H-Infinity Optimal Decentralized Matching Model Is Not Always Rational

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Abstract

We construct structured H-Infinity optimal model matching problems with rational coefficients, in which the optimal solution is not rational, in the sense that the cost does not achieve its maximal lower bound on the set of rational matching models, but the same infimum can be reached by using a continuous non-rational matching model.

Notation and terminology

We use $\mathbb{C}$ to denote the set of all complex numbers, and $\mathbb{C}^{m \times n}$ to denote the set of all $m$-by-$k$ complex matrices. For $M \in \mathbb{C}^{m \times n}$, the element in the $i$-th row and the $r$-th column of $M$ is expressed by $[M]_{i,r}$. For a positive integer $n$, $I_n$ is the $n$-by-$n$ identity matrix, and $J_n$ denotes the $n$-by-$n$ ”order reversal” matrix (i.e. $[J_n]_{i,r} = 1$ when $i + r = n + 1$, and $[J_n]_{i,r} = 0$ otherwise). We use $\mathbb{D} = \{w \in \mathbb{C} : |w| < 1\}$, $\mathbb{T} = \{w \in \mathbb{C} : |w| = 1\}$, $\mathbb{D}_+ = \{w \in \mathbb{C} : |w| > 1\}$ to denote the open unit disc, its boundary, and the complement of its closure. For all $w \in \mathbb{C}$, $\bar{w}$ is the complex conjugate of $w$, and, for $w = re^{j\theta}$, where $-\pi < \theta \leq \pi$, and $\beta > 0$, the power $w^\beta$ is defined by $w^\beta = r^\beta e^{j\beta\theta}$. As a shortcut, we use $\sqrt{w}$ for $w^{1/2}$.

When $\Omega$ is a subset of $\mathbb{C}$, a function $G : \Omega \rightarrow \mathbb{C}$ is said to be real symmetric when $\bar{w} \in \Omega$ and $G(\bar{w}) = G(w)$ whenever $w \in \Omega$. $\mathcal{H}^\infty$ denotes the set of real symmetric analytic functions $G : \mathbb{D} \rightarrow \mathbb{C}$ such that $\sup_{w \in \mathbb{D}} |G(w)| < \infty$, while $\mathcal{A}$ is the subset of functions $G \in \mathcal{H}^\infty$ which can be extended continuously to $\mathbb{D} \cup \mathbb{T}$, and $\mathcal{RA}$ is the subset of rational functions $G \in \mathcal{A}$. Furthermore, $\mathcal{H}^\infty_{m \times k}$, $\mathcal{A}_{m \times k}$, and $\mathcal{RA}_{m \times k}$ denote the sets of $m$-by-$k$ complex matrix-valued functions $G : \mathbb{D} \rightarrow \mathbb{C}^{m \times n}$ for which all components

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$G_{i,r} : z \to [G(z)]_{i,r}$ belong to the classes $H^\infty$, A, and RA, respectively. The elements of $H^\infty_{m \times k}$ will be referred to as stable transfer matrices (or stable transfer functions in the case $m = k = 1$), though it is more common in control systems literature to call so the functions $F : D \to \mathbb{C}$ defined by $F(z) = G(1/z)$ for some $G \in H^\infty_{m \times k}$. Naturally, transfer matrices from $A_{m \times k}$ will be viewed as continuous functions on $D \cup \mathbb{T}$. For $G \in H^\infty_{m \times k}$, the L-Infinity norm $\|G\|_\infty$ is defined as the minimal upper bound of the largest singular number $\sigma_{\text{max}}(G(w))$ of $G(w)$ over $w \in D$. Every function $G \in RA_{m \times k}$ can be represented (in many ways) in the form $G(w) = D + wC(I_n - wA)^{-1}B$, where $A, B, C, D$ are real matrices of dimensions $n$-by-$n$, $n$-by-$k$, $m$-by-$n$, and $m$-by-$k$ respectively (when $n = 0$, the representation is interpreted as $G(w) \equiv D$). The minimal possible value of $n$ in such representation will be referred to as the order of $G$, and the set of all elements $G \in RA_{m \times k}$ of order not larger than $n$ will be denoted by $RA^n_{m \times k}$.

