Brane worlds and dark matter

Shahab Shahidi* and Hamid Reza Sepangi†
Department of Physics, Shahid Beheshti University, G. C., Evin, Tehran (19839) Iran

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Abstract

Two problems related to dark matter is considered in the context of a brane world model in which the confinement of gauge fields on the brane is achieved by invoking a confining potential. First, we show that the virial mass discrepancy can be addressed if the conserved geometrical term appearing in this model is considered as an energy momentum tensor of an unknown type of matter, the so-called X-matter whose equation of state is also obtained. Second, the galaxy rotation curves are explained by assuming an anisotropic energy momentum tensor for the X-matter.

1 Introduction

Over the past decade, model theories in which our 4D space-time (brane) embedded as a hyper-surface in a higher dimensional space-time (bulk) have been proposed and opened a large, relatively unexplored area of research. The interest in the study of gravity in spaces with extra dimensions started to grow rapidly with the appearance of the proposal put forward by Randall and Sundrum (RS) [1, 2]. This was followed by the derivation of the Einstein field equations on the brane through the use of the Gauss-Codazzi equations and Israel junction conditions together with the assumption of $Z_2$ symmetry [3]. The cosmological consequences of this work, such as the additional quadratic density term in the Friedman equations [4, 5] have been widely investigated.

Although use of the Israel junction conditions together with $Z_2$ symmetry is the widely accepted method amongst the workers in the field for confining the gauge fields to the brane and hence obtaining the field equation, there has been concerns expressed over their use in that such junction conditions may not be unique [6]. Another point to note is that if more than one extra dimension is involved, there is, as yet, no way of obtaining viable junction conditions. Eying such concerns, an interesting model which avoids the above questions was proposed in [7]. Here, the confinement of the gauge fields to the brane is done by postulating the existence of a confining potential $V$. One then obtains the field equations on the brane without the use of any junction conditions or the assumption of $Z_2$ symmetry. The field equations obtained in this way contain a term, $Q_{\mu \nu}$, which is an independently conserved quantity and geometrical in nature. This would allow us to attribute it to what is known as the X-matter. The dynamics of test particles confined to such a brane-world at the classical and quantum levels was studied in [8]. The static, spherically symmetric solutions of this model was used subsequently to explain the galaxy rotation curves [9]. Various other cosmological implication of the model were also studied and showed to be consistent with observations, such as the late time accelerated expansion of the universe which is related to dark energy content of the universe. [10, 11].

One of the important observation which necessitates the existence of dark matter is the so-called virial mass discrepancy. The total mass of a cluster of galaxies can be estimated in two ways. Knowing the motions of its member galaxies, the virial theorem gives one estimate $M_V$. The second estimate is obtained by taking the mass of the individual galaxies separately and adding them up to give $M$. Almost without any exception, the virial mass $M_V$ is greater than the total baryonic mass $M$ by 20-30% [12], so that the existence of dark matter is assumed in order to explain this discrepancy. In [13],

*Electronic address: sshahidi@mail.sbu.ac.ir
†Electronic address: hr-sepangi@sbu.ac.ir
this effect is explained within a RS type brane-world together with the fact that the electric part of the
Weyl tensor can be decomposed irreducibly into dark matter energy density and pressure, previously
presented in [14]. The virial mass discrepancy is explained in [15] for a brane-world model theory
discussed above, and also in the context of DGP brane gravity [16].

Recent observations show that the tangential velocity of stars moving around the center of a galaxy
tend to a constant value as we move away from the center of the galaxy. In order to explain such
rotation curves various suggestions have been made. For example, in [17, 18] Modified Newtonian
Dynamics is used to explain the stability of circular orbits, while in [19, 20] the authors use conformal
symmetry in the context of brane-world gravity to explain the constancy of galaxy rotation curves. In
this paper we present similar conclusions within the context of the brane-world scenario proposed here
by invoking the notion of conformal Killing symmetry.

Calculation of the geometrical term obtained in this model needs the specification of the bulk
gometry. One may then compute the tensor $Q_{\mu\nu}$ using the Gauss-Codazzi equation. In the case
that the bulk geometry is unknown, one of the components of the extrinsic curvature tensor is left
arbitrary, which may then be obtained using the Einstein field equations [21]. In this paper however,
we use this arbitrariness and consider the geometric tensor $Q_{\mu\nu}$ as a perfect fluid for which the energy
density and pressure is obtained via the Einstein field equations. As is shown in the sections below,
the following two problems related to dark matter can be explained using different geometrical matter.
The galaxy rotation curves can be accounted for by defining the tensor $Q_{\mu\nu}$ to be anisotropic with
the equation of state of the form (56). The virial mass discrepancy can be explained by introducing
the isotropic geometrical matter. This choice however is somewhat expected in that the virial mass
problem concerns the calculation of the mass of the cluster of galaxies which is a much larger object
than the galaxy itself.

