ALMOST LIE NILPOTENT VARIETIES OF ASSOCIATIVE ALGEBRAS

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Abstract We consider associative algebras over a field. An algebra variety is said to be Lie nilpotent if it satisfies a polynomial identity of the kind $[x_1, x_2, \ldots, x_n] = 0$ where $[x_1, x_2] = x_1x_2 - x_2x_1$ and $[x_1, x_2, \ldots, x_n]$ is defined inductively by

$[x_1, x_2, \ldots, x_n] = [[x_1, x_2, \ldots, x_{n-1}], x_n]$.

By Zorn’s Lemma every non-Lie nilpotent variety contains a minimal such variety, called almost Lie nilpotent, as a subvariety. A description of almost Lie nilpotent varieties for algebras over a field of characteristic 0 was made up by Yu. Mal’cev. We find a list of non-prime almost Lie nilpotent varieties of algebras over a field of positive characteristic.

Introduction

It is well-known that every associative algebra may be considered as a Lie algebra under the Lie multiplication $[x, y] = xy - yx$. Structures of these algebras are closely connected, and properties of one of them provide corresponding properties of other. One of the most natural and intensively studied connections is an influence of Lie algebra nilpotency on the associative structure. The property means a fulfilment of the identity $\ldots [[[x_1, x_2], \ldots, x_n] = 0$, and the associative algebras (or varieties) satisfying the identity are called Lie nilpotent. Lie nilpotent algebras possess many wonderful properties. For example, finitely generated such algebra is finitely presented, residually finite, right and left Noetherian, representable by endomorphisms, etc. (cf. [2], [4]).

Unfortunately, establishing Lie nilpotency by means of explicit deriving of the required identity is, as a rule, a non-trivial and computationally laborious process. To facilitate it one can use a list of minimal w.r.t. inclusion elements in the set of all non-Lie nilpotent varieties. We shall call such minimal varieties almost L.N.. By Zorn’s Lemma every non-Lie nilpotent variety contains as a subvariety at least one almost L.N. variety. Therefore, to prove Lie nilpotency of a variety it is sufficient to verify that no one such varieties is contained in our variety. This approach was successfully realized and corresponding descriptions were found out for many properties such as commutativity [4], [5], Engel condition [1], locally residual finitness, locally weak Noetherian condition [2] and for many others.

A description of almost L.N. varieties of algebras over a field of characteristic zero is obtained in [3]. Note that the found there varieties have an elementary structure, due to which checking for being Lie nilpotent in this case is quite simple. The aim of the present paper is to make more clear a situation with almost L.N. varieties for algebras over a field of positive characteristic.

Henceforth, all algebras are assumed associative.
We adopt the following notation.
Let $F$ be a field. We denote by $F\langle X \rangle$ the free $F$-algebra generated by the countable set $X$. As usual, the elements of $F\langle X \rangle$ are called polynomials. An ideal $I$ of $F\langle X \rangle$ is called a $T$-ideal if it is closed under endomorphisms. Let $A$ be an algebra, $\Sigma$ a set of polynomials, $\mathcal{V}$ a variety. We denote by $\text{var}A$ the variety generated by $A$, and by $\text{var}\Sigma$ the variety defined by $\Sigma$. An identity ideal of a variety $\mathcal{V}$ (or an algebra $A$) is the set of all polynomials $f(x)$ such that $f(x) = 0$ is an identity of $\mathcal{V}$ (or $A$). We denote the identity ideal by $T(\mathcal{V})$ (or $T(A)$, resp.); and we write $T(\Sigma)$ in place of $T(\text{var}\Sigma)$.

We denote by $\mathcal{V}^*$ the variety dual to $\mathcal{V}$; in the case of algebras $A^*$ stands for the algebra anti-isomorphic to $A$. By $\bar{x}$ we denote a tuple of variables $x_1, x_2, \ldots$. It will always be clear from the context whether such is assumed ordered or not.

Let us introduce the notation needed for algebras. Denote by $C$ the algebra over a field generated by $c_1, c_2, \ldots$ with relations
\[
c_i c_j = c_j c_i, \quad c_i^2 = 0, \quad i, j = 1, 2, \ldots
\]
If $p > 0$ is the base field characteristic, then $C$ generates the variety $\text{var}\{[x, y] = 0, x^p = 0\}$.

Letting $U$ be an arbitrary algebra, we put
\[
A(U) \cong \begin{pmatrix} U & U \\ 0 & 0 \end{pmatrix}.
\]

Recall that an algebra variety is said to be prime (or verbally prime) if for its $T$-ideal $T$ every inclusion $I_1 \cdot I_2 \subseteq T$ holding for two $T$-ideals $I_1$ and $I_2$ implies either $I_1 \subseteq T$ or $I_2 \subseteq T$.

