MACROSCOPIC DIMENSION AND FUNDAMENTAL GROUP OF MANIFOLDS WITH POSITIVE ISOTROPIC CURVATURE

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Abstract. We prove a conjecture of Gromov’s to the effect that manifolds with isotropic curvature $K_{isotr}(M) \geq \epsilon^{-2}$ and with bounded geometry are macroscopically 1-dimensional on the scale $>> \epsilon$. As a consequence we prove that compact manifolds with positive isotropic curvature have virtually free fundamental groups. Our main technique is modeled on Donaldson’s version of Hörmander technique to produce (almost) holomorphic sections which we use to construct destabilizing sections.

1. Introduction

Given a Riemannian manifold $(M, g)$, one can extend the metric tensor in two ways to the complexified tangent bundle $TM \otimes \mathbb{C}$: as a complex bilinear $(\cdot, \cdot)$ form or as a Hermitian form $\langle \cdot, \cdot \rangle_{\mathbb{C}}$. A tangent vector $v$ is called isotropic if $(v, v) = 0$, and analogously a 2-plane $\pi \subset TM \otimes \mathbb{C}$ is called totally isotropic if $(u, v) = 0$ for every $u, v \in \pi$.

If we view the Riemannian curvature tensor $R$, as a quadratic form $R_m$ on $\bigwedge^2 TM$ (this is the curvature operator), we can clearly extend it to a quadratic form $R_m \mathbb{C}$ on $\bigwedge^2 (TM \otimes \mathbb{C})$.

The sectional curvature $K(u, v) := \langle R(s, u)u, s \rangle$, inasmuch as the restriction of $R_m$ on bivectors, can then be extended to the complexified tangent bundle. In other words, if we think of the sectional curvature as a function $K$ on $Gr(2, TM)$– the Grassmannian bundle of 2-planes in $TM$– we can extend it as a function $K_{\mathbb{C}}$ to the Grassmannian bundle of complex 2-planes in $TM \otimes \mathbb{C}$ $Gr_{\mathbb{C}}(2, TM \otimes \mathbb{C})$, as follows:

$$K_{\mathbb{C}}(\pi) := R_m(v \land w, v \land w)$$

where $v$ and $w$ are two vectors in $\pi$ which are orthogonal with respect to the Hermitian product $\langle \cdot, \cdot \rangle_{\mathbb{C}}$. Restricting the function $K_{\mathbb{C}}$ to the subbundle of totally isotropic two-planes $Gr_{\mathbb{C}}^{isotr}(2, TM)$ (which is non-empty only if $\dim(M) \geq 4$) we obtain the isotropic curvature $K_{\mathbb{C}}^{isotr}$.

We are now ready for
Definition 1. We say that $M$ has positive isotropic curvature if $K_{isotr} > 0$ and that the isotropic curvature is bounded below by $k$ if $K_{isotr} \geq k$

These conditions are readily seen to be equivalent to the requirement that for any orthonormal 4-frame $\{e_1, e_2, e_3, e_4\}$ one has:

$$R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} > 0 \text{ resp. } > k$$

where $R_{ijkl} = Rm(e_i, e_j, e_k, e_l)$.

Positivity of the isotropic curvature $K_{isotr}$ is implied by the positivity of the complex sectional curvature $K_C$ (which in turn is implied by the strong condition of positivity of the Riemannian curvature operator), and it implies the positivity of the scalar curvature.

Nonetheless there are examples of manifolds with positive sectional curvature, or even positive Ricci curvature, which do not admit positive isotropic curvature, and more importantly there are examples of manifolds which admit positive isotropic curvature, but which cannot admit a metric with positive Ricci curvature. For instance, in dimension $n = 4$, every locally conformally flat manifold with positive scalar curvature has a metric with positive isotropic curvature, as shown in [32]. More explicitly, $M_k := \#_{i=1}^k S^3 \times S^1$ has a metric with positive isotropic curvature for any $k$ (indeed, as proved in [30], the connected sum of any two manifolds with uniformly positive isotropic curvature admits such a metric), but for topological reasons (which are a consequence of the splitting theorem of Cheeger-Gromoll), it cannot support a metric with non-negative Ricci curvature. On the other hand $\mathbb{C}P^2 \#_{i=1}^k \mathbb{C}P^2$ admits no metric such that $K_{isotr} \geq 0$ for $1 \leq k \leq 8$ (cf. [32]), nonetheless it admits a metric with $\text{Ric} > 0$ by Yau’s proof of Calabi’s conjecture (as it is easy to show that the canonical class is positive); in fact, even a Kähler-Einstein one (necessarily with positive scalar curvature) if $k > 2$ (cf. [40]).

The main result in [29] (cf. [3] for a nice survey) is the following strong topological restriction imposed by positive isotropic curvature:

Theorem 2. (Micallef-Moore) A compact manifold $M$ with positive isotropic curvature is such that $\pi_i(M) = (0)$ for $2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$, where $\lfloor x \rfloor$ indicates the integral part of $x$.

The connected sum of manifolds with $K_{isotr} \geq k > 0$ also admits such a metric (cf. [30]). $S^3 \times S^m$ and spherical space forms admit such metrics. The main conjecture for manifolds with positive isotropic curvature is that, so far as the fundamental group is concerned, these connected sums are the only manifolds with positive isotropic curvature. More precisely, the following conjecture has been put forth:
**Conjecture.** The fundamental group of a compact manifold with positive isotropic curvature is residually free.

A particular case of this conjecture, to the effect that the fundamental group of such manifolds cannot contain subgroups abstractly isomorphic to the fundamental group of a Riemann surface of genus \( p \geq 1 \), has been proved by A. Fraser–J.Wolfson (cf. [15]) (the special case \( p = 1 \) had been previously proved by A. Fraser in the foundational paper [14]).

Following Gromov (cf. [21]) we define:

**Definition.** A metric space \( V \) has macroscopic dimension at most \( k \) on the scales \( \gg \epsilon \), if there exists a \( k \)-dimensional polyhedron \( P \) and a continuous map \( \phi : V \to P \), such that for every fiber \( \phi^{-1}(p) \subset V \), \( \text{diam} \phi^{-1}(p) \leq \epsilon \). If this is the case, one writes \( \text{dim}_\epsilon V \leq k \). Moreover, one sets:

\[
\text{dim}_\epsilon V := \sup \{ k : \text{dim}_\epsilon V \leq k \}
\]

The prospective of Gromov’s in [21] on manifolds with positive curvature through his concept of *macroscopic dimension* turns out to be particularly fruitful in answering the above mentioned conjecture, especially when combined with the ideas of Gromov-Lawson (cf. [23]) and the stability inequality due to Micallef and Moore (cf. [29]).

In fact, we can show the following conjecture of Gromov’s (cf. [21], para. 3):

**Theorem 3.** If \((M,g)\) is of bounded geometry and \( K^\text{isotr}_\Sigma(M) \geq \epsilon^{-2} \), \( \text{dim}(M) \geq 4 \). Then \( M \) is macroscopically 1-dimensional on the scale \( \gg \epsilon \).

Gromov remarks that this is enough to show that \( \pi_1(M) \) is residually free. This in fact can be done by applying the estimates to the universal covering, but we will resort to a different approach due to Ramachandran-Wolfson (cf. [34]), as explained below.

The assumption that the manifold has bounded geometry is necessary for the solution of the Plateaux problem. One should remark that the assumption that the isotropic curvature be strictly (and uniformly) positive cannot be relaxed, as shown by the example \( M = \Sigma \times S^k \), which admits a metric with non-negative isotropic curvature—here \( \Sigma \) is a Riemann surface of genus \( p \geq 2 \) and \( S^k \) is the round \( k \)-sphere. This is reflected in the fact that the stability inequality, among other things, yields no information when the isotropic curvature is not strictly (and definitely) positive.

One of the main ingredients—just like in Fraser’s fundamental paper [14]—is the second variation formula of the energy functional, or better
yet its manifestation in the form of the stability inequality (cf. eq. (2)), as described in [29], which we will explain briefly in section 2.

Another major and arguably more important for this paper ingredient comes into play in the construction of the destabilizing sections and is based on using Donaldson’s version of Hörmander techniques to construct suitable almost holomorphic sections (cf. [11] and [12]). Our construction is closely based on Donaldson’s (cf. also Donaldson-Sun), with the difference that we need to make sure that the section also be isotropic. The philosophy is mostly based on Donaldson’s construction of almost holomorphic sections in [11], the major difference being that unlike in [11], we work with integrable complex structures, as in [12], but even in this case we do not need the full blown Hörmander technique as we merely need the sections to be almost holomorphic (and in fact we eventually need to multiply them by cut-off functions, in order to make them compactly supported). We thus effectively first construct highly peaked holomorphic sections (i.e., concentrated on a very small ball) following Donaldson’s argument (the result resembles Tian’s construction of peak sections, cf. [12] but it is remarkably different).

Finally we use a cut-off function argument to render the sections thus constructed compactly supported. The estimate we prove using these ingredients – and itself the major ingredient in proving Theorem 3 is

**Theorem 4.** Assume that $K^{\text{isotr}}_C(M) \geq \epsilon^{-2}$, and that $\dim(M) \geq 4$. Let $f : D \to M$ be a stable, minimal (possibly branched) immersion. Then for every point $p \in D$ there exists a smooth compactly supported isotropic section $\sigma = \sigma_p$ of $E$, and a constant $C$ such that:

$$\int_D |\nabla \frac{\partial}{\partial \bar{z}} \sigma|^2 dV \leq \frac{1}{r^2} K \frac{1}{K^2}.$$

where $r := \text{dist}_D(p, D)$. Furthermore, the constant $K = K(n)$ is computable and it can be taken to be equal to $\frac{9^{3/2} \pi}{4}$.

**Corollary 5.** Assume that $K^{\text{isotr}}_C(M) \geq \epsilon^{-2}$, and that $\dim(M) \geq 4$ and that $M$ has bounded geometry. Then for every closed curve $\gamma$ such that $[\gamma] = 0$ in $H_1(M)$, then:

$$\text{FillRad}\gamma \leq C\epsilon$$

where we can take $C = \sqrt{\frac{9^{3/2} \pi}{4}}.$
An immediate corollary of this coupled with Theorem 1.2 in [34], is the following

**Theorem 6.** If $M$ is a closed manifold such that that $K_{\text{isotr}}(M) \geq \epsilon^{-2}$ then $\pi_1(M)$ is residually free.

Also, using the main theorem of [16] one has the following immediate consequence:

**Theorem 7.** Let $M$ be a closed, orientable Riemannian $n$-manifold with positive isotropic curvature of dimension $n \geq 5$. Then there exists a finite cover of $M$ which is homeomorphic to the connected sum of $k$ copies of $S^{n-1} \times S^1$.

Finally, we would like to point out that in dimension 4 something much stronger is true: using the Ricci flow, R. Hamilton (cf. [25]) and B.-L. Chen and X.P. Zhu (cf. [5]) have been able to prove that a compact 4-manifold with positive isotropic curvature and containing no essential incompressible 3-dimensional space form, is diffeomorphic to $S^4$, $S^3 \times S^1$, $\mathbb{R}P^4$ and $S^3 \tilde{\times} S^1$ and their connected sums (naturally, the last two do not occur in the oriented case). In fact in a recent beautiful paper, Chen, Tang and Zhu (cf. [6]) – using Hamilton’s Ricci flow – have proven the 4-dimensional version of a very far reaching conjecture due to R. Schoen, which claims that the (much stronger) differential version of Theorem 7 should hold.

We finally would like to end the introduction by pointing out an extremely interesting question posed by M. Gromov (cf. [22]) on possible generalizations to the singular setting: “Is there a natural class of singular spaces $X$ with $K_X(X) > 0$ that would satisfy (a suitable version of) the Micallef-Moore and/or La Nave bounds on indices and sizes of harmonic maps of surfaces into $X$?”

1.1. **Outline of proof.** As explained in the introduction, the main ingredients in the proof are the stability inequality of Micallef-Moore (already used also in [14] for similar purposes) and the construction of almost holomorphic sections with controlled $L^2$-norms. The construction of the sections roughly goes as follows. We first show in Proposition 17 that we can reduce to and solve a Dirichlet problem, when the Hermitian metric $H$ and the metric on the disk are controlled in $L^\infty$ with respect to, respectively, the standard Hermitian metric $H_0(v, w) = \sum_i v_i \bar{w}_i$ and the flat metric $g_0 = dx^2 + dy^2$.

Next in section 3.6 we show, using the Dirichlet problem solved in Proposition 17 that in the model situation we can find the sections we seek. By means of rescaling, we then prove in Proposition 27 that in the
case in which the metric on the disk is the flat metric \( g_0 = dx^2 + dy^2 \), we can reduced to the aforementioned model case, with controlled errors –thanks to Proposition 23 which is a direct application of one of the incarnations of the Bochner formula (as explained in [12] in the line bundle case). Then the use of rescaling allows us to construct the desired Gaussian holomorphic \textit{isotropic} sections Theorem 30 with controlled \( L^2 \) norms. Finally we use a cut-off function argument in Proposition 31 to construct the “almost-destabilizing” sections: smooth almost-holomorphic \textit{compactly supported} isotropic sections with controlled \( L^2 \) norms on concentric balls.

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2. The second variation formula

Let \( f : \Sigma \rightarrow M \) be a stable minimal surface in \( M \). Consider the pull back of the tangent bundle with the pull back of the metric and (resp. normal) connection \( \nabla \). Let \( E = f^*TM \otimes \mathbb{C} \) be the complexified bundle. The metric on \( f^*(T_M) \) extends as a complex bilinear form \( \langle \cdot, \cdot \rangle \) or as a Hermitian metric \( \langle \cdot, \cdot \rangle \) on \( E \), and the connection \( \nabla \) (the pull-back via \( f \) of the Levi-Civita connection of \( M \)) and curvature tensor extend complex linearly to sections of \( E \). Moreover the connection is Hermitian with respect to \( \langle \cdot, \cdot \rangle \).

By a well known theorem, (cf. [24] and [1]), there is a unique holomorphic structure on \( E \) such that the \( \bar{\partial} \) operator

\[
\bar{\partial} : \mathcal{A}^{p,q}(E) \rightarrow \mathcal{A}^{p,q+1}(E),
\]

where \( \mathcal{A}^{p,q}(E) \) denotes the space of \( (p,q) \)-forms on \( \Sigma \) with values in \( E \), is given by

\[
\bar{\partial} \omega = (\nabla \omega)_{\bar{z}} dz
\]

where \( \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}) \), for local coordinates \( x, y \) on \( \Sigma \).

