ON A MULTI-POINT INTERPOLATION PROBLEM FOR GENERALIZED SCHUR FUNCTIONS

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Abstract. The nondegenerate Nevanlinna-Pick-Carathéodory-Fejér interpolation problem with finitely many interpolation conditions always has infinitely many solutions in a generalized Schur class \( S^\kappa \) for every \( \kappa \geq \kappa_{\text{min}} \) where the integer \( \kappa_{\text{min}} \) equals the number of negative eigenvalues of the Pick matrix associated to the problem and completely determined by interpolation data. A linear fractional description of all \( S^\kappa_{\text{min}} \) solutions of the (nondegenerate) problem is well known. In this paper, we present a similar result for an arbitrary \( \kappa \geq \kappa_{\text{min}} \).

Dedicated to Professor Joseph Ball on occasion of his 60-th birthday

1. Introduction

Let \( S \) stand for the Schur class of analytic functions mapping the unit disk \( \mathbb{D} \) into \( \mathbb{D} \) and let \( B_\kappa \) be the set of finite Blaschke products of degree \( \kappa \). We denote by \( S^\kappa \) the generalized Schur class of meromorphic functions of the form

\[
f(z) = \frac{s(z)}{b(z)},
\]

where \( s \in S \) and \( b \in B_\kappa \) do not have common zeros (in particular, \( S^0 = S \)). Formula (1.1) is called the Krein-Langer representation of a generalized Schur function \( f \); the entries \( s \) and \( b \) are determined by \( f \) uniquely up to a unimodular constant. Via nontangential boundary limits, the \( S^\kappa \)-functions can be identified with the functions from the unit ball of \( L^\infty(\mathbb{T}) \) which admit meromorphic continuation inside the unit disk with total pole multiplicity equal \( \kappa \). On the other hand, the \( S^\kappa \)-functions \( f \) can be characterized as meromorphic functions on \( \mathbb{D} \) for which the associated kernel

\[
K_f(z, \zeta) := \frac{1 - f(z)f(\zeta)}{1 - z\bar{\zeta}}
\]

has \( \kappa \) negative squares on \( \rho(f) \), the domain of analyticity of \( f \); \( \text{sq}^{-}(K_f) = \kappa \).

The classes \( S_\kappa \) were thoroughly studied in \([9, 10]\), the major interpolation results for \( S_\kappa \)-functions can be found in \([1, 2, 3, 4]\). The objective of this paper is the Nevanlinna-Pick-Carathéodory-Fejér interpolation problem which will be denoted by \( \text{IP}_\kappa \) and which consists of the following:

\( \text{IP}_\kappa \): Given an integer \( \kappa \geq 0 \), distinct points \( z_1, \ldots, z_k \in \mathbb{D} \), a tuple \( n = (n_1, \ldots, n_k) \in \mathbb{N}^k \) and \( |n| := n_1 + \ldots + n_k \) complex numbers \( f_{i,j} \) \((0 \leq j \leq n_i - 1; \ 1 \leq i \leq k)\), find all functions \( f \in S_\kappa \) (if exist) which are analytic at \( z_i \) and satisfy

\[
f^{(j)}(z_i) = j! f_{i,j} \quad (i = 1, \ldots, k; \ j = 0, \ldots, n_i - 1).
\]
Necessary and sufficient conditions for the $\mathbf{IP}_\kappa$ to have a solution can be given in terms of the Pick matrix of the problem which is determined from interpolation data as follows. Let $J_n(z)$ denote the $n \times n$ Jordan block with the number $z$ on the main diagonal and let $E_n$ stand for the column vector of the height $n$ with the first coordinate equals one and other coordinates equal zero:

$$
J_n(z) = \begin{bmatrix}
z & 1 & 0 \\
0 & z & \ddots \\
\vdots & \ddots & \ddots & 1 \\
0 & \ldots & 0 & z
\end{bmatrix}, \quad E_n = \begin{bmatrix}1 \\
0 \\
\vdots \\
0
\end{bmatrix}.
$$

Associated with the tuples $\mathbf{z} = (z_1, \ldots, z_k)$ and $\mathbf{n} = (n_1, \ldots, n_k)$ are the matrices

$$
T = \begin{bmatrix}
J_{n_1}(z_1) & \ldots & 0 \\
0 & \ddots & \ddots \\
\vdots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & J_{n_k}(z_k)
\end{bmatrix}, \quad E = \begin{bmatrix}E_{n_1} \\
\vdots \\
E_{n_k}
\end{bmatrix}, \quad (1.4)
$$

and we arrange the rest of data in the column-vector

$$
C = \begin{bmatrix}
C_1 \\
\vdots \\
C_k
\end{bmatrix}, \quad \text{where} \quad C_i = \begin{bmatrix}f_{i,0} \\
\vdots \\
f_{i,n_i-1}
\end{bmatrix}. \quad (1.5)
$$

Since all the eigenvalues of $T$ fall inside the unit disk, the Stein equation

$$
P - TPT^* = EE^* - CC^* \quad (1.6)
$$

has a unique solution $P$ which is defined via the converging series

$$
P = \sum_{j=0}^{\infty} T^j (EE^* - CC^*) T^{*j} \quad (1.7)
$$

and which is called the Pick matrix of the problem $\mathbf{IP}_\kappa$. A necessary condition for the $\mathbf{IP}_\kappa$ to have a solution can be obtained as follows. Given an $f \in \mathcal{S}_\kappa$, an integer $k \geq 0$ and two $k$-tuples $\mathbf{z} = (z_1, \ldots, z_k) \in \rho(f)^k$ and $\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{N}^k$, we define the column-vector

$$
M_n(f, \mathbf{z}) = \begin{bmatrix}M_{n_1}(f; z_1) \\
\vdots \\
M_{n_k}(f; z_k)
\end{bmatrix}, \quad \text{where} \quad M_{n_i}(f; z_i) = \begin{bmatrix}f(z_i) \\
f'(z_i) \\
\vdots \\
\frac{f^{(n_i-1)}(z_i)}{(n_i-1)!}
\end{bmatrix}. \quad (1.8)
$$

and the $|\mathbf{n}| \times |\mathbf{n}|$ Schwarz-Pick matrix

$$
P_n(f; \mathbf{z}) = -\frac{1}{4\pi^2} \int_{\Gamma} \int_{\Gamma} (\xi - T)^{-1} E K(f, \xi) E^* (\omega - T^*)^{-1} d\xi d\omega, \quad (1.9)
$$

where $T$ and $E$ are given in (1.4) and where $\Gamma \in \mathbb{D}$ is any contour enclosing the points $z_1, \ldots, z_k$ and such that $\text{Int } \Gamma \subset \rho(f)$. Since $\text{sq}_-(Kf) = \kappa$, the standard approximation arguments show that the the matrix $P_n(f; \mathbf{z})$ has at most $\kappa$ negative eigenvalues:
Then the problem conditions (1.3), it follows from (1.5) and (1.10) that \( M \) follows from definitions (1.8) and (1.4) by residue calculus; using this equality, one can readily check that the matrix \( P_\kappa(f; z) \) defined in (1.9) satisfies the Stein equation

\[
P_\kappa(f; z) - TP_\kappa(f; z)T^* = EE^* - M \kappa(f; z)M^* \kappa(f; z)^*.
\]

Now we observe that for every solution \( f \) of the problem \( \text{IP}_\kappa \), the Schwarz-Pick matrix \( P_\kappa(f; z) \) is equal to \( P \), the Pick matrix of the problem (indeed, if \( f \) satisfies interpolation conditions (1.3), it follows from (1.5) and (1.10) that \( M \kappa(f; z) = C \); thus \( P \) and \( P_\kappa(f; z) \) satisfy the same Stein equation which in turn, has a unique solution). In particular, if \( P \) has more than \( \kappa \) negative eigenvalues, the problem \( \text{IP}_\kappa \) has no solutions. Thus, condition \( \kappa \geq \text{sq}_-(P) \) is necessary for the \( \text{IP}_\kappa \) to have a solution. On the other hand, if

\[
\kappa \geq \text{sq}_-(P) \quad \text{and} \quad \det P \neq 0,
\]

then the problem \( \text{IP}_\kappa \) has infinitely many solutions, which are parametrized by a linear fractional transformation. This is the main result of the paper.

