An Elementary Proof for the Double Bubble Problem in $\ell^1$ Norm

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Abstract
We study the double bubble problem with perimeter taken with respect to the $\ell_1$ norm on $\mathbb{R}^2$. We give an elementary proof for the existence of minimizing sets for any volume ratio parameter $0 < \alpha \leq 1$ by direct comparison to a small family of parameterized sets. By simple analysis on this family, we obtain the minimizing shapes found in Morgan et al. in J Geom Anal 8(1):97-*115, 1998.

Keywords Isoperimetry · Double bubble problem · Taxicab metric · Optimization

Mathematics Subject Classification 49Q05 · 49Q20

1 Introduction

In [5] and [6], the double bubble conjecture in $\mathbb{R}^2$ and $\mathbb{R}^3$ was established, stating that the unique perimeter-minimizing double bubble which encloses two fixed volumes consists of three spherical caps whose tangents meet at an angle of 120 degrees. Recently, the Gaussian double bubble conjecture was established by Milman and...
These problems are an extension of the classical isoperimetric problem stating that the perimeter-minimizing shape for a fixed volume is the sphere. The case of the isoperimetric problem in which the perimeter was taken with respect to any norm on $\mathbb{R}^n$ was solved as well. Namely, Taylor [10, 11] proved that the unique solution of the isoperimetric inequality with respect to any norm $\rho$ is the renormalized ball in the dual norm, the so-called Wulff construction [12]. For example, the isoperimetric shape with respect to the $\ell_1$ norm is the $\ell_\infty$ ball $[0, 1]^n$. Such nonisotropic isoperimetric problems arose naturally in the field of probability, mostly in scaling limits of percolation clusters in a lattice [1–4]. In this paper, we study the double bubble problem with respect to the $\ell_1$ norm. The result discussed in this paper was first proved in [9], and is based on previous geometric measure theory results. However, our proof is self-contained and considerably simpler. Moreover, our simple approach that uses no geometric measure theory, seems to be more amenable to generalizations to higher dimensions.

1.1 Notations and Results

For any Lebesgue-measurable set $A \subset \mathbb{R}^2$, let $\mu(A)$ be its Lebesgue measure, $\bar{A}$ its closure, $A^\circ$ its interior, and $\partial A$ its boundary. For a simple curve $\lambda : [a, b] \rightarrow \mathbb{R}^2$, not necessarily closed, where $\lambda(t) = (x(t), y(t))$ define its $\ell^1$ length by

$$\rho(\lambda) = \sup_{N \geq 1} \sup_{a \leq t_1 \leq \ldots \leq t_N \leq b} \sum_{i=1}^N \left( |x(t_{i+1}) - x(t_i)| + |y(t_{i+1}) - y(t_i)| \right).$$

If we wish to measure only a portion of the curve $\lambda$, it will be denoted $\rho(\lambda([t, t']))$, where $[t, t'] \subset [a, b]$. For simplicity, we assume that $[a, b] = [0, 1]$ unless otherwise stated.

We say that two curves $\lambda, \lambda' : [0, 1] \rightarrow \mathbb{R}^2$ intersect nontrivially if there are intervals $[s, s'], [t, t'] \subset [0, 1]$ such that $\lambda([s, s']) = \lambda'([t, t'])$, then their nontrivial intersection can be written as the union of curves $\lambda_i$ such that $\lambda_i([0, 1]) = \lambda([s_i, s_i+1]) \rightarrow \mathbb{R}^2$ for some intervals $[s_i, s_i+1]$, and we define the length of the nontrivial intersection to be $\rho(\lambda \cap \lambda') := \sum_i \rho(\lambda_i)$.

Here, we are interested in the double bubble perimeter of two simply connected open sets $A, B \subset \mathbb{R}^2$ where the boundary of $A$, $\partial A$, is a closed, simple, rectifiable curve, and similarly for $B$, and where the intersection of the boundaries of $A$ and $B$ is a union of disjoint-rectifiable curves. The double bubble perimeter is defined as follows:

$$\rho_{DB}(A, B) = \rho(\lambda) + \rho(\lambda') - \rho(\lambda \cap \lambda'),$$

where $\partial A = \lambda([0, 1])$, and $\partial B = \lambda'([0, 1])$. We will also use the notation $\rho(\lambda) = \rho(\partial A)$.

For $\alpha \in (0, 1)$, define $\gamma_\alpha = \{(A, B) : A, B \subset \mathbb{R}^2, A, B \text{ are disjoint, simply connected open sets, and } \partial A \cap \partial B \supset \partial A \cap \partial B \text{ are unions of closed, continuous, simple, rectifiable curves, with } \mu(A) = 1, \mu(B) = \alpha\}$.

Let

$$\rho_{DB}(\Gamma_\alpha) := \inf \{\rho_{DB}(A, B) : (A, B) \in \gamma_\alpha\}.$$
Fig. 1 Minimizing configurations for different volume ratios

be the infimum of the double bubble perimeter (bounded below by zero).

The main result in this paper is

**Theorem 1** For $0 < \alpha \leq 1$,

I. The set $\Gamma_\alpha := \{(A, B) \in \gamma_\alpha : \rho_{DB}(A, B) = \rho_{DB}(\Gamma_\alpha)\}$ is not empty.

II. The infimum can be expressed as follows:

$$\rho_{DB}(\Gamma_\alpha) = (4\sqrt{1 + \alpha} + 2\sqrt{\alpha})\frac{1}{2}\left[0, \frac{688 - 480\sqrt{2}}{49}\right] + (4 + 2\sqrt{2\alpha})\frac{1}{2}\left[\frac{688 - 480\sqrt{2}}{49}, \frac{1}{2}\right]$$

$$+ (2\sqrt{6(1 + \alpha)})\frac{1}{2}\left[\frac{1}{2}, 1\right].$$

III. For $\alpha = \frac{688 - 480\sqrt{2}}{49}$, we have $|\Gamma_\alpha| \geq 2$, with $\Gamma_\alpha$ containing both sets in Fig. 1a and b. Moreover, for $\alpha \in [1/2, 1]$, $\Gamma_\alpha$ contains Fig. 1c, for $\alpha \in \left(\frac{688 - 480\sqrt{2}}{49}, \frac{1}{2}\right)$, $\Gamma_\alpha$ contains Fig. 1b, and for $\alpha \in \left(0, \frac{688 - 480\sqrt{2}}{49}\right)$, $\Gamma_\alpha$ contains Fig. 1a.

IV. For $\alpha \in (0, \frac{688 - 480\sqrt{2}}{49}) \cup \left[\frac{1}{2}, 1\right]$, $\Gamma_\alpha$ has a unique element up to isometries of the plane. For $\alpha \in \left(\frac{688 - 480\sqrt{2}}{49}, 1/2\right)$, $\Gamma_\alpha$ has a unique element up to moving the set of volume $\alpha$ up or down one side of the set of volume 1, and isometries of these sets. That is, in Fig. 1b, we can move the smaller rectangle up and down the side of the larger rectangle and take isometries of each of these configurations.

Immediately from Theorem 1 part II, we get

**Corollary 1.1** There are two critical $\alpha$’s at which $\rho_{DB}(\Gamma_\alpha)$ undergoes a phase transition. The first, at $\alpha = \frac{688 - 480\sqrt{2}}{49}$, is discontinuous in the first-order derivative, while the second, at $\alpha = 1/2$, is discontinuous in the second-order derivative.

Before we explain the proof strategy, we define in Fig. 2 a finite family of set types abbreviated $\mathcal{F}_\alpha \subset \gamma_\alpha$:

While the general case encapsulates the other cases, the kissing rectangles and embedded rectangle cases are important enough to annotate and include with names. We will be referring to these annotations later in the paper.

The strategy for proving Theorem 1 follows 3 steps:
Kissing rectangles
\[ \rho_{DB} = 2(a + b + c) + d \]
\[ ab = \alpha \text{ or } 1 \]
\[ cd = 1 \text{ or } \alpha \]
\[ b \geq d \]
\[ a, b, c, d > 0 \]

Embedded rectangle
\[ \rho_{DB} = 2(c + d) + a + b \]
\[ ab = \alpha \text{ or } 1 \]
\[ cd = 1 \text{ or } \alpha \]
\[ a, b, c, d > 0 \]

General case
\[ \rho_{DB} = 2(a + b + c + f) + (d + e) \]
\[ ab = \alpha \text{ or } 1 \]
\[ cd = 1 \text{ or } \alpha \]
\[ a, b, c, d, e, f > 0 \]

Fig. 2 Special families within \( F_\alpha \)

1. Begin with any two sets \((A, B) \in \gamma_\alpha\), and find sets \((\tilde{A}, \tilde{B}) \in F_\alpha\) with \( \rho_{DB}(\tilde{A}, \tilde{B}) \leq \rho_{DB}(A, B) \). This part is done in Sect. 2.
2. Since the sets in \( F_\alpha \) are very simple to analyze, and the family is finite, we can show the existence of
\[ \arg \inf \{ \rho_{DB}(A, B) : (A, B) \in F_\alpha \} \]
This part is done in Sect. 3.
3. By the previous points, these sets achieve the infimum over all of \( \gamma_\alpha \) proving the existence of an element in \( \Gamma_\alpha \). Moreover, we get the phase transitions in \( \alpha \) and show nonuniqueness for the first-phase transition. This is done in Sect. 4.
4. Finally, in Sect. 5, we prove that the three configurations found in Sects. 3 and 4, and shown in Fig. 1 are the unique minimizers.

