Generalized “bra-ket” formalism

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Abstract

The Dirac’s bra-ket formalism is generalized to finite-dimensional vector spaces with indefinite metric in a simple mathematical context similar to that of the theory of general tensors where, in addition, scalar products are introduced with the help of a metric operator. The specific calculation rules are given in a suitable intuitive notation. It is shown that the proposed bra-ket calculus is appropriate for the general theory of basis transformations and finite-dimensional representations of the symmetry groups of the metric operators. The presented application is the theory of finite-dimensional representations of the $SL(2,\mathbb{C})$ group with invariant scalar products.

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1 Introduction

The bra-ket calculus invented by Dirac [1] for unitary or Hilbert spaces is one of the most coherent, efficient and elegant formalisms. This is based on the natural relation among vectors and functionals represented by ket and respectively bra vectors. In this way, one obtains simple calculation rules which include the axioms of the scalar product as well as the consequences of the Frechet-Riesz theorem. Grace of these qualities, the Dirac formalism offers us the opportunity of working with general operator relations, independent on the concrete representations that can have very different features in the general case of the rigged Hilbert spaces used in quantum theories.

However, in many problems the vector spaces cannot be organized as unitary, Euclidian or Hilbert spaces. We refer especially to the spaces of finite-dimensional linear representations of the non-compact Lie groups [2, 3] where the bilinear forms we need to construct invariants are not positive definite [4]. In this case one uses the tensor calculus with indices in upper and lower positions and bilinear (or inner) forms defined with the help of a metric tensor. For unitary, Euclidian or Hilbert spaces the tensor calculus is equivalent with the Dirac formalism in a representation given by an orthonormal basis but for the spaces with indefinite metric [5] we have not yet a suitable Dirac formalism which should reproduce all the mechanisms of calculus with covariant and contravariant indices. For this reason we would like to propose here a generalization of the Dirac formalism to vector spaces with indefinite metric. We restrict ourselves only to the finite-dimensional case when these spaces are called semi-unitary or semi-Euclidian [6].

The main problem here is the generalization of the mutual bra-ket relation to spaces with indefinite metric. In our opinion, this problem cannot be solved using only one vector space and its dual space. We mean that starting with a space considered as being of covariant vectors (with contravariant components) and with its dual space of contravariant functionals, we cannot relate covariant vectors with contravariant functionals in a satisfactory bra-ket formalism. This could be achieved only by introducing new ingredients, namely contravariant vectors and covariant functionals, which should allow one to relate vectors and functionals of same kind (covariant or contravariant). Therefore, the generalization of the Dirac formalism requires to double the number of vector spaces like in the theory of general tensors (with simple and dotted indices) where one uses four vector spaces.
associated with the four unequivalent fundamental representations of the group of general linear transformations. This means that we have already the framework we need. It remains only to correctly define mutual bra-ket relations in accordance with the mentioned exigency of a coherent bra-ket mechanism. We show that this can be done with the help of two anti-linear mappings that conserve the covariance. These will relate the spaces of covariant and contravariant ket vectors with their corresponding spaces of bra vectors. Moreover, we couple between themselves the spaces of ket vectors as well as those of bra vectors through an isometry compatible with the considered anti-linear mappings. In these conditions we can use the operator of this isometry as metric operator. This will replace the metric tensor taking over its role in defining scalar products.

Our objective is to generalize the Dirac bra-ket formalism in this mathematical context which combines the framework of the general tensor calculus with the theory of bilinear forms given by metric operators. We establish the basic calculation rules in suitable notations and we verify that these lead to the common tensor calculus in any representation given by a system of dual bases. Actually, we shall see that our formalism recovers all the main results of the theory of the general linear transformations but expressed only in terms of components with usual covariant or contravariant indices. This is because in the bra-ket formalism, where the complex conjugated components appear naturally as the components of bra vectors, the artifice of dotted indices is no more needed. On the other hand, we show that in our approach we have all the technical advantages of the standard bra-ket formalism. One of them is that we can work directly with general relations involving operators instead of their matrix elements in particular representations. Moreover, we can manipulate simultaneously different spectral representations of these operators allowing us to easily study the theory of finite-dimensional representation of the non-compact Lie groups, including those of the symmetry group of the metric operator.

The first step is to precise the mathematical framework of our attempt. This is presented in the second section where we define the mutually related pairs of coupled vector spaces which allows us to introduce the metric operator and correct relations among vectors and functionals. The next section is devoted to the specific symbols and notations we propose for vectors and operators. In section 4 we construct two kind of compatible hermitian forms, called dual forms and scalar products respectively, while in section 5 we define
Section 6 is devoted to the theory of orthogonal projection operators which help us to define pairs of coupled subspaces with hermitian metric operators. The matrix representations in systems of dual bases are studied in section 7 pointing out the advantages of the orthonormal ones. The theory of basis transformations is briefly treated in section 8 where we give the form of the general linear transformations and we study the symmetry transformations that leave invariant the form of the scalar product. The example we propose in section 9 is the theory of finite-dimensional representations of the $SL(2, \mathbb{C})$ group with invariant scalar products.

2 Coupled vector spaces

Our construction is based on a pair of complex vector spaces, $V$ and $\hat{V}$, of covariant vectors, $x, y, ... \in V$, and contravariant vectors, $\hat{x}, \hat{y}, ... \in \hat{V}$, respectively. The dual of $V$, denoted by $\overline{V}$, is the space of contravariant functionals, $\bar{x}, ...$, while the dual of $\hat{V}$, denoted by $\overline{\hat{V}}$, is the space of covariant functionals, $\bar{\hat{x}}, ...$. The values of these functionals are $\bar{x}(y)$ or $\bar{\hat{x}}(\hat{y})$.

The basis vectors and the vector components are labeled by Latin indices $i, j, k, ...$ while the first Latin ones, $a, b, ...$ are held for current needs. We suppose that all the spaces we work are finite-dimensional of dimension $N$ such that $i, j, ... = 1, 2, ..., N$. Furthermore, we consider the covariant basis $\{e_i\} \subset V$, the contravariant basis $\{\hat{e}_i\} \subset \hat{V}$, and the canonical dual bases, from which $\{\bar{e}_i\} \subset \overline{V}$ is the contravariant one while $\{\bar{\hat{e}}_i\} \subset \overline{\hat{V}}$ is the covariant one. Since we assume that these bases satisfy the usual duality conditions,

$$e^i(e_j) = \delta^i_j, \quad \bar{e}_i(\hat{e}^j) = \delta^j_i, \quad (1)$$

we say that these form a system of dual bases.

A coherent bra-ket mechanism requires to mutually relate among themselves the vectors and functionals. To this end, we introduce the anti-linear mappings which conserve the covariance, $\phi : V \rightarrow \overline{V}$ and $\hat{\phi} : \hat{V} \rightarrow \overline{\hat{V}}$, defined by

$$\phi[e_i] = \bar{e}_i, \quad \hat{\phi}[^{\hat{e}}i] = \bar{\hat{e}}^i, \quad (2)$$

such that

$$\phi[x] = (x^i)^*\bar{e}_i, \quad \hat{\phi}[\hat{y}] = (\hat{y}_i)^*\bar{\hat{e}}^i, \quad (3)$$
for any \( x = e_i x^i \in \mathcal{V} \) and \( \hat{y} = \hat{e}^i \hat{y}_i \in \hat{\mathcal{V}} \). Then from (4) we find
\[
\phi[x](\hat{y}) = (x^i)^* \hat{y}_i = \left( \hat{\phi}[\hat{y}](x) \right)^*.
\]

In other respects, equations (3) show that \( \overline{\mathcal{V}} \sim \mathcal{V}^* \) and \( \hat{\mathcal{V}} \sim \mathcal{V}^* \) where \( \mathcal{V}^* \) is the complex conjugate vector space of \( \mathcal{V} \). Therefore, our system of mutually related vector spaces is similar to that of the theory of general tensors, \( (\mathcal{V}, \mathcal{V}^*, \hat{\mathcal{V}}, \hat{\mathcal{V}}^*) \).

