Weyl’s Gauge Invariance:
Conformal Geometry, Spinors,
Supersymmetry, and Interactions

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Abstract
We extend our program, of coupling theories to scale in order to make their Weyl invariance manifest, to include interacting theories, fermions and supersymmetric theories. The results produce mass terms coinciding with the standard ones for universes that are Einstein, but are novel in general backgrounds. They are generalizations of the gravitational couplings of a conformally improved scalar to fields with general scaling and tensor properties. The couplings we find are more general than just trivial ones following from the conformal compensating mechanisms. In particular, in the setting where a scale gauge field (or dilaton) is included, masses correspond to Weyl weights of fields organized in “tractor” multiplets. Breitenlohner–Freedman bounds follow directly from reality of these weights. Moreover, massive, massless and partially massless theories are handled in a uniform framework. Also, \textit{bona fide} Weyl invariant theories (invariant without coupling to scale) can be directly derived in this approach. The results are based on the tractor calculus approach to conformal geometry, in particular we show how to handle fermi fields, supersymmetry and Killing spinors using tractor techniques. Another useful consequence of the construction is that it automatically produces the (anti) de Sitter theories obtained by log-radial reduction of Minkowski theories in one higher dimension. Theories presented in detail include interacting scalars, spinors, Rarita–Schwinger fields, and the interacting Wess–Zumino model.
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1 Introduction

Weyl invariance is often regarded as a symmetry principle obeyed by particular classes of theories. In a recent series of papers \[1, 2\], we argued that, in fact, it should be viewed in the same manner as general coordinate invariance: \textit{all} theories should respect Weyl invariance. As a consequence, physical quantities are then independent of any local choice of units system. Just as one couples to a metric to ensure diffeomorphism invariance, when necessary, theories should be coupled to scale (a not necessarily dynamical dilaton field) to guarantee Weyl invariance. Moreover, analogous to the importance of finding all possible diffeomorphism invariant theories, a crucial problem is to find all possible Weyl and diffeomorphism invariant theories. The solution to this problem not only gives a simple way to find those peculiar theories that are invariant without coupling to scale, but also leads to deep insights into many aspects of those that are not.

Ultimately, we believe that this program will lead to a deeper understanding of quantum effects, in particular the renormalization group and AdS/CFT
correspondence, especially from a holographic renormalization group viewpoint \cite{3}. In the current paper, however, we restrict ourselves to a classical analysis and extend our previous results to (i) interactions, (ii) fermions and (iii) supersymmetry. The Paper is structured as follows: in the next Section, we discuss how to couple scalar fields to scale as well as gravity. Following that, we review the tractor technology required to handle spinors and, in Section \textbf{4} use it to formulate massive and massless spin 1/2 particles in a single, Weyl invariant framework. Extending these ideas, in Section \textbf{5} we couple the spin 3/2 Rarita–Schwinger equation to scale. We then combine bose and fermi models in Section \textbf{6} and describe supersymmetric theories, concentrating on the Wess–Zumino model. Section \textbf{7} describes an interacting Wess–Zumino model coupled to scale. Since tractors are intimately related to the projective approach to conformal geometry, they naturally produce the log-radial reduction procedure \cite{4,5} used to obtain constant curvature theories from flat space ones in a dimension higher. The log-radial reduction for spinors is described in Appendix \textbf{A}.

\section{Coupling to Scale}

\textit{My work always tried to unite the Truth with the Beautiful, but when I had to choose one or the other, I usually chose the Beautiful.}

Hermann Weyl

Classically, all physical theories are required to have a rigid scaling symmetry reflecting the freedom to globally choose any unit system for a given physical quantity. For example, if $x$ is a length, then $x + x^2$ cannot be a sensible answer to a physical question. The scaling properties of physical quantities are encoded by assigning them weights. Then, in an action principle, scale invariance requires

$$S[\Phi_i; \lambda_\alpha] = S[\Omega^{\mu} \Phi_i; \Omega^{\mu} \lambda_\alpha],$$

where $\Omega$ is a rigid parameter, $\{\Phi_i\}$ are the fields of the theory and $\{\lambda_\alpha\}$ are any dimensionful couplings. In fact, only a single dimensionful coupling $\kappa$ (Newton’s constant) is really needed, since all the others become dimensionless upon multiplication by an appropriate power of $\kappa$. 

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The equation (1) simply says that physics is independent of the choice of unit system. Just as physical systems are required to be independent of local changes of coordinates (“diffeomorphisms”), a fundamental physical principle, dating back to Weyl [6], is that physics be independent of local choices of unit systems (“Weyl transformations”) as well. Generically, diffeomorphism invariance is only achieved by introducing a gauge field – the metric tensor $g_{\mu\nu}$. (An exception to the rule, is Chern–Simons theory, for example.)

The same is true for Weyl invariance: generically a gauge field $\sigma$ is necessary although again there are exceptions, notably the conformally improved wave equation, the massless Dirac equation and Maxwell’s equations in four dimensions. The gauge field $\sigma$ is often called a Weyl compensator or dilaton and was employed in this way by Deser and Zumino [7, 8]. We prefer to call $\sigma$ the scale, in concordance with the mathematics literature and also because of its geometric interpretation, which is simply as a local Newton’s “constant” encoding how the choice of unit system varies over space and time.

To efficiently study Weyl transformations, a conformal calculus is needed along the lines of tensor calculus for coordinate transformations. In [1] we explained how the tractor calculus developed by conformal geometers [9, 10] provides exactly such a calculus. We limit ourselves to a few key ideas here and refer the reader to our papers [1, 2] as well as the mathematical literature [10, 11, 12, 13] for further details. Under Weyl transformations, the metric and scale transform according to

$$
g_{\mu\nu} \mapsto \Omega^2 g_{\mu\nu}, \quad \sigma \mapsto \Omega \sigma, \quad \Omega = \Omega(x).$$

In $d$-dimensional tractor theory, fields are arranged in $\mathfrak{so}(d, 2)$ multiplets and, under Weyl transformations, transform under particular $\mathfrak{so}(d, 2)$-valued “tractor” gauge transformations. A fundamental example of a weight zero tractor vector, built only from the scale and the metric is the scale tractor

$$I^M = \begin{pmatrix}
\sigma \\
\partial^m \sigma \\
-\frac{1}{4} \left(\Delta + P\right) \sigma
\end{pmatrix}.$$

Here $M$ is an $\mathfrak{so}(d, 2)$ vector index, the index $\mu$ on $\partial_\mu$ has been flattened with an inverse vielbein and the trace of the Schouten tensor $P$ is proportional to the scalar curvature $R = 2(d - 1) P$. Under Weyl transformations (2), the scale tractor $I^M$ transforms as

$$I^M \mapsto U^M_N I^N.$$
where the $SO(d,2)$-valued matrix $U$ is given by

$$
U = \begin{pmatrix}
\Omega & 0 & 0 \\
\Upsilon^m & \delta^m_n & 0 \\
-\frac{1}{2} \Omega^{-1} \Upsilon_r \Upsilon^r & -\Omega^{-1} \Upsilon_n & \Omega^{-1}
\end{pmatrix},
\quad T_\mu \equiv \Omega^{-1} \partial_\mu \Omega.
$$

Parabolic $SO(d,2)$ transformations of this special form are tractor gauge transformations.

By virtue of the equivalence (2), solving Einstein’s equations means that we need only find a conformally Einstein metric. Then we can arrange for the scale $\sigma$ to be constant and equal $\kappa^{\frac{2}{d-2}}$, which produces the actual Einstein metric from the conformal class of metrics. This amounts identically to requiring that the scale tractor be parallel with respect to the tractor connection defined on a weight zero tractor $T^M$ by

$$
D_\mu \begin{pmatrix} T^+ \\ T^m \\ T^- \end{pmatrix} \equiv \begin{pmatrix} \partial_\mu T^+ - T_\mu \\ \nabla_\mu T^m + P^m_\mu T^+ + e^m_\mu T^- \\ \partial_\mu T^- - P^m_\mu T_m \end{pmatrix},
$$

where the Schouten tensor is the pure trace part of the Riemann tensor, $R_\mu \nu \rho \sigma - W_\mu \nu \rho \sigma = 4 g_{[\mu [\rho} P_{\nu ]] \sigma]}$. I.e., $g_{\mu \nu}$ is conformally Einstein exactly when

$$
D_\mu I^M = 0.
$$

Indeed, at the distinguished choice of scale, the left hand side gives

$$
D_\mu I^M \bigg|_{\sigma = \kappa^{\frac{2}{d-2}}} = \kappa^{\frac{2}{d-2}} \begin{pmatrix} 0 \\ P^m_\mu - \frac{1}{d} e^m_\mu P \end{pmatrix},
$$

which verifies our claim.

