ON GRAVITATIONAL MOTIONS

ANGELO LOINGER

ABSTRACT. 1. Introduction. – 2, 2bis. Exact GR: a new proof of the geodesic character of all motions of bodies that interact only gravitationally. – 3 to 7. Linearized approximation of GR: a detailed illustration of its real meaning and of its shortcomings. – 8. Further considerations. – 9. A significant isomorphism. – Historical appendix.

PACS 04.20 – General relativity.

In memoria di Tullio Levi-Civita (1873-1941).

1. – In a continuous and incoherent “cloud of dust”, the elements of which interact only gravitationally, the motions of these elements are geodesic, as it follows from Einstein field equations (see, e.g., [1]). Consequently, no GW is emitted.

Now, this result can be obtained quite generally, i.e. without specifying the nature of matter tensor $T_{jk}$, $(j, k = 0, 1, 2, 3)$, as I shall now prove (sects. 2, 2bis): all gravitational motions are geodesic.

The sects. 3 to 7 regard an analysis of the linearized approximation of the exact GR. A preliminary and suitable advice: it does not concern the instances of motions of test-particles and light-rays in “external”, “rigidly given”, and weak gravitational fields. For a conceptual dispatch of these cases, the consideration at the end of sect. 3 is clearly sufficient; it embraces also the computations of Thirring-Lense effects.

2. – Let $ds^2 = g_{jk}(x) dx^j dx^k$ be the spacetime interval of the pseudo-Riemannian manifold pertinent to a system of bodies that interact only gravitationally. Denoting with $q^j(\tau)$, $(j = 0, 1, 2, 3)$, the translational coordinates of one of them as functions of proper time $\tau$, we have that

\begin{equation}
(1) \quad \mathcal{L} := g_{jk}(q(\tau)) \frac{dq^j}{d\tau} \frac{dq^k}{d\tau} = c^2
\end{equation}

is a first integral of Lagrange equations

\begin{equation}
(2) \quad \frac{\partial \mathcal{L}}{\partial q^i} - \frac{d}{d\tau} \frac{\partial \mathcal{L}}{\partial (dq^i/d\tau)} = 0.
\end{equation}

Now – as it is well known – eqs. (2) coincide with geodesic equations
\[
\frac{d^2 q^i}{d\tau^2} + \Gamma^i_{mn} \frac{dq^m}{d\tau} \frac{dq^n}{d\tau} = 0 .
\]

In the last analysis, this means that the geodesic character of the motions of our gravitating bodies is implicitly contained in the pseudo-Riemannian structure of \( ds^2 \), from which it follows the existence of the first integral \( \mathcal{L} = c^2 \). This result is independent of the precise expressions of the components \( g_{jk} \)'s of the metric tensor, that are determined by Einstein equations (plus the boundary conditions).

It is certain that in a geodesic motion no GW is sent forth, because, if we employ Riemann-Fermi coordinates \( y^i(\tau) \), any solution of eqs. (3) can be written in the form

\[
y^i(\tau) = a^i \tau + b^i ,
\]

where \( a^i \) and \( b^i \) are constants, and we see that no damping terms due to gravitational radiation are present.

From a conceptual point of view, our conclusion is intuitive: in GR the Newtonian force has been substituted by spacetime geometry, so that the bodies “move freely” in a fourdimensional manifold that is their own creation [2].

If there were also non-gravitational interactions (e.g., electromagnetic interactions), the conclusion of the non-existence of GW’s is still valid. A simple, qualitative proof rests on the fact that the kinematic elements (speeds, accelerations, time derivatives of accelerations, etc.) of the non-geodesic motions are not different from the kinematic elements of suitable purely gravitational (geodesic) motions. (See further \( \alpha \), \( \beta \), \( \gamma \) of [1]).

### 2bis.

Three remarks. i) We have considered the case of a discrete system of bodies. However, our procedure is clearly valid also for a continuous medium. ii) The previous formalism constitutes simply the extension to a system of gravitating masses of the well-known formalism concerning the geodesic motions of test-particles in a “rigidly” assigned potential field \( g_{jk} \). iii) Obviously, formulae [1] \( \div \) [4] hold also for a system of non-gravitating bodies which move freely in a pseudo-Euclidean spacetime referred to general coordinates.

