Dynamical systems analysis of phantom dark energy models

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Abstract. In this work, we study the dynamical systems analysis of phantom dark energy models considering five different potentials. From the analysis of these five potentials we have found a general parametrization of the scalar field potentials which is obeyed by many other potentials. Our investigation shows that there is only one fixed point which could be the beginning of the universe. However, future destiny has many possible options. A detailed numerical analysis of the system has been presented. The observed late time behaviour in this analysis shows very good agreement with the recent observations.

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1 Introduction

Several recent observations [1–5] have confirmed the accelerated expansion of the universe. However, these observations do not offer any clear picture of the driver of this mysterious behaviour of the universe. The cosmological constant [6] is the most popular candidate which can successfully explain the late time acceleration, but it suffers from its own problem which is known as the cosmological constant problem [7]. The alternative approaches [8] to cosmological constant are generally classified into two different classes. In the first approach, the energy-momentum tensor is modified by the introduction of an exotic matter with a negative pressure. These models are called “modified matter models”. The second approach is called “modified gravity models” in which the gravity sector of the Einstein equation is modified. Among the modified matter models the quintessence [9–11] and phantom model [10, 12–14] are very popular as the cosmological dynamics of these models have rich phenomenological behavior. In both quintessence and phantom model, a scalar field is minimally coupled to gravity and the potential supply the sufficient negative pressure to drive the accelerated expansion of the universe. The phantom scalar field has a negative kinetic term which is opposite to the quintessence scalar field model. Present observations suggest that the equation of state of the dark energy is $\omega_\phi < -1$ [1]. In the conventional quintessence scalar field models, $\omega_\phi < -1$ is not achievable as these models are based on the canonical kinetic energy. A phantom field which has non-canonical kinetic term can give us a scenario of $\omega_\phi < -1$ in the evolution of the dark energy. There are some theoretical problems such as violation of some energy conditions [15, 16] which may occur due to the introduction of the phantom field, but one can not deny the fact that it can very well fit the current observations [17, 18]. The literature is full of plenty of work on scalar field dark energy models with different potentials [11, 19–26]. None of these potentials enjoy any kind of preference from either theoretical or observational point of view. For a comprehensive list of potentials, we refer to [27].

Several works have been done to study the scalar field dark energy models using dynamical systems analysis [11, 28–33]. Dynamical system analysis is a very useful method to study the qualitative behaviour of any nonlinear system. For a general discussion on the application of dynamical system analysis in general relativity and cosmology, we refer to [34–36]. The phase space behaviour of the phantom model with different potentials has been studied in [37] using the Hamilton-Jacobi formalism. This work considered a universe in which the phantom scalar field is the only component. In [38], Urena-Lopez studied the attractor behavior of the phantom model with a positive exponential potential. A dynamical system analysis of
a phantom model with a different scalar coupling function and an exponential potential has been done in [39].

In this paper, our aim is to extend the dynamical systems analysis of phantom models beyond exponential potentials. We have performed stability analysis of phantom models for five different potentials. By analyzing these five potentials we have found a general parametrization of the potential variable \( \Gamma = (V \frac{dV}{d\phi})/(\frac{dV}{d\phi})^2 \). It is interesting to see that not only these five potentials, but many other potentials also follow this parametrization.

Using this general parametrization we have tried to make the analysis as general as possible. We have found fixed points and corresponding eigenvalues of the system considering the general form. But it seems difficult to find out the stability conditions for some fixed points as the mathematical expression of the eigenvalues are very long and complicated. So the stability analysis has been done separately for all the five potentials which we have considered to write in the general form. A numerical investigation of the system has been done considering the general form.

A summary of the contents of this paper is as follows: section 2 is a brief introduction to the mathematical background of the phantom model. Section 3 is the stability analysis of the system with different types of potentials. Section 4 is a numerical investigation of the system and discussion about the results obtained.

