Individual attacks with generalized discrimination and inadequacy of some information measures

Alexey E. Rastegin
Department of Theoretical Physics, Irkutsk State University, Gagarin Bv. 20, Irkutsk 664003, Russia

We accomplish studies of properly quantifying eavesdropper’s information gain in individual attacks on the BB84 system of quantum key distribution. A noticeable sensitivity of conclusions to the choice of information measures is brightly revealed, when generalized state discrimination is used at the last stage. To realize her aims, Eve intrudes into the communication channel with some entangling probe. The Fuchs–Peres–Brandt (FPB) probe is known as a powerful individual attack on the BB84 protocol. In the simplest formulation, Eve uses the Helstrom scheme to distinguish output probe states. In conclusive eavesdropping, the unambiguous discrimination is utilized. In the intermediate scenario, she can apply a generalized discrimination scheme that interpolates between the Helstrom and unambiguous discrimination schemes. We analyze Eve’s probe performance from the viewpoint of various measures for quantifying mutual information.

PACS numbers: 03.67.Dd, 03.65.Ta, 03.67.Hk
Keywords: BB84 protocol, Helstrom scheme, unambiguous discrimination, Rényi entropy, mutual information

I. INTRODUCTION

Today, quantum cryptography is considered to be a long-term solution to the problem of communication security [1][2]. It is one of emerging technologies based on new application of quantum phenomena [3][4]. Due to famous Shor’s results [5], we have today a lot of quantum algorithms for algebraic problems [6], including those with an impact on classical cryptography [7]. The BB84 protocol [8] is both the first and most known scheme for quantum key distribution. Another scheme of rather methodological interest is based on two non-orthogonal states [9]. One of basic problems in this field of investigations is to analyze vulnerabilities of quantum cryptography schemes. In principle, there are numerous scenarios of eavesdropper activity. Individual attacks form the simplest and intuitively reasoned way to break a bit sequence shared due to quantum key distribution. Here, Eve intrudes into the channel and entangles each information carrier with her probe. After manipulations, altered carriers are come to the receiver, whereas entangling probes are measured by Eve to recognize the original state of each carriers. The main question of information-theoretic origin is how to characterize quantitatively a performance of quantum-cryptographic probes. Studies of individual attacks on systems of quantum cryptography were initiated by the authors of [10].

During quantum key distribution with eavesdropping, each of the three parties will obtain some string of bits. Treating such strings as random variables, a degree of dependence between two of them is usually measured by the mutual information. This approach used in [11] was motivated due to some results of the paper [12]. The authors of [13] have accomplished a detailed analysis of individual attacks. Together with the standard mutual information, the so-called Rényi mutual information was utilized in [14]. However, the treatment of mutual information quantifiers essentially depends with their properties. Such properties are closely related to the used form of conditional entropies. There is no generally accepted definition of conditional entropy of the Rényi type [15][16]. Existing approaches to this notion do not allow us to protect all the properties of the standard information functions. The question of proper quantifying mutual information in quantum cryptography was addressed in [17]. In this regard, some widely used measures of mutual information are not completely adequate. This fact has been shown by comparing the two cases of state discrimination possible after the action of Eve’s entangling probes. If Eve’s uses conclusive eavesdropping, then measures of mutual information of the Rényi type lead to wrong conclusions about a probe performance.

It must be stressed that conclusive eavesdropping is only one of possible ways, in which Eve may discriminate states of her target qubit. There are intermediate discrimination schemes that interpolates between the Helstrom scheme and the unambiguous one. Such an analysis does not seem to have been previously recognized in the literature, but arises naturally in the problem of probe performance estimation. The aim of the present work is to study measures of mutual information from the viewpoint of such scenarios for eavesdropping. The paper is organized as follows. In Section II, we briefly describe details of the Fuchs–Peres–Brandt probe for an individual attack on the BB84 protocol. Then we recall existing schemes of discrimination between two pure states. Section III is devoted to required information-theoretic notions. In particular, we focus on basic properties of entropic functions of the Rényi type. Entropic uncertainty relation for generalized state discrimination are addressed in Sec. IV. Main results are presented in Sec. V. Namely, we demonstrate some inadequacy of the so-called Rényi α-information as a quantifier of Eve’s probe performance. This feature is brightly revealed by consideration of the scenario with generalized state
II. DETAILS OF THE FPB PROBE

In this section, we first recall the Fuchs–Peres–Brandt probe for an individual attack on the BB84 protocol. Keeping her target carrier, Eve had come across the problem of state discrimination. The second part of this section briefly describes existing schemes of discrimination between two pure states of a qubit. The problem of optimization of probe characteristics was originally examined by Fuchs and Peres [19]. They showed numerically that the optimal detection method for a two-level system can be obtained with a two-dimensional probe. The authors of [19] have accomplished a detailed analysis of the BB84 protocol. Brandt [20, 21] later noted that the obtained probe for attacking the BB84 scheme can be realized with a single CNOT gate. Following [22, 23], this probe will be referred to as the Fuchs–Peres–Brandt (FPB) probe. Note that the analysis of [15] is related to the case, in which the error-discard method is used as a reconciliation procedure. When Alice and Bob use other reconciliation methods, the FPB probe is generally not optimal [24]. We will assume that Bob’s detectors and Eve’s detectors have not failed at all. In this sense, a discussed situation is idealized. Individual attacks have been analyzed in various respects [25–27]. Security of quantum key distribution against collective attacks was examined in [28].

In the BB84 scheme, Alice and Bob use the two polarization bases, \{\ket{h}, \ket{v}\} and \{\ket{r}, \ket{\ell}\}. With respect to the horizontal polarization, the kets \ket{r} and \ket{\ell} of the diagonal basis relate to the angles \(\pi/4\) and \(3\pi/4\), respectively. In each bit interval Alice sends a single photon prepared accordingly. Intercepting this photon, Eve inputs it as the control qubit into her CNOT gate. Computational basis of this gate is defined in terms of the polarization states as

\[
\begin{align*}
\ket{0} &= \cos(\pi/8)\ket{h} + \sin(\pi/8)\ket{v}, \\
\ket{1} &= -\sin(\pi/8)\ket{h} + \cos(\pi/8)\ket{v}.
\end{align*}
\]

Eve prepares her own probe photon in the initial state

\[
\ket{t_{in}} = c\ket{+} + s\ket{-},
\]

where \(c = \sqrt{1 - 2P_E}\), \(s = \sqrt{2P_E}\), and

\[
\ket{\pm} = \frac{\ket{0} \pm \ket{1}}{\sqrt{2}}.
\]

The sense of the parameter \(P_E \in [0; 0.5]\) will be clarified in the following. The state (3) is played as the target qubit of Eve’s CNOT gate controlled by sent Alice’s qubit.

