Harmonic flow of geometric structures

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Abstract
We give a twistorial interpretation of geometric structures on a Riemannian manifold, as sections of homogeneous fibre bundles, following an original insight by Wood (Differ Geom Appl 19:193–210, 2003). The natural Dirichlet energy induces an abstract harmonicity condition, which gives rise to a geometric gradient flow. We establish a number of analytic properties for this flow, such as uniqueness, smoothness, short-time existence, and some sufficient conditions for long-time existence. This description potentially subsumes a large class of geometric PDE problems from different contexts. As applications, we recover and unify a number of results in the literature: for the isometric flow of G₂-structures, by Grigorian (Adv Math 308:142–207, 2017; Calculas Variat Partial Differ Equ 58:157, 2019), Bagaglini (J Geom Anal, 2009), and Dwivedi-Gianniotis-Karigiannis (J Geom Anal 31(2):1855-1933, 2021); and for harmonic almost complex structures, by He (Energy minimizing harmonic almost complex structures, 2019) and He-Li (Trans Am Math Soc 374(9):6179–6199, 2021). Our theory also establishes original properties regarding harmonic flows of parallelisms and almost contact structures.

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Introduction

We formulate a general theory of harmonicity for geometric structures on an oriented Riemannian manifold $(M^n, g)$, with structure group $G \subset SO(m)$, building upon a framework originally outlined in [48, 50], and further considered, for example, in [17]. A geometric structure for our purposes is a smooth section of a tensor bundle $\mathcal{F} \subset T^\bullet(M)$, typically stabilised by a subgroup $H \subset G$, e.g.

- metric-compatible almost complex structures, for $U(n) \subset SO(2n)$;
- metric-compatible almost contact structures, for $U(n) \subset SO(2n + 1)$;
- $G_2$–structures, for $G_2 \subset SO(7)$;
- Spin(7)-structures, for Spin(7) $\subset SO(8)$ etc.

A geometric structure $\xi$ can be viewed as a section of the homogeneous fibre bundle $\pi : N := P/H \to M$, which emerges by reduction of the (oriented) frame bundle $P \to M$, under a one-to-one correspondence [see (5)]:

$$\{\xi : M \to \mathcal{F}\} \leftrightarrow \{\sigma : M \to N\}.$$

We assume throughout that $M$ is closed, although the theory can be easily extended to compactly supported sections over complete manifolds with bounded geometry.

Given a suitable fibre metric $\eta$ on $N$ (see (A) below), a natural Dirichlet energy can be assigned to such sections $\sigma \in \Gamma(\pi)$:

$$E(\sigma) := \frac{1}{2} \int_M |d^V \sigma|^2_\eta,$$

where the vertical torsion $d^V \sigma$ is the projection of $d\sigma$ onto the distribution ker $\pi_*$, tangent to the fibres of $\pi : N \to M$. A geometric structure is defined to be harmonic if $\sigma \in \text{Crit}(E)$, and torsion-free if $d^V \sigma = 0$. The critical set is the vanishing locus of the vertical tension field $\tau^V(\sigma) := \text{tr}_g \nabla^V d^V \sigma$, where $\nabla^V$ is the (pull-back by $\sigma$ of the) vertical part of the Levi–Civita connection of $(N, \eta)$.

Just as in the general theory of harmonic maps, the notion of harmonic geometric structure depends of course on the fixed Riemannian metric. The explicit form of $\tau^V$, in each
particular context, then defines a natural geometric PDE. Such condition is typically weaker than the corresponding notions of ‘integrability’ and ‘torsion-freedom’; thus, in favourable cases harmonicity can still characterise the ‘best’ geometric structure, even when stronger conditions are otherwise obstructed or trivial. In Sect. 1.3, we provide a general method to determine the actual harmonicity condition in any given context, which subsumes several cases previously studied in the literature, e.g. the harmonicity of almost complex structures [46] and almost contact structures [43], and also the most recent div $T = 0$ condition in $G_2$-geometry, originally found by [22] from a rather different perspective. We also propose an original formulation for the problem of harmonic parallelisms, e.g. on spheres.

Starting from any smooth section $\sigma_0 \in \Gamma(\pi)$ of the homogeneous bundle $\pi : N \to M$, the Dirichlet energy gives rise to a natural gradient flow, which we will call the harmonic section flow:

$$\begin{aligned}
\partial_t \sigma_t &= \tau^V(\sigma_t) \\
\sigma_t|_{t=0} &= \sigma_0 \in \Gamma(\pi), \quad \text{on} \quad M_T := M \times [0, T[. \\
\end{aligned}$$

(HSF)

Defining the flow in terms of the vertical tension $\tau^V$ guarantees precisely that a solution $\{\sigma_t\}$ flows among sections, as opposed to just maps from $M$ to $N$ (Proposition 14). The main purpose of this paper is to initiate an analytic theory for (HSF). We establish a priori estimates, uniqueness and short-time existence, and sufficient conditions for long-time existence, regularity and convergence of solutions, which hold regardless of the specific geometric context, assuming the following data:

- $G \subset \text{SO}(m)$ a compact semi-simple Lie group; (A)
- $H \subset G$ a normal reductive Lie subgroup;
- $P$ a principal $G$-bundle over a closed Riemannian manifold $(M^m, g)$;
- $\eta$ an $H$ -- invariant fibre metric on $P$, constructed from a compatible bi-invariant metric on $G$.

Our approach consists of exploiting, as far as possible, classical results and techniques from harmonic map theory, most notably by means of a further one-to-one correspondence [cf. §1.5] between sections $\sigma : M \to N$ and their $G$-equivariant lifts $s : P \to G/H$, defined on the total space of the $G$-bundle $P$ (typically taken to be the oriented frame bundle of $M$). Crucially, under the assumptions (A), the harmonic section flow is in a certain sense equivalent to the harmonic map heat flow for $G$-equivariant lifts [Proposition 20]. In §2.4, we establish:

**Theorem 1** Under the assumptions (A), for each $\sigma_0 \in \Gamma(\pi)$, there exists a maximal time $0 < T_{\text{max}}(\sigma_0) \leq \infty$ such that (HSF) admits a unique smooth (short-time) solution $\{\sigma_t\}$ on $M_{T_{\text{max}}}$.

In §2.5, we prove the following main results. Regarding long-time existence and convergence to a harmonic limit, we obtain an upper estimate for the finite-time blow-up rate and $C^0$-blow-up of density:

**Theorem 2** Under the assumptions (A), let $\{\sigma_t\}$ be a solution of (HSF) on $M_T$, for some $0 < T \leq \infty$, and set

$$\varepsilon(t) := \frac{1}{2} |d^V \sigma_t|^2_\eta \quad \text{and} \quad \bar{\varepsilon}_t := \sup_M \varepsilon(t).$$

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(i) There exist \( C(M, g) > 0 \) and \( 0 < \delta \leq \min\{T, \frac{1}{C_{\bar{e}_0}}\} \) such that
\[
\varepsilon(t) \leq \bar{e}_0 \frac{1}{1 - C_{\bar{e}_0} t}, \quad \forall t \leq \delta.
\]
Moreover, \( C > 0 \) can be chosen so that
\[
\varepsilon(t) \leq 2\bar{e}_0, \quad \forall t \leq \delta.
\]
Conversely, for every \( T > 0 \), there exists \( \gamma(T) > 0 \) such that, for any initial condition satisfying \( \bar{e}_0 < \gamma \), the flow exists on \( M_T \).

(ii) If \( \{\sigma_t\} \) cannot be extended beyond some \( T_{\max} < \infty \), then \( \lim_{t \to T_{\max}} \bar{e}_t = \infty \).

Moreover, the bounded (square) torsion condition, which is sufficient for long-time existence by Theorem 2–(ii), can be slightly weakened into a time-uniform \( L^p \)-bound, for large enough \( p \geq m := \dim M \):

**Theorem 3** Under the assumptions (A), if a solution \( \{\sigma_t\} \) of (HSF), with any initial condition \( \sigma_0 \in \Gamma_1(\pi) \), has \( \varepsilon(t) = |d^V \sigma_t|^2 \) bounded in \( L^m(M) \), uniformly in \( t \), then there exists a sequence \( t_k \nearrow \infty \) along which \( \{\sigma_t\} \) converges smoothly to a section \( \sigma_{\infty} \in \Gamma(\pi) \), defining a harmonic geometric structure.

At this point one might expect that, if the initial energy \( E(\sigma_0) \) is sufficiently small, the flow might actually converge to a torsion-free geometric structure. A preliminary result can indeed be established in that sense, based on the standard theory of harmonic map heat flows [Proposition 29], but its applicability is extremely limited, cf. Remark 30. Moreover, since finding torsion-free geometric structures is in general quite hard, one should expect initial conditions with small energy to be just as difficult to arrange.

An important motivation for this analytic theory stems from recent developments in \( G_2 \)-geometry on 7-dimensional manifolds. In this context, a \( G_2 \)-structure is a section \( \xi = \varphi \) of the bundle of positive 3–forms \( \mathcal{F} = \Omega^3_+(M) \), which induces a \( G_2 \)-metric \( g_\varphi \), and its full torsion tensor \( T := \nabla g_\varphi \) is essentially the same as the vertical torsion \( d^V \sigma \). Distinct \( G_2 \)-structures are isometric if they yield the same \( G_2 \)-metric, and the recent article [22] interprets the divergence-free torsion condition \( \div T = 0 \) as a ‘gauge-fixing’ among isometric \( G_2 \)-structures, by viewing \( \varphi \) as a connection on a certain octonionic bundle. The natural geometric flow \( \dot{\varphi} = \div T \wedge (\ast g_\varphi \varphi) \) associated to this condition has since attracted substantial interest: short-term existence and uniqueness were first established in [3], and sufficient conditions for long-time existence and regularity were studied independently by [12, 23]. We offer an alternative formulation of this equation as a harmonicity condition on sections of the \( \mathbb{R}P^7 \)-bundle \( N = P_{SO(7)}/G_2 \). A number of results in the above literature can then be seen as instances of Theorems 1–3, cf. §6.3]. Moreover, Theorems 2 and 3 offer a new line of attack to the problem of constructing such solutions in on concrete cases, most notably opening a pathway for mass-producing examples over homogeneous manifolds, along the lines of [32], see Afterword.

In addition, Theorems 1–3 generalise some results from a similar recent study by [25, 27] in the context of harmonic almost complex structures, i.e. sections \( \xi = J \) of the subbundle \( \mathcal{F} \subset \text{End}(TM) \) which square to squaring to minus the identity. Then harmonicity is equivalent to the commutation \( [\nabla^* \nabla J, J] = 0 \), and the corresponding geometric flow is shown to always exist at short time, then to blow-up at controlled rates exactly as prescribed by our Theorems, cf. Corollary 34. It should be noted that, just like in the \( G_2 \) case, the authors are
able to achieve analytically much more than our general theory, by mobilising algebraic and differential properties which are contingent to each particular context.

Our results seem to be hitherto unknown in the cases of parallelisms, when $H = \{e\} \subset \text{SO}(n)$ (Corollary 32), and of almost contact structures, when $H = \text{U}(n) \subset \text{SO}(2n + 1)$ (Corollary 33). Therefore in those contexts our conclusions are completely original. Finally, since this text first appeared as a preprint, further studies of harmonicity appeared in the cases of Spin$(7) \subset \text{SO}(8)$ [13] and of homogeneous $G_2$-structures on the sphere $S^7 = \text{Sp}(2)/\text{Sp}(1)$ [34].

Part 1. Harmonicity theory of geometric structures

1 Universal sections on homogeneous fibre bundles

Let $G$ be a Lie group and $p: P \to M$ a principal $G$-bundle. Given a Lie subgroup $H \subseteq G$, denote by $N := P/H$ its orbit space, by $q : P \to N$ the corresponding principal $H$-bundle and by $\pi : N \to M$ the projection on the quotient, so that $p = \pi \circ q$. Denoting by $g := \text{Lie}(G)$ and $h := \text{Lie}(H)$, assume $H$ is naturally reductive, that is, there is an orthogonal complement $m$ satisfying

$$g = h \oplus m \quad \text{and} \quad \text{Ad}_G(H)m \subseteq m.$$ 

1.1 Canonical geometry on homogeneous fibre bundles

Consider a connection $\omega \in \Omega^1(P, g)$ on $P$ (e.g. if $P = \text{Fr}(M)$), inducing the splitting

$$TN = V \oplus H$$

with $V := \ker \pi_* = q_*(\ker p_*)$ and $H := q_*(\ker \omega)$.

Let $m \to N$ be the vector bundle associated to $q$ with fibre $m$. Its points are the $H$-equivalence classes of ‘vectors-in-a-frame’:

$$z \cdot w := [(z, w)]_H = \mathcal{I}(q_*(w^*_z))$$

with

$$w^*_z := \left. \frac{\partial}{\partial t} \right|_{t=0} z \cdot \exp t w \in T_z P,$$

for $z \in P$ and $w \in m$.

Then we have a vector bundle isomorphism

$$\mathcal{I} : \quad \mathcal{V} \xrightarrow{\sim} m \quad \text{with} \quad \mathcal{I}(z \cdot w) = z \cdot w.$$

The $m$-component $\omega_m \in \Omega^1(P, m)$ of the connection is $H$-equivariant and $q$-horizontal, so it projects to a homogeneous connection form $f \in \Omega^1(N, m)$ defined by:

$$f(q_*(Z)) := z \cdot \omega_m(Z) \quad \text{for} \quad Z \in T_z P.$$

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On $\pi$-vertical vectors $f$ coincides with the canonical isomorphism (1), while $\pi$-horizontal vectors are in the kernel:

$$f(v_y) = \mathcal{I}(v_y^V), \quad \text{for } v_y \in T_yN.$$ 

1.2 The universal section of a geometric structure

A vector bundle $\mathcal{F} \to M$ will be called geometric (with respect to $\mathcal{P} \to M$) if there exists a geometric representation $\rho \in \text{Rep}(\mathcal{P}, \mathcal{F})$, i.e. a monomorphism of principal bundles

$$\rho : \mathcal{P} \hookrightarrow \text{Fr}(\mathcal{F}).$$

For simplicity, we may assume $\mathcal{F} \subset \bigoplus T^{p,q}$ is a tensor bundle. Denote by $V = \mathbb{R}^r$ the typical fibre of $\mathcal{F}$, with $r = \text{rank}(\mathcal{F})$. If $\mathcal{F}$ is geometric, then, at each point $x \in M$, the map $\rho$ identifies the element $z_x \in \mathcal{P}_x$ with a frame of $\mathcal{F}_x$, i.e. with a linear isomorphism

$$\rho(z_x) : \mathcal{F}_x \cong V.$$ 

Fixing an element $\xi_0 \in V$, a geometric structure modelled on $\xi_0$ is a section $\xi \in \Gamma(\mathcal{F})$ such that, for each $x \in M$, the induced map $\rho : \mathcal{P}_x \to V$ is surjective at $\xi_0$, i.e. for any $x \in M$, there is always a frame of $T_xM$ whose image by $\rho$ is a frame of $\mathcal{F}_x$ identifying $\xi(x)$ and $\xi_0$. For simplicity, let us just think henceforth of $\rho$ as a fibrewise element of $\text{Hom}(G, \text{GL}(V))$ and omit explicit mention of it.

For each $z \in \mathcal{P}$, the frame $\rho(z)$ defines an isomorphism $\mathcal{F}_{\rho(z)} \cong V$, and the right action of $G$ on $\mathcal{P}$ then carries over to $V$, in the following way. For $u \in V$, consider $b \in \mathcal{F}_x$ ($x = \rho(z)$) the vector of coordinates with coordinates $u$ in the frame $\rho(z)$. Then define $g.u \in V$ to be the coordinates of $b \in \mathcal{F}_x$ in the frame $\rho(z.g^{-1})$. This action is linear, so we can represent $g \in G$ by a matrix $M_g \in \text{GL}(V)$, such that $g.u = M_g(u)$. Differentiating at the identity, we obtain the induced Lie algebra action of $a \in \mathfrak{g}$ on $V$, which we can also identify with the action of a matrix $A \in \text{gl}(V)$: $a.u = Au$. Then

$$g.(a.u) = M_g(Au) = (M_k A M_g^{-1})(M_k u) = ((\text{Ad}_G g) a).(g.u)$$

$$= (g.a). (g.u).$$

Suppose now $H \subseteq G$ fixes the model structure $\xi_0$:

$$H = \text{Stab}(\xi_0).$$

In view of (3), a universal section $\Xi \in \Gamma(N, \pi^*(\mathcal{F}))$ is well-defined by

$$\Xi(y) := y^* \xi_0.$$ 

Explicitly, one assigns to $y \in N$ the vector of $\mathcal{F}_{\pi(y)}$ whose coordinates are given by $\xi_0$ in any frame $\rho(\pi(y))$. Now each section $\sigma \in \Gamma(M, N)$ induces a geometric structure $\xi_\sigma \in \Gamma(M, \mathcal{F})$ modelled on $\xi_0$:

$$\xi_\sigma := \sigma^* \Xi = \Xi \circ \sigma.$$
Since $\pi^*(\mathcal{F})$ is isomorphic to the associated bundle $\pi^*P \times_G V$, there exists a $G$-equivariant map $\tilde{\mathcal{E}} : \pi^*P \to V$ such that

$$\tilde{\mathcal{E}} = \rho \circ \Xi \circ \pi^* p.$$ 

This map $\tilde{\mathcal{E}}$ associates to $(z, y) \in \pi^*P$ the coordinates of the vector $\mathcal{E}(y) \in \mathcal{F}_{\pi(y)}$, but in a frame $\rho(z)$, for $z$ not necessarily in the equivalence class $q^{-1}(y)$. Note that $\tilde{\mathcal{E}}|_P = \xi_0$, by the $H$-equivariant embedding

$$P \hookrightarrow \pi^*P$$

$$z \mapsto (z, q(z)).$$

**Lemma 4** Let $p : P \to M$ be a principal $G$-bundle, $H \subset G$ a naturally reductive Lie subgroup with $g = h \oplus m$, $\pi : N = P/H \to M$ its orbit space and $q : P \to N$ the corresponding principal $H$-bundle. Let $f \in \Omega^1(N, m)$ be the homogeneous connection form associated to the connection $\omega \in \Omega^1(P, g)$ and the splitting $TN = \mathcal{V} \oplus \mathcal{H}$:

$$f(q_*(Z)) = z \cdot \omega_m(Z) \quad \text{for} \quad Z \in T_zP, z \in P.$$ 

Let $\mathcal{F} \to M$ be a geometric vector bundle, such that $H$ fixes an element $\xi_0$ in its typical fibre, cf. (3).

