LEVEL SETS OF CERTAIN SUBCLASSES OF $\alpha$-ANALYTIC FUNCTIONS

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Abstract. For an open set $V \subset \mathbb{C}^n$, denote by $\mathcal{M}_\alpha(V)$ the family of $\alpha$-analytic functions that obey a boundary maximum modulus principle. We prove that, on a bounded domain $\Omega \subset \mathbb{C}^n$, with continuous boundary (that in each variable separately allows a solution to the Dirichlet problem), a function $f \in \mathcal{M}_\alpha(\Omega \setminus f^{-1}(0))$ automatically satisfies $f \in \mathcal{M}_\alpha(\Omega)$, if it is $C^{\alpha_j-1}$-smooth in the $z_j$ variable, $\alpha \in \mathbb{Z}_n^+$ up to the boundary. For a submanifold $U \subset \mathbb{C}^n$, denote by $\mathcal{M}_\alpha(U)$, the set of functions locally approximable by $\alpha$-analytic functions where each approximating member and its reciprocal (off the singularities) obey the boundary maximum modulus principle. We prove, that for a $C^3$-smooth hypersurface, $\Omega$, a member of $\mathcal{M}_\alpha(\Omega)$, cannot have constant modulus near a point where the Levi form has a positive eigenvalue, unless it is there the trace of a polyanalytic function of a simple form.

1. Introduction

A higher order generalization of the holomorphic functions are the solutions of the equation $\frac{\partial^q}{\partial z^q} f = 0$, for a positive integer $q$. These functions are called polyanalytic functions of order $q$ or simply $q$-analytic functions. An excellent introduction to polyanalytic functions can be found in the survey article by Balk [6]. A higher dimensional generalization of $q$-analytic functions is the set of $\alpha$-analytic functions, $\alpha \in \mathbb{Z}_n^+$. In this paper, we consider the set of function which can be locally uniformly approximated by $\alpha$-analytic functions satisfying a boundary maximum modulus principle. We prove an extension of Radó’s theorem for such functions on complex manifolds. Secondly, we prove, for a special subclass on (not necessarily complex) submanifolds, a generalization of the fact that $q$-analytic functions cannot have constant modulus on open sets unless they are of a particularly simple form.

Results. Our main results are Theorem 3.8, Theorem 5.1 and Theorem 5.6. Theorem 5.1 is a generalization of the fact that $q$-analytic functions cannot have constant modulus on open sets unless they are of the form $\lambda Q(z)/Q(z)$ for some polynomial $Q$ of degree $< q$ and some constant $\lambda \in \mathbb{C}$. In particular we consider families, that can be locally uniformly approximated by

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α-analytic functions, which themselves and and their reciprocals (except at singularities) satisfy a boundary maximum modulus principle, and we show that such families cannot have members with constant modulus near points with at least one positive Levi eigenvalue, unless these members are the there the trace of an α-analytic function of the form \( \frac{\lambda Q(z)}{Q(z)} \) for some holomorphic polynomial \( Q(z) \) of degree \( < \alpha_j \) in \( z_j, 1 \leq j \leq n \) and constant \( \lambda \). For hypersurfaces, this is done for the \( C^3 \)-smooth case but we also give a partial generalization for \( C^4 \)-smooth real submanifolds of higher codimension, see Theorem 5.6. Theorem 3.8 is that functions in \( n \) complex variables which are \( C^{\alpha_j-1} \)-smooth in the \( z_j \) variable, and which, off their zero set, are α-analytic functions that obey the boundary maximum modulus principle, extend across their zero set to functions of the same class.

2. Preliminaries

For general background on polyanalytic functions, see Balk [6] and references therein (in particular we want to mention the earlier work by Balk & Zuev [7]).

**Definition 2.1.** Let \( U \subset \mathbb{C} \) be a domain. A function \( f : U \to \mathbb{C} \) is called \( q \)-analytic (or polyanalytic of order \( q \)) if it can be written as

\[
f(z) = \sum_{j=0}^{q-1} a_j(z)\bar{z}^j, a_j \in \mathcal{O}(U).
\]

If \( a_{q-1} \not\equiv 0 \), then \( q \) is called the exact order.

It is well-known, and almost self-evident, that \( f(z) = u(x, y) + iv(x, y) \) of class \( C^q(U) \) is \( q \)-analytic on \( U \) if and only if

\[
\frac{\partial^q f}{\partial \bar{z}^q} = 0 \quad \text{on } U.
\]

(see Balk [6, p. 198]).

The \( q \)-analytic functions behave well under locally uniform convergence.

**Proposition 2.2** (Balk [6, p. 206]). Let \( U \subset \mathbb{C} \) be a domain and let \( \{f_j\}_{j \in \mathbb{N}} \) be polyanalytic functions of the same order \( q \), on \( U \), such that the \( f_j \) converge uniformly on \( U \). Then the limit is a polyanalytic function of order \( q \) on \( U \).

We cannot directly generalize the fact that a holomorphic function with constant modulus on an open subset must reduce to a constant, to the case of \( q \)-analytic functions. A characterization is, however, known due to Balk [5].

**Theorem 2.3** (Balk [5]). A polyanalytic function of order \( q \) in a domain \( \Omega \subset \mathbb{C} \) has constant modulus if and only if \( f \) is representable in the form \( f(z) = \lambda \cdot \frac{P(z)}{Q(z)} \), where \( P(z) \) is a polynomial of degree at most \( q - 1 \), and \( \lambda \in \mathbb{C} \) is a constant.

Note that, in particular, the only entire polyanalytic functions with constant modulus are the constant functions.
Definition 2.4 (See Balk [4]). Let $U \subset \mathbb{C}$ be a domain and let $p \in E \subset U$. By definition the line $\ell := \{z \in \mathbb{C} : z = p + te^{i\theta}, |t| < \infty, t \in \mathbb{R}\}$, $p$ and $\theta$ constants, is a limiting direction of the set $E$ at $p$ if $E$ contains a sequence of points $z_j = p + t_je^{i\theta_j}$, $t_j \to 0, \theta_j \to \theta, t_j \neq 0$. The point $p$ is called a condensation point of order $k$ of $E$ if there are $k$ different lines through $p$ which are limiting directions of $E$.

The following uniqueness property is known.

Theorem 2.5 (Balk [3, p. 202]). Let $U \subset \mathbb{C}$ be a domain and let $f$ and $g$ be polyanalytic functions of order $q$ on $U$. Assume that $E \subset U$ and that $E$ has a condensation point of order $q$. Then $f \equiv g$ on $E$ implies $f \equiv g$ on $U$.

Polyanalytic functions of several variables. Avanissian & Traoré [2], [3] introduced the following definition of a polyanalytic function of order $\alpha$ in several variables.

