CONVERGENCE AND SEMI-CONVERGENCE OF A CLASS OF CONSTRAINED BLOCK ITERATIVE METHODS

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Abstract. In this paper, we analyze the convergence properties of projected non-stationary block iterative methods (P-BIM) aiming to find a constrained solution to large linear, usually both noisy and ill-conditioned, systems of equations. We split the error of the \( k \)th iterate into noise error and iteration error, and consider each error separately. The iteration error is treated for a more general algorithm, also suited for solving split feasibility problems in Hilbert space. The results for P-BIM come out as a special case. The algorithmic step involves projecting onto closed convex sets. When these sets are polyhedral, and of finite dimension, it is shown that the algorithm converges linearly. We further derive an upper bound for the noise error of P-BIM. Based on this bound, we suggest a new strategy for choosing relaxation parameters, which assist in speeding up the reconstruction process and improving the quality of obtained images. The relaxation parameters may depend on the noise. The performance of the suggested strategy is shown by examples taken from the field of image reconstruction from projections.

Key words. Landweber type iteration, Block Iterative Method, Split Feasibility Problem, Relaxation Parameters, Semi-convergence, Constraints, Tomographic Imaging

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1. Introduction. Among reconstruction methods in computerized tomography (CT) iterative methods are particularly suited for their ability to incorporate constraints (e.g. nonnegativity) on the sought solution. This is important in situations with few and/or noisy data, as in limited data CT when exposition to a low dose of X-rays is an issue. After discretization of the underlying integral transform a large, sparse, unstructured and ill-posed (sensitive to noise) linear system occurs:

\[
Ax = b,
\]

where \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m \) are known, \( x \in \mathbb{R}^n \) is the unknown image to be estimated. We will, in connection with each algorithm/problem discuss possible consistency assumptions on the rhs \( b \).

1.1. Some background. The first reported use of iteration in CT seems to be the paper [29]. There, in its basic form, the given method turned out to be identical to the Kaczmarz method [36]. Then for each equation in (1.1) the new iterate is constructed such that the previous equation is satisfied. This is done for each of the \( m \) equations in turn, and then the process is repeated. For more on the history see [28]. Somewhat later in [26] a fully simultaneous method, SIRT, was proposed where the new iterate is constructed by adding to the previous one the gradient of the least squares functional corresponding to (1.1). Later block-iterative versions were developed and analyzed, where, instead of using a single row, blocks of rows are used in each iteration. A very general form of block-iteration is analyzed in [1]. For more on the development of block-iteration see, e.g. the recent survey [20], and further in Subsection 1.4. To cope with constraints it was, early on, suggested to project the
iterate onto the nonnegative orthant, either after each iterative step or after a whole cycle, i.e. after completing a full sweep through the matrix $A$. The order in which the rows/columns are used is called control, and can have quite an impact of the behavior of the method [33]. In cyclic control the rows/columns are picked up in their original order in the matrix. For some other controls, e.g. almost cyclic control see [9]. A recent strategy is stochastic control which has attracted much interest see, e.g. [44, 46, 55]. We will however not consider this control in our paper.

1.2. Simultaneous Algorithms. Let $M$ and $N$ be given symmetric positive definite (SPD) matrices, and denote by $Q^{1/2}$ the square root of an SPD matrix $Q$. Moreover $\{\lambda_k\}_{k=0}^\infty$ is a sequence of positive relaxation parameters, and $\hat{A} = MA^{-1/2}AN^{-1/2}$. The simultaneous iterative reconstruction technique (SIRT) is a class of iterative and gradient-based methods defined as

$$x^{k+1} = x^k + \frac{\lambda_k}{\|A\|^2} NA^T M (b - Ax^k), \quad k = 0, 1, 2, \ldots,$$

see e.g. [32]. Here $\|\hat{A}\| = \|A\|_2$ (note that $\|\hat{A}\|^2 = \|A^T \hat{A}\|$). By premultiplying (1.2) by $N^{-1/2}$ and putting $N^{-1/2}x^k = y^k$ one gets the equivalent iteration

$$y^{k+1} = y^k + \frac{\lambda_k}{\|A\|^2} \hat{A}^T (M^{1/2}b - \hat{A}y^k),$$

which as is well known converges for $\lambda_k \in [\varepsilon, 2 - \varepsilon]$, where $\varepsilon$ is small positive constant. For the stationary case, $\lambda_k = \lambda$ this is also a necessary condition. It follows that the sequence $\{x^k\}$ converges towards an $M$-weighted least squares solution of (1.1) provided $\lambda_k \in [\varepsilon, 2 - \varepsilon]$. Hence no consistency assumption is needed on $b$. For a more general result (for the nonstationary case) see [52]. Both $M$ and $N$ can have an efficacious impact on the initial speed of convergence. A computational analysis of SIRT applied to CT-problems appears in [30].

Let $\text{diag}(B)$ be a diagonal matrix with $B_j$ being the $j$th diagonal element, and let $\text{Id}$ denote the identity matrix of appropriate dimension (or in the case of infinite dimensional Hilbert spaces the identity operator). Further $a^j$ is $i$th row of $A$, $||.||$ is the Euclidean 2-norm, and $||.||_W$ is the weighted Euclidean norm. Here $W$ is a given SPD matrix.

Some well-known choices for $M$ and $N$ yielding fully simultaneous iterations are listed below:

- $N = M = \text{Id}$, lead to the Landweber method [5, 25].
- $N = \text{Id}$ and $M = D_C = \frac{1}{m}\text{diag}\left(\frac{1}{\|a^j\|^2}\right)$ lead to the Cimmino method [19].
- $N = \text{Id}$ and $M = D_W = \frac{1}{m}\text{diag}\left(\frac{1}{\|a^j\|_W}\right)$ lead to CAV (Component Averaging Method) [16]. Here $W = \text{diag}(w_j)$ where $w_j$ equals the number of nonzero elements in the $j$th column of $A$.
- $N = W^{-1}$ and $M = mD_C$ lead to DROP (Diagonally Relaxed Orthogonal Projection) method [14].
- $N = \text{diag}(\text{row sums})^{-1}$ and $M = \text{diag}(\text{column sums})^{-1}$ lead to SART (Simultaneous Algebraic Reconstruction Technique) [37].
- $N = \text{Id}$ and $M = (2 - \lambda)(D + \lambda L^T)D(D + \lambda L)^{-1}$ where $L$ is the left triangular part of $AA^T$, $D$ its diagonal and taking $\lambda_k = \lambda$ fixed, lead to the symmetric Kaczmarz’s method [23].
1.3. Projected SIRT. Consider the linear system of equations (1.1) subject to $x \in C$ where $C$ denotes a closed convex set and $P_C$ the operator of metric projection onto the set $C$. Inserting a projection onto a convex set after each iteration can reduce the error and improve the quality of the reconstructed image, see, e.g., [49, 51] and Figure 2 below. We will now assume that $N = \text{Id}$ (but remark on the case $N \neq \text{Id}$ at the end of Section 2). Put

\begin{equation}
\hat{A} = M^{1/2}A, \quad \hat{b} = M^{1/2}b.
\end{equation}

Then the P(Projected)-SIRT Algorithm is (note that $\|\hat{A}\|^2 = \|A^TMA\|$)

\begin{algorithm}
1: $x^0 \in \mathbb{R}^n$ is an arbitrary starting point.
2: for $k \geq 0$ do $x^{k+1} = U(x^k) = P_C \left(x^k + \frac{\lambda_k}{\|\hat{A}\|^2} \hat{A}^T \left(\hat{b} - \hat{A}x^k\right)\right)$
3: end for
\end{algorithm}

Let $\text{Fix} \hat{U}$ denote the set of fixed points of an operator $\hat{U} : \mathbb{R}^n \to \mathbb{R}^n$. The following characterization (adapted to our problem formulation) is shown in [6, Proposition 2.1] (also given in [9, Proposition 4.7.2]),

\begin{equation}
\text{Fix} U = \arg \min_{x \in C} \|b - Ax\|_M^2.
\end{equation}

We have the following convergence result for Algorithm 1.1.

**Theorem 1.1.** Let $\{x^k\}_{k=0}^\infty$ be the sequence generated by Algorithm 1.1, and let for $k \geq 0$, $\lambda_k \in (\varepsilon, 2 - \varepsilon)$ for some small $\varepsilon > 0$. Then for an arbitrary $x^0 \in \mathbb{R}^n$ the sequence $\{x^k\}_{k=0}^\infty$ converges to a fixed point of $U$ provided such a fixed point exists.

