DECRETION DISKS

1. INTRODUCTION

Astrophysical accretion disks evolve under the action of internal stresses, which transport angular momentum outwards while most of the mass accretes onto the central object (Shakura & Sunyaev 1973; Lynden-Bell & Pringle 1974; Pringle 1974). Excess angular momentum shed by the accreted matter eventually gets transferred to the fluid at the outer edge of the disk, causing its outward spreading. Thus, while the inner parts of the disk get accreted, some gas must also move out to conserve angular momentum globally.

In many systems, such as X-ray binaries or cataclysmic variables, the outward expansion of the disk (fed with mass by the companion) is eventually stopped by the torque exerted on the outer edge of the disk by the non-axisymmetric potential of the companion (Lin & Papaloizou 1979). As a result, the disk attains a steady state, in which the excess angular momentum lost by the gas accreted by the central object gets passed to the binary orbit via the tidal disk-companion coupling.

On the other hand, there are many astrophysically relevant situations, in which the disk is not affected by external torques (at least temporarily). In this case, the disk keeps expanding while losing mass to central accretion. Such freely expanding disks are known as decretion or external disks (Pringle 1991) and are the subject of this study.

Decretion disks readily form when gas is deposited in orbit close to the central object. This situation is natural for the tidal disruption events (Gezari 2012; Komossa 2015), in which the disk is assembled out of stellar debris close to a supermassive black hole (Cannizzo et al. 1990; Shen & Matzner 2014). Another important example is the disks of rapidly spinning Be stars (Okazaki 2007; Rivinius et al. 2013), which have a long and diverse history of observations. Supernova fallback disks, expected to form out of the low angular momentum material that is not energetic enough to become unbound from the supernova remnant (Michel & Dessler 1981; Michel 1988), should also evolve as decretion disks. The disk in which the planetary system around the millisecond pulsar PSR 1257+12 (Wolszczan & Frail 1992) must have originated has also likely undergone viscous expansion prior to planet formation (Phinney & Hansen 1993), regardless of its origin. Post-main-sequence evolution in stellar binaries often results in circumbinary disks fed by the mass outflow through the L2 Lagrange point (Blundell et al. 2008; Dermine et al. 2013; Antoniadis 2014; Pejcha et al. 2016). This disk must also subsequently evolve as a decretion disk torqued by the non-axisymmetric gravitational potential of the binary.

Given their importance in astrophysics, it is not surprising that the characterization of the decretion disk properties has a long history. In their pioneering work, Lynden-Bell & Pringle (1974) derived solutions for the viscous disk evolution, assuming that the kinematic viscosity is independent of the surface density $\Sigma$. They explicitly demonstrated that in the long run the disk structure tends to evolve in a self-similar fashion independent of the initial conditions, i.e., the starting radial distribution of mass in the disk. Transition to this behavior naturally occurs when the disk expands far beyond the characteristic radius, at which most of its mass was concentrated initially. Analytical similarity solutions of Lynden-Bell & Pringle (1974) have been subsequently generalized by Lyubarskij & Shakura (1987) to the nonlinear problem arising when the viscosity in the disk is an explicit function of $\Sigma$.

On the other hand, the actual form of the long-term similarity solution does depend on the inner boundary conditions—the rate of mass accretion and the torque applied to the disk at its center. Lynden-Bell & Pringle (1974) and Lyubarskij & Shakura (1987) were able to find the similarity solutions for two important cases, which are common in nature. First is the disk with zero (or very small) torque at the center, akin to the accretion disk around a black hole (Shakura & Sunyaev 1973),
which features a non-zero mass accretion rate $\dot{M}$ at the origin. Second is the disk with zero mass inflow at the center, $\dot{M}(r = 0) = 0$, in which the non-zero central torque must be present to completely suppresses accretion. This situation is thought to be typical, e.g., for rapidly spinning, magnetized neutron stars that interact with the surrounding disk in the “propeller” regime (Illarionov & Sunyaev 1975; Syunyaev & Shakura 1977). It was also thought, based on one-dimensional models (Chang et al. 2010), that $\dot{M}(r = 0) = 0$ is a natural boundary condition for the circumdiskary disks around stellar binaries (Alexander 2012; Vartanyan et al. 2016) and super-massive black hole binaries (Ivanov et al. 1999; Rafikov 2013) in which the binary torque would strongly suppress the gas inflow.

While these two orthogonal types of central boundary conditions and their corresponding similarity solutions apply to many accreting systems, they certainly do not exhaust all astrophysically relevant possibilities. In fact, there is a number of objects accreting via the disk, in which both $\dot{M}$ and the torque do not vanish at the center, thus representing an intermediate situation when compared to the two known types of similarity solutions. In particular, the magnetic field of the neutron stars in the propeller regime might present an imperfect obstacle to gas inflow. These stars would then accrete at a rate (Arons & Lea 1976), as has been recently proposed to explain the accretion state transitions in Vela X-1 by Doroshenko et al. (2011) and in the ultraluminous X-ray source M82 X-2 (Bachetti et al. 2014) by Tsygankov et al. (2016). Also, recent numerical work on circumbinary disks suggests that the torque produced by the binary may not be efficient at suppressing gas accretion onto the binary (MacFadyen & Milosavljević 2008; D’Orazio et al. 2013; Farris et al. 2014).

Moreover, in some systems, the non-zero central torque is so strong that it results in $\dot{M}(r = 0) < 0$, i.e., mass outflow from the central object to the disk. An obvious example is given by the disks of the Be stars, which are fed by the gas shed from their rapidly spinning hosts (Rivinius et al. 2013). Circumbinary disks of post-main-sequence binaries are also thought to be supplied by the gas lost from the binary, resulting in the central injection of mass (van Winckel 2003; de Ruyter et al. 2006).

The goal of our present work is to explore the behavior of such systems, the evolution of which clearly cannot be captured by the two known decretion disk solutions (Lynden-Bell & Pringle 1974; Pringle 1991). Here we focus on deriving and analyzing the self-similar solutions, which describe the long-term evolution of the decretion disks. We do this for rather general class of the viscosity behaviors, both when $\nu$ is independent of $\Sigma$ (like in Lynden-Bell & Pringle 1974) and when it is an explicit function of $\Sigma$ (like in Pringle 1991). Moreover, our solutions naturally cover the possibility of both the inflow and the outflow at the disk center.

Our work is organized as follows. In Section 2, we convert the disk evolution equations to a form that is particularly well suited for applying the self-similar ansatz (Section 3). After describing the previously known results (Sections 3.1–3.2), we present our new self-similar solutions in Section 4. We emphasize the important connection between the similarity parameter and the degree of the mass accretion suppression in Section 4.3. Time evolution of the global characteristics of the decretion disks is covered in Section 4.4, while their observables are described in Section 5. Finally, in Section 6, we discuss the extraction of the self-similar disk parameters and provide a comparison with previous work (Section 6.4).

2. BASIC EQUATIONS

We consider a thin disk in Keplerian rotation around a central mass $M_c$ (circumbinary disks orbit in the non-Newtonian potential, but far enough from the binary that the rotation profile converges to Keplerian). Local angular frequency of the disk fluid is $\Omega(r) = (GM_c/r^3)^{1/2}$ (we neglect radial pressure support), and $l(r) \equiv \Omega r^2$ is its specific angular momentum. We are interested in the azimuthally averaged, vertically integrated disk structure, so that all disk variables are functions of the radius $r$ and time $t$ only.

In this work, we focus on the evolution driven by the internal (viscous) stresses alone. It is described by a one-dimensional (azimuthally averaged) equation (Lynden-Bell & Pringle 1974)

$$\frac{\partial \Sigma}{\partial t} = \frac{1}{2\pi r} \frac{\partial M}{\partial r}, \quad M = \left(\frac{dl}{dr}\right)^{-1} \frac{\partial l}{\partial r}. \tag{1}$$

Here $M(r)$ is the local value of the mass accretion rate (defined to be positive for inflow), and $T_{\nu 0}$ is the angular momentum flux due to the $r - \phi$ component of the internal stress in the disk (also equal to the total stress exerted by the disk interior to some $r$ on the outer disk, integrated over the circumference and height).