The scope of the paper is as follows. In the next section we obtain the virial mass theorem in
the context of the brane-world model discussed above. In section three the equation of state of the
X-matter is obtained with the specific choice of the energy momentum tensor of the X-matter. Section
four concerns the galaxy rotation curves where we show that the constancy of the velocity is obtained
by the assumption that the brane admits a conformal symmetry. Conclusions are drawn in section
five.

2 Virial theorem

We use the Einstein field equations obtained in [15] in the context of brane-world models without the
$Z_2$ symmetry. Neglecting the global bulk effects, the Einstein equations read

$$G_{\mu\nu} = 8\pi G \tau_{\mu\nu} - \lambda g_{\mu\nu} + Q_{\mu\nu},$$

(1)

where $\tau_{\mu\nu}$ is the energy-momentum tensor of the ordinary matter on the brane and $Q_{\mu\nu}$ is a conserved
gometric quantity which can be considered as an energy-momentum of the X-matter. In this section
we obtain the generalized virial theorem and show that the X-matter mass is linearly related to the
virial mass. We take the brane spherically symmetric and static metric as

$$ds^2 = -e^{\mu(r)}dt^2 + e^{\nu(r)}dr^2 + r^2(d\theta^2 + \sin^2 \theta)d\phi^2.$$  

(2)

For $\tau_{\mu\nu}$, we take

$$\tau_{\mu\nu} = \text{diag} \left(-\rho_{\text{eff}}, p_{\text{eff}}, p_{\text{eff}}, p_{\text{eff}}\right),$$

(3)

while for the X-matter we choose a fluid of the form

$$Q_{\mu\nu} = 8\pi G \text{diag} \left(-\rho_X, p_X, p_X, p_X\right).$$

(4)
Now, substituting the equations above in equation (1) we obtain the following field equations on the brane

\[
\frac{e^{-\nu(r)}}{r^2}(1 - \nu') - \frac{1}{r^2} = -8\pi G(\rho_{\text{eff}} + \rho_x) - \lambda,
\]

\[
\frac{e^{-\nu(r)}}{r^2}(1 + \mu') - \frac{1}{r^2} = 8\pi G(p_{\text{eff}} + p_x) - \lambda,
\]

\[
\frac{e^{-\nu(r)}}{4r}(2\mu' - 2\nu' - \mu'\nu' + 2\mu'' + \mu^2) = 8\pi G(p_{\text{eff}}^L + p_x) - \lambda,
\]

\[
\mu' = -2\frac{d}{dr}p_{\text{eff}}^L + p_x + \frac{2}{2}(p_{\text{eff}}^L - p_{\text{eff}}^X),
\]

where the last equation is obtained from \(\nabla^\mu Q_{\mu\nu} + 8\pi G\tau_{\mu\nu} = 0\) and a prime represents derivative with respect to \(r\).

In order to obtain the virial theorem, it is convenient to work in tetrad formalism. Let us then define the following frame of orthonormal vectors

\[
e^{(a)}_\rho = e^{(1)}_\rho, \quad e^{(1)}_\rho = e^{(1)}_\rho, \quad e^{(2)}_\rho = r\delta^2_\rho, \quad e^{(3)}_\rho = r\sin\theta\delta^3_\rho,
\]

where \(g^{\mu\nu} e^{(a)}_\mu e^{(b)}_\nu = \eta^{(a)(b)}\), and the tetrad indices are indicated by the parenthesis. In tetrad components the 4-velocity \(v^\mu\) of a typical galaxy, with the property \(v^\mu v_\mu = -1\), is given by

\[
v^{(a)} = v^\mu e^{(a)}_\mu, \quad a = 0, 1, 2, 3.
\]

Finally we introduce \(f(x^\mu, v^{(a)})\) as the distribution function of galaxies assumed to be identical and collisionless point particles. The relativistic Boltzmann equation in tetrad components is now given by

\[
v^{(a)} e^{(a)}_\rho \frac{\partial f}{\partial x^\rho} + \gamma^{(a)}_{(b)(c)} v^{(b)} v^{(c)} \frac{\partial f}{\partial v^{(a)}} = 0,
\]

where \(\gamma^{(a)}_{(b)(c)} = e^{(a)}_\rho e^{(b)}_\sigma e^{(c)}_\tau\) are the Ricci rotation coefficients. For our choice of the metric, the Boltzmann equation becomes

\[
v_r \frac{\partial f}{\partial r} + e^{\frac{\nu}{2}} \frac{v_\theta}{r} \frac{\partial f}{\partial \theta} + e^{\frac{\nu}{2}} \frac{v_\phi}{r} \frac{\partial f}{\partial \phi} - \left(\frac{v_r^2}{2} - \frac{v_\theta^2 + v_\phi^2}{r}\right) \frac{\partial f}{\partial v_r} - \frac{v_r}{r} \left(\frac{v_\theta}{v_r} \frac{\partial f}{\partial v_\theta} + \frac{v_\phi}{v_r} \frac{\partial f}{\partial v_\phi}\right)
\]

