COMPLEXES OF DISCRETE DISTRIBUTIONAL DIFFERENTIAL FORMS
AND THEIR HOMOLOGY THEORY

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Abstract. Complexes of discrete distributional differential forms are introduced into finite element exterior calculus. Thus we generalize a notion of Braess and Schöberl, originally studied for a posteriori error estimation. We construct isomorphisms between the simplicial homology groups of the triangulation, the discrete harmonic forms of the finite element complex, and the harmonic forms of the distributional finite element complexes. As an application, we prove that the complexes of finite element exterior calculus have cohomology groups isomorphic to the de Rham cohomology, including the case of partial boundary conditions. Poincaré-Friedrichs-type inequalities will be studied in a subsequent contribution.

1. Introduction

Finite element exterior calculus (FEEC, [2, 4]) recasts finite element theory in the calculus of differential forms and has emerged as a canonical framework for vector-valued finite elements. The classical residual error estimator has been studied recently in the setting of finite element exterior calculus by Demlow and Hirani [21]. On the contrary, implicit error estimators [1, 34, 37] have generally remained restricted to spaces of scalar functions in literature, despite their promising performance in numerical experiments [13]. A notable exception is the equilibrated residual error estimator that Braess and Schöberl [10] have studied for lowest-order Nédélec elements in two and three dimensions.

They have introduced distributional finite element complexes — the major contribution of the present work is to integrate that notion into finite element exterior calculus. We formulate complexes of discrete distributional differential forms and determine their homology spaces. A subsequent work will analyze Poincaré-Friedrichs inequalities of these complexes. The present work serves as a technical preparation for research on a posteriori error estimation, but we derive results of independent interest. For example, we close a gap in literature and derive compatibility on homology for the standard finite element complex, but with partial boundary conditions, which are relevant for the Hodge Laplace equation with mixed boundary conditions.

We outline the essential ideas by an example. Here we employ the formalism of vector calculus, but the remainder of the contribution employs the calculus of differential forms. Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain that has a triangulation $\mathcal{T}$. We denote by $\mathcal{T}^3$, $\mathcal{T}^2$, $\mathcal{T}^1$ and $\mathcal{T}^0$ the sets of tetrahedrons, triangles, edges, and vertices, respectively.

With respect to this triangulation, we consider a lowest-order finite element complex:

$$\mathcal{P}^1(\mathcal{T})_0 \xrightarrow{\text{grad}} \mathcal{N}d(\mathcal{T})_0 \xrightarrow{\text{curl}} \mathcal{RT}(\mathcal{T})_0 \xrightarrow{\text{div}} \mathcal{P}^0_{-1}(\mathcal{T}).$$

Here, $\mathcal{P}^1(\mathcal{T})_0$ is the space of continuous piecewise affine functions satisfying homogeneous boundary conditions, $\mathcal{N}d(\mathcal{T})_0$ is the curl-conforming Nédélec space satisfying homogeneous tangential boundary conditions, $\mathcal{RT}(\mathcal{T})_0$ is the divergence-conforming Raviart-Thomas space satisfying homogeneous normal boundary conditions, and $\mathcal{P}^0_{-1}(\mathcal{T})$ is the space of piecewise constant functions. We refer to [27] for more background on finite element vector analysis.
Following the notation of [10], let $\mathcal{P}^{-1}_1(T)$ be the space of piecewise affine functions; loosely speaking, $\mathcal{P}^1(T)$ without the requirement of boundary conditions and continuity along faces. While the classical gradient is not defined on that space, we can view $\mathcal{P}^{-1}_1(T)$ as a space of functionals on smooth functions and apply the gradient in the sense of distributions. If $u \in \mathcal{P}^{-1}_1(T)$ and $\bar{v} \in C^\infty(\bar{T}, \mathbb{R}^3)$, then the divergence theorem shows:

$$\langle \text{grad} \, u, \bar{v} \rangle := -\sum_{T \in T^3} \int_T u \text{div} \, \bar{v} \, dx$$

$$= \sum_{T \in T^3} \int_T \langle \text{grad} \, u, \bar{v} \rangle \, dx - \sum_{T \in T^3} \sum_{F \in T} \int_F u(\bar{v}, \bar{n}_{T,F}) \, ds,$$

where $\bar{n}_{T,F}$ is the respective outward normal. The distributional gradient maps into a space of distributions over $C^\infty(\bar{T}, \mathbb{R}^3)$, which we denote by $\mathcal{N}d_{-2}(T)$ and which is spanned by two types of functionals: on the one hand, by piecewise Nédélec elements, without tangential continuity along the faces, and, on the other hand, by functionals that act as integration against affine elements over faces. This corresponds to the two final terms in (2).

Next, the curl-operator, in the sense of distributions, maps $\mathcal{N}d_{-2}(T)$ into $\mathcal{RT}_{-3}(T)$, which is another space of distributions on $C^\infty(\bar{T}, \mathbb{R}^3)$, and which is spanned by piecewise Raviart-Thomas elements without normal continuity along faces, by functionals along faces, and functionals along edges. Eventually, the distributional divergence maps $\mathcal{RT}_{-3}(T)$ into $\mathcal{P}^0_{0,1}(T)$, a space of distributions over $C^\infty(\bar{T})$ again, spanned by integral evaluations on cells, faces and edges, and point evaluations. We have thus found a distributional finite element complex [10, Equation (3.18)]:

$$\mathcal{P}^{-1}_1(T) \xrightarrow{\text{grad}} \mathcal{N}d_{-2}(T) \xrightarrow{\text{curl}} \mathcal{RT}_{-3}(T) \xrightarrow{\text{div}} \mathcal{P}^0_{0,1}(T).$$

See also [10, Equations (3.3), (3.5), (3.7), (3.16-3.18)] for similar differential complexes. The publication of Braess and Schöberl considers distributional finite element complexes on local patches of two- and three-dimensional triangulations, based on finite element spaces of lowest order. A generalization and unified treatment of these distributional differential operators and the integration by parts formula (2) is viable with the calculus of differential forms. Thus we extend the basic idea to discrete distributional de Rham complexes of any dimension, over domains of arbitrary topology, and with general partial boundary conditions. The theory includes the spaces of piecewise polynomial differential forms of finite element exterior calculus.

Distributional differential forms and similar ideas appear in different areas of mathematics. De Rham [35] introduced the term “currents” for continuous linear functionals on a class of locally convex spaces of smooth differential forms. Geometric integration theory [28] knows simplicial chain complexes as a specific example of currents; we rediscover this for our homology theory. Christiansen has considered distributional finite element complexes in Regge calculus [17].

The right-hand side of (2) is decomposed into two operators: a piecewise differential operator on the one hand, and the sum of signed traces on the other hand. Similar decompositions hold for the distributional curl and the distributional divergence, and a unified treatment is accessible with the calculus of differential forms.

It is an essential observation of this contribution that both of these operators are constituent for differential complexes. For instance, the standard finite element complex [11] is a subcomplex of the distributional finite element complex (3), and it composed of spaces on which the last term of (2) vanishes. But when identifying simplices with their indicator functions, we furthermore observe that, up to a sign convention, the simplicial chain complex of the triangulation

$$\mathcal{C}_3(T) \xrightarrow{-\partial^3} \mathcal{C}_2(T) \xrightarrow{\partial^2} \mathcal{C}_1(T) \xrightarrow{-\partial^1} \mathcal{C}_0(T)$$
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is a subcomplex of (3) as well. It is composed of spaces on which the piecewise differential vanishes. Indeed, a close look reveals that the “jump term” of (2) resembles the simplicial boundary operator. Given these differential complexes, the incentive question that we address in this contribution is: can we relate their homology spaces?

To answer this question, we adopt an established concept of homological algebra. It is instructive to consider the following diagram:

\[
\begin{array}{cccc}
\mathcal{P}_1(T^3) & \xrightarrow{\text{grad}_T} & \mathcal{N} d_{-1}(T^3) & \xrightarrow{\text{curl}_T} & \mathcal{RT}_{-1}(T^3) & \xrightarrow{\text{div}_T} & \mathcal{P}_0(T^3) \\
\downarrow{\partial^3} & & \downarrow{\partial^3} & & \downarrow{\partial^3} & & \downarrow{\partial^3} \\
\mathcal{P}_1(T^2) & \xrightarrow{\text{grad}_T} & \mathcal{N} d_{-1}(T^2) & \xrightarrow{\text{curl}_T} & \mathcal{RT}_{-1}(T^2) \\
\downarrow{\partial^2} & & \downarrow{\partial^2} & & \downarrow{\partial^2} \\
\mathcal{P}_1(T^1) & \xrightarrow{\text{grad}_T} & \mathcal{N} d_{-1}(T^1) \\
\downarrow{\partial^1} & & \downarrow{\partial^1} \\
\mathcal{P}_1(T^0) \\
\end{array}
\]

The spaces in this diagram are piecewise finite element spaces on lower-dimensional skeletons of the triangulation. For instance, \(\mathcal{N} d_{-1}(T^2)\) is a space of functionals on \(C^\infty(\Omega, \mathbb{R}^3)\) which act by integration against lowest-order Nédélec elements over two-dimensional simplices, and \(\mathcal{P}_1(T^0)\) is the span of point evaluations at vertices of \(T\) acting on \(C^\infty(\Omega)\); similarly for the other spaces of (5). The horizontal mappings are piecewise differential operators, and thus the rows of the diagram are differential complexes by themselves. The rows are furthermore exact sequences for many choices of finite spaces, including the exact sequences of finite element exterior calculus. The vertical mappings correspond to the boundary terms in partial integration formulas like (2), and in the above diagram, we suggestively denote them by the same symbol as the simplicial boundary operator. It is an original observation of this work that these “jump-terms” are operators in their own right and constituent for differential complexes in the columns. The columns are even exact sequences; this uses the geometric decomposition of the finite element spaces and a combinatorial condition on the triangulation. The diagram (5) is a double complex in the sense of homological algebra [25], and we note that the complex (3) corresponds to the sequence of diagonals, also called total complex, of the double complex. Furthermore, our earlier observations transfer: the simplicial chain complex is included in the left-most column, whereas the standard finite element complex is included in the top-most row.

The question regarding the homology spaces can be answered with the adaption of methods that evolved in the treatment of double complexes. We construct isomorphisms between the homology groups of the triangulation, the discrete harmonic forms of the standard finite element complex, and discrete distributional harmonic forms of distributional finite element complexes such as (3). This is an alternative access towards homology theory in a finite element setting, besides smoothed projections [15] and de Rham mappings [14], and, to the author’s best knowledge, the homology theory of discrete de Rham complexes with partial boundary conditions has not been addressed before in literature.

Double complexes are used prominently in differential topology [40]. Falk and Winther have recently introduced a finite element Čech de Rham complex to finite element theory [23], albeit not for questions in homology theory.

The remainder of this contribution is organized as follows. In Section 2 we recall relevant notions on simplicial complexes, differential forms, and Hilbert complexes. Furthermore, the \(L^2\) de Rham complex is introduced as an example application of this contribution’s theory. In Section 3 we introduce discrete distributional differential forms and relevant differential
operators. In Section 3 we study finite element double complexes such as (6). We prove the exactness of the rows and columns under reasonable assumptions. This shows the existence of an isomorphism between the simplicial homology groups and the discrete harmonic forms. In Section 4 we study complexes of discrete distributional differential forms such as (3), and derive isomorphisms between the discrete distributional harmonic forms. This provides also a constructive proof of the result in the previous section.

2. Preliminaries

In this section we gather technical prerequisites and notational conventions from topology, analysis on manifolds, and functional analysis. The core of our results considers finite element spaces on simplices, differential and trace operators between them, and the homology of finite-dimensional Hilbert complexes. Therefore we review simplicial complexes (Subsection 2.1), differential forms on manifolds (Subsection 2.2), and basic aspects of Hilbert complexes (Subsection 2.3). We also give an outline of the $L^2$ de Rham complex of polyhedrally bounded domains with partial boundary conditions (Subsection 2.4) in order to show the connections to the analysis of partial differential equations, and as an example application that is repeatedly addressed in the sequel.

