ON SYMPLECTIC COBDISMS

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Abstract. In this note we make several observations concerning symplectic cobordisms. Among other things we show that every contact 3-manifold has infinitely many concave symplectic fillings and that all overtwisted contact 3-manifolds are “symplectic cobordism equivalent.”

1. Introduction

In this note we make several observations concerning (directed) symplectic cobordisms, Stein cobordisms, and concave symplectic fillings for contact 3-manifolds. Symplectic and Stein cobordisms have recently come to the foreground of symplectic and contact geometry, largely due to the introduction of a symplectic field theory (SFT) by Eliashberg, Hofer and Givental [1]. The goal of SFT is to associate an algebraic structure to a given symplectic cobordism. Though clearly a central notion in symplectic and contact geometry, there is surprisingly little concerning symplectic cobordisms in the literature.

We will assume our 3-manifolds are closed and oriented, and our contact structures are oriented and positive. A contact 3-manifold \((M_1, \xi_1)\) is symplectically (resp. Stein) cobordant to another contact manifold \((M_2, \xi_2)\), if there exists a symplectic (resp. Stein) 4-manifold \((X, \omega)\) with \(\partial X = M_2 - M_1\) and a vector field \(v\) defined on a neighborhood of \((M_1 \cup M_2) \subset X\) for which \(L_v \omega = \omega, v \pitchfork (M_1 \cup M_2)\), and the normal orientation of \(M_1 \cup M_2\) agrees with \(v\). We say \((M_1, \xi_1)\) is the concave end of the cobordism, while \((M_2, \xi_2)\) is the convex end. We denote the existence of such a cobordism by \((M_1, \xi_1) \prec (M_2, \xi_2)\) — in the paper we implicitly assume that \(\prec\) refers to a Stein cobordism, unless specified otherwise. Note that symplectic (and Stein) cobordism is not an equivalence relation. For example, a Stein fillable contact structure \((M, \xi)\) (= one satisfying \(\emptyset \prec (M, \xi)\)) cannot be symplectically cobordant to an overtwisted contact structure, but the opposite is possible. Our first result is:

**Theorem 1.1.** Let \((M_1, \xi_1)\) be a contact 3-manifold. Then there exists a Stein fillable contact 3-manifold \((M_2, \xi_2)\) and a Stein cobordism \((M_1, \xi_1) \prec (M_2, \xi_2)\).
Though this result indicates the overall structure of the partial order on contact 3-manifolds induced by cobordisms, there is very little control over the target contact manifold \((M_2, \xi_2)\). On the other hand, when \((M_1, \xi_1)\) is overtwisted, there is complete freedom in choosing \((M_2, \xi_2)\):

**Theorem 1.2.** Let \((M_1, \xi_1)\) be an overtwisted contact 3-manifold and \((M_2, \xi_2)\) any contact 3-manifold, tight or overtwisted. Then there exists a Stein cobordism \((M_1, \xi_1) \prec (M_2, \xi_2)\).

In particular, all overtwisted contact structures are equivalent under symplectic or Stein cobordism!

It is interesting to compare the previous two theorems with recent work of Epstein-Henkin \([12]\) and de Oliveira \([4]\) which deal with cobordisms between CR-structures. (Here “CR-structure” will mean “strictly pseudoconvex CR-structure”.) On any 3-manifold \(M\), there is a 1-1 correspondence between CR-structures and pairs \((\xi, J)\) consisting of a contact structure \(\xi\) and an almost complex structure \(J\) on \(\xi\). We say a CR-structure \((\xi, J)\) on \(M\) is fillable, if there is a compact, connected, complex manifold \(X\) with \(\partial X = M\), so that the complex tangencies to \(M\) are \(\xi\) and the induced complex structure on \(\xi\) is \(J\). In \([12]\) it was shown that if a CR-manifold \((M_1, \xi_1, J_1)\) is Stein cobordant to a fillable CR-manifold \((M_2, \xi_2, J_2)\), then \((M_1, \xi_1, J_1)\) is also fillable. Here we assume Stein cobordisms of CR-manifolds respect complex structures. Thus, if \((M_1, \xi_1, J_1) \prec (M_2, \xi_2, J_2)\) is a Stein cobordism but \((M_1, \xi_1)\) is not Stein fillable, then \((M_2, \xi_2, J_2)\) cannot be a fillable CR-structure, even if \((M_2, \xi_2)\) is a Stein fillable contact structure. De Oliveira \([4]\) gave some interesting examples of complex (but not Stein) cobordisms from non-fillable CR-structures to fillable ones, thus showing the necessity of having a Stein cobordism in the Epstein-Henkin result.