**Theorem 1** Let $F$ be an infinite field of a positive characteristic, and let $\mathcal{V}$ be a non-prime $F$-algebra variety. Then $\mathcal{V}$ is almost L.N. if and only if $\mathcal{V}$ is generated either by $A(C)$ or by $A(C)^*$. For a finite field $F$ we also introduce the following series of algebras:

\[
B(F, G, \sigma) \cong \begin{pmatrix} b & c \\ 0 & \sigma(b) \end{pmatrix}
\]

where $b, c$ run through a finite extension $G$ of $F$, and $\sigma$ is an automorphism of $G$ such that the invariant field $G^\sigma$ is an unique maximal subfield of $G$ containing $F$.

**Theorem 2** Let $F$ be a finite field, and let $\mathcal{V}$ be a non-prime $F$-algebra variety. Then $\mathcal{V}$ is almost L.N. if and only if $\mathcal{V}$ is generated by one of the following algebras $A(F)$, $A(F)^*$, $A(C)$, $A(C)^*$, $B(F, G, \sigma)$.

Although there is no information about prime almost L.N. varieties in the descriptions, we think that sometimes the results may be quite effective. To demonstrate it we apply the descriptions in a partial case (see **Example**).
Almost L.N. varieties

We prove the Theorems at the end of the section collecting some facts.

**Lemma 1** Let $n \geq 3$ be an integer and $W_n(\bar{x}) = 0$ be an identity of a variety $\mathcal{M}$. Then for every $k \leq n - 3$ $\mathcal{M}$ satisfies the identity

$$W_{n-k}(\bar{y}_1) \cdot W_{n-k}(\bar{y}_2) \cdots W_{n-k}(\bar{y}_{2^k}) = 0.$$  

**Proof.** Let us substitute $x_{n-1} \mapsto zx_{n-1}$ into $W_n$. We obtain

$$0 = [W_{n-2}, zx_{n-1}, x_n] + [W_{n-2}, z][x_{n-1}, x_n] + [W_{n-2}, z, x_n]x_{n-1}.$$  

The first and fourth summands belong to $T(W_n)$. Put $z = W_{n-2}(\bar{u})$ and see that because of Jacobi identity the first commutator in the third summand becomes a consequence of $W_n$. Then we have modulo $T(W_n)$

$$[W_{n-2}(\bar{u}), x_n][W_{n-2}, x_{n-1}] = 0,$$

that is $W_{n-1}(\bar{y}_1) \cdot W_{n-1}(\bar{y}_2) = 0$. As far as $W_s u W_m \in T(W_s \cdot W_m)$ for arbitrary $s, m$ we can repeat the above argument for each of $W_{n-1}$ to obtain a product of 4 factors $W_{n-2}$ and so on.

For the rest of this section we will denote by $\mathcal{V}$ an almost L.N. variety of algebras over a field (finite or infinite). (Recall that an almost L.N. variety is non-Lie nilpotent while its proper subvarieties are all Lie nilpotent.)

**Lemma 2** Let $f(\bar{x}) \notin T(\mathcal{V})$. There exists an integer $n > 0$ such that

$$W_n(\bar{y}) \in T(\{f\}) + T(\mathcal{V}).$$

**Proof.** By the condition the ideal $T(\{f\}) + T(\mathcal{V})$ defines a proper and, hence, Lie nilpotent subvariety of $\mathcal{V}$.

**Lemma 3** If $T(\{f\}) \cdot T(\{g\}) + T(\{g\}) \cdot T(\{f\}) \subseteq T(\mathcal{V})$ then either $f \in T(\mathcal{V})$ or $g(\bar{x}) \cdot g(\bar{y}) \in T(\mathcal{V})$.

**Proof.** Arguing by contradiction, suppose that $f \notin T(\mathcal{V})$ and $g(\bar{x}) \cdot g(\bar{y}) \notin T(\mathcal{V})$. By Lemma 2 we have an identity

$$W_n(\bar{x}) = h(\bar{x})$$

for some $h \in T(\{f\})$. Furthermore, we have an identity

$$W_n(\bar{y}) = t(\bar{y}),$$

where $t(\bar{y})$ is a sum of polynomials of the following kind

$$ug(a_1, a_2, \ldots)g(b_1, b_2, \ldots)v$$
and $u, v$ are monomials, $a_i, b_i$ are polynomials depending on $y_1, y_2, \ldots$. We can obtain the following equations modulo $T(V)$:

$$W_{n+s-1} = W_s(W_n(\bar{x}), y_2, \ldots) = W_s(h(\bar{x}), y_2, \ldots) = t(h(\bar{x}), y_2, \ldots)$$