1 Introduction

Given rational stable transfer matrices $L_0 \in RA_{p \times d}$, $L_1 \in RA_{p \times m}$, $L_2 \in RA_{k \times d}$, the classical H-Infinity model matching problem can be expressed in the form

$$\|L_0 + L_1 Q L_2\|_\infty \to \min_{Q \in H^\infty_{m \times k}}.$$

(1)

In other words, it calls for finding a stable transfer matrix $Q \in H^\infty_{m \times k}$ which minimizes model matching error $\|L_0 + L_1 Q L_2\|_\infty$. In this paper, we only consider the case when the well-posedness assumption

$$L_1(z)'L_1(z) > 0, \quad L_2(z)L_2(z)' > 0 \quad \forall \ z \in \mathbb{T},$$

(2)

is satisfied, thus guaranteeing existence of an optimal $Q \in H^\infty_{m \times k}$.

The classical H-Infinity model matching problem is well studied, as it appears naturally (after applying the so-called ”Youla-”, or ”Q-”, parameterization) as an intermediate step in designing a stabilizing linear time invariant feedback for a given stabilizable finite order linear time invariant plant (with $d$ noise inputs, $m$ actuator inputs, $p$ cost outputs, and $k$ sensor outputs), with an objective of minimizing the L2 induced norm in the closed loop map from the noise inputs to the cost outputs. In particular, when

$$L = \begin{bmatrix} L_0 & L_1 \\ L_2 & 0 \end{bmatrix} \in RA^n_{(p+k) \times (d+m)}$$

has order $n$, restricting $Q$ to be rational of order $n$ does not reduce the best achievable performance, in the sense that

$$\min_{Q \in RA^n_{m \times k}} \|L_0 + L_1 Q L_2\|_\infty = \min_{Q \in H^\infty_{m \times k}} \|L_0 + L_1 Q L_2\|_\infty.$$
In the last decade, breakthrough advances in understanding Q-parameterization (see, for example, [2]) led naturally to a structured version of the model matching problem (1), in which some entries of \( Q \) are constrained to be identically zero. Such formulations are obtained, for plants of a special structure, when there is a need to optimize a decentralized stabilizing linear time invariant feedback.

One basic question associated with this development is whether a rational optimal \( Q \) is guaranteed to exist (subject to assumption (2)) in the problem of minimizing the cost \( \| L_0 + L_1 Q L_2 \|_\infty \) when \( Q \) is restricted to the set of all diagonal stable transfer matrices of appropriate dimension. This paper aims to answer the question (posed to the author by S. Lall) negatively.

Specifically, let \( D \) denote the set of all diagonal stable transfer matrices. We produce triplets \((L_0, L_1, L_2)\) of rational stable transfer matrices \( L_i \in \mathbb{R}A_{2 \times 2} \) satisfying condition (2), such that

\[
\inf_{Q \in \mathbb{R}A_{2 \times 2} \cap D} \| L_0 + L_1 Q L_2 \|_\infty = \min_{Q \in \mathbb{H}_{\infty}^{2 \times 2} \cap D} \| L_0 + L_1 Q L_2 \|_\infty
\]

is achieved at a unique \( Q \in \mathbb{A}_{2 \times 2} \cap D \) which is not a rational function (in fact, the optimal \( Q \) can be computed explicitly). The derivation relies on the conformal mapping technique by Allen Tannenbaum [3].

