Pion form factor from an AdS deformed background

Miguel Angel Martín Contreras¹,∗ Eduardo Folco Capossoli²,†
Danning Li³,‡ Alfredo Vega¹,§ and Henrique Boschi-Filho¶

¹Instituto de Física y Astronomía, Universidad de Valparaíso,
A. Gran Bretaña 1111, Valparaíso, Chile
²Departamento de Física / Mestrado Profissional
em Práticas de Educação Básica (MPPEB),
Colégio Pedro II, 20.921-903 - Rio de Janeiro-RJ - Brazil
³Department of Physics and Siyuan Laboratory,
Jinan University, Guangzhou 510632, China
⁴Instituto de Física, Universidade Federal do Rio de Janeiro,
21.941-972 - Rio de Janeiro-RJ - Brazil

Abstract

We consider a bottom-up AdS/QCD model with a conformal exponential deformation \( e^{k_I z^2} \)
on a Lorentz invariant AdS background, where \( k_I \) stands for the scale \( k_\pi \) that fixes confinement
in the pion case and \( k_\gamma \) for the kinematical energy scale associated with the virtual photon. In
this model we assume the conformal dimension associated with the operator that creates pions
at the boundary as \( \Delta = 3 \), as in the original bottom-up AdS/QCD proposals. Regarding the
geometric slope related to photon field \( k_\gamma \), we analyze two cases: constant and depending on the
transferred momentum \( q \). In these two cases we computed the electromagnetic pion form factor as
well as the pion radius. We compare our results with experimental data as well as other theoretical
(holographic and non-holographic) models. In particular, for the momentum dependent scale, we
find good agreement with the available experimental data as well as non-holographic models.

Keywords: Pion form factor, Gauge/gravity duality, Hadronic Physics

∗Electronic address: miguelangel.martin@uv.cl
†Electronic address: eduardo_capossoli@cp2.g12.br
‡Electronic address: lidanning@jnu.edu.cn
§Electronic address: alfredo.vega@uv.cl
¶Electronic address: boschi@if.ufrj.br
I. INTRODUCTION

Since its proposal and its discovery, the pion still attracts a lot of attention from the high-energy community. The pion is the simplest hadron modeled by QCD and due to the two main features of QCD — asymptotic freedom and confinement — the pion itself plays a role of the most appropriate character to probe the interplay between soft and hard regimes. The soft regime is represented by low-energy scale (small $q^2$) and it can only be investigated by non-perturbative methods. On the other hand, hard regimes belong to the high-energy realm (large $q^2$) can be studied through perturbative techniques. Pions, along with nucleons, are non-perturbative particles responsible for nuclear stability since they are strong force mediators, and also they are the Goldstone bosons of QCD. In particular, the last property has a deeper connection with the mass generation in QCD. Thus, a proper understanding of the pion structure is needed. However, this issue is not accessible to address since its intrinsic non-perturbative nature. Experimentally, the electromagnetic pion form factor was measured in the low $q^2$ region by scattering pions over electrons at CERN, as can be seen in Ref. [1]. In the intermediate $q^2$ range (up to 5 MeV$^2$), the electromagnetic pion form factor is measured from the electroproduction of pions from nucleon scattering processes in facilities as DESY, JLab, and Cornell, for instance in Refs. [2–4], respectively. Regarding theoretical approaches this direction, one can find in literature, we many propositions to compute the pion form factor, such as, QCD sum rules, dispersion relations with QCD constraint, Dyson-Schwinger equation, perturbative QCD (pQCD), light-front quark model (LFQM), and extended vector meson dominance model (extVMD), and Lattice QCD at $N_f = 2$. These approaches are presented in Refs. [5–11], respectively.

The end of the 1990s brought a major breakthrough in theoretical physics. The seminal papers written by Juan Maldacena [12] and soon after by Witten [13, 14] and by Gubser, Klebanov, and Polyakov [15] presented to us how to connect a $(d + 1)$-dimensional gravitational theory to a $d$-dimensional conformal field theory (CFT). Such a relation is known as gauge/gravity correspondence or duality. On the gravity side the theory is established over a high-dimensional curved anti-de Sitter (AdS$_{d+1}$) spacetime (the bulk) meanwhile in the CFT side it is established over a $d$-dimensional flat Minkowski spacetime (the boundary). Although, conceptually speaking, this duality seems extremely abstract, it relationship with high-energy physics became quite straight forward since one can construct a dual gravita-
tional theory in the bulk and related it to a super Yang-Mills theories on the boundary. In particular, and probably the main goal in its use, by breaking the conformal symmetry, one can study QCD-like theories or AdS/QCD models.

The breaking of the conformal invariance is an important subject for the usage of AdS/CFT correspondence since real QCD is not a conformal theory. A very famous AdS/QCD model which breaks conformal symmetry and provides an excellent result for vector meson spectroscopy is the softwall model (SWM) as proposed in the very first time in Ref. [16] which includes a scalar (dilaton) field in the action for the fields.

However, it was argued in Refs. [17, 18] that the original formulation of the SWM, even breaking the conformal invariance, it seems to not work well for the glueball spectroscopy. Besides, in Refs. [19, 20] it was also shown that the original formulation of SWM does not produce a mass gap for the fermionic sector. In order to overcome these mentioned questions one can see in Refs. [21–29] an incomplete list where the authors proposed some modifications in the original SWM. Among many modifications, we rather follow the one proposed in Refs. [30, 31] where a scalar (dilaton) field is introduced in the AdS metric instead of in the action for the fields as done in the original formulation of SWM.

