CARROLLIAN AND CELESTIAL SPACES AT INFINITY

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Abstract. We show that the geometry of the asymptotic infinities of Minkowski spacetime (in \(d+1\) dimensions) is captured by homogeneous spaces of the Poincaré group: the blow-ups of spatial (\(\text{Spi}\)) and timelike (\(\text{Ti}\)) infinities in the sense of Ashtekar–Hansen and a novel space \(\text{Ni}\) fibering over \(\mathcal{I}\). We embed these spaces à la Penrose–Rindler into a pseudo-euclidean space of signature \((d+1,2)\) as orbits of the same Poincaré subgroup of \(O(d+1,2)\). We describe the corresponding Klein pairs and determine their Poincaré-invariant structures: a carrollian structure on \(\text{Ti}\), a pseudo-carrollian structure on \(\text{Spi}\) and a “doubly-carrollian” structure on \(\text{Ni}\). We give additional geometric characterisations of these spaces as grassmannians of affine hyperplanes in Minkowski spacetime: \(\text{Spi}\) is the (double cover of the) grassmannian of affine lorentzian hyperplanes; \(\text{Ti}\) is the grassmannian of affine spacelike hyperplanes and \(\text{Ni}\) fibers over the grassmannian of affine null planes, which is \(\mathcal{J}\). We exhibit \(\mathcal{Ni}\) as the fibred product of \(\mathcal{J}\) and the lightcone over the celestial sphere. We also show that \(\mathcal{Ni}\) is the total space of the bundle of scales of the conformal carrollian structure on \(\mathcal{I}\) and show that the symmetry algebra of its doubly-carrollian structure is isomorphic to the symmetry algebra of the conformal carrollian structure on \(\mathcal{J}\); that is, the BMS algebra. We show how to reconstruct Minkowski spacetime from its asymptotic geometries, by establishing that points in Minkowski spacetime parametrise certain lightcone cuts in the asymptotic geometries. We include an appendix comparing with (A)dS and observe that the de Sitter groups have no homogeneous spaces which could play the rôle that the celestial sphere plays in flat space holography.

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1. Introduction

In the search for a quantum theory of gravity it is by now widely assumed that holography will act as our guide in this endeavour [1, 2]. The benchmark result to which all other instances of holography can be compared is clearly the Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence [3, 4, 5]. This correspondence relates the dynamics of a gravitational theory on \((d + 1)\)-dimensional AdS space to a \(d\)-dimensional CFT on its conformal boundary.

Distilled down to its very essentials, one could argue that the AdS/CFT correspondence is a consequence of the existence of two natural spaces for the same symmetry group \(SO(d, 2)\): on the one hand \((d + 1)\)-dimensional AdS space and on the other hand \(d\)-dimensional Minkowski spacetime on which this group acts (locally) by conformal transformations. Both of these are homogeneous spaces of \(SO(d, 2)\), i.e., they are of the form \(SO(d, 2)/H\) where \(H\) is a closed subgroup of \(SO(d, 2)\). These spaces differ in the choice of \(H\) which is the stabiliser group of points in the respective spacetime.

Of course this is an extreme simplification, for the AdS/CFT correspondence is much more than the mere observation of the existence of a lower-dimensional space with the same symmetry group as AdS. Nevertheless, when trying to generalise the holographic principle to asymptotically flat spacetimes even the simple observation in the last paragraph becomes less obvious. In such a foggy situation, it is often the use of symmetries which shines a light on the forward path. Starting from \((d + 1)\)-dimensional Minkowski spacetime with the Poincaré group as its symmetry group, which other \((d + 1)\)-dimensional AdS spaces share the same symmetries? In analogy to the AdS case, these spaces would be potential candidates on which to define the dual theory of an asymptotically flat spacetime. This leads us to the question: What are the homogeneous spaces of the Poincaré group and their geometric properties?

The homogeneous spaces of the \((d + 1)\)-dimensional Poincaré group are determined locally by a Klein pair \((\mathfrak{iso}(d, 1), \mathfrak{h})\) consisting of the Poincaré Lie algebra \(\mathfrak{iso}(d, 1)\) and a Lie subalgebra \(\mathfrak{h}\). The most obvious example is, of course, Minkowski spacetime \(\mathbb{M}\) with Klein pair \((\mathfrak{iso}(d, 1), \mathfrak{so}(d, 1))\), with \(\mathfrak{so}(d, 1)\) the Lorentz subalgebra. A slightly less obvious example is obtained by instead considering the Klein pair \((\mathfrak{iso}(d, 1), \mathfrak{so}(d - 1, 1))\), i.e., by replacing the Lorentz algebra by the \(d\)-dimensional Poincaré algebra \(\mathfrak{iso}(d - 1, 1)\) which is clearly also a subalgebra of \(\mathfrak{iso}(d, 1)\). In contrast to Minkowski space, the Poincaré group acts on the resulting space in a way that does not allow for the construction of a nondegenerate invariant metric. Instead, one finds a pseudo-carrollian structure consisting of a degenerate Lorentzian metric and a distinguished vector field. As we will explain in more detail below, the resulting \((d + 1)\)-dimensional spacetime fibers over \(d\)-dimensional de Sitter space \(d\mathbb{S}\) and the degenerate metric is the pull-back by the
projection of the constant positive curvature metric on $dS_d$. Although the physical significance of this construction appears rather opaque at first sight, it was observed by Gibbons in [6] that this is precisely the universal structure at spatial infinity $\text{Spi}$ of Ashtekar and Hansen’s (AH) [7]. In a generic asymptotically flat spacetime various physical fields acquire direction-dependent limits at the point $i^0$. One therefore considers a blow-up of $i^0$, such that fields at $i^0$ can be regarded as smooth fields on the blow-up. The blow-up is constructed as the space of space-like geodesics approaching $i^0$ with unit tangent vector. The set of all such curves turns out to be parametrised by the homogeneous space discussed above where the $dS_d$-slices parametrise the choices of tangent vectors and the coordinate along the fibre correspond to the tangential acceleration which is not fixed by the construction of [7]. We will therefore refer to the homogeneous space of the Poincaré group with Klein pair $(\mathfrak{iso}(d,1), \mathfrak{iso}(d-1,1))$ as $\text{Spi}$.

The above construction immediately suggests the existence of another homogeneous space of the Poincaré group corresponding to the universal structure at (either future or past) timelike infinity that we will refer to as $\text{Ti}$. In this case the subgroup is isomorphic to the euclidean group in one lower dimension. The homogeneous space is now equipped with a carrollian structure and fibers over $d$-dimensional hyperbolic space $\mathcal{H}^d$ instead of de Sitter space. In fact, the existence of this space was already revealed in the classification of spatially isotropic homogeneous spacetimes [8] (see also [9]) where it was called the anti-de Sitter–Carroll spacetime (henceforth $\text{AdSC}$) and identified with the carrollian limit of $\text{AdS}$.

Looking at the Penrose diagram (cf. Figure 1) of an asymptotically flat spacetime, the appearance of the universal structure at timelike and spacelike infinities as $(d+1)$-dimensional homogeneous spaces of the Poincaré group further suggests the existence of another homogeneous space related to the universal structure at null infinity. While the latter is indeed described by a homogeneous space of the Poincaré group, namely $\mathcal{I}$, it is only of dimension $d$. The above picture is nevertheless completed by an additional $(d+1)$-dimensional space $^{2}\text{Ni}$ fibering over $\mathcal{I}$. We will see that $\text{Ni}$ also fibers over the light-cone and that both the lightcone and $\mathcal{I}$ fiber over the celestial sphere, resulting in a commuting square of fibrations displayed below together with all the other homogeneous spaces under consideration:

\[ \begin{array}{cccccc}
\mathcal{M} & \text{Spi} & \text{Ti} & \text{Ni} & d + 1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
\mathcal{dS} & \mathcal{H} & \mathcal{I} & \mathcal{L} & d \\
\downarrow & \downarrow & \downarrow & \\
\mathcal{CS} & d - 1 
\end{array} \]

where $\mathcal{L}$ is either the future or past lightcone (without the apex) and $\mathcal{CS}$ is the celestial sphere. To the right of the square we have denoted their dimensions, which shows that $\text{Spi}$ and $\text{Ti}$ do not have the conventional interpretation as a boundary of one lower dimension. As we will see, all manifolds in (1.1) admit transitive actions of the Poincaré group; although the action is not effective for $\mathcal{L}$ and $\mathcal{CS}$, where the translations act trivially. While the Poincaré-invariant structures of $\text{Ti}$ and $\text{Spi}$ are (pseudo)carrollian, that of $\text{Ni}$ is a novel carrollian-like structure which involves two invariant vectors and a corank-two degenerate metric. We tentatively dub this structure a “doubly-carrollian”

\footnote{In the seminal work [10], the corresponding kinematical Lie algebra was termed a “para-Poincaré” algebra, but we will not use that terminology here.}

\footnote{For the avoidance of doubt, let us emphasise that despite the spelling, $\text{Ni}$, just like $\text{Ti}$ and $\text{Spi}$, is pronounced to rhyme with $\mathcal{I}$, and not with “knee” [11].}
structure, by analogy with the fibration $\mathcal{L} \to \mathcal{CS}$. Concretely, one observes that the carrollian structure on $\mathcal{L}$ arises naturally from interpreting $\mathcal{L}$ as the total space of the bundle of scales of the conformal structure of $\mathcal{CS}$. In the same way, the doubly-carrollian structure of $\mathcal{Ni}$ arises naturally from interpreting $\mathcal{Ni}$ as the total space of the bundle of scales of the conformal carrollian structure of $\mathcal{I}$, as discussed in Section 5.5. Consistent with this interpretation is the fact that the symmetries of the doubly-carrollian structure of $\mathcal{Ni}$, determined in Appendix C, agree with the BMS symmetries [14, 15], which are the symmetries of the conformal carrollian structure on $\mathcal{I}$. Indeed, we claim that the symmetries of the Poincaré-invariant structures of $\mathcal{Ni}$, $\mathcal{Spi}$ and $\mathcal{Ti}$ capture precisely the expected asymptotic symmetries of flat space. The explicit Klein pairs of all the aforementioned homogeneous spaces and their symmetries are summarised in Tables 1, 2 and 3, which might provide useful orientation.

Figure 1. The Penrose diagram of Minkowski spacetime $\mathcal{M}$ with its hyperbolic slicing. We have also illustrated how $\mathcal{Ti}$ and $\mathcal{Spi}$ arise as the blow-ups of, respectively, timelike and spacelike infinities, while $\mathcal{Ni}$ fibers over $\mathcal{I}$ and can be understood as the bundle of scales of the conformal carrollian structure of $\mathcal{I}$.

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3The doubly-carrollian structure bears a superficial resemblance to so-called stringy carrollian structures encountered in a “string Carroll geometry” [12], which is the (much less studied) carrollian counterpart of string Newton–Cartan geometry [13]. We stress, however, that they are not the same.
 Whereas $\mathrm{Ti}$, $\mathrm{Spi}$ and $\mathrm{Ni}$ might seem to be rather abstract, remarkably they, together with $\mathbb{M}$, embed simultaneously into a pseudo-euclidean space $\mathbb{E}^{d+1,2}$ of signature $(d + 1, 2)$ as orbits of the same Poincaré subgroup of $O(d + 1, 2)$. This extends the well-known embedding of four-dimensional Minkowski spacetime into $\mathbb{E}^6$ described, for example, in [16, Section 9.2]. Furthermore, any (non-trivial) orbit of the Poincaré group in this pseudo-Euclidean space takes the form of one of the above $(d + 1)$-dimensional homogeneous spaces. This embedding picture provides what is arguably the simplest description of the spaces $\mathbb{M}_{d+1}$, $\mathrm{Ti}_{d+1}$, $\mathrm{Spi}_{d+1}$ and $\mathrm{Ni}_{d+1}$ and also shares intriguing similarities with the embedding picture originally due to Dirac [17] and used recently in the AdS/CFT correspondence. We will elaborate on this in the conclusions.

As we will demonstrate, both $\mathrm{Ti}$ and (a $Z_2$ quotient of) $\mathrm{Spi}$ can also be interpreted as the grassmannians of affine spacelike and lorentzian hyperplanes in Minkowski spacetime, respectively. Mirroring the discussion around (1.1), the grassmannian of affine null hyperplanes is $d$-dimensional and may be identified with $\mathcal{I}$, whereas the $(d + 1)$-dimensional space $\mathrm{Ni}$ can instead be viewed as the space of pairs of null vectors in Minkowski spacetime. Conversely, the embedding picture allows us to show that $\mathbb{M}$ parametrises certain geometrical objects in these other spaces; in other words, we may reconstruct Minkowski spacetime from any of its associated homogeneous geometries. For instance, the embedding space picture allows us to show how certain hypersurfaces in $\mathrm{Ti}$, $\mathrm{Spi}$ and $\mathrm{Ni}$ correspond to points in Minkowski spacetime. This should be compared to the so-called good cuts [18, 19] that allow to reconstruct Minkowski space, or more generally asymptotically flat spacetimes, from certain codimension-one sections of null infinity.

This paper is organised as follows. We start in Section 2 with arguably the simplest description of the $(d + 1)$-dimensional homogeneous spaces $\mathbb{M}$, $\mathrm{Ti}$, $\mathrm{Spi}$ and $\mathrm{Ni}$ of the Poincaré group in terms of their embedding in $\mathbb{E}^{d+1,2}$ as orbits of the same Poincaré subgroup of $O(d + 1, 2)$. Moreover we show that they exhaust the types of nontrivial Poincaré orbits in $\mathbb{E}^{d+1,2}$. In addition we relate $\mathrm{Ni}$ to $\mathcal{I}$ via the passage to the projective space $\mathbb{P}^{d+2}$ of lines through the origin in $\mathbb{E}^{d+1,2}$. Using these embeddings, we show in Section 3 that we may reconstruct Minkowski spacetime from the spaces $\mathrm{Ti}$, $\mathrm{Spi}$ and $\mathrm{Ni}$, as well as $\mathcal{I}$, by exhibiting a bijective correspondence between points in Minkowski spacetime and certain hypersurfaces in these spaces. In Section 4 we proceed to a more algebraic description of these homogeneous spaces in terms of Klein pairs $(g, \mathfrak{h})$, where $g$ is in all cases the Poincaré Lie algebra and $\mathfrak{h}$ is the relevant stabiliser subalgebra. This will allow us to easily determine the Poincaré-invariant structures in the homogeneous spaces. We will see that the lorentzian structure of Minkowski spacetime is replaced by a carrollian structure for $\mathrm{Ti}$, a pseudo-carrollian structure for $\mathrm{Spi}$ and a doubly-carrollian structure for $\mathrm{Ni}$. In Section 5, after a brief review of the basic notions of (affine) grassmannians, we give natural geometric realisations of the Klein pairs for $\mathrm{Ti}$ (resp. $\mathrm{Spi}$) in terms of grassmannians of spacelike (resp. lorentzian) affine hyperplanes in Minkowski spacetime. We then show that $\mathrm{Ni}$ arises as the bundle of scales of $\mathcal{I}$, which we identify with the grassmannian of affine null hyperplanes. We also exhibit $\mathrm{Ni}$ as the fibred product of $\mathcal{I}$ and $\mathcal{I}$ over the celestial sphere $\mathbb{S}^1$. In Section 6 we present our conclusions and describe some potential applications of the results presented here. There are four appendices. In Appendix A, we review briefly the Ashtekar–Hansen construction of $\mathrm{Spi}$ as a blow-up of $i^0$ and how the analogous blow-up of $i^\pm$ gives rise to $\mathrm{Ti}$. In Appendix B, we give a survey of low-dimensional homogeneous spaces of the four-dimensional Poincaré and de Sitter groups. In Appendix C, we determine the symmetry Lie algebra of the Poincaré invariant doubly-carrollian structure on $\mathrm{Ni}$. In Appendix D, we describe an alternative approach to the reconstruction discussed in Section 3 that uses sections corresponding to eigenfunctions of the second Casimir of the Lorentz algebra. Finally, in Appendix E we discuss how our results extend to arbitrary signature, although we concentrate mostly on the Klein space of signature $(2,2)$. 

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2. Embeddings

Although the spaces under consideration were motivated as homogeneous spaces of the Poincaré group, their simplest description turns out to involve their embedding as codimension-2 submanifolds in a pseudo-euclidean space. The use of an auxiliary six-dimensional pseudo-euclidean space to study four-dimensional physics has a long and illustrious pedigree. It was perhaps first used by Dirac [17] in order to discuss conformally invariant wave equations and later by Kasner [20] and Fronsdal [21] in order to embed the Schwarzschild black hole. It appears in Penrose and Rindler [16] in a context very similar to ours in their discussion of the projective model for compactified Minkowski spacetime, and more recently it has become part of the holographic toolkit (see, e.g., [22]).

We will work in general dimension and in this section we will set up our conventions for the pseudo-euclidean space, identify a Poincaré subgroup of isometries and discuss its orbits.

2.1. A Poincaré subgroup of $O(d + 1, 2)$. We start by describing the pseudo-euclidean space $\mathbb{E}^{d+1,2}$. We will be working with global coordinates $x^\mu = (x^0, x^1, \ldots, x^d, x^+, x^-)$ for $\mathbb{E}^{d+1,2}$, closely related to the cartesian coordinates, where $x^0$ is timelike and $x^\pm = \pm \sqrt{x^{d+1} \pm x^{d+2}}$ are null ($x^{d+1}$ and $x^{d+2}$ are spacelike and timelike, respectively). Relative to these coordinates, the metric on $\mathbb{E}^{d+1,2}$ is expressed as

$$g_e = \eta_{AB} dx^A dx^B = -(dx^0)^2 + \sum_{\alpha=1}^d (dx^\alpha)^2 + 2 dx^+ dx^-.$$  \hfill (2.1)

It clearly has signature $(d + 1, 2)$. We will let $\mathbb{R}^{d,1}$ denote the lorentzian vector space $(\mathbb{R}^{d+1}, \tilde{\eta})$, where $\tilde{\eta} = \text{diag}(-1, 1, \ldots, 1)$. A typical point in $\mathbb{E}^{d+1,2}$ is denoted by $(x, x^+, x^-)$ with $x^\mu \in \mathbb{R}$ and $x \in \mathbb{R}^{d,1}$.

We now introduce some algebraic subspaces of $\mathbb{E}^{d+1,2}$. Let $\epsilon \in \mathbb{R}$ and let $\mathcal{Q}_\epsilon$ denote the quadric hypersurface cut out by the equation $\eta_{\alpha\beta} x^\alpha x^\beta = \epsilon$. In particular, if $\epsilon = 0$, we shall call $\mathcal{Q}_0$ the null quadric. These quadrics are preserved by a subgroup $O(d + 1, 2)$ of the isometries of $\mathbb{E}^{d+1,2}$, which acts transitively on every $\mathcal{Q}_{\epsilon \neq 0}$. The null quadric contains a singular point (namely, the origin in $\mathbb{E}^{d+1,2}$) and $O(d + 1, 2)$ acts transitively on the complement.

If $\epsilon = -\rho^2 < 0$, then the induced metric on $\mathcal{Q}_\epsilon$ is lorentzian of constant negative curvature, making $\mathcal{Q}_{\epsilon < 0}$ into the hyperboloid model of $AdS_{d+2}$ with radius of curvature $\rho$. If $\epsilon = \rho^2 > 0$, then the induced metric on $\mathcal{Q}_\epsilon$ has signature $(d, 2)$ and has constant positive curvature, so that $\mathcal{Q}_{\epsilon > 0}$ is a pseudo-sphere of radius of curvature $\rho$, or, equivalently a signature-$(-d, 2)$ version of de Sitter space.

Let $\sigma \in \mathbb{R}$ and let $\mathcal{M}_\sigma$ denote the null hypersurface with equation $x^- = \sigma$. For $\sigma \neq 0$, the subgroup of $O(d + 1, 2)$ which preserves $\mathcal{M}_\sigma$ is isomorphic to the Poincaré group $O(d, 1) \ltimes \mathbb{R}^{d,1}$. It is given explicitly by the following matrices

$$\left\{ \begin{pmatrix} A & 0 & v \\ -v^T \tilde{\eta} A & 1 & -\frac{1}{2} \tilde{\eta}(v,v) \\ 0^T & 0 & 1 \end{pmatrix} \right\} A^T \tilde{\eta} A = \tilde{\eta} \quad \text{and} \quad v \in \mathbb{R}^{d,1}. \quad (2.2)$$

The subgroup of $O(d + 1, 2)$ which preserves $\mathcal{M}_0$ is larger and it includes also “dilatations”. Every matrix in the Poincaré group (2.2) decomposes into a product

$$\begin{pmatrix} A & 0 & v \\ -v^T \tilde{\eta} A & 1 & -\frac{1}{2} \tilde{\eta}(v,v) \\ 0^T & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & v \\ -v^T \tilde{\eta} & 1 & -\frac{1}{2} \tilde{\eta}(v,v) \\ 0^T & 0 & 1 \end{pmatrix} \begin{pmatrix} A & 0 & 0 \\ 0^T & 1 & 0 \\ 0^T & 0 & 1 \end{pmatrix} \quad (2.3)$$

of a Lorentz transformation $A$ fixing the points $(0, x^+, x^-) \in \mathbb{E}^{d+1,2}$ and a translation $v$. 
At the level of the Lie algebra, \( \mathfrak{so}(d+1,2) \) is spanned by the vector fields
\[
\mathcal{M}_{ab} := \eta_{ac} x^c \partial_b - \eta_{bc} x^c \partial_a \in \mathcal{X}(\mathbb{E}^{d+1,2}),
\]
with Lie brackets
\[
[\mathcal{M}_{ab}, \mathcal{M}_{cd}] = \eta_{bc} \mathcal{M}_{ad} - \eta_{ac} \mathcal{M}_{bd} - \eta_{bd} \mathcal{M}_{ac} + \eta_{ad} \mathcal{M}_{bc}.
\]

The Poincaré algebra \( \mathfrak{g} \) is the subalgebra of \( \mathfrak{so}(d+1,2) \) whose vector fields are tangent to the null hypersurfaces \( \mathcal{N}_\sigma \) for any \( \sigma \). It is spanned by
\[
\begin{align*}
L_{ab} &:= M_{ab} = x^a \partial_b - x^b \partial_a \\
P_a &:= M_{a+} = x^a \partial_+ - x^0 \partial_a \\
m_{0a} &:= M_{0a} = -x^0 \partial_a - x^a \partial_0 \\
H &:= M_{0+} = -x^0 \partial_+ - x^+ \partial_0,
\end{align*}
\]
where \( a, b = 1, \ldots, d \). Its Lie brackets are
\[
\begin{align*}
[L_{ab}, L_{cd}] &= \delta_{bc} L_{ad} - \delta_{ac} L_{bd} - \delta_{bd} L_{ac} + \delta_{ad} L_{bc} \quad [B_a, B_b] = L_{ab} \\
[L_{ab}, B_c] &= \delta_{bc} B_a - \delta_{ac} B_b \quad [H, B_a] = -P_a \\
[L_{ab}, P_c] &= \delta_{bc} P_a - \delta_{ac} P_b \quad [B_a, P_b] = \delta_{ab} H.
\end{align*}
\]
If \( \sigma = 0 \), there is an enhancement of symmetry and the subalgebra of \( \mathfrak{so}(d+1,2) \) tangent to \( \mathcal{N}_0 \) has an additional generator: namely, \( D := M_{+-} = x^+ \partial_+ - x^- \partial_- \). This enhances the Poincaré group to the subgroup of \( O(d+1,2) \) consisting of matrices of the form
\[
\begin{pmatrix}
A & 0 & \nu \\
-\nu^T \eta A & a & -a^T \frac{1}{2} \eta(\nu, \nu) \\
0 & 0 & a^{-1}
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 \\
0^T & a & 0 \\
0^T & 0 & a^{-1}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & \nu \\
-\nu^T \eta & 1 & -\frac{1}{2} \eta(\nu, \nu) \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
A & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]
where the additional symmetry is given by nonzero \( a \in \mathbb{R} \).

