Abstract

We consider the Darboux transformation of the Green functions of the regular boundary problem of the one-dimensional stationary Dirac equation. We obtained the Green functions of the transformed Dirac equation with the initial regular boundary conditions. We also construct the formula for the unabridged trace of the difference of the transformed and the initial Green functions of the regular boundary problem of the one-dimensional stationary Dirac equation. We illustrate our findings by the consideration of the Darboux transformation for the Green function of the free particle Dirac equation on an interval.

Keywords: Dirac equation; Green function; Darboux transformation.

1 Introduction

There has been great interest in using the Darboux transformation [1, 2] for the analysis of physical systems [3, 4, 5] and for finding new solvable systems [6, 7, 8]. It has been shown that the transformation method is useful in finding soliton solutions of the integrable systems [9, 10] and constructing supersymmetric quantum
mechanical systems [11, 12]. It is well known that SUSY QM is basically equivalent to the Darboux transformation and the factorization properties of the Schrödinger equation [13, 14, 15]. The Darboux transformation of the one-dimensional stationary Dirac equation is equivalent to the underlying quadratic supersymmetry and the factorization properties [16, 17, 18].

Despite of the growing number of papers in this field many questions still remain open and require further study. In particular, the author are aware of only several papers [19, 20, 21] devoted to the SUSY transformations at the level of the Green functions. In [19] was obtained the integral relation between the Green functions for two SUSY partner Hamiltonians of the one-dimensional Schrödinger equation with discrete spectra. In [20, 21] the integral relation between the Green functions corresponding two SUSY partner Hamiltonians of the one-dimensional Schrödinger equation is generated to the case of continuous spectrum. The exact Green function of the time-dependent Schrödinger equation was obtained in [22, 23].

An interesting open problem is to find the analogous results for the Dirac equation. In this paper, we construct the Darboux transformation of the Green function for the regular boundary problem of the one-dimensional stationary Dirac equation with a generalized form of the potential and obtain formulas for the unabridged trace of the difference of the modified and initial Green functions. The rest of the paper is organized as follows. In Section 2 we construct a Green function for the initial regular boundary problem of the one-dimensional Dirac equation with the generalized form of the potential. In Section 3 we consider the Green function for the transformed regular boundary problem of the one-dimensional Dirac equation. We construct the Darboux transformation of the Green function. In Section 4 we obtain formulas for the unabridged trace of the difference of the modified and the initial Green functions and consider the spectral representation of the corresponding unabridged trace. In Conclusion the summarize our results and speculate about some perspective.

2 Green function of the one-dimensional Dirac equation

The Green function of the one-dimensional Dirac equation

\[(h_0(x) - E)\psi(x) = 0\]  \hspace{1cm} (1)

with the Dirac Hamiltonian of the form \(h_0 = i\sigma_2 \partial_x + V(x)\) [24] is needed for obtaining the solution

\[\Phi(x, E) = \int_a^b G_0(x, y, E)F(y)dy\]  \hspace{1cm} (2)

of the following inhomogeneous equation

\[(h_0(x) - E)\Phi(x, E) = F(x),\]  \hspace{1cm} (3)

\(^1\)Recently a special issue of *Journal of Physics, A* 34 was devoted to research work in supersymmetric quantum mechanics (SUSY QM).
where $a$ and $b$ are the endpoints of an closeg interval.

Earlier this method allowed the authors of the paper [25] to obtain the Green function of the one-dimensional Dirac equation with the potential

$$ U_0 = \begin{pmatrix} p(x) & 0 \\ 0 & q(x) \end{pmatrix}. $$

In the present paper, we will obtain the Green function of the Dirac equation with the generalized form [26] of the potential

$$ V_0(x) = \omega(x)I + (m + S(x))\sigma_3 + q(x)\sigma_1, \quad (4) $$

where $\omega(x)$, $S(x)$ and $q(x)$ are real functions of $x$, $m$ is the mass of a particle, $\sigma_1$, $\sigma_3$ are usual Pauli matrices. The generalized potential is the self-adjoint potential.

