ON CERTAIN MOMENTS OF HARDY’S FUNCTION $Z(t)$ OVER SHORT INTERVALS

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ABSTRACT. Let as usual $Z(t) = \zeta(\frac{1}{2} + it)\chi^{-1/2}(\frac{1}{2} + it)$ denote Hardy’s function, where $\zeta(s) = \chi(s)\zeta(1 - s)$. Assuming the Riemann hypothesis upper and lower bounds for some integrals involving $Z(t)$ and $Z'(t)$ are proved. It is also proved that

$$H(\log T)^k \ll_{k, \alpha} \sum_{T < \gamma \leq T + H} \max_{\gamma \leq \tau \leq \gamma^+} |\zeta(\frac{1}{2} + i\tau)|^{2k} \ll_{k, \alpha} H(\log T)^k.$$

Here $k > 1$ is a fixed integer, $\gamma, \gamma^+$ denote ordinates of consecutive complex zeros of $\zeta(s)$ and $T^\alpha \leq H \leq T$, where $\alpha$ is a fixed constant such that $0 < \alpha \leq 1$. This sharpens and generalizes a result of M.B. Milinovich [17].

1. Introduction

Let the Riemann zeta-function be, as usual,

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \quad (\Re s > 1).$$

For $\Re s \leq 1$ one defines $\zeta(s)$ by analytic continuation (see the monographs of H.M. Edwards [3], the author [11] and E.C. Titchmarsh [20] for the properties of $\zeta(s)$). Here the Riemann Hypothesis (RH), that all complex zeros of $\zeta(s)$ satisfy $\Re s = \frac{1}{2}$, is assumed throughout the paper. The Riemann zeta-function satisfies the functional equation

$$(1.1) \quad \zeta(s) = \chi(s)\zeta(1 - s) \quad (\forall s \in \mathbb{C}), \quad \chi(s) := \frac{\Gamma(\frac{1}{2}(1 - s))}{\Gamma(\frac{1}{2}s)}\pi^{-s/2},$$

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where $\Gamma(s)$ is the familiar gamma-function. One then defines Hardy’s function $Z(t)$ as

$$Z(t) := \zeta(\frac{1}{2} + it)(\chi(\frac{1}{2} + it))^{-1/2},$$

which is real for $t$ real and $|\zeta(\frac{1}{2} + it)| = |Z(t)|$. Thus the real zeros of $Z(t)$ correspond to the zeros of $\zeta(s)$ of the form $\frac{1}{2} + it$, which makes Hardy’s function an invaluable tool in the study of zeros of $\zeta(s)$ on the critical line $\Re s = \frac{1}{2}$. For an extensive account on $Z(t)$ the reader is referred to the author’s monograph [16].

Several papers deal with the estimation of the sum

$$M_k(T) := \frac{1}{N(T)} \sum_{0 < \gamma < T} \max_{\gamma \leq \gamma' \leq \gamma^+} |\zeta(\frac{1}{2} + i\gamma)|^{2k} \equiv \frac{1}{N(T)} \sum_{0 < \gamma < T} \max_{\gamma \leq \gamma' \leq \gamma^+} |Z(\gamma)|^{2k}.$$

Here $k \in \mathbb{N}$ is fixed, and $\gamma, \gamma^+$ denote ordinates of consecutive complex zeros of $\zeta(s)$, ordered according to their size. Also, as usual,

$$N(T) = \sum_{0 < \gamma < T} 1 = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)$$

counts (with multiplicities) the number of zeros of $\zeta(s)$ whose ordinates $\gamma$ satisfy $0 < \gamma \leq T$.

In [2] B. Conrey and A. Ghosh proved, under the RH,

$$M_1(T) = \frac{e^2 - 5}{2} \log T + O(1).$$

Actually, they prove a somewhat stronger result than (1.5), namely

$$\sum_{T < \gamma < T + H} \max_{\gamma \leq \gamma' \leq \gamma^+} |\zeta(\frac{1}{2} + i\gamma)|^2 = \frac{e^2 - 5}{4\pi} H \log^2 T + O(H \log T)$$

with $H = T^{3/4}$. This follows from their proof on noting that (1.4) implies

$$N(T + H) - N(T) \sim \frac{H}{2\pi} \log T \quad (T \to \infty).$$

B. Conrey [1] obtained, also under the RH,

$$\frac{\sqrt{21}}{45\pi} (1 + o(1)) \log^4 T \leq M_2(T) \leq \frac{1 + o(1)}{\pi \sqrt{15}} \log^4 T \quad (T \to \infty),$$
and R.R. Hall [4], [5] obtained some further improvements of (1.7). A general result, due to M.B. Milinovich [17], states that under the RH, for fixed \( k \in \mathbb{N} \),

\[
(1.8) \quad (\log T)^{k^2 - \varepsilon} \ll_{k, \varepsilon} \mathcal{M}_k(T) \ll_{k, \varepsilon} (\log T)^{k^2 + \varepsilon}.
\]

