THE EULER CHARACTERISTIC OF LOCAL SYSTEMS ON THE MODULI OF GENUS 3 HYPERELLIPTIC CURVES

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Abstract. For a partition \( \lambda = \{\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0\} \) of non-negative integers, we calculate the Euler characteristic of the local system \( V_\lambda \) on the moduli space of genus 3 hyperelliptic curves using a suitable stratification. For some \( \lambda \) of low degree, we make a guess for the motivic Euler characteristic of \( V_\lambda \) using counting curves over finite fields.

1. Introduction

Let \( \mathcal{H}_3 \) be the moduli space of genus 3 hyperelliptic curves. It is a 5-dimensional substack of the Deligne-Mumford stack \( \mathcal{M}_3 \) of smooth curves of genus 3. The universal curve \( \pi : \mathcal{M}_{3,1} \to \mathcal{M}_3 \) defines a natural local system \( R^1\pi_* (\mathbb{Q}) \) of rank 6 on \( \mathcal{M}_3 \). It comes with a non-degenerate symplectic pairing. The inclusion morphism \( \iota : \mathcal{H}_3 \to \mathcal{M}_3 \) defines a natural local system \( V := \iota^*(R^1\pi_* (\mathbb{Q})) \) on \( \mathcal{H}_3 \).

For any partition \( \lambda = \{\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0\} \) of weight \( |\lambda| = \lambda_1 + \lambda_2 + \lambda_3 \), consider the irreducible representation of \( \text{Sp}(6, \mathbb{Q}) \) associated with \( \lambda \). Any such representation yields a symplectic local system \( V_\lambda \) on \( \mathcal{H}_3 \), which appears ‘for the first time’ in the decomposition of

\[
\text{Sym}^{\lambda_1 - \lambda_2} V \otimes \text{Sym}^{\lambda_2 - \lambda_3} (\wedge^2 V) \otimes \text{Sym}^{\lambda_3} (\wedge^3 V).
\]

If, for example, \( \lambda = \{\lambda_1 \geq 0 \geq 0\} \), then \( V_\lambda = \text{Sym}^{\lambda_1} (V) \).

The cohomology with compact support of \( \mathcal{H}_3 \) with local coefficients in \( V_\lambda \) is supposed to give interesting motives related to automorphic forms. As a first step in understanding this cohomology one wants to know the Euler characteristic of \( V_\lambda \). This was calculated for genus 2 by Getzler in [4]. In the present paper we calculate the Euler characteristic

\[
e_C(\mathcal{H}_3, V_\lambda) = \sum_{i=0}^{10} (-1)^i \dim H^i_c(\mathcal{H}_3, V_\lambda)
\]

for any local system \( V_\lambda \) on \( \mathcal{H}_3 \). We do this by using a stratification of \( \mathcal{H}_3 \otimes \mathbb{C} \) by a union of quasi-projective varieties \( \Sigma(G) \), where \( G \) is a finite subgroup of \( \text{SL}(2, \mathbb{C}) \times \mathbb{C}^* \), which acts on \( V_\lambda \). By standard properties of the Euler
characteristic of local systems, we thus have
\[ e_c(\mathcal{H}_3, \mathcal{V}_\lambda) = \sum_G e_c(\Sigma(G)) \dim(\mathcal{V}_\lambda^G), \]
where \( e_c(\Sigma(G)) \) is the topological Euler characteristic of \( \Sigma(G) \) and \( \mathcal{V}_\lambda^G \) is the space of \( G \)-invariants. We determine \( e_c(\Sigma(G)) \) via elementary topological arguments and \( \dim(\mathcal{V}_\lambda^G) \) via character theory. Getzler wrote down the generating series of Euler characteristics in [4]; however for genus 2 already this leads to unwieldy rational functions. We give a short algorithm that calculates these numbers efficiently.

This calculation is a step in the program to understand the motivic Euler characteristic
\[ \sum_{i=0}^{10} (-1)^i [H^i_c(\mathcal{H}_3, \mathcal{V}_\lambda)], \]
where \([H^i_c(\mathcal{H}_3, \mathcal{V}_\lambda)]\) is the class of the cohomology with compact support in the Grothendieck ring of mixed \( \mathbb{Q} \)-Hodge structures. The hope is that in analogy to the genus 2 case (cf. [2]), one could use this motivic Euler characteristic to describe properties of Siegel modular forms of genus 3, of which very little is known. In Section 5 we provide some conjectural formulas of the motivic Euler characteristic for specific low values of \(|\lambda|\) based on calculations over finite fields.

Throughout the paper, \( \varepsilon_n \) denotes a primitive \( n \)-th root of unity.

