Can billiard eigenstates be approximated by superpositions of plane waves?

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Abstract
The plane wave decomposition method (PWDM) is one of the most popular strategies for numerical solution of the quantum billiard problem. The method is based on the assumption that each eigenstate in a billiard can be approximated by a superposition of plane waves at a given energy. By the classical results on the theory of differential operators this can indeed be justified for billiards in convex domains. On the contrary, in the present work we demonstrate that eigenstates of non-convex billiards, in general, cannot be approximated by any solution of the Helmholtz equation regular everywhere in $\mathbb{R}^2$ (in particular, by linear combinations of a finite number of plane waves having the same energy). From this we infer that PWDM cannot be applied to billiards in non-convex domains. Furthermore, it follows from our results that unlike the properties of integrable billiards, where each eigenstate can be extended into the billiard exterior as a regular solution of the Helmholtz equation, the eigenstates of non-convex billiards, in general, do not admit such an extension.

1 Introduction
The quantum billiard problem in a domain $\Omega \subset \mathbb{R}^2$ is defined (in units $m=1$) by the Helmholtz equation

$$(-\Delta - k^2)\varphi(x) = 0, \quad E = \hbar^2k^2/2$$

(1)

with Dirichlet boundary conditions

$$\varphi(x)|_{\partial \Omega} = 0.$$  

(2)

The solutions $E_n$, $\varphi_n$ of these equations determine the energy spectrum and the set of eigenstates of $\Omega$. Studying the properties of $(E_n, \varphi_n)$ in quantum billiards
has became a prototype problem in “quantum chaos”. A simple form of eqs. \(1, 2\) suggests a natural way to solve them. First, for a given energy \(E\) one looks for a set of solutions \(\{\psi(n)(k), n \in \mathbb{N}\}\) of the Helmholtz equation \(1\) in the entire plane (without any boundary conditions). For example, \(\{\psi(n)(k)\}\) can be chosen as a set of plane waves: \(\{\exp(ik_n x), |k_n| = k, k_n \in \mathbb{R}^2\}\), or as a set of radial waves: \(\{J_n(kr) \exp(in\theta)\}\). Then regarding \(\{\psi(n)(k)\}\) as a basis one can search for solutions of eqs. \(1, 2\) using the ansatz

\[
\varphi(x) = \sum a_i \psi^{(i)}(k, x).
\]

(3)

As a result, solving eqs. \(1, 2\) is reduced to the algebraic problem of finding the coefficients \(a_i\) such that the linear combination \(3\) vanishes whenever \(x \in \partial \Omega\).

The above approach has been widely used both in analytical and numerical studies of quantum billiards. In particular, it has been suggested by Berry in \[B1\] to use the expansion \(3\) with a Gaussian amplitude distribution to represent eigenfunctions of quantum systems with fully chaotic dynamics. This idea has been applied in numerous works to calculate various quantities associated with eigenfunctions, e.g., autocorrelation functions \[B1\], amplitude distributions \[BS\], statistics of nodal domains \[BGSS\] etc. The same strategy can be also used for a numerical solution of eqs. \(1, 2\). In this context it has been first introduced by Heller \[He\] with the application to the Bunimovich stadium. Since that several modification of the method have been considered in \[LRH\], \[LH\] and in \[CLH\]. Depending on the choice of the basis in the decomposition \(3\) one gets, in general, different numerical methods for solving eqs. \(1, 2\). Here we will single out the basis of plane waves (PW), most often used in applications. For the sake of briefness we will refer to the corresponding numerical method as plane wave decomposition method (PWDM).

As a matter of fact, the whole strategy described above is based on the assumption that the set \(\{\psi(n)(k)\}\) furnishes an appropriate basis for the expansion of solutions of eqs. \(1, 2\) in the entire plane (without any boundary conditions). That means

\[
||\varphi_n - \psi[N]||_{L^2(\Omega)} \to 0, \text{ as } N \to \infty
\]

(4)

for some sequence of the states \(\psi[N]\) which are of the form

\[
\psi[N] = \sum_{i=1}^{N} a_i e^{ik_i x}, \quad k_i \in \mathbb{R}^2, \quad |k_i| = k.
\]

(5)

We will say that the plane wave approximation holds for a state \(\varphi_n\) if the limit \(4\) exists.

Up to now it has been often assumed that the PWDM can be applied to billiards of arbitrary shape. From the results of Malgrange \[Ma\] (see also \[He\]) on the theory of differential operators it is known that any solution of eq. \(1\) regular in a convex open domain can be approximated by superpositions of plane waves with \(k_i \in \mathbb{C}^2\).
Moreover, since each evanescent plane wave \( (\text{Im} k \neq 0) \) can be approximated in a bounded domain by plane waves with real wavenumbers \( \text{B2} \), one immediately gets:

**Proposition 1.** Let \( \Omega \subset \mathbb{R}^2 \) be a convex bounded domain, then any solution of eq. 1 regular in \( \Omega \) can be approximated by plane waves.

This shows that the eigenstates of a quantum billiard \( \Omega \) admit PW approximation inside any convex domain \( \Omega_1 \subset \Omega \), see fig. 1a. Hence, PW approximation always holds for billiard eigenstates in a local sense. Furthermore, if \( \Omega \) is a convex domain one can choose \( \Omega_1 \) in such a way that \( \partial \Omega_1 \) is arbitrary close to \( \partial \Omega \). Consequently, as a simple corollary of Proposition 1 one gets:

**Corollary 1.** Eigenstates of a convex billiard \( \Omega \) can be approximated by superpositions of plane waves.

The question naturally arises whether the same property holds for eigenstates of non-convex billiards, and thus, whether the PWDM can be actually applied to the class of non-convex billiards.

Note that there exists an important link between the PWDM and the problem of eigenstate extension in quantum billiards. Suppose \( \varphi_n \) is an eigenstate of \( \Omega \) which can be extended (as a regular solution of eq. 1) from \( \Omega \) to a convex domain \( \Omega_2 \supset \Omega \). Then it follows immediately by Proposition 1 that PW approximation holds for \( \varphi_n \). The example of a billiard where each eigenstate can be continued in a convex domain is shown in fig 1.b. This is the “cake” billiard whose boundary consists of two concentric circle arcs connected by two segments of radii at an angle \( \alpha < \pi \). In the polar coordinates \( x = (r, \theta) \) the eigenstates of the “cake” billiard can be written explicitly as a sum of Bessel and Neumann functions:

\[
\varphi_n^{(m)}(x) = \left( a_n^m J_{\nu_m}(k_n^{(m)} r) + b_n^m Y_{\nu_m}(k_n^{(m)} r) \right) \sin (\nu_m(\theta - \theta_0)), \quad \nu_m = \frac{\pi m}{\alpha}.
\]

Since the singularity point of \( \varphi_n^{(m)}(x) \) is always at the center \( O \) of the circle arcs it is possible to extend each eigenstate into a convex domain \( \Omega_2 \), see fig 1.b. Accordingly, any eigenstate of the “cake” billiard can be approximated by superpositions of PW.

On the other hand, assume that for a billiard \( \Omega \) an eigenstate \( \varphi_n \) can be expanded in a basis \( \{\psi^{(n)}\} \) (see eq. 3), where \( \psi^{(i)} \)'s are solutions of the Helmholtz equation regular in \( \mathbb{R}^2 \) (e.g., plane waves). If furthermore, the corresponding sum converges everywhere in \( \mathbb{R}^2 \) it makes sense to consider \( \varphi_n(x) \) both inside and outside \( \Omega \). Such extension of \( \varphi_n(x) \) into \( \mathbb{R}^2 \) provides simultaneously solutions for the interior Dirichlet problem (when \( x \in \Omega \)) and for the exterior Dirichlet problem (when \( x \in \Omega^c = \mathbb{R}^2/\Omega \)). Based on this observation a connection (spectral duality) between the interior Dirichlet and the exterior scattering problems has been suggested by Doron and Smilansky in [DS]. The rigorous result has been established by Eckmann and Pillet [EP]. In most general form (weak spectral duality) it
could be stated as follows: $E_n$ is an eigenvalue of the interior problem if and only if there exists an eigenvalue $e^{-i\theta_n}$ of the exterior scattering matrix $S(E)$ such that $\theta_n(E) \to 2\pi$ whenever $E \to E_n$. Moreover, if $\theta_n(E_n) = 2\pi$ (strong spectral duality) then the corresponding interior eigenstate $\varphi_n$ could be extended into $\mathbb{R}^2$ as $L^2$ functions. Therefore if strong form of spectral duality holds for some eigenenergy $E_n$ then PW approximation holds for the corresponding eigenstate $\varphi_n$. It has been explicitly shown that strong form of spectral duality holds for convex integrable billiards [DIS]. However, as has been pointed out in [EP], strong spectral duality cannot hold for billiards in general.

Remark. It should be pointed out that the approximability by PW is much weaker property then strong spectral duality. As has been explained above, strong spectral duality implies PW approximation for the corresponding eigenstate. The opposite, however, is not true: PW approximation for an eigenstate does not imply, in general, strong spectral duality. In fact, in [B2, EP] the examples of convex billiards (in this case the approximation by PW is possible) have been constructed where the eigenstates extension into the exterior domain as $L^2$ functions is not possible.