We consider the following boundary conditions for the components $\Phi_1(x,E)$, $\Phi_2(x,E)$ of the solution (2) to (3):

$$ \Phi_1(a,E) \sin(\alpha) + \Phi_2(a,E) \cos(\alpha) = 0, \quad (5) $$

$$ \Phi_1(b,E) \sin(\beta) + \Phi_2(b,E) \cos(\beta) = 0. \quad (6) $$

We suppose that the solutions $\psi(x)$, $\varphi(x)$ of the Dirac equation with the generalized form of the potential obey the following conditions:

$$ \varphi(a,E) = \begin{pmatrix} \cos \alpha \\ -\sin \alpha \end{pmatrix}, \quad \psi(b,E) = \begin{pmatrix} \cos \beta \\ -\sin \beta \end{pmatrix}. \quad (7) $$

Let us construct the matrix

$$ G_0(x,y,E) = \frac{1}{W(E)} \begin{cases} \psi(x,E)\varphi^T(y,E), & y \leq x, \\ \varphi(x,E)\psi^T(y,E), & x < y, \end{cases} \quad (8) $$

where $W(E) = W\{\varphi(x,E),\psi(x,E)\} = const$ is the Wronskian of the two functions $\psi(x)$ and $\varphi(x)$.

One can reality examine that the vector-function $\Phi(x,E)$ is the solution to the inhomogeneous equation (3) and therefore the matrix (8) is the Green function of the regular boundary problem of the one-dimensional stationary Dirac equation with the generalized form of the potential [1].

Let us construct the spinor $y(x,E) = \begin{pmatrix} y_1(x,E) \\ y_2(x,E) \end{pmatrix}$

$$ y(x,E) = \int_b^a G_0(x,y,E)F(y)dy. \quad (9) $$

In detail, Eq. (8) becomes

$$ G_0(x,y,E) = \frac{1}{W(E)} \begin{cases} \begin{pmatrix} \varphi_1(x,E)\psi_1(y,E) \\ \varphi_2(x,E)\psi_1(y,E) \end{pmatrix}, & x < y, \\ \begin{pmatrix} \varphi_1(y,E)\psi_1(x,E) \\ \varphi_2(y,E)\psi_1(x,E) \end{pmatrix}, & y \leq x. \end{cases} \quad (10) $$

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We write down a detailed form of the components of the spinor

\[ G_0(x, y, E)F(y) = \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix}, \]  

(11)

\[ \Gamma_1 = \frac{1}{W(E)} \begin{cases} \varphi_1(x, E)\psi_1(y, E)F_1(y) + \varphi_1(x, E)\psi_2(y, E)F_2(y), & x < y, \\ \varphi_1(y, E)\psi_1(x, E)F_1(y) + \varphi_2(y, E)\psi_1(x, E)F_2(y), & y \leq x, \end{cases} \]

\[ \Gamma_2 = \frac{1}{W(E)} \begin{cases} \varphi_2(x, E)\psi_1(y, E)F_1(y) + \varphi_2(x, E)\psi_2(y, E)F_2(y), & x < y, \\ \varphi_1(y, E)\psi_2(x, E)F_1(y) + \varphi_2(y, E)\psi_2(x, E)F_2(y), & y \leq x. \end{cases} \]

From these equations, taking into consideration (11) and

\[ \psi_1(y, E)F_1(y) + \psi_2(y, E)F_2(y) = \psi^T(y, E)F(y), \]

\[ \psi_2(y, E)F_1(y) + \varphi_2(y, E)F_2(y) = \varphi^T(y, E)F(y), \]

we find the components of the spinor (9) \( y_1(x, E), y_2(x, E) \).

We consider the components of the spinor (9) in the case \( y \leq x \), then \( y \in [a, x] \), after that in the case \( y > x \), when \( y \in (x, b] \), and as a result, we find the components of the spinor (9) \( y_1(x, E), y_2(x, E) \) in the general case:

\[ y_1(x, E) = \frac{1}{W(E)} \{ \psi_1(x, E) \int_a^x \varphi^T(y, E)F(y)dy + \varphi_1(x, E) \int_x^b \psi^T(y, E)F(y)dy \}, \]

(12)

\[ y_2(x, E) = \frac{1}{W(E)} \{ \psi_2(x, E) \int_a^x \varphi^T(y, E)F(y)dy + \varphi_2(x, E) \int_x^b \psi^T(y, E)F(y)dy \}. \]