2. Stabilizers of hyperelliptic curves

Let \( C \) be a hyperelliptic curve of genus 3 over the field of complex numbers \( \mathbb{C} \). Then \( C \) is a degree two cover of \( \mathbb{P}^1 \) with eight ramification points. It can be given as a curve in the \((X,Y)\)-plane by an equation of the form \( Y^2 = f(X) \), where \( f(X) \) is a polynomial in \( \mathbb{C}[X] \) of degree 7 or 8.

The group \( \text{SL}(2, \mathbb{C}) \times \mathbb{C}^* \) acts on the \((X,Y)\)-plane as follows. An element
\[ (A, \xi) = \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \xi \right) \in \text{SL}(2, \mathbb{C}) \times \mathbb{C}^* \]
acts via
\[ (A, \xi) \cdot (X,Y) := \left( \frac{aX + b}{cX + d}, \xi Y \frac{\xi}{(cX + d)^4} \right). \]
Suppose that \( G \leq \text{SL}(2, \mathbb{C}) \times \mathbb{C}^* \) stabilizes \( C \). Consider the image \( G' \) of \( G \) under the projection of \( \text{SL}(2, \mathbb{C}) \times \mathbb{C}^* \) onto \( \text{SL}(2, \mathbb{C}) \). Clearly, \( G' \) acts as a group of rational transformations on the complex projective line. It also permutes the set of ramification points of \( C \). Note that the kernel of this action is the subgroup generated by the central element \(-I\). By the classification of finite subgroups of \( \text{SL}(2, \mathbb{C}) \) (see [3]), \( G' \) must be isomorphic to one of the following groups:

i) the cyclic group \( C_n \) of order \( n = 2, 4, 14 \);
ii) the quaternionic group \( Q_{4n} \) of order \( 4n = 8, 12, 16, 24, 32 \);
iii) the group $O$ of symmetries of a cube.

For the purposes of what follows, we briefly recall the presentation of the groups in i), ii), iii) as subgroups of SL(2, $\mathbb{C}$). Any cyclic group of order $n$ in SL(2, $\mathbb{C}$) is conjugated to the group generated by the matrix

$$\begin{pmatrix}
\varepsilon_n & 0 \\
0 & \varepsilon_n^{-1}
\end{pmatrix},$$

Any quaternionic subgroup of order 4, $n \geq 2$, is conjugated to the group with generators

$$S = \begin{pmatrix}
\varepsilon_{2n} & 0 \\
0 & \varepsilon_{2n}^{-1}
\end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}.$$

Finally, the group $O$ is conjugated to the group generated by the matrices

$$T = \frac{-1}{\sqrt{2}} \begin{pmatrix}
1 & \varepsilon_8 \\
\varepsilon_8 & 1
\end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}.$$

Remarkably, the isomorphism type of $G'$ determines the whole structure of $G$. Indeed, for any matrix $A \in G'$ there exist two non-zero complex numbers $\pm \xi$ such that

$$\xi^2 Y^2 = (cX + d)^8 f \left( \frac{aX + b}{CX + d} \right),$$

where

$$A = \begin{pmatrix}
a & b \\
c & d
\end{pmatrix}.$$

The assignment

$$u : G' \to \mathbb{C}^*, \quad A \mapsto \xi^2,$$

is a character of a one-dimensional representation of $G'$ because $u(I) = 1$. Thus, the group $G \leq \text{SL}(2, \mathbb{C}) \times \mathbb{C}^*$ contains all pairs $(A, \pm u(A))$, where $A$ varies in one of the groups $G'$ listed in i), ii), iii), and $u$ is a one-dimensional character of $G'$ that satisfies (2.1). Hence, $\#G = 2\#G'$. As a consequence, there are only finitely many non-isomorphic groups $G$ which arise as possible stabilizers of genus 3 hyperelliptic curves. Each of them induces a permutation action on a set of eight points in $\mathbb{P}^1$. Thus, we can deduce a normal form of curves which are stabilized by $G$. Examples and explicit computations can be found, for instance, in [6]. There, the stabilizers are not described as subgroups of $\text{PSL}(2, \mathbb{C}) \times \mathbb{C}^*$. It is however easy to verify a correspondence between the two descriptions.

In Table 1 we list all possible groups in terms of $G'$ and $u$, as well as the associated normal form. To this end, we need to review some conventional notation from character theory. In general, we shall denote by 1 the trivial character of $G'$. If $G'$ is the cyclic group of order $n$, there are $n - 1$ nontrivial characters $\chi^k$ such that

$$\chi^k \left( \begin{pmatrix}
\varepsilon_n & 0 \\
0 & \varepsilon_n^{-1}
\end{pmatrix} \right) = \varepsilon_n^k, \quad 1 \leq k \leq n - 1.$$
On the other hand, the quaternionic group $Q_{4n}$ has only three non-trivial characters of one-dimensional representations, namely:

| $\chi$ | $\chi(S)$ | $\chi(U)$ |
|-------|---------|---------|
| $\chi_0$ | 1 | $-1$ |
| $\chi^+$ | $-1$ | $-i^n$ |
| $\chi^-$ | $-1$ | $i^n$ |

