Emergence of quantum chaos in the quantum computer core and how to manage it

B. Georgeot and D. L. Shepelyansky*
Laboratoire de Physique Quantique, UMR 5626 du CNRS, Université Paul Sabatier, F-31062 Toulouse Cedex 4, France

(Received 2 May 2000)

We study the standard generic quantum computer model, which describes a realistic isolated quantum computer with fluctuations in individual qubit energies and residual short-range interqubit couplings. It is shown that in the limit where the fluctuations and couplings are small compared to the one-qubit energy spacing, the spectrum has a band structure, and a renormalized Hamiltonian is obtained which describes the eigenstate properties inside one band. Studies are concentrated on the central band of the computer (“core”) with the highest density of states. We show that above a critical interqubit coupling strength, quantum chaos sets in, leading to a quantum ergodicity of the computer eigenstates. In this regime the ideal qubit structure disappears, the eigenstates become complex, and the operability of the computer is quickly destroyed. We confirm that the quantum chaos border decreases only linearly with the number of qubits $n$, although the spacing between multiqubit states drops exponentially with $n$. The investigation of time evolution in the quantum computer shows that in the quantum chaos regime, an ideal (noninteracting) state quickly disappears, and exponentially many states become mixed after a short chaotic time scale for which the dependence on system parameters is determined. Below the quantum chaos border an ideal state can survive for long times, and an be used for computation. The results show that a broad parameter region does exist where the efficient operation of a quantum computer is possible.

PACS number(s): 05.45.Mt, 03.67.Lx, 24.10.Cn

I. INTRODUCTION

During the last decade, remarkable progress has been achieved in the fundamental understanding of the main elements necessary for the creation of a quantum computer. Indeed, as stressed by Feynman [1], classical computers have tremendous problems to simulate very common quantum systems, since the computation time grows exponentially with the number of quantum particles. Therefore, for such problems it is natural to envision a computer composed of quantum elements (qubits) which operate according to the laws of quantum mechanics. In any case, such devices will in a sense be unavoidable since technological progress will lead to chips of smaller and smaller size which will eventually reach the quantum scale. At present a quantum computer is viewed as a system of $n$ qubits (two-level quantum systems), with the possibility of switching a coupling between them on and off (see the detailed reviews in Refs. [2–4]). The operation of such computers is based on reversible unitary transformations in the Hilbert space, whose dimension $N_H = 2^n$ is exponentially large in $n$. It was shown that all unitary operations can be realized with two-qubit transformations [5,6]. This makes the existence of a coupling between qubits necessary. Any quantum algorithm will be a sequence of such fundamental transformations, which form the basis of a new quantum logic.

An important next step was the discovery of quantum algorithms which can make certain computations much faster than on a classical computer. The most impressive of these is the problem of factorization of large numbers in prime factors, for which Shor constructed [7] a quantum algorithm which is exponentially faster than the classical ones. It was also shown by Grover [8] that searching for an item in a long list is parametrically much faster on a quantum computer. The recent development of error-correcting codes [9,10] showed that a certain amount of noise due to external coupling could be tolerable in the operation of a quantum computer.

All these exciting developments motivated a great body of experimental proposals to effectively realize such a quantum computer. They include ion traps [11,12], nuclear magnetic resonance systems [13], nuclear spins with interaction controlled electronically [14,15] or by laser pulses [16], electrons [17] or excitons [18] in quantum dots, Cooper pair boxes [19], optical lattices [20] and electrons floating on liquid helium [21]. As a result, a two-qubit gate was experimentally realized with cold ions [22], and the Grover algorithm was performed for three qubits made from nuclear spins in a molecule [23]. However, to have a quantum computer competitive with a classical one will require a much larger number of qubits. For example, the minimal number of qubits for which Shor’s algorithm will become useful is of the order of $n = 1000$ [4]. As a result, a great experimental effort is still needed to achieve quantum computer realization.

A serious obstacle to the physical realization of such computers is quantum decoherence due to couplings with the external world, which gives a finite lifetime to the excited state of a given qubit. This question was discussed by several groups for different experimental qubit realizations [4,6,24,25]. The effects of decoherence and laser pulse shape broadening were numerically simulated in the context of Shor’s algorithm [26,27], and shown to be quite important for the operability of the computer. However, in a number of physical proposals, for example nuclear spins in two-dimensional semiconductor structures, the relaxation time due to this decoherence process can be many orders of mag-
nitude larger than the time required for the gate operation \[2,14,15,25\], so that there are hopes of managing this obstacle.

Here we will focus on a different obstacle to the physical realization of quantum computers that was not stressed up to now. This problem arises even if the decoherence time is infinite and the system is isolated or decoupled from the external world. Indeed, even in the absence of decoherence there are always imperfections in physical systems. Due to this the spacing between the two states of each qubit will fluctuate in some finite detuning interval \(\delta\). Also, some residual static interaction \(J\) between qubits will be unavoidably present (we point out that an interqubit coupling is required to operate the gates). Extensive studies of many-body interacting systems, such as nuclei, complex atoms, quantum dots, and quantum spin glasses \[28–37\], showed that generically in such systems interaction leads to quantum chaos characterized by ergodicity of the eigenstates and level spacing statistics as in random matrix theory. In fact, it was shown that a critical coupling exponentially stronger than \(\Delta_0\) is relatively destroyed since the noninteracting multiqubit states are in such a regime, its operability would be effectively destroyed when noninteracting multiqubit states representing the quantum register states will be eliminated by quantum ergodicity.

