GLOBAL EXISTENCE AND CONVERGENCE FOR THE CR $Q$-CURVATURE FLOW IN A CLOSED STRICTLY PSEUDOCONVEX CR 3-MANIFOLD

*SHU-CHENG CHANG$^1$, *TING-JUNG KUO$^2$, AND TAKANARI SAOTOME$^3$

**Abstract.** In this note, we affirm the partial answer to the long open Conjecture which states that any closed strictly pseudoconvex CR 3-manifold admits a contact form $\theta$ with the vanishing CR $Q$-curvature. More precisely, we deform the contact form according to an CR analogue of $Q$-curvature flow in a closed strictly pseudoconvex CR 3-manifold $(M, J, [\theta_0])$ of the vanishing first Chern class $c_1(T_{1,0}M)$. Suppose that $M$ is embeddable and the CR Paneitz operator $P_0$ is nonnegative with kernel consisting of the CR pluriharmonic functions. We show that the solution of CR $Q$-curvature flow exists for all time and has smoothly asymptotic convergence on $M \times [0, \infty)$. As a consequence, we obtain a contact form of vanishing CR $Q$-curvature.

1. Introduction

A strictly pseudoconvex CR $(2n+1)$-manifold is called pseudo-Einstein if the pseudohermitian Ricci curvature tensor is function-proportional to its Levi metric for $n \geq 2$. This is trivial for $n = 1$. However, this is equivalent to saying that the following quantity is vanishing ($[\text{L}]$, $[\text{H}]$)

$$W_\alpha \equiv (R_\alpha - inA_{\alpha\beta}^\beta, \beta) = 0.$$  

Then the pseudo-Einstein condition can be replaced by ($[\text{L}]$) for all $n \geq 1$. From this, one can define ($[\text{H}]$, $[\text{FH}]$, $[\text{CCC}]$) the CR $Q$-curvature for $n = 1$ by

$$Q := -\frac{4}{3} \text{Re}(R_1 - iA_{11,\overline{1}}) = -\frac{2}{3}(W_{1\overline{1}} + W_{\overline{1}\overline{1}}).$$
J. Lee (L1) showed an obstruction to the existence of a pseudo-Einstein contact form \( \theta \), which is the vanishing of first Chern class \( c_1(T_{1,0}M) \) for a closed strictly pseudoconvex \((2n + 1)\)-manifold \((M, J, \theta)\). Moreover, there is an invariant contact form \((\mathbb{F} \mathbb{H})\) if and only if it is pseudo-Einstein \((L1)\) for all \( n \geq 1 \).

**Conjecture 1.1.** \((L1)\) Any closed strictly pseudoconvex CR \((2n + 1)\)-manifold of the vanishing first Chern class \( c_1(T_{1,0}M) \) admits a global pseudo-Einstein structure.

We observe that the CR \( Q \)-curvature is vanishing when it is pseudo-Einstein. On the other hand, it is unknown whether there is any obstruction to the existence of a contact form \( \theta \) of vanishing CR \( Q \)-curvature \((\mathbb{F} \mathbb{H}), \mathbb{C} \mathbb{C} \mathbb{C})\). We also note that

\[
(1.3) \quad \int_M Q d\mu = 0.
\]

Here \( d\mu = \theta \wedge (d\theta) \) is the volume form. Then we have the following conjecture as well.

**Conjecture 1.2.** \((\mathbb{F} \mathbb{H})\) Let \((M^3, J, \theta_0)\) be a closed strictly pseudoconvex CR 3-manifold. There exists a contact form \( \theta \) in the conformal contact class \([\theta_0]\) with the vanishing CR \( Q \)-curvature.

We will often denote a strictly pseudoconvex CR 3-manifold by \((M, J, [\theta_0])\) with its contact class (or equivalently its underlying contact bundle) indicated. For a real-valued function \( \lambda \) in \( M^3 \), we consider a conformal transformation \( \theta = e^{2\lambda} \theta_0 \). Under this conformal change, it is known that we have the following transformation laws:

\[
(1.4) \quad P = e^{-4\lambda} P_0
\]

and

\[
(1.5) \quad Q = e^{-4\lambda}(Q_0 + 2P_0 \lambda),
\]

where \( P_0 \) and \( Q_0 \) denote the CR Paneitz operator and the CR \( Q \)-curvature with respect to \((M, J, \theta_0)\), respectively. Now we decompose

\[
Q_0 = (Q_0)_{\ker} \oplus Q_0^\perp
\]
with respect to $P_0$, where $Q^\perp_0$ denotes the component of $Q_0$ which is perpendicular to $\ker P_0$ and $(Q_0)_{\ker}$ denotes the component of $Q_0$ in $\ker P_0$. From (1.5), we observe that if $Q = 0$ for $\theta = e^{2\lambda} \theta_0$, then

$$0 = Q_0 + 2P_0 \lambda = (Q_0)_{\ker} \oplus (Q^\perp_0 + 2P_0 \lambda) \in (\ker P_0) \oplus (\ker P_0)^\perp.$$ 

Hence the kernel part of the background CR $Q_0$-curvature must be vanishing:

$$\text{(1.6)} \quad (Q_0)_{\ker} = 0.$$ 

In this paper, via the CR $Q$-curvature flow (1.8) as below, we affirm the partial answer to Conjecture 1.2 in a closed strictly pseudoconvex CR 3-manifold $(M, J, [\theta_0])$ of the vanishing first Chern class $c_1(T_{1,0}M)$. We refer to Theorem 5.1, Theorem 1.1, Theorem 1.3 and Theorem 1.4 for details.

We observe that the vanishing of $c_1(T_{1,0}M)$ is required to ensure that the background assumption (1.6) holds for the flow (1.8) as in Theorem 6.2. Therefore the limit contact form $\theta_\infty = e^{2\lambda_\infty} \theta_0$ has the vanishing CR $Q_\infty$-curvature:

$$Q_\infty = 0.$$ 

We first define the functional $\mathcal{E}$ on a closed strictly pseudoconvex CR 3-manifold $(M, J, [\theta_0])$ by

$$\text{(1.7)} \quad \mathcal{E}(\theta) = \int_M P_0 \lambda \cdot \lambda d\mu_0 + \int_M Q_0 \lambda d\mu_0$$

with $\theta = e^{2\lambda} \theta_0$ and $d\mu_0 = \theta_0 \wedge d\theta_0$. Then, for minimizing $\mathcal{E}(\theta)$ in $[\theta_0]$, we consider the fourth-order CR $Q$-curvature flow in a closed strictly pseudoconvex CR 3-manifold $(M, J, [\theta_0])$:

$$\text{(1.8)} \quad \left\{ \begin{array}{l}
\frac{\partial \lambda}{\partial t} = -(Q^\perp_0 + 2P_0 \lambda) + r, \\
\theta = e^{2\lambda} \theta_0; \ \lambda(p, 0) = \lambda_0(p), \\
\int_M e^{4\lambda_0} d\mu_0 = \int_M d\mu_0.
\end{array} \right.$$
Here
\[ r = \frac{\int_M (Q_0^\perp + 2P_0\lambda) d\mu}{\int_M d\mu}. \]

Note that \( d\mu = e^{4\lambda} d\mu_0 \) and \( V = \int_M d\mu \) is invariant under the flow \((1.8)\). Moreover, \((1.8)\) is the (volume normalized) negative gradient flow of \( E(\theta) \). That is (see Lemma 4.2)
\[ \frac{d}{dt} E(\theta) = -\int_M (Q_0^\perp + 2P_0\lambda)^2 d\mu_0. \]

**Remark 1.1.**

1. Unlike the Riemannian \( Q \)-curvature flow, the space of kernel of the CR Paneitz operator \( P_0 \) is infinite dimensional, containing all CR-pluriharmonic functions (definition 1.1). That is one of reasons why we need to consider the CR \( Q \)-curvature flow \((1.8)\) along direction of the orthogonal complement of \( \ker P_0 \).

2. We recall that \( \theta_0 \) is called an invariant contact form on \( M \) if it is locally given as an invariant contact form for some local embedding of \( M \) into \( \mathbb{C}^2 \). A more intrinsic formulation of an invariant contact form is : if it is locally volume normalized with respect to a closed 2-form on \( M \) \((\text{[L1]})\). In the paper \([\text{FH}]\), the authors proved that the \( Q \)-curvature of an invariant contact form vanishes. Indeed if \( M \) is a real hypersurface in \( \mathbb{C}^2 \), then \( M \) admits an invariant contact form \( \theta_0 \) so that \( Q_0 = 0 \) on \( M \). In general, there is a topological obstruction for the global existence of an invariant contact form \( \theta_0 \) (or pseudo-Einstein contact form on \( M^{2n+1} \) if \( n \geq 2 \)) which implies vanishing of the first Chern class \( c_1(T_{1,0}) \) of the holomorphic tangent bundle \( T_{1,0}(M) \) \((\text{[L1], [FH]})\).

3. The notions of Paneitz operator \( P \) and \( Q \)-curvature were initially introduced on a Riemannian manifold, and were considered as a kind of generalization of Laplacian and Gaussian curvature on a two-dimensional manifold, respectively \((\text{[B], [P]})\).

4. By using the method of Riemannian \( Q \)-curvature flow, the related problems for the \( Q \)-curvature on a Riemannian 4-manifold are also addressed and studied as in \([\text{Br1}, \text{Br2}]\) and \([\text{MS}]\).

Concerning global existence and a priori estimates for the solution of \((1.8)\), we introduced the following notions.
Definition 1.1. ([L1]) (i) Let \((M, J, \theta)\) be a closed strictly pseudoconvex CR 3-manifold. We define

\[(P_1 \varphi)\theta^1 = (\varphi_{11}^1 + iA_{1\bar{1}}\varphi^1)\theta^1\]

which is an operator that characterizes CR-pluriharmonic functions. Here \(P_1 \varphi = \varphi_{11} + iA_{1\bar{1}}\varphi^1\). The CR Paneitz operator \(P\) with respect to \(\theta\) is defined by

\[(1.9) \quad P \varphi = 2 \left( \delta_b((P_1 \varphi)\theta^1) + \bar{\delta}_b((\overline{P_1 \varphi})\theta^1) \right),\]

where \(\delta_b\) is the divergence operator that takes \((1, 0)\)-forms to functions by \(\delta_b(\sigma_1\theta^1) = \sigma_1, 1\), and similarly, \(\bar{\delta}_b(\sigma_1\theta^1) = \sigma_{\bar{1}}\).

