WEAK LIMITS OF FRACTIONAL SOBOLEV HOMEOMORPHISMS ARE ALMOST INJECTIVE: A NOTE

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Abstract. Let $\Omega \subset \mathbb{R}^n$ be an open set and $f_k \in W^{s,p}(\Omega; \mathbb{R}^n)$ be a sequence of homeomorphisms weakly converging to $f \in W^{s,p}(\Omega; \mathbb{R}^n)$. It is known that if $s = 1$ and $p > n - 1$ then $f$ is injective almost everywhere in the domain and the target. In this note we extend such results to the case $s \in (0,1)$ and $sp > n - 1$. This in particular applies to $C^s$-Hölder maps.

1. Introduction and main result

The goal of this note is to prove the following theorem:

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be open and let $f : \Omega \to \mathbb{R}^n$ be a weak $W^{s,p}$-limit of Sobolev homeomorphisms $f_j \in W^{s,p}(\Omega; \mathbb{R}^n)$ with $sp > n - 1$. Then there is a representative $\hat{f}$ and a set $\Gamma \subset \mathbb{R}^n$ of Hausdorff dimension $\frac{n-1}{s}$ such that $(\hat{f})^{-1}(y)$ consists of only one point for every $y \in \hat{f}(\Omega) \setminus \Gamma$.

For definitions we refer to the next section. An immediate corollary of Theorem 1.1 and the embedding $C^s \hookrightarrow W^{s-\varepsilon,p}_{loc}$ for any $\varepsilon > 0$ is the following statement for Hölder maps.

**Corollary 1.2.** Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be open and let $f \in C^s(\Omega; \mathbb{R}^n)$ be the pointwise limit of a sequence of equibounded homeomorphisms $f_j \in C^s(\Omega; \mathbb{R}^n)$. If $s > \frac{n}{n-1}$, then there is a set $\Gamma \subset \mathbb{R}^n$ of Hausdorff dimension $\frac{n-1}{s}$ such that $(f)^{-1}(y)$ consists of only one point for every $y \in f(\Omega) \setminus \Gamma$.

Observe that for $s \leq \frac{n-1}{n}$ the above statements hold trivially.

This note is inspired by the recent work by Bouchala, Hencl, and Molchanova [4] who proved a corresponding result for $s = 1$.

**Theorem 1.3** (Bouchala, Hencl, Molchanova). Let $f : \Omega \to \mathbb{R}^n$ be a weak limit of Sobolev homeomorphisms $f_j \in W^{1,p}(\Omega; \mathbb{R}^n)$ with $p > n - 1$. Then there is a representative $\hat{f}$ and a set $\Gamma \subset \mathbb{R}^n$ of Hausdorff dimension $n - 1$ such that $(\hat{f})^{-1}(y)$ consists of only one point for every $y \in \hat{f}(\Omega) \setminus \Gamma$.

While Theorem 1.3 (and in turn our Theorem 1.1) follows an adaptation of the arguments in the seminal work by Müller and Spector [11], Bouchala, Hencl, and Molchanova [4] also provide an example of the limit case $p = n - 1$, where a theorem such as Theorem 1.3 completely fails. Namely they showed

**Theorem 1.4** (Bouchala, Hencl, Molchanova). For $n \geq 3$ there exists $f : [-1,1]^n \to [-1,1]^n$ and a strong limit of Sobolev homeomorphisms $f_k \in W^{1,n-1}([-1,1]^n, \mathbb{R}^n)$ with $f_k(x) = x$ on the boundary $\partial[-1,1]^n$ and such that there exists a set $\Gamma \subset \mathbb{R}^n$ such that $(\hat{f})^{-1}(y)$ consists of only one point for every $y \in \hat{f}(\Omega) \setminus \Gamma$.
$[-1,1]^n$ of positive Lebesgue measure and $f^{-1}(y)$ is a nontrivial continuum for every $y \in \Gamma$.

As the authors of [4] mention, it may seem surprising that the Hausdorff dimension of the critical set $\Gamma$ seems to suddenly jump from $n - 1$ to $n$ as $p$ changes from $p > n - 1$ to $p = n - 1$. This question served as one motivation to study the situation for fractional Sobolev spaces.

Let us stress that Theorem 1.1 follows a very similar argument as the $s = 1$ proof of Theorem 1.3 in [4], which in turn is a streamlined argument of known results and techniques from earlier works, see [3,11,12]. Indeed, a crucial fact that is used for $s = 1$ is that on “good slices” $\partial B_r$ the $f_k$ converge in $W^{1,p}(\partial B_r)$, and so using Sobolev-Morrey embedding on these $n-1$-dimensional slices the $f_k$ in fact converge uniformly if $p > n - 1$. If $p = n - 1$ this uniform convergence may fail.

The same is true if the $f_k$ converge in $W^{s,p}(\partial B_r)$ for good slices $\partial B_r$ and $s \in (0,1)$: if $sp > n - 1$ then the convergence is uniform on $\partial B_r$, and if $sp = n - 1$ it may not.

But somewhat surprisingly, a result such as Theorem 1.1 and in particular Corollary 1.2 seems to be unknown to some experts, and the authors thought it important to be available in the literature.

We try to keep this note as self-contained as possible. In Section 2 we gather the main results on Sobolev spaces that we work with. In Section 3 we discuss the needed notions of degree, and show monotonicity of the degree for limits of homeomorphisms. In Section 4, we collect the corollaries for the topological image from the previous section. In Section 5 we prove our main theorem.

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2. Preliminaries on Sobolev spaces, capacities etc.

In this section we establish notation. For $s \in (0,1)$ and $p \in (1, \infty)$ we denote the classes of functions $u : \Omega \to \mathbb{R}^n$ for which the Gagliardo seminorm

\begin{equation}
[u]_{W^{s,p}(\Omega)}^p := \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} \, dy \, dx
\end{equation}

is finite as the fractional Sobolev spaces $W^{s,p}(\Omega; \mathbb{R}^n)$, with norm $\|u\|_{W^{s,p}(\Omega)} := \|u\|_{L^p(\Omega)} + [u]_{W^{s,p}(\Omega)}^p$. We denote the $n$-dimensional Lebesgue measure of a set $A \subset \mathbb{R}^n$ by $\mathcal{L}^n(A)$, and for $\beta > 0$ we denote the $\beta$-dimensional Hausdorff measure by $\mathcal{H}^\beta(A)$. We use the convention $A \preceq B$ whenever there exists a constant $C$ such that $A \leq C B$.

Define the precise representative of a measurable function $f$ by

\begin{equation}
f^*(x) := \begin{cases}
\lim_{r \to 0^+} \int_{B_r(x)} f(y) \, dy, & \text{when the limit exists}, \\
0, & \text{otherwise}.
\end{cases}
\end{equation}
Many properties of the precise representative for functions in the Bessel potential spaces are accessible in the literature. The corresponding statements can then be obtained for fractional Sobolev functions via embedding theorems for the Triebel-Lizorkin spaces \( F^{s,p}_{q,q} \); see [14]. For completeness, we gather here a summary of the statements we will need.

We denote the Bessel potential spaces \( H^{s,p} \) by
\[
(2.3) \quad H^{s,p}(\mathbb{R}^n; \mathbb{R}^n) := \left\{ f: \mathbb{R}^n \to \mathbb{R}^n : \|F^{-1}((1 + |\xi|^2)^{s/2}(Ff)(\xi))\|_{L^p(\mathbb{R}^n)} < \infty \right\},
\]
where \( F \) and \( F^{-1} \) denote the Fourier transform and its inverse respectively. The following is a corollary of a classical embedding theorem for the space \( F^{s,p}_{q,q} \); see [14, Section 2.2.3], [15, Theorem 2.14, Remark 2.4]. We are additionally using the identifications \( F^{s,p}_{2,2} = H^{s,p} \) and \( F^{s,p}_{p,p} = W^{s,p} \).