**Theorem 1.1.** Let the Pick matrix \( P \) of the \( \text{IP}_\kappa \) meet conditions (1.11) and let \( \Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \) be the \( 2 \times 2 \) rational matrix-valued function defined by

\[
\Theta(z) = I_2 + (z - 1) \begin{bmatrix} E^* \\ C^* \end{bmatrix} (I - zT^*)^{-1}P^{-1}(I - T)^{-1} \begin{bmatrix} E & -C \end{bmatrix}.
\]

Then all solutions \( f \) of \( \text{IP}_\kappa \) are parametrized by the linear fractional transformation

\[
f(z) = \frac{\Theta_{11}(z)S(z) + \Theta_{12}(z)B(z)}{\Theta_{21}(z)S(z) + \Theta_{22}(z)B(z)},
\]

where the parameters \( S \in \mathcal{S} \) and \( B \in \mathcal{B}_{\kappa - \text{sq}_-(P)} \) do not have common zeros and satisfy conditions

\[
\Theta_{21}(z_i)S(z_i) + \Theta_{22}(z_i)B(z_i) \neq 0 \quad (i = 1, \ldots, k).
\]

Equivalently,

\[
f(z) = T_\Theta[\mathcal{E}] := \frac{\Theta_{11}(z)\mathcal{E}(z) + \Theta_{12}(z)}{\Theta_{21}(z)\mathcal{E}(z) + \Theta_{22}(z)},
\]

where the parameter \( \mathcal{E} \in \mathcal{S}_{\kappa - \text{sq}_-(P)} \) satisfies

\[
\Theta_{21}(z_i)\mathcal{E}(z_i) + \Theta_{22}(z_i) \neq 0 \quad (i = 1, \ldots, k)
\]

or has a pole at \( z_i \) in case \( \Theta_{21}(z_i) \neq \Theta_{22}(z_i) = 0 \).

Equivalence of descriptions (1.13) and (1.15) is established via the Krein-Langer representation \( \mathcal{E} = \frac{S}{B} \) of the function \( \mathcal{E} \in \mathcal{S}_{\kappa - \text{sq}_-(P)} \). If \( \kappa \) is minimally possible (i.e., if \( \kappa = \text{sq}_-(P) \)), then the parameter \( \mathcal{E} \) in (1.15) runs through the Schur class \( \mathcal{S} \); this result can be found in [2, 3, 4]. A somewhat new point presented here is that in case \( \kappa > \text{sq}_-(P) \), some solutions of the problem may arise via formula (1.15) from parameters which are not analytic at interpolation nodes. We illustrate this possibility by a numerical example.
Example 1.2. Let \( z_1 = 0, \) \( z_2 = 1/2, \) \( f_1 = 1 \) and \( f_2 = 1/2 \) so that the interpolation conditions are
\[
f(0) = 1 \quad \text{and} \quad f(1/2) = 1/2. \tag{1.17}
\]
The Pick matrix \( P = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 1 \end{bmatrix} \) of the problem has one negative and one positive eigenvalues and thus, the problem \( \text{IP}_\kappa \) has a solution if and only if \( \kappa \geq 1. \) Furthermore, substituting
\[
P^{-1} = \begin{bmatrix} -4 & 2 \\ 2 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 0 & 0 \\ 0 & 1/2 \end{bmatrix}, \quad E = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \end{bmatrix}
\]
into (1.12) gives
\[
\Theta(z) = \frac{1}{2-z} \begin{bmatrix} 3z - 2 & 2z(1-z) \\ 2(z-1) & z(3-2z) \end{bmatrix}.
\]
By Theorem 1.1 all solutions of the problem \( \text{IP}_1 \) with interpolation conditions (1.17) are parametrized by the formula
\[
f(z) = \frac{(3z - 2)E(z) + 2z(1-z)}{2(z-1)E(z) + z(3-2z)}, \tag{1.18}
\]
where \( E \) belongs to \( S_0 \) and satisfies
\[
E(0) \neq 0 \quad \text{and} \quad E(1/2) \neq 1. \tag{1.19}
\]
To get all solutions of the problem \( \text{IP}_2 \) with interpolation conditions (1.17), we use the formula (1.18) with parameters \( E \in S_1 \) which are analytic at \( z_1 = 0 \) and \( z_2 = 1/2 \) and match constraints (1.19). However, since \( \Theta_{22}(0) = 0 \) and \( \Theta_{21}(0) = -2 \neq 0, \) Theorem 1.1 asserts that any function \( E \) which has a pole at \( z_1 = 0 \) and meets the second constraint in (1.19), also leads via (1.18) to a solution \( f \) of the problem.

Our interest to the “non-minimal” problem \( \text{IP}_\kappa \) (where \( \kappa > \text{sq}_-(P) \)) is motivated by the following reason: if the Pick matrix \( P \) of the problem \( \text{IP}_\kappa \) is singular, then the minimally possible \( \kappa \) for which the \( \text{IP}_\kappa \) has a solution, may be greater than \( \text{sq}_-(P) \). As we will show in the follow-up paper, the description of all solutions for such a degenerate problem can be reduced to a family of nondegenerate “non-minimal” problems at which point Theorem 1.1 will come into play. The proof of Theorem 1.1 is presented in Section 2. In Section 3 we will discuss the divisor-remainder formulation of the problem \( \text{IP}_\kappa \) (also considered in [2, 3, 4] for \( \kappa = \text{sq}_-(P) \)).

2. Proof of Theorem 1.1

We first recall some properties of the function \( \Theta \) defined in (1.12). In what follows, \( N\{g\} \) stands for the total number of zeros of a function \( g \) that fall inside \( \mathbb{D} \).

Lemma 2.1. Let \( T, E, C \) and \( P \) be given by (1.4), (1.5) and (1.7), let \( P \) be invertible and let \( \Theta(z) \) be defined as in (1.12). Then

(1) \( \Theta(t) \) is \( J \)-unitary at every point \( t \in \mathbb{T} \):
\[
\Theta(t)^*J\Theta(t) = \Theta(t)J\Theta(t)^* = J := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.1}
\]
(2) For $z, \zeta \in \mathbb{D}$,
\[
\frac{J - \Theta(z)J\Theta(\zeta)^*}{1 - z\zeta} = \left[ \begin{array}{c} E^* \\ C^* \end{array} \right] (I - zT^*)^{-1}P^{-1}(I - \bar{\zeta}T)^{-1} \left[ \begin{array}{c} E \\ C \end{array} \right] \tag{2.2}
\]
and
\[
\det \Theta(z) = \prod_{i=1}^{k} \left( \frac{(z - z_i)(1 - \bar{z}_i)}{(1 - z_i)(1 - \bar{z}_i)} \right)^{n_i}. \tag{2.3}
\]

(3) The function $(zI - T)^{-1} \left[ \begin{array}{c} E \\ -C \end{array} \right] \Theta(z)$ is analytic on $\mathbb{D}$.
(4) The rational functions $\Theta_{11}$ and $\Theta_{22}$ do not vanish on the unit circle and have respectively $sq_+(P)$ and $sq_-(P)$ zeros in $\mathbb{D}$:
\[
N\{\Theta_{11}\} = sq_+(P) \quad \text{and} \quad N\{\Theta_{22}\} = sq_-(P). \tag{2.4}
\]
(5) For every $z \in \mathbb{C}$,
\[
|\Theta_{21}(z)| + |\Theta_{22}(z)| > 0. \tag{2.5}
\]
(6) For every $E \in S_\kappa$, the function $f = T_\Theta[E] := \begin{pmatrix} \Theta_{11}E + \Theta_{12} \\ \Theta_{21}E + \Theta_{22} \end{pmatrix}$ belongs to $S_\kappa$ with $\kappa \leq \kappa + sq_-(P)$.

