COMPOSITIONAL ABSTRACTION FOR NETWORKS OF CONTROL SYSTEMS: A DISSIPATIVITY APPROACH

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Abstract. In this paper we propose a compositional scheme for the construction of abstractions for networks of control systems using the interconnection matrix and joint dissipativity-type properties of subsystems and their abstractions. In the proposed framework, the abstraction, itself a control system (possibly with a lower dimension), can be used as a substitution of the original system in the controller design process. Moreover, we provide a procedure for constructing abstractions of a class of nonlinear control systems by using the bounds on the slope of system nonlinearities. We illustrate the proposed results on a network of linear control systems by constructing its abstraction in a compositional way without requiring any condition on the number or gains of the subsystems. We use the abstraction as a substitute to synthesize a controller enforcing a certain linear temporal logic specification. This example particularly elucidates the effectiveness of dissipativity-type compositional reasoning for large-scale systems.

1. Introduction

Modern applications, e.g. power networks, biological networks, internet congestion control, and manufacturing systems, are large-scale networked systems and inherently difficult to analyze and control. Rather than tackling the network as a whole, an approach that severely restricts the capability of existing techniques to deal with many numbers of subsystems, one can develop compositional schemes that provide network-level certifications from main structural properties of the subsystems and their interconnections.

In the past few years, there have been several results on the compositional abstractions of control systems. Early results include compositional abstractions of control systems [TPL04, Fre05, Kv10] which are useful for verification rather than synthesis. Those results employ exact notions of abstractions based on simulation relations [Fre05, Kv10] and simulation maps [TPL04], for which constructive methodologies exist only for rather restricted classes of control systems. In contrast to the exact notions, the compositional approximate abstractions were introduced recently which are useful for the controller synthesis. Examples include compositional construction of finite abstractions of linear and nonlinear control systems [TI08, PPD16] and of infinite abstractions of nonlinear control systems [RZ15, RZ16a] and a class of stochastic hybrid systems [ZRMng]. In those works, the abstraction (finite or infinite with possibly a lower dimension) can be used as a substitution of the original system in the controller design process. The proposed results in [TI08, PPD16, RZ15, RZ16a, ZRMng] use the small-gain type conditions to facilitate the compositional construction of abstractions. The resulting small-gain type requirements intrinsically condition the spectral radius of the interconnection matrix which, in general, depends on the size of the graph and can be violated or deteriorated as the number of subsystems grows [DK04].

In this work we propose a novel compositional framework for the construction of infinite abstractions of networks of control systems using dissipativity theory. First, we adapt the notion of storage function from dissipativity theory [AMP16] to quantify the joint dissipativity-type properties of control subsystems and their abstractions. Given a network of control subsystems and their storage functions, we propose conditions based on the interconnection matrix and joint dissipativity-type properties of subsystems and their abstractions guaranteeing that the network of abstractions quantitatively approximate the behaviours of the network of concrete subsystems. The proposed compositionality conditions can enjoy specific interconnection structures and provide scale-free compositional abstractions for large-scale control systems without requiring any
condition on the number or gains of the subsystems; we illustrate this point with an example in Section 3. Furthermore, we provide a geometric approach on the construction of abstractions for a class of nonlinear control systems and of their corresponding storage functions by using the bounds on the slope of system nonlinearities.

**Related Work.** Compositional construction of infinite abstractions of networks of control systems is also proposed in [RZ15, RZ16a]. While in [RZ15, RZ16a] small-gain type conditions are used to facilitate the compositional construction of abstractions, here we use dissipativity-type conditions. The small-gain type requirements inherently condition the spectral radius of the interconnection matrix which, in general, depends on the size of the graph and can be dissatisfied as the number of subsystems grows [DK04]. On the other hand, this is not necessarily the case with broader dissipativity-type conditions and in fact the compositionality requirements may not condition the number or gains of the subsystems at all when the interconnection matrix enjoys some properties (cf. Section 6). Although the results in [RZ15, RZ16a] provide constructive procedures to determine abstractions of linear control systems, we propose techniques on the construction of abstractions for a class of nonlinear control systems by using the bounds on the slope of systems nonlinearities. The results in [RZ15, RZ16a] assume that the internal input and output space dimensions of each component in a network are equal to the corresponding ones of its abstraction which is not the case in this paper. While the interconnection matrix in [RZ15, RZ16a] is a permutation one, the one in this paper can be any general interconnection matrix.

The recent results in [AMP16, MLAP15] establish only stability or stabilizability of networks of control systems compositionally using dissipativity properties of components. On the other hand, the results here provide construction of abstractions of networks of control systems compositionally using abstractions of components and their joint dissipativity-type properties.

2. Control Systems

2.1. Notation. The sets of nonnegative integer and real numbers are denoted by \( \mathbb{N} \) and \( \mathbb{R} \), respectively. Those symbols are footnoted with subscripts to restrict them in the usual way, e.g. \( \mathbb{R}_{>0} \) denotes the positive real numbers. The symbol \( \mathbb{R}^{n \times m} \) denotes the vector space of real matrices with \( n \) rows and \( m \) columns. The symbols \( \mathbf{1}_n, 0_n, I_n, \) and \( 0_{n \times m} \) denote the vector in \( \mathbb{R}^n \) with all its elements to be one, the zero vector, identity and zero matrices in \( \mathbb{R}^n, \mathbb{R}^{n \times n}, \) and \( \mathbb{R}^{n \times m} \), respectively. For \( a, b \in \mathbb{R} \) with \( a \leq b \), the closed, open, and half-open intervals in \( \mathbb{R} \) are denoted by \( [a, b], [a, b), (a, b), \) and \( ]a, b[ \), respectively. For \( a, b \in \mathbb{N} \) and \( a \leq b \), the symbols \( [a, b], ]a, b[, [a, b[, and \( ]a, b] \) denote the corresponding intervals in \( \mathbb{N} \). Given \( N \in \mathbb{N}_{\geq 1} \), vectors \( x_i, n_i \in \mathbb{N}_{\geq 1} \), and \( i \in [1; N] \), we use \( x = [x_1; \ldots; x_N] \) to denote the concatenated vector in \( \mathbb{R}^n \) with \( n = \sum_{i=1}^N n_i \). Given a vector \( x \in \mathbb{R}^n \), \( \| x \| \) denotes the Euclidean norm of \( x \). Note that given any \( x \in \mathbb{R}^N \), \( x \geq 0 \) if \( x_i \geq 0 \) for all \( i \in [1; N] \). Given a symmetric matrix \( A \), \( \lambda_{\text{max}}(A) \) and \( \lambda_{\text{min}}(A) \) denote maximum and minimum eigenvalues of \( A \). We denote by diag\((M_1, \ldots, M_N)\) the block diagonal matrix with diagonal matrix entries \( M_1, \ldots, M_N \).

Given a function \( f : \mathbb{R}^n \to \mathbb{R}^m \) and \( 0_m \in \mathbb{R}^m \), we simply use \( f \equiv 0 \) to denote that \( f(x) = 0_m \) for all \( x \in \mathbb{R}^n \). Given a function \( f : \mathbb{R}_{\geq 0} \to \mathbb{R}_0 \), the (essential) supremum of \( f \) is denoted by \( \| f \|_\infty := \text{(ess)sup}\{\| f(t) \|, t \geq 0 \} \).

A continuous function \( \gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \), is said to belong to class \( K \) if it is strictly increasing and \( \gamma(0) = 0 \); \( \gamma \) is said to belong to class \( K_\infty \) if \( \gamma \in K \) and \( \gamma(r) \to \infty \) as \( r \to \infty \). A continuous function \( \beta \) is said to belong to class \( K\mathcal{L} \) if, for each fixed \( t \), the map \( \beta(r(t)) \) belongs to class \( K \) with respect to \( r \) and, for each fixed nonzero \( r \), the map \( \beta(r, t) \) is decreasing with respect to \( t \) and \( \beta(r, t) \to 0 \) as \( t \to \infty \).

2.2. Control systems. The class of control systems studied in this paper is formalized in the following definition.

**Definition 2.1.** A control system \( \Sigma \) is a tuple \( \Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, U, W, f, \mathbb{R}^q, \mathbb{R}^p, h_1, h_2) \), where \( \mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \mathbb{R}^q, \) and \( \mathbb{R}^{p2} \) are the state, external input, internal input, external output, and internal output spaces, respectively, and
• \( \mathcal{U} \) and \( \mathcal{W} \) are subsets of the sets of all measurable functions of time, from open intervals in \( \mathbb{R} \) to \( \mathbb{R}^m \) and \( \mathbb{R}^p \), respectively;
• \( f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^n \) is a continuous map satisfying the following Lipschitz assumption: for every compact set \( \mathcal{D} \subset \mathbb{R}^n \), there exists a constant \( Z \in \mathbb{R}_{>0} \) such that \( \|f(x, u, w) - f(y, u, w)\| \leq Z\|x - y\| \) for all \( x, y \in \mathcal{D} \), all \( u \in \mathbb{R}^m \), and all \( w \in \mathbb{R}^p \);
• \( h_1 : \mathbb{R}^n \to \mathbb{R}^q \) is the external output map;
• \( h_2 : \mathbb{R}^p \to \mathbb{R}^q \) is the internal output map.