2 Finding the Sets in \( F(\alpha) \)

Our goal in this section is to find elements of \( F_\alpha \) with a smaller double bubble perimeter than given sets \( A \) with \( \mu(A) = 1 \), and \( B \) with \( \mu(B) = \alpha \).

Definition 2.1
\[ A^\Box := [a_{\text{left}}, a_{\text{right}}] \times [a_{\text{bottom}}, a_{\text{top}}], \text{ where} \]
\[ a_{\text{left}} = \inf \{ x : (x, y) \in A \text{ for some } y \in \mathbb{R} \} \]
\[ a_{\text{right}} = \sup \{ x : (x, y) \in A \text{ for some } y \in \mathbb{R} \} \]
\[ a_{\text{bottom}} = \inf \{ y : (x, y) \in A \text{ for some } x \in \mathbb{R} \} \]
\[ a_{\text{top}} = \sup \{ y : (x, y) \in A \text{ for some } x \in \mathbb{R} \} \]

Lemma 1 \( \rho(A^\Box) \leq \rho(A) \text{ and } \mu(A) \leq \mu(A^\Box) \).

Proof By definition, \( A \subset A^\Box \). Therefore, by monotonicity of Lebesgue measure, \( \mu(A) \leq \mu(A^\Box) \).

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Fig. 3 Definition of bounding lines $H_1$, $H_2$, $V_1$, and $V_2$

Now we show that $\rho(A \Box) \leq \rho(A)$. Let $H_1$ be the horizontal line passing through $a_{\text{top}}$, and $H_2$ be the horizontal line passing through $a_{\text{bottom}}$. Similarly define $V_1$ and $V_2$ to be the vertical lines passing through $a_{\text{left}}$ and $a_{\text{right}}$, respectively. Let $D_{H_1,H_2}$ be the distance from $H_1$ to $H_2$, and $D_{V_1,V_2}$ be the distance between $V_1$ and $V_2$. $\partial A$ must touch $H_1$ in at least one point, say $p_1$, and similarly must touch $H_2$ in at least one point, say $p_2$. See Fig. 3 for an illustration of the notations.

Since $A$ is open and $\partial A$ is simple, there must be at least two disjoint paths in $\partial A$ from $p_1$ to $p_2$, abbreviate them $\lambda_i(\cdot) = (x_i(\cdot), y_i(\cdot)) : [0, 1] \to \mathbb{R}^2$, for $i \in \{1, 2\}$. The vertical portion of these paths must be at least $D_{H_1,H_2}$, i.e., for any $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq 1$, and $i \in \{1, 2\}$, we have that

$$\sum_{j=1}^{N} (|y_i(t_{j+1}) - y_i(t_j)|) \geq D_{H_1,H_2}$$

Similarly there must be at least two disjoint paths from a point on $V_1$ to a point on $V_2$, whose horizontal distances must be at least $D_{V_1,V_2}$. We have so far found that the boundary of $A$ must measure at least $2 \cdot D_{V_1,V_2} + 2 \cdot D_{H_1,H_2}$, which is the length of the boundary of $A \Box$. That is, $\rho(A \Box) \leq \rho(A)$, as claimed.

We now show that any configuration $(A, B) \in \gamma_\alpha$ can be transformed into a new configuration $(\tilde{A}, \tilde{B})$ such that $(\tilde{A}, \tilde{B}) \in \mathcal{F}_\alpha$, and $\rho_{DB}(\tilde{A}, \tilde{B}) \leq \rho_{DB}(A, B)$. The transformation we perform depends on whether one, two, or four corners of $B \Box$ are contained in $A \Box$ (or if two or four corners of $A \Box$ are contained in $B \Box$).

**Lemma 2** If $(A, B) \in \gamma_\alpha$, and $B \Box \subset A \Box$, then there exists $(\tilde{A}, \tilde{B}) \in \mathcal{F}_\alpha$, such that $\rho_{DB}(\tilde{A}, \tilde{B}) \leq \rho_{DB}(A, B)$, and $(\tilde{A}, \tilde{B})$ are either kissing rectangles or embedded rectangles. An analogous procedure shows that if $A \Box \subset B \Box$, then we will also end up with the same configurations below but possibly with the roles of $\tilde{A}$ and $\tilde{B}$ reversed.
Proof If $\sqsubseteq B$ is contained in $\sqsubseteq A$, there are several options to consider, namely that one, two, three, or no edges of $\sqsubseteq B$ could touch the same number of edges in $\sqsubseteq A$. Let us take the instance when none of the edges of $\sqsubseteq B$ touch any of the edges of $\sqsubseteq A$, that is, $b_{\text{top}} < a_{\text{top}}$, $b_{\text{right}} < a_{\text{right}}$, $a_{\text{left}} < b_{\text{left}}$, and $a_{\text{bottom}} < b_{\text{bottom}}$. From this case, we can easily derive the results for the other cases. Let $H_1 = \mathbb{R} \times \{a_{\text{top}}\}$, $H_2 = \mathbb{R} \times \{b_{\text{top}}\}$, $H_3 = \mathbb{R} \times \{b_{\text{bottom}}\}$, and $H_4 = \mathbb{R} \times \{a_{\text{bottom}}\}$. Now we define the distance between $H_1$ and $H_2$ to be $D_{H_1,H_2} = a_{\text{top}} - b_{\text{top}}$, the distance between $H_2$ and $H_3$ to be $D_{H_2,H_3} = b_{\text{top}} - b_{\text{bottom}}$, and the distance between $H_3$ and $H_4$ to be $D_{H_3,H_4} = b_{\text{bottom}} - a_{\text{bottom}}$. Similarly, we call the vertical line through $a_{\text{left}}$ $V_1$, the vertical line through $b_{\text{left}}$ $V_2$, the vertical line through $b_{\text{right}}$ $V_3$, and the vertical line through $a_{\text{right}}$ $V_4$, naming the distances between these lines just as before. See Fig. 4 for illustration.

Now, in $\partial A \cup \partial B$, we need to find three paths with vertical lengths of $D_{H_2,H_3}$, and in $\partial A$ two paths with vertical lengths of $D_{H_1,H_2}$ and the same for $D_{H_3,H_4}$. Further, we need these paths to be disjoint so we do not count anything more than once. Then we would do the analogous process for horizontal distances. First, we find points $p_1 \in H_1 \cap \partial A$, and $p_2 \in H_4 \cap \partial A$. There must be two distinct paths in $\partial A$ between

Fig. 4 Definition of $D_{H_1,H_2}$, $D_{H_2,H_3}$, etc. in the fully contained case
these two points, which we call $\lambda_i(t) = (x_i(t), y_i(t)) : [0, 1] \to \mathbb{R}^2$, for $i \in \{1, 2\}$, both of which must have vertical distance of at least $D_{H_1, H_2} + D_{H_2, H_3} + D_{H_3, H_4}$. That is, for any $0 \leq t_1 \leq t_2 \leq \ldots \leq t_N \leq 1$, we have for $i \in \{1, 2\}$,

$$\sum_{j=1}^{N} \left( |y_i(t_{j+1}) - y_i(t_j)| \right) \geq D_{H_1, H_2} + D_{H_2, H_3} + D_{H_3, H_4}.$$  

To complete our search for enough vertical length, it remains to find one final path with vertical length $D_{H_2, H_3}$ that we have yet to count among these crossings. For a path $\gamma : [0, 1] \to \mathbb{R}^2$, we say that a subpath $\gamma[a, b]$ is a crossing of $S := \mathbb{R} \times (b_{\text{bottom}}, b_{\text{top}})$ if $\gamma(a) \in H_2, \gamma(b) \in H_3$ and for all $a < t < b$, $\gamma(t) \notin \{H_2, H_3\}$ (or in the other direction). If $\partial A$ contains more than two crossings then each of these crossings must have vertical length at least $D_{H_2, H_3}$, and we are done. So we may assume that $\partial A$ only contains two crossings of $S$. We denote these two crossings $\xi_1, \xi_2$. Since $\partial A$ contains exactly two crossings of $S$, $S \setminus \xi_1 \cup \xi_2$ consists of three open sets, exactly two of which are unbounded and have joint boundary with both $H_2$ and $H_3$. We call these two open sets $S_1$ and $S_2$. Since $B$ is open and connected and contained in $S$, $B$ must be contained in one of these unbounded open sets, say without loss of generality $S_2$ (note that if there are more than 2 crossings then $B$ can be contained in a bounded set). Since $B$ is open and $\partial B$ is rectifiable, there are at least two distinct paths in $\partial B$ from $H_2$ to $H_3$, which we call $\lambda_3 = (x_3(t), y_3(t))$, and $\lambda_4 = (x_4(t), y_4(t))$. Both of these paths must be in $S_2$. By planarity, only one of these paths might intersect $\xi_2$, say without loss of generality $\lambda_3$. This means that we have not counted the vertical part of $\lambda_4$, and it must have vertical length at least $D_{H_2, H_3}$. That is to say, that for any $0 \leq t_1 \leq t_2 \leq \ldots \leq t_N \leq 1$, we have

$$\sum_{j=1}^{N} \left( |y_4(t_{j+1}) - y_4(t_j)| \right) \geq D_{H_2, H_3}.$$  

This is our third such length, and we are done finding vertical lengths. We need only find horizontal lengths now. But this is done in exactly the same manner as in our search for vertical lengths. We could even rotate our figures 90 degrees either left or right, so that vertical lines become horizontal and vice versa, and perform the exact same proof as above.

