The fixed set of the inverse involution on a Lie group

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Abstract

In [7] we have determined the isomorphism type of the centralizer of an element in a simple Lie group. As a sequel to [7] we present a general procedure to calculate the isomorphism type of the fixed set $\text{Fix}(\gamma)$ of the inverse involution $\gamma$ on a Lie group $G$.

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1 Introduction

Let $G$ be a compact, connected and simple Lie group with group unit $e \in G$. The inverse involution on $G$ is the periodic 2 transformation $\gamma$ sending each group element $g \in G$ to its inverse $g^{-1} \in G$. In this paper we present a general procedure to calculate the isomorphism type of the fixed set

\[ \text{Fix}(\gamma) = \{ g \in G \mid g = g^{-1} \} \]

of the involution $\gamma$.

Given a group element $x \in G$ let $M_x$, $C_x \subset G$ be the adjoint orbit through $x$ and the centralizer of $x$ in $G$, respectively. That is

\[ M_x = \{ gxg^{-1} \in G \mid g \in G \}; \quad C_x = \{ g \in G \mid gx = xg \}. \]

The map $G \to G$ by $g \to gxg^{-1}$ is constant along the left cosets of $C_x$ in $G$, and induces a diffeomorphism from the homogeneous space $G/C_x$ onto the orbit space $M_x$.

\[ f_x : G/C_x \xrightarrow{\cong} M_x, [g] \to gxg^{-1}. \]

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In view of this identification the isomorphism type of the orbit space $M_x$ is completely determined by the centralizer $C_x$. It is crucial to notice that $x \in \text{Fix}(\gamma)$ implies that $M_x \subset \text{Fix}(\gamma)$. Naturally, one asks for a partition of the space $\text{Fix}(\gamma)$ by certain adjoint orbits $M_x$, and determine the isomorphism types of the corresponding centralizers $C_x$.

Concerning the applications of our approach we assume the reader’s familiarity with the classification on Lie groups. In particular, all 1–connected compact simple Lie groups consists of the three infinite families $SU(n + 1), Sp(n), Spin(n + 2), n \geq 2$, of classical groups, and the five exceptional Lie groups $G_2, F_4, E_6, E_7, E_8$. For a classical Lie group $G$ the fixed set $\text{Fix}(\gamma)$ can be easily calculated using linear algebra, see Frankel [8]. For this reason we shall restrict ourself to the simple exceptional Lie groups. Explicitly we shall have

$$G = G_2, F_4, E_6, E_7, E_8$$

where $G^* = G/\mathcal{Z}(G)$ with $\mathcal{Z}(G)$ the center of $G$.

Fix a maximal torus $T$ in $G$ and let $\exp : L(T) \to T$ be the exponential map, where $L(T)$ is the tangent space to $T$ at the unit $e$. In term of a set $\Omega = \{\omega_1, \ldots, \omega_n\} \subset L(T)$ of fundamental dominant weights of $G$ (see Definition 2.3), together with the fundamental Weyl cell $\Delta$ corresponding to $\Omega$, our main result is stated below.

Let $SO(n)$ and $Ss(n)$ be the special orthogonal group and the semispinor group of order $n$, respectively. For a connected Lie group $H$ write $[H]^2$ for the group with two components whose identity component is $H$. For two manifolds $M$ and $N$ denote by $M \coprod N$ their disjoint union.

**Theorem 1.1.** For a simple Lie group $G$ there is a subset $\mathcal{F}_G \subset \Delta$ so that

$$\text{Fix}(\gamma) = \{e\} \bigcup_{u \in \mathcal{F}_G} M_{\exp(u)}.$$  

Moreover, for each exceptional Lie group $G$ the set $\mathcal{F}_G$, as well as the isomorphism type of the adjoint orbit $M_{\exp(u)}$ with $u \in \mathcal{F}_G$, is tabulated below

| $G$  | $\mathcal{F}_G$                                                                 | $M_{\exp(u)} = G/C_{\exp(u)}, u \in \mathcal{F}_G$ |
|------|-------------------------------------------------------------------------------|--------------------------------------------------|
| $G_2$| $\{\frac{\pi}{2}\}$                                                          | $G_2/\text{SO}(4)$                              |
| $F_4$| $\{\frac{\pi}{2}, \omega\}_{k=1,4}$                                         | $F_4/\text{Spin}(9), F_4/\frac{Sp(3) \times Sp(1)}{Z_2}$ |
| $E_6$| $\{\frac{\pi}{2}, \omega_1 = \omega_8\}$                                     | $E_6/\frac{SU(2) \times SU(6)}{Z_2}, E_6/\frac{Spin(10) \times S^1}{Z_2}$ |
| $E_6^*$| $\{\omega_2\}_{k=1,2}$                                                      | $E_6^*/\frac{Spin(10) \times S^1}{Z_2}, E_6^*/\frac{SU(2) \times SU(6)}{Z_2}$ |
| $E_7$| $\{\frac{\pi}{2}, \omega_7\}_{k=1,6}$                                       | $E_7/\frac{Spin(12) \times SU(2)}{Z_2}, E_7/\frac{Spin(12) \times SU(2)}{Z_2}$, $\exp(\omega_7)$ |
| $E_7^*$| $\{\omega_2\}_{k=1,2,7}$                                                    | $E_7^*/\frac{Ss(12) \times SU(2)}{Z_2}, E_7^*/[\frac{SU(8)}{Z_4}]^2, E_7^*/[\frac{E_8 \times S^1}{Z_2}]^2$ |
| $E_8$| $\{\frac{\pi}{2}\}_{k=1,8}$                                                 | $E_8/\text{Ss}(16), E_8/\frac{E_8 \times SU(2)}{Z_2}$ |