\[
- e^{\frac{\nu}{2}} \frac{v_r}{r} \cot \theta \left(\frac{v_\theta}{v_r} \frac{\partial f}{\partial v_\theta} - \frac{v_\phi}{v_r} \frac{\partial f}{\partial v_\phi}\right) = 0,
\]

where

\[
v^{(0)} = v_r, \quad v^{(1)} = v_r, \quad v^{(2)} = v_\theta, \quad v^{(3)} = v_\phi.
\]

If we assume that the coordinate dependence of the distribution function is only through \(r\), equation (12) reduces to

\[
v_r \frac{\partial f}{\partial r} - \left(\frac{v_r^2}{2} \frac{v_\phi^2 + v_\theta^2}{r}\right) \frac{\partial f}{\partial v_r} - \frac{v_r}{r} \left(\frac{v_\theta}{v_r} \frac{\partial f}{\partial v_\theta} + \frac{v_\phi}{v_r} \frac{\partial f}{\partial v_\phi}\right)
\]

\[
- e^{\frac{\nu}{2}} \frac{v_r}{r} \cot \theta \left(\frac{v_\theta}{v_r} \frac{\partial f}{\partial v_\theta} - \frac{v_\phi}{v_r} \frac{\partial f}{\partial v_\phi}\right) = 0.
\]

The spherically symmetric nature of the metric requires that the coefficient of \(\cot \theta\) be zero. Now, multiplying equation (14) by \(m v_r dv_r\), where \(dv = \frac{1}{v_r} dv_r dv_\theta dv_\phi\) is an invariant volume element in the
velocity space and \( m \) is the mass of the galaxy, and integrating over the velocity space and assuming that \( f \) vanishes sufficiently rapidly as the velocities tend to \( \pm \infty \), we obtain
\[
r \frac{\partial}{\partial r} \left[ \rho \langle v_t^2 \rangle \right] + \frac{1}{2} \rho \left[ \langle v_t^2 \rangle + \langle v_r^2 \rangle \right] r \mu' - \rho \left[ \langle v_t^2 \rangle + \langle v_r^2 \rangle - 2 \langle v_r^2 \rangle \right] = 0,
\]
where \( \rho \) is the mass density, and \( \langle \rangle \) represents the average value of the quantity it contains. Multiplying equation (15) by \( 4\pi r^2 \) and integrating over the cluster of galaxies, we obtain
\[
- \int_0^R 4\pi r \left[ \langle v_t^2 \rangle + \langle v_r^2 \rangle + \langle v_r^2 \rangle \right] r^2 dr + \frac{1}{2} \int_0^R 4\pi r^3 \rho \left[ \langle v_t^2 \rangle + \langle v_r^2 \rangle \right] \frac{\partial \mu}{\partial r} dr = 0.
\]
The total kinetic energy of the galaxies is given by
\[
K = \int_0^R 2\pi \rho \left[ \langle v_t^2 \rangle + \langle v_r^2 \rangle + \langle v_r^2 \rangle \right] r^2 dr,
\]
so that equation (16) reduces to
\[
2K = \frac{1}{2} \int_0^R 4\pi r^3 \rho \left[ \langle v_t^2 \rangle + \langle v_r^2 \rangle \right] \frac{\partial \mu}{\partial r} dr.
\]
In terms of the distribution function we can write the energy momentum tensor \( \tau_{\mu\nu} \) as [22]
\[
\tau_{\mu\nu} = \int f m v_\mu v_\nu dv,
\]
leading to
\[
\rho_{\text{eff}} = \rho \langle v_t^2 \rangle, \quad \langle v_r^2 \rangle = \rho \langle v_r^2 \rangle, \quad \langle v_r^2 \rangle = \rho \langle v_r^2 \rangle.
\]
Now, by adding the minus of equation (5) to equation (6) and adding twice of equation (7) to the result we obtain
\[
e^{-\nu} \left( \frac{\mu'}{r} - \frac{\mu' \nu'}{4} + \frac{\mu''}{2} + \frac{\mu^2}{4} \right) = 4\pi G \rho \langle v^2 \rangle + 4\pi G (\rho + 3p) - \lambda,
\]
where we have defined \( \langle v^2 \rangle = \langle v_t^2 \rangle + \langle v_r^2 \rangle + \langle v_r^2 \rangle \). Let us now see what kind of approximations we can make in this formalism. First, we assume that \( \mu(r) \) and \( \nu(r) \) are small so that we can neglect the quadratic terms in equation (21). Second, we assume that the galaxies have velocities much smaller than the speed of light, so \( \langle v_r^2 \rangle, \langle v_r^2 \rangle, \langle v_r^2 \rangle \ll \langle v_t^2 \rangle \approx 1 \). These conditions can be applied to test particles in stable circular motion around galaxies and to the galactic clusters. Equation (21) is then reduced to
\[
4\pi G \rho = \frac{1}{2} \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \mu' \right) + \lambda - 4\pi G (\rho + 3p).
\]
To go any further, we need to specify the equation of the state for the X-matter which we assume to have the following form
\[
p = \alpha \rho X,
\]
where \( \alpha \) is an arbitrary constant. Equation (22) is therefore reduced to
\[
4\pi G \rho = \frac{1}{2} \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \mu' \right) + \lambda - 4\pi G (1 + 3\alpha) \rho X.
\]
Multiplying equation (24) by \( r^2 \) and integrating from 0 to \( r \) we obtain
\[
\frac{1}{2} \frac{1}{r^2} \frac{\partial \mu}{\partial r} - GM(r) + \frac{1}{3} \lambda r^3 - GM_X(r) = 0,
\]
where we have defined

