

\section{Introduction}

\subsection{One-sample \(U\)-statistics and parametric functionals}

Let \(\xi_1, \ldots, \xi_n\) be independent random elements taking values in a measurable space \((X, \mathcal{A})\) and having identical distribution \(P\). Let \(\mathcal{P} = \{P\}\) be some class of probability distributions on \((X, \mathcal{A})\) and let \(\theta(P)\) be a functional on \(\mathcal{P}\).

The functional \(\theta(P)\) is called \textit{regular} \cite{5}, if \(\theta(P)\) can be represented as

\[ \theta(P) = \int_X \ldots \int_X h(x_1, \ldots, x_m) P(dx_1) \ldots P(dx_m) \]  

with some real-valued symmetric Borel function \(h(x_1, \ldots, x_m)\) which is called the kernel, while the integer number \(m \geq 1\) is called the degree of functional \(\theta(P)\).

Halmos and Hoeffding \cite{3}, \cite{4} began to study the class of unbiased estimates of \(\theta(P)\) called \(U\)-statistics, which are defined as follows. Consider a kernel \(h(x_1, \ldots, x_m)\) of parametric functional (1). Then the \textit{\(U\)-statistic of degree} \(m\) is defined as

\[ U_n = \binom{n}{m}^{-1} \sum_J h(\xi_{i_1}, \ldots, \xi_{i_m}), \]

where \(n \geq m\) and \(J = \{(i_1, \ldots, i_m) : 1 \leq i_1 < \ldots < i_m \leq n\}\) is a set of increasing permutations of indices \(i_1, \ldots, i_m\).

It turns out that numerous statistical estimates and test statistics belong to the class of \(U\)-statistics. This entailed the intensive development of the theory, see \cite{5} and \cite{8}.

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\end{itemize}
1.2. **U-max-statistics.** U-max-statistics appear for the description of extreme counterparts of U-statistics, they are given by the formula:

\[ H_n = \max_j h(\xi_{i_1}, \ldots, \xi_{i_m}) . \]

U-min-statistics are defined similarly, they can be reduced to U-max-statistics by changing the sign of the kernel. Here are some examples of U-max and U-min-statistics.

1. Largest interpoint distance \( \max_{1 \leq i < j \leq n} \| \xi_i - \xi_j \| \), where \( \xi_1, \xi_2, \ldots \) are i.i.d. points in the \( d \)-dimensional unit ball \( \mathbb{B}^d, d \geq 2 \).

2. Largest scalar product \( \max_{1 \leq i < j \leq n} \langle \xi_i, \xi_j \rangle \), where \( \xi_1, \xi_2, \ldots \) are i.i.d. points in the ball \( \mathbb{B}^d, d \geq 2 \).

3. Smallest spherical distance: \( \min_{1 \leq i < j \leq n} \beta_{i,j} \), where \( \beta_{i,j} \) denotes the smaller of two central angles between \( U_i \) and \( U_j \). It is assumed that the vertices \( U_1, \ldots, U_n \) are i.i.d. points on the unit sphere \( \mathbb{S}^{d-1}, d \geq 2 \).

4. Largest perimeter \( \max_{1 \leq i < j < l \leq n} \text{peri}(U_i, U_j, U_l) \) and largest area \( \max_{1 \leq i < j < l \leq n} \text{area}(U_i, U_j, U_l) \) among all inscribed triangles, whose vertices are formed by triplets of points taken from a sample \( U_1, \ldots, U_n \) of independent and uniformly distributed points on the unit circumference.

Lao and Mayer [6], [7], [9] initiated the study of U-max-statistics and proved for them the basic limit theorem. They used some modification of a statement on Poisson convergence from the monograph of Barbour, Holst and Janson [1, p.35].

**Lao-Mayer Theorem [6].** Let \( \xi_1, \ldots, \xi_n \) be i.i.d. random elements in some measurable space \((\mathcal{X}, \mathcal{A})\) and let \( h \) be a symmetric Borel function, \( h : \mathcal{X}^m \to \mathbb{R} \). Put

\[ H_n = \max_j h(\xi_{i_1}, \ldots, \xi_{i_m}), \]

and denote for any \( z \in \mathbb{R} \)

\[ p_{n,z} = P\{ h(\xi_1, \ldots, \xi_m) > z \}, \quad \lambda_{n,z} = \binom{n}{m} p_{n,z}, \]

\[ \tau_{n,z}(r) = P\{ h(\xi_1, \ldots, \xi_m) > z, h(\xi_{1+m-r}, \xi_{2+m-r}, \ldots, \xi_{2m-r}) > z \} / p_{n,z}. \]

Then for all \( n \geq m \) and for each \( z \in \mathbb{R} \) we have:

\[ |P\{ H_n \leq z \} - \exp(-\lambda_{n,z})| \leq \left( 1 - \exp(-\lambda_{n,z}) \right) \left( p_{n,z} \left( \frac{n}{m} - \frac{n-m}{m} \right) \sum_{r=1}^{m-1} \binom{m}{r} \binom{n-m}{m-r} \tau_{n,z}(r) \right). \]

Clearly, the result can be reformulated for the minimal value of the kernel by replacing \( h \) with \( -h \).

**Remark 1.** (Lao-Mayer [6]). If the sample size \( n \) tends to infinity, then the error in (2) is asymptotically of order

\[ O \left( p_{n,z} n^{m-1} + \sum_{r=1}^{m-1} \tau_{n,z}(r) n^{m-r} \right), \]

where for \( m > 1 \) the sum is dominating, see [1].
Silverman and Brown [11] formulated the conditions which ensure that the general theorem used in [6] provides a non-trivial Weibull limit law.

**Silverman-Brown Theorem** [11]. In the setting of Lao-Mayer theorem, if for some sequence of transformations \( z_n : T \to \mathbb{R}, T \subset \mathbb{R} \), the conditions:

\[
\lim_{n \to \infty} \lambda_{n,z_n}(t) = \lambda_t > 0, \tag{3}
\]

\[
\lim_{n \to \infty} n^{2m-1} p_{n,z_n}(t) \tau_{n,z_n}(t)(m-1) = 0, \tag{4}
\]

hold for each \( t \in T \), then

\[
\lim_{n \to \infty} P\{H_n \leq z_n(t)\} = \exp\{-\lambda_t\} \tag{5}
\]

for each \( t \in T \).

**Remark 2.** (Lao-Mayer [6]). If \( m > 2 \), then the condition (4) can be replaced by the weaker condition:

\[
\lim_{n \to \infty} n^{2m-r} p_{n,z_n}(t) \tau_{n,z_n}(t)(r) = 0 \tag{6}
\]

for all \( r \in \{1, \ldots, m-1\} \).

**Remark 3.** (Lao-Mayer [6]). Condition (3) implies \( p_{n,z} = O(n^{-m}) \), and therefore (5) is valid with the rate of convergence

\[
O\left(n^{-1} + \sum_{r=1}^{m-1} n^{2m-r} p_{n,z_n}(t) \tau_{n,z_n}(t)(r)\right).
\]

For the perimeter and the area of inscribed triangles (see example 4) Lao and Mayer in [6], [7] and [9] obtained the following results.

**Theorem A** (Perimeter of inscribed triangle). Let \( U_1, U_2, \ldots \) be independent and uniformly distributed points on the unit circumference \( \mathbb{S} \), and let \( \text{peri}(U_i, U_j, U_l) \) be the perimeter of triangle formed by the triplet of points \( U_i, U_j, U_l \). Set \( H_n = \max_{1 \leq i < j < l \leq n} \text{peri}(U_i, U_j, U_l) \). Then for each \( t > 0 \)

\[
\lim_{n \to \infty} P\{n^3(3\sqrt{3} - H_n) \leq t\} = 1 - \exp\left\{-\frac{2t}{9\pi}\right\}.
\]

The rate of convergence is \( O\left(n^{-\frac{1}{2}}\right) \).