Equation (1) assumes that there are no sources (or sinks) of the angular momentum and mass in the disk outside its very central part. In other words, external stress can be applied to the disk only at $r = 0$ ($l = 0$), giving rise to a non-zero value of $T_{\nu 0}(r = 0)$ as a boundary condition for the disk evolution. Similarly, $M(r = 0)$ is also non-zero in general.

Provided that stress is effected by some form of effective viscosity $\nu$, $T_{\nu 0}$ is given by the viscous angular momentum flux $F_J$ (Filipov 1984; Lyubarskij & Shakura 1987; Rafikov 2013)

$$T_{\nu 0} = F_J \equiv -2\pi \nu \Sigma \frac{dl}{dr}, \quad l = 3\pi \alpha c_s^2 \Sigma r^2, \tag{2}$$

where $\nu$ is the kinematic viscosity usually expressed through the dimensionless parameter $\alpha$ and gas sound speed $c_s$ as $\nu = \alpha \Omega^{-1} c_s^2$ (Shakura & Sunyaev 1973). Substituting $F_J$ for $T_{\nu 0}$ in Equation (6), one arrives at the conventional form of the viscous evolution equation with $\Sigma(r, t)$ as the unknown function (Papaloizou & Lin 1995).

Equation (1) can be recast in a particularly simple form by switching from surface density $\Sigma$ to the viscous angular momentum flux $F_J$ and from $r$ to the specific angular momentum $l$ (Lynden-Bell & Pringle 1974; Filipov 1984; Lyubarskij & Shakura 1987; Rafikov 2013):

$$\frac{\partial}{\partial t} \left( \frac{F_J}{D_J} \right) = \frac{\partial^2 F_J}{\partial l^2}, \tag{3}$$

where

$$D_J \equiv -\nu r^2 \frac{dl}{dr} \frac{dl}{dr} = \frac{3}{4} \alpha c_s^2 l \tag{4}$$

is the diffusion coefficient, which depends on $F_J$ if $\nu$ depends on $\Sigma$ (the second equality is for Keplerian $\Omega$). Once $F_J$ is

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1 Eksi (2009) and Eksi (2012) proposed a superposition of these two known solutions to approximately describe the evolution of such general systems; see Section 6.4.
known from Equation (3), the behavior of the surface density is given simply by
\[
\Sigma = \frac{F_J \Omega}{4 \pi D_J},
\]
(specializing to the Keplerian rotation profile). It is also obvious that, in these variables, one can write
\[
\dot{M} = \frac{\partial F_J}{\partial t},
\]
see Equation (1). In particular, a standard constant \( \dot{M} \) accretion disk with zero torques at the center and extending to infinity (Shakura & Sunyaev 1973) is described by a simple solution \( F_J = \dot{M} \).

3. SELF-SIMILAR ANSATZ

Now we consider a (generally nonlinear) problem, which arises when the diffusion \( D_J \) is an explicit function of both \( F_J \) and \( t \). We will focus on the situation when \( F_J \) has a power-law dependence on both \( F_J \) and \( t \):
\[
D_J = D_{J,0} F_J^p t^q,
\]
where \( D_{J,0}, d, \) and \( p \) are constants, which are set by the physics of the problem at hand (the behavior of \( \nu \)) using Equations (2) and (4). This prescription is similar to Pringle (1991), who assumed viscosity to be a power-law function of \( \Sigma \) and \( r \), namely \( \nu \propto \Sigma^m r^n \). To simplify the comparison of the results, we provide the conversion between our variables and those of Pringle (1991) in Appendix A. In particular, Equation (54) makes it clear that whenever \( \nu \) depends on \( \Sigma \) (i.e., \( m \neq 0 \)), one also has \( D_J \) depending on \( F_J \) (i.e., \( d = 0 \)).

The values of \( D_{J,0}, d, \) and \( p \) are determined by the model of the viscosity, namely its dependence on the disk properties. In general, we will constrain \( d \) to satisfy \( 0 \leq d < 1 \); values of \( d > 1 \) result in viscous instability (Lightman & Eardley 1974).

For our adopted \( \alpha \)-model Equation (4) shows that \( D_J \) depends on the disk temperature \( T \) via \( c_v \). Externally illuminated disks typically have their temperature controlled only by the distance to the center; in this case, both \( T \) and \( D_J \) are functions of \( t \) only. This implies that \( d = 0 \), i.e., that \( D_J \) is independent of \( F_J \), making Equation (3) linear. We will often refer to this type of situation as a linear problem. It naturally emerges, e.g., in irradiation-dominated regions of protoplanetary disks, see Vartanyan et al. (2016) for details.

According to the Equation (4), temperature scaling as \( T(r) \propto r^{-k_T} \) results in \( p = 1-2k_T \) (so that \( p < 1 \) for \( T(r) \) decaying with distance). Centrally irradiated disks usually have \( k_T \) close to one-half, so that \( p \approx 0 \). In particular, a passive disk model of Chiang & Goldreich (1997) predicts \( k_T = 3/7 \) for the optically thick part of the disk, resulting in \( p = 1/7 \).

On the other hand, in disks heated by internal dissipation, \( T \) must be self-consistently determined by the local thermal balance and is, in general, a function of \( F_J \) (or \( \Sigma \)). As a result, \( d = 0 \) and Equation (3) becomes nonlinear (we refer to this situation as a nonlinear problem). The effect of the details of the disk thermodynamics, namely the opacity behavior, on the parameters of the ansatz (7) has been previously explored by Lyubarskij & Shakura (1987), Cannizzo et al. (1990), and Lipunova & Shakura (2000). In particular, Filipov (1984) and Rafikov (2013) showed that for the gas-pressure dominated disk with the dominant free-free opacity \( d = 3/10, p = -4/5 \). When the electron scattering opacity dominates \( d = 2/5, p = -6/5 \). We will often use these values of \( d \) and \( p \) when describing our results.

Following Filipov (1984), Lyubarskij & Shakura (1987), and Filipov et al. (1988), we will seek a self-similar solution of the Equation (3) in the form
\[
F_J(l, t) = F_0 \varphi(t) f(\xi), \quad \xi = \frac{l}{l_0 \psi(t)},
\]
where \( \xi \) is the similarity variable, \( \varphi(t) \) and \( \psi(t) \) are the dimensionless scaling functions, and \( l_0 \) and \( F_0 \) are the dimensional scaling factors.

Plugging in this ansatz together with (7) into the Equation (3), we find
\[
\frac{D_0}{l_0^p} \varphi^{1+d+\delta(p-2)} \xi^{\eta d} f''_\xi = (1 - d) \varphi' \left( f - \frac{\psi' \varphi}{\psi \varphi'} \right)
\]
where \( D_0 \equiv D_{J,0} F_0^p l_0^q \) is the value of the viscosity coefficient set by the characteristic values of \( F_J \) and \( l \), and we use \( s' \equiv \partial s/\partial z \), etc. for any function \( s \) and variable \( z \).

Similarity of the solution obviously requires that
\[
\psi = \varphi^\delta,
\]
where \( \delta < 0 \) is a constant (the sign follows from the fact that the characteristic scale of the problem \( l_0 \psi \) must increase with time, while the amplitude \( \varphi \) decreases). We set a constant multiplier in the right-hand side to unity because of the freedom in choosing both \( F_0 \) and \( l_0 \).

Equation (9) then splits into two relations:
\[
\varphi' = \frac{\varphi^{1+d+\delta(p-2)}}{(1 - d) t_0},
\]
which determines time evolution of the scale factor \( \varphi \) (which is explored in detail in Section 4.4), and
\[
\xi^{\eta d} f''_\xi - \delta \xi f'_\xi + f = 0,
\]
which describes the overall spatial distribution of \( F_J \). In Equation (11), we also defined a characteristic time of \( t_0 \equiv l_0^2 / D_0 \).

