Quantitative conditions for right-handedness of flows

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Abstract

We give a numerical condition for right-handedness of a dynamically convex Reeb flow on $S^3$. Our condition is stated in terms of an asymptotic ratio between the amount of rotation of the linearised flow and the linking number of trajectories with a periodic orbit that spans a disk-like global surface of section. As an application, we find an explicit constant $\delta_* < 0.7225$ such that if a Riemannian metric on the 2-sphere is $\delta$-pinched with $\delta > \delta_*$, then its geodesic flow lifts to a right-handed flow on $S^3$. In particular, all finite non-empty collections of periodic orbits of such a geodesic flow bind open books whose pages are global surfaces of section.

Contents

1 Introduction .......................................................... 2
   1.1 Global surfaces of section .................................. 3
   1.2 Reeb flows .................................................... 5

2 Right-handedness from dynamical pinching ....................... 8
   2.1 Transverse rotation numbers ............................... 8
   2.2 Right-handedness .......................................... 10
   2.3 Proof of Theorem 1.13 ..................................... 11
   2.4 Right-handedness on strictly convex energy levels ....... 30

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1 Introduction

Right-handed vector fields, introduced by Ghys in [Ghy09], form a special class of non-singular vector fields on homology 3-spheres. Their flows will also be called right-handed, for simplicity. Roughly speaking, all pairs of trajectories of such a flow link positively. Ghys formalised this definition in terms of positivity of the quadratic linking form, which assigns some kind of linking number to any pair of invariant Borel probability measures; see also [Arn86]. Right-handedness has interesting dynamical implications as the following statement demonstrates.

Theorem 1.1 (Ghys [Ghy09]). Every non-empty finite collection of periodic orbits of a right-handed flow binds an open book whose pages are global surfaces of section.

In particular, the above statement imposes strong restrictions on the periodic orbits that can arise in right-handed flows: knots or links that are not fibred can not be realised. Both the definition of right-handedness (Definition 2.6) and the proof of Theorem 1.1 ([FH21, appendix B]) can actually be given independently of the quadratic linking form.

Right-handedness is difficult to check. It can be checked for certain special flows, like the Hopf flow or other integrable flows. Since in [Ghy09] it is stated that right-handedness is a $C^1$-open condition, one knows that there exist more interesting right-handed flows near these simple examples, but this abstract reasoning does not give them explicitly. It is precisely this general lack of explicit examples that motivates our work. One exception is provided by the work of Dehornoy in [Deh17]. Sometimes it is more natural to talk about left-handedness, which is equivalent to right-handedness when the ambient orientation is reversed. Dehornoy showed that the geodesic flow on the unit tangent bundle of a hyperbolic $n$-conic 2-sphere is left-handed if $n = 3$. Then, using a version of Gromov’s geodesic rigidity [Gro00], the case of arbitrary negative curvature is reduced to the hyperbolic case.

To state our main application, consider the polynomial $P(x) = 4x^3 - 2x^2 - 1$, which has a unique real root $x_*$. It satisfies $0.84 < x_* < 0.85$. Set $\delta_* := x_*^2$. Given $\delta \in (0, 1]$, a Riemannian metric on $S^2$ is said $\delta$-pinched if $\delta \leq K_{\text{min}}/K_{\text{max}}$, where $K_{\text{min}}, K_{\text{max}}$ denote the minimum and the maximum of the Gaussian curvature.

Theorem 1.2. If $\delta > \delta_*$, in particular if $\delta \geq 0.7225$, then the geodesic flow of a $\delta$-pinched Riemannian metric on $S^2$ lifts to a right-handed flow on $S^3$.

As mentioned before, the main motivation for such a statement is that it is not a perturbative result, hence it can be used to check right-handedness in an explicit set of flows that are far from integrable.
Other implications of right-handedness besides Theorem 1.1 have been obtained. For instance, Dehornoy and Rechtman showed in [DR22] that if an invariant measure of a right-handed flow can be approximated by long periodic orbits, then their Seifert genera grow proportionally to the square of the period, the proportionality constant being the helicity of the given invariant measure.

More generally, we look for numerical conditions for right-handedness within the class of Reeb flows of dynamically convex contact forms on $S^3$. This class was introduced by Hofer, Wysocki and Zehnder in [HWZ98]. They showed that this class is rich enough to provide applications: by [HWZ98, Theorem 3.4] the Hamiltonian flow on a strictly convex energy level in a 4-dimensional symplectic vector space is the Reeb flow of a dynamically convex contact form. Harris and Paternain showed in [HP08] that the geodesic flow of a Finsler metric on the two-sphere with reversibility $r$ is dynamically convex if the flag curvatures are pinched by more than $(r/(r + 1))^2$. Theorem 1.2 will be deduced from Theorem 1.13 below, which provides an abstract condition for right-handedness of a dynamically convex Reeb flow on $S^3$ in terms of a dynamical pinching condition. Theorem 1.13 also implies Theorem 1.14 which gives right-handedness on strictly convex energy levels in terms of a relation between the curvatures and the return times of a disk-like global surface of section.

It was explained to us by Dehornoy in private communication that there are Riemannian metrics on $S^2$ with strict positive curvature whose geodesic flows do not lift to right-handed flows on $S^3$. Consider the infimum $\delta_0$ of all $\delta \in (0, 1]$ with the following property: if a Riemannian metric on $S^2$ has curvatures pinched by at least $\delta$ then its geodesic flow lifts to a right-handed flow on $S^3$. Dehornoy’s examples show that $\delta_0 > 0$. Together with Theorem 1.2 we get $0 < \delta_0 \leq \delta_* < 0.7225$. We are led to ask:

**Question.** What is the value of $\delta_0$?

### 1.1 Global surfaces of section

Let $X$ be a smooth vector field on a closed and oriented 3-manifold $M$. Its flow is denoted by $\phi^t$.

**Definition 1.3.** A *global surface of section* for $\phi^t$, or for $X$, is a smooth, embedded and compact surface $\Sigma \hookrightarrow M$, such that $\partial \Sigma$ consists of periodic orbits or is empty, $X$ is transverse to $\Sigma \setminus \partial \Sigma$, and for every $x \in M$ one finds $t_- < 0 < t_+$ such that $\phi^{t_{\pm}}(x) \in \Sigma$.

**Remark 1.4.** The orientation of $M$ and the co-orientation of $\Sigma \setminus \partial \Sigma$ induced by $X$ together orient $\Sigma$. In this paper we always assume that global surfaces of section are oriented in this way.

**Remark 1.5.** From a dynamical perspective a global surface of section is a valuable tool since one can deduce dynamical properties of the flow from those of the associated *first return map*. To define it we need first to consider the *return time function*
\[ \tau : \Sigma \setminus \partial \Sigma \to (0, +\infty), \quad \tau(x) = \min\{t > 0 \mid \phi^t(x) \in \Sigma\}. \] One can use \( \tau \) to define the return map \( \psi : \Sigma \setminus \partial \Sigma \to \Sigma \setminus \partial \Sigma \) by \( \psi(x) = \phi^{\tau(x)}(x) \). It follows from Definition 1.3 that \( \psi \) is a smooth diffeomorphism.

Consider the normal bundle to the flow \( \xi = TM/\mathbb{R}X \to M \), and denote by \( \mathbb{P}_+\xi \) the circle bundle \( (\xi \setminus 0)/\mathbb{R}_+ \to M \). The latter is isomorphic to the unit bundle in \( \xi \) once a choice of metric is fixed. The equivalence class in \( \mathbb{P}_+\xi \) of a non-zero vector \( \nu \in \xi \) will be denoted by \( \mathbb{R}_+\nu \). The linearised flow \( D\phi^t \) induces flows on \( \xi \) and on \( \mathbb{P}_+\xi \), both denoted by \( D\phi^t \) with no fear of ambiguity. These flows cover \( \phi^t \). Both \( \xi \) and \( \mathbb{P}_+\xi \) get oriented as bundles by the flow and the ambient orientation. Now let \( \gamma : \mathbb{R}/T\mathbb{Z} \to M \) be a periodic orbit of \( \phi^t \), where \( T > 0 \) is the primitive period. The total space \( T_\gamma \) of the trivial circle bundle \( \gamma(T) \ast \mathbb{P}_+\xi \to \mathbb{R}/\mathbb{Z} \), which we see as a submanifold of \( \mathbb{P}_+\xi \), is a \( D\phi^t \)-invariant torus. The dynamics of \( D\phi^t \) on \( T_\gamma \) will be referred to as linearised polar dynamics along \( \gamma \). For each boundary orbit \( \gamma \) of a global surface of section \( \Sigma \), consider

\[ \nu_{\Sigma, \gamma} = \{\mathbb{R}_+\nu \mid \nu \in T\Sigma, \nu \text{ outward pointing}\}. \]

It follows that \( \nu_{\Sigma, \gamma}/\mathbb{R}X \) defines the graph of a section of \( \gamma(T \cdot)^*\mathbb{P}_+\xi \) and, as such, determines a smooth submanifold of \( T_\gamma \).

**Definition 1.6.** The global surface of section \( \Sigma \) is called strong if the associated return time function is bounded away from zero and bounded from above. It is called \( \partial \)-strong if \( \nu_{\Sigma, \gamma}/\mathbb{R}X \) is a global surface of section for the linearised polar dynamics along every \( \gamma \subset \partial \Sigma \).

The reader will easily check that \( \partial \)-strong implies strong.

**Remark 1.7.** As suggested by the referee, the notion of \( \partial \)-strong global surface of section could be more succinctly and geometrically described as follows. Consider \( z \in \partial \Sigma \) and \( v \in T_zM \setminus \mathbb{R}X_z \). It follows that \( \Sigma \) is \( \partial \)-strong if, and only if, for any \( z \) and \( v \) as above the curve \( t \mapsto D\phi^t \cdot v \) intersects the 3-dimensional manifold \( T\Sigma|_{\partial \Sigma} \) infinitely many often in the future and in the past, and all such intersections are transverse in the 4-dimensional space \( TM|_{\partial \Sigma} \).

In [Poi12] Poincaré described annular global surfaces of section for certain regimes of the planar circular restricted three-body problem (PCR3BP). In the same paper one finds his last geometric theorem, proved by Birkhoff in [Bir13] and nowadays known as the Poincaré-Birkhoff theorem. Poincaré applied his statement to the return map of the sections he found for the PCR3BP to obtain infinitely many periodic orbits. In [Bir66] Birkhoff explained that positively curved Riemannian geodesic flows on \( S^2 \) always have annulus-like global surfaces of section. Birkhoff’s result admits a generalisation to Reeb flows in dimension three, see [HSW22]. Birkhoff’s section plays an important role in the proof of Theorem 1.2.
1.2 Reeb flows

A contact form $\lambda$ on a 3-manifold $M$ is a 1-form such that $\lambda \wedge d\lambda$ defines a volume form. We always consider $M$ equipped with the orientation induced by $\lambda \wedge d\lambda$. The associated Reeb vector field $X$ is implicitly determined by the equations

$$i_Xd\lambda = 0 \quad i_X\lambda = 1 \quad (1)$$

and its flow $\phi^t$ is referred to as the Reeb flow. The plane field $\xi = \ker \lambda$ is called a contact structure and can be seen as a representation of $TM/\mathbb{R}X$. It becomes a symplectic vector bundle with $d\lambda$. Note that $\phi^t$ preserves $\lambda$, hence also $\xi$, $d\lambda$ and $\lambda \wedge d\lambda$.

A periodic orbit $\gamma$ of $\phi^t$ has a Conley-Zehnder index relative to a symplectic trivialization of $(\xi,d\lambda)|_\gamma$. This index is an integer that can be described in terms of transverse rotation numbers, see Remark 2.1 for details.

**Definition 1.8** (Hofer, Wysocki and Zehnder [HWZ99]). A contact form $\lambda$ on a closed 3-manifold $M$ is dynamically convex if the first Chern class $c_1(\xi,d\lambda)$ vanishes on $\pi_2(M)$, and every contractible periodic Reeb orbit has Conley-Zehnder index at least equal to three in a symplectic frame that extends to a capping disk.

One of the motivations for the above definition is the following remarkable result.

**Theorem 1.9** (Hofer, Wysocki and Zehnder [HWZ98]). The Reeb flow of every dynamically convex contact form on $S^3$ admits a disk-like global surface of section.

Using the methods from [HWZ98] one can prove a characterisation result for periodic Reeb orbits bounding a disk-like global surface of section. Let us recall the self-linking number of a knot $K$ transverse to a contact structure on $S^3$. Consider a knot $K'$ obtained from $K$ by pushing it in the direction of a global trivialization of the contact structure. Choose an orientation for $K$ and orient $K'$ accordingly. The self-linking number of $K$ is the linking number of $K$ and $K'$.

**Theorem 1.10** ([Hry14]). A periodic Reeb orbit of a dynamically convex contact form on $S^3$ bounds a disk-like global surface of section if, and only if, it is unknotted and has self-linking number $-1$.

**Remark 1.11.** The global surfaces of section obtained from [HWZ98, Hry14] are closures of projections of pseudo-holomorphic planes in symplectisations, and as such are smooth in the interior but only $C^1$ up to the boundary. As shown in the proof of Proposition 2.8, they can be $C^1$-perturbed to smooth $\partial$-strong global surfaces of section.

If $M = S^3$ then the contact structure $(\xi,d\lambda)$ is trivial as a symplectic vector bundle. A global symplectic trivialisation $\sigma$ of $(\xi,d\lambda)$ induces a trivialisation $\mathbb{P}_+\xi \simeq S^3 \times \mathbb{R}/2\pi\mathbb{Z}$. The map obtained by composing this diffeomorphism with the projection onto the second factor will be denoted by

$$\Theta_\sigma : \mathbb{P}_+\xi \to \mathbb{R}/2\pi\mathbb{Z}. \quad (2)$$
Let the periodic Reeb orbit $\gamma_0$ bound a $\partial$-strong disk-like global surface of section $\Sigma$. Stokes theorem implies that $\Sigma$ orients $\gamma_0 = \partial \Sigma$ along the flow. For every $x \in S^3 \setminus \gamma_0$ we denote

$$t^+_T(x) = \inf\{t \geq 0 \mid \phi^t(x) \in \Sigma\} \quad t^-_T(x) = \sup\{t \leq 0 \mid \phi^t(x) \in \Sigma\}$$

which are uniformly bounded functions that vanish and are discontinuous on $\Sigma \setminus \gamma_0$, and are non-vanishing and smooth on $S^3 \setminus \Sigma$. Given $T > 0$ consider the interval

$$I(T, x; \Sigma) = [t^+_T(x), T + t^+_T(\phi^T(x))]$$

and denote by $k(T, x; \Sigma) \subset S^3 \setminus \gamma_0$ any loop obtained by concatenating to $\phi^{I(T, x; \Sigma)}(x)$ a path in the interior of $\Sigma$ from $\phi^T + t^+_T(\phi^T(x)) (x)$ to $\phi^T(x) (x)$. For $u \in (\xi \setminus 0)/\mathbb{R}_+$ denote by $t \in \mathbb{R} \mapsto \tilde{\Theta}_\sigma(t, u) \in \mathbb{R}$ a continuous lift of $t \mapsto \Theta_\sigma(D\phi^t \cdot u)$. Finally we define

$$\kappa(\gamma_0) = \liminf_{T \to +\infty} \left( \inf_{x, u} \frac{\tilde{\Theta}_\sigma(T, u) - \tilde{\Theta}_\sigma(0, u)}{\text{link}(k(T, x; \Sigma), \gamma_0)} \right)$$

where the infimum is taken over all pairs $(x, u)$, where $x \in S^3 \setminus \gamma_0$ and $u \in (\xi \setminus 0)/\mathbb{R}_+$.

**Remark 1.12.** The quantity $\kappa(\gamma_0)$ does not depend on the choice of $\partial$-strong disk-like global surface of section spanned by $\gamma_0$. Moreover, since we work on $S^3$, it also does not depend on the chosen global symplectic trivialisation $\sigma$. See [FH21, appendix A] for detailed proofs of these statements.

Our abstract result reads as follows.

**Theorem 1.13.** Let $\lambda$ be a dynamically convex contact form on $S^3$, and $\gamma_0$ be an unknotted periodic Reeb orbit with self-linking number $-1$. If $\kappa(\gamma_0) > 2\pi$ then the Reeb flow of $\lambda$ is right-handed.

As mentioned before, a rich source of dynamically convex contact forms are the strictly convex energy levels in $\mathbb{R}^4$. Consider $\mathbb{R}^4$ equipped with coordinates $(q_1, p_1, q_2, p_2)$ and symplectic form $\omega_0 = d\lambda_0$, where $\lambda_0$ is the 1-form

$$\lambda_0 = \frac{1}{2}(p_1 dq_1 - q_1 dp_1 + p_2 dq_2 - q_2 dp_2).$$

It follows that $\lambda_0$ is a contact form on hypersurfaces that are star-shaped with respect to the origin.

Let $C \subset \mathbb{R}^4$ be a smooth convex body with the origin in the interior. Denote by $\nu_C$ the gauge function of $C$, which is defined to be the unique 1-homogeneous function $\nu_C$ satisfying $\partial C = \nu_C^{-1}(1)$. Note that $\nu_C$ is continuous on $\mathbb{R}^4$ and smooth on $\mathbb{R}^4 \setminus \{0\}$. The Hessian of $\nu_C^2$, denoted by $D^2 \nu_C^2$, defines a 0-homogeneous matrix-valued function on $\mathbb{R}^4 \setminus \{0\}$. Consider

$$K^C_{\text{min}} = \inf_{z \in \mathbb{R}^4 \setminus \{0\}} \min\{\mu \mid \mu \text{ is an eigenvalue of } D^2 \nu_C^2(z) \}.$$
Convexity implies $K_C^{\min} \geq 0$. We call $\partial C$ strictly convex if $K_C^{\min} > 0$. The Hamiltonian vector field $X_H$ of $H = \nu^2$, defined on $\mathbb{R}^4 \setminus \{0\}$ by $i_{X_H} \omega_0 = -dH$, satisfies $i_{X_H} \lambda_0 = H$. Hence $X_H$ restricts to $\partial C$ as the Reeb vector field of the contact form induced by $\lambda_0$. In [HWZ98] Hofer, Wysocki and Zehnder proved that this contact form is dynamically convex provided $\partial C$ is strictly convex. In particular, there are $\partial$-strong disk-like global surfaces of section for the Hamiltonian dynamics on $\partial C$ under the assumption of strict convexity.

**Theorem 1.14.** Assume that $\partial C$ is strictly convex, and let $D \subset \partial C$ be a disk-like $\partial$-strong global surface of section. Denote by $\tau_{\min}(D) > 0$ the infimum of the first return time on $D$. Then $\kappa(D) > 2K_C^{\min} \tau_{\min}(D)$. In particular, if the inequality

$$K_C^{\min} \tau_{\min}(D) > \pi$$

holds, then the Hamiltonian flow on $\partial C$ is right-handed.

**Remark 1.15.** If $C$ is the unit Euclidean ball in $\mathbb{R}^4$, then the gauge function is the Euclidean norm, $K_C^{\min} = 2$, the Reeb flow on $\partial C$ is $\pi$-periodic and equal to the Hopf flow. In particular, $\tau_{\min}(D) = \pi$ for any disk-like global surface of section $D \subset \partial C$. If $C'$ is $C^2$-close to $C$ then it can be proved that the Reeb flow on $\partial C'$ admits a $\partial$-strong disk-like global surface of section $D'$ with return time uniformly close to $\pi$. Since the curvatures of $C'$ are close to those of $C$ we get

$$K_C^{\min} \tau_{\min}(D') \sim K_C^{\min} \tau_{\min}(D) = 2\pi \Rightarrow K_C^{\min} \tau_{\min}(D') > \pi.$$

By Theorem 1.14 the Reeb flow on $\partial C'$ is right-handed, as claimed in [Ghy09].

**Remark 1.16.** In the case of Reeb flows, one gets restrictions of a contact topological nature from Ghys’ Theorem [Ghy09]. For instance, every periodic Reeb orbit defines a transverse knot that satisfies equality in Bennequin’s inequality. Hence, we get restrictions on the transverse knot types defined by periodic Reeb orbits. One also deduces that all finite collections of periodic orbits bind open book decompositions that support the contact structure in the sense of Giroux, with pages that are global surfaces of section.

**Organisation of the paper.** In Section 2 we give the definition of right-handedness and prove Theorem 1.13. We also show how Theorem 1.14 follows from Theorem 1.13. Section 3 is devoted to the study of geodesic flows on $S^2$: we show how the claimed pinching condition enables us to verify the quantitative condition of Theorem 1.13. The proof uses comparison theorems from Riemannian geometry.

