Homology of the curve complex and the Steinberg module of the mapping class group

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Abstract

Harer has shown that the mapping class group is a virtual duality group mirroring the work of Borel-Serre on arithmetic groups in semisimple $\mathbb{Q}$-groups. Just as the homology of the rational Tits building provides the dualizing module for any torsion-free arithmetic group, the homology of the curve complex is the dualizing module for any torsion-free, finite index subgroup of the mapping class group. The homology of the curve complex was previously known to be an infinitely generated free abelian group, but to date, its structure as a mapping class group module has gone unexplored. In this paper we give a resolution for the homology of the curve complex as a mapping class group module. From the presentation coming from the last two terms of this resolution we show that this module is cyclic and give an explicit single generator. As a corollary, this generator is a homologically nontrivial sphere in the curve complex.

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1 Introduction

Let $\Sigma$ denote either the closed genus $g$ surface $\Sigma_g$ or that surface $\Sigma^1_g$ with a marked point. Let $\text{Mod}(\Sigma)$ be the mapping class group of $\Sigma$. Harvey’s [Hv81, §2] complex of curves $\mathcal{C}(\Sigma)$ (see Definition 2.1 below) plays a major role in the study of $\text{Mod}(\Sigma)$ in part because it serves as an analog of the rational Tits building for an arithmetic group in a semisimple $\mathbb{Q}$-group. (See Borel-Serre’s celebrated paper [BS73] and the fundamental work of Harer on homological aspects of the mapping class group, surveyed in [Ha88], for background.) Inspiration for the current study is derived from work by Ash-Rudolph [AR79] and others (cf. [As86], [As94], [Gu00b], [Gu00a], [Tô05]) concerning the dualizing modules of these arithmetic groups.

Harer [Ha86, Theorem 4.1] and Ivanov [Iv87] (in the case of the closed surface) have shown that the mapping class group $\text{Mod}(\Sigma)$ is a virtual duality group in the sense of Bieri-Eckmann [BE73] (see also [Iv02, §6.1], [Br82, §VIII.10], [Bi76, §9]), and that the dualizing module for any torsion-free, finite index subgroup of $\text{Mod}(\Sigma)$ is the reduced homology of the curve complex. The curve complex has the homotopy type of a wedge of infinitely many $(2g-2)$-spheres (see [Ha86, Theorem 3.5] and [IJ07, Theorem 1.4]) so its reduced homology is concentrated in dimension $2g-2$. The left action of the mapping class group on the complex of curves makes the reduced homology of $\mathcal{C}(\Sigma)$ into a $\text{Mod}(\Sigma)$-module. Following Harer [Ha88, §4.1], we will call this $\text{Mod}(\Sigma)$-module the Steinberg module, $\text{St}(\Sigma)$, of the mapping class group (see Definition 2.2 below).

1.1 Summary of results

The purpose of this paper is to initiate an investigation of the (left) $\text{Mod}(\Sigma)$-module structure of $\text{St}(\Sigma)$. For all groups $\Gamma$, we will consider modules (resp. ideals) over the group ring $\mathbb{Z}\Gamma$ to be left $\Gamma$-modules (resp. left ideals). Our results begin in 3 with a resolution of

$$\text{St}(\Sigma^1_g) = \tilde{H}_{2g-2}(\mathcal{C}(\Sigma^1_g); \mathbb{Z})$$

(see Definition 2.2 below).

**Proposition 3.3** Let $(\mathcal{F}_*, \partial)$ be the chain complex defined in equation (5). The exact sequence

$$0 \rightarrow \mathcal{F}_{4g-3} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathcal{F}_2 \xrightarrow{\partial} \mathcal{F}_1 \xrightarrow{\partial} \mathcal{F}_0 \rightarrow \text{St}(\Sigma^1_g) \rightarrow 0$$

is a finite $\text{Mod}(\Sigma^1_g)$-module resolution for the Steinberg module $\text{St}(\Sigma^1_g)$.

From this resolution we have a $\text{Mod}(\Sigma^1_g)$-module isomorphism

$$\text{St}(\Sigma^1_g) \cong \mathcal{F}_0/\partial \mathcal{F}_1$$

which yields the presentation in Proposition 3.5. We use this presentation to prove our main result (see 2-4 below for an explanation of the notation).

**Theorem 4.2** Let $g \geq 1$, and let $\phi_0 \in \mathcal{F}_0$ be the 0-filling system in Figure 5. Let $[\phi_0]$ be the class of $\phi_0$ in $\text{St}(\Sigma^1_g) = \mathcal{F}_0/\partial \mathcal{F}_1$. Then $\text{St}(\Sigma^1_g)$ is generated as a $\text{Mod}(\Sigma^1_g)$-module by $[\phi_0]$. For the closed surface let $[\phi_0]$ be the class of $\phi_0$ in $\text{St}(\Sigma_g) = (\mathcal{F}_0)_p/\partial (\mathcal{F}_1)_p$ (see the proof of Proposition 3.5). Similarly, $\text{St}(\Sigma_g)$ is generated as a $\text{Mod}(\Sigma_g)$-module by $[\phi_0]$. 2
Theorem 4.2 states that the Steinberg module for the mapping class group is a cyclic $\text{Mod}(\Sigma)$-module and gives a specific single generator. It follows immediately that this generator must be nontrivial in $\text{St}(\Sigma) = \tilde{H}_{2g-2}(\mathcal{C}(\Sigma); \mathbb{Z})$.

**Corollary 4.3.** $[\phi_0] \in \text{St}(\Sigma_g^1)$ (resp. $[\phi_0] \in \text{St}(\Sigma_g)$) is nontrivial for $g \geq 1$.

Finally in §5 we consider two examples: the Steinberg modules for the mapping class groups of the surfaces of genus 1 and 2. For these low genus surfaces, Propositions 5.1 and 5.2 convert the multi-generator presentation given in Proposition 3.5 to a presentation based on the single generator from Theorem 4.2.

### 1.2 The curve complex and duality for mapping class groups

The importance of the reduced homology of the curve complex (which we will denote here by $\text{St}$) for the homological structure of the mapping class group comes from fact that it is the dualizing module for every torsion-free, finite index subgroup $\Gamma < \text{Mod}(\Sigma)$. This dualizing module connects the homology and cohomology of $\Gamma$ via the formula

$$H^i(\Gamma; A) \cong H_{d-i}(\Gamma; \text{St} \otimes_\mathbb{Z} A) \quad \text{for any } \Gamma\text{-module } A$$

where $d = 4g - 5$ (resp. $d = 4g - 3$) is the virtual cohomological dimension of the mapping class group $\text{Mod}(\Sigma_g)$ (resp. $\text{Mod}(\Sigma_g^1)$), and $\Gamma$ acts on $\text{St} \otimes_\mathbb{Z} A$ with the diagonal action: $x(s \otimes a) = (xs) \otimes (xa)$ for all $s \in \text{St}$, $a \in A$, and $x \in \mathbb{Z}\text{Mod}(\Sigma)$.

We briefly overview the reason that the reduced homology of the curve complex is the dualizing module for any torsion-free, finite index subgroup of $\text{Mod}(\Sigma)$. The mapping class group acts properly discontinuously on Teichmüller space $\mathcal{T}(\Sigma)$, which is diffeomorphic to $\mathbb{R}^{6g-6+2m}$, where $m$ is the number of marked points on $\Sigma$. For any torsion-free, finite index subgroup $\Gamma < \text{Mod}(\Sigma)$ the quotient of $\mathcal{T}(\Sigma)$ by the properly discontinuous, fixed-point-free action of $\Gamma$ is a manifold. This manifold is not compact but can be compactified either by adding certain “points at infinity” following Ivanov [IV89] or by removing a certain open neighborhood of infinity following Harer [Ha86]. In either case, the universal cover $\tilde{\mathcal{T}}$ of this compactified manifold is called a *bordification* of Teichmüller space. One must then show that $\tilde{\mathcal{T}}$ is contractible and that $\partial \tilde{\mathcal{T}}$ has the homotopy type of a wedge of spheres of dimension $2g - 2$. It then follows [BE73 §6.4] that the cohomological dimension of $\Gamma$ is $6g - 6 + 2m - (2g - 2) - 1$ and the dualizing module of $\Gamma$ is $\tilde{H}_{2g-2}(\partial \tilde{\mathcal{T}}; \mathbb{Z})$. Harer establishes these results for surfaces with boundary or punctures in [Ha86]. In addition, he shows [Ha86 Lemma 3.2] that $\partial \tilde{\mathcal{T}}$ is $\text{Mod}(\Sigma)$-equivariantly homotopy equivalent to the curve complex. Ivanov [Iv87] establishes the same for closed surfaces.

