Baumslag-Solitar relations
in abstract commensurators of free groups

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Abstract

Any non-residually finite Baumslag-Solitar group has a non-residually finite image in the abstract commensurator of a nonabelian free group. This gives a new proof (avoiding Britton’s Lemma) of the classification of residually finite Baumslag-Solitar groups.

1 Introduction

Let $G$ be a group. If $G$ is nice enough to have solvable word problem, then typically this problem is solved by one of two ways: the first method finds an algorithm for rewriting words in the group into a normal form (e.g., free groups have irreducible words, hyperbolic groups have Dehn’s algorithm, Baumslag-Solitar groups have normal forms given by being an HNN Extension, Braid group elements have normal forms as Coxeter groups). The second method embeds the group into a larger group (usually a linear group) where the word problem can be readily solved (e.g., free groups and Braid groups are linear). For non-linear groups, this alternative approach fails as too much information is lost through studying representations of the group, but the abstract commensurator can begin to fill this gap.

The abstract commensurator of $G$, denoted $\text{Comm}(G)$, is the set of equivalence classes of isomorphisms between finite-index subgroups of $G$, where two isomorphisms are equivalent if they agree on a finite-index subgroup. The abstract commensurator is a group with operation given by composition over a commonly defined finite-index subgroup of $G$. Any finitely generated subgroup of an abstract commensurator of a surface group $S$ has solvable word problem [BS, Proposition 6], which gives utility to finding non-trivial images of groups in $\text{Comm}(\pi_1(S, *))$. Moreover, in [BS], these images are used to prove that the intersection of all finite-index subgroups of the Baumslag-Solitar group $BS(2, 3) = \langle a, b : ab^2a^{-1} = b^3 \rangle$ is not trivial. In this short article, we complete the result of [BS] to the entire class of Baumslag-Solitar groups. Recall that a Baumslag-Solitar group is a group with finite presentation $BS(m, n) := \langle a, b : a^{-1}b^ma = b^n \rangle$.

Broadly speaking, this paper pushes the question: Let $H$ and $G$ be groups. What properties of $H$ can we infer from homomorphisms $H \to \text{Comm}(G)$? We focus on two fundamental properties of groups in this article: A group is Hopfian if any endomorphism of it is injective. Let $G$ and $H$ be groups. An element $g \in G$ is detectable by $H$ if there exists a homomorphism $\phi : G \to H$ such that $\phi(g) \neq 1$. A group is residually finite if any element is detectable by some finite group.

Theorem 1.1. Let $G = BS(m, n)$ where $|m| \neq |n|$ and $|m| \neq 1$ and $|n| \neq 1$. Then there exists an element $g$ that is detectable by $\text{Comm}(F_2)$ and is in every finite-index subgroup of $G$. That is, there exists an image of $G$ in $\text{Comm}(F_2)$ that is not residually finite.
Note that if $|m| = |n|$ or $|m| = 1$ or $|n| = 1$, then the group $BS(m,n)$ is residually finite. The proof gives a theoretical reason for why the computations work in [BS]. This is required as there are infinitely many isomorphism (commensurable) classes of Baumslag-Solitar groups. We present two applications of this theorem and its proof. The first is immediate:

**Corollary 1.2.** If $|m| \neq |n|$ and $|m| \neq 1$ and $|n| \neq 1$, then $BS(m,n)$ has a non-residually finite quotient lying in $Comm(F_2)$. 

Note a that this was first proved by Meskin [Mes72] and our proof differs in that we do not use Britton’s Lemma (c.f. [Lev15]).

Our next application concerns abstract commensurators of pro-$p$ completions. Recall that for a profinite group $G$, the abstract commensurator of $G$ is defined as above, where finite-index subgroups are replaced by open subgroups of $G$. This notion was first introduced in the very nice paper [BEW11], where the notion is extensively studied with many examples. Here we show that for any prime $p$, the abstract commensurator of the pro-$p$ completion of any nonabelian free group is not locally residually finite. See §4 for the proof of the following:

**Corollary 1.3.** Let $p$ be a prime. There exists a non-residually finite image of a Baumslag-Solitar group in the abstract commensurator of the pro-$p$ completion of a non-abelian free group.

