ON THE SELF-DUALITY OF RINGS OF INTEGERS IN TAME AND ABELIAN EXTENSIONS

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Abstract. Let \( L/K \) be a tame and Galois extension of number fields with group \( G \). It is well-known that any ambiguous ideal in \( L \) is locally free over \( \mathcal{O}_K G \) (of rank one), and so it defines a class in the locally free class group of \( \mathcal{O}_K G \), where \( \mathcal{O}_K \) denotes the ring of integers of \( K \). In this paper, we shall study the relationship among the classes arising from the ring of integers \( \mathcal{O}_L \) of \( L \), the inverse different \( \mathcal{D}_{L/K}^{-1} \) of \( L/K \), and the square root of the inverse different \( A_{L/K} \) of \( L/K \) (if it exists), in the case that \( G \) is abelian. They are naturally related because \( A_{L/K}^2 = \mathcal{D}_{L/K}^{-1} = \mathcal{O}_L^* \), and \( A_{L/K} \) is special because \( A_{L/K} = A_{L/K}^* \), where \( * \) denotes dual with respect to the trace of \( L/K \).

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Let $L/K$ be a Galois extension of number fields with group $G$. There are two ambiguous ideals in $L$, namely ideals in $L$ which are invariant under the action of $G$, whose Galois module structure has been studied extensively in the literature. The first is the ring of integers $\mathcal{O}_L$ of $L$, the study of which is a classical problem; see [8]. The second is the square root $A_{L/K}$ (if it exists) of the inverse different ideal $\mathfrak{D}_L^{-1}$ of $L/K$, the study of which was initiated by B. Erez in [6]. By Hilbert’s formula [20, Chapter IV, Proposition 4], this ideal $A_{L/K}$ exists when $|G|$ is odd, for example. Also, we note that $A_{L/K}$ is special because it is the unique ideal in $L$ (if it exists) which is self-dual with respect to the trace $\text{Tr}_{L/K}$ of $L/K$.

It is natural to ask whether the Galois module structures of $\mathcal{O}_L$ and $A_{L/K}$ coincide. More specifically, suppose that $L/K$ is tame. Then, any ambiguous ideal $\mathfrak{A}$ in $L$ is locally free over $\mathcal{O}_K G$ of rank one by [26, Theorem 1]. Hence, it determines a class $[\mathfrak{A}]_{ZG}$ in $\text{Cl}(ZG)$ as well as a class $[\mathfrak{A}]$ in $\text{Cl}(\mathcal{O}_K G)$, where $\text{Cl}(-)$ denotes locally free class group. Provided that $A_{L/K}$ exists, we ask:

**Question 1.1.** Does $[O_L]_{ZG} = [A_{L/K}]_{ZG}$ hold in $\text{Cl}(ZG)$?

**Question 1.2.** Does $[O_L] = [A_{L/K}]$ hold in $\text{Cl}(\mathcal{O}_K G)$?

Since $A_{L/K}$ is self-dual with respect to $\text{Tr}_{L/K}$ and $\mathfrak{D}_L^{-1}$ is the dual of $\mathcal{O}_L$ with respect to $\text{Tr}_{L/K}$ by definition, we have that

$$[O_L]_{ZG} = [A_{L/K}]_{ZG} \text{ implies } [O_L]_{ZG} = [\mathfrak{D}_L^{-1}]_{ZG},$$

$$[O_L] = [A_{L/K}] \text{ implies } [O_L] = [\mathfrak{D}_L^{-1}].$$

In other words, for Questions 1.1 and 1.2 to admit an affirmative answer, the ideal $\mathcal{O}_L$ is necessarily *stably self-dual* as a $ZG$-module and an $\mathcal{O}_K G$-module, respectively. It is then natural to also ask:

**Question 1.3.** Does $[O_L]_{ZG} = [\mathfrak{D}_L^{-1}]_{ZG}$ hold in $\text{Cl}(ZG)$?

**Question 1.4.** Does $[O_L] = [\mathfrak{D}_L^{-1}]$ hold in $\text{Cl}(\mathcal{O}_K G)$?

On the one hand, a theorem of M. J. Taylor [22] implies that Question 1.3 admits an affirmative answer; this fact was re-established by S. U. Chase [4].
Using tools from [4], L. Caputo and S. Vinatier showed in [3] that Question 1.1 also admits an affirmative answer as long as $L/K$ is locally abelian.

On the other hand, both Questions 1.2 and 1.4 have never been considered in the literature. The main purpose of this paper is to show that for $K \neq \mathbb{Q}$, they both admit a negative answer in general; see Theorem 1.9 below.

1.1. Basic set-up and notation. Fix a number field $K$ as well as a finite group $G$. Let us define

$$R(\mathcal{O}_K G) = \{[\mathcal{O}_L] : \text{tame } L/K \text{ with } \text{Gal}(L/K) \simeq G\},$$

$$R_{sd}(\mathcal{O}_K G) = \{[\mathcal{O}_L] : \text{tame } L/K \text{ with } \text{Gal}(L/K) \simeq G \text{ and } [\mathcal{O}_L] = [\mathcal{O}_{L/K}^{-1}]\},$$

where “sd” stands for “self-dual”. For $G$ of odd order, further define

$$\mathcal{A}^t(\mathcal{O}_K G) = \{[A_{L/K}] : \text{tame } L/K \text{ with } \text{Gal}(L/K) \simeq G\}.$$

Let us remark that both classes $[\mathcal{O}_L]$ and $[A_{L/K}]$ depend upon the choice of the isomorphism $\text{Gal}(L/K) \simeq G$. For $K \neq \mathbb{Q}$, we shall prove that even the weakened versions of Questions 1.2 and 1.4 below admit a negative answer in general; see Theorem 1.9 below.

**Question 1.5.** Does $R_{sd}(\mathcal{O}_K G) = \mathcal{A}^t(\mathcal{O}_K G)$ hold when $|G|$ is odd?

**Question 1.6.** Does $R(\mathcal{O}_K G) = R_{sd}(\mathcal{O}_K G)$ hold?

In what follows, for simplicity, suppose that $G$ is abelian. We shall implicitly suppose also that $G$ has odd order whenever we write $\mathcal{A}^t(\mathcal{O}_K G)$. Then, the three subsets of $\text{Cl}(\mathcal{O}_K G)$ in question are related to the so-called Adams operations on $\text{Cl}(\mathcal{O}_K G)$ as follows; also see [1] and [2] for other connections between Adams operations and Galois module structures.

For each $k \in \mathbb{Z}$ coprime to $|G|$, the $k$th Adams operation is defined by

$$\Psi_k \in \text{Aut}(\text{Cl}(\mathcal{O}_K G)); \quad \Psi_k([X]) = [X_k],$$

where $X$ denotes an arbitrary locally free $\mathcal{O}_K G$-module of rank one, and $X_k$ denotes the $\mathcal{O}_K$-module $X$ on which $G$ acts via

$$s \ast x = \phi_k^{-1}(s) \cdot x \text{ for } s \in G \text{ and } x \in X,$$
where $\phi_k$ is the automorphism on $G$ given by $\phi_k(s) = s^k$. For example, when $X = \mathcal{O}_L$, where $L/K$ is a tame and Galois extension with $h : \text{Gal}(L/K) \simeq G$, then we have $X_k = \mathcal{O}_{L'}$, where $L' = L$ but with $h' : \text{Gal}(L'/K) \simeq G$ defined by $h' = \phi_k \circ h$; similarly when $X = A_{L/K}$. In the case that $k = -1$, we have

$$\Psi_{-1}([\mathcal{O}_L]) = [\text{Hom}_{\mathcal{O}_K}(\mathcal{O}_L, \mathcal{O}_K)]^{-1},$$

$$\Psi_{-1}([A_{L/K}]) = [\text{Hom}_{\mathcal{O}_K}(A_{L/K}, \mathcal{O}_K)]^{-1},$$

by [7, Appendix IX, Proposition 3]. Let $\ast$ denote dual with respect to $\text{Tr}_{L/K}$. Since $\mathcal{D}_{L/K}^{-1} = \mathcal{O}_L^{\ast}$ and $A_{L/K} = A_{L/K}^\ast$, we then deduce that

$$\Psi_{-1}([\mathcal{O}_L]) = [\mathcal{D}_{L/K}^{-1}]^{-1} \text{ and } \Psi_{-1}([A_{L/K}]) = [A_{L/K}]^{-1},$$

where the latter equality was proven in [23, Theorem 1.2 (a)] as well. In the case that $|G|$ is odd and $k = 2$, we further have

$$[A_{L/K}] = [\mathcal{O}_L]\Psi_2([\mathcal{O}_L]),$$

which was shown in [24, Theorem 1.2.4] and is also essentially a special case of [1, Theorem 1.4].