Let us consider now the isomorphism \( \eta : \mathcal{V} \rightarrow \hat{\mathcal{V}} \), defined by the invertible matrix \( |\eta| \) as
\[
\eta e_i = \hat{e}^j \eta_{ji}.
\]
This couples each covariant vector \( x = e_i x^i \) with the contravariant vector \( \hat{x} = \eta x = \hat{e}^i x_i \) of components \( x_i = \eta_{ij} x^j \). The corresponding isomorphism of the dual spaces, \( \bar{\eta} : \overline{\mathcal{V}} \rightarrow \overline{\mathcal{V}} \), is given by
\[
\bar{\eta} \hat{e}^k = (\eta^{-1})^{kj} \hat{e}_j
\]
so that \( \bar{\eta} \hat{y}(\eta x) = y(x) \). Moreover, it is not difficult to verify that the isomorphism \( \eta \) is compatible with the mappings \( \phi \) and \( \hat{\phi} \) (closing the diagram) only if
\[
\phi[x] = \bar{\eta} \hat{\phi}[\eta x], \quad \forall x \in \mathcal{V}.
\]

**Theorem 1** The condition (7) is accomplished if and only if the matrix \( |\eta| \) is hermitian, i.e. \( \eta_{ij} = (\eta_{ji})^* \).

**Proof:** Let us take \( x = e_i \) and calculate \( \hat{e}_i = \bar{\eta} \hat{\phi}[\eta e_i] \). According to (3) and (4), this gives
\[
\hat{e}_i = (\eta_{ji})^* \bar{\eta} \hat{\phi}[\hat{e}^j] = (\eta^{-1})^{jk} (\eta_{ji})^* \hat{e}_k
\]
from which we obtain the desired result.

In what follows we consider only invertible operators \( \eta \) with hermitian matrices in a given system of dual bases. They will be used as the metric operators that define the bilinear forms of the spaces \( \mathcal{V} \) and \( \hat{\mathcal{V}} \),
\[
\begin{align*}
h(x, y) &= \phi[x](\eta y), \quad x, y \in \mathcal{V}, \\
\hat{h}(\hat{x}, \hat{y}) &= \hat{\phi}[\hat{x}](\eta^{-1} \hat{y}), \quad \hat{x}, \hat{y} \in \hat{\mathcal{V}}.
\end{align*}
\]
A little calculation giving us the bilinear forms in terms of vector components points out that these are hermitian since
\[ h(x, y) = h(y, x)^*, \quad \hat{h}(\hat{x}, \hat{y}) = \hat{h}(\hat{y}, \hat{x})^*. \] (10)
In addition, we can verify that
\[ \hat{h}(\eta x, \eta y) = h(x, y) \] (11)
which means that the isomorphism \(\eta\) is in fact an isometry when \(|\eta| = |\eta|^+\).

**Definition 1** The spaces \(V\) and \(\hat{V}\) isometric through the metric operator \(\eta\) represent a pair of coupled vector spaces (cvs) denoted by \((V, \hat{V}, \eta)\). The corresponding dual cvs are \((\hat{V}, \hat{\omega}, \bar{\eta})\). We say that the mappings \(\phi\) and \(\hat{\phi}\) mutually relate these pairs of cvs.

### 3 Notations

The above defined related pairs of cvs represent the appropriate framework of our generalized Dirac formalism. Let us start with the notations of the basic elements.

**Definition 2** The spaces \(K \equiv V\) and \(\hat{K} \equiv \hat{V}\) are the spaces of ket-down vectors, \(/ x \rangle \in K\), or of ket-up vectors, \(\backslash x \rangle \in \hat{K}\).

Thus in our formalism the covariant vectors appear as ket-down vectors while the contravariant ones as ket-up vectors. The bra vectors mutually related with these ket vectors can be defined with the help of the mappings \(\phi\) and \(\hat{\phi}\).

**Definition 3** The bra vector related to the ket-down vector \(/ x \rangle\) is the bra-down vector \(\langle x \backslash = \phi[\backslash x \rangle] \in \hat{B} \equiv \overline{V}\) while the bra vector related to the ket-up vector \(\hat{x} \rangle\) is the bra-up vector \(\langle \hat{x} / = \hat{\phi}[/ \hat{x} \rangle] \in \hat{B} \equiv \overline{V}\).

In this manner we have related the covariant ket vectors of \(K\) with the covariant bra vectors of \(B\), which are just the covariant functionals defined on the other ket space, \(\hat{K}\). Similarly, the spaces of contravariant vectors, \(\hat{K}\) and \(\hat{B}\), are also related between themselves even though the contravariant functionals of \(\hat{B}\) are defined on \(K\). Of course, the mutually related ket and bra
vectors will be denoted systematically with the same symbol as in the usual bra-ket formalism.

Apparently these crossed bra-ket relations seem to be forced but this is the unique way to obtain well-defined hermitian bilinear forms compatible with the natural duality. In other respects, our bra-ket relations are correct in the sense that the bra vector related with a linear combination of ket vectors is the corresponding anti-linear combination of bra vectors. Obviously, this is because the mappings $\phi$ and $\hat{\phi}$ are anti-linear (e.g. $\phi[\alpha /x\rangle + \beta /y\rangle] = \alpha^* \langle x\backslash + \beta^* \langle y\backslash$).

Let us consider now the linear operators defined on our cvs. In general, we denote by $L(\mathcal{V}, \mathcal{V}')$ the set of the linear operators which map $\mathcal{V}$ onto $\mathcal{V}'$. In our case, the main pieces are the algebras $L(\mathcal{K}, \mathcal{K})$ and $L(\hat{\mathcal{K}}, \hat{\mathcal{K}})$, but we are also interested by the operators which map the spaces $\mathcal{K}$ and $\hat{\mathcal{K}}$ to each other. We start with the observation that it is natural to denote by $\parallel \in L(\mathcal{K}, \mathcal{K})$ and $\backslash \in L(\hat{\mathcal{K}}, \hat{\mathcal{K}})$ the identity operators of these algebras since they act upon the ket-down and respectively ket-up vectors. Moreover, this notation helps us to indicate the action of the operators of different kind by writing them between identity operators. Thus, the operators of $L(\mathcal{K}, \mathcal{K})$ can be denoted either simply by $A, B, ...$ or by $/A/, /B/, ...$ in order to avoid possible confusions with the operators of the other algebra, $\backslash A\backslash, \backslash B\backslash, ... \in L(\hat{\mathcal{K}}, \hat{\mathcal{K}})$, or with those from $L(\mathcal{K}, \hat{\mathcal{K}})$ or $L(\hat{\mathcal{K}}, \mathcal{K})$ that have to be delimited by both identity operators in a suitable order. On the other hand, the notation we propose has the advantage of indicating the allowed algebraic operations. For example, it is clear that the operators $\backslash A\backslash \in L(\mathcal{K}, \hat{\mathcal{K}})$ and $/B/ \in L(\hat{\mathcal{K}}, \mathcal{K})$ can be multiplied to each other while their sum does not make sense.

A special case is that of the metric operators, $\eta \in L(\mathcal{K}, \hat{\mathcal{K}})$ and $\eta^{-1} \in L(\hat{\mathcal{K}}, \mathcal{K})$, which play the central role in our construction. They will be represented by

\[ \eta \equiv \backslash , \quad \eta^{-1} \equiv \backslash \backslash , \]

in order to obtain the intuitive calculation rules

\[ \backslash \backslash = \parallel , \quad \backslash \backslash = \backslash \backslash . \]

With these symbols, the pair of cvs of ket vectors can be denoted now by $(\mathcal{K}, \hat{\mathcal{K}}, \backslash)$ while the related pair of bra cvs is $(\hat{\mathcal{B}}, \mathcal{B}, \backslash)$. The isometry $\backslash : \mathcal{K} \to \hat{\mathcal{K}}$ couples not only the vectors $/x\rangle$ and $\backslash \hat{x}\rangle = \langle \eta x\rangle \equiv \backslash /x\rangle$, but also
couples the operators $/A/ \in L(K, K)$ and $\hat{A} \in L(\hat{K}, \hat{K})$ through

$$\hat{A} = /A/.$$  \hfill (14)

4 Hermitian forms

Here we can define two kinds of brackets. The first one represents the values of covariant or contravariant functionals. We denote by $\langle \hat{x} / y \rangle \equiv \hat{\phi}(\hat{x})(y)$ the value of the contravariant functional $\langle \hat{x} / \rangle$ calculated for the vector $/y\rangle$, and by $\langle y \hat{x} \rangle \equiv \phi(y)(\hat{x})$, the value of the covariant functional $\langle y \rangle$ calculated for $\hat{x}$. We say that the mappings $\langle / \rangle : \hat{K} \times K \to \mathbb{C}$ and $\langle \ \rangle : K \times \hat{K} \to \mathbb{C}$ are dual forms. From (3) it results that the dual forms are hermitian, i.e.