Here \( \ll_{k, \varepsilon} \) means that the constant implied by the \( \ll \)-symbol depends only on \( k \) and \( \varepsilon \), an arbitrarily small positive number, not necessarily the same one at each occurrence. The bounds in (1.8), when \( k = 1, 2 \), are implied by (1.6) and (1.7), respectively.

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2. Statement of results

Milinovich [17] derives (1.8) from upper and lower bounds involving certain integrals with \( Z(t) \) and \( Z'(t) \), which seem to be of independent interest. He investigated integrals over the “long” interval \([0, T]\), but here we are interested in the integrals over the “short” intervals \([T, T + H]\), where \( H = H(T) \) may be much smaller than \( T \). We shall prove here the following theorems.

**THEOREM 1.** Let \( k \geq 2 \) be a fixed integer. Under the RH we have, for \( T^\alpha \leq H = H(T) \leq T, 0 < \alpha \leq 1 \) a fixed constant,

\[
(2.1) \quad \int_T^{T+H} (Z'(t))^2 Z^{2k-2}(t) \, dt \gg_{k, \alpha} H(\log T)^{k^2 + 2}.
\]

**THEOREM 2.** Let \( k \in \mathbb{N} \) be fixed. Under the RH we have, for \( T^\alpha \leq H = H(T) \leq T, 0 < \alpha \leq 1 \) a fixed constant,

\[
(2.2) \quad \int_T^{T+H} |\zeta'(\frac{1}{2} + it)|^{2k} \, dt \ll_{k, \alpha} H(\log T)^{k^2 + 2},
\]

and

\[
(2.3) \quad \int_T^{T+H} (Z'(t))^{2k} \, dt \ll_{k, \alpha} H(\log T)^{k^2 + 2}.
\]

These bounds differ from the analogous results of [17] in two aspects. Firstly, Milinovich has the integrals over \([0, T]\), which corresponds to the case \( H = T \) in our theorems. Indeed, if (2.1)–(2.3) hold with \( H = T \), then replacing \( T \) by \( T/2, T/2^2, \ldots \) etc. and adding up all the results we obtain (2.1)–(2.3) with the interval of integration \([0, T]\). Secondly, in (2.1) Milinovich obtained \( k^2 + 2 - \varepsilon \)}
as the exponent of the logarithm, and in (2.2) and (2.3) he had the exponents $k^2 + 2 + \varepsilon$ of the logarithm. He remarks on page 1122 that one can get rid of the $\varepsilon$’s in his bounds provided that one has

\begin{equation}
\int_0^T |\zeta(\frac{1}{2} + it)|^{2k} \, dt \ll_k T (\log T)^{k^2}
\end{equation}

and

\begin{equation}
\int_0^T |\zeta'(\frac{1}{2} + it)|^{2k} \, dt \ll_k T (\log T)^{k^2 + 2k}.
\end{equation}

The estimates (2.4) and (2.5) do hold indeed. Namely K. Soundararajan [19] proved (2.4) with the exponent of $\log T$ in (2.4) equal to $k^2 + \varepsilon$. The author [15] improved and sharpened Soundararajan’s bound by showing that

\begin{equation}
\int_T^{T+H} |\zeta(\frac{1}{2} + it)|^{2k} \, dt \ll_{k, \alpha} H (\log T)^{k^2 + 1 + O(1/\log_3 T)} \quad \text{(RH)}.
\end{equation}

Here $T^\alpha \leq H \leq T$ where $0 < \alpha \leq 1$ is a fixed number, and

$$
\log_3 T = \log \log \log T = \log (\log_2 T).
$$

Note that [15] appeared before [19] because of the backlog of “Ann. Math.” The key result in [15], which is a proper generalization of the corresponding result in [19], is

