Generic zero-Hausdorff and one-packing spectral measures

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Abstract

For some metric spaces of self-adjoint operators, it is shown that the set of operators whose spectral measures have simultaneously zero upper-Hausdorff and one lower-packing dimensions contains a dense $G_δ$ subset. Applications include sets of limit-periodic operators.

Key words and phrases. Self-adjoint operators, spectral measures, upper-Hausdorff dimension, lower-packing dimension.

1 Introduction

Let $(X, d)$ be a complete metric space of self-adjoint operators acting in a separable Hilbert space $H$, such that convergence in the metric $d$ implies strong resolvent convergence. In three previous papers [2, 3, 4], the present authors have discussed several generic sets of families of self-adjoint operators, in some instances of the space $(X, d)$, in terms of not only spectral properties, but also of dynamical ones. In such works we have gotten, through different grounds, generic sets of operators with one-dimensional packing spectral measures, but an argument for Hausdorff dimensional properties was missing; it is one of the goals of this work to fill up this gap by presenting a result in terms of what we call fractal dimensions of the spectrum: we give contributions related to the upper-Hausdorff and lower-packing dimensions of spectral measures.

Although it is already known that, for some families of self-adjoint operators, a typical (in Baire’s sense) spectral measure has upper-packing dimension equal to one (see Theorem 1.1 in [4]), we improve such result, in the sense that now the same result is valid for the lower-packing dimension; however, as mentioned before, there was no generic result about the (upper or lower) Hausdorff

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dimension. The novel technical argument, encapsulated in Theorem 3.20, gives information about upper-Hausdorff dimensional properties of spectral measures; since it is immediate to adapt such ideas to obtain the counterpart lower-packing properties, we just present the details of the first case. It is also important to underline that every application of the so-called Wonderland Theorem discussed in [14], presenting dense sets of operators with pure point spectrum or absolutely continuous spectrum, can now be converted into a result about the existence of a generic set of operators whose spectral measures are zero upper-Hausdorff and one lower-packing dimensional. The upper-Hausdorff dimension of a Borel measure $\mu$ will be denoted by $\dim_H^+(\mu)$, whereas its lower-packing dimension by $\dim_P(\mu)$ (such concepts are recalled in Definition 2.15).

Next, we present our main result. It should be compared with Theorem 2.1 in [14].

Theorem 1.1 Let $0 \neq \psi \in \mathcal{H}$, let $\emptyset \neq F \subset \mathbb{R}$ be a closed set and suppose that each of the sets

- $C_{\text{out}}^{\psi,F} = \{ T \in X \mid \dim_H^+(\mu_{\psi,F}^T) = 0 \}$,
- $C_{\text{ip}}^{\psi,F} = \{ T \in X \mid \dim_P(\mu_{\psi,F}^T) = 1 \}$,

is dense in $X$. Then, the set $\{ T \in X \mid \dim_H^+(\mu_{\psi,F}^T) = 0 \text{ and } \dim_P(\mu_{\psi,F}^T) = 1 \}$ is generic in $X$.

As an illustration, we consider an application to a class of bounded discrete Schrödinger operators acting on $l^2(\mathbb{Z})$. For a fixed $r > 0$, let $X^r$ be the set of operators $T$ with action

$$ (T\psi)_n = \psi_{n+1} + \psi_{n-1} + V_n \psi_n, \quad (1.1) $$

where the potential $v = (V_n)$ is an arbitrary real bilateral sequence with $|V_n| \leq r$ for every $n \in \mathbb{Z}$. Let $\sigma(T)$ and $\mu_{\psi}^T$ denote the spectrum of $T$ and its spectral measure (associated with the vector $0 \neq \psi \in l^2(\mathbb{Z})$), respectively. By combining Theorem 1.1 with a specific construction presented in the proof of Theorem 4.1 in [14], we obtain the following result.

Theorem 1.2 Fix $r > 0$. The set $\{ T \in X^r \mid \sigma(T) = [-2 - r, 2 + r], \dim_H^+(\mu_{\psi}^T) = 0, \dim_P(\mu_{\psi}^T) = 1, \text{ for all } 0 \neq \psi \in \mathcal{H} \}$ is generic in $X^r$.

Remark 1.3 A well-known fact about discrete Schrödinger operators in $l^2(\mathbb{Z})$, with action (1.1), is the existence of a common set of cyclic vectors $\{ \delta_0, \delta_1 \}$. Now, if for $\zeta \in \{ \delta_0, \delta_1 \}$ the spectral measure $\mu_{\zeta}^T$ is zero upper-Hausdorff dimensional, then $\mu_{\psi}^T$ is zero upper-Hausdorff dimensional for every $\psi \neq 0$ (namely, since $\mu_{\zeta}^T$ is supported on a set of zero Hausdorff dimension and since, for every $\psi \neq 0$, $\mu_{\psi}^T$ is absolutely continuous with respect to $\mu_{\zeta}^T$, then $\mu_{\psi}^T$ is also supported on a set of zero Hausdorff dimension), which implies that $\{ T \in X \mid \sigma(T) \text{ is purely zero upper-Hausdorff dimensional} \}$ is a $G_\delta$ set (the same conclusion is valid for $\{ T \in X \mid \sigma(T) \text{ is purely one lower-packing dimensional} \}$). Thus, the results stated in Theorem 1.2 are obtained after showing that the set $\{ T \in X^r \mid \sigma(T) = [-2 - r, 2 + r], \dim_H^+(\mu_{\psi}^T) = 0 \text{ and } \dim_P(\mu_{\psi}^T) = 1, \text{ for each fixed } 0 \neq \psi \in \mathcal{H} \}$, is generic in $X^r$. This is actually what one gets combining Theorem 1.1 with the aforementioned result in [14].