A locally absolutely continuous curve \( \xi : [a, b] \to \mathbb{R}^n \) is a state trajectory of \( \Sigma \) if there exist input trajectories \( v \in \mathcal{U} \) and \( \omega \in \mathcal{W} \) satisfying:

\[
\Sigma : \begin{cases}
\dot{\xi}(t) = f(\xi(t), v(t), \omega(t)), \\
\zeta_1(t) = h_1(\xi(t)), \\
\zeta_2(t) = h_2(\xi(t)),
\end{cases}
\tag{2.1}
\]

for almost all \( t \in ]a, b[ \). We call the tuple \((\xi, \zeta_1, \zeta_2, v, \omega)\) a trajectory of \( \Sigma \), consisting of a state trajectory \( \xi \), output trajectories \( \zeta_1 \) and \( \zeta_2 \), and input trajectories \( v \) and \( \omega \), that satisfies (2.1). We also denote by \( \xi_{x,v,\omega}(t) \) the state reached at time \( t \) under the inputs \( v \in \mathcal{U}, \omega \in \mathcal{W} \) from the initial condition \( x = \xi_{x,v,\omega}(0) \); the state \( \xi_{x,v,\omega}(t) \) is uniquely determined due to the assumptions on \( f \) [Son98]. We also denote by \( \zeta_{1,v,\omega}(t) \) and \( \zeta_{2,v,\omega}(t) \) the corresponding external and internal output value of \( \xi_{x,v,\omega}(t) \), respectively i.e. \( \zeta_{1,v,\omega}(t) = h_1(\xi_{x,v,\omega}(t)) \) and \( \zeta_{2,v,\omega}(t) = h_2(\xi_{x,v,\omega}(t)) \).

We call \( \zeta_1 \) an external output trajectory, \( \zeta_2 \) an internal output trajectory, \( v \) an external input trajectory, and \( \omega \) an internal input trajectory mainly because \( \zeta_2 \) and \( \omega \) are used only for the interconnection purposes and \( \zeta_1 \) and \( v \) remain available after any interconnection; see Definition 4.1 later for more detailed information.

Remark 2.2. If the control system \( \Sigma \) does not have internal inputs and outputs, the definition of control systems in Definition 2.1 reduces to tuple \( \Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathcal{U}, f, \mathbb{R}^q, h) \) and the map \( f \) becomes \( f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \). Correspondingly, equation (2.1) describing the evolution of system trajectories reduces to:

\[
\Sigma : \begin{cases}
\dot{\xi}(t) = f(\xi(t), v(t)), \\
\zeta(t) = h(\xi(t)).
\end{cases}
\]

3. Storage and Simulation Functions

First, we introduce a notion of so-called storage functions, adapted from the notion of storage functions from dissipativity theory [Wil72, AMP16]. While the notion of storage functions in [Wil72, AMP16] characterizes the correlation of inputs and outputs of a single control system, the proposed notion of storage functions here characterizes the joint correlation of inputs and outputs of two different control systems. In the case that two control systems are the same and have only internal inputs and outputs, our notion of storage functions recovers the one of incremental storage functions introduced in [SS07].

Definition 3.1. Let \( \Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \mathcal{U}, \mathcal{W}, f, \mathbb{R}^q_1, \mathbb{R}^q_2, h_1, h_2) \) and \( \hat{\Sigma} = (\hat{\mathbb{R}}^n, \hat{\mathbb{R}}^m, \hat{\mathbb{R}}^p, \hat{\mathcal{U}}, \hat{\mathcal{W}}, \hat{f}, \hat{\mathbb{R}}^q_1, \hat{\mathbb{R}}^q_2, \hat{h}_1, \hat{h}_2) \) be two control systems with the same external output space dimension. A continuously differentiable function \( V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0} \) is called a storage function from \( \hat{\Sigma} \) to \( \Sigma \) if there exist \( \alpha, \eta \in \mathcal{K}_\infty \), \( \rho_{ext} \in \mathcal{K}_\infty \cup \{0\} \), some matrices \( W, \hat{W}, H \) of appropriate dimensions, and some symmetric matrix \( X \) of appropriate dimension with conformal block partitions \( X^{iq} \), \( i, j \in [1; 2] \), where \( X^{22} \leq 0 \), such that for any \( x \in \mathbb{R}^n \) and \( \hat{x} \in \mathbb{R}^n \) one has

\[
\alpha(||h_1(x) - \hat{h}_1(\hat{x})||) \leq V(x, \hat{x}),
\tag{3.1}
\]
and \( \forall \hat{u} \in \mathbb{R}^m \exists u \in \mathbb{R}^m \) such that \( \forall \hat{w} \in \mathbb{R}^p \ \forall w \in \mathbb{R}^p \) one obtains

\[
\nabla V (x, \hat{x})^T \left[ f(x, u, w) \right] 
\leq -\eta(V(x, \hat{x})) + \rho_{\text{ext}}(\|\hat{u}\|) + \begin{bmatrix} \begin{bmatrix} X^{11} & -X^{12} \\ X^{21} & X^{22} \end{bmatrix} & \begin{bmatrix} W \hat{w} - \hat{W} \hat{w} \\ \hat{h}_2(x) - H\hat{h}_2(\hat{x}) \end{bmatrix} \end{bmatrix}^T. 
\]

We use notation \( \hat{\Sigma} \preceq \Sigma \) if there exists a storage function \( V \) from \( \hat{\Sigma} \) to \( \Sigma \). Control system \( \hat{\Sigma} \) (possibly with \( \hat{u} < n \)) is called an abstraction of \( \Sigma \). There are several key differences between the notion of storage function here and the corresponding one of simulation function in \[GP09\] Definition 2. Definition 2 in \[GP09\] requires internal signals \( w, \hat{w} \) and \( h_2(x), \hat{h}_2(\hat{x}) \) to live in the same spaces, respectively, which is not necessarily the case here. Moreover, the choice of input \( u \) here satisfying (3.2) only depends on \( x, \hat{x} \), and \( \hat{u} \), whereas in \[RZ16a\] Definition 2 it also depends on internal input \( \hat{w} \). Finally, we should point out that if in \[GP09\] Definition 2 \( \mu(s) := s^TPs \), for any \( s \in \mathbb{R}^p \), and some positive definite matrix \( P \), then the simulation function in \[GP09\] Definition 2 is also a storage function as in Definition 3.1 with \( W = \hat{W} = I_p, X^{11} = P \), and the rest of conformal block partitions of \( X \) are zero.

Now, we recall the notion of simulation functions introduced in \[GP09\] with some modifications.

**Definition 3.2.** Let \( \Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathcal{U}, f, \mathcal{R}, h) \) and \( \hat{\Sigma} = (\mathbb{R}^n, \mathbb{R}^m, \hat{\mathcal{U}}, \hat{f}, \mathcal{R}, \hat{h}) \) be two control systems. A continuously differentiable function \( V : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0} \) is called a simulation function from \( \hat{\Sigma} \) to \( \Sigma \) if there exist \( \alpha, \eta \in \mathcal{K}_{\infty} \) and \( \rho_{\text{ext}} \in \mathcal{K}_{\infty} \cup \{0\} \) such that for any \( x \in \mathbb{R}^n \) and \( \hat{x} \in \mathbb{R}^n \) one has

\[
\alpha(||h(x) - \hat{h}(\hat{x})||) \leq V(x, \hat{x}), 
\]

and \( \forall \hat{u} \in \mathbb{R}^m \exists u \in \mathbb{R}^m \) such that

\[
\nabla V (x, \hat{x})^T \left[ f(x, u) \right] 
\leq -\eta(V(x, \hat{x})) + \rho_{\text{ext}}(\|\hat{u}\|).
\]

We use notation \( \hat{\Sigma} \preceq_\Sigma \Sigma \) if there exists a simulation function \( V \) from \( \hat{\Sigma} \) to \( \Sigma \).

Let us point out the differences between Definition 3.2 here and \[GP09\] Definition 1]. Here, for the sake of brevity, we simply assume that for every \( x, \hat{x}, \hat{u} \), there exists \( u \) so that (3.4) holds. Whereas in \[GP09\] Definition 1 the authors use an interface function \( k : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m \) to feed the input \( u = k(x, \hat{x}, \hat{u}) \) enforcing (3.4). Function \( \alpha \) in \[GP09\] Definition 1 is assumed to be the identity. Furthermore, we frame the decay condition (3.4) in so-called “dissipative” form, while in \[GP09\] Definition 1 the decay condition is given in so-called “implication” form.

Note that the notions of storage functions in Definition 3.1 and simulation functions in Definition 3.2 are not comparable in general. The former is defined for control systems with internal inputs and outputs while the latter is defined only for control systems without internal inputs and outputs. One can readily verify that both notions coincide for control systems without internal inputs and outputs.

The next theorem shows the importance of the existence of a simulation function by quantifying the error between the output behaviours of \( \Sigma \) and the ones of its abstraction \( \hat{\Sigma} \).

**Theorem 3.3.** Let \( \Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathcal{U}, f, \mathcal{R}, h) \) and \( \hat{\Sigma} = (\hat{\mathbb{R}}^n, \hat{\mathbb{R}}^m, \hat{\mathcal{U}}, \hat{f}, \mathcal{R}, \hat{h}) \). Suppose \( V \) is a simulation function from \( \hat{\Sigma} \) to \( \Sigma \). Then, there exist a KL function \( \vartheta \) such that for any \( \hat{v} \in \hat{\mathcal{U}}, x \in \mathbb{R}^n \), and \( \hat{x} \in \hat{\mathbb{R}}^n \), there exists \( v \in \mathcal{U} \) such that the following inequality holds for any \( t \in \mathbb{R}_{\geq 0} : 
\)

\[
\|\hat{\varsigma}_{\text{ext}}(t) - \hat{\varsigma}_\vartheta(t)\| 
\leq \alpha^{-1} (2\vartheta (V(x, \hat{x}), t)) + \alpha^{-1} (2\rho_{\text{ext}}(\|\hat{v}\|_\infty))). 
\]

The proof of Theorem 3.3 is similar to the one of Theorem 3.5 in \[ZRMg\] and is omitted due to lack of space.

