Ferromagnetic Transition in Strongly Repulsive One-Dimensional Fermi Gases in Arbitrary Potential with Arbitrary Particle Number

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(Dated: May 29, 2013)

We prove that a one-dimensional strongly repulsive Fermi gas will undergo a ferromagnetic transition at infinite repulsion. The transition is facilitated by the large spin degeneracy at infinite repulsion, which also allows arbitrarily tiny symmetry breaking field to destroy spin conservation to bring the system to the ferromagnetic ground state. The signature of this transition will show up prominently as large spin density fluctuations, as well as formation of large magnetic domains under symmetry breaking fields. We have demonstrated all these effects by exact calculations, and have worked out the quantum mechanical wavefunction that exhibits phase separation.

Ferromagnetism as we know it occurs in solid matters, caused by interactions between localized moments. Itinerant ferromagnetism, or ferromagnetism that emerges in time reversal symmetric systems of mobile fermions, is a novel phenomenon predicted by Stoner 80 years ago using Hartree approximation[1]. The idea was that for sufficiently strong repulsion, the system would rather raise its kinetic energy to be spin polarized so as to eliminate the overwhelming repulsion energy. However, itinerant ferromagnetism has never been found in electronic matters. The question of itinerant ferromagnetism was again raised recently in the context of repulsive three-dimensional(3D) Fermi gas[2]. Despite an initial report of evidence of ferromagnetism in strongly interacting regime[3], it found that such phenomenon in fact did not occur[4]. For strongly repulsive Fermi gas, the issue of ferromagnetism is complicated by severe atom loss due to three-body collisions[5]. This severe loss puts the system out of equilibrium, rendering the question of equilibrium ferromagnetism ill defined.

In a recent paper, we have pointed out by energy analysis[6] that the ferromagnetic transition in fact occurs in the repulsive branch of 1D Fermi gas. The transition is from a spin-singlet to the largest spin state as the coupling constant crosses a resonance from infinite repulsion to infinite attraction. Moreover, near the transition the system is stable against atom loss, and the problem of equilibrium ferromagnetism is well defined. Energetic consideration, however, is not sufficient for experimental realization of ferromagnetic transition. Due to the orthogonality of the distinct spin ground states before and after the transition, the system will remain in a singlet across the transition point with increasing energy rather than switching to the largest spin state. This raises the question of how to facilitate and to identify the signature of this transition.

In separate development, Selim Jochim’s group has recently succeeded in producing small clusters of Fermi gases in 1D harmonic traps in their ground states[7, 8]. Theoretical studies suggest that the clusters in the strongly repulsive regime have tendencies to become ferromagnetic[9, 10]. Since the underlying physics for itinerant ferromagnetism for large and small systems are the same, it is useful to establish rigorous results on the existence and the signature of this transition.

In this work, we first (i) establish a general theorem for ferromagnetic transition at infinite repulsion for a spin-1/2 1D Fermi gas with arbitrary number in arbitrary potential. Although the proof is similar to that in the homogenous case[6], it is important to establish this generalization because of its generality and its relevance to current experiments. (ii) We then propose a number of methods to identify this ferromagnetic transition. The first one is to measure spin susceptibility, or fluctuations in spin density. As the transition point is approached from the repulsive side, susceptibility will diverge linearly with coupling constant[11]. This is completely different from the 3D case where little spin fluctuations are observed[12]. (iii) We show that as the transition point is approached, any tiny perturbation that breaks rotational symmetry in spin space will induce transition to the ferromagnetic state. We shall demonstrate this for bulk system, and illustrate the dramatic development of magnetic domains in a four particle system by exact diagonalization as well as degenerate perturbation theory. (iv) Our studies also uncover the exact wavefunction for phase separation of magnetic domains. As far as we know, this is the first quantum mechanical description of phase separation in terms of wavefunctions.