2.1. Simplicial complexes. We review basic notions of simplicial topology and the homology of simplicial chain complexes. We refer to [31], [36] and [29] for further background.

Let $n \in \mathbb{N}_0$ be fixed. A (closed) $m$-simplex $C$ is the convex closure of a set of $m + 1$ affinely independent points in $\mathbb{R}^n$, called the set of vertices of $C$, and we also write $\dim C = m$. A simplex $F$ is a subsimplex of an $m$-simplex $C$ if the set of vertices of $F$ is a subset of the set of vertices of $C$. Then we write $F \subseteq C$ and call $C$ a supersimplex of $F$. Accordingly, we write $F \subset C$ for $F \subseteq C$ with $F \neq C$, and $F \nsubseteq C$ when $F \subseteq C$ does not hold.

We call a finite set of simplices $\mathcal{T}$ a simplicial complex provided that for each simplex $C \in \mathcal{T}$ all subsimplices of $C$ are included in $\mathcal{T}$ and that any non-empty intersection of two simplices $C, C' \in \mathcal{T}$ is a subsimplex of both $C$ and $C'$. We write $\mathcal{T}^m = \{C \in \mathcal{T} \mid \dim C = m\}$. We say that $\mathcal{T}$ is $p$-dimensional provided that for each $S \in \mathcal{T}$ there exists $C \in \mathcal{T}^p$ with $S \subseteq C$. An $m$-dimensional simplicial subcomplex $\mathcal{U}$ of $\mathcal{T}$ is a subset of $\mathcal{T}$ that is an $m$-dimensional simplicial complex by itself. We write $\mathcal{T}^{[m]}$ for the largest $m$-dimensional simplicial subcomplex of $\mathcal{T}$.

Simplices are orientable compact manifolds with corners [30, Chapter 10]. We henceforth assume that all simplices in $\mathcal{T}$ are oriented, and that this orientation is the Euclidean orientation for $n$-dimensional simplices. Any oriented $m$-simplex $C$ induces an orientation on any $(m - 1)$-dimensional subsimplex $F$, and we then set $o(F, C) = 1$ if either both orientations on $F$ coincide, or $o(F, C) = -1$ if those orientations differ.

The space of simplicial $m$-chains $\mathcal{C}_m(\mathcal{T})$ is the real vector space generated by $\mathcal{T}^m$. It is easy to verify that the simplicial boundary operator

$$\partial_m : \mathcal{C}_m(\mathcal{T}) \rightarrow \mathcal{C}_{m-1}(\mathcal{T}), \quad \partial_m C = \sum_{F \subseteq C \atop F \in \mathcal{T}^{m-1}} o(F, C)F$$

satisfies $\partial_{m-1}\partial_m = 0$. Thus we have a differential complex,

$$
\begin{array}{cccccccc}
0 & \longrightarrow & \mathcal{C}_p(\mathcal{T}) & \overset{\partial_p}{\longrightarrow} & \mathcal{C}_{p-1}(\mathcal{T}) & \overset{\partial_{p-1}}{\longrightarrow} & \ldots & \overset{\partial_1}{\longrightarrow} & \mathcal{C}_0(\mathcal{T}) & \longrightarrow & 0,
\end{array}
$$

called the simplicial chain complex of $\mathcal{T}$.

If $\mathcal{U}$ is a simplicial subcomplex of $\mathcal{T}$, then $\mathcal{C}_m(\mathcal{U})$ is a subspace of $\mathcal{C}_m(\mathcal{T})$, and the simplicial boundary operator $\partial_m : \mathcal{C}_m(\mathcal{U}) \rightarrow \mathcal{C}_{m-1}(\mathcal{U})$ is the restriction of the simplicial boundary operator $\partial_m : \mathcal{C}_m(\mathcal{T}) \rightarrow \mathcal{C}_{m-1}(\mathcal{T})$. In Section 4 we study finite element double complexes such as (6). We prove the exactness of the rows and columns under reasonable assumptions. This shows the existence of an isomorphism between the simplicial homology groups and the discrete harmonic forms. In Section 4 we study complexes of discrete distributional differential forms such as (3), and derive isomorphisms between the discrete distributional harmonic forms. This provides also a constructive proof of the result in the previous section.
operator $\partial_m : C_m(\mathcal{T}) \to C_{m-1}(\mathcal{T})$. This means that the simplicial chain complex of $\mathcal{U}$,

$$
\begin{array}{cccccc}
0 & \longrightarrow & C_p(\mathcal{U}) & \xrightarrow{\partial_p} & C_{p-1}(\mathcal{U}) & \xrightarrow{\partial_{p-1}} \cdots \xrightarrow{\partial_0} C_0(\mathcal{U}) & \longrightarrow 0,
\end{array}
$$

is a differential subcomplex of $\mathbb{B}$. We write $C_m(\mathcal{T}, \mathcal{U}) = C_m(\mathcal{T})/C_m(\mathcal{U})$ for the quotient space, and note that the equivalence classes of the simplices $\mathcal{T}^m \setminus \mathcal{U}^m$ constitute a basis. Now we observe for $C, C' \in C_m(\mathcal{T})$ with $C - C' \in C_m(\mathcal{U})$ that

$$
\partial_mC - \partial_mC' = \partial_m(C - C') \in C_{m-1}(\mathcal{U}).
$$

So the simplicial chain complexes $\mathbb{B}$ and $\mathbb{B}$ induce another differential complex,

$$
\begin{array}{cccccc}
0 & \longrightarrow & C_p(\mathcal{T}, \mathcal{U}) & \xrightarrow{\partial_p} & C_{p-1}(\mathcal{T}, \mathcal{U}) & \xrightarrow{\partial_{p-1}} \cdots \xrightarrow{\partial_0} C_0(\mathcal{T}, \mathcal{U}) & \longrightarrow 0,
\end{array}
$$

called the simplicial chain complex of $\mathcal{T}$ relative to $\mathcal{U}$. Up to elements of $\mathcal{U}$, the differential of that complex is described by the equation

$$
\partial_mC = \sum_{F \in \mathcal{T}^m \setminus \mathcal{U}^m} o(F, C)F, \quad C \in \mathcal{T}^m \setminus \mathcal{U}^m.
$$

Note that $\mathbb{B}$ agrees with $\mathbb{B}$ in the special case $\mathcal{U} = \emptyset$. The simplicial homology spaces $\mathcal{H}_m(\mathcal{T}, \mathcal{U})$ of $\mathcal{T}$ relative to $\mathcal{U}$ are defined as the quotient spaces

$$
\begin{equation}
\mathcal{H}_m(\mathcal{T}, \mathcal{U}) := \frac{\ker \left( \partial_m : C_m(\mathcal{T}, \mathcal{U}) \to C_{m-1}(\mathcal{T}, \mathcal{U}) \right)}{\text{ran} \left( \partial_{m+1} : C_{m+1}(\mathcal{T}, \mathcal{U}) \to C_m(\mathcal{T}, \mathcal{U}) \right)}.
\end{equation}
$$

Their dimensions are of general interest. We call $b_m(\mathcal{U}) := \dim \mathcal{H}_m(\mathcal{T}, \mathcal{U})$ the $m$-th simplicial Betti number of $\mathcal{T}$ relative to $\mathcal{U}$, and we call $b_m(\mathcal{T}) := b_m(\mathcal{T}, \emptyset)$ the $m$-th absolute simplicial Betti number of $\mathcal{T}$.

Let $M$ be a topological manifold with boundary, embedded in $\mathbb{R}^n$, and let $\Gamma$ be a topological submanifold of its boundary manifold $\partial M$. We say a simplicial complex $\mathcal{T}$ triangulates a topological manifold with boundary if that manifold is the union of all simplices in $\mathcal{T}$. The $m$-th topological Betti number $b_m(M, \Gamma) \in \mathbb{N}_0$ is the dimension of the $m$-th singular homology group of $M$ relative to $\Gamma$; we refer to [36, Chapter 4, Section 4] for the details. In case $\Gamma = \emptyset$ we call $b_m(M) := b_m(M, \emptyset)$ the $m$-th absolute topological Betti number of $M$. The following canonical result relates the simplicial and the topological Betti numbers:

**Theorem 2.1** ([36] Chapter 4, Section 6, Theorem 8).

Let $M$ and $\Gamma$ be as in the previous paragraph. Let $\mathcal{T}$ be a simplicial complex triangulating $M$, and $\mathcal{U}$ be a simplicial subcomplex of $\mathcal{T}$ that triangulates $\Gamma$. Then we have

$$
b_m(\mathcal{T}, \mathcal{U}) = b_m(\mathcal{U})
$$

for all $m \in \mathbb{N}_0$. □

**Remark 2.2.**

This result implies that the topological Betti numbers can be computed from the combinatorial structure of any triangulation. They also coincide with the dimensions of the solution spaces of certain homogeneous partial differential equations over $M$, see Subsection 2.4 below.

**Example 2.3.**

The following topological Betti numbers are of frequent interest. All Betti numbers $b_m(B^p)$ of the $p$-ball $B^p$ vanish except for $b_0(B^p) = 1$. All Betti numbers $b_m(S^p)$ of the $p$-sphere $S^p$ vanish except for $b_p(S^p) = b_0(S^p) = 1$. All Betti numbers $b_m(B^p, \partial B^p)$ of the $p$-ball relative to its boundary vanish except for $b_p(B^p, \partial B^p) = 1$. If $D^{p-1} \subsetneq \partial B^p$ is homeomorphic to $B^{p-1}$, then all Betti numbers $b_m(B^p, D^{p-1})$ vanish. △
2.2. Differential forms on domains. We review basic notions of differential forms on Riemannian manifolds with boundary, and some aspects of their $L^2$ theory. We generally refer to [30, 24], and [26] for further background.

Consider an $m$-dimensional open smooth manifold $M$ embedded in $\mathbb{R}^n$ such that its closure $\overline{M}$ is a topological manifold with boundary embedded in $\mathbb{R}^n$. Examples include simplices, polyhedrally bounded domains, Lipschitz domains and $\mathbb{R}^n$ itself.

Let $C^\infty \Lambda^k(M)$ be the space of smooth differential forms on $M$, and let $C^\infty \Lambda^k(\overline{M}) \subseteq C^\infty \Lambda^k(M)$ be the image of the pullback of $C^\infty \Lambda^k(\mathbb{R}^n)$ into $C^\infty \Lambda^k(M)$. We furthermore recall the exterior derivative $d_M^k$ and the $\Lambda$-product:

$$d_M^k : C^\infty \Lambda^k(M) \to C^\infty \Lambda^{k+1}(M),$$

$$\wedge : C^\infty \Lambda^k(M) \times C^\infty \Lambda^l(M) \to C^\infty \Lambda^{k+l}(M).$$

It is known for $\omega \in C^\infty \Lambda^k(M)$ and $\eta \in C^\infty \Lambda^l(M)$ that

$$\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega, \quad d_M^{k+l}(\omega \wedge \eta) = d_M^k \omega \wedge \eta + (-1)^k \omega \wedge d_M^l \eta. \tag{10}$$

The exterior derivative is linear, and it maps $C^\infty \Lambda^k(\overline{M})$ into $C^\infty \Lambda^{k+1}(\overline{M})$. The $\wedge$-product is bilinear, and it maps $C^\infty \Lambda^k(\overline{M}) \times C^\infty \Lambda^l(\overline{M})$ into $C^\infty \Lambda^{k+l}(\overline{M})$.

