Differentially Private Identity and Closeness Testing of Discrete Distributions

Maryam Aliakbarpour
CSAIL, MIT
maryama@mit.edu

Ilias Diakonikolas
CS, USC
diakonik@usc.edu

Ronitt Rubinfeld
CSAIL, MIT & TAU
ronitt@csail.mit.edu

Abstract

We investigate the problems of identity and closeness testing over a discrete population from random samples. Our goal is to develop efficient testers while guaranteeing Differential Privacy to the individuals of the population. We describe an approach that yields sample-efficient differentially private testers for these problems. Our theoretical results show that there exist private identity and closeness testers that are nearly as sample-efficient as their non-private counterparts. We perform an experimental evaluation of our algorithms on synthetic data. Our experiments illustrate that our private testers achieve small type I and type II errors with sample size sublinear in the domain size of the underlying distributions.

1 Introduction

We consider the problem of finding sample-efficient algorithms that allow us to understand properties of distributions over very large discrete domains. Such statistical tests have been traditionally studied in statistics because of their importance in virtually every scientific endeavor that involves data. Recent work in the theoretical computer science community has investigated the case when the discrete domains are large and no a priori assumptions can be made about the distribution (for example, when it cannot be assumed that the distribution is normal, Gaussian, or even smooth). Optimal methods with sublinear sample complexity have been given for testing such properties as whether a distribution is uniform, identical to a known distribution (aka testing “goodness-of-fit”), closeness of two unknown distributions, and independence.

While statistical tests are very important for advancing science, when they are performed on sensitive data representing specific individuals, such as data describing medical or other behavioral phenomena, it may be that the outcomes of the tests reveal information that should not be divulged. Techniques from differential privacy give us hope that one may obtain the scientific benefit of statistical tests without compromising the privacy of the individuals in the study. Specifically, differential privacy requires that similar datasets have statistically close outputs – once this property is achieved, then provable privacy guarantees can be made. Differential privacy is a rich and active area of study, in which techniques have been developed and applied to give private algorithms for a range of data analysis tasks.

Our Contributions In this paper, we study the problem of hypothesis testing in the presence of privacy constraints, focusing on the notion of differential privacy [17]. Our emphasis is on the sublinear regime, i.e., when the number of samples available is sublinear in the domain size of the underlying distribution(s). We leverage recent progress in distribution property testing to obtain sample-efficient private algorithms for the problems of testing the identity and closeness of discrete distributions. The main conceptual message of our results is that we can achieve differential privacy
with only a small increase in the sample complexity compared to the non-private case. We provide sample-efficient testers for identity to a fixed distribution (goodness-of-fit) and equivalence/closeness between two unknown distributions (both given by samples). For the latter problem, we are the first to give such sample-efficient testers with provable privacy guarantees. Our experimental evaluation on synthetic data illustrates that our testers achieve small type I and type II errors with a sublinear number of samples when the domain size is large.

**Technical Overview** We now provide a brief overview of our approach. We start by observing that there is a simple generic method to convert a non-private tester into a private tester with a multiplicative overhead in the sample complexity. This method is well-known in differential privacy, but for the sake of completeness we describe it in Section 3. It is useful to contrast the sample complexity of the generic method with the (substantially smaller) sample complexity of our testers in Sections 4 and 5. For convenience, throughout this paper, we work with testing algorithms that have failure probability at most $1/3$. As we point out in Section 3, this is without loss of generality: a standard amplification method shows that we can always achieve error probability $\delta$ at the expense of a $\log(1/\delta)$ multiplicative factor in the sample complexity, even in the differentially private setting.

Our results for identity testing follow a modular approach: First, we use a recently discovered black-box reduction of identity testing to uniformity testing proposed by Goldreich in [19], building on the result of Diakonikolas and Kane [13]. We point out (Section 3) that this reduction also applies in the private setting. As a corollary, we can translate any private uniformity tester to a private identity tester without increasing the sample size by more than a small constant factor. It remains to develop sample-efficient private uniformity testers. We develop two such private methods (Section 4): Our first method is a private version of Paninski’s uniformity tester [24], which relies on the number of domain elements that appear in the sample exactly once. This statistic has low sensitivity, allowing an easy translation to the private setting. The sample complexity of our aforementioned uniformity tester is

$$O\left(\sqrt{n}/\epsilon^2 + \sqrt{n}/(\epsilon \sqrt{\xi})\right),$$

where $\epsilon$ is the accuracy of the tester and $\xi$ is the privacy parameter. As our experimental results illustrate, this private tester performs very well in the sublinear regime.

On the other hand, it is well-known that Paninski’s uniformity tester only works when the sample size is smaller than the domain size (even in the non-private setting). To obtain a uniformity tester that works for the entire setting of parameters, we develop our second method: a private version of the collisions-based tester first proposed by Goldreich and Ron [20]. The collisions-based tester was recently shown to be sample-optimal in the non-private setting [11]. The main difficulty in turning this into a private tester is that the underlying statistic (number of collisions) has very high worst-case sensitivity. To overcome this issue, we add a simple pre-processing step to our tester that rejects when there is a single element that appears many times in the sample. (We note that a similar idea was independently used in [5], though the details are somewhat different.) This allows us to reduce the effective sensitivity of our statistic and yields a sample-efficient private tester. Specifically, the sample complexity of our collision-based private tester is

$$\tilde{O}\left(\sqrt{n}/\epsilon^2 + \sqrt{n}/(\epsilon \sqrt{\xi}) + 1/(\epsilon^2 \xi)\right).$$

For the problem of closeness testing, we build on the chi-square type optimal tester provided by Chan et al. [9]. A major advantage of this statistic is that it has constant sensitivity. Hence, developing a sample-efficient private version can be achieved by adding Laplace noise. A careful analysis shows that this noisy statistic is still accurate without substantially increasing the sample complexity. Specifically, the sample complexity of our private closeness tester is

$$O\left(\max\left\{\sqrt{n}/\epsilon^2, n^{2/3}/\epsilon^{4/3}, \sqrt{n}/(\sqrt{\xi} \epsilon), 1/(\xi \epsilon^2)\right\}\right).$$

**Related Work** During the past two decades, distribution property testing [3] – whose roots lie in statistical hypothesis testing [23, 22] – has received considerable attention by the computer science community, see [25, 7] for two recent surveys. The majority of the early work in this field has focused on characterizing the sample size needed to test properties of arbitrary distributions of a given support size. After two decades of study, this “worst-case” regime is well-understood: for many properties of interest there exist sample-optimal testers (matched by information-theoretic lower bounds) [24, 10, 9, 26, 15, 14, 2, 13, 6, 11, 8, 16].

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A recent line of work \cite{27, 18, 21, 5} has studied distribution testing with privacy constraints. The majority of these works \cite{27, 18, 21} focus on type I error analysis subject to privacy guarantees. Most relevant to ours is the very recent work by Cai et al. \cite{5} that provides an identity tester with provable sample guarantees and bounded type I and type II errors. Their sample upper bounds for identity testing are comparable to ours (though quantitatively worse in some parameter settings). We also note that Cai et al. \cite{5} do not consider the more general problem of closeness testing. Finally, recent work of Diakonikolas et al. \cite{12} has provided differentially private algorithms for learning various families of discrete distributions. For the case of unstructured discrete distributions, such algorithms inherently require sample size at least linear in the domain size, even for constant values of the approximation parameter.

2 Preliminaries

Notation and Basic Definitions. We use \([n]\) to denote the set \([1, 2, \ldots, n]\). We say \(p\) is a discrete distribution over \([n]\) if \(p : [n] \to [0, 1]\) is a function such that \(\sum_{i=1}^{n} p(i) = 1\), where \(p(i)\) denotes the probability of element \(i\) according to distribution \(p\). For a set \(S \subset [n]\), \(p(S)\) denotes the total probability of the elements in \(S\) (i.e., \(p(S) = \sum_{i \in S} p(i)\)). For any integer \(k > 0\), the \(\ell^k\)-norm of \(p\) is equal to \(\left( \sum_{i=1}^{n} |p(i)|^k \right)^{\frac{1}{k}}\), and it is denoted by \(\|p\|_k\). The \(\ell^k\)-distance between two distributions \(p\) and \(q\) over \([n]\) is equal to \(\left( \sum_{i=1}^{n} |p(i) - q(i)|^k \right)^{\frac{1}{k}}\). We use \(\text{Lap}(b)\) to denote a random variable that is drawn from a Laplace distribution with parameter \(b\) and mean zero.

The problem of identity testing (or goodness-of-fit) is the following: Given sample access to an unknown distribution \(p\) over \([n]\) and an explicit distribution \(q\) over \([n]\), we want to distinguish, with probability at least 2/3, between the cases that \(p = q\) (completeness) and \(\|p - q\|_1 \geq \epsilon\) (soundness). The special case of this problem when \(q = U_n\), the uniform distribution over \([n]\), is called uniformity testing. The generalization of identity testing when both \(p\) and \(q\) are unknown and only accessible via samples is called closeness testing.

Differential Privacy. In our context, a dataset is a multiset of samples drawn from a distribution over \([n]\). We say that \(X\) and \(Y\) are neighboring datasets if they differ in exactly one element.