2.2. Poincaré orbits in \( \mathbb{E}^{d+1,2} \). In discussing the orbits of the Poincaré group on \( \mathbb{E}^{d+1,2} \) we find it convenient to restrict ourselves to the identity component of the Poincaré group, denoted \( G \), and given by
\[
G = \left\{ \begin{pmatrix}
A & 0 & \nu \\
-\nu^T \eta A & 1 & -\frac{1}{2} \eta(\nu, \nu) \\
0 & 0 & 1
\end{pmatrix} \Big| A \in \text{SO}(d,1)_0, \quad \nu \in \mathbb{R}^{d,1} \right\},
\]
where \( \text{SO}(d,1)_0 \) is the identity component of the Lorentz group.

Since \( G \subset O(d+1,2) \), it preserves the quadrics \( \mathcal{Q}_c \) for any \( c \in \mathbb{R} \), and by definition it also preserves the null hyperplanes \( \mathcal{N}_\sigma \) for any \( \sigma \in \mathbb{R} \). Therefore it preserves their intersections
\[
\mathcal{M}_{\epsilon, \sigma} := \mathcal{Q}_\epsilon \cap \mathcal{N}_\sigma.
\]

2.2.1. Embedding Minkowski. Our first observation is that for any \( \epsilon \), provided that \( \sigma \neq 0 \), \( \mathcal{M}_{\epsilon, \sigma} \) is an embedding of Minkowski spacetime \( \mathcal{M}_{d+1} \) in \( \mathbb{E}^{d+1,2} \). Let us first show that \( \mathcal{M}_{\epsilon, \sigma} \) is an orbit of \( G \). Suppose that \( (x, x^+, \sigma) \) is a point in \( \mathcal{M}_{\epsilon, \sigma} \). Because it lies in the quadric \( \mathcal{Q}_\epsilon \) and \( \sigma \neq 0 \), we may solve for \( x^+ \) in terms of \( x \):
\[
x^+ (x) = \frac{\epsilon - \eta(x, x)}{2 \sigma},
\]
so that \( \mathcal{M}_{\epsilon, \sigma} \) is the image of \( \mathbb{R}^{d,1} \) under the embedding \( x \mapsto (x, x^+ (x), \sigma) \). The resulting paraboloid is illustrated in Figure 2 for \( \epsilon = 0 \) and \( \sigma = 1 \).

The action of the Poincaré group on \( (x, x^+, \sigma) \) can be read off from equation (2.9) and we see that it corresponds to \( x \mapsto Ax + \sigma \nu \). This action is transitive on \( \mathbb{R}^{d,1} \) and hence transitive on \( \mathcal{M}_{\epsilon, \sigma} \). Since \( x^- = \sigma \) is a constant, the pull-back to \( \mathcal{M}_{\epsilon, \sigma} \) of the pseudo-euclidean metric \( g_\epsilon \) in
Figure 2. An embedding of \((d + 1)\)-dimensional Minkowski spacetime \(\mathcal{M}_{d+1}\) as the intersection \(\mathcal{D}_0 \cap \mathcal{M}_1\) in the ambient space \(\mathbb{E}^{d+1,2}\)

equation (2.1) agrees with the Minkowski metric, proving that for any \(\epsilon \in \mathbb{R}\) and \(\sigma \neq 0\), \(\mathcal{M}_{\epsilon, \sigma}\) is isometric to \(\mathcal{M}_{d+1}\).

We pick an origin \((0, \frac{\epsilon}{\sigma^2}, \sigma) \in \mathcal{M}_{\epsilon, \sigma}\). The subgroup \(H \subset G\) fixing the origin is the proper orthochronous Lorentz subgroup consisting of matrices of the form

\[
H = \left\{ \begin{pmatrix} A & 0 & 0 \\ 0^T & 1 & 0 \\ 0^T & 0 & 1 \end{pmatrix} \middle| A \in SO(d,1) \right\}.
\] (2.13)

Its Lie algebra \(h\) consists of the span of \(L_{ab}, B_a\), defined in equation (2.6). We write this as \(h = \langle L_{ab}, B_a \rangle\).

If \(\sigma = 0\), the quadric condition does not fix \(x^+\). The Poincaré orbits in \(\mathcal{M}_{\epsilon, 0}\) depend on the sign of \(\epsilon\), so we must distinguish between three cases, depending on whether \(\epsilon > 0\), \(\epsilon < 0\) or \(\epsilon = 0\).

2.2.2. Embedding \(\text{Spi}\). Let \(\epsilon = \rho^2 > 0\). Then \(\mathcal{M}_{\rho^2, 0}\) consists of those points \((x, x^+, 0)\) where \(\bar{\eta}(x, x) = \rho^2\) and \(x^+ \in \mathbb{R}\) is otherwise arbitrary. The condition \(\bar{\eta}(x, x) = \rho^2\) cuts out a one-sheeted hyperboloid in \(\mathbb{R}^{d,1}\), i.e., a \(d\)-dimensional de Sitter space \(dS_d\). The proper orthochronous Lorentz group \(SO(d,1)\) acts transitively on this hyperboloid. The translation \(v\) in \(G\) acts via \((x, x^+, 0) \mapsto (x, x^+ - \bar{\eta}(v, x), 0)\) and hence we see that \(G\) acts transitively on \(\mathcal{M}_{\rho^2, 0}\). Let \(e_d := (0,0, \ldots ,0,1) \in \mathbb{R}^{d,1}\) be an elementary spacelike vector and let us choose an origin \((\rho e_d, 0, 0)\)

\(\text{We can make contact with the hyperboloid picture of } AdS_{d+2} \text{ in the following way. Setting } \epsilon = -\rho^2 \text{ and parametrizing the hyperboloid as } x^0 = y^0 \sigma \rho^{-1}, x^a = \sigma \rho^{-1} y^a, x^- = \sigma \text{ and } x^+ \text{ as in (2.11), the induced metric on the hyperboloid becomes}

\[
ds^2 = \rho^2 \sigma^{-2} \sigma^2 \sigma^2 - \sigma^2 \rho^{-2} \left(\left(dy^0\right)^2 + \left(dy^a\right)\left(dy_a\right)\right).
\] (2.12)

This is the usual Poincaré patch of \(AdS_{d+2}\). For \(\epsilon < 0\) and \(\sigma \neq 0\) the Minkowski spaces \(\mathcal{M}_{\epsilon, \sigma}\) described above therefore correspond to this slicing of \(AdS_{d+2}\). The conformal boundary of \(AdS_{d+2}\) is reached for \(\sigma \to \infty\). The light-like hypersurface \(\sigma = 0\) is not covered by the Poincaré patch coordinates. As we will see below, the corresponding space \(\mathcal{M}_{\rho^2, 0}\) is not Minkowski space but \(Ti^\perp\).
for $\mathscr{M}_{\rho^2,0}$. The subgroup $H \subset G$ which fixes the origin consists of matrices

$$
H = \begin{pmatrix}
A & 0 & v \\
-v^T \bar{\eta} A & 1 & -\frac{1}{2} \bar{\eta} (v, v) \\
0^T & 0 & 1
\end{pmatrix}
$$

(2.14)

where $A \in SO(d,1)_0$ is such that $A e_d = e_d$ and $v \in \mathbb{R}^{d,1}$ is such that $\bar{\eta}(v, e_d) = 0$. This subgroup $H$ is isomorphic to the Poincaré group $SO(d-1,1)_0 \times \mathbb{R}^{d-1,1}$ in one lower dimension. Its Lie algebra $\mathfrak{h}$ can be determined as those vector fields in (2.6) which vanish at the origin and we can see that $\mathfrak{h} = \{\mathfrak{l}_{ij}, B_i, \mathfrak{p}_i, \mathfrak{h}\}$, where $i, j = 1, \ldots, d-1$. For any $\epsilon > 0$, $\mathscr{M}_{\epsilon,0}$ is an embedding in $\mathbb{R}^{d+1,2}$ of the blow-up $\mathfrak{Spi}_{d+1}$ of the spatial infinity $i^0$ of Minkowski spacetime, as reviewed in Appendix A.

This embedding shows that $\mathfrak{Spi}_{d+1}$ fibers over $\partial dS_d$, identifying $\partial dS_d$ with any one of the one-sheeted hyperboloids in $\mathbb{R}^{d,1}$. The projection $\mathfrak{Spi}_{d+1} \to \partial dS_d$ sends $(x, x^+, 0) \mapsto x$. This is a trivial bundle and hence $\mathfrak{Spi}_{d+1} \cong \partial dS_d \times \mathbb{R}$. Every smooth function $f$ on $\partial dS_d$ defines a section $\partial dS_d \to \mathfrak{Spi}_{d+1}$ by $x \mapsto (x, f(x), 0)$. These sections are in one-to-one correspondence with the $\mathfrak{Spi}$-supertranslations, an infinite-dimensional abelian ideal of the Lie symmetries of the pseudo-carrollian structure [6] on $\mathfrak{Spi}_{d+1}$. As we will see in Section 3.1, some of these supertranslations are Poincaré translations and they will thus be associated to points in Minkowski spacetime once we fix an origin and in this way we will reconstruct Minkowski spacetime from its asymptotic geometry $\mathfrak{Spi}$.

2.2.3. Embedding $\mathfrak{Ti}^\pm$. Let $\epsilon = -\rho^2 < 0$. Now $\mathscr{M}_{-\rho^2,0}$ consists of points $(x, x^+, 0)$ where $\bar{\eta}(x, x) = -\rho^2$ and $x^+ \in \mathbb{R}$. The condition $\bar{\eta}(x, x) = -\rho^2$ defines a two-sheeted hyperboloid in $\mathbb{R}^{d,1}$ which is acted on transitively by $SO(d,1)$. Under the identity component $SO(d,1)_0$, each sheet is an orbit. The translation $v$ in $G$ acts via $(x, x^+, 0) \mapsto (x, x^+ - \bar{\eta}(v, x), 0)$ and hence $G$ acts with two orbits on $\mathscr{M}_{-\rho^2,0}$:

$$
\mathscr{M}_{-\rho^2,0} = \mathscr{M}_{-\rho^2,0}^+ \cup \mathscr{M}_{-\rho^2,0}^-,
$$

(2.15)

where

$$
\mathscr{M}_{-\rho^2,0}^\pm = \left\{ \begin{cases} x^+ \\ x^0 \end{cases} \mid \bar{\eta}(x, x) = -\rho^2, \pm x^0 > 0, \text{ and } x^+ \in \mathbb{R} \right\}.
$$

(2.16)

Let $e_0 = (1, 0, \ldots, 0) \in \mathbb{R}^{d,1}$ be an elementary timelike vector and let us fix the origin $(\pm \epsilon e_0, 0, 0) \in \mathscr{M}_{-\rho^2,0}^\pm$. The subgroup $H \subset G$ which fixes the origin is common to both orbits and consists of matrices

$$
H = \begin{pmatrix}
A & 0 & v \\
-v^T \bar{\eta} A & 1 & -\frac{1}{2} \bar{\eta} (v, v) \\
0^T & 0 & 1
\end{pmatrix}
$$

(2.17)

where $A \in SO(d,1)_0$ fixes $e_0$ and $v \in \mathbb{R}^{d,1}$ is perpendicular to $e_0$. This subgroup is isomorphic to the euclidean group $SO(d) \times \mathbb{R}^d$ in one lower dimension, corresponding to the hyperplane of $\mathbb{R}^{d,1}$ perpendicular to $e_0$. Its Lie algebra $\mathfrak{h}$ is spanned by those vector fields in equation (2.6) which vanish at the origin; namely, $\mathfrak{h} = \{\mathfrak{l}_{ab}, \mathfrak{p}_a\}$. As shown in [8] (see also [9]), $\mathcal{M}_{-\rho^2,0}^\pm$ defines embeddings of $AdS_{d+1}$, the carrollian limit of $AdS_{d+1}$. As discussed in Appendix A, $AdS_{d+1}$ is isomorphic (as a homogeneous space of the Poincaré group) to the blow-ups $\mathfrak{Ti}_{d+1}^\pm$ of the timelike infinities $i^\pm$ of Minkowski spacetime. We shall therefore refer to $AdS_{d+1}$ simply as $\mathfrak{Ti}_{d+1}$.

Just as with $\mathfrak{Spi}_{d+1}$, this embedding shows that $\mathfrak{Ti}_{d+1}$ fibers over hyperbolic space $\mathscr{H}^d$, where we identify $\mathscr{H}^d$ with any one of the sheets of the two-sheeted hyperboloids $\bar{\eta}(x, x) = -\rho^2$. The fiberation $\mathfrak{Ti}_{d+1} \to \mathscr{H}^d$ sends $(x, x^+, 0) \mapsto x$. Again this is a trivial bundle and hence $\mathfrak{Ti}_{d+1} \cong \mathscr{H}^d \times \mathbb{R}$. The smooth sections $\mathscr{H}^d \to \mathfrak{Ti}_{d+1}$ can be identified with the smooth functions on $\mathscr{H}^d$ and correspond to the $\mathfrak{Ti}$-supertranslations, the infinite-dimensional abelian ideal of
the Lie symmetries of the carrollian structure of $\text{Ti}_{d+1}(= \text{AdSC})$, which were determined in [23]. Again, some of the supertranslations correspond to Poincaré translations and we will revisit this in Section 3.1 when we discuss the reconstruction of Minkowski spacetime from $\text{Ti}$.

2.2.4. Embedding $\text{Ni}^\pm$. Finally we let $\epsilon = 0$ and consider $\mathcal{M}_{0,0} = \mathcal{Z}_0 \cap \mathcal{N}_0$. The point $[x, x^+, 0]$ lies in $\mathcal{M}_{0,0}$ if and only if $\bar{\eta}(x, x) = 0$, so that $x$ lies on the lightcone in $\mathbb{R}^{d,1}$. Under $\text{SO}(d,1)_0$, the lightcone $\mathcal{L} \subset \mathbb{R}^{d,1}$ breaks up into three orbits:

$$\mathcal{L} = \mathcal{L}^- \cup \{0\} \cup \mathcal{L}^+, \quad (2.18)$$

where $\mathcal{L}^\pm$ are the future/past lightcones with the apex removed. Provided that $x \in \mathcal{L}^\pm$, the translations in $G$ can relate any two values of $x^\pm$, but if $x = 0$, then each of the points $[0, x^+, 0]$ is fixed by $G$. In summary, $\mathcal{M}_{0,0}$ breaks up into two $(d+1)$-dimensional orbits $\mathcal{M}^\pm_{0,0}$ and a line $\ell = \{(0, x^+, 0) : x^+ \in \mathbb{R}\}$ of fixed points under the Poincaré group $G$; that is,

$$\mathcal{M}_{0,0} = \mathcal{M}^+_0 \cup \ell \cup \mathcal{M}^-_0 \quad \text{with} \quad \ell = \bigcup_{x^+ \in \mathbb{R}} \{(0, x^+, 0)\}, \quad (2.19)$$

where

$$\mathcal{M}^\pm_{0,0} = \left\{ \left( \begin{array}{c} x \\ x^+ \\ 0 \end{array} \right) \Bigg| \bar{\eta}(x, x) = 0, \pm x^0 > 0, \text{ and } x^+ \in \mathbb{R} \right\}. \quad (2.20)$$

Let $e_- := \frac{1}{\sqrt{2}}(e_d - e_0) = (-\frac{1}{\sqrt{2}}, 0, \ldots, 0, \frac{1}{\sqrt{2}}) \in \mathbb{R}^{d,1}$ and let us fix the origin $[\mp e_-, x^+, 0] \in \mathcal{M}^\pm_{0,0}$. The subgroup $H \subset G$ which fixes the origin is common to both $\mathcal{M}^\pm_{0,0}$ and consists of matrices

$$H = \left( \begin{array}{ccc} A & 0 & v \\ -v^T \bar{\eta} A & 1 & -\frac{1}{2} \bar{\eta}(v, v) \\ 0 & 0 & 1 \end{array} \right) \quad (2.21)$$

where $A \in \text{SO}(d,1)_0$ fixes $e_-$ and $v \in \mathbb{R}^{d,1}$ is perpendicular to $e_-$. This subgroup is isomorphic to the $d$-dimensional Carroll group. Its Lie algebra $\mathfrak{h}$ is spanned by those vector fields in equation (2.6) which vanish at the origin; namely, $\mathfrak{h} = \{L_i, P_i, L_i d + B_i, H - P_d\}$, for $i, j = 1, \ldots, d - 1$. Again, since the stabiliser subgroup is common to both $\text{Ni}^\pm$, they are isomorphic as homogeneous spaces of $G$. We will therefore refer to either one of these two spaces simply as $\text{Ni}$.

As we will see in Section 5.5, we will identify $\mathcal{M}^\pm_{0,0}$ with $\text{Ni}^\pm$, the bundle of scales of the conformal carrollian structure of $\mathcal{I}^\pm$. This embedding of $\text{Ni}^\pm$ shows that it fibers over the future/past lightcone $\mathcal{L}^\pm$, with the fibration $\text{Ni}^\pm \to \mathcal{I}^\pm$ given simply by $(x, x^+, 0) \mapsto x$. Together with the identification of $\text{Ni}^\pm$ as a bundle over $\mathcal{I}^\pm$, we can see that there is a double fibration

$$\text{Ni}^\pm_{d+1} \quad \mathcal{I}^\pm_d \quad \mathcal{L}^\pm_d \quad \mathbb{C} S^{d-1} \quad (2.22)$$

which allows us to view $\text{Ni}^\pm$ as sitting inside $\mathcal{L}^\pm \times \mathcal{I}^\pm$ as their fibred product over the celestial sphere $\mathbb{C} S$. Said differently, the fibration $\text{Ni}^\pm \to \mathcal{I}^\pm$ is the pull-back fibre bundle of the fibration $\mathcal{L}^\pm \to \mathbb{C} S$ via the fibre bundle $\mathcal{I} \to \mathbb{C} S$.

The fibration $\text{Ni}^\pm \to \mathcal{I}^\pm$ can also be understood from the embedding picture. Let $\mathbb{P}^{d+2}$ be the projective space of $\mathbb{E}^{d+1,2}$. It is the quotient of $\mathbb{E}^{d+1,2} \setminus \{0\}$ by the action of the nonzero reals.
which rescales the nonzero vectors: $x \mapsto \lambda x$ for $x \in \mathbb{R}^{d+1,2} \setminus \{0\}$ and $\lambda \in \mathbb{R}^\times$. As explained in [16, Section 9.2], the image of the null quadric $\mathcal{Q}$ in $\mathbb{P}^{d+2}$ is a conformal compactification $\mathbb{P}^\#_{d+1}$ of Minkowski spacetime $\mathbb{M}^\#_{d+1}$. The image of points in $\mathcal{Q}$ with $x^- \neq 0$ correspond to the interior points of $\mathbb{P}^\#_{d+1}$ (corresponding to Minkowski spacetime itself), whereas the image of points with $x^- = 0$ correspond to the conformal boundary of Minkowski spacetime in this compactification.

The points in $\mathbb{N}_{d+1}$ map to $\mathcal{F}$, which is the identification of $\mathcal{F}^+$ and $\mathcal{F}^-$, which are after all indistinguishable as homogeneous spaces of $G$, whereas the points in the singular line $t$ (except for the origin) get mapped to the same point $I \in \mathbb{P}^{d+2}$ which is the identification of $i^0$, $i^+$ and $i^-$. Hence the fibration $\mathbb{N}_{d+1} \to \mathcal{F}$ to be discussed in Section 4.5 is simply the restriction to $\mathbb{N}_{d+1} \subset \mathbb{E}^{d+1,2}$ of the projection $\mathbb{E}^{d+1,2} \setminus \{0\} \to \mathbb{E}^{d+2}$.

2.2.5. **Summary.** We may summarise the above discussion by explicitly decomposing $\mathbb{E}^{d+1,2}$ in terms of orbits of the connected Poincaré group:

$$\mathbb{E}^{d+1,2} = \left( \bigcup_{\tau, \varepsilon, \sigma \in \mathbb{R} \atop \sigma \neq 0} \mathcal{M}_{\tau, \sigma, 0, \varepsilon} \right) \sqcup \left( \bigcup_{\tau > 0} \mathcal{M}_{\tau, 0, \varepsilon, 0} \right) \sqcup \left( \bigcup_{\tau < 0} \mathcal{M}_{\tau, 0, \varepsilon, 0} \right) \sqcup \left( \bigcup_{\tau \in \mathbb{R}} \mathcal{M}_{0, 0, \varepsilon, 0} \right)$$

$$\sqcup \mathcal{M}_{0, 0, 0, 0} \sqcup \mathcal{M}_{0, 0, 0, 0} \sqcup \left( \bigcup_{x^+ \in \mathbb{R}} \{ (x^+), (0) \} \right)$$

(2.23)

We may now pass to the projective space $\mathbb{P}^{d+2} = (\mathbb{E}^{d+1,2} \setminus \{0\})/\mathbb{R}^\times$ to obtain

$$\mathbb{P}^{d+2} = \left( \bigcup_{\tau \in \mathbb{R}} \mathcal{M} \right) \sqcup \mathcal{S}_i \sqcup \mathcal{TI}_i \sqcup \mathcal{F} \sqcup \{I\},$$

(2.24)

where $\tau = \varepsilon/\sigma^2$ is a projective invariant. Restricting to the projectivised null quadric we obtain the conformal compactification

$$\mathbb{P}^\# = \mathcal{M} \sqcup \mathcal{F} \sqcup \{I\}$$

(2.25)

of Penrose and Rindler [16, Section 9.2]. Although they treat the four-dimensional case ($d = 3$ here), their results are dimension agnostic. Here $\mathcal{F}$ is the identification of $\mathcal{F}^+$ and $\mathcal{F}^-$ and $\{I\}$ is the singleton set obtain by identifying $i^0$ and $i^\pm$. However we see that if we do not restrict to the null quadric, we actually obtain $\mathcal{S}_i$ and $\mathcal{TI}_i$ as limits of a family of embedded Minkowski spacetimes.

3. FABLES OF THE RECONSTRUCTION

The embedding formalism in Section 2 allows us to explain how to reconstruct Minkowski spacetime $\mathbb{M}_{d+1}$ from its asymptotic geometries $\mathbb{S}_i_{d+1}$, $\mathbb{TI}_i_{d+1}$, $\mathbb{Ni}_{d+1}$ and $\mathcal{F}_d$. In all cases, the idea is the same. Every point in Minkowski spacetime is stabilised by a unique Lorentz subgroup of the Poincaré group $G$. Our strategy is to fix an origin in $\mathbb{M}_{d+1}$ and consider the orbits of the corresponding (proper, orthochronous) Lorentz subgroup of $G$ on $\mathbb{S}_i_{d+1}$, $\mathbb{TI}_i_{d+1}$, $\mathbb{Ni}_{d+1}$ and $\mathcal{F}_d$. In all cases the orbits will be hypersurfaces, which turn out to be cut out by a section of the (trivial) fibrations $\mathbb{S}_i_{d+1} \to \mathcal{D}_d$, $\mathbb{TI}_i_{d+1} \to \mathcal{H}_d$, $\mathbb{Ni}_{d+1} \to \mathcal{L}_d$ and $\mathcal{F}_d \to \mathcal{C}^{d-1}$. This means that we may associate one such section to the origin in Minkowski spacetime.