Now we show that $t(h(\bar{x}), y_2, \ldots) \in T(\{f\}) \cdot T(\{g\}) + T(\{g\}) \cdot T(\{f\})$. The substitution $y_1 \mapsto h(\bar{x})$ transforms every summand from $t$ in which $u$ or $v$ contains $y_1$ into a polynomial from $T(\{f\}) \cdot T(\{g\}) + T(\{g\}) \cdot T(\{f\})$. Consider another summands. It is easy to see that $g(a_1, a_2, \ldots) = g(c_1, c_2, \ldots) + d(\bar{y})$ and polynomials $c_1, c_2, \ldots$ do not depend on $y_1$ while every monomial of $d(\bar{y})$ contains $y_1$. Hence, the substitution $y_1 \mapsto h(\bar{x})$ turns all summands of the kind

$$ud(\bar{y})g(b_1, \ldots)v$$

into a polynomial from $T(\{f\}) \cdot T(\{g\})$.

In the other summands $ug(c_1, c_2, \ldots)g(b_1, b_2, \ldots)v$ the variable $y_1$ has to occur in every monomial of $g(b_1, b_2, \ldots)$. Therefore, our substitution turns these summands into polynomials from $T(\{g\}) \cdot T(\{f\})$. Hence, $t(h(\bar{x}), y_2, \ldots) \in T(\{f\}) \cdot T(\{g\}) + T(\{g\}) \cdot T(\{f\})$ and $W_{n+s-1} \in T(V)$. A contradiction.

**Lemma 4** If $V$ is non-prime then it satisfies the identity $[x, y][z, t] = 0$.

**Proof.** As far as $V$ is non-prime, there exist two polynomials $f_1, f_2 \notin T(V)$ such that $T(\{f_1\}) \cdot T(\{f_2\}) \subseteq T(V)$. By Lemma \[2\] $W_n \in T(\{f_1\}) + T(V)$ and $W_s \in T(\{f_2\}) + T(V)$. By Lemma \[1\] we have for some $r$

$$W_3 \cdots W_3 \in (T(\{f_1\}) + T(V))(T(\{f_2\}) + T(V)) \subseteq T(V).$$

Let $r$ be a minimal such number. Put $f = \overline{W_3 \cdots W_3}_{r-1}$ and $g = W_3$. Then, $f \notin T(V)$ and $T(\{f\}) \cdot T(\{g\}) + T(\{g\}) \cdot T(\{f\}) \subseteq T(V)$. By Lemma \[2\] $g(\bar{x}) \cdot g(\bar{y}) \in T(V)$. Therefore, $W_3W_3 = 0$ is an identity of $V$.

Suppose, $[x, y][z, t] \notin T(V)$. By Lemma \[2\] $V$ satisfies for some $m$ an identity

$$W_m(\bar{x}) = g(\bar{x})$$

(1)

where $g$ is a sum of polynomials of the kind $u[x_i, x_j]v[x_i, x_k]w$ and $u, v, w$ are monomials depending on $x_1, \ldots, x_m$. It is easy to see that the substitutions $x_i \mapsto [x_i, y_i] \ (i = 1, \ldots, m)$ turn the right part of \[1\] into a consequence of the polynomial $W_3 \cdot W_3 \in T(V)$. Therefore, the left part $W_m([x_1, y_1], \ldots, [x_m, y_m])$ belongs to $T(V)$ as well. Let $m$ be a minimal number such that

$$W_m([x_1, y_1], \ldots, [x_m, y_m]) \in T(V).$$

We want to prove that $m = 2$. Suppose $m > 2$. By Lemma \[2\] for some $t$ the variety $V$ satisfies an identity

$$W_t(\bar{x}) = h(\bar{x})$$

(2)
where $h$ is a sum of polynomials of the kind
\[ uW_{m-1}([a_1, b_1], \ldots, [a_{m-1}, b_{m-1}])v \]
and $u$, $v$, $a_i$, $b_i$ are monomials depending on $x_1, \ldots, x_m$. It is easy to see that
\[ [h(\tilde{x}), [y, z]] \in T(\{W_m([x_1, y_1], \ldots, [x_m, y_m])\}) + T(\{W_3(\tilde{x})W_3(\tilde{y})\}) \subseteq T(\mathcal{V}). \]

Hence, $[W_3(\tilde{x}), [y, z]] = 0$ is an identity of $\mathcal{V}$. By the assumption, $m > 2$, and we have $t \geq 3$. Substitute $W_i(\tilde{y})$ for $x_1$ in $h(x_1, x_2, \ldots)$. The summands of $h(\tilde{x})$ where $x_1$ occurs in $u$ or $v$ are transformed into polynomials from $T(\{W_3(\tilde{x})W_3(\tilde{y})\})$. Other summands are transformed into sums of polynomials of the kind $\tilde{u}[[c_1, c_2], c_3W_i(\tilde{y})c_4]\tilde{v}$ for some $\tilde{u}, c_1, c_2, c_3, c_4, \tilde{v}$. Clearly,
\[ \tilde{u}[[c_1, c_2], c_3W_i(\tilde{y})c_4]\tilde{v} \in T(\{W_3(\tilde{x})W_3(\tilde{y})\}) + T(\{W_i(\tilde{x}), [y, z]\}). \]

