Nonabelian Monopoles from Matrices:
Seeds of the Spacetime Structure

B. Chen, H. Itoyama
and
H. Kihara
Department of Physics,
Graduate School of Science, Osaka University,
Toyonaka, Osaka 560-0043, Japan

Abstract

We study the expectation value of (the product) of the one-particle projector(s) in
the reduced matrix model and matrix quantum mechanics in general. This quantity is
given by the nonabelian Berry phase: we discuss the relevance of this with regard to
the spacetime structure. The case of the $USp$ matrix model is examined from this re-
spect. Generalizing our previous work, we carry out the complete computation of this
quantity which takes into account both the nature of the degeneracy of the fermions
and the presence of the spacetime points belonging to the antisymmetric representa-
tion. We find the singularities as those of the $SU(2)$ Yang monopole connection as well
as the pointlike singularities in $9 + 1$ dimensions coming from its $SU(8)$ generaliza-
tion. The former type of singularities, which extend to four of the directions lying in
the antisymmetric representations, may be regarded as seeds of our four dimensional
spacetime structure and is not shared by the $IIB$ matrix model. From a mathematical
viewpoint, these connections can be generalizable to arbitrary odd space dimensions
due to the nontrivial nature of the eigenbundle and the Clifford module structure.

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I. Introduction

Continuing studies in matrix models for superstrings and M theory \cite{1, 2, 3, 4, 5} indicate that we are in a stage of obtaining a renewed understanding of old notions such as compactification and spacetime distribution in this constructive framework. These physical quantities are obtained after integrations of matrices and are no longer fixed input parameters or backgrounds. Another feature common to these models is that the actions contain terms bilinear in fermions. This is, of course, related to the D brane in the RR sector as the absence of these bilinears imply the absence of the RR sector of the model \footnote{The reduced matrix model provides a constructive definition to the Green-Schwarz superstrings in the Schild gauge. A subsector of the state space which contains fermion bilinears is able to see the RR sector of the superstrings.}.

The major objective of this paper is to enlighten the spacetime structure and the presence of solitonic objects revealed by the fermionic integrations of the matrix models. These are represented by the behavior of the spacetime points, (or D0 branes in \cite{1}), which are the eigenvalue distributions or the diagonal elements of the bosonic matrices. The effective dynamics of the spacetime points is obtained by carrying out the integrations of the remaining degrees of freedom: we will carry out the half of them represented by the fermions. Our interest, therefore, lies in a collection of individual fermionic eigenmodes obtained from the fermionic part (denoted generically by $S_{fermion}$) of the action upon diagonalization. An object which, we find, plays a role of revealing singularities as those of the bosonic parameters (in particular, those of the spacetime points) is an expectation value of the one-particle projector belonging to each of the fermionic eigenmodes. (See eq. (2.27)).

We see that this expectation value of the projector is generically given by the nonabelian Berry phase \cite{6, 7}. In matrix models of superstrings and M theory, this result offers spacetime interpretation: this is because the parameter space, where the connection one-form lives, is that of the spacetime points or D0 branes. An interesting case that we study as our major example is the $USp$ reduced matrix model \cite{4, 5, 8, 9, 10, 11}. We will carry out an explicit evaluation of the nonabelian Berry phase factor for different types of the projectors. Our computation leads us to the $su(2)$ Lie algebra valued connection one-form known as the Yang monopole \cite{12} in five spatial dimensions and its $su(8)$ generalization in nine spatial dimensions. This latter one, to the best of our knowledge, has not appeared in physics context before. The existence of the nontrivial eigenbundles based on the first quantized hamiltonian with gamma matrices and the orthogonal projection operators ensures that these nonabelian connections are straightforwardly generalized to arbitrary odd spatial dimensions.

The conclusion derived from our computation in the context of the $USp$ matrix model
is that there exist singularities extending to four of the directions of the spacetime points which lie in the antisymmetric representation. These singularities are represented by the Yang monopole. In addition, we find pointlike singularities in $9 + 1$ dimensions which are represented by its $SU(8)$ generalization. The former type of singularities may be regarded as being responsible for our four dimensional spacetime structure and is not shared by the $IIB$ matrix model\footnote{The signature of the four directions which the Yang monopole is extended to is euclidean in the model. We have nothing to say on how to make the signature Minkowskian and on the extension of the spacetime in the $v_0$ direction.}. It is noteworthy that the requirement of having $8 + 8$ supersymmetries brings us such possibility.

Before moving to the next section, let us mention the procedure to formulate the expectation value of the one-particle projector in the reduced matrix models. (See eq. (2.27)). To say things short, these are obtained from short time/infinite temperature limit of the matrix quantum mechanical system, where a path in the parameter space can be readily introduced. This is equivalent to imposing a periodicity with period $R$ on one of the original bosonic matrices, say, $v_0$ and to letting this period go to infinity in the end. Machinery to deal with these situations has been developed in \cite{13}. We mention here the calculation of \cite{14, 15}, which is similar in spirit to ours. There the exact computation of the partition function of the $IIB$ matrix model \cite{2} has been carried out as the limit of infinite temperature of the path integral of the BFSS \cite{1} quantum mechanical model. This is the same limiting procedure as ours although we will measure the one-particle projector instead of unity. (See eq. (2.35).)

In the next section, we describe the formulation and the procedure of our computation indicated above after a brief review on the $USp(2k)$ matrix model. In section III, we decompose the fermionic part of the action into the bases of the adjoint, antisymmetric and fundamental representations appropriate to our computation. This amounts to diagonalizing it for the case that the bosonic matrices are diagonal. Unlike our previous studies \cite{8, 9}, we do not set the spacetime points lying in the antisymmetric representation to zero. This turns out to improve substantially our picture of the spacetime formation suggested by the model. In section IV, we compute the nonabelian Berry phase for three types of actions obtained in section III. The connection one-forms we find are the nonabelian $SU(2)$ Yang monopole in five dimensions and its $SU(8)$ generalization to nine dimensions. The degenerate state space originating from the spinorial space is responsible for making these nonabelian gauge fields. In section V, we summarize the spacetime picture emerging from our computation. In section VI, we discuss the generalization of the Yang monopole in arbitrary odd space dimensions by clarifying the eigenbundle structure associated with the nonabelian Berry connection. Detail of the basis decomposition in section III is collected in the Appendix. Some of the papers on fermionic and bosonic integrations of matrix models are listed in \cite{16}.
II. Nonabelian Berry Phase and Matrix Models

The goal of this section is to establish that the expectation value of the one-particle projector of a fermionic eigenmode is given by the nonabelian Berry phase. This is true in general, in particular, in matrix models (both matrix quantum mechanics and reduced matrix models) containing fermion bilinears. Besides the examples we discuss below, our discussion here will apply to a variety of models obtained, for instance, by a truncation from supersymmetric field theories in various dimensions [17]. As we will occasionally refer to the case of the $USp(2k)$ matrix model—the major example of our paper—already in this section, we will present the brief review of this model in the first subsection and defer the major discussion to the second subsection.

A. some preliminaries

To begin with, the $usp$ Lie algebra is defined by

$$usp(2k) \equiv \{ A \in u(2k) | {}^tAF + FA = 0 \} ,$$  \hspace{1cm} (2.1)

and the antisymmetirc representation is defined by

$$asym(2k) \equiv \{ A \in u(2k) | {}^tAF - FA = 0 \} .$$  \hspace{1cm} (2.2)

Here $F$ is an antisymmetric matrix with nonzero determinant and can be chosen as

$$F = \begin{pmatrix} 0 & -1_k \\ 1_k & 0 \end{pmatrix} .$$  \hspace{1cm} (2.3)

It is easy to recognize

$$u(2k) = usp(2k) \oplus asym(2k) .$$  \hspace{1cm} (2.4)

A representation of eq. (2.1) in accordance with the choice (eq. (2.3)) is

$$A \equiv \begin{pmatrix} H & B \\ -B & -H \end{pmatrix} \in usp(2k) ,$$  \hspace{1cm} (2.5)

$$H^\dagger = H , \quad {}^tB = B ,$$

while that of eq. (2.2) is

$$A \equiv \begin{pmatrix} H & B \\ -B & -H \end{pmatrix} \in asym(2k) ,$$  \hspace{1cm} (2.6)

$$H^\dagger = H , \quad {}^tB = -B .$$
Let us recall here some aspects of the reduced $USp(2k)$ matrix model which are relevant to our discussion in what follows. The definition, the criteria and the rationale leading to the model as descending from typeI superstrings are elaborated fully in ref.[4, 5]. We will therefore not repeat these here.