One can choose a metric \( ds^2 = \lambda(dx^2 + dy^2) \) on \( \Sigma \) compatible with the conformal structure determined by the pull-back of \( G = f^*g \) of the metric \( g \) on \( M \) via the immersion \( f \). Then one has:

\[
\|\nabla \frac{\partial}{\partial \bar{z}} W\|^2 dV_{ds^2} = |\nabla \frac{\partial}{\partial \bar{z}}|^2_H dx \wedge dy
\]
where by $|\nabla \frac{\partial f}{\partial z}|^2_H$ we mean $\langle \nabla \frac{\partial f}{\partial z}, \nabla \frac{\partial f}{\partial z} \rangle$, the Hermitian scalar product induced by $f^*g$ on $E$. Suppose $f : \Sigma \to M$ is a stable minimal immersion. Then the complexified stability inequality (see [38], [28], [29], [3], and [14]) for the Energy functional reads:

$$\int_\Sigma \langle R(s, \frac{\partial f}{\partial z}), \frac{\partial f}{\partial z} \rangle \ dx \wedge dy \leq \int_\Sigma |\nabla \frac{\partial f}{\partial z} s|^2 \ dx \wedge dy$$

for all $s \in A_0^\infty (E)$, the space of smooth sections of $E$ with compact support (if the surface has boundary) or for all $s \in A^\infty (E)$, the space of smooth sections, if $\Sigma$ is closed. Assume now that $s$ is isotropic. Since $f$ is conformal, $\frac{\partial f}{\partial z}$ is isotropic and $\{s, \frac{\partial f}{\partial z}\}$ spans an isotropic two-plane. If the isotropic curvature is such that $K_{\text{isotr}} \geq \epsilon - 2$, we get:

$$\epsilon^{-2} \int_\Sigma |s|^2 \ dV \leq \int_\Sigma |\nabla \frac{\partial f}{\partial z} s|^2 \ dV$$

where $dV$ denotes the area element for the induced metric $f^*g$ on $\Sigma$. Here the norms are $\frac{1}{\sqrt{\lambda}}$ times the corresponding norms coming from $TM \otimes \mathbb{C}$ (cf.[29]). We will also denote by $N := \nu_f \otimes \mathbb{C}$ where $\nu_f$ is the normal bundle of $f$, i.e. the bundle defined by the exact sequence of real bundles:

$$0 \to T\Sigma \to f^*TM \to \nu_f \to 0.$$

The same considerations we did for $E$ hold for $N$ and as observed by A. Fraser in [13], the stability inequality can be formulated as:

$$\epsilon^{-2} \int_\Sigma |s|^2 \ dV \leq \int_\Sigma |\nabla \frac{\partial f}{\partial z} s|^2 \ dV$$

for any compactly supported section $s$ of $N$ and here $\nabla \perp$ is the connection induced to the normal bundle from the Levi-Civita connection.

### 3. The test sections

Throughout this section, $f : D \to M$ will be a stable, minimal (possibly branched) proper immersion from the disk $D$ to $M$. We will also maintain the notation of section 2: $E := f^*(TM \otimes \mathbb{C})$ etc. Let $q_D$ be the quadratic form on $E$ induced from the $\mathbb{C}$-bilinear form $(,).$ If $\gamma$ is a smooth curve in $D$, then we denote by $q_\gamma$ the restriction of $q_D$ to $E|_{F\gamma}$. Furthermore we will call a smooth section $\alpha$ of $E|_{\gamma}$ isotropic if $q_\gamma(\alpha, \alpha) = 0$. 


3.1. Curvature of Hermitian metrics on Riemann surfaces. Recall that given a Hermitian holomorphic bundle \((E, H)\) one has a unique connection \(\nabla\) whose \((0, 1)\)-part \(\nabla^{(0,1)}\) is equal to \(\bar{\partial}\) (the operator determining the integrable complex structure of \(E\)): the Hermitian connection. If \(\{e_1, \cdots, e_n\}\) is a holomorphic frame, and if:

\[ h_{ij} := H(e_i, \bar{e}_j) \]

then the connection 1-form and the curvature 2-form (which is an \(\text{End}(E)\) valued 2-form) are given (respectively) by:

\[ A = \partial h \cdot h^{-1}, \quad \Theta_H := \bar{\partial}H = -\partial \bar{\partial}h \cdot h^{-1} + \partial h \cdot h^{-1} \land \bar{\partial} h \cdot h^{-1} \]

The curvature tensor—which is a section of \(E^* \otimes \bar{E}^* \otimes \Omega_N^{(1,0)} \otimes \Omega_N^{(0,1)}\) where \(\Omega_N^{(p,q)}\) is the space of \((p,q)\)-forms—is given by:

\[ R(H)(v, w, s, \bar{t}) := H(\Theta_H(s \land \bar{t})v, \bar{w}) \]

and in the frame \(\{e_1, \cdots, e_n\}\) and (local) holomorphic coordinates \(z_1, \cdots, z_m\) on \(N\):

\[ R_{\alpha \bar{\beta}} := R(e_\alpha, \bar{e}_\beta, v, \bar{w}) = -\partial^2 h_{ij} \frac{\partial h_{ij}}{\partial z_\alpha} + \frac{\partial h_{ij}}{\partial z_\alpha} h^{st} \frac{\partial h_{st}}{\partial z_\beta} \]

On a Riemann surface, having fixed an holomorphic coordinate \(z\), we simply denote the curvature:

\[ R_{ij} := R(e_i, \bar{e}_j, \partial \frac{\partial}{\partial z}, \partial \frac{\partial}{\partial \bar{z}}) = -\partial^2 h_{ij} \frac{\partial h_{ij}}{\partial z \partial \bar{z}} + \frac{\partial h_{ij}}{\partial z} h^{st} \frac{\partial h_{st}}{\partial \bar{z}} \]

so that:

\[ \Theta_H^i_{\bar{j}} = h^{i\bar{s}} R_{\bar{s} j} dz \land d\bar{z}. \]

One can define th Ricci curvature of \(h\), denoted \(\text{Ric}(h)\), as follows (cf. [43] ch.3 or [26] section 1 for this notion). Let \(\{e_\alpha\}\) be a holomorphic frame for \(E\) and \(\{V_i\}\) any frame field of type \((1,0)\) relative to some fixed Hermitian metric \(g\) on \(N\); then set:

\[ \text{Ric}_h(V_i, V_j) := \sum_{i,\nu} h^{i\bar{\nu}} R(e_\alpha, \bar{e}_\beta, V_i, \bar{V}_j) \]

where \(h_{ij} := h(e_\nu, e_\xi)\). In components, this is simply:

\[ R_{\alpha \bar{\beta}} := \text{Ric}(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\beta}) = h^{i\bar{j}} R_{\alpha \bar{\beta}} \]

We also set:

\[ K_{ij}(g, h) := g^{i\bar{j}} R_{ij \alpha \bar{\beta}} \]

which is referred to as the mean curvature form of \((E, h)\) over \((N, g)\) (cf. [26] section 1). We observe that if \(N = \Sigma\) is a Riemann surface, then the tensor \(K_{ij}\) and the full curvature tensor \(R_{ij \alpha \bar{\beta}}\) for a holomorphic
Hermitian vector bundle \((E, h)\) are equivalent. Note that the notion of Ricci curvature coincides with the standard notion of Ricci curvature, in case \(E\) is the tangent bundle of \(N\).

One has readily:

**Lemma 8.** Let \((M, g)\) has isotropic curvature bounded from below by \(C_I\) (not necessarily positive). Let \(f: D \to M\) be a minimal immersion and let \(E := f^*TM \otimes \mathbb{C}\). then for every holomorphic isotropic section \(\sigma\) of \(E\) such that \(\sigma \wedge \frac{\partial f}{\partial \bar{z}} \neq 0\) one has:

\[
R(H)(\sigma, \sigma) := R(H)(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}, \sigma, \bar{\sigma}) \geq C_I \lambda \|\sigma\|^2
\]

where \(g = \lambda(dx^2 + dy^2)\) is the (necessarily Kähler) metric on \(D\) induced via \(f\) from \((M, g)\).

Moreover, if \((M, g)\) be a Riemannian manifold with \(\text{Ric}(g) \geq -C\) (resp. \(\text{Rm}(g) \geq -C\)). Given a minimal immersion \(f: D \to M\), let \(E = f^*TM \otimes \mathbb{C}\) with the induced holomorphic structure and hermitian structure \(H\). Then the Ricci curvature (resp. curvature tensor) of \((E, H)\):

\[
\text{Ric}(H) \geq -C \quad (\text{resp. } \text{Rm}(h) \geq -C)
\]

**Proof.** The only non-trivial components of the curvature of the Hermitian metric \(H\) on \(E\) are, for any section \(W\) of \(E\):

\[
R(H)(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})W = \left(\nabla_{\frac{\partial}{\partial x}} \nabla_{\frac{\partial}{\partial y}} - \nabla_{\frac{\partial}{\partial y}} \nabla_{\frac{\partial}{\partial x}}\right)W
\]

but in terms of the curvature of \(g\) these are equal to:

\[
R_g(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})W
\]

evaluated along \(f(D)\). Finally, equation \([9]\) is a simple consequence of the computation above and the definition of isotropic curvature on the isotropic plane \(\sigma \wedge \frac{\partial f}{\partial z} \neq 0\). Alternatively, one can see that the \(\mathbb{C}\)-bilinear extension of \(\text{Rm}\) is the curvature of the complex connection obtained by complexifying the Levi-Civita connection. \(\square\)

**Remark 9.** In fact, Micallef-Moore in \([29]\) showed that \((M, g)\) has positive isotropic curvature if and only if \(\text{Rm}_C(v, \bar{v}, w, \bar{w}) > 0\) for every \(v, w \in TM \otimes \mathbb{C}\) such that \(g(v, v) = g(v, w) = g(w, w) = 0\), which are \(\mathbb{C}\)-linearly independent (cf. also Proposition 7.2 in \([2]\)).

Also a very important standard fact is the following (cf. \([19]\) pg. 78-79):
Lemma 10. Let \((E,H)\) be a holomorphic Hermitian vector bundle and \(\pi : E \to Q\) a holomorphic quotient bundle endowed with the quotient Hermitian metric \(H_Q\). Let also \(F \subset E\) an holomorphic sub bundle such that the quotient \(E/F \simeq Q\) and \(H_F\) the induced Hermitian metric. Then for the curvature operator:
\[
\Theta(H_Q) = \Theta(H) |_Q + S \wedge S^* \quad \text{and} \quad \Theta(H_F) = \Theta(H) |_F - S \wedge S^*
\]
where \(S = \nabla_E - \nabla_F\) is the second fundamental form of \(F\) —here \(\nabla_E\) and \(\nabla_F\) are the metric connections of \((E,H)\) and \((F,H_F)\) respectively. In particular:
\[
\Theta(H_Q) \geq \Theta(H) |_Q.
\]

3.2. Bochner technique in Complex Differential Geometry. When working with a vector bundle \(E\) we will often use the norms defined by the rescaled metrics \(R^2g\), which has the effect of rescaling lengths by \(R\) and volumes by \(R^{2n}\).

We will use the notation \(g_R := R^2g\) to denote the rescaled metric and we will further simplify notation by writing \(dV_R\) for \(dV_{g_R}\), the corresponding volume form. Then the scaling weight gives
\[
\|\nabla f\|_{L^2(dV_R)} = R^{n-1}\|\nabla f\|_{L^2(dV_g)} \quad \text{and} \quad \|f\|_{L^2(dV_R)} = R^n\|f\|_{L^2(dV_g)}.
\]

We will make use of the following various forms of Laplacian operators:
\[
\Delta_{\overline{\partial}} = \overline{\partial} \overline{\partial} + \overline{\partial} \overline{\partial},
\]
with adjoints defined using the \(L^2\)-metric induced from \(g\)
\[
\Delta_{R,\overline{\partial}} = \overline{\partial}_R \overline{\partial} + \overline{\partial} \overline{\partial}_R,
\]
with adjoints defined using the rescaled metric \(g_R\).

Given a Hermitian holomorphic vector bundle \((E,H)\), the Laplace-Beltrami operator associated to the connection \(\nabla\) is
\[
\Delta = \nabla \nabla^* + \nabla^* \nabla
\]
where
\[
\nabla^* : C^\infty(M, \Lambda^qT_M^* \otimes E) \to C^\infty(M, \Lambda^{q-1}T_M^* \otimes E)
\]
is the (formal) adjoint of \(\nabla\) with respect to the \(L^2\) inner product.

If \(M\) is a compact complex manifold equipped with a hermitian metric \(\omega = \sum \omega_{j\bar{k}} dz_j \wedge d\bar{z}_k\) and \(E\) is a holomorphic vector bundle on \(M\) equipped with a Hermitian metric, and let \(\nabla = \nabla^{1,0} + \nabla^{0,1}\) be its Chern curvature form (that is to say \(\nabla^{0,1} = \overline{\partial}\)), decomposed in \((1,0)\) and \((0,1)\)
parts respectively. In the same way one forms the Laplace-Beltrami operator $\Delta$, one can form the following complex Laplace operators:

$$\Delta' = \nabla^{1,0} \nabla^{1,0*} + \nabla^{1,0*} \nabla^{1,0}, \quad \Delta'' = \nabla^{0,1} \nabla^{0,1*} + \nabla^{0,1*} \nabla^{0,1}.$$  

Remark that if the connection is Hermitian, then $\nabla^{0,1} = \bar{\partial}$ and thus:

$$\Delta'' = \bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial}.$$

Let also $\Lambda$ be the adjoint of the operator (known as Lefschetz operator) $L = (\omega \wedge) \otimes 1$, defined on $\Omega^{p,q} M \otimes E$, with respect to the Hermitian product on $\Omega^{p,q} M \otimes E$ induced by $g$ and $H$. The main identity we will be using is (cf Corollary 1.4.13 in [27]):

**Theorem 11** (Bochner-Kodaira-Nakano identity). If $(X, \omega)$ is Kähler, the complex Laplace operators $\Delta'$ and $\Delta''$ acting on $E$-valued forms satisfy the identity

$$\Delta'' = \Delta' + \big[ \sqrt{-1} \Theta(h), \Lambda \big].$$  

Another important piece of information is the following (cf. [10], Ch. VII, eq. (7.1)):

**Lemma 12.** If $E$ is a Hermitian vector bundle on a Riemann surface $\Sigma$, then for any smooth section $s$ of $E$, one has:

$$\big[ \sqrt{-1} \Theta(h), \Lambda \big] s = K(g,h)(s)$$

where $K(g,h)$ is defined in equation (8). Furthermore the $\Delta''$ acting on sections $\phi$ of either $\Omega^{1,0}_\Sigma$ or $\Omega^{1,1}_\Sigma$ is:

$$\Delta'' \phi = -\nabla_Z \nabla_{\bar{Z}} \phi$$

where $Z$ is a (local) holomorphic frame of $T^{1,0} \Sigma$. In particular for a smooth section $\sigma$ of $\Omega^{1,0}_\Sigma(E)$ or of $\Omega^{1,1}_\Sigma(E)$ one has:

$$\Delta'' \sigma = \sum_{i=1}^n \left( -\nabla_Z \nabla_{\bar{Z}} \sigma^i \right) e_i - \sum_i \left( \nabla_Z e_i \right) \left( D_Z \sigma^i \right)$$

$$+ \sum_{i,j} e_j \left( i_Z \Omega^j_{i} \wedge i_{\bar{Z}} \sigma^i \right)$$

(11)

where $\{e_i\}$ is a holomorphic frame for $E$ and $\sigma = \sum_i \sigma^i e_i$.

