Upper Embeddability of Graphs and Products of Transpositions Associated with Edges

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Given a graph, we associate each edge with the transposition which exchanges the endvertices. Fixing a linear order on the edge set, we obtain a permutation of the vertices. Dénes proved that the permutation is a full cyclic permutation for any linear order if and only if the graph is a tree.

In this article, we characterize graphs having a linear order such that the associated permutation is a full cyclic permutation in terms of graph embeddings. Moreover, we give a counterexample for Eden’s question about an edge ordering whose associated permutation is the identity.

**Keywords:** full cyclic permutation ordering, upper-embeddable graph, 2-cell embedding, rotation system

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Figure 1: A butterfly graph.  

Figure 2: A dumbbell graph.

1 Introduction

In this article, a graph $G$ stands for a connected multigraph $G = (V_G, E_G, r_G)$, where

- $V_G$ is a finite set of vertices,
- $E_G$ is a finite set of edges,
- $r_G$ is a map from $E_G$ to $\binom{V_G}{2}$, the collection of subsets of $V_G$ consisting of 2 elements.

Note that $r_G(e)$ represents the endvertices of the edge $e$. Also note that loops are not allowed.

Let $n$ be a positive integer and suppose that $V_G = [n] := \{1, 2, \ldots, n\}$. When an edge $e \in E_G$ satisfies $r_G(e) = \{u, v\}$, we associate the transposition $\tau_e := (u \ v) \in S_n$ with the edge $e$, where $S_n$ denotes the symmetric group of degree $n$.

An edge ordering of a graph $G$ is a linear order $\leq_\omega$ on $E_G$, denoted as a sequence $\omega = (e_1, \ldots, e_m)$ in which $e_i <_\omega e_j$ if and only if $i < j$. Given an edge ordering $\omega = (e_1, \ldots, e_m)$, we associate the product $\pi_\omega := \tau_{e_m} \cdots \tau_{e_1} \in S_n$.

Definition 1.1. A permutation $\sigma \in S_n$ is called a full cyclic permutation if $\sigma$ is a cyclic permutation of length $n$. An edge ordering $\omega = (e_1, \ldots, e_m)$ of a graph $G$ is a full cyclic permutation ordering if the corresponding permutation $\pi_\omega = \tau_{e_m} \cdots \tau_{e_1}$ is a full cyclic permutation.

Dénes proved the following theorem to state a connection between labeled trees and factorization of a full cyclic permutation into transpositions.

Theorem 1.2 (Dénes [1]. See also [7, Section 2] and [11, Lemma 2.1]). Given a graph $G$, the following are equivalent.

(i) Any edge ordering of $G$ is a full cyclic permutation ordering.

(ii) $G$ is a tree.

Note that Theorem 1.2 plays an important role in the studies of the chromatic symmetric functions and the chromatic operator for trees [3, 11]. Also, note that recently the second author [16] studied an analogue of Theorem 1.2 for signed graphs and the hyperoctahedral group.
It is a natural question to ask what graphs admit a full cyclic permutation ordering. For example, let $G$ be the butterfly graph pictured in Figure 1 and define the edge ordering $\omega := (e_1, e_2, e_3, e_4, e_5, e_6)$. Then

$$\pi_\omega = \tau_{e_6} \tau_{e_5} \tau_{e_4} \tau_{e_3} \tau_{e_2} \tau_{e_1} = (3 \ 5)(1 \ 3)(4 \ 5)(3 \ 4)(2 \ 3)(1 \ 2) = (1 \ 3 \ 2 \ 5 \ 4).$$

Therefore $\omega$ is a full cyclic permutation ordering of the butterfly graph $G$.

Let $\beta(G) := |E_G| - |V_G| + 1$ denote the Betti number (also called the circuit rank) of the connected graph $G$. When a graph $G$ has a full cyclic permutation ordering, considering the signature of a full cyclic permutation ordering, one can show that the Betti number $\beta(G)$ is even. However, the converse is false. For example, the dumbbell graph (Figure 2) has no full cyclic permutation orderings although its Betti number is 2.

We regard a graph as a topological space by identifying each edge with the unit interval $[0, 1]$ and gluing them at vertices. Then $\beta(G)$ coincides with the Betti number of $G$ as a topological space. In this article, we will show that having a full cyclic permutation ordering is a topological property as follows.