**Proof.** For $\lambda_k = \lambda$ see [6, Theorem 2.1]. For the non-stationary case see [40, 49].

Note that the same interval for the relaxation parameters is valid both for SIRT and P-SIRT.

**Remark 1.2.** When $\ker(A) = \emptyset$ then $U$ has a unique fixed point for any $C$. When $\ker(A)$ is not empty, e.g. when $A$ is underdetermined, then $U$ has a fixed point when $C$ is compact, e.g. $C$ equals the unit hypercube (as in our experiments). One may also note that with $C$ the rangespace of $A^T$ the unique fixed point of $U$ is the pseudo-inverse solution.

1.4. Block-iterative Algorithms. There are two issues related to the practical use of SIRT-methods on large data problems. One is the need for computer memory, and the other the often slow rate of initial rate of convergence. By partitioning the matrix into row-blocks and iterating sequentially over the blocks one can improve on both. A careful study of implementation of block-iterative methods on a modern multicore platform is presented in [57]. We mention also the possibility of performing the iterations in parallel over the blocks [57, 47]. However, here we do not consider this case. The use of block-iteration has been quite successful in CT-applications, where it is often referred to as Ordered Subset Iteration [8, 34]. A thorough convergence analysis of block-iteration used in CT is [35]. A Landweber-Kaczmarz type block-iteration is proposed and analyzed in [31]. Block-iterative versions of the examples listed in Section 1.2 are also presented in the corresponding references.
Let the matrix $A$ be partitioned into $p$ blocks of equations, which may contain common equations but each equation should appear at least in one of the blocks. Choose $A_t$ and $b^t$ as the $t$th row-block of $A$ and $b$, respectively. Let $\{M_t\}_{t=1}^p$ be a set of given SPD weight matrices. Further $\{C_t\}_{t=1}^p$ is a family of closed convex sets. Then (1.1) subject to $x \in C$ is equivalent to the following block problems:

\[
\text{find } x \in C_t \text{ satisfying } A_t x = b^t, \ t = 1, 2, \ldots, p.
\]

Defining $C = \bigcap_{t=1}^p C_t$ problem (1.5) can be also written as find $x \in C$ satisfying $Ax = b$. In our study, we prefer, however, form (1.5), because in algorithms for solving (1.5) we need only a sequential access to the data. Recently, a more general form of this problem has been defined in [53, Eq. (1.3)], see also [61, Algorithm 1].

Define cyclic control of the data by

\[
i_k = [k] := k (\text{mod } p) + 1.
\]

Put

\[
\hat{A}_{[k]} = M_{[k]}^{1/2} A_{[k]}, \ \hat{b}^{[k]} = M_{[k]}^{1/2} b^{[k]}.
\]

With these notations the Projected Block-Iterative Method (P-BIM) is

\begin{algorithm}
\begin{algorithmic}[1]
\State $x^0 \in \mathbb{R}^n$ is an arbitrary starting point
\For{$k \geq 0$}
\State $x^{k+1} = U_k(x^k) = P_{C_{[k]}} \left( x^k + \frac{\lambda_k}{\|A_{[k]}\|^2} \hat{A}_{[k]}^T \left( \hat{b}^{[k]} - \hat{A}_{[k]} x^k \right) \right)$
\EndFor
\end{algorithmic}
\end{algorithm}

Note that the $k$–dependence is in the relaxation parameters, and the cyclic access of the data $\{C_t, A_t, M_t, b^t\}, \ t = 1, 2, \ldots, p$. A special case of the above algorithm has been studied in [48, Eq. (7)], where $C_t = \mathbb{R}^n$, $t = 1, 2, \ldots, p$. A parallel gradient based projection algorithm has been studied in [18, Algorithm 4] and [53]. The difference with respect to our algorithm is that they use the full gradient, whereas in our approach only a part of the gradient (of the underlying least squares functional) is used, which gives rise to the block structure. In the optimization literature such methods are sometimes called incremental methods. The method (1.4) and (1.5) in [45] with $f = \|Ax - b\|_M$, and assuming the relaxation parameter is constant during a whole cycle is identical to P-BIM. In [45] several relaxation strategies are investigated. The one most similar to our choice is using a constant parameter $\lambda_k = \lambda$. Here however, the convergence result is rather weak see [45, Proposition 2.1].

We next characterize the fixed points of the operator $U_k$ in P-BIM. Similarly as in (1.4) we get

\[
\text{Fix } \hat{U}_k = \arg \min_{x \in C_{[k]}} \|b^{[k]} - A_{[k]} x\|_{M_{[k]}}^2.
\]

Put

\[
\hat{U}_t(x) = P_{C_t} \left( x + \frac{\lambda}{\|A_t\|^2} \hat{A}_t^T \left( \hat{b}^t - \hat{A}_t x \right) \right), \ t = 1, 2, \ldots, p.
\]

Then

\[
\text{Fix } \hat{U}_t = \arg \min_{x \in C_t} \|b^t - A_t x\|_{M_t}^2, \ t = 1, 2, \ldots, p.
\]
It holds
\[ \bigcap_{k=0}^{\infty} \text{Fix } U_k = \bigcap_{t=1}^{p} \text{Fix } \hat{U}_t. \]

The following result will be shown in Section 2 in Hilbert space settings.

**Theorem 1.3.** Let \( \varepsilon \in (0,1) \). Assume that \( \lambda_k \in [\varepsilon, 2 - \varepsilon] \). If \( \bigcap_{t=1}^{p} \text{Fix } \hat{U}_t \neq \emptyset \), then the sequence \( \{x^k\}_{k=0}^{\infty} \) produced by the Algorithm P-BIM converges to \( x^* \in \bigcap_{t=1}^{p} \text{Fix } \hat{U}_t \). Further if \( \{C_t\} \) is polyhedral the iteration converges linearly.

**1.5. Noise-error.** A common situation is when the right-hand side \( b \) is corrupted by noise, i.e. \( b = \bar{b} + \delta b \), where \( \bar{b} \) denotes the exact right-hand side. Let \( \bar{x}^k \) be the iteration vector using the exact right-hand side \( \bar{b} \) and let \( x^* \) be a fixed point of the unperturbed iteration. The total error can be decomposed into two terms
\[
(1.9) \quad x^k - x^* = (x^k - \bar{x}^k) + (\bar{x}^k - x^*).
\]

The first term is called the *noise error*, and the second the *iteration error*. During the first iterations of a convergent method the iteration error dominates, and hence the total error decreases – but after a while the noise error starts to grow resulting in so-called semi-convergence, as will be demonstrated in Section 5. This phenomenon originally defined in [43, Section IV.1] is further studied in [43, 25, 5, 33]. It has been observed experimentally that the minimum error, i.e. \( \min_{k \geq 0} \|x^k - x^*\| \), is almost independent of both relaxation parameters and the weight matrix. However the rate of decrease of the iteration error and the rate of increase of the noise error are affected by both. The noise-error measures the deviation due to errors in the data. Here we will only consider errors in the rhs \( b \). Possible errors in \( A \) are usually caused by modelling errors, and should then be dealt with using model refinements. We will study how the noise-error evolves as a function of the iteration index \( k \). Rather trivially it is proportional to \( k \). However we will derive a bound proportional to \( \sqrt{k} \), see (3.16). We next describe some earlier contributions. In [24], an upper bound for the noise error of SIRT with a fixed relaxation parameter was derived. Based on minimizing this upper bound, two strategies for choosing relaxation parameters were suggested. The obtained sequences of relaxation parameters are both non-negative and non-ascending. Later in [21] these strategies were analyzed and used in P-SIRT. An upper bound for the noise error of the block version of SIRT was derived in [48], and the above two strategies for picking relaxation parameters together with a third choice were studied. Recently, the stationary case of SIRT and P-SIRT were studied in [22], [38] when the M-matrix is non-symmetric. This case includes the classical Kaczmarz method [36].