**Proof.** This is a consequence of the definition of the operator $\Lambda$ which in particular implies that if $Z$ is a local orthonormal frame of $T^{(1,0)} \Sigma$, then (cf. [27] section 1.4.3 and in particular formula (1.4.63) therein)

$$\Lambda = -\sqrt{-1} i_Z i_{\bar{Z}}$$

thus for any section of $\Omega^{p,q} M \otimes E$:

$$\big[ \sqrt{-1} \Theta(h), \Lambda \big] s = R(H)(Z, \bar{Z}) \bar{Z} \wedge i_{\bar{Z}} s.$$
and the fact that on a Riemann surface Σ the only nontrivial \( \Omega^{p,q}(E) \)'s are either of the form \( \Omega^{p,0}(E) \) or \( \Omega^{0,q}(E) \). This implies that the curvature of the background metric cancels out.

### 3.3. Rescaling

As before, we are given \((f : D \to M, E, H)\) a triple consisting of a stable minimal immersion \(f : D \to M\), the induced vector bundle \(E := f^*TM \otimes \mathbb{C}\) with induced complex structure and with induced Hermitian metric \(H\), and connection \(A_E\). Also, \(D\) is endowed with the pull-back metric \(g_D = \lambda^2(dx^2 + dy^2)\) (necessarily Kähler by virtue of dimension).

We will actually endow \(E\) with the tensor product metric \(H_R := H \otimes e^{-R^2|z|^2}H_0\) (here \(H_0\) is the Euclidean Hermitian metric on \(E\)) and the tensor product connection: \(A_R := A_E \otimes R^2A\).

We rescale the background metric \(g_D = \lambda^2(dx^2 + dy^2)\) and the (holomorphic) homotetic transformation, for any point \(z_0 \in D\):

\[
\Phi_{R,z_0} : D_R \to D, \text{ defined by } \Phi_{R,z_0}(z) := z_0 + \frac{z}{r}.
\]

and pulling back all the geometric quantities:

\[
g_{D_R} := \Phi_{R,z_0}^*g_D; \quad E_R := \Phi_{R,z_0}^*E; \quad H_R := \Phi_{R,z_0}^*(H); \quad A_R := \Phi_{R,z_0}^*(A_E \otimes R^2A).
\]

The map \(\Phi_R\) and the rescaling associated with it make sense for any vector bundle \(E\), and it is this type of rescaling we will be most interested in. The main elementary observation about rescaling that shall be used in the proof of Theorem 4 is the following:

**Lemma 13.** Let \(\tilde{g}_R := R^2g_{D_R}\). Then for any \(C^1\) section \(\sigma\), one has:

\[
\int_{D_R} |\nabla^{\tilde{g}_R}_w \sigma_R|^2 dV_{\tilde{g}_R} = \int_D |\nabla^{g}_w \sigma|^2 dV_g
\]

where \(\sigma_R := \Phi_{R,z_0}^*\sigma\), \(\nabla^{g_R}\) and \(\nabla^{g}\) are resp. the Levi-Civita connections of \(g_R\) and \(g\) and \(w\) is a holomorphic coordinate on the unit disk \(D\).

**Proof.** The proof— which boils down to showing that \(\int_D |\nabla^{g}_w \sigma|^2 dV_g\) is invariant under conformal transformations— is a consequence of the naturality of the Levi-Civita connection under pull-backs and the integration formula under pull-backs. \(\square\)

The advantage of this Lemma is twofold: it allows one to replace the induced metric on any minimal immersion \(f : D \to M\) by a conformal metric, e.g. the flat metric but also to rescale *ad libitum* without changing the \(L^2\)-norm of \(\nabla^{01}\sigma\).
3.4. **Tweaking the Hermitian metric so it has positive curvature.** This section is not necessary for the proof of the main theorem, but we deem it important for future uses.

For most of our constructions it will be convenient to be able to conformally change –by a uniformly controlled conformal factor– the Hermitian metric on $E$ so that the curvature becomes positive. This is achieved by:

**Proposition 14.** Let $\omega$ be a Kähler form on the disc $D \subset \mathbb{C}$ and let $(E, H)$ be a Hermitian rank $n$ vector bundle over $D$ (hence necessarily trivial), and assume that its curvature satisfies:

$$R(H) \geq -\theta H, \quad \text{(resp.) } \text{Ric}(H) \geq -\theta \omega$$

for some positive number $C$. Then there exists a conformal Hermitian metric $H_\psi := e^{-\psi} H$ on $E$ such that:

$$R_{H_\psi} \geq 2\omega \quad \text{(resp.) } \text{Ric}(H) \geq 2\omega$$

that is to say the bundle $\det(E)$ endowed with metric $h_\psi := \det(e^{-\psi} H)$ has positive curvature. Furthermore, $\psi$ can be chosen so that:

$$\|\psi\|_{C^{k,\alpha}} < C(\theta)$$

for a constant $C = C(\theta)$ depending only on $\theta$ and $\omega$. In particular the oscillation of $\psi$ is uniformly bounded.

**Proof.** This is based on two facts: on the one hand the elementary linear algebra fact that if $A$ is an $n \times n$ symmetric matrix, then there exists a $k \in \mathbb{R}_+$ such that $A + k I_n > 0$ (where $I_n$ is the identity $n \times n$ matrix); on the other hand the fact, discussed in section 3.1 that on a Riemann surface $R(H)$ (or rather $\Theta_H$) is of the same tensorial type as $\omega$.

Since the curvature of $H_\psi := e^{-\psi} H$ is calculated–making use of equation (5) and the fact that $-\partial \bar{\partial} (e^{-\psi}) = (\partial \bar{\partial} \psi - \partial \psi \wedge \bar{\partial} \psi) e^{-\psi}$–as follows (given that $E$ has rank $n$):

$$\Theta(H_\psi) = (\Theta(H) + \partial \bar{\partial} \psi H) e^{-\psi}$$

or in coordinates:

$$R_{ij}(H_\psi) = \left( R_{ij} + \frac{\partial^2 \psi}{\partial z \partial \bar{z}} H_{ij} \right) e^{-\psi}$$

and since $R_{ij}(H_\psi) \geq -\theta H_{ij}$ after tracing with respect to $H$, for the Hermitian metric $H_\psi$ to satisfy the conclusion of the theorem it is (necessary and) sufficient that:

$$\Delta_{flat} \psi = \frac{4}{n} k$$
where \( k \) is a function (which without loss of generality we may assume to have a sign) such that

\[-\theta \omega + k \omega \geq 2\omega\]

and \( \Delta_{\text{flat}} \psi = 4\partial \bar{\partial} \psi \) is the Laplacian with respect to the flat metric \( ds^2 = dx^2 + dy^2 \). On the other hand, by standard elliptic theory, given any smooth \( k \) and any boundary value \( \rho \), we can find a smooth solution to:

\[
\begin{align*}
\Delta_{\text{flat}} \psi &= \frac{4}{n} k \\
\psi \mid_{\partial B} &= \rho
\end{align*}
\]

As for the assertion on the oscillation, observe that by (Calderon-Zygmund) elliptic regularity:

\[
\| \psi \|_{W^{\ell+2,p}(B)} \leq C(B, \ell) \left( \| \psi \|_{L^2} + \frac{1}{n} \| k \|_{W^{\ell+2,p}(B)} + \| \rho \|_{W^{\ell+2,p}(B)} \right)
\]

We remark that since a disk of radius \( R \), \( D_R \subset \mathbb{C} \) is strictly pseudo convex, one can actually choose a plurisubharmonic defining function \( \chi \) (e.g. \( \chi = |z|^2 \)) such that \( \chi \mid_{\partial D_R} = R \) and then by taking a suitable multiple \( \chi = e^{-C|z|^2} \), one can choose \( \rho = CR \) (and in fact \( \psi = C|z|^2 \)) in the above construction.

\[\square\]

**Remark 15.** Our application of this Lemma will be to minimal immersions of the disc into our manifold \( M \). Note that for any (possibly branched) minimal immersion \( f : D \rightarrow M \) the vector bundle \( E \) with the induced (Hermitian) metric from \( M \)–which we indicate by \( H := f^*g \)–has Ricci curvature depending only on the Ricci curvature of the metric \( g \) of \( M \); hence \( \text{Ric}(H) \) is bounded below by some constant \( C_1 \).

We can also prove the following (which is of independent interest and ultimately unnecessary for the proof of the main theorems):

**Lemma 16.** Keeping notation and assumptions as in Proposition 14, if one also has:

\[|\text{Ric}_H| < C\]

then there exists a rank 1 holomorphic line sub-bundle \( L \subset E \) such that, if \( h_L \) denotes the induced metric its curvature satisfies: \( \sup_{D'} R(h_L) > -C_1 \) for any compactly embedded \( D' \subset D \), where \( C_1 \) only depends on \( C \) and the Sobolev constant and \( D' \).
Proof. In order to prove this we notice that there must exist a holomorphic frame \( \{ e_1, \cdots, e_n \} \) such that:

\[
R(h)_{11} := R(h)(e_1, \bar{e}_1, \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}) > -C
\]

Since the Ricci curvature of \( h \) equals:

\[
\text{Ric}(h) = -h^{ij} \frac{\partial^2 h_{ij}}{\partial z \partial \bar{z}} + h^{st} h^{ij} \frac{\partial h_{s\bar{t}}}{\partial z} \frac{\partial h_{i\bar{t}}}{\partial \bar{z}}
\]

–which is clearly an elliptic system– it follows from the Calderon-Zygmund inequality that (cf. \[33\]):

\[
\| h^{ij} \|_{W^{2,p}(D')} \leq C_Z(D') \| \text{Ric}(h) \|_p
\]

and the conclusion follows from Morrey’s version of Sobolev inequality, which implies that:

\[
\| h^{st} \frac{\partial h_{s\bar{t}}}{\partial z} \frac{\partial h_{i\bar{t}}}{\partial \bar{z}} \|_{C^{\alpha,0}(D')} \leq C_S C_Z(D') \| \text{Ric}(h) \|_p
\]

where \( C_S \) is the Sobolev constant of the (compact) embedding \( W^{2,p} \subset C^{1,\alpha} \) when \( \frac{1-\alpha}{2} = \frac{1}{p} \). \( \square \)

3.5. Constructing holomorphic isotropic sections. In this section we fix the standard flat metric \( g_0 = dx^2 + dy^2 \) with Kähler form \( \Omega_0 = \sqrt{-1} dz \wedge d\bar{z} \) on \( D = D_R \subset \mathbb{C} \), the disk of radius \( R \). We also endow \( D_R \) with any metric which is \( L^\infty \)-close to \( g_0 \):

\[
\frac{1}{2} g_0 \leq g \leq 2 g_0
\]

We analyze the datum of a holomorphic vector bundle \( E = D_R \times \mathbb{C}^n \to D_R \) over \( D_R \) and endowed with a Hermitian metric such that:

\[
H \leq \kappa H_0.
\]

where \( H_0 \) is the standard flat metric on the trivial rank \( n \) complex vector bundle \( D \times \mathbb{C}^n \)– i.e., \( H_0(\xi, \bar{\xi}) := \sum_{ij} (H_0)_{ij} \xi_i \bar{\xi}_j = \sum_{ij} \delta_{ij} \xi_i \bar{\xi}_j \), where \( \delta_{ij} \) is the identity matrix. This is the situation that one might achieve by rescaling a small ball centered and contained inside the unit ball.

We intend to show that we can always find holomorphic sections of \( E \)– although clearly they will not have compact support, whence they will not be test sections for the stability inequality– which are also \( g_C \)-isotropic (here \( g_C \) is the \( \mathbb{C} \)-linear extension of the metric corresponding
to the Hermitian metric $H$) and that we can do so with controlled $L^\infty$ and $W^{1,\infty}$ Sobolev-norms. More specifically: Let:

$$\mathcal{OI} := \left\{ s \in C^\infty(E) : \nabla_{\partial} s = 0 \text{ and } g_C(s,s) = 0 \right\}$$

and:

$$\mathcal{I}_{S^1} := \left\{ s \in C^\infty(E \mid_{S^1}) : g_C(s,s) = 0 \right\}$$

**Proposition 17.** Let $H$ and $g_C$ be as above. There exists a surjective map:

$$T : \mathcal{OI} \to \mathcal{I}_{S^1}$$

Furthermore we can find a set of boundary data such that the corresponding counter-images via $T$ are holomorphic isotropic sections $s$ of $E$ such that:

1. $|s(0)| = 1$
2. $|s|^2_H \leq \kappa$
3. $|\partial s|^2_H \leq \frac{\kappa}{R^2}$
4. $|s(z)| \geq \frac{1}{2}$ if $z \in B_{\frac{R}{\sqrt{\kappa}}}(0)$
5. more generally $|s(z)| \geq 1 - a$ if $z \in B_{\frac{R}{\sqrt{\kappa}}}(0)$, for $a \in (0,1)$.