2 Mathematical background

We consider a universe which is homogeneous, isotropic and spatially flat. This type of universe is mathematically represented by flat FRW metric

\[
    ds^2 = dt^2 - a(t)^2(dr^2 + r^2d\Omega^2).
\]  

(2.1)

We also consider in this universe, the matter sector is dominated by a barotropic fluid with an equation of state \( p_m = (\gamma - 1)\rho_m \), where \( \gamma \) is the equation of state parameter, \( \rho_m \) is the energy density of the perfect fluid and \( p_m \) is the corresponding pressure. In addition to the matter, the universe is also filled by a phantom scalar field which is minimally coupled to the gravity. The action of this minimally coupled phantom scalar field can be written as

\[
    S = \int d^4x \sqrt{-g} \left( \frac{1}{2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi) \right),
\]  

(2.2)

where \( \phi \) is the phantom field and \( V(\phi) \) is the scalar potential. By varying the action with respect to the metric one can get the Friedmann equations as

\[
    H^2 = \frac{8\pi G}{3} \left( \rho_m - \frac{1}{2} \dot{\phi}^2 + V(\phi) \right),
\]  

(2.3)

\[
    \dot{H} = -\frac{8\pi G}{2} (\gamma \rho_m - \dot{\phi}^2),
\]  

(2.4)

where, \( H = \frac{\dot{a}}{a} \) is the Hubble parameter, \( a(t) \) is the scale factor of the universe and over dot means the differentiation with respect to time. The conservation equation for the fluid is,

\[
    \dot{\rho}_m = -3\gamma H \rho_m.
\]  

(2.5)

The wave equation is given by,

\[
    \ddot{\phi} + 3H \dot{\phi} = \frac{dV}{d\phi},
\]  

(2.6)
Our aim is to study the phase space behavior of this model and for that purpose the system has to be written as a set of autonomous equations. We define following dimensionless variables as

\[ x = \frac{k\phi'}{\sqrt{6}}, \quad y^2 = \frac{k^2 V}{3H^2}, \]  

(2.7)

where, the prime represents the differentiation with respect to \( N = \ln(\frac{a}{a_0}) \) and \( k^2 = 8\pi G \).

The present value of the scale factor, \( a_0 \), is chosen as unity. The energy density and the effective pressure due to the phantom field can be written as

\[ \rho_{\phi} = -\frac{1}{2}\dot{\phi}^2 + V(\phi), \quad p_{\phi} = -\frac{1}{2}\dot{\phi}^2 - V(\phi). \]  

(2.8)

We also assume that \( p_{\phi} \) and \( \rho_{\phi} \) obey the barotropic relation, \( p_{\phi} = (\gamma_{\phi} - 1)\rho_{\phi} \). Thus the equation of state parameter \( \gamma_{\phi} \) for the scalar field in terms of dimensionless variables can be written as

\[ \gamma_{\phi} = \frac{\rho_{\phi} + p_{\phi}}{\rho_{\phi}} = \frac{-\dot{\phi}^2}{-\frac{x^2}{2} + V} = \frac{-2x^2}{-x^2 + y^2}. \]  

(2.9)

The density parameter \( \Omega_{\phi} \) for the scalar field is given by,

\[ \Omega_{\phi} = \frac{k^2 \rho_{\phi}}{3H^2} = -x^2 + y^2, \]  

(2.10)

which is restricted by the Friedmann constrain equation, (2.3) as,

\[ \Omega_m + \Omega_{\phi} = 1, \]  

(2.11)

\( \Omega_m = \frac{\kappa^2 \rho_m}{3H^2} \), is baryonic energy density parameter.

As we are interested in the late time behavior of the universe, from now onwards, we consider \( \gamma = 1 \), which describes a matter dominated universe. Then the deceleration parameter \( (q = -\frac{\ddot{a}}{a\dot{a}} = -\frac{H^2+\dot{H}}{H^2}) \) can be expressed as,

\[ q = \frac{3}{2}(1 + x^2 - y^2) - 3x^2 - 1. \]  

(2.12)

Using equation (2.3), (2.4), (2.6), the system of equations can be rewritten as an autonomous system in terms of these new variables,

\[ x' = -3x - \lambda \sqrt{\frac{3}{2}y^2 + \frac{3}{2}y[1 - x^2 - y^2]}, \]  

(2.13)

\[ y' = -\lambda \sqrt{\frac{3}{2}xy + \frac{3}{2}y[1 - x^2 - y^2]}, \]  

(2.14)

\[ \lambda' = -\sqrt{6}\lambda^2(\Gamma - 1)x = -\sqrt{6}xf, \]  

(2.15)

where, \( \lambda = -\frac{1}{V} \frac{dV}{d\phi}, \Gamma = V \frac{d^2V}{d\phi^2}/(\frac{dV}{d\phi})^2 \) and \( f = \lambda^2(\Gamma - 1) \). To close the system of equations (2.13), (2.14), (2.15) one needs to know the particular form of \( f \) which depends on the particular choice of the potential. The choice of the potential remains arbitrary until there is a selection of a potential by the fundamental physics or by cosmological observations. So there is no particular form of the potential which is a natural choice. In this work, as for
The fixed points of the system are the simultaneous solutions of the equations $x' = 0$, $y' = 0$, $\lambda' = 0$. The list of the fixed points, their corresponding eigenvalues and corresponding cosmological parameters are given in the table 2 considering the general form of $f(\lambda) = \alpha_1 \lambda^2 + \alpha_2 \lambda + \alpha_3$.

