QUANTUM $\mathfrak{osp}(1|2n)$ KNOT INVARIANTS ARE THE SAME AS QUANTUM $\mathfrak{so}(2n+1)$ KNOT INVARIANTS

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Abstract. We show that the quantum covering group associated to $\mathfrak{osp}(1|2n)$ has an associated colored quantum knot invariant à la Reshetikhin-Turaev, which specializes to a quantum knot invariant for $\mathfrak{osp}(1|2)$, and to the usual quantum knot invariant for $\mathfrak{so}(1+2)$. We then show that these knot invariants are the same, up to a change of variables and a constant factor depending on the knot and weight.

1. Introduction

1.1. Quantum enveloping algebras associated to Kac-Moody Lie algebras are central objects in mathematics, which have many remarkable connections to geometry, combinatorics, mathematical physics, and other areas. One such connection was produced by Reshetikhin and Turaev [Tu, RT] by relating the representation theory of these quantum enveloping algebras to Laurent polynomial knot invariants, such as the (colored) Jones polynomial and the HOMFLYPT polynomial. Many other connections have arisen from the categorification of quantum enveloping algebras and their representations [KL, R]. It was recently shown by Webster [Web] that in fact, one can categorify all Reshetikhin-Turaev invariants using the machinery of categorified quantum enveloping algebras. This procedure generalizes Khovanov’s homological categorification of the Jones polynomial [Kh]. We can summarize some of these connections in the picture in Figure 1, where “Decat.” refers to the appropriate decategorification, “RT” stands for the Reshetikhin-Turaev procedure for constructing the Jones polynomial from the standard quantum $\mathfrak{sl}(2)$ representation, and “Web” stands for Webster’s categorification of RT which produces Khovanov homology.

This beautiful picture recently developed a twist with the discovery of “odd Khovanov homology” [ORS], an alternate homological categorification of the Jones polynomial. This discovery has spurred a program of “oddification”: providing analogues of (categorified) quantum groups for this odd Khovanov homology by developing “odd” analogues of standard constructions [EKL, EL, MW]. In particular, one would like an “odd (categorified) $U_q(\mathfrak{sl}(2))$” which could produce odd Khovanov homology in a similar way to that described in Figure 1. In particular, the decategorified “odd” quantum group should produce the Jones polynomial through some analogue of the Reshetikhin-Turaev procedure. It has been proposed [HW, EL] that such categorifications might naturally arise through categorifying the quantum covering group $U_{q,\pi}(\mathfrak{osp}(1|2))$; in other words, producing a diagram such as in Figure 2.

![Figure 1](image1)

![Figure 2](image2)
This proposal has some heuristic evidence from the work of Mikhaylov and Witten [MW], who have produced candidates for “odd link homologies” categorifying $\mathfrak{so}(1 + 2n)$-invariants via topological quantum field theories using the orthosymplectic supergroups. This suggests that the conjecture represented by Figure 2 should be generalized to include colored link invariants associated to $\mathfrak{osp}(1|2n)$ for any $n \geq 1$. Moreover, it has been shown by Blumen [Bl] that the $\mathfrak{osp}(1|2n)$ and $\mathfrak{so}(2n + 1)$ invariants which are colored by the standard $(2n + 1)$-dimensional representations are relabeled up to a variable substitution. However, it has not been known that the Jones polynomial can be constructed from the Reshetikhin-Turaev procedure on $U_{q,\tau}(\mathfrak{osp}(1|2))$, much less any relation between super and non-super colored knot invariants in higher rank.

1.2. A quantum covering group is an algebra $U = U_{q,\tau}(\mathfrak{g})$ that marries the quantum enveloping superalgebra of an anisotropic Kac-Moody Lie superalgebra (e.g. $\mathfrak{g} = \mathfrak{osp}(1|2n)$) with the quantum enveloping algebra of its associated Kac-Moody Lie algebra, which is obtained by forgetting the parity in the root datum (e.g. $\mathfrak{so}(1 + 2n)$). This is done by introducing a new “half-parameter” $\pi$ satisfying $\pi^2 = 1$, and substituting $\pi$ everywhere a sign associated to the superalgebra braiding should appear; such algebras were defined and studied in detail in the series of papers [CW, CHW1, CHW2, CFLW, C, CH].

These quantum covering groups retain the many nice properties of usual quantum groups, such as a Hopf structure; a quasi-$R$-matrix à la Lusztig [L93, Chapter 4]; a category $\mathcal{O}$; and even canonical bases. A key feature of a quantum covering group is that by specializing $\pi = 1$ (respectively, $\pi = -1$), we obtain the quantum enveloping (super)algebra associated to the Kac-Moody Lie (super)algebra. Moreover, as discovered in [CFLW], the quantum algebra and quantum superalgebra can be identified by a twistor map; that is, an automorphism of (an extension of) the covering quantum group which sends $\pi \mapsto -\pi$ and $q \mapsto t^{-1}q$, where $t^2 = -1$.

In this paper, we use the machinery of covering quantum groups to construct “quantum covering knot invariants”: knot invariants which arise from the representation theory of the finite type quantum covering groups à la Turaev [Tu]. (For our purpose, we do not need the additional ribbon structure of [RT].) To wit, consider the quantum covering group associated to the Lie superalgebra $\mathfrak{osp}(1|2n)$. We first associate a $U$-module homomorphism to each elementary tangle (cups, caps, crossings) such that a straight strand is just the identity map, along with an interpretation of combining tangles (with joining top-to-bottom being composition of the associated maps, and placing along-side being tensor products of the maps). An arbitrary tangle can then be framed and associated with a $U$-module homomorphism by “slicing” the diagram (that is, cutting it into vertical chunks containing at most one elementary diagram alongside any number of straight strands). Each slice corresponds to a $U$-module homomorphism, and the tangle is sent to the composition of these maps. Note that a priori, this assignment is not unique, as many distinct slice diagrams and framings exist for an arbitrary tangle.

We then derive some identities with these maps that are versions of Turaev moves on the associated diagrams. These identities show that the map isn’t dependent on the choice of slice diagram, but factors of $\pi$ keep it from being an invariant of oriented framed tangles. In order to eliminate these factors, we need to expand our base ring to $\mathbb{Q}(q, \tau, t, \pi)$, where $\tau^2 = \pi$, and renormalize the maps corresponding to certain elementary diagrams. Finally, a normalization factor (depending on the writhe of the tangle) yields a oriented tangle invariant (see Theorem 3.8).

In the rank 1 uncolored case, this invariant is simply the (unnormalized) Jones polynomial in the variable $\tau^{-1}q$ (see Example 3.10). This suggests that the $\pi = -1$ (i.e. $\tau = t$) specialization of the knot invariant, viewed as a function of $q$, should be related to the $\pi = 1$ (i.e. $\tau = 1$) specialization, viewed as a function of $t^{-1}q$. To make this connection precise, we further develop the theory of twistors (cf. [CFLW, C]) to define a general operator on tensor powers of $U$ and compatible
operators on its representations. In particular, we show that the twistors $X$ on representations $t$-commute with the maps $S$ representing slices of tangles; that is, $X \circ S = t^x S \circ X$ for some $x \in \mathbb{Z}$.

Once this is done, we obtain the following theorem (combining Theorems 3.8 and 4.24).

**Theorem.** Let $K$ be any oriented knot and $\lambda \in X^+$ a dominant weight. There is a functor from the category $\mathcal{OTAN}$ of oriented tangles modulo isotopy to the category $\mathcal{O}$ of $U$-module representations which sends $K$ to a constant $J_K^\lambda(q, \tau) \in \mathbb{Q}(q, t)^\tau$, which we call the covering knot invariant of $K$. Moreover, let $s_\circ J_K^\lambda(q) = J_K^\lambda(q, 1)$ and $osp J_K^\lambda(q) = J_K^\lambda(q, t)$ denote the specializations of the covering knot invariant to $\tau = 1$ and $\tau = t$. Then

$$osp J_K^\lambda(q) = t^{s(K, \lambda)} s_\circ J_K^\lambda(t^{-1}q),$$

for some $s(K, \lambda) \in \mathbb{Z}$.

In particular, this shows that, after extending scalars, there is indeed a map $RT$ as in Figure 2, and in fact such a map exists for all colored link invariants of any rank. It remains to develop an analogue of the construction in [Web] to complete the picture, though difficulties abound. For example, it is not necessarily clear how to extend the categorification to $\mathbb{Q}(q, t)^\tau$. Moreover, the categorification of covering algebra representations is not yet developed enough to produce the analogous machinery to [Web]. We hope that these results will help cast light on these remaining questions.

1.3. The paper is organized as follows. In Section 2, we recall the definition of quantum covering $osp(1|2n)$, denoted by $U$, and set our conventions. We also develop some additional facts about representations of $U$, specifically about dual modules and (co)evaluation morphisms, and produce a universal-$R$-matrix, which we will simply denote by $R$, from the quasi-$R$-matrix defined in [CHW1]. In Section 3, we use $R$ and the (co)evaluation morphisms to define an associated knot invariant by interpreting the maps in terms of the usual graphical calculus; that is, maps are represented by a finite number of labeled, non-intersecting oriented strands such that the $R$-matrix is a positive crossing, the (co)evaluation morphisms are various cups and caps, and orientation is determined by whether the associated module in the domain/range is the dual module or not. We show that this graphical calculus is almost an framed oriented tangle invariant, and is indeed an oriented tangle invariant after renormalizing these elementary diagrams by an integer power of $\tau$ and a factor depending on the writhe. Finally, in Section 4 we use the twistor maps introduced in [CFLW, C] to relate the morphisms in the $\pi = \pm 1$ cases. In particular, we develop some further details about the Hopf structure and representation theory of the enhanced quantum group $\hat{U}$, and construct twistors on tensor products of simple modules and their duals. We then show that these twistors almost commute (up to an integer power of $t$) with the cups, caps, and crossings, allowing us to relate the $so$ and $osp$ knot invariants.

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2. Quantum covering $osp(1|2n)$

We begin by recalling the definition of quantum covering algebra associated to $osp(1|2n)$ and setting our notations. We then elaborate on the representation theory of this algebra.
Let $\Phi_p \subset 2$ and note that we have $\rho$. We extend $W$ and note that $\lambda$. Let notations be as above. Then $\langle \cdot, \cdot \rangle$, we have $\langle \cdot, \cdot \rangle = \frac{\tau \cdot \tau}{2}$, and $\sigma = \frac{\tau \cdot \tau}{2}$. Then $(\cdot, \cdot)$ is a bar-consistent anisotropic super Cartan datum (see [CHW1]). We extend $\cdot$ to a symmetric bilinear pairing on $\mathbb{Z}[I]$ and $p$ to a parity function $p : \mathbb{Z}[I] \to \mathbb{Z}/2\mathbb{Z}$. Moreover, for $\nu = i_1 + \ldots + i_r \in \mathbb{N}[I]$, we set

$$\text{ht } \nu = t, \quad p(\nu) = \sum_{1 \leq r < s \leq t} p(i_r)p(i_s), \quad \bullet(\nu) = \sum_{1 \leq r < s \leq t} i_r \cdot i_s. \quad (2.1)$$

Let $\Phi^+ \subset \mathbb{N}[I]$ denote the set of positive roots, and set

$$\rho = \sum_{\alpha \in \Phi^+} \alpha = \sum_{i \in I} \rho_i i \in \mathbb{N}[I]. \quad (2.2)$$

Note that we have $i \cdot \rho = i \cdot i$ for all $i \in I$.

Let $Y = \mathbb{Z}[I]$ be the root lattice and $X = \text{Hom}(\mathbb{Z}[I], \mathbb{Z})$ be the weight lattice, and let $\langle \cdot, \cdot \rangle : Y \times X \to \mathbb{Z}$ be the natural pairing. We also identify $\mathbb{Z}[I]$ as a subspace of $X$ so that $\langle \tau, \tau \rangle = 2 \frac{\tau \cdot \tau}{2}$. If $\nu = \sum_{i \in I} \nu_i i \in \mathbb{Z}[I]$, we set

$$\tilde{\nu} = \sum_{i \in I} d_i \nu_i i \in \mathbb{Z}[I] \quad (2.3)$$

and note $\langle \tilde{\nu}, \mu \rangle = \nu \cdot \mu$ for any $\nu, \mu \in \mathbb{Z}[I]$; in particular, observe that for any $i \in I$,

$$\langle \tilde{\rho}, i \rangle = i \cdot i. \quad (2.4)$$

Then $((\cdot, \cdot), X, Y, \langle \cdot, \cdot \rangle)$ is the root datum associated to $\mathfrak{osp}(1|2n)$, and forgetting the parity on the root datum yields the root datum associated to $\mathfrak{so}(1 + 2n)$. As usual, we define the dominant weights to be $X^+ = \{ \lambda \in X \mid \langle i, \lambda \rangle \geq 0 \text{ for all } i \in I \}$.

**Example 2.1.** Throughout the paper, we will discuss some examples in the simplest case: $n = 1$. In this case, we identify $X = \mathbb{Z}$ where $\langle \mathbf{1}, k \rangle = k$ for $k \in \mathbb{Z}$. Then $Y = \mathbb{Z}[\mathbf{1}]$ can be identified with subset $2\mathbb{Z} \subset X$. We will freely use these identifications in later examples.

Note that the weight lattice $X$ doesn’t naturally have a parity grading compatible with that on $\mathbb{Z}[I]$. However, a parity grading on $X$ can be defined as follows. First observe that $X$ carries an action of the Weyl group $W$ of type $B_n$, and that in particular $\lambda - w\lambda \in \mathbb{Z}[I]$ for any $\lambda \in X$. Let $w_0$ denote the longest element of $B_n$. If $\lambda \in X$, then $w_0 \lambda = -\lambda$ hence $2\lambda = \lambda - w_0 \lambda \in \mathbb{Z}[I]$. We write $2\lambda = \sum_{i \in I} (2\lambda)_i i$ and define

$$P(\lambda) = p(2\lambda) \equiv (2\lambda)_{\tau} \pmod{2}. \quad (2.5)$$

This defines a parity grading on $X$, though it is obviously not compatible with the grading on $\mathbb{Z}[I]$ (indeed, for any $i \in I$ we have $P(i) = p(2i) = 2p(i) \equiv 0 \pmod{2}$). In particular, $P$ is constant on cosets $X/\mathbb{Z}[I]$. This parity can be expressed explicitly in terms of the rank and weight as follows.

**Lemma 2.2.** Let notations be as above. Then $P(\lambda) \equiv n \langle \tau, \lambda \rangle \pmod{2}$.

**Proof.** Let $1 \leq s \leq n - 1$ and for convenience set the notation $(2\lambda)_{\tau} = 0$. We have

$$\langle \tau, \lambda \rangle = \frac{1}{2} \langle \sigma, 2\lambda \rangle = \frac{1}{2} \sum_{i \in I} (2\lambda)_i \langle \sigma, i \rangle = (2\lambda)_{\tau} - \frac{1}{2}((2\lambda)_{\tau} + (2\lambda)_{\tau}),$$

$$\langle \tau, \lambda \rangle = (2\lambda)_{\tau} - (2\lambda)_{n-1}.$$
In particular, we see that $\frac{1}{2!}(2\lambda)_{s+1} - (2\lambda)_{s+1} = (2\lambda)_{s+1} - (\pi, \lambda) \in \mathbb{N}$, thus $(2\lambda)_{s+1} = (2\lambda)_{s+1} \equiv (2\lambda)_{s+1}$ modulo 2 for all $1 \leq s \leq n - 1$. Therefore, $(2\lambda)_{s+1} = (2\lambda)_{s+1} \equiv (2\lambda)_{s+1}$ modulo 2 whenever $r \equiv s$ modulo 2.

In particular, since $(2\lambda)_{s+1} \equiv 0$ modulo 2 for each $s \equiv 0$ modulo 2. If $n \equiv 0$ modulo 2, then $P(\lambda) \equiv (2\lambda)_{s+1} \equiv 0$ modulo 2. If $n \equiv 1$ modulo 2, then $(2\lambda)_{s+1} = (\pi, \lambda) - (2\lambda)_{n+1} \equiv (\pi, \lambda) \mod 2$.

**Example 2.3.** When $n = 1$, recall from Example 2.1 that we identify $X = \mathbb{Z}$. Then for any $k \in \mathbb{Z}$, $P(k) = (1, k) \equiv k \mod 2$, hence our $P$-grading is just the natural parity grading on $\mathbb{Z}$.

Throughout, our base ring will be $Q$. Throughout, we will consider objects graded by $\hat{X} = X \times (\mathbb{Z}/2\mathbb{Z})$. If $M$ is $\hat{X}$-graded and $m \in M$ is homogeneous, we let $|m|$ (resp. $|m|; p(m)$) denote its $\hat{X}$-degree (resp. $X$-degree; $\mathbb{Z}/2\mathbb{Z}$-degree or parity). Further, for $\zeta = (\lambda, \epsilon) \in \hat{X}$, we will set $|\zeta| = \lambda$, $p(\zeta) = \epsilon$, and $P(\zeta) = P(\lambda)$. (Note that $P(\zeta)$ is not the same as $p(\zeta)$ in general! They are independent quantities.)

For $\lambda \in \mathbb{X}$, let $\hat{\lambda} = (\lambda, 0) \in \hat{X}$. We will freely identify $\mathbb{Z}[I]$ with $\{(\nu, p(\nu)) \mid \nu \in \mathbb{Z}[I]\} \subset \hat{X}$. In particular, if $\zeta = (\lambda, \epsilon) \in \hat{X}$ and $\nu \in \mathbb{Z}[I]$, then

$$\zeta + \nu = (\lambda + \nu, \epsilon + p(\nu)) \in \hat{X}.$$  

With that in mind, the action of $W$ on $X$ generalizes naturally to $\hat{X}$ by setting

$$s_i(\lambda, \epsilon) = (\lambda, \epsilon) - (i, \lambda) i = (\lambda - (i, \lambda) i, \epsilon - (i, \lambda) p(i))$$

where $i \in I$ and $s_i$ is the corresponding simple reflection.

Lastly, we have the parity swap function $\Pi : \hat{X} \to \hat{X}$ defined by

$$\Pi((\lambda, \epsilon)) = (\lambda, 1 - \epsilon).$$

### 2.2. Parameters.

Let $t \in \mathbb{C}$ such that $t^2 = -1$. Let $q$ be a formal parameter and let $\tau$ be an indeterminate such that

$$\tau^4 = 1.$$  

For convenience, we will also define

$$\pi = \tau^2.$$  

If $R$ is a commutative ring with 1, define the notations

$$R^\tau = R[\tau]/(\tau^4 = 1), \quad R^\alpha = R[\alpha]/(\alpha^2 = 1).$$

Throughout, our base ring will be $Q(q, t)^\tau$, though occasionally we will also refer to the subring generated by $Q(q)$ and $\pi$, which we identify with $Q(q)^\tau$.

We denote by $\tau : Q(q, t)^\tau \to Q(q, t)^\tau$ the $Q(q, t)^\tau$-algebra automorphism satisfying $\tau(q) = \pi q^{-1}$. We also define the $Q(t)$-algebra automorphism $\chi$ given by $\chi(q) = t^{-1} q$ and $\chi(\tau) = t \tau$. We caution the reader that $\tau$ and $\chi$ will be used later to denote extensions of these algebra automorphisms which are defined on $Q(q, t)^\tau$-algebras and $Q(q, t)^\tau$-modules.

