Abstract

On an oriented, compact, connected, real four-dimensional manifold, \( M \), we introduce a topological Lagrangian gauge field theory with a Bogomol’nyi structure that leads to non-singular, finite-Action, stable solutions to the variational field equations. These soliton-like solutions are analogous to the instanton in Yang-Mills theory. Unlike Yang-Mills instantons, however, ‘topological’ instantons are independent of any underlying metric structure, and, in particular, they are independent of the metric signature. We show that when the topology of the underlying manifold, \( M \), is equipped with a complex Kähler structure, and \( M \) is interpreted as space-time, then the moduli space of topological instantons—the space of motions—is a finite-dimensional, smooth, Hausdorff manifold with a natural symplectic structure. We identify space-time topologies which lead to the physical stability of topological instanton field configurations compatible with the additional geometric structures. The spaces of motion for \( U(1) \) topological instantons over either minimal elliptic or algebraic complex space-times with irregularity \( q = 2 \) are examined.

1. Introduction

Witten gave the first examples of topological field theories (TFTs) [W1,W2], and Horowitz proposed a general class of TFTs which included some of Witten’s examples as special cases [H]. To date, most of the interest in TFTs has been in the quantum field theory, where deep connections have been forged with the new smooth-invariants of low dimensional manifolds (e.g., Donaldson invariants) [W1,D1]. In this paper we present and study a class of topological gauge field theories which have a ‘minimising’ Bogomol’nyi structure. The TFTs we introduce below are classical covariant Lagrangian Action functionals with a Bogomol’nyi structure over oriented, compact, connected real four-manifolds. The functionals...
do not depend on the metric structure of the underlying space-time (classical ‘general’ covariance). The Bogomol’nyi structure leads to first-order field equations, the Bogomol’nyi equations, with solutions that automatically satisfy the variational field equations, and, that are stable under perturbation. The Bogomol’nyi equations studied in this paper are metricless versions of the (anti-)self-duality equations of Yang-Mills gauge theory. The TFTs introduced below are Horowitz’s TFTs generalised to strengthen the Bogomol’nyi structure for all ranks of vector bundle. In this paper we shall not discuss the quantization of the TFTs, because we believe that the classical Bogomol’nyi soliton is sufficiently interesting to be examined on its own. Our view comes from recent work which suggests that classical Bogomol’nyi solitons can have remarkable ‘pseudo-quantum’ dynamics [TRA].

We now summarise the sections of this paper. In the next section we introduce classical generally covariant Lagrangian field theories over an oriented, compact, connected real four-manifold with a natural Bogomol’nyi structure. Because of the similarity of the topological Bogomol’nyi equations to the self-duality equations in Yang-Mills theory, the non-singular, finite-Action, stable solutions to the topological Bogomol’nyi equations are called ‘topological instantons’. In section three, we study the line bundle case and introduce further geometrical structure onto the theory: we assume that the underlying compact, connected base-manifold is equipped with a Kähler structure. We obtain a natural correspondence between topological instantons and Einstein-Hermitian vector bundles. This leads to a finite-dimensional, smooth, Hausdorff manifold which we can interpret as the space of motions later, in section five. In section four the appropriate generalisation to vector bundles is presented. In sections three and four we make use of recent results achieved by mathematicians. The differential geometric notion of stability introduced by Kobayashi [Ko1], and the work of Kim [Kim] that studied Einstein-Hermitian moduli spaces, are particularly useful. Section five views the underlying Kähler manifold as space-time, so that the moduli space of Einstein-Hermitian topological instantons becomes the space of motions. We argue that the dynamics of Bogmol’nyi solitons in TFTs may not be trivial, and that diagonal abelian Einstein-Hermitian topological instantons may correspond to either photons, massless neutrinos, or composite lumps of photons or neutrinos. Section six provides a conclusion.
2. Topological instantons