$$\langle y \hat{x} \rangle = \langle \hat{x} / y \rangle^*, \ \forall /y\rangle \in K, \ \hat{x} \in \hat{K}.$$  \hfill (15)

The second kind of brackets are just the hermitian bilinear forms defined by (8). Since the metric operator is invertible, these bilinear forms are nondegenerate and, therefore, can be called scalar products. In our new notation the scalar product of $K$, $\langle / \rangle : K \times K \to \mathbb{C}$, has the values

$$\langle x / y \rangle = \langle x \sqrt{y} \rangle \equiv h(x, y), \ /x\rangle, /y\rangle \in K,$$  \hfill (16)

while that of $\hat{K}$, $\langle \ \rangle : \hat{K} \times \hat{K} \to \mathbb{C}$, gives

$$\langle \hat{x} \sqrt{\hat{y}} \rangle = \langle \hat{x} / \sqrt{\hat{y}} \rangle \equiv \hat{h}(\hat{x}, \hat{y}), \ \hat{x}, \hat{y} \in \hat{K}.$$  \hfill (17)

The equations (10) which show that these scalar products are hermitian take the form

$$\langle x \sqrt{y} \rangle = \langle y \sqrt{x} \rangle^*, \ \langle \hat{x} \sqrt{\hat{y}} \rangle = \langle \hat{y} \sqrt{\hat{x}} \rangle^*.$$  \hfill (18)

Other useful relations can be written starting with the coupled vectors $\hat{x} = \sqrt{x}$ and $\hat{y} = \sqrt{y}$. Thus we can find equivalences among the values of the dual forms and those of the scalar products (e.g. $\langle x \sqrt{y} \rangle = \langle x \sqrt{y} \rangle$, $\langle \hat{x} / \hat{y} \rangle = \langle \hat{x} \sqrt{\hat{y}} \rangle$, etc.) or to recover equation (11) giving us the isometry of cvs,

$$\langle x \sqrt{y} \rangle = \langle \hat{x} \sqrt{\hat{y}} \rangle.$$  \hfill (19)

All these brackets can be imagined as resulting from the traditional “juxtaposition” of the Dirac formalism. For example, we can write $\langle y \sqrt{x} \rangle =
\[(y) (\hat{x}), \langle x \backslash y \rangle = \langle x \backslash (\backslash/y) \rangle = ((x\backslash) /y),\] and so on. Hence the conclusion is that we can combine ket and bra vectors of any kind in order to write brackets. If the slash-lines are parallel we obtain a dual form but when these are not parallel, leaving an empty angle, then we understand that therein is a metric operator giving us a scalar product.

The orthogonality is defined by the scalar product which play the same role as those of unitary or Hilbert spaces with the difference that here the “squared norm” \( \langle x \backslash x \rangle \) can take any real value, including 0 even for vectors \( /x \rangle \neq 0 \). From (19) we see that if two ket-down vectors are orthogonal then their coupled ket-up vectors are also orthogonal. As mentioned before, the cvs are isometric in the sense that for two coupled ket vectors we have \( \langle x \backslash x \rangle = \langle \hat{x} \backslash \hat{x} \rangle \). Consequently, the separation of the orbits for which this number is positive, negative or zero, can be done simultaneously for both cvs.

5 Hermitian and Dirac conjugations

Now we have all the elements for defining the hermitian conjugation that gives us the \textit{hermitian adjoint} operators of our linear operators. First we consider the operators from \( L(K, \hat{K}) \) and \( L(\hat{K}, K) \) and we define:

\textbf{Definition 4} The hermitian adjoint operators of \( /A/ \in L(K, \hat{K}) \) and \( /B\rangle \in L(\hat{K}, K) \) are \( /A^+\rangle \in L(K, \hat{K}) \) and \( /B^+\rangle \in L(\hat{K}, K) \) which satisfy

\begin{align*}
\langle x \backslash A^+ /y \rangle &= \langle y \backslash A /x \rangle^*, \quad \forall \ /x \rangle, /y \rangle \in K, \quad (20) \\
\langle \hat{x} / B^+ \rangle \backslash y \rangle &= \langle \hat{y} / B \rangle \backslash \hat{x} \rangle^*, \quad \forall \ \hat{x} \rangle, \hat{y} \rangle \in \hat{K}. \quad (21)
\end{align*}

If \( A = A^+ \) or \( B = B^+ \) we say that these operators are hermitian.

Obviously, from (19) we see that the metric operators are hermitian,

\[ \backslash^+ = \backslash, \quad \wedge^+ = \wedge. \quad (22)\]

For other operators the situation is more complicated as it results from the following definitions.

\textbf{Definition 5} The hermitian adjoint operator of \( /A/ \in L(K, K) \) is the operator \( (\backslash A) = /A^+\rangle \in L(\hat{K}, \hat{K}) \) which accomplishes

\[ \langle x \backslash A^+ \rangle \backslash y \rangle = \langle y \backslash A /x \rangle^*, \quad \forall \ /x \rangle, /y \rangle \in K. \quad (23)\]
For $B \in L(\hat{\mathcal{K}}, \mathcal{K})$ the hermitian conjugation is defined by

$$\langle \hat{x}/B^+ \rangle = \langle \hat{y}/B \rangle^*, \quad \forall \langle x \rangle, \langle y \rangle \in \hat{\mathcal{K}}, \quad (24)$$

where $B^+ = (\langle B \rangle)^*$ $\in L(\mathcal{K}, \mathcal{K})$.

One can convince ourselves that the mutually related bra vectors with the ket vectors $\langle A/x \rangle$ and $\langle B/y \rangle$ are $\phi[\langle A/x \rangle] = \langle x/A^+ \rangle$ and $\phi[\langle B/y \rangle] = \langle y/B^+ \rangle$ respectively.

**Definition 6** We say that the operators which satisfy

$$\langle \hat{x}/A^+ \rangle = \langle \hat{y}/A \rangle, \quad /B^+ = \langle B \rangle, \quad (25)$$

are hermitian with respect to the metric $\vee$, or simply semi-hermitian.

In other words an operator $\langle A \rangle$ $\in L(\mathcal{K}, \mathcal{K})$ is semi-hermitian if its adjoint operator coincides with its coupled operator $\langle A^+ \rangle$ defined by (14). We specify that these definitions can be formulated in terms of dual form. For example, if we take $\langle y \rangle = \langle y/B \rangle$ then (23) and (24) can be rewritten as

$$\langle \hat{x}/A^+ \rangle = \langle \hat{y}/A/x \rangle^*, \quad \forall \langle x \rangle \in \mathcal{K}, \langle \hat{y} \rangle \in \hat{\mathcal{K}}, \quad (26)$$

$$\langle \hat{x}/B^+ \rangle = \langle \hat{y}/B \rangle^*, \quad \forall \langle y \rangle \in \mathcal{K}, \langle \hat{x} \rangle \in \hat{\mathcal{K}}. \quad (27)$$

Particularly, from (15) we can draw the conclusion that the identity operators are semi-hermitian, $(\langle \rangle)^+ = \langle \rangle$.

In practice it is convenient to introduce another conjugation operation which should unify the above definitions. This is just the generalization of the familiar Dirac conjugation of the theory of four-component spinors.

**Definition 7** Given an operator $X$, the Dirac adjoint operator of $X$ is

$$X = \begin{cases} \langle X^+ \rangle & \text{if } X \in L(\mathcal{K}, \mathcal{K}), \\ \vee X^+ \langle \rangle & \text{if } X \in L(\hat{\mathcal{K}}, \hat{\mathcal{K}}), \\ X^+ & \text{if } X \in L(\mathcal{K}, \hat{\mathcal{K}}) \text{ or } L(\hat{\mathcal{K}}, \mathcal{K}). \end{cases} \quad (28)$$

The operator $X$ is self-adjoint if $X = X^\dagger$.  