The Einstein–Hilbert action also follows simply from the scale tractor; consider the manifestly Weyl invariant action

$$
S(g_{\mu \nu}, \sigma) = \frac{d(d-1)}{2} \int \frac{\sqrt{-g}}{\sigma^d} I^M \eta_{MN} I^N = S(\Omega^2 g_{\mu \nu}, \Omega \sigma).
$$

Here $\eta_{MN} = \begin{pmatrix} 0 & \eta_0^n & \eta_0^m \\ \eta_0^m & 0 & \eta_0^m \\ \eta_0^m & \eta_0^m & 0 \end{pmatrix}$ is the $\mathfrak{so}(d,2)$ invariant metric. At the canonical choice of scale we recover the Einstein–Hilbert action $S(g_{\mu \nu}, \kappa^{\frac{2}{d-2}}) = -\frac{1}{2 \kappa^2} \int \sqrt{-g} R$. 

5
To write physical theories in a Weyl invariant way, an operator taking the place of the Riemannian covariant derivative in formulae is needed. It is provided by the Thomas $D$-operator which covariantly maps weight $w$ tractors to weight $w - 1$ ones and is given by

\[
D^M \equiv \begin{pmatrix} (d + 2w - 2)w \\ (d + 2w - 2)D^m \\ -(D_vD^\nu + w\mathcal{P}) \end{pmatrix}.
\]

Acting on the weight one scale field $\sigma$, the Thomas $D$-operator produces the scale tractor $I^M = \frac{1}{2} D^M \sigma$. Although this operator is not Leibnitzian, its covariance under $\mathfrak{so}(d, 2)$ tractor gauge transformations often makes it a more useful building block than the tractor covariant derivative (3) itself.

Having presented the manifestly locally scale invariant, tractor, formulation of gravity, we now add matter fields and first focus on a single scalar field $\phi$. The standard “massless” scalar field action $S = -\frac{1}{2} \int \sqrt{-g} \nabla_\mu \phi \ g^{\mu\nu} \nabla_\nu \phi$ can easily be reformulated Weyl invariantly using the scale $\sigma$: From $\sigma$ we build the one-form

\[
b = \sigma^{-1} d\sigma,
\]

with the simple transformation rule

\[
b \mapsto b + \Upsilon.
\]

Assigning the weight $w$ to the scalar $\phi$

\[
\phi \mapsto \Omega^w \phi,
\]

then the combination $\tilde{\nabla}_\mu = \nabla_\mu - wb_\mu$ acting on $\phi$ transforms covariantly

\[
\tilde{\nabla}_\mu \phi \mapsto \Omega^w \tilde{\nabla}_\mu \phi.
\]

Hence we find an equivalent, but manifestly Weyl invariant, action principle

\[
S = -\frac{1}{2} \int \frac{\sqrt{-g}}{\sigma^{d+2w-2}} \tilde{\nabla}_\mu \phi \ g^{\mu\nu} \tilde{\nabla}_\nu \phi.
\]

Of course, this is just the result of the “compensating mechanism”, whereby, for any action involving a set of fields $\{\Phi_\alpha\}$ and their derivatives, replacing $\Phi_\alpha \mapsto \Phi_\alpha/\sigma^{w_\alpha}$ and $g_{\mu\nu} \mapsto g_{\mu\nu}/\sigma^2$ yields an equivalent Weyl invariant action.
However, just as one searches for all possible coordinate invariant theories, we should also consider all possible locally scale invariant theories—those independent of local choices of unit system—and then examine their physical consequences. In particular, the set of theories invariant by virtue of the above Weyl compensator trick, does not map out the entire space of possible scalar theories, even at the level of those quadratic in derivatives and fields. Tractor calculus is a very useful tool for such a study.

Motivated by ideas in conformal scattering theory [14] [15] [12], in our recent work [2] we proposed the Weyl invariant theory with action principle

$$S = -\frac{1}{2} \int \frac{\sqrt{-g}}{\sigma^{d+2w-1}} \varphi I^M D_M \varphi .$$

Its difference from the action (7) can also be expressed tractorially as

$$\frac{1}{2} w(d+w-1) \int \frac{\sqrt{-g}}{\sigma^{d+2w}} \varphi I^M I_M \varphi ,$$

which is reminiscent of the tractor Einstein–Hilbert action (5). Our proposal therefore amounted to coupling to the background geometry via the Weyl invariant

$$I^M I_M = -\frac{2 \sigma^2}{d} \left[ P + \nabla^\mu b_\mu - \frac{d-2}{2} b^\mu b_\mu \right].$$

Since the trace of the $P$ tensor is constant in an Einstein background our model yields a regular scalar mass in that case. In fact, since the scale tractor is parallel when the background is conformally Einstein, $I^M I_M$ is then constant for any choice of scale. Moreover, mass is then naturally reinterpreted in terms of the Weyl weight of $\varphi$ according to the mass-Weyl weight relationship

$$m^2 = -\frac{2P}{d} \left[ \left( w + \frac{d-1}{2} \right)^2 - \left( \frac{d-1}{2} \right)^2 \right] .$$

\footnote{Note that since the Thomas $D$-operator is null ($D_M D^M = 0$), $D_M I^M = 0$. The weight one canonical tractor $X^M = \begin{pmatrix} 0 \\ a \\ 1 \end{pmatrix}$ does not produce a new invariant either since $X_M I^M = \sigma$. We also record the component form of the scale tractor

$$I^M = \sigma \begin{pmatrix} 1 \\ b^m \\ -(P + \nabla_n b^n + b_n b^n) \end{pmatrix}.$$}
Additionally, $P$ is negative and constant in anti de Sitter spaces, therefore reality of the Weyl weight $w$ implies the Breitenlohner-Freedman bound \cite{16,17}:
\[ m^2 \geq \frac{P}{2d} (d - 1)^2. \]

Before studying how spinors couple to scale, it is worth noting how Weyl invariant theories without coupling to the scale appear. For that, we need the scale $\sigma$ to decouple: examining the Thomas $D$-operator \eqref{eq:thomas} at the special weight $w = 1 - \frac{d}{2}$, we see that its top and middle slots vanish. In that case the equation of motion $I^M D_M \varphi = 0$ is equivalent to $D_M \varphi = 0$. This implies $\left( \Delta - \frac{d-2}{2} P \right) \varphi = 0$ which is exactly the equation of motion for a conformally improved scalar field.

### 3 Tractor Spinors

The theory of spinors in conformal geometry is a well-developed subject to which tractor calculus can be applied \cite{18,19}. A tractor spinor can be built from a pair of $d$-dimensional spinors. While the latter transform as $\mathfrak{so}(d-1,1)$ representations the tractor spinor is a spinor representation of $\mathfrak{so}(d,2)$; to avoid technical questions on the spinor type in $d$ and $d+2$ dimensions, we do not specify whether the constituent $d$-dimensional spinors are Dirac, Weyl, or Majorana. From the Dirac matrices $\{\gamma^m, \gamma^n\} = 2\eta^{mn}$ in $d$ dimensions, we build $(d+2)$-dimensional Dirac matrices
\[
\Gamma^+ = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix}, \quad \Gamma^- = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}, \quad \Gamma^m = \begin{pmatrix} \gamma^m & 0 \\ 0 & -\gamma^m \end{pmatrix},
\]
subject to
\[
\{\Gamma^M, \Gamma^N\} = 2\eta^{MN}.
\]

A weight $w$ tractor spinor $\Psi = (\psi, \chi)$, built from a pair of $d$-dimensional spinors $\psi$ and $\chi$ is defined by its transformations under tractor gauge transformations
\[
\Psi = \begin{pmatrix} \psi \\ \chi \end{pmatrix} \mapsto \Omega^w \begin{pmatrix} \sqrt{\Omega} \psi \\ \frac{1}{\sqrt{\Omega}} \left[ \chi - \frac{1}{\sqrt{2}} \gamma^m \psi \right] \end{pmatrix}. \tag{10}
\]

The tractor covariant derivative acting on a tractor spinor is defined by
\[
D_\mu \Psi = \begin{pmatrix} \nabla_\mu \psi + \frac{1}{\sqrt{2}} \gamma_\mu \chi \\ \nabla_\mu \chi - \frac{1}{\sqrt{2}} \gamma_\mu \psi \end{pmatrix},
\]
where $\nabla_\mu$ is the standard Levi-Civita connection acting on $d$-dimensional spinors.

