Norm attaining operators and pseudospectrum

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Abstract

It is shown that if $1 < p < \infty$ and $X$ is a subspace or a quotient of an $\ell_p$-direct sum of finite dimensional Banach spaces, then for any compact operator $T$ on $X$ such that $\|I + T\| > 1$, the operator $I + T$ attains its norm. A reflexive Banach space $X$ and a bounded rank one operator $T$ on $X$ are constructed such that $\|I + T\| > 1$ and $I + T$ does not attain its norm.

MSC:

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1 Introduction

All vector spaces in this paper are assumed to be over the field $\mathbb{F}$, being either the field $\mathbb{C}$ of complex numbers or the field $\mathbb{R}$ of real numbers. As usual, $\mathbb{N}$ is the set of positive integers and $\mathbb{R}_+$ is the set of non-negative real numbers. The Banach space of all bounded linear operators from a Banach space $X$ to a Banach space $Y$ is denoted by $L(X, Y)$ and $\mathcal{K}(X, Y)$ stands for the space of compact linear operators $T : X \to Y$. We write $L(X)$ instead of $L(X, X)$, $\mathcal{K}(X)$ instead of $\mathcal{K}(X, X)$ and $X^*$ instead of $L(X, \mathbb{K})$.

We say that $T \in L(X, Y)$ attains its norm on $x \in X$ if $\|x\| = 1$ and $\|Tx\| = \|T\|$. It is said that $T$ attains its norm if there is an $x \in X$ with $\|x\| = 1$ such that $T$ attains its norm on $x$. We would like to mention a few classical results on the norm attaining property. The James theorem [10] says that a Banach space $X$ is reflexive if and only if any $f \in X^*$ attains its norm. As a corollary of the James theorem, we have that $X$ is reflexive if and only if any $T \in \mathcal{K}(X)$ attains its norm. Indeed, if $X$ is non-reflexive, then the James theorem provides a bounded rank one operator which does not attain its norm. On the other hand, if $X$ is reflexive and $T \in \mathcal{K}(X)$, then the function $x \mapsto \|Tx\|$ is weakly sequentially continuous and the closed unit ball of $X$ is weakly sequentially compact and therefore the above function attains its maximum on the unit ball. Below it is shown that the situation with attaining of the norm for operators $I + T$ with compact $T$ is quite different. For further results on the norm attaining operators we refer to [1] [2] [3] [4] [5] [12] [14] [18] [19] and references therein.

The norm attaining property of operators is related to the concept of the pseudospectrum. Let $X$ be a complex Banach space. For $\varepsilon > 0$ and $T \in L(X)$, the $\varepsilon$-pseudospectrum of $T$ is usually defined as

$$\sigma_\varepsilon(T) = \{ \lambda \in \mathbb{C} : \|(T - \lambda I)^{-1}\| > \varepsilon^{-1} \}$$

or as

$$\Sigma_\varepsilon(T) = \{ \lambda \in \mathbb{C} : \|(T - \lambda I)^{-1}\| \geq \varepsilon^{-1} \},$$

where $\|(T - \lambda I)^{-1}\|$ is assumed to be infinite if $\lambda$ belongs to the spectrum $\sigma(T)$ of $T$, see, for instance, [7] [8] [9] [13] [16] [17] [21] [22]. Recently Shargorodsky [21] demonstrated that the level set

$$\Sigma_\varepsilon(T) \setminus \sigma_\varepsilon(T) = \{ \lambda \in \mathbb{C} : \|(T - \lambda I)^{-1}\| = \varepsilon^{-1} \}$$

can have non-empty interior in general, while its interior is empty when the space $X$ or the dual space $X^*$ is complex uniformly convex. It is well-known that

$$\sigma_\varepsilon(T) = \bigcup_{\|A\| < \varepsilon} \sigma(T + A),$$

see, for instance, [9] [11]. This equality is one of the main reasons why many authors prefer $\Sigma_\varepsilon$ rather than $\sigma_\varepsilon$ as the definition of pseudospectrum. We study the question whether the similar equality holds
in the case of non-strict inequalities:
\[ \Sigma_\varepsilon(T) = \Sigma_0^\varepsilon(T), \quad \text{where} \quad \Sigma_0^\varepsilon(T) = \bigcup_{\|A\| \leq \varepsilon} \sigma(T + A). \quad (1.5) \]

It is worth noting that the inclusion
\[ \Sigma_0^\varepsilon(T) \subseteq \Sigma_\varepsilon(T) \quad (1.6) \]
holds for any bounded operator \( T \) on any Banach space \([9, 13]\). It is proved by Finck and Ehrhardt, see [15], that the equality \( (1.5) \) holds if \( X \) is a Hilbert space. Shargorodsky [22] constructed a bounded linear operator \( T \) on the reflexive space \( X = \ell_p \times \ell_q \) with \( 1 < p < q < \infty \) and the norm \( \|(x, y)\| = \|x\|_p + \|y\|_q \) for which \( (1.5) \) fails. He also constructed \( T \in K(\ell_1) \) for which \( (1.5) \) fails. These examples naturally lead to the following question, raised in [22].

**Question 1.1.** Is it true that \( (1.5) \) holds for any compact operator on a reflexive complex Banach space?

We show that, in general, the answer to Question 1.1 is negative and demonstrate that if \( 1 < p < \infty \) and \( X \) is an \( \ell_p \)-direct sum of finite dimensional Banach spaces, then \( (1.5) \) holds for each bounded operator \( T \) on \( X \). In particular, it holds when \( X = \ell_p \) with \( 1 < p < \infty \). It turns out that the validity of \( (1.5) \) for any compact operator \( T \) on a Banach space \( X \) is closely related to the norm-attaining property.

**Proposition 1.2.** Let \( X \) be a complex Banach space, \( T \in K(X) \), \( \varepsilon > 0 \) and \( z \in \Sigma_\varepsilon(T) \). Then the following conditions are equivalent:

1. \( z \in \Sigma_0^\varepsilon(T) \);
2. if \( \|(T - zI)^{-1}\| = \varepsilon^{-1} > |z|^{-1} \), then \((T - zI)^{-1}\) attains its norm.

We use the above proposition in order to prove the following result.

**Proposition 1.3.** Let \( X \) be a complex Banach space. Then the following conditions are equivalent:

1. the equality \( (1.5) \) holds for any \( \varepsilon > 0 \) and any \( T \in K(X) \);
2. for any \( T \in K(X) \) such that \( I + T \) is invertible and \( \|I + T\| > 1 \), the operator \( I + T \) attains its norm.

The above proposition motivates the introduction of the following class of Banach spaces.

**Definition 1.** We say that a Banach space \( X \) belongs to the class \( W \) if for each \( T \in K(X) \) such that \( \|I + T\| > 1 \), \( I + T \) attains its norm.

From Proposition 1.3 it follows that for any \( X \in W \) and any compact operator \( T \) on \( X \), the equality \( (1.5) \) holds. It is also worth noting that the restriction \( \|I + T\| > 1 \) is natural. Indeed, the diagonal operator \( D \) on \( \ell_2 \) with the diagonal entries \( \{1 - 2^{-n}\}_{n \in \mathbb{N}} \) has norm 1 which is not attained and \( D \) is the sum of the identity operator and a compact operator. The following proposition provides a sufficient condition for a Banach space to belong to \( W \).

**Definition 2.** Let \( 1 < p < \infty \). We say that a Banach space \( X \) is a \( p \)-space if \( X \) is reflexive and for any \( x \in X \) and any weakly convergent to zero sequence \( \{u_n\}_{n \in \mathbb{N}} \) in \( X \),
\[ \lim_{n \to \infty} \left( \|x + u_n\| - \|(x)^p + \|u_n\|^p\right)^{1/p} = 0. \quad (1.7) \]

It is easy to see that any Hilbert space is a \( 2 \)-space and that any finite dimensional Banach space is a \( p \)-space for any \( p \). Note that an infinite dimensional Banach space cannot be a \( p \)-space and a \( q \)-space for \( p \neq q \). Recall that if \( 1 \leq p < \infty \) and \( \{X_\alpha\}_{\alpha \in \Lambda} \) is a family of Banach spaces, then their \( \ell_p \)-direct sum is the space
\[ X = \left\{ x \in \prod_{\alpha \in \Lambda} X_\alpha : \sum_{\alpha \in \Lambda} \|x_\alpha\|^p < \infty \right\} \]
endowed with the norm
\[ \|x\| = \left( \sum_{\alpha \in \Lambda} \|x_\alpha\|^p \right)^{1/p}. \]

When the family consists of just 2 spaces \( X \) and \( Y \) we denote its \( \ell_p \)-direct sum by \( X \oplus_p Y \). We also denote \( X \times Y \) with the norm \( \|(x, y)\| = \max\{\|x\|, \|y\|\} \) by the symbol \( X \oplus \infty Y \).
Proposition 1.4. Let \( 1 < p < \infty \). Then any closed linear subspace of an \( \ell_p \)-direct sum of any family of finite dimensional Banach spaces is a \( p \)-space.

The following two theorems provide, in particular, a partial affirmative answer to Question 1.1. The next one extends the validity of (1.5) for any bounded operator \( T \) from just Hilbert spaces to a wider class of Banach spaces.

Theorem 1.5. Let \( 1 < p < \infty \) and \( X \) be an \( \ell_p \)-direct sum of a family of complex finite dimensional Banach spaces. Then (1.5) holds for any \( T \in L(X) \).

In the case of compact operators, we can extend the last theorem.