(ii) On a closed strictly pseudoconvex CR 3-manifold \((M, J, \theta)\), we call the CR Paneitz operator \(P\) with respect to \((J, \theta)\) nonnegative if

\[\int_M P \varphi \cdot \varphi \, d\mu \geq 0,\]

for all real \(C^\infty\) smooth functions \(\varphi\). We call the CR Paneitz operator \(P\) with respect to \((J, \theta)\) essentially positive if there exists a constant \(Y > 0\) such that

\[(1.10) \quad \int_M P \varphi \cdot \varphi \, d\mu \geq Y \int_M \varphi^2 \, d\mu,\]

for all real \(C^\infty\) smooth functions \(\varphi \perp \ker P\) (i.e. perpendicular to the kernel of \(P\) in the \(L^2\)-norm with respect to the volume form \(d\mu\)).

Remark 1.2. 1. Let \((M, J, \theta)\) be a closed strictly pseudoconvex CR 3-manifold. The positivity of \(P\) is a CR invariant in the sense that it is independent of the choice of the contact form \(\theta\) in a fixed contact class \([\theta_0]\).

2. Let \((M, J, \theta)\) be a closed strictly pseudoconvex CR 3-manifold with vanishing pseudohermitian torsion. Then the corresponding CR Paneitz operator is essentially positive ([CCC]).

3. ([H]) For a closed strictly pseudoconvex CR 3-manifold of vanishing pseudohermitian torsion (Sasakian), we have

\[(1.11) \quad \ker P_1 = \ker P_0.\]

In general for non-embeddable CR 3-manifolds, we only have

\[\ker P_1 \subset \ker P_0.\]
4. Let $P_1 \phi = 0$. If $M$ is the boundary of a connected strictly pseudoconvex domain $\Omega \subset \mathbb{C}^2$, then $\phi$ is the boundary value of a pluriharmonic function $u$ in $\Omega$. That is, $\partial \overline{\partial} u = 0$ in $\Omega$. Moreover, if $\Omega$ is simply connected, there exists a holomorphic function $w$ in $\Omega$ such that $\text{Re}(w) = u$ and $u|_M = \phi$.

5. The CR Paneitz operator $P$ with respect to $(J, \theta)$ is defined by

$$P \lambda = \Delta_b^2 \lambda + T^2 \lambda + 4 \text{Im}(A_{1111} \lambda_{11} + A_{1111} \lambda_1)$$

and the CR $Q$-curvature is defined by

$$Q = -\frac{2}{3}(\Delta_b R + 2 \text{Im} A_{1111}),$$

where $\Delta_b$, $T$, $R$, $A_{11}$ denote the sub-Laplacian, the Reeb vector field, the Tanaka-Webster scalar curvature, the pseudo Hermitian torsion with respect to $(J, \theta)$, respectively.

**Definition 1.2.** We say that condition $\ast$ is satisfied on a closed pseudohermitian manifold $(M, J, \theta)$ if there exist constants $0 \leq \epsilon < 1$ and $C(\epsilon) \geq 0$ such that

$$\int_M |\nabla_b^2 \phi|^2 d\mu \leq (2 + \epsilon) \int_M (\Delta_b \phi)^2 d\mu + C(\epsilon) \int_M \phi^2 d\mu,$$

for all real $C^\infty$ function $\phi \in (\ker P)^\perp$. Here we observe that condition $\epsilon < 1$ is crucial. Namely, due to the integral CR Bochner formula (3.2) we always have an inequality $\ast$ for $\epsilon > 1$ if the CR Paneitz operator $P$ is nonnegative.

Note that condition $\ast$ is satisfied on the standard CR 3-sphere $S^3$ with $\epsilon = 0$ and $C(\epsilon) = 0$ (see Lemma 3.1 in Section 3, also [CCC] (1.13)). Now we are ready to state our main results for the paper.

**Theorem 1.1.** Let $(M, J, [\theta_0])$ be a closed strictly pseudoconvex CR 3-manifold of $c_1(T_{1,0}M) = 0$. Suppose that the CR Paneitz operator $P_0$ is essentially positive with kernel consisting of the CR pluriharmonic functions and subelliptic on the orthogonal complement of $\ker P_0$. Then the solution of (1.8) exists on $M \times [0, \infty)$ and converges smoothly to $\lambda_\infty \equiv \lambda(\cdot, \infty)$ as $t \to \infty$. Moreover, the contact form $\theta_\infty = e^{2\lambda_\infty} \theta_0$ has the vanishing CR $Q$-curvature with

$$Q_\infty = 0.$$
Moreover, it follows from Proposition 3.1 and Theorem 1.1 that

**Theorem 1.2.** Let \((M, J, [\theta_0])\) be a closed strictly pseudoconvex CR 3-manifold of \(c_1(T_{1,0}M) = 0\). Suppose that condition (*) is satisfied and the CR Paneitz operator \(P_0\) is essentially positive with kernel consisting of the CR pluriharmonic functions. Then the solution of (1.8) exists on \(M \times [0, \infty)\) and converges smoothly to \(\lambda_\infty \equiv \lambda(\cdot, \infty)\) as \(t \to \infty\). Moreover, the contact form \(\theta_\infty = e^{2\lambda_\infty} \theta_0\) has the vanishing CR \(Q_\infty\)-curvature.

**Remark 1.3.** 1. In the paper of [Br1], S. Brendle considered the \(Q\)-curvature flow on a closed Riemannian manifold with weakly positive Paneitz operator and its kernel consisting of constant functions that corresponds to our assumption (II). However for the CR Paneitz operator \(P\), the space of \(\ker P\) is infinite dimensional ([GL], [CTW]).

2. We observe that a small deformation of the standard CR 3-sphere \((S^3, J, \theta_0)\) may or may not be embeddable but still satisfies condition (*).

However, we observe that if \(M\) is embeddable, it follows from Corollary 3.1 that the CR Paneitz operator \(P_0\) is subelliptic on the orthogonal complement of \(\ker P\). In additional, if the CR Paneitz operator \(P_0\) is nonnegative, then in fact it is essentially positive due to \(P_0\) is a closed operator ([CC]).

**Theorem 1.3.** Let \((M, J, [\theta_0])\) be a closed strictly pseudoconvex CR 3-manifold of \(c_1(T_{1,0}M) = 0\). Suppose that \(M\) is embeddable and the CR Paneitz operator \(P_0\) is nonnegative with kernel consisting of the CR pluriharmonic functions. Then the solution of (1.8) exists on \(M \times [0, \infty)\) and converges smoothly to \(\lambda_\infty \equiv \lambda(\cdot, \infty)\) as \(t \to \infty\). Moreover, the contact form \(\theta_\infty = e^{2\lambda_\infty} \theta_0\) has the vanishing CR \(Q_\infty\)-curvature.

Note that if the pseudohermitian torsion is vanishing, then \(M\) is embeddable and the CR Paneitz operator \(P_0\) is nonnegative ([Le], [CCC]). Then we recapture the first named author’s previous main result ([CCC]) where we made a much stronger assumption of vanishing torsion.

**Corollary 1.1.** Let \((M, J, [\theta_0])\) be a closed strictly pseudoconvex CR 3-manifold of \(c_1(T_{1,0}M) = 0\) and vanishing torsion. Then the solution of (1.8) exists on \(M \times [0, \infty)\)
and converges smoothly to \( \lambda_\infty \equiv \lambda(\cdot, \infty) \) as \( t \to \infty \). Moreover, the contact form \( \theta_\infty = e^{2\lambda_\infty} \theta_0 \) has the vanishing CR \( Q_\infty \)-curvature.

Finally, we recall the following definition.

**Definition 1.3.** The CR Yamabe constant is defined by

\[
\sigma(M, J) = \inf_{\varphi \neq 0} \frac{E_\theta(\varphi)}{(\int_M \varphi^4 d\mu)^{1/2}},
\]

where \( E_\theta \) denoted by

\[
E_\theta(\varphi) = \int_M |\nabla_\theta \varphi|^2 d\mu + \frac{1}{4} \int_M W \varphi^2 d\mu.
\]

Note that \( E_\theta(\varphi) = E_\theta(u\varphi) \) for \( \hat{\theta} = u^2 \theta \). This implies that \( \sigma(M, J) \) is a CR invariant.

In [CCY], the authors showed that under the condition of nonnegativity of \( P_0 \), the embeddability of \( M \) (or equivalently the closedness of the range of \( \Box_\theta \)) follows from the positivity of the Tanaka-Webster curvature. It is known that there always exists a contact form \( \theta_0 \) in \([\theta]\) with the positive Tanaka-Webster curvature if \( \sigma(M, J) > 0 \).

**Theorem 1.4.** Let \((M, J, [\theta_0])\) be a closed strictly pseudoconvex CR 3-manifold of \( c_1(T_{1,0}M) = 0 \) and the positive CR Yamabe constant. Suppose that the CR Paneitz operator \( P_0 \) is nonnegative with kernel consisting of the CR pluriharmonic functions. Then the solution of (1.8) exists on \( M \times [0, \infty) \) and converges smoothly to \( \lambda_\infty \equiv \lambda(\cdot, \infty) \) as \( t \to \infty \). Moreover, the contact form \( \theta_\infty = e^{2\lambda_\infty} \theta_0 \) has the vanishing CR \( Q_\infty \)-curvature.

