ON AN EXPLICIT ZERO-FREE REGION FOR THE DEDEKIND ZETA-FUNCTION

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Abstract. We establish new explicit zero-free regions for the Dedekind zeta-function. Two key elements of our proof are a non-negative, even, trigonometric polynomial and explicit upper bounds for the explicit formula of the so-called differenced logarithmic derivative of the Dedekind zeta-function. The improvements we establish over the last result of this kind come from two sources. First, our computations use a polynomial which has been optimised by simulated annealing for a similar problem. Second, we establish sharper upper bounds for the aforementioned explicit formula.

1. Introduction

Let $K$ be an algebraic number field and $L$ be a normal extension of $K$ with Galois group $G = \text{Gal}(L/K)$. Suppose $d_L, d_K$ denote the absolute values of the respective discriminant, $n_L = [L : \mathbb{Q}]$ and $n_K = [K : \mathbb{Q}]$. The Dedekind zeta-function of $L$ is denoted and defined for $\Re(s) > 1$ by

$$\zeta_L(s) = \sum_{\mathfrak{P}} \frac{1}{N(\mathfrak{P})^s},$$

where $\mathfrak{P}$ ranges over the non-zero ideals of $\mathcal{O}_L$. If $n_L = a + b$, then one can also consider the completed zeta-function

$$\xi_L(s) = s(s - 1)d_L \pi \frac{a}{2} \Gamma \left( \frac{s}{2} \right) \pi \frac{b}{2} \Gamma \left( \frac{s + 1}{2} \right) \zeta_L(s)$$

such that

$$\gamma_L(s) = \pi^{-\frac{a}{2}} \Gamma \left( \frac{s}{2} \right) \pi^{-\frac{b}{2}} \Gamma \left( \frac{s + 1}{2} \right).$$

Here, $\xi_L$ is an entire function satisfying the functional equation $\xi_L(s) = \xi_L(1 - s)$. It can be seen that $\xi_L$ is meromorphic on the complex plane with exactly one simple pole at $s = 1$. Let $\mathcal{P}$ denote a prime ideal of $K$ and $P$ denote a prime ideal of $L$. If $\mathcal{P}$ is unramified in $L$, then the Artin symbol,

$$\left[ \frac{L/K}{\mathcal{P}} \right],$$

denotes the conjugacy class of Frobenius automorphisms corresponding to prime ideals $P|\mathcal{P}$. For each conjugacy class $C \subset G$, the prime ideal counting function is

$$\pi_C(x, L/K) = \# \left\{ \mathcal{P} : \mathcal{P} \text{ unramified in } L, \left[ \frac{L/K}{\mathcal{P}} \right] = C, N_K(\mathcal{P}) \leq x \right\}.$$
In 1926, Chebotarëv [2] proved the Chebotarëv density theorem, which states that
\[
\pi_C(x, L/K) \sim \frac{\#C}{\#G} \text{Li}(x) = \frac{\#C}{\#G} \int_2^x \frac{dt}{\log t} \quad \text{as } x \to \infty.
\]
For example, if \( L = K = \mathbb{Q} \), then the Chebotarëv density theorem restates the prime number theorem. Moreover, if \( \omega = e^{2\pi i/\ell} \) is the \( \ell \)th root of unity, \( K = \mathbb{Q} \) and \( L = \mathbb{Q}(\omega^r) \), then the Chebotarëv density theorem identifies with the Dirichlet theorem for primes in arithmetic progressions.

In 1977, Lagarias–Odlyzko [9] provided explicit estimates for the error term of the Chebotarëv density theorem. There are two results contained therein; one version assumes the generalised Riemann hypothesis (GRH) for \( \zeta_L \) and the other does not. Their error term is effectively computable, dependent only on \( x, n_L, d_L \) and \( \#C/\#G \).

Under the GRH for \( \zeta_L \), one can obtain the best possible effective results. Without assuming the GRH for \( \zeta_L \), the better the zero-free region for \( \zeta_L \) one has, the better the effective result one can achieve. Therefore, the objective of this paper is to improve the best known, explicit zero-free region for \( \zeta_L \), given by Kadiri [6] in 2012.

We recall two famous forms of zero-free regions for the Riemann zeta-function.

**Classical zero-free region.** In 1899, de la Vallée Poussin [3] famously proved that there exists a positive constant \( R \) such that \( \zeta \) is non-zero in the region \( s = \sigma + it \) such that \( t \geq T \) and
\[
\sigma \geq 1 - \frac{1}{R \log t}.
\]
The best known zero-free region for \( \zeta \) of this kind is attributed to Mossinghoff–Trudgian [12], who verified (1) for \( R \approx 5.573 \) and \( T = 2 \).

**Koborov–Vinogradov zero-free region.** In 1958, Koborov [8] and Vinogradov [15] independently demonstrated that there exists a positive constant \( R_1 \) such that \( \zeta \) is non-zero in the region \( s = \sigma + it \) such that \( t \geq T \) and
\[
\sigma \geq 1 - \frac{1}{R_1 (\log t)^{3/2} (\log \log t)^{1/2}}.
\]
The best known zero-free region for \( \zeta \) of this kind is attributed to Ford [4], who has verified (2) for \( R_1 = 57.54 \) and \( T = 3 \). Ford [4] also establishes the zero-free region (2) for large \( t \) with \( R_1 = 49.13 \).

