On geodesics of Berger tangent sphere bundle of Hermitian locally symmetric manifold.

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Abstract

We propose a special deformation of the Sasaki metric on tangent and unit tangent bundle of a Hermitian locally symmetric manifold. Geodesics of this deformed metric have different projections on a base manifold for tangent or unit tangent bundle cases in contrast to usual Sasaki metric. Nevertheless, the projections of geodesics of the unit tangent bundle still preserve the property to have all geodesic curvatures constant.

Keywords: Sasaki metric, Hermitian manifold, geodesics.

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Introduction

Let \((M, g)\) be Riemannian manifold. Denote by \(TM\) and \(T_1M\) tangent and unit tangent bundle of \((M, g)\) with Sasaki metric. It is easy to prove that if \(\pi\) is a bundle projection \(\pi : TM \rightarrow M\) and \(\Gamma(\sigma)\) is geodesic on \(TM\) or \(T_1M\) then the projected curve \(\gamma(\sigma) = \pi \circ \Gamma(\sigma)\) on \(M\) is the same. In other words, geodesic lines on \(TM\) or \(T_1M\) are generated by different vector fields along the same set of curves in a base manifold. A complete description of base curves and vector fields generating geodesics in the case of base manifold of constant curvature one can find in [1] and [2]. It was proved that the projected curves have constant (possibly zero) first and second geodesic curvatures while the others vanish. P.Nagy [3] generalized these results for the case of locally symmetric base and characterized the projected curves by all constant geodesic curvatures.
It present paper we propose a special deformation of Sasaki metric for the case of Hermitian locally symmetric base manifold which distinguish the projections of geodesics in $TM$ and $T_1 M$ cases but preserves the property to have constant geodesic curvatures in $T_1 M$ case.

The general idea is M.Berger-type. Let $S^{2n-1}$ be a unit sphere in Euclidean space $E^{2n}$. Let $J$ be a standard complex structure on $E^{2n}$. If $N$ is a unit normal vector field on $S^{2n-1}$ then $JN$ is a so-called Hopf vector field on $S^{2n+1}$. M.Berger deformation of standard sphere metric assumes its deformation along integral trajectories of the Hopf vector field. Consider a Hermitian manifold $(M^{2n}, g, J)$ and its tangent sphere bundle. Then at each point of $M^{2n}$ the tangent sphere $S^{2n-1}$ carries the Hopf vector field. Applying M.Berger metric deformation to each tangent sphere we get the tangent sphere bundle over $M^{2n}$ with M.Berger metric spheres the fibers. Call it Berger tangent (sphere) bundle. The main result of the paper is the following.

**Theorem 2.1** Let $\gamma = \pi \circ \Gamma$ be a projection of a curve $\Gamma$ on the Berger tangent sphere bundle over Hermitian locally symmetric manifold $M$. Then all geodesic curvatures of $\gamma$ are constant.

If $\Gamma$ is a geodesic on Berger tangent bundle $TM$, then the projected curve $\gamma = \pi \circ \Gamma$ does not posess this property.

In a specific case of $CP^n$ the Theorem 2.1 can be improved.

**Theorem 2.2** Let $\Gamma$ be a geodesic of the Berger tangent sphere bundle over the complex projective space $CP^n$. Then the geodesic curvatures of $\gamma = \pi \circ \Gamma$ are all constant and $k_6 = \cdots = k_{n-1} = 0$.

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1 Some general considerations

Let $(M, g)$ be $n$-dimensional Riemannian manifold with metric $g$. Denote by $\langle \cdot, \cdot \rangle$ a scalar product with respect to $g$. A natural Riemannian metric on the tangent bundle has been defined by S. Sasaki. We describe it briefly in terms of the connection map.

At each point $Q = (q, \xi) \in TM$ the tangent space $T_Q TM$ can be split into the so-called vertical and horizontal parts:

$$T_Q TM = \mathcal{H}_Q TM \oplus \mathcal{V}_Q TM.$$
The vertical part $V_QTM$ is tangent to the fiber, while the horizontal part is transversal to it. Denote $(x^1, \ldots, x^n; \xi^1, \ldots, \xi^n)$ the natural induced local coordinate system on $TM$. Denote $\partial_i = \partial/\partial x^i$, $\partial_{n+i} = \partial/\partial \xi^i$. Then for $\tilde{X} \in T_QTM^n$ we have

$$\tilde{X} = \tilde{X}^i \partial_i + \tilde{X}^{n+i} \partial_{n+i}.$$  

Denote by $\pi : TM \rightarrow M$ the tangent bundle projection map. Then its differential $\pi^* : T_QTM \rightarrow T_qM$ acts on $\tilde{X}$ as

$$\pi^* \tilde{X} = \tilde{X}^i \partial_i$$

and defines a linear isomorphism between $V_QTM$ and $T_qM$.