**Proof.** The map $T$ is simply given by restricting a given section $s$ to the boundary:

$$T(s) := s \mid_\gamma$$

where $\gamma := \partial D \simeq S^1$. In order to show surjectivity, we first show we can solve the $\bar{\partial}$-problem for any boundary condition:

$$\begin{cases}
    \bar{\partial} s = 0 \text{ in } D \\
    s \mid_\gamma = \chi
\end{cases}$$

where $\chi \in C^\infty(D)$ is to be specified later (here $\gamma := \partial D$). That we can solve this equation with any boundary condition is guaranteed by the fact that $E$ is holomorphically trivial on $D$ (cf. Theorem Y pg. 211 in [18]), and therefore the problem reduces to the 1-dimensional Cauchy-Riemann problem for functions on $D$, which can be solved using the Cauchy integral formula:

$$s_i(\zeta) = \frac{1}{2\pi i} \int_{\gamma} \frac{\chi_i(z)}{z - \zeta} \, dz$$

where –after identifying $E$ with $D \times \mathbb{C}^n$– $s_i$ are the components of $s$ and $\chi = (\chi_1, \cdots, \chi_n)$. We next show that if $\chi \in \mathcal{I}_{S^1}$ then the solution of (13) is in $\mathcal{OI}$. In order to make sure that $s$ be isotropic only in
terms of the boundary condition we make use of the fact that if $s$ is a holomorphic section of $E$, then:

$$\bar{\partial}g_C(s, s) = 0$$

i.e., $g_C(s, s)$ is holomorphic, which descends immediately from the fact that:

$$\bar{\partial}g_C(s, s) = g_C(\nabla_{\overline{\partial} s}, s) + g_C(s, \nabla_{\overline{\partial} s}) = 0$$

Indeed, we can choose $\chi = (\chi_1, \ldots, \chi_n)$ such that $g_C|_{\gamma} (\chi, \chi) = 0$, therefore by analytic continuation, also $s$, the solution to the Cauchy problem $\bar{\partial}s = 0$ and $s|_{\partial D} = \chi$, is such that $g_C(s, s) = 0$; i.e., $s$ is an isotropic section. The fact that this can be done is a simple consequence of the fact that we can choose $\chi = \alpha + \sqrt{-1}\beta$ with $\alpha$ and $\beta$ sections of $\partial D \times \mathbb{R}^n$ such that:

$$\|\alpha\|^2_g = \|\beta\|^2_g$$

and

$$\langle \alpha, \beta \rangle_g = 0$$

where $g$ is the real form of the Hermitian metric $H$. This shows the surjectivity of the map $T$.

We now show that we can choose the boundary data $\chi$ so that (1)–(4) hold.

Part (1) and (2) now follow from choosing the $\chi_i$‘s accordingly as follows. In the global trivialization chosen:

$$|s|^2_H(z) = H_{ij}(z)s^is^j$$

$$= H_{ij}(z) \left( \frac{1}{2\pi \sqrt{-1}} \int_{\gamma} \frac{\chi_i(z)}{z - \zeta} \frac{dz}{dz} \right) \left( \frac{1}{2\pi \sqrt{-1}} \int_{\gamma} \frac{\chi_j(z)}{z - \zeta} \frac{dz}{dz} \right)$$

and since in polar coordinates centered at $0 \in D$ (i.e., $z = re^{i\theta}$) there holds $\frac{dz}{\zeta} = \sqrt{-1}d\theta$, in order for (1) to hold it suffices to require:

$$\sum_{i,j=1}^n H_{ij}(0) \left( \frac{1}{2\pi} \int_0^{2\pi} \chi_i(\theta) d\theta \right) \left( \frac{1}{2\pi} \int_0^{2\pi} \chi_j(\theta) d\theta \right) = 1$$

We show the existence of such $\chi$ as follows. We choose a holomorphic frame $e_i$ for $E$ on (an open set containing) $D$ such that at $0$:

$$H(e_i, e_j)(0) = H_{ij}(0) = \delta_{ij}$$

In this frame we choose a smooth section of $E|_{\partial D}$ of the form $\tilde{\chi} = (\tilde{\chi}_1, \ldots, \tilde{\chi}_n)$ and we have chosen $\tilde{\chi}_i$ such that:

$$g_\text{C}(\tilde{\chi}, \tilde{\chi}) = g_{Cij}\tilde{\chi}_i\tilde{\chi}_j = 0.$$

This is tantamount to choosing $\tilde{\chi} = \tilde{\alpha} + \sqrt{-1}\tilde{\beta}$ with $\tilde{\alpha}$ and $\tilde{\beta}$ sections of $\partial D \times \mathbb{R}^n$ such that:

$$\|\tilde{\alpha}\|^2_g = \|\tilde{\beta}\|^2_g$$

and

$$\langle \tilde{\alpha}, \tilde{\beta} \rangle_g = 0.$$
where \( g \) is the real form of the Hermitian metric \( H \) (that is \( H(v, w) = g_C(v, \bar{w}) \) where \( g_C \) is the bilinear extension of \( g \)).

Set \( \chi_i := e^{\sqrt{-1}\lambda\theta} \tilde{\chi}_i \), \( \alpha := e^{\sqrt{-1}\lambda\theta} \tilde{\alpha} \), \( \beta := e^{\sqrt{-1}\lambda\theta} \tilde{\beta} \) and \( \chi := (\chi_i, \ldots, \chi_n) \).

Clearly one still has that:

\[
\|\alpha\|^2_g = \|\tilde{\alpha}\|^2_g = \|\tilde{\beta}\|^2_g = \|\beta\|^2_g \quad \text{and} \quad \langle \alpha, \beta \rangle_g = 0
\]
or equivalently:

\[
g_C(\tilde{\chi}, \tilde{\chi}) = g_C(\chi, \chi) = 0 \quad \text{and} \quad \|\chi\|^2_H = \|\tilde{\chi}\|^2_H.
\]

One can readily show that one can have chosen \( \tilde{\chi} = \tilde{\alpha} + \sqrt{-1}\tilde{\beta} \) satisfying the following:

- the necessary conditions for \( \tilde{\chi} \) (hence for \( \chi \), whence for \( \sigma \)) to be isotropic:
  \[
  \|\tilde{\alpha}\|^2_g = \|\tilde{\beta}\|^2_g \quad \text{and} \quad \langle \tilde{\alpha}, \tilde{\beta} \rangle_g = 0
  \]
- the Euclidean norm of \( \tilde{\chi} \) (whence the one of \( \chi \)) satisfies
  \[
  \sum_{i=1}^n |\tilde{\chi}_i|^2 = 1
  \]

We now prove that we can choose \( \chi \) so that item (1) of the Proposition holds. Since:

\[
I_\lambda := \sum_{i,j=1}^n H_{ij}(0) \left( \frac{1}{2\pi} \int_0^{2\pi} e^{\sqrt{-1}\lambda\theta} \tilde{\chi}_i(\theta) \, d\theta \right) \left( \frac{1}{2\pi} \int_0^{2\pi} e^{-\sqrt{-1}\lambda\theta} \bar{\tilde{\chi}}_j(\theta) \, d\theta \right)
\]
is a continuous expression in \( \lambda \) and since \( \lim_{\lambda \to \infty} I_\lambda = 0 \) (this is a particular case of the fact that the Fourier transform is an automorphism on the space of Schwarz functions, cf. \[32\]) it follows that there exists a choice of \( \lambda \) for which:

\[
\sum_{i,j=1}^n H_{ij}(0) \left( \frac{1}{2\pi} \int_0^{2\pi} e^{\sqrt{-1}\lambda\theta} \chi_i(\theta) \, d\theta \right) \left( \frac{1}{2\pi} \int_0^{2\pi} e^{-\sqrt{-1}\lambda\theta} \bar{\chi}_j(\theta) \, d\theta \right) = 1
\]

That is to say equation (15) holds. Then if \( s \) is the solution of the Dirichlet problem with \( \chi := (\chi_i, \ldots, \chi_n) \) as boundary condition (i.e., \( s = T^{-1}(\chi) \)) one can easily check that there is a choice of \( \lambda \) such that:

\[
|s|^2_{H}(0) = H_{ij}(0)s^i\bar{s}^j = |T^{-1}(\chi)|^2_H(0)
\]

\[
= \sum_{i,j=1}^n H_{ij}(0) \left( \frac{1}{2\pi} \int_0^{2\pi} \chi_i(\theta) \, d\theta \right) \left( \frac{1}{2\pi} \int_0^{2\pi} \bar{\chi}_j(\theta) \, d\theta \right) = 1
\]

which settles (1).

---

1This can be simply achieved by replacing \( \chi \) with \( \frac{\chi}{\sum_{i=1}^n |\chi_i|^2} \), if necessary.
We next prove that (2) holds in two different ways. In the first proof we simply exploit the assumption that $H \leq \kappa H_0$. In the second (which we only sketch) one employs that the curvature of $H$ is bounded (as consequence of the fact that by assumption $H \leq \kappa H_0$ and $\|\nabla H\|_{H_0}, \|\nabla^2 H\|_{H_0} < \kappa$). One easily proves (e.g., choosing a frame at any given $p \in D$ where $H_{ij}(p) = \delta_{ij}$, $\partial H(p) = \bar{\partial} H(p) = 0$ and $\frac{\partial^2 H_{ij}}{\partial \bar{\partial} z^2}(p) = -R(H)_{ij}$) the following Bochner type formula:

$$\partial \bar{\partial} |s|^2_H = -R(H)_{ij} s_i s_j + |\partial s|^2_H$$

We can now proceed by observing that on the one hand:

$$|s|^2_H \leq \kappa |s|^2_{H_0}$$

and that on the other hand $-R(H_0)_{ij} = 0$, therefore, using equation (18) applied to $|s|^2_{H_0}$ and applying the maximum principle to the differential inequality: $\partial \bar{\partial} |s|^2_H = |\partial s|^2_{H_0}$, yields:

$$\sup_D |s|^2_H = \sup_{\partial D} |s|^2_{H_0}$$

whence (coupled with eq. (19)):

$$|s|^2_H \leq \kappa |s|^2_{H_0} \leq \kappa \sup_{\partial D} |s|^2_{H_0} \leq \kappa$$

having used that, by construction $\sup_{\partial D} |s|^2_H = 1$ (as a consequence of eq. (17)). Since by equation (16):

$$\|\alpha\|^2_g = \|\beta\|^2_g = \frac{1}{2}$$

thus, using eq. (14) and eq. (20):

$$|s|^2_H(\zeta) \leq 2 \sup_{\zeta \in \partial D} \left( H_{ij}(\zeta) \left( \frac{1}{2\pi \sqrt{-1}} \int_{\gamma} \frac{\chi_i(z)}{z - \zeta} dz \right) \left( \frac{1}{2\pi \sqrt{-1}} \int_{\gamma} \frac{\chi_j(z)}{z - \zeta} dz \right) \right) \leq 2 \sup_{\partial D} (\|\alpha\|^2_H + \|\beta\|^2_H) = 2$$

Item (3) is a consequence of the inequality (itself a consequence of the Cauchy-integral formula):

$$|\frac{\partial^k}{\partial z^k} s_i|(x_0) \leq \frac{k!}{R^k} \sup_{\partial B(0,R)} |\chi|$$
for any ball $B(x_0, R)$. Therefore on such a ball:
\[
\|\partial s\|_{H^0}^2 \leq \sum_{i=1}^n \sup_{\partial B(0, R)} |\frac{\partial}{\partial z} s_i|^2 \leq \frac{1}{R^2} \sum_{i=1}^n \sup_{\partial B(0, R)} |\chi|^2 = \frac{1}{R^2} \sup_{\partial B(0, R)} \|\chi\|_{H^0}^2
\]
In particular, making use of inequality (20):
\[
(22) \quad |\partial s|^2_H \leq \kappa |\partial s|^2_{H_0} \leq \kappa \frac{1}{R^2} \sup_{\partial D} \|\chi\|_{H_0}^2 \leq \frac{\kappa}{R^2}
\]
where in the next to last inequality we employ the fact that by choice (see footnote on page 13) $\chi = e^{-\phi}v$ for some constant vector $v$.

We now notice that:
\[
|\partial |s|^2_H | \leq 2|\partial s| |s| \quad \text{and} \quad \partial |s|^2_H = 2|s|_{H} \partial |s|_{H}
\]
combining which yields:
\[
\partial |s|_H \leq |\partial s|_H
\]
which, coupled with eq. (22) yields:
\[
\partial |s|_H \leq \sqrt{\frac{\kappa}{R}}
\]
Finally, having bounded the gradient of $|s|_H$, and therefore bounded the Lipshitz constant of $|s|$, item (5) (whence item (4)) follows.

The second proof (which we only sketch) is based on an argument similar to the one employed in the proof of Proposition 14, which we can use to show that we can conformally change $\tilde{H}$ so that $R(e^{-\psi} H)_{\bar{i}j} < 0$ with $\psi$ such that $\|\psi\|_{C^{k, \alpha}} < C$ (cf. (12) in Proposition 14), Bochner formula (eq. (18)), which holds for any holomorphic section $s$) and the maximum principle.

\[\square\]

Remark 18. This proposition is similar in spirit to Lemma 2.1 in [7], except here we make sure we can find solutions with controlled norm (specifically bounded away from zero) at least on a half the disk. Also here we exploit directly the holomorphic triviality of holomorphic vector bundles on the disk (more generally polydisks) rather than solving the Riemann-Hilbert problem for the coupled $\bar{\partial}$ operator.

3.6. The model example: isotropic holomorphic Gaussian sections. In this section, we make use of a slight modification of Donaldson’s technique to construct Gaussian holomorphic sections of the trivial bundle on the ball of radius $R$ in $\mathbb{C}$.

We diverge from Donaldson’s treatment a bit as we need to make sure that the ”local” construction produces isotropic holomorphic sections.

Let then $B_R$ be the ball (i.e., disk) of radius $R > 2$ in $\mathbb{C}$ (in the application we will take $N = 1$) with the standard flat metric and
standard Kähler form: $\Omega_0 = \sqrt{-1} dz \wedge d\bar{z}$. Let $F$ be the trivial rank $n$ holomorphic vector bundle $F = B_R \times \mathbb{C}^n$ with metric conformal to the flat Hermitian metric by the factor $\exp(-|z|^2/2)$. The 1-form:

$$A_k := \frac{k}{2} (z \, d\bar{z} - \bar{z} \, dz)$$

gives rise to a diagonal connection on $F$:

$$A_K := \bigoplus_{i=1}^{n} A_{k_i}$$

for any multi-index $K = (k_1, \cdots, k_n)$ and we will assume $k_i \geq 0$.