To study an adequacy of mutual information quantifiers, probabilities of outcomes should be found. Here, we will express the gate output in the form, in which the first qubit is represented in a proper basis. Let us introduce sub-normalized vectors

\[
\begin{align*}
\ket{t_{\pm}} &= c\ket{+} \pm \frac{s}{\sqrt{2}}\ket{-}, \\
\ket{t_E} &= \frac{s}{\sqrt{2}}\ket{-}.
\end{align*}
\]

When Alice uses the basis \{\ket{h}, \ket{v}\}, Eve’s CNOT gate acts as

\[
\begin{align*}
\ket{h} \otimes \ket{t_{in}} &\longrightarrow \ket{h} \otimes \ket{t_{+}} + \ket{v} \otimes \ket{t_E}, \\
\ket{v} \otimes \ket{t_{in}} &\longrightarrow \ket{v} \otimes \ket{t_{-}} + \ket{h} \otimes \ket{t_E}.
\end{align*}
\]

For states of the basis \{\ket{r}, \ket{\ell}\}, we similarly get

\[
\begin{align*}
\ket{r} \otimes \ket{t_{in}} &\longrightarrow \ket{r} \otimes \ket{t_{+}} - \ket{\ell} \otimes \ket{t_E}, \\
\ket{\ell} \otimes \ket{t_{in}} &\longrightarrow \ket{\ell} \otimes \ket{t_{-}} - \ket{r} \otimes \ket{t_E}.
\end{align*}
\]

These formulas give the ground for analysis of Eve’s intrusion. Note that the vectors \ket{t_{\pm}} and \ket{t_E} in the above formulas are sub-normalized. So, their squared norms are related to the corresponding probabilities. Calculating the inner product \(\langle t_{+}|t_{-}\rangle = 1 - 3P_E\), we further restrict a consideration to the range \(P_E \in [0; 1/3]\).

Suppose that Bob measures the received photon just in the basis that Alice has employed. With no eavesdropping, his outcomes will match what Alice sent. After a sufficiently long session of sending photons by Alice, Bob has
chosen the right basis in a half of cases. Alice and Bob then discard all cases, in which Bob has taken the wrong basis. Hence, Alice and Bob share the sifted bits that should be error-free without eavesdropping. It is seen from the formulas (7)–(10) that Bob’s outcome will sometimes be opposite to what Alice has sent. Such cases are described by the terms which involve the sub-normalized vector \(|t_{E}\rangle\). As its squared norm is equal to \(P_{E}\), the latter quantity gives a fraction of mismatches in those bits that Alice and Bob have used to test eavesdropping. That is, the quantity \(P_{E}\) is the probability of error due to Eve’s activity. To learn shared bit values of Alice and Bob, Eve should distinguish between the sub-normalized outputs \(|t_{+}\rangle\) and \(|t_{-}\rangle\) of the target qubit. At this stage, she could apply suitable schemes of quantum states discrimination.

In general, different quantum states cannot be perfectly discriminated. The following two approaches are most known. In the first approach of Helstrom \cite{29,30}, the optimal measurement is built to minimize the average probability of erroneous answer. The second approach known as unambiguous state discrimination and also as the IDP scheme was developed due to Ivanovich \cite{31}, Dieks \cite{32}, and Peres \cite{33}. Form a general perspective, unambiguous discrimination was discussed in \cite{34}. Its principal point is possibility of inconclusive answer, which allows us to avoid the error of mis-identification. The version of the B92 scheme with unambiguous discrimination on Bob’s side was examined in \cite{13}. Of course, more general strategies of discrimination could be assumed. Concerning the case of linearly independent pure states, such strategies were addressed in \cite{34,50}. The authors of \cite{37} extended this approach to mixed quantum states. Let us describe briefly the generalized scheme of discrimination of two non-orthogonal pure states. This particular case is quite sufficient for our aims.

III. INFORMATION-THEORETIC FUNCTIONS OF THE RÉNYI TYPE

In this section, we recall some information-theoretic notions based on the Rényi entropies. Quantum key distribution is a procedure used by Alice and Bob for obtaining two identical copies of a random and secret sequence of bits. Intruding into a communication channel, Eve try to determine original bits. During the process, each of the three parties will obtain some string of bits. The three strings are typically treated as binary random variables \cite{13}. To measure a dependence between two random variables, one utilizes the concept of mutual information.

Let discrete random variable \(X\) take values on some finite set, and let \(\{p(x)\}\) be the corresponding probability distribution. The Shannon entropy of \(X\) is defined as \cite{38}

\[
H(X) := -\sum_{x} p(x) \log p(x) .
\]  
(11)

The range of summation is usually clear from the context. The logarithm in (11) is taken to the base 2. If \(Y\) is another random variable, then the joint entropy \(H(X, Y)\) is defined by substituting joint probabilities \(p(x, y)\) into (11).

The concept of mutual information is connected with conditional entropies. The conditional form is commonly used in information theory \cite{38} as well as in applied disciplines. The standard conditional entropy is defined by

\[
H(X|Y) := \sum_{y} p(y) H(X|y) = -\sum_{x, y} p(x, y) \log p(x|y) .
\]  
(12)

Together with Bayes’ rule \(p(x|y) = p(x, y)/p(y)\), we used the particular function

\[
H(X|Y) = -\sum_{x} p(x|y) \log p(x|y) .
\]  
(13)

It follows from concavity that \(H(X|Y, Z) \leq H(X|Y)\). In other words, conditioning on more can only reduce the conditional entropy. The quantity (12) leads to one of widely used information distances \cite{39}. The definition (12) is further motivated by the chain rule \cite{38}

\[
H(X, Y) = H(X|Y) + H(Y) = H(Y|X) + H(X) .
\]  
(14)

The relation \(H(X|Y) = H(X, Y) - H(Y)\) reflects remaining lack of knowledge about \(X\) at the given \(Y\).

The mutual information aims to measure how much information \(X\) and \(Y\) have in common \cite{38}, namely

\[
I(X, Y) := H(X) + H(Y) - H(X, Y) .
\]  
(15)

The quantity (15) is clearly symmetric in entries. It follows from the chain rule that

\[
I(X, Y) = H(X) - H(X|Y) = H(Y) - H(Y|X) .
\]  
(16)

So, the mutual information shows a reduction in the uncertainty of one random variable due to knowledge of other \cite{38}. In the context of quantum cryptography, the notion (15) was first applied in \cite{13}. In the following, we will
discuss an extension of the above concepts to entropic functions of the Rényi type. Such extensions typically lose
some of the standard information-theoretic properties. For a common discussion of distinguishability measures in
quantum information science, see [40, 41] and references therein. Entropic functions in application to quantum theory
are reviewed in [42].