In this respect, it is important to stress that the residual interaction \(J\) will unavoidably be much larger than the energy spacing \(\Delta_n\) between adjacent eigenstates of the quantum computer. Indeed, the residual interaction \(J\) is relatively small, so that all \(N_H\) computer eigenenergies are distributed in an energy band of size \(\Delta E \sim n\Delta_0\), where \(\Delta_0\) is the average energy distance between the two levels of one qubit, and \(n\) is the total number of qubits in the computer. As a consequence, the spacing between multiqubit states is \(\Delta_n = \Delta E / N_H = n\Delta_0 2^{-n} \ll \Delta_0\). Let us consider a realistic estimate for \(\Delta_n\) and \(J\) for the case with \(n = 1000\), as required for Shor’s algorithm to be useful. For \(\Delta_n \sim 1\) \(\text{K}\), which corresponds to the typical one-qubit spacing in the experimental proposals \[14,15\], the multiqubit spacing becomes \(\Delta_n \sim 10^3 \times 2^{-10} \Delta_0 \sim 10^{-298}\) \(\text{K}\). This value will definitely be much smaller than any physical residual interaction. In the case of Ref. \[15\], for example, with a distance between donors of \(r = 200\) \(\text{Å}\) and an effective Bohr radius of \(a_B = 30\) \(\text{Å}\) \(\text[Eq. (2)]{15}\), the coupling between qubits (spin-spin interaction) is \(J \sim \Delta_0 \sim 1\) \(\text{K}\). By changing the electrostatic gate potential, the effective electron mass can be modified up to a factor of 2. Since \(J \propto (r/a_B)^{5/2}\) \(\exp(-2r/a_B)\) \(a_B\), and \(a_B\) is inversely proportional to the effective mass, this gives a minimal residual spin-spin interaction of \(J \sim 10^{-5}\) \(\text{K} \gg \Delta_n\). In this situation, one would naturally and naively expect that level mixing, quantum ergodicity of eigenstates, and chaos are unavoidable, since the interaction is much larger than the energy spacing between adjacent levels \((J \gg \Delta_n)\).

In spite of this natural expectation, it was shown recently in Ref. \[40\] that in a quantum computer the quantum chaos sets in only for couplings \(J\) exponentially stronger than \(\Delta_n\). In fact, it was shown that a critical coupling \(J_c\), for the transition to quantum chaos decreases only linearly with the number of qubits \(n\) (for short-range interqubit coupling): \(J_c \sim \Delta_0/n\). This result opens a broad parameter region where a quantum computer can be operated below the quantum chaos border, when noninteracting multiqubit states are very close to the exact quantum computer eigenstates. For example, at \(n = 1000\) and \(\Delta_0 \sim 1\) \(\text{K}\), the critical residual interaction is \(J_c \sim 1\) \(\text{mK}\), compatible with the proposal discussed above \[15\]. We note that for other experimental proposals the value of \(\Delta_0\) might differ \(\text[e.g., 1 eV for excitons in semiconductor quantum dots \[18\]\]), and will accordingly lead to different requirements for the residual interaction.

In the present paper, we study in more detail the transition to chaos, and how it affects the time evolution of the system. The effects of residual interaction in the presence or absence of fine fluctuations of individual qubit energy spacing are analyzed in great detail. The paper is arranged as follows. In Sec. II we describe the standard generic quantum computer model, introduced in Ref. \[40\]. In Sec. III, we present the results of numerical and analytical studies of eigenenergies and eigenstate properties of this model. Section IV is devoted to the analysis of the time evolution of this system, and typical time scales for the development of quantum chaos are presented as a function of the system parameters. We end some concluding remarks in Sec. V.

**II. STANDARD GENERIC QUANTUM COMPUTER MODEL**

In Ref. \[40\] the standard generic quantum computer (SGQC) model was introduced to describe a system of \(n\) qubits containing imperfections which generate a residual interqubit coupling and fluctuations in the energy spacings between the two states of one qubit. The Hamiltonian of this model reads

\[
H = \sum_i \Gamma_I \sigma_i^z + \sum_{i<j} J_{ij} \sigma_i^x \sigma_j^x ,
\]