**Theorem 2.1.** Let \( N \geq 1 \). Let \( p \in (1, \infty) \) and \( s \in (0, 1) \), and suppose that \( t \in (0,1) \) and \( p_t \in (1, \infty) \) satisfy
\[
(2.4) \quad W^{s,p}(\mathbb{R}^N) \hookrightarrow H^{t,p_t}(\mathbb{R}^N), \quad \text{or} \quad [f]_{H^{t,p_t}(\mathbb{R}^N)} \lesssim [f]_{W^{s,p}(\mathbb{R}^N)}.
\]

Note that if we write the definition of \( p_t \) as
\[
(2.5) \quad sp - N = \frac{p}{p_t}(tp_t - N),
\]
then it becomes clear that if \( sp > N \) then \( tp_t > N \) for any \( t \in (0, s) \).

With this embedding we can prove some useful properties of the precise representative:

**Proposition 2.2.** Suppose \( f \in W^{s,p}(\mathbb{R}^n; \mathbb{R}^n) \) with \( sp \in [1,n) \). Let \( p^* = \frac{np}{n-sp} \).

Define
\[
(2.6) \quad A_{sp} := \{ x \in \mathbb{R}^n : x \text{ is not a Lebesgue point of } f \}.
\]

Then the following hold:

(i) \( \dim_{\mathcal{H}}(A_{sp}) \leq n - sp \).

(ii) For any \( x \in \mathbb{R}^n \setminus A_{sp} \),
\[
(2.7) \quad \lim_{r \to 0^+} \frac{1}{B_r(x)} \int_{B_r(x)} |f(y) - f^*(x)|^q \, dy = 0,
\]
for every \( q \in [1,p^*) \).

(iii) If \( \varphi \) is the family of standard mollifiers then
\[
\varphi \ast f(x) \to f^*(x)
\]
for each \( x \in \Omega \setminus A_{sp} \).

**Proof.** Let \( \varepsilon > 0 \) be arbitrary; we will show that
\[
(2.8) \quad \mathcal{H}^{n-sp+\varepsilon}(A_{sp}) = 0,
\]
which will imply (i). We use Theorem 2.1 with \( N = n \); choose \( t \in (0,s) \) so that
\[
n - tp_t = n - sp + \varepsilon;
\]
this is possible since by definition \( n - tp_t > n - sp \) for \( sp \in [1, n) \) and for any \( t \in (0, s) \). Then \( f \in H^{l,p_l}(\mathbb{R}^n; \mathbb{R}^n) \) and so \([1, \text{Proposition } 6.1.2, \text{Theorem } 5.1.13]\) implies \( \mathcal{H}^\beta(A_{sp}) = 0 \) for all \( \beta \geq n - tp_t = n - sp + \varepsilon \), and so (2.8) is established.

To see (ii), use Theorem 2.1 with \( N = n \) again; note that any \( q \in (p, p_s) \) can be written \( q = p_t \) for some \( t \in (0, s) \). Then \( f \in H^{l,p_l}(\mathbb{R}^n; \mathbb{R}^n) \) for every \( t \in (0, s) \), and so \([1, \text{Theorem } 6.2.1]\) applies, which is precisely (ii). We obtain (2.7) for the range \( q \in [1, p] \) using Hölder’s inequality.

For a proof of (iii) see [6, Theorem 4.1, (iv)].

**Lemma 2.3.** Suppose \( f \in W^{s,p}(\mathbb{R}^n; \mathbb{R}^n) \) with \( sp \in [1, n) \), and suppose \( f^*(x) \in E \) for every \( x \in \mathbb{R}^n \setminus M \), where \( \mathcal{L}^n(M) = 0 \) and \( E \subset \mathbb{R}^n \) is a closed set. Then \( f^*(x) \in E \) for every \( x \in \mathbb{R}^n \setminus A_{sp} \).

**Proof.** Suppose to the contrary, that \( f^*(x) \in \mathbb{R}^n \setminus E \) for some \( x \in \mathbb{R}^n \setminus A_{sp} \). Then there exists \( \varepsilon > 0 \) such that \( B(f^*(x), \varepsilon) \subset \mathbb{R}^n \setminus E \). By assumption that \( f^*(y) \in E \) for \( y \in \mathbb{R}^n \setminus M \)

\[
\int_{B(x,r)} |f(y) - f^*(x)|^p \, dy = \int_{B(x,r) \setminus M} |f(y) - f^*(x)|^p \, dy \geq \varepsilon^p
\]

uniformly as \( r \to 0 \), which is a contradiction since \( f^* \) satisfies (2.7) for every \( x \in \mathbb{R}^n \setminus A_{sp} \).

We will need information on the Hausdorff dimension of images of spheres embedded in \( \mathbb{R}^n \). The following is a special case of such a result in \([9]\) for Bessel potential functions, which will then apply to functions in \( W^{s,p} \) via Theorem 2.1:

**Proposition 2.4 (\([9]\), Theorem 1.1).** Let \( N, K \in \mathbb{N}, t \in (0, 1) \) and \( q \in (1, \infty) \) with \( tq > N \) and \( \alpha \in (0, N] \). Define \( \beta := \frac{\alpha q}{tq - N + \alpha} \). Suppose \( g \in H^{t,\alpha}(\mathbb{R}^N; \mathbb{R}^K) \) is a continuous representative and \( A \subset \mathbb{R}^N \) is a set with \( \dim_{\mathcal{H}}(A) \leq \alpha \). Then \( \dim_{\mathcal{H}}(g(A)) \leq \beta \).

We then have as a corollary

**Theorem 2.5.** Let \( n \geq 2, s \in (0, 1), \) and \( p > 1 \) with \( n - 1 < sp < n \). Let \( r > 0 \), \( a \in \mathbb{R}^n \) with \( \partial B \equiv \partial B(a, r) \) and \( g \in W^{s,p}(\partial B; \mathbb{R}^n) \) be a continuous representative. Then \( \dim_{\mathcal{H}}(g(\partial B)) \leq \frac{n-1}{s} \).

**Proof.** It suffices to show that

\[
\mathcal{H}^{\frac{n-1}{s} + \varepsilon}(g(\partial B)) = 0
\]

for arbitrary \( \varepsilon > 0 \) small. Cover \( \partial B \) by sets \( S_i \) diffeomorphic to \( \mathbb{R}^{n-1} \) \((2^n \text{ hemispheres will do})\), and let \( \psi_i : \mathbb{R}^{n-1} \to S_i \) be the corresponding diffeomorphisms. So \( \partial B \subset \bigcup_{i=1}^M S_i \), and the functions

\[
g_i := g \circ \psi_i
\]

belong to \( W^{s,p}(\mathbb{R}^{n-1}; \mathbb{R}^n) \), and hence belong to \( H^{t,p_l}(\mathbb{R}^{n-1}; \mathbb{R}^n) \) by Theorem 2.1 for any \( t \in (s - \frac{n-1}{p}, s) \) and for \( p_l = \frac{(n-1)p}{(n-1)-(s-1)p} \).

Applying Proposition 2.4 to each \( g_i \) with \( q = p_l \) and \( N = \alpha = n - 1 \) gives

\[
\mathcal{H}^\gamma(g_i(\mathbb{R}^{n-1})) = 0, \quad \text{for every } \gamma > \frac{n - 1}{t}, \quad i = \{1, \ldots, M\}.
\]
Choose \( t < s \) close enough to \( s \) so that \( \frac{1}{t} < \frac{1}{s} + \varepsilon \). Then
\[
\mathcal{H}^{\frac{n-1}{t} + \varepsilon}(g(\partial B)) \leq \sum_{i=1}^{M} \mathcal{H}^{\frac{n-1}{s} + \varepsilon}(g(S_i)) = \sum_{i=1}^{M} \mathcal{H}^{\frac{n-1}{s} + \varepsilon}(g(\mathbb{R}^{n-1})) = 0,
\]
as desired. \( \square \)

Throughout this note we additionally require control of fractional Sobolev functions on spheres in \( \mathbb{R}^n \). In the local case, this control is obtained straightforwardly; for example, using Fubini’s theorem for a smooth function \( f \) on \( B(a, r) \)
\[
\int_0^r \int_{\partial B(a, \rho)} |\nabla f(\rho \omega)|^p \, d\mathcal{H}^{n-1}(w) \, d\rho \leq \int_{B(a, r)} |\nabla f(x)|^p \, dx,
\]
where \( \nabla f \) denotes the tangential derivative of \( f|_{\partial B(a, \rho)} \). The following Besov-type inequality serves as a fractional analogue:

**Lemma 2.6.** Let \( B(a, r) \subset \mathbb{R}^n \), with \( p \in [1, \infty) \) and \( s \in (0, 1) \). Then there exists a constant \( C = C(n, s, p) \) such that for every \( f \in W^{s,p}(B(a, r); \mathbb{R}^n) \)
\[\begin{align*}
\int_{r/2}^r \int_{\partial B(a, \rho)} \frac{|f(x) - f(y)|^p}{|x - y|^{n-1+sp}} \, d\mathcal{H}^{n-1}(y) \, d\mathcal{H}^{n-1}(x) \, d\rho &\leq C[f]_{W^{s,p}(B(a, r))}.
\end{align*}\]

These types of estimates are well-known to experts (see for example [5]), but for the sake of completeness we have included the proof in the appendix, see Appendix A. The following corollary to the lemma reveals finer properties of Sobolev functions:

**Corollary 2.7.** Let \( 1 < sp < n \), let \( x_0 \in \Omega \subset \mathbb{R}^n \), and suppose \( f \in W^{s,p}(\Omega; \mathbb{R}^n) \). Then there exists a constant \( C = C(n, s, p) \) such that for every \( r \in (0, \text{dist}(x_0, \partial \Omega)) \)
\[\begin{align*}
\int_{\partial B(x_0, r)} \frac{|f^* - f^*|_{W^{s,p}(\partial B(x_0, r))}}{|x - x_0|^{n-1+sp}} \, d\mathcal{H}^{n-1}(x) \, d\rho &\leq C[f^* - f^*]_{W^{s,p}(\partial B(x_0, r))}.
\end{align*}\]

**Proof.** For \( \varepsilon > 0 \) let \( \varphi^\varepsilon \) be the standard mollifier, and let \( f^\varepsilon := \varphi^\varepsilon \circ f^* \). Then \( f^\varepsilon \) converges to \( f^* \) in \( W^{s,p}(B(x_0, r)) \) for any \( r \in (0, \text{dist}(x_0, \partial \Omega)) \), and by Lemma 2.6
\[\begin{align*}
\int_{r/2}^r \int_{\partial B(x_0, \rho)} \frac{|f^\varepsilon - f^*|_{W^{s,p}(\partial B(x_0, \rho))}}{|x - x_0|^{n-1+sp}} \, d\mathcal{H}^{n-1}(x) \, d\rho &\leq C[f^\varepsilon - f^*]_{W^{s,p}(\partial B(x_0, r))} \to 0 \text{ as } \varepsilon \to 0.
\end{align*}\]

Thus for \( L^1 \)-almost every \( r \in (0, \text{dist}(x_0, \partial \Omega)) \) we have that the smooth functions \( f^\varepsilon|_{\partial B(x_0, r)} \) converge to a function \( g_r \in W^{s,p}(\partial B(x_0, r)) \). On the other hand, Proposition 2.2 applies to \( f \) since we can find a Sobolev extension domain \( K \) satisfying \( B(x_0, r) \subset K \subset \Omega \). Thus since \( sp > 1 \) we have from Proposition 2.2(iii) that for every \( r \in (0, \text{dist}(x_0, \partial \Omega)) \)
\[\begin{align*}
f^\varepsilon(x) \to f^*(x) \text{ on } B(x_0, r) \setminus A_{sp}, \text{ where } \mathcal{H}^{n-1}(A_{sp}) = 0.
\end{align*}\]

Therefore for \( L^1 \)-almost every \( r \in (0, \text{dist}(x_0, \partial \Omega)) \) the functions \( f^\varepsilon|_{\partial B(x_0, r)}(x) \) converge to \( f^*(x) \) for \( \mathcal{H}^{n-1} \)-almost every \( x \in \partial B(x_0, r) \). So for \( L^1 \)-almost every \( r \in (0, \text{dist}(x_0, \partial \Omega)) \) the function \( f^*|_{\partial B(x_0, r)} \) agrees with \( g_r \) up to a set of \( \mathcal{H}^{n-1} \)-measure zero, hence \( f^*|_{\partial B(x_0, r)} \) belongs to \( W^{s,p}(\partial B(x_0, r)) \).

Now if \( sp > n - 1 \), then \( f^\varepsilon \to g \), locally uniformly on \( \partial B(x_0, r) \) by the Sobolev compact embedding theorem (see for example [14, Theorem 2, pg. 82], [16, Lemma 41.4]), and additionally \( \mathcal{H}^1(A_{sp}) = 0 \). Therefore for \( L^1 \)-almost every \( r \in (0, \text{dist}(x_0, \partial \Omega)) \)
the sequence \(f^\varepsilon(x)\) converges to \(f^*(x)\) for every \(x \in \partial B(x_0, r)\), and so \(f^*(x)\) agrees with the continuous function \(g_\varepsilon(x)\) for every \(x \in \partial B(x_0, r)\).

\[\square\]

The following is an adaptation of [10, Proposition 3.1], which in turn is an extension of an argument in [17].

**Proposition 2.8.** Let \(\Omega \subset \mathbb{R}^n\), \(s \in (0, 1)\) and \(p \in (1, \infty)\) with \(n - 1 < sp < n\). Assume that \(f \in W^{s,p}(\Omega; \mathbb{R}^n)\) satisfies the following: for any \(x_0 \in \Omega\) there exists a set \(N_{x_0}\) satisfying \(\mathcal{L}^1(N_{x_0}) = 0\) such that for all radii \(r, \rho \in (0, \text{dist}(x_0, \Omega))\) \(\setminus N_{x_0}\) with \(r < \rho\), there holds for some \(\Lambda \geq 1\) independent of \(r, \rho\) and \(x_0\)

\[
\text{osc}_{\partial B(x_0, r)} f^* \leq \Lambda \text{osc}_{\partial B(x_0, \rho)} f^*,
\]

where \(f^*\) is the continuous representative of \(f\) defined in (2.2). Then there exists a singular set \(\Sigma \subset \Omega\) with \(\mathcal{H}^{(n-sp)+}(\Sigma) = 0\) such that \(f^*\) is continuous on \(\Omega \setminus \Sigma\).

**Proof.** Without loss of generality assume \(sp < n\). The case \(n = sp\) can be found in [10, Proposition 3.1], and \(n < sp\) is obvious by Morrey-Sobolev embedding; see [13].

By Corollary 2.7 for any \(R > 0\) with \(B(x_0, R) \subset \Omega\) and \(\mathcal{L}^1\)-almost any \(r < \rho < R\), the function \(f^*|_{\partial B(x_0, \rho)}\) belongs to \(W^{s,p}(B(x_0, \rho))\). As in [10, Proposition 3.1], by Morrey-Sobolev embedding

\[
(\text{osc}_{\partial B(x_0, r)} f^*)^p \leq \Lambda (\text{osc}_{\partial B(x_0, \rho)} f^*)^p \leq C \rho^{sp-(n-1)} \int_{\partial B(x_0, \rho)} \int_{\partial B(x_0, \rho)} \frac{|f^*(x) - f^*(y)|^p}{|x - y|^{(n-1)+sp}} \, dx \, dy.
\]

Multiplying by \(\rho^{-sp-(n-1)}\) and integrating in \(\rho\) we obtain using Lemma 2.6

\[
c(s, p) \left( R^{n-sp} - r^{n-sp} \right) (\text{osc}_{\partial B(x_0, r)} f^*)^p \leq [f^*]^p_{W^{s,p}(B(x_0, R))}.
\]

In particular we have

\[
(2.10) \quad (\text{osc}_{\partial B(x_0, r)} f^*)^p \leq R^{sp-n} [f^*]^p_{W^{s,p}(B(x_0, R))}
\]

for any \(R \in (0, \text{dist}(x_0, \partial \Omega))\) and for every \(r \in (0, R/2) \setminus N_{x_0}\). Let

\[
(2.11) \quad X := \left\{ x \in \Omega : \limsup_{R \to 0^+} R^{sp-n} [f^*]^p_{W^{s,p}(B(x, R))} > 0 \right\}.
\]

By Frostman’s Lemma (see [18, Corollary 3.2.3]) we have that \(\mathcal{H}^{(n-sp)+}(X) = 0\).