Now, it is known by [16] that $R(\mathcal{O}_K G)$ is a subgroup of $\text{Cl}(\mathcal{O}_K G)$. Writing the operation in $\text{Cl}(\mathcal{O}_K G)$ multiplicatively, we then have well-defined maps

$$\Xi_k : R(\mathcal{O}_K G) \rightarrow R(\mathcal{O}_K G); \quad \Xi_k([X]) = [X]\Psi_k([X]),$$

$$\Xi'_k : R(\mathcal{O}_K G) \rightarrow R(\mathcal{O}_K G); \quad \Xi'_k([X]) = [X]^{-1}\Psi_k([X]),$$

which are in fact homomorphisms because $\text{Cl}(\mathcal{O}_K G)$ is an abelian group. In addition, the above discussion implies that

(1.1) \hspace{1cm} R_{sd}(\mathcal{O}_K G) = \ker(\Xi_{-1}) \text{ and } \mathcal{A}^t(\mathcal{O}_K G) = \text{Im}(\Xi'_2),

which are hence subgroups of $R(\mathcal{O}_K G)$. In particular, we have a chain

(1.2) \hspace{1cm} R(\mathcal{O}_K G) \supset R_{sd}(\mathcal{O}_K G) \supset \mathcal{A}^t(\mathcal{O}_K G)

of subgroups in $\text{Cl}(\mathcal{O}_K G)$. From (1.1), we deduce the following criteria which distinguish classes in these three subgroups.

**Proposition 1.7.** Suppose that $G$ is abelian and let $c \in R(\mathcal{O}_K G)$. 


(a) Assume that $\Psi_{-1}(c) = c$. Then, we have $c \in R_{sd}(O_KG)$ if and only if $|c|$ divides two.

(b) Assume that $|G|$ is odd and that $\Psi_2(c) = c$. Then, we have $c \in A^t(O_KG)$ only if $c^{n_G(2)} = 1$, where $n_G(2)$ is the multiplicative order of 2 mod $|G|$.

Proof. Part (a) follows directly from (1.1). As for part (b), suppose that $|G|$ is odd and that $c \in A^t(O_KG)$. By (1.1), we know that $c = d^{-1}\Psi_2(d)$ for some $d \in R(O_KG)$. This implies that

$$\prod_{j=0}^{n_G(2)-1}\Psi_{2j}(c) = \prod_{j=0}^{n_G(2)-1}\Psi_{2j}(d)^{-1}\Psi_{2j+1}(d) = \Psi_0(d)^{-1}\Psi_{2n_G(2)}(d) = 1.$$

It follows that $c^{n_G(2)} = 1$ whenever $\Psi_2(c) = c$ holds. \hfill $\square$

For notation, let us also define

$$\Psi_Z = \{\Psi_k : k \in \mathbb{Z} \text{ coprime to } |G|\},$$

which is plainly a group isomorphic to $(\mathbb{Z}/|G|\mathbb{Z})^\times$, and

$$\text{Cl}^0(O_KG) = \ker(\text{Cl}(O_KG) \to \text{Cl}(O_K)),$$

where the map is that induced by augmentation. Our idea is to use Proposition 1.7 as well as classes in $\text{Cl}^0(O_KG)_{\Psi Z}$, namely, classes in $\text{Cl}^0(O_KG)$ which are invariant under $\Psi_Z$, to answer Questions 1.5 and 1.6.

Finally, for each $n \in \mathbb{N}$, let $C_n$ denote a cyclic group of order $n$, and let $\zeta_n$ denote a primitive $n$th root of unity. Given any multiplicative group $\Gamma$, write $\Gamma^n$ for the set of $n$th powers of elements in $\Gamma$.

1.2. Statements of the main theorems. First, consider $G = C_p$, where $p$ is an odd prime. For $K \neq \mathbb{Q}$, in order to answer Questions 1.5 and 1.6 in the negative, by (1.2), we must exhibit non-trivial classes in $R(O_KC_p)$. This was done in [10] and a key ingredient is the inclusion

$$\text{(1.3)} \quad (\text{Cl}^0(O_KC_p)^{\Psi Z})^{(p-1)/2} \subset R(O_KC_p)^{\Psi Z}.$$

This was shown in the proof [10, Proposition 4] using the characterization of $R(O_KC_p)$ due to L. R. McCulloh in [15]. Using (1.3), we deduce that:
Proposition 1.8. Let $p$ be an odd prime and let $c \in \text{Cl}^0(\mathcal{O}_K C_p)^{\Psi_z}$.

(a) If $|c|$ does not divide $p - 1$, then $c^{(p-1)/2} \in R(\mathcal{O}_K C_p) \setminus R_{sd}(\mathcal{O}_K C_p)$.

(b) If $|c| = 2$ and $p \equiv -1 \pmod{8}$, then $c \in R_{sd}(\mathcal{O}_K C_p) \setminus \mathcal{A}^t(\mathcal{O}_K C_p)$.

Proof. Observe that $c^{(p-1)/2} \in R(\mathcal{O}_K C_p)$ by (1.3). Part (a) is then clear from Proposition 1.7 (a). As for part (b), suppose that $|c| = 2$. If $p \equiv -1 \pmod{4}$, then $c = c^{(p-1)/2} \in R_{sd}(\mathcal{O}_K C_p)$ by Proposition 1.7 (a). If $p \equiv -1 \pmod{8}$ in addition, then $c \notin \mathcal{A}^t(\mathcal{O}_K C_p)$ by Proposition 1.7 (b), because in this case 2 is a square mod $p$ but $-1$ is not, whence $n_{C_p}(2)$ is necessarily odd. □

Using Proposition 1.8 and some further ideas from (1.3), we shall prove:

Theorem 1.9. Suppose that $K \neq \mathbb{Q}$. Then we have:

(a) $R(\mathcal{O}_K C_p) \supseteq R_{sd}(\mathcal{O}_K C_p)$ for infinitely many odd primes $p$.

(b) $R_{sd}(\mathcal{O}_K C_p) \supseteq \mathcal{A}^t(\mathcal{O}_K C_p)$ for infinitely many odd primes $p$.

Proof. We shall prove part (b) in Subsection 2.1. For part (a), we may deduce it using results in [10] as follows. Let $p$ be an odd prime. Let $T(\mathcal{O}_K C_p)$ denote the Swan subgroup of $\text{Cl}(\mathcal{O}_K C_p)$; see [25] or [5, Section 53] for the definition. Then, as shown in the proof of [10, Proposition 4], we have

$$T(\mathcal{O}_K C_p) \subset \text{Cl}^0(\mathcal{O}_K C_p)^{\Psi_z}. \quad (1.4)$$

Using Chebotarev’s density theorem, it was further shown in [10, Theorem 5 and Proposition 9] that $T(\mathcal{O}_K C_p)$ contains a class of order coprime to $p - 1$ for infinitely many $p$. The claim now follows from Proposition 1.8 (a). □

Since the proof of Theorem 1.9 uses Chebotarev’s density theorem, it does not give explicit primes $p$ satisfying the conclusion. In the special case that $K/\mathbb{Q}$ is abelian with $K$ imaginary, by slightly modifying the proof, we shall give explicit primes $p$ such that $R_{sd}(\mathcal{O}_K C_p) \supseteq \mathcal{A}^t(\mathcal{O}_K C_p)$. See [11] for explicit conditions on $K$, in which $p$ is ramified, such that the $p$-rank of $T(\mathcal{O}_K C_p)$ is at least one, so $R(\mathcal{O}_K C_p) \supseteq R_{sd}(\mathcal{O}_K C_p)$ by (1.4) and Proposition 1.8 (a).

Theorem 1.10. Suppose that $K/\mathbb{Q}$ is abelian with $K$ imaginary, and let $m$ be the conductor of $K$. Then, we have $R_{sd}(\mathcal{O}_K C_p) \supseteq \mathcal{A}^t(\mathcal{O}_K C_p)$ for all primes $p$ satisfying $p \equiv -1 \pmod{8}$ and $p \equiv -1 \pmod{2m}$.
Example 1.11. Consider the special case when \( K = \mathbb{Q}(\sqrt{D}) \), where \( D \) is a negative square-free integer not divisible by \( p \). For simplicity, let us assume that \( D \not\in \{-1, -3\} \). Then, by [12, Lemma 3.2 and Theorem 3.4], we have

\[
T(\mathcal{O}_K C_p) \simeq \begin{cases} 
C_{(p+1)/2} \text{ or } C_{p+1} & \text{if } \left( \frac{D}{p} \right) = -1, \\
C_{(p-1)/2} \text{ or } C_{p-1} & \text{if } \left( \frac{D}{p} \right) = 1,
\end{cases}
\]

where \((\cdot)\) denotes the Legendre symbol. From Proposition 1.8 and (1.4), we then deduce that

\[
\begin{cases} 
R(\mathcal{O}_K C_p) \supset R_{sd}(\mathcal{O}_K C_p) & \text{if } \left( \frac{D}{p} \right) = -1 \text{ and } p \neq 3, \\
R_{sd}(\mathcal{O}_K C_p) \supset \mathcal{A}^t(\mathcal{O}_K C_p) & \text{if } \left( \frac{D}{p} \right) = -1 \text{ and } p \equiv -1 \pmod{8},
\end{cases}
\]

where the second statement may be viewed as a refinement of Theorem 1.10. To see why, note that by quadratic reciprocity, we have

\[
\left( \frac{2}{p} \right) = (-1)^{\frac{p-1}{4}}, \quad \left( \frac{-1}{p} \right) = (-1)^{\frac{p-1}{2}}, \quad \left( \frac{q}{p} \right) = (-1)^{\frac{p-1}{2} \frac{q-1}{2}},
\]

for any odd prime \( q \). Suppose that \( p \equiv -1 \pmod{8} \) and \( p \equiv -1 \pmod{2m} \), where \( m \) is the conductor of \( K \). Since \(|D|\) divides \( m \), we see that any of its prime divisor is a square mod \( p \). It follows that

\[
\left( \frac{D}{p} \right) = \left( \frac{-1}{p} \right) \left( \frac{|D|}{p} \right) = -1, \quad \text{whence } R_{sd}(\mathcal{O}_K C_p) \supset \mathcal{A}^t(\mathcal{O}_K C_p)
\]

by the above, as predicted by Theorem 1.10. Let us note that not much may be deduced from Proposition 1.8 if \( \left( \frac{D}{p} \right) = 1 \), and that the case \( D \in \{-1, -3\} \) may be dealt with analogously.