### 1.3 The Tits building for $\text{SL}(n, \mathbb{Q})$

Throughout this paper we will maintain the viewpoint that the homology of the curve complex should be viewed as an analog for the mapping class group of the homology of the rational Tits building for $\text{SL}(n, \mathbb{Z})$. Briefly, the rational Tits building $\mathcal{B}(n, \mathbb{Q})$ is the simplicial complex of flags
of nontrivial proper subspaces of $Q^n$ (see [Br89, §V.1 Example 1B]). The action of $SL(n,Q)$ on $Q^n$ induces a simplicial action of $SL(n,Q)$ on $B(n,Q)$. For any basis $b$ for $Q^n$ the union of all simplices of $B(n,Q)$ whose vertices are subspaces spanned by nonempty proper subsets of $b$ gives an apartment of $B(n,Q)$. Apartments have the homeomorphism type of an $(n-2)$-sphere. By the Solomon-Tits Theorem (cf. [So69], [Br89, §IV.5 Theorem 2]) the Tits building $B(n,Q)$ has the homotopy type of wedge of infinitely many $(n-2)$-spheres, and $\tilde{H}_{n-2}(B(n,Q);Z)$ is spanned by the homology classes of all apartments. The action of $SL(n,Q)$ is transitive on apartments of $B(n,Q)$ so one sees immediately that $\tilde{H}_{n-2}(B(n,Q);Z)$ is a cyclic $SL(n,Q)$-module.

The Steinberg module for $SL(n,Z)$ is defined [BS73, pg. 437] to be the infinitely generated free abelian group

$$St(n) = \tilde{H}_{n-2}(B(n,Q);Z).$$

Borel and Serre [BS73, Theorem 11.4.2] show that the dualizing module of any torsion-free finite index subgroup of $SL(n,Z)$ is $St(n)$ providing inspiration for Harer’s later work [Ha86] on the mapping class group. The group $SL(n,Z)$ is no longer transitive on apartments of $B(n,Q)$; nevertheless, Ash and Rudolph [AR79] are able to give a reduction argument to show that the homology class of the sphere for any apartment can be written as a linear combination of classes of “integral unimodular” apartments where an integral unimodular apartment is one coming from an integral basis $b$ for $Z^n$. Of course $SL(n,Z)$ is transitive on integral unimodular apartments. It follows that $St(n) = \tilde{H}_{n-2}(B(n,Q);Z)$ is a cyclic $SL(n,Z)$-module. One should view Theorem 4.2 below as an analog for the mapping class group of Ash and Rudolph’s result.

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2 The Steinberg module, the curve complex and the arc complex

Let $\Sigma_g$ be the surface of genus $g$, and let $\Sigma_g^1$ be the surface of genus $g$ with 1 marked point *.

The mapping class group $\text{Mod}(\Sigma_g)$ (resp. $\text{Mod}(\Sigma_g^1)$) is the group of orientation-preserving self diffeomorphisms of $\Sigma_g$ (resp. orientation-preserving self diffeomorphisms of $\Sigma_g^1$ fixing the marked point) modulo diffeomorphisms isotopic to the identity (see [iv02] for a survey). An (essential) curve in $\Sigma_g$ is an isotopy class of the image of an embedding of the circle $S^1$ in $\Sigma_g$ not bounding a disk in $\Sigma_g$. An (essential) curve in $\Sigma_g^1$ is an isotopy class of the image of an embedding of $S^1$ in $\Sigma_g^1 \setminus \{*\}$ not bounding a disk in $\Sigma_g^1$. (The disk may contain the marked point.) Note that for the surface with a marked point, curves may not be isotoped past the marked point. In either surface, a curve system is a set of curves (with isotopy class representatives) which can be made to be disjoint. Since curves are isotopy classes, a curve system cannot have parallel curves. We can partially order curve systems by inclusion.

Definition 2.1 (Harvey [Hv81]). For $g \geq 1$ the curve complex $\mathcal{C}(\Sigma_g)$ (resp. $\mathcal{C}(\Sigma_g^1)$) for the surface $\Sigma_g$ (resp. $\Sigma_g^1$) is the simplicial complex with $n$-simplices corresponding to curve systems with $n+1$
curves and face relation given by inclusion.

Both \( \mathcal{C}(\Sigma_g) \) and \( \mathcal{C}(\Sigma_g^1) \) have the homotopy type of an infinite wedge of spheres of dimension \( 2g - 2 \) (see [Ha86, Theorem 3.5] and [IJ07, Theorem 1.4]). As stated above, the reduced homology of the curve complex (which is concentrated in dimension \( 2g - 2 \)) is fundamental to the structure of the mapping class group as a virtual duality group with virtual cohomological dimension \( 4g - 5 \) for the closed surface and dimension \( 4g - 3 \) for the surface with one marked point [Ha86, Theorem 4.1].

**Definition 2.2.** For \( g \geq 1 \) the *Steinberg module* for the mapping class group \( \text{Mod}(\Sigma_g^1) \) is the \( \text{Mod}(\Sigma_g^1) \)-module

\[
\text{St}(\Sigma_g^1) := \tilde{H}_{2g-2}(\mathcal{C}(\Sigma_g^1); \mathbb{Z}),
\]

and the *Steinberg module* for the mapping class group \( \text{Mod}(\Sigma_g) \) is the \( \text{Mod}(\Sigma_g) \)-module

\[
\text{St}(\Sigma_g) := \tilde{H}_{2g-2}(\mathcal{C}(\Sigma_g); \mathbb{Z}).
\]

Our aim is to investigate the module structure of the Steinberg modules \( \text{St}(\Sigma_g^1) \) and \( \text{St}(\Sigma_g) \). Instead of calculating the homology of the curve complex \( \mathcal{C}(\Sigma_g) \) directly, we will work in the *arc complex* for \( \Sigma_g^1 \) (see Definition 2.5 below). Lemma 2.3 is especially fortunate since no structure analogous to the arc complex is readily available for the surface \( \Sigma_g \).

**Lemma 2.3** (Harer). As \( \text{Mod}(\Sigma_g^1) \)-modules \( \text{St}(\Sigma_g^1) \cong \text{St}(\Sigma_g) \).

This in turn allows an immediate conclusion about the \( \text{Mod}(\Sigma_g^1) \)-module structure of \( \text{St}(\Sigma_g^1) \) which is not at all apparent from direct observation of the action of \( \text{Mod}(\Sigma_g^1) \) on \( \mathcal{C}(\Sigma_g^1) \).

**Corollary 2.4.** The action of \( \text{Mod}(\Sigma_g^1) \) on \( \text{St}(\Sigma_g^1) \) factors through its quotient \( \text{Mod}(\Sigma_g) \).

In light of Lemma 2.3, we will focus exclusively on the \( \text{Mod}(\Sigma_g^1) \)-module structure of \( \text{St}(\Sigma_g^1) \). Instead of calculating the homology of the curve complex \( \mathcal{C}(\Sigma_g^1) \) directly, we will work in the *arc complex* for \( \Sigma_g^1 \) (see Definition 2.5 below). Lemma 2.3 is especially fortunate since no structure analogous to the arc complex is readily available for the surface \( \Sigma_g \).

An (essential) arc in \( \Sigma_g^1 \) is an isotopy class of the image of an embedded loop based at the marked point \( * \) which does not bound a disk in \( \Sigma_g^1 \). An *arc system* is a set of arcs (with isotopy class representatives) which intersect only at the marked point. Arc systems may not have parallel arcs. Henceforth we will make no distinction between an arc system and a set of representatives of each arc intersecting only at the marked point.

**Definition 2.5** (Harer). The *arc complex* \( \mathcal{A} = \mathcal{A}(\Sigma_g^1) \) is the simplicial complex with \( n \)-simplices corresponding to arc systems with \( n + 1 \) arcs and face relation given by inclusion.

An arc system \( \alpha = \{a_0, \ldots, a_n\} \) *fills* \( \Sigma_g^1 \) if the connected components of \( \Sigma_g^1 \setminus \bigcup \alpha \) are all disks. The minimum number of arcs needed to fill \( \Sigma_g^1 \) is \( 2g \). In what follows we will want to keep careful track of the number of arcs in a filling system, so we will say that a filling arc system \( k \)-fills \( \Sigma_g^1 \) if the arc system has \( 2g + k \) arcs. Notice that a \( k \)-filling system cuts the surface into \( k + 1 \) disks.
Definition 2.6 (Harer). The arc complex at infinity \( A_\infty = A_\infty(\Sigma^1_g) \) is the simplicial subcomplex of \( A(\Sigma^1_g) \) which is the union of the simplices of \( A(\Sigma^1_g) \) whose vertex sets do not fill \( \Sigma^1_g \).

Any arc system with fewer than \( 2g \) arcs cannot fill \( \Sigma^1_g \) so \( A_\infty(\Sigma^1_g) \) contains the entire \((2g - 2)\)-skeleton of \( A(\Sigma^1_g) \).

The name “arc complex at infinity” calls for some explanation. A very nice discussion is given in [Ha88, §2]. The idea is this. Using a fundamental construction of Jenkins and Strebel [Sr84] it is possible to associate to each point in Teichmüller space \( T(\Sigma^1_g) \) (the space of finite area marked complete hyperbolic metrics on the surface \( \Sigma^1_g \)) an embedded metric graph in \( \Sigma^1_g \). One can then homeomorphically identify \( T(\Sigma^1_g) \) with \( A(\Sigma^1_g) \) (see [Ha88, §2] for details). Thus one may think of \( A_\infty(\Sigma^1_g) \) as extra “points at infinity” attached to Teichmüller space.

Harer [Ha86, Theorem 3.4] defined a continuous map \( \Psi : A_\infty(\Sigma^1_g) \to C(\Sigma^1_g) \) and showed that it is a homotopy equivalence (see §4.3 below for more on this map). Hence, from Definition 2.2 we get another characterization of the Steinberg module

\[
\text{St}(\Sigma^1_g) \cong \tilde{H}_{2g-2}(A_\infty(\Sigma^1_g); \mathbb{Z}).
\]  

3 A resolution of the Steinberg module

In this section we will give a resolution of the Steinberg module \( \text{St}(\Sigma^1_g) \) as a \( \text{Mod}(\Sigma^1_g) \)-module. The approach here is analogous to Ash’s simplification [As94, §1] of Lee and Szczarba’s resolution of the Steinberg module for \( \text{SL}(n, \mathbb{Z}) \) given in [LS76, §3] (see [Gu00a] for a nice summary).

