Spontaneous breaking of chiral symmetry, and eventually of parity, in a $\sigma$-model with two Mexican hats

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Abstract
A $\sigma$-model with two linked Mexican hats is discussed. This scenario could be realized in low-energy QCD when the ground state and the first excited (pseudo)scalar mesons are included, and where not only in the subspace of the ground states, but also in that of the first excited states, a Mexican hat potential is present. This possibility can change some basic features of a low-energy hadronic theory of QCD. It is also shown that spontaneous breaking of parity can occur in the vacuum for some parameter choice of the model.

1 Introduction
The ‘Mexican hat’ potential allows for a simple and intuitive description of the phenomenon of spontaneous symmetry breaking. For this reason it has been widely used -in a variety of versions- in both condensed matter and hadron physics, see for instance Ref. \cite{1} and refs. therein.

In the context of Quantum Chromodynamics (QCD) nearly massless $N_f^2 - 1$ (3 pions in the case $N_f = 2$, where $N_f$ is the number of light quark flavors) emerge as (quasi) pseudoscalar Goldstone bosons as a consequence of spontaneous breaking of chiral symmetry: $U_R(N_f) \times U_L(N_f) \to SU_V(N_f)$. In the context of a linear $\sigma$-model this spontaneous breaking is induced by a negative squared mass of the scalar and pseudoscalar mesons. This feature is responsible for the typical Mexican hat form of the mesonic potential.

In this work, beyond the ground state (pseudo)scalar mesons, we also consider the first excited (pseudo)scalar states and we investigate the case in which also in this sector a negative squared mass is present. As we shall argue, for some parameter choice this possibility cannot be excluded and leads to a more complicated scenario, in which ground-state and first-excited scalar and pseudoscalar mesons mix. Moreover, for some parameter choice it is possible that also one neutral pseudoscalar pionic field condenses, thus realizing a spontaneous symmetry breaking of parity.

The paper is organized as follows: we first briefly review the properties of the Mexican hat potential and its emergence from an hadronic model of QCD. We then turn to the case of two linked Mexican hats and discuss the consequences of this assumptions. First, the parameter range in which only spontaneous breaking of chiral symmetry take place is studied. Then, the parameter range in which also spontaneous breaking of parity occurs is investigated. In the end, the conclusions are briefly outlined.
2 Mexican hat

In its simplest form the Mexican hat potential is written in terms of two real scalar fields \( \sigma \) and \( \pi \):

\[
V_{\text{MH}} = \frac{\lambda}{4} (\sigma^2 + \pi^2 - F^2)^2 = \frac{\lambda}{4} (\varphi^* \varphi - F^2)^2 ,
\]

where in the last passage the complex scalar field \( \varphi = \sigma + i\pi \) has been introduced. The requirement \( \lambda \geq 0 \) ensures that the potential is bounded from below. Let us assume that -as in QCD, see below- \( \sigma \) represents a scalar field \( (\sigma \equiv \sigma(t,x) \rightarrow \sigma(t,-x) \) under parity transformation \( P) \) while \( \pi \) represents a pseudoscalar field \( (\pi \equiv \pi(t,x) \rightarrow -\pi(t,-x) \) under \( P) \). Note, the quadratic (mass) term of the Mexican hat potential reads \(-\frac{F^2}{2}\varphi^* \varphi \), i.e. it has a negative coefficient as long as \( F \) is a real number, which corresponds to an imaginary mass for both the \( \sigma \) and the \( \pi \) fields. For this reason one can immediately deduce that the point \( \varphi = 0 \) does not correspond to the minimum of the potential. Moreover, an expansion around this point is unstable.

The potential \( V_{MN} \) is symmetric under \( SO(2) \sim U(1) \) (denoted as chiral) transformation, namely:

\[
\begin{pmatrix} \sigma \\ \pi \end{pmatrix} \rightarrow \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \sigma \\ \pi \end{pmatrix} \text{ or } \varphi \rightarrow e^{-i\theta} \varphi .
\]

The model does not have a unique minimum: all the points \( \varphi_{\text{min}} = Fe^{i\theta} \) for each \( \theta \in [0,2\pi) \) are minima. If no other information is given, each one of these minima can be in principle realized. However, we assume that a small perturbation, which breaks chiral symmetry but does not break parity, \( V_{MN} \rightarrow V_{MN} - \varepsilon \varphi \) with \( \varepsilon \in 0^+ \), takes place: as a consequence, the only realized minimum is \( \varphi_{\text{min}} = F \). [A change of sign of \( \varepsilon \) would simply provide the equivalent solution \(-\varphi_{\text{min}}\).] When evaluating the fluctuations around the minimum \( \varphi_{\text{min}} = F \), one obtains a scalar, massive \( \sigma \) meson with \( F^2 = 2\lambda F^2 \) and a pseudoscalar, massless Goldstone boson \( \pi \). The chiral symmetry of the model is not realized as a degeneracy of the particle spectrum because the minimum (i.e. the vacuum) is not left invariant by this transformation: spontaneous breaking of chiral symmetry has taken place and the field \( \pi \) is the corresponding Goldstone boson.