By working within this approach the authors of Ref. [32] could, at same time, break the conformal invariance and compute compatible results for the masses of even and odd spin glueballs, scalar mesons, vector mesons and fermions with spins 1/2, 3/2 and 5/2. This approach was also used in the following incomplete list of Refs. [33–43], where one can check many works dealing with holographic high-energy physics and using some kind of deformation in the AdS space metric.

In this sense the AdS/QCD program is very suitable to study both soft and hard regimes at same time. Since the publication of the pioneer work in Ref. [44] many authors have used some AdS/QCD model to study the pion [45–47] or the pion form factor [48–55].

Here, in this work, by using an AdS/QCD model taking into account a deformed AdS$_5$ space, we will compute the pion electromagnetic form factor, which characterize the interaction of a pion with an external photon, as a function $F_\pi(q^2)$ of the squared four-momentum transfer $q^2$. Notice that our calculation does not define the bulk field conformal dimension, $\Delta$, in terms of the meson constituent number with the twist. We keep the spectroscopic definition of $\Delta$, i.e., defined in terms of the scaling dimension of the operators responsible for creating hadrons at the boundary. This choice implies that at large $q^2$ limit, pion form
factors do not acquire the expected scaling behavior $1/q^2$. However, we propose a solution to this issue by promoting the virtual photon geometric slope to be depending on the transfer momentum. This solution fits well within these models, where each particle is defined with different backgrounds. The confinement scale, related to the pion, is different from the virtual photon scale associated with the kinematics of the scattering process. Finally, we compare our holographic results with the available experimental data as well as other theoretical holographic and non-holographic approaches.

This work is organized as follows: in Section II we present our deformed AdS/QCD model as well as we describe the scalar and the gauge boson fields in the bulk which represent, respectively, the pion and the photon fields at the boundary. In Section III, after a brief review of the pion properties, we present the interaction action of the scalar and gauge fields and compute holographically the pion form factor. In Section IV we present our results in comparison with experimental data and other theoretical approaches. Finally, in Section V we present our conclusions.

II. DEFORMED AdS/QCD MODEL

In this section we present the deformed AdS/QCD model which will be used to calculate the pion form factor. Let us start writing its 5-dimensional action as:

$$ S = \int d^5 x \sqrt{-g} \, \mathcal{L}, $$

(1)

where $\mathcal{L}$ is the Lagrangian density to be detailed below for the scalar and gauge fields, and $g$ is the determinant of the metric $g_{mn}$ of the deformed $AdS_5$ space:

$$ ds^2 = g^I_{mn} dx^m dx^n = e^{k_I z^2} \left( dz^2 + \eta_{\mu\nu} dy^\mu dy^\nu \right), $$

(2)

where the index $I = \pi, \gamma$ is associated with the pion and the photon, respectively. Note that the geometry $g^I_{mn}$ is different for each particle, since they interact differently with the static background. Accordingly, the parameters $k_\pi$ and $k_\gamma$ are related to the pion and the photon, respectively.

To avoid possible misunderstandings one should note that we will use throughout this text the indices $m, n, \cdots$ to refer to the five-dimensional space, separating into $\mu, \nu, \cdots$ for the 4-dimensional Minkowski spacetime and the holographic $z$ coordinate. The Minkowski flat spacetime is endowed with metric $\eta_{\mu\nu}$ with signature $(-, +, +, +)$. 4
The introduction of the conformal exponential factor $e^{k_I z^2}$ in the AdS metric as above defines our deformed AdS space which is asymptotically AdS for $k_I \rightarrow 0$. At the UV boundary $z \rightarrow 0$ the exponential deformation is negligible.

At this point it is very important to make a brief discussion in order to recall how confinement is achieved in our proposed model. When we are dealing with bottom-up models, we consider the duality between perturbative bulk fields living in an AdS space and non-perturbative operators at the boundary. In the pure AdS, normalizable modes coming from the associated holographic potential form a continuum spectrum. Thus, confinement in these sorts of models is achieved by discretizing such a spectrum. This procedure can be done by imposing a hard cutoff (as in the quantum square well) [56, 57] or modifying the holographic potential’s large $z$ behavior (Soft wall model and deformed background). The former defines the so-called hard-wall model. The latter opens the possibility to deform the geometry or use extra auxiliary bulk fields, as in the soft-wall model where a static dilaton field is used. In our particular case, we use a quadratic and static deformation function that induces linear Regge trajectories for baryons and mesons. It is in this frame where we set up our pion field, defined by the Regge slope $k_\pi$. In this case the deformation slope is fixed to get the pion mass which defines the confining IR scale. This also implies that the geometry is not the same for the pion and for the virtual photon. The deformation slope in the case of the virtual photon, $k_\gamma$, is not associated with confinement. This parameter is related to the energy scale of the scattering process, as in the case of deep inelastic scattering in bottom-up models [20, 58, 59].

In the following subsections, we describe the free scalar and gauge fields actions, which represent, respectively, the pion and the photon fields. The interaction action of these fields which account for the electromagnetic pion form factor will be discussed in Section III.