**1.6. Organization.** In Section 2 we define and study a sequential block-iterative iteration (of which P-BIM is a special case) adapted for solving split feasibility problems in Hilbert space. The proof of Theorem 1.3 then follows from a more general result in Section 2. We further study the noise error of P-BIM in Section 3, and derive a new upper bound. In particular we extend the noise error analysis of block-iteration to projected block-iteration, and also to the case when the relaxation parameters are allowed to depend on the noise. In Section 4 we propose a new rule for picking relaxation parameters, and present its properties. We demonstrate the performance of P-BIM using this and other relaxation strategies with examples taken from tomographic imaging in Section 5.
2. Hilbert space analysis. In this section we will first formulate a quite general feasibility problem. Then we continue to study algorithms, and their convergence properties for the problem. It will be shown that P-BIM comes out as a special case.

Let \( \mathcal{H} \) be a real Hilbert and \( C_i \subseteq \mathcal{H}, i = 1, 2, \ldots, m \), be closed convex subsets. The convex feasibility problem (CFP) is to find \( x \in \bigcap_{i=1}^{m} C_i \) if such a point exists. CFP is a classical problem and was studied within the last decades by many researchers, see, e.g., [3, 9, 4] and the references therein, where several methods for solving the CFP and their convergence properties were presented. First we recall some notions regarding operators in a Hilbert space. Let \( H \) be a real Hilbert space. Let \( C \subseteq H \) be a nonempty convex set and \( A : H \to H \) be a nonzero bounded linear operator. The split feasibility problem \( \text{(SFP)} \) is to find \( x \in \mathcal{C} \) such that \( Ax \in \mathcal{Q} \). Here \( \mathcal{C} \) and \( \mathcal{Q} \) are closed convex subsets. The \( \text{CFP} \) and their convergence properties were presented. First we recall some notions for the problem. It will be shown that P-BIM comes out as a special case.

2.2. Block SFP. Let \( \mathcal{H}, \mathcal{H}_i, i \in I := \{1, 2, \ldots, p\} \) be Hilbert spaces. Consider a family of problems:

\[(2.2) \quad \text{find } x \in C_i \text{ such that } A_i x \in Q_i,\]

or, equivalently, to find a fixed point of the operator \( T : \mathcal{H} \to \mathcal{H} \) defined by \( T(x) = P_{C}(x + \frac{1}{\lambda} A^{*} P_{Q}(A x) - A x) \). The SFP was introduced by Censor and Elfving [13] in the finite dimensional settings. Byrne [6] proved the weak convergence of the so-called CQ-method, \( x^{k+1} = T_{\lambda}(x^{k}) \), to \( \text{argmin}_{x \in C} \| P_{Q}(A x) - A x \| \) which is a solution of the SFP. Here \( \lambda \in (0, 2) \) is a relaxation parameter and \( \| A \| \) denotes the spectral norm of \( A \). In the last 20 years the SFP and its variants was intensively studied by many researchers, see, e.g., [41, 60, 59, 10, 12, 54] to mention some of them.
\(i \in I\), where \(C_i \subseteq \mathcal{H}, Q_i \subseteq \mathcal{H}_i\) are nonempty closed convex subsets, and \(A_i : \mathcal{H} \to \mathcal{H}_i\) are nonzero bounded linear operators, \(i \in I\). We call this family of problems, the block SFP. Denote by

\[
F_i := C_i \cap A_i^{-1}(Q_i),
\]

the solution set of problem (2.2), \(i \in I\). Define

\[
\mathcal{H}_0 := \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_p,
\]

and a bounded linear operator \(A : \mathcal{H} \to \mathcal{H}_0\) by

\[
Ax = (A_1x, A_2x, \ldots, A_p x).
\]

Put

\[
Q = Q_1 \times Q_2 \times \cdots \times Q_p, \quad C = C_1 \cap C_2 \cap \cdots \cap C_p,
\]

and suppose that there is a common solution of all problems (2.2), \(i \in I\), i.e.,

\[
\Omega := \bigcap_{i=1}^p F_i = C \cap A^{-1}(Q) \neq \emptyset.
\]

The family of problems (2.2), \(i \in I\), can be written as the following split feasibility problem [6, 9, 15]

\[
\text{find } x \in C \text{ such that } Ax \in Q.
\]

This problem formulation includes problem (1.5) by setting \(Q_i = \{b^i\}, i \in I\). By picking \(Q = \{b\}\) (and using the M-norm) problem (2.1) becomes minimization of \(\|Ax - b\|_M\) subject to \(x \in C\), i.e. problem (1.4). Similarly when \(p > 1\) we retrieve the corresponding block-problem as expressed in Theorem 1.3. In fact one can reformulate the block SFP into an optimization problem (as for the case \(p = 1\), and problem (2.1)) see [15] for details. Several extensions of (2.7) have been proposed, [17, 7]. In, e.g. [17] the sets \(C, Q\) were extended to fixed points sets of certain operators, \(S\) and \(T\). With \(S = P_C, T = P_Q\) the original formulation is retrieved. An interesting extension is compressed sensing, where a sparse solution of a consistent linear system is sought. Here one uses \(l_1\)-regularization which yields the Q-lasso problem \(\min_{x \in C} \| (I - P_Q)Ax \|_2^2 + \gamma \|x\|_1\). With \(C = \mathbb{R}^n\) and \(Q = \{b\}\) the original lasso problem [58] is recovered. For some further work along these lines see [42].

In order to prove Theorem 1.3, we present below some general convergence results which follow from [11]. Let \(S_i : \mathcal{H}_i \to \mathcal{H}_i\) be a cutter with \(\text{Fix } S_i = Q_i, i \in I\). Define an operator \(T_i : \mathcal{H} \to \mathcal{H}\) by

\[
T_i(x) = \mathcal{L}\{S_i\}(x) := x + \frac{1}{\|A_i\|^2} A_i^*(S_i(A_i x) - A_i x),
\]

i.e., \(T_i\) is the Landweber operator corresponding to \(S_i, i \in I\) (cf. [12, Definition 4.1]). Then the \(\lambda\)-relaxation of \(T_i\) is \(T_{i,\lambda} := (1 - \lambda) \text{Id} + \lambda T_i\). Hence

\[
T_{i,\lambda}(x) = \mathcal{L}\{S_{i,\lambda}\}(x) := x + \frac{\lambda}{\|A_i\|^2} A_i^*(S_i(A_i x) - A_i x),
\]
where $\lambda \in (0, 2), i \in I$. By [59, Lemma 3.1] and by (2.6), $T_i$ is a cutter and $\text{Fix} T_i = \text{Fix} T_i = A_i^{-1}(Q_i) \neq \emptyset$. Let $U_i$ be a cutter with $\text{Fix} U_i = C_i, i \in I$, and $U_{i, \mu}$ be the $\mu$-relaxation of $U_i$. Define

$$(2.10) \quad V_i^{\lambda, \mu} := U_{i, \mu} T_i, \lambda,$$

for $i \in I$, where $\lambda, \mu \in (0, 2)$. Let $\{i_k\}_{k=0}^{\infty} \subseteq \{1, 2, ..., p\}$ be a control sequence and $x^k$ be generated by the iteration

$$(2.11) \quad x^{k+1} = U_{i_k, \mu_k} (x^k + \frac{\lambda_k}{\|A_{i_k}\|^2} A_{i_k}^* ((S_{i_k} (A_{i_k} x^k) - A_{i_k} x^k)))$$
or shorter

$$(2.12) \quad x^{k+1} = V_k x^k,$$

where $V_k = V_{i_k, \mu_k}, \lambda_k, \mu_k \in [\varepsilon, 2 - \varepsilon], k \geq 0$, for some $\varepsilon \in (0, 1)$ and $x^0 \in \mathcal{H}$ is arbitrary. Recall that a sequence $\{i_k\}_{k=0}^{\infty} \subseteq I = \{1, 2, ..., p\}$ is called an almost cyclic control if there is a constant $s \geq p$ such that for any $k \geq 0$ the set $\{i_k, i_k+1, ..., i_{k+s}\} \subseteq I$ (cf. [9, Definition 5.6.10]). Theorems 2.1 and 2.3 presented below apply results of [11].

**Theorem 2.1.** Let $S_i$ and $U_i$ be weakly regular cutters, $i \in I, \lambda_k, \mu_k \in [\varepsilon, 2 - \varepsilon], k \geq 0$, for some $\varepsilon \in (0, 1)$ and $\{i_k\}_{k=0}^{\infty}$ be an almost cyclic control. Then the sequence $x^k$ generated by iteration (2.12) converges weakly to some $x^* \in \Omega$.