Definition 2.6 (Avanissian & Traoré [2]). Let $\Omega \subset \mathbb{C}^n$ be a domain and let $z = x + iy$ denote holomorphic coordinates in $\mathbb{C}^n$. A function $f \in C^\infty(\Omega, \mathbb{C})$ is called polyanalytic of order $\alpha$ if there exists a multi-index $\alpha \in \mathbb{Z}_+^n$ such that in a neighborhood of every point of $\Omega$, $(\frac{\partial}{\partial z_j})^{\alpha_j}f(z) = 0, 1 \leq j \leq n$. If the integer $\alpha_j, 1 \leq j \leq n$ is minimal, then $f$ is said to be polyanalytic of exact order $\alpha$.

Definition 2.7. A function $f$ is called countably analytic on an open set $\Omega \subset \mathbb{C}^n$ if $f$ has a local expansion near every point $p \in \Omega$ of the form $f(z) = \sum h_\alpha(z,p)(\bar{z} - \bar{p})^\alpha$ for holomorphic $h_\alpha(z,p)$ (where $p$ is fixed i.e. $h_\alpha$ is holomorphic in the variable $z$).

Definition 2.8. Let $\Omega \subset \mathbb{C}^n$ be an open subset and let $(z_1, \ldots, z_n)$ denote holomorphic coordinates for $\mathbb{C}^n$. A function $f$ is said to be separately $C^k$-smooth with respect to the $z_j$-variable, if for any fixed $(c_1, \ldots, c_{n-1}) \in \mathbb{C}^{n-1}$ the function $f(c_1, \ldots, c_{j-1}, z_j, c_j, \ldots, c_{n-1})$ is $C^k$-smooth with respect to $\text{Re} \ z_j, \text{Im} \ z_j$.

3. $\alpha$-ANALYTICITY ACROSS ZERO SETS FOR A SPECIAL FAMILY

We cannot hope for a strong maximum principle for $q$-analytic functions. Take for example the 2-analytic function $f(z) = 1 - z\bar{z}$ on $\mathbb{C}$, which not only attains a strict local maximum at the origin but in fact vanishes on the boundary of the unit disc, thus it is certainly not determined by its boundary values.

Definition 3.1. Let $\Omega \subset \mathbb{C}^n$ be a submanifold and let $\mathcal{M} \subset C(\Omega, \mathbb{C})$ be a family of functions. Let $D$ denote the unit disc, $D := \{\zeta \in \mathbb{C} : |\zeta| < 1\}$. The family $\mathcal{M}$ is said to have the one dimensional boundary maximum modulus property (1D-BMMP) if given $f \in \mathcal{M}$, then for any $\psi \in \mathcal{O}(D, \Omega) \cap C(\overline{D}, \Omega)$,

$$\max_{\zeta \in \partial D} |f \circ \psi(\zeta)| \leq \max_{\zeta \in \partial D} |f \circ \psi(\zeta)|.$$

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Example 3.2. Let \( \Omega \subset \mathbb{C}^n \) be a complex submanifold. Then \( \mathcal{O}(\Omega) \subset C(\Omega, \mathbb{C}) \) clearly has the one dimensional boundary maximum modulus property.

Using the definition of Avanissian & Traoré [2], together with Theorem 3.7 we obtain the following Corollary to Proposition 2.2.

We shall, in this section, use the following notation: Let \( U \subset \mathbb{C}^n \) be a submanifold. Denote by \( M_\alpha(U) \) the set of restrictions to \( U \) of \( \alpha \)-analytic functions on \( \mathbb{C}^n \) that obey the one dimensional boundary maximum modulus property of Definition 3.1.

An immediate consequence of the fact that the set of holomorphic functions obey a strong maximum principle, is that, if \( U \subset \mathbb{C}^n \) is a complex submanifold (not necessarily of dimension \( n \)), then \( \mathcal{O}(U) \subseteq M_\alpha(U) \), but more can be said (see Example 4.3). For simplicity of notation, let \( \tilde{D} = \frac{\partial}{\partial \bar{z}} \).

A complex function on \( \mathbb{C}^n \) annihilated by the system \( \frac{\partial}{\partial \bar{z}_j} \), \( 1 \leq j \leq n \), on \( \mathbb{C}^n \setminus f^{-1}(0) \) is automatically holomorphic. This theorem was proved for \( n = 1 \) by Radó [18] already in 1924, and generalized to \( n > 1 \) by Cartan [12] in 1952. We shall generalize this result to the subfamilies, \( \alpha \)-analytic functions which obey the boundary maximum modulus principle. Kaufman [14] makes use of a maximum principle and combines this with an approximation property on the boundary to prove Radó’s theorem and we shall use a similar method of proof for the following generalization for the case of bounded domains.

**Theorem 3.3.** Let \( \Omega \subset \mathbb{C} \) be a bounded domain with continuous boundary that allows a solution to the Dirichlet problem. Let \( q \) be a positive integer and let \( f \in C^{q-1}(\Omega) \) such that \( f \in M_q(\Omega \setminus f^{-1}(0)) \). Then \( f \in M_q(\Omega) \).

**Proof.** Note that \( q = 1 \) corresponds to the well-known Radó’s theorem in one complex variable.

**Remark 3.4.** If \( U \subset \mathbb{C}^n \) a submanifold and \( G \) is a continuous function on \( U \) which satisfies the boundary maximum modulus principle on the (necessarily open) subset \( U \setminus G^{-1}(0) \) then \( G \) satisfies the boundary maximum modulus principle on \( U \). Indeed, let \( V \Subset U \), be a domain. If \( V \cap G^{-1}(0) = \emptyset \) then, by continuity, \( \max_{z \in V} |G(z)| = 0 \), so we are done. If instead the (necessarily open) set \( V \cap \{G \neq 0\} \) is nonempty, then \( \max_{z \in V} |G(z)| = \max_{z \in \partial V \cap \{G \neq 0\}} |G(z)| = \max_{z \in \partial V} |G(z)| \).

By Remark 3.4 it is sufficient to show that \( f \) extends to a \( q \)-analytic function on \( \Omega \). For the sake of clarity we first prove the case \( q = 2 \), i.e., assume \( f \in M_q(\Omega \setminus f^{-1}(0)) \) (only small modifications are then required to prove the cases \( 2 < q < \infty \)). By Corollary 4.1 we have that \( f \) is 2-analytic on \( \Omega \setminus f^{-1}(0) \), and by definition we know that \( f \) satisfies the boundary maximum modulus principle on \( \Omega \setminus f^{-1}(0) \). Set \( u(z) := \frac{\partial}{\partial \bar{z}} f(z) = D f(z) \).
Since \( f \in C^{2-1}(\Omega) \), the function \( u \) is well-defined on \( \Omega \). Let \( U = \Omega \setminus f^{-1}(0) \) for convenience of notation. By definition, \( u \) is holomorphic on \( U \). Furthermore, \( u = 0 \) on the interior of \( f^{-1}(0) \) so,
\[
\bar{D}u(z) = 0 \quad \forall z \in \Omega \setminus \partial U.
\]

We shall need the following lemma.