A. Scalar field in the deformed AdS/QCD model

A massive scalar field $X$ in the deformed AdS$_5$ space, Eq. (2), is described by the action:

$$S = \int d^5x \sqrt{-g_\pi} \left[ g_\pi^{mn} \partial_m X \partial^n X + M_5^2 X^2 \right],$$  \hspace{1cm} (3)

where $g_\pi^{mn}$ is the metric defined in Eq. (2) related to the pion.

It is worthwhile to mention that the scalar field in the bulk will represent the mesonic
particles in 4 dimensions. In particular, we are interested in the pion particle.

From the action (3) one can find the following equation of motion:

$$\partial_m [\sqrt{-g_\pi} g_\pi^{mn} \partial_n X] - \sqrt{-g_\pi} M_5^2 X = 0.$$  \hspace{1cm} (4)

Writing $g_\pi^{mn} = e^{-2A_\pi(z)} \eta^{mn}$, with the warp factor $A_\pi(z)$ given by:

$$A_\pi(z) = - \log z + \frac{k_\pi}{2} z^2,$$  \hspace{1cm} (5)

the equation of motion Eq. (4) can then be rewritten as:

$$\partial_m [e^{3A_\pi(z)} \eta^{mn} \partial_n X] - e^{5A_\pi(z)} M_5^2 X = 0,$$  \hspace{1cm} (6)

or defining $B(z) = -3A_\pi(z)$, one has:

$$\partial_m [e^{-B(z)} \eta^{mn} \partial_n X] - e^{-5B(z)/3} M_5^2 X = 0.$$  \hspace{1cm} (7)

Next, we use a plane wave ansatz with amplitude just depending on the $z$ coordinate and propagating in the transverse coordinates $x^\mu$ with momentum $q_\mu$,

$$X(z, x^\mu) = v(z) e^{-iq_\mu x^\mu}.$$  \hspace{1cm} (8)

After some algebraic manipulation and defining $v(z) = \psi(z) e^{B(z)/2}$ one has a “Schrödinger-like” equation:

$$- \psi''(z) + \left[ \frac{B'^2(z)}{4} - \frac{B''(z)}{2} + e^{-2B(z)/3} M_5^2 \right] \psi(z) = -q^2 \psi(z),$$  \hspace{1cm} (9)

where $M_5$ is the scalar field (mesons) mass in five dimensions and $E = -q^2$ are the eigenenergies which represent the mesons masses in four dimensions.

By using the AdS/CFT prescription one can learn how to relate the bulk mass $M_5$ to the conformal dimension $\Delta$ of an operator in four dimensions, so that:

$$M_5^2 = (\Delta - p)(\Delta + p - 4),$$  \hspace{1cm} (10)

where $p$ represents the index of the $p$–form which in this case is associated with the hadronic spin $S$.

From the QCD description we know that the scalar mesons are composed by a bound state of quark-antiquark belonging to a spin singlet with total total angular momentum
\[ J = L + S = 0 \] and for our purposes in this work we will disregard all other meson quantum numbers. Besides scalar mesons are represented, in the boundary, by the operator:

\[ \mathcal{O}_{SM} = \bar{q} D_{J_1 \cdots D_{J_m}} q \quad \text{with} \quad \sum_{i=1}^{m} J_i = J \quad (11) \]

The contribution coming from each quark is \( 3/2 \), then the conformal dimension reads \( \Delta = 3/2 + 3/2 = 3 \) and the bulk mass in Eq. (10) is \( M^2_5 = -3 \). Replacing this result in Eq. (9) one gets:

\[ -\psi''(z) + \left[ \frac{B^2(z)}{4} - \frac{B''(z)}{2} - 3 e^{-B(z)} \right] \psi(z) = -q^2 \psi(z). \quad (12) \]

This equation does not have analytic solutions. Then, solving it numerically with \( k_\pi = -0.0425^2 \) GeV\(^2\) we get \( m_\pi = 0.139 \) GeV which is compatible with the meson \( \pi \) mass [60].

In Fig. 1 (left panel) we present the holographic potential for pions associated with the “Schrödinger-like” equation, Eq. (12). In the right panel of Fig. 1 we present the holographic pion eigenfunctions for the states 1\( S \), 2\( S \) and 3\( S \).

Figure 1: Left panel depicts the holographic potential for bulk eigenmodes dual to pions. Right panel shows the ground state and the first two excited bulk eigenmodes dual to pions.

### B. Gauge boson field in the deformed AdS/QCD model

Here in this section we describe within our deformed AdS space the gauge boson field which represents the physical photon at UV. Such a photon will interact with the pion through an electromagnetic current contributing to the calculation of the pion form factor.
To do so we introduce the action for a five dimensional massless gauge boson field \( \phi^\gamma(x^\mu, z) \), so that:

\[
S = -\frac{1}{c_2^2} \int d^5x \sqrt{-g_\gamma} \frac{1}{4} F_{mn}^\gamma F_{mn}^\gamma ,
\]

(13)

where \( g^\gamma_{mn} \) is the metric presented in Eq. (2) associated with the photon, and the electromagnetic tensor \( F_{mn}^\gamma \) is written as usual as \( F_{mn}^\gamma = \partial^m \phi^n - \partial^n \phi^m \).