Naturally, the closest form of the zero-free region for \( \zeta_L \) will also depend on the extra variables \( d_L \) and \( n_L \). However, the method we adopt is based on de la Vallée Poussin’s method for determining the classical zero-free region for \( \zeta \). One complication is that a so-called exceptional zero could exist inside a zero-free region for \( \zeta_L \). If this exceptional zero exists, then it must be simple and real.

Kadiri [6, Theorem 1.1] was the last to re-purpose de la Vallée Poussin’s proof (using Stečkin’s [13] so-called differencing trick) to obtain a zero-free region for \( \zeta_L \). In this paper, we will establish Theorem [1] a new zero-free region for \( \zeta_L \) which builds upon Kadiri’s zero-free region for \( \zeta_L \). We will also establish Theorem [2] which will reveal more information pertaining to the exceptional zero.
\textbf{Theorem 1.} Suppose \((C_1, C_2, C_3, C_4) = (12.2411, 9.5347, 0.05017, 2.2692)\), then \(\zeta_L(\sigma + it)\) is non-zero for
\[
\sigma \geq 1 - \frac{1}{C_1 \log d_L + C_2 \cdot n_L \log |t| + C_3 \cdot n_L + C_4} \quad \text{and} \quad |t| \geq 1.
\] (3)

\textbf{Theorem 2.} For asymptotically large \(d_L\) and \(R = 12.43436\), \(\zeta_L(\sigma + it)\) has at most one zero in the region
\[
\sigma \geq 1 - \frac{1}{R \log d_L} \quad \text{and} \quad |t| < 1
\] (4)

If this exceptional zero exists, then it is simple and real.

Kadiri [6] established (3) with \((C_1, C_2, C_3, C_4) = (12.55, 9.69, 3.03, 58.63)\). To yield Theorem 1, we will follow a similar process to Kadiri, but observe some improvements. An important step in the proof of Theorem 1 is to choose a polynomial \(p_n(\varphi)\) from the so-called the class of non-negative, trigonometric polynomials of degree \(n\); denoted and defined by
\[
P_n := \left\{ p_n(\varphi) = \sum_{k=0}^{n} a_k \cos(k\varphi) : p_n(\varphi) \geq 0 \text{ for all } \varphi, a_k \geq 0 \text{ and } a_0 < a_1 \right\}.
\]

Whereas Kadiri worked with polynomials from \(P_4\), we will use the same polynomial from \(P_{16}\) as Mossinghoff–Trudgian [12]. This polynomial has been optimised by simulated annealing for computations pertaining to their computations for the zero-free region for \(\zeta\). This amendment contributed all of the improvements that can be seen for \(C_1\) and \(C_2\). In fact, if one re-runs Kadiri’s computations, only updating the polynomial, then this establishes (3) with \((C_1, C_2, C_3, C_4) = (12.2411, 9.5347, 3.3492, 57.7027)\).

Another improvement follows from improvements we have made to [11, Lemma 2] from McCurley. In particular, we improve explicit values for \(S(k)\), a computable constant dependent on \(k \in \mathbb{N}\). These improvements will contribute almost all of the improvement one observes for \(C_3\).

Kadiri [6] also established (4) with \(R = 12.7305\). To yield Theorem 2 we will recycle bounds from [6 §3] and apply the same higher degree polynomial from \(P_{16}\). A corollary of the method we use to establish Theorem 2 is an improvement to a well-known region by Stark [13]. However, because we only update the polynomial for this method, we cannot improve Stark’s result further than [6 Corollary 1.2] already does.

Finally, if an exceptional zero \(\beta_1\) exists, then one can enlarge the zero-free region in Theorem 2 using the Deuring-Heilbronn phenomenon [10]. This was one of the key ingredients in work by Ahn–Kwon [1], Zaman [16] and Kadiri–Ng–Wong [7], which pertains to the least prime ideal in the Chebotarëv density theorem.

\textbf{Remark.} The method of proof which we follow does not use Heath-Brown’s version of Jensen’s formula [5 Lemma 3.2], although this might yield better zero-free regions than those we can obtain using this method. This is partially because there does not exist a general sub-convexity bound for general number fields, so it is
difficult to apply his approach in the number field setting — see Kadiri [6] for an excellent explanation of this.

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2. Proof of Theorem 1

The set-up of our proof for Theorem 1 is the same as that which Kadiri uses in her proof of [6, Theorem 1.1], which has a similar shape to Stečkin’s argument [14] for $\zeta$. Suppose $t \geq 1$. We introduce some definitions, which will hold for the remainder of this paper:

- $\kappa = \frac{1}{\sqrt{5}}$;
- $s_k = \sigma + ikt$ such that $k \in \mathbb{N}, 1 < \sigma < 1 + \varepsilon$ for some $0 < \varepsilon \leq 0.15$;
- $s'_k = \sigma_1 + ikt$ such that $\sigma_1 = \frac{1 + \sqrt{1 + 4\sigma^2}}{2}$. Noting that $\sigma_1$ depends on $\sigma$, so for convenience we will write $\sigma_1(a)$ to denote the value of $\sigma_1$ at $\sigma = a$. To prove Theorem 1 we will isolate a non-trivial zero $\rho = \beta + it$ of $\zeta_L$ such that $\beta > 1 - \varepsilon \geq 0.85$.

