Sampling in the range of the analysis operator of a continuous frame having unitary structure

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Abstract

We establish a regular sampling theory in the range of the analysis operator of a continuous frame having a unitary structure. The unitary structure is related with a unitary representation of a locally compact abelian group on a separable Hilbert space. The samples are defined by means of suitable discrete convolution systems which generalize some usual sampling settings; here regular sampling means that the samples are taken at a countable discrete subgroup.

Keywords: Continuous and discrete frames; Convolution systems; Sampling.

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1 Statement of the problem

In this paper a regular sampling theory in the range space of the analysis operator of a continuous frame is established. To be more specific, the continuous frame has a unitary structure associated with a unitary representation $t \in G \mapsto U(t)$ of an LCA (locally compact abelian) group $G$ on a separable Hilbert space $\mathcal{H}$. The functions to be recovered, at a countable discrete subgroup $H$ of $G$, are associated with functions in a unitary $H$-invariant subspace $\mathcal{H}_\Phi$ of $\mathcal{H}$ (that, eventually, could coincide with $\mathcal{H}$); the subscript $\Phi = \{ \varphi_1, \varphi_2, \ldots, \varphi_N \} \subset \mathcal{H}$ denotes a set of stable generators for $\mathcal{H}_\Phi$. Thus, the functions to be recovered from a sequence of samples at $H$ look like:

$$F_f(t) = \langle f, U(t)\varphi \rangle_{\mathcal{H}}, \ t \in G, \text{ where } f \in \mathcal{H}_\Phi.$$

The set of these functions $F$ (we omit the subscript) forms a reproducing kernel Hilbert space (in short RKHS) of continuous and bounded functions contained in $L^2(G)$ that will be denoted as $\mathcal{H}_{U,\varphi,\Phi}$. For some examples of these spaces we cite, among others, the Paley-Wiener spaces $PW_{\pi,\sigma}$, shift-invariant subspaces $V^2_\Phi$ in $L^2(\mathbb{R}^d)$ and, in the non-abelian case, the range space of the continuous wavelet or Gabor transforms.

The goal in this work is to recover, in a stable way, any $F$ in $\mathcal{H}_{U,\varphi,\Phi}$ from a sequence of samples taken at the subgroup $H$. For instance, for the sequence of pointwise samples

\footnotesize
\begin{itemize}
\item The author is very pleased to dedicate this work to his friend, the mathematician Lance L. Littlejohn on the occasion of his 70th birthday.
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\end{itemize}

\normalsize
\{F(t)\}_{t \in H}$ we obtain a suitable expression as the output of a discrete convolution system (see Eq. (1) in Section 3)

$$F(t) = \sum_{n=1}^{N} a_n *_{H} x_n(t), \ t \in H,$$

where $a_n(t) := \langle \varphi_n, U(t) \phi \rangle _{\mathcal{H}}, \ t \in H$, belongs to $\ell^2(H)$ for $n = 1, 2, \ldots, N$, $x_n \in \ell^2(H)$, $n = 1, 2, \ldots, N$, are the coefficients of the expansion $f = \sum_{n=1}^{N} \sum_{t \in H} x_n(t) U(t) \varphi_n$, and $*_{H}$ denotes the convolution in $\ell^2(H)$. The same occurs for the average samples $\{\mathcal{M}_m F(t)\}_{t \in H}$ defined by

$$\mathcal{M}_m F(t) := \langle f, U(t) \psi_m \rangle _{\mathcal{H}}, \ t \in H, \ \text{for some fixed } \psi_m \in \mathcal{H}.$$ 

The above examples lead us to define, for any $F$ in the space $\mathcal{H}_{U, \phi, \Phi}$, a sequence of generalized samples $\{\mathcal{L}_m F(t)\}_{t \in H; m=1,2,\ldots,M}$, at the subgroup $H$, by means of an $M \times N$ matrix $A = [a_{m,n}]$ with entries in $\ell^2(H)$ as

$$\mathcal{L}_m F(t) := \sum_{n=1}^{N} (a_{m,n} *_{H} x_n)(t), \ t \in H, \ m = 1, 2, \ldots, M.$$ 

Thus, under appropriate conditions (see Definition 1 in Section 3), the main sampling result (see Theorem 1 in Section 3) proves that there exist $M$ sampling functions $S_m$ in $\mathcal{H}_{U, \phi, \Phi}$, $m = 1, 2, \ldots, M$, such that the sequence $\{S_m(\cdot - t)\}_{t \in H; m=1,2,\ldots,M}$ is a frame for $\mathcal{H}_{U, \phi, \Phi}$, and the sampling expansion

$$F(s) = \sum_{m=1}^{M} \sum_{t \in H} \mathcal{L}_m F(t) S_m(s - t), \ s \in G,$$

holds for every $F \in \mathcal{H}_{U, \phi, \Phi}$. In addition, the sampling functions $S_m$, $m = 1, 2, \ldots, M$, can be obtained, via the matrix $A$, by means of an explicit method (see in the end of Section 3).

The work is organized as follows: In Section 2 we include some needed preliminaries on continuous and discrete frames; on Fourier analysis for a countable discrete group, and on convolution systems in the Hilbert space $\ell^2_N(H) := \ell^2(H) \times \cdots \times \ell^2(H)$ ($N$ times); this is an auxiliary space isomorphic to $\mathcal{H}_{U, \phi, \Phi}$.

The used mathematical techniques are similar to those in Ref. [14]. They lie in exploiting the relationship between discrete convolution systems and frames of translates in the product Hilbert space $\ell^2_N(H) := \ell^2(H) \times \cdots \times \ell^2(H)$ ($N$ times); this is an auxiliary space isomorphic to $\mathcal{H}_{U, \phi, \Phi}$.

The work is organized as follows: In Section 2 we include some needed preliminaries on continuous and discrete frames; on Fourier analysis for a countable discrete group, and on convolution systems in the Hilbert space $\ell^2_N(H)$. It is worth to mention the relationship between convolution systems in $\ell^2_N(H)$ and frames of translates in $\ell^2_N(H)$ showing the equivalence of their properties. The needed results have been borrowed from Refs. [14][20]. Section 3 is devoted to introduce the subspace of $L^2(G)$ where the sampling theory will be carried out. Finally, Section 4 contains the main sampling result along with some pertinent comments and remarks. Although the work deals with abelian groups, an example involving semi-direct products of groups is included; this particular non-abelian case will be treated by using the theory developed in Section 4.

It should be noted that working in LCA groups is not just a unified way of dealing with the classical groups $\mathbb{R}^d, \mathbb{Z}^d, \mathbb{T}^d, \mathbb{Z}_s^d$: signal processing often involves products of these groups which are also locally compact abelian groups. For example, multichannel video signal involves the group $\mathbb{Z}^d \times \mathbb{Z}_s$, where $d$ is the number of channels and $s$ the number of pixels of each image. Finally, some companion references in sampling theory are, for instance, Refs. [2][6][9][19][21][23].
2 Some preliminaries

2.1 Continuous and discrete frames

Let \( \mathcal{H} \) be a Hilbert space and let \((\Omega, \mu)\) be a measure space. A mapping \( \psi : \Omega \rightarrow \mathcal{H} \) is a continuous frame for \( \mathcal{H} \) with respect to \((\Omega, \mu)\) if \( \psi \) is weakly measurable, i.e., for each \( x \in \mathcal{H} \), the function \( w \mapsto \langle x, \psi(w) \rangle \) is measurable, and there exist constants \( 0 < A \leq B \) such that

\[
A \|x\|^2 \leq \int_{\Omega} |\langle x, \psi(w) \rangle|^2 \, d\mu(w) \leq B \|x\|^2 \quad \text{for each } x \in \mathcal{H}.
\]

(1)

The constants \( A \) and \( B \) are the lower and upper continuous frame bounds respectively. The mapping \( \psi \) is a tight continuous frame if \( A = B \); a Parseval continuous frame if \( A = B = 1 \). The mapping \( \psi \) is called a Bessel family if only the right-hand inequality holds. Throughout this paper we refer a continuous frame as the mapping \( \psi : \Omega \rightarrow \mathcal{H} \), or as the family \( \{\psi(w)\}_{w \in \Omega} \), or \( \{\psi_w\}_{w \in \Omega} \), in the Hilbert space \( \mathcal{H} \).

