Is there analogy between quantized vortex and black hole?

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ABSTRACT

An attempt is made to promote an analogy between quantized vortex in condensed matter and black hole: both compact objects have fermion zero modes which induce the finite temperature of these objects. The motion of the quantized vortex lines in Fermi superuids and superconductors leads to the spectral flow of fermion zero mode. This results in finite temperature and entropy of the moving vortex. The tunneling transition rate between the fermionic levels under the influence of the vortex motion suggests the effective temperature of the vortex core $T_v = (2\pi \nu_F v_L)$, where $v_L$ is the velocity of the vortex with respect to the heat bath reference frame and $p_F$ the Fermi momentum. This is an analogue of the Unruh temperature of the accelerating object in the relativistic system. For the vortex ring with the radius $R$ this leads to the Hawking type temperature $T_{vortex\ ring} = (\nu_F = 2R) \ln (R=r_c)$, where $v_F$ is the Fermi velocity and $r_c$ is the radius of the vortex core. The corresponding "Hawking" entropy of the vortex ring of radius $R$ and area $A = R^2$ appears to be $S_{vortex\ ring} = (1/6)A p_F^2$. Similar expression but with different numerical factor is obtained for the instanton action for the quantum nucleation of the vortex loop from the homogeneous vacuum, and also for the "Bekenstein" entropy obtained by counting the number of the fermionic bound states which appear when the vortex loop is created. For the super uid $^3$He-$\Lambda$, where some components of the order parameter play the part of the gravitational field, the Fermi momentum $p_F$ corresponds to the Planck scale for this effective gravity. The effective action for the gravity field is obtained after integration over the fermion fields in correspondence with the Sakharov scenario. The integration over the fermions in $^3$He-$\Lambda$ leads to the renormalization of the vortex entropy and the "gravitational constant" while their product remains "fundamental". This is the counterpart of the cancellation of the renormalization corrections to the black hole entropy and to the gravitational constant discussed by Jacobson. The statistical mechanical analysis of the fermions in the vortex core gives however an essentially less dynamical entropy as compared to the "Hawking-Bekenstein" entropy and an essentially larger (by the same factor) temperature of the moving vortex system as compared to the "Hawking-Unruh" temperature.
1 Introduction

Quantized vortices in fermi super uids (and superconductors) provide a simple example of the macroscopic inhomogeneous system, where the fermi ionic excitations move in the collective mean field produced by the motion of other ferm ions. The important property of the mean field potential of the vortex is that it supports the fermi ion zero modes (the gapless or very low-energy excitations [1]). The fermi ion zero modes continuously connect the negative and the positive branches of the energy levels, which opens the possibility for the interaction with the vacuum. Due to this contact some conserved quantities can be transferred from a coherent vacuum motion into incoherent fermi ionic degrees of freedom.

In the high energy physics the only realistic objects, which have similar properties, are the black holes. Other possible objects, such as cosmic strings, walls and magnetic monopoles, are not topologically stable in the electroweak vacuum and can occur only in a very high energy scale of the Grand Unification. Black hole is the macroscopic inhomogeneous system, where the fermi ions (neutrons, quarks, etc.) move in the mean (gravitational) field produced by other particles. The main property of the black hole, which distinguishes it from other astrophysical objects such as neutron star, is the appearance of the fermi ion zero modes, which open a contact with the "bare" vacuum. This constitutes the main analogy with quantized vortices, which we want to explore here.

It will be shown that the vortex moving with respect to the heat bath acquires some effective temperature and entropy (Sec.3), which at first sight (from the generally looking arguments) have a universal form, similar to the Hawking-Unruh temperature and the Hawking-Bekenstein entropy of the black hole [2]. This is supported by the consideration of the tunneling rate of the fermi ions from the moving vortex (Sec.4.1); by the $l=R$ and $R^2$ dependence of the temperature and entropy of the vortex loop of radius $R$ (Sec.4.2); by the instanton action for the quantum nucleation of the vortex loop from the homogeneous vacuum (Sec.4.3); and also by the application to the quantized vortex of the Bekenstein idea on the informational origin of the entropy [3] (Sec.4.4). The information can be lost when the particle leaves the positive energy realm. The Bekenstein entropy is calculated by counting the number of the extra fermionic states which appear when the vortex loop is created from the homogeneous vacuum state.
However the exact statistical mechanics of the fermions within the vortex core (Sec.5) leads to a nonuniversal behavior with much higher temperature and much lower entropy of the vortex excitations. The ordering of the fermions due to successive occupation of the single-particle states occurs independently on the information we have about the particles. The same can occur in the black hole (Sec.6), which can lead to the essential reduction of the entropy compared to the Hawking ansatz.

2 Spectrum of fermions localized on vortices.

The quantized vortex is the topologically stable object of collective motion in superuids and superconductors. The vortex is characterized by the winding of condensate phase $\varphi$ about the vortex axis and by the circulating motion of the superuid component of the liquid around the vortex with the velocity

$$\mathbf{v}_s(r) = \frac{\mathbf{F}}{2} = \frac{N}{2} \mathbf{e} \cdot \mathbf{r};$$

where $\mathbf{F}$ is the circulation quantum: $\mathbf{F} = h=m_3$ for superuid $^3$He and $\mathbf{F} = 2h=m_4$ for superuid $^4$He; $m_3$ and $m_4$ are masses of the $^3$He and $^4$He atoms; $N$ is the (integer) winding number; $r$, $\mathbf{e}$, and $\mathbf{z}$ are cylindrical coordinates with the axis $\mathbf{z}$ along the vortex line.