It still remains open whether there is a non-residually finite Baumslag-Solitar group that embeds inside the abstract commensurator of a nonabelian free group. Please see §5 for additional questions and suggestions for further directions.

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## 2 Preliminaries

### 2.1 Abstract Commensurators

We would like to define a notion of $Comm_p(F_k)$, some restriction on the definition of $Comm(F_k)$ which will enable us to study its local structure. For brevity, we will denote this as $P_{p,k}$.

**Definition 2.1.** The abstract $p$-commensurator $Comm_p(G)$ is the set of equivalence classes of isomorphisms $\phi : H_1 \to H_2$ between finite index subnormal subgroups of $p$-power index $H1, H2$ sn $G$, where two isomorphisms $\phi_1 \sim \phi_2$ are equivalent if $\phi_1 = \phi_2$ on a finite index subgroup of $G$. This is a group under the operation of composition over a commonly defined finite index subnormal subgroup of $p$-power index. We call elements of $Comm_p(G)$ $p$-commensurators of $G$.

**Lemma 2.1.** $Comm_p(G)$ is a group under composition, where the composition is defined as in $Comm(G)$: given two isomorphisms $\Psi : H_1 \to H_1'$ and $\Phi : H_2 \to H_2'$, we define the product $\Phi \circ \Psi : \Psi^{-1}(H_1' \cap H_2) \to \Phi(H_1' \cap H_2)$.

**Proof.** Note that composition respects subnormality and $p$-power index, so the operation is well-defined. □

**Definition 2.2.** Let $F_k$ be the free group of rank $k$. We define $P_{p,k} := Comm_p(F_k)$.

**Corollary 2.2.** $P_{p,k}$ embeds in $Comm(F_k)$.
Proof. Recall that an element of $P_{p,k}$ is an equivalence class of isomorphisms between finite index subnormal subgroups of $p$-power index in $F_k$, and so any $p$-commensurator in $P_{p,k}$ is also a commensurator in $\text{Comm}(F_k)$. Since $F_k$ has the unique root property, two commensurators of $F_k$ are equal if and only if they are equal on some finite index subgroup. If two $p$-commensurators are in the same equal equivalence class in $\text{Comm}(F_k)$, they must agree on some finite index subgroup, and thus they agree on all finite-index subgroups over which they are both defined. Then the $p$-commensurators are equivalent, and so the inclusion is injective. □

3 Non-residually finite Baumslag-Solitar groups

Theorem 3.1. If $m,n$ are integers such that $|m| \neq |n|, |m| \neq 1, |n| \neq 1$, then $BS(m,n)$ is not residually finite.

We will deal with this proof via three lemmas, which deal with the possible prime factorizations of $m$ and $n$. In the first case, $m$ and $n$ are powers of the same prime $p$. If they are not powers of the same prime, then $m$ and $n$ must either have different prime divisors, or different powers of some prime $p$ in their prime factorization.

In each case, we will follow a similar construction. We construct two finite index subgroups of $F_2$, and define isomorphisms between them on their generating elements, which are elements of $\text{Comm}(F_2)$. There is a homomorphism defined on generators from the particular $BS(m,n)$ group under consideration onto these elements of $\text{Comm}(F_2)$. We then show a word $\gamma$ which is the residual finiteness kernel of $BS(m,n)$. This map will produce a non-trivial image of our chosen word $\gamma$, and thus we have that $\gamma$ is not the identity in $BS(m,n)$.

Our proofs rely on work by Meskin [Mes72] and also Bou-Rabee and Studenmund [BS]. As a remark, Meskin defines the Baumslag-Solitar groups via the reverse conjugation. Thus we switch $a^{-1}$ and $a$ in the words given.

Lemma 3.2. If $m,n$ are powers of the same prime and $|m| \neq |n|, |m| \neq 1, |n| \neq 1$, then $BS(m,n)$ is not residually finite.

Proof. Let $m = p^k, n = p^l$, and without loss of generality, we say that $k < l$. From Meskin, $\gamma = [a^{-1}, b, b^p] = [ab^{-1}a^{-1}, b, b^p]$ is in the residual finiteness kernel of $BS(m,n)$. We now show that $\gamma$ is non-trivial via injection into $\text{Comm}(F_2)$.