Any other point in $\mathbb{M}_{d+1}$ is obtained from the origin by a unique translation. Hence to see which sections correspond to points in Minkowski spacetime, we can take the section corresponding to the origin and apply a translation. In this way we will obtain a family of hypersurfaces in each of $\mathbb{S}_i_{d+1}$, $\mathbb{TI}_i_{d+1}$, $\mathbb{Ni}_{d+1}$ and $\mathcal{F}_d$ or, equivalently, a family of sections of each of the trivial fibrations $\mathbb{S}_i_{d+1} \to \mathcal{D}_d$, $\mathbb{TI}_i_{d+1} \to \mathcal{H}_d$, $\mathbb{Ni}_{d+1} \to \mathcal{L}_d$ and $\mathcal{F}_d \to \mathcal{C}^{d-1}$. Being trivial, the sections can
be identified with smooth functions on the base. We will see that the sections corresponding to the points in Minkowski spacetime can be identified with (the restrictions to $dS_d$, $\mathcal{H}^d$ and $\mathcal{L}_d$ of) affine functions on the ambient $\mathbb{R}^{d,1}$ in the first three cases, and from affine functions on the ambient $\mathbb{R}^d$ in the case of $\mathbb{C}S^{d-1}$. An additional freedom in the identification of points of Minkowski space with sections of $\text{Spi}_{d+1}$, $\text{Ti}_{d+1}$, $\mathcal{N}_{d+1}$ and $\mathcal{I}_d$ is fixed by employing the (generalized) light-cone of the ambient space $\mathbb{R}^{d+1,2}$.

The essence of our approach may be described as follows. Once we fix a reference section through $\text{Spi}_{d+1} \to dS_d$, $\text{Ti}_{d+1} \to \mathcal{H}^d$, $\mathcal{N}_{d+1} \to \mathcal{L}_d$ or $\mathcal{I}_d \to \mathbb{C}S^{d-1}$, any other section is obtained from the reference section via the action of a supertranslation in the corresponding symmetry group. Some of the supertranslations are Poincaré translations, and the sections obtained from the reference section via Poincaré translations are in bijective correspondence with points in Minkowski spacetime. This means we reconstruct points in Minkowski spacetime using sections in the asymptotic geometries; something we can interpret as a form of holographic reconstruction of Minkowski spacetime. We will use the conventions introduced in Section 2.

3.1. Reconstructing $\mathcal{M}$ from $\text{Spi}$, $\text{Ti}$ and $\mathcal{N}_i$. Let us first treat the three cases: $\text{Spi}_{d+1}$, $\text{Ti}_{d+1}$ and $\mathcal{N}_{d+1}$. Let $\mathcal{M} := \mathcal{M}_{d,1}$ denote the embedded Minkowski spacetime containing the point $(0,0,0,1)$, which we shall think of as the origin. The origin is stabilised by the subgroup $O(d,1)$ consisting of matrices like those in equation (2.13), whose identity component is

$$H = \left\{ \begin{pmatrix} A & 0 & 0 \\ 0^T & 1 & 0 \\ 0^T & 0 & 1 \end{pmatrix} : A \in SO(d,1) \right\},$$

with $SO(d,1)_0$ the identity component of $O(d,1)$, i.e., they are given by the proper, orthochronous Lorentz transformations parametrised by $A$.

Let $x_0 \in \mathbb{R}^{d,1}$. As one can easily show, the action of $H$ is given by $(x_0, x^+, 0) \mapsto (A x_0, x^+, 0)$. This means the orbit of $(x_0, x^+, 0)$ under $H$ consists of the hypersurface with points $(x, x^+, 0)$ where $x$ is in the (proper, orthochronous) Lorentz orbit of $x_0$. For example, if $(x_0, x^+, 0)$ belongs to $\text{Spi}_{d+1}$ or $\text{Ti}_{d+1}$ or $\mathcal{N}_{d+1}$, its orbit consists of the hypersurface in $\text{Spi}_{d+1}$ or $\text{Ti}_{d+1}$ or $\mathcal{N}_{d+1}$ with $x^+$ constant. We can interpret them as sections of the fibrations $\text{Spi}_{d+1} \to dS_d$, $\text{Ti}_{d+1} \to \mathcal{H}^d$ and $\mathcal{N}_{d+1} \to \mathcal{L}_d$. Since these fibrations are trivial, sections correspond to smooth functions on $dS_d$, $\mathcal{H}^d$ and $\mathcal{L}_d$. For example, if $f \in C^\infty(dS_d)$, then its graph defines a section $dS_d \to \text{Spi}_{d+1}$ consisting of the points $(x, f(x), 0)$. It is then clear that if we take $f$ to be a constant function on $dS_d$, $\mathcal{H}^d$ and $\mathcal{L}_d$, its graph is precisely the section of $\text{Spi}_{d+1} \to dS_d$, $\text{Ti}_{d+1} \to \mathcal{H}^d$ and $\mathcal{N}_{d+1} \to \mathcal{L}_d$ corresponding to fixing $x^+$ to the constant value of $f$. We will now determine the functions giving rise to sections whose corresponding hypersurfaces are parametrised by the points in Minkowski spacetime.

Let us act on these hypersurfaces with the Poincaré translations

$$\begin{pmatrix} \mathbb{I} & 0 & v \\ -v^T \bar{\eta} & 1 & -\frac{1}{2} \bar{\eta}(v, v) \\ 0 & 0 & 1 \end{pmatrix}.$$

Each translation, given by the vector $v$, is identified with a unique point in Minkowski spacetime (once we choose an origin). The components $(v^0, v^1, \ldots, v^d)$ of $v$ are cartesian coordinates for

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5This is to be compared with the reconstruction [24] of four-dimensional (complex) Minkowski spacetime from $\mathcal{I}$ as the space of certain hypersurfaces of $\mathcal{I}$ (the so-called "good cuts"), which arise as sections of the fibre $\mathcal{I} \to \mathbb{C}$. The space of good cuts is an affine space modelled on the kernel of $\mathbb{R}^2$, which for Minkowski spacetime, consists of the spherical harmonics on the sphere with $\ell = 0, 1$. But these are precisely the restriction to the sphere of the affine functions on the ambient three-dimensional euclidean space.
Minkowski spacetime centred at the origin, and hence we can identify Minkowski spacetime with $\mathbb{R}^{d,1}$. The action of the translation is then given by
\[
\begin{pmatrix} x \\ x^+ \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} x \\ x^+ - \bar{\eta}(x,v) \\ 0 \end{pmatrix}.
\] (3.3)

Therefore the action on the hypersurfaces of $\text{Spi}_{d+1}$, $\text{Ti}_{d+1}$ and $\text{Ni}_{d+1}$ corresponds to the sections of the trivial fibrations $\text{Spi}_{d+1} \to \partial S_d$, $\text{Ti}_{d+1} \to \mathcal{H}_d$ and $\text{Ni}_{d+1} \to \mathcal{L}_d$ defined by the restriction to $\partial S_d$, $\mathcal{H}_d$ and $\mathcal{L}_d$ of the affine function $f : \mathbb{R}^{d,1} \to \mathbb{R}$ defined by $f(x) = x^+ - \bar{\eta}(v,x)$.

The space of such affine functions is $(d + 2)$-dimensional: parametrised by $x^+$ and $v \in \mathbb{R}^{d,1}$. Clearly, Minkowski spacetime only knows about $v$, and hence to reconstruct it or, equivalently, to put the hypersurfaces in bijective correspondence with the points of Minkowski spacetime we would either fix $x^+$ or else introduce an equivalence relation between hypersurfaces which are related by a constant shift in $x^+$ and take equivalence classes.

We may choose a value of $x^+$ via the following geometric construction, which is analogous to the one in [18]. An alternate construction is presented in Appendix D. In a nutshell, we will draw a generalised lightcone $L_p \subset E^{d+1,2}$ at every point $p \in \mathcal{M} \subset E^{d+1,2}$ and then study its intersection with $\text{Ti}$, $\text{Spi}$ and $\text{Ni}$. In this way we can associate with every point $p \in \mathcal{M}$ a hypersurface in $\text{Ti}$, $\text{Spi}$ and $\text{Ni}$ and reconstruct $\mathcal{M}$ (and hence Minkowski spacetime) as the parameter space of such hypersurfaces. This is analogous to the identification in [18] of lightcone cuts in $\mathcal{I}^+$ with its intersections with the lightcone at a point in Minkowski spacetime.

Choose a point $p = (v, -\frac{1}{2} \bar{\eta}(v,v), 1) \in \mathcal{M} \subset E^{d+1,2}$ and let $L_p$ denote the null quadric in $E^{d+1,2}$ centred at $p$:
\[
L_p = \left\{ (x, x^+, x^-) \in E^{d+1,2} \mid \bar{\eta}(x - v, x - v) + 2(x^+ + \frac{1}{2} \bar{\eta}(v,v))(x^- - 1) = 0 \right\}.
\] (3.4)

Notice that $L_p$ intersects $\mathcal{M}$ precisely at the lightcone based at $p$. Indeed, if $x^- = 1$, then $\bar{\eta}(x - v, x - v) = 0$, so that $x$ lives in the lightcone of $E^{d,1}$ based at $v$. The value of $x^+$ is undetermined and we can always choose it to be $-\frac{1}{2} \bar{\eta}(x,x)$ so that $(x, x^+, 1) \in \mathcal{M}$. Hence we conclude that $L_p \cap \mathcal{M}$ is the Minkowski lightcone based at the point $p$. We call $L_p$ the “generalised lightcone” based at $p$.

The “light rays” of $L_p$ intersect the asymptotic geometries $\text{Ti}$, $\text{Spi}$ and $\text{Ni}$, which are also embedded in $E^{d+1,2}$. These intersections are easy to determine and we find the following:
\[
\begin{align*}
L_p \cap \text{Spi} &= \left\{ (x, x^+, 0) \mid \bar{\eta}(x,x) = 1 \text{ and } x^+ = \frac{1}{2} - \bar{\eta}(v,x) \right\}, \\
L_p \cap \text{Ti} &= \left\{ (x, x^+, 0) \mid \bar{\eta}(x,x) = -1 \text{ and } x^+ = -\frac{1}{2} - \bar{\eta}(v,x) \right\}, \\
L_p \cap \text{Ni} &= \left\{ (x, x^+, 0) \mid \bar{\eta}(x,x) = 0, \ x^0 > 0 \text{ and } x^+ = -\bar{\eta}(v,x) \right\}. 
\end{align*}
\] (3.5)

The intersections in (3.5) relate a point $x \in \mathbb{R}^{d,1}$ in Minkowski spacetime to smooth functions $x^+(v)$, the last equation in each line in (3.5), on the respective asymptotic geometries. Let us consider the case of $\text{Spi}$, for definiteness. The origin of Minkowski spacetime corresponds to the constant function $x^+(v) = \frac{1}{2}$, which also shows that we have indeed fixed the ambiguity of $x^+$ to $\frac{1}{2}$. A generic point $(v, -\frac{1}{2} \bar{\eta}(v,v),1)$ in Minkowski spacetime leads then to the affine function $x^+(v) = \frac{1}{2} - \bar{\eta}(x,v)$, where $\bar{\eta}(v,v) = 1$, so that $x^+$ is a function on $\partial S$. In this way we have assigned to every point of Minkowski spacetime a function in $\text{Spi}$, which is the restriction of an affine function on $\mathbb{R}^{d,1}$. Conversely, given such a function it is clear we can read off $v$ and therefore find the corresponding point in Minkowski spacetime, establishing a bijection. Similar arguments apply to $\text{Ti}$ and $\text{Ni}$.

One final remark. Whereas the value of $x^+ = 0$ for $\text{Ni}$ is independent of choices, the values $x^+ = \pm \frac{1}{2}$ for $\text{Spi}$ and $\text{Ti}$, respectively, depend on the precise embeddings of $\mathcal{M}_{d+1}$, $\text{Spi}_{d+1}$ and $\text{Ti}_{d+1}$ in $E^{d+1,2}$. Re-embedding $\mathcal{M}_{d+1}$ as the intersection $\mathcal{L}_0 \cap \mathcal{A}_\sigma$, for $\sigma \neq 0$, and similarly $\text{Ti}_{d+1}$
Figure 3. The lightcone $\mathcal{L}$ at $p \in \mathcal{M}$ as the intersection $L_p \cap \mathcal{M}$.

as $\mathcal{L}_c \cap \mathcal{M}_0$ and $\text{Spi}_{d+1}$ as $\mathcal{L}_c \cap \mathcal{M}_0$, for $\epsilon > 0$, we can change $x^+$ to $\pm \frac{\epsilon x^+}{2}$, which can be any nonzero real number. The important thing is that once a choice of $x^+$ has been made and hence one hypersurface chosen, the other translates of that initial hypersurface are in bijective correspondence with the points in Minkowski spacetime. As noted previously in Sections 2.2.2 and 2.2.3, the more general transformations in which $x^+$ is shifted by an arbitrary function $f(x)$ instead of the linear function $-\eta(x, v)$ correspond to $\text{Spi}, \text{Ti}, \text{Ni}$-supertranslations, respectively.

3.2. Reconstructing $\mathcal{M}$ from $\mathcal{I}$. Now we discuss the reconstruction of $\mathcal{M}_{d+1}$ from $\mathcal{I}_d$ along the lines explained above. We recall that $G$ denotes the identity component of the Poincaré group. We shall refer to $(A, v) \in G$ as the Poincaré transformation consisting of a proper, orthochronous Lorentz transformation $A$ and a translation $v$. Let us consider

$$Ni^+ = \left\{ \begin{pmatrix} x \\ x^+ \\ 0 \end{pmatrix} \right\} | x \in \mathcal{L}^+ \; \text{and} \; x^+ \in \mathbb{R},$$

where $\mathcal{L}^+ \in \mathbb{R}^{d+1}$ is the future lightcone. The action of $(A, v) \in G$ on $Ni^+$ is given by

$$(A, v) \cdot \begin{pmatrix} x \\ x^+ \\ 0 \end{pmatrix} = \begin{pmatrix} Ax \\ x^+ - \eta(v, Ax) \\ 0 \end{pmatrix}.$$  

Since the proper orthochronous Lorentz group acts transitively on $\mathcal{L}^+$, it follows that $G$ acts transitively on $Ni^+$ and hence also on $\mathcal{I}^+ = \mathbb{P}_+Ni^+$, where $\mathbb{P}_+$ denotes the ray projectivisation.

$^6$Note that, in the case of $\text{Spi}$ and $\text{Ti}$, Poincaré transformations cannot shift the value of $x^+$ by a constant, so that this statement is indeed Poincaré invariant.
\[ P_+ N^+ \] is the space of orbits of the \( R^+ \) action on \( N^+ \) given by rescaling:

\[
\begin{pmatrix}
  x \\
  x^+ \\
  0
\end{pmatrix} \mapsto \begin{pmatrix}
  \lambda x \\
  \lambda x^+ \\
  0
\end{pmatrix}
\]

for \( \lambda \in R^+ \) a positive real number.

3.2.1. \textit{Orbit decomposition of} \( \mathcal{I}^+ \) \textit{under the Lorentz group.} This section is somewhat outside the main narrative in the paper, but we think it is interesting to point out the fact that although \( \mathcal{I}^+ \) is a homogeneous space of the Poincaré group, it is not far from a homogeneous space of a Lorentz subgroup. Indeed, pick the origin in Minkowski spacetime which is stabilised by a subgroup \( SO(d,1)_0 \) of \( G \). What are the orbits of \( SO(d,1)_0 \) on \( \mathcal{I}^+ \)? We claim that there are three orbits corresponding to those \( \begin{pmatrix}
  x \\
  0
\end{pmatrix} \in \mathcal{I}^+ \) with \( x^+ = 0 \), \( x^+ > 0 \) and \( x^+ < 0 \). The latter two cases are open orbits of the same dimension as \( \mathcal{I}^+ \), whereas the orbit with \( x^+ = 0 \) is the desired hypersurface.

Let \( x^+ > 0 \). We claim that

\[
\begin{pmatrix}
  x \\
  0
\end{pmatrix} \in \mathcal{I}^+
\]

is an orbit of \( SO(d,1)_0 \). It is enough to show that any points in \( \mathcal{O}^+ \) are related by a proper orthochronous Lorentz transformation; that is, that given any two points

\[
\begin{pmatrix}
  x \\
  0
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
  y \\
  0
\end{pmatrix}
\]

with \( x, y \in \mathcal{L}^+ \) and \( x^+, y^+ > 0 \), there exists \( A \in SO(d,1)_0 \) and \( \lambda \in R^+ \) such that \( Ax = \lambda y \) and \( x^+ = \lambda y^+ \). The second relation clearly sets \( \lambda = x^+/y^+ \) and the first equation says that \( Ax = x^+/y^+y \). But \( x^+/y^+y \in \mathcal{L}^+ \) and \( SO(d,1)_0 \) acts transitively, so that there exists some \( A \in SO(d,1)_0 \) sending \( x \) to \( x^+/y^+y \). A similar argument shows that \( \mathcal{O}^- \), defined as \( \mathcal{O}^+ \) but with \( x^+ < 0 \), is an orbit of \( SO(d,1)_0 \).

Suppose now that \( x^+ = 0 \). Then to every \( \begin{pmatrix}
  x \\
  0
\end{pmatrix} \in N^+ \) there corresponds \( [x] \in P_+ \mathcal{L}^+ \cong \mathbb{C} \mathcal{S} \).

Since \( SO(d,1)_0 \) acts transitively on \( \mathcal{L}^+ \), it acts transitively on \( \mathbb{C} \mathcal{S} \) and, indeed, does so via conformal transformations. This orbit is thus a section of the trivial fibration \( \mathcal{I}_d \to \mathbb{C} \mathcal{S}^{d-1} \). The action of the Poincaré translations in this hypersurface is given by

\[
\begin{pmatrix}
  x \\
  0
\end{pmatrix} \mapsto \begin{pmatrix}
  x \\
  -\bar{\eta}(v,x)
\end{pmatrix},
\]

which we may think of as a section of \( \mathcal{I}_d^+ \to \mathbb{C} \mathcal{S}^{d-1} \) associated to a linear function in the ambient euclidean space \( R^d \) into which the sphere embeds.

We think it is curious that to make \( \mathcal{I}^+ \) into a homogeneous space we need to extend the Lorentz group (which acts with three orbits) to the full Poincaré group.
4. Kleln geometries

In this section we will describe the homogeneous spaces of the Poincaré group studied in the previous section as Klein geometries.

Recall that a Klein geometry of a Lie group \( G \) is a homogeneous space of \( G \); that is, a smooth manifold \( M \) on which \( G \) acts smoothly and transitiely. The intuition is that every point of \( M \) "looks the same" through the optics of \( G \). Pick an arbitrary point \( o \in M \) and call it the origin. Let \( H \) be the subgroup of \( G \) consisting of elements which fix the origin. Then \( H \) is a closed subgroup of \( G \) and \( M \) is \( G \)-equivariantly diffeomorphic to the space \( G/H \) of left cosets \( gH \), for \( g \in G \), where the Lie group \( G \) acts on \( G/H \) via left multiplication. Let \( \mathfrak{g} \) and \( \mathfrak{h} \) denote the Lie algebras of \( G \) and \( H \), respectively. Then to a homogeneous space of \( G \) we may assign a Klein pair \(( \mathfrak{g}, \mathfrak{h})\). Conversely a Klein pair \(( \mathfrak{g}, \mathfrak{h})\) is said to be geometrically realisable if there exists a Lie group \( G \) with Lie algebra \( \mathfrak{g} \) such that the connected subgroup \( H \) of \( G \) corresponding to \( \mathfrak{h} \) is closed. Not every Lie pair is geometrically realisable, but it is possible to show that there is a one-to-one correspondence between simply-connected homogeneous spaces and geometrically realisable (effective) Klein pairs. Said differently, such Klein pairs describe homogeneous spaces up to coverings.

In the previous section we described a Poincaré subgroup \( O(d+1,2) \) acting linearly in a pseudo-euclidean space \( \mathbb{E}^{d+1,2} \), which we then decomposed into orbits of the identity component \( G \) of the Poincaré group. Not counting a line of point-like orbits, all other orbits are \((d+1)\)-dimensional and \( G \)-equivariantly diffeomorphic to one of several homogeneous spaces of \( G \): Minkowski spacetime \( \mathbb{M} \) and three other spaces associated to the asymptotic geometry of Minkowski space: \( \mathfrak{Spi}, \mathfrak{Ti}^\pm \) and \( \mathfrak{Ni}^\pm \). As homogeneous spaces of \( G \) there is no distinction between \( \mathfrak{Ti}^+ \) and \( \mathfrak{Ti}^- \) nor between \( \mathfrak{Ni}^+ \) and \( \mathfrak{Ni}^- \), and we will therefore refer to the homogeneous spaces as \( \mathfrak{Ti} \) and \( \mathfrak{Ni} \).

Already in Section 2 we chose some origins in the homogeneous spaces and determined the corresponding stabiliser subgroups and their Lie algebras as subalgebras of the Lie algebra \( \mathfrak{g} \) of the Poincaré group, which are listed in equation (2.6) as vector fields in \( \mathbb{E}^{d+1,2} \). We collect those results here in order to further study the Klein geometries and to determine, in particular, their invariant geometrical structures.

To help to orient the reader let us provide a short overview. In Table 1 we list the Klein pairs \(( \mathfrak{g}, \mathfrak{h})\) in all cases and identify the subalgebra \( \mathfrak{h} \) in the standard basis \( L_{\mu\nu} = -L_{\nu\mu}, P_\mu, \mu, \nu = 0,1, \ldots, d \) for the Poincaré Lie algebra

\[
\begin{align*}
[L_{\mu\nu}, L_{\rho\sigma}] &= \eta_{\nu\rho} L_{\mu\sigma} - \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\sigma} L_{\mu\rho} + \eta_{\mu\sigma} L_{\nu\rho} - 4\eta_{[\rho][\mu\nu]\sigma} \\
[L_{\mu\nu}, P_\rho] &= \eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu = -2\eta_{\rho[m} P_{\nu]} \\
[P_\mu, P_\nu] &= 0,
\end{align*}
\]

related to the one in equation (2.6) by \( B_a = L_{0a} \) and \( H = P_0 \).

The subalgebras of Poincaré which play in \( \mathfrak{Ti}, \mathfrak{Spi} \) or \( \mathfrak{Ni} \) the rôle of the Lorentz subalgebra in \( \mathbb{M} \) admit a uniform description. If we think of Minkowski spacetime as an affine space modelled on a lorentzian vector space \((\mathbb{V}, \eta)\), where \( \eta \) is the Minkowski metric in mostly positive signature, then the subalgebra \( \mathfrak{h} \) in the Klein pairs for \( \mathfrak{Ti}, \mathfrak{Spi} \) and \( \mathfrak{Ni} \) may be described as follows: pick a vector \( v \in \mathbb{V} \) which is, respectively, timelike, spacelike and null and let \( P = v^\mu P_\mu \) be the corresponding momentum generator. The subalgebra \( \mathfrak{h} \) in each of the Klein pairs of \( \mathfrak{Ti}, \mathfrak{Spi} \) and \( \mathfrak{Ni} \) is a semidirect product

\[
\mathfrak{h} = \text{stab}(P) \ltimes P^\perp
\]

of the subalgebra \( \text{stab}(P) \) of \( \mathfrak{so}(\mathbb{V}) \) which fixes \( P \) and the translations perpendicular to \( P \).