The super uid around the vortex line induces the propagation of excitations (quasiparticles) near the vortex due to the Doppler effect. In simplest cases of phonons in superuid $^4$He and fermions in superuid $^3$He-A the classical propagation of these quasiparticles obey the equation of motion of the scalar wave in the metric \[4\]:

$$s^2 = (c^2 - \varphi^2) \left( \frac{dt}{\varphi} \right)^2 \left( \frac{dr}{\varphi} \right)^2 \left( \frac{dz}{\varphi} \right)^2 = \frac{c^2}{\varphi^2} r^2 \mathbf{r}^2.$$ \[2.2\]

Here $c$ is the sound velocity for phonon in $^4$He. For the simplest axisymmetric vortex $^3$He-A vortex the velocity $\mathbf{c} = c_\varphi = -\mathbf{p}_\varphi$ enters which corresponds to the spectrum of fermions in the plane transverse to the vortex axis. Here $\mathbf{p}_\varphi$ is the gap amplitude of fermions in bulk $^3$He-A. The Eq. (2.2) reminds the sonic analogue of black hole discussed in Ref.[3]. The main difference is however that in the Ref.[3] the analog of the gravitational field is produced by the
normal (non-supercritical) motion of the liquid. In our case the gravity is simulated by the non-dissipative supercritical motion with the (supercritical) velocity $v_s$. The normal component of the liquid (the system of the quasiparticles) represents the heat bath and is stationary in the equilibrium, i.e. the normal velocity (the velocity of the normal component or the velocity of heat bath reference frame) $v_n = 0$.

Far from the vortex where $v_s(r)$ is small and can be neglected, this metric corresponds to that of the so-called rotating cosmic string. The spinning cosmic string (see the latest references \[6,7\]) is such a string which has the rotational angular momentum. The metric in Eq. (2.2) corresponds to the string with the angular momentum $J = 8 \, G$ per unit length and with zero mass.

Approaching the vortex axis one crosses the cylindrical surface of the radius

$$r_c = N = 2 \, c,$$  \hspace{1cm} (2.3)

where the metric has a singularity. This singularity cannot be removed by the coordinate transformation to a new reference frame as is usual in black hole physics. This is because the reference is fixed by the stationary heat bath. The energy consideration in the heat bath reference frame shows that within the radius $r_c$ (known as the core radius) the vacuum is unstable and is to be reconstructed. For Fermi system the order parameter field in a new stable vacuum is found in Refs. \[8,9\]. This is made in a self-consistency way, taking into account the modification of the Fermi ion spectrum in a new vacuum. Should the vacuum be reconstructed within the horizon of black hole is an open question.

The quantum mechanical spectrum of single-Fermionic excitations on the background of the vortex contains the states localized near the vortex axis \[1\]. The properties of this spectrum do not depend much on the detailed structure of the order parameter in the vortex core and are mostly determined by the topology, i.e. by the winding number $N$ of the vortex. The spectrum $\{n, p_z; Q\}$ is characterized by the following quantum numbers: $m$ on the angular momentum $p_z$ on the vortex axis; the orbital quantum number $Q$, integer or half of odd integer, which corresponds to the generalized angular momentum conserved in an axisymmetric vortex; $n$ denotes the radial quantum number; the spin $s = 1/2$ quantum number is not indicated.

The interlevel distance of the Fermion bound states $\Omega_n = \Omega = \Omega_n$ is usu-
ally very small compared to the gap amplitude: \( n^2(T) = E_F(T) \), where \( E_F \) is the Fermi energy. Thus for not very small energies, i.e., in the region \( n^2(T) = E_F(T) \), the discrete \( Q \) can be considered as continuous quantum number. This spectrum has anomalous (chiral) branches of fermion zero modes (Fig. 1) whose number \( N_{\text{zm}} \) is related to the vortex winding number \( N_{\text{zm}} = 2N \) according to the index theorem [10]. As a function of (continuous) \( Q \), each anomalous branch crosses zero of energy an odd number of times and runs through both discrete and continuous spectrum from \( n = 1 \) to \( n = +1 \). Any other branch either does not cross zero of energy at all or crosses it an even number of times. For low-energy bound states, the spectrum of the chiral branch is linear in \( Q \). For the most symmetric vortices, for example, this is the branch with \( n = 0 \) [11],

\[
0(p_z; Q) = Q !_0(p_z) ; \quad (2.4)
\]

this spectrum crosses zero as a function of \( Q \) at \( Q = 0 \).

For the continuous vortices in the \( ^3\)He-A the interlevel spacing is inversely proportional to the core radius in Eq. (2.3) [12]:

\[
!_0 \frac{\hbar c}{r_c} ; \quad (2.5)
\]

Due to an odd number of crossings of zero, the spectral ow phenomenon becomes important, which can lead to the creation of the fermions from the vacuum under an external perturbation. The relevant perturbation is the motion of the vortex with respect to the heat bath, which constitutes the normal component of the liquid. When the vortex moves with respect to the heat bath reference frame, the velocity difference \( v_n \), \( v_\nu \) induces a ow of quasiparticles from negative levels to positive levels of the spectrum \( 0(p_z; Q) \). This results in a momentum exchange between the moving vortex and fermions in the heat bath, and thus in the anomalous reactive force between the vortex and the heat bath [10, 13]. This force is a realization of the Callan-Harvey mechanism of the anomaly cancellation in the relativistic quantum field theories [13].