Note that $\gamma \neq 1$ if and only if $b\gamma b^{-1} \neq 1$, and so it suffices to check that $ab^{-1}a^{-1}$ and $b^p$ do not commute.

We now construct a homomorphism from $BS(m,n)$ to $\text{Comm}(F_2)$. Let $F_2 = \langle A, B \rangle$. Let $\pi_1 : F_2 \to \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ be the map given by $A \mapsto (1,0)$ and $B \mapsto (0,1)$. Let $\pi_2 : F_2 \to \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ be the map given by $A \mapsto (0,1)$ and $B \mapsto (1,0)$. Let $\Delta_1 = \ker(\pi_1)$ and $\Delta_2 = \ker(\pi_2)$.

Let $\phi$ be the commensurator with representative $f : F_2 \to F_2$ given by $X \mapsto AXA^{-1}$. Let $\psi$ be the commensurator with representative $g : \Delta_1 \to \Delta_2$, such that $g(A^m) = A^n$. Then the commensurator $\psi \circ \phi^m \circ \psi^{-1}$ has a representative $f = g \circ f^m \circ g^{-1}$, such that for every $\gamma \in \Delta_2$,

$$f(\gamma) = b \circ a^m \circ b^{-1}(\gamma) = b(A^m b^{-1}(\gamma) A^{-m}) = A^n (b \circ b^{-1}(\gamma)) A^n = A^n \gamma A^{-n} = a^n,$$

and so $\psi \circ \phi^m \circ \psi^{-1} = \phi^n$. We now define a homomorphism $\Phi : BS(m,n) \to \text{Comm}(F_2)$ by the map $a \mapsto \psi, b \mapsto \phi$. As we have just verified, $\Phi$ vanishes on the relator.

We now verify that $\Phi(ab^{-1}a^{-1}b^p) \neq \Phi(b^p ab^{-1}a^{-1})$. In our construction, we have specified that $\psi$ is a commensurator with representative $g$ where $g(A^m) = A^n$. We are thus free to specify any such map $g$ which
sends generating elements of $\Delta_1$ to generating elements of $\Delta_2$. By the Nielsen-Schreier algorithm, we can generate the following bases for $\Delta_1$ and $\Delta_2$:

$$S_{\Delta_1} = \{ [A, B], [A, B]^A, \ldots [A, B]^{A^{m-1}}, \}
\{ [A, B^2], [A, B^2]^A, \ldots [A, B^2]^{A^{m-1}}, \}
\{ \ldots \}
\{ [A, B^{n-1}], [A, B^{n-1}]^A, \ldots [A, B^{n-1}]^{A^{m-1}}, \}
\{ B^n, (B^n)^A, \ldots (B^n)^{A^{m-1}}, \}
\{ A^n \}$$

$$S_{\Delta_2} = \{ [A, B], [A, B]^A, \ldots [A, B]^{A^{n-1}}, \}
\{ [A, B^2], [A, B^2]^A, \ldots [A, B^2]^{A^{n-1}}, \}
\{ \ldots \}
\{ [A, B^{m-1}], [A, B^{m-1}]^A, \ldots [A, B^{m-1}]^{A^{n-1}}, \}
\{ B^m, (B^m)^A, \ldots (B^m)^{A^{n-1}}, \}
\{ A^n \}$$

Now we define $g$ as follows: $[A, B] \to [A, B]^A, [A, B]^A \to [A, B]^{A^{p^k}}, [A, B^2]^A \to [A, B], [A, B^2] \to B^m$. Then we have

$$\Phi(ab^{-1}a^{-1}b^{p^k})([A, B]) = gf^{-1}g^{-1}f^{p^k}([A, B])
= gf^{-1}g^{-1}([A, B]^{A^{p^k}})
= gf^{-1}([A, B]^A)
= g([A, B])
= [A, B]^A$$

but

$$\Phi(b^{p^k}ab^{-1}a^{-1})([A, B]) = f^{p^k}gf^{-1}g^{-1}([A, B])
= f^{p^k}gf^{-1}([A, B^2]^A)
= f^{p^k}g([A, B^2])
= f^{p^k}(B^m)
= (B^m)^{A^{p^k}}$$

and so $\Phi(ab^{-1}a^{-1}b^{p^k}) \neq \Phi(b^{p^k}ab^{-1}a^{-1})$. Thus the two words in $\text{BS}(m, n)$ do not commute, and so $\gamma$ in the residual finiteness kernel of $\text{BS}(m, n)$ is non-trivial.