Let \( \pi : P \rightarrow M \) be a principal \( G \)-bundle over an oriented, compact, connected four manifold \( M \), and denote by \( E = P \times_G G \), the vector bundle associated to \( \pi : P \rightarrow M \) by the adjoint representation of \( G \) on the Lie algebra, \( G \). Denote by \( \mathcal{A}(P) \) the space of \( L^2 \) connections on \( P \), and consider two connections, \( A, B \in \mathcal{A}(P) \). \( A \) and \( B \) induce exterior covariant derivatives \( D^A \) and \( D^B \) on the associated vector bundle \( E \). To define the curvatures of \( D^A \) and \( D^B \), let \( s \) be a local frame field of \( E \) over an open set \( U \subset M \). The curvatures \( H^A \) and \( K^B \) are given by \( D^A D^A s = s H^A \) and \( D^B D^B s = s K^B \) and can be interpreted as 2-forms on \( M \) taking values in \( E \), i.e., \( H^A, K^B \in \Omega^2(M, E) \). Finally, we assume that there is an invariant positive-definite inner product \( <,> \) on \( E \) which varies continuously with \( x \in M \).

The topological field theories we study in this paper differ from those introduced by Horowitz in [H] for theories of gravity. The modifications in this paper permit a different geometrical formulation of the problem. The Lagrangian theories of interest to us are given by the functional

\[
L_\pm(A, B) = \int_M <(H^A \otimes I_E) \wedge (I_E \otimes K^B)>
\]

\[
\pm \frac{1}{2} \int_M <(I_E \otimes K^B) \wedge (I_E \otimes K^B)>,
\]

defined on the product space \( \mathcal{A}(P) \times \mathcal{A}(P) \). Interpreting \( H^A \) and \( K^B \) as curvatures in the Lagrangian Action requires that the real dimension of \( M \) be four (in [H], \( K^B \) is viewed as an axion potential). \( I_E \) denotes the identity transformation on the adjoint bundle, \( E \). The brackets \( < > \) in (1) remind us that a choice of adjoint-invariant, real-valued inner product on the adjoint bundle, \( E \), has been made. The variational equations for (1) are

\[
D^A K^B = 0, \quad D^B H^A = 0,
\]

where we have made use of the Bianchi identity \( D^B K^B = 0 \). The set of solutions is clearly neither empty nor entirely trivial.

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The physical stability of a class of nontrivial, nonsingular, finite-Action solutions to the variational equations (3) can be argued when the Lagrangian is rewritten as

\[
2\mathcal{L}_\pm = \pm \int_M <(H^A \otimes I_E \pm I_E \otimes K^B) \wedge (H^A \otimes I_E \pm I_E \otimes K^B)> + \int_M <(H^A \otimes I_E) \wedge (H^A \otimes I_E)> .
\]

The bundle metric defines an adjoint invariant, symmetric, bilinear real-valued function, \(f\), on \(E\)—that is, a Weil polynomial of degree two. When the polynomial is evaluated on the curvature of a vector bundle \(E \rightarrow M\), and integrated over a closed \(M\), the expression becomes independent of the connection and represents a topological invariant for the bundle. Let \(E_A\) and \(E_B\) be the vector bundle \(E\) equipped with either the connection \(A\) or \(B\), respectively. The first term in the Lagrangian \(L_-\) in equation (3) can be interpreted as a topological invariant for the tensor product bundle \(E_A \otimes E_B^*\). We recall that the curvature of \(E_A \otimes E_B^*\) is given by \(\Omega_{E_A \otimes E_B^*} = H^A \otimes I_r - I_r \otimes K^B\). Therefore the Bogomol’nyi equations,

\[
H^A \otimes I_r = I_r \otimes K^B,
\]

(4)
can be viewed as giving a vanishing curvature condition on the tensor bundle \(E_A \otimes E_B^*\). Solutions to (4) automatically satisfy the variational field equations (2).

For line bundles, \(L\), solutions to the Bogomol’nyi equations consist of all pairs of Hermitian connections \((A, B)\) on \(L\) for which \(H^A = K^B\). The Lagrangian in this case reduces to the topological field theory studied by Baulieu and Singer [BS]. For \(r > 1\), an indice computation for the Bogomol’nyi equations (4), \(H^A_{ab} \delta_{cd} = \delta_{ab} K^B_{cd}\), shows that the curvature forms \(H^A\) and \(K^B\) are projectively flat. That is, \(H^A_{aa} = K^B_{cc} = i F\) and \(H^A_{ab} = K^B_{ab} = 0\) for \(a \neq b\), or, equivalently,

\[
H^A = K^B = iFI_r,
\]