Next, we return to an arbitrary abelian group \( G \). Recall that the proof of Theorem 1.9 (a) uses the Swan subgroup \( T(\mathcal{O}_K G) \) of \( \text{Cl}(\mathcal{O}_K G) \). The connection between Question 1.6 and the Swan subgroup was already observed in [4] and [21]; they both used the fact that \( T(\mathbb{Z} C) = 1 \) for all finite cyclic groups \( C \) to answer Question 1.3 in the positive. We shall investigate this connection further as follows.

Observe that the first equality in (1.1) implies that

\[
R(\mathcal{O}_K G)/R_{sd}(\mathcal{O}_K G) \simeq \text{Im}(\Xi_{-1}).
\]
Thus, it suffices to understand \( \text{Im}(\Xi_{-1}) \). In Subsection 3.3, for each subgroup \( H \) of \( G \), we shall define a *generalized Swan subgroup* \( T^*_H(\mathcal{O}_K G) \) of \( \text{Cl}(\mathcal{O}_K G)^\Psi \), such that \( T^*_G(\mathcal{O}_K G) \) is the usual Swan subgroup \( T(\mathcal{O}_K G) \). We shall give lower and upper bounds for \( \text{Im}(\Xi_{-1}) \) in terms of these \( T^*_H(\mathcal{O}_K G) \).

**Theorem 1.12.** Suppose that \( G \) is abelian. Let \( H \) be a cyclic subgroup of \( G \) and let \( n \) denote its order.

(a) We have \( T^*_H(\mathcal{O}_K G)^{d_n(K)} \subset R(\mathcal{O}_K G)^\Psi \), where

\[
d_n(K) = \begin{cases} [K(\zeta_n) : K]/2 & \text{when } (\zeta_n \mapsto \zeta_n^{-1}) \in \text{Gal}(K(\zeta_n)/K), \\ [K(\zeta_n) : K] & \text{when } (\zeta_n \mapsto \zeta_n^{-1}) \notin \text{Gal}(K(\zeta_n)/K). \end{cases}
\]

In particular, we have \( T^*_H(\mathcal{O}_K G)^{2d_n(K)} \subset \text{Im}(\Xi_{-1}) \).

(b) We have \( T^*_H(\mathcal{O}_K G) \subset \text{Im}(\Xi_{-1}) \) if \( n \) is odd and \( \zeta_n \in K^\times \).

**Theorem 1.13.** Suppose that \( G \) is abelian.

(a) We have \( \text{Im}(\Xi_{-1}) \subset T^*_{cyc}(\mathcal{O}_K G) \) if \( \text{Cl}(\mathcal{O}_K) = 1 \), where

\[
T^*_{cyc}(\mathcal{O}_K G) = \prod_{\substack{H \leq G \\ H \text{ cyclic}}} T^*_H(\mathcal{O}_K G).
\]

(b) We have \( \text{Im}(\Xi_{-1}) \neq 1 \) if \( \text{Cl}(\mathcal{O}_K)^{\delta(G)} \neq 1 \) and \( \exp(G) \in K^\times \), where

\[
\delta(G) = \begin{cases} 2 & \text{when } |G| \text{ is a power of two}, \\ 1 & \text{otherwise}, \end{cases}
\]

and \( \exp(G) \) denotes the exponent of \( G \), provided that \( G \neq 1 \).

From Theorems 1.12 and 1.13, as well as (1.6), we deduce that

\[
R(\mathcal{O}_K G) = R_{sd}(\mathcal{O}_K G) \text{ if and only if } \text{Cl}(\mathcal{O}_K) = 1 \text{ and } T^*_{cyc}(\mathcal{O}_K G) = 1,
\]

under the assumption that \( G \) is an abelian group of odd order such that all \( |G| \)th roots of unity are contained in \( K \).

**Example 1.14.** Suppose that \( G = C_p \), where \( p \) is an odd prime. Applying Theorem 1.12 (a) to the full group \( G \), we obtain

\[
T(\mathcal{O}_K C_p)^{d_p(K)} \subset R(\mathcal{O}_K C_p)^\Psi, \text{ where } d_p(K) \text{ divides } (p - 1)/2,
\]
and so we may regard Theorem 1.12 (a) as a refinement of (1.3) and (1.4). By Theorem 1.13 (a), when \( \text{Cl}(\mathcal{O}_K) = 1 \), we then have a chain

\[
T(\mathcal{O}_K \cap_p) \subset T(\mathcal{O}_K \cap_p)^{2d_p(K)} \subset \text{Im}(\Xi_{-1}) \subset T(\mathcal{O}_K \cap_p)
\]

of inclusions. Let us consider a few special examples of \( K \) with \( \text{Cl}(\mathcal{O}_K) = 1 \).

By [12, Lemma 3.2 and Theorem 3.4], we know that

\[
T(\mathcal{O}_K \cap_p) \simeq \begin{cases} 
C_{(p+1)/4} & \text{if } K = \mathbb{Q}(\sqrt{-1}) \text{ and } p \equiv 3 \pmod{8}, \\
C_{(p+1)/6} & \text{if } K = \mathbb{Q}(\sqrt{-3}) \text{ and } p \equiv 5 \pmod{12}.
\end{cases}
\]

By [18], we also know that

\[
T(\mathcal{O}_K \cap_p) \simeq C_p^{\oplus (p-3)/2} \text{ if } K = \mathbb{Q}(\zeta_p) \text{ and } p \in \{3, 5, 7, 11, 13, 17, 19\}.
\]

In all of the above cases, we deduce that \( \text{Im}(\Xi_{-1}) = T(\mathcal{O}_K \cap_p) \), and in particular, from (1.6) we see that the difference between \( R(\mathcal{O}_K \cap_p) \) and \( R_{sd}(\mathcal{O}_K \cap_p) \) becomes bigger as \( p \) increases.

2. **Comparison between \( R_{sd}(\mathcal{O}_K G) \) and \( A^l(\mathcal{O}_K G) \)**

In this section, we shall prove Theorems 1.9 (b) and 1.10, by using Proposition 1.8 (b) to exhibit the existence of a class in \( R_{sd}(\mathcal{O}_K \cap_p) \setminus A^l(\mathcal{O}_K \cap_p) \) for infinitely many odd primes \( p \).

In what follows, let \( p \) be any odd prime. Define

\[
V_p(\mathcal{O}_K) = \frac{(\mathcal{O}_K/p\mathcal{O}_K)^{\times}}{\pi_p(\mathcal{O}_K^{\times})}, \text{ where } \pi_p : \mathcal{O}_K \to \mathcal{O}_K/p\mathcal{O}_K
\]

is the natural quotient map. Then, we have a surjective homomorphism

\[
T(\mathcal{O}_K \cap_p) \longrightarrow V_p(\mathcal{O}_K)^{p-1}
\]

as shown in [10, Theorem 5]. This, together with (1.4), implies that:

**Lemma 2.1.** If \( p \equiv -1 \pmod{4} \) and \( V_p(\mathcal{O}_K) \) has an element of order four, then \( \text{Cl}^0(\mathcal{O}_K \cap_p)^{\Psi_2} \) has an element of order two.

In the case that \( K \) is not totally real, we shall prove Theorem 1.9 (b) using Lemma 2.1. In the case that \( K \) is totally real, however, our method fails in
general; see Remark 2.6. Hence, we must look for elements in \( \text{Cl}^0(\mathcal{O}_KC_p)^{\Psi_Z} \) of order two lying outside of \( T(\mathcal{O}_KC_p) \).

To that end, let \( \mathcal{M}(KC_p) \) denote the maximal order in \( KC_p \), and for convenience, assume that \( p \) is large enough so that \([K(\zeta_p) : K] = p - 1\). Then, we have a natural isomorphism

\[
\mathcal{M}(KC_p) \rightarrow \mathcal{O}_K \times \mathcal{O}_{K(\zeta_p)}; \quad \sum_{s \in C_p} \alpha_s s \mapsto \left( \sum_{s \in C_p} \alpha_s, \sum_{s \in C_p} \alpha_s \chi(s) \right)
\]

where \( \chi \) is a fixed non-trivial character on \( C_p \). This induces an isomorphism

\[
\text{Cl}(\mathcal{M}(KC_p)) \simeq \text{Cl}(\mathcal{O}_K) \times \text{Cl}(\mathcal{O}_{K(\zeta_p)}).
\]

In particular, we have a surjective homormorphism

\[
\text{Cl}^0(\mathcal{O}_KC_p) \rightarrow \text{Cl}(\mathcal{O}_{K(\zeta_p)}),
\]

such that the \( \Psi_Z \)-action on \( \text{Cl}^0(\mathcal{O}_KC_p) \) corresponds precisely to the \( \Gamma_p \)-action on \( \text{Cl}(\mathcal{O}_{K(\zeta_p)}) \), where \( \Gamma_p = \text{Gal}(K(\zeta_p)/K) \). This implies that:

**Lemma 2.2.** If \( \text{Cl}(\mathcal{O}_{K(\zeta_p)})^{\Gamma_p} \) has an element of order two, then \( \text{Cl}^0(\mathcal{O}_KC_p)^{\Psi_Z} \) also has an element of order two.