3 QCD origin of the Mexican hat

For the purpose of this paper we briefly recall how the Mexican hat potential describes the spontaneous breaking of chiral symmetry which is observed in the context of low-energy QCD. The matrix \( \Phi = S + iP \) includes \( N_f^2 \) scalar and \( N_f^2 \) pseudoscalar fields, \( S = S^a t^a \) and \( P = P^a t^a \) where the matrices \( t^a \) with \( a = 1, \ldots, N_f^2 - 1 \) are the generators of \( SU(N_f) \) (with \( \text{Tr}[t^a t^b] = \frac{1}{2} \delta^{ab} \) and \( t^0 = \sqrt{\frac{1}{2N_f^2}} 1_{N_f} \)). Upon chiral transformation \( U_R(N_f) \times U_L(N_f) \) the field \( \Phi \) transforms as \( \Phi \rightarrow L \Phi R^\dagger \) with \( L, R \in U_R(N_f) \) and \( R \in U_R(N_f) \). The transformation in flavor space \( SU_V(N_f) \) is obtained by setting \( L = R = U_V \), where \( U_V \) is a \( SU(N_f) \) matrix. The transformation \( SU_A(N_f) \) is obtained by setting \( L = R^\dagger = U_A \), where \( U_A \) is a \( SU(N_f) \) matrix. (Note, however, that this set of transformations does not form group for \( N_f > 1 \) because two subsequent axial transformations are not an axial transformation). Finally, the \( U_A(1) \) axial transformation is obtained by setting \( L = R^\dagger = e^{-i\alpha}1_{N_f} \). [The \( U_V(1) \) transformation corresponds to \( L = R = e^{i\alpha}1_{N_f} \), thus trivially implying the identity transformation \( \Phi \rightarrow \Phi \).]

The effective potential for the field \( \Phi \) reads [3]

\[
V_{\text{eff}}[\Phi; \mu^2, \gamma, \delta, k, h] = \text{Tr} \left[ \mu^2 \Phi^4 + \gamma (\Phi^4 \Phi^2) + \delta (\text{Tr}[\Phi^4 \Phi])^2 - k (\det \Phi^4 + \det \Phi) - \text{Tr}[h(\Phi^4 + \Phi)] \right] .
\]

The first three terms are invariant upon \( U_R(N_f) \times U_L(N_f) \) transformations. A sufficient condition for the stability of the potential is that \( \gamma > 0 \) and \( \delta > 0 \). The term proportional to \( k \) is not invariant under the \( U_A(1) \) axial transformation and describes the so-called axial anomaly [4]. In the last term the diagonal \( N_f \times N_f \) matrix \( h \) describes the explicit contribution of nonzero current quark masses. It is
not invariant under $SU_A(N_f)$ and $U_A(1)$ transformations, and if $h \neq \text{const} \cdot 1_{N_f}$, it is also not invariant under $SU_V(N_f)$. A first, naive attempt to obtain the Mexican hat of Eq. (1) is to study the case $N_f = 1$ with $\Phi = \sqrt{\frac{1}{2}}(\sigma + i\pi) = \sqrt{\frac{1}{2}}\varphi$. In the chiral limit ($h = 0$) one can easily identify $\lambda = (\gamma + \delta)$ and $\mu^2 = - (\gamma + \delta)F^2 < 0$. The latter is a necessary condition for spontaneous symmetry breaking. However, the anomalous term $-k(\det \Phi^\dagger + \det \Phi) = -\sqrt{2}k\sigma$ breaks explicitly chiral symmetry and cannot be regarded as a small perturbation. This is due to the fact that for $N_f = 1$ the chiral transformation $SU_A(N_f)$ cannot be distinguished from the axial transformation $U_A(1)$. We conclude that, in virtue of the anomaly, the Mexican hat potential cannot be reproduced in the case of one quark flavor only.

When $N_f = 2$ the matrix $\Phi$ reads

$$\Phi = \sum_{a=0}^{3} \phi_a t_a = (\sigma + i\eta) t^0 + (\bar{a}_0 + i\bar{\pi}) \cdot \vec{t},$$

where $\vec{t} = \vec{t}/2$, with the vector of Pauli matrices $\vec{t}$, and $t^0 = 1_2/2$.