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2 Right-handedness from dynamical pinching

2.1 Transverse rotation numbers

Let $\gamma$ be a non-constant periodic orbit of a smooth flow $\phi^t$ defined on an oriented 3-manifold $M$. Denote by $T > 0$ its primitive period, and think of $\gamma$ as a map $\gamma : \mathbb{R}/TZ \to M$. Consider coordinates $(t, z = x + iy = re^{i\theta}) \in \mathbb{R}/TZ \times \mathbb{C}$ defined on a small tubular neighbourhood $N$ of $\gamma$ such that $dt \wedge dx \wedge dy > 0$ and $\phi^t(\gamma(0)) = (t, 0)$. We shall refer to such coordinates as tubular coordinates around $\gamma$. For every $\theta_0 \in \mathbb{R}$ consider the continuous real valued function $\theta(t)$ defined by

$$D\phi^t(0, 0) \cdot (0, e^{i\theta_0}) \in \mathbb{R}(1, 0) + \mathbb{R}+(0, e^{i\theta(t)}) \quad \theta(0) = \theta_0$$

If $y \in H^1(N \setminus \gamma, \mathbb{R})$ is homologous to $p \, dt + q \, d\theta$ then we define

$$\rho^y(\gamma) = \frac{T}{2\pi} \left( p + q \lim_{t \to \infty} \frac{\theta(t)}{t} \right). \tag{8}$$

This number is called the transverse rotation number of $\gamma$ with respect to $y$. The number $\rho^y(\gamma)$ might also be interpreted as a Poincaré translation number of the linearized flow on the unit normal bundle. It turns out that $\rho^y(\gamma)$ does not depend on the choice of tubular coordinates or on the initial condition $\theta_0$; see [Hry20, section 2].

**Remark 2.1.** In the notation above, suppose that $\phi^t$ is the Reeb flow of a contact form $\lambda$, and let $\sigma$ be a local symplectic trivialization of $(\xi, d\lambda)$ along $\gamma$. Denote by $\gamma_\sigma$ an oriented loop in $N \setminus \gamma$ obtained by pushing $\gamma$ in the direction of $\sigma$. If $N$ is small enough then $d\lambda$ defines an area form on any meridional disk $D$. Orient $D$ by $d\lambda$. There is a unique class $y_\sigma \in H^1(N \setminus \gamma, \mathbb{Z})$ determined by $\langle y_\sigma, \partial D \rangle = 1$, $\langle y_\sigma, \gamma_\sigma \rangle = 0$. If no transverse Floquet multiplier of $\gamma$ is a root of unit of order $n \geq 1$ then one says that the $n$-th iterate $\gamma^n$ of $\gamma$ is non-degenerate, and defines

$$\mu_{CZ}^n(\gamma^n) = \begin{cases} 2 \lfloor n2\pi \rho^{y_\sigma}(\gamma) \rfloor + 1 & \text{if } n2\pi \rho^{y_\sigma}(\gamma) \notin \mathbb{Z} \\ 2n2\pi \rho^{y_\sigma}(\gamma) & \text{if } n2\pi \rho^{y_\sigma}(\gamma) \in \mathbb{Z} \end{cases} \tag{9}$$

Note that when $\gamma^n$ is non-degenerate then $n2\pi \rho^{y_\sigma}(\gamma) \in \mathbb{Z}$ precisely when $\gamma^n$ is positive hyperbolic. When $\gamma^n$ is degenerate then $\mu_{CZ}^n(\gamma^n)$ is defined to be the lowest possible value of the right-hand side above, obtained from small $C^2$-perturbations of $\lambda$ that keep $\gamma$ as a periodic Reeb orbit with $\gamma^n$ non-degenerate. The inequality $\mu_{CZ}^n(\gamma) \geq 3$ is equivalent to $\rho^{y_\sigma}(\gamma) > 1$. This follows from (9) in the non-degenerate case, and is also true in general.

Suppose further that $M$ is a homology 3-sphere. Consider any oriented Seifert surface $S$ spanned by $\gamma$. We require that $\partial S = \gamma$ including orientations, when $\gamma$ is oriented by the flow. As before, orient the meridional disk $D \subset N$ by $d\lambda$. Let $S^* \in H^1(M \setminus \gamma, \mathbb{Z})$ denote the class dual to $S$. Since $M$ is a homology 3-sphere, the class $S^*$ is independent of $S$. In fact, $\langle S^*, \beta \rangle = \text{link}(\gamma, \beta)$ for any oriented loop $\beta$ in $M \setminus \gamma$. After restricting to $N \setminus \gamma$ we can view it as a class in $H^1(N \setminus \gamma, \mathbb{Z})$. 


**Definition 2.2.** Under the assumption that $M$ is a homology 3-sphere, we call $\rho^y(\gamma)$ the transverse rotation number of $\gamma$ in a Seifert framing, where $y$ is the cohomology class dual to some (hence any) oriented Seifert surface $S$ spanned by $\gamma$.

From now on assume that $M$ is a homology three-sphere and that $\phi^t$ has no rest points. Let $\gamma_1, \ldots, \gamma_n$ be a collection of periodic orbits oriented along the flow, and consider the oriented link $L = \gamma_1 \cup \cdots \cup \gamma_n$. Let $\Sigma_i$ be an oriented Seifert surface satisfying $\partial \Sigma_i = \gamma_i$, orientations included. Let $\Sigma$ be an oriented Seifert surface satisfying $\partial \Sigma = L$, orientations included. Consider small tubular neighbourhoods $N_i$ of $\gamma_i$ with tubular coordinates as above. Denote by $\Sigma^* \in H^1(M \setminus L, \mathbb{R})$ and $\Sigma_i^* \in H^1(M \setminus \gamma_i, \mathbb{R})$ the classes dual to $\Sigma$ and $\Sigma_i$ respectively. Each $\Sigma_i^*$ restricts to a class in $H^1(N_i \setminus \gamma_i, \mathbb{R})$ still denoted by $\Sigma_i^*$ with no fear of ambiguity. Similarly $\Sigma^*$ restricts to a class in $H^1(N_i \setminus \gamma_i, \mathbb{R})$ for every $i$, all of which are denoted by $\Sigma^*$ with no fear of ambiguity.

**Lemma 2.3.** For every $i$ we have $\rho^{\Sigma^*}(\gamma_i) = \rho^{\Sigma_i^*}(\gamma_i) + \frac{1}{2\pi} \sum_{j \neq i} \text{link}(\gamma_i, \gamma_j)$.

**Proof.** Let $(t, z = |z|e^{i\theta}) \in \mathbb{R}/T_1 \mathbb{Z} \times \mathbb{C}$ be tubular coordinates on a small tubular neighbourhood $N_i$ of $\gamma_i$. With $\epsilon > 0$ small we consider the loops $e_i(t) = (t, \epsilon)$, $f_i(\theta) = (0, \epsilon e^{i\theta})$. Then $\{e_i, f_i\}$ is a basis for $H_1(N_i \setminus \gamma_i, \mathbb{Z})$ whose dual basis is $\{dt/T_i, d\theta/2\pi\}$. Then on $N_i \setminus \gamma_i$ one has

$$\Sigma_i^* \equiv \langle \Sigma_i^*, e_i \rangle \frac{dt}{T_i} + \langle \Sigma_i^*, f_i \rangle \frac{d\theta}{2\pi} = \langle \Sigma_i^*, e_i \rangle \frac{dt}{T_i} + \frac{d\theta}{2\pi}$$

$$\Sigma^* \equiv \langle \Sigma^*, e_i \rangle \frac{dt}{T_i} + \langle \Sigma^*, f_i \rangle \frac{d\theta}{2\pi} = \langle \Sigma^*, e_i \rangle \frac{dt}{T_i} + \frac{d\theta}{2\pi}$$

Denote by $\nu_i$ a section of the normal bundle of $\Sigma_i$ along $\gamma_i$. Let $\gamma'_i$ be the loop obtained by pushing $\gamma_i$ in the direction of $\nu_i$. Note that $\gamma'_i$ gets an orientation from $\gamma_i$. Then

$$\gamma'_i \equiv \left\langle \frac{dt}{T_i}, \gamma_i \right\rangle e_i + \left\langle \frac{d\theta}{2\pi}, \gamma_i \right\rangle f_i = e_i + \left\langle \frac{d\theta}{2\pi}, \gamma_i \right\rangle f_i$$

From these formulas it follows that

$$\text{int}(\gamma'_i, \Sigma) - \text{int}(\gamma'_i, \Sigma_i) = \langle \Sigma^*, \gamma'_i \rangle - \langle \Sigma_i^*, \gamma'_i \rangle = \langle \Sigma^*, e_i \rangle - \langle \Sigma_i^*, e_i \rangle$$

Consider the 2-cycle $S = \Sigma - \Sigma_1 - \cdots - \Sigma_n$. Since the ambient space is a homology sphere (over $\mathbb{Z}$), $S$ is a boundary and we get

$$0 = \text{int}(\gamma'_i, S) = \text{int}(\gamma'_i, \Sigma) - \text{int}(\gamma'_i, \Sigma_i) - \sum_{j \neq i} \text{link}(\gamma'_i, \gamma_j)$$

$$= \langle \Sigma^*, e_i \rangle - \langle \Sigma_i^*, e_i \rangle - \sum_{j \neq i} \text{link}(\gamma_i, \gamma_j)$$

(10)
Hence, if $\theta(t)$ denotes a lift of the polar angle of the linearised flow along $\gamma_i$ in the given tubular coordinates we get

$$\rho^\Sigma(\gamma_i) = \frac{T_i}{2\pi} \left( \langle \Sigma^*_i, e_i \rangle + \frac{1}{2\pi} \lim_{t \to +\infty} \frac{\theta(t)}{t} \right)$$

$$= \frac{T_i}{2\pi} \left( \langle \Sigma^*_i, e_i \rangle + \frac{1}{T_i} \sum_{j \neq i} \text{link}(\gamma_i, \gamma_j) + \frac{1}{2\pi} \lim_{t \to +\infty} \frac{\theta(t)}{t} \right)$$

$$= \rho^\Sigma(\gamma_i) + \frac{1}{2\pi} \sum_{j \neq i} \text{link}(\gamma_i, \gamma_j)$$

(11)

as desired.

**Corollary 2.4.** If $n \geq 2$ and $\text{link}(\gamma_i, \gamma_j) \geq 1$ for all $i \neq j$ then $\rho^\Sigma(\gamma_i) > \rho^\Sigma^*(\gamma_i)$ for every $i$.

**Remark 2.5.** Let the periodic orbit $\gamma$ be oriented by the flow, and let $\Sigma, \hat{\Sigma}$ be oriented Seifert surfaces such that $\partial \Sigma = \gamma, \partial \hat{\Sigma} = \gamma$ (including orientations). Lemma 2.3 proves the previously claimed fact that $\rho^\Sigma(\gamma) = \rho^{\hat{\Sigma}}(\gamma)$.

### 2.2 Right-handedness

Fix a smooth nowhere vanishing vector field $X$ on an oriented homology 3-sphere, and denote its flow by $\phi^t$. Let $\mathcal{P}$ be the set of $\phi^t$-invariant Borel probability measures. Denote by $\mathcal{R}$ the set of recurrent points, and consider the following measurable set:

$$R = \{(x, y) \in \mathcal{R} \times \mathcal{R} \mid \phi^x(x) \cap \phi^y(y) = \emptyset\}.$$  

Let $\mu_1, \mu_2 \in \mathcal{P}$ be ergodic, and denote by $\mu_1 \times \mu_2$ the product measure. There are two cases:

(A) $\mu_1 \times \mu_2(R) = 1$.

(B) $\mu_1 \times \mu_2(R) = 0$ and $\text{supp}(\mu_1) = \text{supp}(\mu_2) = \gamma$ for some periodic orbit $\gamma$ (in particular $\mu_1 = \mu_2$).

Each case needs to be treated separately. Fix an auxiliary Riemannian metric $g$ that near $p$ and $q$ realises pieces of trajectories of $X$ as geodesic arcs.

**Case A.** Consider $(p, q) \in R$ and let $S(p, q)$ denote the set of ordered pairs of sequences $\{T_n\}, \{S_n\}$ satisfying $T_n, S_n \to +\infty, \phi^{T_n}(p) \to p$ and $\phi^{S_n}(q) \to q$. For $n$ large enough let $\alpha_n$ and $\beta_n$ be the (unique) shortest geodesic arcs from $\phi^{T_n}(p)$ to $p$ and from $\phi^{S_n}(q)$ to $q$, respectively. Consider $C^1$-small\(^1\) perturbations $\hat{\alpha}, \hat{\beta}$ of $\alpha_n, \beta_n$, keeping end points fixed, such that the closed loops $k(T_n, p)$ and $k(S_n, q)$ obtained

---

\(^1\)The assumption that $\hat{\alpha}, \hat{\beta}$ are $C^1$-close to $\alpha_n, \beta_n$ is important. Being $C^0$-close is not enough.
by concatenating \(\hat{\alpha}\) to \(\phi^{[0,T_n]}(p)\) and \(\hat{\beta}\) to \(\phi^{[0,S_n]}(q)\), respectively, do not intersect each other. Define

\[
\text{link}_-(\phi^{[0,T_n]}(p), \phi^{[0,S_n]}(q)) = \liminf_{\alpha \to \alpha_n, \beta \to \beta_n} \text{link}(k(T_n, p), k(S_n, q))
\]

(13)

and

\[
\ell(p, q) = \inf_{(T_n, S_n) \in S(p, q)} \liminf_{n \to \infty} \frac{1}{T_n S_n} \text{link}_-(\phi^{[0,T_n]}(p), \phi^{[0,S_n]}(q)).
\]

(14)

Note that (13) and (14) belong to \([-\infty, +\infty]\). One says that \(\mu_1, \mu_2\) are positively linked if for \(\mu_1 \times \mu_2\)-almost all points \((p, q)\) in \(R\) the inequality \(\ell(p, q) > 0\) holds.

**Case B.** One says that \(\mu_1, \mu_2\) are positively linked if the transverse rotation number of the periodic orbit \(\gamma\) corresponding to the supports of \(\mu_1, \mu_2\) computed in a Seifert framing is strictly positive.

**Definition 2.6.** The flow \(\phi^t\) is said to be right-handed if all pairs of ergodic measures in \(\mathcal{P}\) are positively linked.

**Remark 2.7.** The above definition is equivalent to the one explained in [Ghy09]. Of course, Ghys defines right-handedness in a much more elegant way by explaining that in case A the ergodicity assumption can be used to prove that for \(\mu_1 \times \mu_2\)-almost all \((p, q) \in R\), all possible sequences

\[
\text{link}(k(T_n, p), k(S_n, q))
\]

as above will converge to a common limit. This limit is defined to be the value of the quadratic linking form evaluated at the pair \((\mu_1, \mu_2)\). There is also a way of assigning a number in case B. The advantage of the definition explained here is that it avoids dealing with details on the existence of the quadratic linking form.

### 2.3 Proof of Theorem 1.13

We fix a dynamically convex contact form \(\lambda\) on \(S^3\), denote by \(X\) the associated Reeb vector field and by \(\phi^t\) the Reeb flow. The following technical statement is based on the main result from [HWZ98].

**Proposition 2.8.** Let \(\lambda\) be a dynamically convex contact form on \(S^3\), and denote by \(X\) the Reeb vector field of \(\lambda\). Let \(\gamma_0\) be any unknotted periodic Reeb orbit with self-linking number \(-1\). Denote the primitive period by \(T_0 > 0\). There exists a map of class \(C^\infty\)

\[
\Psi : \mathbb{R}/\mathbb{Z} \times D \to S^3
\]

(15)

with the following properties:

(a) \(\Psi(0, e^{is}) = \gamma_0(T_0 s/2\pi)\) for all \(s \in \mathbb{R}/2\pi\mathbb{Z}\), and \(\Psi(0, \cdot) : D \to S^3\) is an embedding that defines a \(\partial\)-strong global surface of section for the flow of \(X\).
(b) \( \Psi \) defines an orientation preserving diffeomorphism \( \mathbb{R}/\mathbb{Z} \times \hat{D} \to S^3 \setminus \gamma_0 \).

(c) There exists a smooth vector field \( W \) on \( \mathbb{R}/\mathbb{Z} \times \hat{D} \) that coincides with the pull-back of \( X \) by \( \Psi|_{\mathbb{R}/\mathbb{Z} \times \hat{D}} \) on \( \mathbb{R}/\mathbb{Z} \times \hat{D} \), and is tangent to \( \mathbb{R}/\mathbb{Z} \times \partial \hat{D} \).

(d) For every \( t \in \mathbb{R}/\mathbb{Z} \) the disk \( \{t\} \times \hat{D} \) is transverse to \( W \) up to the boundary, and defines a global section for the flow of \( W \).

(e) There is a non-vanishing vector field \( Z \) on \( S^3 \) satisfying \( i_Z \lambda = 0 \) with the following property. Let \( Z_0 \) be the unique smooth vector field on \( \hat{D} \) defined by

\[
Z(\Psi(0, z)) - d\Psi(0, z) \cdot Z_0(z) \in \mathbb{R}X(\Psi(0, z)).
\]

If \( \phi : \hat{D} \to \mathbb{R} \) is continuous and satisfies \( Z_0 = |Z_0|e^{i\phi} \), then \( \phi \in L^\infty(\hat{D}) \).

**Proof.** Following [HWZ98, HWZ99, Hry14], there exists an embedded closed disk \( D \hookrightarrow S^3 \) of class \( C^1 \) such that \( \partial D = \gamma_0 \). It is obtained by projecting to \( M \) a special finite energy plane in the symplectisation of \( (M, \xi) \) asymptotic to \( \gamma_0 \). By a careful analysis of the aforementioned embedding, it is also possible to show that there exists a \( C^\infty \) family of \( C^\infty \) disks \( D_\delta \), parametrised by \( 0 < \delta \ll 1 \), such that:

1. Each \( D_\delta \) is a \( \partial \)-strong global surface of section satisfying \( \partial D_\delta = \gamma_0 \).
2. \( D_\delta \to D \) in \( C^1 \) as \( \delta \to 0 \).

For detailed proofs of the claims above we refer to [FH21, appendix C].

We want now to project vector fields on \( D_\delta \setminus \gamma_0 \). Consider a Martinet tube \( F : \mathbb{R}/\mathbb{Z} \times \text{int}(\hat{D}) \subset \mathbb{R}/\mathbb{Z} \times \mathbb{C} \to U \) around \( \gamma_0 \). This means that \( U \) is an open neighborhood of \( \gamma_0 \), and \( F \) is a diffeomorphism satisfying

\[
(\text{MT1}) \quad F(t, 0) = \gamma_0(T_0 t).
\]

\[
(\text{MT2}) \quad \text{There exists a smooth function } f : \mathbb{R}/\mathbb{Z} \times \text{int}(\hat{D}) \to (0, +\infty) \text{ such that } f(\vartheta, 0) \equiv T_0, \quad df(\vartheta, 0) \equiv 0, \quad \text{and } F^*\lambda = f(\vartheta, x + iy)(d\vartheta + xdy).
\]

The vector fields \( \vartheta_x, \vartheta_y \) along \( \gamma_0 \) define a \( d\lambda \)-positive frame of \( \xi \) along \( \gamma_0 \). Fix \( \delta \) small. Near \( \gamma_0 \) we can find a non-vanishing section \( Y_0 \) of \( \xi \) that satisfies \( \mathbb{R}Y_0 = T\bar{D}_\delta \cap \xi \) on points of \( \bar{D}_\delta \) near \( \gamma_0 \). Using the coordinates \( (\vartheta, x + iy) \) in the Martinet tube near \( \gamma_0 \), we can write

\[
Y_0 = a_0 \vartheta_x + b_0 (\vartheta_y - x \vartheta_y)
\]

where the \( \mathbb{C} \)-valued function \( a_0 + ib_0 \) does not vanish. Consider the vector field

\[
Y_1 = a_1 \vartheta_x + b_1 (\vartheta_y - x \vartheta_y) \quad \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} \cos(2\pi \vartheta) & \sin(2\pi \vartheta) \\ -\sin(2\pi \vartheta) & \cos(2\pi \vartheta) \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}.
\]

(16) Since \( \gamma_0 \) has self-linking number \(-1\), and \( Y_1 \) winds \(-1\) with respect to \( Y_0 \) along \( \gamma_0 \), there exists a smooth extension of \( Y_1 \) as a non-vanishing section of \( \xi \to S^3 \). We continue to denote this extension by \( Y_1 \), with no fear of ambiguity.
Recall that $\mathcal{D}_\delta$ is a global surface of section spanned by $\gamma_0$ for small enough $\delta$, and $\mathcal{D}_\delta \to \mathcal{D}$ in $C^1$ as $\delta \to 0$. From now on we will denote by $\hat{\mathcal{D}}_\delta = \mathcal{D}_\delta \setminus \partial \mathcal{D}_\delta = \mathcal{D}_\delta \setminus \gamma_0$ the interior of $\mathcal{D}_\delta$. The projection along the Reeb direction is a smooth vector bundle map

$$P : TS^3|_{\mathcal{D}_\delta} \to T\hat{\mathcal{D}}_\delta$$

characterised by

$$P^2 = P, \quad \ker P = \mathbb{R}X.$$  

Note that $P$ becomes singular on the boundary since $X$ is tangent to $\partial \mathcal{D}_\delta = \gamma_0$. Note also that $P$ defines a vector bundle isomorphism between $\xi|_{\mathcal{D}_\delta}$ and $T\hat{\mathcal{D}}_\delta$, this follows from the transversality between the Reeb vector field and $\hat{\mathcal{D}}_\delta$.