(13)

Now we check that the vector-function \( y(x, E) \) is the solution of the inhomogeneous equation (3)

\[ \partial_x y_2(x, E) - (E - S(x) - m - \omega(x))y_1(x, E) + q(x)y_2(x, E) = F_1(x), \]

(14)

\[ \partial_x y_1(x, E) + (E + S(x) + m - \omega(x))y_2(x, E) - q(x)y_1(x, E) = -F_2(x), \]

(15)

where \( F_1(y), F_2(y) \) are the components of the spinor \( F(y) \).

Let us show that equality (15) is true. For that we differentiate \( y_1(x, E) \):

\[ \partial_x y_1(x, E) = \frac{1}{W(E)} \{ \partial_x \psi_1(x, E) \int_a^x \varphi^T(y, E)F(y)dy + \partial_x \varphi_1(x, E) \int_x^b \psi^T(y, E)F(y)dy \}

- \varphi_1(x, E) \{ \psi_1(x, E)F_1(x) + \psi_2(x, E)F_2(x) \}

+ \varphi_1(x, E) \{ \varphi_1(x, E)F_1(x) + \varphi_2(x, E)F_2(x) \}. \]  

(16)
Since the spinors $\psi(x, E)$ and $\varphi(x, E)$ are the solutions to the Dirac system of equations, the following equalities are correct:

$$\partial_x \psi_1(x, E) = q(x)\psi_1(x, E) - (S(x) + m - \omega(x) + E)\psi_2(x, E),$$  
(17)

$$\partial_x \varphi_1(x, E) = q(x)\varphi_1(x, E) - (S(x) + m - \omega(x) + E)\varphi_2(x, E).$$  
(18)

Substituting equalities (17), (18) into (16), we obtain

$$\partial_x y_1(x, E) = \frac{1}{W(E)}[q(x)\{\psi_1(x, E)\int_a^x \varphi^T(y, E)F(y)dy
\]

$$\varphi_1(x, E)\int_a^b \psi^T(y, E)F(y)dy
\]

$$-(S(x) + m - \omega(x) + E)\{\psi_2(x, E)\int_a^x \varphi^T(y, E)F(y)dy
\]

$$+ \varphi_2(x, E)\int_a^b \psi^T(y, E)F(y)dy\}] - F_2(x).$$  
(19)

From (12), (13) for $y_1(x, E)$, $y_2(x, E)$ we get

$$\partial_x y_1(x, E) = q(x)y_1(x, E) - \{S(x) + m - \omega(x) + E\}y_2(x) - F_2(x).$$  
(20)

Therefore, (15) is correct. Similarly, equality (14) can be proven. The spinor-function $\Phi(x, E) = y(x, E)$ is the solution to (3) with the generalized form of the potential.

Now we demonstrate realization of the regular boundary conditions (5), (6) for spinor-function $\Phi(x, E) = y(x, E)$

$$y_1(a, E)\sin(\alpha) + y_2(a, E)\cos(\alpha) = 0,$$  
(21)

$$y_1(b, E)\sin(\beta) + y_2(b, E)\cos(\beta) = 0,$$  
(22)

where $y_1(x, E)$, $y_2(x, E)$ are the components of the spinor $y(x, E)$. We compute (12), (13) when $x = a$, $x = b$:

$$y_1(a, E) = \frac{1}{W(E)}\{\psi_1(a, E)\int_a^a \varphi^T(y, E)F(y)dy
\]

$$\varphi_1(a, E)\int_a^b \psi^T(y, E)F(y)dy\},$$  
(23)

$$y_2(a, E) = \frac{1}{W(E)}\{\psi_2(a, E)\int_a^a \varphi^T(y, E)F(y)dy
\]

$$+ \varphi_2(a, E)\int_a^b \psi^T(y, E)F(y)dy\},$$  
(24)
$$y_1(b, E) = \frac{1}{W(E)}\{\psi_1(b, E) \int_a^b \varphi^T(y, E)F(y)dy$$
$$+ \varphi_1(b, E) \int_b^a \psi^T(y, E)F(y)dy\},$$
(25)

$$y_2(b, E) = \frac{1}{W(E)}\{\psi_2(b, E) \int_a^b \varphi^T(y, E)F(y)dy$$
$$+ \varphi_2(b, E) \int_b^a \psi^T(y, E)F(y)dy\}.$$  (26)

