the conditions in Theorem 4 meet. Consider that the condition \(\alpha_n/\sqrt{n} \to c_3\) implies that \(\alpha_n \to \infty\). Hence, \(\frac{\sqrt{1+2\alpha_n|b_j|} - 1}{\sqrt{2\alpha_n+1}} \to |b_j|^{1/2}\). Thus, it follows from Theorem 4 that the \(\ell_{1/2}\) penalty can also result in an estimator with the oracle property under the conditions \(\lambda_n/n^{1/4} \to \infty\) and \(\lambda_n/n^{1/2} \to 0\).

On the other hand, \(\lim_{\alpha_n \to 0^+} \frac{\sqrt{1+2\alpha_n|b_j|} - 1}{\sqrt{2\alpha_n+1}} = \lim_{\alpha_n \to 0^+} \frac{\sqrt{1+2\alpha_n|b_j|} - 1}{\alpha_n} = |b_j|\). Thus, it is of great interest to explore the asymptotic property of the sparse estimator when \(\alpha_n \to 0\). In particular, we have the following theorem.

**Theorem 5** Assume \(\lim_{n \to \infty} \alpha_n = 0\). If \(\lim_{n \to \infty} \frac{\eta_n}{\sqrt{n}} = 2c_3 \in [0,\infty)\), then \(\tilde{b}_n \xrightarrow{p} b^*\). Furthermore, if \(\lim_{n \to \infty} \frac{\eta_n}{\sqrt{n}} = 0\), then \(\sqrt{n}(\tilde{b}_n - b^*) \xrightarrow{d} N(0, \sigma^2C^{-1})\).

Recall that the conditions \(\eta_n/\sqrt{n} \to \infty\) and \(\alpha_n/\sqrt{n} \to c_3 \in (0,\infty)\) in Theorem 4 imply that \(\eta_n \to \infty\) and \(\alpha_n \to \infty\). Consequently, \(\eta_n > \frac{1}{\alpha_n}\) when \(n\) is sufficiently large. It then follows from Theorem 2 that the thresholding rule is discontinuous under these conditions. This leads us to an interesting phenomenon; that is, the “oracle properties” do not always
accompany “continuity.” Our following empirical analysis shows that “continuity” is indeed very necessary for the coordinate descent algorithm to achieve good performance. However, the conditions for $\alpha_n$ and $\eta_n$ in Theorem 5 are always able to hold $\eta_n < \frac{1}{\alpha_n}$. For example, we take $\alpha_n = n - \frac{c_4 + 1}{2}$ and $\eta_n = n^{-\frac{1+\gamma}{2}}$ for any $0 \leq c_2 < c_1$, which holds $\eta_n \alpha_n < 1$ true.

In the previous discussion, $p$ is fixed. We also interested in the asymptotic properties when $r$ and $p$ rely on $n$. That is, $r = r_n$ and $p = p_n$ are allowed to grow as $n$ increases. In this case, we are concerned with notion of sign consistency of the estimate $\tilde{\mathbf{b}}_n$ with the true $\mathbf{b}^*$. In particular, it is said that $\tilde{\mathbf{b}}_n$ is equal to $\mathbf{b}^*$ in sign, which is written as $\tilde{\mathbf{b}}_n \overset{s}{=} \mathbf{b}^*$, if and only if $\text{sgn}(\tilde{\mathbf{b}}_n) = \text{sgn}(\mathbf{b}^*)$.

In order to address sign consistency, we consider a so-called strong irrepresentable condition (Zhao and Yu, 2006). Assume that $\mathbf{C}_{11}$ is invertible. The strong irrepresentable condition is that there a positive constant number $\gamma$ such that

$$\|\mathbf{C}_{21}\mathbf{C}_{11}^{-1}\text{sgn}(\mathbf{b}^*_j)\|_\infty \leq 1 - \gamma.$$ 

Following the setting of Zhao and Yu (2006), we further make the following assumptions on $\mathbf{C}_n$, $r_n$ and $\mathbf{b}^*$. Specifically, there exist $M_1 > 0$, $M_2 > 0$ and $M_3 > 0$ as well as $0 \leq c_4 < c_5 \leq 1$ such that

$$\frac{1}{n} \mathbf{x}_i^T \mathbf{x}_i \leq M_1 \quad \text{for } i = 1, \ldots, n,$$

$$\mathbf{a}^T \sum_{i=1}^{r} \mathbf{x}_i \mathbf{x}_i^T \mathbf{a} \geq M_2 \quad \text{for any } \|\mathbf{a}\|_2 = 1.\quad (12)$$

$$r_n = O(n^{c_4}),\quad (13)$$

$$n^{\frac{1-\gamma}{2}} \min_{j=1,\ldots,r} |b^*_j| \geq M_3.\quad (14)$$

The detailed interpretation for these conditions can be also found in Zhao and Yu (2006). Roughly speaking, Conditions 12 and 13 are alternative as the previous assumption on $\frac{1}{n} \mathbf{X}^T \mathbf{X}$ when $p_n$ and $r_n$ are fixed. Condition 14 implies that $r_n/n \to 0$, while Condition 15 shows that there exists a gap of size $n^{c_3}$ between the decay rate of $b^*_1$ and $n^{-\frac{1}{2}}$.

**Theorem 6** Assume $\eta_n < 1/\alpha_n$, $\lim_{n \to \infty} \alpha_n = 0$ and Conditions (12)-(15) are satisfied. Under the strong irrepresentable condition, if $p_n = O(n^{c_6})$ and $\eta_n \propto n^{-\frac{1+c_7}{2}}$ where $c_6 < c_7 < c_5 - c_4$, then

$$\Pr(\tilde{\mathbf{b}}_n \overset{s}{=} \mathbf{b}^*) \geq 1 - o(e^{-n^{c_6}}) \to 1 \quad \text{as} \quad n \to \infty.$$ 

This theorem is similar to Theorem 4 of Zhao and Yu (2006). Consider that $\tilde{\mathbf{b}}_n$ is the solution of the problem in (11). Thus,

$$0 \in (\mathbf{X} \tilde{\mathbf{b}}_n - \mathbf{y})^T \mathbf{x}_j + \frac{\eta_n}{\sqrt{1 + 2\alpha_n |\tilde{b}_{nj}|}} \partial |\tilde{b}_{nj}|, \quad j = 1, \ldots, p. \quad (16)$$

Under the condition $\alpha_n \to 0$, we have

$$0 \in \lim_{n \to \infty} \left\{ (\mathbf{X} \tilde{\mathbf{b}}_n - \mathbf{y})^T \mathbf{x}_j + \frac{\eta_n}{\sqrt{1 + 2\alpha_n |\tilde{b}_{nj}|}} \partial |\tilde{b}_{nj}| \right\} = \lim_{n \to \infty} \left\{ (\mathbf{X} \tilde{\mathbf{b}}_n - \mathbf{y})^T \mathbf{x}_j + \eta_n \partial |\tilde{b}_{nj}| \right\}$$

16
for \( j = 1, \ldots, p \). Since the minimizer of the conventional lasso exists and unique (denote \( \hat{b}_0 \)), the above relationship implies that \( \lim_{n \to \infty} \hat{b}_n = \lim_{n \to \infty} \hat{b}_0 \). Accordingly, based on Theorem 4 of Zhao and Yu (2006), we obtain the result in Theorem 6.