Main difficulties for work of the CR \( Q \)-curvature flow is because the CR Bochner formula (see Lemma 3.1) and CR Paneitz operator are entirely different from the Riemannian version. In order to get uniformly \( S^{kk,2} \)-estimates (see section 2 for the definition of Folland-Stein norms \( S^{k,p} \)), we need an additional analytic condition (*) plus a trick from the essentially positivity of CR Paneitz operator. In fact, Condition (*) will imply the subellipticity of CR Paneitz operator \( P_0 \) on the orthogonal complement of \( \ker P_0 \). Moreover, the essentially positivity of CR Paneitz operator will allow you to obtain the \( L^2 \)-norm estimate for a solution of the flow (1.8) on the orthogonal complement of \( \ker P_0 \).
We briefly describe the methods used in our proofs. In section 2, we will briefly describe the basic notions of a pseudohermitian 3-manifold. In Section 3, we discuss about an analytic property of the CR Paneitz operator $P$ and prove the subellipticity of $P_0$ on $(\ker P_0)\perp$ under the condition ($\ast$). In section 4, we show the CR $Q$-curvature flow (1.8) is a negative gradient flow of $E$. Furthermore, we derive the uniformly $S^{4k,2}$-norm estimate for the solution of (1.8). In section 5, we prove the long time existence and asymptotic convergence of the solution of (1.8) on $M \times [0, \infty)$. Finally, in section 6, we derive the key Bochner-type formulae in a closed strictly pseudo convex CR 3-manifold of the vanishing first Chern class $c_1(T_{1,0}M)$ and then prove the main results.

2. Preliminary

We introduce some basic materials in a pseudohermitian 3-manifold (see [L1], [L2] for more details). Let $M$ be a closed 3-manifold with an oriented contact structure $\xi$. There always exists a global contact form $\theta$ with $\xi = \ker \theta$, obtained by patching together local ones with a partition of unity. The Reeb vector field of $\theta$ is the unique vector field $T$ such that $\theta(T) = 1$ and $\mathcal{L}_T \theta = 0$ or $d\theta(T, \cdot) = 0$. A CR structure compatible with $\xi$ is a smooth endomorphism $J : \xi \rightarrow \xi$ such that $J^2 = -Id$. The CR structure $J$ can extend to $\mathbb{C} \otimes \xi$ and decomposes $\mathbb{C} \otimes \xi$ into the direct sum of $T_{1,0}$ and $T_{0,1}$ which are eigenspaces of $J$ with respect to $i$ and $-i$, respectively. A CR structure is called integrable, if the condition $[T_{1,0}, T_{1,0}] \subset T_{1,0}$ is satisfied. A pseudohermitian structure compatible with $\xi$ is an integrable CR structure $J$ compatible with $\xi$ together with a global contact form $\theta$.

Let $\{T, Z_1, Z_{\bar{1}}\}$ be a frame of $TM \otimes \mathbb{C}$, where $Z_1$ is any local frame of $T_{1,0}$. $Z_1 = \overline{Z_1} \in T_{0,1}$. Then $\{\theta, \theta^1, \theta^{\bar{1}}\}$, the coframe dual to $\{T, Z_1, Z_{\bar{1}}\}$, satisfies

$$d\theta = ih_{1\bar{1}}\theta^1 \wedge \theta^{\bar{1}}$$

for some nonzero real function $h_{1\bar{1}}$. If $h_{1\bar{1}}$ is positive, we call $(M, J, \theta)$ a closed strictly pseudoconvex CR 3-manifold, and we can choose a $Z_1$ such that $h_{1\bar{1}} = 1$; hence, throughout this paper, we assume $h_{1\bar{1}} = 1$.

The pseudohermitian connection of $(J, \theta)$ is the connection $\nabla$ on $TM \otimes \mathbb{C}$ (and extended to tensors) given in terms of a local frame $Z_1 \in T_{1,0}$ by
\[ \nabla Z_1 = \theta_1^1 \otimes Z_1, \quad \nabla \bar{Z}_1 = \theta_1^{\bar{1}} \otimes \bar{Z}_1, \quad \nabla T = 0, \]

where \( \theta_1^1 \) is the 1-form uniquely determined by the following equations:

\[
\begin{align*}
\frac{\partial}{\partial z_1} & = \theta_1^1 \wedge \theta_1^{\bar{1}} + \theta \wedge \tau^1 \\
\tau^1 & \equiv 0 \mod \theta_1^{\bar{1}} \\
0 & = \theta_1^1 + \theta_1^{\bar{1}},
\end{align*}
\]

where \( \tau^1 \) is the pseudohermitian torsion. Put \( \tau^1 = A_1^1 \theta_1^1 \). The structure equation for the pseudohermitian connection is

\[
d\theta_1^1 = R \theta_1^1 \wedge \theta_1^1 + 2i \text{Im}(A_1^1 \theta_1^1 \wedge \theta),
\]

where \( R \) is the Tanaka-Webster curvature.

We will denote components of covariant derivatives with indices preceded by comma; thus write \( A_1^1 \theta_1^1 \wedge \theta \). The indices \( \{0, 1, \bar{1}\} \) indicate derivatives with respect to \( \{T, Z_1, \bar{Z}_1\} \). For derivatives of a scalar function, we will often omit the comma, for instance, \( \phi_1 = Z_1 \phi, \phi_{1\bar{1}} = Z_{\bar{1}} \phi - \theta_1^1(Z_1) Z_{\bar{1}} \phi, \phi_0 = T \phi \) for a (smooth) function.

For a real-valued function \( \phi \), the subgradient \( \nabla_{b\phi} \) is defined by \( \nabla_{b\phi} \in \xi \otimes \mathbb{C} = T_{1,0} \oplus T_{0,1} \) and \(-2d\theta(iZ, \nabla_{b\phi}) = d\phi(Z)\) for all vector fields \( Z \) tangent to contact plane. Locally \( \nabla_{b\phi} = \phi_1 Z_1 + \phi_1 \bar{Z}_1 \).

We can use the connection to define the sub-Hessian \( \nabla_{b\phi}^2 = (\nabla^H)^2 \) as the complex linear map

\[
(\nabla^H)^2 \phi : T_{1,0} \oplus T_{0,1} \to T_{1,0} \oplus T_{0,1}
\]

and

\[
(\nabla^H)^2 \phi(Z) = \nabla_Z \nabla_{b\phi}.
\]

The sub-Laplacian \( \Delta_{b\phi} \) defined as the trace of the subhessian. That is

\[
\Delta_{b\phi} \phi = Tr \left( (\nabla^H)^2 \phi \right) = (\phi_{1\bar{1}} + \phi_{1\bar{1}}).
\]
The Levi form $\langle \cdot, \cdot \rangle_{J,\theta}$ is the Hermitian form on $T_{1,0}$ defined by
\[ \langle Z, W \rangle_{J,\theta} = -i \langle d\theta, Z \wedge \overline{W} \rangle. \]
We can extend $\langle \cdot, \cdot \rangle_{J,\theta}$ to $T_{0,1}$ by defining $\langle Z, W \rangle_{J,\theta}$ for all $Z, W \in T_{1,0}$.

The Levi form induces naturally a Hermitian form on the dual bundle of $T_{1,0}$, and hence on all the induced tensor bundles. Integrating the hermitian form (when acting on sections) over $M$ with respect to the volume form $d\mu = \theta \wedge d\theta$, we get an inner product on the space of sections of each tensor bundle. More precisely, we denote the Levi form $\langle \cdot, \cdot \rangle_{J,\theta}$ by
\[ \langle V, U \rangle_{J,\theta} = 2d\theta(V, JU) = v_1 u_\bar{1} + \bar{v}_1 u_1, \]
for $V = v_1 Z_1 + v_1 Z_1, U = u_1 Z_1 + u_1 Z_1$ in $\xi$ and
\[ \langle V, U \rangle_{J,\theta} = \int_M \langle V, U \rangle_{J,\theta} \theta \wedge d\theta. \]

For a vector $X \in \xi$, we define $|X|^2 \equiv \langle X, X \rangle_{J,\theta}$. It follows that $|\nabla_b \varphi|^2 = 2\varphi_1 \varphi_{1\bar{1}}$ for a real valued smooth function $\varphi$. Also the square modulus of the sub-Hessian $\nabla^2_b \varphi$ of $\varphi$ reads $|\nabla^2_b \varphi|^2 = 2(\varphi_{11} \varphi_{\bar{1}\bar{1}} + \varphi_{1\bar{1}} \varphi_{\bar{1}1})$.

To consider smoothness for functions on strongly pseudo-convex manifolds, we recall below what the Folland-Stein space $S^{k,p}$ is. Let $D$ denote a differential operator acting on functions. We say $D$ has weight $m$, denoted $w(D) = m$, if $m$ is the smallest integer such that $D$ can be locally expressed as a polynomial of degree $m$ in vector fields tangent to the contact bundle $\xi$. We define the Folland-Stein space $S^{k,p}$ of functions on $M$ by
\[ S^{k,p} = \{ \varphi \in L^p \mid D\varphi \in L^p \text{ whenever } w(D) \leq k \}. \]
We define the $L^p$-norm of $\nabla_b \varphi, \nabla^2_b \varphi, \ldots$ to be $(\int |\nabla_b \varphi|^p \theta \wedge d\theta)^{1/p}, (\int |\nabla^2_b \varphi|^p \theta \wedge d\theta)^{1/p}, \ldots$, respectively, as usual. So it is natural to define the $S^{k,p}$-norm $||\varphi||_{S^{k,p}}$ of $\varphi \in S^{k,p}$ as follows:
\[ ||\varphi||_{S^{k,p}} \equiv \left( \sum_{0 \leq j \leq k} ||\nabla^j_b \varphi||_{L^p}^p \right)^{1/p}. \]

The function space $S^{k,p}$ with the above norm is a Banach space for $k \geq 0, 1 < p < \infty$. There are also embedding theorems of Sobolev type. For instance, $S^{2,2} \subset S^{1,4}$ (for
dim \( M = 3 \)). We refer the reader to, for instance, [FS2] and [Fo] for more discussions on these spaces.

3. Subelliptic Estimate for the CR Paneitz Operator

Suppose that \((M^{2n+1}, \theta, J)\) is a closed strictly pseudoconvex CR \((2n+1)\)-manifold. In [Bm], Boutet de Monvel proved that \(M\) can be embedded in \(\mathbb{C}^N\) for some \(N\) if \(n \geq 2\). In the case of \(\text{dim } M = 3\), D. Burns ([Bu]) showed that if the range of \(\bar{\partial}_b\) is closed in \(L^2(M)\), then the Boutet de Monvel’s construction works, and \(M\) can be embedded.