The spectral ow force depends on the quasiparticle kinetics determined by the parameter \( !_0 \), where \( !_0 \) is the lifetime of fermions. In the hydrodynamic limit \( !_0 \to 0 \) the interlevel spacing is smaller than the level width \( 1 = \) and the spectral ow along the anomalous branch \( 0(p_z; Q) \) occurs without
any suppression. When the vortex moves the angular momentum evolves as 
\[ Q + (r(t)p) \cdot 2 = Q + t(v_L v_i) \cdot 2, \]
with the number 
\[ \theta_t Q = (v_L v_i) \cdot (p \cdot 2) \]  
(2.6)
of levels crossing zero energy per unit time. Each level bears the linear
momentum \( p \), therefore, the total flux of the linear momentum from the
vortex to the heat bath is \([10,12]\)
\[ \theta_t P = \prod (\frac{\partial f}{\partial Q})\theta_t Q = \frac{1}{2} \prod_{n \in Q} \frac{\partial f(n)}{\partial Q} \frac{Z}{Z} \frac{\partial p}{\partial \phi} \frac{Z}{Z} \frac{d}{d} \left[ (v_L v_i) \cdot (p \cdot 2) \right] = \]
\[ = N \frac{P^3}{3} \cdot 2 \cdot (v_L v_i); \]  
(2.7)
Here we used that only zero modes contribute the sum \( \prod_{n \in Q} (\partial f(n) = \partial Q) = \)
\( n (f(n = 1) f(i = 1)) = 2N \) as \( Q \) together with \( \partial Q \) run from
1 to +1. Thus the spectral ow force between the moving vortex and
the heat bath is
\[ F_{sp:ow} = N \cdot 2 \cdot C_0 (v_n v_i); 0 \cdot 1 : \]  
(2.8)
The parameter \( C_0 \) does not depend on the details of the core structure: it is
expressed in terms of the bulk liquid parameter, the Fermi momentum \( p_F \):
\[ C_0 = m p_F^3 = 3 \cdot 2; \]  
(2.9)
and coincides with the mass density of the fermi liquid in the normal (non-
super uid) state. This independence on the details demonstrates the topolo-
gical origin of the spectral ow force.

3 Hydrodynamic anomaly due to spectral ow.

In addition to the spectral ow force, there are conventional Magnus forces
which act on the vortex moving with respect to the super uid and normal components:
\[ 2s(T)(v_L v_i (1)) + 2n(T)(v_L v_i); \]  
(3.1)
Here $v_s(1)$ is the constant part of the super uid velocity outside the vortex core: the total super uid velocity around the vortex is

$$v_s(\mathbf{r}) = v_s(1) + N \frac{^\wedge}{2} \mathbf{r} : (3.2)$$

These forces exist even in Bose super uids where fermions and their spectral ow are absent. Adding the spectral ow contribution, one obtains the balance of forces acting on the moving vortex

$$s(v_s(1) \quad v_L) 2 (C_0 \quad n) (v_n \quad v_L) 2 = 0 : (3.3)$$

Here and further we omi t the dissipation, since we are interested in the low temperature regime, where the effects of friction, heat conductivity and other irreversible processes can be neglected.

The hydrodynamic anomaly manifests itself in the so called therm orotation e ect, in which the motion of vortices induces the temperature gradient $\mathbf{T}$. This e ect follows from the hydrodynamic equations and theore dy- namic identities [14]. The role of the spectral ow is that it leads to the temperature gradient even in the absence of dissipation, i.e. in the reversible hydrodynamic motion.

For the hydrodynamic description it is relevant to consider an ensemble of the regularly distributed rectilinear vortices. If $\mathbf{n}$ is the (2-dimensional) density of (singly quantized) vortices then the average vorticity of the super uid velocity is

$$\langle \mathbf{\zeta} \rangle = \frac{1}{2} \langle v_s \quad v_s \rangle = n \mathbf{\zeta} ; (3.4)$$

which means that in average the super uid component rotates as a solid body with the angular velocity:

$$\mathbf{\zeta} = \frac{1}{2} n \mathbf{\zeta} ; (3.5)$$

Then one has the following expression for the therm orotation e ect in the absence of dissipation [3]:

$$S_i \mathbf{T} = n C_0 \mathbf{\zeta} (v_L \quad v_L) : (3.6)$$

Thus, if the anomaly parameter $C_0 \neq 0$, then even in the absence of dissipation the entropy and the temperature of the vortices, which move with respect to the heat bath, should be finite.
The distribution of $T(\tau)$ can be visualized for the compact object (the finite cluster of $N$ singly quantized vortices of radius $R$ (see Fig. 2). According to Eq. (3.5) the radius of the cluster and the angular velocity $s$ of super uid motion are related by

$$2 R^2 s = N$$

(3.7)

In the typical experiments with the vortex cluster the super uid and normal component are initially in equilibrium with the rotating container: in this equilibrium state $\~s = \~n = \~L$. Then the container is suddenly stopped and just after stop one has the situation with $\~n = 0$ and $\~s \neq 0$, while the angular velocity $\~L$ of the vortex lines (with $v_L = \~L$) is expressed through normal and super uid velocities according to the force balance equation (3.3) (see e.g. [15]).