Let $\hat{\Omega}$ be a projection operator acting on $2k \times 2k$ hermitian matrices. For the ten bosonic matrices, $\hat{\Omega}$ projects the $0, 1, 2, 3, 4, 7$ components onto the adjoint representation and the $5, 6, 8, 9$ components onto the antisymmetric representation:

$$v_M = (v_\mu, v_n),$$
$$v_\mu \in usp(2k), \quad \mu = 0, 1, 2, 3, 4, 7,$$
$$v_n \in \text{asym}(2k), \quad n = 5, 6, 8, 9.$$  \tag{2.7}

As for the fermions, $\hat{\Omega}$ splits the thirty two component Majorana-Weyl spinor (sixteen real degrees of freedom) in $9 + 1$ dimensions into an eight component spinor belonging to the adjoint representation and another eight component spinor belonging to the antisymmetric representation:

$$\Psi = \Psi_{\text{adj}} + \Psi_{\text{asym}},$$  \tag{2.8}

where

$$\Psi_{\text{adj}} \equiv t(\lambda_1, 0, \lambda_2, 0, 0, 0, 0, 0, 0, \bar{\lambda}_1, 0, \bar{\lambda}_2, 0, 0, 0),$$
$$\Psi_{\text{asym}} \equiv t(0, 0, 0, 0, \psi_1, 0, \psi_2, 0, 0, 0, 0, 0, \bar{\psi}_1, 0, \bar{\psi}_2).$$  \tag{2.9}

Modulo labelling the indices, these projections are determined by the requirement of having $8 + 8$ supersymmetries. Finally, we add degrees of freedom corresponding to an open string degrees of freedom while preserving supersymmetry. This amounts to adding $n_f = 16$ of the hypermultiplets in the fundamental representation in the $4d$ language. We display these degrees of freedom by the complex $2n_f$ dimensional vectors (see the appendix A of [10])

$$Q \equiv \begin{cases} Q_{(f)}, & f = 1 \sim n_f \\ F^{-1} \tilde{Q}_{(f-n_f)}, & f = n_f + 1 \sim 2n_f \end{cases}, \quad Q^* \equiv \begin{cases} Q^*_{(f)}, & f = 1 \sim n_f \\ \tilde{Q}^*_{(f-n_f)} F, & f = n_f + 1 \sim 2n_f. \end{cases}$$  \tag{2.10}

$$\psi_Q \equiv \begin{cases} \psi_Q(f), & f = 1 \sim n_f \\ F^{-1} \tilde{\psi}_{Q(f-n_f)}, & f = n_f + 1 \sim 2n_f \end{cases}, \quad \psi_Q^* \equiv \begin{cases} \tilde{\psi}_Q(f), & f = 1 \sim n_f \\ \tilde{\psi}_{Q(f-n_f)} F, & f = n_f + 1 \sim 2n_f. \end{cases}$$  \tag{2.11}

Let us turn to the action of the model. It is represented as

$$S_{USp} = S_{\text{closed}} + \Delta S,$$  \tag{2.12}
where
\[ S_{\text{closed}} = \frac{1}{g^2} \text{Tr} \left\{ \frac{1}{4} [v_M, v_N] [v^M, v^N] - \frac{1}{2} \Psi \Gamma^M [v_M, \Psi] \right\} \] (2.13)
is the closed string sector of the model. We denote the fermionic part by
\[ S_{\text{MW}} \equiv -\frac{1}{2g^2} \text{Tr} \left( \Psi \Gamma^M [v_M, \Psi] \right) . \] (2.14)
The remainder of the action
\[ \Delta S = \{ S_{g-s} + \mathcal{V}_{\text{scalar}} + S_{\text{mass}} + S_{g-f} + S_{\text{Yukawa}} \} , \] (2.15)
consists of the parts which depend on the fundamental hypermultiplet. We only spell out the parts relevant to our subsequent discussion.

\[ S_{g-f} \equiv \frac{1}{g^2} \left( \psi_Q^* \sigma^m v_m \cdot \psi_Q + i \sqrt{2} \psi_Q^* \lambda \cdot \psi_Q - i \sqrt{2} \psi_Q^* \bar{\lambda} \cdot Q \right) , \] (2.16)
\[ S_{\text{Yukawa}} \equiv -\frac{1}{g^2} \left( \sum_{(c_1, c_2) = (Q, \tilde{Q}), (Q, \Phi_1), (\Phi_1, \tilde{Q})} \frac{\partial^2 W_{\text{matter}}}{\partial C_1 \partial C_2} \psi_{C_2} \psi_{C_1} + h.c. \right) \]
\[ = \frac{1}{g^2} \left( \frac{1}{2} \psi_Q \cdot \Sigma F \left( \sqrt{2} \Phi_1 + M \right) \psi_Q + \sqrt{2} Q \cdot \Sigma F \psi_\Phi Q + h.c. \right) . \] (2.17)
Here
\[ \Sigma \equiv \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} , \] (2.18)
\[ M \equiv \text{diag} \left( m_{(1)}, \ldots, m_{(n_f)}, -m_{(1)}, \ldots, -m_{(n_f)} \right) , \] (2.19)
\[ W_{\text{matter}} = \sum_{f=1}^{n_f} \left( m_{(f)} \tilde{Q}_{(f)} Q_{(f)} + \sqrt{2} \tilde{Q}_{(f)} Q_{(f)} \right) \] (2.20)
and \( \cdot \) implies the standard inner product with respect to the \( 2n_f \) flavour indices. For a more complete discussion, see [10].

### B. One-particle projector and nonabelian Berry phase

Let us first imagine diagonalizing some action \( S_{\text{fermion}} \) which is bilinear in fermions. In the example of the last subsection, this is given by the fermionic part of the action
\[ S_{\text{fermion}} \equiv S_{\text{MW}} + S_{g-f} + S_{\text{Yukawa}} . \] (2.21)
In general, \( S_{\text{fermion}} \) can be written as
\[
S_{\text{fermion}} = \sum_{\alpha} \sum_{\ell} \lambda_{\ell} \bar{\xi}_{\ell}^{\alpha} \xi_{\ell}^{\alpha} .
\] (2.22)

Here \( \xi_{\ell}^{\alpha} \) is the fermionic eigenmode belonging to an eigenvalue \( \lambda_{\ell} \) and \( \alpha_{\ell} \) labels its degeneracy.

As mentioned in the introduction, we would like to evaluate an expectation value of the one-particle projector
\[
\hat{P}_{\ell}^{\alpha \alpha'} \equiv \xi_{\ell}^{\alpha} \Omega \langle \Omega | \xi_{\ell}^{\alpha'} \rangle ,
\] (2.23)
with \( \Omega \) being the Clifford vacuum which annihilates half of the fermions
\[
\left( b_A, \bar{b}_A \right) ,
\] (2.24)
which are \( \left( \Psi, \bar{\Psi}, \psi_Q \psi_Q^* \right) \) in the model of the last subsection. The eigenmodes \( \xi_{\ell}^{\alpha} \) can be written as
\[
\xi_{\ell}^{\alpha} = \sum_A b_A^{\alpha} \psi_{\ell A}^{\alpha} .
\] (2.25)

The expectation value is defined through the integrations over the fermionic variables of the model. A formula that we find in the end (eq. (2.35) ) and the one we use (eq. (2.38) ) in the subsequent sections are obtained from the short time/infinite temperature limit of the corresponding quantum mechanics, in which the path dependence can be easily introduced.

In order to argue more directly that this quantity can be defined in the reduced models, we start with imposing a periodicity constraint on one of the directions, say, \( v_0 \):
\[
S v_M S^{-1} = v_M + R \delta_{M,0} .
\] (2.26)

The size of the matrices is necessarily infinite dimensional in order to permit solutions to eq. (2.26). Each matrix divides into an infinite number of blocks. The shift operator \( S \) acts on each block and moves it diagonally by one in our situation.

Let us introduce
\[
\langle \langle \hat{P}_{\ell}^{\alpha \alpha'} \rangle \rangle_{\Gamma} \equiv \lim_{R \rightarrow \infty} \int [D\kappa] [DZ] e^{iS(\kappa; Z', Z'')} \langle Z' | \hat{P}_{\ell}^{\alpha \alpha'} | Z'' \rangle .
\] (2.27)

Here \( S(\kappa; Z', Z'') \) and \( \langle Z' | \hat{P}_{\ell} | Z'' \rangle \) are the Grassmann coordinate representation of \( S_{\text{fermion}} \) and that of \( \hat{P}_{\ell}^{\alpha \alpha'} \) respectively. We have indicated the end point constraints and the path \( \Gamma \) in eq. (2.27). These will become clearer shortly. We will also consider
\[
\langle \langle \prod_{\ell \in I} \hat{P}_{\ell}^{\alpha \alpha'} \rangle \rangle_{\Gamma} ,
\] (2.28)
where $\mathcal{I}$ is a subset of all eigenmodes and the case of our interest is the one in which this subset is over the eigenmodes belonging to the positive eigenvalues. This choice is motivated by the Dirac sea filling.

