Approximating the inverse of a balanced symmetric matrix with positive elements

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Abstract
For an \(n \times n\) balanced symmetric matrix \(T = (t_{i,j})\) with positive elements satisfying \(t_{i,i} = \sum_{j \neq i} t_{i,j}\) and certain bounding conditions, we propose to use the matrix \(S = (s_{i,j})\) to approximate its inverse, where \(s_{i,j} = \delta_{i,j}/t_{i,i} - 1/t_\cdot\), \(\delta_{i,j}\) is the Kronecker delta function, and \(t_\cdot = \sum_{i,j=1}^n (1 - \delta_{i,j}) t_{i,j}\). An explicit bound on the approximation error is obtained, showing that the inverse is well approximated to order \(1/(n - 1)^2\) uniformly.

Keywords: Approximation error; Inverse; Symmetric; Positive elements.

1. Introduction

When solving a large system of linear equations, an accurate approximation of the inverse of the coefficient matrix is crucially important in establishing fast convergence rates for iterative algorithms. For extensive reviews, see, for example, [1, 3, 5, 17]. In this paper, we consider the approximation of the inverse of an \(n \times n\) balanced symmetric matrix \(T = (t_{i,j})\) with positive elements, i.e.,

\[ t_{i,j} = t_{j,i} > 0 \quad \text{and} \quad t_{i,i} = \sum_{j=1, j \neq i}^n t_{i,j}, \quad i = 1, \cdots, n. \quad (1) \]

The matrix \(T\) is a special case of the diagonally dominant nonnegative matrix that has received wide attention [6, 8, 10]. It is easy to show that \(T\) must be positive definite. The inverse of a general nonnegative matrix has been extensively studied by [2, 6, 7, 11, 12].
We propose to approximate the inverse of $T$, $T^{-1}$, by the matrix $S = (s_{i,j})$, where

$$s_{i,j} = \frac{\delta_{i,j}}{t_{i,i}} - \frac{1}{t_{..}},$$

where $t_{..} = \sum_{i,j=1}^{n}(1 - \delta_{i,j})t_{i,j}$. An explicit upper bound on the approximation error is given in the following section, which is crucially useful in establishing the asymptotical normality of an estimated vector in the β-model for undirected random graphs with a diverging number of nodes [16].

2. An explicit bound on the approximation error

Let $m := \min_{1 \leq i < j \leq n} t_{i,j}$ and $M := \max_{1 \leq i < j \leq n} t_{i,j}$, and for a matrix $A = (a_{i,j})$, define $||A|| := \max_{1 \leq i < j \leq n} |a_{i,j}|$. We have the following theorem.

Theorem 1.

$$||T^{-1} - S|| \leq \frac{C(m, M)}{(n - 1)^2} + \frac{1}{2m(n - 1)^2},$$

where

$$C(m, M) = \frac{M}{m^2} \times \left[ \frac{nM + (n - 2)m}{2(n - 2)m} \right].$$

Proof. Let $I_n$ be the $n \times n$ identity matrix. Define $F = T^{-1} - S$, $V = (v_{ij}) = I_n - TS$ and $W = (w_{ij}) = SV$. We have the recursion

$$F = T^{-1} - S = (T^{-1} - S)(I_n - TS) + S(I_n - TS) = FV + W. \quad (2)$$

Note that

$$v_{i,j} = \delta_{i,j} - \sum_{k=1}^{n} t_{i,k} s_{k,j}$$

$$= \delta_{i,j} - \sum_{k=1}^{n} t_{i,k} \left( \frac{\delta_{k,j}}{t_{j,j}} - \frac{1}{t_{..}} \right)$$

$$= (\delta_{i,j} - \frac{1}{t_{j,j}}) \frac{t_{i,j}}{t_{j,j}} + \frac{2t_{i,i}}{t_{..}}, \quad (3)$$

2
and

\[
    w_{i,j} = \sum_{k=1}^{n} s_{i,k} v_{k,j} = \sum_{k=1}^{n} \left( \frac{\delta_{i,k}}{t_{i,i}} - \frac{1}{t} \right) \left( \frac{\delta_{k,j} - 1}{t_{j,j}} \right) + \frac{2t_{k,k}}{t} \right] \\
    = \sum_{k=1}^{n} \frac{\delta_{i,k}}{t_{i,i}} \left( \frac{\delta_{k,j} - 1}{t_{j,j}} \right) + \frac{2t_{k,k}}{t} \right] - \frac{1}{t} \sum_{k=1}^{n} \left( \frac{\delta_{k,j} - 1}{t_{j,j}} \right) + \frac{2t_{k,k}}{t} \right] \\
    = \left( \frac{\delta_{i,j} - 1}{t} \right) + \frac{2t_{i,i}}{t} \right] - \frac{1}{t} \left( \frac{t_{i,j}}{t} \right) + 2 \right] \\
    = \left( \frac{\delta_{i,j} - 1}{t} \right) + \frac{1}{t}. \tag{4}
\]

