Casimir effect for a perfectly conducting wedge in terms of local zeta function

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Abstract

The vacuum energy density of electromagnetic field inside a perfectly conducting wedge is calculated by making use of the local zeta function technique. This regularization completely eliminates divergent expressions in the course of calculations and gives rise to a finite expression for the energy density in question without any subtractions. Employment of the Hertz potentials for constructing the general solution to the Maxwell equations results in a considerable simplification of the calculations. Transition to the global zeta function is carried out by introducing a cutoff nearby the cusp at the origin. Proceeding from this the heat kernel coefficients are calculated and the high temperature asymptotics of the Helmholtz free energy and of the torque of the Casimir forces are found. The wedge singularity gives rise to a specific high temperature behaviour $\sim T^2$ of the quantities under consideration. The obtained results are directly applicable to the free energy of a scalar massless field and electromagnetic field on the background of a cosmic string.

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I. INTRODUCTION

The Casimir calculations are usually accomplished by two methods: a global approach, when from the very beginning the total vacuum energy of quantum fields are calculated for given boundary conditions, and a local consideration, when first the space density of the vacuum energy is found. The both methods are in agreement if the integration of the vacuum energy density gives the total Casimir energy. As known, the first example where this condition is not satisfied was a perfectly conducting wedge, i.e., two perfectly conducting plates that are crossed at given angle. The calculation of the Casimir energy for these boundary conditions has the following peculiarities. The spectrum of electromagnetic oscillations inside the wedge is continues and it does not depend explicitly on the opening angle of the wedge. The boundaries are flat therefore there is no problem with additional divergences due to the boundary curvature. However, on the cross line of the plates the smoothness of the boundary is obviously violated due to the cusp. Till now a correct definition of the total vacuum energy of the fields inside the wedge is lacking. It has been calculated only the spatial density of vacuum energy inside the wedge. It turns out that this density possesses a non-integrable singularity $\sim r^{-4}$ in cylindrical coordinates $r, \theta, z$ with the $z$ axes along the cross line of the plates. For the first time these calculations have been done probably in Ref. [1] by making use of a simple method of images and later in Ref. [2] by applying the Green’s function method. In recent publications [3,4] in the framework of the Schwinger source theory it was shown how to avoid some mathematical inconsistency taking place in previous calculations [2,5].

The geometry of space analogous to that between the crossing plates also encounters in other physical problems, in particular, in the cosmic string theory (the space outside the string) and under consideration of quantum fields at finite temperature on the Rindler space-time.

The geometry of space in the background of a cosmic string is determined by the solution of the Einstein equations with the energy-momentum tensor of matter defined by the string. When the string is located along the $z$ axes and possesses the linear mass density $\mu$, then the space outside the string is isomorphic to the manifold $C_{2\pi-\phi} \times \mathbb{R}^1$, where $\mathbb{R}^1$ is an infinite line along the $z$ axes and instead of a plane perpendicular to this axes one has two dimensional cone $C_{2\pi-\phi}$ with the angle deficiency $\phi = 8\pi G \mu$, $G$ being the gravitational constant. The cone of an angle $\beta$ becomes a plane when $\beta = 2\pi$. Certainly the boundary conditions for fields inside the wedge $W_\alpha \times \mathbb{R}^1$ and on the space $C_\beta \times \mathbb{R}^1$ are in the general case different. For the cone $C_\beta$ the periodic boundary conditions are natural. For fields inside the wedge the boundary conditions are determined by the relevant physical origin of the fields and by the physical characteristics of the boundary.

The manifold $C_\beta \times \mathbb{R}^2$ arises under consideration of fields on the Rindler space-time at finite temperature $\beta^{-1}$. In order to calculate the Helmholtz free energy and entropy of fields on this space the local zeta function technique was proposed [6]. A nontrivial problem encountered here is the construction of required analytic continuation of this zeta function. Such a continuation has been accomplished separately for lower and higher eigenvalues of the Laplace operator in the problem at hand.

As far as we know the Casimir energy for a perfectly conducting wedge at finite temperature was considered only in Ref. [7] by the multiple scattering method. In that paper
unusual temperature dependence of the Casimir forces in this problem was noted. These forces, tending to reduce the angle $\alpha$ between the plates, diminish when taking into account the first temperature correction.

The present article seeks to show that the most simple and consistent (from the mathematical point of view) approach to the calculation of the Casimir effect for a perfectly conducting wedge is to use the local zeta function technique. In order to construct such spectral function one should find not only the spectrum of the regarding operator but also its eigenfunctions. When looking for the solutions to the Maxwell equations in confined regions (waveguides or resonators) it is convenient to use the Hertz potentials or Hertz vectors [2,8]. This technique considerably simplifies the calculations, specifically, removing the problem of choosing the gauge. It is this way that we shall follow when constructing the local zeta function for electromagnetic field inside a perfectly conducting wedge $W_\alpha \times \mathbb{R}^1$.

The layout of the paper is as follows. In Section II the Hertz potentials are introduced and the solutions to Maxwell equations inside a perfectly conducting wedge are constructed. These potentials are expressed in terms of two scalar functions which satisfy Dirichlet and Neumann boundary conditions. In Section III the local zeta function in the problem at hand is constructed. By a simple change of angle variable this function reduces to the analogous one for a massless scalar field on the manifold $C_\beta \times \mathbb{R}^1$, when $\beta = 2\alpha$. Therefore the analytic continuation developed in Ref. [3] is also applicable in our case. The local zeta function constructed enables one to calculate the vacuum energy density of the electromagnetic field inside a wedge without subtraction of infinities. In Section IV transition to the global zeta function is carried out by introducing a cutoff nearby the cusp at the origin. Proceeding from this the heat kernel coefficients are calculated and the high temperature asymptotics of the Helmholtz free energy and of the torque of the Casimir forces are found. The wedge singularity gives rise to a specific high temperature behaviour $\sim T^2$ of the quantities under consideration. The obtained results are directly applicable to the free energy of a scalar massless field and electromagnetic field on the background of a cosmic string. In Section V we summarize our results. In the Appendix the local zeta functions are constructed for scalar massless fields inside a wedge with Dirichlet and Neumann boundary conditions.