**Proof:** Identity (2.2) follows by a straightforward calculation (see, e.g., [2, Section 7.1]) based on the identity (1.6). A similar calculation gives
\[
\frac{J - \Theta(\zeta)^*J\Theta(z)}{1 - z\zeta} = \left[ \begin{array}{c} E^* \\ -C^* \end{array} \right] (I - T)^{-1}P^{-1}(I - \bar{\zeta}T)^{-1}P(1 - zT^*)^{-1} \times P^{-1}(I - T)^{-1} \left[ \begin{array}{c} E \\ -C \end{array} \right]. \tag{2.6}
\]

Identities (2.1) follow from (2.2) and (2.6), since $\Theta$ is rational and has no poles on $\mathbb{T}$. Equality (2.3) follows from (1.6) by the standard properties of determinants (see e.g. [5, Lemma 2.2] for the proof). The third statement of the lemma is yet another consequence of identity (1.6) due to which we have
\[
\left[ \begin{array}{c} E \\ -C \end{array} \right] \Theta(z) = \left[ \begin{array}{c} E \\ -C \end{array} \right] + (z - 1) \left[ \begin{array}{c} E \\ -C \end{array} \right] \left[ \begin{array}{c} E^* \\ C^* \end{array} \right] (I - zT^*)^{-1}P^{-1} (I - T)^{-1} \left[ \begin{array}{c} E \\ -C \end{array} \right] \\
\times (I - T)^{-1} \left[ \begin{array}{c} E \\ -C \end{array} \right] = \left(I + (z - 1)(P - TPT^*)(I - zT^*)^{-1}P^{-1}(I - T)^{-1}\right) \left[ \begin{array}{c} E \\ -C \end{array} \right] \\
= (zI - T)P(I - T^*)(I - zT^*)^{-1}P^{-1}(I - T)^{-1} \left[ \begin{array}{c} E \\ -C \end{array} \right].
\]

Therefore,
\[
(zI - T)^{-1} \left[ \begin{array}{c} E \\ -C \end{array} \right] \Theta(z) = P(I - T^*)(I - zT^*)^{-1}P^{-1}(I - T)^{-1} \left[ \begin{array}{c} E \\ -C \end{array} \right]
\]
and the function on the right hand side is analytic on $\mathbb{D}$.

Equalities (2.1) imply in particular
\[
|\Theta_{11}(t)|^2 - |\Theta_{21}(t)|^2 = 1, \quad |\Theta_{21}(t)|^2 - |\Theta_{22}(t)|^2 = -1 \quad (t \in \mathbb{T}) \tag{2.7}
\]
and thus, $\Theta_{11}$ and $\Theta_{22}$ do not vanish on $\mathbb{T}$. For the proof of the second equality in (2.4), we refer to [3, Theorem 13.2.3] or to [7, Lemma 4]. This equality tells us that
if $T$ and $E$ are defined as in (1.4) and $M$ is an arbitrary vector in $\mathbb{C}^{[n]}$ such that the unique solution $R$ (which is necessarily Hermitian) of the Stein equation

$$R - T^*RT = EE^* - MM^*$$

(2.8)
is invertible, then the function

$$F_M(z) = 1 - (z - 1)M^*(I - zT^*)^{-1}R^{-1}(I - T)^{-1}M$$

(2.9)
has $\text{sq}_-(R)$ zeros inside $\mathbb{D}$:

$$N\{F_M\} = \text{sq}_-(R).$$

(2.10)
Let $C$ be the vector associated with the problem $\mathbf{IP}_E$ and decomposed as in (1.5). For an $\varepsilon > 0$, define $C_\varepsilon := C + \varepsilon E$ and the matrix $P_\varepsilon$, a unique solution of the Stein equation

$$P_\varepsilon - TP_\varepsilon T^* = EE^* - C_\varepsilon C_\varepsilon^*.$$ 

(2.11)
Due to the structure (1.4) of $E$, the above perturbation changes only the top entries $f_{i,0}$ in each of the blocks $C_i$ replacing them by $f_{i,0} + \varepsilon$. It is clear that there exists $\varepsilon_0$ so that for every $\varepsilon \in (0, \varepsilon_0)$,

$$\text{sq}_\pm(P_\varepsilon) = \text{sq}_\pm(P) \quad \text{and} \quad f_{i,0} + \varepsilon \neq 0 \quad \text{for} \quad i = 1, \ldots, k.$$

Now we let $C_\varepsilon$ to be the block diagonal matrix with lower triangular Toeplitz diagonal blocks:

$$C_\varepsilon = \begin{bmatrix} C_{\varepsilon,1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & C_{\varepsilon,k} \end{bmatrix}, \quad C_{\varepsilon,i} = \begin{bmatrix} f_{i,0} + \varepsilon & 0 & \cdots & 0 \\ f_{i,1} & f_{i,0} + \varepsilon & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ f_{i,n_i-1} & \cdots & f_{i,1} & f_{i,0} + \varepsilon \end{bmatrix}.$$

It is obvious that $C_\varepsilon$ is invertible and satisfies relations

$$C_\varepsilon E = C_\varepsilon \quad \text{and} \quad C_\varepsilon T = TC_\varepsilon.$$

Multiplying both parts in (2.11) by $C_\varepsilon^{-1}$ on the left and by its adjoint on the right and making use of the two last equalities, we get

$$C_\varepsilon^{-1}P_\varepsilon C_\varepsilon^{-*} - C_\varepsilon^{-1}TP_\varepsilon C_\varepsilon^{-*}T^* = C_\varepsilon^{-1}EE^*C_\varepsilon^{-*} - EE^*$$

which can be written in the form (2.8) upon setting

$$R = -C_\varepsilon^{-1}P_\varepsilon C_\varepsilon^{-*} \quad \text{and} \quad M = C_\varepsilon^{-1}E.$$

For this setting, the formula (2.9) takes the form

$$F_\varepsilon(z) = 1 + (z - 1)E^*C_\varepsilon^{-*}(I - zT^*)^{-1}(C_\varepsilon^{-1}P_\varepsilon C_\varepsilon^{-*})^{-1}(I - T)^{-1}C_\varepsilon^{-1}E$$

$$= 1 + (z - 1)E^*(I - zT^*)^{-1}P_\varepsilon^{-1}(I - T)^{-1}E$$

from which it follows that

$$\lim_{\varepsilon \to 0} F_\varepsilon(z) = 1 + (z - 1)E^*(I - zT^*)^{-1}P^{-1}(I - T)^{-1}E = \Theta_{11}(z).$$

Due to (2.10) we have

$$N\{F_\varepsilon\} = \text{sq}_-(C_\varepsilon^{-1}P_\varepsilon C_\varepsilon^{-*}) = \text{sq}_+(C_\varepsilon^{-1}P_\varepsilon C_\varepsilon^{-*}) = \text{sq}_+(P_\varepsilon) = \text{sq}_+(P).$$
Due to the Jordan structure (2.2) of $T$ which is a contradiction. Thus, inequality (2.5) holds for $z$ solution of the problem and then we have

$$0 = \Theta_{21}(z_1)E^* + \Theta_{22}(z_1)C^*$$

$$= C^* + (z_1 - 1)C^*(I - z_1T^*)^{-1}P^{-1}(I - T)^{-1}(EE^* - CC^*)$$

$$= C^* + (z_1 - 1)C^*(I - z_1T^*)^{-1}P^{-1}(I - T)^{-1}(P - TPT^*)$$

$$= C^*(I - z_1T^*)^{-1}P^{-1}(z_1I - T)(I - T)^{-1}P(I - T^*)$$

which is equivalent to

$$0 = C^*(I - z_1T^*)^{-1}P^{-1}(z_1I - T).$$

Due to the Jordan structure (2.2) of $T$ and since $z_1 \neq z_i$ for $i = 2, \ldots, d$, it follows from the last equality that the row-vector $C^*(I - z_1T^*)^{-1}P^{-1}$ must be of the form

$$C^*(I - z_1T^*)^{-1}P^{-1} = [\alpha \quad 0 \quad \ldots \quad 0]$$

and then we have

$$[0 \quad 0] = \begin{bmatrix} \Theta_{21}(z_1) & \Theta_{22}(z_1) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix} + (z_1 - 1) \begin{bmatrix} \alpha & 0 & \ldots & 0 \end{bmatrix} (I - T)^{-1} \begin{bmatrix} E & -C \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix} - \alpha \begin{bmatrix} 1 & -f_1,0 \end{bmatrix} = [-\alpha \quad 1 + \alpha f_1,0]$$

which is a contradiction. Thus, inequality (2.5) holds for $z = z_1$ and similarly for $z_2, \ldots, z_d$ which completes the proof of the fifth statement.