We will need the following two results of Harer.

Theorem 3.1 (Harer). The arc complex \( A(\Sigma^1_g) \) is contractible.

Theorem 3.2 (Harer). The arc complex at infinity \( A_\infty(\Sigma^1_g) \) is homotopy equivalent to a wedge of spheres of dimension \( 2g - 2 \).

Theorem 3.1 was established in [Ha85, Theorem 1.5] (see [H91] for a concise proof) and Theorem 3.2 was shown in [Ha86, Theorem 3.3] (Hatcher-Vogtmann have simplified the arguments for the key steps in this proof in unpublished work [HV]).

Consider this portion of the long exact sequence of reduced homology groups for the pair of spaces \((A, A_\infty)\).

\[
H_{k+1}(A; \mathbb{Z}) \to H_{k+1}(A/A_\infty; \mathbb{Z}) \to \tilde{H}_k(A_\infty; \mathbb{Z}) \to \tilde{H}_k(A; \mathbb{Z}).
\]  

By Theorem 3.1 the first and last groups in this sequence are trivial for \( k \geq 0 \); consequently,

\[
H_{k+1}(A/A_\infty; \mathbb{Z}) \cong \tilde{H}_k(A_\infty; \mathbb{Z}) \quad \text{for} \quad k \geq 0.
\]  

Now combining equations (1) and (3) we arrive at a very useful description of the Steinberg module,

\[
\text{St}(\Sigma^1_g) \cong H_{2g-1}(A/A_\infty; \mathbb{Z}).
\]
The chain complex \((C_*(\mathcal{A}/\mathcal{A}_\infty; \mathbb{Z}), \partial)\) for cellular homology of the space \(\mathcal{A}/\mathcal{A}_\infty\) will provide us with a resolution for \(\text{St}(\Sigma^1_g)\) as a \(\text{Mod}(\Sigma^1_g)\)-module. We define the chain complex \((\mathcal{F}_*, \partial)\) to be the shifted complex with

\[
\mathcal{F}_k := C_{2g-1+k}(\mathcal{A}/\mathcal{A}_\infty; \mathbb{Z})
\]

and the same boundary maps as \((C_*(\mathcal{A}/\mathcal{A}_\infty; \mathbb{Z}), \partial)\).

**Proposition 3.3.** Let \((\mathcal{F}_*, \partial)\) be the chain complex defined in equation \((5)\). The exact sequence

\[
0 \to \mathcal{F}_{4g-3} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathcal{F}_2 \xrightarrow{\partial} \mathcal{F}_1 \xrightarrow{\partial} \mathcal{F}_0 \to \text{St}(\Sigma^1_g) \to 0
\]

is a finite \(\text{Mod}(\Sigma^1_g)\)-module resolution\(^1\) for the Steinberg module \(\text{St}(\Sigma^1_g)\).

**Proof.** By Theorem 3.2 the arc complex at infinity \(\mathcal{A}_\infty\) has the homotopy type of a wedge of \((2g - 2)\)-dimensional spheres. Therefore

\[
H_k(\mathcal{F}_*) = H_{2g-1+k}(\mathcal{A}/\mathcal{A}_\infty; \mathbb{Z})
\]

\[
\cong H_{2g-2+k}(\mathcal{A}_\infty; \mathbb{Z})
\]

\[
= \begin{cases} 
0, & k > 0 \\
\text{St}(\Sigma^1_g), & k = 0.
\end{cases}
\]

In other words, the chain complex \((\mathcal{F}_*, \partial)\) gives a resolution of \(\text{St}(\Sigma^1_g)\).

The maximum number of arcs in an arc system occurs when the arcs comprise the one-skeleton of a one-vertex triangulation of \(\Sigma^1_g\). Using euler characteristic one may then conclude that an arc system has at most \(6g - 3\) arcs so \(\mathcal{A}\) and hence \(\mathcal{A}/\mathcal{A}_\infty\) is \(6g - 4\) dimensional. It follows that \(\mathcal{F}_k = C_{2g-1+k}(\mathcal{A}/\mathcal{A}_\infty; \mathbb{Z}) = 0\) when \(k > 6g - 4 - 2g + 1 = 4g - 3\).

A \(k\)-filling system corresponds to a unique \((2g + k - 1)\)-cell in \(\mathcal{A}/\mathcal{A}_\infty\) with the cell decomposition inherited from \(\mathcal{A}\). When the orientation of this \((2g + k - 1)\)-cell is important we will specify it via an orientation on the unique \((2g + k - 1)\)-simplex in \(\mathcal{A}\) mapping to it. An oriented \(k\)-filling system will be a \(k\)-filling system together with an order on the set of arcs in the system up to alternating permutations. By definition \(\mathcal{F}_k = C_{2g-1+k}(\mathcal{A}/\mathcal{A}_\infty; \mathbb{Z})\) is the set of finite \(\mathbb{Z}\)-linear combinations of oriented \(k\)-filling systems\(^2\).

There are a finite number of topologically distinct ways to glue the sides of polygons to get a one-vertex cell decomposition of a surface of genus \(g\), and \(\text{Mod}(\Sigma^1_g)\) is transitive on cell decompositions of the same topological type. Consequently, as a \(\text{Mod}(\Sigma^1_g)\)-module, \(\mathcal{F}_k\) is generated by a finite number of oriented \(k\)-filling systems.

Note that in general \(\mathcal{F}_k\) is not quite a free \(\text{Mod}(\Sigma^1_g)\)-module since some filling arc systems have nontrivial (but always finite cyclic) stabilizers in \(\text{Mod}(\Sigma^1_g)\).

**Remark 3.4.** There are two modifications either of which makes the resolution in Proposition 3.3 projective. Firstly, if coefficients are taken in \(\text{QMod}(\Sigma^1_g)\) instead of \(\mathbb{Z}\text{Mod}(\Sigma^1_g)\) then Proposition 3.3 does give a projective resolution of the rational homology of the curve complex (see [Br82], pg.

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\(^1\)Compare with the resolution of the Steinberg module for \(\text{SL}(n, \mathbb{Z})\) given in [As94], §1.

\(^2\)Translational Note: For the mapping class group 0-filling systems play a similar role to that of modular symbols for \(\text{SL}(n, \mathbb{Z})\).
If coefficients in the above presentation are sent to their images under this ring homomorphism then we get

\[ \text{Mod} \sigma \]

By Proposition 3.3 we have the Mod \( \sigma \) module resolution of St(\( \Sigma \)).

For the purposes of this paper we will make do with the resolution as is. Note that the tail of this resolution almost provides us with a Mod(\( \Sigma \))-module presentation for St(\( \Sigma \)). It is not exactly a presentation because \( F_0 \) is not a free Mod(\( \Sigma \))-module. However, it can easily be augmented to give a Mod(\( \Sigma \))-module presentation by artificially adding stabilizer relations which account for the accidental symmetries of certain 0-filling systems.

**Proposition 3.5.** Let \( g \geq 1 \). Choose a set of oriented representatives \( G = \{ \phi_0, \ldots, \phi_n \} \) of each Mod(\( \Sigma \))-orbit of the set of 0-filling systems. For each \( \phi \in G \) let \( h_i \in \text{Mod}(\Sigma) \) be a generator of the stabilizer of the arc system for \( \phi \), and let \( e_i = \pm 1 \) be the sign of the permutation that the mapping class \( h_i \) induces on this set of arcs. Also choose a set of oriented representatives \( \{ \rho_0, \ldots, \rho_m \} \) of each Mod(\( \Sigma \))-orbit of the set of 1-filling systems. St(\( \Sigma \)) has a presentation

\[
\text{St}(\Sigma) \cong \langle \phi_0, \ldots, \phi_n \mid \partial \rho_0, \ldots, \partial \rho_m, (1 - e_0 h_0)\phi_0, \ldots, (1 - e_n h_n)\phi_n \rangle.
\]

Forgetting the marked point gives a surjective ring homomorphism

\[
\mathbb{Z}\text{Mod}(\Sigma) \twoheadrightarrow \mathbb{Z}\text{Mod}(\Sigma).
\]

If coefficients in the above presentation are sent to their images under this ring homomorphism then we get a Mod(\( \Sigma \))-module presentation

\[
\text{St}(\Sigma) \cong \langle \phi_0, \ldots, \phi_n \mid \partial \rho_0, \ldots, \partial \rho_m, (1 - e_0 h_0)\phi_0, \ldots, (1 - e_n h_n)\phi_n \rangle.
\]

**Proof.** By Proposition 3.3 we have the Mod(\( \Sigma \))-module isomorphism

\[
\text{St}(\Sigma) \cong \frac{F_0}{\partial F_1}.
\]

By definition \( F_0 \) is spanned by oriented 0-filling systems as a \( \mathbb{Z} \)-module. Every oriented 0-filling system is of the form \( \pm h_0 \phi_i \) for some \( \phi_i \in G \) and \( h \in \text{Mod}(\Sigma) \). Hence \( G \) spans \( F_0 \) as a Mod(\( \Sigma \))-module. The only \( \mathbb{Z}\text{Mod}(\Sigma) \)-linear dependencies in the set \( G \) arise from stabilizers of 0-filling systems. The arc system for a 0-filling system cuts the surface into a single \( 4g \)-gon so its stabilizer must be a subgroup of the rotational symmetries of the regular \( 4g \)-gon. Hence, we have the presentation

\[
F_0 \cong \langle \phi_0, \ldots, \phi_n \mid (1 - e_0 h_0)\phi_0, \ldots, (1 - e_n h_n)\phi_n \rangle.
\]  

and the presentation for \( \text{St}(\Sigma) \) then follows.