Our last result is:

**Theorem 1.3.** Any contact 3-manifold has infinitely many concave symplectic fillings which are mutually non-isomorphic and are not related to each other by a sequence of blow-ups and blow-downs.

A convex (resp. concave) symplectic filling of \((M, \xi)\) is a symplectic cobordism \((X, \omega)\) from \(\emptyset\) to \((M, \xi)\) (resp. from \((M, \xi)\) to \(\emptyset\)). The phrase “symplectic filling,” without modifiers, is usually reserved for “convex symplectic filling.” Having a (convex) filling is quite restrictive for a contact 3-manifold — for instance, it implies the contact structure is tight. (Note, however, that there are many tight contact structures without such fillings due to Eliashberg \([10]\), Ding-Geiges \([3]\) and Etnyre-Honda \([13]\).) We show that, on the contrary, concave fillings are not restrictive at all. Though this was believed for a long time, and specific isolated contact manifolds with infinitely many such fillings are easy to come by, the degree to which concave fillings are not restrictive is perhaps a little surprising.

We assume the reader is more or less familiar with contact geometry and hence we do not include any background material here. We refer the reader
to [1] for the basics of contact geometry, [7] for Lutz twisting, and [11, 8] for
the notions of Stein and symplectic cobordisms.

2. Legendrian surgeries

In this section we give a description of Legendrian surgery, both on the 3-
manifold level and as a source of Stein filling on the 4-manifold level. There
is some related material in [20] for Legendrian surgeries.

Let \((M, \xi)\) be a contact manifold and \(L \subset M\) a closed Legendrian curve.
Let \(N(L)\) be a standard tubular neighborhood of the Legendrian curve \(L\), with
convex boundary and two parallel dividing curves. Choose a framing for \(L\)
(and a concomitant identification \(\partial N(L) \simeq \mathbb{R}^2/\mathbb{Z}^2\)) so that the meridian has
slope 0 and the dividing curves have slope \(\infty\). With respect to this choice
of framing, a Legendrian surgery is a \(-1\) surgery, where a copy of \(N(L)\)
is glued to \(M \setminus N(L)\) so that the new meridian has slope \(-1\). Here, even
though the boundary characteristic foliations may not exactly match up a priori,
we use Giroux’s Flexibility Theorem [14, 19] and the fact that they
have the same dividing set to make the characteristic foliations agree. This

gives us a new manifold \((M', \xi')\).

The following proposition describes Legendrian surgery on the 4-manifold
level.

**Proposition 2.1.** Let \((M', \xi')\) be a contact manifold obtained by Legendrian
surgery along \(L\) in \((M, \xi)\), in a 3-dimensional manner. Then there exists
a Stein cobordism from \((M, \xi)\) to \((M', \xi')\), obtained by attaching a 2-handle
along \(N(L)\).

**Proof.** We apply Lemma 2.2 below to obtain a Stein cobordism \(X = M \times [0, 1]\). Then Legendrian surgery corresponds to attaching a 2-handle along
\(N(L) \subset M \times \{1\}\) in a Stein (resp. symplectic) manner, which
yields a Stein (resp. symplectic) cobordism from \((M, \xi)\) to \((M', \xi')\). (See Eliashberg [8].)

**Lemma 2.2.** Let \((M, \xi)\) be a contact structure. Then there exists a thick-
ening of \(M\) to \(X = M \times [0, 1]\) and a Stein cobordism from \((M, \xi)\) to itself.