Conversely, using a microlocalization method, J. J. Kohn proved that if \(M^{2n+1}\) is a boundary of a bounded pseudo-convex domain \(\Omega \subset \mathbb{C}^{2n}\) then the range of \(\bar{\partial}_{b}\) is closed in \(L^2(M)\) ([K1], [K2]). In particular, any closed pseudohermitian 3-manifold with vanishing torsion has a tangential CR operator \(\bar{\partial}_{b}\) which has a closed range in \(L^2(M)\) (Theorem 2.1 of [Le]).

We first recall an integral CR Bochner formula as following.

**Lemma 3.1.** ([CCC]) Let \((M, J, \theta)\) be a closed strictly pseudoconvex CR 3-manifold. Then for any \(\lambda \in C^\infty(M)\),

\[
0 = \int_M (\Delta_b \lambda)^2 d\mu - \int_M |\nabla^2_b \lambda|^2 d\mu + 2 \int_M (\lambda_0)^2 d\mu - \int_M R |\nabla_b \lambda|^2 d\mu + 2 \text{Im} \int_M A_\Pi \lambda_1 \lambda_1 d\mu
\]

and

\[
2 \int_M P \lambda \cdot \lambda d\mu = 3 \int_M (\Delta_b \lambda)^2 d\mu - 3 \int_M |\nabla^2_b \lambda|^2 d\mu - \int_M R |\nabla_b \lambda|^2 d\mu_0 - 6 \text{Im} \int_M A_\Pi \lambda_1 \lambda_1 d\mu_0.
\]

**Proposition 3.1.** Let \((M, J, \theta)\) be a closed strictly pseudoconvex CR 3-manifold. Then condition (*) for the CR Paneitz operator \(P\) implies the subellipticity of \(P\) on the orthogonal complement of \(\ker P\). That is

\[
\|\lambda\|^2_{\mathcal{S}_{k+4,2}} \leq C_k (\|P \lambda\|^2_{\mathcal{S}_{k,2}} + \|\lambda\|^2_{L^2}),
\]

for some constant \(C_k\) independent of \(\lambda \in (\ker P)^\perp\).
Proof. By the CR integral Bochner formula (Lemma 3.1), we have
\begin{equation}
(3.3) \quad \int_M |\nabla_b^2 \lambda|^2 \, d\mu = \int_M |\Delta_b \lambda|^2 \, d\mu + 2 \int_M |\lambda_0|^2 \, d\mu - \rho + 2\alpha.
\end{equation}

Here
\[ \alpha = \text{Im} \int_M A^{11} \lambda_1 \lambda_1 \, d\mu \quad \text{and} \quad \rho = \int_M R|\nabla \lambda|^2 \, d\mu. \]

On the other hand, by the definition of CR Paneitz operator
\begin{equation}
(3.4) \quad \int_M P \lambda \cdot \lambda \, d\mu = \int_M |\Delta_b \lambda|^2 \, d\mu - \int_M |\lambda_0|^2 \, d\mu - 8\alpha.
\end{equation}

By Young inequality, we can estimate the term of $\alpha$ and $\rho$ by
\begin{equation}
(3.5) \quad C_1 \alpha + C_2 \rho \leq \epsilon \int_M |\Delta_b \lambda|^2 \, d\mu + C(\epsilon) \int_M |\lambda|^2 \, d\mu.
\end{equation}

It follows from (3.3) and (3.4) that
\[ \int_M |\nabla_b^2 \lambda|^2 \, d\mu = 3 \int_M |\Delta_b \lambda|^2 \, d\mu - 2 \int_M P \lambda \cdot \lambda \, d\mu - \rho - 6\alpha. \]

Hence condition (*) implies
\[ (1 - \epsilon) \int_M |\Delta_b \lambda|^2 \, d\mu - 2 \int_M P \lambda \cdot \lambda \, d\mu - \rho - 6\alpha \leq 0. \]

Therefore from (3.5)
\[ (1 - \epsilon - \epsilon') \int_M |\Delta_b \lambda|^2 \, d\mu - 2 \int_M P \lambda \cdot \lambda \, d\mu - C(\epsilon, \epsilon') \int_M |\lambda|^2 \, d\mu \leq 0. \]

Since we can take $\epsilon'$ and $\epsilon$ arbitrary, we can assume that $0 < \epsilon_0 = (1 - \epsilon - \epsilon') < 1$. It follows that
\[ \int_M |\Delta_b \lambda|^2 \, d\mu \leq \frac{2}{\epsilon_0} \int_M P \lambda \cdot \lambda \, d\mu + C(\epsilon_0, \epsilon') \int_M |\lambda|^2 \, d\mu. \]

Since the sub-Laplacian $\Delta_b$ is subelliptic on $C^\infty(M)$, there exists a constant $C > 0$ such that
\[ (**) \quad ||\lambda||^2_{S^{2,2}} \leq C[(P \lambda, \lambda)_{L^2} + ||\lambda||^2_{L^2}]. \]

for $\lambda \in (\ker P)^\perp$.

Next it follows from [GL] that
\[ P = \Delta_b^2 + T^2 - 2S'. \]

Here
\[ S' \lambda = i((A_{11}^T \lambda^T) - (A_{11} \lambda^1)^1). \]
Indeed,

$$[\Delta_b, P]\lambda = [\Delta_b, \Delta^2_b + T^2 - 2S']\lambda = [\Delta_b, T^2]\lambda - 2[\Delta, S']\lambda.$$ 

Since the order of the operator $S'$ is 2, we can ignore the second part of the above. Moreover, we have

$$[\Delta_b, T^2]\lambda = [\Delta_b, T\lambda + T[\Delta_b, T]]\lambda = [(\text{Im}S)T + T(\text{Im}S)]\lambda,$$

for

$$S\lambda = 2i(A_\Pi \lambda^T)^T.$$ 

Hence we have

(3.6) $$\|[\Delta_b, P]\lambda\|_{L^2} \leq C\|\lambda\|_{S^{1,2}}.$$ 

It follow from (***) and (3.6) that

(3.7) $$\|\Delta_b\lambda\|_{S^{2,2}} \leq C([P \Delta_b\lambda, \Delta_b\lambda]_{L^2} + \|\Delta_b\lambda\|^2_{L^2})$$

$$\leq C((P\lambda, \Delta^2_b\lambda)L^2 + \epsilon\|\lambda\|^2_{S^{1,2}} + C(\epsilon)\|\lambda\|^2_{S^{2,2}})$$

$$\leq C((\epsilon + \epsilon')\|\lambda\|^2_{S^{1,2}} + C(\epsilon')\|P\lambda\|^2_{L^2} + C(\epsilon)\|\lambda\|^2_{S^{2,2}}).$$

These estimates show that

(3.8) $$\|\lambda\|^2_{S^{2,2}} \leq C([P\lambda]^2_{L^2} + \|\lambda\|^2_{L^2}),$$

for some constant $C$ independent of $\lambda \in (\ker P)^\perp$. Therefore one can obtain estimates of higher order derivatives from (3.8) and interpolation inequalities in the Folland-Stein spaces $S^{k,2}$ as following.

$$\|\lambda\|^2_{S^{k+4,2}} \leq C_k([P\lambda]^2_{S^{k,2}} + \|\lambda\|^2_{L^2}),$$

for some constant $C_k$ independent of $\lambda \in (\ker P)^\perp$.

Next, we recall that if the range of $\overline{\partial}_b$ is closed in $L^2(M)$, we have a subelliptic estimate for $\overline{\partial}_b$ on the orthogonal complement of $\ker \Box_b$ (K1, K2).

**Proposition 3.2.** Let $(M, J, \theta)$ be a closed strictly pseudoconvex CR 3-manifold. If the range of $\overline{\partial}_b$ is closed in $L^2(M)$, then the CR Paneitz operator $P$ is subelliptic on the orthogonal complement of $\ker P$. 

Proof. Since the Kohn Laplacian (acting on functions) satisfies \(\Box_b = -\Delta + iT\), we have

\[
\Delta_b^2 + T^2 = \Box_b \Box_b - i[\Delta_b, T] = \Box_b \Box_b + i[\Delta_b, T].
\]

Moreover, since it holds that \(i[\Delta_b, T] \lambda = 2i(A_{17} \lambda^T) + 2i(A_{11} \lambda^1)^{-1}\), the CR Paneitz operator can be written ([GL]) by

\[
P \lambda = \Box_b \Box_b \lambda - 2S \lambda = \Box_b \Box_b \lambda - 2S \lambda,
\]

where \(S \lambda = 2i(A_{17} \lambda^T)\).

We observe that the kernel of \(P_1\) is just the space of all CR pluriharmonic functions (see definition [L1]). Let \(w = u + iv \in \ker \Box_b, u, v \in C^\infty(M; \mathbb{R})\). By Lemma 3.1. of [L1],

\[
P_1 u = P_1 v = 0 \quad \text{and} \quad u, v \in \ker P.
\]

Therefore if \(\lambda \in (\ker P)^\perp \cap C^\infty(M; \mathbb{R})\), then it holds that

\[
(\lambda, w)_{L^2} = (\lambda, u)_{L^2} + i(\lambda, v)_{L^2} = 0.
\]

Since the CR Paneitz operator is a real operator, i.e. \(\overline{P \lambda} = P \lambda\), we have \(\ker P = \{\lambda + i\eta | \lambda, \eta \in \ker P \cap C^\infty(M; \mathbb{R})\}\). This concludes that

\[
(\ker P)^\perp \subset (\ker \Box_b)^\perp.
\]

Let \(\lambda \in (\ker P)^\perp \cap C^\infty(M; \mathbb{R})\). Then

\[
\lambda \in (\ker \Box_b)^\perp \cap (\ker \Box_b)^\perp
\]

so that by the closedness of the range of \(\Box_b\), there exists \(\lambda' \in (\ker \Box_b)^\perp \subset C^\infty(M; \mathbb{C})\) such that \(\lambda = \Box_b \lambda'\). Since \(\ker \Box_b \subset \ker P\), if \(w \in \ker \Box_b\), then we have

\[
(\Box \lambda, w)_{L^2} = (\lambda', \Box_b \Box_b w)_{L^2} = (\lambda', 2S w)_{L^2}.
\]

In particular

\[
(3.9) \quad \Box_b \lambda - 2S \lambda' \in (\ker \Box_b)^\perp.
\]