2 Main results

Let $\Omega$ be a simply connected bounded domain in $\mathbb{R}^2$ with a piecewise smooth boundary $\partial \Omega$. Two different billiard maps can be associated with $\Omega$. First, the standard billiard map $\Psi$ corresponding to the motion of a pointlike particle in the interior domain. Second, the exterior map $\Psi^c$ which corresponds to the scattering off $\Omega$ as an obstacle, see e.g., [Sm]. In order to define the exterior map one can place $\Omega$ on a sphere $S^2$ of “infinite” radius. Then $\Psi^c$ is a standard billiard map corresponding to the motion of a pointlike particle in the domain $S^2/\Omega$. It should be noted that there is an essential difference between convex and non-convex billiards. Whenever $\Omega$ is a convex domain the interior map $\Psi$ determines the same dynamics as the exterior map $\Psi^c$. For any interior trajectory inside $\Omega$ there is a dual trajectory in $\Omega^c$ which travels through the same set of points on the boundary $\partial \Omega$, see fig. 2a. We will refer to this property as interior-exterior duality. In particular, for convex billiards there is one to one correspondence between the interior and exterior periodic trajectories. For each periodic trajectory $\gamma$ its continuation $\gamma^c$ into the exterior domain will be the dual periodic trajectory of the exterior map. On the other hand, it is straightforward to see that in non-convex billiards interior-exterior duality breaks down. Generally, in a non-convex billiard $\Omega$ there exist interior periodic trajectories whose extension into the exterior domain intersects $\Omega$ again, see fig. 2b. Let $\gamma$ be such a trajectory and let $\gamma^c$ be its extension in the exterior. Note that $\gamma \cup \gamma^c$ is a union of straight lines in $\mathbb{R}^2$. Take $l \subset \gamma \cup \gamma^c$ to be a line which intersects the boundary $\partial \Omega$ at $2n$, $n > 1$ points (for the sake of simplicity we will always assume that $n = 2$). Then the intersection $\Omega \cap l$ is the union of two disconnected segments: $\gamma_1 \subset \gamma$ and
If \( \tilde{\gamma}_1 \) does not belong to any periodic trajectory in \( \Omega \), we will refer to \( \gamma \) as *single periodic trajectory* (SPT). By definition any SPT has no dual periodic trajectory in the exterior domain. In what follows we call a non-convex billiard \( \Omega \) as *generic* if it contains at least one stable (elliptic) or unstable (hyperbolic) SPT. According to this terminology the “cake” billiard in fig. 1b is non-generic, since all its periodic trajectories are of neutral (parabolic) type.

We call a smooth function \( \psi(x) \) as a *regular solution* of the Helmholtz equation if it solves eq. 1 everywhere in \( \mathbb{R}^2 \). For a given energy \( E \) we will denote by \( \mathcal{M}(E) \) the set of all regular solutions of eq. 1 and by \( \mathcal{M}_{pu}(E) \subset \mathcal{M}(E) \) the subset of functions which can be represented as linear combinations of finite number of plane waves with real wavenumbers \( k_i \), \( |k_i|^2 = 2E/\hbar^2 \). In particular, \( \mathcal{M}(E) \) includes convergent superpositions of plane waves (also with complex wavenumbers i.e., evanescent modes) and radial waves with the energy \( E \). In its crudest form the main result of the present paper can be formulated in the following way. Based on the breaking of interior-exterior duality we demonstrate that eigenstates of a generic non-convex billiard (in general) cannot be approximated by regular solutions of eq. 1. To illustrate the main idea of our approach it is instructive to consider a non-convex billiard \( \Omega \) with an elliptic SPT \( \gamma \). It is well known that a sequence of quasimodes \( (\tilde{\varphi}_i,k_i) \) associated with \( \gamma \) can be constructed (see e.g., [PU1], [Ba]). Each pair \( (\tilde{\varphi}_n,k_n) \) represents an approximate solution of eqs. 1, 2 such that \( \tilde{\varphi}_n \) is localized along \( \gamma \). Furthermore, in the absence of systematic degeneracies in the spectrum of \( \Omega \) the quasimodes \( (\tilde{\varphi}_n,k_n) \) approximate (in \( L^2 \) sense) a sequence of real solutions \( (\varphi_n,k_n) \) of eqs. 1. For each such eigenstate \( \varphi_n \) let us consider the corresponding Husimi function

\[
H_{\varphi_n}(z) = |\langle z|\varphi_n \rangle|^2, \quad z = (q,p): \quad q \in \Omega, \quad |p| = \hbar k_n,
\]

(6)

where \( \langle z \rangle \) denotes a coherent state localized at the point \( z \) of the phase space of \( \Omega \). By the definition \( H_{\varphi_n}(z) \) is localized along \( \gamma \) and exponentially small everywhere else. On the other hand, assume that \( \varphi_n \) could be approximated by regular solutions of eq. 1. That means for any \( \epsilon > 0 \) there is \( \psi_\epsilon \in \mathcal{M}(E_n) \) such that \( ||\varphi_n - \psi_\epsilon|| < \epsilon \), where \( || \cdot || \) denotes the \( L^2(\Omega) \) norm. Set \( q \) be a point at \( \gamma_1 \) and set \( p \) be directed along \( \gamma_1 \). Then for \( z = (q,p) \) we have

\[
H_{\varphi_n}(z) = \lim_{\epsilon \to 0} |\langle z|\psi_\epsilon \rangle|^2 = \lim_{\epsilon \to 0} |\langle z|e^{-it\Delta/\hbar}\psi_\epsilon \rangle|^2, \quad \epsilon \to 0
\]

(7)

where \( e^{-it\Delta/\hbar} \) is the free evolution operator in \( \mathbb{R}^2 \). Furthermore, in the semiclassical limit the quantum evolution of coherent states is governed by the corresponding classical evolution

\[
e^{-it\Delta/\hbar}|z\rangle = e^{itE/\hbar}|z(t)\rangle + O(\hbar^\infty), \quad z(t) = (q(t),p).
\]

(8)

Plugging (8) into eq. 7 and taking time \( t \) to be such that \( q(t) = q' \in \gamma_1 \) one gets

\[
H_{\varphi_n}(z) - H_{\varphi_n}(z') = O(\hbar^\infty), \quad z' = (q',p).
\]

(9)
This, however, contradicts the fact that the Husimi function \( H_{\phi_n}(z) \) should be exponentially decaying outside \( \gamma \).

The above argument can be extended to the case of hyperbolic SPT \( \gamma \) as follows. Contrary to the elliptic case it is not possible to construct quasimodes concentrated on hyperbolic periodic orbits. Instead, one can use a statistical approach in that case. By the results of Paul and Uribe [PUT] it is known that the average of the Husimi functions \( \langle H_{\phi_n}(z) \rangle \)

\[
\langle H_{\phi_n}(z) \rangle = \frac{1}{\# \mathcal{P}_{ch}} \sum_{E_n \in \mathcal{P}_{ch}} |\langle z|\phi_n \rangle|^2
\]  

(10)

over the energy interval \( \mathcal{P}_{ch} = [E - \hbar, E + \hbar] \), \( c > 0 \) depends in the semiclassical limit \( \hbar \to 0 \) on whether \( z \) belongs to a periodic trajectory or not. On the other hand, as has been explained above, if each \( \phi_n \) could be approximated by a regular solution of eq. 1 then each \( H_{\phi_n}(z) \) (and therefore the average \( \langle H_{\phi_n}(z) \rangle \)) would be (semiclassically) invariant along \( \gamma_1 \cup \bar{\gamma}_1 \).

The preceding discussion provides an intuitive explanation why it is impossible to approximate eigenstates of a generic non-convex billiard by a superposition of plane waves. Speaking informally our argument says that contrary to the real eigenstates of non-convex billiard \( \Omega \), any regular solution of eq. 1 always “preserves” interior-exterior duality. In what follows we consider the \( L^2(\Omega) \) norm

\[
\eta_n(\psi) = ||\phi_n - \psi||,
\]  

(11)

for a solution \( (\phi_n, E_n) \) of eqs. 1, 2 in \( \Omega \) and an arbitrary \( \psi \in \mathcal{M}(E_n) \). By the definition \( \eta_n(\psi) \) measures approximability of \( \phi_n \) by regular solutions of the Helmholtz equation. Recall that a state \( \phi_n \) is approximable by PW if

\[
\inf_{\psi \in \mathcal{M}_{pw}(E_n)} \eta_n(\psi) = 0.
\]

**Remark.** Note that by Proposition 1 for any \( \psi \in \mathcal{M}(E_n) \) and any \( \epsilon > 0 \) one can always find \( \psi_\epsilon \in \mathcal{M}_{pw}(E_n) \) such that \( | \eta_n(\psi) - \eta_n(\psi_\epsilon) | < \epsilon \). In particular this implies

\[
\eta_n^{\min} \equiv \inf_{\psi \in \mathcal{M}(E_n)} \eta_n(\psi) = \inf_{\psi \in \mathcal{M}_{pw}(E_n)} \eta_n(\psi).
\]  

(12)

In other words, an eigenstate \( \phi_n \) can be approximated by \( \psi \in \mathcal{M}(E_n) \) if and only if it can be approximated by PW. Therefore, in what follows one can always assume without lost of generality that \( \psi \) belongs to \( \mathcal{M}_{pw}(E_n) \) rather than to the set \( \mathcal{M}(E_n) \).