Consider the case that $M$ is oriented and equipped with a Riemannian metric. This allows us to define the space $L^2 \Lambda^k(M)$ of $L^2$-integrable differential forms over $M$, and the $L^2$ scalar product $\langle \cdot, \cdot \rangle_{L^2 \Lambda^k(M)}$ of $k$-forms over $M$; see also [20, 26]. The Hodge star operator $\ast_M : L^2 \Lambda^k(M) \to L^2 \Lambda^{n-k}(M)$ is the unique mapping that satisfies

$$\langle \omega, \eta \rangle_{L^2 \Lambda^k(M)} = \int_M \omega \wedge \ast_M \eta, \quad \omega, \eta \in L^2 \Lambda^k(M). \tag{11}$$

For $\omega, \eta \in L^2 \Lambda^k(M)$ the Hodge star satisfies

$$\omega \wedge \ast_M \eta = \eta \wedge \ast_M \omega, \quad \ast_M \ast_M \omega = (-1)^{(m-k)} \omega. \tag{12}$$

The exterior codifferential is defined as

$$\delta_M^k : C^\infty \Lambda^k(M) \to C^\infty \Lambda^{k-1}(M), \quad \omega \mapsto (-1)^{m(k+1)+1} \ast_M d_M^{m-k} \ast_M \omega. \tag{13}$$

Note that $\delta_M^k = (-1)^k \ast_M^{-1} d_M^{m-k} \ast_M$. If $\omega \in L^2 \Lambda^k(M)$ and $\eta \in L^2 \Lambda^{k+1}(M)$, and at least one of these has support compact in $M$, then the integration by parts formula

$$\langle d_M^k \omega, \eta \rangle_{L^2 \Lambda^{k+1}(M)} = \langle \omega, \delta_M^{k+1} \eta \rangle_{L^2 \Lambda^k(M)} \tag{14}$$

holds.

Suppose that $T$ is a simplicial complex in $\mathbb{R}^n$. We recall that we assume all simplices to have a fixed orientation. Furthermore, we assume that $\mathbb{R}^n$ has a Riemannian metric, which induces a Riemannian structure on every $C \subseteq T$.

We consider trace operators between spaces of smooth differential forms. The rigorous definition of traces of differential forms outside the smooth setting is a non-trivial topic; see for example [12, 32, 59, 26]. For the purposes of this presentation, however, manifolds with corners [30, Chapter 10] provide the canonical formalization of the trace. For all $C \subseteq T$ and $F \subseteq C$ we have well-defined tangential trace operators,

$$t_C^k : C^\infty \Lambda^k(C) \to C^\infty \Lambda^k(F). \tag{15}$$

They can be defined via the pullback of the inclusion of manifolds with corners. Outside of traditional notation, we find use for normal trace operators:

$$n_C^k : C^\infty \Lambda^k(C) \to C^\infty \Lambda^{\dim F - \dim C + k}(F), \quad \omega \mapsto \ast_F^{-1} t_C^{\dim C - k} \ast_C \omega. \tag{16}$$
These operators satisfy
\begin{align}
tr^{k+1}_{C,F} \omega & = \delta^k_{C,F} \omega, \\
nm^{k-1}_{C,F} \phi & = (-1)^{\dim C - \dim F} \delta^k_{C,F} \phi
\end{align}
An important appearance of these operators is a variant of Stokes’ theorem,
\begin{equation}
\left< \delta^k \omega, \eta \right>_{L^2 \Lambda^k(C)} - \left< \omega, \delta^{k+1} \eta \right>_{L^2 \Lambda^{k+1}(C)} = \sum_{F \subseteq C} \left< \tr^k_{C,F} \omega, \nm^{k+1}_{C,F} \eta \right>_{L^2 \Lambda^k(F)},
\end{equation}
for \( \omega \in C^\infty \Lambda^k(C) \) and \( \eta \in C^\infty \Lambda^{k+1}(C) \). This generalizes the integration by parts formula \cite{14} above; see also of \cite{22}, Equation (0.2)].

2.3. Hilbert complexes. We review basic notions of Hilbert complexes \cite{11}, so that we can conduct homology theory within an \( L^2 \) setting. Our core results pertain to a class of merely finite-dimensional Hilbert complexes, for which the following definitions simplify considerably, but we frequently revisit an infinite-dimensional example application from partial differential equations on manifolds, which is described in the next subsection below.

A Hilbert complex is a sequence of real Hilbert spaces \( X^i \), typically indexed over non-negative integers, together with a sequence of closed densely-defined linear mappings \( d^i : \text{dom}(d^i) \subseteq X^i \to X^{i+1} \) satisfying \( \text{ran} d^{i-1} \subseteq \ker d^i \).

\[
\begin{array}{cccc}
0 & \longrightarrow & X^0 & \overset{d^0}{\longrightarrow} \ X^1 & \overset{d^1}{\longrightarrow} \ldots \\
\end{array}
\]

Then the adjoint operators \( d^*_i : \text{dom}(d^*_i) \subseteq X^{i+1} \to X^i \) are densely-defined and closed as well. They provide the adjoint complex.

\[
\begin{array}{cccc}
0 & \longleftarrow & X^0 & \overset{d^*_0}{\longleftarrow} \ X^1 & \overset{d^*_1}{\longleftarrow} \ldots \\
\end{array}
\]
We assume that the operators \( d^i \) have closed range. Then \( d^*_i \) has closed range, too. Under this condition, the space \( \mathbf{H}^i = \ker d^i \cap \ker d^*_{i-1} \), which we call i-th harmonic space, satisfies
\[
\mathbf{H}^i = \ker d^i \cap (\text{ran} d^{i-1})^\perp = \ker d^*_{i-1} \cap (\text{ran} d^i)^\perp.
\]
We have
\[
\text{ran} d^i = (\ker d^*_i)^\perp, \quad \text{ran} d^*_i = (\ker d^i)^\perp, \\
X^i = \text{ran} d^{i-1} \oplus \text{ran} d^*_i \oplus \mathbf{H}^i.
\]

The direct decomposition is orthogonal and known as the abstract Hodge decomposition.

2.4. The \( L^2 \) de Rham complex with partial boundary conditions. During the course of this contribution, we develop an example application as a sideline, where we demonstrate the general theory. In this subsection, we introduce this basic example: The \( L^2 \) de Rham complex with partial boundary conditions of a polyhedrally bounded domain. A similar example would be a polyhedral surface embedded in \( \mathbb{R}^n \). Notably our setting also includes polyhedrally bounded domains that are not Lipschitz domains, like the well-known “crossed bricks” polyhedron. Furthermore we remark that our results carry over to non-affine triangulations.

Suppose that \( \Omega \) is a polyhedrally bounded domain, and that \( \overline{\Omega} \) is triangulated by a simplicial complex \( T \). In that case \( \Omega \) is a smooth manifold embedded in \( \mathbb{R}^n \), and \( \overline{\Omega} \) is a topological manifold embedded in \( \mathbb{R}^n \). Furthermore we assume that \( \Omega \) has the Euclidean orientation, and that it inherits the Riemannian structure from the one previously introduced on \( \mathbb{R}^n \).

The tangential trace \( \tr^k_{C} : C^\infty \Lambda^k(\overline{\Omega}) \to C^\infty \Lambda^k(C) \) for \( C \in T \) is well-defined, and we define the normal trace operator as
\begin{equation}
\nm^{k}_{C} : C^\infty \Lambda^k(\overline{\Omega}) \to C^\infty \Lambda^{\dim C - n + k}(C), \quad \omega \mapsto \nm^{k-1}_{C} \tr^k_{C} \star \omega.
\end{equation}
for notational convenience in the sequel.

The $L^2$ de Rham complex on $\Omega$ is a prototypical example of a Hilbert complex. We consider the general case of partial boundary conditions on the background of [26], which is the theoretical setting for the Hodge Laplace equation with mixed boundary conditions.

In order to define partial boundary conditions, we first assume that $\partial \Omega$ is an $(n-1)$-dimensional Lipschitz manifold without boundary. Furthermore $\partial \Omega$ is assumed to be the essentially disjoint union of two $(n-1)$-dimensional Lipschitz manifolds with boundary: the tangential boundary $\Gamma_T$, and the normal boundary $\Gamma_N$. We refer to [26] Subsection 3.5] for the details.

Within the context of our example application, we furthermore assume that there exists a simplicial subcomplex $\mathcal{U}$ of $\mathcal{T}$ that triangulates the normal boundary $\Gamma_N$. Note that this implies that a simplicial subcomplex $\mathcal{V}$ of $\mathcal{T}$ triangulates the tangential boundary $\Gamma_T$.

We introduce spaces of smooth differential $k$-forms over $\Omega$ that satisfy either partial tangential boundary conditions or partial normal boundary conditions:

$$C^k_T \Lambda^k(\Omega) := \{ \omega \in C^\infty \Lambda^k(\Omega) \mid \forall F \in \mathcal{T}, F \subseteq \Gamma_T : \text{tr}_F \omega = 0 \},$$

$$C^\infty_N \Lambda^k(\Omega) := \{ \omega \in C^\infty \Lambda^k(\Omega) \mid \forall F \in \mathcal{T}, F \subseteq \Gamma_N : \text{mm}_F \omega = 0 \}.$$ 

Note that the integration by parts formula (14) holds when pairing these two spaces.

Now, we let $d^k_T$ be the $L^2 \Lambda^k$-closure of the unbounded operator

$$C^\infty_T \Lambda^k(\Omega) \subseteq L^2 \Lambda^k(\Omega) \rightarrow L^2 \Lambda^{k+1}(\Omega), \quad \omega \mapsto d^k_T \omega,$$

and let $\delta^k_N$ be the $L^2 \Lambda^k$-closure of the unbounded operator

$$C^\infty_N \Lambda^k(\Omega) \subseteq L^2 \Lambda^k(\Omega) \rightarrow L^2 \Lambda^{k-1}(\Omega), \quad \omega \mapsto \delta^k_N \omega.$$ 

We define $H_{T} \Lambda^k(\Omega)$ as the domain of $d^k_T$, and $H^{\infty}_{N} \Lambda^k(\Omega)$ as the domain of $\delta^k_N$, each equipped with the graph scalar product. The operators $d^k_T$ and $\delta^k_N$ are densely-defined and closed, and furthermore $d^k_T$ and $\delta^{k+1}_N$ are mutually adjoint. We have $d^k_T(H_{T} \Lambda^k) \subseteq H_{T} \Lambda^{k+1}$ and $\delta^{k}_N(H^{\infty}_{N} \Lambda^k) \subseteq H^{\infty}_{N} \Lambda^{k-1}$ [26] Theorem 4.3, Theorem 4.4. We conclude that we have mutually adjoint closed Hilbert complexes:

$$0 \longrightarrow H_{T} \Lambda^0 \subseteq L^2 \Lambda^0 \xrightarrow{d^0_T} \cdots \xrightarrow{d^{n-1}_T} H_{T} \Lambda^n \subseteq L^2 \Lambda^n \xrightarrow{0}$$

$$0 \longleftarrow H^{\infty}_{N} \Lambda^0 \subseteq L^2 \Lambda^0 \xleftarrow{\delta^0_N} \cdots \xleftarrow{\delta^{n-1}_N} H^{\infty}_{N} \Lambda^n \subseteq L^2 \Lambda^n \longleftarrow 0$$

The spaces of harmonic forms $\mathcal{H}^k(\Omega, \Gamma_T, \Gamma_N)$ of these two complexes,

$$\mathcal{H}^k(\Omega, \Gamma_T, \Gamma_N) = \ker d^k_T \cap \ker \delta^k_N,$$

are finite-dimensional and

$$b_k(\Omega, \Gamma_T) = \dim \mathcal{H}^k(\Omega, \Gamma_T, \Gamma_N) = b_{n-k}(\Omega, \Gamma_N)$$

holds [26] Theorem 5.3].