Rényi entropies form an important family of one-parametric extensions of the Shannon entropy (11). For \(0 < \alpha \neq 1\),
the Rényi \(\alpha\)-entropy is defined as [43]

\[ R_\alpha(X) := \frac{1}{1 - \alpha} \log \left( \sum_x p(x)^\alpha \right). \]  

(17)

It does not increase with growth of \(\alpha\) [16]. If \(p(x) = 1/N\) for all \(x\), then the entropy (17) reaches its maximal value
equal to \(\log N\). In the limit \(\alpha \to 1\), the right-hand side of (17) reduces to (11). The joint \(\alpha\)-entropy \(R_\alpha(X,Y)\) is
defined according to (17) by substituting the joint probabilities. Special choices of the order \(\alpha\) are widely used in
the literature. In the limit \(\alpha \to \infty\), we obtain the min-entropy

\[ R_\infty(X) = -\log(\max p(x)). \]  

(18)

Taking \(\alpha = 2\) gives the so-called collision entropy, namely

\[ R_2(X) = -\log \left( \sum_x p(x)^2 \right). \]  

(19)

For \(\alpha \in [0; 1]\), the Rényi entropy is concave independently of the effective dimensionality [14]. Convexity properties
of \(R_\alpha(X)\) with orders \(\alpha > 1\) actually depend on dimensionality of probabilistic vectors [42, 43]. It is known that the
binary Rényi entropy is concave for \(\alpha \in [0; 2]\) [44].

To quantify mutual information via functions of the Rényi type, the corresponding conditional entropies are needed.
There is no generally accepted definition of conditional Rényi’s entropy [11]. We will use the quantity [46, 47]

\[ R_\alpha^{(1)}(X|Y) := \sum_y p(y) R_\alpha(X|y), \]  

(20)

where

\[ R_\alpha(X|y) := \frac{1}{1 - \alpha} \log \left( \sum_x p(x|y)^\alpha \right). \]  

(21)

Basic properties of (20) are considered in [46, 47]. More special properties of the above conditional entropy were
addressed in [48, 49]. The conditional entropy (20) has been used in formulating the uncertainty principle in successive
measurements [50, 51]. This scenario is very close to that is dealt with in quantum key distribution with eavesdropping.

It turned out that the conditional entropy (20) does not share the chain rule. Instead of (20), other definitions of
conditional Rényi’s entropy were proposed [17]. The authors of [52] proposed the conditional entropy written as

\[ R_\alpha^{(2)}(X|Y) := R_\alpha(X,Y) - R_\alpha(Y). \]  

(22)

Here, the chain rule is posed by definition. However, the right-hand side of (22) cannot be reduced to some form that
is similar to (12) or (20). The authors of [17] reconsidered the notion of conditional Rényi’s entropy including five
known suggestions. The first and second of them are respectively defined as (20) and (22). The forth definition from
the list of [17] is of interest due to its fine properties. For \(0 < \alpha \neq 1\), it is expressed as

\[ R_\alpha^{(4)}(X|Y) := \frac{1}{1 - \alpha} \log \left( \sum_y p(y) \sum_x p(x|y)^\alpha \right). \]  

(23)

The third and fifth definitions from the list of [17] are inconsistent with the standard conditional entropy (12). Further,
we do not consider these definitions.

The notion (20) further leads to the mutual-information quantifier that has found use in studying individual
attacks on quantum cryptographic systems [15, 21–23, 53]. By an analogy with the formula (10), we can define the
\(\alpha\)-information

\[ I_\alpha^{(1)}(X,Y) := R_\alpha(X) - R_\alpha^{(1)}(X|Y). \]  

(24)

It is often interpreted as the Rényi \(\alpha\)-measure of mutual information. Taking \(\alpha = 2\), this quantity has been served
as a measure of probe performance in individual attacks on quantum-cryptographic schemes [15, 21–23, 53]. Security
against collective attacks is mainly examined with using the Holevo information [1]. When \(\alpha \neq 1\), the quantity (24)
differs from (16) in some important respects. First, the function (24) is not generally symmetric in its entries. Second, the parameter \( \alpha \) runs a continuum of values, whence we should make a proper choice of its value. Advantages of an approach with generalized entropic functions include a possibility to vary the used parameter (17). From the other point of view, the conditional \( \alpha \)-entropy (20) does not share the chain rule. So, the right-hand side of (24) is something similar to (16) but is not to similar (15). One fails with interpreting (24) as a reduction in the uncertainty of one random variable due to knowledge of other. For completeness of the presentation, we will also consider the quantities

\[
I_{\alpha}^{(2)}(X,Y) := R_\alpha(X) - R_\alpha^{(2)}(X|Y) = R_\alpha(X) + R_\alpha(Y) - R_\alpha(X,Y),
\]

\[
I_{\alpha}^{(4)}(X,Y) := R_\alpha(X) - R_\alpha^{(4)}(X|Y).
\]

Thus, we actually rewrite the right-hand side of (16) with the corresponding Rényi entropies. The quantity (25) is obviously symmetric in entries. In both the cases, the limit \( \alpha \to 1 \) leads to the standard mutual information (16).

One must be very careful in interpretation of (24) and (25), especially if only one value \( \alpha \neq 1 \) is involved. In the following, we will use already accepted term “Rényi’s mutual information”. However, these quantities are not fully legitimate measures of mutual information. This fact was shown from the viewpoint of analyzing a performance of probes in individual attacks on the BB84 protocol [18]. It was assumed in [18] that Eve can utilize only the two schemes of state discrimination. The first of them is due to Helstrom [29, 30], and the second one is commonly referred to as unambiguous discrimination independently developed by in [31–33]. Focusing on the conclusive eavesdropping allows us to reveal a noticeable inadequacy of (24) served as the measure of mutual information. It is known that more general scenarios of quantum state discrimination are available to Eve.

IV. GENERALIZED DISCRIMINATION AND SOME UNCERTAINTY RELATIONS

To learn an Alice’s qubit, Eve have to distinguish between \(|t_+\rangle\) and \(|t_-\rangle\). Recall that these states are symmetric with respect the basis \( \{|+,|-\rangle\} \). Further, we will refer just to this basis. Using the parameter \( \theta \), we consider the states \(|\theta_+\rangle\) and \(|\theta_-\rangle\) expressed as

\[
|\theta_+\rangle = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \quad |\theta_-\rangle = \begin{pmatrix} \cos \theta \\ -\sin \theta \end{pmatrix}.
\]

(27)

Their inner product is equal to \(|\theta_+\rangle|\theta_-\rangle = \cos 2\theta\). Assuming \( \theta \in (0; \pi/4) \), we focus on non-identical and non-orthogonal states. Due to the results of Davies [54], one can restrict a consideration to rank-one measurement operators. We will express them in terms of three unit kets, namely

\[
|\gamma_+\rangle = \begin{pmatrix} \sin \gamma \\ \cos \gamma \end{pmatrix}, \quad |\gamma_-\rangle = \begin{pmatrix} \sin \gamma \\ -\cos \gamma \end{pmatrix}, \quad |\gamma\rangle = |\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix},
\]