II. SOLUTION TO MAXWELL EQUATIONS IN TERMS OF HERTZ VECTORS

Finding the solutions of Maxwell equations inside confined regions proves to be a rather difficult problem. Mainly it is due to the vector character of the electromagnetic field. A convenient technique for this purpose is to use the electric ($\Pi'$) and magnetic ($\Pi''$) Hertz vectors [8,10]. The Lorentz gauge condition

$$\text{div} \mathbf{A} + \frac{1}{c} \frac{\partial \varphi}{\partial t} = 0$$

is satisfied identically because the Hertz vectors are connected with the potentials $\mathbf{A}$ and $\varphi$ in the following way

$$\mathbf{A} = \frac{1}{c} \frac{\partial \Pi}{\partial t}, \quad \varphi = -\text{div} \Pi.$$ 

For boundary problems with cylindrical symmetry the electric ($\mathbf{E}$) and magnetic ($\mathbf{H}$) fields are expressed in terms of the Hertz vectors having only one nonzero component
\[ \Pi' = (0, 0, U), \quad \Pi'' = (0, 0, V). \] (2.1)

Here cylindrical coordinates \((r, \theta, z)\) are used with \(z\) axes directed along the intersection line of the wedge planes. The scalar functions \(U\) and \(V\) are the eigenfunctions of the Laplace operator and meet, respectively, Dirichlet and Neumann conditions on the boundary \(\partial \Gamma\)

\[ \Delta U = \omega^2 U, \quad U|_{\partial \Gamma} = 0, \] (2.2)

\[ \Delta V = \omega^2 V, \quad \frac{\partial V}{\partial n} \bigg|_{\partial \Gamma} = 0. \] (2.3)

When considering the electromagnetic field inside a perfectly conducting dihedral of opening angle \(\alpha\) (for simplicity, conducting wedge \(W_\alpha\)) the eigenvalues of the operator \((-\Delta)\) are

\[ \omega^2 \equiv \omega^2(k, \lambda) = k^2 + \lambda^2, \quad -\infty < k < \infty, \quad 0 \leq \lambda < \infty, \] (2.4)

with \(k\) being the wave vector along the \(z\) direction. In all the functions we have dropped for simplicity the time-dependent factor \(e^{i\omega t}\). The spectrum (2.4) does not depend explicitly on the opening angle \(\alpha\) of the wedge. This point proves to be a crucial one when attempting to construct a global zeta function in this problem (see Section IV).

The Dirichlet boundary value problem (2.2) has the following normalized eigenfunctions (\(E\)-modes)

\[ u_{\lambda nk}(r, \theta, z) = \frac{e^{ikz}}{\sqrt{\pi \alpha}} J_{np}(\lambda r) \sin(np\theta), \] (2.5)

\[ 0 \leq \theta \leq \alpha, \quad p = \pi/\alpha, \quad n = 1, 2, \ldots, \quad 0 \leq \lambda < \infty. \]

For the Neumann boundary value problem (2.3) the normalized eigenfunctions (\(H\)-modes) have the form

\[ v_{\lambda nk}(r, \theta, z) = \eta_{n0} \frac{e^{ikz}}{\sqrt{\pi \alpha}} J_{np}(\lambda r) \cos(np\theta), \] (2.6)

\[ n = 0, 1, 2, \ldots, \quad 0 \leq \lambda < \infty, \quad \eta_{n0} = \begin{cases} \frac{1}{\sqrt{2}}, & n = 0, \\ 1, & n = 1, 2, \ldots. \end{cases} \]

The eigenfunctions (2.5) and (2.6) meet the orthogonality condition

\[ \int_{-\infty}^{\infty} dz \int_{0}^{\infty} r dr \int_{0}^{\alpha} d\theta f_{\lambda nk}^*(r, \theta, z)f_{\lambda nk}(r, \theta, z) = \delta_{nn'} \frac{1}{\lambda} \delta(\lambda - \lambda') \delta(k - k') \] (2.7)

and the completeness condition

\[ \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} dk \int_{0}^{\alpha} \lambda d\lambda f_{\lambda nk}^*(r', \theta', z')f_{\lambda nk}(r, \theta, z) = \frac{1}{r} \delta(r - r') \delta(z - z') \delta(\theta - \theta'). \] (2.8)
Here \( f_{\lambda nk}(r, \theta, z) \) are the eigenfunctions (2.5) or (2.6). The relations (2.7) and (2.8) can be easily verified if one takes into account the formula [11]
\[
\int_0^\infty \lambda d\lambda J_q(\lambda r) J_q(\lambda r') = \frac{1}{r} \delta(r-r') .
\]

For given Hertzian vectors \( \Pi' \) and \( \Pi'' \) the electric and magnetic fields are constructed by the formulas
\[
\begin{align*}
E &= \nabla \times \nabla \times \Pi', \quad H = -i\omega \nabla \times \Pi' \quad (E\text{-modes}), \\
E &= i\omega \nabla \times \Pi'', \quad H = \nabla \times \nabla \times \Pi'' \quad (H\text{-modes}).
\end{align*}
\]

(2.9)

It has been proved [12] that the superposition of these modes gives the general solution to the Maxwell equations in the problem under consideration.

However in the zeta function formalism there is no need to recover the electric and magnetic fields through these formulas in order to construct the energy density or the energy momentum tensor as one proceeds when the Green’s function technique is used [5–4]. It is an essential merit of the approach considered.

### III. LOCAL ZETA FUNCTION FOR PERFECTLY CONDUCTING WEDGE

At the beginning we remind a few general formulas that define a local zeta function. Let we have a differential operator \( L \) with well posed eigenvalue problem (here it is convenient to use the Dirac bracket notation)
\[
Lf_n(x) = \lambda_n f_n(x) \quad \text{or} \quad L|n> = \lambda_n |n> .
\]

(3.1)

By making use of the completeness of the vector set \( \{ |n> \} \), one can represent (formally) a unity operator \( I \) acting in the linear space of the vectors \( |n> \) as follows
\[
I = \sum_n |n><n| .
\]

(3.2)

In view of this, we have for the inverse operator \( L^{-1} \)
\[
L^{-1} = \sum_n \frac{|n><n|}{\lambda_n} .
\]

(3.3)

It can be easily checked with allowance for Eq. (3.1). For the \( s \)th power of the inverse operator \( L^{-1} \) we can write
\[
L^{-s} = \sum_n \frac{|n><n|}{\lambda_n^s} .
\]

(3.4)

The local zeta function \( \zeta(s, x) \) of the operator \( L \) is a diagonal matrix element of the operator \( L^{-s} \)
\[
\zeta(s, x) = \sum_n \frac{\langle x | n \rangle > n \langle x \rangle}{\lambda_n^s} = \sum_n \lambda_n^{-s} f_n^*(x) f_n(x). \tag{3.5}
\]

The global zeta function
\[
\zeta(s) = \sum_n \lambda_n^{-s} \tag{3.6}
\]
is obtained by integration of \(\zeta(s, x)\) over the whole space
\[
\zeta(s) = \int \zeta(s, x) \, dx. \tag{3.7}
\]
Thus the local zeta function \(\zeta(s, x)\) can be interpreted as a spatial density for the global zeta function \(\zeta(s)\).

