The deformations of symplectic structures by 
moment maps

Tomoya Nakamura *
email: x-haze@ruri.waseda.jp

May 10, 2016

Abstract

We study deformations of symplectic structures on a smooth manifold $M$ via the quasi-Poisson theory. By a fact, we can deform a given symplectic structure $\omega$ to a new symplectic structure $\omega_t$ parametrized by some element $t$ in $\Lambda^2 g$, where $g$ is the Lie algebra of a Lie group $G$. Moreover, we can get a lot of concrete examples for the deformations of symplectic structures on the complex projective space and the complex Grassmannian.

1 Introduction

In the context of symplectic geometry, deformation-equivalence assumptions and conditions are often appeared, for example, in the statement of Moser’s theorem [9] and Donaldson’s four-six conjecture [8]. However, it seems that a method of constructing deformation-equivalent symplectic structures specifically is not well known. In this paper, we construct a method of producing new symplectic structures deformation-equivalent to a given symplectic structure. Our approach to deformations of symplectic structures is to use quasi-Poisson theory which was introduced by Alekseev and Kosmann-Schwarzbach [1], and this approach is carried out by using the fact that a moment map for a symplectic-Hamiltonian action $\sigma$ is also a moment map for a quasi-Poisson action $\sigma$. The former moment map satisfies conditions for only one symplectic structure, whereas the latter does conditions for a family of quasi-Poisson structures parametrized by elements in $\Lambda^2 g$. From

*Department of Mathematics, Waseda University, 3-4-1, Okubo, Shinjuku-ku, Tokyo, Japan
here we call these elements *twists*. Regarding the quasi-Poisson structure induced by a symplectic structure as that with twist 0, which is denoted by $\pi_0$, we can find different quasi-Poisson structures $\pi_t$ which induce symplectic structures $\omega_t$ by the choice of "good" twists $t$. The quasi-Poisson structure inducing a symplectic structure must be a nondegenerate Poisson structure. We describe the conditions for the quasi-Poisson structure with a twist $t$ to be a nondegenerate Poisson structure. Our method of using the family of quasi-Poisson structures is one of interesting geometry frameworks (See [1]).

From here, we explain briefly the difference among moment maps for symplectic- and quasi-Poisson-Hamiltonian actions, and equivariant moment maps for Poisson actions on a smooth manifold (In Poisson geometry, non-equivariant moment maps for Poisson action can be defined [4], [5]).

(I) Symplectic-Hamiltonian actions

In symplectic geometry, a moment map $\mu : M \to g^*$ for a symplectic action $\sigma$ of a Lie group $G$ on a symplectic manifold $(M, \omega)$ is defined with two conditions: one is for the symplectic structure $\omega$,

$$d\mu^X = \iota_{X_M} \omega \quad (X \in g).$$

Here $\mu^X(p) := \langle \mu(p), X \rangle$ and $X_M$ is a vector field on $M$ defined by

$$X_{M,p} := \left. \frac{d}{dt} \sigma_{\exp(tX)}(p) \right|_{t=0}$$

for $p$ in $M$. The other is the $G$-equivariance condition with respect to the action $\sigma$ on $M$ and the coadjoint action $\text{Ad}^*$ on $g^*$,

$$\mu \circ \sigma_g = \text{Ad}^*_g \circ \mu$$

for all $g$ in $G$. In this paper, we call symplectic actions with moment maps *symplectic-Hamiltonian actions* to distinguish it from other actions with moment maps.

(II) Poisson actions with equivariant moment maps

A Poisson Lie group, which was introduced by Drinfel’d [3], is a Lie group with a Poisson structure $\pi$ compatible with the group structure. Namely, the structure $\pi$ satisfies

$$\pi_{gh} = L_{g^*}\pi_h + R_{h^*}\pi_g$$

for any $g$ and $h$ in $G$, where $L_g$ and $R_h$ are the left and right translations in $G$ by $g$ and $h$, respectively. Such a structure is called *multiplicative*. Then the simply connected Lie group $G^*$ called the dual Poisson Lie group is
obtained uniquely from a Poisson Lie group \((G, \pi)\) and a local action of \(G\) on \(G^*\) is defined naturally. We call a multiplicative Poisson structure \(\pi\) on \(G\) complete if the action is global. Then \((G, \pi)\) is called a complete Poisson Lie group. An equivariant moment map \(\mu : M \to G^*\) for a Poisson action \(\sigma\) of a complete Poisson Lie group \((G, \pi)\) on a Poisson manifold \((M, \pi)\) is a generalization of a moment map for a symplectic-Hamiltonian action on a symplectic manifold, which was given by Lu in [4].

(III) Quasi-Poisson-Hamiltonian actions

Quasi-Poisson theory, which was originated with [1] by Alekseev and Kosmann-Schwarzbach, is a generalization of Poisson theory with Poison actions. In quasi-Poisson geometry, quasi-triples \((D, G, h)\) and its infinitesimal version, Manin quasi-triples \((\mathfrak{d}, \mathfrak{g}, \mathfrak{h})\), play important roles. A quasi-triple \((D, G, h)\) defines a quasi-Poisson Lie group \(G^h_D\) and we can obtain the notion of a quasi-Poisson action of such a quasi-Poisson Lie group \(G^h_D\). A moment map \(\mu\) for the action is a map from \(M\) into \(D/G\) and satisfies a condition not for one quasi-Poisson structure but for a family of quasi-Poisson structures parametrized by elements in \(\Lambda^2 \mathfrak{g}\). An equivariant moment map for a Poisson action in (II) is an example of a moment map for a quasi-Poisson-Hamiltonian action. In this paper, we use the moment map theory for quasi-Poisson actions to deform symplectic structures on a smooth manifold.

This paper is constructed as follows. It is contents of Section 2 to review the moment map theory for quasi-Poisson actions. In Section 3, we describe a deformation method of symplectic structures on a smooth manifold via the quasi-Poisson theory. This method is the subject in this paper. Theorem 3.1 gives a condition for a twist to deform a symplectic structure to a new one. In addition, Theorem 3.2 gives a sufficient condition for a twist to satisfy the assumption of Theorem 3.1. In Section 4, we introduce concrete examples for deformations of symplectic structures. We give deformations of the Fubini-Study and the Kirillov-Kostant forms on \(\mathbb{C}P^n\) and the complex Grassmannian, respectively.

2 Moment maps for quasi-Poisson actions on quasi-Poisson manifolds

In this section, we shall recall the quasi-Poisson theory [1]. We start with the definition of quasi-Poisson Lie groups, which is a generalization of Poisson Lie groups.
Definition 1. Let $G$ be a Lie group with the Lie algebra $\mathfrak{g}$. Then a pair $(\pi, \varphi)$ is a quasi-Poisson structure on $G$ if a multiplicative 2-vector field $\pi$ on $G$ and an element $\varphi$ of $\Lambda^3\mathfrak{g}$ satisfy

\begin{align}
\frac{1}{2} [\pi, \pi] &= \varphi^R - \varphi^L, \\
[\pi, \varphi^L] &= [\pi, \varphi^R] = 0,
\end{align}

where the bracket $[\cdot, \cdot]$ is the Schouten bracket on the multi-vector fields on $G$, and $\varphi^L$ and $\varphi^R$ denote the left and right invariant 2-vector fields on $G$ with value $\varphi$ at $e$ respectively. A triple $(G, \pi, \varphi)$ is called a quasi-Poisson Lie group.

Remark 1. In a quasi-Poisson structure $(\pi, \varphi)$ on $G$, the 2-vector field $\pi$ is a multiplicative Poisson structure if $\varphi = 0$. Namely, $(G, \pi)$ is a Poisson Lie group.

We use a ”quasi-triple” to obtain a quasi-Poisson Lie group. To define a quasi-triple, we describe its infinitesimal version, a Man in quasi-triple.

Definition 2. Let $\mathfrak{d}$ be a $2n$-dimensional Lie algebra with an invariant non-degenerate symmetric bilinear form of signature $(n, n)$, which is denoted by $(\cdot|\cdot)$. Let $\mathfrak{g}$ be an $n$-dimensional Lie subalgebra of $\mathfrak{d}$ and $\mathfrak{h}$ be an $n$-dimensional vector subspace of $\mathfrak{d}$. Then a triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ is a Manin quasi-triple if $\mathfrak{g}$ is a maximal isotropic subspace with respect to $(\cdot|\cdot)$ and $\mathfrak{h}$ is an isotropic complement subspace of $\mathfrak{g}$ in $\mathfrak{d}$.

Remark 2. For a given Lie algebra $\mathfrak{d}$ and a Lie subalgebra $\mathfrak{g}$ of $\mathfrak{d}$, a choice of an isotropic complement subspace $\mathfrak{h}$ of $\mathfrak{g}$ in $\mathfrak{d}$ is not unique.