### 3.

The previous considerations regard the exact formulation of GR. However, it could seem that in the linearized approximation of the theory things stand otherwise, and that any variation of a material distribution generates GW’s, which are propagated with the velocity of light in vacuo. Now, this approximation – owing to its formal resemblance to Maxwell theory – has created various misunderstandings, in primis about the physical reality of the GW’s.
Strictly speaking, the proofs that in the exact GR the GW’s are non-existing objects are sufficient to exclude the adequacy of any objection based on the linear version. However, the belief in the physical value of this version is so widespread that a specific indication of its weak points is quite advisable.

As it is well known, in the linearized approximation of GR one sets:

\[ g_{jk} \approx \eta_{jk} + h_{jk} \]

where \( \eta_{jk} \) is the Minkowskian tensor (in the usual diagonal form: \( 1, -1, -1, -1 \)), and the \( h_{jk} \)'s are “small” deviations from the \( \eta_{jk} \)'s. The metric tensor has a covariant character only under Lorentz transformations of the coordinates — and under general, but “small”, coordinate transformations: \( x^j \to x^j + \xi^j(x) \); we have:

\[ h'_{jk} = h_{jk} + \frac{\partial \xi_j}{\partial x^k} + \frac{\partial \xi_k}{\partial x^j}, \]

a formula which can also be interpreted as a gauge transformation. (Clearly, the \( h_{jk} \)'s can be locally transformed into zero by a finite transformation of general coordinates). Remember that all operations of raising and lowering of indices are performed with Minkowski tensor \( \eta_{jk} \).

In a fundamental memoir of 1944 [3], Weyl gave a new, original deduction of the linear version of GR. It was obtained with a fresh start, i.e. by studying and resolving the problem of the theoretical existence of a linear theory of gravitation. As main result, Weyl arrived at gravitational field equations that are identical with the linearized Einsteinian field equations.

If we choose the \( \xi_j \)'s in eqs. (5) so that \( \gamma_{jk} := h'_{jk} - (1/2)h_{nn}^m \eta_{jk} \) satisfies the equations

\[ \frac{\partial \gamma^k_j}{\partial x^k} = 0, \]

the linearized field equations become (as it is known):

\[ \eta^{mn} \frac{\partial^2 \gamma_{jk}}{\partial x^m \partial x^n} = -2\kappa T_{jk}. \]

An immediate consequence of (5') - (5'') is the differential conservation law of material energy-momentum, which yields also the equations of motion of bodies:

\[ \frac{\partial T^k_j}{\partial x^k} = 0; \]

Let us observe explicitly that we have here an ordinary (not a covariant) divergence.

Weyl makes a remark that was never previously made: for a “cloud of dust” we have, with obvious notations, \( T^{jk} = \mu u^j u^k \); accordingly, eqs. (6) give the following law of motion for a “dust” particle:
and we can say: in the linearized approximation of GR the gravitational field does not exert any force on bodies, i.e. is a “powerless shadow”. (Note that the “cloud of dust” is an emblematic instance in GR). This conclusion is perfectly confirmed by the exact theory, for which the gravitational force on bodies appears “only when one continues the approximation beyond the linear stage.” However, the linearized approximation gives the Poisson-Laplace equation, and therefore the Newtonian potential $1/r$. (Of course, it is a partial theory of gravity).

Eqs. (7) are mentioned only in Weyl [3]. They have a devastating effect on the current belief in the existence of GW’s: indeed, if the motions of the gravitationally-interacting “dust” particles satisfy eqs. (7), it is indisputable that no GW can be generated by them – and the customary deduction of the GW’s based on the solution of the linearized homogeneous equations ($T_{jk} = 0$) loses any physical meaning. Eqs. (7), as a consequence of eqs. (6), could have been written immediately after the discovery by Einstein and Grommer in 1927 that the equations of motion of bodies follow from Einstein field equations. We can also say that eqs. (7) are a simple consequence of the fact that – as Weyl pointed out in sect. 32 of Raum-Zeit-Materie [4] – “... wir befinden uns augenblicklich auf dem Boden der speziellen Relativitätstheorie ...”; i.e.: the $ds^2$ of the linearized approximation coincides with the Minkowskian $ds^2$. An assertion which could appear amazing to many physicists, who are misled by the typical instance of a test-particle $T$ (or of a light-ray $L$) in a given, “rigid” field $\eta_{jk} + h_{jk}$; it is clear that the motions of $T$ (or of $L$) are governed by the customary geodesic equations in which the $h_{jk}$ play a decisive role. (Remark, however, that also these kinds of motion do not generate GW’s).