Given an $Q(q, t)^\tau$-module (or algebra) $M$ and $x \in \{\pm 1, \pm t\}$, the $Q(q, t)$-module (or algebra) $M|_{\tau = x} = Q(q, t)_x \otimes_{Q(q, t)} M$, where $Q(q, t)_x = Q(q, t)$ is viewed as a $Q(q, t)^\tau$-module on which $\tau$ acts as multiplication by $x$. We call this the **specialization of $M$ at $\tau = x$**. Moreover, $Q(q, t)^\tau$ has orthogonal idempotents

$$\varepsilon_k = 1 + t^{k+1} \tau + (t^2 k^2 + (t k^3) / 4, \quad 0 \leq k \leq 3$$

such that $Q(q, t)^\tau = Q(q, t)\varepsilon_1 \oplus Q(q, t)\varepsilon_t \oplus Q(q, t)\varepsilon_{-1} \oplus Q(q, t)\varepsilon_{-t}$. In particular, since $t \varepsilon_x = x \varepsilon_x$, we see that for any $Q(q, t)^\tau$-module $M$,

$$M|_{\tau = x} \cong \varepsilon_x M.$$  

For $k \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}$, the $(q, \pi)$-quantum integers, along with quantum factorial and quantum binomial coefficients, are defined as follows (cf. [CHW1]):
\[ [n]_{q, \pi} = \frac{(\pi q)^n - q^{-n}}{\pi q - q^{-1}}, \]
\[ [n]_{q, \pi} = \prod_{i=1}^{n} [i]_{q, \pi}, \]
\[ [n]_{q, \pi}^k = \prod_{i=n-k+1}^{n} (\pi q)^i - q^{-i} \prod_{m=1}^{n-m} (\pi q)^m - q^{-m}. \]

If \( \nu = \sum_{i \in I} \nu_i \in \mathbb{Z}[I] \), we write
\[ q_\nu = \prod_{i \in I} q_i^{\nu_i d_i}, \quad \tau_\nu = \prod_{i \in I} \tau_i^{\nu_i d_i}, \quad \pi_\nu = \prod_{i \in I} \pi_i^{\nu_i d_i} = \pi^{\nu}, \quad t_\nu = \prod_{i \in I} t_i^{\nu_i d_i}. \]

In particular, note that \( q_i = q^{d_i} \) and \( \pi_i = \pi^{d_i} = \pi^i \) and set
\[ [n]_i = [n]_{q_i, \pi_i}, \quad [n]_i^1 = [n]_{q_i, \pi_i^1}, \quad [n]_i^k = [n]_{q_i, \pi_i^k}. \]

2.3. The covering quantum group. The covering quantum group associated to \( \mathfrak{osp}(1|2n) \) (as well as some variants) was introduced and studied in the series of papers starting with \cite{CHW1}. We will recall the necessary definitions and elementary facts now.

**Remark 2.4.** Note that contrary to \cite{CHW1} and further papers in that series, we will take coefficients in the larger ring \( \mathbb{Z}q \mathbb{Z} \supset \mathbb{Q}(q)^{\pi} \). Nevertheless, all of the results until \S3.3 are essentially statements over \( \mathbb{Q}(q)^{\pi} \) which remain true after extending scalars to \( \mathbb{Q}(q, t)^{\pi} \), so the reader may effectively ignore \( \tau \) and \( t \) for the present.

**Definition 2.5.** \cite{CHW1} The half-quantum covering group \( f \) associated to the anisotropic datum \( (I, \cdot) \) is the \( \mathbb{N}[I] \)-graded \( \mathbb{Q}(q, t)^{\pi} \)-algebra on the generators \( \theta_i \) for \( i \in I \) with \( |\theta_i| = i \), satisfying the relations
\[ \sum_{k=0}^{b_{ij}} (-1)^k \pi^k \pi^{(i,j)} [p(i)+k\pi(p(j))[b_{ij}] k \pi^k \pi^{(i,j)} \pi^k \pi^j = 0 \quad (i \neq j), \]

where \( b_{ij} = 1 - \langle i, j \rangle \).

The algebra \( f \) carries a non-degenerate bilinear form \( \langle \cdot, \cdot \rangle \) which satisfies
\[ (1, 1) = 1; \quad (\theta_i, \theta_i) = \frac{1}{1 - \pi_i q_i^2}; \quad (\theta_i x, y) = (\theta_i, \theta_i)(x, \pi(y)); \]

where \( \cdot r : f \to f \) is the \( \mathbb{Q}(q, t)^{\pi} \)-linear map satisfying \( \cdot r(1) = 0, \quad \cdot r(\theta_i) = \delta_{ij}, \quad \cdot r(xy) = i \cdot r(x) y + \pi^{p_i(p_j)} [q_i^{1/2} x \cdot r(y). \quad \rangle \rangle \) (Here, and henceforth, \( \delta_{x,y} \) is set to be \( \delta_{x,y} = 1 \) if \( x = y \) and 0 otherwise.) We define the \( \mathbb{Q}(t)^{\pi} \)-linear bar involution \( \overline{\cdot} \) on \( f \) by
\[ \overline{\theta_i} = \theta_i, \quad \overline{q} = \pi q^{-1}. \]

We also define the \( \mathbb{Q}(q, t)^{\pi} \)-linear anti-involution \( \sigma \) on \( f \) by
\[ \sigma(\theta_i) = \theta_i, \quad \sigma(xy) = \sigma(y) \sigma(x), \]

and the divided powers
\[ \theta_i^{(n)} = \theta_i^n / [n]_i! \].

**Definition 2.6.** \cite{CHW1} The quantum covering group \( U \) associated to \( (I, \cdot), Y, X, \langle \cdot, \cdot \rangle \) is the \( \mathbb{Q}(q, t)^{\pi} \)-algebra with generators \( E_i, F_i, K_\mu, \) and \( J_\mu, \) for \( i \in I \) and \( \mu \in \mathbb{Y} \), subject to the relations:
\[ J_\mu J_\nu = J_{\mu+\nu}, \quad K_\mu K_\nu = K_{\mu+\nu}, \quad K_0 = J_0 = J_0^2 = 1, \quad J_\mu K_\nu = K_\nu J_\mu, \]
\[ J_\mu E_i = \pi^{\mu(i)} E_i J_\mu, \quad J_\mu F_i = \pi^{-\mu(i)} F_i J_\mu, \]
\[ K_\mu E_i = q^{(\mu,i)} E_i K_\mu, \quad K_\mu F_i = q^{-(\mu,i)} F_i K_\mu, \]  
\[ E_i F_j - \pi^p(i)p(j) F_j E_i = \delta_{ij} \frac{J_{d,i}K_{d,i} - K_{-d,i}}{\pi_i q_i - q_i^{-1}}, \]  
\[ \sum_{k=0}^{b_{ij}} (-1)^k \pi^p(i)p(i)+kp(i)p(j) \left[ \frac{b_{ij}}{k} \right]_{q_i,\pi_i} E^{b_{ij}-k}_i E^k_j = 0 \quad (i \neq j), \]  
\[ \sum_{k=0}^{b_{ij}} (-1)^k \pi^p(i)p(i)+kp(i)p(j) \left[ \frac{b_{ij}}{k} \right]_{q_i,\pi_i} F^{b_{ij}-k}_i F^k_j = 0 \quad (i \neq j), \]

for \( i, j \in I \) and \( \mu, \nu \in Y \).

We note that since in this case \( Y = \mathbb{Z}[I] \), \( U \) is actually generated by \( E_i, F_i, K_i, J_i \) for \( i \in I \). For notational convenience, we set \( \tilde{J}_\nu = J_\nu \) and \( \tilde{K}_\nu = K_\nu \) so that (2.17) becomes

\[ E_i F_j - \pi^p(i)p(j) F_j E_i = \delta_{ij} \frac{J_\nu K_\nu - K_\nu^{-1}}{\pi_i q_i - q_i^{-1}}. \]

We also equip \( U \) with a bar involution \( \tau : U \to U \) extending that on \( \mathbb{Q}(q,t)^+ \) by setting \( \bar{E}_i = E_i, \bar{F}_i = F_i, \bar{K}_\mu = J_\mu K_\mu^{-1}, \bar{J}_\nu = J_\nu \).

The algebras \( U \) and \( f \) are related in the following way. Let \( U^- \) be the subalgebra generated by \( F_i \) with \( i \in I \), \( U^+ \) be the subalgebra generated by \( E_i \) with \( i \in I \), and \( U^0 \) be the subalgebra generated by \( K_\nu \) and \( J_\nu \) for \( \nu \in Y \). There is an isomorphisms \( f \to U^- \) (resp. \( f \to U^+ \)) defined by \( \theta_i \mapsto \theta_i^- = F_i \) (resp. \( \theta_i \mapsto \theta_i^+ = E_i \)). We let \( E_i^{(n)} = (\theta_i^{(n)})^+ \) and \( F_i^{(n)} = (\theta_i^{(n)})^- \). As shown in [CHWT], there is a triangular decomposition

\[ U \cong U^- \otimes U^0 \otimes U^+ \cong U^+ \otimes U^0 \otimes U^-. \]

There is also a root space decomposition

\[ U = \bigoplus_{\nu \in \mathbb{Z}[I]} U_\nu, \quad U_\nu = \left\{ x \in U \mid J_\mu K_\nu = \pi^p(\mu,\nu) q^p(\nu,\nu) m \right\}. \]

The root space decomposition induces a parity grading via \( p(u) = p(|u|) \), hence in particular \( U \) is \( X \)-graded.

We say an algebra is a “Hopf covering algebra” if it is a \( \mathbb{Z}/2\mathbb{Z} \)-graded algebra over \( R^* \), for some commutative ring with identity \( R \), with a coproduct, antipode, and counit satisfying the usual axioms of a Hopf superalgebra, but with the braiding replaced by \( x \otimes y \mapsto \pi^p(x)p(y) y \otimes x \). Then the algebra \( U \) is a Hopf covering algebra under the coproduct \( \Delta : U \to U \otimes U \) satisfying

\[ \Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes \tilde{K}_i^{-1} + 1 \otimes F_i, \quad \Delta(K_\mu) = K_\mu \otimes K_\mu, \quad \Delta(J_\nu) = J_\nu \otimes J_\nu; \]

the antipode \( S : U \to U \) satisfying \( S(xy) = \pi^p(x)p(y) S(y) S(x) \) for \( x, y \in U \) and

\[ S(E_i) = -\tilde{J}_i^{-1} \tilde{K}_i E_i, \quad S(F_i) = -F_i \tilde{K}_i, \quad S(K_\mu) = K_\mu^{-1}, \quad S(J_\nu) = J_\nu^{-1}; \]

and the counit \( \epsilon : U \to \mathbb{Q}(q,t)^+ \) satisfying

\[ \epsilon(E_i) = \epsilon(F_i) = 0, \quad \epsilon(K_\mu) = \epsilon(J_\nu) = 1. \]

Moreover, for \( x \in f \), we have that

\[ S^b_1(x^+) = (-1)^{ht\nu} \pi^p(\nu) q^b_{\nu,\nu} \tilde{J}_\nu \tilde{K}_\nu \sigma(x)^+, \quad S^b_1(x^-) = (-1)^{ht\nu} \pi^p(\nu) q^b_{\nu,\nu} \tilde{J}_\nu \tilde{K}_\nu \sigma(x)^-. \]  

(2.20)
2.4. U-modules. In this paper, a weight U-module is a U-module M with a \( X \)-grading compatible with the grading on \( U \), such that
\[
M = \bigoplus_{\lambda \in X} M_{\lambda,0} \oplus M_{\lambda,1}, \quad M_{\lambda,s} = \{ m \in M \mid p(m) = s, \ J_\mu K_\lambda m = \pi^{(\mu,\lambda)} q^{(\nu,\lambda)} m \}
\]
and each \( M_{\lambda,s} \) is a free \( \mathbb{Q}(q,t)^\tau \)-module of finite rank. For \( \lambda \in X \), denote \( M_\lambda = M_{\lambda,0} \oplus M_{\lambda,1} \). We also define the parity-swapped module \( \Pi M \) to be \( M \) as a vector space with the same action of \( U \), but with \( \Pi M_{\lambda,s} = M_{\lambda,-s} \). We let \( \mathcal{O}_{\text{fin}} \) be the category of weight U-modules of finite rank over \( \mathbb{Q}(q,t)^\tau \). Henceforth, we shall always assume our U-modules are in \( \mathcal{O}_{\text{fin}} \).

We define the (restricted) linear dual of a U-module \( M \)
\[
M^* = \bigoplus_{\lambda \in X} (M_{\lambda,0})^* \oplus (M_{\lambda,1})^*, \quad (M_{\lambda,s})^* = \text{Hom}_{\mathbb{Q}(q,t)^\tau} (M_{\lambda,s}, \mathbb{Q}(q,t)^\tau).
\]
This is again a free \( \mathbb{Q}(q,t)^\tau \)-module, which has a \( \mathbb{Z}/2\mathbb{Z} \) grading induced by that of \( V \); namely, \( p(f) = 0 \) if \( f(v) = 0 \) for \( p(v) = 1 \), and vice-versa. Moreover, the Hopf superalgebra structure of \( U \) induces an action of \( U \): for \( f \in V^\ast \) and \( x \in U \), we define \( xf \in V^\ast \) by \( xf(v) = \pi^{p(f)p(x)} f(S(x)v) \). In particular, note that \( V^\ast \) is a U-module with \( (V^\ast)_\lambda = (V_{-\lambda})^* \). While \( V^\ast \) is therefore ambiguous, we will always take it to denote \( (V^\ast)_\lambda \). (In other words, our convention is that taking duals has precedence over taking weight spaces.)

For any U-modules \( V \) and \( W \), we can construct the U-module \( V \otimes W = V \otimes_{\mathbb{Q}(q,t)^\tau} W \) via the coproduct. In particular, we have U-modules \( V^\ast \otimes V \) and \( V \otimes V^\ast \), both of which contain a copy of the trivial module \( V(0) = \mathbb{Q}(q,t)^\tau \) as a direct summand. As the following lemma shows, there are natural projection and inclusion maps to a copy of the trivial module. We borrow notation from \[\text{T}\].

**Lemma 2.7.** Fix a U-module \( V \) and recall the definition of \( \rho \) from \[\text{T}\].

1. Let \( \text{ev}_V : V^\ast \otimes V \to \mathbb{Q}(q,t)^\tau \) be the \( \mathbb{Q}(q,t)^\tau \)-linear map defined by \( v^\ast \otimes w \to v^*(w) \). Then \( \text{ev}_V \) is a U-module epimorphism.
2. Let \( \text{qtr}_V : V \otimes V^\ast \to \mathbb{Q}(q,t)^\tau \) be the \( \mathbb{Q}(q,t)^\tau \)-linear map defined by \( v \otimes w^\ast \to \pi^{p(v)p(w)} q^{-\langle \tilde{p},|v| \rangle} w^*(v) \).
   Then \( \text{qtr}_V \) is a U-module epimorphism.
3. Let \( \text{coev}_V : \mathbb{Q}(q,t)^\tau \to V^\ast \otimes V \) be the \( \mathbb{Q}(q,t)^\tau \)-linear map defined by \( 1 \to \sum_{b \in B} \pi^{p(h)q^{-\langle \tilde{p},|h| \rangle} b \otimes b \) for some homogeneous \( \mathbb{Q}(q,t)^\tau \)-basis \( B \) of \( V \). Then \( \text{coev}_V \) is a U-module monomorphism.
4. Let \( \text{coqtr}_V : \mathbb{Q}(q,t)^\tau \to V \otimes V^\ast \) be the \( \mathbb{Q}(q,t)^\tau \)-linear map defined by \( 1 \to \sum_{b \in B} b \otimes b^\ast \) for some homogeneous \( \mathbb{Q}(q,t)^\tau \)-basis \( B \) of \( V \). Then \( \text{coqtr}_V \) is a U-module isomorphism.

**Proof.** In the proof, we shall suppress the \( V \) subscript on the maps. First note that the maps \( \text{coev} \) and \( \text{coqtr} \) are independent of the choice of basis. It is clear that all these maps are \( \mathbb{Q}(q,t)^\tau \)-linear maps, and it is elementary to verify the claims about surjectivity and injectivity. Moreover, all the maps are clearly homogeneous since \( |v^\ast| = -|v| \) and \( p(v^\ast) = p(v) \); in particular, the maps \( \text{qtr} \) and \( \text{ev} \) are homogeneous since \( v^\ast(w) = 0 \) whenever \( |v| \neq |w| \) or \( p(v) \neq p(w) \), which occurs exactly when \( |v^\ast \otimes w| \neq 0 \) or \( p(v^\ast \otimes w) = 1 \).

Then it remains to show these maps preserve the action of \( E_i \) and \( F_i \) for all \( i \in I \), which is equivalent to showing
\[
\text{ev}(\Delta(E_i)v^\ast \otimes w) = \text{ev}(\Delta(F_i)v^\ast \otimes w) = 0 \quad \text{for all } v, w \in V,
\]
\[
\text{qtr}(\Delta(E_i)v \otimes w^\ast) = \text{qtr}(\Delta(F_i)v \otimes w^\ast) = 0 \quad \text{for all } v, w \in V,
\]
\[
\Delta(E_i) \sum_{b \in B} \pi^{p(h)q^{-\langle \tilde{p},|h| \rangle} b \otimes b} = \Delta(F_i) \sum_{b \in B} \pi^{p(h)q^{-\langle \tilde{p},|h| \rangle} b \otimes b} = 0 \quad \text{for all } b \in B, \quad (*)
\]
\[
\Delta(E_i) \sum_{b \in B} b \otimes b^\ast = \Delta(F_i) \sum_{b \in B} b \otimes b^\ast = 0 \quad \text{for all } b \in B. \quad (**)
\]
We will prove (*) and (**) for the action of \( E_i \); the remaining cases follow from similar arguments.
First, we show \( qtr(\Delta(E_i)v^* \otimes w) = 0 \). From weight considerations we have that \( ev(\Delta(E_i)v^* \otimes w) = 0 \) unless \( |v| + i = |w| \). In this case,

\[
qtr(\Delta(E_i)v \otimes w^*) = qtr(E_i v \otimes w^* + \pi_i^p(v)(\pi_iq_i)(i,v) v \otimes E_i w^*)
\]

\[
= \pi_i^p(E_i v)(w^*) q^{-\langle \rho, \pi_i E_i v \rangle} w^*(E_i v) + \pi_i^p(v)(\pi_iq_i)(i,v) \pi_i^p(E_i w) q^{-\langle \rho, \pi_i E_i w \rangle} (E_i w^*)(v)
\]

\[
= \pi_i^p(v)(w^*) + p(i)(w^*) q^{-\langle \rho, |v| + i \rangle} (w^*(E_i v) - q_i^2(\pi_iq_i)(i,v) w^*(J_i^{-1} K_i^{-1} E_i v))
\]

\[
= \pi_i^p(v)(w^*) + p(i)(w^*) q^{-\langle \rho, |v| + i \rangle} (w^*(E_i v) - w^*(E_i v)) = 0.
\]

Next, we show that \( \Delta(E_i) \sum_{b \in B} \pi_i^p(b)(q^{\langle \rho, |b| \rangle} b^* \otimes b) = 0 \). Set \( B_\lambda = B \cap V_\lambda \), so \( B = \bigsqcup \lambda B_\lambda \). First observe that \( x = \sum v^* \otimes w = 0 \) if and only if \( x(v') := \sum v^*(v')w = 0 \) for all \( v' \in V \). Then setting \( x = \Delta(E_i) \sum_{b \in B} b^* \otimes b \), if

\[
0 \neq x = \sum_{b \in B} \pi_i^p(b)(q^{\langle \rho, |b| \rangle} (E_i b^* \otimes b + \pi_i^p(b)(\pi_iq_i)^{-\langle i, |b| \rangle} b^* \otimes E_i b))
\]

then there must be some \( v \in V \) such that

\[
x(v) = \sum_{b \in B} \pi_i^p(b)(q^{\langle \rho, |b| \rangle} ((E_i b^*)(v)b + \pi_i^p(b)(\pi_iq_i)^{-\langle i, |b| \rangle} b^*(v)E_i b)) \neq 0.
\]

However, if \( b' \in B \),

\[
x(b') = (\pi_iq_i)^{-\langle i, |b'| \rangle} E_i b' + \sum_{b \in B_{|b'|+i}} b^*(-\tilde{J}_i^{-1} K_i^{-1} E_i b')b
\]

\[
= q^{\langle \rho, |b'| \rangle} (\pi_iq_i)^{-\langle i, |b'| \rangle} E_i b' - \sum_{b \in B_{|b'|+i}} q^{\langle \rho, |b| \rangle} (\pi_iq_i)^{-\langle i, |b'|+i \rangle} b^* (E_i b')b
\]

\[
= q^{\langle \rho, |b'| \rangle} (\pi_iq_i)^{-\langle i, |b'| \rangle} \left( E_i b' - \sum_{b \in B_{|b'|+i}} b^* (E_i b')b \right) = 0.
\]

\[\square\]

2.5. Simple modules and their duals. Let \( \lambda \in X^+ \) and recall from [CHW1] that \( V(\lambda) \) is the simple \( U \)-module of highest weight \( \lambda \) such that the highest weight space has even parity. Then \( V(\lambda) \) has finite rank and has the same character as the \( \mathfrak{so}(2n+1) \) module of highest weight \( \lambda \). In particular, using the Weyl character formula for \( V(\lambda) \), the lowest weight vector has weight \( w_0 \lambda = -\lambda \), hence the parity of the lowest weight vector of \( V(\lambda) \) is \( P(\lambda) \). Using standard arguments (for example, analogues of [Jän] §5.3 and §5.16), and considering the above analysis, we obtain the following lemma.

**Lemma 2.8.** For each \( \lambda \in X^+ \), there is an isomorphism \( V(\lambda)^* \cong \Pi^{P(\lambda)} V(\lambda) \) and a natural isomorphism \( V(\lambda)^{**} \to V(\lambda) \).