We also apply our results to a class of limit-periodic operators; these are discrete one-dimensional ergodic Schrödinger operators, denoted by $H^*_{g,\tau}$, acting in $l^2(\mathbb{Z})$, whose action is given by (1.1), with

$$ V_n(\kappa) = g(\tau^n(\kappa)); \quad (1.2) $$
here, \( \kappa \) belongs to a Cantor group \( \Omega \), \( \tau : \Omega \to \Omega \) is a minimal translation on \( \Omega \) and \( g : \Omega \to \mathbb{R} \) is a continuous sampling function, i.e., \( g \in C(\Omega, \mathbb{R}) \), the latter endowed with the norm of uniform convergence. For more details, see [1].

For each \( \kappa \in \Omega \), let \( X_\kappa \) be the set of limit-periodic operators \( H^\kappa_{g,\tau} \) given by (1.1) and (1.2), endowed with the metric
\[
d(H^\kappa_{g,\tau}, H^\kappa'_{g',\tau}) = \|g - g'\|_\infty.
\]
(1.3)

We shall prove the following result.

**Theorem 1.4** For each \( \kappa \in \Omega \), the set \( \{ T \in X_\kappa \mid \sigma(T) \) is purely zero upper-Hausdorff and one lower-packing dimensional\} is generic in \( X_\kappa \).

### 1.1 Countable families of pairwise commuting self-adjoint operators

We remark that is possible to extend the result stated in Theorem 1.1 for countable families of pairwise commuting self-adjoint operators \( T = (T_1, \ldots, T_N) \) acting in a separable Hilbert space \( \mathcal{H} \).

The joint resolution of identity is given by \( E(\cdot) := \prod_{j=1}^N E_j(\cdot) \) over the rectangles of the Borel sets \( \mathcal{B}(\mathbb{R}^N) \); here, \( N \) stands for a natural number or (countable) infinite, and \( E_j(\cdot) \) is the resolution of identity of \( T_j \). For each fixed \( \psi \in \mathcal{H} \) with \( \| \psi \| = 1 \), the support of the spectral measure \( \mu_T^{\psi}(\cdot) := \langle \psi, E(\cdot)\psi \rangle \), denoted by \( \text{supp}(\mu_T^{\psi}) \), is the intersection of all closed subsets of \( \mathbb{R}^N \) with full \( \mu_T^{\psi} \) measure (\( \mathbb{R}^N \) with the product topology). We also set \( J_N = \{1, 2, \ldots, N\} \) if \( N \in \mathbb{N} \), and \( J_N = N \) in case \( N = \infty \).

**Definition 1.5** Let \( K \) denote either \( H \) or \( P \), for Hausdorff or packing, respectively. Let \( \mu \) be a probability product-measure on the Borel sets \( (\mathbb{R}^N; \mathcal{B}(\mathbb{R}^N)) \) given by \( \mu(\cdot) = \prod_{n=1}^N \mu_n(\cdot) \). Let \( I = \prod_{n=1}^N I_n \in \mathcal{B}(\mathbb{R}^N) \) be a measurable rectangle. One says that \( \dim_H^K(\mu) \) is minimal if, for each \( n \in J_N \), \( \dim_H^K(\mu_n) = 0 \). Accordingly, one says that \( \dim_H^K(\mu) \) is maximal if, for each \( n \in J_N \), \( \dim_H^K(\mu^n) = n \), where
\[
\mu^n := \prod_{k=1}^n \mu_k.
\]

Denote by \( X \) the collection of such families of countable sequences of pairwise commuting self-adjoint operators, and let \( d \) be any metric in \( X \) whose convergence implies, for each \( k \in J_N \), strong resolvent convergence; one could set, for instance,
\[
d(T, T') := \sup_{k \in J_N} D(T_k, T'_k),
\]
where
\[
D(T_k, T'_k) := \sum_{l \geq 1} \min(2^{-l}, \| (T_k - T'_k) \xi_l \|)
\]
(\( (\xi_l)_{l \geq 1} \) is an orthonormal basis of \( \mathcal{H} \)). Naturally, \( (X, d) \) is a complete metric space. The following result is the natural extension of Theorem 1.1 to this setting.

**Theorem 1.6** Let \( \psi \in \mathcal{H} \), with \( \| \psi \| = 1 \), let for each \( j \in J_N \), \( \emptyset \neq F_j \) be a closed set and put \( F := \prod_{j=1}^N F_j \). Suppose that each of the sets
\[
\psi; F_{\text{min}} = \{ T \in X \mid \dim_H(\mu_T \psi) \text{ is minimal} \},
\]
\[
\psi; F_{\text{max}} = \{ T \in X \mid \dim_T(\mu_T \psi) \text{ is maximal} \},
\]
is dense in \( X \). Then, the set \( \{ T \in X \mid \dim_H(\mu_T) \text{ is minimal and} \dim_T(\mu_T) \text{ is maximal} \} \) is generic in \( X \).

One can prove Theorem 1.6 using adapted versions of the results stated in Section 3 for functions defined in \( \mathbb{R}^n \), with \( n \in J_N \).

The result stated in Theorem 1.6 is particularly true for the set of normal operators acting in \( \mathcal{H} \), which we denote by \( Y \); recall that a normal operator \( A \) can be written in terms of a pair \( T_1, T_2 \) of commuting self-adjoint operators: \( A = f(T_1, T_2) \), where \( f : \mathbb{R}^2 \to \mathbb{C}, f(x_1, x_2) = x_1 + ix_2 \). This also leads to a version of Simon’s Wonderland Theorem [14] to normal operators.

**Theorem 1.7** Let \( (Y, d) \) be as above, and suppose that each of the sets
\[
\{ A \in Y \mid A \text{ has purely absolutely continuous spectrum} \},
\]
\[
\{ A \in Y \mid A \text{ has pure point spectrum} \}
\]
is dense in \( Y \). Then, the set \( \{ A \in Y \mid A \text{ has purely singular continuous spectrum} \} \) is generic in \( Y \).