**Proof.** Let $\lambda, \mu \in (0, 2)$. Because $T_i$ is a cutter, its $\lambda$-relaxation $T_i, \lambda$ defined by (2.9) is $\alpha$-SQNE, $i \in I$, where $\alpha = (2 - \lambda) / \lambda$ (see, e.g., [9, Theorem 2.1.39]) and $\text{Fix} T_i = A_i^{-1}(Q_i) \neq \emptyset$, because $A_i^{-1}(Q_i) \neq \emptyset, i = 1, 2, ..., p$. Similarly, $U_{i, \mu}$ is $\beta$-SQNE, where $\beta = (2 - \mu) / \mu$. By $C_i \cap A_i^{-1}(Q_i) \neq \emptyset$ (see (2.6)), the operator $V_i^{\lambda, \mu}$ is $\rho$-SQNE, where $\rho = (\alpha^{-1} + \beta^{-1})^{-1}$ and $\text{Fix} V_i^{\lambda, \mu} = C_i \cap A_i^{-1}(Q_i)$ [9, Corollary 2.1.47 and Theorem 2.1.26(ii)], $i \in I$. Thus, one can easily check that $V_{i_k, \mu_k} is \rho_k$-SQNE, where $\rho_k \geq \varepsilon/(4 - 2\varepsilon) > 0$. Let $n_{k-1} = -1$ and

$$(2.13) \quad n_k^i = \min\{n > n_{k-1}^i : i_n = i\},$$

$i \in I, k \geq 0$. Because $\{i_k\}_{k=0}^{\infty}$ is almost cyclic, $n_k^i$ is well defined and there is $s \geq p$ such that $n_{k+1}^i - n_k^i \leq s$ for all $k \geq 0, i = 1, 2, ..., p$. The sequences $\{T_{i, \lambda_{n_k}}\}_{k=0}^{\infty}$ and $\{U_{i, \mu_{n_k}}\}_{k=0}^{\infty}$ are weakly regular [11, Proposition 4.7]. By $V_{n_k} = V_{i_k, \mu_{n_k}} = U_{i, \mu_{n_k}} T_{i_k, \lambda_{n_k}}$, the sequence $\{V_{n_k}\}_{k=0}^{\infty}$ is weakly regular [11, Corollary 5.5]. Now [11, Theorem 6.2(i)] yields the weak convergence of $x^k$ to $x^* \in \Omega$. 

Setting $S_i = P_{Q_i}, x \in \mathcal{H}, U_i = P_{C_i}, \mu_k = 1$ and $\{i_k\}_{k=0}^{\infty}$ a cyclic control, i.e., $i_k = [k] := k \text{(mod } p) + 1$, we obtain the following.

**Corollary 2.2.** Let $\lambda_k \in [\varepsilon, 2 - \varepsilon], k \geq 0$, for some $\varepsilon \in (0, 1)$. Then the sequence $x^k$ generated by iteration

$$(2.14) \quad x^{k+1} = P_{C_{[k]}} (x^k + \frac{\lambda_k}{\|A_{[k]}\|^2} A_{[k]}^* ((P_{Q_{[k]}} (A_{[k]} x^k) - A_{[k]} x^k))),$$

covers weakly to some $x^* \in \Omega$. 

Proof. The metric projection, as a nonexpansive operator is weakly regular [50, Lemma 2]. Thus, the Corollary follows directly from Theorem 2.1.

Now taking \( Q_{[k]} = \{b^{[k]}\} \) and \( A_{[k]} = \hat{A}_{[k]} \) in (2.14) and assuming \( \mathcal{H}_i \) being of finite dimension we retrieve P-BIM and this proves Theorem 1.3.

Next suppose that \( C_i \) is a polyhedral set (the intersection of a finite family of half spaces), \( U_i = P_{C_i} \), \( \mathcal{H}_i \) is finite dimensional, \( Q_i = \{b^i\} \) and \( S_i = P_{Q_i}, i \in I \). Then \( P_{Q_i}(y) = b^i, y \in \mathcal{H}_i \), the range of \( A_i \) is closed, \( A_i^{-1}(Q_i) \) is polyhedral, \( i \in I \), and iteration (2.11) can be written as

\[
x^{k+1} = P_{C_{i,k}^{\lambda_x,\mu_x}}(x^k + \frac{\lambda_k}{\|A_{i,k}\|} A_{i,k}^*(b_{i,k} - A_{i,k}x^k)).
\]

Denote \( b = (b^1, b^2, \ldots, b^p) \in \mathcal{H}_0 \). Then \( A^{-1}(Q) = \{x \in \mathcal{H} : Ax = b\} \) and the solution set has the form \( \Omega = \{x \in \mathcal{C} : Ax = b\} \).

**Theorem 2.3.** Let \( C_i, i \in I \), be polyhedral, \( \lambda_k, \mu_k \in [\varepsilon, 2-\varepsilon] \), \( k \geq 0 \), for some \( \varepsilon \in (0, 1) \) and \( \{i_k\}_{k=0}^{\infty} \subseteq I \) be almost cyclic. Then the sequence \( x^{k^*} \) generated by iteration (2.15) converges linearly to some \( x^* \in \Omega \).

Proof. Similarly as in the proof of Theorem 2.1, \( V_{i,k}^{\lambda_x,\mu_x} \) is \( \rho_k \)-SQNE, where \( \rho_k \geq \varepsilon/(4 - 2\varepsilon) > 0 \). Moreover, the sequences \( \{T_{i,k,\lambda_{i,k}}\}_{k=0}^{\infty} \) and \( \{U_{i,k,\mu_{i,k}}\}_{k=0}^{\infty} \) are linearly regular [11, Proposition 4.7]. Because, \( C_i \) and \( A_i^{-1}(Q_i) \) are polyhedral sets, the family \( \{C_i, A_i^{-1}(Q_i)\} \) is linearly regular, \( i \in I \) [3, Corollary 5.26]. Thus, the sequences \( \{V_{i,k}\}_{k=0}^{\infty}, i \in I \), are linearly regular [11, Corollary 5.5(iii)] (for a definition of a linearly regular sequence of operators, see [11, Definition 4.1]). Because \( F_i := C_i \cap A_i^{-1}(Q_i) \), \( i \in I \), are polyhedral sets, the family \( \{F_i : i \in I\} \) is linearly regular [3, Corollary 5.26]. Now [11, Theorem 6.2(iii)] yields the linear convergence of \( x_k \) to \( x^* \in \Omega \).

We end this section by considering the case \( N \neq \text{Id} \). For ease of notation we only consider the simultaneous case. Let \( A : \mathcal{H}_1 \rightarrow \mathcal{H}_2 \) be a bounded linear operator and \( b \in \mathcal{H}_2 \). Define \( T_{A,b} : \mathcal{H}_1 \rightarrow \mathcal{H}_1 \) by

\[
x^{k} = T_{A,b}(x) = x + \frac{1}{\|A\|^2} A^*(b - Ax),
\]

\( x \in \mathcal{H}_1 \). The operator \( T_{A,b} \) is firmly nonexpansive ([9, Definition 2.2.1]), and \( \text{Fix} T_{A,b} = \{x \in \mathcal{H}_1 : A^*(b - Ax) = 0\} \), see, e.g., [9, Lemma 4.6.2 and Theorem 4.6.3]. In particular, if \( A^{-1}(\{b\}) \neq \emptyset \) then \( \text{Fix} T_{A,b} = A^{-1}(\{b\}) \) [59, Lemma 3.1].

Let \( N : \mathcal{H}_1 \rightarrow \mathcal{H}_1 \) and \( M : \mathcal{H}_2 \rightarrow \mathcal{H}_2 \) be strongly positive symmetric operators and \( N^{\frac{1}{2}} \) and \( M^{\frac{1}{2}} \) their square roots. It is well known that \( N^{\frac{1}{2}} \) and \( M^{\frac{1}{2}} \) are also strongly positive definite symmetric operators [62, page 618]. Let \( \tilde{A} := M^{\frac{1}{2}}AN^{\frac{1}{2}} \) and \( \tilde{b} := M^{\frac{1}{2}}b \). Define \( \tilde{T} := T_{\tilde{A},\tilde{b}} \), i.e.,

\[
\tilde{T}(x) = x + \frac{1}{\|A\|^2} \tilde{A}^* \tilde{b} - \tilde{A} x = x + \frac{1}{\|M^{\frac{1}{2}}AN^{\frac{1}{2}}\|^2} N^{\frac{1}{2}}A^*M(b - AN^{\frac{1}{2}}x).
\]

Clearly, \( \tilde{T} \) is firmly nonexpansive and

\( \text{Fix} \tilde{T} = \{x \in \mathcal{H}_1 : \tilde{A}^* \tilde{b} - \tilde{A} x = 0\} = \{x \in \mathcal{H}_1 : N^{\frac{1}{2}}A^*M(b - AN^{\frac{1}{2}}x) = 0\} \).