In order to motivate partial boundary conditions, we make explicit that the Hodge Laplacian associated to (21) allows us to formulate the Hodge Laplace equation with mixed boundary conditions: Given $f \in L^2 \Lambda^k(\Omega)$, the problem is to find $u \in L^2 \Lambda^k(\Omega)$ and $p \in \mathcal{H}^k(\Omega, \Gamma_T, \Gamma_N)$ such that

$$u \in \text{dom}(d^k_T) \cap \text{dom}(\delta^k_N), \quad \delta^k_N u \in \text{dom}(d^{k-1}_T), \quad d^k_T u \in \text{dom}(d^k_N),$$

$$(\delta^{k+1}_N d^k_T + d^{k-1}_T \delta^k_N) u + p = f, \quad u \perp \mathcal{H}^k(\Omega, \Gamma_T, \Gamma_N).$$
We refer to [26, Section 6] for a rigorous definition of the boundary conditions that \( u \) satisfies; we suggestively write them as
\[
\text{tr}_{\mathcal{T}}^{k-1} u = 0, \quad \text{tr}_{\mathcal{T}}^{n-k} \delta u = 0, \quad \text{tr}_{\mathcal{T}}^{k} \delta^k u = 0, \quad \text{tr}_{\mathcal{T}}^{n-k-1} \delta^k u = 0.
\]
The complexes of finite element differential forms in this contribution pertain to numerical methods for this problem, but we leave that topic for future research.

3. Discrete distributional differential forms

This section introduces spaces of discrete distributional differential forms and operators mapping between those spaces. We assume that \( \mathcal{T} \) is an \( n \)-dimensional simplicial complex and that \( \mathcal{U} \) is a subcomplex of \( \mathcal{T} \). We do not require a priori in this section that \( \mathcal{T} \) triangulates any manifold, although it does in our example application from Subsection 2.4. The subcomplex \( \mathcal{U} \) serves to formalize boundary conditions imposed on spaces of discrete distributional differential forms. However, it affects only a few definitions in this section.

We work with discrete de Rham complexes on simplices. Our assumptions are inspired from abstract frameworks in the theory of finite element differential forms [3, 14]. We assume that for each \( m \)-dimensional simplex \( C \in \mathcal{T} \) we have a discrete de Rham complex of finite-dimensional spaces of smooth differential forms.

\[
\begin{align*}
0 & \longrightarrow \Lambda^0(C) \xrightarrow{d^0} \Lambda^1(C) \xrightarrow{d^1} \cdots \xrightarrow{d^{m-1}} \Lambda^m(C) \longrightarrow 0.
\end{align*}
\]

Furthermore, we assume that the trace operators satisfy \( \text{tr}_{\mathcal{C},F}^{k} \Lambda^k(C) = \Lambda^k(F) \) for any \( F, C \in \mathcal{T} \) with \( F \preceq C \).

**Example 3.1.**

For our example application from Subsection 2.4 consider a finite element complex of Arnold-Falk-Winther-type on \( \mathbb{R}^n \), for example the complex of trimmed polynomial differential forms of degree \( r \).

\[
\begin{align*}
\ldots \xrightarrow{d^{k-1}} \mathcal{P}^{-} \Lambda^{k} (\mathbb{R}^n) \xrightarrow{d^{k}} \mathcal{P}^{-} \Lambda^{k+1} (\mathbb{R}^n) \xrightarrow{d^{k+1}} \ldots
\end{align*}
\]

We then set \( \Lambda^k(C) \) as the trace of \( \mathcal{P}^{-} \Lambda^{k} (\mathbb{R}^n) \) on the simplex \( C \in \mathcal{T} \). This results in discrete de Rham complexes on each simplex, and our assumptions hold in this setting. The other Arnold-Falk-Winther-type complexes can be treated analogously. We refer to [2, 3] for further background on this. A finite element complex of non-uniform polynomial degree is described in [11] and satisfies the above assumptions as well.

When \( C \in \mathcal{T}^m \) and \( \omega_C \in \Lambda^k(C) \), then we call \( \omega_C \) a discrete distributional differential form of degree \( k-n+m \). The following example motivates this terminology.

**Example 3.2.**

We continue with our example application from Subsection 2.4. We show that discrete distributional \( k \)-form is a functional that acts on a test space of smooth \( k \)-forms. Let \( C \in \mathcal{T}^m \) and \( \omega_C \in \Lambda^{k-n+m}(C) \). Then \( \omega_C \) can be interpreted as a linear functional on \( C_N^{\infty} \Lambda^k(\Omega) \) via the bilinear pairing

\[
\langle \omega_C, \phi \rangle := \int_C \omega_C \wedge \ast_C \text{nm}^k \phi, \quad \phi \in C_N^{\infty} \Lambda^k(\Omega),
\]

of \( \Lambda^{k-n+m}(\mathcal{T}^m) \) and \( C_N^{\infty} \Lambda^k(\Omega) \). The exterior derivative can now be applied on \( \Lambda^k(\mathcal{T}^m) \) in the sense of distributions. For \( \Phi \in C_N^{\infty} \Lambda^{k+1}(\Omega) \) we find

\[
(-1)^{n-m} \langle \omega_C, \delta_{\Omega}^{k+1} \Phi \rangle = (-1)^{n-m} \int_C \omega_C \wedge \ast_C \text{nm}^k \delta_{\Omega}^{k+1} \Phi = \int_C \omega_C \wedge \ast_C \delta_{\Omega}^{k+1} \text{nm}^{k+1} \Phi,
\]
chains. The point is that we have well-defined operators \( \Gamma \)

\[
\text{dimensional simplices. The families of operators }
\]

\[
\text{differential form consists of a piecewise exterior derivative, an d the sum of signed traces onto lower}
\]

\[
\text{In Example 3.2 we have seen that the distributional derivati ve of discrete distributional differ-
\]

\[
\text{correspond to these two terms. The discrete distributional exterior derivative on discrete}
\]

\[
\text{distributional } k \text{-forms is a sum of those operators. In order to find a suitable d omain and}
\]

\[
\text{we therefore assume that}
\]

\[
\text{We define the distributional exterior derivative as}
\]

\[
\langle d^k_\Omega \omega_C, \Phi \rangle := \langle \omega_C, \delta^{k+1}_\Omega \Phi \rangle, \quad \Phi \in C^\infty_N \Lambda^{k+1}(\Omega).
\]

\[
\text{We want to define the distributional exterior derivative on discrete distributional differential}
\]

\[
\text{forms without reference to a space of test functions, in purely discrete terms. It is helpful to}
\]

\[
\text{gather } k \text{-forms associated to } m \text{-simplices in the following definition:}
\]

\[
\Lambda^k_{-1}(T^m, \mathcal{U}) := \bigoplus_{C \in T^m \setminus \mathcal{U}^m} \Lambda^k(C).
\]

\[
\text{For } \omega \in \Lambda^k_{-1}(T^m, \mathcal{U}) \text{ then we write } \omega_C \text{ for its component in } \Omega^k(\mathbb{R}^n)
\]

\[
\text{The subcomplex } \mathcal{U} \text{ plays only a minor technical role in our computations. In order to simplify}
\]

\[
\text{we therefore assume that } \mathcal{U} \text{ is understood, and we merely write}
\]

\[
\Lambda^k_{-1}(T^m) = \Lambda^k_{-1}(T^m, \mathcal{U}).
\]

\[
\text{In Example 3.2 we have seen that the distributional derivative of discrete distributional differ-
\]

\[
\text{ent form consists of a piecewise exterior derivative, and the sum of signed traces onto lower}
\]

\[
\text{dimensional simplices. The families of operators } D^m_k \text{ and } T^m_k \text{, defined as}
\]

\[
D^m_k : \Lambda^k_{-1}(T^m) \longrightarrow \Lambda^{k+1}_{-1}(T^m), \quad D^m_k \omega := \sum_{C \in T^m \setminus \mathcal{U}^m} d^k_C \omega_C,
\]

\[
T^m_k : \Lambda^k_{-1}(T^m) \longrightarrow \Lambda^{k}_{-1}(T^{m-1}), \quad T^m_k \omega := \sum_{C \in T^m \setminus \mathcal{U}^m} \sum_{F \in \mathcal{U}^{m-1}} \sum_{F \in \mathcal{U}^{m-1}} o(F, C) \text{ tr}^k_{C, F} \omega_C,
\]

\[
\text{correspond to these two terms. The discrete distributional exterior derivative on discrete}
\]

\[
\text{distributional } k \text{-forms is a sum of those operators. In order to find a suitable domain and}
\]

\[
\Lambda^k_{-b}(T^m) := \bigoplus_{j=0}^{b-1} \Lambda^{k-j}_{-1}(T^{m-j}),
\]

\[
\Gamma^k_{-b}(T^m) := \bigoplus_{j=0}^{b-1} \Lambda^{k+j}_{-1}(T^{m+j}).
\]

\[
\text{Note that } \Gamma^k_{-1}(T^m) = \Lambda^k_{-1}(T^m). \text{ As we see below, the family } \Lambda^k_{-b}(T^m) \text{ generalizes the stan-
\]

\[
\text{differential operators}
\]

\[
d^k_{-b} : \Lambda^k_{-b}(T^m) \rightarrow \Lambda^{k+1}_{-b-1}(T^m),
\]

\[
d^k_{-b} : \Gamma^k_{-b}(T^m) \rightarrow \Gamma^{k-1}_{-b-1}(T^{m-1}),
\]
where $d^k$ is defined by
\begin{equation}
(32) \quad d^k \omega = (-1)^i D^2_{k-i} \omega - (-1)^i T^2_{k-i} \omega, \quad \omega \in \Lambda^k(T^m).
\end{equation}

The differential properties
\begin{equation}
(33) \quad D^{k+1}_m D^k_m = 0, \quad T^{k-1}_m T^k_m = 0, \quad d^{k+1} d^k = 0
\end{equation}
can be verified by direct computation.

The kernels of $D^k_m$ and $T^k_m$ are interesting in their own right. We define
\begin{align}
(34) \quad & \Lambda^k(T^m) := \{ \omega \in \Lambda^k_1(T^m) \mid T^m \omega = 0 \}, \\
(35) \quad & \Gamma^k(T^m) := \{ \omega \in \Lambda^k_1(T^m) \mid D^k_m \omega = 0 \}.
\end{align}

We have well-defined operators
\begin{equation}
(36) \quad d^{k+n-m} : \Lambda^k(T^m) \to \Lambda^{k+1}(T^m), \\
(37) \quad d^{k+n-m} : \Gamma^k(T^m) \to \Gamma^{k}(T^{m-1}).
\end{equation}

We sometimes write $\Lambda^k_0(T^m) = \Lambda^k(T^m)$ and $\Gamma^k_0(T^m) = \Gamma^k(T^m)$ in order to unify the notation.

**Example 3.3.**
We anticipate the meaning of $\Lambda^k(T^n)$ and $\Gamma^k(T^n)$ in our example application.

The standard finite element complexes with partial boundary conditions along $V$ correspond to the spaces $\Lambda^k(T, U)$ in our notation. To see this, note that the fact $T^m \omega = 0$ for $\omega \in \Lambda^k_1(T, U)$ and Euclidean orientation of each $n$-simplex imply that along interior faces the tangential traces of neighboring $n$-simplices coincide, while along faces of $V$ the tangential traces vanish. The space $\Gamma^0(T^n)$ appears in the context of finite volume methods [38], and the the distributional exterior derivative appears in the context of discontinuous Galerkin finite element methods [19].