**Lemma 3.3.** If $m, n$ are not powers of the same prime, do not have the same prime divisors, and $|m| \neq |n|$, $|m| \neq 1, |n| \neq 1$, then $\text{BS}(m, n)$ is not residually finite.
Proof. Without loss of generality, we may assume that \( m < n \) and that there is a prime \( p \) which divides \( m \) but not \( n \). From Meskin, \( \gamma = [aba^{-1}, b^{m/p}] \) is in the residual finiteness kernel of \( BS(m,n) \). We now show that \( \gamma \) is non-trivial via injection into \( \text{Comm}(F_2) \). By definition, \( \gamma \neq 1 \) and only if \( aba^{-1} \) and \( b^{m/p} \) do not commute. Moreover, two terms do not commute if their images under homomorphism do not commute.

We now construct a homomorphism from \( BS(m,n) \) to \( \text{Comm}(F_2) \). Let \( F_2 = \langle A, B \rangle \). Let \( \pi_1 : F_2 \to \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \) be the map given by \( A \mapsto (1,0) \) and \( B \mapsto (0,1) \). Let \( \pi_2 : F_2 \to \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \) be the map given by \( A \mapsto (0,1) \) and \( B \mapsto (1,0) \). Let \( \Delta_1 = \ker(\pi_1) \) and \( \Delta_2 = \ker(\pi_2) \).

Let \( \phi \) be the commensurator with representative \( f : F_2 \to F_2 \) given by \( X \mapsto AXA^{-1} \). Let \( \psi \) be the commensurator with representative \( g : \Delta_1 \to \Delta_2 \), such that \( g(A^n) = A^n \). Then the commensurator \( \psi \circ \phi^m \circ \psi^{-1} \) has a representative \( f = g \circ f^m \circ g^{-1} \), such that for every \( \gamma \in \Delta_2 \),

\[
\begin{align*}
\phi(\gamma) &= b \circ a^m \circ b^{-1}(\gamma) = (A^m b^{-1}(\gamma) A^{-m}) = A^n (b \circ b^{-1}(\gamma)) A^n = A^n \gamma A^{-n} = a^n,
\end{align*}
\]

and so \( \psi \circ \phi^m \circ \psi^{-1} = \phi^m \). We now define a homomorphism \( \Phi : BS(m,n) \to \text{Comm}(F_2) \) by the map \( a \mapsto \psi, b \mapsto \phi \). As we have just verified, \( \Phi \) vanishes on the relator.

We now verify that \( \Phi(aba^{-1}) \neq \Phi(b^{m/p}) \). In our construction, we have specified that \( \psi \) is a commensurator with representative \( g \) where \( g(A^n) = A^n \). We are thus free to specify any map \( g \) which sends generating elements of \( \Delta_1 \) to generating elements of \( \Delta_2 \). By the Nielsen-Schreier algorithm, we can generate the following bases for \( \Delta_1 \) and \( \Delta_2 \):

\[
S_{\Delta_1} = \{ [A, B], [A, B]^A, \ldots, [A, B]^{A^{m-1}}, [A, B^2], [A, B^2]^A, \ldots, [A, B^2]^{A^{m-1}}, \ldots, [A, B^{n-1}], [A, B^{n-1}]^A, \ldots, [A, B^{n-1}]^{A^{m-1}}, B^n, (B^n)^A, \ldots, (B^n)^{A^{m-1}}, A^m \}
\]

\[
S_{\Delta_2} = \{ [A, B], [A, B]^A, \ldots, [A, B]^{A^{n-1}}, [A, B^2], [A, B^2]^A, \ldots, [A, B^2]^{A^{n-1}}, \ldots, [A, B^{m-1}], [A, B^{m-1}]^A, \ldots, [A, B^{m-1}]^{A^{n-1}}, B^m, (B^m)^A, \ldots, (B^m)^{A^{n-1}}, A^n \}
\]