Let \( \tilde{T}_\lambda \) denote the \( \lambda \)-relaxation of \( \tilde{T} \), where \( \lambda \in [0, 2] \), and recall the definition of an averaged operator ([9, Definition 2.2.16]).
Proposition 2.4. The operator $V : \mathcal{H}_1 \to \mathcal{H}_1$ defined by $V = P_C \hat{T}_\lambda$, where $\lambda \in (0, 2)$, is $\frac{2}{1-\lambda}$-averaged.

Proof. $V$ is $\frac{1}{1-\lambda}$-relaxed firmly nonexpansive being a composition of a firmly nonexpansive operator $P_C$ and a $\lambda$-relaxed firmly nonexpansive operator $\hat{T}_\lambda$ [9, Theorem 2.2.37]. Consequently, $V$ is $\frac{2}{1-\lambda}$-averaged [9, Corollary 2.2.17].

Consider now the iterates generated by $x^{k+1} = V x^k$. It follows that $x^k \to x^* \in \text{Fix } V$ provided $\text{Fix } V \neq \emptyset$.

3. P-BIM and its noise error. In this section we will study how errors in the unperturbed right-hand-side $\bar{b}$ affect the iterates in P-BIM. To this end let

$$b = \bar{b} + \delta b,$$

The noise will apart from the iterates also possibly affect the relaxation parameters (depending on their definition). To cope with possible rank-deficiency we will, similarly as in [21, 48] add a regularization term $\alpha \|x\|^2$. Let

$$u^k(x, b[k]) = A_T^k M_k \left( b[k] - A^k x \right) - \alpha x,$$

where the last term is due to regularization. Next we introduce the notations

$$\theta_k = \frac{\lambda_k}{\|A^k\|^2 + \alpha}, \quad \bar{\theta}_k = \frac{\bar{\lambda}_k}{\|A^k\|^2 + \alpha}.$$

We will in the sequel refer to $\theta_k$ and $\bar{\theta}_k$ as the noisy and noise-free relaxation parameter, respectively. Note that these are just a scaled version of the original relaxation parameters $\lambda_k$ and $\bar{\lambda}_k$. For noise-free data $\lambda_k = \bar{\lambda}_k$. Now the noise-free P-BIM becomes

$$\bar{x}^{k+1} = P_{C[k]} \left( x^k + \bar{\theta}_k u^k(\bar{x}^k, \bar{b}[k]) \right).$$

The noisy version of P-BIM is

$$x^{k+1} = P_{C[k]} \left( x^k + \theta_k u^k(x^k, b[k]) \right).$$

Next define

$$\delta = \max_{1 \leq t \leq p} \|A_T^T M_t \delta b^t\|,$$

$$\gamma_k = \max_{0 \leq j \leq k} |\bar{\theta}_j - \theta_j|, \quad \gamma_k = \min_{0 \leq j \leq k} \theta_j, \quad \eta_k = \max_{0 \leq j \leq k} \theta_j.$$

Let $\sigma$ be the smallest nonzero singular value of $M_t^{1/2} A_t$, and define

$$\sigma = \min_{1 \leq t \leq p} \sigma_j, \quad \overline{\sigma} = \max_{1 \leq t \leq p} \|M_t^{1/2} A_t\|,$$

and

$$\hat{u}_k = \max_{0 \leq s \leq k} \|u^s(\hat{x}^s, \hat{b}[s])\|.$$
Theorem 3.1. Assume that $\alpha = \frac{\eta}{2}$. Then the noise-error in P-BIM is bounded above by

$$
(3.10) \quad e_N^k := \|x^k - \tilde{x}^k\| \leq \frac{1}{\alpha} \left(\frac{\gamma_k-1 \bar{u}_{k-1} + \eta_k-1}{\eta_k-1}\right) \Psi^k(\sigma, \bar{\eta}_{k-1}),
$$

where $\Psi^k(x, y)$ is defined by

$$
(3.11) \quad \Psi^k(x, y) \equiv \frac{1 - (1 - yx^2)^k}{x}.
$$

Proof. Since $P_C$ is nonexpansive we get

$$
e_N^k \leq \|x^{k-1} + \theta_k u^{k-1}(x^{k-1}, \tilde{b}^{k|1}) - \left(\bar{x}^{k-1} + \bar{\theta}_k u^{k-1}(\bar{x}^{k-1}, \tilde{b}^{k|1})\right)\|
$$

$$= \|\left(x^{k-1} - \bar{x}^{k-1}\right) + \theta_k(u^{k-1}(x^{k-1}, \tilde{b}^{k|1}) - u^{k-1}(\bar{x}^{k-1}, \tilde{b}^{k|1})) + \left(\theta_k - \bar{\theta}_k\right)u^{k-1}(\bar{x}^{k-1}, \tilde{b}^{k|1})\|.
$$

Now

$$u^{k-1}(x^{k-1}, \tilde{b}^{k|1}) - u^{k-1}(\bar{x}^{k-1}, \tilde{b}^{k|1}) = A^T_{k-1}M_{[k-1]} \left(\tilde{b}^{k|1} - A_{[k-1]}x^{k-1}\right)
$$

$$- A^T_{k-1}M_{[k-1]} \left(\tilde{b}^{k|1} - A_{[k-1]}\bar{x}^{k-1}\right) + \alpha(\bar{x}^{k-1} - x^{k-1})
$$

$$= A^T_{k-1}M_{[k-1]}\delta b^{k|1} + (\alpha I + A^T_{[k-1]}M_{[k-1]}A_{[k-1]})(\bar{x}^{k-1} - x^{k-1}).
$$

Hence

$$e_N^k \leq \left\|\left((1 - \theta_k \alpha)I - \theta_k A^T_{[k-1]}M_{[k-1]}A_{[k-1]}\right)\left(x^{k-1} - \bar{x}^{k-1}\right)\right\|
$$

$$+ \theta_k A^T_{[k-1]}M_{[k-1]}\delta b^{k|1} + \left(\theta_k - \bar{\theta}_k\right)\|u^{k-1}(\bar{x}^{k-1}, \tilde{b}^{k|1})\|.$$

Here

$$Q_k = \left((1 - \theta_k \alpha)I - \theta_k A^T_{[k]}M_{[k]}A_{[k]}\right), \quad q_k = \|Q_k\|.
$$

Then it follows

$$e_N^k \leq q_k e_N^k + |\theta_k - \bar{\theta}_k| \|u^{k-1}(\bar{x}^{k-1}, \tilde{b}^{k|1})\| + \theta_k - \bar{\theta}_k \delta.
$$

Assuming $e_N^0 = x^0 - \tilde{x}^0 = 0$ it follows by induction

$$e_N^k \leq \sum_{s=0}^{k-2} |\theta_s - \bar{\theta}_s| \|u^s(\bar{x}^s, \tilde{b}^s)\| \prod_{j=s+1}^{k-1} q_j
$$

$$+ \delta \sum_{s=0}^{k-2} \bar{\theta}_s \prod_{j=s+1}^{k-1} q_j + |\theta_k - \bar{\theta}_k| \|u^{k-1}(\bar{x}^{k-1}, \tilde{b}^{k|1})\| + \theta_k - \bar{\theta}_k \delta.
$$

With (3.9), and putting $\hat{q}_k = \max_{0 \leq j \leq k} q_j$ we therefore get

$$e_N^k \leq (\gamma_k-2 \hat{u}_{k-2} + \eta_k-2 \hat{\eta}_k) \sum_{s=0}^{k-2} \prod_{j=s+1}^{k-1} \hat{q}_{k-1} + (\gamma_k-1 \hat{u}_{k-1} + \eta_k-1 \hat{\eta}_k)\delta.
$$
Now
\[
\sum_{s=0}^{k-2} \prod_{j=s+1}^{k-1} \hat{q}_{k-1} = (\hat{q}^{k-1}_{k-1} + \hat{q}^{k-2}_{k-1} + \ldots + \hat{q}^2_{k-1} + \hat{q}^1_{k-1}).
\]

It follows from the properties of geometric progression that,
\[
e^k_N \leq (\gamma_{k-1} \hat{u}_{k-1} + \eta_{k-1} \delta) \frac{1 - \hat{q}^k_{k-1}}{1 - \hat{q}_{k-1}}.
\]

We will now use the following result, derived in [48, Lemma 1], for block-iteration, and [21, Lemma 3.9] for simultaneous iteration,
\[
\|Q_k\| = 1 - \theta_k \sigma^2, \text{ assuming } \alpha = \sigma^2.
\]

Using (3.15) it holds \(\hat{q}_{k-1} = \max_{0 \leq j \leq k-1} (1 - \theta_j \sigma^2) = 1 - \eta_{k-1} \sigma^2\). It follows
\[
e^k_N \leq (\gamma_{k-1} \hat{u}_{k-1} + \eta_{k-1} \delta) \frac{1 - (1 - \eta_{k-1} \sigma^2)^k}{\eta_{k-1} \sigma^2}.
\]

Therefore the result follows using (3.11).