(A) Theorem: A 1D repulsive Fermi gas with arbitrary particle number in any trapping potential will undergo a ferromagnetic transition at infinite repulsion. We are interested in a two-component fermion system with the Hamiltonian $H = \sum_{i=1}^{N} h(x_i) + g \sum_{i>j} \delta(x_i - x_j)$, where $h(x) = -(\hbar^2/2M)\partial^2_x + V(x)$, $V(x)$ is an arbitrary trapping potential, $g = \hbar^2 \gamma/M$ is the coupling constant for the 1D gas. The strong coupling regime corresponds to $\gamma/n \gg 1$, where $n$ is the density. The two components will be referred to as $\uparrow$ and $\downarrow$ “spin”. Experimentally, one can vary $-1/g$ continuously from negative to positive through 0, i.e. with $g$ increasing to $+\infty$ and...
then jumping to $-\infty$. Infinite coupling ($g^{-1} = 0$) will be referred to as “resonance”\textsuperscript{[12]}. For strong attraction $g \to -\infty$, two fermions with opposite spin will form a tightly bound state with width $|\gamma|^{-1}$ much less than the inter-particle spacing $n^{-1}$, (or $|\gamma|/n \gg 1$), and the binding energy $\epsilon_{b} = -h^{2} \gamma^{2}/4M$ tends to $-\infty$ near resonance, (as $|\gamma| \to \infty$)\textsuperscript{[13. Due to these deep bound states, a Fermi gas brought across the resonance from the repulsive side will not be able to access the true ground state (consisting of tightly bound pairs) due to energy conservation. The system at $g^{-1} = 0^{-}$ will remain in a gaseous state, with energy and wavefunction being continued analytically from the repulsive side ($g^{-1} = 0^{+}$). A stable Fermi gas that remains in a gaseous state even when the coupling constant is attractive ($g < 0$) will be referred to as a super-Tonk gas, using the nomenclature of the homogeneous case when the system is integrable\textsuperscript{[14].

For repulsive interaction ($g > 0$), the ground state of a spin-1/2 fermion system with equal spin population is a spin singlet, according to Lieb-Mattis theorem\textsuperscript{[15]. Let us consider the general case where the numbers of up and down fermions ($N_{\uparrow}$ and $N_{\downarrow}$) are arbitrary. We denote the energy of the system as $E(N_{\uparrow}, N_{\downarrow}, -1/g) > 0$. According to the Feynman-Hellmann theorem\textsuperscript{[16] (which applies to all eigenstates)

$$\frac{dE}{d(-1/g)} \geq 0. \quad (1)$$

The energy increases as the system approaches $1/g = 0^{+}$, i.e., $E(N_{\uparrow}, N_{\downarrow}, -1/g < 0) < E(N_{\uparrow}, N_{\downarrow}, 0) \equiv E^{*}$; and keeps rising as it is continued to the attractive or super-Tonk side. Hence we have

$$E^{*} < E(N_{\uparrow}, N_{\downarrow}, -1/g > 0). \quad (2)$$

As we shall see, all spin states are degenerate at $g = +\infty$. It in turn implies that $E^{*}$ is given by that of the largest spin state, i.e. total spin $S = (N_{\uparrow} + N_{\downarrow})/2$, which is independent of $N_{\uparrow}$ and $g$. Eq.\textsuperscript{[2]} shows the maximum spin state is the ground state among all gaseous states on the attractive side of the resonance, thus proving the transition to ferromagnetic state at $g = +\infty$.

What remains is to show that all spin states are degenerate at $g = +\infty$. The ground state in this case is

$$|\Psi\rangle = \int D(1, 2, \ldots, N) \chi^{(S, m)}(1, \ldots, N_{\uparrow}|N_{\downarrow} + 1, \ldots, N) \times \prod_{i=1}^{N_{\uparrow}} \psi_{i}^{\dagger}(i) \prod_{j=N_{\downarrow}+1}^{N} \psi_{j}^{\dagger}(j)|0\rangle, \quad (3)$$

where $N_{\uparrow} + N_{\downarrow} = N$, (1, 2, ..) stand the coordinates ($x_{1}, x_{2}, ..$) of the fermions; $f$ means integrating over all $x_{i}$; $\psi_{i}^{\dagger}(i)$ is the field operator that creates a $\uparrow$ fermion at $x_{i}$; $D(1, 2, \ldots, N)$ is a Slater determinant of the coordinates of all fermions (independent of spin) made up of the eigenstate of $h(x)$. It is therefore the wavefunction of a fully spin polarized state. The function $\chi^{(S, m)}(1, \ldots, N_{\uparrow}|N_{\downarrow} + 1, \ldots, N)$ is a spin eigenstate with total spin $S$ and $S_{z} = m$ with down spins located at $(x_{1}, \ldots, x_{N_{\downarrow}})$ and up spins at $(x_{N_{\downarrow}+1}, \ldots, x_{N})$. It is symmetric under the interchanges of the coordinates of up spins and those of down spins respectively. This spin function is independent of the external potential, and is identical to that in the homogenous case, which is a constant in any region ($x_{P1} < x_{P2} < \ldots < x_{P_N}$), where $\{P\}$ stands for a permutation $P$ of the numbers $1, 2, \ldots, N$\textsuperscript{[17]. As a result, Eq.\textsuperscript{[3]} satisfies the Schrodinger equation in this region with energy given by that of a fully polarized state. This shows that all spin states $|S, m\rangle$ are degenerate at infinite repulsion. QED.