The curvature $F_{A_K}$ of $A_K$ is simply $dA_K = -\sqrt{-1} k \Omega_0$ and there is (up to constant rescaling) only one Hermitian metric $H_k$ on $B_R \times \mathbb{C}$ compatible with $A_k$ (i.e., the metric $H_K$ whose curvature is $F_{A_K}$): the metric $H_k = e^{-\frac{k|z|^2}{2}} h_0$ where $h_0$ is the standard flat Hermitian metric on $B_R \times \mathbb{C}$.

Observe that the connection $A_K$ gives rise to a $\bar{\partial}_{A_K}$ operator on $B_R \times \mathbb{C}^n$ as follows:

$$\bar{\partial}_{A_K}(s_1, \cdots, s_n) = (\bar{\partial}_{A_{k_1}} s_1, \cdots, \bar{\partial}_{A_{k_n}} s_n)$$

where $\bar{\partial}_{A_{k_i}} s = \bar{\partial} + A_{k_i}^{0,1} s$ (here $A_{k_i}^{0,1}$ indicates the $(0,1)$-part of $A_{k_i}$). and also to a Hermitian metric on $F$ whose curvature is:

$$dA_K = -\sqrt{-1} \bigoplus_{i=1}^{n} k_i \Omega_0,$$

namely the diagonal metric:

\begin{equation}
H_{K,C} = \bigoplus_{i=1}^{n} C_i e^{-\frac{k_i |z|^2}{2}} h_0
\end{equation}

where $C = (C_1, \cdots, C_n)$ with $C_i \in \mathbb{R}, C_i > 0$ and $h_0(s,s) = s \bar{s}$ on $B_R \times \mathbb{C}$.

**Definition 19.** Let:

$$g_{K,C}(\sigma, \sigma) := \sum C_i \exp(-k_i |z|^2/2) \sigma_i \bar{\sigma}_i$$

where $\sigma = (\sigma_1, \cdots, \sigma_n)$. A section of $F$, $\sigma$ say, is said to be **isotropic** if:

$$g_{K,C}(\sigma, \sigma) = 0.$$

**Remark 20.** Clearly $g_{K,C}$ is the $\mathbb{C}$-linear extension of the metric on $B_R \times \mathbb{R}^n$ whose Hermitian extension is $H_K$. 
In what follows we will find it convenient to switch between different representations (corresponding to different gauges): one which we might view as having fixed the Hermitian metric $H_0$ and having represented the complex structure on $F$ as $\bar{\partial}_A$, or equivalently we will fix the complex structure (which in our trivializing charts is the standard one) and then consider the Hermitian metric on $F$ as being given by $H_{n,k}$ (the equivalence of these views can be seen as an incarnation of the Poincare-Lelong formula for $\partial \bar{\partial} \log |s|^2_H$ for a Hermitian metric $H$ and a holomorphic section $s$). We are now ready to prove:

**Lemma 21.** On the (necessarily) trivial rank $n$ holomorphic bundle $F$ over $B_R$ endowed with the metric $H_{K,C}$ of equation (23), the ball of radius $R$. And assume that:

- $K = (k_1, \cdots, k_n)$ with $k_1 \geq k_2 \geq \cdots k_{n-1} \geq k_n \geq 0$

Let $\kappa := \max \{C_i\} > 0$. Then there exist $R_0 > 0$ and a connection $A_{n,K}$ (with associated Hermitian metric $H_{n,K}$) and smooth section $\sigma$ such that:

1. $F_{A_K} = dA_K + [A_K, A_K] = -\sqrt{-1} \oplus_i k_i \Omega$, where $K = (k_1, \cdots, k_n)$
2. $\bar{\partial}_{A_K} \sigma = 0$
3. $\sigma$ is isotropic: $g_{K,C}(\sigma, \sigma) = 0$.
4. $|\sigma(0)|_{H_{n,K}} = 1$
5. $|\sigma|_{H_{K,C}} = e^{-k_n|z|^2} |\sigma|_{H_0,K}$, with $|\sigma|_{H_0,K} \leq \kappa$
6. $|\sigma(z)|_{H_{n,K}} \geq e^{-k_n a^2 R}(1 - a)$ if $z \in B_{R}(0)$.
7. $\pi < \|\sigma\|_2 < 2\pi$.
8. $\|\sigma\|_{L^2(B_R)}^2 \leq \frac{2\kappa}{1-a} \|\sigma\|_{L^2(B_{\pi R})}^2$

for every $R \geq R_0$.

**Proof.** Set:

$$A_K := \bigoplus_{i=1}^n A_{k_i}$$

as above. Then clearly (since $[A_K, A_K] = 0$):

$$F_{A_K} = dA_K = \bigoplus dA_{k_i} = -\sqrt{-1} \bigoplus k_i \Omega$$

By definition:

$$H_{K,C} = e^{-k_n|z|^2} H_{0,K}$$

where:

$$H_{0,K} := \bigoplus_{i=1}^n C_i e^{\frac{(k_i - k_n)|z|^2}{2}} h_0$$
is such that:

\[ H_{0,K} \leq \kappa H_0 \]

for \( \kappa := \max\{C_i\} > 0 \), since \( k_i - k_n \geq 0 \) for any \( i \). Therefore, we can appeal to Proposition \[17\] and produce a holomorphic isotropic (that is isotropic with respect to \( e^{k_n|z|^2}g_{K,C} \)) section \( \sigma_0 = (\sigma_{0,1}, \cdots, \sigma_{0,n}) \) such that (as constructed in Proposition \[17\]):

- \( |\sigma_0(0)| = 1 \)
- \( |\sigma_0|^2 \leq \kappa \)
- \( |\partial\sigma_0|^2 \leq \kappa \)
- \( |s(z)| \geq 1 - \alpha \) if \( z \in B_{\sqrt{\kappa}}(0) \).

We can now rescale the Euclidean metric \( g_0 \) on \( \mathbb{C} \) if necessary, so that we may assume \( k_n = 1 \). Next, set:

\[ \sigma := \exp(-|z|^2/2)(\sigma_{0,1}, \cdots, \sigma_{0,n}) \]

which manifestly satisfies items (5) and (6). Then \( \sigma \) is \( A_K \)-holomorphic (i.e., item (2) holds):

\[
\bar{\partial}_{A_K} (\exp(-|z|^2/2)\sigma_{0,i}) = \\
= \bar{\partial} (\exp(-|z|^2/2)) \sigma_{0,i} + A^{(0,1)} \exp(-|z|^2/2)\sigma_{0,i} = 0
\]

for every \( i \in \{1, \cdots, n\} \), having used that \( \bar{\partial}\sigma_{0,i} = 0 \). The norm of \( \sigma \) is \( \exp(-|z|^2/4) |\sigma_0|^2 \) and \( \sigma \) is isotropic with respect to \( g_C \) (i.e., item (3) holds). Also:

\[ |\partial\lambda_k (\exp(-|z|^2/2)\sigma_{0,i})| = \]

\[ = |\partial (\exp(-|z|^2/2)) \sigma_{0,i} + \exp(-|z|^2/2)\partial(\sigma_{0,i}) + A^{(1,0)} \exp(-|z|^2/2)\sigma_{0,i}| \]

\[ \leq |\partial (\exp(-|z|^2/2)) |\sigma_{0,i}| + |\exp(-|z|^2/2)| |\partial(\sigma_{0,i})| \]

\[ + |A^{(1,0)}| |\exp(-|z|^2/2)\sigma_{0,i}| \]

Number (8) is a consequence of the fact that for \( R \) sufficiently large:

\[
\int_{B_{\sqrt{\kappa}}(0)} |\sigma|^2_{H_{K,C}} \, dx \wedge dy \geq (1 - \alpha)2\pi \int_0^{\sqrt{\kappa}} e^{r^2/2} d\left(\frac{r^2}{2}\right) \geq \pi(1 - \alpha)
\]

and:

\[
\int_{B_R(0)} |\sigma|^2_{H_{K,C}} \, dx \wedge dy \leq \kappa 2\pi \int_0^R e^{r^2/2} d\left(\frac{r^2}{2}\right) \leq 2\kappa \pi
\]

The rest is straightforward. \( \square \)

\[ \text{Here we tacitly use the fact that the complex structure induced from } \bar{\partial}_{A_K}, - \]

\[ \text{where } K' := (k_n - k_1, \cdots, k_2 - k_1, 0) - \text{ and the standard holomorphic structure are equivalent} \]
Remark 22. It is important to remark that the positivity of the curvature is what produces the Gaussian type holomorphic sections. For instance, on the line bundle $L = D \times \mathbb{C}$ with Hermitian metric $e^{\frac{|z|^2}}h_0$ with negative curvature, the holomorphic sections one produces have exponential growth.

3.7. General facts about holomorphic section of vector bundles. Next we prove some general well known theorems on holomorphic sections of a Hermitian holomorphic vector bundle on a Kähler manifold $(N,h)$ with the extra assumption that the background metric $h$ satisfies a lower bound on the Ricci curvature. This will not apply immediately to our context, but it will apply to the context in which we endow the disk with the flat metric (or a rescaled version of the flat metric). We will then be able to effect the control desired merely because the sections we construct have $L^2$-norm in the background metric which is comparable (by a given and definite amount) to the $L^2$-norm with respect to the flat metric. Our proofs follow the lines of [12], where they discuss the case of line bundles.

**Proposition 23.** Let $(E,H)$ be a holomorphic Hermitian vector bundle on a Kähler manifold $(N,h)$ and $\sigma$ a holomorphic section of $E$ such that following conditions hold:

$$K(H)(\partial \sigma, \partial \sigma) \geq -C_K \quad \text{and} \quad \text{Ric}(h) \geq -C_R$$

where by $K(H)$ we denote the mean curvature of $H$ (cf. section 3.1)

Then:

1. $\|\sigma\|_{L^\infty(H)} \leq \kappa_0 \|\sigma\|_{L^2(H)}$, $\|\nabla \sigma\|_{L^\infty(H)} \leq \kappa_1 \|\sigma\|_{L^2(H)}$ for some uniform constants $\kappa_0$ and $\kappa_1$;

2. $|\sigma(x)| \geq 1/4$ at all points $x$ a distance (in the rescaled metric $h_{R^2} = R^2 h$ on $N$) less than $\min\{R^2, (4\kappa_1)^{-1}\}$ from $0$, for some uniform $\kappa_1$ depending only on $C_K$ and $C_R$;

**Proof.** We produce a uniform derivative estimate first and then use Moser iteration. This produces a uniform estimates since the lower bound on $\text{Ric}(g)$ entails a uniform control on the Sobolev constant. Let $\nabla^*$ and $\overline{\partial}$ be calculated with respect to $h$. First remark that:

$$\nabla^* \nabla s = 2\overline{\partial} \overline{\partial} s + s,$$

therefore in the ball $B_{\frac{R}{2}}$ (here the ball is calculated with respect to $g_R$) –where $s$ is holomorphic– $\nabla^* \nabla s = s$ which implies that (in the sense of barriers):

$$\Delta |s| \leq |s|,$$
since on the one hand:

$$\Delta |s|^2 = 2\langle \nabla^* \nabla s, s \rangle = 2|s|^2$$

and on the other:

$$\Delta |s|^2 = 2|s|\Delta |s| + 2|\nabla |s||^2 \geq 2|s|\Delta |s|.$$  

Now the bound on the $L^\infty$ norm follows from the Moser iteration argument applied to this differential inequality (see [11]). Remark that the Sobolev constant here is uniform because of the lower bound on $\text{Ric}(G_R) = 0$, so the bound obtained from Moser iteration is uniform.

Next we derive the first derivative bound, i.e., the second part in item (3), which in turn implies item (4). Again we restrict ourselves to the ball $B_{\frac{1}{2}}$, where $s$ is a holomorphic section, thus $\overline{\partial}s = 0$ and therefore $\nabla s = \partial s$ where

$$\partial : \Omega^{p,q}(E) \to \Omega^{p+1,q}(E)$$

is defined using the connection. Since $\partial^2 = 0$ we have

$$\Delta_\partial \partial s = \partial \Delta_\partial s,$$

where $\Delta_\partial = \partial^* \partial + \partial \partial^*$. Then for a holomorphic section $s$, $\Delta_\partial s = \nabla^* \nabla s = s$ and

$$\Delta_\partial (\partial s) = \partial s.$$  

The Bochner-Kodaira-Nakano formula (cf. Theorem [11]) involving $\Delta_\partial$ and $\nabla^* \nabla$ on $\Omega^{1,0}(E)$ is:

$$\Delta_\partial = \nabla^* \nabla - 1 + K(H)$$

This yields (using that $R(H_\psi) \geq \omega_G$):

$$\nabla^* \nabla = \Delta_\partial + 1 - K(H) \leq \Delta_\partial + 1$$

so:

$$\langle \nabla^* \nabla (\partial s), \partial s \rangle = \langle \Delta_\partial (\partial s), \partial s \rangle + \langle \partial s, \partial s \rangle - K(H)(\partial s, \partial s)$$

$$= 2|\partial s|^2 - K(H)(\partial s, \partial s) \leq (C_K + 2)|\partial s|^2.$$  

It follows that

$$\Delta |\partial s| \leq (C_K + 2)|\partial s|,$$

and the Moser argument applies as before. Notice that the constants only depend on the lower bound $C_R$ of the Ricci curvature $\text{Ric}(h)$, the Sobolev constant of $h$ and the dimension. Therefore we have shown item (3). Item (4) follows from it, since item (3) bounds the Lipschitz constant of $|s|$.

$\square$
Remark 24. Remark that by Lemma 13 in our application we can take the Kähler metric $h$ on the disk to be the flat metric $dx^2 + dy^2$ or a rescaling of it.

With a stronger hypothesis on the structure of the induced metric on the minimal immersion, we can also prove the following, which shall not be used in the proof of the main theorems but is of independent interest.

Theorem 25. Let $f : \Sigma \to M$ be a minimal immersion of a compact (not necessarily closed) Riemann surface and assume that the induced metric $g_\Sigma$ on $\Sigma$ satisfies:

$$\text{Ric}(g_\Sigma) \geq -C_\Sigma$$

If the isotropic curvature of $(M, g)$ satisfies $K_{\text{isotr}}^g \geq C_K$, then there exists a holomorphic $g$-isotropic section $\sigma$ of $E = f^*TM \otimes \mathbb{C} \to \Sigma$ such that:

1. $1 \leq \|\sigma\|_{L^2} \leq \frac{11}{5} \pi$;
2. $\bar{\partial} \sigma = 0$ on the ball of radius $\frac{R}{2}$ (in the rescaled metric);
3. $\|\sigma\|_{L^\infty(H)} \leq \kappa_0 \|\sigma\|_{L^2(H)}$, $\|\nabla \sigma\|_{L^\infty(H)} \leq \kappa_1 \|\sigma\|_{L^2(H)}$ for some uniform constants $\kappa_0$ and $\kappa_1$;
4. $|\sigma(x)| \geq 1/4$ at all points $x$ a distance (in the rescaled metric $g_R$) less than $\min\{\frac{R}{2}, (4\kappa_1)^{-1}\}$ from 0, for some uniform $\kappa_1$ depending only on $C_\Sigma$.