(5)
where \(F\) is a real-valued two form on \(M\). For the line bundle, equation (5) is also valid. The Bianchi identity for all ranks of vector bundle imposes a simple condition on \(F\), that \(dF = 0\). Therefore \(F\) defines a de Rham cohomology class \([F] \in H^2(M, \mathbb{R})\). Since \(M\)
is compact, $H^2(M, \mathbb{R})$ is of finite dimension. If $F$ is a curvature on $M$, then the second term in (3) is a topological invariant of the underlying four-manifold, $M$. As a result both terms in $\mathcal{L}_-$ are topological invariants and under perturbations the action remains unchanged. Topologically non-trivial solutions to the Bogomol’nyi equations will be said to be ‘physically stable’ if $F$ is a curvature of $M$ and if the solutions have a fixed non-zero action given by

$$\mathcal{L}_- = -\int_M F \wedge F = -24\pi \text{ sgn}(M) \neq 0.$$  \hfill (6)

As a result $\mathcal{L}_-$ is proportional to a topological invariant, the signature of $M$, $\text{sgn}(M)$. For $\mathcal{L}_+$ a similar argument can be made on the tensor bundle $E_A \otimes E_B$ (instead of on $E_A \otimes E^*_B$), however here $H^A$ and $K^B$ must be trivially flat. Flat connections have been well-studied in the literature [W1], therefore we shall say no more about $\mathcal{L}_+$. By analogy with (anti-) self-dual instantons in Yang-Mills theory and using the physical stability of (6):

**Definition.** A physically stable, non-trivial solution $(A, B)$ to the Bogomol’nyi equations (5) on the vector bundle $(E, h)$ is called a topological instanton.

**Remark.**

1. By design, topological instantons are not coupled to the background space-time curvature. Consider $r = 1$, then a topological instanton, $A$, has bundle curvature $H^A = iF$, where $F \in H^2(M, \mathbb{R})$. If we interpret $F$ as a Maxwell field (the Faraday tensor), then $dF = 0$ is the first half of the Maxwell equations. The second half of the Maxwell equations, $\star d\star F = j$, couples electromagnetism (photons) to the space-time curvature (gravity), where $j$ is a four-current and the star operator is defined using the space-time metric structure. Since a Maxwell field is also the field of an $r = 1$ topological instanton (it is closed), it is natural to propose the same equation, $\star d\star F = j$, to couple topological instantons with $r \geq 1$ to the space-time curvature. The Maxwell equation $\star d\star F = j$ cannot be derived from a classically generally covariant TFT, and therefore must be introduced by hand. Although there is no reason to believe that there is a unique coupling, it is sufficient for $F$ to be a Maxwell field for $H^A = I_F E$ ($r > 1$) to satisfy the non-homogeneous Yang-Mills
equations, $\star D^A \star H^A = J \equiv j I_E$.

3. Einstein-Hermitian topological instantons on line bundles

In this section we examine the moduli space of $U(1)$ solutions to the Bogomol’nyi equations (4) when $M$ is a compact, complex Kähler surface. The moduli space will be interpreted as the covariant phase space (the space of motions [Sou]), therefore the moduli space must be topologically well-behaved (e.g., Hausdorff). Good topological behaviour can be achieved by introducing further geometrical structure and by insisting that the solutions correspond to ‘stable vector bundles’. The ‘stability’ of vector bundles, although usually defined algebro-geometrically, can be phrased completely within differential geometry. As we shall see, in differential geometry stable vector bundles are closely related to the topological instantons obtained in the previous section.

We set the notation for a general vector bundle, $E \to M$. The group of bundle automorphisms on $E$ will be denoted by $GL(E)$, and the subgroup of $GL(E)$ preserving the Hermitian structure, $h$, by $U(E, h)$. Let $\mathcal{D}(E, h)$ be the set of connections $D$ on $E$ preserving $h$, i.e., the set of Hermitian connections. We equip $(E, h)$ with a fixed holomorphic structure, $\bar{\partial}_E$. A holomorphic structure, $\bar{\partial}_E$, and a Hermitian structure, $h$, together determine a unique compatible connection, $D = \partial_E + \bar{\partial}_E \in \mathcal{D}(E, h)$, on $(E, \bar{\partial}_E, h)$. Let $\mathcal{D}^{1,1}(E, h)$ denote the set of all holomorphic structures on $(E, h)$, or, equivalently, the subset of Hermitian connections $D = \partial_E + \bar{\partial}_E \in \mathcal{D}(E, h)$ which satisfy $\bar{\partial}_E^2 = 0$. Finally, we recall the definition of an Einstein-Hermitian vector bundle. A holomorphic Hermitian vector bundle $(E, \bar{\partial}, h)$ over a complex Hermitian manifold $(M, g)$ has a weak-Einstein-Hermitian structure if the unique compatible connection, $D$, has mean curvature, $K(D)$, which satisfies