To show that \( \text{Cl}(\mathcal{O}_{K(\zeta_p)})^{\Gamma_p} \) contains an element of order two, we shall need the following so-called Chevalley’s ambiguous class formula.

**Proposition 2.3.** Let \( F/K \) be a cyclic extension. Let \( \Gamma = \text{Gal}(F/K) \) denote its Galois group and let \( N_{F/K} : F \rightarrow K \) denote its norm. Then, we have

\[
|\text{Cl}(\mathcal{O}_F)^\Gamma| = |\text{Cl}(\mathcal{O}_K)| \cdot \frac{2^r \prod_p e_p}{[\mathcal{O}_K^\times : \mathcal{O}_K^\times \cap N_{F/K}(F^\times)][F : K]},
\]

where \( r \) is the number of real places in \( K \) which complexify in \( F/K \). Here \( p \) ranges over the prime ideals in \( K \) and \( e_p \) is its ramification index in \( F/K \).

**Proof.** See [9, Chapter II, Remark 6.2.3].

**Lemma 2.4.** If \( K \neq \mathbb{Q} \) is not totally imaginary, with \([K(\zeta_p) : K] = p - 1\), and \( p \) is totally split in \( K/\mathbb{Q} \), then \( \text{Cl}(\mathcal{O}_{K(\zeta_p)})^{\Gamma_p} \) has an element of order two.
Proof. Assume the hypothesis. Let us write \([K : \mathbb{Q}] = r_1 + 2r_2\), where \(r_1\) and \(2r_2\), respectively, denote the number of real and complex embeddings of \(K\). Applying Proposition 2.3 to the field \(F = K(\zeta_p)\), we then obtain

\[
|\text{Cl}(\mathcal{O}_{K(\zeta_p)})^{\Gamma_p}| = |\text{Cl}(\mathcal{O}_K)| \cdot \frac{2^{r_1}(p - 1)^{r_1 + 2r_2 - 1}}{[\mathcal{O}_K^\times : \mathcal{O}_K^\times \cap N_{K(\zeta_p)/K}(K(\zeta_p)^\times)]}.
\]

Indeed, we have \(r = r_1\) since \(K(\zeta_p)\) is totally imaginary. Further, the prime ideals \(p\) in \(K\) which ramify in \(K(\zeta_p)/K\) are precisely those above \(p\). Since \(p\) is totally split in \(K/\mathbb{Q}\), there are \([K : \mathbb{Q}]\) such \(p\), and \(e_p = [K(\zeta_p) : K] = p - 1\).

Now, by the Dirichlet’s unit theorem, we know that

\[
\mathcal{O}_K^\times = \langle \epsilon_0 \rangle \times \langle \epsilon_1 \rangle \times \cdots \times \langle \epsilon_{r_1 + r_2 - 1} \rangle,
\]

where \(\epsilon_0\) is a root of unity and \(\epsilon_1, \ldots, \epsilon_{r_1 + r_2 - 1}\) are fundamental units. Hence, we have a natural surjective homomorphism

\[
\prod_{j=0}^{r_1 + r_2 - 1} \frac{\langle \epsilon_j \rangle}{\langle \epsilon_j^{p-1} \rangle} \twoheadrightarrow \frac{\mathcal{O}_K^\times}{\mathcal{O}_K^\times \cap N_{K(\zeta_p)/K}(K(\zeta_p)^\times)},
\]

and so the order of the quotient group on the right divides

\[
n_0 \cdot (p - 1)^{r_1 + r_2 - 1}, \quad \text{where } n_0 = [\langle \epsilon_0 \rangle : \langle \epsilon_0^{p-1} \rangle].
\]

Notice that \(n_0\) divides \(p - 1\) and that \(n_0 = 2\) when \(K\) is totally real. We then deduce that \(|\text{Cl}(\mathcal{O}_{K(\zeta_p)})^{\Gamma_p}|\) is divisible by

\[
\frac{2^{r_1}(p - 1)^{r_1 + 2r_2 - 1}}{n_0(p - 1)^{r_1 + r_2 - 1}} = \frac{2^{r_1}(p - 1)^{r_2}}{n_0} = 2^{r_1}(p - 1)^{r_2 - 1} \left(\frac{p - 1}{n_0}\right).
\]

By hypothesis, we have \(r_1 \geq 1\), and \(r_1 \geq 2\) when \(r_2 = 0\). Hence, the number above is always even, and so \(\text{Cl}(\mathcal{O}_{K(\zeta_p)})^{\Gamma_p}\) has an element of order two. \(\square\)

2.1. Proof of Theorem 1.9 (b). Fix an algebraic closure \(K^c\) of \(K\). Let \(\tilde{K}\) denote the Galois closure of \(K\) over \(\mathbb{Q}\) lying in \(K^c\) and let \(K_4\) denote the field obtained by adjoining to \(\tilde{K}\) all fourth roots of elements in \(\mathcal{O}_K^\times\). Notice that \(K_4/\mathbb{Q}\) is a Galois extension.

The next lemma is motivated by [10, Proposition 9] and it allows us to use Chebotarev’s density theorem to prove Theorem 1.9 (b).
Lemma 2.5. Let $\tau \in \text{Gal}(K^c/\mathbb{Q})$ and let $f \in \mathbb{N}$ denote the smallest natural number such that $\tau^f|_K = \text{Id}_K$.

(a) Suppose that

(2.1) \hspace{1em} f \text{ is even, } \tau^f|_{K_4} = \text{Id}_{K_4}, \hspace{1em} \tau|_{\mathbb{Q}(\sqrt{-1})} \neq \text{Id}_{\mathbb{Q}(\sqrt{-1})}, \hspace{1em} \tau|_{\mathbb{Q}(\sqrt{2})} = \text{Id}_{\mathbb{Q}(\sqrt{2})}.

Let $\mathfrak{P}$ be any prime ideal in $K_4(\sqrt{2})$, unramified over $\mathbb{Q}$, such that

$$\text{Frob}_{K_4(\sqrt{2})/\mathbb{Q}}(\mathfrak{P}) = \tau|_{K_4(\sqrt{2})},$$

and let $p\mathbb{Z}$ be the prime lying below $\mathfrak{P}$. Then, we have $p \equiv -1 \pmod{8}$, and the group $V_p(\mathcal{O}_K)$ has an element of order four.

(b) Suppose that

(2.2) \hspace{1em} f = 1, \hspace{1em} \tau|_{\tilde{K}} = \text{Id}_{\tilde{K}}, \hspace{1em} \tau|_{\mathbb{Q}(\sqrt{-1})} \neq \text{Id}_{\mathbb{Q}(\sqrt{-1})}, \hspace{1em} \tau|_{\mathbb{Q}(\sqrt{2})} = \text{Id}_{\mathbb{Q}(\sqrt{2})}.

Let $\mathfrak{P}$ be any prime ideal in $\tilde{K}(\sqrt{-1}, \sqrt{2})$, unramified over $\mathbb{Q}$, such that

$$\text{Frob}_{\tilde{K}(\sqrt{-1}, \sqrt{2})/\mathbb{Q}}(\mathfrak{P}) = \tau|_{\tilde{K}(\sqrt{-1}, \sqrt{2})},$$

and let $p\mathbb{Z}$ be the prime lying below $\mathfrak{P}$. Then, we have $p \equiv -1 \pmod{8}$, and the prime $p$ is totally split in $K/\mathbb{Q}$.

Proof. In both parts (a) and (b), we clearly have $p \equiv -1 \pmod{8}$ because

$$p \equiv -1 \pmod{8} \text{ if and only if } \begin{cases} p \text{ is inert in } \mathbb{Q}(\sqrt{-1}) \\ p \text{ is split in } \mathbb{Q}(\sqrt{2}) \end{cases}$$

by (1.5). In part (b), the prime $p$ is totally split in $\tilde{K}/\mathbb{Q}$ and hence in $K/\mathbb{Q}$.

In part (a), let $\mathfrak{p}_4$ and $\mathfrak{p}$ denote the prime ideals in $K_4$ and $K$, respectively, lying below $\mathfrak{P}$. Note that $f$ is the inertia degree of $\mathfrak{p}$ over $\mathbb{Q}$, and we have

$$\text{Frob}_{K_4/K}(\mathfrak{p}_4) = \tau^f|_{K_4} = \text{Id}_{K_4}.$$

This means that $\mathfrak{p}$ is totally split in $K_4/K$, and so elements in $\mathcal{O}_K^\times$ reduce to fourth powers in $\mathcal{O}_K/\mathfrak{p}$. Hence, we have surjective homomorphisms

$$V_p(\mathcal{O}_K) \longrightarrow (\mathcal{O}_K/\mathfrak{p})^\times / \pi_p(\mathcal{O}_K^\times) \longrightarrow (\mathcal{O}_K/\mathfrak{p})^\times / ((\mathcal{O}_K/\mathfrak{p})^\times)^4,$$

where $\pi_p : \mathcal{O}_K \longrightarrow \mathcal{O}_K/\mathfrak{p}$ is the natural quotient map. But $(\mathcal{O}_K/\mathfrak{p})^\times \simeq C_{p^f-1}$,
and $4$ divides $p^f - 1$ because $f \geq 2$ is even. It follows that the last quotient group above and in particular $V_p(O_K)$ has an element of order four. □

Proof of Theorem 1.9 (b). Let $\sigma_c, \sigma_r : K^c \to \mathbb{C}$ be embeddings such that $$\sigma_c(K) \not\subset \mathbb{R} \text{ and } \sigma_r(K) \subset \mathbb{R},$$ if they exist. Further, define $$\tau_c = \sigma_c^{-1} \circ \rho \circ \sigma_c \text{ and } \tau_r = \sigma_r^{-1} \circ \rho \circ \sigma_r,$$ where $\rho : \mathbb{C} \to \mathbb{C}$ denotes complex conjugation. Observe that:

(i) If $K$ is not totally real, then $\sigma_c$ exists, and $\tau_c$ satisfies (2.1).
(ii) If $K$ is totally real, then $\sigma_r$ exists, and $\tau_r$ satisfies (2.2).