We now provide a brief description of this construction for each of the spaces \( \mathfrak{Ti}, \mathfrak{Spi} \) and \( \mathfrak{Ni} \), while a more thorough analysis follows below. For \( \mathfrak{Ti} \) we take \( P = P_0 \) so that \( P^\perp = (P_0) \cong \mathbb{R}^d \), where the indices \( a, b, \ldots = 1, \ldots, d \) run over the spatial directions. The stabiliser \( \text{stab}(P) \) preserving the timelike momentum is \( \mathfrak{so}(d) \cong (L_{ab}) \). For \( \mathfrak{Spi} \), we pick \( P = P_d \), which leaves
\( P^\perp = \langle P_\alpha \rangle \cong \mathbb{R}^{d-1,1} \), where the \( d \)-dimensional lorentzian indices \( \alpha, \beta, \cdots = 0, \ldots, d-1 \) run over all directions except for the \( d \)-direction. The stabiliser of \( P_\alpha \) is the \( d \)-dimensional Lorentz group \( \mathfrak{so}(d-1,1) \cong \langle L_{\alpha\beta} \rangle \). Finally, for \( N_i \), we choose the null momentum \( P = P_- := \frac{1}{\sqrt{2}}(P_d - P_0) \), and hence \( P^\perp = \langle P_-, P_i \rangle \cong \mathbb{R}^d \), where \( i, j, \cdots = 1, \ldots, d-1 \) are \( (d-1) \)-dimensional spatial indices. The stabiliser of \( P_- \) is \( \mathfrak{iso}(d-1) \cong \langle L_{ij}, L_{-i} \rangle \). These constructions are summarised in Table 1.

We will at various points change basis. In the primed basis the carrollian nature of \( T_i, S_{pi} \) and \( N_i \) is more manifest, with \( B' \) denoting in all cases the (generalised) carrollian boosts and \( P' \) (generalised) carrollian translations. The explicit relations between the primed and unprimed bases for each of the spaces are given in Table 1. The invariants can also be given more uniformly, schematically, by a carrollian metric \( \pi'^2 \) and carrollian vector field(s) \( H' \), and we refer to Table 1 for an overview of these.
Table 1. Overview of the \((d + 1)\)-dimensional homogeneous spaces of the Poincaré group that are covered in this work

| \((g, h)\) Embedding | \(\mathcal{M}_{d+1}\) | \(\mathcal{T}_{d+1} = \text{AdS}_{d+1}\) | \(\mathcal{S}_{d+1}\) | \(\mathcal{N}_{d+1} = (\mathcal{D}_0 \cap \mathcal{N}_0)^+\) |
|----------------------|-----------------|-----------------|-----------------|-----------------|
| \(\mathfrak{e}^\text{orig} \cong \mathfrak{iso}(d, 1)\) | \((\Lambda_{uv}, P_a)_{\mu = 0, \ldots, d, a}\) | \((\Lambda_{ab}, B_a = \Lambda_{a0}, P_a, H = P_0)_{a, b = 0, \ldots, d-1}\) | \((\Lambda_{ab}, B_a = \Lambda_{a0}, P_a, P_d)_{a, b = 0, \ldots, d-1}\) | \((\Lambda_{ij}, L_{ij}, L_{i}, L_{-i}, P_i, P_{-i})_{j = 1, \ldots, d-1}\) |
| \(\mathfrak{h}^\text{orig} \cong \mathfrak{so}(d, 1)\) | \((\Lambda_{uv}, P_a)\) | \((\Lambda_{ab}, P_a)\) | \(\mathfrak{so}(d - 1, 1) \cong \mathfrak{so}(d - 1, 1) \times \mathbb{R}^{d-1,1}\) | \(\mathfrak{iso}(d - 1) \times \mathbb{R}^d \cong \text{Carroll}(d)\) |

The four columns correspond to the Klein pairs \((g, h)\) of the homogeneous spaces of the Poincaré group that we cover in this work. In particular, \(g \cong \mathfrak{iso}(d, 1)\) in all cases, but the subalgebra \(h\) differs between the Klein pairs. Additionally, we recall from Section 2 the embeddings of these spaces into \(\mathbb{E}^{d+1,2}\) as intersections of the quadric \(\mathcal{Q}_d\) with null planes of the form \(\mathcal{N}_0\), and in the case of \(\mathcal{T}\) and \(\mathcal{N}\) with one of the components as explained in Section 2.

In the first section of the Table we provide the decomposition of \(g\) and the subalgebra \(h\) in terms the common Poincaré basis and describe the abstract Lie algebra structure of \(h\).

In the main text of this work we change basis to make their carrollian nature more manifest. In the second section of the Table we provide this change of basis, listing only those elements where the change of basis amounts to more than simply adding a dash to the symbol. These changes are such that the subalgebra is now spanned by (generalised) rotations \(L'\) and (generalised) boosts \(B'\). Additionally we list the invariants of low rank which characterise the class of the geometry.

It is understood that \(\pi'\) is dual to \(P'\). Finally, we list the Lie subalgebra of vector fields which preserve the respective invariants.
TABLE 2. Overview of the d-dimensional homogeneous spaces of the Poincaré group that descend from \( T_{i_d+1} = \text{AdSC}_{d+1} \) and \( \text{Spi}_{d+1} \)

| \( (g, h) \) | \( \mathcal{H}^d \) | \( dS_d \) |
|---|---|---|
| \( g\text{orig} \cong \text{iso}(d, 1) \) | \( (L_{ab}, B_a = L_{0a}, P_a, H = P_0)_{a, b = 1, \ldots, d} \) | \( (L_{ab}, B_a = L_{0a}, P_a, P_0)_{a, b = 0, \ldots, d-1} \) |
| \( \eta\text{orig} \) | \( (L_{ab}, P_a, H) \) | \( (L_{ab}, P_a, P_0) \) |
| \( \eta\text{orig} \cong \mathfrak{so}(d, 1) \) | \( \mathfrak{so}(d) \ltimes \mathbb{R}^d \) | \( \mathfrak{so}(d - 1, 1) \ltimes \mathbb{R}^d \) |
| Effective | No | No |
| Reduction | | |
| \( h \) | \( B'_a = P_a, P'_0 = B_a, H' = -H \) | \( B'_a = P_a, P'_0 = B_a \) |
| Noneff \( \rightarrow \) Eff | \( (4.12) \rightarrow (4.15) \) | \( (4.7) \rightarrow (4.10) \) |
| Invariants | Classical | \( \mathfrak{so}(d, 1) \)
| | Symmetries | riemannian |

This table summarises the \((d-1)\)-dimensional homogeneous spaces of the Poincaré group that descend from \( T_{i_d+1} = \text{AdSC}_{d+1} \) and \( \text{Spi}_{d+1} \), given by \( \mathcal{H}^d \) and \( dS_d \), respectively. Neither is effective and their invariants of low rank are given by nondegenerate metrics, i.e., they are (pseudo-)lorentzian and share the same symmetry algebra \( \mathfrak{so}(d, 1) \). In the “Noneff \( \rightarrow \) Eff” column we link the noneffective and effective Lie pairs.

TABLE 3. Overview of \( d \) and \((d - 1)\)-dimensional homogeneous spaces of the Poincaré group that descend from \( N_{i_{d+1}} \)

| \( [g, b] \) | \( \mathcal{Z}_d \) | \( \mathcal{Z}_d \) | \( \mathcal{CS}^{d-1} \) |
|---|---|---|---|
| \( g\text{orig} \cong \text{iso}(d, 1) \) | \( (L_{ij}, L_{i1}, L_{11}, P_1, \ldots, P_d, P_{d+1}) \) | \( (L_{ij}, L_{1i}, L_{11}, P_1, \ldots, P_d, P_{d+1}) \) | \( (L_{ij}, L_{i1}, L_{11}, P_1, \ldots, P_d, P_{d+1}) \) |
| \( h\text{orig} \cong \mathfrak{so}(d - 1) \ltimes \mathbb{R}^d \) | \( \mathfrak{so}(d - 1) \ltimes \mathbb{R}^d \) | \( \mathfrak{so}(d - 1) \ltimes \mathbb{R}^d \) | \( \mathfrak{so}(d - 1) \ltimes \mathbb{R}^d \) |
| Effective | Yes | No | No |
| Reduction | | | |
| \( b \) | \( L'_{i1} = L_{i1}, B'_{11} = B_{11}, P'_{11} = P_{11}, P'_i = L_{i1}, P'_{d+1} = P_d \) | \( L'_{i1} = L_{i1}, B'_{11} = B_{11}, P'_{11} = P_{11}, P'_i = L_{i1}, P'_{d+1} = P_d \) | \( L'_{i1} = L_{i1}, B'_{11} = B_{11}, P'_{11} = P_{11}, P'_i = L_{i1}, P'_{d+1} = P_d \) |
| Noneff \( \rightarrow \) Eff | \( (4.19) \rightarrow (4.28) \) | \( (4.19) \rightarrow (4.32) \) | \( (4.19) \rightarrow (4.32) \) |
| Invariants | Classical | \( \mathfrak{so}(d, 1) \) | \( \mathfrak{so}(d, 1) \) | \( \mathfrak{so}(d, 1) \)
| | Symmetries | conformal | conformal | conformal |
| | (up to scale) | (up to scale) | (up to scale) |

This table summarises the \( d \) and \((d - 1)\)-dimensional homogeneous spaces of the Poincaré group that descend from \( N_{i_{d+1}} \), where \( i = 1, \ldots, d - 2 \). Notably \( \mathcal{Z} \) is effective, while the lightcone \( \mathcal{Z} \) and the celestial sphere \( \mathcal{CS} \) are not. The lightcone is the only case in possession of low-rank invariants. For the other two cases, the dilatation-like action of \( L_{i1} = P'_i \) only allows for invariants up to scale: conformal carrollian and conformal riemannian, respectively. In the last row we provide for \( \mathcal{Z} \) the symmetries of these invariants, and for the other two cases the conformal symmetries of their respective conformal invariants. In the “Noneff \( \rightarrow \) Eff” column we link from the noneffective to the effective Lie pair or just link to the effective one.
4.1. Minkowski spacetime $\mathcal{M}$. The Klein pair $(\mathfrak{g}, \mathfrak{h}_\mathcal{M})$ for Minkowski spacetime has
\[ \mathfrak{h}_\mathcal{M} = (L_{\mu \nu}), \tag{4.3} \]
which is a Lorentz subalgebra and hence spanned by the rotations and Lorentz boosts. We complete this basis for $\mathfrak{h}_\mathcal{M}$ to a basis for $\mathfrak{g}$ by the addition of translations $P_\mu$. The Poincaré Lie algebra in this basis takes the standard form given in equation (4.1). We observe that the split $\mathfrak{g} = \mathfrak{h}_\mathcal{M} \oplus \mathfrak{m}_\mathcal{M}$, where $\mathfrak{m}_\mathcal{M}$ is the span of the $P_\mu$, is both reductive $([\mathfrak{h}_\mathcal{M}, \mathfrak{m}_\mathcal{M}] \subset \mathfrak{m}_\mathcal{M})$ and symmetric $([\mathfrak{m}_\mathcal{M}, \mathfrak{m}_\mathcal{M}] \subset \mathfrak{h}_\mathcal{M})$. The Poincaré-invariant tensor fields correspond to the Lorentz-invariant tensors of the linear isotropy representation $\mathfrak{m}_\mathcal{M}$: namely, $\eta_{\mu\nu}\pi^\mu\pi^\nu$, corresponding to the Minkowski metric, and $\eta^{\mu\nu}P_\mu P_\nu$, corresponding to its inverse. Here and in what follows, $\pi^\mu$ is the basis of $\mathfrak{m}_\mathcal{M}$ canonically dual to $P_\mu$: that is, $\langle \pi^\mu, P_\nu \rangle = \delta^\mu_\nu$.

The vector fields that preserve the invariant structure $\eta_{\mu\nu}\pi^\mu\pi^\nu$ clearly generate the symmetry algebra of Minkowski spacetime, i.e., the Poincaré algebra.

4.2. Spatial infinity $\text{Spi}$. As already discussed in Section 2.2.2, the Klein pair for $\text{Spi}$ is $(\mathfrak{g}, \mathfrak{h}_{\text{Spi}})$ where
\[ \mathfrak{h}_{\text{Spi}} = (L_{ij}, B_i, P_i, H), \tag{4.4} \]
where $i = 1, \ldots, d-1$. In terms of the semidirect product decomposition (4.2), the subalgebra $\mathfrak{h}_{\text{Spi}}$ is comprised of the stabiliser of the spacelike momentum $P = P_d$, which is $\text{stab}(P) \cong \mathfrak{so}(d-1, 1)$, and the perpendicular translations $P^\perp = (P_\alpha)$ $\cong \mathbb{R}^{d-1,1}$, where $\alpha, \beta, \cdots = 0, 1, \ldots, d-1$. It is convenient to restore some of the symmetry by breaking the manifest Lorentz symmetry in (4.1) via the basis $L_{\alpha\beta}, P_\alpha, B_\alpha := L_{\alpha d}$ and $P_d$ with (nonzero) Lie brackets
\[
\begin{align*}
[L_{\alpha\beta}, L_{\gamma\delta}] &= \eta_{\beta\gamma}L_{\alpha\delta} - \eta_{\alpha\gamma}L_{\beta\delta} - \eta_{\beta\delta}L_{\alpha\gamma} + \eta_{\alpha\delta}L_{\beta\gamma} \\
[L_{\alpha\beta}, B_\gamma] &= \eta_{\beta\gamma}B_\alpha - \eta_{\alpha\gamma}B_\beta \\
[L_{\alpha\beta}, P_\gamma] &= \eta_{\beta\gamma}P_\alpha - \eta_{\alpha\gamma}P_\beta \\
[B_\alpha, B_\beta] &= -L_{\alpha\beta} \\
[B_\alpha, P_\beta] &= -\eta_{\alpha\beta}P_d \\
[B_\alpha, P_d] &= P_\alpha,
\end{align*}
\tag{4.5}
\]
where $\eta_{\alpha\beta}$ is the $d$-dimensional lorentzian inner product with mostly plus signature. In this notation, the Klein pair $(\mathfrak{g}, \mathfrak{h}_{\text{Spi}})$ is such that
\[ \mathfrak{h}_{\text{Spi}} = \langle (L_{\alpha\beta}, B_\alpha) \rangle. \tag{4.6} \]
It is convenient to relabel $L_{\alpha\beta}' = L_{\alpha\beta}, B_\alpha' = P_\alpha, P_\alpha' = B_\alpha$ and $P_d' = P_d$ to arrive at
\[
\begin{align*}
[L_{\alpha\beta}', L_{\gamma\delta}] &= \eta_{\beta\gamma}L_{\alpha\delta}' - \eta_{\alpha\gamma}L_{\beta\delta}' - \eta_{\beta\delta}L_{\alpha\gamma}' + \eta_{\alpha\delta}L_{\beta\gamma}' \\
[L_{\alpha\beta}', B_\gamma'] &= \eta_{\beta\gamma}B_\alpha' - \eta_{\alpha\gamma}B_\beta' \\
[L_{\alpha\beta}', P_\gamma'] &= \eta_{\beta\gamma}P_\alpha' - \eta_{\alpha\gamma}P_\beta' \\
[B_\alpha', B_\beta'] &= \eta_{\alpha\beta}P_d' \\
[B_\alpha', P_d'] &= B_\alpha' \\
[P_\alpha', P_d'] &= -L_{\alpha\beta}'.
\end{align*}
\tag{4.7}
\]
As in the case of Minkowski spacetime, the split $\mathfrak{g} = \mathfrak{h}_{\text{Spi}} \oplus \mathfrak{m}_{\text{Spi}}$, where
\[ \mathfrak{h}_{\text{Spi}} = \langle (L_{\alpha\beta}', B_\alpha') \rangle \tag{4.8} \]
and $\mathfrak{m}_{\text{Spi}}$ is now the span of $P_\alpha'$ and $P_d'$. It is both reductive and symmetric.

The computation of the low-rank invariants is formally identical to those of Ti (AdSC in [8]), changing $\delta \mapsto \eta$. They result in a pseudo-carrollian structure consisting of a nowhere-vanishing
vector field corresponding to $P_\mu'$ and a degenerate lorentzian metric corresponding to $\eta_{\alpha\beta}\pi^{\alpha}\pi^{\beta}$, allowing us to conclude that $\text{Spi}$ is a pseudo-carrollian symmetric space [6].

In parallel to the symmetries of carrollian structure of $\mathfrak{T}_i$, which were worked out in [23], one finds that the vector fields preserving the pseudo-carrollian structure of $\text{Spi}$ generate the algebra $\mathfrak{so}(d,1) \ltimes C^\infty(dS_d)$, i.e., a semi-direct product of the Lorentz group, the Killing vectors of $dS_d$ and $\text{Spi}$-supertranslations. As in the case of AdS, the action of the Killing vectors of $dS_d$ on $C^\infty(dS_d)$ is not as functions, but as sections on the density line bundle. These results essentially go back to [7].

We can uncover the asymptotic geometry of $\text{Spi}$. We add $P_\mu'$ to $\mathfrak{h}_{\text{Spi}}$ to obtain

$$\mathfrak{h}_{\text{AS}} = \langle L_{\alpha\beta}, P_\alpha, P_\mu \rangle = \langle L'_{\alpha\beta}, B_\alpha', P'_\mu \rangle. \quad (4.9)$$

We observe that the Klein pair $(\mathfrak{g}, \mathfrak{h}_{\text{AS}})$ is not effective because $\mathfrak{t} = \langle P_\mu \rangle = \langle B_\alpha', P'_\mu \rangle$ is an ideal of $\mathfrak{g}$ contained in $\mathfrak{h}_{\text{AS}}$. Quotienting by $\mathfrak{t}$, we arrive at an effective Klein pair $(\mathfrak{g}/\mathfrak{t}, \mathfrak{h}/\mathfrak{t})$. The quotient Lie algebra $\mathfrak{g}/\mathfrak{t}$ is spanned by the image of $L'_{\alpha\beta}, P'_\alpha$ in $\mathfrak{g}/\mathfrak{t}$. If we again use the same notation for their images, we obtain

$$[L'_{\alpha\beta}, L'_\gamma] = \eta_{\beta\gamma}L'_\alpha - \eta_{\alpha\gamma}L'_\beta - \eta_{\delta\gamma}L'_\alpha + \eta_{\delta\alpha}L'_\beta,$$

$$[L'_{\alpha\beta}, P'_\gamma] = \eta_{\beta\gamma}P'_\alpha - \eta_{\alpha\gamma}P'_\beta,$$

$$[P'_\alpha, P'_\beta] = -L'_{\alpha\beta}. \quad (4.10)$$

from where we see that $(\mathfrak{g}/\mathfrak{t}, \mathfrak{h}_{\text{AS}}/\mathfrak{t}) \cong (\mathfrak{so}(d,1), \mathfrak{so}(d-1,1))$ is the Klein pair of de Sitter spacetime $dS_d$. In terms of the original basis, $L'_{\alpha\beta} = L_{\alpha\beta}$ and $P'_\alpha = P_\alpha$, so that the Minkowski translations act trivially on the asymptotic geometry.

4.3. Timelike infinity $\mathfrak{T}_i \cong \text{AdS}$. In Section 2.2.3, we determined that the Klein pair $(\mathfrak{g}, \mathfrak{h}_{\mathfrak{T}_i})$ is such that

$$\mathfrak{h}_{\mathfrak{T}_i} = \langle L_{ab}, P_a \rangle. \quad (4.11)$$

This subalgebra consists of the stabiliser of the timelike momentum $P = P_0$, which is isomorphic to $\mathfrak{so}(d)$, and the perpendicular momenta $P^\perp = \langle P_a \rangle \cong \mathbb{R}^d$, where the indices $a, b, \ldots = 1, \ldots, d$ run over the spatial directions. To emphasise that $P_a$ should actually be understood as Carroll boosts we relabel the basis so that the Carroll boosts are $B'_a = P_a$. We relabel the rest of the generators as $L'_{ab} = L_{ab}, P'_a = B_a$ and $H' = -H$. In the new basis, the nonzero Lie brackets then become:

$$[L'_{ab}, L'_c] = \delta_{bc}L'_a - \delta_{ac}L'_b - \delta_{bd}L'_c + \delta_{ad}L'_b,$$

$$[L'_{ab}, B'_c] = \delta_{bc}B'_a - \delta_{ac}B'_b,$$

$$[L'_{ab}, P'_c] = \delta_{bc}P'_a - \delta_{ac}P'_b,$$

$$[B'_a, P'_b] = \delta_{ab}H',$$

$$[H', P'_a] = B'_a,$$

$$[P'_a, P'_b] = L'_{ab}. \quad (4.12)$$

This means that the isotropy subalgebra is now

$$\mathfrak{h}_{\mathfrak{T}_i} = \langle L'_{ab}, B'_a \rangle \quad (4.13)$$

and $m_{\mathfrak{T}_i}$ is the span of $P'_a, H'$. We see that the split $\mathfrak{g} = \mathfrak{h}_{\mathfrak{T}_i} \oplus m_{\mathfrak{T}_i}$ is again both reductive and symmetric, consistent with the fact that $\mathfrak{T}_i$ is a carrollian symmetric space.

One can now calculate the Poincaré-invariant tensor fields of this spacetime. This has been done in, e.g., [8], where it was shown that they are given by a nowhere-vanishing vector field corresponding to $H' \in m_{\mathfrak{T}_i}$ and a degenerate metric corresponding to the $\mathfrak{h}_{\mathfrak{T}_i}$-invariant bilinear form $\delta_{ab}\pi^{\alpha a}\pi^{\beta b} \in \otimes^2 m_{\mathfrak{T}_i}^*$, where $\langle \pi^{\alpha a}, P'_b \rangle = \delta_{b}^{\alpha}$. 

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As in the case of \( \text{Spi} \), the symmetry algebra of \( \text{Ti} \) is determined by the algebra of vector fields preserving the above invariant structure. In [23] the symmetry algebra was determined to be the infinite-dimensional algebra \( \frak{so}(d, 1) \times \mathbb{C}^{\infty}(\mathcal{M}^{d}) \), where \( \mathbb{C}^{\infty}(\mathcal{M}^{d}) \) are the smooth functions on \( d \)-dimensional hyperbolic space, although the action of \( \frak{so}(d, 1) \) on \( \mathbb{C}^{\infty}(\mathcal{M}^{d}) \) is that of Killing vectors on \( \mathcal{M}^{d} \) not on functions but on sections of the density line bundle. The symmetry algebra consists of Lorentz transformations and \( \text{Ti} \)-supertranslations.

