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Features of composite beams bending analysis by energy methods

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Abstract. Simple analytical calculations of rectangular-sectioned composite console beams loaded with concentrated force, spread load or dead load are presented. The simplicity of originate expression for maximum deflections by energy methods is shown.

1. Features of strength analysis by set level of stored elastic strain energy

Figure 1 shows the diagram of half of a few-leaf spring of \( n \) equal leaves loaded with end force \( P \). The term “few-leaf” is applicable to a spring compiled of equal-length leaves \( 2l \), uniform strength of which is provided by shaping and not by leaves lengths change according to linear law as in typical steel multi-leaf springs with fixed dimensions of leaves cross sections (\( w \) – width, \( t \) – thickness).

Elastic members – energy storage unit such as springs, leaf springs, torsional spring – should meet three main conflicting requirements: by strength, by stored energy, and by durability. The inconsistency of these requirements lies in the fact that by set length \( 2l \) for the provision of requirements on strength and durability, a beam should be as thicker than \( t \) as possible, for the provision of requirements on stored energy – as thinner than \( t \) as possible. To sort out these differences leaves \( n \) multiplication is typically applied, but as it will be shown further, this is not an exclusive and better way as applied to composite materials of quasi-unidirectional glass-fiber-reinforced plastic type. The requirement on stored energy should be met accurately, i. e. a spring loaded by the set static end force \( P_s \) should have the set deflection \( \nu_{st} \), and the requirements on strength and durability should be met on the safe side. The lesser stresses \( \sigma_{max} \) occur by the set maximum load \( P_{max} \) and the greater...
number of cycles \( N \) a spring will bear at the set range \( \Delta P_c \) of cyclic loads, the greater will be fault-free haul as applied to an automotive vehicle.

At the Technical assignment for the spring design calculation, initial camber and its change are typically set: static \( \nu_{st} \), maximum \( \nu_{max} \), and cyclic \( \Delta \nu_{c} \) deflections. Stiffness \( C = \frac{P_{st}}{\nu_{st}} \) is defined via a static deflection, and maximum and cyclic loads are defined via stiffness:

\[
P_{max} = k_d P_{st} \approx C \nu_{max}; \quad P_c = \nu_{st} \pm C \Delta \nu_c,
\]

where \( k_d \) is the dynamic factor depending on operational conditions, for example, of a light motor vehicle or load-carrying vehicle. For SUV, \( k_d = 1.4 \), for a mine dump truck, \( k_d = 2.7 \). As can be seen from the above, for the benchmark related to shaping influence analysis, to set the requirements by load-bearing capacity (maximum load \( P_{max} \)) and by stored energy \( U_{max} \) (or stiffness \( C \)) is enough. The estimation of cyclic stresses in beam approximation is carried out by the same formulas with \( P_{max} \) replaced by \( P_c \).

To generalize the assignment, let us assume that console beam is loaded by bending moment changing along \( x \) coordinate as a power function (figure 2):

\[
M(\bar{x}) = M(1)\bar{x}^{\gamma}; \quad \bar{x} = \frac{x}{l} \in [0;1].
\]

the shearing force is expressed by a derivative of a moment, and spread load – by a derivative of the shearing force:

\[
P(x) = \frac{dM}{dx}; \quad p(x) = \frac{dP}{dx}.
\]

Consequently, loading by end force corresponds to \( \gamma = 1 \), spread load – \( \gamma = 2 \), linearly changing spread load – \( \gamma = 3 \) etc.