We now construct our figure. First, we have found a total length of

$$2 \cdot (D_{H_1, H_2} + D_{H_3, H_4} + D_{V_1, V_2} + D_{V_3, V_4}) + 3 \cdot (D_{h_2, H_3} + D_{V_2, V_3}).$$  

This gives us enough length to construct $A^\square$ and still have left over a total length of $D_{H_2, H_3} + D_{V_2, V_3}$. With these lengths, we construct a box in the corner of $A^\square$ with measure $D_{H_2, H_3} \times D_{V_2, V_3}$. This box, which we call $\tilde{B}$ in the corner has volume the same as $B^\square$, and, therefore, volume at least $\alpha$. We can shrink it easily so that it has volume exactly $\alpha$, and by abuse of notation still call, this possibly smaller rectangle $\tilde{B}$. On the other hand, $A \cup B \subset A^\square$, and therefore $\mu(A^\square) \geq 1 + \alpha$. Therefore,
\( \mu(A^{\square} \setminus \tilde{B}) \geq 1 + \alpha - \alpha \). So we can easily move the sides of \( A^{\square} \) that do not share joint boundary with \( \tilde{B} \) inwards until the volume is exactly 1. The interior of this set we call \( \tilde{A} \) and have completed our construction.

Now, suppose that \( b_{\text{bottom}} = a_{\text{bottom}} \), or \( b_{\text{left}} = a_{\text{left}} \), etc. That is one of the sides of \( B^{\square} \) is contiguous with one of the lines of \( A^{\square} \). The process would be as above, except we would have \( H_3 = H_4 \). This means we would not have to find two paths with vertical length \( D_{H_3, H_4} \). The rest of the proof would be the same. Similarly, if two sides of \( B^{\square} \) are contiguous with two sides of \( A^{\square} \), say \( a_{\text{bottom}} = b_{\text{bottom}} \) and \( a_{\text{left}} = b_{\text{left}} \), then we would not have to find paths with vertical length \( D_{H_3, H_4} \) and we would not have to find paths with horizontal length \( D_{V_1, V_2} \). The rest of the proof would be the same. 

The previous lemma took into account all of the cases when all four corners of \( B^{\square} \) are contained in \( A^{\square} \). The only other two options are if two or one corner of \( B^{\square} \) is contained in \( A^{\square} \).

**Lemma 3** If \((A, B) \in \gamma_\alpha\), and exactly one corner of \( B^{\square} \) is contained in \( A^{\square} \), then there exists \((\tilde{A}, \tilde{B}) \in \mathcal{F}_\alpha\), such that \( \rho_{DB}(\tilde{A}, \tilde{B}) \leq \rho_{DB}(A, B) \).

**Proof** For the one corner case, we argue that we can find sets, with the same or better double bubble perimeter and at least as much joint volume as the original shapes that looks like the general case of Fig. 2:

Here, the rectangle can be either \( A^{\square} \) or \( B^{\square} \), say \( A^{\square} \), and the other set is \( B^{\square} \setminus A^{\square} \). Once we create these two sets, we are not necessarily done because the volumes may not be correct. This causes more of a problem than in other cases, but the method remains similar.

In this proof, we are assuming that \( \mu(A) = 1 \), and \( \mu(B) = \alpha \). Since \( A^{\square} \) and \( B^{\square} \) only intersect in one corner, either \( a_{\text{top}} > b_{\text{top}} \), or \( b_{\text{top}} > a_{\text{top}} \). We can suppose without loss of generality that \( a_{\text{top}} > b_{\text{top}} \). Let \( H_1 = \mathbb{R} \times \{a_{\text{top}}\} \) be the horizontal line passing through \( a_{\text{top}} \), \( H_2 = \mathbb{R} \times \{b_{\text{top}}\} \) be the horizontal line passing through \( b_{\text{top}} \), \( H_3 = \mathbb{R} \times \{a_{\text{bottom}}\} \) be the horizontal line passing through \( a_{\text{bottom}} \), and \( H_4 = \mathbb{R} \times \{b_{\text{bottom}}\} \) be the horizontal line passing through \( b_{\text{bottom}} \). Note that the height of \( A^{\square} \cap B^{\square} \) is the same as the distance between \( H_2 \) and \( H_3 \), which we will call \( D_{H_2, H_3} \). Furthermore, let the distance between \( H_1 \) and \( H_2 \) be \( D_{H_1, H_2} \), and the distance between \( H_3 \) and
Fig. 5 Definition of $D_{H_1,H_2}, D_{H_2,H_3}$, etc. in the one corner containment case

$H_4$ be $D_{H_3,H_4}$. Now, let $V_1 = \{b_{\text{left}}\} \times \mathbb{R}$ be the vertical line passing through $b_{\text{left}}$, $V_2 = \{a_{\text{left}}\} \times \mathbb{R}$ be the vertical line passing through $a_{\text{left}}$, $V_3 = \{b_{\text{right}}\} \times \mathbb{R}$ be the vertical line passing through $b_{\text{right}}$, and $V_4 = \{a_{\text{right}}\} \times \mathbb{R}$ be the vertical line passing through $a_{\text{right}}$. We define the distance between $V_1$ and $V_2$ to be $D_{V_1,V_2}$, the distance between $V_2$ and $V_3$ to be $D_{V_2,V_3}$, and the distance between $V_3$ and $V_4$ to be $D_{V_3,V_4}$. See Fig. 5.

Notice that to construct (the interiors of) $A^\square$ and $B^\square \setminus A^\square$ in this way without increasing double bubble perimeter, we need to find in $\partial A \cup \partial B$ two lengths of $D_{H_1,H_2}$, $D_{H_3,H_4}$, $D_{V_1,V_2}$, and $D_{V_3,V_4}$, as well as three lengths of $D_{H_2,H_3}$, and $D_{V_2,V_3}$. In other words, for $\rho_{DB}\left((A^\square)^\circ, (B^\square \setminus A^\square)^\circ\right)$ to be at most $\rho_{DB}(A,B)$, we need to find two vertical lengths of $D_{H_1,H_2}$ in $\partial A$, two vertical lengths of $D_{H_3,H_4}$ in $\partial B$, and three vertical lengths of $D_{H_2,H_3}$ between $\partial A$ and $\partial B$. Similarly for horizontal lengths.

First, there must be a point $p_1 \in H_1 \cap \partial A$, and another point $p_2 \in H_3 \cap \partial A$. Between these two points, there must be at least two disjoint paths, which we call $\lambda_i(t) = (x_i(t), y_i(t)) : [0, 1] \to \mathbb{R}^2$, $i \in \{1, 2\}$, in $\partial A$, both of which have vertical length at least $D_{H_1,H_2} + D_{H_2,H_3}$. That is, as before, for any $0 \leq t_1 \leq t_2 \leq \ldots \leq t_N \leq 1$,

$$\sum_{j=1}^{N} (|y_i(t_{j+1}) - y_i(t_j)|) \geq D_{H_1,H_2} + D_{H_2,H_3}$$
Similarly in $\partial B$, we can find two disjoint paths of vertical length at least $DH_{2, H_3} + DH_{H_3, H_4}$. Since there can be no joint boundary below $H_3$, the portions of these paths that measure at least $DH_{3, H_4}$ have not been counted yet. It remains to find a path in $\partial B$ with vertical length at least $DH_{H_3, H_3}$ that we have yet to count.