Theorem 1.6. Let \( 1 < p \leq q < \infty \), \( X \) be a \( p \)-space and \( Y \) be a \( q \)-space. Then for any \( J \in L(X, Y) \) and \( T \in K(X, Y) \) such that \( \| J + T \| > \| J \| \), the operator \( J + T \) attains its norm. In particular, any \( p \)-space belongs to \( \mathcal{W} \).

Theorem 1.6 and Proposition 1.3 imply the following corollary.

Corollary 1.7. Let \( 1 < p < \infty \) and \( X \) be a complex \( p \)-space. Then (1.5) holds for any \( T \in K(X) \).

Even a slight perturbation of the norm destroys the above results. The following theorem provides a negative answer to Question 1.1.

Theorem 1.8. Let \( 1 < p < \infty \), \( 1 \leq q \leq \infty \), \( q \neq p \) and \( X = \mathbb{K} \oplus_q \ell_p \). Then there exists a compact operator \( T \) on \( X \) such that \( I + T \) is invertible, \( \| I + T \| > 1 \) and \( I + T \) does not attain its norm.

Proposition 1.3 and Theorem 1.8 imply that for \( 1 < p < \infty \) and \( 1 \leq q \leq \infty \), \( q \neq p \) there are \( \varepsilon > 0 \) and \( T \in K(C \oplus_q \ell_p) \) such that (1.5) fails. Since \( \mathbb{K} \oplus_q \ell_p \) is isomorphic to \( \ell_p \), we see that belonging to \( \mathcal{W} \) and validity of (1.5) are renorming sensitive properties. In particular, \( \mathbb{K} \oplus_p \ell_2 \) is isomorphic to the Hilbert space \( \ell_2 \) for any \( 1 \leq p \leq \infty \) and belongs to \( \mathcal{W} \) if and only if \( p = 2 \). However, for the spaces \( \mathbb{K} \oplus_p \ell_2 \) the situation improves if we consider finite rank operators instead of compact ones.

Proposition 1.9. Let \( 1 \leq p \leq \infty \), \( Y \) be a finite dimensional Banach space and \( X = Y \oplus_p \ell_2 \). Then for any bounded finite rank operator \( T \) on \( X \), the operator \( I + T \) attains its norm.

The last proposition suggests that the answer to Question 1.1 might be affirmative if we replace the compactness condition by the stronger one of \( T \) having finite rank. Unfortunately this is not the case.

Proposition 1.10. There exists a norm \( \| \cdot \| \) on \( \ell_2 \), equivalent to the original norm \( \| \cdot \|_2 \), and a rank one operator \( T \) on \( \ell_2 \) such that \( T^2 = 0 \), \( \| I + T \| = 2 \) (with respect to the norm \( \| \cdot \| \)) and the norm of \( I + T \) is not attained.

The equality \( T^2 = 0 \) for the operator from the above proposition ensures invertibility of \( T - I \) and the equality \( I + T = -(T - I)^{-1} \). Thus \( (T - I)^{-1} \) does not attain its norm and \( \|(T - I)^{-1}\| = 2 > 1 \). By Proposition 1.2, \( -1 \in \Sigma_{1/2}(T) \setminus \Sigma^0_{1/2}(T) \) and (1.5) fails for \( T \) with \( \varepsilon = 1/2 \).

2 Proof of Propositions 1.2 and 1.3

The following lemma is a known fact [9, 13]. For convenience of the reader we reproduce its short proof.

Lemma 2.1. Let \( X \) be a complex Banach space \( \varepsilon > 0 \) and \( T \in L(X) \). Assume also that \( z \in \Sigma_{\varepsilon}(T) \setminus \sigma(T) \) and \( (T - zI)^{-1} \) attains its norm. Then there is \( A \in L(X) \) such that \( \| A \| \leq \varepsilon \) and \( z \in \sigma(T + A) \).

Proof. Since \( (T - zI)^{-1} \) attains its norm, there exist \( x, y \in X \) such that \( \| y \| = \| x \| = 1 \) and \( (T - zI)^{-1}x = cy \), where \( c = \|(T - zI)^{-1}\| \). Using the Hahn–Banach theorem, we can pick \( f \in X^* \) for which \( \| f \| = f(y) = 1 \). Consider the operator

\[ A \in L(X), \quad Au = -c^{-1}f(u)x. \]

Clearly \( \| A \| \leq c^{-1} \). Since \( z \in \Sigma_{\varepsilon}(T) \), we have \( c^{-1} \leq \varepsilon \). Thus \( \| A \| \leq \varepsilon \). Moreover, \( Ay = -c^{-1}x \). From the equality \( (T - zI)^{-1}x = cy \) it follows that \( Ty = zy + c^{-1}x \). Hence \( (T + A)y = zy \) and \( z \in \sigma(T + A) \).
2.1 Proof of Proposition [1.2]

If $X$ is finite dimensional, then any $S \in L(X)$ attains its norm and according to Lemma 2.1 both (1.2.1) and (1.2.2) are satisfied. Thus for the rest of the proof, we can assume that $X$ is infinite dimensional.

Assume that (1.2.2) is satisfied. Since $X$ is infinite dimensional and $T$ is compact, $\| (T - zI) -1 \| \geq |z|^{-1}$.

If the relation $\| (T - zI) -1 \| = \varepsilon^{-1} > |z|^{-1}$ fails, then either $\| (T - zI) -1 \| > \varepsilon^{-1}$ or $\| (T - zI) -1 \| = \varepsilon^{-1} = |z|^{-1}$. If $\| (T - zI) -1 \| = \varepsilon^{-1} = |z|^{-1}$, then $z \in \sigma(T)$ and, according to (1.4), $z \in \Sigma_0(T)$. If $\| (T - zI) -1 \| = \varepsilon^{-1} = |z|^{-1}$, then $\| A \| = \varepsilon$, where $A = zI$. Since $X$ is infinite dimensional and $T$ is compact, we have $0 \in \sigma(T)$.

Hence $z \in \sigma(T + zI) = \sigma(T + A)$. Thus $z \in \Sigma_0(T)$. It remains to consider the case when $\| (T - zI) -1 \| = \varepsilon^{-1} > |z|^{-1}$ and $(T - zI) -1$ attains its norm. In this case, from Lemma 2.1 it follows that $z \in \Sigma_0(T)$. The implication (1.2.2) $\Rightarrow$ (1.2.1) is verified.

Assume now that (1.2.1) is satisfied. That is, there exists $A \in L(X)$ such that $\| A \| \leq \varepsilon$ and $z \in \sigma(T + A)$. Hence $0 \in \sigma(T - zI + A)$. Suppose that (1.2.2) fails. Then $\| (T - zI) -1 \| = \varepsilon^{-1} > |z|^{-1}$ and the norm of $(T - zI) -1$ is not attained. Since $\| A \| \leq \varepsilon$ and $|z| > \varepsilon$, the operator $-zI + A$ is invertible. Then $T - zI + A$ is a Fredholm operator of index zero as a compact operator $T$ and an invertible operator $-zI + A$. Since $0 \in \sigma(T - zI + A)$, $T - zI + A$ is non-invertible and therefore, being a Fredholm operator of index zero, it has non-trivial kernel. Thus we can pick $x \in X$ such that $\| x \| = 1$ and $(T - zI + A)x = 0$. It follows that $-Ax = (T - zI)x$ and therefore $x = -(T - zI) -1 Ax$. Using the relations $\| (T - zI) -1 \| = \varepsilon^{-1}$ and $\| A \| \leq \varepsilon$, we obtain

$$1 = \| x \| = \| -(T - zI) -1 Ax \| \leq \varepsilon^{-1} \| Ax \| \leq \varepsilon^{-1} \| A \| \| x \| \leq \| x \| = 1.$$ 

Obviously, all inequalities in the above display should be equalities which can only happen if $\| A \| = \| Ax \| = \varepsilon$. Then $\| y \| = 1$, where $y = -e^{-1} Ax$. Since $-Ax = (T - zI) -1 Ax = x$, we obtain $\| T - zI -1 \| = \varepsilon^{-1}$. Thus $\| (T - zI) -1 y \| = \varepsilon^{-1} = \| (T - zI) -1 \|$. That is, $(T - zI) -1$ attains its norm on $y$. This contradiction completes the proof of the implication (1.2.1) $\Rightarrow$ (1.2.2) and that of Proposition 1.2.

2.2 Proof of Proposition [1.3]

First, assume that (1.3.2) is satisfied. Let also $T \in K(X)$, $\varepsilon > 0$ and $z \in \Sigma_\varepsilon(T)$. According to (1.7), it suffices to show that $z \in \Sigma_\varepsilon^0(T)$. By Proposition 1.2 the latter happens if and only if (1.2.2) is satisfied. Assume that it is not the case. Then $\| (T - zI) -1 \| = \varepsilon^{-1} > |z|^{-1}$ and the norm of $(T - zI) -1$ is not attained. On the other hand, $(T - zI) -1 = -z^{-1} (I + S)$, where $S = z (T - zI) -1 I$ is compact. Moreover $\| I + S \| > 1$ since $\| (T - zI) -1 \| > |z|^{-1}$. By (1.3.2), $I + S$ attains its norm and therefore so does $(T - zI) -1$. This contradiction proves the implication (1.3.2) $\Rightarrow$ (1.3.1).

Next, assume that (1.3.1) is satisfied, $T \in K(X)$, $I + T$ is invertible and $c = \| I + T \| > 1$. Let $S = (I + T) -1 I$. Clearly $S$ is compact and $I + T = (S + I) -1$. Let $\varepsilon = c^{-1}$. Since $\| (S + I) -1 \| = \| I + T \| = \varepsilon^{-1} > 1$, we have $-1 \in \Sigma_\varepsilon(S)$. According to (1.3.2), $-1 \in \Sigma_\varepsilon^0(S)$. Proposition 1.2 implies now that $(S + I) -1 = I + T$ attains its norm. This completes the proof of the implication (1.3.1) $\Rightarrow$ (1.3.2) and that of Proposition 1.3.