Hence, $h(W_i(\tilde{y}), x_2, \ldots) \in T(\mathcal{V})$. Therefore, replacing $x_1$ by $W_i(\tilde{y})$ in \( (3) \), we obtain an identity of $\mathcal{V}$ on the right and
\[ W_{2t-1} = W_t(W_i(\tilde{y}), x_2, \ldots, x_t) \]
on the left. The contradiction shows that $m = 2$, i.e. $[[x_1, y_1], [x_2, y_2]] = 0$ is an identity of $\mathcal{V}$.

Substituting $y_2 \mapsto y_2z$, we find the consequence
\[ [x_1, y_1, y_2][x_2, z] + [x_2, y_2][x_1, y_1, z] = 0. \]

Substituting $y_2 \mapsto y_2t$ in the identity and using $W_3 \cdot W_3 = 0$, we obtain the consequence $[x_2, y_2][x_1, y_1, z, t] = 0$. By Lemma 3 we conclude that $[x_2, y_2][x_3, y_3] \in T(\mathcal{V})$.

The next property will help us to simplify some identities. We say that a variety $\mathcal{M}$ possesses **Property Z** if $\mathcal{M}$ satisfies the following condition:

*for any polynomials $f$ and $g$, if $f(h(\tilde{i}), \tilde{x}) \in T(\mathcal{M})$ holds for all $h \in T(\{g\})$, then either $g \in T(\mathcal{M})$ or $f([y, z], \tilde{x}) \in T(\mathcal{M})$.***

**Lemma 5** Let $\mathcal{V}$ satisfies an identity $[x, y][z, t] = 0$. Then $\mathcal{V}$ possesses Property Z.

**Proof.** Let $f$ and $g$ be polynomials from Property Z condition. Suppose that neither $g$ nor $f([y, z], \tilde{x})$ belongs to $T(\mathcal{V})$. By Lemma 2 $\mathcal{V}$ satisfies two identities
\[ W_r(\tilde{i}) = h(\tilde{i}), \quad (3) \]
where $h(\tilde{i})$ is a consequence of $g$; and
\[ W_s(\tilde{u}) = c(\tilde{u}), \quad (4) \]
where $c(\tilde{u})$ is a consequence of $f([y, z], \tilde{x})$. Clearly, $c(\tilde{u})$ is a sum of summands of the two types:

A: $vf(\sum_i[a_i, b_i], \ldots)w$ and $u_1$ occurs in every monomial $a_i$,
B: $a[v, w]b$ and $u_1$ occurs either in $a$ or in $b$.

The substitution $u_1 \mapsto W_r(\bar{t})$ in (4) transforms all summands of the type B into consequences of $[x, y][z, t]$ and all summands of the type A into polynomials from $T(\{f(T(\{g\}), \ldots)\})$ (see [3]). Therefore, under this substitution we obtain modulo $T(\mathcal{V})$

$$W_{r+\tau-1} = W_s(W_r(\bar{t}), u_2, \ldots) = 0.$$ 

The contradiction shows that either $g$ or $f([y, z], \bar{x})$ belongs to $T(\mathcal{V})$.

In detail varieties satisfying Property Z are considered in [1]. In the next item we give some necessary facts concerning such varieties.

Let $H$ be a relatively free countably generated $F$-algebra of $\text{var}\{[x, y][z, t] = 0\}$.

Denote by $\Lambda$ the set of free generators for $H$. For convenience, the elements of $H$ will be called polynomials.

For an arbitrary monomial $f(x_1, x_2, \ldots)$ put $S_{x_i}(f) = \{m\}$ if $m$ is the number of occurences of the letter $x_i$ in $f$ (in other words, the degree of $f$ over $x_i$). If $f(\bar{x}, \bar{t}) = f_1(\bar{x}, \bar{t}) + \cdots + f_n(\bar{x}, \bar{t})$ is the sum of monomials $f_i$, we put

$$S_{\bar{u}}(f) = \bigcup_{i=1}^{n} \bigcup_{u \in \bar{u}} S_u(f_i).$$

Similarly, for every polynomial $f$ in $[H, H]$ we define $D(f)$, the set of bilateral degrees. First, let $f(x, \bar{t})$ be a commutator monomial of the form