**Example 2.9.** In the case \( n = 1 \), the module \( V = V(m) \) for \( m \in \mathbb{Z}_{\geq 0} \) has basis \( v_{m-2k} = F^{(k)} v_m \) with \( 0 \leq k \leq m \), where \( v_m \) is a choice of highest weight vector. Note that by convention \( p(v_m) = 0 \), so \( p(v_{m-2k}) \equiv k \) (mod 2). The dual module \( V(m)^* \) has a dual basis \( v_{m-2k}^*, 0 \leq k \leq m \), and the actions of \( E = E_T \) and \( F = F_T \) are given by

\[
Ev_{m-2k}^* = -\pi^k(pq)^{m-2k} [n+1-k] v_{m-2(k+1)}^*
\]

\[
Fv_{m-2k}^* = -\pi^k(pq)^{m-2k+2} [k] v_{m-2(k-1)}^*
\]

In particular, this is a simple module generated by the highest weight vector \( v_m^* \), where \( |v_m^*| = -|v_{-m}| = m \) and \( p(v_{-m}^*) = p(v_m) \equiv m \) (mod 2), hence we have an isomorphism \( V(m)^* \cong \Pi^m V(m) \).
For convenience, we will use the notation
\[ V(-\lambda) = V(\lambda)^*, \quad \lambda \in X^+. \] (2.21)

We denote the maps in Lemma 2.7 in the case \( V = V(\lambda) \) with the subscript \( \lambda \) instead of \( V(\lambda) \); for instance, \( \text{ev}_\lambda = \text{ev}_{V(\lambda)} \). Note that
\[ \text{ev}_\lambda \circ \text{coev}_\lambda = \sum_{\nu \in \mathbb{N}[I]} \text{rank}_\mathbb{Q}(q,t)^{\nu} (V_{\lambda-\nu})_{\nu} q^{(\delta, \lambda-\nu)} = \pi^{P(\lambda)} \text{qtr}_\lambda \circ \text{coqtr}_\lambda. \]

**Example 2.10.** For \( n = 1 \), we have \( \rho = \tilde{\rho} = 1 \) hence for \( \lambda = m \), \( \langle \tilde{\rho}, \lambda \rangle = m \). Then
\[ \text{ev}_m \circ \text{coev}_m = q^m + \pi q^{-m} + \ldots + \pi^m q^{-m} = \pi^m |m + 1| = \pi^m \text{qtr}_m \circ \text{coqtr}_m. \]

**2.6. Further properties of the quasi-\( \mathcal{R} \)-matrix.** Let us recall the quasi-\( \mathcal{R} \)-matrix from [CHW1, §4]

**Proposition 2.11.** [CHW1] Let \( B \) be any \( \mathbb{Q}(q,t)^{\nu} \)-basis of \( \mathfrak{f} \) such that \( B_{\nu} = \mathfrak{f} \cap B_{\nu} \) is a basis of \( B_{\nu} \) for any \( \nu \in \mathbb{N}[I] \), with \( B_0 = \{1\} \). Let \( B^* = \{b^* \mid b \in B\} \) be the basis of \( \mathfrak{f} \) dual to \( B \) under \( (\cdot, \cdot) \). Define
\[ \Theta_\nu = (-1)^{ht\nu} \pi^{P(\nu)} \pi_\nu q_{\nu} \sum_{b \in B_{\nu}} b^- \otimes (b^*)^+ \in U_{-\nu} \otimes U_{\nu}^+. \]

Then if \( M, M' \) are integrable modules of \( U \), then \( \Theta = \sum_\nu \Theta_\nu \) is a well defined operator on \( M \otimes M' \) which satisfies \( \Delta(u) \Theta = \Theta \Delta(u) \) as endomorphisms of \( M \otimes M' \), where \( \Delta(u) = \Delta(\overline{u}) \). Moreover, \( \Theta \) is independent of the choice of basis \( B \), and is invertible with inverse \( \overline{\Theta} \).

In particular, note that all modules considered in this paper are of finite rank over \( \mathbb{Q}(q,t)^{\nu} \), hence are integrable.

**Example 2.12.** When \( n = 1 \), the quasi-\( \mathcal{R} \)-matrix \( \Theta \) can be explicitly given by the formula
\[ \Theta = \sum_{n \geq 0} (-1)^n (\pi q)^\Delta \cdot (\pi q - 1)^n F(n) \otimes E(n) = 1 - (\pi q - 1) F \otimes E + \ldots. \]

(NB. there is a typo in the power of \( \pi q \) in [CHW1, Example 3.1.2].)

While \( \overline{\Theta} \) can be evaluated easily, it will be more convenient to have the following alternate description of \( \overline{\Theta} \) using the properties of the bilinear form on \( \mathfrak{f} \) (cf. [CHW1, §1.4]).

**Lemma 2.13.** With the same notations as in Proposition 2.11, \( \overline{\Theta} = \sum_\nu \overline{\Theta}_\nu \) is given by
\[ \overline{\Theta}_\nu = \pi_\nu q^{\frac{\Delta}{2}} \sum_{b \in B_{\nu}} b^- \otimes \sigma(b^*)^+ \in U_{-\nu} \otimes U_{\nu}^+. \]

**Proof.** Let \( \overline{B} = \{ \overline{b} \mid b \in B \} \), with dual basis \( \overline{B}^* \). Then since \( \Theta \) is independent of the choice of basis, we see that for \( \nu \in \mathbb{N}[I] \), \( \Theta_\nu = (-1)^{ht\nu} \pi^{P(\nu)} \pi_\nu q_{\nu} \sum_{b \in B_{\nu}} \overline{b}^- \otimes (\overline{b}^*)^+ \). We have \( \overline{\Theta}_\nu = (-1)^{ht\nu} \pi^{P(\nu)} q^{-\nu} \sum_{b \in B_{\nu}} (\overline{b}^- \otimes (\overline{b}^*)^+), \) and note that \( (\overline{\pi})^\pm = (\pi^\pm), \) so \( (\overline{b}^-) = b^- \).

On the other hand, recall from [CHW1, §1.4] the variant bilinear form \( \{ - , - \} \) defined by \( \{ x, y \} = (x, \overline{y}) \). Note that by construction, \( \langle \overline{b}, \overline{b} \rangle = \delta_{b,b'} \). Then for any \( b, b' \in B \), we apply Lemma 1.4.3 (b) of loc. cit. to deduce that
\[ \delta_{b,b'} = \langle \overline{b}, \overline{b} \rangle = \{ \overline{b}, \overline{b} \} = (-1)^{ht\nu} \pi^{P(\nu)} \pi_\nu q^{\frac{\Delta}{2}} \pi_{-\nu} q_{-\nu} (\overline{b}, \sigma(b')). \]

(We note that while the power of \( \pi \) appears different from that in loc. cit., it is equivalent.) Therefore, we have
\[ \overline{b} = (-1)^{ht\nu} \pi^{P(\nu)} q^{\frac{\Delta}{2}} \pi_\nu q_{\nu} \sigma(b)^*. \]

Then the lemma follows from the observation that since \( (\sigma(x), \sigma(y)) = (x, y) \), \( \sigma(b)^* = \sigma(b^*) \). \( \square \)
Now we will proceed to use $\Theta$ to define a universal map $\mathcal{R} : M \otimes N \rightarrow N \otimes M$ for any modules $M$ and $N$. These constructions will be modified versions of the standard arguments in the non-super case; cf. [Jän] SS7.3-7.6 or [L93] §4.2 and Chapter 32.

For $1 \leq s < t \leq 3$, let $\Theta_{\nu}^{st} \in U \otimes U \otimes U$ be defined by $\Theta_{\nu}^{st} = (-1)^{ht}\nu \pi^{(v)}_{\pi_{\nu}}q_{\nu}\sum_{b \in B_v} b_1 \otimes b_2 \otimes b_3$ where $b_s = b^-b_1 = (b^*)^+$, and $b_m = 1$ for $m \neq s, t$.

**Proposition 2.14.** We have the following identities.

\[
(\Delta \otimes 1)(\Theta_{\nu}) = \sum_{\nu' + \nu'' = \nu} \Theta_{\nu'}^{23}(1 \otimes \tilde{K}_{-\nu''} \otimes 1)\Theta_{\nu''}^{13}.
\]

\[
(\bar{\Delta} \otimes 1)(\Theta_{\nu}) = \sum_{\nu' + \nu'' = \nu} \Theta_{\nu'}^{13}(1 \otimes \tilde{J}_{\nu'} \otimes 1)\Theta_{\nu''}^{23}.
\]

\[
(1 \otimes \Delta)(\Theta_{\nu}) = \sum_{\nu' + \nu'' = \nu} \Theta_{\nu'}^{12}(1 \otimes \tilde{J}_{\nu''} \otimes 1)\Theta_{\nu''}^{13}.
\]

\[
(1 \otimes \bar{\Delta})(\Theta_{\nu}) = \sum_{\nu' + \nu'' = \nu} \Theta_{\nu'}^{13}(1 \otimes \tilde{K}_{-\nu''} \otimes 1)\Theta_{\nu'}^{12}.
\]

**Proof.** These identities are proved exactly as in [L93] §4.2. We will prove the first identity here. For $x \in f$ and $b_1, b_2 \in B$, define $f(x, b_1, b_2), f'(x, b_1, b_2) \in \mathbb{Q}(q, t)^{\mathbb{N}}$ via

\[
r(x) = \sum_{b_1, b_2 \in B} f(x, b_1, b_2) b_1 \otimes b_2,
\]

\[
\pi(x) = \sum_{b_1, b_2 \in B} f'(x, b_1, b_2) b_1 \otimes b_2.
\]

Then it suffices to show that

\[
\sum_{b_1, b_2 : |b_1| + |b_2| = |b| = \nu} f'(b, b_1, b_2) b_1^* \otimes \tilde{K}_{b_1} b_2^* \otimes (b^*)^+ = \sum_{b_1, b_2 : |b_1| + |b_2| = \nu} \pi^{(b_1)p(b_2)}_{\nu} b_1^* \otimes b_2^* \tilde{K}_{-|b_1|} \otimes (b^*_1 b^*_2)^+.
\]

In particular, it is enough to show that for all $b_1, b_2 \in B$ such that $|b_1| + |b_2| = \nu$, we have

\[
\sum_{b_1, b_2 : |b_1| = \nu} f'(b, b_1, b_2) b^* = \pi^{(b_1)p(b_2)}_{\nu} q^{-|b_1| - |b_2|} b^*_1 b^*_2.
\]

This follows from the equalities

\[
\pi^{(b_1)p(b_2)}_{\nu} q^{-|b_1| - |b_2|} f'(b, b_1, b_2) = f(b, b_2, b_1) = (r(b), b^*_2 \otimes b^*_1) = (b, b^*_2 b^*_1),
\]

which in turn follow from elementary properties of $f$; cf. [CHW] Lemmas 1.4.1, 1.4.3.

To construct a universal $U$-module homomorphism from $\Theta$, we will need some additional maps. The first is the swap map; that is, the algebra $U \otimes U$ is equipped with an involution $s$ defined by $s(x \otimes y) = \pi^{(x)p(y)} y \otimes x$. This induces involutions on $U^\otimes m$ by applying $s$ to sequential pairs of tensor factors; specifically, these involutions are the maps $s_{t,t+1} = 1^\otimes t-1 \otimes s \otimes 1^\otimes m-t-1$, and it is not hard to see they satisfy the braid relations $s_{t-1,t} s_{t+1,t} s_{t-1,t} = s_{t+1,t} s_{t-1,t} s_{t+1,t-1}$. In particular, we see that to each element $\gamma$ of the permutation group $S_m$, there is an automorphism $s_{\gamma}$ of $U^m$; for example $s_{(23)} = s_{2,3}$ and $s_{(123)} = s_{1,2} s_{2,3}$. Similarly, to any tensor product of modules $N = \bigotimes_{i=1}^{m} M_i$ and $\gamma \in S_m$, we can define $N_\gamma = \bigotimes_{i=1}^{m} M_{\gamma(i)}$ and a map $s_{\gamma} : N \rightarrow N_\gamma$ given by $s_{\gamma}(v) = \pi^{\gamma(v)}_{\gamma(1)} v_{\gamma}$, where $v = v_1 \otimes \cdots \otimes v_m$, $v_{\gamma} = v_{\gamma(1)} \otimes \cdots \otimes v_{\gamma(m)}$, and

\[
p(\gamma, v) = \sum_{1 \leq s < t \leq n} p(v_s)p(v_t).
\]

where $v = v_1 \otimes \cdots \otimes v_m$, $v_{\gamma} = v_{\gamma(1)} \otimes \cdots \otimes v_{\gamma(m)}$, and

\[
p(\gamma, v) = \sum_{1 \leq s < t \leq n} p(v_s)p(v_t).
\]
These maps are compatible in the sense that for \( v = \bigotimes_{i=1}^{n} v_i \in N \) and \( u \in U \),
\[
s_{\gamma}(\Delta^{m-1}(u)v) = s_{\gamma}(\Delta^{m-1}(u))s_{\gamma}(v).
\]
When \( m = 2 \), we will just write \( s = s_{1,2} \).

The other ingredient is a weight-renormalization operator. This operator is induced by the weight function defined in the following lemma.

**Lemma 2.15.** There exists a function \( f : X \times X \to (\mathbb{Q}(q, t)^*)^X \) satisfying
\[
f(\zeta + \mu', \zeta' + \nu')f(\zeta, \zeta')^{-1} = (\pi q)^{-(\hat{\mu}, \zeta')q^{-\nu}(\hat{\nu}, \zeta) - \nu \mu - (\hat{\nu}, \zeta) - (-\nu)\mu}
\]
for \( \zeta, \zeta' \in X \) and \( \mu, \nu \in \mathbb{Z}[I] \). Moreover,
1. The function \( \nu(\zeta, \zeta') = f(\zeta, \zeta')f(\zeta - \zeta') \) satisfies \( \nu(\zeta + \mu, \zeta' + \nu) = \nu(\zeta, \zeta') \) for any \( \mu, \nu \in \mathbb{Z}[I] \).
2. The function \( \nu(\zeta, \zeta') = f(\zeta, \zeta')f(\zeta - \zeta') \) satisfies \( \nu(\zeta + \mu, \zeta' + \nu) = \nu(\zeta, \zeta') \) for any \( \mu, \nu \in \mathbb{Z}[I] \).
3. We have \( f(\zeta, \zeta')f(\zeta - \zeta')^{-1} = \pi^{\nu(\zeta, \zeta')} \); in particular, \( \nu(\zeta, \zeta') = \pi^{\nu(\zeta, \zeta')} \).

**Proof.** It is easy to verify that such a function \( f \) exists by choosing a set of coset representatives \( R \) for \( \mathbb{Z}[I] \) in \( X \). It is similar to verify (1) and (2), so let us show (1). Let \( \xi = \zeta + \nu \) and \( \xi' = \zeta' + \mu \) for some \( \mu, \nu \in \mathbb{Z}[I] \) and \( \zeta, \zeta' \in X \). Then
\[
f(\xi, \xi')f(\xi, -\xi') = f(\xi, \xi')f(\xi, -\xi')(\pi q)^{\nu(\zeta, \zeta')} = f(\zeta, \zeta')(f(\zeta - \zeta')^{-1})
\]
Finally, let \( \zeta, \zeta' \in X \). Then \( -\zeta = \zeta - 2\zeta, -\zeta' = \zeta' - 2\zeta' \) so
\[
f(\zeta, \zeta')f(-\zeta, -\zeta')^{-1} = f(-\zeta + 2\zeta, -\zeta' + 2\zeta')(f(-\zeta, -\zeta')^{-1}) = \pi^{-(\hat{\nu}, \zeta) - \nu \mu - (\hat{\nu}, \zeta) - (-\nu)\mu}
\]
Now note that for any \( \eta, \eta' \in X \), we have \( -\langle \tilde{2}\eta, \tilde{-2}\eta' \rangle = \frac{1}{2}(2\eta) \cdot (2\eta') \). Moreover, by (2.5) and Lemma 2.2, we see that \( \langle \tilde{2}\eta, \tilde{-2}\eta' \rangle = \langle 2\eta, 2\eta' \rangle \equiv (2\eta)\eta' \equiv n(2\eta)\eta' \equiv 2(n, \eta') \equiv 2p(\eta)p(\eta') \) mod 2. Therefore, we see that
\[
f(-\zeta + 2\zeta, -\zeta' + 2\zeta')f(-\zeta, -\zeta')^{-1} = \pi^{p(\eta)p(\eta')}
\]
This finishes the proof. \( \square \)

**Example 2.16.** Let us consider the case \( n = 1 \). Then the function \( f \) is determined by the values
\[
f(0, 0), f(0, 1), f(1, 0), \text{ and } f(1, 1).
\]
Then for any \( \epsilon_1, \epsilon_2 \in \{0, 1\},
\[
f(\epsilon_1 + 2s, \epsilon_2 + 2t) = f(\epsilon_1, \epsilon_2)q^{s_1s_2 - 2at}.
\]
By direct computation, one finds the corresponding coset functions to be
\[
\nu(\epsilon_1 + 2s, \epsilon_2 + 2t) = f(\epsilon_1, \epsilon_2)q^{s_1s_2}.
\]

Given \( U \)-modules \( M, M' \), define the \( \mathbb{Q}(q, t)^* \)-linear bijection \( \tilde{\delta} : M \otimes M' \to M \otimes M' \) by \( \tilde{\delta}(m \otimes m') = f(|m|, |m'|)m \otimes m' \). For \( 1 \leq s < t \leq 3 \), we define \( \tilde{\delta}^{st} \) on \( M_1 \otimes M_2 \otimes M_3 \) via \( \tilde{\delta}^{st}(m_1 \otimes m_2 \otimes m_3) = f(|m_s|, |m_t|)m_1 \otimes m_2 \otimes m_3 \). Let \( \tilde{\Theta}^{st} = \Theta^{st} \circ \tilde{\delta}^{st} \).

**Proposition 2.17** (Yang-Baxter equation). As operators on \( M_1 \otimes M_2 \otimes M_3 \),
\[
\tilde{\delta}^{12}\Theta^{13} \circ \tilde{\delta}^{13}\Theta^{23} \circ \tilde{\delta}^{23}\Theta^{12} = \tilde{\delta}\Theta^{23} \circ \tilde{\Theta}^{13} \circ \tilde{\delta}\Theta^{12}
\]
**Proof.** First note that the maps \( \tilde{\delta}^{st} \) are bijections which commute with one another. One verifies directly that, as operators on \( M_1 \otimes M_2 \otimes M_3 \),
\[
\tilde{\delta}^{12}\Theta^{13} = \Theta^{13}_\nu(1 \otimes \tilde{J}_\nu K_\nu \otimes 1)\tilde{\delta}^{12}, \quad \tilde{\delta}^{23}\Theta^{13} = \Theta^{13}_\nu(1 \otimes K_{-\nu} \otimes 1)\tilde{\delta}^{23}, \quad \tilde{\delta}^{12}\Theta^{13} = \Theta^{13}_\nu(1 \otimes \tilde{J}_\nu \otimes 1)\tilde{\delta}^{12}.
\]
In particular, it suffices to show that
\[ \Theta^{12} \left( \sum_{\nu} \Theta^{13}_{\nu} (1 \otimes J_{\nu} K_{\nu} \otimes 1) \right) \Theta^{23} = \Theta^{23} \left( \sum_{\nu} \Theta^{13}_{\nu} (1 \otimes K_{-\nu} \otimes 1) \right) \Theta^{12}. \]

Writing \( \Theta^{12} = \sum_{\mu} \Theta^{12}_{\mu} \), we have
\[ \Theta^{12} \left( \sum_{\nu} \Theta^{13}_{\nu} (1 \otimes J_{\nu} K_{\nu} \otimes 1) \right) = \sum_{\mu, \nu} \Theta^{12}_{\mu} (1 \otimes J_{\nu} K_{\nu} \otimes 1) \Theta^{13}_{\nu} = \sum_{\nu} (1 \otimes \Delta)(\Theta_{\nu}), \]
and similarly
\[ \left( \sum_{\nu} \Theta^{13}_{\nu} (1 \otimes K_{-\nu} \otimes 1) \right) \Theta^{12} = \sum_{\nu} (1 \otimes \Delta)(\Theta_{\nu}). \]

Then we are reduced to showing the equality
\[ \sum_{\nu} (1 \otimes \Delta)(\Theta_{\nu}) \Theta^{23} = \Theta^{23} \sum_{\nu} (1 \otimes \Delta)(\Theta_{\nu}), \]
which follows from the defining property of \( \Theta \).

**Proposition 2.18.** Define \( \mathcal{R} : M \otimes M' \to M' \otimes M \) by \( \mathcal{R} = \Theta \circ \tilde{\ell} \circ \sigma \). Then \( \mathcal{R} \) is a \( U \)-module isomorphism.