where \(\sigma_i\) are the Pauli matrices for the qubit \(i\), and the second sum runs over nearest-neighbor qubit pairs on a two-dimensional lattice with periodic boundary conditions applied. The energy spacing between the two states of a qubit is represented by \(\Gamma_I\), randomly and uniformly distributed in the interval \([\Delta_0 - \delta/2, \Delta_0 + \delta/2]\). The detuning parameter \(\delta\) gives the width of the distribution near the average value \(\Delta_0\), and may vary from 0 to \(\Delta_0\). Fluctuations in the values of \(\Gamma_I\) appear generally as a result of imperfections. For example, in the framework of the experimental proposals \[14,15\], the detuning \(\delta\) will appear for nuclear spin levels as a result of local magnetic fields and density fluctuations. For electrons floating on liquid helium \[21\], it will appear due to fluctuations of the electric field near the surface. The couplings \(J_{ij}\) represent the residual static interaction between qubits which is always present for reasons explained in Sec. I. They can originate from spin-exciton exchange \[14,15\], Coulomb interaction \[11\], dipole-dipole interaction \[21\], etc. To catch the general features of the different proposals, we chose \(J_{ij}\) randomly and uniformly distributed in the interval \([-J, J]\). We note that a similar Hamiltonian, but without coupling or detuning fluctuations, was discussed for a quantum computer based on optical lattices \[20,41\]. This SGQC model describes the quantum computer hardware, while the gate operation in time should include additional time-dependent
terms in Hamiltonian (1), and will be studied separately. At
\( J = 0 \) the noninteracting eigenstates of the SGQC model can
be presented as \( |\psi_j\rangle = |\alpha_1, \ldots, \alpha_n\rangle \), where \( \alpha_k = 0 \) or 1
marks the polarization of each individual qubit. These are the
ideal eigenstates of a quantum computer, and we will call
them quantum register states. For \( J \neq 0 \), these states are no
longer eigenstates of the Hamiltonian, and the new eigen-
states are now linear combinations of different quantum reg-
ister states. We will use the term multiqubit states to denote
the eigenstates of the SGQC model with interaction, but also
for the case \( J = 0 \).

While in Ref. [40] the main studies concentrated on the
case where \( \delta \) is relatively large and comparable to \( \Delta_0 \), here
we will focus on the case \( \delta \ll \Delta_0 \), which corresponds to the
situation where fluctuations induced by imperfections are
relatively weak. In this case, the unperturbed energy spec-
trum of Eq. (1) (corresponding to \( J = 0 \)) is composed of \( n + 1 \)
well separated bands, with interband spacing \( 2\Delta_0 \). An
dexample of the density of multiqubit states \( \rho_n = 1/\Delta_n \) in this
situation is presented in Fig. 1. Since \( \Gamma \), randomly fluctuate
in an interval of size \( \delta \), each band at \( J = 0 \), except the ex-
reme ones, have a Gaussian shape with width \( \approx \sqrt{n} \delta \). The
number of states in the band \( j \) is equal to the binomial coeffi-
cient \( \binom{n}{j} \) whose value is approximately \( N_{\text{int}}/n \) in the central
bands, so that the energy spacing between adjacent multiq-
bit states inside one band is exponentially small (\( \delta_n \sim n^{3/2}2^{-n} \delta \)), in line with the general estimate in Sec. I.

In the presence of a residual interaction \( J \sim \delta \), the spec-
trum will still have the above band structure with exponen-
tially large density of states. For \( J \sim \delta \ll \Delta_0 \), the interband
coupling is very weak and can be neglected. In this situation,
the SGQC Hamiltonian [Eq. (1)] is to a good approximation
described by the renormalized Hamiltonian

\[
H_P = \sum_{k=1}^n \hat{P}_k H \hat{P}_k
\]

where \( \hat{P}_k \) is the projector on the \( k \)th band, so
that qubits are coupled only inside one band. We will there-
fore concentrate our studies on the band nearest to \( E = 0 \).
For an even \( n \) this band is centered exactly at \( E = 0 \), while for
odd \( n \) there are two bands centered at \( E = \pm \Delta_0 \), and we will
use the one at \( E = -\Delta_0 \). Such a band corresponds to the
highest density of states, and in a sense represents the quan-
tum computer core. It is clear that quantum chaos and ergo-
dicity will first appear in this band, which will therefore set
the limit for operability of the quantum computer. Inside this
band, the system is described by a renormalized Hamiltonian
\( H_P \) which depends only on the number of qubits \( n \) and the
dimensionless coupling \( J/\delta \).

III. QUANTUM COMPUTER EIGENENERGIES
AND EIGENSTATES

The first investigations in Ref. [40] showed that the quan-
tum chaos border in the SGQC model [Eq. (1)] corresponds
to a critical interaction \( J_c \), given by

\[
J_c \approx \frac{C \delta}{n},
\]

where \( C \) is a numerical constant. This border is exponentially
larger than the energy spacing between adjacent multiqubit
states \( \Delta_n \). The physical origin of this difference is due to the
fact that the interaction is of a two-body nature. As a result,
one noninteracting multiqubit state \( |\psi\rangle \) has nonzero cou-
pling matrix elements only with \( 2n \) other multiqubit states
[this is for nearest-neighbor interaction; if all qubits are
coupled, this number becomes \( n(n-1)/2 \)]. In the basis of
quantum register states \( |\psi\rangle \), the Hamiltonian is represented
by a very sparse nondiagonal matrix with only \( 2n + 1 \) non-
zero matrix elements by line of length \( N_{\text{int}} = 2^n \) (one diagonal
element plus \( 2n \) coupled states). For \( \delta = \Delta_0 \) all these transi-
tions take place in an energy interval \( B \) of width of order
\( 6\Delta_0 \), since flipping two qubits changes the energy by the
order of \( \pm 3\Delta_0 \). Therefore, the energy spacing between di-
rectly coupled states is \( \Delta_n \approx B/2n \sim 3\Delta_0/n \). According to
studies of quantum chaos in many-body systems [29,32–
37,40], the transition to chaos takes place when the matrix
elements become larger than the energy spacing between di-
rectly coupled states. This gives \( J \sim \Delta_c \), which leads to rela-
tion (2). For the case \( \delta \ll \Delta_0 \) on which we focus here, still in
the renormalized Hamiltonian \( H_P \), the number of nonzero
matrix elements in one line is of the order of \( n \), and \( B \sim \delta \), so
that \( \Delta_n \sim \delta/n \), that leads to result (2) [42].