7. Experimental Analysis

In Section 7.1 we conduct a simulation analysis of KEP in sparsity modeling. This analysis is based on Theorem 5. In Sections 7.2 and 7.3 we evaluate the performance of the KEP-based coordinate descent algorithm given in Algorithm 1 in linear regression problems on simulated data and real data, respectively. We also conduct comparisons with the coordinate descent algorithms based on the \( \ell_1 \)-norm, \( \ell_{1/2} \)-norm and MCP, respectively.

7.1 Simulation Analysis

In this simulation analysis, we use a data model same to that in Mazumder et al. (2011). In particular, we generate data from the following model:

\[
y = x^T b + \sigma e
\]

where \( e \sim N(0, 1) \), and \( b \) is a \( p \)-dimensional vector with only 10 nonzero elements: \( b_i = b_{i+p/2} = 0.2i, \ i = 1, \ldots, 5 \). Each data point \( x \) is sampled from a multivariate normal distribution with zero mean and covariance matrix \( \Sigma = \{0.7^{i-j}\}_{1 \leq i,j \leq p} \). We choose \( \sigma \) such that the Signal-to-Noise Ratio (SNR), which is

\[
\text{SNR} = \frac{\sqrt{b^T \Sigma b}}{\sigma},
\]

is a specified value. Let \( \hat{b} \) denote the solution obtained from each algorithm. We use a standardized prediction error (SPE) and a feature selection error (FSE) as measure metrics. The SPE is defined as

\[
\text{SPE} = \frac{\sum_{i=1}^{m} (y_i - x_i^T \hat{b})^2}{ma^2}
\]

and the FSE is proportion of coefficients in \( \hat{b} \) which is wrongly set to zero or nonzero based on the true \( b \).

In this simulation we prespecify the values of hyperparameters \( \eta_n \) and \( \alpha_n \). Based on Theorem 5, we particularly set \( \eta_n = n^{1/4} \) and \( \alpha_n = n^{-1/2} \). Clearly, in this setting we always have \( \eta_n \alpha_n < 1 \). We also implement the MCP-based coordinate descent method with the same setting, and the lasso-based coordinate descent method with \( \lambda_n = \eta_n = n^{1/4} \). Our simulation analysis is performed on the training datasets with different sizes (\( n \)) and a fixed \( p \) (that is, \( p = 200 \)). But all the corresponding test datasets include \( m = 1000 \) samples.

We use different settings of \( n \) and SNR to generate the training datasets. Tables 1-4 report the results over 20 repeats for each setting. We can see that when \( n \) takes a smaller value, the performance of the KEP penalty is significantly better than that of MEP and of the \( \ell_1 \)-norm. As \( n \) takes a larger value, the performances of all the three penalties become better. Especially, the KEP and MCP are both competitive. Moreover, the three penalties can almost fully capture the model sparsity for a large \( n \). Thus, in this case, \( \eta_n = n^{1/4} \) and \( \alpha_n = n^{-1/2} \) are good choices for KEP and MCP.
Additionally, for a larger SNR, the performances of the MCP and \( \ell_1 \)-norm become worse. In contrast, the KEP penalty still works well. This shows that KEP is more robust than the MCP and \( \ell_1 \)-norm. Finally, Figure 5 depicts the convergence procedure of the coordinate descent algorithm. As we see, the algorithms with the KEP, MCP and \( \ell_1 \)-norm are efficient, because they get convergence after about 10 steps.

Table 1: Simulation results on datasets with \( p = 200 \) and \( SNR = 3.0 \) under the setting \( \eta_n = n^{1/4} \) and \( \alpha_n = n^{-1/2} \) for MCP and KEP, and \( \lambda_n = n^{1/4} \) for Lasso.

|       | n=100 | n=200 | n=400 | n=1600 | n=6400 | n=12800 |
|-------|-------|-------|-------|--------|--------|---------|
| SPE   | “FSE” | SPE   | “FSE” | SPE    | “FSE” | SPE     | “FSE” |
| KEP   | 1.979 | 0.020 | 1.210 | 0.010  | 1.098  | 0.000   | 1.065   | 0.000  | 1.045  | 0.000  | 1.031  | 0.000  |
| MCP   | 2.310 | 0.040 | 1.397 | 0.020  | 1.196  | 0.010   | 1.126   | 0.005  | 1.046  | 0.000  | 1.030  | 0.000  |
| Lasso | 2.826 | 0.020 | 1.789 | 0.010  | 1.528  | 0.010   | 1.389   | 0.005  | 1.331  | 0.000  | 1.267  | 0.000  |

Table 2: Simulation results on datasets with \( p = 200 \) and \( SNR = 6.0 \) under the setting \( \eta_n = n^{1/4} \) and \( \alpha_n = n^{-1/2} \) for MCP and KEP, and \( \lambda_n = n^{1/4} \) for Lasso.

|       | n=100 | n=200 | n=400 | n=1600 | n=6400 | n=12800 |
|-------|-------|-------|-------|--------|--------|---------|
| SPE   | “FSE” | SPE   | “FSE” | SPE    | “FSE” | SPE     | “FSE” |
| KEP   | 2.102 | 0.010 | 1.679 | 0.000  | 1.311  | 0.005   | 1.191   | 0.000  | 1.090  | 0.000  | 1.061  | 0.000  |
| MCP   | 3.779 | 0.020 | 2.594 | 0.020  | 2.415  | 0.005   | 2.123   | 0.010  | 1.084  | 0.000  | 1.062  | 0.000  |
| Lasso | 6.042 | 0.010 | 4.863 | 0.010  | 3.013  | 0.010   | 2.506   | 0.010  | 2.231  | 0.005  | 1.948  | 0.000  |