**Proof.** That \( \mathcal{R} \) is bijective and homogeneous in parity is clear from construction. Note that
\[ \Delta(u)\mathcal{R}(m \otimes m') = \Theta(\Delta(u) \tilde{\ell} \circ \sigma(m \otimes m')) = \Theta(f(|m'|, |m|)\pi^{p(m)p(m')}, \Delta(u)(m' \otimes m)), \]
so it suffices to show
\[ \tilde{\ell} \circ \sigma(\Delta(u)m \otimes m') = f(|m'|, |m|)\pi^{p(m)p(m')}\Delta(u)(m' \otimes m) \]
for all \( u \in U \), hence it is enough to show this equality holds when \( u \) is a generator. For \( u = J_{\nu}, K_{\nu} \), this is straightforward. The cases \( u = E_i \) and \( u = F_i \) are similar, so we shall prove the first case:
\[ \tilde{\ell} \circ \sigma(\Delta(E_i)m \otimes m') = f(|m'|, i + |m|)\pi^{p(i)p(m') + p(m')}m' \otimes E_i m \]
\[ + f(i + |m'|, |m|)\pi^{p(m)p(m')}\pi^{d_i(i, |m|)}E_i m' \otimes m \]
\[ = f(|m'|, |m|)\pi^{p(m)p(m')} (E_i m' \otimes m + \pi^{p(i)p(m')}q^{d_i(i, |m'|)}m' \otimes E_i m). \]

We thus obtain the following crucial property of \( \mathcal{R} \).

**Proposition 2.19.** For any modules \( M_1, M_2, \) and \( M_3 \), let \( \mathcal{R}_{st} = \tilde{\ell} \Theta^{st} \circ \sigma(st) \). Then
\[ \mathcal{R}_{12}\mathcal{R}_{23}\mathcal{R}_{12} = \mathcal{R}_{23}\mathcal{R}_{12}\mathcal{R}_{23} : M_1 \otimes M_2 \otimes M_3 \to M_3 \otimes M_2 \otimes M_1. \]

**Proof.** First note that if \( \sigma(s) < \sigma(t) \), \( \sigma \circ \tilde{\ell} \Theta^{st} = \tilde{\ell} \Theta^{s|t|} \sigma \). Therefore we have \( \sigma_{(12)} \tilde{\ell} \Theta^{23} = \tilde{\ell} \Theta^{13} \sigma_{(12)} \), and \( \sigma_{(13)} \tilde{\ell} \Theta^{12} = \tilde{\ell} \Theta^{23} \sigma_{(12)} \), hence in particular
\[ \mathcal{R}_{12}\mathcal{R}_{23}\mathcal{R}_{12} = \tilde{\ell} \Theta^{12} \circ \tilde{\ell} \Theta^{13} \circ \tilde{\ell} \Theta^{23} \circ \sigma_{(13)} \]
Similar manipulations of the right-hand side yield the equality
\[ \mathcal{R}_{23}\mathcal{R}_{12}\mathcal{R}_{23} = \tilde{\ell} \Theta^{23} \circ \tilde{\ell} \Theta^{13} \circ \tilde{\ell} \Theta^{12} \circ \sigma_{(13)}. \]

Since \( s_{13} \) is a bijection, the proposition follows from the Yang-Baxter equation. □
Remark 2.20. In [L93, §32], it is shown that for \( g = \mathfrak{sl}(2) \), which we can view as the \( \pi = 1 \) specialization of Example 2.12, we can extend our field \( \mathbb{Q}(q) \) to \( \mathbb{Q}(\sqrt{q}) \) and normalize so that \( f(m + a, n + b) = f(m, n)f(a, b)f(a, n)f(a, b) \). This is necessary for the maps \( R \) to satisfy the Hexagon Identities and thus define a braiding on the category of finite dimensional modules.

Note that Example 2.12 shows such a renormalization is impossible in general in the \( \pi = -1 \) case, so in particular the maps \( R \) can not be normalized to define a braiding on the category of finite dimensional weight modules. It is possible to overcome this difficulty by restricting the class of modules to those of "even" highest weight, or by expanding the definition of \( f \) to a function on \( \hat{X} \times \hat{X} \), but we shall not need this at present.

3. Diagrammatic Calculus and Knot invariants

We will now interpret the \( U \)-module homomorphisms in terms of planar diagrams. At first, these diagrams should be interpreted as slice diagrams; that is, diagrams together with vertical slices at various heights such that between consecutive slices is an elementary diagram corresponding to a \( U \)-module homomorphism. However, we will ultimately see that diagrams which can be identified by planar isotopies yield the same morphisms.

3.1. Cups, caps, and crossings. Recall that \( \text{coqtr}_\lambda, \text{coev}_\lambda, \text{qtr}_\lambda, \text{ev}_\lambda \) are the maps defined in Lemma 2.7 where \( V = V(\lambda) \). Likewise, let \( R_{\pm \lambda, \pm \mu} : V(\pm \lambda) \otimes V(\pm \mu) \to V(\pm \mu) \otimes V(\pm \lambda) \) be the map defined in Proposition 2.18. Furthermore, we will use the notation \( 1_{\pm \lambda} = 1_{V(\pm \lambda)} \).

We will now begin to represent our maps via a graphical calculus in anticipation of constructing tangle invariants. Specifically, we follow [Tu, ADO] and interpret maps between tensor products of the modules \( V(\pm \lambda) \) for various \( \lambda \in X^+ \) as sliced oriented tangle diagrams with \( X^+ \)-labeled strands; a concise exposition of this approach is laid out in [Oht, Chapter 3]. The elementary oriented tangle diagrams are interpreted as follows. (Note that while sideways-oriented crossings aren’t considered elementary, we include them here for convenience in later arguments.)

\[
\begin{align*}
1_\lambda &= \begin{array}{c}
\lambda \\
\end{array} & 1_{-\lambda} &= \begin{array}{c}
\lambda \\
\end{array} \\
\text{coqtr}_\lambda &= \begin{array}{c}
\lambda \\
\end{array} & \text{coev}_\lambda &= \begin{array}{c}
\lambda \\
\end{array} & \text{qtr}_\lambda &= \begin{array}{c}
\lambda \\
\end{array} & \text{ev}_\lambda &= \begin{array}{c}
\lambda \\
\end{array} \\
R_{\lambda, \mu} &= \begin{array}{c}
\lambda \\
\mu \\
\end{array} & R_{-\lambda, -\mu} &= \begin{array}{c}
\lambda \\
\mu \\
\end{array} & R_{\lambda, -\mu} &= \begin{array}{c}
\lambda \\
\mu \\
\end{array} & R_{-\lambda, \mu} &= \begin{array}{c}
\lambda \\
\mu \\
\end{array} \\
R_{\lambda, \mu}^{-1} &= \begin{array}{c}
\lambda \\
\mu \\
\end{array} & R_{-\lambda, -\mu}^{-1} &= \begin{array}{c}
\lambda \\
\mu \\
\end{array} & R_{\lambda, -\mu}^{-1} &= \begin{array}{c}
\lambda \\
\mu \\
\end{array} & R_{-\lambda, \mu}^{-1} &= \begin{array}{c}
\lambda \\
\mu \\
\end{array}
\end{align*}
\]

We construct more general diagrams from these elementary ones by the following constructions. If \( T \) is some diagram denoting the morphism \( \phi \) and \( S \) is some diagram denoting the morphism \( \psi \), then we can combine them as

- the horizontal composition \( T S \) which denotes the tensor product \( \phi \otimes \psi \)
• the vertical composition \[ T \circ S \] which denotes the composition \( \phi \circ \psi \), or zero if this composition is undefined (which is to say, when the strands on top of S don’t match the number and labelling of the strands on the bottom of T).

We will say two diagrams are equal if the corresponding morphisms agree. Note that, by construction and by Lemma 2.19, the following diagrams are equal for any choice of orientation and labeling of strands:

\[ T = T = T = T \]

\[ \text{Diagram 1} \]

\[ \text{Diagram 2} \]

In 3.1, the symbols \[ T \] and \[ S \] stand for arbitrary sub-diagrams with an arbitrary number of strands protruding from the top and bottom and with an arbitrary labeling of strands.

3.2. Graphical identities. Now we shall prove some more substantial diagrammatic identities.

Lemma 3.1. We have an equality of diagrams

\[ \text{Diagram 3} \]

For any choice of orientation or labeling of the strand.

Proof. This follows by choosing a homogeneous basis for the module and applying the definitions; we will prove the equality

\[ \text{Diagram 4} \]

in detail, as the other cases are similar. In terms of morphisms, we wish to show \((1_{-\lambda} \otimes \text{qtr}_\lambda) \circ (\text{coev}_\lambda \otimes 1_{-\lambda}) = 1_{-\lambda}\). Let \( B \) be a homogeneous basis of \( V(\lambda) \) and \( B^* \) the dual basis of \( V(\lambda)^* \). Then for any \( b_0 \in B \),

\[ (1_{-\lambda} \otimes \text{qtr}_\lambda)(\text{coev}_\lambda \otimes 1_{-\lambda})(b_0^*) = \sum_{b \in B} \pi^{(b)} q_b^{(\rho, [b])} (1_{-\lambda} \otimes \text{qtr}_\lambda)(b^* \otimes b \otimes b_0^*) = \sum_{b \in B} b_0^*(b)b^* = b_0^*. \]

\[ \square \]
Lemma 3.2. For $\lambda \in X^+$, we have an equality of diagrams

(a) $f(\lambda, \lambda) q^{-\langle \tilde{\rho}, \lambda \rangle} = \pi^{P(\lambda)}$

(b) $f(\lambda, \lambda) q^{-1} q^{\langle \tilde{\rho}, \lambda \rangle} = \pi^{P(\lambda)}$

(c) $f(\lambda, \lambda) q^{-\langle \tilde{\rho}, \lambda \rangle} = \pi^{P(\lambda)}$

(d) $f(\lambda, \lambda) q^{-1} q^{\langle \tilde{\rho}, \lambda \rangle} = \pi^{P(\lambda)}$

Proof. The proofs of (a)-(d) are all similar, so we will only prove (a). First, let us denote

$\phi = (1_\lambda \otimes \text{qtr}_\lambda) \circ (R_{\lambda, \lambda} \otimes 1_\lambda) \circ (1_\lambda \otimes \text{coqtr}_\lambda)$,

$\psi = (\text{ev}_\lambda \otimes 1_\lambda) \circ (1_\lambda \otimes R_{\lambda, \lambda}) \circ (\text{coev}_\lambda \otimes 1_\lambda)$.

Since $\phi$ and $\psi$ are $U$-module homomorphisms from $V(\lambda)$ to $V(\lambda)$, $\phi$ and $\psi$ must each be a multiple of the identity which is completely determined by the image of an extremal weight vector, so let $v_\lambda \in V(\lambda)_\lambda$ and $v_{-\lambda} \in V(\lambda)_{-\lambda}$ be nonzero highest- and lowest-weight vectors. Then if $B(\lambda)$ is a homogeneous basis of $V(\lambda)$, then

$\phi(v_\lambda) = (1_\lambda \otimes \text{qtr}_\lambda) \circ (R_{\lambda, \lambda} \otimes 1_\lambda) \left( \sum_{v \in B(\lambda)} v_\lambda \otimes v \otimes v^* \right)$

$= (1_\lambda \otimes \text{qtr}_\lambda) \left( \sum_{v \in B(\lambda)} f(v, \lambda) v \otimes v_\lambda \otimes v^* \right) = f(\lambda, \lambda) q^{-\langle \tilde{\rho}, \lambda \rangle} v_\lambda$,

and thus $\phi = f(\lambda, \lambda) q^{-\langle \tilde{\rho}, \lambda \rangle} 1_\lambda$. Likewise, we compute

$\psi(v_{-\lambda}) = (\text{ev}_\lambda \otimes 1_\lambda) \circ (1_\lambda \otimes R_{\lambda, \lambda}) \left( \sum_{v \in B(\lambda)} \pi^{P(v)} q^{\langle \tilde{\rho}, |v| \rangle} v^* \otimes v \otimes v_{-\lambda} \right)$
We shall compare the images of our three maps on tensor factors on the right: these tensor factors when computing these maps. First, we have the coevaluation which adds two
representation tells us which tensor factors are impacted at each step, so we restrict our view to
Let
\[ v_{\lambda, \mu} = (\pi_{p(\mu)p(\lambda)} \varphi(\mu, \lambda) \psi) \]
\begin{align*}
\coev(1) &= \sum_{v \in B(\lambda)} \pi_{\nu} q^{0, \nu} v^* \otimes v.
\end{align*}
Next, we apply the quantum trace to the two tensor factors on the left, hence we need to compute
\[ \text{qtr}(v_0 \otimes \sigma(b^*)^+ v^*). \]
Since \( x^* (y) = 0 \) unless \( ||x|| = ||y|| \) (that is, unless \( x \) and \( y \) have the same weight and parity), we can assume \( |v| = \kappa + \nu \) and \( p(v) = p(v_0) + p(\nu) \). Then we have
\[ \text{qtr}_{\lambda}(v_0 \otimes \sigma(b^*)^+ v^*) = \pi_{p(v)} q^{-\widehat{\rho}, |v|}(\sigma(b^*)^+ v^*)(v_0) \]
\[ = (-1)^{htv} \pi_{p(v)+p(v_0)+p(\nu)p(v_0)+p(\nu)} q^{-\widehat{\rho}, \nu}(\pi q)^{-\widehat{\rho}, \nu} q_{(b^*)^+ v^*}((b^*)^+ v_0). \]
Putting these computations together, we see that
\[
\phi(v_0 \otimes w_0^*) = \sum_{v \in B(\lambda)} \sum_{\nu} \sum_{b \in B_{\nu}} \pi^{p(v_0) + p(\nu)} q^{\delta, \kappa + \nu} \times f(-\xi - \nu, -\kappa)^{-1} \pi^{p(w_0) p(v_0) + p(\nu) p(v_0) + p(\nu)} q^{\frac{\nu}{\kappa}} \times (-1)^{ht \nu} \pi^{p(\nu)} q^{\nu} \sum_{b \in B_{\nu}} b^{-v_0^*} \otimes v
\]
\[
= \sum_{\nu} (-1)^{ht \nu} f(-\xi - \nu, -\kappa)^{-1} (\pi q)^{\delta, \kappa} \sum_{b \in B(\lambda)} b^{-v_0^*} \otimes \left( \sum_{v \in B(\lambda)} v^* ((b^*)^+ + v_0^*) \right).
\]

But note that \(f(-\xi - \nu, -\kappa) (\pi q)^{\delta, \kappa} = f(-\xi, -\kappa), q^{\delta, \nu} = q_\nu^2\), and \(\sum_{v \in B(\lambda)} v^* ((b^*)^+ + v_0) = (b^*)^+ + v_0\).

Therefore, we have
\[
\phi(v_0 \otimes w_0^*) = f(-\xi, -\kappa)^{-1} \pi^{p(v_0) p(w_0)} \sum_{\nu} (-1)^{ht \nu} \pi^{p(\nu)} \sum_{b \in B_{\nu}} q^{\nu} q_{\nu} \pi^{p(w_0) p(\nu)} b^{-w_0^*} \otimes (b^*)^+ + v_0
\]
\[
= \varphi(-\xi, -\kappa)^{-1} R_{\lambda, -\mu} (v_0 \otimes w_0^*).
\]

Finally, since \(-\xi \in \mu + \mathbb{Z}[I]\) and \(\kappa \in \lambda + \mathbb{Z}[I]\), we can apply Lemma 2.15(1) to conclude that \(\phi = \varphi(\mu, \lambda)^{-1} R_{\lambda, -\mu}\). A similar computation shows that \(\psi = \Pi(\mu, \lambda)^{-1} R_{\lambda, -\mu}\), and the result then follows from Lemma 2.15 \(\Box\)

Note that by identifying inverse maps in Lemma 2.23(a) and (b), we obtain the following corollary.

**Corollary 3.4.** We have an equality of diagrams

![Diagram](image)

Finally, we show a somewhat more involved identity, which will lead us to our our final result.

**Lemma 3.5.** We have an equality of diagrams

![Diagram](image)

for any choice of orientation.

**Proof.** In order to prove the identity without referring to a particular orientation, it will be convenient to introduce the following notation. Suppose \(m \in V(\zeta)\) and \(n \in V(-\zeta)\) for some \(\zeta \in X^+\).

Let us denote by \((n, m)\) (respectively \(m, n\)) the evaluation \(ev_\zeta(n \otimes m)\) (respectively, the quantum trace \(qt_\zeta(m \otimes n)\)). In particular, one may think of \((-,-)\) as a pairing on \(V(\zeta) \oplus V(-\zeta)\) satisfying, for \(v, w \in V(\zeta),\)

\[
(v, w) = (v^*, w^*) = 0, \quad (v, w^*) = \pi^{p(v) p(w)} q^{\delta, \nu} (w^*, v),
\]
\[
(uv, w^*) = \pi^{p(u) p(v)} (v, S(u) w^*), \quad (uw^*, v) = \pi^{p(u) p(w)} (w^*, S(u) v).
\]

(3.4)
Thus to see that

\[ (E_i^v, w^*) = \pi^{p(v)p(i)}(v, S(E_i)w^*) \]

which follows from a simple calculation on the generators: for example,

\[ (E_i^v, w^*) = \pi^{p(v)p(w)}q^{-\langle \bar{r}, |v| \rangle}q^{-1}(E_i\bar{J}_i^{-1}K_i^{-1}w^*)(v) = \pi^{p(v)p(i)}(v, S(E_i)w^*) \]

In this proof we will use the notation \((-\cdot, -\cdot)\) as shorthand for \(ev_\zeta\) and \(qtr_\tau\) for both \(\zeta = \lambda, \mu\) with the intended map (and highest weight) being clear from context. Using this notation, the diagram equality is equivalent to showing that the maps

\[
\psi = (-, -) \circ (1_{sp} \otimes (-, -) \otimes 1_{-sp}) \circ (R_{s\lambda,t\mu} \otimes 1_{-s\lambda} \otimes 1_{-t\mu})
\]

\[
\phi = (-, -) \circ (1_{t\lambda} \otimes (-, -) \otimes 1_{-t\lambda}) \circ (1_{s\lambda} \otimes 1_{t\mu} \otimes R_{-s\lambda,-t\mu})
\]

are \(\pi^{p(\mu)p(\lambda)}\) multiples of each other for any choice of \(s, t \in \{1, -1\}\).

Let \(w \in V(s\lambda), x \in V(t\mu), y \in V(-s\lambda), \) and \(z \in V(-t\mu), \) where \(V(-\xi) = V(\xi)^*\) for \(\xi \in X^+.\)

Then on one hand,

\[
\psi(w \otimes x \otimes y \otimes z) = \sum_{\nu} \sum_{b \in B_{r\nu}} \pi^{p(\nu)p(w)}f(|x|, |w|)(-1)\nu \pi^{p(\nu)p(x)}(b^{-x}, b)(b^* + w, y).
\]

On the other hand, using the representation of \(\Theta\) in the basis \(\sigma(B)\),

\[
\phi(w \otimes x \otimes y \otimes z) = \sum_{\nu} \sum_{b \in B_{r\nu}} \pi^{p(\nu)p(z)}f(|z|, |y|)(-1)\nu \pi^{p(\nu)p(x)}(b, b^{-z} + w, \nu).
\]

Thus to see that \(\psi(w \otimes x \otimes y \otimes z) = \pi^{p(\mu)p(\lambda)}\phi(w \otimes x \otimes y \otimes z)\), and hence that \(\psi = \pi^{p(\mu)p(\lambda)}\phi\) since \(w, x, y, z\) are arbitrary, it is enough to show that \(l = \pi^{p(\mu)p(\lambda)}r\), where

\[
l = \pi^{p(\nu)p(z)}f(|z|, |y|)(x, \sigma(b^{-z} + w, \nu)).
\]

\[
r = \pi^{p(\nu)p(x)}f(|x|, |w|)(b^{-x}, b^* + w, y).
\]

Using the properties of \((-\cdot, -\cdot)\) (see (3.3) and \(S\) (see (2.20)), we see that

\[
(x, \sigma(b^{-z} + w, \nu)) = \pi^{p(x)p(v)p(w)}q^{-\nu \nu + \langle \bar{v}, |x| \rangle}(\pi q)^{-\langle \bar{v}, |w| \rangle}(b^{-x}, b^* + w, y).
\]

Note that \(l, r\) are both zero unless \(-||x|| = ||z|| - \nu\) and \(-||w|| = ||y|| + \nu\). In particular, \(l\) and \(r\) are both zero unless \(p(y) = p(w) + p(x)\), \(p(z) = p(x) + p(y)\), in which case

\[
p(y)p(z) + p(x)p(x) + p(w)p(y) \equiv p(w)p(x) + p(w)p(y) \pmod{2}.
\]

Likewise, \(l, r\) are both zero unless \(-||y|| = |w| + \nu, -||z|| = |x| - \nu\), in which case

\[
f(|z|, |y|)q^{-\nu \nu + \langle \bar{v}, |x| - |w| \rangle} = f(-|x|, -|w|).
\]

Finally, note that \(f(-|x|, -|w|) = \pi^{p(-|x|)p(-|w|)}f(|x|, |w|)\). Putting these observations together,

\[
l = \pi^{p(x)p(x) + p(w)p(x) + p(-|x|)p(-|w|)}f(|x|, |w|)(b^{-x}, b^* + w, y) = \pi^{p(-|x|)p(-|w|)}r
\]

Since parity in \(X\) only depends on the \(X/Z[I]\) cosets and we have \(-|x| \in \mu + Z[I]\) and \(-|w| \in \lambda + Z[I]\), the result follows.