Definition 2.1. A randomized algorithm \(A : [n]^* \to \mathcal{R}\), is \(\xi\)-differentially private if for any \(S \subseteq \mathcal{R}\) and any neighboring datasets \(X, Y\), we have that \(\Pr[A(X) \in S] \leq e^{\xi} \cdot \Pr[A(Y) \in S]\).

We will say that a tester is \((\epsilon, \xi)\)-private, to mean that \(\epsilon\) is the accuracy parameter, \(\xi\) is the privacy parameter, and the tester outputs the right answer with probability at least 2/3. For conciseness, we use the term \(\xi\)-private instead of \(\xi\)-differentially private. We provide more details about general techniques in differential privacy in Appendix A.

3 Private Identity Testing: Reduction to Private Uniformity Testing

In this section, we provide a simple black-box reduction of private identity testing (against an arbitrary fixed distribution) to its special case of private uniformity testing. Specifically, we point out that a recent reduction of (non-private) identity testing to (non-private) uniformity testing works in the private setting as well.

In recent work \cite{19}, Goldreich provides a reduction from identity testing to uniformity testing with only a constant multiplicative overhead in the query complexity. Here, we show that the same reduction works in the private setting.

Suppose we want to test identity between an unknown distribution \(p\) over \([n]\) and an explicit distribution \(q\). The reduction of \cite{19} transforms the distribution \(p\) into a new distribution \(p'\) over a domain of size \(O(n)\), such that if \(p = q\) then \(p'\) is the uniform distribution, and if \(p\) is far from \(q\), \(p'\) is also far from uniform. Specifically, the reduction defines a randomized mapping of a sample \(i \in [n]\) from \(p\) to a sample \((j, a)\) from \(p'\) that depends only on the explicit distribution \(q\). This property is crucial as it allows us to show that the transformation preserves differential privacy, as the following theorem states.

Theorem 3.1. Given an \((\epsilon, \xi)\)-private uniformity tester using \(s(n, \epsilon, \xi)\) samples, there exists an \((\epsilon, \xi)\)-private tester for identity using \(s = s(6n, \epsilon/3, \xi)\) samples.
In the proof, we give a reduction from identity testing to uniformity testing. The mapping itself depends only on $q$. This fact implies that it suffices to use any private tester for uniformity. The detailed proof of the theorem is in Appendix D.

4 Private Uniformity Testing

In this section, we provide two sample-efficient private uniformity testers. Our testers are private versions of two well-studied (non-private) testers, due to Goldreich and Ron [20] and Paninski [24]. Paninski’s uniformity tester [24] relies on the number of unique elements in the sample, while [20] relies on the number of collisions. Both testers are known to be sample-optimal in the non-private setting [24, 11].

We give private versions of both of these algorithms. The sample complexity of our private Paninski uniformity tester is $O(\sqrt{n/\epsilon^2} + \sqrt{n/(\epsilon\xi)})$. Therefore, as long as $\xi = \Omega(\epsilon^2)$, the privacy requirement increases the sample complexity by at most a constant factor.

Unfortunately, the aforementioned tester only succeeds when its sample size is smaller than the domain size $n$. To be able to handle the entire range of parameters, we develop a private version of the collisions-based tester from [20]. Our private version of the collisions tester has sample complexity $\tilde{O}(\sqrt{n/\epsilon^2} + \sqrt{n/(\epsilon\xi)} + 1/(\epsilon^2))$. Similarly, the effect of the privacy is mild as long as $\xi = \Omega(\epsilon)$.

4.1 Private Uniformity Tester based on Unique Elements

Here, we provide a private tester for uniformity based on the number of unique elements. The number of unique elements is (negatively) related to the number of collisions and the $\ell^2$-norm of the distribution. Therefore, the greater the number of unique elements is, the more the distribution appears uniform. To make the algorithm private, we use the Laplace mechanism which adds a small amount of noise to the number of unique elements. Then, we compare the number of unique elements with a threshold to decide if the distribution is uniform or far from uniform. The noise is big enough to make the algorithm private, but it does not detract much from the accuracy of the tester. We prove this formally in Theorem 4.1.

Algorithm 1 The Private Algorithm for the uniformity test

1: procedure PRIVATE-UNIFORMITY-TEST($\epsilon, \xi$)
2: \hspace{1cm} $s \leftarrow 5\sqrt{n}/(\epsilon\sqrt{\xi}) + 6\sqrt{n}/\epsilon^2$
3: \hspace{1cm} $C \leftarrow \frac{s\epsilon^2}{\sqrt{n}}$
4: \hspace{1cm} $x_1, x_2, \ldots, x_s \leftarrow s$ samples drawn from $p$
5: \hspace{1cm} $K \leftarrow$ the number of unique elements in \{ $x_1, x_2, \ldots, x_s$ \}
6: \hspace{1cm} $K' \leftarrow K + \text{Lap}(2/\xi)$
7: \hspace{1cm} if $K' < \mathbb{E}_d[K] - C^2/(2\epsilon^2)$ then
8: \hspace{1cm} Output reject.
9: \hspace{1cm} Output accept.

Theorem 4.1. Given $s = O(\sqrt{n}/(\epsilon\sqrt{\xi}) + \sqrt{n}/\epsilon^2)$ samples from distribution $p$ over $[n]$, Algorithm 1 is an $(\epsilon, \xi)$-private tester for uniformity if $s$ is sufficiently smaller than $n$.

By the properties of the Laplace mechanism, we know that $K$ is private. Then, we show that adding noise to $K$ does not harm the accuracy of the tester because the variance of the noise is small. Using the Chebyshev inequality, we show that, with high probability, $Z$ concentrates well around its expected value given the size of the sample set. Thus, since there is a large enough gap between the expected values of $K$ which case the samples came from. The proof of the theorem is in Appendix E.

4.2 Private Uniformity Tester via Collisions

In this subsection, we describe the private version of our collisions-based uniformity tester. The main difficulty in turning this into a private tester is that the underlying statistic (number of collisions) has very high worst-case sensitivity. Specifically, if the sample set contains $s$ copies of a given domain element, by changing just one of the copies to another element, the number of collisions drops by
an additive $s$. So, if we add enough noise to the statistic to cover the sensitivity of $s$, it substantially degrades the tester accuracy.

To overcome this issue, we add a simple pre-processing step to our tester. We notice that the sensitivity of the number of collisions, $f(X)$, for sample set $X$, depends on the maximum frequency of the element in the sample set. Let $n_i(X)$ denote the number of occurrences of element $i$ in the sample set $X$, and let $n_{\text{max}}(X)$ denote the maximum $n_i(X)$. We note that for two neighboring sample sets $X$ and $Y$, the difference of the number of collisions, $|f(X) - f(Y)|$, is at most $n_{\text{max}}(X)$. Therefore, the sensitivity of $f$ is high on $X$'s with large $n_{\text{max}}(X)$. However, if the underlying distribution is uniform, we do not expect any particular element to show up very frequently. Hence, if $n_{\text{max}}(X)$ is high, the algorithm can output reject regardless of $f(X)$. So, the final output of the algorithm does not change drastically on $X$ and $Y$, while the number of collisions varies a lot.

This simple observation forms the basis for our modified algorithm. The algorithm uses two statistics: $n_{\text{max}}$ and $f$. If $n_{\text{max}}$ is too large, it outputs reject. Otherwise, $f(X)$ determines the output. In the second case, since $n_{\text{max}}$ is not too large, $f$ has bounded sensitivity. Therefore, we can make it private by adding a small amount of noise to it. The detailed procedure is explained in Algorithm 2. We show correctness in Theorem 4.2. We prove this result in Appendix F.

### Algorithm 2 Private uniformity tester based on the number of collisions

1: procedure PRIVATE-UNIFORMITY-TEST($\epsilon, \xi$)
2: $s \leftarrow \Theta \left( \frac{\sqrt{n}}{\epsilon^2} + \frac{n \log n}{\epsilon \xi^{1/2}} + \frac{\sqrt{n \max(1, \log 1/\xi)}}{\epsilon \xi} + \frac{1}{\epsilon^2} \xi \right)$.
3: Let $X = \{x_1, x_2, \ldots, x_s\}$ be a multiset of $s$ samples drawn from $p$.
4: $n_i(X) \leftarrow |\{j : x_j \in x \text{ and } x_j = i\}|$.
5: $n_{\text{max}}(X) \leftarrow \max_i n_i(X)$.
6: $\hat{n}_{\text{max}}(X) \leftarrow n_{\text{max}}(X) + \text{Lap}(2/\xi)$.
7: $f(X) \leftarrow \text{collisions}(X)$.
8: $\eta_f \leftarrow \max \left( \frac{3\epsilon}{2n}, 12 \epsilon^2 \ln 24 n \right) + (2 \ln 12) / \xi + 2 \max(\ln 3, \ln n) / \xi$.
9: $T \leftarrow \max \left( \frac{3\epsilon}{2n}, 12 \epsilon^2 \ln 24 n \right) + (2 \ln 12) / \xi$.
10: $\hat{f}(X) \leftarrow f(X) + \text{Lap}(2 \eta_f / \xi)$.
11: if $\hat{n}_{\text{max}}(X) < T$ and $\hat{f}(X) < \frac{6 + \epsilon^2}{n \max} (s)$ then
12: $O \leftarrow \text{accept}$.
13: else
14: $O \leftarrow \text{reject}$.
15: With probability $1/6$, $O \leftarrow \{\text{accept, reject}\} \setminus O$. (Flip the answer with probability $1/6$.)
16: Output $O$.