A Manin quasi-triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ defines the decomposition $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{h}$. Then the linear isomorphism

\[ j : \mathfrak{g}^* \to \mathfrak{h}, \quad (j(\xi)|x) := \langle \xi, x \rangle \quad (\xi \in \mathfrak{g}^*, x \in \mathfrak{g}) \]

is determined by the decomposition. We denote the projections from $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{h}$ to $\mathfrak{g}$ and $\mathfrak{h}$ by $p_\mathfrak{g}$ and $p_\mathfrak{h}$ respectively. We introduce an element $\varphi_\mathfrak{h}$ in $\Lambda^3\mathfrak{g}$ which is defined by the map from $\Lambda^2\mathfrak{g}^*$ to $\mathfrak{g}$, denoted by the same letter,

\[ \varphi_\mathfrak{h}(\xi, \eta) = p_\mathfrak{g}([j(\xi), j(\eta)]), \]

for any $\xi, \eta$ in $\mathfrak{g}^*$. We define the linear map $F_\mathfrak{h} : \mathfrak{g} \to \Lambda^2\mathfrak{g}$ by setting

\[ F_\mathfrak{h}^*(\xi, \eta) = j^{-1}(p_\mathfrak{h}([j(\xi), j(\eta)])) \]

4
for any $\xi, \eta$ in $\mathfrak{g}^*$, where $F^*_h : \Lambda^2 \mathfrak{g}^* \to \mathfrak{g}^*$ is the dual map of $F_h$. These elements will be used later to define a quasi-Poisson structure and a quasi-Poisson action respectively.

Next we define a quasi-triple $(D, G, \mathfrak{h})$ and construct a quasi-Poisson structure on $G$ using $(D, G, \mathfrak{h})$.

**Definition 3.** Let $D$ be a connected Lie group with a bi-invariant scalar product with the Lie algebra $\mathfrak{d}$ and $G$ be a connected closed Lie subgroup of $D$ with the Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}$ be a vector subspace of $\mathfrak{d}$. Then a triple $(D, G, \mathfrak{h})$ is a quasi-triple if $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ is a Manin quasi-triple.

A method of constructing a quasi-Poisson structure by a quasi-triple is as follows. Let $(D, G, \mathfrak{h})$ be a quasi-triple with a Manin quasi-triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$. Using the inverse $j^{-1} : \mathfrak{h} \to \mathfrak{g}^*$ of the linear isomorphism (5), we identify $\mathfrak{d}$ with $\mathfrak{g} \oplus \mathfrak{g}^*$. Consider the map

$$r_h : \mathfrak{d}^* \to \mathfrak{d}, \quad \xi + X \mapsto \xi,$$

for any $\xi$ in $\mathfrak{g}^*$ and $X$ in $\mathfrak{g}$. This map defines an element $r_h \in \mathfrak{d} \otimes \mathfrak{d}$ which we denote by the same letter. We set

$$\pi^h_D := r^L_h - r^R_h,$$

where $r^L_h$ and $r^R_h$ is denoted as the left and right invariant 2-tensors on $D$ with value $r_h$ at the identity element $e$ in $D$ respectively, and we can see that it is a multiplicative 2-vector field on $D$. Furthermore, the 2-vector field $\pi^h_D$ and the element $\varphi_h$ defined by (6) satisfy (3) and (4). We set

$$\pi^h_{G,g} := \pi^h_{D,g}$$

(8)

for any $g$ in $G$. Then we can see that $\pi^h_{G}$ is well-defined and that $\pi^h_{G}$ is a multiplicative 2-vector field on $G$. Moreover, $\pi^h_{G}$ and $\varphi_h$ satisfy (3) and (4). Therefore $(G, \pi^h_{G}, \varphi_h)$ is a quasi-Poisson Lie group. We sometimes denote a Lie group with such a structure by $G^h_D$.

From here, we consider only connected quasi-Poisson Lie group $G^h_D$ defined as above by a quasi-triple $(D, G, \mathfrak{h})$. For a smooth manifold $M$ with a 2-vector field $\pi_M$, a quasi-Poisson action is defined as follows. It is a generalization of Poisson actions of connected Poisson Lie groups [6].

**Definition 4.** Let $G^h_D$ be a connected quasi-Poisson Lie group acting on a smooth manifold $M$ with a 2-vector field $\pi_M$. The action $\sigma$ of $G$ on $M$ is a
quasi-Poisson action if for each \( X \) in \( \mathfrak{g} \),

\[
\frac{1}{2} [\pi_M, \pi_M] = (\varphi_h)_M, \tag{9}
\]

\[
L_{X_M} \pi_M = F_h(X)_M, \tag{10}
\]

where \( x_M \) is a fundamental multi-vector field for any \( x \) in \( \wedge^* \mathfrak{g} \). Here \( F_h \) is the dual of the map (7). Then a 2-vector field \( \pi_M \) is called a quasi-Poisson \( G_D^h \)-structure on \( M \) and \((M, \pi_M)\) is called a quasi-Poisson \( G_D^h \)-manifold.

**Remark 3.** A quasi-Poisson Lie group \( G_D^h \) with the natural left action is not a quasi-Poisson \( G_D^h \)-manifold. In fact, \( (\varphi_h)_G = \varphi_h^R \).

Finally we define a moment map for a quasi-Poisson action to carry out the deformation of symplectic structures using the moment map theory for quasi-Poisson actions in Section 3. We need some preliminaries to define a moment map. For any quasi-triple \((D, G, \mathfrak{h})\), since \( G \) is a closed subgroup of \( D \), the quotient space \( D/G \) is a smooth manifold, which is the range of moment maps. The action of \( D \) on itself by left multiplication induces an action of \( D \) on \( D/G \). We call it dressing action of \( D \) on \( D/G \) and denote the corresponding infinitesimal action by \( X \mapsto X_{D/G} \) for \( X \) in \( \mathfrak{d} \). Let \( p_{D/G} : D \to D/G \) be the natural projection. Then

\[
\pi_{D/G}^h := p_{D/G*} \pi_D^h
\]

is a 2-vector field on \( D/G \). We consider the dressing action on \( D/G \) restricted to \( G \), and can see that \( \pi_{D/G}^h \) satisfies (9) and (10). Therefore \((D/G, \pi_{D/G}^h)\) is a quasi-Poisson \( G_D^h \)-manifold. The following definition is one of the important notions to define moment maps.

**Definition 5.** An isotropic complement \( \mathfrak{h} \) of \( \mathfrak{g} \) in \( \mathfrak{d} \) is called admissible at a point \( s \) in \( D/G \) if the infinitesimal dressing action restricted to \( \mathfrak{h} \) defines an isomorphism from \( \mathfrak{h} \) onto \( T_s(D/G) \), that is, the map \( \mathfrak{h} \to T_s(D/G) \), \( \xi \mapsto \xi_{D/G,s} \) is an isomorphism. A quasi-triple \((D, G, \mathfrak{h})\) is complete if \( \mathfrak{h} \) is admissible everywhere on \( D/G \).

It is clear that any isotropic complement \( \mathfrak{h} \) of \( \mathfrak{g} \) is admissible at \( eG \) in \( D/G \). If the complement \( \mathfrak{h} \) is admissible at a point \( s \) in \( D/G \), then it is also admissible on some open neighborhood \( U \) of \( s \). For a quasi-triple \((D, G, \mathfrak{h})\), we assume that \( \mathfrak{h} \) is admissible on an open subset \( U \) of \( D/G \). Then for any \( X \) in \( \mathfrak{g} \), we define the 1-form \( \hat{X}_h \) on \( U \) by

\[
< \hat{X}_h, \xi_{D/G} > = (X | \xi) \tag{11}
\]
for any \( \xi \) in \( \mathfrak{h} \). If a quasi-triple \((D,G,\mathfrak{h})\) is complete, then \( \tilde{X}_h \) is a global 1-form on \( D/G \). Next we define a twist between isotropic complement sub-spaces \( \mathfrak{h} \) and \( \mathfrak{h}' \) of \( \mathfrak{g} \) in \( \mathfrak{d} \). Twists also play an important role in the moment map theory for quasi-Poisson actions. Let \( j \) and \( j' \) be the linear isomorphism \([5]\) defined by Manin quasi-triples \((\mathfrak{d},\mathfrak{g},\mathfrak{h})\) and \((\mathfrak{d},\mathfrak{g},\mathfrak{h}')\) respectively. Consider the map

\[
t := j' - j : \mathfrak{g}^* \to \mathfrak{d}.
\]