3bis. – If $\Phi_k$ is the vector potential of the e.m. field $f_{jk}$, the sum

$$
\frac{\partial \Phi_k}{\partial x^j} + \frac{\partial \Phi_j}{\partial x^k}
$$

can locally be transformed into zero by means of a gauge transformation of $\Phi_k$. In the gravitational case all derivatives $\partial h_{jk}/\partial x^m$ can locally be transformed into zero by virtue of relations (5).

Whereas in Maxwell theory we have an energy-momentum tensor of the e.m. field which depends quadratically on the components $f_{jk}$’s, no tensor (different from zero) depending quadratically on derivatives $\partial h_{jk}/\partial x^m$ exists if the gauge relation (5) of the linear version $(L)$ of GR is required. As it was emphasized by Weyl [3], with a consideration that is just the counterpart in $(L)$ of a remark by Levi-Civita (1917) regarding the exact theory.
3ter. – Let us recall that if we express the equations \( R_{jk} - (1/2)g_{jk}R = -\kappa T_{jk} \) in a system of harmonic coordinates, that are characterized by the relations \( \partial(\sqrt{-g}g^{jk})/\partial x^k = 0 \), and put \( g_{jk} \approx \eta_{jk} + \gamma_{jk} \), eqs. \((5'')\) and \((5''')\) are obtained immediately.

The precise status of the linearized approximation with respect to the exact theory can be made evident by the following comparison: i) Exact GR: the covariant divergences of both sides of Einstein equations are equal to zero; the potential field \( g_{jk} \) is the spacetime – ii) Fundamental property of the linearized approximation: the ordinary divergences of both sides of its field equations are equal to zero; the approximate version describes a potential field of a Minkowskian spacetime, referred to a Lorentzian system of coordinates.

This comparison is useful for rendering intuitive the solution of the problem of the non-geodesic motions (see further \( \alpha \), \( \beta \), \( \gamma \) of \([1]\)). Assume that the “dust” is electrically charged, with a charge density \( \rho \). In the linearized theory we have the equations of motion

\[
(7') \quad \mu \frac{du^j}{d\tau} = \rho f^{jk}u_k ,
\]

i.e., motions that do not generate GW’s. In the exact theory the particles of an electrically neutral “dust” obey geodesic equations that are the strict analogue of eqs. \((7)\); for a charged “dust” the motions of the particles are non-geodesic, and correspond strictly with the motions described by the above eqs. \((7)\) – and it is intuitive that no GW is emitted.

4. – As it follows from eqs. \((5''')\), the linearized homogeneous \((T_{jk} = 0)\) equations are

\[
(9) \quad \eta^{mn} \frac{\partial^2 \gamma_{jk}}{\partial x^m \partial x^n} = 0 ,
\]

i.e. the customary homogeneous d’Alembert equations in Lorentzian coordinates; the “waves” given by \([9]\) represent undulatory fields in a Minkowskian spacetime. However, the curvature tensor \( R_{jklm} \) of the so-called “TT-waves” (see in the sequel) is different from zero – and this fact seems to confer them a particular reality. Now, this curvature tensor has an invariant meaning only under Lorentz transformations, and under infinitesimal transformations of general coordinates. The curvature tensor of a Minkowskian entity – as a \( \gamma_{jk} \)-wave – is a hybrid notion that mixes linearized and exact formulations.

5. – Let us consider now the “proof” of the famous quadrupole formula, e.g. in the detailed treatment given by Landau and Lifshitz in sects. 101 and 104 of their book \([5]\). The central point of their argumentation is the use of their variant of pseudo energy-momentum tensor \( t^{jk} \) of the gravitational
field. In the exact GR this mathematical object is a false (pseudo) tensor, because it has a covariant character only under linear transformations (in particular, Lorentz transformations); it can be transformed into zero at any spacetime point with a suitable transformation of general coordinates. An analogous conclusion holds in the linearized approximation, by virtue of eqs. (5). In spite of this fact, Landau and Lifshitz wrote (sect. 101 of [5]): “Possédant une énergie déterminée, l’onde gravitationnelle crée elle-même autour d’elle un certain champ de gravitation.” A striking example of a senseless statement, because only a true energy-momentum tensor can create a gravitational field. (A brilliant proof of the inadequacy of the very notion of pseudo energy-momentum tensor of a gravitational field was given by H. Bauer [Phys. Z., 19 (1918) 163], who showed – with reference to the first proposed variant of $t^{jk}$ – that it is possible to introduce spatial coordinates in a pseudo-Euclidean world for which the $t^{jk}$’s are different from zero, and the total “energy” is infinite. As an example, Bauer considers the following coordinates: $\xi^1 = (1/3)r^3; \xi^2 = -\cos \vartheta; \xi^3 = \varphi$, where $r, \vartheta, \varphi$ are the usual polar coordinates).