By Corollary 1, \( \eta_n^{\min} = 0 \) for any eigenstate of a convex billiard. On the contrary, in the body of the paper we show that for a generic non-convex billiard the average of \( \eta_n^{\min} \) over an energy interval is bounded from below by a strictly positive constant:
Proposition 2. Let $\Omega$ be a non-convex billiard with at least one stable or unstable SPT and let $(\varphi_n, E_n)$, $n = 1, 2, \ldots, \infty$ denote the eigenstates and eigenenergies of the corresponding quantum billiard. For any set of approximating functions $\{\psi_i \in M(E_i), i \in \mathbb{N}\}$ the average of $\eta_n = \eta_n(\psi_n)$ over the energy interval $P_{ch} = [E - \bar{c}h, E + \bar{c}h]$, satisfies

$$\langle \eta_n \rangle > C(h),$$

where

$$B = \lim_{\bar{h} \to 0} C(h)/\bar{h}$$

is strictly positive and independent of $\psi_i$'s. Moreover, if $\Omega$ contains a SPT $\gamma$ of elliptic type then (provided the spectrum of $\Omega$ has no systematic degeneracies) there exists an infinite subsequence $\mathcal{S}_\gamma = \{(\varphi_{jm}, E_{jm}), m \in \mathbb{N}\}$ (of a positive density, i.e.,

$$\lim_{N \to \infty} \frac{\#\{(j_m|j_m < N)\}}{N} > 0$$

such that for any $(\varphi_n, E_n) \in \mathcal{S}_\gamma$ and any regular solution $\psi \in M(E_n)$

$$\eta_n(\psi) > C_\gamma + O(\bar{h}^{1/2}),$$

where $C_\gamma$ is a strictly positive constant independent of $\psi$ and $\bar{h}$.

From (13,14) one immediately obtains the corollary:

**Corollary 2.** For a generic non-convex billiard $\Omega$ there exists an infinite subsequence of eigenstates $\{\varphi_n, n \in \mathbb{N}\}$ such that: 1) $\eta_n^{\min} > 0$; 2) $\varphi_n$ cannot be extended into the domain $\Omega^c$ (as a regular solution of eq. [7]).

Obviously, this implies the following properties of a generic non-convex billiard:

- In general, eigenstates of non-convex billiards do not admit approximation by PW and PWDM cannot be used in that case;
- The spectral duality for a generic non-convex billiard holds only in the weak form.

The paper is organized as follows. In the next section we collect several necessary facts about coherent states. In Sec. 4 the case of elliptic SPT’s is considered. First, using the coherent states we construct a family of quasimodes $(\tilde{\varphi}_n, \tilde{E}_n)$ associated with such trajectories. Then, we show that the lower bound (14) holds for the eigenstates $\varphi_n$ approximated by $\tilde{\varphi}_n$. The case of hyperbolic SPT’s is considered in Sec. 5. Here we use the results of Paul and Uribe to estimate the average $\langle \eta_n \rangle$ over an energy interval. Finally in Sec. 6 we discuss our results and consider possible generalizations.

### 3 Coherent states

**Definition of coherent states.** The coherent states have been introduced already in the beginning of quantum mechanics and have been used in many areas since
then. The basic idea is to build a complete set of vectors of Hilbert space localized in the phase space both in $q$ and $p$ directions at the scale $\sqrt{\hbar}$. The standard example of such states in $\mathbb{R}^d$ is given by the Gaussians:

$$u_\sigma^z(x) = (\det \Im \sigma)^{-\frac{1}{2}} (\hbar \pi)^{-\frac{d}{4}} e^{\frac{i}{\hbar} \frac{1}{2} (x - q)^2 + \frac{i}{\hbar} (x - q, \sigma(x - q))}, \quad z = (q, p), \quad \Im \sigma > 0. \quad (15)$$

In the present work we will consider a slightly more general class of coherent states. (For a more general definition of coherent states see e.g., [PU1].) Let $\rho^\varepsilon_q(\cdot)$ be a $C^\infty_0$ function in $\mathbb{R}^d$ equal to one in a neighborhood of the point $q$ and zero outside the sphere of radius $\varepsilon$ centered at $q$. A coherent state at $z = (q, p)$ is the vector

$$\phi^\varepsilon_z(x) = \rho^\varepsilon_q(x) u_\sigma^z(x). \quad (16)$$

It is easy to see that the coherent states (16) are semiclassically orthogonal:

$$||\phi^\varepsilon_z||^2 = 1 + O(\hbar), \quad \langle \phi^\varepsilon_z | \phi^\varepsilon_{z'} \rangle = O(\hbar^\infty) \text{ if } z \neq z'. \quad (17)$$

The role of the cut-off $\rho^\varepsilon_q(x)$ is rather technical, it allows to define coherent states inside compact domains. To use the vectors (16) as coherent states inside a billiard domain $\Omega$ one needs that

$$\text{supp}[\rho^\varepsilon_q(x)] \subset \Omega. \quad (18)$$

**Propagation of coherent states.** An important property of coherent states is that their quantum evolution in the semiclassical limit is completely determined by the corresponding classical evolution. Let $H = -\hbar^2 \Delta/2 + v(x)$ be the operator of symbol $H = p^2/2 + v(x)$ inducing the flow $\Psi_t : V \to V$ on the phase space $V$. Then, as it is well known, for any time $t$ the propagation of the coherent state $\phi^\varepsilon_z$ localized at $z \in V$ is given by

$$e^{-itH/\hbar} \phi^\varepsilon_z = e^{i(S(t)/\hbar + \mu(t))} \phi^\varepsilon_{\zeta(t)} + O(\hbar^{1/2}), \quad (19)$$

where $S(t) = \int_0^t (p\dot{q} - H(p, q)) \, dt$ is the classical action along the path $\zeta(t)$ and $\mu(t)$ is the Maslov index. The parameters $z(t) = \zeta(t) \cdot z$, $\sigma(t) = D\Psi_t : \sigma$ in eq. (19) are determined by the evolution of the initial data $z$, $\sigma$ under the flow $\Psi_t : z \to z(t)$ and its derivative

$$D\Psi_t : \sigma \to \sigma(t) = \frac{a \sigma + b}{c \sigma + d}, \quad (20)$$

where $d \times d$ matrices $a, b, c, d$ are the components of $D\Psi_t$ in a given coordinate system:

$$D\Psi_t = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \quad$$

It is convenient to chose two of $2d$ coordinates in the phase space $V$ to be along the flow and along the line orthogonal to the energy surface. Then the matrix $\sigma$ can be
decomposed into $\sigma = \sigma^0 \oplus \sigma^1$, where the scalar part $\sigma^0$ corresponds to the above two directions and $(d-1) \times (d-1)$ matrix $\sigma^1$ corresponds to the orthogonal subspace. It is straightforward to see that in such a basis $D\Psi^t$ acts separately on $\sigma^1$ and $\sigma^0$. In particular, $D\Psi^t \cdot \sigma^0$ is given by a linear transformation:

$$D\Psi^t \cdot \sigma^0 = \frac{\sigma^0}{u\sigma^0 + 1}. \quad (21)$$

In the present paper we will use the above results for two types of two-dimensional flows: free evolution on $\mathbb{R}^2$ under the Hamiltonian $H_0$ ($v(x) = 0$) and the evolution induced by the billiard Hamiltonian $H_{\Omega}$ ($v(x) = 0$ if $x \in \Omega$ and $v(x) = \infty$ otherwise). Let us consider in some detail the evolution of coherent states in billiards. Set $\Omega$ be the billiard domain. We will denote by $\Psi_{\Omega}^t : V \to V$ the billiard flow, whose action is on the standard phase space $V$ of $\Omega$. It should be pointed out that one can use the coherent states (16) for the point $z = (q,p) \in V$ only if $q$ is sufficiently far away from the boundary $\partial \Omega$. Indeed, to satisfy the condition (18) $q$ has to be at the distance larger than $\varepsilon$ from the boundary. For the sake of simplicity, we will not consider a generalized class of coherent states defined in the whole domain $\Omega$, rather we will use the states (16) but only for the interior points of $\Omega$. For this purpose let us define the inner domain $\Omega_\varepsilon \subset \Omega$ which contains all the points $q$ of $\Omega$ such that the distance between $q$ and $\partial \Omega$ is larger then $\varepsilon$: $\text{dist}(q,\partial \Omega) \geq \varepsilon$, see fig. 3. In what follows we will fix $\varepsilon$ to be a small compare to linear sizes of the billiard (but large compare to $\hbar^{1/2}$) and consider the coherent states propagation under the condition that at the initial moment $t_1 = 0$ and the final moment $t_2 = t$ the points $z(0), z(t)$ belong to the domain $\Omega_\varepsilon$. Whenever this condition is fulfilled one can use the formula (19), where the states $\phi^z_\sigma, \phi^{\sigma(t)}_z$ are both of the form (16). Furthermore, if $q(t) \in \Omega_\varepsilon$ for all $t \in [t_1,t_2]$ (i.e., there is no collisions with the boundary between the times $t_1$ and $t_2$) then the reminder term in (19) is of the order $O(\hbar^\infty)$.