The so-called connection map $K : T_QTM \rightarrow T_qM$ acts on $\tilde{X}$ by the rule

$$K \tilde{X} = (\tilde{X}^{n+i} + \Gamma^i_{jk} \xi^j \tilde{X}^k) \partial_i$$

and defines a linear isomorphism between $H_QTM$ and $T_qM$. The images $\pi^* \tilde{X}$ and $K \tilde{X}$ are called horizontal and vertical projections of $\tilde{X}$, respectively. It is easy to see that $V_Q = \ker \pi^*_Q$, $H_Q = \ker K|_Q$.

Let $X, Y \in T_qM$. The standard Sasaki metric on $TM$ is defined by the following scalar product

$$\langle \langle X, Y \rangle \rangle_q = \langle \pi^* \tilde{X}, \pi^* \tilde{Y} \rangle_q + \langle K \tilde{X}, K \tilde{Y} \rangle_q$$

at each point $Q = (q, \xi)$. Horizontal and vertical subspaces are mutually orthogonal with respect to Sasaki metric.

The operations inverse to projections are called lifts. Namely, if $X \in T_qM^n$, then

$$X^h = X^i \partial_i - \Gamma^i_{jk} \xi^j X^k \partial_{n+i}$$

is in $H_QTM$ and is called the horizontal lift of $X$, and

$$X^v = X^i \partial_{n+i}$$

is in $V_QTM$ and is called the vertical lift of $X$.

The Sasaki metric can be completely defined by scalar product of various combinations of lifts of vector fields from $M$ to $TM$ as

$$\langle \langle X^h, Y^h \rangle \rangle = \langle X, Y \rangle, \quad \langle \langle X^h, Y^v \rangle \rangle = 0, \quad \langle \langle X^v, Y^v \rangle \rangle = \langle X, Y \rangle.$$
Consider now as a base manifold $M$ a Hermitian manifold $(M^{2n},g,J)$. Define a deformation of Sasaki metric along the $J\xi$ directions in each fiber of the form
\[
\begin{align*}
\langle\langle X^h, Y^h \rangle \rangle &= \langle X, Y \rangle, \\
\langle\langle X^h, Y^v \rangle \rangle &= 0, \\
\langle\langle X^v, Y^v \rangle \rangle &= \langle X, Y \rangle + \delta^2 \langle X, J\xi \rangle \langle Y, J\xi \rangle,
\end{align*}
\]
where $J$ is an almost complex structure on $M$ and $\delta$ is some constant.

Geometrically, this deformation means that we deform each tangent sphere $S^{2n-1} \subset T_qM$ along the fibers of standard Hopf fibration of $S^{2n-1}$ at each point $q \in M$. We will refer to the tangent (sphere) bundle with the metric (1) as Berger tangent (sphere) bundle.

In what follows we suppose that $M$ is Hermitian locally symmetric space. In this case $M$ is necessarily Kahlerian, i.e. $\nabla J = 0$, and locally symmetric as a Riemannian manifold [5, Proposition 9.1].

The following formulas are independent on the choice of tangent bundle metric and are known as Dombrowski formulas.

**Lemma 1.1** At each point $(q, \xi) \in TM$ the brackets of lifts of vector fields from $M$ to $TM$ are
\[
\begin{align*}
[X^h, Y^h] &= [X, Y]^h - (R(X, Y)\xi)^v, \\
[X^h, Y^v] &= (\nabla_X Y)^v, \\
[X^v, Y^h] &= 0,
\end{align*}
\]
where $\nabla$ is the connection on $M$ and $R$ its curvature tensor.

Denote by $\tilde{\nabla}$ the Levi-Civita connection of the metric (1). The following Kowalski-type lemma is the main tool for further considerations.

**Lemma 1.2** The Levi-Civita connection of metric (1) is completely defined at each point $(q, \xi) \in TM$ by
\[
\begin{align*}
\tilde{\nabla}_{X^h} Y^h &= (\nabla_X Y)^h - \frac{1}{2} (R(X, Y)\xi)^v, \\
\tilde{\nabla}_{X^h} Y^v &= \frac{1}{2} \left( R(\xi, Y)X + \delta^2 \langle Y, J\xi \rangle R(\xi, J\xi)X \right)^h + \left( \nabla_X Y \right)^v, \\
\tilde{\nabla}_{X^v} Y^h &= \frac{1}{2} \left( R(\xi, X)Y + \delta^2 \langle X, J\xi \rangle R(\xi, J\xi)Y \right)^h \\
\tilde{\nabla}_{X^v} Y^v &= \delta^2 \left( \langle X, J\xi \rangle JY + \langle Y, J\xi \rangle JX - \frac{\delta^2}{1 + \delta^2|\xi|^2} \left( \langle Y, \xi \rangle \langle X, J\xi \rangle + \langle X, \xi \rangle \langle Y, J\xi \rangle J\xi \right) J\xi \right)^v,
\end{align*}
\]
where $\nabla$ is the Levi-Civita connection on $M$ and $R$ is its curvature tensor.
Proof. To prove this lemma we will need the useful formulas which we naturally gather in a separate sublemma.