In terms of quark degrees of freedom, the scalar isotriplet $\bar{a}_0$ and the pseudoscalar pion $\bar{\pi}$ are given by $\bar{u}d$, $\sqrt{\frac{1}{2}}(\bar{u}u - \bar{d}d)$, $\bar{d}u$, while the $\sigma$ and the $\eta$ mesons by $\sqrt{\frac{1}{2}}(\bar{u}u + \bar{d}d)$. The identification of the pion triplet with the experimentally very well known resonances $\pi^\pm(139)$ and $\pi^0(135)$ listed in the Particle Data Group (PDG) [5] is straightforward. In the pseudoscalar-isoscalar channel, one has in Ref. [5] two resonances $\eta(547)$ and $\eta(958)$, which are a combination of the bare contributions $\eta \equiv \sqrt{\frac{1}{2}(\bar{u}u + \bar{d}d)}$ entering in Eq. (1) and the $s$-quark counterpart $\bar{s}s$. The physical field $\eta(547)$ reads $\eta(547) = \cos(\varphi_P)\sqrt{\frac{1}{2}(\bar{u}u + \bar{d}d)} + \sin(\varphi_P)\bar{s}s$ where $\varphi_P \simeq -35^\circ$ [6], while $\eta(958)$ is the corresponding orthogonal combination. (One can also ‘unmix’ the two physical fields and obtain that, in a hypothetical $N_f = 2$ world without $s$ quark, the $\eta \equiv \sqrt{\frac{1}{2}(\bar{u}u + \bar{d}d)}$ would have a mass of about 700 MeV [6]). The identification of the fields $\sigma$ and $\bar{a}_0$ is more complicated and addresses the problem of the identification of scalar mesons in low-energy QCD. Two set of candidates are the resonances $\{\eta(600), a_0(980)\}$ and $\{f_0(1370), a_0(1450)\}$. A detailed description of this issue is not relevant for the scope of this paper, see however Ref. [7] and refs. therein.

We assume that the charged fields $\pi^1$, $\pi^2$, $a_0^1$, $a_0^2$ do not condense. In this case they are not relevant in the study of the minima of the potential and we set their mean value to zero. We are therefore left with the diagonal matrix

$$\Phi = \frac{1}{2} \begin{pmatrix} \sigma + a_0 + i(\eta + \pi) & 0 \\ 0 & \sigma - a_0 + i(\eta - \pi) \end{pmatrix}$$

(5)

where $a_0$ and $\pi$ refer to the neutral $a_0^3$ and $\pi^3$ mesons.

The anomaly term of the potential reads explicitly in the case $N_f = 2$

$$- k(\det \Phi^\dagger + \det \Phi) = -\frac{k}{2}(\sigma^2 + \pi^2) + \frac{k}{2}(a_0^2 + \eta^2).$$

(6)

For the case $k > 0$, the absolute minimum is found for a nonzero expectation value of the field $\sigma$ (or $\pi$) and not for a nonzero value of $\eta$ (or $a_0$). This is thus the physically interesting case because a condensation of $\eta$ (or $a_0$) would imply a parity (or isospin) breaking which is not observed in the processes listed in the PDG [5].

By further setting $a_0 = \eta = 0$ the potential [3] reduces exactly to Eq. (1) by identifying

$$\lambda = \left(\frac{i}{2} + \delta\right), \quad \mu^2 = - \left(\frac{i}{2} + \delta\right)F^2 + k, \quad \varepsilon = 0.$$ 

(7)
Note, the choice $\varepsilon = 0$ is denoted as the chiral limit. The explicit inclusion of a breaking term proportional to $h = \varepsilon 1_2 \neq 0$ plays the role of the small external perturbation, which induces the condensation of the $\sigma$ field and not of $\pi$. By further setting the mean value of $\pi$ to zero, the potential in terms of the field $\sigma$ only reads
\begin{equation}
V(\sigma) = \frac{1}{2} (\mu^2 - k) \sigma^2 + \frac{1}{4} \left( \frac{\gamma}{2} + \delta \right) \sigma^4 - \varepsilon \sigma \, .
\end{equation}

The minimum of the latter is realized by a nonzero value $\sigma = \phi \neq 0$ if the quantity $\mu^2 - k$ is a negative number (at zeroth order in $\varepsilon$ one has $\phi = F$). In this case spontaneous breaking of chiral symmetry takes place and the pions emerge as (quasi) Goldstone bosons.

As long as $\min \sigma_a \neq 0$ while the latter is clearly heavier than the pion fields. It is also renowned that the axial current reads for instance, in Ref. [8]. The condition $\lambda_1 = 2$ effective potential by identifying $\sigma$ as the scalar-isoscalar field and $\pi$ as the pseudoscalar neutral member of the isotriplet field.