Then, the covariant derivative of the universal section $\mathcal{E} \in \Gamma(N, \pi^*(\mathcal{F}))$, defined by (4), is

$$\nabla_Y \mathcal{E} = f(Y).\mathcal{E}, \quad \text{for} \quad Y \in TN.$$ 

**Proof** Let $Z \in TP$ be a lift of $Y \in TN$, i.e. $dq(Z) = Y$. Applying the exterior covariant derivative $D$ for $V$-valued differential forms on $\pi^*P$, in the direction $Z$:

$$D\tilde{\mathcal{E}}(Z) = d\tilde{\mathcal{E}}(Z) + \omega(Z).\tilde{\mathcal{E}} = \omega(Z).\tilde{\mathcal{E}},$$

since $\tilde{\mathcal{E}}|_P = \xi_0$ and $h.\xi_0 = 0$, by (3). Under the $H$-equivariant embedding $P \hookrightarrow \pi^*P$, the bundle $q$ may be seen as a principal $H$-subbundle of $\pi^*P$, i.e. the $G$-action on $\pi^*P$ restricts to an $H$-action on $q$. Under the $\cdot$ operation (which we perform with the group $H$), we obtain:

$$\nabla_Y \mathcal{E} = (z, q(z)) \cdot (\omega_m(Z).\xi_0) \in P \times_H V, \quad \text{for} \quad Y = q_*(Z) \quad \text{and} \quad (z, q(z)) \in P \hookrightarrow \pi^*P.$$ 

Now we compute:

$$(z, q(z)) \cdot (\omega_m(Z).\xi_0) = [(z, q(z))h, h^{-1}(\omega_m(Z).\xi_0)]_H$$

$$= [(z, q(z))h, ((Ad_Gh^{-1})\omega_m(Z)).(h^{-1}.\xi_0)]_H$$

$$= [(z, q(z))h, (h^{-1}.\omega_m(Z)).\xi_0]_H$$

$$= [(z, q(z))h, (h^{-1}.\omega_m(Z))]_H \cdot \Xi$$

$$= f(Y).\mathcal{E}.$$ 

$\square$
1.3 Determining the vertical tension field

In any context of interest, one would like to determine the vertical tension explicitly, thus obtaining a natural geometric PDE in terms of the original objects. Here is a practical procedure to do this, mobilising the definitions of Sect. 1.1.

1.3.1 Vertical torsion in \( m \)

We first use the universal structure to determine the homogeneous connection form (2), which gives an isomorphism an \( m \)-valued one-form \( f: T_N \to m \). Given vector fields \( X \in \Gamma_1(TM) \) and \( Y := d\sigma(X) \in \Gamma(\sigma^*TN) \), we invert (6) to obtain an expression of \( f(Y) \in m \).

We can now compute the vertical tension field of a section \( \sigma: M \to N \). Pulling back \( f \) to \( TM \), we obtain the image of the vertical torsion under \( I: V \to m \):

\[
(\sigma^* f)(X) = I(dV \sigma)(X)
\]

since \( I(dV \sigma) = \sigma^* f = f(d\sigma) \).

1.3.2 Natural connection \( \nabla^\omega \) on \( g \)

The connection \( \nabla^\omega \) on \( g \) is obtained as follows. Let \((\mathcal{E}, (\cdot, \cdot), \nabla)\) be an oriented Riemannian vector bundle of rank \( k \) over \( M \). For example, take \( G = SO(k) \) and \( P \to M \) to be the principal \( SO(k) \)-frame bundle of \( E \), such that \( E = P \times_{SO(k)} \mathbb{R}^k \) and \( g \) is its associated bundle.

Identifying the Lie algebra \( \mathfrak{gl}(k) \) with a subset of \( (\mathbb{R}^k)^* \otimes \mathbb{R}^k \), we have the bundle inclusion:

\[
\mathfrak{g} = P \times_{SO(k)} \mathfrak{gl}(k) \subset P \times_{SO(k)} (\mathbb{R}^k)^* \otimes P \times_{SO(k)} \mathbb{R}^k = \mathcal{E}^* \otimes \mathcal{E},
\]

and the connection \( \nabla^\omega \) on \( g \) is the restriction projection of the tensor product connection on \( \mathcal{E}^* \otimes \mathcal{E} \). Alternatively, identifying \( \mathfrak{gl}(k) \subset (\mathbb{R}^k)^* \otimes (\mathbb{R}^k)^* \), we have \( g \subset \wedge^2 \mathcal{E} \) and the connection is induced by restriction projection accordingly.

1.3.3 Vertical second fundamental form in \( m \)

Let \( \nabla^\omega \) denote the connection on \( \pi^*g \), pulled back from the natural connection (also denoted by \( \nabla^\omega \)) on the bundle \( g \) associated to \( P \to M \).

**Lemma 5** Acting on diagonal pairs \( (X, X) \in \Gamma(TM) \times \Gamma(TM) \), the (vertical) second fundamental form \( \nabla^V dV \sigma \) is expressed in \( m \) by:

\[
I((\nabla^V dV \sigma)(X, X)) = (\nabla^\omega (\sigma^* f))(X, X) \in m.
\]

**Proof** Combining Theorems 3.4 and 3.5 from [50], we have:

\[
I(\nabla^V dV \sigma) = \nabla^c (\sigma^* f) + \frac{1}{2} \sigma^* f^* B,
\]

where \( \nabla^c \) is the covariant derivative on \( m \) (inherited from the principal \( H \)-bundle \( P \to N \)) and \( B = [\cdot, \cdot]_m \) is the \( m \)-component of the Lie bracket on \( g \), inherited from \( g \), since we are in the naturally reductive case. Furthermore, the canonical connection \( \nabla^c \) acts on a section \( \alpha \in \Gamma(m) \) by [50, Prop. 2.7]

\[
\nabla^c \alpha = \nabla^m \alpha - [f, \alpha]_m = (\nabla^\omega \alpha)_m - [f, \alpha]_m = \nabla^\omega \alpha - [f, \alpha],
\]

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where $\nabla_m$ is the $m$-component of the pullback connection $\nabla^\omega$ on $\pi^*g$.

Since $H = \ker I$ and $\sigma^* \circ \pi^* g = (\pi \circ \sigma)^* g = g$, the above relations yield:

$$I(\nabla d^\sigma) = I(\nabla d^\sigma) = (\nabla^\omega (\sigma^* f))_m - [\sigma^* f, \sigma^* f]_m + \frac{1}{2}\sigma^* f^* B$$

using $[50, (2.6)]$. Taking diagonal terms in (8):

$$I((\nabla d^\sigma)(X, X)) = (\nabla^\omega (\sigma^* f))(X, X) \in m.$$

since all the other terms are skew-symmetric.

Finally, taking $\text{tr}_g$ of the expression from Lemma 5 for $(\nabla^\omega (\sigma^* f))(X, X)$, we obtain the representation in $m$ of the vertical tension field $\tau^V(\sigma)$.

1.4 The Dirichlet energy of a section

Suppose in addition that $G$ is semi-simple, so it admits a bi-invariant metric $\eta$, which naturally descends to each homogeneous fibre of $N \cong P \times_G G/H$, and let $\nabla^\eta$ be its Levi-Civita connection. Using the metrics $(M, g)$ and $(N, \eta)$, there is an induced metric $(\cdot, \cdot)$ on $T^* M \otimes \sigma^* TN$, compatible with the orthogonal splitting $TN = V \oplus H$. This setup admits a (total) Dirichlet action on sections of $N$:

$$\tilde{E}(\sigma) := \frac{1}{2} \int_M |d\sigma|^2.$$

Lemma 6 Up to the constant $b_M := \frac{1}{2}(\dim M)(\text{vol} M)$, $\tilde{E}(\sigma)$ is completely determined by its vertical component:

$$\tilde{E}(\sigma) = E(\sigma) + b_M \text{ with } E(\sigma) := \frac{1}{2} \int_M |d^V \sigma|^2.$$

Proof Since $\pi$ is a Riemannian submersion, for a vector field $X$ on $M$, the norm $|d\sigma(X)|^2$ splits into $|d^V \sigma(X)|^2 + |d^H \sigma(X)|^2$ and, as the horizontal part of the metric on $N$ is the pull-back by $\pi$ of the metric $g$ on $M$, the second term is

$$|d^H \sigma(X)|^2 = g(d\pi \circ d\sigma(X), d\pi \circ d\sigma(X)) = g(X, X).$$

Seeing a section $\sigma \in \Gamma(\pi)$ as a map from $M$ to $N$, we define its tension field to be $\tau(\sigma) = tr_g \nabla d\sigma$, whereas its vertical tension field, now as a section, is $\tau^V(\sigma) := tr_g \nabla d^V \sigma$, denoting by $V$ the pull-back by $\sigma$ of the Levi-Civita connection of $(N, \eta)$. Moreover, the vertical part of the tension field is precisely the vertical tension field (this is a non-trivial statement, see [50]), so sections of $\pi : N \to M$ with $\tau^V(\sigma) = 0$ are exactly the critical points of the functional $E(\sigma)$ (and indeed of $\tilde{E}(\sigma)$) for variations through sections of $\pi$. We have the first variation formula:

Proposition 7 $\text{Crit}_{\Gamma(\pi)}(E) = \{ \sigma \in \Gamma(\pi) : \tau^V(\sigma) := tr_g \nabla d^V \sigma = 0 \}$.

Proof Let $\mathcal{F} : M \times [-\epsilon, \epsilon] \to N$ be a local variation of $\sigma := \mathcal{F}(\cdot, 0)$ as a section, with (vertical) tangent field
\[ V(x) := \frac{\partial}{\partial t}|_{t=0} \mathcal{F}(x, t) \in V_{\sigma(x)} \subset T_{\sigma(x)}N \simeq (\sigma^*TN)_x \]
given therefore by \( \mathcal{F}(x, t) = \exp_{\sigma(x)}^{-1} V(x) \) for small enough \( \epsilon \). Then
\[
\frac{\partial}{\partial t}|_{t=0} E(\mathcal{F}(t)) = \frac{1}{2} \int_M \frac{\partial}{\partial t}|_{t=0} |d^\nu \mathcal{F}_t(x, t)|^2 dx = \int_M \left( \nabla V, d^\nu \sigma \right)
\]
and integrating the divergence of the vector field \( X := \{ V, d^\nu \sigma \}_\eta \) yields the claim, since
\[
\text{div } X = \left( \nabla V, d^\nu \sigma \right) + \left( V, \text{tr } \nabla d^\nu \sigma \right)
\]
and \( \mathcal{H} = \ker \nabla^\omega \) is the orthogonal complement of \( \mathcal{V} \) in \( TN \).
\[\square\]

This harmonicity condition is elliptic, in the following sense:

**Proposition 8** The vertical tension field \( \tau^\nu : \Gamma(\pi) \to \Gamma(\mathcal{V}) \) is described locally by an elliptic system.

**Proof** The homogeneous bundle \( \pi : N \to M \) is equipped with a Sasaki-type metric \( g = g^\mathcal{H} + g^\mathcal{V} \) with \( g^\mathcal{H} \) the pull-back of the metric on \( M \). Let us denote its typical fibre by \( F \simeq G/H \). Let \((U, x^i)\) be a coordinate neighbourhood of \( x \in M \).

Given a section \( \sigma \in \Gamma(\pi) \), let \( V \subset N \) be an open neighbourhood of \( \sigma(x) \). Since \( N \) is a fibre bundle, we can assume that there exist \( V_1 \subset M \) in the base manifold and \( V_2 \subset F \) in the fibre such that \( V = V_1 \times V_2 \). We take local coordinates \((x^k)\) on \( V_1 \subset M \) and \((v^\alpha)\) on \( V_2 \subset F \). The horizontal and vertical lifts of the vector fields \((\frac{\partial}{\partial x^i})\) and \((\frac{\partial}{\partial v^\alpha})\) give a local frame on \( V \). Then
\[
d^\nu \sigma\left( \frac{\partial}{\partial x^i} \right) = \sum_\alpha \frac{\partial \sigma^\alpha}{\partial x^i} (\frac{\partial}{\partial v^\alpha})^\nu \circ \sigma,
\]
and
\[
(\nabla^\nu d^\nu \sigma)\left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right)
= \nabla^\nu \left( \sum_\alpha \frac{\partial \sigma^\alpha}{\partial x^i} \frac{\partial}{\partial v^\alpha} \right) - d^\nu \sigma \left( \nabla^M \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} \right)
= \nabla^\nu \left( \sum_\alpha \frac{\partial^2 \sigma^\alpha}{\partial x^i \partial x^j} \frac{\partial}{\partial v^\alpha} \right) \circ \sigma + \sum_\alpha \frac{\partial \sigma^\alpha}{\partial x^i} \nabla^\nu \left( \frac{\partial}{\partial v^\alpha} \right) \circ \sigma - \Gamma^k_{ij} d^\nu \sigma \left( \frac{\partial}{\partial x^k} \right)
= \nabla^\nu \left( \sum_\alpha \frac{\partial^2 \sigma^\alpha}{\partial x^i \partial x^j} \frac{\partial}{\partial v^\alpha} \right) \circ \sigma + \sum_\alpha \frac{\partial \sigma^\alpha}{\partial x^i} \nabla^\nu \left( \frac{\partial}{\partial v^\alpha} \right) \circ \sigma - \Gamma^k_{ij} \frac{\partial \sigma^\gamma}{\partial x^k} \left( \frac{\partial}{\partial v^\gamma} \right) \circ \sigma
+ \sum_\alpha \frac{\partial \sigma^\alpha}{\partial x^i} \frac{\partial \sigma^\beta}{\partial x^j} \left( \nabla^\nu \left( \frac{\partial}{\partial v^\alpha} \right) \right) \circ \sigma - \Gamma^k_{ij} \frac{\partial \sigma^\gamma}{\partial x^k} \left( \frac{\partial}{\partial v^\gamma} \right) \circ \sigma
= \nabla^\nu \left( \sum_\alpha \frac{\partial^2 \sigma^\alpha}{\partial x^i \partial x^j} \frac{\partial}{\partial v^\alpha} \right) \circ \sigma + \sum_\alpha \frac{\partial \sigma^\alpha}{\partial x^i} \frac{\partial \sigma^\beta}{\partial x^j} G^\nu_{\alpha \beta} \left( \frac{\partial}{\partial v^\gamma} \right) \circ \sigma
+ \sum_\alpha \frac{\partial \sigma^\alpha}{\partial x^i} \frac{\partial \sigma^\beta}{\partial x^j} G^\nu_{\alpha \beta} \left( \frac{\partial}{\partial v^\gamma} \right) \circ \sigma - \Gamma^k_{ij} \frac{\partial \sigma^\gamma}{\partial x^k} \left( \frac{\partial}{\partial v^\gamma} \right) \circ \sigma
= \sum_\alpha \left( \frac{\partial^2 \sigma^\alpha}{\partial x^i \partial x^j} + \frac{\partial \sigma^\alpha}{\partial x^i} \frac{\partial \sigma^\beta}{\partial x^j} G^\nu_{\alpha \beta} \left( \frac{\partial}{\partial v^\gamma} \right) \right) \circ \sigma.
This concludes the proof. Notice that $G^y_{\alpha\beta}$ and $G^y_{k\beta}$ do not commute are not symmetric in their lower indices. \hfill $\square$

To obtain the horizontal part of $\tau(\sigma)$, we take the second covariant derivative of $\pi \circ \sigma = \text{id}$ and use standard properties of Riemannian submersions with totally geodesic fibres:

$$\begin{align*}
-2g(\langle \nabla d\pi((\nabla d\sigma)(X,Y)), Z \rangle) & = 2g \left( (\nabla d\pi)(d^y\sigma(X),d^H\sigma(Y)) + (\nabla d\pi)(d^H\sigma(X),d^y\sigma(Y)), Z \right) \\
& = \langle (\sigma^* f)(X), f[d^H\sigma(Y),d^H\sigma(Z)] \rangle + \langle (\sigma^* f)(Y), f[d^H\sigma(X),d^H\sigma(Z)] \rangle,
\end{align*}$$

since, using O’Neill’s equations,

$$
2g \left( (\nabla d\pi)(d^H\sigma(X),d^y\sigma(Y)), Z \right) = -2g \left( \langle \nabla d\pi X, d^y\sigma(Y) \rangle, Z \right) = 2\langle \nabla d\pi X, d^H\sigma(Z) \rangle = \langle \nabla [d^H\sigma(X),d^H\sigma(Z)], d^y\sigma(Y) \rangle = \langle [d^H\sigma(X),d^H\sigma(Z)], \nabla d^y\sigma(Y) \rangle.
$$

On the other hand, the structure equation for $\omega$, projected on $N$, implies that

$$-f[H,K] = F(H,K), \quad \forall H,K \in \mathcal{H},$$

where $F$ is the $m$-valued 2-form on $N$ obtained as the projection of the $m$-component of the curvature form of $p : P \to M$. Therefore

$$2g(\langle \nabla d\pi((\nabla d\sigma)(X,Y)), Z \rangle) = \langle (\sigma^* f)(X), f(\sigma(Y),\sigma(Z)) \rangle + \langle (\sigma^* f)(Y), f(\sigma(X),\sigma(Z)) \rangle.$$ 

The vanishing of the horizontal part of the tension field of $\sigma$ will therefore be given by the condition

$$\sum_{i=1}^7 \langle (\sigma^* f)(e_i), (\sigma^* F)(e_i, X) \rangle = 0, \quad \forall X \in TM.$$ 

### 1.5 Equivariant lifts from sections to maps

There is a natural $1-1$ correspondence among sections of $\pi : N \to M$ and $G$-equivariant maps from $P$ to $G/H$, due to the isomorphism between $P/H$ and the associated bundle $P \times_G G/H$, given by

$$[(z.gH)]_G \in P \times_G G/H \mapsto [z.g]_H \in P/H.$$ (9)
Lemma 9  For each \( z \in P \), the map (cf. §1.1)

\[
\mu_z : G/H \to N_{p(z)} \subset P \times_G G/H \quad a \mapsto z \cdot a
\]

defines an isometry of \( G/H \) onto the fibre of \( \pi \) over \( p(z) \), with respect to the bi-invariant metric on \( G \).