From the above action one derives the equations of motion:

\[
\partial_m[\sqrt{-g_\gamma} F_{mn}^\gamma] = 0 .
\]

(14)

Since the metric \( g^\gamma_{mn} \), Eq. (2), is diagonal, for \( n = z \) one gets

\[
\Box \phi_z - \partial_z[\partial_\mu \eta^{\mu\nu} \phi_\nu] = 0 ,
\]

(15)

or simply

\[
\Box \phi_z - \partial_z (\partial_\mu \phi^\mu) = 0 ,
\]

(16)

while for \( n = \mu \) one has

\[
e^{-A_\gamma(z)} \partial_z[e^{A_\gamma(z)} \partial_\mu \phi_\nu] + \Box \phi_\nu - \partial_\nu \left( e^{-A_\gamma(z)} \partial_z[e^{A_\gamma(z)} \phi_z] + \partial_\alpha \phi^\alpha \right) = 0 ,
\]

(17)

where \( \Box \equiv \eta^{\alpha\beta} \partial_\alpha \partial_\beta \). The electromagnetic field profile comes from the solutions of Eqs. (16) and (17). Choosing the gauge:

\[
\partial_\nu \left( e^{-A_\gamma(z)} \partial_z[e^{A_\gamma(z)} \phi_z] + \partial_\alpha \phi^\alpha \right) = 0 ,
\]

(18)

the Eq. (17), written in Fourier space, reduces to

\[
-q^2 \phi_\mu + A'_\gamma \partial_z \phi_\mu + \partial_z^2 \phi_\mu = 0 ,
\]

(19)

where \( A_\gamma = A_\gamma(z) \) is analogous to Eq.(5) but now related to the photon, such as

\[
A_\gamma(z) = - \log z + \frac{k_\gamma}{2} z^2 ,
\]

(20)

and prime denotes derivative with respect to \( z \). This kind of gauge fixing was used in Refs. [19, 20, 58, 59, 61, 62], when discussing deep inelastic scattering (DIS) holographically. Of course the gauge choice will not affect the physical results as, for instance, the pion form factor.
Further, we will consider a photon with a transversal polarization $\eta$ such that $\eta \cdot q = 0$. In this sense Eq. (16) will not contribute for our calculations, and only the electromagnetic field component $\phi^\mu$ will be relevant to the pion form factor. Such a consideration was also done in Refs. \[19, 20, 58, 59, 61, 62\], in holographic DIS studies to calculate structure functions.

The general solution to equation (19) has the following form:

$$
\phi^\mu(z, q) = C_{1\mu}^1(q) e^{iqy} G_{1,2}^{2,0} \left( \frac{k_\gamma z^2}{2}, \frac{q^2}{2k_\gamma} + 1 \right) - \frac{1}{2} C_{2\mu}^2(q) e^{iqy} k_\gamma z^2 F_1 \left( 1 - \frac{q^2}{2k_\gamma}; 2; -\frac{k_\gamma z^2}{2} \right) \tag{21}
$$

where

$$
G_{p,q}^{m,n} \left( z \left| \begin{array}{c} a_1 \cdots a_p \\ b_1 \cdots b_q \end{array} \right. \right) \quad \text{and} \quad F_1(a; b; z)
$$

are the the Meijer G function and the Kummer confluent hypergeometric function, respectively. By imposing the boundary condition $\phi^\mu(z, q)|_{z=0} = \eta^\mu e^{iqy}$, that implies $C_{2\mu}^2(q) = 0$, and considering normalizable (square integrable) solutions, one can write:

$$
\phi^\mu(z, q) = -\frac{\eta^\mu e^{iqy}}{2} k_\gamma z^2 \Gamma \left[ 1 - \frac{q^2}{2k_\gamma} \right] U \left( 1 - \frac{q^2}{2k_\gamma}; 2; -\frac{k_\gamma z^2}{2} \right)
$$

$$
\equiv -\frac{\eta^\mu e^{iqy}}{2} B(z, q), \tag{22}
$$

where $\Gamma[a]$ is the Gamma function and $U(a, b, z)$ is the Tricomi hypergeometric function \[63\]. This equation represents the solution for the electromagnetic field that will be used to compute the pion form factor.

### III. PION FORM FACTOR

The pion form factor is one of the most valuable QCD quantities related to the transition from the non-perturbative to the perturbative regime, appearing at large transferred momentum, $q$. In the electromagnetic case, the pion form factor comes from the annihilation or scattering of leptons interacting with charged pions. Specifically, it is defined from the photon-charged pions three-body vertex. Supposing a lepton scattering, we can write the corresponding amplitude as

$$
\mathcal{M} = \frac{1}{q^2} i Q \bar{u}(k_2) \gamma_\mu u(k_1) \langle \pi^\pm(p_2) | J_\pi^\mu(0) | \pi^\pm(p_1) \rangle, \tag{23}
$$
where $Q$ stands for the lepton electric charge, the four-vectors $k_i$ and $p_i$ labels the leptons and pions momenta, $q = p_2 - p_1$ is the virtual photon momentum which is the momentum transfer of the process, and $J_\pi^\mu$ is pion EM current. The matrix element $\langle \pi^\pm(p_2) | J_\pi^\mu(0) | \pi^\pm(p_1) \rangle$ describes the pion-photon vertex, and it has a general Lorentz structure defined in terms of the pions momenta as follows

$$
\langle \pi^\pm(p_2) | J_\pi^\mu(0) | \pi^\pm(p_1) \rangle = c_{\pi^\pm} (p_1 + p_2)^\mu F_\pi(q^2).
$$

(24)

with $c_{\pi^\pm}$ the vertex coupling constant that normalizes the pion form factor $F_\pi(q^2)$. Notice this structure ensures that gauge, time reversal, parity, and Lorentz invariance is fulfilled.