$$K(D) = \varphi I_E,$$

where $\varphi$ is a real function defined on $M$. Equation (7) is called the weak-Einstein condition. If $\varphi = c$ in (7) with $c$ a constant, then $D$ is said to satisfy the Einstein condition and $(E, \bar{\partial}_E, h)$ is an Einstein-Hermitian vector bundle. Let $\mathcal{E}(E, h)$ denote the set of Einstein-Hermitian connections, that is, the connections in $\mathcal{D}^{1,1}(E, h) \subset \mathcal{D}(E, h)$ which satisfy equation (7) with $\varphi = c$. The moduli space of holomorphic structures on $(E, h)$ is given by
The moduli space of Einstein-Hermitian structures on \((E, h)\) is given by \(\mathcal{M}_r \equiv \mathcal{E}(E, h)/U(E, h)\).

Although the notation set above is that for a general vector bundle, we now specialize to the line bundle case. Let \((L, \bar{\partial}_L, h) \to (M, g, \Phi)\) be a holomorphic Hermitian line bundle over a compact, complex Kähler surface. Then, all connections in \(\mathcal{D}^{1,1}(L, h)\) are Einstein-Hermitian up to a conformal transformation of the Hermitian structure. To see this, we note that every holomorphic Hermitian line bundle \((L, \bar{\partial}_L, h)\) over \((M, g, \Phi)\) is weak-Einstein-Hermitian for any metric \(g\), since \(\Omega = R_{\alpha \beta} dz^\alpha \wedge d\bar{z}^\beta\). Furthermore, there exists a conformal transformation of the Hermitian structure, \(h \to h' = ah\), which makes \((L, \bar{\partial}_L, h') \to (M, g, \Phi)\) an Einstein-Hermitian vector bundle, that is, \(K(D) = c [\text{Ko2}]\). The constant \(c\) is given by

\[
c = \frac{2\pi}{V} \deg(L),
\]

where \(V = \text{Vol}(M)\) and \(\deg(L) = \int_M c_1(L) \wedge \Phi\). The constant, \(c\), is a topological invariant when \(M\) is Kähler (when \(\Phi\) is closed), and depends only on the cohomology classes of \(\Phi\) and \(c_1(L)\). Therefore all holomorphic Hermitian line bundles \((L, h) \to M\) with compatible connection are Einstein-Hermitian, up to a conformal transformation of the Hermitian structure, \(h\). If, in addition, the second cohomology class of \(M\) is non-zero, then from (6) all connections \(D \in \mathcal{D}^{1,1}(L, h)/U(E, h)\) are physically stable, and the converse is trivial: every Einstein-Hermitian line bundle admits a topological instanton. We now introduce two types of topological instantons, both compatible with the Kähler and holomorphic structures. The set of holomorphic Kähler topological instantons on \((L, h)\) is defined by

\[
\mathcal{I}^{1,1}(L, h) = \{(D^A, D^B) \in \mathcal{D}^{1,1}(L, h) \times \mathcal{D}^{1,1}(L, h) | (D^A)^2 = (D^B)^2 = iF, \ F \in \Omega^{1,1}(M)\}.
\]

Also define the diagonal holomorphic Kähler topological instantons by

\[
\mathcal{I}^{1,1}_{\text{dia}}(L, h) = \{(D^A, D^A) \in \mathcal{D}^{1,1}(L, h) \times \mathcal{D}^{1,1}(L, h) | (D^A)^2 = iF, \ F \in \Omega^{1,1}(M)\}.
\]

From the discussion above, we conclude that

\[
\mathcal{I}^{1,1}(L, h) \simeq \mathcal{E}(L, h) \times \mathcal{E}(L, h), \quad \mathcal{I}^{1,1}_{\text{dia}}(L, h) \simeq \mathcal{E}(L, h).
\]
up to a conformal transformation of the Hermitian structure, \( h \). The isomorphism suggests that we rename \( T^{1,1} \) and \( T^{1,1}_{\text{dia}} \), Einstein-Hermitian instantons and diagonal Einstein-Hermitian instantons, respectively.