For simplicity let us suppose that the entropy is linear in $T$, i.e. $S(T) = T$: this situation with a finite density of states at zero energy, which is typical for the Landau Fermi-liquid, is also valid for the fermion zero modes. Then one has $S \cdot T = (l=2)F(ST)$ which gives the following distribution of $T(\tau)S(\tau)$ in the cluster:

$$S(\tau)T(\tau) = N_1 C_0 \frac{R^2}{R^2} \tau^2$$

(3.8)

Here it was assumed that $T = 0$ outside the cluster. Thus, if the vortex cluster rotates, i.e. $L \neq 0$, the product of the temperature and the entropy density increases towards the center of the cluster reaching the maximum value

$$S(0)T(0) = N_1 C_0$$

(3.9)

at the center.

This equation does not indicate how $ST$ is distributed between $T$ and $S$. One should calculate the parameter in the relation $S(T) = T$ from the microscopic analysis. This will be considered in Sec. 5, but before that let us speculate on the temperature and the entropy of the vortex using some plausible arguments.

4 Speculations on vortex entropy and temperature.
4.1 Radiation from the moving vortex.

First let us try to relate the vortex temperature with the quantum tunneling of the fermions from the vortex during the vortex motion \(^2\). The Hamiltonian, which describes the problem at low \(T\) is related only with the low-energy anomalous branch:

\[
H = \mathcal{Q} \, \partial_0 (\partial_x) + \partial_0 (\partial_x) t(\psi_L \, \partial) \, \mathcal{Z} : \quad (4.1.1)
\]

Here the second term comes from the change of the angular moment \(\mathcal{Q}\) due to the vortex motion relative to the heat bath, \(\psi_L \, \partial\) (we choose \(\psi_n = 0\)). The operators of \(Z\) component of the angular moment \(\mathcal{Q}\), and transverse linear moment \(\partial \), do not commute:

\[
[\mathcal{Q} ; \partial Z] = i \partial Z \quad : \quad (4.1.2)
\]

In terms of the matrix elements between the states with different \(\mathcal{Q}\):

\[
H_{0\mathcal{Q}} = \mathcal{Q} \, \partial_0 (\partial_x) + \partial_0 (\partial_x) t(\partial \, \psi) \quad < \mathcal{Q} \, \partial \mathcal{Q} > \quad : \quad (4.1.3)
\]

Here

\[
< \mathcal{Q} \, \partial \mathcal{Q} > = \frac{1}{2} \partial^2 \left( (\psi + i\partial) \, \partial_0 \mathcal{Q}_0 + (\psi - i\partial) \, \partial_0 \mathcal{Q}_1 \right) \quad : \quad (4.1.4)
\]

Let us use the semiclassical approach, which becomes valid, when the vortex velocity \(v_L\) is small compared to \(\partial_0 = \partial\). In this case the level overlap is determined by the exponentially small transition probability between two neighboring levels. Let us find this exponent. The Hamiltonian for two level system, \(\mathcal{Q} + 1\) and \(\mathcal{Q}\) is

\[
H = (\mathcal{Q} + \frac{1}{2}) \, \partial_0 (\partial_x) + \frac{1}{2} \, \partial_0 (\partial_x) \, \frac{1}{v_L \partial Z} \, \partial \, \partial Z : \quad (4.1.5)
\]

The square of the energy counted from the position in the middle between the states is

\[
E \left( \mathcal{Q} + \frac{1}{2} \partial_0 (\partial_x) \right)^2 = \frac{1}{4} \partial_0^2 (\partial_x) (1 + \partial \, \partial Z^2) \quad : \quad (4.1.6)
\]
The trajectory $t = i$ in the imaginary time axis, which connects two states, gives the following transition probability between the states in the exponential approximation:

$$w = \exp \left[ 2 \text{Im } S \right]; \quad \text{Im } S = 2 \int_0^1 \frac{z}{2} (\mathcal{P}_z) - \frac{s}{2} \theta; \quad \theta = \frac{1}{v_L p_T};$$

This gives

$$w = \exp \frac{1}{v_L p_T} (\mathcal{P}_z);$$

which is equivalent to the thermal distribution of quasiparticles on the levels of the anomalous branch of the spectrum with the effective temperature

$$T_{\text{eff}} = \frac{2}{v_L p_T};$$

It is more pronounced in the 2-dimensional system, where there is no dependence on $p_z$ and $p_T = p_F$ is constant. This temperature equals the energy of the created quasiparticle averaged over the azimuthal angle:

$$T_{\text{eff}} = \frac{2}{v_L p_T} \int_0^{2\pi} \frac{2}{v_L p_T} \cos = \frac{2}{v_L p_T};$$

If one considers this temperature seriously, then from Eq. (3.9) it follows that the total entropy of the $N$-vortex cluster is

$$S = R^2 L S (0) N A p_F^2;$$

where $L$ is the length of the cluster along $z$ and $A = 2 R L$ is the surface area of the cluster. Thus in this reasoning the average entropy of one (singly quantized) vortex in the cluster, $S / p_F^2 A$, corresponds to the area $A$ swept by the vortex in its circular motion.

### 4.2 Entropy and temperature of the vortex ring.

For the single closed loop of the $N$-quantum vortex with radius $R$ and area $A = R^2$ the corresponding entropy is similar to the Hawking entropy of the black hole with the same radius $r_g = R$ of the event horizon [4]

$$S_{\text{vortex ring}} = \frac{1}{6} N A p_F^2 = \frac{1}{4} N A < p_F^2 >;$$

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If one takes into account that the vortex ring velocity $v_L$ and the radius of the ring are related by $v_L = (N = 4 R) \ln(R = R_c)$ one obtains that the temperature $T_{\text{eff}}$ of the vortex ring is

$$T_{\text{vortex ring}} = \frac{N \rho_F}{2 R} \ln \frac{R}{R_c} : \quad (4.2.2)$$

It is inversely proportional to the radius $R$ of the vortex ring in the same manner as the Hawking temperature of the black hole is inversely proportional to the radius $r_g$ of the event horizon. Similar analogy between the closed string loop and the black hole was suggested in [14].