Define \(\Sigma = A_{sp} \cup X\), where \(A_{sp}\) is defined in Proposition 2.2. Let \(x_0 \in \Omega \setminus \Sigma\) and let \(\varepsilon > 0\). Observe that if for some \(R > 0\)

\[
R^{sp-n} [f^*]^p_{W^{s,p}(B(x_0, R))} < \varepsilon
\]

do not hold for some \(y_0 \in B(x_0, R/2)\),

\[
(R/2)^{sp-n} [f^*]^p_{W^{s,p}(B(y_0, R/2))} < C_{sp,n} \varepsilon.
\]

That is, from (2.10) and the definition of \(X\) there must be some \(R = R(x_0, \varepsilon) > 0\) such that

\[
(2.12) \quad \sup_{r \in (0, R/2) \setminus N_{x_0}} \text{osc}_{\partial B(y_0, r)} f^* < \varepsilon \quad \forall y_0 \in B(x_0, R/2).
\]

This implies

\[
(2.13) \quad \text{osc}_{B(x_0, R/4)} f^* < 2\varepsilon,
\]
which is continuity. To see (2.13), without loss of generality let \( x_0 = 0 \). Let \( x \) and \( y \) be any two points in \( B(0, R/4) \setminus \{0\} \). Then there exist \( r \in (0, |x|) \) and \( t \in (0, |y|) \) such that

\[
osc_{\partial B}(r, \frac{x}{|x|} |x| - r)f^* < \varepsilon, \quad osc_{\partial B}(t, \frac{y}{|y|} |y| - t)f^* < \varepsilon, \quad \text{and} \quad \partial B\left(\frac{x}{|x|} |x| - r\right) \cap \partial B\left(\frac{y}{|y|} |y| - t\right) \neq \emptyset.
\]

If this is not the case, then by (2.12) and since the maps \( r \mapsto \partial B(rx/|x|, |x| - r) \) and \( t \mapsto \partial B(ty/|y|, |y| - t) \) are continuous it follows that some open interval must reside within the set \( N_{x_0} \), a contradiction. Now let \( z \in B(0, R/4) \) be a point in the intersection; we have

\[
|f^*(x) - f^*(y)| \leq |f^*(x) - f^*(z)| + |f^*(z) - f^*(y)| \leq osc_{\partial B}(r, \frac{x}{|x|} |x| - r)f^* + osc_{\partial B}(t, \frac{y}{|y|} |y| - t)f^* < 2\varepsilon.
\]

This holds for any \( x \) and \( y \) not equal to zero. If one of the two points is the center of \( B(0, R/4) \) (without loss of generality \( y = 0 \)) then repeat the argument with the set \( \partial B(t\frac{x}{|x|}, t) \) for \( t \in (0, |x|) \) in place of \( \partial B(t\frac{x}{|x|}, |y| - t) \). Thus (2.13) is proved.

\[ \square \]

3. Degree and Monotonicity estimates

Let \( B = B(x_0, r) \subset \mathbb{R}^n \) and let \( f : \partial B \to \mathbb{R}^n \) be continuous. For \( p \notin f(\partial B) \) define the degree

\[
\deg(f, \partial B, p) := \deg_{S^{n-1}}(\psi)
\]

where

\[
\psi := \frac{f \left( \frac{x - x_0}{r} \right) - p}{\left| f \left( \frac{x - x_0}{r} \right) - p \right|} : S^{n-1} \to S^{n-1}
\]

and \( \deg_{S^{n-1}} \) computes the homotopy group of \( \psi \) in \( \pi_{n-1}(S^{n-1}) = \mathbb{Z} \).

The main topological ingredient is the following lemma (which is well-known). Items (i) and (iii) are essentially a rewritten version of [4, Lemma 5.1], and (ii) is a consequence of (i) motivated by [7, 10, 17].

**Lemma 3.1.** Let \( \Omega \subset \mathbb{R}^n \) be an open set.

Assume that \( B_1 := B(x_1, r_1) \) and \( B_2 := B(x_2, r_2) \subset \subset \Omega \) are two open balls and \( f, f_k : \partial B_1 \cup \partial B_2 \to \mathbb{R}^n \) be continuous maps, \( k \in \mathbb{N} \) such that \( f_k \) uniformly converges to \( f \) on \( \partial B_1 \cup \partial B_2 \).

If for any \( k \in \mathbb{N} \), the map \( f_k \) can be extended to a homeomorphism \( F_k : \Omega \to \mathbb{R}^n \) then the following hold:

(i) If \( B_1 \subset B_2 \) then

\[
f(\partial B_1) \cup \{ p \in \mathbb{R}^n \setminus f(\partial B_1) : \deg(f, \partial B_1, p) \neq 0 \} \\
\subset f(\partial B_2) \cup \{ p \in \mathbb{R}^n \setminus f(\partial B_2) : \deg(f, \partial B_2, p) \neq 0 \}
\]

(ii) If \( B_1 \subset B_2 \) then we have monotonicity of oscillation,

\[
osc_{\partial B_1} f \leq 8 osc_{\partial B_2} f
\]

and

\[
diam \{ p \in \mathbb{R}^n \setminus f(\partial B_1) : \deg(f, \partial B_1, p) \neq 0 \} \leq 8 osc_{\partial B_2} f.
\]
We use a contradiction argument; assume that $\mathrm{deg}(f, \partial B_p) = 0$. By uniform convergence $p \not\in f(\partial B_2)$ for all large $k$.

We use a contradiction argument; assume that $\mathrm{deg}(f, \partial B_2, p) = 0$. By the uniform convergence and since $p \not\in f_k(\partial B_2)$ we have that $\mathrm{deg}(f_k, \partial B_2, p) = 0$ for large $k$.

Let $F_k : \Omega \to \mathbb{R}^n$ be a homeomorphism such that $f_k = F_k\big|_{\partial B_2}$. Then $\mathrm{deg}(f_k, \partial B_2, p) = 0$ implies that $p \not\in F_k(\overline{B_2})$. Since $\overline{B_1} \subset B_2$ this implies that $p \not\in F_k(\overline{B_1})$ and thus

$$\mathrm{deg}(f_k, \partial B_1, p) = 0 \quad \text{for large } k.$$  

This leads to a contradiction as $k \to \infty$ unless $p \in f(\partial B_1)$. However since $F_k : \overline{B_2} \to \mathbb{R}^n$ is a homeomorphism, it is an open map so if $p \in (\partial B_1) \setminus F_k(\overline{B_2})$ there must be $q_k \in \partial B_2$ such that

$$\mathrm{dist}(p, F_k(\overline{B_2})) = |p - f_k(q_k)|.$$

Since $p \not\in f(\partial B_2)$, we conclude via uniform convergence that

$$\liminf_{k \to \infty} \mathrm{dist}(p, F_k(\overline{B_2})) > 0$$

and thus

$$\mathrm{dist}(p, f(\partial B_1)) = \liminf_{k \to \infty} \mathrm{dist}(p, f_k(\partial B_1)) \geq \liminf_{k \to \infty} \mathrm{dist}(p, F_k(\overline{B_2})) > 0,$$

consequently $p \not\in f(\partial B_1)$.

To prove (ii), we have that

$$f(\partial B_1) \subset f(\partial B_2) \cup \{p \in \mathbb{R}^n \setminus f(\partial B_2) : \mathrm{deg}(f, \partial B_2, p) \neq 0\}.$$  

Let $D := \mathrm{diam}(f(\partial B_2))$ and pick any $x_0 \in \partial B_2$. then $f(\partial B_2) \subset B(f(x_0), 3D)$. Moreover, let $\pi : \mathbb{R}^n \to B(f(x_0), 4D)$ be Lipschitz such that $\pi\big|_{B(f(x_0), 3D)} = \mathrm{id}$.