We have now assembled the ingredients required to compute the Thomas $D$-operator \( \Box \) acting on spinors. Of particular interest is the “Dirac–Thomas $D$-operator”

$$\Gamma.D \Psi = \begin{pmatrix}
(d + 2w - 2)\nabla + \frac{1}{\sqrt{2}}(d + 2w)(d + 2w - 2) \\
-\sqrt{2}\nabla^2 \end{pmatrix} \Psi
$$

$$= \begin{pmatrix}
(d + 2w - 2)[\nabla \psi + \frac{1}{\sqrt{2}}(d + 2w)\chi] \\
-(d + 2w)\nabla \chi - \sqrt{2}[\Delta - \frac{p}{2}(d - 1)]\psi
\end{pmatrix}.$$

Here we have denoted the contraction of tractor vector indices by a dot and it is worth bearing in mind that the $d$-dimensional Weitzenbock identity acting on spinors is $\nabla^2 = \Delta - \frac{p}{2}(d - 1)$.

It is well known that the massless Dirac equation is Weyl covariant in any dimension. This follows naturally from tractors: Observe that we can use the canonical tractor to produce a weight $w + 1$ tractor spinor from $\Psi$

$$\Gamma.X \Psi = \begin{pmatrix}
0 \\
\sqrt{2}\psi
\end{pmatrix},$$

where $\psi$ has the transformation rule

$$\psi \mapsto \Omega^{w+\frac{1}{2}}\psi.$$

Now acting with the Dirac–Thomas $D$-operator yields

$$\Gamma.D \Gamma.X \Psi = (d + 2w + 2) \begin{pmatrix}
(d + 2w)\psi \\
-\sqrt{2}\nabla \psi
\end{pmatrix}.$$

Hence assigning $\Psi$ the weight $w = -\frac{d}{2}$ so that $\psi \mapsto \Omega^{1-\frac{d}{2}}\psi$, it follows from the tractor spinor gauge transformation rule \([10]\), that

$$\nabla \psi \mapsto \Omega^{-\frac{d+1}{2}}\nabla \psi,$$

which proves the covariance of the Dirac operator. We are now suitably armed to construct fermionic tractor theories.
Fermionic theories pose some interesting puzzles for our tractor approach. Firstly, since the tractor approach is based on arranging fields in $\mathrm{so}(d,2)$ multiplets, we might generically expect a doubling of degrees of freedom. This can be seen from the previous Section where tractor spinors were constructed from pairs of spacetime spinors. Secondly, the mass-Weyl weight relationship \((9)\) relates the mass squared to the scalar curvature. However, massive spinor theories depend linearly on the mass, and therefore the square root of the scalar curvature. It is not immediately obvious how this square root could arise. We will solve both of these puzzles by employing several principles: To construct tractor-spinor and tractor-spinor-vector theories

1. We search for massive wave equations whose masses are related to Weyl weights by an analog of the scalar relationship \((\ref{mass-Weyl})\).

2. We require that, in a canonical choice of scale, these theories match those found by the log-radial dimensional reduction of $d+1$ dimensional massless Minkowski theories to $d$ dimensional constant curvature ones described in \([4,5]\) and Appendix A.

3. We will impose as many constraints as consistent with the above requirements so as to find a “minimal covariant field content”.

These principles will become clearer through their applications, so let us provide the details.

In our previous work, we used the fact that (from an ambient viewpoint as described in \([13,12]\) and further studied in \([20]\)), the contraction of the scale tractor and Thomas $D$-operator $I.D$ generates bulk evolution, and therefore searched for wave equations $(I.D + \text{more})V^\bullet = 0$, where the terms “+more” were chosen on the grounds of gauge invariance. In addition we imposed the most general field constraints, linear in the Thomas $D$-operator, on the bosonic gauge fields $V^\bullet$ that were consistent with gauge invariance. In this picture the tractor weight $w$ of the gauge field $V^\bullet$, controls the mass, save at special weights where the theory becomes massless or partially massless. In addition at the special weight $w = 1 - d/2$ the scale tractor decouples from the equations of motion and conformal wave equations result (a comprehensive study of higher spin conformal wave equations may be found in \([21,22]\), see also references therein).
Along the same lines we propose the spinor equation of motion and field constraint for a weight $w$ tractor spinor $\Psi$

\[
I.D \Psi = 0, \\
\Gamma.D \Psi = 0.
\]

(13)

We can view the second equation as a scale covariant constraint eliminating the lower component of $\Psi$. Its solution is

\[
\Psi = \begin{pmatrix}
\psi \\
-\sqrt{\frac{2}{d+2w}} \nabla \psi
\end{pmatrix}.
\]

In turn, the $I.D$ field equation, in the canonical scale $\sigma = \text{constant}$, implies the massive wave equation

\[
\left[ \Delta + \frac{2\mathcal{P}}{d} (w^2 + wd + \frac{d}{4}) \right] \psi = 0.
\]

(14)

Defining the squared mass as the eigenvalue of $\Delta$ (note that $\mathcal{P}$ is constant in an Einstein background) gives the spinorial mass-Weyl weight relationship

\[
m^2 = -\frac{2\mathcal{P}}{d} \left[ \left( w_{\psi} + \frac{d-1}{2} \right)^2 - \frac{d(d-1)}{4} \right],
\]

(15)

analogous to its bosonic counterpart (9). Here we have defined $w_{\psi} \equiv w + \frac{1}{2}$ because under Weyl transformations $\psi$ transforms according to (11). Observe that reality of the weight $w$ for spaces with negative scalar curvature implies a Breitenlohner–Freedman type bound [16, 17] on the mass parameter $m^2 \geq \frac{1}{2} \mathcal{P}(d - 1)$. Before analyzing this system further, let us present an alternate formulation.

The Thomas $D$-operator is second order in its lowest slot. For Fermi systems, we would like to find a set of first order field equations. To that end, we recall the double $D$-operator defined by

\[
(d + 2w - 2)D^{MN} = X^N D^M - X^M D^N.
\]

(16)

\footnote{The value $w = -d/2$ is distinguished here, as in fact is the value $w = -d/2 + 2$. In the first case we cannot solve the constraint in (13). Also, in deriving (14), we have dropped an overall factor $(d + 2w - 2)/(d + 2w)$. However, below we give a second formulation of the system that still predicts (14) at $w = -d/2 + 2$.}
At generic weights it obeys the identity

$$[X^M, D^N] = 2D^{MN} - (d + 2w) \eta^{MN},$$

so the double-\(D\) operator essentially amounts to the commutator of the Thomas \(D\)-operator and the canonical tractor. (An ambient interpretation of this algebra is explored in [20].) In components it is given by

$$D^{MN} = \begin{pmatrix}
0 & 0 & w \\
0 & 0 & D^m \\
-w & -D^n & 0
\end{pmatrix}.$$  \hspace{1cm} (17)

In these terms, we propose the Dirac-type equation\(^3\)

$$I^M \Gamma^N D_{MN} \Psi = 0.$$ \hspace{1cm} (18)

In the canonical choice of scale it reads

$$-\sigma \begin{pmatrix}
\nabla \psi + \frac{d+2w}{\sqrt{2}} \chi \\
-\nabla \chi + \frac{(d+2w)p}{\sqrt{2d}} \psi
\end{pmatrix} = 0.$$

Firstly, when \(d + 2w \neq 0, 2\), it is easy to verify that these equations are equivalent to (13). In general, they are more fundamental because (at \(d + 2w \neq 2\)) the equation (14) follows as an integrability condition. Moreover, even at \(d + 2w = 2\) we can still define the double-\(D\) operator by (17) and then have well-defined system (that implies the massive wave equation (14)).

The weight \(d + 2w = 0\), has a special physical significance because at that weight we expect to find a scale invariant theory as explained in Section 3. Although it is not true that the scale decouples from the equation (18), the modified equation

$$\sigma^{-1} \Gamma.X I^M \Gamma^N D_{MN} \Psi = 0,$$

is in fact independent of the scale at \(w = -d/2\). It is then equivalent to the equation \(\Gamma.X \Gamma^N D_{MN} \Psi = 0\) (just as for scalars in Section 2). In components this amounts simply to the Dirac equation \(\nabla \psi = 0\).