The logarithmic correction in Eq. (4.2.2) disappears in the 2+1 dimensional case, where the counterpart of the vortex loop is the pair of oppositely oriented point vortices. The vortex pair moves with the velocity $v_L = (N = 2 R)$, where $R$ now is the distance between the vortices in the pair. The temperature $T_{\text{eff}}$ of the vortex pair is thus

$$T_{\text{vortex pair}} = \frac{N \rho_F}{2 R} = \frac{N \hbar v_f}{R} : \quad (4.2.3)$$

where $v_f$ is the Fermi velocity. This can be compared with the temperature of the black hole with $R = r_g$.

$$T_{\text{black hole}} = \frac{\hbar c}{4 R} : \quad (4.2.4)$$

The energy of the vortex ring is

$$E_{\text{vortex ring}} = \frac{1}{2} N^2 \rho_F R \ln \frac{R}{R_c} : \quad (4.2.5)$$

where the mass density of the superfluid Fermi liquid is very close to $C_0 = m_p^2 = 3$. As a result:

$$E_{\text{vortex ring}} = \frac{1}{2} T_{\text{vortex ring}} S_{\text{vortex ring}} : \quad (4.2.6)$$

In some cases one can relate the Fermi momentum $p_F$ in Eq. (4.2.1) for the vortex ring entropy to the Planck momentum $p_{\text{Planck}} = \hbar c$ in the black hole entropy.

$$S_{\text{BH}} = \frac{1}{4} A p_{\text{Planck}}^2 : \quad (4.2.7)$$
This can be done for example in super uid \( ^3 \text{He} - \text{A} \), where some components of the order parameter play the part of the gravitational field (see [17,18]). The effective action for the gravity field is obtained after integration over the fermion fields. This corresponds to the Sakharov scenario of the effective gravity [19]. The integration over the fermions gives some combination which is equivalent to the Einstein-Hilbert term \((1=16 \quad )^\text{p} \quad gR\) with the value of

\[
\frac{\hbar}{2p_F c_s^2} : \quad (4.2.8)
\]

The factor \(c_s^3\) is absorbed into the metric tensor: \(c_s c_s^2 = 1 = p_F \quad g\), where \(c_s = v_F\) and \(c_s = -p_F\) are the velocities of light propagating along the axis of the axi symmetric vortex and in the transverse plane correspondingly (see [17,18]). Thus the Fermi momentum \(p_F\) corresponds to the Planck scale for the effective gravity in \( ^3 \text{He} - \text{A} \). This makes the analogy more close.

Note that the equation for the entropy in terms of \(S\) does not depend on the number of species of fermions, since they are absorbed in \(c_s^3\). This corresponds to the scenario of Jackobson [20], in which the renormalization perturbs both the black hole entropy and the gravitational constant, while their product remains fundamental.

### 4.3 Instanton action and Hawking entropy of the vortex ring.

Let us apply to the vortex loop the instanton interpretation of the black hole entropy. According to [21], the instanton action for the tunneling creation of the pair of the black holes is proportional to \(e^{S_{\text{BH}}}\). Let us estimate the tunneling rate of the nucleation of the vortex loop. The latter is created from the super uid vacuum state if the super uid velocity \(v_s\) deviates from the velocity \(v_n\) of the heat bath. The effective energy of the vortex loop in the presence of such counter \(v_n\) or \(v_s\) of the super uid and normal components of the liquid is (at low temperature)

\[
E_{\text{vortex ring}} = E_{\text{vortex ring}} \quad R_{\text{vortex ring}}(v_s \quad v_n) = \quad \\
= \frac{1}{2} N^2 \quad 2 \quad R \ln \frac{R}{R_c} \quad N \quad R^2 (v_s \quad v_n) : \quad (4.3.1)
\]
where \( p_{\text{vortex ring}} \) is the momentum of the vortex loop which is proportional to its area. We consider the transition rate between the vacuum state without the vortex loop \( (R = 0) \) and the vortex state with the same energy \( E = 0 \), i.e., with the vortex loop of radius

\[
R = \frac{N}{2} \left( \frac{v_s}{v_t} \right) \ln \frac{R}{r_c} \quad : \quad (4.3.2)
\]

Such instanton was calculated in [22] for the case when the mass of vortex line was neglected and the quantum nucleation of the vortex loop was mediated by the irregularity on the surface of container. Here we follow the arguments of Ref. [23] where the quantum nucleation of vorticity was considered in the homogeneous vacuum and the existence of the vortex inertial mass \( m_{\text{vortex ring}} \) is important. The result for the semi-classical tunneling is

\[
I = f R^0 \frac{q}{2m_{\text{vortex ring}}(R^0)E_{\text{vortex ring}}(R^0)} \quad : \quad (4.3.3)
\]

For the inertial mass of the vortex loop we take the value

\[
m_{\text{vortex ring}} = \frac{E_{\text{vortex ring}}}{s^2} \quad ; \quad (4.3.4)
\]

discussed in [23, 24], where \( s \) is the sound velocity determined by the compressibility of the liquid with \( s = v_F = p/\rho \) for the Fermi gas. The integration in Eq. (4.3.3) gives again the area law for the entropy of the vortex loop with the radius \( R \) but with different factor:

\[
I = A p_F^2 \quad ; \quad = \frac{N^2}{5} \left( \frac{s}{27} \right) \ln \frac{R}{r_c} \quad : \quad (4.3.5)
\]

This nevertheless confirms the suggestion made in previous subsection that the vortex entropy can be proportional to the area \( A \) of some membrane term inating on the vortex loop.