Proof. According to Lemma 26 we can find a rescaling $\Psi_R : B_R \to B$ so that (for $R$ sufficiently large) we can get $E_R \to B_R$ to satisfy the hypotheses of Theorem 30.

Let $\Delta_\bar{\partial} = \bar{\partial} \cdot \bar{\partial} + \bar{\partial} \bar{\partial}$, with adjoints defined using the rescaled metric $g_R := \Phi^*_R g = \Phi^*_R (\lambda(dx^2 + dy^2))$, then for all $\phi$

$$(\Delta_{\bar{\partial}}^{-1}\phi, \phi)_{g_R} \leq \frac{R}{R-C} \|\phi\|^2_{L^2(g_R)}$$

In fact by Kodaira-Nakano (after using the natural isometry $\Omega^{0,q} \otimes E \simeq \Omega^{n-q} \otimes \bar{E} \otimes K_{\Sigma}^*$):

$$(27) \quad \Delta_{\bar{\partial}} = (\nabla^{(0,1)})^* \nabla^{(0,1)} + K(H, g_{\Sigma,R}) + \text{Ric}(g_\Sigma)$$

where $g_{\Sigma,R}$ is the result of rescaling $g_\Sigma$ and $K(H, g_{\Sigma,R})_{\alpha\bar{\beta}} = g^{\bar{\gamma}\bar{\delta}}_{\Sigma,R} R(H)_{\gamma\delta \alpha \bar{\beta}}$. Whence:

$$\Delta_{\bar{\partial}} \geq \frac{C_K - C_\Sigma}{R}$$

in the operator sense (since $\text{Ric}(g_R) \geq -\frac{C_\Sigma}{R}$ and $K(H, g_{\Sigma,R}) \geq \frac{C_\Sigma}{R}$, the latter when restricted to isotropic sections).

Remark 24. Remark that by Lemma 13 in our application we can take the Kähler metric $h$ on the disk to be the flat metric $dx^2 + dy^2$ or a rescaling of it.

With a stronger hypothesis on the structure of the induced metric on the minimal immersion, we can also prove the following, which shall not be used in the proof of the main theorems but is of independent interest.

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1. $1 \leq \|\sigma\|_{L^2} \leq \frac{11}{5} \pi$;
2. $\bar{\partial} \sigma = 0$ on the ball of radius $\frac{R}{2}$ (in the rescaled metric);
3. $\|\sigma\|_{L^\infty(H)} \leq \kappa_0 \|\sigma\|_{L^2(H)}$, $\|\nabla \sigma\|_{L^\infty(H)} \leq \kappa_1 \|\sigma\|_{L^2(H)}$ for some uniform constants $\kappa_0$ and $\kappa_1$;
4. $|\sigma(x)| \geq 1/4$ at all points $x$ a distance (in the rescaled metric $g_R$) less than $\min\{\frac{R}{2}, (4\kappa_1)^{-1}\}$ from 0, for some uniform $\kappa_1$ depending only on $C_\Sigma$.

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Let $\Delta_{\bar{\partial}} = \bar{\partial} \cdot \bar{\partial} + \bar{\partial} \bar{\partial}$, with adjoints defined using the rescaled metric $g_R := \Phi^*_R g = \Phi^*_R (\lambda(dx^2 + dy^2))$, then for all $\phi$

$$(\Delta_{\bar{\partial}}^{-1}\phi, \phi)_{g_R} \leq \frac{R}{R-C} \|\phi\|^2_{L^2(g_R)}$$

In fact by Kodaira-Nakano (after using the natural isometry $\Omega^{0,q} \otimes E \simeq \Omega^{n-q} \otimes \bar{E} \otimes K_{\Sigma}^*$):

$$(27) \quad \Delta_{\bar{\partial}} = (\nabla^{(0,1)})^* \nabla^{(0,1)} + K(H, g_{\Sigma,R}) + \text{Ric}(g_\Sigma)$$

where $g_{\Sigma,R}$ is the result of rescaling $g_\Sigma$ and $K(H, g_{\Sigma,R})_{\alpha\bar{\beta}} = g^{\bar{\gamma}\bar{\delta}}_{\Sigma,R} R(H)_{\gamma\delta \alpha \bar{\beta}}$. Whence:

$$\Delta_{\bar{\partial}} \geq \frac{C_K - C_\Sigma}{R}$$

in the operator sense (since $\text{Ric}(g_R) \geq -\frac{C_\Sigma}{R}$ and $K(H, g_{\Sigma,R}) \geq \frac{C_\Sigma}{R}$, the latter when restricted to isotropic sections).
Thus, if we set $s = \sigma - \tau$ where $\tau = \Delta^{-1}_\sigma \partial \bar{\partial} \sigma$ then clearly $\bar{\partial} s = 0$. Also:
\[
\|\tau\|_{L^2,R} = \langle \Delta^{-1}_\sigma \partial \bar{\partial} \Delta^{-1}_\sigma \partial \bar{\partial} \sigma, \partial \bar{\partial} \Delta^{-1}_\sigma \partial \bar{\partial} \sigma \rangle = \langle \Delta^{-1}_\sigma \partial \bar{\partial} \sigma, \partial \sigma \rangle,
\]
since $\partial \bar{\partial} \sigma = 0$. Thus
\[
(28) \quad \|\tau\|_{L^2,R} \leq \sqrt{\frac{R}{R-C_K}} \|\partial \sigma\|_{L^2,R}
\]
Hence in particular, for $R$ sufficiently big:
\[
\|s\|_{L^2,R} \leq \|\sigma\|_{L^2,R} + \|\tau\|_{L^2,R} \leq \frac{11}{10} 2\pi
\]
Also, since on the ball of radius $\frac{R}{2}$, $\sigma$ is holomorphic, it follows that $s = \sigma$ on $B_{\frac{R}{2}}$ (since $\partial \bar{\partial} \sigma = 0$ implies $\tau = 0$) and therefore $s$ is isotropic there since $\sigma$ is. Since $s$ is holomorphic everywhere so is $g_C(s,s)$ but since $s$ is isotropic on $B_{\frac{R}{2}}$, which is equivalent to $g_C(s,s) = 0$, it follows by analytic continuation that $s$ is isotropic everywhere.

The rest is Proposition 23.

3.8. Making the complex structure and the bundle almost standard. Let $B \subset \mathbb{C}$ be the unit ball and let $E \rightarrow D$ a holomorphic vector bundle endowed with a Hermitian metric $H$.

Since $E \rightarrow D$ is a holomorphic line bundle we can infer the fact that $E$ is holomorphically trivial on $D$ (cf. [18]), that is to say there is an isomorphism:
\[
\phi : E \rightarrow D \times \mathbb{C}^n
\]
Clearly the map $\phi$ above is not an isometry of bundles. Nonetheless in order to apply Donaldson’s philosophy we merely need to construct an almost isometry.

More precisely, fix an integer $k$ and rescale the background metric $g$ on $\Omega$ by a factor of $R^2 = k$ and consider the Hermitian metric $H^k$ on $E$—here $H^k$ is calculated by diagonalizing $H$ and then taking the $k$-th power. To set this up formally, we introduce the following notation. Denote by:
\[
\Psi_R : D_R \rightarrow B
\]
the standard dilation by $R$:
\[
\Psi_R(z) = w_0 + \frac{z}{R}
\]
and let $E_k := \Psi^*_R E$, $g_R := R^2 \Psi^*_R g = \lambda(w_0 + \frac{z}{R}) |dz|^2$ (where $g = \lambda |dw|^2$) and $H_k := \Psi^*_R (H^k)$ and also $\phi_R := \Psi^*_R \phi$. 

Observe that for the curvature form of $H_R$ one has:

$$
\Theta(H_R)^i = H_R^{i\bar{j}} R(H_R)_{i\bar{j}}(z) dz \wedge d\bar{z} \\
= \phi^* \left( \Theta(H)^i \left( w_0 + \frac{z}{R} \right) \right) dz \wedge d\bar{z}
$$

where $R(H)_{i\bar{j}} := R(H)(e_i, \bar{e}_j, \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}})$ are the components of the curvature of $H$ and $\Theta(H)$ is the curvature 2-form. It is thus manifest that measured with respect to the rescaled metrics, for any $\epsilon > 0$ there exists $R_0$ such that:

$$
\|\Theta(H_R) - \Theta_0\| < \epsilon
$$

for any $R \geq R_0$, where $\Theta_0 = \Theta(H)(w_0) dz \wedge d\bar{z}$.

Let $\Lambda := (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}_+^n$ an n-tuple of positive numbers and consider the endomorphism of $\mathbb{C}^n$:

$$
\Lambda Id_{\mathbb{C}^n} : \mathbb{C}^n \to \mathbb{C}^n \quad \Lambda Id_{\mathbb{C}^n}(z_1, \cdots, z_n) = (\lambda_1 z_1, \cdots, \lambda_n z_n)
$$

Observe that, for any $\epsilon > 0$, there exists an $R$ sufficiently big, such that:

$$
\|K - \Lambda Id_{\mathbb{C}^n}\|_{C^\infty} < \epsilon
$$

where $\Omega_0$ denotes the (Kähler form associated to the) flat metric on $\mathbb{C}^n$. The following fact is now obvious:

**Lemma 26.** Up to a constant endomorphism of $E$, the Hermitian bundle $(E, H_k)$ is nearly isometric, via $\Phi_R$, to the bundle $B_R \times \mathbb{C}^n$ endowed with the Hermitian metric (defined up to a constant endomorphism of $D \times \mathbb{C}^n$):

$$
H_\Lambda = \bigoplus_{i=1}^n e^{-\frac{\lambda_i |z|^2}{2}} h_0
$$

whose curvature form is:

$$
\Omega_\Lambda := \oplus_i \lambda_i \Omega_0
$$

**Proof.** First observe that by equation (29) we may assume that in some scale $R_{ij}(H)$ and and $\Lambda Id_{\mathbb{C}^n}$ are $\epsilon$-close. Since we are on a Riemann surface, the curvature of the metric $H$ takes the form (cf. formula (6))

$$
R_{ij} := R(e_i, \bar{e}_j, \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}) = -\frac{\partial^2 H_{ij}}{\partial z \partial \bar{z}} + \frac{\partial H_{ik}}{\partial z} H^{kj} \frac{\partial H_{kj}}{\partial \bar{z}}
$$

By elliptic regularity, given a function $K_{ij}$ and a metric on the boundary $h_{ij} \in C^{k,\alpha}(\partial D)$, there exists a metric $H_{ij} \in C^{k+2,\alpha}(D)$ with $K_{ij}$ as
POSITIVE ISOTROPIC CURVATURE: MACROSCOPIC DIM. AND $\pi_1(M)$

Let $D \subset \mathbb{C}$ be the unit disk endowed with the flat metric $g_0 = dx^2 + dy^2$. Let $(E, H)$ be a holomorphic Hermitian bundle over $D$ with associated connection 1-form $\Lambda$.

Assume that $E = F \otimes \mathbb{C}$ for some real bundle $F$ and assume further that $F$ is endowed with an inner product structure $g$ and that its complex bilinear extension $g_C$ is such that $H(v, w) := g_C(v, \bar{w})$. Let $w_0 \in D$ any point and $r > 0$ such that $B_r(w_0) \subset D$.

If the curvature of $H$ is positive on isotropic two planes, then there exist an $R_0 > 0$ and a smooth section $\sigma$ of $E$ such that, if $H_R := kH$ and $R := \sqrt{k}$, for $r \geq R_0$:

1. $\bar{\partial}_A \sigma = 0$, i.e. $\sigma$ is holomorphic;
2. $\sigma$ is isotropic: $g_C(\sigma, \sigma) = 0$.
3. $|\sigma(w_0)|_{H_R} = 1$
4. $|\sigma(z)|_{H_R} \geq e^{-\frac{|z|^2}{4a}} (1 - a)$ if $z \in B_{\frac{a}{2}}(w_0) \subset D$.
5. $\pi \leq \|\sigma\|_{H_{kL^2(B_R)}} \leq 2\pi$
6. $\|\sigma\|^2_{H_{kL^2(B_{\frac{a}{2}})}} \leq \frac{2\kappa}{1-a} \|\sigma\|^2_{H_{kL^2(B_{\frac{a}{2\sqrt{k}}})}}$

This is automatic on the disk $D$.
The same holds true if one merely assumes that $K_{\text{iso}}^C \geq -C$, but the constants will depend on $C$.

Proof. Let $w_0 \in D$ any point in the interior. We choose $k = R^2$ by requiring that after rescaling, using the map $\Phi_R$ defined at the beginning of this section, on $D_R$ we can achieve:

$$\sup_{D_R} \| \Theta(H_k) - \Theta(H_k)(w_0) \| < \epsilon$$

for any given $\epsilon > 0$. Therefore, according to Lemma 26 we have:

$$\sup_{D_R} \| H_k - H_\Lambda \| < \epsilon$$

where $\Lambda = (\lambda_1, \cdots, \lambda_n)$,

$$\Theta(H_R)(w_0) = \bigoplus_i \lambda_i \Omega_0$$

and

$$H_\Lambda = \bigoplus_{i=1}^n e^{-\frac{\lambda_i |z|^2}{2}} \hat{h}_0$$

Next remark that the complex structure on $E \to D$ is rigid, up to equivalence, thanks to Theorem Y pg. 211 in [18], as already remarked in Proposition 17. Since $H_\Lambda \leq H_0$, where $H_0$ is the Euclidean Hermitian metric on $E \simeq D \times \mathbb{C}^n$ we can choose $R$ sufficiently large so that

$$\frac{100}{81} H_0 \leq H_R \leq \frac{100}{81} H_0.$$  

Whence, according to Lemma 21 we can produce a holomorphic isotropic section $\sigma_0 = (\sigma_{0,1}, \cdots, \sigma_{0,n})$ of $(E, H_R)$ satisfying (here the constant $\kappa$ appearing in the hypotheses of Lemma 17 is $\kappa = \frac{100}{81}$):

- $|\sigma_0(0)| = 1$
- $|\sigma_0|_{H_R}^2 \leq 2$
- $|\sigma_0|_{H_R}^2 \leq 2$
- $|\sigma_0(z)|_{H_{n,k}} \geq e^{-k_n \frac{a^2R}{\kappa}} (1 - a)$ if $z \in B_{\frac{a}{\sqrt{n}}} \hat{h}_0(0)$.
- $\pi < \| \sigma_0 \|_2 < 2\pi$.
- $\| \sigma_0 \|_{L^2(B_R)} \leq \frac{2\kappa}{1-a} \| \sigma \|_{L^2(B_{\frac{a}{\sqrt{n}}})}^2$

Where the $L^2$-norms are taken with respect to volume form $dx \wedge dy$.