Let $A_\delta \subset \mathcal{D}_\delta$ be a small compact neighborhood of $\gamma_0 = \partial \mathcal{D}_\delta$ in $\mathcal{D}_\delta$, small enough so that it is contained in the domain of definition of $Y_0$. Denote $A_\delta = A_\delta \setminus \gamma_0$. We can equip $A_\delta$ with polar coordinates $(\rho, \vartheta)$, coherent with the Martinet tube in the sense that $\vartheta$ is the same $\mathbb{R}/\mathbb{Z}$-coordinate in the domain of $F$, so that $A_\delta$ corresponds to $\{ (\rho, \vartheta) \in [1-\epsilon, 1] \times \mathbb{R}/\mathbb{Z} \}$ for $\epsilon > 0$ small enough, and, moreover, $A_\delta$ corresponds to $\{ (\rho, \vartheta) \in [1-\epsilon, 1) \times \mathbb{R}/\mathbb{Z} \}$. The local slice disk $\{ \vartheta \equiv 0 \mod \mathbb{Z} \}$ intersects $A_\delta$ in a smooth arc $\eta$ transverse to $\partial A_\delta$ defining a generator of $H_1(A_\delta, \partial A_\delta)$.

By (16), $Y_0$ and $Y_1$ are positively collinear only at $\{ \vartheta \equiv 0 \mod \mathbb{Z} \}$. Hence on $\hat{A}_\delta$ the vector fields $P(Y_1)$ and $Y_0 = P(Y_0)$ are positively collinear only at $\eta \setminus \gamma_0$. The universal covering of $A_\delta$ can be given coordinates $(\rho, \tilde{\vartheta}) \in [1-\epsilon, 1) \times \mathbb{R}$, where $\vartheta \equiv \tilde{\vartheta} \mod \mathbb{Z}$. Hence the universal covering of $A_\delta$ is $[1-\epsilon, 1) \times \mathbb{R}$. The disk $\mathcal{D}_\delta$ can be equipped with global coordinates $u + iv \in \mathbb{D}$ such that $u + iv = re^{i2\pi \vartheta}$ near the boundary. We can write

$$\begin{align*}
Y_0 &= R_0(\cos \varphi_0 \partial_u + \sin \varphi_0 \partial_v) \quad \text{on } A_\delta \\
P(Y_1) &= R_1(\cos \varphi_1 \partial_u + \sin \varphi_1 \partial_v) \quad \text{on } \mathcal{D}_\delta
\end{align*}$$

in polar coordinates, where

$$\varphi_0 : [1-\epsilon, 1] \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}/2\pi\mathbb{Z} \quad \varphi_1 : \mathcal{D}_\delta \to \mathbb{R}/2\pi\mathbb{Z}$$

are smooth. Choose smooth lifts

$$\tilde{\varphi}_0 : [1-\epsilon, 1] \times \mathbb{R} \to \mathbb{R} \quad \text{and} \quad \tilde{\varphi}_1 : \mathcal{D}_\delta \to \mathbb{R}$$

of $\varphi_0$ and $\varphi_1$, respectively. We will also write $\tilde{\varphi}_1(\rho, \tilde{\vartheta})$ to denote the corresponding lift of the restriction of $\varphi_1$ to $\hat{A}_\delta$, with no fear of ambiguity. Note that

$$\begin{cases}
\tilde{\varphi}_0(\rho, \tilde{\vartheta} + 1) = \tilde{\varphi}_0(\rho, \tilde{\vartheta}) + 2\pi, \\
\tilde{\varphi}_1(\rho, \tilde{\vartheta} + 1) = \tilde{\varphi}_1(\rho, \tilde{\vartheta}).
\end{cases} \quad (17)$$

Note also that $\tilde{\varphi}_0$ is bounded on any compact subset of $[1-\epsilon, 1] \times \mathbb{R}$ since $Y_0$ is smooth on $A_\delta$. As observed before,

$$\tilde{\varphi}_1(\rho, \tilde{\vartheta}) - \tilde{\varphi}_0(\rho, \tilde{\vartheta}) \in 2\pi\mathbb{Z} \quad \iff \quad \tilde{\vartheta} \in \mathbb{Z}. \quad (18)$$
The lifts of $\eta$ divide the universal covering of $\hat{A}_\delta$ in fundamental domains. It follows from (18) that on each such fundamental domain $[1 - \epsilon, 1) \times [k, k + 1]$ the function $\hat{\varphi}_1 - \hat{\varphi}_0$ takes values on $[2\pi(m - 1), 2\pi m]$ for some $m \in \mathbb{Z}$, in particular it is bounded there. Hence the function $\hat{\varphi}_1 = \hat{\varphi}_0 + \hat{\varphi}_1 - \hat{\varphi}_0$ is bounded on each fundamental domain $[1 - \epsilon, 1) \times [k, k + 1]$. In view of the second equation in (17) we can conclude that $\hat{\varphi}_1$ is bounded on $\hat{A}_\delta$. By compactness of the closure of $\hat{D}_\delta \setminus \hat{A}_\delta$, we get that $\hat{\varphi}_1 \in L^\infty(\hat{D}_\delta)$.

This shows that the desired vector field $Z$ satisfying property (e) in Proposition 2.8 can be taken as $Z = Y_1$.

We can now conclude our proof. Using the Martinet tube $F$ chosen above, we can consider the space

$$M = \left( S^3 \setminus \gamma_0 \sqcup \mathbb{R}/\mathbb{Z} \times [0, 1) \times \mathbb{R}/2\pi \mathbb{Z} \right) / \sim$$

(19)

where a point $F(\vartheta, r e^{i\theta}) \in U \setminus \gamma_0$ is identified with $(\vartheta, r, \theta) \in \mathbb{R}/\mathbb{Z} \times (0, 1) \times \mathbb{R}/2\pi \mathbb{Z}$. One gets a differentiable structure on $M$ which makes $M$ diffeomorphic to $\mathbb{R}/\mathbb{Z} \times \mathbb{D}$ in such a way that $r e^{i2\pi \vartheta}$ are polar coordinates near the boundary of the $\mathbb{D}$-factor. The smooth embedded disk $D_\delta \subset S^3$ is such that $\hat{D}_\delta = D_\delta \setminus \partial D_\delta$ is the interior of a smoothly embedded meridional disk $D \subset M$ intersecting $\partial M$ cleanly. Moreover, it is possible to prove that $X|_{S^3 \setminus \gamma_0}$ extends to a smooth vector field $W$ on $M$ and such that, in particular, $W$ is tangent to $\partial M$. Since $D_\delta$ is $\partial$-strong, the embedded disk $D$ is transverse to $W$ up to the boundary and $D \cap \partial M = \partial D$ is a global surface of section for the flow of $W$ on $\partial M$. Since $D \setminus \partial M = \hat{D}_\delta$ is also a global surface of section for the flow of $W$ on $M \setminus \partial M$, one gets that $D$ is a global section for the flow of $W$ on $M$. Using the flow of $W$ to deform $D$ one constructs a smooth foliation $\{D_t\}_{t \in \mathbb{R}/\mathbb{Z}}$ of $M$ by embedded disks in such that $D = D_0$ and $D_t$ are transverse to $W$. It follows that all $D_t$ are global surfaces of section for the flow of $W$ on $M$ since $D = D_0$ is. Hence we get a diffeomorphism

$$M \simeq \mathbb{R}/\mathbb{Z} \times \mathbb{D}, \quad D_t \simeq \{t\} \times \mathbb{D}$$

Properties (a)-(d) follow from this construction.

We fix coordinates $u + iv$ on $D_0 \simeq \{0\} \times \mathbb{D}$. The aforementioned discussion has shown the existence of a non-vanishing vector field $Z$ on $S^3$ tangent to the contact structure $\xi = \ker \lambda$ that projects to $\hat{D}_\delta \simeq D_0 \setminus \partial D_0 \simeq \{0\} \times \hat{\mathbb{D}}$ along the direction of $X \simeq W$ to a non-vanishing vector field on $\hat{\mathbb{D}}$ for which any continuous choice of argument (in the coordinates $u + iv$) defines a function in $L^\infty(\hat{\mathbb{D}})$. This is property (e).

Denote by $\varphi^t$ the flow of $W$, so that on $\mathbb{R}/\mathbb{Z} \times \hat{\mathbb{D}}$ we have

$$\varphi^t = \Psi^{-1} \circ \phi^t \circ \Psi.$$

The first return time back to $\{0\} \times \hat{\mathbb{D}}$ is

$$\tau : \hat{\mathbb{D}} \to (0, +\infty), \quad \tau(z) = \inf \{t > 0 \mid \varphi^t(0, z) \in \{0\} \times \hat{\mathbb{D}}\}\)
It follows from (c) and (d) that $\tau$ is bounded away from zero and bounded from above. Denote

$$0 < \tau_{\min} = \min \tau \quad \tau_{\max} = \max \tau < +\infty$$

The return map is denoted by

$$h : \mathbb{D} \to \mathbb{D} \quad \quad (0, h(z)) = \varphi^z(0, z) \quad \quad (21)$$

**Lemma 2.9.** For every $c \in (0, 1)$ there exists $t : [0, 1] \times \mathbb{D} \to [0, +\infty)$ smooth such that:

(i) $t(0, z) = 0$ and $t(1, z) = \tau(z)$ for all $z \in \mathbb{D}$.

(ii) $D_1 t(s, z) \geq c \tau_{\min}$ for all $(s, z) \in [0, 1] \times \mathbb{D}$.

(iii) $\exists \epsilon > 0$ such that $D_1 t(s, z) = \tau_{\max}$ if $s \in [0, \epsilon) \cup (1 - \epsilon, 1]$, for all $z \in \mathbb{D}$.

**Proof.** Fix $c \in (0, 1)$ and let $\delta := c \tau_{\min}$. Consider the following set

$$\Omega := \{(t, z) \mid z \in \mathbb{D}, \quad 0 \leq t \leq \tau(z)\} \subset \mathbb{R} \times \mathbb{D},$$

and define the piecewise smooth function $s : \Omega \to [0, 1]$ by

$$s(t, z) := \begin{cases} \frac{t}{\tau_{\max}} & 0 \leq t \leq \tau(z) - \delta \\ \frac{\tau(z) - \delta}{\tau_{\max}} + \frac{\tau_{\max} - \tau(z) + \delta}{\delta \tau_{\max}} (t - \tau(z) + \delta) & \text{for } \tau(z) - \delta \leq t \leq \tau(z). \end{cases}$$

Let us now define $\hat{t} : [0, 1] \times \mathbb{D} \to [0, +\infty)$ by the identity $\hat{t}(s(t, z), z) = t$. Observe that $\hat{t}(0, z) = 0$ and $\hat{t}(1, z) = \tau(z)$. For every $(s, z) \in [0, 1] \times \mathbb{D}$ it holds, denoting by $D_1^+$ one-sided derivatives in the first variable,

$$\min \{D_1^+ \hat{t}(s, z), D_1^- \hat{t}(s, z)\} \geq \min \left\{\frac{\delta \tau_{\max}}{\tau_{\max} - \tau(z) + \delta}, \frac{\tau_{\max}}{\tau_{\max} - \tau(z) + \delta}\right\} > \delta = c \tau_{\min}.$$  

Using a parametrised splining method we get the a smooth $t : [0, 1] \times \mathbb{D} \to [0, +\infty)$ satisfying $D_1 t(s, z) \in [D_1^+ \hat{t}(s, z), D_1^- \hat{t}(s, z)] \geq c \tau_{\min}$, $t(0, z) = 0$ and $t(1, z) = \tau(z)$.

Extend $t$ to $[0, +\infty) \times \mathbb{D}$ by

$$t(s, z) = \sum_{j=0}^{[s]-1} \tau \circ h^j(z) + \frac{1}{s} \left(s - [s], h^{[s]}(z)\right) \quad \quad (s > 1) \quad (22)$$

It follows that $t$ is continuous on $[0, +\infty) \times \mathbb{D}$ and smooth on $[n, n + 1] \times \mathbb{D}$ for each $n \geq 0$, that $D_1 t$ has only discontinuities possibly at $\mathbb{N} \times \mathbb{D}$, and that $D_1 t \geq c \tau_{\min}$ on $[n, n + 1] \times \mathbb{D}$ for every $n \geq 0$. Again by a parametrized splining method, we can ask also that the extended function $t$ is smooth at $\mathbb{N} \times \mathbb{D}$ and that there exists $0 < \epsilon \ll 1$ such that $D_1 t(s, z) = \tau_{\max}$ if $s \in [0, \epsilon) \cup (1 - \epsilon, 1]$. 

Consider disks $D_s$ defined by

$$D_s = \{ \varphi^{t(s,z)}(0,z) \mid z \in \mathbb{D} \}$$

(23)

where $s$ varies on $[0,1]$. Note that $D_1 = D_0 = \{0\} \times \mathbb{D}$ in view of (21). Hence we get a continuous $\mathbb{R}/\mathbb{Z}$-family $\{D_s\}_{s \in \mathbb{R}/\mathbb{Z}}$ of smooth disks. Actually, the family $\{D_s\}_{s \in \mathbb{R}/\mathbb{Z}}$ defines a smooth foliation of $\mathbb{R}/\mathbb{Z} \times \mathbb{D}$.

Lemma 2.10. In addition to properties (a)-(e) from Proposition 2.8 one can assume that the map $\Psi$ in (15) satisfies in addition the following property.

(f) If $\varphi^t$ denotes the flow of $W$ then for every $c \in (0,1)$ there exists a smooth function $t : [0,1] \times \mathbb{D} \rightarrow [0,\infty)$ satisfying properties (i)-(iii) in Lemma 2.9, and a smooth isotopy $\{h_s : \mathbb{D} \rightarrow \mathbb{D}\}_{s \in [0,1]}$ such that

$$\varphi^{t(s,z)}(0,z) = (s,h_s(z))$$

(24)

for all $(s,z) \in [0,1] \times \mathbb{D}$.

Proof. As a consequence of Lemma 2.9 and since $\{D_s\}_{s \in \mathbb{R}/\mathbb{Z}}$ is a smooth foliation of $\mathbb{R}/\mathbb{Z} \times \mathbb{D}$, one finds an orientation preserving diffeomorphism $\Phi : \mathbb{R}/\mathbb{Z} \times \mathbb{D} \rightarrow \mathbb{R}/\mathbb{Z} \times \mathbb{D}$ satisfying $\Phi(\{s\} \times \mathbb{D}) = D_s$ for all $s \in \mathbb{R}/\mathbb{Z}$. If $\tilde{W} = \Phi^*W$ and $\varphi^t$ denotes the flow of $\tilde{W}$ then we find a smooth isotopy $\{h_s\}_{s \in [0,1]}$ of self-diffeomorphisms of $\mathbb{D}$ satisfying $h_0 = id$, $h_1 = h$, uniquely determined by $\varphi^{t(s,z)}(0,z) = (s,h_s(z))$. Here $h$ is the return map (21). The proof is concluded if we revert the notation from $\varphi^t$, $\tilde{W}$ and $\Phi \circ \Psi$ back to $\varphi^t$, $W$ and $\Psi$. \hfill $\Box$

Notation 2.11. If $z : [a,b] \rightarrow \mathbb{C} \setminus \{0\}$ is continuous then we denote

$$\text{wind}_{[a,b]}(z) = \frac{\theta(b) - \theta(a)}{2\pi}$$

(25)

where $\theta : [a,b] \rightarrow \mathbb{R}$ is continuous and satisfies $z(s) = |z(s)|e^{i\theta(s)}$.

Up to rotating the coordinate system on $S^3 \setminus \gamma_0$ obtained via the map $\Psi$, i.e. up to changing $\Psi(t,z)$ by $\Psi(t,e^{2\pi k t}z)$ for some $k \in \mathbb{Z}$, we may assume that the following property holds: for any pair of loops $\alpha, \beta : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{D}$ satisfying $\alpha(t) \neq \beta(t) \forall t$, the linking number in $S^3$ between the loops $t \mapsto \Psi(t, \alpha(t))$ and $t \mapsto \Psi(t, \beta(t))$ is equal to $\text{wind}_{[0,1]}(\beta(t) - \alpha(t))$. From now on we assume that $\Psi$ satisfies this property.

Let $I = \{f_t\}_{t \in [0,1]} \subset \text{Diff}^1(\mathbb{D})$ be a $C^1$-isotopy joining $f_0 = id_{\mathbb{D}}$ to $f_1 = f$. Extend the isotopy on $\mathbb{R}_+ = [0,\infty)$ by asking that $f_{1+t} = f_t \circ f$.

Theorem 2.12 ([Flo19b]). Let $x,y \in \mathbb{D}$, $x \neq y$, and $T \geq 0$ be fixed arbitrarily. Then

$$\text{wind}_{[0,T]}(f_t(y) - f_t(x)) = \text{wind}_{[0,T]}(Df_t(z)(y-x))$$

holds for some $z$ in the segment $[x,y]$ joining $x$ to $y$.  

\hfill 16
From the standing assumption that \( \kappa(\gamma_0) > 2\pi \) we can choose \( 0 < \epsilon \ll 1 \) such that
\[
\kappa(\gamma_0) - \epsilon > 2\pi.
\] (26)

Our main technical statement reads as follows.

**Proposition 2.13.** Let \( p, q \in \Psi(\{0\} \times \mathbb{D}) \) be recurrent points for \( \phi^t \) in distinct trajectories. Then
\[
\ell(p, q) \geq \frac{\kappa(\gamma_0) - \epsilon - 2\pi}{2\pi \tau_{\max}^2} > 0
\] (27)

where \( \ell(p, q) \) is the number (14).

**Key ideas of the proof of Proposition 2.13**

We resume here the main ideas and steps of the proof of Proposition 2.13; the proof can be found in the next subsection. For simplicity, consider periodic points \( p, q \in \Psi(\{0\} \times \mathbb{D}) \) of the flow \( \{\phi^t\} \) with period \( T_p, T_q \) respectively. The quantity we want to estimate is \( \ell(p, q) = \frac{\text{link}(\gamma_p, \gamma_q)}{T_p T_q} \) where \( \gamma_p = \phi^{[0,T_p]}(p), \gamma_q = \phi^{[0,T_q]}(q) \). We now explain why these periodic orbits are positively linked when \( \kappa > 2\pi \) further assuming that both have the same linking number \( n \) with \( \gamma_0 \), and that \( n \) is large enough. This means that \( \Psi^{-1}(p) \) and \( \Psi^{-1}(q) \) have the same period \( n \in \mathbb{Z} \) with respect to the flow \( \phi^t = \varphi^s \) parametrised by the “flipping pages” parameter \( s \).

Recall that \( \{\varphi^s\} \) induces a smooth isotopy \( \{h_s\} \) of self-diffeomorphisms of \( \mathbb{D} \). Because of the transversality of the flow to each page \( \{s\} \times \mathbb{D} \), the linking number can be computed in terms of winding numbers with respect to the isotopy \( \{h_s\} \), if we assume that \( \Psi \) satisfies the following normalisation condition: given loops \( c_1, c_2 : \mathbb{R}/\mathbb{Z} \to \mathbb{D} \) satisfying \( c_1(s) \neq c_2(s) \ \forall s \), the loops \( s \mapsto \Psi^{-1}(s, c_1(s)) \) and \( s \mapsto \Psi^{-1}(s, c_2(s)) \) in \( S^3 \) have linking number equal to \( \text{wind}_{s \in [0,1]}(c_2(s) - c_1(s)) \).

Consider the sets \( \mathcal{P} := \Psi^{-1}(\gamma_p) \cap \{0\} \times \mathbb{D} \) and \( \mathcal{Q} := \Psi^{-1}(\gamma_q) \cap \{0\} \times \mathbb{D} \). Then one chooses \( (0, y_0) \in \mathcal{Q} \) arbitrarily to get
\[
\text{link}(\gamma_p, \gamma_q) = \sum_{(0,x) \in \mathcal{P}} \sum_{(0,y) \in \mathcal{Q}} \text{wind}_{s \in [0,1]}(h_s(y) - h_s(x))
= \sum_{(0,x) \in \mathcal{P}} \text{wind}_{s \in [0,1]}(h_s(y_0) - h_s(x)).
\] (28)

By Theorem 2.12, we can replace each winding number with a linearised one, that is, for each \( (0, x) \in \mathcal{P} \) there exists \( z(x) \in \mathbb{D} \) such that
\[
\text{wind}_{s \in [0,1]}(h_s(y_0) - h_s(x)) = \text{wind}_{s \in [0,1]}(Dh_s(z(x))(y_0 - x)).
\]

Consider the (trivial) rank-2 vector bundle over the 3-manifold \( \mathbb{R}/\mathbb{Z} \times \mathbb{D} \) whose fiber over the point \( (s, z) \) is \( T_z \mathbb{D} = \mathbb{R}^2 \). We can choose a special frame of this vector bundle with the following property: its push forward by \( \Psi \) projects along the Reeb
vector field to a frame of \( \xi \) on \( S^3 \setminus \gamma_0 \) that extends to a global frame of \( \xi \) on \( S^3 \). By definition
\[
\text{wind}_{s \in [0,n]}(Dh_s(z(x))(y_0 - x)) \geq \frac{\kappa(\gamma_0) - \epsilon}{2\pi} n,
\]
where \( \text{wind}_{s \in [0,n]}^\text{Global} \) denotes the winding number with respect to a special frame mentioned above, provided that \( 0 < \epsilon < \kappa(\gamma_0) - 2\pi \) and that \( n \) is large enough. Denoting then by \( X = \{X_s\} \) a non-autonomous vector field on \( \mathbb{D} \) that is constant in the above mentioned special frame, we have
\[
\text{wind}_{s \in [0,n]}(Dh_s(z(x))(y_0 - x)) = \text{wind}_{s \in [0,n]}^\text{Global}(Dh_s(z(x))(y_0 - x)) + \text{wind}_{s \in [0,n]}(X(h_s(z(x)))).
\]
Informally speaking, at each turn of the solid torus the above mentioned special frame rotates roughly once in the negative sense with respect to the Seifert frame.