As a comment to this result, we note that among all triangles inscribed in the unit circumference, the regular triangle has the maximal value of perimeter equal to \( 3\sqrt{3} \). It is a classical and well-known result, see [12]. Clearly, the maximal perimeter \( H_n \) of random triangle tends to this value. The theorem gives the required normalization for this convergence, describes the limit distribution and establishes the rate of convergence.

Theorem A is proved by means of the following Lemma A.

**Lemma A.** Let \( U_1, U_2, U_3 \) be independent and uniformly distributed points on the unit circumference \( \mathbb{S} \). Then

\[
\lim_{s \to +0} s^{-1} P\{\text{peri}(U_1, U_2, U_3) \geq 3\sqrt{3} - s\} = \frac{4}{3\pi}.
\]

Now we proceed to random areas.
**Theorem B** (Area of inscribed triangle). Let $U_1, U_2, \ldots$ be independent and uniformly distributed points on the unit circumference $S$. Set $G_n = \max_{1 \leq i < j < l \leq n} \text{area}(U_i, U_j, U_l)$, where $\text{area}(U_i, U_j, U_l)$ is the area of random triangle formed by the triplet of points $U_i, U_j, U_l$. Then for each $t > 0$

$$\lim_{n \to \infty} P \left\{ n^3 \left( \frac{3\sqrt{3}}{4} - G_n \right) \leq t \right\} = 1 - \exp \left\{ -\frac{2t}{9\pi} \right\}.$$ 

The rate of convergence is $O \left( n^{-\frac{1}{2}} \right)$.

Commenting this result, we note that the area of the triangle inscribed into the unit circumference has the maximal value $\frac{3\sqrt{3}}{4}$ when its vertices are the vertices of regular triangle $[0, 12]$. The following Lemma B plays an important role in the proof of Theorem B.

**Lemma B.** Let $U_1, U_2, U_3$ be independent and uniformly distributed points on the unit circumference $S$. Then

$$\lim_{s \to +0} s^{-1} P \left\{ \text{area}(U_1, U_2, U_3) \geq \frac{3\sqrt{3}}{4} - s \right\} = \frac{4}{3\pi}.$$ 

In this paper we consider the limit behavior of more general $U$-max - statistics of this type related to $m$-polygons, $m \geq 3$. In the sequel $C_1, C_2, \ldots$ denote positive constants depending only on $m$.

2. **INSCRIBED POLYGON**

2.1. **Perimeter of inscribed polygon.** The result of this section is the generalization of Theorem A for inscribed triangles to the case of convex $m$-polygons, $m \geq 3$. We underline that the proof of this theorem in [6] turned out to be inapplicable for random perimeters and areas of $m$-polygons with $m > 3$. Therefore we had to use some new ideas.

**Theorem 1.** Let $U_1, U_2, \ldots$ be independent and uniformly distributed points on the unit circumference $S$, and let

$$P^m_n = \max_{1 \leq i_1 < \ldots < i_m \leq n} \text{peri}(U_{i_1}, \ldots, U_{i_m})$$

be the maximal perimeter among the perimeters $P^m$ of all convex $m$-polygons, generated by $m$ points from $U_1, \ldots, U_n$, $m \geq 3$. Then for each $t > 0$ we have

$$\lim_{n \to \infty} P \left\{ n^{2m-1} \left( 2m \sin \frac{\pi}{m} - P^m_n \right) \leq t \right\} = 1 - \exp \left\{ -\frac{t \frac{m-1}{2}}{K_{1m}} \right\},$$

where $K_{1m} = m^{\frac{3}{2}} \left( \pi \sin \frac{\pi}{m} \right)^{m-1} \Gamma \left( \frac{m+1}{2} \right)$. The rate of convergence is $O \left( n^{-\frac{1}{2}} \right)$.

Next Lemma is crucial in the proof of Theorem 1.

**Lemma 2.1.** Let $U_1, \ldots, U_m$ be $m$ independent and uniformly distributed points on the unit circumference. Consider the convex inscribed $m$-polygon with such vertices and with the perimeter $P^m = \text{peri}(U_1, \ldots, U_m)$. We have the following limit relation:

$$\lim_{s \to +0} s^{-\frac{m-1}{2}} P \left\{ P^m \geq 2m \sin \frac{\pi}{m} - s \right\} = \frac{\Gamma(m+1)}{K_{1m}},$$

(8)
where the constant $K_{1m}$ is from Theorem 1.

Proof of Lemma 2.1. For $m = 3$ Lemma 2.1 coincides with the result of Lao and Mayer [6]. The perimeter $P^m$ is maximal [12] for the regular $m$-polygon (its side is $2 \sin \frac{\pi}{m}$) and this maximal value equals $2m \sin \frac{\pi}{m}$. Consider for $i = 1, \ldots, m - 1, m \geq 3$, the central angle $\beta_i = \angle U_iOU_{i+1}$. By rotational symmetry, these angles are independent and uniformly distributed on the interval $[0, 2\pi]$. Let show that the points $U_i$ can be taken in order of increasing angles, so that our $m$-polygon corresponds to the following figure.

Consider the permutation of angles with increasing indices $\{\beta_1 \leq \beta_2 \leq \ldots \leq \beta_{m-1}\}$. Let prove for any $0 < z < 2m \sin \frac{\pi}{m}$ the equality of probabilities

$$P\{P^m \geq z, \beta_{i_1} \leq \beta_{i_2} \leq \ldots \leq \beta_{i_{m-1}}\} = P\{P^m \geq z, \beta_1 \leq \beta_2 \leq \ldots \leq \beta_{m-1}\},$$

where $\{i_1, \ldots, i_m\}$ is any permutation of indices $\{1, \ldots, m\}$ and $\beta_{i_s}$ is the central angle defined by the points $U_{i_s}$ and $U_{i_{s+1}}, s = 1, \ldots, m - 1$.

Let $C_{m-1}$ be the $(m - 1)$-dimensional cube with the edge $2\pi$. Note that the sides of $m$-polygon are calculated by the law of cosines:

$$|U_lU_{l+1}| = 2 \sin \frac{\beta_l - \beta_{l-1}}{2}, 1 \leq l \leq m - 1; |U_mU_1| = 2 \sin \frac{\beta_{m-1}}{2}.$$

We have

$$P\{P^m \geq z, \beta_{i_1} \leq \beta_{i_2} \leq \ldots \leq \beta_{i_{m-1}}\} =$$

$$P\left\{ \sum_{k=1}^{m-1} |U_{i_k}U_{i_{k+1}}| + |U_{i_m}U_1| \geq z, \beta_{i_1} \leq \beta_{i_2} \leq \ldots \leq \beta_{i_{m-1}} \right\} =$$

$$\frac{1}{(2\pi)^{m-1}} \int_{C_{m-1}} \left\{ \sum_{k=1}^{m-1} |U_{i_k}U_{i_{k+1}}| + |U_{i_m}U_1| \geq z, \beta_{i_1} \leq \beta_{i_2} \leq \ldots \leq \beta_{i_{m-1}} \right\} d\beta_{i_1} \ldots d\beta_{i_{m-1}}.$$

After the change of variables $\beta_{i_1} = \beta_1, \ldots, \beta_{i_{m-1}} = \beta_{m-1}$ the last integral becomes

$$\frac{1}{(2\pi)^{m-1}} \int_{C_{m-1}} \left\{ |U_1U_2| + \ldots + |U_mU_1| \geq z, \beta_1 \leq \ldots \leq \beta_{m-1} \right\} d\beta_1 \ldots d\beta_{m-1} =$$

$$= P\{P^m \geq z, \beta_1 \leq \beta_2 \leq \ldots \leq \beta_{m-1}\},$$

as required. □
Our arguments imply that
\[ P\{P^m \geq 2m \sin \frac{\pi}{m} - s\} = (m - 1)!P\{P^m \geq 2m \sin \frac{\pi}{m} - s, \ \beta_1 \leq \beta_2 \leq \ldots \leq \beta_{m-1}\}. \] (9)

The condition \( \beta_1 \leq \beta_2 \leq \ldots \leq \beta_{m-1} \) provides that the points \( U_1, \ldots, U_m \) stand in increasing order just as in Fig.1.

