Constraints over Cosmological Constant and Quintessence Fields in an Accelerating Universe

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Abstract

A brief account of the current cosmological observations is given and their implications for QCDM and ΛCDM cosmologies are discussed. The nucleosynthesis and the galaxy formation constraints have been used to put limits on Ωφ during cosmic evolution, and develop a realistic approach to the tracking behaviour of quintessence fields. The astrophysical constraints are applied to interpolate the value of the tracking parameter ε ≃ 0.75 at the present epoch and also to find the lower and the upper limits for Λ in the accelerating universe. It is shown that the transition from deceleration to acceleration in the cosmic expansion occurs earlier in ΛCDM cosmology compared to QCDM cosmology.

PACS numbers: 98.80.Cq, 04.50.+h

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1 Introduction

The observational view of the universe has drastically changed during the last ten years. Until a decade ago, the universe was supposed to be matter dominated and the cosmic expansion was understood to be slowing down; consequently, the Einstein de Sitter model was taken to be the standard model of the observable universe. The latest cosmological observations reveal a low mass density, spatially flat universe with accelerating expansion. We shall briefly survey the current observations and apply them to investigate the 'missing mass' problem in cosmology.

In cosmological theories, the Hubble Constant is one of the most important observational parameters due to its sensitivity to the variations in the cosmic energy density and the spatial curvature of the universe. According to the current estimates [1, 2, 3], $H_0 = 65 \pm 10 \, \text{km/Mpc/s}$. As regards the curvature of the universe, it was long predicted by the inflationary scenarios that the observable universe must be spatially flat. In the angular power spectrum of CMB [4], the location of the first Doppler peak near $l \approx 200$ fortifies this view. At the same time the precise measurements [3, 4, 5, 6] of the density of matter derived from cluster abundances, baryon fraction in clustered matter indicate a low mass density with $\Omega_m \simeq 0.35$. It gives rise to the question as to how to account for 65% 'missing mass' in the universe, sometimes, referred to as 'dark energy'. The recent studies undertaken by Supernova Cosmology Project Team [9] and High Red-shift Search Team [10] reveal that the distant SNe are fainter and thus more distant than expected for a decelerating universe. It implies that the rate of cosmic expansion is accelerating which in turn provides empirical evidence of the existence of dark energy with negative pressure. For a spatially flat universe, the best values derived from the analysis of combined results of [9, 10] are approximately $\Omega_m = 0.25 \pm 0.1$ and for dark energy $\Omega_X = 0.75 \pm 0.1 (1\sigma)$. These values are in excellent agreement with $\Omega_m$ derived from the baryon fraction in clustered matter as discussed above. The most probable candidates for the dark energy are the cosmological constant $\Lambda$ and the Quintessence fields – the scalar fields with evolving equation of state which, during the matter dominated phase, acquire negative pressure and behave like $\Lambda_{eff}$.

As for the break-up of the material content of the universe, the baryonic matter (BM) contributes only $\Omega_{BM} = 0.05$ as determined from the precise measurement of deuterium abundance [11, 12] in very distant hydrogen clouds. The neutrinos might contribute a small fraction $\Omega_\nu \geq 0.003$. The remaining contribution $\Omega_{DM} \approx 0.29$ comes from the dark matter (DM) which determines the hierarchy of the structure formation in the universe. The cold...
dark matter (CDM) consists of particles like axions and neutralinos which move slowly and cannot remove lumpiness on small scale; as such in CDM cosmology, the structure formation follows bottom-up sequence i.e. the galaxies are formed first followed by clusters and super-clusters. On the contrary, the fast moving particles like neutrinos constitute hot dark matter (HDM) which can remove lumpiness on small scale; as such HDM supports top-down sequence of structure formation in the universe. The astronomical observations made by the Keck 10-meter Telescope and the Hubble Space Telescope reveal that most of the galaxies in the universe formed between redshifts 2 to 4, that clusters formed at redshift $z \leq 1$ and the superclusters are forming to-day. These observations rule out HDM cosmology, restrict the contribution of neutrinos to about 0.3% as stated above and favour a CDM cosmology for the observable universe.

It is important to distinguish between the dark matter and dark energy. The dark matter may consist of exotic particles like axions and neutralinos but it attracts and clumps like ordinary matter whereas the dark energy does not clump; it repels matter. In the present paper, we consider the role of two leading candidates of dark energy under the sections QCDM cosmology and ΛCDM cosmology and discuss those astrophysical/cosmological observations which constrain the magnitude of the dark energy at the nucleosynthesis epoch, during galaxy formation era and at the end of matter dominated (i.e. at the onset of acceleration) era and their implications for the quintessence field and their tracking behaviour. These constraints have also been used to put theoretical limits on the magnitude of the cosmological constant $\Lambda$.