There are a lot of examples of continuous frames in the mathematical/physics literature. For instance: the family \( \{k_w\}_{w \in \Omega} \) of reproducing kernels of a RKHS \( \mathcal{H}_k \) contained in \( L^2(\Omega, \mu) \) is a continuous Parseval frame with respect to \((\Omega, \mu)\); a Gabor system \( \{M_\xi T_x g : (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d\} \) is a tight continuous frame for \( L^2(\mathbb{R}^d) \) with respect to \((\mathbb{R}^d \times \mathbb{R}^d, dx \, d\xi)\), where \( g \in L^2(\mathbb{R}^d) \) is a fixed non zero function, and \( M_\xi \) and \( T_x \) denote the modulation and translation operators in \( L^2(\mathbb{R}^d) \) respectively; a wavelet system \( \{T_b D_a \psi : (a, b) \in (\mathbb{R} \setminus \{0\}) \times \mathbb{R}\} \) is a tight continuous frame for \( L^2(\mathbb{R}) \) with respect to \((\mathbb{R} \setminus \{0\}) \times \mathbb{R}, \frac{da \, db}{a^2}\), where \( T_b \) and \( D_a \) denote the translation and dilation operators in \( L^2(\mathbb{R}) \) respectively, and \( \psi \in L^2(\mathbb{R}) \) is an admissible function, i.e., a function for which the constant \( c_\psi := \int_{\mathbb{R}} \frac{|\hat{\psi}(w)|^2}{|w|} \, dw < +\infty \); coherent states in physics, etc. (see, for instance, Refs. [4] [8] [10]).

The operator \( T_\psi : L^2(\Omega, \mu) \rightarrow \mathcal{H} \) weakly defined for each \( f \in L^2(\Omega, \mu) \) by

\[
\langle T_\psi f, x \rangle = \int_{\Omega} f(w) \langle \psi(w), x \rangle \, d\mu(w), \quad x \in \mathcal{H},
\]

is linear and bounded; it is called the synthesis operator of \( \{\psi_w\}_{w \in \Omega} \). Its adjoint operator \( T_\psi^* : \mathcal{H} \rightarrow L^2(\Omega, \mu) \) is given by \( (T_\psi^* x)(w) = \langle x, \psi(w) \rangle \), \( w \in \Omega \), and it is called the analysis operator of \( \{\psi_w\}_{w \in \Omega} \). The continuous frame operator \( S_\psi = T_\psi T_\psi^* \) is a bounded, self-adjoint, positive and invertible operator in \( \mathcal{H} \). For any \( x \in \mathcal{H} \) we have the weak representations

\[
x = \int_{\Omega} \langle x, \psi(w) \rangle S_\psi^{-1} \psi(w) \, d\mu(w) = \int_{\Omega} \langle x, S_\psi^{-1} \psi(w) \rangle \psi(w) \, d\mu(w).
\]

The counting measure \( \mu \) on \( \Omega = \mathbb{N} \) gives the classical definition of discrete frame \( \{x_n\}_{n=1}^\infty \); there exist two constants \( 0 < A \leq B \) such that

\[
A \|x\|^2 \leq \sum_{n=1}^\infty |\langle x, x_n \rangle|^2 \leq B \|x\|^2 \quad \text{for each } x \in \mathcal{H}.
\]

(2)

Given a discrete frame \( \{x_n\}_{n=1}^\infty \) for \( \mathcal{H} \), its preframe (synthesis) operator \( T : \ell^2(\mathbb{N}) \rightarrow \mathcal{H} \) is defined by \( T\{c_n\}_{n=1}^\infty = \sum_{n=1}^\infty c_n \, x_n \). Its adjoint operator \( T^* : \mathcal{H} \rightarrow \ell^2(\mathbb{N}) \) is given by
The frame operator $S$ is defined by $S(x) := \langle x, x \rangle$ and it is called its analysis operator. The frame operator $S$ is defined by $S(x) := x S^{-1} x = \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n$, $x \in \mathcal{H}$; it is a bounded, invertible, positive and self-adjoint operator in $\mathcal{H}$. The sequence $\{S^{-1} x_n\}_{n=1}^{\infty}$ is also a frame for $\mathcal{H}$ called the canonical dual frame. For each $x \in \mathcal{H}$ we have the expansions

$$x = \sum_{n=1}^{\infty} \langle x, x_n \rangle S^{-1} x_n = \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n.$$

As a consequence, given a frame $\{x_n\}_{n=1}^{\infty}$ for $\mathcal{H}$ the representation property of any vector $x \in \mathcal{H}$ as a series $x = \sum_{n=1}^{\infty} c_n x_n$ is retained, but, unlike the case of Riesz (orthonormal) bases, the uniqueness of this representation is sacrificed. Suitable frame coefficients $\{c_n\}$ which depend continuously and linearly on $x$ are obtained by using the dual frames $\{y_n\}_{n=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$, i.e., $\{y_n\}_{n=1}^{\infty}$ is another frame for $\mathcal{H}$ such that

$$x = \sum_{n=1}^{\infty} \langle x, y_n \rangle x_n = \sum_{n=1}^{\infty} \langle x, x_n \rangle y_n \quad \text{for each } x \in \mathcal{H}.$$

In particular, frames include orthonormal and Riesz bases for $\mathcal{H}$. For more details and proofs on discrete and continuous frames, see, for instance, Refs. [4, 8, 10, 13, 22].

Assume that $\{\psi(w)\}_{w \in \Omega}$ is a continuous frame for a Hilbert space $\mathcal{H}$ with respect to $(\Omega, \mu)$ such that the mapping $w \mapsto \psi(w)$ is weakly continuous, i.e., for each $x \in \mathcal{H}$ the function $w \mapsto \langle x, \psi(w) \rangle$ is continuous. Its analysis operator $T_{\psi}^* : \mathcal{H} \to L^2(\Omega, \mu)$ is a bounded and boundedly invertible operator on its range denoted as $\mathcal{H}_{\psi} := \text{Range } T_{\psi}^*$. This is a closed subspace of $L^2(\Omega, \mu)$ described as the functions $F_x$ such that

$$\mathcal{H} \to \mathcal{H}_{\psi}, \quad x \mapsto F_x : F_x(w) = \langle x, \psi(w) \rangle_{\mathcal{H}}, \quad w \in \Omega.$$

Besides $\mathcal{H}_{\psi}$ is a RKHS (of continuous functions in $\Omega$) whose reproducing kernel is given by

$$k_{\psi}(u, v) = \langle \psi(v), S_{\psi}^{-1} \psi(u) \rangle_{\mathcal{H}}, \quad u, v \in \Omega,$$

where $S_{\psi}^{-1}$ denotes the inverse of the frame operator $S_{\psi}$ associated to $\{\psi(w)\}_{w \in \Omega}$. That is, for each $F_x \in \mathcal{H}_{\psi}$ we have the reproducing property

$$F_x(u) = \int_{\Omega} F_y(v) k_{\psi}(u, v) d\mu(v) = \langle F_x, k_{\psi}(\cdot, u) \rangle_{L^2(\Omega, \mu)}, \quad u \in \Omega.$$

### 2.2 Discrete convolution systems and frames of translates

Let $(H, +)$ be a countable discrete abelian group and let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the unidimensional torus. A character $\xi$ of $H$ is a homomorphism $\xi : H \to \mathbb{T}$, i.e., $\xi(h + h') = \xi(h) \xi(h')$ for all $h, h' \in H$; we denote $\xi(h) = (h, \xi)$. By defining $(\xi + \xi')(h) = (\xi(h) \xi'(h))$, the set of characters $\hat{H}$ is a group, called the dual group of $H$; since $H$ is discrete, the group $\hat{H}$ is compact [11, Prop. 4.4].

For $x \in \ell^1(H)$ its Fourier transform is defined by

$$\hat{x}(\xi) := \sum_{h \in H} x(h) \overline{\xi(h)} = \sum_{h \in H} x(h) \xi(-h), \quad \xi \in \hat{H}.$$
The Plancherel theorem extends uniquely the Fourier transform on $\ell^1(H) \cap \ell^2(H)$ to a unitary isomorphism from $\ell^2(H)$ to $L^2(\hat{H})$. For the details see, for instance, Refs. [11, 12].

Let $H$ be a countable discrete group and let consider the product Hilbert space $\ell^2_N(H) := \ell^2(H) \times \cdots \times \ell^2(H)$ ($N$ times). For a matrix $A = [a_{m,n}] \in \mathcal{M}_{M \times N}(\ell^2(H))$, i.e., an $M \times N$ matrix with entries in $\ell^2(H)$, and $x \in \ell^2_N(H)$, the matrix convolution $A \star x$ in $H$ is defined by

$$(A \star x)(h) := \sum_{h' \in H} A(h - h') x(h'), \quad h \in H.$$ 

Note that the $m$-th entry of $A \star x$ is $\sum_{n=1}^{N} (a_{m,n} \ast x_n)$, where $x_n$ denotes the $n$-th entry of $x \in \ell^2_N(H)$. The usual properties of a discrete convolution can be found in Refs. [11, 12].