Table 1. The fixed sets of the inverse involution on exceptional Lie groups

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Historically, the problem of determining the isomorphism type of the fixed set $Fix(\gamma)$ of a simple Lie group $G$ has been studied by Frankel [8] for the classical Lie groups, and by Chen, Nagano [6, 11], Yokota [12, 13, 14] for the exceptional Lie groups, see Remark 4.4. These works rely largely on the specialities of each individual Lie group and the calculations were performed case by case. In comparison, our approach is free of the types of simple Lie groups, and is ready to extend to general cases, see Corollaries 5.1–5.2 of Section 5.

The paper is arranged as follows. Section §2 contains a brief introduction to the roots and weight systems of simple Lie groups. In Section §3 the set $\mathcal{F}_G$ specifying the partition on $Fix(\gamma)$ in formula (1.2) is largely determined by Lemma 3.3. Combining Lemma 3.3 with the algorithm calculating the isomorphism type of a centralizer $C_x$ obtained in [7], Theorem 1.1 is established in Section 4. Finally, general structure of the fixed set $Fix(\gamma)$ of the inverse involution $\gamma$ on an arbitrary Lie group $G$ is discussed briefly in Section 5.

2 Geometry of roots and weights

For a simple Lie group $G$ with Lie algebra $L(G)$ and a maximal torus $T$, the dimension $n = \dim T$ is called the rank of $G$, and the subspace $L(T)$ of $L(G)$ is called the Cartan subalgebra of $G$. Equip $L(G)$ with an inner product $(,)$ so that the adjoint representation acts on $L(G)$ as isometries, and let

$$d : G \times G \to \mathbb{R} \quad (\text{resp. } d : T \times T \to \mathbb{R})$$

be the induced metric on $G$ (resp. on $T$).

The restriction of the exponential map $\exp : L(G) \to G$ to $L(T)$ defines a set $S(G) = \{L_1, \cdots, L_m\}$ of $m = \frac{1}{2}(\dim G - n)$ hyperplanes in $L(T)$, namely, the set of singular hyperplanes through the origin in $L(T)$ [2, p.168]. Let $l_k \subset L(T)$ be the normal line of the plane $L_k$ through the origin, $1 \leq k \leq m$. Then the map $\exp$ carries $l_k$ onto a circle subgroup of $G$.

**Definition 2.1.** Let $\pm \alpha_k \in l_k$ be the non–zero vectors with minimal length so that $\exp(\pm \alpha_k) = e$, $1 \leq k \leq m$. The subset

$$\Phi = \{\pm \alpha_k \in L(T) \mid 1 \leq k \leq m\}$$

of $L(T)$ is called the root system of $G$.

The Weyl group of $G$, denoted by $W$, is the subgroup of $\text{Aut}(L(T))$ generated by the reflections $r_k$ in the hyperplane $L_k$, $1 \leq k \leq m$. □

**Remark 2.2.** We point out that the root system $\Phi$ by Definition 2.1 is dual to those that are commonly used in literatures, e.g. [1, 10].
particular, the symplectic group $Sp(n)$ is of the type $B_n$, while the spinor group $Spin(2n+1)$ is of the type $C_n$. □

The planes in $S(G)$ divide $L(T)$ into finitely many convex open cones, called the Weyl chambers of $G$. Fix once and for all a regular point $x_0 \in L(T) \setminus \bigcup_{1 \leq k \leq m} L_m$, and let $F(x_0)$ be the closure of the Weyl chamber containing $x_0$. Assume that $L(x_0) = \{L_1, \cdots, L_n\}$ is the subset of $S(G)$ consisting of the walls of $F(x_0)$, and let $\alpha_i \in \Phi$ be the root normal to the wall $L_i \in L(x_0)$ and pointing toward $x_0$. Then the subset $\Delta(x_0) = \{\alpha_1, \cdots, \alpha_n\}$ of $\Phi$ is called the system of simple roots of $G$ relative to $x_0$.

**Definition 2.3** ([10, p.67]). Each root $\alpha \in \Phi$ gives rise to a linear map $\alpha^* : L(T) \to \mathbb{R}$ by $\alpha^*(x) = 2(x, \alpha)/(|\alpha|^2)$, called the inverse root of $\alpha$. The weight lattice of $G$ is the subset of $L(T)$

$$\Lambda = \{x \in L(T) \mid \alpha^*(x) \in \mathbb{Z} \text{ for all } \alpha \in \Phi\},$$

whose elements are called weights. Elements in the subset of $\Lambda$

$$\Omega = \{\omega_i \in L(T) \mid \alpha_j^*(\omega_i) = \delta_{i,j}, \alpha_j \in \Delta(x_0)\}$$

are called the fundamental dominant weights of $G$ relative to $x_0$, where $\delta_{i,j}$ is the Kronecker symbol. □

To be precise we adopt the convention that for each simple group $G$ with rank $n$ its fundamental dominant weights $\omega_1, \cdots, \omega_n$ are ordered by the order of their corresponding simple roots pictured as the vertices in the Dynkin diagram of $G$ in [10, p.58]. Useful properties of the weights are:

**Lemma 2.4.** Let $\Omega = \{\omega_1, \cdots, \omega_n\}$ be the set of fundamental dominant weights relative to the regular point $x_0$. Then,

i) $\Omega = \{\omega_1, \cdots, \omega_n\}$ is a basis for $\Lambda$ over $\mathbb{Z}$;

ii) for each $1 \leq i \leq n$ the half line $\{t\omega_i \in L(T) \mid t \in \mathbb{R}^+\}$ is the edge of the Weyl chamber $F(x_0)$ opposite to the wall $L_i$;

iii) if $G$ is simple, $(\omega_i, \omega_j) > 0$ for all $1 \leq i, j \leq n$. □

**Proof.** Property i) is well known. By (2.2) each weight $\omega_i \in \Omega$ is perpendicular to $\alpha_j$ (i.e. $\omega_i \in L_j$) for all $j \neq i$, $1 \leq j \leq n$. This verifies ii). For iii) we refer to [10, p.72, Exercise 8]. □

Let $Z(G)$ be the center of the group $G$, and let $\Lambda_c = \exp^{-1}(e) \subset L(T)$ be the unit lattice. The set $\Delta(x_0) = \{\alpha_1, \cdots, \alpha_n\}$ of simple roots spans also a lattice $\Lambda_r$ on $L(T)$, known as the root lattice of $G$.

**Lemma 2.5** ([7, (3.3)]). In the Euclidean space $L(T)$ one has
\[ \Lambda = \exp^{-1}(Z(G)); \quad \text{ii) } \Lambda_r \subseteq \Lambda_e \subseteq \Lambda, \]
where in ii), the first equality holds if and only if \( G \) is 1–connected, and the second equality holds if and only if \( Z(G) = \{0\}. \]

For a simple Lie group \( G \) the quotient group \( \Lambda/\Lambda_e \) is always finite (see [10, p.68]). As a result we can introduce the \textit{deficiency function} on the weight lattice

\begin{equation}
\kappa : \Lambda \to \mathbb{Z}, \quad x \mapsto \kappa_x,
\end{equation}

by letting \( \kappa_x \) be the least positive integer so that \( \kappa_x x \in \Lambda_e, \ x \in \Lambda \). This function provides us with a partition \( \Omega = \Omega_1 \sqcup \Omega_2 \) with

\[ \Omega_1 = \{ \omega \in \Omega \mid \kappa_\omega = 1 \}, \quad \Omega_2 = \{ \omega \in \Omega \mid \kappa_\omega \geq 2 \}. \]

\textbf{Example 2.6.} Let \( G \) be a simple Lie group.

If \( Z(G) = \{0\} \) we get from \( \Lambda_e = \Lambda \) by Lemma 2.5 that \( \Omega = \Omega_1 \).

If \( G \) is 1–connected with \( Z(G) \neq \{0\} \), we have \( \Lambda_e = \Lambda_r \) by Lemma 2.5. From the expressions of the fundamental dominant weights by simple roots in [10, p.69] one determines the subset \( \Omega_1 \), consequently \( \Omega_2 \), as that tabulated below:

| \( G \) | \( A_n \) | \( Sp(n) \) | \( Spin(2n + 1) \) | \( Spin(2n) \) | \( E_6 \) | \( E_7 \) |
|-------|--------|--------|--------|--------|--------|--------|
| \( \Omega_1 \) | \( \emptyset \) | \( \{ \omega_i \}_{i=1}^{n} \) | \( \{ \omega_i \}_{i=2k} \) | \( \{ \omega_i \}_{i=2k} \) | \( \{ \omega_i \}_{i=2,4} \) | \( \{ \omega_i \}_{i=1,3,4,6} \) |

The set \( \Delta(x_0) \) of simple roots is a basis for both \( L(T) \) and the root lattice \( \Lambda_r \). Using this basis a partial order \( \prec \) on \( L(T) \) (hence on \( \Phi \subset L(T) \)) can be introduced by the following rule:

\[ v \prec u \text{ if and only if the difference } u - v \text{ is a sum of elements of } \Delta(x_0). \]

As in [10, p.67] we put \( \Lambda^+ := \Lambda \cap \mathcal{F}(x_0) \). An element \( \omega \in \Lambda^+ \) is called minimal if \( \omega \succ \omega' \in \Lambda^+ \) implies that \( \omega = \omega' \).

\textbf{Lemma 2.7.} Let \( G \) be an 1–connected simple Lie group, and let \( \Pi_G \subset \Lambda^+ \) be the subset of all non–zero minimal weights. Then \( \Pi_G \subset \Omega_2 \). Moreover,

i) for each \( \omega \in \Omega_2 \) there is precisely one weight \( \omega' \in \Pi_G \) so that \( \omega \succ \omega' \); 

ii) the set of all non–trivial elements in \( Z(G) \) are given without repetition by \( \{ \exp(\omega) \in Z(G) \mid \omega \in \Pi_G \} \).

\textbf{Proof.} See [10, P.92]. \( \square \)

In view of Lemma 2.7 we can introduce a \textit{retraction} \( r : \Omega_2 \to \Pi_G \) and an \textit{involution} \( \tau : \Pi_G \to \Pi_G \) respectively by the rules:

a) \( \omega \succ r(\omega) \in \Pi_G, \omega \in \Omega_2 \) (by i) of Lemma 2.7);

b) \( \tau(\omega) + \omega \in \Lambda_e, \omega \in \Pi_G \) (by ii) of Lemma 2.7).