**Husimi functions.** Let $\varphi_n$ be an eigenstate of $H$ with the eigenenergy $E_n$. Given a coherent state $\phi^z_\sigma$ one can construct the corresponding Husimi function:

$$H_n(z) = |\langle \phi^z_\sigma | \varphi_n \rangle|^2 \quad z = (q,p); \quad \sigma = (\sigma^0, \sigma^1), \quad -i\sigma^0 = \beta > 0. \quad (22)$$

Based on the propagation formula (19) the following average over Husimi functions

$$\sum_n f(\omega_n) |\langle \varphi_n | \phi^z_\sigma \rangle|^2 = \sum_l d_l \hbar^{\frac{1}{2}+l}, \quad \omega_n = \frac{E_n - E}{\hbar}, \quad E = p^2 / 2 \quad (23)$$

has been calculated to the leading order by Paul and Uribe [PU1]. It turns out that the result depends on whether the classical trajectory through $z$ is periodic or not. With the application to the Hamiltonian $H_{\Omega}$ the results in [PU1] read as follows. Let $\tilde{f}(\cdot)$ be the Fourier transform of $f$. If $z$ is not periodic under the flow $\Psi_{\Omega}^t$ then

$$d_0 = \left( \frac{1}{\beta E} \right)^{1/2} \tilde{f}(0). \quad (24)$$
Alternatively, if $z$ belongs to a periodic trajectory the additional terms (of the same order in $\hbar$) arise. In particular, for a hyperbolic periodic trajectory $\gamma$ with the period $T_\gamma$ the leading term in (23) is given by

$$d_0 = \left( \frac{1}{\beta E} \right)^{1/2} \left( \sum_{l=-\infty}^{+\infty} \tilde{f}(lT_\gamma) \frac{e^{il(S_\gamma/h+\mu_\gamma)}}{\cosh^{1/2}(l\lambda_\gamma)} \right),$$

(25)

where

$$S_\gamma = 2ET_\gamma, \quad \mu_\gamma, \quad \lambda_\gamma$$

are the action, Maslov index and Laypunov exponent of $\gamma$.

4 PW approximation for eigenstates of non-convex billiards (elliptic case)

Let $\gamma$ be a periodic orbit in the billiard $\Omega$ and let $\Gamma(E)$ be the “lift” of $\gamma$ to the phase space $V$ at the energy $E$. This means $\Gamma(E)$ is a set of the points $z = (q, p) \in V$ such that $q \in \gamma$, $p^2 = 2E$ and the vector $p$ is directed along $\gamma$. Obviously, for any $z \in \Gamma(E)$, $\Psi_{\tilde{T}_\gamma} \cdot z = z$, where $\tilde{T}_\gamma$ is the period of the trajectory. We will make use of the letter $\varepsilon$ to denote the restriction of $\gamma$, $\Gamma(E)$ to the domain $\Omega_\varepsilon$ i.e.,

$$\gamma^\varepsilon = \{q \in \gamma \cap \Omega_\varepsilon\}, \Gamma^\varepsilon(E) = \{z = (q, p) \in \Gamma(E) : q \in \Omega_\varepsilon\}.$$

Provided that $\gamma$ is elliptic a set of approximate solutions (quasimodes) $\tilde{\varphi}_n(x)$ of eqs $1$, $2$ associated with $\gamma$ can be constructed. The possibility of quasimode construction on elliptic periodic orbits is well known. In the following we will follow the approach developed in [PU1, PU2] (see also [Sc], [Pa] and the references there).

Before we turn to the construction of the states $\tilde{\varphi}_n(x)$ in billiards let us recall a general definition for quasimodes.

**Definition.** Let $H$ be a Hilbert space and $\mathcal{H}$ be a self adjoint operator with the domain $D(H)$. A pair $(\tilde{\varphi}_n, \tilde{E}_n)$ with $\tilde{\varphi}_n \in D(H)$, $||\tilde{\varphi}_n|| = 1$ and $\tilde{E}_n \in \mathbb{R}$ is called a quasimode with the discrepancy $\delta_n$, if

$$(H - \tilde{E}_n)\tilde{\varphi}_n = r_n, \quad \text{with} \quad ||r_n|| = \delta_n.$$  

(26)

By a general theory (see e.g., [La]) the quasimodes $(\tilde{\varphi}_n, \tilde{E}_n)$ should be close to an exact solution $(\varphi_n, E_n)$ of the equation

$$(H - E)\varphi = 0$$

(27)

in the following sense. If $(\tilde{\varphi}, \tilde{E})$ is a quasimode with the discrepancy $\delta$ then there exists at least one eigenvalue of $H$ in the interval

$$\mathcal{P}_\delta = [\tilde{E} - \delta, \tilde{E} + \delta].$$

(28)
Furthermore, let \( \nu \) be the distance between \( \tilde{E} \) and an eigenvalue \( E_i \) of \( H \) outside \( \mathcal{P}_\delta \), then
\[
||\tilde{\varphi} - \pi_\nu \tilde{\varphi}|| \leq \frac{\delta}{\nu},
\]
(29)
where \( \pi_\nu \) denotes the spectral projection operator on the part of the spectrum \( \{E_n\} \) inside the interval \((\tilde{E} - \nu, \tilde{E} + \nu)\).

**Remark.** In general, the formula (29) implies that any state \( \tilde{\varphi} \) approximates a superposition of eigenstates \( \varphi_n \). In order to approximate individual eigenstates of \( H \), \( \delta_n \) should be much less than the energy intervals: \( \Delta E_n = |E_n - E_{n+1}|, \Delta E_{n-1} = |E_n - E_{n-1}| \). For two dimensional billiards \( \langle \Delta E_n \rangle \sim \bar{\hbar}^2 \), so the approximation of \( \varphi_n \) by \( \tilde{\varphi}_n \) becomes semiclassically \( \bar{\hbar} \to 0 \) meaningful only if the spectrum of \( \Omega \) has no systematic degeneracies and quasimodes with discrepancy \( \delta \sim \bar{\hbar}^\alpha, \alpha > 2 \) can be constructed. For the quantum billiard problem a quasimode construction providing \( \delta = O(\hbar^{\infty}) \) is known to exist [CP] and for the rest of this section we will assume that the billiard spectrum has no systematic degeneracies.

### 4.1 Quasimode construction

We will now schematically describe the construction of quasimodes concentrated on elliptic periodic orbits. The basic idea is to launch a coherent state along the orbit and average over the time. As it can be shown, this procedure yields an approximately invariant state if the initial state is chosen in the right way, see e.g., [PU1, Sc]. Let \( \phi_z, z = (q, p) \in \Gamma(E) \) be a coherent state localized on the periodic orbit \( \gamma \). We will associate with \( \gamma \) the state
\[
|\Phi^\sigma_{\Gamma(E)}\rangle = \frac{1}{C} \int_0^{T_\gamma} e^{i(E-H_\Omega)/\hbar} |\phi_z^\sigma\rangle \, dt,
\]
(30)
where \( C \) is fixed by the normalization condition \( ||\Phi^\sigma_{\Gamma(E)}|| = 1 \) and \( T_\gamma \) is the period of the classical evolution along \( \gamma \): \( z(T_\gamma) = z \). The propagation formula (19) yields
\[
(E - H_\Omega)|\Phi^\sigma_{\Gamma(E)}\rangle = r_\gamma, \quad Cr_\gamma = i\hbar \left( e^{i(S_\gamma/\hbar + \mu_\gamma)} \phi_z^{\sigma(T_\gamma)} - \phi_z^{\sigma} \right) + O(\hbar^{3/2}),
\]
(31)
where \( S_\gamma \), \( \mu_\gamma \) are the classical action and Maslov index after one period. Therefore, \( Cr_\gamma = O(\hbar^{3/2}) \) provided that the following conditions are satisfied:

**Condition 1:** \( \sigma(T_\gamma) = \sigma \); **Condition 2:** \( S_\gamma/\hbar + \mu_\gamma = 2\pi n \) for some integer \( n \).

For each \( n \) let \( \mathcal{E}_n, \sigma_n = (\sigma_n^0, \sigma_n^1) \) denote solutions of Conditions 1, 2. It is possible to show (see e.g., [PU1]) that the first condition can be satisfied if and only if \( \sigma_n^0 = 0 \) and \( \gamma \) is an elliptic periodic orbit. The second condition impose the Bohr-Sommerfeld quantization on the quasienergy \( \mathcal{E}_n \). When both conditions are satisfied
the corresponding pair \((\mathcal{E}_n, \Phi_{\Gamma(\mathcal{E}_n)}^\sigma)\) provides the quasimode with the discrepancy \(\delta_\gamma = O(h^{3/2})/C\).

**Remark.** It should be noted that a much wider class of quasimodes concentrated on \(\gamma\) can be constructed by this method if one uses in \((30)\) coherent states with transverse excitations \([PU1, Pa]\). For simplicity of exposition, we restrict our consideration only to the quasimodes without transverse excitations, whose leading order is determined by eq. \((30)\).