### 2 Main Results

We will use the standard expression for the conformal map of the open unit disc \( \mathbb{D} \) to the open ”lens” region

\[
\Omega_\gamma = \{ s \in \mathbb{C} : |1 - s| < \gamma, |1 + s| < \gamma \} \quad (1 < \gamma < \infty).
\]

**Lemma 1** For every \( \alpha \in (0, \pi/2) \) and \( \gamma = \frac{1}{\cos \alpha} \in (1, \infty) \) the function \( F_\gamma : \mathbb{D} \to \mathbb{C} \) defined by

\[
F_\gamma(w) = j \tan(\alpha) \cdot \frac{1 - \left( \frac{1+jw}{1-jw} \right)^{2\alpha/\pi}}{1 + \left( \frac{1+jw}{1-jw} \right)^{2\alpha/\pi}} \quad (w \in \mathbb{D})
\]

belongs to class \( \mathbb{A} \), establishes a bijection between \( \mathbb{D} \) and \( \Omega_\gamma \), and satisfies the condition \( \dot{F}_\gamma(0) = 2\alpha \tan(\alpha)/\pi \).

The following statement provides a simple example of a structured H-Infinity optimal model matching problem with \( d = p = m = k = 2, L_2 = I_2 \), such that the optimal \( Q \in \mathbb{H}_{\infty}^{2 \times 2} \cap D \) is unique, belongs to the class \( \mathbb{A}_{\infty}^{2 \times 2} \), but is not a rational function.
Theorem 1  Equalities

\[
\inf_{Q \in RA_{2 \times 2} \cap D} \|L_0 + L_1 Q\|_\infty = \min_{Q \in H_{2 \times 2}^\infty \cap D} \|L_0 + L_1 Q\|_\infty = \sqrt{2}
\]

hold for

\[
L_0(w) = I_2 + 0.5wJ_2 = \begin{bmatrix} 1 & 0.5w \\ 0.5w & 1 \end{bmatrix}, \quad L_1(w) = w^2J_2 = \begin{bmatrix} 0 & w^2 \\ w^2 & 0 \end{bmatrix}.
\]

Moreover, the only \(Q^* \in H_{2 \times 2}^\infty \cap D\) such that \(\|L_0 + L_1 Q^*\|_\infty = \sqrt{2}\) is given by \(Q^*(w) = S^*(w)I_2\), where \(S^* \in A\) is defined by \(0.5w + w^2S^*(w) = F_{\sqrt{2}}(w)\), and \(F_{\gamma} \in A\) is defined in Lemma 7.

A proof of Theorem 1 is given in the Appendix section below.

The optimization task described in Theorem 1 is actually a special case of a slightly more general class of structured model matching problems in which the optimal diagonal \(Q\) is guaranteed to be continuous but not rational.

Theorem 2  Let \(a, b \in RA\) be such that \(b(w) \neq 0\) for all \(w \in \mathbb{T}\), and \(a + bq\) is not constant for every \(q \in RA\). Then, for \(L_0(w) = I_2 + a(w)J_2, L_1(w) = b(w)J_2\), the only \(Q^* \in H_{2 \times 2}^\infty \cap D\) satisfying

\[
\|L_0 + L_1 Q^*\|_\infty = \inf_{Q \in RA_{2 \times 2} \cap D} \|L_0 + L_1 Q\|_\infty = \min_{Q \in H_{2 \times 2}^\infty \cap D} \|L_0 + L_1 Q\|_\infty = \gamma > 1
\]

is given by \(Q^*(w) = S^*(w)I_2\), where \(S^* \in A\) is defined by \(a(w) + b(w)S^*(w) = F_{\gamma}(p(w))\) for some non-constant \(p \in RA\) satisfying \(|p(w)| = 1\) for all \(w \in \mathbb{T}\), and \(F_{\gamma} \in A\) is defined in Lemma 7.

Theorem 1 corresponds to a special case of Theorem 2 with

\[
a(w) = 0.5w, \quad b(w) = w^2, \quad \gamma = \sqrt{2}, \quad p(w) = w.
\]

A sketch of a proof of Theorem 1 is given in the Appendix section below.