THEOREM A. Let $H = T^\theta$ where $0 < \theta \leq 1$ is a fixed number, and let $\mu(T, H, V)$ denote the measure of points $t$ from $[T - H, T + H]$ such that

$$
\log |\zeta(\frac{1}{2} + it)| \geq V, \quad 10 \sqrt{\log_2 T} \leq V \leq \frac{3 \log 2T}{8 \log_2(2T)}.
$$

Then, under the RH, for $10 \sqrt{\log_2 T} \leq V \leq \log_2 T$ we have

$$
\mu(T, H, V) \ll H \frac{V}{\sqrt{\log_2 T}} \exp \left( - \frac{V^2}{\log_2 T} \left( 1 - \frac{7}{2\theta \log_3 T} \right) \right),
$$

for $\log_2 T \leq V \leq \frac{1}{2} \theta \log_2 T \log_3 T$ we have

$$
\mu(T, H, V) \ll H \exp \left( - \frac{V^2}{\log_2 T} \left( 1 - \frac{7V}{4\theta \log_2 T \log_3 T} \right)^2 \right),
$$
and for \( \frac{1}{2} \theta \log_2 T \log_3 T \leq V \leq \frac{3 \log 2T}{8 \log_2(2T)} \) we have
\[
\mu(T, H, V) \ll H \exp\left(-\frac{1}{20} \theta V \log V\right).
\]

Later A. Harper \cite{9} (RH) improved the upper bound in \cite{19} by establishing (2.4) for the long interval \([0, T]\). As remarked in \cite{9} on p. 4, the method of \cite{15} leading to (2.6), i.e., Theorem A, can be combined with that of \cite{9} to produce the sharp upper bound over the short interval \([T, T + H]\), namely
\[
(\text{2.7}) \quad \int_T^{T+H} |\zeta\left(\frac{1}{2} + it\right)|^{2k} dt \ll_{k, \alpha} H(\log T)^{k^2} \text{ (RH)}.
\]
The bound in (2.7) is the key ingredient in the proof of our results. It is, up to the values of the \( \ll \)-constants, best possible, since long ago it was shown by R. Balasubramanian and K. Ramachandra (see the latter’s monograph \cite{18}, in particular the remark on p. 45) that, if \( k \geq 1 \) is a fixed integer, then for \( C(\varepsilon, k) \log \log T \leq H \leq T / 2 \) we have
\[
(\text{2.8}) \quad \int_T^{T+H} |\zeta\left(\frac{1}{2} + it\right)|^{2k} dt \geq (C_k^\prime - \varepsilon)H(\log H)^{k^2},
\]
where
\[
C_k^\prime = \frac{1}{2\Gamma(k^2 + 1)} \prod_p \left\{ (1 - p^{-1})^{k^2} \sum_{m=0}^{\infty} \left( \frac{\Gamma(k + m)}{\Gamma(k)m!} \right)^2 p^{-m} \right\}.
\]
We note that the lower bound in (2.8) is \emph{unconditional}, with a very wide range for \( H \). As for (2.5), it will be shown later that a corresponding result holds over \([T, T + H]\).

\textbf{THEOREM 3.} \( 1 < k \in \mathbb{N} \) be fixed, \( \gamma, \gamma^+ \) denote ordinates of consecutive complex zeros of \( \zeta(s) \) and \( T^{\alpha} \leq H = H(T) \leq T \), where \( \alpha \) is a fixed constant such that \( 0 < \alpha \leq 1 \). Under the RH we have then
\[
(\text{2.9}) \quad H(\log T)^{k^2} \ll_{k, \alpha} \sum_{T < \gamma \leq T + H} \max_{\gamma \leq \tau_\gamma \leq \gamma^+} |\zeta\left(\frac{1}{2} + i\tau_\gamma\right)|^{2k} \ll_{k, \alpha} H(\log T)^{k^2}.
\]

\textbf{Remark.} The case \( k = 1 \) was treated in \cite{2} (see (1.6)) and is not covered by Theorem 3. This is because Theorem 1 does not cover the case \( k = 1 \). The method of its proof (see (3.4)) does not work in obtaining a lower bound for
\[
\int_T^{T+H} |Z'(t)Z(t)| dt.
\]
3. Proof of Theorem 1

As was also done in [Mil], we follow Conrey and Ghosh [2], and introduce the analytic function

\[ Z_1(s) := \zeta'(s) - \frac{\chi'(s)}{2\chi(s)}\zeta(s). \]