The emergence of ferromagnetism at negative coupling constant may appear at first sight contradictory to the Stoner idea. The reason is that without access to tightly bound molecules, the system at $g < 0$ is in fact a super-Tonk gas with an interaction energy even more strongly repulsive than at $g = +\infty$, as indicated by Eq.\textsuperscript{[1]. That the system would rather remain in the ferromagnetic state instead of being a super-Tonk gas is precisely the Stoner argument.

(B) Large spin density fluctuations near ferromagnetic transition: In homogeneous repulsive Fermi gas, it has been shown from Bethe Ansatz calculation that spin susceptibility $\chi$ diverges linearly as the coupling constant $g$ in the strongly repulsive regime, $\chi = (2M/h^{2})(3/4\pi^{4})(\gamma/n^{2})$\textsuperscript{[11]. Since spin susceptibility is related to the fluctuations in spin polarization, it can be easily detected. In a trapped gas, the local susceptibility $T\chi(x) \equiv T\frac{d\langle s(x)\rangle}{dx} = \int dx'[\langle s_{z}(x)s_{z}(x')\rangle - \langle s_{z}(x)\rangle\langle s_{z}(x')\rangle]$, where $s_{z}(x)$ is the spin density operator at $x$ and $T$ is temperature. At elevated temperatures where correlation length becomes shorter, $T\chi(x)$ can be approximated by the local spin density fluctuation $\langle s_{z}(x)^{2}\rangle - \langle s_{z}(x)\rangle^{2}$, for which we can apply the result of the homogenous system through local density approximation. These fluctuations will also be magnified near the low density region since the interaction effect (characterized by the ratio $\gamma/n$) increases with decreasing density. They are in stark contrast with the similar measurements in the 3D case\textsuperscript{[4].

(C) Demonstration of ferromagnetic transition through phase separation: In this section, we study the formation of magnetic domains associated with ferromagnetic transition. As a theoretical device, we consider the response of the system to a tiny “field gradient” $V_{G}(x) = -G\sigma_{z}$, where $G$ is so small to cause any significant shift in the spin density profile of a free fermion system, (see Fig.1a), i.e., the energy change due to the field gradient is negligible compared to the total energy of the unperturbed system. The discussion of the realization of $V_{G}$ and other similar perturbations will be presented in Section (E).

To give an explicit example, we consider four fermions (two $\uparrow$, two $\downarrow$) in a harmonic trap with frequency $\omega$ in a tiny field gradient. We study its ground state by exact diagonalization. For free fermions, the tiny $G$ will cause
essentially no change in the spin density profile, as shown in Fig. 1a. As repulsion increases beyond a certain value \( g_c \), the two spin populations suddenly separate into two domains as shown in Fig. 1b. Moreover, the separated spin density profile for \( g > g_c \) quickly turns into that at \( g = +\infty \), indicating the quantum state after \( g > g_c \) settles into the infinite interaction fixed point. Conversely, if we start with zero field gradient (\( G = 0 \)) at large repulsion, we find that the ground state is indeed a spin singlet as dictated by Lieb-Mattis theorem. The density profiles for both spin components are identical as shown in Fig. 1c. However, as \( G \) increases beyond a critical value \( G_c \), \((G_c \to 0 \text{ as } g^{-1} \to 0)\), the system phase separates into two magnetic domains as shown in Fig. 1b. Yet its number density profile is essential the same as that at infinite repulsion with \( G = 0 \) (Fig. 1c).

Due to the large spin degeneracy, the ground state at \( g = +\infty \) can also be obtained using degenerate perturbation theory. In our four fermion case, there are six degenerate states: two \( \chi^{(0,0)} \), three \( \chi^{(1,0)} \), and one \( \chi^{(2,0)} \) state. (See Section (D) below). The ground state, which has a spin density distribution shown in Fig. 1b, is superposition of these six states. Remarkably, as shown in Section (D) below, this superposition produces a very simple yet nontrivial wavefunction. It is the \( N_1 = N_\downarrow = 2 \) case of the many-body wavefunction

\[
\chi^{PS}(1, 2, \ldots, N_\downarrow | 1', 2', \ldots, N_\uparrow')
\]

where \( j' = N_\downarrow + j, N_\uparrow' = N_\downarrow + N_\uparrow = N \), and \( i \) and \( j' \) denote the coordinates \( x_i \) and \( x_{N_\downarrow+j} \) of down and up spins respectively. Eq. (4) describes a state with all the down spins to the left of the up spins, though the location domain wall is unspecified. On the other hand, the determinant \( D \) gives rise to a density \( n(x) \). The location of the domain wall \((x_o)\) is then given by \( \int_{-\infty}^{x_o} dx n(x) = N_\downarrow \).