Now let show that for small \( s > 0 \) and under the condition
\[ P^m \geq 2m \sin \frac{\pi}{m} - s \] (10)
al all central angles of our \( m \)-polygon differ very little from \[ 2\pi \frac{m}{m} - 2\psi \] and \[ 2\pi + 2\varphi, \ \varphi > \psi > 0. \]

One can displace these sides so that they are adjacent. Their lengths are equal to \( 2\sin((\frac{\pi}{m} + \varphi) \) and \( 2\sin((\frac{\pi}{m} - \psi) \). Now let make the so-called ”symmetrization” [12] replacing our two sides by two equal sides having the common vertex on the circle in the center of the arc subtending the sum of the angles. Then the length of new two sides will increase [12, Probl. 55b] and is equal to \( 4\sin((\frac{\pi}{m} + \varphi - \psi) \). The increment of the perimeter is just
\[ \Delta = 4\sin\left(\frac{\pi}{m} + \frac{\varphi - \psi}{2}\right)\left(1 - \cos\frac{\varphi + \psi}{2}\right), \ \varphi - \psi > 0. \]

Suppose the initial perimeter of our \( m \)-polygon was \( 2m \sin \frac{\pi}{m} - \sigma \) with \( \sigma \leq s \). Consequently
\[ 2m \sin \frac{\pi}{m} - \sigma + \Delta \leq 2m \sin \frac{\pi}{m}. \]
whence it follows that \( \Delta \leq s \) and therefore
\[ 1 - \cos\frac{\varphi + \psi}{2} \leq \frac{s}{4\sin\left(\frac{\pi}{m} + \frac{\varphi - \psi}{2}\right)}. \] (11)

Clearly, the sum of the largest and smallest central angles \[ \frac{4\pi}{m} + 2\varphi - 2\psi \] is smaller than \( 2\pi \). Hence \( 0 < \frac{\pi}{m} + \frac{\varphi - \psi}{2} < \frac{\pi}{2} \) and we have from (11) the inequality
\[ 1 - \cos\frac{\varphi + \psi}{2} \leq \frac{s}{4\sin\left(\frac{\pi}{m}\right)} = C_1 s, \]
which implies that for small \( s > 0 \)
\[ \varphi + \psi \leq 2\arccos(1 - C_1 s) < C_2 \sqrt{s}. \]

Consequently all random central angles differ from the angle \[ \frac{2\pi}{m} \] by no more than \( O(\sqrt{s}) \).

Let estimate the deviation of the angles \( \beta_k \) from \( \beta_{k-1}, k = 2, \ldots, m - 1, \) under the condition (10). We introduce the auxiliary random angles \( \alpha_0 = 0, \alpha_1, \ldots, \alpha_{m-1}, \alpha_m = 0 \) such that
\[ \beta_k = \frac{2\pi k}{m} + \alpha_k, \ k = 1, \ldots, m. \] (12)
The random angles \( \alpha_1, \ldots, \alpha_{m-1} \) are independent and each \( \alpha_k \) is uniformly distributed on \( [-\frac{2\pi}{m}, 2\pi - \frac{2\pi}{m}] \). In terms of \( \alpha_k \) we have
\[ P^m = 2\sum_{k=1}^{m} \sin\left(\frac{\pi}{m} + \frac{\alpha_k - \alpha_{k-1}}{2}\right), \]
and the inequality (10) takes the form
\[
2 \sum_{k=1}^{m} \sin \left( \frac{\pi}{m} + \frac{\alpha_k - \alpha_{k-1}}{2} \right) \geq 2m \sin \frac{\pi}{m} - s. \tag{13}
\]

The argument given above shows that any central angle differs from the expected value \( \frac{2\pi}{m} \) by no more than \( O(\sqrt{s}) \). Hence we have under (10)
\[
\max_{1 \leq k \leq m} |\alpha_k - \alpha_{k-1}| \leq C_3 \sqrt{s}. \tag{14}
\]
and consequently
\[
\max_{1 \leq k \leq m-1} |\alpha_k| \leq C_4 \sqrt{s}. \tag{15}
\]

Let return to the formula (13). As the differences \( |\alpha_k - \alpha_{k-1}| \) are small, we can expand in (13) the sine function in Taylor series with the remainder term. Hence we obtain for some small random angles \( \eta_k, k = 1, \ldots, m-1 \), that
\[
2 \sum_{k=1}^{m} \sin \left( \frac{\pi}{m} + \frac{1}{2} (\alpha_k - \alpha_{k-1}) \right) =
2m \sin \frac{\pi}{m} - \frac{1}{4} \sin \frac{\pi}{m} \sum_{k=1}^{m} (\alpha_k - \alpha_{k-1})^2 + \frac{1}{24} \sum_{k=1}^{m} \cos \left( \frac{\pi}{m} + \eta_k \right) (\alpha_k - \alpha_{k-1})^3,
\]
and therefore (13) is equivalent to
\[
\sum_{k=1}^{m} (\alpha_k - \alpha_{k-1})^2 - \frac{1}{6 \sin \frac{\pi}{m}} \sum_{k=1}^{m} \cos \left( \frac{\pi}{m} + \eta_k \right) (\alpha_k - \alpha_{k-1})^3 \leq \frac{4s}{\sin \frac{\pi}{m}}. \tag{16}
\]

Clearly
\[
| \sum_{k=1}^{m} \cos \left( \frac{\pi}{m} + \eta_k \right) (\alpha_k - \alpha_{k-1})^3 | \leq \max_{1 \leq k \leq m} |\alpha_k - \alpha_{k-1}| \sum_{k=1}^{m} (\alpha_k - \alpha_{k-1})^2,
\]
hence, using (16) and (14), we may write the inequalities
\[
P \left\{ \mathcal{P}^m \geq 2m \sin \frac{\pi}{m} - s, \ \beta_1 \leq \beta_2 \leq \ldots \leq \beta_{m-1} \right\} \leq P \left\{ \sum_{k=1}^{m} (\alpha_k - \alpha_{k-1})^2 \leq \frac{24s}{6 \sin \frac{\pi}{m} - \max_{1 \leq k \leq m} |\alpha_k - \alpha_{k-1}|} \right\} \leq P \left\{ \sum_{k=1}^{m} (\alpha_k - \alpha_{k-1})^2 \leq \frac{4s_1}{\sin \frac{\pi}{m}} \right\},
\]
where \( s_1 \to s \) as \( s \to 0 \). Quite analogously
\[
P \left\{ \mathcal{P}^m \geq 2m \sin \frac{\pi}{m} - s, \ \beta_1 \leq \beta_2 \leq \ldots \leq \beta_{m-1} \right\} \geq P \left\{ \sum_{k=1}^{m} (\alpha_k - \alpha_{k-1})^2 \leq \frac{4s_2}{\sin \frac{\pi}{m}} \right\},
\]
where \( s_2 \to s \) as \( s \to 0 \).