Finally, let us observe that the function $f = T\Theta[\mathcal{E}]$ is well defined, i.e., that the function $G_{\mathcal{E}} = \Theta_{21}\mathcal{E} + \Theta_{22}$ does not vanish identically for any generalized Schur function $\mathcal{E}$. Indeed, if $G_{\mathcal{E}} \equiv 0$, then the function $-\frac{\Theta_{22}}{\Theta_{21}} = \mathcal{E}$ belongs to a generalized Schur class which is impossible, since due to the second equality in (2.7), $\left|\frac{\Theta_{22}(t)}{\Theta_{21}(t)}\right| > 1$ at every $t \in \mathbb{T}$. Now the last statement in the lemma follows from the identity

$$G(z)K_f(z, \zeta)G(\zeta)^* = K_\mathcal{E}(z, \zeta) + \left[\mathcal{E}(\zeta)^* \quad 1 - \Theta(\zeta)^* J\Theta(z) \right] \left[\begin{array}{c} \mathcal{E}(z) \\frac{1 - \Theta(\zeta)^* J\Theta(z)}{1 - z\zeta} \end{array}\right]$$

since $\text{sq}_-(\frac{J - \Theta(\zeta)^* J\Theta(z)}{1 - z\zeta}) \leq \text{sq}_-(P)$, by (2.6). 

For notational convenience, in what follows we will often write $f^*$ rather than $\overline{f}$.

**Theorem 2.2.** Let $P$ satisfy conditions (1.11), let $\Theta$ be given by (1.12) and let $f$ be a solution of the problem $\text{IP}_\kappa$. Then

1. The kernel

$$K_f(z, \zeta) = \left[\begin{array}{c} P \quad (I - \zeta T)^{-1}(E - Cf(z)^*) \end{array} \right]$$

has $\kappa$ negative squares on $\rho(f)$. 


(2) The function $f$ is of the form (1.15) for some $E \in S_{\kappa-sq_{-}(P)}$.

**Proof:** Let $\Gamma \in \mathbb{D}$ be any contour enclosing the points $z_1, \ldots, z_k$ and such that $\text{Int } \Gamma \subset \rho(f)$. Since $sq_{-}(K_f) = \kappa$, the standard approximation arguments show that the kernel

$$
\tilde{K}_f(z, \zeta) = -\frac{1}{4\pi^2} \int_{\Gamma} \int_{\Gamma} \left[ (\xi - T)^{-1}E \right]_{\kappa} K_f(\xi, \omega) \left[ (\omega - \zeta)^{-1} \right] d\xi d\omega
$$

has at most $\kappa$ negative squares. Since $f$ is a solution of the problem $IP_{\kappa}$, we have $P_n(f; \mathbf{z}) = P$ and $M_n(f; \mathbf{z}) = C$ which together with (1.9) and (1.10) lead us to

$$
P = -\frac{1}{4\pi^2} \int_{\Gamma} \int_{\Gamma} (\xi - T)^{-1} EK_f(\xi, \omega)E^* (\omega - T^*)^{-1} d\xi d\omega
$$

and

$$
C = \frac{1}{2\pi i} \int_{\Gamma} (\xi - T)^{-1} Ef(\xi)d\xi.
$$

Using the latter equality along with (1.2) gives

$$
-\frac{1}{4\pi^2} \int_{\Gamma} \int_{\Gamma} (\xi - T)^{-1} EK_f(\xi, \omega) \frac{d\xi d\omega}{\omega - \zeta} = \frac{1}{2\pi i} \int_{\Gamma} (\xi - T)^{-1} E \frac{1-f(\xi)f(\xi)^*}{1-\xi \zeta} d\xi = (I - \bar{\zeta}T)^{-1} E - (I - \bar{\zeta}T)^{-1} C f(\xi)^*.
$$

Finally,

$$
-\frac{1}{4\pi^2} \int_{\Gamma} \int_{\Gamma} \frac{K_f(\xi, \omega) d\xi d\omega}{(\xi - z)(\omega - \zeta)} = K_f(z, \zeta).
$$

Substituting the two last equalities and (2.14) into (2.13) and comparing the resulting matrix with (2.12) we conclude that $K_f(z, \zeta) = \tilde{K}_f(z, \zeta)$. Therefore, $sq_{-}(K_f) \leq \kappa$. On the other hand, it follows from (2.12) that $sq_{-}(K_f) \geq sq_{-}(F_f) = \kappa$ which completes the proof of the first statement of the theorem. To prove the second, we first note that the kernel

$$
S(z, \zeta) = K_f(z, \zeta) - (E^* - f(z)C^*)(I - zT^*)^{-1}P^{-1}(I - \bar{\zeta}T)^{-1} (E - Cf(\xi)^*)
$$

is the Schur complement of the block $P$ in $K_f(z, \zeta)$ and since $sq_{-}(K_f) = \kappa$ by the first part, it follows that

$$
sq_{-}(S) = sq_{-}(K_f) - sq_{-}(P) = \kappa - sq_{-}(P).
$$

Making use of relation

$$
K_f(z, \zeta) = \left[ \begin{array}{c} 1 - f(z) \\ 1 - z \zeta \end{array} \right] \left[ \begin{array}{c} 1 \\ -f(\xi)^* \end{array} \right] \text{ where } J := \left[ \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right],
$$

we represent $S$ in the form

$$
S(z, \zeta) = \left[ \begin{array}{c} 1 - f(z) \\ 1 - z \zeta \end{array} \right] \left\{ \begin{array}{c} J \\ \bar{\zeta} \end{array} \right\} \left[ \begin{array}{c} E^* \\ C^* \end{array} \right] (I - zT^*)^{-1}P^{-1}(I - \bar{\zeta}T)^{-1} \left[ \begin{array}{cc} E & C \end{array} \right] \left[ \begin{array}{c} 1 \\ -f(\xi)^* \end{array} \right].
$$
which in turn, can be written as
\[
S(z, \zeta) = \left[1 - f(z) \right] \frac{\Theta(z)}{1 - z} \left[1 - f(z) \right]^{*} - f(z)^{*},
\]
due to (2.2). The last equality can be written in terms of the pair \(\{u, v\}\) defined by
\[
[v(z) - u(z)] = \left[1 - f(z) \right] \Theta(z)
\]
as
\[
S(z, \zeta) = \left[v(z) - u(z) \right] \frac{J}{1 - z} \left[v(\zeta)^{*} - u(\zeta)^{*} \right] = \frac{v(z)v(\zeta)^{*} - u(z)u(\zeta)^{*}}{1 - z}.
\]

Let us show that \(v(z) \neq 0\). Indeed, by the first equality in (2.7), \(\left|\Theta_{11}(t) \right| > 1\) for every \(t \in \mathbb{T}\). Thus, \(\Theta_{11}(t) \not\in S_{\kappa}\). However, if \(v = \Theta_{11} - f\Theta_{21} \equiv 0\) we have that \(f = \frac{\Theta_{11}}{\Theta_{21}} \in S_{\kappa}\) which is a contradiction. Thus, \(v \neq 0\) and the function \(E = \frac{u}{v}\) is meromorphic on \(\mathbb{D}\).

Equality (2.18) can be written in terms of this function as
\[
S(z, \zeta) = v(z) \frac{1 - E(z)E(\zeta)^{*}}{1 - z} v(\zeta)^{*} = v(z)K_{E}(z, \zeta)v(\zeta)^{*}
\]
which together with (2.15) implies \(sq_{-}(K_{E}) = \kappa - sq_{-}(P)\) so that \(E \in S_{\kappa - sq_{-}(P)}\).

Finally, it follows from (2.17) that
\[
f = \frac{\Theta_{11}u + \Theta_{12}v}{\Theta_{21}u + \Theta_{22}v} = \frac{\Theta_{11}E + \Theta_{12}}{\Theta_{21}E + \Theta_{22}}.
\]
Thus, \(f\) is of the form (1.15) with \(E \in S_{\kappa - sq_{-}(P)}\) which completes the proof of the theorem.