In principle, one can diagonalize eq. (2.21) for general $v_M$ and $Q$. We will, however, restrict ourselves to the case

$$v_M = X_M = \text{diagonal}, \quad Q = 0.$$  

Explicit diagonalization of eq. (2.21) to the form of eq. (2.22) in this case will be carried out in the next section.

Let us convert eq. (2.27) into the Fourier transformed variables and this helps us understand the limiting procedure in eq. (2.27) better.

$$\langle\langle \hat{P}_\ell^\alpha \rangle\rangle_{\Gamma} = \lim_{\beta \to 0} \int dz' dz'' F(z'; \beta | z''; 0) \Gamma \langle z' | \hat{P}_\ell^\alpha \alpha' | z'' \rangle ,$$  

$$\mathcal{F} = \int [D_K(\cdot)] [D_z(\cdot)] e^{i \int_0^\beta d\beta L_{\text{fermion}}(\kappa(\beta), z(\beta); z', z'', X_M = X_M(\beta))} .$$  

Here

$$\beta = 2\pi/R .$$  

and $\mathcal{L}(\cdots)$ is the Grassman coordinate representation of the matrix quantum mechanics Lagrangian which is obtained from $S_{\text{fermi}}$ by the susbstitution $^4$

$$v_0 \to i \frac{d}{dt} .$$  

Here we have chosen the $v_0 = 0$ gauge. In the operator representation with corresponding Hamiltonian denoted by

$$H(\beta') \equiv H \left[ b^A, \bar{b}^A, | \Gamma; X_M = X_M(\beta') \right] ,$$  

Eq. (2.30) is written as

$$\langle\langle \hat{P}_\ell^\alpha \alpha' \rangle\rangle_{\Gamma} = \lim_{\beta \to 0} Tr_{\text{fermion}} \left( (-)^F e^{-i \int_0^\beta d\beta H(\beta')} \mathcal{P}_\ell^\alpha \alpha' \right) .$$  

Reducing this expression into that of the first quantized quantum mechanics, we find that this quantity is nothing but the time evolution of the $\ell$-th degenerate eigenfunction $\psi_\ell^\alpha$. (Note that this $\psi_\ell^\alpha$ is the same as the one appearing in the original expression eq. (2.23).) The generic expression is known to consist of the energy dependent dynamical phase and

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$^4$ An infinite normalization $\delta(0)$ is involved in the relation $S_{\text{fermi}} \sim \mathcal{L}_{\text{fermion}}$, which is related to the problem of the scaling limit. We will simply absorb this in the coupling $g^2$. 
the nonabelian Berry phase [3, 5]. This latter phase factor is given by the path-ordered exponential of the loop integral of the connection one-form $A_\ell (X_M)$. We obtain

$$\lim_{\beta \to 0} \left( P \exp \left[ -i \int_0^\beta d\beta' E_\ell (X_M(\beta')) - i \oint_\Gamma A_\ell (X_M) \right] \right)^{\alpha \alpha'} .$$

Here

$$A_\ell (X_M) = -i \psi_\ell^\dagger d\psi_\ell = -i \sum_A \psi_\ell^{\alpha A} d\psi_\ell^{\alpha'} .$$

and the nonabelian gauge field $A_\ell (X_M)$ originates from the degenerate eigenfunction.

Finally letting $R$ large or the time period $\beta$ short, we find

$$\langle \langle \hat{P}^{\alpha \alpha'} \rangle \rangle_\Gamma = P \exp \left[ -i \oint_\Gamma A_\ell (x_M) \right]^{\alpha \alpha'} ,$$

separating the phase of topological origin from the dynamical phase.

III. Complete decomposition of the fermionic part of the action

In this section, we will show how to decompose the fermionic part of the action. The diagonal part of the fermions has no contribution to the action in the present treatment. We only need to focus on the off-diagonal part. After expanding by the bases of Lie algebra generators of $USp(2k)$, the fermionic part of the action finally can be classified into three types.

A. the case of the IIB matrix model

Let us consider the fermionic part of the action of IIB reduced matrix model:

$$S = \frac{1}{2} \text{Tr} \bar{\Psi} \Gamma^M [X_M, \Psi] .$$

The matrices are all $u(2k)$ Lie algebra valued. Here, for brevity, we ignore the coupling and the minus sign in the action. These will be put back in the next section.

$$X_M = \begin{pmatrix} x_M^1 & 0 & & \\ & \ddots & & \\ 0 & & \ddots & \\ & & & x_M^N \end{pmatrix} .$$

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5 With regard to the last footnote, we have adopted here the normalization of the energies/eigenvalues as quantum mechanics.
Using the set of bases defined by

\[(E_{ab})_{ij} \equiv \delta_{ai}\delta_{bj} \,, \quad (3.3)\]

we can write the bases of the Lie algebra generators of \(U(N)\), which are Hermitian matrices, as

\[H_a = E_{a,a} - E_{a+1,a+1} \quad (a = 1, \cdots, N - 1) \,, \quad (3.4)\]

\[S_{a,b} = E_{a,b} + E_{b,a} \quad (a < b) \,, \quad (3.5)\]

\[T_{a,b} = -i(E_{a,b} - E_{b,a}) \quad (a < b) \, . \quad (3.6)\]

Here

\[
H_a = \begin{pmatrix} 1 & \cr & -1 \end{pmatrix} \,, \quad S_{a,b} = \begin{pmatrix} 1 & \cr & 1 \end{pmatrix} \,, \quad T_{a,b} = \begin{pmatrix} -i & \cr & i \end{pmatrix} \, . \quad (3.7)
\]

Let us decompose \(\Psi\) into the diagonal part \(\psi\) and the off-diagonal part \(\chi\):

\[\Psi = \psi + \chi \, . \quad (3.8)\]

The component expansion of \(\chi\) reads

\[\chi = \sum_{a<b} \left( \chi^{ab}_S S_{ab} + \chi^{ab}_T T_{ab} \right) \, . \quad (3.9)\]

Now the action depends only on \(\chi_S\) and \(\chi_T\):

\[S = i \sum_{a<b} \left\{ -\chi^{ab}_S \Gamma^M (x^a_M - x^b_M) \chi^{ab}_T + \chi^{ab}_T \Gamma^M (x^a_M - x^b_M) \chi^{ab}_S \right\} \, . \quad (3.10)\]

where \(a, b = 1, \cdots, N\). Introducing

\[\chi^{ab}_U \equiv \chi^{ab}_S - i\chi^{ab}_T \, , \quad \chi^{ab}_L \equiv \chi^{ab}_S + i\chi^{ab}_T \, , \quad \mathcal{M}^{ab} = \Gamma^M (x^a_M - x^b_M) \, . \quad (3.11)\]

we find

\[S = \frac{1}{2} \sum_{a<b} \left\{ \chi^{ab}_U \mathcal{M}^{ab}_U \chi^{ab}_U - \chi^{ab}_L \mathcal{M}^{ab}_L \chi^{ab}_L \right\} \, . \quad (3.12)\]

This action belongs to the type I action in the next subsection.
B. the case of the $USp(2k)$ matrix model

Along the same line, the fermionic action in $USp(2k)$ matrix model can be decomposed. Here we pay most of our attention to the closed string sector of $USp(2k)$ model. The sector which the fundamental representation belongs to has been discussed in [9], and will be reviewed briefly at the end of this subsection. The fermionic part of the action takes the same form as eq. (3.1). But now the matrix fermion $\Psi$ is decomposed into the adjoint and the antisymmetric representation as is described in eqs. (2.8), (2.9). As for $X_M$, 

$$X_M = diag(x_M^1, \cdots, x_M^k, \rho(x_M^1), \cdots, \rho(x_M^k)) ,$$  \hspace{1cm} (3.13)

where the $\rho$ is a projection:

$$\rho: x_\mu \rightarrow -x_\mu , \hspace{0.5cm} \mu = 0, 1, 2, 3, 4, 7 ,$$  \hspace{0.5cm} \hspace{0.5cm} \hspace{0.5cm} (3.14)

The generators of $usp(2k)$ excluding the Cartan subalgebras are

$$S_{ab} - S_{a+k,b+k} , \hspace{0.5cm} T_{ab} + T_{a+k,b+k} , \hspace{0.5cm} a < b \in \{1, 2, \cdots, k\}$$

$$S_{a,b+k} + S_{b,a+k} , \hspace{0.5cm} T_{a,b+k} + T_{b,a+k} , \hspace{0.5cm} a < b \in \{1, 2, \cdots, k\}$$

$$S_{a,a+k} , \hspace{0.5cm} T_{a,a+k} , \hspace{0.5cm} a = 1, 2, \cdots, k$$  \hspace{1cm} (3.15)

while the generators of $asym(2k)$ excluding the diagonal ones are

$$S_{ab} + S_{a+k,b+k} , \hspace{0.5cm} T_{ab} - T_{a+k,b+k} , \hspace{0.5cm} a < b \in \{1, 2, \cdots, k\}$$

$$S_{a,b+k} - S_{b,a+k} , \hspace{0.5cm} T_{a,b+k} - T_{b,a+k} , \hspace{0.5cm} a < b \in \{1, 2, \cdots, k\} .$$  \hspace{1cm} (3.16)

The fermions $\Psi_{adj}$ and $\Psi_{asym}$ are expanded respectively by the bases eq. (3.15) and eq. (3.16).