Alternatively, the element \( \tau(\omega) \) is characterized by the relation
\[ \exp(\omega) \exp(\tau(\omega)) = e. \]

**Example 2.8.** Assume that \( G \) is simple and 1–connected with \( Z(G) \neq \{0\} \).

a) The set \( \Pi_G \) of minimal weights is given by (see [10, P.92]):

\[
\begin{array}{cccc}
G & SU(n) & Sp(n) & Spin(2n+1) & Spin(2n) & E_6 & E_7 \\
\Pi_G & \{\omega_i \}_{1 \leq i \leq n} & \{\omega_n\} & \{\omega_1, \omega_n\} & \{\omega_1, \omega_n\} & \{\omega_1, \omega_6\} & \{\omega_7\} \\
\end{array}
\]

b) The set \( \Omega_2 \), as well as the composition \( \tau \circ r : \Omega_2 \to \Pi_G \), is given by

\[
\begin{array}{cccc}
G & SU(n) & Sp(n) & Spin(2n+1) & Spin(2n) & E_6 & E_7 \\
\Omega_2 & \{\omega_k \}_{1 \leq k \leq n} & \{\omega_n\} & \{\omega_{2k+1}\} & \{\omega_k \}_{k=1,3,5,6} & \{\omega_k \}_{k=2,5,7} \\
\tau \circ r(\omega_k) & \omega_{n+1-k} & \omega_n & \omega_1 & \omega_k & \omega_7 \\
\end{array}
\]

and for \( G = Spin(2n) \), by \( \Omega_2 = \{\omega_k, \omega_{n-1}, \omega_n \mid k \leq n - 2 \ \text{odd}\} \),

\[
\tau \circ r(\omega_k) = \begin{cases} 
\omega_1 & \text{if } k \leq n - 2; \\
\omega_n & \text{if either } n \text{ is odd, } k = n - 1, \text{ or } n \text{ is even, } k = n; \\
\omega_{n-1} & \text{if either } n \text{ is odd, } k = n, \text{ or } n \text{ is even, } k = n - 1. 
\end{cases}
\]

\[ \square \]

### 3 Computation in the fundamental Weyl cell

For a simple Lie group \( G \) elements in the root system \( \Phi \) has at most two lengths. Let \( \beta \) be the **maximal short root** relative to the partial order \( \prec \) on the set \( \Phi^+ \) of positive roots [10, p.55]. The **fundamental Weyl cell** is the simplex in the Weyl chamber \( \mathcal{F}(x_0) \) defined by

\[ \Delta = \{ u \in \mathcal{F}(x_0) \mid \beta^*(u) \leq 1 \}. \]

Let \( d : T \times T \to \mathbb{R} \) be the distance function on \( T \). It is well known that

**Lemma 3.1 ([11, 5]).** Let \( G \) be a simple Lie group. Then

i) the equation \( d(e, \exp(u)) = \|u\| \) holds if and only if \( \|u\| \leq \|u - v\| \) holds for all \( v \in \Lambda_e \);

ii) if \( G \) is 1–connected, then \( u \in \Delta \) implies that \( d(e, \exp(u)) = \|u\|. \)

It is well known that every element \( x \in G \) is conjugate under \( G \) to an element of the form \( \exp(u) \in G \) with \( u \in \Delta \) and \( d(e, \exp(u)) = \|u\| \).

Moreover, if \( x \in Fix(\gamma) \) then \( 2u \in \Lambda_e \). This implies that

**Lemma 3.2.** For a simple Lie group \( G \) with fundamental Weyl cell \( \Delta \) set
(3.1) $K_G = \{ u \in \Delta \mid 2u \in \Lambda_e, \ d(e, \exp(u)) = \|u\| \}$.

Then

(3.2) $\text{Fix}(\gamma) = \{ e \} \bigcup_{u \in K_G} M_{\exp(u)}$ with $d(e, x) = \|u\|$ for all $x \in M_{\exp(u)}$. □

Comparing (3.2) with (1.2) we emphasise that the decomposition (3.2) on $\text{Fix}(\gamma)$ may not be disjoint, as overlap like $M_{\exp(u)} = M_{\exp(v)}$ may occur for some $u, v \in K_G$ with $u \neq v$. However, based on the relation (3.2) our approach to $\text{Fix}(\gamma)$ consists of three steps:

i) find a general expression for elements in $K_G$;  
ii) specify a subset $F_G \subseteq K_G$ so that the relation (3.2) can be refined as $\text{Fix}(\gamma) = \{ e \} \bigcup_{u \in F_G} M_{\exp(u)}$;  
iii) decide the isomorphism types of $M_{\exp(u)}$ for all $u \in F_G$.

In this section we accomplish step i) in the next result.