**Lemma 3.5.** Let \( g \) be a function continuous on \( U \) and holomorphic on \( U \). Then for all \( z \in U \),
\[
|g(z)| \leq \sup_{\zeta \in \partial U \cap \partial \Omega} |g(\zeta)|.
\]

**Proof.** First of all, \( \sup_{\partial U \cap \partial \Omega} |f| = 0 \) since \( f = 0 \) on the given set. Secondly, note that \( f \) and \( g^j \) each satisfies the boundary maximum modulus principle applied to \( U \). Thus for any \( z \in U \),
\[
|f(z)| |g(z)|^j \leq \left( \sup_{\partial U \cap \partial \Omega} |f| \right) \cdot \left( \sup_{\partial U \cap \partial \Omega} |g| \right)^j \leq \left( \sup_{\partial U \cap \partial \Omega} |f| \right) \cdot \left( \sup_{\partial U \cap \partial \Omega} |g| \right)^j,
\]
which in turn implies, after taking 1/jth power and the limit as \( j \to \infty \),
\[
|g(z)| \leq \sup_{w \in \partial U \cap \partial \Omega} |g(w)|, z \in \Omega \setminus \partial U.
\]

By Lemma 3.5,\( (1) \)
\[
|u(z)| \leq \sup_{w \in \partial U \cap \partial \Omega} |u(w)|, z \in U.
\]

Also, we know that \( |u(z)| = 0 \) for all \( z \) in the interior of \( f^{-1}(0) \). In fact, given Lemma 3.5, a verbatim repetition of an argument which can be found in e.g. Kaufmann [14], proves that \( U \) must be a dense subset of \( \Omega \), in particular \( f^{-1}(0) \) must have empty interior. This together with Equation (1) gives,
\[
(2) \quad |u(z)| \leq \sup_{w \in \partial U \cap \partial \Omega} |u(w)|, z \in \Omega \setminus \partial U.
\]

Next we show that \( \partial u / \partial \bar{z} \) is harmonic, and since it is zero on a dense open subset of \( \Omega \) it vanishes identically. To see that \( \partial u / \partial \bar{z} \) is harmonic, set \( u = w + iv \), and show that \( w, v \) are harmonic as follows: First of all we can find a sequence of complex polynomials whose real parts converge uniformly on \( \partial \Omega \) to \( u \) (a procedure for doing this is described in e.g. Boivin & Gauthier [11], p.123. In short one solves the Dirichlet problem for the boundary, in order to obtain a harmonic function, say \( W \), in \( \Omega \) whose continuous boundary values agree with \( u \). Of course it is important to refer to the condition that the boundary be given by a continuous function. Then \( W \) can be complemented with its harmonic conjugate in \( \Omega \), to a holomorphic function on \( \Omega \). The latter can the be approximated by complex polynomials). So let \( \{ P_j \}_{j \in \mathbb{N}} \), be a sequence of holomorphic polynomials such that \( \Re(P_j - u) \to 0 \) on \( \partial \Omega \). We have \( |e^{P_j - u}| = e^{\Re(P_j - u)} \), and also that \( e^{P_j - u}, e^{u - P_j} \) are holomorphic on
U. Thus the maximum principle of Equation (2) applies to both $e^{P_j-u}$ and $e^{u-P_j}$. We can choose $P_j$ such that $|P_j - u| < \frac{1}{j}$. Consequently,

\[
\left| e^{(P_j(z)-u(z))} \right| < e^{\frac{1}{j}} \quad \text{and} \quad \left| e^{(u(z)-P_j(z))} \right| < e^{\frac{1}{j}}, \quad z \in \partial \Omega,
\]

and by the maximum principle of Equation (2), the inequalities (3) continue to hold in $\Omega$. This however implies that

\[
e^{\frac{1}{j}} > \left| e^{P_j(z)-u(z)} \right| = e^{Re(P_j(z)-u(z))}, \quad \forall z \in \Omega.
\]

After taking logarithms,

\[
\frac{1}{j} > Re(P_j(z) - u(z)), \quad \forall z \in \Omega.
\]

Since the real part of each $P_j$ is harmonic, this uniform convergence implies that $Re u(z) = w(z)$ is harmonic on $\Omega$. Analogously one shows that $v$ is harmonic. It then follows that $\partial u/\partial \bar{z}$ has harmonic real and imaginary parts since, $\Delta \frac{\partial u}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}}(\Delta w) - \frac{\partial}{\partial \bar{z}}(\Delta v) + i \left( \frac{\partial}{\partial \bar{z}}(\Delta w) - \frac{\partial}{\partial \bar{z}}(\Delta v) \right) = 0$. A harmonic function (in particular the real and imaginary part respectively of $\frac{\partial u}{\partial \bar{z}}$) vanishing on a dense open subset vanishes identically thus $\frac{\partial u}{\partial \bar{z}}$ vanishes identically on $\Omega$ i.e. $u$ is holomorphic on $\Omega$ meaning that,

\[0 = \bar{D} u(z) = \bar{D}^2 f(z), \quad \forall z \in \Omega,\]

i.e. $f$ is bianalytic on $\Omega$. This proves Theorem 3.3 for $q = 2$.

Now we can easily adapt the proof to the cases $q > 2$. Assume $f \in C^{q-1}(\Omega)$ is $q$-analytic on $U$ and as before let $u(z) := \bar{D}^{q-1} f(z)$. If $f^{-1}(0) \cap \Omega$ has nonempty interior then also $\bar{D}^q f(z) = 0$ on $(\Omega \cap f^{-1}(0))^0$, thus $\bar{D} u(z) = 0$ on $\Omega \setminus (\partial f^{-1}(0))$, which is a dense open subset of $\Omega$. Applying the same arguments to $u$ as for the case $q = 2$ we obtain that $\bar{D} u(z)$ vanishes identically on $\Omega$, in particular $u$ is differentiable\footnote{In fact, $u$ must be $C^{\infty}$-smooth due to a well-known property of the $\bar{D}$-operator, see e.g. Krantz \cite{Krantz} p. 200.} on $\Omega$ thus $\bar{D}^{q-1} f(z)$ is well-defined and differentiable on $\Omega$, meaning that we can write $\bar{D}^q f = \frac{\partial}{\partial \bar{z}} (\bar{D}^{q-1} f(z)) = 0, \forall z \in \Omega$. This completes the proof.

Note the importance of the starting function $f$ to be $C^{q-1}(\Omega)$ instead of merely continuous, namely, we need in the proof for $q > 2$ that $u$ be continuous on $\Omega$.

**Example 3.6.** Let $\Omega := \{|z| < 2\} \subset \mathbb{C}$, and set,

\[
f(z) := \begin{cases} 1 - \frac{z \bar{z}}{|z|}, & |z| \leq 1, \\ \frac{1}{z \bar{z} - 1}, & 1 < |z| < 2. \end{cases}
\]
The function $f$ is clearly $2$-analytic on the open subset $\Omega \setminus f^{-1}(0)$ (which consists of two disjoint domains), and we have $f \in C^0(\Omega)$, $0 = q - 2$. However $f$ is not $2$-analytic at any point of $\{|z| = 1\}$.