After some algebraic manipulation which we leave in the appendix, we find that the fermionic action of $USp(2k)$ reduced model is expressed in terms of three types of actions

$$S = \frac{1}{2} \sum_{a<b} \mathcal{L}_I \left( \left( \Psi^{DU}_{adj} \right)^{ab} ; \left( \Psi^{DU}_{asym} \right)^{ab} ; x^a_M, x^b_M \right)$$

$$+ \sum_{a<b} \mathcal{L}_I \left( \left( \Psi^{DL}_{adj} \right)^{ab} ; \left( \Psi^{DL}_{asym} \right)^{ab} ; -x^a_M, -x^b_M \right)$$

$$+ \sum_{a<b} \mathcal{L}_{II} \left( \left( \Psi^{OU}_{adj} \right)^{ab} ; \left( \Psi^{OU}_{asym} \right)^{ab} ; x^a_M, x^b_M \right)$$

$$+ \sum_{a<b} \mathcal{L}_{II} \left( \left( \Psi^{OL}_{adj} \right)^{ab} ; \left( \Psi^{OL}_{asym} \right)^{ab} ; -x^a_M, -x^b_M \right)$$

$$+ \sum_a \mathcal{L}_{III} \left( \left( \Psi^{ODU}_{adj} \right)^a ; x^a_M - x^{a+k}_M \right)$$

$$+ \sum_a \mathcal{L}_{III} \left( \left( \Psi^{ODL}_{adj} \right)^a ; -x^a_M + x^{a+k}_M \right) \right) ,$$  \hspace{1cm} (3.17)
where

\[ \mathcal{L}_I (\Lambda, \Phi; x_M, y_M) \equiv 2 \left( \Lambda + \Phi \right) \Gamma^M (x_M - y_M) (\Lambda + \Phi) , \quad (3.18) \]

\[ \mathcal{L}_{II} (\Lambda, \Phi; x_M, y_M) \equiv 2 \left( \Lambda + \Phi \right) \Gamma^M (x_M - \rho (y_M)) (\Lambda + \Phi) , \quad (3.19) \]

\[ \mathcal{L}_{III} (\Lambda; x_\mu) \equiv \overline{\Lambda} \Gamma^\mu x_\mu \Lambda . \quad (3.20) \]

We call \( \mathcal{L}_I, \mathcal{L}_{II} \) and \( \mathcal{L}_{III} \) type I, type II and type III action respectively. See the appendix for detail of our notation.

It is obvious that only components from the diagonal blocks \( D \) contribute to type I action, while only components from \( O \) contribute to type II action. As for type III action, only components from \( OD \) contribute and they are all in the adjoint representation. We will find that the contribution from the fermions in the fundamental representation has the same form as type III action. We see that the part of the adjoint fermions and all of the antisymmetric fermions form Majorana-Weyl fermions while the remainder of the adjoint fermions decouple from the spacetime points lying in the antisymmetric representation. We indicate below the parts of the matrix degrees of freedom of the fermion \( \Psi \) contributing to \( \mathcal{L}_I, \mathcal{L}_{II} \) and \( \mathcal{L}_{III} \) by \( \bullet, \circ, \text{ and } \star \) respectively.

\[
\begin{pmatrix}
0 & \bullet & \bullet & \star & \circ & \circ \\
\bullet & 0 & \bullet & \circ & \star & \\
\bullet & \bullet & 0 & \circ & \circ & \star \\
\star & \circ & \circ & 0 & \bullet & \\
\circ & \star & \circ & \bullet & 0 & \\
\circ & \circ & \star & \bullet & 0 & 
\end{pmatrix}
\] \quad (3.21)

Apart from the fermions in the closed string sector, we also have the fermions belonging to the fundamental representation in eqs. (2.16) and (2.17). The action, up to the prefactor, reads

\[ S_F = \psi_Q \sigma^m X_m \cdot \psi_Q + \frac{1}{2} (\psi_Q \cdot \Sigma F (X_4 + iX_7 + M) \psi_Q + h.c.) . \quad (3.22) \]

As is already discussed in [8, 9], this action does not depend on \( X_n \) \( n = 5, 6, 8, 9 \) and is in fact type III;

\[ S_F = \sum_{a, f, \pm} \mathcal{L}_{III} \left( \psi_{Q(f)a}; x_\mu^a \pm m(f) \delta_{\mu,4} \right) . \quad (3.23) \]

See [8, 9] for more detail. The fermionic part of the action reads

\[ S_{fermion} \sim S + S_F . \quad (3.24) \]
IV. Computation of nonabelian Berry phase

We now proceed to the computation of the nonabelian Berry phase. Eqs. (4.17) are our results.

A. type III case

The generic type III action \( L_{\text{III}} \) in the last section do not really depend on the spacetime points \( x_n \) belonging to the antisymmetric representation. Let us first compute the nonabelian Berry phase for the Hamiltonian of this type. Putting back the prefactor discarded in the last section, we obtain the corresponding first quantized Hamiltonian:

\[
\mathcal{H} = \frac{1}{g^2} \sum_{\mu=1,2,3,4,7} x_\mu \gamma^\mu ,
\]

where \( \gamma^\mu \)'s are the five dimensional gamma matrices obeying the Clifford algebra. We take the following representation:

\[
\begin{align*}
\gamma^1 &= \sigma^1 \otimes \sigma^3 , & \gamma^2 &= \sigma^2 \otimes \sigma^3 , & \gamma^3 &= \sigma^3 \otimes \sigma^3 , \\
\gamma^4 &= 1^2 \otimes \sigma^1 , & \gamma^7 &= 1^2 \otimes \sigma^2 .
\end{align*}
\]

where \( \sigma^i \) are the Pauli matrices.

The eigenvalues of eq. (4.1) are \( \pm \frac{|x|}{g^2} \) with \( |x| \equiv \sqrt{\sum_{\mu=1,2,3,4,7} x_\mu^2} \) and each one is doubly degenerate. We focus on the two dimensional subspace of the one-particle states which belongs to the positive eigenvalue. The nonabelian Berry connection obtained will be \( su(2) \) Lie algebra valued. The eigenstates can be obtained with the help of projection operators, which are defined by

\[
P_{\pm} \equiv \frac{1}{2} (1 + y_\nu \gamma^\nu) ,
\]

where

\[
y_\nu \equiv \frac{x_\mu}{|x|}.
\]

parametrize \( S^4 \). These projection operators satisfy

\[
P_{\pm}^2 = P_{\pm} \quad P_{\pm}^1 = P_{\pm} \quad \mathcal{H} P_{\pm} = \pm \frac{|x|}{g^2} P_{\pm} .
\]

Let us denote by \( e_\alpha \) \((\alpha = 1, 2, 3, 4)\) the component representation of the unit vector in the \( i \)-th direction, i.e., the one nonvanishing only at the \( i \)-th position

\[
e_\alpha \equiv \left( \cdots, 0, 1, 0, \cdots \right) .
\]
The normalized eigenvectors with plus eigenvalue are

$$\psi_{\alpha} = \frac{1}{N_\alpha} P_+ e_{\alpha}. \quad (4.7)$$

Here, $$\psi_{\alpha}$$, $$\alpha = 1, 4$$ form a two-dimensional eigenspace well-defined around the north pole $$x_3 = |x|$$ while $$\psi_{\alpha}, \alpha = 2, 3$$ the one around the south pole $$x_3 = -|x|$$. We see that the origin of the degeneracy is in the spinor index. The $$N_\alpha$$ are the normalization factors:

$$N ≡ N_1 = N_4 = \sqrt{1 + y_3^2}, \quad N' ≡ N_2 = N_3 = \sqrt{1 - y_3^2}. \quad (4.8)$$