Moreover, since \(\lambda' \in (\ker \Box_b)^\perp\),

\[
\|S \lambda'\|_{L^2} \leq C'\|\lambda'\|_{S^2} \leq C'\|\lambda\|_{L^2}.
\]
Therefore
\[ \| □_b λ - 2Sλ' \|_{S^{2,2}}^2 \leq 2(\| □_b λ \|_{S^{2,2}}^2 + 4\| Sλ' \|_{S^{2,2}}^2) \]
\[ \leq C'(\| □_b λ \|_{S^{2,2}}^2 + \| λ \|_{S^{2,2}}^2) \]
\[ \leq C''\| □_b λ \|_{S^{2,2}}^2. \]
(3.10)

Similarly
\[ \| □_b λ \|_{S^{2,2}}^2 \leq C\| □_b λ - 2Sλ' \|_{S^{2,2}}^2. \]

It follows from (3.9) and the subelliptic estimates for □_b, □_b on their orthogonal complement of the kernel, for λ ∈ (ker P)^⊥,
\[ \| λ \|_{S^{4,2}} \leq C_1\| □_b λ \|_{S^{2,2}} \leq C_2\| □_b λ - 2Sλ' \|_{S^{2,2}} \]
\[ \leq C_3\| \Box_b(□_b λ - 2Sλ') \|_{L^2} \]
\[ \leq C_4(\| □_b □_b λ \|_{L^2} + \| □_b Sλ' \|_{L^2}) \]
\[ \leq C_5(\| □_b □_b λ \|_{L^2} + \| λ \|_{L^2}). \]

Hence □_b □_b has the subellipticity on (ker P)^⊥. Since P = □_b □_b − 2S and S is a lower order operator, we obtain the subelliptic estimate for P on (ker P)^⊥.

As a consequence of Proposition 3.2 and results of J. J. Kohn ([K1], [K2]), one obtains

**Corollary 3.1.** Let (M, θ, J) be a closed strictly pseudoconvex CR 3-manifold. Suppose that M is embeddable, then the CR Paneitz operator P is subelliptic on the orthogonal complement of ker P.

### 4. A Priori Uniformly Estimates

Let T be the maximal time for a solution of the flow (1.8) on M × [0, T). We will derive the uniformly S^{4k,2}-norm estimate for λ under the flow (1.8) for all \( t \geq 0 \). It then follows that we have the long-time existence and asymptotic convergence for solutions of (1.8) on M × [0, ∞) as in section 5.

First we recall a pseudohermitian Moser inequality. Let \( \| \cdot \|_p \) denote the L^p-norm with respect to the volume form dμ₀.
Lemma 4.1. Let \((M, J, \theta_0)\) be a closed strictly pseudoconvex CR 3-manifold. Then there exist constants \(C, \kappa,\) and \(\nu\) such that for all \(\varphi \in C^\infty(M),\) there holds
\[
\int_M e^\varphi \, d\mu_0 \leq C \exp \left( \kappa \| \nabla b \varphi \|^4 + \nu \| \varphi \|^4 \right).
\]

Lemma 4.2. Let \((M, J, [\theta_0])\) be a closed strictly pseudoconvex CR 3-manifold. Let \(\lambda \) be a solution of the flow (1.8) on \(M \times [0, T)\). Then there exists a positive constant \(\beta = \beta(Q_0, \theta_0)\) such that
\[
\mathcal{E}(\theta) = \int_M P_0 \lambda \cdot \partial \lambda + \int_M Q_0 \lambda \, d\mu_0 \leq \beta^2,
\]
for all \(t \geq 0.\)

Proof. It follow from (1.3) and (1.5) that
\[
\frac{d}{dt} \mathcal{E}(\theta) = 2 \int_M P_0 \lambda \cdot \partial \lambda + \int_M Q_0 \partial \lambda \, d\mu_0 - 2 \int_M (Q_0 + 2P_0 \lambda - r(t))P_0 \lambda \, d\mu_0 - \int_M Q_0 (Q_0 + 2P_0 \lambda - r(t)) \, d\mu_0 \\
= -2 \int_M (Q_0^2 + 2P_0 \lambda) \, d\mu_0 - \int_M Q_0^2 \, d\mu_0 = -2 \int_M (Q_0^2 + 2P_0 \lambda)^2 \, d\mu_0.
\]

We also observe that

Lemma 4.3. Let \(f : [0, T) \rightarrow \mathbb{R}\) be a \(C^1\) smooth function satisfying
\[
f' \leq -L_1 f + L_2
\]
for some positive constants \(L_1, L_2 > 0.\) Then
\[
f(t) \leq f(0) e^{-L_1 t} + \frac{L_2}{L_1}
\]
for \(t \in [0, T).\)
We will often use the above lemma (whose proof is left to the reader) to obtain the higher order estimates. Now we write \( \lambda = \lambda_{\text{ker}} + \lambda^\perp \). \( Q_0 = (Q_0)_{\text{ker}} + Q_0^\perp \) with respect to \( P_0 \). Comparing both sides of the following formula:

\[
\frac{\partial \lambda_{\text{ker}}}{\partial t} + \frac{\partial \lambda^\perp}{\partial t} = \frac{\partial \lambda}{\partial t} = -(Q_0^\perp + 2P_0\lambda) + r(t).
\]

we obtain

(4.4) \[
\frac{\partial \lambda^\perp}{\partial t} = -(Q_0^\perp + 2P_0\lambda^\perp)
\]

and

(4.5) \[
\frac{\partial \lambda_{\text{ker}}}{\partial t} = r(t).
\]

From now on, \( C \) or \( C_j \) denotes a generic constant which may vary from line to line.

**Proposition 4.1.** Let \((M, J,[\theta_0])\) be a closed strictly pseudoconvex CR 3-manifold. Suppose that the CR Paneitz operator \( P_0 \) is subelliptic on the orthogonal complement of \( \ker P_0 \) and essentially positive. Then under the flow (1.8), there exists a positive constant \( C(k, \Upsilon, \theta_0, \beta) > 0 \) such that

\[
||\lambda||_{S^4k,2} \leq C(k, \Upsilon, \theta_0, \beta)
\]

for all \( t \geq 0 \).

**Proof.** From (4.5), we have

(4.6) \[
\lambda_{\text{ker}}(t, p) - \lambda_{\text{ker}}(0, p) = \int_0^t r dt.
\]

Since \( (P_0 \) being self-adjoint)

\[
\int_M P_0 \lambda^\perp \cdot \lambda_{\text{ker}} d\mu_0 = 0
\]

and

\[
\int_M Q_0 \lambda_{\text{ker}} d\mu_0
\]

\[
= \int_M Q_0[\lambda_{\text{ker}}(0, p) + \int_0^t r dt] d\mu_0
\]

\[
= \int_M Q_0 \lambda_{\text{ker}}(0, p) d\mu_0
\]

\[
\geq -C.
\]
We compute
\[\int_M P_0 \lambda \cdot \lambda d\mu_0 + \int_M Q_0 \lambda d\mu_0 \]
\[= \int_M P_0 \lambda^\perp \cdot (\lambda_{\text{ker}} + \lambda^\perp) d\mu_0 + \int_M Q_0 (\lambda_{\text{ker}} + \lambda^\perp) d\mu_0 \]
\[\geq \int_M P_0 \lambda^\perp \cdot \lambda d\mu_0 + \int_M Q_0 \lambda^\perp d\mu_0 - C.\]

It then follows from Lemma 4.2 that
\[\int_M P_0 \lambda^\perp \cdot \lambda d\mu_0 + \int_M Q_0 \lambda^\perp d\mu_0 \leq (\beta^2 + C)\]
for all \(t \geq 0\). Since the CR Paneitz operator \(P_0\) is essentially positive in the sense that
\[\int_M P_0 \lambda^\perp \cdot \lambda^\perp d\mu_0 \geq \Upsilon \int_M (\lambda^\perp)^2 d\mu_0.\]

All these imply that there exists a positive constant \(C(\Upsilon, Q_0, \theta_0) > 0\) such that
\[(\beta^2 + C) \geq \int_M P_0 \lambda^\perp \cdot \lambda^\perp d\mu_0 + \int_M Q_0 \lambda^\perp d\mu_0 \]
\[\geq \frac{\Upsilon}{2} \int_M (\lambda^\perp)^2 d\mu_0 - C(\Upsilon, Q_0^\perp, \theta_0)\]
for all \(t \geq 0\). So there exists a positive constant \(C(\Upsilon, Q_0^\perp, \theta_0, \beta)\) such that
\[(4.7)\]
\[\int_M (\lambda^\perp)^2 d\mu_0 \leq C(\Upsilon, Q_0^\perp, \theta_0, \beta)\]
for all \(t \geq 0\).

Next we compute, for all positive integers \(k\),
\[
\frac{d}{dt} \int_M |P_0^k \lambda^\perp|^2 d\mu_0
\]
\[= 2 \int_M (P_0^k \lambda^\perp) \left( P_0^k \frac{\partial \lambda^\perp}{\partial t} \right) d\mu_0\]
\[= -2 \int_M (P_0^k \lambda^\perp) P_0^k (Q_0^\perp + 2P_0 \lambda^\perp) d\mu_0\]
\[= -2 \int_M (P_0^k \lambda^\perp)(P_0^k Q_0^\perp) d\mu - 4 \int (P_0^k \lambda^\perp)(P_0^{k+1} \lambda^\perp) d\mu_0.\]

By essential positivity of \(P_0\), we obtain
\[(4.9)\]
\[\int_M (P_0^k \lambda^\perp)(P_0^{k+1} \lambda^\perp) d\mu_0 = \int_M (P_0^k \lambda^\perp)(P_0 P_0^k \lambda^\perp) d\mu_0 \geq \Upsilon \int_M (P_0^k \lambda^\perp)^2 d\mu_0.\]

Therefore there exists a constant \(C = C(k, Q_0^\perp, \Upsilon)\) such that
\[(4.10)\]
\[\frac{d}{dt} \int_M |P_0^k \lambda^\perp|^2 d\mu_0 \leq -3\Upsilon \int_M |P_0^k \lambda^\perp|^2 d\mu_0 + C(k, Q_0^\perp, \Upsilon).\]
By applying Lemma 4.3 to the O.D.E. \( f'(t) \leq -3\Upsilon f(t) + C(k) \) from (4.10), we obtain
\[
\int_M |P^k_0 \lambda^\perp|^2 d\mu_0 \leq C(k, Q_0^\perp, \Upsilon).
\]

By the subelliptic estimate of \( P_0 \) and (4.7), we obtain
\[
||\lambda^\perp - \overline{\lambda}||_{S^{4k,2}} \leq C(k, Q_0^\perp, \Upsilon)
\]
for all \( t \geq 0 \).