Coming back to (28), and since the cardinality of \( \mathcal{P} \) is \( n \), we obtain that
\[
\frac{\text{link}(\gamma_p, \gamma_q)}{T_p T_q} \geq \frac{n^2}{T_p T_q} (\kappa(\gamma_0) - \epsilon - 2\pi).
\]
By the control of the flipping pages parameter \( s \) with respect to time parameter \( t \) of the original Reeb flow, it holds \( n^2/T_p T_q \geq 1/\tau_{\max}^2 \). One gets \( \ell(p, q) \geq \frac{\kappa(\gamma_0) - \epsilon - 2\pi}{2\pi \tau_{\max}^2} \), which is strictly positive. When \( n \) is not large enough, or when the linking numbers \( n_p \) and \( n_q \) of \( \gamma_p \) and \( \gamma_q \) with \( \gamma_0 \), respectively, do not coincide, then one applies the above arguments with a very large common multiple of \( n_p \) and \( n_q \) in the place of \( n \).

In the sequel, we formalise these ideas and prove Proposition 2.13.

**Proof of Proposition 2.13**

From now on we are concerned with the proof of Proposition 2.13. We define loops \( k(T_n, p), k(S_n, q) \) as in case \( \text{A} \) of subsection 2.2, but make here a more general construction. We denote by \( (s, z = x + iy) \) the coordinates on the domain \( \mathbb{R}/\mathbb{Z} \times \mathbb{D} \) of the map \( \Psi \). The points \( p, q \) are assumed to belong to \( \Psi(\{0\} \times \mathbb{D}) \). Choose an auxiliary Riemannian metric \( g \) on \( S^3 \) that is Euclidean in local coordinates near \( p \) and \( q \) with respect to which the vector field \( X \) is constant. It can also be assumed that \( g \) induces the metric \( dx^2 + dy^2 \) tangentially to \( \Psi(\{0\} \times \mathbb{D}) \). The metric \( g \) can easily be constructed by considering flow boxes for \( X \) near \( p \) and \( q \) obtained by flowing small disks in \( \Psi(\{0\} \times \mathbb{D}) \) centred at these points. Let \( \mathcal{W}_p \) and \( \mathcal{W}_q \) be small disjoint \( g \)-convex neighbourhoods of \( p \) and \( q \), respectively, where the metric \( g \) has the above properties. We can assume that \( \mathcal{W} = \mathcal{W}_p \cup \mathcal{W}_q \subset \Psi([-1/4, 1/4] \times \mathbb{D}). \)
Consider shortest geodesic paths $\alpha_n \subset W_p$ and $\beta_n \subset W_q$ from $\phi^{T_n}(p)$ to $p$ and from $\phi^{S_n}(q)$ to $q$, respectively. The sequences $T_n, S_n$ are assumed to satisfy

$$T_n, S_n \to +\infty \quad \phi^{T_n}(p) \in W_p \quad \phi^{S_n}(q) \in W_q.$$ 

Note that such sequences exist if $p$ and $q$ are recurrent points, but here we only assume that $p$ and $q$ lie in distinct orbits, and that $T_n, S_n$ as above exist. We find real numbers $s_n, \hat{s}_n$ and $z_n, \hat{z}_n \in \mathbb{D}$ such that $\phi^{T_n}(p) = \Psi(s_n, z_n)$, $\phi^{S_n}(q) = \Psi(\hat{s}_n, \hat{z}_n)$, $\max(|s_n|, |\hat{s}_n|) \leq 1/4$.

There exist $m_n(p), m_n(q) \geq 1$ and

$$t_0^p = 0 < t_1^p < \cdots < t_{m_n(p)}^p \leq T_n \quad t_0^q = 0 < t_1^q < \cdots < t_{m_n(q)}^q \leq S_n$$

classified by

$$\{t_0^p, \ldots, t_{m_n(p)}^p\} = \{t \in [0, T_n] \mid \phi^t(p) \in \Psi(\{0\} \times \mathbb{D})\}$$

$$\{t_0^q, \ldots, t_{m_n(q)}^q\} = \{t \in [0, S_n] \mid \phi^t(q) \in \Psi(\{0\} \times \mathbb{D})\}.$$ 

Let $t_{m_n(p)+1}^p$ denote the next hitting time following $t_{m_n(p)}^p$. We have estimates

$$m_n(p)\tau_{\min} \leq t_{m_n(p)}^p \leq T_n < t_{m_n(p)+1}^p \leq t_{m_n(p)}^p + \tau_{\max} \leq (m_n(p)+1)\tau_{\max}$$

and it follows that

$$\frac{m_n(p)}{T_n} \leq \frac{1}{\tau_{\min}}, \quad \frac{T_n}{\tau_{\max}} - 1 \leq m_n(p). \quad (29)$$

Analogously

$$\frac{m_n(q)}{S_n} \leq \frac{1}{\tau_{\min}}, \quad \frac{S_n}{\tau_{\max}} - 1 \leq m_n(q). \quad (30)$$

The loop $k(T_n, p) = \phi^{[0,T_n]}(p) + \hat{\alpha}_n$ is obtained by concatenating to $\phi^{[0,T_n]}(p)$ a $C^1$-small perturbation $\hat{\alpha}_n$ of $\alpha_n$ with fixed endpoints, and $k(S_n, q) = \phi^{[0,S_n]}(q) + \hat{\beta}_n$ is obtained by concatenating to $\phi^{[0,S_n]}(q)$ a $C^1$-small perturbation $\hat{\beta}_n$ of $\beta_n$ with fixed endpoints. The $\hat{\alpha}_n, \hat{\beta}_n$ are chosen so that $k(T_n, p)$ and $k(S_n, q)$ do not intersect. Observe that different choices of $\hat{\alpha}_n, \hat{\beta}_n$ determine an ambiguity by an additive integer in $[-m_n(p) - m_n(q), m_n(p) + m_n(q)]$ of the value of $\text{link}(k(T_n, p), k(S_n, q))$. This is so because $\hat{\alpha}_n, \hat{\beta}_n$ are $C^1$-close enough to $\alpha_n, \beta_n$. As in subsection 2.2, we continue to denote

$$\text{link}_-(\phi^{[0,T_n]}(p), \phi^{[0,S_n]}(q)) = \liminf_{\alpha_n, \beta_n \to \hat{\alpha}_n, \hat{\beta}_n} \text{link}(k(T_n, p), k(S_n, q)).$$

In view of (29)-(30) and of the fact that $\hat{\alpha}_n, \hat{\beta}_n$ are $C^1$-close enough to $\alpha_n, \beta_n$, we deduce the estimate

$$\left| \frac{\text{link}_-(\phi^{[0,T_n]}(p), \phi^{[0,S_n]}(q))}{T_nS_n} - \frac{\text{link}(k(T_n, p), k(S_n, q))}{T_nS_n} \right| \leq \frac{1}{\tau_{\min}} \left( \frac{1}{S_n} + \frac{1}{T_n} \right). \quad (31)$$
We now study the effect of replacing $T_n$ be a different sequence $T'_n \to +\infty$ such that \( \{ t \in \mathbb{R} \mid \phi'(p) \in W_p \} \) contains the closed interval whose endpoints are $T_n, T'_n$. Let $\alpha'_n$ be the shortest geodesic arc from $\phi^{T'_n}(p)$ to $p$, which must again satisfy $\alpha'_n \subset W$. Let $\hat{\alpha}'_n$ be a small $C^1$-perturbation of $\alpha'_n$ with fixed endpoints, so that the corresponding loop $k(T'_n, p)$ as described before does not intersect $k(S_n, q)$. Construct the loop $c = (-\hat{\alpha}'_n) + \phi^{T'_n \to T_n}(p) + \hat{\alpha}_n$. For $n$ large we estimate

\[
|\text{link}(c, k(S_n, q))| \leq 2m_n(q) \quad (32)
\]

provided $\hat{\alpha}'_n, \hat{\alpha}_n$ are $C^1$-close enough to $\alpha'_n, \alpha_n$. The identity $k(T'_n, p) + c = k(T_n, p)$ implies that

\[
e_n = \frac{\text{link}(k(T_n, p), k(S_n, q))}{T_n S_n} - \frac{\text{link}(k(T'_n, p), k(S_n, q))}{T'_n S_n}
\]

can be estimated by

\[
|e_n| = \left| \frac{\text{link}(k(T_n, p), k(S_n, q))}{T_n S_n} - \frac{\text{link}(k(T'_n, p), k(S_n, q))}{T'_n S_n} \right|
\]

\[
\leq 1 - \frac{T_n}{T'_n} \left| \frac{\text{link}(k(T_n, p), k(S_n, q))}{T_n S_n} \right| + \frac{\text{link}(c, k(S_n, q))}{S_n T'_n}
\]

\[
\leq 1 - \frac{T_n}{T'_n} \left| \frac{\text{link}(k(T_n, p), k(S_n, q))}{T_n S_n} \right| + \frac{2m_n(q)}{S_n T'_n}
\]

provided $\hat{\alpha}'_n, \hat{\alpha}_n, \hat{\beta}_n$ are $C^1$-close enough to $\alpha'_n, \alpha_n, \beta_n$. Together with (29) and (31)
this implies 

\[
\frac{\text{link}_-(\phi^{[0,T_n]}(p), \phi^{[0,S_n]}(q))}{T_n'S_n} \\
\geq \frac{\text{link}(k(T_n, p), k(S_n, q))}{T_n'S_n} - \frac{1}{\tau_{\text{min}}} \left( \frac{1}{S_n} + \frac{1}{T_n'} \right) \\
\geq \frac{\text{link}(k(T_n, p), k(S_n, q))}{T_n'S_n} - \left| e_n \right| \frac{1}{\tau_{\text{min}}} \left( \frac{1}{S_n} + \frac{1}{T_n'} \right) \\
\geq \frac{\text{link}(k(T_n, p), k(S_n, q))}{T_n'S_n} - \left| 1 - \frac{T_n}{T_n'} \right| \left| \frac{\text{link}(k(T_n, p), k(S_n, q))}{T_n'S_n} \right| \\
- \frac{2}{\tau_{\text{min}}} \left( \frac{1}{S_n} + \frac{1}{T_n'} \right) \\
- \frac{1}{\tau_{\text{min}}} \left( \frac{1}{S_n} + \frac{1}{T_n'} \right) - \frac{1}{\tau_{\text{min}}} \left( \frac{1}{S_n} + \frac{1}{T_n'} \right) \left| 1 - \frac{T_n}{T_n'} \right| \\
- \frac{1}{\tau_{\text{min}}} \left( \frac{1}{S_n} + \frac{1}{T_n'} \right)
\] 

(33)

provided \(\hat{\alpha}_n', \hat{\alpha}_n, \hat{\beta}_n\) are \(C^1\)-close enough to \(\alpha_n', \alpha_n, \beta_n\). Estimate (33) and an analogous argument interchanging the roles of \(p\) and \(q\) show the following statement. If there exists \(a > 0\) such that for every pair of sequences \(T_n, S_n\) satisfying \(T_n \to +\infty, S_n \to +\infty, \phi^{T_n}(p) \in W_p\) and \(\phi^{S_n}(q) \in W_q\), one finds sequences \(T_n', S_n'\) satisfying the same properties and, in addition, \(\{t \in \mathbb{R} \mid \phi^t(p) \in W_p\}\) contains the closed interval bounded by \(T_n\) and \(T_n'\), \(\{t \in \mathbb{R} \mid \phi^t(q) \in W_q\}\) contains the closed interval bounded by \(S_n\) and \(S_n'\),

\[
\sup_n |T_n - T_n'| + |S_n - S_n'| < +\infty, \quad \liminf_{n \to +\infty} \frac{\text{link}_-(\phi^{[0,T_n']})(p), \phi^{[0,S_n]}(q))}{T_n'S_n} \geq a
\]

then

\[
\ell(p, q) \geq a.
\]

It follows from these arguments that \(\phi^{T_n}(p), \phi^{S_n}(q)\) can be assumed to belong to \(\Psi(\{0\} \times \tilde{D})\). We proceed under this assumption.

Since \(\phi^{T_n}(p) \in W_p \cap \Psi(\{0\} \times \tilde{D}), \phi^{S_n}(q) \in W_q \cap \Psi(\{0\} \times \tilde{D})\), and the metric has a special form on \(W\), we have that \(\alpha_n, \beta_n\) are images under \(\Psi\) of straight line segments in \(\{0\} \times \tilde{D}\). Denote by \(z_{0}^p, \ldots, z_{m_n(p)}^p\) and by \(z_{0}^q, \ldots, z_{m_n(q)}^q\) the points in \(\tilde{D}\) uniquely determined by

\[
\Psi(0, z_{i}^p) = \phi^i(p), \quad \Psi(0, z_{j}^q) = \phi^j(q).
\]

It follows that

\[
\alpha_n = \Psi(\{0\} \times [z_{m_n(p)}^p, z_{0}^p]), \quad \beta_n = \Psi(\{0\} \times [z_{m_n(q)}^p, z_{0}^q])
\]
where \([z, w]\) denotes the path \(u \in [0, 1] \mapsto (1 - u)z + uw\). By transversality of the flow to \(\Psi(\{0\} \times \mathbb{D})\) we can choose \(\hat{\alpha}_n, \hat{\beta}_n\) to be equal to \(\Psi(\{0\} \times e_n^p)\) and \(\Psi(\{0\} \times e_n^q)\), respectively, for some paths \(e_n^p, e_n^q : [0, 1] \to \mathbb{D}\), and so contained in \(\Psi(\{0\} \times \mathbb{D})\).

Define paths
\[
k^p_0, \ldots, k^p_{m_n(p)-1}, k^q_0, \ldots, k^q_{m_n(q)-1} : [0, 1] \to \hat{\mathbb{D}}
\]
by
\[
\Psi(s, k^p_i(s)) = \phi^t(s, z^p_i) (\Psi(0, z^p_i)) \quad \Psi(s, k^q_j(s)) = \phi^t(s, z^q_j) (\Psi(0, z^q_j))
\]
where \(t(s, z)\) is the function given by Lemma 2.9. It follows that
\[
i \in \{0, \ldots, m_n(p) - 1\} \quad \Rightarrow \quad k^p_i(0) = z^p_i, \quad k^p_i(1) = z^p_{i+1}
\]
\[
j \in \{0, \ldots, m_n(q) - 1\} \quad \Rightarrow \quad k^q_j(0) = z^q_j, \quad k^q_j(1) = z^q_{j+1}
\]
Fix \(0 < \delta < 1\). Consider paths
\[
\hat{k}^p_i, \hat{k}^q_j : [0, 1] \to \hat{\mathbb{D}} \quad (i, j) \in \{0, \ldots, m_n(p) - 1\} \times \{0, \ldots, m_n(q) - 1\}
\]
defined by
\[
\text{If } 0 \leq i \leq m_n(p) - 2 : \quad \hat{k}^p_i(s) = \begin{cases} 
  k^p_i \left( \frac{s}{1-\delta} \right) & \text{if } s \in [0, 1 - \delta] \\
  z^p_{i+1} & \text{if } s \in [1 - \delta, 1]
\end{cases}
\]
\[
\text{If } 0 \leq j \leq m_n(q) - 2 : \quad \hat{k}^q_j(s) = \begin{cases} 
  k^q_j \left( \frac{s}{1-\delta} \right) & \text{if } s \in [0, 1 - \delta] \\
  z^q_{j+1} & \text{if } s \in [1 - \delta, 1]
\end{cases}
\]
and
\[
\hat{k}^p_{m_n(p)-1}(s) = \begin{cases} 
  k^p_{m_n(p)-1} \left( \frac{s}{1-\delta} \right) & \text{if } s \in [0, 1 - \delta] \\
  c_n^p \left( \frac{s}{1+\delta} \right) & \text{if } s \in [1 - \delta, 1]
\end{cases}
\]
\[
\hat{k}^q_{m_n(q)-1}(s) = \begin{cases} 
  k^q_{m_n(q)-1} \left( \frac{s}{1-\delta} \right) & \text{if } s \in [0, 1 - \delta] \\
  c_n^q \left( \frac{s}{1+\delta} \right) & \text{if } s \in [1 - \delta, 1]
\end{cases}
\]
In the following we extend \(k^p_i, \hat{k}^p_i\) to all \(i \in \mathbb{Z}\) in a \(m_n(p)\)-periodically way. Similarly, we extend \(k^q_j, \hat{k}^q_j\) to all \(j \in \mathbb{Z}\) in a \(m_n(q)\)-periodically way. With this convention we have
\[
\hat{k}^p_i(1) = \hat{k}^p_{i+1}(0) \quad \forall i \quad \hat{k}^q_j(1) = \hat{k}^q_{j+1}(0) \quad \forall j
\]
Let \(L = \text{lcm}(m_n(p), m_n(q))\) and \(r_n(p), r_n(q) \in \mathbb{N}\) satisfy
\[
L = r_n(p)m_n(p) = r_n(q)m_n(q)
\]
Let us denote by \(k(T_n, p)^{r_n(p)}, k(S_n, q)^{r_n(q)}\) the \(r_n(p)\)-fold iteration of \(k(T_n, p)\) and the \(r_n(q)\)-fold iteration of \(k(S_n, q)\), respectively. Then, by the transversality of \(\phi^t\) to each \(\Psi(\{s\} \times \mathbb{D})\), it holds
\[
\text{link}(k(T_n, p)^{r_n(p)}, k(S_n, q)^{r_n(q)}) = \sum_{i,j=0}^{L-1} \text{wind}_{s \in [0, 1]} (\hat{k}^p_i(s) - \hat{k}^q_j(s)). \quad (35)
\]
With $i, j \in \{0, \ldots, L - 1\}$ consider

$$e_{ij} = \text{wind}_{s \in [1-\delta,1]}(\hat{k}_i^p(s) - \hat{k}_j^q(s)).$$  \hfill (36)

For each pair $(i, j)$ there are two cases to be studied separately:

(i) $m_n(p)$ does not divide $i + 1$ and $m_n(q)$ does not divide $j + 1$.

(ii) $m_n(p)$ divides $i + 1$ or $m_n(q)$ divides $j + 1$.