It is easy to show that \( t \) takes values in \( \mathfrak{g} \) and that it is anti-symmetric, so that the map \( t \) defines an element \( t \) in \( \Lambda^2 \mathfrak{g} \) which we denote by the same letter. The element \( t \) is called the twist from \( \mathfrak{h} \) to \( \mathfrak{h}' \). Fix a quasi-triple \((D,G,\mathfrak{h})\). Let \( \mathfrak{h}_t \) be an isotropic complement of \( \mathfrak{g} \) with a twist \( t \) from \( \mathfrak{h} \) to \( \mathfrak{h}' \). Then we can represent the elements \( \varphi_{\mathfrak{h}_t}, F_{\mathfrak{h}_t} \) and \( \pi_{\mathfrak{h}_t}^G \) defined by a quasi-triple \((D,G,\mathfrak{h}_t)\) as follows:

\[
\varphi_{\mathfrak{h}_t} = \varphi_h + \frac{1}{2} [t,t] + \varphi_t, \tag{12}
\]

\[
F_{\mathfrak{h}_t} = F_h + F_t, \tag{13}
\]

\[
\pi_{\mathfrak{h}_t}^G = \pi_h^G + t^L - t^R, \tag{14}
\]

where \([t,t] := [t^L,t^L]_e\), \( \varphi_t(\xi) := \text{ad}_T^2 t \) and \( F_t(X) := \text{ad}_X t \). Here \( \text{ad} \) denotes the adjoint action of \( \mathfrak{g} \) on \( \Lambda^2 \mathfrak{g} \) and \( \text{ad}_T^2 t \) denotes the projection of \( \text{ad}_T^2 t \) onto \( \Lambda^2 \mathfrak{g} \subset \Lambda^2 \mathfrak{d} \), where \( \mathfrak{d}^* \) including \( \mathfrak{g}^* \) acts on \( \Lambda^2 \mathfrak{d} \) by the coadjoint action. Let \( \{e_i\} \) be a basis on \( \mathfrak{g} \) and \( \{\varepsilon^i\} \) be the basis on \( \mathfrak{h} \) identified with the dual basis of \( \{e_i\} \) on \( \mathfrak{g}^* \) by \( j^{-1} \). Then the basis \( \{\varepsilon_t^i\} \) on \( \mathfrak{h}_t \) identified with the dual basis of \( \{e_i\} \) on \( \mathfrak{g}^* \) by \( j'^{-1} \) can be written by

\[
\varepsilon_t^i = \varepsilon^i + t^{ij} e_j, \tag{15}
\]

where \( t = \frac{1}{2} t^{ij} e_i \wedge e_j \). Moreover components of \( \varphi_t \) with respect to the basis \( \{\varepsilon^i\} \) are written as

\[
\varphi_t^{ij} = (F_h t)^{jk} i/i - (F_h t)^{jk} i/j. \tag{16}
\]

This indication is useful later. Let \((M,\pi_M^h)\) be a quasi-Poisson \( G_D^h \)-manifold. We set that \( \pi_M^{\mathfrak{h}_t} := \pi_M^h - t_M \). Then it follows that \((M,\pi_M^{\mathfrak{h}_t})\) is a quasi-Poisson \( G_D^{\mathfrak{h}_t} \)-manifold. Now we define moment maps for quasi-Poisson actions.

**Definition 6.** Let \( G_D^h \) be a connected quasi-Poisson Lie group defined by a quasi-triple \((D,G,\mathfrak{h})\) and \((M,\pi_M^h)\) be a quasi-Poisson \( G_D^h \)-manifold. Then a map \( \mu : M \to D/G \) which is equivariant with respect to the \( G \)-action on
M and the dressing action on $D/G$ is a moment map for the quasi-Poisson action of $G_D^h$ on $(M, \pi_M^h)$ if for any open subset $\Omega \subset M$ and any isotropic complement $h'$ admissible on $\mu(\Omega)$,

$$\left(\pi_M^{h'}\right)^\sharp (\mu^* (\hat{X}_{h'})) = X_M$$

on $\Omega$ for any $X$ in $\mathfrak{g}$. Here $\langle (\pi_M^{h'})^\sharp (\alpha), \beta \rangle := \pi_M^{h'} (\alpha, \beta)$. We call a quasi-Poisson action with a moment map a quasi-Poisson-Hamiltonian action.

Actually we need not impose the equation (17) on all admissible complements because we have the following proposition.

**Proposition 2.1** ([1]). Let $\mathfrak{h}$ and $\mathfrak{h}'$ be two complements admissible at a point $s$ in $D/G$, and $p$ in $M$ be such that $\mu(p) = s$. Then, at the point $p$, conditions (17) for $\mathfrak{h}$ and $\mathfrak{h}'$ are equivalent, namely

$$\left(\pi_M^{\mathfrak{h}}\right)^\sharp (\mu^*(\hat{X}_h))_p = \left(\pi_M^{\mathfrak{h}'}\right)^\sharp (\mu^*(\hat{X}_{h'}))_p.$$

For a quasi-Poisson manifold with a quasi-Poisson-Hamiltonian action, the following theorem holds.

**Theorem 2.2** ([1]). Let $(M, \pi_M^h)$ be a quasi-Poisson manifold on which a quasi-Poisson Lie group $G_D^h$ defined by a quasi-triple $(D, G, \mathfrak{h})$ acts by a quasi-Poisson-Hamiltonian action $\sigma$. For any $p$ in $M$, if both $\mathfrak{h}'$ and $\mathfrak{h}''$ are admissible at $\mu(p)$ in $D/G$, then

$$\text{Im}\left(\pi_M^{\mathfrak{h}'}\right)^\sharp = \text{Im}\left(\pi_M^{\mathfrak{h}''}\right)^\sharp,$$

where $\mu$ is a moment map for $\sigma$.

Now we show important examples for quasi-Poisson-Hamiltonian actions.

**Example 1** (Poisson manifolds [1],[2],[6]). Let $(M, \pi)$ be a Poisson manifold on which a connected Poisson Lie group $(G, \pi_G)$ acts by a Poisson action $\sigma$. Then $(M, \pi)$ is a quasi-Poisson $(G, \pi_G, 0)$-manifold and $\sigma$ is a quasi-Poisson action on $(M, \pi)$. In fact, the Manin triple $(\mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)$ corresponding to $(G, \pi_G)$ is a Manin quasi-triple and the multiplicative 2-vector field $\pi_G^\sigma$ on $G$ coincides with the 2-vector field $\pi_G^{\mathfrak{g}^*}$ defined by the corresponding quasi-triple $(D, G, \mathfrak{g}^*)$. Since $[\pi, \pi] = 0$ and the Poisson action $\sigma$ satisfies

$$\mathfrak{L}_{X_M} \pi = F_{\mathfrak{g}^*}(X)_M$$

for any $X$ in $\mathfrak{g}$, the action $\sigma$ is a quasi-Poisson action by Definition 7. Here the dual of $F_{\mathfrak{g}^*}$ coincides with the bracket on $\mathfrak{g}^*$ defined by $(G, \pi_G)$. 

8
We assume that $\pi_G$ is complete and that there exists a $G$-equivariant moment map $\mu : M \to G^*$ for the Poisson action $\sigma$, where $G^*$ is the dual Poisson Lie group of $(G,\pi_G)$ and $G$ acts on $G^*$ by the dressing action in the sense of Lu and Weinstein [6]. Then $\sigma$ is a quasi-Poisson-Hamiltonian action. Actually, by the definition, the map $\mu$ satisfies

$$\pi^*(\mu^*(X^L)) = X_M \quad (19)$$

for any $X$ in $\mathfrak{g}$, where $X^L$ is a left-invariant 1-form on $G^*$ with value $X$ at $e$ in $G^*$. The quotient manifold $D/G$ is diffeomorphic to $G^*$ as a manifold. The quasi-triple $(D,G,\mathfrak{g}^*)$ is complete since $\pi_G$ is complete. Then 1-form $\hat{X}_{\mathfrak{g}^*}$ defined by (11) is global for any $X$ in $\mathfrak{g}$. Furthermore the 1-form $\hat{X}_{\mathfrak{g}^*}$ on $D/G \cong G^*$ coincides with $X^L$. The complement $\mathfrak{g}^*$ is admissible at any point in $D/G$, so that the map $\mu : M \to G^* \cong D/G$ is a moment map for the quasi-Poisson action $\sigma$ because of (19) and Proposition [2.1].

Example 2 (symplectic manifolds [1],[9]). Let $(M,\omega)$ be a symplectic manifold on which a connected Lie group $G$ acts by a symplectic-Hamiltonian action $\sigma$. Since the symplectic structure $\omega$ induces a Poisson structure $\pi$, the pair $(M,\pi)$ is a Poisson manifold. Then the action $\sigma$ is a Poisson action of a trivial Poisson Lie group $(G,0)$ on $(M,\pi)$. The trivial Poisson structure $0$ on $G$ is complete and the quasi-triple corresponding to $(G,0)$ is $(T^*G,G,\mathfrak{g}^*)$, where $T^*G \cong G \times \mathfrak{g}^*$ is the cotangent bundle of $G$ equipped with the group structure of a semi-direct product with respect to coadjoint action of $G$ on $\mathfrak{g}^*$ (See [1]). The dual group $G^*$ of $(G,0)$ is the additive group $\mathfrak{g}^*$ and the moment map $\mu$ for symplectic action $\sigma$ is $G$-equivariant with respect to $\sigma$ on $M$ and $Ad^*$ on $\mathfrak{g}^*$ by the definition. Furthermore the dressing action of $G$ on $G^* = \mathfrak{g}^*$ coincides with the coadjoint action $Ad^*$. Thus the map $\mu : M \to \mathfrak{g}^* = G^*$ is a moment map for the Poisson action $\sigma$. Therefore, by Example [1] the map $\mu : M \to \mathfrak{g}^* \cong T^*G/G$ is a moment map for the quasi-Poisson action $\sigma$ on the quasi-Poisson $(G,0,0)$-manifold $(M,\pi)$.