For a plane undulation which is propagated in the direction $x^1 \equiv x$, we can choose in eqs. (9) the coordinates in such a way that $\gamma^{jk}$ is only function of $x$ and $x^0 = ct$. Accordingly, eqs. (9) give:

$$
(10) \left( \frac{\partial^2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \gamma^{jk} = 0 ;
$$

it can now be proved that our plane wave is characterized by two components only, i.e. $\gamma^{23}$ and $\gamma^{22} = -\gamma^{33}$, which are functions of $(x - ct)$: they are the transverse-transverse (TT) components. The other components: longitudinal-longitudinal (LL) components and longitudinal-transverse (LT) components can be eliminated with an infinitesimal coordinate transformation.

Then, the authors compute the pseudo energy-momentum tensor $t^{jk}$ of the above plane wave, and find that the only component different from zero is $t^{01}$ (a dot denotes a time derivative):

$$
(11) t^{01} = \frac{c^2}{16 \pi G} \left( (\dot{\gamma}^{23})^2 + \frac{1}{4} (\dot{\gamma}^{22} - \dot{\gamma}^{33})^2 \right).
$$

By virtue of eqs. (5), all these derivatives can be locally transformed into zero.

In sect. 104 of [5] Landau and Lifshitz investigate the weak gravitational field generated by bodies in slow motions. They write (cf. our eqs. (5m), (5n) and (6)):

$$
(12) \frac{1}{2} \eta^{mn} \frac{\partial^2 \gamma_{jk}}{\partial x^m \partial x^n} = -\frac{8 \pi G}{c^4} (\mu u_j u_k) ;
$$

$$
(13) \frac{\partial \gamma^{jk}}{\partial x^k} = 0 = \frac{\partial}{\partial x^k} (\mu u^j u^k) .
$$
ON GRAVITATIONAL MOTIONS

The authors are interested in the following solution of eqs. (12) – the meaning of symbols is obvious:

\[ \gamma^{jk} = \frac{4G}{c^4} \int \left[ \mu u^j u^k \right]_{t-R/R_0} \mathrm{d}V ; \]

by taking into account the smallness of the speeds of the bodies of our physical system, eqs. (14) can be approximated, for the very far field, as follows:

\[ \gamma^{jk} = \frac{4G}{c^4} \frac{1}{R_0} \int \left[ \mu u^j u^k \right]_{t-R_0/c} \mathrm{d}V , \]

if \( R_0 \) is the distance from the coordinate origin, situated in a point within the spatial region of the system. Using the second set of eqs. (13), one can compute various relations among the integrals of (15); one finds \( \alpha, \beta = 1, 2, 3 \):

\[ \gamma_{\alpha\beta} = \frac{2G}{c^4 R_0} \frac{\partial^2}{\partial t^2} \int \mu \left( x, t - \frac{R_0}{c} \right) x_\alpha x_\beta \mathrm{d}V . \]

At large distances from the physical system, and inside small regions, the GW is practically a plane wave. Accordingly, we can compute the energy flow emitted by the system in \( x_1 \)-direction by means of eq. (11); from eqs. (16) we have:

\[ \gamma_{23} = \frac{2G}{3c^4 R_0} \ddot{D}_{23} ; \quad \gamma_{22} - \gamma_{23} = \frac{2G}{c^4 R_0} (\dddot{D}_{22} - \dddot{D}_{33}) , \]

where

\[ D_{\alpha\beta} := \int \mu \left( x, t - \frac{R_0}{c} \right) [3 x_\alpha x_\beta - \delta_{\alpha\beta} x_\gamma x_\gamma] \mathrm{d}V \]

is the quadrupole moment of masses. Then, the pseudo tensor \( t^{01} \) of eq. (11) can be written:

\[ c t^{01} = \frac{G}{36 \pi c^5 R_0^5} \left[ \left( \dddot{D}_{22} - \dddot{D}_{33} \right)^2 - \left( \ddot{D}_{23} \right)^2 \right] . \]