We now consider the bundle $S$ whose sheaf of sections is:

$$\{ \alpha \in C^\infty(E) : \bar{\partial}_A \alpha = 0 \text{ and } K(H)(\alpha, \alpha) \geq 0 \}$$

which is clearly non-empty since isotropic sections belong to it by assumption.
It is now easy to show that after pulling back via $\psi_R: D_R \to D$ and rescaling the metric $g_R = R^2\psi^*g$, the second fundamental form of $S$ becomes negligible and therefore, by Lemma 10 we may assume that $(S, H_S)$ has non-negative curvature (up to rescaling) (alternatively, we can endow $S$ with the complex structure determined by the connection $\nabla_E$ of $E$ and then argue that, for $R$ sufficiently large, the induced sub-bundle complex structure and this complex structure are arbitrarily close).

But in fact we can do better. Since every vector bundle splits holomorphically on the disk $\Delta$, thanks to Theorem Y pg. 211 in [18], it is also the case that every short exact sequence of vector bundles must split. Thus, we can view $S$ as being, in a natural way, a quotient of $E$: $E \to S \to 0$. Thus, appealing to Lemma 10 yields that $S$ has non-negative curvature operator, with respect to the induced Hermitian metric and induced Hermitian structure (so that one does not have to work with errors). We then make use of Theorem 30 (see below) to argue that we can confuse the induced complex structure with the complex structure for which $S$ has non-negative curvature operator.

Whichever way, we can next apply the arguments of Lemma 21 to the holomorphic section $e^{-\lambda_n|z|^2}\sigma_0$ to prove (1)-(3), (5) and (6). We now achieve the uniform estimate in item (4) by applying Proposition 23 (with $h = g_0 = dx^2 + dy^2$) to control uniformly the $L^\infty$-norm of $\|\nabla s\|$ and therefore the Lipschitz constant of $|\sigma|$.

Next we need (in the spirit of Property (H) in [12]):

**Proposition 28.** In the same assumptions as above, there exists an $R > 0$, a smooth isotropic section, $s$ of $E$ such that:

1. $\pi < \|s\|_{L^2} < 2\pi$;
2. $|s(0)| = 1$;
3. For any smooth section $\tau$ of $E$ over a neighborhood of $\overline{D}$ we have
   $$|\tau(0)| \leq C \left( \|\bar{\partial}\tau\|_{L^p(D)} + \|\tau\|_{L^2(D)} \right);$$
   where the volume form is the Euclidian one.
4. $\|\bar{\partial}s\|_{L^2} < \frac{3}{R} \left( \|s\|_{L^2(B_R)} + e^{-\frac{|z|^2}{4}}2\pi R \right)$;
5. $|\sigma(z)|_{H_R} \geq e^{-\frac{|z|^2}{4}} (1 - a)$ if $z \in B_{\frac{1}{2}}(w_0) \subset D$.
6. $\|\sigma\|^2_{H_R,L^2(B_R)} \leq \frac{2\pi}{1 - a} \|\sigma\|^2_{H_R,L^2(B_{\frac{aR}{2\pi}})}$.

**Proof.** For simplicity of notation we set $B \equiv B_R$. First of all, item (3) holds for a $C$ independent of $R$. In fact, given $B' \subset B$ some interior
domain containing 0, the standard elliptic estimate
\[(31) \quad \| \tau \|_{L^p(B')} \leq C_e \left( \| \overline{\partial \tau} \|_{L^p(B)} + \| \tau \|_{L^2(B')} \right),\]
coupled with the Sobolev inequality
\[|\tau(u_*)| \leq C_S \| \tau \|_{L^p_1(B')} .\]
yield (3).

Let \( \eta_R = \eta_R(|z|) \) be a cut-off function:
\[
\eta_R(|z|) = \begin{cases} 
1 & \text{if } |z| \leq R/2 \\
0 & \text{if } |z| \geq 9R/10
\end{cases}
\]
such that:
\[|\eta_R'| \leq \frac{3}{R}.
\]

Let \( \sigma \) be as in Lemma 21. Define:
\[s = \eta_R \sigma.\]

Then we have \( \overline{\partial s} = (\overline{\partial \eta_R}) \sigma \), therefore:
\[(32) \quad \| \overline{\partial s} \| = \| \overline{\partial \eta_R} \sigma \| = \| \overline{\partial \eta_R} \| \| \sigma \|
\]
whence (using that \( \| \sigma \|_{L^2(B_R)} = e^{-\frac{R^2}{2}} \| \sigma_0 \|_{L^2(B_R)} \) and that \( \| \sigma_0 \|_{L^2(B_R)} \leq 2\kappa \pi \)):
\[
\| \overline{\partial s} \|_{L^2(B_R)} \leq \frac{3}{R} \| \sigma \|_{L^2(B_R)} \leq \frac{3}{R} \left( \| \sigma \|_{L^2(B_{R/2})} + \| \sigma \|_{L^2(B_R \setminus B_{R/2})} \right)
\leq \frac{3}{R} \left( \| \sigma \|_{L^2(B_{R/2})} + e^{-\frac{R^2}{4}} \| \sigma_0 \|_{L^2(B_{R/2})} \right) \leq \frac{3}{R} \left( \| \sigma \|_{L^2(B_{R/2})} + e^{-\frac{R^2}{4}} \kappa \pi R \right)
\leq \frac{3}{R} \left( \| s \|_{L^2(B_{R/2})} + e^{-\frac{R^2}{4}} \kappa \pi R \right)
\]
which proves item (4). In the second inequality of the series of inequalities above we used the fact that:
\[\| \sigma \|_{L^2(B_{R/2})}^2 = \| \sigma \|_{L^2(B_{R/2})}^2 + \| \sigma \|_{L^2(B_R \setminus B_{R/2})}^2
\]
and that if \( x, y \geq 0 \) then \( \sqrt{x^2 + y^2} \leq x + y \).

Next, one easily verifies that:
\[\| s \|_{L^2} = 2\pi - \delta \quad \| s \|_{L^2(G)} = 2\pi - \delta'
\]
for some small, \( \delta, \delta' > 0 \)– which proves item (1)– and also, trivially:
\[|s(0)| = 1.\]

This proves item (2). \( \square \)
The next Proposition is due to Donaldson and Song (cf. [12])

**Proposition 29.** The properties (1)-(6) in Proposition 28 are open with respect to variations in \((g, J, A)\) (for fixed \((B, D, 0, F)\)) and the topology of convergence in \(C^0\) on compact subsets of \(U\).

We are now ready to prove the main result of this section. First we define the following sets:

\[
\mathcal{M}(\epsilon, C) := \{ (M, g) : K_{\text{isotr}}^c(M) \geq \epsilon^{-2} \}
\]

and

\[
\mathcal{K} := \left\{ f : D \to (M, g) : f \text{ is a minimal proper immersion and } (M, g) \in \mathcal{M}(\epsilon, C) \right\}
\]

where \(D \subset \mathbb{R}^2\) is the unit disc.

**Theorem 30.** Suppose that \((D_R, D, 0, F := D_R \times \mathbb{C}^n)\) are as above and the datum \(g_0, J_0, A_0, H_0\) satisfies properties (1)-(6) in Proposition 28. Also assume that the Hermitian metric \(H_0\) is such that \(H_0 \leq H_e\) where \(H_e\) is the standard flat Hermitian metric. Then there is some \(\epsilon_0 > 0\) such that if \((D, f, M, g)\) is in \(\mathcal{K}\) and we can find \(R > 0\), a scaling \(\Phi_R : D \to B\) with \(\Phi_R(0) = 0\) and a bundle isomorphism \(\tilde{\Phi}_R : F \to E := f^*(TM \otimes \mathbb{C})\) such that

\[
\|\tilde{\Phi}_R^*(J) - J_0\|_U, \|\tilde{\Phi}_R^*(g) - g_0\|_U, \|\tilde{\Phi}_R^*(A) - A_0\|_U \leq \epsilon, \|\tilde{\Phi}_R^*H - H_0\| < \epsilon
\]

with \(\epsilon \leq \epsilon_0\) then there is a smooth section \(s\) of \(E\) such that:

1. \(\pi < \|s\|_{L^2} < 2\pi\);
2. \(|s(0)| = 1|\)
3. For any smooth section \(\tau\) of \(F\) over a neighborhood of \(\overline{D}\) we have

\[
|\tau(0)| \leq C \left( \|\tilde{\partial} \tau\|_{L^p(D)} + \|\tau\|_{L^2(D)} \right);
\]

4. \(\|\tilde{\partial}s\|_{L^2} < \frac{2\pi}{\pi} \left( \|s\|_{L^2(B_R)} + e^{-\frac{a^2}{4}} \right) < \frac{9}{\pi} \|s\|_{L^2} ;
\]

5. \(|\sigma(z)|_{H_R} \geq e^{-\frac{|z|^2}{2(1-a)}} \) if \(z \in B_{\frac{1}{2}}(w_0) \subset D\).

6. \(|\sigma(z)|_{H_R} \leq \frac{2\kappa}{1-a} \|\sigma\|_{H_{\text{R}}, L^2(B_{\frac{1}{2}}(w_0))}^2 \)

**Proof.** Note that the Hermitian metric \(H\) on \(E\) corresponding to the datum \((A, J, g)\) satisfies:

\[
\|H - H_0\|_{H_0} < \epsilon
\]

where \(H_0\) is the Hermitian metric \(H\) on \(E\) corresponding to the datum \((A_0, J_0, g_0)\). Since by assumption \(H_0 \leq H_e\), we may assume \(H \leq 2H_e\).
Hence, according to Proposition 17 we can find a holomorphic isotropic section \( s \) satisfying:

1. \( |s(0)| = 1 \)
2. \( |s|_{H}^{2} \leq 2 \)
3. \( |\partial s|_{H}^{2} \leq 2 \)
4. \( |s(z)| \geq \frac{1}{2} \) if \( z \in B_{\frac{2}{\pi}}(0) \)
5. \( \pi \leq \|s\|_{L^{2}}^{2} \leq 2\pi \)

The rest of the proof goes like the proof of Proposition 27. Items (1)-(4) are a straightforward consequence of Proposition 29 and Proposition 28. Item (5) follows from Proposition 23 since the \( L^{2} \)-norm of with respect to \( \Phi_{R}^{*}g \) (where \( g \) is the metric induced from the embedding) is comparable to the \( L^{2} \)-norm calculated with respect to the \( \Phi_{R}^{*}G \), where \( G = dx^{2} + dy^{2} \).

Let us elaborate item (4) a bit. We have:

\[ \|\bar{\partial}s\|_{L^{2}} < \frac{3}{R}(\|s\|_{L^{2}(B_{R})} + e^{-\frac{R^{2}}{2}}2\pi R); \]

therefore, using that \( \|s\| > \pi \) and that \( 2xe^{-\frac{x^{2}}{2}} \leq \frac{2}{\sqrt{e}} < 2 \) (so that \( e^{-\frac{R^{2}}{2}}2\pi R < 2e^{-\frac{R^{2}}{2}}R \|s\| < 2\|s\|)\)

\[ \|\bar{\partial}s\|_{L^{2}} \leq \frac{9}{R}\|s\|_{L^{2}} \]

\[ \square \]

3.9. The destabilizing section and the Main theorem. In this section we fix the flat metric \( G := dx^{2} + dy^{2} \) on \( D \) and the metric \( f^{*}g \) on \( E \) (here we abuse notation in writing \( f^{*}g \), meaning the Hermitian metric induced by \( H := f^{*}g \) on \( E := f^{*}TM \otimes \mathbb{C} \)). We will also denote by \( \Delta \) the flat Laplacian on \( D \) (i.e., the Laplacian on functions associated to \( G \)).

In order to prove the Main theorem we will need:

**Proposition 31.** For any point \( p \in f(D) \) such that \( r := \text{dist}_{f(D)}(p, \partial f(D)) \) there exists a compactly supported \( g \)-isotropic section \( s = s_{p} = \eta\sigma \) of \( E \to D \), where \( \sigma \) is isotropic and holomorphic and \( \eta \) is compactly supported smooth function, such that:

1. The support of \( s \) is contained in the ball of radius \( r \) centered at \( p \): \( B_{r}(p) \);
2. \( \pi < \|s\|_{L^{2}} < 2\pi \);
3. \( \|\bar{\partial}s\|_{L^{2}}^{2} < \frac{9}{R}\|s\|_{L^{2}(B_{R})}^{2} \);
4. \( \frac{81n\pi}{4} \|s\|_{L^{2}(B_{r})}^{2} \geq \|\sigma\|_{L^{2}(B_{r})}^{2} \)
Proof. We first rescale (and translate) so that $B_r(p)$ is the unit ball centered at $0$. According to Lemma 26 we can find a rescaling $\Psi_k : B_k \to B$ so that (for $k$ sufficiently large) we can get $E_k \to B_k$ to satisfy the hypotheses of Theorem 30. We also choose $R$ sufficiently large so that $g_R := R^2 \Psi_k^* g = \lambda \left( x_0 + \frac{x}{R}, y_0 + \frac{y}{R} \right) (dx^2 + dy^2)$, where $w_0 = (x_0, y_0)$, is such that:

$$\frac{1}{2} (dx^2 + dy^2) \leq g_R \leq 2(dx^2 + dy^2).$$

We then get, by a straightforward application of Proposition 27, a holomorphic section $\sigma_k$ of $(E, H_k)$ satisfying (in the rescaled metric):

- $\pi < \|\sigma\|_{L^2, g_0, H_k} < 2\pi$;
- $|\sigma(0)|_{H_k} = 1$;
- $\pi < \|\sigma\|_{L^2, g_0, H_k} < 2\pi$;
- $\|\sigma\|^2_{L^2(B_R), g_0, H_k} \leq \frac{2\kappa}{1-a} \|\sigma\|^2_{L^2(B_{2\lambda R}), g_0, H_k}$, with $\kappa = \frac{100}{91}$;
- $|\sigma(z)| \geq e^{-\lambda |z|^2} (1-a)$ if $z \in B_{2\lambda R}(0)$.