The discrete convolution system with associated matrix $A = [a_{m,n}] \in \mathcal{M}_{M \times N}(\ell^2(H))$ given by

$$A : \ell^2_N(H) \to \ell^2_M(H), \quad x \mapsto A(x) = A \star x$$

is a well defined bounded operator if and only if $\hat{A} \in \mathcal{M}_{M \times N}(L^\infty(\hat{H}))$, where $\hat{A}(\xi) := [\hat{a}_{m,n}(\xi)]$ denotes the transfer matrix of $A$ (see Refs. [14, 20]). We identify the matrix $\hat{A}$ with entries in $L^\infty(\hat{A})$ and the essentially bounded matrix-valued function $\hat{A}(\xi)$, a.e. $\xi \in \hat{A}$.

Its adjoint operator $A^* : \ell^2_M(H) \to \ell^2_N(H)$ is also a bounded convolution system with associated matrix $A^* = [a_{m,n}^*]^\top \in \mathcal{M}_{N \times M}(\ell^2(H))$, where $a_{m,n}^*$ denotes the involution $a_{m,n}^*(h) := a_{m,n}(-h^\dagger)$, $h \in H$ (Refs. [14, 20]). Its transfer matrix is $\hat{A^*}(\xi)$ is just the transpose conjugate of $\hat{A}(\xi)$, i.e., $\hat{A}(\xi)^*$, a.e. $\xi \in \hat{H}$ (Refs. [14, 20]).

The bounded operator $A$ is injective with a closed range if and only if the operator $A^* A$ is invertible; equivalently, the constant $\delta_A := \text{ess inf}_{\xi \in \hat{H}} \det[\hat{A}(\xi)^* \hat{A}(\xi)] > 0$ (see Ref. [14]).

Let $a_{m,n}^*$ denote the $m$-th column of the matrix $A^*$, then the $m$-th component of $A(x)$ is

$$[A \star x]_m(h) = \sum_{n=1}^{N} (a_{m,n} \ast_H x_n)(h) = \langle x, T_h a_{m,n}^* \rangle_{\ell^2_N(H)}, \quad h \in H,$$

where $T_h$ denotes the translation operator by $h \in H$ in $\ell^2_N(H)$, i.e., for $a \in \ell^2_N(H)$, $T_h a(g) = a(g - h)$, $g \in H$, and $\ast_H$ is the convolution indexed by $H$. In other words, the operator $A$ is the analysis operator of the sequence $\{T_h a_{m,n}^*\}_{h \in H; m=1,2,\ldots,M}$ in $\ell^2_N(H)$. Since the sequence $\{T_h a_{m,n}^*\}_{h \in H; m=1,2,\ldots,M}$ is a frame for $\ell^2_N(H)$ if and only if its (bounded) analysis operator is injective with a closed range (see Ref. [3]). Hence, the sequence $\{T_h a_{m,n}^*\}_{h \in H; m=1,2,\ldots,M}$ will be a frame for $\ell^2_N(H)$ if and only if the constant $\delta_A > 0$.

Concerning the duals of $\{T_h a_{m,n}^*\}_{h \in H; m=1,2,\ldots,M}$ having the same structure, consider two matrices $\hat{A} \in \mathcal{M}_{M \times N}(L^\infty(\hat{H}))$ and $\hat{B} \in \mathcal{M}_{N \times M}(L^\infty(\hat{H}))$, and let $b_m$ denote the $m$-th column of the matrix $B$ associated to $\hat{B}$. Then, the sequences $\{T_h a_{m,n}^*\}_{h \in H; m=1,2,\ldots,M}$ and $\{T_h b_m\}_{h \in H; m=1,2,\ldots,M}$ form a pair of dual frames for $\ell^2_N(H)$ if and only if $\hat{B}(\xi) \hat{A}(\xi) = I_N$, a.e. $\xi \in \hat{H}$; equivalently, if and only if $B A = T_{e_N}(H)$, i.e., the convolution system $B$ is a
left-inverse of the convolution system $A$ (see Ref. [14]). Thus in $\ell^2_N(H)$ we have the frame expansion

$$x = \sum_{m=1}^{M} \sum_{h \in H} \langle x, T_h a_m^* \rangle \ell^2_N(H) T_h b_m \quad \text{for each } x \in \ell^2_N(H).$$

Finally to remind that the convolution system $A$ in (3) is an isomorphism if and only if $M = N$ and $\text{ess inf}_{\xi \in \hat{H}} |\det \tilde{A}(\xi)| > 0$ (see Refs. [14] [20]). Thus, for the case $M = N$ the sequence $\{T_h a_m^*\}_{h \in H; m=1,2,...,N}$ is a Riesz basis for $\ell^2_N(H)$. The square matrix $\tilde{A}(\xi)$ is invertible, a.e. $\xi \in \hat{H}$, and from the columns of $\tilde{A}(\xi)^{-1}$ we get its dual Riesz basis $\{T_h b_m\}_{h \in H; m=1,2,...,N}$.

3 The subspace of $L^2(G)$ where the sampling is carried out

Let $G \ni t \mapsto U(t) \in \mathcal{U}(H)$ be a unitary representation of a LCA group $(G, +)$ on a separable Hilbert space $H$. Recall that $\{U(t)\}_{t \in G}$ is a family of unitary operators in $H$ satisfying: $U(t)U(t') = U(t + t')$ for $t, t' \in G$; $U(0) = I_H$; and $\langle U(t)\varphi, \phi \rangle_H$ is a continuous function of $t$ for any $\varphi, \phi \in H$. Note that $U(t)^{-1} = U(-t)$, and since $U(t)^* = U(t)^{-1}$ we have $U(t)^* = U(-t)$.

Assume that for a fixed $\phi \in H$ the family $\{U(t)\phi\}_{t \in G}$ is a continuous frame for the Hilbert space $H$ with respect to $(G, \mu_G)$, where $\mu_G$ denotes the Haar measure on $G$. Let $H$ be a countable discrete subgroup of $G$. For a stable set of generators $\Phi = \{\varphi_1, \varphi_2, \ldots, \varphi_N\} \subset H$ we consider the $U$-invariant subspace in $H$ generated by $\Phi$:

$$\mathcal{H}_\Phi = \left\{ \sum_{n=1}^{N} x_n(h)U(h)\varphi_n : x_n \in \ell^2(H), \ n = 1, 2, \ldots, N \right\}. $$

We are assuming that the sequence $\{U(h)\varphi_n\}_{h \in H; n=1,2,...,N}$ is a Riesz sequence in $H$, i.e., a Riesz basis for $\mathcal{H}_\Phi$. Finally, we define the subspace in $L^2(G)$ given by

$$\mathcal{H}_{U,\phi,\Phi} := \left\{ F_f : G \rightarrow \mathbb{C} : F_f(t) = \langle f, U(t)\phi \rangle_H, \ t \in G, \text{ where } f \in \mathcal{H}_\Phi \right\}. $$

Notice that the mapping $\mathcal{H}_\Phi \ni f \mapsto F_f \in \mathcal{H}_{U,\phi,\Phi}$ is an isomorphism between Hilbert spaces, and the space $\mathcal{H}_{U,\phi,\Phi}$ is a RKHS of continuous and bounded functions in $L^2(G)$.

Besides, the space $\mathcal{H}_{U,\phi,\Phi}$ is an $H$-shift-invariant subspace of $L^2(G)$; indeed, for each $t \in H$ we have that the function $F_f(s-t) = F_{U(t)f}(s)$, $s \in G$, and consequently it belongs to $\mathcal{H}_{U,\phi,\Phi}$.

For instance, if $\phi = \text{sinc}$ is the cardinal sine function then $\{U(t)\phi = \text{sinc}(\cdot - t)\}_{t \in \mathbb{R}}$ is a continuous frame for $H = PW_\pi$ with respect to $[\mathbb{R}, dx]$ and the space $\mathcal{H}_{U,\phi,\Phi}$ coincides with $PW_\pi$.

From now on we will omit the subscript $f$ in the notation of $F_f$. The aim of this work is to obtain stable sampling results for the functions $F \in \mathcal{H}_{U,\phi,\Phi}$; for instance, the stable recovery of any $F \in \mathcal{H}_{U,\phi,\Phi}$ from the sequence of its samples $\{F(t)\}_{t \in H}$ taken at the countable subgroup $H$ of $G$ and/or other sequences of samples $\{L_mF(t)\}_{t \in H}$ introduced in next section.
3.1 Sampling data as a filtering process

Let $F$ be a function in $\mathcal{H}_{U,\phi,\Phi}$. For each $t \in H$, we have for the sample $F(t)$ the expression

$$F(t) = \langle f, U(t) \phi \rangle_{\mathcal{H}} = \left\langle \sum_{n=1}^{N} \sum_{h \in H} x_n(h) U(h) \varphi_n, U(t) \phi \right\rangle_{\mathcal{H}}$$

$$= \sum_{n=1}^{N} \sum_{h \in H} x_n(h) \langle \varphi_n, U(t-h) \phi \rangle_{\mathcal{H}} = \sum_{n=1}^{N} (a_n *_{H} x_n)(t), \quad t \in H,$$

where $a_n(s) := \langle \varphi_n, U(s) \phi \rangle_{\mathcal{H}}, s \in H$, and $*_{H}$ denotes the convolution in $\ell^2(H)$.