The moduli space of Einstein-Hermitian structures on Hermitian vector bundles has been studied extensively by H.J. Kim [Kim,Ko2]. Again restricting to the line bundle, \((L, h)\), Kim proves that the moduli space \( \mathcal{M} = \mathcal{E}(L, h)/U(L, h) \) is a nonsingular Kähler manifold in a neighbourhood of \([D] \in \mathcal{E}(L, h)/U(L, h)\), if \( H^2(M, \text{End}^0(E)) = 0 \), and when

\[
\text{deg}(M) \equiv \int_M c_1(M) \wedge \Phi \geq 0. \tag{8}
\]

It is not surprising that the moduli space \( \mathcal{M} \) forms a well-behaved manifold, because the concept of Einstein-Hermitian vector bundle was originally introduced by Kobayashi as the differential geometric counterpart to vector bundle stability [Ko1]. When the inequality (8) is satisfied, the complex dimension of the moduli space of abelian diagonal Einstein-Hermitian instantons, if non-empty, can be computed to give

\[
\dim_{\mathbb{C}}(\mathcal{M}) = q, \tag{9}
\]

where \( q \equiv h^{(0,1)} \) is the irregularity of \( M \). The Hodge numbers, \( h^{(p,q)} \), are defined by \( h^{(p,q)} \equiv \dim_{\mathbb{C}} H^{p,q}(M) \).

Remarks.

1. Let \( M \) be a compact, complex surface with \( c_1(M) = 0 \) and \( b_1(M) = 0 \) (a K3 surface). We argue the physical stability of Einstein-Hermitian instantons from the Noether formula:

\[
c_1^2(M) + c_2(M) = 12(1 - q + p_g). \tag{10}
\]

The geometric genus, \( p_g \), in equation (10) is defined as the Hodge number \( h^{(0,2)} \). The irregularity \( q \), the geometric genus \( p_g \), and the Betti numbers \( (b_i \equiv H_i(M, \mathbb{R})) \) are the homological invariants of complex analytic surfaces. Since \( c_1(M) = 0 \), \( q = 0 \), and \( p_g = 1 \) for a K3 surface, from (10) we conclude that \( c_2(M) = 24 \). The topology of \( M \) therefore makes non-trivial topological instantons physically stable. From equation (9) we
see that the moduli space of Einstein-Hermitian topological instantons, $\mathcal{M}_r$, on $E \to K3$ with topology such that $c_2(E \otimes E^*) = 0$, is of dimension zero.

2. Let the flat complex two-torus, $T^2_\mathbb{C} = \mathbb{C}^2/\Gamma$, be the set of equivalence classes with respect to a given lattice $\Gamma$. For $T^2_\mathbb{C}$, the irregularity and geometric genus take the values $q = 2$, $p_g = 1$, and the canonical line bundle, $K_M$, is trivial. From (9) it follows that the moduli space of holomorphic connections is of complex dimension two. Using the Noether formula (10), the topological invariants of $T^2_\mathbb{C}$, and $c_1(M) = -c_1(K_M) = 0$ we find that $c_2(M) = 0$. As a result, the topological instantons on the flat complex two-torus are not topologically fixed away from the trivially flat topological instanton—they are potentially unstable under perturbations, and therefore are not of interest to us.

3. When the canonical line bundle $K_M$ is non-trivial, we turn to the Enriques-Kodaira classification of compact, complex surfaces (free from exceptional curves). There are two types of compact, complex surfaces which admit Kähler structures and lead to moduli spaces of physically stable Einstein-Hermitian topological instantons with $\dim_{\mathbb{R}}(\mathcal{M}_r) \geq 4$ [Kod,Bar]:

(a) minimal elliptic surfaces with $c_1^2(M) = 0$, $c_2(M) \geq 0$, and

(b) minimal algebraic surfaces of ‘general type’ with $c_1^2(M) > 0$, $c_2(M) \geq 0$.