Table 3: Simulation results on datasets with \( p = 200 \) and \( SNR = 9.0 \) under the setting \( \eta_n = n^{1/4} \) and \( \alpha_n = n^{-1/2} \) for MCP and KEP, and \( \lambda_n = n^{1/4} \) for Lasso.

|       | n=100 | n=200 | n=400 | n=1600 | n=6400 | n=12800 |
|-------|-------|-------|-------|--------|--------|---------|
| SPE   | “FSE” | SPE   | “FSE” | SPE    | “FSE” | SPE     | “FSE” |
| KEP   | 2.892 | 0.010 | 2.379 | 0.020  | 1.675  | 0.005   | 1.327   | 0.000  | 1.201  | 0.000  | 1.119  | 0.000  |
| MCP   | 6.564 | 0.025 | 5.123 | 0.030  | 2.669  | 0.001   | 1.405   | 0.005  | 1.186  | 0.000  | 1.116  | 0.000  |
| Lasso | 9.472 | 0.050 | 8.479 | 0.010  | 6.404  | 0.005   | 4.444   | 0.010  | 3.760  | 0.010  | 3.143  | 0.000  |
Table 4: Simulation results on datasets with $p = 200$ and $SNR = 12$ under the setting $\eta_n = n^{1/4}$ and $\alpha_n = n^{-1/2}$ for MCP and KEP, and $\lambda_n = n^{1/4}$ for Lasso.

| $n$ | SPE “FSE” | SPE “FSE” | SPE “FSE” | SPE “FSE” | SPE “FSE” | SPE “FSE” |
|-----|-----------|-----------|-----------|-----------|-----------|-----------|
| 100 | 3.972     | 0.010     | 7.309     | 0.020     | 6.341     | 0.020     |
| 200 | 3.381     | 0.010     | 2.138     | 0.005     | 1.898     | 0.005     |
| 400 | 3.138     | 0.005     | 1.748     | 0.000     | 1.303     | 0.000     |
| 1600| 1.334     | 0.000     | 1.233     | 0.000     | 1.206     | 0.000     |
| 6400| 1.206     | 0.000     | 1.206     | 0.000     | 1.206     | 0.000     |
| 12800| 1.206 | 0.000     | 1.206     | 0.000     | 1.206     | 0.000     |

Figure 5: Convergence procedures for coordinate descent iterations with the $\ell_1$-norm, KEP and MCP over the datasets with $p = 200$ and $n = 12800$. Here $\eta_n = n^{1/4}$ and $\alpha_n = n^{-1/2}$ for KEP and MCP, and $\lambda_n = n^{1/4}$ for Lasso.

7.2 Linear Regression on Simulated data

In this paper our principal focus has been to provide KEP with which the iteratively reweighted $\ell_q$ method of Daubechies et al. (2010) can be derived. We particularly study the case that $q = 1$, because the corresponding KEP is nonconvex and has strong ability in sparsity modeling. Moreover, we have also proposed the coordinate descent (CD) algorithm based on KEP in Section 4.2. Thus, it is interesting to conduct empirical comparison of our CD with the iterative reweighted (IR) method (see Section 4.2). For description simplicity, we denote them by KEP-CD and KEP-IR.

We now conduct comparisons of the methods based on KEP with the Lasso, the adaptive Lasso (AdaLasso) (Zou, 2006), the method based on the $\ell_{1/2}$-norm penalty, and the SparseNet based on the MCP penalty (Mazumder et al., 2011). All these methods are solved by using the coordinate descent algorithm. Moreover, the hyperparameters ($\eta$, $\alpha$ or $\lambda$) involved in all the methods are selected via cross validation. The experiments are also implemented over the previous simulation data model. In particular, we generate 12 datasets based on $SNR = 3.0, 6.0, 9.0, 12$ and $p = 50, 200, 500$ to implement the simulation.
Our experimental analysis is performed on the training datasets of \( n = 100 \) samples and the corresponding test datasets of \( m = 1000 \) samples.

Tables 5-7 report the average results over 20 repeats. From them, we can see that the KEP penalty function is competitive with the MCP penalty, \( \ell_{1/2} \)-norm and \( \ell_1 \)-norm in both prediction accuracy and feature selection accuracy. In most cases, KEP can lead to more accurate prediction results than the rest three penalty functions. Additionally, we see that nonconvex penalization outperforms convex penalization in sparsity, and almost outperforms convex penalization in regression accuracy. Thus, nonconvex penalization is an effective approach for high-dimensional data modeling.

Comparing KEP-CD and KEP-IR, we can see that their performances are competitive. However, KEP-CD is computationally more efficient than KEP-IR. KEP-CD usually takes about 10 iterations to get convergence (see Figure 6 for illustration). As discussed in Section 4.2, KEP-IR uses two nested loops to iterate over all the elements of \( b \). In the inner loop, KEP-IR uses a LLA of the original problem in each coordinate, which needs to take several iterations to get convergence. In contrast, KEP-CD just takes one step to obtain the exact solution of the original problem in each coordinate.

From the experimental results, we see that the \( \ell_{1/2} \)-norm and MCP are slightly stronger than KEP in sparsity ability. This makes sense because KEP with \( q = 1 \) bridges the \( \ell_1 \)-norm and the \( \ell_{1/2} \) norm, and we have \( M(|b|) \leq K(|b|) \leq |b| \) (see Section 5). However, the \( \ell_{1/2} \)-norm indeed suffers from the numerical instable problem. It is seen from Tables 5-7 that relative to the other methods, the prediction performance with the \( \ell_{1/2} \)-norm becomes worse as \( p \) grows. As for the MCP-based method, Mazumder et al. (2011) showed that a recalibration strategy can improve performance.

|               | SNR=3.0 | SNR=6.0 | SNR=9.0 | SNR=12.0 |
|---------------|---------|---------|---------|-----------|
|               | SPE     | “FSE”   | SPE     | “FSE”    | SPE     | “FSE”   |
| KEP-CD        | 1.245   | 0.071   | 1.215   | 0.069     | 1.169   | 0.048   | 1.147   | 0.014     |
| KEP-IR        | 1.264   | 0.088   | 1.243   | 0.056     | 1.199   | 0.030   | 1.148   | 0.032     |
| MCP           | 1.269   | 0.040   | 1.237   | 0.053     | 1.203   | 0.028   | 1.169   | 0.025     |
| \( \ell_{1/2} \) | 1.276   | 0.085   | 1.255   | 0.047     | 1.196   | 0.032   | 1.252   | 0.012     |
| AdaLasso      | 1.275   | 0.096   | 1.291   | 0.123     | 1.215   | 0.058   | 1.175   | 0.030     |
| Lasso         | 1.361   | 0.166   | 1.337   | 0.160     | 1.253   | 0.130   | 1.220   | 0.139     |