**Lemma 1.3** The following rules of differentiations are true:

\[
X^h \langle \langle Y^h, Z^h \rangle \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle,
\]

\[
X^h \langle Y^v, Z^v \rangle = \langle \langle (\nabla_X Y)^v, Z^v \rangle \rangle + \langle \langle Y^v, (\nabla_X Z)^v \rangle \rangle,
\]

\[
X^v \langle \langle Y^h, Z^h \rangle \rangle = 0,
\]

\[
X^v \langle \langle Y^v, Z^v \rangle \rangle = \delta^2 \left( \langle Y, JX \rangle \langle Z, J\xi \rangle + \langle Y, J\xi \rangle \langle Z, JX \rangle \right),
\]

where \( \langle \cdot, \cdot \rangle \) means the scalar product with respect to metric of the base manifold.

**Proof.**

i) Indeed, keeping in mind (1), we have

\[
X^h \langle \langle Y^h, Z^h \rangle \rangle = X^h \langle Y, Z \rangle = \langle \nabla_X Y \rangle + \langle Y, \nabla_X Z \rangle.
\]

ii) In a similar way

\[
X^h \langle \langle Y^v, Z^v \rangle \rangle = X^h \left( \langle Y, Z \rangle + \delta^2 \langle Y, J\xi \rangle \langle Z, J\xi \rangle \right) = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle + \delta^2 X^h \left( \langle Y, J\xi \rangle \langle Z, J\xi \rangle \right).
\]

Since \( M \) is Kahlerian, \( \nabla_X J = 0 \) and we have

\[
X^h \langle Y, J\xi \rangle = -X^h \langle JY, \xi \rangle = -X^i \partial_i \langle JY, \xi \rangle + \Gamma^s_{jk} \xi^j X^k \partial_n \langle JY, \xi \rangle = -X^i \langle J\nabla_i Y, \xi \rangle - X^i \xi^k \langle JY, \Gamma^s_{kl} \partial_s \rangle + \Gamma^s_{kl} \xi^k X^i \langle JY, \partial_s \rangle = \langle \nabla_X Y, J\xi \rangle.
\]

Therefore,

\[
X^h \langle \langle Y^v, Z^v \rangle \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle + \delta^2 \langle \nabla_X Y, J\xi \rangle \langle Z, J\xi \rangle + \delta^2 \langle Y, J\xi \rangle \langle \nabla_X Z, J\xi \rangle = \langle \langle (\nabla_X Y)^v, Z^v \rangle \rangle + \langle \langle Y^v, (\nabla_X Z)^v \rangle \rangle.
\]

iii) Rather evident, that \( X^v \langle \langle Y^h, Z^h \rangle \rangle = X^v \langle Y, Z \rangle = 0 \).

iv) Finally, it is easy to see that \( X^v \langle Y, J\xi \rangle = X^i \partial_{n+i} \langle Y, J\xi \rangle = \langle Y, JX \rangle \) and therefore

\[
X^v \langle \langle Y^v, Z^v \rangle \rangle = X^v \left( \langle Y, Z \rangle + \delta^2 \langle Y, J\xi \rangle \langle Z, J\xi \rangle \right) = \delta^2 \left( \langle Y, JX \rangle \langle Z, J\xi \rangle + \langle Y, J\xi \rangle \langle Z, JX \rangle \right).
\]
Now we can prove the lemma relatively easy applying the Kozsu l formula for the Levi-Civita connection

\[
2 \langle \nabla_A B, C \rangle = A \langle B, C \rangle + B \langle A, C \rangle - C \langle A, B \rangle + \\
\frac{1}{2} \left( \langle [A, B], C \rangle + \langle [C, A], B \rangle - \langle [B, C], A \rangle \right)
\]

and Dombrowski formulas to the metric (1).