3 Main Result

In this section, we carry out deformations of symplectic structures on a smooth manifold. We use the moment map theory for quasi-Poisson actions for it. A moment map for the quasi-Poisson action on a quasi-Poisson $G^h_D$-manifold $(M,\pi^h_M)$ are defined with the conditions for the family of quasi-Poisson $G^h_D$-structures $\{\pi^h_M\}_{b'}$ on $M$. For each complement $b'$, there exists
a twist $t$ in $\Lambda^2\mathfrak{g}$ such that $\mathfrak{h}' = h_t$, so that the family $\{\pi_M^{h_t'}\}_{h_t'}$ is regarded as the family parametrized by twist, $\{\pi_M^{h_t}\}_{t \in \Lambda^2\mathfrak{g}}$. When the quasi-Poisson $G^\mathfrak{h}_D$-structure with twist $t = 0$ is induced by a given symplectic structure, we will give the method of finding a quasi-Poisson $G^\mathfrak{h}_D$-structure which induced a symplectic structure in $\{\pi_M^{h_t}\}_t$. That is, we can deform a given symplectic structure to a new one by a twist $t$. This deformation can be carried out due to using the family $\{\pi_M^{h_t}\}_t$ as moment map conditions for quasi-Poisson actions. In this regard, it is described as follows in [1]: It would be interesting to find a geometric framework for considering the family $\{\pi_M^{h_t}\}_t$. Our deformation is one of the answers for this proposal.

Let $(M,\omega)$ be a symplectic manifold on which an $n$-dimensional connected Lie group $G$ acts by symplectic-Hamiltonian action $\sigma$ with a moment map $\mu: M \to g^*$. Let $\pi$ be the nondegenerate Poisson structure on $M$ induced by $\omega$. Then $\mu$ is a moment map for the quasi-Poisson-Hamiltonian action $\sigma$ of $(G, 0, 0)$ on $(M, \pi)$ by Example 2 in Section 2.

Let $(\mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)$ be the Manin triple corresponding to the trivial Poisson Lie group $(G, 0)$, where $\mathfrak{g} \oplus \mathfrak{g}^*$ has the Lie bracket

\[
[X, Y] = [X, Y]_\mathfrak{g}, \quad [X, \xi] = \text{ad}_X^* \xi, \quad [\xi, \eta] = [\xi, \eta]_{\mathfrak{g}^*} = 0
\]

for any $X, Y$ in $\mathfrak{g}$ and $\xi, \eta$ in $\mathfrak{g}^*$. Here the bracket $[\cdot, \cdot]_\mathfrak{g}$ and $[\cdot, \cdot]_{\mathfrak{g}^*}$ are the brackets on $\mathfrak{g}$ and $\mathfrak{g}^*$ respectively. Then the Manin (quasi-)triple $(\mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)$ defines $F := F_{\mathfrak{g}^*} = 0$ and $\varphi := \varphi_{\mathfrak{g}^*} = 0$ (see (6) and (7)). Since the corresponding quasi-triple $(T^*G, G, \mathfrak{g}^*)$ is complete by Example 1 and 2, an isotropic complement $\mathfrak{g}^*$ is admissible at any $\xi$ in $\mathfrak{g}^*$ by Definition 5, and hence it is admissible at any $\xi$ in $\mu(M)$.

Let $\mathfrak{g}^*_t$ be an isotropic complement of $\mathfrak{g}$ in $\mathfrak{g} \oplus \mathfrak{g}^*$ with a twist $t$ in $\Lambda^2\mathfrak{g}$ from $\mathfrak{g}^*$. When we deform $\pi$ to $\pi_M^t := \pi - t_M$ by a twist $t$, the quasi-Poisson Lie group $(G, 0, 0)$ is deformed to $(G, \pi^t_G, \varphi_{\mathfrak{g}^*_t})$, where $\pi^t_G = t^L - t^R$ and $\varphi_{\mathfrak{g}^*_t} = \frac{1}{2}[t, t] + \varphi_t$ by (12) and (13). Moreover it follows from $F = 0$ and (16) that $\varphi_t = 0$. So $\varphi_{\mathfrak{g}^*_t} = \frac{1}{2}[t, t]$.

On the other hand, it follows from Definition 4 that the quasi-Poisson $(G, \pi^t_G, \varphi_{\mathfrak{g}^*_t})$-manifold $(M, \pi^t_M)$ satisfies

\[
\frac{1}{2} [\pi^t_M, \pi^t_M] = (\varphi_{\mathfrak{g}^*_t})_M, \quad (21)
\]

\[
\Sigma_{X_M} \pi^t_M = F_{\mathfrak{g}^*_t}(X)_M. \quad (22)
\]
If \((\varphi_{G^*})_M = 0\), i.e., \([t, t]_M = 0\), then the 2-vector field \(\pi^t_M\) is a Poisson structure on \(M\) by (21).

Assume that a twist \(t \in \Lambda^2 g\) is an \(r\)-matrix, namely that \([t, t]\) is \(\pi\)-invariant. Then \(\pi^t_G = t^L - t^R\) is a multiplicative Poisson structure (see [6]). Therefore \((G, \pi^t_G)\) is a Poisson Lie group. Then it follows that \(F_{G^*}\) coincides with the dual of the bracket map \([\cdot, \cdot]_{\pi^t_G} : g^* \wedge g^* \rightarrow g^*\) on \(g^*\) defined by the Poisson Lie group \((G, \pi^t_G)\). In fact, by the relation (19), we have

\[
F_{G^*}^t(\xi, \eta) = \text{ad}_{\partial_t(\xi)} \eta - \text{ad}_{\partial_t(\eta)} \xi,
\]

where \(\langle t^\xi(\xi), \eta \rangle := t(\xi, \eta)\). And the bracket on \(g^*\) induced by a multiplicative Poisson structure defined by an \(r\)-matrix is represented by the right-hand side of (23) (see [5], Ex.2.19). Therefore, since \(G\) is connected, the condition (22) means that the action \(\sigma\) is a Poisson action of \((G, \pi^t_G)\) on \((M, \pi^t_M)\) under the assumption that \(t\) is an \(r\)-matrix and that \([t, t]_M = 0\).

Let \(\{e_i\}\) be a basis on \(g\), a set \(\{\epsilon^i\}\) be the dual basis of \(\{e_i\}\) on \(g^*\). Then we can write by (15),

\[
g^* = \text{span}\{\epsilon^i + t^{ij} e_j \mid i = 1, \ldots, n\},
\]

where \(t = \frac{1}{2} t^{ij} e_i \wedge e_j \in \Lambda^2 g\). If \(g^*_t\) is admissible at any point in \(\mu(M)\), then it satisfies \(\text{Im}(\pi^t_p) = \text{Im}(\pi^t_M)_p\) for any \(p \in M\) by Theorem 2.2. The nondegeneracy of \(\pi\) means that \(\text{Im}(\pi^t_p) = T_p M\) for any \(p \in M\). Therefore, by the fact that \(\text{Im}(\pi^t_M)_p = T_p M\) for any \(p \in M\), a quasi-Poisson structure \(\pi^t_M\) is also nondegenerate.

Here we shall examine the condition for a isotropic complement to be admissible at a point in \(g^*\) in more detail. Let \((\xi_i)\) be the linear coordinates for \(\{\epsilon^i\}\). Then it follows that for \(i = 1, \ldots, n\),

\[
(\epsilon^i + t^{ij} e_j)_{g^*} = -\frac{\partial}{\partial \xi_i} + t^{ij} c_{ij}^k \xi_k \frac{\partial}{\partial \xi_i} = -t^{ij} \sum_{l \neq i} c_{lj}^k \xi_k \frac{\partial}{\partial \xi_i} - (1 + t^{ij} c_{ij}^k \xi_k) \frac{\partial}{\partial \xi_i},
\]

where \(X \mapsto X_{g^*}\), for \(X\) in \(g \oplus g^*\), is the infinitesimal action of the dressing action on \(g^* \cong T^* G / G\). The quasi-triple \((T^* G, G, g^*_t)\) is complete if and only if the elements (25) form a basis on \(T_{\xi}(g^*) \cong g^*\) for any \(\xi = (\xi_1, \ldots, \xi_n)\). Hence this means that the matrix

\[
A_t(\xi) := \begin{pmatrix}
-1 - t^{ij} c_{ij}^k \xi_k & -t^{ij} c_{ij}^k \xi_k & \cdots & -t^{ij} c_{ij}^k \xi_k \\
-t^{ij} c_{ij}^k \xi_k & -1 - t^{ij} c_{ij}^k \xi_k & \cdots & -t^{ij} c_{ij}^k \xi_k \\
\vdots & \vdots & \ddots & \vdots \\
-t^{ij} c_{ij}^k \xi_k & -t^{ij} c_{ij}^k \xi_k & \cdots & -1 - t^{ij} c_{ij}^k \xi_k
\end{pmatrix}
\]

(26)
is regular for any $\xi$. Therefore this is equivalent to $f_t(\xi_1, \ldots, \xi_n) := \det A_t(\xi) \neq 0$. Since the constant term of $f_t$ is $(-1)^n$ and since coefficients of the rest of the term include $t^{ij}$'s, a family of hypersurfaces $\{f_t = 0\}_{t \in \Lambda^2 g}$ in $g^*$ diverges to infinity as $t$ approaches the origin $0$ in $\Lambda^2 g$. If $M$ is compact, then $\mu(M)$ is bounded. So it follows that an intersection of $\{f_t = 0\}$ and $\mu(M)$ is empty for a twist $t$ close sufficiently to the origin. Therefore since $g_t^*$ is admissible on $\mu(M)$, the 2-vector field $\pi^t_M$ is nondegenerate.