(28)

where \( \gamma \) is some angle. The latter is chosen so that \(|\gamma_+\rangle\) is closer to \(|\theta_+\rangle\) and \(|\gamma_-\rangle\) is closer to \(|\theta_-\rangle\). The measurements operators are proportional to the operators \(|\gamma_\pm\rangle\langle\gamma_\pm|\) and \(|\gamma\rangle\langle\gamma|\). To provide the completeness relation,

\[
M_+ + M_- + M_\gamma = 1,
\]

(29)

coefficients should obey certain conditions. Here, we present only the final expressions,

\[
M_+ = \frac{1}{1 + \cos 2\gamma} |\gamma_+\rangle\langle\gamma_+|, \quad M_- = \frac{2\cos 2\gamma}{1 + \cos 2\gamma} |\gamma_-\rangle\langle\gamma_-|.
\]

(30)

For each of the two inputs \(|\theta_+\rangle\) and \(|\theta_-\rangle\), one may give three outcomes, namely the success answer, the erroneous one and the inconclusive one. If the inputs have equal prior probabilities, then the average probabilities of the corresponding outcomes are written as

\[
Q_s = \frac{\sin^2(2\theta + \phi)}{1 + \cos 2\gamma}, \quad Q_e = \frac{\sin^2 \phi}{1 + \cos 2\gamma}, \quad Q_? = \frac{2\cos 2\gamma \cos^2 \theta}{1 + \cos 2\gamma}.
\]

(31)

Here, we introduce auxiliary angle \( \phi \) such that \( 0 \leq \phi \leq \pi/4 - \theta \) and \( \gamma = \theta + \phi \). It should be pointed out that the above expressions saturate the lower bound (35)

\[
Q_e \geq \frac{1}{2} \left( 1 - Q_? - \sin 2\theta \sqrt{1 - \frac{Q_?}{\cos^2 \theta}} \right).
\]

(32)
Note also that the measurement operators \( \mathbf{30} \) are optimal with respect to the symmetric case of equiprobable inputs. For more general cases, see the papers \( \mathbf{35} \) \( \mathbf{37} \). We ask for a performance the FPB probe, when Eve uses the described intermediate scheme of state discrimination.

In the following, we will refer to the particular cases of the above formulas. In the Helstrom scheme, we take \( \phi = \pi/4 - \theta \), whence \( Q_\theta = 0 \) and

\[
Q_S = \frac{1 + \sin 2\theta}{2} = \frac{1 + \sqrt{1 - (\theta_+ \theta_-)^2}}{2},
\]

\[
Q_E = \frac{1 - \sin 2\theta}{2} = \frac{1 - \sqrt{1 - (\theta_+ \theta_-)^2}}{2}.
\]

The IDP scheme is obtained for \( \phi = 0 \). In this case, we have \( Q_\theta = \cos 2\theta = \langle \theta_+ | \theta_- \rangle \), \( Q_S = 1 - Q_\theta \) and \( Q_E = 0 \). If we restrict a consideration to the error-free sifted bits, then the term \( Q_\theta \) gives relative frequency of the inconclusive answer in Eve’s results. Conclusive eavesdropping was originally considered by Brandt \( \mathbf{53} \).

Thus, POVM measurements of the form \( \mathbf{30} \) can be used for generalized discrimination including both the Helstrom and IDP schemes as least cases. The IDP scheme of unambiguous discrimination of the states \( |\theta_\pm\rangle \) is given by substituting \( \gamma = \theta \). In the following, we will use some entropic uncertainty bounds for measurements of the form \( \mathbf{30} \).

In the case of finite-dimensional systems, entropic uncertainty relations were developed due to Deutsch \( \mathbf{52} \) and Maassen and Uffink \( \mathbf{54} \). Many important results obtained in this way are reviewed in \( \mathbf{57} \) \( \mathbf{58} \). Let us consider orthonormal bases \( G = \{|g_i\}\) and \( G' = \{|g'_i\}\) in \( d \) dimensions. If the pre-measurement state is described by \( d \times d \) matrix \( \rho \), then the generated probabilities are equal to \( \langle g_i | \rho | g_i \rangle \) and \( \langle g'_i | \rho | g'_i \rangle \), respectively. Substituting these probabilities into (17), we have the corresponding entropies \( R_\alpha (G; \rho) \) and \( R_\beta (G'; \rho) \). The Maassen–Uffink uncertainty relation is posed as follows. One should take the unitary matrix with entries \( w_{ij} = \langle g_i | g'_j \rangle \). For \( 1/\alpha + 1/\beta = 2 \), we have

\[
R_\alpha (G; \rho) + R_\beta (G'; \rho) \geq -2 \ln s_{\max},
\]

where

\[
s_{\max} := \max_{ij} |w_{ij}|.
\]

Maassen and Uffink presented the relation with Shannon entropies, viz. \( \mathbf{56} \)

\[
H(G; \rho) + H(G'; \rho) \geq -2 \ln s_{\max},
\]

but their method is actually sufficient for \( \mathbf{30} \). It must be stressed that the result \( \mathbf{57} \) was initially conjectured by Kraus \( \mathbf{59} \). The author of \( \mathbf{60} \) mentioned how the above relations can be applied to a single POVM. It follows from the fact that the given POVM measurement can be realized as a projective one with different Naimark extensions. Adding third component to kets \( \{|\gamma_\pm\rangle, |\gamma\rangle\} \), we can build an orthonormal basis with the unitary Gramian matrix of size 3. Vectors of this basis are expressed as

\[
|\gamma_\pm(\varphi)\rangle = \frac{1}{\sqrt{1 + \eta}} \left( \sin \gamma \pm \cos \gamma \sqrt{\eta} e^{i \varphi} \right),
\]

\[
|\gamma(\varphi)\rangle = \frac{1}{\sqrt{1 + \eta}} \left( \sqrt{2 \eta} \sin \frac{\varphi}{2} \right),
\]

where \( \eta = \cos 2\gamma \). The phase factor \( e^{i \varphi} \) shows a unitary freedom in the ancillary one-dimensional space. Varying \( \varphi \), we obtain different extensions of the original POVM. We now apply \( \mathbf{35} \) and \( \mathbf{37} \) to the considered POVM by inspecting matrix elements of the form \( \langle \gamma_\gamma(\varphi) | \gamma_\gamma(\varphi') \rangle \), so that

\[
s_{\max}(\varphi, \varphi') := \max_{ij} |\langle \gamma_\gamma(\varphi) | \gamma_\gamma(\varphi') \rangle|.
\]