Choose a polynomial $p_n(\varphi)$ from $P_n$, and consider the function

$$S(\sigma, t) = \sum_{k=0}^{n} a_k f_L(\sigma, kt),$$

such that

$$f_L(\sigma, kt) = -\Re \left( \frac{\zeta_L'(s_k)}{\zeta_L(s_k)} - \kappa \frac{\zeta_L'(s'_k)}{\zeta_L(s'_k)} \right) = \sum_{0 \neq P \subset O_L} \Lambda(P)(N(P)^{-\sigma} - \kappa N(P)^{-\sigma_1}) \cos(kt \log(N(P))).$$

It follows that

$$S(\sigma, t) = \sum_{0 \neq P \subset O_L} \Lambda(P)(N(P)^{-\sigma} - \kappa N(P)^{-\sigma_1}) p_n(t \log(N(P))) \geq 0.$$

On the other hand, we can utilise the explicit formula \[8, (8.3)]\;

$$- \frac{\zeta_L'(s_k)}{\zeta_L(s_k)} = \frac{\log d_L}{2} + \frac{1}{s_k} + \frac{1}{s_k - 1} + \frac{\gamma_L'(s_k)}{\gamma_L(s_k)} - \frac{1}{2} \left( \sum_{\varphi \in Z(\zeta_L)} \left( \frac{1}{s_k - \varphi} + \frac{1}{s_k - \overline{\varphi}} \right) \right). \tag{5}$$

Here, $Z(\zeta_L)$ denotes the set of non-trivial zeros of $\zeta_L$. One can use (5) to show

$$0 \leq S(\sigma, t) \leq S_1 + S_2 + S_3 + S_4,$$ 

(6)
where $F(s, z) = \Re \left( \frac{1}{s-z} + \frac{1}{s+1+z} \right)$ such that

\[
S_1 = -\sum_{k=0}^{n} a_k \sum_{\ell \in Z(\zeta_L)} \Re \left( \frac{1}{s_k - \varrho} - \frac{\kappa}{s_k' - \varrho} \right),
\]

\[
S_2 = \frac{1 - \kappa}{2} \left( \sum_{k=0}^{n} a_k \right) \log d_L,
\]

\[
S_3 = \sum_{k=0}^{n} a_k (F(s_k, 1) - \kappa F(s_k', 1)), \quad \text{and}
\]

\[
S_4 = \sum_{k=0}^{n} a_k \Re \left( \frac{\gamma'_L(s_k)}{\gamma_L(s_k)} - \kappa \frac{\gamma'_L(s_k')}{\gamma_L(s_k')} \right).
\]

We will choose $n = 16$, so that we can apply Mossinghoff–Trudgian’s polynomial $p_{16}(\varphi) \in P_{16}^\ast$ from [12]. Taking $n = 16$, $S_2$ is directly computable, and we find upper bounds for $S_1$, $S_3$, and $S_4$ in Sections 2.1, 2.2, and 2.3. The resulting upper bound for $S_1 + S_2 + S_3 + S_4$ will depend on $\beta$, $\sigma$, $t$, the coefficients of $p_{16}(\varphi)$ and $\varepsilon$, therefore we may use [8] and rearrange the inequality to obtain Theorem 1 in Section 2.4.

### 2.1. Upper bound for $S_1$.

**Lemma 3** (Stečkin [14]). Suppose $s = \sigma + it$ with $1 < \sigma \leq 1.25$ and $z \in \mathbb{C}$. If $0 < \Re(z) < 1$, then

\[
F(s, z) - \kappa F(s'_1, z) \geq 0.
\]

Moreover, if $\Im(z) = \Im(s) = t$ and $\frac{t}{2} \leq \Re(z) < 1$, then

\[
\Re \left( \frac{1}{s - 1 + \bar{z}} \right) - \kappa F(s'_1, z) \geq 0.
\]

Note that $\kappa$ is the largest value such that (7) holds. This subsection is not an improvement on [6] Lemma 2.3), rather a repeat for the purpose of clarity. By the positivity condition (7) in Lemma 3 we have

\[
\ell(s_k) := \sum_{\ell \in Z(\zeta_L)} \Re \left( \frac{1}{s_k - \varrho} - \frac{\kappa}{s_k' - \varrho} \right) \leq \kappa F(s_k', \rho) - F(s_k, \rho).
\]

If $k = 1$, then (8) implies that

\[
\ell(s_1) \leq -\frac{1}{\sigma - \beta} - \frac{1}{\sigma - 1 + \beta} + \frac{\kappa}{\sigma_1 - \beta} + \frac{\kappa}{\sigma_1 - 1 + \beta} = -\frac{1}{\sigma - \beta} + g(\sigma, \beta).
\]

We see that $g(\sigma, \beta) < g(1, 1)$ and $g(1, 1)$ is small and negative, so $\ell(s_1) \leq -\frac{1}{\sigma - \beta}$. Moreover, if $k \neq 1$, then (8) implies that $\ell(s_k) \leq 0$ by (7). One can package the preceding observations into the following lemma.

**Lemma 4.** Isolate a zero $\rho = \beta + it \in Z(\zeta_L)$ such that $\beta \geq 1 - \varepsilon \geq 0.85$, then

\[
\ell(\sigma + itk) \leq \begin{cases} 
-\frac{1}{\sigma - \beta} & \text{if } k = 1, \\
0 & \text{if } k \neq 1.
\end{cases}
\]

Therefore, $S_1 \leq -\frac{a_1}{\sigma - \beta}$. 
2.2. Upper bound for $S_3$. Suppose that

$$
\Sigma_k(\sigma, t) := F(\sigma + ikt, 1) - \kappa F(\sigma_1 + ikt, 1)
= \frac{\sigma}{\sigma^2 + k^2t^2} + \frac{\sigma - 1}{(\sigma - 1)^2 + k^2t^2} - \kappa \frac{\sigma_1}{\sigma_1^2 + k^2t^2} - \kappa \frac{\sigma_1 - 1}{(\sigma_1 - 1)^2 + k^2t^2}.
$$