In both cases, let $p \equiv -1 \pmod{8}$ be a prime given as in Lemma 2.5. Then, we deduce from Lemmas 2.1, 2.2, and 2.4 that $\text{Cl}^0(O_KC_p)^{\Psi_z}$ has an element of order two. The claim now follows from Proposition 1.8 (b) and Chebotarev’s density theorem. □

Remark 2.6. Suppose that $K$ is a real quadratic field such that its fundamental unit $\epsilon$ has norm $-1$ over $\mathbb{Q}$. For any odd prime $p$ which is inert in $K/\mathbb{Q}$, we then have $\epsilon^{p+1} \equiv -1 \pmod{pO_K}$, as shown in [13, (1.0.1)]. Letting $n_p(\epsilon)$ denote the multiplicative order of $\epsilon \mod pO_K$, this implies that $$|V_p(O_K)| = \frac{|(O_K/pO_K)^\times|}{|\pi_p(O_K^\times)|} = \frac{p^2 - 1}{n_p(\epsilon)} = \frac{2(p + 1)}{n_p(\epsilon)} \cdot \frac{p - 1}{2}.$$ The first quotient is an odd integer by [13, Theorem 1.3], so then $V_p(O_K)$ has odd order when $p \equiv -1 \pmod{4}$. This means that we cannot use Lemma 2.5 (a) to find primes $p \equiv -1 \pmod{8}$ such that $V_p(O_K)$ has an element of order four.

2.2. Proof of Theorem 1.10. First, we need the following group-theoretic lemmas.

Lemma 2.7. Let $\Gamma$ be a finite abelian $p$-group, where $p$ is a prime. Let $\Delta$ be any cyclic subgroup of $\Gamma$ whose order is maximal among all cyclic subgroups of $\Gamma$. Then, there exists a subgroup $\Delta'$ of $\Gamma$ such that $\Gamma = \Delta \times \Delta'$. 
Proof. See the proof of [14, Chapter I, Theorem 8.2], for example. □

Lemma 2.8. Let $\Gamma$ be a group isomorphic to $k$ copies of $C_n$, where $k, n \in \mathbb{N}$, and let $\Delta$ be any cyclic subgroup of order $n$. Then, there exists a subgroup $\Delta'$ of $\Gamma$ such that $\Gamma = \Delta \times \Delta'$. Moreover, for any $x \in \Gamma$, there exists a surjective homomorphism from $\Gamma/\langle x \rangle$ to $k-1$ copies of $C_n$.

Proof. The first claim is a direct consequence of Lemma 2.7 and plainly $\Delta'$ is necessarily isomorphic to $k-1$ copies of $C_n$. The second claim follows as well because any $x \in \Gamma$ is contained in some cyclic subgroup $\Delta$ of order $n$. □

Proof of Theorem 1.10. By Proposition 1.8 (b) and Lemma 2.1, it is enough to show that $V_p(\mathcal{O}_K)$ has an element of order four.

Set $d = [K : \mathbb{Q}]$ and note that $K \subset \mathbb{Q}(\zeta_m)$ by hypothesis. First, since $K$ is imaginary, by the Dirichlet’s unit theorem, we know that

$$\mathcal{O}_K^\times = \langle \epsilon_0 \rangle \times \langle \epsilon_1 \rangle \times \cdots \times \langle \epsilon_{d/2-1} \rangle,$$

where $\epsilon_0$ is a root of unity and $\epsilon_1, \ldots, \epsilon_{d/2-1}$ are fundamental units. Now, the hypothesis $p \equiv -1 \pmod{m}$ implies that $p$ is unramified in $\mathbb{Q}(\zeta_m)/\mathbb{Q}$ and

$$\text{Frob}_{\mathbb{Q}(\zeta_m)/\mathbb{Q}}(p) = \text{complex conjugation}.$$

Since $K$ is imaginary, the inertia degree of $p$ in $K/\mathbb{Q}$ is equal to two, and so

$$(\mathcal{O}_K/p\mathcal{O}_K)^\times \simeq \prod_{p|\mathfrak{p}}(\mathcal{O}_K/p)^\times \simeq C_{p^2-1} \times \cdots \times C_{p^2-1} \quad (d/2 \text{ copies}).$$

From Lemma 2.8, we then deduce that there is a surjective homomorphism

$$\left(\frac{\mathcal{O}_K}{p\mathcal{O}_K}\right)^\times \to \pi_p(\langle \epsilon_1, \ldots, \epsilon_{d/2-1} \rangle) \to C_{p^2-1}.$$

Let $\delta = 2$ if $m$ is odd, and $\delta = 1$ if $m$ is even. Then, the order of $\langle \epsilon_0 \rangle$ divides $\delta m$, and we see that there are surjective homomorphisms

$$V_p(\mathcal{O}_K) \to C_{p^2-1}[\langle \epsilon_0 \rangle] \to C_{p^2-1}/\delta m.$$

The last cyclic group has order dividing four, because

$$\frac{p^2 - 1}{\delta m} = \left(\frac{p+1}{2\delta m}\right) \cdot 2(p-1), \text{ and } p \equiv -1 \pmod{2\delta m}.$$
by hypothesis. Thus, indeed $V_p(\mathcal{O}_K)$ has an element of order four. □

3. Comparison between $R(\mathcal{O}_K G)$ and $R_{sd}(\mathcal{O}_K G)$

In this section, we shall prove Theorems 1.12 and 1.13. A key ingredient is the characterization of $R(\mathcal{O}_K G)$ due to L. R. McCulloh [16], which works for all abelian groups $G$; see Subsection 3.2 below. We remark that the proof of (1.3) given in [10] uses his older characterization of $R(\mathcal{O}_K G)$ from [15], which works only for elementary abelian groups $G$.

In the subsequent subsections, except in Subsection 3.3, we shall assume that $G$ is abelian. We shall further use the following notation.

**Notation 3.1.** Let $M_K$ denote the set of finite primes in $K$. The symbol $F$ shall denote either $K$ or the completion $K_v$ of $K$ at some $v \in M_K$, and

$$
\mathcal{O}_F = \text{the ring of integers in } F,
$$
$$
F^c = \text{a fixed algebraic closure of } F,
$$
$$
\mathcal{O}_{F^c} = \text{the integral closure of } \mathcal{O}_F \text{ in } F^c,
$$
$$
\Omega_F = \text{the Galois group of } F^c/F.
$$

For each $v \in M_K$, we shall regard $K^c$ as lying in $K_v^c$ via a fixed embedding $K^c \hookrightarrow K_v^c$ extending the natural embedding $K \hookrightarrow K_v$.

### 3.1. Locally free class group.

The class group $\text{Cl}(\mathcal{O}_K G)$ admits an idelic description as follows; see [5, Chapter 6], for example.

Let $J(KG)$ denote the restricted direct product of $(K_v G)\times$ with respect to the subgroups $(\mathcal{O}_{K_v} G)\times$ for $v \in M_K$. We have a surjective homomorphism

$$
j : J(KG) \longrightarrow \text{Cl}(\mathcal{O}_K G); \quad j(c) = [\mathcal{O}_K G \cdot c],
$$

where we define

$$
\mathcal{O}_K G \cdot c = \bigcap_{v \in M_K} (\mathcal{O}_{K_v} G \cdot c_v \cap K G).
$$

This in turn induces an isomorphism

$$
(3.1) \quad \text{Cl}(\mathcal{O}_K G) \simeq \frac{J(KG)}{(KG)\times U(\mathcal{O}_K G)}, \quad \text{where } U(\mathcal{O}_K G) = \prod_{v \in M_K} (\mathcal{O}_{K_v} G)\times.
$$
Each component \((K_vG)\times\) as well as \((KG)\times\) also admit a Hom-description as follows. Write \(\hat{G}\) for the group of irreducible \(K^c\)-valued characters on \(G\). We then have canonical identifications

\[
(F^cG)\times = \text{Map}(\hat{G}, (F^c)^\times) = \text{Hom}(\mathbb{Z}\hat{G}, (F^c)^\times),
\]

\[
(FG)\times = \text{Map}_{\Omega_F}(\hat{G}, (F^c)^\times) = \text{Hom}_{\Omega_F}(\mathbb{Z}\hat{G}, (F^c)^\times),
\]

induced by the association \(\alpha \mapsto (\chi \mapsto \alpha(\chi))\), where we define \(\alpha(\chi) = \sum_{s \in G} \alpha_s \chi(s)\) for \(\alpha = \sum_{s \in G} \alpha_s s\).