Its usefulness comes from the fact that differentiation of (1.2) gives

\[ Z'(t) = i \left\{ \zeta'\left(\frac{1}{2} + it\right) - \frac{1}{2} \frac{\chi'(\frac{1}{2} + it)}{\chi(\frac{1}{2} + it)} \zeta(\frac{1}{2} + it) \right\} \chi^{-1/2}(\frac{1}{2} + it). \]

This gives

\[ |Z'(t)| = |Z_1(\frac{1}{2} + it)|. \]

Using (3.3) and \(|Z(t)| = |\zeta(\frac{1}{2} + it)|\) one may write

\[ \int_T^{T+H} (Z'(t))^2 Z^{2k-2}(t) \, dt = \int_T^{T+H} \left| Z_1(\frac{1}{2} + it) \zeta(\frac{1}{2} + it)^{k-1} \right|^2 \, dt. \]

The basic idea is to use the inequality

\[ \left| \int_T^{T+H} Z_1(\frac{1}{2} + it) \zeta(\frac{1}{2} + it)^{k-1} A(t) \, dt \right|^2 \leq \int_T^{T+H} (Z'(t))^2 Z^{2k-2}(t) \, dt \cdot \int_T^{T+H} |A(t)|^2 \, dt. \]

This comes on using (3.4) and the Cauchy-Schwarz inequality for integrals with a suitably chosen function \(A(t)\). Following [Mil] we set

\[ A(t) := A(\frac{1}{2} + it), \quad A(s) = A(s; k, \xi) := \sum_{n \leq \xi} d_k(n)n^{-s}, \]

where \(d_k(n)\) (generated by \(\zeta^k(s)\) for \(\Re s > 1\)) is the (generalized) divisor function which represents the number of ways \(n\) can be written as a product of \(k\) fixed factors (see e.g., Chapter 13 of [11] for more properties). The parameter \(\xi\) is given by \(\xi = T^\theta, 0 < \theta < 1\). The function \(A(t)\) has the property that in mean square it behaves like \((\log \xi)^{k^2}\) (see Chapter 13 of [11]). Therefore, by the mean value theorem for Dirichlet polynomials (see e.g., Theorem 5.2 of [11]), we have

\[ \int_T^{T+H} |A(t)|^2 \, dt = H \sum_{n \leq \xi} d_k^2(n)n^{-1} + O\left( \sum_{n \leq \xi} d_k^2(n) \right) \]

\[ = H(C_k + o(1))(\log \xi)^{k^2} + O(\xi(\log \xi)^{k^2-1}) \]
Moments of Hardy’s function over short intervals

for \(2 \leq \xi \leq T\) and a positive constant \(C_k\), which may be made explicit. It remains to estimate from below

\[
\int_{T}^{T+H} Z_1\left(\frac{1}{2} + it\right)\zeta\left(\frac{1}{2} + it\right)^{k-1} \tilde{A}(t) \, dt
\]

\[
= \frac{1}{i} \int_{1/2+iT}^{1/2+iH} Z_1(s)\zeta^k(s)\mathcal{A}(1-s) \, ds
\]

\[
= \int_{a+iT}^{a+iH} \frac{1}{i} Z_1(s)\zeta^k(s)\mathcal{A}(1-s) \, ds + O_\varepsilon(T^\varepsilon \xi). \tag{3.7}
\]

Here we used Cauchy’s theorem and set \(a := 1 + 1/\log T\). We also used standard consequences of the RH (see Chapter 12 of [20]):

\[\zeta(s) \ll \varepsilon, |t|, \quad \zeta'(s) \ll \varepsilon, |t| \quad (s = \sigma + it, \sigma \geq \frac{1}{2}),\]

as well as the unconditional, elementary bound \(d_k(n) \ll_{\varepsilon,k} n^\varepsilon\) and

\[
\frac{\chi'\left(\sigma + it\right)}{\chi(\sigma + it)} = -\log \frac{t}{2\pi} + O\left(\frac{1}{t}\right). \tag{3.8}
\]

One obtains (3.8) by logarithmic differentiation of (1.1) and the use of Stirling’s formula for the gamma-function. It is valid for \(\frac{1}{2} \leq \Re s \leq 2\), and the \(O\)-term in (3.8) in fact admits a full asymptotic expansion in terms of negative exponents of \(t\), and the left-hand side of (3.8) can be further differentiated. The integral on the right-hand side of (3.7) is written as \(J_1 + J_2\), where

\[
J_1 := \frac{1}{i} \int_{a+iT}^{a+iH} \zeta'(s)\zeta^{k-1}\mathcal{A}(1-s) \, ds,
\]

\[
J_2 := -\frac{1}{2i} \int_{a+iT}^{a+iH} \frac{\chi'(s)}{\chi(s)}\zeta^k(s)\mathcal{A}(1-s) \, ds.
\]