**Case I.** If $k = 0$, then $\Sigma_k$ is only dependent on $\sigma$, with a singularity occurring at $\sigma = 1$. In fact,

$$
\Sigma_0(\sigma, t) = \frac{1}{\sigma} + 1 \frac{1}{\sigma - 1} - \kappa \frac{1}{\sigma_1} - \kappa \frac{1}{\sigma_1 - 1} := \frac{1}{\sigma - 1} + h(\sigma).
$$

We observe that $h(\sigma)$ increases as $\sigma$ increases, so for $\alpha_\varepsilon = \hat{h}(1 + \varepsilon) < 0.021467$, we have

$$
\Sigma_0(\sigma, t) \leq \frac{1}{\sigma - 1} + \alpha_\varepsilon.
$$

**Case II.** Suppose $1 \leq k \leq 16$, then $\Sigma_k(\sigma, t)$ depends on $\sigma$ and $t$. For each $\sigma$, $\Sigma_k(\sigma, t)$ decreases as $t$ increases, because the derivative of $\Sigma_k(\sigma, t)$ with respect to $t$ is negative for all $t \geq 1$. Therefore, $\Sigma_k(\sigma, t) \leq \Sigma_k(\sigma, 1)$, which increases as $\sigma$ increases, because the derivative of $\Sigma_k(\sigma, 1)$ with respect to $\sigma$ is positive for all $1 \leq \sigma \leq 1.15$. It follows that

$$
\Sigma_k(\sigma, t) \leq \Sigma_k(1 + \varepsilon, 1) < B_\varepsilon(k),
$$

where admissible values for $B_\varepsilon(k)$ are easily computed using a computer. To further verify this bound, the Maximize command in Maple confirms that the maximum of $\Sigma_k(\sigma, t)$ occurs at $\sigma = 1 + \varepsilon$ and $t = 1$. For example, if $\varepsilon = 0.15$ or $\varepsilon = 0.01$, then admissible values of $B_{0.15}(k)$ and $B_{0.01}(k)$ are given in Table 1 and Table 2 respectively. Note that we round up at 8 decimal places, to account for any possible rounding errors.

| $k$ | $B_{0.15}(k)$ | $k$ | $B_{0.15}(k)$ | $k$ | $B_{0.01}(k)$ | $k$ | $B_{0.01}(k)$ |
|-----|---------------|-----|---------------|-----|---------------|-----|---------------|
| 1   | 0.23445352    | 9   | 0.00235718    | 1   | 0.10919579    | 9   | 0.00029396    |
| 2   | 0.06869804    | 10  | 0.00188669    | 2   | 0.03040152    | 10  | 0.00021655    |
| 3   | 0.02783858    | 11  | 0.00154513    | 3   | 0.00958566    | 11  | 0.00016557    |
| 4   | 0.01427867    | 12  | 0.00128917    | 4   | 0.00384196    | 12  | 0.00013046    |
| 5   | 0.0085573     | 13  | 0.0010924     | 5   | 0.00185609    | 13  | 0.00010535    |
| 6   | 0.00568194    | 14  | 0.00093759    | 6   | 0.00102853    | 14  | 0.00008684    |
| 7   | 0.00404715    | 15  | 0.00081374    | 7   | 0.00063099    | 15  | 0.00007282    |
| 8   | 0.00303134    | 16  | 0.00071303    | 8   | 0.00041809    | 16  | 0.00006196    |

**Table 1.** Admissible values for $B_{0.15}(k)$.

**Table 2.** Admissible values for $B_{0.01}(k)$.

Now, we can collect the preceding observations to yield Lemma

**Lemma 5.** For $0 \leq k \leq 16$, we have that

$$
\Sigma_k(\sigma, t) \leq \begin{cases} 
\frac{1}{\sigma - 1} + \alpha_\varepsilon & \text{if } k = 0, \\
B_\varepsilon(k) & \text{if } k \neq 0.
\end{cases}
$$
Under a choice of polynomial from $P_{16}$, it follows that

$$S_3 \leq a_0 \left( \frac{1}{\sigma - 1} + \alpha \right) + \sum_{k=1}^{16} a_k B_{\varepsilon}(k).$$

Remark. The benefits of Lemma 5 over [6, Lemma 2.4] lie in the computed constants $B_{\varepsilon}(k)$. That is, Kadiri established $\Sigma_k(\sigma, t) \leq 1.6666$ for $1 \leq k \leq 4$.

2.3. Upper bound for $S_4$. We bring forward an observation from Kadiri [6, §2.4],

\[
\Re \left( \frac{\gamma_L'(s_k)}{\gamma_L(s_k)} - \frac{\gamma_L'(s_k')}{\gamma_L(s_k')} \right) \leq - \frac{1 - \kappa}{2} \cdot \log \pi \cdot n_L + \frac{n_L}{2} \max_{\delta \in \{0, 1\}} \left\{ \Re \left( \frac{\Gamma'(s_k + \delta)}{\Gamma'} - \frac{\Gamma'(s_k' + \delta)}{\Gamma} \right) \right\}.
\]