Let us illustrate the importance of the existence of a simulation function, correspondingly inequality (3.5), on a simple example. Assume we are given a control system \( \Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathcal{U}, f, \mathcal{R}, h) \) and interested in
computing a control input \( v \) to keep the output \( \zeta_{xv} \) always inside a safe set \( D \subset \mathbb{R}^q \). Instead, one can compute a control input \( \hat{v} \) for the abstraction \( \hat{\Sigma} \) keeping the output \( \zeta_{x\hat{v}} \) always inside \( D \) which is potentially easier due to a lower dimension of \( \hat{\Sigma} \). The existence of a simulation function from \( \hat{\Sigma} \) to \( \Sigma \) and, hence, the inequality (3.5) imply that there exists control input \( v \) such that \( \zeta_{xv} \) is always inside \( D^\varepsilon \), where \( \varepsilon = \alpha^{-1}(2\vartheta(V(x, \hat{x}, 0)) + \alpha^{-1}(2\eta^{-1}(2\rho_{\text{ext}}(\|\hat{v}\|_\infty)))) \) and \( D^\varepsilon = \{ y \in \mathbb{R}^p \mid \inf_{\hat{y} \in D} \|y - \hat{y}\| \leq \varepsilon \} \). Note that one can choose initial conditions \( x \in \mathbb{R}^n \) and \( \hat{x} \in \mathbb{R}^n \) to minimize the first term in \( \varepsilon \) and, hence, to have a smaller error in the satisfaction of the desired property.

**Remark 3.4.** Note that if \( \alpha^{-1} \) and \( \eta^{-1} \) satisfy the triangle inequality (i.e., \( \alpha^{-1}(a + b) \leq \alpha^{-1}(a) + \alpha^{-1}(b) \) and \( \eta^{-1}(a + b) \leq \eta^{-1}(a) + \eta^{-1}(b) \) for all \( a, b \in \mathbb{R}_{\geq 0} \)), one can divide all the coefficients 2, appearing in the right hand side of (3.5), by factor 2 to get a less conservative upper bound.

**Remark 3.5.** Note that if one is given an interface function \( k : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^\bar{n} \to \mathbb{R}^m \) that maps every \( x, \hat{x}, \hat{\hat{u}} \) to an input \( u = k(x, \hat{x}, \hat{\hat{u}}) \) such that (4.4) is satisfied (similar to [GP09, Definition 1]), then input \( v \) realizing (3.5) is readily given by \( v = k(\xi, \hat{\xi}, \hat{\hat{v}}) \). In Section 5 we show how the map \( k \) can be constructed for a class of nonlinear control systems.

## 4. Compositionality Result

In this section, we analyze networks of control systems and show how to construct their abstractions together with the corresponding simulation functions by using storage functions for the subsystems. The definition of the network of control systems is based on the notion of interconnected systems described in [AMP16].

### 4.1. Interconnected control systems

Here, we define the **interconnected control system** as the following.

**Definition 4.1.** Consider \( N \in \mathbb{N}_{\geq 1} \) control subsystems \( \Sigma_i = (\mathbb{R}^{n_i}, \mathbb{R}^{m_i}, \mathbb{R}^{p_i}, \mathcal{U}_i, \mathcal{W}_i, f_i, \mathbb{R}^{u_i}, \mathbb{R}^{w_i}, h_{11}, h_{2i}), \) \( i \in [1, N] \), and a static matrix \( M \) of an appropriate dimension defining the coupling of these subsystems. The interconnected control system \( \Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathcal{U}, f, \mathbb{R}^u, h) \), denoted by \( \mathcal{I}(\Sigma_1, \ldots, \Sigma_N) \), follows by \( n = \sum_{i=1}^N n_i, m = \sum_{i=1}^N m_i, q = \sum_{i=1}^N q_{ii}, \) and functions

\[
\begin{align*}
  f(x, u) &:= [f_1(x_1, u_1, w_1); \ldots; f_N(x_N, u_N, w_N)], \\
  h(x) &:= [h_{11}(x_1); \ldots; h_{1N}(x_N)],
\end{align*}
\]

where \( u = [u_1; \ldots; u_N], x = [x_1; \ldots; x_N] \) and with the internal variables constrained by

\[
[w_1; \ldots; w_N] = M[h_{21}(x_1); \ldots; h_{2N}(x_N)].
\]

An interconnection of \( N \) control subsystems \( \Sigma_i \) is illustrated schematically in Figure 1.

![Figure 1](image-url)
4.2. Composing simulation functions from storage functions. We assume that we are given \( N \) control subsystems \( \Sigma_i = (R^{n_i}, R^{m_i}, \mathcal{U}_i, \mathcal{W}_i, f_i, \eta_i, h_{i1}, h_{i2}) \), together with their corresponding abstractions \( \hat{\Sigma}_i = (R^{\hat{n}_i}, R^{\hat{m}_i}, \hat{\mathcal{U}}_i, \hat{\mathcal{W}}_i, \hat{f}_i, \hat{\eta}_i, \hat{h}_{i1}, \hat{h}_{i2}) \) and with storage functions \( \hat{V}_i \) from \( \hat{\Sigma}_i \) to \( \Sigma_i \). We use \( \alpha_i, \eta_i, \rho_i, H_i, W_i, \hat{x}_i, X_i, X_i^{11}, X_i^{12}, X_i^{21}, \) and \( X_i^{22} \) to denote the corresponding functions, matrices, and their corresponding conformal block partitions appearing in Definition 3.1.

The next theorem, one of the main results of the paper, provides a compositional approach on the construction of abstractions of networks of control systems and that of the corresponding simulation functions.

**Theorem 4.2.** Consider the interconnected control system \( \Sigma = \mathcal{I}(\Sigma_1, \ldots, \Sigma_N) \) induced by \( N \in \mathbb{N}_{\geq 1} \) control subsystems \( \Sigma_i \) and the coupling matrix \( M \). Suppose each control subsystem \( \Sigma_i \) admits an abstraction \( \hat{\Sigma}_i \) with the corresponding storage function \( \hat{V}_i \). If there exist \( \mu_i \geq 0, i \in [1; N] \), and matrix \( M \) of appropriate dimension such that the matrix (in)equality

\[
\begin{bmatrix}
WM \\
I_q
\end{bmatrix}^T X(\mu_1 X_1, \ldots, \mu_N X_N) \begin{bmatrix}
WM \\
I_q
\end{bmatrix} \leq 0,
\]

are satisfied, where \( \hat{q} = \sum_{i=1}^{N} q_{2i} \) and

\[
W := \text{diag}(W_1, \ldots, W_N), \quad \hat{W} := \text{diag}(\hat{W}_1, \ldots, \hat{W}_N), \quad H := \text{diag}(H_1, \ldots, H_N),
\]

then

\[
V(x, \hat{x}) := \sum_{i=1}^{N} \mu_i V_i(x_i, \hat{x}_i)
\]

is a simulation function from the interconnected control system \( \hat{\Sigma} = \mathcal{I}(\hat{\Sigma}_1, \ldots, \hat{\Sigma}_N) \), with the coupling matrix \( M \), to \( \Sigma \).

**Proof.** First we show that inequality (3.3) holds for some \( \mathcal{K}_\infty \) function \( \alpha \). For any \( x = [x_1; \ldots; x_N] \in \mathbb{R}^n \) and \( \hat{x} = [\hat{x}_1; \ldots; \hat{x}_N] \in \mathbb{R}^n \), one gets:

\[
\| h(x) - \hat{h}(\hat{x}) \| = \| [h_{11}(x_1); \ldots; h_{1N}(x_N)] - [\hat{h}_{11}(\hat{x}_1); \ldots; \hat{h}_{1N}(\hat{x}_N)] \|
\leq \sum_{i=1}^{N} \| h_{1i}(\hat{x}_i) - \hat{h}_{1i}(x_i) \| \leq \sum_{i=1}^{N} \alpha_i^{-1}(V_i(x_i, \hat{x}_i)) \leq \bar{\alpha}(V(x, \hat{x})),
\]

where \( \bar{\alpha} \) is a \( \mathcal{K}_\infty \) function defined as

\[
\bar{\alpha}(s) := \begin{cases}
\max_{\hat{s} \geq 0} \sum_{i=1}^{N} \alpha_i^{-1}(s_i) \\
s.t. \quad \mu^T \hat{s} = s,
\end{cases}
\]

where \( \hat{s} = [s_1; \ldots; s_N] \in \mathbb{R}^N \) and \( \mu = [\mu_1; \ldots; \mu_N] \). By defining the \( \mathcal{K}_\infty \) function \( \alpha(s) = \bar{\alpha}^{-1}(s), \forall s \in \mathbb{R}_{\geq 0} \), one obtains

\[
\alpha(\| h(x) - \hat{h}(\hat{x}) \|) \leq V(x, \hat{x}),
\]

satisfying inequality (3.3). Now we show that inequality (3.4) holds as well. Consider any \( x = [x_1; \ldots; x_N] \in \mathbb{R}^n \), \( \hat{x} = [\hat{x}_1; \ldots; \hat{x}_N] \in \mathbb{R}^n \), and \( \hat{u} = [\hat{u}_1; \ldots; \hat{u}_N] \in \mathbb{R}^n \). For any \( i \in [1; N] \), there exists \( u_i \in \mathbb{R}^{m_i} \), consequently,
a vector $u = [u_1; \ldots; u_N] \in \mathbb{R}^n$, satisfying (3.2) for each pair of subsystems $\Sigma_i$ and $\hat{\Sigma}_i$ with the internal inputs given by $[u_1; \ldots; u_N] = M[h_21(x_1); \ldots; h_{2N}(x_N)]$ and $[\hat{u}_1; \ldots; \hat{u}_N] = M[\hat{h}_21(\hat{x}_1); \ldots; \hat{h}_{2N}(\hat{x}_N)]$. We derive the following inequality

$$
\dot{V}(x, \hat{x}) = \sum_{i=1}^{N} \mu_i \dot{V}_i(x_i, \hat{x}_i) 
\leq \sum_{i=1}^{N} \mu_i \left( -\eta_i(V_i(x_i, \hat{x}_i)) + \rho_{\text{ext}}(\|\hat{u}_i\|) \right) + \left[ \begin{array}{c}
W_{ij} w_i - \hat{W}_{ij} \hat{w}_i \\
h_{21}(x_1) - H_1 \hat{h}_{21}(\hat{x}_1) \\
\vdots \\
h_{2N}(x_N) - H_N \hat{h}_{2N}(\hat{x}_N)
\end{array} \right]^T X(\mu_1 X_1, \ldots, \mu_N X_N) \left[ \begin{array}{c}
W_{ij} w_i - \hat{W}_{ij} \hat{w}_i \\
h_{21}(x_1) - H_1 \hat{h}_{21}(\hat{x}_1) \\
\vdots \\
h_{2N}(x_N) - H_N \hat{h}_{2N}(\hat{x}_N)
\end{array} \right].
$$