Since any nondegenerate Poisson structure on $M$ defines a symplectic structure on $M$, the following theorem holds.

**Theorem 3.1.** Let $(M, \omega)$ be a symplectic manifold on which a connected Lie group $G$ with the Lie algebra $g^*$ acts by a symplectic-Hamiltonian action $\sigma$ and $\mu$ be a moment map for $\sigma$. Then the following holds:

1. If a twist $t$ in $\Lambda^2 g$ satisfies that $[t, t]_M = 0$, then $t$ deforms the Poisson structure $\pi$ induced by $\omega$ to a Poisson structure $\pi^t_M := \pi - t_M$. Moreover, if $t$ is an $r$-matrix, then $\sigma$ is a Poisson action of $(G, \pi^t_G)$ on $(M, \pi^t_M)$, where $\pi^t_G = t^L - t^R$.

2. For a twist $t$ in $\Lambda^2 g$, if the isotropic complement $g^*_t$ is admissible on $\mu(M)$, then $t$ deforms the nondegenerate 2-vector field $\pi$ induced by $\omega$ to a nondegenerate 2-vector field $\pi^t_M$. In particular, if $M$ is compact, then a 2-vector field $\pi^t_M$ is nondegenerate for a twist $t$ close sufficiently to the origin $0$ in $\Lambda^2 g$.

Therefore, if a twist $t$ satisfies the assumptions of both 1 and 2, then $t$ deforms $\omega$ to a symplectic structure $\omega'$ induced by the nondegenerate Poisson structure $\pi^t_M$.

The following theorem gives a sufficient condition for a twist to deform a symplectic structure.

**Theorem 3.2.** Let $(M, \omega)$ be a symplectic manifold on which an $n$-dimensional connected Lie group $G$ acts by a symplectic-Hamiltonian action $\sigma$. Assume that $X, Y$ in $g$ satisfy $[X, Y] = 0$. Then the twist $t = \frac{1}{2} X \wedge Y$ in $\Lambda^2 g$ deforms the symplectic structure $\omega$ to a symplectic structure $\omega_t$. For example, a twist $t$ in $\Lambda^2 h$, where $h$ is a Cartan subalgebra of $g$, satisfies the assumption of the theorem.

**Proof.** For $X$ and $Y$ in $g$, we set

$$X = X^i e_i, \quad Y = Y^j e_j,$$
where \( \{ e_i \}_{i=1}^n \) is a basis on the Lie algebra \( g \). Then since \([X, Y] = X^i Y^j c^k_{ij} e_k = 0\), we obtain the following conditions:

\[
X^i Y^j c^k_{ij} = 0 \quad \text{for any } k,
\]

where \( c^k_{ij} \) are the structure constants of \( g \) with respect to the basis \( \{ e_i \} \).

Moreover, since we have

\[
[t, t] = \left[ \frac{1}{2} X \wedge Y, \frac{1}{2} X \wedge Y \right] = \frac{1}{2} [X, Y] \wedge Y = 0,
\]

the twist \( t \) is an r-matrix such that \([t, t]_M = 0\) obviously. Hence \( \pi^t_M \) is a Poisson structure, and if \( \pi^t_M \) is nondegenerate, then twist \( t \) induces the symplectic structure \( \omega_t \).

We shall show the nondegeneracy of \( \pi^t_M \). Let \( \mu \) be the moment map for a given symplectic-Hamiltonian action \( \psi \). Then the nondegeneracy of \( \pi^t_M \) means that \( g^* \) is admissible at any point in \( \mu(M) \). We prove a stronger condition that the quasi-triple \((T^*G, G, g^*_t)\) is complete.

Let \( \{ \epsilon^i \} \) be the dual basis of \( \{ e_i \} \) on \( g^* \) and \( (\xi_i) \) be the linear coordinates for \( \{ \epsilon^i \} \). Since \( t = \frac{1}{2} X^i Y^j e_i \wedge e_j \), \( g^*_t = \text{span}\{ \epsilon^i + X^i Y^j e_j | i = 1, \ldots, n \} \).

Then it follows that for \( i = 1, \ldots, n \),

\[
(\epsilon^i + X^i Y^j e_j)_{g^*} = -X^i Y^j \sum_{l \neq i} c^k_{lj} \xi_k \frac{\partial}{\partial \xi_l} - (1 + X^i Y^j c^k_{ij} \xi_k) \frac{\partial}{\partial \xi_i}. \tag{27}
\]

The quasi-triple \((T^*G, G, g^*_t)\) is complete if and only if the elements \( (27) \) form a basis on \( T_\xi(g^*) \cong g^* \) for any \( \xi = (\xi_1, \ldots, \xi_n) \). Therefore we shall prove that the matrix

\[
\begin{pmatrix}
-1 - X^1 Y^j c^k_{1j} \xi_k & -X^1 Y^j c^k_{2j} \xi_k & \ldots & -X^1 Y^j c^k_{nj} \xi_k \\
-X^2 Y^j c^k_{1j} \xi_k & -1 - X^2 Y^j c^k_{2j} \xi_k & \ldots & -X^2 Y^j c^k_{nj} \xi_k \\
\vdots & \vdots & \ddots & \vdots \\
-X^n Y^j c^k_{1j} \xi_k & -X^n Y^j c^k_{2j} \xi_k & \ldots & -1 - X^n Y^j c^k_{nj} \xi_k
\end{pmatrix} \tag{28}
\]

is regular. In the case of \( X = 0 \), this matrix is equal to the opposite of the identity matrix, so that it is regular. In the case of \( X \neq 0 \), using \( X^i Y^j c^k_{ij} = 0 \), we can transform the matrix to the opposite of the identity matrix. Thus the matrix \( (28) \) is regular. Therefore \( g^*_t \) is admissible at any point in \( g^* \). That is, \((T^*G, G, g^*_t)\) is complete. \( \square \)
Remark 4. We try to generalize the assumption of Theorem 3.2 and consider $X, Y$ in $\mathfrak{g}$ such that $[X, Y] = aX + bY$ $(a, b \in \mathbb{R})$, that is, the subspace spanned by $X, Y$ is also a Lie subalgebra. We set $t = \frac{1}{2}X \wedge Y$ in $\Lambda^2 \mathfrak{g}$. Since $[t, t] = \frac{1}{2}X \wedge [X, Y] \wedge Y = \frac{1}{2}X \wedge (aX + bY) \wedge Y = 0$,

the twist $t$ is an $r$-matrix such that $[t, t]_M = 0$. Therefore the symplectic action $\psi$ is a Poisson action of $(G, \pi^t_G)$ on $(M, \pi^t_M)$. Then we research whether $\mathfrak{g}^*_t$ is admissible at all points in $\mathfrak{g}^*$. Similarly to the proof of Theorem 3.2, a matrix to check the regularity can be deformed to

$$
\begin{pmatrix}
-1 - (aX^k + bY^k)\xi_k & 0 & \cdots & 0 \\
0 & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & -1
\end{pmatrix}.
$$

Therefore this matrix is regular if and only if

$$-1 - (aX^k + bY^k)\xi_k \neq 0.$$

In the case of $[X, Y] = 0$, by Theorem 3.2, the space $\mathfrak{g}^*_t$ is admissible at all points in $\mathfrak{g}^*$. In the case of $[X, Y] \neq 0$, we shall denote by $\cdot$ the standard inner product on $\mathfrak{g}^* \cong \mathbb{R}^n$. Then the above condition means

$$\eta_{[X, Y]} \cdot \xi \neq -1,$$

where $\eta_X = \sum_k X^k \varepsilon^k$ for $X = X^k e_k$ in $\mathfrak{g}$. Let $\xi'$ be an element which is not orthogonal to $\eta_{[X, Y]}$. By setting

$$\xi := -\frac{\xi'}{\eta_{[X, Y]} \cdot \xi'},$$

we obtain $\eta_{[X, Y]} \cdot \xi = -1$, so that $\mathfrak{g}^*_t$ is not admissible at $\xi$. Eventually, to make sure of the admissibility of $\mathfrak{g}^*_t$, we need check whether such a point $\xi$ is included in $\mu(M)$.