The Fig. 2 represents the Feynman diagram of a scattering between a pion and a lepton through the exchange of a virtual photon.

![Feynman diagram representing the scattering pions and leptons via the exchange of a virtual photon. The shaded blob represents the effective vertex used to define the electromagnetic pion form factor.](image)

Figure 2: Feynman diagram representing the scattering pions and leptons via the exchange of a virtual photon. The shaded blob represents the effective vertex used to define the electromagnetic pion form factor.

At the holographic level, we will suppose this pionic matrix element lives at the conformal boundary; thus, we will follow the AdS/CFT standard technics to write down an expression for the pion form factor using the deformed geometric model described above.

Let us focus on the holographic calculation: the three-point effective vertex in the bulk, dual to the two-pion-photon vertex, is defined via the following structure when the minimal coupling is imposed [64]:
\[ S_{\text{eff}} = g_{\text{eff}} \int d^5x \sqrt{-g} g_\mu^m \phi_m(x, z) \left[ X_{p_1}(x, z) \partial_m X_{p_2}^*(x, z) - X_{p_2}^*(x, z) \partial_m X_{p_1}(x, z) \right] , \]  

(25)

where the bulk fields \( X(x, z) \) and \( \phi_m(x, z) \), dual to the pion and the virtual photon, are defined in terms of the Eqns. (6) and (19). The coupling \( g_{\text{eff}} \) is a constant that fixes units in the effective action. These bulk fields can be spanned in terms of waves in AdS as

\[ X(x, z) = e^{-ip \cdot x} v(z), \]

\[ \phi_\mu(x, z) = \eta_\mu e^{-iq \cdot x} B(z). \]

Putting these definitions into the effective action and after some calculations, we will obtain

\[ S_{\text{eff}} = i (2 \pi)^4 \delta^4(q - (p_2 - p_1)) \eta^{\mu \nu} (p_1 + p_2)_\nu \ g_{\text{eff}} \int dz e^{3A(z)} v(z) B(z, q) v(z), \]

(26)

where the delta appears due to the four-momentum conservation at the vertex, and \( A(z) \) is the warp factor defined in Eq. (5).

Finally, from the effective action, isolating all of the terms associated with momentum conservation, we can extract the pion matrix element defined in Eq. (24), allowing us to write the pion form factor as

\[ F_\pi(q^2) = \int dz e^{3A(z)} v(z) B(z, q^2) v(z). \]

(27)

with \( v(z) \) is the scalar normalizable mode dual to pion. The field \( B(z, q) \) is the vector non-normalizable mode (bulk-to-boundary propagator) dual to the virtual photon.

Notice that under the boundary conditions, we expect that \( \phi_m(x, z \rightarrow 0) = \eta_m e^{-i q \cdot x} \), yielding \( B(z \rightarrow 0, q^2) = B(z, q^2 \rightarrow 0) = 1 \). These conditions imply naturally that \( F(q^2 \rightarrow 0) = 1 \), since \( v(z) \) is normalized. Therefore, we conclude that \( c_{\pi \pm} = g_{\text{eff}} = 1 \) in our approach.

We can go further by introducing the Schrodinger-like modes for the bulk scalar field, i.e., \( v(z) = e^{-3A(z)/2} \psi(z) \). Since we consider that the charged pion at the boundary is dual to the scalar field ground state, we will restrict ourselves to consider only \( \psi_1(z) \) (See figure 1). Thus, the final expression for the form factor in our case is

\[ F_\pi(q^2) = \int dz \psi_1(z) B(z, q^2) \psi_1(z). \]

(28)
This integral can not be computed by analytical approaches and then we will solve it numerically. After computing the pion form factor in Eq. (28), one can get the pion radius which is given by:

$$\langle r_\pi^2 \rangle = -6 \left. \frac{dF_\pi(q^2)}{dq^2} \right|_{q^2=0}.$$ 

(29)

In the next section we will present our results in comparison with the experimental data and some theoretical works (holographic and nonholographic).

**IV. NUMERICAL RESULTS FOR THE PION FORM FACTOR**

In this section we will explore and comment on our numerical results obtained from our deformed AdS/QCD model and compare them with experimental data as well as some theoretical approaches.

Just before to present our results it is worthwhile to make some comments on previous results for pion form factor which will guide us in order to explain our results. Let us start our discussion by experimental data on pion form factor. As one can see in Refs. [1–4, 65, 66] there is vast collection of those data obtained by renowned collaborations in the last 40 years. However one should notice that mostly of these data are related to soft processes (low $q^2$), or intermediate ones, and the few data related to hard processes (large $q^2$), so far, are not reliable.