Self-similar ansatz (8) together with Equations (2), (4), (7), and (10) allows us to write the instantaneous gas mass \( M(\dot{M}) \) as
\[
M_d = 2 \pi \int_0^\infty \Sigma(r) r dr = \int_0^\infty \frac{F_J}{D_J} dl = \int_0^\infty \frac{F_J^{1-d} l^{-p}}{D_{J,0}} dl = \frac{F_0 l_0}{D_0} \varphi^{1-d+\delta(1-p)} I_M,
\]
where we defined a new constant
\[
I_M \equiv \int_0^\infty f^{1-d} \xi^{-p} d\xi.
\]

The full angular momentum of the disk is given by
\[
L_d = \frac{F_0 l_0^2}{D_0} \varphi^{1-d+\delta(2-p)} I_L, \quad I_L \equiv \int_0^\infty f^{1-d} \xi^{1-p} d\xi.
\]
Finally, with the help of Equation (6), the mass accretion rate becomes
\[ M(l, t) = \frac{F_0}{\ell_0} \phi^\ell \xi^\ell \lambda f_{\xi}^\ell (\alpha). \] (16)

By assumption, no torque is applied to the disk at its outer edge \( \xi = \xi_{\text{out}} \) (\( \xi_{\text{out}} \) may be equal to infinity), where \( f(\xi_{\text{out}}) = 0 \). It is easy to see that \( M \) must also vanish at this radius, meaning that \( f_{\xi}^\ell (\xi_{\text{out}}) = 0 \) according to Equation (16).

Mass conservation then implies that \( M(l = 0, t) = -dM_d/dt \). Taking the time derivative of Equation (13) and using Equation (11), we find that the requirement of mass conservation reduces to
\[ f_{\xi}^\ell (0) = \frac{1 - d + \delta(1 - p)}{1 - d} I_M. \] (17)

Analogously, global angular momentum conservation implies that \( F_l(l = 0, t) = dL_d/dt \), which can be written as
\[ f(0) = \frac{1 - d + \delta(2 - p)}{1 - d} I_L. \] (18)

It is important to note that Equations (17) and (18) should be viewed as consistency relations rather than the boundary conditions for Equation (12). Indeed, they directly follow from Equation (12): multiplying it by \( f^{\ell - d} \xi^{-p} \) (or \( f^{-d} \xi^{1-p} \)), integrating from \( \xi = 0 \) to \( \xi = \xi_{\text{out}} \), and using \( f(\xi_{\text{out}}) = 0, f_{\xi}^\ell (\xi_{\text{out}}) = 0 \) one immediately retrieves relation (17) (or (18)). Thus, these relations do not additionally constrain \( f(\xi) \), but any solution of Equation (12) must satisfy them.

In the following instead of \( \delta \) we will use a new similarity parameter \( \lambda \), defined as
\[ \lambda \equiv 1 + \delta \frac{1 - p}{1 - d}, \quad \Rightarrow \quad \delta = -\frac{(1 - \lambda)(1 - d)}{1 - p}. \] (19)

With this new parameter, Equation (12) transforms to
\[ \xi^{\ell - d} f_{\xi}^{\ell - d} + \frac{(\lambda - 1)(1 - d)}{1 - p} \xi^\ell f_{\xi}^\ell + f = 0, \] (20)

subject to conditions
\[ f_{\xi}^\ell (0) = \lambda I_M, \quad f(0) = \frac{1 - \lambda(2 - p)}{1 - p} I_L. \] (21)

Equation (20) possesses an important symmetry property: if some function \( g(\xi) \) is its solution, then a function \( f(\xi) = k^{(p - 2)/d} g(k\xi) \) also satisfies Equations (20)–(21). This rescaling also results in
\[ I_M(f) = k^{1 + (p - 2)/d} I_M(g), \] (22)
\[ I_L(f) = k^{(p - 2)/d} I_L(g). \] (23)

This gauge freedom affects the choice of the dimensional parameters \( F_0 \) and \( \ell_0 \) characterizing the amplitude and scale of the density distribution in the disk. For a disk with fixed mass and angular momentum, this follows immediately from Equations (13) and (15). To fix this gauge dependence in what follows, we constrain the solution \( f \) of Equation (20) to satisfy the condition \( I_M = 1 \). The value of \( I_L \) then follows from Equation (23).

Another benefit of the scaling symmetry property of the Equation (20) is that it allows its order to be lowered (Lyubarskij & Shakura 1987). However, this provides useful insights in only a couple of cases considered below.

3.1. Previously Known Solutions: No Central Torque, \( \lambda = (2 - p)^{-1} \)

The problem of the decretion disk evolution is known to admit two types of analytical solutions. One of them corresponds to the standard assumption commonly used in modeling accretion disks (Shakura & Sunyaev 1973)—that of zero (or very small) central torque, when \( F_l(r = 0) = 0 \) or \( f(\xi = 0) = 0 \) and the angular momentum of the disk is conserved, \( L_d(t) = \text{const} \). It is obvious that the zero central torque assumption must correspond to
\[ \lambda = \lambda_0 \equiv (2 - p)^{-1}. \] (24)

Indeed, the second constraint in (21) can be written as
\[ f(0) = (1 - \lambda/\lambda_0)(1 - p)^{-1} I_L, \quad \text{so that} \quad \lambda = \lambda_0 \quad \text{naturally reduces it to} \quad f(0) = 0. \]

The similarity solution for the linear problem (\( d = 0 \)) with no central torque was derived in the pioneering study of Lynden-Bell & Pringle (1974):
\[ f(\lambda = \lambda_0, \xi) = \frac{\xi}{2 - p} \exp \left[ -\frac{\xi^2 - p}{(2 - p)^2} \right], \] (25)

where we set the normalization to guarantee that \( I_M = 1 \). This solution is shown by the black solid curve in Figure 1.

In the nonlinear case (\( d > 0 \)), the analytical similarity solution conserving \( L_d \) was derived by Lyubarskij & Shakura...
(1987), using the results of Barenblatt & Zel’dovich (1957) in the field of gas filtration (Barenblatt 1996), and subsequently by Cannizzo et al. (1990). It follows from Equation (20) that such a solution has a form of (Pringle 1991)

\[ f(\xi) = (2 - p)^{-1} \xi (1 - c_1 \xi^{2-p})^{1/d}, \]

with a constant factor \( c_1 \) given by Equation (63) to guarantee \( I_M = 1 \). This solution clearly satisfies the first consistency relation (21). Its central mass accretion rate is non-zero and given by \( M(r = 0, t) = (2 - p)^{-1} (F_0 J_0)|\varphi(t)|^{3-d}, \) see Equation (16). The value of \( I_L \) for this \( \lambda \) is given by Equation (67). The shape of this solution is illustrated in Figures 2 and 3 (black solid curve).

3.2. Previously Known Solutions: No Central Inflow, \( \lambda = 0 \)

The second previously known analytical self-similar solution corresponds to zero mass flux at the center, i.e., \( M(r = 0) = 0 \) or \( f_\xi'(0) = 0 \). According to the relation (21), this requires \( \lambda = 0 \). In this case, the total mass of the disk is conserved, \( M_d(t) = \text{const.} \)

Linear \( (d = 0) \) similarity solutions of this kind were again obtained by Lynden-Bell & Pringle (1974):

\[ f(\lambda = 0, \xi) = c_2 \exp \left[ -\frac{\xi^{2-p}}{(1-p)(2-p)} \right], \]

with \( c_2 \) given by Equation (60) to ensure \( I_M = 1 \).

In the nonlinear case \( (d > 0) \), the mass-conserving \( (\lambda = 0) \) solution of the Equation (20) was derived by Lyubarskij & Shakura (1987), using the results obtained by Zel’dovich & Kompaneets, (1950) and Barenblatt (1952) in their studies of the nonlinear heat conduction and gas filtration (Barenblatt 1996). It reads (subject to the constraint \( I_M = 1 \))

\[ f(\xi) = c_3 (1 - c_4 \xi^{2-p})^{1/d}, \]

in agreement with Pringle (1991). The constant factors \( c_3 \) and \( c_4 \) are given by Equations (64) and (65) and yield \( I_M = 1 \). As the disk viscously spreads while preserving its mass, the central torque varies as \( F_J(r = 0, t) = c_3 F_0 \varphi(t) \). The value of \( I_L \) for this \( \lambda \) is given by Equation (66). The shape of this solution is shown by the dashed black curve in Figures 2 and 3.

4. NEW SIMILARITY SOLUTIONS

To the best of our knowledge, Equations (25)–(28) represent the only two self-similar decetration disk solutions that have been discussed in the literature. In the space of possible values of the similarity parameter \( \lambda \) they cover just two discrete points, leaving the continuum of other values of \( \lambda \) unaddressed. This is illustrated in Figure 4, which displays various possibilities related to different values of \( \lambda \). Our goal here is to provide a description of the self-similar solutions for all possible values of \( \lambda \) and to connect them to the physical properties of the specific systems.