\(^3\)This equation is similar in spirit to Dirac's proposal for writing four dimensional conformal wave equations by employing the six dimensional Lorentz generators [23]. Of course here, we also describe massive systems that are not invariant without coupling to scale.
In the above formulation, \( w = -d/2 \) is the only value at which multiplication by a factor \( \Gamma_X \) yields a consistent system, at other values the field \( \chi \) enters on the right hand side of the Dirac equation. The presence of the second spinor \( \chi \) is undesirable, because it doubles the degrees of freedom of the \( d \)-dimensional theory. We next explain how to obtain a tractor theory of a single \( d \)-dimensional spinor.

Firstly observe that a massive Dirac equation is linear in the mass parameter, whereas according to (15) the constant scalar curvature is proportional to the square of the mass. Therefore we need a tractor mechanism that somehow introduces the square root of the scalar curvature while at the same time relating the pair of spinors \( \chi \) and \( \psi \). Examining our spinorial wave equations (18) in their canonical component form, we see that a relationship \( \chi = \alpha \psi \) means that this pair of equations are equivalent only when \( \alpha^2 = -P/d \). This relationship can be imposed tractorially using the projectors

\[
\Pi_\pm \equiv \frac{1}{2} \left[ 1 \pm \frac{\Gamma_I}{\sqrt{I.I}} \right].
\]

(Recall that in a conformally Einstein background, \( I^M \) is tractor parallel, so that \( I.I \) is constant. Note that \( I.I \) is positive for negative scalar curvature.) Hence we propose the tractor Dirac equations

\[
I^M \Gamma^N D_{MN} \Psi = 0 = \Pi_+ \Psi,
\]

We could equally well multiply the first of these equations by \( \Gamma_X \) since the its bottom slot is a consequence of the top one. (The choice of \( \Pi_+ \) rather than \( \Pi_- \) corresponds to the sign of the Dirac mass term.) In canonical components these imply the massive curved space Dirac equation

\[
\left[ \nabla - \sqrt{-P} \frac{1}{2d} (d + 2d) \right] \psi = 0.
\]

(19)

Its mass is again related to the weight of \( \psi \) by (15).

In summary, the irreducible tractor Dirac equation for a tractor spinor \( \Psi \) (subject to the “Weyl”-like condition\( \Pi_\pm \Psi = 0 \)) is given by

\[
\Gamma_X I^M \Gamma^N D_{MN} \Psi = 0.
\]

\(^4\text{We cannot help but remark that this conditions melds two of Weyl’s seminal contributions to physics – the Weyl spinor and Weyl symmetry.}\)
This equation of motion follows from an action principle which we now describe. To that end we need to introduce the tractor Dirac conjugate spinor, which is defined as

$$\Psi \equiv \begin{pmatrix} \psi \\ \chi \end{pmatrix} = i \Psi^\dagger \Gamma^0 = (\bar{\chi} \bar{\psi})$$,

where $\bar{\psi}$ and $\bar{\chi}$ are the standard $d$-dimensional Dirac conjugates of $\psi$ and $\chi$, and $\Gamma^0$ obeys the following properties (because it derives from the product of the two timelike Dirac matrices of $\mathfrak{so}(d,2)$):

$$(\Gamma^0)^2 = -1, \quad \Gamma^0\dagger = -\Gamma^0, \quad \Gamma^M\dagger = -\Gamma^0\Gamma^M\Gamma^0.$$ 

Then the required action principle is

$$S = \frac{1}{\sqrt{2}} \int \sqrt{-g} \sigma^{d+2w+1} \bar{\Psi} \Gamma \cdot X I^M \Gamma^N D_{MN} \Psi,$$  

where the tractor spinor $\Psi$ obeys $\Pi_- \Psi = 0$. This action is hermitean. Since it is useful to possess the tractor machinery required to vary actions of this type, let us prove this. Firstly, the double-$D$ operator $D_{MN}$ is Leibnitzian. Moreover $\int \sqrt{-g} D_{MN} \Xi^{MN} = 0$ (up to surface terms) for any $\Xi^{MN}$ of weight zero. This allows us to integrate $D_{MN}$ by parts. Therefore, to verify $S = S^\dagger$ we need to compute $D_{MN} \left[ \frac{1}{\sigma^{d+2w+1}} \bar{\Psi} \Gamma \cdot X I^M \Gamma^N \right]$ which requires the following identities

$$D_{MN}\sigma = X_N I_M - X_M I_N,$$
$$D_{MN} I^N = 0,$$
$$D_{MN} X^R = X_N \delta^R_M - X_M \delta^R_N,$$
$$X^M D_{MN} = w X_N.$$  

Orchestrating these, we find

$$S - S^\dagger = \frac{d + 2w}{\sqrt{2}} \int \sqrt{-g} \sigma^{d+2w+1} \bar{\Psi} (\Gamma \cdot X \Gamma \cdot I - \sigma) \Psi.$$ 

For generic weights $w$ this is non-vanishing, however using the condition $\Pi_- \Psi = 0$ to conclude that $\Psi$ is in the image of $\Pi_-$ along with the facts that $\Pi_+ \Psi = \bar{\Psi} \Pi_-$ and $\Pi_- (\Gamma \cdot X \Gamma \cdot I - \sigma)\Pi_- = 0$, shows that $S = S^\dagger$. A similar computation implies that the above action implies the field equations quoted.
Our final computation is to write out the action principle in components. Rather than working at the canonical scale, let us give the general result, namely

\[ S = - \int \frac{\sqrt{-g}}{\sigma^{d+2w}} \bar{\psi} \left( \nabla - \frac{1}{2} (d + 2w) \left( \hat{b} + \sqrt{-2(P + \nabla \cdot b - \frac{d-2}{d} b \cdot b)} \right) \right) \psi. \]

Each term has a simple interpretation. The \( \hat{b} \) contribution covariantizes the leading Dirac operator with respect to scale transformations so, \( \psi \mapsto \Omega^{w+\frac{1}{2}} \psi \) implies \( [\nabla - \frac{1}{2} (d+2w) \hat{b}] \psi \mapsto \Omega^{w+\frac{1}{2}} [\nabla - \frac{1}{2} (d+2w) \hat{b}] \psi \). These terms also follow from the standard Weyl compensator mechanism. The square root factor is the mass term which equals (up to a factor \( \sigma \)) \( \sqrt{I.I} \), and is therefore constant for conformally Einstein backgrounds. The prefactor \( (d + 2w) \) calibrates the mass to the square of the scale tractor and implies the mass-Weyl weight relationship \( (15) \). When \( w = -\frac{d}{2} \), the scale \( \sigma \) decouples from the action and we obtain the Weyl invariant curved space Dirac equation discussed in Section 6.

5 Tractor Rarita–Schwinger Equation

In the spinor models we have encountered so far there have been choices for mass terms: we could have used a “gravitational mass term” \( (8) \) or a compensated mass term. The gravitational mass term is proportional to \( I.I \), while the compensated mass term is obtained by using the scale \( \sigma \) to compensate a standard mass term (for example, we could add a term \( \frac{1}{2} \int \frac{\sqrt{-g}}{\sigma^{d+2w}} \phi^2 \) to the scalar action principle). However, once we study models with spins \( s \geq 1 \), gauge invariances are necessary to ensure that only unitary degrees of freedom propagate. In our previous work \( [1, 2] \), we showed how higher spin gauge invariant tractor models described bosonic massless, partially massless \( [24, 25, 26] \) and massive models in a single framework. In particular, they implied “gravitational mass terms” (rather than compensated ones) with masses dictated by Weyl weights. We now extend those results to the higher spin \( s = 3/2 \) Rarita–Schwinger system. The following analysis closely mirrors the tractor Maxwell system studied in \( [1, 2] \) so we keep details to a minimum.