Now let introduce the quadratic form \( Q(\alpha) = Q(\alpha_1, \ldots, \alpha_{m-1}) \) by
\[
Q(\alpha) = \frac{1}{2} \sum_{k=1}^{m} (\alpha_k - \alpha_{k-1})^2 = \sum_{k=1}^{m-1} \alpha_k^2 - \sum_{k=1}^{m-1} \alpha_k \alpha_{k-1}.
\]

We have by (9) for small $s_1 > s_2, s_1 \to s, s_2 \to s$ as $s \to 0$:
\[
P \left\{ Q(\alpha) \leq \frac{2s_2}{\sin \frac{\pi}{m}} \right\} \leq \frac{1}{(m-1)!} P \left\{ \mathcal{P}^m \geq 2m \sin \frac{\pi}{m} - s \right\} \leq P \left\{ Q(\alpha) \leq \frac{2s_1}{\sin \frac{\pi}{m}} \right\}. \quad (17)
\]

Now we proceed to the calculation of the probability in the right-hand side of (17). The quadratic form $Q(\alpha)$ has the following matrix of size $(m-1) \times (m-1)$:
\[
B = \begin{pmatrix}
1 & -1/2 & 0 & 0 & 0 & \ldots & 0 \\
-1/2 & 1 & -1/2 & 0 & 0 & \ldots & 0 \\
0 & -1/2 & 1 & -1/2 & 0 & \ldots & 0 \\
0 & 0 & -1/2 & 1 & -1/2 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & \ldots & \ldots & -1/2 & 1 & -1/2 \\
0 & \ldots & \ldots & \ldots & 0 & -1/2 & 1
\end{pmatrix}.
\]

This matrix is symmetric tridiagonal. The spectrum of such matrices is known, see [2, p.137] or [13], so that all eigenvalues of the matrix $B$ are
\[
\lambda_k = 1 - \cos \frac{\pi k}{m}, \quad k = 1, \ldots, m - 1.
\]

Then using the appropriate orthogonal transformation, we can replace the quadratic form $\sum_{k=1}^{m-1} \alpha_k^2 - \sum_{k=1}^{m-1} \alpha_k \alpha_{k-1}$ by the quadratic form
\[
\sum_{k=1}^{m-1} \left( 1 - \cos \frac{\pi k}{m} \right) Y_k^2
\]
in new variables $Y_k$. Now set for brevity
\[
l_k^2 = \frac{2s_1}{\sin \frac{\pi}{m}} \left( 1 - \cos \frac{\pi k}{m} \right), \quad k = 1, \ldots, m - 1.
\]

As the Jacobian of the orthogonal transformation is 1, we get by (17) that
\[
P \left\{ \mathcal{P}^m \geq 2m \sin \frac{\pi}{m} - s \right\} \leq \frac{(m-1)!}{(2\pi)^{m-1}} \int_{\left\{ \sum_{k=1}^{m-1} y_k^2 \leq \frac{2s_1}{\sin \frac{\pi}{m}} \right\}} dx_1 \ldots dx_{m-1} =
\]
\[
= \frac{(m-1)!}{(2\pi)^{m-1}} \int_{\left\{ \sum_{k=1}^{m-1} (1-\cos \frac{\pi k}{m}) y_k^2 \leq \frac{2s_1}{\sin \frac{\pi}{m}} \right\}} dy_1 \ldots dy_{m-1} =
\]
\[
= \frac{(m-1)!}{(2\pi)^{m-1}} \int_{\left\{ \sum_{k=1}^{m-1} \left( \frac{\alpha_k}{l_k} \right)^2 \leq 1 \right\}} dy_1 \ldots dy_{m-1} = \frac{(m-1)!}{(2\pi)^{m-1}} b_{m-1} \Pi_{m-1}(s_1).
\]

Here $b_{m-1}$ is a volume of $(m-1)$-dimensional ball of unit radius, and $\Pi_{m-1}(s_1)$ is a product of semi-axes of the small ellipsoid
\[
\left\{ (y_1, \ldots, y_{m-1}) : \sum_{k=1}^{m-1} \frac{y_k^2}{l_k^2} \leq 1 \right\},
\]
which is equal to
\[
\Pi_{m-1}(s_1) = \prod_{k=1}^{m-1} l_k = \prod_{k=1}^{m-1} \sqrt{\frac{2s_1}{\sin \frac{\pi}{m} \left(1 - \cos \left(\frac{\pi}{m}\right)\right)}} = \left(\frac{s_1}{\sin \frac{\pi}{m}}\right)^{\frac{m-1}{2}} \prod_{k=1}^{m-1} \left(\sin \left(\frac{\pi k}{2m}\right)\right)^{-1}.
\]

To simplify this expression we use the identity from [10, formula (6.1.2.3)]:
\[
\prod_{k=1}^{m-1} \sin \left(\frac{\pi k}{2m}\right) = \frac{\sqrt{m}}{2^{m-1}}.
\]

Hence the product of semi-axes of the ellipsoid is equal to
\[
\Pi_{m-1}(s_1) = \left(\frac{s_1}{\sin \frac{\pi}{m}}\right)^{\frac{m-1}{2}} \frac{2^{m-1}}{\sqrt{m}} = \frac{2^{m-1} s_1^{\frac{m-1}{2}}}{\sqrt{m} \left(\sin \frac{\pi}{m}\right)^{\frac{m-1}{2}}}.
\]

The volume of \((m-1)\)-dimensional ball of the unit radius is well-known and equals
\[
b_{m-1} = \left(\frac{\sqrt{\pi}}{\Gamma\left(\frac{m+1}{2}\right)}\right)^{m-1}.
\]

Therefore the volume of the \((m-1)\)-dimensional ellipsoid is equal to
\[
V_{m-1}(s_1) = b_{m-1} \Pi_{m-1}(s_1) = \frac{\left(\frac{\sqrt{\pi}}{\Gamma\left(\frac{m+1}{2}\right)}\right)^{m-1} 2^{m-1} s_1^{\frac{m-1}{2}}}{\sqrt{m} \left(\sin \frac{\pi}{m}\right)^{\frac{m-1}{2}}},
\]
from where we finally obtain:
\[
P\left\{\mathcal{P}^m \geq 2m \sin \frac{\pi}{m} - s\right\} \leq \frac{(m-1)!}{(2\pi)^{m-1}} \frac{1}{\sqrt{m} \left(\sin \frac{\pi}{m}\right)^{\frac{m-1}{2}}} = \frac{s_1^{\frac{m-1}{2}} \Gamma(m)}{\sqrt{m} \left(\sin \frac{\pi}{m}\right)^{\frac{m-1}{2}} \Gamma\left(\frac{m+1}{2}\right)} = \frac{\Gamma(m\!+\!1)}{s_1^{\frac{m-1}{2}} K_{1m}}.
\]

In the same manner we get from (17) the opposite inequality
\[
P\left\{\mathcal{P}^m \geq 2m \sin \frac{\pi}{m} - s\right\} \geq \frac{s_2^{\frac{m-1}{2}} \Gamma(m\!+\!1)}{K_{1m}}.
\]

Lemma 2.1 immediately follows from these two inequalities. \(\square\)

**Proof of Theorem 1.** Consider the transformation
\[
z_n(t) = 2m \sin \frac{\pi}{m} - s = 2m \sin \frac{\pi}{m} - tn^{-\frac{2m}{m-1}},
\]
and denote
\[
\lambda_{n,z_n}(t) = \left(\frac{n}{m}\right) P\left\{\mathcal{P}_n^m > z_n(t)\right\} = \frac{n!}{m!(n-m)!} P\left\{\mathcal{P}_n^m > z_n(t)\right\},
\]
where \(\mathcal{P}_n^m\) is the maximal perimeter from (7).
As \( s = tn^{-\frac{2m}{m-1}} \), we note that \( n^m s^{\frac{m-1}{2}} = t^{\frac{m-1}{2}} \). Since for fixed \( m \) and large \( n \) it holds \( \frac{n!}{(n-m)!} \sim n^m \), then we have from Lemma 2.1 that

\[
\lim_{n \to \infty} \lambda_{n,z_n(t)} = \frac{1}{m!} \lim_{n \to \infty} n^m s^{\frac{m-1}{2}} \cdot s^{\frac{m-1}{2}} P\{P_n > 2m \sin \frac{\pi}{m} - s\} = \\
= \frac{1}{m!} \frac{m-1}{m} \lim_{n \to \infty} \left( tn^{-\frac{2m}{m-1}} \right)^{\frac{m-1}{2}} P\{P_n > 2m \sin \frac{\pi}{m} - tn^{-\frac{2m}{m-1}}\} = \\
= \frac{1}{m!} t^{\frac{m-1}{2}} \Gamma(m+1) K_{1m} = \frac{m-1}{K_{1m}} := \lambda_t > 0.
\]

Thus, the condition (3) from Silverman-Brown Theorem holds true.