First of all, from the boundary conditions (21) it is clear that unless

\[ \lambda \leq \lambda_0 = (2 - p)^{-1} \]

the central torque on the disk would become negative. This is not possible since, in our setup, \( F_J \) is directly related to \( \Sigma \), meaning that \( f(0) < 0 \) would imply \( \Sigma(r = 0) < 0 \). Thus, physically meaningful similarity solutions are possible only for the values of \( \lambda \) satisfying the constraint (29).

Boundary conditions (21) also make it clear that the negative values of \( \lambda \) result in mass outflow at the center, because \( M(r = 0) \propto \lambda \). This situation describes the mass injection by

![Figure 2](image1)

Figure 2. Same as Figure 1 but for a disk with opacity dominated by the electron scattering \((d = 2/5, p = -6/5)\), when disk evolution is a nonlinear problem. Note the finite extent in \( \xi \) of the solutions for different \( \lambda \). Black solid and dashed curves correspond to solutions (26) and (28).

![Figure 3](image2)

Figure 3. Same as Figure 2 but for a disk with the free-free opacity \((d = 3/10, p = -4/5)\).
the central object into the disk, as expected, e.g., in decretion disks of Be stars.

4.1. Linear Problem

We first describe a similarity solution for the linear ($D_j$ independent of $F_j$, $d = 0$) viscous evolution problem, when the diffusion coefficient $D_j$ is a function of $l$ only. In Appendix B, we show that the general linear solution of the Equation (20) can be expressed analytically as

$$f(\lambda, \xi) = c_5 e^{-\kappa \xi^2 - \psi} U(b - a, b, \kappa \xi^2 - \psi),$$

(30)

where constant factors $\kappa(\lambda), a, b$, and $c_5(\lambda)$ are given by Equations (56), (58), and (59), respectively, and $U(c, q, t)$ is the Tricomi confluent hypergeometric function (Abramowitz & Stegun 1972). This solution is plotted in Figure 1 for different values of $\lambda$.

One can easily show that this solution naturally satisfies the constraints (21) at the origin. Far from the origin, $f(\xi)$ rapidly decays as

$$f(\xi) \sim \xi^{(1 - d)/2(1 - \lambda)} e^{-\kappa \xi^2 - \psi}, \quad \xi \to \infty.$$  

(31)

Thus, the solution for $\Sigma$-independent viscosity extends to infinity for any $\lambda$.

Using basic properties of the Tricomi function (Abramowitz & Stegun 1972), one can easily show that in the disk with zero torque the solution (30) reduces to Equation (25), while for $\lambda = 0$ (no central inflow) one reproduces Equation (27).

A nice feature of the linear problem is that many of its properties can be written explicitly. For example, Equations (61) and (62) provide analytical expressions for the angular momentum integral $I_L$ and the degree $\bar{m}$ to which the central accretion is suppressed, see Section 4.3. Behavior of these and some other characteristics of the linear solutions are shown in Figure 5 as functions of $\lambda$.

4.2. Nonlinear Problem

We next explore the case of the nonlinear viscosity, $d > 0$. Restricting ourselves to the range (29), we numerically solve Equation (20) for different values of $\lambda$ subject to the additional constraint $I_M = 1$. Our results are shown in Figure 2 for $d = 2/5, p = -6/5$ ($\kappa_{es}$ regime) and Figure 3 for $d = 3/10, p = -4/5$ ($\kappa_{gs}$ regime). In Figures 6 and 7, we display the behavior of various characteristics of these
nonlinear solutions—$L$, $f_0$ (directly related to the amplitude of the central torque), etc.—as functions of $\lambda$.

Note that all solutions of the nonlinear problem have $F_f$ and, consequently, $\Sigma$ vanishing at a finite radius. This is a characteristic feature of the nonlinear diffusion, in which the speed of signal propagation is limited, unlike the linear problem of Section 4.1, in which the (exponentially suppressed) tail of $\Sigma$ distribution extends out to infinity almost instantaneously.

Solutions for other values of $d$ and $p$ can be found analogously, by numerically solving Equation (20). We were unable to identify other obvious analytical solutions of this nonlinear equation, apart from the known results described in Sections 3.1 and 3.2.

Comparing Figures 5–7, one can notice several features of our new solutions common to both the linear (Section 4.1) and the nonlinear cases. First, decreasing $\lambda$ and $f'_0(0)$ always results in the monotonic increase of $f(0)$. This is expected since more severe suppression of accretion (lower $\lambda$) requires stronger central torque. Central mass outflow like in Be stars requires even higher levels of the angular momentum injection by the central object. A unique relation between $\lambda$, which characterizes the efficiency of accretion (see Section 4.3), and $f(0)$, which sets the central torque, is one of the most important properties of the new self-similar solutions.

Second, the $\xi$-extent of our solutions slowly decreases as $\lambda$ is lowered. This is a consequence of our constraint $I_M = 1$ for all $\lambda$, which forces higher amplitude solutions to occupy a smaller interval of $\xi$.

4.3. Suppression of Accretion

Central torque acting on the disk suppresses mass accretion onto the central object or even reverses it to an outflow. We quantify the degree of this suppression via a dimensionless parameter $\dot{m}(\lambda)$ defined as

$$m(\lambda) = \frac{\dot{M}(\lambda, M_d, L_d)}{\dot{M}(\lambda_0, M_d, L_d)}.$$  \hspace{1cm} (32)

Here $\dot{M}(\lambda, M_d, L_d)$ is the central accretion rate for a solution corresponding to a disk with total mass $M_d$, angular momentum $L_d$, and a given value of $\lambda$; $\dot{M}(\lambda_0, M_d, L_d)$ is the value of that rate for a solution with no central torque (see Section 3.1), when $\lambda = \lambda_0$ (see Equation (24)) and central $M$ attains its maximum possible value for the fixed $M_d$ and $L_d$.

Using Equations (13)–(16), (21), and keeping in mind our constraint $I_M = 1$, one can easily show that

$$\dot{M}(\lambda, M_d, L_d) = \lambda \left[ D_f M_d \left( \frac{L_f(\lambda) M_d}{L_d} \right)^{2-p-d} \right]^{1/(1-d)}.$$  \hspace{1cm} (33)

Thus, for a given $\lambda$, the central mass accretion rate is uniquely set by the total disk mass $M_d$ and angular momentum $L_d$.

Plugging this into definition (32), we immediately find that

$$\dot{m}(\lambda) = (2-p) \lambda \left[ \frac{L_f(\lambda)}{L_f(\lambda_0)} \right]^{2-p-d/(1-d)},$$  \hspace{1cm} (34)

independent of $M_d$ and $L_d$ and with $L_f(\lambda_0)$ given by Equation (61) in the linear and (67) in the nonlinear case. By construction, it is always true that $\dot{m}(\lambda_0) = 1$, i.e., mass inflow is completely unsuppressed for a disk with zero central torque ($\lambda = \lambda_0$). It is also clear that $\dot{m} = 0$ for a disk with $\lambda = 0$, which has fully suppressed accretion, $\dot{M}(\lambda = 0) = 0$.

Thus, $\dot{m}$ indeed represents a convenient dimensionless variable for characterizing the degree of the central inflow suppression for a solution with a given $\lambda$, which is completely independent of the current disk characteristics ($M_d$ and $L_d$). It monotonically varies from 1 to 0 as we move away from the zero torque solution (26) toward the zero inflow solution (3.2), and then becomes negative for outflow solutions ($\lambda < 0$); see Figure 4. Given this, it is useful to show different properties of our solutions as functions of $\dot{m}$ rather than $\lambda$ (see Section 6.1 for a discussion of this issue), and this is done in Figures 8–10.