Proof  Since this is a crucial fact, we give a little more detail for this statement found in [48]. Given \( z \in P \), the differential of \( \mu_z \) at \( a \in G/H \) is:

\[
(d\mu_z)_a : T_a(G/H) \to T_{z \cdot a}N
\]

\[
X_a \mapsto [X_a]_{\text{Ad}(G)}.
\]

Now, the metric \( \eta \) on \( G/H \) is \( G \)-invariant, by assumption, so it goes over identically to \( \text{Ad}(G) \)-orbits. \( \square \)

Then, to any section \( \sigma \in \Gamma(\pi) \), seen as a section of \( P \times_G G/H \), we can associate bijectively the \( G \)-equivariant map \( s : P \to G/H \) defined by:

\[
\sigma(x) = \mu_z(s) := z \cdot s(z) \quad \text{with} \quad p(z) = x.
\]

One can easily check that this does not depend on the choice of \( z \in P \), and that \( s \) is equivariant, i.e. \( s(zg) = g^{-1}s(z) \). Conversely, since \( \mu_z \) is onto the fibre of \( \pi \) over \( p(z) \), the inverse association is

\[
s(z) = \mu_z^{-1}(\sigma(x)) \quad \text{with} \quad p(z) = x.
\]

In summary, the maps

\[
(5) \quad \xi_\sigma = \Xi \circ \sigma
\]

\[
(9) \quad [(z, gH)]_G \leftrightarrow [z \cdot g]_H
\]

\[
(12) \quad \sigma(x) = z \cdot s(z) = [(z, s(z))]_G
\]

enable us to associate to a \( H \)-structure \( \xi_\sigma \in \Gamma(M, \mathcal{F}) \) a \( G \)-equivariant map \( s : P \to G/H \), through a section \( \sigma \), interpreted under two guises, as \( \sigma : M \to N = P/H \) and \( \sigma : M \to P \times_G G/H \):

\[
\{\xi_\sigma : M \to \mathcal{F}\} \xleftarrow{(5)} \{\sigma : M \to P/H\} \xleftrightarrow{(9)} \{\sigma : M \to P \times_G G/H\}
\]

\[
\xleftrightarrow{(12)} \{s : P \to G/H, \ G\text{-eq.}\}
\]

Lemma 10  Under the correspondences (5), (9) and (12), between geometric structures \( \xi : M \to \mathcal{F} \), sections \( \sigma : M \to N \), and \( G \)-equivariant maps \( s : P \to G/H \),

\[
ds = 0 \quad \Rightarrow \quad d^\mathcal{V}\sigma = 0 \quad \Rightarrow \quad \nabla \xi_\sigma = 0.
\]

Proof  Since \( \mu \) is an isometry, the first implication follows directly from the formula in [48, Lemma 1]:

\[
d\mu_z \circ ds(Z) = d^\mathcal{V}\sigma(X),
\]

for a vector \( X \in TM \) and any \( Z \) in the horizontal distribution of \( p : P \to M \), such that \( p_*Z = X \).

For the second implication, consider the universal section \( \Xi : N \to \pi^*(\mathcal{F}) \) and the \( H \)-structure \( \xi_\sigma = \Xi \circ \sigma : M \to \mathcal{F} \). Since \( \sigma^*(\pi^*(\mathcal{F})) = (\pi \circ \sigma)^*\mathcal{F} = \mathcal{F} \), the expression (6)
of the covariant derivative of a universal structure in terms of the homogeneous connection form \( f \) yields:

\[
\nabla_X^E \xi_\sigma = \nabla_X^{\sigma^*(\pi^*(\mathcal{F}))} (\Xi \circ \sigma) = (\nabla_{d\sigma(X)}^{\sigma^*(\mathcal{F})} \Xi) \circ \sigma = (f(d\sigma(X)).\Xi) \circ \sigma,
\]

(13)

since the horizontal distribution of \( \pi \) lies in \( \ker f \).

\[\square\]

**Remark 11** If \( m \) is an irreducible representation of \( H \) then, by Schur’s lemma, there exists an isomorphism mapping \((f(d^V\sigma(X)).\xi_\sigma)\) to \( d^V\sigma(X) \), so the last implication of Lemma 10 becomes an equivalence. Moreover, if, in the irreducible decomposition of the 2-symmetric powers of \( m \) under \( H \)-action, the trivial representation appears only once, then the norms (in their respective spaces) of \( \nabla_X^E \xi_\sigma \) and \( d^V\sigma(X) \) are proportional, by a multiplicative constant.

Comparing the natural harmonicity theories of a section and its corresponding map \( s = \mu^{-1}(\sigma) \), one easily obtains:

**Lemma 12** In terms of the constants \( a_P := \text{vol}G \) and \( b_P := \frac{1}{2} \dim G\text{vol}P \), for every \( z \in P \) and \( \sigma = \mu_{z}(s) \),

\[
E(s) := \frac{1}{2} \int_P |ds|^2 = a_P E(\sigma) + b_P.
\]

**Proof** Let \( \{X_i\} \) be an orthonormal frame of \( M \) and \( \{\tilde{X}_i, a_j^*\} \) an orthonormal frame of \( P \) such that each \( \tilde{X}_i \) is the horizontal lift of \( X_i \) and \( a_j^* \) the (vertical) fundamental vector field generated by \( a_j \in \mathfrak{g} \). Then, in terms of the fibre metric \( |\cdot| = |\cdot|_\eta \),

\[
E(s) = \frac{1}{2} \int_P |ds|^2 = \frac{1}{2} \int_P |ds(\tilde{X}_i)|^2 + |ds(a_j^*)|^2
\]

\[
= \frac{1}{2} \int_P |d^V(\sigma(X_i))|^2 + |d^V(\sigma(a_j^*))|^2, \quad \text{by [Woo1997, Lemma 1]}
\]

\[
= \frac{1}{2} \text{vol}(G) \int_M |d^V(\sigma(X_i))|^2 + \frac{1}{2} \int_P |d^V(\sigma(a_j^*))|^2, \quad \text{by the co-area formula}
\]

But [48, Proposition 1] shows that \( d^V(\sigma(dp(Z)) = d\mu_q(dp(Z) + \omega(Z)\gamma(p)) \), for any \( Z \in TP \), so for the vertical vector fields \( a_j^* \), we have \( \sum_j |ds(a_j^*)|^2 = \sum_j |a_j|^2 = \dim G \).

\[\square\]

**Remark 13** We learn in [48, Theorem 1] that \( \sigma \) is harmonic (as a section) if, and only if, \( s \) is horizontally harmonic, i.e. the trace, over the horizontal distribution, of its second fundamental form vanishes. This corrects a statement which originally appeared in [45], by organising it as two distinct sets of conditions: if \( G \) is unimodular and \( G/H \) compact with non-positive Ricci curvature, or if \( G/H \) is a normal \( G \)-homogeneous manifold (cf. Definition 19) and the metric on \( P \) is constructed from any compatible metric on \( G \), then \( \sigma \) is a harmonic section if and only if \( s \) is a harmonic map.

## 2 The harmonic section flow of geometric structures

In the setup of Assumptions (A), suppose \( G/H \) is a normal \( G \)-homogeneous manifold, and the metric on \( P \) is constructed from any compatible metric on \( G \), cf. [4, Def. 7.8 & 7.86].
Given a smooth initial section $\sigma_0$ of the fibre bundle $\pi : N \to M$, consider the natural harmonic section flow (HSF):

$$\begin{align*}
\frac{\partial}{\partial t}\sigma_t &= \tau'_t(\sigma_t), &&\text{on } M_T := M \times [0, T[.
\end{align*}$$

The main purpose of this paper is to initiate an analytic theory for this flow.

### 2.1 Motivation for the vertical flow

A first simple fact behind the formulation of (HSF) is the property that the usual harmonic map heat flow starting from a section $\sigma_0 : M \to N$ remains among sections for as long as it exists:

**Proposition 14** Let $\sigma_0 : M \to N$ be a section of $\pi$ and assume that for all $x \in M$, $N_x = \pi^{-1}(x)$ is a totally geodesic submanifold of $N$. Let $I$ be an interval and $\{u_t\}_{t \in I} \subset C^\infty(M, N)$ a solution of the harmonic map heat flow

$$\frac{\partial}{\partial t}u_t = \tau(u_t), \quad u_0 = \sigma_0.$$ 

Then $u_t$ is a section of $\pi$, for all $t \in I$.

**Proof** Use Nash’s isometric embedding theorem to see $N$ as a submanifold of a Euclidean space $\mathbb{R}^m$. Then the tension field of a map $u : M \to N \subset \mathbb{R}^m$ is expressed in terms of the second fundamental form of $N$ by $\tau(u) = \Delta u - \Pi_N(du, du)$. Since the base manifold $M$ and the fibres $N_x$ are compact, for all $x \in M$, there exists a uniform $\epsilon > 0$ such that every $N_x$ admits a tubular $\epsilon$-neighbourhood $N^\epsilon_x$, together with a submersion $pr_x : N^\epsilon_x \to N_x$. Each $pr_x(y)$ can be assumed to be the orthogonal projection of $y \in N^\epsilon_x$ onto $N_x$, and the collection of such difference vectors in $\mathbb{R}^m$ defines a function

$$\rho_x : N^\epsilon_x \to \ker(pr_x) \subset \mathbb{R}^m$$

$$y \mapsto \rho_x(y) = y - pr_x(y).$$

For small enough $t > 0$, define the squared-distance $\ell_t(x) = |\rho_x(u(x, t))|^2$ on $M \times I$. Clearly $\ell_0 \equiv 0$, since $u_0$ is a section. Then

$$\begin{align*}
\frac{\partial}{\partial t}\ell &= \frac{\partial}{\partial t}(\rho_x(u), \rho_x(u)) = 2\langle d\rho_x(\partial_t u), \rho_x(u) \rangle \\
&= 2\langle d\rho_x(\Delta u - \Pi_N(du, du)), \rho_x(u) \rangle.
\end{align*}$$

On the other hand,

$$\begin{align*}
\Delta \ell &= \Delta \langle \rho_x(u), \rho_x(u) \rangle \\
&= 2\langle \Delta \rho_x(u), \rho_x(u) \rangle + 2|\nabla \rho_x(u)|^2.
\end{align*}$$

Now, $\rho_x = \id - pr_x$ implies $\nabla d\rho_x = -\nabla dpr_x$. Moreover, since the submersion $pr$ defines an immersion of each fibre $N_x$ into $\mathbb{R}^m$, each trace $\nabla dpr_x$ is the second fundamental form of $N_x \subset \mathbb{R}^m$, so

$$\Delta \rho_x(u) = d\rho_x(\Delta u) + (\nabla d\rho_x)(du, du)$$

$$= d\rho_x(\Delta u) - (\Pi_{N_x})(du, du),$$

In conclusion,

$$\Delta \ell = 2\langle d\rho_x(\Delta u), \rho_x(u) \rangle - 2\langle (\Pi N_x)(du, du), \rho_x(u) \rangle + 2|\nabla \rho_x(u)|^2.$$
Furthermore, \( N_x \) is totally geodesic in \( N \), so \( \Pi_{N_x} = \Pi_N \) and, since \( d\rho_x \) is the identity on normal vectors,
\[
\partial_t \ell - \Delta \ell = -2|\nabla \rho_x(u)|^2.
\]
We conclude that \( \ell_T \equiv 0 \) by the divergence theorem:
\[
\partial_t \int_M \ell_t = \int_M \partial_t \ell_t = -2 \int_M |\nabla \rho_x(u(x,t))|^2 \leq 0.
\]
\[
\Rightarrow 0 \leq \int_M \ell_T \leq \int_M \ell_0 = 0, \quad \forall T > 0.
\]
\[ \square \]

As a result, the harmonic map heat flow will evolve an initial section through sections, hence \( \partial_t u_t \) will be vertical. We may therefore relax that flow and work just with the vertical part, which motivates our formulation of (HSF).

### 2.2 A priori estimates along the harmonic section flow

We consider a flow of geometric structures corresponding to sections \( \sigma \in \Gamma(\pi) \) as in (HSF). A number of preliminary properties can be derived from the outset, reflecting the close resemblance between (HSF) and the classical harmonic map flow, cf. [41]. We adopt the following conventions for norm estimates in this time-dependent context:
\[
\sigma \in L^p(M_T) \Leftrightarrow \sigma(\cdot, t) \in L^p(M), \quad \forall t \in [0, T].
\]
For notational convenience, let us introduce the \textit{vertical torsion}
\[
T := d^V \sigma.
\]
We define respectively the \textit{Dirichlet action} and the \textit{kinetic energy}, by
\[
E(t) := \frac{1}{2} \int_M |T(t)|^2_{g,\eta} \quad \text{and} \quad K(t) := \frac{1}{2} \int_M |\tau^V(\sigma(t))|^2_{g,\eta},
\]
as well as their pointwise densities
\[
\epsilon(t) := \frac{1}{2}|T(t)|^2_{g,\eta} \quad \text{and} \quad \kappa(t) := \frac{1}{2}|\tau^V(\sigma(t))|^2_{g,\eta}.
\]
Since (HSF) is the gradient flow of \( E(t) \), we should expect the energy to be non-increasing, and indeed, a direct computation gives:

**Lemma 15** \( E'(t) = -2K(t) \leq 0, \) hence \( E(t) \leq E_0 := E(0) \) and therefore \( T = d^V \sigma \) is uniformly bounded in \( L^2 \).

Along the flow (HSF), the densities \( \epsilon \) and \( \kappa \) are subsolutions of non-homogeneous heat equations:

**Proposition 16** Denote the heat operator on \( M_T \) by
\[
\mathcal{H} := \partial_t - \Delta.
\]
There exist constants \( C_1, C_2 > 0 \), depending only on \( (M, g) \), such that any solution of (HSF) satisfies the following inequalities:
\[
\mathcal{H}(\epsilon) = (\partial_t - \Delta) \epsilon \leq C_1 (\epsilon^2 + \epsilon + 1) - |\nabla^o T|_{\eta}^2, \quad (i)
\]
\[ \mathcal{H}(\kappa) = (\partial_t - \Delta) \kappa \leq C_2 \epsilon \kappa - |\nabla^\omega \tau^\psi(\sigma)|^2. \] (ii)

**Proof** Following [41, Theorem 4.2] and [50, Theorem 3.5], we apply Schwartz’s lemma for \(C^\infty\)-sections \(\sigma \in \Gamma(\pi)\), in local normal coordinates \(g_{ij} = \delta_{ij}\) on \(M\):

\[
\partial_t \epsilon = \frac{1}{2} \partial_t \left( dV^\sigma, dV^\sigma \right)_{g, \eta} = \sum_{j=1}^m \left( \nabla^\omega_j \sigma, dV^\sigma \right)_{\eta} + \left( \sigma^* f^* B(\partial_t, \partial_j), dV^\sigma \right).
\] (14)

On the other hand, the Laplacian of \(\epsilon\) is expanded as follows:

\[
\Delta \epsilon = \text{tr}_{g} \nabla \nabla \epsilon = \frac{1}{2} \sum_{i=1}^m \nabla_i \left( \nabla_i \left( dV^\sigma, dV^\sigma \right)_{g, \eta} \right) = \sum_{i,j=1}^m \nabla_i \left( \nabla_i dV^\sigma, dV^\sigma \right)_{\eta} + |\nabla dV^\sigma|^2_{\eta}.
\] (15)

Adopting the curvature convention

\[ R(X, Y) = [\nabla X, \nabla Y] - \nabla_{[X, Y]}, \]

the Ricci identity for the product connection \(\nabla := \nabla^g \otimes \sigma^* \nabla^\eta\) on \(TM \otimes \sigma^* TN\) reads [41, pp.125,161] and [13, Section 2]:

\[
\nabla_i \nabla_i dV^\sigma = \nabla_j \nabla_i dV^\sigma + (R_M)_{ij}(dV^\sigma) - \left( (R_N)(dV^\sigma, dV^\sigma) \right) dV^\sigma
+ \nabla_i (\sigma^* f^* B)_{ji} - dV^\sigma (\nabla_i \nabla_i \sigma), \]

(17)

and \(\nabla_i dV^\sigma = \tau^\psi(\sigma) = \sigma\) along the flow. This yields the first claim, since both \(M\) and \(N\) are compact, and hence have bounded curvature. The second claim is proved in a similar manner [ibid.]. \(\square\)

As an immediate consequence, we have a first a priori regularity estimate for both \(T\) and \(\tau^\psi\):

**Proposition 17** If a solution to (HSF) exists on \(M_T\) and the kinetic energy \(K(t) = \|\kappa(t)\|_{L^4(M)}\) is bounded uniformly in \(t\), then:

(i) \(\|\nabla T\|_{L^2(M)}^2 \lesssim \|T\|_{L^4(M)}^4\).