Now we will take a closer look at the numerator and the denominator in the linear fractional formula (1.15). Let \(E\) be a fixed function from \(S_{\kappa}\) with the coprime factorization
\[
E(z) = \frac{S(z)}{B(z)}, \quad S \in S, \quad B \in B_{\kappa},
\]
and let \(\Theta\) be the rational matrix function defined as in (1.12). Let
\[
U_{S, B} = \Theta_{11}S + \Theta_{12}B, \quad V_{S, B}(z) = \Theta_{21}S + \Theta_{22}B,
\]
so that (1.15) takes the form
\[
f(z) = \frac{U_{S, B}(z)}{V_{S, B}(z)}.
\]

For the rest of the paper we assume that \(V_{S, B}\) has zeros at \(z_{i}\) of respective multiplicities \(m_{i} \geq 0\), i.e., that
\[
V_{S, B}(z_{i}) = \ldots = V_{S, B}^{(m_{i})-1}(z_{i}) = 0 \quad \text{and} \quad V_{S, B}^{(m_{i})}(z_{i}) \neq 0 \quad (i = 1, \ldots, k).
\]

Theorem 2.3. Let \(P\) be invertible, let \(S \in S\), \(B \in B_{\kappa}\), let \(\Theta\), \(U_{S, B}\) and \(V_{S, B}\) be given as in (1.12) and (2.20). Then

1. \(N\{V_{S, B}\} = sq_{-}(P) + \kappa\). If in addition, \(S\) is a finite Blaschke product of degree \(m\) (i.e., if \(S \in B_{m}\)), then \(N\{U_{S, B}\} = sq_{+}(P) + m\).
(2) $U_{S,B}$ and $V_{S,B}$ can have a common zero at no point inside $\mathbb{D}$, but $z_1, \ldots, z_d$.
(3) $U_{S,B}$ and $V_{S,B}$ cannot have a common zero at $z_j$ of multiplicity greater than $n_j$.
(4) If $V_{S,B}$ has the zero of multiplicity $m_j > n_j$ at $z_j$, then $U_{S,B}$ has the zero of multiplicity $n_j$ at $z_j$.
(5) If $V_{S,B}$ has the zero of multiplicity $m_j \leq n_j$ at $z_j$, then $U_{S,B}$ has the zero of multiplicity at least $m_j$ at $z_j$.

**Proof:** By the second equality in (2.7), $|\Theta_{22}(t)| > |\Theta_{21}(t)|$ on $\mathbb{T}$ and since $S \in \mathcal{S}$ and $B$ is unimodular on $\mathbb{T}$, it follows that

$$|\Theta_{22}(t)B(t)| > |\Theta_{21}(t)S(t)|$$

at almost every point $t \in \mathbb{T}$. Then, by Rouche’s theorem, the functions $V_{S,B} = \Theta_{21}S + \Theta_{22}B$ and $\Theta_{22}B$ have the same number of zeros in the disk $\{ z : |z| < r \}$ for every $r$ close enough to 1. Since the rational function $\Theta_{22}$ and the finite Blaschke product $B$ have finitely many zeros in $\mathbb{D}$, we let $r \to 1$ to conclude that

$$N\{V_{S,B}\} = N\{\Theta_{22}B\} = N\{\Theta_{22}\} \cdot N\{B\} = \text{sq}_- (P) + \kappa,$$

where the last equality holds since $N\{\Theta_{22}\} = \text{sq}_- (P)$ (see (2.4)) and since $N\{B\} = \kappa$. Furthermore, $|\Theta_{11}(t)| > |\Theta_{12}(t)|$ on $\mathbb{T}$ by the first equality in (2.7) and if $S$ is a finite Blaschke product, we have

$$|\Theta_{11}(t)S(t)| > |\Theta_{12}(t)B(t)|$$

at every $t \in \mathbb{T}$. Then we use the preceding arguments to conclude

$$N\{U_{S,B}\} = N\{\Theta_{11}S\} = N\{\Theta_{11}\} \cdot N\{S\} = \text{sq}_+ (P) + m,$$

where the last equality holds since $N\{\Theta_{11}\} = \text{sq}_+ (P)$ (see (2.4)) and since $N\{S\} = m$. This completes the proof of the first statement.

To prove the second statement, we write (2.20) in the matrix form as

$$\begin{bmatrix} U_{S,B}(z) \\ V_{S,B}(z) \end{bmatrix} = \Theta(z) \begin{bmatrix} S(z) \\ B(z) \end{bmatrix}$$

(2.23)

and assuming that $U_{S,B}(w) = V_{S,B}(w) = 0$ at some point $w \in \mathbb{D}$, we get

$$\Theta(w) \begin{bmatrix} S(w) \\ B(w) \end{bmatrix} = 0$$

which implies, since $|S(w)| + |B(w)| > 0$, that $\det \Theta(w) = 0$. But by (2.3), $z_1, \ldots, z_k$ are the only zeros of $\det \Theta$, which completes the proof of the second statement.

Assuming that $U_{S,B}$ and $V_{S,B}$ have the common zero of order $m_j > n_j$ at $z_j$, we conclude by (2.23) that the vector valued function $\Theta(z) \begin{bmatrix} S(z) \\ B(z) \end{bmatrix}$ has the zero of multiplicity $m_j > n_j$ at $z_j$. But then, $\det \Theta(z)$ has the zero of multiplicity $m_j > n_j$ at $z_j$, which contradicts to (2.3) and completes the proof of the third statement.

By statement (3) in Lemma 2.1, the function

$$Q(z) := (zI - T)^{-1} \begin{bmatrix} E & -C \end{bmatrix} \Theta(z) \begin{bmatrix} S(z) \\ B(z) \end{bmatrix} = (zI - T)^{-1} \begin{bmatrix} E & -C \end{bmatrix} \begin{bmatrix} U_{S,B}(z) \\ V_{S,B}(z) \end{bmatrix}$$
To this end, take from the definition of $M$ where $IP$ analytic at $z_i$ and to the conclusion that $Q_i(z)$ is analytic at $z_i$ for $i = 1, \ldots, k$. It is readily seen from the definition of $Q_i$ that the residue of $Q_i$ at $z_i$ is equal to

$$\text{Res}_{z=z_i} Q_i(z) = M_{n_i}(U_{S,B}; z_i) - \begin{bmatrix} f_{i,0} & 0 & \cdots & 0 \\ f_{i,1} & f_{i,0} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ f_{i,n_i-1} & \cdots & f_{i,1} & f_{i,0} \end{bmatrix} M_{n_i}(V_{S,B}; z_i)$$

where $M_{n_i}(U_{S,B}; z_i)$ and $M_{n_i}(V_{S,B}; z_i)$ are defined in accordance to (1.8). Since $Q_i$ is analytic at $z_i$ and therefore, $\text{Res}_{z=z_i} Q_i(z) = 0$, the last displayed equality implies

$$\begin{bmatrix} f_{i,0} \\ f_{i,1} \\ \vdots \\ f_{i,n_i-1} \end{bmatrix} = \begin{bmatrix} V_{S,B}(z_i) \\ V_{S,B}'(z_i) \\ \vdots \\ V_{S,B}^{(n_i-1)}(z_i) \end{bmatrix}_{\frac{1}{(n_i-1)!}} = \begin{bmatrix} U_{S,B}(z_i) \\ U_{S,B}'(z_i) \\ \vdots \\ U_{S,B}^{(n_i-1)}(z_i) \end{bmatrix}_{\frac{1}{(n_i-1)!}}.$$ 

Thus, if $m_i \leq n_i$, then conditions (2.22) force

$$U_{S,B}^{(j)}(z_i) = 0 \quad \text{for } j = 0, \ldots, m_i - 1,$$

which means that $U_{S,B}$ has the zero at $z_i$ of at least the same multiplicity as $V_{S,B}$ does. If $m_i > n_i$, then the same arguments show that $U_{S,B}$ has zero of multiplicity $\bar{m}_i \geq n_i$ at $z_i$. If $\bar{m}_j > n_i$, then $z_i$ is a common zero of $U_{S,B}$ and $V_{S,B}$ of multiplicity greater than $n_j$, which is impossible, by Statement 3. Thus, $\bar{m}_i = n_i$, which completes the proof of the theorem.