Since the degree depends only on the boundary values, for any $p \not\in f(\partial_2)$,

$$\mathrm{deg}(f, \partial B_2, p) = \mathrm{deg}(\pi \circ f, \partial B_2, p).$$

Since a necessary condition for the degree to be nonzero in a point $p$ is that $p$ belongs to the image, we conclude that

$$\{p \in \mathbb{R}^n \setminus f(\partial B_2) : \mathrm{deg}(f, \partial B_2, p) \neq 0\} \subset B(f(x_0), 4D).$$

In conclusion, we have shown

$$f(\partial B_1) \subset B(f(x_0), 4D)$$

and thus

$$\mathrm{diam}(f(\partial B_1)) \leq 8D = 8\mathrm{diam}(f(\partial B_2)).$$

For (iii), assume that $p \in \mathbb{R}^n \setminus (f(\partial B_1) \cup f(\partial B_2))$ and

$$\mathrm{deg}(f, \partial B_1, p) \neq 0, \quad \mathrm{deg}(f, \partial B_2, p) \neq 0.$$
By uniform convergence, \( p \in \mathbb{R}^n \setminus (f_k(\partial B_1) \cup f_k(\partial B_2)) \) for eventually all \( k \in \mathbb{N} \), and 
\[
\deg(f_k, \partial B_1, p) \neq 0, \quad \deg(f_k, \partial B_2, p) \neq 0.
\]
This means that \( p \in F_k(B_1) \cap F_k(B_2) \) which is a contradiction to \( F_k \) being a homeomorphism.

\[ \square \]

4. Corollaries for Limits of Homeomorphisms

We need the following result, which is a fractional analogue of [11, Lemma 2.9]:

**Lemma 4.1.** Let \( n \geq 2 \), and let \( p \in (1, \infty) \) and \( s \in (0, 1) \). Suppose that \( \Omega \subset \mathbb{R}^n \) is a bounded domain, and let
\[
f_k \rightharpoonup f \text{ in } W^{s,p}(\Omega; \mathbb{R}^n).
\]
Let \( x_0 \in \Omega \), and define \( r_{x_0} := \text{dist}(x_0, \partial \Omega) \). Then there exists a set \( N_{x_0} \subset \mathbb{R} \) with \( \mathcal{L}^1(N_{x_0}) = 0 \) such that for any \( r \in (0, r_{x_0}) \setminus N_{x_0} \) there exists a subsequence \( f_k \) such that
\[
f^*_k \to f^* \text{ in } W^{s,p}(\partial B(x_0, r); \mathbb{R}^n).
\]
If \( sp > n - 1 \) then
\[
f^*_k \rightharpoonup f^* \text{ on } \partial B(x_0, r).
\]
In general the subsequence depends on \( r \).

**Proof.** First, by compact embedding there is a subsequence \( f_k \to f \) in \( L^p(B(x_0, r_{x_0}); \mathbb{R}^n) \) and so Fubini’s theorem implies
\[
f^*_k \to f^* \text{ in } L^p(\partial B(x_0, r); \mathbb{R}^n), \quad \text{for every } r \in (0, r_{x_0}) \setminus N_1 \text{ with } \mathcal{L}^1(N_1) = 0.
\]
Next, define
\[
\Phi_k(r) := \int_{\partial B(x_0, r)} \int_{\partial B(x_0, r)} \frac{|f^*_k(x) - f^*_k(y)|^p}{|x - y|^{n-1+sp}} \, d\mathcal{H}^{n-1}(y) \, d\mathcal{H}^{n-1}(x),
\]
with
\[
\Phi(r) := \liminf_{k \to \infty} \Phi_k(r).
\]
Then by Fatou’s Lemma and by Lemma 2.6
\[
\int_{r/2}^r \Phi(r) \, dr \leq \liminf_{k \to \infty} \int_{r/2}^r \Phi_k(r) \, dr \leq \liminf_{j \to \infty} [f^*_k]_{W^{s,p}(B(x_0, r))}^p < \infty
\]
for every \( r \in (0, r_{x_0}) \). Define \( N_2 := \{ r \in (0, r_{x_0}) : \Phi(r) = \infty \} \), and define \( N_{x_0} := N_1 \cup N_2 \); note \( \mathcal{L}^1(N_{x_0}) = 0 \). Then let \( r \in (0, r_{x_0}) \setminus N_{x_0} \), and choose a subsequence (not relabeled) satisfying
\[
\Phi(r) = \lim_{k \to \infty} \Phi_k(r).
\]
Then \( f^*_k \to f^* \) strongly in \( L^p(\partial B(x_0, r); \mathbb{R}^n) \) and \( \lim_{k \to \infty} [f^*_k]_{W^{s,p}(\partial B(x_0, r))}^p < \infty \), and so (4.2) follows.

In the event that \( sp > n - 1 \) the uniform convergence follows from the compact Sobolev embedding theorem. \[ \square \]
The following is a corollary of the Sobolev compact embedding theorem, Lemma 3.1 and Proposition 2.8:

**Corollary 4.2.** Let \( f_k \in W^{s,p}(\Omega; \mathbb{R}^n) \) be a sequence of homeomorphisms weakly converging in \( W^{s,p}(\Omega; \mathbb{R}^n) \) to \( f \). If \( sp > n - 1 \) there exists a set \( \Sigma \subset \Omega \) with \( \mathcal{H}^{n-sp}(\Sigma) = 0 \) such that

1. \( f^* \) is continuous in \( \Omega \setminus \Sigma \), and
2. The set \( \{f^*(x)\} \) coincides with the topological image \( (f^*)^T(x) \) for every \( x \in \Omega \setminus \Sigma \), where \( (f^*)^T(x) \) is defined as

\[
(f^*)^T(x) := \bigcap_{r \in (0,r_x) \setminus N_x} f^*(\partial B(x,r)) \cup \{p \in \mathbb{R}^n \setminus f^*(\partial B(x,r)) : \text{deg}(f^*, B(x,r), p) \neq 0\},
\]

and \( r_x \) and \( N_x \) have been defined in Lemma 4.1.

**Proof.** By Lemma 4.1 and Corollary 2.7, the assumptions of Lemma 3.1 are satisfied for every \( x_1 \) and \( x_2 \in \Omega \) and for almost every \( r_1 \in (0, r_{x_1}) \setminus N_{x_1} \) and \( r_2 \in (0, r_{x_2}) \setminus N_{x_2} \). It follows that the assumptions of Proposition 2.8 are satisfied, and so \( f^* \) is continuous on a \( \mathcal{H}^{n-sp} \)-null set \( \Sigma \), where \( \Sigma = A_{sp} \cup X \); see (2.6) and (2.11) for the sets’ definitions. Thus (i) is proven.

To prove (ii) it suffices to show that

1. \( f^*(x) \in (f^*)^T(x) \) for every \( x \in \Omega \setminus \Sigma \), and
2. the diameter of the set \( (f^*)^T(x) \) is zero for every \( x \in \Omega \setminus \Sigma \).

To see (a) we start by proving the following stronger statement:

For every \( x_0 \in \Omega \) and \( r \in (0, r_{x_0}) \setminus N_{x_0} \),

(a') \( f^*(x) \in f^*(\partial B(x_0, r)) \cup \{p \in \mathbb{R}^n \setminus f^*(\partial B(x_0, r)) : \text{deg}(f^*, B(x_0, r), p) \neq 0\} \)

for every \( x \in B(x_0, r) \setminus \Sigma \).

Then (a) follows easily from (a') by choosing \( x_0 \in \Omega \setminus \Sigma \). By definition of \( \Sigma \) and by Lemma 2.3 it in turn suffices to show that

For every \( x_0 \in \Omega \) and \( r \in (0, r_{x_0}) \setminus N_{x_0} \),

(a'') \( f^*(x) \in f^*(\partial B(x_0, r)) \cup \{p \in \mathbb{R}^n \setminus f^*(\partial B(x_0, r)) : \text{deg}(f^*, B(x_0, r), p) \neq 0\} \)

for every \( x \in B(x_0, r) \setminus M \) with \( \mathcal{L}^n(M) = 0 \).