Lastly, note that Lemmas (3.5) and (3.1) immediately imply the following corollary.

**Corollary 3.6.** We have an equality of diagrams

\[
\pi^{p(\mu)p(\lambda)}
\]

for any \(\lambda, \mu \in X^+.\)
3.3. Renormalization. In the previous section, we deduced a number of identities between various slice diagrams. These identities are almost the Turaev moves for (framed) oriented tangles, except for factors of $\pi$. Now we shall correct these factors.

As noted in Remark 2.4, all of the previous statements about $U$ and its modules hold verbatim over the subring $\mathbb{Q}(q^\pi)$ of $\mathbb{Q}(q,t)^\tau$. Now we will use the fact that $\pi = \tau^2$ to renormalize our maps. These renormalized $U$-module homomorphisms will always be represented by a diagrammatic calculus with red strands and labels to differentiate them.

Remark 3.7. We make two remarks about the red diagrammatic calculus.

(1) We observe that whenever $P(\lambda) = 0$, the maps represented by the red and black diagrams are the same. By Lemma 2.2, this holds whenever $\lambda$ is an even weight $(\langle n,\lambda \rangle \in 2\mathbb{N})$ or $n$ is even, thus in these cases we can work over $\mathbb{Q}(q)^\pi$.

(2) Note that we don’t define sideways-oriented crossings in the red strands. This can be done using these renormalizations and Lemma 3.3 but we shall not need these diagrams here.

Recall that the writhe $\text{wr}(T)$ of an oriented tangle $T$ is defined by forgetting the orientation and setting

$$
\text{wr} \left( \begin{array}{c}
\lambda \\
\end{array} \right) = 1, \quad \text{wr} \left( \begin{array}{c}
\mu \\
\end{array} \right) = -1, \quad \text{wr}(T) = \sum \text{wr}(X),
$$

where the sum is over all crossings $X$ in $T$.

Theorem 3.8. Let $T$ be an oriented tangle, and $\lambda \in X^+$ be a dominant weight. For any slice diagram $S(T)$ of $T$, let $S(T)_\lambda$ be the associated map defined by the red diagrammatic calculus with strands colored by $\lambda$. Then $S(T)_\lambda$ is independent of the choice of slice diagram, and $T_\lambda = S(T)_\lambda$ is an isotopy invariant of oriented framed tangles. Moreover, if $J^\lambda_T = (\pi P(\lambda) \{ \langle \lambda, \lambda \rangle^{-1} q^{\langle \rho, \lambda \rangle} \}^{\text{wr}(T)} T_\lambda$, then $J^\lambda_T$ is independent of the framing, hence is an invariant of $T$.

Proof. To prove the theorem, it suffices to show that the maps $S(T)_\lambda$ (resp. $J^\lambda_T$) are invariant under the Turaev moves (cf. [Tu, Theorem 3.2], [Oht, Theorem 3.3, Equations (3.9)-(3.16)]) for framed (resp. unframed) oriented tangles. First, observe that the identities (3.1) and Lemma 3.1 hold for red strands as well. We also see that (3.2) holds for red strands which all have the same orientation. (In fact, if we define sideways-oriented crossings of red strands as described in Remark 3.3 (2), then (3.2) would hold for red strands with any orientation.)
Furthermore, applying the normalizations and rearranging the $\tau$ factors in Lemma 3.2 shows that, for either orientation, we have

\[ \lambda = \pi^P(\lambda) f(\lambda, \lambda) q^{-\langle \tilde{\rho}, \lambda \rangle} = \lambda \]

\[ \lambda = \pi^P(\lambda) f(\lambda, \lambda)^{-1} q^{\langle \tilde{\rho}, \lambda \rangle} = \lambda \]

Similarly, we see that Corollaries 3.4 and 3.6 gives us the identities

\[ = = \]

for any choice of labeling of the strands. In particular, we see that the Turaev moves for oriented framed tangles are satisfied, which proves that $T_{\lambda}$ is indeed an isotopy invariant of oriented framed tangles. Moreover, note that $J_{\lambda}$ then satisfies the Turaev moves for oriented unframed tangles, since the only Turaev move that changes the writhe is Reidemeister 2 (which is to say the move straightening the crossings in Lemma 3.2). \(\square\)

We note that the proof of Theorem 3.8 actually implies a more general result, though we first need to recall some notions. The category of $X^+$-colored oriented tangles is the strict monoidal category whose objects are finite sequences of pairs $(\lambda, s)$ where $\lambda \in X^+$ and $s \in \{\pm 1\}$, and whose morphisms from $(\lambda_a, s_a)_{1 \leq a \leq b}$ to $(\mu_c, s_c)_{1 \leq c \leq d}$ are tangle diagrams where the labeling and orientation of the $r^{th}$ strand from the left at the lower (respectively, upper) boundary corresponds to $(\lambda_r, s_r)$ (respectively, $(\mu_c, s_c)$); c.f. [Tu, ADO] for more details. In particular, morphisms in this category (and thus colored tangles) are generated from the elementary morphisms

\[ \nearrow \lambda, \quad \searrow \lambda, \quad \swarrow \lambda, \quad \nwarrow \lambda, \quad (\nabla^\pm)_{\lambda, \mu} \]

subject to relations which are simply colored versions of the Turaev moves.

We can extend Theorem 3.8 to framed multicolored tangles with the same proof. To obtain the unframed invariant, the normalization constant is replaced by $\prod_{\lambda \in X^+} (\pi^P(\lambda) f(\lambda, \lambda)^{-1} q^{\langle \tilde{\rho}, \lambda \rangle})^{\text{wt}_\lambda(T)}$, where $\text{wt}_\lambda$ is defined to be the writhe where we exclude from the sum any crossings where there is a strand not labeled by $\lambda$. Therefore, we obtain the following corollary.
Corollary 3.9. There exists a covariant functor $J$ from the category of $X^+$-colored oriented tangles modulo isotopy to $O_{2n}$ which sends the object $((\lambda_1, s_1), \ldots, (\lambda_r, s_r))$ to the module $V(s_1 \lambda_1) \otimes \ldots \otimes V(s_r \lambda_r)$ and is given on morphisms by

$$\circlearrowleft_{\lambda} \mapsto \tau^{P(\lambda)} \operatorname{ev}_{\lambda}, \quad \circlearrowright_{\lambda} \mapsto \operatorname{qtr}_{\lambda}, \quad \leftarrow_{\lambda} \mapsto \operatorname{coqtr}_{\lambda}, \quad \rightarrow_{\lambda} \mapsto \tau^{-P(\lambda)} \operatorname{coev}_{\lambda},$$

$$\bigotimes_{\lambda, \mu} \mapsto (\tau^{P(\lambda)} f(\lambda, \lambda)^{-1} q^{\bar{\rho}(\lambda)})^{\pm \delta_{\lambda, \nu}} \tau^{P(\mu)} \rho_{\lambda, \mu},$$

where $\rho_{\lambda, \mu} = \tau^{P(\lambda)} f(\lambda, \lambda)^{-1} q^{\bar{\rho}(\lambda)}$ and is given on morphisms by $\bigotimes_{\lambda, \mu} \mapsto (\tau^{P(\lambda)} f(\lambda, \lambda)^{-1} q^{\bar{\rho}(\lambda)})^{\pm \delta_{\lambda, \nu}} \tau^{P(\mu)} \rho_{\lambda, \mu}$. In particular, if $L$ is an oriented colored link, then $J(L) \in \mathbb{Q}(q, t)^{\tau}$ is the associated quantum covering $\mathfrak{osp}(1|2n)$ colored link invariant.

Example 3.10. Let’s take $n = 1$ and $\lambda = 1$. Fix $f(1, 1) = 1$, and note that $\langle \bar{\rho}, \lambda \rangle = 1$ and $p(\lambda) = 1$. We can explicitly compute the maps represented by our diagrams on $V(1) \otimes V(1)$. Let $v_1, v_{-1}$ be the basis of $V(1)$ from Example 2.9. Then with respect to the ordered basis $\{v_1 \otimes v_1, v_1 \otimes v_{-1}, v_{-1} \otimes v_1, v_{-1} \otimes v_{-1}\}$ of $V(1) \otimes V(1)$, we have

$$\Theta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q^{-1} - \pi q & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \tilde{s} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & \pi q & 0 \\ 0 & 0 & 0 & \pi \end{bmatrix}, \quad s = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \pi \end{bmatrix}$$

and thus

$$\begin{bmatrix} \tau & 0 & 0 & 0 \\ 0 & \tau q & 0 & 0 \\ 0 & \tau^3 q & \tau - \tau^3 q^2 & 0 \\ 0 & 0 & 0 & \tau \end{bmatrix}, \quad \begin{bmatrix} \tau^3 & 0 & 0 & 0 \\ 0 & \tau^3 - \tau^3 q^{-2} & \tau q^{-1} & 0 \\ 0 & \tau^3 q^{-1} & 0 & 0 \\ 0 & 0 & 0 & \tau^3 \end{bmatrix}, \quad \begin{bmatrix} \tau & 0 & 0 & 0 \\ 0 & \tau q & 0 & 0 \\ 0 & \tau^3 q & \tau - \tau^3 q^2 & 0 \\ 0 & 0 & 0 & \tau \end{bmatrix}$$

Note that $\pi^{p(\lambda)} q^{\bar{\rho}(\lambda)} = \pi q$. Then it is easy to verify directly that

$$\begin{bmatrix} \tau & 0 & 0 & 0 \\ 0 & \tau q & 0 & 0 \\ 0 & \tau^3 q & \tau - \tau^3 q^2 & 0 \\ 0 & 0 & 0 & \tau \end{bmatrix} = (\tau - \tau^3 q^2) \begin{bmatrix} \tau & 0 & 0 & 0 \\ 0 & \tau q & 0 & 0 \\ 0 & \tau^3 q & \tau - \tau^3 q^2 & 0 \\ 0 & 0 & 0 & \tau \end{bmatrix}$$

Now let $T$ be an unframed oriented link with all strands colored by $\lambda$, and fix a subdiagram which consists of two strands with either no crossing or a single crossing. Since $T^2$ is isotopy invariant and independent of framing, we may assume that the strands are directed upward. Let $T_+$ (resp. $T_0$, $T_-$) be $T$ with the subdiagram replaced by $\begin{bmatrix} \tau & 0 & 0 & 0 \\ 0 & \tau q & 0 & 0 \\ 0 & \tau^3 q & \tau - \tau^3 q^2 & 0 \\ 0 & 0 & 0 & \tau \end{bmatrix}$ (resp. $\begin{bmatrix} \tau & 0 & 0 & 0 \\ 0 & \tau q & 0 & 0 \\ 0 & \tau^3 q & \tau - \tau^3 q^2 & 0 \\ 0 & 0 & 0 & \tau \end{bmatrix}$). Then using the above relation and the definition in Theorem 3.8,

$$(\pi q^{-1}) J^1_{T_+} - (\pi q^3) J^1_{T_-} = (\tau - \tau^3 q^2) J^1_{T_0}$$

hence

$$(\pi q^3)^{-1} J^1_{T_+} - \pi q^2 J^1_{T_-} = (\tau q^{-1} - \tau^3 q) J^1_{T_0}.$$
4.1. Definition of Twistors. An enhancer $\phi$ is a function $\phi: \mathbb{Z}[I] \times X \to \mathbb{Z}$ satisfying

$$\phi(\nu, \lambda + \mu) \equiv \phi(\nu, \mu) + \phi(\nu, \lambda) \mod 4 \text{ for } \nu, \mu \in \mathbb{Z}[I]$$

$$\phi(\nu + \mu, \lambda) \equiv \phi(\nu, \lambda) + \phi(\nu, \mu) \mod 4 \text{ for } \nu, \mu \in \mathbb{Z}[I]$$

$$\phi(i, i) = d_i \text{ and } \phi(i, j) \in 2\mathbb{Z} \text{ for } i \neq j \in I.$$  \hfill (4.1)

Note that $\phi(i, i) - \phi(i, i) = 0 \equiv i \cdot i + 2p(i)p(i)$ modulo 4 since $i \cdot i = 2d_i$ and $2p(i)p(i) = 2p(i) = 2d_i$. In particular, note that these congruences imply that

$$\phi_4: \mathbb{Z}[I] \times \mathbb{Z}[I] \to \mathbb{Z}/4\mathbb{Z} \text{ defined by } \phi_4(\mu, \nu) = \phi(\mu, \nu) \mod 4 \text{ is a } \mathbb{Z}\text{-bilinear map}$$

and

$$\phi(\mu, \nu) \equiv \phi(\nu, \mu) + \mu \cdot \nu + 2p(\mu)p(\nu) \mod 4 \text{ for } \mu, \nu \in \mathbb{Z}[I].$$  \hfill (4.2)

Note that an enhancer can always be defined on $\mathbb{Z}[I] \times \mathbb{Z}[I]$ by defining it for $I$ and extending in $\mathbb{Z}$-bilinearly, and then it can be extended to $\mathbb{Z}[I] \times X$ by translation along a transversal of $X/\mathbb{Z}[I]$.

When $I$ has a unique odd element, as in the present case, the enhancer is closely related to the usual pairing.

**Lemma 4.1.** Let $\phi$ be an enhancer. Then $\phi(\mu, \nu) + \phi(\nu, \mu) \equiv \mu \cdot \nu \mod 4$.

**Proof.** First set $(, ,)_{\phi}, (, ,)_{\phi} : \mathbb{Z}[I] \times \mathbb{Z}[I] \to \mathbb{Z}/4\mathbb{Z}$ by $(\mu, \nu)_{\phi} = \phi_4(\mu, \nu) + \phi_4(\nu, \mu)$ and $(\mu, \nu)_{\bullet} = \mu \cdot \nu \mod 4$. Both maps are $\mathbb{Z}$-bilinear, so it suffices to show they take the same values on $I \times I$. Well, if $i \neq j$, then at least one of $i$ or $j$ is even and thus $2p(i)p(j) = 0$ since $|I_1| = 1$. On the other hand, $\phi(i, j) \in 2\mathbb{Z}$ so $\phi(i, j) + \phi(j, i) \equiv \phi(i, j) - \phi(j, i) \equiv i : j + 2p(i)p(j) = i \cdot j$. Finally, note that $\phi(i, i) + \phi(i, i) = 2d_i = i \cdot i$ for any $i \in I$. \hfill $\square$

The $\phi$-enhanced quantum covering group $\tilde{U}$ associated to $U$ and the enhancer $\phi$ is the semidirect product of $U$ with the algebra $\mathbb{Q}(q, t)[T_\mu, Y_\mu \mid \mu \in \mathbb{Z}[I]]$ subject to the relations

$$T_{\mu}T_{\nu} = T_{\mu + \nu}, \quad Y_\mu Y_\nu = Y_{\mu + \nu}, \quad T_0 = Y_0 = T^4_\nu = Y^4_\nu = 1, \quad T_\mu Y_\nu = Y_\nu T_\mu, \quad (4.3)$$

$$T_\mu u = t^{-|u|} u T_\mu, \quad u \in U, \quad \mu \in \mathbb{Z}[I]$$

$$Y_\mu u = t^{-|u|} u Y_\mu, \quad u \in U, \quad \mu \in \mathbb{Z}[I]$$

See [CFLW] for a more formal definition. The enhanced quantum covering group has a useful $\mathbb{Q}(t)$-linear automorphism called a twistor. There are several ways to define such a twistor; we will need the following.

**Proposition 4.2.** [CFLW] Theorems 4.3, 4.12] Define a product $\ast$ on $f$ by the following rule: if $x$ and $y$ are homogeneous elements of $f$, let $x \ast y = t^{\phi(|x|, |y|)}xy$. Let $(f, \ast)$ denote $f$ with this multiplication.

1. Then there is a $\mathbb{Q}(t)$-linear algebra isomorphism $\mathcal{X}: f \to (f, \ast)$ defined by

$$\mathcal{X}(t_i) = t_i, \quad \mathcal{X}(q) = t^{-1} q, \quad \mathcal{X}(\tau) = t^{-1} \tau.$$  

2. Let $B$ be the canonical basis of $f$ (cf. [CHW]). Then $\mathcal{X}$ on $f$ satisfies $\mathcal{X}(b) = t^{\ell(b)} b$ for all $b \in B$, where $\ell(b)$ is some integer depending on $b$.

3. There is a $\mathbb{Q}(t)$-algebra automorphism $\mathcal{X}: \tilde{U} \to \tilde{U}$ defined by

$$\mathcal{X}(E_i) = t_i^{-1} T_i Y_i E_i, \quad \mathcal{X}(F_i) = F_i Y_{-i}, \quad \mathcal{X}(K_\mu) = T_{-\mu} K_\mu, \quad \mathcal{X}(J_\mu) = T_{2\mu} J_\mu,$$

$$\mathcal{X}(T_\mu) = T_\mu, \quad \mathcal{X}(Y_\mu) = Y_\mu, \quad \mathcal{X}(q) = t^{-1} q, \quad \mathcal{X}(\tau) = t^{-1} \tau,$$

where if $\mu = \sum_{i \in I} \mu_i t^i$, $\bar{T}_\mu = \prod_{i \in I} T_{\mu_i} t^i$.

4. For $x \in f[t]$, we have

(a) $\mathcal{X}(x^+) = t_i^2 \mathcal{X}(x) + \bar{T}_i X_{|x|}$

(b) $\mathcal{X}(x^-) = \mathcal{X}((x)^-) + Y_{-|x|}$
Later on, we will need some alternate versions of the results in Proposition 4.2 which we shall prove now. First, we note the following analogue of Proposition 4.2 (2) for the dual canonical basis.

**Lemma 4.3.** Let $(-, -)$ be the bilinear form on $f$ defined in (2.13). Then
\[
X^{-1}((X(x), X(y))) = (−1)^{p(|x|)}(x, y).
\]
In particular, $X(b^*) = (−1)^{p(b)}t^{-\ell(b)}b^*$ for any $b \in B$.

**Proof.** Let $(x, y)X = X^{-1}((X(x), X(y)))$ and observe that this is a $\mathbb{Q}(q, t)^{-}$-bilinear form on $f[t]$. Moreover, note that $ir(X(w)) = t^{\phi(i, w)}(x(r(X(w)))$ by the same proof as [CFLW, Proposition 3.2(b)]. To show $(x, y)X = (−1)^{p(|x|)}(x, y)$, we proceed by induction on the height. First note that
\[
(1, 1)X = 1 = (1, 1), \quad (\theta_i, \theta_i)X = X^{-1}\left(\frac{1}{1 - \pi_i q_i^{-2}}\right) = (\theta_i, \theta_i).
\]
Now if $x \in f[t]_{\nu_i}$ and $y \in f[t]_{\nu_j}$ for some $i \in I$ and $\nu \in \mathbb{Z}_{\geq 0}[I]$ with $\text{ht}(\nu) > 1$, we have
\[
(\theta_i, x, y)X = t^{\phi(i, \nu_i)}(\theta_i, X(x, y)) = t^{\phi(i, \nu_i)}(\theta_i, \theta_i)X^{-1}((X(x), i r(X(y))))
\]
\[
= t^{2\phi(i, \nu_i)}(\theta_i, \theta_i)X^{-1}((X(x), i r(X(y))))
\]
\[
= (−1)^{p(\nu_i) + \phi(i, \nu_i)}(\theta_i, \theta_i)(x, i r(y))
\]
\[
= (−1)^{p(\nu_i) + p(\nu_i)\pi_i}(\theta_i, \theta_i)(x, y) = (−1)^{p(\theta_i, x, y)}
\]
where in the last equality, note that if $\nu = \sum_{i \in I} \nu_i i$ then we have $\phi(i, \nu - \nu) \equiv (\nu_i - 1)d_i$ modulo 2. The proof is finished by observing that $(\nu_i - 1)d_i \equiv (\nu - \nu_i)p(i)$ for any $i \in I$, since if $i \neq \pi_i$ both sides are 0 modulo 2, and if $i = \pi_i$ both sides are equivalent to $\nu_i - 1$ modulo 2.