An elliptic surface is a complex surface, $M$, with a holomorphic projection, $\pi: M \to \Delta$, onto a non-singular algebraic curve, $\Delta$, such that $\pi^{-1}(u)$ for $u \in \Delta$ is (generically) an elliptic curve. An algebraic surface is a compact complex surface that can be holomorphically embedded in a projective space, $\mathbb{CP}^n$. Assume that $M$ is one or the other of these two cases, i.e., (a) or (b). Then the Kähler metric on $M$ defines a Hodge star operator, $\star$, that splits $H^2(M, \mathbb{R})$ into $(\pm 1)$-eigenspaces, say $H^2_{\pm}(M, \mathbb{R})$ of dimension $b^\pm$, where $b_2 = b^+ + b^-$. The signature of $M$ is defined as $\text{sgn}(M) = b^+ - b^-$. The Hodge identities tell us that the first Betti number $b_1$ is even, $b^+ = 2p_g + 1$, and $h^{(1,0)} = h^{(0,1)}$ [We]. We can now rewrite equation (10) using the identity $c_1^2(M) = 3\text{sgn}(M) + 2\chi(M)$, where the Euler characteristic can be written as $\chi(M) = 2 - 2b_1 + b_2$ (using Poincaré duality). For $c_1^2(M) = 0$, the Noether formula (10) is simply

$$c_2(M) = 12(p_g - q + 1). \quad (10')$$
If $c_1^2(M) > 0$, then the Noether formula (10) becomes

$$c_2(M) = b_2 + 2(1 - b_1). \quad (10'')$$

We return to these remarks in the next section.

4. Einstein-Hermitian topological instantons on vector bundles

The trivial relationship between abelian topological instantons and Einstein-Hermitian line bundles in the previous section generalises in a not so trivial way for the vector bundle. Using the notation in section three, $E$ is a $C^\infty$ complex vector bundle of rank $r$ with a Hermitian structure, $h$, over a compact complex Kähler surface, $(M, g, \Phi)$. Recall from section two that topological instantons on vector bundles with rank greater than one are projectively flat. Projectively flat connections are again closely related to Einstein-Hermitian connections, as we shall see. If $E$ is a projectively flat complex vector bundle of rank $r > 1$, then $E$ must satisfy $c_2(E \otimes E^*) = 0$. This leads to a simple topological condition relating the first and second Chern classes:

$$c_2(E) = \frac{r - 1}{2r} c_1^2(E). \quad (11)$$

All vector bundles in this section will be assumed to satisfy this topological condition. Now, every projectively flat Hermitian vector bundle on $(M, g, \Phi)$ satisfies the weak-Einstein condition. To see this, recall that a Kähler operator, $\Lambda: \Omega^{(p,q)} \to \Omega^{(n-p,n-q)}$, can be defined [We]. The Kähler operator, $\Lambda$, is the adjoint operator to the wedge multiplication of forms by the Kähler form, $\Phi$. The mean curvature, $K(D)$, can be written using the Kähler operator,

$$K(D) = i\Lambda H^A.$$

It is not difficult to show that all projectively flat connections are weak-Einstein:

$$K(D) = i\Lambda H^A = i\Lambda (F I_E) = (i\Lambda F) I_E = \varphi I_r,$$

where we define $\varphi \equiv i\Lambda F$. Moreover, for every projectively flat Hermitian vector bundle $(E, h)$ there is a conformal transformation of the Hermitian structure, $h' = ah$, for which
the projectively flat Hermitian vector bundle \((E, h')\) is Einstein-Hermitian [Ko1]. The converse is also true when (11) is satisfied. This follows from the Lübke inequality [L],

\[
\int_M \left\{ (r - 1)c_1(E)^2 - 2rc_2(E) \right\} \wedge \Phi \leq 0.
\] (12)

The equality in (12) holds if and only if \((E, h)\) is projectively flat [L]. Therefore the topological condition (11) implies that Einstein-Hermitian vector bundles are projectively flat.