**Theorem 4.2.** Algorithm 2 is an $(\epsilon, \xi)$-private tester for uniformity.

To preserve the accuracy of the tester, we add a tiny amount of noise to $f$, which is not sufficient to make $\hat{f}$ fully private. However, we observe that the sensitivity of $f$ is high when $n_{\text{max}}$ is high. So, the algorithm is likely to output reject because of high $n_{\text{max}}$ and regardless of $f$. We show that effect of $f$ on the output is small enough that the algorithm remains private. The proof of the theorem is in Appendix F.

### 5 Private Closeness Testing

In this section, we give a private algorithm for testing closeness of two unknown discrete distributions. Our tester relies on the chi-squared type sample-optimal (non-private) closeness tester proposed in [1] and analyzed in [9]. The closeness tester relies on the following statistic:

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\[ Z := \sum_i \frac{(X_i - Y_i)^2 - X_i - Y_i}{X_i + Y_i}, \]

where \( X_i \) is the number of occurrences of element \( i \) in the sample set from \( p \), and \( Y_i \) is the number of occurrences of element \( i \) in the sample set from \( q \). The statistic \( Z \) is chosen in a way so that its expected values in the completeness and soundness cases differ substantially. The challenging part of the analysis involves a tight upper bound on the variance, which allows to show that \( Z \) is well-concentrated after an appropriate number of samples. More precisely, the following statements were shown in [9]:

\[ \mathbb{E}[Z] = \sum_i \left( \frac{p(i) - q(i)}{p(i) + q(i)} \right)^2 \frac{m}{m(p(i) + q(i))} \geq \frac{m^2}{4n + 2m} \| p - q \|_1^2. \]

(1)

and

\[ \text{Var}[Z] \leq 2 \min\{m, n\} + \sum_i 5m \left( \frac{p(i) - q(i)}{p(i) + q(i)} \right)^2 \frac{p(i) + q(i)}{p(i) + q(i)}. \]

(2)

The private version of the above statistic is simple: We add noise to the random variable \( Z \) and work with the noisy statistic, denoted by \( Z' \). We need to show that we still can infer the correct answer from \( Z' \), and the noise does not incapacitate our tester. The main reason that this is indeed possible is because the statistic \( Z \) has bounded sensitivity.

Algorithm 3 is our private closeness tester and we prove its correctness in Theorem 5.1.

**Algorithm 3** The private tester for closeness of two unknown distributions

1: procedure PRIVATE-CLOSENESS-TEST\((\epsilon, \xi)\)
2: \( m \leftarrow C \cdot \max \left( \frac{\sqrt{n}}{\epsilon^2}, \sqrt{\frac{n}{\xi}}, \sqrt{\frac{1}{\epsilon \xi}}, \sqrt{\frac{1}{\xi \epsilon}} \right) \)
3: Draw \( m \) samples from distributions \( p \) and \( q \).
4: \( X_i \leftarrow \text{the number of occurrences of the } i\text{-th element in the samples from } p \)
5: \( Y_i \leftarrow \text{the number of occurrences of the } i\text{-th element in the samples from } q \)
6: \( Z \leftarrow \sum_i \frac{(X_i - Y_i)^2 - X_i - Y_i}{X_i + Y_i} \) \( \langle \text{for } X_i + Y_i \neq 0. \rangle \)
7: \( \eta \leftarrow \text{Lap}(8/\xi) \)
8: \( Z' = Z + \eta \)
9: \( T \leftarrow \frac{m^2 \epsilon^2}{8n + 4m} \)
10: if \( Z' \leq T \) then
11: \( \text{Accept} \)
12: else
13: \( \text{Reject} \)

**Theorem 5.1.** Given sample access to two distributions \( p \) and \( q \). Algorithm 3 is an \((\epsilon, \xi)\)-private tester for closeness of \( p \) and \( q \).

Since the sensitivity of \( Z \) is small, we can add a small amount of noise to it to make it private, using the Laplace mechanism. Then, we show that adding the noise to \( Z \) does not increase its variance drastically. Finally, we prove by the Chebyshev inequality that, with high probability, \( Z \) concentrates well around its expected value given the size of the sample set. Thus, since we have shown a large enough gap between the expected values of \( Z \) the in accept and reject cases, we can distinguish between a pair of identical distributions and a pair of distributions that are \( \epsilon \)-far from each other. The proof of the theorem is in Appendix G.

6 Experiments

We provide an empirical evaluation of the proposed algorithms on synthetic data. All experiments were performed on a computer with a 1.6 GHz Intel(R) Core(TM) i5-4200U CPU and 3 GB of RAM.
The focus of the experiments is to find the minimum number of samples such that the type I and type II errors are small. In our synthetic trials, we show that for sufficiently large domain size $n$, our algorithms is likely to succeed with a sublinear number of samples.

Specifically, given a domain size $n$, we find the (approximately) minimum number of samples such that the type I and type II errors are less than $1/3$. First, we pick a distribution (or a pair of distributions) that should be accepted with probability $2/3$, and another distribution that should be rejected with probability $2/3$. Then, we start with an initial number of samples $s$. For each case, we run the algorithm $r$ times on a sample set of size $s$. Then we estimate the accuracy of the algorithm for these sample sets. If the empirical accuracy is less than $2/3$ for either of the distributions, we increase $s$ appropriately and repeat the process until we find $s$ that results in an accuracy of at least $2/3$.

Private Uniformity Testing. We implemented Algorithm 1 to test the uniformity of a distribution in $\ell^1$-distance. Let $p$ be a distribution that has probability $(1 + \epsilon)/n$ on half of the domain and probability $(1 - \epsilon)/n$ on the other half. Clearly, $p$ is $\epsilon$-far from uniform. Since $p$ can be used to generate a tight sample lower bound [24], $p$ is in some sense the hardest instance to distinguish from the uniform distribution. We run the algorithm using samples from the uniform distribution and from $p$ with the following parameters: $\epsilon = 0.3$, $r = 300$, and $\xi = 0.2$. We determine the number of samples required for this tester to have accuracy at least $2/3$ for domain sizes $n$ ranging from 1 million to 2 million (increasing $n$ by 10000 at each step). The experimental results are shown in Figure 1.

![Figure 1: The sample complexity of Algorithm 1 for private uniformity testing.](image)

Private Identity Testing. Testing uniformity is a special case of testing identity of distributions, and it is known to be essentially the hardest instance of the more general problem. Similarly to [5], we consider testing identity to a distribution $q$, where $q$ is uniform on two disjoint subsets of the domain, of sizes $n_1 = n/1000$ and $n_2 = 999n/1000$. The total probability mass of the first subset is 0.6 and the mass of the second one is 0.4. The distribution $q$ can be viewed as a distribution which is “heavy” on a small number of elements and “light” on the rest of the elements. To build a distribution $p$ which is $\epsilon$-far from $q$, we tweak the probability of the elements in the second subset by $\pm \epsilon/n_2$. As explained in Section 5, to implement the identity test, we map our sample set $S$ to another sample set $S'$ on a slightly larger domain. Then, we use Algorithm 1 to test the uniformity on the new domain using samples in $S'$. We set $\epsilon = 0.3$, $r = 200$, and $\xi = 0.2$. We find the required number of samples of this tester in order to have accuracy at least $2/3$, for $n$ from 1 million to 2 million (increasing $n$ by 10000 at each step). The result is shown in Figure 2.

Private Closeness Testing. We implemented Algorithm 3 to test closeness of two unknown distributions. Let $p$ be a distribution such that $n^{2/3}$ of the domain elements have probability $(1 - \epsilon/2)/n^{2/3}$ (the “heavy elements”) and $n/4$ “light” elements have probability $2\epsilon/n$. Let $q$ be a distribution that has probability $(1 - \epsilon/2)/n^{2/3}$ on the same set of heavy elements as $p$, and for a disjoint set of $n/4$ light elements assigns probability $2\epsilon/n$. Since the light elements are disjoint, it is clear that $p$ is $\epsilon$-far from $q$. It has been shown in [4] and [9], that this pair of distributions yields a family of pairs of distributions (via randomly permuting the names of the elements) which can be used to give a tight lower bound on the sample complexity for the problem of testing closeness.
To evaluate the accuracy of our algorithm, we use the tester to distinguish the following pairs: \((q, q)\) and \((p, q)\). We set \(\epsilon = 0.3, r = 200,\) and \(\xi = 0.2\). We find the required number of samples of this tester in order to have accuracy at least 2/3, for \(n\) ranging from 1 million to 2 million (increasing \(n\) by 10000 at each step). The result is shown in Figure 3.

Figure 2: The sample complexity of private identity testing.

Figure 3: The sample complexity of private closeness testing.