Lemma 3.3. Let $G$ be a simple Lie group. Then $u \in K_G$ implies that

(3.3) $u = \begin{cases} \frac{1}{2}\omega_k \text{ for some } \omega_k \in \Omega_1, \\ \frac{1}{2}(\omega_k + \tau \circ r(\omega_k)) \text{ for some } \omega_k \in \Omega_2. \end{cases}$

Proof. For an $u \in K_G$ we get from $u \in \Delta$ and $2u \in \Lambda_e$ that

$u = \lambda_{k_1}\omega_{k_1} + \cdots + \lambda_{k_t}\omega_{k_t}$ with $\lambda_{k_s} > 0$ and $2\lambda_{k_s} \in \mathbb{Z}$

by Lemma 2.4, where $\{k_1, \cdots, k_t\} \subseteq \{1, \cdots, n\}$. This implies that

(3.4) $2u - \omega_{k_1} = a \in \Lambda^+$.  

The formula (3.3) will be deduced from the second constraint $d(e, \exp(u)) = \|u\|$ on $u \in K_G$ in (3.1).

If $\omega_{k_1} \in \Omega_1$ then $\omega_{k_1} \in \Lambda_e$ implies that $\|u\| \leq \|u - \omega_{k_1}\|$ by i) of Lemma 3.1. That is

$\| \frac{1}{2}a + \frac{1}{2}\omega_{k_1} \|^2 \leq \| \frac{1}{2}a - \frac{1}{2}\omega_{k_1} \|^2$

by (3.4). However, since $(\omega_i, \omega_j) > 0$ by iii) of Lemma 2.4 and since $a \in \Lambda^+$, this is possible if and only if $a = 0$. That is

(3.5) $u = \frac{1}{2}\omega_{k_1}$.  

If $\omega_{k_1} \in \Omega_2$ we have $a \in \Lambda^+$ but $a \not\in \Lambda_e$ by (3.4). According to Lemma 2.7 there is precisely one weight $\omega_s \in \Pi_G$ so that $2u - \omega_{k_1} = a = \omega_s + b$ with $b$ a sum of elements of $\Delta(x_0)$. From $2u, b \in \Lambda_e$ we find that $\omega_{k_1} + \omega_s \in \Lambda_e$ and therefore $\|u\| \leq \|u - \omega_{k_1} - \omega_s\|$ by i) of Lemma 3.1. That is
\[
\left\| \frac{1}{2}(\omega_k + \omega_s) + \frac{1}{2}b \right\|^2 \leq \left\| \frac{1}{2}(\omega_k + \omega_s) - \frac{1}{2}b \right\|^2.
\]
Again, since \((\omega_i, \omega_j) > 0\) by iii) of Lemma 2.4 and since \(b \in \Lambda^+\), this is possible if and only if \(b = 0\). We obtain from (3.4) that \(u = \frac{1}{2}(\omega_k + \omega_s)\), \(\omega_s \in \Pi_G\). Furthermore, from the calculation
\[
e = \exp(2u) = \exp(\omega_k) \exp(\omega_s) = \exp(r(\omega_k)) \exp(\omega_s)
\]
(since \(\omega_k > r(\omega_k) \in \Pi_G\)) as well as the definition of \(\tau\) we get \(\omega_s = \tau \circ r(\omega_k)\).
This shows that
\[
(3.6) \ u = \frac{1}{2}(\omega_k + \tau \circ r(\omega_k)) \text{ with } \omega_k \in \Omega_2.
\]
The proof of (3.3) has now been completed by (3.5) and (3.6). □

4 Proof of Theorem 1.1

Assume that \(G\) is an exceptional Lie group and the expression of its maximal short root \(\beta\) in term of the simple roots is \(\beta = m_1\alpha_1 + \cdots + m_n\alpha_n\) (see [10, p.66]). By the definition (2.2) of the fundamental dominant weights
\[
(4.1) \ \beta^\ast\left(\frac{\omega}{2}\right) = m_1\|\alpha_1\|^2; \quad \beta^\ast\left(\frac{\omega + \omega_j}{2}\right) = m_1\|\alpha_1\|^2 + m_j\|\alpha_j\|^2.
\]
Let \(K'_G \subset \Delta\) be the subset of the vectors \(u\) satisfying (3.3). Combining (3.3) and (4.1), together with computations in Examples 2.6 and 2.8, one determines the set \(K'_G\) for each exceptional \(G\), as that presented in the second column of Tables 2 below.

In [7] an explicit procedure to calculate the isomorphism type of the centralizer \(C_{\exp(u)} \subset G\) in term of \(u \in \Delta\) is obtained. As applications those centralizers \(C_{\exp(u)}\) with \(u \in K'_G\) are determined and presented in the third column of Table 2 (see also in [7, Theorem 4.4, Theorem 4.6]).

In general \(K_G \subseteq K'_G\) by Lemma 3.3. However, the centralizers \(C_{\exp(u)}\) recorded in Table 2 are useful for us to specify the desired subset \(K_G\) from \(K'_G\). To explain this we observe that if \(u \in K'_G\) is a vector with \(u \notin K_G\), then \(d(e, \exp(u)) < \|u\|\) implies that there exists a vector \(v \in L(T)\) satisfying
\[
\exp(v) = \exp(u) \text{ and } d(e, \exp(v)) = \|v\|.
\]
Take a Weyl group element \(w \in W\) so that \(v' = w(v) \in F(x_0)\). The relations \(2v' \in \Lambda_+\) and \(d(e, \exp(v')) = \|v'\|\) indicate that \(v' \in K_G\) by (3.1). In particular we have