As was noted in \( \mathbf{61} \), we can further optimize the lower bound \(-2 \ln s_{\max}(\varphi, \varphi')\) with respect to freely variable parameters. Calculations are somewhat long, though simple in matter. We refrain from presenting the details here. The uncertainty relation for the POVM \( \mathcal{M} = \{ M_+, M_-, M_\varphi \} \) reads as

\[
R_\alpha (\mathcal{M}; \rho) + R_\beta (\mathcal{M}; \rho) \geq 2 \ln f(\eta),
\]

where we put the function

\[
f(\eta) := \begin{cases} 
\frac{1 + \eta}{1 - \eta}, & \text{if } \eta \in [0; 0.2], \\
\sqrt{\frac{2 - \eta}{1 - \eta}}, & \text{if } \eta \in [0.2; 0.5], \\
\sqrt{\frac{2 - \eta}{\eta}}, & \text{if } \eta \in [0.5; 1].
\end{cases}
\]
The authors of [61] proposed an improvement of (57). In our notation, their result is expressed as

$$H(G; \rho) + H(G'; \rho) \geq -2 \ln s_2 + (1 - s_2) \ln \left(\frac{s_2}{s_2^2}\right),$$  \hspace{1cm} (42)

where $s_2$ is the second largest value among moduli $|w_{ij}|$ of entries $w_{ij} = \langle g_i | g_j' \rangle$. Applying (42) to the considered POVM, we have

$$H(M; \rho) \geq \ln f(\eta) + \eta \frac{U(0.2 - \eta)}{2(1 + \eta)} \ln \left(\frac{1 - \eta}{4\eta}\right).$$  \hspace{1cm} (43)

By $U(x)$, we denote here the unit step function, which is 0 for $x < 0$ and 1 for $x > 0$. In the right-hand side of (43), the correction term is non-zero only for $\eta < 0.2$. It follows from the fact that, for $\eta > 0.2$, optimizing the term $-2 \ln s_{\max}(\varphi, \varphi')$ results in $s_{\max} = s_2$. The lower bound (43) is simply expressed in terms of the parameter $\eta = \cos 2\gamma$. It will be used for estimating the standard mutual information of Eve from above.

V. MEASURES OF MUTUAL INFORMATION IN THE SCENARIO WITH GENERALIZED DISCRIMINATION

In this section, we will consider quantities of the form (24) as quantifiers of a performance of the FPB quantum-cryptographic probe. In the question considered, we focus on the error-free sifted bits shared by Alice and Bob [22, 23]. By the prime sign, we will mean events related to the error-free sifted bits. Using the formulas for probabilities of different outcomes, we should remember that the states $|t_{\pm}\rangle$ are sub-normalized, since $\langle t_+ | t_+ \rangle = \langle t_- | t_- \rangle = 1 - P_E$. Then the angle $\theta \in [0; \pi/4]$ is defined as such that

$$\cos 2\theta = \frac{1 - 3P_E}{1 - P_E}, \quad \sin 2\theta = \frac{\sqrt{4P_E(1 - 2P_E)}}{1 - P_E}. \hspace{1cm} (44)$$

Thus, $\theta$ is a function of the error probability $P_E \in [0; 1/3]$. The following fact should be emphasized. We will focus on the case, when the legitimate users set the values 0 and 1 to be equally likely. Hence, the binary variable $B'$ has the uniform distribution, so that $R_{\alpha}(B') = \log 2 = 1$ irrespectively to $\alpha$. For $\alpha > \beta$, we then obtain

$$I_{\alpha}^{(1)}(B', E') \geq I_{\beta}^{(1)}(B', E'). \hspace{1cm} (45)$$

It follows from the fact that $R_{\alpha}(B'|E') \leq R_{\beta}(B'|E')$ for $\alpha > \beta$.

Eve can use the generalized scheme to discriminate between $|t_+\rangle$ and $|t_-\rangle$. In this scenario, we have the two parameters $P_E$ and $\phi$ describing Eve’s activity. For the given $P_E \in [0; 1/3]$, the angle $\phi$ ranges between 0 and $\pi/4 - \theta$. For $j = 0, 1$, we now write the following conditional probabilities,

$$p(e' = j | b' = j) = Q_S, \hspace{1cm} (46)$$
$$p(e' = 0 | b' = j) = p(e' = 1 | b' = 0) = Q_E, \hspace{1cm} (47)$$
$$p(e' = ? | b' = j) = Q_?, \hspace{1cm} (48)$$

where the values of $Q_S$, $Q_E$, and $Q_?$ are given by (31). Multiplying the above expressions by 1/2, we obtain the corresponding joint probabilities. Hence, we also have

$$p(e' = j) = \frac{1 - Q_?}{2}, \quad p(e' = ?) = Q_. \hspace{1cm} (49)$$

Then the Shannon entropy $H(E') = 1 - Q_? + h(Q_?)$, where $h(Q_?)$ is the binary entropy. The final expressions for the conditional probabilities $p(b'|e')$ appear as

$$p(b' = j | e' = j) = \frac{Q_S}{1 - Q_?}, \hspace{1cm} (50)$$
$$p(b' = 0 | e' = 1) = p(b' = 1 | e' = 0) = \frac{Q_E}{1 - Q_?}, \hspace{1cm} (51)$$
$$p(b' = j | e' = ?) = \frac{1}{2}. \hspace{1cm} (52)$$
Due to (52), one gets \( R_\alpha(B'|e' = ?) = \log 2 = 1 \). For \( R_\alpha(B'|e' = j) \) with \( j = 0, 1 \), we obtain more complicated formula. The resulting expression for the first-type \( \alpha \)-measure of information reads as

\[
I^{(1)}_\alpha(B', E') = (1 - Q_\xi) \left[ 1 - \frac{\log(Q_S + Q_E)}{1 - \alpha} + \frac{\alpha \log(1 - Q_\xi)}{1 - \alpha} \right].
\]  

(53)

In the particular cases, we have the following expressions. The standard mutual information is written as

\[
I(B', E') = 1 - Q_\phi - (1 - Q_\phi) \log(1 - Q_\phi) + Q_S \log Q_S + Q_E \log Q_E.
\]  

(54)

Further, for \( \alpha = \infty \) we obtain

\[
I^{(1)}_\infty(B', E') = 1 - Q_\xi - (1 - Q_\xi) \log(1 - Q_\xi) + (1 - Q_\xi) \log Q_S.
\]  

(55)

This also follows from (53) by substituting \( \alpha = \infty \). We prefer to write the above expressions explicitly due to the fact that the quantity \( \xi \) have found wide use in considering individual attacks.