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A  General Techniques in Differential Privacy

A standard mechanism in the privacy literature, the Laplace mechanism, perturbs the output of an algorithm by adding Laplace noise to make the output private. Assume the algorithm computes a function \( f : [n]^a \rightarrow \mathbb{R} \). The amount of noise required depends on the privacy parameter, \( \xi \), and how much \( f \) varies over two neighboring datasets. More precisely, this variation of \( f \) is called sensitivity of the function and it is defined as:

\[
\Delta f = \max_{\text{neighboring } x, y} |f(x) - f(y)|.
\]

The noise is drawn from a Laplace distribution with parameter \( b = \Delta f / \xi \). We denote the noise by \( \text{Lap}(b) \). More precisely,

\[
\Pr[\text{Lap}(b) = x] = \frac{1}{2b} \exp \left( -\frac{|x|}{b} \right).
\]

The following is well-known:

**Lemma A.1** (The Laplace mechanism (Theorem 3.6 in [17])). Assume there is an algorithm \( A \) that on input \( x \), outputs \( f(x) + \text{Lap}(\Delta f / \xi) \). Then \( A \) is \( \xi \)-private.

Note that the expected value of \( \text{Lap}(b) \) is zero. Therefore, the expected value of the output remains \( E[f(x)] \). Since we draw the noise independently from \( x \), the variance of the output is increased by \( \text{Var}[\text{Lap}(b)] = 2b^2 \).

Moreover, the following lemmas help us understand how the privacy guarantee changes if we process the output of one or more private algorithm.

**Lemma A.2** (Post-processing (Proposition 2.1 in [17])). Assume \( A \) is a \( \xi \)-private algorithm. Any algorithm that on input \( x \) outputs a function \( f(A(x)) \) is also \( \xi \)-private.

**Lemma A.3** (Composition Theorem (Theorem 3.16 in [17])). Let \( A_i : [n]^a \rightarrow \mathbb{R} \) be a \( \xi_i \)-private algorithm for \( i = 1, \ldots, k \). Any algorithm that on input \( x \) outputs a function \( f(A_1(x), A_2(x), \ldots, A_k(x)) \) is \( (\sum_{i=1}^k \xi_i) \)-private.

B  Generic Differentially Private Tester

In this section, we describe a simple generic method to convert a non-private tester into a private tester with a multiplicative overhead in the sample complexity. While this method is known in the differential privacy community, it is useful to contrast its sample complexity with the (substantially smaller) sample complexity of our testers in Sections 3, 4, and 5.

Assume \( A \) is a tester that draws \( s(n, \epsilon) \) samples. The idea is to draw \( m \cdot s(n, \epsilon) \) samples for a sufficiently large \( m \), and from this sample, to pick a random subset of size \( s(n, \epsilon) \) samples. Then, the new tester runs \( A \) on the randomly chosen subset and outputs \( A \)'s output. Given two sample sets that differ in one sample, the new private tester will give the same output whenever a chunk that does not contain the differing sample is chosen, which happens with probability at most \( 1/m \). This reduction to a non-private tester is described in Algorithm 4. We formally show its correctness in Theorem B.1.

**Algorithm 4** Reduction to a non-private tester

1: procedure GENERAL-PRIVATE-TESTER(\( \epsilon, \xi \))
2: \( m \leftarrow \lceil \frac{l}{\xi} \rceil \)
3: \( s' \leftarrow m \cdot s(n, \epsilon) \)
4: \( x_1, x_2, \ldots, x_{s'} \leftarrow s' \) samples from \( p \).
5: \( r \leftarrow \text{Pick a random number from } [m] \).
6: \( O \leftarrow A \{ x_{(r-1)s+1}, x_{(r-1)s+2}, \ldots, x_{rs} \} \).
7: With probability \( 1/6 \), \( O \leftarrow \{ \text{accept, reject} \} \setminus O \). \( \langle \text{flip the answer with probability } 1/6 \rangle \)
8: Output \( O \).
Theorem B.1. Let \( A \) be an \( \epsilon \)-tester for property \( P \) that uses \( s(n, \epsilon) \) samples from distribution \( p \) over \([n]\). Algorithm 4 is an \((\epsilon, \xi)\)-private property tester for property \( P \) using \( O(s(n, \epsilon)/\xi) \) samples.

Proof: Suppose \( A \) is an \( \epsilon \)-tester for property \( P \) that uses \( s(n, \epsilon) \) samples. Without loss of generality, assume the tester \( A \) errs with probability at most \( 1/4 \). Since the output of \( A \) is then flipped with probability 1/6, by the union bound, the probability that Algorithm 4 errs is at most 1/3, and it is thus an \( \epsilon \)-tester for uniformity.

To prove the privacy guarantee, let \( m \) be \( \lceil 6/\xi \rceil \), and let \( X = \{x_1, x_2, \ldots, x_m\} \) and \( Y = \{y_1, y_2, \ldots, y_m\} \) be two sample sets of size \( s' := m \cdot s(n, \epsilon) \) that differ in exactly one sample. Without loss of generality, we assume they differ in the first sample: \( x_i = y_i \) for \( i > 1 \) and \( x_1 \neq y_1 \).

Algorithm 4 picks a random number, \( r \), where \( \xi < e \). The new sample complexity grows by at most a constant multiplicative factor.

Proof: First, we show that algorithm 5 is an \( \epsilon \)-private algorithm. More formally, we have

\[
\Pr[T(X) = \text{reject}] = \sum_{i=1}^{m} \Pr[T(X) = \text{reject}|r = i] \cdot \Pr[r = i] = \frac{1}{m} \sum_{i=1}^{m} \Pr[T(X) = \text{reject}|r = i]
\]

\[
= \frac{1}{m} \sum_{i=2}^{m} \Pr[T(Y) = \text{reject}|r = i] + \frac{1}{m} \Pr[T(X) = \text{reject}|r = 1] 
\]

\[
\leq \frac{1}{m} \sum_{i=1}^{m} \Pr[T(Y) = \text{reject}|r = i] + \frac{1}{m} 
\]

\[
\leq \Pr[T(Y) = \text{reject}] + \frac{1}{m} .
\]

Since we change the output of \( A \) with probability 1/6, it is not hard to see that \( \Pr[T(Y) = \text{accept}] \) is at least 1/6 for any input \( y \). Thus,

\[
\frac{\Pr[T(X) = \text{reject}]}{\Pr[T(Y) = \text{reject}]} \leq 1 + \frac{6}{m} \leq 1 + \xi \leq e^\xi .
\]

Similarly, we can show the above inequality when the output is accept. Thus, the algorithm is \( \xi \)-private. \( \square \)

C Amplification of Confidence Parameter in the Private Setting

For convenience, throughout this paper we work with testing algorithms that have failure probability at most 1/3. Here we point out that this is without loss of generality, since a standard amplification method also succeeds in the differentially private setting.

Theorem C.1. Given \( A \), an \((\epsilon, \xi)\)-private tester for property \( P \), such that \( A \) uses \( s(n, \epsilon, \xi) \) samples for any input distribution \( p \) over \([n]\). Algorithm 4 is an \((\epsilon, \xi)\)-private tester for property \( P \), using \( O(\log 1/\delta \cdot s(n, \epsilon, \xi)) \) samples from \( p \), that outputs the correct answer with probability \( 1 - \delta \).

Proof: First, we show that algorithm 5 is \( \xi \)-private: Let \( X \) and \( Y \) be two sample sets of size \( m \cdot s \) (where \( m \) and \( s \) are as defined in algorithm 5) that differ only in one sample. Without loss of generality, assume they differ in the first sample. Therefore, \( X^{(i)} \) and \( Y^{(i)} \) differ in only one sample, and for \( i > 1 \), \( X^{(i)} \) and \( Y^{(i)} \) are identical. Hence, the distribution of the output of \( A \) in all of the iterations except the first one is identical for both \( X \) and \( Y \). For the first iteration, the distribution over the output of \( A \) cannot change drastically, because \( A \) is a \( \xi \)-private algorithm. More formally, we have the following:

\[
\Pr[A(X^{(i)}) = \text{accept}] = \Pr[A(Y^{(i)}) = \text{accept}] \quad \text{for } i > 1,
\]

\footnote{This can be achieved by the standard amplification method (i.e., running the tester \( O(1) \) times and taking the majority answer). The new sample complexity grows by at most a constant multiplicative factor.}
Algorithm 5 Amplified confidence parameter

1: procedure AMPLIFIER($n, \epsilon, \xi$)
2: \hspace{1em} $m \leftarrow 18 \lceil \ln \frac{1}{\delta} \rceil + 1$
3: \hspace{1em} $s \leftarrow s(n, \epsilon, \xi)$
4: \hspace{1em} $c \leftarrow 0$
5: \hspace{2em} for $i = 1, \ldots, m$
6: \hspace{3em} $X^{(i)} \leftarrow$ a set of $s$ samples from $p$
7: \hspace{3em} Run $A$ using samples in $X^{(i)}$.
8: \hspace{2em} if $A$ accepts then
9: \hspace{3em} $c \leftarrow c + 1$
10: \hspace{2em} if $c \geq m/2$ then
11: \hspace{3em} Output accept.
12: \hspace{2em} else
13: \hspace{3em} Output reject.

\[ \Pr[A(X^{(1)}) = \text{accept}] \leq e^\xi \cdot \Pr[A(Y^{(1)}) = \text{accept}] . \]