\[ M(r) = 4\pi \int_0^r \rho(r') r'^2 \, dr', \tag{26} \]

and

\[ M_x(r) = 4\pi(1 + 3\alpha) \int_0^r \rho_x(r') r'^2 \, dr'. \tag{27} \]

Finally, multiplying equation (25) by \( \frac{dM(r)}{r} \) and integrating from 0 to \( R \) and using equation (18) we obtain

\[ W + 2K + \frac{1}{3}M + W_x = 0, \tag{28} \]

where

\[ W = - \int_0^R \frac{GM(r)}{r} dM(r), \tag{29} \]

\[ W_x = - \int_0^R \frac{GM_x(r)}{r} dM(r), \tag{30} \]

and

\[ I = \int_0^R r^2 dM(r), \tag{31} \]

is the moment of inertia of the system. Equation (28) is the generalized virial theorem on the brane in the presence of the brane cosmological constant and X-matter. If \( W_x = 0 \), equation (28) reduces to the virial theorem with a cosmological constant [23]. In what follows we assume that \( \alpha \neq -\frac{1}{3} \). In order to obtain the ratio of the X-matter mass to the virial mass, we introduce the following radii [23]

\[ R_v = \frac{M^2}{\int_0^R \frac{M(r)}{r} dM(r)}, \tag{32} \]

\[ R_i = \frac{\int_0^R r^2 dM(r)}{M(r)}, \tag{33} \]

\[ R_X = \frac{M^2_X}{\int_0^R \frac{M_x(r)}{r} dM(r)}, \tag{34} \]

where \( R_v \) is the virial radius and \( R_X \) is the radius defined by the X-matter. If we define the virial mass as

\[ 2K = \frac{GM^2}{R_v}, \tag{35} \]

and use the following relations

\[ W = -\frac{GM^2}{R_v}, \quad W_x = -\frac{GM^2}{R_X}, \quad I = MR_i^2, \tag{36} \]

the generalized virial theorem (28) reduces to

\[ \left( \frac{M_v}{M} \right)^2 = 1 + \left( \frac{M_x}{M} \right)^2 \left( \frac{R_v}{R_X} \right) + \frac{\lambda}{3G} \frac{R_i^2 R_v}{M}. \tag{37} \]

Since the contribution of a cosmological constant to the mass of the galaxy is several order of magnitude smaller than the observed mass, we can neglect it in equation (37). Moreover we can neglect the unity in (37) since \( M_v \) is much greater than \( M \) for most galaxies. Therefore, the virial mass in our model is given by

\[ M_v \simeq M_x \sqrt{\frac{R_v}{R_X}}, \tag{38} \]

showing that the virial mass is proportional to the X-matter mass.
2.1 Estimating the virial mass

In this section we obtain $M_X$ as a function of $r$ and consequently see that the solution is linearly increasing with $r$, pointing to a possible explanation of the mass discrepancy in cluster s of galaxies. We start with Einstein equations (5)-(8) and since most of the baryonic mass in clusters is in the gas form, we assume that the effective energy-density and pressure in $\tau_{\mu\nu}$ is that of a gas and therefore set

$$\rho_{\text{eff}} = \rho_g(r), \quad p_{\text{eff}} = p_g(r).$$

In majority of clusters most of the baryonic mass is in the form of the intra-cluster gas. The gas density $\rho_g$ can be fitted with the observational data by using the following radial baryonic mass distribution $\rho_g(r) = \rho_0 \left(1 + \frac{r^2}{r_c^2}\right)^{-\frac{3\beta}{2}}$,

where $r_c$ is the core radius, and $\rho_0$ and $\beta$ are cluster independent constants. A static spherically symmetric system of collisionless particles that is in equilibrium, can be described by the Jean’s equation