Now for any \( r \in \{1, \ldots, m-1\} \) denote by \( P^{m; i_1, \ldots, i_m} \) the perimeter of the convex inscribed \( m \)-polygon based on the random points \( U_{i_1}, \ldots, U_{i_m}; i_1 < \ldots < i_m \), on the unit circumference. We must verify the condition (6):

\[
\lim_{n \to \infty} n^{2m-r} P\{P^{m; 1, \ldots, m} > z_n(t), P^{m; m-r+1, \ldots, 2m-r} > z_n(t)\} = 0
\]

for each \( r \in \{1, \ldots, m-1\} \).

**Lemma 2.2** (verification of the condition (6)). For each \( r \in \{1, \ldots, m-1\} \) it holds

\[
\lim_{n \to \infty} n^{2m-r} P\{P^{m; 1, \ldots, m} > z_n(t), P^{m; m-r+1, \ldots, 2m-r} > z_n(t)\} = 0.
\]

**Proof of Lemma 2.2.** Using same arguments as in the proof of Lemma 2.1, we may assume that \( \beta_1 < \beta_2 < \ldots < \beta_{m-1} \), so that the points \( U_j \) follow one after one in increasing order. Then the condition \( P^{m; 1, \ldots, m} > z_n(t) \) implies by (15) that for some constant \( C_5 > 0 \)

\[
|\alpha_1| < C_5 \sqrt{s}, \ldots, |\alpha_{m-1}| < C_5 \sqrt{s}.
\]

The second condition \( P^{m; m-r+1, \ldots, 2m-r} > z_n(t) \) implies analogously that

\[
|\alpha_{m-r+1}| < C_5 \sqrt{s}, \ldots, |\alpha_{2m-r-1}| < C_5 \sqrt{s}.
\]

The intersection of these two events is the event

\[
\bigcap_{j=1}^{2m-r-1} \{\alpha_j < C_5 \sqrt{s}\}.
\]

All angles \( \alpha_j \) are independent and have the uniform distribution on the intervals \((\frac{-2\pi}{m}, 2\pi - \frac{2\pi}{m})\) of length \(2\pi\). Then the expression of interest can be estimated for each \( r = 1, \ldots, m-1 \), as follows

\[
n^{2m-r} P\{P^{m; 1, \ldots, m} > z_n(t), P^{m; m-r+1, \ldots, 2m-r} > z_n(t)\} \leq \\
\leq n^{2m-r} \left(C_5 \sqrt{s}/2\pi\right)^{2m-r-1} = O(n^{2m-r-\frac{(2m-r-1)m}{m-1}}) = O(n^{\frac{-m}{m-1}}) = o(1).
\]

By Remark 3 it follows that the rate of convergence in the worst case \( r = m-1 \) is

\[
O\left(n^{m+1} p_{n,z_n}(t) T_{n,z_n}(m-1)\right) = O\left(n^{m+1} n^{-\frac{2m}{m-1}}\right) = O\left(n^{-\frac{m}{m-1}}\right).
\]

We see that this result coincides with the result of Lao and Mayer in Theorem A for \( m = 3 \) but deteriorates when \( m \) grows.
Continuation of the proof of Theorem 1. Next we apply the relation (5), see also [6], and obtain:
\[
\lim_{n \to \infty} P \left\{ P^m_n < 2m \sin \frac{\pi}{m} - t \frac{2m}{m-1} \right\} = \exp \left\{ -t \frac{m-1}{K_{1m}} \right\}.
\]
Consequently for each \( t > 0 \) we have:
\[
\lim_{n \to \infty} P \left\{ \frac{2m}{m-1} \left( 2m \sin \frac{\pi}{m} - P^m_n \right) \leq t \right\} = 1 - \exp \left\{ -t \frac{m-1}{K_{1m}} \right\},
\]
where
\[
K_{1m} = m^{\frac{3}{2}} \left( \pi \sin \frac{\pi}{m} \right)^{\frac{m-1}{2}} \Gamma \left( \frac{m+1}{2} \right) .
\]
\( \square \)

The limit distribution is the Weibull distribution which is expected for the distribution of a maximum. For \( m = 3 \) we get the result from [6], i.e. Theorem A from Introduction.

2.2. Area of inscribed polygon.

**Theorem 2.** Let \( U_1, U_2, \ldots \) be independent and uniformly distributed points on the unit circumference \( S \), \( A^m_n = \max_{1 \leq i_1 < \ldots < i_m \leq n} A_m(U_{i_1}, \ldots, U_{i_m}) \) is a maximal area among all areas of convex \( m \)-polygons with the vertices \( U_{i_1}, \ldots, U_{i_m} \). Then for each \( t > 0 \) it is true that
\[
\lim_{n \to \infty} P \left\{ \frac{2m}{m-1} \left( \frac{m}{2} \sin 2\frac{\pi}{m} - A^m_n \right) \leq t \right\} = 1 - \exp \left\{ -t \frac{m-1}{K_{2m}} \right\},
\]
where
\[
K_{2m} = m^{\frac{3}{2}} \left( \pi \sin \frac{2\pi}{m} \right)^{\frac{m-1}{2}} \Gamma \left( \frac{m+1}{2} \right). \tag{20}
\]
The rate of convergence is \( O \left( n^{-\frac{1}{m-1}} \right) \).

**Remark 4.** Curiously, the limit constants \( K_{1m} \) and \( K_{2m} \) are very similar, but yet differ by the argument of the sine function. At the end of the paper we will discuss the asymptotic behavior of them and of similar constants from limit theorems for the metric characteristics of circumscribed polygons.

**Lemma 2.3.** Let \( m \) independent and uniformly distributed points \( U_1, \ldots, U_m \) be chosen on the unit circumference. Consider the convex \( m \)-polygon with such vertices having the area \( A^m = \text{area}(U_1, \ldots, U_m) \). Then the following limit relation holds:
\[
\lim_{s \to 0} s^{-\frac{m-1}{2}} P \left\{ A^m \geq \frac{m}{2} \sin \frac{2\pi}{m} - s \right\} = \frac{\Gamma(m+1)}{K_{2m}},
\]
where \( K_{2m} \) is given by (20).

**Proof of Lemma 2.3.** The area of \( m \)-polygon inscribed into the unit circle has the maximal value for the regular \( m \)-polygon [12] and equals \( A^m = \frac{m}{2} \sin \frac{2\pi}{m} \). As in the proof of Lemma 2.1, consider the angles \( \beta_i \).

The area of inscribed \( m \)-polygon is the sum of the areas \( S_i \) of triangles, formed by the triples of points \( U_i, U_{i+1}, O \), where \( O \) is the center of the circle, see Fig.2.
Given that the angle between $U_k$ and $U_{k+1}$ equals $\beta_k - \beta_{k-1}$, the area of the triangle $\triangle U_kOU_{k+1}$ is $\frac{1}{2} \sin(\beta_k - \beta_{k-1})$. Hence

$$A^m = \sum_{k=1}^{m} S_k = \frac{1}{2} \sum_{k=1}^{m} \sin(\beta_k - \beta_{k-1}).$$

We have, assuming that $\beta_0 = 0, \beta_m = 2\pi$,

$$P \left\{ A^m \geq \frac{m}{2} \sin \frac{2\pi}{m} - s \right\} =
= (m - 1)! P \left\{ \frac{1}{2} \sum_{k=1}^{m} \sin(\beta_k - \beta_{k-1}) \geq \frac{m}{2} \sin \frac{2\pi}{m} - s, \beta_1 \leq \beta_2 \leq \ldots \beta_{m-1} \right\}.$$

Next we pass by (12) from random angles $\beta_k$ to random angles $\alpha_k$, which are independent and uniformly distributed on $[-\frac{2\pi k}{m}, 2\pi - \frac{2\pi k}{m}]$.