4 Deformations of the canonical symplectic form on $\mathbb{C}P^n$ and $\text{Gr}(r; \mathbb{C}^n)$

In this section, we compute specifically which element $t$ in $\Lambda^2 \mathfrak{g}$ defines a different symplectic structure $\omega_t$ from given one $\omega$ on a smooth manifold. One
example is the complex projective space \((\mathbb{C}P^k, \omega_{FS})\), where \(\omega_{FS}\) is the Fubini-Study form, with an action of \(\text{SU}(k + 1)\), \(k = 1, 2\). Another is \((\mathbb{C}P^n, \omega_{FS})\) with an action of \(\text{SU}(n + 1)\). The other is the complex Grassmannian \((\text{Gr}(r; \mathbb{C}^n), \omega_{KK})\), where \(\omega_{KK}\) is the Kirillov-Kostant form, with an action of \(\text{SU}(k + 1)\).

First we review the relation between \(\text{SU}(n + 1)\) and \(\mathbb{C}P^n\). For any \([z_1 : \cdots : z_{n+1}]\) in \(\mathbb{C}P^n\) and \(g = (a_{ij})\) in \(\text{SU}(n + 1)\), the action is given by

\[
g \cdot [z_1 : \cdots : z_{n+1}] := \left[ \sum_{j=1}^{n+1} a_{1j}z_j : \cdots : \sum_{j=1}^{n+1} a_{n+1,j}z_j \right].
\]

The isotropic subgroup of \([1 : 0 : \cdots : 0]\) is

\[
\text{S(U}(1) \times \text{U}(n)) = \left\{ \begin{pmatrix} e^{i\theta} & O \\ O & B \end{pmatrix} \in \text{SU}(n+1) \middle| \theta \in \mathbb{R}, B \in \text{U}(n) \right\}.
\]

Therefore it follows

\[
\text{SU}(n+1)/\text{S(U}(1) \times \text{U}(n)) \cong \mathbb{C}P^n.
\]

The complex projective space \(\mathbb{C}P^n\) has the coordinate neighborhood system \(\{(U_i, \varphi_i)\}_i\) consisting of \(n + 1\) open sets \(U_i\) given by

\[
U_i := \{[z_1 : \cdots : z_{n+1}] \in \mathbb{C}P^n | z_i \neq 0\},
\]

\[
\varphi_i : U_i \rightarrow \mathbb{C}^n \cong \mathbb{R}^{2n},
\]

\[
[z_1 : \cdots : z_{n+1}] \mapsto \left( \frac{z_1}{z_i}, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_{n+1}}{z_i} \right)
\]

\[
\mapsto \left( \frac{\text{Re}z_1}{z_i}, \frac{\text{Im}z_1}{z_i}, \ldots, \frac{\text{Re}z_{n+1}}{z_i}, \frac{\text{Im}z_{n+1}}{z_i} \right),
\]

for \(i = 1, \ldots, n + 1\). By using this coordinate system, the Fubini-Study form \(\omega_{FS}\) on \(\mathbb{C}P^n\) is defined by setting

\[
\varphi_i^* \left( \frac{i}{2} \partial \bar{\partial} \log \left( \sum_j |z_j|^2 + 1 \right) \right)
\]

on each \(U_i\).

The action of \(\text{SU}(n+1)\) on \((\mathbb{C}P^n, \omega_{FS})\) is a symplectic-Hamiltonian action and its moment map \(\mu\) satisfies

\[
<\mu([z_1 : \cdots : z_{n+1}]), X> = \frac{1}{2} \text{Im} \frac{\langle z_1, \ldots, z_{n+1} \rangle \langle \bar{z}_1, \ldots, \bar{z}_{n+1} \rangle - \langle z_1, \ldots, \bar{z}_{n+1} \rangle \langle \bar{z}_1, \ldots, z_{n+1} \rangle}{\langle z_1, \ldots, z_{n+1} \rangle \langle \bar{z}_1, \ldots, \bar{z}_{n+1} \rangle}
\]
for any \([z_1: \cdots : z_{n+1}]\) in \(\mathbb{CP}^n\) and \(X\) in \(\mathfrak{su}(n + 1)\).

We use

\[
X_{ij} : \text{the (i, j)-element is 1, the (j, i)-element is } -1, \text{ and the rest are 0},
\]

\[
Y_{ij} : \text{the (i, j)- and (j, i)-elements are } i, \text{ and the rest are 0},
\]

\[
Z_k : \text{the } (k, k)\text{-element is } i, \text{ and the } (n + 1, n + 1)\text{-element is } -i
\]

for \(1 \leq i < j \leq n + 1\) and \(k = 1, \ldots, n\), as a basis of \(\mathfrak{su}(n + 1)\) which is defined by a Chevalley basis of the complexified Lie algebra \(\mathfrak{sl}(n + 1, \mathbb{C})\) of \(\mathfrak{su}(n + 1)\). The subspace spanned by \(Z_k\)'s is a Cartan subalgebra of \(\mathfrak{su}(n + 1)\).

We consider the case of \(n = 1\). The complex projective line \(\mathbb{CP}^1\) has the coordinate neighborhood system \(\{(U_1, \varphi_1), (U_2, \varphi_2)\}\) given by

\[
U_i : = \{[z_1 : z_2] \in \mathbb{CP}^1 | z_i \neq 0 \} \ (i = 1, 2),
\]

\[
\varphi_1 : U_1 \to \mathbb{C} \cong \mathbb{R}^2, \ [z_1 : z_2] \mapsto \frac{z_2}{z_1} \mapsto \left( \frac{\text{Re } z_2}{z_1}, \frac{\text{Im } z_2}{z_1} \right),
\]

\[
\varphi_2 : U_2 \to \mathbb{C} \cong \mathbb{R}^2, \ [z_1 : z_2] \mapsto \frac{z_1}{z_2} \mapsto \left( \frac{\text{Re } z_1}{z_2}, \frac{\text{Im } z_1}{z_2} \right).
\]

The Fubini-Study form \(\omega_{FS}\) on \(\mathbb{CP}^1\) is

\[
\omega_{FS} = \frac{dx_1 \wedge dy_1}{(x_1^2 + y_1^2 + 1)^2}
\]

on \(U_1\), where \((x_1, y_1) := \left( \frac{\text{Re } z_2}{z_1}, \frac{\text{Im } z_2}{z_1} \right)\). Then a moment map \(\mu : \mathbb{CP}^1 \to \mathfrak{su}(2)^*\) for the natural action of \(SU(2)\) on \((\mathbb{CP}^1, \omega_{FS})\) is defined by

\[
< \mu([z_1 : z_2]), X > = -\frac{1}{2} \text{Im } \frac{\langle (z_1, z_2), X(z_1, z_2) \rangle}{\langle (z_1, z_2), (z_1, z_2) \rangle}
\]

for any \([z_1 : z_2]\) in \(\mathbb{CP}^1\) and \(X\) in \(\mathfrak{su}(2)\). Then \(e_1 := X_{12}, e_2 := Y_{12}\) and \(e_3 := Z_1\) form a basis of \(\mathfrak{su}(2)\). Let \(\{\varepsilon^i\}\) be the dual basis of \(\mathfrak{su}(2)^*\). We obtain

\[
\mu(x_1, y_1) = \frac{y_1}{1 + x_1^2 + y_1^2} \varepsilon^1 + \frac{x_1}{1 + x_1^2 + y_1^2} \varepsilon^2 + \frac{1 - x_1^2 - y_1^2}{2(1 + x_1^2 + y_1^2)} \varepsilon^3.
\]

Hence \(\mu(\mathbb{CP}^1) \subset \mathfrak{su}(2)^*\) is the 2-sphere with center at the origin and with radius \(\frac{1}{2}\).

Let \((\xi_i)\) be the linear coordinates for \(\{\varepsilon^i\}\). We set \(\mathfrak{g} := \mathfrak{su}(2)\). Any twist \(t\) is an \(r\)-matrix on \(\mathfrak{g}\) because \(e_1 \wedge e_2 \wedge e_3\) is \(\text{ad}\)-invariant. Since \(\mathbb{CP}^1\)
is 2-dimensional, it follows that \([t, t]_{\mathbb{CP}^1} = 0\). Therefore we can deform the Poisson structure \(\pi_{\mathbb{FS}}\) induced by \(\omega_{\mathbb{FS}}\) to a Poisson structure \(\pi^t_{\mathbb{FS}}\) on \(\mathbb{CP}^1\) by \(t\) and the natural action is a Poisson action of \((\text{SU}(2), t^L - t^R)\).