As field content, we take a weight \( w \) tractor vector-spinor \( \Psi^M \) subject to
the gauge invariance
\[ \delta \Psi^M = D^M \Xi, \]
where \( \Xi \) is a weight \( w + 1 \) tractor spinor parameter. Since the Thomas \( D \)-operator is null, we may consistently impose the field constraint
\[ D_M \Psi^M = 0. \tag{22} \]
We assume that the background is conformally flat, so that Thomas \( D \)-operators commute. We now observe that the quantity
\[ \mathcal{R}^{MNR} = 3D^{[MN}\Psi^{R]}, \]
is gauge invariant by virtue of the identity (16) and use it to construct a set of tractor Rarita–Schwinger equations coupled to the scale tractor
\[ \mathcal{R}_M \equiv \Gamma_{MNR} I_S \mathcal{R}^{SNR} = 0. \tag{23} \]
The final requirement we impose is the projective one found for spinors
\[ \Pi_+ \Psi^M = 0, \quad \Pi_- \Xi = 0. \tag{24} \]
To verify that the set of equations (22,23,24) are the desired ones, we write them out explicitly in canonical components. This computation is lengthy but straightforward. The field constraint (22) and projective condition (24) eliminate most of the field content leaving only the top spinorial components \( \psi^+ \), and middle vector slots \( \psi^m \), independent. Since the system will describe both massive and massless excitations, the spinor \( \psi^+ \) plays the rôle of a Stückelberg field. The Rarita–Schwinger type equation \( \mathcal{R}^M = 0 \) in (23) then yields the independent field equations
\[ \gamma^{\mu \nu \rho} \tilde{\nabla}_\nu \psi_\rho + \sqrt{-2/P} \gamma^{\mu \nu} \left( [w + 1] \psi_\nu - \tilde{\nabla}_\nu \psi^+ \right) = 0. \]
Here the operator
\[ \tilde{\nabla}_\mu \equiv \nabla_\mu - \sqrt{-P/2d} \gamma_\mu \]
\[ ^5 \text{We leave an investigation of whether non-minimal couplings could relax this restriction to future work. Any such study will be highly constrained by existing results for gravitational spin 3/2 couplings, see [27].} \]
is the modification of the covariant derivative acting on spinors found quite some time ago in a cosmological supergravity context [28]. Its distinguishing property is that $[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu]$ vanishes on spinors (but not vector-spinors). The above equation of motion enjoys the gauge invariance

$$
\delta \psi_\mu = (d + 2w)\tilde{\nabla}_\mu \varepsilon, \\
\delta \psi^+ = (d + 2w)(w + 1)\varepsilon.
$$

(25)

We include the factor $(d + 2w)$ to synchronize the component transformations with the tractor ones $\delta \Psi^M = D^M \Xi$. Notice they imply that $\psi^+$ is an auxiliary St"uckelberg field at generic $w \neq 1$, which can be gauged away leaving a massive Rarita–Schwinger field $\psi_\mu$. When $w$ does equal $-1$, the field $\psi^+$ is gauge inert and we may impose the additional constraint $\psi^+ = 0$ (in fact, a careful analysis shows that this field decouples completely at $w = -1$). That leaves the massless Rarita–Schwinger equation in AdS with standard gauge invariance

$$
\gamma^{\mu\nu\rho} \tilde{\nabla}_\nu \psi_\rho = 0, \quad \delta \psi_\mu = \tilde{\nabla}_\mu \varepsilon.
$$

Returning to generic $w$, we may rewrite the above equation in the standard massive form

$$
\gamma^{\mu\nu\rho} \nabla_\nu \psi_\rho + m \gamma^{\mu\nu} \psi_\nu = 0.
$$

The integrability conditions for this system imply the usual constraints $\nabla^\mu \psi_\mu = 0 = \gamma^\mu \psi_\mu$, and in turn $(\nabla - m)\psi_\mu = 0$. The mass $m$ is here given in terms of weights by the mass-Weyl weight relationship

$$
m = \sqrt{-\frac{P}{2d}} (d + 2w).
$$

Via the spin 3/2 Weitzenbock identity, this implies a wave equation $(\Delta - \mu^2)\psi_\mu = 0$ where $\mu^2$ obeys a Weyl weight relationship highly reminiscent of the spin 0 and 1/2 ones above

$$
\mu^2 = -\frac{2P}{d} \left[ \left( w_{\psi_m} + \frac{d - 1}{2} \right)^2 - \frac{d(d - 1)}{4} - 1 \right] .
$$

(26)

Here $w_{\psi_m} = w + \frac{1}{2}$ because, in the St"uckelberg gauge $X, \Psi = 0$, we have the Weyl transformation rule $\psi_\mu \mapsto \Omega^{w+3/2}\psi_\mu$. We end by observing, that this result implies a Breitenlohner–Freedman type bound for massive gravitini.
\( \mu^2 \geq \frac{b}{2d} [d(d - 1) + 4] \). Following the procedure outlined at the end of Section 4, we write (23) at arbitrary scale

\[
\gamma^{\mu\nu\rho} \nabla_\nu \psi_\rho + \frac{1}{2} (d + 2w) \gamma^{\mu
u} \left( b + \frac{\sqrt{I}}{\sigma} \right) \psi_\nu - (w + 1) \gamma^\mu b \cdot \psi = 0 .
\]

To understand the above expression, let us define the Weyl-covariantized Rarita-Schwinger operator

\[
R^\mu \equiv \gamma^{\mu\nu\rho} \nabla_\nu \psi_\rho + \frac{1}{2} (d + 2w) \gamma^{\mu\nu} b \psi_\nu - (w + 1) \gamma^\mu b \cdot \psi , \quad \gamma \cdot \psi = 0 . \quad (27)
\]

The \( b \) contribution covariantizes the Rarita-Schwinger operator with respect to scale transformations such that \( \psi_\mu \mapsto \Omega^{w+\frac{3}{2}} \psi_\mu \) implies \( R^\mu \mapsto \Omega^{w-\frac{3}{2}} R^\mu \), modulo the condition \( \gamma \cdot \psi = 0 \). This operator also follows from the standard Weyl compensator mechanism. As before, the square root factor is the mass term, and the prefactor \((d + 2w)\) calibrates the mass to the square of the scale tractor and implies the mass-Weyl weight relationship (26). In \( d = 2 \), when \( w = -\frac{d}{2} = -1 \), the scale \( \sigma \) decouples from the equation of motion and we obtain the Weyl invariant curved space Rarita-Schwinger equation.

In fact, in arbitrary dimensions \( d \) it is possible to write down a Weyl invariant Rarita–Schwinger system [24]. We can obtain that theory from our tractor one as follows: Consider a new field equation \( \tilde{R}^\mu = R^\mu - \frac{1}{d} \gamma^\mu (\gamma \cdot R) = 0 \), or explicitly

\[
\tilde{R}^\mu = \nabla^\mu \psi^\mu - \frac{2}{d} \gamma^\mu \nabla \cdot \psi + \frac{d + 2w}{2} \left[ \gamma^{\mu\nu} b \psi_\nu - \frac{2(d - 1)}{d} \gamma^\mu b \cdot \psi \right] . \quad (28)
\]

When \( w = -\frac{d}{2} \), the scale dependence through the composite gauge field \( b \) decouple completely, and we are left with the Weyl invariant Rarita-Schwinger system of [24] generalized to arbitrary dimensions

\[
\nabla^\mu \psi^\mu - \frac{2}{d} \gamma^\mu \nabla \cdot \psi = 0 = \gamma \cdot \psi . \quad (29)
\]

We can derive the same results efficiently using tractors. This requires imposing two additional constraints

\[
X \cdot \Psi = 0 , \quad \Gamma X \cdot \Gamma \Psi = 0 , \quad (30)
\]

which in components read

\[
\psi^+ = X^+ = \gamma \cdot \psi = 0 . \quad (31)
\]
As argued before, at $w = \frac{d}{2}$ the compensator field $\sigma$ can be safely eliminated without compromising the Weyl invariance. At this special value of the weight, the tractorial expression describing Weyl invariant Rarita-Schwinger equation is:

$$\tilde{R}^M = \Gamma \cdot X [R^M - \frac{d - 2}{d(d - 6)} \Gamma^M (\Gamma \cdot R)] = 0,$$

which in components exactly matches (29).