To find the stability of the system, we need to find out the fixed points of the system. The fixed points of the system are the simultaneous solutions of the equations $x' = 0$, $y' = 0$, $\lambda' = 0$. The list of the fixed points, their corresponding eigenvalues and corresponding cosmological parameters are given in the table 2 considering the general form of $f(\lambda) = \alpha_1 \lambda^2 + \alpha_2 \lambda + \alpha_3$.

Here, $\lambda_\pm = \frac{1}{2\alpha_1} (-\alpha_2 \pm \sqrt{\alpha_2^2 - 4\alpha_1 \alpha_3})$ are the two solutions of the quadratic equation $f(\lambda) = \alpha_1 \lambda^2 + \alpha_2 \lambda + \alpha_3 = 0$.

One can see from the table 2 that the fixed point $a$ is a set of non isolated fixed points as $\lambda$ is undetermined. The eigenvalues of this fixed point is independent of the choice of potentials hence stability of it can be studied without considering any particular form of the potential. In order to check the nature of the fixed point $a$, we have plotted projection of the 3d phase space on $x - y$ plane and on $x - \lambda$ plane for $\alpha_1 = \alpha_2 = \alpha_3 = 1$. Figure 1(a) is the
projection of the phase space on the $x-y$ plane which indicates that the fixed point $a$ is a saddle in nature. Figure 1(b) is the projection of the phase space on $x-\lambda$ plane. In this plot one can see that the $\lambda$ axis is an attractor line. The trajectories approaching $x=0, y=0$ do not converge for any particular range of $\lambda$, rather the whole $\lambda$ axis is an attractor line. Hence, fixed point $a$ is saddle in nature. As it does not depend on the particular choice of potential the behaviour of this fixed point remains same for any other choice of $\alpha$ parameters.

Cosmologically, this fixed point is matter dominated and decelerated.

Fixed point $b$ and $c$ are always dark energy dominated and accelerated. For $\alpha_3 \neq 0$, they are always saddle in nature. For $\alpha_3 = 0$ these fixed points become non-hyperbolic as the eigenvalues are $(-3, -3, 0)$. The main tool to study these type of nonhyperbolic fixed point is center manifold theory. In this work we will use another method to study the stability of these fixed points. We will numerically investigate the stability. This approach is widely used in many dynamical systems analysis of dark energy models [28, 30, 47]. Two eigenvalues of the fixed points $d$ to $g$ are denoted by $m$ and $n$ as they are very big and complexed (please see appendix). It is very difficult to draw any conclusion about the stability of these fixed points without considering any particular form of the potential. In the next section, we consider the same potentials from the table 1 as for example to study the stability of these fixed points.

3 Stability analysis with different potentials

A: $V(\phi) = V_0 \phi^n$. For this potential $f(\lambda) = -\frac{\lambda^2}{n}$. Though all the fixed points are allowed for this potential but $b, d, e$ $(0, 1, 0)$ and $c, f, g$ $(0, -1, 0)$ becomes indistinguishable. Basically, there are three fixed points $a, b$ and $c$. The eigenvalues of the fixed points for this potential is given in table 3. All of these fixed points are nonhyperbolic in nature as out of the three eigenvalues one eigenvalue is zero. The stability of the fixed point $a$ has been already discussed in the previous section. For other two fixed points, we cannot use the standard linear stability analysis as these are nonhyperbolic in nature. We will study the stability
of these fixed points by studying the evolution of the perturbations near these fixed points. In order to do that, generally the system is perturbed from the fixed point and allowed to evolve numerically. If the system comes back to the fixed point, the fixed point is stable; otherwise unstable.