$$a(x, \bar{t})[t_i, t_j]b(x, \bar{t})$$

with $S_x(a) = \{k\}$ and $S_x(b) = \{l\}$. Then we put $D_x(f) = \{(k, l)\}$. If $f(x, \bar{t}) = a(x, \bar{t})[x, t_i]b(x, \bar{t})$ and $a, b$ are as above, we put

$$D_x(f) = \{(k + 1, s), (k, s + 1)\}.$$

Finally, if

$$f(\bar{x}, \bar{t}) = f_1(\bar{x}, \bar{t}) + \cdots + f_n(\bar{x}, \bar{t})$$

is the sum of commutator monomials $f_i$, we define $D_x(f)$, the set of bilateral degrees of a polynomial $f$ over variables $x_1, x_2, \ldots$, as follows

$$D_x(f) = \bigcup_{i=1}^{n} \bigcup_{u \in \bar{x}} D_u(f_i).$$

For instance, let

$$f(x, y, z) = x[x, y]x^2 + y^5[y, z];$$

then $S_{\{x, y\}}(f) = \{4, 1, 0, 6\}$ and

$$D_{\{x, y\}}(f) = \{(2, 2), (1, 3), (1, 0), (0, 1), (0, 0), (6, 0), (5, 1)\}.$$
Remark 1  Generally, the sets $S_x(f)$ and $D_x(f)$ are not uniquely defined for $f$ and depend on its particular representation — as a sum of monomials or commutator monomials. Below, by writing $S_x(f) = M$ ($D_x(f) = M$) we mean that $f$ is given by a representation such that $S_x(f)$ ($D_x(f)$ resp.) coincides with the set $M$. Moreover, we assume that $D_x(f)$ and $S_x(f)$ for a polynomial $f \in [H, H]$ are expressed via the same representation, that is, $S_x(f) = \{i_1 + j_1, \ldots, i_m + j_m\}$ if $D_x(f) = \{(i_1, j_1), \ldots, (i_m, j_m)\}$.

Recall that polynomial $f(\bar{x}, \bar{t})$ is said to be essential in variables from $\bar{x}$ if every $x_i$ occurs in every monomial of $f$.

Proposition 1  Let $\mathcal{M}$ be a $F$-algebra variety possessing Property $Z$ and satisfying $[x, y][z, t] = 0$. Suppose that a polynomial $f(\bar{x}, \bar{t}) \in H$ essential in all $x_i$ has the form

$$f(\bar{x}, \bar{t}) = w(\bar{x}, \bar{t}) + \sum_i g_i(\bar{x}, \bar{t}) + \sum_i h_i(\bar{x}, \bar{t}),$$

where $h_i(\bar{x}, \bar{t})$ are monomials, $g_i(\bar{x}, \bar{t})$ are commutator monomials, and $w(\bar{x}, \bar{t}) \in [H, H]$. Also, assume that there exist finite sets

$$A \subseteq (\mathbb{N} \cup \{0\}) \times (\mathbb{N} \cup \{0\}) \text{ and } B \subseteq \mathbb{N} \cup \{0\}$$

satisfying the following conditions:

1) $A \cap D_x(w(\bar{x}, \bar{t})) = \emptyset$ and $B \cap S_x(w(\bar{x}, \bar{t})) = \emptyset$;

2) for every $i$, there exist $j$ and $k$ such that $D_x(g_i) \subseteq A$ and $S_x(h_i) \subseteq B$.

If $f \in T(\mathcal{M})$ and $w(\bar{x}, \bar{t}) \notin T(\mathcal{M})$ then $\mathcal{M}$ satisfies an identity

$$x^l[y, z]x^m = x^r[y, z]x^s,$$

where $(l, m) \in D_x(w(\bar{x}, \bar{t}))$ and $(r, s) \in A$ or $r + s \in B$.

(The set $A$, like $B$, may be empty, in which case all $g_i$ (resp., $h_i$) will be zero.)

The proof of Proposition [1] can be found in [1].

Let us recall that by $\mathcal{V}$ we denote an almost L.N. variety.

Lemma 6  Let $\mathcal{V}$ be generated by nilpotent algebras and $[x, y][z, t] \in T(\mathcal{V})$. Then $\mathcal{V}$ satisfies either the identity $x[y, z] = 0$ or the identity $[y, z]x = 0$.