To construct quasimodes with discrepancies of higher order in \(\bar{h}\) one has to consider the time evolution of coherent states of a more general type. This leads to transport equations whose solvability pose additional conditions on the quasienergies, see \([Sc]\). From the results of Cardoso and Popov \([CP]\) the possibility to construct quasimodes \((\tilde{\mathcal{E}}_n, \tilde{\varphi}_n)\) in billiards having discrepancy \(\delta_\gamma = O(h^\infty)\) is known. Let \((s,y)\) be a coordinate system in a neighborhood of \(\gamma\) such that \(s\) is a coordinate along the trajectory and \(y\) is a coordinate in the orthogonal direction. Using these coordinates the leading order of \((\tilde{\mathcal{E}}_n, \tilde{\varphi}_n)\) can be written as follows \([Ba, Sc]\):

\[
\tilde{\mathcal{E}}_n = \mathcal{E}_n + O(h^2), \quad \tilde{\varphi}_n(x) = e^{iv(x)/\bar{h}}u(x) + O(\bar{h}), \quad (32)
\]

where

\[
u(s, y) = v_0(s)y^2 + O(y^3), \quad u(s, y) = u_0(s) + O(y^2)\]

and the parameters \(v_0(s), u_0(s)\) are determined by Conditions 1, 2:

\[
\Phi_{\Gamma(\mathcal{E}_n)}^\sigma(x) = e^{iv_0(s)y^2/\bar{h}}u_0(s), \quad x = (s, y). \quad (33)
\]

As has been explained before, in the absence of systematic degeneracies in the billiard spectrum one can expect that, in general, a state \(\tilde{\varphi}_n\) approximates an individual eigenstate of the billiard \(\Omega\). In what follows we will denote by \(\tilde{\mathcal{S}}_\gamma\) the set of quasimodes for which \(\tilde{\varphi}_n\) approximates some eigenstate \(\varphi_n\) (rather than a linear combination of \(\varphi_n\)’s) and by \(\mathcal{S}_\gamma\) the set of true solutions of eqs. \([1, 2]\) corresponding to \(\tilde{\mathcal{S}}_\gamma\). Then by eq. \((29)\) for each \((\tilde{\varphi}_i, \tilde{E}_i) \in \tilde{\mathcal{S}}_\gamma\) and \((\varphi_i, E_i) \in \mathcal{S}_\gamma\) we have

\[
C_i^1 = ||\tilde{\varphi}_i - \varphi_i|| = O(h^\infty), \quad |\tilde{E}_i - E_i| = O(h^\infty). \quad (34)
\]

### 4.2 A lower bound for the approximation of eigenstates

The quasimode construction described in the previous section is quite general and can be applied to an arbitrary elliptic periodic trajectory. In the present section we will consider eigenstates of the billiard \(\Omega\) from the subset \(\mathcal{S}_\gamma\), where \(\gamma\) is an elliptic SPT. We show that for \((\varphi_n, E_n) \in \mathcal{S}_\gamma\) and any regular solution \(\psi \in \mathcal{M}(E_n)\) of eq. \((1)\) in \(\mathbb{R}^2\) the norm

\[
\eta_n(\psi) = ||\varphi_n - \psi|| \quad (35)
\]
is bounded from below by

$$\eta_n(\psi) \geq C_\gamma + C_n^1 + O(h^{1/2}),$$  \hspace{1cm} (36)$$

where $C_\gamma$ is a positive constant determined only by geometrical parameters of the periodic orbit. Since $C_n^1 = O(h^\infty)$, this implies the inequality [14] holds for any $(\varphi_n, E_n) \in \mathcal{S}_\gamma$.

Let $\gamma$ be an elliptic SPT and let $\gamma_1$, $\bar{\gamma}_1$ be as defined in Sec. 2, see fig. 3. Now fix the parameter $\varepsilon$ to be sufficiently small such that $\gamma_1^\varepsilon \equiv \gamma_1 \cap \Omega_\varepsilon \neq \emptyset$, $\bar{\gamma}_1^\varepsilon \equiv \bar{\gamma}_1 \cap \Omega_\varepsilon \neq \emptyset$. We will denote by the capital letters $\Gamma_1(E)$, $\bar{\Gamma}_1(E)$ (resp. $\Gamma_1^\varepsilon(E)$, $\bar{\Gamma}_1^\varepsilon(E)$) the corresponding “lifts” of $\gamma_1$, $\bar{\gamma}_1$ (resp. $\gamma_1^\varepsilon$, $\bar{\gamma}_1^\varepsilon$) into the phase space $V$ at the energy shell $E$. Recall that the main idea behind the quasimode construction (30) is to use coherent states propagating along a periodic orbit. By analogy, one can construct states localized on $\gamma_1$ and $\bar{\gamma}_1$. Let $z(0) = z \in \Gamma_1(E)$. Consider the classical evolution (both for positive and negative time) of $z$ under the free flow $\Psi_0^\sigma : z \rightarrow z(t) = (q(t), p(t))$ in $\mathbb{R}^2$. Obviously, as time evolves, the point $q(t)$ successively crosses the boundary of $\Omega_\varepsilon$ at the sequence of points $q_1, q_2, q_1, q_2$, see fig. 3. We will denote by $t_1, t_2, \tilde{t}_1, \tilde{t}_2$ the corresponding time moments: $q_1 = q(t_1), q_2 = q(t_2), \tilde{q}_1 = q(\tilde{t}_1), \tilde{q}_2 = q(\tilde{t}_2)$. Then the states localized along $\gamma_1$ and $\bar{\gamma}_1$ are given by

$$|\Phi_{\Gamma_1(E)}^\sigma\rangle = \int_{t_2}^{t_1} e^{i t(E-H_0)/\hbar} |\phi_{z_1}^\sigma\rangle \, dt, \hspace{1cm} (37)$$

$$|\Phi_{\bar{\Gamma}_1(E)}^\sigma\rangle = \int_{\tilde{t}_2}^{\tilde{t}_1} e^{i t(E-H_0)/\hbar} |\phi_{\tilde{z}_2}^\sigma\rangle \, dt. \hspace{1cm} (38)$$

Note, that under the free evolution $e^{-i t H_0/\hbar}$ the support of $\phi_z^\sigma$ is not preserved inside $\Omega$, and therefore the supports of $\Phi_{\Gamma_1}^\sigma$, $\Phi_{\bar{\Gamma}_1}^\sigma$ do not belong to the billiard domain. However, one can slightly modify the definition of the states $\Phi_{\Gamma_1}^\sigma, \Phi_{\bar{\Gamma}_1}^\sigma$ to make them admissible as billiard states in $\Omega$. Let $z = z_1$, $\sigma = \sigma_1$ be as before and set $\tau$ be such that under the classical evolution $\Psi_0^\sigma : z_1 \rightarrow z(\tau)$ the point $z(\tau) = z_2$ belongs to $\Gamma_1$. Set $\phi_{z_2}^\sigma(x) = e^{-i t H_0/\hbar} \phi_{z_1}^\sigma(x) + O(\hbar^\infty)$ be the coherent state in $\Omega$, whose parameters are given by: $(\sigma_2, z_2) = (D \Psi_0^\tau \cdot \sigma_1, \Psi_0^\tau \cdot z_1)$. Then the states

$$|\Phi_{\Gamma_1(E)}^\sigma\rangle = \int_{t_2}^{t_1} e^{i t(E-H_0)/\hbar} |\phi_{z_1}^\sigma\rangle \, dt, \hspace{1cm} (39)$$

$$|\Phi_{\bar{\Gamma}_1(E)}^\sigma\rangle = \int_{\tilde{t}_2-\tau}^{\tilde{t}_1-\tau} e^{i t(E-H_0)/\hbar} |\phi_{z_2}^\sigma\rangle \, dt. \hspace{1cm} (40)$$

have their supports in $\Omega$ and satisfy

$$|\Phi_{\Gamma_1(E)}^\sigma\rangle = |\Phi_{\Gamma_1(E)}^\sigma\rangle + O(\hbar^\infty), \hspace{1cm} |\Phi_{\bar{\Gamma}_1(E)}^\sigma\rangle = |\Phi_{\Gamma_1(E)}^\sigma\rangle + O(\hbar^\infty). \hspace{1cm} (41)$$

To get the lower bound (30) we are going first to construct a state $\Phi$ with the property

$$\langle \psi | \Phi \rangle = 0 + O(\hbar^\infty), \hspace{1cm} (42)$$
Then it follows immediately from eqs. 43, 45 that the state \( \Phi = \Phi_\sigma \) for any \( \psi \omega \). Furthermore, let us introduce the states corresponding quasimode, whose leading order parameters determined by Conditions 1, 2, see eqs. 32, 33. Now fix the energy parameters in an arbitrary real positive number. We will make use of the state \( \eta \) in order to get a lower bound on \( \Phi \). Using the triangle inequality where
\[
\langle \psi | \Phi_1^\sigma(E) \rangle_{R^2} = \int_{t_2}^{t_1} e^{i(t(E-E')/\hbar} \langle \psi | \phi_z^\sigma \rangle dt = C_1(\omega) \langle \psi | \phi_z^\sigma \rangle,
\]
we have
\[
\langle \psi | \Phi_1^\sigma(E) \rangle_{R^2} = C_2(\omega) \langle \psi | \phi_z^\sigma \rangle,
\]
with
\[
C_2(\omega) = \exp \left( \frac{i(\bar{t}_1 + \bar{t}_2)\omega}{2} \frac{2\sin(\omega T_{\gamma_1}/2)}{\omega} \right), \quad T_{\gamma_1} = |\bar{t}_1 - \bar{t}_2|.
\]
Furthermore, let us introduce the states
\[
|\Phi_1^\sigma(E, E')\rangle = \frac{1}{C_1(\omega)} |\Phi_1^\sigma(E)\rangle, \quad |\Phi_2^\sigma(E, E')\rangle = \frac{1}{C_2(\omega)} |\Phi_2^\sigma(E)\rangle.
\]
Then it follows immediately from eqs. 43-45 that the state \( \Phi = \Phi^\sigma(E, E') \)
\[
|\Phi^\sigma(E, E')\rangle = |\Phi_1^\sigma(E, E')\rangle - |\Phi_2^\sigma(E, E')\rangle
\]
satisfies orthogonality condition 12.