(ii) If \(T = \infty\), there exist \(C, \tau^\infty\) such that \(\|\nabla \tau^\psi(\tau^\infty)\|_{L^2(M)} \leq C\).

**Proof** Integrating over \(M\) the inequalities from Proposition 16, and applying Lemma 15, we have respectively:

(a) \(\|\nabla T(t)\|_{L^2(M)}^2 \lesssim \|T(t)\|_{L^4(M)}^4 + K(t) + E_0\).

(b) \(K(t) \leq E_0 \sup_{M} \kappa(t) - \|\tau^\psi(t)\|_{L^2(M)}^2\).

For instance,

\[
-2K(t) = E'(t) = \int_M (\partial_t - \Delta) \epsilon \lesssim \int_M \left( e^2 + \epsilon + 1 - |\nabla T|^2_{\eta} \right) = 4\|T(t)\|_{L^4(M)}^4 + 2E(t) + \text{vol}(M) - \|\nabla T(t)\|_{L^2(M)}^2 \implies \|\nabla T(t)\|_{L^2(M)}^2 \lesssim \|T(t)\|_{L^4(M)}^4 + 2E_0 + 2K(t).
\]

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Now, assertion (i) is immediate from (a) and our hypothesis on $K(t)$. For (ii), let $B := \sup K(t)$, and consider the ‘doubling’ intervals

$$I_j := [2^j, 2^{j+1}] .$$

Suppose, by contradiction, that for every $C > 0$ there exists $j > 0$ such that

$$\| \tau^{V_j}(t) \|_{L_1^2(M)} > C, \quad \forall t \in I_j .$$

This has to be the case, for if the inequality failed so much as at a single instant $t_j$ in each $I_j$, the sequence $\{t_j\}$ would satisfy the assertion. Then, for $C$ sufficiently large, inequality (b) above implies $K'|I_j \lesssim -C$, and so:

$$-B \leq K(2^{j+1}) - K(2^j) = \int_{I_j} K'(t) dt \lesssim -2^j C ,$$

which is eventually false, for $j \gg 0$.

\( \square \)

### 2.3 The harmonic map heat flow of $G$-equivariant maps

Following an insight formulated in [45], we will see that the harmonic section flow (HSF) is closely related to the natural harmonic map heat flow of a given $G$-equivariant map $s_0 : P \to G/H$:

$$\begin{cases}
\partial_t s_t = \tau(s_t) := \text{tr } \nabla s \\
s_t\big|_{t=0} = s_0
\end{cases} \quad \text{on } P_T := P \times [0, T], \quad (\text{HMHF})$$

An immediate question in this context is whether this flow preserves $G$-equivariance, or whether it flows some initial $s_0$ merely as a map from $P$ to $G/H$. Recall that a family $f := \{f_t\} \in C^1(X \times \mathbb{R}, Y)$ of maps between $G$-manifolds $X$ and $Y$ is equivariant if, and only if, each $f_t$ is equivariant, i.e. $g.f = f$. The induced action on vector fields is:

$$(g.V)(x) = dI_g(V(g^{-1}.x)), \quad \forall g \in G ,$$

where $I_g$ is the diffeomorphism corresponding to $g$, which we declare to be an isometry.

**Lemma 18** ([45, Lemma 2.1]) If $P$ and $Q$ are (left) $G$-manifolds, then the action extends naturally to $C^1(P, Q)$ by

$$(g.s)(x) = g.s(g^{-1}.x), \quad g \in G , \ x \in P$$

and trivially to families $s := \{s_t\} \in C^1(P \times \mathbb{R}, Q)$. Given an equivariant family $s \in C^1(P \times \mathbb{R}, Q)$,

$$\mathcal{L}(g.s) = g.\mathcal{L}(s) \quad \forall g \in G ,$$

where $\mathcal{L}(s) := \tau(s) - \partial_t s$ is the ‘heat’ operator.

This lemma implies that the flow with equivariant initial value remains equivariant, in particular, if $s_t$ is a heat flow then $g.s_t$ is also a heat flow. When both $M$ and the structure group $G$ are compact, hence also the total space of the principal $G$-bundle $P \to M$, and assuming that the quotient $G/H$ has non-positive sectional curvature, then one could consider invoking the celebrated Eells–Sampson Theorem [15] to obtain a limiting equivariant harmonic map $s_\infty : P \to G/H$, which, by uniqueness, is also equivariant. It corresponds therefore also to
a harmonic section $\sigma_\infty : M \to N$ under the isometry $\mu$ in (10), by Lemma 10. That occurs indeed if, and only if, $\mathfrak{g} = \text{Lie}(G)$ is a NC algebra [1].

Unfortunately, that is seldom the case among the relevant groups in the context of geometric structures. We are, quite often, in the exactly opposite case, e.g. when $G/H$ is assumed normal, as in our assumptions (A), and therefore has nonnegative sectional curvature. We must therefore address the flow (HSF) in full generality.

2.4 Correspondence between (HSF) and (HMHF)

In the absence of curvature assumptions, one can still prove several analytic properties of the harmonic section flow, using the fact that the target space is a homogeneous fibre bundle, i.e. a ‘bundle of homogeneous spaces’. The crucial metric compatibility hypothesis in our assumptions (A) is the normality of the induced metric on the fibres of $N = P/H$:

**Definition 19** [4, Def. 7.86] A G-homogeneous Riemannian manifold $(G/H, \eta)$ is called normal if there exists an $\text{Ad}(G)$-invariant scalar product $\tilde{\eta}$ on $\mathfrak{g}$ such that, if $\mathfrak{m} \subset \mathfrak{g}$ is the $\tilde{\eta}$-orthogonal complement of $\mathfrak{h}$, then $\eta$ coincides with the restriction $\tilde{\eta}|_{\mathfrak{m}}$.

Normal metrics are naturally reductive [30]*Cor. 3.6 and have non-negative sectional curvature. In particular, given from the outset a left-invariant metric on $G$, one can take directly $\eta$ as its restriction to $\mathfrak{m} := \mathfrak{h}^\perp$. The following result effectively proves Theorem 1:

**Proposition 20** Under the assumptions (A), a family $\{\sigma_t \} \subset \Gamma(\pi)$ is a solution of the harmonic section flow (HSF), for some $T > 0$, if, and only if, the corresponding family of $G$-equivariant lifts $s_t = \mu^{-1}(\sigma_t) : P \to G/H$ in (10) is a solution of the harmonic map heat flow (HMHF) with initial condition $s_0 := \mu^{-1}(\sigma_0)$.

In particular, any initial condition $\sigma_0 \in \Gamma(\pi)$ determines a (short) time $T > 0$ and a unique solution $\{\sigma_t \} \subset \Gamma(\pi)$ of (HSF) on $M_T$.

**Proof** Let $s_0 : P \to G/H$ be the $G$-equivariant lift of $\sigma_0$. Then there exists a (short) time $T > 0$ such that, for all $t \in [0, T)$, $\{s_t : P \to G/H\}$ is a solution to (HMHF), cf. [41, Theorem 4.10]. By equivariance from Lemma 18 and uniqueness of the short-time harmonic map heat flow for smooth initial data [37], $s_t$ and $gs_t$ are harmonic flows with the same initial value, hence equal. Moreover, the family $\{s_t\}$ is equivariant, and it gives rise to sections $\sigma_t : M \to G/H$, for $t \in [0, T]$.

We now pull-back the $G$-equivariant heat flow equation $\partial_1 s_t = \tau(s_t)$ to the flow satisfied by $\sigma_t$. For $z \in P$, the isometry $\mu_z : G/H \to P$ relates the corresponding tension fields by

$$\tau^Y(\sigma_t) = d\mu_z(\partial_1 s_t).$$

To further express the vertical torsion in terms of $\partial_1 \sigma_t$, we consider the maps $\Sigma(x, t) = \sigma_t(x)$ and $S(x, t) = s_t(x)$, defined on the Cartesian product $M_T$ of $M$ by a real interval, and denote by $\partial_t \in T(M_T)$ the vector field along the ‘time’ direction. Then $\partial_1 s_t = dS(\partial_t)$ and $\partial_1 \sigma_t = d\Sigma(\partial_t)$.

Extending trivially the $H$-action along $I = [0, T]$, we have a natural homogeneous bundle construction on $M_T$ (with time-dependent versions of the maps on page 4), since $(P \times \mathbb{R}^* I)/H = P/H \times \mathbb{R}^* I = N \times \mathbb{R}^* I$. Define $\Sigma : M_T \to N \times \mathbb{R}^* I$ by

$$(x, t) \mapsto \Sigma(x, t) = (\sigma_t(x), rt)$$

to be a variation of $\sigma$ through sections of the homogeneous bundle $\pi : N \to M$. 
Therefore, identifying $\pi$ and $\pi$, $d(\pi \circ \Sigma)(x, t) = x$, $\forall t \in [0, T]$, hence the differential of $\pi \circ \Sigma$ with respect to the variable $t$ vanishes, that is $d\pi(d\Sigma(\partial_t)) = 0$. So, $d\Sigma(\partial_t) \in V$ and, since $q_*$ is the identity on the second factor of $P \times \mathbb{R}^*I$ (in particular, vectors tangent to the second factor of $P \times \mathbb{R}^*I$ are horizontal for $p$), the formula

$$
\mu_z(r)(DS(Z)) = d\Sigma(q_*Z), \quad \forall Z \in TP \times \mathbb{R},
$$
implies, for $Z = \partial_t$, that

$$
d\mu_z(r)(DS(\partial_t)) = d\Sigma(q_*\partial_t) = d\Sigma(\partial_t) = \partial_t \sigma_t.
$$

The pull-back of the heat flow equation for $\sigma_t$ is therefore a solution of $(HSF)$. □

In summary, the relation $\mu_z(s_t(z)) = (\sigma_t \circ p)(z)$ from (10) gives a 1-1 correspondence

$$
\{\sigma_t : M \to N \mid \text{solution of (HSF)}\} \leftrightarrow \{s_t : P \to G/H \mid \text{solution of (HMHF)}\}
$$

between sections of $N$ and $G$-equivariant maps. Since $\mu$ is an isometry [Lemma 9], the flow $(HSF)$ inherits some well-known properties from $(HMHF)$, as stated in Theorem 2. Indeed, the elementary finite-time blow-up result Theorem 2–(i) is a direct consequence of [37, Theorem 5.2.1] (quoted there from [15, pp.154-155]), applied to a solution $\{s_t\}$ of $(HMHF)$; and Theorem 2–(ii) follows immediately from the Weitzenböck formula in Proposition 16–(i) and the maximum principle, see, for example, the proof of [12, Proposition 3.2].

**Remark 21** Huang and Wang [28] show smooth regularity and uniqueness of the heat flow of harmonic maps into a general closed Riemannian manifold, with initial data in $L^2$ for Serrin’s $(p, q)$-solutions, i.e. under a small parabolic Morrey norm condition. This, combined with the correspondence between sections and $G$-equivariant maps of section 1.5, could pave the way for a study of the harmonic evolution of weak geometric structures.

### 2.5 Conditions for long-time existence

#### 2.5.1 Regularity of solutions to a parabolic flow

Our next result exploits the regularising nature of parabolic flows, by means of the Harnack–Moser estimate:

$$
(z, rt) \in P \times \mathbb{R}^*I
$$
Lemma 22 [36, §5.3.4] Let \( D \subset \mathbb{R}^{n+1} \) be a domain containing the parabolic cylinder
\[ \Sigma_R P_R(x_0, t_0) := \{ (x, t) \in \mathbb{R}^n \times \mathbb{R} \mid |x - x_0| < R, \quad t_0 - R^2 \leq t \leq t_0 \}, \]
and let \( g \in C^\infty(D, \mathbb{R}^+) \) be a subsolution of the nonhomogeneous heat equation with linear source:
\[ \mathcal{H}(g) \leq C g. \]
Then
\[ g(x_0, t_0) \lesssim \frac{1}{R^{n+2}} \int_{\Sigma_R P_R} g. \]

Proposition 23 If a solution to (HSF) exists for all \( t < T \) and it has \( \sup_M \epsilon(t) \) bounded, uniformly in \( t \), then the following hold:

(i) All the derivatives \( \nabla^m \tau^V(t, m \geq 1) \) are uniformly bounded in \( t \).
(ii) If \( T = \infty \), then \( \kappa \to 0 \) uniformly in \( t \).

Proof (i) Let \( \kappa_m(t) := \frac{1}{2} |\nabla^m \tau^V(t)|^2 \). Under our assumption of a uniform energy density bound, we assert that \( \kappa_m \) are subsolutions of the heat equation with linear source:
\[ \mathcal{H}(\kappa_m) \leq C_m \kappa_m, \]
for constants \( C_m > 0 \). That indeed implies the claim, by a standard argument comparing \( \kappa_m \) to a solution of the corresponding heat equation, with same initial condition, via the maximum principle.

For each integer \( m \), we compute:
\[ \nabla^m \nabla_k \nabla_k \tau^V = \nabla_k \nabla_k \nabla^m \tau^V + \mathcal{R}_m(\tau^V, \nabla \tau^V, \ldots, \nabla^m \tau^V) \]
where \( \mathcal{R}_m \) is an algebraic function with coefficients determined by the geometry of \((M, g)\), we deduce that
\[ \Delta \kappa_m = \left( \Delta \nabla^m \tau^V, \nabla \nabla^m \tau^V \right) + \left( \nabla \nabla^m \tau^V \right)^2 \]
\[ = \left( \nabla \nabla^m \tau^V, \nabla \nabla^m \tau^V \right) + \left( \mathcal{R}_m(\tau^V, \nabla \tau^V, \ldots, \nabla^m \tau^V), \nabla \nabla^m \tau^V \right) \]
\[ + |\nabla \nabla^m \tau^V|^2, \]
\[ \partial_t \kappa_m = \left( \nabla^m \partial_t \tau^V, \nabla^m \tau^V \right). \]

On the other hand, setting \( \Sigma(x, t) = \sigma_t(x) \) as a map on \( M_T \) along the flow (HSF), we have
\[ \Delta \tau^V = \Delta \partial_t \sigma = \nabla \Sigma \nabla^e_i \Sigma^e d(\partial_t) = \nabla \Sigma \left\{ \nabla^\Sigma d \Sigma(e_i) + d \Sigma([e_i, \partial_t]) \right\} \]
\[ = \nabla \Sigma \nabla^e_i d \Sigma(e_i) + R_{\Sigma \times M}(e_i, \partial_t) d \Sigma(e_i) + \nabla \Sigma (d \Sigma([e_i, \partial_t])) \]
\[ = \nabla \Sigma \nabla^e_i d \Sigma(e_i) = \partial_t \tau^V. \]

Therefore the corresponding terms cancel out in the expression of the heat operator:
\[ (\partial_t - \Delta) \kappa_m = -\left( \mathcal{R}_m(\tau^V, \nabla \tau^V, \ldots, \nabla^m \tau^V), \nabla \nabla^m \tau^V \right) - \left( \nabla \nabla^m \tau^V \right)^2. \]
(ii) This is a direct consequence of the parabolic cylinder estimate of Lemma 22, by exhaustion of the initial energy $E_0$. Let us suppose not, i.e. that there exist $x \in M$, $\delta > 0$ and $t_n \to \infty$ such that

$$\kappa(x, t_n) > \delta, \quad \forall n \in \mathbb{N}.$$  

By Proposition 16, $\kappa$ is bounded on the unit-length cylinder, so:

$$0 < \delta < \kappa(x, t_n) \leq C \int_{\Sigma_1(x, t_n)} \kappa \, d\hat{t} \leq C \int_{t_n-1}^{t_n} K(i) \, d\hat{t}$$

$$= \frac{C}{2} (E(t_n-1) - E(t_n)),$$

again by Lemma 15. Hence $E(t_n) < E_0 - n \frac{2\delta}{C}$, which would lead to negative energy, for $n \gg 0$. \hfill \Box

Corollary 24 In the context of Proposition 23–(i), if $T = \infty$, there exists a sequence $t_n \to \infty$ along which the solution to the flow $(\text{HSF})$ converges to a smooth section $\sigma_\infty \in \Gamma(\pi)$, defining a harmonic geometric structure:

$$\tau^V(\sigma_\infty) = 0.$$  

Proof Differentiating successively the Laplacian of $\epsilon$ in (15), we have:

$$|\nabla^m \Delta \epsilon| \leq \sum_{i=0}^{m} \binom{m}{i} \left| \left( \nabla^i \tau^V, \nabla^{m-i} T \right) \right| + \mathcal{R}_m(|\tau^V|, |T|, |\nabla T|, \ldots, |\nabla^{m+1} T|)$$

$$\lesssim \sum_{i, j=0}^{m+1} \left( |\nabla^i \tau^V|^2 + |\nabla^i T|^2 \right),$$

where $\mathcal{R}_m$ is an algebraic function with coefficients determined by the Riemannian geometry of $(M, g)$, and we used Cauchy–Schwartz at the last step.