Proof. Suppose that $x[y, z] \notin T(\mathcal{V})$. Then, by Lemma [2] $\mathcal{V}$ satisfies for some $n$ an identity

$$W_n(\bar{x}) = h(\bar{x}),$$

where $h$ is a sum of polynomials of the kind $ux_i[x_j, x_k]v$ and $u, v$ are monomials (maybe, empty). Substituting $[y, z]$ for $x_1$ we transform all summands where $x_1$ does not occur in commutators into consequences of $[x, y][z, t]$. Thus, under this substitution we get an identity

$$[y, z]x_2x_3 \cdots x_n + g(y, z, \bar{x}) = 0,$$
where \( g = \sum_{i \geq 1} a_{ij} x_i [y, z] b_{ij} \). By Lemma 5, \( V \) possesses Property Z and satisfies all conditions of Proposition 1. Put

\[
w = [y, z] x_2 x_3 \cdots x_n,
\]

\[
A = \{(n, k) | n \geq 1\} \cap D_{x_2, x_3, \ldots, x_n}(g), \quad B = \emptyset.
\]

Clearly, \( D_{x_2, \ldots, x_n}(w) = \{(0, 1)\} \) and \( D_{x_2, \ldots, x_n}(w) \cap A = \emptyset \). It is easy to see that our \( w, A \) and \( B \) satisfy the conditions of Proposition 1. Thus, \( V \) satisfies either the identity \( w = 0 \) or an identity \( [y, z] x = x^r [y, z] x^s \) where \( r, s \) are integer and \( r \geq 1 \).

Suppose, at first, that \( w = 0 \). We can assume that \( n \) is a minimal such number, that is \( [y, z] x_2 x_3 \cdots x_{n-1} \notin T(V) \). Put \( f(t, x) = tx \) and \( g = [y, z] x_2 x_3 \cdots x_{n-1} \). Clearly, \( f(h, x) \in T(V) \) for every \( h \in T(\{g\}) \). Hence, we have \( f([y, z], x) \in T(V) \) because \( V \) satisfies the property Z and \( g \notin T(V) \). Thus, \( [y, z] x \in T(V) \). In this case Lemma 6 is proved.

Now suppose that \( [y, z] x = x^r [y, z] x^s \) is an identity of \( V \). We can assume \( s = 0 \); otherwise, we obtain the identity \( [y, z] x = 0 \) because \( V \) is generated by nilpotent algebras. Then, we see that \( r \neq 1 \), because \( [[y, z], x] \notin T(V) \). Thus, an identity \( [y, z] x = x^r [y, z] \) with \( r > 1 \) holds in \( V \).

Furthermore, assuming, in addition, that \( [y, z] x \notin T(V) \) we find by the dual argument an identity \( x[y, z] = [y, z] x^k \) for an integer \( k > 1 \). In every nilpotent algebra two identities \( [y, z] x = x^r [y, z] \) and \( x[y, z] = [y, z] x^k \) imply \( x[y, z] = 0 \) and \( [y, z] x = 0 \) which contradicts the assumption because \( V \) is generated by nilpotent algebras.

Below, \( p > 0 \) is the characteristic of the base field.

**Lemma 7** If \( [y, z] x \in T(V) \) and \( V \) is generated by nilpotent algebras then \( y^p x \in T(V) \).

**Proof.** It is easy to see that modulo \( T([y, z] x) \) the polynomial \( W_n(\bar{x}) \) is equal to \( x_3 \cdots x_n[x_1, x_2] \) and \( (u + v)^p x \) is equal to \( u^p x + v^p x \). Suppose that \( y^p x \notin T(V) \). By Lemma 2 and the above observation, \( V \) satisfies an identity

\[
x_3 \cdots x_n[x_1, x_2] = h(\bar{x})
\]

where \( h(\bar{x}) \) is a sum of monomials of the kind \( ab^p c \). Clearly, the length of every such monomial is greater than \( n \). Substituting \( [x_1, y] \) for \( x_1 \) in the identity, we obtain modulo \( [y, z] x \)

\[
x_2 x_3 \cdots x_n[x_1, y] = \sum_i u_i [y, x_1],
\]

where the length of every monomial \( u_i \) is greater than \( n - 1 \). Since \( V \) is generated by nilpotent algebras, it satisfies the identity \( x_2 x_3 \cdots x_n[x_1, y] = 0 \) and, hence, \( W_{n+1}(y, x_1, \ldots, x_n) = 0 \). A contradiction.

Let us denote by \( A_p \) the variety given by \( [y, z] x = 0 \) and \( y^p x = 0 \). Obviously, \( A_p^* \) is given by \( x[y, z] = 0 \) and \( xy^p = 0 \). Consider these varieties. First of all, they are not Lie nilpotent because of a following easily verified lemma.
Lemma 8 The algebras $A(C)$ and $A(C)^*$ are not Lie nilpotent. The algebra $A(C)$ belongs to $A_p$, and $A(C)^*$ belongs to $A_p^*$. Now, to show that $A_p$ and $A_p^*$ are almost L.N it suffices to prove the following lemma and its dual analogue.

Lemma 9 Every proper subvariety of $A_p$ is Lie nilpotent.