Figure 6 depicts the convergence procedure of the coordinate descent iterations with KEP, \( \ell_{1/2} \) and MCP, respectively. This figure shows that the coordinate descent algorithm is appropriate for nonconvex penalty functions. Furthermore, it is seen that the convergence speedups with KEP and \( \ell_{1/2} \) are competitive, but they faster than MCP. Specifically, to achieve convergence, MCP usually needs to take about 50 steps while both KEP and \( \ell_{1/2} \) need to take about 10 steps. In summary, the KEP function with \( q = 1 \) is a good choice in nonconvex penalization and the KEP-CD method is an efficient approach for solving the corresponding nonconvex optimization problem.
Table 6: Simulation results on dataset with $p = 200$ and $n = 100$

|                  | SNR=3.0 |     | SNR=6.0 |     | SNR=9.0 |     | SNR=12.0 |     |
|------------------|---------|-----|---------|-----|---------|-----|----------|-----|
|                  | SPE     | "FSE" | SPE     | "FSE" | SPE     | "FSE" | SPE     | "FSE" |
| KEP-CD           | 1.248   | 0.038 | 1.224   | 0.024 | 1.197   | 0.018 | 1.179   | 0.009 |
| KEP-IR           | 1.240   | 0.035 | 1.225   | 0.024 | 1.203   | 0.009 | 1.181   | 0.007 |
| MCP              | 1.246   | 0.020 | 1.235   | 0.040 | 1.219   | 0.015 | 1.196   | 0.015 |
| $\ell_{1/2}$-CD  | 1.296   | 0.021 | 1.253   | 0.015 | 1.233   | 0.016 | 1.215   | 0.011 |
| AdaLasso         | 1.347   | 0.041 | 1.274   | 0.035 | 1.261   | 0.020 | 1.203   | 0.013 |
| Lasso            | 1.356   | 0.078 | 1.368   | 0.069 | 1.280   | 0.072 | 1.300   | 0.063 |

Table 7: Simulation results on dataset with $p = 500$ and $n = 100$

|                  | SNR=3.0 |     | SNR=6.0 |     | SNR=9.0 |     | SNR=12.0 |     |
|------------------|---------|-----|---------|-----|---------|-----|----------|-----|
|                  | SPE     | "FSE" | SPE     | "FSE" | SPE     | "FSE" | SPE     | "FSE" |
| KEP-CD           | 1.327   | 0.023 | 1.273   | 0.013 | 1.247   | 0.002 | 1.215   | 0.009 |
| KEP-IR           | 1.319   | 0.014 | 1.292   | 0.008 | 1.242   | 0.003 | 1.225   | 0.009 |
| MCP              | 1.338   | 0.016 | 1.284   | 0.012 | 1.260   | 0.014 | 1.195   | 0.010 |
| $\ell_{1/2}$-CD  | 1.383   | 0.051 | 1.360   | 0.003 | 1.272   | 0.002 | 1.251   | 0.003 |
| AdaLasso         | 1.360   | 0.029 | 1.310   | 0.021 | 1.285   | 0.011 | 1.295   | 0.019 |
| Lasso            | 1.356   | 0.040 | 1.404   | 0.028 | 1.434   | 0.034 | 1.372   | 0.043 |

7.3 Linear Regression on Real Datasets

In this experiment, we apply our methods to real regression problems on the cookie (Near-Infrared (NIR) Spectroscopy of Biscuit Doughs) dataset (Osborne et al., 1984). We follow the setup of the original dataset: 39 instances for training and 31 instances for the test. Note that the original dataset consists of 72 instances, but two instances were suggested by Brown et al. (2001) to be excluded as outliers. We train a model for each response among the four responses ("fat," "sucrose," "dry flour" and "water") in the experiment.

We report the root mean square error (RMSE) on the test set and the model sparseness (proportion of zero coefficients in $\hat{b}$) in Table 8. We can see that all the methods are competitive in prediction accuracy. But in most cases the nonconvex methods have strong ability in feature selection. We can also see that performance of the method with KEP is stable, while it is instable for the method with the $\ell_{1/2}$-norm penalty. This agrees with the theoretical analysis in Section 4.1.
8. Conclusion

In this paper we have studied sparse penalized learning problems. We have focused on the iteratively reweighted $\ell_q$ method of Daubechies et al. (2010) and developed the kinetic energy plus (KEP) penalty function. In particular, we have illustrated that KEP can be defined as a concave conjugate of the nonnegative of a $\chi^2$-distance function. We have thus rederived the iteratively reweighted $\ell_q$ method of Daubechies et al. (2010).

Under the setting of $q = 1$, we have derived the thresholding operator for the KEP penalized univariate least-squares problem. Accordingly, we have devised a coordinate descent algorithm. We have validated that this algorithm is effective and feasible in theoretically and empirically. Additionally, we have investigated the relationship of KEP with the $\ell_1$ and
Table 8: Root Mean Square Error (RMSE) and Model Sparseness “SPR” on real datasets NIR where $n = 39$ and $p = 700$.

|       | fat        | sucrose    | flour      | water      |
|-------|------------|------------|------------|------------|
|       | RMSE “SPR”| RMSE “SPR” | RMSE “SPR”| RMSE “SPR” |
| KEP-CD| 0.4478     | 0.9914     | 1.1174     | 0.9871     |
| KEP-IR| 0.5172     | 0.9829     | 1.0677     | 0.9929     |
| MCP   | 0.5170     | 0.9871     | 1.2163     | 0.9929     |
| $\ell_{1/2}$ | 0.6767 | 0.9700 | 1.6353 | 0.9671 |
| AdaLasso | 0.5331 | 0.9843 | 0.8780 | 0.9929 |
| Lasso | 0.8177 | 0.9786 | 1.3601 | 0.9557 |

$\ell_{1/2}$ penalties. That is, the limiting cases are the $\ell_1$ and $\ell_{1/2}$ penalties. Moreover, we have uncovered an interesting connection between the KEP and MCP functions. Specifically, the MCP function can be also defined as the concave conjugate of the $\chi^2$-distance function. The difference between both them is due to asymmetricity of the $\chi^2$-distance function.