Let $\Sigma$ be an orientable closed surface and $\iota: G \to \Sigma$ an embedding. We call a connected component of the complement of the image of $\iota$ a face. An embedding $\iota$ is a 2-cell embedding if every face is homeomorphic to an open disk. For any 2-cell embedding $\iota: G \to \Sigma$,

$$|V_G| - |E_G| + f_\iota = 2 - 2g_\Sigma,$$

where $f_\iota$ denotes the number of faces of the 2-cell embedding $\iota$ and $g_\Sigma$ the genus of $\Sigma$. Then

$$2g_\Sigma + f_\iota = \beta(G) + 1.$$

Therefore maximizing the genus $g_\Sigma$ is equivalent to minimizing the number of faces $f_\iota$. Hence the maximum genus $\gamma_{\max}(G)$, the maximum of genus $g_\Sigma$ such that there exists a 2-cell embedding $G \to \Sigma$, satisfies

$$\gamma_{\max}(G) \leq \left\lfloor \frac{\beta(G)}{2} \right\rfloor,$$

where $\lfloor \rfloor$ denotes the floor function. If the equality holds, then we say that $G$ is upper embeddable.

Upper embeddable graphs are well-studied objects by many researchers [4, 5, 8–10, 12–15, 17, 18]. Jungerman and Xuong gave a combinatorial characterization of upper embeddability independently.

**Theorem 1.3** (Jungerman [5, Theorem 2], Xuong [18, Theorem A]). A connected graph $G$ with even (odd) Betti number is upper embeddable if and only if there exists a spanning tree $T$ of $G$ such that all (all but one) connected components of $G \setminus T$ consists of an even number of edges.
Here is the main theorem of this article.

**Theorem 1.4.** Given a graph $G$, the following are equivalent.

1. $G$ has a full cyclic permutation ordering.
2. There exists a 2-cell embedding $\iota : G \to \Sigma$ such that $f_\iota = 1$.
3. The Betti number $\beta(G)$ is even and $G$ is upper embeddable, that is, $\beta(G) = 2\gamma_{\text{max}}(G)$.
4. There exists a spanning tree $T$ of $G$ such that every connected components of $G \setminus T$ consists of an even number of edges.

Note that the conditions (2), (3), and (4) are equivalent by the definition of upper embeddability and Theorem 1.3. Figure 3 shows how the butterfly graph can be embedded into a torus with exactly one face.

The organization of this article as follows. In Section 2, we will prove that (3) implies (1) and give an example of constructing a full cyclic permutation ordering. In Section 3, we will review the relation of 2-cell embeddings and rotation systems and we will prove that (1) implies (2). Combining the proofs in Section 2 and Section 3, we will complete the proof of Theorem 1.4.

In Section 4, we will study another extreme condition, that is, edge orderings $\omega$ such that $\pi_\omega$ is the identity permutation, which Eden [2] studied. Eden gave necessary conditions for such orderings and asked whether the condition is also sufficient. We will give a counterexample for this question.

## 2 Proof that (3) implies (1)

First, we introduce the following lemma.

**Lemma 2.1.** Let $\pi$ be a full cyclic permutation in $S_n$. If the distinct numbers $u, v, w$ appear in $\pi$ in this cyclic order, then the product $(u \ v)(v \ w)\pi$ is a full cyclic permutation.

**Proof.** By the assumption, we can write

$$\pi = (u \ a_1 \ \cdots \ a_r \ v \ b_1 \ \cdots \ b_s \ w \ c_1 \ \cdots \ c_t),$$

Figure 3: A 2-cell embedding of the butterfly graph into a torus with exactly one face.
where \( a_i, b_i, c_i \) denote distinct numbers in \([n] \setminus \{u, v, w\}\). Then we obtain
\[
(u \ v)(v \ w)\pi = (u \ v)(v \ w)(u \ a_1 \ \cdots \ a_r \ v \ b_1 \ \cdots \ b_k \ w \ c_1 \ \cdots \ c_t)
\]
\[
= (u \ a_1 \ \cdots \ a_r \ w \ c_1 \ \cdots \ c_t \ v \ b_1 \ \cdots \ b_k),
\]
which is a full cyclic permutation ordering.

We say that two edges \( e \) and \( e' \) are adjacent if \( r_G(e) \cap r_G(e') \neq \emptyset \), that is, they have a common endvertex. Note that the case \( r_G(e) = r_G(e') \) is allowed.