Furthermore, when \( i \neq j \),

\[
    0 < \frac{1}{t} \leq \frac{1}{mn(n-1)}, \\
    0 < \frac{t_{i,j}}{t_{i,i}t_{j,j}} \leq \frac{M}{m^2(n-1)^2},
\]

and it is easy to show, when \( i, j, k \) are different from each other,

\[
    |w_{i,i}| \leq \frac{1}{mn(n-1)}, \\
    |w_{i,j}| \leq \frac{1}{mn(n-1)}, \\
    |w_{i,j} - w_{i,k}| \leq \frac{M}{m^2(n-1)^2}, \\
    |w_{i,i} - w_{i,k}| \leq \frac{M}{m^2(n-1)^2}.
\]

It follows that

\[
    \max(|w_{i,j}|, |w_{i,j} - w_{i,k}|) \leq \frac{M}{m^2(n-1)^2} \text{ for all } i, j, k. \tag{5}
\]

Next we use the recursion (2) to obtain a bound of the approximate error \( ||F|| \). Let \( a = \frac{M}{m^2(n-1)^2} \). By (2) and (3), for any \( i \), we have

\[
    f_{i,j} = \sum_{k=1}^{n} f_{i,k} \left[ (\delta_{k,j} - 1) \frac{t_{k,j}}{t_{j,j}} + \frac{2t_{k,k}}{t} \right] + w_{i,j}, \quad j = 1, \ldots, n. \tag{6}
\]
Thus, to prove Theorem 1, it is sufficient to show that \(|f_{i,j}| \leq C(M,m)/(n-1)^2\) for any \(i,j\). Fixing any \(i\), let \(f_{i,\alpha} = \max_{1 \leq k \leq n} f_{i,k}\) and \(f_{i,\beta} = \min_{1 \leq k \leq n} f_{i,k}\).

First, we will show that \(f_{i,\beta} \leq 1/t.. \leq 1/(m(n-1)^2)\). A direct calculation gives that

\[
\sum_{k=1}^{n} f_{i,k} t_{k,i} = \sum_{k=1}^{n} (T_{i,k}^{-1} - (\frac{\delta_{i,k}}{t_{i,i}} - \frac{1}{t..})) t_{k,i} = 1 - (1 - \sum_{k=1}^{n} \frac{t_{k,i}}{t..}) = \sum_{k=1}^{n} \frac{t_{k,i}}{t..}.
\]

(7)

Thus, \(f_{i,\beta} \sum_{k=1}^{n} t_{k,i} \leq \sum_{k=1}^{n} f_{i,k} t_{k,i} = \sum_{k=1}^{n} \frac{t_{k,i}}{t..}\). It follows that \(f_{i,\beta} \leq 1/t..\) and, similarly, \(f_{i,\alpha} \geq 1/t..\).

Note that \(f_{i,\beta} = -\sum_{k=1}^{n} f_{i,\beta} (\delta_{k,\alpha} - 1) \frac{t_{k,\alpha}}{t_{\alpha,\alpha}}\). Thus,

\[
f_{i,\alpha} + f_{i,\beta} = \sum_{k=1}^{n} (f_{i,k} - f_{i,\beta}) (\delta_{k,\alpha} - 1) \frac{t_{k,\alpha}}{t_{\alpha,\alpha}} + \sum_{k=1}^{n} f_{i,k} (2t_{k,k} \frac{1}{t..}) + w_{i,\alpha}.
\]

(8)

Similarly, we have that

\[
f_{i,\beta} + f_{i,\beta} = \sum_{k=1}^{n} (f_{i,k} - f_{i,\beta}) (\delta_{k,\beta} - 1) \frac{t_{k,\beta}}{t_{\beta,\beta}} + \sum_{k=1}^{n} f_{i,k} (2t_{k,k} \frac{1}{t..}) + w_{i,\beta}.
\]

(9)

