Turbulence phenomena for viscous fluids by a phase field model. Vortices and instability

Mauro Fabrizio

1 Department of Mathematics, University of Bologna, Italy

May 18, 2018

Abstract

Through Ginzburg-Landau and Navier-Stokes equations, we study turbulence phenomena for viscous incompressible and compressible fluids by a second order phase transition. For this model, the velocity is defined by the sum of classical and whirling components. Moreover, the laminar-turbulent transition is controlled by rotational effects of the fluid. Hence, the thermodynamic compatibility of the differential system is proved.

This model can explain the turbulence by instability effects motivated by a double well potential of the Ginzburg-Landau equation. The same model is used to understand the origins of tornadoes and the birth of the vortices resulting from the fall of water in a vertical tube. Finally, we demonstrate how the weak Coriolis force is able to change the direction of rotation of the vortices by modifying the minima of the phase field potential.

Key words: Viscous fluids, phase transitions, turbulence, instability.

1 Introduction

Turbulence phenomena in a viscous fluid was for a long time a controversial problem for the study and modeling of the transition from laminar flow to turbulent behavior [1], [2], [3], [15], [19], [21]. In many papers, it has been tried a modeling using only the classical Navier Stokes equations. In other works, the turbulence is studied by a stochastic cascade model. More recently, the turbulence is described by a phase transition, but not always with a suitable connection with Navier Stokes equations. In a previous work [5], we have presented a transition model by a system, where the Navier Stokes equation are associated with the Ginzburg-Landau equation [8], [13], [15] and where the transition is described by a phase field $\varphi$ and controlled not by the velocity $v$ of the fluid, but by its $|\nabla \times v|$. This choice is motivated by the results presented in many articles [10], [11], [19], [22], in which it is observed that the roughness of the walls or the obstacles inside a channel can anticipate the transition to turbulence, because they produce vortices. Indeed, it seems to us not convenient to believe that a laminar
flow in a pipe, consisting only of perfectly parallel velocities, can be transformed into a turbulent flow, when the velocity exceeds a given value predicted by the Reynolds number. On the other hand, it seems reasonable to assume that an appropriate (even small) disorder flow is needed for the transition.

In the paper we also want to emphasize, as the model proposed be convenient to describe the instability evident in phenomena of turbulence transitions. Unluckily, this phenomenon was not enough considered in literature. In this paper, we recall the model of a viscous incompressible fluid studied in \cite{5} with some corrections and improvements. In particular, we introduce a small (but important) modification, because now the phase \( \varphi \in (-1, 1) \), unlike of the previous work, where \( \varphi \in [0, 1) \). By this change, the instability effect is now described through the bifurcation, which is manifest in the Landau potential \cite{13}, when the transition triggers. Because in such a case the potential goes from a minimum to a double well potential.

Hence, in the paper we present an extension to compressible viscous fluids and its thermodynamic compatibility. As in \cite{5}, for these fluids we suppose the threshold, that identifies the transition, given by a suitable value of \( |\nabla \times \mathbf{v}| \).

Moreover, there is a great similarity between turbulence phenomena for viscous fluids and superfluids in Helium II (see \cite{1},\cite{3},\cite{12},\cite{13},\cite{16},\cite{17},\cite{20}) and superconductivity \cite{9}.

Finally, as a consequence of this model, it follows that the differential system can be well-posed problem only if we are in the laminar phase. Otherwise, when we are in turbulence flow, the instability of the model makes the system ill posed.

In the last part, we consider phenomena for which the transition does not produce turbulent effects, because the vortices have ample dimensions as in the tornadoes or in the fall of water in the hole of a sink. To describe these new effects, we considered the same differential systems, but with a different symmetry of Ginzburg-Landau potentials. So for these phenomena, the weak force of Coriolis plays an important role, naturally not generating the vortices, but indirectly influencing the direction of rotation, because it is able to modify the minima of the Ginzburg-Landau potential.

2 Incompressible fluids, turbulence and instable behavior

In this paper, the mathematical model proposed for a turbulence flow in a viscous fluid is confined in a smooth domain \( \Omega \subset IR^3 \) Hence, we suppose the phenomenon consequence of a phase transition, such that the transfer from laminar to turbulent flow is checked by a order parameter (or phase field) \( \varphi \in (-1, 1) \).