The masses of all the fields, as calculated from Eq. (8) as second derivatives around the minimum $\sigma = \phi \neq 0$, read:
\begin{equation}
M_\sigma^2 = \mu^2 - k + \left( \frac{\gamma}{2} + \delta \right) \phi^2 = \frac{\varepsilon}{\phi}, \quad M^2 = \mu^2 + k + \left( \frac{\gamma}{2} + \delta \right) \phi^2
\end{equation}
\begin{equation}
M^2 = \mu^2 - k + 3 \left( \frac{\gamma}{2} + \delta \right) \phi^2, \quad M^2 = \mu^2 + k + \left( \frac{3}{2} \gamma + \delta \right) \phi^2
\end{equation}

It is clear that $M^2$ receive a positive contribution form the anomalous term $k > 0$; this also explains while the latter is clearly heavier than the pion fields. It is also renowned that the axial current reads $J_{A_{\mu}} = \phi \partial_\mu \pi^a$: the constant $\phi$ can then be set equal to the pion decay constant $f_\pi = 92.4$ MeV (details, for instance, in Ref. [5]).

As a last remark we note that, in the limit of Eq. (3), the matrix $\Phi$ can be written as: $\Phi = \frac{1}{2} \text{diag}(\sigma + i \pi, \sigma - i \pi) = \sigma t^0 + i \pi t^3$. Therefore, a $SU_A(2)$ transformation $\Phi \to U_A \Phi U_A$ with $U_A \in SU_A(2)$ in the third isospin direction, i.e. $U_A = e^{-i \alpha t^3}$, is such that $\Phi \to U_A \Phi U_A = U^2 \Phi = e^{-2i \alpha t^3} \Phi$. The latter reduces exactly to the transformation of Eq. (2), i.e. $\varphi = \sigma + i \pi \to e^{-i \theta} \varphi$ by identifying $\alpha = \theta/2$. We thus obtain the simple Mexican hat potential in Eq. (1) as a special case of the general $N_f = 2$ effective potential by identifying $\sigma$ as the scalar-isoscalar field and $\pi$ as the pseudoscalar neutral member of the isotriplet field.

4 Two Mexican hats

Let us now turn to the case of interest of this work: two linked Mexican hats. A ‘double Mexican hat potential’ is introduced in terms of the complex fields $\varphi_1 = \sigma_1 + i \pi_1$ and $\varphi_2 = \sigma_2 + i \pi_2$:
\begin{equation}
V_{\text{DMH}} = \frac{\lambda_1}{4} (\varphi_1^* \varphi_1 - F^2_1)^2 + \frac{\lambda_2}{4} (\varphi_2^* \varphi_2 - F^2_2)^2 + \frac{c}{2} \left[ (\varphi_2^* \varphi_1)^2 + (\varphi_1^* \varphi_2)^2 \right] .
\end{equation}

As long as $F_1$ and $F_2$ are real numbers, it constitutes of two distinct Mexican hats for $\varphi_1$ and $\varphi_2$, and a $c$-term, which mixes them [9]. The fields $\sigma_1$ and $\sigma_2$ are assumed to have positive parity, while the fields $\pi_1$ and $\pi_2$ negative parity.

The model of Eq. (11) is manifestly invariant under the “chiral” $U(1)$ transformation applied to both fields:
\begin{equation}
\varphi_1 \to e^{-i \theta} \varphi_1, \quad \varphi_2 \to e^{-i \theta} \varphi_2 .
\end{equation}

The condition $\lambda_1, \lambda_2 > 0$ is obviously necessary to guarantee the stability of the potential. Simple algebra shows that a further constraint is needed: the parameter $c$ must be such that $|c| < \text{min}\{ \frac{\lambda_1 + \lambda_2}{4}, \frac{\lambda_1 - \lambda_2}{4} \}$. We also set, for definiteness, $F_1 < F_2$.

Note that if $c = 0$ the model reduces to two decoupled linear sigma models. The symmetry is in this limit larger: $U(1)^{1} \times U(2)^{1}$, i.e. it is invariant under $\varphi_1 \to e^{-i \theta_1} \varphi_1$ or $\varphi_2 \to e^{-i \theta_2} \varphi_2$ separately. Two Goldstone bosons $\pi_1$ and $\pi_2$ and two massive $\sigma_1$ and $\sigma_2$ fields with $M^2 = 2 \lambda_1 F^2_1$ and $M^2 = 2 \lambda_2 F^2_2$ are obtained.
In terms of the fields \((\sigma_1, \pi_1)\) and \((\sigma_2, \pi_2)\) the potential \(V_{\text{DMH}}\) takes the form

\[
V_{\text{DMH}} = \frac{\lambda_1}{4} (\sigma_1^2 + \pi_1^2 - F_1^2)^2 + \frac{\lambda_2}{4} (\sigma_2^2 + \pi_2^2 - F_2^2)^2 + c \left[ (\sigma_1^2 - \pi_1^2)(\sigma_2^2 - \pi_2^2) + 4\sigma_1\pi_1\sigma_2\pi_2 \right].
\]  

As usual, the minima of the model must be identified. The sign of the parameter \(c\) plays an important role: the cases \(c \leq 0\) and \(c > 0\) are studied separately later on, after that we have related the potential to a generalized hadronic model.