Since $$\int_a^a f(y)dy = 0$$ and $$\int_b^b f(y)dy = 0$$, where $$f(y)$$ is the arbitrary function of $$y$$, the equalities (23), (24), (25), (26) take the following for ms:

$$y_1(a, E) = \frac{1}{W(E)}\{\varphi_1(a, E) \int_a^b \psi^T(y, E)F(y)dy\},$$
(27)

$$y_2(a, E) = \frac{1}{W(E)}\{\varphi_2(a, E) \int_a^b \psi^T(y, E)F(y)dy\},$$  (28)

$$y_1(b, E) = \frac{1}{W(E)}\{\psi_1(b, E) \int_a^b \varphi^T(y, E)F(y)dy\},$$
(29)

$$y_2(b, E) = \frac{1}{W(E)}\{\psi_2(b, E) \int_a^b \varphi^T(y, E)F(y)dy\}.  \tag{30}$$

Now, taking into account (27), (28), (29), (30), we check the conditions (21), (22):

$$y_1(a, E) \sin(\alpha) + y_2(a, E) \cos(\alpha)$$

$$= \frac{1}{W(E)}\{\varphi_1(a, E) \sin(\alpha) + \varphi_2(a, E) \cos(\alpha)\} \int_a^b \psi^T(y, E)F(y)dy,$$
(31)

$$y_1(b, E) \sin(\beta) + y_2(b, E) \cos(\beta)$$

$$= \frac{1}{W(E)}\{\psi_1(b, E) \sin(\alpha) + \psi_2(b, E) \cos(\alpha)\} \int_a^b \varphi^T(y, E)F(y)dy.$$
(32)

Due to the boundary conditions (7) for the functions $$\psi(x, E)$$, $$\varphi(x, E)$$ the relations are (21), (22) are valid. Hence, the matrix $$G_0(x, y, E) \tag{8}$$, is the Green function for the regular boundary problem of the one-dimensional stationary Dirac equation with the generalized form of the potential.

The matrix (8) can be written in the following form:

$$G_0(x, y, E) = (\psi(x)\varphi^T(y)\Theta(x - y) + \varphi(x)\psi^T(y)\Theta(y - x))\big/ (W\{\varphi(x), \psi(x)\}),$$
(33)
where $\Theta(x - y), \Theta(y - x)$ are the Heaviside step functions. Thus, we have constructed the Green function of the initial problem.

**Example 1** Let us calculate the Green function of the initial problem for the Dirac free particle equation with the $h_0 = i\sigma_2 \partial_x + V_0(x), V_0 = m\sigma_3$.

We first chose the two linearly independent solutions $\varphi, \psi$ for the free particle case:

$$\psi = \begin{pmatrix} \cos (kx) \\ k \sin (kx)/(E + m) \end{pmatrix}, \quad \varphi = \begin{pmatrix} \cos k(x - 1) \\ k \sin k(x - 1)/(E + m) \end{pmatrix} \quad (34)$$

$$k^2 = E^2 - m^2, \quad W\{\varphi, \psi\} = -k \sin (k)/(E + m). \quad (35)$$

Next we apply the boundary conditions to functions $\psi, \varphi$ for the case $a = 0, b = 1$:

$$\psi \big|_{x=1} = \pm \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \varphi \big|_{x=0} = \pm \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (36)$$

From (34) it follows that $k = \pi n, n \in Z$.

Also, we find the final expression for the Green function of the initial regular boundary problem (5), (6) with $\alpha = \beta = \pm \pi n$:

$$G_0(x, y, E) = \frac{A^0 \Theta(x - y) + B^0 \Theta(y - x)}{W\{\varphi, \psi\}}, \quad (37)$$

$$A^0 = \begin{pmatrix} \cos (kx) \cos (ky - k) & k \cos (kx) \sin (ky - k)/(E + m) \\ k \sin (kx) \cos (ky - k)/(E + m) & k^2 \sin (kx) \sin (ky - k)/(E + m)^2 \end{pmatrix}$$

$$B^0 = (A^0)^T (x \leftrightarrow y).$$

### 3 Darboux transformed Green function

Before constructing Darboux transformed Green function, let us shortly consider the Darboux transformation method for the Dirac equation.