Note that this example break both the regularity assumption and the boundary maximum modulus principle required in Theorem 3.3. We are not aware of an example of a function that only fails one of these two assumptions and that cannot be extended polyanalytically across its zero set.

As was pointed out by Cartan [12], an extension of Radó’s theorem to the case of several variables is easy in the presence of Hartogs’ theorem on separately holomorphic functions. It turns out that such a multi-variable Hartogs theorem is indeed known for polyanalytic functions in the sense of Definition 2.6.

**Theorem 3.7** (Avanissian & Traoré [3, Theorem 1.3, p.264]). Let $\Omega \subset \mathbb{C}^n$ be a domain and let $z = (z_1, \ldots, z_n)$, denote holomorphic coordinates in $\mathbb{C}^n$ with $\Re z = x, \Im z = y$. Let $f$ be a function which, for each $j$, is polyanalytic of order $\alpha_j$ in the variable $z_j = x_j + iy_j$ (in such case we shall simply say that $f$ is separately polyanalytic of order $\alpha$). Then $f$ is jointly smooth with respect to $(x, y)$ on $\Omega$ and furthermore is polyanalytic of order $\alpha = (\alpha_1, \ldots, \alpha_n)$ in the sense of Definition 2.6.

We immediately obtain the following consequence of Theorem 3.3.

**Theorem 3.8.** Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with continuous boundary that in each variable separately allows a solution to the Dirichlet problem. Let $(z_1, \ldots, z_n) \in \mathbb{C}^n$ denote holomorphic variables. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$. Then any function $f$ which is $C^{\alpha_j-1}$-smooth in the $z_j$ variable, for each $j$, up to the boundary and which is a member of $\mathcal{M}_\alpha(\Omega \setminus f^{-1}(0))$ is automatically a member of $\mathcal{M}_\alpha(\Omega)$.

**Proof.** Recall that when $\Omega$ is a complex manifold of same dimension as the ambient space, $\mathcal{M}_\alpha$ reduces to a subspace of $\alpha$-analytic functions which satisfy the boundary maximum modulus principle. Denote for a fixed $c \in \mathbb{C}^{n-1}$, $\Omega_{c,k} := \{z \in \Omega : z_j = c_j, j < k, z_j = c_{j-1}, j > k\}$. Consider the function $f_c(z_k) := f(c_1, \ldots, c_{k-1}, z_k, c_k, \ldots, c_{n-1})$. Note that the restriction of a function in $n$ complex variables which satisfies the boundary maximum modulus principle, to a complex one-dimensional submanifold, must also satisfy the boundary maximum modulus principle. Clearly, $f_c$ is $\alpha_k$-analytic on $\Omega_{c,k} \setminus f^{-1}(0)$ for any $c \in \mathbb{C}^{n-1}$. Since $f_c^{-1}(0) \subseteq f^{-1}(0)$, Theorem 3.3 applies to $f_c$, meaning that $f$ is separately polyanalytic of order $\alpha_j$ in the variable $z_j, 1 \leq j \leq n$. By Theorem 3.7 the function $f$ must be polyanalytic of order $\alpha$ (in the sense of Definition 2.6) on $\Omega$. By Remark 3.4, $f$ satisfies the boundary maximum modulus principle. This completes the proof. \qed
4. A special subfamily, \( \mathcal{M}_\alpha(U) \), of local limits on submanifolds \( U \subset \mathbb{C}^n \)

**Corollary 4.1** (to Proposition 2.2). Let \( \Omega \subset \mathbb{C}^n \) be a domain and let \( \{f_j\}_{j \in \mathbb{N}} \) be a sequence of polyanalytic functions of order \( \alpha \in \mathbb{Z}_+^n \), such that the \( f_j \) converge uniformly on \( \Omega \). Then the limit is a polyanalytic function of order \( \alpha \) on \( \Omega \).

**Proof.** Let \( (z_1, \ldots, z_n) \in \mathbb{C}^n \) denote holomorphic variables with respect to which being polyanalytic is defined. Let \( 1 \leq k \leq n \). Fixing all variables except \( z_k \), say \( (z_1, \ldots, z_n) = (c_1, \ldots, c_{k-1}, z_k, c_k, \ldots, c_{n-1}) \), for some \( c \in \mathbb{C}^{n-1} \), we know that for any \( j \in \mathbb{N} \), the restriction of \( f_j \) becomes an \( \alpha_k \)-analytic function in the variable \( z_k \) on the set \( \Omega_{c,k} := \{ z \in \mathbb{C}^n : z_i = c_i, i < k, z_i = c_{i-1}, i > k \} \). By Proposition 2.2, this implies that the uniform limit function, which we denote \( f \), of the sequence \( \{f_j\}_{j \in \mathbb{N}} \), is separately polyanalytic of order \( \alpha \). Thus by Theorem 3.7 \( f \) is polyanalytic of order \( \alpha \) (in the sense of Definition 2.6) on \( \Omega \). This completes the proof. \( \square \)

We shall in this section be interested in the following families of functions, which in particular includes \( C^q \)-smooth boundary values of special \( \alpha \)-analytic functions \((|\alpha| \leq q)\).

**Definition 4.2.** Let \( V \subset \mathbb{C}^n \) be a domain. Denote by \( \mathcal{M}(V) \) the set of countably analytic functions \( g \) which obey the one dimensional boundary maximum modulus property of Definition 3.1 and such that \( 1/g \) has the 1D-BMMP on \( V \setminus g^{-1}(0) \). Let \( U \subset \mathbb{C}^n \) be a real submanifold. Denote by \( \mathcal{M}_\alpha(U) \) the set of functions \( f \) defined on \( U \) with the property that for every \( p \in U \) there exists an open set \( V_p \subset \mathbb{C}^n \) such that \( f \) can be uniformly approximated on \( U \cap V_p \) by \( \alpha \)-analytic functions in \( \mathcal{M}(V_p)^2 \).

**Example 4.3.** Clearly if \( M \subset \mathbb{C}^n \) is an open subset then so is any open \( U \subset M \) thus, by Proposition 2.2, \( \mathcal{M}_\alpha(U) \) coincides with the set of \( \alpha \)-analytic functions on \( U \) which can be locally uniformly approximated by \( \alpha \)-analytic functions that together with their reciprocals (where defined) satisfy the one dimensional boundary maximum modulus property. If \( \Omega \subset \mathbb{C}^n \) is a complex submanifold (of dimension \( \leq n \)) then \( \mathcal{M}(\Omega) \subset \mathcal{M}_\alpha(\Omega) \).