Using conditions (4.1) and (4.2) and the definition of matrices $W$, $\hat{W}$, $H$, and $X$ in (4.3) and (4.4), the inequality (4.5) can be rewritten as

$$
\dot{V}(x, \hat{x}) \leq \sum_{i=1}^{N} -\mu_i \eta_i(V_i(x_i, \hat{x}_i)) + \sum_{i=1}^{N} \mu_i \rho_{\text{ext}}(\|\hat{u}_i\|) 
+ \left[ \begin{array}{c}
h_{21}(x_1) - H_1 \hat{h}_{21}(\hat{x}_1) \\
\vdots \\
h_{2N}(x_N) - H_N \hat{h}_{2N}(\hat{x}_N)
\end{array} \right]^T X(\mu_1 X_1, \ldots, \mu_N X_N) \left[ \begin{array}{c}
h_{21}(x_1) - H_1 \hat{h}_{21}(\hat{x}_1) \\
\vdots \\
h_{2N}(x_N) - H_N \hat{h}_{2N}(\hat{x}_N)
\end{array} \right].
$$

Define the functions

$$
\eta(s) := \min_{s \geq 0} \sum_{i=1}^{N} \mu_i \eta_i(s_i) \quad \text{s.t.} \quad \mu^T \bar{s} = s, \tag{4.6a}
$$

$$
\rho_{\text{ext}}(s) := \max_{s \geq 0} \sum_{i=1}^{N} \mu_i \rho_{\text{ext}}(s_i) \quad \text{s.t.} \quad \|\bar{s}\| = s, \tag{4.6b}
$$

where $\eta \in \mathcal{K}_\infty$ and $\rho_{\text{ext}} \in \mathcal{K}_\infty \cup \{0\}$. By construction, we readily have

$$
\dot{V}(x, \hat{x}) \leq -\eta(V(x, \hat{x})) + \rho_{\text{ext}}(\|\hat{u}\|),
$$

which satisfies inequality (3.4). Hence, we conclude that $V$ is a simulation function from $\hat{\Sigma}$ to $\Sigma$. \hfill \Box

Remark 4.3. Let us assume, for each subsystem $\Sigma_i$ is single-input-single-output. Under these assumptions, analytical feasibility conditions for matrix inequality (4.1) can be derived for special interconnection matrices $M$ including negative and positive feedback interconnection,
skew symmetric interconnection, negative feedback cyclic interconnection, and finally extension to cactus graphs as provided in details in [AMP16, Chapter 2].

5. Abstraction Synthesis for a Class of Nonlinear Control Systems

In this section, we concentrate on a specific class of nonlinear control systems $\Sigma$ and quadratic storage functions $V$. In the first part, we formally define the specific class of nonlinear control systems with which we deal in this section. In the second part, we assume that an abstraction $\hat{\Sigma}$ is given and we provide conditions under which $V$ is a storage function. In the third part it is shown geometrically how to construct the abstraction $\hat{\Sigma}$ together with the storage function $V$. Finally, we discuss the feasibility of a key condition based on which the results of this section hold.

5.1. A class of nonlinear control systems. The class of nonlinear control systems, considered in this section, is given by

$$
\dot{\xi} = A\xi + E\varphi(F\xi) + Bv + D\omega,
\begin{align*}
\zeta_1 &= C_1\xi, \\
\zeta_2 &= C_2\xi,
\end{align*}
$$

(5.1)

where $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ satisfies

$$
a \leq \frac{\varphi(v) - \varphi(w)}{v - w} \leq b, \ \forall v, w \in \mathbb{R}, v \neq w,
$$

(5.2)

for some $a \in \mathbb{R}$ and $b \in \mathbb{R}_{>0} \cup \{\infty\}$, $a \leq b$, and

$$
A \in \mathbb{R}^{n \times n}, E \in \mathbb{R}^{n \times 1}, F \in \mathbb{R}^{1 \times n}, B \in \mathbb{R}^{n \times m}, D \in \mathbb{R}^{n \times p}, C_1 \in \mathbb{R}^{q_1 \times n}, C_2 \in \mathbb{R}^{q_2 \times n}.
$$

We use the tuple

$$
\Sigma = (A, B, C_1, C_2, D, E, F, \varphi),
$$

to refer to the class of control systems of the form (5.1).

Remark 5.1. If $\varphi$ in (5.1) is linear including the zero function (i.e. $\varphi \equiv 0$) or $E$ is a zero matrix, one can remove or push the term $E\varphi(F\xi)$ to $A\xi$ and, hence, the tuple representing the class of control systems reduces to the linear one $\Sigma = (A, B, C_1, C_2, D)$. Therefore, every time we use the tuple $\Sigma = (A, B, C_1, C_2, D, E, F, \varphi)$, it implicitly implies that $\varphi$ is nonlinear and $E$ is nonzero.
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\[
\begin{bmatrix}
(A + BK)^T \hat{M} + \hat{M} (A + BK) & \hat{M} Z (BL_1 + E) \\
Z^T \hat{M} & 0 & 0 \\
(BL_1 + E)^T \hat{M} & 0 & 0 \\
\end{bmatrix} \preceq
\begin{bmatrix}
-\hat{\kappa} M + C_2^T X^{22} C_2 & C_2^T X^{21} & -F^T \\
X^{12} C_2 & X^{11} & 0 \\
-F & 0 & \frac{2}{b} \\
\end{bmatrix}
\]

(5.5)

Similar to what is shown in [AK01a], without loss of generality, we can assume \( a = 0 \) in (5.2) for the class of nonlinear control systems in (5.1). If \( a \neq 0 \), one can define a new function \( \tilde{\varphi}(r) := \varphi(r) - ar \) which satisfies (5.2) with \( \tilde{a} = 0 \) and \( \tilde{b} = b - a \), and rewrite (5.1) as

\[
\Sigma : \begin{cases}
\dot{\xi} = \tilde{A}\xi + E \tilde{\varphi}(F\xi) + B\upsilon + D\omega, \\
\zeta_1 = C_1\xi, \\
\zeta_2 = C_2\xi,
\end{cases}
\]

where \( \tilde{A} = A + aEF \).

**Remark 5.2.** For simplicity of derivations, we restrict ourselves to systems with a single nonlinearity as in (5.1). However, it would be straightforward to obtain analogous results for systems with multiple nonlinearities as

\[
\Sigma : \begin{cases}
\dot{\xi} = A\xi + \sum_{i=1}^{M} E_i \varphi_i(F_i\xi) + B\upsilon + D\omega, \\
\zeta_1 = C_1\xi, \\
\zeta_2 = C_2\xi,
\end{cases}
\]

where \( \varphi_i : \mathbb{R} \to \mathbb{R} \) satisfies (5.2) for some \( a_i \in \mathbb{R} \) and \( b_i \in \mathbb{R}_{>0} \cup \{\infty\} \), \( E_i \in \mathbb{R}^{n \times 1} \), and \( F_i \in \mathbb{R}^{1 \times n} \), for any \( i \in [1; M] \). Furthermore, the proposed results here can also be extended to systems with multivariable nonlinearities satisfying a multivariable sector property along the same lines as in [FA03] in the context of observer design.

Note that the class of nonlinear control systems in (5.1) and Remark 5.2 has been used widely to model many physical systems including active magnetic bearing [AK01a], flexible joint robot [FA03], fuel cell [AGPV03], the power generators [Sch04], underwater vehicles [AAFK01], and so on.

### 5.2. Quadratic storage functions

Here, we consider a quadratic storage function of the form

\[
V(x, \dot{x}) = (x - P\hat{x})^T \hat{M}(x - P\hat{x}),
\]

(5.3)

where \( P \) and \( \hat{M} \succ 0 \) are some matrices of appropriate dimensions. In order to show that \( V \) in (5.3) is a storage function from an abstraction \( \hat{\Sigma} \) to a concrete system \( \Sigma \), we require the following key assumption on \( \Sigma \).

**Assumption 1.** Let \( \Sigma = (A, B, C_1, C_2, D, E, F, \varphi) \). Assume that for some constant \( \hat{\kappa} \in \mathbb{R}_{>0} \) there exist matrices \( \hat{M} \succ 0, K, L_1, Z, W, X^{11}, X^{12}, X^{21}, \) and \( X^{22} \preceq 0 \) of appropriate dimensions such that the matrix equality

\[
D = ZW,
\]

(5.4)

and inequality (5.5) hold, where \( 0 \)'s in (5.5) denote zero matrices of appropriate dimensions.

The next rather straightforward result provides a necessary and sufficient geometric condition for the existence of matrix \( W \) appearing in condition (5.4).

**Lemma 5.3.** Given \( D \) and \( Z \), condition (5.4) is satisfied for some matrix \( W \) if and only if

\[
\text{im } D \subseteq \text{im } Z.
\]

(5.6)
Before providing the proof, we point out that there always exist matrices \( X \) satisfying (5.3) for subsystems of the form (5.1). In particular, assume we are given \( N \) control subsystems \( \Sigma_i = (A_i, B_i, C_{i1}, C_{i2}, D_i, E_i, F_i, \varphi_i) \), \( i \in [1; N] \). For any \( i \in [1; N] \), one can consider matrices \( X_i \) in the LMI (4.1) as decision variables instead of being fixed and \( \mu_i = 1 \) without loss of generality, and solve the combined feasibility problems (5.5) and (4.1). Although the combined feasibility problem may be huge for large networks and solving it directly may be intractable, one can use the alternating direction method of multipliers (ADMM) to solve the feasibility problem in a distributed fashion along the same lines proposed in [MLAP15].