Lemma 4.1. If \(u \in K'_G \setminus K_G\) then there exists an element \(vt \in K_G\) so that
\[C_{\exp(u)} \cong C_{\exp(v')}\text{ and } \|v'\| < \|u\|. \square\]
The centralizer $C_{\text{exp}}(u)$ with $u \in \mathcal{K}'_G$

| $G$ | $\mathcal{K}'_G$ | $SO(4)$ | $Spin(9), Sp(3) \times Sp(1)$ |
|-----|-----------------|---------|--------------------------------|
| $G_2$ | $\{ \frac{1}{2} \omega_1 \}$ | $SU(2) \times SU(6)$ | $Spin(10) \times S^1$ |
| $F_4$ | $\{ \frac{1}{2} \omega_k \}_{k=1,4}$ | $Z_2$ | $Z_4$ |
| $E_6$ | $\{ \frac{\omega}{2}, \omega_1 + \omega_6 \}$ | $Spin(10) \times S^1$ for $k = 1, 6$; $SU(2) \times SU(6)$ for $k = 2, 3, 5$ |
| $E_6^*$ | $\{ \frac{1}{2} \omega_k \}_{k=1,2,3,5,6}$ | $Z_2$ | $Z_4$ |
| $E_7$ | $\{ \frac{1}{2} \omega_j \}_{j=1,6}$ | $Spin(12) \times SU(2)$, $Spin(12) \times SU(2)$ | $E_7$ |
| $E_7^*$ | $\{ \frac{1}{2} \omega_k \}_{k=1,2,6,7}$ | $Z_2$ | $Z_4$ |
| $E_8$ | $\{ \frac{1}{2} \omega_k \}_{k=1,8}$ | $Spin(12) \times SU(2)$, $Spin(12) \times SU(2)$, $[E_6 \times S^1]^2$ |

Table 2. The set $\mathcal{K}'_G$ as well as the centralizers $C_{\text{exp}}(u)$ with $u \in \mathcal{K}'_G$.

**Proof of Theorem 1.1.** The proof will be divided into two cases, depending on whether $G$ is 1–connected. Concerning the use of Lemma 4.1 in the forthcoming arguments, we note that in view of the presentation of the fundamental dominant weights with respect to appropriate Euclidean coordinates $\{ \varepsilon_1, \cdots, \varepsilon_m \}$ on $L(T)$ in the standard reference [1] p.265-277, the length $\|u\|$ for a vector $u \in \mathcal{K}'_G$ can be easily evaluated.

**Case I.** $G = G_2, F_4, E_6, E_7, E_8$. Since $G$ is 1–connected, we have

$$K_G = \mathcal{K}'_G$$

by ii) of Lemma 3.1. Consequently, it follows from (3.2) that

$$\text{Fix}(\gamma) = \{ e \} \bigcup_{u \in \mathcal{K}_G} M_{\text{exp}}(u) \text{ with } M_{\text{exp}}(u) = G / C_{\text{exp}}(u).$$

Assume in the decomposition (4.3) on $\text{Fix}(\gamma)$ that the relation $M_{\text{exp}}(u) = M_{\text{exp}}(v)$ holds for some $u, v \in \mathcal{K}_G$ with $u \neq v$. By (3.2) one has

i) $C_{\text{exp}}(u) \cong C_{\text{exp}}(v)$ and ii) $\|u\| = \|v\|$. However, in view of the groups $C_{\text{exp}}(u)$ presented in Table 2 the only possibility for i) to hold is when $G = E_7$ and $(u, v) = (\frac{\omega}{2}, \frac{\omega_7}{2})$, but in this case the calculation

$$\| \frac{\omega}{2} \| = \frac{1}{\sqrt{2}} < \| \frac{\omega_7}{2} \| = 1 \text{ (see [1] p.280)},$$

shows that the relation ii) does not hold. Summarizing, taking $\mathcal{F}_G = \mathcal{K}_G$ then the decomposition (1.2) on $\text{Fix}(\gamma)$ is given by (4.3), and the proof of Theorem 1.1 is completed by the corresponding items in Table 2.

**Case II.** $G = E_6^*, E_7^*$. This case is slightly delicate because, instead of the equality (4.2) one has $\mathcal{K}_G \subseteq \mathcal{K}'_G$ by Lemma 3.3. Nevertheless, granted with Lemma 4.1 and results in Table 2 we shall show that

$$K_G = \begin{cases} \{ \frac{\omega}{2} \}_{k=1,2,6} & \text{if } G = E_6^* \\ \{ \frac{\omega}{2} \}_{k=1,2,7} & \text{if } G = E_7^* \end{cases}$$
(4.5) for \( u, v \in K_G \) the overlap \( M_{\exp(u)} = M_{\exp(v)} \) (see (3.2)) happens if and only if \( G = E_6^* \) and \( (u, v) = (\frac{\omega_1}{2}, \frac{\omega_6}{2}) \).

Consequently, setting

\[
\mathcal{F}_G = \begin{cases} 
\{ \frac{\omega_k}{2} \}_{k=1,2} & \text{if } G = E_6^* \\
\{ \frac{\omega_k}{2} \}_{k=1,2,7} & \text{if } G = E_7^*
\end{cases}
\]

the proof of Theorem 1.1 for this case is completed by (4.4) and (4.5), and the relevant items in Table 2.