3 $\ell_p$-direct sums of finite dimensional Banach spaces

Throughout this section $1 < p < \infty$ and $X$ is the $\ell_p$-direct sum of a family $\{ X_\alpha : \alpha \in \Lambda \}$ of finite dimensional Banach spaces. For $x \in X$, the support of $x$ is the set

$$\text{supp}(x) = \{ \alpha \in \Lambda : x_\alpha \neq 0 \}.$$ 

From the definition of the $\ell_p$-direct sum it follows that the support of any element of $X$ is at most countable. For a subset $B$ of $\Lambda$, we consider $P_B \in L(X)$ defined by the formula

$$P_B x_\alpha = \begin{cases} x_\alpha & \text{if } \alpha \in B, \\ 0 & \text{if } \alpha \notin B. \end{cases}$$

(3.1)

Clearly $P_B$ is a linear projection and $\| P_B \| = \| I - P_B \| = 1$ if $B$ is non-empty and $B \neq \Lambda$. 

4
Lemma 3.1. Let \( \{x_n\} \) be a sequence in \( X \) weakly convergent to zero and \( \{\varepsilon_k\} \) be a sequence of positive numbers. Then there exist a strictly increasing sequence \( \{n_k\} \) of positive numbers and a sequence \( \{u_k\} \) of elements of \( X \) such that \( \|x_{n_k} - u_k\| < \varepsilon_k \) for each \( k \in \mathbb{N} \) and the sets \( \text{supp}(u_k) \) are finite and pairwise disjoint.

Proof. We construct the required sequences inductively. On the first step we take \( n_1 \) = 1, pick a finite subset \( B \) of \( \Lambda \) such that \( \|x_1 - P_B x_1\| < \varepsilon_1 \) and put \( u_1 = P_B x_1 \).

Assume now that \( k \geq 2 \), \( n_1 < \ldots < n_{k-1} \), \( u_1, \ldots, u_{k-1} \) are vectors in \( X \) with pairwise disjoint finite supports such that \( \|x_{n_j} - u_j\| < \varepsilon_j \) for \( 1 \leq j \leq k - 1 \). Let now \( C \) be the union of \( \text{supp}(u_j) \) for \( 1 \leq j \leq k - 1 \). Since \( C \) is finite, \( P_C \) is a compact operator and therefore \( \|P_C x_n\| \to 0 \) as \( n \to \infty \). Thus we can pick \( n_k > n_{k-1} \) such that \( \|P_C x_{n_k}\| < \varepsilon_k/2 \). Next, choose a finite subset \( A \) of \( \Lambda \) such that \( C \subset A \) and \( \|x_{n_k} - P_A x_{n_k}\| < \varepsilon_k/2 \) and put \( u_k = P_A x_{n_k} - P_C x_{n_k} \). Clearly \( \text{supp}(u_k) \subset A \setminus C \) and therefore \( u_k \) has finite support and the supports of \( u_1, \ldots, u_k \) are pairwise disjoint. Finally,

\[
\|x_{n_k} - u_k\| \leq \|P_C x_{n_k} + (P_{n_k} x_{n_k}) - P_A x_{n_k}\| + \varepsilon_k/2 + \varepsilon_k/2 = \varepsilon_k.
\]

The description of the inductive construction of sequences \( \{n_k\} \) and \( \{u_k\} \) is now complete. \( \square \)

Lemma 3.2. \( X \) is a \( p \)-space.

Proof. Since \( 1 < p < \infty \), \( X \) is reflexive as an \( \ell_p \)-direct sum of reflexive Banach spaces. Let \( x \in X \) and \( \{u_n\} \) be a sequence in \( X \) weakly convergent to 0. Let also \( \varepsilon > 0 \). Pick a finite subset \( B \) of \( \Lambda \) such that \( \|x - P_B x\| < \varepsilon \). Since \( P_B \) is a compact operator and \( \|u_n\| \) converges weakly to 0, we have \( \|P_B u_n\| \to 0 \) as \( n \to \infty \). Let \( x_n = u_n - P_B u_n \). Then \( \|x_n - u_n\| \to 0 \) as \( k \to \infty \) and supports of \( x_n \) do not meet \( B \). Since the support of \( P_B x \) is contained in \( B \), the supports of \( x_n \) do not intersect the support of \( P_B x \) and from the definition of the norm on \( X \) it follows that

\[
\|P_B x + x_n\| = \|P_B x\| + \|x_n\|/p^{1/p}.
\]

Since \( \|x - P_B x\| < \varepsilon \) and \( \|x_n - u_n\| \to 0 \) as \( n \to \infty \), we see that

\[
\|P_B x + x_n\| - \|x + u_n\| < \varepsilon \quad \text{and} \quad \left|\frac{(\|x\|/p + \|u_n\|/p^{1/p}) - (\|P_B x\| + \|x_n\|/p^{1/p})}{p}\right| < \varepsilon
\]

for all sufficiently large \( n \). From the last two displays it follows that

\[
\|x + u_n\| - (\|x\|/p + \|u_n\|/p^{1/p}) < 2\varepsilon
\]

for all sufficiently large \( n \). Since \( \varepsilon > 0 \) is arbitrary, the equality \[1.27\] follows. \( \square \)

3.1 Proof of Proposition 1.4

By Lemma 3.2 the class of \( p \)-spaces contains \( \ell_p \)-direct sums of finite dimensional Banach spaces. From the definition it immediately follows that (closed linear) subspaces of \( p \)-spaces are \( p \)-spaces. Hence closed linear subspaces of \( \ell_p \)-direct sums of finite dimensional Banach spaces are \( p \)-spaces.

3.2 Operators on \( p \)-spaces

Lemma 3.3. Let \( 1 < p < q < \infty \), \( Y \) be a \( p \)-space, \( Z \) be a \( q \)-space, \( T \in L(Y, Z) \), \( \{x_n\} \in Y \) be a sequence in \( Y \) such that \( \|x_n\| \to 1 \) and \( \|T x_n\| \to \|T\| \) as \( n \to \infty \) and \( \{x_n\} \) is weakly convergent to \( x \in X \). Then \( \|T x\| = \|T\| \|x\| \).

Proof. Let \( u_n = x_n - x \). Then \( \{u_n\} \) is weakly convergent to 0. Since \( T \) is linear and bounded, \( T \) is also continuous with respect to the weak topology and therefore \( \{T u_n\} \) is weakly convergent to 0. For brevity denote \( c = \|T\| \). Since \( Y \) is a \( p \)-space and \( Z \) is a \( q \)-space, we have

\[
1 = \lim_{n \to \infty} \|x_n\| = \lim_{n \to \infty} \|x + u_n\| = \lim_{n \to \infty} \|\|x\|/p + \|u_n\|/p^{1/p}\|,
\]

\[
c = \lim_{n \to \infty} \|T x_n\| = \lim_{n \to \infty} \|T x + T u_n\| = \lim_{n \to \infty} \|\|T x\|/q + \|T u_n\|/q\|^{1/q}.
\]

(3.2)
Clearly \( \|Tu_n\| \leq c\|u_n\| \) for each \( n \in \mathbb{N} \). Assume that \( \|Tx\| \neq c\|x\| \). Then \( \|Tx\| < c\|x\| \). Using these inequalities together with \((3.2)\) and \((3.3)\) and taking into account that \( p \leq q \), we obtain
\[
c = \lim_{n \to \infty} (\|Tx\|^q + \|Tu_n\|^q)^{1/q} < \lim_{n \to \infty} (c^q\|x\|^q + c^q\|u_n\|^q)^{1/q} \leq c \lim_{n \to \infty} (\|x\|^p + \|u_n\|^p)^{1/p} = c.
\]
This contradiction proves the equality \( \|Tx\| = c\|x\| \).

Recall that \( X \) is the \( \ell_p \)-direct sum of the family \( \{X_\alpha : \alpha \in \Lambda\} \) of finite dimensional Banach spaces.

**Lemma 3.4.** Let \( T \in L(X) \) be such that
\[
\inf_{\|x\|=1} \|Tx\| = c > 0.
\]
Then there exists \( S \in L(X) \) such that \( \|S\| = c \) and
\[
\inf_{\|x\|=1} \|(T + S)x\| = 0.
\]

**Proof.** Pick a sequence \( \{x_n\}_{n \in \mathbb{N}} \) in \( X \) such that \( \|x_n\| \to 1 \) and \( \|Tx_n\| \to c \) as \( n \to \infty \). Since \( X \) is reflexive, we can choose such a sequence \( \{x_n\} \) being weakly convergent to \( x \in X \). Clearly \( \|x\| \leq 1 \).