The transition to quantum chaos and ergodicity can be
clearly seen in the change of the spectral statistics of the
system. One of the most convenient is the level spacing sta-
tistics \( P(s) \), which gives the probability of finding two adja-
cent levels whose spacing is in \([s, s + ds]\). Here \( s \) is the en-
ergy spacing measured in units of average level spacing. It is
well known that while the average density of states is not
sensitive to the presence or absence of chaos, fluctuations of
the energy spacings between adjacent levels around the mean
value, determined by \( P(s) \), are sensitive to it. In the presence
of chaos, eigenstates are ergodic, overlap of wave functions
gives a finite coupling matrix element between nearby states
and the spectral statistics \( P(s) \) follows the Wigner-Dyson
(WD) distribution \( P_W(s) = (\pi s/2)\exp(-\pi s^2/4) \) typical of
random matrices. This distribution \( P_W(s) \) shows level repul-
sion at small \( s \), due to the fact that overlap matrix elements
between adjacent levels tend to move them away from each
other. Conversely, in the integrable case at \( J = J_c \), the over-
In this way the number of qubits increases and the statistical fluctuations, we averaged over several independent realizations of $P_p(s)$ and the Wigner-Dyson distribution $P_W(s)$; $N_d=8$ and $N_s>1.2 \times 10^3$.

In the SGQC model, we expect a transition from $P_p(s)$ at small $J$ to $P_W(s)$ above the quantum chaos border [Eq. (2)]. An example of such a transition is shown in Fig. 2. To decrease the statistical fluctuations, we averaged over several independent realizations of $P_p(s)$, which is the standard procedure used in random matrix theory [38,39]. We used up to $N_d=5 \times 10^4$ realizations so that the total statistics $1.5 \times 10^5 \gg N_d \gg N_s>1.2 \times 10^4$. It is interesting to note that in the limit $J/\delta \to \infty$ the system remains in the regime of quantum chaos with WD statistics [43], as illustrated in Fig. 3. This means that in the absence of individual qubit energy fluctuations, the residual coupling alone leads to chaotic eigenstates.

To characterize the variation of $P(s)$ from one limiting distribution to another it is convenient to use the parameter $\eta= \int_0^{s_0} (P_p(s) - P_W(s))ds/\int_0^{s_0} (P_p(s) - P_W(s))ds$ [33], where $s_0=0.4729\ldots$ is the intersection point of $P_p(s)$ and $P_W(s)$. In this way $P_p(s)$ corresponds to $\eta=1$, and $P_W(s)$ to $\eta=0$. Studies of different systems have already shown that this parameter well characterizes the transition from one statistics to the other [33,35,37,40]. Indeed, according to the data of Fig. 4, $\eta$ changes from 1 at small $J$ to $\eta \approx 0$ at large $J$. To characterize this transition, we chose the critical value $J_c$ by the condition $\eta(J_c)=0.3$. The dependence of $\eta$ on the rescaled coupling strength $J/J_c$ shows that the transition becomes sharper and sharper when $n$ increases (Fig. 4).

The dependence of the critical coupling strength $J_c$ on the number of qubits $n$ is shown in Fig. 5. It clearly shows that this critical strength decreases linearly with $n$, and follows the theoretical border [Eq. (2)] with $C \approx 3$. For comparison, in the same figure we also show the dependence of the multiquitbit spacing $\Delta_n$ (computed numerically) on $n$. This definitely demonstrates that $J \gg \Delta_n$.

The transition in the level spacing statistics reflects a qualitative change in the structure of the eigenstates. While for $J < J_c$, the eigenstates are expected to be very close to the quantum register states $|\psi_i\rangle$, for $J > J_c$ each eigenstate $|\phi_m\rangle$ becomes a superposition of an exponential number of states $|\psi_i\rangle$. It is convenient to characterize the complexity of an eigenstate $|\phi_m\rangle$ by the quantum eigenstate entropy $S_q = - \sum_{|\phi_m\rangle} W_{im} \log W_{im}$, where $W_{im}$ is the quantum probability to find the quantum register state $|\psi_i\rangle$ in the eigenstate $|\phi_m\rangle$ of the Hamiltonian ($W_{im}=|\langle \psi_i | \phi_m \rangle|^2$). In this way $S_q=0$ if $|\phi_m\rangle$ is one quantum register state ($J=0$), $S_q=1$ if $|\phi_m\rangle$ is equally composed of two $|\psi_i\rangle$’s, and the maximal value is

![Figure 2](image1.png)  
**FIG. 2.** Transition from Poisson to Wigner-Dyson statistics for the renormalized Hamiltonian of the SGQC model in the central band. The statistics is obtained for the states in the middle of the energy band ($\pm 6.25\%$ around the center) for $n=16$: $J/\delta = 0.05, \eta = 0.99$ (dashed line histogram); $J/\delta = 0.32, \eta = 0.047$ (full line histogram). Full curves show the Poisson distribution $P_p(s)$ and the Wigner-Dyson distribution $P_W(s)$; $N_d=8$ and $N_s>1.2 \times 10^4$.