To estimate the critical value \( G_c \) for given repulsion \( g \) close to resonance, let us consider the case \( N_\uparrow = N_\downarrow = N/2 \). The ground state \(|\Psi_g\rangle\) is a singlet. Its energy near resonance is \( E(N, -1/g < 0) = E(N, 0)(1 - \alpha n/\gamma) \). where \( \alpha > 0 \) is a constant, and \( E(N, 0) \equiv E^* \) is the energy of a spin polarized fermi gas. The singlet state clearly does not take advantage of the field gradient, \(|\langle G_\uparrow | |\Psi_g\rangle = 0 \). In contrast, the total energy of the spin segregated state \(|\Psi^{PS}\rangle\) is \( E^{PS}(N) = E^* - GX_\uparrow + GX_\downarrow \), where \( X_\uparrow = \int d\mathbf{x} n_{\uparrow}(x) \), which scale as sample size \( L \) and particle number \( N \). Hence, we write \((X_\uparrow - X_\downarrow) = \beta NL \), where \( \beta > 0 \) is a constant. The energy difference between \(|\Psi_g\rangle\) and \(|\Psi^{PS}\rangle\) is then \( \Delta E = -E^* \alpha n/\gamma + \beta NL \). Spin segregation will occur if \( G > G_c \) is incomplete\[19\]. It is sufficient to find all the states with maximum spin projection \( \chi^{(S, S)} \); for other \( S_z \) states can be obtained using spin lowering operator. The procedure to construct spin eigenstates is (i) to construct the maximum spin state \( \chi^{(N/2, N/2)} \) for a system of \( N \) fermions,  

(D) The wavefunction of the phase separated ground state at infinite repulsion: To show how Eq. (4) results from a degenerate perturbation calculation, we first consider all the spin eigenstates \( \chi^{(S, m)} \). Some of these spin states have been given in ref. [17]. However, the list there is incomplete\[19\]. It is sufficient to find all the states with maximum spin projection \( \chi^{(S, S)} \), for other \( S_z \) states can be obtained using spin lowering operator. The procedure to construct spin eigenstates is (i) to construct the maximum spin state \( \chi^{(N/2, N/2)} \) for a system of \( N \) fermions,
(ii) contract fermions pairs with opposite spins into singlets to construct \(\chi^{(S,S)}\) with \(S < N/2\), and (iii) maintain the permutations symmetry in the spatial coordinates of all up spins and all down spins respectively. In the four particle case, we find \(\chi^{(2,2)}(1, 2, 3, 4) = 1,\)

\[
\begin{align*}
\chi_{3}^{(1,1)}(1|2, 3, 4) & = [(1-P_{12}) + (1-P_{13}) + (1-P_{14})][S(12)S(13)S(14)] \\
\chi_{2}^{(1,1)}(1|2, 3, 4) & = (1-P_{12})[S(12)(S(13) + S(14))] + (1-P_{13})[S(13)(S(12) + S(14))] \\
& + (1-P_{14})[S(14)(S(12) + S(13))] \\
\chi_{1}^{(1,1)}(1|2, 3, 4) & = (1-P_{12})S(12) + (1-P_{13})S(13) + (1-P_{14})S(14) \\
\chi_{2}^{(0,0)}(1, 2|3, 4) & = [(1-P_{13})(1-P_{24}) + (1-P_{14})(1-P_{23})][S(13)S(14)S(23)S(24)] \\
\chi_{2}^{(0,0)}(1, 2|3, 4) & = [1 - P_{13}](1-P_{24})S(13)S(24) + (1-P_{14})(1-P_{23})][S(14)S(23)].
\end{align*}
\]

where \(P_{ij}\) denotes the permutation the coordinates \(x_i\) and \(x_j\), and \(S(ij)\) is the sign of \((x_i - x_j)\). To be clear, the spin eigenstate states associated with \(\chi_{3}^{(1,1)}\) are