In case (ii) both paths $\hat{k}_i^p |_{[1-\delta,1]}$ and $\hat{k}_j^q |_{[1-\delta,1]}$ are constant paths, hence $e_{ij} = 0$. In case (ii) there are two subcases: either one of the paths is a $C^1$-perturbation of a straight line segment in $\mathbb{D}$ and the other is a point in the complement, or both are contained in disjoint open balls (assuming $n$ is large enough). In both subcases we get $|e_{ij}| \leq 1$. The number of multiples of $m_n(p)$ in $[0, L-1]$ is $r_n(p)$, and the number of multiples of $m_n(q)$ in $[0, L-1]$ is $r_n(q)$. Hence, there at most $L(r_n(p) + r_n(q))$ pairs $(i, j)$ falling in case (ii), and this yields the estimate

$$\sum_{i,j=0}^{L-1} |e_{ij}| \leq L(r_n(p) + r_n(q)).$$  \hfill (37)

Now note that

$$\sum_{i,j=0}^{L-1} \text{wind}_{s \in [0,1]}(\hat{k}_i^p(s) - \hat{k}_j^q(s)) = \sum_{i,j=0}^{L-1} \text{wind}_{s \in [0,1]}(k_i^p(s) - k_j^q(s)) + e_{ij}.$$  \hfill (38)

Combining (35), (37) and (38)

$$\left| \text{link}(k(T_n,p)^{r_n(p)}, k(S_n,q)^{r_n(q)}) - \sum_{i,j=0}^{L-1} \text{wind}_{s \in [0,1]}(k_i^p(s) - k_j^q(s)) \right| \leq L(r_n(p) + r_n(q)).$$  \hfill (39)

Now define maps

$$\Gamma^p : \mathbb{R}/L\mathbb{Z} \to \mathbb{D} \quad \Gamma^p(s) = k^p_{[s]}(s - \lfloor s \rfloor)$$

$$\Gamma^q : \mathbb{R}/L\mathbb{Z} \to \mathbb{D} \quad \Gamma^q(s) = k^q_{[s]}(s - \lfloor s \rfloor)$$  \hfill (40)

Note that $\Gamma^p$ is discontinuous, unless $p$ lies on a periodic orbit. Analogously, $\Gamma^q$ is discontinuous, unless $q$ lies on a periodic orbit. For each $j \in \mathbb{Z}/L\mathbb{Z}$ define:

$$j \ast \Gamma^q(s) = \Gamma^q(s + j).$$  \hfill (41)

Define

$$E_p = \{0, m_n(p), \ldots, (r_n(p) - 1)m_n(p)\},$$

$$E_q = \{0, m_n(q), \ldots, (r_n(q) - 1)m_n(q)\}.$$
We see \( E_p, E_q \) as subsets of \( \mathbb{Z}/L\mathbb{Z} \). The set of discontinuity points of \( \Gamma^p \) is contained in \( E_p \) whose cardinality is equal to \( r_n(p) \). The set of discontinuity points of \( j \ast \Gamma^q \) is contained in \( E_q - j \) whose cardinality is \( r_n(q) \). We then write

\[
\sum_{i,j=0}^{L-1} \text{wind}_{x \in [0,1]}(k_i^p(s) - k_j^q(s)) = \sum_{j=0}^{L-1} \text{wind}_{x \in [0,L]}(\Gamma^p(s) - j \ast \Gamma^q(s)). \tag{42}
\]

Note that the set \( E_p \cup (E_q - j) \) divides the circle \( \mathbb{R}/L\mathbb{Z} \) into \( N_j \) closed intervals \( I_1, \ldots, I_{N_j} \). Note that \( N_j \leq r_n(p) + r_n(q) \). We denote the end points of these intervals by \( I_j^\mathcal{A} = [a_j^\mathcal{A}, b_j^\mathcal{A}] \) and their lengths by \( |I_j^\mathcal{A}| = b_j^\mathcal{A} - a_j^\mathcal{A} \in \mathbb{N} \). The end points of these intervals belong to \( \mathbb{Z}/L\mathbb{Z} \). For each \( I_j^\mathcal{A} = [a_j^\mathcal{A}, b_j^\mathcal{A}] \) we find points \( \zeta_\mathcal{A}, \zeta_\mathcal{A}' \in \hat{\mathbb{D}} \) such that

\[
h_s(\zeta_\mathcal{A}) = \Gamma^p(s + a_j^\mathcal{A}) \quad h_s(\zeta_\mathcal{A}') = j \ast \Gamma^q(s + a_j^\mathcal{A}) \quad \forall s \in [0, b_j^\mathcal{A} - a_j^\mathcal{A}] \quad \tag{43}
\]

where \( h_s \) is the unique extension of the isotopy (24) to \( s \in [0, +\infty) \) determined by the identity

\[
h_s+1 = h_s \circ h \quad s \in [0, +\infty). \tag{44}
\]

Since \( \{D_s\}_{s \in \mathbb{R}/\mathbb{Z}} \) is a smooth foliation, the isotopy (44) is smooth. In particular, observe that, for every \( j = 0, \ldots, L - 1 \), it holds

\[
\text{wind}_{x \in [0,L]}(\Gamma^p(s) - j \ast \Gamma^q(s)) = \sum_{j=1}^{N_j} \text{wind}_{x \in [0,b_j^\mathcal{A} - a_j^\mathcal{A}]}(h_s(\zeta_\mathcal{A}) - h_s(\zeta_\mathcal{A}')). \tag{45}
\]

Recall that, from (26), \( \kappa(\gamma_0) - \varepsilon > 2\pi \). Note that, for \( x \in S^3 \setminus \gamma_0 \),

\[
\text{link}(k(T, x; D), \gamma_0) = \# \{ t \in [0, T] \mid \phi^t(x) \in \Psi(\{0\} \times \hat{\mathbb{D}}) \} + \ell,
\]

for some \( \ell \in \{-1, 0, 1\} \), where \( \# \{ t \in [0, T] \mid \phi^t(x) \in \Psi(\{0\} \times \hat{\mathbb{D}}) \} \) is the number of times that \( \phi^{[0,T]}(x) \) meets \( \Psi(\{0\} \times \hat{\mathbb{D}}) \). Observe that

\[
\kappa(\gamma_0) = \lim_{T \to +\infty} \inf_{x \in S^3 \setminus \gamma_0} \inf_{u \in \xi_x \setminus 0} \frac{\Delta \tilde{\Theta}_\sigma(T; x, u)}{\# \{ t \in [0, T] \mid \phi^t(x) \in \Psi(\{0\} \times \hat{\mathbb{D}}) \}}.
\]

Thus, there exists \( M = M(\varepsilon) > 0 \) such that for every \( x \in S^3 \setminus \gamma_0 \) and every \( u \in \xi_x \setminus 0 \),

\[
T \geq M \quad \Rightarrow \quad \frac{\Delta \tilde{\Theta}_\sigma(T; x, u)}{\# \{ t \in [0, T] \mid \phi^t(x) \in \Psi(\{0\} \times \hat{\mathbb{D}}) \}} > \kappa(\gamma_0) - \varepsilon. \tag{46}
\]

Observe that, by the extension of the function \( t \) in (22) and by (ii) of Lemma 2.9, for every \( z \in \mathbb{D} \), if \( s \geq \frac{M}{c_{\min}} \), then \( t(s, z) \geq M \). According to Lemma 2.9 the constant \( c \) can be any number in \((0, 1)\) fixed \textit{a priori}, independent of \( p \) and \( q \). Let us define

\[
\mathcal{M} := \sup_{z \in \mathbb{D}} \max_{1 \leq i \leq \frac{M}{c_{\min}} + 1} |\text{wind}_{s \in [0,1]}(Dh_s(z)u)| < +\infty. \tag{47}
\]
Remark 2.14. For every $\lambda$ such that $|I_\lambda|^2 < \frac{M}{c_{\min}}$, by Theorem 2.12 and by (47), we deduce that

$$\text{wind}_{s \in [0,b^\lambda_\lambda - a^\lambda_\lambda]} (h_s(\zeta_\lambda) - h_s(\zeta'_\lambda)) \geq -M.$$ 

Lemma 2.15. There exists a constant $C > 0$ independent of $p, q, \{S_n\}, \{T_n\}$ such that for each $\lambda \in \{1, \ldots, N\}$ satisfying $|I_\lambda|^2 = b^\lambda_\lambda - a^\lambda_\lambda \geq \frac{M}{c_{\min}}$, the estimate

$$\text{wind}_{s \in [0,b^\lambda_\lambda - a^\lambda_\lambda]} (h_s(\zeta_\lambda) - h_s(\zeta'_\lambda)) \geq (\kappa(\gamma_0) - \epsilon - 2\pi) \frac{|I_\lambda|^2}{2\pi} - C$$

holds.

Proof. We start by introducing some notation. Let $\tilde{X}_1, \tilde{X}_2$ be the nowhere vanishing vector fields given by $\sigma \circ \tilde{X}_j \equiv e_j$ where $e_1 = (1,0), e_2 = (0,1)$. We choose $\sigma$ so that $\tilde{X}_1$ is equal to the vector field $Z$ given by (e) in Proposition 2.8. Observe that $i_{\tilde{X}_j \lambda} \equiv 0$. Denote by $\tilde{X}_1, \tilde{X}_2$ the pull-back of $X_{1}|_{\mathbb{C} \setminus \gamma_0}, X_{2}|_{\mathbb{C} \setminus \gamma_0}$ respectively under the diffeomorphism $\Psi|_{\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}}$. For every $s \in \mathbb{R}/\mathbb{Z}$ let $\Pi^s: T(\mathbb{R}/\mathbb{Z} \times \mathbb{R})|_{\{s\} \times \mathbb{R}/\mathbb{Z}} \to T\mathbb{R}$ be the vector bundle map determined by projecting along the direction of $W$, where $\{s\} \times \mathbb{R}$ is identified with $\mathbb{R}$. Then $X^*_1, X^*_2$ denote the images of $\tilde{X}_1|_{\{s\} \times \mathbb{R}}, \tilde{X}_2|_{\{s\} \times \mathbb{R}}$ under $\Pi^s$, respectively. We get two smooth families $X^*_1, X^*_2$ of smooth vector fields on $\mathbb{R}/\mathbb{Z}$ parametrized by $s \in \mathbb{R}/\mathbb{Z}$. At times it might be convenient to think of $s$ as variable in $\mathbb{R}$ and the families $X^*_1, X^*_2$ as 1-periodic in $s$. By Theorem 2.12, there exists $z_{\lambda} \in [\zeta'_\lambda, \zeta_\lambda]$ such that

$$\text{wind}_{s \in [0,b^\lambda_\lambda - a^\lambda_\lambda]} (h_s(\zeta_\lambda) - h_s(\zeta'_\lambda)) = \text{wind}_{s \in [0,b^\lambda_\lambda - a^\lambda_\lambda]} (Dh_s(z_{\lambda})(\zeta_\lambda - \zeta'_\lambda)). \quad (48)$$

By definition we have

$$\text{wind}_{s \in [0,b^\lambda_\lambda - a^\lambda_\lambda]} (Dh_s(z_{\lambda})(\zeta_\lambda - \zeta'_\lambda)) = \frac{\theta(b^\lambda_\lambda - a^\lambda_\lambda) - \theta(0)}{2\pi}$$

where $\theta : [0,b^\lambda_\lambda - a^\lambda_\lambda] \to \mathbb{R}$ is a continuous argument of $Dh_s(z_{\lambda})(\zeta_\lambda - \zeta'_\lambda)$. Consider now the smooth path $c : [0,b^\lambda_\lambda - a^\lambda_\lambda] \to \mathbb{C} \setminus \{0\}$

$$s \mapsto c(s) := X^*_1(s, h_s(z_{\lambda}))$$

and let $\Theta : [0,b^\lambda_\lambda - a^\lambda_\lambda] \to \mathbb{R}$ be a continuous choice of argument of $c(s)$. Thus we can write $\theta(s) = \vartheta(s) + \Theta(s)$, where $\vartheta : [0,b^\lambda_\lambda - a^\lambda_\lambda] \to \mathbb{R}$ is a unique choice of continuous angular coordinate of the vector $Dh_s(z_{\lambda})(\zeta_\lambda - \zeta'_\lambda)$ in the frame $\{c(s), ic(s)\}$ determined by $\theta$ and $\Theta$. Hence

$$\frac{\theta(b^\lambda_\lambda - a^\lambda_\lambda) - \theta(0)}{2\pi} = \frac{\Theta(b^\lambda_\lambda - a^\lambda_\lambda) - \Theta(0)}{2\pi} + \frac{\vartheta(b^\lambda_\lambda - a^\lambda_\lambda) - \vartheta(0)}{2\pi}$$

$$= \text{wind}_{s \in [0,b^\lambda_\lambda - a^\lambda_\lambda]} (c(s)) + \frac{\vartheta(b^\lambda_\lambda - a^\lambda_\lambda) - \vartheta(0)}{2\pi} \quad (49)$$

25
(I) Lower bound for \( \text{wind}_{s \in [0,b_{\frac{q}{p}} - a_{\frac{q}{p}}]}(c(s)) \). Consider a path \( \rho : [0,1] \to \hat{\mathbb{D}} \) of the form 
\[ \rho(0) = h_{b_{\lambda} - a_{\lambda}}(z_\lambda) \quad \text{and} \quad \rho(1) = z_\lambda. \]
Build the loop \( \Gamma : [0, b_{\lambda} - a_{\lambda} + 1] \to \mathbb{R}/\mathbb{Z} \times \hat{\mathbb{D}} \)
\[ \Gamma(s) := \begin{cases} 
(s, h_s(z_\lambda)) & s \in [0, b_{\lambda} - a_{\lambda}] \\
(0, \rho(s - b_{\lambda} + a_{\lambda})) & s \in [b_{\lambda} - a_{\lambda}, b_{\lambda} - a_{\lambda} + 1]
\end{cases} \]
Observe that \( \text{link}(\Psi \circ \Gamma, \gamma_0) = |I_i| = b_{\lambda} - a_{\lambda} \). We associate to \( \Gamma \) the path 
\[ D\Gamma : [0, b_{\lambda} - a_{\lambda} + 1] \to \mathbb{C} \setminus \{0\} \]
defined by 
\[ D\Gamma(s) := \begin{cases} 
X_i^s(\Gamma(s)) & s \in [0, b_{\lambda} - a_{\lambda}] \\
X_i^s(\Gamma(s)) & s \in [b_{\lambda} - a_{\lambda}, b_{\lambda} - a_{\lambda} + 1].
\end{cases} \]
Then it holds that 
\[ \text{wind}_{s \in [0,b_{\frac{q}{p}} - a_{\frac{q}{p}}]+1}(D\Gamma(s)) = \text{wind}_{s \in [0,b_{\frac{q}{p}} - a_{\frac{q}{p}}]}(c(s)) + \text{wind}_{s \in [0,1]}(X_i^s(\rho(s))). \]
Since \( \hat{X}_1 \) satisfies (e) of Proposition 2.8 we deduce that there exists a constant 
\( C_1 > 0 \), independent from \( p, q, \{T_n\}, \{S_n\} \), such that 
\[ |\text{wind}_{s \in [0,1]}(X_i^s(\rho(s)))| \leq C_1. \tag{50} \]

Brouwer's translation theorem gives a fixed point \( \Psi(0, \hat{\gamma}) \) of the first return map to \( \Psi([0] \times \hat{\mathbb{D}}) \) corresponding to a periodic Reeb orbit \( \hat{\gamma} \subset S^3 \setminus \gamma_0 \). We denote by \( \hat{\gamma}^{[I_i]} \) the \( |I_i| \)-fold iteration of \( \hat{\gamma} \). The loop \( \beta := \Psi^{-1}(\hat{\gamma}^{[I_i]}) \) is homotopic to \( \Gamma \) in \( \mathbb{R}/\mathbb{Z} \times \hat{\mathbb{D}} \) since they have the same linking number with \( \gamma_0 \). The loop \( \beta \) can be parametrised by \( s \in [0, b_{\lambda} - a_{\lambda}] \) so that \( \beta(s) = (s, \hat{\beta}(s - |s|)) \) where \( \hat{\beta} : [0,1] \to \hat{\mathbb{D}} \) satisfies \( \hat{\beta}(1) = \beta(0) \). Then 
\[ \text{wind}_{s \in [0,b_{\lambda} - a_{\lambda}]+1}(D\Gamma(s)) = \text{wind}_{s \in [0,b_{\lambda} - a_{\lambda}]}(X_i^s(\hat{\beta}(s - |s|))) \]
\[ = (b_{\lambda} - a_{\lambda}) \text{ wind}_{s \in [0,1]}(X_i^s(\hat{\beta}(s))). \tag{51} \]
The quantity \( \text{wind}_{s \in [0,1]}(X_i^s(\hat{\beta}(s))) \) is equal to the self-linking number of \( \hat{\gamma} \), see Definition 1.5 in [Hry14]. By [Hry14, Theorem 1.8], the self-linking number of \( \hat{\gamma} \) is \(-1\). Thus, from (51) we get 
\[ \text{wind}_{s \in [0,b_{\lambda} - a_{\lambda}]+1}(D\Gamma(s)) = -(b_{\lambda} - a_{\lambda}). \tag{52} \]
From (50) and (52) we deduce that 
\[ \text{wind}_{s \in [0,b_{\lambda} - a_{\lambda}]}(c(s)) \geq -(b_{\lambda} - a_{\lambda}) - C_1 = -|I_i|-C_1. \tag{53} \]
Let $v \in \xi_{\Psi(0,z)}$ be uniquely characterised by 
\[
D\Psi(0, z)(0, \zeta - \zeta') \in v + \mathbb{R}X.
\]

From Lemma 2.9, the strictly increasing function $t(\cdot, z) : [0, b^i_\lambda - a^i_\lambda] \to [0, +\infty)$ satisfies 
\[
t(0, z) = 0 \quad \Psi(s, h_s(z)) = \phi^{t(s,z)}(\Psi(0, z)),
\]
with 
\[
t(s, z) = \sum_{k=0}^{|s| - 1} t(1, h^k(z)) + t(s - |s|, h^{|s|}(z)).
\]

See also (22). The coordinates of the vector $Dh_s(z)(\zeta - \zeta')$ in the basis (54) are the same of 
\[
D\phi^{t(s,z)}(\Psi(0, z)) \cdot v
\]
in the basis $\{\hat{X}_1, \hat{X}_2\}$. Recall the $\mathbb{R}/2\pi\mathbb{Z}$ coordinate $\Theta_\sigma$ in $(\xi \setminus 0)/\mathbb{R}_+$ induced by the frame $\sigma$. If $p \in S^3$, $v \in \xi_p \setminus 0$, we denote by $t \mapsto \Theta_\sigma(t, p, w)$ a continuous lift to $\mathbb{R}$ of $t \mapsto \Theta_\sigma(D\phi^t(p)w)$. Thus 
\[
\hat{\phi}(b^i_\lambda - a^i_\lambda) - \hat{\phi}(0) = \hat{\Theta}_\sigma(t(b^i_\lambda - a^i_\lambda, z), \Psi(0, z), v) - \hat{\Theta}_\sigma(0, \Psi(0, z), v) =: \Delta \Theta_\sigma(t(b^i_\lambda - a^i_\lambda, z), \Psi(0, z), v).
\]

Since, by assumption, $b^i_\lambda - a^i_\lambda \geq \frac{M}{c_{\min}}$, then $t(b^i_\lambda - a^i_\lambda, z) \geq M$ and, from (46), it holds that 
\[
\Delta \Theta_\sigma(t(b^i_\lambda - a^i_\lambda, z), \Psi(0, z), v) > (\kappa(\gamma_0) - \epsilon) \#\{t \in [0, t(b^i_\lambda - a^i_\lambda, z)] \mid \phi^t(\Psi(0, z)) \in \Psi(\{0\} \times \mathbb{D})\}
\]
\[
= (\kappa(\gamma_0) - \epsilon) (|I^i_\lambda| + 1).
\]

\(^2\text{Comparing } \hat{\phi} \text{ and } \hat{\phi} \text{ corresponds to compare oriented angles with respect to different Riemannian metrics.}\)
Consequently
\[
\frac{\vartheta(b^j_\lambda - a^j_\lambda) - \vartheta(0)}{2\pi} > \frac{(\kappa(\gamma_0) - \epsilon) (|I^j_\lambda| + 1)}{2\pi} - 1
\]  
(59)

With the help of (I) and (II) we can conclude the proof, since from (48), (49), (53), (59) we have

\[
\text{wind}_{s \in [0,b^j_\lambda - a^j_\lambda]}(h_s(\zeta_\lambda) - h_s(\zeta'_\lambda)) > \left(\kappa(\gamma_0) - \epsilon - 2\pi\right) \frac{|I^j_\lambda|}{2\pi} - C,
\]

where \(C := C_1 + 1 - \frac{(\kappa(\gamma_0) - \epsilon)}{2\pi}\).

We now give the final estimate to complete the proof of Proposition 2.13. For every \(j = 0, \ldots, L - 1\) denote

\[
G^j = \left\{ \lambda \in \{1, \ldots, N_j\} \mid |I^j_\lambda| \geq \frac{M}{c_{\min}} \right\},
\]

\[
B^j = \left\{ \lambda \in \{1, \ldots, N_j\} \mid |I^j_\lambda| < \frac{M}{c_{\min}} \right\}.
\]

With Lemma 2.15 and Remark 2.14 we can estimate (42) as (see (45))

\[
\sum_{i,j=0}^{L-1} \text{wind}_{s \in [0,1]}(k^p_i(s) - k^q_j(s)) =
\]

\[
> \sum_{j=0}^{L-1} \left( \sum_{\lambda \in G^j} \left( \kappa(\gamma_0) - \epsilon - 2\pi \right) \frac{|I^j_\lambda|}{2\pi} - C \right) - \sum_{\lambda \in B^j} \mathcal{M}
\]

and, denoting \(\tilde{C} := \max(C, \mathcal{M})\), we have

\[
\sum_{i,j=0}^{L-1} \text{wind}_{s \in [0,1]}(k^p_i(s) - k^q_j(s)) \geq \sum_{j=0}^{L-1} \left( \frac{\kappa(\gamma_0) - \epsilon - 2\pi}{2\pi} \sum_{\lambda \in G^j} |I^j_\lambda| - \sum_{\lambda = 1}^{N_j} \tilde{C} \right)
\]

\[
\geq \sum_{j=0}^{L-1} \left( \frac{\kappa(\gamma_0) - \epsilon - 2\pi}{2\pi} \sum_{\lambda \in G^j} |I^j_\lambda| \right) - \tilde{C} L \left( r_n(p) + r_n(q) \right).
\]

For every \(\lambda \in B^j\) recall that \(|I^j_\lambda| < \frac{M}{c_{\min}}\). Observe now that

\[
0 \leq \sum_{\lambda \in B^j} |I^j_\lambda| < \frac{M}{c_{\min}} N^j \leq \frac{M}{c_{\min}} \left( r_n(p) + r_n(q) \right) = L \frac{M}{c_{\min}} \frac{m_n(p) + m_n(q)}{m_n(q)m_n(p)}.
\]
Moreover, it holds that $\sum_{\lambda \in \mathfrak{A}} |I_{\lambda}^1| + \sum_{\lambda \in \mathfrak{A}} |I_{\lambda}^2| = L$. Consequently
\[
\sum_{\lambda \in \mathfrak{A}} |I_{\lambda}^1| > \left(1 - \frac{M}{c\tau_{\min}} \frac{m_n(p) + m_n(q)}{m_n(q)m_n(p)}\right) L.
\]
(60)

Therefore, we have
\[
\sum_{i,j=0}^{L-1} \text{wind}_{s \in [0,1]}(k_i^p(s) - k_i^q(s))
> \frac{\kappa(\gamma_0) - \epsilon - 2\pi}{2\pi} \left(1 - \frac{M}{c\tau_{\min}} \frac{m_n(p) + m_n(q)}{m_n(q)m_n(p)}\right) L^2 - \tilde{C} L (r_n(p) + r_n(q)),
\]
which together with (39) yields
\[
\text{link}(k(T_n,p)^{r_n(p)}, k(S_n, q)^{r_n(q)})
\geq \frac{\kappa(\gamma_0) - \epsilon - 2\pi}{2\pi} \left(1 - \frac{M}{c\tau_{\min}} \frac{m_n(p) + m_n(q)}{m_n(q)m_n(p)}\right) L^2 - \tilde{C} (r_n(p) + r_n(q)).
\]
Combining with (29)-(30) we get
\[
\frac{\text{link}(k(T_n,p), k(S_n, q))}{T_n S_n} = \frac{\text{link}(k(T_n,p)^{r_n(p)}, k(S_n, q)^{r_n(q)})}{r_n(p)T_n r_n(q)S_n}
\geq \frac{(\kappa(\gamma_0) - \epsilon - 2\pi)}{2\pi} \frac{m_n(p)m_n(q)}{T_n S_n}
- \frac{M}{c\tau_{\min}} \frac{(\kappa(\gamma_0) - \epsilon - 2\pi)}{2\pi} \frac{m_n(p) + m_n(q)}{T_n S_n}
- L(1 + \tilde{C}) \frac{r_n(p) + r_n(q)}{T_n S_n} \frac{m_n(q)m_n(p)}{L^2}
\geq \frac{(\kappa(\gamma_0) - \epsilon - 2\pi)}{2\pi \tau_{\min}^2} \frac{m_n(p)m_n(q)}{(m_n(p) + 1)(m_n(q) + 1)}
- \frac{M}{c\tau_{\min}} \frac{(\kappa(\gamma_0) - \epsilon - 2\pi)}{2\pi \tau_{\min}^2} \left(\frac{1}{m_n(q)} + \frac{1}{m_n(p)}\right)
- \frac{L(1 + \tilde{C})}{\tau_{\min}^2} \frac{1}{m_n(p)m_n(q)} \left(\frac{1}{r_n(q)} + \frac{1}{r_n(p)}\right)
\]
(61)
from where it follows that
\[
\liminf_{n \to \infty} \frac{\text{link}(k(T_n,p), k(S_n, q))}{T_n S_n} \geq \frac{\kappa(\gamma_0) - \epsilon - 2\pi}{2\pi \tau_{\max}^2} > 0
\]
(62)
Here we made use of (26). The proof of Proposition 2.13 is complete.