Same arguments as in the proof of Lemma 2.1 show that for small $s$ and under the condition

$$A^m \geq \frac{m}{2} \sin \frac{2\pi}{m} - s$$

all central angles of the inscribed polygon differ from $\frac{2\pi}{m}$ at most by $O(\sqrt{s})$ and the inequalities (14) and (15) hold. Expanding the sine function in the expression of area for small $\alpha_k, k = 1, \ldots, m - 1$, we get in the same way as above the following inequalities for small $s$:

$$P \left\{ \sum_{k=1}^{m-1} \alpha_k^2 - \sum_{k=1}^{m-1} \alpha_k \alpha_{k-1} \leq \frac{2s_4}{\sin \frac{2\pi}{m}} \right\} \leq$$

$$\leq P \left\{ \frac{1}{2} \sum_{k=1}^{m} \sin(\beta_k - \beta_{k-1}) \geq \frac{m}{2} \sin \frac{2\pi}{m} - s, \beta_1 \leq \beta_2 \leq \ldots \beta_{m-1} \right\} \leq$$

$$\leq P \left\{ \sum_{k=1}^{m-1} \alpha_k^2 - \sum_{k=1}^{m-1} \alpha_k \alpha_{k-1} \leq \frac{2s_3}{\sin \frac{2\pi}{m}} \right\},$$

where $s_3 > s_4$, and $s_3 \rightarrow s, s_4 \rightarrow s$ as $s \rightarrow 0$. 

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The probabilities in the left and in the right have been calculated above (with another constant in the right-hand side), so we get analogously
\[
\frac{s_{m-1}}{K_{2m}} \Gamma(m+1) \leq P \left\{ A^m \geq \frac{m}{2} \sin \frac{2\pi}{m} - s \right\} \leq \frac{s_{m-1}}{K_{2m}} \Gamma(m+1),
\]
where
\[
K_{2m} = m^{3/2} \left( \pi \sin \frac{2\pi}{m} \right)^{m-1} \Gamma \left( \frac{m+1}{2} \right).
\]
Lemma 2.3 follows from this as \( s \to 0 \).

Proof of Theorem 2 is quite analogous to the proof of Theorem 1, but with the transformation \( z_n(t) = \frac{h}{m} \sin \frac{2\pi}{m} - t n^{\frac{m-1}{m}} \).

3. Circumscribed polygon

We proceed to the perimeter and area of random *circumscribed* polygons. In this case we are of course interested in minimal perimeter and area. As far as we know this problem has never been studied.

### 3.1. Perimeter of circumscribed polygon

**Theorem 3.** Let \( U_1, U_2, \ldots \) be independent and uniformly distributed points on the circumference \( S \) of unit radius. For each set of points \( U_{i_1}, \ldots, U_{i_m} \) on the circle draw the tangents at these points, denoting the intersection of these tangents at the points \( V_{i_k}, k = 1, \ldots, m \), correspondingly. Let \( \text{peri}_m = \min_{1 \leq i_1 < \ldots < i_m \leq n} \text{peri}_m(V_{i_1}, \ldots, V_{i_m}) \) be minimal of perimeters among all circumscribed \( m \)-polygons with the vertices \( V_{i_1}, \ldots, V_{i_m} \). Then for each \( t > 0 \) we have
\[
\lim_{n \to \infty} P \left\{ n^{2m-1} \left( \text{peri}_m - 2m \tan \frac{\pi}{m} \right) \leq t \right\} = 1 - \exp \left\{ -\frac{K_{3m} t^{m-1}}{\Gamma(m+1)} \right\},
\]
where
\[
K_{3m} = m^{3/2} \left( 2\pi \left( 1 + \tan^2 \frac{\pi}{m} \right) \tan \frac{\pi}{m} \right)^{m-1/2} \Gamma \left( \frac{m+1}{2} \right).
\]
The rate of convergence is \( O \left( n^{-m/3} \right) \).

Clearly, the minimal perimeter is achieved at the regular circumscribed \( m \)-polygon, whence the centering constant \( 2m \tan \frac{\pi}{m} \) appears. The following Lemma 3.1 plays a key role in a proof of Theorem 3.

**Lemma 3.1.** Let choose \( m \) independent and uniformly distributed points \( U_1, \ldots, U_m \) on the unit circle. Draw the tangents at these points on the circle and denote the points of intersection of the tangents by \( V_k, k = 1, \ldots, m \), correspondingly. Consider the convex circumscribed \( m \)-polygon with the vertices \( V_1, \ldots, V_m \), and perimeter \( \text{peri}_m(V_1, \ldots, V_m) \). Then the limit relation holds:
\[
\lim_{s \to +0} s^{-m/2} P \left\{ \text{peri}_m(V_1, \ldots, V_m) \leq 2m \tan \frac{\pi}{m} + s \right\} = \frac{\Gamma(m+1)}{K_{3m}},
\]
where
\[
K_{3m} = m^{3/2} \left( 2\pi \left( 1 + \tan^2 \frac{\pi}{m} \right) \tan \frac{\pi}{m} \right)^{m-1/2} \Gamma \left( \frac{m+1}{2} \right).
\]
Proof of Lemma 3.1. In the sequel denote for brevity \( \text{peri}_m(V_1, \ldots, V_m) := \text{peri}_m \). Consider the angles \( \beta_k, k = 1, \ldots, m - 1 \), which are uniformly distributed on \([0, 2\pi]\), where

\[
\angle U_k O_k = \beta_{k-1}, k = 2, \ldots, m - 1, \quad \angle U_m O_1 = 2\pi - \beta_{m-1}.
\]

From this we get \( \angle U_k O_{k+1} = \beta_k - \beta_{k-1} \). Next we obtain by same arguments as in the proof of Lemma 2.1 that

\[
P\left\{ \text{peri}_m \leq 2m \tan \frac{\pi}{m} + s \right\} = (m - 1)! P\left\{ \text{peri}_m \leq 2m \tan \frac{\pi}{m} + s, \beta_1 \leq \ldots \leq \beta_{m-1} \right\}.
\]

Therefore we may assume that the points \( U_k \) and therefore the points \( V_k \) go in increasing order. Denote the segments of the tangents to the circumference drawn from the point \( V_k \) by:

\[
|U_k V_k| = |V_k U_{k+1}| = a_k, k = 1, \ldots, m - 1, |U_m V_m| = |V_m U_1| =: a_m.
\]

Hence the perimeter of circumscribed \( m \)-polygon is given by

\[
\text{peri}_m = \text{peri}_m(V_1, \ldots, V_m) = 2 \sum_{k=1}^{m} a_k.
\]

Performing some elementary calculations, we obtain

\[
a_k = \tan \left( \frac{\beta_k - \beta_{k-1}}{2} \right), \quad \text{peri}_m = 2 \sum_{k=1}^{m} \tan \left( \frac{\beta_k - \beta_{k-1}}{2} \right).
\]

Analogously to the proof of Lemma 2.1, we pass from the angles \( \beta_k \) to the auxiliary angles \( \alpha_k, k = 1, \ldots, m - 1 \), by (12):

\[
\frac{\beta_k - \beta_{k-1}}{2} = \frac{\pi}{m} + \frac{\alpha_k - \alpha_{k-1}}{2}.
\]

As in the proof of Lemma 1.1, one can show that under the condition

\[
\text{peri}_m \leq 2m \tan \frac{\pi}{m} + s
\]

the sides of our \( m \)-polygon differ from the regular \( m \)-polygon at most at \( O(\sqrt{s}) \).

Without loss of generality we may assume that the largest and the smallest central angles, say \( \angle U_{k-1} O_{k} = \frac{2\pi}{m} + 2\varphi \) and \( \angle U_k O_{k+1} = \frac{2\pi}{m} + 2\varphi, \ \varphi > \psi > 0 \) on Fig. 3.

Fig. 3
adjacent. Then the perimeter of the broken line \( U_{k-1}V_{k-1}U_kV_kU_k \) is equal to \( 2 \tan\left( \frac{\pi}{m} + 2\varphi \right) + 2 \tan\left( \frac{\pi}{m} - 2\psi \right) \). Making the symmetrization, we displace the point \( U_k \) in the center of the arc \( U_{k-1}U_{k+1} \) and consider the resulting new circumscribed \( m \)-polygon.