Now we define \( \Phi \) conditional on \( m/p \) (note the change in the image of \( [A, B] \)). If \( m/p \neq 1, [A, B] \to [A, B]^{A^{m/p}}, [A, B]^A \to [A, B]^A, [A, B^2] \to [A, B], [A, B^2]^A \to B^m \). Then we have

\[
\Phi(aba^{-1}b^{m/p})([A, B]) = gfg^{-1}f^{m/p}([A, B])
\]

\[
= gfg^{-1}([A, B]^{A^{m/p}})
\]

\[
= g([A, B]^A)
\]

\[
= [A, B]^A
\]

but
\[ \Phi(b^{m/p}aba^{-1})([A, B]) = f^{m/p}gfg^{-1}([A, B]) \]
\[ = f^{m/p}gf([A, B^2]) \]
\[ = f^{m/p}g([A, B^2]^A) \]
\[ = f^{m/p}(B^m) \]
\[ = (B^m)^{A^{m/p}}. \]

If \( m/p = 1, [A, B] \rightarrow [A, B]^{A^{m/p}}, [A, B]^A \rightarrow [A, B]^A^2, [A, B^2] \rightarrow [A, B], [A, B^2]^A \rightarrow B^m. \)

\[ \Phi(aba^{-1}b^{m/p})([A, B]) = gfg^{-1}f^{m/p}([A, B]) \]
\[ = gfg^{-1}([A, B]^{A^{m/p}}) \]
\[ = gf([A, B]) \]
\[ = g([A, B]^A) \]
\[ = [A, B]^{A^2} \]

but

\[ \Phi(b^{m/p}aba^{-1})([A, B]) = f^{m/p}gfg^{-1}([A, B]) \]
\[ = f^{m/p}gf([A, B^2]) \]
\[ = f^{m/p}g([A, B^2]^A) \]
\[ = f^{m/p}(B^m) \]
\[ = (B^m)^{A^{m/p}}. \]

and so in both cases \( \Phi(ab^{-1}b^{m/p}) \neq \Phi(b^{m/p}aba^{-1}) \). Thus the two words in \( BS(m, n) \) do not commute, and so \( \gamma \) in the residual finiteness kernel of \( BS(m, n) \) is non-trivial.

**Lemma 3.4.** If \( m, n \) are not powers of the same prime, have the same prime divisors, and \( |m| \neq |n|, |m| \neq 1, |n| \neq 1 \), then \( BS(m, n) \) is not residually finite.

**Proof.** There exists some \( k \) which divides both \( m \) and \( n \), such that \( m/k \) and \( n/k \) do not have the same prime divisors. Once again, without loss of generality, we may assume that there is a prime \( p \) which divides \( m/k \) but not \( n/k \). From Meskin, \( \gamma = [ab^{k}a^{-1}, b^{m/p}] \) is in the residual finiteness kernel of \( BS(m, n) \). We now show that \( \gamma \) is non-trivial via injection into \( \text{Comm}(F_2) \). By definition, \( \gamma \neq 1 \) if and only if \( ab^{k}a^{-1} \) and \( b^{m/p} \) do not commute. Moreover, two terms do not commute if their images under homomorphism do not commute.

We now construct a homomorphism from \( BS(m, n) \) to \( \text{Comm}(F_2) \). Let \( F_2 = \langle A, B \rangle \). Let \( \pi_1 : F_2 \rightarrow \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \) be the map given by \( A \mapsto (1, 0) \) and \( B \mapsto (0, 1) \). Let \( \pi_2 : F_2 \rightarrow \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \) be the map given by \( A \mapsto (0, 1) \) and \( B \mapsto (1, 0) \). Let \( \Delta_1 = \ker(\pi_1) \) and \( \Delta_2 = \ker(\pi_2) \).