The argument above has a simple consequence.
Lemma 2.16. If $\kappa(\gamma_0) > 2\pi$ then every periodic orbit has strictly positive transverse rotation number in a Seifert framing.

Proof. We draw freely from the notation established in the proof of Proposition 2.13. In (61) consider here the case where the point $p \in \Psi(\{0\} \times \mathbb{D})$ belongs to a periodic orbit $\gamma \subset S^3 \setminus \gamma_0$ with primitive period $T$. Let $q$ be a recurrent point in $\Psi(\{0\} \times \mathbb{D}) \setminus \{p\}$. If $q$ is closed enough to $p$ then $q \notin \gamma$. It follows from (61) that if $T_n = nT$ and $S_n$ are larger than some constant $\bar{T} > 0$, which is fixed large enough independently of $q$, then

$$\text{link}(\gamma_n, k(S_n, q)) \geq nTS_n(\kappa(\gamma_0) - \epsilon - 2\pi) / (2\pi \tau_{\max}) .$$

The existence of $\bar{T}$ relies crucially on the fact that the constants $M, c, \tilde{C}, C, \mathcal{M}$ appearing in the previous proof do not depend on $p$ and $q$. Dividing by $n$ we get

$$\text{link}(\gamma, k(S_n, q)) \geq TS_n \frac{\kappa(\gamma_0) - \epsilon - 2\pi}{2\pi \tau_{\max}} > 0 .$$

Letting $q$ converge to $p$ arbitrarily along recurrent points in $\Psi(\{0\} \times \mathbb{D})$, and comparing the trajectory $\varphi_{[0,S_n]}(q)$ with the linearised flow along $p$ over the fixed interval $[0,S_n]$ independent of $q$, we conclude that the transverse rotation number of $\gamma$ on a Seifert framing is positive.

By Proposition 2.13 and Lemma 2.16 the assumption $\kappa(\gamma_0) > 2\pi$ implies right-handedness of the Reeb flow. The proof of Theorem 1.13 is complete.

2.4 Right-handedness on strictly convex energy levels

We start by studying criteria to estimate the invariant $\kappa$ in (5). Let the contact form $\lambda$ on $S^3$ be dynamically convex. The contact structure and the Reeb flow are $\xi$ and $\phi^t$, respectively. Consider

$$K_\sigma := \inf_{\mathbb{P}_+ \xi} i_{X_\sigma} d\Theta_\sigma$$

(63)

where $\sigma$ is the global frame, $\Theta_\sigma : \mathbb{P}_+ \xi \to \mathbb{R} / 2\pi \mathbb{Z}$ is the global angle coordinate induced by $\sigma$ as in (2), and $X_\sigma$ is the vector field generating the linearised Reeb flow on $\mathbb{P}_+ \xi$. Let $\gamma_0$ be an unknotted periodic orbit with self-linking number $-1$, and let $D \subset S^3$ be the disk-like global surface of section given by Theorem 1.7 in [Hry14]. Denote $0 < \tau_{\min}(D) \leq \tau_{\max}(D) < +\infty$ the infimum and the supremum of the return time function on $D$.

Theorem 2.17. If $K_\sigma \tau_{\min}(D) > 2\pi$ then the Reeb flow of $\lambda$ is right-handed.

Proof. We show that the hypothesis implies $\kappa(\gamma_0) > 2\pi$ and then apply Theorem 1.13. By the fundamental theorem of calculus, for every $x \in S^3 \setminus \gamma_0$ and
Observe that
\[ \Theta_\sigma(T, u) - \Theta_\sigma(0, u) = \int_0^T \frac{d}{dt} \Theta_\sigma(\mathbb{R}_+ D\phi^t(x) u) dt \]
\[ \geq T \inf_{x \in S^3 \setminus \gamma_0} \{ \tau_{\min}(D) \} = T K_\sigma. \] (64)

Moreover
\[ \text{link}(k(T, x; D), \gamma_0) = \# \{ t \in [T^-_-(x), T + D^+ D(\phi^t(x))] | \phi^t(x) \in D \} - 1. \] (65)

Observe that
\[ \# \{ t \in [T^-_-(x), T + D^+ D(\phi^t(x))] | \phi^t(x) \in D \} - 1 \leq \frac{T + D^+ D(\phi^t(x)) - D^t(x)}{\tau_{\min}(D)}. \] (66)

Hence
\[ \frac{\Theta_\sigma(T, u) - \Theta_\sigma(0, u)}{\text{link}(k(T, x; D), \gamma_0)} \geq \frac{TK_\sigma}{T + D^+ D(\phi^t(x)) - D^t(x)} = K_\sigma \tau_{\min}(D) \frac{T}{T + D^+ D(\phi^t(x)) - D^t(x)}. \]

Since \( 0 < D^+ D(\phi^t(x)) - D^t(x) \leq 2\tau_{\max}(D) \) for every \( x \in S^3 \setminus \gamma_0 \), we finally get
\[ \frac{\Theta_\sigma(T, u) - \Theta_\sigma(0, u)}{\text{link}(k(T, x; D), \gamma_0)} \geq K_\sigma \tau_{\min}(D) \frac{T}{T + 2\tau_{\max}(D)}. \]

It follows immediately by first taking the infimum on \( x, u \) and then taking the limit as \( T \to +\infty \) that \( \kappa(\gamma_0) \geq K_\sigma \tau_{\min}(D) > 2\pi. \)

Our final task in this section is to prove Theorem 1.14. Let \( J_0 : \mathbb{R}^4 \to \mathbb{R}^4 \) be the complex structure defined by the matrix
\[
J_0 = \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\] (67)

with respect to coordinates \((q_1, q_2, p_1, p_2)\) of \( \mathbb{R}^4 \). Then \( \omega_0(u, v) = \langle u, J_0 v \rangle \) holds for every \( u, v \in \mathbb{R}^4 \), where \( \langle \cdot, \cdot \rangle \) is the standard Euclidean inner product. We will denote as \( \| \cdot \| \) the inherited norm. As explained in the introduction, the Hamiltonian vector field of \( H = u_2^2 \), defined as \( i_{X_H} \omega_0 = -dH \), is the Reeb vector field of the contact form on \( \partial C \) induced by \( \lambda_0 \). The associated contact structure is denoted by \( \xi \subset T \partial C \). Denote as \( \varphi^t_H \) the Hamiltonian flow on \( \partial C \), and \( X_\varphi \) the vector field on \( \mathbb{P}^+ \xi \) generating the linearised flow. Together with the matrix \( J_0 \), consider the following
\[
J_1 = \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix}, \quad J_2 = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\]
Then $J_i^2 = -I$ and $J_i^T = -J_i$ hold for all $i = 0, 1, 2$ where $I$ denotes the identity matrix. Moreover, $J_0J_1 = J_2$, $J_1J_2 = J_0$ and $J_2J_0 = J_1$. On $\partial C$ consider the following vector fields

$$X_0 = \frac{\nabla H}{\|\nabla H\|}, \quad X_1 = J_2X_0, \quad X_2 = J_1X_0, \quad X_3 = -J_0X_0. \quad (68)$$

The frame $\{X_1, X_2, X_3\}$ is an orthonormal global frame of $T\partial C$. Observe that the vector field $X_3$ is positively colinear with $X_H$ and so it is tangent to the trajectories of the Reeb flow on $\partial C$. Let $\sigma$ the global symplectic trivialisation of $(\xi, d\lambda_0)$ determined by projecting $X_1, X_2$ onto $\xi$ along the Reeb direction $\mathbb{R}X_3$. As explained in the introduction, it induces a trivialisation $\mathbb{P}_+\xi \simeq \partial C \times \mathbb{R}/2\pi\mathbb{Z}$ of circle bundles. Denote by $\Theta_\sigma$ the $\mathbb{R}/2\pi\mathbb{Z}$-component of this map. Theorem 1.14 follows immediately from Theorem 2.17 and the following statement.

**Lemma 2.18 ([GRS06]).** The inequality $i_{X_3}d\Theta_\sigma \geq 2K_{\min}^C$ holds everywhere on $\mathbb{P}_+\xi$.

## 3 Pinched two-spheres

The aim of this section is to prove Theorem 1.2. The strategy is, first, to lift the geodesic flow on the unit tangent bundle of a Riemannian two-sphere $(S^2, g)$ to the Reeb flow of a contact form $f_g\lambda_0$ on $S^3$ for some smooth $f_g : S^3 \to (0, +\infty)$, where $\lambda_0$ is the restriction to $S^3 \subset \mathbb{C}^2$ of the standard Liouville form $\frac{1}{4i}(zd\bar{z} - \bar{z}d z + \bar{w}dw - wd\bar{w})$. Here $(z, w)$ denote the complex coordinates on $\mathbb{C}^2$. The second step is to use the pinching condition on the curvatures to show that the assumptions of Theorem 1.13 are fulfilled.

### 3.1 Lifting geodesic flows on $S^2$ and Birkhoff annuli

Here we establish notation, and recall well-known facts about geodesic flows. Let $(S^2, g)$ be an oriented Riemannian two-sphere. The foot-point projection of its tangent bundle is denoted by $\pi : TS^2 \to S^2$. Consider the unit tangent bundle $T^1gS^2$ defined as the set of $(x, v) \in TS^2$ such that $g(x)(v, v) = 1$.

There are two important vector bundle maps covering $\pi$, namely, the differential of the projection $d\pi : TTS^2 \to TS^2$ and the connection map $K : TTS^2 \to TS^2$. There is a splitting

$$TTS^2 = \mathcal{H} \oplus \mathcal{V} \quad \mathcal{H} = \ker K, \quad \mathcal{V} = \ker d\pi$$

where both $\mathcal{H}, \mathcal{V}$ are fiberwise two-dimensional. For all $v, w \in T_xS^2$, the horizontal lift $w^{\text{hor}} \in \mathcal{H}(x, v)$ and the vertical lift $w^{\text{vert}} \in \mathcal{V}(x, v)$ are both well-defined: we refer to [HP08, HS13] for further details. The so-called Hilbert form $\sum_{ijkl} g_{ij}(x) v^i dx^j$ restricts to a contact form $\lambda_g$ on $T^1gS^2$ whose Reeb flow coincides with the geodesic flow. For every $(x, v) \in T^1gS^2$ denote by $v^\perp \in T_xS^2$ the unique vector such that
$g(x)(v^\perp, v^\perp) = 1$, $g(x)(v^\perp, v) = 0$, and \{v, v^\perp\} is a positive basis. It follows that \{v^\text{hor}, (v^\perp)^\text{vert}, (v^\perp)^\text{hor}\} defines a basis of $T_{(x,v)}T^1_gS^2$, where $v^\text{hor}$ is the Reeb vector field at $v$ and \{(v^\perp)^\text{vert}, (v^\perp)^\text{hor}\} is a $d\lambda_g$-symplectic frame of the contact structure $\ker \lambda_g$. In fact, we see from this that $\ker \lambda_g$ is trivial as a (symplectic) vector bundle since it admits a global frame

$$\sigma_g : v \mapsto \{(v^\perp)^\text{vert}, (v^\perp)^\text{hor}\}$$

which we call the \textit{geodesic frame}. If $h$ is another Riemannian metric on $S^2$, then there exists a contactomorphism between $T^1_hS^2$ and $T^1_gS^2$. Denote by $g_0$ the round metric on $S^2$. There is a double covering map

$$D_0 : S^3 \to T^1_{g_0}S^2 \quad (70)$$

such that, following [HP08], it holds

$$D_0^* \lambda_{g_0} = 4 \lambda_0 \quad (71)$$

where $\lambda_0$ denotes the restriction to $S^3$ of the 1-form

$$\frac{1}{2}(xdy - ydx + udv - vdu). \quad (72)$$

In other words, the geodesic flow of $g_0$ lifts to the Hopf Reeb flow on $S^3$ up to a constant time reparametrisation; to explain the factor 4 note that a Hopf fibre has Reeb flow period $\pi$ with respect to $\lambda_0$, and is the lift of a great circle of length $2\pi$ prescribed twice. Given any other metric $g$ on $S^2$, using the contactomorphism between $T^1_{g_0}S^2$ and $T^1_gS^2$, we get a double covering map

$$D_g : S^3 \to T^1_gS^2 \quad (73)$$

respecting contact structures, i.e.

$$D_g^* \lambda_g = f_g \lambda_0 \quad (74)$$

for some smooth $f_g : S^3 \to \mathbb{R} \setminus \{0\}$. The covering (73) is the universal covering and the group of deck transformations is $\mathbb{Z}_2$ generated by the antipodal map. If $g = g_0$ then $f_{g_0} \equiv 4$ by (71). The following special case of a result from [HP08] relates Gaussian curvature to dynamical convexity; see also [HS13] for an alternative proof.

\textbf{Theorem 3.1} (Harris and Paternain). If $(S^2, g)$ is $\delta$-pinched with $\delta > 1/4$, then the Reeb flow of $D^*_g \lambda_g$ is dynamically convex.

\textbf{Remark 3.2}. In [HP08] one finds a version of the above theorem for Finsler metrics. Dynamical convexity is ensured once the flag curvatures are pinched by more than $(r/(r + 1))^2$ where $r \geq 1$ is the reversibility parameter. In [HS13] one finds examples showing that this pinching condition is sharp for dynamical convexity.
From now on we denote
\[ \phi_g^t \quad \text{the geodesic flow on } T_g^1 S^2 \]
\[ \phi^t \quad \text{the lift of } \phi_g^t \text{ to } S^3 \text{ via } D_g \]
so that the identity
\[ \phi_g^t \circ D_g = D_g \circ \phi^t \]
holds. The generating vector field \( X \) of \( \phi^t \) is the Reeb vector field of the contact form \( f_g \lambda_0 \). Let \( \tilde{\sigma}_g \) be the lift of the frame \( \sigma_g \) (69) to \( S^3 \) by \( D_g \). Then \( \tilde{\sigma}_g \) is a \( d(f_g \lambda_0) \)-symplectic global frame of \( \xi = \ker \lambda_0 \), and as in (2) we get a well-defined global circle coordinate \( \Theta : (\xi \setminus 0)/\mathbb{R}_+ \to \mathbb{R}/2\pi \mathbb{Z} \). The linearised flow \( D\phi^t \) on \( \xi \) induces a flow on \( (\xi \setminus 0)/\mathbb{R}_+ \), still denoted \( D\phi^t \), whose generating vector field is denoted by \( \tilde{X} \).

The following elementary lemma can be proved with the arguments presented in [HS13, section 2.4.2].

**Lemma 3.3.** We have \( \min\{1, K_{\min}\} \leq i_X d\Theta \sigma_g \leq \max\{1, K_{\max}\} \) where we denote by \( K_{\min}, K_{\max} \) the minimum and the maximum of the Gaussian curvature of \( (S^2, g) \).

Let \( c : \mathbb{R}/L \mathbb{Z} \to (S^2, g) \) be a unit speed smooth immersion. It induces a smooth immersion \( s \in \mathbb{R}/L \mathbb{Z} \mapsto (c(s), \dot{c}(s)) \in T_g^1 S^2 \) such that \( \lambda_g \cdot \frac{d}{ds}(c, \dot{c}) = g(c, \dot{c}) = 1 \). If \( c \) has no positive self-tangencies then \( (c, \dot{c}) \) defines a knot on \( T_g^1 S^2 \). Note that \( \pi_1(T_g^1 S^2, \text{pt}) \) is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \), and a generator can be taken as \( s \mapsto (c(s), \dot{c}(s)) \) for an embedded unit speed loop \( s \mapsto c(s) \) in \( S^2 \).

**Lemma 3.4** ([HS13], subsection 3.1). Let \( c : \mathbb{R}/L \mathbb{Z} \to S^2 \) be a smooth unit speed embedding, and denote by \( \gamma_c : \mathbb{R}/2L \mathbb{Z} \to S^3 \) a lift to \( S^3 \) by \( D_g \) of the double cover of \( s \mapsto (c(s), \dot{c}(s)) \). Then \( \gamma_c \) is unknotted with self-linking number \(-1\).

Suppose now that \( c : \mathbb{R}/L \mathbb{Z} \to S^2 \) is a unit speed embedding. Choose a lift \( \gamma_c : \mathbb{R}/2L \mathbb{Z} \to S^3 \) of the double cover of \( s \mapsto (c(s), \dot{c}(s)) \) to \( S^3 \) by \( D_g \). The Birkhoff annulus \( A_c \subset T_g^1 S^2 \) associated to \( c \) is parametrised by
\[
a : \mathbb{R}/L \mathbb{Z} \times [0, \pi] \to T_g^1 S^2, \quad a(s, \theta) = (c(s), \cos \theta \dot{c}(s) + \sin \theta \dot{c}(s)^\perp). \tag{75}\]

It follows from this formula that \( a \) admits a unique double lift to \( S^3 \)
\[
\tilde{a} : \mathbb{R}/2L \mathbb{Z} \times [0, \pi] \to S^3 \tag{76}\]
fixed by requiring that \( D_g \circ \tilde{a}(s, \theta) = a(s \mod L, \theta) \) holds identically, together with the boundary condition \( \tilde{a}(s, 0) = \gamma_c(s) \). The image \( \tilde{A}_c \) of \( \tilde{a} \) is an embedded annulus in \( S^3 \) and will be referred to as the *lifted Birkhoff annulus* associated to \( c \). If \( A_c \subset T_g^1 S^2 \) is the image of \( a \) then as sets we have \( \tilde{A}_c = D_g^{-1}(A_c) \) which holds because \( \tilde{A}_c \) is antipodal symmetric and satisfies \( D_g(\tilde{A}_c) = A_c \). When \( \tilde{A}_c \) is oriented by \( ds \land d\theta \) the oriented boundary \( \partial \tilde{A}_c \) consists of \( \gamma_c \) together with a lift \( \tilde{\gamma}_c : \mathbb{R}/2L \mathbb{Z} \to S^3 \) of the double cover of \( s \mapsto (c(-s), -\dot{c}(-s)) \).

34
Lemma 3.5. The identity

\[ \text{link}(\beta, \gamma_c) + \text{link}(\beta, \hat{\gamma}_c) = \text{int}(\beta, \hat{A}_c) \]

holds for every loop \( \beta \) on \( S^3 \setminus (\gamma_c \cup \hat{\gamma}_c) \).

Proof. Let \( D \) and \( \hat{D} \) be oriented disks spanned by \( \gamma_c \) and \( \hat{\gamma}_c \), respectively, in such a way that the boundary orientations coincide with the flow orientation. Then \( C = D + \hat{D} - \hat{A}_c \) is a 2-cycle on \( S^3 \). Thus \( 0 = \text{int}(\beta, C) \) holds for every loop \( \beta \) on \( S^3 \). The conclusion follows. \( \square \)

From now on we make the standing assumption that \( c \) is an embedded closed geodesic. We consider covering maps

\begin{align*}
P : \mathbb{R}/2L\mathbb{Z} \times [0, \pi] &\to \mathbb{R}/L\mathbb{Z} \times [0, \pi] \quad (s, \theta) \mapsto (s \mod L, \theta), \\
P_{\infty} : \mathbb{R} \times [0, \pi] &\to \mathbb{R}/2L\mathbb{Z} \times [0, \pi] \quad (s, \theta) \mapsto (s \mod 2L, \theta).
\end{align*}

We have an identity \( Dg \circ \tilde{a} = a \circ P \). Let \( \Psi : A_c \to A_c \) be the return map, which exists by the result of Birkhoff [Bir66]. Note that, in principle, this return map would only be defined on the interior of \( A_c \) but then it can be extended smoothly up to the boundary by taking second conjugate points. We proceed assuming that the map has been extended in this manner. Then \( \hat{A}_c \) is a global surface of section for the lifted flow on \( S^3 \) and the return map \( \tilde{\Psi} : \tilde{A}_c \to \tilde{A}_c \) satisfies \( Dg \circ \tilde{\Psi} = \Psi \circ Dg \).