Finally, we note that via \((3.1)\) and \((3.2)\), for each \(k \in \mathbb{Z}\) coprime to \(|G|\), the \(k\)th Adams operation \(\Psi_k\) on \(\text{Cl}(O_K G)\) is induced by \(\chi \mapsto \chi^k\) on \(\hat{G}\).

### 3.2. McCulloh’s characterization.

The characterization of \(R(O_K G)\) due to L. R. McCulloh \[16\] is given in terms of the so-called Stickelberger transpose. We shall recall its definition below.

**Definition 3.2.** Let \(G(-1)\) denote the group \(G\) on which \(\Omega_F\) acts by

\[\omega \cdot s = s^{\kappa(\omega^{-1})}\]

for \(s \in G\) and \(\omega \in \Omega_F\),

where \(\kappa(\omega^{-1}) \in \mathbb{Z}\), which is unique modulo \(\exp(G)\), is such that

\[\omega^{-1}(\zeta) = \zeta^{\kappa(\omega^{-1})}\]

for all \(\zeta \in F^c\) with \(\zeta^{\exp(G)} = 1\).

Note that if \(\zeta_n \in F\), then \(\Omega_F\) fixes all elements in \(G(-1)\) of order dividing \(n\).

**Definition 3.3.** Given \(\chi \in \hat{G}\) and \(s \in G\), define

\[
\langle \chi, s \rangle \in \left\{ \frac{0}{|s|}, \frac{1}{|s|}, \ldots, \frac{|s| - 1}{|s|} \right\}
\]

to be such that \(\chi(s) = (\zeta_{|s|})^{s|\langle \chi, s \rangle|}\).

Extend this to a pairing \(\langle \cdot, \cdot \rangle : \mathbb{Q}\hat{G} \times \mathbb{Q}G \rightarrow \mathbb{Q}\) via \(\mathbb{Q}\)-linearity, and define

\[
\Theta : \mathbb{Q}\hat{G} \rightarrow \mathbb{Q}G(-1); \quad \Theta(\psi) = \sum_{s \in G} \langle \psi, s \rangle s,
\]

called the *Stickelberger map*.

As shown in \[16, Proposition 4.5\], the Stickelberger map preserves the \(\Omega_F\)-action. Set \(A_{\hat{G}} = \Theta^{-1}(\mathbb{Z}G)\). Then, applying the functor \(\text{Hom}(-, (F^c)^\times)\) and
taking $\Omega_F$-invariants yield a homomorphism

$$\Theta^t : \text{Hom}_{\Omega_F}(\mathbb{Z}G(-1), (F^c) \times) \rightarrow \text{Hom}_{\Omega_F}(A_\widehat{G}, (F^c) \times); \ g \mapsto g \circ \Theta.$$  

This is the Stickelberger transpose map defined in [16].

For brevity, define

$$(3.3)\quad \Lambda(FG) \times = \text{Map}_{\Omega_F}(G(-1), (F^c) \times) \quad (= \text{Hom}_{\Omega_F}(\mathbb{Z}G(-1), (F^c) \times)).$$

Observe that we have a diagram

$$(FG) \times \xymatrix{\rightarrow \ar[r]^{\text{rag}} & \text{Hom}_{\Omega_F}(A_\widehat{G}, (F^c) \times) \ar[u]^\Theta^t} \Lambda(FG) \times,$$

where \text{rag} is restriction to $A_\widehat{G}$ via the identification (3.2).

Now, let $J(\Lambda(KG))$ denote the restricted direct product of $\Lambda(K_vG) \times$ with respect to the subgroups $\text{Map}_{\Omega_F}(G(-1), \mathcal{O}_{F^c}^\times)$ for $v \in M_K$. We then have the following partial characterization of $R(\mathcal{O}_KG)$; see [16] for the full characterization.

**Lemma 3.4.** Given $c = (c_v) \in J(KG)$, if there exists $g = (g_v) \in J(\Lambda(KG))$ such that \text{rag}(c_v) = \Theta^t(g_v)$ for all $v \in M_K$, then $j(c) \in R(\mathcal{O}_KG)$.

**Proof.** This follows directly from [16, Theorem 6.17].

For each $v \in M_K$, fix a uniformizer $\pi_v$ of $K_v$. We shall also need:

**Lemma 3.5.** Let $L/K$ be a tame and Galois extension with $\text{Gal}(L/K) \simeq G$. Then, for each $v \in M_K$, there exists $s_v \in G$ whose order is the ramification index of $L/K$ at $v$, such that $\Xi_{-1}(\mathcal{O}_L) = j(c_L)$, where $c_L = (c_{L,v}) \in J(KG)$ is defined by $c_{L,v}(\chi) = \pi_v^{(\chi, s_v) + (\chi, s_v^{-1})}$ for $\chi \in \hat{G}$ via the identification (3.2).

**Proof.** We have $L = KG \cdot b$ for some $b \in L$ by the Normal Basis Theorem, and since $\mathcal{O}_L$ is locally free over $\mathcal{O}_KG$ of rank one, for $v \in M_K$, we have

$$\mathcal{O}_{K_v} \otimes_{\mathcal{O}_K} \mathcal{O}_L = \mathcal{O}_{K_v}G \cdot a_v$$

for some $a_v \in \mathcal{O}_{K_v} \otimes_{\mathcal{O}_K} \mathcal{O}_L$. 
Following the notation in [16, Section 1], put
\[ r_G(b) = \sum_{s \in G} s(b)s^{-1} \] and \[ r_G(a_v) = \sum_{s \in G} s(a_v)s^{-1}. \]

Then, by [16, Proposition 5.4], we may choose \( a_v \) to be such that
\[(3.4) \quad r_G(a_v)(\chi) = \pi_b^{(\chi,s_v)} \text{ for all } \chi \in \hat{G}.\]

By [16, Proposition 3.2] and the discussion following it, there exists \( c = (c_v) \in J(KG) \) such that \( r_G(a_v) = c_v \cdot r_G(b) \) and \( j(c) = [\mathcal{O}_L] \).

Write \([-1]\) for the involution on \((K^*_vG)^\times\) induced by the involution \( s \mapsto s^{-1} \) on \( G \). Since \( r_G(b)r_G(b)^{-1} \in (KG)^\times \), we then deduce that
\[ \Xi_{-1}([\mathcal{O}_L]) = j((c_vc_v^{-1})) = j((r_G(a_v)r_G(a_v)^{-1})). \]

The claim now follows immediately from (3.4). \( \square \)

### 3.3. Generalized Swan subgroups

Let \( H \) be a subgroup of \( G \). Following the definition of the Swan subgroup \( T(O_KG) \) given in [25], we shall define a generalized Swan subset/subgroup associated to \( H \) as follows.

For each \( r \in O_K \) coprime to \(|H|\), define
\[ (r, \Sigma_H) = O_KG \cdot r + O_KG \cdot \Sigma_H, \text{ where } \Sigma_H = \sum_{s \in H} s. \]

The next proposition, which generalizes [25, Proposition 2.4 (i)], shows that \((r, \Sigma_H)\) is locally free over \( O_KG \) of rank one and so it defines a class \([ (r, \Sigma_H) ] \) in \( \text{Cl}(O_KG) \). Define
\[ T_G^*(O_KG) = \{ [(r, \Sigma_H)] : r \in O_K \text{ coprime to } |H| \} \]

to be the collection of all such classes. It follows directly from the definition that \( T_G^*(O_KG) \) is equal to \( T(O_KG) \).

**Proposition 3.6.** Let \( r \in O_K \) be coprime to \(|H|\). For each \( v \in M_K \), define
\[ c_{H,r,v} = \begin{cases} 1 & \text{if } v \nmid r, \\ r + \frac{1-r}{|H|}\Sigma_H & \text{if } v \mid r, \end{cases} \]
and set $c_{H,r} = (c_{H,r,v})$. Then we have $\mathcal{O}_K G \cdot c_{H,r} = (r, \Sigma_H)$.

**Proof.** For each $v \in M_K$, we need to show that

$$\mathcal{O}_{K_v} G \cdot c_{H,r,v} = \mathcal{O}_{K_v} G \cdot r + \mathcal{O}_{K_v} G \cdot \Sigma_H.$$ 

For $v \nmid r$, we have $r \in \mathcal{O}_{K_v}^\times$, and this is clear. For $v \mid r$, we have $|H| \in \mathcal{O}_{K_v}^\times$ because $r$ is coprime to $|H|$, and so

$$\mathcal{O}_{K_v} G \cdot \left(r + \frac{1-r}{|H|} \Sigma_H \right) \subset \mathcal{O}_{K_v} G \cdot r + \mathcal{O}_{K_v} G \cdot \Sigma_H.$$ 

The reverse inclusion also holds because

$$r = \left(1 + \frac{r-1}{|H|} \Sigma_H \right) \left(r + \frac{1-r}{|H|} \Sigma_H \right) \text{ and } \Sigma_H = \Sigma_H \left(r + \frac{1-r}{|H|} \Sigma_H \right).$$

We then see that the claim holds. $\square$

In what follows, for simplicity, let us assume that

(3.5) $H$ is normal in $G$ and the quotient $G/H$ is abelian.