According to relation (1) for the provision of equal stress (strength uniformity), section modulus should also be changed as a power function. Let us assume that thickness \( t \) and width \( w \) of rectangular sections are changed as a power function:

\[
w(\bar{x}) = w(1)\bar{x}^\alpha; \quad t(\bar{x}) = t(1)\bar{x}^\beta; \quad \bar{x} \in [\bar{a};1].
\]

Strength uniformity condition from the expressions (1) and (2) will be as follows:

\[
\frac{6M(1)\bar{x}^\gamma}{nw(\bar{x})^{-1}} = \frac{6M(1)}{nw(1)^{-1}} \implies \alpha + 2\beta = \gamma.
\]
The requirement on strength for loading by end force is expressed as follows:

\[
\frac{6M(1)}{nw(1)r^2(1)} = \frac{6P_{\text{max}}l}{nw(1)r^2(1)} = \sigma_{\text{max}} \leq \sigma.
\]  

(4)

The requirement on the maximum stored energy \(U_{\text{max}}\) can be expressed by Castigliano theorem:

\[
U_{\text{max}} = \frac{6E_{\text{w}}}{\delta_{\text{U}}} \left[ \int_0^1 \frac{M(x)}{w(x)} r^3(x) d\alpha \right]^2 = \frac{6M^2(1)v}{Enw(1)r^2(1)} \left[ \int_0^1 \frac{X_{\text{w}}}{X_{\text{w}}^2 + \beta} d\alpha \right] = U_0 \delta_{\text{U}}, \quad \delta_{\text{U}} = \frac{1 + 2\gamma}{1 + 2\gamma - \alpha - 3\beta},
\]

where \(U_0\) is the energy stored by the same moment in \(n\) equal rectangular leaves with the same fixed dimensions of root section \(w(1), t(1)\); \(\delta_{\text{U}} = U_{\text{max}} / U_0 = \nu_{\text{max}} / \nu_0\) – shape factor by energy in case of end force is the ratio between the deflection at the end of shaped beam to the deflection at the end of the rectangular beam.

By accurate carrying out the requirement (5) on energy, the necessary thickness of root section (in case of loading by end force) is found:

\[
t^3(1) = \frac{6M^2(1)v}{Enw(1)U_{\text{max}}} \delta_{\text{U}} = \frac{6P_{\text{max}}^2l^3}{Enw(1)U_{\text{max}}} \delta_{\text{U}},
\]

and by accurate carrying out the condition (4) on strength – the necessary number of leaves:

\[
n = \frac{6P_{\text{max}}E^2U_{\text{max}}^2}{\sigma^2 w(1)t^4 \delta_{\text{U}}^2}.
\]  

(6)

The relation (6) shows that even considering the strength of unidirectional glass-fiber-reinforced plastic equal to steel strength (although it can be higher), and not considering the useful contribution of shaping (the increase of \(\delta_{\text{U}}\)), it is possible to reduce the number of leaves by a factor of 20 by elasticity modulus reduction only (210 GPa in steel, 45 GPa in glass-fiber-reinforced plastic). At which point, virtually any steel multi-leaf spring can be replaced by glass-fiber-reinforced plastic single-leaf spring, as it will come into account further considering \(n = 1\).

2. The possibility justification of approximate estimates

Let us explain the possibility of passing by the influence of low inter-laminar shearing modulus and low shearing strength for the sufficiently long beams of glass-fiber-reinforced plastic. The presence of shear stresses results in an additional deflection \(v_2\), which can be estimated considering that work done by force at this deflection is spent on shear strain elastic energy:

\[
\frac{1}{2} P_{v_2} = \frac{1}{2} \int_0^w \int_0^l \int_0^{t/2} \tau_{\text{sc}}(z) dxdydz = \frac{1}{2} \frac{\alpha_o P^2l}{4G_{\text{sc}} w t},
\]

where \(\alpha_o = 6/5\) for the transverse bending of a beam with a rectangular section.

The distribution of shear stresses by the beam height is considered as parabolic:

\[
\tau_{\text{sc}} = \frac{3P}{th} \left( \frac{h^2}{4} - z^2 \right),
\]

and independent of the beam length and axial coordinate \(x\). Across the width, along the axis \(y\), the
distribution of the shear stresses is considered uniform at the first approximation. Numerical analysis by finite-element method (FEM) showed that for beams with the relation of span length to the height more than 5, a hypothesis for such type of distribution diagram of shear stresses introduces at most 3% error.