**Remark 4.4.** Though Lemma 4.3 as stated requires $|I_1| = 1$, a version of it also holds for arbitrary enhanced quantum covering algebras. Indeed, if $|I_1| > 1$, then $X^{-1}((X(x), X(y))) = t^{(|x|)}(x, y)$, where $|x| = \nu = \sum_{i \in I} \nu_i i$ and $(|x|) = \sum_{i \in I} (\nu_i) d_i$.

It will also be more convenient to have the following variant of Proposition 4.2 (4b).

**Lemma 4.5.** We have
\[
X(x^+) = t_{|x|}^{-1}T_{|x|}Y_{|x|}X(x)^{+}.
\]

**Proof.** This is true if $x = \theta_i$. It suffices to show if it is true for $x$, then it is true for $\theta_i x$.
\[
X(\theta_i x^+) = X(\theta_i x)^{+} = t_{i}^{-1}T_i Y_i E_i t_{\nu_i}^{-1}T_{\nu_i}Y_{\nu_i}X(x)^{+}
\]
\[
= t_{i}^{-1}t^{-\nu - \phi(\nu, i)}T_i Y_i E_i X(x)^{+} = t_{|x|}^{-1}t^{-\nu - \phi(\nu, i) - \phi(i, \nu)}T_{i} Y_{i} Y_{\nu} X(x)^{+}
\]
But then by Lemma 4.1
\[
-\nu - \phi(\nu, i) - \phi(i, \nu) \equiv_{4} -2i \cdot \nu \equiv_{4} 0.
\]

\[
4.2. \hat{U}\text{-modules and Hopf structure.} \quad \text{Let } M \text{ be a } U\text{-weight module. Then } M \text{ is canonically a } \hat{U}\text{-module by defining}
\]
\[
T_{\mu}m = t^{(\mu, \lambda)}m, \quad \mu \in \mathbb{Z}[I], m \in M_{\lambda}; \quad \text{(4.6)}
\]
\[
Y_{\mu}m = t^{(\mu, \lambda)}m, \quad \mu \in \mathbb{Z}[I], m \in M_{\lambda}. \quad \text{(4.7)}
\]
To that end, we will call any $\hat{U}$-module which restricts to a $U$-weight module and satisfies (4.6) a $\hat{U}$-weight module. If it additionally satisfies (4.7), we shall call it a canonical $\hat{U}$-weight module.

In particular, any tensor product of $\hat{U}$-modules can be given a canonical $\hat{U}$-weight module structure. However, such a procedure forgets the action of the $Y$ elements on the factors due to the lack of additivity in the second component of $\phi$. 

Example 4.6. Consider the case \( n = 1 \). Then \( \mathbb{U} \) has the canonical weight module \( \mathbb{V}(1) = \mathbb{Q}(q,t) v_1 \oplus \mathbb{Q}(q,t) v_{-1} \) which is isomorphic to \( \mathbb{V}(1) \) as a \( \mathbb{U} \)-module and satisfies \( T_i v_1 = t v_1 \) and \( \mathbb{Y}_{T_i} v_1 = t^{\phi(T_i)} v_1 \). Then \( \mathbb{V}(1) \otimes \mathbb{V}(1) \) is a \( \mathbb{U} \)-weight module hence has a canonical \( \mathbb{U} \)-module structure, but note that
\[
\mathbb{Y}_{T_i} v_1 \otimes v_1 = t^{\phi(T_i)} v_1 \otimes v_1
\]
and by the definition of \( \phi \), we have \( \phi(T,2) = \phi(T,T) = 1 \neq \phi(T,1) + \phi(T,1) \).

In particular, canonical module structures will be too naive for our purposes. Instead, we will introduce Hopf structure which will inform our classes of weight modules.

Proposition 4.7. The algebra \( \mathbb{U} \) has a Hopf covering algebra structure given by the following:
\begin{enumerate}
\item A coassociative coproduct \( \Delta : \mathbb{U} \to \mathbb{U} \otimes Q(q,t)^\ast \mathbb{U} \) extending \( \Delta : \mathbb{U} \to \mathbb{U} \otimes Q(q,t)^\ast \mathbb{U} \) such that \( \Delta (T_\mu) = T_\mu \otimes T_\mu \) and \( \Delta (\mathbb{Y}_\mu) = \mathbb{Y}_\mu \otimes \mathbb{Y}_\mu \) for \( \mu \in \mathbb{Z}[I] \). In particular, we inductively define \( \Delta^t = (\Delta \otimes 1)^t \circ 1 = 1 \) for any integer \( t > 1 \).
\item An antipode \( S : \mathbb{U} \to \mathbb{U} \) extending \( S : \mathbb{U} \to \mathbb{U} \) such that \( S(T_\mu) = T_{-\mu} \) and \( S(\mathbb{Y}_\mu) = \mathbb{Y}_{-\mu} \) for \( \mu \in \mathbb{Z}[I] \).
\item A counit map \( \epsilon : \mathbb{U} \to \mathbb{U} \) extending \( \epsilon : \mathbb{U} \to \mathbb{U} \) such that \( \epsilon (T_\mu) = \epsilon (\mathbb{Y}_\mu) = 1 \) for \( \mu \in \mathbb{Z}[I] \).
\end{enumerate}

Proof. To show that these maps define a Hopf structure, we need only check that these morphisms respect (1.3) - (1.5). This is obvious for (1.3), and can be quickly verified for (1.4) and (1.5) by checking it for the generators of \( \mathbb{U} \). For instance,
\[
\Delta(\mathbb{Y}_\mu) \Delta (E_1) = (\mathbb{Y}_\mu \otimes \mathbb{Y}_\mu) (E_1 \otimes 1 + \mathbb{J}_i \mathbb{K}_i \otimes E_i)
\]
\[
= \mathbb{Y}_\mu E_1 \otimes \mathbb{Y}_\mu + \mathbb{Y}_\mu \mathbb{J}_i \mathbb{K}_i \otimes T_\mu E_i
\]
\[
= t^{\phi(\mu,i)} E_i \mathbb{Y}_\mu \otimes \mathbb{Y}_\mu + \mathbb{J}_i \mathbb{K}_i \mathbb{Y}_\mu \otimes t^{\phi(\mu,i)} E_i \mathbb{Y}_\mu
\]
\[
= t^{\phi(\mu,i)} E_i \Delta (E_i) \Delta (\mathbb{Y}_\mu);
\]
\[
S(E_1) S(\mathbb{Y}_\mu) = -\mathbb{J}_i \mathbb{K}_i \mathbb{Y}_\mu - t^{\phi(\mu,i)} \mathbb{Y}_{-\mu} \mathbb{J}_i \mathbb{K}_i E_i = t^{\phi(\mu,i)} S(\mathbb{Y}_\mu) S(E_i);
\]
\[
\epsilon (E_i) \epsilon (\mathbb{Y}_\mu) = (0)(1) = t^{\phi(\mu,i)}(1)(0) = t^{\phi(\mu,i)} \epsilon (E_i) \epsilon (\mathbb{Y}_\mu).
\]

Finally, the co-associativity of \( \Delta \) on \( \mathbb{U} \) follows immediately from the co-associativity of \( \Delta \) on \( \mathbb{U} \) and the fact that \( T_\mu \) and \( \mathbb{Y}_\mu \) are grouplike elements.

The coproduct gives us another way to define an action of \( \mathbb{U} \) on tensor products of canonical \( \mathbb{U} \)-weight modules. Henceforth, given \( \mathbb{U} \)-weight modules \( M \) and \( N \), we let \( M \otimes N \) denote the space \( M \otimes Q(q,t)^\ast N \) with the \( \mathbb{U} \)-weight module structure induced by the coproduct on \( \mathbb{U} \). (Note that in general, the module \( M \otimes N \) is not canonical!!)

Example 4.8. Continuing the previous example, the action of \( \mathbb{Y}_\mu \) on \( \mathbb{V}(1) \otimes \mathbb{V}(1) \) is given by
\[
\Delta(\mathbb{Y}_\mu) v_1 \otimes v_1 = t^{\phi(\mathbb{T},1)} v_1 \otimes v_1.
\]

Another natural module to consider is the following. Given a canonical \( \mathbb{U} \)-weight module \( M \), we can construct the restricted linear dual \( M^\ast \). This space is naturally a \( \mathbb{U} \)-weight module as in \( \ref{2.4} \), hence has a canonical \( \mathbb{U} \) structure. On the other hand, let \( M^\ast \) denote the space \( M^\ast \) with the action of \( \mathbb{U} \) defined by \( (uf)(x) = \pi^{\phi(f)} f(S(u)x) \). Note that \( M^\ast \) is not canonical: if \( f \in (M, s)^\ast \), then \( |f| = -\lambda \) but nevertheless
\[
\mathbb{Y}_\mu f = t^{-\phi(\mu,\lambda)} f.
\]

Since modules with these unorthodox actions of the \( \mathbb{Y}_\mu \) will be of primary importance, we give the following definitions.
Therefore, it is determined by the images of the generators, which are defined by the following formula:

\[ (X^i)_{\lambda} = \{ (\lambda_1, \ldots, \lambda_t) \in X^t \mid \lambda = \lambda_1 + \cdots + \lambda_s \}, \]

\[ M(\lambda) = \left\{ m \in M \mid \Upsilon_m m = t^{\sum_{1 \leq i \leq t} c_i \phi(\mu, \lambda_s)} m \text{ for all } \mu \in \mathbb{Z}[I] \right\}. \]

We say this is the signature of \( M \), and denote it by \( \text{sig}(M) = c \).

**Remark 4.10.** We note that just as any weight \( U \)-module can be given a canonical \( \hat{U} \)-module structure, it can also be given an anti-canonical \( \hat{U} \)-module structure. Indeed, suppose \( M \) is a weight \( U \)-module and define \( T_i \) for all \( i \in I \) and \( \Upsilon_m = t^{-\phi(i, -|m|)} \). Then this defines an action of \( \hat{U} \), since for any \( i \in I \) and \( u \in U \), \( \Upsilon_i \), \( \Upsilon_i \) is defined by descent from the highest weight vector. To obtain a convenient action of \( \hat{U} \), we consider the \( U \)-module and define \( \tilde{T}_i \), \( \Upsilon_i \), \( \Upsilon_i \) such that \( \Upsilon_i \), \( \Upsilon_i \) is an algebra automorphism of \( \hat{U} \).

In addition to classifying modules by the action of the \( \Upsilon \) elements, another property of \( \hat{U} \)-weight modules which will be important to us is their interaction with the twistor map \( \hat{X} : \hat{U} \to \hat{U} \).

**Definition 4.11.** Let \( M \) be a \( \hat{U} \)-weight module. We say \( M \) carries a twistor \( \mathcal{X} \) (or \( \mathcal{X} \) is a twistor on \( M \)) if there exists a homogeneous \( \mathbb{Q}(t) \)-linear bijection \( \mathcal{X} : M \to M \) such that \( \mathcal{X}(um) = \mathcal{X}(u) \mathcal{X}(m) \).

Modules which carry twistors are not hard to find. Indeed, the simple \( U \)-modules \( V(\lambda) \) are themselves examples when given canonical (or anti-canonical) actions of \( \hat{U} \).

**Lemma 4.12.** [\( \mathbb{C} \) Lemma 6.9] Let \( \lambda \in X^+ \). Let \( \hat{V}(\lambda) \) be the space \( V(\lambda) \) with the canonical action of \( \hat{U} \). There is a \( \mathbb{Q}(t) \)-linear map \( \hat{X} : \hat{V}(\lambda) \to \hat{V}(\lambda) \) which satisfies \( \hat{X}(v) = v \) and \( \hat{X}(um) = \hat{X}(u) \hat{X}(m) \) for all \( u \in \hat{U} \) and \( m \in \hat{V}(\lambda) \).

In light of Lemma 2.8, it follows that the \( U \)-module \( V(\lambda) \), viewed as a canonical \( \hat{U} \)-module, also carries a twistor. A similar argument to [\( \mathbb{C} \) Lemma 6.9] can be used to construct a twistor on \( V(\lambda) \) with an anti-canonical action of \( U \), hence the \( \hat{U} \)-module \( \hat{V}(\lambda) \) carries a twistor. However, this construction is not very compatible with the dual basis, since it relies on an isomorphism \( V(\lambda) \to \Pi^{p(\lambda)}(\lambda) \) and is defined by descent from the highest weight vector. To obtain a convenient definition of a twistor on the dual modules, we will define a map directly on \( \hat{V}(\lambda) \).

Define the dual twistor on \( \hat{U} \) to be the map \( \hat{X}^\sharp(u) = S \circ \hat{X} \circ S^{-1}(u) \). This map is clearly a bijection, and for any \( u, v \in \hat{U} \) we have

\[ \hat{X}^\sharp(1u) = S(\hat{X}(S^{-1}(u))) \]
\[ = t^{2|u|}S(\hat{X}(S^{-1}(u)))S(\hat{X}(S^{-1}(v))) \]
\[ = t^{2|u|}S(\hat{X}(u))\hat{X}^\sharp(v). \]

Therefore, it is determined by the images of the generators, which are

\[ \hat{X}^\sharp(E_i) = t_i E_i \gamma_{-i}, \quad \hat{X}^\sharp(F_i) = \gamma_i F_i \tilde{T}_i, \quad \hat{X}^\sharp(K_{\mu}) = T_{-\mu}K_{\mu}, \quad \hat{X}^\sharp(J_{\mu}) = T_{2\mu}J_{\mu} \]
\[ \hat{X}^\sharp(q) = t^{-1}q, \quad \hat{X}^\sharp(\tau) = t \tau. \]

In particular, note that

\[ \hat{X}^\sharp(x^{-}) = \gamma_{-\nu} \hat{X}(x)^{-} \tilde{T}_\nu. \]

While \( \hat{X}^\sharp \) is not an algebra automorphism of \( \hat{U} \), it shares many properties with \( \hat{X} \). In particular, we have a version of Lemma 4.12

**Lemma 4.13.** Let \( \lambda \in X^+ \). There is a \( \mathbb{Q}(t) \)-linear map \( \hat{X}^\sharp : \hat{V}(\lambda) \to \hat{V}(\lambda) \) which satisfies \( \hat{X}^\sharp(v) = v \) and \( \hat{X}^\sharp(um) = t^{2|u|}S(\hat{X}(u))\hat{X}^\sharp(m) \) for all \( u \in \hat{U} \) and \( m \in \hat{V}(\lambda) \).
Proof. This follows from more or less the same proof as [C] Lemmas 6.8, 6.9. To wit, we can identify the Verma module of highest weight \( \lambda \) for \( U \) with \( f \) (cf. loc. cit for details), and in particular this is naturally a canonical \( \hat{U} \)-module. Then we define a map \( \mathfrak{X}_\lambda \) such that \[ \mathfrak{X}_\lambda(E_i x) = t^* E_i \mathfrak{X}_\lambda(x) \] for any positive integers \( U \). Proposition 4.15. treat them en suite as an operator on \( \hat{\mathfrak{X}} \). Finally, we note that the kernel of the projection \( f \to \hat{V}(\lambda) \) is trivially preserved by \( \mathfrak{X}_\lambda \), hence it descends to a map on \( \hat{V}(\lambda) \).

The dual twistor \( \mathfrak{X} \) is what will allow us to define a convenient twistor map on dual modules, as follows. Recall that \( V(-\lambda) \) denotes the \( U \)-module \( V(\lambda)^* \). We will adapt this notation to \( \hat{V}(\lambda)^* \).

Lemma 4.14. For \( \lambda \in X^+ \), let \( \hat{V}(\lambda) = \hat{V}(\lambda)^* \); that is, the space \( \hat{V}(\lambda)^* \) with the action of \( \hat{U} \) induced by the antipode \( S : \hat{U} \to \hat{U} \). Define a map \( \mathfrak{X} \) on \( \hat{V}(\lambda) \) by \( \mathfrak{X}(f)(x) = t^2p(a)p(x)\mathfrak{X}(f(\mathfrak{X}^{-1}(x))) \) for homogeneous \( x \in \hat{V}(\lambda) \) and \( f \in \hat{V}(\lambda) \). Then \( \mathfrak{X}(uf) = \mathfrak{X}(u)\mathfrak{X}(f) \) for all \( u \in \hat{U} \) and \( f \in \hat{V}(\lambda) \).

Proof. Let \( f \in \hat{V}(\lambda) \) and \( x \in \hat{V}(\lambda) \) be homogeneous. First, observe that since \( \mathfrak{X} \) preserves the \( \hat{X} \)-grading, \( \mathfrak{X}(f)(x) = 0 \) unless \(|x| = |f|\). Moreover, if \( a \in \mathbb{Q}(q, t) \),

\[ \mathfrak{X}(f)(ax) = t^{2p(a)p(x)}\mathfrak{X}(f(\mathfrak{X}^{-1}(ax))) = t^{2p(a)p(x)}\mathfrak{X}(\mathfrak{X}^{-1}(a)f(\mathfrak{X}^{-1}(x))) = a\mathfrak{X}(f)(x), \]

so \( \mathfrak{X}(f) \) is indeed an element of \( \hat{V}(\lambda) \).

Now suppose \( u \in \hat{U} \). We compute that

\[ \mathfrak{X}(uf)(x) = t^{2p(u)p(x)}\mathfrak{X}((uf)(\mathfrak{X}^{-1}(x))) = t^{2p(u)p(x)+2p(a)p(x)}\mathfrak{X}(\pi(p(a)p(f))f(S(u)\mathfrak{X}^{-1}(x))), \]

\[ \mathfrak{X}(u)\mathfrak{X}(f)(x) = t^{2p(f)p(ax)}\pi(p(a)p(f))\mathfrak{X}(f(\mathfrak{X}^{-1}(S(\mathfrak{X}(u))))x)) \]

\[ = t^{2p(f)p(ax)+2p(a)p(x)+2p(u)p(x)}\mathfrak{X}(f(\mathfrak{X}^{-1}(S(\mathfrak{X}(u))))\mathfrak{X}^{-1}(x))) \]

\[ = t^{2p(f)p(ax)+2p(a)p(x)}\pi(p(a)p(f))f(S(u)\mathfrak{X}^{-1}(x)). \]

Therefore, \( \mathfrak{X}(uf) = \mathfrak{X}(u)\mathfrak{X}(f) \).

4.3. Twistor on tensor products. Now let us return to the question of relating the \( \mathfrak{osp}(1|2) \) and \( \mathfrak{sl}(2) \) link invariants. Since the invariants arise from maps between tensor products of simple modules and their duals, we shall also need variants of the twistor maps on the corresponding \( \hat{U} \)-modules. In the following, we shall define a number of versions of \( \mathfrak{X} \) in different settings. However, they will all be compatible in natural ways, so rather than label these maps differently, we shall treat them en suite as an operator on \( \hat{U} \) and its modules.

The following proposition takes the first step in this direction by showing that there is a natural extension of the twistor maps to tensor powers of \( U \).

Proposition 4.15. For each positive integer \( t \), there exists a \( \mathbb{Q}(t) \)-algebra automorphism \( \mathfrak{X} \) of \( \hat{U}^\otimes t+1 \) which satisfies

\[ \mathfrak{X}(x \otimes y) = \mathfrak{X}(x)\Delta^t(\mathfrak{Y}(y)) \otimes \Delta^t(\mathfrak{T}(\mathfrak{Y}(y)) \mathfrak{X}(y) \]

for any positive integers \( s, s' \) satisfying \( s + s' = t + 1 \), \( x \in \hat{U}^\otimes s \), and \( y \in \hat{U}^\otimes s' \). Moreover, \( \Delta^t(\mathfrak{X}(x)) = \mathfrak{X}(\Delta^t(x)) \) for any \( x \in \hat{U} \).
Proof. Define \( X' : \hat{U}^{\otimes t+1} \rightarrow \hat{U}^{\otimes t+1} \) as follows: for \( x = \bigotimes_{s=1}^{t+1} x_s \in \hat{U}^{\otimes t+1} \), let \( X(x) = \bigotimes_{s=1}^{t+1} X(x_s) \) where
\[
X(x)_s = \tilde{T}_{[x_1]+\ldots+[x_{s-1}]} Y_{[x_1]+\ldots+[x_{s-1}]} X(x_s) Y_{[x_{s+1}]+\ldots+[x_{t+1}]}. \tag{4.9}
\]
It is elementary to check that
\[
X(x \otimes y) = X(x) \Delta'(T \{ y \}) \otimes \Delta' \tilde{T}[x] Y X(y)
\]
for any positive integers \( s, s' \) satisfying \( s + s' = t \), \( x, y \in \hat{U}^{\otimes s} \), and \( y \in \hat{U}^{\otimes s'} \). Moreover, since \( X \) on \( \hat{U} \) is a bijection, it is easy to see that so is \( X \) on \( \hat{U}^{\otimes t+1} \).