![Figure 3](image2.png)  
**FIG. 3.** Level spacing statistics for the renormalized Hamiltonian of the SGQC model in the central band for $\delta=0$. The statistics is obtained for the states in the middle of the energy band ($\pm 6.25\%$ around the center) for $n=15$: $\eta = 0.023$ (histogram). Full curves show $P_p(s)$ and $P_W(s)$; $N_d=20$ and $N_s>1.6 \times 10^4$.

![Figure 4](image3.png)  
**FIG. 4.** Dependence of $\eta$ on the rescaled coupling strength $J/J_c$ for the states in the middle of the energy band for $n = 6$ (*), 9 (o), 12 (triangles), 15 (squares), and 16 (diamonds).
Indeed, the entropy gives magnitude. This is confirmed by the data in Fig. 5, which shows a small dispersion near different values of mixed states. These data show that the critical coupling has a large value corresponding to an exponential number of corrections

\[ J_c = \frac{C}{\ln(n)} \] with \( C = 3.3 \); the solid line is \( J_c = 0.41 \delta_n/n \); the dotted curve is drawn to guide the eye for (+). Logarithms are decimal.

\[ S_q = n \] if all \( 2^n \) states contribute equally to \( |\phi_m\rangle \). We average \( S_q \) over the states in the center of the energy band, and \( N_d \) realizations of \( \Gamma_i \) and \( J_{ij} \).

The variation of this average \( S_q \) as a function of \( J \) for different values of \( n \) is shown in Figs. 6 and 7. This shows that indeed the entropy \( S_q \) grows with \( J \) until it saturates to a large value corresponding to an exponential number of mixed states. These data show that the critical coupling \( J_{cs} \) at which \( S_q = 1 \) (two states mixed) is proportional to \( J_c \). Indeed, Fig. 7 shows a small dispersion near \( S_q = 1 \) when \( n \) changes from 6 to 16, while \( \Delta_n \) varies by three orders of magnitude. This is confirmed by the data in Fig. 5, which give \( J_{cs} \approx 0.13 \) and \( J_c \approx 0.4\delta/n \). This result is in agreement with the results [40] obtained by direct diagonalization of the

\[ \rho_W(E - E_i) = \sum_m W_{im} \delta(E - E_m). \] (3)

The function \( \rho_W \) characterizes the average probability distribution of \( W_{im} \) (see a numerical example in Fig. 3 of Ref. [40]). For moderate coupling strength, \( \rho_W \) is well described by the well-known Breit-Wigner distribution

\[ \rho_{BW}(E - E_i) = \frac{\Gamma}{2 \pi ((E - E_i)^2 + \Gamma^2/4)}, \] (4)

where \( \Gamma \) is the width of the distribution. This expression is valid when \( \Gamma \) is smaller than the bandwidth \( \Gamma < \sqrt{\delta} \), and many levels are contained inside this width. In this regime, the Breit-Wigner width \( \Gamma \) is given by the Fermi golden rule,

\[ \Gamma = 2\pi U_i^2 \rho_c, \] where \( U_i \) is the root mean square of the transition matrix element, and \( \rho_c \) is the density of directly coupled states. The validity of this formula was well checked in many-body systems with quantum chaos [30,35,36,39]. In our case \( U_i \sim J \) and \( \rho_c \sim n/\delta \), so that

\[ \Gamma \sim \frac{J^2 n}{\delta}. \] (5)

This dependence is confirmed by the data in Fig. 8. However, for large \( J \), when \( \Gamma > \sqrt{\delta} \), the shape of \( \rho_W \) becomes non-Lorentzian, and is well fitted by a Gaussian distribution. The width of this modified distribution grows like \( \Gamma \sim J \). This scaling naturally appears in the limit \( \delta = 0, J \ll \Delta_0 \), since the noninteracting part of the Hamiltonian is simply a constant commuting with the perturbation. The change from one dependence to the other takes place for \( J > \delta n^{1/4} \). Above this limit \( \Gamma \) is still weakly dependent on the number of qubits \( n \).
We expect that for $J \gg \delta$ the energy width of one band is $\Gamma \sim J \sqrt{n}$ (an effective frequency of the sum of $n$ Rabi frequencies with random signs), and have checked this law numerically for $\delta = 0$ (data not shown).