As studied above in the presence of only one complex scalar field \(\varphi\), the potential \(V_{\text{DMH}}\) may arise as a special case of a more general \(N_f = 2\) QCD effective theory in which one starts from two matrices \(\Phi_1\) and \(\Phi_2\), each one made of \(N_f^2\) scalar and \(N_f^2\) pseudoscalar fields as in Eq. ([14]). The matrix \(\Phi_k\) represents the ground state (pseudo)scalar fields, while \(\Phi_2\) the first radial excitation. The effective potential reads

\[
V_{\text{eff}}[\Phi_1, \Phi_2] = V_{\text{eff}}^{(1)}[\Phi_1] + V_{\text{eff}}^{(2)}[\Phi_2] + 2c\text{Tr} \left[ (\Phi_1^2\Phi_1)^2 + (\Phi_2^2\Phi_2)^2 \right],
\]

where \(V_{\text{eff}}^{(1)}[\Phi_1]\) and \(V_{\text{eff}}^{(2)}[\Phi_2]\) read as in Eq. ([3]):

\[
V_{\text{eff}}^{(1)}[\Phi_1] = V_{\text{eff}}[\Phi_1; \mu_1^2, \gamma_1, \delta_1, k_1, h_1] = \varepsilon_1 \mathbf{1}_{2}, \quad V_{\text{eff}}^{(2)}[\Phi_1] = V_{\text{eff}}[\Phi_2; \mu_2^2, \gamma_2, \delta_2, k_2, h_2] = \varepsilon_2 \mathbf{1}_{2}.
\]

The \(U_R(N_f) \times U_L(N_f)\) chiral transformation implies the simultaneous transformation of both fields

\[
\Phi_1 \rightarrow L\Phi_1 R^\dagger, \quad \Phi_2 \rightarrow L\Phi_2 R^\dagger.
\]

By performing the same steps as before, we reduce the matrices \(\Phi_1, \Phi_2\) to their diagonal form \(\Phi_1(2) = \frac{1}{2}\text{diag}\{\sigma_1(2), i\pi_1(2), \sigma_2(2) - i\pi_2(2)\}\). A \(SU_4(2)\) chiral transformation in the third isospind direction reduces to Eq. ([12]). The identification of the parameters of Eq. ([11]) with those of Eq. ([14]) leads to

\[
\begin{align*}
\lambda_1 &= \left(\frac{\gamma_1}{2} + \delta_1\right), \quad \mu_1^2 - k_1 = -\left(\frac{\gamma_1}{2} + \delta_1\right)F_1^2, \quad \varepsilon_1 = 0, \\
\lambda_2 &= \left(\frac{\gamma_2}{2} + \delta_2\right), \quad \mu_2^2 - k_2 = -\left(\frac{\gamma_2}{2} + \delta_2\right)F_2^2, \quad \varepsilon_2 = 0.
\end{align*}
\]

Two Mexican hats are present as long as \(F_1\) and \(F_2\) are real numbers, i.e. if the quantities \(\mu_1^2 - k_1\) and \(\mu_2^2 - k_2\) are negative real numbers. In this case, one has a Mexican hat for \(V_{\text{eff}}[\Phi_1] = \frac{1}{2}\text{diag}\{\sigma_1 + i\pi_1, \sigma_1 - i\pi_1\}, \Phi_2 = 0\) (in the subspace of the ground state fields \(\{\sigma_1, \pi_1\}\)) and also for \(V_{\text{eff}}[\Phi_1] = 0, \Phi_2 = \frac{1}{2}\text{diag}\{\sigma_2 + i\pi_2, \sigma_2 - i\pi_2\}\) (in the subspace of \(\{\sigma_2, \pi_2\}\)).

Note, in Ref. ([10]) a Lagrangian with (an infinity of) linked \(\Phi_k\) has been introduced, but only one Mexican hat is present: while \(\mu_1^2 - k_1 < 0\), one has \(\mu_2^2 - k_2 > 0\) for \(p = 2, 3, \ldots\) Similarly, in the \(N_f = 3\) models of Refs. ([11]) an additional nonet of scalar and pseudoscalar tetraquark mesons is introduced, but the Mexican hat is present only in the subspace of the ground-state quark-antiquark (pseudo)scalar mesons. In the recent work of Ref. ([12]) two multiplets \(\Phi_1\) and \(\Phi_2\) are considered in a general fashion, but the attention is focused on parity breaking at nonzero temperatures/densities.

More in general, we also refer to Higgs sector of supersymmetric models (Ref. ([13]) and refs. therein) and to works on superconductivity (Refs. ([14]) and refs. therein) where scalar theories, their mixing and spontaneous symmetry breaking are studied.