6 Supersymmetry

Given a tractor description of spinors and scalars, it is natural to search for a supersymmetric combination of the two. Here we study global supersymmetry. In a curved background, globally supersymmetric theories require a generalization of the constant spinors employed as parameters of supersymmetry transformations in flat space. A possible requirement is to search for covariantly constant spinors, although most backgrounds do not admit such special objects. Focusing on conformally flat backgrounds, a more natural condition is to require that the background possess a Killing spinor $\varepsilon$ defined by

$$\nabla_\mu \varepsilon = -\sqrt{-P} \frac{2}{d} \gamma_\mu \varepsilon.$$

As a consequence it follows that $\bar{\varepsilon} \varepsilon$ is constant. This condition can be neatly expressed in tractors in terms of what we shall call a “scale spinor”

$$\Xi = \left( \begin{array}{c} \varepsilon \\ \eta \end{array} \right), \quad \Pi \cdot \Xi = 0.$$

Here $\eta$ is determined by the projective condition. The Killing spinor condition for $\varepsilon$ is now imposed by requiring the weight $w = 0$ tractor spinor $\Xi$ to be tractor parallel

$$D_\mu \Xi = 0.$$

From the scale spinor, we can form the scale tractor as

$$I^M = \frac{\sigma \Xi^M \Xi}{\Xi \Gamma \cdot X \Xi}.$$

Note that there actually no pole in this expression in six dimensions as evidenced by the component expression (29).
which justifies its name.

Having settled upon the global supersymmetry parameters, we specify the field content as a weight \( w + 1 \) scalar \( \varphi \) and a weight \( w \) tractor spinor \( \Psi \) subject to

\[
\Pi_+ \Psi = 0 .
\]

We have chosen the tuning between weights of fermionic and bosonic fields in order to preserve supersymmetry. The supersymmetry transformations are given by\(^7\)

\[
\begin{align*}
\delta \varphi &= \Re \left( \Xi \Gamma . X \Psi \right), \\
\delta \Psi &= \frac{1}{d + 2w} \left[ (\Gamma . D - \frac{1}{\sigma} \Gamma . X I . D) \varphi \right] \Xi .
\end{align*}
\]

(33) \hspace{1cm} (34)

Here we take \( \varphi \) to be real, but make no assumption for reality conditions for the spinors. If the underlying \( d \)-dimensional spinors are Majorana, there is no need to take the real part in the supersymmetry transformations of the bosons. For the independent bosonic and fermionic field components, these transformations amount to

\[
\begin{align*}
\delta \varphi &= \Re (\sqrt{2} \bar{\varepsilon} \psi), \\
\delta \psi &= \left[ \left( \bar{\Psi} + \sqrt{\frac{-2 \varsigma}{d}} \right) (w + 1) \varphi \right] \varepsilon .
\end{align*}
\]

The invariant tractor action for this system is the sum of the Bose and Fermi actions discussed in previous Sections

\[
S = \int \frac{\sqrt{-g}}{\sigma^{d+2w+1}} \left\{ \bar{\Psi} \Gamma . X \Gamma^M I^N D_{MN} \Psi + \varphi I . D \varphi \right\} .
\]

To verify the invariance of this action one first uses the identity

\[
\Gamma . X \Gamma^M I^N D_{MN} = \frac{\sigma}{d + 2w - 2} \Gamma . X \Gamma . D,
\]

so that

\[
\Gamma . X \Gamma^M I^N D_{MN} \delta \Psi = - \left( \frac{\sigma}{(d + 2w - 2)(d + 2w)} \Gamma . X \Gamma . D \Gamma . X \frac{1}{\sigma} I . D \varphi \right) \Xi .
\]

\(^7\)There is no pole in the fermionic variation at \( w = -d/2 \); this can be checked explicitly from a component computation.
Then the identity
\[ \Gamma.X \Gamma.D = \frac{d + 2w - 2}{d + 2w + 2} \Gamma.D \Gamma.X + (d + 2w)(d + 2w - 2), \]
yields
\[ \Psi \Gamma.X \Gamma.M I N D_{M N} \delta \Psi = - (\Psi \Gamma.X \Xi) I.D \varphi. \]
Comparing the last expression with the bosonic variation in (33) completes our invariance proof.

7 Interactions

To add interactions, we begin by closing the supersymmetry algebra off-shell with the aid of an auxiliary field. In curved backgrounds, the square of a supersymmetry transformation yields an isometry as the generalization of translations in flat space. Therefore we also need to explain how to handle isometries with tractors. On the bosonic field \( \varphi \), the supersymmetry algebra closes without any auxiliary field and the algebra of two supersymmetry transformations is given by

\[ [\delta_1, \delta_2] \varphi = \Re(\Xi_1 \Gamma^{MN} \Xi_2) D_{MN} \varphi. \]

The adjoint tractor \( \Re(\Xi_1 \Gamma^{MN} \Xi_2) \) is an example of what we shall call a “Killing tractor” [29, 12]. Let us make a brief aside to describe these objects: Suppose that \( \xi^\mu \) is any vector field. Then we can form a weight \( w = 1 \) tractor

\[ V^M = \begin{pmatrix} 0 \\ \xi^m \\ -\frac{1}{d} \nabla_\mu \xi^\mu \end{pmatrix} \]

subject to \( X.V = D.V = 0 \). In turn we may build an adjoint tractor

\[ V^{MN} = \frac{1}{d} D^{[M [V^N]} = \begin{pmatrix} 0 & \xi^n & -\frac{1}{d} \nabla_\mu \xi^\mu \\ a/s & \nabla^{[m} \xi^{n]} & \frac{1}{2d} \left( [\Delta + P] \xi^m - \frac{d+2}{d} [\nabla^m \nabla_\mu + d P_\mu^m] \xi^\mu \right) \\ a/s & a/s & 0 \end{pmatrix}. \]

The operator

\[ \frac{1}{2} V^{MN} D_{NM} = \xi^\mu D_\mu - \frac{w}{d} (\nabla_\mu \xi^\mu), \]
may be viewed as a tractor analog of the vector field $\xi^\mu \partial_\mu$. Notice that acting on weight $w$ scalars, it gives the correct transformation law for a conformal isometry

$$\delta \varphi = (\xi^\mu \partial_\mu - \frac{w}{d}[\nabla_\mu \xi^\mu]) \varphi.$$ 

It is not difficult to verify that $\Re(\Xi_1 \Gamma^{MN} \Xi_2)$ corresponds to $V^{MN}$ with $\xi^\mu$ given by the Killing vector $\sqrt{2}\Re(\bar{\epsilon}_1 \gamma^\mu \bar{\epsilon}_2)$. This shows that acting on $\varphi$, the supersymmetry algebra closes onto isometries.

To close the algebra on the fermions we need first to understand how (conformal) isometries act on (tractor) spinors. In the work [20], the double $D$-operator was related to the generators of ambient Lorentz transformations. This suggests that, acting on tractors of arbitrary tensor type, we should introduce the operator

$$\mathcal{L} = \frac{1}{2} V^{MN} \left[ D_{NM} + S_{MN} \right],$$

where $S_{MN}$ are the ambient intrinsic spin generators. On spinors we have

$$S_{MN} = \frac{1}{2} \Gamma_{MN}.$$ 

Indeed, the transformation rule $\delta \Psi = \mathcal{L} \Psi$ for a weight $w$ tractor spinor $\Psi$ with top slot $\psi$ implies

$$\delta \psi = (\mathcal{L}_\xi - \frac{w\psi}{d}[\nabla_\mu \xi^\mu]) \psi,$$

where the Lie derivative on spinors is $\mathcal{L}_\xi \psi = (\xi^\mu \nabla_\mu + \frac{1}{4} \gamma^{\mu\nu}[\nabla_\mu \xi_\nu]) \psi$.

To obtain a closed, offshell supersymmetry algebra for the fermions we need to introduce auxiliary fields. Since the off-shell bosonic and fermionic field contents must balance, the details will depend on the dimensionality. Therefore, for simplicity, we now restrict ourselves to a four dimensional chiral multiplet with $(z, \psi, F)$ where $z$ and $F$ are complex scalars and $\psi$ is a Majorana spinor. We represent the scalars $z$ and $F$ by weight $w + 1$ and $w$ tractor scalars with the same names while $\psi$ is the top slot of a weight $w$ tractor spinor $\Psi$ subject to $\Pi_+ \Psi = 0$. Notice, this implies that independent spinor field content is characterized by

$$\Gamma.X \Psi = \sqrt{2} \begin{pmatrix} 0 \\ \psi \end{pmatrix}.$$ 

---

*It could be interesting and natural in our framework to study extensions where the supersymmetry algebra closes onto conformal isometries.*
It is therefore sufficient (and simplifying) to specify the transformation rules of \( \Gamma X \Psi \) in what follows (note also that the operators \( \Gamma X \) and \( \xi \) commute).