We will prove that \( X \) is an isomorphism by induction. Since \( X \) on \( \hat{U} \) is an isomorphism, let us assume \( X \) on \( \hat{U}' \) is an isomorphism. Then for \( x, w \in \hat{U}^{\otimes t} \) and \( y, z \in \hat{U} \),
\[
X(x \otimes y)X(w \otimes z) = (X(x)Y_{[y]} \otimes \tilde{T}[x] X(y))(X(w)Y_{[z]} \otimes \tilde{T}[w] X(z))
\]
\[=
\pi^p(y)p(w)X(x)Y_{[y]}X(w)Y_{[z]} \otimes \tilde{T}[x] \tilde{T}[w] X(y)X(z)
\]
\[=
\pi^p(y)p(w)\phi(|y|,|w|) - \phi(|y|,|y|) + \phi(|w|,|w|)X(xw)Y_{[yw]} \otimes \tilde{T}[yw] X(yz)
\]
\[=\]
\[
X((x \otimes y)(w \otimes z)) = X(xw \otimes yz)
\]
This completes the induction showing \( X \) on \( \hat{U}^{t+1} \) is an isomorphism as claimed. Finally, showing that \( X \) commutes with \( \Delta' \) is straightforward using (4.9) and checking on the generators.

Now that we have a viable twistor map on tensor powers of \( \hat{U} \), we need an analogue on the tensor powers of modules. In particular, suppose we have a collection of \( \hat{U} \) modules which are canonical or anticanonical, and which carry twistors. We will produce a twistor on the tensor product of these modules.

As might be suggested by (4.9), this is not as simple as taking the tensor power of the twistors. A version of such a twistor is produced in [C, Proposition 6.11] by rescaling the tensor product of twistors by a power of \( t \) given by a function of the weights of the tensor factors. We will do something similar, but it turns out that we will need functions which depend not only on the weights of tensor factors but also their parities, as well as the signature of the tensor product.

Lemma 4.16. Let \( c = (c_1, c_2) \) where \( c_1, c_2 \in \{1, -1\} \). There exists a function \( \kappa_c : \hat{X}^2 \rightarrow \mathbb{Z} \) satisfying \( \kappa((0, 0), \zeta) \equiv \kappa_c(0, 0) \equiv 0 \) modulo 4 and
\[
\kappa_c(\zeta + \mu, \zeta' + \nu) - \kappa_c(\zeta, \zeta') \equiv c_1 |\phi(\mu, c_2 |\zeta'|) + c_2 |\phi(\mu, c_2 |\zeta|) + 2p(\zeta)p(\nu) + c_1 |\phi(\mu, c_2 |\zeta|) + \mu \cdot \nu + |\phi(\mu, \nu) \mod 4
\]
for all \( \zeta, \zeta' \in \hat{X} \) and \( \mu, \nu \in \mathbb{Z}[I] \).

Proof. Fix \( c = (c_1, c_2) \) where \( c_1, c_2 \in \{1, -1\} \). Note that it suffices to show such a function \( \kappa = \kappa_c \) exists on each coset of \( \mathbb{Z}[I] \times \mathbb{Z}[I] \) (where as in (2.6), we view \( \mathbb{Z}[I] \) as a subset of \( \hat{X} \)), so fix a set of representatives \( C \) of \( \hat{X}/\mathbb{Z}[I] \). For \( \zeta_0, \zeta_1 \in C \), set
\[
\kappa(\zeta_0 + \mu, \zeta_1 + \nu) = \langle \mu^*, |\zeta_1| \rangle + c_2 |\phi(\mu, c_2 |\zeta_1|) + 2p(\zeta_0)p(\nu) + c_1 |\phi(\mu, c_2 |\zeta_1|) + \mu \cdot \nu + |\phi(\mu, \nu) \mod 4.
\]
It is elementary to verify that this has the desired properties.

We henceforth suppose we have fixed choices of \( \kappa_c \) for each \( c \in \{1, -1\}^2 \). We can extend \( \kappa \) naturally to larger powers of \( \hat{X} \). Let \( t > 1 \) be a positive integer and fix a sequence \( c = (c_s) \in \{\pm 1\}^t \). Let \( \kappa_c : \hat{X}^t \rightarrow \mathbb{Z} \) be the function defined by
\[
\kappa_c(\zeta) = \sum_{1 \leq s \leq t} \kappa_{(c_s, c_s)} (\zeta_s, \zeta_s), \quad \zeta = (\zeta_s) \in \hat{X}^t.
\]
Then if \( \zeta = (\zeta_s), \zeta' = (\zeta'_s) \in \hat{X}^t \) with \( \zeta'_s = \zeta_s + \delta_{r,s} \) for some \( 1 \leq r \leq t \), then
\[
\kappa(\zeta') - \kappa(\zeta) = \sum_{r < s \leq t} \left( \langle \tilde{\iota}_r |\zeta_s| \rangle + c_s |\phi(\iota_r, c_s |\zeta_s|) \right) + \sum_{1 \leq s' < r} (2p(\zeta_{s'})p(i) + c_s' |\phi(i, c_s' |\zeta_{s'}|)) \mod 4.
\]
We can observe some convenient properties of the maps \( \kappa_c \).

**Lemma 4.17.** Let \( c = (c_t) \in \{ \pm 1 \}^t \) and \( \zeta = (\zeta_t), \zeta' = (\zeta'_t) \in \hat{X}^t \).

1. Let \( 1 \leq r \leq t \), and define \( c_{<r} = (c_1, \ldots, c_r), \ c_{>r} = (c_{r+1}, \ldots, c_t) \). Likewise, define \( \zeta''_{<r} = (\zeta''_1, \ldots, \zeta''_r) \) and \( \zeta''_{>r} = (\zeta''_{r+1}, \ldots, \zeta''_t) \) for any \( \zeta'' = (\zeta''_t) \in \hat{X}^t \). Then
   \[
   \kappa_c(\zeta, \zeta') = \kappa_{c_{<r}}(\zeta_{<r}, \zeta''_{<r}) + \kappa_{c_{>r}}(\zeta_{>r}, \zeta''_{>r}) + \sum_{1 \leq s \leq r < r' \leq t} \kappa_{(c_{<s}, c_{s})}(\zeta_s, \zeta_{s'})
   \]

2. Suppose that there exists \( 1 \leq r < t \) such that \( \zeta_r = \zeta'_r + \nu, \ z_{r+1} = \zeta'_{r+1} - \nu \), and \( \zeta_s = \zeta'_s \) for \( s \neq r, r + 1 \) and some \( \nu \in \mathbb{Z}[I] \). Then
   \[
   \kappa_c(\zeta) - \kappa_c(\zeta') = (\hat{\nu}, \zeta_{r+1}) + c_{r+1}\varphi(\nu, c_{r+1}\zeta_{r+1}) + 2p(\nu)p(\zeta_r) - c_r\varphi(\nu, c_r\zeta_r) - \nu \cdot \nu - \varphi(\nu, \nu)
   \]

3. For any \( \zeta \in \hat{X} \) and \( c_1 = \pm 1 \), we have
   \[
   \kappa_{c_1, \pm 1, \pm 1}(\zeta + \hat{\nu}, (\pm \lambda, 0), (\mp \lambda, 0)) = \kappa_{c_1, \pm 1, \mp 1}(\zeta, (\pm \lambda, 0), (\mp \lambda, 0))
   \]
   \[
   \kappa_{\mp 1, \pm 1, c_1}(((\pm \lambda, 0), (\mp \lambda, 0), \zeta + \hat{\nu}) = \kappa_{\pm 1, \mp 1, c_1}((\pm \lambda, 0), (\mp \lambda, 0), \zeta)
   \]

**Proof.** We note that (1) is an immediate consequence of the definition of \( \kappa_c \). On the other hand, (2) and (3) both follow from direct computations and the definition.

The functions \( \kappa_c \) allows us to define a twistor on tensor product modules as follows.

**Proposition 4.18.** Let \( M_1, M_2, \ldots, M_t \) be canonical or anti-canonical \( \hat{U} \)-modules carrying twistors and let \( M = M_1 \otimes M_2 \otimes \ldots \otimes M_t \) be the \( \hat{U} \)-module (and hence a mixed \( U \)-module via \( \Delta^{(-1)} \)) with the natural action. Set \( c = \text{sig}(M) = (c_1, \ldots, c_t) \). Then the automorphism
   \[
   \mathcal{X}(m_1 \otimes \ldots \otimes m_t) = t^{c_{(\sum|m_i|)}}\mathcal{X}(m_1) \otimes \ldots \otimes \mathcal{X}(m_t)
   \]
satisfies
   \[
   \mathcal{X}(u \otimes m) = \mathcal{X}(u) \mathcal{X}(m)
   \]
In particular,
   \[
   \mathcal{X}(um) = \mathcal{X}(u)\mathcal{X}(m)
   \]
   for \( u \in \hat{U} \) and \( m \in M \).

**Proof.** First, observe it is enough to show
   \[
   \mathcal{X}(1^{s-1} \otimes x_s \otimes 1^{-s}) = \mathcal{X}(x_s \otimes 1^{t-s})\mathcal{X}(m_1 \otimes \ldots \otimes m_t)
   \]
where \( 1 \leq s \leq t \) and \( x_s \) is a generator of \( \hat{U} \). This is trivial when \( x_s = K_\mu, J_\mu, T_\mu \) and \( \Upsilon_\mu \) for some \( \mu \in \mathbb{Z}[I] \) so it suffices to check the case \( x_s = E_i, F_i \) for \( i \in I \). To do this, let us make our equations more compact with the following notations: for \( m_1 \otimes \ldots \otimes m_t \in M \), let
   \[
   m_{<s} = m_1 \otimes \ldots \otimes m_{s-1}, \ m_{>s} = m_{s+1} \otimes \ldots \otimes m_t
   \]
   \[
   \mathcal{X}(m)_{<s} = \mathcal{X}(m_1) \otimes \ldots \otimes \mathcal{X}(m_{s-1}), \ \mathcal{X}(m)_{>s} = \mathcal{X}(m_{s+1}) \otimes \ldots \otimes \mathcal{X}(m_t),
   \]
   \[
   \mathcal{X}(m)_{<s} = (||m_1||, \ldots, ||m_{s-1}||), \ \mathcal{X}(m)_{>s} = (||m_{s+1}||, \ldots, ||m_t||)
   \]
   \[
   \phi'(i, m_{<s}) = \sum_{1 \leq r < s} c_r \varphi(i, c_r m_r), \ \phi''(i, m_{>s}) = \sum_{s < r \leq t} c_r \varphi(i, c_r m_r)
   \]
Using these notations, we compute that
   \[
   \mathcal{X}(1^{s-1} \otimes E_i \otimes 1^{-s})(m_{<s} \otimes m_s \otimes m_{>s}) = \mathcal{X}(\tau^{\hat{m}_{<s}} m_{<s} \otimes E_i m_s \otimes m_{>s})
   \]
   \[
   = t^{2p(i)p(m_{<s}) + \kappa_{\pm 1, \pm 1, c_1}(||m_{<s}||, ||m_s||, ||m_{>s}||)} \mathcal{X}(m)_{<s} \otimes \mathcal{X}(E_i m_s) \otimes \mathcal{X}(m)_{>s}
   \]
   \[
   = t^{\kappa_{\pm 1, \pm 1, c_1}(||m_{<s}||, ||m_s||, ||m_{>s}||)} \mathcal{X}(E_i m_s)
   \]
   \[
   = \mathcal{X}(1^{s-1} \otimes E_i \otimes 1^{-s})(m_{<s} \otimes m_s \otimes m_{>s}).
   \]
The case \( x_s = F_i \) proceeds similarly. \( \square \)

We now have defined a family of compatible twistor maps on (anti-)canonical modules and their tensor products. Moreover, the twistor maps on tensor products of modules are compatible with one another in the following sense. Let \( M_1, \ldots, M_t, c_1, \ldots, c_s \) and \( M \) be as in Proposition 4.18. Fix \( 1 \leq r \leq t \) and set \( m_{\leq r} = m_1 \otimes \ldots \otimes m_r \) and \( m_{> r} = m_{r+1} \otimes \ldots \otimes m_t \). Then by Lemma 4.17(1),
\[
\mathfrak{X}(m_{\leq r} \otimes m_{> r}) = \left( \prod_{1 \leq s < s' \leq t} t^{\kappa_{r,s,r',s'}}(||m_s||,||m_{s'}||) \right) \mathfrak{X}(m_{\leq r}) \otimes \mathfrak{X}(m_{> r})
\] (4.11)

4.4. Twisting the crossings, caps, and cups. We have now lain the groundwork for studying the atomic maps in our graphical calculus from \( \mathfrak{X} \) under the twistor functor. Specifically, we will show that the twistor almost commutes with caps, caps, and crossings up to a factor of an integral power of \( t \), where the power depends on the map. We begin by considering the cups and caps on their domains of definition.

Proposition 4.19. Let \( \lambda \in X^+ \). Then the map \( \text{ev}_\lambda \) (respectively, \( \text{qtr}_\lambda, \text{coev}_\lambda, \) and \( \text{coqtr}_\lambda \)) viewed as a function \( \hat{V}(\lambda) \otimes \hat{V}(\lambda) \rightarrow \mathbb{Q}(q,t)^r \) (resp. \( \hat{V}(\lambda) \otimes \hat{V}(\lambda) \rightarrow \mathbb{Q}(q,t)^r \), \( \hat{V}(\lambda) \otimes \hat{V}(\lambda) \rightarrow \hat{V}(\lambda) \otimes \hat{V}(\lambda) \)) is a \( \hat{U} \)-module homomorphism. Moreover, we have
\[
\begin{align*}
(1) \quad & \text{ev}_\lambda \mathfrak{X} = t^{\kappa_{(i,0)}((\lambda,0),(-\lambda,0))} \mathfrak{X} \text{ev}_\lambda; \\
(2) \quad & \text{qtr}_\lambda \mathfrak{X} = t^{\kappa_{(1,-1)}((\lambda,0),(-\lambda,0))} \mathfrak{X} \text{qtr}_\lambda; \\
(3) \quad & \text{coev}_\lambda \mathfrak{X} = t^{\kappa_{(i,0)}((\lambda,0),(-\lambda,0))} \mathfrak{X} \text{coev}_\lambda; \\
(4) \quad & \text{coqtr}_\lambda \mathfrak{X} = t^{\kappa_{(1,-1)}((\lambda,0),(-\lambda,0))} \mathfrak{X} \text{coqtr}_\lambda.
\end{align*}
\]

Proof. First, observe that since these maps are \( \hat{U} \)-module homomorphisms, they preserve weight spaces hence preserve the action of \( T_i \) for \( i \in I \). Therefore, it only remains to check that they commute with the action of \( T_i \) for \( i \in I \), As the arguments are all similar, let us show this for \( \text{ev}_\lambda \). Let \( f \in \hat{V}(\lambda) \) and \( x \in \hat{V}(\lambda) \). Then
\[
Y_i \text{ev}_\lambda(f \otimes x) = t^{\phi(i,0)} \text{ev}_\lambda(f \otimes x) = f(x).
\]
On the other hand, \( Y_i(f \otimes x) = (Y_i f) \otimes (Y_i x) = t^{-\phi(i,-|f|)+\phi(i,|x|)} f \otimes x \) hence
\[
\text{ev}_\lambda(Y_i(f \otimes x)) = t^{\phi(i,|x|)-\phi(i,-|f|)} f(x).
\]
However, since \( f(x) = 0 \) if \( |f| \neq |x| \), we see that \( \text{ev}_\lambda(Y_i(f \otimes x)) = t^{\phi(i,|x|)-\phi(i,|x|)} f(x) = f(x) = Y_i \text{ev}_\lambda(f \otimes x) \).

To verify (1)-(4), it suffices to compute the images \( \mathfrak{X}(b^{-v}_\lambda \otimes (b^{-v}_\lambda)^*) \) and \( \mathfrak{X}((b^{-v}_\lambda)^* \otimes b^{-v}_\lambda) \) for \( b \in \hat{B}_\nu = \mathbb{B} \cap \mathfrak{f}_\nu \). We compute directly that
\[
\mathfrak{X}(b^{-v}_\lambda) = t^{\ell(b)-\phi(\nu,\lambda)} b^{-v}_\lambda,
\]
\[
\mathfrak{X}^2(b^{-v}_\lambda) = t^{\ell(b)+\phi(\nu,\lambda)} b^{-v}_\lambda.
\]
This implies that for any \( b, b' \in \hat{B}_\nu \),
\[
\mathfrak{X}((b^{-v}_\lambda)^*)(b^{-v}_\lambda) = t^{2p(\nu)} \mathfrak{X}((b^{-v}_\lambda)^* \mathfrak{X}^{-1}(b^{-v}_\lambda)) = t^{2p(\nu)-\ell(b)-\phi(\nu,\lambda)} b_{b,b'}
\]
and hence \( \mathfrak{X}((b^{-v}_\lambda)^*) = t^{2p(\nu)-\ell(b)-\phi(\nu,\lambda)} b^{-v}_\lambda \). In particular, for \( c = (1, -1) \) observe that
\[
\mathfrak{X}((b^{-v}_\lambda) \otimes (b^{-v}_\lambda)^*) = t^{2p(\nu)-\ell(b)-\phi(\nu,\lambda)} b^{-v}_\lambda \otimes (b^{-v}_\lambda)^*.
\]
It is easy to verify that \( \nu/2 = p(\nu) \) modulo 2 by induction, hence we see that
\[
\mathfrak{X}((b^{-v}_\lambda) \otimes (b^{-v}_\lambda)^*) = t^{\kappa_{(1,-1)}((\lambda,0),(-\lambda,0))} (b^{-v}_\lambda) \otimes (b^{-v}_\lambda)^*.
\]
A similar computation shows that
\[
\mathfrak{X}(b^- v_\lambda) \ast \otimes (b^- v_\lambda) = t^{\kappa(-1, 1)}((-\lambda, 0), (\lambda, 0))(b^- v_\lambda)^{\ast} \otimes (b^- v_\lambda).
\]
Note that in either case, the power of \( t \) is independent of \( b \in \mathcal{B} \), and applying this to the definition of the maps proves (1) and (4). For (2) and (3), also note that \( \pi^{p(v)} q^{\pm(\hat{\rho}, \lambda)} = \pi^{p(v)} q^{\pm(\hat{\rho}, \lambda)} \), and we compute that \( \mathfrak{X}(\pi^{p(v)} q^{\pm(\hat{\rho}, \lambda)}) = t^{\pm(\hat{\rho}, \lambda)} t^{\pi(\hat{\rho}, \lambda)} \), the result follows.

**Example 4.20.** Consider the case \( n = 1 \) and \( \lambda = m \). As noted in Example 2.10, \( (\hat{\rho}, \lambda) = m \) and \( ev_m \circ coev_m = \pi^m | m + 1 \). Then we have \( ev_m \circ coev_m \circ \mathfrak{X}(1) = \pi^m | m + 1 \), and
\[
\mathfrak{X} \circ ev_m \circ coev_m(1) = \mathfrak{X}(\pi^m | m + 1) = t^{-m} \pi^m | m + 1 = t^{-m} ev_m \circ coev_m \circ \mathfrak{X}(1).
\]
Note that this is consistent with Proposition 4.19 as we see that
\[
ev_m \circ coev_m \circ \mathfrak{X} = t^{-\kappa(-1, 1)}((-\lambda, 0), (\lambda, 0))(\hat{\rho}, \lambda) ev_m \circ \mathfrak{X} \circ coev_m = t^{-m} \mathfrak{X} \circ ev_m \circ coev_m.
\]

The last elementary diagram to consider is the crossing, which is to say the automorphism \( R = \Theta \mathfrak{g} \mathfrak{s} \) of a tensor product of two modules. In order to have a concrete comparison of \( R X \) and \( \mathfrak{X} R \) on tensor products twisting by \( T \), it will be necessary to have a precise description of \( \mathfrak{X}(f(\zeta, \eta)) \) for any \( \zeta, \eta \in X \). To that end, let us once and for all fix a transversal \( T \) of \( X/\mathbb{Z}[I] \) and note that \( \tilde{T} = \{(\zeta, 0), (\zeta, 1) \mid \zeta \in T\} \) is a transversal of \( \tilde{X}/\mathbb{Z}[I] \). Then for \( \zeta_0, \zeta_1 \in T \), we shall henceforth require that
\[
f((\zeta_0), (\zeta_1)) = 1.
\]
(4.12)

Then we have the following proposition.