The decomposition above suggests another Klein pair where one adds

\[
\frak{b}_{\mathcal{M}} = \langle \mathcal{L}_{ab}, \mathcal{P}_{a}, H \rangle = \langle \mathcal{L}_{ab}', \mathcal{B}_{a}', H' \rangle. \tag{4.14}
\]

We observe that the Klein pair \( (\frak{g}, \frak{h}_{\mathcal{M}}) \) is not effective, as \( \frak{t} = \langle P_{a} \rangle = \langle \mathcal{B}_{a}', H' \rangle \) is an ideal of \( \frak{g} \) contained in \( \frak{h}_{\mathcal{M}} \). We may quotient by \( \frak{t} \) to arrive at an effective Klein pair \( (\frak{g}/\frak{t}, \frak{h}_{\mathcal{M}}/\frak{t}) \), where the quotient Lie algebra \( \frak{g}/\frak{t} \) is spanned by the images of \( \mathcal{L}_{ab}', \mathcal{P}_{a}' \). Letting them stand for their images in \( \frak{g}/\frak{t} \), the brackets of \( \frak{g}/\frak{t} \) are given by

\[
\begin{align*}
[\mathcal{L}_{ab}', \mathcal{L}_{cd}'] &= \delta_{bc} \mathcal{L}_{ad}' - \delta_{ac} \mathcal{L}_{bd}' - \delta_{ad} \mathcal{L}_{bc}' + \delta_{bd} \mathcal{L}_{ac}' \\
[\mathcal{L}_{ab}', \mathcal{P}_{c}'] &= \delta_{bc} \mathcal{P}_{a}' - \delta_{ac} \mathcal{P}_{b}' \\
[\mathcal{P}_{a}', \mathcal{P}_{b}'] &= \delta_{ab}.
\end{align*}
\tag{4.15}
\]

The Klein pair \( (\frak{g}/\frak{t}, \frak{h}_{\mathcal{M}}/\frak{t}) \cong (\frak{so}(d, 1), \frak{so}(d)) \) is the infinitesimal description of \( d \)-dimensional hyperbolic space \( \mathcal{M}^{d} \), whose invariant riemannian metric can then be understood as the asymptotic geometry at \( \text{Ti} \) (in analogy to the one of \( \text{Spi} \) as described, e.g., in [25]). The Killing symmetries of this metric are again \( \frak{so}(d, 1) \) which is isomorphic to a Lorentz subalgebra of the original Poincaré algebra. This algebra is spanned by the Minkowski rotations \( \mathcal{L}_{ab} \) and Minkowski boosts \( \mathcal{B}_{a} \), so that the translations of Minkowski spacetime (corresponding to the ideal \( \frak{t} \)) act trivially on the asymptotic geometry of \( \text{Ti} \).

### 4.4. The doubly-carrollian manifold \( \text{Ni} \)

In Section 2.2.4, after choosing a suitable origin for \( \text{Ni} \), we determined the stabiliser subgroup. The corresponding Lie pair is \( (\frak{g}, \frak{h}_{\text{Ni}}) \), where

\[
\frak{h}_{\text{Ni}} = \langle \mathcal{L}_{ij}, P_{1}, L_{1d} + B_{i}, P_{d} + H \rangle, \tag{4.16}
\]

for \( i, j = 1, \ldots, d - 1 \). It is convenient to change basis to \( P_{1} \) and \( P_{\pm} := \frac{1}{\sqrt{2}} (P_{d} \pm P_{0}) \), and where \( i = 1, \ldots, d - 1 \) now, with \( \eta_{ij} = \delta_{ij} \) and \( \eta_{++} = \eta_{--} = 1 \) with other components zero. The Lorentz generators break up as \( \mathcal{L}_{ij}, L_{1+i}, L_{-i} \) and \( L_{++} \). In this basis,

\[
\frak{h}_{\text{Ni}} = \langle \mathcal{L}_{ij}, P_{1}, L_{1+i}, P_{-} \rangle, \tag{4.17}
\]

and the (nonzero) Lie brackets (4.1) of the Poincaré Lie algebra are given by

\[
\begin{align*}
[\mathcal{L}_{ij}, \mathcal{L}_{kl}] &= \delta_{ik} \mathcal{L}_{jl} - \delta_{jk} \mathcal{L}_{il} - \delta_{il} \mathcal{L}_{jk} + \delta_{kl} \mathcal{L}_{ij} \\
[\mathcal{L}_{ij}, \mathcal{L}_{\pm k}] &= \delta_{ik} \mathcal{L}_{\pm j} - \delta_{jk} \mathcal{L}_{\pm i} + \delta_{\pm j} \mathcal{L}_{ik} \\
[\mathcal{L}_{1+i}, P_{1}] &= -\mathcal{L}_{ij} - \delta_{ij} \mathcal{L}_{1+i} \\
[\mathcal{L}_{1+i}, L_{-i}] &= \pm L_{1+i} \\
[\mathcal{L}_{1+i}, L_{\pm i}] &= \pm L_{\pm i} \tag{4.18}
\end{align*}
\]

The subalgebra \( \frak{h}_{\text{Ni}} \) is a semidirect product (cf., (4.2)) of the stabiliser of \( P = P_{-} \), which is isomorphic to \( \frak{iso}(d - 1) \), and the perpendicular translations \( P^{\perp} = \langle P_{-}, P_{1} \rangle \cong \mathbb{R}^{d} \).

Let us already make some observations. Firstly, in contrast to \( \frak{M}, \frak{Ti} \) and \( \frak{Spi} \), we keep only manifest symmetry under \( \frak{so}(d - 1) \) rather than the larger \( \frak{so}(d) \) or \( \frak{so}(d - 1, 1) \). This means that \( \text{Ni} \) has three \( \frak{so}(d - 1) \) vectors \( \langle L_{1+i}, P_{1} \rangle \) and three \( \frak{so}(d - 1) \) scalars \( \langle L_{+-}, P_{\pm} \rangle \) rather than two vectors and one scalar, as is the case in \( \text{Ti} \) and \( \frak{Spi} \).

Secondly, the \( \frak{so}(d, 1) \) Lorentz subalgebra spanned by \( \langle L_{ij}, L_{\pm i}, L_{+-} \rangle \), i.e., the first four brackets of (4.18), are written in such a way that the relation to euclidean conformal field theory in \( d - 1 \) dimensions is manifest. Indeed, both have \( \frak{so}(d - 1) \) symmetries, however acting differently on the underlying manifold. The dilatations are given by \( L_{++} \), and the translations and special
conformal transformations are given by $L_{ij}$. This observation is one of the motivations to study flat space holography from the point of view of $(d-1)$-dimensional celestial conformal field theory, see, e.g., [26, 27] for a review.

The Klein pair $(g, h_{\text{NI}})$ is non-reductive. In other words, we cannot find a complement $m_{\text{NI}}$ to $h_{\text{NI}}$ in $g$ for which $[h, m_{\text{NI}}] \subset m_{\text{NI}}$. This means that we must make a choice. We choose $m_{\text{NI}}$ to be the subspace of $g$ spanned by $P_+, L_{++}, L_{+-}$. Let us relabel the basis so that $P'_+ = P_+, P'_- = L_{+-}$ and $P'_i = L_{++}, L'_i = L_{--}, B'_i = P_i$ and $B'_- = P_-$, in terms of which the (nonzero) Poincaré Lie brackets (4.18) are given by

$$
\begin{align*}
[L'_{ij}, L'_{kl}] &= \delta_{ik}L'_{jl} - \delta_{jk}L'_{il} - \delta_{il}L'_{jk} + \delta_{jl}L'_{ik} \\
[L'_{ij}, L'_{ik}] &= \delta_{ik}L'_{j} - \delta_{jk}L'_{i} \\
[L'_{ij}, B'_{k}] &= \delta_{ik}B'_{j} - \delta_{jk}B'_{i} \\
[L'_{ij}, P'_{k}] &= \delta_{ik}P'_{j} - \delta_{jk}P'_{i} \\
[L'_{ij}, P'_{j}] &= -L'_{ij} + \delta_{ij}P' \\
[B'_i, P'_j] &= -\delta_{ij}P'_+ \\
\end{align*}
$$

and

$$
[B'_+, P'_+] = B'_- \\
B'_-, P'_- = P'_+.
$$

In this basis, we define

$$
h_{\text{NI}} = \langle L_{ij}, L_{-i}, P_i, P_j \rangle = \langle L'_{ij}, L'_{i}, B'_i, B'_j \rangle. \tag{4.20}
$$

The subalgebra $h_{\text{NI}}$ can be seen to be isomorphic to the $d$-dimensional Carroll algebra, when $L'_i$ are interpreted as spatial translations and $B'_-\,$ as time translations. This has a simple geometric explanation: the orbits in Minkowski spacetime of the subgroup of the Poincaré group generated by $h_{\text{NI}}$ are null hyperplanes, which as shown, e.g., in [8, Section 4.2.5.], are homogeneous spacetimes of the Carroll group; namely, copies of the $d$-dimensional carrollian spacetime $\mathbb{C}_d$.

Since the Klein pair $(g, h_{\text{NI}})$ is non-reductive, the linear isotropy representation is the quotient $g/h_{\text{NI}}$ and its dual is the annihilator of $h_{\text{NI}}$ in $g^*$. This space is spanned by $\langle \pi^+, \pi'^+, \pi^\gamma \rangle$ and the action of $h_{\text{NI}}$ is given by

$$
\begin{align*}
L'_{ij} \cdot \pi^k &= \delta_{ik} \delta_{lm} \pi^m - \delta_{im} \delta_{lj} \pi^m \\
L'_{ij} \cdot \pi^\gamma &= -\delta_{ij} \pi^\gamma \\
B'_i \cdot \pi'^+ &= \delta_{ij} \pi^\gamma.
\end{align*}
$$

We illustrate how these are obtained for the last of the expressions above. The action of $h_{\text{NI}}$ on its annihilator is the restriction of the coadjoint action of $g$. This is a linear map $g \to \mathfrak{g}(g^*)$, which, when restricted to $h_{\text{NI}}$, leaves invariant the annihilator of $h_{\text{NI}}$. This action is defined as follows: if $\alpha \in g^*$ and $X \in h_{\text{NI}}$, $\langle X, \alpha \rangle = -\alpha \circ \alpha$ on $\mathfrak{g}$, $\alpha \circ \alpha$. Applying this to $X = B'_i$ and $\alpha = \pi'^+$, we find that for $X \in g$, $B'_i \cdot \pi'^+ (X) = -\langle \pi'^+, [B'_i, X] \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the dual pairing. The only contributing bracket is $[B'_i, P'_+] = -\delta_{ij}P'_+$, which gives rise to $B'_i \cdot \pi'^+ (P'_+) = \delta_{ij}$, allowing us to conclude that $B'_i \cdot \pi'^+ = \delta_{ij} \pi^\gamma$.

It follows that the Poincaré-vanishing vector fields up to second rank on $N_\text{I}$ are generated by two nowhere-vanishing vector fields corresponding to the $h_{\text{NI}}$-invariant vectors $P_{+} \in g/h_{\text{NI}}$ and the $h_{\text{NI}}$-invariant symmetric bilinear form $\delta_{ij} \pi^\alpha \pi^\beta$. In addition, let us highlight the existence of the invariant volume form

$$
\epsilon_{a_1 \cdots a_{d-1}} \pi^{a_1} \cdots \pi^{a_{d-1}}. \quad (4.22)
$$

An invariant structure of the above type, i.e., two nowhere-vanishing vector fields together with a doubly degenerate metric, differs from previously known structures such as carrollian, galilean, aristotelian or their stringy versions. We will refer to such a structure tentatively as \textit{doubly-carrollian}. The justification is that carrollian structures arise naturally in the bundle of
scales of a conformal manifold and this structure arises naturally in the bundle of scales of a conformal carrollian manifold.

The Lie algebra of symmetries of the Poincaré-invariant tensors in Ni is calculated in Appendix C. Summarising the results, we find that for \( d \geq 3 \), the symmetry algebra is \( \mathfrak{so}(d, 1) \times C^\infty(\mathbb{C}S^{d-1}) \), whereas for \( d = 2 \) it is isomorphic to \( \mathcal{K}(\mathbb{C}S^1) \times C^\infty(\mathbb{C}S^1) \), where \( \mathcal{K}(\mathbb{C}S^1) \) is the Lie algebra of smooth vector fields on the (celestial) circle. In all cases, the abelian ideals of smooth functions on \( \mathbb{C}S^{d-1} \) and the action of the Lie algebra of conformal Killing vectors (\( \mathfrak{so}(d, 1) \) or \( \mathcal{K}(\mathbb{C}S^1) \)) on them suggest that they should be interpreted as sections of the density line bundle, as explained in [23, §10].

These symmetry algebras are precisely BMS\(_{d+1}\) which suggests that Ni is closely related to null infinity. We will see in the next section that this is indeed the case.

4.5. \( \mathcal{I} \) and \( \mathcal{I}' \): two ways to the celestial sphere \( \mathcal{CS} \). In Sections 4.2 and 4.3, we saw that the respective Klein pairs \((\mathfrak{g}, \mathfrak{h})\) of Spi and Ti allowed for enlargements of the subalgebra \( \mathfrak{h} \) that led to the naturally related lower-dimensional homogeneous spaces (4.15) and (4.10). Similarly, we find in the following that there are several lower-dimensional Klein geometries \((\mathfrak{g}, \mathfrak{h})\) which can be obtained from \( \mathrm{Ni} \) by enlarging the stabiliser subalgebra \( \mathfrak{h}_{\mathrm{Ni}} \subset \mathfrak{h} \) while keeping \( \mathfrak{g} \) fixed as the Poincaré algebra. Geometrically this corresponds to viewing \( \mathrm{Ni} \) as the total space of a principal bundle over some lower-dimensional homogeneous spaces.

4.5.1. Null infinity \( \mathcal{I} \). Still working in the basis (4.19) for the Poincaré algebra, we may add \( P'_i \) to \( \mathfrak{h}_{\mathrm{Ni}} \) to arrive at

\[
\mathfrak{h}_{\mathcal{I}} = \langle L_{ij}, L_{-\cdot}, P_{\pm}, L_{+-} \rangle = \langle L'_{ij}, B'_i, B'_{-\cdot}, P'_i \rangle.
\]

(4.23)

Since \( \mathfrak{h}_{\mathcal{I}} \) does not contain a nonzero ideal of \( \mathfrak{g} \), the resulting Klein pair \((\mathfrak{g}, \mathfrak{h}_{\mathcal{I}})\) is effective. The simply-connected homogeneous space based on this Klein pair

\[
\mathrm{ISO}(d, 1)_0/ (\{\mathrm{ISO}(d - 1) \times \mathbb{R}^d \} \times \mathbb{R})
\]

(4.24)

can be identified with \( \mathcal{I} \), i.e., the null boundary of \((d + 1)\)-dimensional Minkowski spacetime as discussed, e.g., in [28]. As we will see in Section 5.4 we can also view it as the grassmannian of affine null hyperplanes in Minkowski spacetime.

The dual of the linear isotropy representation of \( \mathfrak{h}_{\mathcal{I}} \) is given relative to the basis \( \langle \pi'^i, \pi'^+ \rangle \) by

\[
\begin{align*}
L'_{ij} \cdot \pi'^k &= \delta^{k}_{j} \delta_{lm} \pi'^{lm} - \delta^{k}_{l} \delta_{jm} \pi'^{lm} \\
B'_i \cdot \pi'^+ &= \delta_{ij} \pi'^{j} \\
P'_i \cdot \pi'^i &= -\pi'^i \\
P'_i \cdot \pi'^+ &= -\pi'^+.
\end{align*}
\]

(4.25)

Since \( P'_i \) is now in \( \mathfrak{h}_{\mathcal{I}} \) and it acts like a dilatation, there are no invariant tensors of low rank. However, it is natural to look for invariant conformal classes of tensors. They turn out to be a vector field \( P'_i \) with conformal weight \(-1\) and a degenerate metric \( \pi^2 = \delta_{ij} \pi'^{i} \pi'^{j} \) with conformal weight 2. This space thus admits a conformal carrollian structure. The symmetries of this structure give rise to the BMS group in \((d + 1)\) dimensions [25, 29]

\[
\begin{align*}
\mathfrak{so}(d, 1) \times C^\infty(\mathbb{C}S^{d-1}) &\cong \text{BMS}_{d+1} \quad (d \geq 3) \\
\mathcal{K}(\mathbb{C}S^1) \times C^\infty(\mathbb{C}S^1) &\cong \text{BMS}_2
\end{align*}
\]

(4.26)
as is expected from the identification of this homogeneous space with \( \mathcal{I} \).
4.5.2. **Lightcone \( \mathcal{L} \).** Starting again from the Klein pair of \( \mathfrak{Ni} \), we may alternatively add \( P'_+ \) to \( \mathfrak{h}_\mathcal{N} \) to arrive at

\[
\mathfrak{h}_\mathcal{L} = \langle L_{ij}, L_{-i}, P_i, P_{-i}, P_+ \rangle = \langle L'_{ij}, L'_i, B'_i, B'_+, P'_{+} \rangle.
\]  
(4.27)

Since \( \mathfrak{h}_\mathcal{L} \) now contains the ideal \( \mathfrak{t} = \langle P_\mu \rangle = \langle B'_i, B'_+, P'_+ \rangle \), the Klein pair is not effective as we may quotient both \( \mathfrak{g} \) and \( \mathfrak{h}_\mathcal{L} \) by this ideal. Therefore \( \{ \mathfrak{g}, \mathfrak{h}_\mathcal{L} \} \) reduces to \( \{ \mathfrak{g}/\mathfrak{t}, \mathfrak{h}_\mathcal{L}/\mathfrak{t} \} \), where \( \mathfrak{g}/\mathfrak{t} \) is spanned by the residue classes modulo \( \mathfrak{t} \) of \( L'_{ij}, L'_i, P'_+, P'_{+} \) subject to the following Lie brackets, which we obtain from those in \( \mathfrak{g} \) simply by dropping any terms in \( \mathfrak{t} \):

\[
[L'_{ij}, P'_j] = -L'_{ij} + \delta_{ij} P'_+, \quad [L'_i, P'_+] = L'_i \quad \text{and} \quad [P'_+, P'_+] = P'_+,
\]  
(4.28)

in addition to those brackets involving \( L'_{ij} \) which simply say that \( L'_{ij} \) spans a subalgebra isomorphic to \( \mathfrak{so}(d-1) \) relative to which \( L'_i, P'_+ \) are vectors and \( P'_+ \) is a scalar. We recognise this Lie algebra as the Lorentz algebra \( \mathfrak{so}(d, 1) \) and \( \mathfrak{h}_\mathcal{L}/\mathfrak{t} \) is isomorphic to a euclidean algebra \( \mathfrak{iso}(d-1) \). We thus recognise the resulting Klein pair as the homogeneous space of the \( \mathfrak{so}(d-1) \) Lorentz group describing the \( d \)-dimensional lightcone

\[
\text{SO}(d, 1)_0/\text{ISO}(d-1)
\]  
(4.29)

in \( (d+1) \)-dimensional Minkowski spacetime \([8]\).

The dual of the linear isotropy representation of \( \mathfrak{h}_\mathcal{L}/\mathfrak{t} \) is given in the basis \( \langle \pi^a, \pi^a' \rangle \) by

\[
L'_{ij} \cdot \pi^a = \delta^a_{ij} \pi^m \pi^m - \delta^a_{ij} \pi^m \pi^m, \\
L'_i \cdot \pi^a' = -\delta_{ij} \pi^a'.
\]  
(4.30)

We see that the invariant tensors on \( \mathcal{L} \), as already discussed in \([8]\), correspond to the invariant carrollian structure: a nowhere vanishing vector field corresponding to the \( \mathfrak{h}_\mathcal{L} \)-invariant vector \( \mathfrak{P}_- \) (where the overbar denotes the image of \( \mathfrak{P}_+ \) in \( \mathfrak{g}/\mathfrak{t} \)) and the symmetric tensor \( \delta_{ij} \pi^a \pi^a' \). The algebra of vector fields leaving this structure invariant was determined in \([8]\). It yields the \( (d+1) \)-dimensional Lorentz algebra \( \mathfrak{so}(d, 1) \).

4.5.3. **Celestial sphere \( \mathcal{CS} \).** Returning again to the Klein pair of \( \mathfrak{Ni} \), we find that the two cases discussed above are the only \( d \)-dimensional homogeneous spaces that can be obtained from an enlargement of the subalgebra \( \mathfrak{h}_\mathcal{N} \). In contrast to \( \mathfrak{Ti} \) and \( \mathfrak{Sp} \) one finds however, that we can construct a \( (d-2) \)-dimensional homogeneous space by adding both \( P'_e \) to \( \mathfrak{h}_\mathcal{N} \). We arrive at

\[
\mathfrak{h}_\mathcal{CS} = \langle L_{ij}, L_{-i}, P_i, P_{-i}, L_{+e}, P_{+e} \rangle = \langle L'_{ij}, L'_i, B'_i, B'_+, P'_{+}, P'_{e} \rangle.
\]  
(4.31)

Since both \( \mathfrak{h}_\mathcal{S} \) and \( \mathfrak{h}_\mathcal{L} \) are contained in \( \mathfrak{h}_\mathcal{CS} \), it contains once more the ideal \( \mathfrak{t} \) spanned by \( B'_i, B'_+, P'_+ \), resulting in a non-effective Klein pair \( \{ \mathfrak{g}, \mathfrak{h}_\mathcal{CS} \} \). The reduced Klein pair \( \{ \mathfrak{g}/\mathfrak{t}, \mathfrak{h}_\mathcal{CS}/\mathfrak{t} \} \) is effective, as can be seen from the following brackets:

\[
[L'_{ij}, L'_{kl}] = \delta_{ik} L'_{jl} - \delta_{lk} L'_{ij} - \delta_{il} L'_{jk} + \delta_{lj} L'_{ik}, \\
[L'_{ij}, L'_k] = \delta_{jk} L'_{il} - \delta_{ik} L'_j, \\
[L'_i, P'_e] = \delta_{je} P'_i - \delta_{ie} P'_j. \\
[L'_i, P'_+] = -L'_{ij} + \delta_{ij} P'_+, \\
[P'_+, L'_i] = -L'_i, \\
[P'_+, P'_+] = P'_+.
\]  
(4.32)

The Lie algebra \( \mathfrak{g}/\mathfrak{t} \) is again \( \mathfrak{so}(d, 1) \), whereas now \( \mathfrak{h}_\mathcal{CS}/\mathfrak{t} \) is the parabolic subalgebra spanned by \( L'_{ij}, L'_i, P'_+ \). We recognise the resulting Klein pair as that corresponding to the sphere as a flat conformal geometry. In the present case, it is more appropriate to call it the celestial sphere \( \mathcal{CS}^{d-1} \). In summary, the Lorentz group acts transitively on the celestial sphere via conformal transformations.
The dual of the linear isotropy representation is given by
\[ L'_ij \cdot \pi^k = \delta^k_j \delta_{im} \pi^m - \delta^k_i \delta_{jm} \pi^m, \]
\[ P'_i \cdot \pi^i = -\pi^i. \] (4.33)

There are again no invariant tensors due to the fact that \(-P'_i\) acts as a dilatation relative to which the linear isotropy representation has weight \(-1\). Nevertheless, there is an invariant conformal class of metric (with conformal weight 2) associated to \(\delta_{ij}\pi^i\pi^j\) and its inverse (with conformal weight \(-2\)) \(\delta^{ij}P'_iP'_j\). As is well known, the vector fields preserving this conformal structure, i.e., the conformal Killing vectors of the \((d-1)\)-sphere, generate the \((d+1)\)-dimensional Lorentz algebra \(\mathfrak{so}(\mathbb{S}^{d-1}) \cong \mathfrak{so}(d,1)\) for \(d \geq 3\), whereas for \(d = 2\) every vector field on the circle is conformal Killing.