When calculating the vacuum energy of a relativistic quantum field \(E_0\) by making use of the zeta function regularization \([13,14]\), we have
\[
E_0 = \frac{1}{2} \zeta \left( s = -\frac{1}{2} \right), \tag{3.8}
\]
with the operator \(L\) being the spatial part of the relevant differential operator defining the field dynamics in the problem at hand. In this case \(\lambda_n = \omega_n^2\), where \(\omega_n\) are the classical characteristic frequencies of the field under study. In view of Eqs. (3.7) and (3.8), the quantity \((1/2)\zeta(-1/2, x)\) can be interpreted as the vacuum energy density calculated in the framework of the zeta function regularization. Besides that, it is easy to show that the vacuum expectation value of the canonical energy-momentum tensor of non-interacting scalar field can be represented in the form
\[
<0|T_{00}(x)|0> = \frac{1}{2} \sum_n \lambda_n^{1/2} f_n^*(x) f_n(x) = \frac{1}{2} \zeta \left( -\frac{1}{2}, x \right). \tag{3.9}
\]

In this case the operator \(L\) is simply \(-\Delta\).

The main difficulty in obtaining the local zeta function (3.3), as well as the global one (3.6), is an analytic continuation of these formula to the region \(\text{Re} \ s < 1\).

It is remarkable that for constructing the local zeta function \(\zeta(s, x)\) only the spectrum and the eigenfunctions of the relevant operator are needed. The explicit form of the regarding physical fields (for example, electromagnetic fields) is not involved in these formulas. Therefore we shall not use Eq. (2.9) determining the electric and magnetic fields. Instead of that we proceed from the spectrum (2.4) in the problem at hand and the respective eigenfunctions (2.5) and (2.6).

Substituting Eqs. (2.4), (2.3) and (2.6) into the definition of the local zeta function (3.5) we obtain
\[
\zeta(s, r, \theta, z) = \frac{1}{\pi \alpha} \int_{-\infty}^{\infty} dk \int_0^\infty \frac{\lambda d\lambda}{(k^2 + \lambda^2)^s} \left\{ \frac{1}{2} J_0^2(\lambda r) + \sum_{n=1}^\infty J_{np}^2(\lambda r) \right\} \\
= \frac{1}{\pi \alpha} \int_{-\infty}^{\infty} dk \int_0^\infty \frac{\lambda d\lambda}{(k^2 + \lambda^2)^s} \left[ \frac{1}{2} J_0^2(\lambda r) + \sum_{n=1}^\infty J_{np}^2(\lambda r) \right]. \tag{3.10}
\]
Thus the local zeta function in the problem at hand is independent of the angle variable \( \theta \) and, that is natural, of \( z \) due to the homogeneity of the system in the \( z \)-direction. For simplicity, we shall drop these arguments in what follows.

By making use of the value of the integral

\[
\int_{-\infty}^{\infty} \frac{dk}{(k^2 + \lambda^2)^s} = \frac{\sqrt{\pi} \lambda^{1-2s} \Gamma(s-1/2)}{\Gamma(s)}, \quad \text{Re } s > \frac{1}{2},
\]

we cast Eq. (3.10) into the form

\[
\zeta(s, r) = \frac{1}{\sqrt{\pi} \alpha} \frac{\Gamma(s-1/2)}{\Gamma(s)} \sum_{n=0}^{\infty} \int_{0}^{\infty} \lambda^{2-2s} J_{np}^2(\lambda r) d\lambda .
\]

The prime on the summation sign means that the \( n = 0 \) term is taken with half weight.

Now we have to use the following integration formula (see equation 6.574.2 with \( \mu = \nu \) in Ref. [15])

\[
\int_{0}^{\infty} t^{-\lambda} J_{\nu}^2(\alpha t) dt = \frac{\alpha^{\lambda-1} \Gamma(\lambda) \Gamma\left(\frac{2\nu - \lambda + 1}{2}\right)}{2^{\lambda} \Gamma^2\left(\frac{\lambda + 1}{2}\right) \Gamma\left(\frac{2\nu + \lambda + 1}{2}\right)}, \quad \text{Re } (2\nu + 1) > \text{Re } \lambda > 0, \quad \alpha > 0.
\]

Applying this formula we can accomplish the integration in each term in Eq. (3.12) for the respective values of the variable \( s \) restricted by the equalities

\[
1 < \text{Re } s < np + \frac{3}{2}, \quad n = 0, 1, 2, \ldots.
\]

In these regions the integral (3.13) converges. Therefore the objections against using this formula brought up in Ref. [3] are obviously not applicable here. As a result, we obtain

\[
\zeta(s, r) = \frac{1}{2\pi \alpha r^{3-2s}} \frac{\Gamma(s-1)}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{\Gamma(np + 3/2 - s)}{\Gamma(np + s - 1/2)}.
\]

When deriving this equation the formula for the doubling of the gamma function argument [15]

\[
\Gamma(2z) = (2\pi)^{-1/2} 2^{2z-1/2} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right)
\]

has been used.

By making use of the asymptotics of the ratio of two gamma functions [16], one can easily show (see below) that the series (3.15) converges in the region

\[
\text{Re } s > \frac{3}{2}.
\]
while the first term with \( n = 0 \) in the sum (3.15) is defined in the region

\[
1 < \text{Re} \ s < \frac{3}{2}.
\]

Thus in the problem under study, there is no representation for the local zeta function (3.15) determining it as an analytic function of the complex variable \( s \) in a finite region. The first term in Eq. (3.15) is defined in the region (3.18) but the rest of the sum is defined in the strip

\[
\frac{3}{2} < \text{Re} \ s < \frac{3}{2} + 1.
\]

Cheeger has proposed to overcome this difficulty in the following way [17]. The analytic continuation should be constructed separately for the first term with \( n = 0 \) in Eq. (3.15) and for the rest sum in this equation. Further the obtained results should be added. Obviously, such a continuation is unique.