A proof of this fact appears in [3].

3. Open book decompositions

Recall an open book decomposition of a 3-manifold \(M\) consists of a link
\(K\), called the binding, and a fibration \(f: (M \setminus K) \to S^1\) such that each fiber
\(F\) in the fibration is a Seifert surface for \(K\). The manifold \(M \setminus K\) is obtained by taking \(F \times [0, 1]\) with coordinates \((x, t)\) and identifying \((x, 0) \sim (\phi(x), 1)\) via the monodromy map \(\phi: F \to F\).

Following Thurston and Winkelkenkemper [25], we construct a contact structure on \(M\) from an open
book decomposition: Let \(\lambda\) be a primitive for an area form on \(F\) and let
\(\lambda_t = t \cdot \lambda + (1 - t) \cdot \phi^* \lambda\), \(t \in [0, 1]\). The 1-form \(\alpha = dt + \lambda_t\) is a contact 1-form
on $F \times [0, 1]$ which glues to give a contact structure on $M \setminus K$. One easily checks that $\alpha$ extends over $K$. If $(M, \xi)$ is obtained in this manner, then we say that the open book decomposition of $M$ is adapted to $\xi$. We now have the following recent result of Giroux [15]:

**Theorem 3.1.** Any contact structure $\xi$ on a closed 3-manifold $M$ admits an open book decomposition of $M$ which is adapted to $\xi$.

The following lemma (and more importantly its converse) is due to the efforts of many people, beginning with the work of Loi and Piergallini [22] (also see Mori [24] for an earlier effort), and recently culminating in the works of Giroux [15], Akbulut-Ozbagci [2], and Matveyev [23].

**Lemma 3.2.** If the monodromy $\phi : F \to F$ for an open book can be expressed as a product of positive Dehn twists, then the adapted contact structure is Stein fillable.

**Proof.** If $\phi = \text{id}$, then the manifold $M_n$ is simply the connected sum of several copies of $S^1 \times S^2$. There is a unique tight contact structure $\xi_n$ on $M_n = \#_n(S^1 \times S^2)$, and it is Stein fillable. The uniqueness of $\xi_n$ on $M_n$ follows from the unique connect sum decomposition theorem of Colin [3] and the uniqueness on $S^1 \times S^2$ due to Eliashberg [9].

Assume $\phi$ consists of a single positive Dehn twist along $\gamma \subset F$. Then the manifold $M$ is obtained from $M_n$ by a Dehn surgery along $\gamma$ with surgery coefficient one less than the framing induced on $\gamma$ by the fiber. But we can also make $\gamma$ a Legendrian curve in $F$ so that the framings given by the contact structure and the fibers agree. (In other words, the twisting number of $\gamma$ relative to $F$ is zero.) This is made possible by applying the Legendrian Realization Principle. Note that to apply the Legendrian Realization Principle, a fold may be necessary (for details see [19]). Thus $(M, \xi)$ is obtained from $(M_n, \xi_n)$ by a Legendrian surgery and hence is Stein fillable. Now, if $\phi$ is the product of $k > 1$ positive Dehn twists, we perform $k$ Legendrian surgeries on different leaves, in order.

We are now ready to prove Theorem 1.1. It should be pointed out that the strategy of proof is similar to the proof strategy in [5], where it is proved that “most” universally tight contact contact structures on torus bundles over the circle are not (strongly) symplectically fillable.