**Lemma 2.2.** Let \( e \) and \( e' \) be two adjacent edges in a graph \( G \). If \( G \setminus \{e, e'\} \) has a full cyclic permutation ordering, then \( G \) has a full cyclic permutation ordering.

**Proof.** Let \( \omega' = (e_1, \ldots, e_m) \) be a full cyclic permutation ordering of \( G \setminus \{e, e'\} \). If \( r_G(e) = r_G(e') \), then \( \omega := (e_1, \ldots, e_m, e', e) \) is a full cyclic permutation ordering of \( G \) since \( \pi_\omega = \pi_{\omega'} \).

Now, suppose that \( r_G(e) = \{u, v\} \) and \( r_G(e') = \{v, w\} \) with \( u \neq w \). If the cycle order of \( u, v, w \) in \( \pi_{\omega'} \) is \( u, v, w \), then \( \omega := (e_1, \ldots, e_m, e', e) \) is a full cyclic permutation ordering of \( G \) by Lemma 2.1. In a symmetrical manner, if the cycle order is \( w, v, u \), then let \( \omega := (e_1, \ldots, e_m, e, e') \).

The following lemma is required.

**Lemma 2.3** (Young [17, Lemma 3]). Suppose that \( G \) is upper embeddable and \( \beta(G) \) is even. If \( G \) is not a tree, then there exist two adjacent edges \( e \) and \( e' \) such that \( G \setminus \{e, e'\} \) is upper embeddable.

**Proof that (3) implies (1) in Theorem 1.4.** We will show that \( G \) has a full cyclic permutation ordering by induction on the Betti number \( \beta(G) \). When \( \beta(G) = 0 \), \( G \) is a tree and has a full cyclic permutation ordering by Dénes’ theorem (Theorem 1.2).

Assume \( \beta(G) > 0 \). By Lemma 2.3, there exist two adjacent edges \( e \) and \( e' \) such that \( G' := G \setminus \{e, e'\} \) is upper embeddable. By the induction hypothesis, \( G' \) has a full cyclic permutation ordering. By Lemma 2.2, \( G \) has a full cyclic permutation ordering.

**Example 2.4.** Consider the wheel graph \( W_5 \) pictured in Figure 4. The Betti number of \( W_5 \) is 4. Let \( T \) be the spanning tree of \( W_5 \) consisting of the edges \( 12, 23, 34, 45 \). Then \( W_5 \setminus T \) is connected and consisting of 4 edges. Therefore \( W_5 \) satisfies the condition (4) in Theorem 1.4 and hence has a full cyclic permutation ordering. We will construct a full cyclic permutation ordering following by the proof of Lemma 2.2.

We partition the edges of \( W_5 \setminus T \) into two adjacent pairs \( \{25, 35\} \) and \( \{14, 15\} \). Define the edge ordering \( \omega_1 \) of \( T \) by \( \omega_1 := (12, 23, 34, 45) \). By Dénes’ theorem (Theorem 1.2), \( \pi_{\omega_1} \) is a full cyclic permutation. Indeed we have
\[
\pi_{\omega_1} = (4 \ 5)(3 \ 4)(2 \ 3)(1 \ 2) = (2 \ 1 \ 5 \ 4 \ 3).
\]

Next we define the edge ordering \( \omega_2 \) of \( T \cup \{25, 35\} \). Following the proof above, define \( \omega_2 \) by \( \omega_2 := \omega_1 * (35, 25) \), where * denotes the concatenation. Then
\[
\pi_{\omega_2} = (2 \ 5)(3 \ 5)(2 \ 1 \ 5 \ 4 \ 3) = (4 \ 2 \ 1 \ 3 \ 5).
\]
Similarly, define $\omega_3$ by $\omega_3 := \omega_2 \ast (15, 14)$. Then

$$\pi_{\omega_3} = (1 4)(1 5)(4 2 1 3 5) = (4 2 5 1 3).$$

Thus we obtain a full cyclic permutation ordering $\omega_3 = (1 2 3 4 5)$ of $W_5$.

### 3 Proof that (1) implies (2)

Let $I_G(v)$ denote the set of edges of $G$ incident to a vertex $v \in V_G$. A rotation system of $G$ is a collection $\rho = (\rho_v)_{v \in V_G}$ consisting of cyclic orders $\rho_v$ on $I_G(v)$, where a cyclic order on $I_G(v)$ is an equivalence class of linear orders on $I_G(v)$ obtained by identifying $(e_1, e_2, \ldots, e_s)$ with its circular shift $(e_2, \ldots, e_s, e_1)$, denoted by $[e_1, \ldots, e_s]$.