**Proof of Theorem 1.1**: By Theorem 2.2 every solution $f$ of the problem $\text{IP}_\kappa$ is of the form (1.15), which is equivalent to representation (1.13). Thus, to prove Theorem 1.1 it suffices to show that a function $f$ of the form (1.13) is a solution of the problem $\text{IP}_\kappa$ problem if and only if the parameters $S \in S$ and $B \in B_\kappa$ meet conditions (1.14). To this end, take $f$ in the form (2.21) and represent the function $Q_i$ from (2.24) as

$$Q_i(z) = (zI - J_{n_i}(z_i))^{-1} [E_{n_i}f(z) - C_i] V_{S,B}(z).$$
If conditions (1.14) are satisfied, i.e., if $V_{S,B}(z_i) \neq 0$ for $i = 1, \ldots, k$, then $f$ is analytic at $z_1, \ldots, z_d$ and the residue of $Q_i$ at $z_i$ equals

$$0 = \text{Res}_{z=z}Q_i(z) = \begin{bmatrix} r_{i,0} & 0 & \ldots & 0 \\ r_{i,1} & r_{i,0} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ r_{i,n_i-1} & \ldots & r_{i,1} & r_{i,0} \end{bmatrix} \begin{bmatrix} V_{S,B}(z_i) \\ V_{S,B}'(z_i) \\ \vdots \\ V_{S,B}'(n_i-1)(z_i) \end{bmatrix}$$

(2.25)

where

$$r_{i,j} = f_{i,j} - \frac{f(j)(z_i)}{j!} \quad (j = 0, \ldots, n_i - 1).$$

(2.26)

Note that analyticity of $f$ at $z_i$ is required to establish the second equality in (2.25); the first holds in any event since $Q_i$ is analytic at $z_i$. Since $V_{S,B}(z_i) \neq 0$, it follows from (2.25) that

$$r_{i,j} = 0 \quad (j = 0, \ldots, n_i - 1; i = 1, \ldots, k),$$

which is equivalent to (2.31), by (2.26). Furthermore, $V_{S,B}(z)$ has sq$_-\epsilon(\gamma)$ zeros inside $D$ by Theorem 2.3 (part (1)) and none of them are canceled by zeros of $U_{S,B}(z)$ by statement (2) in the same Theorem 2.3 (part (2)). Therefore, $f$ has $\kappa$ poles inside $D$ and it is a generalized Schur function by Lemma 2.1 (part (4)). Therefore, $f$ belongs to $S_{\kappa}$ and since it satisfies conditions (2.31), it solves the problem $\nabla P_{\kappa}$.

Let us assume that at least one of the conditions (1.14) fails, i.e., that $V_{S,B}(z_i) = 0$ for some $i \in \{1, \ldots, k\}$. Then $V_{S,B}(z_i) = 0$, by statement (5) in Theorem 2.3 and after cancellation, it turns out that $f$ has $\kappa' < \kappa$ poles inside $D$ and therefore, it does not belong to $S_{\kappa}$. This is one reason why $f$ is not a solution of the $\nabla P_{\kappa}$ problem. Besides, a function $f \in S_{\kappa'}$ cannot satisfy all the interpolation conditions in (2.31). If it had, then by virtue of Theorem 2.2 it would be of the form $f = T_0[\Theta']$ for some $\Theta' \in S_{\kappa'-\text{sq}_-\epsilon}$. Since the map $E \rightarrow T_0[\Theta']$ is invertible, we would have $E \equiv \Theta'$ which is impossible since the latter functions have different numbers of poles in $D$. This completes the proof of Theorem 1.1. □

In conclusion we consider the functions $f$ obtained via the formula (1.13) from the parameters $\{S, B\}$ which fail to satisfy all the conditions (1.14). We will be interested in two questions: how many negative squares $f$ may lose and which interpolation conditions it still satisfies. In additional to the tuple $n = (n_1, \ldots, n_k) \in \mathbb{Z}_+^k$ associated with the problem $\nabla P_{\kappa}$ we consider another tuple $m = (m_1, \ldots, m_k) \in \mathbb{Z}_+^k$ and introduce

$$\mathcal{I}_+ := \{i \in \{1, \ldots, k\} : n_i > m_i\}, \quad \mathcal{I}_- := \{i \in \{1, \ldots, k\} : n_i < m_i\}, \quad \mathcal{I}_0 = \{i \in \{1, \ldots, k\} : n_i = m_i\}, \quad \gamma_m := \sum_{i=1}^k \min\{m_i, n_i\}.$$  

(2.27)

(2.28)

**Theorem 2.4.** Let $f$ be of the form (1.13) with $S \in S$ and $B \in B_{\kappa}$ having no common zeros on $D$ and such that

$$(\Theta_1 S + \Theta_2 B)^{(j)}(z_i) = 0 \quad (i = 1, \ldots, k; j = 0, \ldots, m_i - 1)$$

(2.29)
and
\[(\Theta_{21} S + \Theta_{22} B)^{(m_i)}(z_i) \neq 0 \quad (i = 1, \ldots, k). \tag{2.30}\]

Then \(f\) belongs to the class \(S_{\tilde{\kappa} + \text{sq}_-(P) - \gamma_m}\), where \(\gamma_m\) is given in (2.25). Furthermore, \(f\) has a pole of multiplicity \(m_i - n_i\) at \(z_i\) if \(i \in \mathcal{I}_-\), and satisfies interpolation conditions
\[f^{(j)}(z_i) = j! f_{i,j} \quad (i \in \mathcal{I}_4; \ j = 0, \ldots, n_i - m_i - 1). \tag{2.31}\]

**Proof:** We take \(f\) in the form (2.21) with \(U_{S,B}\) and \(V_{S,B}\) defined as in (2.20). Conditions (2.29), (2.30) say that \(V_{S,B}\) has zeros of order \(m_i\) at \(z_i\) for \(j = 1, \ldots, k\). If \(m_i \leq n_i\), then \(U_{S,B}\) has zero of order at least \(m_i\) at \(z_i\) (statement (5) in Theorem 2.3) and therefore \(f\) admits an analytic continuation to \(z_i\). If \(m_i > n_i\), then the same arguments show that \(U_{S,B}\) has zero of multiplicity \(n_i\) at \(z_i\) (statement (4) in Theorem 2.3) and after cancellation \(f\) will have a pole of multiplicity \(m_i - n_i\) at \(z_i\). By statement (1) in Theorem 2.3, the total number of zeros of \(V_{S,B}\) inside \(D\) is equal to \(\tilde{\kappa} + \text{sq}_-(P)\). Therefore \(\tilde{\kappa} + \text{sq}_-(P) - |m|\) zeros fall into \(D \setminus \{z_1, \ldots, z_k\}\) and cannot be canceled by zeros of \(U_{S,B}\) by statement (2) in Theorem 2.3. After all zero cancellations, the function \(V_{S,B}\) will have \(m_i - n_i\) zeros every \(z_i\) for \(i \in \mathcal{I}_-\) and still \(\tilde{\kappa} + \text{sq}_-(P) - |m|\) zeros in \(D \setminus \{z_1, \ldots, z_k\}\). Thus, the function \(f = \frac{U_{S,B}}{V_{S,B}}\) will have
\[\tilde{\kappa} + \text{sq}_-(P) - |m| + \sum_{i \in \mathcal{I}_-} (m_i - n_i) = \tilde{\kappa} + \text{sq}_-(P) - \gamma_m\]
poles inside \(D\). By statement (4) in Lemma 2.1, \(f\) is a generalized Schur function and therefore, it belongs to \(S_{\tilde{\kappa} + \text{sq}_-(P) - \gamma_m}\). Furthermore, if \(i \in \mathcal{I}_+\), then \(f\) is analytic at \(z_i\) and therefore equality (2.25) holds. Since \(V_{S,B}^{(m_i)}(z_i) \neq 0\) by (2.30), it follows from (2.25) that
\[r_{i,j} = 0 \quad (i \in \mathcal{I}_+; \ j = 0, \ldots, n_i - m_i - 1),\]
which is equivalent to (2.31), by (2.26).