According to the results obtained from many-body systems [35], the number of quantum register states mixed inside the width $\Gamma$ is of the order of $\Gamma \rho_n$, and is exponentially large. However, this assumes that $J > J_c$, and that the system is already in the quantum chaos regime. In this case the quantum eigenstate entropy $S_q$ is large [$S_q \sim \log(\Gamma \rho_n)$], and the operability of the computer is quickly destroyed, since many quantum register states become mixed. The pictorial view of the quantum computer melting is shown in Fig. 9. This image is qualitatively similar to the one in Ref. [40] (Fig. 5 there), which was obtained for the SGQC model at $\delta = \Delta_0$. In Fig. 9 the melting goes in a smoother way, since all the states belong to the same central band (quantum computer core).

The effect of quantum chaos melting in the quantum register representation is shown in Fig. 10 for $J > J_c$. The ideal register structure is manifestly washed out. Conversely, below the chaos border ($J < J_c$), only a few quantum register states are mixed. For comparison, Fig. 11 shows the same part of the register in the regime $J \ll J_c$ (no mixing of states) and Fig. 12 that in the regime $J \sim J_c$ (few states are mixed).

IV. TIME EVOLUTION IN THE SGQC MODEL

In Sec. III we determined the properties of eigenstates of the quantum computer in the presence of residual interqubit coupling. In the presence of this coupling the quantum register states $|\psi_i\rangle$ are no longer stationary states, and therefore it is natural to analyze how they evolve in time. Indeed, if at time $t = 0$ an initial state is $|\psi_i\rangle$, corresponding to the quantum register state $i_0$, then with time the probability will spread over the register and at a time $t$ the projection probability on the register state $|\psi_i\rangle$ will be $\rho_{W}^{\psi_i}(t)$.

FIG. 8. Dependence of the Breit-Wigner width $\Gamma$ on the coupling strength $J$ for $n = 15$ for the states in the middle of the energy band. The straight lines show the theoretical dependence [Eq. (5)] with $\Gamma = 1.3J^2n^2\delta$ and the strong coupling regime with $\Gamma \sim J$: $N_{\rho} = 20$. Logarithms are decimal. Lower inset: example of the local density of states $\rho_{W}$ [Eq. (3)] for $J/\delta = 0.08$; the full line shows the best fit of the Breit-Wigner form [Eq. (4)] with $\Gamma = 0.10\delta$. Upper inset: example of the local density of states $\rho_{W}$ [Eq. (3)] for $J/\delta = 0.4$; the full line shows the best Gaussian fit of width $\Gamma = 0.64\delta$.

FIG. 9. Melting of the quantum computer core generated by the interqubit coupling. Grayness represents the level of quantum eigenstate entropy $S_q$, from gray ($S_q = 12$) (top) to black ($S_q = 0$) (bottom). The horizontal axis is the scaled energy $E/\delta$ of the computer eigenstates in the central band counted from the band bottom to the top ($E/\delta = \pm \sqrt{n}$). The vertical axis is the value of $J/\delta$, varying from 0 to 0.5. Here $n = 16$, $J/\delta = 0.22$, and one random realization is chosen. A color figure is available on http://xyz.lanl.gov/format/quant-ph/0005015.

FIG. 10. Quantum chaos in the quantum register: Grayness represents the value of the projection probability $W_{im}$ of the quantum register states on the eigenstates of the Hamiltonian, from gray (maximal value) to black (minimal value). The horizontal axis corresponds to 150 quantum register states, and the vertical axis represents the nearest 150 computer eigenstates (both ordered in energy). Here $n = 16$, $J/\delta = 0.4$ ($J/\delta > J_c/\delta = 0.22$), and one random realization is chosen. A color figure is available on http://xyz.lanl.gov/format/quant-ph/0005015.
where $A_{i,m} = \langle \psi_i | \phi_m \rangle$, $E_m$ is the energy of the stationary state $|\phi_m\rangle$, and we chose $\hbar = 1$. For $J < J_c$, the probability $F_{i|0}(t)$ is very close to 1 for all times, since the states are not mixed by the interaction. This means that all quantum register states $|\psi_i\rangle$ remain well defined, and the computer can operate properly. For $J < J_c$, only a few states $|\psi_i\rangle$ are mixed by the interaction, and $F_{i|0}(t)$ oscillates in time regularly around an average value of order 1/2. These oscillations are similar to the Rabi oscillations between two levels with frequency $\Omega \sim J$. An example is presented in Fig. 13. In this regime, we expect that error-correcting codes [9,10] may efficiently correct the spreading over few quantum register states.