Proof. Let $M$ be a proper subvariety of $A_p$. Then, there exists a polynomial $f$ from $T(M) \setminus T(A_p^*)$. We want to find a consequence of $f$ of the form

$$u_1^{s_1} \cdots u_m^{s_m} [y, z], \quad s_1 < p, \ldots, s_m < p. \quad (5)$$

Obviously, its full linearization and the polynomial $[y, z]x$ have as a consequence the desired polynomial $W_n(\bar{x})$ for $n = (p - 1)m + 2$.

Since $[u, v]x_1 + [x_1, u]v + [v, x_1]u = x_1[u, v] + v[x_1, u] + u[v, x_1]$ in every associative algebra, we have modulo $T(\{[y, z]x\})$

$$x_1[u, v] = -v[x_1, u] - u[v, x_1].$$

Moreover, we have modulo $T(A_p^*)$

$$x_1^{p-1}[x_1, u] = ux_1^p.$$

Therefore, $f$ can be written modulo $T(A_p)$ in a following form

$$f(\bar{x}) = \sum_{\bar{k}: k_1 < p, \ldots, k_n < p} \alpha_{\bar{k}} x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} + \sum_{i, \bar{s}: s_1 < p-1, s_2 < p, \ldots, s_n < p} \beta_{i, \bar{s}} x_1^{s_1} x_2^{s_2} \cdots x_n^{s_n} [x_1, x_i] + \sum_{i, \bar{t}: t_1 < p, \ldots, t_n < p} \gamma_{i, \bar{t}} t_1^{t_1} \cdots t_n^{t_n} x_1^p$$

If for some $\bar{k}$ we have $\alpha_{\bar{k}} \neq 0$, we multiply $f$ by $[y, z]$ from the left and obtain modulo $T(A_p)$

$$\sum_{\bar{k}: k_1 < p, \ldots, k_n < p} \alpha_{\bar{k}} x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} [y, z] \in T(M).$$

Since degrees of variables are smaller than $p$, every polyhomogeneous summand of the last polynomial belongs to $T(M)$. It remains to see that all of them are of the form (5).

If $\alpha_{\bar{k}} = 0$ for all $\bar{k}$ and $\beta_{i, \bar{s}} \neq 0$ for some pair $\bar{s}, i$, we substitute $x_1 + [y, z]$ for $x_1$ in $f$ and select all summands with $[y, z]$ to obtain modulo $T(A)$

$$\sum_{\bar{s}: s_1 < p-1, s_2 < p, \ldots, s_n < p} \beta_{i, \bar{s}} x_1^{s_1+1} x_2^{s_2} \cdots x_n^{s_n} [y, z] \in T(M).$$

Repeating the above argument we obtain the desired polynomial (5).
Finally, if all \( \alpha \) and \( \beta \) equal zero, we substitute \( x_1 + [y, z] \) for \( x_1 \) in \( f \) and select all summands with \([y, z]\) to obtain modulo \( T(\mathcal{A})\)

\[
\sum_{t, t_2 < p, \ldots, t_n < p} \gamma t x_1^{p-1} x_2^{t_2} \cdots x_n^{t_n} [y, z] \in T(\mathcal{M}).
\]

As above, every polyhomogeneous summand of the polynomial belongs to \( T(\mathcal{M}) \) and it has the form \([5]\).

Hence, every proper subvariety of \( \mathcal{A}_p \) is Lie nilpotent.

Now we are ready to prove the first main result.

**Proof of Theorem 1.**

Necessity. Let \( \mathcal{V} \) be a non-prime almost L.N. variety. It is well known that every variety of algebras over an infinite field is generated by its nilpotent algebras. Hence, by Lemmas \([1, 6, 7]\) and dual to \([7]\) \( \mathcal{V} \) is contained in \( \mathcal{A}_p \) or in \( \mathcal{A}_p^* \). By Lemmas \([8, 9]\) and by their dual variants \( \mathcal{A}_p \) and \( \mathcal{A}_p^* \) are almost L.N. Hence, \( \mathcal{V} \) coincides with one of them. It remains to see that an almost L.N. variety is generated by any its non L.N. algebra. By Lemma \([8]\) we can give \( A(C) \) and \( A(C)^* \) as such algebras.

Sufficiency follows from Lemmas \([8, 9]\) and their dual analogues.

To prove Theorem 2 we need in the discription of almost non-Engel varieties. Recall that a variety is said to be Engel if for some natural \( n \) it satisfies an identity \( W_{n+1}(x, y, \ldots, y) = 0 \). A variety is called almost Engel if it is itself non-Engel but its proper subvarieties are all Engel.

**Proposition 2** *(Theorem 2, \([7]\)) A variety of algebras over a finite field \( F \) is almost Engel if and only if it is generated by one of the algebras \( A(F) \), \( A(F)^* \), or \( B(F, G, \sigma) \).