Similarly, we can define generalized average samples as follows: Given $M$ fixed elements $\psi_1, \psi_2, \ldots, \psi_M$ in $\mathcal{H}$, for any $F \in \mathcal{H}_{U,\phi,\Phi}$ we define the samples at $H$ as

$$\mathcal{M}_m F(t) := \langle f, U(t) \psi_m \rangle_{\mathcal{H}}, \quad t \in H, \text{ and } m = 1, 2, \ldots, M.$$  \hfill (5)

Proceeding as in Eq. (4), for each $m = 1, 2, \ldots, M$ we get

$$\mathcal{M}_m F(t) = \sum_{n=1}^{N} (a_{m,n} *_{H} x_n)(t), \quad t \in H,$$

where $a_{m,n}(s) := \langle \varphi_n, U(s) \psi_m \rangle_{\mathcal{H}}, s \in H$. The above examples together with the results in Section 2.2 lead us to define in $\mathcal{H}_{U,\phi,\Phi}$ a generalized stable sampling procedure at the subgroup $H$ as follows:

**Definition 1.** Let $F(s) = \langle f, U(s) \phi \rangle_{\mathcal{H}}, s \in G$, be a function in $\mathcal{H}_{U,\phi,\Phi}$ and suppose that $f = \sum_{n=1}^{N} \sum_{h \in H} x_n(h) U(h) \varphi_n$ in $\Phi$. A generalized stable sampling procedure $\mathcal{L}_A$ at $H$ in $\mathcal{H}_{U,\phi,\Phi}$ is defined for each $F \in \mathcal{H}_{U,\phi,\Phi}$ by

$$\mathcal{L}_A F(t) := (A *_{H} x)(t), \quad t \in H,$$  \hfill (6)

where $\mathcal{L}_A F(t) := (\mathcal{L}_1 F(t), \mathcal{L}_2 F(t), \ldots, \mathcal{L}_M F(t))^\top, x(t) = (x_1(t), x_2(t), \ldots, x_N(t))^\top \in \ell^2_N(H)$, and $A$ denotes a matrix $[a_{m,n}] \in \mathcal{M}_{M \times N}(\ell^2(H))$ such that:

1. Its transfer matrix $\mathcal{A} \in \mathcal{M}_{M \times N}(L^\infty(\widehat{H}))$, and

2. the constant $\delta_A := \text{ess inf}_{\xi \in \widehat{H}} \det[\mathcal{A}(\xi)^* \mathcal{A}(\xi)] > 0$.

Definition 1 is, of course, equivalent to the classical one that states the existence of two positive constants $0 < c < C$ such that

$$c \|F\|^2 \leq \sum_{m=1}^{M} \sum_{t \in H} |\mathcal{L}_m F(t)|^2 \leq C \|F\|^2 \text{ for any } F \in \mathcal{H}_{U,\phi,\Phi}.$$  

See Notes 2 and 7 in Section 4.

The definition of stable sampling as stated above shows, in a explicit way, the relationship between the stable samples and their associated sampling formulas. Indeed, as it will be
proved in Theorem 1 (see Section 2), once a generalized stable sampling procedure \( \mathbf{L}_A \) is given in \( \mathcal{H}_{U,\phi,\Phi} \), there exists a method to obtain the associated stable sampling formulas of the form:

\[
F(s) = \sum_{m=1}^{M} \sum_{t \in H} \mathcal{L}_m F(t) S_m(s - t), \quad s \in G,
\]

for every function \( F \in \mathcal{H}_{U,\phi,\Phi} \). Namely:

- Compute a matrix \( \hat{B}(\xi) \in \mathcal{M}_{N \times M}(L^\infty(\hat{H})) \) such that \( \hat{B}(\xi) \hat{A}(\xi) = I_N \), a.e. \( \xi \in \hat{H} \).
- Compute the matrix \( B \in \mathcal{M}_{N \times M}(\ell^2(H)) \) such that \( \hat{B}(\xi) \) is its transfer matrix.
- Let \( b_m = (b_{1,m}, b_{2,m}, \ldots, b_{N,m})^\top \) be the \( m \)-th column of the matrix \( B, m = 1, 2, \ldots, M \).
- Finally, we obtain the sampling functions as \( S_m(s) = \langle \beta_m, U(s)\phi \rangle_{\mathcal{H}^*} \), \( s \in G \), where 
  \[
  \beta_m = \sum_{n=1}^{N} \sum_{h \in H} b_{n,m}(h) U(h) \varphi_n, \quad m = 1, 2, \ldots, M.
\]

Condition 2. in Definition 1 implies necessarily that \( M \geq N \), i.e., the number \( N \) of stable generators used in the auxiliary space \( \mathcal{H}_\Phi \) determines the minimal number \( M \) of sequences of samples at \( H \) that should be considered. In next section we go into the details.

4 The main sampling result and consequences

In this section we state and prove a general sampling result for \( \mathcal{H}_{U,\phi,\Phi} \) associated with a stable sampling procedure \( \mathbf{L}_A \) at a subgroup \( H \). We will see that other usual sampling results can be deduced from it.

**Theorem 1.** Assume that a generalized sampling procedure \( \mathbf{L}_A \) at \( H \), as in Definition 1, has been defined in \( \mathcal{H}_{U,\phi,\Phi} \). Then, there exist \( M \) sampling functions \( S_m, m = 1, 2, \ldots, M \), in \( \mathcal{H}_{U,\phi,\Phi} \) such that the sequence \( \{S_m(\cdot - t)\}_{t \in H; m=1,2,\ldots,M} \) is a frame for \( \mathcal{H}_{U,\phi,\Phi} \), and the sampling expansion

\[
F(s) = \sum_{m=1}^{M} \sum_{t \in H} \mathcal{L}_m F(t) S_m(s - t), \quad s \in G,
\]

holds for every \( F \in \mathcal{H}_{U,\phi,\Phi} \). The pointwise convergence of the above series is absolute in \( G \) and uniform on \( G \). It also converges in the \( L^2(G) \)-norm sense.

**Proof.** Consider \( F(s) = \langle f, U(s)\phi \rangle_{\mathcal{H}^*} \), \( s \in G \), a function in \( \mathcal{H}_{U,\phi,\Phi} \) and suppose that for the corresponding \( f \) in \( \mathcal{H}_\Phi \) we have the expansion \( f = \sum_{n=1}^{N} \sum_{h \in H} x_n(h) U(h) \varphi_n \). According to the results in Section 2.2 since the matrix \( \hat{A} \) has entries in \( L^\infty(\hat{H}) \) and \( \delta_A > 0 \), the sequence \( \{T_t a_m^*\}_{t \in H; m=1,2,\ldots,M} \) is a frame for \( \ell^2_N(H) \) where 

\[
a_m^* = (a_{m,1}^*, a_{m,2}^*, \ldots, a_{m,N}^*)^\top \in \ell^2_N(H)
\]

denotes the \( m \)-th column of the matrix \( A^* = [a_{m,n}^*] \in \mathcal{M}_{N \times M}(\ell^2(H)) \) whose entries are the involutions \( a_{m,n}^*(h) = a_{m,n}(-h) \), \( h \in H \). Moreover, \( \mathcal{L}_m F(t) = \langle x, T_t a_m^* \rangle_{\ell^2_N(H)} \), \( t \in H \) and \( m = 1, 2, \ldots, M \).