One derives easily from (19) the radiation emitted in an arbitrary direction, and then the total radiation emitted in all directions, i.e. the energy lost by our system in a second:

\[ - \frac{\mathrm{d}E}{\mathrm{d}t} = \frac{G}{45 c^5} D_{\alpha\beta} D_{\alpha\beta} . \]

I have reported almost literally, with inessential modifications, some passages of the treatment by Landau and Lifshitz [5].
The authors give now the result of a computation which anticipates the computations concerning the celebrated binary B PSR1913+16. They consider two bodies which interact according to Newton law and describe circular orbits. If \( m_1, m_2 \) are their masses, \( r \) their distance, and \( T = 2\pi/\omega \) their revolution period, we obtain:

\[
-\frac{dE}{dt} = \frac{32G}{5c^5} \left( \frac{m_1m_2}{m_1 + m_2} \right)^2 r^4 \omega^6;
\]

at this end, it is necessary to perform in eqs. (18) the passage from the continuous \( \mu[x, t - (R_0/c)] \) to its corresponding discrete expression by means of Dirac’s delta-distributions (masses \( m_1, m_2 \) are considered as pointlike).

Since, clearly:

\[
\omega^2r^3 = G(m_1 + m_2) \quad ; \quad E = -\frac{Gm_1m_2}{2r},
\]

we have:

\[
\frac{dr}{dt} = \frac{2r^2}{Gm_1m_2} \frac{dE}{dt} = -\frac{64G^3m_1m_2(m_1 + m_2)}{5c^3r^3},
\]

which gives the approaching speed of the two bodies as it follows from the energy loss due to gravitational radiation.

6. – The results of previous sect. 5 are physically meaningless. Indeed, since we know that the equations

\[
\frac{\partial}{\partial x^k} (\mu u^j u^k) = 0 ;
\]

have as a consequence \( du^j/d\tau = 0 \) (sect. 3), it is not allowed to postulate in the linearized approximation of GR the existence of the action on bodies of Newton force; actually, the gravitational force on them appears only beyond the linear stage. Further, eqs. (24) tell us that no gravitational motion generates GW’s, and therefore the undulatory solutions of eqs. (9) are destitute of a physical reality.

In the exact formulation of GR the “dust” particles describe geodesic lines – and therefore cannot emit GW’s (cf. sect. 2). Further, as it was pointed out by Weyl – see p.268 of [3] – it is always possible to choose a coordinate system for which two gravitating bodies in relative motion are both at rest. This Weylian observation clarifies very well the conceptual meaning of the computations by Thirring and Lense: instances of effects of Einsteinian “dragging” forces, not very different in nature from the Newtonian centrifugal and Coriolis forces. In general relativity no kinematic parameter (velocity, acceleration, time derivative of acceleration, etc.) has an invariant character. On the contrary, in Maxwell theory only the reference frames in rectilinear and uniform relative motions are physically equivalent.
7. – If we apply the treatment of sect. 2 to the linear version of GR, where \( ds^2 \) coincides with Minkowskian \( ds^2 \), we find immediately, in lieu of eqs. (3), the simple equations

\[
\frac{d^2 q^i}{d\tau^2} = 0 ,
\]

i.e., eqs. (7).

8. – Sects. 3 to 7 (with the exception of the formulae by Landau and Lifshitz [5]) will result unpalatable to many readers, owing to a diffuse opinion that the Weylian linear version of GR [3] does not coincide exactly with the current linearized approximation. Of course, this is not true, as a simple inspection of the corresponding formulae can easily show. (The only, inessential, difference is that in [3] the “smallness” of the \( h_{jk} \’s \) is not postulated).

I emphasize again that there are two different applications of the approximate equality \( g_{jk} \approx \eta_{jk} + h_{jk} \). The application No.1 concerns the computations of the geodesic motions of test-particles and light-rays in “external” manifolds that are weakly pseudo-Riemannian. The application No.2 regards the approximation of the Einsteinian field equations; it gives eqs. (5′′) and (5′′′): i.e., the linear version of GR (Minkowskian spacetime).