The supersymmetry transformations of our tractor chiral multiplet then read

\[
\delta z = \Xi \Gamma X L \Psi,
\]

\[
\delta (\Gamma X L \Psi) = \Gamma X L \left( F + \frac{1}{8} \Gamma Dz \right) \Xi,
\]

\[
\delta F = -\frac{1}{d+2w+2} \Xi \Gamma D\Gamma X \Psi.
\]

In components, these transformation rules agree with the usual ones for a massive Wess-Zumino model in an AdS background. It is important to note, however, since the tractor system treats the massive and massless systems on the same footing, the auxiliary field used here differs from the standard one by terms linear in the complex scalar \( z \).

The rules for the complex conjugates are given by replacing \( L \rightarrow R \), or explicitly

\[
L,R = \frac{1}{2} (1 \pm \gamma^7), \quad L,R = (L,R, R,L).
\]

In components, these transformation rules agree with the usual ones for a massive Wess-Zumino model in an AdS background. It is important to note, however, since the tractor system treats the massive and massless systems on the same footing, the auxiliary field used here differs from the standard one by terms linear in the complex scalar \( z \).

The supersymmetry algebra closes. After some algebra it follows that

\[
[\delta_1, \delta_2] (\Gamma X L \Psi) = \xi \Gamma X L \Psi,
\]

\[
[\delta_1, \delta_2] F = \xi F,
\]

proving that the supersymmetry algebra closes.

With a closed supersymmetry algebra, an invariant action principle is easily obtained in tractors:

\[
S = \int \sqrt{-g} \sigma d^{d+2w} (L \text{kin} + L \text{int})
\]

(36)

where
\[ \mathcal{L}_{\text{kin}} = \frac{1}{2\sigma} \bar{\Psi} \Gamma.X I_M \Gamma N D^{MN} \Psi - \frac{1}{\sigma} \bar{z} I.Dz + |F|^2 - 2a(w + 2)[(F - \bar{F})(z - \bar{z}) + a(2w + 5)(z - \bar{z})^2], \tag{37} \]

\[ \mathcal{L}_{\text{int}} = \frac{1}{2} \bar{\Psi} \Gamma.X (\mathcal{L}W'' + \mathcal{R}\bar{W}'')\Psi - FW' - \bar{F}\bar{W}' - 2a(w + 1)(zW' + \bar{z}\bar{W}') - 6a(W + \bar{W}). \tag{38} \]

Notice the appearance of the weight $-1$ scalar $a$ in the weight $2w$ lagrangian density. At arbitrary scales

\[ a = \frac{\sqrt{\mathcal{T} \mathcal{I}}}{2\sigma}, \]

while in a canonical choice of scale it is related to the four dimensional cosmological constant by $12a^2 = -\Lambda$.

The action is split into a kinetic and interacting pieces. The latter depends linearly on a weight $2w + 1$ holomorphic potential $W = W(z, \sigma, a)$. Since our tractor theories describe massive and massless excitations uniformly in terms of weights, the above splitting is not the canonical one into a massless action plus potential terms, but rather uses the freedom of the function $W$ to split the action into free (generically massive) and interacting pieces. At $w = -2$, the Lagrangian density $\mathcal{L}_{\text{kin}}$, expressed in components at the canonical choice of scale recovers the massless part of the supersymmetric AdS Wess-Zumino model quoted in [31, 32].

### 8 Conclusions

It should by now be clear that free, interacting, and supersymmetric classical field theories can be manifestly formulated independently of choices of local unit systems using Weyl invariance. This viewpoint clarifies the origins of masses in field theories, particular in curved spaces where it is necessary to survey all possible couplings to the background geometry and scale (rather than just the compensating mechanism alone) to obtain the theories we have described here.

The above results are all classical, but in fact the greatest impact of our ideas may be to quantization. At the quantum level, scale invariance is
anomalous while our classical approach gauges this symmetry and promotes curved Riemannian backgrounds $g_{\mu\nu}$ to conformal equivalence classes $[g_{\mu\nu}, \sigma]$. There are strong reasons to believe that this approach can be very fruitful based on ideas coming from the AdS/CFT correspondence [33, 34, 35, 36] where renormalization group flows can be formulated holographically [3] and scale or Weyl anomalies become geometric [37]. Indeed conformal geometry computations of Poincaré metrics [38] can be used to obtain physical information about these anomalies [3]. At the very least the tractor techniques provide a powerful machinery for these types of computations, and optimally can provide deep insights into the AdS/CFT correspondence itself.

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A Doubled Reduction

There is a rather explicit relationship between the tractor theories we write down (when the background is conformally flat) and log radial reductions from massless theories in $(d + 1)$-dimensional flat Lorentzian spaces to massive ones in $d$-dimensional (anti) de Sitter spaces [4, 5]. The mathematical underpinning of this relationship is the connection between conformal and projective structures [10]. In fact, the independent field content of our tractor models, (which typically inhabit the top and middle slots of tractor fields) corresponds precisely to that of these log radial reductions. Therefore, for completeness, in this Appendix we present the log radial reduction for spinor theories.

For spinor theories there are two possible reduction schemes depending on how Dirac matrices in adjacent dimensions are handled. One approach is to write the $(d + 1)$-dimensional Dirac matrices as $\Gamma^M = (-i\Gamma^{d+1}\gamma^m, \Gamma^{d+1})$ where $\gamma^m$ then obey a $d$-dimensional Clifford algebra. Alternatively, beginning with the $d$-dimensional Dirac matrices $\gamma^m$, the $(d+1)$-dimensional Dirac matrices are then built by doubling, namely $\Gamma^M = (\sigma_z \otimes \gamma^m, \sigma_z \otimes 1)$. Irreducibility of the spinor representations produced in these ways depends on the dimensionality $d$ and metric signature, but these details do not concern
us here. Either approach can be related to tractors, but the latter approach (which we adopt here) produces a doubled set of equations for which this relationship is simplest—we shall call it a “doubled reduction”.

We start by writing the flat metric in log radial coordinates

$$ds_{\text{flat}}^2 = dX^M G_{MN} dX^N = e^{2u}(du^2 + ds_{\text{dS}}) = E^A \eta_{AB} E^B.$$  \hspace{1cm} (39)

Here, we consider the case where the underlying manifold is de Sitter for reasons of simplicity (the corresponding AdS computation is not difficult either). Notice that the indices $M, N, \ldots$ and $A, B, \ldots$ are not tractor indices but rather $(d+1)$-dimensional curved and flat ones, respectively, while $\mu, \nu, \ldots$ and $m, n, \ldots$ are $d$-dimensional. Note that in this background $P = \frac{d}{2}$. The $(d+1)$-dimensional vielbeine are

$$E^5 = e^u du, \quad E^m = e^u e^m,$$  \hspace{1cm} (40)

where $e^m$ is the $d$-dimensional de Sitter vielbein. The de Sitter spin connection $\omega_{mn}$ obeys

$$de^m + \omega^{mn} \wedge e_n = 0,$$  \hspace{1cm} (41)

in terms of which the $(d+1)$-dimensional flat spin connection reads

$$\Omega^{mn} = \omega^{mn}, \quad \Omega^5 = -e^m.$$  \hspace{1cm} (42)

We use the label 5 to stand for the reduction direction so that $A = (m, 5)$ and $M = (\mu, u)$. The $(d+1)$-dimensional Dirac matrices and covariant derivative acting on spinors are

$$\Gamma^A = \left( \begin{array}{cc} \gamma^m & 0 \\ 0 & -\gamma^m \end{array} \right), \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\nabla_M = \left( \begin{array}{cc} \nabla^\mu & \frac{1}{2} \gamma^\mu \\ -\frac{1}{2} \gamma^\mu & \nabla^\mu \end{array} \right), \quad \begin{pmatrix} \partial_\mu & 0 \\ 0 & \partial_u \end{pmatrix},$$  \hspace{1cm} (43)

where $\nabla_\mu$ on the right hand side of the second line is the de Sitter covariant derivative and $\gamma^m$ are the $d$-dimensional Dirac matrices. For future use we call

$$\tilde{\nabla}_\mu = \left( \begin{array}{cc} \nabla^\mu & \frac{1}{2} \gamma^\mu \\ -\frac{1}{2} \gamma^\mu & \nabla^\mu \end{array} \right).$$  \hspace{1cm} (44)
Acting on spinors $[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu] = 0$. The Dirac conjugate is defined as $\bar{\Psi} = i\Psi^\dagger \Gamma^0$. We now have enough technology to start the doubled reduction. As a warm up, we consider a Dirac spinor

$$\Psi = \begin{pmatrix} \psi \\ \chi \end{pmatrix}, \quad (45)$$

with action principle

$$S_{1/2} = -\int \sqrt{-G} \, d^{d+1}x \, \bar{\Psi} \Gamma^M \nabla_M \Psi. \quad (46)$$

Upon making the field redefinition

$$\Psi = e^{-\frac{ud}{2}} \begin{pmatrix} \psi \\ \chi \end{pmatrix}, \quad (47)$$

all explicit $u$ dependence disappears and the action becomes

$$S_{1/2} = -\int dud^dx \sqrt{-g} (\bar{\psi}, -\bar{\chi}) \left( \frac{\bar{\Psi}}{\partial_u} \frac{\partial_u}{-\bar{\Psi}} \right) (\psi, \chi). \quad (48)$$

Varying this action, defining ($a l\'a$ Scherk and Schwarz [39])

$$\partial_u = w + \frac{d}{2},$$

and redefining $\chi$ by multiplying it with $\sqrt{2}$, we recover the pair of equations of motion following from the tractor computation (18) in a canonical choice of scale. Having warmed up on spin 1/2, our next task is the spin 3/2 doubled reduction.