Remark 3.2. In the proof of Theorem 3.1, we assumed cyclic control. However, provided the same control sequence \(\{i_k\}\) (where \(1 \leq i_k \leq p\)) is used in both the noise-free (3.4), and the noisy (3.5) iteration it is obvious that Theorem 3.1 also holds. Examples of other controls are almost cyclic control as defined above and stochastic control [39, Definition 5.1]. Stochastic control has proven to be quite successful and we refer to [55, 44, 46] for some recent developments.

Remark 3.3. When the regularization parameter \(\alpha\) is chosen positive the convergence of P-BIM follows from the contraction mapping theorem as follows. By (3.15) \(Q_k\) is a \(\rho_k\)-contraction with \(\rho_k = 1 - \theta_k \sigma^2\). Since \(P_k\) is a nonexpansive it follows that \(U_k\) (defined in Algorithm 1.2) also is a contraction. Because \(\rho_k < \rho < 1\), it follows that \(U_k\) is a \(\rho\)-contraction. Thus, assuming feasibility, \(x^k\) converges to \(x^* \in \cap_{k=1}^\infty \text{Fix } U_k\).

Remark 3.4. If \(\lambda_k = \lambda_1\), i.e. the relaxation parameters do not depend on the noise, then \(\gamma_k = 0\), and \(\eta_k = \min_{0 \leq j \leq k} \theta_j, \overline{\eta}_k = \max_{0 \leq j \leq k} \theta_j\) so that
\[
\|x^k - \bar{x}^k\| \leq \frac{1}{\overline{\sigma}} \left( \frac{\overline{\eta}_{k-1} \delta}{\eta_{k-1}} \right) \Psi^k(\overline{\sigma}, \eta_{k-1})
\]

This bound, assuming decreasing relaxation parameters \(\theta_k\), coincides with the bound given in [48, Theorem 3].

Next we remind the reader of the inequality \((1 - (1 - x^2)^k)/x \leq \sqrt{k}, x \in (0, 1)\). It follows
\[
\Psi^k(\overline{\sigma}, \eta_{k-1}) = \frac{1 - (1 - \eta_{k-1} \sigma^2)^k}{\sqrt{\eta_{k-1} \sigma}} \leq \sqrt{\eta_{k-1} \sigma},
\]

provided \(\sigma \leq 1/\eta_{k-1}\). Hence we arrive at the bound (with \(c\) a small constant)
\[
\|x^k - \bar{x}^k\| \leq \frac{c}{\overline{\sigma}} \sqrt{k}, \quad \sigma \leq 1/\eta_{k-1},
\]
cf [22, 38] for Kaczmarz’s method and [25] for SIRT. Due to the factor $1/\sigma$ this bound is unrealistically large, but we have not been able to sharpen it. See also [20] for a more detailed discussion.

We next give an alternative upper bound for the noise error presented in Theorem 3.1, when the relaxation parameters are decreasing, i.e. $0 < \theta_{k+1} < \theta_k$. At first, following [24, Propositions 2.3, 2.4], we consider

$$g_{k-1}(y) = (2k - 1)y^{k-1} - (y^{k-2} + \cdots + y + 1)$$

which has a unique root $\zeta_k \in (0, 1)$ and satisfies $0 < \zeta_k < \zeta_{k+1} < 1$ and $\lim_{k \to \infty} \zeta_k = 1$.

**Theorem 3.5.** If the relaxation parameters are decreasing and satisfy

$$0 < \theta_k \leq \frac{1}{\|M_t^{1/2} A_t\|^2},$$

then we have

$$\|x^k - \tilde{x}^k\| \leq \frac{1}{2} \left( \frac{(\theta_0 - \theta_{k-1})\hat{u}_{k-1} + \theta_0 \delta}{\sqrt{\theta_{k-1}}} \right) \frac{1 - \zeta_k^k}{\sqrt{1 - \zeta_k}}.$$  

**Proof.** It holds $\bar{\eta}_{k-1} = \min \theta_s = \theta_{k-1}, \bar{\eta}_{k-1} = \max \theta_s = \theta_0$ and $\gamma_{k-1} \leq |\theta_0 - \theta_{k-1}|$.

Based on [24, Proposition 2.4], we have

$$\max_{0 < \sigma \leq 1/\sqrt{\theta_{k-1}}} \Psi^k(\sigma, \theta_k) \leq \sqrt{\theta_{k-1}} \frac{1 - \zeta_k^k}{\sqrt{1 - \zeta_k}}$$

where $\zeta_k \in (0, 1)$ is the unique root of (3.18). By applying (3.21) in (3.10) we obtain (3.20).

**4. Relaxation parameters.** In [48] three step-size rules for use in block iterative methods were studied (the first two were originally proposed in [24], and in [21] (constrained case) for simultaneous iteration). The main idea with these rules is to control the propagated noise component of the error. Here we use the formulation (3.4) for P-BIM (to conform with the notations in the preceding Section and in [24], [21]).

$$\Psi_1 - \text{rule} \quad \theta_k = \begin{cases} \sqrt{2\sigma}^{-2}, & \text{for } k = 0, 1 \\ 2\sigma^{-2}(1 - \zeta_k), & \text{for } k \geq 2, \end{cases}$$

$$\Psi_2 - \text{rule} \quad \theta_k = \begin{cases} \sqrt{2\sigma}^{-2}, & \text{for } k = 0, 1 \\ 2\sigma^{-2} \frac{1 - \zeta_k}{(1 - \zeta_k^k)^2}, & \text{for } k \geq 2, \end{cases}$$

$$\Psi_3 - \text{rule} \quad \theta_k = \begin{cases} \sqrt{2\sigma}^{-2}, & \text{for } k = 0, 1 \\ 2\sigma^{-2} \frac{(1 - \zeta_k^k)^2}{(1 - \zeta_k)^{1-r}}, & \text{for } k \geq 2. \end{cases}$$

Here $1 < r \leq 2$. All three rules are descending, i.e. $0 < \theta_{k+1} < \theta_k, k \geq 2$. Also for these three rules the relaxation parameters do not depend on the noise.
The following upper bounds for the noise error were derived in [48] (for the simultaneous case see also [21]).

\[
\| x^k - \bar{x}^k \| \leq \begin{cases} 
\frac{\beta_{\delta b}}{\sigma} (1 - \zeta_k^k) / (1 - \zeta_k), & \Psi_1 \text{ - rule} \\
\frac{\beta_{\delta b}}{\sigma} (1 - \zeta_k^k)^2 / (1 - \zeta_k), & \Psi_2 \text{ - rule} \\
\frac{\beta_{\delta b}}{\sigma} (1 - \zeta_k), & \Psi_3 \text{ - rule}
\end{cases}
\]

Here

\[
\beta_{\delta b} = \max_{1 \leq t \leq p} \| M^T_{[t]} \delta b[t] \|, \quad \beta_b = \max_{1 \leq t \leq p} \| M^T_{[t]} b[t] \|.
\]