The group $O$ has a unique 1-dimensional character $\rho$, which is not trivial.

$$Y^2 = f(X)$$

| name | $(G', u)$ | Normal Form $Y^2 = f(X)$ |
|------|----------|--------------------------|
| $G_1$ | $(C_2, 1)$ | $(X^2 - 1)(X^6 + \sum_{i=1}^{5} a_i X^{6-i} + 1)$ |
| $G_2$ | $(C_4, 1)$ | $X^8 + b_1 X^6 + b_2 X^4 + b_3 X^2 + 1$ |
| $G_3$ | $(Q_8, 1)$ | $(X^4 + c_1 X^2 + 1)(X^8 + c_2 X^4 + 1)$ |
| $G_4$ | $(C_4, \chi^2)$ | $X(X^6 + d_1 X^4 + d_2 X^2 + 1)$ |
| $G_5$ | $(Q_{16}, 1)$ | $X^8 + fX^4 + 1$ |
| $G_6$ | $(Q_8, \chi_0)$ | $(X^4 - 1)(X^4 + LX^2 + 1)$ |
| $G_7$ | $(Q_{12}, 1)$ | $X(X^6 + mX^3 + 1)$ |
| $G_8$ | $(Q_{32}, \chi_-)$ | $X^8 - 1$ |
| $G_9$ | $(O, 1)$ | $X^8 + 14 X^4 + 1$ |
| $G_{10}$ | $(Q_{24}, \chi_-)$ | $X(X^6 - 1)$ |
| $G_{11}$ | $(C_{14}, \chi^6)$ | $X^7 - 1$ |

**Table 1: Groups and Associated Normal Forms**

We remark that the normal forms in Table 1 are equivalent to the equations given in [6], Table 3. For example, the map

$$(X, Y) \mapsto \left(\frac{-iX + i}{X + 1}, \frac{\sqrt{8} \varepsilon_8}{\sqrt{2} - l(X + 1)^4}\right), \quad l \neq 2,$$

transforms the normal form associated with $G_6$ to

$$Y^2 = X(X^2 - 1)(X^4 + LX^2 + 1),$$

where $L = -(12 + 2l)/(2 - l)$. Additionally, the character $u$ changes too. However, this does not affect the calculation of $e_v(\mathcal{H}_3, \mathcal{V}_\lambda)$ - see Section 4.

**3. The stratification of $\mathcal{H}_3$**

For each group $G_i$ in Table 1, define $\Sigma_i$ to be the locally closed sublocus of $\mathcal{H}_3$ that contains all curves $C$ whose stabilizer is exactly $G_i$. As seen in Section 2, the corresponding group $G_i'$ is a permutation group on a set of eight elements. We thus obtain a stratification of $\mathcal{H}_3$ if the relation $G_i' \leq G_j'$ is interpreted as an inclusion of permutation groups. In other words, $G_i'$ is a subgroup of $G_j'$, and any set of eight elements, which is permuted by $G_i'$,
can be decomposed in $G'_j$-orbits. All possible relations are displayed in the diagram below.

From this diagram, to be justified later, we also deduce information on the strata $\Sigma_i$. Actually, we have:

1. $\mathcal{H}_3 = \bigcup_{i=1}^{11} \Sigma_i$;
2. $\Sigma_i \cap \Sigma_j = \emptyset$ for $i \neq j$;
3. $\Sigma_j \subseteq \Sigma_i$ whenever $G'_i \leq G'_j$.

As explained in Section 1, we need to calculate the topological Euler characteristic $e_c$ of all the strata. Since $e_c(\mathcal{H}_3) = 1$, we work out $e_c(\Sigma_i)$, $i = 2, \ldots, 10$ and we deduce $e_c(\Sigma_1)$.

**0-dimensional strata.** The stratum $\Sigma_i$ for $i = 8, 9, 10, 11$ is clearly 0-dimensional and irreducible, so its Euler number is 1.

**1-dimensional strata.** The strata corresponding to $G_5$, $G_6$, $G_7$ are 1-dimensional. Moreover, let us consider the following subsets of $\mathbb{P}^1$:

$\mathcal{O}_1 := \{ \varepsilon^k_8; 0 \leq k \leq 7 \},$

$\mathcal{O}_2 := \{ 0, \infty, \pm 1, \pm \varepsilon_3, \pm \varepsilon_3^2 \},$

$\mathcal{O}_3 := \{ \pm \alpha_1, \pm i \alpha_1, \pm 1/\alpha_1, \pm i/\alpha_1 \},$

where $\alpha_1$ is a root of the polynomial $X^2 - (i + 1)X - i$.