Similarly as in [17] one shows that

\[
J_1 = -H \sum_{n \leq \xi} \bar{d}_k(n)d_k(n)n^{-1} + O(T^\varepsilon \xi)
\]

with

\[\bar{d}_k(n) := \sum_{\delta|n} d_{k-1}(\delta) \log \frac{n}{\delta} \leq d_k(n) \log n,\]

and

\[
J_2 = \frac{1}{2} H \left(\log \frac{T}{2\pi e}\right) \sum_{n \leq \xi} d_k^2(n)n^{-1} + O(T^\varepsilon \xi).
\]
This yields
\[
\int_{J}^{T+H} Z_1(\frac{1}{2} + it)\zeta(\frac{1}{2} + it)^{k-1} dA(t) = J_1 + J_2 + O(T^\varepsilon\xi)
\]
\[
= -H \sum_{n \leq \xi} \tilde{d}_k(n)d_k(n)n^{-1} + O(T^\varepsilon\xi) + \frac{1}{2}H \log \frac{T}{2\pi} (1 + o(1)) \sum_{n \leq \xi} d_k^2(n) n^{-1}
\]
\[
\geq -H \sum_{n \leq \xi} d_k^2(n) n^{-1} \log n + \frac{1}{2}H \log \frac{T}{2\pi} (1 + o(1)) \sum_{n \leq \xi} d_k^2(n) n^{-1} + O(T^\varepsilon\xi)
\]
\[
\geq \left\{ \frac{1}{2}H (1 + o(1)) \log \frac{T}{2\pi} - H \log \xi \right\} \sum_{n \leq \xi} d_k^2(n) n^{-1} + O(T^\varepsilon\xi)
\]
\[
\geq A_k H \log T \cdot (\log \xi)^k^2
\]
for \( \xi = T^\theta, \theta = \frac{1}{2}\alpha \) and \( \varepsilon \) sufficiently small. From (3.5) and (3.6) we finally gather that
\[
H^2 \log^2 T (\log T)^{2k^2} \ll_{k, \alpha} \int_{J}^{T+H} (Z'(t))^2 Z^{2k-2}(t) d \cdot H (\log T)^{k^2},
\]
and (2.1) of Theorem 1 follows.

4. PROOF OF THEOREM 2

First note that, for \( 0 < R \leq \frac{1}{2}, T \leq t \leq 2T \), by Cauchy’s integral formula we have
\[
\zeta'(\frac{1}{2} + it) = \frac{1}{2\pi i} \int_{|z|=R} \frac{\zeta(\frac{1}{2} + it + z)}{z^2} dz.
\]
This yields
\[
(4.1) \int_{J}^{T+H} |\zeta'(\frac{1}{2} + it)|^{2k} dt = \frac{1}{(2\pi)^{2k}} \int_{J}^{T+H} \left| \int_{|z|=R} \frac{\zeta(\frac{1}{2} + it + z)}{z^2} dz \right|^{2k} dt.
\]
By Hölder’s inequality for integrals the right-hand side of (4.1) does not exceed
\[
\frac{1}{(2\pi)^{2k}} \int_{J}^{T+H} \left\{ \int_{|z|=R} |\zeta(\frac{1}{2} + it + z)|^{2k} |dz| \right\} \cdot \left\{ \frac{|dz|}{|z|^{4k/(2k-1)}} \right\}^{2k-1} dt
\]
\[
\leq \frac{1}{(2\pi)^{2k}} \int_{J}^{T+H} \left\{ \int_{|z|=R} |\zeta(\frac{1}{2} + it + z)|^{2k} |dz| \right\} (2\pi R)^{2k-1} R^{-4k}
\]
\[
\leq \frac{1}{R^{2k}} \max_{0 \leq \theta \leq 2\pi} \int_{J}^{T+H} |\zeta(\frac{1}{2} + it + Re^{i\theta})|^{2k} dt.
\]
Therefore
\[ \int_T^{T+H} |\zeta'(\frac{1}{2} + it)|^{2k} dt \leq \frac{1}{R^{2k}} \max_{0 \leq \theta \leq 2\pi} \int_T^{T+H} |\zeta(\frac{1}{2} + it + Re^{i\theta})|^{2k} dt. \]

As in [Mil], we could have obtained an inequality for the \( \ell \)-th derivative of \( \zeta(\frac{1}{2} + it) \), but this is not necessary for our purposes.