In what follows, we use the remark above in order to extend the result stated in Theorem 3.1 in [14] to normal operators. Let \( a := (a_1, a_2) \) be such that \( a_1, a_2 > 0 \), and set \( Y^a = \{ A \in Y \mid \| T_1 \| \leq a_1, \| T_2 \| \leq a_2 \} \).

**Theorem 1.8** Let \( \psi \in \mathcal{H} \) with \( \| \psi \| = 1 \) and set \( R := [-a_1, a_1] \times [-a_2, a_2] \). Then, the set \( \{ A \in Y^a \mid \text{supp}(\mu_A) = R, \dim_H(\mu_A) \text{ is minimal,} \dim_T(\mu_A) \text{ is maximal} \} \) is generic in \( Y^a \).

### 1.2 Organization

In Section 2 we recall important decompositions of Borel measures on \( \mathbb{R} \) with respect to Hausdorff and packing dimensions, along with the corresponding spectral decompositions of self-adjoint operators. Section 3 is dedicated to the construction of suitable \( G_\delta \) sets. In Section 4 we present the proofs of Theorems 1.1 and 1.4.

Now some words about notation. \( \mathcal{H} \) will always denote a complex separable Hilbert space. \( B(\mathbb{R}) \) denotes the collection of Borel sets in \( \mathbb{R} \); \( \mu \) will always indicate a finite nonnegative Borel measure on \( \mathbb{R} \), and its restriction to the Borel set \( A \) will be indicated by \( \mu_A(\cdot) := \mu(A \cap \cdot) \). The adjective absolutely continuous without specification means that \( \mu \) is absolutely continuous with respect to Lebesgue measure on \( \mathbb{R} \). A nonnegative Borel measure \( \nu \) on \( \mathbb{R} \) is supported on a Borel set \( S \) if \( \nu(\mathbb{R} \setminus S) = 0 \). Finally, it will also be convenient to use the symbol \( K \) to refer to either \( H \) or \( P \), which stands for Hausdorff and packing properties, respectively.
2 Preliminaries

2.1 Hausdorff and packing measures

Let us recall the definitions of Hausdorff and packing measures on $\mathbb{R}$.

**Definition 2.9** Let $A \subset \mathbb{R}$. By a $\delta$-covering of $A$ we mean any countable collection $\{E_k\}$ of subsets of $\mathbb{R}$ such that $A \subset \bigcup_{k \geq 1} E_k$ and $\text{diam}(E_k) := \sup_{x, y \in E_k} |x - y| \leq \delta$. For each $\alpha \in [0, 1]$, the $\alpha$-dimensional (exterior) Hausdorff measure of $A$ is defined as

$$h^\alpha(A) = \lim_{\delta \downarrow 0} \inf \left\{ \sum_{k=1}^{\infty} (\text{diam}(E_k))^\alpha \mid \{E_k\} \text{ is a } \delta\text{-covering of } A \right\}.$$  

The Hausdorff dimension of the set $S$, here denoted by $\text{dim}_H(S)$, is defined as the infimum of all $\alpha$ such that $h^\alpha(S) = 0$; note that $h^\alpha(S) = \infty$ if $\alpha < \text{dim}_H(S)$.

A $\delta$-packing of an arbitrary set $A \subset \mathbb{R}$ is a countable disjoint collection $(\bar{B}(x_k; r_k))_{k \in \mathbb{N}}$ of closed balls centered at $x_k \in A$ and radii $r_k \leq \delta/2$, so with diameters at most of $\delta$. Define $P^\alpha_\delta(A)$, $\alpha \in [0, 1]$, as

$$P^\alpha_\delta(A) = \sup \left\{ \sum_{k=1}^{\infty} (2r_k)^\alpha \mid (\bar{B}(x_k; r_k))_k \text{ is a } \delta\text{-packing of } A \right\},$$

that is, the supremum is taken over all $\delta$-packings of $A$. Then, take the decreasing limit

$$P^\alpha_0(A) = \lim_{\delta \downarrow 0} P^\alpha_\delta(A)$$

which is a pre-measure.

**Definition 2.10** The $\alpha$-packing (exterior) measure $P^\alpha(A)$ of $S$ is given by

$$P^\alpha(A) := \inf \left\{ \sum_{k=1}^{\infty} P^\alpha_0(E_k) \mid S \subset \bigcup_{k=1}^{\infty} E_k \right\}.$$  

The packing dimension of the set $A$, here denoted by $\text{dim}_P(A)$, is defined (in analogy to $\text{dim}_{H}(A)$) as the infimum of all $\alpha$ such that $P^\alpha(A) = 0$, which coincides with the supremum of all $\alpha$ so that $P^\alpha(A) = \infty$.

It is known [11] that $\text{dim}_H(A) \leq \text{dim}_P(A)$, and this inequality is in general strict. It is also important to mention that $P^\alpha$ and $h^\alpha$ are Borel (regular) measures; furthermore, $P^0 \equiv h^0$, $P^1 \equiv h^1$, and they are equivalent, respectively, to the counting measure (which assigns to each set $S$ the number of elements it has) and the Lebesgue measure.

**Definition 2.11** Let $\alpha \in [0, 1]$. A finite nonnegative Borel measure $\mu$ on $\mathbb{R}$ is called:

1. $\alpha$-K continuous, denoted $\alpha Kc$, if $\mu(S) = 0$ for every Borel set $S$ such that $K^\alpha(S) = 0$.
2. $\alpha$-K singular, denoted $\alpha Ks$, if it is supported on some Borel set $S$ with $K^\alpha(S) = 0$. 


3. 0-K dimensional, denoted $0\text{Kd}$, if it is supported on a Borel set $S$ with $\dim_K(S) = 0$.