The moduli space, \(\mathcal{M}_r = \mathcal{E}(E, h)/U(E, h)\), of irreducible diagonal Einstein-Hermitian \(U(r)\) topological instantons is a nonsingular Kähler manifold when

\[
\deg(M) \equiv \int_M c_1(M) \wedge \Phi \geq 0,
\]

and \(H^2(M, \text{End}^0(E)) = 0 [\text{Kim}, \text{Ko1}]\). The complex dimension of the moduli space \(\mathcal{M}_r\), if non-empty, depends on whether the canonical line bundle, \(K_M \equiv \Lambda^n(T^*M)\), is trivial or non-trivial. If \(K_M\) is trivial, then

\[
\dim\mathbb{C}(\mathcal{M}_r) = 2 + r^2(q - 2),
\]

where again \(q\) is the irregularity and \(r\) is the rank of the vector bundle. If \(K_M\) is non-trivial, then the complex dimension of the moduli space is given by

\[
\dim\mathbb{C}(\mathcal{M}_r) = 1 + r^2(q - 1).
\]

A useful account of Kim’s work on the moduli space of Einstein-Hermitian connections can be found in [Ko1].

Finally, it is interesting to note that Nahm duality (‘reciprocity’ in [CG]), a property usually associated with (anti-)instanton solutions to the (anti-)self-duality equations, is also present for Einstein-Hermitian connections over algebraic surfaces [Mu2,BvB,Sc]. It is this fact that led us to the topological instanton construction in this paper.

5. The space of motions
It is often stated that TFTs are physically trivial because they have ‘no dynamics’. This is due to the invariance of the topological Lagrangian under Diff($M$), so that all point-particle world-paths can be deformed using Diff($M$) to the static point-particle world-line. This may not be true for the dynamics of Bogomol’nyi solitons, however, because there may not exist a geometrically compatible, static, topologically non-trivial, everywhere non-singular, finite-Action field configuration that satisfies the Bogomol’nyi field equations. Put another way, the internal structure of the soliton may provide an obstruction to trivial dynamics where a structureless point-particle would not. Without explicitly solving the dynamical field equations, we can only argue that obstructions to trivial dynamics occur by examining whether a static ‘motion’ is compatible with the dimension of the space of motions. In this section we discuss the two simplest examples of non-trivial Einstein-Hermitian topological instanton dynamics.

We begin by requiring the physical stability of the instantons under perturbation. Assume that $M$ has a non-trivial canonical bundle, $K_M$, so that $c_1(K_M) = -c_1(M) \neq 0$ (cf., the third remark in section three). If $c_1(M) > 0$, then from Kodaira’s vanishing theorem and Serre duality one concludes that $q = h^{(0,1)} = 0$. From the dimension formula (9) the space of motions, $\mathcal{M}$, is either empty or of complex dimension zero. Therefore, in order to obtain a space of motions of sufficient size, we require that $M$ have the property that $c_1(M) < 0$, in addition to satisfying the inequality (8). Now, let $M$ be a minimal elliptic ($c_1^2(M) = 0$), or, minimal algebraic ($c_1^2 > 0$) surface with irregularity $q = 2$. The two cases: (a) $c_1^2(M) = 0$, and (b) $c_1^2(M) > 0$, which correspond to equations (10’) and (10’’), respectively, give physical stability with $q = 2$, when (a) $p_g > 1$, and (b) $b_2 > 6$, respectively. With $M$ taken to be space-time, the moduli space of diagonal Einstein-Hermitian instantons, $\mathcal{M}_{\text{dia}}$, can be interpreted as the space of motions. From the dimension formula (9) the space of motions, when non-empty, is a real four-dimensional manifold. Moreover, $\mathcal{M}_{\text{dia}}$ inherits a natural holomorphic symplectic structure from space-time $(M, g, \Phi)$ [Mu]:

$$\Theta(a, b) = \int_M \text{tr}(a \wedge b) \wedge \Phi,$$

where $a, b \in T\mathcal{M}_{\text{dia}}$. 

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Non-trivial instanton solutions to the self-duality equations in Yang-Mills theory exist only for imaginary time. Topological instantons differ significantly from Yang-Mills instantons since the signature of the space-time metric has no effect on the existence of solutions. Therefore topological instantons can have real-time dynamics. The real dimension of the \( q = 2 \) abelian diagonal Einstein-Hermitian instanton space of motions (four) considered above suggests that these topological instantons are massless free solitons in \( \mathbb{R}^3 \) with a constant velocity \([\text{Wo}]\); the dimension of the space of motions is the same dimension as the space of (oriented) geodesics in \( \mathbb{R}^3 \). If the velocity were zero, the space of motions would be equivalent to the configuration space, which would be three dimensional and not four. Since the constant velocity of the diagonal Einstein-Hermitian instanton appears not to vanish, the dynamics cannot be viewed as trivial. In effect, the topology of space-time and the geometry of the line bundle restricts the possible field configurations. Diagonal Einstein-Hermitian topological instantons might be interpreted as photons, massless neutrinos, or stable structures made up of photons or neutrinos (ball lightening, perhaps).