An analogous argument holds when the output is reject. Let $T(X)$ indicate the output of Algorithm 5 on input $X$. Let $\sigma(X^{(i)})$ be an indicator variable that is one if $A$ outputs accept on input $X^{(i)}$ and zero otherwise. Since iterations of the algorithm are independent, we have:

\[
\Pr[T(X) = \text{accept}] = \Pr \left[ \sum_{i=1}^{m} \sigma(X^{(i)}) \geq 9 \lceil \ln \frac{1}{\delta} \rceil + 1 \right] \\
= \Pr[\sigma(X^{(1)}) = 1] \cdot \Pr \left[ \sum_{i=2}^{m} \sigma(X^{(i)}) = 9 \lceil \ln \frac{1}{\delta} \rceil + 1 \right] + \Pr \left[ \sum_{i=2}^{m} \sigma(X^{(i)}) \geq 9 \lceil \ln \frac{1}{\delta} \rceil + 1 \right] \\
\leq e^\xi \cdot \Pr[\sigma(Y^{(1)}) = 1] \cdot \Pr \left[ \sum_{i=2}^{m} \sigma(Y^{(i)}) = 9 \lceil \ln \frac{1}{\delta} \rceil + 1 \right] + \Pr \left[ \sum_{i=2}^{m} \sigma(Y^{(i)}) \geq 9 \lceil \ln \frac{1}{\delta} \rceil + 1 \right] \\
\leq e^\xi \cdot \Pr \left[ \sum_{i=1}^{m} \sigma(Y^{(i)}) \geq 9 \lceil \ln \frac{1}{\delta} \rceil + 1 \right] \\
\leq e^\xi \cdot \Pr[T(Y) = \text{accept}] .
\]

An analogous inequality holds for the case where the output is reject. Therefore, Algorithm 5 is $\xi$-private. Moreover, the output of the algorithm is wrong only if the majority of the invocations of $A$ return the wrong answer (i.e. more than $9 \lceil \ln 1/\delta \rceil$ times). However, $A$ errs with probability at most $1/3$ by definition. By the Hoeffding bound, the probability of outputting the wrong answer is

\[ \Pr[T(X) \text{ is wrong}] \leq e^{-2m/36} \leq \delta . \]

Thus, the total error probability is at most $\delta$. Therefore, Algorithm 5 is an $(\epsilon, \xi)$-private tester that outputs the correct answer with probability $1 - \delta$. \hfill \Box

D Proof of Theorem 3.1

Theorem 3.1. Given an $(\epsilon, \xi)$-private uniformity tester using $s(n, \epsilon, \xi)$ samples, there exists an $(\epsilon, \xi)$-private tester for identity using $s = s(6n, \epsilon/3, \xi)$ samples.

Proof: Given $s$ samples from $p$, we map them to $s$ samples from $p'$ using the following mapping:

1. Given sample $i$ from $p$, the process $F_1(i)$ flips a fair coin. If the coin is Heads, $F_1(i)$ outputs $i$, otherwise, $F_1(i)$ outputs $j$ drawn uniformly from $[n]$. Let $p_1$ denote the output distribution of $F_1(i)$'s. It is clear that $p_1(i) = (1/2)p(i) + 1/(2n)$. We define $q_1(i)$ similarly.
2. Let $m_i = \lceil 3n(q(i) + 1/n) \rceil$. Given $j$ and the output of process $F_1(i)$ where $i$ is drawn from $p$, process $F_2(i)$ outputs $j$ with probability $m_i/(3n(q(i) + 1/n))$ and $n + 1$ otherwise. Let $p_2$ denote the output distribution of the $F_2(i)$'s. It is not hard to see that
\[ p_2(j) = p_1(j) \cdot \frac{m_j}{3n(q(j) + 1/n)} = \frac{1}{2} \left( p(j) + \frac{1}{n} \right) \cdot \frac{m_j}{3n(q(j) + 1/n)} \]

for all \( i \in [n] \), and \( p_2(n + 1) = 1 - \sum_{i=1}^{n} p_2(\ell) \). We define \( q_2(i) \) similarly.

3. Given \( k \), the output of process \( F_2(i) \) where \( i \) is drawn from \( p \), we output \( F_3(i) = (k, a) \) such that \( a \) is uniformly chosen from \( \{6nq_2(k)\} \). Note that for \( k \in [n], 6nq_2(k) \) is equal to \( m_k \) and it is an integer, so the set \( \{6nq_2(k)\} \) is well-defined. We denote the distribution of \( F_3(k) \)'s as \( p' \). It is not hard to see that if \( p = q \), then

\[ p'( (k, a) ) = \frac{1}{2} \left( q(j) + \frac{1}{n} \right) \cdot \frac{m_j}{3n(q(j) + 1/n)} \cdot \frac{1}{m_j} = \frac{1}{6n} \]

for \( j \in [n] \). For \( k = n + 1 \),

\[ 6nq_2(n + 1) = 6n - 6n \sum_{\ell=1}^{n} \frac{m_\ell}{6n} = 6n - \sum_{\ell=1}^{n} m_\ell . \]

is also an integer. Therefore, \( p'( (n + 1, a) ) \) is also \( q_2(n + 1)/(6n q_2(n + 1)) = 1/6n \).

Thus, if \( p = q \), then \( p' \) will be a uniform distribution. Similarly, if \( \| p - q \|_1 \geq \epsilon \) then \( \| p' - U \|_1 \geq \epsilon/3 \). For a detailed proof, see [19].

Then, we run the private uniformity tester using the samples from \( p' \), and output the answer of the tester. As shown in [19], if \( p \) is \( \epsilon \)-far from \( q \), then \( p' \) is \( \epsilon/3 \)-far from uniform; and if \( p \) is identical to \( q \), then \( p' \) is uniform. Therefore, the algorithm is an \( \epsilon \)-tester for identity. It suffices to show that the algorithm preserves differential privacy.

Assume \( X \) is the set of samples drawn from \( p \), and denote by \( \pi \) the bits of randomness that the mapping used to build \( X'_\pi \), the set of samples from \( p' \). Assume \( Y \) is a sample set from \( p \) that differs from \( X \) in exactly one location. Then \( X'_\pi \) also differs from \( X'_\pi \) in at most one location, because each sample from \( p \) is used in generating exactly one sample from \( p' \). Let \( A \) be the \( (\epsilon, \xi) \)-private uniformity tester and denote by \( \mathcal{A}(X'_\pi) \) the output of the tester on input \( X'_\pi \). Since the algorithm is \( \xi \)-private, we have:

\[ \Pr[\mathcal{A}(S'_\pi) = \text{accept}] \leq e^\xi \cdot \Pr[\mathcal{A}(Y) = \text{accept}] . \]

Let \( T(X) \) denote the output of our algorithm. By construction, we have

\[ \frac{\Pr[T(X) = \text{accept}]}{\Pr[T(Y) = \text{accept}]} \leq \frac{\sum_\pi \Pr[\mathcal{A}(X'_\pi) = \text{accept}] \cdot \Pr[\pi]}{\sum_\pi \Pr[\mathcal{A}(Y'_\pi) = \text{accept}] \cdot \Pr[\pi]} \leq \frac{\sum_\pi e^\xi \cdot \Pr[\mathcal{A}(Y'_\pi) = \text{accept}] \cdot \Pr[\pi]}{\sum_\pi \Pr[\mathcal{A}(Y'_\pi) = \text{accept}] \cdot \Pr[\pi]} \leq e^\xi . \]

By the same argument, we can show the above inequality holds when the output is reject. Therefore, our algorithm is an \( (\epsilon, \xi) \)-private tester. \( \square \)

### E Proof of Theorem 4.1

**Theorem 4.1.** Given \( s = O(\sqrt{n}/(\epsilon \sqrt{\xi}) + \sqrt{n}/\epsilon^2) \) samples from distribution \( p \) over \([n]\), Algorithm \( I \) is an \((\epsilon, \xi)\)-private tester for uniformity if \( s \) is sufficiently smaller than \( n \).

**Proof:** Algorithm \( I \) draws \( s \) samples from the underlying distribution \( \mathcal{P} \). We use the Laplace mechanism to make the algorithm private: Let \( K \) be the number of unique elements in the sample set. Since changing one sample in the sample set can change the number of unique elements by no more than two, adding Laplace noise with parameter \( 2/\xi \) to \( K \) makes it \( \xi \)-private. Using the composition theorem \( [A.3] \) the algorithm is \( \xi \)-private.
To show the algorithm is an $\epsilon$-tester, we prove the statistic $K'$ concentrates well around its expected value in both the soundness and completeness cases. Using Lemmas 1 and 2 in [24], we have the following inequalities for the number of unique elements:

\[ \mathbb{E}_d[K] - \mathbb{E}_P[K] \geq \frac{s^2\|p - U_n\|^2}{n} \]  

and

\[ \text{Var}[K] \leq \mathbb{E}_d[K] - \mathbb{E}_P[K] + \frac{s^2}{n}. \]  

First, we show the algorithm is an $\epsilon$-tester for uniformity. Then, we prove that it is $\xi$-private.

