On the Metric Structure of Space-Time*

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Abstract

I present an analysis of the physical assumptions needed to obtain the metric structure of space-time. For this purpose I combine the axiomatic approach pioneered by Robb with ideas drawn from works on Weyl’s Raumproblem. The concept of a Lorentzian manifold is replaced by the weaker concept of an ‘event manifold’, defined in terms of volume element, causal structure and affine connection(s). Exploiting properties of its structure group, I show that distinguishing Lorentzian manifolds from other classes of event manifolds requires the key idea of General Relativity: namely that the manifold’s physical structure, rather than being fixed, is itself a variable.

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1 Introduction

In General Relativity, space-time is assumed to be a Lorentzian manifold. The metric field determines the volume element and the causal structure. Conversely, given a Lorentzian manifold, the causal structure and the volume element uniquely specify the metric field. But why does one start from a Lorentzian manifold in the first place? Why are light signals described by a quadratic equation $g_{ab}dx^a dx^b = 0$? Why is the light ‘cone’ not a pyramid standing on its top? One might argue that the quadratic term is the lowest order contribution to the Taylor expansion of some distance function; but it is unclear both how this distance function is motivated and, if it is, whether the quadratic term is necessarily non-trivial. Indeed there have been suggestions that rather than being Lorentzian, the space-time geometry might be Finslerian [1]. So why do Lorentzian manifolds nevertheless play a privileged role?

This is a particular case of Weyl’s space problem. It was first solved by Weyl [2], who also placed it in the context of General Relativity. A more elegant treatment was given by Cartan [3]. Since then their result, known as Weyl-Cartan theorem, has been reviewed by various authors [4]. A very different line of reasoning was initiated by Robb, who for the case of Special Relativity derived the Minkowskian metric from properties of the causal relations before and after [5]. More recent attempts to axiomatize General Relativity in Robb’s spirit are based on such notions as signals, light rays, freely falling particles, or clocks [6]; some even invoke quantum mechanics [7].

A synthesis of these two kinds of approaches is the aim of the present paper. Its purpose is not so much an actual derivation as it is an analysis: which physical assumptions are being tacitly made whenever one postulates the existence of a Lorentzian metric? Only after these assumptions are exhibited can one start to systematically relax them; thus, answers to the above question may be helpful for the study of more general space-time structures.

Primitive concepts are taken to be events, counting of events, causal relationships and the ability to compare measurements; the corresponding mathematical structures are a differentiable manifold, volume element, causal vectors and affine connection(s), leading to the notion of an ‘event manifold’. The key assumption, which I will call ‘deformability’, is that the event manifold’s physical structure is allowed to vary freely. The proof of the Weyl-Cartan theorem is then reviewed to establish the result that any deformable
event manifold must be Lorentzian.

2 Event Manifolds

I assume that space-time is a connected $n$-dimensional differentiable manifold $M$. At $x \in M$, local measurements (e.g., evaluating vector fields) are performed using a basis of the tangent space $T_x M$. In order to have a means to compare local measurements at different points, I require the manifold to be endowed with an affine connection. The connection is assumed to be torsion-free. Space-time is also endowed with a physical structure $\Phi$ reflecting, e.g., causal relationships between events. Details of this structure do not
matter at this point; all I assume is that (a) \( \Phi \) induces in each \( T_x M \) a local physical structure \( T_x \Phi \), and (b) this local structure can be measured with a basis of \( T_x M \). In order for an observer at \( x \in M \) to be able to determine the local structure not only at \( x \) itself but also elsewhere on the manifold, the connection must preserve the local physical structure.

A frame field provides a map from the local physical structure in \( T_x M \) to some structure in \( \mathbb{R}^n \) for every \( x \in M \). The images in \( \mathbb{R}^n \) will generally vary from point to point. However, since the connection is structure-preserving, one can use parallel transport to construct a special frame field called an \( n \)-bein such that the image of the local physical structure is the same everywhere. This enables one to fix a standard structure \( \eta \) in \( \mathbb{R}^n \) and then describe the local physical structure by the \( n \)-bein. However, there may be several \( n \)-beins associated with the same local physical structure. Such a redundancy in the mathematical description of a physical structure gives rise to a gauge theory. Its symmetry group is the Lie subgroup \( G \) of \( GL(n, \mathbb{R}) \) which leaves \( \eta \) invariant. Hence, the local physical structure on the manifold \( M \) corresponds to an entire subbundle of the frame bundle \( F(M) \), with (reduced) structure group \( G \). This is a \( G \)-structure on \( M \). It must admit a connection, called a \( G \)-connection. In contrast to Weyl, I do not assume the \( G \)-connection to be determined uniquely.