For $J > J_c$, quantum chaos sets in, and with time the probability spreads over more and more quantum register states until a quasistationary regime is reached where an exponentially large number of states is mixed. The probability $F_{i|0}(t)$ drops approximately to zero, as shown in Fig. 14. The chaotic time scale for this decay $\tau_\chi$ can be estimated as $\tau_\chi \sim 1/\Gamma$, where $\Gamma$ is the width determined in Sec. III. This estimate is very natural in the Fermi golden rule regime, with the Breit-Wigner local density of states [Eq. (4)] since $F_{i|0}(t)$ is essentially the Fourier transform of the local density of states $p_W$, and therefore decreases as $\exp(-\Gamma t)$. We note that the decay in this regime was recently discussed in

FIG. 11. Same as Fig. 10 below the quantum chaos border, $J/\delta=0.001$ ($J/\delta \approx J_c/\delta = 0.026$).

\begin{equation}
F_{i|0}(t) = |\langle \psi_i | \chi(t) \rangle|^2 = \sum_{m,m'} A_{i,m}^* A_{i,m'}^* A_{i,m'} \exp[i(E_{m'} - E_m)t],
\end{equation}

where $A_{i,m} = \langle \psi_i | \phi_m \rangle$, $E_m$ is the energy of the stationary state $|\phi_m\rangle$, and we chose $\hbar = 1$. For $J < J_c$, the probability $F_{i|0}(t)$ is very close to 1 for all times, since the states are not mixed by the interaction. This means that all quantum register states $|\psi_i\rangle$ remain well defined, and the computer can operate properly. For $J < J_c$, only a few states $|\psi_i\rangle$ are mixed by the interaction, and $F_{i|0}(t)$ oscillates in time regularly around an average value of order 1/2. These oscillations are similar to the Rabi oscillations between two levels with frequency $\Omega \sim J$. An example is presented in Fig. 13. In this regime, we expect that error-correcting codes [9,10] may efficiently correct the spreading over few quantum register states.

For $J > J_c$, quantum chaos sets in, and with time the probability spreads over more and more quantum register states until a quasistationary regime is reached where an exponentially large number of states is mixed. The probability $F_{i|0}(t)$ drops approximately to zero, as shown in Fig. 14. The chaotic time scale for this decay $\tau_\chi$ can be estimated as $\tau_\chi \sim 1/\Gamma$, where $\Gamma$ is the width determined in Sec. III. This estimate is very natural in the Fermi golden rule regime, with the Breit-Wigner local density of states [Eq. (4)] since $F_{i|0}(t)$ is essentially the Fourier transform of the local density of states $p_W$, and therefore decreases as $\exp(-\Gamma t)$. We note that the decay in this regime was recently discussed in

FIG. 12. Same as Fig. 10 for $J/\delta=0.01$ ($J/\delta \sim J_c/\delta = 0.026$).

FIG. 13. Time dependence of the probability to remain in the same quantum register state for $n=16$ and $J=0.01-J_c=0.026$ ($J_c/\delta = 0.22$); one random realization is chosen.

FIG. 14. Time dependence of the probability to remain in the same quantum register state for $n=16$ and $J=0.4-J_c=0.59$ ($J_c/\delta = 0.59$). An average is made over 200 states randomly chosen in the central band. The inset shows the chaotic time scale $\tau_\chi$ [defined by $F_{i|0}(\tau_\chi)=1/2$] as a function of $1/\Gamma$; the straight line is $\tau_\chi = 1.27/\Gamma$. 

FIG. 15. Same as Fig. 10 above the quantum chaos border, $J/\delta=0.001$ ($J/\delta \approx J_c/\delta = 0.026$).
FIG. 15. Time dependence of the quantum entropy $S(t)$ for $J / \delta = 0.4 \approx J_c / \delta$; symbols are as in Fig. 14. An average is made over 200 initial states randomly chosen in the central band. The inset shows the same curves normalized to their maximal value.

Ref. [45]. According to our data, when $\Gamma$ becomes comparable to the energy bandwidth $\sqrt{\hbar \delta}$, $\rho_{i0}$ is close to a Gaussian distribution of width $\Gamma$, and its Fourier transform $F_{i0}(t)$ is also a Gaussian of width $1/\Gamma$. Therefore, in both regimes we expect the time scale $\tau_\chi$ for the decay of $F_{i0}(t)$ to be $\tau_\chi \sim 1/\Gamma$. The data shown in Fig. 14 correspond to the saturation regime for large values of $n$, and the inset shows that $\tau_\chi \sim 1/\Gamma$ is still valid. In fact the curve for $n = 16$ in Fig. 14 is already close to the limiting decay curve at $\delta = 0$ (data not shown).

At the same time scale $\tau_\chi$ the quantum entropy $S(t)$ is large but still growing. It reaches its maximal value on a larger time scale which seems independent of $n$. At this stage, an initial quantum register state is now spread over most of the register [here $S(t) = -\Sigma_i F_{i0}(t) \log F_{i0}(t)$]. This process is shown in Fig. 15. This maximal value of $S(t)$ is approximately given by $S_q$ (see Fig. 6), and accordingly decreases with decreasing $J_c$ as illustrated in Fig. 16.

Figure 17 illustrates this mixing process in the quantum register representation, evolving in time. The quantum computer hardware becomes quickly destroyed due to the interqubit coupling. It is necessary to decrease the coupling strength below the quantum chaos border to get obtain well-defined quantum register states for $t > 0$, as illustrated in Fig. 18. The obtained data clearly show that exponentially many quantum register states become mixed after the finite chaotic time scale $\tau_\chi \approx 1/\Gamma$.