Appendix A. The Proof of Theorem 2

**Proof** The first-order derivative of (10) w.r.t. $b$ is

$$\text{sgn}(b) \left( |b| + \eta(2\alpha|b| + 1)^{-\frac{1}{2}} \right) - z.$$  

Let $g(|b|) = |b| + \eta(2\alpha|b| + 1)^{-\frac{1}{2}}$. It is clear that $|z| < \min_{b \neq 0} \{g(|b|)\}$, the resulting estimator is 0; namely, $\hat{b} = 0$. We now check the minimum value of $g(s) = s + \eta(2\alpha s + 1)^{-\frac{1}{2}}$ for $s \geq 0$.

Taking the first-order derivative of $g(s)$ w.r.t. $s$, we have

$$g'(s) = 1 - \eta\alpha(2\alpha s + 1)^{-\frac{3}{2}}.$$  

Thus, if $\eta \leq \frac{1}{\alpha}$, $g(s)$ attains its minimum value $\eta$ at $s^* = 0$. Otherwise, $g(s)$ attains its minimum value when $s^* = \frac{1}{2\alpha}[\eta\alpha^{2/3} - 1]$; that is,

$$g(s^*) = \frac{3}{2\alpha}(\alpha\eta)^{\frac{2}{3}} - \frac{1}{2\alpha} \left( > \frac{1}{\alpha} \right).$$  

First, we consider the case that $\eta > \frac{1}{\alpha}$. In this case, the resulting estimator is 0 when $|z| \leq \frac{3}{2\alpha}(\alpha\eta)^{\frac{2}{3}} - \frac{1}{2\alpha}$. If $z > \frac{3}{2\alpha}(\alpha\eta)^{\frac{2}{3}} - \frac{1}{2\alpha}$, then the resulting estimator should be the positive root of the equation $b + \eta(2\alpha b + 1)^{-\frac{1}{2}} - z = 0$ in $b$. Let $u = (2\alpha b + 1)^{\frac{1}{2}}$. We denote

$$h(u) = u^3 - (2\alpha z + 1)u + 2\eta\alpha.$$
Since $h((2\alpha z + 1)^{1/2}) = 2\eta \alpha > 0$, $h(((2\alpha z + 1)/3)^{1/2}) = -2((2\alpha z + 1)/3)^{3/2} + 2\eta \alpha < -2\alpha \eta + 2\eta \alpha = 0$, $h(0) = 2\eta \alpha > 0$, and

$$h\left(\frac{2}{\sqrt{3}}(2\alpha z+1)^{1/2}\right) = -\frac{2}{3\sqrt{3}}(2\alpha z+1)^{3/2} + 2\eta \alpha < 0,$$

we have that cubic equation $h(u) = 0$ has three real roots. Moreover, the largest root (denoted $u_0$) is in $(((2\alpha z + 1)/3)^{1/2}, (2\alpha z + 1)^{1/2})$, which implies that $1 \leq u_0 \leq (2\alpha z + 1)^{1/2}$. As a result, the resulting estimator is $\hat{b} = \frac{u_0-1}{2\alpha} \leq z$. Based on the trigonometric (and hyperbolic) method (Nickalls, 1993), $u_0$ is specified by

$$u_0 = 2\sqrt{\frac{2\alpha z+1}{3}} \cos \left[\frac{1}{3} \arccos \left(-\alpha \eta \left(\frac{3}{2\alpha z+1}\right)^{3/2}\right)\right].$$

Similarly, if $z < -\frac{3}{2\alpha}(\alpha \eta)^{2} + \frac{1}{2\alpha}$, we can derive the analytic expression of the resulting estimator, which is given in (i). Note that in this case of $\eta > \frac{1}{\alpha}$, the second largest root $u_1 \in (0, ((2\alpha z + 1)/3)^{1/2})$. This implies that $u_1 > 1$ is possible. If so, however, the second root should correspond to the maximum value of the original problem. Therefore, in this case, we still can prove the existence and uniqueness of the estimator $\hat{b}$.

Next, we consider the case that $\eta \leq \frac{1}{\alpha}$. In this case, the resulting estimator is 0 when $|z| \leq \eta$. If $z > \eta$, then the resulting estimator should be the positive root of the equation $b + \eta (2\alpha b + 1)^{-\frac{1}{2}} - z = 0$ in $b$. Accordingly, we study the roots of $h(u) = 0$. Note that

$$\Delta = -4(2\alpha z + 1)^{3} + 27t^{2} \leq -4(t+1)^{3} + 27t^{2} = -(4t^{3} - 15t^{2} + 12t + 4) = -(t-2)^{2}(4t+1) \leq 0$$

where $t = 2\alpha \eta \geq 0$. Thus, cubic equation $h(u) = 0$ has three real roots. In fact, we further have $h((2\alpha z + 1)^{1/2}) = 2\eta \alpha > 0$, $h(1) = -2\alpha z + 2\alpha \eta < 0$, $h(0) = 2\eta \alpha > 0$, and

$$h\left(-\left(1 + (2\alpha z + 1)^{1/2}\right)\right) = -[1 + (2\alpha z + 1)^{1/2}]^{3} + (2\alpha z + 1)[1 + (2\alpha z + 1)^{1/2}] + 2\eta \alpha \leq -[3(2\alpha z+1)^{1/2} + 4\alpha z + 1] < 0.$$

This implies that $h(u) = 0$ has one and only one root greater than 1, which belongs to $(1, (2\alpha z + 1)^{1/2})$. Consequently, the resulting estimator $0 < \hat{b} < z$ when $z > \eta$. Similarly, we can obtain that $z < \hat{b} < 0$ when $z < -\eta$. Using the trigonometric theory, we can also obtain an analytic formula for this root which is given the second part of the theorem. As stated in Fan and Li (2001), a sufficient and necessary condition for “continuity” is the minimum of $|b| + \Psi'(|b|)$ is attained at 0. This implies that that the resulting estimator is continuous. In fact, the continuity of the resulting estimator can also be obtained from Lemma 7-(ii) which is given below.

**Lemma 7** Given a $t \geq 0$, we define

$$\varphi(u) = 2\sqrt{\frac{u+1}{3}} \cos \left[\frac{1}{3} \arccos \left(-t \left(\frac{u+1}{3}\right)^{-\frac{3}{2}}\right)\right]$$

for $u \geq 3t^{2} - 1$. Then,
(i) \( \varphi(u) \) and \( \varphi^2(u) \) are strictly increasing on \([3t^{2/3} - 1, \infty)\).

(ii) If \( 0 \leq t \leq 1 \), then \( \varphi(2t) \equiv 1 \).

(iii) If \( 0 \leq t < 1 \), then \( \varphi^2(u) \) is Lipschitz continuous on \([2t, \infty)\).