### 4.4 Bekenstein entropy of the fermions bound to the vortex core.

Let us consider now the contribution to the entropy from the elementary excitations in the vicinity of the string loop. We are interested on the change
of the entropy of the fermion zero modes during creation or annihilation of the vortex line.

Let us eliminate the vortex loop in the following manner: First one changes the phase field in such a way that everywhere one has $\phi = 0$ except in the region within the membrane. The phase changes by $2N$ when the membrane is crossed. In this case the membrane represents the $2N$ soliton (Fig 3). On the second stage one eliminates the soliton together with the vortex loop and the homogeneous vacuum state is achieved.

The main property of the soliton is that it gives rise to the fermion zero modes whose number is proportional to $N$ \cite{25,26,27,28}. The total number of these fermionic bound states

$$\frac{1}{4}N A p_F^2 ; \quad (4.4.1)$$

which is $N = 2$ multiplied by the number of the quantum states of the motion along the soliton plane: $2^R \int dA \int d^2p = (2)^2 = p_F^2 A = 2$.

This can be considered as an extra number of states which appeared in the system when the vortex loop of the area $A$ is created from the vacuum. Each zero energy state can be either empty or occupied, which gives the following Bekenstein \cite{3} entropy of fermion zero modes:

$$S / N p_F^2 A ; \quad (4.4.2)$$

We considered the case when the vortex loop (or the vortex-antivortex pair) annihilates via the soliton wall bounded by the loop. One can also apply the Bekenstein arguments using the scenario in which the vortex loop or the vortex pair annihilates by shrinking. Let us consider a pair of rectilinear vortex lines with opposite winding numbers $N$ and $-N$. The anomalous branch, which enters the vacuum in one vortex, returns back to the positive energy world in another vortex. The number of the negative levels is thus $\sqrt{A / N p_F R}$, where $R$ is the distance between the vortex and antivortex. Let us suppose that we lose information on the particle when it enters the negative level state. This is not so crazy since in the presence of the spectral wave, the number of particles occupying the anomalous branch depend on the prehistory. Since each of these $p_F R$ states can be either empty or occupied, the entropy $/ p_F R \ln 2$. This should be multiplied by the number $p_F L = \text{of longitudinal } p_z \text{-states, where } L \text{ is the length of the vortex. This again gives}
the estimation for the entropy in terms of the area \( A \):

\[
S / N p^2 R L \ln 2 / N p^2 A \quad \text{(4.43)}
\]

This can be considered as the area of the surface swept by two vortices if they move to each other until complete annihilation; this is just another way of the elimination of the vortex pair or of the vortex loop.

5  **Microscopic analysis for the vortex temperature.**

However all the arguments above can be applied only in a (semi)classical macroscopic picture. This means that we neglected the energy difference between the quantum levels of fermions in the core, i.e. put \( \epsilon_0 = 0 \). Now let us consider an exact statistical-mechanical problem taking into account the finiteness of the interlevel distance. In this case one is not lacking the information on the particles since they successively occupy the lowest energy levels. This should essentially reduce the estimated temperature of the vortex. The same arguments possibly can be applied to the black hole entropy, which can be essentially reduced as compared to the Bekenstein-Hawking entropy.

Since there are no excitations in the bulk liquid, the temperature gradient in the vortex cluster should be produced by the vortex core excitations (fermion zero modes). Since the vortices rotate with the angular velocity \( \Omega \), there is an interaction \( Q \Omega L \) of the rotation velocity with the orbital momentum \( Q \) of the fermion. As a result the distance between the neighbouring \( Q \) levels is

\[
\epsilon_0 + \Omega L : \quad (5.1)
\]

The density of the one-dimensional fermionic states on the anomalous branch in a single vortex is (if one neglects the dependence of \( \epsilon_0 \) on \( p_z \))

\[
\frac{p_x L}{\epsilon_0 + \Omega L} : \quad (5.2)
\]

The density of states per unit volume is obtained when the Eq. (5.2) is multiplied by the vortex density \( n = (2m_s = \hbar) s \). This gives for the density of
states per unit volume

\[ N(0) = \frac{np_F}{(l_0 + L)} : \quad (5.3) \]

The energy density of the fermions is thus

\[ E = N(0) \int_0^1 df(T) = nN(0)T^2 \frac{2}{12} = nT^2p_F \frac{1}{12(l_0 + L)} : \quad (5.4) \]

This gives the parameter in the relation between the entropy density and the temperature

\[ S = T ; \quad np_F \frac{1}{12(l_0 + L)} : \quad (5.5) \]

Thus the rhs of Eq. (3.9) can be distributed between \( S \) and \( T \) in a different way depending on ratio \( l_0 = L \).