Let us examine the Rényi \( \alpha \)-measures of mutual information as characteristics of a performance of the FBP probe. Such measures should be compared with the standard mutual information. Among these measures, we will focus on the measures \( I^{(1)}_\alpha(B', E') \), \( I^{(2)}_\alpha(B', E') \), and \( I^{(4)}_\alpha(B', E') \). Following [15], many authors have used \( I^{(1)}_2(B', E') \) as the basis for analyzing a performance of entangling probes. We will also use \( I^{(1)}_2(B', E') \), since it gives the upper bound in the sense of (54). This approach was realized in [18], but only with the two schemes of state discrimination. It is useful to compare different information measures in the context of generalized scenarios of discrimination on Eve’s side. In particular, we will more brightly reveal an inadequacy of \( \alpha \)-measures of mutual information in the considered context. To do so, we choose several intermediate values of \( \phi \) that discretize the allowed interval \( 0 \leq \phi \leq \pi/4 - \theta \). For definiteness, we will use the characteristic ratio

\[
\xi_\phi := \frac{\phi}{\pi/4 - \theta}.
\]  

(56)

It ranges between the values 0 and 1 that respectively correspond to the unambiguous discrimination and the Helstrom scheme. The parameter \( \xi_\phi \) also characterizes the amount of erroneous outcomes in Eve’s results. Together with the values 0 and 1, we visualize measures of mutual information for intermediate ones, namely \( \xi_\phi = 0.25, 0.50, 0.75 \).

It was already reasoned in [18] that only the standard mutual information gives a reasonable ground for studies of a performance of the FBP probe. In the right plot of Fig. 1, we show \( I(B', E') \) as a function of \( P_E \) for several values of \( \xi_\phi \). Independently of \( P_E \), the value of mutual information is maximal for \( \xi_\phi = 1 \), when Eve uses the Helstrom scheme. If Eve allows some fraction of inconclusive answers, then her mutual information is slightly reduced. We see in Fig. 1 that for relatively small values of \( P_E \) corresponding curves come very closely to the curve \( \xi_\phi = 1 \). Even in the case of conclusive eavesdropping, Eve’s mutual information is quite comparable with this top curve. The latter conclusion was already emphasized in [18]. The maximal difference between the two curves for \( \xi_\phi = 1 \) and \( \xi_\phi = 0 \) takes place for \( P_E \approx 0.227 \). Note also that the relative difference between these curves is observed for negligible values of \( P_E \). We have discussed the picture based on the standard mutual information. It will be used as a benchmark for testing quantities (21) - (26) as characteristics of Eve’s mutual information.

Using the uncertainty relation (13), we can estimate from above the standard mutual information \( I(B', E') \). Although this estimate is not very tight, it will describe a range of possible variations of information measures. Applying (15) to \( H(E'|b') \) for every \( b' \), we finally get

\[
I(B', E') \leq H(E') - \ln f(\eta) - \frac{\eta U(0.2 - \eta)}{2(1 + \eta)} \ln \left( \frac{1 - \eta}{4\eta} \right).
\]  

(57)

Here, the parameter \( \eta \) characterizes Eve’s discrimination scheme. In Fig. 2, we present the mutual information \( I(B', E') \) and the right-hand side of (57) for the two discrimination schemes on Eve’s side. For convenience of comparison with other figures, the ordinate values are restricted to be less than 1. Hence, a little element of the upper bound is not shown explicitly.

Let us proceed to information measures of the form \( I^{(1)}_\alpha(B', E') \). As was mentioned above, Rényi’s mutual information of order \( \alpha = 2 \) has found use in studying a performance of Eve’s entangling probes. In the left plot of Fig. 3, we picture \( I^{(1)}_2(B', E') \) as a function of \( P_E \) for the fifth values of \( \xi_\phi \). First of all, the curve for \( \xi_\phi = 0 \) goes just along its track in Fig. 3. Indeed, for conclusive eavesdropping we have \( I^{(1)}_2(B', E') = I(B', E') \). This principal observation made in [18] has lead to the conclusion that the so-called Rényi mutual information is not a completely legitimate measure. Using the curves drawn in Fig. 3, we further support the mentioned conclusion. We see in Fig. 4 that the
The conditional probability 
\[ p(b' = j | e' = j) \]
and the standard mutual information 
\[ I(B', E') \]
versus \( P_E \) for the five discrimination schemes on Eve’s side. Note that unambiguous discrimination, i.e., \( \xi_\phi = 0 \), implies \( p(b' = j | e' = j) = 1 \).

It is instructive to inspect the situation for other values of \( \alpha \). In the right plot of Fig. 3 we picture \( I^{(1)}_\infty(B', E') \) as a function of \( P_E \) for the fifth values of \( \xi_\phi \). Curves for the value \( \alpha = \infty \) are very demonstrative due to the property (45). We observe very essential buckling of the curves with a little fraction of inconclusive answers. If our approach is based on the measure (55), then we should accept an essential weakness of scenarios with a large fraction of inconclusive answers. So, we could underestimate the degree of vulnerability with respect to conclusive eavesdropping and close scenarios. As is seen in the plots of Fig. 3 wrong pictures are especially noticeable for relatively small values of \( P_E \). Dealing with communication security, we cannot be guided by false proposals. At least in the case of individual attacks, \( \alpha \)-functions of mutual information of the Rényi type are not completely adequate measures.
Thus, we have additionally confirmed that the standard mutual information is the only reliable base for estimating quantum-cryptographic probes.

The mentioned distinction is most essential for “weak” eavesdropping, when Eve attempts to get a small amount of information while causing only a slight disturbance. It is sufficiently typical that an opposite party try to cloak its activity against legitimate users. If we abandon the standard measure, then a quality of the usual FPB probe in comparison with conclusive ones may be illusory up to several times. This example illustrates a conclusion that security requirement could be overstated spuriously on the base of inadequate measures. Such wrong conclusions about too strong probe performance may lead to rejecting some feasible realizations. They could seemingly be evaluated as very vulnerable, even though they are suitable in other respects. As was emphasized in section VI.L of [2], the infinite security will demand the infinite cost. In practice, characteristics of a quantum-cryptographic system are actually determined by some compromise between several conflicting requirements. Hence, we wish to avoid wrong conclusions of any kind.

Let us consider briefly the quantity (25). It can be treated as a candidate to quantify Eve’s mutual information about the error-free sifted bits. In comparison with (24), the quantity (25) is symmetric. If $\alpha$ is very close to 1, then
the picture is quite similar to that is shown in Fig. 1. It is not amazing since the right-hand side of (25) tries to reproduce the right-hand side of (15). When $\alpha$ deviates from 1, the corresponding curves are changed essentially. To illustrate this fact, we present the cases $\alpha = 2$ and $\alpha = 10$ in Fig. 4. The resulting curves reveal a behavior quite different from the curves shown in Figs. 1 and 4. For relatively small values of $\xi = 1$, we see an $s$-like shape. That is, for schemes with sufficient number of inconclusive answers the measure (25) is even less suitable than (24). We also note the following. In the right plot of Fig. 5, the lines for $\xi = 1$ and $\xi = 0.75$ are clearly intersected with an intermediate value $P_E \approx 0.108$. No such intersection takes place in Figs. 1 and 3. For completeness, we also touch (26) related to more sophisticated version (23) of conditional Rényi’s entropy. The cases $\alpha = 2$ and $\alpha = 10$ are shown in Fig. 5. Although the left plot of Fig. 5 is similar to the left plot of Fig. 3, the right one is very different. For $\alpha = 10$, the lines for various values of $\xi$ intersect many times. These observations witness that the $\alpha$-measures of information are adequate only when $\alpha$ is very close to 1. However, the latter inevitably implies the use of the standard measure $I(B', E')$.