**Case** \( x = 0 \). That is, \( \{x_n\} \) weakly converges to 0. By Lemma 3.1 we can find a strictly increasing sequence \( \{n_k\}_{k \in \mathbb{N}} \) of positive integers and a sequence \( \{y_k\}_{k \in \mathbb{N}} \) of elements of \( X_0 \) such that the supports of \( y_k \) are pairwise disjoint and
\[
\|x_{n_k} - y_k\| < 2^{-k} \text{ for any } k \in \mathbb{N}.
\]
Since the sequence \( \{x_{n_k}\} \) weakly converges to 0, formula \((3.6)\) implies that \( \{y_k\} \) also weakly converges to 0. Since \( T \in L(X) \), the sequence \( \{Ty_k\} \) weakly converges to 0. Using Lemma 3.1 once again, we see that there exist a strictly increasing sequence \( \{k_m\}_{m \in \mathbb{N}} \) of positive integers and a sequence \( \{w_m\}_{m \in \mathbb{N}} \) in \( X_0 \) such that the supports of \( w_m \) are pairwise disjoint and
\[
\|Ty_{k_m} - w_m\| < 2^{-m} \text{ for any } m \in \mathbb{N}.
\]
From \((3.6)\) it follows that \( \|Tx_{n_k} - Ty_k\| < 2^{-k}\|T\| \) for any \( k \in \mathbb{N} \). Since \( \|Tx_{n_k}\| \to c \) as \( k \to \infty \), we have \( \|Ty_k\| \to c \) as \( k \to \infty \). Now by \((3.5)\) and \((3.6)\) we obtain
\[
\lim_{m \to \infty} \|y_{k_m}\| = 1 \text{ and } \lim_{m \to \infty} \|w_m\| = c.
\]

For each \( m \in \mathbb{N} \) let \( A_m = \text{supp}(y_{k_m}), P_m = P_{A_m} \) and \( X_m = P_m(X) \). By Hahn–Banach theorem, for any \( m \in \mathbb{N} \), we can find \( \varphi_m \in X_m^* \) such that \( \|\varphi_m\| = 1 \) and \( \varphi_m(y_{k_m}) = \|y_{k_m}\| \). Consider the operator \( S \in L(X) \) defined by the formula
\[
Su = -c \sum_{m=1}^{\infty} \frac{\varphi_m(P_m u)}{\|w_m\|} w_m.
\]
From the equalities \( \|\varphi_m\| = 1 \), pairwise disjointness of \( A_m \) and pairwise disjointness of \( \text{supp}(w_m) \) it immediately follows that \( \|S\| = c \). On the other hand, by the definition of \( S \)
\[
Sy_{k_m} = -c \frac{\|y_{k_m}\|}{\|w_m\|} w_m \text{ for any } m \in \mathbb{N}.
\]
According to \((3.2)\) we have \( \|Sy_{k_m} + w_m\| \to 0 \) as \( m \to \infty \). Thus by \((3.7)\), \( \|(T + S)y_{k_m}\| \to 0 \) as \( m \to \infty \). From \((3.5)\) it follows that \( \|y_{k_m}\| \to 1 \) as \( m \to \infty \). Hence \((3.5)\) is satisfied.

**Case** \( x \neq 0 \). Let \( Y = T(X) \). According to \((3.4)\), \( Y \) is a closed linear subspace of \( X \) and \( T : X \to Y \) is invertible. Consider \( R \in L(Y,X) \) being the inverse of \( T : X \to Y \). From \((3.4)\) it follows that \( \|R\| = c^{-1} \). It is also clear that the sequence \( u_n = c^{-1}Tx_n \) is weakly convergent to \( c^{-1}Tx \) and \( \|u_n\| \to 1 \) as \( n \to \infty \). Moreover \( Ru_n = c^{-1}x_n \) for any \( n \in \mathbb{N} \) and therefore \( Ru_n \) weakly converges to \( c^{-1}x \) and \( \|Ru_n\| \to c^{-1} = \|R\| \) as \( n \to \infty \). By Proposition 1.3 \( X \) and \( Y \) are \( p \)-spaces. Hence, according to Lemma 3.3 \( \|R||c^{-1}Tx\| = \|c^{-1}RTx\| \). Taking into account that \( RTx = x \) and \( \|R\| = c^{-1} \), we have \( \|Tx\| = c\|x\| \). By Hahn–Banach theorem, we can find \( \varphi \in X^* \) such that \( \|\varphi\| = 1 \) and \( \varphi(x) = \|x\| \). Let now \( S \in L(X), Su = -\|x\|^{-1}\varphi(u)x \). Since \( \|Tx\| = c\|x\| \), we have \( \|S\| \leq c \). Moreover \( (T + S)x = Tx - Tx = 0 \) and therefore \( T + S \) has non-trivial kernel. Hence \((3.5)\) is satisfied. \( \square \)
3.3 Proof of Theorem 1.5

Let $T \in L(X)$, $\varepsilon > 0$ and $z \in \Sigma_1(T)$. In view of (1.6), it suffices to show that $z \in \Sigma_z(T)$. Since $z \in \Sigma_z(T)$, we have $\|(T - zI)^{-1}\| \geq \varepsilon^{-1}$. If $\|(T - zI)^{-1}\| > \varepsilon^{-1}$, the inclusion $z \in \Sigma_z(T)$ follows from (1.2). It remains to consider the case $\|(T - zI)^{-1}\| = \varepsilon^{-1}$. In this case

$$
\varepsilon = \inf_{\|x\|=1} \|(T - zI)x\|.
$$

By Lemma 3.4, we can find $S \in L(X)$ such that $\|S\| \leq \varepsilon$ and

$$
0 = \inf_{\|x\|=1} \|(T - zI + S)x\|.
$$

The last display implies that $T + S - zI$ is not invertible. Hence $z \in \sigma(T + S)$. Since $\|S\| \leq \varepsilon$, we obtain the required inclusion $z \in \Sigma_z(T)$.

3.4 Proof of Theorem 1.6

Lemma 3.5. Let $Y$ and $Z$ be Banach spaces, $J \in L(Y,Z)$ and $T \in K(Y,Z)$ be such that $\|J + T\| > \|J\|$. Assume also that $\{x_n\}_{n \in \mathbb{N}}$ is a sequence of vectors in $Y$ weakly convergent to $x \in Y$ for which $\|x_n\| \to 1$ and $\|(J + T)x_n\| \to \|J + T\|$ as $n \to \infty$. Then $x \neq 0$.

Proof. Since $T$ is compact, $\|Tx_n - Tx\| \to 0$ as $n \to \infty$. Hence $\lim_{n \to \infty} \|Jx_n + Tx\| = \|J + T\|$. On the other hand, $\lim_{n \to \infty} \|Jx_n\| \leq \|J\|$. Thus using the triangle inequality, we obtain $\|Tx\| \geq \|J + T\| - \|J\|$. It follows that $\|x\||T|| \geq \|Tx\| \geq \|J + T\| - \|J\| > 0$. Hence, $x \neq 0$.

We are ready to prove Theorem 1.6. Pick a sequence $\{x_n\}_{n \in \mathbb{N}}$ of elements of $X$ such that $\|x_n\| = 1$ for any $n \in \mathbb{N}$ and $\|(J + T)x_n\| \to \|J + T\|$ as $n \to \infty$. Since $X$ is reflexive, we, passing to a subsequence, if necessary, may assume that $\{x_n\}$ weakly converges to $x \in X$. By Lemma 3.3, $x \neq 0$. According to Lemma 3.3, $\|Jx + Tx\| = \|x||J + T\|$. Hence $J + T$ attains its norm on the vector $x/\|x\|$.

4 Operators on $\mathbb{K} \oplus_q \ell_p$

We start by a series of elementary observations.

Lemma 4.1. Let $X$ and $Y$ be Banach spaces and $T \in L(X,Y)$ be an operator attaining its norm. Then the dual operator $T^* \in L(Y^*,X^*)$ attains its norm.

Proof. Since $T$ attains its norm, there exists $x \in X$ such that $\|x\| = 1$ and $\|Tx\| = \|T\|$. By Hahn–Banach theorem, we can pick $\varphi \in Y^*$ such that $\|\varphi\| = 1$ and $\varphi(Tx) = \|Tx\|$. Since $\varphi(Tx) = (T^*\varphi)(x)$, we have $\varphi(Tx) \leq \|T^*\varphi\||x| = \|T^*\varphi||$. Since $\|Tx\| = \|T\| = \|T^*\|$, we see that $\|T^*\varphi\| \geq \|T^*\|$ and $\|\varphi\| = 1$. Thus $\|T^*\varphi\| = \|T^*\|$ and therefore $T^*$ attains its norm at $\varphi$.

The above lemma immediately implies the following corollary.

Corollary 4.2. Let $X$ be a reflexive Banach space and $T \in L(X)$. Then $T$ attains its norm if and only if $T^*$ attains its norm.

In particular, using the facts that an operator is compact if and only if its dual is compact and an operator is invertible if and only if its dual is invertible, we have the following result.

Corollary 4.3. Let $X$ be a reflexive Banach space. Then $X \in W$ if and only if $X^* \in W$. Moreover, the following two statements are equivalent:

1) there is a compact operator $T$ on $X$ such that $I + T$ is invertible, $\|I + T\| > 1$ and $I + T$ does not attain its norm;

2) there is a compact operator $S$ on $X^*$ such that $I + S$ is invertible, $\|I + S\| > 1$ and $I + S$ does not attain its norm.
4.1 Proof of Theorem 1.8

Let $1 < p < \infty$, $1 \leq q \leq \infty$, $p \neq q$ and $X = \mathbb{K} \oplus_q \ell_p$. Clearly $X$ is reflexive and $X^*$ is naturally isometrically isomorphic to $\mathbb{K} \oplus_q \ell_{p'}$, where $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$. It is also easy to see that $p' > q'$ if $p < q$ and $p' < q'$ if $p > q$. According to Corollary 4.3 it is enough to prove Theorem 1.8 in the case $p < q$. Thus from now on, we assume that $p < q$.