Furthermore, there exists a matrix \( \hat{B}(\xi) \in \mathcal{M}_{N \times M}(L^\infty(\hat{H})) \) such that \( \hat{B}(\xi) \hat{A}(\xi) = I_N \), a.e. \( \xi \in \hat{H} \); it suffices to take \( \hat{B}(\xi) = \hat{A}(\xi)^\dagger := [\hat{A}(\xi)^* \hat{A}(\xi)]^{-1} \hat{A}(\xi)^* \), the Moore-Penrose pseudo-inverse of \( \hat{A}(\xi) \). Besides, the sequence \( \{T_t b_m\}_{t \in H; m=1,2,\ldots,M} \) in \( \ell^2_N(H) \) is a dual frame of
\[ \{ T_t a_m^* \}_{t \in \ell^r} \] (8) in the \( \ell^r \) shows that the due to the inequality (9) is absolute due to the unconditional character of a frame expansion. It is uniform on \( S^\beta \) where \( T \) which satisfies the shifting property \( H \) group transfer matrix is \( \hat{T} \). Now consider the isomorphism \( T_\Phi \) defined as
\[
T_\Phi : \ell^r \rightarrow \mathcal{H}_\Phi
\]
x \mapsto f = \sum_{n=1}^{N} \sum_{h \in H} x_n(h) U(h) \Phi_n,
\]
which satisfies the shifting property \( T_\Phi(T_t b) = U(t)(T_\Phi b) \), \( t \in H \) and \( b \in \ell^r \). Applying the isomorphism \( T_\Phi \) in Eq. (8), we get
\[
f = \sum_{m=1}^{M} \sum_{t \in \ell^r} L_m F(t) U(t)(T_\Phi b_m) = \sum_{m=1}^{M} \sum_{t \in \ell^r} L_m F(t) U(t) \beta_m, \text{ in } \mathcal{H}_\Phi,
\]
where \( \beta_m = T_\Phi b_m = \sum_{n=1}^{N} \sum_{h \in H} b_n m(h) U(h) \phi_n \in \mathcal{H}_\Phi \). Finally, for \( F \in \mathcal{H}_{U,\phi,\Phi} \) we obtain
\[
F(s) = \langle \sum_{m=1}^{M} \sum_{t \in \ell^r} L_m F(t) U(t) \beta_m, U(s) \phi \rangle = \sum_{m=1}^{M} \sum_{t \in \ell^r} L_m F(t) \langle \beta_m, U(s-t) \phi \rangle_{\mathcal{H}}
\]
\[
= \sum_{m=1}^{M} \sum_{t \in \ell^r} L_m F(t) S_m(s-t), \quad s \in G,
\]
where \( S_m(s) = \langle \beta_m, U(s) \phi \rangle_{\mathcal{H}} \), \( s \in G \), and \( m = 1,2,\ldots,M \). The pointwise convergence in (9) is absolute due to the unconditional character of a frame expansion. It is uniform on \( G \) due to the inequality \( |F(s)| \leq \|f\| \|U(s)\phi\| \leq \|f\| \|\phi\| \) for all \( s \in G \). The composition of isomorphisms
\[
\ell^r \rightarrow \mathcal{H}_\Phi \rightarrow \mathcal{H}_{U,\phi,\Phi}
\]
shows that the \( \ell^r \)-convergence in expansion (8) implies the convergence of expansion in (9) in the \( L^2(G) \)-norm sense; besides, the sequence \( \{ S_m(\cdot - t) \}_{t \in \ell^r; m=1,2,\ldots,M} \) is a frame for \( \mathcal{H}_{U,\phi,\Phi} \).

As a consequence of the above theorem we obtain a Shannon-type sampling formula for the space \( \mathcal{H}_{U,\phi,\Phi} \):

**Corollary 2.** In order to recover any \( F \in \mathcal{H}_{U,\phi,\Phi} \) from its samples \( \{ F(t) \}_{t \in H} \), at the subgroup \( H \), necessarily \( N = 1 \). Under conditions in Definition 4, i.e., \( \hat{a} \in L^\infty(\hat{H}) \) and \( \text{ess inf}_{\xi \in \hat{H}} |\hat{a}(\xi)| > 0 \), where \( a(s) = \langle \varphi, U(s) \phi \rangle_{\mathcal{H}} \), \( s \in H \), there exists a unique sampling function \( S_a \in \mathcal{H}_{U,\phi,\Phi} \) such that the sampling expansion
\[
F(s) = \sum_{t \in H} F(t) S_a(s-t), \quad s \in G,
\]
holds in \( \mathcal{H}_{U,\phi,\Phi} \). The sequence \( \{ S_a(\cdot - t) \}_{t \in H} \) is a Riesz basis for \( \mathcal{H}_{U,\phi,\Phi} \).
Proof. In this scalar case, there exists a unique \( \hat{b}(\xi) \in L^\infty(\hat{H}) \) such that \( \hat{b}(\xi) \hat{a}(\xi) = 1 \), a.e. \( \xi \in \hat{H} \). As a consequence, the associated sampling function \( S_n(s) = \langle \beta, U(s)\phi \rangle_{\mathcal{H}}, \ s \in G \), where \( \beta = \sum_{h \in H} b(h) U(h)\phi \) belongs to \( \mathcal{H}_\phi \) and it is unique. \( \square \)

In case \( N > 1 \) we must add \( M - 1 \) sequences \( \{ \mathcal{L}_m F(t) \}_{t \in H} \) of samples, \( m = 2, \ldots, M \), with \( M \geq N \), to the sequence \( \{ F(t) \}_{t \in H} \) such that the corresponding matrix \( A = [a_{m,n}] \) in \( \mathcal{M}_{M \times N}(\ell^2(H)) \) satisfies the conditions in Definition [1]. Thus, by using Theorem [1] there exist \( M \) sampling functions \( S_m \in \mathcal{H}_{U,\phi,\Phi}, m = 1, 2, \ldots, M \), such that \( \{ S_m(t - t) \}_{t \in H; m=1,2,\ldots,M} \) is a frame for \( \mathcal{H}_{U,\phi,\Phi} \), and the sampling expansion

\[
F(s) = \sum_{t \in H} F(t) S_1(s - t) + \sum_{m=2}^M \sum_{t \in H} \mathcal{L}_m F(t) S_m(s - t), \quad s \in G, \tag{12}
\]

holds for every \( F \in \mathcal{H}_{U,\phi,\Phi} \).

4.1 Sampling at a subgroup \( R \) with finite index in \( H \)

The result in Theorem [1] can be easily modified in order to take just samples at a subgroup \( R \) with finite index in \( H \). Indeed, let \( R \) be a subgroup of \( H \) with finite index \( L \). We pick a set \( \{ h_1, h_2, \ldots, h_L \} \) of representatives of the cosets of \( R \), i.e., the group \( H \) can be decomposed as

\[
H = (h_1 + R) \cup (h_2 + R) \cup \ldots \cup (h_L + R) \text{ with } (h_l + R) \cap (h_{l'} + R) = \emptyset \text{ for } l \neq l'.
\]

The space \( \mathcal{H}_\Phi \) can be written as

\[
\mathcal{H}_\Phi = \left\{ \sum_{n=1}^N \sum_{h \in H} x_n(h) U(h)\varphi_n : x_n \in \ell^2(H) \right\} = \left\{ \sum_{n=1}^N \sum_{l=1}^L \sum_{r \in R} x_n(h_l + r) U(h_l + r)\varphi_n \right\}
\]

\[
= \left\{ \sum_{n=1}^N \sum_{l=1}^L \sum_{r \in R} x_{nl}(r) U(r)\varphi_{nl} : x_{nl} \in \ell^2(R) \right\},
\]

with \( x_{nl}(r) := x_n(h_l + r) \) and \( \varphi_{nl} := U(h_l)\varphi_n \), where the new index \( nl \) goes from 11 to \( NL \). Thus our subspace \( \mathcal{H}_\Phi \) can be rewritten as \( \mathcal{H}_\Phi \) with \( NL \) generators \( \Phi = \{ \varphi_{nl} \} \) and coefficients \( x_{nl} \) in \( \ell^2(R) \).

For instance, concerning the samples \( \{ F(r) \}_{r \in R} \) we have

\[
F(r) = \langle f, U(r)\phi \rangle_{\mathcal{H}} = \sum_{n=1}^N \sum_{l=1}^L \sum_{s \in R} x_{nl}(s) U(s)\varphi_{nl}, U(r)\phi \rangle_{\mathcal{H}}
\]

\[
= \sum_{n=1}^N \sum_{l=1}^L \sum_{s \in R} x_{nl}(s) \langle \varphi_{nl}, U(r - s)\phi \rangle_{\mathcal{H}} = \sum_{n=1}^N \sum_{l=1}^L (a_{1,nl} * \kappa x_{nl})(r), \quad r \in R, \tag{13}
\]

where \( a_{1,nl}(s) = \langle \varphi_{nl}, U(s)\phi \rangle_{\mathcal{H}}, \ s \in R \), and \( *_{\kappa} \) denotes the convolution in \( \ell^2(R) \). The new index runs as \( nl = 11, 12, \ldots, 1L, \ldots, N1, N2, \ldots, NL \). In general, we could consider a stable sampling procedure \( \mathcal{L}_A \) at \( R \) with associated matrix \( A = [a_{m,nl}] \in \mathcal{M}_M^{M \times NL}(\ell^2(R)) \) as in Definition [1]. Thus we have:
Corollary 3. Let $A = [a_{m,n}] \in \mathcal{M}_{MN \times N}(\ell^2(\mathbb{R}))$ be a matrix associated to a stable sampling procedure $\mathcal{L}_A$ at $R$ as in Definition 4. Then, there exist $M \geq NL$ sampling functions $S_m \in \mathcal{H}_{U,\phi,\Phi}$, $m = 1,2,\ldots, M$, such that the sampling expansion

$$F(s) = \sum_{m=1}^{M} \sum_{r \in R} \mathcal{L}_m F(r) S_m(s-r), \quad s \in G.$$  \hfill (14)

holds in $\mathcal{H}_{U,\phi,\Phi}$. The sequence $\{S_m(\cdot - r)\}_{r \in R; m=1,2,\ldots, M}$ is a frame for $\mathcal{H}_{U,\phi,\Phi}$.