Also the results of sects. 2 and 2bis can give rise to some perplexity, because they are at variance with current ideas on GW’s. As a matter of fact, GR does not admit the physical existence of GW’s – as it was first proved by Levi-Civita in 1917. And experience continues to confirm the validity of this theorem [6]. A very simple and qualitative proof of it runs as follows (for an analytical support see [7]). Consider the general concept of wave (without any adjective). There exist waves that are undulatory perturbations of material media (as air, water, etc.; formerly, cosmic ether), and waves that are undulatory perturbations of fields in vacuo with respect to an infinite class of physically privileged reference frames, as the Lorentzian frames of special relativity. More generally, we can have field waves in vacuo with respect to an infinite class of physically privileged coordinate systems of a “rigidly given” pseudo-Riemannian manifold, endowed with uniformity properties. Now, in the exact formulation of GR the metric tensor \( g_{jk} \) is the spacetime; thus, an undulatory \( g_{jk} \) lacks of any spatio-temporal substrate (as Minkowski spacetime or uniform pseudo-Riemannian manifold) through which it can be propagated. Consequently, it is doomed to be a property of some systems of general coordinates; a change of coordinates can impair its undulatory character, and give it an arbitrary velocity of propagation.

In short: it is only a mathematical object with a zero energy and without physical reality – even if its curvature tensor is different from zero.

9. – Let us consider in Minkowski spacetime of special relativity a tensor field \( \varphi_{jk}(x) [= \varphi_{kj}(x)] \), \((j, k = 0, 1, 2, 3)\), satisfying the following equations:
\begin{align}
\eta^{mn} \frac{\partial^2 \varphi_{jk}}{\partial x^m \partial x^n} &= S_{jk}, \\
\frac{\partial \varphi_{jk}}{\partial x_k} &= 0, \quad \left( \Rightarrow \frac{\partial S_{jk}}{\partial x_k} = 0 \right);
\end{align}

\varphi_{jk} and its source \( S_{jk} \) are of an indefinite physical nature. Eqs. (26) and (27) are invariant under the gauge transformations

\begin{equation}
\varphi_{jk} \rightarrow \varphi'_{jk} = \varphi_{jk} + \frac{\partial \xi_j}{\partial x_k} + \frac{\partial \xi_k}{\partial x_j} - \eta_{jk} \frac{\partial \xi_m}{\partial x_m},
\end{equation}

where the four functions \( \xi_j(x) \) satisfy the homogeneous d’Alembert equations

\begin{equation}
\frac{\partial^2 \xi_j}{\partial x_k \partial x^k} = 0;
\end{equation}

(Of course, \( S_{jk} \) is a gauge invariant quantity, as the current \( s_k \) of Maxwell theory).

If we put \( \varphi_{jk} = \gamma_{jk} \), and \( S_{jk} = -2\kappa T_{jk} \), we re-obtain the equations of the linearized approximation of GR. We see in the clearest way that in this approximation we are dealing with the special relativity \[4\].

Some authors affirm that the nonlinearity of the Einsteinian field equations is the only responsible for the fact that these equations have as a consequence the equations of motions of bodies. A false opinion: indeed, such a consequence exists also in the linearized approximation of the theory.

\begin{center}
\textit{Historical appendix}
\end{center}

In the current literature – even in a literature of historical character (see, e.g., \[3\] – Weyl’s memoir \[3\] is strangely ignored. An exiguous minority of theoreticians have read it, and have concluded erroneously that Weyl’s linear theory contradicts the usual linearized approximation of GR. Maybe, Weyl’s observation that in the linear version of GR the gravitational field is a powerless shadow (because it does not exert any force on bodies) can have misled many cursory readers. In reality – as I have pointed out in sect. 3 –, Weyl’s result could have been discovered immediately after 1927, when the geodesic principle became the geodesic theorem.

Another strange fact is the current overlooking of Hilbertian repulsive effect \[9\], with its important consequences on the behaviour of geodesic lines of test-particles and light-rays in a given Schwarzschild manifold. An extraneous factor could have played a role in this neglect: the diffuse belief in surprising properties of a well-known “soft” geometrical singularity.

It is interesting that \textit{Hilbert} \[9\] \textit{does not mention the GW’s}, and utilizes the linearized approximation of GR only for a perturbative proof of Serini’s
theorem [10]: the non-existence of *regular* time-independent solutions of \( R_{jk} = 0 \), that become pseudo-Euclidean at spatial infinity; *i.e.*, the unique regular solution of this kind is the Minkowskian one.

Finally, I wish to recall a significant remark by Hilbert [9] on the physical meaning of any statement (*Aussage*) in general relativity. He emphasized that in GR a given statement has a real physical meaning *only if* it has an invariant character under any whatever transformation of general coordinates. An analogous criterion holds obviously for the properties (*Eigenschaften*). (Thus, *e.g.*, the wave character of a \( g_{jk} \), and its velocity of propagation, are unphysical properties, see sect. 8).