Following \cite{1}, the velocity \( \mathbf{v} \) of fluid is composed by the normal velocity \( \mathbf{v}_n \) and the rotational component \( \mathbf{v}_s \). Moreover, for incompressible fluids \( \mathbf{v}_n \) and \( \mathbf{v}_s \) are related by the constraints

\begin{equation}
\mathbf{v}_s = \nu \nabla \times \mathbf{v}_n , \quad \nabla \cdot \mathbf{v}_n = 0
\end{equation}
with $\nu$ scalar and positive coefficient. Moreover, we assume the velocity $v$ given by the following relationship between $v_n$ and $v_s$

$$v = v_n + \varphi v_s$$

moreover, the phase $\varphi \in (-1, 1)$ satisfies the Ginzburg-Landau equation

$$\rho_0 \dot{\varphi} = \nabla \cdot (L \nabla \varphi) - NF'(\varphi) + \alpha G'(\varphi)v_s^2$$

where $\rho_0 > 0$ is the density of the incompressible fluid, $\alpha > 0$ and $L(x) > 0$ for all $x \in \Omega$. Besides, the two potentials $F$ and $G$, are such that $F$ is given by a parabolic function with minimum for $\varphi = 0$ and $G$ by a function with two well potential and maximum in $\varphi = 0$. As an example we propose for $F$ and $G$ the functions

$$F(\varphi) = \frac{\varphi^2}{2}, \quad G(\varphi) = \frac{\varphi^4}{4} + \frac{b\varphi^3}{3} - \frac{\varphi^2}{2} - b\varphi, \quad \text{with } \varphi \in (-1, 1)$$

Finally, the velocity $v_n$ satisfies an extension of Navier-Stokes equation

$$\rho_0 \dot{v}_n = -\nabla p - \mu \nabla \times \nabla \times v_n + \gamma \nabla \times (\dot{G}(\varphi)v_s) + \rho_0 b$$

with the scalar $\gamma > 0$ and the body force $b$.

Now, we introduce the boundary conditions on the smooth domain $\Omega \in IR^3$ related with the system on two domains $\partial \Omega_1 \neq 0$ and $\partial \Omega_2$

$$\nabla \times v_n \times n|_{\partial \Omega_1} = 0$$
$$v_n|_{\partial \Omega_2} = 0$$

and

$$\nabla \varphi \cdot n|_{\partial \Omega} = 0$$

In this section, we suppose in $b = 0$, so we have that the laminar-turbulent transition occurs when $\left(\frac{2}{\rho_0} v_s^2 - 1\right)$ changes sign. Indeed, for $\frac{2}{\rho_0} v_s^2 > 1$, we are in turbulent phase. Otherwise, if $\frac{2}{\rho_0} v_s^2 < 1$ we are in laminar flows.

![Fig.1 - The two pictures describe the potential $F(\varphi)$ of the first and the potential $G(\varphi)$ with $b = 0$ the second.](image)
The turbulence behavior is related with the double well function of $G(\varphi)$ of (4), because the two minima are to the same height. So that, in such a case the system is unable to select between the two minima. So in any point, we can have a different choose of the minimum. So this instability produces a turbulence behavior of the fluid.

Even if our system can be instable, anyway we are always able to obtain the compatibility of the system with Thermodynamics, which for isothermal processes assumes the form of Dissipation Principle [3],[7],[8].

There exists a state function $\psi(\varphi, \nabla \varphi)$, called internal energy, such that:

$$\rho_0 \dot{\psi}(\varphi, \nabla \varphi) \leq \mu (\nabla \times \mathbf{v})^2 + \rho_0 \dot{\varphi}^2 + \frac{1}{2} \frac{d}{dt} (L(\nabla \varphi)^2 + 2N F(\varphi))$$

(7)

From the system (3-5), with the restriction (1-2) and the boundary conditions (6), we obtain the internal mechanical and structural power respectively

$$P_{m}^{i} = \mu (\nabla \times \mathbf{v}_n)^2 + \alpha \dot{G}(\varphi) \mathbf{v}_s^2$$

$$P_{s}^{i} = \rho_0 \dot{\varphi}^2 + \left(\frac{L}{2} (\nabla \varphi)^2\right) + N \dot{F}(\varphi) - \alpha \dot{G}(\varphi) \mathbf{v}_s^2$$

Finally, by Dissipation Principle

$$\rho_0 \dot{\psi}(\varphi, \nabla \varphi) \leq P_{m}^{i} + P_{s}^{i}$$

we obtain the inequality (7) with free energy

$$\psi(\varphi, \nabla \varphi) = \frac{1}{2\rho_0} (L(\nabla \varphi)^2 + 2N F(\varphi)).$$

3 A generalized Navier-Stokes and Ginzburg-Landau equations for turbulence compressible fluids

As in the previous section, we suppose the velocity of a compressible fluid composed by a normal velocity $\mathbf{v}_n$ and the rotational component $\mathbf{v}_s$ by the same equation (2), while the equation (1) is replaced by the following

$$\rho(x,t)\mathbf{v}_s(x,t) = \tilde{\lambda} \nabla \times \rho(x,t)\mathbf{v}_n(x,t)$$