**Proof of Theorem 1.1.** If $(M, \xi)$ is Stein fillable, then we are done by Lemma 2.2. Therefore, let $(M, \xi)$ be a contact structure which is not Stein fillable. By Theorem 3.1 there exists an open book decomposition for $M$ which is adapted to $\xi$. Let $K$ be the binding, $f : (M \setminus K) \to S^1$ the fibering of the complement, $F$ the fiber, and $\phi$ the monodromy map. Since $(M, \xi)$ is not Stein fillable, any product decomposition of $\phi$ into Dehn twists must contain some negative Dehn twists. We view each Dehn twist as being done on a separate fiber. On a fiber just after one on which a negative Dehn twist was done along $\gamma$, we can take a parallel copy of $\gamma$ and perform a positive
Dehn twist, which is tantamount to a Legendrian surgery. If a compensatory positive Dehn twisted is added whenever there is a negative Dehn twist, then we will have a new monodromy map $\phi'$ with only positive Dehn twists. Of course $\phi'$ will define a different manifold $M'$ and a different contact structure $\xi'$. However, since the difference in between the monodromy for $M$ and for $M'$ is just several positive Dehn twists, we can get from $(M, \xi)$ to $(M', \xi')$ by a sequence of Legendrian surgeries. Thus we have a Stein cobordism from $(M, \xi)$ to $(M', \xi')$. □

4. Overtwisted Contact Structures

In this section we prove Theorem 1.2. The proof will be broken down into two propositions.

**Proposition 4.1.** Any overtwisted contact manifold is Stein cobordant to any overtwisted contact manifold.

**Proof.** Let $(M_i, \xi_i), i = 1, 2$ be two overtwisted contact manifolds. It is a well-known fact in 3-manifold topology that we can find a link $L$ in $M_1$ such that a certain integer Dehn surgery on $L$ will yield $M_2$. Thus we can construct a topological cobordism $X$ from $M_1$ to $M_2$ by attaching 2-handles with the appropriate framing to $M_1 \times [0, 1]$. Moreover, one can adapt the proof of Lemma 4.4 in [18] to show that we may assume that $X$ has an almost complex structure with complex tangencies $\xi_i$ on $M_i$. We now apply the following theorem of Eliashberg (Theorem 1.3.4 in [8]):

**Theorem 4.2** (Eliashberg). Let $(X, J)$ be a compact, almost complex (real) 4-manifold with boundary $\partial X = M_2 - M_1$. Assume $M_1$ is $J$-concave, $J$ is integrable near $M_1$, and the corresponding contact structure $(M_1, \xi_1)$ is overtwisted. If the cobordism $(X, J)$ from $M_1$ to $M_2$ consists of only 2-handle attachments, then there exists a deformation of $J$ (rel $M_1$) to an integrable complex structure $\tilde{J}$ on $X$.

Using this theorem, we obtain a Stein structure on $X$ for which the complex tangencies on $M_1$ are $\xi_1$ and on $M_2$ are some contact structure $\xi'$ homotopy equivalent to $\xi_2$. Now, we are done if $\xi'$ is overtwisted, since overtwisted contact structures are classified by their 2-plane field homotopy type [1]. But we can easily ensure that the contact structure on $M_2$ is overtwisted by adding some extra Lutz twists to $(M_1, \xi_1)$ that are disjoint from the regions where the 2-handles are attached. □

**Proposition 4.3.** Given a tight contact manifold $(M, \xi)$, there exists an overtwisted contact structure $\xi'$ on $M$ in the same homotopy class as $\xi$ and which satisfies $(M, \xi') \prec (M, \xi)$.

**Proof.** Given $(M, \xi)$, take a Legendrian curve $L \subset M$ and its standard neighborhood $N(L)$. Choose a framing as in Section 2 so that the slope of the dividing set of $\partial N(L)$ is $\infty$. Now, identify slopes $s \in \mathbb{R} \cup \{\infty\}$ with
their respective “angles”, \([\theta_s] \in \mathbb{R}/\pi \mathbb{Z}\). In order to distinguish the different amounts of “wrapping around”, we will choose a lift \(\theta_s \in \mathbb{R}\) instead. There exists an exhaustion of \(N(L)\) by concentric \(T^2\), where the angles of the dividing curves on the tori monotonically increase over the interval \([\frac{\pi}{2}, \pi]\) as the \(T^2\) move towards the core.