**Example 4.4.** For any open subset \( V \subset \mathbb{C}^n \),

\[
\mathcal{M}_{\{1, \ldots, 1\}}(V) = \mathcal{O}(V)
\]

and, with the usual partial ordering of multi-indices (i.e. \( \alpha \leq \beta \) if \( \alpha_j \leq \beta_j, 1 \leq j \leq n \)), we have for any real submanifold \( U \subset \mathbb{C}^n \),

\[
\alpha, \beta \in \mathbb{Z}_+^n, \alpha \leq \beta \Rightarrow \mathcal{M}_\alpha(U) \subseteq \mathcal{M}_\beta(U).
\]

In particular, the set of restrictions of holomorphic functions to \( U \) belong to \( \mathcal{M}_\alpha(U) \), for any \( \alpha \in \mathbb{Z}_+^n \).

\(^2\)Note that the reciprocals of the approximating functions need not be \( \alpha \)-analytic, merely countably analytic away from their singularities.
Example 4.5. If $U \subset \mathbb{C}^n$ is a generic CR submanifold (see the Appendix) then, due to a theorem of Baouendi & Treves [8] (the special case we need is formulated more directly in Boggess & Polking [10], Theorem 2.1, p.761),

$$\mathcal{M}_{(1,\ldots,1)}(U) = \{\text{continuous CR functions on } U\},$$

(for the definition of continuous CR functions on $U$, see Appendix).

In contrast to the case of $\alpha$-analytic functions where $\alpha_j \leq 1, 1 \leq j \leq n$, we know that there exists nonconstant $\alpha$-analytic functions which are real valued as soon as at least one $\alpha_j > 1$.

Example 4.6. Let $U \subset \mathbb{C}^n$ be a domain, let $(z_1, \ldots, z_n)$ denote holomorphic coordinates in $\mathbb{C}^n$ with respect to which being polyanalytic is defined and let $\alpha \in \mathbb{Z}_+^n$ such that at least one $\alpha_j > 1$. Then $z_\alpha z_j$ is a real-valued function belonging to $\mathcal{M}_\alpha(U)$. More generally let $P(z) = \sum_{|\beta| \leq q} a_\beta z^\beta$. Then the function $|P(z)|^2 = \overline{P}(z) \cdot P(z)$ is a polyanalytic function of order $\alpha$ for some $\alpha$ with $|\alpha| \leq q$. Since holomorphic polynomials obey the maximum principle (and so do their reciprocals where well-defined) on any complex submanifold of $U$ we obtain $|P(z)|^2 \in \mathcal{M}_\alpha(U)$.

Example 4.7. It is also clear that $\mathcal{M}_\alpha(U)$ always contains functions which are neither holomorphic nor plurisubharmonic, e.g. the restriction of $(z^3 e^{z^2})\overline{z}^3$ to any submanifold $U \subset \mathbb{C}$ belongs to $\mathcal{M}_4(U)$.

Example 4.8. For $\alpha \in \mathbb{Z}_+^n$ such that at least one $\alpha_j > 1$, $\mathcal{M}_\alpha$ is in general not closed under addition. Take e.g. $\alpha = (2, 3), n = 2$. Then $g(z_1, z_2) = (z_1 z_3^2)z_1 z_2^2 \in \mathcal{M}_{(2,3)}(\mathbb{C}^2)$, but $(1 - g) \notin \mathcal{M}_{(2,3)}(\mathbb{C}^2)$.

Example 4.9. The phenomenon that for $U \subset \mathbb{C}^n$, $\mathcal{M}_{(1,\ldots,1)}(U)$ may contain elements that are not the restrictions of holomorphic functions, also holds for $\alpha \geq (1, \ldots, 1) \in \mathbb{Z}_+^n$. Let $(z_1, \ldots, z_n) \in \mathbb{C}^n, n > 1$, be complex coordinates with respect to which being $\alpha$-analytic is defined. $U$ be the real analytic hypersurface $U := \{z \in \mathbb{C}^n : \text{Im } z_n = 0\}$. It is known (see e.g. Boggess [9], p.109) that any continuous function $f$ on $U$ which is holomorphic with respect to $z_1, \ldots, z_{n-1}$, can be locally uniformly approximated on $U$ by entire functions. For example given any point $p \in U$ there exists an open neighborhood $p \in V_p \subset \mathbb{C}^n$ together with a sequence $\{E_{j,p}\}_{j \in \mathbb{N}}$ of holomorphic functions such that $E_{j,p} \to f$ on $V_p \cap U$. Take e.g. the continuous function $f(z) := |\text{Re } z_n| \cdot \exp(\sum_{j=1}^{n-1} z_j), z \in U$ and set $g(z) = f(z) \cdot \overline{P(z)}$, $z \in U$ where $P(z) := \overline{P}|_U$ for a holomorphic polynomial $P(z)$ of highest power $(\alpha_j - 1)$ in the variable $z_j, 1 \leq j \leq n$. Then on any $V_p \cap U$ as above the function $g$ is the uniform limit of $\{E_{j,p} \cdot \overline{P}\}_{j \in \mathbb{N}}$ where clearly each $E_{j,p} \cdot \overline{P}$ is $\alpha$-analytic on $V_p$, thus $g \in \mathcal{M}_\alpha(U)$ since $|E_{j,p} \cdot \overline{P}| = |E_{j,p}| \cdot |P|$, and both $E_j$ and $P$ are.

\[\text{3We mention that it is straightforward to further define } \mathcal{M}_{(\infty,\ldots,\infty)}(U) \text{ for countably analytic functions and in that case, } \{g : g = |f(z)|^k, \text{ for some } f \in \theta(U) \} \subset \mathcal{M}_{(\infty,\ldots,\infty)}(U). \text{ We shall however only need } \mathcal{M}_\alpha(U) \text{ for finite order } \alpha \text{ in this text.}\]
holomorphic on $V_p$. However the factor $|\text{Re} \, z_n|$ implies that $g(z)$ is not the restriction to $U$ of an $\alpha$-analytic function on any neighborhood of the origin.

5. Implication of constant modulus

We shall now point out that some results for the space $\mathfrak{M}_\alpha(M)$ where $M \subset \mathbb{C}^n$ is a generic CR submanifold (see Appendix), follow immediately from the construction of one dimensional manifolds attached near a reference point and which cover an open subset. For clarity we shall begin with the easier case of hypersurfaces and then give a generalization to higher codimension. In the case of hypersurfaces, the main tool follows from a result of Lewy [17] and in higher codimension, the generalization (involving the so called Levi cone) is given by Boggess & Polking [10] (see Boggess [9] for textbook version).

5.1. The case of $C^3$-smooth hypersurfaces near 1-convex points.

**Proposition 5.1.** Let $M \subset \mathbb{C}^n$, $n \geq 2$, be a $C^3$-smooth hypersurface such that the Levi form of $M$ at the origin has at least one positive eigenvalue. Let $f \in \mathfrak{M}_\alpha(\Omega)$ on a domain $\Omega \subset M$ containing the origin. Then,

(i) There exists an open subset of $\mathbb{C}^n$ containing an open $M$-neighborhood of the origin in its closure, to which $f$ extends to an $\alpha$-analytic function (i.e. $f$ can be identified near the origin as the trace values of an $\alpha$-analytic function).