The distributional exterior derivative generalizes the classical exterior derivative. Since the families of operators $D^k_m$, $T^k_m$, and $d^k$ satisfy the differential properties [39], we are motivated to consider differential complexes of discrete distributional differential forms. But differential complexes are defined in purely algebraic terms, whereas in a finite element setting, a scalar product usually is a relevant additional structure. In order to take this into account, we utilize the theory of Hilbert complexes. We therefore assume that the spaces $\Lambda^k_1(T^m)$ are Hilbert spaces. The choice of the scalar product determines the harmonic spaces of the Hilbert complexes, but that choice depends on the respective application. △

**Example 3.4.**
As a possible choice we consider the mesh-dependent scalar product
\begin{equation}
(38) \quad \langle \omega, \eta \rangle := \sum_{C \in \mathcal{T}^- \cup \mathcal{U}^m} h_C^{n-m} \langle \omega_C, \eta_C \rangle_{L^2 \Lambda^k(C)}, \quad \omega, \eta \in \Lambda^k_1(T^m).
\end{equation}

Here, $h_C$ denotes the diameter of $C$ if $\dim C \geq 1$, and, say, the average diameter of all adjacent edges if $C$ is a vertex. This scalar product appears in literature on residual error estimators [33] [34], but also in relation to discontinuous Galerkin methods [32 Equation (2.2)]. The dimension-dependent exponents of the weighting factors allow for scaling arguments. △

**Remark 3.5.**
Our notion of discrete distributional differential form is similar but different from the notion of currents [35] introduced by de Rham. Currents in the sense of de Rham over an $n$-dimensional manifold are functionals on smooth differential $k$-forms, and they generalize differential $(n-k)$-forms via the pairing of $(n-k)$-forms with $k$-forms. Thus their definition only involves the oriented smooth structure on the manifold, whereas our notion of discrete distributional
differential form is based on the $L^2$ pairing, which requires additionally a Riemannian structure on the manifold.

4. HORIZONTAL AND VERTICAL HOMOLOGY THEORY

In this section we analyze Hilbert complexes with the differential operators $D_k^n$ and $T_k^n$. Under conditions that hold in applications, we provide a new proof for the spaces of discrete harmonic $k$-forms of the standard finite element complex to have dimension equal to the $k$-th Betti numbers of the triangulation. Since we allow for partial boundary conditions, we also close a gap in the literature on finite element differential forms.

We introduce the horizontal and vertical Hilbert complexes: one with the differentials $D_k^n$ for $m$ fixed, and one with the differentials $T_k^n$ for $k$ fixed.

(39) \[
\begin{array}{ccccccccc}
0 & \rightarrow & \Lambda^0(T^m) & \xrightarrow{D^0_k} & \Lambda^1(T^m) & \xrightarrow{D^1_k} & \ldots & \xrightarrow{D^{m-1}_k} & \Lambda^m(T^m) & \rightarrow & 0,
\end{array}
\]

(40) \[
\begin{array}{ccccccccc}
0 & \rightarrow & \Lambda^k(T^n) & \xrightarrow{T_k^n} & \Lambda^k(T^{n-1}) & \xrightarrow{T_{k}^{n-1}} & \ldots & \xrightarrow{T_{k+1}^{n-k}} & \Lambda^k(T^k) & \rightarrow & 0.
\end{array}
\]

We write $\delta_H^k(T^m)$ for the harmonic spaces of the Hilbert complex (39), called horizontal harmonic spaces, and we write $\delta_V^k(T^n)$ for the harmonic spaces of the Hilbert complex (40), called vertical harmonic spaces.

$\delta_H^k(T^m) := \{ \omega \in \Lambda^k(T^m) \mid \omega \in \ker D_k^n, \omega \perp D_k^{n-1} \Lambda^k(T^m) \}$

$\delta_V^k(T^n) := \{ \omega \in \Lambda^k(T^n) \mid \omega \in \ker T_k^n, \omega \perp T_k^{n-1} \Lambda^k(T^{n-1}) \}$

Note that by definition $\delta_H^0(T^m) = \Gamma^0(T^m)$ and $\delta_V^k(T^n) = \Lambda^k(T^n)$. More information about these complexes is deduced under additional assumptions on the finite element spaces and the combinatorial properties of $\mathcal{T}$, to be described below.

First, we consider the horizontal harmonic spaces $\delta_H^k(T^m)$. We recall that the complex

(41) \[
\begin{array}{ccccccccc}
0 & \rightarrow & \Gamma^0(T^m) & \rightarrow & \Lambda^0(T^m) & \xrightarrow{D^0_k} & \ldots & \xrightarrow{D^{m-1}_k} & \Lambda^m(T^m) & \rightarrow & 0,
\end{array}
\]

is the direct sum of the simplex-wise complexes

(42) \[
\begin{array}{ccccccccc}
0 & \rightarrow & \ker d_C^0 & \rightarrow & \Lambda^0(C) & \xrightarrow{d_C^0} & \ldots & \xrightarrow{d_C^{m-1}} & \Lambda^m(C) & \rightarrow & 0
\end{array}
\]

over $m$-simplices $C \in \mathcal{T}^m$.

**Definition 4.1.**

We say that the local exactness condition holds if for each $C \in \mathcal{T}$, the sequence (42) is exact, and if furthermore $\ker d_C^0$ is spanned by the indicator function $1_C$ of $C$.

This condition implies that $\delta_H^k(T^m)$ is trivial for $k \geq 1$, and that $\delta_H^0(T^m)$ is spanned by the local indicator functions $1_C, C \in \mathcal{T}^m$. It is also another way of saying that (24) realizes the absolute cohomology on each cell. If the local exactness condition holds, then also $\delta_H^0(T^m) \simeq C_m(\mathcal{T}, \mathcal{U})$ by identifying each simplex with its local indicator function.

**Example 4.2.**

We continue our example application. The finite element complexes of finite element exterior calculus [2] satisfy the local exactness condition. The finite element complexes of non-uniform polynomial degree developed in [11] satisfy the local exactness condition as well.

Next, we study the vertical harmonic spaces $\delta_V^k(T^n)$. We want the sequence

(43) \[
\begin{array}{ccccccccc}
0 & \rightarrow & \Lambda^k(\mathcal{T}) & \rightarrow & \Lambda^k(T^m) & \xrightarrow{T_k^n} & \ldots & \xrightarrow{T_{k+1}^{n-k}} & \Lambda^k(T^k) & \rightarrow & 0
\end{array}
\]
to be exact. We show that this complex is the direct sum of complexes associated to local element patches, after assuming a condition on the spaces $\Lambda^k(C)$. Then we show exactness of those local sequences, which requires another condition on the triangulation. The exactness of (43) then follows.

In order to derive local vertical complexes associated to patches, we use the concept of geometric decomposition of finite element spaces. This is a basic concept of finite element theory, although our notation is slightly different.

For $F \in T_m$ let $\hat{\Lambda}^k(F)$ denote the subspace of $\Lambda^k(F)$ whose members have vanishing traces on the boundary simplices of $F$.

**Definition 4.3.** We say the geometric decomposition condition holds if we have linear extension operators

$$\text{ext}^k_{F,C} : \hat{\Lambda}^k(F) \rightarrow \Lambda^k(C),$$

for $F,C \in T$ with $F \subseteq C$, that satisfy

$$\begin{align*}
\text{tr}^k_{C,F} \text{ext}^k_{F,C} &= \text{Id}_{\hat{\Lambda}^k(F)}, \\
\text{tr}^k_{C,G} \text{ext}^k_{F,C} &= \text{ext}^k_{F,G}, \quad F \subseteq G \subseteq C, \\
\text{tr}^k_{C,G} \text{ext}^k_{F,C} &= 0, \quad F, G \subseteq C, F \nsubseteq G,
\end{align*}$$

for $F,G,C \in T$.

Under this conditions one can derive the eponymous geometric decomposition of the distributional finite element spaces:

$$\Lambda^k(T^m) = \bigoplus_{C \subseteq T^m \setminus \cup^m \forall \subseteq C} \bigoplus_{F \subseteq C} \text{ext}^k_{F,C} \hat{\Lambda}^k(F).$$

This follows, for example, from a careful reading of [3]. The authors make stronger assumptions in that publication, but their derivation of (44) only requires the conditions that we assume in this work. We refer to [18, Proposition 2.2] for a similar result.

**Example 4.4.** We continue our example application. The geometric decomposition condition holds for the ansatz spaces of finite element exterior calculus. See Theorem 4.3, Theorem 7.3 and Theorem 8.3 of [3]. It is also used implicitly for the complexes of non-uniform polynomial degree in [41], and in the exposition [20].

Next we introduce the local vertical complexes. They are assembled from the spaces

$$\Gamma^m_k(F) := \bigoplus_{C \subseteq T^m \setminus \cup^m \forall \subseteq C} \text{ext}^k_{F,C} \hat{\Lambda}^k(F), \quad F \in T,$$

and it is easy to verify:

**Lemma 4.5.** Assume the geometric decomposition condition holds. Then the complex

$$0 \rightarrow \Lambda^k(T) \rightarrow \Lambda^k(T^m) \rightarrow \ldots \rightarrow \Lambda^k_{k+1}(T^m) \rightarrow 0$$

is the direct sum of complexes

$$0 \rightarrow \Gamma^m_k(F) \cap \ker \nabla^m_k \rightarrow \Gamma^m_k(F) \rightarrow \ldots \rightarrow \Gamma^m_{k+1}(F) \rightarrow 0$$

over all $F \in T$. 

vanishing trace of the global finite element space $\Lambda M$.

Note that, if the local patch condition holds for an

Remark 4.8.

If $0 \leq d < \dim F$, we observe that $\mathcal{M}_F^d = \mathcal{N}_F^d$ by definition. So in these cases we have $C_d(M_F, N_F) = 0$, and accordingly $b_d(M_F, N_F) = 0$.

Now consider the case $d = \dim F$. If $F \notin U$, then $\mathcal{M}_F^{\dim F} \setminus \mathcal{N}_F^{\dim F} = \{ F \}$, and any simplex in $\mathcal{M}_F$ of dimension $\dim F + 1$ has $F$ as a face; if instead $F \in U$, then $C_{\dim F} (\mathcal{M}_F, N_F) = 0$. This means that (47) is always exact at $0 \leq d \leq \dim F$.

Lastly, we treat the cases $\dim F < d < n$. Note that $\mathcal{M}_F$ always triangulates a topological ball containing $F$. If $F$ is not a boundary simplex, then $\mathcal{N}_F$ is just the boundary of that ball, and the exactness of (47) at $\dim F < d < n$ follows from Example 2.3. Note that $b_n(M_F, N_F) = 1$ corresponds to the contribution of $\hat{\Lambda}^k(F)$ to the global finite element space $\hat{\Lambda}^k(T, U)$. If instead $F$ is a boundary simplex with $F \notin V$, then the same argument applies. If $F$ is a boundary simplex with $F \in V$, then $\mathcal{N}_F$ is a ball-shaped patch on the boundary of $\mathcal{M}_F$, and $b_d(M_F, N_F)$ for $\dim F < d \leq n$. Note that $b_n(M_F, N_F) = 0$ corresponds to the vanishing trace of the global finite element space $\hat{\Lambda}^k(T, U)$ on simplices of $V$ in that case. △

Remark 4.8.

Note that, if the local patch condition holds for an $n$-dimensional simplicial complex $T$ relative to an $(n - 1)$-dimensional subcomplex $U$, then it holds for $T^{[n-1]} \setminus U^{[n-2]}$ relative to $U^{[n-2]}$. The underlying idea of this condition is that $M_F$ triangulates the element patch around $F$, and that $N_F$ triangulates its boundary, with modifications at $U$. Our requirement on their combinatorial structure is:

**Definition 4.6.**

We say that $T$ satisfies the local patch condition relative to $U$, if for all $F \in T$ the complex

\[
\begin{array}{ccccccc}
0 & \longrightarrow & C_n(M_F, N_F) & \xrightarrow{\partial_n} & \ldots & \xrightarrow{\partial_1} & C_0(M_F, N_F) & \longrightarrow & 0
\end{array}
\]

has vanishing homology spaces at indices $n - 1, \ldots, 0$.

**Example 4.7.**

We continue with our example application, and show that the local patch condition holds. Recall that we assume the boundary $\Omega$ to be decomposed into a normal boundary part $\Gamma_N$, triangulated by $U$, and a tangential boundary part $\Gamma_T$, triangulated by $V$.

For $0 \leq d < \dim F$, we observe that $\mathcal{M}_F^d = \mathcal{N}_F^d$ by definition. So in these cases we have $C_d(M_F, N_F) = 0$, and accordingly $b_d(M_F, N_F) = 0$.