To find this path, we define the infinite strip of height $DH_{2, H_3}$, $S := \mathbb{R} \times (a_{\text{bottom}}, b_{\text{top}})$. If $\partial A$ has more than two crossings of $S$, then each of these crossings has vertical length of at least $DH_{2, H_3}$, as above, and we are done. So we may assume that $\partial A$ contains exactly two crossings of $S$ (it cannot be less than two as we noted above). Abbreviate these crossings $\xi_1, \xi_2$. In this case, consider $S \setminus (\xi_1 \cup \xi_2)$. This consists of three open sets, only two of which are unbounded, and it is in one of these open sets that we find $B \cap S$. We call these sets $S_1$ and $S_2$ such that $\xi_1 \subset \partial S_1$ and $\xi_2 \subset \partial S_2$. Suppose without loss of generality that $B \cap S \subset S_2$. There is at least one point $p_3 \in \partial B \cap H_2$, and at least one point $p_4 \in \partial B \cap H_3$, and there must be two distinct paths in $\partial B$ from $p_3$ to $p_4$. We call these paths $\lambda_i(t) = (x_i(t), y_i(t)) : [0, 1] \to \mathbb{R}^2$, $i = 3, 4$. By planarity, only one of the paths, either $\lambda_3$ or $\lambda_4$, can have joint boundary with $\xi_2$, say $\lambda_3$. This means that we have yet to include the vertical length of $\lambda_4$, and we have just found our third vertical length of $DH_{H_2, H_3}$. That is, for any $0 \leq t_1 \leq t_2 \leq \ldots \leq t_N \leq 1$,

$$\sum_{j=1}^{N} (|y_4(t_{j+1}) - y_4(t_j)|) \geq DH_{H_2, H_3}.$$  

Finding the three horizontal lengths of $DV_{2, V_3}$ follows the same argument. The reason we can be sure that we would not double count anything is because inside $A \square \cap B \square$ we have only counted vertical distance, and in the argument to find our three horizontal lengths of $DV_{2}, V_3$, we would only count horizontal distance. So, we have proved that $\rho_{DB}(A \square, B \square \setminus A \square) \leq \rho_{DB}(A, B)$. It is clear that $\mu(A \cup B) \leq \mu(A \square \cup B \square)$. Since $A \subset A \square$, $1 = \mu(A) \leq \mu(A \square)$. However, it is possible that $\mu(B \square \setminus A \square) < \mu(B) = \alpha$. This we must correct. For this purpose, let us refer to the notation we established in the introduction for the general case.

The situation here is somewhat delicate and we may have to perform different procedures to ensure that we can find two new sets of the correct volume that correspond to the general case in Figure 2 without increasing double bubble perimeter. We look
first at the two intervals of lengths $c, f$ and assume without loss of generality that $f \geq c$ (see Fig. 6).

(1) Suppose first that $c + d + e \leq a$ and that $d \geq f$. The first problem we want to rectify is that it may be the case that the volume of $B \setminus A < \alpha$. We do this by moving the line with length $d$ down, eating up volume of $A$, until we have one set of volume 1 and another of volume $\alpha$ as in Fig. 7:

Referring to Fig. 7, we now have that $\mu(\tilde{A}) = 1$, and $\mu(\tilde{B}) = \alpha$. If after this procedure, we have that $\tilde{B} = B \setminus A$, then we are done because that result is covered under the general case of Figure 2. Otherwise, we remove the rectangle from $\tilde{B}$ measuring $\ell$ by $f$. In other words, we move the line in Fig. 7 of length $f$ up until $\tilde{B}$ is a rectangle. We have lost a volume of $f \cdot \ell$ and gained perimeter of length $\ell$ with which to work. We can then increase the height of $\tilde{B}$ by $\frac{\ell}{2}$. The situation is illustrated in Fig. 8:

As can be seen in Figures 7 and 8, we lost a volume of $\ell \cdot f$, and gained a volume of $\frac{\ell}{2} (d + f) \geq \frac{\ell}{2} (f + f) = \ell \cdot f$. In other words, this process has given us a new configuration such that the double bubble perimeter has not increased, and the volume has not decreased. If the volume of $\tilde{B}$ is more than $\alpha$, then we can move its top line down, its left side to the right, and possibly its right side to the left in such a way as to ensure a general case configuration, until the volume is $\alpha$. Note that this can only reduce double bubble perimeter. If $\mu(\tilde{A}) > 1$, we can also easily reduce its volume without increasing double bubble perimeter, and while preserving the general form. Thus, we have found a configuration that is of the general form in Fig. 2 as desired and whose double bubble perimeter is no more than our original $(A, B)$.

Now, we move to the case where $c + d + e \leq a$, $d < f$, and $e \geq c$. Again, beginning with Fig. 6, we want to adjust the volume of $B \setminus A$ so that the result has volume $\alpha$. This time we move the shared boundary of $A$ and $B \setminus A$ that has length $e$ to the left until the appropriate volumes are achieved. This result is as in Fig. 9:

Here $\mu(\tilde{B}) = \alpha$, and $\mu(\tilde{A}) \geq 1$, and we have not increased the double bubble perimeter. As indicated in Figure 9, we will remove the rectangle measuring $c$ by $\ell$. This will remove a volume of $c \cdot \ell$ and we will gain a length of $\ell$. If $\ell = 0$, then we have recovered $A \setminus \tilde{B}$ and $B \setminus \tilde{B}$, and we are done because this configuration is covered in the general case of figure 2. We then move the right side of $\tilde{B}$ farther to the right by a distance of $\frac{\ell}{2}$ as in Figure 10:

The volume we lost from $\tilde{B}$ was $\ell \cdot c$, and the volume we gain from this procedure is $\frac{\ell}{2} (c + e) \geq \frac{\ell}{2} (c + c) = \ell \cdot c$. If necessary we can reduce the volume of $\tilde{B}$ by moving its right side to the left, its bottom side up, and possibly its top side down. Similarly, if $\mu(\tilde{A}) > 1$, we can reduce the volume without increasing double bubble perimeter, and while preserving the general form.

Next, we must consider when $c + d + e \leq a$, $d < f$, and $e < c$. Recall also from above that since either $c > f$ or $f > c$, and we assumed without loss of generality that $f > c$. Again we begin with the configuration in Fig. 6.

Here our procedure is a little different. Notice first that the volume of $B \setminus A$ could be less than $\alpha$. In the previous two procedures we began by adjusting the
volumes so that one set had volume 1 and the other volume $\alpha$. This time, we will not adjust the volumes as a first step. Therefore, in the end we will have to ensure that what will become $\tilde{B}$ recovers this lost volume, which (referring to Figure 6) is at most $c \cdot d$.

Let’s begin by moving $B \Box \setminus A \Box$ down a distance of $c$ and rename this adjusted set as $\tilde{B}$ (i.e., so that the top of $\tilde{B}$ is at the same height as the top of $A \Box$). In doing so, we lose a volume of $c \cdot d$, and we gain a total boundary length of $c + d$ to work with. The result will be like Fig. 11:

Now, we must show that we can regain any volume that was lost from $B$ (at most $c \cdot d + d \cdot e$) to ensure that $\mu(\tilde{B}) = \alpha$ while adding at most $c + d$ to the double bubble perimeter. If $2c + d + e \leq a$, then we can just add $c + d$ to the right side of $\tilde{B}$, which will add a volume of $f \cdot (c + d) = f c + e \cdot c + f. d \geq d c + e d$ (recall our assumptions that $f \geq d$, and $f \geq c > e$). That is, we can add at least as much volume to $\tilde{B}$ as was missing. Figure 12 illustrates this situation.

At this point, if $\mu(\tilde{B}) > \alpha$, then we can move the right side of $\tilde{B}$ to the left, which will reduce its volume, until the desired volume of $\alpha$ is obtained. Similarly, if $\mu(A \Box) > 1$, we can move its left side to the right until the desired volume is reached.

On the other hand, however, if $2c + d + e > a$, then there is some $l \in [0, c)$ such that $c + d + e + l = a$, and so we add the length $l + d$ to the right side of $\tilde{B}$, which will add $(l + d) f$ volume (which is at least as much as $dc$). The volume we removed from $B \Box \setminus A \Box$ was $dc$, and now we have two kissing rectangles of the same height whose total volume is at least $1 + \alpha$. See Fig. 13 for illustration:

By adjusting the shared boundary (i.e., the center line shared by both rectangles) we can ensure that one set has volume 1, which means that the other set has volume at least $\alpha$. Again, we can adjust the volume of this set while decreasing double bubble perimeter in a similar way as above.

Thus far, we have considered all possibilities when $c + d + e \leq a$ and have been able to construct a configuration that is in $\mathcal{F}_\alpha$ without increasing double bubble perimeter. We now turn to the case when $c + d + e > a$.

(2) Again, we begin with a configuration as in Figure 6 and assume that $c + d + e > a$.

If $d$ is large enough (i.e., if $d \geq a - e$) let $d' = d - (a - e)$, and we can move the line of length $e$ to the right by $d - d' = a - e$. This perimeter that we gain $(d - d')$ we can use to elongate the right side of $B \Box \setminus A \Box$ to get the configuration in Fig. 14. Since $f \geq c$ (we made this assumption, without loss of generality, above), this procedure has not decreased the volume. If $\mu(A \Box) > 1$, we can move its right side to the left (decreasing $d'$) until $\mu(A \Box) = 1$. The result would look either like Fig. 14 again (but $d'$ would have gotten smaller), or like Fig. 15. If the resulting volume of $\tilde{B}$ is too large, we could reduce it by moving its right side to the left, and possibly its top side down.