We naturally interpret $X$ as a space of sequences $x = \{x_n\}_{n \geq 0}$, where $x_0$ and $\{x_n\}_{n \in \mathbb{N}}$ correspond to the $\mathbb{K}$-component and the $\ell_p$-component in the decomposition $X = \mathbb{K} \oplus_q \ell_p$ respectively. For any $x \in X$, we denote

$$\alpha(x) = |x_0|, \quad \beta(x) = |x_1| \quad \text{and} \quad \gamma(x) = \left(\sum_{n=2}^{\infty} |x_n|^p\right)^{1/p}. \quad (4.1)$$

Clearly, for $x \in X$,

$$\|x\| = f(\alpha(x), \beta(x), \gamma(x)), \quad \text{where} \quad f(\alpha, \beta, \gamma) = \left\{ \begin{array}{ll}
\left(\alpha^q + (\beta^p + \gamma^q)q/p\right)^{1/q} & \text{if } q < \infty, \\
\max\{\alpha, (\beta^p + \gamma^q)^{1/p}\} & \text{if } q = \infty.
\end{array} \right. \quad (4.2)$$

Consider the operator $S \in L(X)$ defined by the formula $(Sx)_0 = x_1$, $(Sx)_1 = x_0$ and $(Sx)_n = \frac{p}{n+1}$ if $n \geq 2$. That is,

$$Sx = \left(x_1, x_0, \frac{2x_2}{3}, \frac{3x_3}{4}, \ldots \right).$$

Clearly $T = S - I$ is compact. Thus in order to verify that $T$ satisfies the required conditions, it suffices to show that $S = I + T$ is invertible, $\|S\| > 1$ and $S$ does not attain its norm. Invertibility of $S$ is obvious. Indeed, the operator $R \in L(X)$ defined as $Rx = (x_1, x_0, 3x_2/2, 4x_3/3, \ldots)$ is the inverse of $S$. Next, let $x \in X$ be such that $\|x\| = 1$ and let $\alpha(x)$, $\beta(x)$ and $\gamma(x)$ be the numbers defined in (4.1). It is clear that $\alpha(Sx) = \beta(x)$, $\beta(Sx) = \alpha(x)$, $\gamma(Sx) \leq \gamma(x)$. Moreover, $\gamma(Sx) < \gamma(x)$ if $\gamma(x) > 0$. Thus according to (4.2),

$$f(\alpha(x), \beta(x), \gamma(x)) = 1 \quad \text{and} \quad \|Sx\| = f(\beta(x), \alpha(x), \gamma(Sx)) \leq f(\beta(x), \alpha(x), \gamma(x)). \quad (4.4)$$

Moreover, since $\gamma(Sx) < \gamma(x)$ when $\gamma(x) > 0$, we have

$$\|Sx\| < f(\beta(x), \alpha(x), \gamma(x)) \quad \text{if } q < \infty \text{ and } \gamma(x) > 0$$
$$\text{and} \quad \text{if } q = \infty, \gamma(x) > 0 \text{ and } \beta(x) < (\alpha(x)p + \gamma(x)p)^{1/p}. \quad (4.5)$$

According to (4.4), $\|Sx\| \leq C$, where

$$C = \sup\{f(\beta, \alpha, \gamma) : (\alpha, \beta, \gamma) \in K\} \quad \text{and} \quad K = \{(\alpha, \beta, \gamma) \in \mathbb{R}_+^3 : f(\alpha, \beta, \gamma) = 1\}.$$ 

Since $K$ is compact and $f$ is continuous, the supremum in the definition of $C$ is attained. Using, for instance, the Lagrange multipliers technique, one can easily see that the function $(\alpha, \beta, \gamma) \mapsto f(\beta, \alpha, \gamma)$ from $K$ to $\mathbb{R}_+$ attains its maximal value $C = 2^{\frac{1}{q} - \frac{1}{p}}$ in exactly one point being $(2^{-1/q}, 0, 2^{-1/q})$. From (4.3) it now follows that

$$\|Sx\| < C = 2^{\frac{1}{q} - \frac{1}{p}} \quad \text{whenever } \|x\| = 1. \quad (4.6)$$

Now consider the sequence $x_n = 2^{-1/q}e_0 + 2^{-1/q}e_n$, $n \in \mathbb{N}$, where $\{e_k\}$ is the canonical basis in the sequence space $X$. Clearly $\|x_n\| = 1$ for each $n \in \mathbb{N}$. On the other hand, for any $n \geq 2$, $Sx_n = 2^{-1/q}e_1 + \frac{n}{n+1}e_n$ and therefore

$$\|Sx_n\| = 2^{-1/q}\left(1 + \left(\frac{n}{n+1}\right)^p\right)^{1/p} \to 2^{\frac{1}{p} - \frac{1}{q}} \quad \text{as } n \to \infty. \quad (4.7)$$

From (4.6) and (4.7) it follows that $\|S\| = 2^{\frac{1}{p} - \frac{1}{q}} > 1$ and the norm of $S$ is not attained. The proof of Theorem 1.8 is now complete.
5 Proper extensions of Hilbert spaces and finite rank operators

In this section we prove a theorem slightly stronger then Proposition 1.9. We need some preparation. Throughout this section \( \mathcal{H} \) is a Hilbert space and \( n \in \mathbb{N} \). We say that \( X = \mathbb{K}^n \times \mathcal{H} \) is a proper extension of \( \mathcal{H} \) if \( X \) is endowed with a norm such that

\[
\|(t, x)\| = \varphi(t, \|x\|) \quad \text{for any } t \in \mathbb{K}^n, \ x \in \mathcal{H},
\]

(5.1)

where \( \varphi : \mathbb{K}^n \times \mathbb{R}_+ \to \mathbb{R}_+ \) is a function and \( \varphi(0, 1) = 1 \). The fact that \((t, x) \mapsto \varphi(t, \|x\|)\) is a norm on \( X \) immediately that \( \varphi \) is Lipschitzian, convex, \( \varphi(t, a) > 0 \), whenever \((t, a) \neq (0, 0)\) and \( \varphi(st, sa) = s\varphi(t, a) \) for any \( s, a \in \mathbb{R}_+ \) and \( t \in \mathbb{K}^n \). The normalization condition \( \varphi(0, 1) = 1 \) implies that \( \|0, x\| = \|x\|_{\mathcal{H}} \) for any \( x \in \mathcal{H} \). Thus \( \mathcal{H} \) is naturally isometrically embedded into \( X \). Since \( \mathcal{H} \) has finite codimension in \( X \), we see that \( X \) is a Banach space and admits an equivalent norm which turns it into a Hilbert space. In particular, \( X \) is reflexive.

**Theorem 5.1.** Let \( X = \mathbb{K}^n \times \mathcal{H} \) be a proper extension of a Hilbert space \( \mathcal{H} \). Then for any bounded finite rank operator \( T \) on \( X \), \( I + T \) attains its norm.

**Proof.** Let \( \varphi : \mathbb{K}^n \times \mathbb{R}_+ \to \mathbb{R}_+ \) be a function defining the norm on \( X \) according to (5.1). If \( \mathcal{H} \) is finite dimensional, the result becomes trivial. Thus we can assume that \( \mathcal{H} \) is infinite dimensional. Pick a sequence \( \{\xi_k = (t_k, x_k)\} \subseteq X \) of elements of \( X \) such that \( \|\xi_k\| \to 1 \) and \( \|(I + T')\xi_k\| \to c \) as \( k \to \infty \).

Since \( X \) is reflexive, we can, passing to a subsequence, if necessary, assume that \( \{\xi_k\} \) converges weakly to \( \xi = (t, x) \in X \). Since \( T \) has finite rank, \( \{T\xi_k\} \) is norm convergent to \( T\xi = (s, y) \). Next, since weak and norm convergences on a finite dimensional Banach space coincide, we see that \( \{t_k\} \) converges to \( t \) in \( \mathbb{C}^n \). Passing to a subsequence again, if necessary, we can assume that \( \|x_k - x\| \to \alpha \in \mathbb{R}_+ \).

Since \( x \neq x_k \), \( x \) is weakly convergent to zero in the Hilbert space \( \mathcal{H} \) and any Hilbert space is a 2-space, we see that

\[
\lim_{k \to \infty} \|x_k\| = (\|x\|^2 + \alpha^2)^{1/2} \quad \text{and} \quad \lim_{k \to \infty} \|x_k + y\| = (\|x + y\|^2 + \alpha^2)^{1/2}.
\]

(5.2)

Since \( \|\xi_k\| \to 1 \), \( \|(I + T')\xi_k\| \to c \), \( t_k \to t \) and \( \|T\xi_k - (s, y)\| \to 0 \), we have

\[
1 = \lim_{k \to \infty} \|\xi_k\| = \lim_{k \to \infty} \|(t_k, x_k)\| = \lim_{k \to \infty} \|(t, x)\|
\]

\[
c = \lim_{k \to \infty} \|(I + T')\xi_k\| = \lim_{k \to \infty} \|\xi_k + (s, y)\| = \lim_{k \to \infty} \|(t_k + s, x_k + y)\| = \lim_{k \to \infty} \|(t + s, x_k + y)\|.
\]

Using (5.1), (5.2) and continuity of \( \varphi \), we obtain

\[
\varphi(t, (\|x\|^2 + \alpha^2)^{1/2}) = 1 \quad \text{and} \quad \varphi(t + s, (\|x + y\|^2 + \alpha^2)^{1/2}) = c.
\]

(5.3)

Since \( \mathcal{H} \) is infinite dimensional and \( T \) has finite rank, the linear subspace

\[
L = \{v \in \mathcal{H} : T(0, v) = 0, \langle u, x \rangle = \langle u, y \rangle = 0\}
\]

has finite codimension and therefore is non-trivial. Hence we can pick \( u \in L \) such that \( \|u\| = \alpha \). Since \( u \) is orthogonal to both \( x \) and \( y \), we see that \( \|x + u\| = (\|x\|^2 + \alpha^2)^{1/2} \) and \( \|x + y + u\| = (\|x + y\|^2 + \alpha^2)^{1/2} \).