As can be seen from the above, the general deflection can be expressed as follows

\[ \nu = v_1 + v_2 = \frac{P l^3}{4E_I w t^4} + \frac{\alpha_0 P l}{4G_{xz} w t} = v_1 \left(1 + \alpha_0 \frac{E_x h^2}{G_{xz} l^2}\right). \]

By relation of longitudinal Young's modulus \( E_x \) to inter-laminar shearing modulus \( G_{xz} \) of glass-fiber-reinforced plastic around five, a correction to deflection does not exceed 1% for sufficiently long beams \( l/t > 20 \). Consequently, by further calculations of the stored elastic energy we were restricted to accounting only the deflections from normal stresses, i.e. without sacrificing the plane-sections hypothesis.

By bending short composite beams, failure in the form of delamination by the action of normal and shear stresses combination is possible:

\[ \sigma_x + m \tau_{xz} = c, \]

where \( m, c \) are the experimentally determined material parameters.

In addition, conditional shearing strength becomes dependent on the ratio of span to beam thickness.

\[ \tau_0 = \frac{cm}{m^2 + l^2/t^2}. \]

But analytical estimations and experiments show that delamination occurs in glass-fiber-reinforced plastics at \( l/t \leq 12 \) only, and for sufficiently long beams \( l/t > 20 \), the influence of shear stresses on strength can be neglected, which is considered in this article.

The glass-fiber-reinforced plastic effect in a leaf spring

Let us analyze how modulus reduction allows increasing cyclic durability for carrying out one of the requirements. For the set cyclic load and stored energy, from the relation (6), the fulfilment of conditions on strength and stored energy results in the following expression for cyclic stresses range:

\[ \sigma_c = \frac{1}{l} \left(6P E^2 U_{max} / m w (1 \delta_U^c)\right)^{1/3}. \] (7)

Formula (7) reflects the main features of strength analysis at the set elastic energy. For example, for the given maximum load \( P_{max} \) (or \( P_c \)), a long (!) beam occurs to be less stressed than a short one. To increase the operational life, it is required to reduce maximum stress in the cycle \( \sigma_c \) for the set cyclic load \( P_c \). According to expression (7), this can be achieved by five methods:

1) increase leaves number \( n \) (typical approach);
2) increase width \( w(1) \) (that is why wide cross springs are effective);
3) increase length \( l \) (let us remember old-fashioned carriages with a spring exceeding the carriage sizes);
4) increase shape factor \( \delta_U \) by shping of uniform strength;
5) use glass-fiber-reinforced plastic with elasticity modulus \( E \) lower than elasticity modulus of steel.

Wohler curves describing the cyclic durability are often considered as straight lines in semilog coordinates: \( \sigma_c = \sigma_c (1-k \lg N) \), from which
As an example, let us assume that $k_c = 0.1$, which approximately corresponds to Wohler curves slope for unidirectional composites and for a variety of steels. Let the cyclic stress amounts 40% from static strength: $\sigma_c/\sigma_* = 0.4$ is a national norm adopted for composite aeronautical structures, so saying “ignorance factor”. Then, from relation (8), the critical number of cycles $N = 10^6$. According (7), the elasticity modulus reduction by 4 times under otherwise equal conditions will result in the cyclic stresses reduction by 2.5 times, which from expression (8) is equivalent to the durability increase by more than 100 times:

$$\lg N = \frac{1 - \frac{0.4}{2.5}}{0.1} = 8.4.$$ 

This conclusion is based on data close to reality and given for illustration purposes only. Consequently, it can not be recommended for use as a strict result.

For the single-leaf version, from relations (3), (4) and (5), sizes of root section by the strict fulfilment of conditions on strength and stored energy can be specified:

$$t(1) = \frac{\sigma_c M(1)}{U_E(1 + 2\gamma - \alpha - 3\beta)} = t_0 \delta_U;$$

$$w(1) = \frac{6U_{max} E^2 (1 + 2\gamma - \alpha - 3\beta)^2}{\sigma_c^2 M(1) l^2} = w_0 \delta^2_U.$$ 

Where $t_0, w_0$ are the sizes of rectangular beam root section meeting the same conditions (4)-(5).