Several final remarks are related to a possible usage of generalized information functions. The above discussion does not aim to criticize extended information-theoretic functions in general. In effect, generalized entropies were fruitfully used in many important questions. In future, such functions may found novel interesting applications. We only wish to emphasize the necessity of a certain circuspection with their handling in new areas. It is known that generalized entropic functions do not succeed all the properties of the standard functions. For instance, the definition (24) is based on the conditional entropic form, which does not share the chain rule. The authors of (52) have treated the chain rule just as the definition of conditional Rényi’s entropy. However, we have above seen an inadequacy of several $\alpha$-measures of mutual information. Other forms of conditional Rényi’s entropy, even more sophisticated, will hardly be able to improve such things. Dealing with generalized information functions, the following extreme viewpoints should be avoided. The first of them disregards any study with generalized entropies. The second one widely uses such entropies without a critical evaluation of the conclusions obtained. It would be more attractive to use a balanced approach, within which generalized information functions are allowed in the presence of sufficient ground.

VI. CONCLUSIONS

We have examined scenarios with generalized state discrimination after the action of the FPB probe. From this viewpoint, we further studied different quantifiers of an amount of Eve’s mutual information about the error-free sifted bits. The so-called Rényi information of order $\alpha = 2$ was widely adopted for estimation of a performance of quantum-cryptographic probes. It turned out that this quantity is not a completely appropriate measure. At least in the context of individual attacks on the BB84 scheme, wrong conclusions are inspired by means of the first-type 2-information and related measures. We have also examined another families of information measures inspired by conditional Rényi entropies. In our opinion, measures of the second and forth types seem to be even worse. They are quite inadequate from the viewpoint of considered questions. The presented results gave an evidence that a performance of quantum-
cryptographic entangling probes should be evaluated on the base of the standard information functions. In this sense, using generalized discrimination hardly adds anything essentially new to a probe performance.

On the other hand, the standard information functions may also be less suitable in some circumstances. In quantum information theory, we have come across numerous questions of different kind. The authors of the papers addressed the problem of properly quantifying informational content of an unknown quantum state. It is natural to adopt mutually unbiased measurements for such purposes. As was shown in, the sum of corresponding Shannon entropies has several counter-intuitive properties. More appropriate measure has been proposed and motivated in. This approach is immediately connected with generalized entropies of order 2. So, we see that the standard information functions should sometimes be replaced with other functions. A priori, no quantifier can be recognized as the only justified one. Rather, we shall apply several approaches and compare the conclusions obtained. Among existing questions, using the so-called fake states to detect an opposite activity has received less attention than it deserves. In any case, a validity of information measures in application to quantum cryptography should be further studied.