It is easy to verify that $\mathcal{O}_1$ is a $G'_8$-orbit, a union of two $G'_5$-orbits and a union of three $G'_6$-orbits. On the other hand, $\mathcal{O}_2$ is a union of three $G'_7$-orbits, a union of three $G'_6$-orbits and a union of two $G'_{10}$-orbits. Finally, $\mathcal{O}_3$ is a full $G'_9$-orbit, a union of two $G'_7$-orbits and a union of three $G'_7$-orbits. This justifies the lower row of directed edges in the above diagram.
As for the Euler number \( e_i \), the following holds.

**Proposition 3.1.** The topological Euler characteristic of \( \Sigma_i \), \( i = 5, 6, 7 \), is equal to \(-2\).

**Proof.** We just prove the statement for \( \Sigma_5 \), the other cases being similar. For \( f \in \mathbb{C} - \{\pm 2\} \), consider the set of hyperelliptic curves \( C_f \) with equation \( Y^2 = X^8 + fX^4 + 1 \). By direct inspection, two such curves \( C_{f_1} \) and \( C_{f_2} \) are isomorphic if and only if \( f_1 = \pm f_2 \). Note that \( \Sigma_8 \) and \( \Sigma_9 \) are the isomorphism classes of \( C_0 \) and \( C_{14} \), respectively. Therefore, there exists an isomorphism \( \Phi : \Sigma_5 \cup \Sigma_8 \cup \Sigma_9 \rightarrow \mathbb{C} - \{4\} \) which maps the orbit of \( C_f \) to \( f^2 \). Accordingly, the topological Euler characteristic of \( \Sigma_5 \) is \(-2\).

**2-dimensional strata.** As readily checked from Table 1, the strata corresponding to \( G_3 \) and \( G_4 \) have dimension two. It is easy to deduce from the ramification sets in \( \mathbb{P}^1 \) that the following holds:

\[
\Sigma_5 \subset \Sigma_4, \quad \Sigma_6 \subset \Sigma_3, \\
\Sigma_6 \subset \Sigma_4, \quad \Sigma_7 \subset \Sigma_4.
\]

On the other hand, note that \( \Sigma_5 \) does not lie in the closure of \( \Sigma_4 \). Equivalently, there is no set \( S \) of eight elements which is both a union of \( G'_1 \)-orbits and \( G'_5 \)-orbits. Indeed, any set \( S \subset \mathbb{P}^1 \) has always two orbits of length one under the action of \( G'_1 \). Conversely, the permutation action of \( G'_5 \) does not have any fixed point.

**Proposition 3.2.** The topological Euler characteristic of \( \Sigma_3 \) is 1.

**Proof.** The group \( G_3 \) corresponds to the pair \( (G'_3, 1) \), where \( G'_3 \) is the quaternionic group \( Q_4 \cong C_2 \times C_2 \). The group \( G'_3 \) induces a permutation action on \( \mathbb{P}^1 \) via the group \( V_4 \) generated by the transformations \( x \mapsto -x \) and \( x \mapsto 1/x \). Denote by \( V(x) \) the orbit of \( x \) under \( V_4 \). Note \( \#V(a) = 4 \) unless \( a \in \{0, \infty, 1, -1, i, -i\} \).

We recall that the normal form associated with \( G_3 \) is

\[
Y^2 = f(X) = (X^4 + c_1X^2 + 1)(X^4 + c_2X^2 + 1).
\]

Moreover, we have

\[
\{x : f(x) = 0\} = \{\pm q_1, \pm 1/q_1, \pm q_2, \pm 1/q_2\},
\]

for distinct \( q_1, q_2 \) such that \( \#V(q_1) = \#V(q_2) = 4 \). Note that \( c_i = -q_i^2 - 1/q_i^2 \) for \( i = 1, 2 \).

Let \( \{Y^2 = f_1(X)\} \) and \( \{Y^2 = f_2(X)\} \) be two curves with stabilizer \( G_3 \). They are isomorphic if and only if there exists a rational transformation that maps \( \{z : f_1(z) = 0\} \) to \( \{z : f_2(z) = 0\} \). All such transformations commute with the elements of \( V_4 \). Therefore, two curves are isomorphic if and only if there exists an automorphism of \( \mathbb{P}^1/V_4 \) which preserves the set \( E := \{V(0), V(1), V(i)\} \), i.e. the ramification set of \( \mathbb{P}^1 \rightarrow \mathbb{P}^1/V_4 \). Observe that the map \( \mathbb{P}^1 \rightarrow \mathbb{P}^1/V_4 \) sends \( y \) to \((y^2 + 1/y^2)/2\).

A curve \( C \) with equation \( (3.1) \) has a larger stabilizer than \( G_3 \) if and only if there exists \( M \in \text{SL}(2, \mathbb{C}) \) - not in \( G'_3 \) - which induces a permutation of
and a permutation of the set \(\{0, \infty, 1, -1, i, -i\}\). By direct inspection, there is only one possible \(M\), namely:

\[
M = \begin{pmatrix}
\varepsilon_8 & 0 \\
0 & \varepsilon_8^{-1}
\end{pmatrix}.
\]

In this case, \(M\) induces the automorphism \(x \mapsto ix\) on \(\mathbb{P}^1\) and the automorphism \(z \mapsto -z\) on \(\mathbb{P}^1 / V_4\). Its fixed points on \(\mathbb{P}^1 / V_4\) are \(V(0)\) and \(V(\varepsilon_8)\).