Combining the above two equations, it yields

\[
f_{i,\alpha} - f_{i,\beta} = \sum_{k=1}^{n} (f_{i,k} - f_{i,\beta}) ([\delta_{k,\alpha} - 1] \frac{t_{k,\alpha}}{t_{\alpha,\alpha}} - (\delta_{k,\beta} - 1) \frac{t_{k,\beta}}{t_{\beta,\beta}}] + w_{i,\alpha} - w_{i,\beta}.
\]

(10)

Let \(\Omega = \{k : (1 - \delta_{k,\beta}) t_{k,\beta}/t_{\beta,\beta} \geq (1 - \delta_{k,\alpha}) t_{k,\alpha}/t_{\alpha,\alpha}\}\) and let \(|\Omega| = \lambda\). Note that \(1 \leq \lambda \leq n - 1\). Then,

\[
\sum_{k=1}^{n} (f_{i,k} - f_{i,\beta}) ([\delta_{k,\alpha} - 1] \frac{t_{k,\alpha}}{t_{\alpha,\alpha}} - (\delta_{k,\beta} - 1) \frac{t_{k,\beta}}{t_{\beta,\beta}}] \leq \sum_{k \in \Omega} (f_{i,k} - f_{i,\beta}) ([1 - \delta_{k,\beta}] \frac{t_{k,\beta}}{t_{\beta,\beta}} - (1 - \delta_{k,\alpha}) \frac{t_{k,\alpha}}{t_{\alpha,\alpha}}]
\]

\[
\leq (f_{i,\alpha} - f_{i,\beta}) \frac{\sum_{k \in \Omega} t_{k,\beta}}{t_{\beta,\beta}} - \sum_{k \in \Omega} [(1 - \delta_{k,\alpha}) t_{k,\alpha}]
\]

\[
\leq (f_{i,\alpha} - f_{i,\beta}) \frac{\lambda M}{\lambda M + (n - 1 - \lambda)m} - \frac{(\lambda - 1)m}{(\lambda - 1)m + (n - \lambda)M}.
\]

(11)
Let 
\[ f(\lambda) = \frac{\lambda M}{\lambda M + (n - 1 - \lambda)m} - \frac{(\lambda - 1)m}{(\lambda - 1)m + (n - \lambda)M}. \]

There are two cases to consider the maximum of \( f(\lambda) \) in the range of \( \lambda \in [1, n - 1] \).

Case I: When \( M = m \), it is easy to show \( f(\lambda) = 1/(n - 1) \).

Case II: \( M \neq m \). Since 
\[
 f'(\lambda) = \frac{(n-1)Mm}{[\lambda M + (n-1-\lambda)m]^2} - \frac{(n-1)Mm}{[(\lambda-1)m+(n-\lambda)M]^2} = \frac{(n-1)Mm[(\lambda-1)m+(n-\lambda)M]^2}{(n-1)Mm[(n-2\lambda)(M-m)][\lambda M + (n-1-\lambda)m+(\lambda-1)m+(n-\lambda)M]} 
\]
and
\[
 f''(\lambda) = -2(M - m)Mm(n - 1) \left( \frac{1}{[\lambda M + (n-1-\lambda)m]^3} + \frac{1}{[(\lambda-1)m+(n-\lambda)M]^3} \right),
\]
\( f(\lambda) \) takes its maximum at \( \lambda = n/2 \) when \( 1 \leq \lambda \leq n - 1 \). A direct calculation gives that
\[
 f\left(\frac{n}{2}\right) = \frac{nM - (n - 2)m}{nM + (n - 2)m}.
\] (12)

Combining (10), (11) and (12), it yields
\[
 f_{i,\alpha} - f_{i,\beta} \leq \frac{nM - (n - 2)m}{nM + (n - 2)m} \times (f_{i,\alpha} - f_{i,\beta}) + a,
\]
so that
\[
 f_{i,\alpha} - f_{i,\beta} \leq \left[ \frac{2(n - 2)m}{nM + (n - 2)m} \right]^{-1} \times a = C(M, m)/(n - 1)^2.
\]

Note that
\[
 \max_{j=1,\ldots,n} |f_{i,j}| \leq f_{i,\alpha} - f_{i,\beta} + \frac{1}{t_i} \leq \left[ \frac{nM + (n - 2)m}{2(n - 2)m} \right] \times \frac{M}{m^2(n - 1)^2} + \frac{1}{2m(n - 1)^2}.
\]

This completes the proof.
3. Discussion

In many applications, it is important to closely approximate the inverse of a matrix when its explicit form is unavailable. For example, when an algorithm involves solving a matrix in each iteration, its convergence rate is often related to the approximate inverse it uses. On the other hand, in some statistical applications, an accurate approximation of the inverse of the Fisher information matrix is critical in establishing the theoretical properties of the maximum likelihood estimates. For instance, Simons and Yao [15] obtained a good approximation of the inverse of a symmetric positive definite matrix with negative off-diagonal elements. This result is crucial in establishing their most surprising result that the maximum likelihood estimates of the merit parameters in the Bradley-Terry model for paired comparisons retain good asymptotic properties even when the number of subjects goes to infinity. Similarly, our results can be readily used to prove the asymptotic normality of the maximum likelihood estimate in the $\beta$-model with a diverging dimension [16] since the Fisher information matrix of the $\beta$ model is a diagonally dominant nonnegative matrix.