\[
\begin{align*}
|\Psi_{3}^{(S=1, S_z=1)}\rangle & = \int D(1, 2, 3, 4)\chi_{3}^{(1,1)}(1|2, 3, 4)\psi_{1}^{d}(1)\psi_{1}^{d}(2)\psi_{1}^{u}(3)\psi_{1}^{u}(4)\rangle_0 \\
|\Psi_{3}^{(S=1, S_z=0)}\rangle & = \int D(1, 2, 3, 4)|\chi_{3}^{(1,1)}(1|2, 3, 4) + \chi_{3}^{(1,1)}(2|1, 3, 4)|\psi_{1}^{d}(1)\psi_{1}^{d}(2)\psi_{1}^{u}(3)\psi_{1}^{u}(4)\rangle_0.
\end{align*}
\]

To show that \(|\Psi_{3}^{(S=1, S_z=1)}\rangle\) is state of total spin \(S = 1\), we note that it is annihilated by \(S_{+} = S_{x} + iS_{y}\), which then implies \(S_{z} = \sum_{i=1}^{4} S_{z+i}\). Noting that under the integral sign in Eq. (10), the factor \(1-P_{12}\) can be eliminated by turning \(\psi_{1}^{d}(1)\psi_{1}^{d}(2)\) into a singlet pair \(\psi_{1}^{d}(1)\psi_{1}^{u}(2) - \psi_{1}^{d}(1)\psi_{1}^{u}(2)\). This state can therefore be annihilated by \(S_{1+} + S_{2+}\). Since both 3 and 4 are spin up, the state is annihilated by total \(S_{+} = \sum_{i=1}^{4} S_{i+}\).

Performing degenerate perturbations on the six states \(|S, 0\rangle\), we find ground state to be of the form

\[
|G\rangle = \sqrt{\frac{1}{5}} |\Psi_{2}^{(2,0)}\rangle + \sqrt{\frac{1}{5}} |\Psi_{3}^{(1,0)}\rangle \sqrt{\frac{5}{8}} |\Psi_{1}^{(1,0)}\rangle + \sqrt{\frac{1}{12}} |\Psi_{2}^{(0,0)}\rangle + \frac{1}{2} |\Psi_{1}^{(0,0)}\rangle,
\]

where the over-bar means the state is normalized. From Eq. (12), the ground state is found to be

\[
\chi_{G}^{(12; 34)} = 1 + [12; 34] + [1; 234] + [2; 134] + S(13)S(24) + S(14)S(23) + S(41) + S(31) + S(42) + S(32),
\]

where \([12; 34] = S(13)S(14)S(23)S(24), \ [1; 234] = S(12)S(13)S(14).\) By evaluating Eq. (12) in all regions \(x_{P1} > x_{P2} > x_{P3} > x_{P4},\) one can see that it is the function Eq. (4), which gives rise to the spin density distribution shown by dashed lines in Fig.1b.

(E) Experimental demonstrations of ferromagnetic transition: In the absence of any symmetry breaking fields, ferromagnetic transition can be identified by the large number fluctuations in each spin component, as discussed in Section (B). Such fluctuations also imply the appearance of large magnetic domains. To realize of the field gradient term discussed in Section (C), it is necessary to stay in low field regime where the magnetic moments of different spin states are different. In the case of \(40\)K, there are spin states with Feshbach resonance near \(B = 200\)G, and with magnetic moments different enough to realize the field gradient term. The presence of a Feshbach resonance allows one to tune the 3D scattering length close to the confinement length, which in turn allows one to achieve infinite 1D coupling through the confinement-induced resonance. For fermions that do not have a Feshbach resonance at low fields, as long as they have a positive scattering length, strong repulsion \((\gamma/n \gg 1)\) can be achieved by making the system sufficiently dilute. Finally, another way to demonstrate ferromagnetism is to use a very small rf field to couple the two spin states. This introduces a symmetry breaking field \(\epsilon \int \psi_{1}^{d}(x)\psi_{1}^{u}(x)dx + h.c.\) and will align all the spins along \(\hat{x}\) in the strongly repulsive regime, even \(\epsilon\) is so small to cause any noticeable polarization in the non-interacting regime.

We would like to thank Randy Hulet for discussions.