(ii) $|f|$ cannot be constant on a domain $0 \in \omega \subset \Omega$ unless $f$ is near 0 the trace of some $\alpha$-analytic function of the form $\lambda \cdot \frac{Q(z)}{Q(z)}$ for some constant $\lambda$ and holomorphic polynomial $Q(z)$ of degree $< \alpha_j$ in $z_j$, $1 \leq j \leq n$.

**Proof.** Let $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ denote the complex variables with respect to which being $\alpha$-analytic is defined. Given a unitary matrix $U$, the linear coordinate transformation $z \mapsto Uz$ map polynomials in the components of $\bar{z}$ with holomorphic coefficients to polynomials in the components of $U\bar{z}$ with holomorphic coefficients. In particular for any fixed multi-index $\alpha \in \mathbb{Z}^n_+$ and any unitary matrix $U$, there is a fixed $\beta \in \mathbb{Z}^n_+$ such that every function $P$ that is $\alpha$-analytic with respect to $z$ becomes $\beta$-analytic with respect to $Uz$ and $U^T$ transforms the coordinates back to $z$ making $P(z) = P(U^TUz)$, $\alpha$-analytic. The proof is based on first finding analytic discs attached to $\omega$ and then filling out a one-sided open subset of $\mathbb{C}^n$ containing $\omega$ in its closure. To prove the existence of the discs we shall first make a linear coordinate transformation using a unitary matrix. Once we have the discs, an inverse transformation to $z$ will preserve the existence of discs attached near a reference point, because a complex one dimensional manifold attached to $\omega$ remains a one dimensional manifold attached to $\omega$ after a linear change of coordinates. Then we shall use the properties of $f$ (in the $z$-variables) in order to obtain that a sequence uniformly approximating $f$ near 0 also converges on the one-sided open subset. We will need the following lemma.
Lemma 5.2. There is a domain $V \subset \Omega$ (chosen sufficiently small) containing 0, decomposed by $M$ into $V^+, V \cap M, V^-$ with $\omega = V \cap M$ containing 0 such that:

1. There exists a function $F$ which is $\alpha$-analytic on $V^+$ such that $F|_\omega = f$.
2. $\max_{\omega} |f| \geq \max_{V^+} |F|$. (In particular if $|f| \equiv C$ on $\omega$ then $\max_{V^+} |F| \leq C$ on $V^+$).

Proof. The method of proof is that of Lewy [17] and uses filling by interiors of analytic discs attached to $M$ near $p$. For further details, see e.g. Boggess [9, p.209].

We start from the local graph representation of $M \cap V = \{ \text{Im} z_n = h(z_1, \ldots, z_{n-1}, \text{Re} z_n) \} \subset V$ for a sufficiently small open neighborhood $V$ of the origin in $\mathbb{C}^n$. After a change of coordinates, $z \mapsto Uz = (w, \bar{z})^T \in \mathbb{C}^{n-1} \times \mathbb{C}$, we can diagonalize the Hermitian symmetric matrix $S := \left[ \frac{\partial^2 h(0)}{\partial w_j \partial w_k} \right]_{(n-1) \times (n-1)}$ such that $\Lambda := U^T SU = \text{diag}(\lambda_1, \ldots, \lambda_{n-1})$ for some unitary matrix $U$. We may assume that

$$h(w_1, 0, \ldots, 0, 0) = \lambda_1 |w_1|^2 + o(|w|^2, |x|^2)$$

(where $o(|w|^2, |x|^2)$ denote terms depending on both $w, x$ which vanish to third order at 0). Since the Levi form at 0 has at least one positive eigenvalue we can assume that $\lambda_1 > 0$ (after a reordering of the $w$ coordinates). The manifold $M$ divides $V$ into $V \cap M, V \cap \{ y > h(w, x) \}$ and $V \cap \{ y < h(w, x) \}$.

For a domain $0 \in \omega \subset M$, a sufficiently small translation of the complex line $\mathbb{C} \times 0 \subset \mathbb{C} \times \mathbb{C}^{n-1}$ in the positive $y$-direction intersects $\{ y > h(w, x) \}$ in a simply connected open subset. More precisely, there exists $\epsilon, \delta > 0$ such that

$$A_{x,y,w_2,\ldots,w_{n-1}} := \{ (\zeta, w_2, \ldots, w_{n-1}, \bar{z}) : \zeta \in \mathbb{C} \cap \{ |y| < \epsilon, |x|, |w_2|, \ldots, |w_{n-1}| < \delta \}$$

is simply connected with its boundary contained in $\omega$. Finally, the union of the $A_{x,y,w_2,\ldots,w_{n-1}} \cap \{ y > h(w, x) \}$ cover an ambient open subset (see example in Figure 1) which we denote by $\hat{U}$. Now we transform the coordinates back to $(z_1, \ldots, z_n)$ by applying the linear transformation given by the matrix $U^T$. The image of each $A_{x,y,w_2,\ldots,w_{n-1}} \cap \{ y > h(w, x) \}$ is again a complex one-dimensional submanifold attached to $\omega$. Furthermore the image of $\hat{U}$ is an open subset of $\mathbb{C}^n$ containing $\omega$ in its closure. Without loss of generality, we may assume that this open subset is $V^+$.

Now by definition, any function in $\mathfrak{M}_\alpha$ can be approximated locally uniformly on $\omega$ by functions, say $\{ P_j \}_{j \in \mathbb{N}}$, that are $\alpha$-analytic on an open (in $\mathbb{C}^n$) neighborhood of $\omega$. Furthermore, the restriction of these functions to
each analytic disc $\psi(\mathcal{D})$ is a function $P_j \circ \psi$ that obeys the maximum modulus principle. (Here, by slight abuse of notation, $\psi(\mathcal{D})$ is the image, under the reverse transformation with matrix the $U^T$, of the closure of sets $A_{x,y,w,2,...,w_{n-1}} \cap \{y > h(w,x)\}$.)

Thus the $P_j$ converge uniformly on the union of the interiors of the analytic discs, which contains the open one-sided neighborhood $V^+$ (or in the $Uz$-variables, the part part of $V$ belonging to $\{y > h(w,x)\}$) and by Theorem 2.2 the limit function, which we denote by $F$, is $\alpha$-analytic on $V^+$. Also, the inequalities $\max_{\mathcal{D}} |P_j| \leq \max_{\omega} |P_j|$, $\forall j \in \mathbb{N}$, immediately imply that $\max_{\mathcal{D}} |F| \leq \max_{\omega} |f|$. This completes the proof of Lemma 5.2. □

Part (1) of Lemma 5.2 proves statement (i). Also Lemma 5.2 immediately takes care of the case $C = 0$. We may therefore assume that $C > 0$. We continue to work with $(z_1, \ldots, z_n) \in \mathbb{C}^n$ as the coordinates in $\mathbb{C}^n$ with respect to which being $\alpha$-analytic is defined.

**Lemma 5.3.** The extension $F$ in (1) of of Lemma 5.2 must have constant modulus as soon as its trace $F|_\omega$ has constant modulus.