(i) Setting \( A = X^h, B = Y^h, C = Z^h \) we see that

\[
2 \langle \langle \tilde{\nabla}_{X^h} Y^h, Z^h \rangle \rangle = 2 \langle \nabla_X Y, Z \rangle = 2 \langle \langle \nabla_X Y^h, Z^h \rangle \rangle.
\]

Setting \( A = X^h, B = Y^h, C = Z^v \), we have

\[
2 \langle \langle \tilde{\nabla}_{X^h} Y^h, Z^v \rangle \rangle = - Z^v \langle \langle X^h, Y^h \rangle \rangle + \langle \langle [X^h, Y^h], Z^v \rangle \rangle = \\
- \langle \langle (R(X, Y) \xi)^v, Z^v \rangle \rangle.
\]

Hence

\[
\tilde{\nabla}_{X^h} Y^h = (\nabla_X Y)^h - \frac{1}{2} (R(X, Y) \xi)^v
\]

(ii) Set \( A = X^h, B = Y^v, C = Z^h \). Then

\[
2 \langle \langle \tilde{\nabla}_{X^h} Y^v, Z^h \rangle \rangle = \langle \langle [Z^h, X^h], Y^v \rangle \rangle = \langle \langle (R(X, Z) \xi)^v, Y^v \rangle \rangle = \\
\langle R(X, Z) \xi, Y \rangle + \delta^2 \langle R(X, Z) \xi, J \xi \rangle \langle Y, J \xi \rangle = \\
\langle R(\xi, Y) X, Z \rangle + \delta^2 \langle Y, J \xi \rangle \langle R(\xi, J \xi) X, Z \rangle = \\
\langle \langle (R(\xi, Y) X + \delta^2 \langle Y, J \xi \rangle R(\xi, J \xi) X)^h, Z^h \rangle \rangle
\]

Set \( A = X^h, B = Y^v, C = Z^v \). Then, applying lemma \[\text{1.3}\] we have

\[
2 \langle \langle \tilde{\nabla}_{X^h} Y^v, Z^v \rangle \rangle = \\
X^h \langle \langle Y^v, Z^v \rangle \rangle + \langle \langle [X^h, Y^v], Z^v \rangle \rangle + \langle \langle [Z^v, X^h], Y^v \rangle \rangle = \\
\langle \langle (\nabla_X Y)^v, Z^v \rangle \rangle + \langle \langle Y^v, (\nabla_X Z)^v \rangle \rangle + \langle \langle (\nabla_X Y)^v, Z^v \rangle \rangle - \\
\langle \langle (\nabla_X Z)^v, Y^v \rangle \rangle = 2 \langle \langle (\nabla_X Y)^v, Z^v \rangle \rangle.
\]

So we see that

\[
\tilde{\nabla}_{X^h} Y^v = \frac{1}{2} (R(\xi, Y) X + \delta^2 \langle Y, J \xi \rangle R(\xi, J \xi) X)^h + (\nabla_X Y)^v.
\]
(iii) Set $A = X^v, B = Y^h, C = Z^h$. Then

$$2\langle\langle \nabla_{X^v} Y^h, Z^h \rangle\rangle = X^v\langle\langle Y^h, Z^h \rangle\rangle + \langle\langle [X^v, Y^h], Z^h \rangle\rangle + \langle\langle [Z^h, X^v], Y^h \rangle\rangle - \langle\langle [Y^h, Y^h], X^v \rangle\rangle = \langle\langle (R(Y, Z)\xi)^v, X^v \rangle\rangle = R(Y, Z)\xi, X\rangle + \delta^2 \langle\langle R(Y, Z)\xi, J\xi \rangle\rangle, X, J\xi \rangle = \langle\langle R(\xi, X)Y, Z \rangle\rangle + \delta^2 \langle\langle X, J\xi \rangle\rangle R(\xi, J\xi)Y, Z \rangle = \langle\langle (R(\xi, X)Y + \delta^2 <X, J\xi>)R(\xi, J\xi)Y^h, Z^h \rangle\rangle$

Set $A = X^v, B = Y^h, C = Z^v$. Then

$$2\langle\langle \nabla_{X^v} Y^h, Z^v \rangle\rangle = Y^h\langle\langle Z^v, X^v \rangle\rangle + \langle\langle [X^v, Y^h], Z^v \rangle\rangle - \langle\langle [Y^h, Z^v], X^v \rangle\rangle = \langle\langle (\nabla_Y Z)^v, X^v \rangle\rangle + \langle\langle Z^v, (\nabla_Y X)^v \rangle\rangle - \langle\langle (\nabla_Y Z)^v, X^v \rangle\rangle = 0$$

So, we have

$$\nabla_{X^v} Y^h = \frac{1}{2} \left( R(\xi, X)Y + \delta^2 \langle X, J\xi \rangle R(\xi, J\xi)Y \right)^h$$