Let \( (\varphi, E_n) \in \mathcal{S}_\gamma \) be a solution of eqs. 12 and let \( (\tilde{\varphi}, \tilde{E}_n) \in \tilde{\mathcal{S}}_\gamma \) be the corresponding quasimode, whose leading order parameters \( E_n, \sigma_n = (\sigma_n^0, \sigma_n^1) \) are determined by Conditions 1, 2, see eqs. 32, 33. Now fix the energy parameters in eq. 48 by \( E = E_n, E' = E_n \) and put \( \sigma = \bar{\sigma}_n \), where \( \bar{\sigma}_n = (i\beta, \sigma_n^1) \) and \( \beta \) is an arbitrary real positive number. We will make use of the state
\[
|\Phi_n\rangle = |\Phi^\sigma_n(E_n, E_n)\rangle
\]
in order to get a lower bound on \( \eta_n \). For any \( \psi \in \mathcal{M}(E_n) \) we have
\[
||\tilde{\varphi}_n - \psi|| ||\Phi_n|| \geq |\langle \tilde{\varphi}_n - \psi |\Phi_n\rangle| = |\langle \tilde{\varphi}_n |\Phi_n\rangle| + O(h^\infty).
\]
Using the triangle inequality
\[
||\tilde{\varphi}_n - \varphi_n|| + ||\varphi_n - \psi|| \geq ||\tilde{\varphi}_n - \varphi_n + \varphi_n - \psi|| = ||\tilde{\varphi}_n - \psi||
\]
one gets immediately from 19
\[
\eta_n(\psi) = ||\varphi_n - \psi|| \geq \frac{|\langle \tilde{\varphi}_n |\Phi_n\rangle|}{||\Phi_n||} - C_n^1 + O(h^\infty).
\]
It remains to estimate the scalar product $|\langle \tilde{\varphi}_n | \Phi_n \rangle|$ and the norm of the state $\Phi_n$. First, consider the norm $||\Phi_n||$. Since $\gamma_1 \cap \bar{\gamma}_1 = \emptyset$ one has from the definition of $\Phi_n$

$$
\langle \Phi_n | \Phi_n \rangle = \frac{1}{|C_1(\omega_n)|^2} \langle \Phi_1^{\sigma_n} | \Phi_1^{\sigma_n} \rangle + \frac{1}{|C_1(\omega_n)|^2} \langle \Phi_1^{\sigma_n} | \Phi_1^{\sigma_n} \rangle + O(h^\infty),
$$

(52)

with $\omega_n = (E_n - \mathcal{E}_n)/\hbar$. The calculations of the scalar products performed in Appendix give:

$$
\langle \Phi_{1,\gamma}^{\sigma_n} | \Phi_{1,\gamma}^{\sigma_n} \rangle = T_{\gamma_1} \left( \frac{2\pi\hbar}{\beta E_n} \right)^{1/2} + O(h),
$$

(53)

$$
\langle \Phi_{1,\gamma}^{\sigma_n} | \Phi_{1,\gamma}^{\sigma_n} \rangle = T_{\gamma_1} \left( \frac{2\pi\hbar}{\beta E_n} \right)^{1/2} + O(h)
$$

and for the leading order of $C_1(\omega_n)$, $C_2(\omega_n)$ one has from eqs. (53) and (54)

$$
|C_2(\omega_n)| = T_{\gamma_1} + O(h), \quad |C_1(\omega_n)| = T_{\gamma_1} + O(h).
$$

(54)

Combining (53) and (54) together one finally gets

$$
\langle \Phi_n | \Phi_n \rangle = \left( \frac{2\pi\hbar}{\beta E_n} \right)^{1/2} \left( \frac{1}{T_{\gamma_1}} + \frac{1}{T_{\gamma_1}} \right) + O(h).
$$

(55)

In the same way for the scalar product $\langle \tilde{\varphi}_n | \Phi_n \rangle$ we have by (52)

$$
|\langle \tilde{\varphi}_n | \Phi_n \rangle| = \left| \langle \Phi_{1,\gamma}^{\sigma_n} | \Phi_{1,\gamma}^{\sigma_n} \rangle \right| + O(h) = \frac{T_{\gamma_1}}{T_{\gamma}} \left| \langle \Phi_{1,\gamma}^{\sigma_n} | \Phi_{1,\gamma}^{\sigma_n} \rangle \right|^{1/2} \left| \langle \Phi_{1,\gamma}^{\sigma_n} | \Phi_{1,\gamma}^{\sigma_n} \rangle \right|^{1/2} + O(h)
$$

$$
= \frac{1}{T_{\gamma}} \left( \frac{2\pi\hbar}{\beta E_n} \right)^{1/2} + O(h).
$$

(56)

The estimation (56) follows now immediately after inserting eqs. (55) and (56) into (51). The resulting constant $C_\gamma$, which determines the lower bound on $\eta_n$ in the semiclassical limit reads as

$$
C_\gamma = \sqrt{\frac{T_{\gamma_1} T_{\gamma_1}}{(T_{\gamma_1} + T_{\gamma_1}) T_{\gamma}}} = \sqrt{\frac{\ell_{\gamma_1} \ell_{\gamma_1}}{(\ell_{\gamma_1} + \ell_{\gamma_1}) \ell_{\gamma}}} + O(\varepsilon),
$$

(57)

where $\ell_{\gamma_1}, \ell_{\gamma_1}, \ell_{\gamma}$ are the lengths of $\gamma_1, \gamma_1$ and $\gamma$ respectively.

5 PW approximation for eigenstates of non-convex billiards (hyperbolic case)

In the present section we consider the case of a hyperbolic SPT $\gamma$. Let as before $\{\varphi_n(x)\}$ be the set of eigenfunctions in $\Omega$ approximated by regular solutions $\{\psi_n(x)\}$
of eq. 1. For an arbitrary set of \( \psi_n(x) \in \mathcal{M}(E_n) \), \( n = 1, 2, ... \infty \) we will estimate the average of

\[
\eta_n \equiv \eta_n(\psi_n) = ||\varphi_n - \psi_n||
\]  

over an energy interval. Our objective is to show that independently of the choice of \( \psi_n \)’s, in the limit \( \hbar \to 0 \) the average \( \langle \eta_n \rangle \) is bounded from below by a strictly positive constant.

Let \( \Phi_1'(E, E'), \Phi_2'(E, E'), \Phi_\sigma (E, E') \) be as in the previous section with the parameter \( \sigma \) of the form \( \sigma = (i\beta, \sigma^1) \), \( \beta > 0 \). For each integer \( n \) we will consider the states

\[
|\Phi_{n,1}\rangle = |\Phi_1'(E, E_n)\rangle, \quad |\Phi_{n,2}\rangle = |\Phi_2'(E, E_n)\rangle
\]

and their difference

\[
|\tilde{\Phi}_n\rangle = |\Phi_{n,1}\rangle - |\Phi_{n,2}\rangle = |\Phi_\sigma (E, E_n)\rangle.
\]

which is orthogonal to any \( \psi \in \mathcal{M}(E_n) \) up to the term \( O(\hbar^\infty) \) (see eq. 42). In addition, it will be also useful to introduce the state

\[
|\tilde{\Phi}'_n\rangle = |\Phi_{n,1}\rangle + |\Phi_{n,2}\rangle.
\]

Note that \( \tilde{\Phi}'_n \) is orthogonal to \( \tilde{\Phi}_n \) in the semiclassical limit.

Similarly to the case of elliptic SPT’s, one can make use of the state \( \tilde{\Phi}_n \) to get a lower bound on \( \eta_n \):

\[
\eta_n \geq \frac{|\langle \tilde{\Phi}_n | \varphi_n - \psi_n \rangle|}{||\Phi_n||} = \frac{|\langle \tilde{\Phi}_n | \varphi_n \rangle|}{||\Phi_n||} + O(\hbar^\infty). \quad (62)
\]

In order to estimate the right side of this inequality let us consider the following difference

\[
D_n = |\langle \Phi_{n,1} | \varphi_n \rangle|^2 - |\langle \Phi_{n,2} | \varphi_n \rangle|^2. \quad (63)
\]

Using the states \( \tilde{\Phi}_n, \tilde{\Phi}'_n \) one can rewrite \( D_n \) as

\[
D_n = \text{Re} \left( \langle \tilde{\Phi}'_n | \varphi_n \rangle \langle \tilde{\Phi}_n | \varphi_n \rangle^* \right). \quad (64)
\]

Hence, the following inequality follows immediately

\[
|D_n| \leq |\langle \tilde{\Phi}_n | \varphi_n \rangle| \leq |\langle \tilde{\Phi}'_n | \varphi_n \rangle| \leq ||\tilde{\Phi}'_n|| \langle \tilde{\Phi}_n | \varphi_n \rangle. \quad (65)
\]