## 2 QCDM Cosmology

Let us first consider CDM cosmology with Quintessence – the rolling scalar fields, with evolving equation of state, which acquire repulsive character (owing to negative pressure) during the late evolution of the universe. The quintessence in the present day observable universe, behaves like $\Lambda_{\text{eff}}$ and may turn out to be the most likely form of dark energy which induces acceleration in the cosmic expansion.

Consider the homogeneous scalar field $\phi(t)$ which interacts with matter only through gravity. The energy density $\rho_\phi$ and the pressure $p_\phi$ of the field are given by

$$\rho_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi)$$

$$p_\phi = \frac{1}{2} \dot{\phi}^2 - V(\phi)$$
The equation of motion of the scalar field

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0, \quad V'(\phi) \equiv \frac{dV}{d\phi}$$  \hspace{1cm} (3)

leads to the energy conservation equation

$$\dot{\rho}_\phi + 3H(1 + w_\phi)\rho_\phi = 0$$ \hspace{1cm} (4)

where $w_\phi \equiv \frac{p_\phi}{\rho_\phi}$ and $H \equiv \frac{\dot{a}}{a}$ is the Hubble constant. Accordingly, $\rho_\phi$ scales down as

$$\rho_\phi \sim a^{-3(1+w_\phi)}, \quad -1 \leq w_\phi < \frac{1}{3}$$  \hspace{1cm} (5)

Obviously, the scaling of $\rho_\phi$ gets slower as the potential energy $V(\phi)$ starts dominating over the kinetic energy $\frac{1}{2}\dot{\phi}^2$ of the scalar field and $w_\phi$ turns negative.

Since there is minimal interaction of the scalar field with matter and radiation, It follows from Eq.(4) that the energy of matter and radiation is conserved separately as

$$\dot{\rho}_n + 3H(1 + w_n)\rho_n = 0$$  \hspace{1cm} (6)

Accordingly

$$\rho_n \sim a^{-3(1+w_n)}$$  \hspace{1cm} (7)

where $\rho_n$ is the energy density of the dominant constituent (matter or radiation) in the universe with the equation of state $p_n = w_n \rho_n$ where $w_n = \frac{1}{3}$ for radiation and $w_n = 0$ for matter.

Although, the scalar field is non-interactive with matter, it affects the dynamics of cosmic expansion through the Einstein field equations. Assuming large scale spatial homogeneity and isotropy of the universe, the field equations for a flat Friedmann model are

$$H^2 = \frac{\rho_n + \rho_\phi}{3M_p^2}$$  \hspace{1cm} (8)

and

$$\frac{2\ddot{a}}{a} = -\rho_n + \rho_\phi + 3p_n + 3p_\phi$$  \hspace{1cm} (9)

where $M_p = 2.4 \times 10^{18}$ GeV is the reduced Planck mass.

Denoting the fractional density of the scalar field by $\Omega_\phi \equiv \frac{\rho_\phi}{\rho_n + \rho_\phi}$ and that of the matter/radiation field by $\Omega_n \equiv \frac{\rho_n}{\rho_n + \rho_\phi}$, equations (8) and (9) may be rewritten as

$$\Omega_n + \Omega_\phi = 1$$  \hspace{1cm} (10)
and

\[ \frac{2}{a} \frac{\dot{a}}{a} = - \frac{\rho_n}{3M_p^2} \left[ (1 + 3w_n) + (1 + 3w_\phi) \frac{\Omega_\phi}{\Omega_n} \right] \]  

(11)

The relative growth of \( \Omega_\phi \) versus \( \Omega_n \) during the cosmic evolution is given by

\[ \frac{\Omega_\phi}{\Omega_n} = \left( \frac{a}{a_0} \right)^{3\epsilon} \]  

(12)

where the tracking parameter \( \epsilon \equiv w_n - w_\phi \) and \( \Omega^0_\phi, \Omega^0_n \) denote the values of \( \Omega_\phi \) and \( \Omega_n \) at the present epoch \((a = a_0)\). As discussed in section 1, the SNeIa observations suggest that \( \Omega^0_\phi \approx 2\Omega^0_n \); consequently Eq.(12) may be expressed in terms of the red-shift \( z \) as below

\[ 2(\Omega^{-1}_\phi - 1) = (1 + z)^{3\epsilon} \]  