4.6. Summary. All \((d+1)\)-dimensional homogeneous spaces, \(T_i, \text{Spi}, \text{Ni}\), discussed in this section allow for the construction of a related lower-dimensional homogeneous space that is obtained by adding one scalar to the respective subalgebra \(\mathfrak{h}\). Interestingly, the space \(\text{Ni}\) allows for a richer structure of lower-dimensional spaces summarised in this diagram:

\[ \text{Ni}_{d+1} \longrightarrow \mathcal{L}_d \]
\[ ?_{d+1} \longrightarrow \mathcal{I}_d \longrightarrow \mathbb{S}^{d-1} \] (4.34)

where horizontal/diagonal arrows represent additions of elements of \(P_\mu\) and \(L_{\mu\nu}\), respectively, to the subalgebra \(\mathfrak{h}\). One notices that \(\mathcal{I}\) constructed in this way is somewhat special when compared to the way \(\mathcal{L}/\mathbb{S}/\mathcal{H}\) arise from \(\text{Ni}/\text{Spi}/\text{Ti}\). In the case of \(\mathcal{I}\) the additional generator is an element of the Lorentz transformations \(\mathfrak{so}(d,1)\), whereas in all the other cases the additional generator is a light-/space-/timelike element of the translations. One might wonder if there exists a \((d+1)\)-dimensional space “\(?_{d+1}\)” such that \(\mathcal{I}\) can be constructed in analogy with \(\mathcal{L}/\mathbb{S}/\mathcal{H}\), i.e., by adding a generator of \(P_\mu\) to the subalgebra \(\mathfrak{h}\) of this putative space. Inspection of (4.18), however, reveals that this is not possible.

5. Geometric realisations

Now we will describe explicit geometric realisations of the Klein pairs discussed above in terms of Minkowski spacetime. We will see that \(T_i\) and \((a \mathbb{Z}_2\)-quotient of) \(\text{Spi}\) can be described as grassmannians of affine spacelike and lorentzian hyperplanes in Minkowski spacetime, respectively; whereas \(\text{Ni}\) fibers over the grassmannian of affine null hyperplanes, which is \(\mathcal{I}\). This provides, after we recall some basic notions about grassmannians, a comparably simple geometric and coordinate-independent realisation of these spaces.

5.1. Grassmannians and affine grassmannians. The classical grassmannians are the spaces \(\text{Gr}(p,n)\) of \(p\)-dimensional vector subspaces of \(\mathbb{R}^n\); equivalently, \(p\)-dimensional planes through the origin. We shall simply call them \(p\)-planes from now on. Every point in \(\text{Gr}(p,n)\) corresponds to such a \(p\)-plane. Duality allows us to identify \(\text{Gr}(p,n)\) and \(\text{Gr}(n-p,n)\), so that we can also think of every point in \(\text{Gr}(p,n)\) as an \((n-p)\)-plane. If we put a euclidean inner product on \(\mathbb{R}^n\) we can visualise this duality as simply taking perpendicular complements. Notice that \(\text{Gr}(n-1,n)\) is the space of hyperplanes in \(\mathbb{R}^n\) and since every hyperplane has a perpendicular line, we see that \(\text{Gr}(n-1,n)\) can be identified with \(\text{Gr}(1,n)\), which is the projective space \(\mathbb{P}^{n-1}\).

Let us again put a euclidean inner product on \(\mathbb{R}^n\). Then if \(\Pi \subset \mathbb{R}^n\) is a \(p\)-plane, we can choose an orthonormal basis for \(\Pi\) and complete it to an orthonormal basis for \(\mathbb{R}^n\). Conversely
the first p vectors in any orthonormal basis of $\mathbb{R}^n$ span a p-plane. Since $O(n)$ acts transitively on orthonormal bases, it acts transitively on p-planes. The subgroup of $O(n)$ which stabilises a given p-plane $\Pi$ is isomorphic to $O(p) \times O(n-p)$, which are the changes of orthonormal bases for $\Pi$ and for its perpendicular complement $\Pi^\perp$. In other words, $O(p)$ is the subgroup of orthogonal transformations which map the p-plane $\Pi$ into itself and $O(n-p)$ is the subgroup of orthogonal transformations which act trivially on $\Pi$. In summary, $Gr(p, n)$ can be thought of as a homogeneous space

$$Gr(p, n) \cong O(n)/(O(p) \times O(n-p)),$$

from where it follows that $\dim Gr(p, n) = p(n-p)$. Notice that putting $p = 1$ we get that $\dim Gr(1, n) = n - 1$, consistent with $Gr(1, n) \cong \mathbb{R}^{n-1}$.

If we instead put a lorentzian inner product on $\mathbb{R}^n$, we may refine the notion of Grassmannian by keeping track of the causal character of the planes. We will be solely interested in hyperplanes below. If $\mathcal{H} \subset \mathbb{R}^n$ is a hyperplane (through the origin), then its perpendicular $\mathcal{H}^\perp$ relative to a lorentzian inner product is a line which can be either timelike, spacelike or null. In the former two cases, $\mathbb{R}^n = \mathcal{H} \oplus \mathcal{H}^\perp$, whereas in the null case $\mathcal{H}^\perp \subset \mathcal{H}$. We may therefore partition the Grassmannian of hyperplanes $Gr(n-1, n)$ into three sub-Grassmanns: the Grassmannians of spacelike, timelike and null hyperplanes; i.e.,

$$Gr(n-1, n) = Gr(n-1, n)^{\text{spacelike}} \cup Gr(n-1, n)^{\text{timelike}} \cup Gr(n-1, n)^{\text{null}}. \quad (5.2)$$

Equivalently, this corresponds to partitioning the projective space $\mathbb{RP}^{n-1}$ of lines in $\mathbb{R}^n$ into three: the projective spaces of timelike, spacelike or null lines, respectively. We will discuss the three Grassmannians of hyperplanes in more detail in Sections 5.2, 5.3, and 5.4. Illustrations of the Grassmannians of timelike and spacelike hypersurfaces can be found in Figures 4 and 5, respectively.

Now let’s go back to the general discussion of Grassmannians, not necessarily in a Lorentzian vector space. A closely related notion is the Grassmannian of affine p-planes in $\mathbb{R}^n$, denoted $Graff(p, n)$, which are translates of the p-planes which pass through the origin. If we again put a Euclidean inner product on $\mathbb{R}^n$, then it is clear that the Euclidean group $E(n)$ (with Lie algebra $iso(n) = so(n) \ltimes \mathbb{R}^n$) acts transitively: we can use the translations to bring an affine p-plane to the origin, on which orthogonal transformations act transitively as we saw above. This allows us to describe $Graff(p, n)$ as a homogeneous space of the Euclidean group, a first hint that the Lorentzian generalisation could be related to some of the spaces we have already discussed. The stabiliser of a given affine p-plane $\Pi$ are the longitudinal translations along $\Pi$, the orthogonal transformations of $\Pi$ and the orthogonal transformations of $\Pi^\perp$; in other words,

$$Graff(p, n) \cong E(n)/[E(p) \times O(n-p)], \quad (5.3)$$

from where we see that $\dim Graff(p, n) = (n-p)(p+1)$. In the case of hyperplanes (or dually lines), $\dim Graff(n-1, n) = n$, which is one dimension higher than $Gr(n-1, n)$. This is easy to visualise. Since translating by a vector tangent to a hyperplane does not move the hyperplane, we need only translate along the perpendicular line. This exhibits $Graff(n-1, n)$ as a principal $\mathbb{R}$-bundle over $Gr(n-1, n)$: the fibre at a hyperplane $\mathcal{H}$ is precisely $\mathcal{H}^\perp$. If we identify $Gr(n-1, n)$ with the projective space $\mathbb{RP}^{n-1}$, then $Graff(n-1, n)$ is the total space of the tautological line bundle of $\mathbb{RP}^{n-1}$, so called because the fibre at a point in $\mathbb{RP}^{n-1}$ is precisely the line to which that point corresponds.

Now let us see what happens when we put a Lorentzian inner product on $\mathbb{R}^n$ and let us restrict to hyperplanes. We may now partition the affine Grassmannian by the causal type of the affine hyperplane. Since translations do not alter the causal type, we see that all translates of a timelike (resp. spacelike or null) hyperplane will be timelike (resp. spacelike or null) affine hyperplanes. Translating an affine hyperplane back to the origin gives a hyperplane of the same causal type.
and hence the tautological fibration $\text{Graff}(n-1,n) \to \text{Gr}(n-1,n)$ restricts to the submanifolds of timelike, spacelike or null hyperplanes in $\text{Gr}(n-1,n)$ to give three principal $\mathbb{R}$-bundles:

$$
\text{Graff}(n-1,n)_{\text{timelike}} \to \text{Gr}(n-1,n)_{\text{timelike}}
$$

$$
\text{Graff}(n-1,n)_{\text{spacelike}} \to \text{Gr}(n-1,n)_{\text{spacelike}}
$$

$$
\text{Graff}(n-1,n)_{\text{null}} \to \text{Gr}(n-1,n)_{\text{null}}.
$$

The first two are easy to visualise: they correspond to translating a given timelike (resp. spacelike) hyperplane along its perpendicular spacelike (resp. timelike) line. In the null case, the perpendicular lies on the plane and this description is not accurate. We can of course, translate by a line not on the null plane, which we could choose to be a null line without loss of generality.

We will now proceed to describe $\text{Ti}$, $\text{Spi}$ and $\text{Ni}$ geometrically in this language.

5.2. $\text{Ti} \cong \text{AdSC as a grassmannian}$. An explicit geometric realisation of the Klein pair $(\mathfrak{g}, \mathfrak{h})$ for $\text{Ti} \cong \text{AdSC}$ is provided by the grassmannian of spacelike affine hyperplanes in Minkowski spacetime $\mathbb{M}$. The purpose of this section is to prove this. We will also see that this description explains the structure of the BMS-like algebra of symmetries of AdSC determined in [23].

Recall that Minkowski spacetime $\mathbb{M}$ is an affine space modelled on a lorentzian vector space $V$. Affine hyperplanes are codimension-one affine subspaces of $\mathbb{M}$. They are all of the form $p + W$, where $p$ is a point on $\mathbb{M}$ and $W \subset V$ is a spacelike hyperplane of $V$. Equivalently, the perpendicular line $W^\perp$ is timelike, so it is contained in the interior of the lightcone of $V$. The Lorentz group $O(V)$ acts transitively on timelike lines and hence it acts transitively on spacelike hyperplanes of $V$. The translation subgroup acts transitively on Minkowski spacetime, hence any two spacelike affine hyperplanes $p + W$ and $p' + W'$ are related by a Poincaré transformation. This shows that the Poincaré group acts transitively on the space of spacelike affine hyperplanes on $\mathbb{M}$.

Now, fix one such affine spacelike hyperplane: $p + W$. What is its stabiliser subgroup? The translations which preserve $p + W$ are precisely translations by vectors in $W$. And the Lorentz transformations preserving $W$ are the ones which fix the timelike line $W^\perp$, which is a subgroup isomorphic to the group $O(W)$ of orthogonal transformations of $W$. In other words, the stabiliser subgroup is the group of euclidean transformations of $W$, which is isomorphic to the euclidean group $O(d) \ltimes \mathbb{R}^d$ for $\text{Ti}_{d+1}$.

This description shows that $\text{Ti}_{d+1}$ fibers over $d$-dimensional hyperbolic space $\mathbb{H}^d$. This is nothing but the natural fibration of the grassmannian of affine spacelike hyperplanes over the grassmannian of spacelike hyperplanes, which admits a dual description as the projective space of timelike lines. Indeed, as we now explain, the space of timelike lines in $V$ is naturally identified with $\mathbb{H}^d$. Recall that one model for hyperbolic space is given by any one of the two sheets of the hyperboloid $\eta(x, x) = -\ell^2$, where $\eta$ is the lorentzian inner product on $V$ and $\ell$ is the radius of curvature of hyperbolic space. A timelike line will hit any one of those hyperbolic spaces at exactly one point, as illustrated in Figure 4. Hence $\text{Ti}$ fibers over hyperbolic space: the map sends the hyperplane $p + W$ to the point in hyperbolic space which the line $W^\perp$ hits. The fibre is identified with $W^\perp$ itself, since these are all the translates of $W$. So we conclude that $\text{Ti}$ is the total space of a line bundle over hyperbolic space, which is tautological when we view hyperbolic space as the space of timelike lines. Furthermore, the carrollian degenerate metric is the pullback via the projection of the hyperbolic metric on $\mathbb{H}^d$.

This allows us to understand the structure of the symmetry algebra of $\text{Ti}$, namely those vector fields which preserve the carrollian structure. As shown in [23] and discussed in Section 4.3, the Lie algebra of carrollian Killing vector fields on $\text{Ti} \cong \text{AdSC}$ is isomorphic to the semidirect product $\mathfrak{so}(d,1) \ltimes C^\infty(\mathbb{H}^d)$ of the Lorentz Lie algebra with the smooth functions on hyperbolic space. Some of the functions on $\mathbb{H}^d$ correspond to the Poincaré translations, but the rest are the so-called “supertranslations”. The emergence of the supertranslations is clear in this geometric
realisation. Neither the carrollian vector field nor the degenerate metric depend on the fibre coordinate, which explains the symmetry of moving along each fibre at will. These are the \( \mathfrak{Ti} \) supertranslations: they are sections of a line bundle and hence they are abelian. This line bundle is trivialisable and hence its sections can be identified with the smooth functions on the base; although this description hides the geometry.
5.3. \( \text{Spi}/\mathbb{Z}_2 \) as a grassmannian. A very similar picture exists for \( \text{Spi} \), except that now it is a double cover of the grassmannian of affine lorentzian hyperplanes in Minkowski spacetime. The discussion mimics that of \( \text{Ti} \), so we will be brief. An affine lorentzian hyperplane is again of the form \( p + W \), but where now \( W \subset V \) is a lorentzian hyperplane. Its perpendicular line \( W^\perp \) is now spacelike and shown in Figure 5 such a line intersects any one of the one-sheeted hyperboloids \( \eta(x,x) = \ell^2 \) at precisely two points. The induced metric on the hyperboloid \( \eta(x,x) = \ell^2 \) is now that of \( d \)-dimensional de Sitter spacetime with radius of curvature \( \ell \). The space of spacelike lines in \( V \) is then diffeomorphic to a \( \mathbb{Z}_2 \) quotient of de Sitter spacetime \( dS_d \), known as elliptic de Sitter spacetime [30] and which we denote by \( dS_d/\mathbb{Z}_2 \) in this paper. The action of \( \mathbb{Z}_2 \) is easy to understand in the ambient vector space \( V \): it changes the sign of all coordinates simultaneously. That is clearly an isometry of the ambient metric and since it preserves the hyperboloid it is also an isometry of \( dS_d \). Hence \( dS_d/\mathbb{Z}_2 \) inherits a metric from \( dS_d \), making it locally isometric to \( dS_d \).

[Figure 5. \( \text{Spi}/\mathbb{Z}_2 \) fibering over elliptic de Sitter spacetime.]

As argued for \( \text{Ti} \), the fact that the Lorentz group \( \text{O}(V) \) acts transitively on spacelike lines shows that the Poincaré group acts transitively on the space of affine lorentzian hyperplanes and the Poincaré transformations which preserve such an affine hyperplane \( p + W \) are the translations along \( W \) and the Lorentz transformations \( \text{O}(W) \); in other words, the \( d \)-dimensional Poincaré group of \( W \) with the induced lorentzian inner product.

Similarly to \( \text{Ti} \), also \( \text{Spi}/\mathbb{Z}_2 \) is seen to fiber over the projective space of spacelike lines, which is elliptic de Sitter spacetime, and hence it is the total space of a tautological line bundle whose fibre at a point in the projective space of spacelike lines is the corresponding spacelike line. The pseudo-carrollian degenerate metric is the pull-back via the projection of the elliptic de Sitter metric on \( dS_d/\mathbb{Z}_2 \). As discussed in Section 4.2, the symmetry algebra of \( \text{Spi}/\mathbb{Z}_2 \), consisting of those vector fields which preserve the pseudo-carrollian structure is then isomorphic to \( \mathfrak{so}(d,1) \times \mathbb{C}^\infty(dS_d/\mathbb{Z}_2) \), where \( \mathbb{C}^\infty(dS_d/\mathbb{Z}_2) \) are the \( \mathbb{Z}_2 \)-invariant functions in \( \mathbb{C}^\infty(dS_d) \). The “supertranslations” are again
more properly interpreted as sections of the tautological line bundle, which is trivialisable and hence can be identified with the functions on $dS_d/\mathbb{Z}_2$.

5.4. Geometric description of $N_i$, $\mathcal{F}$, $\mathcal{L}$ and $CS$. Given the interpretations of $T_i$ and $Sp_i/\mathbb{Z}_2$ as the Grassmannians of affine spacelike and lorentzian hyperplanes in Minkowski spacetime, respectively, one might be forgiven for thinking that $N_i$ is the Grassmannian of affine null hyperplanes. However this is easily seen not to be the case. An affine null hyperplane is of the form $p + W$ where $W \subset V$ is a null hyperplane. The hyperplane is determined by the null line $W^\perp$, since $W = (W^\perp)^\perp$. Unlike the case of spacelike or timelike hyperplanes, $W^\perp$ is actually contained in $W$. The space of null lines is the projectivised lightcone or, in other words, the celestial sphere. Therefore the Grassmannian of null hyperplanes (dually, the projective space of null lines) is $(d-1)$-dimensional and since affine null hyperplanes are obtained by translating a null hyperplane by a vector $p$ with nonzero inner product with any generator of the null line, the Grassmannian of affine null hyperplanes is $d$-dimensional. Of course, it is also a homogeneous space of the Poincaré group, namely the identification of future and past null infinity, $\mathcal{F}$. This is straightforward to see using the dual picture of light-like lines. The projectivised lightcone through the origin maps to a sphere both at future and past null infinity. Acting with a translation on the origin we can clearly reach any point on both future and past null infinity.

The $(d + 1)$-dimensional $N_{d+1}$ is actually a bundle over the Grassmannian of affine null hyperplanes, i.e., a bundle over $\mathcal{L}_d$. The difference is that there is a Lorentz boost which preserves the null line but rescales the points in the null line. Let us choose a null frame $e_+, e_-, e_i$ for $V$. By a Lorentz transformation we can bring the null line to be the line $Re_-$ spanned by $e_-$. The Lie algebra of the stabiliser of $Re_-$ includes the boost $L_{\lambda -}$, which rescales $e_-$. Then $L_{\lambda +}$ is in the stabiliser of the null line, but not of any null vector generating that line. The stabiliser of $e_-$ in the Lorentz algebra consists of $L_{ij}, L_{-},$ whereas the translations perpendicular to $e_-$ are spanned by $P_i$ and $P_-$. Hence the Grassmannian of affine null hyperplanes in Minkowski spacetime is described infinitesimally by the Klein pair $(g, h)$, where $\mathfrak{h} = (L_{ij}, L_{-}, P_-)$, whereas that of $N_{d+1}$ is $(g, \mathfrak{h})$ with $\mathfrak{h} = (L_{ij}, L_{-}, P_-, P_i)$.

5.5. $N_i$ as the bundle of scales of the conformal carrollian structure on $\mathcal{F}$. In the following we want to further clarify the nature of the homogeneous space $N_i$, in addition to the geometric perspective provided above. In order to do this, let us first consider the two spaces on the right in the diagram (4.34), the light cone $\mathcal{L}_d$ and the celestial sphere $CS^{d-1}$. The action of the Poincaré group is not effective since the translations act trivially on both $\mathcal{L}_d$ and $CS^{d-1}$. It is the quotient by the translation ideal, isomorphic to the Lorentz group, which acts effectively and it does so leaving invariant a carrollian structure on $\mathcal{L}_d$ and a conformal riemannian structure on $CS^{d-1}$. Indeed, it can be characterised as the symmetry group of such structures; cf. the last rows in Table 3 which shows that they are isomorphic. The fact that conformal symmetries of $CS^{d-1}$ correspond to carrollian symmetries of $\mathcal{L}_d$ can be explained by the fact that $\mathcal{L}_d$ is the total space of the bundle of scales of the conformal manifold $CS^{d-1}$, as we will explain momentarily.

The Lorentz-invariant conformal structure on $CS^{d-1}$ consists of all the metrics on $CS^{d-1}$ which are conformal to the round metric $g$; that is, $[g] = \{ \Omega^2 g \mid \Omega \in C^\infty(CS^{d-1}) \}$. Pick a point $x \in CS^{d-1}$, which we view as a unit-norm vector in $\mathbb{R}^d$. Evaluating the round metric at $x$, we get $g_x \in \otimes^2 T_x^* CS^{d-1}$ and hence a ray $Q_x = \{ \lambda^2 g_x \mid \lambda \in \mathbb{R}^+ \} \subset \otimes^2 T_x^* CS^{d-1}$. Let $Q = \cup_{x \in CS^{d-1}} Q_x$ and define $\pi : Q \rightarrow CS^{d-1}$ by sending $\lambda^2 g_x$ to $x$. This then becomes a principal $R^+$-bundle over $CS^{d-1}$ with $\sigma \in R^+$ acting on $Q$ via $\lambda^2 g_x \rightarrow \sigma^2 \lambda^2 g_x$. Since the round metric defines a global section of $Q$, it is a trivial bundle, so that $Q$ is diffeomorphic to $R^+ \times CS^{d-1}$. By its very definition, every section $CS^{d-1} \rightarrow Q$ defines a metric in the conformal class $[g]$. 
Figure 6. Fibering of $N_i$ over both the future-pointing lightcone and the space of affine null hyperplanes

The (proper, orthochronous) Lorentz group $\text{SO}(d-1,1)_0$ acts transitively on $\mathbb{C}S^{d-1}$ via conformal transformations and therefore it acts on $Q$: if $A \in \text{SO}(d-1,1)_0$, then $A \cdot (\lambda^2 g_x) = (\sigma(A,x)^2 \lambda^2 g_{Ax})$, for some function $\sigma: \text{SO}(d-1,1)_0 \times \mathbb{C}S^{d-1} \to \mathbb{R}^+$ whose explicit form is of no consequence. This action is also transitive and the stabiliser of the point $g_x \in Q$ is the subgroup of the stabiliser of $x \in \mathbb{C}S^{d-1}$ which preserves $g_x$; that is, a subgroup isomorphic to $\text{ISO}(d-1)$. This shows that $Q$ is isomorphic to the future (deleted) lightcone $\mathcal{L}_d$ as a homogeneous space of the (proper, orthochronous) Lorentz group: the diffeomorphism $Q \to \mathcal{L}_d$ sends $\lambda^2 g_x \in Q$ to $(\lambda, \lambda x) \in \mathcal{L}_d$. This diffeomorphism is equivariant under both the action of $\mathbb{R}^+$ and that of $\text{SO}(d-1,1)_0$ and makes the following triangle commute:

$$Q \xrightarrow{\sim} \mathcal{L}_d \xleftarrow{\mathbb{C}S^{d-1}}$$

so that is both a bundle isomorphism and an isomorphism of homogeneous spaces of $\text{SO}(d-1,1)_0$. For more details and the relation of this construction to tractor calculus we refer the reader to [31].