The evolution of the perturbations around these fixed points $b$ and $c$ are plotted. It is very difficult to draw any physical conclusion from the 3D plot as it is very obscure, so for the sake of simplicity, we have shown the evolution of the variables $x, y, \lambda$ individually. From figure 2 one can see that the perturbations around fixed point $b$ indeed come back to the fixed point $b$. In figure 3 the perturbations around the fixed point $c$ also show similar behavior. These plots are for $n = 1$. For $n = 2$ the qualitative behavior of the plots remains the same. Therefore, we conclude that the fixed point $b$ and $c$ are stable fixed points. We have also drawn the projections of the 3d phase trajectories of the system on $x - y$ plane and $x - \lambda$ plane (figure 4) which confirms our finding from the perturbation technique. One can also see from the phase plot figure 4(a) that the fixed point $a$ is saddle and $b, c$ are attractors. Figure 4(b) is the 2d phase plot around the fixed point $b$ on $x - \lambda$ axis where $y = 1$ and it also confirms that the fixed point $b$ is an attractor. We find similar phase plot for the fixed point $c$. Fixed point $a$ is a matter dominated point and always decelerated. Fixed point $b$ and $c$ are scalar field dominated and the universe at these points always expands with a constant acceleration $q = -1$. Due to the saddle nature of the fixed point $a$, it could be the beginning of the universe which means that the universe started from a matter dominated decelerated state. Fixed point $b$ and $c$ are late time attractors and these points may be the ultimate fate of the universe which indicates a universe in future completely dominated by the dark energy and ever accelerating.
Figure 3. Projection of perturbations along $x,y,\lambda$ axis for potential $A$: $V(\phi) = V_0\phi^n$ with $\gamma = 1$, $n = 1$ around the fixed point $c$.

Figure 4. 2d Phase plot for the potential $A$: $V(\phi) = V_0\phi^n$ with $\gamma = 1, n = 1$.

**B: $V(\phi) = V_0e^{-k\phi} + V_1$.** From the expression of $f$, we get

$$\lambda_{\pm} = -\frac{k \pm \sqrt{k^2}}{2},$$

$$\lambda_+ = -k,$$

$$\lambda_- = 0. \quad (3.1)$$

The fixed points and the eigenvalues are given in table 4. Fixed point $a$ is saddle in nature. $b,c$ are nonhyperbolic and need to be dealt with numerically. Other fixed points are hyperbolic fixed points. One can see from the figure 5 and figure 6 that some perturbations of the system do not come back to the fixed points $b$ and $c$. Apparently, it seems that these fixed points are saddle in nature. We have plotted 2d phase plot of the system for this potential in figure 7. Figure 7(a) is the phase plot on the $x - y$ plane. From this plot it is clear that the
| Fixed points | Eigenvalues |
|-------------|-------------|
| $a$         | $0, \frac{3}{2}, -\frac{3}{2}$ |
| $b, c$      | $-3, 0, -3$ |
| $d$         | $-k^2, -\frac{1}{2}(6 + k^2), -(k^2 + 3)$ |
| $e, g$      | $-3, 0, -3$ |
| $f$         | $-k^2, -\frac{1}{2}(6 + k^2), -(k^2 + 3)$ |

Table 4. Fixed points and the corresponding eigenvalues of the potential $B: V(\phi) = V_0 e^{-k\phi} + V_1$.

Figure 5. Projection of perturbations along $x, y, \lambda$ axis for potential $V(\phi) = V_0 e^{-k\phi} + V_1$ with $\gamma = 1, k = 1$ around the fixed point $b$.

fixed point $a$ is a saddle in nature, but it seems that the fixed point $b$ and $c$ are attractor which apparently contradicts our finding from the perturbation technique. To resolve this contradiction in figure 7(b), we have plotted the trajectories near the fixed point $b$ on $x - \lambda$ plane where $y = 1$. It can be clearly seen from figure 7(b) that the phase plot around the fixed point $b$ is a saddle in nature. For the fixed point $c$ the behaviour of the phase plot is same. Hence, fixed point $b$ and $c$ are saddle in nature which agrees with the finding using the perturbation technique. For different values of $k$ nature of these fixed points remains same. Fixed point $d$ and $f$ are always attractor and dominated by the dark energy. For this potential, there are three possible beginnings of the universe, fixed point $a, b$ and $c$. Fixed points $b$ and $c$ are saddle and therefore, a heteroclinic solution may start from these fixed points, but these fixed points are accelerating and dark energy dominated. A dark energy dominated accelerated beginning of the universe is not supported by the observations. So from the observational point of view, fixed point $a$ remains as the best choice to be the beginning of the universe. Like the power law potentials, this potentials also have similar behavior which is a decelerated matter dominated beginning and a dark energy dominated ever-accelerating future.