Let \(g^*_t\) be the space twisted \(g^*\) by \(t\) in \(\Lambda^2 g\). We consider what is the condition for \(t\) under which \(g^*_t\) is admissible on \(\mu(\mathbb{CP}^1)\). For any twist\n
\[
    t = \sum_{i < j} \frac{1}{2} \lambda_{ij} e_i \wedge e_j \in \Lambda^2 g \ (\lambda_{ij} \in \mathbb{R}),
\]
we obtain\n
\[
    g^*_t = \text{span}\{\varepsilon^1 + \lambda_{12} e_2 + \lambda_{13} e_3, \varepsilon^2 - \lambda_{12} e_1 + \lambda_{13} e_3, \varepsilon^3 - \lambda_{13} e_1 - \lambda_{23} e_2\}.
\]

We calculate as\n
\[
\begin{align*}
    (\varepsilon^1 + \lambda_{12} e_2 + \lambda_{13} e_3)_{g^*} &= -(1 + 2\lambda_{12} \xi_3 - 2\lambda_{13} \xi_2) \frac{\partial}{\partial \xi_1} - 2\lambda_{13} \xi_1 \frac{\partial}{\partial \xi_2} + 2\lambda_{12} \xi_1 \frac{\partial}{\partial \xi_3}, \\
    (\varepsilon^2 - \lambda_{12} e_1 + \lambda_{13} e_3)_{g^*} &= 2\lambda_{23} \xi_2 \frac{\partial}{\partial \xi_1} - (1 + 2\lambda_{12} \xi_3 + 2\lambda_{23} \xi_1) \frac{\partial}{\partial \xi_2} + 2\lambda_{12} \xi_3 \frac{\partial}{\partial \xi_3}, \\
    (\varepsilon^3 - \lambda_{13} e_1 - \lambda_{23} e_2)_{g^*} &= 2\lambda_{23} \xi_3 \frac{\partial}{\partial \xi_1} - 2\lambda_{13} \xi_3 \frac{\partial}{\partial \xi_2} - (1 - 2\lambda_{13} \xi_2 + 2\lambda_{23} \xi_1) \frac{\partial}{\partial \xi_3}.
\end{align*}
\]

Then \(g^*_t\) is admissible at \(\xi = (\xi_1, \xi_2, \xi_3)\) in \(g^*\) if and only if the matrix\n
\[
    A_t(\xi) = \begin{pmatrix}
        1 + 2\lambda_{12} \xi_3 - 2\lambda_{13} \xi_2 & 2\lambda_{13} \xi_1 & -2\lambda_{12} \xi_1 \\
        -2\lambda_{23} \xi_2 & 1 + 2\lambda_{12} \xi_3 + 2\lambda_{23} \xi_1 & -2\lambda_{12} \xi_2 \\
        -2\lambda_{23} \xi_3 & 2\lambda_{13} \xi_3 & 1 - 2\lambda_{13} \xi_2 + 2\lambda_{23} \xi_1
    \end{pmatrix}
\]

is regular. By computing the determinant of the matrix, we have\n
\[
    f_t(\xi) = \det A_t(\xi) = (1 + 2\lambda_{23} \xi_1 - 2\lambda_{13} \xi_2 + 2\lambda_{12} \xi_3)^2.
\]

So the complement \(g^*_t\) is admissible at \(\xi = (\xi_1, \xi_2, \xi_3)\) if and only if \(1 + 2\lambda_{23} \xi_1 - 2\lambda_{13} \xi_2 + 2\lambda_{12} \xi_3 \neq 0\).

Therefore \(g^*_t\) is admissible on \(\mu(\mathbb{CP}^1)\) if and only if the "non-admissible surface" \(\{\xi = (\xi_1, \xi_2, \xi_3) \in g^* | 1 + 2\lambda_{23} \xi_1 - 2\lambda_{13} \xi_2 + 2\lambda_{12} \xi_3 \neq 0\}\) for \(g^*_t\) and the image \(\mu(\mathbb{CP}^1)\) have no common point. Since \(\mu(\mathbb{CP}^1)\) is the 2-sphere with center at the origin and with radius \(\frac{1}{2}\), we can see that this condition is equivalent to the condition\n
\[
    \lambda_{12}^2 + \lambda_{13}^2 + \lambda_{23}^2 < 1.
\]

From the above discussion, we obtain the following theorem.
Theorem 4.1. If a twist \( t := \sum_{i<j} \frac{1}{2} \lambda_{ij} e_i \wedge e_j \) satisfies \( \lambda_{12}^2 + \lambda_{13}^2 + \lambda_{23}^2 < 1 \), then the Fubini-Study form \( \omega_{FS} \) on \( \mathbb{C}P^1 \) can be deformed by \( t \) in the sense of Section 3.

We shall see an example of twists giving symplectomorphisms on \( \mathbb{C}P^1 \).

Example 3. We use a twist \( t = \frac{1}{2} X_{12} \wedge Y_{12} \) in \( \Lambda^2 \mathfrak{su}(2) \) and a real number \( \lambda \), where \( -1 < \lambda < 1 \). The symplectic structure \( \omega_{FS}^\lambda \) deformed \( \omega_{FS} \) by \( \lambda t \) is written by

\[
\omega_{FS}^\lambda = \left\{ \left( 1 + \frac{1}{2} \lambda \right) (x_1^2 + y_1^2)^2 + 2(x_1^2 + y_1^2) + \left( 1 - \frac{1}{2} \lambda \right) \right\}^{-1} dx_1 \wedge dy_1
\]
on \( U_1 \). Then it follows from an elementary calculation that the symplectic volume \( \text{Vol}(\mathbb{C}P^1, \omega_{FS}^\lambda) \) of \( (\mathbb{C}P^1, \omega_{FS}^\lambda) \) is

\[
\text{Vol}(\mathbb{C}P^1, \omega_{FS}^\lambda) = \begin{cases} 
\pi & (\lambda = 0) \\
\frac{\pi}{\lambda} \log \left| \frac{2 + \lambda}{2 - \lambda} \right| & (\lambda \neq 0).
\end{cases}
\]

Next, we consider a cohomology class of each \( \omega_{FS}^\lambda \). Since \( H^2_{\text{DR}}(\mathbb{C}P^1) = \mathbb{R} \), there exists a real number \( k_\lambda \in \mathbb{R} \) such that \([\omega_{FS}^\lambda] = k_\lambda [\omega_{FS}]\). By integrating, we obtain

\[
k_\lambda = \frac{1}{\lambda} \log \left| \frac{2 + \lambda}{2 - \lambda} \right|, \quad \lambda \neq 0.
\]

where \( \lambda \neq 0 \). Since the function \( k_\lambda \) of \( \lambda \) is smooth, two symplectic structures \( \omega_{FS} \) and \( k_\lambda \omega_{FS}^\lambda \) are cohomologous. In particular, \( (\mathbb{C}P^1, \omega_{FS}) \) and \( (\mathbb{C}P^1, k_\lambda \omega_{FS}^\lambda) \) are symplectomorphic by Moser’s theorem.

Next we shall see deformations of symplectic structures in the case of \( \mathbb{C}P^n \).

Example 4. We consider the case of \( n = 2 \). Since \([Y_{23}, 2Z_1 - Z_2] = 0 \) in \( \Lambda^2 \mathfrak{su}(3) \), we use the twist \( t = \frac{1}{2} \lambda Y_{23} \wedge (2Z_1 - Z_2) \) (\( \lambda \in \mathbb{R} \)) to deform \( \omega_{FS} \). Then \( \omega_{FS} \) is deformed to

\[
\omega_{FS}^t = \omega_{FS} + \frac{\lambda}{\sum_k (x_k^2 + y_k^2) + 1} \left\{ (x_1 y_2 - x_2 y_1) dx_1 \wedge dy_1 \\
+ (x_1^2 - x_2^2) dx_1 \wedge dx_2 + (x_1 y_1 - x_2 y_2) dx_1 \wedge dy_2 \\
+ (x_1 y_1 - x_2 y_2) dy_1 \wedge dx_2 + (y_1^2 - y_2^2) dy_1 \wedge dy_2 \\
- (x_1 y_2 - x_2 y_1) dx_2 \wedge dy_2 \right\}
\]
on \( U_1 \), where \( x_i := \text{Re} \frac{z_i + \bar{z}_i}{2} \) and \( y_i := \text{Im} \frac{z_i - \bar{z}_i}{2} \).
Example 5. The next example is a symplectic toric manifold $\mathbb{C}P^n$ with the torus action:

$$(e^{i\theta_2}, e^{i\theta_3}, \ldots, e^{i\theta_{n+1}}) \cdot [z_1 : \cdots : z_{n+1}] := [z_1 : e^{i\theta_2}z_2 : \cdots : e^{i\theta_{n+1}}z_{n+1}]$$

for any $\theta_i$ in $\mathbb{R}$. The moment map $\mu : \mathbb{C}P^n \to \mathbb{R}^n$ for this action on $(\mathbb{C}P^n, \omega_{FS})$ is

$$
\mu([z_1 : \cdots : z_{n+1}]) := -\frac{1}{2} \left( \frac{|z_2|^2}{|z|^2}, \ldots, \frac{|z_{n+1}|^2}{|z|^2} \right),
$$