Hence, according to (5.1) and (5.3)

\[
\|(t, x + u)\| = \varphi(t, \|x + u\|) = 1 \quad \text{and} \quad \|(t + s, x + y + u)\| = \varphi(t, \|x + y + u\|) = c.
\]

Finally, since \( T(0, u) = 0 \), we have \( T(t, x + u) = T(t, x) = (s, y) \). Hence

\[
(I + T)(t, x + u) = (t + s, x + y + u).
\]

Since \( c = \|I + T\| \), from the last two displays it follows that \( I + T \) attains its norm on the vector \((t, x + u)\).

Theorem 1.3 follows from Theorem 5.1 since \( Y \oplus_p \ell_2 \) for a finite dimensional Banach space \( Y \) is a particular case of a proper extension.

\[\square\]
6 Examples with rank one operators

As was already mentioned in the introduction, Shargorodsky \[22\] constructed \(T \in \mathcal{K}(\ell_1)\) such that \[155\] fails for \(T\) for one prescribed \(\varepsilon > 0\). We shall demonstrate that for \(X = \ell_1\) and \(X = c_0\) one can find a rank 1 operator \(T\) for which \[150\] fails for any \(\varepsilon > 0\). As usual, we denote the canonical basis in \(c_0\) or \(\ell_1\) by \(\{e_n\}_{n \geq 0}\).

**Example 6.1.** Let \(T \in L(c_0)\),

\[Tx = \left[\sum_{n=1}^{\infty} 2^{-n}x_n\right]e_0.\]

Then \(T\) has rank 1, \(T^2 = 0\) and for any \(z \in \mathbb{K}\), \(\|T + zI\| = 1 + |z|\) and the operator \(T + zI\) does not attain its norm.

**Proof.** Obviously, \(T\) has rank 1, \(\|T\| = 1\) and \(T^2 = 0\). Let \(z \in \mathbb{K}\) and \(r = |z|\). Since \(\|T\| = 1\), we have \(\|T + zI\| \leq 1 + r\). For \(n \in \mathbb{N}\), consider \(x_n = (r/z)e_0 + e_1 + e_2 + \ldots + e_n\). Clearly \(\|x_n\| = 1\) and the \(e_0\)-coefficient of \((T + zI)x_n\) equals \(r + 1 - 2^{-n}\). Hence \(\|T + zI\| \geq r + 1 - 2^{-n}\) for any \(n \in \mathbb{N}\). Thus \(\|T + zI\| \geq 1 + r\). Since the opposite inequality is also true, \(\|T + zI\| = 1 + r\). It remains to show that \(T + zI\) does not attain its norm. Assume the contrary. Then there exists \(x \in c_0\) such that \(\|x\| = 1\) and \(\|y\| = 1 + r\), where \(y = zx + Tx\). Since \(Tx\) is a scalar multiple of \(e_0\), we have \(y_n = zx_n\) for \(n \in \mathbb{N}\). Hence \(|y_n| \leq r\) for \(n \in \mathbb{N}\). Thus \(1 + r = \|y\| = |y_0|\). Using the definition of \(T\) we obtain \(y_0 = zx_0 + \sum_{n=1}^{\infty} 2^{-n}x_n\). Hence

\[1 + r = |y_0| \leq |z||x_0| + \sum_{n=1}^{\infty} 2^{-n}|x_n| \leq r + \sum_{n=1}^{\infty} 2^{-n} = 1 + r.\]

The latter is possible only if \(|x_j| = 1\) for any \(j\) which contradicts the inclusion \(x \in c_0\).

**Example 6.2.** Let \(T \in L(\ell_1)\),

\[Tx = \left[\sum_{n=1}^{\infty} (1 - 2^{-n})x_n\right]e_0.\]

Then \(T\) has rank 1, \(T^2 = 0\) and for any \(z \in \mathbb{C}\), \(\|T + zI\| = 1 + |z|\) and the operator \(T + zI\) does not attain its norm.

**Proof.** Obviously, \(T\) has rank 1, \(\|T\| = 1\) and \(T^2 = 0\). Let \(z \in \mathbb{K}\) and \(r = |z|\). For \(n \in \mathbb{N}\), we have \((T + zI)e_n = (1 - 2^n)e_0 + ze_n\). Hence \(\|T + zI\| = 1 + r - 2^{-n}\). Since \(\|e_n\| = 1\), we see that \(\|T + zI\| \geq 1 + r\). Since the opposite inequality is also true, \(\|T + zI\| = 1 + r\). It remains to show that \(T + zI\) does not attain its norm. Assume the contrary. Then there exists \(x \in \ell_1\) such that \(\|x\| = 1\) and \(\|y\| = 1 + r\), where \(y = zx + Tx\). By definition of \(T\),

\[1 + r = \|y\| = \|zx_0 + \sum_{n=1}^{\infty} (1 - 2^{-n})x_n + r \sum_{n=1}^{\infty} x_n| \leq r|x_0| + \sum_{n=1}^{\infty} (1 + r - 2^{-n})|x_n| < (1 + r)\|x\| = 1 + r.\]

The latter inequality is due to the fact that the coefficients \(r\) and \(1 + r - 2^{-n}\) in the last sum are strictly less than \(1 + r\). This contradiction completes the proof.

The following Proposition clarifies the situation with the above two operators and formula \[155\].

**Proposition 6.3.** Let \(T\) be the operator from either Example 6.1 or Example 6.2 in the case \(\mathbb{K} = \mathbb{C}\). Then for any \(\varepsilon > 0\), \(\Sigma^0_\varepsilon(T) = \sigma_\varepsilon(T) \neq \Sigma_\varepsilon(T)\).

**Proof.** Since \(T^2 = 0\), \(\sigma(T) = \{0\}\) and for any \(z \in \mathbb{C} \setminus \{0\}\), \((T - zI)^{-1} = (-z^{-2})(T + zI)\). Thus \((T - zI)^{-1}\) attains its norm if and only if so does \(T + zI\) and \(\|T - zI\|^{-1} = |z|^{-2}\|T + zI\| = |z|^{-1} + |z|^{-2} > |z|^{-1}\). Since \(T + zI\) never attains its norm, we, applying Proposition \[12\] see that

\[\Sigma^0_\varepsilon(T) = \sigma_\varepsilon(T) = \{z \in \mathbb{C} : |z|^{-1} + |z|^{-2} > \varepsilon^{-1}\} = \{z : |z| < \varepsilon + \sqrt{4\varepsilon + \varepsilon^2}\} \text{ for any } \varepsilon > 0.\]

On the other hand,

\[\Sigma_\varepsilon(T) = \{z \in \mathbb{C} : |z|^{-1} + |z|^{-2} \geq \varepsilon^{-1}\} = \{z : |z| \leq \varepsilon + \sqrt{4\varepsilon + \varepsilon^2}\} \text{ for any } \varepsilon > 0.\]

Clearly \(\Sigma^0_\varepsilon(T) \neq \Sigma_\varepsilon(T)\) for each \(\varepsilon > 0\).
6.1 Proof of Proposition 1.10

Recall that a subset $A$ of a vector space $X$ is called balanced if $\lambda x \in A$ whenever $x \in A$, $\lambda \in \mathbb{K}$ and $|\lambda| \leq 1$. A set is called absolutely convex if it is convex and balanced. By $\overline{\operatorname{aconv}}(A)$ we denote the absolutely convex hull of $A$, being the minimal absolutely convex set containing $A$. Clearly

$$\overline{\operatorname{aconv}}(A) = \left\{ \sum_{j=1}^{n} \lambda_j x_j : n \in \mathbb{N}, x_1, \ldots, x_n \in A, \lambda_1, \ldots, \lambda_n \in \mathbb{K}, \sum_{j=1}^{n} |\lambda_j| \leq 1 \right\}. \quad (6.1)$$

For a subset $A$ of a topological vector space $X$, $\overline{\operatorname{aconv}}(A)$ stands for the closure of $\operatorname{aconv}(A)$. We recall two elementary properties of absolutely convex hulls. The proof of the first one can be found in virtually any book on topological vector spaces, see, for instance, [20]. The second one is proved in [6]. For a different proof see [24].

**Lemma 6.4.** Let $n \in \mathbb{N}$ and $K_1, \ldots, K_n$ be compact convex subsets of a Hausdorff topological vector space $X$. Then

$$\overline{\operatorname{aconv}} \left( \bigcup_{j=1}^{n} K_j \right) = \overline{\operatorname{aconv}} \left( \bigcup_{j=1}^{n} K_j \right) = \left\{ \sum_{j=1}^{n} \lambda_j x_j : \lambda_j \in \mathbb{K}, x_j \in K_j, \sum_{j=1}^{n} |\lambda_j| \leq 1 \right\}. \quad (6.3)$$

Moreover, the above set is compact.

**Lemma 6.5.** Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence of elements of a sequentially complete locally convex Hausdorff topological vector space $X$, converging to $x \in X$ as $n \to \infty$. Then

$$\overline{\operatorname{aconv}}(A) = \left\{ \alpha_0 x + \sum_{n=1}^{\infty} \alpha_n x_n : \alpha \in \ell_1, \|\alpha\|_1 \leq 1 \right\}, \quad \text{where } A = \{x_n : n \in \mathbb{N}\}. \quad (6.4)$$

Moreover, $\overline{\operatorname{aconv}}(A)$ is metrizable and compact.