C: $V(\phi) = \cosh(\xi \phi) - 1$. For this potential, $\lambda_+ = \xi$ and $\lambda_- = -\xi$. All the fixed points are allowed and the eigenvalues are listed in the table 5. Fixed point $a$ is saddle. Fixed point $b$ and $c$ have same eigenvalues and these fixed points are also always saddle. Fixed points $d, f, e,$ and $g$ are always stable. As the fixed point $d, f, e,$ and $g$ are attractor so the universe may be attracted towards these fixed points which are dark energy dominated and ever accelerating.
Figure 6. Projection of perturbations along $x, y, \lambda$ axis for potential $V(\phi) = V_0 e^{-k\phi} + V_1$ with $\gamma = 1, k = 1$ around the fixed point $c$.

Figure 7. 2d Phase plot for the potential $V(\phi) = V_0 e^{-k\phi} + V_1, \gamma = 1, k = 1$.

| Fixed points | Eigenvalues |
|--------------|-------------|
| $a$          | $0, \frac{3}{2}, -\frac{3}{2}$ |
| $b$          | $-3, -\frac{1}{2} \left(3 + 3\sqrt{3(3 + 2\xi^2)}\right), -\frac{1}{2} \left(3 - 3\sqrt{3(3 + 2\xi^2)}\right)$ |
| $c$          | $-3, -\frac{1}{2} \left(3 + 3\sqrt{3(3 + 2\xi^2)}\right), -\frac{1}{2} \left(3 - 3\sqrt{3(3 + 2\xi^2)}\right)$ |
| $d, f$       | $-\xi^2, -\frac{1}{2} (6 + \xi^2), -(3 + \xi^2)$ |
| $e, g$       | $-\xi^2, -\frac{1}{2} (6 + \xi^2), -(3 + \xi^2)$ |

Table 5. Fixed points and the corresponding eigenvalues of the potential $C: V(\phi) = \cosh(\xi \phi) - 1$. 
Fixed points | Eigenvalues
--- | ---
\(a\) | \(0, \frac{3}{2}, -\frac{3}{2}\)
\(b\) | \(-3, \frac{1}{2} \left( 3 + \sqrt{3(3 - 4\alpha\beta^2)} \right), \frac{1}{2} \left( 3 - \sqrt{3(3 - 4\alpha\beta^2)} \right)\)
\(c\) | \(a, b\)
\(d, e, f, g\) | \(2\alpha\beta^2, -\frac{1}{2} (6 + \alpha^2\beta^2), -(\alpha^2\beta^2 + 3)\)

Table 6. Fixed points and the corresponding eigenvalues of the potential D: \(V(\phi) = V_0 \sinh^{-\alpha}(\beta \phi)\).

| Fixed points | Eigenvalues |
--- | --- |
\(a\) | \(0, \frac{3}{2}, -\frac{3}{2}\)
\(b\) | \(-\frac{1}{2} (3 + \sqrt{3(3 + \frac{2}{\alpha^2})}), -\frac{1}{2} (3 - \sqrt{3(3 + \frac{2}{\alpha^2})}), -3\)
\(c\) | \(-\frac{1}{2} (3 + \sqrt{3(3 + \frac{2}{\alpha^2})}), -\frac{1}{2} (3 - \sqrt{3(3 + \frac{2}{\alpha^2})}), -3\)

Table 7. Fixed points and the corresponding eigenvalues of the potential E: \(V(\phi) = 2M^2 \cos(\frac{\phi}{M^2})^2\).

D: \(V(\phi) = V_0 \sinh^{-\alpha}(\beta \phi)\). Here, \(\lambda_+ = \alpha\beta, \lambda_- = -\alpha\beta\). All the fixed points exist for this potential. The corresponding eigenvalues of the fixed points are given in table 6. The fixed points \(b\) and \(c\) are stable for \(\frac{2}{3} < \alpha\beta^2\) and these fixed points are spiral attractor for \(\alpha\beta^2 > \frac{3}{4}\). Fixed point \(d, e, f, g\) are stable only if \(\alpha < 0\). The stability analysis of this potential also indicates a decelerated matter dominated beginning of the universe and an accelerated dark energy dominated future.

E: \(V(\phi) = 2M^2 \cos(\frac{\phi}{M^2})^2\). Only fixed points \(a, b\) and \(c\) exist for this potential, where the saddle nature of the fixed point \(a\) remains same. Other two fixed points have same eigenvalues and they are always saddle, see table 7. The cosmological dynamics, in this case, is very simple. Fixed point \(a\) could be the beginning of the universe and there is no late time attractor for this potential.