We deduce the following estimates by induction on $m$, restricting ourselves to the sequence $\{t_n\}$ from Proposition 17–(ii), which ensures the first step $m = 0$, then using elliptic regularity and the uniform bounds on all derivatives of $\tau^V$ from Proposition 23–(i):

$$\| \epsilon(t_n) \|_{L^2_{m+2}} \leq \| \Delta \epsilon(t_n) \|_{L^2_m} \leq \| \tau^V(t_n) \|_{L^2_{m+1}} \quad \forall m \geq 0.$$  

Hence, $\| T(t_n) \|_{C^\infty(M)}$ is also bounded, and $\sigma_\infty := \lim \sigma(t_n)$ is smooth. \hfill \Box

Remark 25 In particular, when $M = K/L$ is itself a homogeneous manifold, then we should expect long-time existence of the flow among homogeneous geometric structures, since in that case the $L^2$-norm $E(t)$ and the pointwise density $\epsilon(t)$ are proportional and the former is always uniformly bounded, by Lemma 15. The Gromov–Hausdorff limit in this case can exhibit some quite non-trivial behaviour, following the theory by J. Lauret [31, 32].

2.5.2 Energy density $L^m$–bounded for all time

By a standard property of the heat kernel, the condition of uniformly bounded energy density can actually be slightly weakened into a sufficient uniform $L^p$-bound, which can be shown to be exactly $p = m := \text{dim} M$.

The following instance of [20, Theorem 1.1] stems from a long series, going back to Nash (1958) and Aronson (1971) [op.cit.], of generalised ‘Gaussian’ upper bounds in terms of the geodesic distance $r$, for the heat kernel $H_r$ of a Riemannian manifold:
Theorem 26 Let $M$ be an arbitrary connected Riemannian $m-$manifold, $x, y \in M$ and $0 \leq T \leq \infty$; if there exist suitable [see below] real functions $f$ and $g$ such that the heat kernel $H_t$ satisfies the ‘diagonal’ conditions

$$H_t(x, x) \leq \frac{1}{f(t)} \quad \text{and} \quad H_t(y, y) \leq \frac{1}{g(t)}, \quad \forall t \in ]0, T[,$$

then, for any $C > 4$, there exists $\delta = \delta(C) > 0$ such that

$$H_t(x, y) \leq \frac{(\text{cst.})}{\sqrt{f(\delta t)g(\delta t)}} \exp \left\{ -\frac{r(x, y)^2}{Ct} \right\}, \quad \forall t \in ]0, T[,$$

where (cst.) depends on the Riemannian metric only.

For the present purposes, one may assume simply $f(t) = g(t) = t^{\frac{m}{2}}$, but in fact $f$ and $g$ can be much more general [Op. cit. p.37]. In particular,

$$\|H_t(x, \cdot)\|_{L^p(M)} \lesssim \frac{1}{t^{\frac{m}{2}}} \left( \int_M \exp \left\{ -\frac{p(r(x, \hat{x})^2)}{Ct} \right\} d\hat{x} \right)^{\frac{1}{p}}.$$

For $q \gg 0$, we may passing pass from $L^q$ to $C^0$-bounds for subsolutions of the nonhomogeneous heat equation:

Proposition 27 Let $\mathcal{H} := \partial_t - \Delta$ denote the heat operator, and let $P(g)$ be a polynomial of degree $k$; then a heat subsolution $g$ satisfying $\mathcal{H}(g) \leq P(g)$ is bounded in $C^0(M_T)$, provided it is bounded in $L^{k + \frac{m}{2}}(M_T)$.

Proof By Young’s convolution inequality, we have

$$g(t) \lesssim \int_0^t |H_{t-i} * P(g(\hat{t}))| d\hat{t} \lesssim \int_0^t \|H_t\|_{L^p(M)} \|P(g(\hat{t}))\|_{L^{p^*}(M)} d\hat{t}, \quad \text{for} \quad \frac{1}{p} + \frac{1}{p^*} = 1.$$

On a complete $m-$dimensional Riemannian manifold $M$, the heat kernel $H_t$ satisfies [15, §9] the diagonal condition of Theorem 26:

$$H_t(x, x) \lesssim \frac{1}{t^{\frac{m}{2}}}, \quad \forall x \in M.$$

So, fixing $C > 4$ and denoting by $r(\cdot, \cdot)$ the geodesic distance, we have

$$H_t(x, \hat{x}) \lesssim \frac{1}{t^{\frac{m}{2}}} \exp \left\{ -\frac{r(x, \hat{x})^2}{Ct} \right\}, \quad \forall x, \hat{x} \in M.$$

Hence, for each $x \in M$,

$$\|H_t(x, \cdot)\|_{L^p(M)} \lesssim t^{-\frac{m}{2}} \left( \int_M \exp \left\{ -\frac{p(r(x, \hat{x})^2)}{Ct} \right\} d\hat{x} \right)^{\frac{1}{p}} \lesssim t^{-\frac{m}{2}} \left( \int_0^\infty \exp \left\{ -\frac{p}{Ct} \hat{r}^2 \right\} \hat{r}^{m-1} d\hat{r} \right)^{\frac{1}{p}} \lesssim t^{-\frac{m}{2}} \left( \int_0^\infty \left( \frac{Ct}{p} \right)^{\frac{m}{2}} \exp\{\hat{u}^2\hat{u}^{m-1} d\hat{u} \right)^{\frac{1}{p}}.$$
\[
\lesssim t^{\frac{m}{2} \left( 1 - \frac{1}{p} \right)}.
\]

Now,
\[
m \left( \frac{1}{p} - 1 \right) > -1 \iff p < \frac{m}{m-2} \iff p^* > \frac{m}{2},
\]
in which case, for some constant \(c_p(T) > 0\) depending on the diameter of \((M, g)\),
\[
\int_0^T \| H_t(x, \cdot) \|_{L^p} \, dt \leq c_p(T).
\]

**Corollary 28** If \(\epsilon \in L^m(M_T)\), for \(m = \dim \mathbb{R} M\), then actually \(\epsilon \in C^0(M_T)\).

**Proof** Using Gaussian bounds on the heat kernel from Proposition 27:
\[
\epsilon(t) \lesssim 1 + \int_0^t \| H_t \|_{L^p} \| C_1 \epsilon + C_2 \epsilon^2 \|_{L^{p^*}} \, dt
\]
This implies the assertion, provided \(\epsilon \in L^{2p^*}(M_T)\). \(\square\)

Together, Corollaries 24 and 28 prove Theorem 3.

### 2.5.3 Long-time existence from small initial energy

We introduce the following small adaptation of a fundamental result by Chen–Ding, which establishes long-time existence of the HMHF for sufficiently small initial energy:

**Proposition 29** [8, Corollary 1.1] Let \((P, \eta)\) and \((Q, \tilde{\eta})\) be compact Riemannian manifolds without boundary, \(\dim P \geq 3\). For each \(c > 0\), there exists a constant \(e(c) > 0\) such that, if \(s_0 \in C^{2,\alpha}(P, Q)\) satisfies
\[
\begin{align*}
(i) & \quad |ds_0(z)| \leq c, \quad \forall z \in P, \\
(ii) & \quad E(s_0) < e(c),
\end{align*}
\]
then there exists a solution \([s_t]\) to the harmonic map flow (HMHF) with initial data \(s_0\), defined for all \(t > 0\). Moreover, \(s(t)\) converges to a constant map as \(t \to \infty\).

**Remark 30** Since the Dirichlet energies of a section \(\sigma\) and its corresponding \(G\)-equivariant map \(s\) are proportional, up to the constants from Lemma 12, at this point one might be led to ask whether, perhaps in a very optimistic context, the small-energy condition goes over from the former to the latter, thus satisfying the hypotheses of Proposition 29. In other words, whether there exists a uniform constant \(c_0 > 0\) such that, if the initial condition \(\sigma_0 \in C^\infty(M, N)\) satisfies \(\tilde{\sigma} \leq c_0\), then there exists a unique smooth solution to the flow (HSF) for all \(t \geq 0\). Notice that if that was indeed the case, then we know from Lemma 10 that constant \(G\)-equivariant maps are precisely the lifts of torsion-free sections, so the limit would be a torsion-free structure.

However, that intuition should be taken with scepticism, because quite often \(E(s_0)\) is definitely not small. Looking closely at the proof of [8, Theorem 1.1 & Corollary 1.1], on the one hand, there exists a uniform small constant \(\delta_0 > 0\), such that a maximum time must be at least
\[
T > \delta_0 \ln \left( 1 + \frac{1}{2c^2} \right).
\]
On the other hand, denoting by $\rho > 0$ the injectivity radius of $(P, \eta)$, there exists a uniform constant $\gamma > 0$ such that the following implication holds:

$$E(s_0) < \gamma \rho^{m-2} \Rightarrow T < \rho^2.$$ 

Together with the previous inequality, this implies

$$c > \frac{1}{\sqrt{2 \exp \frac{\rho^2}{\gamma} - 1}}.$$ 

Moreover, $E(s_0) = a_P E(\sigma_0) + b_P$, so set $c_0 := \frac{\gamma \rho^{m-2} - b_P}{a_P \text{vol} M}$. It is then clear that, whenever $\gamma \rho^{m-2} > b_P$, the desired assertion would indeed make sense. Unfortunately, there is no immediate constructive way to determine the constant $\gamma$, which comes from a Moser iteration argument, so in practice this procedure cannot be expected to work. Besides, a map $s : P \to G/H$ at once constant and equivariant might very well not exist at all, for topological reasons.

### 2.6 A pseudo-monotonicity formula for the harmonic section flow

We formulate a straightforward argument leading to a pseudo-monotonicity formula for a natural entropy functional along the harmonic section flow (HSF). It likely underlies an $\epsilon$-regularity theory for $G$-equivariant maps along the lines of [5, 8, 24], which will be the object of subsequent work, see Afterword.

Let $Q$ denote the Riemannian manifolds $M$, the total space of $P$ or its fibre $G$, and set

$$d_Q := \dim(Q), \quad v_Q := \text{vol}(Q) \quad \text{for} \quad Q = P, M, G.$$ 

Following [5], we define the system of flows

$$\begin{aligned}
\partial_t v_t &= -1 \\
v_t|_{t=t_0} &= v_0
\end{aligned} \quad (19)$$

and

$$\begin{aligned}
\partial_t \theta^Q_t &= (|\nabla^Q \theta^Q_t|^2 - \Delta^Q \theta^Q_t) + (d_Q)/(2v_t) \\
\theta^Q_t|_{t=t_0} &= \theta^Q_0
\end{aligned} \quad (20)$$

such that $\Theta^Q_t = (4\pi v_t)^{-\frac{d_Q}{2}} e^{-\theta^Q_t}$ is a solution of the backward heat equation on $Q$:

$$\begin{aligned}
\partial_t \Psi^Q_t &= -\Delta^Q (\Psi^Q_t) \\
\Psi^Q_t|_{t=t_0} &= \Psi^Q_0
\end{aligned} \quad (21)$$

for $0 \leq t < t_0$. As $\pi$ is a Riemannian submersion with minimal fibres, the solutions to (20) and (21) on $P$ and $M$ are related, respectively, by

$$\begin{aligned}
\theta^P_t &= -(d_G/2) \ln v_t + \theta^M_t \circ \pi, \\
\Theta^P_t &= (4\pi)^{-\frac{d_G}{2}} \Theta^M_t \circ \pi.
\end{aligned}$$

The functionals

$$\mathcal{F}_s^P(t) = \frac{T-t}{2} \int_P \Theta^P_t[d\sigma_t]^2 \quad \text{and} \quad \mathcal{F}_s^M(t) = \frac{T-t}{2} \int_M \Theta^M_t[d\nu_{\sigma_t}]^2$$

\(\circ\) Springer
are then related by
\[ F_s^p(t) = (4\pi)^{-dG/2} v_G F_s^m(t) + \frac{T-t}{2} d_G \]
with the arguments of Lemma 12, assuming the normalisation
\[ \int_P \Theta_0^P = 1. \]
This allows us to apply Hamilton [24, Theorem A] to \( F_s^p \), to have, for \( T - 1 \leq \tau \leq t \leq T \) and the harmonic \( G \)-equivariant map \( s : P \to G/H \),
\[ F_s^p(t) \leq C_P F_s^p(\tau) + C_P (t-\tau) \int_P |ds_0|^2 \]
or, in terms of sections, a pseudo-monotonicity formula for \( \sigma \):

**Proposition 31** Under the assumptions (A), a solution \( \{\sigma_t\} \subset \Gamma(\pi) \) of the flow (HSF) satisfies, for \( T - 1 \leq \tau \leq t \leq T \),
\[ (4\pi)^{-dG/2} F_s^m(t) + \frac{T-t}{2} d_G \leq C_P (4\pi)^{-dG/2} F_s^m(\tau) + \frac{T-\tau}{2} d_G + (t-\tau)(E(\sigma_0) + \frac{d_G v_M}{2}), \]
where \( v_P = v_G v_M \).

Almost monotonicity formulas of this form can be used to establish \( \epsilon \)-regularity mechanisms. Under favourable circumstances, those may lead to long-time existence and convergence of the flow, as well as inform on the structure of finite-time singularities, see, for example, [12].

**Part 2. Instances of harmonic geometric structures**

### 3 Parallelisms on the 3-sphere

Let us begin to illustrate the abstract framework of Part 1 in the case of parallelisms, corresponding to the simplest possible choice of the (trivial) subgroup \( H = \{e\} \subset \text{SO}(m) \). The outcome will be a natural harmonicity condition, together with its associated flow, which to our knowledge has not so far been explicitly studied. Qualitatively, while the very existence of a parallelism has the topological interpretation of trivialising \( TM \), harmonicity is a weaker condition than covariant constancy and therefore has the potential to distinguish global frames for instance on manifolds known not to admit a Lie group structure.

In this case, the bundles \( P = P_{\text{SO}(m)} = N \to M \) coincide, and their vertical and horizontal distributions are the same, since the quotient map \( q \) is the identity. The Lie algebra \( \mathfrak{h} \) is trivial, its complement is \( m = \mathfrak{g} \) and the associated bundle \( \mathfrak{m} \) is \( P \times \mathfrak{g} \). The canonical isomorphism becomes
\[ f(q_*(E)) = f(E) = (z, \omega(E)) \in P \times \mathfrak{g}. \]
A section \( \sigma : M \to N = P \) corresponds to a global frame field, i.e. a parallelism on \( M \). Then \( \omega \) is the Maurer–Cartan form of \( \mathfrak{g} \), since \( d^V \sigma_t(X) \in \mathcal{V}_s \) and the typical fibre of \( P \) is the group \( G \) itself, and the connection 1-form \( f \) acts on the vertical torsion by:
\[ f(d\sigma(X)) = \omega(d^V \sigma(X)) \in P \times \mathfrak{g}, \quad \text{for} \quad X \in TM. \]
To the best of our knowledge, there are no results in the literature regarding harmonic flows of parallelisms; therefore, the statements of Theorems 1-3 in this context seem to be original.
3.1 Torsion of a parallelism on $\mathbb{S}^3$

To make matters concrete, let us carry out the detailed derivation of the harmonicity condition for a parallelism on $M = \mathbb{S}^3$, so that $P = \text{SO}(4) \to \mathbb{S}^3$ is an SO(3)-bundle. It is rather clear how to extend this procedure to $(\mathbb{S}^m, g)$ with more general metrics, for $m = 3, 7$.