4.4. Time Evolution

We now go back to Equation (11) and determine the time evolution of our solutions by solving it:

$$\varphi(t) = \left[ 1 + \frac{k^{-1} \varphi}{1 - d \varphi} \right]^{-k},$$  \hspace{1cm} (35)

$$k = \frac{d + (1-d)(1-d)(2-p)}{1-p}.$$  \hspace{1cm} (36)

We set $\varphi(0) = 1$ since we have the freedom to choose this value due to its degeneracy with $F_0$. The scaling exponent $k_{\varphi} = 1$ for $\lambda = (2-p)^{-1}$, i.e., for a disk with zero torque at the center. It monotonically decreases for lower $\lambda$, and $k_{\varphi} \to 0$ as $\lambda \to -\infty$. This behavior is illustrated in Figures 5–7. By considering $\lambda$ as a free parameter, we naturally recover the existing results for $\lambda = \lambda_0$ and $\lambda = 0$ (Pringle 1991).

Since $k_{\varphi} > 0$, one can see that $\varphi \to \infty$ at a finite time $t = -(1-d)k_{\varphi}t_0$ in the past, reflecting our similarity assumption.
Spatial scale of the mass distribution in the disk is set by the condition \( x \sim 1 \), corresponding to a characteristic value of \( y = l_0 t_0 \), varying in time as \( y = \frac{k_y - 1}{1 - \lambda} \tfrac{d}{1 - d} t \) with \( d = -\frac{k_\varphi}{k_\Psi} \), or

\[
  k_\Psi = \left[ 2 - p + \frac{d(1 - p)}{(1 - \lambda)(1 - d)} \right]^{-1},
\]

where we used definition (19) to eliminate \( \delta \). Figures 5–10 demonstrate a rather weak dependence of \( k_\Psi \) on \( \lambda \) or \( \dot{m} \).

Plugging result (35) into Equations (8), (13), (15), and (16), we can determine the time evolution of other disk characteristics. In particular, for \( t > t_0 \), the central torque scales as

\[
  F_j(0, t) \propto f(0) t^{-k_\varphi},
\]

In the same limit the central inflow rate behaves as

\[
  \dot{M}(t) \propto \lambda t^{-k_M},
\]

\[
  k_M = k_\varphi \left[ 1 + \frac{(1 - \lambda)(1 - d)}{1 - p} \right],
\]

where we used Equation (21). Figures 5–7 show that \( k_M \) monotonically decreases from \( k_M = (3 - d - p)/(2 - p) \) for \( \lambda = \lambda_0 \) to \( k_M \to (2 - p)^{-1} \) as \( \lambda \to -\infty \).

The total disk mass and angular momentum vary as

\[
  M_d(t) = M_d(0) [\varphi(t)]^{\lambda(1-d)} \propto t^{-\lambda k_\varphi (1-d)},
\]

\[
  L_d(t) = L_d(0) [\varphi(t)]^{-k_\varphi} \propto t^{1 - k_\varphi},
\]

where the explicit scalings with time pertain to \( t > t_0 \).

According to these scalings, a disk with \( \lambda = \lambda_0 \) has a constant \( L_d \) (since \( k_\varphi \to 1 \)). Using the conversion in Appendix A, one can easily show that the behavior (39)–(40) in such a disk coincides with the one obtained by Pringle (1991) in the case of a vanishing central torque. Analogously, a disk with \( \lambda = 0 \) preserves its total mass, while the scaling (38) of its central torque agrees with the Pringle’s result for the case of zero mass inflow.

More generally, it is clear that the disk angular momentum can only increase with time, irrespective of \( \lambda > \lambda_0 \). Similarly, central torque always decreases. At the same time, disk mass
grows for \( \lambda < 0 \) (decretion) and decays for \( \lambda > 0 \) (accretion). Thus, depending on the value of \( \lambda \), there could be four possible initial states for the self-similar disk evolution at \( t = -(1 - d) k \omega t_0 \):

1. \( \lambda = \lambda_0 \): infinite \( M_d(0) \), and finite \( L_d(0) \) that stays constant through the evolution,
2. \( 0 < \lambda < \lambda_0 \): infinite \( M_d(0) \), and \( L_d(0) = 0 \),
3. \( \lambda = 0 \): finite \( M_d(0) \) that remains fixed through the evolution, and \( L_d(0) = 0 \),
4. \( \lambda < 0 \): \( M_d(0) = 0 \) and \( L_d(0) = 0 \).

In Figure 11, we illustrate the self-similar time evolution of a couple of solutions for a disk with the electron scattering opacity. One is an accreting solution with \( \lambda = 0.2 \), corresponding to a 58% suppression of central \( \dot{M} \) compared to the zero-torque case, i.e., \( \dot{m} = 0.4 \), see Figure 6. Another describes a decretion disk solution with \( \lambda = -0.2 \), for which \( \dot{m} = -0.19 \).

Quite naturally, one finds that near the origin \( F_j(l) \) develops a flat profile (as long as \( \lambda < \lambda_0 \)), the amplitude of which goes down with time. In addition, the radial profile of the surface density \( \Sigma \) attains a universal slope for \( l < l_0 \psi(t) \). This naturally follows from Equation (5), which predicts that

\[
\Sigma(r) \propto r^{-(p+3)/2}, \quad r \to 0,
\]

since \( F_j(r) \to \text{const} \) when there is a non-zero torque at the disk center. Note that this universal result holds for both linear \( (d = 0) \) and nonlinear \( (d = 0) \) settings, and for systems with both accretion \( (\lambda > 0) \) and ejection \( (\lambda < 0) \) of mass by the central object.

Over time, the distributions of both \( F_j \) and \( \Sigma \) extend over larger and larger distance \( r \) (or \( l \)), while at the same time going down in amplitude. Note that at a given radius \( \Sigma \) decreases noticeably slower for the solution with \( \lambda < 0 \), simply because it represents a decretion disk gaining mass at the center. Since \( d \neq 0 \) for \( \kappa = \kappa_{\text{esc}} \), these solutions are always truncated at finite radii.

5. OBSERVATIONAL SIGNATURES

Viscous stresses driving the outward expansion of the disk inevitably result in energy dissipation, heating the disk and giving rise to observable signatures. The thermal state of the decretion disks is often determined by some external agents, e.g., via the irradiation by the central object. This is expected to be the case in the Be disks (Rivinius et al. 2013) or in the outer parts of the protoplanetary disks around young stellar binaries (Vartanyan et al. 2016). In this regime, internal dissipation is not expected to appreciably affect the disk spectrum.

However, in many systems, thermodynamics of the disk is dominated by the internal heating. This is likely to be true for the disks orbiting compact objects (e.g., neutron stars in the propeller regime), inner regions of the circumbinary protoplanetary disks (Vartanyan et al. 2016), circumbinary disks...
One can see that in the low frequency limit $\nu \ll \tilde{\nu}$, the scaling of the shape function $\Phi$ with $\nu$ is universal and independent of $\lambda$. This can be easily understood from the asymptotic behavior of $\Phi$, which is discussed in Appendix D. For $\nu \ll \tilde{\nu}(t)$, one finds

$$\Phi(\lambda, z) \to c_6(\lambda) z^2, \quad z \ll 1,$$

with $c_6$ given by Equation (69). This behavior describes the Raileigh–Jeans tail of the disk emission and is robust for all $\lambda$, as Figure 12 shows. The $\lambda$-dependence of the amplitude of this asymptotic is also rather weak.

Things are different in the high-frequency limit, $\nu \gg \tilde{\nu}(t)$. There, as long as $\lambda = \lambda_0$, one finds

$$\Phi(\lambda, z) \to c_7 [f(\lambda, 0)]^{1/7} z^{12/7}, \quad z \gg 1,$$

with $c_7$ given by Equation (70). Because of this scaling, we chose to divide $\nu \mathcal{L}_\nu$ by $\nu^{12/7}$ in Figure 12 to better illustrate the SED dependence on $\lambda$ (the inset of that figure shows SED for two values of $\lambda$ without such division).

However, for the disk without any torque at the center ($\lambda = \lambda_0$ and $f(\lambda_0, 0) = 0$), the SED behavior is qualitatively different:

$$\Phi(\lambda_0, z) \to c_8 (2 - p)^{-2/3} z^{4/3}, \quad z \gg 1,$$

with $c_8$ given by Equation (71). Asymptotic behaviors (48)–(50) are illustrated in the inset in Figure 12.