(13)

If we insist that the scalar field, regardless of its initial value, should behave like \( \Lambda_{eff} \) today, it must obey tracking conditions [13, 14, 15] with wide ramifications for quintessence fields already discussed in detail [16-21]. In nutshell, tracking consists in synchronised scaling of \( \rho_\phi \) and \( \rho_n \) along a common evolutionary track so as to ensure the restricted growth of \( \Omega_\phi \) during the cosmic evolution in accordance with the observational (both astrophysical and cosmological) constraints. As discussed in [15], the tracking parameter \( \epsilon \) plays a vital role in monitoring the desired growth of \( \Omega_\phi \). The existence theorem for tracker fields [15] requires \( \epsilon \) to satisfy the condition \( \frac{\Omega_\phi}{2(1+w_\phi)} < 1 \) and also to conform to cosmological constraints mentioned therein. Here we investigate the consequent limits on the tracking parameter \( \epsilon \) imposed by these constraints to ensure tracking by a quintessence field.

According to the Eq. (13), \( \Omega_\phi \) depends both upon the redshift \( z \) and the tracking parameter \( \epsilon \). From a general functional form \( \Omega_\phi = f(z, \epsilon) \), it is difficult to chart out \( \Omega_\phi - z \) variation through tracking unless we are able to fix the value of \( \epsilon \) corresponding to known \( \Omega_\phi \) at certain points in the phase space of \( z - \epsilon \). This is achieved with the help of the astrophysical constraints discussed in this paper. Having evaluated \( \Omega_\phi \) at some typical points \((z_0, \epsilon_0)\), we can interpolate \( \Delta \Omega_\phi \) at neighbouring points \((z_0 + \Delta z_0, \epsilon_0 + \Delta \epsilon_0)\). Using this technique, realistic tracking diagrams of \( \Omega_\phi - z \) variation, \( w_\phi - z \) variation and \( \epsilon - z \) variation may be drawn as shown in the figures 1, 2 and 3.

In this connection, the following differential relation, derived from Eq. (13)

\[ - \frac{\Delta \Omega_\phi}{3\epsilon \Omega_n \Omega_\phi} = \frac{\Delta z}{1 + z} + \frac{\Delta \epsilon}{\epsilon} \ln(1 + z). \]  

(14)
Figure 1: Variation of $\Omega_\phi$ versus Redshift $z$ in QCDM Cosmology assuming $H_0 = 65$ Km/Mpc/s.

Figure 2: Evolution of the equation of state of the quintessence field in QCDM Cosmology ($H_0 = 65$ Km/Mpc/s).
is found quite useful in interpolating the increment $\Delta \Omega_\phi$ in terms of the increments $\Delta z$ and $\Delta \epsilon$. It is noteworthy that the contribution of the term $\frac{\Delta \epsilon}{1+z}$ is very small compared to the contribution of $\frac{\Delta z}{1+z}$ at high redshifts.

Let us now reconsider the astrophysical constraints discussed in our previous paper [15], try to refine them and examine their implications for quintessence fields.

1. The Nucleosynthesis Constraint. The first constraint on $\Omega_\phi$ during the cosmic evolution comes from the helium abundance at the nucleosynthesis epoch ($z \sim 10^{10}$). The presence of an additional component of energy in the form of quintessence field with energy density $\rho_\phi$ results in an increase in the value of the Hubble constant $H$ as given by the differential of the Friedmann equation

$$\frac{2\delta H}{H} = \frac{\delta \rho}{\rho} = \frac{\rho_\phi}{\rho}. \quad (15)$$

This, in turn, yields a higher ratio of neutrons to protons at the freeze-out temperature (1 MeV) of the weak interactions and a consequent higher percentage of the helium abundance in the universe. Assuming the existence of three known species of neutrinos, the

Figure 3: Variation of the tracking parameter $\epsilon$ versus Redshift $z$ in QCDM Cosmology ($H_o = 65$ Km/Mpc/s).
Figure 4: The important cosmic events corresponding to the astrophysical constraints are marked by circles on the thick line (representing spatially flat universe) in QCDM Cosmology.

nucleosynthesis calculation [22] yields

$$\frac{\delta \rho}{\rho} = \frac{7(N_\nu - 3)}{43}$$  \hspace{1cm} (16)