$$\frac{d}{dr} \left[ \rho_g \sigma_r^2 \right] + \frac{2 \rho_g(r)}{r} \left( \sigma_r^2 - \sigma_{\theta,\phi}^2 \right) = -\rho_g(r) \frac{d\Phi}{dr},$$

where $\Phi(r)$ is the gravitational potential, and $\sigma_r$ and $\sigma_{\theta,\phi}$ are the mass-weighted velocity dispersions, respectively, in the radial and tangential directions. We assume that the gas is isotropically distributed inside the cluster, so $\sigma_r = \sigma_{\theta,\phi}$. The gas pressure is related to the velocity dispersion and gas density by $p_g = \rho_g \sigma_r^2$. By assumption that the gravitational field is weak so that it satisfies the usual Poisson equation $\Delta \Phi \approx 4\pi \rho_{\text{tot}}$, where $\rho_{\text{tot}}$ is the energy density including $\rho_g$ and other forms of matter, like luminous matter and the X-matter, etc., the Jean’s equation becomes

$$\frac{dp_g(r)}{dr} = -\rho_g(r) \frac{d\Phi}{dr} = -\frac{GM_{\text{tot}}}{r^2} \rho_g(r),$$

where $M_{\text{tot}}(r)$ is the total mass inside the radius $r$. The observed X-ray emission from the hot ionized intra-cluster gas is usually interpreted by assuming that the gas is in isothermal equilibrium. So we assume that the gas is in equilibrium state having the equation of state $p_g = k_B T_g \mu m_p \rho_g$,

where $\mu = 0.61$ is the mean atomic weight of the particles in the gas cluster, and $m_p$ is the proton’s mass. Equation (42) then becomes

$$M_{\text{tot}}(r) = -\frac{k_B T_g}{\mu m_p G} r^2 \frac{d}{dr} \ln \rho_g = \frac{3k_B T_g \beta}{\mu m_p G} r^3 \left( \frac{r^2}{r_c^2} + 1 \right)^2,$$

where the second equality is obtained using equation (40). Using equations (26) and (27) we can obtain another expression for $M_{\text{tot}}$

$$\frac{dM_{\text{tot}}}{dr} = 4\pi r^2 \rho_g + 4\pi (1 + 3\alpha) r^2 \rho_X.$$

Combining equations (44) and (45) we obtain the following expression for $\rho_X$,

$$\rho_X = \frac{1}{1 + 3\alpha} \left[ \frac{3k_B T_g \beta}{4\pi \mu m_p G} \left( \frac{r^2}{r_c^2} + \frac{2\beta}{2} \right) - \rho_0 \left(1 + \frac{r^2}{r_c^2}\right)^{-\frac{3\beta}{2}} \right].$$
In the regime $r \gg r_c$, equation (46) is reduced to
\[
\rho_X \simeq \frac{1}{1 + 3\alpha} \left[ \frac{3k_B T g \beta}{4\pi \mu m_p G} - \rho_0 r_c^{3\beta} r^{2-3\beta} \right] \frac{1}{r^2}, \tag{47}
\]
and by equation (27) we obtain
\[
M_X \simeq \left[ \frac{3k_B T g \beta}{\mu m_p G} - \frac{4\pi \rho_0 r_c^{3\beta}}{3(1-\beta)} r^{2-3\beta} \right] r. \tag{48}
\]
We now consider the limit of large $r$. For most clusters the parameter $\beta$ has the value $\beta \geq \frac{2}{3}$ [24]. Because of the smallness of the second term in (48) in this case, we have
\[
M_X \approx \frac{3k_B T g}{\mu m_p G} r. \tag{49}
\]
As can be seen, the X-matter mass increases linearly with $r$ and this is similar to the behavior of dark matter in the cluster of galaxies. Let us now make some estimates. First we note that $k_B T g \approx 5$ KeV for most clusters. The virial radius of clusters is usually assumed to be $r_{200}$, indicating the radius for which the energy density $\rho_{200}$ of the cluster becomes $\rho_{200} = 200\rho_{\text{univ}}$, where $\rho_{\text{univ}} \approx 4.6975 \times 10^{-30} h_{50}^2 g/cm^3$ [24]. We can estimate the critical radius of the X-matter by determining the distance in which the X-matter density is equal to $\rho_{200}$. Using (47) and noting that the last term may be dropped in our limit we find
\[
r_{cr} = 1.82 \sqrt{\beta} \left( \frac{k_B T g}{5 \text{ KeV}} \right)^{\frac{1}{2}} h_{50}^{-1} \text{ Mpc}. \tag{50}
\]
We then have
\[
M_{X}^{cr} = 9.8 \times 10^{14} \beta^{\frac{1}{2}} \left( \frac{k_B T g}{5 \text{ KeV}} \right)^{\frac{1}{2}} h_{50}^{-1} M_\odot. \tag{51}
\]
For $\beta < \frac{2}{3}$ the X-matter mass has a maximum at
\[
R_{\text{max}} = \left( \frac{3k_B \beta T g}{4\pi G \rho_0 \mu m_p r_c^{3\beta}} \right)^{\frac{1}{2-3\beta}}. \tag{52}
\]
The maximum value of the X-matter mass is given by
\[
G M_X = \left( \frac{3k_B \beta T g}{\mu m_p G} \right) \left( \frac{2-3\beta}{3(1-\beta)} \right) R_{\text{max}}. \tag{53}
\]