Returning to $N_i$ we can mirror the above discussion and interpret $N_i$ as the bundle of scales of the conformal carrollian structure on $\mathscr{F}$. Again the symmetries of the doubly-carrollian structure of $N_{d+1}$ and the conformal carrollian symmetries of the carrollian structure of $\mathscr{F}$ are isomorphic and given by $\text{BMS}_{d+1}$; cf. the last rows in Table 1 and 3. Let us consider $\mathscr{F}$ with its Poincaré-invariant conformal carrollian structure $[[\xi, h]]$ consisting of all carrollian structures $(\Omega^{-1} \xi, \Omega^2 h)$, for $\Omega \in C^\infty_\mathbb{C}(\mathscr{F})$ a positive smooth function. Any carrollian structure in that class, say $(\xi, h)$, defines a ray sub-bundle $P \subset T\mathscr{F} \oplus \mathbb{C}^2 T^* \mathscr{F}$, where if $p \in \mathscr{F}$, $P_p = \{ (\lambda^{-1} \xi_p, \lambda^2 h_p) \in T_p \mathscr{F} \oplus \mathbb{C}^2 T^*_p \mathscr{F} \mid \lambda \in \mathbb{R}^+ \}$. 


Let $\pi : \mathbb{P} \to \mathcal{J}$ be the restriction to $\mathbb{P}$ of the projection $T_0 \oplus \circ \mathbb{R}^* \times \mathcal{J} \to \mathcal{J}$. Then $\pi : \mathbb{P} \to \mathcal{J}$ is a principal $\mathbb{R}^+$ bundle, where $\sigma \in \mathbb{R}^+$ acts on $\mathbb{P}$ via $(\lambda^{-1} \mathbf{ξ}_p, \lambda^2 h_p) \mapsto (\sigma^{-1} \lambda^{-1} \mathbf{ξ}_p, \sigma^2 \lambda^2 h_p)$. Since any carrollian structure in the conformal class defines a section $\mathcal{J} \to \mathbb{P}$, we see that this is a trivial bundle and hence $\mathbb{P} \cong \mathcal{J} \times \mathbb{R}^+$.

The (proper, orthochronous) Poincaré group $\text{ISO}(d,1)_0$ acts on $\mathcal{J}$ preserving the conformal carrollian structure and hence it acts on $\mathbb{P}$: if $A \in \text{ISO}(d,1)_0$ then

$$A \cdot (\mathbf{ξ}_I, \mathbf{h}_p) = (\sigma(A, p)^{-1} \mathbf{ξ}_{A \cdot p}, \sigma(A, p)^2 \mathbf{h}_{A \cdot p}),$$

for some function $\sigma : \text{ISO}(d,1)_0 \times \mathcal{J} \to \mathbb{R}^+$ whose explicit expression is not needed. This action is transitive and the stabiliser of $(\mathbf{ξ}_I, \mathbf{h}_p)$ is the subgroup of the stabiliser of $p \in \mathcal{J}$ which leaves invariant the carrollian structure, not just its conformal class. We may choose $p$ to be the point of $\mathcal{J}$ with stabiliser subalgebra $\mathfrak{h}_{\mathcal{J}} = \langle L_{ij}, L_{-i}, P_i, P_-, L_{++} \rangle = \langle L_{ij}', L_i', B_i', B'_- \rangle$, whose invariant conformal carrollian structure is the one corresponding to $P_1' \mod \mathfrak{h}_{\mathcal{J}}$ and $\delta_1 \pi^d \pi^0$.

The subalgebra $(L_{ij}, L_{-i}, P_i, P_-) = \langle L_{ij}', L_i', B_i', B'_- \rangle$ of $\mathfrak{h}_{\mathcal{J}}$ leaves the carrollian structure invariant not just up to conformal rescaling and we see that it is isomorphic to $\mathfrak{n}_I$ as a homogeneous space of the Poincaré group, which allows us to identify $\mathbb{N}$ as the bundle of scales of the Poincaré-invariant conformal carrollian structure of $\mathcal{J}$.

The interpretation of $\mathcal{L}_d$ as the bundle of scales of $\mathcal{C}_{d-1}$ can be seen rather explicitly by embedding $\mathcal{L}_d$ in $(d+1)$-dimensional Minkowski spacetime. In the same way, we can see explicitly the relation between $\mathcal{J}$ and $\mathbb{N}$ from the embedding $\mathbb{N}_{d+1} \subset \mathbb{E}^{d+1,2}$ and the projection down to $\mathcal{J}_d \subset \mathbb{P}^{d+2}$ described in Section 2.2.4.

6. Conclusion and outlook

Before we provide a few words concerning the relation to holography in (anti-)de Sitter space and provide a list of some of the intriguing questions for further studies, let us conclude and summarise the results of this work.

6.1. Conclusion. With the aim to improve our understanding of flat holography we have studied homogeneous spaces of the Poincaré group which are relevant to asymptotically flat spacetime: see Figure 1 for an overview.

A concrete way to understand these spaces is provided in Section 2, where we embed Minkowski spacetime $\mathbb{M}_{d+1}$ together with $\mathfrak{t}_{d+1}$ and $\mathfrak{spi}_{d+1}$, the blow-ups of timelike/spatial infinities [7], and the novel space $\mathbb{N}_{d+1}$ into the pseudo-euclidean space $\mathbb{E}^{d+1,2}$. We also showed that $\mathbb{N}_{d+1}$ fibers over $\mathcal{J}_d$ and the lightcone $\mathcal{L}_d$ which in turn fiber over the celestial sphere $\mathbb{S}^{d-1}$ (see (2.22)). The celestial sphere plays a distinguished rôle as the unique two-dimensional manifold admitting a transitive (albeit non-effective) action of the Poincaré group in $3+1$ dimensions. This fact, however, also provides some challenges for nonzero cosmological constant as we will discuss in Section 6.2. Let us emphasise that $\mathfrak{spi}_{d+1}$ and $\mathfrak{t}_{d+1}$ are, in contradistinction to null infinity $\mathcal{J}_d$, of the same dimension as the bulk Minkowski spacetime $\mathbb{b}_{d+1}$ itself. The novel space $\mathbb{N}_{d+1}$ is the natural $(d+1)$-dimensional lift of $\mathcal{J}_d$ that fills this gap.

In Section 3 we reconstructed points of Minkowski spacetime form intrinsic properties of these carrollian-like geometries, roughly speaking from functions on $d\mathbb{S}_d$, $\mathcal{H}_d$, $\mathcal{L}_d$ and $\mathbb{S}^{d-1}$. This can be seen as a form of “holography”. It also provides a generalisation of the “good cut” equation to generic dimension and is not necessarily tied to $\mathcal{J}$.

In Section 4 we introduced these spaces as homogeneous spaces of the Poincaré group. This means that they are characterised as quotients of the Poincaré group by different subgroups, which implies that they have different geometrical and physical interpretations, e.g., their invariants and, notably, their dimensions differ. We also studied their invariant structures: a carrollian structure for $\mathfrak{t}_I$, a pseudo-carrollian [6] structure for $\mathfrak{spi}$, and a novel doubly-carrollian structure
for Ni, intimately linked to the conformal carrollian structure of \( \mathcal{I} \), in analogy to how the carrollian structure of the lightcone \( \mathcal{L} \) is linked to the conformal riemannian structure of the celestial sphere \( \mathcal{CS} \). See Tables 1, 2 and 3 for an overview of the homogeneous spaces and their properties. Through the looking-glass, these spaces provide glimpses of an interesting and rich carrollian-like world that lies beyond the by now well-studied flat carrollian space. The subgroups and homogeneous spaces of this work have also appeared in various other contexts: e.g., in relation to Dirac’s form of relativistic dynamics\(^7\) to the subalgebra of \( \mathcal{I} \) in the context of the infinite momentum frame (see, e.g., [33]), and to induced representations as we will comment on in Section 6.3.

As we also showed, the spaces \( \text{Spi}_{d+1}, \text{Ti}_{d+1} \) and \( \text{Ni}_{d+1} \) have the following intriguing property: the symmetries of their invariant structure match precisely the asymptotic symmetries one expects from asymptotic flat spacetimes, e.g., BMS symmetries for \( \text{Ni}_{d+1} \). This should be contrasted with the conformal carrollian symmetries of \( \mathcal{I}_d \), which are also given by the very same BMS symmetries. The underlying reason is that \( \text{Ni}_{d+1} \) can be seen, as we prove in Section 5.5, as the bundle of scales of the conformal carrollian structure of \( \mathcal{I}_d \), in complete analogy to how the lightcone is the bundle of scales of the conformal structure on the celestial sphere.

Finally, in Section 5, we have developed a simple, explicit and coordinate-independent geometric realisation of Ti, Spi, Ni and \( \mathcal{I} \) in terms of grassmannians of hyperplanes in Minkowski spacetime.

6.2. Relation to (anti-)de Sitter holography. In this work we have focused on the Poincaré group and the asymptotic infinities of flat space. Let us briefly discuss some aspects of the relation to (anti-)de Sitter space. For simplicity we will restrict to \( 3+1 \) dimensions. These observations concerning the subalgebras are based on [34], of which we have summarised the relevant details in Appendix B.

Let us first observe that the asymptotic structure of asymptotically flat space has more boundaries (\( \mathcal{I}^\pm, i^\pm \) (or \( \text{Ti}^\pm \)), \( i^0 \) (or \( \text{Spi} \))) than their curved counterparts. Anti-de Sitter space has one and de Sitter space has two: past and future infinity. We have shown that the asymptotic geometries of flat space are captured by homogeneous spaces of the Poincaré group. Remarkably, this also generalises to the (anti-)de Sitter groups, although the situation there is even simpler. Since the boundaries in these cases are not singular we will focus on three-dimensional homogeneous spaces, which can roughly be thought of as the boundaries of (anti-)de Sitter space. In this sense they are close in spirit to \( \mathcal{I} \).

Upon inspection of the three-dimensional homogeneous spaces of the de Sitter groups (see Appendix B for details), we obtain the conformal symmetries relevant for AdS/CFT [3, 4, 5] and the (euclidean) conformal symmetries of dS/(E)CFT [35]. For AdS there exists a second homogeneous three dimensional space, but remarkably not more. For de Sitter space the three dimensional geometry is unique. This should be contrasted to the infinitely many three-dimensional spaces of the Poincaré group [36] (again, we refer to Appendix B for the details).

Reducing the homogeneous spaces by another dimension, i.e., looking at putative holographic correspondences to theories on a two-dimensional geometry one finds that the de Sitter groups do not possess homogeneous spaces of the same dimension as the celestial sphere; only spaces of higher dimension appear, where their putative dual three-dimensional (E)CFTs live. This means there exists no homogeneous space with (A)dS symmetry which could play the rôle of the celestial sphere in flat space holography, and consequentially there is no flat limit. This is a precise statement based only on symmetries and presumably quite robust.

6.3. Outlook. There are several interesting open questions which deserve further exploration, some of which we list in the following.

\(^7\)More precisely Ti relates to the instant form, \( \mathcal{I} \) to the front form and \( \mathbb{M} \) to the point form of [32].
**Embedding formalism and holography:** The embeddings we discuss in Section 2 can also be viewed as a generalisation of the embedding space formalism used in the $\text{AdS}_{d+1}/\text{CFT}_d$ correspondence (see, for example, [17, 22, 37]) to flat space. This is a generalisation in the sense that both the bulk Minkowski spacetime as well as the “boundaries” embed in a higher dimensional space. Let us emphasise that our space $E^{d+1,2}$ is one dimension higher than that commonly used for $\text{AdS}_{d+1}/\text{CFT}_d$, which opens the possibility to also embed these spaces and study their limits in $E^{d+1,2}$.

A concrete way to do this would be to use the common embedding formalism and intersect it with a null hyperplane in the ambient space, as in Section 2, in which case one could hope to obtain aspects of flat holography from $\text{AdS}/\text{CFT}$ in one dimension higher by restricting to the null hyperplane.

**Reconstruction of Minkowski space:** In Section 3 we employed our embedding space picture to show how points of Minkowski space can be related to certain sections of $\mathcal{T}_i, \mathcal{S}_i, \mathcal{N}_i,$ and $\mathcal{J}$. It would be interesting to relate this more explicitly to an intrinsically minkowskian construction that uses time-/space-/light-like curves to reconstruct a point in Minkowski space from a given section of the above spaces. We leave a more comprehensive discussion of this to future studies.

**Gauging and Cartan geometry:** Homogeneous spaces are the flat models of Cartan geometries (see, e.g., [38]). The so-called gauging procedure may be re-interpreted as the construction of a Cartan geometry, with the gauge field defining a Cartan connection. For the case of Minkowski spacetime this leads to pseudo-riemannian geometry and consequently to general relativity and for the de Sitter spaces to MacDowell–Mansouri gravity [39].

The study of the Cartan geometries modelled on $\mathcal{S}_i$, $\mathcal{T}_i$ and $\mathcal{N}_i$ via the gauging procedure will be the subject of a forthcoming paper. In [40] a Chern–Simons action for Cartan geometries based on $\mathcal{L}$ was written down, and it was shown that the so obtained geometries reproduced certain features of the asymptotic structure of asymptotically flat spacetimes. Cartan geometries modelled on $\mathcal{J}$ have recently been discussed by Herfray in [28] and related to the geometry of asymptotically flat spacetimes [41] (see [42] for a review), and it would be interesting to extend those results to $\mathcal{S}_i$, $\mathcal{T}_i$ and $\mathcal{N}_i$.

**Lower dimensional theories:** In 2+1 dimensions we can write down Chern–Simons theories for the homogeneous spaces discussed in this work. They are homogeneous spaces of the Poincaré group in 2+1 dimensions, which admits a bi-invariant metric [43], equivalently the Poincaré Lie algebra admits an ad-invariant scalar product. One can then generalise what was already done for $\text{AdS} \cong \text{Ti Chern–Simons theory}$ [44] (see [45] for the supergravity generalisation) and write down actions with an interpretation suited for the homogeneous spaces of this work.

Similar remarks apply to (1+1)-dimensional generalisations of JT gravity, as well as their associated BF theory and dilaton gravity analogues [46, 47].

**Relation to novel (induced) representations of the Poincaré group:** The homogeneous spaces we discuss have another interesting interpretation in the theory of induced representations, where one induces a representation of the Poincaré group using one of its subgroups; see [48, Section 3] for a review. The connection to our work comes from looking at the momentum orbits of the Poincaré particles that are given by homogeneous spaces of the Poincaré group (see, e.g., [48, Section 4.2]). The momentum orbit of massive particles are related to $\mathcal{J}$, massless orbits to $\mathcal{L}$, and the tachyonic ones to $d\mathcal{S}$.

Besides these well-known Wigner momentum eigenstate representations, other interesting representations of the Poincaré group have recently been put forward [49, 50, 51]. They have played a central role in advances in (celestial) holography, and it might be
interesting to clarify if and how they are related to the homogeneous spaces described in this work. To our understanding, representations induced by $\mathbb{I}$ have already been considered in [51], but we have discussed other interesting subgroups (see also Appendix B for additional subalgebras).

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Appendix A. $\text{Spi}$ and $\text{Ti}$ as blow-ups of spatial and timelike infinity

The following discussion is based on the original work of Ashtekar–Hansen (AH) [7]. Although a detailed discussion of the AH construction lies beyond the scope of this work, we will summarise the salient features in the following. In a conformal compactification of Minkowski spacetime the conformal boundary at spatial infinity is given by a single point $i^0$. This remains true for more general asymptotically flat spacetimes in the definition of AH. However, various physical fields, e.g., the connection coefficients, admit only direction-dependent limits at $i^0$. One therefore constructs a blow-up of $i^0$, such that fields at $i^0$ can be regarded as smooth fields on a blow-up manifold $\text{Spi}$.

The blow-up manifold is constructed using the behaviour of certain inextensible spacelike curves approaching $i^0$. The AH definition gives rise to a universal lorentzian metric at $i^0$ that is used to demand that these curves have unit tangent vector at $i^0$. Such tangent vectors form a hyperboloid in the tangent space of $i^0$, with induced metric being $dS_2$. This defines the asymptotic geometry at spatial infinity in the sense of [25]. However, the differentiability conditions in the AH definition allow also to define (direction-dependent) connection coefficients at $i^0$. Using these one demands that the spacelike curves be geodesics of the original asymptotically flat manifold. This requirement leaves undefined the component of the acceleration along the tangent vector of the curve at $i^0$. This additional parameter, taking values in the real numbers, can therefore be used to distinguish between asymptotic spacelike curves and thus becomes an additional coordinate on $\text{Spi}$.

From the above construction it is apparent that $\text{Spi}$ has the structure of a fibre bundle. The base space is the (one-sheeted) unit hyperboloid with fibre $\mathbb{R}$. There are two natural tensor fields defined on $\text{Spi}$: a nowhere vanishing vector field $n \in \mathfrak{X}(\text{Spi})$ that generates diffeomorphism of the fibre and a corank-one $\gamma \in \Gamma(\otimes^2 T^*\text{Spi})$ of lorentzian signature with constant positive curvature (the pullback of $\gamma$ to the base space $dS_2$ is the metric on $dS_2$), which furthermore satisfies $\gamma(n, -) = 0$. This is exactly the invariant structure that the Klein pair of $\text{Spi}$ gives rise to, and we therefore recognise the AH construction of $\text{Spi}$ as the (simply-connected) homogeneous space of this Klein pair. This observation was first made in [6].

This construction is applicable, mutatis mutandis, to future/past timelike infinity. Here, the AH construction leads a universal riemannian metric at $i^\pm$, which is now used to demand that
timelike curves approaching (or emanating from) \( i^\pm \) have unit tangent vector at \( i^\pm \), with those tangent vectors now giving the tangent space of \( i^\pm \) the structure of hyperbolic space \( \mathbb{H}_\delta \). Exactly as for \( \mathfrak{Sp}i \), the component of the acceleration along the tangent vector of a curve at \( i^\pm \) can be used to distinguish between asymptotic timelike curves and is taken to be an additional coordinate on \( Ti \). Hence \( Ti \) is a (trivial) line bundle over hyperbolic space, whose invariant structure is a carrollian structure, consisting of a nowhere vanishing \( \xi \in \mathcal{Z}(Ti) \) and a corank-one positive semi-definite \( h \in \Gamma(\mathbb{R}^2Ti) \) of constant negative curvature whose kernel is spanned by \( \xi \): \( h(\xi, -) = 0 \). This is precisely our space \( Ti \cong \text{AdS}_C \).

**Appendix B. Low-dimensional homogeneous spaces of \( \text{ISO}(3, 1) \), \( \text{SO}(3, 2) \) and \( \text{SO}(4, 1) \)**

As already discussed, up to coverings, a homogeneous space is characterised by a Klein pair \((g, h)\) consisting of a Lie algebra \( g \) and a Lie subalgebra \( h \). This implies that the classification of Lie subalgebras of the Poincaré algebra \( \mathfrak{so}(3, 1) \), see \([52, 36]\) and references therein, contains the classification of homogeneous spaces of the Poincaré group. (Not every Klein pair need be geometrically realisable, so it could be that there are more subalgebras than homogeneous spaces.) Although it might be interesting to study the geometry of all the homogeneous spaces of the Poincaré group, in this work we concentrate on \( Ti \), \( Spi \) and \( Ni \) and their descendants as depicted in (1.1), which can be defined in any dimension, and, in particular, capture the asymptotic structure of Minkowski spacetime at infinity.

In this appendix we restrict to \( 3 + 1 \) dimensions and we will comment on homogeneous spaces of dimension four or lower, i.e., Lie subalgebras of dimension six or higher. In the main part we were not exhaustive with regard to three- and four-dimensional spaces (and ignored the higher-dimensional ones). Here we want to provide additional useful information, and relate and contextualise the spaces of this work to the classification of \([36]\). In this way, we can read off from \([36]\) the (generalised) invariants of the Lie subalgebras.

A cursory glance at the classification of subalgebras of the Poincaré Lie algebra in \([36, \text{Table VI}]\) shows 10 six-dimensional subalgebras of the Poincaré algebra, albeit that one of them has a parameter: an angle \( 0 < c < \pi \). A slightly less cursory glance shows that for three of the putative homogeneous four-dimensional spaces (\( P_{8, 1} \), \( P_{9, 1} \), \( P_{10, 1} \) in the notation of \([36]\)), the action of the Poincaré group is not effective. The homogeneous spaces of this work are related in the following way to the subalgebras \([36, \text{Table VI}]\): \( \mathfrak{m}_4 \leftrightarrow P_{1, 2}, \mathfrak{t}_4 \leftrightarrow P_{2, 2}, \mathfrak{sp}_4 \leftrightarrow P_{4, 2}, \mathfrak{n}_4 \leftrightarrow P_{6, 2} \). It is interesting to note that \( \mathfrak{n}_4 \) can be seen as the endpoints \( c = 0, \pi \) of the one-parameter family.

If we look at three-dimensional homogeneous spaces, i.e., subalgebras of dimension seven \([36, \text{Table VII}]\), we find that there exist 6 cases of which one is a one-parameter family \( 0 < c < \pi \). The lightcone \( L_3 \) corresponds to the endpoints \( c = 0, \pi \) of this family. The subalgebras relate to the spaces of our work as \( \mathcal{H}^3 \leftrightarrow P_{3, 1}, \mathfrak{s} \mathcal{D}_3 \leftrightarrow P_{4, 1}, \mathcal{J}_3 \leftrightarrow P_{2, 2}, \mathcal{L}_3 \leftrightarrow P_{6, 1} \). It is interesting to note that \( \mathcal{J}_3 \) is the unique effective three-dimensional homogeneous space of the Poincaré group.

The unique two-dimensional Klein pair of the Poincaré Lie algebra is not effective and yields, upon reduction, a Klein pair for the celestial sphere \( CS_2 \leftrightarrow \mathbb{P}_2 \).

This discussion of homogeneous spaces of the Poincaré group should be contrasted with the case of the de Sitter groups \([34]\). We will again restrict to \( 3 + 1 \) dimensions, but will now only discuss subalgebras of dimension seven or higher. Since the relevant Lie algebras, \( \mathfrak{so}(3, 2) \) and \( \mathfrak{so}(4, 1) \), are simple, there are no non-effective Klein pairs and because of dimension, it follows that there are no two-dimensional spaces on which the corresponding groups can act transitively.

Indeed, for anti de-Sitter space the relevant Lie algebra is \( \mathfrak{so}(3, 2) \) and it follows from the classification in \([34]\) that there are precisely two seven dimensional subalgebras, \( \mathfrak{s}_2,1 \) in Table IV and \( \mathfrak{b}_7,1 \) in Table V, which are maximal. The Klein pair \( (\mathfrak{so}(3, 2), \mathfrak{s}_2,1) \) is conformally compactified Minkowski space and hence of relevance in the AdS/CFT correspondence. The Klein pair \( (\mathfrak{so}(3, 2), \mathfrak{b}_7,1) \) can be interpreted as the grassmannian of maximally isotropic planes in \( \mathbb{R}^{3,2} \) and
has still to find relevance in holography. It admits a (non-metric) conformal structure whose
bundle of scales is the null quadric $Q_0 \subseteq \mathbb{L}^3$ introduced (in general dimension) in Section 2.2.
This interpretation of the null quadric reveals that it has an $O(3,2)$-invariant pseudo-carrollian
structure. There are no (proper) subalgebras of $\mathfrak{so}(3,2)$ of dimension higher than 7. In other
words, there are no two-dimensional spaces on which $O(3,2)$ acts transitively.

The relevant Lie algebra for de Sitter space is now $\mathfrak{so}(4,1)$, and it follows from the classification
in [34, Table XI] there is now a unique 7-dimensional subalgebra $\mathfrak{a}_{7,1}$ and again no subalgebra of
higher dimension. This implies that there is a unique three-dimensional homogeneous space of
$O(4,1)$ with Klein pair $(\mathfrak{so}(4,1), \mathfrak{a}_{7,1})$ corresponding to the celestial $3$-sphere with its invariant
conformal structure. Its bundle of scales is the four-dimensional lightcone, with Klein pair
$(\mathfrak{so}(4,1), \mathfrak{a}_{0,1})$ in the notation of [34, Table XI]. No lower-dimensional homogeneous spaces (or
even Klein pairs) exist.