The so-called *Darboux transformation operator* $L$ and the potential $V_1$ of the transformed Dirac Hamiltonian $h_1$ have respectively the following form:

$$L = \partial_x - u_x u^{-1}, \quad (38)$$

$$V_1 = V_0 + [i\sigma_2, u_x u^{-1}]. \quad (39)$$

The so-called *transformation function* $u$ is $2 \times 2$ matrix consists from two solutions $u = (u_1, u_2)$ of the initial Dirac equation with $E = \lambda_1, \lambda_2$, where $\lambda_1, \lambda_2$ are the neighboring energy levels.
If the function $\psi$ is solution of initial equation and $\psi \neq u_1, u_2$, then the function $\tilde{\psi} = L\psi$ is solution of the transformed Dirac equation

$$(h_1(x) - E)\tilde{\psi}(x) = 0, \quad h_1 = i\sigma_2 + V_1.$$  \hfill (40)

The Green function of the equation (40) with the regular boundary conditions for the components $\tilde{\Phi}_1(x, E), \tilde{\Phi}_2(x, E)$

$$\tilde{\Phi}_1(a, E) \sin(\tilde{\alpha}) + \tilde{\Phi}_2(a, E) \cos(\tilde{\alpha}) = 0,$$  \hfill (41)

$$\tilde{\Phi}_1(b, E) \sin(\tilde{\beta}) + \tilde{\Phi}_2(b, E) \cos(\tilde{\beta}) = 0$$  \hfill (42)

of the spinor $\tilde{\Phi}(x, E) = \int_a^b G_1(x, y, E)F(y)dy$ one can represent in the form:

$$G_1(x, y, E) = (\tilde{\psi}(x)\tilde{\varphi}^T(y)\Theta(x - y) + \tilde{\varphi}(x)\tilde{\psi}^T(y)\Theta(y - x))/(W\{\tilde{\varphi}(x), \tilde{\psi}(x)\}).$$  \hfill (43)

(i) If $\tilde{\alpha} = \alpha$, $\tilde{\beta} = \beta$, we have the Green function of the Dirac equation with the transformed potential and the initial regular boundary conditions.

(ii) If $\tilde{\alpha} \neq \alpha$, $\tilde{\beta} \neq \beta$, we have the Green function of the Dirac equation with the transformed potential and the modified regular boundary conditions.

Further we consider only the case when $\tilde{\alpha} = \alpha$, $\tilde{\beta} = \beta$.

Now let us consider the construction of the Darboux transformed Green function on the basis of the concrete example.

**Example 2** Let us first chose the transformation functions as follows:

$$u = \begin{pmatrix} 1 & \cos (k_1 x) \\ 0 & k_1 \sin (k_1 x)/ (\lambda + m) \end{pmatrix}. $$  \hfill (44)

Next, obtain from the free particle case the transformed potential

$$V_1 = -\lambda \sigma_3 + k_1 \cot (k_1 x) \sigma_1,$$  \hfill (45)

where

$$\lambda_1 = m, \quad \lambda_2 = \lambda = \pm \sqrt{m^2 + \pi^2}. $$  \hfill (46)

It can be checked that only these two transformation matrices (differing from each other only by sign of $\lambda$) provide the conservation of the initial regular boundary condition.