(iv) Setting $A = X^v, B = Y^v, C = Z^h$, we have

$$2\langle\langle \nabla_{X^v} Y^v, Z^h \rangle\rangle = -Z^h\langle\langle X^v, Y^v \rangle\rangle + \langle\langle [Z^h, X^v], Y^v \rangle\rangle - \langle\langle [Y^v, Z^h], X^v \rangle\rangle = -\langle\langle (\nabla_Z X)^v, Y^v \rangle\rangle - \langle\langle X^v, (\nabla_X Y)^v \rangle\rangle + \langle\langle (\nabla_Z X)^v, Y^v \rangle\rangle + \langle\langle (\nabla_Z Y)^v, X^v \rangle\rangle = 0$$

Set, finally, $A = X^v, B = Y^v, C = Z^v$. Then

$$2\langle\langle \nabla_{X^v} Y^v, Z^v \rangle\rangle = X^v\langle\langle Y^v, Z^v \rangle\rangle + Y^v\langle\langle X^v, Z^v \rangle\rangle - Z^v\langle\langle X^v, Y^v \rangle\rangle = \delta^2 \left( \langle Y, JX \rangle \langle Z, \xi \rangle + \langle Y, J\xi \rangle \langle Z, JX \rangle + \langle X, JY \rangle \langle Z, J\xi \rangle \right. + \langle X, J\xi \rangle \langle JX, Z \rangle + \langle X, J\xi \rangle \langle JY, Z \rangle \rangle = 2\delta^2 \left( \langle Y, J\xi \rangle \langle JX, Z \rangle + \langle X, J\xi \rangle \langle JY, Z \rangle \right)$$

Thus, we see that

$$\langle\langle \nabla_{X^v} Y^v, Z^v \rangle\rangle = \delta^2 \left( \langle Y, J\xi \rangle \langle JX, Z \rangle + \langle X, J\xi \rangle \langle JY, Z \rangle \right)$$

On the other hand,

$$\langle\langle (JY)^v, Z^v \rangle\rangle = \langle JY, Z \rangle + \delta^2 \langle Y, \xi \rangle \langle Z, J\xi \rangle$$
and
\[ \langle\langle (J\xi)^v, Z^v \rangle \rangle = \langle J\xi, Z \rangle + \delta^2 \langle Z, J\xi \rangle |\xi|^2 = (1 + \delta^2 |\xi|^2) \langle Z, J\xi \rangle. \]

Therefore,
\[ \langle Z, J\xi \rangle = \frac{1}{1 + \delta^2 |\xi|^2} \langle\langle (J\xi)^v, Z^v \rangle \rangle \]
and as a consequence
\[ \langle JY, Z \rangle = \langle\langle (JY)^v, Z^v \rangle \rangle - \delta^2 \langle Y, \xi \rangle \frac{1}{1 + \delta^2 |\xi|^2} \langle\langle (J\xi)^v, Z^v \rangle \rangle = \langle\langle (JY)^v - \delta^2 \frac{1}{1 + \delta^2 |\xi|^2} \langle Y, \xi \rangle (J\xi)^v, Z^v \rangle \rangle. \]

So we have
\[ \langle\langle \tilde{\nabla}_X Y^v, Z^v \rangle \rangle = \delta^2 \langle\langle \left[ \langle X, J\xi \rangle (JY - \delta^2 \frac{1}{1 + \delta^2 |\xi|^2} \langle Y, \xi \rangle J\xi) + \langle Y, J\xi \rangle (JX - \delta^2 \frac{1}{1 + \delta^2 |\xi|^2} \langle X, \xi \rangle J\xi) \right]^v, Z^v \rangle \rangle. \]

Finally, we conclude that
\[ \tilde{\nabla}_X Y^v = \delta^2 \left( \langle X, J\xi \rangle JY + \langle Y, J\xi \rangle JX - \frac{\delta^2}{1 + \delta^2 |\xi|^2} \left( \langle Y, \xi \rangle \langle X, J\xi \rangle + \langle X, \xi \rangle \langle Y, J\xi \rangle \right) J\xi \right)^v. \]

2 Geodesics of the deformed metric.

Consider a curve $\Gamma$ on the tangent bundle with the metric (1). Geometrically, $\Gamma = \{ x(\sigma), \xi(\sigma) \}$, where $x(\sigma)$ is a curve on $M$ and $\xi(\sigma)$ is a vector field along this curve. Let $\sigma$ be an arc length parameter on $\Gamma$. Then $\Gamma' = \left( \frac{dx}{d\sigma} \right)^h + \left( \nabla_{\frac{dx}{d\sigma}} \xi \right)^v$. Introduce the notations $x' = \frac{dx}{d\sigma}$ and $\xi' = \nabla_{\frac{dx}{d\sigma}} \xi$. Then
\[ \Gamma' = (x')^h + (\xi')^v. \]