**Proof** The first-order derivative of \( \varphi(u) \) w.r.t. \( u \) is

\[
\varphi'(u) = \frac{1}{3} \left( \frac{u+1}{3} \right)^{-1/2} \cos \left[ \frac{1}{3} \arccos \left( -t \left( \frac{u+1}{3} \right)^{-3} \right) \right] + \frac{\frac{4}{3} \left( \frac{u+1}{3} \right)^{-2}}{\sqrt{1 - \left( \frac{u+1}{3} \right)^{-3}}} \sin \left[ \frac{1}{3} \arccos \left( -t \left( \frac{u+1}{3} \right)^{-3} \right) \right],
\]

which is greater than 0. Additionally, \( \frac{d\varphi^2(u)}{du} = 2\varphi(u)\varphi'(u) > 0 \). Thus, \( \varphi(u) \) and \( \varphi^2(u) \) are strictly increasing.

For \( 0 \leq t \leq 1 \), it is directly verified that \( 1 \leq \sqrt{\frac{3}{2t+1}} \leq \sqrt{3} \) and

\[-1 \leq -t \left( \frac{3}{2t+1} \right)^{3/2} \leq 0.\]

We thus can assume that \( \cos(\theta) = \frac{1}{2} \sqrt{\frac{3}{2t+1}} \) where \( \theta \in [\pi/6, \pi/3] \). Since

\[\cos(3\theta) = 4 \cos^3(\theta) - 3 \cos(\theta) = -t \left( \frac{3}{2t+1} \right)^{3/2}\]

and \( 3\theta \in [\pi/2, \pi] \), we have

\[\frac{1}{2} \sqrt{\frac{3}{2t+1}} = \cos(\theta) = \cos \left[ \frac{1}{3} \arccos \left( -t \left( \frac{3}{2t+1} \right)^{3/2} \right) \right].\]

Finally, we have

\[
\frac{d\varphi^2(u)}{du} = \frac{2}{3} \left( 1 + \cos \left[ \frac{2}{3} \arccos \left( -t \left( \frac{u+1}{3} \right)^{-3} \right) \right] \right) + \frac{\frac{2}{3}}{\sqrt{1 - \left( \frac{u+1}{3} \right)^{-3}}} \sin \left[ \frac{2}{3} \arccos \left( -t \left( \frac{u+1}{3} \right)^{-3} \right) \right].
\]

Note that \( \frac{1}{3} \left( \frac{u+1}{3} \right)^3 - 1 \geq \frac{1}{3} \left( \frac{2t+1}{3} \right)^3 - 1 \) for \( u \in [2t, \infty) \). Moreover, \( \frac{1}{3} \left( \frac{2t+1}{3} \right)^3 - 1 \) is strictly decreasing for \( 0 \leq t < 1 \). Thus, we have

\[
\left| \frac{d\varphi^2(u)}{du} \right| \leq \frac{4}{3} + \frac{2}{3} \frac{1}{\sqrt{1 - \left( \frac{2t+1}{3} \right)^3}}.
\]
Appendix B. The Proof of Theorem 3

Let $b_n^{(1)} = b^* + \frac{u}{\sqrt{n}}$ and

$$\hat{u} = \arg\min_u \left\{ G_n(u) := \left\| y - X (b^* + \frac{u}{\sqrt{n}}) \right\|^2 + \sum_{j=1}^p \omega_j^{(0)} |b_j^* + \frac{u_j}{\sqrt{n}}| \right\},$$

where $\omega_j^{(0)} = \frac{\eta_j}{\sqrt{1+2\alpha_n|b_j^{(0)}|}}$. Consider that

$$G_n(u) - G_n(0) = u^T \left( \frac{1}{n} X^T X \right) u - 2 \frac{\varepsilon^T X}{\sqrt{n}} u + \sum_{j=1}^p \omega_j^{(0)} \left\{ |b_j^* + \frac{u_j}{\sqrt{n}}| - |b_j^*| \right\}.$$

We know that $X^T X/n \to C$ and $\frac{X^T \varepsilon}{\sqrt{n}} \to_d N(0, \sigma^2 C)$. We thus only consider the third term of the right-hand side of the above equation. If $b_j^* = 0$, then $\sqrt{n}(|b_j^* + \frac{u_j}{\sqrt{n}}| - |b_j^*|) = |u_j|$. And since $\sqrt{n}\omega_j^{(0)} = O_p(1)$, we have $\alpha_n |b_j^{(0)}| = (\alpha_n/\sqrt{n})\sqrt{n}|b_j^{(0)}| = O_p(1)$. Hence,

$$\frac{\omega_j^{(0)}}{\sqrt{n}} \to_p \frac{\eta_j}{n^{3/4}} \to \infty$$

due to $\alpha_n/\sqrt{n} \to c_1 \in (0, \infty)$. If $b_j^* \neq 0$, then

$$\frac{\omega_j^{(0)}}{\sqrt{n}} = \frac{\eta_j}{n^{3/4}} \frac{1}{\sqrt{1+2\alpha_n|b_j^{(0)}|}} \to_p \frac{\eta_j}{n^{3/4}} \to 0$$

and $\sqrt{n}(|b_j^* + \frac{u_j}{\sqrt{n}}| - |b_j^*|) \to u_j \text{sgn}(b_j^*)$. Thus, $\frac{\omega_j^{(0)}}{\sqrt{n}} \sqrt{n}(|b_j^* + \frac{u_j}{\sqrt{n}}| - |b_j^*|) \to_p 0$. The remaining parts of the proof can be immediately obtained via some slight modifications to that in Zou (2006) or Zou and Li (2008). We here omit them.

Appendix C. The Proof of Theorem 4

Let $b_n = b^* + \frac{u}{\sqrt{n}}$ and

$$\hat{u} = \arg\min_u \left\{ G_n(u) := \left\| y - X (b^* + \frac{u}{\sqrt{n}}) \right\|^2 + \frac{\eta_n}{\alpha_n} \sum_{j=1}^p \left[ \sqrt{1+2\alpha_n|b_j^* + \frac{u_j}{\sqrt{n}}|} - 1 \right] \right\}.$$

Consider that

$$G_n(u) - G_n(0) = u^T (X^T X/n) u - 2 \frac{\varepsilon^T X}{\sqrt{n}} u + \frac{\eta_n}{\alpha_n} \sum_{j=1}^p \left[ \sqrt{1+2\alpha_n|b_j^* + \frac{u_j}{\sqrt{n}}|} - \sqrt{1+2\alpha_n|b_j^*|} \right].$$

Clearly, $X^T X/n \to C$ and $\frac{X^T \varepsilon}{\sqrt{n}} \to_d z \overset{d}{=} N(0, \sigma^2 C)$. We now discuss the limiting behavior of the third term of the right-hand side. We partition $z$ into $z^T = (z_1^T, z_2^T)$ where $z_1 = \{ z_j : j \in A \}$ and $z_2 = \{ z_j : j \notin A \}$. 