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According to this definition all the hermitian and semi-hermitian operators are self-adjoint, including the metric and the identity operators,

\[
\nabla = \nabla, \quad \Lambda = \Lambda, \quad \mathbb{I} = \mathbb{I}, \quad \mathbb{\Lambda} = \mathbb{\Lambda}.
\]  

(29)

Furthermore, it is not difficult to demonstrate that

\[
\overline{(X)} = X,
\]  

(30)

and that for two operators, \(A\) and \(B\), from the same linear space \(L\) and \(\alpha, \beta \in \mathbb{C}\) we have

\[
\overline{(\alpha A + \beta B)} = \alpha^* A + \beta^* B.
\]  

(31)

If the multiplication of two operators makes sense, e.g. \(A, B \in L(\mathcal{K}, \mathcal{K})\) or \(A \in L(\mathcal{K}, \hat{\mathcal{K}})\) and \(B \in L(\hat{\mathcal{K}}, \mathcal{K})\), etc, then we can show that

\[
\overline{AB} = \overline{BA}.
\]  

(32)

Thus we obtain a simple and homogeneous set of calculation rules for all the linear operators we manipulate. However, despite of this advantage, we prefer to use here the hermitian conjugation rather than the Dirac one since we are interested to follow the coherence of the presented formalism in its all details.

6 Projection operators

In the case of our cvs, where the orthogonality is defined by the metric operator, the problem of the decomposition in orthogonal subspaces is more complicated than that of unitary spaces \(\mathbb{K}\) and, therefore, the theory of projection operators needs some specifications.

Let us consider the cvs \(\mathcal{K}, \hat{\mathcal{K}}, \nabla\) and an idempotent operator \(\mathbb{P}\), satisfying \(P^2 = P\), coupled with \(\mathbb{P}^\dagger = \nabla P\). Then \(P\) is the projection operator on the subspace \(PK \subset \mathcal{K}\) while \(\hat{P}\) is the projection operator on the subspace \(\hat{P}\hat{\mathcal{K}} \subset \hat{\mathcal{K}}\). Two projection operators, \(\mathbb{P}_1\) and \(\mathbb{P}_2\), are additive if \(\mathbb{P}_1 \mathbb{P}_2 = \mathbb{P}_2 \mathbb{P}_1 = 0\) since then \(\mathbb{P}_1 + \mathbb{P}_2\) is a projection operator too. The operators \(\mathbb{P}_a, a = 1, 2, ..., n\), which satisfy

\[
\mathbb{P}_a \mathbb{P}_b = \delta^b_a \mathbb{P}_a, \quad \forall b,
\]  

(33)
form a set of additive projection operators. This set is called complete if

$$\sum_a P_a = \mathbb{1}.$$  \hfill (34)

However, these projection operators do not have good orthogonality properties since the additive ones generally are not orthogonal. For this reason one prefers the term perp instead of orthogonal [6].

**Definition 8** Two projection operators, $P_1$ and $P_2$, are perp if they satisfy

$$P_1^\perp \cap P_2 = 0 \quad (35)$$

Then the projection subspaces are called perp to each other.

The projection subspaces of these projection operators can not be coupled anytime because there is the risk to find that the restriction of the metric operator to these subspaces is no more hermitian. Two projection subspaces, $PK$ and $\hat{P}\hat{K}$, can be coupled only if their metric operator $\hat{P}\perp P$ is hermitian. This requires $\hat{P} = P^+$ which means that $P$ must be semi-hermitian (self-adjoint). Then this has the property

$$P^+ \perp = \perp P,$$  \hfill (36)

which indicates that the subspaces $PK$ and $P^+\hat{K}$ are invariant subspaces of the metric operators $\perp$ and $\cap$. In these conditions the operators

$$\perp_P = P^+ \perp P = P^+ \perp = \perp P, \quad \cap_P = P \cap P^+ = P \cap = \cap P^+, \quad (37)$$

can be considered as metric operators since they are hermitian and invertible in the sense that

$$\perp_P \cap_P = P^+, \quad \perp_P P^+ = P.$$  \hfill (38)

**Definition 9** The invariant subspaces $PK$ and $P^+\hat{K}$ isometric through $\perp_P$ represent a pair of coupled subspaces (css) denoted by $(PK, P^+\hat{K}, \perp_P) \subset (\mathcal{K}, \hat{\mathcal{K}}, \perp)$.

The related css, $(\mathcal{B}P, B^+P^+, \cap_P)$, can be defined using the restrictions to $PK$ and $P^+\hat{K}$ of the mappings $\phi$ and $\hat{\phi}$.

Now the theory of orthogonal decomposition can be done in terms of css determined by semi-hermitian projection operators.
Definition 10  If two semi-hermitian projection operators are perp then they as well as their projection css are called orthogonal.

This definition is justified by the fact that the semi-hermitian projection operators have similar properties as the usual hermitian ones.

Theorem 2  Two semi-hermitian projection operators are orthogonal if and only if they are additive.

Proof:  Let us consider that $P_1$ and $P_2$ are semi-hermitian and additive, satisfying $P_1 P_2 = 0$. Then, according to (36), we can write $0 = \sqrt{P_1 P_2} = P_1^+ \sqrt{P_2}$ which means that these projection operators are orthogonal. From the same relation it results that the orthogonal projection operators must be additive since, by hypothesis, $\sqrt{\cdot}$ is invertible. The consequence is that the css $(\mathcal{K}_3, \hat{\mathcal{K}}_3, \sqrt{3})$ defined by the projection operator $P_3 = P_1 + P_2$ is the direct sum

$$(\mathcal{K}_3, \hat{\mathcal{K}}_3, \sqrt{3}) = (\mathcal{K}_1, \hat{\mathcal{K}}_1, \sqrt{1}) \oplus (\mathcal{K}_2, \hat{\mathcal{K}}_2, \sqrt{2})$$

(39)

where

$$\sqrt{3} = \sqrt{P_1 + P_2} = \sqrt{1} + \sqrt{2}.$$  

(40)

If $P_2 = / - P_1$ then the css defined by $P_1$ and $P_2$ are orthogonal complements.

In general, a complete set of additive projection operators is a complete set of orthogonal projection operators if all of these operators are semi-hermitian.

7  Matrix representations

7.1  Dual bases

According to the usual terminology, any system of dual bases defines a matrix representation of the cvs of ket and bra vectors. In our new notation the vectors of the dual bases we have introduced in section 2 are

$$/(i) \equiv e_i \in \mathcal{K}, \quad \langle (i) \rangle \equiv \bar{e}^i \in \hat{\mathcal{K}}, \quad \langle (i)/ \equiv \bar{e}^i \in \hat{\mathcal{B}}, \quad \langle (i)\rangle \equiv \bar{e}_i \in \mathcal{B}.$$  

(41)

We consider that the indices $\langle (i)$ or $(i)/$ are in upper position while $/(i)$ or $(i)$\ are in lower position and we use the summation convention over dummy
indices in opposite positions (e.g. we sum over $i$ in expressions where we find $\ldots (i) \ldots (i) \ldots$ or $\ldots / (i) \ldots (i) \ldots$).

In a given representation, the main tools of our formalism are the duality conditions (1), written now as

$$\langle (i) / \rangle = \delta_j^i, \quad \langle (i) \rangle = \delta_i^j,$$

(43)

and the completeness relations

$$/ (i) \rangle \langle (i) / = \parallel, \quad \langle \langle (i) \rangle \langle (i) \rangle = \parallel,$$

(44)

which show that the sets of elementary projection operators

$$/ P_i / \rangle = / (i) \rangle \langle (i) \rangle \Sigma, \quad i = 1, 2, \ldots, N,$$

(45)

and respectively $P_i^+, i = 1, 2, \ldots, N$, are complete sets of additive projection operators.

These formulas contain all the information concerning the vector calculus with upper and lower indices, allowing us to express the final results in terms of vector components and matrix elements. A special role play the matrix elements of the metric operator,

$$\langle (i) \rangle = (i) \rightarrow /\rangle / /\rangle = / (i) \rangle \langle (i) \rangle \equiv \eta_{ij}, \quad (i) \langle (j) \rangle = (i) \rangle \langle (j) \rangle \equiv (\eta^{-1})_{ij},$$

(46) (47)

which change the positions of indices. The components of ket-down vectors are $\langle (i) \rangle / \rangle \equiv x^i$ or $\langle (i) \rangle / \rangle \equiv x_i = \eta_{ij} x^j$, and similarly for the ket-up vectors. The components of the corresponding bra vectors have to be obtained through complex conjugation, according to the usual properties of the dual forms or scalar products.