Now, it is possible to give an alternative description of \(\Sigma_3\), which contains all curves whose stabilizer is exactly \(G_3\). Denote by \(\Delta\) the diagonal in \((\mathbb{P}^1 / V_4 - E) \times (\mathbb{P}^1 / V_4 - E)\). Define a group \(W_4\) of transformations of \(\mathbb{P}^1 / V_4 \times \mathbb{P}^1 / V_4\) as follows: \(W_4\) is generated by \(\tau\), which interchanges both factors and \(i\), which simultaneously multiplies both factors by \(i\). Note that \(W_4\) is isomorphic to the Klein four group. Therefore, \(\Sigma_3\) can be parametrized as

\[
((\mathbb{P}^1 / V_4 - E) \times (\mathbb{P}^1 / V_4 - E) - \Delta - Z) / W_4,
\]

where

\[
Z := \{(V(a), V(ia) : a \in (\mathbb{P}^1 / V_4 - E) - V(\varepsilon_8))\}.
\]

For the Euler number we get:

\[
e_\varepsilon(\Sigma_3) = \frac{1}{4}((-1) \times (-1) - (-1) - (-2)) = 1.
\]

\(\square\)

**Proposition 3.3.** The topological Euler characteristic of \(\Sigma_4\) is 1.

**Proof.** The group \(G_4\) corresponds to the pair \((G_4', \chi^2)\), where \(G_4'\) is cyclic of order 2. Now \(G_4'\) induces a permutation action on \(\mathbb{P}^1\) via the transformation \(x \mapsto -x\). Denote by \(\sigma(x)\) the orbit of \(x\) under such transformation.

We recall that the normal form associated with \(G_4\) is

\[
Y^2 = f(X) = X(X^5 + d_1X^4 + d_2X^2 + 1).
\]

Moreover, we have

\[
\{\infty\} \cup \{z : f(z) = 0\} = \{\infty, 0, \pm a, \pm b, \pm c\}
\]

for some distinct \(a, b, c \in \mathbb{C}^*\). Therefore, any equation of the form \(\{x\}\) corresponds to the 5-point set \(\{\sigma(0), \sigma(\infty), \sigma(a), \sigma(b), \sigma(c)\}\) on the \(\mathbb{P}^1\) which parametrizes the orbits \(\{\sigma(x) : x \in \mathbb{P}^1\}\).

Let \(\{Y^2 = f_1(X)\}\) and \(\{Y^2 = f_2(X)\}\) be two curves with stabilizer \(G_4\). They are isomorphic if and only if there exists a rational transformation that maps \(\{\infty\} \cup \{z : f_1(z) = 0\}\) to \(\{\infty\} \cup \{z : f_2(z) = 0\}\) and fixes 0 and \(\infty\). Such a transformation commutes with \(x \mapsto -x\). Consequently, \(\{Y^2 = f_1(X)\}\) and \(\{Y^2 = f_2(X)\}\) are isomorphic if and only if the associated 5-point sets are mapped one onto the other by a rational transformation which preserves \(\sigma(0)\) and \(\sigma(\infty)\) and permutes the other three points. In other words, an isomorphism class of curves with stabilizer \(G_4\) defines an element in \(\mathcal{M}_{0,5}/\mathfrak{S}_3\), where \(\mathcal{M}_{0,5}\) is the moduli space of rational 5-pointed curves.
and $\mathcal{S}_3$ is the symmetric group of degree three. Conversely, any element in $\mathcal{M}_{0,5}/\mathcal{S}_3$ determines an equivalence class of curves with stabilizer $G_4$.

Note that elements in $\mathcal{M}_{0,5}/\mathcal{S}_3$ can be written as $(0, \infty, 1, \sigma(u), \sigma(v))$ for some distinct $u, v \in \mathbb{P}^1 - \{0, \infty, \pm 1\}$. The corresponding curve in $\Sigma_4$ have a bigger stabilizer if and only if $\sigma(u)\sigma(v) = 1$. As a consequence, $\Sigma_4$ can be identified with $\mathcal{M}_{0,5}/\mathcal{S}_4 - Y$, where $Y$ is the image of

$$X := \{(0, \infty, 1, \sigma(u), 1/\sigma(u))\} \subset \mathcal{M}_{0,5}$$

under the quotient map onto $\mathcal{M}_{0,5}/\mathcal{S}_3$. Thus, we have

$$e_c(\Sigma_4) = e_c(\mathcal{M}_{0,5}/\mathcal{S}_3) - e_c(Y) = 1 - e_c(Y)$$

and

$$e_c(X) = 6e_c(Y) - r.$$  

Note that $e_c(X) = 2 - 4 = -2$ since $\sigma(u) \notin \{\sigma(0), \sigma(i), \sigma(1), \sigma(\infty)\}$. Additionally, $r = 2$ since the quotient map onto $\mathcal{M}_{0,5}/\mathcal{S}_3$ is ramified over $X$ when $\sigma(u)$ is the orbit of a primitive third root of unity. Hence, the statement is completely proved.