The difference in the high-$\nu$ SED scaling between the standard accretion disk (50) and the disk with some non-zero central torque ($\lambda$) was previously noted in Syunyaev & Shakura (1977), Syer & Clarke (1995), and Rafikov (2013). Figure 12 clearly shows that as long as $\lambda$ even slightly deviates from $\lambda_0$, the high-frequency asymptotic of $\Phi$ follows the behavior (49). Already at $\lambda = 0.35$ (different from $\lambda_0$ by only 0.07) $\Phi(\lambda, z)$ clearly tends to converge to $\nu^{12/7}$ scaling. At smaller values of $\lambda$, including the negative ones, the convergence is faster and only the amplitude of the scaling depends on $\lambda$.

Temporal evolution of the SED accompanying the viscous spreading of the disk is illustrated in Figure 13. There, we show $\nu \mathcal{L}_\nu$ at several moments of time for a self-luminous decretion disk with $\kappa = \kappa_\text{ff} (d = 3/10, p = -4/5)$ and $\lambda = 0.3$ (for which accretion is suppressed by 35%, i.e., $\dot{m} \approx 0.65$, see Figure 5). Over time, the spectrum of the disk shifts toward lower frequencies, while maintaining its overall self-similar shape.

One can see that as time goes by, the spectral power above the characteristic frequency $\tilde{\nu}$ always decreases. Indeed, using Equation (49), one finds $\nu \mathcal{L}_\nu(t) \propto (\nu/\nu_0)^{12/7} [\varphi(t)]^{1/7}$ for $\nu \gg \tilde{\nu}(t)$, meaning the decay of $\mathcal{L}_\nu$ for $\nu > \tilde{\nu}$.

On the contrary, below $\tilde{\nu}$, the amplitude of $\mathcal{L}_\nu$ grows with time. This can be understood by combining Equations (44)–(46) and (48) to find that $\nu \mathcal{L}_\nu(t) \propto (\nu/\nu_0)^{-3} [\varphi(t)]^{4/7}$ for $\nu \ll \tilde{\nu}$. As $\delta < 0$, this implies that $\nu \mathcal{L}_\nu$ increases with time at a fixed frequency, as long as $\nu$ stays below $\tilde{\nu}(t)$.

From Equation (68), it is easy to see that the bolometric luminosity $\mathcal{L}$ of a decretion disk is proportional to $F_1(r \to 0)$, as long as the latter is non-zero (to obtain $\mathcal{L}$ the integral of $d\mathcal{L}_\nu/dr$ has to be truncated at some inner radius). As a result, $\mathcal{L} \propto \varphi(t) \propto t^{-k_v}$ for such disks. This luminosity evolution is different from that of the disks with zero central torque ($\lambda = \lambda_0$), which have $\mathcal{L} \propto \dot{M}(t) \propto t^{-k_v}$, see Equation (39).
The difference between the high-frequency spectra given by Equations (49) and (50) can be used as an observational probe of the presence of the non-zero torque at the center of a decretion disk. Rafikov (2013) suggested utilizing this feature as a way of inferring the presence of the binary supermassive black holes from the quasar spectra. Based on Figure 12, we expect \( \lambda \) to have a steeper \( \nu \)-dependence (49) at high frequencies as long as there is even weak central torque at the center. The spectrum of a disk with no central torque whatsoever would follow the shallower frequency dependence (50). However, in has to be remembered that this distinction applies only to a purely self-luminous disk radiating as a blackbody. Any deviation from this regime (e.g., illumination by the central object, strong emission lines, etc.) could easily affect this observational probe of the central torque.

6. DISCUSSION

Our results in Section 4 clearly show that to fully specify the self-similar disk evolution one must provide the values of the three constants—\( \lambda, F_0 \), and \( l_0 \). There are different ways in which they can be fixed by the physics of the problem at hand, which we discuss in Sections 6.1–6.3.

Once this is done, one obtains a lot of information about the integral characteristics of the decretion disk evolution, for a given value of \( \lambda \). In particular, one finds a unique relation between the central \( M(0, t) \) and torque \( F_J(0, r) \) (Figures 5–7), determines time evolution of the total angular momentum \( L_d \) and mass \( M_d \) of an evolving disk, the rate at which it expands, and so on (Section 4.4). We illustrate the use of these results in Rafikov (2016), where we employ our understanding of the decretion disk evolution to constrain physical mechanisms of the eccentricity excitation in the post-main-sequence binaries.

6.1. Solution Determination: Fixed Degree of the Suppression of Accretion

We now discuss how one can uniquely determine \( \lambda, F_0 \) and \( l_0 \) using physical arguments relevant for different astrophysical objects.

The value of \( \lambda \) can be fixed if one expects the torque exerted by the central object to suppress \( \dot{M} \) at the origin by a prescribed amount \( \dot{\dot{m}} \) compared to the accretion rate in the absence of the central torque. A direct and monotonic relation between \( \lambda \) and \( \dot{\dot{m}} \) established in Section 4.3 then allows one to determine the former once the latter is fixed. In particular, in the linear case, one would invert analytical formula (62) for that purpose, using the definition (58). In the nonlinear case \( (d = 0) \) one would use the numerical calculations such as described in Section 4.3 and shown in Figures 8–10.

Once \( \lambda \) is fixed, the values of \( F_0 \) and \( l_0 \) are uniquely specified by the total mass \( M_d(0) \) and angular momentum \( L_d(0) \) of the disk at \( t = 0 \). This is shown mathematically via Equations (72)–(73) in Appendix E, similar to the procedure in Ertan et al. (2009). Thus, the knowledge of \( M_d \) and \( L_d \) at some moment in time (which can always be set to \( t = 0 \)) fully specifies the subsequent self-similar evolution of the disk.

Note that there are other ways of fixing \( F_0 \) and \( l_0 \) for a given \( \lambda \). For example, instead of \( L_d(0) \) one may choose to specify the characteristic radius enclosing a given fraction of the disk mass—obviously, it is directly related to \( l_0 \). There are many other similar choices, which we do not discuss here.

Fixing the value of \( \lambda \) via known \( \dot{m} \) is a very simple and attractive way of specifying the disk evolution. For example, recent simulations of the circumbinary disks (MacFadyen & Milosavljević 2008; D’Orazio et al. 2013; Farris et al. 2014) provide a measurement of the accretion rate in the presence of the binary torque, resulting in the estimate of \( \dot{m} \) and, thus, \( \lambda \). Motivated by these results, Martin et al. (2013) explored one-dimensional viscous evolution of the circumbinary disks including a model with a fixed non-zero value of \( \dot{m} \), similar to what we have described.

6.2. Solution Determination: Known Physics of the Central Barrier

The problem with the approach outlined in Section 6.1 is that there is usually no a priori reason why one should expect \( \dot{m} \) to be constant in time. Indeed, the central \( M \) is set by the physics of the central barrier to accretion (details of the torque exerted on the disk by the accreting object, local gas density, etc.), while \( M(\lambda = \lambda_0) \) is set by the global structure of the disk.

At the same time, it is still possible to find a unique value of the similarity parameter \( \lambda \) using the knowledge of what sets \( M \) at the disk center. It is reasonable to expect that \( M \) should be proportional to the amount of mass in the inner disk, i.e., to \( \Sigma(r \to 0) \). Since \( \Sigma \) is related to \( F_J \) via definition (2), we will consider a simple boundary condition for \( M \) in the power-law form

\[
\dot{M} = K[F_J(r \to 0)]^\eta,
\]

with constant \( K \) and \( \eta > 0 \). In other words, the larger is the inner torque \( F_J(r \to 0) \), the more mass accumulates near the origin, the higher \( \Sigma \) is there, and the larger \( \dot{M} \) is.

Using Equations (8), (16), and (19), one can see that with this prescription for the physics of the inner barrier the self-
similarity uniquely determines
\[ \lambda = 1 + (1 - \eta) \frac{1 - p}{1 - d}. \quad (52) \]

The same procedure also yields an algebraic relation between \( F_0 \) and \( I_0 \). To separately determine their values, one needs to supply additional information as discussed in Section 6.1. We will assume here that we know the disk mass \( M_d \) at time \( t = 0 \). Then, using Equation (13) with \( \phi(0) = I_M = 1 \), one finds that \( F_0 \) and \( I_0 \) are given by Equations (74) and (75). Given the constraint (29), our derived value (52) of \( \lambda \) implies
\[ \eta \geq 1 + \frac{1 - d}{2 - p}, \quad (53) \]
i.e., that self-similarity is possible only for steep enough dependence of the central \( M \) on \( F_I(r \to 0) \), certainly faster than linear. Thus, the inner barrier should be less effective at suppressing gas inflow as more gas accumulates near the origin.