V. CONCLUSIONS

The results presented in this paper show that residual interqubit coupling can lead to quantum chaos and very complicated ergodic eigenstates of the quantum computer. We have shown that in this regime quantum register states disintegrate quickly in time over an exponentially large number of states, and that computer operability is destroyed. We determined the dependence of the chaotic time scale $\tau_\chi$ of this process on coupling strength $J$, detuning fluctuations $\delta$ of one-qubit energy spacing, and the number of qubits $n$. After a time $\tau_\chi$ the quantum computer hardware is melted. To prevent this melting one needs to introduce an efficient error-correction code which operates on a time scale much shorter than $\tau_\chi$, and suppresses the development of quantum chaos.

FIG. 16. Time dependence of the quantum entropy $S(t)$ for different values of $J$, $n = 16$, $J_c / \delta = 0.22$, and $J_c / \delta = 0.026$; one random realization is chosen: $J / \delta = 0.001 \approx J_c / \delta$ (disks), $J / \delta = 0.01 < J_c / \delta$ (crosses), $J / \delta = 0.03 \approx J_c / \delta$ (squares), $J / \delta = 0.2 \approx J_c / \delta$ (diamonds), and $J / \delta = 0.4 > J_c / \delta$ (triangles). The inset gives the probability of remaining in the same quantum register state for the same values of $J / \delta$. Averages are made over 200 states randomly chosen in the central band.

FIG. 17. Time explosion of quantum chaos in the quantum register: grayness represents the value of the projection probability $|\langle \psi | \chi(t) \rangle|^2$ of an initial state on the quantum register states ordered in energy, from white (maximal value) to black (minimal value). The horizontal axis corresponds to 150 states, and the vertical axis to 150 time steps from $t \delta = 0$ to $t \delta = 2$. At $t \delta = 0$, the chosen initial state is the superposition of two quantum register states. Here $n = 16$, $J / \delta = 0.4 / (J / \delta > J_c / \delta = 0.22)$, and one random realization is chosen. A color figure is available on http://xyz.lanl.gov/format/quant-ph/0005015.
To avoid the quantum chaos regime dangerous for quantum computing, one should engineer the quantum computer in the integrable regime below the quantum chaos border $J/\delta = 0.001$ ($J/\delta \ll J_{ci}/\delta = 0.026$).

FIG. 18. Same as Fig. 17 below the quantum chaos border.

Our main conclusion is that although in the quantum chaos regime a quantum computer cannot operate for long, fortunately the border for this process happens to be exponentially larger than the spacing between adjacent computer eigenstates, and therefore a broad parameter region remains available for realization of a quantum computer. Another possibility is to operate the quantum computer in the regime of quantum chaos. However, here one should keep in mind that after the chaotic time scale $\tau_\chi$, the computer hardware will melt due to interqubit coupling and quantum chaos. Therefore, the computer operability in this regime is possible only if many gate operations can be realized during the finite time $\tau_\chi$ (in a sense it becomes similar to the decoherence time). It is clear that the most preferable regime corresponds to quantum computer operation below the quantum chaos border.

ACKNOWLEDGMENTS

We thank O.P. Sushkov for stimulating discussions, and the IDRIS in Orsay and the CICT in Toulouse for access to their supercomputers. One of us (D.L.S.) thanks the Gordon Godfrey Foundation at the University of New South Wales in Sydney for their hospitality during the final stage of this work. This research was carried out partially within the frame of EC program RTN1-1999-00400.
[34] A.D. Mirlin and Y.V. Fyodorov, Phys. Rev. B 56, 13 393 (1997).
[35] B. Georgeot and D.L. Shepelyansky, Phys. Rev. Lett. 79, 4365 (1997).
[36] D. Weinmann, J.-L. Pichard, and Y. Imry, J. Phys. I 7, 1559 (1997).
[37] B. Georgeot and D.L. Shepelyansky, Phys. Rev. Lett. 81, 5129 (1998).

Chaos and Quantum Physics, edited by M.-J. Giannoni, A. Voros, and J. Zinn-Justin, Les Houches Lecture Series Vol. 52 (North-Holland, Amsterdam, 1991).

[39] T. Guhr, A. Müller-Groeling, and H.A. Weidenmüller, Phys. Rep. 299, 189 (1999).
[40] B. Georgeot and D.L. Shepelyansky, e-print quant-ph/9909074.

[41] A. Sørensen and K. Mølmer, Phys. Rev. Lett. 83, 2274 (1999).

[42] This result is valid for short-range couplings. In the case of long-range couplings where all qubits are coupled, the system is similar to the one studied in Ref. [37], $\Delta \sim \delta/n^2$, and we will have $J_c \sim C' \delta/n^2$, where $C'$ is a numerical constant. We note that long-range couplings can appear in systems where collective modes are used in the design of the quantum computer [11,12,18,20,41].

[43] For even $n$ and $\delta=0$, the system has an additional symmetry corresponding to the inversion of all $\sigma_i \rightarrow -\sigma_i$ (spin inversion). However, inside one symmetry class the spacing statistics is still close to $P_w(s)$ according to our numerical data (not shown).

[44] E.P. Wigner, Ann. Math. 62, 548 (1955); 65, 203 (1957).
[45] V.V. Flambaum, e-print quant-ph/9911061.