**Proof.** Assume that there exists an open subset $W \subset V^+$ such $|F| \equiv C$ on $W$. By Theorem 2.3 the restriction of $F$ to any

$$W_{k,c} := W \cap \{z_j = c_j, j < k, z_j = c_j-1, j > k\}$$

for a fixed $c \in \mathbb{C}^{n-1}$, is a function of the form $\lambda_{k,c} \cdot \frac{Q_{k,c}(z_k)}{Q_{k,c}(z_k)}$, where $Q_{k,c}$ is a polynomial of degree less than $\alpha_k$ and $\lambda_{k,c}$ is a complex constant. By continuity of $|F|$ up to $\omega$ we deduce that $|\lambda_{k,c}|$ must equal $C$ independently of $k,c$. Now any open subset has a condensation point of order $\alpha_k$ so by
Theorem 2.5 the restriction of $F$ extends as $\lambda_{k,c} \cdot \frac{Q_{k,c}(z_k)}{Q_{k,c}(z_k)}$, to the intersection of any $\{z_j = c_j, j < k, z_j = c_j-1, j > k\}$, $1 \leq k \leq n$ with $V^+$ which passes $W$. Repeated application of Theorem 2.5 starting from that intersection we obtain (using a union of intersections of $V^+$ with polydiscs) that the restriction of $F$ to $V^+ \cap \{z_j = c_j, j < k, z_j = c_j-1, j > k\}$, $1 \leq k \leq n$, has the form $\lambda_{k,c} \cdot \frac{Q_{k,c}(z_k)}{Q_{k,c}(z_k)}$. Hence $|F| \equiv C$ on $V^+$ and $F$ is an $\alpha$-analytic extension of $f$ from $\omega$ so we are done under the assumption that $|F| \equiv C$ on some open subset of $V^+$.

Now, we assume instead that $|F| < C$ on a dense (necessarily open) subset of $V^+$. This implies that there exists an open subset $W \subset V^+$ on which $\frac{C}{2} < |F| < C$ such that $\partial W \cap \omega$ is relatively open. In particular $\frac{C}{2}$ is well-defined on $W$ and since $\frac{C}{2} > 0$ we can choose a subsequence $(P_{ji})_{j \in \mathbb{N}}$ such that each $P_{ji}$ is nowhere zero on $W$ and $1/P_{ji} \rightarrow 1/F$ on $W$. Hence,

\begin{equation}
\max_{z \in W} \left| \frac{1}{F(z)} \right| \leq \frac{1}{C}.
\end{equation}

But we already know $\max_{z \in W} |F(z)| \leq C$ (where we are using $W \subset V^+$ together with (2) of Lemma 5.2). So if there were to exist a point $z_0 \in W$ such that $|F(z_0)| \neq C$, then necessarily $|F(z_0)| < C$ which implies $\left| \frac{1}{F(z_0)} \right| > \frac{1}{C}$ (thus incompatible with Equation (4)). This yields $|F| \equiv C$ on $W$, which is a contradiction. This completes the proof of Lemma 5.3. \hfill \Box

Finally we need to verify that the $\alpha$-analytic extension $F$ which has constant modulus in fact must have the form required in (ii).

**Lemma 5.4.** Let $V \subset \mathbb{C}^n$ be an open subset with variables $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. Let $F(z)$ be a function $\alpha$-analytic with respect to $z$ such that $|F| \equiv C$ on $V$ for some constant $C > 0$. Then $F(z) = \lambda \cdot \frac{Q(z)}{Q(z)}$ for some constant $\lambda$ and holomorphic polynomial $Q(z)$.

**Proof.** Let $V_{c,k} = V \cap \{z_j = c_j, j < k, z_j = c_j-1, j > k\}$. By Theorem 2.3 the restriction to each $V_{c,k}$ satisfies $F(z) = \lambda_{c,k} Q_{c,k}(z_k)/Q_{c,k}(z_k)$, $|\lambda_{c,k}| = C$, where each $Q_{c,k}(z_k)$ is a holomorphic polynomial in one variable of degree $< \alpha_k$. Obviously the function $Q(z)$ defined by,

$$Q(z) = Q_{c,k}(z_k), \forall z \in V_{c,k}, \forall c \in \mathbb{C}^{n-1}, 1 \leq k \leq n,$$

is locally bounded and separately a holomorphic polynomial in each variable, in particular of order $\alpha_k - 1$ in the $z_k$-variable, $1 \leq k \leq n$. This implies that $Q(z)$ is jointly a polynomial, i.e. that it has the representation

$$Q(z) = \sum_{\{\beta \in \mathbb{N}^n; \beta_j < \alpha_j, 1 \leq j \leq n\}} a_{\beta} z^\beta, z \in V, \text{ some constants } a_\beta.$$

Similarly we conclude that the function,

$$\lambda(z) = \lambda_{c,k}, \forall z \in V_{c,k}, \forall c \in \mathbb{C}^{n-1}, 1 \leq k \leq n,$$
must be holomorphic and since $|\lambda| \equiv C$ on an open subset, $\lambda \equiv$ constant on $\mathcal{V}$. This means that the function $F(z)$ coincides pointwise with the function $\lambda \cdot \frac{Q(z)}{Q'(z)}$ on the open $\mathcal{V}$. This proves Lemma 5.4.

This also completes the proof of Proposition 5.1.

5.2. $C^4$-smooth CR submanifolds of $\mathbb{C}^n$ near points with a Levi cone condition. The proof of Proposition 5.1 relies heavily upon Lemma 5.2. The adaptation to higher codimension follows from a difficult technical improvement of Lewy’s theorem (see Boggess & Polking [10]), but for our purposes we only require the existence of a family of analytic discs attached near a reference point. This result is known (we have chosen to place this citation in the Appendix) in the case of a $C^4$-smooth CR submanifold $M \subset \mathbb{C}^n$ near a reference point where the Levi cone has nonempty interior.

A consequence of Lemma A.2 is the following.

Remark 5.5. If the convex hull of the image of the Levi form at $p \in M$ contains an interior point, there exists an open subset $\mathcal{V} \subset \mathbb{C}^n$ and a domain $p \in \omega \subset M$ such that (i) $\omega \subset \mathcal{V}$, (ii) $\mathcal{V}$ can be covered by the interiors of analytic discs whose boundaries are attached to $\omega$. This immediately yields for $f \in \mathcal{M}_\alpha(U)$ with a domain $U \subset M$, that there exists a subdomain $\omega \subset U$ with $p \in \omega$ and an associated $\mathcal{V}$ such that $\max_{z \in \mathcal{V}} |f(z)| \geq \max_{z \in \mathcal{V}} |F(z)|$, where $F$ is $\alpha$-analytic on $\mathcal{V}$ and $F|_\omega = f$.