We focus our attention on the sections near the north pole. The Berry connection [3, 7] is

$$iA = \begin{pmatrix} \psi_1 \\ \psi_4 \end{pmatrix} d(\psi_1, \psi_4) = E \mathcal{M}^T \mathcal{E}, \quad (4.9)$$

where

$$\mathcal{M} ≡ \frac{1}{N} P^+ d \frac{1}{N} P_+, \quad (4.10)$$

$$\mathcal{E} = t(e_1, e_4). \quad (4.11)$$

Introducing

$$C_{\mu\nu} ≡ (y_\mu dy_\nu - y_\nu dy_\mu), \quad (4.12)$$

we obtain

$$\mathcal{M} = \frac{1}{1 + y_3} \left( \frac{1}{2} dy_\mu \gamma^\mu + \frac{1}{4} C_{\mu\nu} \gamma^\mu \gamma^\nu - \frac{1}{1 + y_3} dy_3 P_+ \right), \quad (4.13)$$

$$A(y_i) = \frac{1}{2(1 + y_3)} \mathbf{B} \cdot \sigma, \quad (4.14)$$

where

$$\mathbf{B} ≡ \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{bmatrix} y_7 dy_1 - y_1 dy_7 - y_2 dy_4 + y_4 dy_2 \\ y_1 dy_4 - y_4 dy_1 - y_2 dy_7 + y_7 dy_2 \\ y_4 dy_7 - y_7 dy_4 - y_2 dy_1 + y_1 dy_2 \end{bmatrix}. \quad (4.15)$$

Observe that $$y_3$$ appears only in the overall scale factor. Define

$$T ≡ \frac{1}{\sqrt{1 - y_3^2}} (y_2 \mathbf{1}_2 + i \mathbf{y} \cdot \sigma). \quad (4.16)$$

The Berry connection can then be rewritten as

$$A(y_\nu) = \frac{1 - y_3}{2} dTT^{-1}. \quad (4.17)$$
Eq. (4.17) is the form we have obtained in [9]. We now discuss our renewed understanding. We will show below that, by a further change of coordinates, the above non-Abelian connection $A$ can be brought exactly to the form of the BPST instanton configuration [18].

We are in the five dimensional space with coordinates $x_1, x_2, x_3, x_4, x_7$, and in this coordinate system the Berry connection seems to have a prefactor depending on $x_3$. As is seen in [12], however, the connection is independent of the radius if we work in the polar coordinate system. This means that we can consider the problem on $S^4$. From this point of view alone, $A$ is a nontrivial $SU(2)$ bundle over $S^4$ with second Chern number $±1$. In other words, it should be the BPST self-dual or anti-selfdual (ASD) instanton connection with instanton number $±1$. To show this explicitly, we make the following series of change of coordinates: first change to the polar coordinate system on $S^4$ and, via the stereographic projection, change to the orthogonal coordinate system on $R^4$ where the connection is made manifestly the ASD $SU(2)$ connection. The transformations which realize these are found to be

$$y_i = \frac{2z_i}{1 + z^2}, \quad i = 1, 2, 4, 7, \quad y_3 = \frac{1 - z^2}{1 + z^2}, \quad (4.18)$$

where $\{z_i\}$ parameterize $R^4$, and

$$T = \frac{1}{|z|} (z_2 \mathbf{1}_2 + iz \cdot \sigma) = \frac{1}{|z|} \hat{z}, \quad (4.19)$$

$$dT T^{-1} = \frac{1}{2|\hat{z}|^2} (d\hat{z} \bar{\hat{z}} - \hat{z} d\bar{\hat{z}}). \quad (4.20)$$

Here $\hat{z}$ is a quaternion

$$\hat{z} \equiv z_2 \mathbf{1}_2 + iz \cdot \sigma. \quad (4.21)$$

We obtain

$$A = \frac{1 - y_3}{2} dTT^{-1}$$

$$= \frac{z^2}{1 + z^2} dTT$$

$$= \frac{1}{1 + \hat{z}^2} \cdot \frac{1}{2} (d\hat{z} \bar{\hat{z}} - \hat{z} d\bar{\hat{z}}), \quad (4.22)$$

which is exactly the gauge connection for the ASD instanton on $R^4$ [19].

In terms of $z_i$, the Berry connection can also be written as

$$A = \frac{1}{2} \cdot \frac{1}{1 + z^2} (z_i dz_j - z_j dz_i) \sigma^{ij}, \quad (4.23)$$

---

6 In our notation $z_i$ are real numbers.
where
\[ \sigma^{ij} \equiv \frac{i}{2} \mathcal{E}[\gamma^i, \gamma^j] \mathcal{E} , \tag{4.24} \]
with components
\[ \sigma^{12} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} , \quad \sigma^{14} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} , \tag{4.25} \]
\[ \sigma^{17} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \quad \sigma^{24} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \tag{4.26} \]
\[ \sigma^{47} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} , \quad \sigma^{27} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} . \tag{4.27} \]

The curvature two-form of \( \mathcal{A} \) is
\[ \mathcal{F} = \frac{1}{(1 + z^2)^2} dz_i \wedge dz_j \sigma^{ij} . \tag{4.28} \]

As expected, the curvature satisfies the anti-selfdual condition
\[ \mathcal{F}_{\mu\nu} + \frac{1}{2} \epsilon_{\mu\nu\sigma\rho} \mathcal{F}^{\sigma\rho} = 0 , \tag{4.29} \]
and the instanton number, i.e., the second Chern class \( k = -C_2(P) \) is
\[ k = -\frac{1}{8\pi^2} Tr \int \mathcal{F} \wedge \mathcal{F} = -1 . \tag{4.30} \]

In the same way, the Berry connection corresponding to the minus eigenvalue can be evaluated. It is a selfdual instanton with \( k = 1 \).

For the sake of our discussion, it is more appropriate to regard this as pointlike nonabelian singularity located at the origin of five space dimensions. In fact, this is what is sometimes called Yang monopole [12] or an SU(2) monopole in five-dimensional flat space or four-dimensional spherical space. It has a nontrivial Chern number on \( S^4 \) and its topological stability is summarized by \( \pi_3(SU(2)) = \mathbb{Z} \) as is the case for the BPST instanton.

**B. type I, II**

Let us turn to the evaluation of the nonabelian Berry phase associated with the action of type I, II. The generic action and its Hamiltonian are respectively
\[ \mathcal{L}_{I,II} = -\frac{1}{g^2} \bar{\Psi} \Gamma^M x_M \Psi , \quad \mathcal{H}_F = \frac{1}{g^2} \bar{\Psi} \Gamma^M x_M \Psi . \tag{4.31} \]
Here $\Psi$ is the ten-dimensional Majorana-Weyl spinor. Working in the $v_0 = 0$ gauge, the relevant first quantized Hamiltonian is
\[
H = \frac{1}{g^2} \sum_{i=1,2,\ldots,9} \Gamma^0 \Gamma^i x_i .
\] (4.32)

As in the last subsection, we define a projection operator
\[
P_{\pm} \equiv \frac{1}{2} (1 \pm \frac{1}{|x|} \Gamma^0 \Gamma^i x_i) , \quad |x| \equiv \sqrt{\sum_{i=1}^9 (x_i)^2} .
\] (4.33)

This time, the projection operator acts on the Weyl projected sixteen dimensional space. The gamma matrices are understood to act on this space and are regarded as $16 \times 16$ matrices. With the help of this projection operator, the orthogonal eigenvectors can be constructed. In the case of the positive eigenvalue, the eigenvectors are
\[
\psi_\alpha = \frac{1}{N_\alpha} P_+ e_\alpha ,
\] (4.34)

\[
N^2 = N'^2 = \frac{1}{2} \left( 1 \pm \frac{x_3}{|x|} \right) , \quad \text{for } \alpha = 1, 3, 5, 7, 10, 12, 14, 16 ,
\]

Here the second line is well-defined around the north pole while the third line is well-defined around the south pole.