3-dimensional strata. There is only a 3-dimensional stratum, namely $\Sigma_2$. As readily checked, both $\Sigma_3$ and $\Sigma_4$ lie in the closure of $\Sigma_2$.

**Proposition 3.4.** The topological Euler characteristic of $\Sigma_2$ is 2.

**Proof.** The group $G_2$ corresponds to the pair $(G'_2, 1)$, where $G'_2$ is cyclic of order two. As in Proposition 3.3, $G'_2$ induces a permutation action on $\mathbb{P}^1$ via the transformation $x \mapsto -x$. Again, denote by $\sigma(x)$ the orbit of $x$ under such transformation.

We recall that the normal form associated with $G_2$ is

$$Y^2 = f(X) = X^8 + b_1X^6 + b_2X^4 + b_3X^2 + 1.$$  

Moreover, we have

$$\{z : f(z) = 0\} = \{\pm p_1, \pm p_2, \pm p_3, \pm p_4\}$$

for some distinct $p_1, p_2, p_3, p_4 \in \mathbb{C}^*$. Therefore, any equation of the form (3.4) corresponds to the 4-point set $\{\sigma(p_1), \sigma(p_2), \sigma(p_3), \sigma(p_4)\}$ on the $\mathbb{P}^1$ which parametrizes the orbits $\{\sigma(x) : x \in \mathbb{P}^1\}$.

Let $\{Y^2 = f_1(X)\}$ and $\{Y^2 = f_2(X)\}$ be two curves with stabilizer $G_2$. They are isomorphic if and only if there exists a rational transformation that maps $\{z : f_1(z) = 0\}$ to $\{z : f_2(z) = 0\}$. All such possible transformations commute with $x \mapsto -x$. Consequently, $\{Y^2 = f_1(X)\}$ and $\{Y^2 = f_2(X)\}$ are isomorphic if and only if the associated 4-point sets are mapped one onto the other by a rational transformation. In other words, equivalence of curves with equation (3.4) corresponds to equivalence of 4-tuples of points in $\mathbb{P}^1$ under the action of $\text{SL}(2, \mathbb{C})$ and the symmetric group of degree 4. Thus, an isomorphism class of curves stabilized by $G_2$ defines a point in $\mathcal{M}_{0,4}/\mathcal{S}_4$, where $\mathcal{M}_{0,4}$ is the moduli space of 4-pointed rational curves and $\mathcal{S}_4$ is the symmetric group of order 4. Note that $e_c(\mathcal{M}_{0,4}/\mathcal{S}_4) = 1$: see, for instance, [1].
We finally observe that $\Sigma_2$ is not the whole $M_{0,4}/\mathfrak{S}_4$. In fact, we need to disregard all curves with extra automorphisms, i.e., the ones in lower dimensional strata. Therefore,

$$e_c(\Sigma_2) = e_c(M_{0,4}/\mathfrak{S}_4) - \sum_{i=3}^{10} e_c(\Sigma_i)$$

$$= 1 - (-6 + 2 + 3) = 2.$$

In Table 2, we list the dimension and the topological Euler characteristic of all the strata in $H_3$.

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|-----|---|---|---|---|---|---|---|---|---|----|----|
| dim($\Sigma_i$) | 5 | 3 | 2 | 2 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| $e_c(\Sigma_i)$ | -1 | 2 | 1 | 1 | -2 | -2 | -2 | 1 | 1 | 1 | 1 |

Table 2: Some Topological Invariants of the Strata $\Sigma_i$.

4. The Calculation of $e_c(H_3, V_\lambda)$

Let $\gamma_j : \Sigma_j \to H_3$ be the embedding of $\Sigma_j$ in $H_3$. By the properties of the Euler characteristic of local systems, we have

$$e_c(H_3, V_\lambda) = \sum_{j=1}^{11} e_c(\Sigma_j, \gamma_j^*(V_\lambda)).$$

On the other hand, $\gamma_j^*(V_\lambda)$ is a local system on $\Sigma_j$ with respect to $G_j$. Hence, (4) can be written as

$$e_c(H_3, V_\lambda) = \sum_{j=1}^{11} e_c(\Sigma_j) \dim(V_\lambda^{G_j}),$$

where $V_\lambda^{G_j}$ is the space of $G_j$-invariants. In Section 3 we computed $e_c(\Sigma_j)$. Now, we work out the dimension of the corresponding invariant subspaces.