The $(d + 1)$-dimensional Rarita-Schwinger action is

$$S_{3/2} = -\int \sqrt{-G} \, d^{d+1}x \, \bar{\Psi}_M \Gamma^{MNR} \nabla_N \Psi_R. \quad (49)$$

Using the log radial coordinates (39) and rescaling fields

$$\Psi_M = e^{-\frac{u(d-2)}{2}} \begin{pmatrix} \psi_M \\ \chi_M \end{pmatrix}, \quad (50)$$
the action becomes devoid of explicit $u$ dependence and implies the doubled set of equations of motion

\[
\left[\begin{array}{cc}
\gamma_{\mu\rho} & 0 \\
0 & -\gamma_{\mu\rho}
\end{array}\right] \hat{\nabla}_\nu - \left(\begin{array}{cc}
0 & \gamma_{\mu\rho} \\
\gamma_{\mu\rho} & 0
\end{array}\right) \left(\partial_u - \frac{d}{2} + 1\right) \left(\begin{array}{c}
\psi_{\rho} \\
\chi_{\rho}
\end{array}\right) = 0. \tag{51}
\]

(Note that the $\Psi_u$ variation simply implies the constraint that is a consequence of the above equation.) Redefining the $\chi$ field as before, and equating $\partial_\mu = w + \frac{d}{2}$, the above equation agrees with the tractor Rarita-Schwinger (23) explicated in the canonical choice of scale.

References

[1] A. R. Gover, A. Shaukat and A. Waldron, Nucl. Phys. B 812, 424 (2009) [arXiv:0810.2867 [hep-th]].

[2] A. R. Gover, A. Shaukat and A. Waldron, Phys. Lett. B 675, 93 (2009) [arXiv:0812.3364 [hep-th]].

[3] J. de Boer, E. P. Verlinde and H. L. Verlinde, JHEP 0008, 003 (2000) [arXiv:hep-th/9912012].

[4] T. Biswas and W. Siegel, JHEP 0207, 005 (2002) [arXiv:hep-th/0203115].

[5] K. Hallowell and A. Waldron, Nucl. Phys. B 724, 453 (2005) [arXiv:hep-th/0505255].

[6] H. Weyl, *Gravitation und Elektrizität*. Sitzungsber. Akademie der Wissenschaften Berlin, 1918.

[7] B. Zumino, “Effective Lagrangians and Broken Symmetries”, Lectures on Elementary Particles and Quantum Field Theory, Brandeis University Summer Institute, 2, 437 (1970).

[8] S. Deser, Ann. Phys. 59, 248 (1970).

[9] T. Y. Thomas, Proc. N.A.S., 12, 352 (1926); The Differential Invariants of Generalized Spaces, Cambridge University Press, Cambridge, 1934.
[10] T. N. Bailey, M. G. Eastwood and A. R. Gover, Rocky Mtn. J. Math. 24, 1 (1994).
[11] A. R. Gover, Adv. Math. 163, 206 (2001).
[12] A. R. Gover and L. J. Peterson, Commun. Math. Phys. 235, 339 (2003).
[13] A. Čap and A. R. Gover, Ann. Glob. Anal. Geom. 24, 231 (2003).
[14] C.R. Graham and M. Zworski, Invent. Math. 152, 89, (2003) [arXiv:math/0109089v1].
[15] A. R. Gover, SIGMA 3, 21 (2007).
[16] P. Breitenlohner and D. Z. Freedman, Annals Phys. 144, 249 (1982); Phys. Lett. B 115, 197 (1982).
[17] L. Mezincescu and P. K. Townsend, Annals Phys. 160, 406 (1985).
[18] H. Friedrich, Gen. Rel. and Grav. 8, 303 (1977).
[19] T. Branson, “Clifford bundles and Clifford algebras” in Lectures on Clifford (geometric) algebras and applications, Birkhäuser Boston, 2004; “Conformal structure and spin geometry” in Dirac operators: yesterday and today Int. Press, Somerville 2005.
[20] A. R. Gover and A. Waldron, “The so(d+2,2) Minimal Representation and Ambient Tractors: the Conformal Geometry of Momentum Space,” arXiv:0903.1394 [hep-th].
[21] M. A. Vasiliev, “Bosonic conformal higher-spin fields of any symmetry,” arXiv:0909.5226 [hep-th].
[22] X. Bekaert and M. Grigoriev, “Manifestly conformal descriptions and higher symmetries of bosonic singletons,” arXiv:0907.3195 [hep-th].
[23] P. A. M. Dirac, Annals Math. 37, 429 (1936).
[24] S. Deser and R. I. Nepomechie, Phys. Lett. B 132, 321 (1983); Annals Phys. 154, 396 (1984).
[25] A. Higuchi, Nucl. Phys. B 282, 397 (1987); Nucl. Phys. B 325, 745 (1989); J. Math. Phys. 28, 1553 (1987) [Erratum-ibid. 43, 6385 (2002)].
[26] S. Deser and A. Waldron, Phys. Rev. Lett. 87, 031601 (2001) [arXiv:hep-th/0102166]; Nucl. Phys. B 607, 577 (2001) [arXiv:hep-th/0103198]; Phys. Lett. B 508, 347 (2001) [arXiv:hep-th/0103255]; Phys. Lett. B 513, 137 (2001) [arXiv:hep-th/0105181]; Nucl. Phys. B 662, 379 (2003) [arXiv:hep-th/0301068];

[27] S. Deser and A. Waldron, Nucl. Phys. B 631, 369 (2002) [arXiv:hep-th/0112182].

[28] P. K. Townsend, Phys. Rev. D 15, 2802 (1977).

[29] A. Čap J. Eur. Math. Soc. 10, 415(2008) 415. F. Leitner, “A remark on conformal SU(p,q) holonomy”, in “Symmetries and overdetermined systems of partial differential equations” Ed. Michael Eastwood, Willard Miller, Springer 2007.

[30] E. A. Ivanov and A. S. Sorin, “Wess-Zumino Model As Linear Sigma Model Of Spontaneously Broken Conformal Sov. J. Nucl. Phys. 30, 440 (1979) [Yad. Fiz. 30, 853 (1979)].

[31] D. W. Dusedau and D. Z. Freedman, Phys. Rev. D 33, 395 (1986).

[32] C. J. C. Burges, D. Z. Freedman, S. Davis and G. W. Gibbons, Annals Phys. 167, 285 (1986).

[33] J. M. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998) [Int. J. Theor. Phys. 38, 1113 (1999)] [arXiv:hep-th/9711200].

[34] E. Witten, Adv. Theor. Math. Phys. 2, 253 (1998) [arXiv:hep-th/9802150].

[35] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. B 428, 105 (1998) [arXiv:hep-th/9802109].

[36] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, Phys. Rept. 323, 183 (2000) [arXiv:hep-th/9905111].

[37] M. Henningson and K. Skenderis, JHEP 9807, 023 (1998) [arXiv:hep-th/9806087].
[38] C. Fefferman and C.R. Graham, Conformal Invariants, in Elie Cartan et les Mathématiques aujourd'hui (Astérisque, 1985) 95.

[39] J. Scherk and J. H. Schwarz, Nucl. Phys. B 153, 61 (1979).