For the closed surface \( \Sigma \) we do not have a resolution for \( \text{St}(\Sigma) \) as a Mod(\( \Sigma \))-module; however, we do get a Mod(\( \Sigma \))-module presentation for \( \text{St}(\Sigma) \) by taking co-invariants (defined below) in the presentation for \( \text{St}(\Sigma) \). Let \( P \subset \text{Mod}(\Sigma) \) be the point pushing subgroup of Mod(\( \Sigma \)); that is, the kernel of \( \mu \) in the Birman Exact Sequence [Bi69]

\[
1 \rightarrow \pi_1(\Sigma) \rightarrow \text{Mod}(\Sigma) \overset{\mu}{\twoheadrightarrow} \text{Mod}(\Sigma) \rightarrow 1 \quad (\text{for } g \geq 2)
\]  

\[ (8) \]
or for $g = 1$ the trivial kernel of $\mu$ in the exact sequence

$$1 \to \text{Mod}(\Sigma_1^1) \xrightarrow{\mu} \text{Mod}(\Sigma_1) \to 1$$

(8)

The $P$-co-invariants $M_P$ of a $\text{Mod}(\Sigma_1^g)$-module $M$ are the quotient of $M$ by the $P$-submodule generated by the set $\{pm - m | p \in P, m \in M\}$. The $P$-co-invariants $M_P$ have a $\text{Mod}(\Sigma_1^g)$-module structure [Br82, §II.2 Problem 3].

The exact sequence

$$F_1 \to F_0 \to \text{St}(\Sigma_1^1) \to 0$$

remains exact [Br82, §II.2] after taking $P$-co-invariants to get

$$(F_1)_P \to (F_0)_P \to \text{St}(\Sigma_1^1)_P \to 0.$$ 

By Corollary 2.4 and Lemma 2.3 we have $\text{Mod}(\Sigma_1^g)$-module isomorphisms

$$\text{St}(\Sigma_1^1)_P \cong \text{St}(\Sigma_1^1) \cong \text{St}(\Sigma_1).$$

Hence, as a $\text{Mod}(\Sigma_1^g)$-module $\text{St}(\Sigma_1)$ satisfies

$$\text{St}(\Sigma_1) \cong \frac{(F_0)_P}{\partial(F_1)_P}.$$ 

All that is left is to observe that taking the $P$-co-invariants of $F_0$ achieves the same effect as sending the coefficients in the presentation in $\mathbb{C}$ to their images in $\mathbb{Z}\text{Mod}(\Sigma_1^g)$. 

4 The Steinberg module is cyclic

Solomon-Tits proved two theorems [So69]. First the Tits building $B(n, \mathbb{Q})$ for $\text{SL}(n, \mathbb{Q})$ has the homotopy type of a wedge of spheres of dimension $n - 2$. Second, the $\text{SL}(n, \mathbb{Q})$-module $\text{St}(n) = \tilde{H}_{n-2}(B(n, \mathbb{Q}); \mathbb{Z})$ is a cyclic $\text{SL}(n, \mathbb{Q})$-module generated by the $(n - 2)$-sphere coming from a single apartment. In analogy with the first Solomon-Tits Theorem, Harer has shown that the curve complex has the homotopy type of a wedge of $(2g - 2)$-spheres. In this section we will prove the analog of the second in Theorem 4.2 below which states that as a $\text{Mod}(\Sigma)$-module the Steinberg module $\text{St}(\Sigma)$ is generated by a single element.

Actually Theorem 4.2 more closely resembles a result of Ash-Rudolph [AR79, Theorem 4.1] which implies that the reduced homology of the Tits building $B(n, \mathbb{Q})$ for $\text{SL}(n, \mathbb{Q})$ is a cyclic $\text{SL}(n, \mathbb{Z})$-module. The action of $\text{SL}(n, \mathbb{Z})$ is no longer transitive on apartments of $B(n, \mathbb{Q})$, so one must rely on a reduction process to rewrite the homology class of the sphere for an arbitrary apartment as a sum of homology classes of spheres of “integral unimodular” apartments.

The arc complex at infinity does not come with any apartment structure, but 0-filling systems come in a finite number of types based on their $\text{Mod}(\Sigma_1^1)$-orbits. Proposition 4.6 below gives a reduction algorithm to write the class of any oriented 0-filling system in the Steinberg module for the mapping class group as a linear combination of classes of oriented 0-filling systems of a single type.

In order to state and prove Theorem 4.2 we will first discuss a notational convenience which will allow us to keep track of complicated filling arc systems.
4.1 Chord diagrams

Already in genus two it is difficult to keep track of filling arc systems and their symmetries when drawn on surfaces. Chord diagrams provide a convenient notation for this. (See [Mo94, Figure 9] and the related discussion for an introduction.)

An (unlabelled) chord diagram (see Figure 2, left) will be a regular $2n$-gon (which will always be depicted as a circle) with vertices paired off so that no two adjacent vertices are paired. The pairing will be indicated by $n$ chords in the chord diagram. We will call an edge of the $2n$-gon an outer edge of the chord diagram. A labelled chord diagram (see Figure 1 right) will be a chord diagram in which each chord is labelled on one side by an element of $\pi_1(\Sigma_g, \ast)$ and on the other side by its inverse. In practice we will only label one side of each chord with the assumption that the other side is labelled with the inverse element in $\pi_1(\Sigma_g, \ast)$. Two labelled or unlabelled chord diagrams are the same if they can be made to agree after rotation of the $2n$-gon. A labelling of a chord diagram will be called proper if it comes from a filling arc system in the surface $\Sigma_g$. We implicitly require all labellings of chord diagrams to be proper.

In an arc system on the surface $\Sigma_g$, arcs leave and return to a neighborhood of the marked point $\ast$ in a certain cyclic order. To each filling arc system in $\Sigma_g$ with $n$ arcs, associate the labelled chord diagram got by placing a $2n$-gon in a neighborhood of the marked point so that each arc enters and leaves the $2n$-gon at a vertex. For each arc in the arc system connect the two vertices of the $2n$-gon on that arc with a chord. The two sides of the chord correspond to the two sides of the arc. Label each side of the chord by the element of $\pi_1(\Sigma_g, \ast)$ got by following the arc on the corresponding side in a counter-clockwise direction (see Figure 1).

![Figure 1: A filling arc system (left) and the corresponding labelled chord diagram (right).](image)

We will do most of our calculations using chord diagrams, so it will be convenient to be able to recover certain characteristics of the surface and embedded filling arc system corresponding to a given chord diagram. Firstly we briefly explain how to reconstruct the surface. From an $n$-chord diagram we may construct a surface with a certain embedded, one-vertex graph as follows. A fat graph is a regular neighborhood of a graph embedded in a surface together with the embedding of the graph, or equivalently, a graph together with for each vertex of the graph a cyclic order on the termini of the edges incident with that vertex. A chord diagram specifies a one-vertex fat graph with one edge for each chord where the cyclic order on the edge termini corresponds to the cyclic order on endpoints of the chords in the diagram (see Figure 2).
Now we may glue disks to each of the $b$ boundary components of the fat graph to get a closed surface in which the edges of the fat graph form a filling arc system. Note that one generally considers the fat graph dual to an arc system [PM07, ABP07], but here our arc system and fat graph agree. It will be useful to know the genus $g$ of this closed surface, which is easily calculated using the number $n$ of edges in the fat graph and the number $b$ of boundary components of the fat graph, to be

$$g = \frac{n + 1 - b}{2}. \quad (9)$$

In particular, this shows that a chord diagram with $2g$ chords corresponds to a 0-filling system in a surface of genus $g$ precisely when the corresponding fat graph has one boundary component.

In light of equation (9) we would like to be able to quickly read off the number $b$ of boundary components in the fat graph corresponding to a given chord diagram. In fact it is easy to see that $b$ is the number of cycles in the chord diagram. A cycle in a chord diagram (see Figure 3) is an alternating sequence of chords and outer edges of the chord diagram got by starting at a point just inside an outer edge of the diagram and walking along in a clockwise direction keeping the outer edge on one’s left until a chord is encountered, turning right and then following the chord keeping it on the left until an outer edge is encountered, turning right and repeating until one returns to the starting point. Notice that for each cycle in a labelled chord diagram the products of the labels in the cycle must be $1 \in \pi_1(\Sigma_g)$, because the corresponding loop in the surface bounds a disk.

Two chords are parallel if along with two outer edges they bound a rectangular cycle. (The left chord diagram in Figure 3 has two parallel chords.) Since we will only be concerned with arc
systems with no parallel arcs, we will not consider chord diagrams with parallel chords.