Let \( \phi \) be the commensurator with representative \( f : F_2 \rightarrow F_2 \) given by \( X \mapsto AXA^{-1} \). Let \( \psi \) be the commensurator with representative \( g : \Delta_1 \rightarrow \Delta_2 \), such that \( g(A^m) = A^n \). Then the commensurator \( \psi \circ \phi^m \circ \psi^{-1} \) has a representative \( f = g \circ f^m \circ g^{-1} \), such that for every \( \gamma \in \Delta_2 \),
and so \( \psi \circ \phi^m \circ \psi^{-1} = \phi^n \). We now define a homomorphism \( \Phi : BS(m, n) \to \text{Comm}(F_2) \) by the map \( a \mapsto \psi, b \mapsto \phi \). As we have just verified, \( \Phi \) vanishes on the relator.

We now verify that \( \Phi(ab^ka^{-1}) \neq \Phi(b^{m/p}) \). In our construction, we have specified that \( \psi \) is a commensurator with representative \( g \) where \( g(A^m) = A^n \). We are thus free to specify any map \( g \) which sends generating elements of \( \Delta_1 \) to generating elements of \( \Delta_2 \). By the Nielsen-Schreier algorithm, we can generate the following bases for \( \Delta_1 \) and \( \Delta_2 \):

\[
S_{\Delta_1} = \{ [A, B], [A, B]^A, \ldots [A, B]^{A^{m-1}}, [A, B^2], [A, B^2]^A, \ldots [A, B^2]^{A^{m-1}}, \ldots [A, B^{m-1}], [A, B^{m-1}]^A, \ldots [A, B^{m-1}]^{A^{m-1}}, B^m, (B^m)^A, \ldots (B^m)^{A^{m-1}}, A^m \}
\]

\[
S_{\Delta_2} = \{ [A, B], [A, B]^A, \ldots [A, B]^{A^{n-1}}, [A, B^2], [A, B^2]^A, \ldots [A, B^2]^{A^{n-1}}, \ldots [A, B^{m-1}], [A, B^{m-1}]^A, \ldots [A, B^{m-1}]^{A^{n-1}}, B^m, (B^m)^A, \ldots (B^m)^{A^{n-1}}, A^n \}
\]

Now we define \( g \) as follows: \([A, B] \to [A, B]^{A^{m/p}}, [A, B]^A \to B^m, [A, B^2] \to [A, B], [A, B^2]^A \to [A, B^2] \). Then we have

\[
\Phi(ab^ka^{-1})([A, B]) = g f^k g^{-1} f^{m/p}([A, B])
\]

\[
= g f^k g^{-1}([A, B]^{A^{m/p}})
\]

\[
= g f^k([A, B])
\]

\[
= g([A, B]^A)
\]

\[
= B^m
\]

but

\[
\Phi(b^{m/p}ab^ka^{-1})([A, B]) = f^{m/p} f^k g^{-1}([A, B])
\]

\[
= f^{m/p} f^k([A, B^2])
\]

\[
= f^{m/p}([A, B^2]^A)
\]

\[
= f^{m/p}([A, B^2])
\]

\[
= [A, B^2]^{A^{m/p}}
\]

and so \( \Phi(ab^ka^{-1}b^{m/p}) \neq \Phi(b^{m/p}ab^ka^{-1}) \). Thus the two words in \( BS(m, n) \) do not commute, and so \( \gamma \) in the residual finiteness kernel of \( BS(m, n) \) is non-trivial. \( \square \)
4 Abstract commensurators of pro-\(p\) completions

By construction, the images of \(BS(p, p^2)\) produced above lie inside the subgroup of \(\text{Comm}(F_2)\) consisting of isomorphisms between \(p\)-power subnormal subgroups of \(F_2\). That is, \(BS(p, p^2)\) embeds inside \(P_{p,k} = \text{Comm}_p(F_k)\) for any prime \(p\) and any natural number \(k > 2\). By the following lemma, we have that \(\text{Comm}(\hat{F}_2^p)\) contains a non-residually finite quotient of \(BS(p, p^2)\). We refer the reader to [Seg83] and [Wil98] for the basics on profinite groups and pro-\(p\) completions of groups.