As above, this map exists and is smooth up on the closed annulus \( \hat{A}_c \).

Later we will use the geometry to choose an appropriate smooth lift

\[ \tilde{\Psi} : \mathbb{R} \times [0, \pi] \to \mathbb{R} \times [0, \pi] \]

of \( \Psi \), i.e. a map that satisfies \( \tilde{a} \circ P_{\infty} \circ \tilde{\Psi} = \tilde{\Psi} \circ \tilde{a} \circ P_{\infty} \), and hence makes the following diagram commute

\[ \begin{diagram}
\mathbb{R} \times [0, \pi] & \xrightarrow{P_{\infty}} & \mathbb{R}/2L\mathbb{Z} \times [0, \pi] & \xrightarrow{\tilde{a}} & \tilde{A}_c & \xrightarrow{Dg} & A_c \\
\tilde{\Psi} & | \downarrow & \tilde{\Psi} & | \downarrow \Psi & & & \\
\mathbb{R} \times [0, \pi] & \xrightarrow{P_{\infty}} & \mathbb{R}/2L\mathbb{Z} \times [0, \pi] & \xrightarrow{\tilde{a}} & \tilde{A}_c & \xrightarrow{Dg} & A_c
\end{diagram} \]

3.2 Asymptotic estimates on linking and intersection numbers

Denote by \( \delta > 0 \) the pinching factor. We may assume that

\[ \delta = K_{\min} \leq K_{\max} = 1 \]

where \( K_{\min}, K_{\max} \) denote the minimum and the maximum of the Gaussian curvature. Recall that if \( \delta > 1/4 \), then \( \phi^t_g \) is dynamically convex ([HP08]), hence so is \( \phi^t \).
Lemma 3.4 and Theorem 1.10 imply that both $\gamma_c$ and $\tilde{\gamma}_c$ span disk-like global surfaces of section. Denote by $\hat{D}$ a $\partial$-strong disk-like surface of section spanned by $\tilde{\gamma}_c$; see Remark 1.11.

We write $\Theta = \Theta_{\sigma_0}$ for simplicity. With $u \in \xi$, $u \neq 0$, arbitrary we denote

$$\Delta \Theta(T, u) = \hat{\Theta}(T, u) - \hat{\Theta}(0, u)$$

where $t \mapsto \hat{\Theta}(t, u)$ is a continuous lift of $t \mapsto \Theta(D\phi^t(u))$. It does not depend on the choice of lift.

Let $T \geq 0$, $x \in S^3 \setminus \tilde{\gamma}_c$. Recall that $k(T, x; \hat{D})$ is a loop obtained by closing the piece of trajectory $\phi^j(T, x; \hat{D})$ with a path $\alpha$ in $\hat{D} \setminus \tilde{\gamma}_c$. From now on we shall refer to $\alpha$ as a closing path for $(T, x; \hat{D})$. Clearly, $\text{link}(k(T, x; \hat{D}), \tilde{\gamma}_c)$ does not depend on the choice of $\alpha$. Suppose further that $x \in S^3 \setminus (\gamma_c \cup \tilde{\gamma}_c)$. The number $\text{int}(k(T, x; \hat{D}), A_\gamma)$ might not be well-defined since $\alpha$ could go through the (unique) intersection point of $\gamma_c$ and $\hat{D}$. Even if it does not touch this point, $\alpha$ can be chosen in such a way that $\text{int}(k(T, x; \hat{D}), A_\gamma)$ is any integer. Below we might write $k_\alpha(T, x; \hat{D})$ if the dependence on $\alpha$ needs to be made explicit.

**Lemma 3.6.** There exists $C > 0$ with the following property. For every point $x \in S^3 \setminus (\gamma_c \cup \tilde{\gamma}_c)$ and every $T > 0$ there exists a closing path $\alpha$ for $(T, x; \hat{D})$ contained in $\hat{D} \setminus (\gamma_c \cup \tilde{\gamma}_c)$ such that

$$|\text{int}(k_\alpha(T, x; \hat{D}), A_\gamma) - \# \{ t \in [0, T] \mid \phi^t(x) \in A_\gamma \}| \leq C. \quad (80)$$

**Proof.** Since $\hat{D}$ is $\partial$-strong, it is possible to slightly $C^\infty$-perturb $\hat{D}$ to a new $\partial$-strong disk-like global surface of section $\bar{D}$ satisfying $\partial \bar{D} = \tilde{\gamma}_c$ and such that $\bar{D} \setminus \partial \bar{D}$ intersects $\bar{A}_\gamma \setminus \partial \bar{A}_\gamma$ transversely, and the manifolds $\{ \mathbb{R}_+ \nu \mid \nu \in T \bar{D} \setminus \gamma \}$ is outward pointing and $\{ \mathbb{R}_+ \nu \mid \nu \in T \bar{A}_\gamma \}$ is outward pointing induce loops in the 2-torus $\mathbb{R}_+ \xi_{\tilde{\gamma}_c}$ that intersect transversely. Denote by $K$ the closure of $(\bar{D} \setminus \partial \bar{D}) \cap (\bar{A}_\gamma \setminus \partial \bar{A}_\gamma) = (\bar{D} \cap \bar{A}_\gamma) \setminus (\gamma_c \cup \tilde{\gamma}_c)$. Then, $\tilde{K}$ is a smooth compact 1-manifold, $\partial \tilde{K} = K \cap (\gamma_c \cup \tilde{\gamma}_c)$, and at the boundary points of $\tilde{K}$ the 1-manifolds $\gamma_c \cup \tilde{\gamma}_c$ and $K$ are not tangent.

Recall the interval $I(T, x; \hat{D})$ defined in (4). Denote by $\tilde{x}_0 \in \hat{D}$ and $\tilde{x}_1 \in \hat{D}$ the initial and end points of the trajectory $\phi^{j(T, x; \hat{D})}(x)$. We consider the small transfer map $\tilde{\psi} : p \in \hat{D} \mapsto \phi^{g(p)}(p) \in \hat{D}$ where $g$ is a smooth function on $\hat{D}$, up to the boundary, $C^\infty$-close to zero, satisfying $g(p) \in \hat{D}$ for every $p \in \bar{D}$. Define $\tilde{x}_0 = \tilde{\psi}^{-1}(\tilde{x}_0)$, $\tilde{x}_1 = \tilde{\psi}^{-1}(\tilde{x}_1)$. There is an induced interval $\bar{I}$ whose end points are close to those of $I(T, x; \hat{D})$ such that $\phi^{j}(x)$ is a piece of trajectory from $\tilde{x}_0$ to $\tilde{x}_1$. The Lebesgue measure of the symmetric difference between $\bar{I}$ and $I(T, x; \hat{D})$ is not larger than $2\|g\|_{L^\infty} \ll 1$.

Assume $\tilde{x}_0 \neq \tilde{x}_1$. Consider a smooth immersion $\alpha_0 : [0, 1] \to \hat{D} \setminus (\gamma_c \cup \tilde{\gamma}_c)$ from $\tilde{x}_1 = \alpha_0(0)$ to $\tilde{x}_0 = \alpha_0(1)$ that is transverse to $K$. Let $\beta$ be a connected component of $K$, and suppose $t_0 < t_1$ satisfy $\alpha_0(t_0), \alpha_0(t_1) \in \beta$. Consider the piecewise smooth arc $\tilde{\alpha}_0$ obtained from $\alpha_0$ by replacing $\alpha_0|_{[t_0, t_1]}$ with the arc in $\beta$ from $\alpha_0(t_0)$ to $\alpha_0(t_1)$. A further small perturbation of $\tilde{\alpha}_0$ creates a smooth immersed
arc $\alpha : [0, 1] \to \tilde{D} \setminus (\gamma_c \cup \gamma_c)$ from $\tilde{x}_1$ to $\tilde{x}_0$, transverse to $K$, such that the number of intersection points of $\alpha$ and any component of $K$ is no larger than that of $\alpha_0$, and the number of intersection points of $\alpha_1$ with $\beta$ is strictly less than the number of intersection points of $\alpha_0$ with $\beta$. Proceeding inductively in this way, after a finite number of steps we end up with an immersed arc $\tilde{\alpha}$ in $\tilde{D} \setminus (\hat{\gamma}_c \cup \gamma_c)$ from $\tilde{x}_1$ to $\tilde{x}_0$ which is transverse to $K$ and intersects each connected component of $K$ at most once. Consider the loop $\tilde{k}$ obtained by concatenating $\phi^t(x)$ with $\tilde{\alpha}$. Then

$$\left| \text{int}(\tilde{k}, \tilde{A}_c) - \#\{ t \in \tilde{I} \mid \phi^t(x) \in \tilde{A}_c \} \right| \leq N + 2 \tag{81}$$

where $N$ is the number of connected components of $K$. The image $\alpha = \psi(\tilde{\alpha}) \subset \hat{D}$ is an arc from $\hat{x}_1$ to $\hat{x}_0$. By construction $k_\alpha(T,x;\hat{D})$ is homotopic to $\tilde{k}$ on $S^3 \setminus (\gamma_c \cup \hat{\gamma}_c)$. Hence

$$\text{int}(\tilde{k}, \tilde{A}_c) = \text{int}(k_\alpha(T,x;\hat{D}), \tilde{A}_c) \tag{82}$$

Finally, consider $\tau_{\text{max}}(\hat{D})$ the supremum of the return time of $\hat{D}$ and $\tau_{\text{min}}(\tilde{A}_c) > 0$ the infimum of the return time of $\tilde{A}_c$. Note that the Lebesgue measure of the symmetric difference of $\tilde{I}$ and $[0, T]$ is not larger than $2(\tau_{\text{max}}(\hat{D}) + \| g \|_{L^\infty})$. Note also that for any interval $J$ one estimates

$$\#\{ t \in J \mid \phi^t(x) \in \tilde{A}_c \} \leq \frac{|J|}{\tau_{\text{min}}(\tilde{A}_c)} + 2$$

Hence

$$\left| \#\{ t \in \tilde{I} \mid \phi^t(x) \in \tilde{A}_c \} - \#\{ t \in [0, T] \mid \phi^t(x) \in \tilde{A}_c \} \right| \leq \frac{2(\tau_{\text{max}}(\hat{D}) + \| g \|_{L^\infty})}{\tau_{\text{min}}(\tilde{A}_c)} + 2 \tag{83}$$

Combining (81), (82) and (83) we arrive at (80) with

$$C = N + \frac{2(\tau_{\text{max}}(\hat{D}) + \| g \|_{L^\infty})}{\tau_{\text{min}}(\tilde{A}_c)} + 4$$

as desired.

The case $\tilde{x}_0 = \tilde{x}_1$ is left to the reader. \qed

Next we recall some definitions and facts about Riemannian geometry that will be useful all along this section; see [ABHS17].

**Definition 3.7.** A set $D \subset S^2$ is said to be a geodesic polygon if it is the closure of an open disk bounded by a simple closed unit speed broken geodesic $\gamma : \mathbb{R}/L\mathbb{Z} \to S^2$. A corner of $\gamma : \mathbb{R}/L\mathbb{Z} \to S^2$ is a point $\gamma(t)$ such that $\gamma'_\pm(t) \notin \mathbb{R}^+\gamma'_\pm(t)$, where $\gamma'_\pm$ denote one-sided derivatives. The corners of $\gamma$ are called vertices of $D$, and a side of $D$ is a smooth geodesic arc contained in $\partial D$ connecting two adjacent vertices.
For $p \in S^2$ and $u, v \in T_p S^2$ non-colinear vectors, consider the sets

$$\Delta(u, v) := \{su + tv | s, t \geq 0\}, \quad \Delta_r(u, v) := \{w \in \Delta(u, v) | |w| < r\}$$

for some $r > 0$. We denote by inj$_p$ the injectivity radius at $p$, and inj = inf$_p$ inj$_p$ the injectivity radius of $g$.

**Definition 3.8.** A geodesic polygon $D \subset S^2$ is convex if for every corner $p = \gamma(t)$ of $\partial D$ we find $0 < r < \text{inj}_p$ small enough such that $D \cap B_r(p) = \exp(\Delta_r(-\gamma'(t), \gamma'(t)))$.

![Figure 1: Examples of geodesic polygons, convex (a) and non convex (b).](image)

**Theorem 3.9** (Corollary of Toponogov’s Theorem [ABHS17, Theorem A.12]). If $D$ is a convex geodesic polygon in $(S^2, g)$ then $|\partial D| \leq 2\pi / \sqrt{\delta}$ where $|\partial D|$ denotes the perimeter of $D$.

For the following theorem, recall that we are assuming $K_{\text{max}} = 1$, see (79).

**Theorem 3.10** ([Kli59],[Kli82, Theorem 2.6.9]). The injectivity radius satisfies $\text{inj} \geq \pi$.

In particular it follows from theorems 3.9 and 3.10 that if $D \subset S^2$ is a convex geodesic bi-gon for the metric $g$ then

$$2\pi \leq |\partial D| \leq \frac{2\pi}{\sqrt{\delta}}.$$  (84)

**Notation 3.11.** Let $(x, v) \in T^*_x S^2$ be such that $x$ belongs to the embedded closed geodesic $c$. If $v$ is not tangent to $c$ then define

$$\tau_+(x, v) := \min\{t > 0 | \pi \circ \phi^t_g(x, v) \in c\}.$$  (85)

That is, $\tau_+(x, v)$ is the first positive time when the geodesic ray with initial conditions $(x, v)$ meets again $c$. Since $\delta > 1/4$, by [ABHS17, Lemma 3.9] the geodesic arc
\{ \pi \circ \phi^t_y(x, v) \mid t \in [0, \tau_+(x, v)] \} \) is embedded, in particular \( x \neq \pi \circ \phi^\tau_y(x, v)(x, v) \). If \( v \) is tangent to \( c \) then define \( \tau_+(x, v) \) as the time to the first conjugate point of \( (x, v) \). Denote by \( \alpha_+(x, v) \) the path contained in \( c \) joining \( x \) to \( \pi \circ \phi^{\tau_+(x, v)}(x, v) \) following the orientation of \( c \). See Figure 2.

**Lemma 3.12.** If \( \delta > 1/4 \) then there exists a unique lift \( \bar{\Psi} : \mathbb{R} \times [0, \pi] \rightarrow \mathbb{R} \times [0, \pi] \) such that

\[
L + 2\pi \left( 1 - \frac{1}{\sqrt{\delta}} \right) \leq p_1 \circ \bar{\Psi}(S, \theta) - S \leq L + 2\pi \left( \frac{1}{\sqrt{\delta}} - 1 \right)
\]

holds for every \( (S, \theta) \in \mathbb{R} \times [0, \pi] \).

**Proof.** Any lift of \( \bar{a}^{-1} \circ \bar{\Psi} \circ \bar{a} \) preserves the 2-form \( \sin \theta \, ds \wedge d\theta \), maps each boundary component into itself, and commutes with the translation \( (S, \theta) \mapsto (S + L, \theta) \). Let \( (s, \theta) \in \mathbb{R} / LZ \times [0, \pi] \). By [ABHS17, Lemma 3.9] the geodesic arc \( \phi^0_y(s, \theta) \) is embedded. Denote by \( \ell \) its length, and by \( \eta = \eta(s, \theta) \in (0, L) \) the length of the arc \( \alpha_+(a(s, \theta)) \). In particular, \( \eta \) is a continuous function of \( (s, \theta) \). Consider the convex geodesic polygon bounded by \( \phi^0_y(s, \theta) \) and \( \alpha_+(a(s, \theta)) \). By (84) we have

\[
2\pi \leq \eta + \ell \leq \frac{2\pi}{\sqrt{\delta}}.
\]

Similarly, we consider the convex geodesic polygon whose boundary consists of the arcs \( \phi^0_y(s, \theta) \) and \( c \setminus \alpha_+(a(s, \theta)) \). Observe that the length of \( c \setminus \alpha_+(a(s, \theta)) \) is \( L - \eta \in (0, L) \). Again by (84)

\[
2\pi \leq L - \eta + \ell \leq \frac{2\pi}{\sqrt{\delta}}.
\]
Thus, considering the difference between (86) and (87), one gets
\[ \frac{L}{2} + \pi \left( 1 - \frac{1}{\sqrt{\delta}} \right) \leq \eta \leq \frac{L}{2} + \pi \left( \frac{1}{\sqrt{\delta}} - 1 \right). \]

We repeat the same argument, but now using the length \( \nu = \nu(s, \theta) \) of the arc \( \alpha_+ (\phi^\tau_+ (a(s, \theta)) (a(s, \theta))) \), and add to obtain
\[ L + 2\pi \left( 1 - \frac{1}{\sqrt{\delta}} \right) \leq \eta + \nu \leq L + 2\pi \left( \frac{1}{\sqrt{\delta}} - 1 \right). \] (88)

The endpoints of the arc obtained by concatenating the arc \( \alpha_+ (a(s, \theta)) \) with the arc \( \alpha_+ (\phi^\tau_+(a(s, \theta)) (a(s, \theta))) \) are \( c(s) \) and \( \tilde{c}(\hat{p}_1 \circ \Psi \circ a(s, \theta)) \), where we denote by \( \hat{p}_1 : \mathbb{R}/L\mathbb{Z} \times [0, \pi] \to \mathbb{R}/L\mathbb{Z} \) the projection onto the first coordinate. In particular, the \( \mathbb{R} \)-components of a lift of \( \tilde{a}^{-1} \circ \Psi \circ \tilde{a} \) and of the identity map differ by \( \eta + \nu + kL \), with some \( k \in \mathbb{Z} \). Here we see \( \eta \) and \( \nu \) as continuous functions of \( (S, \theta) \) that are \( L \)-periodic in \( S \). Choose then the lift \( \tilde{\Psi} \) such that \( k = 0 \). From (88) we conclude that
\[ L + 2\pi \left( 1 - \frac{1}{\sqrt{\delta}} \right) \leq \tilde{p}_1 \circ \tilde{\Psi}(S, \theta) - S \leq L + 2\pi \left( \frac{1}{\sqrt{\delta}} - 1 \right) \]
holds for every \( (S, \theta) \in \mathbb{R} \times [0, \pi] \), as desired.

According to [HSW22, Appendix B], let \( (\Pi, \gamma_c \cup \hat{\gamma}_c) \) be the open book decomposition of \( S^3 \) such that \( \hat{A}_c \) is one of its pages. It holds that
\[ H_1(S^3 \setminus (\gamma_c \cup \hat{\gamma}_c)) \cong H_1(\hat{A}_c) \oplus \langle e \rangle \cong \mathbb{Z} \oplus \mathbb{Z} \]
where \( e \) is a loop such that \( \Pi_* e \) is the positive generator of \( H_1(\mathbb{R}/\mathbb{Z}) \). In particular, we can choose \( e \) so that \( \text{link}(e, \gamma_c) = 0 \) and \( \text{link}(e, \hat{\gamma}_c) = 1 \). Choose now a loop \( f \) lying in \( \hat{A}_c \setminus (\gamma_c \cup \hat{\gamma}_c) \) which is a generator of \( H_1(\hat{A}_c) \). We can choose \( f \) so that \( \text{link}(f, \gamma_c) = 1 \) and \( \text{link}(f, \hat{\gamma}_c) = -1 \). See Figure 3.

\[ \begin{array}{c}
\gamma_c \\
\hat{\gamma}_c \\
\hat{A}_c \\
e \\
f
\end{array} \]

Figure 3: The chosen loops \( e \) and \( f \).
**Lemma 3.13.** The loop $f$ is homotopic to the loop $s \in [0, 2L] \mapsto \bar{a} \circ P_{\infty}(-s, \theta_s)$ in $S^3 \setminus (\gamma_c \cup \hat{\gamma}_c)$, for any $\theta_s \in (0, \pi)$.

**Proof.** The free homotopy classes of loops in $S^3 \setminus (\gamma_c \cup \hat{\gamma}_c)$ are determined by the linking numbers with $\gamma_c$ and $\hat{\gamma}_c$. Note that $\bar{a} \circ P_{\infty}(-s, 0)$ is $-\gamma_c$, and that $\bar{a} \circ P_{\infty}(-s, \pi)$ is $\hat{\gamma}_c$, where $s \in [0, 2L]$. Hence the loop $\beta : s \in [0, 2L] \mapsto \bar{a} \circ P_{\infty}(-s, \theta_s)$ is homotopic to $-\gamma_c$ in $S^3 \setminus \hat{\gamma}_c$, and is homotopic to $\hat{\gamma}_c$ in $S^3 \setminus \gamma_c$. We conclude that $	ext{link}(\beta, \gamma_c) = \text{link}(-\gamma_c, \hat{\gamma}_c) = -1$ and $	ext{link}(\beta, \gamma_c) = \text{link}(\hat{\gamma}_c, \gamma_c) = 1$. Hence $\beta$ and $f$ are homotopic in $S^3 \setminus (\gamma_c \cup \hat{\gamma}_c)$.