Proof. Consider a matrix $B \in \mathcal{M}_{N, M \times M}(\ell^2(\mathbb{R}))$ such that such that $\hat{B}(\xi) \in \mathcal{M}_{N, M}(L^\infty(\mathbb{R}))$ and $\hat{B}(\xi) \hat{A}(\xi) = I_{NL}$, a.e. $\xi \in \hat{R}$. The sampling functions are $S_m(s) = \langle \beta_m, U(s) \phi \rangle_{\mathcal{H}}$, $s \in G$, where $\beta_m = T_\Phi b_m$ belongs to $\mathcal{H}_\Phi$, and $b_m \in \ell^2_{NL}(R)$ is the $m$-th column of the matrix $B$ with transfer matrix $\hat{B}$, $m = 1,2,\ldots, M$. \hfill $\square$

4.2 Additional notes and remarks

Next we include some comments and remarks enlightening the above results:

1. In Section 3 we are assuming that the sequence $\{U(h) \varphi_n\}_{n \in H; n=1,2,\ldots, N}$ is a Riesz sequence in $\mathcal{H}$, i.e., a Riesz basis for $\mathcal{H}_\Phi$. Necessary and sufficient conditions for sequences having this unitary structure can be found in Refs. [1,5,7,16,18,20].

2. For any stable sampling procedure $\mathcal{L}_A$ at $H$ as in Definition 4 there exist positive constants $0 < c \leq C$ such that

$$c\|F\|^2 \leq \sum_{m=1}^{M} \sum_{t \in H} |\mathcal{L}_m F(t)|^2 \leq C\|F\|^2 \quad \text{for every } F \in \mathcal{H}_{U,\phi,\Phi}. \hfill (15)$$

Indeed, it follows since the sequence $\{T_t a_m^*\}_{t \in H; m=1,2,\ldots, M}$ is a frame for $\ell^2_N(H)$ and from the isomorphism in Eq. (12).

3. In case $M = N$ in Theorem 4 the sequence $\{T_t a_m^*\}_{t \in H; m=1,2,\ldots, N}$ is a Riesz basis for $\ell^2_N(H)$ and the square matrix $\hat{A}(\xi)$ is invertible; proceeding as in the proof of Theorem 4 its inverse gives the dual Riesz basis $\{T_t b_m\}_{t \in H; m=1,2,\ldots, N}$. As a consequence, the sequence $\{S_n(\cdot - t)\}_{t \in H; n=1,2,\ldots, N}$ is a Riesz basis for $\mathcal{H}_{U,\phi,\Phi}$. The uniqueness of the coefficients in a Riesz basis expansion gives the interpolation property.

$$\mathcal{L}_n S_n'(t - t') = \delta_{n,n'} \delta_{t,t'}, \quad \text{where } t,t' \in H \quad \text{and } n,n' = 1,2,\ldots, N.$$

4. The use of multiple generators for the auxiliary space $\mathcal{H}_\Phi$ might have advantages against the single generator case. For instance, in Corollary 2 the conditions of stable sampling in Definition 4 for a single generator $\varphi$ are: $\hat{a} \in L^\infty(H)$ and $\text{ess inf}_{\xi \in \hat{H}} |\hat{a}(\xi)| > 0$, where $a(s) = \langle \varphi, U(s) \phi \rangle_{\mathcal{H}}$, $s \in H$. A way to overcome the restrictive second condition is to consider a subgroup $R$ of $H$ with finite index $L > 1$, and then consider $M \geq L$ sequences of samples taken at the subgroup $R$. 

11
5. In Theorem 1 whenever $M > N$, there exist infinite sampling functions $S_m$ coming from the different dual frames $\{ T_l b_m \}_{l \in H; m=1,2,...,M}$ of $\{ T_l a^*_m \}_{l \in H; m=1,2,...,M}$. All these duals come from the left-inverses $B(\xi)$ of $A(\xi)$ which are obtained, from the Moore-Penrose pseudo-inverse $\hat{A}(\xi)^\dagger = [\hat{A}(\xi)^*\hat{A}(\xi)]^{-1}\hat{A}(\xi)^*$, by means of the $N \times M$ matrices

$$B(\xi) := \hat{A}(\xi)^\dagger + C(\xi)[I_M - \hat{A}(\xi)\hat{A}(\xi)^\dagger],$$

where $C(\xi)$ denotes any $N \times M$ matrix with entries in $L^\infty(G)$. Indeed, it is straightforward to check that any matrix having this form is a left-inverse of $\hat{A}(\xi)$. Moreover, any left-inverse $\hat{B}(\xi)$ of $\hat{A}(\xi)$ belongs to the above family; it suffices to take $C(\xi) = \hat{B}(\xi)$.

6. The sequence $\{ T_l a^*_m \}_{l \in H; m=1,2,...,M}$ is a Bessel sequence in $\ell^2_N(H)$ if and only if the convolution system in (5) is bounded, i.e., if and only if the transfer matrix $\hat{A}(\xi)$ belongs to $\mathcal{M}_{M \times N}(L^\infty(\hat{H}))$. Moreover, having in mind the equivalence between the spectral and Frobenius norms for matrices (see Ref. [17]), it is equivalent to the condition

$$\beta_A := \sup_{\xi \in \hat{H}} \lambda_{\max}[\hat{A}(\xi)^*\hat{A}(\xi)] < +\infty,$$

where $\lambda_{\max}$ denotes the largest eigenvalue of the positive semidefinite matrix $\hat{A}(\xi)^*\hat{A}(\xi)$.

7. Under the hypothesis $\hat{A} \in \mathcal{M}_{M \times N}(L^\infty(\hat{H}))$, the condition $\delta_A > 0$ in Definition 1 is equivalent to the condition $\alpha_A := \inf_{\xi \in \hat{H}} \lambda_{\min}[\hat{A}(\xi)^*\hat{A}(\xi)] > 0$, where $\lambda_{\min}$ denotes the smallest eigenvalue of the positive semidefinite matrix $\hat{A}(\xi)^*\hat{A}(\xi)$. Indeed, it comes from the inequalities:

$$\alpha_A^N \leq \delta_A \leq \alpha_A \beta_A^{N-1},$$

where $\beta_A$ was introduced in the above note.

8. Assuming that $\hat{A} \in \mathcal{M}_{M \times N}(L^\infty(\hat{H}))$, it is worth to mention that the condition $\delta_A > 0$ is also necessary for the existence of a frame expansion (7) as those in Theorem 1. Indeed, suppose that, for each $F \in \mathcal{H}_{U,\phi,\Phi}$, a frame expansion $F = \sum_{m=1}^M \sum_{l \in H} E_m F(t) S_m(\cdot - t)$ holds, and let $\mathcal{T}$ denote the isomorphism in Eq. (10). Then we have

$$x = \sum_{m=1}^M \sum_{l \in H} \langle x, T_l a^*_m \rangle_{\ell^2_N(H)} \mathcal{T}^{-1} S_m(\cdot - t) \quad \text{for all } x \in \ell^2_N(H).$$

Since $\{ T_l a^*_m \}_{l \in H; m=1,2,...,M}$ is a Bessel sequence, the sequences $\{ T_l a^*_m \}_{l \in H; m=1,2,...,M}$ and $\{ \mathcal{T}^{-1} S_m(\cdot - t) \}_{l \in H; m=1,2,...,M}$ form a pair of dual frames in $\ell^2_N(H)$ (see Ref. [8]). In particular, according to a result in Section 2.2 since $\{ T_l a^*_m \}_{l \in H; m=1,2,...,M}$ is a frame for $\ell^2_N(H)$ we get $\delta_A > 0$.

9. Whenever the entries $a_{m,n}$ of the matrix $A$ belong to $\ell^1(H)$, their Fourier transforms $\hat{a}_{m,n}$ are continuous and consequently they belong to $L^\infty(\hat{H})$. In this case the condition $\delta_A > 0$ is equivalent to $\det[\hat{A}(\xi)^*\hat{A}(\xi)] \neq 0$ for all $\xi \in \hat{H}$.