Let \( \delta > 0 \) be arbitrary. Then by the Sobolev compact embedding theorem and by Egorov’s theorem there exists a subsequence (not relabeled) \( f_k \) converging uniformly to \( f^* \) on \( B(x_0, r) \setminus M_\delta \) with \( \mathcal{L}^n(M_\delta) < \delta \). Now let \( x \in \Omega \setminus M_\delta \). It suffices to show that if \( f^*(x) \notin f^*(\partial B(x_0, r)) \) then \( \text{deg}(f^*, B(x_0, r), f^*(x)) = 0 \). Since \( f_k \Rightarrow f^* \) on \( \partial B(x_0, r), f^*(x) \notin f_k(\partial B(x_0, r)) \) for all \( k \) sufficiently large. So there exists \( \varepsilon > 0 \) such that \( B(f^*(x), \varepsilon) \) does not intersect \( f^*(\partial B(x_0, r)) \) or \( f_k(\partial B(x_0, r)) \) for \( k \) sufficiently large. Then since the \( f_k \) are homeomorphisms, it must be that \( \text{deg}(f_k, \partial B(x_0, r), p) \) is a nonzero constant for all \( k \) sufficiently large and for all \( p \in B(f^*(x), \varepsilon) \). In addition, \( f_k \Rightarrow f^* \) on \( B(x_0, r) \setminus M_\delta \) so \( f_k(x) \in B(f^*(x), \varepsilon) \) for \( k \) sufficiently large, uniformly in \( x \). Thus the continuity of the degree yields

\[
\text{deg}(f^*, B(x_0, r), f^*(x)) = \lim_{k \to \infty} \text{deg}(f_k, B(x_0, r), f_k(x)).
\]
Since \( \text{deg}(f_k, B(x_0, r), f_k(x)) \) is a nonzero constant for all \( k \) sufficiently large, we have proved that

For every \( x_0 \in \Omega \) and \( r \in (0, r_{x_0}) \setminus N_{x_0} \),

\[
f^*(x) \in f^*(\partial B(x_0, r)) \cup \{ p \in \mathbb{R}^n \setminus f^*(\partial B(x_0, r)) : \text{deg}(f^*, B(x_0, r), p) \neq 0 \}
\]

for every \( x \in B(x_0, r) \setminus M_5 \) with \( \mathcal{L}^n(M_5) < \delta \).

Since \( \delta > 0 \) is arbitrary (a”) is proved.

To see (b), let \( x_0 \in \Omega \setminus \Sigma \), and let \( \varepsilon > 0 \). Then by definition of the set \( X \) there exists \( R = R(x_0, \varepsilon) \in (0, r_{x_0}) \) such that

\[
R^{s-p-n}[f^*]_{W^{s,p}(B(x_0, R))} < \varepsilon.
\]

So by Lemma 3.1 (ii) and (2.10)

\[
\text{diam}(f^*)^T(x_0) \leq \text{diam} \left( f^*(\partial B(x_0, r)) \cup \{ p : \text{deg}(f^*, B(x_0, r), p) \neq 0 \} \right) < C \varepsilon
\]

for every \( r \in (0, R/4) \setminus N_{x_0} \). Therefore by definition \( \text{diam}(f^*)^T(x_0) < \varepsilon \). The proof is complete.

Remark 4.3. We can define a representative \( \hat{f} \) of \( f \) as

\[
\hat{f}(x) := \begin{cases} f^*(x), & x \in \Omega \setminus \Sigma, \\ \text{any element of } f^T(x), & \text{otherwise}, \end{cases}
\]

Then \( \hat{f} \) agrees with \( f^* \) everywhere outside \( \Sigma \), and \( \hat{f} \) has the added property that \( \hat{f}(x) \in (\hat{f})^T(x) \) for every \( x \in \Omega \).

\[ \square \]

5. PROOF OF THEOREM 1.1

Proof of Theorem 1.1. We proceed identically to [4]. Assume that \( f = \hat{f} \). We argue by contradiction; suppose that there is a \( \delta > 0 \) such that the set

\[
\Gamma := \{ y \in \mathbb{R}^n : \text{diam}(f^{-1}\{y\}) > 0 \}
\]

satisfies \( \mathcal{H}^{\frac{n-1}{s}+\delta}(\Gamma) > 0 \). Then there exists \( K \in \mathbb{N} \) such that the set

\[
\Gamma_K := \left\{ y \in \mathbb{R}^n : \text{diam}(f^{-1}\{y\}) > \frac{1}{K} \right\}
\]

satisfies \( \mathcal{H}^{\frac{n-1}{s}+\delta}(\Gamma_K) > 0 \), since \( F = \bigcup_{k \in \mathbb{N}} \Gamma_k \). For each \( x \) there exists \( r < \frac{1}{2K} \) such that \( f|_{\partial B(x, r)} \in W^{s,p}(\partial B(x, r); \mathbb{R}^n) \cap C^0(\partial B(x, r); \mathbb{R}^n) \) by Corollary 4.1. Then choosing a covering of \( \Omega \) with such a collection \( \mathcal{B} := (B(x_i, r_i))_{i=1}^\infty \), by Theorem 2.5 we have \( \dim_{\mathcal{H}}(f(\partial B(x_i, r_i))) < \frac{n-1}{s} \), so \( \mathcal{H}^{\frac{n-1}{s}+\delta}(f(\partial B(x_i, r_i))) = 0 \), Therefore, the set

\[
E := \bigcup_{i=1}^\infty f(\partial B(x_i, r_i))
\]

satisfies \( \mathcal{H}^{\frac{n-1}{s}+\delta}(E) = 0 \). We will show that \( \Gamma_K \subset E \), which contradicts the statement \( \mathcal{H}^{\frac{n-1}{s}+\delta}(\Gamma_K) > 0 \).
Assume \( y \in \Gamma_K \setminus E \). Then there must exist \( z_1 \) and \( z_2 \) in \( \Omega \) with \( f(z_1) = f(z_2) = y \), with \( \text{dist}(z_1, z_2) > \frac{1}{\lambda} \). Fix an element \( B(x_i, r_i) \) from the collection \( B \) with \( z_1 \in B(x_i, r_i) \) and \( z_2 \notin B(x_i, r_i) \). Combining Lemma 3.1(i) with the fact that
\[
f(x) \in f'(x) \subset f(\partial B(x, r)) \cup \{p \in \mathbb{R}^n \setminus f(\partial B(x, r)) : \deg(f, \partial B(x, r), p) \neq 0\}
\]
for all \( x \in \Omega \) and for \( r \in (0, \text{dist}(x, \partial \Omega)) \setminus N_x \), we get
\[
y = f(z_1) \in f(\partial B(x_i, r_i)) \cup \{p \in \mathbb{R}^n \setminus f(\partial B(x_i, r_{x_i})) : \deg(f, B(x_i, r_{x_i}), p) \neq 0\}.
\]
However \( y \notin E \) so \( y \notin f(\partial B(x_i, r_i)) \), and thus
\[
y = f(z_1) \in \{p \in \mathbb{R}^n \setminus f(\partial B(x_i, r_{x_i})) : \deg(f, B(x_i, r_{x_i}), p) \neq 0\},
\]
At the same time, a similar argument using Lemma 3.1(iii) gives
\[
y = f(z_2) \in f'(z_2) \subset \mathbb{R}^n \setminus \{p \in \mathbb{R}^n \setminus f(\partial B(x_i, r_{x_i})) : \deg(f, B(x_i, r_{x_i}), p) \neq 0\},
\]
which is a contradiction. \( \square \)

**APPENDIX A. PROOF OF LEMMA 2.6**

**Proof.** It suffices to prove (2.9) for \( a = 0 \) and \( r = 1 \). In the case of general \( a \) and \( r \) we can apply (2.9) for \( a = 0 \), \( r = 1 \) to the function
\[
g(x) := f(a + rx) \in W^{s,p}(B(0,1))
\]
and obtain (2.9) for general \( a \) and \( r \) by change of variables.

Since the function \( f - (f)_B \) also belongs to \( W^{s,p}(B(0,1)) \) we can assume without loss of generality that
\[
\int_{B(0,1)} f \, dx = 0.
\]
Thus by the Poincaré inequality it suffices to show that there exists a constant
\[
C = C(n, s, p) > 0 \text{ such that } (2.9)
\]
(A.1)
\[
\int_{1/2}^1 \int_{S^{n-1}} \int_{S^{n-1}} \rho^{n-1-sp} \frac{|f(\rho x) - f(\rho y)|^p}{|x - y|^{n+1+sp}} \, d\mathcal{H}^{n-1}(y) \, d\mathcal{H}^{n-1}(x) \, d\rho \leq C \|f\|^p_{W^{s,p}(B(0,1))};
\]

note that we used polar coordinates to rewrite the integral.

We prove (A.1) by splitting the domain of the left-hand side integral and estimating each piece. Each domain of integration is locally homeomorphic to a Euclidean ball in \( \mathbb{R}^{n-1} \), which allows us to apply translation arguments in the spirit of [2, Lemma 7.44]. Any local diffeomorphism between \( S^{n-1} \) and \( \mathbb{R}^{n-1} \) will do, but we make this argument explicit by using stereographic projection.