We may identify $k$-filling systems for the surface $\Sigma 1$ with (properly) labelled chord diagrams with $2g + k$ chords, $k + 1$ cycles and no parallel chords. When the choice of orientation for a $k$-filling system expressed as a chord diagram is important, we will indicate it by marking a starting point on the edge of the diagram with a “•” and then ordering the chords as they are first encountered traveling clockwise around the edge of the diagram. This convention has the disadvantage of sometimes only allowing one of the two possible orientations of a $k$-filling system to be specified. However, we will use it due to its notational economy.

As an oriented $k$-filling system gives an oriented cell in $\mathcal{A}/\mathcal{A}_{\infty}$, we may take its boundary in the chain complex $F_\ast$ by taking the boundary of the corresponding oriented simplex in $\mathcal{A}$ and projecting that boundary back to $\mathcal{A}/\mathcal{A}_{\infty}$. The boundary of a simplex is an alternating sum of the codimension-1 faces. Thus the boundary of a $k$-filling system $\alpha$ will be a linear combination of all the $(k - 1)$-filling systems got from $\alpha$ by removing one arc. Removing certain arcs from $\alpha$ may leave an arc system which no longer fills $\Sigma 1$. The simplex of $\mathcal{A}$ for such an arc system is contained in $\mathcal{A}_{\infty}$ and hence is trivial in $F_{k-1}$. In the language of chord diagrams we will be able to tell when this has happened by counting cycles and applying equation (9).

By the Dehn-Nielsen-Baer Theorem [Iv02], the mapping class group $\text{Mod}(\Sigma 1_g)$ acts as the index 2 subgroup $\text{Aut}(\pi_1(\Sigma))$ contained in the full automorphism group of the fundamental group of the closed surface of genus $g$. This gives a left action of $\text{Mod}(\Sigma 1_g)$ on labelled chord diagrams by modifying labels. This action of $\text{Mod}(\Sigma 1_g)$ preserves the underlying unlabelled chord diagram for a $k$-filling system. Conversely, any homeomorphism of fat graphs extends to a homeomorphism of the surface so the mapping class group is transitive on proper labellings for a fixed unlabelled chord diagram. Thus the $\text{Mod}(\Sigma 1_g)$-orbits of $k$-filling systems correspond precisely to unlabelled chord diagrams with $2g + k$ chords and $k + 1$ cycles.

Example 4.1. Figure 4 depicts the 4 orbits of the action of $\text{Mod}(\Sigma 1_g)$ on the set of 0-filling systems for the surface $\Sigma 1$. These are all 4-chord diagrams with one cycle and (redundantly) no parallel chords.

![Figure 4](image-url): The unlabelled 4-chord diagrams corresponding to the four $\text{Mod}(\Sigma 1_g)$-orbits of 0-filling systems in the genus 2 surface with one marked point.
4.2 A singleton generating set for the Steinberg module

As shown in §3, we have the $\text{Mod}(\Sigma^1_g)$-module isomorphism from Proposition 3.5

$$\text{St}(\Sigma^1_g) \cong \frac{\mathcal{F}_0}{\partial \mathcal{F}_1}.$$ 

Although $\mathcal{F}_0$ and $\mathcal{F}_1$ are finitely generated $\text{Mod}(\Sigma^1_g)$-modules, the number of $\text{Mod}(\Sigma^1_g)$-orbits of 0-filling systems grows very quickly as a function of $g$ (see [HZ86]). It is therefore somewhat surprising that the Steinberg module is generated by the class of a single 0-filling system.

**Theorem 4.2** (The Steinberg module is cyclic). Let $g \geq 1$, and let $\phi_0 \in \mathcal{F}_0$ be the 0-filling system in Figure 5. Let $[\phi_0]$ be the class of $\phi_0$ in $\text{St}(\Sigma^1_g) = \mathcal{F}_0 / \partial \mathcal{F}_1$. Then $\text{St}(\Sigma^1_g)$ is generated as a $\text{Mod}(\Sigma^1_g)$-module by $[\phi_0]$. For the closed surface let $[\phi_0]$ be the class of $\phi_0$ in $\text{St}(\Sigma_g) = (\mathcal{F}_0)_P / \partial (\mathcal{F}_1)_P$ (see the proof of Proposition 3.5). Similarly, $\text{St}(\Sigma_g)$ is generated as a $\text{Mod}(\Sigma_g)$-module by $[\phi_0]$.

Theorem 4.2 will be proved using Propositions 4.6 and 4.5 below.

**Corollary 4.3.** $[\phi_0] \in \text{St}(\Sigma^1_g)$ (resp. $[\phi_0] \in \text{St}(\Sigma_g)$) is nontrivial for $g \geq 1$.

**Proof of Corollary 4.3.** If $[\phi_0]$ were trivial then by Theorem 4.2 the Steinberg module $\text{St}(\Sigma^1_g)$ would be trivial. But then $\text{Mod}(\Sigma^1_g)$ would have a finite index subgroup $\Gamma$ whose dualizing module is trivial. This would imply that $\Gamma$ must be the trivial group and hence that $\text{Mod}(\Sigma^1_g)$ is finite (see [IJ07] for more).

Figure 5: The generator for $\text{St}(\Sigma^1_g)$

**Remark 4.4.** A natural first guess at a single generator for $\text{St}(\Sigma^1_g)$ is the arc system coming from the standard identification of the 4g-gon which corresponds to the labelled chord diagram in Figure 6, but in fact by Proposition 4.5 below, the class of this 0-filling system is trivial in $\text{St}(\Sigma^1_g)$.

A filling arc system describes a decomposition of the surface $\Sigma^1_g$ into polygons. If any of the polygons has more than 3 vertices, one may add an arc connecting two non-adjacent vertices of that polygon to get a filling arc system with one more arc. This same process can be done from

---

x_1  
x_2  
x_3  
x_4  
\vdots

\phi_0

---
the point of view of chord diagrams. Given a \( k \)-filling system \( \alpha \) in \( \Sigma_g^1 \) represented as a labelled chord diagram, the outer edges of \( \alpha \) are partitioned into \( k + 1 \) disjoint sets (corresponding to the polygons mentioned above) according on which of the \( k + 1 \) cycles they belong to. A \((k + 1)\)-filling system can be created by adding a new chord (not parallel to any of the chords of \( \alpha \)) connecting any two outer edges of \( \alpha \) in the same cycle. Note that since a 0-filling system has a single cycle, any new (non-parallel) chord gives a 1-filling system.

Two chords in a chord diagram cross if the cyclic order on their endpoints forces them to. The finest partition of the set of chords in which each pair of crossing chords is in the same set will give the connected components of the chord diagram. For example the chord diagram in Figure 5 is connected, while the chord diagram in Figure 6 has \( g \) connected components and so is disconnected for \( g > 1 \).

**Proposition 4.5.** If \( \alpha \) is a disconnected 0-filling system then \( [\alpha] \) is trivial in \( \text{St}(\Sigma_g^1) = F_0/\partial F_1 \).

**Proof.** Add a chord \( c \) to \( \alpha \) which does not cross any of the chords of \( \alpha \) and such there are nonempty connected components of \( \alpha \) on both sides of \( c \). Let \( \alpha_c \) be the resulting chord diagram. Firstly, we claim that \( \alpha_c \) is a 1-filling system. This will be the case unless the chord \( c \) is parallel to some chord \( c' \) of \( \alpha \). Suppose that such a chord \( c' \) exists. Then \( c' \) crosses the same set of chords as \( c \) so \( c' \) also divides the chord diagram \( \alpha \) into two halves with no communicating chords. It follows that the two sides of \( c' \) must be in different cycles of \( \alpha \) contradicting the fact that 0-filling systems have exactly one cycle.

Now since \( \alpha_c \) is a 1-filling system we may orient it and take its boundary. Next we claim

\[
\partial \alpha_c = \pm \alpha.
\]

By definition \( \partial \alpha_c \) is a linear combination of the 0-filling systems that one can get by removing a single chord of \( \alpha_c \). The chord \( c \) cuts the chord diagram into two halves with no communicating chords, so by the argument above, any diagram with a subset of the chords of \( \alpha_c \) which contains \( c \) has at least two cycles. Equation (10) follows establishing the proposition.

Now we will introduce a subset of the 0-filling systems whose classes generate \( \text{St}(\Sigma_g^1) \) and for which, by Proposition 4.5, the class of each element, save one, is trivial. A chord diagram will
be called *salient* if each of its connected components is of the form given in Figure 7. Of course $\phi_0$ from Figure 5 is the unique connected, salient 0-filling system.

**Proposition 4.6.** $\text{St}(\Sigma^1_g)$ is generated by salient 0-filling systems.

*Proof.* For $n \geq 0$ a chord diagram will be said to have a *salient tail of length* $n$ if a neighborhood of some segment in the boundary of the diagram is of the form pictured in Figure 8. Given a 0-filling system $\alpha$ with a salient tail of length $n$, we can add a chord $c$ to the far right of the salient tail to get a 1-filling system $\alpha_c$ with a salient tail of length $n + 1$. Consider the relation

$$\partial \alpha_c = \pm \alpha + \sum_{i=1}^{2g} \pm \beta_i.$$  

Notice that if any single chord in the salient tail of $\alpha_c$ other than $c$ is removed from $\alpha_c$ then the resulting chord diagram is disconnected, and so by Proposition 4.5 the class of that 0-filling system is trivial in $\text{St}(\Sigma^1_g)$. Thus for every $i$ with $1 \leq i \leq 2g$ the class of the 0-filling system $\beta_i$ is either trivial in $\text{St}(\Sigma^1_g)$ or the chord diagram for $\beta_i$ has a salient tail of length $n + 1$. Repeating this process no more than $2g$ times we may write the class of any 0-filling system as a linear combination of classes of salient 0-filling systems.