Now one can apply this to the rotating vortex cluster, discussed in Sec 3. Equating \( S(T(0)) = T^2(0) \) to the rhs of Eq. (3.8), i.e. to \( p_F^2nR^2L = 3 \), one obtains the temperature in the center of the cluster and the entropy of the cluster:

\[ T(0) = 2Rdp_F \frac{q}{L(l_0 + L)} ; \quad S = np_F^2A \frac{s}{l_0 + L} : \quad (5.6) \]

In conventional situation one has \( l_0 \approx L \) and the entropy is essentially reduced as compared with the Bekenstein value. However, in the (very non-physical) limit of the small interlevel distance, \( l_0 \approx 0 \), the temperature in the center of the vortex cluster, expressed in terms of the linear velocity of vortices on the periphery of the cluster,

\[ T(0) = \frac{2}{v_L(R)p_F} : \quad (5.7) \]

reminds the vortex temperature in Eq. (4.1.10), while the entropy tends to its Bekenstein limit.
6 Statistics of fermions in black holes and vortices.

The apparent analogy between black holes and vortices is in the fermion distribution of the fermions. In both cases the vacuum is "open", i.e., the fermion zero modes appear which connect positive and negative energy levels. The fermion zero modes are concentrated within the horizon of the black hole and in the core of the vortex and define all the quantum statistics of the object. The fermions are in the field of the other fermions, which produce the mean field potential: gravitational field in the black hole and the order parameter distribution in the vortex core. The fermion zero modes in black holes can be found in a semi-classical approximation, which is valid for calculations because the short wave lengths are mostly important.

The radial action, which quantization gives the energy levels \((n;L)\) of the fermionic excitations with the bare mass \(m\), angular momentum \(L\) and radial quantum number \(n\), is [23]

\[
S_r = \int_0^{r_g} p_r dr = 2 \int_0^{r_g} \frac{\sqrt{\frac{2}{c^2(1 - \frac{r_g}{r})}}}{\frac{m^2 c^2 + \frac{L(L+1)}{r^2}}{1 - \frac{r_g}{r}}} \, dr : \quad (6:1)
\]

Here \(r_g = 2M/c^2\) is the Schwarzschild radius of a black hole with mass \(M\).
At small energy of the excitation \(m \ll c\) the integral is concentrated in two regions within the horizon: (1) The vicinity of the horizon, where the integral is logarithmically divergent at small distances, gives the contribution

\[
S_r^{(1)} = 2 r_g \ln \frac{r_g}{r_0} ;
\]

where \(r_0\) is the cut-off parameter (GUT or Planck size). (2) In the region far from the horizon the term with the energy \(m c^2\) can be neglected as well as that with \(m c^2\) and one has (for \(L \ll m c r_g\))

\[
S_r^{(2)} = 2 L \int_0^{r_g} \frac{dr}{r(r_g - r)} = 2L : \quad (6:3)
\]

Thus the energy levels are given by

\[
(n;L) = \mathcal{L} (n \quad L) ; \quad \mathcal{L} = \frac{\hbar c}{r_g \ln \frac{m c}{r_0}} = \frac{4}{\ln \frac{r_g}{r_0}} \frac{\hbar c}{T_{\text{Hawking}}} ; \quad (6:4)
\]
The Eq. (6.4) describes the fermionic zero modes. The interlevel spacing is similar to the Eq. (2.5) for that of fermions in the vortex core. As distinct from the Eq. (2.4) for the fermion zero modes in the vortex, which is linear in the generalized angular momentum \( Q \), the zero modes in black holes are linear in two discrete parameters \( L \) and \( n \). The fermion energy becomes exactly zero at \( n = L \), while in the vortex it is zero at \( Q = 0 \). Also the branches are symmetric, i.e., their number is even, as distinct from the excitations on vortices, which have an asymmetric branch.

The thermodynamics of the black-hole fermions depends on whether the fermion number is conserved or not.

(i) If the fermion charge \( N_F \) is fixed, then it determines the mass of the black hole, which equals the total energy of the fermions. At zero temperature the fermions occupy the lowest energy levels below the Fermi energy \( E_F \) and the total energy is

\[
E = M c^2 = \sum_{n \mu_1 \mu_2; < E_F} (2L + 1) ;
\]

where \( E_F \) is determined by the fermionic charge \( N_F \):

\[
N_F = \sum_{n \mu_1 < E_F} (2L + 1) ;
\]

This gives

\[
N_F = L_{\text{max}}^2 \frac{E_F}{E_0} ;
\]

where the maximum momentum is \( L_{\text{max}} \), \( \pi r_g = r_g = r_0 \), and the Fermi energy is

\[
E_F = N_F \frac{L_{\text{max}}^2}{r_g^2} N_F \hbar \frac{r_0^2}{r_g^3} ;
\]

From the total energy

\[
E = M c^2 = N_F E_F + N_F^2 \hbar \frac{r_0^2}{r_g^3} ;
\]

where

\[
T_{\text{Hawking}} = \frac{\hbar c^3}{8 M} = \frac{\hbar c}{4 r_g} ;
\]

is the Hawking temperature.