The perimeter of the broken line \( U_{k-1}V_{k-1}U_kV_kU_k \) is equal to \( 4 \tan\left( \frac{\pi}{m} + \varphi - \psi \right) \), and the difference of two perimeters is \( 2 \sin^2(\varphi + \psi) / \cos(\frac{\pi}{m} + 2\varphi) + \cos(\frac{\pi}{m} - 2\psi) \). It follows from this that \( |\sin(\varphi + \psi)| < C_6 \sqrt{s} \), hence \( \varphi + \psi < C_7 \sqrt{s} \). It follows that the inequalities (14) and (15) take place, possibly with other constants.

Now decompose the tangent of the sum of two angles

\[
\tan\left( \frac{\pi}{m} + \frac{\alpha_k - \alpha_{k-1}}{2} \right) = \frac{\tan\frac{\pi}{m} + \tan\frac{\alpha_k - \alpha_{k-1}}{2}}{1 - \tan\frac{\pi}{m} \tan\frac{\alpha_k - \alpha_{k-1}}{2}}. 
\]

As the differences \( |\alpha_k - \alpha_{k-1}| \) are small and as \( \delta \to 0 \)

\[
\tan \delta = \delta + \frac{1}{3} \delta^3 + O(\delta^4), \quad \frac{1}{1 - \delta} = 1 + \delta + \delta^2 + O(\delta^3),
\]

we obtain, denoting temporarily \( \delta := \frac{\alpha_k - \alpha_{k-1}}{2} \), that

\[
\frac{\tan\frac{\pi}{m} + \tan\frac{\alpha_k - \alpha_{k-1}}{2}}{1 - \tan\frac{\pi}{m} \tan\frac{\alpha_k - \alpha_{k-1}}{2}} = \tan\frac{\pi}{m} + \delta \left( \tan^2\frac{\pi}{m} + 1 \right) + \delta^2 \left( \tan^2\frac{\pi}{m} + 1 \right) \tan\frac{\pi}{m} + O(\delta^3).
\]

By (21) and (22), we get the following expression:

\[
peri_m = 2m \tan\frac{\pi}{m} + \frac{1}{2} (1 + \tan^2\frac{\pi}{m}) \tan\frac{\pi}{m} \sum_{k=1}^{m-1} (\alpha_k - \alpha_{k-1})^2 + O\left( \sum_{k=1}^{m-1} (\alpha_k - \alpha_{k-1})^3 \right).
\]

By means of same reasoning as above we have

\[
P \left\{ \sum_{k=1}^{m-1} \alpha_k^2 - \sum_{k=1}^{m-1} \alpha_k \alpha_{k-1} \leq \frac{s_5 \cot\frac{\pi}{m}}{(1 + \tan^2\frac{\pi}{m})} \right\} \leq \\
\leq P \left\{ peri_m \leq 2m \tan\frac{\pi}{m} + s, \beta_1 \leq \beta_2 \leq \ldots \beta_{m-1} \right\} \leq \\
\leq P \left\{ \sum_{k=1}^{m-1} \alpha_k^2 - \sum_{k=1}^{m-1} \alpha_k \alpha_{k-1} \leq \frac{s_6 \cot\frac{\pi}{m}}{(1 + \tan^2\frac{\pi}{m})} \right\},
\]

where \( s_6 > s_5 \), and \( s_5 \to s, s_6 \to s \) as \( s \to 0 \).

Now denote for brevity

\[
w_k^2 = \frac{s_6 \cot\frac{\pi}{m}}{(1 + \tan^2\frac{\pi}{m}) (1 - \cos\left(\frac{k\pi}{m}\right))}, \quad k = 1, \ldots, m-1.
\]
Using the orthogonal transform, we reduce the quadratic form under the sign of probability to the form (18), and obtain for small $s$:

$$P\{ \text{peri}_m \leq 2m \tan \frac{\pi}{m} + s \} \leq \frac{\Gamma(m)}{(2\pi)^{m-1}} \int_{\sum_{k=1}^{m-1} \{ (1 - \cos \frac{\pi k}{m}) y_k^2 \leq \frac{s_k \cot \frac{s}{m}}{1 + \tan^2 \frac{s}{m}} \}} dy_1 \ldots dy_{m-1} = \frac{\Gamma(m)}{(2\pi)^{m-1}} \int_{\sum_{k=1}^{m-1} \{ y_k/w_k \leq 1 \}} dy_1 \ldots dy_{m-1} = \frac{\Gamma(m)}{(2\pi)^{m-1}} b_{m-1} \Pi''_{m-1}(s_6),$$

where $b_{m-1}$ is from (19), and the product of semi-axes of the new $(m - 1)$-dimensional ellipsoid equals

$$\Pi''_{m-1}(s_6) = \prod_{k=1}^{m-1} w_k = s_6^{\frac{m-1}{2}} \left( \frac{\cot \frac{\pi}{m}}{2 \left( 1 + \tan^2 \frac{\pi}{m} \right)} \right)^{\frac{m-1}{2}} \frac{2^{m-1}}{\sqrt{m}}.$$

Thus, we obtain the following estimate from above:

$$s^{-\frac{m-1}{2}} P\{ \text{peri}_m \geq 2m \tan \frac{\pi}{m} + s \} \leq (s_6/s)^{\frac{m-1}{2}} \frac{\Gamma(m)}{(2\pi)^{m-1}} \left( \frac{\cot \frac{s}{m}}{2 \left( 1 + \tan^2 \frac{s}{m} \right)} \right)^{\frac{m-1}{2}} \frac{2^{m-1}}{\sqrt{m}} \frac{\Gamma \left( \frac{m}{2} + 1 \right)}{\Gamma \left( \frac{m}{2} \right)} = (s_6/s)^{\frac{m-1}{2}} \frac{\Gamma(m)}{\sqrt{m}! \left( \frac{m}{2} + 1 \right)} \left( \frac{\cot \frac{s}{m}}{2 \pi \left( 1 + \tan^2 \frac{s}{m} \right)} \right)^{\frac{m-1}{2}}.$$

Taking the limit as $s \to 0$, we obtain the upper estimate for the probability of interest. The lower estimate can be obtained analogously. The conclusion of Lemma 3.1 follows.

□

**Proof of Theorem 3.** The proof is carried out similarly to the proof of Theorem 1 and Theorem 2 by means of the transformation

$$z_n(t) = 2m \tan \frac{\pi}{m} + t n^{2m-r}.$$

In this case we verify the condition (6) for each $r \in \{1, \ldots, m - 1\}$:

$$\lim_{n \to \infty} n^{2m-r} P\{ \text{peri}^{m-1} \cdots \text{peri}^{m} < z_n(t), \text{peri}^{m+1-r} \cdots \text{peri}^{2m-r} < z_n(t) \} = 0,$$

which can be proved analogously to Lemma 2.2.

Basing on the conclusion of Silverman and Brown’s Theorem (5) and considering the estimate (2) from Lao-Mayer Theorem, which can be reformulated in terms of $U$-min-statistic by replacement $h$ on $-h$, i.e. $H_n = - \min(-h_n)$, we have:

$$\lim_{n \to \infty} P \left\{ \text{peri}^{m} > 2m \tan \frac{\pi}{m} + t n^{2m-r} \right\} = \exp \left\{ - \frac{t \frac{m-1}{2}}{K_{3m}} \right\}.$$

Thus for each $t > 0$

$$\lim_{n \to \infty} P \left\{ n^{2m-r} (\text{peri}^{m} - 2m \sin \frac{\pi}{m}) \leq t \right\} = 1 - \exp \left\{ - \frac{t \frac{m-1}{2}}{K_{3m}} \right\},$$
where

\[ K_{3m} = m^3 \left( 2\pi \left( 1 + \tan^2 \frac{\pi}{m} \right) \tan \frac{\pi}{m} \right)^{m-1} \Gamma \left( \frac{m+1}{2} \right). \]

The next result is a corollary of Theorem 3.