Case I. If $k = 0$, then we directly compute that

\[
\frac{1}{2} \max_{\delta \in \{0, 1\}} \left\{ \Re \left( \frac{\Gamma'}{\Gamma} \left( s_k + \frac{\delta}{2} \right) - \frac{\Gamma'}{\Gamma} \left( s_k' + \delta \right) \right) \right\} \leq d_{\varepsilon}(0),
\]

where $d_{\varepsilon}(0)$ is the maximum of the functions such that $\sigma = 1 + \varepsilon$. To see this, one can observe that the left-hand side of (9) is maximised at $\sigma = 1 + \varepsilon$ visually or use the Maximize command in Maple. For example, if $\varepsilon = 0.01$, then

\[
d_{0.01}(0) = -0.2500763736.
\]

Case II. Suppose $1 \leq k \leq 16$. McCurley [11, Lemma 2] establishes that

\[
\frac{1}{2} \Re \left( \frac{\Gamma'}{\Gamma} \left( s_k + \frac{\delta}{2} \right) - \frac{\Gamma'}{\Gamma} \left( s_k' + \frac{\delta}{2} \right) \right) = \frac{1 - \kappa}{2} \log \frac{kt}{2} + \Xi(\sigma, k, t, \delta) + \frac{\theta_1}{2k} \left( \pi - \arctan \left( \frac{1 + \delta}{k} \right) \right) + \frac{\theta_2}{2k} \left( \pi - \arctan \left( \frac{\sigma_1(1) + \delta}{k} \right) \right),
\]

where $|\theta_i| \leq 1$ and

\[
\Xi(\sigma, k, t, \delta) = \frac{1}{4} \log \left[ 1 + \left( \frac{\sigma + \delta}{kt} \right)^2 \right] - \frac{\kappa}{4} \log \left[ 1 + \left( \frac{\sigma_1 + \delta}{kt} \right)^2 \right] - \frac{\sigma + \delta}{2((\sigma + \delta)^2 + k^2t^2)} + \frac{\sigma_1 + \delta}{2((\sigma_1 + \delta)^2 + k^2t^2)}.
\]

Next, we will bound $\Xi(\sigma, k, t, \delta)$ using two different methods, then choose the best bound for each $k$.

Method I. For any $t > 0$, we have

\[
\Xi_1(\sigma, k, t, \delta) := -\frac{\sigma + \delta}{2((\sigma + \delta)^2 + k^2t^2)} + \frac{\sigma_1 + \delta}{2((\sigma_1 + \delta)^2 + k^2t^2)} \leq \frac{\kappa(\sigma_1 + \delta) - \sigma - \delta}{2((\sigma_1 + \delta)^2 + k^2t^2)} \leq 0,
\]
because \( \sigma < \sigma_1 \) and \( \kappa(\sigma_1 + \delta) - \sigma - \delta \leq 0 \). Moreover, for fixed \( \sigma \), observe that

\[
\Xi_2(\sigma, k, t, \delta) := \frac{1}{4} \log \left[ 1 + \left( \frac{\sigma + \delta}{kt} \right)^2 \right] - \frac{\kappa}{4} \log \left[ 1 + \left( \frac{\sigma_1 + \delta}{kt} \right)^2 \right]
\]

is positive for \( t \geq 1 \) and decreases as \( t \) increases, because the derivative of \( \Xi_2(\sigma, k, t, \delta) \) with respect to \( t \) is negative for all \( t \geq 1 \). Therefore,

\[
\Xi_2(\sigma, k, t, \delta) \leq \Xi_2(\sigma, k, 1, \delta)
\]

for \( t \geq 1 \), which increases as \( \sigma \) increases in the range \( 1 \leq \sigma \leq 1.15 \), because the derivative of \( \Xi_2(\sigma, k, 1, \delta) \) with respect to \( \sigma \) is positive for \( 1 \leq \sigma \leq 1.15 \). Hence, for each \( k \),

\[
\Xi_2(\sigma, k, t, \delta) \leq \Xi_2(1 + \varepsilon, k, 1, \delta).
\]

To verify the preceding bound, the Maximize command in Maple confirms that the maximum of \( \Xi_2(\sigma, k, t, \delta) \) occurs at \( \sigma = 1 + \varepsilon \) and \( t = 1 \). It follows that \( \Xi(\sigma, k, t, \delta) \leq \Xi_2(1 + \varepsilon, k, 1, \delta) \) for each \( k \) and

\[
\frac{1}{2} \max_{\delta \in \{0, 1\}} \left\{ \arg \left( \frac{\Gamma' \left( \frac{s_k + \delta}{2} \right)}{\Gamma_1 \left( \frac{s_k + \delta}{2} \right)} \right) \right\} \leq \frac{1 - \kappa}{2} \log t + S_1(k, \varepsilon),
\]

where \( S_1(k, \varepsilon) = \max_{\delta \in \{0, 1\}} \{ C_1(k, \delta, \varepsilon) \} \) such that

\[
C_1(k, \delta, \varepsilon) := \frac{1 - \kappa}{2} \log \frac{k}{2} + \Xi_2(1 + \varepsilon, k, 1, \delta)
+ \frac{1}{2k} \frac{\pi}{2} - \arctan \left( \frac{1 + \delta}{k} \right)
+ \frac{\kappa}{2k} \frac{\pi}{2} - \arctan \left( \frac{\sigma_1(1) + \delta}{k} \right).
\]

**Method II.** We will verify that for \( 0 < \varepsilon \leq 0.15 \),

\[
\Xi(\sigma, k, t, \delta) \leq A(k, \delta, \varepsilon) := \begin{cases} 0 & \text{if } \delta = 0, \\ 0 & \text{if } \delta = 1 \text{ and } k \not\in \{1, 2\}, \\ \Xi(1 + \varepsilon, k, 1, 1) & \text{if } \delta = 1 \text{ and } k = 1, \\ \Xi(1.15, k, 1, 1) & \text{if } \delta = 1 \text{ and } k = 2. \end{cases}
\]