Now consider the case $d = \dim F$. If $F \notin U$, then $\mathcal{M}_F^{\dim F} \setminus \mathcal{N}_F^{\dim F} = \{ F \}$, and any simplex in $\mathcal{M}_F$ of dimension $\dim F + 1$ has $F$ as a face; if instead $F \in U$, then $C_{\dim F} (\mathcal{M}_F, N_F) = 0$. This means that (47) is always exact at $0 \leq d \leq \dim F$.

Lastly, we treat the cases $\dim F < d < n$. Note that $\mathcal{M}_F$ always triangulates a topological ball containing $F$. If $F$ is not a boundary simplex, then $\mathcal{N}_F$ is just the boundary of that ball, and the exactness of (47) at $\dim F < d < n$ follows from Example 2.3. Note that $b_n(M_F, N_F) = 1$ corresponds to the contribution of $\hat{\Lambda}^k(F)$ to the global finite element space $\hat{\Lambda}^k(T, U)$. If instead $F$ is a boundary simplex with $F \notin V$, then the same argument applies. If $F$ is a boundary simplex with $F \in V$, then $\mathcal{N}_F$ is a ball-shaped patch on the boundary of $\mathcal{M}_F$, and $b_d(M_F, N_F)$ for $\dim F < d \leq n$. Note that $b_n(M_F, N_F) = 0$ corresponds to the vanishing trace of the global finite element space $\hat{\Lambda}^k(T, U)$ on simplices of $V$ in that case. △
Lemma 4.9.

Assume the geometric decomposition condition and the local patch condition hold. Then the sequence

\[ 0 \longrightarrow \Gamma_k^n(C) \cap \ker T_k^n \longrightarrow \Gamma_k^n(C) \xrightarrow{T_k^n} \cdots \xrightarrow{T_k^{k+1}} \Gamma_k^k(C) \longrightarrow 0 \]

is exact for each \( C \in \mathcal{T} \).

**Proof.** Consider the differential complex

\[ \cdots \xrightarrow{\partial_{m+1} \otimes \text{Id}} C_m(\mathcal{M}_F, \mathcal{N}_F) \otimes \hat{A}^k(F) \xrightarrow{\partial_m \otimes \text{Id}} C_{m-1}(\mathcal{M}_F, \mathcal{N}_F) \otimes \hat{A}^k(F) \xrightarrow{\partial_{m-1} \otimes \text{Id}} \cdots \]

The homology space of (49) at \( C_m(\mathcal{M}_F, \mathcal{N}_F) \otimes \hat{A}^k(F) \) is isomorphic to \( H_m(\mathcal{M}_F, \mathcal{N}_F) \otimes \hat{A}^k(F) \), as follows, for example, by the universal coefficient theorem [31 Theorem 11.1].

It is easy to see that the linear mapping \( \Theta^m_k \) defined by

\[ \Theta^m_k : \Gamma_k^m(\mathcal{F}) \rightarrow C_l(\mathcal{M}_F, \mathcal{N}_F) \otimes \hat{A}^k(F), \quad \text{ext}_{F,C}^k \omega \mapsto C \otimes \omega \]

is bijective, and that

\[ \begin{array}{ccc}
\Gamma_k^m(C) & \xrightarrow{T_k^m} & \Gamma_k^m(C) \\
\Theta^m_k & \downarrow & \Theta^m_k \downarrow \\
C_m(\mathcal{M}_F, \mathcal{N}_F) \otimes \hat{A}^k(F) & \xrightarrow{\partial_m \otimes \text{Id}} & C_{m-1}(\mathcal{M}_F, \mathcal{N}_F) \otimes \hat{A}^k(F)
\end{array} \]

is a commuting diagram. So \( \Theta^m_k \) is an isomorphism of differential complexes from (48) to (49), and induces isomorphisms on homology. The desired result now follows by the local patch condition. \( \square \)

The main result of this section is based on the notion of double complexes in homological algebra. We consider the diagram

\[ \begin{array}{ccccccccc}
0 & \longrightarrow & \Lambda^0(\mathcal{T}^n) & \xrightarrow{D^0_0} & \Lambda^1(\mathcal{T}^n) & \xrightarrow{D^1_1} & \Lambda^2(\mathcal{T}^n) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Lambda^0(\mathcal{T}^{n-1}) & \xrightarrow{D^0_0} & \Lambda^1(\mathcal{T}^{n-1}) & \xrightarrow{D^1_1} & \Lambda^2(\mathcal{T}^{n-1}) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & & \cdots & & \cdots & & \cdots & & \\
\end{array} \]

where the left-most horizontal and the top-most vertical arrows denote the respective inclusion morphisms. Note that the choice of signs in (50) is motivated by (52). The identities

\[ D^m_{k+1} D^m_k = 0, \quad T^m_{k-1} T^m_k = 0, \quad T^m_k D^m_k - D^m_{k-1} T^m_k = 0, \]

which have been observed before, imply that (50) constitutes a double complex in the sense of [25 Chapter 1, §3.5]. The previous results on the homology of the horizontal and vertical complexes allow us to relate the harmonic spaces of two further families of differential complexes:

\[ \begin{array}{cccc}
0 & \longrightarrow & \Lambda^0(\mathcal{T}^m) & \xrightarrow{D^m_0} & \cdots & \xrightarrow{D^m_{m-1}} & \Lambda^m(\mathcal{T}^m) & \longrightarrow & 0, \\
\downarrow & & & & & & & & \\
0 & \longrightarrow & \Lambda^k(\mathcal{T}^m) & \xrightarrow{T^m_k} & \cdots & \xrightarrow{T^m_{k+1}} & \Lambda^k(\mathcal{T}^k) & \longrightarrow & 0.
\end{array} \]

The complex (51) resembles the standard finite element complex, and for \( m = n \), it coincides with the standard finite element complex of finite element exterior calculus in our example.
application. Similarly, the complex \((52)\) resembles the chain complex of \(T\) relative to \(U\), and for \(k = 0\), those two complexes coincide, up to a sign convention, in our example application.

We denote the harmonic spaces of the Hilbert complex \((51)\) by \(\mathcal{H}^k(T^n)\), and the harmonic spaces of the Hilbert complex \((52)\) by \(\mathcal{C}^k(T^n)\).

\[
\mathcal{H}^k(T^n) := \{ \omega \in \Lambda^k(T^n) \mid \omega \in \ker D_k, \omega \perp D_{k-1}^m \Lambda^k(T^n) \}
\]

\[
\mathcal{C}^k(T^n) := \{ \omega \in \Gamma^k(T^n) \mid \omega \in \ker T_k, \omega \perp T_{k+1}^m \Gamma^k(T^{m+1}) \}
\]

**Theorem 4.10.**

Suppose that the rows and columns of the double complex \((50)\) are exact sequences, with the possible exception of the top-most row and the left-most column. Then \(\mathcal{H}^{n-k}(T, U) \cong \mathcal{C}^0(T^{n-k}) \cong \mathcal{H}^k(T^n)\).

**Proof.** This is a standard result in homological algebra; see for example Proposition 3.11 of [33], Chapter 9.2 of [7] or Corollary 6.4 of [6].

**Corollary 4.11.**

Suppose that the local exactness condition, the geometric decomposition condition, and the local patch condition hold. Then we have isomorphisms between harmonic spaces:

\(\mathcal{H}_{n-k}(T, U) \cong \mathcal{E}^0(T^{n-k}) \cong \mathcal{H}^k(T^n)\).

**Example 4.12.**

We continue our example application. The complex

\[
\begin{array}{cccccc}
0 & \longrightarrow & \Lambda^0(T^n) & \xrightarrow{d^0} & \ldots & \Lambda^n(T^n) & \longrightarrow 0,
\end{array}
\]

is a complex of finite element differential forms whose traces on simplices of \(V\) vanish. This means that it is a conforming discretization of the \(L^2\) de Rham complex with partial tangential boundary conditions along \(\Gamma_T\). The discrete harmonic forms of the finite element complex are \(\mathcal{H}^k(T^n)\). We have

\[
\dim \mathcal{H}^k(T) = b_{n-k}(\Omega, \Gamma_N) = \dim \mathcal{H}^{n-k}(\Omega, \Gamma_N, \Gamma_T) = \dim \mathcal{H}^k(\Omega, \Gamma_T, \Gamma_N) = b_k(\Omega, \Gamma_T)
\]

This includes the special cases \(\Gamma_T = \emptyset\) and \(\Gamma_T = \Gamma\), which have been treated earlier in the literature [15].

**Remark 4.13.**

Arnold, Falk and Winther have derived the discrete de Rham cohomology without boundary conditions from the \(L^2\) de Rham complex [2]; Christiansen and Winther have extended this to the case of essential boundary conditions [13]. Christiansen has also derived the discrete de Rham cohomology without boundary conditions within the framework of element systems via de Rham mappings [14]; the approach can easily be extended to general partial boundary conditions. With different techniques, we have derived the discrete de Rham cohomology without reference to the \(L^2\) de Rham complex.

## 5. Harmonic Forms of Discrete Distributional de Rham Complexes

In this section we introduce discrete distributional de Rham complexes and construct isomorphisms between their harmonic spaces. Our main result, Theorem 5.11 below, generalizes Corollary 4.11 of the previous section. In particular, we construct an isomorphism between \(\mathcal{H}_{n-k}(T, U)\) and \(\mathcal{H}^k(T^n)\).

For our definitions, we continue to assume that \(T\) is an \(n\)-dimensional simplicial complex, and that \(U\) is a simplicial subcomplex.
Consider again the complex
\begin{equation}
0 \longrightarrow \Lambda^0(T^n) \xrightarrow{d^0} \ldots \xrightarrow{d^{n-1}} \Lambda^n(T^n) \longrightarrow 0,
\end{equation}
which resembles the standard finite element complex. This complex might be “redirected” at any index \(k\), in the sense that we replace \(\Lambda^k(T^n)\) with \(\Lambda^k_{n-1}(T^n)\), and continue the complex with the spaces \(\Lambda^{k+1}_{n-2}(T^n), \Lambda^{k+2}_{n-3}(T^n)\), and so forth, with the exterior derivative applied in the distributional sense.

\begin{equation}
\ldots \xrightarrow{d^{k-2}} \Lambda^{k-1}(T^n) \xrightarrow{d^{k-1}} \Lambda^k_{n-1}(T^n) \xrightarrow{d^k} \Lambda^{k+1}_{n-2}(T^n) \xrightarrow{d^{k+1}} \ldots
\end{equation}

We see that the original complex is already trivially redirected at the 0-forms, noting \(\Lambda^n(T^n) = \Lambda^n_{n-1}(T^n)\). This is a subcomplex of the complex redirected at the \((n-1)\)-forms. We proceed in this manner, until we eventually have a “maximal” complex that is redirected already at the 0-forms at the very beginning. We observe a sequence of complexes, from the original complex, noting \(\Lambda^n(T^n) = \Lambda^n_{n-1}(T^n)\), to a “maximal” complex:

\begin{equation}
0 \longrightarrow \Lambda^0_{n-1}(T^n) \xrightarrow{d^0} \ldots \xrightarrow{d^{n-1}} \Lambda^n_{n-1}(T^n) \longrightarrow 0.
\end{equation}
The same arguments can be applied analogously to the complex

\begin{equation}
0 \longrightarrow \Gamma^0(T^n) \xrightarrow{d^0} \ldots \xrightarrow{d^{n-1}} \Gamma^0(T^0) \longrightarrow 0.
\end{equation}

At any index \(m\), it can be redirected,

\begin{equation}
\ldots \xrightarrow{d^{k-2}} \Gamma^0_{n-k+1}(T^n) \xrightarrow{d^{k-1}} \Gamma^0_{n-k}(T^n) \xrightarrow{d^k} \Gamma^0_{n-k-1}(T^n) \xrightarrow{d^{k+1}} \ldots
\end{equation}

and we obtain again a sequence of complexes. The maximal example of these complexes,

\begin{equation}
0 \longrightarrow \Gamma^0_{n-1}(T^n) \xrightarrow{d^0} \ldots \xrightarrow{d^{n-1}} \Gamma^0_{n-1}(T^0) \longrightarrow 0,
\end{equation}
is, in fact, identical to (56). Our goal is to construct isomorphisms between the harmonic spaces of all these complexes. There are also complexes that do not start with 0-simplices, and we analyze their harmonic spaces as well:

\begin{equation}
\ldots \xrightarrow{d^{k+n-m-2}} \Lambda^{k-1}(T^m) \xrightarrow{d^{k+n-m-1}} \Lambda^k(T^m) \xrightarrow{d^{k+n-m}} \Lambda^{k+1}(T^m) \xrightarrow{d^{k+n-m+1}} \ldots
\end{equation}

\begin{equation}
\ldots \xrightarrow{d^{k+n-m-2}} \Gamma^k(T^{m+1}) \xrightarrow{d^{k+n-m-1}} \Gamma^k(T^m) \xrightarrow{d^{k+n-m}} \Gamma^k(T^{m-1}) \xrightarrow{d^{k+n-m+1}} \ldots
\end{equation}

We denote the harmonic spaces of the complexes of distributional differential forms by

\(\mathcal{S}^k_b(T^m) := \{\omega \in \Lambda^k_b(T^m) \mid d^{k+n-m} b \omega = 0, \quad \omega \perp d^{k+n-m-1} b \Lambda^k_{b+1}(T^m)\}\),

\(\mathcal{C}^k_b(T^m) := \{\omega \in \Gamma^k_b(T^m) \mid d^{k+n-m} b \omega = 0, \quad \omega \perp d^{k+n-m-1} b \Gamma^k_{b+1}(T^{m+1})\}\).