Every embedding of $G$ on an orientable closed surface defines a rotation system with the clockwise ordering for each vertex. Conversely, from a rotation system, we can obtain a 2-cell embedding of $G$ on an orientable closed surface as follows.

Define $D_G$ by

$$D_G := \{(e, u) \in E_G \times V_G \mid u \in r_G(e)\}.$$  

We call an element of $D_G$ a dart. When $r_G(e) = \{u, v\}$, the dart $(e, u)$ shows an orientation of the edge $e$ from the source $u$ to the target $v$. Define the involution $\alpha$ on $D_G$ by $\alpha(e, u) := (e, v)$.

Given a rotation system $\rho = (\rho_v)_{v \in V_G}$, we will define bijections $\sigma$ and $\phi$ from $D_G$ to itself. Suppose that $\rho_v = [e_1, \ldots, e_s]$. Define $\sigma$ by $\sigma(e, v) := (e_{i+1}, v)$, where we consider $e_{s+1} = e_1$. Let $\phi := \sigma \circ \alpha$.

For every dart $d$, the target of $d$ coincides with the source of $\phi(d)$. Therefore each orbit in $D_G/\langle \phi \rangle$ determines a closed walk on $G$ and we can make a polygon whose sides are formed by the darts in the orbit. Gluing the sides of the polygons obtained from the orbits in $D_G/\langle \phi \rangle$ by the involution $\alpha$, we obtain an embedding of $G$ on a closed surface. One can show that this surface is actually orientable (See [6, Subsection 3.2]) and hence this embedding is the desired 2-cell embedding.

**Theorem 3.1** (See [6, Theorem 3.2.4]). Given a graph $G$, there exists a one-to-one correspondence between rotation systems of $G$ and 2-cell embeddings of $G$ on oriented closed surfaces up to orientation-preserving homeomorphism.
Note that, from the construction, the number of the faces of the embedding corresponding to a rotation system is equal to the number of the orbits in $D_G/\langle \phi \rangle$.

Let $\omega$ be an edge ordering of $G$. For each $v \in V_G$, let $\omega_v$ denote the linear order on $I_G(v)$ induced by $\omega$. Moreover, let $\rho_{v,\omega}$ be the cyclic order on $I_G(v)$ determined by $\omega_v$. Thus we obtain the rotation system $\rho_\omega := (\rho_{v,\omega})_{v \in V_G}$ from an edge ordering $\omega$ and hence the corresponding bijections $\sigma_\omega$ and $\phi_\omega = \sigma_\omega \circ \alpha$.

**Lemma 3.2.** Let $\omega = (e_1, \ldots, e_m)$ be an edge ordering of a graph $G$. For each $v \in V_G$, let $f_v$ denote the minimal edge in $I_G(v)$ with respect to $\omega$. Define a map $\Psi : V_G/\langle \pi_\omega \rangle \to D_G/\langle \phi_\omega \rangle$ by

$$\Psi([v]) := [(f_v, v)],$$

where the brackets denote equivalence classes. Then $\Psi$ is a bijection.

**Proof.** First, we will show that the map $\Psi$ is well-defined. It is sufficient to show that $[(f_v, v)] = [(f_{\pi_\omega(v)}, \pi_\omega(v))]$ for each $v \in V_G$.

Fix $v \in V_G$. The edge ordering $\omega = (e_1, \ldots, e_m)$ defines the set $T_v$ as follows.

$$T_v := \{ e_i \in E_G \mid (\tau_1 \tau_{i-1} \cdots \tau_1)(v) \neq (\tau_{i-1} \cdots \tau_1)(v) \},$$

where we agree with $(\tau_{i-1} \cdots \tau_1)(v) = v$ if $i = 1$. Suppose that $T_v = \{ e_{j_1}, e_{j_2}, \ldots, e_{j_s} \}$ with $j_1 < j_2 < \cdots < j_s$. Then $(e_{j_1}, \ldots, e_{j_s})$ is a trail from $v$ to $\pi_\omega(v)$. Let $v_0 := v$ and for $k \in \{1, \ldots, s\}$ define $v_k$ recursively as the endvertex of $e_{j_k}$ other than $v_{k-1}$. Note that $v_s = \pi_\omega(v)$.