4. 1-K dimensional, denoted $1\text{Kd}$, if $\mu(S) = 0$ for any Borel set $S$ with $\dim_K(S) < 1$.

**Remark 2.12**

1. $\mu$ is $0\text{Kd}$ if, and only if, it is $\alpha\text{Ks}$ for each $\alpha \in (0, 1]$. Equivalently, $\mu$ is $1\text{Kd}$ if, and only if, it is $\alpha\text{Kc}$ for each $\alpha \in [0, 1)$.

2. It follows from Definition 2.11 that $\mu$ is $0\text{Kd}$ if it is pure point, whereas $\mu$ is $1\text{Kd}$ if it is absolutely continuous.

**Definition 2.13** Let $\mu$ be a finite nonnegative Borel measure on $\mathbb{R}$ and $x \in \mathbb{R}$. Set $B(x; \varepsilon) = \{y \in \mathbb{R} \mid |x - y| < \varepsilon\}$, i.e., the open ball of radius $\varepsilon > 0$ centered at $x$, and

\[
D^H_\mu(x) := \limsup_{\varepsilon \downarrow 0} \frac{\mu(B(x; \varepsilon))}{(2\varepsilon)^\alpha}, \quad D^P_\mu(x) := \liminf_{\varepsilon \downarrow 0} \frac{\mu(B(x; \varepsilon))}{(2\varepsilon)^\alpha}.
\]

The following density results [8, 13] relate the continuity of $\mu$, with respect to Hausdorff (packing) dimension, to its local scaling behavior as probed by $D^\mu_\alpha$.

**Theorem 2.14** Let $\mu$ be as above and let $\alpha \in [0, 1]$. Let

\[
K_{\alpha\text{Kc}} := \{x \in \mathbb{R} \mid D^\mu_\alpha(x) < \infty\}, \quad K_{\alpha\text{Ks}} := \{x \in \mathbb{R} \mid D^\mu_\alpha(x) = \infty\}.
\]

Then, these are Borel sets, $\mu_{\alpha\text{Kc}}(\cdot) := \mu(K_{\alpha\text{Kc}} \cap \cdot)$ is $\alpha\text{Kc}$, $\mu_{\alpha\text{Ks}}(\cdot) := \mu(K_{\alpha\text{Ks}} \cap \cdot)$ is $\alpha\text{Ks}$, $\mu_{0\text{Kd}}(\cdot) := \mu((\bigcap_{k \geq 1} K_{(1/k)\text{Ks}}) \cap \cdot)$ is $0\text{Kd}$, and $\mu_{1\text{Kd}}(\cdot) := \mu((\bigcap_{k \geq 1} K_{(1-1/k)\text{Kc}}) \cap \cdot)$ is $1\text{Kd}$.

**Proof.** See Section 4 in [10] for the Hausdorff case; the packing case follows analogously. \qed

By following [8, 13], we recall the upper and lower dimensions of a finite Borel measure $\mu$.

**Definition 2.15** Let $\mu$ be as above, and let $I \subset \mathbb{R}$ be a Borel set. The K upper dimension of $\mu$ restricted to $I$, denoted by $\dim^+_K(\mu, I)$, is defined as

\[
\dim^+_K(\mu, I) := \inf\{\dim_K(S) \mid \mu(I \setminus S) = 0, S \text{ a Borel subset of } I\},
\]

and the K lower dimension of $\mu$ restricted to $I$, denoted by $\dim^-_K(\mu, I)$, as

\[
\dim^-_K(\mu, I) := \sup\{\alpha \mid \mu(S) = 0 \text{ if } \dim_K(S) < \alpha, S \text{ a Borel subset of } I\}.
\]

When $I = \mathbb{R}$, we simply denote $\dim^+_K(\mu, I)$ by $\dim^+_K(\mu)$.

**Proposition 2.16** Let $\mu$ be as above, let $I$ be a Borel subset of $\mathbb{R}$, and let $\alpha \in (0, 1)$. Then,

1. $\alpha \leq \dim^-_K(\mu, I)$ if, and only if, for each $\varepsilon \in (0, \alpha]$, $\mu, I$ is $(\alpha - \varepsilon)\text{Kc}$;

2. $\dim^+_K(\mu, I) \leq \alpha$ if, and only if, for each $\varepsilon \in (0, 1 - \alpha]$, $\mu, I$ is $(\alpha + \varepsilon)\text{Ks}$.

**Proof.** See Section 1 in [2]. \qed
3 \ G_0 \ sets

Let \((X, d)\) be as in the Introduction, let \(\emptyset \neq O \subset \mathbb{R}\) be an open set, and let

\[
\mathcal{M}_+(O) := \left\{ \mu \in \mathcal{M}(O) \mid 0 \leq \mu \leq 1 \right\},
\]

that is, the set of positive measures on \(O\) with total mass less than or equal to one. We endow such set with the weak topology, i.e., the topology of the weak convergence of measures \((\mu_n)\) converges weakly to \(\mu\) if for each \(f \in C_b(O)\), \(\int f(x)d\mu_n(x) \to \int f(x)d\mu(x)\); here, \(C_b(O)\) denotes the set of bounded continuous functions defined on \(O\). Recall that such topology is metrizable (since \(O\) is a Polish space): take, for instance, the Lévy-Prohorov metric, which will be denoted by \(\rho\) (see Appendix 2 in \([7]\) for details).

Let also, for each \(T \in X\) and each \(0 \neq \psi \in \mathcal{H}\), \(\zeta_\psi : X \to \mathcal{M}_+(O)\) be defined by the law \(\zeta_\psi(T) := \mu^T_{\psi,O}\), where \(\mu^T_{\psi,O}(\cdot) := \mu^T_{\psi}(O \cap \cdot)\). It follows from the functional calculus for self-adjoint operators that \(\zeta_\psi\) is a continuous function: if \(\lim_{m \to \infty} d(T_m, T) = 0\), then \(\lim_{m \to \infty} \rho \left( \mu^T_{\psi,O}, \mu^T_{\psi,O} \right) = 0\).