Step 1: To this end, define for each \( \mu \in [0, 1) \) the spherical cap \( H_\mu := \{x \in S^{n-1} : x_n < \mu\} \). We will show that for every \( \mu \in [0, 1) \) there exists a constant \( C = C(n, s, p) \) such that
\[
(A.2)
\int_{1/2}^1 \int_{H_\mu} \int_{H_\mu} \rho^{n-1-sp} \frac{|f(\rho x) - f(\rho y)|^p}{|x - y|^{n+1+sp}} \, d\mathcal{H}^{n-1}(y) \, d\mathcal{H}^{n-1}(x) \, d\rho \leq C \left(\frac{1 + \mu}{1 - \mu}\right)^{1+sp} \|f\|^p_{W^{s,p}(B(0,1))}.
\]
Throughout the proof we write \( B_{n-1}(0, \lambda) \) for any \( \lambda > 0 \) as the ball in \( \mathbb{R}^{n-1} \) centered at 0 of radius \( \lambda \). We next establish notation for the stereographic projection \( \psi : \)}
\[ \psi(x_1, \ldots, x_{n-1}) := \left( \frac{2x_1}{1 + |x|^2}, \ldots, \frac{2x_{n-1}}{1 + |x|^2}, \frac{|x|^2 - 1}{1 + |x|^2} \right) \]

so that we have the correspondence of domains

\[ \psi(B_{n-1}(0, \lambda)) = H_\mu, \quad \text{where} \quad \lambda = \sqrt{\frac{1 + \mu}{1 - \mu}}. \]

The formula for the Jacobian \( J_\psi(x) := \left( \frac{2}{1 + |x|^2} \right)^{n-1} \) will be used throughout in order to ensure that quantities such as \( J_\psi(x - y) \) and \( |J_\psi(x) - J_\psi(y)| \) remain bounded above and below uniformly for \( x \) and \( y \) in \( B_{n-1}(0, \lambda) \), with bounds depending only on \( n \) and \( \lambda \).

To prove (A.2) we need to show that for every \( \lambda \in [1, \infty) \) and for every ball \( B_{n-1}(0, \lambda) \subset \mathbb{R}^{n-1} \)

\[ \int_{1/2}^1 \int_{B_{n-1}(0, \lambda)} \int_{B_{n-1}(0, \lambda)} \rho^{n-1-sp} \left| f(\rho \psi(x)) - f(\rho \psi(y)) \right|^p \frac{d\rho}{|\psi(x) - \psi(y)|^{n-1+sp}} \times J_\psi(x) J_\psi(y) \, d\rho \, d\sigma \leq \zeta_{n,s,p} \lambda^{2+2sp} \|f\|_{W^{s,p}(B(0,1))}^p. \]

We proceed using a technique found in [2, Lemma 7.44]. Let \( \sigma \in [1/2, 1] \), and integrate the inequality

\[ |f(\rho \psi(x)) - f(\rho \psi(y))|^p \leq \zeta_p \left| f(\rho \psi(x)) - f(\sigma \psi(\frac{x+y}{2})) \right|^p \]

\[ + \left| f(\sigma \psi(\frac{x+y}{2})) - f(\rho \psi(y)) \right|^p \]

with respect to \( \sigma \) over the ball \( B(r, \frac{|\psi(x)-\psi(y)|}{2}) \cap [1/2, 1] \subset \mathbb{R} \) to get

\[ \left| f(\rho \psi(x)) - f(\rho \psi(y)) \right|^p \leq \zeta_p \left| \frac{1}{\psi(x) - \psi(y)} \int_{|\sigma - \rho| \leq \frac{|\psi(x) - \psi(y)|}{2}} |f(\rho \psi(x)) - f(\sigma \psi(\frac{x+y}{2}))|^p \, d\sigma \right| \]

\[ + \frac{1}{\psi(x) - \psi(y)} \int_{|\sigma - \rho| \leq \frac{|\psi(x) - \psi(y)|}{2}} \left| f(\sigma \psi(\frac{x+y}{2})) - f(\rho \psi(y)) \right|^p \, d\sigma. \]

Set for \( \eta \in \mathbb{R}^{n-1} \)

\[ \Upsilon(\eta) := \int_{|\sigma - \rho| \leq \frac{|\psi(x) - \psi(y)|}{2}} \left| f(\rho \psi(\eta)) - f(\sigma \psi(\frac{x+y}{2})) \right|^p \, d\sigma; \]

therefore

\[ \int_{1/2}^1 \rho^{n-1-sp} \int_{B_{n-1}(0, \lambda)} \int_{B_{n-1}(0, \lambda)} \left| f(\rho \psi(x)) - f(\rho \psi(y)) \right|^p \frac{d\rho}{|\psi(x) - \psi(y)|^{n-1+sp}} \times J_\psi(x) J_\psi(y) \, d\rho \, d\sigma \leq \zeta_p \int_{1/2}^1 \rho^{n-1-sp} \int_{B_{n-1}(0, \lambda)} \int_{B_{n-1}(0, \lambda)} \Upsilon(x) + \Upsilon(y) \left| \psi(x) - \psi(y) \right|^{n+sp} J_\psi(x) J_\psi(y) \, d\rho \, d\sigma \leq I + II. \]

Now by change of variables and using the formula for \( J_\psi \) as well as the formula

\[ |\psi(x) - \psi(y)| = \frac{2|x - y|}{(1 + |x|^2)^{1/2}(1 + |y|^2)^{1/2}}, \]
which is valid for all \( x, y \in \mathbb{R}^{n-1} \), we have

\[
I = \int_{1/2}^{1} \int_{B_{n-1}(0, \lambda)} \int_{B_{n-1}(0, \lambda)} \int_{[\sigma - x] \leq |x| \leq (1/2) \cap [1/2, 1]} \rho^{n-1-\sigma} \frac{|f(\rho \psi(x) - f(\sigma \psi(z))|^p}{|\psi(x) - \psi(y)|^{n+\sigma}} J_\psi(x) J_\psi(y) \, d\sigma \, dy \, d\rho
\]

\[
= \int_{1/2}^{1} \int_{1/2}^{1} \int_{B_{n-1}(0, \lambda)} \int_{[2z-x] \leq |z| \leq (2z-x) \cap [1/2, 1]} \rho^{n-1-\sigma} \frac{|f(\rho \psi(x) - f(\sigma \psi(z))|^p}{|\psi(x) - \psi(2z-x)|^{n+\sigma}} J_\psi(x) J_\psi(2z-x) \, dz \, d\sigma \, d\rho
\]

\[
= \frac{1}{2^{(n+\sigma)/2}} \int_{1/2}^{1} \int_{1/2}^{1} \int_{B_{n-1}(0, \lambda)} \int_{[2z-x] \leq |z| \leq (2z-x) \cap [G(x,z), |z|=\sigma - \rho]} \rho^{n-1-\sigma} \frac{|f(\rho \psi(x) - f(\sigma \psi(z))|^p}{|\psi(x) - \psi(z)|^{n+\sigma}} J_\psi(x) J_\psi(z) G(x,z)^{2+\sigma-n} \, dz \, d\sigma \, d\rho,
\]

where

\[
G(x,z) := \left( \frac{1 + |2z-x|^2}{1 + |x|^2} \right)^{1/2}.
\]

Now, since \( |x| \leq \lambda \) and \( |2z-x| \leq \lambda \) the uniform bound \( \frac{1}{\sqrt{1+\lambda^2}} \leq G(x,z) \leq \sqrt{1+\lambda^2} \) holds, and so I can be majorized by

\[
C(n, s, p)\lambda^{2+\sigma-n} \int_{1/2}^{1} \int_{1/2}^{1} \int_{B_{n-1}(0, \lambda)} \int_{[2z-x] \leq |z| \leq (2z-x) \cap [G(x,z), |z|=\sigma - \rho]} \rho^{n-1-\sigma} \frac{|f(\rho \psi(x) - f(\sigma \psi(z))|^p}{|\psi(x) - \psi(z)|^{n+\sigma}} J_\psi(x) J_\psi(z) \, dz \, d\sigma \, d\rho.
\]