A discrete rather than continuous set of events can be characterized by the number of events and their mutual causal relationships. By analogy one expects that on a continuous space-time manifold, the physical structure \( \Phi \) should be a pair \((\mu, \leq)\) consisting of a volume element and a causal relation. A differentiable map \( \gamma : [0, 1] \to M \) with the property that \( \gamma(t_1) \leq \gamma(t_2) \) iff \( t_1 \leq t_2 \) is called a causal curve. \( \Phi \) then induces in each tangent space \( T_x M \) a local physical structure \( T_x \Phi \), consisting of (a) a volume element and (b) the set of vectors, called causal vectors, which are tangent to a causal curve through \( x \). Correspondingly, the standard structure \( \eta \) in \( \mathbb{R}^n \) consists of (a) a volume element and (b) the image \( j^+ \) of the set of causal vectors.

About the sets \( j^+ \) and \( j^- := -j^+ \) I make the following four assumptions. (i) Except for its direction, the parametrization of a causal curve is irrelevant; hence \( j^+ \) is scale invariant. (ii) But the well-defined direction of causal curves distinguishes locally between past and future; therefore \( j^+ \cap j^- = \{0\} \) (‘antisymmetry’). (iii) As there is no torsion, two causal vectors at \( x \in M \) can be used to construct an infinitesimal closed geodesic parallelogram; and as the connection is structure-preserving, all its four sides must be parts of causal curves. Requiring transitivity of the causal relation, its (geodesic)
diagonal, too, is part of a causal curve; thus \( j^+ \) is convex. (iv) Let \( \theta \) be the coframe field dual to the \( n \)-bein. ‘No torsion’ implies that for any vector fields \( X, Y \),
\[
\nabla_X \theta(Y) - \nabla_Y \theta(X) = \theta([X,Y]) \tag{1}
\]
Defining \( S := \text{span}\{j^+\} \), the requirement that the connection be structure-preserving implies \( \nabla_U S \subseteq S \) for any vector field \( U \). Choosing \( X,Y \in \theta^{-1}(S) \) in (1) one thus obtains
\[
[\theta^{-1}(S),\theta^{-1}(S)] \subseteq \theta^{-1}(S) \tag{2}
\]
By Frobenius’ theorem, the causal curves mesh to form a foliation of \( M \), each leaf having dimension at most \( \text{dim} \ S \). Different leaves would represent ‘separated worlds’, a situation I want to exclude. Hence \( \text{dim} \ S = n \), and \( j^+ \) spans the entire \( \mathbb{R}^n \).

The above properties of the local physical structure are sufficient to determine the structure group \( G \) for \( n = 2 \). In this case, the boundary \( \partial j^+ \) consists of two straight rays. Choosing basis vectors on these rays, any symmetry transformation \( \Lambda \) has the form \( \Lambda = \text{diag} (\lambda, 1/\lambda) \) with \( \lambda > 0 \). The group of these transformations is isomorphic to \( SO(1,1) \). Although for \( n \geq 3 \) the structure group \( G \) is not yet determined, one may already infer some properties: as a structure group, \( G \) is a Lie group; it preserves the volume and is therefore a subgroup of \( SL(n,\mathbb{R}) \); it preserves \( j^+ \) and thus may not contain reflections: \( -1 \notin G \); finally,
\[
\text{dim} \ G \leq n(n-1)/2 \tag{3}
\]

**Proof.** I prove by induction that any invertible linear map \( g : \mathbb{R}^n \to \mathbb{R}^n \) preserving \( \eta \) is determined by at most \( n(n-1)/2 \) parameters. (i) The proposition holds for \( n = 2 \). (ii) Assume it is proven for \( n \). In \( n + 1 \) dimensions, \( g \) is specified by the images of \( n + 1 \) linearly independent vectors \( v_1, \ldots, v_{n+1} \). These vectors can be chosen such that \( v_{n+1} \) is the unique intersection \( v_{n+1} = \partial j^+ \cap (v_1 + \partial j^-) \cap \ldots \cap (v_n + \partial j^-) \). Since \( g \) preserves \( \partial j^\pm \), the image of \( v_{n+1} \) is uniquely determined by the images of \( v_1, \ldots, v_n \). It is therefore sufficient to consider the restriction of \( g \) to \( S := \text{span}\{v_1, \ldots, v_n\} \).