Since the number of neutrino species $N_\nu < 4$, we arrive at the constraint $\frac{\delta \rho}{\rho} < \frac{7}{43}$. If the contribution $\delta \rho$ comes from the quintessence field instead, the above constraint translates into nucleosynthesis constraint on $\Omega_\phi$ as follows

$$\Omega_\phi = \frac{\rho_\phi}{\rho_n + \rho_\phi} < \frac{7}{50} = 0.14$$  \hspace{1cm} (17)

The corresponding value of the tracking parameter is $\epsilon \leq 0.035$. 
2. **Galaxy Formation Constraint.** According to the current estimates in CDM cosmology, the galactic structure formation takes place between the redshift $z = 4$ and $z = 2$. The clumping of matter into galaxies demands the dominance of gravitational attraction during this period. Therefore, the repulsive force of quintessence must be relatively weak and $\Omega_\phi$ must be reasonably less than 0.5 during the galaxy formation era. Interpolation from Eqs. (13) and (14) shows that $0.33 \leq \epsilon \leq 0.5$ during the galaxy formation era.

3. **Present Epoch.** Two recent surveys [9, 10] based on $SN_\text{Ia}$ measurements predict accelerating cosmic expansion with $\Omega_\phi \simeq 0.65 \pm 0.05$ at the present epoch ($z=0$). The constraint $\ddot{a} > 0$ inserted in Eq. (11) leads to $\epsilon > 0.5$ at the present epoch. Interpolation with the help of Eq.(14) places $\epsilon$ around 0.75.

4. **Transition to Accelerated Expansion (Quintessence Dominated Era).** The onset of acceleration ($\ddot{a} \geq 0$) in the observable universe takes place around the value of $\Omega_\phi \geq 0.5$ which corresponds to $\epsilon \sim 0.66$ from Eq.(11) at a redshift of $z = 0.419$ from Eq.(13).

In figure 4, the cosmic events are depicted sequentially by the thick line in the $\Omega_n - \Omega_\phi$ diagram. The events corresponding to the above constraints are marked by circles.

3 **ΛCDM Cosmology**

In this section, we regard the universe to be filled up with a mixture of cold dark matter and the vacuum energy represented by the cosmological constant $\Lambda$ which plays the role of dark energy and tends to accelerate the expansion of the universe.

In the presence of $\Lambda$ term, the Einstein field equations for a spatially flat Friedmann universe are given by

$$H^2 = \frac{8\pi G}{3}(\rho_n + \rho_\Lambda)$$

and

$$\frac{2\ddot{a}}{a} = -\frac{8\pi G}{3}(\rho_n + 3p_n/c^2 - 2\rho_\Lambda)$$

where $\rho_\Lambda = \frac{\Lambda c^2}{8\pi G}$ and the Newtonian Gravitational constant $G = 6.6 \times 10^{-8}$ cgs units. In case of matter dominated universe ($p_n \sim 0$), the Eqs. (18) and (19) may be rewritten as

$$\Omega_n + \Omega_\Lambda = 1$$

and

$$q = \frac{1}{2}\Omega_n - \Omega_\Lambda$$
where $q \equiv -\frac{\ddot{a}}{a H^2}$ is the deceleration parameter and $\Omega_\Lambda \equiv \frac{\rho_\Lambda}{\rho_\Lambda + \rho_n} = \frac{\Lambda c^2}{3H^2}$.

It is obvious from Eq. (21) that the cosmic expansion slows down as long as $\Omega_n > 2\Omega_\Lambda$ (i.e. $\rho_n > 2\rho_\Lambda$); the clumping of matter into galaxies takes place during this period. Transition to accelerated expansion occurs when the deceleration parameter $q \leq 0$ which corresponds to $\Omega_\Lambda \geq \frac{1}{3}$. It marks the beginning of the $\Lambda$-dominated phase during which the universe goes on expanding faster and faster and ultimately enters the de Sitter phase of exponential expansion. This is borne out clearly from the analytical solution of the field equations (18) and (19) during matter dominated phase

$$a \sim \sinh^{2/3}\left(\frac{3}{2}\sqrt{\Lambda/3}ct\right)$$  \hspace{1cm} (22)

It is noteworthy that this solution reduces to Einstein de Sitter solution $a \sim t^{2/3}$ in the limit $\Lambda \rightarrow 0$ and goes back to de Sitter expansion as $\rho_n \rightarrow 0$.