Finally, since \( ||\tilde{\Phi}_n|| - ||\tilde{\Phi}'_n|| = O(\hbar^\infty) \), we get by (62) and (65)

\[
\eta_n \geq \frac{|D_n|}{||\tilde{\Phi}_n|| \langle \tilde{\Phi}_n | \varphi_n \rangle} + O(\hbar^\infty) = \frac{|\langle \Phi_{n,1} | \varphi_n \rangle|^2 - |\langle \Phi_{n,2} | \varphi_n \rangle|^2}{\langle \tilde{\Phi}_n | \varphi_n \rangle} + O(\hbar^\infty). \quad (66)
\]

We will now use this inequality to get a lower bound for the sum of \( \eta_n \) over the energy interval \( \mathcal{P}_c = [E - c\hbar, E + c\hbar] \), where \( c \) is a positive constant. One has straightforwardly from (66)
Furthermore, the definition of the states $\Phi_{n,1}$, $\Phi_{n,2}$ implies

$$
|\langle \Phi_{n,1}|\varphi_n\rangle|^2 = |\langle \phi_{z_1}^{\sigma_1}|\varphi_n\rangle|^2, \ z_1 \in \Gamma_1^\circ; \ |\langle \Phi_{n,2}|\varphi_n\rangle|^2 = |\langle \phi_{z_2}^{\sigma_2}|\varphi_n\rangle|^2, \ z_2 \in \Gamma_1^\circ,
$$

(68)

where $(z_1, \sigma_1) = (z, \sigma)$ and $(z_2, \sigma_2) = (z(\tau), \sigma(\tau))$ are related by the free classical evolution as in the previous section. As a result, the inequality (67) reads as

$$
\sum_{E_n \in \mathcal{P}_{ch}} \eta_n > \sum_{E_n \in \mathcal{P}_{ch}} \left| \sum_{E_n \in \mathcal{P}_{ch}} \frac{|\langle \Phi_{n,1}|\varphi_n\rangle|^2}{\langle \Phi_n|\Phi_n\rangle} - \sum_{E_n \in \mathcal{P}_{ch}} \frac{|\langle \Phi_{n,2}|\varphi_n\rangle|^2}{\langle \Phi_n|\Phi_n\rangle} \right| + O(h^\infty).
$$

(67)

with $\omega_n = (E - E_n)/\hbar$ and

$$
f(\omega_n) = \begin{cases} 
1/|\langle \Phi_n|\Phi_n\rangle| & \text{if } \omega_n \in [-c, c] \\
0 & \text{otherwise}.
\end{cases}
$$

The elementary calculations (see Appendix) provide the leading order of the function $f(\omega_n)$, $\omega_n \in [-c, c]$:}

$$
f(\omega_n) = \frac{1}{\langle \Phi_{n,1}|\Phi_{n,1}\rangle + \langle \Phi_{n,2}|\Phi_{n,2}\rangle} + O(h^\infty)
= \frac{2|p|}{(\pi\hbar\beta)^{1/2}} \left( \frac{\omega_n^2 T_{\gamma_1}}{\sin^2(\omega_n T_{\gamma_1}/2)} + \frac{\omega_n^2 T_{\gamma_1}}{\sin^2(\omega_n T_{\gamma_1}/2)} \right)^{-1} + O(h^0).
$$

(70)

Now we can apply to (67) the results of Paul and Uribe (see Sec. 3). Taking into account that $z_1 \in \Gamma$ while $z_2$ does not belong to any periodic trajectory, we get by eqs. 24 and 25 the following estimation for the average of $\eta_n$:

$$
\langle \eta_n \rangle \equiv \frac{1}{\# \mathcal{P}_{ch}} \sum_{E_n \in \mathcal{P}_{ch}} \eta_n > \frac{1}{\# \mathcal{P}_{ch}} \left| \sum_{l \neq 0} \bar{F}(lT_{\gamma}) \frac{e^{il(S_{\gamma}/h + \mu_{\gamma})}}{\cosh^{1/2}(l\lambda_{\gamma})} \right| + O(h^{3/2}),
$$

(71)

where $\bar{F}(\cdot)$ is the Fourier transform of the function

$$
F(x) = \begin{cases} 
\left( \frac{8}{\pi} \right)^{1/2} \frac{\left( \frac{x^2 T_{\gamma_1}}{\sin^2(x T_{\gamma_1}/2)} + \frac{x^2 T_{\gamma_1}}{\sin^2(x T_{\gamma_1}/2)} \right)^{-1}}{\sin^2(x T_{\gamma_1}/2)} & \text{if } x \in [-c, c] \\
0 & \text{otherwise}
\end{cases}
$$

and $\# \mathcal{P}_{ch}$ is the number of eigenstates in the interval $\mathcal{P}_{ch}$ whose leading order for a billiard of area $A$ is given by the Weyl formula:

$$
\# \mathcal{P}_{ch} = Ac/2\pi\hbar + O(h^0).
$$

Consequently, if

$$
Y = \left| \sum_{l \neq 0} \bar{F}(lT_{\gamma}) \frac{e^{il(S_{\gamma}/h + \mu_{\gamma})}}{\cosh^{1/2}(l\lambda_{\gamma})} \right| \neq 0
$$

(72)
one has from (71)

\[ \langle \eta_n \rangle > Bh + O(h^{3/2}), \quad B = 2\pi Y/cA > 0. \] (73)

If moreover one assumes that \( T_{\gamma_1}c, T_{\gamma_1}c << 1 \), the function \( F(x) \) takes a simple form:

\[ F(x) \approx \begin{cases} \left( \frac{1}{2\pi} \right)^{1/2} \frac{T_{\gamma_1}T_{\bar{\gamma}_1}}{T_{\gamma_1} + T_{\bar{\gamma}_1}} & \text{if } x \in [-c, c] \\ 0 & \text{otherwise} \end{cases} \]

and the constant \( B \) can be written explicitly:

\[ B \approx \frac{\sqrt{2\pi}}{A} \left( \frac{T_{\gamma_1}T_{\bar{\gamma}_1}}{T_{\gamma_1} + T_{\bar{\gamma}_1}} \right) \left| \sum_{l \neq 0} \frac{\sin(lcT_{\gamma})}{lcT_{\gamma}} \frac{e^{i[l(S_{\gamma}/h + \mu_{\gamma})]}}{\cosh^{1/2}(l\lambda_{\gamma})} \right|. \] (74)

Note, that the lower bound (73) has been obtained using only one SPT. In the case of hyperbolic dynamics, however, the periodic orbits (and, in particular, SPT’s) proliferate exponentially. Therefore, one can improve the estimation (73) making use of a state \( \Phi_{\text{sum}}^n \) which is concentrated on a set of SPT’s \( \{ \gamma \} \) and satisfies eq. 42. A simple way to construct such a state is to define it as the superposition:

\[ \Phi_{\text{sum}}^n = \sum_{\gamma} \Phi_n(\gamma), \] (75)

where \( \Phi_n(\gamma) \) stands for the state (60) associated with a SPT \( \gamma \).

Finally, let us mention that the statistical estimation (73) can be straightforwardly generalized to the case of elliptic SPT’s. In that case one should use the analogs of eqs. 24, 25 (which are known to exist [PU1]) for stable periodic trajectories.

### 6 Discussion and conclusions

Speaking informally, Proposition 2 implies that there is no on-shell basis of regular solutions of the Helmholtz equation which can be used to approximate all eigenstates of a generic non-convex billiard. That means any linear combination of plane waves, radial waves etc., with the same energy fails to approximate real eigenstates of non-convex billiards. In fact, a stronger result can be shown. Let \( \Omega \) be a generic non-convex billiard and let \( \Omega' \) be a domain (not necessarily convex) which properly contains \( \Omega : \Omega' \supset \Omega, \partial \Omega' \cap \partial \Omega = \emptyset \). Denote by \( M_{\Omega'}(E) \) the set of all solutions of eq. 1 regular in \( \Omega' \) (note, that \( M_{\Omega'}(E) \supseteq M(E) \)). Let us argue that the eigenstates of \( \Omega \) cannot be approximated, in general, by states belonging to \( M_{\Omega'}(E) \). Let \( \gamma \) be a SPT and let \( l, \gamma_1, \bar{\gamma}_1 \) be as defined before. Furthermore, assume that the segment of the line \( l \) between \( \gamma_1 \) and \( \bar{\gamma}_1 \) is entirely in \( \Omega' \), see fig. 4. (It seems to be a natural assumption that in a generic case one can always find such a
SPT, provided \( \Omega' \) properly contains \( \Omega \). Then take \( \Omega_0 \subset \Omega \) to be a convex domain satisfying: \( \Omega_0 \cap \gamma_1 \neq \emptyset, \Omega_0 \cap \tilde{\gamma}_1 \neq \emptyset \). Now, suppose an eigenstate \( \varphi_n \) of \( \Omega \) can be approximated by states \( \psi'(x) \) from \( \mathcal{M}_{\Omega'}(E_n) \). According to Proposition 1 \( \psi'(x) \) can be approximated in \( \Omega_0 \) by regular solutions of eq. \( \mathbb{I} \) and thus for any \( \epsilon > 0 \) there exists \( \psi_\epsilon \in \mathcal{M}(E_n) \) such that \( \| \varphi_n(x) - \psi_\epsilon(x) \|_{L^2(\Omega_0)} < \epsilon \). Therefore, applying the same arguments as in Sec. 2 we get

\[
H\varphi_n(z_1) - H\varphi_n(z_2) = \lim_{\epsilon \rightarrow 0} |\langle z_1 | \psi_\epsilon \rangle|^2 - \lim_{\epsilon \rightarrow 0} |\langle z_2 | \psi_\epsilon \rangle|^2 = O(h\infty),
\]

where \( z_1 = (q_1, p) \in \Gamma_1(E_n), q_1 \in \gamma_1 \cap \Omega_0 \) and \( z_2 = (q_2, p) \in \tilde{\Gamma}_1(E_n), q_2 \in \tilde{\gamma}_1 \cap \Omega_0 \). However, as has been pointed out before, this cannot be true for each \( n \) since \( z_2 \notin \Gamma \).