We next derive a new relaxation parameter rule which will be based on the bound (3.20). We will then use the following heuristic \( \hat{u}_k \leq \hat{u}_0 \). This was fulfilled in all our numerical experiments (but we have no formal proof). Also we use, for simplicity \( x^0 = 0 \) (as in the experiments). Hence

\[
\hat{u}_k \approx \hat{u}_0 \leq \max_{1 \leq t \leq p} \| A_t M_t b^t \| \leq \sigma \beta_b.
\]

Further using (3.6) \( \hat{\delta} \leq \sigma \beta_{\delta b} \). Summarizing we get

\[
(4.6) \quad \hat{u}_k \approx \sigma \beta_b, \quad \hat{\delta} \approx \sigma \beta_{\delta b}.
\]

Again the basic idea is to control the noise error. Therefore consider the function

\[
(4.7) \quad \frac{\beta_{\delta b}}{\sigma} (1 - \zeta_k^k)^{\frac{1}{2}} (1 - \zeta_k)^{\frac{\sigma}{1 - \zeta_k}}
\]

We now enforce that the noise error bound (3.20) equals (4.7). It follows, also using (4.6)

\[
\sigma \left( \frac{(\theta_0 - \theta_{k-1}) \beta_b + \theta_0 \beta_{\delta b}}{\sqrt{\theta_{k-1}}} \right) \frac{1 - \zeta_k^k}{\sqrt{1 - \zeta_k}} = \frac{\beta_{\delta b}}{\sigma} (1 - \zeta_k^k)^{\frac{1}{2}} (1 - \zeta_k)^{\frac{\sigma}{1 - \zeta_k}}
\]

After simplification we get

\[
\sigma \left( \frac{(\theta_0 - \theta_{k-1}) \beta_b + \theta_0 \beta_{\delta b}}{\sqrt{\theta_{k-1}}} \right) = \frac{\beta_{\delta b} (1 - \zeta_k^k)^{\frac{1}{2}}}{\sqrt{1 - \zeta_k}}.
\]

By solving for \( \theta_{k-1} \), and using that \( \theta_0 = \sqrt{2/\sigma^2} \) one gets (after elementary calculations)

\[
\theta_{k-1} = \frac{B + Z_{k,r}^2 \beta_b^2 - Z_{k,r} \beta_{\delta b} \sqrt{Z_{k,r}^2 \beta_b^2 + 2B}}{2 \sigma^2 \beta_b^2},
\]

where

\[
B = 2 \sqrt{2} \beta_b (\beta_b + \beta_{\delta b}), \quad Z_{k,r} = \frac{(1 - \zeta_k^k)^{\frac{1}{2}}}{\sqrt{1 - \zeta_k}}.
\]
Summarizing we have the following rule

\[(4.8)\]

\[
\Gamma - \text{rule} \quad \theta_k = \begin{cases} 
\frac{\sqrt{\beta}}{\alpha}, & \text{for } k = 0, 1 \\
\frac{B + Z_{k,r}^2 - Z_{k,r} \beta_{bh} \sqrt{Z_{k,r}^2 + 2B}}{2\sigma^2 \beta_{bh}^2}, & \text{for } k \geq 2,
\end{cases}
\]

where \(1 < r \leq 2\). Note that the parameters depend on the noise. With no noise present \(\theta_k = \theta_0\), \(k \geq 1\). We mention here the recent paper [2, Proposition 5] where also knowledge of \(\|\delta b\|\) is utilized. By construction the noise upper bound for the \(\Gamma\)-rule is

\[(4.9)\]

\[
\|x^k - \bar{x}^k\| \leq \frac{\beta_{bh}}{\alpha} \left(1 - \zeta_k \right)^{\frac{1}{2}} \left(1 - \zeta_k \right)^{-\frac{1}{2}}, \quad \Gamma - \text{rule}.
\]

Let \(\|x^k - \bar{x}^k\|_\Omega\) denote the upper bound for the noise error when \(\Omega\)-rule is used for choosing relaxation parameters. Then the following inequalities obviously hold,

\[(4.10)\]

\[
\|x^k - \bar{x}^k\|_\Omega \leq \|x^k - \bar{x}^k\|_{\Psi_3} \quad \text{and} \quad \|x^k - \bar{x}^k\|_{\Psi_2} \leq \|x^k - \bar{x}^k\|_{\Psi_1} \quad \text{for} \quad k \geq 0.
\]

The bounds for the \(\Psi_3\) and the \(\Psi_2\) rules respectively may intersect see [48, Figure 2]. We next show that the \(\Gamma\)-rule also is a diminishing step-size strategy.

**Lemma 4.1.** Let \(r \in (1, 2)\), \(k \geq 2\). The relaxation parameters in the \(\Gamma\)-rule are descending, i.e. \(0 < \theta_{k+1} < \theta_k\).

**Proof.** Since \(\zeta_k \in (0, 1)\), we have

\[
\frac{dZ_{k,r}}{d\zeta_k} = \left( \frac{r - 1}{(1 - \zeta_k)} + \frac{k\zeta_k^{k-1}}{(1 - \zeta_k)} \right) \left(1 - \zeta_k\right)^{-\frac{1}{2}} \geq 0.
\]

Showing that \(Z_{k,r}(\zeta_k)\) is increasing. Next

\[
\frac{d\theta_k}{dZ_{k,r}} = \frac{1}{2\sigma^2 \theta_0 \sqrt{\beta_{bh}^2 Z_{k,r}^2 + 2B}} \tau_{k,r},
\]

with

\[
\tau_{k,r} = \beta_{bh} Z_{k,r} \sqrt{\beta_{bh}^2 Z_{k,r}^2 + 2B - (\beta_{bh}^2 Z_{k,r}^2)}.
\]

We now show that \(\tau_{k,r} < 0\). To this end assume the contrary, i.e.

\[
\beta_{bh} Z_{k,r} \sqrt{\beta_{bh}^2 Z_{k,r}^2 + 2B} \geq (\beta_{bh}^2 Z_{k,r}^2),
\]

After squaring and simplifying one gets \(B^2 \leq 0\), a contradiction. Hence \(\frac{d\theta_k}{dZ_{k,r}} < 0\). As noted previously \(\zeta_k(k)\) is increasing. Therefore the result follows by the chain rule. \(\square\)

In Section 5, in addition to the above rules also the '\(\theta\)-opt' rule is used. This means finding that constant value of \(\theta\) which give rise to the smallest relative error within a fixed number of cycles (\(cnax\)). Here a cycle denotes one pass through all \(p\) row-blocks. This value of \(\theta\) is found by searching over the interval \(0 < \theta_{\text{opt}} < 2/\sigma^2\). This strategy requires knowledge of the exact solution, so for real data one would first need to train the algorithm using simulated data, see [32, Section 6]. In our tests we took \(cnax = 100\) and used the exact phantom for training. This is clearly unrealistic but we think it is of interest to include this rule for comparison with our other rules.
5. Numerical Results. We will report on tests using examples from the field of image reconstruction from projections. To create the projection matrix $A$ and the right hand side $b$, the paralleltoomo function in the MATLAB package AIR tools [32] is used. We take the Shepp-Logan phantom as the original image $x^*$ discretized into $365 \times 365$ square pixels. In the first test problem we use 88 views uniformly distributed over 180 degrees, and 516 projections per view. Since zero rows do not contribute to the reconstruction, after identifying and removing these, the resulting matrix $A$ has dimension $40796 \times 133225$ (case-one). Although in our application iterative methods usually are more competitive the more underdetermined the system is this might not be the case in other applications. Therefore we also consider taking more rays (264) per projection leading to a matrix of dimension $122388 \times 133225$ (case-two).

Apart from using noise-free data we added independent Gaussian noise of mean 0 and relative noise-level ($\parallel \delta b \parallel / \parallel b \parallel$) 2% and 5% respectively. In the experiments we set $x^0 = 0$ and $C_{[t]} = [0, 1]^n$ for $t = 1, \ldots, p$. We further partition the matrix $A$ and the right hand side $b$ into 8 and 22 blocks, respectively. The largest and smallest (nonzero) singular values of each block is estimated using the power method [56]. We further used Cimmino’s $M$-Matrix. The error metric is Relative Error defined by

$$\text{Relative Error} = \frac{\parallel x^k - x^* \parallel}{\parallel x^* \parallel}.$$ 

All codes are written in MATLAB(R2015a) and conducted on a PC with a Intel Core i7-7700K CPU @ 4.2 GHz and 16 GB RAM.