In the framework of theoretical works, we will start with the iconic papers in Refs. [67, 68], Brodsky-Farrar and Matveev-Muradian-Tavkhelidze, respectively, predicted a scaling law at large transverse momentum meaning for large $q^2$ regime, pion form factor should behave as $F_\pi(q^2 \to \infty) \sim q^{-2}$. Another important prediction related to the pion form factor at at moderately large $q^2$ can be seen in Ref. [69] where Efremov and Radyushkin, taking into account quark counting rule (QCR) argue the the pion form factor behaves as $F_\pi(q^2 \to \infty) \sim (q^{-2})^{n-1}$, where $n$ corresponds the number of the valence quarks in a composed system. Also in Ref. [69] the authors mention that in this moderately large $q^2$ region the contribution coming from Feynman mechanism is damped by the Sudakov form factor meaning an abrupt decreasing of the pion form factor while $q^2$ increasing. For large $q^2$ regimes the process recovers $F_\pi(q^2 \to \infty) \sim q^{-2}$ behaviour. It is worthwhile to remember that other important theoretical works were cited in Sec. I. In particular, for the comparison with the results achieved in this work, we will focus in approaches showed in Refs. [5–10].
Studies related to the pion form factor within AdS/QCD program were motioned in Sec. I. For sake of completeness we will present them again here (see Refs. [48–55]). Such works achieved a good agreement with others found in the literature specially for soft or intermediate processes. In particular, the Refs. [48, 49] addressed two important questions related to bottom-up models, as the softwall one. The first question was presented in Ref. [48] where the authors have noticed that if one considers the conformal dimension as $\Delta = 3$, the behaviour for the corresponding form factors for vector mesons scales as $1/q^4$, which is inconsistent with the QCD sum rules. The second question was brought in Ref. [49], where the authors proposed that the conformal dimension associated with the bulk scalar field should be reinterpreted as the twist, i.e., the number of hadronic constituents, to get the correct results. Such a reinterpretation implies that the near-to-the-boundary behavior of the bulk modes also changes.

The motivation for our work can be enclosed in a single question: why the conformal dimension used in holography at constant time is interpreted as scaling dimension, while when we deal with form factors in light-front holography, it should be reinterpreted as the number of constituents?. In order to answer this question, our work explores another possibility: instead of reformulating the meaning of the conformal dimension, we modify the energy scale associated with the virtual photon in the scattering process. Recall that the photon energy scale is not related with the pion confinement process, since both particles are described by two different geometries. The photon scale measures the momentum transfer between the virtual photon and the pion at the interaction vertex. Thus, we can expect this scale to be a function of the transferred momentum, $q$.

After this brief digression we are able to present our numerical results within our deformed AdS/QCD model and compare it with the available experimental data, as well as other non-holographic and holographic results. In Section IV A and IV B we will present our results for both pion form factor and pion radius considering the canonical conformal dimension as $\Delta = 3$. In Section IV A, we take $k_{\gamma}$ independent of $q$, and in Section IV B, the energy scale $k_{\gamma}$ associated with the virtual photon in the scattering process is assumed to be a function of $q$. In particular, the results of section IV B for pion form factor and pion radius are in agreement with those found in the literature.
A. Pion form factor and pion radius for $\Delta = 3$

In this section we will present the results achieved within our deformed model considering the scaling dimension $\Delta = 3$ for the operators which will represent the pion at the boundary.

Computing numerically the integral in Eq. (28), one get the pion form factor, as a function of the squared four-momentum, in comparison with available experimental data, presented in the upper panel of Fig. 3.

Instead to fix only one value for the parameter $k$ and gets a single curve, we rather choose to present not only one curve, but a region (colored). Inside this region, our model works in agreement with the experimental data. In particular, this agreement is for small $q^2$.

In the lower panels in Fig. 3 we compare our results with other theoretical non-holographic approaches found in the literature (left panel), as well as with other holographic ones (right panel).

It is worthwhile to mention that the pion radius computed by using our model, from Eq. (29), considering $\Delta = 3$, is $r_\pi = 0.458$ fm with an error around 30% compared to the experimental one [60]. Besides one can also notice that our model with this consideration does not seem to capture the predicted scaling law $F_\pi(q^2 \to \infty) \sim q^{-2}$. In the appendix A we provide a brief review on this scaling law within softwall context.

In the next section we will propose a modification in our model, different from the one proposed by Brodsky and Teramond in Ref. [49], in order to accommodate the experimental/theoretical results for the pion radius and the scaling law $F_\pi(q^2 \to \infty) \sim q^{-2}$.

B. Pion form factor and pion radius for $\Delta = 3$ and $k$ dependent of the momentum

In ref. [49], the authors are using the light-front approach, based on in the softwall model with the dilaton field $\Phi(z) = \kappa^2 z^2$, which obeys to the AdS/CFT dictionary, where the conformal dimension of the operator which represents the scalar field should be $\Delta = 3$. However, their solution was based on considering their conformal dimension as the twist dimension ($\Delta = \tau = 2$), for the scalar particles which have spin $J = 0$. The twist is related to the number of the constituents of the hadron. By doing this they achieved the correct scaling law as can seen through Eq. A9 in appendix A.
Figure 3: The upper panel compares our results for the pion form factor with the available experimental data [1–4, 65, 66]. In the lower panels we have a comparison of our results with non-holographic models (left panel) such as BSE [7], perturbative QCD [8], dispersion relations [6], sum rules [5], and LFQCD [9]. In the lower right panel, we depict a comparison with other holographic models such as hardwall and softwall with $\Delta = 2$ [49], and Sakai-Sugimoto/extrapolated Sakai-Sugimoto [52]. In our results we have taken $k_\gamma = -3.8$ GeV$^2$.

Here, in this work, we will not resort to the twist dimension. This means that we will consider $\Delta = 3$. We will propose that our free parameter could depends on the scale of energy in the scattering process with profile $k_\gamma \rightarrow k_\gamma(q) = q k_\gamma$.