On the other hand, if $d < a - e$, then we can move the line of length $c$ in Fig. 6 to the right by a length of $d$ (reducing the double bubble perimeter by $d$), add this length to the right side of $\tilde{B}$, and move the resulting rectangle down so that its bottom side is at the same height as the bottom side of the rectangle $A \Box$. The
result will look much like Figure 15. We now have kissing rectangles, and it is an easy process to fix the volumes so that one set has volume 1 and the other has volume $\alpha$. The result will still look like Fig. 15. Both of these configurations (in Figures 14 and 15) are of the general form.

Now we need only deal with the case when two corners of $B^\Box$ are in $A^\Box$.

**Lemma 4** If $(A, B) \in \gamma_\alpha$, and two corners of $B^\Box$ are contained in $A^\Box$, then there exists $(\tilde{A}, \tilde{B}) \in \mathcal{F}_\alpha$, such that $\rho_{DB}(\tilde{A}, \tilde{B}) \leq \rho_{DB}(A, B)$.

**Proof** Here we must have one of the following: $b_{\text{top}} > a_{\text{top}}$, $b_{\text{right}} > a_{\text{right}}$, $b_{\text{bottom}} < a_{\text{bottom}}$, or $b_{\text{left}} < a_{\text{left}}$. Let’s suppose, without loss of generality, that $b_{\text{right}} > a_{\text{right}}$. 

$\square$
**Fig. 10** Volume correction if \( e \geq c \), necessary in general after the construction of moving \( e \) leftwards.

**Fig. 11** Result obtained from starting configuration after moving \( B \square \setminus A \square \) down by \( c \).

**Fig. 12** Volume correction should \( d < f \), \( e < c \), and \( 2c + d + e \leq a \): necessary after moving \( B \square \setminus A \square \) down by \( c \).

**Fig. 13** Volume correction should \( d < f \), \( e < c \), and \( 2c + d + e > a \): necessary after moving \( B \square \setminus A \square \) down by \( c \).
We construct our configuration in a similar way as before. First, since two corners of \(B\square\) are in \(A\square\), it follows that \(a_{\text{left}} < b_{\text{left}} \leq a_{\text{right}}\). However, if \(b_{\text{left}} = a_{\text{right}}\), we can just replace \(A\) with (the interior of) \(A\square\) and \(B\) with (the interior of) \(B\square\). This will increase the volume of both \(A\) and \(B\), and increase their joint boundary. Then we can reduce the volumes as necessary, which will only decrease the double bubble perimeter. So, we can assume that \(a_{\text{left}} < b_{\text{left}} < a_{\text{right}}\). Let \(H_1\) be the horizontal line passing through \(a_{\text{top}}\), \(H_2\) be the horizontal line passing through \(b_{\text{top}}\), \(H_3\) be the horizontal line passing through \(b_{\text{bottom}}\), and \(H_4\) the horizontal line passing through \(a_{\text{bottom}}\). We define \(D_{H_i, H_{i+1}}\) as before, \(i = 1, 2, 3\). Similarly define \(V_1, V_2, V_3, V_4\) as the vertical lines passing through \(a_{\text{left}}, b_{\text{left}}, a_{\text{right}}, b_{\text{right}}\), respectively, and \(D_{V_i, V_{i+1}}\) be defined as before, \(i = 1, 2, 3\). Notice that we haven’t eliminated the possibility that \(H_1 = H_2\), or \(H_3 = H_4\), or both. See Fig. 16.

We wish to find two disjoint paths with vertical lengths at least \(D_{H_1, H_2}\), two disjoint paths with vertical length as least as long as \(D_{H_3, H_4}\), three disjoint paths with vertical lengths at least as long as \(D_{H_2, H_3}\), two disjoint paths with horizontal length at least as long as \(D_{V_1, V_2}\), three disjoint paths with horizontal lengths at least as long as \(D_{V_2, V_3}\), and two disjoint paths with horizontal lengths at least as long as \(D_{V_3, V_4}\). We achieve this much the same as in the previous Lemma.

Finding the horizontal lengths is nearly identical. We can then proceed to construct our sets. If we move \(B\square\) up until \(H_1 = H_2\) we reduce the double bubble perimeter and we get a shape of the general case type. Now we can fix the volumes in the same way as in the previous Lemma. \(\square\)
3 KKT Analysis

In the previous section, we constructed a finite list of set types $\mathcal{F}_\alpha$ in which from any configuration $(A, B) \in \gamma_\alpha$ we obtain a configuration $(\tilde{A}, \tilde{B}) \in \mathcal{F}_\alpha$ so that $\rho_{DB}(\tilde{A}, \tilde{B}) \leq \rho_{DB}(A, B)$. In the upcoming analysis, however, it is convenient to note that all the cases in $\mathcal{F}_\alpha$ can be represented by the 6 parameter configuration we called “General Case” given in Fig. 17.

The other cases, kissing rectangles for instance, occur when some of the parameters defining the geometry of the configuration are set to zero. For kissing rectangles, this would consist of setting $e$ and $f$ to zero (or $c = d = 0$) in Fig. 17. Using this configuration we have made a geometric problem into the minimization of a hyperplane with algebraic inequality constraints. We want to minimize the $\rho_{DB}$ in Figure 17, while remaining within the inequality constraints.

The Karush Kuhn Tucker method [7] is one method used for calculating minima in such problems; with some care the global minimum of each element of the above
list can be found. In the previous lemmas, it was sometimes ambiguous which of the
two shapes in a configuration had unit volume and which \( \alpha \). For our purposes, that
just means we must alternate which shape has volume \( \alpha \) and analyze both.

The original problem we would have to solve is 6 dimensional, but our two equality
constraints reduce it to four. These are that \( ab = \alpha \) or 1 and \( cd + cf + ef = 1 \) or \( \alpha \). Using these constraints, our problem becomes to minimize

\[
2 \left( a + \frac{\alpha \text{ or } 1}{a} + c + \frac{(1 \text{ or } \alpha) - cd}{c + e} \right) + (d + e),
\]

subject to

\[
(\alpha \text{ or } 1) \geq ad, a \geq e, a \geq 0, c \geq 0, d \geq 0, e \geq 0, (1 \text{ or } \alpha) \geq cd.
\]

The variable \( a \) cannot be zero, so we can exclude its inequality Lagrange multiplier. The \( (\alpha \text{ or } 1) \geq ad \) constraint comes from \( \frac{(1 \text{ or } \alpha) - cd}{c + e} = f \geq 0 \). Note that the conventional conditions required for KKT (equality constraints be affine, etc.) are not satisfied here. In this case, however, we can constrain our variable space in the upper octant and inside some hypercube. This is because if one of the variables becomes larger than some example double bubble perimeter, it is not optimal. A final precaution we address is that due to the exclusion of the constraint \( a \geq 0 \) from our list of inequality constraints under analysis, our space of variables is not truly closed and not truly compact. In the upcoming analysis, other portions of the variable space will similarly be excluded (it will be explicitly mentioned when a portion of the variable space is excluded) because they are not valid configurations. However at some finite distance in variable space from these coordinates, the fact that \( a \) (as an example) becomes small forces another dimension to grow as \( a^{-1} \). This growth eventually pushes \( \rho_{DB} \) over the perimeter of that example configuration we used to build the hypercube. Therefore in a similar manner we bound our domain in these degenerate cases by curves a finite distance from the degenerate case; the boundary here does not require explicit checking because, by construction, it is too large.

Thus our modified variable space is truly closed and bounded and therefore compact. The global minimum is either along the boundary or is the lowest local minimum inside the domain itself. The KKT method checks all of this by implementing the inequality constraints which define the boundary.

Our above \( \rho_{DB} \) yields the Lagrangian

\[
L = 2 \left( a + \frac{\alpha \text{ or } 1}{a} + c + \frac{(1 \text{ or } \alpha) - cd}{c + e} \right) + (d + e) + ((\alpha \text{ or } 1) - ad)\mu_1
+ (a - e)\mu_2
+ c\mu_3 + d\mu_4 + e\mu_5 + ((1 \text{ or } \alpha) - cd)\mu_6.
\]


Let's call $\beta = \alpha$ or 1, $\gamma = 1$ or $\alpha$, then the gradient of $L$ becomes

\[
\left(2 \left(1 - \frac{\beta}{a^2}\right) - d \mu_1 + \mu_2, 2 \left(1 - \frac{\gamma + ed}{(c + e)^2}\right) + \mu_3 - d \mu_6, 2 \left(-\frac{c}{c + e}\right) \right.
\]
\[+ 1 + \mu_4 - c \mu_4, ...
\]
\[...2 \left(-\frac{\gamma - cd}{(c + e)^2}\right) + 1 - a \mu_1 - \mu_2 + \mu_5\].