Now, we provide one of the main results of this section showing under which conditions \( V \) in (5.3) is a storage function.

**Theorem 5.5.** Let \( \Sigma = (A, B, C_1, C_2, D, E, F, \varphi) \) and \( \hat{\Sigma} = (\hat{A}, \hat{B}, \hat{C}_1, \hat{C}_2, \hat{D}, \hat{E}, \hat{F}, \varphi) \) with \( q_1 = \hat{q}_1 \). Suppose Assumption (4.1) holds and that there exist matrices \( P, Q, H, L_1, \) and \( W \) such that

\[
\begin{align*}
AP &= P\hat{A} - BQ \\
C_1P &= \hat{C}_1 \\
X^{12}C_2P &= X^{12}H\hat{C}_2 \\
X^{22}C_2P &= X^{22}H\hat{C}_2 \\
FP &= \hat{F} \\
E &= P\hat{E} - B(L_1 - L_2) \\
P\hat{D} &= Z\hat{W},
\end{align*}
\]

hold. Then, function \( V \) defined in (5.3) is a storage function from \( \hat{\Sigma} \) to \( \Sigma \).

Before providing the proof, we point out that there always exist matrices \( \{\hat{A}, \hat{B}, \hat{C}_1, \hat{C}_2, \hat{D}, \hat{E}, \hat{F}\} \) satisfying (5.8) if \( P = I_n \) implying that \( \hat{n} = n \). Naturally, it is better to have the simplest abstraction \( \hat{\Sigma} \) and, therefore, one should seek a \( P \) with \( \hat{n} \) as small as possible. We elaborate on the construction of \( P \) satisfying (5.8) in details in the next subsection.
Proof. From (5.8b) and for all \( x \in \mathbb{R}^n, \dot{x} \in \mathbb{R}^n \), we have \( \|C_1 x - \dot{C}_1 \dot{x}\|^2 = (x - P \dot{x})^T C_1^T C_1 (x - P \dot{x}) \). It can be readily verified that \( \frac{\lambda_{\min}(\widehat{M})}{\lambda_{\max}(C_1^T C_1)} \|C_1 x - \dot{C}_1 \dot{x}\|^2 \leq V(x, \dot{x}) \) holds for all \( x \in \mathbb{R}^n, \dot{x} \in \mathbb{R}^n \) implying that inequality (3.1) holds with \( \alpha(r) = \frac{\lambda_{\min}(\widehat{M})}{\lambda_{\max}(C_1^T C_1)} r^2 \) for any \( r \in \mathbb{R}_{\geq 0} \). We proceed with showing that the inequality (3.2) holds. Note that
\[
\frac{\partial V(x, \dot{x})}{\partial x} = 2(x - P \dot{x})^T \widehat{M}, \quad \frac{\partial V(x, \dot{x})}{\partial \dot{x}} = -2(x - P \dot{x})^T \widehat{M} P.
\] (5.9)

Given any \( x \in \mathbb{R}^n, \dot{x} \in \mathbb{R}^n \), and \( \dot{u} \in \mathbb{R}^m \), we choose \( u \in \mathbb{R}^m \) via the following linear interface function:
\[
u = k(x, \dot{x}, \dot{u}) := K(x - P \dot{x}) + Q \dot{x} + \bar{R} \dot{u} + L_1 \varphi(Fx) - L_2 \varphi(FP \dot{x}),
\] (5.10)
for some matrix \( \bar{R} \) of appropriate dimension.

By using the equations (5.8a), (5.8c), and (5.8g) and the definition of the interface function in (5.10), we get
\[
Ax + E \varphi(Fx) + Bk(x, \dot{x}, \dot{u}) + D \dot{w} + P(\dot{A} \dot{x} + \dot{E} \varphi(\dot{F} \dot{x}) + \dot{B} \dot{u} + \dot{D} \dot{w}) =
(A + BK)(x - P \dot{x}) + (D \dot{w} - PD \dot{w}) + (B \bar{R} - P \bar{B}) \dot{u} + (E + BL_1)(\varphi(Fx) - \varphi(FP \dot{x})).
\]
Using (5.9), (5.4), and (5.8g), we obtain the following expression for \( \dot{V}(x, \dot{x}) \):
\[
\dot{V}(x, \dot{x}) = 2(x - P \dot{x})^T \widehat{M} [(A + BK)(x - P \dot{x}) + (ZW \dot{w} - \dot{W} \dot{w}) + (B \bar{R} - P \bar{B}) \dot{u} + (E + BL_1)(\varphi(Fx) - \varphi(FP \dot{x}))].
\]
From the slope restriction (5.2), one obtains
\[
\varphi(Fx) - \varphi(FP \dot{x}) = \delta(Fx - FP \dot{x}) = \delta F(x - P \dot{x}),
\] (5.11)
where \( \delta \) is a constant and depending on \( x \) and \( \dot{x} \) takes values in the interval \([0, b]\). Using (5.11), the expression for \( \dot{V}(x, \dot{x}) \) reduces to:
\[
\dot{V}(x, \dot{x}) = 2(x - P \dot{x})^T \widehat{M} [(A + BK) + \delta(E + BL_1)F](x - P \dot{x}) + Z(W \dot{w} - \dot{W} \dot{w}) + (B \bar{R} - P \bar{B}) \dot{u}].
\]
Using Young’s inequality [Yon12] as
\[
\frac{ab}{2} \leq \frac{a^2}{2} + \frac{b^2}{2},
\]
for any \( a, b \in \mathbb{R} \) and any \( \epsilon > 0 \), and with the help of Cauchy-Schwarz inequality, (5.5), (5.8e), and (5.8a), one gets the following upper bound for \( \dot{V}(x, \dot{x}) \):
\[
\dot{V}(x, \dot{x}) = 2(x - P \dot{x})^T \widehat{M} [(A + BK) + \delta(E + BL_1)F](x - P \dot{x}) + Z(W \dot{w} - \dot{W} \dot{w}) + (B \bar{R} - P \bar{B}) \dot{u}]
\]
\[
= \left[ \begin{array}{c} x - P \dot{x} \\ Ww - \dot{W} \dot{w} \\ \delta F(x - P \dot{x}) \end{array} \right]^T \left[ \begin{array}{ccc} (A + BK)^T \widehat{M} + \widehat{M} (A + BK) & \widehat{M} Z & \widehat{M} (BL_1 + E) \\ Z^T \widehat{M} & 0 & 0 \\ (BL_1 + E)^T \widehat{M} & 0 & 0 \end{array} \right] \left[ \begin{array}{c} x - P \dot{x} \\ Ww - \dot{W} \dot{w} \\ \delta F(x - P \dot{x}) \end{array} \right]
+ 2(x - P \dot{x})^T \widehat{M} (B \bar{R} - P \bar{B}) \dot{u}
\]
\[
\leq \left[ \begin{array}{c} x - P \dot{x} \\ Ww - \dot{W} \dot{w} \\ \delta F(x - P \dot{x}) \end{array} \right]^T \left[ \begin{array}{ccc} -\bar{\kappa} \widehat{M} + C_{2}^T X_{2}^2 C_2 & C_{2}^T X_{2}^1 & -F^T \\ X_{2}^1 C_2 & 0 & 0 \\ -F & 0 & \frac{2\pi}{b} \end{array} \right] \left[ \begin{array}{c} x - P \dot{x} \\ Ww - \dot{W} \dot{w} \\ \delta F(x - P \dot{x}) \end{array} \right] + 2(x - P \dot{x})^T \widehat{M} (B \bar{R} - P \bar{B}) \dot{u}
\]
\[
= -\bar{\kappa} V(x, \dot{x}) - 2 \delta(1 - \frac{\delta}{b})(x - P \dot{x})^T F^T F(x - P \dot{x}) + \left[ \begin{array}{c} Ww - \dot{W} \dot{w} \\ C_{2}^X - H \bar{C}_{2} \dot{x} \end{array} \right]^T \left[ \begin{array}{ccc} X_{11} & X_{12} \\ X_{21} & X_{22} \end{array} \right] \left[ \begin{array}{c} Ww - \dot{W} \dot{w} \\ C_{2}^X - H \bar{C}_{2} \dot{x} \end{array} \right] + 2(x - P \dot{x})^T \bar{\kappa} \widehat{M} (B \bar{R} - P \bar{B}) \dot{u}
\]
\[
\leq - (\bar{\kappa} - \pi) V(x, \dot{x}) + \frac{\|\sqrt{\overline{\lambda_{\min}(B \bar{R} - P \bar{B})}}\|}{\pi} \|\dot{u}\|^2 + \left[ \begin{array}{c} Ww - \dot{W} \dot{w} \\ C_{2}^X - H \bar{C}_{2} \dot{x} \end{array} \right]^T \left[ \begin{array}{ccc} X_{11} & X_{12} \\ X_{21} & X_{22} \end{array} \right] \left[ \begin{array}{c} Ww - \dot{W} \dot{w} \\ C_{2}^X - H \bar{C}_{2} \dot{x} \end{array} \right],
\] for any positive constant \( \pi < \bar{\kappa} \).
Using this computed upper bound, the inequality (5.2) is satisfied with the functions \( \eta \in \mathcal{K}_\infty, \rho_{\text{ext}} \in \mathcal{K}_\infty \cup \{0\} \), and the matrix \( X \), as \( \eta(s) := (\tilde{\kappa} - \pi)s, \rho_{\text{ext}}(s) := \sqrt{\frac{M(B\hat{R} - P\hat{B})^T}{\pi}} s^2, \forall s \in \mathbb{R}_{\geq 0} \), and \( X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \). \( \square \)

The next result shows that conditions (5.8a) - (5.8f) are actually necessary for (5.3) being a storage function from \( \hat{\Sigma} \) to \( \Sigma \) provided that the structure of the interface function is as in (5.10) for some matrices \( K, Q, \hat{R}, L_1, \) and \( L_2 \) of appropriate dimension.