5 Condensation with no spontaneous breaking of parity

We study the minima of the potential \(V_{\text{DMH}}\) of Eq. ([11]) for \(-c_{\text{max}} < c \leq 0\). One absolute minimum of the potential \(V_{\text{DMH}}\) is given by

\[
(\pi_1 = \pi_2 = 0, \quad \sigma_1 = A_1, \quad \sigma_2 = A_2) \leftrightarrow (\varphi_1 = A_1, \varphi_2 = A_2),
\]
Due to the form of the potential this minimum is not unique. All other minima can be obtained by applying a chiral $U(1)$ transformation to Eq. (19):

$$\left(\varphi_{1,\min}, \varphi_{2,\min}\right) = \left(A_1 e^{i\theta}, A_2 e^{i\theta}\right) \text{ with } \theta \in [0, 2\pi) .$$

The minimum of Eq. (19) is unequivocally realized if we add to the potential the following parity-conserving but chirally breaking terms

$$V_{\text{DMH}} \rightarrow V_{\text{DMH}} - \varepsilon_1 \sigma_1 - \varepsilon_2 \sigma_2 \text{ with } \varepsilon_1, \varepsilon_2 \in 0^+ .$$

Note, the latter shift corresponds to small but nonzero current quark masses, $h_1 = \varepsilon_1 1_2$, $h_2 = \varepsilon_2 1_2$, in Eq. (14).

Clearly, the minimum of Eq. (19) is parity-conserving because two scalar fields condense. Being not invariant under chiral transformation, a spontaneous breaking of this symmetry occurs in the vacuum. The behavior of the condensates as function of the parameter $c$ is reported in Fig. 1, left panel ($c \leq 0$) for a paradigmatic numerical choice.

The mass matrices in both the scalar and the pseudoscalar sectors are obtained by calculating second-order derivatives evaluated at the point given in Eq. (19). They explicitly read [15]:

$$M_{\sigma_1}^2 = 3\lambda_1 A_1^2 - \lambda_1 F_1^2 + 2c A_2^2$$

$$M_{\sigma_1}^2 = 3\lambda_2 A_2^2 - \lambda_2 F_2^2 + 2c A_1^2$$

$$M_{\pi_1}^2 = \lambda_1 (A_1^2 - F_1^2) - 2c A_2^2$$

$$M_{\pi_2}^2 = \lambda_2 (A_2^2 - F_2^2) + 2c A_1^2$$

The 'physical masses' $M_{\sigma_1'}, M_{\sigma_2'}, M_{\pi_1'}, M_{\pi_2'}$ (the first two states with positive parity, the latter two with negative parity) are obtained in the standard way as eigenvalues of the latter two matrices. The spectrum of the system consists of two massive scalar fields, one massive pseudoscalar field and one massless pseudoscalar Goldstone boson. In fact, one eigenvalue of the pseudoscalar matrix of Eq. (24) vanishes, therefore realizing the Goldstone theorem. In Fig. 1, right panel, the masses are plotted as function of $c < 0$ for a particular numerical choice. Obviously, no mass degeneracy is present due to the fact that chiral symmetry is spontaneously broken. Notice also that the mass of the massive pseudoscalar meson $M_{\sigma_1'}$ vanishes for $c \rightarrow 0^-$, in agreement with the fact that a second Goldstone boson exists due to the larger, spontaneously broken symmetry in this limit.

Some considerations are in order:

(i) If the parameter $F_2$ instead of being real is a purely imaginary number (i.e. if $\mu_2^2 - k_2 > 0$) the model has different properties: only one Mexican hat in the subspace of $\sigma_1$ and $\pi_1$ is present. As a consequence, only the field $\sigma_1$ condenses to $F_1$ (chiral condensate) and $\pi_1$ is the Goldstone boson [16]. Denoting $F_2 = i\alpha$ one obtains: $M_{\sigma_1}^2 = 2\lambda_1 F_1^2$, $M_{\sigma_1} = 0$, $M_{\pi_1}^2 = \lambda_2 \alpha^2 + c F_1^2$ and $M_{\pi_2}^2 = \lambda_2 \alpha^2 - c F_1^2$. The mass splitting between $\sigma_2$ and $\pi_2$ is generated by the chiral condensate $\sigma_1 = F_1$. This is the typical picture for low-energy QCD effective theories, in which the fields $\sigma_2, \pi_2$ are interpreted as the radial excitations of the ground state $\sigma_1, \pi_1$ [10]. If, for heavier multiplets $\Phi_N$, one has smaller and smaller $c$, one recovers the degeneracy of the chiral partners. For a more detailed description of chiral symmetry restoration see Ref. [16] and refs. therein.