Following the paper [3], we introduce the function

\[
G_{2\alpha}(s) = \sum_{n=1}^{\infty} \frac{\Gamma(np - s + 1)}{\Gamma(np + s)}, \quad \text{Re} \ s > 1.
\]

(3.20)

For \( \alpha = \pi \) \((p = 1)\) we have

\[
G_{2\pi}(s) = \sum_{n=1}^{\infty} \frac{\Gamma(n - s + 1)}{\Gamma(n + s)} = \frac{\Gamma(1 - s)}{\Gamma(s)} \left[ 1 + \frac{1 - s}{s} + \frac{(1 - s)(2 - s)}{s(1 + s)} + \ldots - 1 \right] = \frac{\Gamma(1 - s)}{\Gamma(s)} \left[ F(1 - s, 1; s; 1) - 1 \right], \quad \text{Re} \ s > 1,
\]

where \( F(\alpha, \beta; \gamma; z) \) is the hypergeometric function. Equation 9.122.1 from Ref. [15] gives

\[
F(1 - s, 1; s; 1) = \frac{\Gamma(s)\Gamma(2s - 2)}{\Gamma(2s - 1)\Gamma(s - 1)}, \quad \text{Re} \ s > 1.
\]

Finally we obtain

\[
G_{2\pi}(s) = -\frac{\Gamma(1 - s)}{2\Gamma(s)}, \quad \text{Re} \ s > 1.
\]

(3.21)

The right-hand side of this equation can naturally be considered as an analytic continuation of the function \( G_{2\pi}(s) \) to all over the complex plane \( s \). On the other hand, the expression in the right-hand side of Eq. (3.21) for \( 1/2 < \text{Re} \ s < 1 \) is nothing else as the first term with \( n = 0 \) in the sum in Eq. (3.15). Thus the analytical continuation of the first term in the sum (3.13) is trivial.

The basic idea of the analytical continuation of the function \( G_{2\alpha}(s) \) for \( \alpha \neq \pi \) proposed by Cheeger [17] is also simple. It relies on the fact that the \( n \)-th term of the series (3.20) has a polynomial asymptotics for large \( n \).
\[ a_n = \frac{\Gamma(np - s + 1)}{\Gamma(np + s)} \sim (np)^{1-2s} \sum_{j=0}^{\infty} c_j(s)(np)^{-2j}, \]  
where the functions \( c_j(s) \) are calculated exactly upon substituting in Eq. (3.22) the known asymptotics for the gamma function [8,10]. From Eq. (3.22) it follows immediately that the series in Eq. (3.20) determining the function \( G_{2\alpha}(s) \) is convergent for

\[ \text{Re } s > 1. \]  

When summing over \( n \), the terms in the asymptotics (3.22) with

\[ \text{Re } (2s + 2j - 1) \leq 1 \]  

lead to the divergences. Therefore, it is natural to substitute these sums by respective values of the Riemann zeta function \( \zeta_R(s) \) [8]

\[ G_{2\alpha}(s) = p^{1-2s} \sum_{0 \leq j \leq 1 - \text{Re } s} p^{-2j}c_j(s)\zeta_R(2s + 2j - 1) + \text{analytical part}, \]  

where the analytical part is the sum of the differences between \( a_n \) and the asymptotic terms in Eq. (3.19) picked out in the way described above. In the general case, it is impossible to calculate this analytical part, i.e., to express it in terms of known analytical functions. However, for

\[ s = \frac{1}{2}, 0, -\frac{1}{2}, -1, -\frac{3}{2}, \ldots, -\frac{k}{2} \]  

this part vanishes because for these values of \( s \) the ratio \( a_n \) in Eq. (3.22) reduces to the finite polynomial in \( n \). The positive integer \( k \) in Eq. (3.26) obviously depends on the number of asymptotic terms in Eq. (3.20) summed by the Riemann zeta function in the course of analytical continuation. Thus for values of the variable \( s \) listed in Eq. (3.20) there is an explicit representation of the function \( G_{2\alpha}(s) \) in terms of the Riemann zeta function. For the boundary value \( s = 1 \) the series defining the function \( G_{2\alpha}(s) \) diverges

\[ G_{2\alpha}(1) = \sum_{n=1}^{\infty} \frac{\Gamma(np)}{\Gamma(np + 1)} = \frac{1}{p} \sum_{n=1}^{\infty} \frac{1}{n}. \]  

It leads to a simple pole at the point \( s = 1 \) which acquires the analytic continuation of the function \( G_{2\alpha}(s) \) to the region \( \text{Re } s < 1 \).

In view of Eqs. (3.20) and (3.21), the zeta function (3.14) can be represented in the form

\[ \zeta(s,r) = \frac{1}{2\pi \alpha r^{3-2s}} \frac{\Gamma(s-1)}{\Gamma(s)} [G_{2\alpha}(s-1/2) - G_{2\alpha}(s-1/2)] = \frac{1}{2\sqrt{\pi \alpha r^{3-2s}}} \frac{I_{2\alpha}(s-1/2)}{\Gamma(s)}, \]

where, following Ref. [3], we have introduced the function
\[ I_{2\alpha}(s) = \frac{\Gamma(s - 1/2)}{\sqrt{\pi}} [G_{2\alpha}(s) - G_{2\pi}(s)]. \] (3.29)

This function, as well as \( G_{2\alpha}(s) \), possesses only a simple pole at the point \( s = 1 \). At the first sight the gamma function \( \Gamma(s - 1/2) \), entering in the definition of \( I_{2\alpha}(s) \), leads to the poles for \( s = 1/2, -1/2, -3/2, \ldots \). However, it can be shown that the difference \( G_{2\alpha}(s) - G_{2\pi}(s) \) vanishes at these points.

The local zeta function (3.28) for electromagnetic field inside a perfectly conducting dihedral of opening angle \( \alpha \) is equal to twice the corresponding zeta function for a massless scalar field on the space \( C_\beta \times \mathbb{R}^1 \) with \( \beta = 2\alpha \), where \( C_\beta \) is a cone of angle \( \beta \). This conclusion is obvious if one takes into account the relation between corresponding boundary value problems (see Eq. (12) in Ref. [18]). It is also easy to arrive at this conclusion directly writing out a complete set of the eigenfunction for a scalar massless field on the manifold \( C_\beta \times \mathbb{R}^1 \):

\[ f_{\lambda nk}(r, \theta, z) = \eta_n e^{ikz} J_{np}(\lambda r) \sin^n(\lambda^2) \cos^{np}(\lambda^2), \quad n = 0, 1, 2, \ldots, \quad p = \frac{2\pi}{\beta}. \] (3.30)

In this case all the eigenvalues with \( n = 1, 2, \ldots \) turn out to be double degenerate, each of them has two eigenfunctions proportional to \( \sin(\lambda^2) \) or \( \cos(\lambda^2) \). The local zeta functions for scalar fields inside a wedge with Dirichlet and Neumann boundary conditions are constructed in The Appendix.

