Optimal Hölder Regularity of Solution Operator to the $\bar{\partial}$-equation on Product Domains

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Abstract

This note seeks to prove the existence of a canonical solution operator to the $\bar{\partial}$-equation that preserves Hölder regularity on product domains. It is a well known fact that such solution operators do not in general gain Hölder regularity, and as such, our solution operator is optimal in this regard.

1 Introduction and Main Result

It is a classical problem in complex analysis to describe solutions to the $\bar{\partial}$-equation with estimates in prescribed normed function spaces. The most complete result on problems of this type was given by Sergeev and Henkin in [SH], giving uniform estimates for the $\bar{\partial}$-equation in any pseudoconvex polyhedron. Recently, the Hölder spaces $C^{k+\alpha}$ on product domains in $\mathbb{C}^n$ have been given some attention, and some results have been published on this matter. In [PZ1], [PZ2], a solution operator which loses arbitrarily small amounts of Hölder regularity was found, while in [Zhang] a solution operator which preserves Hölder regularity was found in the case $n = 2$. This note seeks to improve on those results and show that optimal Hölder regularity can be achieved in any product domain in $\mathbb{C}^n$. Indeed, the following theorem holds:

**Theorem 1.** Let $D_1, ..., D_n \subseteq \mathbb{C}$ be bounded open subsets with $C^{k+1+\alpha}$ boundary, $D = \prod_{j=1}^n D_j$, and $\Lambda^{k+\alpha}_{(0,1)}(D) \subset C^{k+\alpha}(D)$ be the subspace of $\bar{\partial}$-closed forms in $D$. Then the equation $\bar{\partial}u = f$ admits a bounded linear solution operator $\mathcal{L}$ such that,

$$\mathcal{L} : \Lambda^{k+\alpha}_{(0,1)}(D) \to C^{k+\alpha}(D)$$

for any integer $k \geq 0$, and $\alpha \in (0, 1)$.

**Remark 1.** When $k = 0$ we always view both $\mathcal{L}[f]$ as solution and $\bar{\partial}f = 0$ in the distributional sense. $\mathcal{L}$ is canonical in the sense that by defining the generalised torus integral as

$$\mathcal{C}[u](z) = \frac{1}{(2\pi i)^n} \int_{\prod_{j=1}^n (\partial D_j)} \frac{u(\zeta)}{\prod_{j=1}^n (\zeta_j - z_j)} d\zeta_1 \wedge ... \wedge d\zeta_n$$

$\mathcal{L}$ is the unique solution operator $\mathcal{L} : \Lambda^{k+\alpha}_{(0,1)}(D) \to C(\bar{D})$ such that $\mathcal{C}\mathcal{L} \equiv 0$. 

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The canonical nature of $L$ will follow from the following lemma:

**Lemma 1.** Given $u \in C(\overline{D})$, $f \in C_{(0,1)}(\overline{D})$ and $\overline{\partial} u = f$ in the distributional sense if necessary, there exists a linear operator $\Phi$ depending only on $D$ and $n$ such that

$$u(z) = C[u] + \Phi[f]$$

**Remark 2.** We will prove this lemma in the last section of this note. The case $n = 1$ amounts to the fact that subject to the conditions above, $u(z) = \frac{1}{2\pi i} \int_{\partial D} u(\zeta) (\zeta - z) d\zeta$, while the case $n = 2$ appears implicitly in [HC].

## 2 Definitions and Notation

**Notation.** *(Symbols)*

Let $z = (z_1, ..., z_n) \in \mathbb{C}^n$, $f = \sum_j f_j dz_j$. Fix $D_1, ..., D_n$ bounded open subsets of $\mathbb{C}^n$ with $C^{k+1+\alpha}$ boundary, and define $D = \prod_{j=1}^n D_j$. Let $d^n z = dz_1 \wedge ... \wedge dz_n$. For any subset $\{r_1, ..., r_s\}$ of $\{1, ..., n\}$, let

$$d^{n-s} z_{r_1, ..., r_s} = \bigwedge_{j=1, j \notin \{r_1, ..., r_s\}}^n dz_j$$

$$W_j = \begin{cases} \partial D_j & \text{if } j \in \{r_1, ..., r_s\} \\ D_j & \text{otherwise} \end{cases}$$

$$\tilde{D}_{r_1, ..., r_n} = \prod_{j=1}^n W_j$$

Finally, for any integer $k \geq 0$, $\alpha \in (0, 1)$, and $\beta, \gamma$ $n$-tuple multi-indices of order $n$, we let

$$\partial^\beta := \frac{\partial^{|\beta|}}{\partial^{\beta_1} z_1, ..., \partial^{\beta_n} z_n}$$

$$\bar{\partial}^\gamma := \frac{\partial^{|\gamma|}}{\partial^{\gamma_1} \bar{z}_1, ..., \partial^{\gamma_n} \bar{z}_n}$$

$$\partial_j = \frac{\partial}{\partial z_j}, \quad \bar{\partial}_j = \frac{\partial}{\partial \bar{z}_j}, \quad j = 1, ..., n$$

We now define the function spaces that will be considered in this note.

**Notation.** *(Function Spaces)*

We denote the standard space of Hölder-$(k + \alpha)$ functions with domain $D$ by $C^{k+\alpha}(D)$ and the space of Hölder-$(k + \alpha)$, $(0, 1)$- forms with domain in $D$ by $C^{k+\alpha}_{(0,1)}(D)$. We denote their norms by $\| \cdot \|_{C^{k+\alpha}}$ and $\| \cdot \|_{C^{k+\alpha}((0,1))}$ respectively, where $\|f\|_{C^{k+\alpha}((0,1))} = \max_j \|f_j\|_{C^{k+\alpha}}$, and denote the subspace of $C^{k+\alpha}_{(0,1)}(D)$ consisting of $\overline{\partial}$-closed $(0, 1)$ forms by

$$\Lambda^{k+\alpha}_{(0,1)}(D) = \{ f \in C^{k+\alpha}_{(0,1)}(D) : \overline{\partial} f = 0 \}$$

We let $C^{k+\alpha}_j(D)$ denote the space of functions in $D$ that are Hölder-$(k + \alpha)$ in the $j$'th coordinate $z_j$ with
uniform estimates in $D$. More specifically, for $j = 1, \ldots, n$, let

$$
\|g\|_{C_j^{k+\alpha}} = \sum_{r+s \leq k} \sup_{z \in D} \left| \partial_r^s \tilde{\partial}_j \sigma g(z) \right| + \sum_{r+s=k} \sup_{z \in D, w \in D_j \setminus z_j} \frac{|\partial_r^s \tilde{\partial}_j \sigma g(z_1, \ldots, z_j, \ldots, z_n) - \partial_r^s \tilde{\partial}_j \sigma g(z_1, \ldots, w_j, \ldots, z_n)|}{|z_j - w_j|^{\alpha}}
$$

And define,

$$
C_j^{k+\alpha}(D) = \{ g \in C^k(D) : \|g\|_{C_j^{k+\alpha}} < \infty \}
$$

**Definition 1.** *(Operators)*

For any $g \in C^{k+\alpha}(D)$, $z \in D$ let

$$
T_j[g](z) = \frac{1}{2\pi i} \int_{D_j} \frac{g(z_1, \ldots, \zeta_j, z_{j+1}, \ldots, z_n)}{\zeta_j - z} d\zeta_j
$$

$$
S_j[g](z) = \frac{1}{2\pi i} \int_{\partial D_j} \frac{g(z_1, \ldots, \zeta_j, z_{j+1}, \ldots, z_n)}{\zeta_j - z} d\zeta_j
$$

For any permutation $\sigma$ of $\{1, \ldots, n\}$, let

$$
\tilde{S}_\sigma(1) = id, \quad \tilde{S}_\sigma(k) = S_{\sigma(k-1)} \ldots S_{\sigma(1)}, \quad \text{for all } 1 < k \leq n
$$

Define

$$
L_\sigma[f] = \sum_{j=1}^n T_{\sigma(j)} \tilde{S}_\sigma(j) f_{\sigma(j)}
$$

**Remark 3.** When $\sigma = id$ we write $L := L_{id}$. In [NW] it was shown that given any $\bar{\partial}$-closed $f \in C^{\infty}_{(0,1)}(D)$, $L[f]$ is a solution to $\bar{\partial}u = f$ on product domains with $C^1$ boundary. This same solution operator appears in [Zhang], which was written whilst the author was working on this note. It can also be shown that $L[f]$ is a solution to $\bar{\partial}u = f$ (in the distributional sense if necessary) on the product of domains with at least $C^{1+\alpha}$ boundary when $f \in \Lambda^{k+\alpha}_{(0,1)}(D)$. Likewise, $L_\sigma[f]$ is a solution to $\bar{\partial}u = f$ for any $\sigma$, a permutation of $\{1, \ldots, n\}$. In fact, the case $n = 2$ follows from manipulating Henkin’s formulæ for the bi-disk in [HC]. This will be proved at the start of the next section.

### 3 Proof of the Main Result from Lemma 1

Following Remark 3, we prove the following Lemma.

**Lemma 2.** Let $f \in \Lambda^{k+\alpha}_{(0,1)}(D)$ Then $\bar{\partial}L_\sigma[f] = u$ for any permutation $\sigma$.

**Proof.** When $k \geq 1$ the claim follows easily by differentiation. It therefore suffices to consider the case $k = 0$. We note that when $D_j$ has at least $C^2$ boundary for all $j = 1, \ldots, n$, it is easy to show that $L_\sigma$ is a solution operator in the distributional sense by smooth approximation. However, in order to relax this condition so that $L_\sigma$ is a solution operator in the distributional sense when $D_j$ has $C^{1+\alpha}$ boundary for all $j = 1, \ldots, n$, we take a more convoluted approach below. This approach however, has the advantage that it illustrates the main ideas in the proofs contained in the rest of this note.

We begin with the case $n = 2$. From [PZ2], $\bar{\partial}u = f$ admits a uniformly continuous solution $u$ in the distributional sense. By considering $v(z) = u(z) - C[u](z)$, $v(z)$ is a solution in the distributional sense as
Hence we obtain \( v(z) = \frac{1}{4\pi^2}(v_0 + v_1 - v_2) \) where

\[
v_0 = -\int_D \frac{f_1(\zeta_1)\bar{z_1} + f_2(\zeta_2)(\bar{z}_2 - \bar{z}_1)}{|\zeta - z|^4} d\bar{\zeta}_1 \wedge d\zeta_1 \wedge d\bar{z}_2 \wedge d\zeta_2
\]

\[
v_1 = -\int_{\partial D_1} \frac{f_2(\zeta)(\bar{z}_2 - \bar{z}_1)}{|\zeta - z|^2(\zeta_2 - z_2)} d\bar{\zeta}_2 \wedge d\zeta_2 \wedge d\zeta_1
\]

\[
v_2 = \int_{D_2} \frac{f_1(\zeta)\bar{z}_1}{|\zeta - z|^2(z_2 - z_2)} d\bar{\zeta}_1 \wedge d\zeta_1 \wedge d\bar{z}_2 \wedge d\zeta_2
\]

Then by Stokes' Theorem, or by examining the calculations in [HC] we obtain

\[
v_2 - v_0 = -4\pi^2T_1[f_1] + \int_{\partial D_1} \frac{f_1(\zeta)(\bar{z}_1 - \bar{z}_1)}{|\zeta - z|^2(\zeta_2 - z_2)} d\bar{\zeta}_2 \wedge d\zeta_2 \wedge d\zeta_1 = -4\pi^2T_1[f_1] - 4\pi^2T_2S_1[f_2] + v_1
\]

Hence \( v(z) = \frac{1}{4\pi^2}(v_2 - v_0 - v_1) = T_1[f_1] + T_2S_1[f_2] \) is a solution of the equation \( \bar{\partial}u = f \) in the distributional sense. By using a similar identity for \( v_1 + v_0 \), [HC] we may obtain further that \( v(z) = T_2[f_2] + T_1S_2[f_1] \) as well. This concludes the proof of the case \( n = 2 \). The general case follows easily from this.

Let \( n \geq 2 \) and without loss of generality, \( \sigma = id \). For any \( C^\alpha \) functions \( g_j \) and \( g_r \) such that \( \bar{\partial}_r g_j = \bar{\partial}_j g_r \) in the distributional sense, and test function \( \phi \in C_c^\infty(D) \), we deduce from the \( n = 2 \) case the identity

\[
\langle \bar{\partial}_r T_j g_j, \phi \rangle = \langle g_r, \phi \rangle - \langle S_j, \phi \rangle.
\]

Coupling this with the identities \( \bar{\partial}_j T_j = id, \bar{\partial}_j S_j = 0 \) and using Fubini’s theorem we see that

\[
\langle T_j \bar{S}_j [f_j], -\bar{\partial}_r \phi \rangle = \begin{cases} 0 & \text{if } j > r \\ \langle \bar{S}_r [f_r], \phi \rangle & \text{if } j = r \\ \langle \bar{S}_k [f_r] - \bar{S}_{k+1} [f_r], \phi \rangle & \text{if } j < r \end{cases}
\]

Hence,

\[
\langle L[f], -\bar{\partial}_r \phi \rangle = \sum_{j=1}^n \langle T_j \bar{S}_j [f_j], \phi \rangle = \sum_{j=1}^{r-1} \langle \bar{S}_j [f_r] - \bar{S}_{j+1} [f_r], \phi \rangle + \langle \bar{S}_r [f_r], \phi \rangle
\]

Where the RHS = \( \langle f_r, \phi \rangle \) by telescoping the sum. Thus \( \bar{\partial}L[f] = f \) in the distributional sense.

**Remark 4.** The \( n = 2 \) case above shows that Henkin and Chirka’s formula in [HC] yields the same solution as Nijenhuis and Woolf’s formula applied to the bi-disk. Since [Zhang] showed that said formula preserves Hölder-\((k + \alpha)\) regularity on product domains in \( \mathbb{C}^2 \), this solution operator in fact preserves both uniform continuity and Hölder-\((k + \alpha)\) regularity on the bi-disk.

Furthermore, it also shows that \( L[f] = L_{(12)}[f] \). In fact, a similar statement holds for all \( n \). Indeed, the following proposition follows immediately from Lemma 1 and Lemma 2.

**Proposition 1.** There is a unique solution operator \( L : \Lambda^{2^k+\alpha}_{(0,1)}(D) \to C(\bar{D}) \) such that \( CL \equiv 0 \). Moreover, we may take \( L = L = L_\sigma \) for any \( \sigma \) a permutation of \( \{1, ..., n\} \).

**Proof.** If \( L_1 \) is any such solution operator such that \( CL_1 \equiv 0 \), then by Lemma 1 we have \( L_1[f] = CL_1[f] + \Phi[f] = \Phi[f] \). Thus such a solution operator is unique. Note that \( L_\sigma(\Lambda^{2^k+\alpha}_{(0,1)}(D)) \subset C(\bar{D}) \) for any \( \sigma \). Therefore, since \( S_j S_j \equiv S_j T_j \equiv 0, C = S_1, ..., S_j, [\text{Vekua}], \) by Fubini’s Theorem, we see that \( CL_\alpha \equiv 0 \). By Lemma 2,
\[ \bar{\partial}L_\sigma[f] = f \text{ for any } \sigma, f \in \Lambda^{k+\alpha}_{(0,1)}(D) \text{ so that } L_\sigma \equiv L \text{ on } \Lambda^{k+\alpha}_{(0,1)}(D) \text{ for any permutation } \sigma \text{. Thus we may take } \mathcal{L} = L = L_\sigma : \Lambda^{k+\alpha}_{(0,1)}(D) \rightarrow C(D) \text{.} \]

We may now obtain Theorem 1 as a corollary of Proposition 1.

**Proof.** (Proof of Theorem 1) As noted in [PZ2] or in [Vekua], for any \( g \in C^{k+\alpha}(D) \) we have \( T_j[g] \in C^{k+\alpha}(D) \cap C^{k+1+\alpha}_{j}(D) \) and \( S_j[g] \in C^{k+\alpha'}(D) \cap C^{k+\alpha}(D) \) for all \( \alpha' \in (0, \alpha) \) where

\[
\|T_j[g]\|_{C^{k+\alpha}} \lesssim \|g\|_{C^{k+\alpha}}
\]

\[
\|T_j[g]\|_{C^{k+\alpha}} \lesssim \|g\|_{C^{k+\alpha}}
\]

\[
\|S_j[g]\|_{C^{k+\alpha}} \lesssim \|g\|_{C^{k+\alpha}}
\]

By iterating these estimates we see that \( T_{\sigma(j)} S_{\sigma(j)}[f_{\sigma(j)}] \in C^{k+\alpha}(D) \cap C^{k+\alpha'}(D) \) for all \( j = 1, \ldots, n \) and any \( \alpha' \in (0, \alpha) \). Summing over \( j \), we obtain \( L_\sigma[f] \in C^{k+\alpha}(D) \cap C^{k+\alpha}(D) \). As noted in [PZ2] and [Zhang], an example in [Tumanov] shows that the operator \( S \) does not in general preserve \( C^{k+\alpha}(D) \), losing arbitrarily small amounts of Hölder regularity in parameters. Therefore each term \( T_{\sigma(j)} S_{\sigma(j)}[f_{\sigma(j)}] \) is not necessarily \( C^{k+\alpha}(D) \) and this method of iterated estimates cannot be used to show directly that \( L_\sigma[f] \) is \( C^{k+\alpha}(D) \).

However, in light of Proposition 1, by taking \( \sigma = (1, j), j = 1, \ldots, n \) we may conclude that since \( \mathcal{L} = L_\sigma \) for any \( \sigma \), we have \( \mathcal{L}[f] \in \cap_{j=1}^n C^{k+\alpha}(D) \) with \( \|\mathcal{L}[f]\|_{C^{k+\alpha}} \lesssim \|f\|_{C^{k+\alpha}} \) for all \( j = 1, \ldots, n \). It is a well known result that separate Hölder smoothness with uniform estimates imply Hölder smoothness. Indeed, since \( D \) is bounded and has (at least) Lipschitz boundary, the conditions of Nikol’ski’s Separate Hölder Smoothness theorem (See [Nikol’ski] and [Krantz]) apply so that \( \mathcal{L}[f] \in C^{k+\alpha}(D) \) where

\[
\|\mathcal{L}[f]\|_{C^{k+\alpha}} \lesssim \max_{j=1, \ldots, n} \|\mathcal{L}[f]\|_{C^{k+\alpha}_j} \lesssim \|f\|_{C^{k+\alpha}_{(0,1)}}
\]

\[\square\]

### 4 Proof of Lemma 1

We now begin the proof of Lemma 1 that was stated at the end of Section 1.

**Proof.** On \( \{1, \ldots, n\} \) we work modulo \( n \) so that \( n + r = r \). In all that follows we let \( C \) be an arbitrary constant depending only on \( n \), and \( \Psi \) be an arbitrary linear operator depending only on \( n \). The same symbols will be used for different constants and operators to simplify the exposition. By the Bochner-Martinelli formula [HL], given uniformly continuous \( u, f \) on \( D \), \( \bar{\partial}u = f \) in the distributional sense, we have

\[
u(z) = C \int_{\partial D} \sum_{j=1}^{n} (-1)^{j-1} u(\zeta) \frac{\bar{\zeta}_j - \bar{z}_j}{|\zeta - z|^{2n}} d^{n-1} \bar{\zeta}_j \wedge d^n \zeta_j + \Psi[f] =: C \sum_{j=1}^{n} u_j(z) + \Psi[f]
\]

Observe that

\[
\bar{\zeta}_j - \bar{z}_j \quad |\zeta - z|^{2n} = \frac{-1}{n-1} \frac{\partial}{\partial \bar{\zeta}_{j+1}} \frac{\bar{\zeta}_j - \bar{z}_j}{|\zeta - z|^{2n-2}}
\]

\[\square\]
Therefore
\[
u_j(z) = C \cdot (-1)^{j-1}(-1)^{j-1} \int_{D_j} u(\zeta) \frac{\partial}{\partial \zeta_{j+1}} \frac{\tilde{\zeta}_j - \tilde{z}_j}{|\zeta - z|^{2n-2}\zeta_{j+1} - z_{j+1}} \, d^{n-2}\tilde{\zeta}_{j,j+1} \wedge d^n\zeta + \Psi[f]
\]

For fixed $z$, our integrand is a uniformly continuous form in $\zeta$ with uniformly continuous differential (possibly in the distributional sense). Thus by Stokes’ theorem, we obtain
\[
u_j(z) = C \cdot (-1)^{j-1}(-1)^{j-1} \int_{D_{j,j+1}} u(\zeta) \frac{\partial}{\partial \zeta_{j+1}} \frac{\tilde{\zeta}_j - \tilde{z}_j}{|\zeta - z|^{2n-2}(\zeta_{j+1} - z_{j+1})} \, d^{n-2}\tilde{\zeta}_{j,j+1} \wedge d^n\zeta + \Psi[f]
\]

where the domain of integration is $\tilde{D}_{j,j+1}$ since the pullback of the form in the integral vanishes on the other components of $\partial \tilde{D}_j$. By repeating this procedure $n - 2$ more times, each time observing that
\[
\frac{\tilde{\zeta}_j - \tilde{z}_j}{|\zeta - z|^{2n-2} \prod_{r=1}^s (\zeta_{j+r} - z_{j+r})} = \frac{1}{(n-s-1)} \frac{\partial}{\partial \zeta_{s+1}} \frac{\tilde{\zeta}_j - \tilde{z}_j}{|\zeta - z|^{2n-2-2} \prod_{r=1}^{s+1} (\zeta_{j+r} - z_{j+r})}
\]

We obtain
\[
u_j(z) = C \cdot (-1)^{(j-1)(j-1)-(n-j)(n-1)} \int_{T_j} u(\zeta) \frac{\tilde{\zeta}_j - \tilde{z}_j}{|\zeta - z|^{2n-2} \prod_{r \neq j} (\zeta_r - z_r)} \, d^n\zeta + \Psi[f]
\]

Where $T_j = \prod_{r=1}^n \partial D_j$ with the orientation induced by successive applications of the boundary operator. It is clear from inspection that $T_j = (-1)^{(j-1)(n-j+1)} T_1^n$. Thus we have
\[
u_j(z) = C \int_{T_1} u(\zeta) \frac{\tilde{\zeta}_j - \tilde{z}_j}{|\zeta - z|^{2n} \prod_{r \neq j} (\zeta_r - z_r)} \, d^n\zeta + \Psi[f]
\]

Summing over $j$ and noting that
\[
\sum_{j=1}^n \frac{\tilde{\zeta}_j - \tilde{z}_j}{|\zeta - z|^{2n} \prod_{r \neq j} (\zeta_r - z_r)} = \frac{1}{\prod_{r=1}^n (\zeta_r - z_r)}
\]

We see that
\[
u(z) = C \cdot C[u](z) + \Psi[f]
\]

Since this equation holds for any $u$, by considering for example $u = 1$, we see that $C = 1$. By taking $\Phi = \Psi$, this concludes the proof.

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