For \( G = E_7^* \) we have \( K'_E = \{ \frac{1}{2}\omega_k \}_{k=1,2,6,7} \) by Table 2. Since \( \omega_7 \in \Lambda_e \) by Example 2.6 and since

\[
\sqrt{\frac{1}{2}} \neq \| \omega_7 - \frac{1}{2}\omega_6 \| < \| \frac{1}{2}\omega_6 \| = 1 \quad (\text{see } [1, \text{p.280}])
\]

we have \( \frac{1}{2}\omega_6 \notin K_{E_7^*} \) by i) of Lemma 3.1. The proof of (4.4) for \( G = E_7^* \) is done by Lemma 4.1, together with the groups \( C_{\exp(u)} \) with \( u \in \{ \frac{1}{2}\omega_k \}_{k=1,2,7} \) given in Table 2.

Similarly, for \( G = E_6^* \) we have \( \omega_1, \omega_6 \in \Lambda_e \) by Example 2.6, but

\[
\sqrt{\frac{1}{2}} \neq \| \omega_1 - \frac{1}{2}\omega_3 \| < \| \frac{1}{2}\omega_3 \| = \sqrt{\frac{5}{3}},
\]

\[
\sqrt{\frac{1}{2}} \neq \| \omega_6 - \frac{1}{2}\omega_5 \| < \| \frac{1}{2}\omega_5 \| = \sqrt{\frac{5}{3}}, \text{ see [1, p.276].}
\]

We get \( \frac{1}{2}\omega_3, \frac{1}{2}\omega_5 \notin K_{E_6^*} \) from i) of Lemma 3.3. The proof of (4.4) for \( G = E_6^* \) is done by Lemma 4.1, together with the groups \( C_{\exp(u)} \) with \( u \in \{ \frac{1}{2}\omega_k \}_{k=1,2,6} \) given in Table 2.

For (4.5) assume that in the decomposition (3.2) on \( Fix(\gamma) \) the relation \( M_{\exp(u)} = M_{\exp(v)} \) holds for some \( u, v \in K_G \) with \( u \neq v \). By (3.2) one has

i) \( C_{\exp(u)} \cong C_{\exp(v)} \); ii) \( \| u \| = \| v \| \).

In view of the groups \( C_{\exp(u)} \) with \( u \in K_G \) presented in the last column of Table 2 the only possibility for both i) and ii) to hold is when \( G = E_6^* \) and \( (u, v) = (\frac{\omega_1}{2}, \frac{\omega_6}{2}) \). Let \( w_0 \) be the unique longest element of the Weyl group of \( E_6 \) [1, p.171]. Then

\[
w_0(\frac{1}{2}\omega_1) = -\frac{1}{2}\omega_6 \quad (\text{see } [1, \text{p.276}]).
\]

This implies that \( M_{\exp(\frac{\omega_1}{2})} = M_{\exp(-\frac{1}{2}\omega_6)} \). We get (4.5) from the general relation \( \exp(-\frac{1}{2}u) = \exp(\frac{1}{2}u), \ u \in \Lambda, \) which holds in all groups \( G \) with trivial center. This completes the proof. \( \square \)

**Remark 4.2.** For a vector \( u \in \Delta \) let \( C^0_{\exp(u)} \) be the identity component of the centralizer \( C_{\exp(u)} \). Indeed, for \( G = E_6^* \) or \( E_7^* \) the main result in
Theorem 3.7] is applicable to determine the isomorphism type of $C_{\exp(u)}^0$ instead of the whole group $C_{\exp(u)}$. Therefore, additional explanation for the groups $C_{\exp(u)}$ corresponding to $G = E_6^*$ or $E_7^*$ in Table 2 is requested.

In general, let $p : \tilde{G} \to G$ be the universal covering of a simple Lie group $G$ and let $T' \subset \tilde{G}$ be the maximal torus of $\tilde{G}$ corresponding to $T$ in $G$. With respect to the standard identification $L(G^*) = L(G)$ (resp. $L(T^*) = L(T)$) the exponential map exp of $G$ factors through that exp of $\tilde{G}$ in the fashion

$$\exp = p \circ \exp^* : L(G^*) \to G^* \to G \text{ (resp. } L(T^*) \to T^* \to T).$$

Since for $u \in \mathcal{F}_G$, the subspace $M_{\exp^*}(u)$ of $\tilde{G}$ is 1–connected [3, Corollary 3.4, p.101], $p$ restricts to a universal covering $p_u : M_{\exp^*}(u) \to M_{\exp(u)}$.

On the other hand, as the set of all non–trivial covering transformations of $p$ are in one to one correspondence with the set $\Pi_{G^*}$ of minimal weights (see Example 2.8) in the fashion

$$\tilde{g} \to \exp^*(\omega_s) \cdot \tilde{g}, \text{ } \tilde{g} \in \tilde{G}, \omega_s \in \Pi_{G^*},$$

the set $\Pi_u$ of nontrivial covering transformations of $p_u$ can be shown to be

(4.6) $\Pi_u = \{\omega_s \in \Pi_{G^*} \mid \omega_s + u - w(u) \in \Lambda_r \text{ for some } w \in W\},$

where $\Lambda_r$ is the root lattice of $G^*$. Based on this formula a direct calculation in the vector space space $L(T^*)$ shows that

(4.7) $\Pi_u = \begin{cases} \{\omega_7\} \text{ for } G = E_7^* \text{ and } u \in \{\frac{\omega_k}{2}\}_{k=2,7} \\ \emptyset \text{ otherwise.} \end{cases}$