For the KKT method, the inequality Lagrangian multipliers are either positive and their conditions applied (i.e., if $\mu_1 > 0$ then $\beta = ad$) or they are zero. This means without symmetry arguments that there are $2^6$ systems of nonlinear equations to check for minima. If $\mu_6 > 0$ then $cd = \gamma$, then from our second equality constraint we have that $(c + e) f + cd = 1$ or $a$ and $(c + e) f = 0$, or $c = e = 0$ or $f = 0$. The first is impossible and the second becomes kissing rectangles. As one may see, in parts of this calculation we will eliminate some of the 64 possible systems of equations by demonstrating geometrically what they resolve to, and then doing that case only once. In this case, this simple calculation got rid of 31 systems and left us only needing to calculate the general kissing rectangles case. Similarly if $\mu_3 > 0, \mu_4 > 0$, or $\mu_5 > 0$ then $c = 0, d = 0$, or $e = 0$ and again we have kissing rectangles. This means we only have to calculate it with $\mu_1$ and $\mu_2$ possibly positive, leaving 4 systems of equations and the kissing rectangles to analyze. If we remember that $\mu_1$ corresponds to $b \geq d$, then by symmetry we realize that $\mu_1$ and $\mu_2$ being activated alone have the same effect by symmetry of the figure, so we need only check $\mu_1 > 0$ and both $\mu_1 > 0$ and $\mu_2 > 0$. So we have only 3 systems and kissing rectangles to calculate.

### 3.1 Kissing Rectangles

For kissing rectangles we let the sides of one rectangle be $a, b$, and the other $c, d$ (notation is changed for convenience, see Fig. 2). So $ab = \beta$ while $cd = \gamma$. And taking the $b-d$ edge to be the kissing side, we then take $b$ to be the larger size ($b \geq d$ or $\frac{\beta}{a} \geq \frac{\gamma}{c}$). Our Lagrangian becomes

\[
L = 2 \left(a + \frac{\beta}{a} + c\right) + \frac{\gamma}{c} + \mu_1 (c \beta - a \gamma)
\]

where the inequality constraint $\mu_1$ is $b \geq d$ or in terms of $a$ and $c$ $c \beta \geq a \gamma$. The gradient is

\[
\left(2 \left(1 - \left(\frac{\beta}{a^2}\right)\right) - \gamma \mu_1, 2 - \frac{\gamma}{c^2} + \mu_1 \beta\right) = 0.
\]
The unconstrained minimum has \( a = \sqrt{\beta} = b, \ c = \sqrt{\gamma}, \ d = \sqrt{2\gamma} \) and

\[
\rho_{DB} = 2\left(2\sqrt{\beta} + \sqrt{\gamma}\right) + \sqrt{2\gamma}.
\]

The perimeter becomes either \( 4\sqrt{\alpha} + 2\sqrt{2} \) or \( 4 + 2\sqrt{2\alpha} \). This first equation again has \( \gamma = 1, \ \beta = \alpha, \ a = \sqrt{\alpha} = b, \ \) and \( d = \sqrt{2} \). Now we need \( b \geq d \), which would mean \( \alpha \geq 2 \). This is a contradiction, because \( \alpha \leq 1 \). The second equation, in the exact same way, forces us instead to have \( \sqrt{\alpha} \leq \sqrt{1/2} \implies \alpha \leq 1/2 \). For \( \alpha > 1/2 \) we need to apply the constraint. We know now that \( \gamma = \alpha, \ \beta = 1, \) so \( c = a\alpha \), and our gradient becomes

\[
\left(\left(2 - \frac{2}{a^2}\right) - \alpha \mu_1, 2 - \frac{\alpha}{c^2} + \mu_1\right) = 0.
\]

Multiplying the second equation in the gradient by \( \alpha \) and adding it to the first we have \( 2\alpha - \frac{1}{a^2} + 2 - \frac{2}{a^2} = 0 \), so \( 2(\alpha + 1) = \frac{3}{a^2}, \ a = \sqrt{\frac{3}{2(\alpha + 1)}}, \ c = a\alpha \). Plugging these into our equation for \( \rho_{DB} \) we obtain the kissing rectangle solution

**Lemma 5** For any \( \alpha \in (0, 1) \) the infimum among all \( (A, B) \in \mathcal{F}_\alpha \) of the kissing rectangles type is achieved and admits:

\[
\inf\{\rho_{DB}(A, B) : (A, B) \in \mathcal{F}_\alpha \text{ are kissing rectangles}\} = \left(4 + 2\sqrt{2\alpha}\right)1_{(0, \frac{1}{2})} + \left(2\sqrt{6(1 + \alpha)}\right)1_{[\frac{1}{2}, 1]}
\]

**3.2 Embedded Rectangles**

Now we go on to the case of \( \min(\mu_1, \mu_2) > 0 \). This means \( b = d \) and \( a = e \) which becomes an “Embedded rectangle” type. Again it becomes useful to briefly depart from our original notation. We can label this structure with \( a, b \) being the inner rectangle’s sides, \( c \) and \( d \) the outer (See Fig. 2). This means \( ab = \beta \) the volume of the inner rectangle, \( cd = \gamma + \beta \) the volume of the inner rectangle and the outer piece, and our Lagrangian becomes

\[
2(c + d) + a + b + \lambda_1(ab - \beta) + \lambda_2(cd - (\gamma + \beta))
\]

or in an unrestrained form

\[
2\left(\frac{\beta + \gamma}{d} + d\right) + a + \frac{\beta}{a}
\]
The gradient of this is
\[ \left( 1 - \frac{\beta}{a^2}, 1 - \frac{\beta + \gamma}{d^2} \right), \]
so \( a = b = \sqrt{\beta}, \ c = d = \sqrt{\gamma + \beta} \), and altogether the perimeter is
\[ \rho_{DB} = 2\sqrt{\beta} + 4\sqrt{\gamma + \beta}, \]
which is either \( 2\sqrt{\alpha} + 4\sqrt{1 + \alpha} \) or \( 2 + 4\sqrt{1 + \alpha} \). This second one is too large and is proven by our example in the paper to be suboptimal. So for this case we have

**Lemma 6** For any \( \alpha \in (0, 1] \) the infimum among all \((A, B) \in \mathcal{F}_\alpha\) of the embedded rectangles type is achieved and admits:

\[ \inf\{\rho_{DB}(A, B) : (A, B) \in \mathcal{F}_\alpha \text{ are embedded rectangles} \} = 2\sqrt{\alpha} + 4\sqrt{1 + \alpha}. \]

### 3.3 General Case

We now check the unconstrained case where all \( \mu_i = 0 \). For this purpose we go back to the original notation we established.

Assume all \( \mu_i = 0 \). Then we have the following gradient
\[ \left( 2 \left( 1 - \frac{\beta}{a^2} \right), 2 \left( 1 - \frac{\gamma + ed}{(c + e)^2} \right), 2 \left( \frac{-c}{c + e} \right) + 1, 2 \left( \frac{-\gamma - cd}{(c + e)^2} \right) + 1 \right) = 0 \]

So \( a = \sqrt{\beta}, \ c = e, \ d = \frac{\gamma}{3c} \). The second constraint gives us \((c + e)^2 = \gamma + ed\). Plugging in our expression for \( e \) and \( d \) in terms of \( c \), we have \( c = \sqrt{\frac{\gamma}{3}} \).

Now that we know \( a \) and \( c \) in terms of \( \beta \) and \( \gamma \), we can obtain the perimeter from the first expression in the KKT analysis (i.e., the perimeter for the general case in terms of our variables).

We obtain the following as perimeter:
\[ 2 \left( 2\sqrt{\beta} + 2\sqrt{\frac{\gamma}{3}} \right) + 2\sqrt{\frac{\gamma}{3}} = 4\sqrt{\beta} + 6\sqrt{\frac{\gamma}{3}}, \]
which is \( 4\sqrt{\alpha} + 6\sqrt{\frac{1}{3}} \) or \( 4 + 6\sqrt{\frac{\alpha}{3}} \). For the perimeter to represent a valid shape, we need \( b \geq d \iff \sqrt{\alpha} \geq \sqrt{3}, \) which is never true. The second double bubble perimeter is never optimal as can be seen by comparing it to the double bubble perimeter of the sets in Theorem 1 part III.

Now to the last case where only \( \mu_1 > 0 \). So \( \beta = ad \) and we have the following gradient
\[
\left( 2 \left( 1 - \frac{\beta}{a^2} \right) - d \mu_1, 2 \left( 1 - \frac{\gamma + ed}{(c + e)^2} \right), 2 \left( \frac{-c}{c + e} \right) + 1, 2 \left( \frac{-\gamma - cd}{(c + e)^2} \right) \right) + 1 - a \mu_1 \right) = 0
\]
So $c = e$ and this gradient becomes (by reducing the dimension and removing $e$)

$$
2 \left(1 - \frac{\beta}{a^2}\right) - d \mu_1, 2 \left(1 - \frac{\nu + cd}{4c^2}\right), 2 \left(-\frac{\nu - cd}{4c^2}\right)
+ 1 - a \mu_1 = 0
$$

Multiply the first equation by $a$, the third by $d$, and subtract the third by first to get

$$
2 \left(a - \frac{\beta}{a} + \frac{\nu d - cd^2}{4c^2}\right) - d. \quad \text{Next from the second equation we get } 4c^2 = \gamma + cd,
$$

so $d = \frac{\nu d + cd^2}{4c^2}$. Applying this to the derived equation, we get $d = 2 \left(\frac{\beta}{a} - a\right)$.