We note that
\[
\lambda - \overline{\lambda} = (\lambda^\perp - \overline{\lambda}) + [\lambda_{\ker}(0, p) - \overline{\lambda}_{\ker}(0, p)],
\]
and hence
\[
(4.11) \quad ||\lambda - \overline{\lambda}||_{S^{4k,2}} \leq C(\lambda_{\ker}(0, p), k) + ||\lambda^\perp - \overline{\lambda}||_{S^{4k,2}} \leq C(k, Q_0^\perp, \Upsilon)
\]
for all \( t \geq 0 \). Recall that the average \( \overline{f} \) of a function \( f \) is defined by \( \overline{f} = \frac{\int_M f d\mu}{\int_M d\mu} \). In particular, there holds
\[
||\lambda - \overline{\lambda}||_{S^{4,2}} \leq C(Q_0^\perp, \Upsilon).
\]
Therefore by the Sobolev embedding theorem, we have \( S^{4,2} \subset S^{1,8} \) and
\[
(4.12) \quad ||\lambda - \overline{\lambda}||_{S^{1,8}} \leq C(Q_0^\perp, \Upsilon).
\]

Now using Lemma 4.1 (pseudohermitian Moser inequality), we get
\[
\int_M e^{4(\lambda - \overline{\lambda})} d\mu_0 \leq C \exp(C||\lambda - \overline{\lambda}||_{S^{1,4}}) \leq C(Q_0^\perp, \Upsilon).
\]
Together with \( \int_M e^{4\lambda} d\mu_0 \) being invariant under the flow, we conclude that
\[
(4.13) \quad C \geq \overline{\lambda} \geq -C(Q_0^\perp, \Upsilon).
\]
Here the upper bound is obtained by observing that \( \int \lambda d\mu_0 \leq \int e^{4\lambda} d\mu_0 \).

We finally obtain
\[
(4.14) \quad ||\lambda||_{S^{4k,2}} \leq C(Q_0^\perp, \Upsilon)
\]
for all \( t \geq 0 \). \( \square \)
5. Long-Time Existence and Asymptotic Convergence

In the previous section, we obtain an a priori $S^{4k,2}$ uniformly estimate (i.e., independent of the time) of the solution of (1.8) on $M \times [0, \infty)$ if the CR Paneitz operator $P_0$ is subelliptic on the orthogonal complement of $\ker P_0$ and essentially positive. Therefore we are able to prove the long time existence and asymptotic convergence of solutions of (1.8).

**Theorem 5.1.** Let $(M, J, [\theta_0])$ be a closed strictly pseudoconvex CR 3-manifold. Suppose that the CR Paneitz operator $P_0$ is essentially positive and subelliptic on the orthogonal complement of $\ker P_0$. Then the solution of (1.8) exists on $M \times [0, \infty)$ and converges smoothly to $\lambda_\infty \equiv \lambda(\cdot, \infty)$ as $t \to \infty$. Moreover, the contact form $\theta_\infty = e^{2\lambda_\infty} \theta_0$ has the CR $Q$-curvature with

$$Q_\infty = e^{-4\lambda_\infty} (Q_0)_{\ker}.$$  

In addition if $(Q_0)_{\ker} = 0$, then $Q_\infty = 0$.

**Proof.** Since $P_0$ is nonnegative and subelliptic on $(\ker P_0)^\perp$. It follows that there exists a unique $C^\infty$ smooth solution $\lambda^\perp$ of (4.4) for a short time. On the other hand, we can apply the contraction mapping principle to show the short time existence of a unique $C^\infty$ smooth solution $\lambda_{\ker}$ to (4.5). The long time solution then follows from Proposition 4.1, the Sobolev embedding theorem for $S^{k,2}$, and the standard argument for extending the solution at the maximal time $T$.

Starting from (4.3), we compute

$$\frac{d^2}{dt^2} \mathcal{E}(\theta) = -4 \int_M (Q^\perp_0 + 2P_0 \lambda) P_0 \frac{\partial \lambda}{\partial t} d\mu_0$$

$$= 4 \int_M (Q^\perp_0 + 2P_0 \lambda) P_0 (Q^\perp_0 + 2P_0 \lambda) d\mu_0$$

$$\geq 4\Upsilon \int_M (Q^\perp_0 + 2P_0 \lambda)^2 d\mu_0$$

by essential positivity of $P_0$. Therefore $\frac{d}{dt} \mathcal{E}(\theta)$ is nondecreasing, and hence

$$\int_M e^{4\lambda}[Q - e^{-4\lambda}(Q_0)_{\ker}]^2 d\mu \ (\geq 0)$$

is nonincreasing.
By (4.14), we can find a sequence of times \( t_j \) such that \( \lambda_j \equiv \lambda(\cdot, t_j) \) converges to \( \lambda_\infty \) in \( C^\infty \) topology as \( t_j \to \infty \). On the other hand, integrating (4.3) gives

\[
\mathcal{E}(\lambda_\infty) - \mathcal{E}(\lambda_0) = - \int_0^\infty \int_M e^{4\lambda} [Q - e^{-4\lambda}(Q_0)_{\ker}]^2 \, d\mu dt.
\]

In view of (5.3), (5.4), we obtain

\[
0 = \lim_{t \to \infty} \int_M e^{4\lambda} [Q - e^{-4\lambda}(Q_0)_{\ker}]^2 \, d\mu = \int_M e^{4\lambda_\infty} [Q_\infty - e^{-4\lambda_\infty}(Q_0)_{\ker}]^2 \, d\mu_\infty
\]

where \( Q_\infty \) denotes the Q-curvature with respect to \( (J, \theta_\infty) \), \( \theta_\infty = e^{2\lambda_\infty} \theta_0 \), and \( d\mu_\infty = e^{4\lambda_\infty} \, d\mu_0 \). It follows that

\[
Q_\infty = e^{-4\lambda_\infty}(Q_0)_{\ker}.
\]

That is

\[
2P_0 \lambda_\infty^\perp + Q_0^\perp = 0.
\]

In the following, we are going to prove the smooth convergence for all time. First we want to prove that \( \lambda \) converges to \( \lambda_\infty \) in \( L^2 \). Write \( \lambda_\infty = \lambda_\infty^\perp + (\lambda_\infty)_{\ker} \). Observe that \( (|| \cdot ||_2 \) denotes the \( L^2 \) norm with respect to the volume form \( d\mu_0 \))

\[
||\lambda_\infty^\perp - \lambda^\perp||_2 \leq ||\lambda_\infty^\perp - \lambda^\perp||_{S^1,2} \leq C||2P_0(\lambda_\infty^\perp - \lambda^\perp)||_2 = C||2P_0 \lambda^\perp + Q_0^\perp||_2
\]

by the subellipticity of \( P_0 \) on \( (\ker P_0)^\perp \) and \( 0 = 2P_0 \lambda_\infty^\perp + Q_0^\perp \). We compute

\[
| \mathcal{E}(\lambda_\infty^\perp) - \mathcal{E}(\lambda^\perp) | = \left| \int^1_0 \frac{d}{ds} \mathcal{E}(\lambda^\perp + s(\lambda_\infty^\perp - \lambda^\perp)) ds \right| \leq \int^1_0 \left| \int_M [2P_0(\lambda^\perp + s(\lambda_\infty^\perp - \lambda^\perp)) + Q_0^\perp] \cdot (\lambda_\infty^\perp - \lambda^\perp) \, d\mu_0 ds \right| \leq \int^1_0 ||2P_0(\lambda^\perp + s(\lambda_\infty^\perp - \lambda^\perp)) + Q_0^\perp||_2 ||\lambda_\infty^\perp - \lambda^\perp||_2 ds \leq C_1||2P_0 \lambda^\perp + Q_0^\perp||^2_2
\]

by the Cauchy inequality and (5.6). Let \( \vartheta \) be a number between 0 and \( \frac{1}{2} \). It follows from (5.7) that

\[
| \mathcal{E}(\lambda_\infty^\perp) - \mathcal{E}(\lambda^\perp) |^{1-\vartheta} \leq C_2||2P_0 \lambda^\perp + Q_0^\perp||^{2(1-\vartheta)} \leq C_2||2P_0 \lambda^\perp + Q_0^\perp||^2_2
\]
for \( t \) large by noting that \( 2(1 - \vartheta) > 1 \) and \( \|2P_0{\lambda}^\perp + Q_0^\perp\|_2 \) tends to 0 as \( t \to \infty \) by (5.3). Next we compute

\[
-d\frac{dt}{dt}(\mathcal{E}(\lambda^\perp) - \mathcal{E}(\lambda^\perp_\infty))^{\vartheta} = -\vartheta(\mathcal{E}(\lambda^\perp) - \mathcal{E}(\lambda^\perp_\infty))^{\vartheta-1}d\frac{dt}{dt}(\mathcal{E}(\lambda^\perp) - \mathcal{E}(\lambda^\perp_\infty)) \]

\[
+ \vartheta(\mathcal{E}(\lambda^\perp) - \mathcal{E}(\lambda^\perp_\infty))^{\vartheta-1}\|2P_0{\lambda}^\perp + Q_0^\perp\|_2|\dot{\lambda}^\perp|_2 \\
\geq \vartheta C_2^{-1}|\dot{\lambda}^\perp|_2
\]

by (4.3), (5.8), and noting that \( \dot{\lambda}^\perp = -(2P_0{\lambda}^\perp + Q_0^\perp) \) (see (4.4)) and hence the left side is nonnegative. We learned the above trick of raising the power to \( \vartheta \) from \([S]\). Integrating (5.9) with respect to \( t \) gives

\[
(5.10) \quad \int_0^\infty |\dot{\lambda}^\perp|_2 dt < +\infty.
\]