Using the Lemma 1.2 we can easily derive the differential equations of geodesic lines of the metric (1).
Lemma 2.1 Let \((M^{2n}, g, J)\) be Hermitian locally symmetric manifold and \(TM\) its Berger tangent bundle. A curve \(\Gamma = \{x(\sigma), \xi(\sigma)\}\) is a geodesic on \(TM\) if \(x(\sigma)\) and \(\xi(\sigma)\) satisfy the equations
\[
\begin{align*}
    x'' + R(\xi', \xi) x' &= 0, \\
    \xi'' + 2\delta^2 \langle \xi', J\xi \rangle \left( J\xi' - \frac{\delta^2}{1+\delta^2|\xi|^2} \langle \xi', \xi \rangle J\xi \right) &= 0,
\end{align*}
\]
where \(R(\xi, \xi') = R(\xi, \xi') + \delta^2 \langle \xi', J\xi \rangle R(\xi, J\xi)\) and \(R\) is the curvature operator of the base manifold \(M\).

Consider now the tangent sphere bundle \(T_1M\). The unit normal to \(T_1M\) is \(\xi_v\). Indeed, with respect to metric \((\ref{eq:metric})\) we have
\[
\begin{align*}
    \langle \langle X^h, \xi_v \rangle \rangle &= 0 \text{ for all } X \text{ tangent to } M, \\
    \langle \langle X^v, \xi_v \rangle \rangle &= 0 \text{ for all } X \in \xi^\perp.
\end{align*}
\]
So, to obtain the equations of geodesics for \(T_1M\), it is sufficient to set \(|\xi| = 1\) in \((\ref{eq:geodesic_equations})\) and to suppose the second equation left-hand side of \((\ref{eq:geodesic_equations})\) to be proportional to \(\xi\). Thus, we get

Lemma 2.2 Let \((M^{2n}, g, J)\) be Hermitian locally symmetric manifold and \(T_1M\) its Berger tangent sphere bundle. Set \(c = |\xi'|, \mu = \langle \xi', J\xi \rangle\). A curve \(\Gamma = \{x(\sigma), \xi(\sigma)\}\) is a geodesic on \(T_1M\) if and only if (a) \(c = \text{const}, \mu = \text{const}\); (b) \(x(\sigma)\) and \(\xi(\sigma)\) satisfy the equations
\[
\begin{align*}
    x'' + R(\xi, \xi') x' &= 0, \\
    \xi'' + c^2\xi + 2\delta^2 \mu (J\xi' + \mu \xi) &= 0,
\end{align*}
\]
where \(R(\xi, \xi') = R(\xi, \xi') + \delta^2 \langle \xi', J\xi \rangle R(\xi, J\xi)\) and \(R\) is the curvature operator of the base manifold \(M\).

Proof. Set \(|\xi| = 1\) in \((\ref{eq:geodesic_equations})\) and suppose that
\[
\xi'' + 2\delta^2 \langle \xi', J\xi \rangle J\xi' = \rho \xi,
\]
where \(\rho\) is some function.

Set \(c = |\xi'|.\) Then \(c = \text{const},\) since directly from \((\ref{eq:orthogonality})\) we see that \(\langle \xi'', \xi' \rangle = 0.\) Set \(\mu = \langle \xi', J\xi \rangle.\) Then \(\mu = \text{const}\) since \(\mu' = \langle \xi'', J\xi \rangle = 0\) by the similar reason. Multiplying \((\ref{eq:orthogonality})\) by \(\xi,\) we found that \(-\rho = c^2 + 2\delta^2 \mu^2 = \text{const}.\) After substitution of \(\rho\) into \((\ref{eq:orthogonality})\) we get what was claimed.

The difference in description of solutions of \((\ref{eq:geodesic_equations})\) and \((\ref{eq:geodesic_equations2})\) becomes clear because of different behaviour of the operator \(R(\xi, \xi')\) along the \(\pi \circ \Gamma.\)
Proposition 2.1 Let $\gamma = \pi \circ \Gamma$ be a projection of a curve $\Gamma$ on the Berger tangent (sphere) bundle over Hermitian locally symmetric manifold $M$. Then $R(\xi, \xi')$ is parallel along $\gamma$ for the case of $T_1M$ and non-parallel for the case of $TM$.