Appendix C. Symmetries of the doubly-carrollian structure of $\mathbf{N}_i$

In this appendix we work out the Lie algebra of vector fields preserving the “doubly-carrollian”
structure of $\mathbf{N}_i$. We recall from Section 4.4 that $\mathbf{N}_i$ is one of the two smooth components of the
intersection of the null quadric $Q_0$ with the null hyperplane $\mathcal{M}_0$ in $\mathbb{L}^{d+1,2}$, so that it consists of the
points $(r, x^a, x^+, 0) \in \mathbb{L}^{d+1,2}$, where $r = \sqrt{\sum_{a=1}^{d} (x^a)^2} > 0$. This shows that $\mathbf{N}_i$ is diffeomorphic to
$N := (\mathbb{R}^d \setminus \{0\}) \times \mathbb{R}$ and the explicit embedding $j : \mathbf{N}_i \rightarrow \mathbb{L}^{d+1,2}$ is defined by $j(x^a, x^+) = (r, x^a, x^+, 0)$.

The Poincaré generators on $\mathbb{L}^{d+1,2}$ restricted to $\mathbf{N}_i$ are the image under the embedding of the
following vector fields on $N$:

$$
L_{ab} := x^a \partial_b - x^b \partial_a, \quad B_a = -r \partial_a, \quad P_a = x^a \partial_+ \quad \text{and} \quad H = -r \partial_+.
$$

(C.1)

The vector fields on $\mathbf{N}_i$ which commute with the Poincaré generators form a two-dimensional
nonabelian Lie algebra with basis

$$
\xi_+ = \partial_+ \quad \text{and} \quad \xi_- = r \partial_r + x^+ \partial_+,
$$

(C.2)

with Lie bracket $[\xi_+, \xi_-] = \xi_+$. The Poincaré-invariant $(0,2)$-tensor is the pull-back to $\mathbf{N}_i$ of the
pseudo-euclidean metric on $\mathbb{L}^{d+1,2}$:

$$
j^* g_{\mathbb{E}} = -dr^2 + \sum_{a=1}^{d} (dx^a)^2.
$$

(C.3)

Using spherical polar coordinates in $\mathbb{R}^d \setminus \{0\}$,

$$
\sum_{a=1}^{d} (dx^a)^2 = dr^2 + r^2 g_s,
$$

(C.4)

where $g_s$ is the round metric on the unit sphere in $\mathbb{R}^{d+1}$, so that $j^* g_{\mathbb{E}} = r^2 g_s$. In summary, the
doubly-carrollian structure on $\mathbf{N}_i$ is given by the data $(\partial_+, r \partial_r + x^+ \partial_+, r^2 g_s)$.

We now determine the Lie algebra of symmetries of the doubly-carrollian structure of $\mathbf{N}_i$. Let
$\zeta \in \mathcal{X}(N)$ be a vector field on $\mathbf{N}_i$, which we choose to decompose as

$$
\zeta = \zeta^r \partial_r + \zeta^+ \partial_+ + \zeta^a \partial_a,
$$

(C.5)

where the vector field $\zeta^a$ is tangent to the spheres (so in the kernel of $dr$ and $dx^+$), but depends a
priori on all the coordinates. Demanding that $[\zeta, \xi_+] = 0$ says that the functions $\zeta^r$, $\zeta^+$ and the
vector field $\zeta^a$ do not depend on $x^+$. The Lie bracket $[\xi_-, \zeta]$ is given by

$$
[\xi_-, \zeta] = (r \partial_r \zeta^r - \zeta^r) \partial_r + (r \partial_r \zeta^+ - \zeta^+) \partial_+ + r \partial_\zeta^a,
$$

(C.6)

which vanishes provided that

$$
\zeta^r = r \partial_r \quad \text{and} \quad \zeta^+ = r \partial_+ \quad \text{and} \quad \zeta^a \in \mathcal{X}(S^{d-1}),
$$

(C.7)
where $f^r, f^+ \in C^\infty(S^{d-1})$. Demanding that
\[
\zeta = rf^r \partial_r + rf^+ \partial_+ + \zeta^\perp
\]
leaves invariant $r^2 g_S$ results in
\[
\mathcal{L}_{\zeta^\perp} g_S = -2rf^r g_S,
\]
so that $\zeta^\perp$ is a conformal Killing vector on $S^{d-1}$ and $f^r$ is related to its divergence by
\[
f^r = \frac{1}{r^2} \text{div} \zeta^\perp.
\]

The function $f^+$ is unconstrained and hence we find that as a vector space, the symmetry Lie algebra of $N_i$ (as a doubly-carrollian manifold) is $\mathfrak{ctv}(S^{d-1}) \oplus C^\infty(S^{d-1})$, with $\mathfrak{ctv}(S^{d-1})$ the Lie algebra of conformal Killing vectors for the round metric on $S^{d-1}$. For $d \geq 3$, $\mathfrak{ctv}(S^{d-1}) \cong \mathfrak{so}(d,1)$, whereas for $d = 2$, $\mathfrak{ctv}(S^1) = \mathfrak{X}(S^1)$ since every smooth vector field on the circle is conformal Killing.

To understand the Lie algebra structure, let us write for $(X,f) \in \mathfrak{ctv}(S^{d-1}) \oplus C^\infty(S^{d-1})$, the corresponding vector field as
\[
\zeta_{(X,f)} = X - \frac{i\nu X}{d-1} r \partial_r + rf \partial_+
\]
and we calculate
\[
[\zeta_{(X,0)}, \zeta_{(Y,0)}] = \zeta_{(X,Y,0)},
[\zeta_{(X,0)}, \zeta_{(0,t)}] = \zeta_{(0,X,t)}
[\zeta_{(0,t)}, \zeta_{(0,s)}] = 0,
\]
where $[X,Y]$ is the Lie bracket in $\mathfrak{ctv}(S^{d-1})$ and
\[
X \cdot f = X^i \frac{df}{dr} - \frac{\text{div}_+ X}{d-1} f.
\]

So the Lie algebra of symmetries of $N_i$ is a semidirect product with $C^\infty(S^{d-1})$ an abelian ideal and the action of $X \in \mathfrak{ctv}(S^{d-1})$ on $f \in C^\infty(S^{d-1})$ is such that $f$ does not transform as a function, but as a section of the density line bundle in a conformal geometry.

We should contrast these results with those in [23, Section 10] for the conformal symmetries of carrollian spacetimes at level $N = 2$. We find that the symmetry algebra of $N_{d+1}$ is isomorphic to the conformal symmetry algebra of $dSC_d$ which, as shown there, is itself isomorphic to the conformal symmetry algebra of $Z_d$.

**Appendix D. Another choice of sections for reconstruction**

In Section 3.1 we discussed how to interpret Minkowski spacetime as the parameter space of certain hypersurfaces of $\mathcal{S}_{i}$, $\mathcal{T}_i$ and $N_i$ which arise as sections of the fibrations $\mathcal{S}_{d+1} \rightarrow dS_d$, $\mathcal{T}_{d+1} \rightarrow \mathcal{M}^{d}$ and $N_{d+1} \rightarrow Z_d$. More concretely we showed that once we pick one such section, any other such section is related to it by a Poincaré translation. The choice of the initial section, and hence all the sections which correspond to points in $\mathcal{M}$, does not follow from the formalism, but we presented a geometric construction, analogous to the interpretation of the good cuts (sections of $\mathcal{S} \rightarrow \mathcal{C}$) in [18], which exhibits the desired sections as intersections with (generalised) lightcones in the pseudo-euclidean space $\mathcal{E}$ based at the points of the embedded Minkowski spacetime. In this appendix we give an alternative construction which results in another choice of section; although both constructions agree for the case of $N_i$.

In the construction in this appendix, the choice $x^+ = 0$ would seem to be preferred by the fact that the resulting linear functions defining the sections are eigenfunctions of the second Casimir of the Lorentz algebra. We assume that $d > 1$ in what follows, since only for $d > 1$ is the Lorentz algebra $\mathfrak{so}(d,1)$ semisimple.
Let \( L_{\mu \nu} = -L_{\nu \mu} \) be generators of the Lorentz algebra, thought of as vector fields on \( \mathbb{R}^{d,1} \). Relative to the cartesian coordinates,
\[
L_{\mu \nu} = \eta_{\mu p} x^p \partial_\nu - \eta_{\nu p} x^p \partial_\mu.
\]
(D.1)
The Lorentz algebra is semisimple and hence the Killing form
\[
K_{\mu \nu, \rho \sigma} = \text{Tr}(\text{ad}_{L_{\mu \nu}} \circ \text{ad}_{L_{\rho \sigma}})
\]
(D.2)
is non-degenerate. Up to a dimension-dependent proportionality constant, it is given by
\[
K_{\mu \nu, \rho \sigma} = \eta_{\mu p} \eta_{\nu q} - \eta_{\nu p} \eta_{\mu q}.
\]
(D.3)
with inverse (again up to a dimension-dependent multiplicative factor)
\[
K^{\mu \nu, \rho \sigma} = \eta^{\mu p} \eta^{\nu q} - \eta^{\nu p} \eta^{\mu q}.
\]
(D.4)
The second Casimir element is then given (up to normalisation) by
\[
C_2 = \frac{1}{4} K^{\mu \nu, \rho \sigma} L_{\mu \nu} L_{\rho \sigma},
\]
(D.5)
which becomes the following second-order differential operator on \( \mathbb{R}^{d,1} \):
\[
C_2 = x^2 \nabla^2 + (2 - d) E - E^2,
\]
(D.6)
where \( x^2 = \eta_{\mu \nu} x^\mu x^\nu \). \( E = x^\mu \partial_\mu \) is the Euler vector field and \( \nabla^2 = \eta^{\mu \nu} \partial_\mu \partial_\nu \) is the D'Alembertian.

Acting on an affine function \( f(x) = x^+ - \bar{\eta}(v, x) \), we find
\[
C_2 f = (d - 1) \bar{\eta}(v, x),
\]
so that if \( x^+ = 0 \) then the resulting linear function \( f(x) = -\bar{\eta}(v, x) \) is an eigenfunction of \( C_2 \) with nonzero eigenvalue, since we assumed that \( d > 1 \).

**Appendix E. Other signatures**

In this appendix we indicate how the embedding formalism and results of Section 2 extend to other signatures. Due to their relevance for scattering amplitudes in quantum field theory we put particular emphasis on the case of the euclidean \((4,0)\)-signature and split \((2,2)\)-signature version of Minkowski spacetime. The conformal compactified split signature case was already discussed in [53] and is called Klein space in [54].

Let us consider \( \mathbb{E} := \mathbb{R}^{p+1,q+1} \), where \( p, q \geq 0 \), with global coordinates \( x^A = (x^\mu, x^+, x^-) \) where \( x^\mu \) are coordinates on \( \mathbb{R}^{p,q} \), relative to which the flat pseudo-euclidean metric is given by
\[
g = \eta_{AB} dx^A dx^B = \eta_{\mu \nu} dx^\mu dx^\nu + 2 dx^+ dx^-,
\]
(E.1)
with \( \eta_{\mu \nu} \) of signature \((p, q)\). We define quadrics for \( \epsilon \in \mathbb{R} \) by
\[
\mathcal{Q}_\epsilon = \{ x \in \mathbb{E} \mid \eta_{AB} x^A x^B = \epsilon \}
\]
(E.2)
and hyperplanes for \( \sigma \in \mathbb{R} \) by
\[
\mathcal{M}_\sigma = \{ x \in \mathbb{E} \mid x^- = \sigma \}
\]
(E.3)
A subgroup \( O(p+1, q+1) \subseteq \text{GL}(p+q+2, \mathbb{R}) \) preserves every \( \mathcal{Q}_\epsilon \) and acts transitively on \( \mathcal{Q}_\epsilon \) for \( \epsilon \neq 0 \). If \( \epsilon = 0 \), \( \mathcal{Q}_0 \) contains the origin \( x = 0 \), which is a point-like orbit and \( O(p+1, q+1) \) acts transitively on the complement.

The subgroup \( \mathbb{G} \subseteq O(p+1, q+1) \) which preserves \( \mathcal{M}_\sigma \) (for \( \sigma \neq 0 \)) consists of matrices formally identical to those in equation (2.2) except that \( \mathbf{v} \in \mathbb{R}^{p+q} \) and \( A \in O(p,q) \). It follows that \( \mathbb{G} \cong O(p, q) \times \mathbb{R}^{p+q} \). If \( \sigma = 0 \) there is an enhancement to a \( \text{CO}(p,q) \times \mathbb{R}^{p+q} \) subgroup of \( O(p+1, q+1) \). The formulae are *mutatis mutandis* as in Section 2.

Let \( \mathcal{G} \) denote the identity component of \( \mathbb{G} \) and let us decompose \( \mathbb{E} \) into \( \mathcal{G} \)-orbits. By construction, \( \mathcal{G} \) preserves every \( \mathcal{M}_{\epsilon, \sigma} = \mathcal{Q}_\epsilon \cap \mathcal{M}_\sigma \).
If \( \sigma \neq 0 \), then we may solve for \( x^+ \) as in equation (2.11) and we find that \( \mathcal{M}_{\epsilon, \sigma \neq 0} \) is an embedding of \( \mathbb{R}^{p,q} \) into \( E \):

\[
\mathbb{R}^{p,q} \longrightarrow \mathcal{M}_{\epsilon, \sigma \neq 0}
\]

(E.4)

which is \( G \)-equivariant under the \( G \)-action \( x \mapsto Ax + v \) on \( \mathbb{R}^{p,q} \) and the linear action of \( G \) on \( E \). The pull-back to \( \mathbb{R}^{p,q} \) of the metric (E.1) on \( E \) is \( \eta_{\mu\nu}dx^\mu dx^\nu \), so that the embedding is isometric relative to the pseudo-euclidean metric on \( \mathbb{R}^{p,q} \).

Pick as the origin of \( \mathcal{M}_{\epsilon, \sigma \neq 0} \) the point with coordinates \((0, \frac{x_0}{\sigma}, \sigma)\). Its stabiliser is the copy of the identity of component of \( O(p,q) \) which is formally the same as the subgroup \( H \) in equation (2.13), except that \( A \in SO(p,q) \).

So far this is \textit{mutatis mutandis} as in Section 2. The only changes, albeit minor, arise when \( \sigma = 0 \). In this case we have \( \mathcal{M}_{\epsilon,0} \) and we consider three cases depending on whether \( \epsilon > 0 \), \( \epsilon = 0 \) or \( \epsilon < 0 \).

The case \( \epsilon = \rho^2 > 0 \). Here,

\[
\mathcal{M}_{\rho^2,0} = \left\{ \left( \begin{array}{c} x \\ x^+ \\ 0 \end{array} \right) \bigg| \eta(x,x) = \rho^2 \mbox{ and } x^+ \in \mathbb{R} \right\}. \tag{E.5}
\]

Let us break up \( x = (y,z) \in \mathbb{R}^q \), so that \( \eta(x,x) = |y|^2 - |z|^2 = \rho^2 \), so that \( |y|^2 = |z|^2 + \rho^2 \). We have several cases to consider:

- If \( p = 0 \), \( y = 0 \) and there are no solutions: \( \mathcal{M}_{\rho^2,0} = \emptyset \).
- If \( p = 1 \), \( y = y \in \mathbb{R} \) and \( y^2 = \rho^2 + |z|^2 \), so \( y = \pm \sqrt{\rho^2 + |z|^2} \). This breaks up into two subcases depending on whether \( q > 0 \) or \( q = 0 \):
  - if \( q > 0 \), then the equation \( y = \pm \sqrt{\rho^2 + |z|^2} \) defines a two-sheeted hyperboloid \( \mathcal{H}_p = \mathcal{H}_p^+ \cup \mathcal{H}_p^- \) and, since \( G \) is connected, \( \mathcal{M}_{\rho^2,0} \) decomposes into two \( G \)-orbits:
    \[
    \mathcal{M}_{\rho^2,0} = (\mathcal{H}_p^+ \times \mathbb{R}) \cup (\mathcal{H}_p^- \times \mathbb{R}); \tag{E.6}
    \]
  - whereas if \( q = 0 \), then we have two points \( y = \pm \rho \) and hence \( \mathcal{M}_{\rho^2,0} \) decomposes into two \( G \)-orbits:
    \[
    \mathcal{M}_{\rho^2,0} = ([\rho] \times \mathbb{R}) \cup ([-\rho] \times \mathbb{R}). \tag{E.7}
    \]
- Finally, if \( p > 1 \) then \( |y|^2 = \rho^2 + |z|^2 \) is connected and \( \mathcal{M}_{\rho^2,0} \) is its own \( G \)-orbit.

The case \( \epsilon = -\rho^2 < 0 \). This case is virtually identical to the previous case interchanging \( p \leftrightarrow q \) and \( y \leftrightarrow z \).

The case \( \epsilon = 0 \). Now

\[
\mathcal{M}_{0,0} = \left\{ \left( \begin{array}{c} x \\ x^+ \\ 0 \end{array} \right) \bigg| x \in \mathbb{R}^{p,q}, \ \eta(x,x) = 0 \mbox{ and } x^+ \in \mathbb{R} \right\}. \tag{E.8}
\]

We again decompose \( x = (y,z) \) with \( y \in \mathbb{R}^p \) and \( z \in \mathbb{R}^q \) and now \( |y|^2 = |z|^2 \). We have several cases:
• If either \( p = 0 \) or \( q = 0 \), then \( x = 0 \) and \( M_{0,0} \) is a line of point-like orbits:

\[
M_{0,0} = \bigcup_{x^+ \in \mathbb{R}} \left\{ \begin{pmatrix} 0 \\ x^+ \\ 0 \end{pmatrix} \right\}.
\]  
(E.9)

• If either \( p = 1 \) or \( q = 1 \) we are in the situation discussed in Section 2:

\[
M_{0,0} = M_{0,0}^+ \cup \bigcup_{x^+ \in \mathbb{R}} \left\{ \begin{pmatrix} 0 \\ x^+ \\ 0 \end{pmatrix} \right\} \cup M_{0,0}^{-},
\]  
(E.10)

with

\[
M_{0,0}^\pm = \left\{ \begin{pmatrix} x \\ x^+ \\ 0 \end{pmatrix} \mid x \in \mathcal{L}^\pm \text{ and } x^+ \in \mathbb{R} \right\}.
\]  
(E.11)

• Finally if \( p > 1 \) and \( q > 1 \), we have that

\[
M_{0,0} = M_{0,0}^+ \cup \bigcup_{x^+ \in \mathbb{R}} \left\{ \begin{pmatrix} 0 \\ x^+ \\ 0 \end{pmatrix} \right\},
\]  
(E.12)

where

\[
M_{0,0}^+ = \left\{ \begin{pmatrix} x \\ x^+ \\ 0 \end{pmatrix} \mid \eta(x, x) = 0, \ x \neq 0 \text{ and } x^+ \in \mathbb{R} \right\}.
\]  
(E.13)

Let us contrast the three cases \( (p, q) \in \{(4, 0), (3, 1), (2, 2)\} \). The case \( (p, q) = (3, 1) \) is as in Section 2:

\[
\mathbb{E}^{4,2} = \bigcup_{\epsilon, \sigma \in \mathbb{R}, \sigma \neq 0} M_{\epsilon, \sigma} \cup \bigcup_{\epsilon > 0} \bigcup_{\epsilon < 0} \left( M_{\epsilon, 0}^+ \cup M_{\epsilon, 0}^- \right) \cup \bigcup_{\epsilon > 0} \bigcup_{x^+ \in \mathbb{R}} \left\{ \begin{pmatrix} 0 \\ x^+ \\ 0 \end{pmatrix} \right\},
\]  
(E.14)

giving, in order of appearance, embeddings of Minkowski spacetime, \( \mathcal{S}_1, \mathcal{T}_1, \mathcal{T}_1^- \), \( \mathcal{N}_1, \mathcal{N}_1^- \) and the line of fixed points \( (0, x^+, 0) \).

The case \( (p, q) = (4, 0) \) gives

\[
\mathbb{E}^{5,1} = \bigcup_{\epsilon, \sigma \in \mathbb{R}, \sigma \neq 0} M_{\epsilon, \sigma} \cup \bigcup_{\epsilon > 0} \bigcup_{\epsilon < 0} \bigcup_{x^+ \in \mathbb{R}} \left\{ \begin{pmatrix} 0 \\ x^+ \\ 0 \end{pmatrix} \right\},
\]  
(E.15)

giving, in order of appearance, embeddings of euclidean space, cylinders and the line of fixed points \( (0, x^+, 0) \). We may think of each cylinder as the euclidean version of \( \mathcal{S}_1 \): the blow-up of the point at infinity where all geodesics end. Of course in euclidean signature there are no timelike nor null infinities, which explains the absence of orbits corresponding to \( \mathcal{T}_1^\pm \) or \( \mathcal{N}_1 \).

Finally, if \( (p, q) = (2, 2) \) we have

\[
\mathbb{E}^{3,3} = \bigcup_{\epsilon, \sigma \in \mathbb{R}, \sigma \neq 0} M_{\epsilon, \sigma} \cup \bigcup_{\epsilon \neq 0} \bigcup_{\epsilon > 0} \bigcup_{x^+ \in \mathbb{R}} \left\{ \begin{pmatrix} 0 \\ x^+ \\ 0 \end{pmatrix} \right\},
\]  
(E.16)

giving, in order or appearance, embeddings of the Klein space \( \mathbb{R}^{2,2} \), a pseudo-carrollian three-dimensional manifold which is the blow-up of either timelike (\( \epsilon < 0 \)) or spacelike (\( \epsilon > 0 \)) infinities, as described in [54, Section 2], and \( M_{0,0} \) which, just like \( \mathcal{N}_1^\pm \) to \( \mathcal{S}_1^\pm \), can be interpreted as the bundle of scales of the null infinity of the Klein space which is connected, unlike for Minkowski spacetime. The final expression is again the line of fixed points \( (0, x^+, 0) \).
Under the projection $\mathbb{P}^{1,q+1} \setminus \{0\} \rightarrow \mathbb{P}^{p+q+1}$, what happens to the null quadric $\mathcal{Q}_0 \setminus \{0\}$ now? Let us again contrast $(p, q) \in \{(4, 0), (3, 1), (2, 2)\}$.

The case $(p, q) = (3, 1)$ is as in Section 2 and gives a conformal compactification of Minkowski spacetime $\mathbb{M} = \mathbb{R} \cup \mathcal{I} \cup \{l\}$. Here $\mathcal{I}$ is antipodally identified $\mathcal{I} \equiv \{l\}$ is a point where $i^\pm$ and $i^0$ are identified, see [16, Section 9.2] for more details. This compactification has topology $S^3 \times S^1$ with boundary $S^2 \times S^1$ where $S^2$ is the celestial sphere.

For $(p, q) = (4, 0)$ we get the one-point compactification of euclidean space: namely, $S^4 = \mathbb{R}^4 \cup \{\infty\}$, where $\infty$ is the projective image of the line of fixed points (minus the origin). The boundary consists of one point.

Finally, for $(p, q) = (2, 2)$ the conformal compactification of the Klein space has topology $S^2 \times S^2 / \mathbb{Z}_2$ [53] with a boundary of topology $S^3$ [54]. It might be interesting to see if the celestial tori can be understood from the point of view of the Reeb foliation of $S^3$.

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