The two properties of generic non-convex billiards follows immediately from the above analysis. First, it is not possible to approximate eigenstates of a generic non-convex billiard \( \Omega \) also if one includes in the basis \( \{ \psi^{(n)}(k) \} \) singular solutions of eq. \( \mathbb{I} \), e.g., the Hankel functions

\[
\{ H_n^\pm(k|x - x_i|)e^{i\theta(x,x_i)}, n \in \mathbb{N} \},
\]

with a finite number of singularity points \( x_i \). Second, there exists an infinite sequence of eigenstates which do not admit extension into any large domain \( \Omega' \) properly containing \( \Omega \). That means the continuation of the interior eigenstates of a generic non-convex billiard into the exterior domain should be (in general) impossible because of singularities which occur arbitrary close to the billiard’s boundary. It remains as an open problem what is the exact nature of such singularities. (For example, whether one can, in principal, extend eigenstates beyond the boundary of a generic non-convex billiard.) It should be also mentioned that the problem of the eigenstates extension in convex billiards is beyond the scope of the present paper. It would become a natural question to inquire about the relation between the billiard shape and the type of singularities arising for the extended eigenstates. In particular, it would be interesting to know whether the strong form of spectral duality (when it is possible to extend eigenfunctions in \( \mathbb{R}^2 \) as regular solutions of the Helmholtz equation) holds exclusively for integrable billiards.

Further, let us stress an important difference between the cases of elliptic and hyperbolic dynamics. The counting function \( \mathcal{N}^*(k) = \# \{ k_n < k \} \) for quasimodes \( (\tilde{\varphi}_n, \tilde{k}_n) \) which can be constructed on an elliptic periodic trajectory is known to be of the same asymptotic form \( \mathcal{N}^*(k) = \alpha k^2 + O(k) \), \( \alpha > 0 \) as the counting function \( \mathcal{N} = Ak^2/4\pi + O(k) \) for the real spectrum \( \{ k_n \} \), see \([\text{CP}]\). Therefore, in a generic case, if an elliptic SPT \( \gamma \) exists the subsequence \( \{ \varphi_{j_n}, n \in \mathbb{N} \} \) of billiard eigenstates approximated by the quasimodes concentrated on \( \gamma \) should be of the positive density:

\[
\lim_{N \rightarrow \infty} \frac{1}{N} \# \{ j_n | j_n \leq N \} = \lim_{k \rightarrow \infty} \frac{\mathcal{N}^*(k)}{\mathcal{N}(k)} > 0.
\]

Since for each \( \varphi_{j_n} \) the estimation \( (14) \) holds, that means there exists a subsequence of eigenstates with a positive density which do not admit approximation by plane
waves. In the case of hyperbolic dynamics the statistical lower bound (13) implies, in fact, only a weaker result. It says that an infinite sequence (possibly of zero density) of such states exists. However, if one assumes that all eigenstates of fully chaotic billiards have “uniform properties” the inequality (13) suggests a natural conjecture:

**Conjecture.** For a non-convex billiard with fully chaotic dynamics the set of states which can be approximated by PW is of density zero.

Note, that it is impossible to exclude the possibility of existence of “exceptional” eigenstates (the eigenstates which can be approximated by PW) in non-convex billiards. Indeed, one can take a finite superposition of plane waves $\psi^{[N]}$ and set a (non-convex) nodal domain of $\psi^{[N]}$ to be the billiard’s boundary. Then $\psi^{[N]}$ itself is the eigenstate of this billiard which can be approximated by PW.

Finally, the study of the present paper is restricted to the two-dimensional simply connected domains with Dirichlet boundary conditions. However, it is easy to see that presented results allow several rather straightforward generalizations. First, higher dimensional billiards and different types of boundary conditions can be treated in the same way. Second, billiards in multiply connected domains (fig. 5) have the same properties as non-convex billiards. Consequently, all the results obtained for non-convex billiards hold for multiply connected billiards as well. Third, we conjecture that our results can be generalized to the billiards on non-compact manifolds with non-trivial metrics (also in the presence of a potential) e.g., billiards on the hyperbolic plane. In such a case, one needs to adjust the notion of domain’s “convexity” to the corresponding classical dynamics. In other words, a domain should be defined as “convex” if the interior-exterior duality holds and defined as “non-convex” if it breaks dawn.

**Acknowledgments**

I am grateful to R. Schubert, U. Smilansky, A. Voros and S. Nonnenmacher for useful discussions.

**Appendix**

**Proposition 3.** Let $\Phi_{T}^{\sigma}$, $\Phi_{T}^{\bar{\sigma}}$ be the states:

$$|\Phi_{T}^{\sigma}\rangle = \frac{1}{C_{1}} \int_{0}^{T} e^{i(E-H_{0})t/\hbar} |\phi_{2}^{\sigma}\rangle dt, \quad \sigma = (\sigma^{0}, \sigma^{1})$$

$$|\Phi_{T}^{\bar{\sigma}}\rangle = \frac{1}{C_{2}} \int_{0}^{T} e^{i(E-H_{0})t/\hbar} |\phi_{2}^{\bar{\sigma}}\rangle dt, \quad \bar{\sigma} = (\bar{\sigma}^{0}, \bar{\sigma}^{1})$$

(76)
localized along the path $\Gamma = \Gamma(E)$, $\Gamma(E) = \{\Psi^t \cdot z = (q(t), p(t)), \ t \in [0, T], E = p^2/2\}$ with $\sigma^0 = i\beta_1, \ \bar{\sigma}^0 = i\beta_2; \ \beta_1, \beta_2 > 0$ and $\sigma^1 = \bar{\sigma}^1$. Then

$$\langle \Phi_\Gamma^\sigma | \Phi_\Gamma^\bar{\sigma} \rangle = \frac{T}{C_1^2} \left( \frac{2\pi \hbar}{\beta_1 E} \right)^{1/2} O(h); \quad \langle \Phi_\Gamma^\sigma | \Phi_\Gamma^\bar{\sigma} \rangle = \frac{T}{C_2^2} \left( \frac{2\pi \hbar}{\beta_2 E} \right)^{1/2} O(h), \quad (77)$$

$$\langle \Phi_\Gamma^\sigma | \Phi_\Gamma^\bar{\sigma} \rangle = \langle \Phi_\Gamma^\sigma | \Phi_\Gamma^\bar{\sigma} \rangle^{1/2} \langle \Phi_\Gamma^\sigma | \Phi_\Gamma^\bar{\sigma} \rangle^{1/2} + O(h). \quad (78)$$

Proof. The inner product

$$\langle \Phi_\Gamma^\sigma | \Phi_\Gamma^\bar{\sigma} \rangle = \frac{1}{C_1^2 C_2^2} \int_0^T \int_0^T \langle \phi_2^\sigma | e^{i(E-H_0)(t_1-t_2)/\hbar} | \phi_2^\bar{\sigma} \rangle dt_1 dt_2 \quad (79)$$

can be written as

$$\langle \Phi_\Gamma^\sigma | \Phi_\Gamma^\bar{\sigma} \rangle = \frac{1}{2C_1 C_2} \left( \int_0^T (T-t) H(t) dt + \int_0^T (T-t) H(-t) dt \right)$$

$$= \frac{1}{2C_1 C_2} \int_{-T}^T (T-|t|) H(t) dt, \quad (80)$$

where

$$H(t) = \langle \phi_2^\sigma | e^{i(E-H_0)t/\hbar} | \phi_2^\bar{\sigma} \rangle. \quad (81)$$

By the propagation formula \[19\] we get for \[81\]

$$H(t) = e^{i(S(t)+E)/h+i\mu(t)} \langle \phi_2^\sigma | \phi_2^{\sigma(t)} \rangle + O(h)$$

$$= \det \left( \frac{4i \text{Im} \sigma \text{Im} \bar{\sigma}^* (t)}{(\sigma - \bar{\sigma}^* (t))^2} \right)^{1/4} \exp \left( -\frac{i t^2}{2 \hbar} \langle p, \bar{\sigma}^* (t) \frac{1}{\sigma - \bar{\sigma}^* (t)} \sigma p \rangle \right) + O(h)$$

$$= \left( \frac{(\beta_2 \beta_1)^{1/4}}{(\beta_2 + \beta_1)^{1/4}} + O(t) \right) \exp \left( -\frac{t^2 p^2 \beta_2 \beta_1}{2 \hbar (\beta_2 + \beta_1)} + O(t^3) \right) + O(h). \quad (82)$$

After inserting this expression into eq. \[80\] and applying the stationary phase approximation to the integral one gets \[77\]. Finally, let us note that eq. \[78\] remains true also when $\beta_1$ or $\beta_2$ equals zero.

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FIGURES.

(a)

(b)

Figure 1:
Figure 2:
