The nucleon as a relativistic quark-diquark bound state with an exchange potential *).

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Abstract
Treating the quark and diquark as elementary particles, the Bethe-Salpeter equation for the nucleon is solved numerically. The dependence of the mass on the diquark mass and on the coupling constants is investigated. The resulting relativistic quark-diquark-nucleon vertex is presented and discussed.

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1 Introduction

The quark-diquark picture has been successful in describing a variety of nucleon properties \[1\]. It has been used to relate such different phenomena as the $\Delta$-$N$ mass difference \[2\], the neutron formfactor \[3\] and the ratios of structure functions in deep inelastic scattering \[4, 5\]. In most applications one uses purely phenomenological quark-diquark wave functions.

In order to actually calculate the dynamics of the nucleon, some authors have solved the Bethe-Salpeter equation for the quark-diquark system, with a local (quenched or static) quark-diquark interaction \[6\]. The integral equation is then separable and can be solved analytically. The exchanged quark does not propagate in this approximation.

In this letter, the quark-diquark-nucleon (qdN) vertex is calculated, using the Bethe-Salpeter equation (BSE) for the quark-diquark system, with a propagating quark exchange potential. The properly normalized qdN vertex is shown to lead to the correct nucleon charges. Similar calculations, without inclusion of vector diquarks, have been presented in \[7\].

The three-quark problem with a quark exchange potential has recently been solved by Ishii, Bentz and Yazaki \[8\]. However, they did not handle the problem of normalization and did not present results for the vertices (see our figure (3)), which are important e. g. for the calculation of formfactors and structure functions \[9\].

2 The model

We start from the following model Lagrangian:

\[
L = i \bar{\psi} \gamma^\mu \partial_\mu \psi - m_q \bar{\psi} \psi + (\partial_\mu \chi)^\dagger (\partial^\mu \chi) - m_s^2 \chi^\dagger \chi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m_v^2 \chi^\mu \chi^\mu \tag{1}
\]

where $\psi$ is the quark field and $m_q$ is the (constituent) quark mass. The scalar and vector diquark fields are denoted by $\chi$ and $\chi^\mu$. The vector diquark field-strength tensor is denoted by $F_{\mu\nu}$, with $F_{\mu\nu} \equiv \partial_\mu \chi^\nu - \partial_\nu \chi^\mu$. The isospin index of the vector diquark is $i = -, 0, +$, and $\tau^- = (1 - \tau_3)/2$, $\tau^0 = 1/\sqrt{2}$ and $\tau^+ = (1 + \tau_3)/2$ are the corresponding combinations of isospin $SU(2)$ matrices for the quarks. The strengths of the quark couplings to the scalar and axial vector diquarks are $g_s$ and $g_v$, respectively. The charge conjugation operator is $C = i \gamma^2 \gamma^0$.

The Bethe-Salpeter amplitude of the quark-diquark system has scalar and vector components $\phi(p, P)$ and $\phi^\mu(p, P)$. Their mixing in the BSE is shown in fig. (1). The total and relative momenta in terms of the quark momentum $p_a$ and the diquark momentum $p_b$ are:

\[
p = \eta_b p_a - \eta_a p_b \tag{2}
\]
\[
P = p_a + p_b \tag{3}
\]

where $\eta_a = m_a/(m_a + m_b)$ and $\eta_b = m_b/(m_a + m_b)$ with the quark mass $m_a \equiv m_q$, and $m_b$ is either one of the diquark masses $m_s$ or $m_v$. We redefine the Bethe-Salpeter amplitudes as follows:

\[
\tilde{\phi}(p) = [iS(p_a) iS(p_b)]^{-1} \phi(p) \tag{4}
\]
\[
\tilde{\phi}^\mu(p) = [iS(p_a) iS^{\mu\nu}(p_b)]^{-1} \phi^\nu(p) \tag{5}
\]
The amplitudes (4,5) satisfy the following equations:

\[ \tilde{\phi}(p) = 12i|g_s|^2 \int \frac{d^4p'}{(2\pi)^4} S(q)S(p'_a)S(p'_b)\tilde{\phi}(p') \]  
\[ -18ig_s g_v^* \int \frac{d^4p'}{(2\pi)^4} \gamma^\mu \gamma^5 S(q)S(p'_a)S_{\mu\nu}(p'_b)\tilde{\phi}^\nu(p'), \]  
\[ \tilde{\phi}^\mu(p) = 12ig_s g_v \int \frac{d^4p'}{(2\pi)^4} S(q)\gamma^5\gamma^\mu S(p'_a)S(p'_b)\tilde{\phi}(p') \]  
\[ +6i|g_v|^2 \int \frac{d^4p'}{(2\pi)^4} \gamma^\lambda \gamma^5 S(q)\gamma^5\gamma^\mu S(p'_a)S_{\lambda\rho}(p'_b)\tilde{\phi}^\rho(p'). \]  

Here, \( \tilde{\phi}^\mu \) is defined such that it involves the isospin " + " component of the diquark in the proton (the component with two up quarks in it). The vector-diquarks in the proton can have isospin " + " (up-up) or "0" (up-down) components. The corresponding vertices are related through isospin symmetry: \( \tilde{\phi}^{\mu^+} = -\sqrt{2}\tilde{\phi}_0^\mu \equiv \phi_\mu \). The isospin " - " and "0" vertices in the neutron satisfy: \( \tilde{\phi}_\mu^- = -\sqrt{2}\tilde{\phi}_0^\mu = -\phi_\mu \). The momentum of the exchanged quark is denoted by \( q \).

3 Structure of the solution

The equations (12) describe an on-shell nucleon, with its spin \( \frac{1}{2} \). Therefore the vertex functions \( \tilde{\phi} \) and \( \tilde{\phi}^\mu \) can be written as:

\[ \tilde{\phi} = \Psi U(P, S) \]  

Figure 1: The mixing of scalar \( \phi \) and vector \( \phi^\mu \) quark-diquark-nucleon vertices in the Bethe Salpeter equation.

thereby removing the external quark and diquark propagators:

\[ S(p_a) = \frac{\hat{p}_a + m_a}{p_a^2 - m_a^2}, \]  
\[ S(p_b) = \frac{1}{p_b^2 - m_b^2}, \]  
\[ S_{\mu\nu}(p_b) = -g_{\mu\nu} + \frac{p_\mu p_\nu}{p_b^2 - m_b^2}. \]
\[ \tilde{\phi}^{\mu} = \Psi^{\mu} U(P, S), \] (14)

where \( U(P, S) \) is the nucleon spinor and \( \Psi \) and \( \Psi^{\mu} \) are Dirac matrices. Multiplying the equations (12) by \( \bar{U}(P, S) \) and summing over the nucleon spin orientation \( S \), it is seen that only the projections:

\[ \chi \equiv \Psi \Lambda^{+}, \] (15)

\[ \chi^{\mu} \equiv \Psi^{\mu} \Lambda^{+}. \] (16)

occur, where \( \Lambda^{+} = (\mathcal{P} + M)/(2M) \) is the positive energy projection operator. The on-shell vertex functions \( \chi \) and \( \chi^{\mu} \) are Dirac operators that satisfy:

\[ \chi = \chi \Lambda^{+}, \] (17)

\[ \chi^{\mu} = \chi^{\mu} \Lambda^{+}. \] (18)

They depend on the total 4-momentum \( P \) of the nucleon and the relative 4-momentum \( p \) of quark and diquark. Introducing the transverse relative momentum \( p_T^{\mu} = p^{\mu} - P^{\mu} \frac{p \cdot P}{M^2} \), one can make the following decomposition into scalar functions:

\[ \chi = S_1 \left( 1 + \frac{P}{M} \right) + S_2 \left( \gamma_T - \frac{i}{M} p \cdot \sigma \cdot P \right) \] (19)

for the scalar diquarks (\( p \cdot \sigma \cdot P = p^{\mu} \sigma_{\mu\nu} P^{\nu} \)), and:

\[ \chi^{\mu} = V_1 \gamma^5 p_T^{\mu} \left( 1 + \frac{P}{M} \right) + V_2 \gamma^5 P^{\mu} \left( 1 + \frac{P}{M} \right) + \] (20)

\[ \frac{V_3}{M^2} \left( M e^{\mu\nu\rho\sigma} \gamma_\nu p_\rho P_\sigma + M^2 \gamma^5 \sigma^{\mu\nu} p_T^{\nu} + \gamma^5 p \cdot \sigma \cdot P P^{\mu} \right) + \]

\[ V_4 \gamma^5 \left( \gamma_T^{\mu} - \frac{i}{M} \sigma^{\mu\nu} P_\nu \right) + V_5 \gamma^5 p_T^{\mu} \left( \gamma_T - \frac{i}{M} p \cdot \sigma \cdot P \right) + \]

\[ V_6 \gamma^5 P^{\mu} \left( \gamma_T - \frac{i}{M} p \cdot \sigma \cdot P \right) \]

for the vector diquarks. The eight scalar functions \( S_{1,2} \) and \( V_{1-6} \), depend only on the scalars \( p \cdot p, p \cdot P \) and \( P \cdot P = M^2 \) (which is fixed). In the static limit, the vertices do not depend on the relative momentum. Therefore only the functions \( S_1, V_2 \) and \( V_4 \) survive in this limit. In the non-relativistic limit, these functions should dominate, since the remaining functions are multiplied by positive powers of the relative 3-momentum.

### 4 Bethe-Salpeter norm and nucleon charge

From the inhomogeneous BSE, one can derive a normalization condition for the homogeneous BSE [10]. Defining the quark and diquark parts of the norm in the S(calar) and V(ector) channels:

\[ N_{qS} = i \int \frac{d^4 p}{(2\pi)^4} \frac{1}{2} \mathrm{tr} \bar{\chi} S(p_a) \frac{\gamma^5}{M} S(p_a) S(p_b) \chi \] (21)

\[ N_{dS} = i \int \frac{d^4 p}{(2\pi)^4} \frac{1}{2} \mathrm{tr} \bar{\chi} S(p_a) \frac{2 P \cdot p_b}{M} S^2(p_b) \chi \] (22)
\begin{align}
N_{qV} & \equiv i \int \frac{d^4p}{(2\pi)^4/2} \text{tr} \bar{\chi}^{\alpha} S(p_a) \frac{P}{M} S(p_a) S(p_b) \chi^{\beta} S(p_b), \\
N_{dV} & \equiv i \int \frac{d^4p}{(2\pi)^4/2} \text{tr} \bar{\chi}^{\alpha} S_{\alpha\mu}(p_b) \frac{1}{M} \left(-2P \cdot p_b g^{\mu\nu} + p_b^{\mu} P^{\nu} + P^{\mu} p_b^{\nu} \right) S_{\nu\beta}(p_b) \chi^{\beta},
\end{align}

the BS-norm of the nucleon can be expressed as a linear combination of these one-loop integrals:

\[ \eta^a_{\alpha} N_{qS} + \eta^b_{\alpha} N_{dS} + \frac{3}{2} (\eta^v_{\alpha} N_{qV} + \eta^v_{\beta} N_{dV}) = 1. \]

Calculating the electric formfactors of proton and neutron at zero momentum transfer, one finds for the charges of proton and neutron:

\begin{align}
Q_p &= \frac{2}{3} N_{qS} + \frac{1}{3} N_{dS} + \frac{3}{2} N_{dV}, \\
Q_n &= -\frac{1}{3} N_{qS} + \frac{1}{2} N_{qV} + \frac{1}{3} N_{dS} - \frac{1}{2} N_{dV}.
\end{align}

If the vertex functions (19, 20) are normalized according to eq. (25), then these charges should be 1 and 0.

## 5 Results and discussion

Figure 2: The mass \( M \) and binding energy \( E = M - m_a - N_s m_s - N_v m_v \) of the nucleon as a function of the scalar diquark mass, with \( m_v = m_s + 0.2 \text{ GeV} \) and \( m_a \equiv m_q = 0.45 \text{ GeV} \). Dashed line: \( g_s = 2.17, g_v = 1.64 \), in accordance with [11, 12]. Solid line: \( g_s = 2.0, g_v = 1.51 \).

In the rest frame of the nucleon, the scalar vertex functions depend on \(|p|\) and \(p_0\). By taking appropriate traces of the BSE (12), one obtains coupled integral equations for the eight scalar vertex functions.

The integrand of these kernels depend on \(|\vec{p}|\), \(p_0\) and on the angle \(\theta\), defined by \(\vec{p} \cdot \vec{p}' = |\vec{p}| |\vec{p}'| \cos \theta\). This last dependence enters only through the potential and the corresponding integral can be done independently of the form of the eight scalar vertex functions. The integrands do not depend on the azimuthal angle \(\phi\). Poles in the propagators are avoided by performing a Wick-rotation: \(p^0 \to i\gamma^0\). Now, one can start with eight arbitrarily chosen functions of 2 variables that are defined on some \(N_0 \otimes N_3\) grid. Applying the BSE repeatedly one finds a stable solution if \(M\) has the correct value. In this sense an eigenvalue problem for the relativistic energy is solved.
The parameters of the model are the scalar and vector diquark masses \( m_s \) and \( m_v \), the scalar and vector coupling constants \( g_s \) and \( g_v \), the quark mass \( m_a \) and the Euclidean cutoff \( \Lambda \). We take \( m_a = 0.45 \) GeV for the quark mass, suggested by \[5\] and \( m_v - m_s = 0.2 \) GeV for the difference of the vector and scalar diquark masses, which reproduces the \( \Delta - N \) mass difference \[2\]. The cutoff is taken to be \( \Lambda = 0.9 \) GeV.

In figure (2) the mass \( M \) and binding energy \( E \) of the nucleon as a quark-diquark bound state are shown as a function of \( m_s \). The binding energy is defined as \( E = M - m_a - N_s m_s - N_v m_v \), where \( N_s = \eta_a N_{qS} + \eta_b N_{dS} \) and \( N_v = \frac{3}{2} (\eta_a N_{qV} + \eta_b N_{dV}) \) are weighting factors for the scalar and vector channels. The nucleon mass is reached for a scalar diquark mass of about 0.6 GeV. This result however depends on the parameters of the model. In order to constrain these parameters further, it is necessary also to calculate other physical observables. Still it is interesting to see that the binding energy hardly depends on the constituent masses and is relatively low. Evidently in this picture most of the nucleon mass is already given by the sum of quark and diquark quasiparticle masses.

If one puts the scalar coupling \( g_s \) to zero by hand, it turns out that the potential in the vector channel is repulsive. However, if the scalar coupling is turned on, the potential is attractive in both the scalar and vector channels.

The resulting Bethe-Salpeter vertex functions are plotted in figure (3) as functions of the absolute value of the relative 3-momentum \( |\mathbf{p}| \) for different values of the relative energy \( p_0 \). This second variable does not have a non-relativistic analogue; the n.r. wave function is obtained from the relativistic vertex by integrating over the relative energy. The plotted functions include characteristic factors \( |\mathbf{p}|^l \) where \( l = 0, 1, 2 \) correspond to s-, p- and d-waves. It is seen that the s-wave contributions \( S_1, V_1 \) and \( V_4 \) dominate, but the p-wave contributions \( S_2, V_2, V_5 \) and \( V_6 \) are also important. The d-wave term \( V_3 \) can be neglected. In the static limit, only the s-waves survive.

The values of the norm in the various channels are:

\[
N_{qS} = 0.80, \quad N_{dS} = 0.76, \quad N_{qV} = 0.17, \quad N_{dV} = 0.14.
\]

One observes that the conditions \( Q_p = 1 \) and \( Q_n = 0 \) for the proton and neutron charge (see eqs.(26, 27)) are very well realized.

It is interesting to see that the charges carried by the S and V diquark channels in the proton, 0.79 and 0.20 respectively, differ substantially from the values obtained in the non-relativistic \( SU(2)_{\text{spin}} \otimes SU(2)_{\text{isospin}} \) limit where these charges are \( \frac{1}{2} \) for each channel. This result indicates that the scalar diquark component of the quark-diquark-nucleon vertex is more important than the vector diquark component.

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Figure 3: The Bethe-Salpeter vertex functions as functions of the absolute value of the relative 3-momentum $p$ for three values of the quark-diquark relative energy $p^0 = 0.05 \text{ GeV}$ (solid), $p^0 = 0.50 \text{ GeV}$ (dashed) and $p^0 = 1.61 \text{ GeV}$ (dotted).
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