Assume that the underlying distribution is the uniform distribution. Note that $\mathbb{E}[K] = \mathbb{E}[K']$. Then, by the Chebyshev inequality and Equation (4) we have that:

\[
\Pr[|K' - \mathbb{E}_d[K]| \geq \frac{C^2}{2\epsilon^2}] = \Pr[|K' - \mathbb{E}_d[K']| \geq \frac{C^2}{2\epsilon^2}] \\
\leq \frac{4\epsilon^4}{C^4} \text{Var}[K'] \\
\leq \frac{4\epsilon^4}{C^4} (\text{Var}[K] + \text{Var}[\text{Lap}(2/\xi)]) \\
\leq \frac{4\epsilon^4}{C^4} \left( \frac{s^2}{n} + \frac{8}{\xi^2} \right) \\
\leq \frac{4}{C^2} + \frac{32\epsilon^4}{C^4\xi^2} \\
\leq \frac{1}{3},
\]

where the last inequality comes from the fact that $C \geq \max(3.73\epsilon/\sqrt{\xi}, 4.9)$. Thus, the probability of rejecting $\mathcal{P}$ is less than $1/3$.

Now suppose $\mathcal{P}$ is a distribution which is $\epsilon$-far from uniform. Again by the Chebyshev inequality and Equation (4) we have that:

\[
\Pr[|K' - \mathbb{E}_P[K]| \geq (\mathbb{E}_d[K] - \mathbb{E}_P[K])/2] = \Pr[|K' - \mathbb{E}_P[K']| \geq (\mathbb{E}_d[K] - \mathbb{E}_P[K])/2] \\
\leq \frac{4\text{Var}[K']}{(\mathbb{E}_d[K] - \mathbb{E}_P[K])^2} \\
= \frac{4(\text{Var}[K] + \text{Var}[\text{Lap}(2/\xi)])}{(\mathbb{E}_d[K] - \mathbb{E}_P[K])^2} \\
= \frac{4(\text{Var}[K] + 8\xi^{-2})}{(\mathbb{E}_d[K] - \mathbb{E}_P[K])^2} \\
\leq \frac{4(\mathbb{E}_d[K] - \mathbb{E}_P[K] + s^2/n + 8\xi^{-2})}{(\mathbb{E}_d[K] - \mathbb{E}_P[K])^2} \\
\leq \frac{4}{\mathbb{E}_d[K] - \mathbb{E}_P[K]} + \frac{4s^2/n + 32/\xi^2}{(\mathbb{E}_d[K] - \mathbb{E}_P[K])^2}.
\]

On the other hand by Equation [3] $\mathbb{E}_d[K] - \mathbb{E}_P[K]$ is at least $C^2/\epsilon^2$. Thus,

\[
\Pr[|K' - \mathbb{E}_P[K]| \geq (\mathbb{E}_d[K] - \mathbb{E}_P[K])/2] \leq \frac{4\epsilon^2}{C^2} + \frac{4s^2\epsilon^4}{C^4n} + \frac{32\epsilon^4}{C^4\xi^2} \\
\leq \frac{1}{3},
\]

where the last inequality is true when $C \geq \max \left( 6\epsilon, 6.412\epsilon/\sqrt{\xi} \right)$. Thus, the probability of accepting is less than $1/3$. \hfill \square
F Proof of Theorem 4.2

Theorem 4.2. Algorithm 2 is an \((\epsilon, \xi)\)-private tester for uniformity.

Proof: Let \(X = \{x_1, \ldots, x_s\}\) be a set of \(s\) samples from \(p\). Let \(f(X)\) be the number of collisions in \(X\). All variables are as defined in Algorithm 2. First, we show that \(\hat{f}(X)\) and \(n_{\text{max}}(X)\) concentrate well around their expected values.

Lemma F.1. If \(s = \Theta\left(\frac{\sqrt{n}}{\epsilon^2} + \sqrt{\frac{n \log n}{\epsilon^2}} + \frac{\sqrt{n \max(1, \log 1/\xi)}}{\epsilon \xi} + \frac{1}{\epsilon^2 \xi}\right)\), the following holds with probability at least \(11/12\):

- If \(p\) is the uniform distribution, then \(\hat{f}(X)\) is less than \(\frac{1 + \epsilon^2 / 6}{n} (\frac{s}{2})\).
- If \(p\) is \(\epsilon\)-far from uniform, then \(\hat{f}(X)\) is greater than \(\frac{1 + \epsilon^2 / 6}{n} (\frac{s}{2})\).

Proof: First, we compute the expected value of \(\hat{f}(X)\). Since the expected value of the noise is zero, \(E[f(X)]\) is equal to \(E[f(X)]\). So, if \(p\) is uniform, then \(E[f(X)]\) is \((\frac{s}{2}) / n\), and if \(p\) is \(\epsilon\)-far from uniform \(E[f(X)]\) is at least \((1 + \epsilon^2) (\frac{s}{2}) / n\). Let \(\alpha\) satisfy \(\|p\|^2 = (1 + \alpha) / n\) and \(\sigma\) be the standard deviation of \(f(X)\). We make an assumption that \(|\epsilon^2 / \alpha| (\frac{s}{2}) / n\) is at least \(\sqrt{12} \sigma\). Below, this assumption concludes the statement of the lemma. Later, we prove that the assumption holds for sufficiently large \(s\).

The conditions of the lemma hold if \(\hat{f}(X)\) is closer to its expected value than the distance of the threshold, \(\frac{1 + \epsilon^2 / 6}{n} (\frac{s}{2})\), to its expected value. Using the Chebyshev inequality, the probability that the conditions do not hold is at most

\[
\Pr\left[|\hat{f}(X) - E[f(X)]| > \frac{1 + \epsilon^2 / 6}{n} (\frac{s}{2}) - E[f(X)]\right] = \Pr\left[|\hat{f}(X) - E[f(X)]| > \frac{\epsilon^2 / n - \alpha}{n} (\frac{s}{2})\right] \\
\leq \Pr\left[|\hat{f}(X) - E[f(x)]| \geq \sqrt{12} \sigma\right] \leq \frac{1}{12}.
\]

Thus, it is sufficient to show that

\[
\frac{|\epsilon^2 / 6 - \alpha|}{n} (\frac{s}{2}) \geq \sqrt{12} \sigma.
\]

(5)

Recall that \(\sigma^2\) is equal to \(\text{Var}[f(X)] + \text{Var}[\text{Lap}(2\eta f / \xi)]\), so \(\sigma\) is at most \(\sqrt{2 \max(\text{Var}[f(x)], 8n \eta f^2 / \xi^2)}\). Hence, we prove two stronger inequalities that yield to Equation (5):

\[
s \geq \sqrt{\frac{20 n \sqrt{\text{Var}[f(X)]}}{\epsilon^2 / 6 - \alpha}}.
\]

(6)

and

\[
s \geq \sqrt{\frac{28 n \eta f}{\xi \epsilon^2 / 6 - \alpha}}.
\]

(7)

Using a similar proof to the proof of Lemma 4 in [11], the inequality of Equation (6) holds for \(s = c \sqrt{n} / \epsilon^2\) for sufficiently large constant \(c\). Now, we focus on Equation (7). If \(p\) is a uniform distribution, \(\alpha\) is zero, and if \(p\) is \(\epsilon\)-far from being uniform, then \(\alpha\) is at least \(\epsilon^2\). Therefore, the denominator is at least \(\epsilon^2 / 6\). Solving Equation (7) for \(s\), we have:

\[
s \geq c' \cdot \left(\frac{1}{\epsilon^2 \xi} + \frac{\sqrt{n \log n}}{\epsilon \xi^{1/2}} + \frac{\sqrt{n \max(1, \log 1/\xi)}}{\epsilon \xi}\right).
\]

Hence, for sufficiently large constant \(c'\), Equation (5) holds and the proof is complete. \(\square\)

We have the following lemma:
Lemma F.2. Let $X$ be a sample set of size $s$ from the uniform distribution over $[n]$. With probability $11/12$, we have

$$\hat{n}_{\text{max}} \leq \max \left( \frac{3}{2}, \frac{s}{n}, 12 \frac{e^2 \ln 24n}{e} \right) + \frac{2 \ln 12}{\xi}.$$  

Proof: First, we show that $n_{\text{max}}(X)$ is at most $\max \left( \frac{3s}{2n}, \frac{12e^2 \ln 24n}{e} \right)$ with probability at least $23/24$. It suffices to show that all of the $n_i(X)$’s are smaller than this bound. Consider the following cases: First, assume $s$ is at most $12n \cdot \ln(24n)$. Let $k := 12e^2 \cdot \ln(24n) \geq e^2 s/n$. If $s \leq k$, then $n_{\text{max}}(X)$ is at most $\max \left( \frac{3s}{2n}, \frac{12e^2 \ln 24n}{e} \right)$. Otherwise,

$$\Pr[n_i(X) > k] \leq \left( \frac{s}{k} \right) \cdot \frac{1}{n^k} \leq \left( \frac{s}{e} \right)^k \frac{1}{n^k} \leq e^{-k} \leq \frac{1}{24n}.$$  

Second, assume $s$ is greater than $12n \cdot \ln(24n)$. By the Chernoff bound, we have

$$\Pr[n_i(X) > \frac{s}{n} \left( 1 + \frac{1}{2} \right)] \leq \exp(-\frac{s}{12n}) \leq \frac{1}{24n}.$$  

Thus,

$$\Pr[n_i(X) > \max \left( \frac{3s}{2n}, \frac{12e^2 \ln 24n}{e} \right)] \leq \frac{1}{24n}.$$  

Using the union bound, with probability $23/24$ all the $n_i(X)$’s, and consequently $n_{\text{max}}(X)$, are smaller than $\max \left( \frac{3s}{2n}, \frac{12e^2 \ln 24n}{e} \right)$. Moreover, based on the properties of the Laplace distribution, we have

$$\Pr \left[ \text{Lap}(2/\xi) \geq \frac{2 \ln 12}{\xi} \right] \leq \frac{\exp(-\ln 12)}{2} \leq \frac{1}{24}.$$  

By the union bound, $n_{\text{max}}(X)$ and $\text{Lap}(2/\xi)$ are not exceeding the aforementioned bounds with probability $11/12$. Therefore, we have

$$\Pr \left[ \hat{n}_{\text{max}} < \max \left( \frac{3s}{2n}, \frac{12e^2 \ln 24n}{e} \right) + \frac{2 \ln 12}{\xi} \right] \geq \frac{11}{12}.$$  

Thus, the proof is complete. 