However, the most important is that now we are able to explicitly use spectral representations. For example, the operator $/ A /$ can be written as

$$/ A / = / (i) \rangle \langle (i) / A / (j) \rangle \langle (j) / = / (i) \rangle A^i_j \langle (j) /$$

(48)

where $A^i_j$ are the matrix elements in usual notation. The adjoint operator of $A$ is

$$\langle A^+ \rangle = \langle (i) \rangle \langle (i) \rangle A^i_j \langle (j) \rangle \langle (j) \rangle = \langle (i) \rangle (A^j_i)^* \langle (j) \rangle \langle (j) \rangle$$

(49)
since from (26) we have

$$(A^+)_{ij} = \langle (i) \backslash A^+ \backslash (j) \rangle = \langle (j) \backslash A^d (i) \rangle^* \equiv (A^d_{ij})^*.$$  \hspace{1cm} (50)

If, in addition, $A$ is semi-hermitian then $\langle (i) \backslash A^+ \backslash (j) \rangle = \langle (j) \backslash A^d \backslash (i) \rangle$ and using (50) we recover the well-known property $\eta_{ik} A^k_i (\eta^{-1})^j_l = (A^d_{ij})^*.$

In any system of dual bases the traces of the operators $A \in L(\mathcal{K}, \mathcal{K})$ and $B \in L(\hat{\mathcal{K}}, \hat{\mathcal{K}})$ are defined by

$$\text{Tr}(A) = \langle (k) \backslash A \backslash (k) \rangle \equiv A^k_k, \quad \text{Tr}(B) = \langle (k) \backslash B \backslash (k) \rangle \equiv B^k_k. \hspace{1cm} (51)$$

It is easy to show that the coupled operators (14) have the same trace.

### 7.2 Orthonormal bases

In applications one prefers the representations given by the orthonormal dual bases where the matrix of the metric operator is diagonal,

$$\eta_{ij} \equiv \langle (i) \backslash (j) \rangle = \eta_i \delta_{ij},$$

since there the “squared norms” have the simplest expressions,

$$\langle x \backslash x \rangle = \sum_{i=1}^N \eta_i \langle (i) \backslash x \rangle^2.$$  \hspace{1cm} (53)

The numbers $\eta_i$ may take $n_+$ times the value 1 and $n_- = N - n_+$ times the value $-1$. Thus in orthonormal bases the metric operator is defined by its signature that can be given either explicitly as a sequence of signs or simply as $(n_+, n_-)$. We note that $n_-$ is called the index of the metric operator \[3\].

The main advantage of the orthonormal bases is that there the elementary projection operators (43) are semi-hermitian and, consequently, they are orthogonal to each other. Of a particular interest are the orthogonal projection operators

$$P_+ = \sum_{\eta=1} P_\eta, \quad P_- = \sum_{\eta=-1} P_\eta,$$

which satisfy $\parallel P_+ \parallel + \parallel P_- \parallel = \parallel$. They split the space $\mathcal{K}$ into the pair of unitary spaces $\mathcal{K}_+ = P_+ \mathcal{K}$ and $\mathcal{K}_- = P_- \mathcal{K}$, of dimensions $\text{dim} \mathcal{K}_+ = n_+$ and
dim $\mathcal{K}_- = n_-$. Since the coupled space can be split in the same manner we have

$$ \langle \mathcal{K}, \hat{\mathcal{K}}, \sqrt{\cdot} \rangle = \langle \mathcal{K}_+, \hat{\mathcal{K}}_+, \sqrt{\cdot}_+ \rangle \oplus \langle \mathcal{K}_-, \hat{\mathcal{K}}_-, \sqrt{\cdot}_- \rangle \quad (55) $$

where

$$ \sqrt{\cdot}_+ = \sqrt{\cdot}_p + \sqrt{\cdot}_-; \quad \sqrt{\cdot}_+ = \sqrt{\cdot}_p, \quad \sqrt{\cdot}_- = \sqrt{\cdot}_p \quad (56) $$

Obviously, the same decomposition can be done for the related cvs of bra vectors.

Finally we specify that in the particular case of the metric operators with signature $(N, 0)$ or $(0, N)$, the matrix $|\eta|$ in orthonormal bases coincides up to sign with the unit matrix. Then our scalar products become usual inner forms (i.e. scalar products in the sense of the theory of Hilbert spaces) and, therefore, $\mathcal{K}$ and $\hat{\mathcal{K}}$ will be unitary spaces. In this situations we have two options. The first one is to keep the cvs structure if this is appropriate for our problem. The second option is to consider only one ket space $\mathcal{K} \equiv \hat{\mathcal{K}}$ in usual Dirac formalism, with $\langle \rangle = / \rangle = \rangle \rangle$ and $\sqrt{\cdot} = \langle \rangle = / = \rangle = \rangle$, where $I$ is the identity operator on $\mathcal{K}$.

### 8 Basis transformations

#### 8.1 General linear transformations

The change of the system of dual bases of our related pairs of cvs can be done using for each basis an arbitrary linear transformation, but then it is possible to obtain new bases which do not satisfy the canonical duality conditions or giving a non-hermitian matrix for the metric operator. In order to avoid these unwanted situations, we consider only the transformations which preserve the cvs structure in the sense that (i) leave invariant the duality conditions and (ii) transform the hermitian matrix $|\eta|$ into another hermitian matrix, $|\eta'|$.

**Theorem 3** The general form of a transformation which satisfies the conditions (i) and (ii) is

$$ \langle (i) \rangle \rightarrow \langle (i)' \rangle = \langle T^{-1} (i) \rangle, $$

$$ (i) \rightarrow |(i)'\rangle = |(T^{-1})^+ (i)\rangle, \quad (57) $$

$$ \langle i \rangle / \rightarrow \langle i \rangle' / = \langle i \rangle / T^{-1} /, $$

$$ \langle i \rangle \ \rightarrow \ \langle i \rangle / = \langle i \rangle / T^+ \ /.$$

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where \( T \in \text{Aut}(\mathcal{K}) \subset L(\mathcal{K}, \mathcal{K}) \).

**Proof:** Let us start with the following linear transformation of the ket vectors

\[
/\langle i' \rangle = /T /\langle i \rangle , \quad \langle i' \rangle = \langle i \rangle /T ^{-1}\langle i \rangle ,
\]

(58)
given by two operators \( T \in \text{Aut}(\mathcal{K}) \) and \( \tilde{T} \in \text{Aut}(\mathring{\mathcal{K}}) \) arbitrarily chosen. The transformations of the bra vectors which conserve the duality conditions are

\[
\langle i' \rangle = \langle i /T ^{-1} \rangle , \quad \langle i' \rangle = \langle i \rangle /\tilde{T} ^{-1} /\langle i \rangle .
\]

(59)

In these new bases the matrix of the metric operator remains hermitian only if \( \tilde{T} = (T ^{-1})^+ \) since then we have

\[
\eta _{ij} \equiv \langle i \rangle \langle j \rangle = \langle i \rangle \langle T ^{-1} \rangle (\langle j \rangle ) = (T _{ik} ^{-1})^* T _{ij} ^{-1} \eta _{kl} .
\]

(60)

Consequently, the transformations (58) and (59) take the form (57).

The conclusion is that \( T \) is the operator of a general linear transformation of the group \( GL(N, \mathbb{C}) \subset L(\mathcal{K}, \mathcal{K}) \). This means that our formalism is equivalent with the representation theory of general linear transformations in the vector spaces of the theory of general tensors. Indeed, in a given system of dual bases, the components of the ket-down and respectively ket-up vectors transform according to a pair of unequivalent fundamental representations of the \( GL(N, \mathbb{C}) \) group while the bra components transform according to the corresponding complex conjugate representations. Moreover, we observe that the bra components of our formalism, \( \langle x \rangle \langle k \rangle \equiv \langle x^k \rangle ^* \) and \( \langle \hat{y} \rangle \langle k \rangle \equiv \langle \hat{y}^k \rangle ^* \), are just those replaced in the theory of general tensors by components carrying dotted indices in opposite positions. With this specification, it is clear that the four transformations laws (57) correspond to the four unequivalent fundamental representations of the \( GL(N, \mathbb{C}) \) group [4].