3. **The divisor-remainder version**

The problem \(\text{IP}_\kappa\) can be formulated in the divisor-remainder form (3.3) as follows. Let \(H^\infty_\kappa\) be the set of all functions \(f\) of the form (1.1) where \(s \in H^\infty\) and \(b \in \mathcal{B}_\kappa\) may have common zeros. From this definition it follows that \(\mathcal{S}_\kappa = (H^\infty_\kappa \setminus H^\infty_{\kappa-1}) \cap \mathcal{B}L^\infty\) where \(\mathcal{B}L^\infty\) denotes the unit ball of \(L^\infty(\mathbb{T})\). Let \(\varphi \in H^\infty\) be any function satisfying interpolation conditions (1.3):
\[\varphi^{(j)}(z_i) = j! f_{i,j} \quad (i = 1, \ldots, k; \ j = 0, \ldots, n_i - 1), \tag{3.1}\]
and let \(\theta\) be a finite Blaschke product defined by
\[\theta(z) = \prod_{i=1}^k \left( \frac{z - z_i}{1 - z z_i} \right)^{n_i}. \tag{3.2}\]
Proposition 3.1. A function $f$ is a solution of the problem $IP_\kappa$ if and only if it belongs to $S_\kappa$ and admits a representation

$$f(z) = \varphi(z) + \theta(z)h(z) \quad \text{for some} \quad h \in H_\kappa^\infty.$$  \hfill (3.3)

Proof: If $f = s/b$ (where $s \in S$ and $b \in B_\kappa$) belongs to $S_\kappa$ and satisfies conditions (1.3), then by (3.1), the function $s - \varphi b$ belongs to $H_\kappa^\infty$ and satisfies the homogeneous conditions

$$(s - \varphi b)^{(j)}(z_i) = 0 \quad (i = 1, \ldots, k; \ j = 0, \ldots, n_i - 1)$$  \hfill (3.4)

By the maximum modulus principle, the function $(s - \varphi b)/\theta$ belongs to $H_\kappa^\infty$ where $\theta$ is defined in (3.2). Since $b \in B_\kappa$, the function $h := \frac{s - \varphi b}{\theta b}$ belongs to $H_\kappa^\infty$ and therefore $f$ can be represented as in (3.3), since

$$f = \frac{s}{b} = \varphi + \theta \cdot \frac{s - \varphi b}{\theta b} = \varphi + \theta h.$$  

Conversely, let $f \in S_\kappa$ be of the form (3.3). Since $f$ has $\kappa$ poles and $\varphi \in H_\kappa^\infty$, it follows that $h$ has $\kappa$ poles in $\mathbb{D}$ none of which are in $\{z_1, \ldots, z_k\}$, the zero set of $\theta$. Therefore, $h$ is analytic at $z_1, \ldots, z_k$. Therefore, the function $f - \varphi = \theta h$ satisfies the homogeneous conditions (3.4), so that $f$ satisfies (1.3) due to (3.3).

By Proposition 3.1, the solution set for the problem $IP_\kappa$ is equal to $(\varphi + \theta H_\kappa^\infty) \cap S_\kappa$. Thus if the Pick matrix $P$ of the problem meets conditions (1.11), then the set $(\varphi + \theta H_\kappa^\infty) \cap S_\kappa$ is not empty (and is parametrized as in Theorem 1.1) and therefore, a larger set

$$\Omega_\kappa(\varphi, \theta) := (\varphi + \theta H_\kappa^\infty) \cap B^\infty = (\varphi + \theta H_\kappa^\infty) \cap \left( \bigcup_{\alpha=0}^\kappa S_\alpha \right)$$ \hfill (3.5)

is not empty. The second equality in (3.5) is easily verified: since $\varphi, \theta \in H_\kappa^\infty$, it follows that $\varphi + \theta H_\kappa^\infty \subset H_\kappa^\infty$ and on the other hand, $H_\kappa^\infty \cap B^\infty = \cup_{\alpha=0}^\kappa S_\alpha$. Clearly, the elements of $\Omega_\kappa(\varphi, \theta)$ are solutions of certain $L^\infty$-norm constraint interpolation problem; the next theorem characterizes $\Omega_\kappa(\varphi, \theta)$ in terms of the kernel $K_f$ defined in (2.12) as well as in terms of the linear fractional transformation $T_\Theta$ defined in (1.15).

Theorem 3.2. Let the Pick matrix $P$ defined in (1.7) be invertible and let $\kappa \geq \text{sq}_-(P)$. Let $\Theta$ be given by (1.12), let $\varphi$ be an $H_\kappa^\infty$-function satisfying conditions (5.1), let $\theta$ be given by (3.2) and let $f$ be a function meromorphic on $\mathbb{D}$. The following are equivalent:

1. $f$ belongs $(\varphi + \theta H_\kappa^\infty) \cap B^\infty$.
2. $\text{sq}_-(K_f) \leq \kappa$ where the kernel $K_f$ is defined in (2.12).
3. $f = T_\Theta[E]$ for some $E \in H_\kappa^\infty - \text{sq}_-(P) \cap B^\infty$.

Proof: (1) $\Rightarrow$ (2). Let us assume that $f$ is of the form (3.3) and belongs to $S_\kappa$ for some $\kappa \leq \kappa$. Let us assume that the function $h$ in representation (3.3) has poles of multiplicities $m_i$ at $z_i$ for $i = 1, \ldots, k$ (the case where $m_i = 0$ is not excluded). Let $I_+^r$, $I_0$ and $\gamma_m$ be defined as in (2.27), (2.28). Since $h \in H_\kappa^\infty$, it may have at most $\kappa - |m|$
poles outside the set \( \{z_1, \ldots, z_k\} \). After cancellation of the poles of \( h \) with the zeros of \( \theta \), we obtain the following representation for \( \tilde{f} \):

\[
\tilde{f}(z) = \varphi(z) + \tilde{\theta}(z)\tilde{h}(z),
\]

where

\[
\tilde{\theta}(z) = \prod_{i \in \mathcal{I}^+} \left( \frac{z - z_i}{1 - z z_i} \right)^{m_i - n_i} \quad \text{and} \quad \tilde{h}(z) = h(z) \cdot \prod_{i \in \mathcal{I}^-} \left( \frac{z - z_i}{1 - z z_i} \right)^{m_i - n_i}.
\]

(3.7)

It is clear that \( \tilde{h} \) has poles of multiplicities \( m_i - n_i \) at \( z_i \) for every \( i \in \mathcal{I}^- \) and at most \( \kappa - |\mathbf{m}| \) poles in \( \mathbb{D} \setminus \{z_1, \ldots, z_k\} \). It follows from (2.28) that \( f \) has the same poles and since \( f \) belongs to \( \mathcal{S}_\kappa \), we get

\[
\tilde{\kappa} \leq \sum_{i \in \mathcal{I}^-} (m_i - n_i) + \kappa - |\mathbf{m}| = \kappa - \gamma_\mathbf{m}.
\]

(3.8)

On the other hand, \( \tilde{h} \) is analytic at every \( z_i \) for \( i \in \mathcal{I}^+ \) and therefore, \( f \) of the form (3.6) satisfies interpolation conditions

\[
f^{(j)}(z_i) = \varphi^{(j)}(z_i) = j! f_{i,j} \quad (i \in \mathcal{I}^+; \ j = 0, \ldots, n_i - m_i - 1).
\]

(3.9)

Therefore, \( f \) is a solution of the problem \( \text{IP}_\kappa \) with interpolation conditions (3.9). The Pick matrix \( \tilde{P} \) of this interpolation problem is a principal submatrix of the Pick matrix \( P \) of the original \( \text{IP}_\kappa \), and for a suitable permutation matrix \( U \), we have

\[
UPU = \begin{bmatrix} P_1 & P_2^* \\ P_2 & \tilde{P} \end{bmatrix}.
\]

(3.10)

Furthermore, associating the matrices \( \tilde{T}, \tilde{E} \) and \( \tilde{C} \) with the problem \( \text{IP}_\kappa \) via formulas (1.4) and (1.7), it is easy to check the block decompositions

\[
UTU = \begin{bmatrix} T_1 & 0 \\ T_2 & \tilde{T} \end{bmatrix}, \quad UE = \begin{bmatrix} E_1 \\ E \end{bmatrix}, \quad UE = \begin{bmatrix} C_1 \\ C \end{bmatrix}
\]

(3.11)

conformal with (3.10). By Theorem 2.2 applied to the problem \( \text{IP}_\kappa \), the kernel

\[
\tilde{K}_f(z, \zeta) = \begin{bmatrix} \tilde{P} & (I - \tilde{\zeta} \tilde{T})^{-1} (\tilde{E} - \tilde{C} f(\zeta)^*) \\ (\tilde{E}^* - f(z)\tilde{C}^*) (I - z\tilde{T}^*)^{-1} & K_f(z, \zeta) \end{bmatrix}
\]

(3.12)

has \( \tilde{\kappa} \) negative squares on \( \rho(f) \). Due to (3.10)–(3.12), the kernel \( K_f \) defined in (2.12) can be represented as

\[
\begin{bmatrix} U & 0 \\ 0 & 1 \end{bmatrix} K_f(z, \zeta) \begin{bmatrix} U^* & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} P_1 & B(\zeta)^* \\ B(z) & \tilde{K}_f(z, \zeta) \end{bmatrix},
\]