Proof. Consider the case of $T_1M$ first. Then using (3) we get

$$R' (\xi, \xi') = R(\xi, \xi'') + \delta^2 \mu R(\xi', J \xi) + \delta^2 \mu R(\xi, J \xi') = -2 \delta^2 \mu R(\xi, J \xi') - \delta^2 \mu R(J \xi', \xi) + \delta^2 \mu R(\xi, J \xi') = 0$$

Here we also used the fact that $R(J X, J Y) = R(X, Y)$.

A similar but slightly longer calculation shows that for the case of $TM$

$$R' (\xi, \xi') = 2 \delta \langle \xi', J \xi \rangle \langle \xi', \xi \rangle (1 - |\xi|^2) / (1 + \delta^2 |\xi|^2) R(\xi, J \xi)$$

which completes the proof.

Theorem 2.1 Let $\gamma = \pi \circ \Gamma$ be a projection of a curve $\Gamma$ on the Berger tangent sphere bundle over Hermitian locally symmetric manifold $M$. Then all geodesic curvatures of $\gamma$ are constant.

Proof. For the case of $T_1M$ the proposition 2.1 imply that if $\Gamma$ is geodesic on $T_1M$ than along each curve $\gamma = \pi \circ \Gamma$

$$x^{(p+1)}(\sigma) = -R(\xi, \xi') x^{(p)}(\sigma) \quad p \geq 1, \quad (5)$$

or, continuing the process,

$$x^{(p+1)}(\sigma) = (-1)^p R^p(\xi, \xi') x'(\sigma) \quad p \geq 1. \quad (6)$$

On the other hand, rather evident that

$$\langle R(\xi, \xi') X, Y \rangle = - \langle R(\xi, \xi') Y, X \rangle.$$

This fact and (5) imply

$$|x^{(p)}(\sigma)| = const \quad \text{for all } p \geq 1 \quad (7)$$
Indeed,

\[ \frac{d}{d\sigma} |x^{(p)}(\sigma)|^2 = 2 \langle x^{(p+1)}(\sigma), x^{(p)}(\sigma) \rangle = -2 \langle \mathcal{R}(\xi, \xi') x^{(p)}(\sigma), x^{(p)}(\sigma) \rangle = 0. \]

Denote \( s \) the natural parameter on \( \gamma \). Then \( x'_{\sigma} = x'_{s} \frac{ds}{d\sigma} \) and therefore

\[ 1 = \|\Gamma'\|^2 = \left| \frac{ds}{d\sigma} \right|^2 + |\xi'|^2 + \delta^2 \langle \xi', J\xi \rangle^2 = \left| \frac{ds}{d\sigma} \right|^2 + c^2 + \delta^2 \mu^2. \]

From this we get

\[ \frac{ds}{d\sigma} = \sqrt{1 - c^2 - \delta^2 \mu^2} = \sqrt{1 - \lambda^2}, \quad (8) \]

where we set \( \lambda^2 = c^2 + \delta^2 \mu^2 = \text{const} \).

Denote \( \nu_1, \ldots, \nu_{2n-1} \) the Frenet frame along \( \gamma \) and \( k_1, \ldots, k_{2n-1} \) the geodesic curvatures of \( \gamma \). Then, keeping in mind \( (8) \), we have

\[ x' = \sqrt{1 - \lambda^2} \nu_1, \]
\[ x'' = (1 - \lambda^2)k_1 \nu_2. \]

Now \( (7) \) imply \( k_1 = \text{const} \). So next we have

\[ x^{(3)} = (1 - \lambda^2)^{3/2} k_1(-k_1 \nu_1 + k_2 \nu_3) \]

and \( (7) \) imply again \( k_2 = \text{const} \). Continuing the process we finish the proof.

As it was proved in [4], for the case of \( T_1 CP^n \) and \( TCP^n \) with Sasaki metric the curvatures of \( \gamma = \pi \circ \Gamma \) are zeroes starting from \( k_6 \). It is rather remarkable that this property still valid for the case of Berger tangent sphere bundle over \( CP^n \).

**Theorem 2.2** Let \( \Gamma \) be a geodesic of the Berger tangent sphere bundle over the complex projective space \( CP^n \). Then the geodesic curvatures of \( \gamma = \pi \circ \Gamma \) are all constant and \( k_6 = \cdots = k_{2n-1} = 0 \).