Proposition 5.6. Let $M \subset \mathbb{C}^n, n \geq 2$, be a $C^4$-smooth CR submanifold and let $p \in M$ such that the image of the convex hull of the Levi cone at $p$ has nonempty interior. Let $U \subset M$ be an open neighborhood of $p$ and $f \in \mathcal{M}_\alpha(U)$. Then $|f|$ cannot be constant on a neighborhood of $p$ unless $f$ extends to an $\alpha$-analytic function of constant modulus (in particular of the form $\lambda \frac{Q(z)}{Q'(z)}$ for some holomorphic polynomial $Q(z)$ of degree $< \alpha_j$ in $z_j, 1 \leq j \leq n$) on an open $\mathcal{V} \subset \mathbb{C}^n$ such that $\mathcal{V}$ contains an $M$-neighborhood of $p$.

Proof. By Lemma A.2 we obtain that Lemma 5.2 still holds if we replace $M$ by a $C^4$-smooth generic submanifold of $\mathbb{C}^n$ (of arbitrary positive codimension) and replace $p$ by a point in $M$ such that the Levi cone at $p$ has nonempty interior. The remaining arguments of the proof can now be repeated analogously to the proof of Proposition 5.1.

Example 5.7. Let $M \subset \mathbb{C}^n$ be a generic $C^4$-smooth submanifold and let $p \in M$ be such that the interior of the Levi cone at $p$ is nonempty. Proposition 5.6 shows that every continuous CR function $f$ whose modulus is constant on a neighborhood of a point $p \in M$ must have a holomorphic extension (to some open subset, but not necessarily a full neighborhood of $p$), again of constant modulus. Since holomorphic functions of constant modulus must

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4This generalizes the case of a 1-convex point (by which we mean that the Levi form has at least one positive eigenvalue) for hypersurfaces, see Appendix.
be constant, this implies that \( f \) must reduce to a constant near \( p \) on \( M \). This result is known due to Range [19] for \( C^{\infty} \)-smooth \( M \) and for \( p \) of so-called finite type\(^5\) but with a different proof. We mention that Stoll [20] also deduces a version of this result for \( C^{\infty} \)-smooth boundaries but where the method of proof depends on a result of Hakim & Sibony [13], which in turn depends on working with functions \( C^{\infty} \) on \( \Omega \) and in this proof, the condition on the boundary being \( C^{\infty} \)-smooth is necessary, see Krantz [16, p.5].

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\(^5\)If a generic CR submanifold \( M \subset \mathbb{C}^n \) has real codimension \( d \) then there is associated to every point \( p \in M \) a set of \( d \) integers \( m_1, \ldots, m_d \) called Hörmander numbers (whose definition involves higher order Lie brackets). \( p \) is said to be of finite type if all the Hörmander numbers are finite, see e.g. Boggess [9, p.181].
Appendix A. Preliminaries for Lewy’s theorem

A complex structure $J$ on $T\mathbb{C}^n$ is defined as a real linear map $J : T\mathbb{C}^n \to T\mathbb{C}^n$ such that $J^2 = -Id_2$, specifically $J$ is defined fiberwise on the tangent vector spaces by $\mathbb{R}$-linear maps $J_p : T_p\mathbb{C}^n \to T_p\mathbb{C}^n$. If $M \subset \mathbb{C}^n$ is a submanifold, $T^c_p M := T_p M \cap J_p(T_p M)$ is called the holomorphic tangent space of $M$ at $p$. $J$ maps each $T^c_p M$ to itself thus defines a complex structure on $T^c_p M$. If $T^c_p M$ has constant dimension ($CR$ dimension) for every $p$ then $M$ is called a $CR$ manifold. $M$ is called generic if $T_p\mathbb{C}^n = T_p M \oplus J_p(T_p M/T^c_p M)$.

The $\mathbb{R}$-linear maps $J_p : T^c_p M \to T^c_p M$ have eigenvalues $\pm i$. $J$ extends to a $\mathbb{C}$-linear map on the bundle $\mathbb{C} \otimes T^c M = \bigcup_{p \in M} \mathbb{C} \otimes T^c_p M$ such that the extension again has eigenvalues $\pm i$. This decomposes $\mathbb{C} \otimes T^c M = H^{1,0} M \oplus H^{0,1} M$ namely a $\mathbb{C}$-linear and anti-$\mathbb{C}$-linear part, where we denote by $H^{0,1} M$ the anti-$\mathbb{C}$-linear part.

Definition A.1. A differentiable function $f$ on $M$ is called $CR$ if it is annihilated by any $C^1$-section $X$ of $H^{0,1} M$ over $M$. A distribution $f$ is called $CR$ if $X f = 0$ in the weak sense i.e. $\langle f, X^{\text{adj}} \phi \rangle = 0, \forall \phi \in C^\infty_c(M)$, where $X^{\text{adj}}$ denotes the adjoint.

The Levi form at $p$ of $M$, denoted $\mathcal{L}_p$ is defined as the map $H^{0,1}_p M \to T_p M/T^c_p M$, $\mathcal{L}_p(X) = \frac{1}{2i} [\tilde{X}, \tilde{X}]|_p \mod \mathbb{C} \otimes T^c_p M, X \in H^{0,1}_p M$, where $\tilde{X}$ is any local ambient extension with $\tilde{X}|_p = X$. However for practical reasons one usually identifies the image of the Levi form as a subspace of $N_p M := J(T_p M/T^c_p M)$. $N_p M$ is called the normal space at $p$ and is a real manifold with dimension the same as the real codimension of $M$. Denote by $\Gamma_p (\subseteq N_p M)$ the cone which constitutes the convex hull of the image of $\mathcal{L}_p$ (see Boggess & Polking [10]). For two subcones $\Gamma_1, \Gamma_2$ of $\Gamma_p$ we say that $\Gamma_1$ is smaller that $\Gamma_2$ if $\Gamma_1 \cap S_p$ (where $S_p$ denotes the unit sphere in $N_p M$) is a compact subset of the (relative) interior of $\Gamma_2 \cap S_p$. Boggess & Polking [10] proved a theorem on holomorphic extension of continuous $CR$ functions near a point such that the Levi cone at $p$ has nonempty interior. The domain of extension has the shape of the product of an open set with a cone. The proof involves explicit construction of families of analytic discs by solving a Bishops equation, such that the center of these discs pass each point of an open subset of the given normal cone and simultaneously are
attached sufficiently close to $p$. For our purposes it is precisely the existence of such families of discs which is useful.

Lemma A.2 (Boggess [9, p.207]). Let $M \subset \mathbb{C}^n$ be a $C^l, l \geq 4$, generic embedded CR submanifold and let $p \in M$ be a point such that the Levi cone at $p$ has nonempty interior. Then for every open neighborhood $\omega$ of $p$ and for each cone $\Gamma < \Gamma_p$, there is a neighborhood $\omega_T \subset \omega$ and a positive number $\epsilon_T$ such that each point in $\omega_T + \{\Gamma \cap B_{\epsilon_T}\}$ is contained in the image of an analytic disc whose boundary image is contained in $\omega$.