We sometimes write \(\mathcal{S}^k(T^m) = \mathcal{S}^k_0(T^m)\) and \(\mathcal{C}^k(T^m) = \mathcal{C}^k_0(T^m)\) for notational reasons. We call the elements of these spaces \emph{discrete distributional harmonic forms}.

**Remark 5.1.**
The results in this section generalize ideas of [9], in particular the proofs of their Lemma 3, Theorem 5 and Theorem 7. But these complexes can also be interpreted within the theory of double complexes. We identify the maximal complex (56) as the total complex of the double complex (50), skipping the left-most column and the top-most-row of that diagram. The two sequences of broken complexes, (55) and (58), exemplify the two canonical filtrations of the total complex. We refer to [7] for more background on this perspective. Although the underlying ideas are similar, our presentation is specifically tailored towards finite element analysis and addresses the harmonic spaces of the broken complexes explicitly.

The above sequences can be defined in the setting of Section 3 but in order to derive the desired isomorphisms between the harmonic spaces, we utilize the additional assumptions of Section 4. This means that in the sequel, we assume that the local exactness condition, the
geometric decomposition condition, and the local patch condition of $\mathcal{T}$ relative to $\mathcal{U}$ hold.

For the construction of the isomorphisms we assume that we have chosen right-inverses of the operators $D_k^m$ and $T_k^m$. This means that we choose operators

$$E_k^m : \Lambda_{k-1}(T^{m-1}) \to \Lambda_{k-1}(T^m), \quad P_k^m : \Lambda_{k+1}^\omega(T^m) \to \Lambda_k^\omega(T^m).$$

that satisfy $T_k^m = T_k^m E_k^m T_k^m$ and $D_k^m = D_k^m P_k^m D_k^m$. The Moore-Penrose pseudoinverses of $D_k^m$ and $T_k^m$ are a possible choice. It is notationally convenient to introduce the following operators as well. We consider, on the one hand,

$$R_{k,m} : \Lambda_{k-1}(T^{m-1}) \to \Lambda_k^\omega(T^m), \quad \omega \mapsto D_{k-1}^m E_{k-1}^m \omega,$$

and on the other hand,

$$S_{k,m} : \Gamma_{k-1}^{m+1}(T^{m+1}) \to \Gamma_k^0(T^m), \quad \omega \mapsto T_k^{m+1} P_k^{m+1} \omega,$$

$$S_{n,b} : \Gamma_{n,b}^0(T^n) \to \Gamma_{n,b}^0(T^n), \quad \omega \mapsto \omega + (-1)^{b-1} d_{n-m}^{n-m-1} P_{b-2}^{m+b-1} \omega.$$

All the following statements appear in pairs. For each result on the broken complexes generalizing finite element complexes [55], there is an analogous result on the broken complexes generalizing simplicial chain complexes [58]. Each time we only give the proof for the first of those statements, since the proof of the other statement is analogous.

Our first observation is, loosely speaking, that the images of discrete distributional differential forms under the discrete distributional exterior derivative always have preimages that are “more regular” than those images. The idea follows the following intuition: Suppose that $\omega \in \Lambda^k_\omega(T^n)$ with $d^k \omega \in \Lambda^{k+1}_\omega(T^n)$. Then $\omega \in \Lambda^k_\omega(T^n)$ by definition, so $d^k \omega$ even has a preimage in $\Lambda^k_\omega(T^n)$. Completely analogously, suppose that $\omega \in \Gamma_0_{k-1}(T^m)$ with $d_{n-m} \omega \in \Gamma_0_{k-1}(T^{m-1})$. Then $\omega \in \Gamma_k(T^m)$ by definition.

More generally, a discrete distributional differential forms in $\Lambda^k_{b} (T^n)$ that has a preimage under $d^k$ in $\Lambda_{b-1}^k (T^n)$ already has a preimage in $\Lambda_{b-1}^k (T^n)$. But in the general case the construction is more complicated and involves the operators $R_{k,b}$ and $S_{m,b}$, which, in this sense, can be seen as regularizers of preimages. The following lemmas generalize these observations.

**Lemma 5.2.**

Suppose that $b > 1$, and that $\omega \in \Lambda^k_{b} (T^n)$ with $d^k \omega \in \Lambda_{b-1}^{k+1} (T^n)$. Then $d^k R_{k,b} \omega = d^k \omega$ and $R_{k,b} \omega \in \Lambda^k_{b-1} (T^n)$.

**Proof.** Let $\omega = \omega^0 + \cdots + \omega^{b-1} \in \Lambda^k_{b-1} (T^n)$ with $\omega^j \in \Lambda_{k-1}^{j-1} (T^{n-j})$. From $d^k \omega \in \Lambda_k^{k+1} (T^n)$ we see $T_{k-b} \omega = 0$. So $T_{k-b} E_{k-b}^{n-b-1} \omega^{b-1} = 0$. We conclude that $R_{k,b} \omega = \omega^0 + \cdots + \omega^{b-1} = (-1)^{b-1} d_{b-1} E_{k-b}^{n-b} \omega^{b-1}$, so $R_{k,b} \omega \in \Lambda^k_{b-1} (T^n)$. Furthermore,

$$d^k R_{k,b} \omega = d^k \omega - (-1)^{b-1} d^k E_{k-b+1}^{n-b} \omega^{b-1} = d^k \omega.$$  

This completes the proof. \qed

**Lemma 5.3.**

Suppose that $b > 1$, and that $\omega \in \Gamma^0_{b} (T^n)$ with $d_{n-m} \omega \in \Gamma_{-b}^{n-m} (T^{m-1})$. Then $d_{n-m} S_{m,b} \omega = d_{n-m} \omega$ and $S_{m,b} \omega \in \Gamma^0_{b-1} (T^m)$.
Another auxiliary result restricts the class of discrete distributional differential forms that are candidates for being discrete distributional harmonic forms. The result implies that an element of $\mathcal{H}_k^{b}(T^n)$ is not contained in $\mathcal{H}_k^{-b+1}(T^n)$.

**Lemma 5.4.**
Suppose that $\omega \in \Lambda_k^{b-1}(T^n)$ with $d^k\omega = 0$. Then $\omega$ is not orthogonal to $d^{k-1}\Lambda_k^{b-1}(T^n)$.

**Proof.** Suppose that $\omega = \omega^0 + \cdots + \omega^{b-2} \in \Lambda_k^{b-2}(T^n)$ with $\omega^i \in \Lambda^{b-1}(T^{n-i})$ and $d^i\omega = 0$. We then set $\xi^0 := P_{k-1}^n\omega^0$, and define recursively
\[
\xi^i := (-1)^i P_{k-1}^{n-i} \left( \omega^i - (-1)^{i+1} T_k^{n-i-1} \xi^{i-1} \right), \quad 1 \leq i \leq b-2.
\]
We clearly have $d^{k-1}\xi^0 = \omega^0 - T_k^{n-k}\xi^0$, since $D_k^n\omega^0 = 0$. Now assume that we have
\[
d^{k-1}(\xi^0 + \cdots + \xi^j) = \omega^0 + \cdots + \omega^j + (-1)^{j+1} T_k^{n-j-1} \xi^{j+1}
\]
for $j < b-2$. This implies that
\[
d^{k-1}(\xi^0 + \cdots + \xi^j) = \omega^j - (-1)^{j+1} T_k^{n-j-1} \xi^{j+1}
\]
So we find that
\[
(-1)^{j+1} D_{k-j-1}^{n-j-1} \left( \omega^j - (-1)^{j+1} T_k^{n-j-1} \xi^{j+1} \right)
\]
\[
= (-1)^{j+1} D_{k-j-1}^{n-j-1} \omega^j + D_k^{n-j-1} T_k^{n-j-1} \xi^{j+1}
\]
\[
= (-1)^{j+1} D_{k-j-1}^{n-j-1} \omega^j - T_k^{n-j-1} D_k^{n-j-1} \xi^{j+1}
\]
\[
= (-1)^{j+1} D_{k-j-1}^{n-j-1} \omega^j + (-1)^{j+1} T_k^{n-j-1} \xi^{j+1} + T_k^{n-j-1} T_k^{n-j} \xi^{j+1}
\]
\[
= (-1)^{j+1} D_{k-j-1}^{n-j-1} \omega^j + (-1)^{j+1} T_k^{n-j-1} \omega^j = 0
\]
This implies
\[
d^{k-1}(\xi^0 + \cdots + \xi^j + \xi^{j+1})
\]
\[
= \omega^0 + \cdots + \omega^j + (-1)^{j+1} T_k^{n-j-1} \xi^{j+1} + \omega^0 + (-1)^{j+1} D_k^{n-j-1} \xi^{j+1} + (-1)^{j+1} T_k^{n-j-1} \xi^{j+1}
\]
\[
= \omega^0 + \cdots + \omega^j + (-1)^{j+1} T_k^{n-j-1} \xi^{j+1} + \omega^j + (-1)^{j+1} T_k^{n-j-1} \xi^{j+1} + (-1)^{j+1} D_k^{n-j-1} \xi^{j+1} + (-1)^{j+1} T_k^{n-j-1} \xi^{j+1}
\]
\[
= \omega^0 + \cdots + \omega^j + \omega^j + (-1)^{j+1} T_k^{n-j-1} \xi^{j+1} = (-1)^{j+1} T_k^{n-j-1} \xi^{j+1}
\]
So eventually we see that
\[
d^{k-1}(\xi^0 + \cdots + \xi^{b-2}) = \omega^0 + \cdots + \omega^{b-2} + (-1)^{b-2} + (-1)^{b-2} + T_k^{n-b+2} \xi^{b-2}.
\]
This leads to
\[
\langle d^{k-1}(\xi^0 + \cdots + \xi^{b-2}), \omega \rangle = \|\omega\|^2,
\]
which proves the claim. \qed

**Lemma 5.5.**
Suppose that $\omega \in \Gamma_{b+1}^{n-k-1}(T^m)$ with $d^{n-m}\omega = 0$. Then $\omega$ is not orthogonal to $d^{n-m} \Gamma_{b+1}^{n-k-1}(T^{m+1})$.