Given $X$, we define two probabilistic events, $E_1(X)$ and $E_2(X)$, to be

$$E_1(X) : \hat{n}_{\text{max}} < T \quad E_2(X) : \hat{f}(X) < \frac{6 + e^2 s}{6n} \left( \frac{s}{2} \right),$$

where the probability is taken over the randomness of the noise. Observe that $E_1(X)$ and $E_2(X)$ are independent. We use $\overline{E_1(X)}$ and $\overline{E_2(X)}$ to indicate the complementary events. Let $\mathcal{M}(X)$ denote the output of the algorithm when the input sample set is $X$. We set the output, $O$, to accept, if both $E_1(X)$ and $E_2(X)$ are true, and at the end of the algorithm we may flip the output with small probability. Here, we prove the probability of outputting the correct answer is at least $2/3$. Consider two following cases:

(i) $p$ is uniform: Using Lemma F.2, with probability at least $11/12$ we have that $\hat{n}_{\text{max}}$ is less than $T$. By lemma F.1, $\hat{f}$ is less than $\left( \frac{s}{2} \right) \left( 1 + e^2 / 6 \right) / n$ with probability at least $11/12$. Therefore, $\Pr[\overline{E_1(X)}]$ and $\Pr[\overline{E_2(X)}]$ are at most $1/12$. At the end of the algorithm, we flip the output with probability at most $1/6$. Using the union bound, we have

$$\Pr[\mathcal{M}(X) = \text{accept}] \geq 1 - \Pr[\overline{E_1(X)}] - \Pr[\overline{E_2(X)}] - \frac{1}{6} > \frac{2}{3}.$$  

(ii) $p$ is $\epsilon$-far from uniform: By lemma F.1, $\hat{f}(X)$ is greater than $\left( \frac{s}{2} \right) \left( 1 + e^2 / 6 \right) / n$ with probability at least $11/12$, so $\Pr[\overline{E_2(X)}]$ is at most $1/12$. We flip the output of the algorithm with probability at most $1/6$. As a result, we have

$$\Pr[\mathcal{M}(X) = \text{reject}] \geq 1 - \Pr[\overline{E_2(X)}] - \frac{1}{6} \geq \frac{2}{3}.$$  

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Thus, with probability at least 2/3 we output the correct answer.

In the rest of the proof, we focus on proving the privacy guarantee. It is not hard to see that \(|n_{\max}(X) - n_{\max}(Y)|\) is at most one. By the properties of the Laplace mechanism in Lemma A.1, \(n_{\max}(X)\) is \(\xi/2\)-private. Assume \(|f(X) - f(Y)|\) is at most \(\eta_f\). Then, \(\hat{f}(X)\) is \(\xi/2\)-private as well. Since privacy preserved after post-processing (Lemma A.2), both \(E_1(X)\) and \(E_2(X)\) are \(\xi/2\)-private. Using the composition lemma A.3 the output is \(\xi\)-private (by Lemma A.3).

Now, assume \(|f(X) - f(Y)|\) is greater than \(\eta_f\). In this case, we show that \(n_{\max}(X)\) has to be large. Therefore, the output is reject with high probability regardless of \(\hat{f}(X)\). Although \(\hat{f}(X)\) is not private, it cannot affect the output drastically and the output remains private. We prove this formally below. Without loss of generality, assume we replace a sample \(i\) in \(X\) with \(j\) to get \(Y\). Thus, we have

\[
|f(X) - f(Y)| = \left|\left(n_i(X) - 1 - n_j(X)\right)\right|
\]

where the inequality comes from the assumption that there is at least one copy of \(i\) in \(X\). Therefore, \(n_{\max}(X)\) is greater than \(\eta_f\) as well. Since \(T\) is even smaller than \(\eta_f\), it is very unlikely that \(n_{\max}\) be smaller than the threshold \(T\). More formally, by the properties of the Laplace distribution, we have:

\[
\Pr[E_1(X)] = \Pr\left[n_{\max}(X) \leq T\right] \leq \Pr[n_{\max}(X) - n_{\max}(X) \leq T - \eta_f]
\]

\[
\leq \Pr\left[\text{Lap}(2/\xi) \leq -\frac{2\max(\ln 3, \ln 3/\xi)}{\xi}\right]
\]

\[
\leq \exp\left(-\max(\ln 3, \ln 3/\xi)/2\right) \leq \min(1/6, \xi/6).
\]

Now, consider the case that the algorithm output accept on input \(X\). It is not hard to see that

\[
\Pr[M(X) = \text{accept}] = (5/6) \cdot \Pr[E_1(X) \land E_2(X)] + (1/6) \cdot (1 - \Pr[E_1(X) \land E_2(X)])
\]

\[
= (2/3) \cdot \Pr[E_1(X) \land E_2(X)] + 1/6
\]

\[
= (2/3) \cdot \Pr[E_1(X)] \cdot \Pr[E_2(X)] + 1/6.
\]

Observe that since we flip the answer with probability 1/6 at the end, \(\Pr[M(X) = \text{accept}]\) and \(\Pr[M(Y) = \text{accept}]\) are at least 1/6. By this fact, Equation (8), and Equation (9), we have:

\[
\frac{\Pr[M(X) = \text{accept}]}{\Pr[M(Y) = \text{accept}]} \leq \frac{\Pr[E_1(X)] + 1/6}{1/6} \leq \xi + 1 < e^\xi.
\]

Now, consider the case where the output of the algorithm is reject on the input \(X\). Similar to Equation (8), we can prove \(\Pr[E_1(Y)]\) is at most \(\min(1/6, \xi/6)\). Similar to Equation (9), it is not hard to see that

\[
\Pr[M(X) = \text{reject}] = (2/3) \cdot \left(\Pr[F_1(X) \lor F_2(X)] + 1/6\right).
\]

If \(\Pr[M(X) = \text{reject}]\) is at most \(\Pr[M(Y) = \text{reject}]\), then clearly, we have:

\[
\frac{\Pr[M(X) = \text{reject}]}{\Pr[M(Y) = \text{reject}]} \leq 1 < e^\xi.
\]

Thus, assume \(\Pr[M(X) = \text{reject}]\) is less than \(\Pr[M(Y) = \text{reject}]\). Then, we have:

\[
\frac{\Pr[M(X) = \text{reject}]}{\Pr[M(Y) = \text{reject}]} = \frac{(2/3) \cdot \left(\Pr[F_1(X) \lor F_2(X)] + 1/6\right)}{(2/3) \cdot \left(\Pr[F_1(Y) \lor F_2(Y)] + 1/6\right)}
\]

\[
\leq \frac{\Pr[F_1(X) \lor F_2(X)]}{\Pr[F_1(Y) \lor F_2(Y)]} \leq \frac{1}{1 - \Pr[E_1(Y)]}
\]

\[
\leq \frac{1}{1 - \min(1/6, \xi/6)} < 1 + \xi < e^\xi.
\]

The second to last inequality is true since we showed previously that \(\Pr[E_1(Y)]\) is at most \(\min(1/6, \xi/6)\). Hence, the proof is complete.
Proof of Theorem 5.1

Theorem 5.1. Given sample access to two distributions $p$ and $q$, Algorithm 3 is an $(\epsilon, \xi)$-private tester for closeness of $p$ and $q$.

Proof: Our proof has two main parts. First, we show that the algorithm outputs the correct answer with probability $2/3$. Second, we show that the algorithm is private.

Proof of Correctness: First, assume $p$ and $q$ are equal. In the algorithm, we compute $Z$ and add Laplace noise, $\eta$, to it. Then we compare it to threshold $T := \epsilon^2 m^2/(8n + 4m)$. Based on Equation (1), we have

$$E[Z'] = E[Z] + E[\eta] = E[Z].$$

Using the Chebyshev inequality and Equation (2), we obtain

$$\Pr[\text{outputting reject}] = \Pr[Z > T] \leq \frac{\text{Var}[Z']}{T^2} \leq \frac{\text{Var}[Z] + \text{Var}[\eta]}{T^2} \leq \frac{2 \min\{m, n\} + 128/\xi^2}{T^2} \leq \frac{1}{3},$$

where the last inequality is true for a sufficiently large universal constant $C$.