26
First, consider the case that $b^*_j = 0$. In this case, we have

$$\frac{\eta_n}{\alpha_n} \left[ \sqrt{1+2|u_j|} \frac{\alpha_n}{\sqrt{n}} - 1 \right] = \frac{\eta_n}{\sqrt{n}} \frac{2|u_j|}{\sqrt{1+|u_j|^{2\alpha_n}/\sqrt{n} + 1}} \to +\infty.$$ 

Second, we assume that $b^*_j \neq 0$. Subsequently,

$$\frac{\eta_n}{\alpha_n} \left[ \sqrt{1+2\alpha_n b^*_j + \frac{u_j}{\sqrt{n}}} - \sqrt{1+2\alpha_n b^*_j} \right]$$

$$\to \frac{\eta_n}{\alpha_n} \left[ \sqrt{1+2\alpha_n (b^*_j + \frac{u_j}{\sqrt{n}}) \text{sgn}(b^*_j) - \sqrt{1+2\alpha_n b^*_j \text{sgn}(b^*_j)}} \right]$$

$$= \frac{\eta_n}{\alpha_n} \frac{2u_j \text{sgn}(b^*_j)}{\sqrt{[1+2\alpha_n b^*_j \text{sgn}(b^*_j)]/\sqrt{n}} + \sqrt{[1+2\alpha_n (b^*_j + \frac{u_j}{\sqrt{n}}) \text{sgn}(b^*_j)]/\sqrt{n}}}$$

$$\to 0.$$ 

By Slutsky’s theorem, we have

$$G_n(u) - G_n(0) \xrightarrow{d} \left\{ \begin{array}{ll}
\mathbf{u}_T \mathbf{C}_{11} \mathbf{u}_1 - 2\mathbf{u}_1^T \mathbf{z}_1 & \text{if } u_j = 0 \forall j \notin A,
\infty & \text{otherwise.}
\end{array} \right.$$ 

This implies that $G_n(u) - G_n(0)$ converges in distribution to a convex function, whose unique minimum is $(\mathbf{C}_{11}^{-1} \mathbf{z}_1, 0)^T$. It then follows from epiconvergence (Geyer, 1994, Knight and Fu, 2000) that

$$\hat{\mathbf{u}}_1 \xrightarrow{d} \mathbf{C}_{11}^{-1} \mathbf{z}_1 \text{ and } \hat{\mathbf{u}}_2 \xrightarrow{d} \mathbf{0}. \quad (16)$$

This proves asymptotic normality due to $\mathbf{z}_1 \xrightarrow{d} N(\mathbf{0}, \sigma^2 \mathbf{C}_{11})$.

Recall that $\tilde{b}_{nj} \xrightarrow{p} b^*_j$ for any $j \in A$, which implies that $\Pr(j \in A_n) \to 1$. Thus, for consistency in Part (1), it suffices to obtain $\Pr(l \in A_n) \to 0$ for any $l \notin A$. For such an event “$l \in A_n$,” it follows from the KKT optimality conditions that $2\mathbf{x}_l^T (\mathbf{y} - \mathbf{X} \tilde{b}_n) = \frac{\eta_n}{\sqrt{1+2\alpha_n |b_{nl}|}}$ where $\mathbf{x}_l$ is the $l$th column of $\mathbf{X}$. Note that

$$\frac{2\mathbf{x}_l^T (\mathbf{y} - \mathbf{X} \tilde{b}_n)}{\sqrt{n}} = 2\mathbf{x}_l^T \mathbf{X} \sqrt{n} (\mathbf{b}^* - \tilde{b}_n) + 2\mathbf{x}_l^T \epsilon$$

and $\lim_{n \to \infty} \frac{\eta_n}{\sqrt{n} \sqrt{1+2\alpha_n |b_{nl}|}} \to \infty$ due to $\sqrt{n} |\tilde{b}_{nj}| \xrightarrow{p} 0$ by (16) and Slutsky’s theorem. Accordingly, we have

$$\Pr(l \in A_n) \leq \Pr \left[ \frac{2\mathbf{x}_l^T (\mathbf{y} - \mathbf{X} \tilde{b}_n)}{\sqrt{n}} = \frac{\eta_n}{\sqrt{1+2\alpha_n |b_{nl}|}} \right] \to 0.$$ 

**Appendix D. The Proof of Theorem 5**

As for the proof of Theorem 5, we consider the case that $\lim \alpha_n = 0$. In this case, we have

$$\lim_{n \to \infty} \sqrt{1 + (2\alpha_n/\sqrt{n})} - 1 = 1 \text{ and } \lim_{n \to \infty} \frac{\sqrt{1 + 2\alpha_n} - 1}{\alpha_n} = 1.$$ 

27
Assume that \( \lim_{n \to \infty} \eta_n / \sqrt{n} = 2c_3 \in [0, \infty] \). Then

\[
\eta_n \frac{\sqrt{1 + 2|u_j| \alpha_n / \sqrt{n} - 1}}{|u_j| \alpha_n / \sqrt{n}} = |u_j| \eta_n \frac{\sqrt{1 + 2|u_j| \alpha_n / \sqrt{n} - 1}}{|u_j| \alpha_n / \sqrt{n}} \to 2c_3 |u_j|
\]

when \( u_j \neq 0 \). If \( b_j^* \neq 0 \), then

\[
\eta_n \sqrt{1 + 2(\alpha_n |b_j^*| + \frac{u_j}{\sqrt{n}}) - 1} = \eta_n \frac{\sqrt{1 + 2\alpha_n |b_j^*|}}{\sqrt{n}} \to 2c_3 u_j |b_j^*|.
\]

We now first consider the case that \( c_3 = 0 \). In this case, we have

\[
G_n(u) - G_n(0) \xrightarrow{d} u^T Cu - 2u^T z,
\]

which is convex w.r.t. \( u \). Then the minimizer of \( u^T Cu - 2u^T z \) is \( u^* \) if and only if \( Cu^* - z = 0 \). Since \( \hat{u} \xrightarrow{d} u^* \) (by epiconvergence), we obtain \( \sqrt{n}(\hat{b}_n - b^*) = \hat{u} \xrightarrow{d} N(0, \sigma^2 C^{-1}) \).