3.2. Area of circumscribed polygon. Theorem 4. Let \( U_1, U_2, \ldots \) be independent and uniformly distributed points on the circumference of unit radius \( \mathbb{S} \). For each set of the points \( U_1, U_2, \ldots, U_m \) on the circle draw tangents at these points, denoting intersection of the tangents at the points \( U_k \) and \( U_{k+1} \) by \( V_{ki}, k = 1, \ldots, m \), correspondingly. Let \( \text{area}_m = \min_{1 \leq i_1 < \ldots < i_m \leq n} \text{area}_m(V_1, \ldots, V_m) \) be minimal of areas among all circumscribed \( m \)-polygons, formed by points \( V_1, \ldots, V_m \). Then for each \( t > 0 \)

\[ \lim_{n \to \infty} P \left\{ n^{m-1} \left( \text{area}_m - m \tan \frac{\pi}{m} \right) \leq t \right\} = 1 - \exp \left\{ -\frac{t^{m-1}}{K_{4m}} \right\}, \]

where

\[ K_{4m} = m^3 \left( \pi \left( 1 + \tan^2 \frac{\pi}{m} \right) \tan \frac{\pi}{m} \right)^{m-1} \Gamma \left( \frac{m+1}{2} \right). \]

The rate of convergence is \( O \left( n^{-\frac{m-1}{2}} \right) \).

Proof of Theorem 4. Note that the minimal area and the minimal perimeter of circumscribed \( m \)-polygon in the case of unit circle are related by the equality \( \text{area}_m = \frac{1}{2} \text{peri}_m^m \).

Hence the following chain of equalities follows from Theorem 3:

\[ \lim_{n \to \infty} P \left\{ n^{m-1} (\text{peri}_m^m - 2m \tan \frac{\pi}{m}) \leq t \right\} = \lim_{n \to \infty} P \left\{ n^{m-1} (2 \text{area}_m - 2m \tan \frac{\pi}{m}) \leq t \right\} = \lim_{n \to \infty} P \left\{ n^{m-1} (\text{area}_m - m \tan \frac{\pi}{m}) \leq t/2 \right\}. \]

On the other hand, by Theorem 3 the right-hand side of the last equality is

\[ 1 - \exp \left\{ -\frac{t^{m-1}}{K_{3m}} \right\} = 1 - \exp \left\{ -\frac{(t/2)^{m-1}}{2^{-\frac{m-1}{2}} K_{3m}} \right\} = 1 - \exp \left\{ -\frac{(t/2)^{m-1}}{K_{4m}} \right\}, \]

where \( K_{4m} = 2^{-\frac{m-1}{2}} K_{3m} \), whence we get the conclusion of Theorem 4. \qed

4. Asymptotic behavior of constants

In this section we investigate the asymptotics of the limit constants \( K_{im}, i = 1, 4 \), as \( m \to \infty \), using the well-known Stirling’s formula: as \( t \to +\infty \)

\[ \Gamma(t+1) \sim \sqrt{2\pi t} \left( \frac{t}{e} \right)^t, \quad t \to +\infty. \]

The constants from previous sections are as follows:

\[ K_{1m} = m^3 \left( \pi \sin \frac{\pi}{m} \right)^{\frac{m-1}{2}} \Gamma \left( \frac{m+1}{2} \right), \]
\[ K_{2m} = m^3 \left( \pi \sin \frac{2\pi}{m} \right)^{\frac{m-1}{2}} \Gamma \left( \frac{m+1}{2} \right) = (2 \cos \frac{\pi}{m})^{\frac{m-1}{2}} K_{1m}, \]
\[ K_{3m} = m^3 \left( 2\pi \left( 1 + \tan^2 \frac{\pi}{m} \right) \tan \frac{\pi}{m} \right)^{\frac{m-1}{2}} \Gamma \left( \frac{m+1}{2} \right) = 2^{\frac{m-1}{2}} K_{4m}, \]
\[ K_{4m} = m^3 \left( \pi \left( 1 + \tan^2 \frac{\pi}{m} \right) \tan \frac{\pi}{m} \right)^{\frac{m-1}{2}} \Gamma \left( \frac{m+1}{2} \right). \]
Applying Stirling’s formula, we get

\[
\Gamma\left(\frac{m+1}{2}\right) \sim \frac{\sqrt{2\pi \left(\frac{m-1}{2}\right) \left(\frac{m-1}{2}\right)^{\frac{m-1}{2}}}}{e^{\frac{m}{2}}} = \frac{\sqrt{\pi (m-1)}^{\frac{m}{2}}}{(2e)^{\frac{m-1}{2}} e^{\frac{m}{2}}} = \frac{\sqrt{\pi m}^{\frac{m}{2}}}{2^{\frac{m-1}{2}} e^{\frac{m}{2}}}, \quad m \to \infty.
\]

The following relations clearly hold as \(m \to \infty\):

\[\sin \frac{\pi}{m} \sim \frac{\pi}{m}, \quad \tan \frac{\pi}{m} \sim \frac{\pi}{m}, \quad 1 + \tan^2 \frac{\pi}{m} \sim 1.\]

From here we easily deduce as \(m \to \infty\):

\[
K_{1m} = m^{\frac{3}{2}} \left(\pi \sin \frac{\pi}{m}\right)^{\frac{m-1}{2}} \Gamma \left(\frac{m+1}{2}\right) \sim m^{\frac{3}{2}} \left(\pi \cdot \frac{\pi}{m}\right)^{\frac{m-1}{2}} \frac{\sqrt{\pi m}^{\frac{m}{2}}}{2^{\frac{m-1}{2}} e^{\frac{m}{2}}} = \frac{m^{\frac{3}{2}} m^{\frac{m}{2}}}{2^{\frac{m-1}{2}} e^{\frac{m}{2}}} := \tilde{K}_{1m},
\]

\[
K_{2m} = (2 \cos \frac{\pi}{m})^{\frac{m-1}{2}} K_{1m} = m^{\frac{3}{2}} \left(\pi \sin \frac{2\pi}{m}\right)^{\frac{m-1}{2}} \Gamma \left(\frac{m+1}{2}\right) \sim \frac{m^{\frac{3}{2}} m^{\frac{m-1}{2}}}{2^{\frac{m-1}{2}} e^{\frac{m}{2}}} := \tilde{K}_{2m} = 2^{\frac{m-1}{2}} \tilde{K}_{1m},
\]

\[
K_{3m} = m^{\frac{3}{2}} \left(2 \pi \left(1 + \tan^{2} \frac{\pi}{m}\right) \tan \frac{\pi}{m}\right)^{\frac{m-1}{2}} \Gamma \left(\frac{m+1}{2}\right) \sim \frac{m^{\frac{3}{2}} m^{\frac{m}{2}}}{2^{\frac{m}{2}} e^{\frac{m}{2}}} := \tilde{K}_{3m} = \tilde{K}_{2m},
\]

\[
K_{4m} = 2^{\frac{m-1}{2}} \tilde{K}_{3m} \sim \frac{m^{\frac{3}{2}} m^{\frac{m}{2}}}{2^{\frac{m-1}{2}} e^{\frac{m}{2}}} := \tilde{K}_{4m} = \tilde{K}_{1m}.
\]

In total:

\[
\tilde{K}_{2m} = \tilde{K}_{3m} = 2^{\frac{m-1}{2}} \tilde{K}_{1m} = 2^{\frac{m-1}{2}} \tilde{K}_{4m} = \pi^{\frac{3}{2}} m^{\frac{m}{2}} e^{\frac{m}{2}}.
\]

Hence the asymptotic analysis shows the coincidence of the asymptotic constants for the maximal area of inscribed random polygon and for the minimal perimeter of circumscribed random polygon \(\tilde{K}_{2m} = \tilde{K}_{3m}\), and also their coincidence for the maximal perimeter of inscribed random polygon and for the minimal area of circumscribed random polygon. This seems to be the unexpected and curious observation. It would be interesting to understand if this follows from some general fact.

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