In order to find in the approach developed the spatial distribution of the vacuum energy inside the conducting wedge (see Eqs. (3.8) and (3.18)) the value of \( G_{2\alpha}(-1) \) should be calculated. Here we show how to do this applying the procedure of analytic continuation of the function \( G_{2\alpha}(s) \) described above.

From Eq. (3.24) it follows that for \( s = -1 \) three terms with \( j = 0, 1, 2 \) in the asymptotics (3.22) should be summed by making use of the Riemann zeta function. For \( s = -1 \) Eq. (3.22) gives

\[ \frac{\Gamma(np + 2)}{\Gamma(np - 1)} = (np)^3 \left[ 1 - \frac{1}{(np)^2} \right]. \] (3.31)

Comparing Eqs. (3.31) and (3.22) we infer

\[ c_0(-1) = 1, \quad c_1(-1) = -1, \quad c_2(-1) = 0. \] (3.32)

According to Eq. (3.23) the analytic continuation of the function \( G_{2\alpha}(s) \) at the point \( s = -1 \) has the value

\[ G_{2\alpha}(-1) = \lim_{s \to -1} p^{1 - 2s} \sum_{j=0}^{2} c_j(s) p^{-2j} \zeta_R(2j + 2s - 1) \]

\[ = p^3 \left[ c_0(-1)\zeta_R(-3) + c_1(-1)p^{-2}\zeta_R(-1) + p^{-4} \lim_{s \to -1} c_2(s)\zeta_R(2s + 3) \right]. \] (3.33)

In order to find the limit in the last term in Eq. (3.33), the exact form of the polynomial \( c_2(s) \) should be used [6].
\[ c_2(s) = \frac{s}{18} \left( s^2 - \frac{1}{4} \right) (s^2 - 1) \left( s - \frac{6}{5} \right), \]
as well as the behaviour of the Riemann zeta function near the pole \[ \zeta_R(2s + 3) \simeq \frac{1}{2s + 2} + \gamma + \ldots, \quad s \to -1, \]
where \( \gamma \) is the Euler constant. It gives
\[ \lim_{s \to -1} p^{-4} c_2(s) \zeta_R(2s + 3) = -\frac{11}{120} p^{-4}. \]  
(3.34)

Substituting Eq. (3.34) and the values of the Riemann zeta function
\[ \zeta_R(-1) = -\frac{1}{12}, \quad \zeta_R(-3) = \frac{1}{120} \]
into Eq. (3.33) one obtains
\[ G_{2\alpha}(-1) = \frac{1}{120} \left( p^3 + 10p - \frac{11}{p} \right) = \frac{1}{120p} (p^2 - 1)(p^2 + 11). \]  
(3.35)

Equation (3.21) gives for \( G_{2\pi}(-1) \)
\[ G_{2\pi}(-1) = 0. \]  
(3.36)

Let us remind that upon analytic continuation of the function \( G_{2\pi}(s) \) the right-hand side of Eq. (3.21) is considered all over the plane of complex variables \( s \). Substituting Eqs. (3.35) and (3.36) into Eq. (3.28) and taking into account that
\[ \frac{\Gamma(-3/2)}{\Gamma(-1/2)} = \frac{2}{3}, \]
we find
\[ \zeta \left( -\frac{1}{2}, r \right) = -\frac{1}{360\pi^2 r^4} (p^2 - 1)(p^2 + 11) \]  
(3.37)
and respectively for the density of the vacuum energy
\[ \rho(r) = \frac{1}{2} \zeta \left( -\frac{1}{2}, r \right) = -\frac{1}{720\pi^2 r^4} (p^2 - 1)(p^2 + 11), \quad p = \frac{\pi}{\alpha}. \]  
(3.38)

For \( \alpha < \pi \) \( (p > 1) \) the vacuum energy density is negative, and it vanishes for \( \alpha = \pi \) \( (p = 1) \), when the wedge transforms into semispace. Putting in Eq. (3.38) \( \alpha \to 0 \) and \( r \to \infty \) in such a way that \( r\alpha = d = \text{constant} \), one arrives at the vacuum energy density between two parallel perfectly conducting plates
\[ \rho_{plates} = -\frac{\pi^2}{720 d^4}. \]  
(3.39)
Multiplying this expression by the volume of space bounded by plates $S d$, where $S$ is the area of the plate surface, we obtain the well known Casimir energy \[ E_C = -\frac{\pi^2 S}{720 d^3}. \] (3.40)

The vacuum energy density (3.38) possesses non-integrable singularity at the point $r = 0$. Here we shall not touch the physical origin of this singularity. It is discussed in the literature for a long time (see, for example, Refs. \[13,20\]). When defining the global quantities in the problem at hand, we simply introduce a cutoff at the lower limit of integration over $r$. Thus for the total energy we have

$$ E = \int_\varepsilon^\infty r dr \int_0^\alpha d\theta \rho(r) = -\alpha \frac{(p^2 - 1)(p^2 + 11)}{1440\pi^2 \varepsilon^2}. $$

(3.41)

Having calculated the energy density for a conducting wedge, one can immediately recover all the components of the relevant energy-momentum tensor $T_{\mu\nu}$ \[4\]. In fact, taking into account the symmetries of the problem and the vanishing of the trace and divergence of $T_{\mu\nu}$ one finds

$$ T_{\mu\nu} = \frac{1}{r^4} f(\theta, \alpha) \text{diag}(1, -3, 1, 1) = (T_{rr}, T_{\theta\theta}, T_{zz}, -\rho) $$

(3.42)

with some function $f(\theta, \alpha)$, vanishing when $\alpha = \pi$. Substituting $\rho$ on the right-hand side of Eq. (3.42) by Eq. (3.38) we obtain

$$ f(\alpha) = \frac{(p^2 - 1)(p^2 + 11)}{720\pi^2}. $$

Thus all the components of the energy-momentum tensor are found. Specifically, we conclude that the surface density of the Casimir force $F(r)$ acting on the wedge faces is

$$ F(r) = -T_{\theta\theta}(r) = \frac{(p^2 - 1)(p^2 + 11)}{240\pi^2 r^4}. $$

(3.43)

These forces tend to diminish the opening angle $\alpha$ between the wedge plates.

The torque of the Casimir forces about the origin is

$$ M = \int_\varepsilon^\infty F(r) r dr = \frac{(p^2 - 1)(p^2 + 11)}{480\pi^2 \varepsilon^2} = -\frac{3E}{\alpha}. $$

(3.44)

Certainly it depends on the cutoff parameter $\varepsilon$. We again recall that all the quantities found in our study are related to a unit length along the $z$ direction.