With Propositions 4.6 and 4.5 established, the proof of Theorem 4.2 is immediate.

**Remark 4.7.** As a consequence of Theorem 4.2 there is a left ideal

$$\mathcal{J} \subset \mathbb{Z} \text{Mod}(\Sigma^g)$$

of Theorem 4.2. By Proposition 4.6 $\text{St}(\Sigma^1_g)$ is generated by salient 0-filling systems. The element $\phi_0 \in \mathcal{F}_0$ is a representative of the unique $\text{Mod}(\Sigma^1_g)$-orbit of connected, salient 0-filling systems. Thus by Proposition 4.5 $[\phi_0]$ generates $\text{St}(\Sigma^1_g)$.

For the closed surface without a marked point $\Sigma_g$ we have $\text{St}(\Sigma_g) \cong \text{St}(\Sigma^1_g)$ as $\text{Mod}(\Sigma^1_g)$-modules so $[\phi_0]$ generates $\text{St}(\Sigma_g)$ as a $\text{Mod}(\Sigma^1_g)$-module. On the other hand, the $\text{Mod}(\Sigma^1_g)$-module structure of $\text{St}(\Sigma_g)$ factors through $\text{Mod}(\Sigma_g)$ so $[\phi_0]$ generates $\text{St}(\Sigma_g)$ as a $\text{Mod}(\Sigma_g)$-module.

**Remark 4.7.** As a consequence of Theorem 4.2 there is a left ideal

$$\mathcal{J} \subset \mathbb{Z} \text{Mod}(\Sigma_g)$$

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such that the \( \text{Mod}(\Sigma_g) \)-module sequence
\[
0 \rightarrow \mathcal{J} \rightarrow \mathbb{Z}\text{Mod}(\Sigma_g) \rightarrow \text{St}(\Sigma_g) \rightarrow 0 \tag{11}
\]
is exact. In theory for a fixed genus \( g \) the presentation in Proposition 3.3 gives enough information to calculate a finite generating set for \( \mathcal{J} \). We will perform this calculation for the surfaces \( \Sigma_1 \) and \( \Sigma_2 \) in \( \Sigma_g \) below. A more "geometric" understanding of \( \mathcal{J} \) would certainly advance the understanding of \( \text{St}(\Sigma_g) \). At minimum one would like to know the stabilizer in \( \text{Mod}(\Sigma_g) \) of the class \( [\phi_0] \in \text{St}(\Sigma_g) \).

### 4.3 Spheres in the curve complex

As illustrated above it is useful to view the Steinberg module as the reduced homology of the arc complex at infinity. On the other hand, the curve complexes \( C(\Sigma_g) \) and \( C(\Sigma_g^1) \) are more widely studied so we will now explain Harer’s homotopy equivalence \( \Psi \) between the arc complex at infinity and the curve complex. We will use this map to get an explicit nontrivial 2-sphere in the curve complex.

Harer [Ha86, Theorem 3.4] gives a homotopy equivalence from the arc complex at infinity to the curve complex. Harer’s map
\[
\Psi : A_\infty^g(\Sigma_g^1) \rightarrow C^0(\Sigma_g^1)
\]
is simplicial where \( A_\infty^g(\Sigma_g^1) \) denotes the second barycentric subdivision of \( A_\infty(\Sigma_g^1) \) and \( C^0(\Sigma_g^1) \) denotes the first barycentric subdivision of \( C(\Sigma_g^1) \). The map \( \Psi \) is defined on the vertices of \( A_\infty^g(\Sigma_g^1) \), as follows. A vertex \( v \) of \( A_\infty^g(\Sigma_g^1) \) is a nested sequence of non-filling arc systems \( \beta_1 \subset \beta_2 \subset \cdots \subset \beta_k \). For \( 1 \leq i \leq k \) if one removes a small open regular neighborhood of the union of the arcs in \( \beta_i \) from \( \Sigma_g^1 \), one is left with a surface \( \Sigma(i) \subset \Sigma_g^1 \) with nonempty boundary. After omitting redundancies and trivial curves, the boundary components of \( \Sigma(i) \) give a curve system \( C_i \) in \( \Sigma_g^1 \). We define \( \Psi(v) := \bigcup_{i=1}^{k} C_i \) again omitting redundancies. In fact, \( \Psi(v) \) is a curve system since if \( \beta_i \subset \beta_j \) then we can arrange that \( \Sigma(i) \supset \Sigma(j) \) and hence the curves of \( C_i \) can be taken to be disjoint from the curves of \( C_j \). Of course \( \Psi(v) \) is a vertex of \( C^0(\Sigma_g^1) \) whose vertices are curve systems. We then extend \( \Psi \) simplicially; that is, the simplex with vertices \( v_1, \ldots, v_n \) is mapped linearly to the simplex with vertices \( \Psi(v_1), \ldots, \Psi(v_n) \).

The 0-filling system \( \phi_0 \) in Figure 5 gives an arc system whose class in \( H_{2g-1}(A/A_\infty; \mathbb{Z}) \) is nontrivial. From the long exact sequence for the homology of the pair of spaces \( (A, A_\infty) \) we have the isomorphism
\[
H_{2g-1}(A/A_\infty; \mathbb{Z}) \xrightarrow{\partial} \overline{H}_{2g-2}(A_\infty; \mathbb{Z}).
\]
In \( A(\Sigma_g^1) \) the arc system for \( \phi_0 \) gives a \( (2g-1) \)-simplex all of whose proper surfaces are contained in \( A_\infty(\Sigma_g^1) \). The class of \( \partial[\phi_0] \in H_{2g-2}(A_\infty; \mathbb{Z}) \) is represented by the boundary of this \( (2g-1) \)-simplex.

For the surface \( \Sigma_g^1 \) we may compute the image of \( \partial\phi_0 \) under \( \Psi \) directly. Figure 9 gives the image of the vertices of the first barycentric subdivision of \( \partial\phi_0 \). One may easily fill in the images.
for vertices in the second barycentric subdivision by taking unions of the curve systems at the vertices of the first subdivision. The sphere in Figure 9 is nontrivial, but it is slightly unsatisfying in that it is specified in $C^0(\Sigma^1)$ and not $C(\Sigma^1)$. Figure 10 and Proposition 4.8 below provide a homotopic sphere with a nicer description. The reader may recognize the shape in Figure 10 to be the boundary of the dual of the 3-dimensional associahedron (see [Le89]).

**Proposition 4.8.** The $\text{Mod}(\Sigma^1)$-orbit of the homology class of the 2-sphere in $C(\Sigma^1)$ pictured in Figure 10 generates $H_2(C(\Sigma^1);\mathbb{Z})$. Forgetting the marked point gives a 2-sphere in $C(\Sigma_2)$ whose $\text{Mod}(\Sigma_2)$-orbit generates $H_2(C(\Sigma_2);\mathbb{Z})$.

**Proof.** The arc system for the generator $[\phi_0] \in \text{St}(\Sigma^1)$ from Theorem 4.2 is pictured in Figure 1. One may use Harer’s map directly to show that the vertices in the first barycentric subdivision of $\partial \phi_0$ are mapped under $\Psi$ to the curve systems pictured in Figure 9. We will give a simplicial sphere in the unbarycentricly subdivided curve complex which is homotopic to the sphere pictured in Figure 9. One may construct a homotopy equivalence

$$f : C(\Sigma^1) \rightarrow C(\Sigma^2)$$

as follows. Let $Y$ be the set of exactly those curves appearing in Figure 9. Fix some linear order on $Y$ so that each of the curves that appear in Figure 10 is greater than all of those that do not. Let
Figure 10: This is a homologically nontrivial sphere in $C(\Sigma_2)$. Forgetting the marked point gives a homologically nontrivial sphere in $C(\Sigma_2)$.

Let $f : C(\Sigma_2) \rightarrow C(\Sigma_2)$ be the map satisfying the following three properties:

1. The map $f$ fixes the vertices of $C(\Sigma_2)$.
2. For any simplex $\sigma$ of $C(\Sigma_2)$ whose vertices are disjoint from $Y$ the map $f|_{\sigma} : \sigma \rightarrow \sigma$ is the identity.
3. If at least one vertex of the simplex $\sigma$ of $C(\Sigma_2)$ is in the set $Y$ then $f|_{\sigma} : \sigma \rightarrow \sigma$ is the map which is linear on the barycentric subdivision of $\sigma$, fixes the vertices of $\sigma$ and sends the barycenter of $\sigma$ to the greatest vertex of $\sigma$ in the order on $Y$.

The map $f : C(\Sigma_2) \rightarrow C(\Sigma_2)$ is homotopic to the identity. In fact, sending each point of a simplex to weighted averages of itself and its image under $f$ gives the homotopy between $f$ and the identity. It follows that the sphere $f \Psi(\partial \phi_0)$ is homotopic to $\Psi(\partial \phi_0)$. Some of the vertices in Figure 9 coalesce under $f$ and we get the simpler homotopic sphere pictured in Figure 10. We may then conclude that the Mod$(\Sigma_2)$-orbits of sphere in Figure 10 generate $H_2(C(\Sigma_2); \mathbb{Z})$. Forgetting the marked point gives a 2-sphere in $C(\Sigma_2)$ whose Mod$(\Sigma_2)$-orbit generates $H_2(C(\Sigma_2); \mathbb{Z})$ [Ha86, Lemma 3.6].