6.3. Solution Determination: Prescribed Central \( M \) or Torque \( F_I \)

It is also possible that the system imposes boundary conditions on the disk that enforce self-similarity of its evolution. For example, consider a central object that ejects mass at a rate that asymptotically scales as a power law of time, \( M \propto t^{-1/\mu} \), where \( \mu_M \) is a constant determined by the physics of the ejection process. An example of such a system could be a Be star or a post-main-sequence binary losing mass via its outer Lagrange point. Then Equation (39) implies that \( k_M = \mu_M \), which, according to Equations (36) and (40), uniquely determines the value of the similarity parameter \( \lambda \) as a function of \( \mu_M \) (see the discussion in Section 6.4).

This, in turn, fixes the value of \( k_\psi \) (Equation (36)) and, according to Equation (38), sets the time evolution of the central torque \( F_I(0, t) \). Then a natural question to ask is whether one would expect \( F_I(0, t) \) to follow this particular unique behavior, as required by the similarity of the solution. At least in some cases the answer is yes.

For example, a stellar binary losing mass through its L2 Lagrange point exerts gravitational torque on the escaping gas at particular resonant locations in the disk (Goldreich & Tremaine 1980). The amplitude of this torque is proportional to the disk surface density at the resonant radii and should naturally self-regulate to follow the behavior (38) in the following way. If \( F_I(0, t) \) grows above the value needed for the self-similar expansion of the disk, the inner disk will absorb excess angular momentum, driving its expansion. This will reduce \( \Sigma \) at the resonant locations until \( F_I(0, t) \) is brought back in accord with the global viscous evolution of the disk.

On the contrary, if at any point in time \( F_I(0, t) \) becomes lower than the self-similar value (38), the mass will be less readily evacuated from the central object’s vicinity, causing gas pileup at the resonant locations and the return of the central torque to the behavior (38). This is how the central torque would self-regulate to ensure the self-similar behavior determined by the exponent \( \eta_M \).

Determination of the values of \( F_0 \) and \( I_0 \) is possible in this case via the normalization of \( M \), which provides an algebraic relation between these variables. Another relation can be obtained, e.g., through the knowledge of the total angular momentum of the disk \( L_d \) at some moment of time. Then, similar to Sections 6.1–6.2, one would uniquely determine both \( F_0 \) and \( I_0 \). We do not show the resulting expressions due to their complexity even though they could be easily derived as just described.

Another possibility for governing the self-similar evolution is via the prescribed central torque on the disk, which may be more typical for accreting objects (i.e., \( M > 0 \)). If \( F_I(0, t) \propto t^{-\eta_I} \) with a constant \( \eta_I \), then Equation (38) immediately implies \( k_\psi = \eta_I \), thus fixing the value of \( \lambda \) via Equation (36). Provided that the central \( M \) self-regulates to obey Equation (39), the self-similar evolution would again be possible.

6.4. Comparison with the Existing Studies

Following the pioneering work of Lynden-Bell & Pringle (1974), a number of authors have explored viscous evolution of the decretion disks in a variety of contexts. Self-similar solutions, which are the focus of our work, were first discussed for the linear problem \( (d = 0) \) in Lynden-Bell & Pringle (1974); see Sections 3.1 and 3.2. Self-similar ansatz for the nonlinear problem was considered in Filipov (1984), but the detailed analysis of this problem had to wait until Lyubarskij & Shakura (1987) obtained the two known nonlinear solutions without either the central torque (26) or the central inflow (28). These solutions were also discussed in Filipov (1988), Filipov et al. (1988), Cannizzo et al. (1990), and Pringle (1991). Self-similar solutions with somewhat different boundary conditions were studied by Lipunova & Shakura (2000).

Our work extends these past studies by also exploring a much more general class of astrophysical systems in which neither the central inflow nor the central torque vanish. Some qualitative discussions of the decretion disk evolution in this case can, however, be found in Vartanyan et al. (2016). Eksi (2009) and Eksi (2012) also made an attempt to describe evolution of viscously spreading disks with “partial” accretion at the center by a particular superposition of the two known analytic solutions (Lynden-Bell & Pringle 1974; Lyubarskij & Shakura 1987). Unlike our self-similar solutions, in that case, the time evolution of the central torque and \( M \) is prescribed by \( d \) and \( p \) only and is independent of the degree of the accretion suppression. Even though such solutions do not, in general, satisfy the original viscous evolution equation and are not truly self-similar (over time they evolve to either \( \lambda = 0 \) or \( \lambda = \lambda_0 \) solutions), they nevertheless match the numerical results for the disk evolution with properly designed boundary conditions rather well.

In his study of the circumbinary disks around the supermassive black hole binaries, Rafikov (2013) found self-similar solutions with both \( M(r \to 0) = 0 \) and \( F_I(r \to 0) = 0 \) for accretion disks externally supplied at a fixed \( M \), generalizing the previous result of Ivanov et al. (1999), which was limited to \( M(r \to 0) = 0 \). This is a qualitatively different setup compared to the decretion disks studied here, for which no self-similar solutions with such general inner boundary conditions have been explored until now.

Our results also apply to systems, in which a decretion disk is fed with mass ejected by the central object, such as the disks around Be stars. We are not aware of any existing self-similar solutions applicable to disks with central mass injection, which makes our results particularly valuable for understanding disks of Be stars and mass-losing post-main-sequence binaries.
For example, in his study of the Be disks, Okazaki (2007) numerically calculated viscous evolution of an isothermal decretion disk \((c_s = \text{const})\) fed at a constant injection rate \(M\). In our self-similar ansatz (7), such a disk would correspond to \(d = 0\) and \(p = 1/2\), as follows from Equation (4) for constant \(c_s\). However, Equations (36), (39), and (40) demonstrate that the assumption of time invariant \(M\) is incompatible with the self-similarity of the disk evolution: it would require \(k_M = 0\), which is impossible; see the discussion after Equation (40). This expectation agrees with the numerical results of Okazaki (2007), which indeed do not exhibit the development of a self-similar profile of the surface density.

At the same time, Okazaki (2007) found the convergence of \(\Sigma(r)\) to \(r^{-2}\) profile previously suggested by Bjorkman & Carciofi (2005), which is what Equation (43) predicts for \(p = 1/2\). However, Equation (43) does not require similarity and follows simply from the fact that \(F_j(r) \rightarrow \text{const}\) as \(r \rightarrow 0\) (naturally fulfilled for any disk with central mass source) as discussed earlier in Section 4.4.

7. SUMMARY

Our work provides a general understanding of the decretion disk evolution in the late time asymptotic limit, when the viscous stresses drive the disk structure toward the self-similarity. Going beyond the existing studies, we calculate the self-similar viscous evolution of the most general decretion disks that feature both the non-zero accretion (or decretion) rate at the center and the non-zero central torque. This situation naturally arises in a number of real astrophysical objects—accreting neutron stars, post-main-sequence binaries, disks of Be stars, etc.

The variety of diverse evolutionary pathways of decretion disks, both linear and nonlinear, is shown to be a function of a single similarity parameter \(\lambda\). The two previously known similarity solutions (Lynden-Bell & Pringle 1974; Pringle 1991) correspond to the two discrete values of this parameter (see Sections 3.1 and 3.2). With our new results, we have now covered a continuum of other possible values of \(\lambda\), relevant for both accretion and ejection of mass by the central object. We have also shown that \(\lambda\) is closely related to the degree \(\dot{m}\), to which the non-zero central torque suppresses accretion by the central object (Section 4.3).

Our calculations reveal the intimate connection between the central torque acting on the disk and the central accretion rate, which is closely related to the value of \(\lambda\). Once the latter is known, the self-similar ansatz uniquely predicts in a transparent way the time evolution of the main disk properties—its total mass and angular momentum, radial scale, central torque, and mass accretion rate. We calculate observational signatures of the self-luminous decretion disks and show that their spectra are different from the SEDs of the conventional accretion disks with zero central torque. This is also true for the evolution of their bolometric luminosity.