**IV. HIGH TEMPERATURE ASYMPTOTICS OF THE CASIMIR EFFECT FOR A CONDUCTING WEDGE**

In this section we briefly consider the application of the constructed zeta function to investigation of the temperature behavior of the problem under consideration. In the general
case, to find the temperature dependence of the local characteristics proves to be a rather complicated problem that still waits its complete solution [21]. Therefore we address the global thermodynamical function of electromagnetic field inside a perfectly conducting wedge (the Helmholtz free energy). For this purpose we need the global zeta function. However Eq. (3.7) cannot be used for this purpose directly due to non-integrable singularity at \( r = 0 \) of the local zeta function (3.28). We again introduce a cutoff in the \( r \) integral 

\[
\zeta(s) = \int_{\varepsilon}^{\infty} r dr \int_{0}^{\alpha} d\theta \zeta(s, r)
= \frac{1}{2\pi \varepsilon^{1-2s}(1-2s)} \Gamma(s-1) \Gamma(s) \left[ G_{2\alpha}(s-1/2) - G_{2\pi}(s-1/2) \right]
= \frac{I_{2\alpha}(s-1/2)}{2\sqrt{\pi} \varepsilon^{1-2s}(1-2s) \Gamma(s)}.
\]

(4.1)

The appearance of the cutoff here can be formally explained in the following way. In the general case, the local zeta function (3.5) is, from the mathematical standpoint, a distribution. Therefore its complete definition requires specification of the relevant set of test functions. In the problem at hand one can choose the test functions proportional to the step function \( \theta(r - \varepsilon) \). Certainly this consideration is a formal one, and the problem of constructing the global zeta function for boundary conditions in question still waits its solution.

One could try to substitute the known spectrum (2.4) directly in the definition of the global zeta function (3.7). However, without introducing a cutoff the integration cannot be done here. Really,

\[
\tilde{\zeta}(s) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{0}^{\infty} \frac{\lambda d\lambda}{(k^2 + \lambda^2)^s} = \frac{1}{2\pi^{s/2}} \Gamma(s-1/2) \Gamma(s) \Gamma(3-2s) \varepsilon^{3-2s}.
\]

(4.2)

On introducing the exponential regularization in the \( \lambda \)-integral we obtain

\[
\int_{0}^{\infty} \lambda^{2-2s} d\lambda \rightarrow \int_{0}^{\infty} e^{-\varepsilon \lambda} \lambda^{2-2s} d\lambda = \frac{\Gamma(3-2s)}{\varepsilon^{3-2s}}.
\]

(4.3)

Finally the zeta function \( \tilde{\zeta}(s) \) acquires the form

\[
\tilde{\zeta}(s) = \frac{1}{2\pi^{s/2}} \Gamma(s-1/2) \Gamma(s) \Gamma(3-2s) \varepsilon^{3-2s}.
\]

(4.4)

This expression disagrees with Eq. (4.1). The substantial shortcoming of the formula (4.4), in comparison with Eq. (4.1), is independence of the opening angle \( \alpha \). Therefore we shall use further the zeta function (4.1).

The global zeta function (4.1) enables one to construct immediately the high temperature asymptotics of the thermodynamical functions in the problem at hand [22]. Here it is worth reminding that in our consideration we are dealing only with the spatial part of the differential operator governing the fluctuation dynamics of electromagnetic field (see Eqs. (2.2) and (2.3)). Therefore the corresponding global zeta function enables one to construct, in a direct way, only the high temperature asymptotics of the thermodynamical
characteristics in the problem under study. In order to investigate complete temperature behavior of these characteristics a complete spectral zeta function is needed that involves also the summation over the Matsubara frequencies.

The high temperature expansion for the Helmholtz free energy has the form \( F(T) \approx \frac{T}{2} \zeta'(0) + \sum_{n=0,1/2,1,3/2,...} a_n T^{1-2n} c_n, \quad T \to \infty. \) (4.5)

Here \( c_n \)'s are numerical coefficients

\[
    c_0 = -\frac{\pi^2}{90}, \quad c_{1/2} = -\frac{\zeta_R(3)}{4\pi^{3/2}}, \quad c_1 = -\frac{1}{24}, \quad \cdots,
\] (4.6)
safe for \( c_{3/2} \) and \( c_2 \), which also depend on \( \ln T \), and \( a_n \)'s are the heat kernel coefficients. The latter are determined by the zeta function through the relation

\[
    \frac{a_n}{(4\pi)^{3/2}} = \lim_{s \to \frac{3}{2}-n} \left( s + n - \frac{3}{2} \right) \zeta(s)\Gamma(s), \quad n = 0, 1/2, 1, \ldots.
\] (4.7)

As follows from Eq. (4.1) the zeta function \( \zeta(s) \) has simple poles at the point \( s = 1/2 \) (due to the multiplier \( (1-2s) \) in the denominator) and at the point \( s = 3/2 \) because the function \( I_{2m}(z) \) has such a pole at the point \( z = 1 \). Thus, only the heat kernel coefficients \( a_0 \) and \( a_1 \) do not vanish in the problem under study. This is in full accord with the general properties of the heat kernel expansion [24]. Indeed, for flat manifolds without boundary or with flat boundary all the heat kernel coefficients except for \( a_0 \) are equal to zero.

And what is more, one can show in the general case that the heat kernel coefficient \( a_0 \) is equal to the volume of the system under consideration. The corresponding term in the high temperature expansion (4.7)

\[
    -a_0 \frac{\pi^2}{90} T^4
\] is the Stefan-Boltzmann law for the free energy of electromagnetic field in the space restricted by the boundaries

\[
    \frac{F}{V} = -\frac{\pi^2}{90} T^4 = -\frac{2}{3} \sigma T^4,
\] (4.8)

where \( \sigma = \pi^2 k_B^4 / (60 c^2 \hbar^2) \) is the Stefan-Boltzmann constant.

When the system has an infinite volume (just that very case is considered here), a special limiting procedure should be used in calculations of \( a_0 \) (see, for example, Ref. [25]). In the problem at hand this implies the introduction of a cutoff also at large values of \( r: \varepsilon \leq r \leq R \). In view of this the substitution of the zeta function (4.1) into Eq. (4.7) gives

\[
    a_0 = (R^2 - \varepsilon^2)(\alpha - \pi).
\] (4.9)

In the problem at hand we are dealing with two independent scalar fields. Therefore the value (4.9) should be compared with doubled volume of the system under study.
2V = 2\pi(R^2 - \varepsilon^2)\frac{\alpha}{2\pi} = (R^2 - \varepsilon^2)\alpha. \quad (4.10)

It is worth noting once more that all our formulas give the quantities per unit length along the axes OZ. The distinction between Eqs. (4.9) and (4.10) are obviously attributed to a special condition satisfied by the zeta function (4.1), namely, \(\zeta(s) = 0\) for \(\alpha = \pi\).