**Theorem 5.6.** Let \( \Sigma = (A, B, C_1, C_2, D, E, F, \varphi) \) and \( \hat{\Sigma} = (\hat{A}, \hat{B}, \hat{C}_1, \hat{C}_2, \hat{D}, \hat{E}, \hat{F}, \varphi) \) with \( q_1 = \hat{q}_1 \). Suppose that \( V \) defined in (5.3) is a storage function from \( \hat{\Sigma} \) to \( \Sigma \) with the interface \( k \) given in (5.10). Then, equations (5.8a) - (5.8f) hold.

**Proof.** Since \( V \) is a storage function from \( \hat{\Sigma} \) to \( \Sigma \), there exists a \( \mathcal{K}_\infty \) function \( \alpha \) such that \( \|C_1 \hat{x} - \hat{C}_1 \hat{x}\| \leq \alpha^{-1}(V(x, \hat{x})) \). From (5.3), it follows that \( \|C_1 P \hat{x} - \hat{C}_1 \hat{x}\| \leq \alpha^{-1}(V(P \hat{x}, \hat{x})) = 0 \) holds for all \( \hat{x} \in \mathbb{R}^n \) which implies (5.8b).

Let us consider the inputs \( \hat{\nu} \equiv 0, \omega \equiv 0 \), and \( \hat{\omega} \equiv 0 \). Since \( X^{22} \preceq 0 \), inequality (5.2) reduces to

\[
\dot{V}(x, \hat{x}) \leq -\eta(V(x, \hat{x})) + (h_2(x) - H\dot{h}_2(\hat{x}))^T X^{22} (h_2(x) - H\dot{h}_2(\hat{x})) \leq -\eta(V(x, \hat{x})),
\]  

(5.12) for any \( x \in \mathbb{R}^n \) and \( \hat{x} \in \mathbb{R}^n \). Using the results in Lemma 4.4 in [LSW96] or Lemma 3.6 in [ZRMng], inequality (5.12) implies the existence of a \( \mathcal{KL} \) function \( \vartheta \) such that

\[
\dot{V}(\xi(t), \dot{\xi}(t)) \leq \vartheta(V(\xi(0), \dot{\xi}(0)), t),
\]

(5.13) holds, where \( \hat{\nu} = 0, \omega = 0, \hat{\omega} = 0 \), and \( v \) is given by the interface function \( k \) in (5.10). Then, for all \( \xi(0) = P\xi(0), t \geq 0 \), and using (5.13), we obtain \( \dot{V}(\xi(t), \dot{\xi}(t)) = 0 \). Since \( M \) is positive definite, we have

\[
\xi(t) = P\xi(t) \quad \text{and} \quad \dot{\xi}(t) = P\dot{\xi}(t),
\]

from which we derive that

\[
AP\hat{x} + BQ\hat{x} + (E + B(L_1 - L_2))\varphi(FP\hat{x}) = P\hat{A}\hat{x} + P\hat{E}\varphi(\hat{F}\hat{x})
\]

holds for all \( \hat{x} \in \mathbb{R}^n \) and, hence, (5.8a), (5.8e), and (5.8f) hold. First assume \( X^{22} \neq 0 \). Since \( V(P \hat{x}, \hat{x}) = V(P \hat{x}, \hat{x}) = 0 \) and using the first inequality in (5.12), one gets

\[
(C_2 P \hat{x} - H\hat{C}_2 \hat{x})^T X^{22} (C_2 P \hat{x} - H\hat{C}_2 \hat{x}) \geq 0,
\]

for any \( \hat{x} \in \mathbb{R}^n \). Since \( X^{22} \preceq 0 \) and by assumption \( X^{22} \neq 0 \), one obtains \( X^{22}(C_2 P - H\hat{C}_2) = 0 \) which implies (5.8d). Now, let us consider the inputs \( \hat{\nu} \equiv 0, \omega \equiv 0 \), and \( \hat{\omega} \equiv 0 \). Therefore, inequality (5.2) reduces to

\[
\dot{V}(x, \hat{x}) \leq -\eta(V(x, \hat{x})) + (Ww - \hat{W}\hat{w})^T X^{11} (Ww - \hat{W}\hat{w}) + 2(Ww - \hat{W}\hat{w})^T X^{12} (C_2 x - H\hat{C}_2 \hat{x}),
\]

(5.14) for any \( x \in \mathbb{R}^n, \hat{x} \in \mathbb{R}^n, w \in \mathbb{R}^p \), and \( \hat{w} \in \mathbb{R}^n \). From (5.14) and by choosing \( x = 0_n \) and \( \hat{x} = 0_n \), one can readily verify that \( X^{11} \succeq 0 \). Then, for all \( x = P \hat{x}, \) we obtain

\[
(Ww - \hat{W}\hat{w})^T X^{11} (Ww - \hat{W}\hat{w}) + 2(Ww - \hat{W}\hat{w})^T X^{12} (C_2 P - H\hat{C}_2) \hat{x} \geq 0,
\]

for any \( w, \hat{w}, \) and \( \hat{x} \), which implies \( X^{12}(C_2 P - H\hat{C}_2) = 0 \) and, hence, (5.8e) holds. \( \square \)

**Remark 5.7.** Note that matrix \( \hat{R} \) is a free design parameter in the interface function (5.10). Using the results in [GP09] Proposition 1, we choose \( \hat{R} \) to minimize function \( \rho_{\text{ext}} \) for \( V \) and, hence, reduce the upper bound in (3.5) on the error between the output behaviors of \( \Sigma \) and \( \hat{\Sigma} \). The choice of \( R \) minimizing \( \rho_{\text{ext}} \) is given by

\[
\tilde{R} = (B^T \hat{M} B)^{-1} B^T \hat{M} \hat{B}.
\]

(5.15)
So far, we extracted various conditions on the original system matrices \( \{A, B, C_1, C_2, D, E, F\} \), the abstraction matrices \( \{\hat{A}, \hat{B}, \hat{C}_1, \hat{C}_2, \hat{D}, \hat{E}, \hat{F}\} \), and the ones appearing in (5.3) and (5.10). Those conditions ensure that \( V \) in (5.3) is a storage function from \( \Sigma \) to \( \Sigma \) with the corresponding interface function in (5.10) refining any control signal designed for \( \hat{\Sigma} \) to the one for \( \Sigma \). Apparently, those requirements do not enforce any condition on matrix \( \hat{B} \). For example, one can select \( \hat{B} = I_n \) making the abstract system \( \hat{\Sigma} \) fully actuated. On the other hand, one can ask not only for the existence of a storage function from \( \Sigma \) to \( \Sigma \), but additionally require that all the controllable behaviors (in the absence of internal inputs) of the concrete system \( \Sigma \) are preserved over the abstraction \( \hat{\Sigma} \). We refer the interested readers to [GP09, Subsection 4.1] and [PLS00, Section V] for more details on what we mean by preservation of controllable behaviors.

The next theorem requires a condition on \( \hat{B} \) in order to guarantee the preservation of controllable behaviors of \( \Sigma \) over \( \hat{\Sigma} \).

**Theorem 5.8.** Let \( \Sigma = (A, B, C_1, C_2, D, E, F) \) and \( \hat{\Sigma} = (\hat{A}, \hat{B}, \hat{C}_1, \hat{C}_2, \hat{D}, \hat{E}, \hat{F}) \) with \( q_1 = \hat{q}_1 \). Suppose that there exist matrices \( P, Q, L_1, \) and \( L_2 \) satisfying (5.8a) and (5.8f), and that matrix \( \hat{B} \) is given by

\[
\hat{B} = [PB \quad PAG],
\]

where \( \hat{P} \) and \( G \) are assumed to satisfy

\[
\begin{align*}
C_1 &= \hat{C}_1 \hat{P} \tag{5.17a} \\
I_n &= \hat{P}^2 + GT \tag{5.17b} \\
L_n &= \hat{P}P \tag{5.17c} \\
F &= \hat{F} \hat{P}. \tag{5.17d}
\end{align*}
\]

for some matrix \( T \). Then, for every trajectory \( (\xi, \zeta_1, \zeta_2, v, 0) \) of \( \Sigma \) there exists a trajectory \( (\hat{\xi}, \hat{\zeta}_1, \hat{\zeta}_2, \hat{v}, 0) \) of \( \hat{\Sigma} \) where \( \hat{\xi} = \hat{P}\xi \) and \( \hat{\zeta}_1 = \hat{\zeta}_1 \) hold.

**Proof.** Let \( (\xi, \zeta_1, \zeta_2, v, 0) \) be a trajectory of \( \Sigma \). We are going to show that \( (\hat{P}\xi, \zeta_1, \zeta_2, \hat{v}, 0) \) with

\[
\hat{v} = \left[ v - Q\hat{P}\xi - (L_1 - L_2)\varphi(F\xi) \right]
\]

is a trajectory of \( \hat{\Sigma} \). We use (5.17b) and derive

\[
\dot{\hat{P}}\xi = \hat{P}A\xi + \hat{P}E\varphi(F\xi) + \hat{P}Bv = \hat{P}AP\xi + \hat{P}(I_n - \hat{P})\xi + \hat{P}E\varphi(F\xi) + \hat{P}Bv
\]

for some matrix \( T \). Then, for every trajectory \( (\xi, \zeta_1, \zeta_2, v, 0) \) of \( \Sigma \) there exists a trajectory \( (\hat{\xi}, \hat{\zeta}_1, \hat{\zeta}_2, \hat{v}, 0) \) of \( \hat{\Sigma} \) where \( \hat{\xi} = \hat{P}\xi \) and \( \hat{\zeta}_1 = \hat{\zeta}_1 \) hold.