It leads to

$$H = c\sqrt{\Lambda/3} \coth\left(\frac{3}{2}\sqrt{\Lambda/3}ct\right)$$  \hspace{1cm} (23)

and

$$q = \frac{1}{3} - \tanh^2\left(\frac{3}{2}\sqrt{\Lambda/3}ct\right)$$  \hspace{1cm} (24)

The astrophysical constraints discussed in the previous section apply to $\Lambda$ as well. Since $\Lambda$ remains constant throughout, these constraints put theoretical limits on the plausible values of $\Lambda$. For instance, the clumping of matter into galaxies can take place during the cosmic deceleration phase ($q > 0$) when the gravitational attraction is dominant over cosmic repulsion. It follows from Eq. (24) that during the galaxy formation era

$$\tanh^2\left(\frac{3}{2}\sqrt{\Lambda/3}ct\right) < \frac{1}{3}$$  \hspace{1cm} (25)

Expressed in terms of redshift $z$ with age $t_0$ of the universe taken as 13 billion years, we get

$$\Lambda < 4.2 \times 10^{-57}(1 + z)^3$$  \hspace{1cm} (26)

Assuming that the galaxy formation continues up to redshift $z = 2$, we derive the upper limit of $\Lambda$

$$\Lambda < 33.5 \times 10^{-57}$$  \hspace{1cm} (27)

The transition to accelerated expansion takes place at $t = t_c$ corresponding to redshift $z = z_c$ as given by Eq. (24) with $q = 0$. At the present epoch ($z = 0$), the cosmic expansion is accelerating ($q < 0$). This yields the lower limit of $\Lambda$ as given by

$$\Lambda > 4.2 \times 10^{-57}$$  \hspace{1cm} (28)
According to the constraint 4 in Section 2, it was found that the transition to accelerated expansion (scalar field dominated) phase in QCDM cosmology occurs at \( z_c \sim 0.419 \). Since the transition in \( \Lambda \)CDM cosmology takes place earlier, we can take \( z_c \sim 0.5 \). This corresponds to material density \( \rho_n = 7.415 \times 10^{-30} \text{ gm/cm}^3 \). At the point of onset of acceleration \( (q = 0) \), we have by Eq. (21)

\[
2\rho_\Lambda = \rho_n = 8.43 \times 10^{-30} \text{ gm/cm}^3
\]

(29)

It yields the value

\[
\Lambda = 6.99 \times 10^{-57}
\]

(30)

which is in good agreement with the observational estimate \( \Lambda = \frac{3H_0^2\Omega_\Lambda}{c^2} = 7.74 \times 10^{-57} \) based on the value of the Hubble constant \( H_0 = 65 \text{ km/Mpc/s} \) and \( \Omega_\Lambda = 0.65 \).

4 Conclusions

The astrophysical constraints on \( \Omega_\phi \), discussed in Section 2, enable us to present a realistic picture of the scaling of the quintessence energy during tracking and plot the variation of \( \Omega_\phi \) versus \( \epsilon, w_\phi \) and \( z \) (redshift). Whereas both QCDM cosmology and \( \Lambda \)CDM cosmology are fairly compatible with the recent CMB observations and power spectrum of galaxy clustering, QCDM has the advantage in fitting constraints from high redshift supernovae, gravitational lensing and structure formation at large redshifts [17]. Another distinguishing feature of QCDM cosmology is that the onset of cosmic acceleration takes place when \( \Omega_\phi = 0.5 \) whereas in \( \Lambda \)CDM cosmology, \( \Omega_\Lambda = \frac{1}{3} \) marks the onset of acceleration. Taking the observational value \( \Lambda = 7.74 \times 10^{-57} \), the transition to the accelerating expansion phase in \( \Lambda \)CDM cosmology occurs when \( \rho_m = 9.3 \times 10^{-30} \text{ gm/cm}^3 \) which corresponds to \( z_c \sim 0.54 \). In QCDM cosmology, as discussed in Section 2, the transition occurs at \( z_c = 0.419 \). It means that the transition to accelerating expansion occurs earlier in \( \Lambda \)CDM cosmology than in QCDM cosmology. Again, in \( \Lambda \)CDM cosmology, the universe ends up in inflationary phase with exponential expansion whereas in QCDM cosmology, the ultimate fate of the universe is inflation with exponential or hyperbolic expansion, depending upon the form of the scalar potential.

Acknowledgments This work was supported in part by UGC grant from India. The author acknowledges useful discussions and valuable help of Keith Olive and Panagiota Kanti and hospitality of Theoretical Physics Institute, University of Minnesota.
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