The advantage of our formalism is that we work directly with the operators \( /A\) and \( \backslash B\) transform as

\[
A _i ^j \rightarrow A _i ^{i'} \equiv \langle i' \rangle /A /\langle j \rangle \equiv \langle i \rangle /T ^{-1} \langle i \rangle /A ^{-1} /\langle j \rangle \rangle = (T ^{-1})_{ik} A _i ^{i'} T _{ij} ^{-1}
\]

(61)

\[
B _i ^j \rightarrow B _i ^{i'} \equiv \langle i' \rangle \backslash B /\langle j \rangle \rangle = \langle i \rangle \backslash T ^{-1} \langle i \rangle \backslash B ^{-1} /\langle j \rangle \rangle = (T _{ik} ^{-1})^* B _i ^{i'} [(T ^{-1})_{ik} ^{-1}] ^* \]

(62)
while those of the operators from \( L(\mathcal{K}, \hat{\mathcal{K}}) \) and \( L(\hat{\mathcal{K}}, \mathcal{K}) \) transform like the matrices of the metric operators, according to equation (60). These transformations leave invariant the traces of the operators from \( L(\mathcal{K}, \mathcal{K}) \) and \( L(\hat{\mathcal{K}}, \hat{\mathcal{K}}) \). Moreover, one can show that any system of dual bases can be transformed at any time in a system of othonormal bases using a suitable general transformation.

### 8.2 Symmetry transformations

The general transformations change the form of all the operator matrices including that of the metric operator. However, there is a special case of some transformations which do not change the matrix of the metric operator.

**Definition 11** The general transformations that leave invariant the matrix \( |\eta| \) are called symmetry transformations.

These transformations are of the form (57) but their operators have special properties.

**Theorem 4** The operators \( U \) of the symmetry transformations must satisfy the semi-unitarity condition,

\[
U^+ \sqrt{U} = \sqrt{1},
\]

which can be written as \( \overline{U} = U^{-1} \).

**Proof:** A transformation (57) generally change the matrix \( |\eta| \) according to (60). This is a symmetry transformation only if \( \langle (i)' \sqrt{i} (j)' \rangle = \langle (i) \sqrt{i} (j) \rangle \). Hereby it results (63).

Consequently, the symmetry transformations have the form

\[
\langle (i) \rangle \rightarrow \langle (i)' \rangle = \langle U \sqrt{i} \rangle,
\]

\[
\langle (i) \rangle \rightarrow \langle (i)' \rangle = \langle \hat{U} \sqrt{i} \rangle,
\]

\[
\langle (i) \rangle \rightarrow \langle (i)' \rangle = \langle (i) \sqrt{U^{-1}} \rangle,
\]

\[
\langle (i) \rangle \rightarrow \langle (i)' \rangle = \langle (i) \sqrt{\hat{U}^{-1}} \rangle,
\]

where we recall that \( \hat{U} = \sqrt{U} \) is the operator coupled with \( U \). The main virtue of these transformations is that they do not change the expressions of
scalar product in terms of vector components. In other words these leave in-
variant not only the expressions of the dual forms but also those of the scalar
products. Moreover, when we work with orthonormal bases the symmetry
transformations conserve the orthogonality.

The semi-unitary operators $U$ which accomplish the condition (58) form
a subgroup of $GL(N, \mathbb{C})$, namely the maximal symmetry group or the gauge
group of the metric operator $\sqrt{\cdot}$. Any system of dual bases defines a pair of
coupled fundamental representations of this group and its algebra in the car-
rier spaces $\mathcal{K}$ and $\hat{\mathcal{K}}$. Obviously, these representations are equivalent through
the metric operator.

A given pair of ket cvs can be used as carrier spaces for the coupled
semi-unitary representations of any subgroup of the gauge group. In general,
these representations are reducible in usual sense but it is not sure that
their subspaces can be correctly coupled. For this reason we consider here a
modified definition of reducibility.

**Definition 12** The coupled representations are irreducible if their generators
and the metric operator have no common non-trivial invariant subspaces.
Otherwise the representations are reducible.

Since these representations are semi-unitary one can show that, like in the
unitary case, the reducible representations are decomposable (i.e. fully re-
ducible). Consequently, the original cvs can be written as a direct sum of css
carrying irreducible representations. Thus, using this definition we preserve
the important advantage of the css structure that guarantees the existence
of invariant scalar products.

**8.3 Group generators**

The properties of the $GL(N, \mathbb{C})$ group and its subgroups are well-studied but
it is interesting to review few among them in our formalism where we can
work directly with the spectral representations of the group generators.

Let us consider the related pairs of cvs carrying the fundamental repre-
sentations of the $GL(N, \mathbb{C})$ group (57). In any system of dual bases (11) and
(12) the usual parametrisation of the operators $T \in GL(N, \mathbb{C}) \subset L(\mathcal{K}, \mathcal{K})$
reads

$$T(\omega) = e^{i\omega ij}X_{ij}, \quad X_{ij} = \langle (i) \rangle \langle (j) \rangle \sqrt{\cdot},$$

(65)
where $\omega^{ij}$ are arbitrary c-numbers and $X_{ij}$ are the “real generators” (having real matrix elements in orthonormal bases) which satisfy
\[ (X_{ij})^+ = \sqrt{X_{ji}}/ \ (X_{ij} = X_{ji}) \quad . \tag{66} \]

Hereby we can separate the $SL(N, \mathbb{C})$ generators
\[ H_{ij} = X_{ij} - \frac{1}{N} \eta_{ji} // \quad , \tag{67} \]
which have the properties
\[ \langle (i) \wedge (j) \rangle H_{ij} = 0, \quad \text{Tr}(H_{ij}) = 0 \quad . \tag{68} \]

If we consider only the gauge group of the metric operator $\sqrt{\cdot}$ then we have to use the same generators but with restrictions imposed upon the parameter values. From equations (63) and (67) we obtain that the parameters of the gauge group must satisfy the condition $\omega^{ij} + (\omega^{ji})^* = 0$, which means that
\[ \Re \omega^{ij} = -\Re \omega^{ji}, \quad \Im \omega^{ij} = \Im \omega^{ji} \quad . \tag{69} \]

There are $N^2$ real parameters as it was expected since the gauge group of the metric operator $\sqrt{\cdot}$ is the semi-unitary group $U(n_+, n_-) = U(1) \otimes SU(n_+, n_-)$. When one uses the parameters $\omega^{ij}$ then the operators (67) are considered as “real” $SU(n_+, n_-)$ generators. However, the canonical parametrisation with the real parameters (69) reads $\omega^{ij} H_{ij} = -i(\Re \omega^{ij} A_{ij} + \Im \omega^{ij} S_{ij})$ involving the semi-hermitian (self-adjoint) $SU(n_+, n_-)$ generators defined as
\[ A_{ij} = \frac{i}{2} (X_{ij} - X_{ji}) + \frac{1}{N} \Im \eta_{ji} // \quad , \]
\[ S_{ij} = -\frac{1}{2} (X_{ij} + X_{ji}) + \frac{1}{N} \Re \eta_{ji} // \quad . \tag{70} \]

It is clear that the antisymmetric ones, $A_{ij}$, are the generators of the subgroup $SO(n_+, n_-) \subset SU(n_+, n_-)$ [3]. In the particular case of real cvs the gauge group reduces to $O(n_+, n_-)$. On the other hand, with the help of the generators (70) one can show that the group $SL(N, \mathbb{C})$ is the complexification of the $SU(n_+, n_-)$ group in the sense that in a parametrisation with real numbers the $SL(N, \mathbb{C})$ generators are $A_{ij}, S_{ij}, iA_{ij}$ and $iS_{ij}$.

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An interesting problem is how transform the generators $X_{ij}$ when we change the bases through a general linear transformation \((57)\). In our formalism it is easy to show that, according to \((65)\), the transformed generators are

$$X'_{ij} = (i') \langle (j) \rangle \langle \rangle = TX_{ij} \langle T^+ \rangle = TX_{ij} T.$$  \hspace{1cm} (71)

Particularly, if we consider only symmetry transformations, $T = U$, then from \((63)\) we recover the usual transformation law of the $SU(n_+, n_-)$ generators,

$$H'_{ij} = UH_{ij}U^{-1} = U^k_i (U^l_j)^* H_{kl},$$  \hspace{1cm} (72)

which indicates that they transform according to the adjoint representation of $SU(n_+, n_-)$.