**Remark 5.9.** Note that the previous result establishes that \( \hat{\Sigma} \) (in the absence of internal inputs) is \( \hat{P} \)-related to \( \Sigma \) as in [GP09, Definition 3]. We refer the interested readers to [PLS00] for more details about properties (e.g. controllability) of \( \Phi \)-related systems for some surjective smooth map \( \Phi \).
5.3. Construction of abstractions. Here, we provide several straightforward sufficient and necessary geometric conditions on matrices appearing in the definition of $\Sigma$, of storage function and its corresponding interface function. The proposed geometric conditions facilitate the constructions of such matrices. First, we recall [GP09, Lemma 2] providing necessary and sufficient conditions for the existence of matrices $\hat{A}$ and $Q$ appearing in condition (5.8a).

Lemma 5.10. Consider matrices $A$, $B$, and $P$. There exist matrices $\hat{A}$ and $Q$ satisfying (5.8a) if and only if
\[ \text{im} \ AP \subseteq \text{im} \ P + \text{im} \ B. \] (5.18)

Now, we give necessary and sufficient conditions for the existence of matrices $\hat{C}_2$, $\hat{E}$, and $L_2$ appearing in conditions (5.8c), (5.8d), and (5.8f), respectively.

Lemma 5.11. Given $P$, $C_2$, and $X^{12}$ (resp. $X^{22}$), there exists matrix $\hat{C}_2$ satisfying (5.8c) (resp. (5.8d)) if and only if
\[ \text{im} \ X^{12}C_2P \subseteq \text{im} \ X^{12}H, \ (\text{resp.} \ \text{im} \ X^{22}C_2P \subseteq \text{im} \ X^{22}H) \] (5.19)
for some matrix $H$ of appropriate dimension.

Lemma 5.12. Given $P$, $B$, and $L_1$, there exist matrices $\hat{E}$ and $L_2$ satisfying (5.8f) if and only if
\[ \text{im} \ E \subseteq \text{im} \ P + \text{im} \ B. \] (5.20)

Lemmas 5.10, 5.11, and 5.12 provide necessary and sufficient conditions on $P$ and $H$ resulting in the construction of matrices $\hat{A}$, $\hat{C}_2$, and $\hat{E}$ together with the matrices $Q$ and $L_2$ appearing in the definition of the interface function in (5.10). Matrices $\hat{F}$ and $\hat{C}_1$ are computed as $\hat{F} = FP$ and $\hat{C}_1 = C_1P$. The next lemma provides a necessary and sufficient condition on the existence of matrix $\hat{D}$ appearing in condition (5.8g).

Lemma 5.13. Given $Z$, there exists matrix $\hat{D}$ satisfying (5.8g) if and only if
\[ \text{im} \ Z\hat{W} \subseteq \text{im} \ P, \] (5.21)
for some matrix $\hat{W}$ of appropriate dimension.

Although condition (5.21) is readily satisfied by choosing $\hat{W} = 0$, one should preferably aim at finding a nonzero $\hat{W}$ to smooth later the satisfaction of compositionality condition (4.2).

As we already mentioned, the choice of matrix $\hat{B}$ is free. One can also construct $\hat{B}$ as in (5.16) ensuring preservation of all controllable behaviors of $\Sigma$ over $\Sigma$ under extra conditions given in (5.17). Lemma 3 in [GP09], as recalled next, provides necessary and sufficient conditions on $P$ and $C_1$ for the existence of $\hat{P}$, $G$, and $T$ satisfying (5.17a), (5.17b), and (5.17c).

Lemma 5.14. Consider matrices $C_1$ and $P$ with $P$ being injective and let $\hat{C}_1 = C_1P$. There exists matrix $\hat{P}$ satisfying (5.17a), (5.17b), and (5.17c), for some matrices $G$ and $T$ of appropriate dimensions, if and only if
\[ \text{im} \ P + \text{ker} \ C_1 = \mathbb{R}^n. \] (5.22)

Similar to Lemma 5.14, we give necessary and sufficient conditions on $P$ and $F$ for the existence of $\hat{P}$ satisfying (5.17d).

Lemma 5.15. Consider matrices $F$ and $P$ with $P$ being injective and let $\hat{F} = FP$. There exists matrix $\hat{P}$ satisfying (5.17d) if and only if
\[ \text{im} \ P + \text{ker} \ F = \mathbb{R}^n. \] (5.23)

Note that conditions (5.5), (5.6), and (5.18)-(5.21) (resp. (5.5), (5.6), and (5.18)-(5.23)) complete the characterization of mainly matrices $P$ and $Z$ which together with the matrices $\{A, B, C_1, C_2, D, E, F\}$ result in the
construction of matrices \(\{\hat{A}, \hat{B}, \hat{C}_1, \hat{C}_2, \hat{D}, \hat{E}, \hat{F}\}\), where \(\hat{B}\) can be chosen freely with appropriate dimensions (resp. \(\hat{B}\) is computed as in (5.16)).

We summarize the construction of the abstraction \(\hat{\Sigma}\), storage function \(V\) in (5.3), and its corresponding interface function in (5.10) in Table 1.

**Table 1.** Construction of \(\hat{\Sigma} = (\hat{A}, \hat{B}, \hat{C}_1, \hat{C}_2, \hat{D}, \hat{E}, \hat{F}, \phi)\), the corresponding storage function \(V\) in (5.3), and interface function in (5.10) for a given \(\Sigma = (A, B, C_1, C_2, D, E, F, \phi)\).

1. Compute matrices \(\hat{M}, K, L_1, Z, X^{11}, X^{12}\), and \(X^{22}\) satisfying (5.18) and (5.21) (resp. (5.18)-(5.23));
2. Pick an injective \(P\) with the lowest rank satisfying (5.8a);
3. Compute \(\hat{A}\) and \(Q\) from (5.8a);
4. Compute \(\hat{E}\) and \(L_2\) from (5.8a);
5. Compute \(\hat{F} = FP\);
6. Compute \(\hat{C}_1 = CP\);
7. Compute \(\hat{C}_2\) satisfying \(H\hat{C}_2 = CP\) for some \(H\);
8. Compute \(\hat{D}\) satisfying \(P\hat{D} = Z\hat{W}\) for some (rather nonzero) \(\hat{W}\);
9. Choose \(\hat{B}\) freely (resp. \(\hat{B} = [\hat{P}B \hat{P}AG]\));
10. Compute \(\hat{R}\), appearing in (5.10), from (5.15).

### 5.4. Feasibility of LMI (5.5)

In this subsection we discuss sufficient and necessary feasibility conditions for the LMI (5.5) in the restrictive case of \(X^{12} = 0\) and \(X^{11} \succeq \frac{Z^T\hat{M}Z}{\pi}\) for any positive constant \(\pi < \hat{\kappa}\), where 0 denotes a zero matrix of appropriate dimension. To do so, we convert the feasibility conditions for the restricted version of LMI (5.5) into the ones for two dual control problems. When \(b = \infty\) in (5.5), the feasibility of restricted (5.5) is dual to the one of designing a controller rendering a linear system strictly positive real (SPR) [AK01b]. When \(b < \infty\), the duality is with a linear \(L_2\)-gain assignment control problem [Isi99 Section 13.2].

When \(b = \infty\), the restricted version of LMI (5.5) reduces to

\[
(A + BK)^T\hat{M} + \hat{M}(A + BK) \prec 0, \tag{5.24}
\]

\[
\hat{M}(BL_1 + E) + F^T = 0. \tag{5.25}
\]

By virtue of the Positive-Real Lemma [Yak62], conditions (5.24) and (5.25) mean that the linear control system

\[
\Sigma : \begin{cases}
\dot{x} = Ax + Bu + E\omega, \\
\zeta = -Fx\xi,
\end{cases} \tag{5.26}
\]

is enforced SPR from the disturbance \(\omega\) to the output \(\zeta\) by the control law

\[
v = Kx + L_1\omega. \tag{5.27}
\]

Therefore, when \(b = \infty\), the feasibility of the restricted version of LMI (5.5) is dual to the feasibility of the control problem in which the system (5.26) is enforced SPR by the control law (5.27).

When \(b < \infty\), using the Schur complement of \(-2/b\), one can readily verify that the restricted version of LMI (5.5) is equivalent to

\[
\left(A + BK + \frac{b}{2}(BL_1 + E)F\right)^T\hat{M} + \hat{M}\left(A + BK + \frac{b}{2}(BL_1 + E)F\right) + \frac{b}{2}\hat{M}(BL_1 + E)(BL_1 + E)^T\hat{M} + \frac{b}{2}F^TF \prec 0,
\]

By virtue of the Positive-Real Lemma [Yak62], conditions (5.24) and (5.25) mean that the linear control system

\[
\Sigma : \begin{cases}
\dot{x} = Ax + Bu + E\omega, \\
\zeta = -Fx\xi,
\end{cases} \tag{5.26}
\]

is enforced SPR from the disturbance \(\omega\) to the output \(\zeta\) by the control law

\[
v = Kx + L_1\omega. \tag{5.27}
\]

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When \(b < \infty\), using the Schur complement of \(-2/b\), one can readily verify that the restricted version of LMI (5.5) is equivalent to

\[
\left(A + BK + \frac{b}{2}(BL_1 + E)F\right)^T\hat{M} + \hat{M}\left(A + BK + \frac{b}{2}(BL_1 + E)F\right) + \frac{b}{2}\hat{M}(BL_1 + E)(BL_1 + E)^T\hat{M} + \frac{b}{2}F^TF \prec 0,
\]
which means that the $L_2$-gain of the dual system (5.26) from input $\tilde{w} := \omega + (b/2)\zeta$ to output $\zeta$ is enforced to be strictly less than $2/b$ by the control law $v = K\zeta + L_1\omega$ [Isi99, Section 13.2].

We refer the interested readers to [AK01a, Theorem 3] deriving sufficient and necessary feasibility conditions for the restricted version of LMI (5.5) by looking into the corresponding dual control problems, namely, enforcing SPR and assigning a linear $L_2$-gain.