3 X-matter solutions on the brane

In this section we find the vacuum solutions of the theory and obtain the equation of motion of the X-matter. Assuming that $\tau_{\mu\nu} = 0$, the effective vacuum equation (1) is reduced to
\[
G_{\mu\nu} = -\lambda g_{\mu\nu} + Q_{\mu\nu}. \tag{54}
\]
Let us assume that $Q_{\mu\nu}$ has the form of an anisotropic perfect fluid
\[
Q^\mu_\nu = 8\pi G \text{ diag} \left( -\rho_X, p_X^\parallel, p_X^\perp, p_X^\perp \right), \tag{55}
\]
with an equation of state
\[
\rho_X = p_X^\parallel. \tag{56}
\]
Taking metric \( (2) \), the vacuum field equations on the brane become
\[
e^{-\nu(r)} \left( 1 - \frac{1}{r^2} \right) = -\lambda - 8\pi\rho_x \tag{57}
\]
\[
e^{-\nu(r)} \left( 1 - \frac{1}{r^2} \right) = -\lambda + 8\pi\rho_p \tag{58}
\]
\[
e^{-\nu(r)} \left( 2\mu' - 2\nu' - \mu'\nu' + 2\mu''r + \mu'^2r \right) = -\lambda + 8\pi\rho_p \tag{59}
\]
The equations above contain 4 unknown quantities so that an extra relation would be desirable. Such an equation can be provided by assuming that the brane admits a one parameter group of conformal motions so that
\[
\mathcal{L}_\xi g_{\mu\nu} = \phi(r) g_{\mu\nu}, \tag{60}
\]
where \( \phi(r) \) is the conformal factor and \( \mathcal{L} \) represents the Lie derivative. Moreover, we assume the following general form for the vector field \( \xi \)
\[
\xi = \xi^0(t, r) \frac{\partial}{\partial t} + \xi^1(t, r) \frac{\partial}{\partial r} + \xi^2(\theta, \phi) \frac{\partial}{\partial \theta} + \xi^3(\theta, \phi) \frac{\partial}{\partial \phi}. \tag{61}
\]
Use of equation (60) then leads to
\[
\mu'\xi^1 + 2\frac{\partial \xi^0}{\partial t} = \phi(r), \tag{62}
\]
\[
\nu'\xi^1 + 2\frac{\partial \xi^1}{\partial r} = \phi(r), \tag{63}
\]
\[
e^\nu \frac{\partial \xi^1}{\partial t} = e^\mu \frac{\partial \xi^0}{\partial r}, \tag{64}
\]
\[
\frac{1}{r} \xi^1 + \frac{\partial \xi^2}{\partial \theta} = \frac{\phi(r)}{2}, \tag{65}
\]
\[
\frac{1}{r} \xi^1 + \cot \theta \xi^2 + \frac{\partial \xi^3}{\partial \phi} = \frac{\phi(r)}{2}, \tag{66}
\]
\[
\sin^2 \theta \frac{\partial \xi^3}{\partial \theta} = -\frac{\partial \xi^2}{\partial \phi}. \tag{67}
\]
In order to solve these equations, we see from equation (65) that
\[
\xi^2 = \frac{dF(\varphi)}{d\varphi}, \tag{68}
\]
\[
\xi^1 = \frac{r}{2} \phi, \tag{69}
\]
where we have set the arbitrary separation constant equal to zero (this can be done easily by shifting \( \phi \) by a constant) and \( F(\varphi) \) is an arbitrary function. From equation (64) we have \( \xi^0 = \xi^0(t) \) which, upon using equation (62), we obtain
\[
\xi^0 = \frac{k}{2} t + A, \tag{70}
\]
where \( k \) is an arbitrary separation constant and \( A \) is some other constant. Now, using equation (66) we obtain
\[
\xi^3 = -\cot \theta F(\varphi) + G(\theta), \tag{71}
\]
where \( G(\theta) \) is some arbitrary function. Since \( \frac{\partial}{\partial \varphi} \) is a killing vector field we can set \( A = 0 \) without loss of generality, so that the form of the conformal killing vector field \( \xi \) is given by
\[
\xi = \frac{k}{2} \frac{\partial}{\partial t} + \frac{r \phi}{2} \frac{\partial}{\partial r} + \frac{dF(\varphi)}{d\varphi} \frac{\partial}{\partial \theta} - [\cot \theta F(\varphi) - G(\theta)] \frac{\partial}{\partial \varphi}. \tag{72}
\]