We then consider the case that \( c_3 \in (0, \infty) \). Right now we have

\[
G_n(u) - G_n(0) \xrightarrow{d} u^T Cu - 2u^T z + 2c_3 \sum_{j \in A} u_j sgn(b_j^*) + 2c_3 \sum_{j \notin A} |u_j| \triangleq H_2(u).
\]

\( H_2(u) \) is convex in \( u \). Let the minimizer of \( H_2(u) \) be \( u^* \). Then

\[
Cu^* - z + c_3 s = 0
\]

where \( s^T = (\text{sgn}(b_1^*), v^T) \) and \( v \in \mathbb{R}^{p^2} \) with \( \max_j |v_j| \leq 1 \). Thus, we have \( u^* \xrightarrow{d} N(t, \sigma^2 \Theta) \) where \( t = (t_1, \ldots, t_p)^T = -c_3 C^{-1} s \) and \( \Theta = [\theta_{ij}] = C^{-1} \). For any \( \epsilon > 0 \), when \( n \) is significantly large and using Chebyshev’s inequality, we have that

\[
\Pr \left[ |u_j^*| / \sqrt{n} \geq \epsilon \right] = \Pr \left[ |u_j^*| \geq \sqrt{n} \epsilon \right] 
\leq \Pr \left[ |u_j^* - t_j| \geq \sqrt{n} \epsilon - |t_j| \right] \leq \frac{\sigma^2 \theta_{jj}}{(\sqrt{n} \epsilon - |t_j|)^2} \to 0
\]

for \( j = 1, \ldots, p \). Consequently, \( |u_j^*| / \sqrt{n} \to 0 \); that is, \( \hat{b}_n \xrightarrow{p} b^* \).
References

P. S. Bradley and O. L. Mangasarian. Feature selection via concave minimization and support vector machines. In The 26th International Conference on Machine Learning, pages 82–90. Morgan Kaufmann Publishers, San Francisco, California, 1998.

P. J. Brown, T. Fearn, and M. Vannucci. Bayesian wavelet regression on curves with application to a spectroscopic calibration problem. Journal of the American Statistical Association, 96:398–408, 2001.

E. J. Candès, M. B. Wakin, and S. P. Boyd. Enhancing sparsity by reweighted $\ell_1$ minimization. The Journal of Fourier Analysis and Applications, 14(5):877–905, 2008.

R. Chartrand and W. Yin. Iteratively reweighted algorithms for compressive sensing. In The 33rd IEEE International Conference on Acoustics, Speech, and Signal Processing (ICASSP), 2008.

I. Daubechies, R. Devore, M. Fornasier, and C. S. Güntürk. Iteratively reweighted least squares minimization for sparse recovery. Communications on Pure and Applied Mathematics, 63(1):1–38, 2010.

J. Fan and R. Li. Variable selection via nonconcave penalized likelihood and its Oracle properties. Journal of the American Statistical Association, 96:1348–1361, 2001.

C. Gao, N. Wang, Q. Yu, and Z. Zhang. A feasible nonconvex relaxation approach to feature selection. In Proceedings of the Twenty-Fifth National Conference on Artificial Intelligence (AAAI’11), 2011.

C. J. Geyer. On the asymptotics of constrained M-estimation. The Annals of Statistics, 22:1993–2010, 1994.

D. Hunter and R. Li. Variable selection using MM algorithms. The Annals of Statistics, 33(4):1617–1642, 2005.

K. Knight and W. Fu. Asymptotics for lasso-type estimators. The Annals of Statistics, 28:1356–1378, 2000.

K. Lange, D. R. Hunter, and I. Yang. Optimization transfer using surrogate objective functions (with discussion). Journal of Computational and Graphical Statistics, 9(1):1–59, 2000.

R. Mazumder, J. Friedman, and T. Hastie. SparseNet: Coordinate descent with nonconvex penalties. Journal of the American Statistical Association, 106(495):1125–1138, 2011.

R. W. D. Nickalls. A new approach to solving the cubic: Cardan’s solution revealed. The Mathematical Gazette, 77(480):354–359, 1993.

B. G. Osborne, T. Fearn, A. R. Miller, and S. Douglas. Application of near-infrared reflectance spectroscopy to compositional analysis of biscuits and biscuit dough. Journal of the Science of Food and Agriculture, 35(1):99–105, 1984.
J. A. Palmer, D. P. Wipf, K. Kreutz-Delgado, and B. D. Rao. Variational EM algorithms for non-Gaussian latent variable models. In Advances in Neural Information Processing Systems 18, 2006.

R. Tibshirani. Regression shrinkage and selection via the lasso. Journal of the Royal Statistical Society, Series B, 58:267–288, 1996.

D. Wipf and S. Nagarajan. A new view of automatic relevance determination. In Advances in Neural Information Processing Systems 20, 2008.

D. Wipf and S. Nagarajan. Iterative reweighted $\ell_1$ and $\ell_2$ methods for finding sparse solutions. IEEE Journal of Selected Topics in Signal Processing, 4(2):317–329, 2010.

Z. Xu, X. Chang, F. Xu, and H. Zhang. $l_{1/2}$ regularization: a thresholding representation theory and a fast solver. IEEE Transactions on Neural Networks and Learning Systems, 23(7):1013–1027, 2012.

C.-H. Zhang. Nearly unbiased variable selection under minimax concave penalty. The Annals of Statistics, 38:894–942, 2010a.

C.-H. Zhang and T. Zhang. A general theory of concave regularization for high dimensional sparse estimation problems. Statistical Science, 27(4):576–593, 2012.

S. Zhang, H. Qian, W. Chen, and Z. Zhang. A concave conjugate approach for nonconvex penalized regression with the mcp penalty. In In Proceedings of the Twenty-Seventh National Conference on Artificial Intelligence (AAAI’13), 2013.

T. Zhang. Analysis of multi-stage convex relaxation for sparse regularization. Journal of Machine Learning Research, 11:1081–1107, 2010b.

Z. Zhang and B. Tu. Nonconvex penalization using Laplace exponents and concave conjugates. In NIPS 26, 2012.

P. Zhao and B. Yu. On model selection consistency of lasso. Journal of Machine Learning Research, 7:2541–2563, 2006.

H. Zou. The adaptive lasso and its Oracle properties. Journal of the American Statistical Association, 101(476):1418–1429, 2006.

H. Zou and R. Li. One-step sparse estimates in nonconcave penalized likelihood models. The Annals of Statistics, 36(4):1509–1533, 2008.