Now from our $\mu_1$ constraint we know $ad = \beta$ or $2(\beta - a^2) = \beta$, so we get $a = \sqrt{\frac{\beta}{2}}$ and $d = \sqrt{2\beta}$. Now plugging $d$ back into $4c^2 = \gamma + cd$ we get

$$4c^2 - c\sqrt{2\beta} - \gamma = 0. \quad \text{So from the quadratic equation we have } c = e = \frac{\sqrt{2\beta} + \sqrt{2\beta + 16\gamma}}{8} \quad \text{(the minus solution is negative so it can’t work). This obtains all the relevant variables. From this we can plug into the perimeter and obtain: }
\rho_{\text{DB}} = 2 \left(\sqrt{\frac{\beta}{2}} + \frac{\beta}{\sqrt{2\beta}} + \frac{\sqrt{2\beta} + \sqrt{2\beta + 16\gamma}}{8} + \frac{\sqrt{2\beta} + \sqrt{2\beta + 16\gamma}}{8}\right) \left(\sqrt{2\beta} + \frac{\sqrt{2\beta + 16\gamma}}{8}\right)$$

For this if let $\gamma = \alpha$, it is always more than double bubble perimeter of the sets in Theorem 1 part III. If $\beta = \alpha$ and $\gamma = 1$, then we do get valid and smaller answers for $\alpha < 0.12$. But we also need $e \leq a$, or $\sqrt{\frac{2\alpha + \sqrt{2\alpha + 16}}{8}} \leq \sqrt{\frac{\alpha}{2}}$ which is only true outside of the range from 0 to 1. Therefore this is an invalid answer and we obtain that:

**Lemma 7** The minimum of the double bubble perimeter over every configuration in $\mathcal{F}_\alpha$ is the minimum between the kissing rectangles and embedded rectangle given in Lemmas 5 and 6.

**Remark 3.1** The two graphs for $Vol(A) = 1, Vol(B) = \alpha$ and $Vol(A) = \alpha, Vol(B) = 1$ are different because in one case the encased rectangle has volume 1 and never gets small enough to be absorbed into the bigger rectangle like in the embedded rectangle case, so we only see the kissing rectangles case for it. The other case exhibits all portions of the minimum. See figure 18 for a comparison of the two cases.

### 4 Proof of Theorem 1

In this section, we collect the results of the previous sections to prove our main result.

In Lemmas 5, 6, and 7, we analyze each of the possible elements of $\mathcal{F}_\alpha$ using the KKT method and by comparing them we can find a global minimizer in $\mathcal{F}_\alpha$ which we call here $\chi_\alpha \in \mathcal{F}_\alpha$ satisfying

**Lemma 8** I. For any $0 < \alpha \leq 1$ there is a $\chi_\alpha \in \mathcal{F}_\alpha$ such that for any $(A, B) \in \mathcal{F}_\alpha$, $\rho_{\text{DB}}(\chi_\alpha) \leq \rho_{\text{DB}}((A, B))$.

II.

$$\rho_{\text{DB}}(\chi_\alpha) = \left(4\sqrt{1 + \alpha + 2\sqrt{\alpha}}\right) \mathbb{1}_{\left[0, \frac{688 - 480\sqrt{2}}{49}\right]}(\alpha) + \left(4 + 2\sqrt{2}\right) \mathbb{1}_{\left(\frac{688 - 480\sqrt{2}}{49}, \frac{1}{2}\right]}(\alpha)$$
Fig. 18 Perimeters for the two choices for Vol(A) and Vol(B)

\[ \text{Fig. 19 Minimizing configurations for different volume ratios} \]

\[ + \left( 2\sqrt{6(1+\alpha)} \right) \left[ \frac{1}{2}, 1 \right](\alpha) \]

III. For \( \alpha = \frac{688-480\sqrt{2}}{49} \) we can choose for \( \chi_\alpha \) either Fig. 19a or b, for \( \alpha \in [1/2, 1] \)
\( \chi_\alpha \) satisfies Fig. 19c, for \( \alpha \in \left( \frac{688-480\sqrt{2}}{49}, \frac{1}{2} \right) \) \( \chi_\alpha \) satisfies Fig. 1b, and for \( \alpha \in \left( 0, \frac{688-480\sqrt{2}}{49} \right) \) \( \chi_\alpha \) satisfies Fig. 19a.

**Proof of Theorem 1** First let \( 0 < \alpha \leq 1 \). Take some sequence \( \chi_i \in \gamma_\alpha \) such that \( \lim_{i \to \infty} \rho_{DB}(\chi_i) = \rho_{DB}(\Gamma_\alpha) \). By Lemmas 2, 3 and 4, for each element of this sequence we obtain an element \( \tilde{\chi}_i \in \mathcal{F}_\alpha \) such that \( \rho_{DB}(\tilde{\chi}_i) \leq \rho_{DB}(\chi_i) \). By Lemma 8 part I there is a \( \chi_\alpha \in \mathcal{F}_\alpha \) satisfying for any \( i \in \mathbb{N} \),

\[ \rho_{DB}(\chi_\alpha) \leq \rho_{DB}(\tilde{\chi}_i) \leq \rho_{DB}(\chi_i). \]

We have that

\[ \rho_{DB}(\Gamma_\alpha) \leq \rho_{DB}(\chi_\alpha) \leq \lim_{i \to \infty} \rho_{DB}(\chi_i) = \rho_{DB}(\Gamma_\alpha), \]
Fig. 20  The unique four corner 
(A, B) configuration

and thus $\chi_\alpha \in \Gamma_\alpha$ and $\Gamma_\alpha$ is non empty, establishing part I of Theorem 1. Since we have that $\chi_\alpha \in \Gamma_\alpha$, Lemma 8 parts II and III establishes Theorem 1 parts II and III. \(\square\)

5 Proof of Theorem 1 Part IV

We will now show that the minimizing configurations found in the previous section are unique. Since we know that $\Gamma_\alpha \neq \emptyset$ (i.e., minimizing sets exist), we can take $(A, B) \in \Gamma_\alpha$ and follow the procedures described in Lemmas 2, 3, and 4 which give us new sets $\tilde{A}, \tilde{B}$ that are of our general form seen in Fig. 2, and such that $\rho_{DB}(\tilde{A}, \tilde{B}) \leq \rho_{DB}(A, B)$. Since $(\tilde{A}, \tilde{B}) \in \Gamma_\alpha$ and is of the general form, it must look like one of the sets given in Figure 1. Therefore, if we go through the transformations described in Lemmas 2, 3, and 4, and the result is a configuration that is of the general form, but not the exact form of one of the minimizing configurations shown in Figure 1, then we could not have started out with $(A, B) \in \Gamma_\alpha$.

In the Lemma 9, we will show that if four corners of $B\Box$ are contained in $A\Box$, then the only possible minimizer is Fig. 1a.

Lemma 9 If $(A, B) \in \Gamma_\alpha$, and $B\Box \subset A\Box$, i.e., four corners of $B\Box$ are contained in $A\Box$, then $(A, B)$ is of the form of Fig. 1a.

Proof We begin with $(A, B) \in \Gamma_\alpha$. Recall from Lemma 2, when $B\Box \subset A\Box$, we can construct another minimizing configuration that is either similar to the embedded rectangles (see Fig. 1a) but not necessarily with the same side lengths for both rectangles, or to the kissing rectangles (see Fig. 1c) but, again, not necessarily with the same side lengths.

First, let us assume that $(A, B)$ is such that either zero, one, or two edges of $B\Box$ is contiguous with the same number of edges of $A\Box$. Recall from Lemma 2 that in these cases, we could construct a new set $\hat{B}$, which was just a shifted version of $B\Box$ and put it in one of the corners of $A\Box$, and a new set $\tilde{A}$ which was the interior of $A\Box \setminus \hat{B}\Box$. These sets had the property that $\rho_{DB}(\hat{A}, \hat{B}) \leq \rho_{DB}(A, B)$, $\mu(\hat{B}) \geq \alpha$, and $\mu(\hat{A} \cup \hat{B}) \geq 1 + \alpha$. The picture looks like Fig. 20.

Now, if $\mu(B\Box) > \alpha = \mu(B)$, then $\mu(\hat{B}) = \mu(B\Box) > \alpha$. So, to adjust the volume we can move the left side of $\hat{B}$ to the right until the volume is correct. This, however, would strictly reduce the double bubble perimeter, contradicting that $(A, B) \in \Gamma_\alpha$. 