We focus on the eigenspace well defined around the north pole with the positive eigenvalue. The eigenspace forms an eight dimensional vector space. The nonabelian Berry phase is $su(8)$ Lie algebra valued one form and is given by
\[
A = -i \begin{pmatrix}
\psi_1 \\
\psi_3 \\
\vdots \\
\psi_{14} \\
\psi_{16}
\end{pmatrix} d (\psi_1, \psi_3, \cdots \psi_{14}\psi_{16}) ,
\]

\[
= E O^t E ,
\] (4.35)

where
\[
E \equiv \begin{pmatrix}
e_1, e_3, e_5, e_7, e_{10}, e_{12}, e_{14}, e_{16}
\end{pmatrix} ,
\] (4.36)

and
\[
O = -i \frac{1}{N} P_+ d \frac{1}{N} P_+ ,
\] (4.37)

\[
= -i \frac{1}{N^2} \left( -\frac{1}{2N^2} d (N)^2 P_+ + P_+ d P_+ \right) .
\]
Introduce the coordinates
\[ y_i \equiv \frac{x_i}{|x|} \] (4.38)
which parametrize \( S^8 \). We find
\[
\Omega = -i \frac{1}{1+y_3} \left( -\frac{1}{(1+y_3)} P_+ dy_3 + P_+ \Gamma^0 \Gamma^i dy_i \right) ,
\]
\[
= -i \frac{1}{2(1+y_3)} \left\{ \frac{1}{(1+y_3)} \left( -1_{16} dy_3 - \Gamma^0 \Gamma^i y_i dy_3 + (1+y_3) \Gamma^0 \Gamma^i dy_i \right) 
- \frac{1}{4} [\Gamma^i, \Gamma^j] C_{ij} \right\} .
\]
(4.39)
Here we have introduced
\[ C_{ij} \equiv y_i dy_j - y_j dy_i \] (4.40)
and used
\[ \sum_{i=1,2,\ldots,9} y_i dy_i = 0 \] (4.41)
Observe that
\[
-1_{16} dy_3 - \Gamma^0 \Gamma^i y_i dy_3 + (1+y_3) \Gamma^0 \Gamma^i dy_i
= (1_{16} + \Gamma^0 \Gamma^3) dy_3 + \sum_{A=1,2,4,5,6,7,8,9} \Gamma^0 \Gamma^A (dy_A + C_{3A}) ,
\]
(4.42)
as well as
\[
\mathbf{E} \left\{ (1_{16} + \Gamma^0 \Gamma^3) dy_3 + \sum_{A} \Gamma^0 \Gamma^A (dy_A + C_{3A}) \right\} \mathbf{E} = 0 ,
\]
(4.43)
due to the orthogonality of different eigenvectors. By the same reason, the last line in eq. (4.33) with \( y_3 \) term involved will not contribute to the Berry connection either. We conclude that only the last term of \( \Omega \) with \( i, j \neq 3 \) contributes to the nonabelian Berry phase.

Notice that the vector \( \mathbf{E} \) satisfies
\[
\begin{align*}
\mathbf{t} \mathbf{E} \mathbf{E} &= \frac{1}{2} \left( 1 + \Gamma^0 \Gamma^3 \right) , \\
\Gamma^0 \Gamma^i \mathbf{t} \mathbf{E} \mathbf{E} + \mathbf{t} \mathbf{E} \mathbf{E} \Gamma^0 \Gamma^i &= \Gamma^0 \Gamma^i , \text{ for } i = 1, 2, 4, 5, 6, 7, 8, 9 .
\end{align*}
\]
(4.44)
(4.45)
Defining \( 8 \times 8 \) matrix
\[
\Sigma^{ij} \equiv \frac{i}{2} \mathbf{E} [\Gamma^i, \Gamma^j] \mathbf{E} ,
\]
(4.46)
we obtain
\[
\mathcal{A} = \frac{1}{4(1+y_3)} C_{ij} \Sigma^{ij} .
\]
(4.47)
Here the indices $i, j$ run over the eight directions. The generators $\Sigma^{ij}$ is found to form an $so(8)$ algebra:

\[
[\Sigma^{ij}, \Sigma^{kl}] = 2i \left( \delta^{jk} \Sigma^{il} - \delta^{jl} \Sigma^{ik} - \delta^{ik} \Sigma^{jl} + \delta^{il} \Sigma^{jk} \right).
\] (4.48)

The curvature is

\[
\mathcal{F} = dA - iA \wedge A,
\]
\[
= \frac{1}{4} dy^i \wedge dy^j \Sigma^{ij} - \frac{1}{4(1 + y_3^2)} dy_3 \wedge C_{ij} \Sigma^{ij}.
\] (4.49)

As in the case of the $SU(2)$ Berry connection, we change our coordinate system to that on $\mathbb{R}^8$:

\[
y_i = \frac{2z_i}{1 + z^2}, \quad y_3 = \frac{1 - z^2}{1 + z^2}.
\] (4.50)

Here $z_i, (i = 1, 2, 4, \cdots, 9)$ are the orthogonal coordinates on $\mathbb{R}^8$. The Berry connection can be rewritten as

\[
A = \frac{1}{2(1 + z^2)} (z_i dz_j - z_j dz_i) \Sigma^{ij}.
\] (4.51)

The curvature of this connection is

\[
\mathcal{F} = \frac{1}{(1 + z^2)^2} dz_i \wedge dz_j \Sigma^{ij}.
\] (4.52)

Let us see if this eigenbundle is nontrivial. The nontrivial $SU(8)$ bundle on $S^8$ is characterized by the seventh homotopy group

\[
\Pi_7(SU(8)) = \mathbb{Z}.
\] (4.53)

This leads us to compute the fourth Chern number using eq. (4.52) derived from our eigenbundle:

\[
C_4(P) = \int c_4(P) = \left( \frac{i}{2\pi} \right)^4 \cdot \frac{1}{4!} \int Tr(\mathcal{F}^4) = 1.
\] (4.54)

Our eigenbundle is in fact nontrivial and has the fourth Chern number 1.

V. Spacetime picture emerging from our computation

Let us recall the generic formula (eq. (2.38)) stated in the end of section two:

\[
\langle\langle \hat{T}^{\alpha\alpha'} \rangle\rangle_\Gamma = P \exp \left[ -i \oint_\Gamma A_\ell(x_M) \right]^{\alpha\alpha'}.
\] (5.1)
The results from our computation in the last section are summarized as eqs. (4.17), (4.23):

\[ A_{SU(2)} = \frac{1 - y_3}{2} \mathrm{d}TT^{-1}, \]

\[ A_{SU(2)} = \frac{1}{2} \cdot \frac{1}{1 + z^2} (z_i \mathrm{d}z_j - z_j \mathrm{d}z_i) \sigma^{ij}, \]

\[ \sigma^{ij} \equiv \frac{i}{2} \mathcal{E}[\gamma^i, \gamma^j] \mathcal{E}, \]

for the generic type III action \( L_{III} \), giving the \( SU(2) \) monopole and as eqs. (4.47), (4.51):

\[ A_{SU(8)} = \frac{1}{4(1 + y_3)} C_{ij} \Sigma^{ij}, \]

\[ A_{SU(8)} = \frac{1}{2} \cdot \frac{1}{1 + z^2} (z_i \mathrm{d}z_j - z_j \mathrm{d}z_i) \Sigma^{ij}, \]

\[ \Sigma^{ij} \equiv \frac{i}{2} \mathcal{E}[\Gamma^i, \Gamma^j] \mathcal{E}, \]

for the generic type I,II action \( L_{I,II} \), giving the \( SU(8) \) monopole. Putting these together, we state that the expectation value of the projector of a fermionic eigenmode is given by the path-ordered exponential of the integration of the connection one-form and that this factor in the case of \( USp \) matrix model is controlled by the \( SU(2) \) or the \( SU(8) \) nonabelian monopole singularity sitting at the origin of the parameter space \( X_M \). In the case of the \( SU(8) \) monopole, it is a pointlike singularity in nine dimensions while in the case of the \( SU(2) \) Yang monopole it is a singularity which does not depend on the four antisymmetric directions. The latter one, viewed as a singularity in the entire space, is not pointlike but is actually a four dimensionally extended object. The emergence of these interesting objects, albeit being aposteriori, justifies the study of this expectation value rather than of the fermionic part of the partition function.

The matrix models in general contain many species of fermions, which couple to different spacetime points or \( D0 \) branes. They provide a collection of nonabelian Berry phases rather than just one. With this respect, it is more appropriate to consider the manybody counterpart indicated in section two (eq. (2.28)):

\[
\langle \langle \prod_{\ell \in I_+} \sum_\alpha \hat{\mathcal{P}}_{\ell}^{\alpha} \rangle \rangle_{\Gamma} = \langle \langle \prod_{\ell \in I_+} \sum_\alpha \hat{\mathcal{P}}_{\ell}^{\alpha} \rangle \rangle_{\Gamma} = \prod_{\ell \in I_+} \mathrm{tr}_{\ell} P \exp \left[ -i \int_{\Gamma} \mathcal{A}_{\ell}(x_M) \right], \tag{5.2}
\]

where the subset \( I_+ \) of all eigenmodes is taken over the eigenmodes belonging to the positive eigenvalues.