10. For the samples given in Eqs. (13), the entries $a_{m,n}$ of the matrix $A$ depend on the unitary representation $U(t)$ of the group $G$, $\psi$ and $\Phi$. For a general stable sampling procedure $\mathcal{L}_A$ at $H$, by changing $\psi$ and $\Phi$ we could recover a function $F$, in different spaces $\mathcal{H}_{U,\phi,\Phi}$, from the same sequence of samples.
11. We can relax the initial assumptions in this section by assuming that \( \{U(t)\phi\}_{t \in G} \) is just a complete Bessel family for \( \mathcal{H} \) with respect to \((G, \mu_G)\). In this case the mapping \( f \in \mathcal{H}_\Phi \mapsto F_f \in \mathcal{H}_{U,\phi,\Phi} \) is injective and continuous but not necessarily an isomorphism (the subspace \( \mathcal{H}_{U,\phi,\Phi} \) is not necessarily closed in \( L^2(G) \)). Under the hypotheses in Theorem 4, the sampling formula holds but the left-hand inequality in (15) does not hold.

4.3 The case of a semi-direct product of groups

The case where the group \( G \) is the semi-direct product of two groups can be easily reduced to the situation described in Section 4 under appropriate conditions. Let \( G = K \rtimes \sigma \) \( H \) be the semi-direct product of the LCA group \((K, +)\) and a not necessarily abelian group \((H, \cdot)\), where \( \sigma \) denotes the action of the group \( H \) on the group \( K \), i.e., a homomorphism \( \sigma : H \to \text{Aut}(K) \) mapping \( h \mapsto \sigma_h \). The composition law in \( G \) is \((k_1, h_1)(k_2, h_2) := (k_1 + \sigma_h(k_2), h_1h_2)\) for \((k_1, h_1), (k_2, h_2) \in G\). In general, the group \( G = K \rtimes \sigma \) \( H \) is not abelian. In case \( \sigma_h \equiv \text{Id}_K \) for each \( h \in H \) we recover the direct product group \( G = K \times H \).

Now we consider a subgroup \( G' = K' \ltimes \sigma \) \( H' \) where \((K', +)\) is a countable discrete group and \((H', \cdot)\) is a finite group of order \( N \) (we will write \( H' = \{1_H, h_2, \ldots, h_N\} \)), and such that \( \sigma_h(K') = K' \) for each \( h \in H' \). Suppose that \((k, h) \mapsto U(k, h)\) is a unitary representation of the group \( G = K \rtimes \sigma \) \( H \) on a separable Hilbert space \( \mathcal{H} \). For a fixed \( \varphi \in \mathcal{H} \), the corresponding \( \mathcal{H}_\varphi \) subspace can be written as

\[
\mathcal{H}_\varphi = \left\{ \sum_{(k, h) \in G'} x(k, h) U(k, h) \varphi : \{x(k, h)\} \in \ell^2(G') \right\}
= \left\{ \sum_{n=1}^{N} \sum_{k \in K'} x(k, h_n) U(k, 1_H) U(0_K, h_n) \varphi \right\}
= \left\{ \sum_{n=1}^{N} \sum_{k \in K'} x_n(k) U(k, 1_H) \varphi_n : \{x_n\} \in \ell^2(K') \right\},
\]

where \( x_n(k) := x(k, h_n), k \in K' \), and \( \varphi_n := U(0_K, h_n) \varphi, n = 1, 2, \ldots, N \). That is, \( \mathcal{H}_\varphi \equiv \mathcal{H}_\Phi \) where \( \Phi = \{\varphi_1, \varphi_2, \ldots, \varphi_N\} \) is a set of \( N \) generators in \( \mathcal{H} \). Assume that \( \{U(s, t)\phi\}_{(s, t) \in G} \) is a continuous frame for \( \mathcal{H} \) with respect to \((G, \mu_G)\), and consider the corresponding \( \mathcal{H}_{U,\phi,\Phi} \) space. For any function \( F(s, t) = \langle f, U(s, t)\phi \rangle_{\mathcal{H}} \), \( (s, t) \in G \), in \( \mathcal{H}_{U,\phi,\Phi} \) we define a sampling procedure at \( K' \) by

\[
\mathcal{L}_A F(k) = (\mathcal{L}_1 F(k), \mathcal{L}_2 F(k), \ldots, \mathcal{L}_M F(k))^\top := (A *_{K'} x)(k), \quad k \in K',
\]

where \( A = [a_{m,n}] \in M_{M \times N} (\ell^2(K')) \) and \( x = (x_1, x_2, \ldots, x_N)^\top \in \ell^2_N(K') \). Under conditions in Definition 4 there exist \( M \) elements \( \beta_m \in \mathcal{H}_\Phi, m = 1, 2, \ldots, M \), such that

\[
f = \sum_{m=1}^{M} \sum_{k \in K'} \mathcal{L}_m F(k) U(k, 1_H) \beta_m \quad \text{in} \ \mathcal{H}_\Phi .
\]

Remind that the elements \( \beta_m \in \mathcal{H}_\Phi, m = 1, 2, \ldots, M \), are obtained from a left-inverse \( \tilde{B}(\xi) \)
of \( \hat{A}(\xi) \), a.e. \( \xi \in \hat{K}' \), as in Theorem \( \text{[11]} \). Hence, for each \( F \in \mathcal{H}_{U,\phi,\psi} \) we have

\[
F(s,t) = \langle f, U(s,t) \phi \rangle_{\mathcal{H}} = \sum_{m=1}^{M} \sum_{k \in K'} \mathcal{L}_m F(k) \langle U(k,1_H) \beta_m, U(s,t) \phi \rangle_{\mathcal{H}}
\]

\[
= \sum_{m=1}^{M} \sum_{k \in K'} \mathcal{L}_m F(k) (\beta_m, U[(k,1_H)^{-1}(s,t)] \phi)_{\mathcal{H}}
\]

\[
= \sum_{m=1}^{M} \sum_{k \in K'} \mathcal{L}_m F(k) S_m(s - k,t), \quad (s,t) \in G,
\]

where \( S_m(s,t) = \langle \beta_m, U(s,t) \phi \rangle_{\mathcal{H}}, (s,t) \in G, m = 1,2,\ldots,M \). Notice that \( (k,1_H)^{-1}(s,t) = (-k,1_H)(s,t) = (s - k,t) \), for \( s \in K, k \in K' \) and \( t \in H \).

### 4.3.1 Euclidean motion group and crystallographic subgroups

An example of the above setting is given by crystallographic groups as subgroups of the Euclidean motion group \( E(d) \). This group is the semi-direct product \( \mathbb{R}^d \rtimes O(d) \) corresponding to the homomorphism \( \sigma : O(d) \to \text{Aut}(\mathbb{R}^d) \) given by \( \sigma_{\gamma}(x) = \gamma x \), where \( \gamma \in O(d) \) and \( x \in \mathbb{R}^d \); \( O(d) \) denotes the orthogonal group of order \( d \). The composition law on \( E(d) = \mathbb{R}^d \rtimes O(d) \) reads \( (x,\gamma) \cdot (x',\gamma') = (x + \gamma x', \gamma \gamma') \).

The subgroup \( G' \) would be the crystallographic group \( C_{P,\Gamma} := P \mathbb{Z}^d \rtimes \Gamma \) where \( P \) is a non-singular \( d \times d \) matrix and \( \Gamma \) is a finite subgroup of \( O(d) \) of order \( N \) such that \( \gamma(P \mathbb{Z}^d) = P \mathbb{Z}^d \) for each \( \gamma \in \Gamma \). We will denote \( \{\gamma_1 = I, \gamma_2, \ldots, \gamma_N\} \) the elements of the group \( \Gamma \). In this example we consider the quasi regular representation (see Ref. \[5\]) on \( L^2(\mathbb{R}^d) \):

\[
U(s,\gamma)f(t) = f[\gamma^\top (t - s)], \quad t, s \in \mathbb{R}^d, \gamma \in O(d) \text{ and } f \in L^2(\mathbb{R}^d).
\]

Assume that \( \phi \in L^2(\mathbb{R}^d) \) is a function such that the family \( \{U(s,\gamma)\phi\}_{(s,\gamma) \in E(d)} \) is a continuous frame for a closed subspace \( \mathcal{H} \) of \( L^2(\mathbb{R}^d) \) (containing \( \mathcal{H}_\phi \) with respect to \( (E(d),ds \, d\mu(\gamma)) \)), where \( d\mu(\gamma) \) denotes the left Haar measure on the group \( O(d) \). For instance we could take \( \phi \) a bandlimited function to a compact set \( \Omega \subset \mathbb{R}^d \) and consider \( \mathcal{H} := PW_{\Omega} \); the details are similar to those in Ref. \[15\]. See Ref. \[5\] for the details on the left Haar measure in semi-direct products of groups.