Our goal is to construct the discrete distributional harmonic forms of the distributional complexes, and to find isomorphisms between the harmonic spaces. To begin with, the harmonic forms of the spaces $\Lambda_k^b(T^n)$ and $\Gamma_{b+1}^{n-k-1}(T^n)$ are easily described:

**Lemma 5.6.**
We have $\mathcal{H}_k^{b}(T^n) = \mathcal{H}_k^{b-1}(T^n)$.

**Proof.** We know that $\omega \in \Lambda_k^b(T^n)$ with $d^k\omega = 0$, is equivalent to $\omega \in \Lambda_k^b(T^n)$ with $d^k\omega = 0$. The equality $\mathcal{H}_k^{b}(T^n) = \mathcal{H}_k^{b-1}(T^n)$ now follows from definitions. \qed
Lemma 5.7.  
We have $C^0(T^n) = C^0_1(T^n)$.  

The harmonic spaces $\tilde{H}^k_{b}(T^n)$ and $C^0_{b}(T^n)$ for $b \geq 2$ are constructed in a recursive manner.

Lemma 5.8.  
Suppose that $b \geq 2$, and let $P_{\ker d^k}$ denote the orthogonal projection onto the kernel of the operator $d^k : \Lambda^k_{b}(T^n) \rightarrow \Lambda^{k+1}_{b+1}(T^n)$. Then the operator $P_{\ker d^k} R^*_{k,b}$ acts as an isomorphism from $\tilde{H}^k_{b+1}(T^n)$ to $\tilde{H}^k_{b}(T^n)$.

Proof. Let $\omega = \omega^0 + \cdots + \omega^{b-1} \in \Lambda^k_{b}(T^n)$ with $\omega^j \in \Lambda^{k-j}_{-j}(T^{n-j})$. We have by construction that $\omega - R_{k,b}\omega \in d^{k-1}\Lambda^{k-1}_{b+1}(T^n)$. This implies in particular that

$$d^k\omega = 0 \iff d^k R_{k,b}\omega = 0,$$

$$\omega \in d^{k-1}\Lambda^{k-1}_{b+1}(T^n) \iff R_{k,b}\omega \in d^{k-1}\Lambda^{k-1}_{b+2}(T^n).$$

From the last equivalence and the abstract Hodge decomposition, we conclude that

$$d^k\omega = 0, \ \omega \perp \tilde{H}^k_{b}(T^n) \iff d^k R_{k,b}\omega = 0, \ R_{k,b}\omega \perp \tilde{H}^k_{b}(T^n).$$

Now $d^k R_{k,b}\omega = 0$ implies that $R_{k,b}\omega \in \Lambda^k_{b+1}(T^n)$ by Lemma 5.2. So

$$R_{k,b}\omega \in d^{k-1}\Lambda^{k-1}_{b+1}(T^n) \iff R_{k,b}\omega \in d^{k-1}\Lambda^{k-1}_{b+2}(T^n)$$

by Lemma 5.2 again. We derive from this that

$$R_{k,b}\omega \in d^{k-1}\Lambda^{k-1}_{b+2}(T^n) \iff d^k R_{k,b}\omega = 0, \ R_{k,b}\omega \perp \tilde{H}^k_{b+1}(T^n) \iff d^k\omega = 0, \ \omega \perp R_{k,b}^* \tilde{H}^k_{b+1}(T^n).$$

We conclude that the projection of $R^*_{k,b}\tilde{H}^k_{b+1}(T^n)$ onto $\ker d^k$ equals $\tilde{H}^k_{b}(T^n)$. Furthermore, we observe for $p \in \tilde{H}^k_{b+1}(T^n)$ that

$$\langle p, P_{\ker d^k} R^*_{k,b}p \rangle = \langle p, R^*_{k,b}p \rangle = \langle p, p \rangle.$$  

This is a consequence of Lemma 5.3. We conclude that $P_{\ker d^k} R^*_{k,b}$ defines an isomorphism from $\tilde{H}^k_{b+1}(T^n)$ onto $\tilde{H}^k_{b}(T^n)$. $\square$

Lemma 5.9.  
Suppose that $b \geq 2$, and let $P_{\ker d^{n-m}}$ denote the orthogonal projection onto the kernel of the operator $d^{n-m} : \Gamma^{0}_{m}(T^{m-1}) \rightarrow \Gamma^{0}_{b+1}(T^{m-1})$. Then the operator $P_{\ker d^{n-m}} S_{m,b}$ acts as an isomorphism from $C^0_{b+1}(T^m)$ to $C^0_{b}(T^m)$.

Remark 5.10.  
If $p \in \tilde{H}^k_{b}(T^n)$, then generally $R^*_{k,b+1} p \notin \tilde{H}^k_{b}(T^n)$. This is unsatisfying, because the $R_{k,b}$ are local operations, whereas projections appear not to have an explicit description. If the projections are left out, then recursive application of the $R_{k,b}$ yields a completely local construction, but in general it does not provide the discrete distributional harmonic forms. Instead, the construction can be interpreted as a criterion to decide whether a discrete distributional differential form in $\Lambda^k_{b}(T^n)$ with vanishing distributional exterior derivative lies in $d^{k-1}\Lambda^{k-1}_{b+1}(T^n)$. However, we have $d^k\Lambda^n_{b}(T^n) = 0$, so the projection is redundant in the special case $k = n$. The local construction thus delivers the discrete distributional harmonic $n$-forms explicitly. An analogous statement holds for the operators $S_{m,b}$.  

The main result of this contribution is now evident from Lemmas 5.6 and 5.7, and the repeated application of Lemmas 5.8 and 5.9. It generalizes of Corollary 4.11.
Theorem 5.11.
Under the assumptions of this section, we have isomorphisms between harmonic spaces:
\[
\mathcal{C}^0(T^{n-k}) = \mathcal{C}^0_{-1}(T^{n-k}) \simeq \cdots \simeq \mathcal{C}^0_{k-1}(T^{n-k})
\]
\[
= \mathcal{F}_{-k-1}(T^n) \simeq \cdots \simeq \mathcal{F}_{k-1}(T^n) = \mathcal{F}_k(T^n).
\]
Furthermore, \(\mathcal{C}^0(T^{n-k}) \simeq \mathcal{H}_{n-k}(T, \mathcal{U})\) holds. \(\square\)

We have studied the harmonic spaces \(\mathcal{F}_{n-k}(T^n)\) and \(\mathcal{C}^0_{n-k}(T^{n-k})\). The harmonic spaces \(\mathcal{F}_{n-k}(T^{n-1})\) can be studied in a similar manner. One merely replaces \(T^n\) by \(T^n \setminus \mathcal{U}^{n-1}\) and \(\mathcal{U}\) by \(\mathcal{U}^{n-2}\), and uses the arguments of the previous and this section. Note that the local patch condition is then satisfied; see Remark 4.8 above. Repeating this idea for \(0 \leq m < n\), we eventually obtain isomorphisms
\[
\mathcal{C}^0(T^{m-k}) = \mathcal{C}^0_{-1}(T^{m-k}) \simeq \cdots \simeq \mathcal{C}^0_{-k-1}(T^{m-k})
\]
\[
= \mathcal{F}_{k-1}(T^m) \simeq \cdots \simeq \mathcal{F}_{k-1}(T^m) = \mathcal{F}_k(T^m), \quad k > 0.
\]
Furthermore, the harmonic space in the 0-forms is the orthogonal sum of \(\mathcal{C}^0(T^m)\) and \(T^m_{k+1} \mathcal{F}^0(T^m_{k+1})\). While this is sufficient to determine the dimensions of the distributional harmonic forms of \(\mathcal{C}^0(T^m)\), the following result is of ancillary interest:

Lemma 5.12.
The orthogonal projection from \(\Lambda^k_{n-k}(T^n)\) onto the kernel of \(d^k : \Lambda^{k+1}(T^{n-1}) \rightarrow \Lambda^k(T^{n-1})\) induces an isomorphism from \(\mathcal{F}^k_{n-k}(T^n)\) to \(\mathcal{F}^k_{k+1}(T^{n-1})\) for \(k \geq 2\).

Proof. Suppose that \(\omega \in d^{k+1} \Lambda^k(T^{n-1})\), with \(\xi \in \Lambda^k(T^{n-1})\) such that \(d^{k+1} \xi = \omega\). Then \(R_{k+1,2} \xi \in \Lambda^k_{n-k+1}(T^n)\) with \(d^{k+1} R_{k+1,2} \xi = \omega\). We conclude that
\[
d^{k+1} \omega = 0, \quad \omega \perp \mathcal{F}_k(T^{n-1})
\]
\[
\iff \omega \in d^{k+1} \Lambda^k(T^{n-1})
\]
\[
\iff \omega \in d^{k+1} \Lambda^k_{n-k+1}(T^n)
\]
\[
\iff d^{k+1} \omega = 0, \quad \omega \perp \mathcal{F}_k(T^n)
\]
Thus we see that the orthogonal projection of \(\mathcal{F}^{k+1}_{n-k+2}(T^n)\) onto \(\Lambda^k_{n-k+1}(T^{n-1})\) yields \(\mathcal{F}_k(T^{n-1})\). Furthermore, from Lemma 5.4 we conclude that this mapping is not only onto, but also one-to-one. This describes the spaces \(\mathcal{F}^{k+1}_{n-k+2}(T^n)\) for \(k \geq 2\). \(\square\)

To study the harmonic spaces the complexes \(\mathcal{C}^k(T^m)\), one does not consider the lower dimensional skeletons \(T^m\). Instead, one leaves out the spaces \(\Lambda^k(T^m)\) with \(k\) below a certain degree. We do not describe this in detail, since the differences are mostly notational; it is possible to show that
\[
\mathcal{C}^k(T^m) = \mathcal{C}^k_{-1}(T^m) \simeq \cdots \simeq \mathcal{C}^k_{k+1-n(m-1)}(T^m)
\]
\[
= \mathcal{F}_{-k+n-m}(T^n) \simeq \cdots \simeq \mathcal{F}_{k+1-n(m-1)}(T^n) = \mathcal{F}_{k+1-n(m-1)}(T^n), \quad m < n.
\]
The harmonic space in the \(k\)-forms is the orthogonal sum of \(\mathcal{F}_k(T^n)\) and \(D^0_{n-k} \Lambda^k(T^n)\).

6. Conclusions

Complexes of discrete distributional differential forms have been introduced into finite element exterior calculus. We have analyzed their homology theory in this contribution. We will analyze Poincaré-Friedrichs inequalities in a subsequent contribution.

Applications in a posteriori error estimation motivate this research, but distributional finite elements appear in other facets of computational partial differential equations as well. Most prominently, these include non-conforming methods like discontinuous Galerkin finite element
methods and finite volume methods. Furthermore, a distributional elasticity complex in three dimensions appears in the context of Regge calculus [17]. Our example application concerns the $L^2$ de Rham complex on a triangulated manifold, but the discrete theory applies to a larger class of triangulations. For example, such instances of non-manifold triangulations appear in the numerical treatment of the Electric Field Integral Equation; see also [16, Section 5.2] for more details.

The idea of distributional de Rham complexes emerged several times in analysis, and in finite element analysis within at least one other context: as observed in [14] and [18] within the framework of element systems, the degrees of freedom in finite element exterior calculus constitute a differential complex by themselves. The complexes of discrete distributional differential forms emerge in that context again; for example, the complex

\[
\begin{align*}
P^{-1} \Lambda^0(T^n) \xrightarrow{d^0} P^{-1} \Lambda^1(T^n) \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} P^{-1} \Lambda^{n-1}(T^n)
\end{align*}
\]

is isomorphic to the complex of degrees of freedom of the finite element complex

\[
\begin{align*}
P^1 \Lambda^n(T) \xleftarrow{d^{n-1}} P^2 \Lambda^{n-1}(T) \xleftarrow{d^{n-2}} \cdots \xleftarrow{d^0} P_{n+1} \Lambda^0(T).
\end{align*}
\]

Further exploration of this relation will contribute to finite element theory in general.

We have considered only finite-dimensional spaces of distributional differential forms. The ideas of this contribution can be possibly be extended to Hilbert complexes of infinite-dimensional spaces. Such a generalization might be contributive to the convergence analysis of finite element methods.

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