**Proposition 4.21.** Let \( \lambda, \lambda' \in X^+ \cup -X^+ \). Let \( \hat{\zeta}, \hat{\zeta}' \in \hat{T} \) be the corresponding coset representatives of \((\lambda, 0)\) and \((\lambda', 0)\) in \( \tilde{X}/\mathbb{Z}[I] \) and let \((c_1, c_2) = \text{sig} (\tilde{V}(\lambda) \tilde{\otimes} \tilde{V}(\lambda')) \). Let \( R : \tilde{V}(\lambda) \tilde{\otimes} \tilde{V}(\lambda') \to \tilde{V}(\lambda') \tilde{\otimes} \tilde{V}(\lambda) \) be the map described in Proposition 2.18. Then \( R \) is a \( \tilde{U} \)-module homomorphism. Moreover, as maps on \( V(\lambda) \tilde{\otimes} \tilde{V}(\lambda') \), we have
\[
\mathfrak{X} R = t^{\kappa(c_2, c_1)}(\hat{\zeta}', \hat{\zeta}) - \kappa(c_1, c_2) (\hat{\zeta}, \hat{\zeta}') + 2p(\hat{\zeta})p(\hat{\zeta}') R X.
\]

**Proof.** Recall that \( R = \Theta \mathfrak{g} \mathfrak{s} \) by definition. It is easy to see that \( R \) is a \( \tilde{U} \)-module homomorphism: indeed, since \( \mathfrak{s} \) preserve weight-spaces, it commutes with the action of the \( T_i \) for \( i \in I \); moreover, \( \mathfrak{g} \) obviously commutes with the diagonal action of \( \mathfrak{g} \), and it is easy to check directly that \( \Theta \mathfrak{g} \mathfrak{s} \Delta(\mathfrak{g}) = \Delta(\mathfrak{g}) \Theta \mathfrak{g} \mathfrak{s} \). We will prove the remainder of the proposition in two steps.

First we shall show that \( \mathfrak{X}(\Theta \nu) = \Theta \nu \) for any \( \nu \in \mathbb{Z}_{\geq 0}[I] \), and thus \( \mathfrak{X} \Theta = \Theta \mathfrak{X} \) as maps on \( V(\lambda) \tilde{\otimes} V(\lambda') \). This is straightforward: applying Lemmas 4.3 and Proposition 4.15 to the expression for \( \Theta \nu \) in terms of the canonical basis \( \mathcal{B} \), we compute that
\[
\mathfrak{X}(\Theta \nu) = (-1)^{ht \nu} t^{p(v)} t^{\pi(\nu)} t_{\nu}^{-1} q_{\nu} \sum_{b \in \mathcal{B}_v} \mathfrak{X}(b^-) Y_{\nu} \otimes \tilde{T}_{-\nu} Y_{\nu} \mathfrak{X}(b^+) = \Theta \nu
\]
\[
= (-1)^{ht \nu + p(\nu)} t^{\pi(\nu)} t_{\nu} q_{\nu} \sum_{b \in \mathcal{B}_v} (\mathfrak{X}(b^-) Y_{\nu} \otimes \tilde{T}_{-\nu} Y_{\nu} (t_{\nu}^{-1} \tilde{T}_{\nu} Y_{\nu} \mathfrak{X}(b^+))
\]
\[
= (-1)^{ht \nu} t^{\pi(\nu)} t_{\nu} q_{\nu} \sum_{b \in \mathcal{B}_v} (t^b b^- \otimes (t^b b^-) = \Theta \nu
\]

Now it remains to show that we have \( \mathfrak{X} \mathfrak{g} \mathfrak{s} = t^{\kappa(c_2, c_1)}(\hat{\zeta}', \hat{\zeta}) - \kappa(c_1, c_2) (\hat{\zeta}, \hat{\zeta}') + 2p(\hat{\zeta})p(\hat{\zeta}') \mathfrak{g} \mathfrak{s} \mathfrak{X} \) as maps on \( V(\lambda) \tilde{\otimes} V(\lambda') \). Set \( c = (c_1, c_2) \), and \( \hat{c} = (c_2, c_1) \). Let \( m \in V(\lambda) \) and \( n \in V(\lambda') \). Then we see directly that
\[
\mathfrak{X} \mathfrak{g} \mathfrak{s}(m \otimes n) = t^{2p(m)p(n) + \kappa(||n||, ||m||)} \mathfrak{X}(c(n), m) \mathfrak{X}(n, m),
\]
The proposition then follows by verifying that
\[ t^{2p(m)p(n)+\kappa_ε(||m||,||n||)} X(\hat{f}(|n|,|m|)) = t^{\kappa_ε(\hat{\mu},\hat{\nu})+2p(\hat{\mu})+2p(\hat{\nu})+\kappa_ε(||m||,||n||)} f(|n|,|m|). \]

Note that \( \hat{\mu} = ||m|| + \mu \) and \( \hat{\nu} = ||n|| + \nu \) for some \( \mu, \nu \in \mathbb{Z}[I] \). Let \( \hat{\zeta} = |\hat{\zeta}| \in X \) and \( \hat{\zeta}' = |\hat{\zeta}'| \in X \). Then in particular, (4.12) implies
\[ f(|n|,|m|) = (\pi q)\hat{\nu} q(\hat{\mu},\hat{\zeta})^m, \]
so \( X(\hat{f}(|n|,|m|)) = t^{(\hat{\mu},\hat{\nu})^m} f(|n|,|m|) \). Therefore, we are reduced to showing that \( \ell \equiv r \mod 4 \), where
\[ \ell = 2p(m)p(n) + \langle \hat{\nu}, \hat{\zeta} \rangle - \langle \hat{\mu}, \hat{\zeta}' \rangle + \mu \cdot \nu + \kappa_ε(||m||,||n||), \]
\[ r = 2p(\hat{\mu})+2p(\hat{\nu}) + \kappa_ε(||m||,||n||) \mod 4. \]

We compute directly that
\[ \kappa_ε(||m||,||n||) - \kappa_ε(\hat{\mu},\hat{\zeta}) + \kappa_ε(\hat{\mu},\hat{\zeta}') - \kappa_ε(\hat{\mu},\hat{\zeta}') \mod 4 \]
\[ = \kappa_ε(\hat{\mu},\hat{\zeta}') - \kappa_ε(\hat{\mu},\hat{\zeta}) + \kappa_ε(\hat{\mu},\hat{\zeta}') - \kappa_ε(\hat{\mu},\hat{\zeta}') \mod 4 \]
\[ = (\pi q)\hat{\nu} q(\hat{\mu},\hat{\zeta})^m \mod 4. \]

We have seen that the twistor map commutes (up to an integral power of \( t \)) with the elementary functions in our graphical calculus. However, note that in Theorem 3.3 the typical comapnd of a tangle invariant is not just one of these maps, but in fact is a tensor product of these maps with various identities. It is important to note that a consequence of Proposition 4.12 is that the twistor maps on tensor products are not local, since the power of \( t \) in the construction depends on the weight and signature of each tensor factor. Nevertheless, we can extend Propositions 4.12 and 4.21 to this more general setting.

**Proposition 4.22.** Let \( M_1, \ldots, M_t \) be \( \hat{U} \)-modules such that for each \( 1 \leq s \leq t \), \( M_s = \hat{V}(\mu_s) \) for some \( \mu_s \in X^+ \cup -X^+ \). Let \( M = M_1 \otimes \ldots \otimes M_t \) and let \( e = (c_1, \ldots, c_t) = \mathrm{sig}(M) \). For any \( \lambda \in X^+ \) and \( 0 \leq r \leq t \), we define \( M_{\leq r} = M_1 \otimes \ldots \otimes M_r, M_{> r} = M_{r+1} \otimes \ldots \otimes M_t \), and
\[ M(r, \pm \lambda) = M_{\leq r} \otimes \nabla(\pm \lambda) \otimes M_{> r}. \]

1. Let \( R_s = 1_{M_{\leq r}} \otimes R \otimes 1_{M_{> r}}, : M \to M \) for some \( 1 \leq s \leq t - 1 \). Then as maps on \( M \), \( X R_s \) and \( R_s X \) are proportional up to an integral power of \( t \).
2. Let \( \ev(M,r,\lambda) = 1_{M_{\leq r}} \otimes \ev_A \otimes 1_{M_{> r}} \) for some \( 1 \leq r \leq t \). Then as maps on \( M(r,-\lambda), X \ev(M,r,\lambda) \) and \( \ev(M,r,\lambda) X \) are proportional up to an integral power of \( t \).
3. Let \( \qtr(M,r,\lambda) = 1_{M_{\leq r}} \otimes \qtr_A \otimes 1_{M_{> r}} \) for some \( 1 \leq r \leq t \). Then as maps on \( M(r,\lambda), X \qtr(M,r,\lambda) \) and \( \qtr(M,r,\lambda) X \) are proportional up to an integral power of \( t \).
4. Let \( \cov(M,r,\lambda) = 1_{M_{\leq r}} \otimes \cov_A \otimes 1_{M_{> r}} \) for some \( 1 \leq r \leq t \). Then as maps on \( M \), \( X \cov(M,r,\lambda) \) and \( \cov(M,r,\lambda) X \) are proportional up to an integral power of \( t \).
5. Let \( \coqtr(M,r,\lambda) = 1_{M_{\leq r}} \otimes \coqtr_A \otimes 1_{M_{> r}} \) for some \( 1 \leq r \leq t \). Then as maps on \( M \), \( X \coqtr(M,r,\lambda) \) and \( \coqtr(M,r,\lambda) X \) are proportional up to an integral power of \( t \).
Remark 4.23. The precise constants of proportionality can be determined directly as in Propositions 3.39 and 4.21 (and can be worked out from the following proof), but we leave them out of the statement of Proposition 4.22 because they are not particularly illuminating, and are not necessary for Theorem 4.23.

Proof. As the proofs of (2)-(5) are similar, we shall only prove (1) and (2) here in detail.

We will begin with the proof of (1), which is essentially the same as the proof of Proposition 4.21. To wit, we first observe that for any $a, b \geq 0$ and $\nu \in \mathbb{N}[I]$, 
\[
\mathcal{X}(1^{\otimes a} \otimes \Theta_{\nu} \otimes 1^{\otimes b}) = (\mathcal{Y}_{\otimes \nu})^{\otimes a} \otimes \mathcal{X}(\mathcal{Y}_{\otimes \nu}) \otimes (\mathcal{Y}_{\otimes \nu} \mathcal{T}_{\otimes b} \mathcal{T}_{\otimes \nu})^{\otimes b} = ,
\]
and the result follows from the observation that $|\Theta_{\nu}| = \nu - \nu = 0$. Then $\mathcal{X}R_{s} = (1^{\otimes s-1} \otimes \Theta \otimes 1^{\otimes 1-s})\mathcal{X}\mathcal{S}_{s}. Then we verify directly that $\mathcal{X}\mathcal{S}_{s}\mathcal{S}_{s} = t^{\kappa(c_{s+1-c_{s}})(\zeta',\zeta) - \kappa(c_{s-c_{s+1}})(\zeta',\zeta')+2p(\zeta)p(\zeta')}\mathcal{S}_{s}\mathcal{S}_{s}$, where $\zeta$ (resp. $\zeta'$) is the coset representative for $(\mu_{s},0)$ (resp. $(\mu_{s+1},0)$).

Now, we shall prove (2). Note that an arbitrary element of $M(r,-\lambda)$ is a linear combination of simple tensors of the form $x = x_{m_{s-1}} \otimes (b^{-v_{\lambda}}) \otimes \otimes (b^{-v_{\lambda}}) \otimes m_{m_{s}}$, where $b, b' \in \mathcal{B}, m_{s-1} \otimes \otimes m_{m_{s}} \in M_{s-1}$ and $m_{s-1} \otimes \otimes m_{m_{s}} \in M_{s-1}$, hence we need only prove (1) holds when evaluating both sides at such elements. Since $ev_{\lambda}(b^{-v_{\lambda}}) \otimes (b^{-v_{\lambda}}) = \delta_{b,b'}$, note that (1) is trivially true when $b \neq b'$, so let’s assume $b = b' \in \mathcal{B_{r}}$. Then 
\[
ev(M,r,\lambda)\mathcal{X}(x) = t^{\diamondsuit(m_{1},\ldots,m_{s})+\bullet\mathcal{X}(m_{s}) \otimes ev_{\lambda}\mathcal{X}(b^{-v_{\lambda}}) \otimes (b^{-v_{\lambda}}) \otimes \mathcal{X}(m_{m_{s}})}
\]
where we set 
\[
\diamondsuit(m_{1},\ldots,m_{s}) = \sum_{s < s' < r} \kappa(c_{s+1-c_{s}})(\|m_{s}\|,\|m_{s}'\|)
\]
\[
\bullet = \sum_{s < r} \kappa(c_{s-1})(\|m_{s}\|,(-\lambda,0) + \hat{\nu}) + \kappa(c_{s,1})(\|m_{s}\|,(\lambda,0) - \hat{\nu})
\]
\[
+ \sum_{s > r} \kappa(c_{s+1-c_{s}})(\|m_{s}\|,(-\lambda,0) + \hat{\nu}),\|m_{s}\|) + \kappa(c_{s,1})(\|m_{s}\|,(\lambda,0) - \hat{\nu}),\|m_{s}\|)
\]
Now $\bullet$ can be simplified. Note that 
\[
\kappa(c_{s-1})(\|m_{s}\|,(-\lambda,0) + \nu) = \kappa(c_{s-1})(\|m_{s}\|,(-\lambda,0)) + 2c_{s}\phi(\nu,\|m_{s}\|) + 2p(\nu)p(m_{s})
\]
\[
\kappa(c_{s,1})(\|m_{s}\|,(\lambda,0) - \nu) = \kappa(c_{s,1})(\|m_{s}\|,(\lambda,0)) - 2c_{s}\phi(\nu,\|m_{s}\|) + 2p(\nu)p(m_{s})
\]
hence $\kappa(c_{s-1})(\|m_{s}\|,(-\lambda,0) + \nu) + \kappa(c_{s,1})(\|m_{s}\|,(\lambda,0) - \nu) = \kappa(c_{s-1})(\|m_{s}\|,(-\lambda,0)) + \kappa(c_{s,1})(\|m_{s}\|,(\lambda,0)).$ Moreover, note that $\|m_{s}\| = (\mu_{s},0) + \mu_{s}$ for some $\mu_{s} \in \mathbb{Z}[I]$, and so 
\[
\kappa(c_{s-1})(\|m_{s}\|,(-\lambda,0)) = \kappa(c_{s-1})(\mu_{s},0),(-\mu_{s},\lambda)) - (\hat{\nu},\lambda) - 2\phi(\mu_{s},\lambda),
\]
\[
\kappa(c_{s,1})(\|m_{s}\|,(\lambda,0)) = \kappa(c_{s,1})(\mu_{s},0),(\mu_{s},\lambda)) + \hat{\nu},\lambda) + 2\phi(\mu_{s},\lambda),
\]
hence 
\[
\kappa(c_{s-1})(\|m_{s}\|,(-\lambda,0)) + \kappa(c_{s,1})(\|m_{s}\|,(\lambda,0)) = \kappa(c_{s-1})(\mu_{s},0),(-\mu_{s},\lambda)) + \kappa(c_{s,1})(\mu_{s},0),(\mu_{s},\lambda))
\]
Similar applies to the sum over $s > r$ in $\bullet$, hence 
\[
\bullet = \sum_{s < r} \kappa(c_{s-1})(\mu_{s},0),(-\mu_{s},\lambda)) + \kappa(c_{s,1})(\mu_{s},0),(\mu_{s},\lambda))
\]
\[
+ \sum_{s > r} \kappa(c_{s+1-c_{s}})(\|m_{s}\|,(-\lambda,0),\mu_{s},0)) + \kappa(c_{s,1})(\mu_{s},0),(\mu_{s},\lambda))
\]
Note that $\bullet$ is independent of $x$. Then 
\[
ev(M,r,\lambda)\mathcal{X}(x) = t^{\diamondsuit(m_{1},\ldots,m_{s})+\bullet\mathcal{X}(m_{s}) \otimes ev_{\lambda}\mathcal{X}(b^{-v_{\lambda}}) \otimes (b^{-v_{\lambda}}) \otimes \mathcal{X}(m_{m_{s}})}
\]
\[
= t^{\diamondsuit(m_{1},\ldots,m_{s})+\bullet+\kappa(c_{s-1})(\|m_{s}\|,(-\lambda,0),\mu_{s},0)) \mathcal{X}(m_{m_{s}}) \otimes \mathcal{X}(m_{m_{s}})}
\]
\[
= t^{\bullet+\kappa(c_{s-1})(\|m_{s}\|,(-\lambda,0),\mu_{s},0)) \mathcal{X}(m_{m_{s}}) \otimes \mathcal{X}(m_{m_{s}})}
\]
Since $\mathcal{X}(m_{\leq r} \otimes m_{> r}) = \mathcal{X}(\text{ev}_\lambda(x))$ and the exponent of $t$ is independent of $x$, this completes the proof of (2).

We now arrive at the final result of this paper.

**Theorem 4.24.** Let $K$ be any oriented knot, and let $J^\lambda_K(q, \tau) \in \mathbb{Q}(q, t)^+$ be the $\lambda$-colored knot invariant defined in Theorem 4.8. Let $\ast_0 J^\lambda_K(q) = J^\lambda_K(q, 1)$ and $\ast_0 J^\lambda_K(q) = J^\lambda_K(q, t)$. Then

$$\ast_0 J^\lambda_K(q) = t^{(K, \lambda)} \ast_0 J^\lambda_K(t^{-1}q),$$

for some $*(K, \lambda) \in \mathbb{Z}$.

**Proof.** Let $J = J^\lambda_K(q, \tau)$. First, observe that $J$ can be thought of as a function $\mathbb{Q}(q, t)^+ \to \mathbb{Q}(q, t)^+$, and in that spirit $\mathcal{X}(J)$ is $\mathcal{X}(J(1))$. On the other hand, $J = W_K \circ S$, where $W_K = (f(\lambda, \tau)^{-1}(1 - \pi P(\lambda)q(\rho, \lambda))^{\text{wrt}(K)}$ (interpreted as a function $\mathbb{Q}(q, t)^+ \to \mathbb{Q}(q, t)^+$) and $S$ is a slice diagram of $K$ interpreted as a composition of morphisms as described in Section 3 (with strands colored by $\lambda$). In particular, observe that by (4.12) we have $\mathcal{X}(f(\lambda, \tau)) = t^x f(\lambda, \tau)$ for some $x \in \mathbb{Z}$ depending on the coset representative of $\lambda$ in $X/\mathbb{Z}[I]$, and that $\mathcal{X}(\pi P(\lambda)q(\rho, \lambda)) = \pi P(\lambda)q(\rho, \lambda)$). Then in particular we see that $\mathcal{X}W = t^{-x \text{wrt}(K)}tWX$.

Likewise, note that $S$ can be written as a composition of maps of the form $\text{ev}(M, r, \lambda)$, $\text{coev}(M, r, \lambda)$, $\text{qrt}(M, r, \lambda)$, $\text{cq}_{\text{qrt}}(M, r, \lambda)$, and $R_s : M \to M$ for various $r, s \in \mathbb{N}$ with all notations being the same as in Proposition 4.22. In particular, we see that $\mathcal{X} = t^y S \circ \mathcal{X}$ for some $y \in \mathbb{Z}$, and thus

$$\mathcal{X}(J) = \mathcal{X} \circ c \circ S(1) = t^{-x \text{wrt}(K)}t^{y + 1} S \circ \mathcal{X}(1) = t^{-x \text{wrt}(K)}t^{-y}J.$$

On the other hand, observe that $\mathcal{X}(J^\lambda_K(q, \tau)) = J^\lambda_K(t^{-q}t^{-1} \tau)$, and so

$$t^{y - x \text{wrt}(K)} J^\lambda_K(t^{-q}t^{-1} \tau) = J^\lambda_K(q, \tau).$$

The theorem follows from specializing $\tau = t$.

**Remark 4.25.** Note that since $\ast_0 J^\lambda_K(q) \in \mathbb{Z}[q, q^{-1}]$, Theorem 4.24 implies that (after a renormalization) $\ast_0 J^\lambda_K(q) = \ast_0 J^\lambda_K(v) \in \mathbb{Z}[v, v^{-1}]$ where $v = q^{t^{-1}}$. Furthermore, note that when $n$ or $(\overline{n}, \lambda)$ is even, $\ast_0 J^\lambda_K(q) \in \mathbb{Q}(q)$ (cf. Remark 4.7 (1)), thus in this case $\ast_0 J^\lambda_K(q) \equiv \ast_0 J^\lambda_K(q)$ modulo 2.

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