Assuming this simple profile we could accommodate experimental/theoretical results for small, intermediate and large $q^2$. Besides, the value we have found for the pion radius, $r_\pi = 0.671$ fm, is in agreement with the experimental data [60] presenting a relative error
Figure 4: The upper panel compares our results for the pion form factor with the available experimental data [1–4, 65, 66] using the proposed scaling for $k_\gamma$. In the lower panels we depict a comparison of our results with non-holographic models (left panel) such as BSE [7], perturbative QCD [8], dispersion relations [6], sum rules [5], and LFQCD [9]. In the lower right panel, we show a comparison with other holographic models such as hardwall and softwall with $\Delta = 2$ [49], and Sakai-Sugimoto/extrapolated Sakai-Sugimoto [52]. In our results, for lower panels we have taken $k_\gamma = -2.8$ GeV$^2$.

of 2.0 %, and the scaling law $F_\pi(q^2 \to \infty) \sim q^{-2}$ is recovered. These results can be seen in Fig. 4.

The motivation behind this particular choice of $k_\gamma$ comes from the analysis of the pion form factor $F_\pi(q)$ in the softwall model context, with quadratic dilaton $\Phi(z) = |k_\gamma| z^2$, see appendix B. Then, eq. (B6) has the following form:
\[ F_\pi(q) = \frac{32 k_\gamma^2}{(q^2 + 4 |k_\gamma|)(q^2 + 8 |k_\gamma|)}, \]  

(30)

where \(|k_\gamma|\) is our deformation slope in energy squared units.

In general, this behavior is not restricted to the softwall model only. It is expected to appear in any other AdS/QCD model that has quadratic static dilaton or geometric deformation. Therefore we propose a rescaling of our photon slope as \( k_\gamma \rightarrow k_\gamma(q) = q k_\gamma \), that recovers the expected phenomenological behavior \( F_\pi|_{q^2 \to \infty} \rightarrow q^{-2} \), produces good agreement with the experimental/theoretical data and gives a very low relative error for the pion radius.

V. CONCLUSIONS

In this work we have discussed the pion form factor calculation in the context of a geometric deformed AdS/QCD model, considering the conformal dimension \( \Delta = 3 \) associated with the operator that creates pions in the boundary.

In this formalism with fixed \( k_\gamma \), the scaling law for the pion form factor at \( q^2 \rightarrow \infty \) is not captured properly. Despite that numerical results are reasonable for the intermediate \( q^2 \) region, as Figure 3 shows, the pion radius is also not well fitted.

In order to improve our results, we still consider \( \Delta = 3 \), and propose a rescaling in the parameter \( k_\gamma \) with the transferred momentum \( q \), i.e., \( k_\gamma \rightarrow k_\gamma(q) = q k_\gamma \) that fixes the form factor behavior at large \( q^2 \).

This assumption seems to be natural since, as we discussed in appendix A, large \( q^2 \) behavior comes from the analysis of the pion eigenmodes near the conformal boundary, where the dilaton or the deformation do not have any effect.

The approach exposed here is another possibility rather than the proposal presented by Brodsky and de Teramond [49], where they consider \( \Delta = 2 \) in light-front softwall model.

It is worth to mention that our approach seems to present a Sudakov suppression similar to the one pointed out in Ref. [69], which is not manifest in other bottom-up AdS/QCD models, as the hardwall and the softwall [49].
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Appendix A: Large $q^2$ analysis in the AdS deformed background

Let us consider the photon bulk to boundary propagator given by eqn. (22). Following [49], we found an integral representation for the Tricomi function

$$\Gamma(a) \mathcal{U}(1, b, z) = \int_0^\infty dt e^{-zt} t^{a-1} (1 + t)^{b-a-1}, \quad (A1)$$

which in our particular case reads as

$$\Gamma\left(1 + \frac{q^2}{2|k|}\right) \mathcal{U}\left(1 + \frac{q^2}{2|k|}, 2, \frac{|k| z^2}{2}\right) = \int_0^\infty dt e^{-\frac{|k| z^2}{2}} \left(1 + \frac{t}{t}\right)^{-\frac{q^2}{2|k|}}, \quad (A2)$$

where we have redefined $k \equiv -|k|$, since it is defined as a negative quantity per se. Therefore, the propagator $\mathcal{B}(z, q^2)$ is written as

$$\mathcal{B}(z, q^2) = \frac{|k| z^2}{2} \int_0^\infty dt e^{-\frac{|k| z^2}{2}} \left(1 + \frac{t}{t}\right)^{-\frac{q^2}{2|k|}}. \quad (A3)$$

In order to consider large $q^2$ limit, we need to do the transformation $t = \frac{q^2}{2|k|} \mu$ in the integral representation defined above (See Appendix C in Ref. [49]), obtaining

$$\mathcal{B}(z, q^2) = \frac{|k| z^2}{4} \int_0^\infty d\mu e^{-\frac{q^2 z^2}{2|k|} \mu} \left(1 + \frac{q^2}{2|k|} \mu \right)^{-\frac{q^2}{2|k|}}. \quad (A4)$$

Now, we can perform the large $q^2$ limit, yielding
\[
\mathcal{B}(z, q^2 \to \infty) = \frac{q^2 z^2}{4} \int_0^\infty d\mu \, e^{\frac{q^2 \mu}{4}} e^{-\frac{z \mu}{\mu}}. \tag{A5}
\]