Let us introduce some notation needed for the statement of Lemma 3.14 below. Let $T > 0$ and $x \in S^3 \setminus (\gamma_c \cup \hat{\gamma}_c)$, and consider the loop $k(T, x; \hat{D})$ in $S^3 \setminus (\gamma_c \cup \hat{\gamma}_c)$. There is a choice of closing path in $\hat{D} \setminus \hat{\gamma}_c$ which is not made explicit in the notation. Assume that this closing path does not intersect $\gamma_c$, and denote

$$M(T,x) = \text{link}(k(T,x;\hat{D}),\hat{\gamma}_c), \quad N(T,x) = \text{link}(k(T,x;\hat{D}),\gamma_c).$$

Note that $M$ depends only on $(T, x)$ and that, in contrast, $N$ highly depends on the choice of closing path; again we do not make this dependence explicit in the notation, for simplicity. By Lemma 3.5 we write in $H_1(S^3 \setminus (\gamma_c \cup \hat{\gamma}_c))$

$$k(T,x;\hat{D}) = (M(T,x) + N(T,x))c + N(T,x)f.$$  

Let $t \geq 0$ and $x \in \hat{A}_c \setminus (\gamma_c \cup \hat{\gamma}_c)$ be defined by

$$t = \min \{ t \geq 0 \mid \phi^{t+\hat{D}}(x) \in \hat{A}_c \}, \quad x = \phi^{t+\hat{D}}(x).$$

Note that $t$ and $x$ depend only on $x$, $\hat{A}_c$ and $\hat{D}$.

**Lemma 3.14.** Suppose that $\delta > 1/4$. There exists $C \geq 0$, which depends only on $\hat{D}$ and $\hat{A}_c$, with the following significance. Let $T > 0$ and $x \in S^3 \setminus (\gamma_c \cup \hat{\gamma}_c)$ be arbitrary, and let $(s_0, \theta_0) \in \mathbb{R} \times [0, \pi]$ satisfy $\bar{a} \circ P_{\infty}(s_0, \theta_0) = x$. Then there exists a closing path $\alpha$ in $\hat{D} \setminus \hat{\gamma}_c$ such that

$$\left| \frac{p_1 \circ \bar{a} M(T,x) + N(T,x)(s_0,\theta_0) - s_0 - M(T,x)}{2L} \right| \leq C$$

where $p_1 : \mathbb{R} \times [0, \pi] \to \mathbb{R}$ is the projection onto the first coordinate, and such that the conclusion of Lemma 3.6 holds, i.e.

$$|\text{int}(k_\alpha(T,x;\hat{D}),\hat{A}_c) - \# \{ t \in [0,T] \mid \phi^t(x) \in \hat{A}_c \}| \leq C.$$

**Proof.** Denote $p = \phi^{t+\hat{D}}(x)$. Consider

$$m(T,x) = \# \{ t \in [0,T+t_D^\hat{D}(\phi^T(x))-t_D^\hat{D}(x)] \mid \phi^t(p) \in \hat{A}_c \}. $$

41
For each \( i = 0, \ldots, m(T, x) - 1 \) let \( t_i \) be defined by

\[
\phi^{t_i}(p) \in \bar{A}_c, \quad 0 \leq t_0 = \ell < t_1 < \cdots < t_{m(T, x) - 1} \leq T + t^{D}(\phi^T(x)) - t^{D}(x).
\]

If \( (s_0, \theta_0) \in \mathbb{R} \times [0, \pi] \) satisfies \( \bar{a} \circ P_\infty(s_0, \theta_0) = x \) and \( (s_i, \theta_i) := \Phi(s_0, \theta_0) \) then we have

\[
\bar{a} \circ P_\infty(s_i, \theta_i) = \phi^{t_i}(p)
\]

and by Lemma 3.12

\[
s_{i+1} - s_i > 0
\]

for every \( i = 0, \ldots, m(T, x) - 1 \). Let \( k \in \mathbb{Z} \) be such that

\[
k - 1 < \frac{S_{m(T, x) - 1} - s_0}{2L} \leq k.
\]

We introduce a path \( \varepsilon : [0, 1] \to \mathbb{R} \times [0, \pi] \) from \( \varepsilon(0) = (s_{m(T, x) - 1}, \theta_{m(T, x) - 1}) \) to \( \varepsilon(1) = (s_0 + 2Lk, \theta_0) \) defined in the following way:

\[
\varepsilon(\tau) = \begin{cases} 
(s_{m(T, x) - 1} + 2\tau(s_0 + 2Lk - s_{m(T, x) - 1}), \theta_{m(T, x) - 1}) & 0 \leq \tau \leq 1/2, \\
(s_0 + 2Lk, \theta_{m(T, x) - 1} + (2\tau - 1)(\theta_0 - \theta_{m(T, x) - 1})) & 1/2 \leq \tau \leq 1.
\end{cases}
\]

See Figure 4.

For every \( i = 1, \ldots, m(T, x) - 1 \) construct a path \( \ell_i : [0, 1] \to \mathbb{R} \times [0, \pi] \) as follows. If \( i \leq m(T, x) - 2 \) then

\[
\ell_i(\tau) = \begin{cases} 
(s_i + 2\tau(s_{i-1} - s_i), \theta_i) & 0 \leq \tau \leq 1/2, \\
(s_{i-1}, \theta_i + (2\tau - 1)(\theta_{i-1} - \theta_i)) & 1/2 \leq \tau \leq 1.
\end{cases}
\]

The path \( \ell_i \) is defined similarly, but following \( \epsilon \) with a path from \( \varepsilon(1) = (s_0 + 2Lk, \theta_0) \) to \( (s_{m(T, x) - 2}, \theta_{m(T, x) - 2}) \), see Figure 4. Note that \( \ell_i \) connects \( (s_i, \theta_i) \) to \( (s_{i-1}, \theta_{i-1}) \).

Denote by \( \mathcal{L} \) the concatenation of the path \( \tau \mapsto \epsilon(1 - \tau) \) with all the paths \( \ell_i, i = 1, \ldots, m(T, x) - 1 \). By Lemma 3.13, the path \( \bar{a} \circ P_\infty(\mathcal{L}) \) is a loop homotopic to \( kf \) in \( S^3 \setminus (\gamma_c \cup \tilde{\gamma}_c) \). Recall \( p = \phi^{t^{D}(x)}(x) \). For \( i = 1, \ldots, m(T, x) - 1 \) denote by \( \nu_i \) the loop

\[
\nu_i = \phi^{[t_{i-1}, t_i]}(p) + \bar{a} \circ P_\infty(\ell_i).
\]

See Figure 5. In \( S^3 \setminus (\gamma_c \cup \tilde{\gamma}_c) \), since \( \bar{a} \circ P_\infty(\mathcal{L}) \) is homotopic to \( kf \), we have that

\[
\phi^{[t_0, t_{m(T, x) - 1}]}(p) + \bar{a} \circ P_\infty(\varepsilon) + kf
\]

is homologous to

\[
\nu_1 + \cdots + \nu_{m(T, x) - 1}.
\]
This implies
\[
\text{link}\left(\phi_{[t_0,m(T,x)-1]}(p) + \tilde{a} \circ P_\infty(\varepsilon), \hat{\gamma}_c\right) - k = \text{link}\left(\phi_{[t_0,m(T,x)-1]}(p) + \tilde{a} \circ P_\infty(\varepsilon) + kf, \hat{\gamma}_c\right)
\]
\[
= \sum_{i=1}^{m(T,x)-1} \text{link}(\nu_i, \hat{\gamma}_c).
\]

\textbf{Claim 1.} For every } i = 1, \ldots, m(T,x) - 1 \text{ it holds } \text{link}(\nu_i, \hat{\gamma}_c) = 0.

\textbf{Proof of Claim 1.} The loop } \nu_i \subset S^3 \setminus (\gamma_c \cup \hat{\gamma}_c) \text{ is made up of a trajectory of the lifted geodesic flow } \phi' \text{ and a path contained in the annulus } \tilde{A}_c. \text{ Consider the loop } D_g(\nu_i) \subset T^1 S^2, \text{ see Figure 6. This loop consists of the velocity vectors of a geodesic starting at } c \text{ up to the second hitting point with } c \text{ (first hit with } A_c\text{), and then of velocity vectors obtained by parallel transport back the initial point, and then of a vertical deformation. In particular, } D_g(\nu_i) \text{ does not intersect the set } c \cup (-c) = D_g(\gamma_c \cup \hat{\gamma}_c). \text{ The path } D_g(\nu_i) \text{ can be deformed in a continuous way into a loop } \Gamma \text{ contained in } c \text{ as depicted in Figure 6: the initial velocity vector of } \pi \circ D_g(\nu_i) \text{ becomes more and more positively tangent to } c, \text{ the second hitting point converges to second conjugate point, and one parallel transport back by vec-
tors positively tangent to $c$. Such a deformation can be realised within $T^1_gS^2 \setminus (-\dot{c})$. Moreover, observe that the loop $\Gamma \subset \dot{c}$ is homotopic to a point in $\dot{c}$. Thus, $D_g(\nu_i)$ is contractible to a point in $T^1_gS^2 \setminus (-\dot{c})$. By the homotopy lifting property, also the loop $\nu_i$ is contractible to a point in $S^3 \setminus \hat{\gamma}_c$. Hence $\text{link}(\nu_i, \hat{\gamma}_c) = 0$ and Claim 1 is proved.

From (93) and Claim 1 we deduce that
\[
\text{link}(\phi^{[0,t_m(T,x)-1]}(p) + \tilde{a} \circ P_\infty(\varepsilon), \hat{\gamma}_c) = k.
\] (94)

CLAIM 2. $\text{link}(\phi^{[0,t_m(T,x)-1]}(p) + \tilde{a} \circ P_\infty(\varepsilon), \hat{\gamma}_c) = M(T,x) + O(1)$.

Proof of Claim 2. Recall that $k(T,x; \hat{D})$ was defined as the concatenation of the piece of trajectory $\phi^{[0,T+t^D_\alpha (\phi^x(\varepsilon))-t^D_\alpha (x)]}(p)$ with a path $\alpha \subset \hat{D}$, called a closing path.
Claim 3. There exists $C \geq 0$ independent of $(T, x)$ such that the inequality $|m(T, x) - M(T, x) - N(T, x)| \leq C$ holds for the closing path chosen in Lemma 3.6 defining $k(T, x; \hat{D})$.

Proof of Claim 3. Note that $\tilde{A}_c$ is a $\partial$-strong global surface of section. This is true by positivity of the curvature. Hence, the return time back to the interior of $\tilde{A}_c$ is bounded away from zero and bounded from above. We find $\rho > 0$ depending only on $\tilde{A}_c$ such that $\# \{t \in [a, b] \mid \phi^t(q) \in \tilde{A}_c\} \leq \rho(b - a)$ holds for every $a < b$ and
\( q \in S^3 \setminus \partial \tilde{A}_c \). Now note that \( \| \tilde{D} \|_{L^\infty} < \infty \). This is true because \( \tilde{D} \) is \( \partial \)-strong. With 
\[ L = \max\{\| \tilde{D} \|_{L^\infty}, \| \tilde{D}^+ \|_{L^\infty}\} \] 
we get
\[
0 \leq m(T, x) - \#\{t \in [0, T] \mid \phi^t(x) \in \tilde{A}_c\} \\
\leq \#\{t \in [\tilde{D}(x), 0] \mid \phi^t(x) \in \tilde{A}_c\} \\
+ \#\{t \in [0, \tilde{D}(\phi^T(x))] \mid \phi^{t+T}(x) \in \tilde{A}_c\} \\
\leq 2L\rho.
\]
By Lemma 3.5 we have
\[
M(T, x) + N(T, x) = \int(k(T, x; \tilde{D}), \tilde{A}_c),
\]
and by Lemma 3.6 there exists \( C' \geq 0 \) independent of \((T, x)\) such that
\[
|\#\{t \in [0, T] \mid \phi^t(x) \in \tilde{A}_c\} - \int(k(T, x; \tilde{D}), \tilde{A}_c)| \leq C'
\]
holds for some particular choice of closing path. The proof is complete if we set 
\( C = C' + 2L\rho \).

By Claim 2 and (94)
\[
M(T, x) = k + O(1)
\]
and (92) gives
\[
k = \frac{p_1 \circ \tilde{\Psi}^{m(T, x)-1}(s, \theta) - s}{2L} + O(1).
\]
Thus
\[
\left| M(T, x) - \frac{p_1 \circ \tilde{\Psi}^{m(T, x)-1}(s, \theta) - s}{2L} \right| = O(1)
\]
Up to now the constants \( O(1) \) are independent of \((T, x)\) and also of choice of closing paths. By Claim 3 we find
\[
|m(T, x) - M(T, x) - N(T, x)| = O(1)
\]
where this last constant is still independent of \((T, x)\), although it depends on the particular choice of closing path. Hence
\[
\left| \frac{p_1 \circ \tilde{\Psi}^{m(T, x)-1}(s, \theta) - s}{2L} - \frac{p_1 \circ \tilde{\Psi}^{M(T, x)+N(T, x)}(s, \theta) - s}{2L} \right| = O(1).
\]
All of this implies that
\[
\left| M(t, x) - \frac{p_1 \circ \tilde{\Psi}^{M(T, x)+N(T, x)}(s, \theta) - s}{2L} \right| = O(1)
\]
and the proof is complete. \( \square \)
We now complete the proof of Theorem 1.13. With \( x \in S^3 \setminus (\gamma_c \cup \hat{\gamma}_c) \) and \( T > 0 \) arbitrary, below we shall write \( k(T, x; \hat{D}) \) to denote a loop obtained with a closing path given by Lemma 3.14. Combining Lemma 3.12 with Lemma 3.14
\[
\int(k(T, x; \hat{D}), \hat{\gamma}_c) = \frac{M(T, x) + N(T, x)}{M(T, x)} = \frac{M(T, x) + N(T, x)}{p_1 \circ \hat{\Psi}^{M(T, x) + N(T, x)}(s, \theta) - s + O(1)} \geq \frac{2L}{L + 2\pi \left( \frac{1}{\sqrt{\delta}} - 1 \right)} + O \left( \frac{1}{M(T, x) + N(T, x)} \right).
\]  
(96)

On the other hand by Lemma 3.3 and Lemma 3.14
\[
\frac{\Delta \Theta(T, u)}{\int(k(T, x; \hat{D}), \hat{\gamma}_c)} \geq \frac{T \delta}{\frac{T \delta}{\tau_{\min}} + O(1)} = \delta \tau_{\min} + O \left( \frac{1}{T} \right)
\]  
(97)

where \( \tau_{\min} \) denotes the infimum of the return time function of \( \hat{A}_c \). By Lemma 3.14 there is \( \chi > 0 \) independent of \( x \) such that \( \chi^{-1}T \geq M(T, x) + N(T, x) \geq \chi T \). Hence,
\[
O \left( \frac{1}{M(T, x) + N(T, x)} \right) = O \left( \frac{1}{T} \right).
\]

Plugging this together with (96) and (97) we finally arrive at
\[
\frac{\Delta \Theta(T, u)}{\text{link}(k(T, x; \hat{D}), \hat{\gamma}_c)} \geq \delta \tau_{\min} \frac{2L}{L + 2\pi \left( \frac{1}{\sqrt{\delta}} - 1 \right)} + O \left( \frac{1}{T} \right). \tag{98}
\]

Let us consider an auxiliary constant \( \mu > 1 \) and look for \( 0 < \delta < 1 \) such that
\[
\frac{2L}{L + 2\pi \left( \frac{1}{\sqrt{\delta}} - 1 \right)} > \frac{\mu}{\delta} \iff L(2\delta - \mu) > 2\pi \mu \left( \frac{1}{\sqrt{\delta}} - 1 \right).
\]

Since by (84) we have \( L \geq 2\pi \), it suffices to ask for
\[
\mu < 2\delta \sqrt{\delta}
\]

Under this condition on \( \mu \), by (98) we need
\[
\tau_{\min} \mu > 2\pi
\]

in order to get \( \kappa(\hat{\gamma}_c) > 2\pi \) and complete the proof. Toponogov’s Theorem yields
\[
\tau_{\min} \geq 2\pi \left( 2 - \frac{1}{\sqrt{\delta}} \right).
\]
hence we need
\[ \mu \left( 2 - \frac{1}{\sqrt[2]{\delta}} \right) > 1. \]

Combining all of the above, we need to ask
\[ 2\delta \sqrt[2]{\delta} > \mu > \frac{1}{2 - \frac{1}{\sqrt[2]{\delta}}} = \frac{\sqrt[2]{\delta}}{2\sqrt[2]{\delta} - 1}. \tag{99} \]

Note that \( \frac{\sqrt[2]{\delta}}{2\sqrt[2]{\delta} - 1} \geq 1 \) as long as \( \frac{1}{4} < \delta \leq 1 \). Under this condition, a choice of \( \mu \) as in (99) is possible if
\[ 2 \frac{\delta \sqrt[2]{\delta}}{2\sqrt[2]{\delta} - 1} \Leftrightarrow 2\delta (2\sqrt[2]{\delta} - 1) > 1. \tag{100} \]

Introducing the variable \( x = \sqrt[2]{\delta} \) we get the inequality \( P(x) := 2x^2(2x - 1) - 1 > 0 \). The polynomial \( P(x) \) has a unique real root \( x_* \) because it is negative at its critical points 0 and \( \frac{1}{3} \). This root \( x_* \) lies in \( (\frac{1}{3}, 0.85) \) because \( P(0.85) > 0 \). Hence we get \( \delta_* = x_*^2 < 0.7225 \) and (100) holds for all \( \delta \in (\delta_*, 1] \), as claimed.

References

[ABHS17] A. Abbondandolo, B. Bramham, U. Hryniewicz, and P. A. S. Salomão, A systolic inequality for geodesic flows on the two-sphere, Math. Ann. 367 (2017), no. 1-2, 701–753.

[Arn86] V. I. Arnold, The asymptotic Hopf invariant and its applications, Selecta Math. Soviet. 5 (1986), no. 4, 327–345. Selected translations.

[Bir13] G. D. Birkhoff, Proof of Poincaré’s geometric theorem, Trans. Amer. Math. Soc. 14 (1913), no. 1, 14–22.

[Bir66] , Dynamical systems, With an addendum by Jurgen Moser. AMS Colloquium Publications, Vol. IX, American Mathematical Society, Providence, R.I., 1966.

[Deh17] P. Dehornoy, Which geodesic flows are left-handed?, Groups Geom. Dyn. 11 (2017), no. 4, 1347–1376.

[DR22] P. Dehornoy and A. Rechtman, Vector fields and genus in dimension 3, Int. Math. Res. Not. IMRN 5 (2022), 3262–3277.

[FH21] A. Florio and U. Hryniewicz, Quantitative conditions for right-handedness, arXiv:2106.12512 v1 (2021).

[Flo19a] A. Florio, Asymptotic Maslov indices, 2019. PhD thesis, Avignon Université.

[Flo19b] , Torsion and linking number for a surface diffeomorphism, Math. Z. 292 (2019), no. 1-2, 231–265.

[Ghy09] E. Ghys, Right-handed vector fields & the Lorenz attractor, Jpn. J. Math. 4 (2009), no. 1, 47–61.

[Gro00] M. Gromov, Three remarks on geodesic dynamics and fundamental group, Enseign. Math. (2) 46 (2000), no. 3-4, 391–402.
[GRS06] C. Grotta-Ragazzo and P. A. S. Salomão, *The Conley-Zehnder index and the saddle-center equilibrium*, J. Differential Equations **220** (2006), no. 1, 259–278.

[HP08] A. Harris and G. P. Paternain, *Dynamically convex Finsler metrics and J-holomorphic embedding of asymptotic cylinders*, Ann. Global Anal. Geom. **34** (2008), no. 2, 115–134.

[Hry14] U. Hryniewicz, *Systems of global surfaces of section for dynamically convex Reeb flows on the 3-sphere*, J. Symplectic Geom. **12** (2014), no. 4, 791–862.

[Hry20] ______, *A note on Schwartzman-Fried-Sullivan theory, with an application*, J. Fixed Point Theory Appl. **22** (2020), no. 1, Paper No. 25, 20.

[HS13] U. Hryniewicz and P. A. S. Salomão, *Global properties of tight Reeb flows with applications to Finsler geodesic flows on S^2*, Math. Proc. Cambridge Philos. Soc. **154** (2013), no. 1, 1–27.

[HSW22] U. Hryniewicz, P. A. S. Salomão, and K. Wysocki, *Genus zero global surfaces of section for Reeb flows and a result of Birkhoff*, J. Eur. Math. Soc. (2022).

[HWZ98] H. Hofer, K. Wysocki, and E. Zehnder, *The dynamics on three-dimensional strictly convex energy surfaces*, Ann. of Math. (2) **148** (1998), no. 1, 197–289.

[HWZ99] ______, *A characterization of the tight 3-sphere. II*, Comm. Pure Appl. Math. **52** (1999), no. 9, 1139–1177.

[Kli59] W. Klingenberg, *Contributions to Riemannian geometry in the large*, Ann. of Math. (2) **69** (1959), 654–666.

[Kli82] ______, *Riemannian geometry*, de Gruyter Studies in Mathematics, vol. 1, Walter de Gruyter & Co., Berlin-New York, 1982.

[Poi12] H. Poincaré, *Sur un théorème de géométrie*, Rend. Circ. Mat. Palermo **33** (1912), 375–407.

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49