First, for fixed \( \sigma \) and \( \delta = 0 \), the derivative of \( \Xi(\sigma, k, t, \delta) \) with respect to \( t \) is positive for \( t \geq 1 \), so \( \Xi(\sigma, k, t, 0) \) is increasing as \( t \to \infty \). Therefore, for each \( \sigma \in [1, 1.15] \),

\[
\Xi(\sigma, k, t, 0) \leq \lim_{t \to \infty} \Xi(\sigma, k, t, 0) = 0.
\]

Next, for fixed \( \sigma \) and \( \delta = 1 \), the derivative of \( \Xi(\sigma, k, t, \delta) \) with respect to \( t \) is positive for \( t \geq 1 \) whenever \( k \not\in \{1, 2, 3\} \), so \( \Xi(\sigma, k, t, 1) \) is increasing as \( t \to \infty \) for \( k \not\in \{1, 2, 3\} \). Therefore, for each \( k \not\in \{1, 2, 3\} \) and \( 1 \leq \sigma \leq 1.15 \),

\[
\Xi(\sigma, k, t, 1) \leq \lim_{t \to \infty} \Xi(\sigma, k, t, 1) = 0.
\]

To completely verify (11), we now establish bounds for the special cases \( \delta = 1 \) and \( k \in \{1, 2, 3\} \). Observe that for each \( t \geq 1 \), the derivative of \( \Xi(\sigma, k, t, 1) \) with respect to \( \sigma \) is positive for \( 1 \leq \sigma \leq 1.15 \) whenever \( k \in \{1, 2, 3\} \), so

\[
\Xi(\sigma, k, t, 1) \leq \Xi(1 + \varepsilon, k, t, 1).
\]

Suppose that \( k \in \{1, 2, 3\} \) and observe that in the range \( t \geq 1 \), \( \Xi(1 + \varepsilon, k, t, 1) \) either has one minimum point at \( t = t_k(\varepsilon) \) or increases as \( t \to \infty \). Here, \( t_k(\varepsilon) \) equals the only root of the derivative of \( \Xi(1 + \varepsilon, k, t, 1) \) with respect to \( t \) in the range \( t \geq 1 \). If this root does not exist, then set \( t_k(\varepsilon) = 1 \) for convenience. For example,
Under a choice of polynomial from $P_{16}$, we say that $S(t)$ decreases for $1 \leq t \leq t_k(\varepsilon)$ and $\Xi(1+\varepsilon, k, t, 1)$ increases for $t > t_k(\varepsilon)$, so

$$\Xi(1+\varepsilon, k, t, 1) \leq \begin{cases} \Xi(1+\varepsilon, k, 1, 1) & \text{if } 1 \leq t \leq t_k(\varepsilon), \\ \lim_{t \to \infty} \Xi(1+\varepsilon, k, t, 1) & \text{if } t > t_k(\varepsilon), \end{cases}$$

in which $\lim_{t \to \infty} \Xi(1+\varepsilon, k, t, 1) = 0$ for each $k$. If $k = 1$, then for $t \geq 1$, we have

$$\Xi(1+\varepsilon, 1, t, 1) \leq \max \{ \Xi(1+\varepsilon, 1, 1, 1), 0 \} = \Xi(1+\varepsilon, 1, 1, 1).$$

Observe that $\Xi(1+\varepsilon, 1, 2, 1)$ increases as $0 < \varepsilon \leq 0.15$ increases. So, if $k = 2$, then for $t \geq 1$, we have

$$\Xi(1+\varepsilon, 2, t, 1) \leq \max \{ \Xi(1+\varepsilon, 2, 1, 1), 0 \} \leq \max \{ \Xi(1+\varepsilon, 2, 1, 1), 0 \} \leq \Xi(1+\varepsilon, 2, 1, 1).$$

In this case, the final bound is convenient and not too wasteful, because $\Xi(1+\varepsilon, 2, 1, 1)$ is small. Finally, if $k = 3$, then for $t \geq 1$, we have

$$\Xi(1+\varepsilon, 3, t, 1) \leq \max \{ \Xi(1+\varepsilon, 3, 1, 1), 0 \} = 0.$$
Remark. The benefits of Lemma 6 over [6, Lemma 2.5] lie in the computed constants $d_\varepsilon(0)$ and $S(k, \varepsilon)$. Kadiri imports results from McCurley [11, Lemma 2] for her bound, so the improvements we see follow from our observations pertaining to McCurley’s work.