Now, let \((M, \xi')\) be the overtwisted 3-manifold obtained by performing a full Lutz twist along \(L\). This replaces \(N(L)\) by the solid torus \(N\), where the angles of the dividing curves of an exhaustion by tori monotonically increase over the interval \([\frac{3\pi}{2}, \frac{5\pi}{2}]\). We claim that a full Lutz twist \((M, \xi) \rightarrow (M, \xi')\) is the inverse process of a sequence of Legendrian surgeries along the same core. In fact, take a Legendrian curve \(K\) in \((M, \xi')\) in the same isotopy class as \(L\), whose standard neighborhood \(N(K) \subset N\) has an exhausting set of tori which spans the interval \([3\pi - \frac{3\pi}{2}, 3\pi]\). After Legendrian surgery, the new \(N\) “rotates” in the interval \([\frac{3\pi}{2}, \frac{5\pi}{2}]\). Repeated application (total of 4 times) of Legendrian surgery will get us back to \((M, \xi)\). Note, however, that the intermediate manifolds are not necessarily diffeomorphic to \(M\).

Combining Propositions 4.1 and 4.3, we immediately get Theorem 1.2.

5. CONCAVE FILLINGS

In this section we prove Theorem 1.3. Before we set out on the proof, we give a straightforward proof of this theorem for overtwisted contact structures.

**Lemma 5.1.** Theorem 1.3 is true for any overtwisted contact structure.

*Proof.* Given any overtwisted contact structure \((M, \xi)\), we know by Theorem 1.2 that there is a Stein cobordism \((X, \omega)\) from \((M, \xi)\) to \((S^3, \xi_{std})\). Let \((Y, \omega')\) be any closed symplectic 4-manifold. Use Darboux’s theorem to excise a small standard ball around a point in \(Y\) and obtain a manifold \(Y'\) with concave boundary \((S^3, \xi_{std})\). We then obtain a concave filling of \((M, \xi)\) by gluing \((X, \omega)\) to \((Y', \omega'|_{Y'})\). It is clear that there are infinitely many choices for \((Y, \omega')\) that will yield infinitely many different concave fillings for \((M, \xi)\).

**Lemma 5.2.** Theorem 1.3 is true for any Stein fillable contact structure.

*Proof.* Let \((M, \xi)\) be Stein filled by \((X, \omega)\). According to Corollary 3.3 in [21], there is a symplectic embedding of \((X, \omega)\) into a compact Kähler minimal surface \(S\) of general type. If we take \(Y = S \setminus X\), then \((Y, \omega|_Y)\) will be a concave symplectic filling of \((M, \xi)\).

A slight modification of the above argument will produce infinitely many concave fillings. Specifically, in a small standard 3-ball \((B^3, \xi_{std}) \subset (M, \xi)\), there exist a right-handed Legendrian trefoil knot with \(tb = 1\) and a linking Legendrian unknot with \(tb < 0\). If we add 2-handles to \(X\) along these Legendrian knots, we obtain a new Stein manifold \((X', \omega')\). Embed \(X'\) in
a compact Kähler surface \( S \) and remove \( X \) to obtain a concave symplectic filling \( (Y', \omega') \) of \( (M, \xi) \). In the layer \( X' \setminus X \) in \( Y' \) there exists a symplectically embedded torus \( T \) (see [16]). Let \( E(n) \) be the elliptic surface obtained by taking the normal sum \([17]\) of \( n \geq 1 \) copies of the rational elliptic surface along regular fibers. Then consider the symplectic manifold \( Y_n = E(n) \#_T Y' \), obtained by taking the normal sum of \( Y' \) along \( T \) and \( E(n) \) along a regular fiber. These concave fillings of \( (M, \xi) \) are not related by blowing up and down, since if they were then the compact manifolds \( S_n \), obtained from \( S \) by normal summing with \( E_n \), would also be so related. However, this is not the case, as \( b_2^+ (S_n) = b_2^+ (S) + 2n \) and \( b_2^+ \) is unchanged by blowing up and down.

Theorem 1.3 now follows from Lemma 5.2 and Theorem 1.1.

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