(ii) The scenario of two Mexican hats together with $c < 0$ cannot be excluded as an effective theory of QCD. Although a phenomenological study in the framework of a realistic potential should be performed to investigate this possibility, here we simply note that the case with two Mexican hats (with $c < 0$) is in agreement with all the symmetries and constraints imposed by QCD.

(iii) The case $c = 0$ is interesting. It implies that a larger symmetry group is realized for the effective theory than at the fundamental level. In fact, in this limit the effective theory of Eq. (13) is invariant.

\[ A_1 = \sqrt{\frac{F_1^2 - \frac{2c}{\lambda_1} F_1^2}{1 - \frac{2c}{\lambda_1} F_1^2}}, \quad A_2 = \sqrt{\frac{F_2^2 - \frac{2c}{\lambda_2} F_2^2}{1 - \frac{2c}{\lambda_2} F_2^2}}. \]
the question why this should be the case is interesting. Is there some yet unknown motivations which
situation as described by the potential (3), or its reduced form (1). In the scenario of two Mexican hats
it is therefore necessary to take into account both multiple ts $\Phi$ easier’ if such a scenario is not realized and if only the ground state
of more Mexican hats, one is obliged to include all of them in a linear hadronic theory of QCD. Needles
refers to the $k$-th excited (pseudo)scalar matrix $\Phi$ for $\Lambda^2 = F_2$, where $F_2$ is imaginary when axial transformations in the third isospin direction are considered.)
If $F_2$ is a real number, this would imply the presence of two Goldstone bosons, an eventuality which is not seen in the real world. Indeed, the parameter $c$ should also not be too small, otherwise a second, light pseudoscalar meson would be present in the spectrum, see Fig. 1, right panel, what is excluded by experimental data (the second pionic excitation has a mass of about 1.3 GeV [5]). If $F_2$ is imaginary as described in the point (ii), the condition $c = 0$ implies the degeneracy $M^2_{\sigma_2} = \lambda_2 \alpha^2 + cF_2^2$ and $M^2_{\phi_2} = \lambda_2 \alpha^2 - cF_2^2$. In the context of the already mentioned effective restoration of chiral symmetry, where for heavier multiplets a degeneracy is postulated, one indeed would have an approximately higher symmetry, corresponding to a product of $U^R_\nu(N_f) \times U^L_\nu^{(k)}(N_f)$ for different values of $k$, where $k$ refers to the $k$-th excited (pseudo)scalar matrix $\Phi_k$.
(iv) A generalization to more than 2 Mexican hats can also be easily performed. However, in order to avoid a proliferation of undesired light pseudoscalar mesons, the mixing among the different $\Phi_k$ should be large. We regard this possibility as remote for QCD, see next point.

(v) QCD in the chiral limit has only dimensional parameter, the Yang-Mills scale $\Lambda_{QCD}$. By varying it, it is -although speculative- conceivable that different phases are realized: a phase in which no Mexican hat is present ($F_1$ and $F_2$ both purely imaginary, with no spontaneous chiral symmetry breaking and no Goldstone boson(s)) obtained for $0 < \Lambda_{QCD} \leq \Lambda_2$ [11], a phase in which only for the ground state mesons one has a Mexican hat ($F_1$ real and $F_2$ purely imaginary, which is the standard scenario) for $\Lambda_1 \leq \Lambda_{QCD} \leq \Lambda_2$, a phase in which two Mexican hats are present ($F_1$ and $F_2$ both real) for $\Lambda_2 \leq \Lambda_{QCD} \leq \Lambda_3$, and so on and so forth. The case $\Lambda_2 \leq \Lambda_{QCD} \leq \Lambda_3$ is the one described by the potential of Eq. (14) when both $F_1$ and $F_2$ are real numbers [15].

(vi) In the case of a double Mexican potential ($F_1$ and $F_2$ real), it is not possible to obtain a simple situation as described by the potential [14], or its reduced form [15]. In the scenario of two Mexican hats it is therefore necessary to take into account both multiplets $\Phi_1$ and $\Phi_2$. More in general, in the presence of more Mexican hats, one is obliged to include all of them in a linear hadronic theory of QCD. Needless to say, a double (or multiple) Mexican hat would correspond to a substantial complication. ‘Life is easier’ if such a scenario is not realized and if only the ground state $\sigma \equiv \sigma_1$ condenses. Nevertheless, the question why this should be the case is interesting. Is there some yet unknown motivations which
forbids the emergence of a second (or more) Mexican hat(s)? Can it be an accidental fact, which depends only on the value of $\Lambda_{QCD}$ as describe above?