**Lemma 3.17** Let \(\emptyset \neq O \subset \mathbb{R}\) be an open set and let, for each \(t > 0\), \(V_t(\cdot, \cdot) : \mathcal{M}_+(O) \times O \to [0,1]\) be defined by the law \(V_t(\mu, x) := \int f_{t,x}(y)d\mu(y)\), where \(f_{t,x} : O \to [0,1]\) is given by

\[
f_{t,x}(y) := \begin{cases} 1, & \text{if } |x-y| \leq 1/t, \\ -t|x-y| + 2, & \text{if } 1/t \leq |x-y| \leq 2/t, \\ 0, & \text{if } |x-y| \geq 2/t. \\
\end{cases}
\]

Let also, for each \(0 \neq \psi \in \mathcal{H}\), \(U_{t,\psi}(\cdot, \cdot) : X \times O \to [0,1]\) be defined by the law

\[
U_{t,\psi}(T, x) := (\psi, f_{t,x}(T)\psi) = \int f_{t,x}(y)d\mu_{\psi,O}(y).
\]

Then, \(U_{t,\psi}(T, x) = V_t(\zeta_\psi(T), x)\) and

\[
(D^{K,\alpha} \mu^T_{\psi,O})(x) = \lim_{t \to \infty} K t^\alpha U_{t,\psi}(T, x).
\]

Furthermore, for each \(t > 0\), the function \(V_t : \mathcal{M}_+(O) \times O \to [0,1]\) is jointly continuous.

**Proof.** It follows from the Spectral Theorem that, for each \(x \in O\), each \(t > 0\) and each \(0 \neq \psi \in \mathcal{H}\),

\[
\mu^T_{\psi,O}(B_{1/t}(x)) \leq U_{t,\psi}(T, x) = \int f_{t,x}(y)d\mu_{\psi,O}(y) \leq \mu^T_{\psi,O}(B_{2/t}(x)).
\]

Then, one has \(t^\alpha \mu^T_{\psi,O}(B(x,1/t)) \leq t^\alpha U_{t,\psi}(T, x) \leq t^\alpha \mu^T_{\psi,O}(B(x,2/t))\), which proves the first assertion.

Note that, for each \(x \in O\) and each \(t > 0\), \(f_{t,x} : O \to \mathbb{R}\) is a continuous function such that, for each \(y \in O\), \(\chi_{B_{1/t}(y)}(y) \leq f_{t,x}(y) \leq \chi_{B_{2/t}(y)}(y)\). Given that each \(f_{t,x}(y)\) depends only on \(|x-y|\), it is straightforward to show that for each \(t > 0\), \(f_{t,x}\) converges uniformly to \(f_{t,x}\) on \(O\) when \(x \to x\).

We combine this remark with Theorems 2.13 and 2.15 in \([9]\) in order to prove that \(V_t(\mu, x)\) is jointly continuous. Let \((\mu_m)\) and \((x_l)\) be sequences in \(\mathcal{M}_+(O)\) and \(O\), respectively, such that \(\rho(\mu_m, \mu) \to 0\) and \(x_l \to x\). Firstly, we show that

\[
\lim_{m \to \infty} \lim_{l \to \infty} U_{t,\psi}(\mu_m, x_l) = \lim_{l \to \infty} \lim_{m \to \infty} \int f_{t,x}(y)d\mu_m(y) = V_t(\mu, x).
\]
Since, for each $y \in \mathbb{R}$, $|f_{t,x}(y)| \leq 1$, it follows from dominated convergence that, for each $m \in \mathbb{N}$, 
\[ \lim_{m \to \infty} \int f_{t,x}(y) d\mu_m(y) = \int f_{t,x}(y) d\mu(y). \]
Now, since $f_{t,x}$ is continuous and convergence in the metric $\rho$ implies weak convergence of measures, one has
\[ \lim_{m \to \infty} \lim_{l \to \infty} \int f_{t,x_l}(y) d\mu_m(y) = \lim_{m \to \infty} \int f_{t,x}(y) d\mu_m(y) = V_t(\mu, x). \]

The next step consists in showing that, for each $l \in \mathbb{N}$, the function $\varphi_l : \mathbb{N} \to \mathbb{R}$, defined by the law $\varphi_l(m) := V_t(\mu_m, x_l)$, converges uniformly to $\varphi(m) := \lim_{l \to \infty} V_t(\mu_m, x_l) = \int f_{t,x}(y) d\mu(y)$. Let $\delta > 0$. Since, for each $t > 0$, $f_{t,x_l}$ converges uniformly to $f_{t,x}(y)$, there exists $N \in \mathbb{N}$ such that, for each $l \geq N$ and each $y \in \mathbb{R}$, $|f_{t,x_l}(y) - f_{t,x}(y)| < \delta$. Then, one has, for each $l \geq N$ and each $m \in \mathbb{N}$,
\[ |\varphi_l(m) - \varphi(m)| = \left| \int f_{t,x_l}(y) d\mu_m(y) - \int f_{t,x}(y) d\mu_m(y) \right| \leq \int |f_{t,x_l}(y) - f_{t,x}(y)| d\mu_m(y) < \delta. \]

It follows from Theorem 2.15 in [9] that $\lim_{l,m \to \infty} V_t(\mu_m, x_l) = V_t(\mu, x)$. Given that $\lim_{l,m \to \infty} V_t(\mu_m, x_l) = \int f_{t,x}(y) d\mu_m(y)$ and that $\lim_{m \to \infty} V_t(\mu_m, x_l) = \int f_{t,x}(y) d\mu(y)$ exist for each $m \in \mathbb{N}$ and each $l \in \mathbb{N}$, respectively, Theorem 2.13 in [9] implies that
\[ \lim_{l \to \infty} \lim_{m \to \infty} V_t(\mu_m, x_l) = \lim_{m \to \infty} \lim_{l \to \infty} V_t(\mu_m, x_l) = \lim_{l,m \to \infty} V_t(\mu_m, x_l) = V_t(\mu, x). \]

Hence, if $(\mu_l, x_l)$ is some sequence in $\mathcal{M}_+(O) \times O$ (endowed with the product topology) such that $(\mu_l, x_l) \to (\mu, x) \in \mathcal{M}_+(O) \times O$, then $\lim_{l \to \infty} V_t(\mu_l, x_l) = V_t(\mu, x)$, showing that $V_t(\cdot, \cdot)$ is jointly continuous at $(\mu, x)$. 