We then discuss a variety of ways in which the characteristics of our new self-similar solutions—their amplitude, radial scale, value of \(\lambda\)—can be constrained for different astrophysical objects. Our results are applicable to understanding viscous evolution of decretion disks in various astrophysical settings, as demonstrated in Rafikov (2016).

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APPENDIX A

CONNECTION TO THE NOTATION OF PRINGLE (1991)

Pringle (1991) studied the nonlinear viscous spreading problem assuming viscosity in the form of \(\nu \propto \Sigma^m r^p\). Using definitions (2) and (4), one can show that such scaling implies

\[
d = \frac{m}{m + 1}, \quad p = \frac{2n - 3m - 2}{m + 1}
\]

in our notation. Also, Pringle (1991) wrote down the evolution equation not for \(F_j\) but for

\[
S \propto \Sigma r^{3/2} \propto F_j^{1/(m+1)} f^{3m+2-2n)/(m+1)}
\]

as a function of time and \(r \propto \bar{L}\). These relations allow one to convert analytical solutions (26) and (28) into Pringle’s notation. Note that in Pringle (1991) these solutions are not normalized to satisfy \(I_M = 1\).

APPENDIX B

DETAILS OF THE LINEAR SOLUTION

Change of variables

\[
z = -\kappa(\lambda) \xi^{2-p}, \quad \kappa(\lambda) = \frac{1 - \lambda}{(1 - \lambda)(2 - p)}, \quad (56)
\]

converts Equation (20) with \(d = 0\) into Kummer’s equation (Abramowitz & Stegun 1972a)

\[
f_{zz}^p + (b - z)f_z - af = 0, \quad (57)
\]

\[
a(\lambda) \equiv \frac{1}{(1 - \lambda)(2 - p)}, \quad b \equiv \frac{1}{2 - p}. \quad (58)
\]

Its solutions can be generally expressed via the confluent hypergeometric function. A particular solution that describes the disk with a finite mass (Abramowitz & Stegun 1972b) and satisfies the condition \(I_M = 1\) is given by Equation (30) with

\[
c_s(\lambda) = \frac{(2 - p)(\kappa(\lambda))^p \Gamma(1 + b - a(\lambda))}{\Gamma(b)} \cdot (59)
\]

where \(\Gamma(t)\) is the \(\gamma\)-function (Gradsteyn & Ryzhik 1994). In a disk with zero inflow at the center (\(\lambda = 0\)), this pre-factor becomes

\[
c_2 = \left[\frac{2 - p}{(1 - p)^{-p}}\right]^{1/(2-p)} \left[\Gamma\left(1 - \frac{p}{2 - p}\right)\right]^{1-p} \cdot (60)
\]

For the linear problem, the angular momentum integral \(I_L\) can be expressed as an analytic function of the disk parameters:

\[
I_L(\lambda) = \kappa^{-1/(2 - p)} \frac{\Gamma(1 + b - a(\lambda))\Gamma(2 - b)}{\Gamma(2 - a(\lambda))\Gamma(b)} \cdot (61)
\]

Using Equation (34), we find the degree to which accretion is suppressed for a given \(\lambda\) in the linear case as

\[
\dot{m}(\lambda) = \frac{\lambda}{2 - p} \left[\frac{\Gamma(1 + b - a(\lambda))}{\Gamma(2 - a(\lambda))}\right]^{2-p} \cdot (62)
\]
APPENDIX C
DETAILS OF THE NONLINEAR SOLUTIONS

Here we provide expressions for the various constant factors relevant for the nonlinear problem ($d = 0$):

\[ c_1 = \frac{d(2 - p)^{d-1}}{2 - p - d}, \]  
\[ c_3 = \left[ \left( \frac{d}{1 - p} \right)^{1-p} \frac{2 - p}{B^{2-p}} \right]^{1/(2-p-d)}, \]  
\[ c_4 = \left[ \left( \frac{d}{1 - p} \right)^{1-d} B^d \right]^{2/(2-p-d)}, \]

where $B = B((1 - p)/(2 - p), d^{-1})$ is the $\beta$-function. Also angular momentum integrals $\int d\ell \xi^d$ for zero inflow and zero torque cases are given by

\[ I_L(0) = \frac{d}{2 - p - c_4}, \]
\[ I_L(\lambda) = \frac{(2 - p)^{d-1}}{(2 - p - d) c_4} \left[ \left( \frac{2 - p - d}{2 - p - d} \right)^{d-1} \right]. \]

In the linear case, Equation (61) should be used instead.

APPENDIX D
SED CALCULATION

To compute the SED of a self-luminous decretion disk, we use the following relation between the viscous energy dissipation rate per unit radius $dE_v/dr$, effective temperature of the disk $T_e$, and $F_2$ (Rafikov 2013):

\[ \frac{dE_v}{dr} = 4\pi r^3 \sigma T_e^4 = \frac{3}{2} \frac{F_2 \Omega}{r}, \]

where we assumed Keplerian rotation. Given the self-similar behavior of $F_2$ in the form (8) with the known $\varphi(t)$, $\psi(t)$, and $f(\xi)$, one immediately finds the behavior of $T_e$ as a function of $r$ and $t$.

We compute disk SED as $\nu L_\nu = 2\pi \nu \int_0^\infty 2\pi r B_\nu (T_e(r, t), \nu) dr$, where $B_\nu$ is a Planck function. After a series of straightforward transformation, we find the SED to be given by the Equations (44)–(47), where we used Equation (10) to express $\psi$ via $\varphi$.

Asymptotic behavior of the SED shape function $F(\nu)$ can be easily derived in the limit $z \ll 1$ (Equation (48)) by expanding the argument of the exponential in the denominator of the Equation (48), and in the limit $z \gg 1$ (Equation (49)) by noticing that the integral is dominated by $\xi \ll 1$ and setting $f(\xi) \rightarrow f(0)$. The corresponding behaviors are characterized by the constants

\[ c_6(\lambda) = \frac{45}{\pi} \int_0^\infty [f(\xi) \xi^4]^2/4 d\xi, \]
\[ c_7 = \frac{180}{7\pi^2} \Gamma \left[ \frac{16}{7} \right] \zeta \left[ \frac{16}{7} \right] \approx 0.439782, \]

where $\zeta(x)$ is a Riemann’s $\zeta$-function (Abramowitz & Stegun 1972).

The high-frequency asymptotic changes in the case of a disk with no central torque ($\lambda = \lambda_0$), as then we cannot set $f(\xi) \rightarrow f(0) = 0$. Instead, we use the fact that $f(\lambda_0, \xi) \rightarrow \lambda_0 \xi$ as $\xi \rightarrow 0$ for $\lambda = \lambda_0$ and substitute this behavior in the integrand. As a result, we arrive at the Equation (50) with $\lambda$ given by Equation (24) and $c_8$ given by

\[ c_8 = \frac{30}{\pi^4} \Gamma \left[ \frac{8}{3} \right] \zeta \left[ \frac{8}{3} \right] \approx 0.595066. \]

APPENDIX E
DETAILS OF THE SOLUTION DETERMINATION

Here we provide some details of the similarity solution determination covered in Sections 6.2–6.3. In the case when $\lambda$ is fixed (e.g., through $m$, see Section 6.1), the knowledge of $M_d$ and $I_d t$ at time $t = 0$ allows one to find, using Equations (13) and (15), that

\[ F_0 = [D_2 t_0 M_4^{2-\eta} (I_4 t_4 L_4 (0))^{\eta-1}]^{(1-d)}, \]
\[ l_0 = I_4 t_4 L_4 (0) / M_4 (0). \]

Note that the dependence on $\lambda$ enters only through $I_4 (\lambda)$.

For the model, in which the physics of the central barrier is adequately characterized by the Equation (51), one finds

\[ F_0 = \left[ D_2 t_0 M_4 (0) \left( \frac{K(f(\lambda, 0))}{f_\xi^2 (\lambda, 0)} \right)^{\eta-1} \left( \frac{\lambda}{1-d} \right) \right] \left( \frac{1}{f_\xi (\lambda, 0)} \right)^\eta, \]

where $\lambda$ is given by Equation (52) and $f(\lambda, 0), f_\xi (\lambda, 0)$ are evaluated for this particular value of the similarity parameter.

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