Now the Darboux transformed functions $\tilde{\psi}, \tilde{\varphi}$ one can represent in the form:

$$\tilde{\psi} = \frac{k}{E + m} \begin{pmatrix} (\lambda - E) \sin kx \\ k \cos kx - k_1 \cot k_1 x \sin kx \end{pmatrix},$$  \hfill (47)

$$\tilde{\varphi} = \frac{k}{E + m} \begin{pmatrix} (\lambda - E) \sin (k(x - 1)) \\ k \cos (kx - k) - k_1 \cot (k_1 x) \sin (kx - k) \end{pmatrix}.$$  \hfill (48)
Finally, the Green functions of the Dirac equation with the transformed potential (45) may be given by the formula:

\[ G_1(x, y, E) = \frac{A^{(1)}(x - y) + B^{(1)}(y - x)}{W\{\tilde{\varphi}, \tilde{\psi}\}}, \]

\[ W\{\tilde{\varphi}, \tilde{\psi}\} = (E - \lambda)(E - m)W\{\varphi, \psi\}, \]

where

\[ A^{(1)} = \frac{k^2}{E + m^2} \begin{pmatrix} A_{11}^{(1)} & A_{12}^{(1)} \\ A_{21}^{(1)} & A_{22}^{(1)} \end{pmatrix}, \quad B^{(1)} = (A^{(1)})^T(x \leftrightarrow y), \]

\[ A_{11}^{(1)} = (\lambda - E)^2 \sin (kx) \sin (ky - k), \]
\[ A_{21}^{(1)} = (\lambda - E)(k \cos (kx) - k_1 \cot (k_1 x) \sin (kx)) \sin (ky - k)), \]
\[ A_{12}^{(1)} = (\lambda - E) \sin (kx)(k \cos (ky - k) - k_1 \cot (k_1 y) \sin (ky - k))}, \]
\[ A_{22}^{(1)} = (k \cos (kx) - k_1 \cot (k_1 x) \sin (kx)) \]
\[ \times (k \cos (ky - k) - k_1 \cot (k_1 y) \sin (ky - k))}. \]

Thus, we have constructed the Green function of the transformed Dirac equation with the potential (45) and the initial regular boundary conditions.

### 4 Unabridged trace for the difference of the modified and the initial Green functions

In this section we calculate the unabridged trace of the difference of the transformed and the initial Green functions.
Firstly, let us consider the unabridged trace of the transformed Green function 
\[ \text{tr} \int_a^b G_1(x, y, E) \Big|_{x=y} dy. \]
It can be represented in the form:
\[ \text{tr} \int_a^b G_1(x, y, E) \Big|_{x=y} dy = \frac{\text{tr} \int_a^b \tilde{\psi} \tilde{\phi}^T \Big|_{x=y} dy}{W\{\tilde{\phi}\}}. \] (52)

It is known that the action of the transformation operator and the conjugate transformation operator on the spinors may be written in the following way [16]:
\[ L\psi = u \frac{d}{dx}(u^{-1}\psi), \] (53)
\[ L^+\tilde{\psi} = -(u^+)^{-1} \frac{d}{dx}(u^+\tilde{\psi}). \] (54)

Also, accounting these properties, we can rewrite (52) as:
\[ \text{tr} \int_a^b G_1(x, y, E) \Big|_{x=y} dy = \text{tr} \int_a^b \frac{d}{dx}(u^{-1}\psi)\tilde{\phi}^T u \Big|_{x=y} dy / W\{\tilde{\phi}\}. \] (55)

Next let us consider the unabridged trace of the difference of the transformed and the initial Green functions. It is obvious [25] that the spectral representation for the initial and the transformed Green functions correspondingly look like as
\[ G_0(x, y, E) = \sum_{n_0} \frac{\psi_{n_0}(x)\psi_{n_0}^T(y)}{E_{n_0} - E}, \] (56)
\[ G_1(x, y, E) = \sum_{n_1} \frac{\phi_{n_1}(x)\phi_{n_1}^T(y)}{E_{n_1} - E}, \] (57)
where the functions
\[ \phi_{n_1} = ((E_{n_1} - \lambda_1)(E_{n_1} - \lambda_2))^{-1}L\psi_{n_1}L, \] (58)
form an orthonormal set. The completeness of (58) will be shown in examples 3 and 4.