9 The semi-unitary representations of the $SL(2, \mathbb{C})$ group

The application presented in order to illustrate how works our formalism is the problem of the finite-dimensional representations of the $SL(2, \mathbb{C})$ group with invariant scalar products. It is well-known that the finite-dimensional irreducible representations of the $sl(2, \mathbb{C})$ algebra can be constructed with the help of those of the $su(2)$ algebra \[4\]. However, in general, these do not have invariant scalar products under $SL(2, \mathbb{C})$ transformations. In practice these scalar products are defined in each particular case of physical interest separately starting with a suitable representation which is often reducible. In this section we would like to present the general theory of the semi-unitary and irreducible finite-dimensional representations of $sl(2, \mathbb{C})$, in coupled carrier spaces where the invariant scalar products are well-defined.

9.1 The representations of $su(2)$ in cvs

The problem of the irreducible representations of $su(2)$ in cvs reduces to that of the canonical irreducible representations in unitary spaces $K^j \sim \hat{K}^j$ of weight $j$ \[3\]. Therefore the non-trivial cvs may be carrier spaces only for reducible coupled representations. These cvs have the general structure

$$(K, \hat{K}, \langle \rangle) = \bigoplus_{j \in J} (K^j, \hat{K}^j, \langle j \rangle)$$  \hspace{1cm} (73)
where
\[ \sqrt{\mathcal{V}} = \sum_{j \in \mathcal{J}} \sqrt{j} \] (74)
and \( \mathcal{J} \) is an arbitrary set of weights. In each css we consider the system of canonical bases of ket vectors, \( \{ / j, \lambda \} \subset \mathcal{K}^j \) and \( \{ \backslash j, \lambda \} \subset \check{\mathcal{K}}^j \), \( \lambda = -j, -j+1, ..., j \), and the related bases of bra vectors satisfying
\[ \langle j, \lambda / j', \lambda' \rangle = \delta_j^{j'} \delta_{\lambda \lambda'} , \quad \langle j, \lambda \backslash j', \lambda' \rangle = \delta_j^{j'} \delta_{\lambda \lambda'} . \] (75)

The metric operators of css,
\[ \sqrt{i} = \epsilon_j \sum_{\lambda} \langle j, \lambda \rangle \langle j, \lambda \rangle \] (76)
are defined by the set of numbers \( \epsilon_j = \pm 1, j \in \mathcal{J} \), that gives the signature of the whole metric operator \( \sqrt{\mathcal{V}} \) of the cvs (73). For each css the metric operator \( \sqrt{i} \) couples the subspaces \( \mathcal{K}^j = P_j \mathcal{K} \) and \( \check{\mathcal{K}}^j = P_j^+ \check{\mathcal{K}} \) given by the semi-hermitian projection operator
\[ P^j = \sum_{\lambda} / j, \lambda \rangle \langle j, \lambda / \] (77)

The spectral representations of the projections of the operators \( X \in su(2) \) are
\[ X^j = P^j X P^j = \sum_{\lambda \lambda'} / j, \lambda \rangle X_{\lambda \lambda'}^j \langle j, \lambda' / \] (78)
where \( X_{\lambda \lambda'}^j \) are the usual matrix elements of \( X \) in the canonical basis of the unitary irreducible representation of weight \( j \). Hereby it is easily to verify that the generators of the irreducible representations with values in \( L(\mathcal{K}^j, \check{\mathcal{K}}^j) \), denoted by \( / J^j_a \), \( a = 1, 2, 3 \), are semi-hermitian. Consequently, the generators of the coupled representation are \( \check{J}^j_a = (J^j_a)^+ \in L(\check{\mathcal{K}}^j, \check{\mathcal{K}}^j) \).

### 9.2 Finite-dimensional representations of \( sl(2, \mathbb{C}) \)

The usual non-covariant generators of the \( sl(2, \mathbb{C}) \) algebra are the rotation generators, \( I_a, a = 1, 2, 3 \), and the Lorentz boosts, \( K_a \). From their well-known commutation rules it results that the algebras generated by
\[ M_a = \frac{1}{2} (I_a + iK_a) , \quad N_a = \frac{1}{2} (I_a - iK_a) \] (79)
are two $su(2)$ algebras commuting with each other \(\mathfrak{g}\). Notice that these algebras can not be seen as ideals of $sl(2, \mathbb{C})$ since their generators are complex linear combinations of the generators of a real algebra.

Our aim is to construct the semi-unitary finite-dimensional representations of $sl(2, \mathbb{C})$ using our formalism. This means to consider from the beginning that the generators of the coupled representations are semi-hermitian satisfying $I_a^+ = \sqrt{I_a}\sqrt{} = \hat{I}_a$ and $K_a^+ = \sqrt{K_a}\sqrt{} = \hat{K}_a$ (or $\overline{T}_a = I_a$ and $\overline{K}_a = K_a$). Consequently, we must have

$$M_a^+ = \sqrt{N_a}\sqrt{} = \hat{N}_a, \quad N_a^+ = \sqrt{M_a}\sqrt{} = \hat{M}_a, \quad (\overline{M}_a = N_a).$$ \hspace{1cm} (80)

The general solution of this problem can be written starting with the space of the reducible representation $(j_1, j_2) \oplus (j_2, j_1)$ of the $sl(2, \mathbb{C})$ algebra \(\mathfrak{g}\), $\mathcal{K} = (\mathcal{K}^{j_1} \otimes \mathcal{K}^{j_2}) \oplus (\mathcal{K}^{j_2} \otimes \mathcal{K}^{j_1})$, with $j_1 \neq j_2$. We find that the generators

$$M_a = J_a^{j_1} \otimes P^{j_2} + J_a^{j_2} \otimes P^{j_1}, \hspace{1cm} (81)$$

$$N_a = P^{j_1} \otimes J_a^{j_2} + P^{j_2} \otimes J_a^{j_1}, \hspace{1cm} (82)$$

and the metric operator

$$\sqrt{} = \epsilon_{j_1, j_2} \sum_{\lambda \lambda'} (\langle j_1, \lambda \rangle \otimes \langle j_2, \lambda' \rangle \langle j_2, \lambda' \rangle \otimes \langle j_1, \lambda \rangle / \langle j_1, \lambda \rangle \langle j_2, \lambda' \rangle \rangle \rangle$$

$$+ \langle j_2, \lambda' \rangle \otimes \langle j_1, \lambda \rangle \langle j_1, \lambda \rangle \langle j_2, \lambda' \rangle \rangle \rangle$$ \hspace{1cm} (83)

with $\epsilon_{j_1, j_2} = \pm 1$, satisfy equations (51). Hereby it results that the vector space coupled with $\mathcal{K}$ must be $\hat{\mathcal{K}} = (\mathcal{K}^{j_2} \otimes \mathcal{K}^{j_1}) \oplus (\mathcal{K}^{j_1} \otimes \mathcal{K}^{j_2})$.

These semi-unitary representations in cvs $(\mathcal{K}, \hat{\mathcal{K}}, \sqrt{})$ will be denoted by $[j_1, j_2]$. They can be considered irreducible in the sense of definition (12) since the metric operator $\sqrt{}$ and the generators $M_a$ and $N_a$ do not have common non-trivial invariant subspaces. We can convince that with the help of the chiral projection operators, $/P_L/ = P^{j_1} \otimes P^{j_2}$ and $/P_R/ = P^{j_2} \otimes P^{j_1}$, which generalize the familiar left and right-handed ones of the theory of Dirac spinors. These projection operators are just those of the invariant subspaces of the generators $M_a$ and $N_a$. They form a complete set of additive projection operators,

$$P_LP_R = 0, \quad P_L + P_R = \parallel, \hspace{1cm} (84)$$

but they are not semi-hermitian (or self-adjoint) operators since

$$P_L^+ = \sqrt{P_R}\sqrt{}, \quad P_R^+ = \sqrt{P_L}\sqrt{}, \quad (\overline{P}_L = P_R). \hspace{1cm} (85)$$

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Therefore, the subspaces \( P_L \mathcal{K} \) and \( P_R \mathcal{K} \) are not invariant subspaces of \( \mathcal{K} \) and the coupled representations \([j_1, j_2]\) are irreducible from our point of view.