From now on in this section, by $\| \cdot \|_2$ we denote the canonical norm on $\ell_2$. We use the same symbol to denote the standard Euclidean norm on $\mathbb{K}^2$:

$$\|(t,s)\|_2 = (|t|^2 + |s|^2)^{1/2}. \quad (6.5)$$

Let also $\{e_n\}_{n \in \mathbb{N}}$ be the canonical orthonormal basis in $\ell_2$. For $x \in \ell_2$ we denote

$$x' = x - x_1 e_1 - x_2 e_2.$$ 

That is, $x'$ is the orthogonal projection of $x$ onto the closed linear span of the vectors $e_3, e_4, \ldots$. Fix a sequence $\{q_n\}_{n \in \mathbb{N}}$ of positive numbers such that $1/2 < q_n < 1/\sqrt{2}$ for each $n \in \mathbb{N}$ and $\lim_{n \to \infty} q_n = 1/2$. Consider the set $B \subset \ell_2$,

$$B = \left\{ x + \sum_{n=1}^{\infty} (\alpha_n (e_2 + e_{n+2}) + \beta_n q_n (e_1 + e_2 + e_{n+2})) : \|x'\|_2 + \|x_1, x_2\|_2 + \|\alpha\|_1 + \|\beta\|_1 \leq 1 \right\}. \quad (6.6)$$

where $x \in \ell_2$, $\alpha, \beta \in \ell_1$ and $\| \cdot \|_1$ is the canonical norm in $\ell_1$. Obviously, $B$ is absolutely convex. Taking into account that $\|e_2 + e_{n+2}\|_2 = \sqrt{2}$ and $\|q_n (e_1 + e_2 + e_{n+2})\|_2 = q_n \sqrt{3} \leq \sqrt{3}/2$, we see that

$$\|u\|_2 \leq \sqrt{2} \quad \text{for any } u \in B. \quad (6.7)$$

Now if $\|u\|_2 \leq 1/2$, then $\|u'\|_2^2 + \|u_1\|^2 \leq 1/4$. An elementary application of the Cauchy inequality gives $\|u'\|_2 + \|(u_1, u_2)\|_2 \leq 1$. Taking $\alpha = \beta = 0$ and $x = u$, we see then that $u \in B$. Thus

$$u \in B \quad \text{if } \|u\|_2 \leq 1/2. \quad (6.8)$$

We consider the norm $\| \cdot \|$ on $\ell_2$ being the Minkowski functional of the set $B$. Formulae $(6.3)$ and $(6.4)$ imply that it is indeed a norm and that it is equivalent to the Hilbert space norm $\| \cdot \|_2$:

$$2^{-1/2} \|u\|_2 \leq \|u\|_2 \leq 2 \|u\|_2 \quad \text{for all } u \in \ell_2.$$
In particular, ℓ₂ endowed with the norm ∥·∥ is a reflexive Banach space. Using the definition of the Minkowski functional, we have that for u ∈ ℓ₂,

\[ \|u\| = \inf \left\{ \|x\|_2 + \|(x_1, x_2)\|_2 + \|\alpha\|_1 + \|\beta\|_1 : u = x + \sum_{n=1}^{\infty} (\alpha_n (e_2 + e_{n+2}) + \beta_n q_n (e_1 + e_2 + e_{n+2})) \right\}. \quad (6.5) \]

We shall show that B coincides with the closed unit ball with respect to the norm ∥·∥. Since B is bounded and absolutely convex, it suffices to show that B is closed in ℓ₂. First, note that the set

\[ B_1 = \{ x ∈ ℓ₂ : \|x\|_2 + \|(x_1, x_2)\|_2 ≤ 1 \} \quad (6.6) \]

is weakly compact and B₁ ⊆ B. Next, let B₂ = \( \overline{\text{aconv}} \{ e_2 + e_{n+2} : n ∈ \mathbb{N} \} \). Since the sequence \( e_2 + e_{n+2} \) converges weakly to \( e_2 \), Lemma 6.5 implies that \( B_2 \) is weakly compact and

\[ B_2 = \left\{ se_2 + \sum_{n=1}^{\infty} \alpha_n (e_2 + e_{n+2}) : |s| + \|\alpha\|_1 ≤ 1 \right\}. \quad (6.7) \]

It follows that \( B_2 ⊆ B \). Indeed, for \( u = se_2 + \sum_{n=1}^{\infty} \alpha_n (e_2 + e_{n+2}) \in B_2 \), one has just to take \( x = se_2 \) and \( \beta = 0 \) to see that \( u \in B \). Similarly, let \( B_3 = \overline{\text{aconv}} \{ q_n (e_1 + e_2 + e_{n+2}) : n ∈ \mathbb{N} \} \). Since the sequence \( q_n (e_1 + e_2 + e_{n+2}) \) converges weakly to \( (e_1 + e_2)/2 \), Lemma 6.5 implies that \( B_3 \) is weakly compact and

\[ B_3 = \left\{ \frac{t}{2} (e_1 + e_2) + \sum_{n=1}^{\infty} \beta_n q_n (e_1 + e_2 + e_{n+2}) : |t| + \|\beta\|_1 ≤ 1 \right\}. \quad (6.8) \]

As above, it is clear that \( B_3 ⊆ B \). Since B is absolutely convex, we have

\[ B_0 = \overline{\text{aconv}}(B_1 ∪ B_2 ∪ B_3) ⊆ B. \]

By Lemma 6.4 \( B_0 \) is weakly compact and

\[ B_0 = \{ ax + by + cw : x ∈ B_1, y ∈ B_2, w ∈ B_3, |a| + |b| + |c| ≤ 1 \}. \quad (6.9) \]

From formulae (6.9), (6.11) and (6.2) it follows that \( B ⊆ B_0 \). Hence \( B = B_0 \) and therefore \( B \) is weakly compact. Thus \( B \) is closed in ℓ₂ which ensures that \( B \) is the closed unit ball for the norm (6.5). It follows that the infimum in (6.5) is always attained and that we can write

\[ \|u\| = \min \left\{ \|x\|_2 + \|(x_1, x_2)\|_2 + \|\alpha\|_1 + \|\beta\|_1 : u = x + \sum_{n=1}^{\infty} (\alpha_n (e_2 + e_{n+2}) + \beta_n q_n (e_1 + e_2 + e_{n+2})) \right\}. \quad (6.10) \]

**Lemma 6.6.** The norm on ℓ₂ defined by (6.5) satisfies the following conditions:

1. \( \|e_2 + e_{n+2}\| = 1 \) for any \( n ∈ \mathbb{N} \);
2. \( q_n \|e_1 + e_2 + e_{n+2}\| = 1 \) for any \( n ∈ \mathbb{N} \).

**Proof.** Taking \( x = 0, \beta = 0 \) and \( \alpha = e_n \), we see that \( e_2 + e_{n+2} ∈ B \). Hence \( \|e_2 + e_{n+2}\| ≤ 1 \). Assume that \( \|e_2 + e_{n+2}\| < 1 \). Then there exist \( x ∈ ℓ₂ \) and \( \alpha, \beta ∈ ℓ₁ \) such that

\[ \|x\|_2 + \|(x_1, x_2)\|_2 + \|\alpha\|_1 + \|\beta\|_1 < 1 \quad (6.11) \]

and

\[ e_2 + e_{n+2} = x + \sum_{k=1}^{\infty} (\alpha_k (e_2 + e_{k+2}) + \beta_k q_k (e_1 + e_2 + e_{k+2})). \]

Taking the inner product of both sides of the above equality with \( e_2 \), we obtain

\[ 1 = x_2 + \sum_{k=1}^{\infty} (\alpha_k + q_k \beta_k). \]
we see arrive to the following equality in $K$

On the other hand, $q$

Hence $|x_2| + \sum_{k=1}^{\infty}(|\alpha_k| + q_k|\beta_k|) \geq 1$,

which contradicts (6.11). This contradiction proves (6.13).

Taking $x = 0$, $\alpha = 0$ and $\beta = e_n$, we see that $q_n(e_1 + e_2 + e_{n+2}) \in B$. Hence $q_n\|e_1 + e_2 + e_{n+2}\| \leq 1$. Assume that $q_n\|e_1 + e_2 + e_{n+2}\| < 1$. Then there exist $x \in \ell_2$ and $\alpha, \beta \in \ell_1$ such that

$$q_n(e_1 + e_2 + e_{n+2}) = x + \sum_{k=1}^{\infty} (\alpha_k(e_2 + e_{k+2}) + \beta_k q_k(e_1 + e_2 + e_{k+2})).$$

and (6.11) is satisfied. Taking the inner product of both sides of the above equality with $e_{n+2}$, $e_1$ and $e_2$ we obtain the following equality in $K^3$:

$$q_n(1 - \beta_n)(1, 1, 1) = (x_{n+2}, x_1, x_2) + \alpha_n(1, 0, 1) + \sum_{k \neq n} (\alpha_k(0, 0, 1) + \beta_k q_k(0, 1, 1)) =$$

$$= (x_{n+2}, x_1, x_2) + \alpha_n(1, 0, 1) + \sum_{k \neq n} (\alpha_k(0, 1) + \beta_k q_k(1, 1)).$$  (6.12)

Note that $\|(0, 1)\|_2 = 1$ and $\|q_k(1, 1, 1)\|_2 \leq 1$ since $q_k \leq 1/\sqrt{2}$. Using (6.12) and the triangle inequality, we obtain

$$\|(\tau, \sigma)\|_2 \leq \|(x_1, x_2)\|_2 + \sum_{k \neq n} (|\alpha_k| + |\beta_k|).$$

From the last display together with (6.11) and (6.12), it follows that

$$q_n(1 - \beta_n)(1, 1, 1) = ((x_{n+2} + \alpha_n)e_{n+2}, \tau, \sigma + \alpha_n), \text{ where } |x_{n+2}| + |\alpha_n| + |\beta_n| + \|(\tau, \sigma)\|_2 < 1.$$

Dividing by $1 - \beta_n$ and denoting $y = x_{n+2}/(1 - \beta_n)$, $a = \alpha_n/(1 - \beta_n)$, $r = \tau/(1 - \beta_n)$ and $p = \sigma/(1 - \beta_n)$ we see arrive to the following equality in $K^3$:

$$(q_n, q_n, q_n) = (y + a, r, p + a), \text{ where } |y| + |a| + \sqrt{|r|^2 + |p|^2} < 1.$$

Hence $r = q_n, p = y = q_n - a$ and

$$|a| + |q_n - a| + \sqrt{|q_n|^2 + |q_n - a|^2} < 1.$$  (6.13)

On the other hand, $q_n > 1/2$ and therefore $|a| + |q_n - a| \geq q_n > 1/2$ and $\sqrt{|q_n|^2 + |q_n - a|^2} \geq q_n > 1/2$. Hence $|a| + |q_n - a| + \sqrt{|q_n|^2 + |q_n - a|^2} > 1$ which contradicts (6.13). This contradiction completes the proof of (6.12).