Before we present our main result, some preparation is required. Let, for each $\alpha \in (0, 1)$, $\beta_{\mu}^{H, \alpha} : E \times \mathbb{N} \to [0, +\infty)$ be defined by the law $\beta_{\mu}^{H, \alpha}(x, s) := \sup_{t \geq s} t^\alpha V_t(\mu, x)$, where for each $t > 0$, $V_t(\cdot, \cdot) : \mathcal{M}_+(E) \times E$ is defined as in the statement of Lemma 3.17.

**Remark 3.18** The proof that, for each $t > 0$, the mapping $V_t(\cdot, \cdot) : \mathcal{M}_+(E) \times E$, $V_t(\mu, x) = \int f_{t,x}(y) d\mu(y)$, is jointly continuous if $(E, d)$ is a Polish metric space is identical to the proof of Lemma 3.17 in the definition of $f_{t,x}$, just replace the euclidean metric in $\mathbb{R}$ by $d$.

**Lemma 3.19** Let $E$ be a Polish metric space and let $\alpha \in (0, 1)$. Then, for each $\delta > 0$ and each $r, s \in \mathbb{N}$,
\[ \mathcal{M}_{r,s}(\delta) := \{ \mu \in \mathcal{M}_+(E) \mid \mu(\mathcal{Z}_\mu(r, s)) \geq \delta \} \]
is a closed subset of $\mathcal{M}(E)$, where $\mathcal{Z}_\mu(r, s) := \{ x \in E \mid \beta_{\mu}^{H, \alpha}(x, s) \leq r \}$.

**Proof.** **Claim 1.** For each $r, s \in \mathbb{N}$ and each $\mu \in \mathcal{M}_+(E)$, $Z_r(r, s)$ is a closed subset of $E$.

Let $\{w_i\}$ be a sequence in $\mathcal{Z}_\mu(r, s)$ such that $\lim w_i = w$. Since, for each $t > 0$, $f_{t,w_i} \to f_{t,w}$ pointwise, it follows from Remark 3.18 that the mapping $x \to \beta_{\mu}^{H, \alpha}(x, s)$ is lower semi-continuous. Hence, $\beta_{\mu}^{H, \alpha}(w, s) \leq r$, which means that $w \in \mathcal{Z}_\mu(r, s)$.
Claim 2. For each \( s \in \mathbb{N} \), \( W_{r,s} = \{(\nu, x) \in \mathcal{M}(E) \times E \mid \beta^{H,\alpha}_\nu(x, s) > r\} \) is open.

This is a consequence of the fact that, by Remark 3.18, the mapping \( \mathcal{M}_+(E) \times E \ni (\nu, x) \mapsto \beta^{H,\alpha}_\nu(x, s) \) is lower semi-continuous.

Now, we show that \( \mathcal{M}_{r,s}(\delta) \) is closed. Let \( \mu_m \) be a sequence in \( \mathcal{M}_{r,s}(\delta) \) such that \( \mu_m \to \mu \).

Suppose, by absurd, that \( \mu \notin \mathcal{M}_{r,s}(\delta) \); we will find that \( \mu_m \notin \mathcal{M}_{r,s}(\delta) \) for \( m \) sufficiently large, a contradiction.

If \( \mu \notin \mathcal{M}_{r,s}(\delta) \), then \( \{\mu\} \times A \subset W_r \). Since \( \mu \) is a tight measure on \( E \) (\( \mu \) is a Borel measure and the space \( X \) is Polish; see Proposition A.2.2.5 in [7]), there exists a compact \( C \subset A \) such that \( \mu(C) > \mu(E) - \delta \) (note that, by Claim 1, \( A \) is open).

Now, we construct a suitable subset of \( W_{r,s} \) that contains a neighborhood of \( \{\mu\} \times C \). Let, for each \( x \in C \), \( V_x \subset W_{r,s} \) be an open neighborhood of \( (\mu, x) \) (such open set exists, by Claim 2); that is, \( V_x := B((\mu, x); \varepsilon) = \{\nu \in \mathcal{M}_+(E) \times E \mid \max\{\rho(\nu, \mu), d(x, y)\} < \varepsilon\} \), for some suitable \( \varepsilon > 0 \). Then, \( \{V_x\}_{x \in C} \) is an open cover of \( \{\mu\} \times C \), and since \( \{\mu\} \times C \) is a compact subset of \( \mathcal{M}_+(E) \times E \), it follows that one can extract from \( \{V_x\}_{x \in C} \) a finite subcover, \( \{V_{x_i}\}_{i=1}^n \).

We affirm that there exists an \( \ell \in \mathbb{N} \) (which depends on \( C \)) such that \( \{\mu_n\}_{n \geq \ell} \subset \bigcap_i (\pi_1(V_{x_i})) \).

Namely, for each \( i \), there exists an \( \ell_i \) such that \( \{\mu_n\}_{n \geq \ell_i} \subset \pi_1(V_{x_i}) \); set \( \ell := \max\{\ell_i \mid i \in \{1, \ldots, n\}\} \), and note that for each \( i \), \( \{\mu_n\}_{n \geq \ell_i} \subset \pi_1(V_{x_i}) \). Set also \( \mathcal{I} := \bigcap_i (\pi_1(V_{x_i})) \) and \( \mathcal{O} := \bigcup_i (\pi_2(V_{x_i})) \).