The construction
\[ \text{tr} \int_a^b (G_1(x, y, E) - G_0(x, y, E)) \big|_{x=y} dy \] (59)
is called the unabridged trace of the difference of the transformed and the initial Green functions. The spectral representation of (59) is as follows:
\[ \text{tr} \int_a^b (G_1(x, y, E) - G_0(x, y, E)) \big|_{x=y} dy = \sum_{n_1} \frac{1}{E_{n_1} - E} - \sum_{n_0} \frac{1}{E_{n_0} - E}, \] (60)
where \( E_{n_0}, E_{n_1} \) are discrete eigenvalues of \( h_0 \) and \( h_1 \) respectively.
If the set (58) is complete then the formula (59) one can represent in the form:

\[ tr \int_a^b (G_1(x, y, E) - G_0(x, y, E))|_{x=y}dy = \frac{1}{E - \lambda_1} + \frac{1}{E - \lambda_2}. \]  

(61)

Now let us show another derivation of the trace formula for the Green functions difference. Let us integrate (55) by parts and apply the trace property

\[ trAB = trBA \]  

(62)
to obtain the relation

\[ tr \int_a^b G_1(x, y, E)|_{x=y}dy = \frac{tr\psi\tilde{\varphi}^T|_a^b}{W\{\tilde{\varphi}\}} + \int_a^b tr\psi(L^1 L\varphi)^T|_{x=y}dy. \]  

(63)

Due to (50) and the factorization property from [16] we obtain:

\[ \int_a^b trG_1(x, y, E)|_{x=y}dy = \frac{tr\psi\tilde{\varphi}^T|_a^b}{W\{\tilde{\varphi}\}} + \int_a^b trG_0(x, y, E)|_{x=y}dy. \]  

(64)

Similarly, we would like to write

\[ \int_a^b trG_1(x, y, E)|_{x=y}dy = \frac{tr\tilde{\psi}\varphi^T|_a^b}{W\{\tilde{\varphi}\}} + \int_a^b trG_0(x, y, E)|_{x=y}dy. \]  

(65)

Finally, from (64), (65) we find:

\[ \begin{align*}
tr \int_a^b (G_1(x, y, E) - G_0(x, y, E))|_{x=y}dy &= \frac{1}{W\{\tilde{\varphi}, \psi\}} tr(\psi\varphi^T) |_a^b, \\
tr \int_a^b (G_1(x, y, E) - G_0(x, y, E))|_{x=y}dy &= \frac{1}{W\{\tilde{\varphi}, \psi\}} tr(\tilde{\psi}\varphi^T) |_a^b. 
\end{align*} \]  

(66), (67)

Here \(a, b\) are limits of the integration. As shown in Appendix, right parts of (66), (67) are equal.

**Example 3** In this example we calculate (59) with the help of (67).

More precisely, take into account (35), (17), (50), we calculate \( tr(\tilde{\psi}\varphi^T)/W\{\tilde{\varphi}, \psi\}|_a^b \),

where \(a = 0, b = 1\):

\[ \frac{tr(\tilde{\psi}\varphi^T)}{W\{\tilde{\varphi}, \psi\}} \bigg|_0^1 = \frac{1}{- (E - m)(E - \lambda) \sin(k)} \left( \begin{array}{c} \lambda - E \cos(kx) \cos(kx - k) + \frac{k^2}{E + m} \\
\cos(kx) \sin(kx - k) - \frac{k^2}{E + m} \cot(k_1x) \sin(kx) \sin(kx - k) \end{array} \right) \bigg|_0^1. \]

Using the L’Hospital’s rule for evaluating of indeterminations, we obtain

\[ \frac{tr(\tilde{\psi}\varphi^T)}{W\{\tilde{\varphi}, \psi\}} \bigg|_0^1 = \frac{1}{E - m} + \frac{1}{E - \lambda}. \]  

(68)
Due to (67) we get:
\[ tr \int_0^1 (G_1(x, y, E) - G_0(x, y, E))|_{x=y} dy = \frac{1}{E - \lambda_1} + \frac{1}{E - \lambda_2}, \]

(69)
where \( \lambda_1 = m, \lambda_2 = \lambda \). This result indicates that the spectrum of the transformed Dirac Hamiltonian differs from the spectrum of the initial Dirac operator. The spectral lines \( E = \lambda, m \) disappear from the transformed spectrum and the set of functions \( \phi_{n1} \) is complete.