Recall that, for $z \in \text{SO}(4),$

$$T_z \text{SO}(4) = \left\{ \frac{1}{2} (M - z M^T z) : M \in \mathcal{M}_4(\mathbb{R}) \right\} = \{ z X : X + X^T = 0 : X \in \mathcal{M}_4(\mathbb{R}) \} ,$$

(22)

The projection

$$p : \text{SO}(4) \to \mathbb{S}^3$$

$$z = (x, z_1, z_2, z_3) \mapsto x$$

is $\text{SO}(4)$-equivariant, i.e. $p(\tilde{z} z) = \tilde{z} p(z)$, for all $\tilde{z} \in \text{SO}(4)$, so the vertical and horizontal distributions are related by

$$\mathcal{V}_z = \tilde{z} \mathcal{V}_{\tilde{z}} \quad \text{and} \quad \mathcal{H}_z = \tilde{z} \mathcal{H}_{\tilde{z}} .$$

The differential $p_*$ at the identity maps the matrix $M = (M_0, M_1, M_2, M_3)$ to $M_0$, so we deduce from the anti-symmetry in $\mathfrak{so}(4)$ that

$$\mathcal{V}_{id} = \left\{ M \in \mathfrak{so}(4) : M = \begin{pmatrix} 0 & 0 \\ 0 & M \end{pmatrix}, \tilde{M} \in \mathfrak{so}(3) \right\} \quad \text{and} \quad \mathcal{V}_z = z \mathcal{V}_{id} .$$

A section of the orthonormal frame bundle

$$\sigma : \mathbb{S}^3 \to \text{SO}(4)$$

$$x \mapsto \sigma(x) = (x, \sigma_1(x), \sigma_2(x), \sigma_3(x))$$

has, at each $x \in \mathbb{S}^3$, the linearisation

$$d\sigma_x : T_x \mathbb{S}^3 \to T_{\sigma(x)} \text{SO}(4)$$

$$X \mapsto d\sigma_x(X) = : \text{Proj}_{T_{\sigma(x)} \text{SO}(4)} (MM_\sigma)$$

with $M_\sigma = (X, d\sigma_1(X), d\sigma_2(X), d\sigma_3(X))$. Therefore,

$$d\sigma_x(X) = \frac{1}{2} \left( M_\sigma - \sigma(x) M_\sigma^T \sigma(x) \right) = \sigma(x), \frac{1}{2} \left( \sigma(x)^T M_\sigma - M_\sigma^T \sigma(x) \right)$$

$$= \sigma(x) \begin{pmatrix} 0 & -v^T \\ v & M_\sigma \end{pmatrix} , \quad \text{with} \quad \tilde{M}_\sigma \in \mathfrak{so}(3) \quad \text{and} \quad v \in \mathbb{R}^3 ,$$

$$\Rightarrow \quad d^3 \sigma_x(X) = \sigma(x) \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} .$$

Then

$$\sigma(x)^T M_\sigma = \begin{pmatrix} \langle x, X \rangle & \langle x, d\sigma_1(X) \rangle & \langle x, d\sigma_2(X) \rangle & \langle x, d\sigma_3(X) \rangle \\ \langle \sigma_1, X \rangle & \langle \sigma_1, d\sigma_1(X) \rangle & \langle \sigma_1, d\sigma_2(X) \rangle & \langle \sigma_1, d\sigma_3(X) \rangle \\ \langle \sigma_2, X \rangle & \langle \sigma_2, d\sigma_1(X) \rangle & \langle \sigma_2, d\sigma_2(X) \rangle & \langle \sigma_2, d\sigma_3(X) \rangle \\ \langle \sigma_3, X \rangle & \langle \sigma_3, d\sigma_1(X) \rangle & \langle \sigma_3, d\sigma_2(X) \rangle & \langle \sigma_3, d\sigma_3(X) \rangle \end{pmatrix}$$
and since
\[ \langle x, X \rangle = \langle \sigma_i, d\sigma_i(X) \rangle = 0, \quad \langle x, d\sigma_i(X) \rangle = -\langle \sigma_i, X \rangle \]
\[ \langle \sigma_i, d\sigma_j(X) \rangle = \langle \sigma_i, \nabla^3_X \sigma_j \rangle, \quad \text{for } i, j = 1, 2, 3, \]
we obtain, cf. (22):
\[ d\sigma_x(X) = \sigma(x) \begin{pmatrix} 0 & -\langle \sigma_1, X \rangle & -\langle \sigma_2, X \rangle & -\langle \sigma_3, X \rangle \\ \langle \sigma_1, X \rangle & 0 & \langle \sigma_1, \nabla^3_X \sigma_2 \rangle & \langle \sigma_1, \nabla^3_X \sigma_3 \rangle \\ \langle \sigma_2, X \rangle & \langle \sigma_2, \nabla^3_X \sigma_1 \rangle & 0 & \langle \sigma_2, \nabla^3_X \sigma_3 \rangle \\ \langle \sigma_3, X \rangle & \langle \sigma_3, \nabla^3_X \sigma_1 \rangle & \langle \sigma_3, \nabla^3_X \sigma_2 \rangle & 0 \end{pmatrix} \in T_{\sigma(x)}SO \quad (4) \]
\[ \Rightarrow \quad d^V\sigma_x(X) = \sigma(x) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \langle \sigma_1, \nabla^3_X \sigma_2 \rangle & \langle \sigma_1, \nabla^3_X \sigma_3 \rangle \\ 0 & \langle \sigma_2, \nabla^3_X \sigma_1 \rangle & 0 & \langle \sigma_2, \nabla^3_X \sigma_3 \rangle \\ 0 & \langle \sigma_3, \nabla^3_X \sigma_1 \rangle & \langle \sigma_3, \nabla^3_X \sigma_2 \rangle & 0 \end{pmatrix} \]
\[ = \left( \sigma(x), \begin{pmatrix} 0 & \langle \sigma_1, \nabla^3_X \sigma_2 \rangle & \langle \sigma_1, \nabla^3_X \sigma_3 \rangle \\ \langle \sigma_2, \nabla^3_X \sigma_1 \rangle & 0 & \langle \sigma_2, \nabla^3_X \sigma_3 \rangle \\ \langle \sigma_3, \nabla^3_X \sigma_1 \rangle & \langle \sigma_3, \nabla^3_X \sigma_2 \rangle & 0 \end{pmatrix} \right) \in P \times \mathfrak{so}(3). \quad (23) \]

### 3.2 Harmonic parallelisms on $S^3$

Recall that $\nabla^\omega$ is the connection on $T^*S^3 \times TS^3$, i.e. the connection on endomorphisms of $S^3$. Since $p^*\mathfrak{g} = m \oplus \mathfrak{h}$, where $\mathfrak{h} = P \times_H \mathfrak{h}$, we identify $\mathcal{I}(d^V\sigma_x(X))$ with the endomorphism of $S^3$ defined by the matrix of (23) in the frame $\sigma(x)$. Since $\mathcal{I}$ only sees the $V$-component, by Lemma 5 we have:
\[ \mathcal{I}(d^V\sigma_x(X)) = \omega(d^V\sigma_x(X)) = \tilde{\sigma}(x)(d^V\sigma_x(X)) = (\nabla^\omega(\sigma^* f))(X, X). \]

Specialising to vectors of the frame itself,
\[ \mathcal{I}(d^V\sigma_x(X)) \langle \sigma_i \rangle = (\sigma^* f)(X) = \nabla^3_X \sigma_i, \quad \text{for } i = 1, 2, 3. \]

If $X$ is a vector field on $S^3$ such that $\nabla^3_X X = 0$, then
\[ (\nabla^\omega(\sigma^* f))(X, X) = \nabla^\omega_X((\sigma^* f)(X)) \in T^*S^3 \times TS^3, \]
and its evaluation on the column vectors of $(\sigma_1(x), \sigma_2(x), \sigma_3(x))$ yields (without summing on repeated indices $j$ or $k$)
\[ (\nabla^\omega(\sigma^* f))(X)(\sigma_i) = \nabla^3_X ( (\sigma^* f)(X)(\sigma_i) - (\sigma^* f)(X)(\nabla^3_X \sigma_i) ) 
\[ = \nabla^3_X \nabla^3_X \sigma_i - \langle \nabla^3_X \sigma_i, \sigma_j \rangle \nabla^3_X \sigma_j - \langle \nabla^3_X \sigma_i, \sigma_k \rangle \nabla^3_X \sigma_k \]
for $\{i, j, k\} = \{1, 2, 3\}$. Taking traces over the positions of bullet points,
\[ \mathcal{I}(\tau^V(\sigma))(\sigma_i) = \nabla^* \nabla \sigma_i - \text{tr}(\nabla^3 \sigma_i, \sigma_j) \nabla^3 \sigma_j - \text{tr}(\nabla^3 \sigma_i, \sigma_k) \nabla^3 \sigma_k, \]
taking $\nabla^* \nabla \sigma_i = \text{tr} \nabla^2 \sigma_i$. One can easily check that 
\[ \langle I(\tau^V(\sigma))(\sigma_i), \sigma_i \rangle = 0 \quad \text{and} \quad \langle I(\tau^V(\sigma))(\sigma_i), \sigma_j \rangle = -\langle I(\tau^V(\sigma))(\sigma_j), \sigma_i \rangle, \]
so $I(\tau^V(\sigma))$ is indeed in $\sigma^*(P \times_{SO(3)} so(3))$. One can rewrite
\[
I(\tau^V(\sigma))(\sigma_i) = \nabla^* \nabla \sigma_i + |\nabla S^3 \sigma_i|^2 \sigma_i - \text{tr}(\nabla S^3 \sigma_i, \sigma_k)\langle \nabla S^3 \sigma_k, \sigma_j \rangle \sigma_j \\
- \text{tr}(\nabla S^3 \sigma_i, \sigma_j)\langle \nabla S^3 \sigma_j, \sigma_k \rangle \sigma_k \\
= \nabla^* \nabla \sigma_i + |\nabla S^3 \sigma_i|^2 \sigma_i + \langle \nabla S^3 \sigma_i, \nabla S^3 \sigma_j \rangle \sigma_j + \langle \nabla S^3 \sigma_i, \nabla S^3 \sigma_k \rangle \sigma_k. \quad (24)
\]
Recalling our curvature convention (16) and since $\nabla^* \nabla \sigma_i = \text{tr} \nabla^2 \sigma_i$, it is not hard to check that, for example, the Hopf vector fields (i.e., $x_i x_j$) are a solution, since they are unit harmonic vector fields. Indeed, they are Killing vector fields satisfying $\nabla S^3 \sigma_i = 0$ and $\nabla S^3 \sigma_j = \pm \sigma_k$, for $i \neq j \neq k \neq i$, and $\text{Ric}^S = 2 \text{id}$ [44, 49].

More generally, if $\sigma_1, \sigma_2, \sigma_3$ are orthogonal unit harmonic vector fields on $S^3$, then the orthonormal frame $\sigma = \{\sigma_1, \sigma_2, \sigma_3\}$ is a harmonic parallelism if, and only if, the $(1, 1)$-tensors $\nabla S^3 \sigma_i$ and $\nabla S^3 \sigma_j$ are mutually orthogonal.

### 3.3 The harmonic parallelism flow from (HSF)

A natural harmonic flow for parallelisms, e.g., on $S^3$, stems from applying the canonical isomorphism of Sect. 1.1 to both sides of the equation (HSF):
\[ I(\partial_t \sigma) = I(\tau^V(\sigma)). \]

While $I(\tau^V(\sigma))$ was computed in (24), in order to deduce $I(\partial_t \sigma)$ from Sect. 3.1, we extend the $G$-action trivially on $I = [0, T]$ and apply the constructions from Sects. 1.1 and 1.2 to $M_T := M \times I$. Define
\[
\Sigma : M_T \to N \\
(x, t) \mapsto \Sigma(x, t) = \sigma_t(x),
\]
so that $d\Sigma(\partial_t) = \partial_t \sigma$. Since the initial data $\sigma_0$ of (HSF) is a section, $\sigma_t$ remains a section for all $t \in I$. Hence $(\pi \circ \Sigma)(x, t) = x$ and, in particular,
\[ d\pi(d\Sigma(\partial_t)) = 0, \]
i.e., $d\Sigma(\partial_t)$ is vertical, and therefore lies in the domain of $I$.

The formulas of Sect. 3.1 extend to $M \times I$, $\Sigma$ and $\partial_t$ (cf. Proposition 20) and we deduce that
\[
I(d\Sigma(x, t))(\partial_t) = I(\partial_t \sigma) \\
= \left( \Sigma(x, t), \begin{pmatrix}
0 & \langle \sigma_1, \partial_t \sigma_2 \rangle & \langle \sigma_1, \partial_t \sigma_3 \rangle \\
\langle \sigma_2, \partial_t \sigma_1 \rangle & 0 & \langle \sigma_2, \partial_t \sigma_3 \rangle \\
\langle \sigma_3, \partial_t \sigma_1 \rangle & \langle \sigma_3, \partial_t \sigma_2 \rangle & 0
\end{pmatrix} \right).
\]
The harmonic flow for parallelisms of $S^3$ is therefore given by the following matrix equation in $so(3)$:
structures are characterised by the condition $D_{\maximal}$.

$\nabla^* \nabla$ will be denoted by $H$.

The general approach of harmonicity via reduction of the structure group applies to almost contact structure $\eta$. Where $\eta(\xi)$ is determined by $\eta(\xi) = 1$ and metric compatibility is taken as a blanket assumption [6, 18]. The distribution orthogonal to the $\xi$-direction will be denoted by $\mathcal{D}$, and the component of $\theta$ transverse to $\xi$ by

$$J := \theta|_{\mathcal{D}}.$$  

The induced connection and curvature on $\mathcal{D}$ will be denoted by $\tilde{\nabla}$ and $\tilde{R}$, respectively. Contact structures are characterised by the condition $\eta \wedge (d\eta)^n \neq 0$, i.e. the contact sub-bundle $\mathcal{D}$ is maximally non-integrable.

### 4.1 Torsion of an almost contact structure

The general approach of harmonicity via reduction of the structure group applies to almost contact structures [43], for $H = U(n) \subset SO(2n + 1)$, embedded by

$$A + iB \mapsto \begin{pmatrix} A - B & 0 \\ B & A \end{pmatrix}.$$
Then \( U(n) = \{ A \in \text{SO}(2n+1) : A\phi_0 A^{-1} = \phi_0 \} \), with
\[
\phi_0 := \begin{pmatrix}
\mathbb{O}_n & -I_n \\
I_n & \mathbb{O}_n \\
0 & \cdots & 0
\end{pmatrix} \in \mathfrak{g} = \mathfrak{so}(2n+1),
\]
where \( \mathbb{O}_n \) and \( I_n \) are the \( n \times n \) zero and identity matrices, respectively. Its Lie algebra is
\[
\mathfrak{h} = u(n) = \{ a \in \mathfrak{g} : [a, \phi_0] = 0 \}.
\]
The orthogonal complement of \( u(n) \) in \( \mathfrak{so}(2n+1) \), with respect to the Killing form, splits into \( m_1 \) and \( m_2 \):
\[
m_1 = \{ a \in \mathfrak{so}(2n+1) : [a, \phi_0] = 0 \}, \quad m_2 = \{ [a, \eta_0 \otimes \xi_0] : a \in \mathfrak{so}(2n+1) \},
\]
where \( \xi_0 = (0, \ldots, 0, 1) \in \mathbb{R}^{2n+1} \) and \( \eta_0 \) is the dual of \( \xi_0 \). Then \( \mathfrak{so}(2n+1) = u(n) \oplus m_1 \oplus m_2 \) is an \( \text{Ad}(U(n)) \)-invariant splitting with:
\[
[u(n), m_i] \subset m_i \quad (i = 1, 2), \quad [m_1, m_1] \subset u(n), \quad [m_2, m_2] \subset u(n) \oplus m_1, \quad [m_1, m_2] \subset m_2.
\]
If \( a \in \mathfrak{so}(2n+1) \) then \( a = a_{u(n)} + a_{m_1} + a_{m_2} \) with
\[
a_{u(n)} = -\frac{1}{2} (\phi_0 [a, \phi_0] + a \circ (\eta_0 \otimes \xi_0)), \quad a_{m_1} = \frac{1}{2} (\phi_0 [a, \phi_0] - a \circ (\eta_0 \otimes \xi_0)), \quad a_{m_2} = \{ a, \eta_0 \otimes \xi_0 \},
\]
and \( \xi_0 \) and \( \phi_0 \) induce a universal almost contact structure defined by \( \xi \) and \( \Phi \) on \( \mathcal{F} = \pi^* TM \).