On the other hand, we have the following integral representation for Bessel functions

\[
K_\nu(z) = \frac{1}{2} \left( \frac{z}{2} \right)^\nu \int_0^\infty dx \, e^{-x - \frac{x^2}{4}}. \tag{A6}
\]

If we fix \( \nu = -1 \) we arrive to

\[
\mathcal{B}(z, q^2 \to \infty) = q z K_1(q z), \tag{A7}
\]

since parity of Bessel functions: \( K_{-\nu}(z) = K_\nu(z) \). From holographic grounds, taking \( q^2 \to \infty \) implies \( z \to 0 \), since we have \( z \to 1/q \). In this limit, the form factor is written as

\[
F_\pi(q^2) \bigg|_{q^2 \to \infty} = \int_{z \to 1/q} \frac{dz}{z^3} \left( z^\Delta \right)^2 q z K_1(q z)
= \int_{z \to 1/q} \frac{dz}{z^3} \left( z^\Delta \right)^2 \left[ 1 + \frac{q^2}{4} \left( -1 + 2 \gamma_e + 2 \log \frac{q z}{2} \right) z^2 + \mathcal{O}(z^4) \right]
= \frac{1}{8 \Delta^2 (\Delta - 1)} \left( \frac{1}{q^2} \right)^{\Delta - 1} \left[ 1 + \gamma_e (\Delta - 1) \Delta - \Delta^2 (\log 4 - 3) + \Delta \log 4 \right]
\]

where we have taken the low \( z \)-limit of the pion eigenmodes and the warp factor, and also the power series of the Bessel function \( K_\nu(z) \). Finally, that the form factor in this case scales as

\[
F_\pi(q^2) \bigg|_{q^2 \to \infty} \to \left( \frac{1}{q^2} \right)^{\Delta - 1}. \tag{A9}
\]

Notice this scaling is expected from the solutions obtained for the \( \Delta = 3 \) within the SWM. Recall the dilaton or the deformation do not affect the low-\( z \) behavior of the bulk eigenmodes.

**Appendix B: Pion form factor in the original softwall model**

Let us consider the softwall model (SWM) [16], with \( \Phi(z) = \kappa^2 z^2 \), applied to scalar mesons in order to compute the pion form factor, following the same idea proposed in Ref.
[49], but without considering the twist dimension. In general, for the scalar mesons, the scaling dimension $\Delta = 3$ fixes the bulk mass to be $M^2 R^2 = -3$. In such a case, solutions are written as associated Laguerre polynomials [70]. In this scenario, the pion is associated with the lowest eigenmode, which is $n = 0$, giving a cubic polynomial in $z$, i.e.,

$$\psi_0(z) = \sqrt{\frac{2 \kappa^4}{R^3}} z^3,$$

where $\kappa$ stands for the dilaton slope that defines the scalar meson linear Regge trajectory and $R$ is the AdS curvature radius. The bulk-to-boundary propagator $\mathcal{V}(z, q)$ associated with the photon in the softwall model (SWM obeys the following equation of motion

$$\partial_z \left[ e^{-\kappa^2 z^2} z \mathcal{V}'(z) \right] + (-q^2) e^{-\kappa^2 z^2} z \mathcal{V}(z) = 0.$$  

The solution of this equation, in general, is written in terms of Tricomi hyperconfluent functions as

$$\mathcal{V}(z, q) = \Gamma \left( 1 + \frac{q^2}{4 \kappa^2} \right) U \left( \frac{q^2}{4 \kappa^2}, 0, \kappa^2 z^2 \right) = \kappa^2 z^2 \int_0^1 \frac{dx}{(1-x)^2} e^{-\frac{\kappa^2 x}{1-x}} x^{\frac{q^2}{4 \kappa^2}},$$

where we have used the integral representation for the bulk-to-boundary propagator, suggested by Grigoryan and Radyushkin, in Ref. [71], in the right part of the equation. From this integral definition for the pion form factor, we have

$$F_\pi(q^2) = R^3 \int_0^\infty dz \frac{e^{-\kappa^2 z^2}}{z^3} \psi_0^*(z) \mathcal{V}(z, q) \psi_0(z)$$

$$= 2 \kappa^6 \int_0^1 \frac{dx}{(1-x)^2} x^{\frac{q^2}{4 \kappa^2}} \int_0^\infty dz z^5 e^{-\frac{\kappa^2 x}{1-x}}$$

$$= \frac{32 \kappa^4}{(q^2 + 4 \kappa^2) (q^2 + 8 \kappa^2)}$$

As we are considering from the beginning of this section that the conformal dimension $\Delta = 3$, one should notice that the form factor in this scenario suggests a $q^{-4}$ behavior which is not expected from the known particle phenomenology [67]. Moreover, since any deformation or dilaton used in the AdS background does not modify the UV behavior of the bulk solutions, we can say that this is a general feature of the softwall-like AdS/QCD models.
However if we consider in Eq. (B6), $\kappa \to \sqrt{q} \kappa$, meaning the dilaton slope is now depending on $q$ or the energy scale in the process. And then, For large $q^2$, Eq. (B6), behaves as:

$$F_\pi(q^2) \sim \frac{1}{q^2},$$

(B7)

fulfilling the expected scaling law even considering $\Delta = 3$. Another possibility is to consider $\Delta = 2$ as done in Ref. [49] in the context of Light-Front holography.

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