6 Condensation with spontaneous breaking of parity

We now study $V_{DMH}$ for $0 < c < c_{\text{max}}$. One absolute minimum is given by

$$\left(\pi_1 = \sigma_2 = 0, \sigma_1 = B_1, \pi_2 = B_2\right) \leftrightarrow \left(\varphi_1 = B_1, \varphi_2 = B_2 e^{i\pi/2}\right), \quad (25)$$

$$B_1 = \sqrt{\frac{F_1^2 + 2cF_2^2}{1 - \frac{4c^2}{\lambda_1^2\lambda_2^2}}}, \quad B_2 = \sqrt{\frac{F_2^2 + 2cF_1^2}{1 - \frac{4c^2}{\lambda_1^2\lambda_2^2}}} \quad \text{with} \quad (26)$$

The pseudoscalar field $\pi_2$ assumes a nonzero vacuum expectation value. This minimum is not unique: the full set of minima is obtained by performing a $U(1)$ rotation of Eq. (25):

$$\left(\varphi_1 = B_1 e^{i\theta}, \varphi_2 = B_2 e^{i(\theta + \pi/2)}\right) \text{with} \quad 0 \leq \theta < 2\pi. \quad (27)$$

Each of these minima breaks parity because $\pi_1$ and $\pi_2$ never vanish simultaneously. By adding to the system the parity conserving but chirally breaking term $V_{DMH} \rightarrow V_{DMH} - \varepsilon_1 \sigma_1 - \varepsilon_2 \sigma_2$, Eq. (25) is the univocally selected minimum: in fact, this is the point at which $\sigma_1$ is maximal for the assumed ordering $F_1 < F_2$. Note that, although a parity conserving perturbation has been added, still the realized vacuum breaks parity. We conclude that in the proposed model, besides spontaneous breaking of chiral symmetry, also a spontaneous breaking of parity takes place in the vacuum for $c > 0$. In Fig. 1, left panel, the condensate of Eq. (25) are plotted for $c > 0$.

The determination of the physical masses is obtained in the standard fashion. The crucial difference with respect to the case $c < 0$ is that the states of opposite parity $\sigma_1$ and $\pi_2$ mix, thus originating two massive physical states $\sigma_1'$ and $\pi_2'$ which are not eigenstates of parity. At the same time also the states of opposite parity $\sigma_2$ and $\pi_1$ mix, out of which one massless and one massive bosons $\pi_1'$ and $\sigma_2'$ -both with undefined parity- are obtained. Numerically, one has a mirror-like picture for $c > 0$ with respect to the parity conserving case, as depicted in Fig. 1, right panel.

Obviously, the here outlined scenario for $c > 0$ cannot describe QCD, where parity is conserved in the vacuum. The Vafa-Witten theorem states that spontaneous parity violation does not occur in theories containing vector-like fermions. Thus, if this theorem holds, the model of Eq. (11) with $c > 0$ cannot be an effective description of QCD even when varying $\Lambda_{QCD}$: it is not possible that $F_1$ and $F_2$ are real numbers and that at the same time $c$ is negative. However, the validity of the Vafa-Witten theorem has been questioned in a variety of works (see the discussion in Ref. [20] and refs. therein). If it is not valid, it is still conceivable that for a different value of $\Lambda_{QCD}$, spontaneous breaking of parity takes place in the vacuum: in this case the here outlined model -with real $F_2$ and negative $c-$ would correspond to its low-energy hadronic (confined) realization.

More in general, the original constrain that the charged components of the pion field can also be released. All the present treatment is still valid upon replacing $\pi_2$ with $|\vec{\pi}|$. We have in this case the condition $|\vec{\pi}| = B_2$: as soon as also $\pi^1 \neq 0$ and/or $\pi^2 \neq 0$ not only parity, but also charge conjugation is spontaneously broken. However, it is enough that a further, small perturbation, which originates from other interactions and is invariant under change conjugation, is present: then this additional perturbation generates a condensation of $\pi^1 \equiv \pi^0$ only, in line with the discussion of the present paper.

7 Conclusions

The main interest of this paper has been the possibility that an hadronic, $\sigma$-model for QCD is effectively described by a ‘double’ Mexican hat effective potential. In this scenario not only in the subspace of
the neutral ground state (pseudo)scalar mesons $\sigma \equiv \sigma_1$ and $\pi \equiv \pi_1$ fields, but also in the subspace of the first excited (pseudo)scalar mesons $\sigma \equiv \sigma_2$ and $\pi \equiv \pi_2$ fields, a typical Mexican hat form is present. Mixing among these bare configurations arise: in the case that no spontaneous parity breaking occurs (here for $c < 0$) the outlined effective model is in agreement with all the constraints imposed by QCD. In the case that parity symmetry breaking occurs ($c > 0$) the described model can provide an effective description of a underlying QCD-like theory only if the Vafa-Witten theorem does not strictly hold. More in general, the here presented model can also be conceived as an ‘elementary’ model of (pseudo)scalar fields which generates parity breaking for some choices of the parameters and may play a role in the early Universe.

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