Where for emphasis we have indicated by $\|\beta\|_{L^2, g_0, H_k}$ the $L^2$-norm of a section $\beta$ calculated using the Euclidean metric $g_0$ and the Hermitian metric $H_k$.

We now set $\sigma_k := (\sigma_1^k, \cdots, \sigma_n^k)$ –which is defined in the interior of $B_{\frac{r}{10}}$ by the last item above, since it shows there are no zero in the interior of the ball $B_{\frac{r}{10}}$– and we observe that:

$$\|\sigma\|^2_{H_k} \leq \|\sigma_k\|^2_{H_k} \leq n \|\sigma\|^2_{H_k}$$

as one can easily prove by diagonalizing $H$ (hence $H_k$) at a point (and using that $(\sum_i a_i^2)^{\frac{1}{2}} \leq \sum_i a_i^k \leq n (\sum_i a_i^2)^{\frac{1}{2}}$ for any $(a_1, \cdots, a_n) \in \mathbb{R}^n$).

Next, let $\eta_r = \eta_r(|z|)$ be a standard cut-off function:

$$\eta_r(|z|) = \begin{cases} 
1 & \text{if } |z| \leq r/2 \\
0 & \text{if } |z| \geq 9r/10
\end{cases}$$

such that:

$$|\eta_r'| \leq \frac{3}{r}$$

Let $\sigma$ be as above and define:

$$s = \eta_r \sigma.$$

Then we have $\partial \bar{s} = (\partial \eta_r) \sigma$, therefore:

$$\|\partial \bar{s}\| = \|\partial \eta_r \sigma\| = |\partial \eta_r| \|\sigma\|$$
which proves item (4). As for item (5), equation (33) and the fact that 
\[ \|\sigma\|^2_{L^2(B_R),g_0,H_k} \leq \frac{2\kappa}{1 - a} \|\sigma\|^2_{L^2(B_{\frac{g_0}{2\kappa}},g_0,H_k)} \]
yield:

(35) \[ \|\sigma\|^2_{L^2(B_R),g_R,H_k} \leq n \|\sigma\|^2_{L^2(B_R),g_R,H_k} \leq 2n \frac{200}{91(1 - a)} \|\sigma\|^2_{L^2(B_{\frac{g_0}{2\kappa}},g_0,H_k)} \leq \frac{4n}{91(1 - a)} \|\sigma\|^2_{L^2(B_{\frac{g_0}{2\kappa}},g_R,H_k)} \]

Next observe that:

(36) \[ \|\sigma\|^2_{L^2(B_{\frac{g_0}{2\kappa}}),g_R,H_k} = \left( \|\sigma\|^2_{L^2(B_{\frac{g_0}{2\kappa}}),g_R,H_k} \right)^{\frac{k-1}{k}} \|\sigma\|^2_{L^2(B_{\frac{g_0}{2\kappa}}),g_R,H_k} \leq (2\pi)^{\frac{k-1}{k}} \|\sigma\|^2_{L^2(B_{\frac{g_0}{2\kappa}}),g_R,H_k} \leq 2\pi \|\sigma\|^2_{L^2(B_{\frac{g_0}{2\kappa}}),g_R,H_k} \]

Therefore putting together equations (35) and (36):

\[ \|\sigma\|^2_{L^2(B_R),g_R,H_k} \leq \frac{800}{91(1 - a)} 2n \pi \|\sigma\|^2_{L^2(B_{\frac{g_0}{2\kappa}}),g_R,H_k} \]

Noticing that \( \frac{800}{91} < 9 \) and using once again eq. (33), we infer:

\[ \|\sigma\|^2_{L^2(B_R),g_R,H} \leq \frac{18n\pi}{1 - a} \|\sigma\|^2_{L^2(B_{\frac{g_0}{2\kappa}}),g_R,H} \]

Thus, choosing \( a = \frac{5}{9} \), we have:

\[ \|\sigma\|^2_{L^2(B_R),g_R,H} \leq \frac{81n\pi}{4} \|\sigma\|^2_{L^2(B_{\frac{g_0}{2\kappa}}),g_R,H} \]

Scaling back to \( g \), we prove item (4). \( \Box \)

Remark 32. It is important to notice that, since we don’t really apply the full Hörmander technique to find holomorphic sections – as we only care about almost holomorphic ones– and since we ultimately simply apply rescalings, we can afford to achieve point 4 in Proposition 31 with constants (and radii) that are independent of the full sectional curvature. This could also be achieved by insisting that \( \partial\sigma_R \) be isotropic too. Ultimately this point is irrelevant for the topological application on the fundamental group, but it is important to prove Gromov’s conjecture as originally stated.

We can now prove our main estimate:

**Theorem 33.** Let \( f : D \to M \) be a stable, minimal (possibly branched) immersion. Then for every point \( p \in D \) there exists a smooth isotropic
section $\sigma = \sigma_p$ of $E$ which is perpendicular to $\frac{\partial f}{\partial z}$, and a constant $C$ such that:

$$
\left(37\right) \quad \frac{\int_{D} |\nabla_{\sigma} \sigma|^2 dV}{\int_{D} |\sigma|^2 dV} \leq C \frac{1}{r^2},
$$

where $r := \text{dist}_D(p, \partial D)$. Furthermore, the constant $C$ is computable and it can be chosen to be $\frac{9}{4} \pi n$.

Proof. Let us fix notation first. We will view $f^*g$ simultaneously as inducing a Hermitian metric on the vector bundle $E$ and as inducing a (necessarily) Kähler metric on $D$. In its incarnation as the latter, we will write it as $g = \lambda G = \lambda (dx^2 + dy^2)$. As before, we denote the Hermitian metric induced on $E$ by $H$.

We will in fact consider the vector bundle $N := \nu_f \otimes \mathbb{C}$ (here $\nu_f$ is the normal bundle of the map $f$, cf. section 2) with the complex structure compatible with $\nabla^\perp$ (whose existence is guaranteed by Koszul-Malgrange theorem). Let $H_n$ be the hermitian metric induced by the quotient map $E \to N$.

Since the curvature of $g$ is positive on totally isotropic 2-planes, we infer that for any isotropic section $s$ of $N$ (so that $s$ and $\frac{\partial}{\partial z}$ are independent) we have that:

$$
K(H_n)(\frac{\partial}{\partial z}, s, \frac{\partial}{\partial \bar{z}}, s) > 0
$$

therefore a straightforward application of Lemma 10 yields that for any isotropic section $s$ of $N$:

$$
K(H_n)(\frac{\partial}{\partial z}, s, \frac{\partial}{\partial \bar{z}}, s) > 0
$$

Given a point $p \in D$ at distance $r$ from the boundary, as in the hypothesis, we consider the ball $B_r(p)$ centered at $p$ of radius $r$. We then rescale the induced metric $g = \lambda (dx^2 + dy^2)$ on $D$ so that $B_r(p)$ becomes the unit disk $D$ centered at the origin. It now suffices to prove that the inequality (37) in the theorem holds with $r = 1$.

Whence, according to Proposition 31 we can find a smooth $g_C$-isotropic section $\sigma$ of $E \to B$ such that:

1. The support of $s$ is contained in the ball of radius $r$ centered at $p$: $B_r(p)$;
2. $\pi < \|s\|_{L^2} < 2\pi$;
3. $\|\bar{s}\|_{L^2} \leq \frac{2}{\pi} \|s\|_{L^2(B_r)}$;
4. $\frac{81n^4}{4} \|s\|^2_{L^2(B_r)} \geq \|\sigma\|^2_{L^2(B_r)}$.
Note that item (2) and (4) imply that:

\[ \frac{\int_D |\nabla \tilde{\omega}|^2 \sigma^2\frac{1}{H^2} dV_g}{\int_D |\sigma|^2_{H^2} dV_g} \leq \frac{9^3 n \pi}{4r^2} \]

\[ \square \]

Finally we remark that with Theorem 4– which we just proved– in hand, the proof of Theorem 3 is a mere application of the techniques of Gromov-Lawson in [23], and in particular of the implication that Theorem 10.2 therein implies Theorem 10.7.

REFERENCES

[1] M. F. Atiyah, N. J. Hitchin, and I. M. Singer, Self-duality in four-dimensional Riemannian geometry, Proc. Royal Soc. London A 362 (1978), 425-461
[2] S. Brendle, Ricci flow and the Sphere Theorem, AMS, G. S. in Mathematics vol 111.
[3] S. Brendle and R. Schoen, Sphere theorems in geometry, Surveys in Differential Geometry XIII.
[4] J. Cheeger Degeneration of Riemannian metrics under Ricci curvature bounds, Lectures at Scuola Normale di Pisa.
[5] B.-L. Chen and X.-P. Zhu, Ricci Flow with Surgery on Four-manifolds with Positive Isotropic Curvature, J. Diff. Geom. 74 (2006), 177264.
[6] B.-L. Chen, S.-H. Tang and X.-P. Zhu Complete classification of Compact Four-manifolds with Positive Isotropic Curvature, J. Diff. Geom 91 (2012) 41-80
[7] J. Chen and A. Fraser On stable minimal disks in manifolds with non-negative isotropic curvature, J. reine angew. Math. 643 (2010), 2137
[8] S.Y. Cheng -S.T. Yau Differential equations on Riemannian manifolds and their geometric applications, Comm. Pure Appl. Math. 28 (1975), no. 3, 333–354
[9] T. Colding and W. Minicozzi, Minimal Surfaces, Courant Lecture Notes, 4, 1999
[10] J. P. Demailly Complex Analytic and Differential Geometry, [http://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf](http://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf)
[11] S. K. Donaldson, Symplectic submanifolds and almost-complex geometry, J. Differential Geom. 44 (1996), 666–705.
[12] S. K. Donaldson, S. Sun, Gromov-Hausdorff limits of Kahler manifolds and algebraic curvature, [http://arxiv.org/pdf/1206.2609.pdf](http://arxiv.org/pdf/1206.2609.pdf)
[13] A. Fraser, On the free boundary variational problem for minimal disks, Comm. Pure Appl. Math. 53 (2000), 931–971.
[14] A. Fraser, Fundamental groups of manifolds with positive isotropic curvature, Ann. of Math.
[15] A. Fraser and J. Wolfson The fundamental group of manifolds of positive isotropic curvature and surface groups, Duke
S. Gadgil and H. Seshadri On the topology of manifolds with positive isotropic curvature, Proceedings of the A.M.S.

Gilbarg and Trudinger Elliptic partial differential equations Springer.

P. Griffiths Topics in algebraic and analytic geometry, Mathematical Notes, No. 13. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1974

P. Griffiths and J. Harris Principles of Algebraic Geometry

M. Gromov Filling Riemannian manifolds J. Differential Geom. Volume 18, Number 1 (1983), 1-147.

M. Gromov Positive curvature, Macroscopic dimension, Spectral gaps and Higher Signatures Functional analysis on the eve of the 21st century, Vol. II (New Brunswick, NJ, 1993), 1213, Progr. Math., 132, Birkhuser Boston, Boston, MA, 1996.

M. Gromov Dirac and Plateau Billiards in Domains with Corners, 2013 http://www.ihes.fr/~gromov/PDF/Plateauhedra_modified_apr23.pdf

M Gromov and H. B. Lawson Positive scalar curvature and teh Dirac operator on complete Riemannian manifolds, Publ. Math. de l’I.H.E.S, tome 58, p.83-196.

J. L. Koszul and B. Malgrange, Sur certaines fibrées complexes, Arch. Math. 9 (1958).

R. Hamilton, Four manifolds with positive isotropic curvature, Comm. Anal. Geom., 5 (1997) 1-92.

S. Kobayashi, Differential geometry of complex vector bundles, Mathematical Society of Japan, No 15.

X. Ma and G. Marinescu Holomorphic Morse inequalities and Bergman Kernels, Progress in Mathematics vol. 254, Birkhäuser

M. Micallef Stable minimal surfaces in Euclidean space, J. Differential Geom. 19 (1984), 57–84.

M. Micallef and J. D. Moore, Minimal two-spheres and the topology of manifolds with positive curvature on totally isotropic two-planes, Ann. of Math. 127 (1988), 199–227.

M. Micallef and M. Wang, Metrics with nonnegative isotropic curvature, Duke Math. J. 72 (1992), 649–672.

J. Sacks and K. Uhlenbeck, The existence of minimal immersions of 2-spheres, Ann. of Math. 113 (1981), 1–24.

M. Micallef and J. Wolfson, The second variation of area of minimal surfaces in four-manifolds, Math. Ann. 295 (1993), 245-267.

C. B. Morrey, Multiple Integrals in the Calculus of Variations Springer, New York 1966

M. Ramachandran and J. Wolfson, Fill radius and the fundamental group. J. Topol. Anal. 2 (2010), no. 1, 99107.

J. Sacks and K. Uhlenbeck, Minimal immersions of closed Riemann surfaces, Trans. Amer. Math. Soc. 271 (1982), 639–652.

R. Schoen and S.-T. Yau, Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with nonnegative scalar curvature, Ann. of Math. 110 (1979), 127–142.

R. Schoen and S.-T. Yau, Lectures on Harmonic maps, International Press (1997)
[38] Y.-T. Siu and S.-T. Yau, *Compact Kähler manifolds of positive bisectional curvature*, Invent. Math. **59** (1980), 189–204.

[39] Stein, Elias; Weiss, Guido, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, 1971.

[40] Tian, G., *On Calabi’s conjecture for complex surfaces with positive first Chern class*, Invent. Math., 101, 1990, 1 101–172.

[41] G. Tian *Kähler-Einstein metrics on algebraic manifolds. Proc. of Int. Congress Math. Kyoto, 1990*

[42] G. Tian, *On a set of polarized Kähler metrics on algebraic manifolds*, J. Differential Geometry 32 (1990), 99-130.

[43] Hung-Hsi Wu, *The Bochner Technic in Differential Geometry*, Mathematical Reports, Volume **3**, Part 2, Harwood Academic Publishers, (1988)

[44] S.-T. Yau, *Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry*, Indiana Univ. Math. J., **25**, (1976), 659-670.

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