3 Appendix

The appendix contains proof of the main results (Theorems 1 and 2), as well as that of the (well known) statement of Lemma 1.
3.1 Proof of Lemma

For $\theta \in (0, \pi)$ let
\[
\mathbb{C}_\theta = \{ re^{jt} : r > 0, |t| < \theta \}, \quad \hat{\mathbb{C}}_\theta = \{ re^{jt} : r \geq 0, |t| \leq \theta \} \cup \{ \infty \}
\]
denote the "open angle $2\theta$ cone" in $\mathbb{C}$ and its closure in $\mathbb{C} \cup \{ \infty \}$. By definition, $F_\gamma = U_\alpha \circ R_\alpha \circ V$ is a composition of one power function $R_\alpha : \mathbb{C}_{\pi/2} \to \mathbb{C}_\alpha$ and two Möbius transformations $V : \mathbb{D} \to \mathbb{C}_{\pi/2}, U_\alpha : \mathbb{C}_\alpha \to \Omega_\gamma$ defined by
\[
V(w) = \frac{1+jw}{1-jw}, \quad R_\alpha(s) = s^{2\alpha/\pi}, \quad U_\alpha(y) = j \tan(\alpha) \cdot \frac{1-y}{1+y}.
\]
Since each function $V, U_\alpha, R_\alpha$ is a holomorphic bijection, $F_\gamma$ is a holomorphic bijection, too. Moreover, since $V, U_\alpha, R_\alpha$ have continuous extensions $\hat{V} : \mathbb{D} \cup \mathbb{T} \to \hat{\mathbb{C}}_{\pi/2}, \hat{U}_\alpha : \hat{\mathbb{C}}_\alpha \to \hat{\Omega}_\gamma, \hat{R}_\alpha : \hat{\mathbb{C}}_{\pi/2} \to \hat{\mathbb{C}}_\alpha$ (where $\hat{\Omega}_\gamma$ is the closure of $\Omega_\gamma$), $F_\gamma$ has a continuous extension $\hat{F}_\gamma : \mathbb{D} \cup \mathbb{T} \to \hat{\Omega}_\gamma$. In addition, while the maps $V$ and $U_\alpha$ do not have real symmetry, they satisfy conditions
\[
V(\bar{w}) = (V(w))^{-1}, \quad R_\alpha(1/\bar{s}) = (R_\alpha(s))^{-1}, \quad U_\alpha(1/\bar{y}) = U_\alpha(y),
\]
which proves that the total composition $F_\gamma = U_\alpha \circ R_\alpha \circ V$ is real symmetric. Finally, the expression for $\hat{F}_\gamma(0)$ follows from the observation that
\[
\hat{V}(0) = 2j, \quad V(0) = 1, \quad \hat{R}_\alpha(1) = 2\alpha/\pi, \quad R_\alpha(1) = 1, \quad \hat{U}_\alpha(1) = -0.5j \tan(\alpha).
\]

3.2 Proof of Theorem

Since
\[
L_0(w) + L_1(w)Q(w) = \begin{bmatrix} 1 & 0.5w + w^2S_2(w) \\ 0.5w + w^2S_1(w) & 1 \end{bmatrix}, \quad \text{for } Q = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix},
\]
the set of all transfer matrices $L_0 + L_1Q$ with $Q \in \mathbb{H}_{2 \times 2}^\infty \cap \mathcal{D}$ can be represented in the form
\[
\{ L_0 + L_1Q : Q \in \mathbb{H}_{2 \times 2}^\infty \cap \mathcal{D} \} = \{ H[G_1, G_2] : G_1, G_2 \in \mathbb{X} \},
\]
where
\[
\mathbb{X} = \left\{ G \in \mathbb{H}^\infty : G(0) = 0, \dot{G}(0) = 0.5 \right\}, \quad H[G_1, G_2] = \begin{bmatrix} 1 & G_2 \\ G_1 & 1 \end{bmatrix}.
\]
Theorem claims that the infimum of $\|H[G_1, G_2]\|_\infty$ over $G_1, G_2 \in \mathbb{X} \cap \mathbb{R} \mathbb{A}$ equals $\sqrt{2}$, same as the minimum of $\|H[G_1, G_2]\|_\infty$ over $G_1, G_2 \in \mathbb{X}$, which in turn is achieved at a unique pair $G_1 = G_2 = F_{\sqrt{2}}$. The proof is presented in several steps.
Step 1. Note that $X$ is an affine subspace in $H^\infty$, i.e. $G = 0.5(G_1 + G_2) \in X$ whenever $G_1, G_2 \in X$. Moreover, since