Observing that \( \frac{d}{dt}({\lambda}^\perp - {\lambda}^\perp_\infty) = \dot{\lambda}^\perp \) and \( -\frac{d}{dt}|{\lambda}^\perp - {\lambda}^\perp_\infty|_2^2 \leq C|\dot{\lambda}^\perp|_2 \), we can then deduce

\[
(5.11) \quad \lim_{t \to \infty} |{\lambda}^\perp - {\lambda}^\perp_\infty|_2^2 = 0
\]

by (5.10). On the other hand, we have estimate \( |r(t)| \leq C|2P_0{\lambda}^\perp + Q_0^\perp|_2 = C|\dot{\lambda}^\perp|_2 \). It follows from (5.10) that

\[
(5.12) \quad \int_0^\infty |r(t)| dt < +\infty.
\]

So in view of (4.6) and (5.12), \( \lambda_\text{ker} \) converges to \( (\lambda_\infty)_{\text{ker}} = (\lambda_{\text{ker}})_\infty \) as \( t \to \infty \) (not just a sequence of times). Since \( \lambda_\text{ker} - (\lambda_\infty)_{\text{ker}} = -\int_t^\infty r(t) dt \) is a function of time only, we also have

\[
(5.13) \quad \lim_{t \to \infty} \|\lambda_\text{ker} - (\lambda_\infty)_{\text{ker}}\|_{S^{k,2}} = 0
\]

for any nonnegative integer \( k \). With \( \lambda^\perp \) replaced by \( \lambda^\perp - \lambda^\perp_\infty \) in the argument to deduce (4.10), we obtain

\[
(5.14) \quad \frac{d}{dt} \int_M (P_0^k(\lambda^\perp - \lambda^\perp_\infty))^2 d\mu_0 \leq -4\Upsilon \int_M (P_0^k(\lambda^\perp - \lambda^\perp_\infty))^2 d\mu_0 + C(k)|\lambda^\perp - \lambda^\perp_\infty|_2.
\]
By Lemma 4.3 and (5.11), we get
\[
(5.15) \quad \lim_{t \to \infty} \int_M (P_k^0(\lambda^\perp - \lambda^\perp_\infty))^2 d\mu_0 = 0.
\]
Hence by the subellipticity of $P_k^0$, we conclude from (5.15) and (5.11) that
\[
\lim_{t \to \infty} ||\lambda^\perp - \lambda^\perp_\infty||_{S^{4k,2}} = 0.
\]
Together with (5.13), we have proved that $\lambda$ converges to $\lambda_\infty$ smoothly as $t \to \infty$.

\[ \square \]

6. Bochner-Type Formulae for the CR Q-Curvature

In this section, let $(M, J, \theta)$ be a closed strictly pseudoconvex CR 3-manifold of $c_1(T_{1,0}M) = 0$. We first recall J. J. Kohn’s Hodge theory for the $\overline{\partial}_b$ complex ([K3]).

Give $\eta \in \Omega^{0,1}(M)$, a smooth $(0,1)$-form on $M$ with
\[
\overline{\partial}_b \eta = 0,
\]
then there are a smooth complex-valued function $\varphi = u + iv \in C^\infty(M)$ and a smooth $(0,1)$-form $\gamma \in \Omega^{0,1}(M)$ for $\gamma = \gamma_{\overline{T}}^T$ such that
\[
(6.1) \quad (\eta - \overline{\partial}_b \varphi) = \gamma \in \ker(\Box_b),
\]
where $\Box_b = 2(\overline{\partial}_b \overline{\partial}_b^* + \overline{\partial}_b^* \overline{\partial}_b)$ is the Kohn-Rossi Laplacian. If we assume $c_1(T_{1,0}M) = 0$ and then there is a pure imaginary 1-form $\sigma = \sigma_\overline{T}^T - \sigma_1^T + i\sigma_0^T$ with
\[
(6.2) \quad d\theta_1^1 = d\sigma
\]
for the pure imaginary Webster connection form $\theta_1^1$.

Lemma 6.1. ([L1]) Let $(M, J, \theta)$ be a closed strictly pseudoconvex CR 3-manifold of $c_1(T_{1,0}M) = 0$. Then there is a pure imaginary 1-form
\[
\sigma = \sigma_\overline{T}^T - \sigma_1^T + i\sigma_0^T
\]
with $d\theta_1^1 = d\sigma$ such that
\[
(6.3) \quad \begin{cases} 
R = \sigma_{\overline{T},1} + \sigma_{1,\overline{T}} - \sigma_0 \\
A_{11}^1 = \sigma_{1,0} + i\sigma_{0,1} - A_{11}^1
\end{cases}
\]
Lemma 6.2. If \((M, J, \theta)\) is a closed strictly pseudoconvex CR 3-manifold of \(c_1(T_{1,0}M) = 0\). Then there are \(u \in C^\infty_R(M)\) and \(\gamma = \gamma_T^\theta \in \Omega^{0,1}(M)\) such that
\[(6.4)\]
\[W_1 = 2P_1 u + i (A_{11} \gamma_T - \gamma_{1,0})\]
and
\[\gamma_{T,1} = 0.\]

Proof. By choosing
\[\eta = \sigma_T \theta^T,\]
as in \((6.1)\), where \(\sigma\) is chosen from Lemma 6.1, there exists
\[\varphi = u + iv \in C^\infty_C(M)\]
and
\[\gamma = \gamma_T^\theta \in \Omega^{0,1}(M) \cap \ker(\Box_b)\]
such that
\[(6.5)\]
\[\sigma_T = \varphi_T + \gamma_T.\]

Note that
\[(6.6)\]
\[\Box_b \gamma = 0 \implies \bar{\partial}_b \gamma = 0 = \bar{\partial}_b^T \gamma \implies \gamma_{T,1} = 0\]
and
\[(6.7)\]
\[\sigma_1 = (\overline{\varphi})_1 + \gamma_1.\]

Here \(\gamma_1 = \overline{\gamma_T}\). From the first equality in \((6.3)\),
\[(6.8)\]
\[R = \sigma_{T,1} + \sigma_{1,T} - \sigma_0.\]

Therefore
\[\sigma_{1,T_1} = (\overline{\varphi})_{1,T_1} + \gamma_{1,T_1}\]
\[= (\overline{\varphi})_{1T_1}\]
\[= (\overline{\varphi})_{T_1} + i(\overline{\varphi})_{01}\]
\[= (\overline{\varphi})_{T_1} + i \left[ (\overline{\varphi})_{10} + A_{11}(\overline{\varphi}), \tau \right].\]
The first equality due to \((6.7)\) and second equality due to \((6.6)\). But from \((6.6)\) and \((6.5)\)
\[\sigma_{T,11} = \varphi_{T,11} + \gamma_{T,11} = \varphi_{T,11}.\]
These and \((6.3), (6.8)\) imply
\[
W_1 = (R_{11} - iA_{11,T})
\]
\[
= \sigma_{T,11} + \sigma_{1,T1} - i\sigma_{1,0} + iA_{11}\sigma_T
\]
\[
= \varphi_{T11} + (\overline{\varphi})_{T11} + iA_{11}(\overline{\varphi})_T - i\gamma_{1,0} + iA_{11} (\varphi_T + \gamma_T).
\]
\[
= 2 (u_{T11} + iA_{11}u_T) + i (A_{11}\varphi_T - \gamma_{1,0})
\]
\[
= 2P_1 u + i (A_{11}\varphi_T - \gamma_{1,0}).
\]

Next we come out with the following key Bochner-type formula.

**Theorem 6.1.** Let \((M, J, \theta)\) be a closed strictly pseudoconvex CR 3-manifold of \(c_1(T_{1,0}M) = 0\). Then the following equality holds
\[
\int_M (R - \frac{1}{2} Tor) (\gamma, \gamma) d\mu + \int_M |\gamma_{1,1}|^2 d\mu
\]
\[
- \frac{1}{2} \int_M Tor' (\gamma, \gamma) d\mu + \frac{1}{4} \int_M (3Q + 2P u) u d\mu = 0.
\]  
(6.9)

Here \(Tor (\gamma, \gamma) := i(A_{TT1}\gamma_{1,1} - A_{11}\gamma_T)\), \(Tor' (\gamma, \gamma) := i(A_{TT1}\gamma_{1,1} - A_{11}\gamma_T)\).

**Proof.** From the equality \(6.4\)
\[
W_1 = 2P_1 u + i (A_{11}\varphi_T - \gamma_{1,0}),
\]
we are able to get
\[
(R_{11} - iA_{11,T}) \gamma_T = W_1 \gamma_T
\]
\[
= 2 (u_{T11} + iA_{11}u_T) \gamma_T + i (A_{11}\varphi_T - \gamma_{1,0}) \gamma_T
\]
\[
= 2 (u_{T11} + iA_{11}u_T) \gamma_T + iA_{11}\varphi_T - (\gamma_{1,1} - \gamma_{1,T1} - R\gamma_1) \gamma_T.
\]

Taking the integration over \(M\) of both sides and its conjugation, we have, by the fact that \(\gamma_{1,T1} = 0\),
\[
\int_M \left( R - \frac{1}{2} Tor - \frac{1}{2} Tor' \right) (\gamma, \gamma) d\mu + \int_M |\gamma_{1,1}|^2 d\mu - \int_M Tor (d\bar{u}, \gamma) d\mu = 0.
\]  
(6.10)

Here \(Tor (d\bar{u}, \gamma) = i(A_{TT1}u_{1,1} - A_{11}\varphi_T)\). On the other hand, it follows from the equality \(6.4\) that
\[
(R_{11} - iA_{11,T}) u_T = W_1 u_T = [2P_1 u + i (A_{11}\varphi_T - \gamma_{1,0})] u_T.
\]  
(6.11)
By the fact that $\gamma_{1,T} = 0$ again, we see that

$$
\int_M \gamma_{1,0} u_\mathcal{T} d\mu = - \int_M \gamma_1 u_{T0} d\mu = \frac{1}{2} \int_M \gamma_1 (u_{0T} - A_{TT} u_1) d\mu = \int_M A_{TT} u_1 \gamma_1 d\mu.
$$

(6.12)

It follows from (6.11) and (6.12) that

$$
\frac{3}{4} \int_M Qu d\mu + \frac{1}{2} \int_M (Pu) u d\mu = i \int_M [(A_{11} u_\gamma T - A_{TT} u_1 \gamma_1) - \text{con} \mu] d\mu = -2 \int_M \text{Tor} (d_b u, \gamma) d\mu.
$$

That is

$$
\frac{3}{4} \int_M Qu d\mu + \frac{1}{2} \int_M (Pu) u d\mu = - \int_M \text{Tor} (d_b u, \gamma) d\mu.
$$

(6.13)

Thus by (6.10)

$$
\int_M \left( R - \frac{1}{2} \text{Tor} - \frac{1}{2} \text{Tor}' \right) (\gamma, \gamma) d\mu + \int_M |\gamma_{1,1}|^2 d\mu + \frac{3}{4} \int_M Q u d\mu + \frac{1}{2} \int_M (Pu) u d\mu = 0.
$$

(6.14)

\[ \square \]

**Theorem 6.2.** Let $(M, J, \theta)$ be a closed strictly pseudoconvex CR 3-manifold of $c_1(T_{1,0} M) = 0$ and the CR Paneitz operator $P$ with kernel consisting of the CR pluriharmonic functions. Then

$$
Q_{\ker} = 0.
$$

(6.15)

Here $Q = Q_{\ker} + Q^\perp$. $Q^\perp$ is in $(\ker P)^\perp$ which is perpendicular to the kernel of self-adjoint Paneitz operator $P$ in the $L^2$ norm with respect to the volume form $d\mu = \theta \wedge d\theta$.