[1] Lomonaco, S.J.: A quick glance at quantum cryptography. Cryptologia 23, 1–41 (1999)
[2] Gisin, N., Ribordy, G., Tittel, W., Zbinden, H.: Quantum cryptography. Rev. Mod. Phys. 74, 145–195 (2002)
[3] van Assche, G.: Quantum Cryptography and Secret-Key Distillation. Cambridge University Press, Cambridge (2006)
[4] Scarani, V., Bechmann-Pasquinucci, H., Cerf, N.J., Dušek, M., Lütkenhaus, N., Peev, M.: The security of practical quantum key distribution. Rev. Mod. Phys. 81, 1301–1350 (2009)
[5] Nielsen, M.A., Chuang, I.L.: Quantum Computation and Quantum Information. Cambridge University Press, Cambridge (2000)
[6] Galindo, A., Martín-Delgado, M.A.: Information and computation: classical and quantum aspects. Rev. Mod. Phys. 74, 347–423 (2002)
[7] Barnett, S.M.: Quantum Information. Oxford University Press, Oxford (2009)
[8] Shor, P.W.: Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer. SIAM J. Comput. 26, 1484–1509 (1997)
[9] Childs, A.M., van Dam, W.: Quantum algorithms for algebraic problems. Rev. Mod. Phys. 82, 1–52 (2010)
[10] Childs, A.M., Ivanyos, G.: Quantum computation of discrete logarithms in semigroups. J. Math. Cryptol. 8, 405–416 (2014)
[11] Bennett, C.H., Brassard, G.: Quantum cryptography: public key distribution and coin tossing. In: Proceedings of IEEE International Conference on Computers, Systems, and Signal Processing, pp. 175–179. IEEE, New York (1984)
[12] Bennett, C.H.: Quantum cryptography using any two nonorthogonal states. Phys. Rev. Lett. 68, 3121–3124 (1992).
[13] Ekert, A.K., Huttner, B., Palma, G.M., Peres, A.: Eavesdropping on quantum-cryptographical systems. Phys. Rev. A 50, 1047–1056 (1994)
[14] Csiszár, I., Körner, J.: Broadcast channels with confidential messages. IEEE Trans. Inf. Theory 24, 339–348 (1978)
[15] Slutsy, B.A., Rao, R., Sun, P.-C., Fainman, Y.: Security of quantum cryptography against individual attacks. Phys. Rev. A 57, 2383–2398 (1998)
[16] Teixeira, A., Matos, A., Antunes, L.: Conditional Rényi entropies. IEEE Trans. Inf. Theory 58 (2012), 4273–4277.
[17] Fehr, S., Berens, S.: On the conditional Rényi entropy. IEEE Trans. Inf. Theory 60 (2014), 6801–6810.
[18] Rastegin, A.E.: On conclusive eavesdropping and measures of mutual information in quantum key distribution. Quantum Inf. Process. 15 (2016), 1225–1239.
[19] Fuchs, C.A., Peres, A.: Quantum state disturbance versus information gain: uncertainty relations for quantum information. Phys. Rev. A 53, 2038–2045 (1996)
[20] Brandt, H.E.: Quantum-cryptographic entangling probe. Phys. Rev. A 71, 042312 (2005)
[21] Brandt, H.E.: Optimum probe parameters for entangling probe in quantum key distribution. Quantum Inf. Process. 2, 37–79 (2003)
[22] Shapiro, J.H.: Performance analysis for Brandt’s conclusive entangling probe. Quantum Inf. Process. 5, 11–24 (2006)
[23] Shapiro, J.H., Wong, F.N.C.: Attacking quantum key distribution with single-photon two-qubit quantum logic. Phys. Rev. A 73, 012315 (2006)
[24] Herberts, I.M., Bettelli, S., Hübel, H., Peev, M.: On the optimality of individual entangling-probe attacks against BB84 quantum key distribution. Eur. Phys. J. D 46, 395–406 (2008)
[25] Wals, E., Takesue, H., Yamamoto, Y.: Security of differential-phase-shift quantum key distribution against individual attacks. Phys. Rev. A 73, 012344 (2006)
[26] Vasiliev, E.V.: Non-coherent attack on the ping-pong protocol with completely entangled pairs of qutrits. Quantum Inf. Process. 10, 189–202 (2011)
[27] Bartkiewicz, K., Černoch, A., Lemr, K., Miranowicz, A., Nori, F.: Temporal steering and security of quantum key distribution with mutually unbiased bases against individual attacks. Phys. Rev. A 93, 062345 (2016)
[28] Biham, E., Boyer, M., Brassard, G., van de Graaf, J., Mor, T.: Security of quantum key distribution against all collective attacks. Algorithmica 34, 372–388 (2002)
[29] Helstrom, C.W.: Detection theory and quantum mechanics. Inform. Control 10, 254–291 (1967)
[30] Helstrom, C.W.: Quantum Detection and Estimation Theory. Academic Press, New York (1976)
[31] Ivanovic, I.D.: How to differentiate between non-orthogonal states. Phys. Lett. A 123, 257–259 (1987)
[32] Dieks, D.: Overlap and distinguishability of quantum states. Phys. Lett. A 126, 303–306 (1988)
[33] Peres, A.: How to differentiate between non-orthogonal states. Phys. Lett. A 128, 19 (1988)
[34] Huttner, B., Muller, A., Gautier, J.D., Zbinden, H., Gisin, N.: Unambiguous quantum measurement of nonorthogonal states. Phys. Rev. A 54, 3783–3789 (1996)
[35] Chefles, A., Barnett, S.M.: Strategies for discriminating between non-orthogonal quantum states. J. Mod. Opt. 45, 1295–1302 (1998)
[36] Zhang, C.-W., Li, C.-F., Guo, G.-C.: General strategies for discrimination of quantum states. Phys. Lett. A 261, 25–29 (1999)
[37] Fiurášek, J., Ježek, M.: Optimal discrimination of mixed quantum states involving inconclusive results. Phys. Rev. A 67, 012321 (2003)
[38] Cover, T.M., Thomas, J.A.: Elements of Information Theory. John Wiley & Sons, New York (1991)
[39] Bennett, C.H., Gács, P., Li, M., Vitányi, P.M.D., Zurek, W.H.: Information distance. IEEE Trans. Inf. Theory 44, 1407–1423 (1998)
[40] Fuchs, C.A., van de Graaf, J.: Cryptographic distinguishability measures for quantum mechanical states. IEEE Trans. Inf. Theory 45, 1216–1227 (1999)
[41] Cachin, C.: Entropy Measures and Unconditional Security in Cryptography. Ph.D. thesis, Swiss Federal Institute of Technology, Zürich (1997)
[42] Bengtsson, I., Życzkowski, K.: Geometry of Quantum States: An Introduction to Quantum Entanglement. Cambridge University Press, Cambridge (2006)
[43] Rényi, A.: On measures of entropy and information. In: Neuman, J. (ed.) Proceedings of 4th Berkeley Symposium on Mathematical Statistics and Probability, vol. 1, pp. 547–561. University of California Press, Berkeley (1961)
[44] Jizba, P., Arimitsu, T.: The world according to Rényi: thermodynamics of multifractal systems. Ann. Phys. 312, 17–59 (2004)
[45] Ben-Bassat, M., Raviv, J.: Rényi’s entropy and error probability. IEEE Trans. Inf. Theory 24, 324–331 (1978)
[46] Cachin, C.: Entropy Measures and Unconditional Security in Cryptography. Ph.D. thesis, Swiss Federal Institute of Technology, Zürich (1997)
[47] Kamimura, R.: Minimizing α-information for generalization and interpretation. Algorithmica 22, 173–197 (1998)
[48] Rastegin, A.E.: Convexity inequalities for estimating generalized conditional entropies from below. Kybernetika 48, 242–253 (2012)
[49] Rastegin, A.E.: Further results on generalized conditional entropies. RAIRO-Theor. Inf. Appl. 49, 67–92 (2015)
[50] Zhang, J., Zhang, Y., Yu, C.-S.: Rényi entropy uncertainty relation for successive projective measurements. Quantum Inf. Process. 14, 2239–2253 (2015)
[51] Rastegin, A.E.: Entropic uncertainty relations for successive measurements of canonically conjugate observables. Ann. Phys. (Berlin) 528, 835–844 (2016)
[52] Golshani, L., Pasha, E., Yari, G.: Some properties of Rényi entropy and Rényi entropy rate. Inf. Sci. 179, 2426–2433 (2009)
[53] Brandt, H.E.: Unambiguous state discrimination in quantum key distribution. Quantum Inf. Process. 4, 387–398 (2005)
[54] Davies, E.B.: Information and quantum measurement. IEEE Trans. Inf. Theory 24, 596–199 (1978)
[55] Deutsch, D.: Uncertainty in quantum measurements. Phys. Rev. Lett. 50, 631–633 (1983)
[56] Maassen, H., Uffink, J.B.M.: Generalized entropic uncertainty relations. Phys. Rev. Lett. 60, 1103–1106 (1988)
[57] Wehner, S., Winter, A.: Entropic uncertainty relations – a survey. New J. Phys. 12, 025009 (2010)
[58] Coles, P.J.; Berta, M., Tomamichel, M., Wehner, S.: Entropic uncertainty relations and their applications. Rev. Mod. Phys. 89, 015002 (2017)
[59] Kraus, K.: Complementary observables and uncertainty relations. Phys. Rev. D 35, 3070–3075 (1987)
[60] Massar, S.: Uncertainty relations for positive-operator-valued measures. Phys. Rev. A 76, 042114 (2007)
[61] Coles, P.J., Piani, M.: Improved entropic uncertainty relations and information exclusion relations. Phys. Rev. A 89, 022112 (2014)
[62] Brukner, Č., Zeilinger, A.: Operationally invariant information in quantum measurements. Phys. Rev. Lett. 83, 3354–3357 (1999)
[63] Brukner, Č., Zeilinger, A.: Conceptual inadequacy of the Shannon information in quantum measurements. Phys. Rev. A 63 (2001), 022113.
[64] Rastegin, A.E.: On the Brukner–Zeilinger approach to information in quantum measurements. Proc. R. Soc. A 471 (2015), 20150435.