A geometric comparison: in the differential geometry of curves and surfaces a given statement, or a given property, have a real geometric meaning *only if* they are independent of the choice of the coordinates.

Hilbert wrote [9]: “Dem Wesen des neuen Relativitätsprinzipes [*i.e.*, of GR] entsprechend müssen wir [...] die Invarianz nicht nur für die allgemeinen Gesetze der Physik verlangen, sondern auch jeder Einzelaussage [*to any single statement*] in der Physik den invarianten Charakter zusprechen, falls sie einen physikalischen Sinn haben soll – im Einklang damit, daß jede physikalische Tatsache letzten Endes durch Lichtuhren, d.h. durch Instrumente von *invarianten* Charakter feststellen sein muß. Genau so wie in der Kurven- und Flächentheorie eine Aussage, für die die Parameterdarstellung der Kurve oder Fläche gewählt ist, für die Kurve oder Fläche selbst keinen geometrischen Sinn hat, wenn nicht die Aussage gegenüber einer beliebigen Transformation der Parameter invariant bleibt oder sich in eine variable Form bringen läßt, so müssen wir auch in der Physik eine Aussage, die nicht gegenüber jeder beliebigen Transformation des Koordinatensystems invariant bleibt, als *physikalisch sinnlos* bezeichnen.”

References

[1] A. Loinger, *Nuovo Cimento* B, 115 (2000) 679; also in arXiv:astro-ph/0003230 (March 16th, 2000). For a generalization to not purely gravitational motions, see in particular: a) arXiv:physics/0106052 (June 17th, 2001); b) ibid. /0202065 (February 27th, 2002); c) ibid. /0606019 (June 2nd, 2006).
[2] H. Weyl, *Mathematische Analyse des Raumproblems* (J. Springer, Berlin) 1923. Siebente Vorlesung. For an Italian translation by A. Loinger, see H. Weyl, *Analisi matematica del problema dello spazio* (Zanichelli, Bologna) 1991. See also A. Loinger, *Riv. Nuovo Cimento*, 11 (no.8) (1988) 1.
[3] H. Weyl, *Amer. J. Math.*, 66 (1944) 591. See also the Appendix in: A. Loinger, arXiv:physics/0407134 (July 27th, 2004).
[4] H. Weyl, *Raum-Zeit-Materie*, Siebente Auflage (Springer-Verlag, Berlin, etc.) 1988. See also W. Pauli, *Teoria della Relatività* (Boringhieri, Torino) 1958, sect. 60.
[5] L. Landau et E. Lifchitz, *Théorie du Champ* (Editions Mir, Moscou) 1966.
[6] The recent papers on the experimental search of GW’s are lists of hopes of future detections. See, *e.g.*, i) “Host Galaxies Catalog Used in LIGO Searches for Compact Binary Coalescence” – by R.K. Kopparapu *et alii*, to appear in *Astrophys. J.*, March 20th, 2008; arXiv:0706.1283 [astro-ph]; ii) “Search method for coincident events from LIGO and IceCube detectors” – by Y. Aso *et alii*, arXiv:0711.0107 [astro-ph] Jan 31th, 2008.
[7] A. Loinger, arXiv:gr-qc/9909091 (September 30th, 1999); and references therein.

[8] D. Kennefick, *Traveling at the Speed of Thought* (Princeton University Press, Princeton and Oxford) 2007. A not too objective panorama (1916-1997) of papers about GW’s.

[9] D. Hilbert, *Mathem. Annalen*, 92 (1924) 1; also in *Gesammelte Abhandlungen*, Dritter Band (J. Springer, Berlin) 1935, p.258. This memoir reproduces three previous communications (1915, 1916, 1918) with some slight modifications. – See also A. Loinger and T. Marsico, arXiv:0803.0050 [physics.gen-ph] 1 Mar 2008; and references therein.

[10] W. Pauli, *Teoria della Relatività* (Boringhieri, Torino) 1958, sect. 62; A. Loinger, arXiv:physics/0504171 (April 23rd, 2005); and the reference therein.

A.L. – Dipartimento di Fisica, Università di Milano, Via Celoria, 16 - 20133 Milano (Italy)

E-mail address: angelo.loinger@mi.infn.it