Consequently

$$C_{\exp(u)} = \begin{cases} [C_{\exp(u)}^0]^2 \text{ for } G = E_7^* \text{ and } u \in \{\frac{\omega_k}{2}\}_{k=2,7} \\ C_{\exp(u)}^0 \text{ otherwise.} \end{cases}$$

This justify the groups $C_{\exp(u)}$ corresponding to $G = E_6^*$ or $E_7^*$ in Table 2.$\square$

**Remark 4.3.** With the preliminary data for $G = SU(n+1), Sp(n), Spin(n+2), n \geq 2$, recorded in Example 2.8, one can obtain the fixed set $Fix(\gamma)$ for the simple Lie groups of the classical types (see [8]) by the same argument as that used to establish Theorem 1.1.$\square$

**Remark 4.4.** It is clear that $x \in Fix(\gamma)$ implies that $x^2 = e$. Consequently, the map $\sigma : G \to G$ by $\sigma(g) = xgx^{-1}$ is an involutive automorphism of $G$ with fixed subgroup $C_x$, the centralizer at $x$. This indicates that the orbit spaces $M_{\exp(u)}$ in the decomposition (1.2) are all global Riemannian symmetric spaces of $G$ in the sense of E. Cartan. However, the existing
theory of symmetric spaces \([9, 11, 12, 13, 14]\) does not constitute a solution to our problem for the following reasons:

i) not every symmetric space of \(G\) can appear as a component of \(\text{Fix}(\gamma)\);

ii) if a symmetric space of \(G\) happens to be a component of \(\text{Fix}(\gamma)\), it may occur twice (see in (1.2) for the case \(G = E_7\));

iii) in view of the relation \(M_{\exp(u)} = G/C_{\exp(u)}\), a complete characterization of the symmetric space \(M_{\exp(u)}\) amounts to the determination of the centralizer \(C_{\exp(u)}\), which is a delicate issue absent in the classical theory of Lie groups \([9]\), and has recently been made explicit in our paper \([7]\).

As a witness of i)–iii), for the exceptional Lie groups Nagano \([11]\) stated the list of all the symmetric spaces which are connected components of \(\text{Fix}(\gamma)\) without specifying their embedding in \(G\). He did not write a proof in his other papers, although he promised to do so in \([11]\). In addition, the cases \(G = E^*_6\) or \(E^*_7\) were not considered in the papers \([12, 13, 14]\).

Summarizing, without resorting to the theory of symmetric spaces and by a unified approach, we have enumerated all the symmetric spaces of an exceptional \(G\) that are components of \(\text{Fix}(\gamma)\), and presented concrete realization of these spaces as the adjoint orbits of \(G\).

5 Generalities

Result on \(\text{Fix}(\gamma)\) for the simple Lie groups (i.e. Theorem 1.1 and \([8]\)) is fundamental in understanding the general structure of the fixed set \(\text{Fix}(\gamma)\) for the inverse involution \(\gamma\) on an arbitrary Lie group \(G\). Fairly transparent in our context we have the next result which indicates how Theorem 1.1 could be extended to general settings.

**Corollary 5.1.** For any semi–simple Lie group \(G\) with a maximal torus \(T\), there is a finite subset \(\mathcal{F}_G \subset L(T)\) so that

i) \(\|u\| = d(e, \exp(u))\) for all \(u \in \mathcal{F}_G\);

ii) \(\text{Fix}(\gamma) = \{e\} \coprod M_{\exp(u)}\).

In particular, if \(G = G_1 \times \cdots \times G_k\) with all the factor groups \(G_i\) exceptional, one can take \(\mathcal{F}_G = \mathcal{F}_{G_1} \times \cdots \times \mathcal{F}_{G_k}\) with \(\mathcal{F}_{G_i}\) being given by the second column of Table 1. Consequently, for an \(u = (u_1, \cdots, u_k) \in \mathcal{F}_G\) with \(u_i \in \mathcal{F}_{G_i}\), one has

\[
M_{\exp(u)} = M_{\exp(u_1)} \times \cdots \times M_{\exp(u_k)}.
\]

A homomorphism \(h : G \to G'\) of two semisimple Lie groups \(G\) and \(G'\) clearly satisfies the relation \(h(\text{Fix}(\gamma)) \subseteq \text{Fix}(\gamma')\). This indicates that Corollary 5.1 can play a role in the representation theory of Lie groups. More precisely

**Corollary 5.2.** A group homomorphism \(h : G \to G'\) determines uniquely a correspondence \(h^\circ : \mathcal{F}_G \to \mathcal{F}_{G'} \cup \{0\}\) so that

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Moreover, if $h: G \to G'$ is the inclusion of a totally geodesic subgroup, then

\begin{equation}
\|h^g(u)\| = \|u\| \text{ for all } u \in F_G. \quad \Box
\end{equation}

Specifying a subset $F_G \subset L(T)$ with properties (1.2) amounts to an explicit characterization of the embedding $\text{Fix}(\gamma) \subset G$. Apart from the general fact that the choice of $F_G$ may not be unique, our proof of Theorem 1.1 implies that, if $G$ is 1–connected, there exists a unique set $F_G$ satisfying the relation $F_G \subset \Delta$. Geometrically

**Corollary 5.3.** If $G$ is simple and 1–connected, each adjoint orbit $M_{\exp(u)}$ in $\text{Fix}(\gamma)$ meets the subspace $\exp(\Delta)$ of $G$ exactly at one point. $\Box$

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