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Therefore, $\mu(B) = \mu(B^{\square})$. Since $B \subset B^{\square}$, it follows that $\tilde{B} = B^{\square}$. Similarly, if $\mu(\tilde{A} \setminus \tilde{B}) = \mu(A^{\square} \setminus B^{\square}) > 1$, then we can move the left side of $\tilde{A}$ to the right (and possibly the bottom side of $\tilde{A}$ up) until we obtain the correct volume. Doing so, however, would strictly reduce the double bubble perimeter. This would, again, contradict that $(A, B) \in \Gamma_\alpha$. So, if $\mu(\tilde{A} \cup \tilde{B}) = 1 + \alpha$, we know that $\mu(A \cup B) = 1 + \alpha$ because otherwise the procedure to reduce the volumes would strictly reduce the double bubble perimeter. Therefore, $\mu(A^{\square}) = 1 + \alpha$. Since $A^{\square} \supset A \cup B$, it then follows that $(A^{\square})^\circ = A \cup B$. Recall that there must be at least three disjoint paths in $\partial A \cup \partial B$ the vertical length of which is at least the distance between the top of $A^{\square}$ and the bottom of $A^{\square}$. In $A^{\square}$, there are two such distances, which means that to make $A \cup B$, there must be exactly one more such vertical length. That is, we must have started out with $(A, B)$ in the configuration of kissing rectangles, contradicting that $B^{\square} \subset A^{\square}$. So, if $(A, B) \in \Gamma_\alpha$, and $B^{\square} \subset A^{\square}$, then $(A, B)$ must be the exact configuration shown in Figure 1 (a), or some isometry of this figure. □

We now turn our attention to when two corners of $B^{\square}$ are contained in $A^{\square}$.

**Lemma 10** Let $(A, B) \in \Gamma_\alpha$. If two corners of $B^{\square}$ are contained in $A^{\square}$, then $(A, B)$ is of the form of kissing rectangles given in Fig. 1c.

**Proof** In Lemma 4, the process we followed showed that $(A, B)$ could be transformed into $(\tilde{A}, \tilde{B})$ such that $\rho_{DB}(\tilde{A}, \tilde{B}) \leq \rho_{DB}(A, B)$, and $(\tilde{A}, \tilde{B})$ was one of the following two forms:

In the figure on the left, the construction was such that $\mu(B^{\square}) = \mu(\tilde{B})$. Since $(A, B) \in \Gamma_\alpha$, $(\tilde{A}, \tilde{B}) \in \Gamma_\alpha$. So, the configuration on the left in Fig. 21 is not possible unless $\tilde{A} \cap \tilde{B} = \emptyset$, which means $(\tilde{A}, \tilde{B})$ is the same as Fig. 1 (b), with the smaller rectangle shifted up. If $\mu(B) > \alpha = \mu(B)$, then we could make $\tilde{B}$ smaller by moving its right side to the left an appropriate amount. But this would strictly reduce the double bubble perimeter. Therefore, $\mu(B) = \mu(B^{\square})$. Similarly, if $\mu(\tilde{A} \setminus \tilde{B}) = \mu(A^{\square} \setminus B^{\square}) > 1$, then we can move the left side of $\tilde{A}$ to the right (and possibly the bottom side of $\tilde{A}$ up) until we obtain the correct volume. Doing so, however, would strictly reduce the double bubble perimeter. This would, again, contradict that $(A, B) \in \Gamma_\alpha$. So, if $\mu(\tilde{A} \cup \tilde{B}) = 1 + \alpha$, we know that $\mu(A \cup B) = 1 + \alpha$ because otherwise the procedure to reduce the volumes would strictly reduce the double bubble perimeter. Therefore, $\mu(A^{\square}) = 1 + \alpha$. Since $A^{\square} \supset A \cup B$, it then follows that $(A^{\square})^\circ = A \cup B$. Recall that there must be at least three disjoint paths in $\partial A \cup \partial B$ the vertical length of which is at least the distance between the top of $A^{\square}$ and the bottom of $A^{\square}$. In $A^{\square}$, there are two such distances, which means that to make $A \cup B$, there must be exactly one more such vertical length. That is, we must have started out with $(A, B)$ in the configuration of kissing rectangles, contradicting that $B^{\square} \subset A^{\square}$. So, if $(A, B) \in \Gamma_\alpha$, and $B^{\square} \subset A^{\square}$, then $(A, B)$ must be the exact configuration shown in Figure 1 (a), or some isometry of this figure. □

![Fig. 21](image-url) The unique two corner $(A, B)$ configuration
bubble perimeter resulting in $\rho_{DB}(\tilde{A}, \tilde{B}) < \rho_{DB}(A, B)$, a contradiction. Therefore, 
$\mu(B) = \mu(\tilde{B}) = \mu(B^\square)$. Similarly, $\mu(A) = \mu(A^\square)$. Since $B^\square \supseteq B$, and $A^\square \supseteq A$, it follows then that $A^\square = A \cup \partial A$, and $B^\square = B \cup \partial B$. If $A \cap B \neq \emptyset$, then $\mu(A \cap B) > 0$, meaning that either $\mu(A) < \mu(A^\square)$, or $\mu(B) < \mu(B^\square)$, both contradictions. Therefore, $A \cap B = \emptyset$, and we must have started with a configuration like Fig. 1 (b) but possibly with the small rectangle shifted up or down, or touching a different side.

If, on the other hand, the process described in Lemma 4 results in the configuration on the right side of Fig. 21, then $\mu(\tilde{A} \cup \tilde{B}) = \mu(A^\square \cup B^\square) = \mu(A \cup B)$ (otherwise, as before, we could shrink the size of $\tilde{A} \cup \tilde{B}$ to the correct volume and strictly reduce the double bubble perimeter). So, $(A \cup B) = (\tilde{A} \cup \tilde{B}) = A^\square \cup B^\square$. Notice here that since $A^\square$ and $B^\square$ are closed, $A^\square \cup B^\square$ is its own box. That is, $A^\square \cup B^\square = (A^\square \cup B^\square)^\square$. Therefore, we have that $\partial (A \cup B) = \partial (\tilde{A} \cup \tilde{B})$ and so, $\rho(\partial A \cup B) = 2(a_{\text{top}} - a_{\text{bottom}}) + 2(b_{\text{right}} - a_{\text{left}})$. Now, $\rho_{DB}(\tilde{A}, \tilde{B}) = 3(a_{\text{top}} - a_{\text{bottom}}) + 2(b_{\text{right}} - a_{\text{left}})$. This means that the boundary of $A \cup B$ is longer than it should be by $a_{\text{top}} - a_{\text{bottom}}$. It must be a single path since there are only two sets, and it must intersect the horizontal lines passing through both $a_{\text{top}}$ and $a_{\text{bottom}}$. This leaves one possibility, which is a single vertical line as in the right side of Fig. 21. Furthermore, since this line must divide the interior of $A^\square \cup B^\square$ into two sets whose volumes are 1 and $\alpha$, respectively. There are, therefore, only two places to position such a line, each being a mirror of the other. Therefore, $A \cup B$ must be the same as $\tilde{A} \cup \tilde{B}$, or its mirror image. $\square$

**Lemma 11** Let $(A, B) \in \gamma\alpha$. If exactly one corner of $B^\square$ is contained in $A^\square$, then $(A, B) \notin \Gamma\alpha$.

**Proof** Suppose that $(A, B) \in \Gamma\alpha$, and that exactly one corner of $B^\square$ is contained in $A^\square$. Recall that in Lemma 3, we could follow several different procedures to construct different configurations whose double bubble perimeter did not increase, and whose volumes were still correct. Two of these configurations are shown in Figs. 8 and 10. These figures are both in the general form, and the analysis of the previous section shows that neither of these are minimizing shapes.

Now, consider Fig. 11. We arrived at this figure by assuming that $c + d + e \leq a$, $d < f$, and $e < c < f$. The volume of $\tilde{B}$ is short of $\alpha$ by at most $c \cdot d + d \cdot e$. We then increased the volume of $\tilde{B}$ by $f \cdot (c + d) > d \cdot (e + c)$ (see Fig. 12). Therefore, we have too much volume in $\tilde{B}$, and by moving its right side to the left to decrease the volume until it is $\alpha$, we strictly decrease the double bubble perimeter.

The other option here was that our procedure resulted in Figure 13, where we added $(d + l) \cdot f > dc$ volume to $\tilde{B}$. Now the total volume of $A^\square \cup \tilde{B} > 1 + \alpha$, and we can again adjust the volumes so that $A^\square$ has volume 1 and $\tilde{B}$ has volume $\alpha$ in a way that strictly reduces the double bubble perimeter.

In all of the above cases, if one corner of $B^\square$ is contained in $A^\square$, then the configuration cannot minimize the double bubble perimeter. There is a final case to consider which, referring to Fig. 6, is when $c + d + e > a$. In this situation, we can create one of
the configurations in either Fig. 14, or Fig. 15. However, both of these configurations are covered in the general form, over which we already minimized, and we saw that neither of them is a minimizer. Therefore, \((A, B)\) could not have been a minimizing configuration. \(\square\)

**Declarations**

**Conflict of interest** The authors have no conflicts of interest.

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