From the definition of $T_v$, for each $k \in \{1, \ldots, s-1\}$, $e_{j_k+1}$ covers $e_{j_k}$ in $I_G(v_k)$ with respect to $\omega$. Therefore $\phi_\omega(e_{j_k}, v_{k-1}) = (e_{j_k+1}, v_k)$. Since $e_{j_k}$ is the maximal element in $I_G(v_k)$ with respect to the order $\omega$, we have $\phi_\omega(e_{j_k}, v_{s-1}) = (f_v, v_s) = (f_{\pi_\omega(v)}, \pi_\omega(v))$. Thus $\Psi$ is well-defined.

Next, to prove the surjectivity, take an orbit $W \in D_G/\langle \phi_\omega \rangle$. Let $f$ be the minimal element in

$$\{ e \in E_G \mid (e, v) \in W \text{ for some } v \in V_G \}$$

with respect to $\omega$. Suppose $(f, v) \in W$. Assume $f$ is not minimal in $I_G(v)$ with respect to $\omega_v$. Then $f' := \sigma_\omega^{-1}(f)$ is less than $f$ in $I_G(v)$ with respect to $\omega$. Let $r_G(f') = \{ v, v' \}$. Then $\phi_\omega(f', v') = (f, v)$ and hence $(f', v') \in W$. This contradicts to the minimality of $f$. Therefore $f$ is minimal in $I_G(v)$ and hence $f = f_v$. Hence $\Psi([v]) = [(f, v)] = W$.

Finally, we prove the injectivity. Let $u, v \in V_G$ and suppose that $\Psi([u]) = \Psi([v])$. Then we have $\phi_\omega(u, f_u) = (f_v, v)$ for some $s \in \mathbb{Z}$. Without loss of generality, we can assume that $s > 0$.

Recall the edges of $G$ are ordered by $\omega = (e_1, \ldots, e_m)$. We can write the edge $f_u$ as $f_u = e_{j_0}$ with some $j_0 \in \{1, \ldots, m\}$. Moreover we can obtain the walk $(e_{j_0}, e_{j_1}, \ldots, e_{j_k})$ by $(e_{j_k}, v_k) := \phi_\omega(e_{j_{k-1}}, v_{k-1})$ for $k \in \{1, \ldots, s\}$, where $v_0 := u$. Note that $(e_{j_0}, v_0) = (f_u, u)$ and $(e_{j_k}, v_s) = (f_v, v)$. Suppose that

$$\{ k \in \{1, \ldots, s\} \mid j_{k-1} \geq j_k \} = \{ p_1, \ldots, p_t \}$$
with \( p_1 < \cdots < p_t = s \). Then \( \pi^i_\omega(v_0) = v_{p_i} \) for \( i \in \{1,\ldots,t\} \). In particular, \( \pi^t_\omega(u) = \pi^t_\omega(v_0) = v_{p_t} = v_s = v \). Thus \([u] = [v]\) and hence \(f\) is injective. \(\square\)

Now we are ready to prove that (1) implies (2).

**Proof that (1) implies (2) in Theorem 1.4.** Let \( \omega \) be a full cyclic permutation ordering of \( G \). Then the number of faces of the 2-cell embedding corresponding to the rotation system \( \rho_\omega \) is equal to \(|D_G/\langle \phi_\omega \rangle| = |V_G/\langle \pi_\omega \rangle| = 1\). \(\square\)

## 4 Identity permutation ordering

In this section, a graph is not necessarily connected. We say that an edge ordering \( \omega \) of a graph \( G \) is an **identity permutation ordering** if \( \pi_\omega = \varepsilon \), where \( \varepsilon \) denotes the identity permutation. Every edgeless graph vacuously has an identity permutation ordering. The minimal example of a non-trivial graph having an identity permutation ordering is the 2-cycle \( C_2 \). Eden \[2\] studied simple graphs that have an identity permutation ordering and mentioned the complete graph \( K_4 \) is the minimal example. Figure 5 shows identity permutation orderings of \( C_2 \) and \( K_4 \).

Eden gave necessary conditions (without proof) for simple connected graphs having an identity permutation ordering as follows.

**Proposition 4.1** (Eden \[2, P. 130\]). Let \( G \) be a simple connected graph on \( n \) vertices with \( m \) edges. If \( G \) has an identity permutation ordering, then the following conditions hold.

(1) \( m \) is even.