4 Numerical investigation

In this section, we discuss the numerical integration of the system. One can see from the table 1 that fixed point \(a\) is always saddle, matter dominated and decelerated. This nature of the fixed point \(a\) is same for all kinds of potentials which follows the general parameterization of \(f\) as the stability condition of the fixed point does not depend on the particular form of the potentials.

As it is matter dominated and decelerated from the observational point of view fixed point \(a\) is the best choice as the beginning of the universe. In a phase space, a heteroclinic solution joins an unstable or saddle fixed points to a stable fixed point. There will be no evolution of the system if we start exactly from the fixed point. So in this numerical investigation, we have allowed the system to evolve from the neighborhood of the fixed point \(a\). The numerical integration is very stiff and we found it to be integrable for a very small parameter range \([\alpha_1, -1 : 0.6], [\alpha_2, 0 : 4], [\alpha_3, -2 : 0]\). It deserves mention that this range of parameters includes all the possibilities to get back the potentials in the table 1. So this
Figure 8. (a) Plot of $x(N)$, $y(N)$ and $\lambda(N)$ against $N = \ln a$ from the fixed point $a$. (b) Plot of density parameters $\Omega_m$ and $\Omega_\phi$ against $N = \ln a$ from the fixed point $a$. (c) Plot of deceleration parameter against $N = \ln a$ from the fixed point $a$. (d) Plot of equation of state parameter $w_\phi = \gamma_\phi - 1$ against $N = \ln a$ from the fixed point $a$.

numerical analysis represents a general analysis of all the potentials listed in table 1. From figure 8(a), one can see that the solutions around the fixed point $a$ are attracted towards the fixed point $b$. Figure 8(b), is the plot of the density parameters and it can be seen that the universe is now dark energy dominated and $\Omega_\phi \simeq 0.67$. The plot of the deceleration parameter $q$ in the figure 8(c) also shows a that the universe has smoothly entered into an accelerated expansion phase from a decelerated expansion phase around $z \simeq 0.55$ which is in good agreement with the observations. Figure 8(d) shows that the equation of state of the scalar field is slightly lower than $-1$ and which is much supported by observations [1]. These solutions are originated from the fixed point $a$ and it is interesting to see that these solutions can describe the accelerated expansion of the universe and satisfy the current observations for a general class of potentials which obey $f(\lambda) = \alpha_1 \lambda^2 + \alpha_2 \lambda + \alpha_3$. 
5 Discussions

In this work, we have performed dynamical systems analysis of phantom dark energy models with five different potentials. The study of these potentials leads to a general parameterization of the potential function $\Gamma$. This general parameterization is not only valid for these five potentials but also applicable to a class of potentials for which $f$ can be written in our generalized form. We have tried to keep the analysis as much general as possible. Stability and the cosmological behaviors of the fixed point $a$ are independent of the choice of the potentials and it deserves to be the best choice as the beginning of the universe, whereas the stability of other fixed points depends on the choice of potentials but their cosmological behavior is generic. Each one of them is dark energy dominated and decelerated. So every potential which follows our general parameterization scheme for phantom dark energy models has the same beginning and same ultimate fate.

We have performed a detail numerical analysis of the system. As the fixed point $a$ is the most favorable to the beginning of the universe we have allowed the system to evolve from the surrounding of this fixed point. In this numerical analysis, the parameter range has been chosen in such a way that includes all the potentials in table 1. However, this numerical analysis is not only restricted to these five potentials. The solutions around the fixed point $a$ evolved to the fixed point $b$. The numerical solutions show that the universe started from a matter dominated, decelerated saddle point and very recently around $z \simeq 0.55$. It smoothly transits from the decelerated expansion state to an accelerated state. It also shows that the universe is presently dark energy dominated as $\Omega_\phi \simeq 0.67$ and the equation of state of the scalar field is $w_\phi < -1$. All these above findings agree well with the current cosmological observations and dynamical systems analysis of the system.

This analysis does not show any favor to a particular form of the potential. It shows that a class potentials are allowed to describe the accelerated expansion of the universe. So the arbitrariness of the choice of potentials remains the same.

Though this analysis is done by choosing five different potentials one can consider more potentials to do the same. We have restricted our self to these five examples so that the analysis does not become unnecessarily long. It is also interesting to note that this general parameterization of $f$ does not depend on the particular scalar field dark energy model; rather it comes from the definition of $\Gamma$ and $\lambda$. So this general parameterization of $f$ will also be valid for other scalar field dark energy models like quintessence, quintom etc where the dynamical system’s variables can be considered in similar fashion.

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