Now, from equations (62) and (63) we can write the metric components in terms of the conformal factor

\[ e^{\mu(r)} = C^2 r^2 e^{\exp \left[ -2k \int \frac{dr}{r \phi} \right] }, \]  

\[ e^\nu(r) = \frac{B^2}{\phi}, \]  

(73)

(74)

where C and B are arbitrary integration constants. Upon substitution of these equations into the field equations (57)-(59) we obtain the following system of equations for \( \phi(r) \), \( \rho_X \) and \( p^\perp_X \)

\[ \frac{1}{r^2} \frac{\phi^2}{B^2} \left( 1 + 2r \frac{\phi'}{\phi} \right) - \frac{1}{r^2} = -\lambda - 8\pi G \rho_X, \]  

(75)

\[ \frac{1}{r^2} \frac{\phi^2}{B^2} \left( 3 - 2k \frac{k}{\phi} \right) - \frac{1}{r^2} = -\lambda + 8\pi G \rho_X, \]  

(76)

\[ \frac{1}{r^2} \frac{\phi^2}{B^2} \left( 2r \frac{\phi'}{\phi} + \frac{(\phi - k)^2}{\phi^2} \right) = -\lambda + 8\pi G p^\perp_X. \]  

(77)

Assuming \( \lambda = 0 \) and equating equations (75) and (76) we obtain

\[ r \phi \phi' + 2 \phi^2 - k \phi - B^2 = 0. \]  

(78)

Now, if one expresses \( r \) in terms of \( \phi \), one finds

\[ r^2 = r_0^2 \frac{f(\phi)}{\sqrt{2 \phi^2 - k \phi - B^2}}, \]  

(79)

where \( r_0 \) is an integration constant and \( f(\phi) \) is defined as

\[ f(\phi) = \exp \left\{ \frac{k}{\sqrt{8B^2 + k^2}} \tanh^{-1} \left[ \frac{-k + 4 \phi}{\sqrt{8B^2 + k^2}} \right] \right\}. \]  

(80)

The energy density and pressure of the X-matter can now be obtained in terms of \( \phi \)

\[ 8\pi G \rho_X = 8\pi G p^\perp_X = -\frac{1}{B^2 r_0^2} \frac{\sqrt{2 \phi^2 - k \phi - B^2}}{f(\phi)} (B^2 - 3 \phi^2 + 2k \phi), \]  

(81)

and

\[ 8\pi G p^\perp_X = -\frac{1}{B^2 r_0^2} \frac{\sqrt{2 \phi^2 - k \phi - B^2}}{f(\phi)} (3 \phi^2 - 2B^2 - k^2). \]  

(82)

The equation of state of the X-matter can be obtained from equations (81) and (82)

\[ 8\pi G (p^\perp_X + \rho_X) = -\frac{1}{B^2 r_0^2} (k^2 + 2k \phi + B^2). \]  

(83)

We see that in the limit \( r \to \infty \) the X-matter has an equation of state \( p^\perp_X = -\rho_X \).

### 4 Galaxy rotation curves

The results of the previous section is in agreement with the observed behavior of the tangential velocity of galaxies; we know from observations that the rotational velocities increase almost linearly from the center of a galaxy and approaches a constant value of about 200 km/s as one moves away from the center [12]. In this section we consider the tangential velocity of a test particle which moves in a circular time-like geodesic orbit. The Lagrangian of the system is given by [25]

\[ 2L = \left( \frac{ds}{d\tau} \right)^2 - e^{\mu(r)} \left( \frac{dt}{d\tau} \right)^2 + e^\nu(r) \left( \frac{dr}{d\tau} \right)^2 + r^2 \left( \frac{d\Omega}{d\tau} \right)^2, \]  

(84)
where $\tau$ is the affine parameter along the geodesic. From equation (84) we find that the energy $E = e^{\nu} \dot{t}$, the $\varphi$ component of the angular momentum of the particle $l_\varphi = r^2 \sin^2 \theta \dot{\varphi}$, where a dot denotes differentiation with respect to $\tau$ and the total angular momentum of the particle $l^2 = l_\varphi^2 + (l_\varphi/\sin \theta)^2$ are conserved quantities. The total angular momentum of the particle can be written in terms of the solid angle as $l^2 = r^4 \Omega^2$ [25]. The equation of the geodesic orbits can then be written as

$$\dot{r}^2 + V(r) = 0,$$

where

$$V(r) = -e^{-\nu} \left( E^2 e^{-\mu} - \frac{l^2}{r^2} - 1 \right).$$

For stable circular orbits, we must have

$$\dot{r} = 0, \quad \frac{\partial V}{\partial r} = 0, \quad \frac{\partial^2 V}{\partial r^2} > 0,$$