(8)

where $\rho$ denotes the density of the fluid and $\tilde{\lambda}$ a positive coefficient. So, from [8] we obtain

$$\nabla \cdot (\rho \mathbf{v}_s) = 0$$

so from continuity equation and by the equation (2)

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}) = -\nabla \cdot (\rho \mathbf{v}_n) - \rho \mathbf{v}_s \cdot \nabla \varphi$$

(9)
hence, we have the motion equation

$$\rho \dot{v}_n = -\nabla (p(\rho) + (2\mu + \lambda) \nabla \cdot v_n - \mu \nabla \times \nabla \times v_n + \gamma \nabla \times \nabla \times (\rho \hat{G}(\varphi) v_s) + \rho b$$  \hspace{1cm} (10)$$

where \( \mu \) and \( \gamma \) are positive constants, such that \( \lambda \gamma = \alpha \) and \( b \) the body force.

Finally, as in the previous section, we consider the Ginzburg-Landau equation

$$\rho \dot{\varphi} = \nabla \cdot \rho L \nabla \varphi - \rho N F'(\varphi) + \alpha \rho G'(\varphi) v_s^2$$  \hspace{1cm} (11)$$

where the coefficients \( N, L \) and the potentials \( F(\varphi) \) and \( G(\varphi) \) are the same of the previous section, defined by (4).

So, the internal power \( P_m, P_s \) are given by

$$P_m = \mu (\nabla \times v_n)^2 + (2\mu + \lambda)(\nabla^2 v_n)^2 + \alpha \rho \hat{G}(\varphi) v_s^2$$

$$P_s = \rho \dot{\varphi}^2 + (\rho \frac{L}{2}(\nabla \varphi)^2) + \rho N \dot{F}(\varphi) - \alpha \rho \hat{G}(\varphi) v_s^2$$

Hence, the internal energy \( \psi(\varphi, \nabla \varphi) \) satisfies the inequality

$$\rho \dot{\psi}(\varphi, \nabla \varphi) \leq P_m + P_s =$$

$$\mu (\nabla \times v_n)^2 + (2\mu + \lambda)(\nabla^2 v_n)^2 + \rho \dot{\varphi}^2 + \rho \frac{d}{dt}(\frac{L}{2}(\nabla \varphi)^2 + N F(\varphi))$$  \hspace{1cm} (12)$$

4  Vortices and tornadoes

In this last section, we observe how the same model considered in the sect. 2, can study new phenomena like tornadoes or water vorticity, when it falls into a vertical deep hole. For these models, we consider the equations (1-3) and (4) with \( b \neq 0 \).

a . Water vorticity  It is known that usually the water that descends into a sink or more generally into a eddy, rotates in one direction in the northern hemisphere and in the other direction on the south hemisphere. In some papers, this different behavior (as well as for tornadoes) is explained by the Coriolis force. However, it is often observed that this opposite trend can not be directly attributed to the Coriolis force, since the rotation speed of the earth (hence the Coriolis force) is very weak. Otherwise in our model, this force can work indirectly by the coefficient \( b \) of \( G(\varphi) \) in (4). So that, a suitable definition of \( b \) could be

$$b = \tau \omega \times v_r \cdot t$$  \hspace{1cm} (13)$$

where \( t \) is the tangent to the parallel of the earth and \( v_r \) is the velocity relative to the terrestrial surface, while \( \omega \) is the angular velocity of the earth and \( \tau \) an appropriate constant coefficient.
In fact, to explain the rotation of the fluid that is generated during the fall in a vertical channel, we can use the equation (2) associated with (1) and (3) with the potential $F(\varphi)$ and $G(\varphi)$ of (4), where $b \neq 0$ can be defined by (13).

Now we consider the trajectory of a water particle when it enters the sink tube. Then, if the speed is small enough the motion is laminar. So the water falls inside the tube without any rotational component. If instead the speed is sufficiently high, then we observe, as a result of equation (2), that the motion does not remain on the same vertical plane, but a rotational component is generated around the tube axis. In other words, this effect is due to variation in speed when water enters the pipe. In fact, because of the trajectory we have a rotary motion with axis tangent to the edge of the tube, which on the basis of equation (2) generates a new rotational component $\varphi v_s$ around the tube axis, whose rotation sense is consequent to the sign of $\varphi$. Therefore, it is not the direct action of the Coriolis force that causes the variation of the water rotation in the two hemispheres, but the structural change caused by the transition described by eq. (3), which modifies the sign of $b$ of the eq. (4) and therefore the absolute minimum $\varphi_m$ of the potential (4) changes with the sign of $b$, which is related with the angular velocity $\omega$, which conditions the rotation of water according to the hemisphere in which the phenomenon occurs.