**Lemma 4.1.** \(P_{p,k}\) embeds inside \(\text{Comm}(\hat{F}_2^p)\).

**Proof.** Any non-identity commensurator acts non-trivially on any finite-index subgroup that it is defined on, because \(F_2\) has the unique root property [BB10, Lemma 2.2] and [BB10, Lemma 2.4]). Hence, it suffices to show that any isomorphism between \(p\)-power index subnormal subgroups of \(F_k\) extends to an isomorphism between closed subgroups of \(\hat{F}_k^p\).

Note that a basis for the topology of \(\hat{F}_k^p\) is given by the closure of the sets \(\Lambda_{p,k}\) defined to be the intersection of all normal subgroups of \(F_k\) of index \(p^k\). Using this basis, it is clear that any isomorphism between finite-index \(p\)-power subnormal subgroups of \(F_k\) is continuous under the subspace topology of \(F_k\) induced by \(\hat{F}_k^p\). Thus, as any finite-index \(p\)-power subnormal subgroups of \(F_k\) extends to an isomorphism between closed subgroups of \(\hat{F}_k^p\).

5 Further directions

Our past experience with algebraic groups guides our study of \(\text{Comm}(F_2)\). Recall that the Chinese Remainder Theorem shows that arithmetic subgroups of a fixed Chavelley group decompose into local parts, which are generally easier to work with. For example, \(\text{SL}_k(\mathbb{Z}/n\mathbb{Z}) \cong \prod_{p^k|n} \text{SL}_k(\mathbb{Z}/p^k\mathbb{Z})\), where \(p^k|\n\) is the largest prime power of \(p\) that divides \(n\). Our suggested questions ask whether some of the useful properties in the linear group setting hold in the abstract commensurator.

First, note that the definition of the local parts of \(\text{Comm}(F_k)\) seems to depend on \(k\). However, the groups \(F_2\) and \(F_k\) are abstractly commensurable, so \(\text{Comm}(F_2) \cong \text{Comm}(F_k)\) for \(k \geq 2\). So we ask: Is the same true of \(P_{p,k}\) for fixed \(p\)? Second, from work by Bartholdi and Bogopolski, \(\text{Comm}(F_2)\) is not finitely generated [BB10]. Are the corresponding \(P_{p,k}\) finitely generated? We note that their proof fails for \(P_{p,k}\), as the natural infinite generating set candidate that the proof provides contains a finite generating set. Third, does the collection of all local parts of \(\text{Comm}(F_k)\) generate \(\text{Comm}(F_k)\)?
6 Appendix A

Given a prime $p$, the GAP [GAP20] code below defines maps $\phi$ and $\psi$ from the proof of Theorem 1.1. The code verifies that the word $\Phi(\gamma)$ is not equal to the identity map on at least one of the generators of $\Delta_2$.

```gap
# Specify p
p := 2;
# Define the groups
f := FreeGroup("A", "B");
A := DirectProduct(CyclicGroup(p), CyclicGroup(p^2));;
# Create list objects of the generators of the previous groups
GenK1 := GeneratorsOfGroup(K1);;
GenA := GeneratorsOfGroup(A);;
# Create conjugation function
psi := GroupHomomorphismByImages( A, A, GenA, ConjGenf);;
phi := GroupHomomorphismByImages ( A, A, GenA, ConjGenf)();
# Create generator lists for K1, K2
GenK1 := List(GeneratorsOfGroup(K1));;
GenK2 := List(GeneratorsOfGroup(K2));;
# Create permuted generator lists for K1, K2
PermGenK2 := Permuted(ShortGenK2, perm);
# Evaluate the word $w$ in the residual finiteness kernel of BS(p,p^2):
WordX := Image(phi, Word2); WordY := Image(phi, Word3);
WordZ := Image(phi, Word4); WordB := Image(phi, Word5);
WordC := Image(phi, Word6); WordD := Image(phi, Word7);
WordE := Image(phi, Word8); WordF := Image(phi, Word9);
WordG := Image(phi, Word10); WordH := Image(phi, Word11);
if IsOne(Word13*Word^(-1)) = false then
pass := 0;
fi;
# Print(debug, "+", perm, "+\n");
```

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