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\]

From the total energy

\[
E = M c^2 = N_F E_F + N_F^2 \hbar \frac{r_0^2}{r_g^3} ;
\]
one obtains the relation between the fermion number and the mass of the black hole:

\[ M = \left( \frac{\hbar c r_0^2 c^4 N^2}{3} \right)^{1/4} \]  

(6.11)

(ii) Since the vacuum is open, the situation can be the same as in the super uid, where within the vortex core there is an exchange of the fermionic charge between the vacuum and the heat bath of fermions, and thus the fermion number is not conserved. In this case the fermion number and the total energy are nonzero only at nonzero temperature. The total energy of the system at temperature \( T \), which should be equal the mass \( M \) of the black hole, is given by

\[ E = X \sum_{n,\mu,\nu} f (-T) = X(2L + 1) \sum_{n,\mu} T_{\text{max}}^2 n g \ln \frac{r_g}{r_0} : \]  

(6.12)

It was assumed that \( T = \hbar c = (r_g \ln (r_g = r_0)) \), i.e. \( T_{\text{Hawking}} \) otherwise the dependence is exponential. Thus the estimation of the energy of the black hole with the temperature \( T \) is

\[ E = M c^2 + T^2 r_g^2 \frac{1}{r_0^2 \hbar c} - T^2 + 3M^3 \frac{1}{3 \hbar c^3} r_0^2 \]  

(6.13)

where the temperature of the black hole of mass \( M \) is

\[ T_{\text{BH}} (M) = \frac{r_0^2 c^4}{M} \frac{\hbar c}{3} \frac{\hbar c^2}{M} r_0 \eta \text{ Planck} ; \]  

(6.14)

where \( \eta \text{ Planck} = \hbar c^2 \) is the Planck momentum. This qualitatively agrees with the Hawking temperature only if the cut-off \( r_0 \) is on the Planck scale. In all realistic cases \( r_0 \eta \text{ Planck} \leq 1 \) and \( T_{\text{BH}} \geq T_{\text{Hawking}} \), which was assumed at derivation. Maybe this means that in real situation the Hawking limit, where the fundamental temperature is given by Eq.(6.5), is never reached. Just in the same manner as in Eq.(6.6) the real fermion temperature of the vortex always exceeds the fundamentally looking temperature (4.1.10).

The entropy of the black hole also appears to be much less than the Hawking-Bekenstein entropy. It is zero in the case (i), while in the case (ii)

\[ S_{\text{BH}} = S_{\text{Hawking}} \frac{1}{r_0 \eta \text{ Planck}} ; \]  

(6.15)
while $ST$, like in vortices, remains to be invariant and independent on the cut-off $r_0$, since it is defined by the black hole mass. The information on the particles behind the horizon is not lost because of the fermi statistics, which allows only one quantum state per each fermion. Each fermion entering the black hole eventually finds a well defined empty state with lowest energy.

The black hole dynamical entropy was discussed in [30, 31, 32]. Their result $S_{\text{FN B Hawking}} (l=0\text{Plank})^2$ differs both from the Eq. (6.15) and the Hawking entropy. It is worthwhile to note that their entropy can be derived from the same thermal energy $E_\text{in}$ in Eq(6.13) as $S = 2E = T$, if one uses the Hawking temperature for $T$. However, it seems that the much higher temperature in Eq.(6.14) should be more relevant.

7 Conclusion.

Two systems, black hole and condensed matter vortices, have some similar features.

(1) In both systems the vacuum is "open", i.e., they contain the fermion zero modes, which could lead to the nonconservation of the fermion charge due to the spectral flow of the fermion from the vacuum to the heat bath.

(2) The fermion zero modes determine the thermodynamic properties of the objects: they are responsible for the nonzero temperature of the vortex, if it has a finite velocity with respect to the heat bath, and of the black hole with a finite mass. The black hole moving with respect to the heat bath is to be considered.

(3) In both systems the entropy is proportional to the area $A$, where $A$ is the area of horizon in the black hole and in the case of the vortex loop it is the area of the loop, while the temperature is $1/A$.

(4) Both systems have a limiting case (though possibly not achievable) in which the temperature and the entropy are given by the fundamental equations. The temperature is defined by the tunneling of the fermions, while the entropy corresponds to extra fermion degrees of freedom which appear when the object is created from the vacuum state. In the case of the black hole these are the Hawking-Unruh temperature and the Bekenstein-Hawking entropy. Similar expression for the entropy is obtained for the instanton action for the quantum nucleation of the vortex loop and the pair of black holes from the homogeneous vacuum.
(5) For the super uid $^3$He-A, where some components of the order parameter plays the part of the gravitational field, one may obtain the dependence of the vortex entropy on the cut-off parameter. The Fermi momentum $p_F$, which is the largest momentum in the Fermi liquid theory appears to correspond to the Planck scale for the effective gravity in $^3$He-A. The integration over the fermions in $^3$He-A leads to the renormalization of the vortex entropy and of the "gravitational constant" in such a way that their product remains "fundamental". This is the counterpart of the cancellation of the renormalization corrections to the black hole entropy and to the gravitational constant discussed by Jacobson.
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Figure 1: Spectrum $n(p_z; Q)$ of the fermions localized in the condensed matter vortex in terms of the generalized angular momentum $Q$ and for given $Q$ in terms of the linear momentum $p_z$ along the vortex axis. The anomalous branch $0(p_z; Q)$, which crosses as a function of $Q$ the zero energy level, is the source of the anomaly in the vortex dynamics.

Figure 2: Cluster of the vortex lines. The container and the heat bath rotate with the angular velocity $n$. The super uid velocity within the cluster simulates the solid body rotation of the vacuum (the super uid component of the liquid) with the average velocity $<v_s> = \sim_s r$. If $s \not\in n$, the vortices move with respect to the heat bath: they rotate as a solid body with velocity $L \not\in n$. Due to the spectral flow of fermion zero modes this gives rise to the finite temperature of the cluster which increases towards the center of the cluster.

Figure 3: The soliton membrane between the vortex and antivortex. Outside the soliton the phase $= 0$: the winding of the phase is concentrated within the soliton. This winding gives rise to the fermion zero modes within the soliton.
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