\begin{table}[h]
\centering
\begin{tabular}{c|c|c|c}
$k$ & $S_1(k, 0.15)$ & $S_2(k, 0.15)$ & $S(k, 0.15)$ \\
\hline
1 & 0.3784516540 & 0.3249009026 & 0.3249009026 \\
2 & 0.3839873212 & 0.3763572015 & 0.3763572015 \\
3 & 0.4018562060 & 0.4004551145 & 0.4004551145 \\
4 & 0.4238223974 & 0.4236306767 & 0.4236306767 \\
5 & 0.4467597648 & 0.4468482525 & 0.4468482525 \\
6 & 0.4693610537 & 0.4695098183 & 0.4695098183 \\
7 & 0.4910902618 & 0.4912403488 & 0.4912403488 \\
8 & 0.5117562107 & 0.5118920810 & 0.5118920810 \\
9 & 0.5313238925 & 0.5314428586 & 0.5314428586 \\
10 & 0.5498280118 & 0.5499312088 & 0.5499312088 \\
11 & 0.5673323540 & 0.5674218683 & 0.5674218683 \\
12 & 0.5839104248 & 0.5839883668 & 0.5839883668 \\
13 & 0.5996362678 & 0.5997044990 & 0.5997044990 \\
14 & 0.6145802698 & 0.6146403531 & 0.6146403531 \\
15 & 0.6288074426 & 0.6288606647 & 0.6288606647 \\
16 & 0.6423769295 & 0.6424243440 & 0.6424243440 \\
\hline
\end{tabular}
\caption{Computed values for $S_1(k, 0.15)$, $S_2(k, 0.15)$ and $S(k, 0.15)$.}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{c|c}
$a_0$ & 1 \\
$a_1$ & 1.74126664022806 \\
$a_2$ & 1.12828282804652 \\
$a_3$ & 0.506527232186642 \\
$a_4$ & 0.1253566902628852 \\
$a_5$ & 2.372710620 \cdot 10^{-26} \\
$a_6$ & 2.818732841 \cdot 10^{-22} \\
$a_7$ & 0.01201214561729989 \\
$a_8$ & 0.006875849760911001 \\
$a_9$ & 2.064157910 \cdot 10^{-23} \\
$a_{10}$ & 6.601587090 \cdot 10^{-11} \\
$a_{11}$ & 0.001608306592372963 \\
$a_{12}$ & 0.001017994683287104 \\
$a_{13}$ & 6.728831293 \cdot 10^{-11} \\
$a_{14}$ & 3.682448595 \cdot 10^{-11} \\
$a_{15}$ & 2.949853019 \cdot 10^{-6} \\
$a_{16}$ & 0.00003713656497 \\
\hline
\end{tabular}
\caption{Table of coefficients for Mossinghoff–Trudgian’s polynomial $p_{16}(\varphi) \in P_{16}$.}
\end{table}

2.4. Computations. As declared in the introduction, we will choose the polynomial $p_{16}(\varphi) \in P_{16}$ from [12], whose coefficients are given in Table 4. Suppose $r > 0$
and $\sigma$ is chosen such that $\sigma - 1 = r(1 - \beta)$ where $\rho = \beta + it \in Z(\zeta_L)$ is an isolated zero such that $\beta \geq 1 - \varepsilon \geq 0.85$. Applying the upper bounds for each $S_i$, which can be found in Lemmas 4, 5 and 6, then rearranging inequality (6) will yield

$$\beta \leq 1 - \frac{\frac{a_1}{1 + r} - \frac{a_0}{r}}{c_1 \log d_L + c_2 n_L \log t + c_3 n_L + c_4},$$

where

$$c_1 = \frac{1 - \kappa}{2} \sum_{k=0}^{16} a_k,$$

$$c_2 = \frac{1 - \kappa}{2} \sum_{k=1}^{16} a_k,$$

$$c_3 = a_0 \left( d(0) - \frac{1 - \kappa}{2} \log \pi \right) + \sum_{k=1}^{16} a_k \left( \frac{1 - \kappa}{2} \log \left( \frac{k}{\pi} \right) + S(k, \varepsilon) \right)$$

and

$$c_4 = \alpha \varepsilon a_0 + \sum_{k=1}^{16} a_k B_\varepsilon(k).$$

For the remainder of this proof, we replicate the process which Kadiri [6] followed. The maximum value of $\frac{a_1}{1 + r} - \frac{a_0}{r}$ occurs at $r = \sqrt[4]{a_1 - a_0}$. Therefore, dividing the numerator and denominator of (16) by

$$M = \frac{\sqrt{a_1}}{\sqrt[4]{a_1 - a_0}} - \frac{a_0}{\sqrt[4]{a_1}},$$

we see that

$$\beta \leq 1 - \frac{1}{\frac{a_1}{M} \log d_L + \frac{a_0}{M} n_L \log t + \frac{c_3}{M} n_L + \frac{c_4}{M}}.$$ 

In Table 5, we present the constants for two choices of $\varepsilon$. Observing the values for $\varepsilon = 0.01$, inequality (17) will yield the explicit zero-free region (3) for $t \geq 1$, which completes the proof of Theorem 1.

| $\varepsilon$ | $M$ | $\frac{c_1}{M}$ | $\frac{c_2}{M}$ | $\frac{c_3}{M}$ | $\frac{c_4}{M}$ |
|---------------|-----|----------------|----------------|----------------|----------------|
| 0.15          | 0.1021253857 | 9.534650638 | 0.444485082 | 5.123026304 | 2.269182727 |
| 0.01          | 0.1021253857 | 9.534650638 | 0.050168175 | 2.269182727 | 5.123026304 |

Table 5. Constants for the explicit zero-free region in Theorem 1 given $\varepsilon = 0.15$ or $\varepsilon = 0.01$. 
3. Proof of Theorem 2

Theorem 2 is an improvement of part of [6, Theorem 1.2]. We will recycle Kadiri’s proof, except we use the polynomial $p_{16}(\varphi)$ in place of a polynomial from $P_4$. Suppose $\log d_L$ is asymptotically large and consider three regions,

$$I_A = \left(0, \frac{d_1}{\log d_L}\right], I_B = \left(\frac{d_1}{\log d_L}, \frac{d_2}{\log d_L}\right], I_C = \left(\frac{d_2}{\log d_L}, 1\right),$$

where $d_1, d_2$ are constants to be chosen. Suppose further, that

$$\sigma - 1 = \frac{r}{\log d_L} \quad \text{and} \quad 1 - \beta = \frac{c}{\log d_L}.$$