**Proof.** For the case of \( CP^n \) we have

\[ R(X, Y)Z = \frac{m}{4} \left( \langle Y, Z \rangle X - \langle X, Z \rangle Y + \langle JY, Z \rangle JX - \langle JX, Z \rangle JY + 2\langle X, JY \rangle JZ \right). \]
Therefore, $R(\xi, J\xi) = -2J$ and for the case of Berger tangent sphere bundle
\[ R(\xi, \xi') = R(\xi, \xi') - 2\delta^2 J. \]
Since $R$ and $J$ commute, i.e. $RJ = JR$, the operators $R$ and $J$ also commute. Using this fact one can relatively easy to find expression for powers of the operator $R$ along $\gamma$. Indeed, in [4] it was proved that the powers of the curvature operator of $CP^n$ satisfy the relations
\[ R^{2q} = \text{Lin} (R^2, JR, E), \quad R^{2q+1} = \text{Lin} (JR^2, R, J), \]
where Lin means a linear combination of corresponding tensors and $E$ means the identity operator. It is elementary to see that
\[
\begin{align*}
\text{Lin} (R, J) \cdot \text{Lin} (R, J) &= \text{Lin} (R^2, JR, E) \\
\text{Lin} (R^2, JR, E) \cdot \text{Lin} (R, J) &= \text{Lin} (JR^2, R, J) \\
\text{Lin} (JR^2, R, J) \cdot \text{Lin} (R, J) &= \text{Lin} (R^2, JR, E).
\end{align*}
\]
Therefore, for the operator $R$ we also have
\[ R^{2q} = \text{Lin} (R^2, JR, E), \quad R^{2q+1} = \text{Lin} (JR^2, R, J). \] (9)
On the other hand, $RJ = JR$ and $R = R - 2\delta^2 J$. Therefore,
\[ R^2 = \text{Lin} (R^2, JR, E) \quad JR = \text{Lin} (JR, R) \quad JR^2 = \text{Lin} (JR, JR, J). \]
Taking this into account, we may rewrite (9) as
\[ R^{2q} = \text{Lin} (R^2, JR, E), \quad R^{2q+1} = \text{Lin} (JR^2, R, J). \] (10)
Using now (5), (6) and (10), we get
\[
\begin{align*}
x^{(2q)} &= \text{Lin} (Jx'''', x''', Jx') \\
x^{(2q+1)} &= \text{Lin} (x'''', Jx'', x')
\end{align*}
\] (11)
for $q \geq 2$. On the other hand, Frenet formulas yield
\[ x' = \sqrt{1 - \lambda^2} \nu_1 \quad x'' = (1 - \lambda^2)k_1 \nu_2, \quad x''' = (1 - \lambda^2)^{3/2}(-k_1^3 \nu_1 + k_1 k_2 \nu_3) \]
and in general,
\[
\begin{align*}
x^{(2q)} &= \text{Lin} (\nu_2, \ldots, \nu_{2q-2}) + (1 - \lambda^2)^q k_1 \ldots k_{2q-1} \nu_{2q} \\
x^{(2q+1)} &= \text{Lin} (\nu_1, \ldots, \nu_{2q-1}) + (1 - \lambda^2)^{q+1/2} k_1 \ldots k_{2q} \nu_{2q+1}
\end{align*}
\] (12)
Thus, (11) takes the form

\[ x^{(2q)} = \text{Lin} (J \nu_1, J \nu_3, \nu_2), \quad x^{(2q+1)} = \text{Lin} (\nu_1, \nu_3, J \nu_2). \]

Comparing the results, for all \( q \geq 2 \) we have

\[
\begin{align*}
\text{Lin} (J \nu_1, J \nu_3, \nu_2) &= \text{Lin} (\nu_2, \ldots, \nu_{2q-2}) + (1 - \lambda^2)^{q} k_1 \ldots k_{2q-1} \nu_{2q}, \\
\text{Lin} (\nu_1, \nu_3, J \nu_2) &= \text{Lin} (\nu_1, \ldots, \nu_{2q-1}) + (1 - \lambda^2)^{q+1/2} k_1 \ldots k_{2q} \nu_{2q+1}.
\end{align*}
\]

Setting \( q = 2 \) and \( q = 3 \), from the second equation above, we get

\[
\begin{align*}
\text{Lin} (\nu_1, \nu_3, J \nu_2) &= (1 - \lambda^2)^{5/2} k_1 \ldots k_4 \nu_5 \\
\text{Lin} (\nu_1, \nu_3, \nu_5, J \nu_2) &= (1 - \lambda^2)^{7/2} k_1 \ldots k_6 \nu_7,
\end{align*}
\]

and therefore, in general,

\[ \text{Lin} (\nu_1, \nu_3, \nu_5) = k_1 \ldots k_6 \nu_7. \]

Since \( \nu_1, \ldots, \nu_7 \) are linearly independent, we conclude that \( k_6 = 0 \).

\[ \blacksquare \]

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