XC acknowledges the support of NSFC under Grant No. 11104158. TLH acknowledges the support by DARPA under the Army Research Office Grant Nos. W911NF-07-1-0464, W911NF0710576, by the Institute for Advanced Study of Tsinghua University through the Qian Ren Program.

\[\text{[1]}\text{ E. Stoner, Philos. Mag. 15, 1018 (1933).}\]

\[\text{[2]}\text{ R. A. Duine and A. H. MacDonald, Phys. Rev. Lett. 95, 230403 (2005); L. J. LeBlanc, J. H. Thywissen, A. A.}\]
Burkov and A. Paramekanti, Phys. Rev. A 80, 013607 (2009); S. Pilati, G. Bertaina, S. Giorgini, and M. Troyer, Phys. Rev. Lett. 105, 030405 (2010); S.-Y. Chang, M. Randeria, and N. Trivedi, Proc. Natl. Acad. Sci. 108, 51 (2011).

[3] G.-B. Jo, Y.-R. Lee, J.-H. Choi, C. A. Christensen, T. H. Kim, J. H. Thywissen, D. E. Pritchard and W. Ketterle, Science 325, 1521 (2009).

[4] C. Sanner, E. J. Su, W. Huang, A. Keshet, J. Gillen, and W. Ketterle, Phys. Rev. Lett. 108, 240404 (2012).

[5] D.S. Petrov, M. Holzmann and G.V. Shlyapnikov, Phys. Rev. Lett. 84, 2551 (2000).

[6] X. Cui and T.-L. Ho, Phys. Rev. Lett., 110, 165302 (2013).

[7] F. Serwane, G. Zürn, T. Lompe, T. B. Ottenstein, A. N. Wenz, S. Jochim, Science 325, 1521 (2009).

[8] G. Zürn, F. Serwane, T. Lompe, A. N. Wenz, M. G. Ries, J. E. Bohn, S. Jochim, Phys. Rev. Lett. 108, 075303 (2012).

[9] E. J. Lindgren, J. Rotureau, C. Forssén, A. G. Volosniev and N. T. Zinner, [arXiv:1304.2992]

[10] P.O. Bugnion and G.J. Conduit, [arXiv:1304.3299]

[11] J. Y. Lee, X. W. Guan, K. Sakai, and M. T. Batchelor, Phys. Rev. B 85, 085141 (2012).

[12] M. Olshanii, Phys. Rev. Lett. 81, 938 (1998). In quasi-1D systems, $\gamma = 2/a_\perp (a_\perp/a_s - C)^{-1}$, where $C = 1.4603$ and $a_\perp$ is the transverse confinement length. The confinement induced resonance, $g \sim \gamma = \infty$, occurs at $a_\perp/a_s = C$.

[13] In quasi-1D system, the molecule near the confinement-induced resonance ($g = \infty$) is also tightly bound with its size of the order of confinement length. See T. Berge

man, M. G. Moore and M. Olshanii, Phys. Rev. Lett. 91, 163201 (2003). The deep molecule effectively suppresses the decay process of the repulsive upper branch at $g = \infty$; instead, the severe decay occurs when approaching $g = 0^-$ limit. See X. Cui, Phy. Rev. A 86, 012705 (2012).

[14] L. Guan and S. Chen, Phys. Rev. Lett., 105, 175301 (2010). Analytic continuation from Tonks to super-Tonks regime has been shown for a homogenous Fermi gas.

[15] E. H. Lieb and D. Mattis, Phys. Rev. 125, 164 (1962).

[16] H. Hellmann, Einhrung in die Quantenchemie. Leipzig: Franz Deuticke. p. 285, (1937). R.P. Feynman, Phys. Rev. 56, 340, (1939).

[17] L. Guan, S. Chen, Y. Wang, and Z.-Q. Ma, Phys. Rev. Lett. 102, 160402 (2009).

[18] N. Oelkers, M. T. Batchelor, M. Bortz, and X.-W. Guan, J. Phys. A 39, 1073 (2006).

[19] The total number of all spin states $\{|S,m\rangle\}$ for given $N_\uparrow$ and $N_\downarrow$ is $N!/ (N_\uparrow! N_\downarrow! )$.

[20] Note that all six states are orthogonal to each other except for $|\Psi_{1,0}^{(1,0)}\rangle$ and $|\Psi_{1,1}^{(1,0)}\rangle$. They can be combined as $|\Psi_{1,0}^{(1,0)}\rangle \pm |\Psi_{1,1}^{(1,0)}\rangle$ to ensure orthogonality when performing degenerate perturbations.

[21] In the high field regime, all hyperfine states with the same electron spin have the same magnetic moment, and their energy differences will not be affected by magnetic field gradient.

[22] M. Koschorreck, D. Pertot, E. Vogt, M. Köhl, arxiv: 1304.4980.