Since for each \( i \), \( V_{x_i} = \pi_1(V_{x_i}) \times \pi_2(V_{x_i}) \), and given that

\[
\{\mu_n\}_{n \geq \ell} \times \mathcal{O} \subset \mathcal{I} \times \mathcal{O} \subset \bigcup_i (\pi_1(V_{x_i}) \times \pi_2(V_{x_i})) = \bigcup_i V_{x_i} \subset W_{r,s},
\]

it follows that, for each \( n \geq \ell \) and each \( y \in \mathcal{O} \), \( \beta^{H,\alpha}_{\mu_n}(y, s) > r \). Moreover, \( \mathcal{O} \) is an open set that contains \( C \).

On the other hand, weak convergence implies that

\[
\limsup_{m \to \infty} \mu_m(E \setminus \mathcal{O}) \leq \mu(E \setminus \mathcal{O}) \leq \mu(E \setminus C) < \delta,
\]

from which follows that there exists an \( \ell_1 \geq \ell \) such that, for \( m \geq \ell_1 \), \( \mu_m(E \setminus \mathcal{O}) < \delta \).

Combining the last results, one concludes that, for \( m \geq \ell_1 \), \( \mu_m(E \setminus \mathcal{O}) < \delta \), and for each \( x \in \mathcal{O} \), \( \beta^{H,\alpha}_{\mu_m}(x, s) > r \), so

\[
\mu_m(Z_{\mu_m}(r, s)) \leq \mu_m(E \setminus \mathcal{O}) < \delta;
\]

this contradicts the fact that, for each \( m \in \mathbb{N} \), \( \mu_m \in \mathcal{M}_{r,s}(\delta) \). Hence, \( \mu \notin \mathcal{M}_{r,s}(\delta) \), and \( \mathcal{M}_{r,s}(\delta) \) is a closed subset of \( \mathcal{M}_+(E) \).

\[\Box\]

Define, for \( \alpha \in (0, 1) \) and \( s \in \mathbb{N} \), \( \gamma^{H(P),\alpha}_{\psi,T}(x, s) := \sup(\inf)_{\ell \geq x} \ell^\alpha U_{\psi,T}(T, x) \). Then, by Lemma 3.17, one has, for each \( x \in O \), \( \lim_{s \to \infty} \gamma^{K,\alpha}_{\psi,T}(x, s) = (D^{K,\alpha}_{\psi,T})_{(\psi, \cdot)}(x) \). By definition, for each \( x \in O \), \( \mathbb{N} \ni s \mapsto \gamma^{H(P),\alpha}_{\psi,T}(x, s) \in [0, +\infty) \) is a nonincreasing (nondecreasing) mapping.

**Theorem 3.20** Let \( \emptyset \neq F \subset \mathbb{R} \) be a closed subset, let \( 0 \neq \psi \in \mathcal{H} \), and \( \mu^{\psi,T}_{F}(\cdot) := \mu^{\psi,T}_{F}(F \cap \cdot) \). Then, each of the sets \( C^{\psi,F}_{0T} := \{T \in X \mid \dim^{\psi}_{T}(\mu^{\psi,F}) = 0\} \) and \( C^{\psi,F}_{1T} := \{T \in X \mid \dim^{\psi}_{T}(\mu^{\psi,F}) = 1\} \) is a \( G_\delta \) set in \( X \).
Proof. Since the arguments in both proofs are analogous, we just prove the statement for $C_{\alpha}^{0,Hc}$. Note that for each closed set $F$, there exists a countable family of open sets, $\{A_i\}$, such that $F = \bigcap_{i \geq 1} A_i$ (each closed set $F$ is a $G_\delta$ set); thus, one just has to prove the result for $C_{\alpha}^{0,Hc}$, where $\emptyset \neq O \subset \mathbb{R}$ is an open set.

If, for each $T \in X$, $\mu_T^O(\mathbb{R}) = 0$, then $C_{\alpha}^{0,Hc} = \emptyset$ is a $G_\delta$ subset of $X$. Thus, suppose that $T \in X$ such that $\mu_T^O(\mathbb{R}) > 0$.

Set, for each $\alpha \in (0,1)$, $C_{\alpha}^{0,Hc} := \bigcup_{p \geq 1} C_{\alpha}^{0,Hc}(p)$, where $C_{\alpha}^{0,Hc}(p) = \{T \in X \mid \mu_T^O(\{x \in \mathbb{R} \mid H^{H,\alpha}(\mu_T^O)(x) < p\}) > 0\}$. Now, by Theorem 2.14 and Proposition 2.16,

$$C_{\alpha}^{0,Hc} = \bigcap_{k \geq 1} \bigcup_{p \geq 1} (C_{\alpha}^{0,Hc}(1/k))^{c} = \bigcap_{k \geq 1} \bigcap_{p \geq 1} (C_{\alpha}^{0,Hc}(1/k))^c. \quad (3.4)$$

Claim 1. For each $\alpha \in (0,1)$ and each $p \in \mathbb{N},$

$$(C_{\alpha}^{0,Hc}(p))^c = \bigcap_{s \in \mathbb{N}} \{T \in X \mid \mu_T^O(\gamma_{H,\alpha}^{H,T}(x,s)) \geq p\}.$$ 

Let $T \in (C_{\alpha}^{0,Hc}(p))^c$. Since, for each $x \in \mathbb{R}$, $\nabla H^{H,\alpha}(x,s) \in [0,\infty)$ is a nonincreasing function, it follows that, for each $s \in \mathbb{N}$, $\mu_T^O(\gamma_{H,\alpha}^{H,T}(x,s)) \geq p$.