The number of vertices in the boundary of the first barycentric subdivision of the $(2g - 1)$-
simplex is $2^g - 2$. Hence, for large $g$ the direct approach to producing homologically nontrivial spheres in $\mathcal{C}(\Sigma_1^g)$ given in the proof of Proposition 4.8 is impractical. Moreover, the asymmetry of the arc system for $\phi_0$ is likely to yield a sphere in $\mathcal{C}(\Sigma_1^g)$ which is difficult to even describe in any concise form.

![Diagram](image)

Figure 11: Using these $2g + 2$ curves one can construct a map from the boundary of the dual of the $(2g - 1)$-dimensional associahedron into $\mathcal{C}(\Sigma_1^g)$. Is that map homologically nontrivial?

Using the curves in Figure 11 there is a simple construction of a map of a $(2g - 2)$-sphere into $\mathcal{C}(\Sigma_1^g)$. Let $\Theta$ be the boundary of the dual of the $(2g - 1)$-dimensional associahedron which we now define. Fix a regular $(2g + 2)$-gon $K$. By definition $\Theta$ is the simplicial complex whose vertices are diagonals of $K$ and whose simplices are given by sets of disjoint diagonals in $K$. It is well-known that $\Theta$ is a $(2g - 1)$-dimensional sphere [Le89]. Associate the vertices of $K$ with the $2g + 2$ curves in Figure 11 so that adjacent vertices of $K$ correspond to intersecting curves. Each pair of nonintersecting curves in Figure 11 corresponds to a diagonal of $K$. Let

$$q' : \Theta \to \mathcal{C}^g(\Sigma_1^g)$$

be the simplicial map sending each pair of nonintersecting curves in Figure 11 to the curve system (with at most 4 nontrivial curves) consisting of the boundary curves of the surface got by removing an open regular neighborhood of the union of all the other curves. For each vertex $v$ of $\Theta$ choose a curve $c_v \in q'(v)$ and let

$$q : \Theta \to \mathcal{C}(\Sigma_1^g)$$

be the simplicial map sending $v$ to $c_v$. One may construct a homotopy like the one in the proof of Proposition 4.8 above to show that $q$ and $q'$ are homotopic and hence the homotopy class of $q$ is independent of the choices of $c_v \in q'(v)$.

**Conjecture 4.9.** For $g \geq 1$ the class $[q] \in \tilde{H}_{2g-2}(\mathcal{C}(\Sigma_1^g); \mathbb{Z})$ is nontrivial. (When $g = 1$ the proper picture for Figure 11 consists of 2 pairs of parallel curves.)

By Proposition 4.8 above the conjecture holds for $g = 2$. It also holds for $g = 1$.

\[^3\]Of course the mapping class groups of the surfaces of genus 1 and 2 are somewhat exceptional. (For instance they have nontrivial centers [Iv02] Theorem 7.5.D.)
One-generator presentations for two examples of low genus

Theorem 4.2 shows that the Steinberg module is generated by a single element; however, the presentation of the Steinberg module given in Proposition 3.5 has multiple generators. For the surfaces of genus one and two (with or without marked points) we will give presentations for the Steinberg module based on the single generator from Theorem 4.2.

5.1 The Steinberg module in genus one

A natural first example to consider is $\text{St} (\Sigma_1^1)$. For the surface of genus 1 we have $\pi_1 (\Sigma_1) = \langle x, y | xyx^{-1}y^{-1} \rangle \cong \mathbb{Z}^2$. Identify $x$ with the column vector $(1,0)$ and $y$ with $(0,1)$. When $g = 1$ we do not have the Birman Exact Sequence (7). In fact we have $\text{Mod} (\Sigma_1^1) \cong \text{Mod} (\Sigma_1) \cong \text{Aut}^+ \mathbb{Z}^2$ which we may identify with $\text{SL}(2, \mathbb{Z})$. While the dualizing modules for torsion-free, finite index subgroups of $\text{SL}(2, \mathbb{Z})$ are already well-known, the reader may find the proof of Proposition 5.1 to be illustrative of the content of Proposition 3.5. For this case we have the good fortune to be able to draw the entire arc complex $A (\Sigma_1^1)$ which is a 2-complex with 1-skeleton the Farey graph (see Figure 12). The arc complex at infinity $A_\infty (\Sigma_1^1)$ is the 0-skeleton of this complex.

Figure 12: The arc complex for the surface of genus 1 with 1 marked point

**Proposition 5.1.** Identify the groups

$$\text{Mod} (\Sigma_1) \cong \text{Mod} (\Sigma_1^1) \cong \text{SL}(2, \mathbb{Z})$$

The Steinberg module $\text{St} (\Sigma_1) \cong \text{St} (\Sigma_1^1)$ has a $\text{SL}(2, \mathbb{Z})$-module presentation with free generator $f_0$ and
relations

\[
\begin{bmatrix}
0 & -1 \\
1 & 1
\end{bmatrix} - \begin{bmatrix}
1 & -1 \\
0 & 1
\end{bmatrix} + 1 f_0
\begin{bmatrix}
1 + \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix} f_0.
\]

Proof. By equations (3) and (4) we have

\[ \text{St}(\Sigma_1^1) \cong \tilde{H}_0(A_\infty; \mathbb{Z}) \cong H_1(A/A_\infty; \mathbb{Z}). \]

The resolution of \( \text{St}(\Sigma_1^1) \) given in Proposition 3.3 is of the form

\[ 0 \to F_1 \to F_0 \to \text{St}(\Sigma_1^1) \to 0 \]

where \( F_0 \) is generated by the 0-filling system \( \phi_0 \) in Figure 13 and \( F_1 \) is generated by the 1-filling system \( \rho_0 \). Recall from §4.1 above that we specify an orientation on a \( k \)-filling system by marking a starting point on the edge of a chord diagram with a "•" and then ordering the chords as they are first encountered traveling clockwise around the edge of the diagram.

![Figure 13: The generator (left) and relation (right) for St(\Sigma_1) labelled with π_1(\Sigma_1) = \mathbb{Z}^2.](image)

Proposition 3.5 gives a presentation for \( \text{St}(\Sigma_1^1) \) with the single generator \( \phi_0 \) and two relations. We get the first relation by taking the boundary of \( \rho_0 \).

\[
\partial \begin{array}{c}
\phi_0
\end{array} = \begin{array}{c}
\phi_0
\end{array} - \begin{array}{c}
\phi_0
\end{array} + \begin{array}{c}
\phi_0
\end{array}
\]

\[ 0 = \begin{bmatrix}
0 & -1 \\
1 & 1
\end{bmatrix} [\phi_0] - \begin{bmatrix}
1 & -1 \\
0 & 1
\end{bmatrix} [\phi_0] + [\phi_0] \]

The other relation comes from the fact that \( F_0 \) is not a free \( \text{Mod}(\Sigma_1^1) \)-module. In fact, the arc system for the 0-filling system \( \phi_0 \) has a cyclic stabilizer of order 4 so \( \phi_0 \) satisfies the relation

\[
\phi_0 = - \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix} \phi_0.
\]
The proposition follows. \hfill \Box

### 5.2 The Steinberg module in genus two

Now we will use the same techniques as in the previous section to convert the 4-generator presentations for \(\text{St}(\Sigma_1^1)\) and \(\text{St}(\Sigma_2^1)\) given in Proposition 3.5 to the relatively efficient one-generator presentations given in Proposition 5.2.

Before we can proceed to write down an explicit \(\text{Mod}(\Sigma_1^1)\)-module presentation for \(\text{St}(\Sigma_2^1)\) we must give a set of mapping classes \(\{R, T_0, S_0\}\) which generate \(\text{Mod}(\Sigma_1^1)\) along with their action on

\[\pi_1(\Sigma_2) = \langle x, y, z, w \mid xzw^{-1}z^{-1}y^{-1}x^{-1}yw \rangle\]

where \(x, y, z, w\) are as in Figure 1.