**Remark 2** In fact, every algebra in the list of Proposition \([2]\) generates a non-Engel almost commutative variety (it is easy to verify immediately or see \([5]\)). Hence, it generates an almost L.N. variety.

**Proof of Theorem 2.**

Necessity. Let \( \mathcal{V} \) be almost L.N. Assume, at first, that \( \mathcal{V} \) is non-Engel. Then, by Zorn’s Lemma, it contains some almost Engel variety as a subvariety. Hence, by Proposition \([2]\) and Remark \( 2 \) \( \mathcal{V} \) coincides with one of the varieties \( \var A(F) \), \( \var A(F)^* \), or \( \var B(F, G, \sigma) \).

Assume now, that \( \mathcal{V} \) is non-prime and Engel. It remains to show that in this case \( \mathcal{V} \) is generated by nilpotent algebras. Then, we shall be able to repeat word for word the argument of Theorem 1 to verify that \( \mathcal{V} \) is generated by \( A(C) \) or \( A(C)^* \). So, being Engel \( \mathcal{V} \) satisfies an identity \( W_{n+1}(x, y, \ldots, y) = 0 \) Without loss of generality, we can assume that \( n = p^k \) where \( p \) is the characteristic of the base field. It is easy to prove that for every \( n \)

\[
W_{n+1}(x, y, \ldots, y) = \sum_{k=0}^{n} C_n^k (-1)^k y^k x y^{n-k}. 
\]
Therefore, for \(n = p^t\) we have

\[ W_{n+1}(x, y, y, \ldots, y) = [x, y^n]. \]

Thus, \(V\) satisfies the identity \([x, y^{p^t}] = 0\). Hence, the center of every algebra \(A \in V\) contains all subalgebras of \(A\) which are finite fields. Moreover, \(V\) is locally residual finite (see [2]). Hence, it is generated by its finite-dimensional algebras. Consider an arbitrary such algebra \(A\). The variety \(V\) does not contain full matrix algebras because each of them is not Engel. Hence, \(A = B + J(A)\) and \(B\) is a finite direct sum of finite fields, \(J(A)\) is a nilpotent radical. By above remark, \(B\) is contained in the center of \(A\). Therefore, \(W_k(A, \ldots, A) \subseteq W_k(J(A), J(A), \ldots, J(A))\) for every \(k\).

Suppose now that all nilpotent algebras of \(V\) generate a proper subvariety. Since \(V\) is an almost L.N. variety, this proper subvariety satisfies an identity \(W_k(\bar{x}) = 0\). This identity holds in all algebras \(J(A)\). Therefore, by the last inclusion it also holds in all finite-dimensional algebras \(A\). Thus, \(W_k(\bar{x}) = 0\) is an identity of \(V\). The contradiction shows that \(V\) is generated by nilpotent algebras.

The sufficiency follows from Remark 2, Lemmas 8, 9 and their dual analogues.

We found descriptions of almost L.N. varieties modulo prime varieties. Nevertheless, we hope to show below that the descriptions are quite useful.

**Example**

Let \(p\) be the characteristic of a field \(F\). Denote by \(M\) the variety of \(F\)-algebras given by two identities

\[ [x^p, y] = 0, \quad \text{(6)} \]

and

\[ W_{n_1}(\bar{x}_1) \cdot W_{n_2}(\bar{x}_2) \cdots W_{n_s}(\bar{x}_s) = 0 \quad \text{(7)} \]

for some natural \(n_1, \ldots, n_s\). We state that \(M\) is Lie nilpotent.

Indeed, assume, on the contrary, that \(M\) is not Lie nilpotent. By Zorn’s Lemma \(M\) contains an almost L.N. variety \(N\) as a subvariety. The subvariety is not verbally prime because of (7). Hence, \(N\) is generated by one of the algebras from Theorem lists. Algebras \(A(F), A(F)^*, B(F, G, \sigma)\) from Theorem 2 are non-Engel (see Proposition 2). Therefore, they do not satisfy Identity (6), that is \(p\)-Engel condition, and can not belong to \(M\).

For \(A(C)\) put

\[ x = \begin{pmatrix} c_1 + \cdots + c_{p-1} & c_p \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} c_{p+1} & 0 \\ 0 & 0 \end{pmatrix}. \]

It is easy to prove that \(x^p = \begin{pmatrix} 0 & (p-1)! \cdot c_1 \cdots c_{p-1}c_p \\ 0 & 0 \end{pmatrix}\), and

\[ [x^p, y] = x^py - yx^p = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & (p-1)! \cdot c_1 \cdots c_{p+1} \\ 0 & 0 \end{pmatrix} \neq 0. \]

Thus, \(A(C)\) and \(A(C)^*\) (by dual reason) also do not belong to \(M\). None of the algebras can generate the variety \(N\). The contradiction shows that \(M\) is Lie nilpotent.

11
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