The \( U(n) \)-modules \( u(n) \), \( m_1 \) and \( m_2 \) are, respectively, the typical fibres of the vector bundles \( u(n) \), \( m_1 \) and \( m_2 \), associated to \( q : N \rightarrow \mathcal{F} \), which, respectively, commute with \( \Phi \), anti-commute with \( \Phi \) and interchange \( \text{im} \Phi \) and \( \langle \xi \rangle \). Then, the homogeneous connection form is the \( m_1 \oplus m_2 \)-valued 1-form \( f \) on \( N \) obtained by projecting the \( (m_1 + m_2) \)-component of the connection form \( \omega \in \Omega^1(P, g) \) and its \( m_1 \) - and \( m_2 \)-components are [43, Lemma 2.1]:
\[
f_1 = \frac{1}{2} \Phi \circ (\nabla \Phi)_1; \quad f_2 = (\Phi, (\nabla \Phi)_2).
\]

Finally, in order to express the torsion in terms of \( J \) and \( \eta \), we further need to apply the ‘hat’ isomorphisms: [43, Proposition 2.2]
\[
\begin{align*}
(I_1 d^V \sigma(X))^\flat &= \frac{1}{2} J \nabla_X J \\
(I_2 d^V \sigma(X))^\flat &= \nabla_X \xi.
\end{align*}
\]
4.2 Harmonic almost contact structures

From [50, (3.2)], the canonical vector bundle isomorphism $\mathcal{I} : \mathcal{V} = \ker d\pi \to \mathfrak{m}_1 \oplus \mathfrak{m}_2$ sends $d^\mathcal{V}\sigma$ to $\psi = \sigma^* f$ and

$$\mathcal{I}(\psi) = \nabla^E_i \psi(E_i).$$

Projecting onto Skew $\mathbb{R}^{2n}$ and $\mathbb{R}^{2n}$, respectively, the vertical tension field $\tau^\mathcal{V}(\sigma) = \text{tr} \nabla^\mathcal{V} d^\mathcal{V}\sigma$ gives rise to the two harmonic section equations of [43, Theorem 3.2]:

$$\begin{cases}
(\mathcal{I}_1 \tau^\mathcal{V}(\sigma))^\dagger = \frac{1}{4} [\tilde{\nabla}^* \tilde{\nabla} J, J] \\
(\mathcal{I}_2 \tau^\mathcal{V}(\sigma))^\dagger = \nabla^* \nabla \xi + |\nabla \xi|^2 \xi - \frac{1}{2} \text{tr} \tilde{\nabla} J \otimes \nabla \xi,
\end{cases} \quad (27)$$

still with the conventions that $\tilde{\nabla}^* \tilde{\nabla} J = \text{tr} \tilde{\nabla}^2 J$ and $\nabla^* \nabla \xi = \text{tr} \tilde{\nabla}^2 \xi$. For the harmonic map equation, we use [50] to obtain:

$$\frac{1}{4} \langle R(E_i, X), J \tilde{\nabla} E_i, J \rangle + \langle R(E_i, X) \xi, \nabla E_i \xi \rangle = 0.$$ 

The energy functional of an almost contact structure can then be computed to be

$$E(\sigma) = \frac{\dim(M)}{2} + \frac{1}{2} \int_M \frac{1}{4} |\tilde{\nabla} J|^2 + |\nabla \xi|^2 v_g$$

and, since variations are considered among unit vector fields and $(1,1)$-tensors related by (8), (25), this could be seen as a constrained variational problem.

Examples of harmonic almost contact structures are rather numerous, e.g. (i) the canonical structure of a hypersurface of a Kähler manifold is harmonic, whenever the Reeb vector field is harmonic; (ii) on the unit sphere $S^{2n-1} \subset \mathbb{C}^n$, it is moreover a harmonic map; (iii) and the latter indeed holds on any Sasakian manifold; (iv) among hyperspheres of the nearly Kähler $\mathbb{S}^6$, the equator is the only one to define a harmonic section, and it is also a harmonic map; (v) the property (iv) was further extended to all nearly cosymplectic manifolds, i.e. when $\nabla \phi$ is skew-symmetric (cf. [35] for further details).

4.3 The harmonic almost contact structure flow from (HSF)

In view of (26) and (27), applying the canonical isomorphism $\mathcal{I}$ to both sides of the harmonic section flow equation (HSF) yields the harmonic almost contact structure flow:

$$\begin{align*}
\partial_t J &= \frac{1}{2} (\tilde{\nabla}^* \tilde{\nabla} J + J (\tilde{\nabla}^* \tilde{\nabla} J) J), \\
\partial_t \xi &= \nabla^* \nabla \xi + |\nabla \xi|^2 \xi - \frac{1}{2} \text{tr} \tilde{\nabla} J \otimes \nabla \xi,
\end{align*} \quad \text{on } M_T. \quad (\text{HACtSF})$$

In this case, just as for parallelisms, the analysis of the corresponding harmonic flow seems hitherto uncharted, so Theorems 1–3 bear the following original consequence:

**Corollary 33** Let $(M^{2n+1}, g, \xi_0, \theta_0)$ be a closed almost contact manifold. For fixed $g$, suppose the torsion $(\frac{1}{2} J \nabla J, \nabla \xi)$ from (26) is uniformly $L^{4n+2}(M)$-bounded along the harmonic almost complex structure flow (HACtSF), with initial condition defined by $\sigma_0$ under the correspondence (5). Then the flow (HACtSF) admits a continuous solution $(\xi, J(t))$ for all time. Moreover, there exists a strictly increasing sequence $\{t_j\} \subset [0, +\infty]$ such that $(\xi, J(t_j)) \xrightarrow{C^\infty} (\xi_\infty, J_\infty) \in \Gamma(TM) \times \text{End}(TM)$, and the limiting almost contact structure satisfies the harmonicity condition:

$$[\tilde{\nabla}^* \tilde{\nabla} J_\infty, J_\infty] = 0 \quad \text{and} \quad \nabla^* \nabla \xi_\infty + |\nabla \xi_\infty|^2 \xi_\infty - \frac{1}{2} J_\infty \text{tr} \tilde{\nabla} J_\infty \otimes \nabla \xi_\infty = 0.$$
5 Almost complex structures

Let \((M^{2n}, g)\) be an even-dimensional Riemannian manifold. A compatible almost complex structure on \(M\) is a \((1, 1)\)-tensor field \(J\) such that \(J^2 = -\text{id}\) and \(g(J\cdot, J\cdot) = g(\cdot, \cdot)\). This is one of the most studied and fruitful types of geometric structures, especially in the Kähler case. Nonetheless, many central questions remain unanswered, for example the existence of an integrable almost complex structure on (non-standard) \(S^6\), or a topological characterisation of manifolds admitting almost complex structures. The introduction of a variational problem will hopefully shed a new light on these objects. This case comes first, historically and in importance, and its study was pioneered by [46, 47].

5.1 Torsion of an almost complex structure

On an even-dimensional manifold, consider the classical situation of twistor spaces, in which \(H = U(n) \subset SO(2n)\) is embedded by 

\[ A + iB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix}. \]

The Lie algebra \(\mathfrak{h} = u(n)\) consists of anti-symmetric matrices commuting with 

\[ J_0 = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix}, \]

and \(\mathfrak{m}\) consists of anti-symmetric matrices anti-commuting with \(J_0\). On the vector bundle \(\pi^*TM \to N\), denote by \(\mathcal{E} = \mathcal{J}\) the universal almost complex structure introduced in Sect. 1.2. At each \(U(n)\)-class of frames \(y \in N\), the tensor \(\mathcal{J}_y \in \text{End}(T_{\pi(y)}M)\) is modelled on \(J_0\), with respect to any orthonormal frame of \(p^{-1}(y)\). Any element \(\beta \in \mathfrak{g} = so(2n) = u(n) \oplus \mathfrak{m}\) decomposes as:

\[ \frac{1}{2}J_0[\beta, J_0] + \frac{1}{2}J_0\{\beta, J_0\}, \]

where \(\{A, B\} = AB + BA\) is the anti-commutator. Using (6) to differentiate the universal structure, then \(\{\mathcal{J}, J_0\} = 0\) and \(\mathcal{J}^2 = -\text{id}\), we have

\[ \nabla_A \mathcal{J} = [f(A), \mathcal{J}] = -2\mathcal{J} \circ f(A) \]

\[ \Leftrightarrow f(A) = \frac{1}{2} \mathcal{J} \circ \nabla_A \mathcal{J}. \]

Applying this to the pull-back \(\sigma^* f\), we recover the torsion, as first obtained in [50, Theorem 4.2]:

\[ \mathcal{I}(d^\nabla \sigma) = \mathcal{I}(\sigma^* f) = \frac{1}{2}J \nabla J. \tag{28} \]

One can easily check that the energy functional is

\[ E(\sigma) = \frac{\dim(M)}{2} + \frac{1}{2} \int_M \frac{1}{4} |\nabla J|^2 v_g, \]

so that Kähler structures are absolute minimisers.
5.2 Harmonic almost complex structures

In the case at hand, since \([m, m] \subset u(n)\) the connections \(\nabla^c\) and \(\nabla^m\) of Lemma 5 coincide (equivalently \(G/H\) is a symmetric space). Moreover, since \(\nabla^m\) is the connection on tensors, the \(m\)-component of the derivative of a section \(\alpha\) of \(m\) is

\[
\nabla^c_A \alpha = \frac{1}{2} J[\nabla_A \alpha, J],
\]

where \(J\) is the universal almost complex structure defined above. Taking \(\alpha = \sigma^* f = \frac{1}{2} J \nabla J\) to be the torsion and tracing, we obtain the corresponding Euler–Lagrange equation \([46]\):

\[
I(\nabla^V(\sigma)) = -\frac{1}{4} [\nabla^* \nabla J, J],
\]

with the convention that \(\nabla^* \nabla J = \text{tr} \nabla^2 J\). So \(J\) is a harmonic section if and only if it commutes with its rough Laplacian. Moreover, it is a harmonic map if \([50]\)

\[
g(R(E_i, Z) J, \nabla E_i J) = 0, \quad \forall Z \in TM.
\]

The first examples of harmonic sections, but also of harmonic maps, have been nearly Kähler structures \([46]\), largely because of their interesting curvature identities \([19]\). On the other hand, \((1, 2)\)-symplectic structures are harmonic sections if, and only if, the operator \(\text{Ric}^* = \text{tr} R(X, .) J\) is symmetric. The Calabi–Eckmann complex structure on the product of odd-dimensional spheres and the Abbena–Thurston almost-Kähler structure are harmonic sections, but not critical for the volume functional. Davidov and Muskarov \([11, 14]\) show the harmonicity, as sections or maps, of the Atiyah–Hitchin–Singer and Eells–Salamon almost Hermitian structures on the twistor space of an oriented Riemannian four-manifold.

5.3 The harmonic almost complex structure flow from \((\text{HSF})\)

As for the others cases, the torsion equation \((28)\) can be extended to the manifold \(M_T\), thus allowing us to compute that \(I(\partial_t \sigma_t) = \frac{1}{2} J \partial_t J\). Applying the canonical isomorphism \((1)\) to the flow \((\text{HSF})\), we recover the harmonic almost complex structure flow recently introduced in \([27]\):

\[
\partial_t J = \frac{1}{2} \left( \nabla^* \nabla J + J(\nabla^* \nabla J) J \right) \quad \text{(HACxSF)}
\]

As an immediate consequence of Theorems 1–3, we have therefore:

**Corollary 34** Let \((M^{2n}, g, J_0)\) be a closed almost Hermitian manifold. For fixed \(g\), suppose the torsion \((28)\) is uniformly \(L^{4n}(M)\)-bounded along the harmonic almost complex structure flow \((\text{HACxSF})\), with initial condition defined by \(J_0 =: \sigma_0^* J\) under the correspondence \((5)\). Then \((\text{HACxSF})\) admits a continuous solution \(J(t)\) for all time. Moreover, there exists a strictly increasing sequence \(\{t_j\} \subset [0, +\infty)\) such that \(J(t_j) \xrightarrow{C^\infty} J_\infty \in \text{End}(TM)\), and the limiting almost complex structure satisfies the harmonicity condition:

\[
[\nabla^* \nabla J_\infty, J_\infty] = 0.
\]

In \([27]\), He and Li study the harmonic flow for almost complex structures and prove general short-time existence (Theorem 2.1) and long-time existence for initial data with small energy with convergence to a Kähler structure (Theorem 2). In \([25]\), weak almost complex structures are introduced, an \(\epsilon\)-regularity is proved and energy-minimising structures are studied.
6 G2-Structures on 7-manifolds

G2-structures on an oriented and spin Riemannian manifold \((M^7, g)\) are reductions from SO(7) to G2, i.e. principal G2-subbundles of the frame bundle \(p : P \to M\). This is encoded by a 3-form \(\varphi \in \Lambda^3 T^* M\) which is modelled, pointwise, on the canonical vector cross-product \(\varphi_0 \in \Lambda^3 (\mathbb{R}^7)^*\), which is stabilised by the standard action of G2. Precise conditions for the existence of a G2-structure are known to be purely topological: orientability and spinability, cf. [7] for Gray’s argument. A G2-structure induces a G2-ϕ by a 3-form \(\varphi\) that \(G2\)-structures could be regarded as sections of an \(G2\)-structure are known to be purely topological: orientability and spinability, while the canonical isomorphism \(q^* \varphi_0\) becomes a Riemannian submersion, and the article [22] interprets the divergence-free torsion condition \(\text{div}\ T = 0\) as a ‘gauge-fixing’ among isometric G2-structures. Any G2-structure can be deformed to become coclosed, so far we only know that one may deform into a closed one if \(M\) is an open manifold, cf. [9].

In the language of the present paper, the universal G2-structure mediates the one-to-one correspondence between G2-structures on \((M^7, g)\) and sections \(\sigma\) of the fibre bundle \(\pi : N := P/G2 \to M\), with fibre SO(7)/G2 \(\simeq \mathbb{R}P^7\). The principal G2-bundle \(q : P \to N\) is such that \(\pi \circ q = p\). Moreover, \(\pi\) is isomorphic to the associated bundle \(P \times_{SO(7)} SO(7)/G2\). That G2-structures could be regarded as sections of an \(\mathbb{R}P^7\)-bundle was indeed already known to Bryant [7, Remark 4].

6.1 Torsion of a G2-structure

Let \(V = (\mathbb{R}^7)^*\) and \(\varphi_0 \in \Lambda^3 V\), \(\psi_0 = \ast \varphi_0 \in \Lambda^4 V\), be the standard G2-structure on \(\mathbb{R}^7\). We identify the Lie algebras \(\mathfrak{so}(7) \simeq \Lambda^2 V = \mathfrak{g}_2 \oplus \mathfrak{m}\), with
\[
\mathfrak{g}_2 = \{ \eta \in \Lambda^2 V : \ast (\eta \wedge \varphi_0) = \eta \}, \\
\mathfrak{m} = \{ \eta \in \Lambda^2 V : \ast (\eta \wedge \varphi_0) = -2 \eta \}.
\]

One easily checks that \(\text{Ad}_{SO(7)}(G2) \mathfrak{m} \subset \mathfrak{m}\), since the G2-action preserves \(\varphi_0\), so \(SO(7)/G2\) is reductive. Using the standard inner-product of \(SO(7)\), we have an invariant Riemannian metric on \(SO(7)/G2\) and an \(\text{Ad}_{SO(7)}(G2)\)-invariant inner-product \(\langle , \rangle\) on \(\mathfrak{m}\) and the canonical bundle of Sect. 1.1 is now the vector bundle \(\mathfrak{m} = P \times_{G2} \mathfrak{m} \to N\) while the canonical isomorphism \(\mathcal{I} : V \to \mathfrak{m}\) maps \(q_*(a^\sharp(z)) \in V\) to \(z \bullet a = [(z, a)]_{G2}\).

We define a Riemannian metric on \(N\) by \(h = \pi^* g + (f, f)\), so the map \(\pi : (N, h) \to (M, g)\) becomes a Riemannian submersion, and the universal G2-structure assigns to each class of frames \(y \in N\) the 3-form in \((\Lambda^3 T^* M)_{\pi(y)}\) given by \(\varphi_0\) in any frame of \(q^{-1}(y)\):
\[
\Phi \in \Gamma(N, \pi^* (\Lambda^3 T^* M)), \quad \Phi(y) = y^* \varphi_0.
\]
Since $\pi^*(\Lambda^3 T^* M) \cong \pi^* P \times_{SO(7)} \Lambda^3 V$, the geometric representation $\rho : \pi^*(\Lambda^3 T^* M) \to \Lambda^3 V$ induces a $SO(7)$-equivariant map

$$\tilde{\Phi} := \rho \circ \Phi \circ \pi^* P : \pi^* P \to \Lambda^3 V,$$

The existence of a $G_2$-structure implies a splitting of $\pi^*(\Lambda^3 T^* M)$ by definition, the pull-back by $\tilde{\Phi}$.