$$\|H[G_2, G_1]\|_\infty = \|H[G_1, G_2]\|_\infty, \quad H[G, G] = 0.5(H[G_1, G_2] + H[G_2, G_1]),$$

convexity of the H-Infinity norm function implies $\|H[G_1, G_2]\|_\infty \leq \|H[G, G]\|_\infty$. In other words, $\|H[G_1, G_2]\|_\infty \leq \gamma$ for $G_1, G_2 \in X$ implies $\|H[G, G]\|_\infty \leq \gamma$ for $G = 0.5(G_1 + G_2) \in X$.

Step 2. Since

$$\sigma_{\text{max}} \left( \begin{bmatrix} 1 & g \\ g & 1 \end{bmatrix} \right) = \max \{|1-g|, |1+g|\} \quad \forall \ g \in \mathbb{C},$$

the inequality $\|H[G, G]\|_\infty \leq \gamma$, where $\gamma > 1$ and $G \in H^\infty$ is not constant (note that all $G \in X$ are not constant) holds if and only if $G(w) \in \Omega_\gamma$ for all $w \in \mathbb{D}$.

Step 3. Whenever $G \in X$ is such that $G(w) \in \Omega_\gamma$ for all $w \in \mathbb{D}$, the composition $p = F^{-1}_\gamma \circ G$ satisfies conditions

$$p \in H^\infty, \quad \|p\|_\infty \leq 1, \quad p(0) = 0, \quad \dot{p}(0) = \frac{1}{2F_\gamma(0)} = \frac{\pi \cos \alpha}{4\sin \alpha} \quad (0 < \alpha < \pi/2, \ \cos \alpha = 1/\gamma) \tag{5}$$

whenever $\|H[G, G]\|_\infty \leq \gamma$.

Step 4. Since the Cauchy integral identity yields

$$\dot{p}(0) = \frac{1}{2r\pi} \int_{-\pi}^{\pi} e^{-j \gamma} p(re^{j \theta}) \, d\theta$$

for every $p \in H^\infty$ and $r \in (0, 1)$, it follows that $|\dot{p}(0)| \leq 1$ whenever $p$ satisfies conditions (4), with equality $\dot{p}(0) = 1$ possible only when $p(w) \equiv w$. Hence $\alpha \leq \pi/4$ in (5), i.e. $\gamma \geq \sqrt{2}$ whenever $\|H(G_1, G_2)\|_\infty \leq \gamma$ for $G_1, G_2 \in X$, with equality $\|H(G_1, G_2)\|_\infty = \sqrt{2}$ possible only when $0.5(G_1 + G_2) = F_{\sqrt{2}}$. In particular, $F_{\sqrt{2}} \in X$, and $\|H(F_{\sqrt{2}}, F_{\sqrt{2}})\|_\infty = \sqrt{2}$.

Step 5. As established at step 3, the functional $\|H[G_1, G_2]\|_\infty$ achieves its minimal value over $G_1, G_2 \in X$ when $G_1 = G_2 = F_{\sqrt{2}}$. To show that this is the only argument of minimum, let $G_1, G_2 \in X$ be any pair satisfying $\|H[G_1, G_2]\|_\infty = \sqrt{2}$. Then $G =$
0.5(G₁ + G₂) = F_{\sqrt{2}}. Let T₁ = H[G₁, G₂], T₂ = H[G₂, G₁], D = 0.5(G₁ − G₂). Applying matrix identity

\[ M'_a M_a + M'_d M_d = \frac{M'_1 M_1 + M'_2 M_2}{2}, \quad \text{where} \quad M_a = \frac{M_1 + M_2}{2}, \quad M_d = \frac{M_1 - M_2}{2} \]

to \( M_1 = T₁(w) \) and \( M_2 = T₂(w) \), with \( w \in \mathbb{D} \), in which case the diagonal elements of \( M'_1 M_1 \) are not larger than 2, the diagonal elements of \( M'_a M_a \) equal 1 + \(|w|^2\), and the diagonal elements of \( M'_d M_d \) equal \(|D(w)|^2\), we conclude that \(|D(w)|^2 \leq 1 - |w|^2\) which, due to the maximum modulus principle, implies \( D(w) \equiv 0 \), i.e. \( G₁ = G₂ = F_{\sqrt{2}} \).