In the particular case of \( j_1 = j_2 = j \) the solution is simpler. The space \( \mathcal{K} = \mathcal{K}^j \otimes \mathcal{K}^j \) is just that of the irreducible representation \((j, j)\) of \( sl(2, \mathbb{C}) \) \[4\]. The generators have the form

\[
M_a = J^j_a \otimes P^j, \quad N_a = P^j \otimes J^j_a
\]

while the metric operator reads

\[
\mathcal{K} = \epsilon j \sum_{\lambda, \lambda'} \langle j, \lambda \rangle \otimes \langle j, \lambda' \rangle \langle j, \lambda' \rangle \otimes \langle j, \lambda \rangle.
\]

These irreducible representations will be denoted by \([j]\).

### 9.3 Rotation bases

Now we can introduce the system of rotation dual bases in which the operators \( I^2 = (I_1)^2 + (I_2)^2 + (I_3)^2 \) and \( I_3 \) as well as their coupling partners are diagonal. Since \( I_a = M_a + N_a \), the vectors of these bases can be constructed with the help of the Clebsh-Gordan coefficients of the \( SU(2) \) group \[8\]. The ket-down vectors of the rotation basis of the subspace \( \mathcal{K}^{j_1} \otimes \mathcal{K}^{j_2} \subset \mathcal{K} \) are defined as

\[
\langle j_1, j_2 \rangle s, \sigma \rangle = \sum_{\lambda, \lambda' = \sigma - \lambda} \langle j_1, \lambda \rangle \otimes \langle j_2, \lambda' \rangle \langle j_1, \lambda; j_2, \lambda' \rangle s, \sigma \rangle
\]

such that

\[
I^2 \langle j_1, j_2 \rangle s, \sigma \rangle = s(s+1) \langle j_1, j_2 \rangle s, \sigma \rangle,
\]

\[
I_3 \langle j_1, j_2 \rangle s, \sigma \rangle = \sigma \langle j_1, j_2 \rangle s, \sigma \rangle.
\]

The rotation bases of the other spaces of our cvs have to be introduced in the same manner. Then by taking into account that \[8\]

\[
\langle j_1, \lambda; j_2, \lambda' \rangle s, \sigma \rangle = (-1)^{s-j_1-j_2} \langle j_2, \lambda'; j_1, \lambda \rangle s, \sigma \rangle
\]
and using the orthogonality relations of these coefficients, we find the final forms of the metric operators in rotation bases. When \( j_1 \neq j_2 \) this is

\[
\sqrt{\epsilon} = \epsilon_{j_1, j_2} (-1)^{-j_1 - j_2} \sum_{s = |j_1 - j_2|}^{j_1 + j_2} (-1)^s \sum_{\sigma = -s}^{s} \langle (j_1, j_2) s, \sigma \rangle \langle (j_2, j_1) s, \sigma \rangle,
\]

while for \( j_1 = j_2 = j \) we have

\[
\sqrt{\epsilon} = \epsilon_j (-1)^{2j} \sum_{s=0}^{2j} (-1)^s \sum_{\sigma = -s}^{s} \langle (j, j) s, \sigma \rangle \langle (j, j) s, \sigma \rangle.
\]

In our opinion a good choice of the factors \( \epsilon \) could be

\[
\epsilon_{j_1, j_2} = (-1)^{j_1 + j_2 - |j_1 - j_2|}, \quad \epsilon_j = (-1)^{2j}.
\]

Thus we obtain the spectral representations of the metric operators in rotation dual bases. We observe that these bases are orthonormal only for \( j_1 = j_2 \). In the general case of \( j_1 \neq j_2 \) the ket-down vectors of the orthonormal basis of \( \mathcal{K} \) are given by the linear combinations

\[
/(\pm) s, \sigma \rangle = \frac{1}{\sqrt{2}} (/(j_1, j_2) s, \sigma \rangle \pm /(j_2, j_1) s, \sigma \rangle).
\]

Similarly we get the ket or bra vectors of the other orthonormal bases where the metric operator can be represented as

\[
\sqrt{\epsilon} = \epsilon_{j_1, j_2} (-1)^{-j_1 - j_2} \sum_{s = |j_1 - j_2|}^{j_1 + j_2} (-1)^s \sum_{\sigma = -s}^{s} \langle (j_1, j_2) s, \sigma \rangle \langle (j_1, j_2) s, \sigma \rangle
\]

From this formula we see that for \( j_1 \neq j_2 \) the metric operator has the symmetric signature \((n, n)\) with \( n = (2j_1 + 1)(2j_2 + 1) \). The signatures of the metric operators of the representations \([j]\) result from (94) and (95) to be either \((n, m)\) for integer \( j \) or \((m, n)\) if \( j \) is a half-integer, where \( n = (j + 1)(2j + 1) \) and \( m = j(2j + 1) \).

Hence for each pair of coupled irreducible representations of \( sl(2, \mathbb{C}) \) the metric operator has a well-determined signature, \((n_+, n_-)\), which shows us
that the corresponding maximal symmetry group is just $U(n_+, n_-)$. This result can be important from the physical point of view since the transformations of this group are semi-unitary, leaving invariant the scalar product of the carrier spaces of the $sl(2, \mathbb{C})$ representations. For example, in the theory of Dirac spinors the metric operator of the representation $[1/2, 0]$ has the signature $(2, 2)$ which explains why the whole algebra of $\gamma$ matrices and $sl(2, \mathbb{C})$ generators is just the $u(2, 2)$ algebra. We note that the corresponding $U(2, 2)$ group was recently considered as an extended gauge group of the theory of the Dirac field in curved spacetime, obtaining thus new interesting results.

10 Concluding remarks

The presented formalism is the natural generalization of the Dirac’s bra-ket calculus to spaces with indefinite metric. The main point of our proposal is to organize the four different spaces of the theory of general tensors into a pair of coupled ket spaces mutually related with a pair of coupled bra one. Then the metric operator can be correctly introduced in accordance with the natural duality such that compatible scalar products and dual forms do co-exist. In this way we recover the results of the theory of the general linear transformations formulated simply in terms of operators, independent on the concrete representations given by systems of dual bases. For this reason, our approach helps one to precise the nature of the different mathematical objects, avoiding the risk to confuse among themselves those which accidently could have the same components or matrix elements in several representations (e.g. the components of second rank tensors and the operator matrix elements). Moreover, in this framework some special operations used up to now only in particular problems get a general meaning. We refer especially to the Dirac adjoint which in our formalism can be defined for all the types of involved operators, taking over the role of the Hermitian adjoint from the usual unitary case.

The presented example points out few of these advantages. Using our general bra-ket calculus we are able to write the spectral representations in different basis of the $sl(2, \mathbb{C})$ generators and the metric operators of the finite-dimensional irreducible representations with invariant scalar products. The study of basis transformations, the generalization of the chiral (left and
right-handed) projection operators and the analyze of the gauge group of the metric operator can be easily done in this context.

We tried here to remain in the spirit of the original Dirac’s bra-ket calculus, imagining a formalism with simple calculation rules but having the “memory” of the definitions and properties of its basic elements. We hope that this should be appropriate for different applications including algebraic programming on computers.

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References

[1] P. A. M. Dirac, The Principles of Quantum Mechanics (Oxford: Clarendon 1947)

[2] R. Gilmore, Lie Groups, Lie Algebras and Some of Their Applications (New York: Wiley 1974)

[3] A. O. Barut and R. Račzka, Theory of Group Representations and Applications (Warszawa: PWN 1977)

[4] W.-K. Tung, Group Theory in Physics (Singapore: World Scientific 1985)

[5] J. Bognar, Indefinite Inner Product Spaces (Berlin: Springer 1974)

[6] B. O’Neill, Semi-Riemannian Geometry (New York: Academic 1983)

[7] K. L. Nagy, Vector Spaces with Indefinite Metric in Quantum Field Theory (Budapest: Akadémiai Kiadó 1966)

[8] A. Böhm, Quantum mechanics (Berlin: Springer 1979)

[9] T. Yao, J. Math. Phys. 12, 315 (1971)
[10] F. Finster, *J. Math. Phys.* **39**, 6276 (1998)
F. Finster, J. Smoller and S.-T. Yau, *Phys. Rev. D* **59**, 104020 (1999)
F. Finster, J. Smoller and S.-T. Yau, *J. Math. Phys.* **41**, 2173 (2000)