Case 1: Consider the case $m \leq n$. Then,

$$\frac{2 \min\{m, n\}}{T^2} = \frac{2 m (8n + 4m)^2}{m^4 \epsilon^4} \leq \frac{2 m (12n)^2}{m^4 \epsilon^4} \leq 288 \left(\frac{n^{2/3}}{\epsilon^{4/3}} \frac{1}{m}\right)^3 \leq \frac{288}{C^3} \leq \frac{1}{6},$$

where the last inequality is true for $C$ greater than 12. Moreover,

$$\frac{128 \xi^2}{T^2} \leq \frac{128 (8n + 4m)^2}{\xi^2 m^4 \epsilon^4} \leq \frac{128 (12 n)^2}{\xi^2 m^4 \epsilon^4} \leq 18432 \left(\frac{\sqrt{n}}{\sqrt{\xi} \epsilon} \frac{1}{m}\right)^4 \leq \frac{18432}{C^4} \leq \frac{1}{6},$$

where the last inequality is true for $C$ greater than 19. Thus, for sufficiently large $C$, the probability of rejecting two identical distribution $p$ and $q$ is less than $1/3$.

Case 2: Consider the case $n < m$. Then,

$$\frac{2 \min\{m, n\}}{T^2} = \frac{2 n (8n + 4m)^2}{m^4 \epsilon^4} \leq \frac{2 n (12n)^2}{m^4 \epsilon^4} \leq 288 \left(\frac{n}{\epsilon^2} \frac{1}{m}\right)^2 \leq \frac{288}{C^2} \leq \frac{1}{6},$$

where the last inequality is true for $C$ greater than 42. Moreover,

$$\frac{128 \xi^2}{T^2} \leq \frac{128 (8n + 4m)^2}{\xi^2 m^4 \epsilon^4} \leq \frac{128 (12n)^2}{\xi^2 m^4 \epsilon^4} \leq 18432 \left(\frac{1}{\xi \epsilon} \frac{1}{m}\right)^2 \leq \frac{18432}{C^2} \leq \frac{1}{6},$$

where the last inequality is true for $C$ greater than 136. Thus, for sufficiently large $C$ the probability of rejecting two identical distribution $p$ and $q$ is less than $1/3$.

Now, suppose $p$ and $q$ are at least $\epsilon$-far from each other in $\ell^1$-distance. We show that in this case $Z'$ is greater than $T$ with high probability using Chebyshev’s inequality. Based on Equation (2), we bound the variance of $Z'$ in terms of the expected value of $Z'$. First, observe that, by Equation (1), we have $E[Z']$ is at least $C/6$ for any setting of parameters. Thus, for sufficiently large $C$, we can assume $E[Z']$ is at least 360. Let $I_1$ be the set of all indices $i$ such that $(1 - (1 - e^{-m(p_i + q_i)})/(m(p_i + q_i)))$ is greater $1/2$, and let $I_2$ be the set of remaining indices, i.e., $I_2 = [n] \setminus I_1$. By Equation (4), we have

$$E[Z']^2 = \left(\sum_i \frac{(p_i - q_i)^2}{p_i + q_i} m \left(1 - \frac{1 - e^{-m(p_i + q_i)}}{m(p_i + q_i)}\right)\right)^2 \geq 360 \sum_i \frac{(p_i - q_i)^2}{p_i + q_i} m \left(1 - \frac{1 - e^{-m(p_i + q_i)}}{m(p_i + q_i)}\right) \geq 36 \sum_{i \in I_2} \frac{(p_i - q_i)^2}{p_i + q(i)}.$$
We use a superscript (C) where the last inequality is true for sufficiently large $Z$. By Equation (1), the expected value of $X_i$ and $Y_i$ is at least $p(i) + q(i)$. Therefore, $m - \frac{(p(i) - q(i))^2}{p(i) + q(i)}$ is at most 2. Thus, $\sum_{i \in I_2} 5m \frac{(p(i) - q(i))^2}{p(i) + q(i)}$ is at most $10n$. Since $\frac{(p(i) - q(i))^2}{p(i) + q(i)}$ is less than $p(i) + q(i)$, $\sum_{i \in I_2} 5m \frac{(p(i) - q(i))^2}{p(i) + q(i)}$ is also less than $10m$. Hence, we have

$$
\text{Var}[Z] \leq 2 \min\{m, n\} + \sum_{i \in I_2} 5m \frac{(p(i) - q(i))^2}{p(i) + q(i)}
$$

$$
\leq 2 \min\{m, n\} + \sum_{i \in I_2} 5m \frac{(p(i) - q(i))^2}{p(i) + q(i)} + \sum_{i \in I_2} 5m \frac{(p(i) - q(i))^2}{p(i) + q(i)}
$$

$$
\leq 12 \min\{m, n\} + \frac{\text{E}[Z']^2}{36}.
$$

By Equation (1), the expected value of $Z'$ is at least 2$T$. Using Chebyshev’s inequality, we obtain

$$
\text{Pr[outputting “Accept”] = Pr[Z' \leq T] \leq Pr[\text{E}[Z'] - Z' \geq \text{E}[Z'] - T] \leq \text{Pr} \left[ \text{E}[Z'] - Z' \geq \frac{\text{E}[Z']}{2} \right]
$$

$$
\leq \frac{4 \text{Var}[Z']}{\text{E}[Z']^2} \leq \frac{4(\text{Var}[Z] + \text{Var}[Y])}{\text{E}[Z']^2} \leq \frac{48 \min\{m, n\}}{\text{E}[Z']^2} + \frac{10}{A} + \frac{512}{\text{E}[Z']^2} \xi^2
$$

$$
\leq \frac{48 \min\{m, n\}(4n + 2m)^2}{m^4 \epsilon^4} + \frac{1}{9} + \frac{512(4n + 2m)^2}{m^4 \epsilon^4} \xi^2 \leq \frac{1}{3},
$$

where the last inequality is true for sufficiently large $C$.

**Proof of Privacy Guarantee:** First, observe that the value of $Z$ does not change drastically over two neighboring datasets. More formally, we have the following simple lemma:

**Lemma G.1.** The sensitivity of the statistic $Z$ is at most 8.

**Proof:** Assume two neighboring dataset $x$ and $y$. Let $Z(x)$ and $Z(y)$ be the statistic for $x$ and $y$ respectively. We define $Z_i$ as follows:

$$
Z_i := \begin{cases} 
\frac{|X_i + Y_i| - X_i - Y_i}{X_i + Y_i} & \text{if } X_i + Y_i \neq 0 \\
0 & \text{otherwise.}
\end{cases}
$$

We use a superscript $(x)$ or $(y)$ for $X_i$, $Y_i$, $Z_i$ to indicate the corresponding dataset we calculate them from. Since $x$ and $y$ are two neighboring datasets, there is a sample $i$ in the $x$ which has been replaced by $j$. Without loss of generality, assume $i$ was a sample from $p$. This implies that $X_i^{(x)} - X_i^{(y)} = 1$ and $Y_i^{(x)} = Y_i^{(y)}$. 

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If $X_i^{(y)} + Y_i^{(y)}$ is zero, then $Z_i^{(x)}$ is one. Thus, the difference of $Z_i^{(x)}$ and $Z_i^{(y)}$ is one. Now, assume $X_i^{(y)} + Y_i^{(y)}$ is at least one. Then, we have

$$
\left| Z_i^{(x)} - Z_i^{(y)} \right| = \left| \frac{(X_i^{(x)} - Y_i^{(x)})^2}{X_i^{(x)} + Y_i^{(x)}} - \frac{(X_i^{(y)} - Y_i^{(y)})^2}{X_i^{(y)} + Y_i^{(y)}} \right|
$$

$$
= \left| \frac{(X_i^{(y)} - Y_i^{(y)} + 1)^2}{X_i^{(y)} + Y_i^{(y)} + 1} - \frac{(X_i^{(y)} - Y_i^{(y)})^2}{X_i^{(y)} + Y_i^{(y)}} \right|
$$

$$
= \left| \frac{(X_i^{(y)} - Y_i^{(y)})^2 + 2(X_i^{(y)} - Y_i^{(y)}) + 1}{X_i^{(y)} + Y_i^{(y)} + 1} - \frac{(X_i^{(y)} - Y_i^{(y)})^2}{X_i^{(y)} + Y_i^{(y)}} \right|
$$

$$
= \left| \frac{2(X_i^{(y)} - Y_i^{(y)}) + 1}{X_i^{(y)} + Y_i^{(y)} + 1} - \frac{(X_i^{(y)} - Y_i^{(y)})^2}{(X_i^{(y)} + Y_i^{(y)} + 1)(X_i^{(y)} + Y_i^{(y)})} \right|
$$

$$
\leq 2 + 1 + 1 \leq 4 .
$$

Similarly, we can show $|Z_j^{(x)} - Z_j^{(y)}|$ is at most four. Hence, we can conclude that $|Z^{(x)} - Z^{(y)}|$ is at most eight.

Therefore, using the property of the Laplace mechanism (Lemma A.1), $Z'$ is $\xi$-private. Using Lemma A.2 and the fact that the output of the algorithm is a function of $Z'$, we conclude the algorithm is $\xi$-private. \[\square\]