Now, let $T \in \bigcap_{s \in \mathbb{N}} \{U \in X \mid \mu_U^{(p,s)}(\gamma_{H,\alpha}^{H,T}(x,s)) \geq p\}$. Then, for each $s \in \mathbb{N}$, there exits a Borel set $A_s \subset \mathbb{R}$ with $\mu_O(A_s) = 1$, such that for each $x \in A_s$, $\gamma_{H,\alpha}^{H,T}(x,s) \geq p$. Let $A := \bigcap_{s \geq 1} A_s$; then, for each $x \in A$, one has $(H^{H,\alpha}(\mu_T^O))(x) = \lim_{s \to \infty} \gamma_{H,\alpha}^{H,T}(x,s) \geq p$, given that $\mu_T^O(A) = 1$, we are done.

Let, for each $\alpha \in (0,1)$ and each $p, q, s, l \in \mathbb{N},$

$$C_{\alpha}^{0,Hc}(p - 1/q, s, l) := \{T \in X \mid \mu_T^O(A_{\alpha}^{H,T}(p - 1/q, s)) \geq 1/l\},$$

where $A_{\alpha}^{H,T}(p - 1/q, s) := \{x \in O \mid \gamma_{H,\alpha}^{H,T}(x,s) \leq p - 1/q\}$. Thus, according to Claim 1 and (3.4),

$$C_{\alpha}^{0,Hc} = \bigcap_{k \geq 1} \bigcap_{p \geq 1} (C_{\alpha}^{0,Hc}(1/k))^c,$$

and one just needs to show that, for each $\alpha \in (0,1)$ and each $r, s, l \in \mathbb{N}$, $C_{\alpha}^{0,Hc}(r, s, l)$ is closed in $X$.

Claim 2. For each $\delta > 0$ and each $r, s \in \mathbb{N}$, $\{\mu \in M_{\alpha}(O) \mid \mu(Z_\mu(r,s)) \geq \delta\}$ is a closed subset of $M_{\alpha}(O)$, where $Z_\mu(r,s) = \{x \in \mathbb{R} \mid \beta_\mu^{H,\alpha}(x,s) \leq r\}$.

Here, we use the fact that $O$ can be isometrically embedded in $\overline{O}$, which is a Polish metric space. Thus, for each $\mu \in M_{\alpha}(O)$ can be identified with the measure $\tilde{\mu} \in M_{\alpha}(\overline{O})$ defined by $\tilde{\mu}(A) = \mu(A \cap O)$ for each $A \in B(\overline{O})$, and $M_{\alpha}(O)$ can be identified with a subset of $M_{\alpha}(\overline{O})$, namely, the set $\{\tilde{\mu} \in M_{\alpha}(\overline{O}) \mid \tilde{\mu}(\overline{O}) = \mu(O)\}$. Then, the induced topology in $M_{\alpha}(O)$ by the Polish space $M_{\alpha}(\overline{O})$ coincides with the weak topology in $M_{\alpha}(O)$ (see Section 6 in [12] for details).

Moreover, for each $\mu \in M_{\alpha}(O)$ and each $r, s \in \mathbb{N}$, $Z_\mu(r,s) = \{x \in \overline{O} \mid \beta_\mu^{H,\alpha}(x,s) \leq r\} \cap O$, so for each $\delta > 0$,

$$\{\mu \in M_{\alpha}(O) \mid \mu(Z_\mu(r,s)) \geq \delta\} = \{\tilde{\mu} \in M_{\alpha}(\overline{O}) \mid \tilde{\mu}(Z_\tilde{\mu}(r,s)) = \mu(Z_\mu(r,s)) \geq \delta\} \cap M_{\alpha}(O).$$
Recall that by the functional calculus, for each \( \neq \psi \in H \), the mapping \( \zeta_\psi : X \to M_+(O) \), \( \zeta_\psi(T) = \mu_\psi^{-\mu_\psi}(O) \), is continuous (since convergence in \( X \) implies strong resolvent convergence), and note that for each \( (x,s) \in O \times N \), \( \gamma_\psi^{H,\alpha}(x,s) = \beta_\psi^{H,\alpha}(T,x,s) \).

Thus, it follows that for each \( l, r, s \in N \), \( C_\psi^{O}(r,s,l) = (\gamma_\psi)^{-1}(M_{r,s}(1/l)) \), and therefore, by Claim 2, \( C_\psi^{O}(r,s,l) \) is a closed subset of \( X \).

\[ \square \]

4 Proof of Theorems 1.1 and 1.4

**Proof.** (Theorem 1.1) The result is a direct consequence of the hypotheses, Theorem 3.20, and the fact that the intersection of a countable family of generic sets is still a generic set. \( \square \)

In order to prove Theorem 1.4, we need the following result.

**Theorem 4.21 (Theorems 1.1 in [5] and 1.3 in [6])** Suppose that \( \Omega \) is a Cantor group and that \( \tau : \Omega \to \Omega \) is a minimal translation. Then, there exist dense sets of \( g \in C(\Omega, \mathbb{R}) \) such that, for each \( \kappa \in \Omega \),

1. the spectrum of \( H_\kappa^{\tau} \) is purely absolutely continuous;
2. the spectrum of \( H_\kappa^{\tau} \) is zero-Hausdorff dimensional.

**Proof.** (Theorem 1.4) Fix \( \kappa \in \Omega \) and let \( \tau : \Omega \to \Omega \) be a minimal translation of the Cantor group \( \Omega \).

It follows from Theorem 4.21 that each of the sets \( C^\kappa_{Tpa} \supset C^\kappa_{ac} := \{ T \in X_\kappa \mid \sigma(T) \) is purely absolutely continuous \} and \( C^\kappa_{Ohd} \) is dense in \( X_\kappa \).

The result is now a consequence of Theorem 1.1 and Remark 1.3. \( \square \)

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