And since (by a suitable choice of the \( v_i \)) \( \partial j^+_S := S \cap \partial j^+ \) can be made to have all the properties of a ‘light cone’ in \( S \), it is sufficient to determine the images only of vectors that lie on \( \partial j^+_S \). To specify \( g(\partial j^+_S) = \partial j^+ \cap g(S) \) requires at most \( n \) parameters; to specify how vectors transform within \( \partial j^+_S \) requires, by assumption, at most \( n(n-1)/2 \); so altogether at most \( (n+1)n/2 \). Q.E.D.
3 Deformability

So far my considerations have been very general, and the symmetry group $G$ is by no means uniquely determined. Only now the key idea of General Relativity comes into play: rather than being fixed as in Newtonian theory, the local physical structure on the space-time manifold is itself a *variable*; it depends on the distribution of matter in the universe (and on boundary conditions). Whenever the local physical structure is thus allowed to vary freely I call the event manifold ‘deformable’. However, the ‘nature’ of the physical structure — embodied by $\eta$ — and hence the symmetry group $G$ must remain unchanged. Provided the local physical structure reflects the distribution of matter, deformability amounts to the requirement that arbitrary matter distributions be allowed.

Mathematically, varying the $G$-structure corresponds to varying the $n$-bein; or, equivalently, its dual $\theta$. By assumption, any choice of the $G$-structure, and hence of $\theta$, must admit a torsion-free $G$-connection. If expressed with respect to $\theta$, the connection 1-form $\omega$ takes values in $\text{Lie}(G)$ and thus has $n \cdot \dim G$ degrees of freedom. The requirement that it be torsion-free reads

$$d\theta + \omega \wedge \theta = 0. \quad (4)$$

Since $\omega \wedge \theta$ is an $\mathbb{R}^n$-valued 2-form, this requirement imposes $n \cdot n(n-1)/2$ constraints on $\omega$ which for arbitrary $\theta$ can only be satisfied if $n(n-1)/2 \leq \dim G$. Together with (3) one thus obtains

$$\dim G = n(n-1)/2. \quad (5)$$

This result implies that the connection on $M$ is indeed unique, which Weyl had assumed without proof.

Let us assume that $G$ leaves a proper subspace $S \subset \mathbb{R}^n$ invariant. This again leads to (2) which for $\dim S \geq 2$ is an undue restriction on $\theta$. Now suppose $S$ is spanned by a vector $r \in \mathbb{R}^n$. The invariance of $S$ implies that there is a 1-form $\alpha$ such that $\omega r = r \otimes \alpha$. Defining $X := \theta^{-1}(r)$ and using (4) this yields

$$L_X \theta + \omega(X)\theta = r \otimes \alpha. \quad (6)$$

Here $L$ denotes the Lie derivative. Defining the maps

$$\Lambda : \mathbb{R}^n \to \mathbb{R}^n, \quad \Lambda u = -(L_X \theta)(\theta^{-1}(u)) \quad (7)$$

$$\xi : \mathbb{R}^n \to \mathbb{R}, \quad \xi(u) = \alpha(\theta^{-1}(u)) \quad (8)$$
one obtains
\[ \omega(X) - r \otimes \xi = \Lambda, \]  
(9)
a relation among endomorphisms of \( R^n \). Since \( \theta \) may be varied freely, \( \Lambda \) can be chosen freely with the sole constraint that \( \Lambda r = 0 \). Thus, restricting (9) to an \((n-1)\)-dimensional space \( S \) complement to \( S \) yields an equation which in general has a solution only if \( \omega(X)|_S \) and \( \xi|_S \) together have at least as many degrees of freedom as \( \Lambda|_S \). Hence \( \dim G + (n-1) \geq n(n-1) \) and therefore \( \dim G \geq (n-1)^2 \), a condition which is compatible with \( \dim G = n(n-1)/2 \) only if \( n = 2 \). Thus for \( n \geq 3 \), \( G \) acts irreducibly on \( R^n \).

I have now established several important properties of the symmetry group \( G \) which are summarized in figure 1. It can be shown that these properties uniquely determine the Lorentz group, which in turn implies the existence of an invariant metric with signature \( n - 2 \).

4 Conclusion

Any deformable event manifold is Lorentzian.

This result has a nice physical interpretation if one assumes a one-to-one correspondence between the manifold’s local physical structure and the distribution of matter: out of all possible event manifolds, only Lorentzian manifolds admit arbitrary matter distributions; any non-metric structure imposes undue restrictions.

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