**Step 6.** Finally, to show that the maximal lower bound of \( \|H[G₁, G₂]\|_∞ \) over \( G₁, G₂ \in RA \cap X \) equals \( \sqrt{2} \), note that \( F_{\sqrt{2}} \) (as any other function from class \( A \)) can be approximated arbitrarily well by polynomials, i.e. for every \( \delta > 0 \) there exists a polynomial \( R_δ \in RA \) such that \( \|R_δ - F_{\sqrt{2}}\|_∞ < \delta \). Then \( R_δ(0) \to 0 \) and \( \dot{R}_δ(0) \to 0.5 \) as \( \delta \to 0 \), hence \( \|F_{\sqrt{2}} - G_δ\|_∞ \to 0 \) and \( \|H(G_δ,G_δ)\|_∞ \to \sqrt{2} \) for

\[ G_δ = \frac{R_δ - R_δ(0)}{2\dot{R}_δ(0)} \in X \quad (0 < \delta < 0.5, \ \delta \to 0). \]

### 3.3 Proof of Theorem 2 (a sketch)

We follow the lines of the proof of Theorem 1 with some minor modifications. We redefine \( X = \{a + bS : S \in H^∞\} \). Theorem 2 claims that the infimum \( \gamma \) of \( \|H[G₁, G₂]\|_∞ \) over \( G₁, G₂ \in X \cap RA \) is always greater than 1, and equals the minimum of \( \|H[G₁, G₂]\|_∞ \) over \( G₁, G₂ \in X \), which in turn is achieved at a unique pair \( G₁ = G₂ = F_γ \circ p \), where \( p \in RA \) is not constant, and satisfies \( |p(z)| = 1 \) for all \( z \in T \).

The reduction of the task of minimizing \( \|H[G₁, G₂]\|_∞ \) over \( G₁, G₂ \in X \) to the minimization task

\[ r \to \min, \quad \text{subject to} \quad G \in X, \quad G(w) \in \Omega_r, \quad \forall \ w \in \mathbb{D} \]  \hspace{1cm} (6)

is done the same way as in the proof of Theorem 1. Since the sets \( \{G \in H^∞ : \|G\|_∞ \leq R\} \) are compact in the topology of uniform convergence on all compact subsets of \( \mathbb{D} \), and since the function \( G \to \|G\|_∞ \) is lower semi-continuous in this topology, there exists an optimal \( G = G_* \in X \). Then, for \( \gamma = \min r \), the function \( p = F_γ⁻¹ \circ G_* \in H^∞ \) satisfies \( |p(w)| < 1 \) for all \( w \in \mathbb{D} \), i.e. \( \|p\|_∞ \leq 1 \). Moreover, there exist no \( S \in H^∞ \) such that \( \|p + bS\|_∞ \leq 1 \), because otherwise \( G = F_γ \circ (p + bS) \in X \) would satisfy the constraints in (6) for some \( r < \gamma \). According to the standard theory of frequency-domain H-Infinity optimization (see, for example, [1]), \( p \) is a rational function of order smaller than the number of roots of \( b \) in \( \mathbb{D} \) (counting multiplicity), which satisfies the condition \( |p(z)| = 1 \) for all \( z \in T \), and is unique in the sense that \( \|p + bS\|_∞ > 1 \) for all non-zero \( S \in H^∞ \). Since every other minimizer \( G \) in (6) will satisfy \( S = (F_γ⁻¹ \circ G - p)/b \in H^∞ \), this confirms \( G = G_* = F_γ⁻¹ \circ p \) as the unique minimizer in (6).
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