By definition, the fibre of the local system $V_{(1,0,0)}$ over a curve $C$ is given by the cohomology group $H^1(C; \mathbb{Q})$. $V_\lambda$ is thus obtained from the Sp$(6, \mathbb{Q})$-module $V_{(1,0,0)}$ by standard construction in representation theory (cfr. [3]). Obviously, any group $G$ in Table 1 acts on $V_{(1,0,0)}$. This action yields a homomorphism $\eta : G \to \text{Sp}(6, \mathbb{Q})$. Let $(A, \xi)$ be an element in $G$, where $A$ is a matrix with eigenvalues $a$ and $a^{-1}$. By Corollary 3 in [4], the eigenvalues of $\eta(g)$ are given by

$$a^2\xi, \quad a^{-2}\xi^{-1}, \quad a^{-2}\xi, \quad a^2\xi^{-1}, \quad \xi, \quad \xi^{-1}.$$

As a consequence, it is possible to compute the dimension of the $G$-invariant subspace of $V_\lambda$ by elementary character theory. More specifically, let $J_d$ be the symmetric function

$$J_d(x_1, x_2, x_3) = h_d(x_1, x_1^{-1}, x_2, x_2^{-1}, x_3, x_3^{-1}),$$
where \( h_d \) is the complete symmetric function in six variables. Moreover, for any \( \{ \lambda = \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0 \} \), we denote by \( J_\lambda \) the determinant of the 3 \( \times \) 3 matrix whose \( i \)-th row is

\[
(J_{\lambda_i} - i + 2) J_{\lambda_i} - i + J_{\lambda_i} - i + 3 + J_{\lambda_i} - i - 1).
\]

By Proposition 24.22 in [3], the following holds:

\[
\dim (\mathcal{Y}_{G}) = \frac{1}{\# G} \sum_{a, \xi, \eta} J_\lambda(a^2 \xi, a^{-2} \eta, \xi).
\]

For each of the groups \( G \), we can list the pairs \((a, \xi)\) that occur as \( g \) runs through \( G \). If \((a, \xi)\) occurs, then \((-a, -\xi)\) and \((a, -\xi)\) occur too. For each \( G \) in Table 3 we give a set \( Y_i \) of pairs \((a, \xi)\) with multiplicity (indicated by an exponent). The set \( Y_i \) has the following property. If we replace \((a, \xi) \in Y_i \) by the 4 elements \((\pm a, \pm \xi)\) we get all the pairs with multiplicity corresponding to the \( g \in G \). This is indicated by the notation \((\pm a, \pm \xi) \in Y_i \).

**Theorem 4.1.** The Euler characteristic \( e_c(\mathcal{H}_3, \mathcal{V}_\lambda) \) is given by

\[
e_c(\mathcal{H}_3, \mathcal{V}_\lambda) = \sum_{i=1}^{11} \frac{\# G_i}{\# G} \sum_{(\pm a, \pm \xi) \in Y_i} J_\lambda(a^2 \xi, a^{-2} \eta, \xi),
\]

where the Euler numbers \( e(\Sigma_i) \) and the sets \( Y_i \) are given in Tables 2 and 3.

| \( Y_1 \) | \((1, 1)\) |
| \( Y_2 \) | \((1, 1), (i, 1)\) |
| \( Y_3 \) | \((1, 1), (i, 1)^3\) |
| \( Y_4 \) | \((1, 1), (i, i)\) |
| \( Y_5 \) | \((1, 1), (i, i, i)\) |
| \( Y_6 \) | \((1, 1), (i, 1, i)\) |
| \( Y_7 \) | \((1, 1), (i, 1, i)^3\) |
| \( Y_8 \) | \((1, 1), (i, i, i)\) |
| \( Y_9 \) | \((1, 1), (i, 1)^9, (i, 1)^2, (i, 1)^4, (i, 1)^4, (i, 1)^4, (i, 1)^4, (i, 1)^4, (i, 1)^4, (i, 1)^4\) |
| \( Y_{10} \) | \((1, 1), (i, i)_9, (i, i, i)_4, (i, i, i)_4, (i, i, i)_4, (i, i, i)_4, (i, i, i)_4, (i, i, i)_4, (i, i, i)_4, (i, i, i)_4\) |
| \( Y_{11} \) | \((1, 1), (i, i)_9, (i, i, i)_4, (i, i, i)_4, (i, i, i)_4, (i, i, i)_4, (i, i, i)_4, (i, i, i)_4, (i, i, i)_4, (i, i, i)_4\) |