Put $Q = G/H$, and let $H_1, \ldots, H_q$ denote all the distinct cosets of $H$ in $G$. Notice that we have an augmentation homomorphism

$$\epsilon : \mathcal{O}_K G \rightarrow \mathcal{O}_K Q; \quad \epsilon \left( \sum_{s \in G} \alpha_s s \right) = \sum_{i=1}^q \left( \sum_{s \in H_i} \alpha_s \right) H_i.$$ 

Then, we have a fiber product diagram of rings, given by

$$\begin{array}{ccc}
\mathcal{O}_K G & \xrightarrow{\epsilon} & \mathcal{O}_K Q \\
\downarrow & & \downarrow \pi \\
\Gamma_H & \xrightarrow{\varpi} & \Lambda_{|H|} Q
\end{array}$$

where

$$\begin{cases}
\Gamma_H = \mathcal{O}_K G / (\Sigma_H), \\
\Lambda_{|H|} = \mathcal{O}_K / |H| \mathcal{O}_K.
\end{cases}$$

Here the vertical maps are the canonical quotient maps, and $\varpi$ is the homomorphism induced by $\epsilon$. We then have the identification

(3.7) $\mathcal{O}_K G = \{(x, y) \in \mathcal{O}_K Q \times \Gamma_H : \pi(x) = \varpi(y)\}$.

In particular, writing

(3.8) $x = \sum_{i=1}^q x_i H_i, \quad y = \tilde{y} + (\Sigma_H), \quad \epsilon(\tilde{y}) = \sum_{i=1}^q \tilde{y}_i H_i,$
the corresponding element in $\mathcal{O}_K G$ is given by
\[ \tilde{y} + \left( \sum_{i=1}^{q} \left( \frac{x_i - \tilde{y}_i}{|H|} \right) s_i \right) \Sigma_H, \]
where $s_i \in H_i$ is fixed.

Since $Q$ is abelian, from the Mayer-Vietoris sequence (see [5, Section 49B] or [19, (1.12), (4.19), (4.21)]) associated to (3.6), we obtain a homomorphism
\[ \partial_H : (\Lambda_{|H|} Q)^{\times} \longrightarrow D(\mathcal{O}_K G); \quad \partial_H(\eta) = [(\mathcal{O}_K G)(\eta)], \]
where $D(\mathcal{O}_K G)$ denotes the kernel group in $\text{Cl}(\mathcal{O}_K G)$ defined as in [19], and
\[ (\mathcal{O}_K G)(\eta) = \{(x, y) \in \mathcal{O}_K Q \times \Gamma_H : \pi(x) = \overline{c}(y)\eta\} \]
is equipped with the obvious $\mathcal{O}_K G$-module structure via (3.7).

The next proposition, which generalizes [25, Proposition 2.7], shows that
\[ T^*_H(\mathcal{O}_K G) = \partial_H(\Lambda_{|H|}^{\times}), \]
where $\Lambda_{|H|}$ is regarded as a subring of $\Lambda_{|H|} Q$ in the obvious way (cf. the set $T_H(\mathcal{O}_K G)$ defined in [17]). This means that under the assumption (3.5), the set $T^*_H(\mathcal{O}_K G)$ is in fact a subgroup of $\text{Cl}(\mathcal{O}_K G)$.

**Proposition 3.7.** Let $r \in \mathcal{O}_K$ be coprime to $|H|$. Then we have
\[ \partial_H((r + |H|\mathcal{O}_K)H) = [(r, \Sigma_H)]. \]

*Proof.* For brevity, put $\eta = (r + |H|\mathcal{O}_K)H$. Note that by definition, we have

\[ \eta = \pi(rH) = \overline{c}(r + (\Sigma_H)). \]

Via the identification (3.7), we may define an $\mathcal{O}_K G$-homomorphism
\[ \varphi : (\mathcal{O}_K G)(\eta) \longrightarrow \mathcal{O}_K G; \quad \varphi(x, y) = (x, y(r + (\Sigma_H))). \]

Below, we shall show that $\text{Im}(\varphi) = (r, \Sigma_H)$ and $\text{ker}(\varphi) = 0$. This would imply that $(\mathcal{O}_K G)(\eta)$ and $(r, \Sigma_H)$ are isomorphic as $\mathcal{O}_K G$-modules, from which the claim follows. Given $(x, y) \in (\mathcal{O}_K G)(\eta)$, in the notation of (3.8), we have

\[ (x, y(r + (\Sigma_H))) = \tilde{y}r + \left( \sum_{i=1}^{q} \left( \frac{x_i - \tilde{y}_i r}{|H|} \right) s_i \right) \Sigma_H. \]
First, from (3.9), we immediately see that \( \text{Im}(\varphi) \subset (r, \Sigma_H) \), as well as

\[ \varphi((rH, 1 + (\Sigma_H)) = r \text{ and } \varphi(|H|H, (\Sigma_H)) = \Sigma_H, \]

whence \( \text{Im}(\varphi) \supset (r, \Sigma_H) \) holds also. Next, suppose that \((x, y) \in \ker(\varphi)\). It is clear from the definition of \(\varphi\) that \(x = 0\). Then, we deduce from (3.9) that

\[ \tilde{y}r - \left( \sum_{i=1}^{q} \frac{\tilde{y}_i r}{|H|s_i} \right) \Sigma_H = 0 \text{ and hence } \tilde{y} \in (\Sigma_H). \]

This shows that \(y = 0\), and so \(\ker(\varphi) = 0\), as desired. \(\square\)

3.4. Preliminaries. Let \(H\) be a subgroup of \(G\) and let \(r \in \mathcal{O}_K\) be coprime to \(|H|\). Then, via the isomorphism (3.1), we have

\[ j(c_{H,r}) = [(r, \Sigma_H)], \text{ where } (c_{H,r}) = (c_{H,r,v}) \in J(KG) \]

is defined as in Proposition 3.6. Also, note that for \(v \mid r\), we have

\[ c_{H,r,v}(\chi) = \begin{cases} 1 & \text{if } \chi(H) = 1 \\ r & \text{if } \chi(H) \neq 1 \end{cases} \]

for \(\chi \in \widehat{G}\) via the identification (3.2). This immediately implies that:

**Proposition 3.8.** We have \(T^*_H(\mathcal{O}_K G) \subset \text{Cl}(\mathcal{O}_K G)^{\Psi_2}\).

**Proof.** This follows from (3.11) and the fact that

\[ \chi^k(H) = 1 \text{ if and only if } \chi(H) = 1 \]

for any \(k \in \mathbb{Z}\) coprime to \(|H|\). \(\square\)

To make connections between \(T^*_H(\mathcal{O}_K G)\) and \(R(\mathcal{O}_K G)\), we shall use Lemmas 3.4 and 3.5. We shall also need the following definitions.

Fix a prime \(v \in M_K\). Recall from (3.2) and (3.3) that

\[ (K_v G)^\times = \text{Map}_{\Omega_{K_v}}(\widehat{G}, (K_v^c)^\times) \text{ and } \Lambda(K_v G)^\times = \text{Map}_{\Omega_{K_v}}(G(-1), (K_v^c)^\times). \]

Given \(t \in G\) with \(t \neq 1\) and \(x \in K_v^\times\), define

\[ c_{t, v, x, 1}(\chi) = x^{(\chi, t) + (\chi, t^{-1})} \text{ and } c_{t, v, x, 2}(\chi) = x^{2(\chi, t) - (\chi, t^2)}. \]
for $\chi \in \hat{G}$, where both exponents are integers by Definition 3.3. In the case that $|t| = 2$ and $|t| > 2$, respectively, define

$$g_{t,v,x,1}(s) = \begin{cases} x^2 & \text{if } s = t \\ 1 & \text{otherwise} \end{cases}$$

and

$$g_{t,v,x,1}(s) = \begin{cases} x & \text{for } s \in \{t, t^{-1}\} \\ 1 & \text{otherwise} \end{cases}$$

for $s \in G(-1)$. In the case that $|t|$ is odd, further define

$$g_{t,v,x,2}(s) = \begin{cases} x^2 & \text{if } s = t \\ x^{-1} & \text{for } s = t^2 \\ 1 & \text{otherwise} \end{cases}$$

for $s \in G(-1)$. We have the following lemmas.

**Lemma 3.9.** We have $c_{t,v,x,1} \in (K_vG)^\times$.

**Proof.** The map $c_{t,v,x,1}$ preserves the $\Omega_{K_v}$-action because

$$\langle \chi, s \rangle + \langle \chi, s^{-1} \rangle = \begin{cases} 0 & \text{if } \chi(s) = 1 \\ 1 & \text{if } \chi(s) \neq 1 \end{cases}$$

for all $\chi \in \hat{G}$ and $s \in G$ by Definition 3.3. □

**Lemma 3.10.** Suppose that $\zeta_{|t|} \in K_v^\times$. Then we have

$$c_{t,v,x,2} \in (K_vG)^\times \text{ and } g_{t,v,x,1}, g_{t,v,x,2} \in \Lambda(K_vG)^\times.$$  

Moreover, for both $i = 1, 2$, we have

$$r\text{ag}(c_{t,v,x,i}) = \Theta^t(g_{t,v,x,i}).$$

**Proof.** Since $\zeta_{|t|} \in K_v^\times$, we easily see that $c_{t,v,x,2}$, $g_{t,v,x,1}$, and $g_{t,v,x,2}$ indeed all preserve the $\Omega_{K_v}$-action. Since

$$\Theta^t(g)(\psi) = \prod_{s \in G} g(s)^{\langle \psi, s \rangle} \text{ for } g \in \Lambda(K_vG)^\times \text{ and } \psi \in A\hat{G}$$

by definition, the second also holds by a simple verification. □

3.5. **Proof of Theorem 1.12.** Let $H$ be a cyclic subgroup of $G$ of order $n$ and let $r \in \mathcal{O}_K$ be coprime to $n$. Recall (3.10) and that $j(c_{H,r}) \in \text{Cl}(\mathcal{O}_K G)^{\Psi_2}$
by Proposition 3.8. We need to show that \( j(c_{H,r})^{d_n(K)} \in R(\mathcal{O}_K G) \) in part (a), and that \( j(c_{H,r}) \in \text{Im}(\Xi_{-1}) \) in part (b). We shall do so using Lemma 3.4.