In the regions $I_B$ and $I_C$, we impose further restrictions. Suppose $0 < c, r < 1$ such that

$$a_0 - a_0 c < r \quad \text{and} \quad d_2 > \sqrt{r(r + c)}.$$

Combining analogous arguments to those results in [6, §3.2, §3.3, §3.4], one can easily establish that

$$0 \leq \frac{1}{r} - 2 \frac{r + c}{(r + c)^2 + d_1^2} + \frac{1 - \kappa}{2}$$

in the region $I_A$,

$$0 \leq \mathcal{E}_B(d_1, d_2, r, c)$$

$$:= \frac{a_0}{r} - \frac{a_1}{r + c} + \frac{a_1 r}{r^2 + d_1^2} - \frac{a_0(r + c)}{(r + c)^2 + d_1^2}$$

$$- \frac{a_0(r + c)}{(r + c)^2 + d_2^2} - \frac{a_1(r + c)}{(r + c)^2 + 4d_2^2} + \frac{1 - \kappa}{2} \sum_{k=0}^{16} a_k$$

$$+ \sum_{k=2}^{16} a_k \left(\frac{r}{r^2 + k^2 d_1^2} - \frac{r + c}{(r + c)^2 + (k - 1)^2 d_2^2} - \frac{r + c}{(r + c)^2 + (k + 1)^2 d_2^2}\right)$$

in the region $I_B$ and

$$0 \leq \mathcal{E}_C(d_2, r, c)$$

$$:= \frac{a_0}{r} - \frac{a_1}{r + c} + \frac{a_1 r}{r^2 + d_2^2} - \frac{a_0(r + c)}{(r + c)^2 + d_2^2} + \frac{1 - \kappa}{2} \sum_{k=0}^{16} a_k$$

in the region $I_C$. Suppose $d_1$ and $r$ are fixed. The admissible values of $c$ which one can input into (18) are those $c$ such that

$$c \geq \sqrt{r^2 - d_1^2 \left(1 + \frac{r}{2} \right)^2 - \frac{1 - \kappa}{2} r^2}.$$

(21)

Denote the smallest value for $c$ in (21) by $c_A$. Next, let $c_B$ denote the root of $\mathcal{E}_B(d_1, d_2, r, c)$, where $r$ is chosen such that the root $c_B$ is as small as possible. Similarly, let $c_C$ denote the smallest root of $\mathcal{E}_C(d_2, r, c)$ for some optimally chosen $r$. It follows that $\zeta_L$ has at most one zero in the region $s = \sigma + it$ such that $t < 1$ and

$$\sigma \geq 1 - \frac{1}{R \log d_L}.$$
such that $R = \max \left( \frac{1}{c_A}, \frac{1}{c_B}, \frac{1}{c_C} \right)$. Moreover, if an exceptional zero exists then it is real and simple by [6 §3.5]. To complete our proof of Theorem 2 it will suffice to show that $R = 12.43436$ is an admissible value.

First, suppose that we choose the same values that Kadiri chose; $d_1 = 1.021$ and $d_2 = 2.374$. One can establish that $\frac{1}{c_A} = 12.5494$ when $r = 2.1426$. Moreover, using our higher degree polynomial, we can compute the roots of $E_B(1.021, 2.374, r, c)$ and $E_C(2.374, r, c)$ over a selection of $r$. The results of these computations are presented below.

| Root of $E_B(1.021, 2.374, r, c)$ | $r$ | $\frac{1}{c}$ |
|---------------------------------|-----|---------------|
| $E_B(1.021, 2.374, r, c)$       | 0.2366 | 12.43922 |
| $E_C(2.374, r, c)$             | 0.2477 | 12.42548 |

Therefore, these choices of $d_1$ and $d_2$ would yield Theorem 2 with $R = \max (12.5494, 12.43922, 12.42548) = 12.5494$.

Above, the limiting factor appears to be the value for $\frac{1}{c_A}$. We can reduce the value of $\frac{1}{c_A}$ by decreasing the value of $d_1$, however, we are also limited by the sizes of $\frac{1}{c_B}$ and $\frac{1}{c_C}$ which we can obtain. Therefore, we only need to choose $d_1$ such that $\frac{1}{c_A}$ is small enough. The cost of choosing $d_1$ too small is a larger interval $I_B$, which might not be ideal.

Given $d_1$, to find a good enough choice for $d_2$, we have tested many values for $d_2$ and computed the optimal outcomes in each case. If one chooses $d_1 = 1.0015$, then we found (to 3 decimal places) that $d_2 = 2.318$ yielded the best results. For this $d_1$, one can determine that $\frac{1}{c_A} = 9.7946$ when $r = 2.1163$. The results of the remaining computations for $\frac{1}{c_B}$ and $\frac{1}{c_C}$ are presented below.

| Root of $E_B(1.0015, 2.318, r, c)$ | $r$ | $\frac{1}{c}$ |
|---------------------------------|-----|---------------|
| $E_B(1.0015, 2.318, r, c)$       | 0.2363 | 12.43355 |
| $E_C(2.318, r, c)$             | 0.2473 | 12.43436 |

Therefore — as required — these choices of $d_1$ and $d_2$ will yield Theorem 2 with $R = \max (9.7946, 12.43355, 12.43436) = 12.43436$.

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