Pulling back by $\tilde{\Phi}$ yields

$$\pi^* \Lambda^3 T^* M \leftarrow \Lambda^3 V \xrightarrow{\rho} \pi^* \Lambda^3 T^* M.$$ 

Let $A \in TN$ and $E \in TP$ a lift of $A$, i.e. $q^*(E) = A$, from Eq. (6), we have

$$\nabla_A \Phi = f(A) \cdot \Phi. \tag{29}$$

In order to work with this formula, we need to understand the action of $m$ on $\Lambda^3 T^* M$, resorting to the representation theory of $G_2$. Let us recall the well-known irreducible bundle splittings

$$\Lambda^2 V = \Lambda^2_7 \oplus \Lambda^2_{14}, \quad \text{with} \quad \Lambda^2_7 = \left\{ \eta \in \Lambda^2 : *(\varphi_0 \wedge \eta) = -2\eta \right\}, \tag{30}$$

and

$$\Lambda^3 V = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}, \quad \text{with} \quad \Lambda^3_1 = \left\{ f \varphi_0 : f \in C^\infty(\mathbb{R}^7) \right\}, \tag{31}$$

The natural action of $\text{gl}(7, \mathbb{R})$ on $\Lambda^3 V$ descends to an action of $\text{so}(7) \cong \Lambda^2 V$ on $\Lambda^3 V$. The map $\beta \mapsto \beta.\varphi_0$ has kernel isomorphic to the subspace $\Lambda^2_{14} \cong \text{g}_2$, so for $\beta \in \Lambda^2_7$ in its complement, there exists $X \in \Gamma(\mathbb{R}^7)$ such that $\beta = X \downarrow \varphi_0$. On the other hand, $\beta.\varphi_0 \in \Lambda^3_7$, so $\beta.\varphi_0 = Y \downarrow \psi_0$ for some $Y \in \Gamma(\mathbb{R}^7)$, and elementary representation theory shows that $Y = -3X$, cf. [29]. Furthermore,

$$(X \downarrow \psi_0) \downarrow \psi_0 = -24X^b$$

hence, for $\beta \in \Lambda^2_7$,

$$(\beta.\varphi_0) \downarrow \psi_0 \downarrow \varphi_0 = 72\beta. \tag{32}$$

The existence of a $G_2$-structure implies a splitting of $\Lambda^2 T^* M$ and $\Lambda^3 T^* M$ according to (30) and (31). Pulling back (32) by $\pi$ and combining with Eq. (29), we find

$$f(A) = \frac{1}{72} (\nabla_A \Phi \downarrow \Psi) \downarrow \Phi, \quad \text{with} \quad \Psi := *\Phi \in \pi^*(\Lambda^4 T^* M).$$

Let $\sigma : M \to N$ be a section of $\pi$ and $\varphi \in \Lambda^3 T^* M$ the corresponding $G_2$-structure. By definition, the pull-back by $\sigma$ of the homogeneous connection form gives the vertical component of $d\sigma$:

$$(\sigma^* f)(X) = f(d\sigma(X)) = \mathcal{I}(d\sigma^\mathcal{V}(X)), \quad \forall X \in TM.$$ 

Plugging this into (32) yields

$$f(d\sigma(X)) = \frac{1}{72} \left( (\nabla_{d\sigma(X)} \Phi) \downarrow \Psi \right) \downarrow \Phi \circ \sigma = \frac{1}{72} \left( (\nabla_{d\sigma(X)} \Phi) \circ \sigma \downarrow \Psi \circ \sigma \right) \downarrow \Phi \circ \sigma$$
\[
\frac{1}{72} (\nabla_X (\Phi \circ \sigma) \lhd \psi \circ \sigma) \lhd \Phi \circ \sigma \\
= \frac{1}{72} ((\nabla_X \psi) \lhd \psi) \lhd \psi,
\]
with \( \psi = *\varphi \in \Lambda^4 T^* M \). So
\[
\mathcal{I}(d^V \sigma(X)) = \frac{1}{72} ((\nabla_X \psi) \lhd \psi) \lhd \psi,
\]
or indeed, since \( \nabla_X \psi \in \Lambda_3^3 \), in terms of the (full) torsion tensor \( \nabla_X \psi =: T(X) \lhd \varphi \),
\[
\mathcal{I}(d^V \sigma(X)) = -\frac{1}{3} T(X) \lhd \varphi. \quad (33)
\]

Knowing the homogeneous connection form, we can compute the vertical energy density of \( \sigma : M \to N \):
\[
|d^V \sigma|^2 = \sum_{i=1}^{7} |d^V \sigma(e_i)|^2 = \sum_{i=1}^{7} h \left( (d^V \sigma)(e_i), (d^V \sigma)(e_i) \right)
\]
\[
= \sum_{i=1}^{7} (f(d^V \sigma(e_i)), f(d^V \sigma(e_i))) = \frac{1}{9} \sum_{i=1}^{7} (T(e_i) \lhd \varphi, T(e_i) \lhd \varphi)
\]
\[
= \frac{2}{3} |T|^2,
\]

since \( \langle X \lhd \varphi, X \lhd \varphi \rangle = 6|X|^2 \), by [29, Lemma A8]. Therefore (assuming \( M \) compact),
\[
E(\sigma) = \frac{1}{2} \int_M |d^V \sigma|^2 = \frac{1}{2} \int_M |T|^2, \quad (34)
\]
and, as \( \pi : (N, \eta) \to (M, g) \) is a Riemannian submersion, the (full) energy functional is
\[
\tilde{E}(\sigma) = \frac{7}{2} + \frac{1}{3} \int_M |T|^2 v_g.
\]

### 6.2 Harmonicity of a \( G_2 \)-structure

To determine the vertical tension field of \( \sigma : M \to N \), we take \( X, Y \in TM \) and express
\[
\mathcal{I}(\nabla^V \sigma(X, Y)), \quad \text{where} \quad \nabla^V \text{ is the vertical component of the Levi-Civita connection } \nabla^\eta \text{ of } (N, \eta), \quad \text{in terms of the connection } \nabla^\omega \text{ on } \mathfrak{so}(7) \text{ inherited from the curvature form of } \omega \text{ on } P.
\]
It is easy to verify, using [29, Proposition 2.5], that the homogeneous space \( \text{SO}(7)/G_2 \) is indeed naturally reductive (but not a symmetric space), i.e. geodesics are exactly given by orbits of the exponential map of \( \text{SO}(7) \). Therefore, by [50, Corollary 2.5], the \( m \)-component of the structure equation for \( \omega \) implies that \( \nabla^V \) and the connection \( \nabla^c \) on \( m \), induced by \( \omega_{G_2} \) on \( q : P \to N \), are related by:
\[
\mathcal{I} \left( \nabla^c_k V \right) = \mathcal{I} \left( \nabla^c_k (\mathcal{I} V) \right) + \frac{1}{2} [f A, \mathcal{I} V]_m,
\]
which is a concrete instance of (7). Indeed, we know from Lemma 5 that
\[
\mathcal{I} \left( \tau^V(\sigma) \right) = \text{tr } \nabla^\omega(\sigma^* f).
\]

Identifying \( \mathfrak{so}(7) \simeq \Lambda^2 V \), the associated bundle is \( \mathfrak{so}(7) := P \times_{\text{SO}(7)} \Lambda^2 V \simeq \Lambda^2 T^* M \), and the connection \( \nabla^\omega \) is exactly the standard one on \( \Lambda^2 T^* M \), inherited from the Levi-Civita...
connection of \((M, g)\). Furthermore, if \(\{e_i\}_{i=1,\ldots,7}\) is an orthonormal frame field:

\[
\mathcal{I}(\tau^V(\sigma)) = \sum_{i=1}^{7} \left( \nabla^\omega (\sigma^* f)(e_i, e_i) \right) = \sum_{i=1}^{7} \nabla_{e_i} \left( ((\sigma^* f)(e_i)) - (\sigma^* f)(\nabla_{e_i} e_i) \right)
\]

\[
= \sum_{i=1}^{7} \nabla_{e_i} \left( f(d\sigma(e_i)) - f(d\sigma(\nabla_{e_i} e_i)) \right)
\]

\[
= -\frac{1}{3} \sum_{i=1}^{7} \nabla_{e_i} (T(e_i) \lrcorner \varphi) - T(\nabla_{e_i} e_i) \lrcorner \varphi
\]

\[
= -\frac{1}{3} \sum_{i=1}^{7} (\nabla_{e_i} T)(e_i) \lrcorner \varphi - T(e_i) \lrcorner \nabla_{e_i} \varphi
\]

\[
= -\frac{1}{3} \sum_{i=1}^{7} \nabla_{e_i} T(e_i) \lrcorner \varphi - T(e_i) \lrcorner (T(e_i) \lrcorner \psi)
\]

\[
= -\frac{1}{3} (\text{div } T) \lrcorner \varphi,
\]

by skew-symmetry of \(\psi\). So

\[
\mathcal{I}(\tau^V(\sigma)) = -\frac{1}{3} (\text{div } T) \lrcorner \varphi.
\]  

(35)

**Remark 35** As was done by Wood [50] and Vergara-Díaz–Wood [43] for almost complex and almost contact structures, respectively, we characterise sections \(\sigma : M \to N\) which are moreover harmonic maps, in addition to being harmonic sections. We follow the blue-print of Sect. 1.4 and need to consider the \(m\)-component of \(\Omega\), the curvature form of \(\omega\), \(\Omega_m = \frac{1}{3} \Omega - \frac{1}{3} \ast (\varphi_0 \wedge \Omega)\), and observe that it is the restriction to \(P\) of the two-form \(\frac{1}{3} \Omega - \frac{1}{3} \ast (\Phi \wedge \Omega)\) on \(\pi^*P\), \(\Omega\) being the pull-back of \(\Omega\). As, \(\Omega = \pi_* R\), on \(N\), for vectors \(A, B \in TN\):

\[
F(A, B) = \frac{1}{3} \left( \pi_* R - \frac{1}{3} \ast (\Phi \wedge \pi_* R) \right)(A, B).
\]

Composing with a harmonic section \(\sigma\) and applying \(\pi \circ \sigma = \text{id}\), we have

\[(\sigma^* F)(X, Y) = \frac{1}{3} \left( R - \ast (\varphi \wedge R) \right)(X, Y).
\]

Since \((\sigma^* f)(e_i) \in \mathfrak{m}\),

\[
\langle \sigma^* f, \sigma^* F \rangle(X) = \sum_{i=1}^{7} \langle (\sigma^* f)(e_i), R_{m}(e_i, X) \rangle = \sum_{i=1}^{7} \langle (\sigma^* f)(e_i), R(e_i, X) \rangle
\]

\[
= -\frac{1}{3} \sum_{i,j,k=1}^{7} (T(e_i) \lrcorner \varphi)(e_k, e_j)R(e_i, X)(e_k, e_j),
\]

and, taking \(X = e_p\) in local coordinates, the harmonicity condition for \(\sigma\) becomes

\[
-3 \langle \sigma^* f, \sigma^* F \rangle(e_p) = \sum_{i,j,k,l=1}^{7} T_{ij} \varphi_{jkl} R_{ipkl} = 0, \quad p = 1, \ldots, 7.
\]  

(36)
The Bianchi-type identity [29, Theorem 4.2] states that
\[
\frac{1}{2} R_{ipkl} \varphi_{jkl} = \nabla_i T pj - \nabla_p T_{ij} - T_{ik} T_{pl} \varphi_{klj},
\]
hence, by skew-symmetry of \( \varphi \), (36) becomes
\[
0 = \sum T_{ij} \left( \nabla_i T_p j - \nabla_p T_{ij} \right) - T_{ik} T_{pl} \varphi_{klj}
= \sum T_{ij} \left( \nabla_i T_p j - \nabla_p T_{ij} \right).
\] (37)

**Example 36** A G\(_2\)-structure is called nearly-G\(_2\) if \( (\nabla_X \varphi)(X) = 0, \forall X \in TM, \nabla \varphi = \lambda \psi \) for a nonzero constant \( \lambda \). Many examples can be found in [16], in particular the squashed seven-sphere, SU(3)/S\(^1\), SO(5)/SO(3) or principal S\(^1\)-bundles over Kähler-Einstein manifolds, such as S\(^2\) \( \times \) S\(^2\) \( \times \) S\(^2\), CP\(^2\) \( \times \) S\(^2\), F(1, 2) or S\(^2\) times a del Pezzo surface.

**Corollary 37** (Nearly-G\(_2\) structures as harmonic maps) A nearly-G\(_2\) structure defines a harmonic map from \( (M, g) \) to \( (N, h) \).

**Proof** The nearly-G\(_2\) condition can easily be shown to be equivalent to \( T(X) = \lambda X \), for some \( \lambda \in \mathbb{R} \). Then, immediately, \( \text{div} T = 0 \), so all nearly-G\(_2\) structures are harmonic sections.

For the harmonicity condition, by [29, Corollary 4.4], on a nearly-G\(_2\) structure, the Bianchi-type identity becomes
\[
\sum_{a,b=1}^7 R_{abij} \varphi_{abj} = 0, \quad i = 1, \ldots, 7,
\]
since \( T = \lambda \text{id}_7 \), so Eq. (36) is satisfied (alternatively, one could use \( \nabla T = 0 \) and the formulation of Eq. (37)).

\( \square \)

### 6.3 The div \( T \)-flow of G\(_2\)-structures from (HSF)

Over the last decade, the field has incorporated an important analytic aspect in the use of natural geometric flows, outlined by the seminal works of Bryant [7] and Hitchin [26], to produce special G\(_2\)-metrics with ‘the least possible torsion’. Moreover, Lotay and Wei developed the analytic theory for the Laplacian flow of closed G\(_2\)-structures, studied its soliton solutions, and proved long-time existence and stability results [38–40]. The Laplacian flow and coflow are manifestations of the gradient flow of Hitchin’s volume functional [26], according to the initial value’s irreducible decomposition. While both should optimistically converge to G\(_2\)-structures with less torsion (ideally torsion-free), there occurs a trade-off between the abundance of coclosed structures, relative to closed ones, and the failure of parabolicity of the coflow, which hinders important properties such as even short-time existence. In some sense, restoring parabolicity of the coflow is equivalent to being able to solve div \( T = 0 \) in each isometric class [21, 22]. While it is not reasonable to expect such a problem to be solvable in general, we believe our theory sheds new light onto this problem, by identifying the divergence-free torsion condition as actual harmonicity with respect to the Dirichlet energy, and indeed the isometric G\(_2\)-flow (see below) as the harmonic map heat flow of the associated SO(7)-equivariant maps.

Let \( (M, \varphi_0) \) be a closed 7-manifold with G\(_2\)-structure. The divergence-free torsion condition from (33) and (35) motivates a natural geometric flow, which evolves \( \varphi_0 \) along isometric G\(_2\)-structures:
\[
\begin{array}{ll}
\partial_t \varphi = (\text{div} T)_\varphi & \text{on } M_T := M \times [0, T], \\
\varphi(0) = \varphi_0 & 
\end{array}
\] (38)
Of course, up to conventions, this is literally \((\text{HSF})\) for \(H = G_2 \subset \text{SO}(7)\).

From its inception by Grigorian [22], the so-called isometric \(G_2\)-flow, or ‘\(\text{div} \, T\)-flow’, (38) has attracted interest and triggered some rapid developments [3, 12, 23], see also [10]. Let us briefly relate some of their interesting results to the general theory of the harmonic section flow. For uniqueness and short-time existence, our Theorem 1 generalises [3], [23, Theorem 5.1], and [12, Theorem 2.12]. We know immediately, from Lemma 15, that the torsion \(T\) remains uniformly \(L^2\)-bounded along such solutions, cf. [23, Lemma 5.3] and [12, Proposition 2.5]. Further regularity of solutions is then inferred from Proposition 23, for so long as \(|T(t)|\) remains bounded, cf. [23, Theorems 5.7 & 5.8], and [12, Theorem 3.7], and subsequential convergence to a smooth harmonic limit follows from Corollary 24 and Proposition 27:

**Corollary 38** Let \((M, \varphi_0)\) be a closed 7-manifold with \(G_2\)-structure. Suppose the torsion (33) is uniformly \(L^{14}(M)\)-bounded along the harmonic section flow (HSF), with initial condition defined by \(\varphi_0 =: \sigma_0^* \Phi\) under the correspondence (5). Then the isometric \(G_2\)-flow (38) admits a continuous solution \(\varphi(t)\) for all time. Moreover, there exists a strictly increasing sequence \(\{t_j\} \subset [0, +\infty)\) such that \(\varphi(t_j) \xrightarrow{C^\infty} \varphi_\infty \in \Omega^3_+(M)\), and the limiting \(G_2\)-structure is harmonic, which is equivalent to:

\[
\text{div} \, T_\infty = 0.
\]

Bounded torsion is a reasonable assumption for short time, in view of the growth estimate of Theorem 2–(i), and indeed a necessary and sufficient condition for long-time existence, by Theorem 2–(ii), cf. [23, Theorem 5.4], and [12, Theorem 3.8].

**Afterword: further developments**

We sketch some interesting outcomes of the proposed theory of harmonic section flows, to be addressed in subsequent articles, besides of course working out the particular manifestations of (HSF) for the reader’s favourite normal reductive pair \(H \subset G\).

\(\epsilon\)-regularity: One important set of questions regarding sufficient conditions for long-term existence and convergence remains untreated in the present paper, namely whether a small ‘entropy’ condition suffices to ensure eventually \(C^0\)-bounded torsion and hence convergence to a harmonic (or even torsion-free) limit. Comparing for instance with what is known for \(G_2\)-structures, e.g. [23, Theorems 6.1 & 7.5] and [12, Theorems 5.3 & 5.7], the (HSF) then manifests itself as the isometric ‘\(\text{div} \, T\)-flow’ and the methods from [5] can be adapted to that specific context. Very similar results can be found also for almost complex structures in [25, 27]. But we now see that these flows are closely related to the (HMHF) for \(G\)-equivariant sections \(s_t\) under the isometry \(\mu\), so analogous formulations of monotonicity and \(\epsilon\)-regularity should hold in general for equivariant maps \(s_t\) onto say homogeneous spaces. One should then be able to study the formation of singularities, generalising, for example, [12, §6]. This next step will require a theory along the lines of [8] for equivariant maps, which, to our best knowledge, has not yet been developed in the literature.

**Harmonic homogeneous geometric structures** In view of Remark 25, in the particular context of homogeneous geometric structures on \(M = K/L\), Corollary 38 gives somewhat automatically the long-time existence and regularity of the flow. This could lead to an interesting classification programme based on the Gromov–Hausdorff limits by [31, 32], especially in cases in which the existence of (non-flat) torsion-free geometric structures is known to be obstructed.
For instance, in the particular context of homogeneous $G_2$-structures, torsion-free structures would be Ricci-flat and hence downright flat. On the other hand, coclosed structures are always harmonic, and compact homogeneous 7-manifolds admitting such structures have been completely classified, independently by [33, 42]. A finer question would then be whether these spaces admit homogeneous harmonic $G_2$-structures which are not coclosed, which could be addressed by examining closely the set $\text{Crit}(E)$. A complementary question would be whether those spaces which do not possess homogeneous coclosed structures admit any harmonic ones, and hence could be classified by the limits of the harmonic section flow.

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