Actually, \( S_{\text{fermion}} \sim S + S_F \) of the \( USp \) matrix model contains many terms consisting of the generic \( L_{I,II} \) type action as well as the generic \( L_{III} \) type action. Listing the parameters
which they depend on, we obtain from eq. (3.17)

\[ i) \quad x^a_M - x^b_M, - (x^a_M - x^b_M) \]
\[ ii) \quad x^a_M - \rho (x^b_M), - (x^a_M - \rho (x^b_M)) \]
\[ iii) \quad 2x^a_\mu, -2x^a_\mu. \] (5.3)

Singularities occur when the two points \( x^a_M \) and \( x^b_M \) collide in the case of the first line, when \( x^a_M \) and \( \rho (x^b_M) \) collide in the case of the second line and when \( x^a_M \) lies in the orientifold surface. Similarly from eq. (3.23), we obtain

\[ iv) \quad x^a_\mu \pm m(f) \delta_{\mu 0} \] (5.4)

The singularities occur when \( x^a_\mu \) is away from the orientifold surface in the \( x_4 \) direction by \( \pm m(f) \). The situation is depicted in Figure 1.

We denote the first six cases of eq. (5.3) by \( x^{ab(K)}_M \) \( K = 1, \ldots 6. \) and the last two cases of eq. (5.3) and the case of eq. (5.4) (sum over \( a \) and \( f \)) by \( x^{a(K')}_\mu \). In the case where more than two spacetime points collide, we will obtain an enhanced symmetry which supports a nonabelian monopole.

As for eq. (5.2), the subset \( I_+ \) is taken over the eigenmodes seen in eq. (5.3) and eq. (5.4). To write this more explicitly,

\[ \prod_K \prod_{a < b} \text{tr}_8 P \exp \left[ -i \oint_{\Gamma} A_{SU(8)} (x^{ab(K)}_M) \right] \prod_K \prod_a \text{tr}_2 P \exp \left[ -i \oint_{\Gamma} A_{SU(2)} (x^{a(K')}_\mu) \right]. \] (5.5)

In the context of a matrix model for unified theory of superstrings, measuring eq. (2.28), or eq. (2.38) will provide means to examine spacetime formation suggested by the model.
The presence of colliding singularities of spacetime points in general means a dominant probability to such configurations. As we have said, there are two varieties of singularities we have exhibited in this paper. The singularities of the $SU(8)$ monopoles appear to be evenly distributed in all directions as long as the off-diagonal bosonic integrations are ignored. This type is present both in the $IIB$ matrix model and the $USp$ matrix model. The singularities of the $SU(2)$ monopoles appear to be a string soliton of four dimensional extension present in the USp matrix model and is not shared by the IIB matrix model. It is tempting to think that the four dimensional structure is formed by a collection of the $SU(2)$ monopoles. We put the spacetime picture emerging from contributions of various eigenmodes of the $USp$ matrix model as a table.

|                  | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|------------------|---|---|---|---|---|---|---|---|---|
| adjoint(D) + antisymmetric(D) | × | × | × | × | × | × | × | × | × |
| adjoint(O) + antisymmetric(O) | × | × | × | × | × | × | × | × | × |
| adjoint(OD)       | × | × | × | □ | □ | × | □ | □ | □ |
| fundamental      | × | × | × | □ | □ | × | □ | □ | □ |
| orientifold       | × | × | × | □ | □ | × | □ | □ | □ |

As for the $v_0$ direction, we have used this to parametrize the path in the remaining nine directions. The price we have to pay is that we have nothing to say on the distributions of the spacetime points in this direction and this is closely related to the problem of the scaling limit. Further progress and understanding of the spacetime formation of matrix models require overcoming this point.

VI. Generalization of the Yang monopole to odd dimensions

In section IV, we have discussed the two kinds of nonabelian Berry connections which have led us to the nontrivial $SU(2)$ and $SU(8)$ vector bundles over the parameter space. The tools which have brought us to these bundles are the first quantized Hamiltonian with gamma matrices and the orthogonal projection operators. Letting aside the physics context, these are certainly generalizable to arbitrary odd dimensions. In fact, the relevant mathematical discussion of such eigenbundle can be found for instance in [20].

Let $\Gamma : \mathbb{R}^{m+1} \to End(C^k)$ be a linear map. We assume

$$\Gamma(x)^2 = |x|^2 \cdot I_k \quad \text{and} \quad \Gamma(x)^* = \Gamma(x).$$

Let

$$\Gamma(x) = x_j \Gamma^j.$$
Such $\Gamma^i$ satisfy the Clifford commutation rules:

$$\Gamma^i \Gamma^j + \Gamma^j \Gamma^i = 2\delta^{ij}. \quad (6.3)$$

That is, $\Gamma^i$ are in fact gamma matrices. It is said that $\Gamma^i$ give a Cliff$(R^{m+1})$ module structure to $C^k$.

Let

$$P_\pm = \frac{1}{2} (1 \pm \Gamma(x)) \quad \text{for} \quad |x| = 1. \quad (6.4)$$

be an orthogonal projection onto the $\pm 1$ eigenspace of $\Gamma(x)$. Let

$$\Pi^\Gamma_\pm = \{(x, \nu) \in S^m \times C^k : \Gamma(x) \nu = \pm \nu\} \quad (6.5)$$

be the corresponding eigenbundle. Clearly

$$S^m \times C^k = \Pi^\Gamma_+ \bigoplus \Pi^\Gamma_- \quad (6.6)$$

It is obvious that the orthogonal projection defined here is exactly the projection operator defined from our first quantized Hamiltonian. The Berry connection is just the connection on eigenbundle $\Pi^\Gamma_+$ or $\Pi^\Gamma_-$. It has been shown that the Chern number of this eigenbundle is

$$\int_{S^2} c_j(\Pi^\Gamma_+) = i^j 2^{-j} \text{Tr}(\Gamma^0 \cdots \Gamma^{2j}). \quad (6.7)$$

From this formula on the Chern number, we see that when $m$ is odd, $\text{Tr}(\Gamma^0 \cdots \Gamma^{m})$ vanishes and the eigenbundle is trivial. When $m$ is even, we find

$$\text{Tr}(\Gamma^0 \cdots \Gamma^{m}) = 2^m (-i)^{m} \quad (6.8)$$

Therefore the $\frac{m^2}{2}$-th Chern number should be 1. We have proven this explicitly by deriving the $SU(2)$ Berry connection on $\mathbb{R}^5$ (the $m = 4$ case), and the $SU(8)$ Berry connection on $\mathbb{R}^9$ (the $m = 8$ case), and the curvature two-forms associated with them. The $U(1)$ Berry connection on $\mathbb{R}^3$ (the $m = 2$ case) is well known. The Berry connection on $\mathbb{R}^7$ (the $m = 6$ case) may be related to some configuration in the reduced matrix model.

Having formulated the eigenbundles associated with the nonabelian Berry phase in arbitrary odd dimensions, we can safely state that the connection one form and the curvature two form on $\mathbb{R}^{m+1}$ are given by

$$\mathcal{A} = \frac{1}{2(1+z^2)} (z_i dz_j - z_j dz_i) \Sigma^{ij} \quad (6.9)$$

and

$$\mathcal{F} = \frac{1}{(1+z^2)^2} dz_i \wedge dz_j \Sigma^{ij} \quad (6.10)$$

respectively. Here $z_i$ are orthogonal coordinates on $\mathbb{R}^m$. 