**Table 3. The Sets \( Y_i \)**

For example, the elements of the group \( G_1 \) are \((\pm \text{Id}, \pm 1)\). If \( \lambda = (k, 0, 0) \), then the contribution from this group yields

\[
\dim(\mathcal{Y}_{G_1}^{(k, 0, 0)}) = \frac{1}{4} \{ 2h_k(1, 1, 1, 1, 1, 1) + 2h_k(-1, -1, -1, -1, -1, -1) \}
\]

\[
= \frac{1}{2} \binom{k + 5}{k} (1 + (-1)^k).
\]
In the following table we give the values of $e_c(\mathcal{H}_3, \mathcal{V}_\lambda)$ for all $\lambda$ of weight $\leq 10$. Note that because of the hyperelliptic involution $e_c(\mathcal{H}_3, \mathcal{V}_\lambda) = 0$ if the weight is odd.

| $(\lambda_1, \lambda_2, \lambda_3)$ | $e_c(\mathcal{H}_3, \mathcal{V}_\lambda)$ | $(\lambda_1, \lambda_2, \lambda_3)$ | $e_c(\mathcal{H}_3, \mathcal{V}_\lambda)$ |
|----------------------------------|-----------------------------------|----------------------------------|-----------------------------------|
| (0, 0, 0)                        | 1                                 | (5, 2, 1)                        | -10                               |
| (2, 0, 0)                        | -1                                | (4, 4, 0)                        | -5                                |
| (1, 1, 0)                        | 0                                 | (4, 3, 1)                        | -4                                |
| (4, 0, 0)                        | -1                                | (4, 2, 2)                        | -7                                |
| (3, 1, 0)                        | 0                                 | (3, 3, 2)                        | -2                                |
| (2, 2, 0)                        | -1                                | (10, 0, 0)                       | -17                               |
| (2, 1, 1)                        | 0                                 | (9, 1, 0)                        | -22                               |
| (6, 0, 0)                        | -5                                | (8, 2, 0)                        | -43                               |
| (5, 1, 0)                        | -2                                | (8, 1, 1)                        | -8                                |
| (4, 2, 0)                        | -5                                | (7, 3, 0)                        | -34                               |
| (4, 1, 1)                        | 0                                 | (7, 2, 1)                        | -32                               |
| (3, 3, 0)                        | 0                                 | (6, 4, 0)                        | -37                               |
| (3, 2, 1)                        | 0                                 | (6, 3, 1)                        | -26                               |
| (2, 2, 2)                        | -3                                | (6, 2, 2)                        | -27                               |
| (8, 0, 0)                        | -7                                | (5, 5, 0)                        | -6                                |
| (7, 1, 0)                        | -8                                | (5, 4, 1)                        | -22                               |
| (6, 2, 0)                        | -13                               | (5, 3, 2)                        | -12                               |
| (6, 1, 1)                        | -2                                | (4, 4, 2)                        | -15                               |
| (5, 3, 0)                        | -10                               | (4, 3, 3)                        | 0                                 |

Table 4: Some Values of $e_c(\mathcal{H}_3, \mathcal{V}_\lambda)$

5. Some Remarks on the motivic Euler characteristic

For partitions of small degree $|\lambda|$ it is not unreasonable to expect that all cohomology of $\mathcal{V}_\lambda$ is Tate, i.e., that the motivic Euler characteristic

$$E_c(\mathcal{H}_3, \mathcal{V}_\lambda) = \sum_{i=0}^{10} (-1)^i [H^i_c(\mathcal{H}_3, \mathcal{V}_\lambda)]$$

is a polynomial in $L$, the Tate motive of weight 2. It is well known that $E_c(\mathcal{H}_3, \mathcal{V}_0) = L^5$. One can calculate the trace of Frobenius on the $\ell$-adic variant of $\mathcal{V}_\lambda$ in characteristic $p$ on $\mathcal{H}_3 \otimes \mathbb{F}_p$ by summing

$$\sum_C \text{Tr}(F, \mathcal{V}_\lambda(H^1))/\#\text{Aut}_{\mathbb{F}_p}(C),$$

where $C$ runs over a complete set of representatives of the $\mathbb{F}_p$-isomorphism classes of hyperelliptic curves of genus 3 over $\mathbb{F}_p$. We found that the following guesses for the motivic Euler characteristic are compatible with these traces for $p = 2, 3$ and 5 and with the values of $e_c(\mathcal{H}_3, \mathcal{V}_\lambda)$. 
\[ \lambda \quad \quad E_c(\mathcal{H}_3, \mathbb{V}_\lambda) \\
(0, 0, 0) \quad L^5 \\
(2, 0, 0) \quad -1 \\
(1, 1, 0) \quad L^6 - L^5 \\
(4, 0, 0) \quad L^2 - 2 \\
(3, 1, 0) \quad L^2 - L \\
(2, 2, 0) \quad L^7 - 2L^6 + L^2 - 1 \\
(2, 1, 1) \quad -L^6 + 2L^5 - L^4 - L^3 + L^2 - L + 1 \\

Table 5. Motivic Euler Characteristics

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