Henceforth let \( R = 1/\log T \) in (4.2). The integral on the right-hand side of (4.2) equals
\[ \int_T^{T+H} |\zeta(\frac{1}{2} + R \cos \alpha + i(t + R \sin \alpha))|^{2k} dt. \]

Recall that, under the RH (see [Tit]),
\[ \zeta(\sigma + it) \ll \exp\left(C \frac{\log t}{\log \log t}\right) \quad (\frac{1}{2} \leq \sigma \leq 1, C > 0, |t| \geq 2). \]

When \( \cos \alpha \geq 0 \) in (4.3), we use (4.4) to obtain that the integral in (4.3) is equal to
\[ \int_T^{T+H} |\zeta(\frac{1}{2} + R \cos \alpha + it)|^{2k} dt + o(T). \]

When \( \pi/2 \leq \theta \leq 3\pi/2 \) in (4.5), that is, when \( \cos \alpha \leq 0 \), we use the functional equation (1.1). In this case we have, with \( \sigma = \Re s = \frac{1}{2} + R \cos \alpha, R = 1/\log T, \)
\[ \chi(s) \ll |t|^{1/2-\sigma} \ll T^{-\cos \alpha/\log T} \ll 1, \]
thus we reduce the estimation of our integral to the case when \( \cos \alpha \geq 0 \). For this we shall use a convexity result which shows that essentially the integral in question is bounded by the \( 2k \)-th moment of \( |\zeta(\frac{1}{2} + it)| \) over a short interval. More precisely, let \( 1/2 \leq \sigma \leq 3/4, k > 0, t \geq 2, \)
\[ J_k(\sigma) := \int_{-\infty}^{\infty} |\zeta(\sigma + it)|^{2k} w_k(t) dt, \quad w_k(t) := \int_T^{T+H} e^{-2k(t-\tau)^2} d\tau. \]

Then
\[ J_k(\sigma) \ll T^{\sigma-1/2} \left(J_k(\frac{1}{2})\right)^{3/2-\sigma} + e^{-kT^2/4}. \]

The bound in (4.7) is the analogue of Lemma 4.2 of [17] for short intervals. This in turn is a result of D.R. Heath-Brown [10], which is also expounded in [12], pp.
In the original version the interval of integration in the kernel function $w_k(t)$ was $[T, 2T]$. However, the change made in (4.6) does not affect the proof, and one obtains (4.7). Now note that $w_k(t) \gg 1$ for $t \in [T, T + H]$, so that

$$\int_T^{T + H} |\zeta(\sigma + it)|^{2k} \, dt \ll J_k(\sigma).$$

(4.8) We have the bound $w_k(t) \ll \exp(-2kH^2)$ when $t \leq T - H$ or $t \geq T + 2H$. On the other hand $w_k(t) \ll \exp(-kt^2)$ for $t < 0$ or $t > 3T$. Thus combining (4.7) and (4.8) it follows that

$$\int_T^{T + H} |\zeta(\sigma + it)|^{2k} \, dt \ll 1 + \int_{T - H}^{T + 2H} |\zeta(\frac{1}{2} + it)|^{2k} \, dt \ll_{k, \alpha} H(\log T)^k,$$

(4.9) where in the last step (2.7) was used. Inserting (4.9) in (4.2) the bound in (2.2) follows. The estimate (2.3) easily follows from (2.2), (2.9) and (3.2). Theorem 2 is proved.