(A.5)

Finally, on the domain of integration in (A.5) the estimate

\[
|\rho \psi(x) - \sigma \psi(z)| \leq \rho |\psi(x) - \psi(z)| + |\psi(z)||\rho - \sigma|
\]

\[
\leq |\psi(x) - \psi(z)| + (1 + \lambda^2)^{1/2} |\psi(x) - \psi(z)| \leq 2\lambda |\psi(x) - \psi(z)|
\]

holds, and since \( \rho \) and \( \sigma \) are bounded away from zero

\[
I \leq C \lambda^{2+\sigma} \int_{1/2}^{1} \int_{1/2}^{1} \int_{B_{n-1}(0, \lambda)} \int_{[2z-x] \leq |z| \leq (2z-x) \cap [G(x,z), |z|=\sigma - \rho]} \rho^{n-1} \sigma^{-1}\frac{|f(\rho \psi(x) - f(\sigma \psi(z))|^p}{|\psi(x) - \psi(z)|^{n+\sigma}} J_\psi(x) J_\psi(z) \, dz \, d\sigma \, d\rho
\]

\[
\leq C \lambda^{2+\sigma} \int_{0}^{1} \int_{R^{n-1}} \int_{R^{n-1}} \rho^{n-1} \sigma^{-1}\frac{|f(\rho \psi(x) - f(\sigma \psi(z))|^p}{|\rho \psi(x) - \sigma \psi(z)|^{n+\sigma}} J_\psi(x) J_\psi(z) \, dz \, d\sigma \, d\rho
\]

\[
= C(n, s, p)\lambda^{2+\sigma} \int_{B(0,1)} \int_{B(0,1)} \frac{|f(x) - f(y)|^p}{|x - y|^{n+\sigma}} \, dy \, dx.
\]

A similar estimate holds for the quantity II in (A.4). Therefore (A.3), and thus (A.2), is proved.
Step 2: We conclude the proof. Split the integral on the left-hand side of (A.1) via change of variables and symmetry as
\[
\int_{1/2}^1 \int_{S^{n-1}} \int_{S^{n-1}} \rho^{n-1-sp} \frac{|f(\rho x) - f(\rho y)|^p}{|x - y|^{n-1+sp}} \, d\mathcal{H}^{n-1}(y) \, d\mathcal{H}^{n-1}(x) \, d\rho
\]
(A.6)
\[
= \int_{1/2}^1 \int_{H_0} \int_{H_0} \cdots + \int_{1/2}^1 \int_{S^{n-1}\setminus H_0} \int_{S^{n-1}\setminus H_0} \cdots + 2 \int_{1/2}^1 \int_{S^{n-1}\setminus H_0} \int_{H_0} \cdots
\]
\[
:= I + II + III.
\]
Clearly by (A.2) with \( \mu = 0 \)
\[
(A.7) \quad I \lesssim_{n,s,p} \|f\|_{W^{s,p}(B(0,1))}^p.
\]
Now, let \( Q : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1} \) be the matrix \( \text{diag}(1,1,\ldots,1,-1) \). Setting \( h(x) = f(Qx) \) for any \( x \in B_1(0) \), a change of variables gives
\[
II = \int_{1/2}^1 \int_{H_0} \int_{H_0} \rho^{n-1-sp} \frac{|h(\rho x) - h(\rho y)|^p}{|x - y|^{n-1+sp}} \, d\mathcal{H}^{n-1}(y) \, d\mathcal{H}^{n-1}(x) \, d\rho.
\]
Thus by (A.2) with \( \mu = 0 \) and by another change of variables
\[
(A.8) \quad II \leq C \int_{B(0,1)} \int_{B(0,1)} \frac{|h(x) - h(y)|^p}{|x - y|^{n+sp}} \, dy \, dx = C \int_{B(0,1)} \int_{B(0,1)} \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} \, dy \, dx.
\]
For the last integral, we have
\[
III = 2 \int_{1/2}^1 \int_{H_{1/2}\setminus H_0} \int_{H_0} \cdots + 2 \int_{1/2}^1 \int_{S^{n-1}\setminus H_{1/2}} \int_{H_0} \cdots
\]
\[
:= III_1 + III_2.
\]
Using that \( H_0 \subset H_{1/2} \) along with (A.2) for \( \mu = 1/2 \),
\[
(A.9) \quad III_1 \leq \int_{1/2}^1 \int_{H_{1/2}} \int_{H_{1/2}} \rho^{n-1-sp} \frac{|f(\rho x) - f(\rho y)|^p}{|x - y|^{n-1+sp}} \, d\mathcal{H}^{n-1}(y) \, d\mathcal{H}^{n-1}(x) \, d\rho \leq \|f\|_{W^{s,p}(B(0,1))}^p.
\]
Since \( \text{dist}(\mathbb{S}^{n-1}\setminus H_{1/2}, H_0) = C(n) > 0 \), we have that \( |x - y| \geq C(n) > 0 \) for all \( x \in \mathbb{S}^{n-1}\setminus H_{1/2} \) and for all \( y \in H_0 \), and so the integral \( III_2 \) can be estimated by
\[
(A.10) \quad III_2 \lesssim_{n,s,p} \int_{1/2}^1 \int_{\mathbb{S}^{n-1}} |f(\rho x)|^p \, d\mathcal{H}^{n-1}(x) \, d\rho \lesssim_{n,s,p} \|f\|_{L^p(B(0,1))}.
\]
Combining (A.6) with estimates (A.7), (A.8), (A.9) and (A.10) gives (A.1). \( \square \)

References
[1] D. R. Adams and L. I. Hedberg. *Function spaces and potential theory*, volume 314. Springer Science & Business Media, 2012.
[2] R. A. Adams. *Sobolev spaces*, volume 65 of *Pure and Applied Mathematics*. Academic, New York-London, 1975.
[3] M. Barchiesi, D. Henao, and C. Mora-Corral. Local invertibility in Sobolev spaces with applications to nematic elastomers and magnetoelasticity. *Arch. Ration. Mech. Anal.*, 224(2):743–816, 2017.
[4] O. Bouchala, S. Hencl, and A. Molchanova. Injectivity almost everywhere for weak limits of sobolev homeomorphisms, 2019.
[5] J. Bourgain, H. Brezis, and P. Mironescu. Lifting, degree, and distributional Jacobian revisited. *Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences*, 58(4):529–551, 2005.

[6] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. CRC press, 2015.

[7] P. Goldstein, P. Hajlasz, and M. R. Pakzad. Finite distortion Sobolev mappings between manifolds are continuous. *Int. Math. Res. Not. IMRN*, (14):4370–4391, 2019.

[8] L. Grafakos. *Classical Fourier Analysis, 2nd Edition*, volume 249. 2008.

[9] S. Hencl and P. Honzík. Dimension distortion of images of sets under Sobolev mappings. *Ann. Acad. Sci. Fenn. Math* 40, (1):427–442, 2015.

[10] S. Li and A. Schikorra. $W^{s, s}$-maps with positive distributional Jacobians. *Pot.A. (accepted)*, 2019.

[11] S. Müller and S. J. Spector. An existence theory for nonlinear elasticity that allows for cavitation. *Arch. Rational Mech. Anal.*, 131(1):1–66, 1995.

[12] S. Müller, S. J. Spector, and Q. Tang. Invertibility and a topological property of Sobolev maps. *SIAM J. Math. Anal.*, 27(4):959–976, 1996.

[13] E. D. Nezza, G. Palatucci, and E. Valdinoci. Hitchhiker’s guide to the fractional Sobolev spaces. *Bull. Sci. Math.*, 136(5):521–573, 2012.

[14] T. Runst and W. Sickel. *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*, volume 3. Walter de Gruyter, 2011.

[15] Y. Sawano. *Theory of Sesov spaces*, volume 56. Springer, 2018.

[16] L. Tartar. *An introduction to Sobolev spaces and interpolation spaces*, volume 3. Springer Science & Business Media, 2007.

[17] V. Šverák. Regularity properties of deformations with finite energy. *Arch. Rational Mech. Anal.*, 100(2):105–127, 1988.

[18] W. P. Ziemer. *Weakly differentiable functions*, volume 120 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1989. Sobolev spaces and functions of bounded variation.

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