Note that in the context of observer design and observer-based control, the feasibility of those dual control problems have been investigated for several physical problems in [AK01a, FA03, AGPV03, Sch04].

6. Example

Consider a linear control system $\Sigma = (-L, I_n, C)$ satisfying

$$\dot{\xi} = -L\xi + v,$$
$$\zeta = C\xi,$$

for some matrix $C \in \mathbb{R}^{q \times n}$ and $L \in \mathbb{R}^{n \times n}$. Assume $L$ is the Laplacian matrix [GR01] of an undirected graph, e.g., for a complete graph:

$$L = \begin{bmatrix}
  n - 1 & -1 & \cdots & \cdots & -1 \\
  -1 & n - 1 & -1 & \cdots & -1 \\
  -1 & -1 & n - 1 & \cdots & -1 \\
  \vdots & \ddots & \ddots & \ddots & \vdots \\
  -1 & \cdots & \cdots & -1 & n - 1
\end{bmatrix},$$

and $C$ has the following block diagonal structure

$$C = \text{diag}(C_{11}, \ldots, C_{1N}),$$

where $C_{1i} \in \mathbb{R}^{q_1 \times n_i}$. We partition $\xi$ as $\xi = [\xi_1; \ldots; \xi_N]$ and $v$ as $v = [v_1; \ldots; v_N]$ where $\xi_i$ and $v_i$ are both taking values in $\mathbb{R}^{n_i}, \forall i \in [1; N]$. Now, by introducing $\Sigma_i = (0_{n_i}, I_{n_i}, C_{1i}, I_{n_i}, I_{n_i})$ satisfying

$$\Sigma_i : \begin{cases}
\dot{\xi}_i = \omega_i + v_i, \\
\zeta_{1i} = C_{1i}\xi_i, \\
\zeta_{2i} = \xi_i,
\end{cases}$$

one can readily verify that $\Sigma = I(\Sigma_1, \ldots, \Sigma_N)$ where the coupling matrix $M$ is given by $M = -L$.

Our goal is to aggregate each $\xi_i$ taking values in $\mathbb{R}^{n_i}$ into a scalar-valued $\hat{\xi}_i$, governed by $\hat{\Sigma}_i = (0, 1, C_{1i}1_{n_i}, 1, 1)$ which satisfies:

$$\hat{\Sigma}_i : \begin{cases}
\dot{\hat{\xi}}_i = \hat{\omega}_i + \hat{v}_i, \\
\hat{\zeta}_{1i} = C_{1i}1_{n_i}\hat{\xi}_i, \\
\hat{\zeta}_{2i} = \hat{\xi}_i.
\end{cases}$$

One can readily verify that, for any $i \in [1; N]$, conditions (5.4) and (5.5) are satisfied with $\hat{M}_i = I_{n_i}, K_i = -\lambda I_{n_i}$, for some $\lambda > 0$, $\hat{\omega}_i = 2\lambda$, $Z_i = I_{n_i}, L_{1i} = 0$, $W_i = I_{n_i}, X_{11} = 0, X_{22} = 0, X_{12} = X_{21} = I_{n_i}$, where 0 denotes zero matrices of appropriate dimensions. Moreover, for any $i \in [1; N], P_i = 1_{n_i}$ satisfies conditions (5.8) with $Q_i = L_{2i} = 0_{n_i}, H = 1_{n_i}$, and $W_i = 1_{n_i}$. Hence, function $V_i(x_i, \hat{x}_i) = (x_i - 1_{n_i}\hat{x}_i)^T(x_i - 1_{n_i}\hat{x}_i)$ is a storage function from $\hat{\Sigma}_i$ to $\Sigma_i$ satisfying condition (3.1) with $\alpha_i(r) = \frac{1}{\lambda_{\text{max}}(P_i\hat{G}_i)}r^2$ and condition (3.2) with $\eta(r) = -2\lambda r, \rho_{\text{ext}}(r) = 0, \forall r \in \mathbb{R}^{\geq 0}, W_i = I_{n_i}, \hat{W}_i = H_i = 1_{n_i}$, and

$$X_i = \begin{bmatrix}
0 & I_{n_i} \\
I_{n_i} & 0
\end{bmatrix},$$

where the input $u_i \in \mathbb{R}^{n_i}$ is given via the interface function in (5.10) as $u_i = -\lambda(x_i - 1_{n_i}\hat{x}_i) + 1_{n_i}\hat{u}_i$. Note that $\hat{R}_i = 1_{n_i}$ was computed as in (5.15).
Now, we look at $\Sigma = \mathcal{I}(\hat{\Sigma}_1, \ldots, \hat{\Sigma}_N)$ with a coupling matrix $\hat{M}$ satisfying condition (4.2) as follows:

$$-L \text{diag}(1_{n_1}, \ldots, 1_{n_N}) = \text{diag}(1_{n_1}, \ldots, 1_{n_N}) \hat{M}. \quad (6.3)$$

Note that the existence of $\hat{M}$ satisfying (6.3) for a graph Laplacian $L$ means that the $N$ subgraphs form an equitable partition of the full graph [GR01]. Although this restricts the choice of a partition in general, for the complete graph (6.1) any partition is equitable.

Choosing $\mu_1 = \cdots = \mu_N = 1$ and using $X_i$ in (6.2), matrix $X$ in (4.4) reduces to

$$X = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix},$$

and condition (4.1) reduces to

$$\begin{bmatrix} -L^T \\ I_n \end{bmatrix} X \begin{bmatrix} -L \\ I_n \end{bmatrix} = -L - L^T \preceq 0$$

which always holds without any restrictions on the size of the graph. In order to show the above inequality, we used $L = L^T \succeq 0$ which is always true for Laplacian matrices of undirected graphs.

For the sake of simulation, we fix $n = 9$ and

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

where

$$C_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, C_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, C_{13} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

Let us now synthesize a controller for $\Sigma$ via the abstraction $\hat{\Sigma}$ to enforce the specification, defined by the LTL formula [BK08]

$$\psi = \Box S \land \left( \bigwedge_{i=1}^{5} \Box(\neg O_i) \right) \land \Box \diamond T_1 \land \Box \diamond T_2, \quad (6.4)$$

which requires that any output trajectory $\zeta$ of the closed loop system evolves inside the set $S$, avoids sets $O_i$, $i \in \{1; 5\}$, indicated with blue boxes in Figure 4 and visits each $T_i$, $i \in \{1; 2\}$, indicated with red boxed in Figure 5 infinitely often. We use SCOTS [RZ16b] to synthesize a controller for $\hat{\Sigma}$ to enforce (6.4). In the synthesis process we restricted the abstract inputs to $\hat{u}_1, \hat{u}_2, \hat{u}_3 \in [-14, 14]$. Given that we can set the initial states of $\Sigma$ to $x_i = P_i \hat{x}_i$, so that $V_i(x_i, \hat{x}_i) = 0$, and since $\rho_{\text{ext}}(r) = 0, \forall r \in \mathbb{R}_{\geq 0}$, we obtain $\|\zeta(t) - \zeta(t)\| = 0$ for all $t \geq 0$. A closed-loop output trajectory of $\Sigma$ is illustrated in Figure 6. Note that it would not have been possible to synthesize a controller using SCOTS for the original 9-dimensional system $\Sigma$, without the 3-dimensional intermediate approximation $\hat{\Sigma}$.

**Remark 6.1.** This scale-free result highlights the advantage of dissipativity-type over small-gain type conditions proposed in [RZ15, RZ16a]: the storage function $V_i$ from $\Sigma_i$ to $\Sigma_i$ in this example also satisfies the requirements of a simulation function defined in [RZ15, RZ16a]; however, the resulting small-gain type condition, e.g., for $L$ in (6.1) reduces to $\frac{n-1}{n-1+\lambda} < 1$ which involves the spectral radius $\rho$ of $L$ ($\rho(L) = n$). Hence, using the results in [RZ15, RZ16a], one can readily verify that as the number of components increases, e.g., $n \to \infty$, the quality of approximation deteriorates unless the interface gain $\lambda$ is increasing with $n$ which is not desirable because it results in high amplitude inputs $u_i$.

---

1The spectral radius of a square matrix $A \in \mathbb{R}^{n \times n}$, denoted by $\rho(A)$, is defined as $\rho(A) := \max\{|\lambda_1|, \cdots, |\lambda_n|\}$ where $\lambda_1, \ldots, \lambda_n$ are eigenvalues of $A$. 
The specification with closed loop output trajectory of $\Sigma$. The sets $S$, $O_i$, $i \in [1; 5]$, and $T_i$, $i \in [1; 2]$ are given by: $S = [0, 10]^3$, $T_1 = [1, 2]^3$ and $T_2 = [8, 9]^3$, $O_1 = [4, 6]^3$, $O_2 = [7, 9] \times [1, 3] \times [0, 10]$, $O_3 = [2, 3] \times [7, 8] \times [0, 10]$, $O_4 = [1, 2] \times [1, 2] \times [5, 10]$, and $O_5 = [8, 9] \times [8, 9] \times [0, 5]$.

7. Conclusion

In this paper, we proposed for the first time a notion of so-called storage function relating a concrete control system to its abstraction by quantifying their joint input-output correlation. This notion was adapted from the one of storage function from dissipativity theory. Given a network of control subsystems together with their corresponding abstractions and storage functions, we provide compositional conditions under which a network of abstractions approximate the original network and the approximation error can be quantified compositionally using the storage functions of the subsystems. Finally, we provide a procedure for the construction of abstractions together with their corresponding storage functions for a class of nonlinear control systems by using the bounds on the slope of system nonlinearities. One of the main advantages of the proposed results here based on a dissipativity-type condition in comparison with the existing ones based on a small-gain type condition is that the former can enjoy specific interconnection matrix and provide scale-free compositional conditions (cf. Section 6).

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