When using the $\Gamma$-rule we need to estimate $\parallel M^{\frac{1}{2}} \delta b \parallel$. Therefore we randomly generated a new vector $\overline{\delta b}$ analogous with original $\delta b$. To show that the estimate $\parallel M^{\frac{1}{2}} \overline{\delta b} \parallel$ does not strongly influence the final results, for each case, three random $\delta b$ with different noise levels are generated as follows. We first generated $e = \text{randn(size(b))}$, then put $\overline{\delta b} = g \parallel b \parallel e / \parallel e \parallel$ where $g$ denotes guessed noise level. We take $g = 0.01, 0.02$ and 0.03 for 2% noise and $g = 0.03, 0.05$ and 0.07 for 5% noise. Table 1 shows exact (i.e. $\parallel M^{\frac{1}{2}} \delta b \parallel$) and estimated values (i.e. $\parallel M^{\frac{1}{2}} \overline{\delta b} \parallel$). It will be seen from Figures below that the error-curves are quite robust versus the estimates of the noise.

Following [48], we take $r = 1.5$ in the $\Psi_3$ rule. As noted above $r \in (1, 2]$ in the $\Gamma$-rule. In Figure 1 the effect of picking different $r$-values is displayed. As seen, the choice of proper value of $r$ can have efficacious impact on the rate of convergence. The value of $r$ should also be chosen in such a way that the property (3.19) is satisfied.

| Problem/ noise | # of blocks | Exact value | Estimated value |
|---------------|-------------|-------------|-----------------|
| g=1%          |             |             | g=1% g=2% g=3% |
| case-one/ 2%  | 8           | 14.04       | 5.07 10.14 15.22|
|               | 22          | 19.71       | 8.70 17.40 26.10|
| case-two/ 2%  | 8           | 8.75        | 4.87 9.56 14.33 |
|               | 22          | 15.82       | 7.59 15.17 22.77|
| g=3%          |             |             | g=3% g=5% g=7% |
| case-one/ 5%  | 8           | 28.36       | 19.29 32.14 45.01|
|               | 22          | 45.91       | 31.55 52.59 73.64|
| case-two/ 5%  | 8           | 15.10       | 8.95 14.91 20.88 |
|               | 22          | 21.57       | 13.48 22.47 31.46|
at least after a few iterations. To insure this, we take \( r = 1.5 \) and \( 1.75 \) for 2% and 5% noise, respectively. Then (3.19) will be satisfied, for \( k \geq 2 \). We further remark that our computational experience shows that the unregularized problem \( (\alpha = 0) \) gives results that are indistinguishable from those of the regularized problem with \( \alpha = \sigma^2 \). In the sequel we study relative error curves during \( c_{max} = 100 \) cycles, but will also extend to \( c_{max} = 500 \) cycles to study semiconvergence. Figure 2 illustrates the importance of incorporating constraints, especially for highly underdetermined systems, during the iterations.

We next discuss Figures 3 and 4 where we display error-curves, and relaxation parameters for both case-one and case-two using 2% noise-level. Since the relaxation parameters showed a similar behavior for the same noise-level we only display these for case-one. We observe that the relative error is much smaller for the \( \Gamma \)-rule than for the best \( \Psi \)-rule \((\Psi_3)\). The reason for this behavior could be that the corresponding relaxation parameters are bigger using the \( \Gamma \)-rule than using the \( \Psi \)-rule.

In Figures 5 and 6 we show error curves and relaxation parameters (for case-two) using 5% noise. It is seen that now there is smaller difference in relative error between the \( \Gamma \)-rule and the \( \Psi_3 \)-rule. We see from the figures that the relaxation parameters using the two rules are quite close in value. We have also listed the minimal error and corresponding cycle number in Tables 2 and 3.

We note from Figures 3, 4, 5 and 5 that semi-convergence has hardly begun for the \( \Gamma \)-rule. To see more clearly this effect we extend in Figure 7 the error curves (for two
Table 2
Minimum relative error and corresponding cycle number, when 2% noise applied.

| Test Problem | # of blocks | $\theta_{opt}$ | $\Psi_3$ | Strategy | $\Gamma$ |
|--------------|-------------|----------------|----------|----------|----------|
|              |             |                |          | $g = 0.01$ | $g = 0.02$ | $g = 0.03$ |
| Case One     | 8           | (66, 0.1531)   | (100, 0.2914) | (100, 0.1543) | (100, 0.1622) | (100, 0.1706) |
|              | 22          | (29, 0.1538)   | (100, 0.2295) | (100, 0.1530) | (100, 0.1567) | (100, 0.1613) |
| Case Two     | 8           | (40, 0.1221)   | (100, 0.2715) | (100, 0.1265) | (100, 0.1449) | (100, 0.1597) |
|              | 22          | (15, 0.1219)   | (100, 0.2128) | (64, 0.1217)  | (100, 0.1237) | (100, 0.1300) |
cases) to $c_{\text{max}} = 500$. In this Figure we have marked the point where the error has its minimum, i.e. the cycle number where semi-convergence starts. We see that the $\theta-$opt rule has the fastest convergence (note again that we used the exact phantom for training). For this rule the semi-convergence behavior is quite pronounced, and hence it requires a reliable stopping criterion (the choice of stopping criterion is an interesting issue but is not addressed in this paper). In contrast the $\Psi-$ and $\Gamma-$ rules show little effect of semi-convergence, and therefore it is less critical where the iterations are stopped.
Table 3

Minimum relative error and corresponding cycle number, when 5% noise applied.

| Test Problem | # of blocks | Strategy | $\theta_{opt}$ | $\Psi_3$ | $\Gamma$ |
|--------------|-------------|----------|---------------|----------|----------|
|              |             |          |               |          | $g = 0.03$ | $g = 0.05$ | $g = 0.07$ |
| Case One     | 8           | $\Psi_1$ | (12, 0.2383)  | (100, 0.2914) | (100, 0.2439) | (100, 0.2666) | (100, 0.2866) |
|              | 22          | $\Psi_2$ | (5, 0.2392)   | (100, 0.2557) | (97, 0.2398)  | (100, 0.2495) | (100, 0.2639) |
| Case Two     | 8           | $\Psi_3$ | (15, 0.1947)  | (100, 0.2769) | (100, 0.2606) | (100, 0.2356) | (100, 0.2408) |
|              | 22          | $\Gamma$ | (6, 0.1948)   | (100, 0.2559) | (100, 0.1952) | (100, 0.2200) | (100, 0.2313) |

The behavior of the relative noise error $\|x^k - \bar{x}^k\|/\|\bar{x}^k\|$, and the relative iteration error $\|\bar{x}^k - x^*\|/\|x^*\|$ for case-one is shown in Figure 8. Figure 9 shows the phantom and reconstructions using the $\Psi_3$ and the $\Gamma$-rule respectively for case-one. To better judge the quality of the two reconstructions we display also the corresponding difference images. These are defined as the difference between the phantom and the respective reconstruction. More artifacts and noise can be seen in the left image ($\Psi_3$) than in the right image ($\Gamma$).

6. Conclusion. We define a sequential block-iterative iteration (2.12) for solving split feasibility problems in Hilbert space. In this respect it compliments the simultaneous block iteration given in [15] (defined in the finite dimensional case). The basic operators involved are weakly regular cutters which includes e.g. metric projections. A complete convergence analysis is provided. When the projecting sets are polyhedral it is also shown that the iterates converge linearly. The projected Block Iterative method (P-BIM) Algorithm 1.2 is a special case of our general method. We also consider the noise error of P-BIM, and derive a new upper bound. This bound generalizes earlier bounds given in [21, 24, 22, 48]. In particular we extend the noise error analysis of block-iteration to projected block-iteration, and also to the case when the relaxation parameters are allowed to depend on the noise. Based on the new bound a new rule for picking relaxation parameters is derived. We demonstrate the performance of P-BIM using this and other relaxation parameter rules on examples taken from tomographic imaging.

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Fig. 8. Relative noise error $\|x^k - \bar{x}^k\|/\|x^*\|$ (dashed), and the relative iteration error $\|\bar{x}^k - x^*\|/\|x^*\|$.

Fig. 9. (a) The original image (left), the reconstructed image using $\Psi_3$ strategy (middle) and the reconstructed image using the $\Gamma$ strategy, and (b) the difference images for the $\Psi_3$ strategy (left) and $\Gamma$ strategy (right).

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