In what follows, let \( t \) be a fixed generator of \( H \).

**Proof of Theorem 1.12 (a).** Let \( D \) be the subgroup of \( (\mathbb{Z}/n\mathbb{Z})^\times \) such that
\[
D \simeq \text{Gal}(K(\zeta_n)/K) \text{ via } i \mapsto (\zeta_n \mapsto \zeta_n^i).
\]
Define \( g = (g_v) \in J(\Lambda(KG)) \) by setting \( g_v = 1 \) for \( v \nmid r \), and
\[
g_v(s) = \begin{cases} r & \text{if } s \in \{t^i, t^{-i}\} \text{ for some } i \in D \\ 1 & \text{otherwise} \end{cases}
\]
for \( v \mid r \). It is easy to see that \( g_v \) preserves the \( \Omega_{K_v} \)-action.

Observe that for all \( v \in M_K \), we have
\[
(3.14) \quad r a g((c_{H,r,v})^{d_n(K)}) = \Theta^t(g_v).
\]
Indeed, for \( v \nmid r \), this is clear. As for \( v \mid r \), we have from (3.13) that
\[
\Theta^t(g_v)(\chi) = \begin{cases} (r) \frac{1}{2} \sum_{i \in D} (\langle \chi, t^t \rangle + \langle \chi, t^{-t} \rangle) & \text{if } -1 \in D \\ (r) \sum_{i \in D} (\langle \chi, t^t \rangle + \langle \chi, t^{-t} \rangle) & \text{if } -1 \not\in D \end{cases}
\]
and from (3.12) that
\[
\sum_{i \in D} \left( \langle \chi, t^t \rangle + \langle \chi, t^{-t} \rangle \right) = \begin{cases} 0 & \text{if } \chi(t) = 1 \\ |D| & \text{if } \chi(t) \neq 1 \end{cases}
\]
for any \( \chi \in \hat{G} \). The equality (3.14) then follows from (3.11). Hence, we have \( j(c_{H,r})^{d_n(K)} \in R(\mathcal{O}_K G) \) by Lemma 3.4, as desired. \( \square \)

**Proof of Theorem 1.12 (b).** Suppose that \( n \) is odd and that \( \zeta_n \in K^\times \). Then, by Lemma 3.10, we may define \( c = (c_v) \in J(KG) \) by setting \( c_v = 1 \) for \( v \nmid r \), and \( c_v = c_{t,v,r,2} \) for \( v \mid r \). Also, we have \( j(c) \in R(\mathcal{O}_K G) \) by Lemma 3.4.

Below, we shall show that \( j(c_{H,r}) = \Xi_{-1}(j(c)) \), whence \( j(c_{H,r}) \in \text{Im}(\Xi_{-1}) \). To that end, let \( v \in M_K \) and \( \chi \in \hat{G} \). It suffices to show that
\[
(3.15) \quad c_{H,r,v}(\chi) = c_v(\chi)c_v(\chi^{-1}).
\]
For \( v \nmid r \), this is clear. For \( v \mid r \), observe that
\[
\chi(t) = 1 \text{ if and only if } \chi(t^2) = 1
\]
because \( |t| \) is odd. It then follows from (3.12) that
\[
c_v(\chi)c_v(\chi^{-1}) = r^2(\langle \chi, t \rangle + \langle \chi^{-1}, t \rangle - \langle \chi, t^2 \rangle + \langle \chi^{-1}, t^2 \rangle) = \begin{cases} 1 & \text{for } \chi(t) = 1, \\ r & \text{for } \chi(t) \neq 1. \end{cases}
\]

From (3.11), we then see that (3.15) indeed holds. \( \Box \)

3.6. Proof of Theorem 1.13. In what follows, for each \( v \in M_K \), let \( \pi_v \) be a fixed uniformizer of \( K_v \).

Proof of Theorem 1.13 (a). Suppose that \( \text{Cl}(\mathcal{O}_K) = 1 \). For each \( v_0 \in M_K \), we may then choose \( \pi_{v_0} \) to be an element of \( \mathcal{O}_K \). Then, for any cyclic subgroup \( H \) of \( G \) of order coprime to \( v_0 \), it makes sense to write
\[
[(\pi_{v_0}, \Sigma_H)] = j(c_{H,\pi_{v_0}}), \text{ where } c_{H,\pi_{v_0}} = (c_{H,\pi_{v_0},v}) \in J(KG)
\]
is as in Proposition 3.6. Plainly \( c_{H,\pi_{v_0},v} = 1 \) for all \( v \neq v_0 \).

Now, let \( L/K \) be any tame and Galois extension with \( \text{Gal}(L/K) \cong G \), and we shall use the notation as in Lemma 3.5. Let \( V \) denote the subset of \( M_K \) consisting of the primes which ramify in \( L/K \). Then, we have
\[
s_v = 1 \text{ for } v \notin V, \text{ and so } \Xi_{-1}([\mathcal{O}_L]) = j(c_L) = \prod_{v_0 \in V} j(c_{L,v_0}),
\]
where we regard \( c_{L,v_0} \) as an element of \( J(KG) \) whose components outside of \( v_0 \) are all 1. For each \( v_0 \in V \), take \( H_{v_0} = \langle s_{v_0} \rangle \), whose order is coprime to \( v_0 \) because \( L/K \) is tame. By (3.11) and (3.12), we have
\[
c_{L,v_0}(\chi) = \pi_{v_0}^{\langle \chi, s_{v_0} \rangle + \langle \chi^{-1}, s_{v_0}^{-1} \rangle} = c_{H_{v_0},\pi_{v_0},v_0}(\chi) \text{ for all } \chi \in \hat{G}.
\]
By definition, we also have \( c_{L,v_0,v} = c_{H_{v_0},\pi_{v_0},v} = 1 \) for \( v \neq v_0 \). It follows that \( j(c_{L,v_0}) = j(c_{H_{v_0},\pi_{v_0}}) \), which is an element of \( T_{H_{v_0}}^*(\mathcal{O}_KG) \). This implies that
\[
\Xi_{-1}([\mathcal{O}_L]) \in \prod_{v_0 \in V} T_{H_{v_0}}^*(\mathcal{O}_KG) \subset T_{\text{cyc}}^*(\mathcal{O}_KG),
\]
as claimed. \( \Box \)
Proof of Theorem 1.13 (b). Suppose that \( G \neq 1 \). Then, fix an element \( t \in G \) with \( t \neq 1 \), whose order shall be assumed to be odd when \( \delta(G) = 1 \), and fix a character \( \chi \in \hat{G} \) such that \( \chi(t) \neq 1 \). Now, suppose that \( \zeta_{\exp(G)} \in K^\times \). Then, via (3.1) and (3.2), evaluation at \( \chi \) induces a surjective homomorphism

\[
\xi_\chi : \text{Cl}(\mathcal{O}_K G) \longrightarrow \text{Cl}(\mathcal{O}_K).
\]

Below, we shall show that

\[
(3.16) \quad \xi_\chi(\text{Im}(\Xi_{-1})) \supset \text{Cl}(\mathcal{O}_K)^{\delta(G)},
\]

from which the claim would follow.

Now, every class in \( \text{Cl}(\mathcal{O}_K) \) may be represented by a prime ideal \( p_0 \) in \( \mathcal{O}_K \), corresponding to \( v_0 \in M_K \), say. Since \( \zeta_{\exp(G)} \in K^\times \), by Lemmas 3.9 and 3.10, we may define \( c = (c_v) \in J(KG) \) by setting

\[
c_{v_0} = \begin{cases} 
  c_{t,v,\pi,v_0,1} & \text{if } \delta(G) = 2 \\
  c_{t,v,\pi,v_0,2} & \text{if } \delta(G) = 1
\end{cases}
\]

and \( c_v = 1 \) for \( v \neq v_0 \). Note that \( j(c) \in R(\mathcal{O}_K G) \) by Lemmas 3.4 and 3.10, whence \( \Xi_{-1}(j(c)) \in \text{Im}(\Xi_{-1}) \). Also, we have

\[
c_{v_0}(\chi)c_{v_0}(\chi^{-1}) = \begin{cases} 
  \pi_{v_0}^{2(\langle \chi, t \rangle + \langle \chi^{-1}, t \rangle)} & \text{if } \delta(G) = 2 \\
  \pi_{v_0}^{2(\langle \chi, t \rangle + \langle \chi^{-1}, t \rangle) - (\langle \chi, t^2 \rangle + \langle \chi^{-1}, t^2 \rangle)} & \text{if } \delta(G) = 1
\end{cases}
\]

by (3.12). We then deduce that

\[
\xi_\chi(\Xi_{-1}(j(c))) = [p_0]^{\delta(G)} \text{ in } \text{Cl}(\mathcal{O}_K).
\]

This proves the desired inclusion \((3.16)\). \( \square \)

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