6. Conclusion

We have shown that there are classical generally covariant topological field theories with a natural Bogomol’nyi structure. The TFTs generalise Maxwell’s equations and the Yang-Mills equations. When the underlying base manifold, \( M \), is taken to be a compact Kähler space-time, then a natural correspondence has been established between topological instantons and Einstein-Hermitian connections. The moduli space of solutions to the Bogomol’nyi equations can be interpreted as the space of motions of the gauge particles. The Einstein-Hermitian condition is precisely the condition needed to insure that this space is well-behaved. The dimension of the space of motions depends on only the topology of space-time and the rank of the vector bundle. By assuming that space-time has no exceptional curves, we have shown that space-time is either a minimal elliptic complex surface, or, a minimal algebraic complex surface with certain extra conditions (e.g., \( c_1(M) < 0 \)) in order to admit Einstein-Hermitian topological solitons with a topologically well-behaved space of motions. We suggest that these solitons represent \( n \)-photon or massless \( n \)-neutrino states.
References

[Bar] Barth, W., Peters, C., Van der Ven, A.: Compact Complex Surfaces. Ergebnisse der Mathematik und ihrer Grenzgebiete: 3. Folge, Bd. 4. Berlin: Springer-Verlag, 1984.

[B] Beauville, A.: Variétés kählériennes dont la première classe de Chern est null. J. Differ. Geom. 18 755-782 (1983).

[BS] Baulieu, Singer: Topological Yang-Mills Symmetry. In proceedings of conformal field theory and related topics. (Annecy, France, March 1988).

[BvB] Braam, P., van Baal, P.: Nahm’s transformation for instantons. Comm. Math. Phys. 122 267-280 (1989).

[CG] Corrigan, E., Goddard, P.: Construction of instanton and monopole solutions and reciprocity, Ann. Phys. 154 253-279 (1984).

[D1] Donaldson, S.K.: An application of gauge theory to the topology of four manifolds. J. Differ. Geom. 18 269 (1983).

[H] Horowitz, G.T.: Exactly soluble diffeomorphism invariant theories. Comm. Math. Phys. 125 417-437 (1989).

[Kim] Kim, H.J.: Moduli of Hermite-Einstein vector bundles. Math. Z. 195 143-150 (1987).

[Ko1] Kobayashi, S.: Differential Geometry of Complex Vector Bundles. Princeton: Princeton U.P., 1987.

[Ko2] Kobayashi, S.: Curvature and stability of vector bundles. Proc. Japan Acad. 58 158-162 (1982).

[Kod] Kodaira, K.: On the structure of compact complex analytic surfaces. Am. J. Math. 84 751-798 (1964).

[L] Lübke, M.: Chernklassen von Hermite-Einstein-Vektorbündeln. Math. Ann. 260 133-141 (1982).

[Mu] Mukai, S.: Symplectic structure of the moduli space of sheaves on an abelian or K3 surface. Invent. Math. 77 101-116 (1984).

[Mu2] Mukai, S.: Fourier functor and its application to the moduli of bundles on a abelian variety. Adv. Studies in Pure Math. 10 (1987).

[Sc] Schenk, H.: On a generalised Fourier transform for instantons over flat tori. Comm. Math. Phys. 116 177-183 (1988).

[Sou] J-M Souriau: Structure des systèmes dynamique. Paris: Dunod, 1970.
[TRA] Temple-Raston, M., Alexander, D.: Differential cross-sections and escape plots for solitonic BPS $SU(2)$ magnetic monopoles. *Nucl. Phys.* B 397 195-213 (1993).

[W1] Witten, E.: Topological Quantum Field Theory. *Comm. Math. Phys.* 117 353-386 (1988).

[W2] E. Witten: 2+1 dimensional gravity as an exactly soluble system. *Nucl. Phys.* B 311 46-78 (1988).

[We] Wells, R.O.: Differential Analysis on Complex Manifolds. New York: Springer-Verlag, 1980.

[Wo] Woodhouse, N.M.J.: Geometric Quantization, 2 ed.. Oxford: Clarendon Press, 1992.