(3.13)

where \( B(z) \) is analytic on \( \rho(f) \) (the explicit formula for \( B \) is not that important for now). Then the number of negative squares of the kernel on the right hand side of (3.13) can be estimated as follows

\[
\text{sq}_{-} \left( \begin{bmatrix} P_1 & B(\zeta)^* \\ B(z) & \tilde{K}_f(z, \zeta) \end{bmatrix} \right) \leq \text{sq}_{-}(\tilde{K}_f) + d = \tilde{\kappa} + d,
\]

(3.14)
where \( d \) is the size of the square matrix \( P_1 \) (see [6, Proposition 4.1] for the proof). The number \( d \) is equal to the difference between the sizes of the Pick matrices \( P \) and \( \tilde{P} \) or, which is the same, to the difference between the numbers of interpolation conditions in (1.3) and (3.9). Thus,

\[
d = |n| - \sum_{i \in I_+} (n_i - m_i) = \sum_{i \in I_- \cup I_0} n_i + \sum_{i \in I_+} m_i = \gamma m
\]  

(3.15)

Now we combine (3.8), (3.14) and (3.15) to conclude from (3.13) that

\[
\text{sq}_-(K_f) = \text{sq}_- \left( \begin{bmatrix} P_1 & B(z) \\ B(z) & K_f(z, \zeta) \end{bmatrix} \right) \leq \bar{\kappa} + d = \bar{\kappa} + \gamma m \leq \kappa - \gamma m + \gamma m = \kappa,
\]

which completes the proof of the implication (1) \( \Rightarrow \) (2).

(2) \( \Rightarrow \) (3). Let us assume that \( \text{sq}_-(K_f) = \bar{\kappa} \leq \kappa \). Then by the arguments used in the proof of Theorem 2.2 to derive statement (2) from statement (1) we conclude that \( f \) is of the form \( f = \Theta [E] \) for some \( \Theta \in \mathcal{S}_{\bar{\kappa} - \text{sq}_-(P)} \). Since \( \mathcal{S}_{\bar{\kappa} - \text{sq}_-(P)} \subset H^{\infty}_{\kappa - \text{sq}_-(P)} \cap BL^{\infty} \subset H^{\infty}_{\kappa - \text{sq}_-(P)} \cap BL^{\infty} \), the proof is completed.

(3) \( \Rightarrow \) (1). Let \( f \) be of the form \( f = \Theta [E] \) for some \( \Theta \in \mathcal{S}_{\bar{\kappa}} \) where \( \bar{\kappa} \leq \kappa - \text{sq}_-(P) \). Then we can take \( f \) in the form (1.13) with \( S \in \mathcal{S} \) and \( B \in \mathcal{B}_{\tilde{\kappa}} \), having no common zeros on \( \mathbb{D} \). Let \( m_1, \ldots, m_k \) be the integers uniquely determined from conditions (2.29) and (2.30). Then we conclude by Theorem 2.4 that \( f \) satisfies interpolation conditions (2.31) (or (3.9) which is the same) and belongs to the class \( \mathcal{S}_{\kappa_1} \), where

\[
\kappa_1 = \bar{\kappa} + \text{sq}_-(P) - \gamma m.
\]

(3.16)

Thus, \( f \) solves the problem \( \text{IP}_{\kappa_1} \) with interpolation conditions (3.9) and therefore by virtue of Proposition 3.1 it admits a representation (3.6) with \( \Theta(z) \) defined as in (3.7) and some \( \tilde{h} \in H^{\infty}_{\kappa_1} \). From (3.2) and (3.7) we observe that the ratio \( \theta_1 := \theta / \tilde{\theta} \) is a finite Blaschke product of degree

\[
\deg \theta_1 = \sum_{i \in I_+} m_i + \sum_{i \in I_- \cup I_0} n_i = \gamma m
\]

and therefore the function \( h := \tilde{h} / \theta_1 \) belongs to \( H^{\infty}_{\kappa_2} \) where \( \kappa_2 = \kappa_1 + \gamma m \) and due to (3.16), we have \( \kappa_2 = \bar{\kappa} + \text{sq}_-(P) \leq \kappa \), so that \( h \in H^{\infty}_{\bar{\kappa}} \). Now we get from (3.6)

\[
f = \varphi + \tilde{\theta} \cdot \tilde{h} = \varphi + \Theta \cdot \frac{\tilde{h}}{\theta_1} = \varphi + \theta h.
\]

Since \( f \in \mathcal{S}_{\kappa^*} \) and \( h \in H^{\infty}_{\bar{\kappa}} \), it follows that

\[
f \in (\varphi + \Theta H^{\infty}_{\kappa}) \cap \mathcal{S}_{\kappa^*} \subset (\varphi + \Theta H^{\infty}_{\kappa}) \cap BL^{\infty}
\]

which completes the proof of the theorem.

As corollary, we obtain the following “if and only if” version of Theorem 2.2.

**Theorem 3.3.** Let \( P \) satisfy conditions (1.11), let \( \Theta \) be given by (1.12) and let \( f \in \mathcal{S}_{\kappa} \). The following are equivalent:

1. \( f \) is a solution of the problem \( \text{IP}_{\kappa} \).
(2) \( \text{sq}_- (K_f(z, \zeta)) = \kappa \) where the kernel \( K_f \) is defined in (2.12).

(3) \( f = T_\Theta [\mathcal{E}] \) for some \( \mathcal{E} \in S_{\kappa - \text{sq}_- (P)} \).

**Proof:** Implications \((1) \Rightarrow (2) \Rightarrow (3)\) are proved in Theorem 2.2. If \( f = T_\Theta [\mathcal{E}] \) for some \( \mathcal{E} \in S_{\kappa - \text{sq}_- (P)} \), then \( f \in (\varphi + \theta H^\infty_\kappa) \cap BL^\infty \), by Theorem 3.2 where \( \varphi \) and \( \theta \) are the functions associated with the problem \( \text{IP}_\kappa \). By the assumption of the theorem, \( f \in S_\kappa \) and therefore

\[
f \in (\varphi + \theta H^\infty_\kappa) \cap BL^\infty \cap S_\kappa = (\varphi + \theta H^\infty_\kappa) \cap S_\kappa
\]

and the latter set coincides with the solution set for the problem \( \text{IP}_\kappa \), by Proposition 3.1. This completes the proof of the implication \((3) \Rightarrow (1)\) and therefore, of the theorem. \(\square\)

**References**

1. V.M. Adamjan, D.Z. Arov and M.G. Krein, *Analytic properties of the Schmidt pairs of a Hankel operator and the generalized Schur-Takagi problem*, Mat. Sb. 86 (1971), 34–75.
2. J. A. Ball, *A non-Euclidean Lax-Beurling theorem with applications to matricial Nevanlinna-Pick interpolation*, Toeplitz centennial (Tel Aviv, 1981), pp. 67–84, Operator Theory: Adv. Appl., OT 4, Birkhäuser, 1982.
3. J. A. Ball, I. Gohberg, and L. Rodman, *Interpolation of rational matrix functions*, OT45, Birkhäuser Verlag, 1990.
4. J. A. Ball, J. W. Helton, *A Beurling-Lax theorem for the Lie group \( U(m, n) \) which contains most classical interpolation theory*, J. Operator Theory 9 (1983), no. 1, 107–142.
5. V. Bolotnikov, *On Carathéodory–Fejér problem for generalized Schur functions*, Integral Equations Operator Theory, 50 (2004), no.1, 9–41.
6. V. Bolotnikov and A. Kheifets, *Boundary Nevanlinna–Pick interpolation problems for generalized Schur functions*, Operator Theory: Advances and Applications OT 165 (2006), 67–119.
7. L. B. Golinskii, *A generalization of the matrix Nevanlinna–Pick problem*, Izv. Akad. Nauk Armyan. SSR Ser. Mat. 18 (1983), 187–205. (Russian).
8. J. W. Helton, *The distance of a function to \( H^\infty \) in the Poincare metric; electrical power transfer*, J. Funct. Anal. 38 (1980), no. 2, 273–314.
9. M. G. Krein and H. Langer, *Über die verallgemeinerten Resolventen und die charakteristische Funktion eines isometrischen Operators im Raum \( \Pi_n \)*, Colloq. Math. Soc. János Bolyai 5 (1972), 353–399.
10. M. G. Krein and H. Langer, *Über einige Fortsetzungsprobleme, die eng mit der Theorie hermitescher Operatoren im Raum \( \Pi_n \) zusammenhängen. I. Einige Funktionenklassen und ihre Darstellungen* Math. Nachr. 77 (1977), 187–236.

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