For any function \( F(s,\gamma) = \langle f(\cdot,\phi(\gamma^\top(\cdot - s))) \rangle_{L^2(\mathbb{R}^d)} \), \( (s,\gamma) \in E(d) \), with \( f \in \mathcal{H}_\phi \), we consider a stable sampling procedure \( \mathcal{L}_A F(p) := (A * P_{\mathbb{Z}^d} \chi)(p), p \in P \mathbb{Z}^d \), defined at the lattice \( P \mathbb{Z}^d \) as in Definition \[11\]. Remind that in this particular example the space defined in \( \text{[16]} \) is:

\[
\mathcal{H}_\phi = \left\{ \sum_{n=1}^{N} \sum_{p \in P \mathbb{Z}^d} x_n(p) \varphi_n(t-p) : \{x_n\} \in \ell^2(P \mathbb{Z}^d) \right\},
\]

with \( N \) generators \( \varphi_n(t) = \varphi(\gamma_n^\top t), n = 1,2,\ldots,N, \) in \( L^2(\mathbb{R}^d) \). The corresponding sampling formula \( \text{[17]} \) reads:

\[
F(s,\gamma) = \sum_{m=1}^{M} \sum_{p \in P \mathbb{Z}^d} \mathcal{L}_m F(p) S_m(s - p,\gamma), \quad (s,\gamma) \in E(d),
\]
where $S_m(s, \gamma) = \langle \beta_n(\cdot), \phi(\gamma^r \cdot - s) \rangle_{L^2(\mathbb{R}^d)}$, $(s, \gamma) \in E(d)$, for some functions $\beta_n \in \mathcal{H}_\Phi$, $m = 1, 2, \ldots, M$, obtained from a left-inverse of the matrix $\tilde{A}(\xi)$ as in Theorem 4.1.

Whenever $f = \sum_{n=1}^N \sum_{p \in \mathbb{P} \mathbb{Z}^d} x_n(p) \varphi_n(t - p) \in \mathcal{H}_\Phi$, for the pointwise samples $F(p, I) = \langle f(\cdot), \varphi(\cdot - p) \rangle_{L^2(\mathbb{R}^d)}$, $p \in \mathbb{P} \mathbb{Z}^d$ we have the expression $F(p, I) = \sum_{n=1}^N (a_{1, n} * x_n)(p)$, $p \in \mathbb{P} \mathbb{Z}^d$, where

$$a_{1, n}(k) = \langle \varphi(\cdot), \phi(\gamma_n t - k) \rangle_{L^2(\mathbb{R}^d)} = \langle \varphi(t), \psi_\gamma \phi \rangle_{L^2(\mathbb{R}^d)}, k \in \mathbb{P} \mathbb{Z}^d \text{ and } n = 1, 2, \ldots, N.$$  

### 4.4 Some final comments

The theory obtained in this work relies on the use of an LCA group $G$. We have considered non-abelian groups which are semi-direct product of groups; the case treated here can be reduced to the abelian case by increasing the number of generators in the auxiliary space. Formally, the general non-abelian case can be handled in the same way but necessarily it will need other additional mathematical tools (see, for instance, Refs. [5, 24]).

As an example, consider the (positive) affine group $G_+ = \{(a, b) : a > 0, b \in \mathbb{R}\}$ with composition law $(a, b) \cdot (a', b') = (aa', a + b')$, and its unitary representation $(a, b) \mapsto U(a, b)$ on $L^2(\mathbb{R})$ given by

$$[U(a, b)f](t) = \frac{1}{\sqrt{a}} f\left(\frac{t - b}{a}\right), \quad t \in \mathbb{R}, \quad \text{where } f \in L^2(\mathbb{R}).$$

The non-abelian group $G_+$ is non-unimodular with left Haar measure $d\mu = \frac{da db}{a^2}$.

Let $\phi$ be a function in $L^2(\mathbb{R})$ such that $\{U(a, b)\phi\}_{(a, b) \in G_+}$ is a continuous frame for $L^2(\mathbb{R})$, and let $\{\psi_{m, n}\}_{m, n \in \mathbb{Z}}$ be an orthonormal basis of wavelets for $L^2(\mathbb{R})$ where we use the notation

$$\psi_{m, n}(t) = 2^{-m/2} \psi\left(\frac{t - n}{2^m}\right), m, n \in \mathbb{Z}.$$  

Now we sample any function $F(a, b) = \langle f, U(a, b)\phi \rangle_{L^2(\mathbb{R})}$, $(a, b) \in G_+$, where $f \in L^2(\mathbb{R})$, at the subspace $\Gamma := \{(2^m, n) : m, n \in \mathbb{Z}\}$ of $G_+$. The functions $F$ defined above form a RKHS contained in $L^2(\mathbb{R}^+ \times \mathbb{R}; \frac{da db}{a^2})$.

As in Section 3.1 the samples $\{F(2^m, n)\}_{m, n \in \mathbb{Z}}$ can be expressed as a discrete convolution in $\ell^2(\Gamma)$. A straightforward computation gives:

$$F(2^m, n) = \sum_{p, q} \langle \psi, U \left[\left(2^p, q\right)^{-1} (2^m, n)\right] \phi \rangle_{L^2(\mathbb{R})} \langle f, \psi_{p, q} \rangle_{L^2(\mathbb{R})} = (a * \Gamma b)(2^m, n), (2^m, n) \in \Gamma,$$

where $a(m, n) = \langle \psi, U (m, n) \phi \rangle_{L^2(\mathbb{R})}$ and $b(m, n) = \langle f, \psi_{m, n} \rangle_{L^2(\mathbb{R})}$, $(2^m, n) \in \Gamma$. The mathematical techniques used in Section 2.2 do not work for the non-abelian group $G_+$, and other mathematical techniques are necessary (see, for instance, Refs. [5, 24]).

Another related classical problem is the following: let $\mathcal{H}_k$ be a RKHS of continuous functions $f : \mathbb{R} \to \mathbb{C}$ contained in $L^2(\mathbb{R})$ with reproducing kernels $\{k_x\}_{x \in \mathbb{R}}$. Assume that there exists a Riesz basis for $\mathcal{H}_k$ having the form $\{\varphi_n(t - kN)\}_{k \in \mathbb{Z}, n = 1, 2, \ldots, N}$, and we want to recover any function $f \in \mathcal{H}_k$ from the sequences of its samples $\{f(kN + r)\}_{k \in \mathbb{Z}, r = 0, 1, \ldots, N - 1}$.
in a stable way. For the sample \( f(kN + r) = \langle f, kN + r \rangle_{L^2(\mathbb{R})} \), \( k \in \mathbb{Z} \) and \( r = 0, 1, \ldots, N - 1 \), we have the expression:

\[
\begin{align*}
f(kN + r) &= \langle f, kN + r \rangle_{L^2(\mathbb{R})} = \left\langle \sum_{n=1}^{N} \sum_{m \in \mathbb{Z}} x_n(mN) \varphi_n(\cdot - mN), kN + r \right\rangle_{L^2(\mathbb{R})} \\
&= \sum_{n=1}^{N} \sum_{m \in \mathbb{Z}} x_n(mN) \varphi_n(kN - mN) = \sum_{n=1}^{N} (a_{r,n} \ast_{\mathbb{Z}} x_n)(kN),
\end{align*}
\]

where \( a_{r,n}(mN) = \varphi_n(mN - r) \), \( m \in \mathbb{Z} \), for \( n = 1, 2, \ldots, N \). Under the conditions in Definition 1 for the \( N \times N \) matrix \( A = [a_{r,n}] \), there exist \( N \) sampling functions \( S_r \in \mathcal{H}_k \), \( r = 0, 1, \ldots, N - 1 \), such that the sampling formula

\[
f(t) = \sum_{r=0}^{N-1} \sum_{k \in \mathbb{Z}} f(kN + r) S_r(t - kN), \quad t \in \mathbb{R},
\]

holds in \( \mathcal{H}_k \). The sequence \( \{S_r(\cdot - kN)\}_{k \in \mathbb{Z}, r = 0, 1, \ldots, N-1} \) is a Riesz basis for \( \mathcal{H}_k \).

Finally to say that there is some affinity of the approach followed in this work with the topic of dynamical sampling (see, for instance, Ref. [3] and references therein). Indeed, from the correlation between a continuous frame \( \{U(t)\phi\}_{t \in \mathbb{G}} \) and an element \( f \) in a suitable Hilbert space \( \mathcal{H}_\Phi \) we obtain a function \( F(t) \) in \( L^2(\mathbb{G}) \). In the case studied here, assuming that the space \( \mathcal{H}_\Phi \) has a discrete unitary structure and under appropriate conditions, the function \( F \) can be recovered, in a stable way, from a finite number of data sequences. An important difference is that dynamical sampling approach relies on a semigroup structure rather than on a group one.

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