The matrix $S$ which we use to approximate the inverse of the matrix $T$ takes the form of $I + H_c$, where each element of $H_c$ is $c$. If $c > 0$, then $S$ is a class of preconditioners for $M$-matrices [17]. In our situation, $c < 0$ since $S$ is a matrix with non-negative elements. The bound on the approximation error in Theorem 1 depends on $m$, $M$ and $n$. When $m$ and $M$ are bounded by a constant, all the elements of $T^{-1} - S$ are of order $O(1/(n-1)^2)$ as $n \to \infty$, uniformly.

We illustrate by an example that the bound on the approximation error in Theorem 2.1 is optimal in the sense that any bound in the form of $C(m, M)/f(n)$ requires $f(n) = O((n-1)^2)$ as $n \to \infty$. Assume that the matrix $T$ consists of the elements: $t_{i,i} = (n-1)M, i = 1, \ldots, n-1; t_{n,n} = (n-1)m$ and $t_{i,j} = m, i, j = 1, \ldots, n; i \neq j$, which satisfies (1). By the Sherman-Morrison formula, we have

\[
(T^{-1})_{i,j} = \frac{\delta_{i,j}}{(n-1)M-m} - \frac{m}{((n-1)M-m)^2} \times \left(1 + \frac{(n-1)m}{(n-1)M-m} + \frac{1}{(n-2)}\right)^{-1}, i, j = 1, \ldots, n-1
\]

\[
(T^{-1})_{n,j} = \frac{\delta_{n,j}}{(n-2)m} - \frac{m}{((n-2)m)^2} \times \left(1 + \frac{(n-1)m}{(n-1)M-m} + \frac{1}{(n-2)}\right)^{-1}, j = 1, \ldots, n-1
\]

\[
(T^{-1})_{n,n} = \frac{1}{(n-2)m} - \frac{1}{(n-2)^2m} \times \left(1 + \frac{(n-1)m}{(n-1)M-m} + \frac{1}{(n-2)}\right)^{-1}
\]
In this case, the elements of \( S \) are
\[
S_{i,j} = \frac{\delta_{i,j}}{(n-1)M} - \frac{1}{n(n-1)m}, \quad i, j = 1, \ldots, n-1; i \neq j,
\]
\[
S_{n,j} = \frac{\delta_{n,j}}{(n-1)m} - \frac{1}{n(n-1)m}, \quad j = 1, \ldots, n.
\]

It is easy to show that the bound of \( ||T^{-1} - S|| \) is \( \frac{1}{(n-1)^2m} + o\left(\frac{1}{(n-1)^2}\right) \). This suggests that the rate \( 1/(n-1)^2 \) is optimal. On the other hand, there is a gap between \( 1/m \) and \( C(m, M) = \frac{(M+m)M}{2m^3} + \frac{1}{m} + o(1) \) which implies that there might be space for improvement.

Finally, we discuss some extension to diagonally dominant case, where we still use \( S \) to approximate the inverse of \( T \). In this case, some \( t_{i,i} \) may be greater than the corresponding row sum without the diagonal element. Denote \( \Delta_i := t_{i,i} - \sum_{j \neq i} t_{i,j} \) and redefine \( m := \min_{1 \leq i < j \leq n} t_{i,j} \) and \( M := \max\{ \max_{1 \leq i < j \leq n} t_{i,j}, \max_{1 \leq i \leq n} \Delta_i \} \). With a similar argument, we can prove the new upper bound of the approximation errors is
\[
\frac{1}{(n-1)^2} \times \left[ \frac{2(n-2)m}{m(n-1)} - \frac{M}{(n-2)m + M} \right]^{-1} \times \left[ \frac{(n-2)Mm}{m^2} + \frac{4M}{m^2 n} + \frac{1}{mn(n-1)} \right].
\]
for large \( n \) when \( M/m = o(n) \).

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