**Remark.** In a similar way one can show that $\|u\| = \|u\|_2$ if either $u_1 = u_2 = 0$ or $u$ belongs to the linear span of $e_1$ and $e_2$.

Now we consider the operator $S \in L(\ell_2)$ defined by the formula $Su = u + u_2 e_1$. Clearly $S$ is the sum of the identity operator and a bounded rank 1 operator $T u = u_2 e_1$. Obviously, $T^2 = 0$. Proposition 1.10 will be proved if we verify that $\|S\| = 2$ and $S$ does not attain its norm.

**Lemma 6.7.** For any non-zero $u \in \ell_2$, $\|Su\| < 2\|u\|$.

**Proof.** Let $u \in \ell_2$ be such that $\|u\| = 1$. It suffices to show that $\|Su\| < 2$. Since $\|u\| = 1$, from (6.10) it follows that there are $x \in \ell_2$ and $\alpha, \beta \in \ell_1$ such that

$$u = x + \sum_{n=1}^{\infty} (\alpha_n(e_2 + e_{n+2}) + \beta_n q_n(e_1 + e_2 + e_{n+2})), \quad (6.14)$$

$$\|x\|_2 + \|(x_1, x_2)\|_2 + \|\alpha\|_1 + \|\beta\|_1 = 1.$$  (6.15)
Next, from (6.13) and the definition of $S$, we obtain that

$$Su = x + \tau e_1 + \sum_{n=1}^{\infty} q_n (\beta_n + q_n^{-1} \alpha_n) (e_1 + e_2 + e_{n+2}),$$

where $\tau = x_2 + \sum_{n=1}^{\infty} q_n \beta_n$.

Using (6.5), we see that

$$\|Su\| \leq \|x'\|_2 + \|(x_1 + \tau, x_2)\|_2 + \sum_{n=1}^{\infty} |\beta_n + q_n^{-1} \alpha_n|.$$  

From the definition of $\tau$ it follows that

$$\|Su\| \leq \|x'\|_2 + \|(x_1 + x_2, x_2)\|_2 + \sum_{n=1}^{\infty} ((1 + q_n)|\beta_n| + q_n^{-1} |\alpha_n|).$$

Taking into account that the norm of the operator with the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ acting on the 2-dimensional Hilbert space $\mathbb{K}^2$ equals $\left(\frac{3+\sqrt{5}}{2}\right)^{1/2} < \frac{5}{4}$, we have $\|(x_1 + x_2, x_2)\|_2 \leq \frac{5}{4} \|(x_1, x_2)\|_2$. Substituting this into the last display and taking into account that $1 + q_n \leq 1 + 2^{-1/2} < \frac{7}{4}$, we obtain

$$\|Su\| \leq \|x'\|_2 + \frac{5}{3} \|(x_1, x_2)\|_2 + \frac{7}{4} ||\beta||_1 + \sum_{n=1}^{\infty} q_n^{-1} |\alpha_n|.$$  

Since the coefficients in the above display in front of $\|x'\|_2$, $\|(x_1, x_2)\|_2$, $||\beta||_1$ and each $|\alpha_n|$ are all strictly less than 2, formula (6.13) implies that $\|Su\| < 2$. \qed

Now, observe that $S(e_2 + e_{n+2}) = e_1 + e_2 + e_{n+2}$. By Lemma 6.6 $\|e_2 + e_{n+2}\| = 1$ and $\|e_1 + e_2 + e_{n+2}\| = q_n^{-1}$. Hence $\|S\| \geq q_n^{-1}$ for any $n \in \mathbb{N}$. Since $q_n^{-1} \to 2$ as $n \to \infty$, we have $\|S\| \geq 2$. Thus from Lemma 6.7 it follows that $\|S\| = 2$ and $S$ does not attain its norm. This completes the proof of Proposition 1.10.

7 Concluding remarks

1. A more general approach to study the class $W$ is to consider the following property.

**Definition 3.** We say that a Banach space $X$ is tame if for any $y \in X$, $x \in X \setminus \{0\}$ and any sequence $\{u_n\}_{n \in \mathbb{N}}$ in $X$ weakly convergent to zero,

$$\lim_{n \to \infty} \frac{\|y + u_n\|}{\|x + u_n\|} \leq \max \left\{ 1, \frac{\|y\|}{\|x\|} \right\}. \quad (7.1)$$

It is easy to see that $p$-spaces for $1 < p < \infty$ are tame.

**Proposition 7.1.** Let $X$ be a reflexive Banach space such that either $X$ or $X^*$ is tame. Then $X \in W$.

**Proof.** According to Corollary 4.3, it is sufficient to consider the case, when $X$ is tame. Let $T \in K(X)$ and $\|I + T\| = c > 1$. Since $X$ is reflexive, we can pick a sequence $\{x_n\}_{n \in \mathbb{N}}$ of elements of $X$ such that $\|x_n\| \to 1$, $\|(I + T)x_n\| \to c$ and $\{x_n\}$ is weakly convergent to $x \in X$. By Lemma 3.5 $x \neq 0$. Since the sequence $u_n = x_n - x$ is weakly convergent to 0 and $T$ is compact, $Tx_n$ is norm-convergent to $Tx$. Hence $\|x + Tx + u_n\| = \|x + Tx\| \to c$. Since $X$ is tame, we have

$$c = \lim_{n \to \infty} \frac{\|x + Tx + u_n\|}{\|x + u_n\|} \leq \max \left\{ 1, \frac{\|x + Tx\|}{\|x\|} \right\}.$$  

The inequality $c > 1$ and the last display imply that $\|x + Tx\| \geq c|x|$. Taking into account that $c = \|I + T\|$, we see that $I + T$ attains its norm on $x/\|x\|$. \qed
Unfortunately, it seems there are no known examples of tame Banach spaces which are not \( p \)-spaces. This naturally leads to the problem of characterizing the tame spaces.

2. Analyzing the proof of Theorem 1.6, one can easily see that if \( 1 < p < \infty \), \( X \) is a \( p \)-space and \( T \in \mathcal{K}(X) \) is such that \( \|I + K\| > 1 \), then whenever \( \{x_n\}_{n \in \mathbb{N}} \) is a sequence of elements of \( X \) weakly converging to \( x \in X \) and satisfying \( \|x_n\| \to 1 \), \( \|(I + T)x_n\| \to \|I + T\| \) as \( n \to \infty \), then \( \{x_n\} \) is norm convergent to \( x \) and \( I + T \) attains its norm on \( x \).

3. In a recent paper [25], Shargorodsky and the author constructed a strictly convex reflexive Banach space \( X \) and \( S \in L(X) \) such that for some \( \varepsilon > 0 \), the level set \( \Sigma_x(S) \setminus \sigma_x(S) \) has non-empty interior. The space \( X \) constructed in [25] is an \( \ell_2 \)-direct sum of a countable family of finite dimensional Banach spaces. Thus by Theorem 1.5 and 1.6, any subspace of any \( \ell_2 \)-direct sum of \( X \) attains its norm on \( x \) for any \( T \in L(X) \). This observation shows that there is no relation between validity of (1.5) and meagreness of the level sets of the norm of the resolvent.

4. Let \( 1 < p < \infty \). As it follows from Proposition 1.3 and Theorem 1.6, any subspace of any \( \ell_p \)-direct sum \( X \) of a family of finite dimensional Banach spaces belongs to \( \mathcal{W} \). Applying Corollary 1.3, one can easily see that same holds true for quotients of \( X \) as well. Indeed, \( X^* \) is naturally isometrically isomorphic to any \( \ell_p \)-direct sum of a family of finite dimensional Banach spaces, where \( \frac{1}{p} + \frac{1}{q} = 1 \). Moreover, for any closed linear subspace \( Y \) of \( X \), \( (X/Y)^* \) is naturally isometrically isomorphic to a subspace of \( X^* \).

5. It would be interesting to figure out which classical Banach spaces do belong to the class \( \mathcal{W} \). A good starting point would be to address the spaces \( L_p[0, 1] \) for \( 1 < p < \infty \). There is a strong indication against their membership in \( \mathcal{W} \) for \( p \neq 2 \). Namely, it is easy to show that \( L_p[0, 1] \) for \( p \neq 2 \) is not tame.

6. The compactness condition in Propositions 1.3 and 1.2 can be replaced by the weaker condition of \( T \) being strictly singular. The proofs work without any changes.

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