**Proof.** Note that it follows from (L1) that a real-valued smooth function $u$ is said to be CR-pluriharmonic if, for any point $x \in M$, there is a real-valued smooth function $v$ such that

$$
\overline{\partial}_b (u + iv) = 0.
$$

(6.16)
Then the equality (6.4) still holds if we replace $u$ by $(u+ CQ_{\ker})$. It follows from the Bochner-type formula (6.9) that

$$\int_M (R - \frac{1}{2} T \text{or}) \langle \gamma, \gamma \rangle \, d\mu - \frac{1}{2} \int_M T \text{or}' \langle \gamma, \gamma \rangle \, d\mu$$

$$+ \int_M |\gamma|_1^2 \, d\mu + \frac{1}{2} \int_M (Pu) \, ud\mu + \frac{3}{4} \int_M Qu \, d\mu + \frac{3}{4} C \int_M (Q_{\ker})^2 \, d\mu$$

$$= 0.$$ 

However, if $\int_M (Q_{\ker})^2 \, d\mu$ is not zero, this will lead to a contradiction by choosing the constant $C << -1$ or $C >> 1$. Then we are done. 

Proof of Main Theorem 1.1: It follows easily from Theorem 5.1 and Theorem 6.2.

References

[A] T. Aubin, Some Nonlinear Problems in Riemannian Geometry, Springer-Verlag, Berlin, 1998.

[B] T. Branson, Sharp Inequalities, the Functional Determinant, and the Complementary Series, Trans. Amer. Math. Soc. 347 (1995), 3671-3742.

[Bm] L. Boutet and De Monvel, Integration des equations de Caushy-Riemann induites formelles, Seminaire Goulaouic-Lions-Schwartz, Expose IX, 1974-1975.

[Br1] S. Brendle, Global Existence and Convergence for a Higher-Order Flow in Conformal Geometry, Ann. of Math., Vol. 158 (2003), 323–343.

[Br2] S. Brendle, Convergence of the $Q$-curvature Flow on $S^4$, Advances in Mathematics, 205 (2006), 1-32.

[Bu] D. Burns, Global Behavior of Some Tangential Cauchy-Riemann Equations, Partial Differential Equations and Geometry (Proc. Conf., Park City, Utah, 1977), Dekker, New York, 1979, 51-56.

[CC] J. Cao and S.- C. Chang, Pseudo-Einstein and $Q$-flat Metrics with Eigenvalue Estimates on CR-hypersurfaces, Indiana Univ. Math. Journal, 56, No.6, (2007), 2839-2857.

[CCh] S.- C. Chang and J.- H. Cheng, The Harnack Estimate for the Yamabe Flow on $CR$ Manifolds of Dimension 3, Annals of Global Analysis and Geometry, 21 (2002), 111-121.

[CCC] S.- C. Chang and J.- H. Cheng and H.-L. Chiu, A Fourth Order Curvature Flow on a CR 3-Manifold, Indiana Univ. Math. Journal, 56, No.4, (2007), 1793-1825.

[CCY] S. Chanillo and H.-L. Chiu, and P. Yang, Embeddability for Three-dimensional Cauchy-Riemann Manifolds and CR Yamabe Invariants, arxiv1007.5020v3, Duke Math. Journal.

[CH] S.- S. Chern and R. Hamilton, On Riemannian Metrics Adapted to Three-dimensional Contact Manifolds, Lecture Notes in Math., 1111, 279-305, Springer-Verlag, 1984.
[Chi] H.-L. Chiu, The Sharp Lower Bound for the First Positive Eigenvalue of a Sub-Laplacian on a Three-Dimensional Pseudo-hermitian Manifold, Annals of Global Analysis and Geometry 30 (2006), 81-96.

[CL] W. S. Cohn and G. Lu, Best Constants for Moser-Trudinger Inequalities on the Heisenberg Group, Indiana Univ. Math. J. 50 (2001), 1567-1591.

[CLe] J.-H. Cheng and J. M. Lee, The Burns-Epstein Invariant and Deformation of CR Structures, Duke Math. J., 60 (1990), 221-254.

[CS] S.-C. Chen and M.-C. Shaw, Partial Differential Equations in Several Complex Variables, Studies in Advan. Math., 19, AMS/IP, 2001.

[CSa] S.-C. Chang and T. Saotome, The Q-curvature flow in a closed CR 3-manifold, Proceedings of the 15th International Workshop on Differential Geometry and the 4th KNUGRG-OCAMI Differential Geometry Workshop [Volume 15], 57–69, Natl. Inst. Math. Sci. (NIMS), Taejŏn, 2011.

[CTW] S.-C. Chang, Jingzhu Tie and C.-T. Wu, Subgradient Estimate and Liouville-type Theorems for the CR Heat Equation on Heisenberg groups $\mathbb{H}^n$, Asian J. Math., Vol. 14, No. 1 (2010), 041–072.

[CW] S.-C. Chang and C.-T. Wu, The Fourth-Order Q-Curvature Flow on Closed 3-Manifolds, Nagoya Math. J., Vol. 185 (2007), 1-15.

[DT] S. Dragomir, G. Tomassini, Differential Geometry and Analysis on CR Manifolds. Progress in mathematics 246, Birkhäuser, 2006.

[FH] C. Fefferman and K. Hirachi, Ambient Metric Construction of Q-Curvature in Conformal and CR Geometries, Math. Res. Lett., 10, No. 5-6 (2003), 819-831.

[Fo] G. B. Folland, Subelliptic Estimates and Function Spaces on Nilpotent Lie Groups, Arkiv for Mat. 13 (1975), 161-207.

[FK] G. B. Folland, J. J. Kohn, The Neumann Problem for the Cauchy-Riemann Complex. Ann. of Math. Studies, No 75, Princeton University Press, New Jersey, 1972.

[FS1] G. B. Folland and E. M. Stein, Hardy spaces on homogeneous groups, Princeton U. Press, 1980.

[FS2] G. B. Folland and E. M. Stein, Estimates for the $\bar{\partial}_b$ Complex and Analysis on the Heisenberg Group, Comm. Pure Appl. Math., 27 (1974), 429-522.

[GG] A. R. Gover and C. R. Graham, CR Invariant Powers of the Sub-Laplacian, preprint.

[GL] C. R. Graham and J. M. Lee, Smooth Solutions of Degenerate Laplacians on Strictly Pseudoconvex Domains, Duke Math. J., 57 (1988), 697-720.

[H] K. Hirachi, Scalar Pseudo-hermitian Invariants and the Szegő Kernel on 3-dimensional CR Manifolds, Lecture Notes in Pure and Appl. Math. 143, pp. 67-76, Dekker, 1992.
[JL] D. Jerison and J. M. Lee, The Yamabe Problem on CR Manifolds, J. Diff. Geom. 25 (1987), 167-197.

[K1] J. J. Kohn, Estimates for $\partial_b$ on Compact Pseudoconvex CR Manifolds, Proc. of Symposia in Pure Math., 43 (1985), 207–217.

[K2] J. J. Kohn, The Range of the Tangential Cauchy-Riemann Operator, Duke. Math. J., 53 (1986) 525–545.

[K3] J.J. Kohn, Boundaries of Complex Manifolds, Proc. Conf. on Complex Analysis, Minneapolis,1964, Springer-Verlag, 81–94 (1965).

[L1] J. M. Lee, Pseudo-Einstein Structure on CR Manifolds, Amer. J. Math. 110 (1988), 157-178.

[L2] J. M. Lee, The Fefferman Metric and Pseudohermitian Invariants, Trans. A.M.S. 296 (1986), 411-429.

[Le] L. Lempert, On Three-Dimensional Cauchy-Riemann Manifolds. J. of Amer. Math. Soc., 5 (1992), 923–969.

[MS] A. Malchiodi and M. Struwe, Q-curvature flow on $S^4$, J. Differential Geom. 73 (2006), no. 1, 1–44.

[P] S. Paneitz, A Quartic Conformally Covariant Differential Operator for Arbitrary Pseudo-Riemannian Manifolds, preprint, 1983.

[S] L. Simon, Asymptotics for a Class of Nonlinear Evolution Equations, with Applications to Geometric Problems, Ann. of Math., 118 (1983), 525-571.

[SC] Sánchez-Calle A., Fundamental Solutions and Geometry of the Sum of Squares of Vector Fields, Invent. Math., 78 (1984), 143–160.

\[1\] Department of Mathematics, National Taiwan University, Taipei 10617, Taiwan

E-mail address: scchang@math.ntu.edu.tw

\[2\] Department of Mathematics, National Taiwan Normal University, Taipei 11677, Taiwan

E-mail address: tjkuo1215@ntnu.edu.tw

\[3\] National Center for Theoretical Sciences, National Taiwan University, Taipei 10617, Taiwan, R.O.C.

E-mail address: tksaotome@gmail.com