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Appendix

In this appendix, we give some details of the derivation of the component expression of the action (eq. (3.17)). Let us first expand $\Psi_{\text{adj}}$ and $\Psi_{\text{asym}}$ by the generators stated respectively in eq. (3.15) and in eq. (3.16):

\[
\Psi_{\text{adj}} = \sum_{a=1}^{k} \left\{ \left( \Psi_{\text{adj}}^{ODS} \right)^a S_{a,a+k} + \left( \Psi_{\text{adj}}^{ODT} \right)^a T_{a,a+k} \right\} + \sum_{a<b} \left\{ \left( \Psi_{\text{adj}}^{DS} \right)^{ab} (S_{ab} - S_{a+k,b+k}) + \left( \Psi_{\text{adj}}^{DT} \right)^{ab} (T_{ab} + T_{a+k,b+k}) \right\} 
+ \left( \Psi_{\text{adj}}^{OS} \right)^{ab} (S_{a,b+k} + S_{b,a+k}) + \left( \Psi_{\text{adj}}^{OT} \right)^{ab} (T_{a,b+k} + T_{b,a+k}) \right\} , \quad (A.1)
\]

\[
\Psi_{\text{asym}} = \sum_{a<b} \left\{ \left( \Psi_{\text{asym}}^{DS} \right)^{ab} (S_{ab} + S_{a+k,b+k}) + \left( \Psi_{\text{asym}}^{DT} \right)^{ab} (T_{ab} - T_{a+k,b+k}) \right\} 
+ \left( \Psi_{\text{asym}}^{OS} \right)^{ab} (S_{a,b+k} - S_{b,a+k}) + \left( \Psi_{\text{asym}}^{OT} \right)^{ab} (T_{a,b+k} - T_{b,a+k}) \right\} . \quad (A.2)
\]

Let us explain our notation more carefully. $\Psi_{\text{adj}}$ is expanded by eq. (3.15) which consists of three sets of generators: the first set of generators (the first line of eq. (3.15)) is in the off-diagonal elements of the diagonal blocks; the second set of generators (the second line of eq. (3.15)) is in the off-diagonal elements of off-diagonal blocks; and the third set of generators (the third line in eq. (3.15)) is in the diagonal elements of the off-diagonal blocks. We, therefore, distinguish these generators by $D$ (the diagonal blocks), $O$ (off-diagonal elements of the off-diagonal blocks) and $OD$ (diagonal elements in the off-diagonal blocks) respectively. In each set of the generators, there are contributions from both the real part $S$ and the purely imaginary part $T$. We distinguish these by the superscript $S$ and $T$.

As for $\Psi_{\text{asym}}$, the bases are $O$ type and $D$ type only. In the above expansion, (eqs. (A.1), (A.2)), the component fields are specified by the superscript specifying the species of the generators and the subscript specifying the representation. For example, $\left( \Psi_{\text{adj}}^{ODS} \right)$ are the component fields of the adjoint fermions which correspond to the expansion coefficients of the $S$ type $OD$ generators. The same is true for the other components.

Let us denote

\[-i(\bar{\psi}^S \mathcal{M} \psi^T - \bar{\psi}^T \mathcal{M} \psi^S) = 2\text{Im}(\bar{\psi}^S \mathcal{M} \psi^T) . \quad (A.3)\]

The component action can then be written as

\[
S = 4\text{Im} \sum_{a<b} \left\{ \left( \Psi_{\text{adj}}^{DS} \right)^{ab} + \left( \Psi_{\text{asym}}^{DS} \right)^{ab} \right\} \mathcal{M}^{ab} \left\{ \left( \Psi_{\text{adj}}^{DT} \right)^{ab} + \left( \Psi_{\text{asym}}^{DT} \right)^{ab} \right\} 
+ 4\text{Im} \sum_{a<b} \left\{ \left( \Psi_{\text{adj}}^{DS} \right)^{ab} + \left( \Psi_{\text{asym}}^{DS} \right)^{ab} \right\} \mathcal{M}^{a+k,b+k} \left\{ \left( \Psi_{\text{adj}}^{DT} \right)^{ab} - \left( \Psi_{\text{asym}}^{DT} \right)^{ab} \right\} 
\]
Here $M^{ab} = x^a - x^b$. Let us redefine the component fields, introducing complex notation.

$$
\Psi_{adj}^{DU} \equiv \Psi_{adj}^{DS} - i\Psi_{adj}^{DT} , \quad \Psi_{adj}^{DL} \equiv \Psi_{adj}^{DS} + i\Psi_{adj}^{DT} , \quad \Psi_{adj}^{OU} \equiv \Psi_{adj}^{OS} - i\Psi_{adj}^{OT} , \quad \Psi_{adj}^{OL} \equiv \Psi_{adj}^{OS} + i\Psi_{adj}^{OT} .
$$

(A.5)

We obtain

$$
S = \sum_{a<b} \left\{ \left(\Psi_{adj}^{DU}\right)^{ab} + \left(\Psi_{asym}^{DU}\right)^{ab} \right\} M^{ab} \left\{ \left(\Psi_{adj}^{DU}\right)^{ab} + \left(\Psi_{asym}^{DU}\right)^{ab} \right\}
+ \sum_{a<b} \left\{ -\left(\Psi_{adj}^{DL}\right)^{ab} + \left(\Psi_{asym}^{DL}\right)^{ab} \right\} M^{a+k,b+k} \left\{ \left(\Psi_{adj}^{DL}\right)^{ab} - \left(\Psi_{asym}^{DL}\right)^{ab} \right\}
- \sum_{a<b} \left\{ -\left(\Psi_{adj}^{DL}\right)^{ab} + \left(\Psi_{asym}^{DL}\right)^{ab} \right\} M^{a+k,b+k} \left\{ \left(\Psi_{adj}^{UL}\right)^{ab} - \left(\Psi_{asym}^{UL}\right)^{ab} \right\}
+ \sum_{a} \left(\Psi_{adj}^{ODU}\right)^{a} M^{a} \left(\Psi_{adj}^{ODU}\right)^{a}
- \sum_{a} \left(\Psi_{adj}^{ODL}\right)^{a} M^{a} \left(\Psi_{adj}^{ODL}\right)^{a}
+ \sum_{a<b} \left\{ \left(\Psi_{adj}^{OU}\right)^{ab} + \left(\Psi_{asym}^{OU}\right)^{ab} \right\} M^{a,b+k} \left\{ \left(\Psi_{adj}^{OU}\right)^{ab} + \left(\Psi_{asym}^{OU}\right)^{ab} \right\}
+ \sum_{a<b} \left\{ \left(\Psi_{adj}^{OL}\right)^{ab} - \left(\Psi_{asym}^{OL}\right)^{ab} \right\} M^{a,b+k} \left\{ \left(\Psi_{adj}^{OL}\right)^{ab} - \left(\Psi_{asym}^{OL}\right)^{ab} \right\}
- \sum_{a<b} \left\{ \left(\Psi_{adj}^{OL}\right)^{ab} - \left(\Psi_{asym}^{OL}\right)^{ab} \right\} M^{a,b+k} \left\{ \left(\Psi_{adj}^{OL}\right)^{ab} - \left(\Psi_{asym}^{OL}\right)^{ab} \right\} .
$$

(A.6)

Introduce three types of actions consisting of fermion bilinears:

$$
\mathcal{L}_I(\Lambda, \Phi; x_M, y_M) \equiv (\Lambda + \Phi) \Gamma^M (x_M - y_M) (\Lambda + \Phi)
$$
\[ - (\Lambda - \Phi) \Gamma^M (\rho(x_M) - \rho(y_M)) (\Lambda - \Phi) \quad , \tag{A.7} \]

\[ \mathcal{L}_{II}(\Lambda, \Phi; x_M, y_M) \equiv \left( \Lambda + \Phi \right) \Gamma^M (x_M - \rho(y_M)) (\Lambda + \Phi) \]

\[ - (\Lambda - \Phi) \Gamma^M (\rho(x_M) - y_M) (\Lambda - \Phi) \quad , \tag{A.8} \]

\[ \mathcal{L}_{III}(\Lambda; x_M) \equiv \Lambda \Gamma^\nu x_\nu \Lambda \ . \tag{A.9} \]

Notice the relation

\[ \Gamma^0 \Gamma^M (x_M - y_M) - \Gamma^0 H \Gamma^M (\rho(x_M) - \rho(y_M)) H = 2 \Gamma^0 \Gamma^M (x_M - y_M) \quad , \tag{A.10} \]

with

\[ H \equiv \text{diag}(1, 1, 1, -1, -1, -1, 1, 1, 1, -1, -1, -1) \] . \tag{A.11} \]

The above three types are simplified to be

\[ \mathcal{L}_I \equiv 2 \left( \Lambda + \Phi \right) \Gamma^M (x_M - y_M) (\Lambda + \Phi) \quad , \tag{A.12} \]

\[ \mathcal{L}_{II} \equiv 2 \left( \Lambda + \Phi \right) \Gamma^M (x_M - \rho(y_M)) (\Lambda + \Phi) \quad , \tag{A.13} \]

\[ \mathcal{L}_{III} \equiv \Lambda \Gamma^\nu x_\nu \Lambda \ . \tag{A.14} \]

It is easy to see that the action \( \mathcal{S} \) can be written in terms of \( \mathcal{L}_I, \mathcal{L}_{II}, \) and \( \mathcal{L}_{III} \), as is seen in the text (eq. (3.17)).
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