5. Proof of Theorem 3

It is in the folklore that $Z(t)$, for $t \geq 14$, cannot have a negative local maximum or a positive local minimum under the RH. For this, see [3], or [13], [14], [6]. In other words, the zeros of $Z(t)$ and $Z'(t)$ are interlacing. Thus if $\gamma, \gamma^+$ are consecutive zeros of $Z(t)$, there is a unique point $\lambda_\gamma \in [\gamma, \gamma^+]$ for which $Z'(\lambda_\gamma) = 0$ (this is trivially true if $\gamma = \gamma^+$, that is, if $\gamma$ is a multiple zero of $Z(t)$). Therefore

$$\max_{\gamma \leq \gamma \leq \gamma^+} |\zeta(\frac{1}{2} + i\tau_\gamma)|^{2k} = Z^{2k}(\lambda_\gamma).$$

(5.1) Then, since $Z(t)$ is positive in $(\gamma, \lambda_\gamma)$ and negative in $(\lambda_\gamma, \gamma^+)$, or conversely, we have

$$\int_{\gamma}^{\gamma^+} |Z'(t)Z^{2k-1}(t)| \, dt = \left| \int_{\lambda_\gamma}^{\gamma} Z'(t)Z^{2k-1}(t) \, dt - \int_{\gamma}^{\lambda_\gamma} Z'(t)Z^{2k-1}(t) \, dt \right| = \frac{1}{k} Z^{2k}(\lambda_\gamma).$$

(5.2) Therefore (5.1) and (5.2) give, in view of (4.4),

$$\sum_{T < \gamma \leq T + H} \max_{\lambda_\gamma \leq \gamma \leq \gamma^+} |\zeta(\frac{1}{2} + i\tau_\gamma)|^{2k} = k \int_{T}^{T + H} |Z'(t)Z^{2k-1}(t)| \, dt + O_{k, \varepsilon}(T^\varepsilon).$$

(5.3)
Assume that \( k \geq 2 \), so that \( 2k - 2 \geq 2 \). To bound the integral on the right-hand side of (5.3) from below, note that
\[
|(Z')^2 Z^{2k-2}| = |Z'|^{1/2} |Z|^{k-1/2} \cdot |Z'|^{3/2} \cdot |Z|^{k-3/2}.
\]
Thus Hölder’s inequality for integrals shows that
\[
\int_T^{T+H} (Z'(t))^2 Z^{2k-2}(t) \, dt \leq
\left( \int_T^{T+H} (|Z'|^{1/2} |Z|^{k-1/2})^p \, dt \right)^{\frac{1}{p}} \left( \int_T^{T+H} |Z'|^{3/2} \, dt \right)^{\frac{1}{q}} \left( \int_T^{T+H} |Z|^{r(k-3/2)} \, dt \right)^{\frac{1}{r}}
\]
with \( p, q, r > 0, 1/p + 1/q + 1/r = 1 \). Take
\[
\frac{1}{p} = \frac{1}{2}, \quad \frac{1}{q} = \frac{3}{4k}, \quad \frac{1}{r} = \frac{1}{2} - \frac{3}{4k}.
\]
Then the right-hand side is, on using (2.3) and (2.4),
\[
\ll_{k, \alpha} \left( \int_T^{T+H} |ZZ|^{2k-1} \, dt \right)^{\frac{1}{2}} \left( \int_T^{T+H} |Z'|^{2k} \, dt \right)^{\frac{3}{4k}} \left( \int_T^{T+H} |Z|^{2k} \, dt \right)^{\frac{1}{2} - \frac{3}{4k}}.
\]
This gives, on using (2.1),
\[
H(\log T)^{k^2 + 2} \ll_{k, \alpha} I^{1/2} \left( H(\log T)^{k^2 + 2k} \right)^{\frac{3}{4k}} \left( H(\log T)^{k^2} \right)^{\frac{1}{2}} \cdot
\]
which on simplifying yields
\[
I := \int_T^{T+H} |Z'(t)Z^{2k-1}(t)| \, dt \gg_{k, \alpha} H(\log T)^{k^2 + 1}.
\]
In view of (5.3) this proves the lower bound in (2.9) of Theorem 3.

As for the upper bound, the integral in (5.3) does not exceed, by Hölder’s inequality for integrals,
\[
\left| \int_T^{T+H} (Z'(t))^{2k} \, dt \right|^{1/(2k)} \left| \int_T^{T+H} |\zeta(\frac{1}{2} + it)|^{2k} \, dt \right|^{1-1/(2k)}
\ll_{k, \alpha} \left\{ H(\log T)^{k^2 + 2k} \right\}^{1/(2k)} \left\{ H(\log T)^{k^2} \right\}^{1-1/(2k)} = H(\log T)^{k^2 + 1},
\]
which finishes the proof of Theorem 3 when \( k \geq 2 \). Here we used (2.3) and (2.4), and we note that the bound in (5.4) holds also for \( k = 1 \). It is the lower bound in this case which is problematic.
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