All the terms in the asymptotics (4.5), safe for the term with \(n = 0\), describe the deviation from the Stefan-Boltzmann law. In the problem in question the coefficient \(a_1\) does not vanish due to the discontinuity of the boundary at the point \(r = 0\) (wedge or cone singularity). There is a voluminous literature concerning with such a singularity. we mention here only a few papers [6,17,26–29] where further references can be found.

Taking into account the behaviour of the gamma function at the origin \(\Gamma(s) \sim s^{-1}, \quad s \to 0\) we deduce from Eq. (4.1)

\[
\zeta'(0) = \frac{I_{2\alpha}(-1/2)}{2\sqrt{\pi\varepsilon}}. \quad (4.11)
\]

The explicit formulas derived in Ref. [6] give

\[
I_{2\alpha}(-1/2) = -\frac{\zeta'(-2)}{2\sqrt{\pi}} = \frac{\zeta_R(3)}{8\pi^2\sqrt{\pi}}. \quad (4.12)
\]

Hence

\[
\zeta'(0) = \frac{\zeta_R(3)}{16\pi^3\varepsilon}. \quad (4.13)
\]

Substituting the zeta function (4.1) in Eq. (4.7) we find

\[
\frac{a_1}{(4\pi)^{3/2}} = \lim_{s \to 1/2} (s - 1/2)\zeta(s)\Gamma(s) = -\frac{I_{2\alpha}(0)}{4\sqrt{\pi}}. \quad (4.14)
\]

In order to calculate \(I_{2\alpha}(0)\) we take, in accord with relation (3.24), two terms in Eq. (3.25) with \(j = 0, 1\). It gives

\[
G_{2\alpha}(0) = \lim_{s \to 0} p^{1-2s}[c_0(s)\zeta_R(2s - 1) - p^{-2}c_1(s)\zeta_R(2s + 1)]. \quad (4.15)
\]

The coefficients \(c_0(s)\) and \(c_1(s)\) are explicitly written, for example, in Ref. [16]

\[
c_0(s) = 1, \quad c_1(s) = \frac{s}{3}(s - 1/2)(s - 1). \quad (4.16)
\]

With allowance for the behaviour of the Riemann zeta function close by pole

\[
\zeta_R(1 + 2s) \simeq \frac{1}{2s} + \gamma + \cdots, \quad s \to 0
\]

we obtain

\[
G_{2\alpha}(0) = \frac{1}{12}(p^{-1} - p). \quad (4.17)
\]
Form Eq. (3.21) it follows that

$$G_{2\pi}(0) = 0.$$  \hfill (4.18)

Finally we have

$$I_{2\alpha}(0) = \frac{1}{6} (p - p^{-1}) = \frac{1}{6} \left( \frac{\pi}{\alpha} - \frac{\alpha}{\pi} \right)$$

and

$$a_1 = -2\pi I_{2\alpha}(0) = -\frac{\pi}{3} \left( \frac{\pi}{\alpha} - \frac{\alpha}{\pi} \right). \hfill (4.19)$$

Now we are in position to write the high temperature corrections to the Stefan-Boltzmann law inside a conducting dihedral angle

$$F_{cor}(T) \simeq -\frac{T \zeta_R(3)}{32\pi^3} + \frac{\pi^2 - \alpha^2}{72\alpha^2 \hbar c} T^2, \quad T \to \infty. \hfill (4.20)$$

Here the constants $\hbar$ and $c$ are explicitly restored in order to separate a pure classical contribution (the first term on the right hand side of Eq. (4.20)) and the quantum correction (the second term in Eq. (4.20)).

The authors of Ref. [7] argue that the high temperature asymptotics of the free energy of electromagnetic field inside a perfectly conducting wedge should be the same as that for smooth conducting surfaces with nonzero curvature

$$F(T) \simeq CT \ln(T/Q), \quad T \to \infty, \hfill (4.21)$$

with $C$ and $Q$ being some constants. The specific term proportional to $T^2$ in the asymptotics (4.21) was not brought out.

Making use of Eq. (4.20) one can try to estimate, at least in kind, the behaviour of the torque of the Casimir forces at high temperature. For this purpose we substitute the total energy $E$ in Eq. (3.44) by the asymptotic expression for the free energy in the problem under study

$$M \simeq \frac{3\zeta_R(3)}{32\pi^3\alpha\varepsilon} - \frac{\pi^2 - \alpha^2}{24\alpha^2 \hbar c} T^2, \quad T \to \infty. \hfill (4.22)$$

The book [1] is a sole source where the high temperature behaviour of the torque $M$ has been considered. The authors have used the method of images for constructing the Green’s functions of electromagnetic field inside the dihedral, the photon energy being cut in ultra-violet region. In terms of the notations used in the present paper, the high temperature asymptotics of $M$ found in Ref. [1] reads

$$M \simeq \frac{\pi T}{12\alpha^2\varepsilon}, \quad T \to \infty. \hfill (4.23)$$

\footnote{The examples of such asymptotics can be found in Ref. [23].}
Here we have also to ascertain disagreement with our calculations.

In the preceding section it was shown that the zeta function of electromagnetic field inside a perfectly conducting dihedral of opening angle $\alpha$ is equal to double zeta function for a massless scalar field on the manifold $C_\beta \times \mathbb{R}^1$ with $2\alpha = \beta$, where $C_\beta$ is a cone of angle $\beta$. Therefore all the results obtained in our consideration are directly applicable to the massless scalar field on the manifold $C_\beta \times \mathbb{R}^1$, in particular to the cosmic string background. Really, the general solution to the Maxwell equations in the presence of a cosmic string are expressed in terms of two scalar massless fields that satisfy the periodicity conditions in angle variable with a period $\beta = 2\pi - \phi$, where $\phi$ is the deficiency angle of the metric due to the cosmic string, $\phi = 8\pi G\mu$. Most simply one can come to this conclusion in the following way. Let us assume that the string has a finite transverse size, $r_{str}$, and its surface is perfectly conducting. Obviously, the reasonings of the Section II apply to this boundary configuration, two scalar functions $U$ and $V$ being periodic in angle variable $\theta$ and meeting the Dirichlet and Neumann boundary conditions at $r = r_{str}$. Letting $r_{str}$ tend to zero, we substitute the Dirichlet and Neumann boundary conditions at $r = r_{str}$ by requirement that the relevant solutions to be bounded everywhere including the origin: $0 \leq r < \infty$. The periodicity condition in angle variable $\theta$ apparently remains. From here we infer that the local zeta function for electromagnetic field in the presence of cosmic string is obtained from the relevant zeta function for the conducting wedge (3.28) by the substitution $2\alpha = 2\pi - 8\pi G\mu$. Specifically, for the vacuum energy density in this case we have again Eq. (3.39) with the same substitution. The same result was obtained in Ref. [30] by making use of the Greens function method. Further, the high temperature corrections to the Stefan-Boltzmann law for the free energy of electromagnetic field on the background of a cosmic string are given by the formula

$$F(T) \simeq \frac{T \zeta_R(3)}{32\pi^3 \varepsilon} + \frac{T^2}{72\hbar c} \frac{\phi 4\pi - \phi}{2\pi - \phi}, \quad T \to \infty. \quad (4.24)$$