Let \(R \in \text{Mod}(\Sigma_1^1)\) be the mapping class of order 5

\[
R : \begin{align*}
x &\mapsto y, \\
y &\mapsto z, \\
z &\mapsto w, \\
w &\mapsto z^{-1}x^{-1}w^{-1}y^{-1}, \\
z^{-1}x^{-1}w^{-1}y^{-1} &\mapsto x
\end{align*}
\]

(12)

The mapping class \(R\) generates the stabilizer of the arc system for a certain 1-filling system in the \(\text{Mod}(\Sigma_1^1)\)-orbit depicted in the top row and third column of Figure 16. Let \(T_0\) be the Dehn twist indicated in Figure 14. Its action on \(\pi_1(\Sigma_2)\) is given by

\[
T_0 : y \mapsto x^{-1}y
\]

(13)

where \(T_0\) fixes generators not listed. Let \(S_0\) be the Dehn twist indicated in Figure 14. Its action on \(\pi_1(\Sigma_2)\) is

\[
S_0 : x \mapsto ywx.
\]

(14)

For convenience, for \(0 \leq i \leq 4\) set

\[
T_i = R^iT_0R^{-i} \quad \text{and} \quad S_i = R^iS_0R^{-i}.
\]

(15)

**Proposition 5.2.** For mapping class group of the surface of genus 2 with one marked point the Steinberg module \(\text{St}(\Sigma_1^1)\) has a \(\text{Mod}(\Sigma_1^1)\)-module presentation with generator \(f_0\) and relations

\[
(1 + S_0^{-1}R)f_0, \\
(1 + R + R^2 + R^3 + R^4)f_0, \\
(1 - S_0^{-1}T_0S_0T_0T_2S_4^{-1}S_1S_4R)f_0, \\
(1 + S_3T_2T_0^{-1}R^{-1})T_0^{-1}(1 - T_2^{-1})(1 - S_2^{-1})f_0
\]

where the mapping classes \(R, S_i, T_i\) are defined in equations (12-15). Taking the image in \(\mathbb{Z}\text{Mod}(\Sigma_2)\) of the coefficients of the relations above, one gets a \(\text{Mod}(\Sigma_2)\)-module presentation for \(\text{St}(\Sigma_2)\).
Proof. Firstly, by Proposition 4.5 the connected 0-filling systems in Figure 15 provide a sufficient generating set for $St(\Sigma_2)$, and since disconnected 1-filling systems provide relations between disconnected (and hence trivial) 0-filling systems, we need only consider relations coming from connected 1-filling systems. All orbits under the action of $\text{Mod}(\Sigma_2)$ of all such 1-filling systems are pictured in Figure 16.

Let $\phi_0$, $\phi_1$, and $\phi_2$ be the 0-filling systems pictured in Figure 15. Note that $\phi_0$ and $\phi_1$ are free generators since any mapping class fixing the corresponding arc systems must be the identity. On the other hand, $\phi_2$ is not a free generator. Setwise, its corresponding arc system has stabilizer a cyclic subgroup of $\text{Mod}(\Sigma_2)$ of order 8.
Figure 17: A smaller set of $\text{Mod}(\Sigma_1^2)$-orbits of 1-filling systems whose boundaries still provide a sufficient set of relations for $\text{St}(\Sigma_1^2)$ with the generating set from Figure 15.

Figure 18: $\text{Mod}(\Sigma_1^2)$-orbits of 2-filling systems (top rows) exhibiting redundancy of the relation coming from the $\text{Mod}(\Sigma_1^2)$-orbit of 1-filling systems below it (bottom rows).

We could fix a proper labelling for each of the chord diagrams in Figure 16 and compute a presentation from the resulting relations. However, before we introduce the added complexity of labels we will use the exactness of the sequence

$$\mathcal{F}_2 \xrightarrow{\partial} \mathcal{F}_1 \xrightarrow{\partial} \mathcal{F}_0 \quad (16)$$

to eliminate much of the redundancy in the relations induced by these 1-filling systems. (Actually we will only use the fact that $\partial^2 = 0$.) In fact, the smaller set of relations coming from representatives of the $\text{Mod}(\Sigma_1^2)$-orbits of 1-filling systems in Figure 17 suffice which we now show.

Firstly, using each 2-filling system in the top row of Figure 18 (i) we can exhibit the 1-filling system below it as a linear combination of the 1-filling systems in our smaller set in Figure 17.
Now using each 2-filling system in the top row of Figure [18]ii) we can exhibit the 1-filling system below it as a linear combinations of 1-filling systems in Figure [17] and Figure [18]i). Continuing in this manner in for Figure [18]iii-vi) we can derive the remaining relations as consequences of the relations coming from preceding 1-filling systems.

We note that the elimination of redundant relations that lead to the list in Figure [17] was performed in a completely ad hoc manner. It is not clear if this set of relations is minimal or if there are further redundancies. Ideally, one would like a systematic method for eliminating redundant relations. Such a method might yield a one-generator presentation for the Steinberg module for every genus.

Now we will assign arbitrary proper labels to our sufficient set of 1-filling systems given in Figure [17] and use the action of $\text{Mod}(\Sigma_2)$ on $\pi_1(\Sigma_2)$ to convert them into explicit $\text{Mod}(\Sigma_2)$-module relations.
In order to get a proper presentation for \( \text{St}(\Sigma^2_1) \) as a quotient of a free \( \text{Mod}(\Sigma^2_1) \)-module we need one more relation to account for the fact that the 0-filling system \( \phi_2 \) has a nontrivial (but cyclic) stabilizer:

\[
\phi_2 = -S_3 T_2 T_0^{-1} R^{-1} \phi_2
\]

Notice that this relation is satisfied by \( \phi_2 \in \mathcal{F}_0 \) itself, not just the class \( [\phi_2] \in \text{St}(\Sigma^2_1) \).

In summary, we now have the \( \text{Mod}(\Sigma^2_1) \)-module presentation with free generators

\[
f_0, f_1, f_2
\]

and relations

\[
\begin{align*}
(\S_0^{-1} R + 1) f_0, \\
(-R - 1 - R^2 - R^3 - R^4) f_0, \\
(1 - \S_0^{-1} T_0 S_0 T_0 T_2 S_4^{-1} S_1 S_4 R) f_0, \\
f_2 = (T_0^{-1} - T_0^{-1} T_1^{-1}) f_1, \\
f_1 = (1 - S_1^{-1}) f_0, \\
(1 + S_3 T_2 T_0^{-1} R^{-1}) f_2.
\end{align*}
\]

Combining the last three relations to eliminate the generators \( f_1 \) and \( f_2 \) we get the desired one-generator presentation.
6 Questions for further study

The negative of the identity matrix \(-I \in \text{SL}(n, \mathbb{Z})\) stabilizes every subspace of \(\mathbb{Q}^n\) so \(-I\) is in the kernel of the action of \(\text{SL}(n, \mathbb{Z})\) on its Steinberg module \(\text{St}(n)\). Consequently the action of \(\text{SL}(n, \mathbb{Z})\) on \(\text{St}(n)\) factors though an action of the simple group \(\text{PSL}(n, \mathbb{Q})\) (see [La02, Theorem 9.3]). It follows that \(\text{St}(n)\) is a faithful \(\text{PSL}(n, \mathbb{Z})\)-module.

As we have seen in Corollary 2.4 for \(g \geq 2\) the kernel of the action of \(\text{Mod}^1(\Sigma_g)\) on the Steinberg module contains the infinite point pushing subgroup \(P \leq \text{Mod}^1(\Sigma_g)\). For the closed surface without a marked point there is some hope that the action on the mapping class group (modulo its center) on the Steinberg module is faithful. When \(g \in \{1, 2\}\), the hyperelliptic involution generates the center \(Z = Z(\text{Mod}(\Sigma_g))\) of the mapping class group and acts trivially on the curve complex. For \(g \geq 3\) the center \(Z = Z(\text{Mod}(\Sigma_g))\) is trivial [Iv02, Theorem 7.5.D]. Thus for \(g \geq 1\) the action of \(\text{Mod}(\Sigma_g)\) on \(\text{St}(\Sigma_g)\) factors through \(\text{Mod}(\Sigma_g)/Z\).

**Question 6.1.** For \(g \geq 1\) let \(Z = Z(\text{Mod}(\Sigma_g))\) be the center of \(\text{Mod}(\Sigma_g)\). Is \(\text{St}(\Sigma_g)\) a faithful \(\text{Mod}(\Sigma_g)/Z\)-module? If not what is the kernel of the action?

The Solomon-Tits Theorem (cf. [So69], [Br89, §IV.5 Theorem 2]) shows that the Tits building \(B(n, \mathbb{Q})\) has the homotopy type of a wedge of spheres, that its reduced homology is a cyclic module (over the associated \(\mathbb{Q}\)-group), and further gives a \(\mathbb{Z}\)-basis for the reduced homology. Harer has shown that the curve complex has the homotopy type of a wedge of spheres. Theorem 4.2 above shows that the reduced homology of the curve complex is a cyclic \(\text{Mod}(\Sigma_g)\)-module. The full “Solomon-Tits Theorem” for the mapping class group should also provide the following.

**Problem 6.2.** Give a \(\mathbb{Z}\)-basis for \(\text{St}(\Sigma_g) \cong \text{St}(\Sigma_g^1)\).

**Steinberg modules for automorphism groups of free groups.** Let \(F_n\) be the free group on \(n\) generators, and let \(\text{Aut}(F_n)\) and \(\text{Out}(F_n)\) denote its automorphism group and outer automorphism group respectively (see [Vo02] for a survey of these groups). Many of the results on the homological structure of the mapping class group have analogs for \(\text{Out}(F_n)\) and \(\text{Aut}(F_n)\). For example, Bestvina and Feighn [BF00] have shown that \(\text{Out}(F_n)\) and \(\text{Aut}(F_n)\) are virtual duality groups. To date, the module structure of the dualizing modules for (torsion-free, finite index subgroups of) these groups has not been studied in depth.

In the case of \(\text{Aut}(F_n)\) Hatcher and Vogtmann [HV98] have proposed a likely candidate for this dualizing module. They define a simplicial complex based on the poset of free factors of \(F_n\) and show that it has the homotopy type of a wedge of spheres. They define its reduced homology group to be the Steinberg module of \(\text{Aut}(F_n)\), and ask if this module provides the dualizing module for finite index, torsion-free subgroups of \(\text{Aut}(F_n)\). It is possible that the approach of the current paper might be used to address this question.

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