According to relation (2), the volume of a shaped beam is defined by integration:

$$V = \int_0^l w(\bar{x}) t(1) d\bar{x} = t(1) \int_0^l \bar{x}^{\alpha+\beta} d\bar{x} = V_0 \delta_v;$$

$$\delta_v = \frac{1}{1 + \alpha + \beta},$$

where $\delta_v$ is the shape factor by volume equal to the relation of volumes of shaped and rectangular beams with similar sizes of root section. Now, from expression (9) and (10), it is possible to estimate the required weight of spring leaf via density $\rho$:

$$m = \rho w(1) t(1) l \delta_v = \rho w_0 t_0 l \delta_v \delta_U = m_0 \delta_s;$$

$$\delta_s = \frac{\delta_v}{\delta_U},$$

where $\delta_s$ is the aggregate coefficient factor of weight reduction; $m_0$ is the rectangular leaf weight meeting the same conditions on strength and energy; its section sizes differ from $w(1), t(1).$ From relations (11), it follows that by low density (2500 kg/m$^3$ in glass-fiber-reinforced plastic; 7800 kg/m$^3$ in steel) and much lower elasticity modulus, the weight of glass-fiber-reinforced plastic spring (ideally) can be at 15 times lesser than the weight of steel spring with the same service properties. This makes the direct effect evident. The constructive effect follows from relation (6) and shows the possibility of leaves number reduction by a factor of 20. Finally, technological effect involves the possibility to shape composite beams and, as it follows from formulas (5), (10) and (11), by carrying out the condition of uniform strength (3) $\alpha = \gamma - 2\beta$: 

$$1 - \frac{\sigma_c}{\sigma_*} = \frac{1}{k_c}.$$
As can be seen from the above, according to relation (12), the efficiency factor of shaping of uniform strength depends on the velocity of bending moment change (1): at loading by end force, any "ideal" equistrong beam is 3 times lighter than rectangular one, at spread load – 5 times lighter, at linear load (see figure 2, c) – 7 times lighter.

A real beam can have neither zero nor infinite sizes at the end. Consequently, due to shearing force resistance conditions, it is required to leave the end area of some length \( a \) with fixed dimensions (figure 3).

In calculating of energy (5) and volume (10), the integration should be carried out by two areas: with variable and fixed sizes of the section, and as a result, the expressions for shape factor will be as follows:

\[
\begin{align*}
\delta^*_U &= \delta_U + \left( 1 - \delta_U \right) \bar{a}^{3/\delta_U}; \\
\delta^*_V &= \delta_V + \left( 1 - \delta_V \right) \bar{a}^{1/\delta_V}; \\
\bar{a} &= a / l.
\end{align*}
\]  

(13)

Formula (13) should be used for the calculation of the required section sizes (9) and weight (11), but the result will no longer be that sort of decisive as (12).

Four design factors \( \alpha, \beta, w(1), t(1) \) can not be found definitely from three conditions: strength (4), uniform strength (3) and stored energy (5). Consequently, one more relation linking the section sizes is required. Fig. 4 gives the most common types of beams of uniform strength for the most important case of loading by end force. This is either triangle beam of uniform thickness: \( \alpha = 1, \beta = 0, t(\bar{x}) = const, \) or parabolic beam of uniform width: \( \alpha = 0, \beta = 1/2, w(\bar{x}) = const, \) or constant-area beam with the cross section fixed area:

\[
w(\bar{x}) \times t(\bar{x}) = const \Rightarrow \alpha + \beta = 0 \Rightarrow u_3 \quad (3) \quad \alpha = -1, \beta = 1.
\]  

(14)
Figure 4. Equistrong beams for the case of loading by end force: 1 – triangle beam of uniform thickness; 2 – parabolic beam of uniform width; 3 – constant-area beam with the uniform cross section area.

The last-named beam type is the most convenient for composites as it allows saving the initial number of uncut fibres, which is of critical importance for strength implementation. However, in addition, the beam width increases unrestrictedly and the significant fibre misalignment occurs. Consequently, to apply to nature for experience and to observe how tree crown increases its flexibility to withstand wind loads is reasonable.

So, simultaneous branching and shaping have the most positive effects for composite leaf-spring design.

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