V. CONCLUSION

We have shown that the technique of the local zeta function enables one to carry out, in a consistent way, the calculation of the vacuum energy density of electromagnetic field inside a conducting wedge without dealing with obvious divergences. This regularization method leads to a finite expression for the energy density under study without any subtractions. Our consideration is a natural addition to studies of the Casimir effect for a wedge accomplished in recent papers [3,4] by making use of Green’s function method. Employment of the Hertz potentials for constructing the general solution of the Maxwell equations results in a considerable simplification of the calculations of the local zeta function in the problem at hand. Transition to the global zeta function has been carried out by introducing a cutoff for small $r$. On this basis the nonzero heat kernel coefficients $a_0$, $a_1$, and the zeta determinant were calculated. Remarkably that the coefficient $a_1$ does not depend on the cutoff and it is completely determined by the wedge singularity of the boundary. Proceeding from this the high temperature asymptotics of the Helmholtz free energy and of the torque of the Casimir forces are found. The wedge singularity gives rise to a specific high temperature behaviour.
\[ \sim T^2 \] of the quantities under consideration. The found results are directly applicable to the free energy of electromagnetic field on the background of a cosmic string.

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**APPENDIX: LOCAL ZETA FUNCTIONS FOR SCALAR MASSLESS FIELDS INSIDE DIHEDRAL ANGLE WITH DIRICHLET AND NEUMANN BOUNDARY CONDITIONS**

First we consider the Dirichlet boundary conditions with the eigenfunctions (2.5). The local zeta function is given by

\[
\zeta_D(s, r, \theta) = \frac{1}{\pi \alpha} \int_{-\infty}^{\infty} dk \int_{0}^{\infty} \frac{\lambda d\lambda}{(k^2 + s^2)^s} \sum_{n=1}^{\infty} J_{np}^2(\lambda r) \sin^2(n \theta) = \frac{\Gamma(s - 1/2)}{2\sqrt{\pi \alpha} \Gamma(s)} \int_{0}^{\infty} d\lambda \lambda^{2-2s} \sum_{n=1}^{\infty} J_{np}^2(\lambda r) [1 - \cos(2np)]. \quad (A1)
\]

By means of the formula 8.531.3 from [13]

\[ J_0(z \sin \alpha) = J_0^2 \left( \frac{z}{2} \right) + 2 \sum_{k=1}^{\infty} J_k^2 \left( \frac{z}{2} \right) \cos(2k \alpha) \]

the dependence on the angle \( \theta \) in Eq. (A1) can be stored in a single term

\[
\zeta_D(s, r, \theta) = \frac{\Gamma(s - 1/2)}{2\sqrt{\pi \alpha} \Gamma(s)} \int_{0}^{\infty} d\lambda \lambda^{2-2s} \left[ \frac{1}{2} J_0^2(\lambda r) + \sum_{n=1}^{\infty} J_{np}^2(\lambda r) - \frac{1}{2} J_0(2\lambda r \sin p \theta) \right] \]

\[
= \frac{1}{2} \zeta(s, r) - \frac{\Gamma(s - 1/2)}{4\sqrt{\pi \alpha} \Gamma(s)} \int_{0}^{\infty} d\lambda \lambda^{2-2s} J_0(2\lambda r \sin(p \theta)), \quad (A2)
\]

where \( \zeta(s, r) \) is the local zeta function for electromagnetic field in a conducting wedge defined in Eq. (3.10). By making use of the integration formula 6.561.14 from [13]

\[
\int_{0}^{\infty} x^\mu J_{\nu}(ax)dx = 2^\mu a^{-\mu-1} \frac{\Gamma \left( \frac{1}{2} + \frac{1}{2} \nu + \frac{1}{2} \mu \right)}{\Gamma \left( \frac{1}{2} + \frac{1}{2} \nu - \frac{1}{2} \mu \right)} - \text{Re} \nu - 1 < \text{Re} \mu < \frac{1}{2}, \quad a > 0
\]

we cast Eq. (A2) into the final form.

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\[
\zeta_D(s, r, \theta) = \frac{1}{2} \zeta(s, r) + \frac{1}{8 \sqrt{\pi} \alpha (r \sin(p\theta))^{3-2s}} \frac{\Gamma(3/2 - s)}{\Gamma(s)}. \tag{A3}
\]

The last term in Eq. (A3) is first defined in the strip

\[\frac{3}{4} < \text{Re } s < \frac{3}{2}, \tag{A4}\]
determined by the conditions of applicability of Eq. (A3). As usual under the analytic continuation, we assume that outside this strip the same equation holds.

Proceeding along similar lines we obtain the local zeta function for the Neumann boundary conditions

\[
\zeta_D(s, r, \theta) = \frac{1}{2} \zeta(s, r) - \frac{1}{8 \sqrt{\pi} \alpha (r \sin(p\theta))^{3-2s}} \frac{\Gamma(3/2 - s)}{\Gamma(s)}. \tag{A5}
\]

In particular the vacuum energy densities for massless scalar fields subjected to the Dirichlet and Neumann boundary conditions on the wedge sites are

\[
\rho_D(r, \theta) = \rho_-(r, \theta), \quad \rho_N(r, \theta) = \rho_+(r, \theta), \tag{A6}
\]

where

\[
\rho_\pm(r, \theta) = \frac{1}{2} \rho(r) \pm \frac{1}{16 \pi \alpha r^4 \sin^4(p\theta)}, \tag{A7}
\]

and \(\rho(r)\) is the vacuum energy density of electromagnetic field, defined in Eq. (3.38).

Thus the vacuum energy densities derived have nonintegrable singularities near by the wedge sites. It is a typical behaviour of spatial distribution of the Casimir energy (see, for example, Ref. [31]).

The local zeta function technique, consider in the present paper, cannot directly be applied to the scalar field with conformal coupling. The point is that in this case the relation (3.9) does not hold.
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