Therefore, according to the sign of $b$, we obtain the following two graphs.

![Graphs](image)

Fig. 2 - The two pictures are obtained with two different value of the coefficient $b$. In one we have $b > 0$ and in the second $b < 0$.

**b - Tornadoes** Then as a consequence of the wind action, we suppose to have in the atmosphere clouds vortices. So, when the vorticity $\mathbf{v}_v = \nu \nabla \times \mathbf{v}_n$ is quite high and its modulus exceeds the critical value, as a consequence of equation (2) we observe that the component $\varphi \mathbf{v}_s$ of the velocity is normal to vortices. Then, it begin the tornado effect towards the earth.

It should be noted that this model is able to explain the rotation of the tornado vortexes in the two hemispheres of the earth. In fact, to explain this different behavior we have to use again the coefficient $b$ defined in (13). In fact, when $b$ is positive, the graph of the function $G(\varphi)$ in (4) has an absolute minimum for $\varphi > 0$ and therefore a direction at the term $\varphi \mathbf{v}_s$, if instead $b < 0$ we have the absolute minimum for $\varphi < 0$ and therefore an opposite direction to the previous one.
Finally, in agreement with the sign of $\beta$, we have two similar graphs to the Fig. 2.

References

[1] I. Aranson and V. Steinberg, Spin-up and nucleation of vortices in superfluid $^4$He. Phys. Rev. B 54 (1996) 13072-13082.

[2] B. S. Chandrasekhar, Early experiments and phenomenological theories, Superconductivity (I). (Edited by R.D. Parks), Dekker (1969).

[3] M. Fabrizio, Ginzburg-Landau equations and first and second order phase transitions, Int. J. Engng. Sci. 44, (2006) 529–539.

[4] M. Fabrizio, A Ginzburg-Landau model for the phase transition in Helium II, Zeitschrift fur Angewandte Mathematik und Physik. 61, (2010) 329-340.

[5] M. Fabrizio, Critical phenomena in laminar–turbulence transitions by a mean field model, Meccanica 49 (9), (2014) 2079-2086.

[6] A.P. Finne, T. Araki, R. Blaauwgeers and V.B. Eltsov, An intrinsic velocity-independent criterion for superfluid turbulence - Nature, 2003.

[7] M. Frémond, Non-smooth thermomechanics. Springer-Verlag, Berlin (2002).

[8] E. Fried and M. Gurtin, Continuum theory of thermally induced phase transitions based on a order parameter, Physica D 68, (1993) 326-343.

[9] V. L. Ginzburg and L. D. Landau, On the theory of superconductivity, Zh. Eksp. Teor. Fiz. 20, (1950) 1064–1082.

[10] N. Goldenfeld. Roughness-Induced Critical Phenomena in a Turbulent Flow, Phys. Rev. Lett. 96, (2006) 044503.

[11] B. Hof, A. Juel, and T. Mullin, “Scaling of the turbulence transition threshold in a pipe,” Phys. Rev. Lett. 91 (2003) 014501.

[12] P. Kapitza,., Viscosity of Liquid Helium Below the $\lambda$-Point. Nature, 141 (1938) 3558.

[13] L. D. Landau, On the theory of superfluidity of helium II, J. Phys. USSR. 5, (1941) 71-77.

[14] L.D. Landau and E.M. Lifshitz, Fluid Mechanics. Pergamon Press. Oxford (1987).

[15] E.M. Lifsic and L.P. Pitaevskii, Statistical Physics, Part 2 : Volume 9, Pergamon Press. Oxford (1980).

[16] F. London, Superfluids II, John Wiley & Sons, New York (1950).
[17] K. Mendelssohn, Liquid Helium, in S. Flugge (ed.), Handbuch Physik, Springer Verlag, Berlin (1956) 370–461.

[18] D. Ruelle and F. Takens. On the nature of turbulence. Comm. Math. Phys., 20 (3) (1971), 167-192.

[19] R. Stevens, et al., Transitions between turbulent states in rotating Rayleigh-Bénard convection, Phys. Rev. Lett 103, (2009) 024503.

[20] L. Tisza. The Theory of Liquid Helium. Phys. Rev. 72 (9) (1947) 838–854

[21] A. P. Willis and R. R. Kerswell, Critical behavior in the relaminarization of localized turbulence in pipe flow, Phys. Rev. Lett. 98 (2007) 014501.

[22] D. Zubarev, V. Morozov, and O. V. Troshkin, Turbulence as a nonequilibrium phase transition, Theor. Math. Phys. 92 (1992) 896–908.