**Example 4** In this example we calculate (59) in a direct way.

First, we consider
\[ trG_0(x, x, E) = \frac{E}{k} \cot(k) - \frac{m \cos(2kx - k)}{k \sin(kx)}. \]

(70)
Next, we integrate the expression (70)
\[ tr \int_0^1 G_0(x, x, E) dx = -\frac{E}{k} \cot k - \frac{m}{k^2}, \]
\[ k = \pi n, \quad \cot(k) - k^{-1} = \Sigma_{n=1}^{\infty} 2k/(E^2 - E_n^2), \]
\[ tr \int_0^1 G_0(x, x, E) dx = -\Sigma_{n=1} (E + E_n)^{-1} - \Sigma_{n=1}^{\infty} (E - E_n)^{-1} - (E - m)^{-1}. \]

(71)
Similarly, we calculate
\[ tr \int_0^1 G_1(x, x, E) = -\Sigma_{n=1} (E + E_n)^{-1} - \Sigma_{n=1}^{\infty} (E - E_n)^{-1} + (E - \lambda)^{-1} \]
and obtain that
\[ tr \int G_1(x, x, E) - G_0(x, x, E) dx = \frac{1}{E - \lambda} + \frac{1}{E - m}. \]

(72)
(73)
(74)
Thus, we can conclude that the set of functions \( \phi_{n1} \) is complete.

**5 Conclusion**

In this paper, we have studied the Darboux transformation of the Green functions of the regular boundary problem corresponding to the initial and the transformed potentials of the one-dimensional Dirac equation for the case of the Dirac Hamiltonians with discrete spectrum. The main results of the paper are the construction of the Darboux transformed Green function with initial regular boundary conditions and the formulae for an unabridged trace (66), (67). For the checking of formulae (66) and (67) we consider the unabridged trace of difference between transformed and initial Green functions by usual possible and by formula (66), both results are equal. The all results of this paper are studied only for discrete spectrum and regular boundary problem. An interesting question for the future is generation of these results to the case of continuous spectrum of the Dirac equation and the case of problem on the real line and a half-line. We believe that these problem will investigate in a separate publication.
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Appendix. Equality of traces

In this Appendix, taking into account (34), (35), (47), (48), we show the equivalence of expressions for right parts of (66), (67)

\[
\text{tr}(\psi \tilde{\varphi}^T) |^b_a = (\psi_1 \varphi_1 + \psi_2 \varphi_2) |^b_a = \psi_1(b) \varphi_1(b) \\
+ \psi_2(b) \varphi_2(b) - \psi_1(a) \varphi_1(a) - \psi_2(a) \varphi_2(a),
\]

(75)

\[
\text{tr}(\tilde{\psi}\varphi^T) |^b_a = (\tilde{\psi}_1 \varphi_1 + \tilde{\psi}_2 \varphi_2) |^b_a = \tilde{\psi}_1(b) \varphi_1(b) \\
+ \tilde{\psi}_2(b) \varphi_2(b) - \tilde{\psi}_1(a) \varphi_1(a) - \tilde{\psi}_2(a) \varphi_2(a),
\]

(76)

where \(a = 0, \ b = 1\).

In the explicit form (75), (76) look like as follows:

\[
\text{tr}(\psi \tilde{\varphi}^T) |^1_0 = \{(\lambda - E) \cos(kx) \sin(kx - k) + (E - m) \sin(kx) \cos(kx - k) \\
- \frac{k k_1}{E + m} \cot(k_1 x) \sin(kx) \sin(kx - k)\} |^1_0,
\]

(77)

\[
\text{tr}(\tilde{\psi}\varphi^T) |^1_0 = \{(\lambda - E) \sin(kx) \cos(kx - k) + (E - m) \cos(kx) \sin(kx - k) \\
- \frac{k k_1}{E + m} \cot(k_1 x) \sin(kx) \sin(kx - k)\} |^1_0.
\]

(78)

Since

\[
\text{tr}(\psi \tilde{\varphi}^T) |^1_0 - \text{tr}(\tilde{\psi}\varphi^T) |^1_0 = \{(\lambda - E) \sin(-k) + (E - m) \sin(k)\} |^1_0 = 0
\]

(79)

we finally obtain

\[
\text{tr}(\psi \tilde{\varphi}^T) |^1_0 = \text{tr}(\tilde{\psi}\varphi^T) |^1_0.
\]

(80)

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