where $z = (z_1, \ldots, z_{n+1})$ in $\mathbb{C}^n$. We set $X_1 := (1,0,\ldots,0), \ldots, X_n := (0,\ldots,0,1)$. Since $\mathbb{T}^n$ is commutative, the brackets $[X_i, X_j]$ vanish for all $i$ and $j$. Hence for any $\lambda_{12}$ in $\mathbb{R}$, the twist $t_{12} := \lambda_{12}X_1 \wedge X_2$ deforms $\omega_{FS}$ to a symplectic structure $\omega_{FS}^{t_{12}}$ induced by a Poisson structure $\pi_{FS}^{t_{12}} := \pi_{FS} - (t_{12})_{\mathbb{C}P^n}$ by Theorem 3.2. On the other hand, it follows that the trivial Poisson structure on $\mathbb{T}^n$ is invariant and this action is a symplectic-Hamiltonian action with the same moment map $\mu$. Therefore, by Theorem 3.2 again, the twist $t_{13} := \lambda_{13}X_1 \wedge X_3$ deforms $\omega_{FS}^{t_{12}}$ to $\omega_{FS}^{t_{12}t_{13}} = \omega_{FS}^{t_{12}+t_{13}}$ induced by $(\pi_{FS}^{t_{12}})^{t_{13}} = \pi_{FS}^{t_{12}+t_{13}}$. Then we see that the trivial Poisson structure on $\mathbb{T}^n$ is invariant and that the action is a symplectic-Hamiltonian action with $\mu$. By repeating this operation, it follows that we can deform $\omega_{FS}$ to $\omega_{FS}^t$ for any twist $t = \sum_{i<j} \lambda_{ij}X_i \wedge X_j$ and that the action is a symplectic-Hamiltonian action with $\mu$. On $U_1$, since we obtain

$$(X_i \wedge X_j)_{\mathbb{C}P^n} = y_iy_j \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} - y_ix_j \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j} - x_iy_j \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j} + x_ix_j \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j} \quad (1 \leq i < j \leq n),$$

where $x_i := \text{Re}\frac{z_{i+1}}{z_1}$ and $y_i := \text{Im}\frac{z_{i+1}}{z_1}$, it follows that

$$
\omega_{FS}^t = \omega_{FS} + \sum_{i<j} \frac{\lambda_{ij}}{\left( \sum_k (x_k^2 + y_k^2) + 1 \right)^3} (x_ix_j dx_i \wedge dx_j + x_iy_j dx_i \wedge dy_j + y_ix_j dy_i \wedge dx_j + y_iy_j dy_i \wedge dy_j),
$$

The last example is the complex Grassmannian Gr$(r; \mathbb{C}^n):= SU(n)/(SU(r) \times U(n-r))$ with the Kirillov-Kostant form $\omega_{KK}$. With respect to $\omega_{KK}$, the natural SU$(n)$-action is symplectic-Hamiltonian (9).
Then we consider the following r-matrix of $\text{su}(n)$:

$$t = \frac{1}{4n} \sum_{1 \leq i < j \leq n} X_{ij} \wedge Y_{ij},$$

where the r-matrix $t$ is the canonical one defined on any compact semi-simple Lie algebra over $\mathbb{R}$ (for example, see [3]). This is an r-matrix such that $[t, t] \neq 0$. We show that it satisfies $[t, t]_M = 0$, where $M := \text{Gr}(r; \mathbb{C}^n)$. Since $t$ is an r-matrix, the element $[t, t]$ is ad-invariant by the definition. Therefore $[t, t]$ is Ad-invariant because $\text{SU}(n)$ is connected. By the definition of the $\text{SU}(n)$-action on $\text{Gr}(r; \mathbb{C}^n)$, it follows that

$$[t, t]_M, m = p_* [t, t]^R = p_* R_{gs}[t, t].$$

Because of the Ad-invariance of $[t, t]$, we obtain

$$p_* R_{gs}[t, t] = p_* L_{gs} L_{gs^{-1}} R_{gs}[t, t] = p_* L_{gs} \text{Ad}_{gs^{-1}}[t, t] = p_* L_{gs}[t, t].$$

Let $\mathfrak{h}$ be the Lie algebra of $H$. For any $X$ in $\mathfrak{h}$ and $g$ in $\text{SU}(n)$, we compute

$$p_* L_{gs} X = p_* L_{gs} \frac{d}{ds} \exp sX \bigg|_{s=0} = \frac{d}{ds} (g \exp sX) H \bigg|_{s=0} = \frac{d}{ds} g H \bigg|_{s=0} = 0,$$

where we have used that $\exp sX$ is in $H$ in the third equality. Therefore it holds that $[t, t]_M = 0$ if each term of $[t, t]$ includes elements in $\mathfrak{h}$ as follows.

We notice that

$$\mathfrak{h} = \text{span}_\mathbb{R}\{X_{ij}, Y_{ij}, Z_k | 1 \leq i < j \leq r \text{ or } r+1 \leq i < j \leq n, \text{ and } k = 1, \ldots, n\}.$$

If $X_{ij}, Y_{ij} \in \mathfrak{h}$, then

$$[\cdot, X_{ij} \wedge Y_{ij}] = [\cdot, X_{ij}] \wedge Y_{ij} - X_{ij} \wedge [\cdot, Y_{ij}].$$

So these terms include an element in $\mathfrak{h}$. Hence we investigate terms of the form

$$[X_{ij} \wedge Y_{ij}, X_{kl} \wedge Y_{kl}] = -[X_{ij}, X_{kl}] \wedge Y_{ij} \wedge Y_{kl} - X_{ij} \wedge [Y_{ij}, X_{kl}] \wedge Y_{kl} - Y_{ij} \wedge [X_{ij}, Y_{kl}] \wedge X_{kl} - X_{ij} \wedge X_{kl} \wedge [Y_{ij}, Y_{kl}],$$

20
where $X_{ij}, Y_{ij}, X_{kl}$ and $Y_{kl}$ are not in $\mathfrak{h}$. In the case of $i = k$ and $j = l$, we get

$$[X_{ij}, X_{ij}] = [Y_{ij}, Y_{kl}] = 0,$$

$$[X_{ij}, Y_{ij}] = 2(Z_i - Z_j) \in \mathfrak{h},$$

where $Z_n = 0$. In the case of $i = k$ and $j < l$ (resp. $l < j$), since it follows that $r + 1 \leq j, l \leq n$, we obtain

$$[X_{ij}, X_{kl}] = [Y_{ij}, Y_{kl}] = -X_{jl} (\text{resp. } X_{ij}) \in \mathfrak{h},$$

$$[Y_{ij}, X_{kl}] = [Y_{kl}, X_{ij}] = -Y_{jl} (\text{resp. } Y_{ij}) \in \mathfrak{h}.$$

We can also show the case of $i < k$ (resp. $k < i$) and $j = l$ in the similar way. Therefore all terms of $[t, t]$ include elements in $\mathfrak{h}$, so that $[t, t]_M = 0$. Since $\text{Gr}(r; \mathbb{C}^n)$ is compact, for sufficiently small $|\lambda|$, a 2-vector field $\pi^K_{KK}$ is nondegenerate by Theorem 3.1 where $\pi_{KK}$ is the Poisson structure induced by $\omega_{KK}$. Example 3 is the special case of this example. From the above discussion, we obtain the following.

**Theorem 4.2.** Let $t$ be the above $r$-matrix of $\mathfrak{su}(n)$. Then there exists sufficiently small number $\lambda$ such that the Kirillov-Kostant form $\omega_{KK}$ on $\text{Gr}(r; \mathbb{C}^n)$ can be deformed by a twist $\lambda t$ in the sense of Section 3.

**acknowledgments**

I would like to express my deepest gratitude to Yuji Hirota for leading me into the study of quasi-Poisson theory and my supervisor Yasushi Homma for his helpful advice.

**References**

[1] A. Alekseev and Y. Kosmann-Schwarzbach. Manin pairs and moment maps. *J. Diff. Geom.* **56** (2000) 133–165.

[2] H. Bursztyn and M. Crainic. Dirac geometry, quasi-Poisson actions and $D/G$-valued moment maps. *J. Diff. Geom.* **82** 3 (2009) 501–566.

[3] V. G. Drinfel'd. Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of the classical Yang-Baxter equations. *Soviet Math. Dokl.* **27** (1983) 68–71.
[4] J. -H. Lu. Momentum mappings and reduction of Poisson actions, in Symplectic Geometry, Groupoids and Integrable Systems, P. Dazord and A. Weinstein, eds. *Springer* (1991) 209–226.

[5] J. -H. Lu. Multiplicative and Affine Poisson structures on Lie groups. Berkeley Thesis (1991).

[6] J. -H. Lu and A. Weinstein. Poisson-Lie groups, dressing transformations and Bruhat decompositions. *J. Diff. Geom.* 31 (1990) 501–526.

[7] J. E. Marsden and T. S. Ratiu. *Introduction to Mechanics and Symmetry Second Edition.* (Springer, 2003).

[8] D. Salamon. Uniqueness of Symplectic Structures. arXiv:1211.2940v5

[9] A. C. da Silva. *Lectures on Symplectic Geometry.* (Springer-Verlag, 2006).

[10] I. Vaisman. Lectures on the Geometry of Poisson Manifold. *Birkhaeuser* (1994).