(2) There exist a set \( C \) consisting of closed trails and a map \( \psi : V_G \to C \) such that the following conditions hold.

(i) \( \psi \) is bijective.

(ii) Every \( v \in V_G \) belongs to the closed trail \( \psi(v) \).

(iii) The sum of the number of edges of closed trails in \( C \) is \( 2m \).

(iv) Each edge of \( G \) belongs to exactly two closed trails in \( C \).
For any $\psi$ to $\psi$ be the bijection considered in Lemma \ref{thm:1}. Note that $V_G = V_G/\langle \pi_\omega \rangle$ since $\pi_\omega = \varepsilon$. For each $v \in V_G$, there exists no dart $d \in D_G$ such that both $d$ and $\alpha(d)$ belong to $\Psi(v)$ since $\pi_\omega$ is the identity. Thus, forgetting the direction of each dart in $\Psi(v)$, we obtain the closed trail $\psi(v) \subseteq E_G$. Letting $W := \{ \psi(v) \mid v \in V_G \}$, we have a surjection $\psi : V_G \rightarrow W$.

We will prove $\psi$ is injective. Assume that there exist distinct vertices $u, v$ such that $\psi(u) = \psi(v)$. Let $\omega' = (f_1, \ldots, f_r)$ be the induced order of $\omega$ on $\psi(u)$. Since $\pi_\omega(u) = u$, the edges $f_1$ and $f_r$ are incident to $u$. Also, $f_1$ and $f_r$ are incident to $v$ by the same reason. Hence $f_1$ and $f_r$ are parallel edges between $u$ and $v$. This contradicts that $G$ is simple. Therefore $\psi$ is injective and hence bijective.

By the definition of maps $\Psi$ and $\psi$, every $v \in V_G$ belongs to $\psi(v)$. Moreover,

$$\sum_{v \in V_G} |\psi(v)| = \sum_{v \in V_G} |\Psi(v)| = |D_G| = 2m.$$ 

For any $e \in E_G$, the two darts on $e$ belongs distinct orbits $\Psi(u)$ and $\Psi(v)$. Then $e$ belongs to $\psi(u)$ and $\psi(v)$ and the other trails do not contain $e$. Thus the map $\psi : V_G \rightarrow C$ has the desired properties. \hfill \square

**Example 4.2.** For the complete graph $K_4$ in Figure 5, the following map $\psi$ satisfies the conditions in Proposition 4.1.

$$\psi(1) = \{e_1, e_4, e_5\}, \quad \psi(2) = \{e_1, e_3, e_6\}, \quad \psi(3) = \{e_2, e_4, e_6\}, \quad \psi(4) = \{e_2, e_3, e_5\}.$$

Eden asked whether the necessary conditions in Proposition 4.1 are also sufficient. We will give a counterexample for this question. Let $G$ be the graph on 12 vertices with 20 edges pictured in Figure 6. Define a map $\psi$ by

$$\begin{align*}
\psi(v_1) &= v_1v_2v_10v_1, \quad \psi(v_2) = v_2v_9v_10v_2, \quad \psi(v_3) = v_3v_10v_4v_3, \quad \psi(v_4) = v_4v_10v_11v_4, \\
\psi(v_5) &= v_5v_6v_12v_5, \quad \psi(v_6) = v_6v_9v_12v_6, \quad \psi(v_7) = v_7v_8v_12v_7, \quad \psi(v_8) = v_8v_11v_12v_8, \\
\psi(v_9) &= v_9v_2v_1v_10v_9, \quad \psi(v_{10}) = v_{10}v_3v_4v_11v_{10}, \quad \psi(v_{11}) = v_{11}v_8v_7v_12v_{11}, \quad \psi(v_{12}) = v_{12}v_5v_6v_9v_{12}.
\end{align*}$$
The conditions in Proposition 4.1 are satisfied. Suppose that $G$ has an identity permutation ordering $\omega$. By Lemma 3.2 there exists a 2-cell embedding $\iota$ with $f_\iota = |D_G/\langle \phi_\omega \rangle| = |V_G| = 12$ faces. Then the genus $g_\iota$ satisfies

$$2 - 2g_\iota = |V_G| - |E_G| + f_\iota = 12 - 20 + 12 = 4.$$ 

Therefore $g_\iota = -1$, which is a contradiction.

**Problem 4.3.** Characterize graphs that have an identity permutation ordering.

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