so that the potential describes a minimum of the motion. These conditions lead to the following expressions for the energy and total angular momentum as [19]

$$E^2 = \frac{2e^{\mu}}{2 - r \mu'}, \quad l^2 = \frac{r^3 \mu'}{2 - r \mu'}.$$

On the other hand the line element (2) can be written in terms of the velocity, measured by an inertial observer far from the source as [25]

$$ds^2 = -dt^2 \left( 1 - \frac{v^2}{c^2} \right),$$

where

$$v^2 = e^{-\mu} \left[ e^{\nu} \left( \frac{dr}{dt} \right)^2 + r^2 \left( \frac{d\Omega}{dt} \right)^2 \right].$$

In the case of stable circular orbits we have the following expression for the tangential velocity [26]

$$v_{tg}^2 = r^2 e^{-\mu} \left( \frac{d\Omega}{dt} \right)^2.$$

Using the expressions for the conserved quantities $E$ and $l^2$, we obtain

$$v_{tg}^2 = e^{\mu} \frac{l^2}{r^2 E^2}.$$
Use of equation (88) then leads to
\[ v_{tg}^2 = \frac{r \mu'}{2}, \]  
(93)
showing that \( v_{tg} \) depends only on the time-time component of the metric. Using equation (73), the tangential velocity in terms of \( \phi \) is given by
\[ v_{tg}^2 = 1 - \frac{k}{\phi}. \]  
(94)
Although we cannot express \( v_{tg} \) explicitly in terms of \( r \), we can obtain the value of the tangential velocity at infinity from equation (79). We see from this equation that in the case of \( \phi \to \phi_{1,2} \) where
\[ \phi_{1,2} = \frac{k \pm \sqrt{k^2 + 8B^2}}{4}, \]  
(95)
we have \( r \to \infty \). If we use the upper sign the tangential velocity at infinity is given by
\[ v_{tg\infty} = \sqrt{1 - \frac{4k}{k + \sqrt{8B^2 + k^2}}}. \]  
(96)
We only need to fix the ratio of the constants in equation (96) to be consistent with the observational data. For example with the choice \( B/k = 1.000000667 \) we have \( v_{tg\infty} = 200 \text{km/s} \). Figure 1 shows a plot of \( v_{tg} \) as a function of \( r \).

Let us obtain the X-matter mass of the galaxy and its dependence on \( r \). After integrating equation (57), we obtain
\[ e^{-\nu} = 1 - \frac{2G}{r} M_X(r), \]  
(97)
where we have defined
\[ M_X(r) = 4\pi \int_0^r \rho_X(r') r'^2 dr'. \]  
(98)
That is equation (27) with \( \alpha = 0 \). Now define a boundary \( R \) of the galaxy where \( \rho_b \approx 0 \). We then have
\[ e^\mu = e^{-\nu} = 1 - \frac{2GM_b}{R}, \]  
(99)
where
\[ M_b = 4\pi \int_0^R \rho(r') r'^2 dr', \]  
(100)
is the total mass of the galaxy. By equations (74), (94) and (99), we can find the relation between constants of the model in terms of the tangential velocity at the boundary
\[ \frac{B^2}{k^2} = \frac{1 - \frac{2GM_b}{R}}{[1 - v_{tg\infty}(R)]^2}. \]  
(101)
Now, substituting equations (74) and (94) into equation (97) and using equation (101), we obtain
\[ M_X = \frac{M_b}{R} r, \]  
(102)
where we have used the fact that the tangential velocity is much smaller than the speed of light. The X-matter mass depends linearly on \( r \) even after the boundary of the galaxy is reached, and has a value of the order of the baryonic mass. This is in agreement with the observational data.
5 Conclusions

In this paper we have considered the problems of virial mass and rotation curves of galaxies in the context of a brane-world scenario which uses a confining potential instead of $\mathbb{Z}_2$ symmetry and the Israel junction conditions in order to confine the gauge fields to the brane. We have shown that these problems can be adequately addressed in this model by identifying the conserved geometric quantity $Q_{\mu\nu}$ with a new kind of matter, also known as the X-matter. The virial theorem was then obtained, assuming a perfect fluid form for $Q_{\mu\nu}$ and shown to be related linearly to a geometrical mass obtained by $Q_{\mu\nu}$. We also assumed an anisotropic form for $Q_{\mu\nu}$ in order to find the energy-density and the pressure of the X-matter and obtained the tangential velocity of a point particle in the galaxy, showing that it behaves linearly with respect to $r$. The X-matter mass of the galaxy was then calculated and shown to be of the order of the baryonic mass of the galaxy which extends beyond its boundary linearly with $r$.

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