QCD AND EXPERIMENT ON MULTIPLICITY DISTRIBUTIONS

I.M. DREMIN
Lebedev Physical Institute, Moscow 117924, Russia

Abstract

The solution of QCD equations for generating functions of parton multiplicity distributions reveals new peculiar features of cumulant moments oscillating as functions of their rank. It happens that experimental data on hadron multiplicity distributions in $e^+e^-$, $hh$, $AA$ collisions possess the similar features. However, the "more regular" models like $\lambda\phi^3$ behave in a different way. Evolution of the moments at smaller phase space bins and zeros of the truncated generating functions are briefly discussed.

1 Introduction and main results

Since the space-time (space in proceedings and time at oral presentation) is very limited I present from the beginning both short review of the problem and the main results obtained, leaving their justification and brief discussion for the next section. Those interested in more detailed description should use the list of references for further reading (in particular, the review paper [1]).

I would like to stress that QCD predicts the distributions of partons (quarks and gluons) while in experiment one gets the distributions of final hadrons. Therefore no quantitative comparison has been attempted. However, the qualitative features of both distributions are so spectacular and remind each other that one is tempted to confirm once again that QCD is a powerful tool for predicting new features of hadron distributions as well.

For a long time, the phenomenological approach dominated in description of multiplicity distributions in multiparticle production [2]. The very first attempts to apply QCD formalism to the problem failed because in the simplest double-logarithmic approximation it predicts an extremely wide shape of the distribution [3] that contradicts to experimental data. Only recently it became possible [4] to get exact solutions of QCD equations for the generating functions of multiplicity distributions which revealed much
narrower shapes and such a novel feature of cumulant moments as their oscil-
lations at higher ranks [5],[6]. The similar oscillations have been found in ex-
periment for the moments of hadron distributions [7]. Their pattern differs
drastically from those of the popular phenomenological distributions [11] and of the "non-singular" $\lambda\phi^3$-model [8].

These findings have several important implications [1]. They show that:

1. the QCD distribution belongs to the class of non-infinitely-divisible ones.

Two corollaries of this statement follow immediately:

a. The Poissonian cluster models (e.g., the multiperipheral cluster model) are ruled out by QCD.

b. The negative binomial distribution (so popular nowadays in phenomeno-
logical fits) can not be valid asymptotically.

2. the new expansion parameter appears in description of multiparticle pro-
cesses.

Since this parameter becomes large when large number of particles are in-
volved, it asks for the search of some collective effects and of more convenient
basis than the common particle number representation.

QCD is also successful in qualitative description of evolution of multiplicity distributions with decreasing phase space bins which gives rise to notions of intermittency and fractality [1],[10],[11]. However, there are some new problems with locations of the minimum of cumulants at small bins [12],[13].

The experimentally defined truncated generating functions possess an intrigu-
ing pattern of zeros in the complex plane of an auxiliary variable [13],
[14],[15]. It recalls the pattern of Lee-Yang zeros of the grand canonical partition function in the complex fugacity plane related to phase transition [16],[17] and asks for some collective effects to be searched for [18],[19].

2 Technicalities

Let us define the multiplicity distribution

$$P_n = \sigma_n / \sum_{n=0}^{\infty} \sigma_n,$$  \hspace{1cm} (1)
where $\sigma_n$ is the cross section of $n$-particle production processes, and the generating function

$$G(z) = \sum_{n=0}^{\infty} P_n (1 + z)^n.$$  \hspace{1cm} (2)

The (normalized) factorial and cumulant moments of the $P_n$ distribution are

$$F_q = \frac{\sum_n P_n n(n-1)\ldots(n-q+1)}{(\sum_n P_n)^q} = \frac{1}{\langle n \rangle^q} \frac{d^q G(z)}{dz^q} \bigg|_{z=0},$$  \hspace{1cm} (3)

$$K_q = \frac{1}{\langle n \rangle^q} \frac{d^q \ln G(z)}{dz^q} \bigg|_{z=0},$$  \hspace{1cm} (4)

where $\langle n \rangle = \sum_n P_n n$ is the average multiplicity. They describe full and genuine $q$-particle correlations, correspondingly. Let us point out here that the moments are defined by the derivatives at the origin and are very sensitive to any nearby singularity of the generating function.

First, let us consider QCD without quarks, i.e. gluodynamics. The generating function of the gluon multiplicity distribution in the full phase-space volume satisfies the equation

$$\frac{\partial G(z, Y)}{\partial Y} = \int_0^1 dx K(x) \gamma_0^2 [G(z, Y + \ln x)G(z, Y + \ln(1 - x)) - G(z, Y)].$$  \hspace{1cm} (5)

Here $Y = \ln(p\theta/Q_0)$, $p$ is the initial momentum, $\theta$ is the angular width of the gluon jet considered, $p\theta \equiv Q$ where $Q$ is the jet virtuality, $Q_0 =$const,

$$\gamma_0^2 = \frac{6\alpha_S(Q)}{\pi},$$  \hspace{1cm} (6)

$\alpha_S$ is the running coupling constant, and the kernel of the equation is

$$K(x) = \frac{1}{x} - (1 - x)[2 - x(1 - x)].$$  \hspace{1cm} (7)

It is the non-linear integro-differential equation with shifted arguments in the non-linear part which take into account the conservation laws. , and with the initial condition

$$G(z, Y = 0) = 1 + z,$$  \hspace{1cm} (8)

and the normalization

$$G(z = 0, Y) = 1.$$  \hspace{1cm} (9)
The condition (9) normalizes the total probability to 1, and the condition (8) declares that there is a single particle at the very initial stage.

After Taylor series expansion at large enough $Y$ and differentiation in eq. (5), one gets the differential equation

$$
(\ln G(Y))'' = \gamma_0^2 [G(Y) - 1 - 2h_1 G'(Y) + h_2 G''(Y)],
$$

where $h_1 = 11/24; h_2 = (67 - 6\pi^2)/36 \approx 0.216$, and higher order terms have been omitted.

Leaving two terms on the right-hand side, one gets the well-known equation of the double-logarithmic approximation which takes into account the most singular components. The next term, with $h_1$, corresponds to the modified leading-logarithm approximation, and the term with $h_2$ deals with next-to-leading corrections.

The straightforward solution of this equation looks very problematic. However, it is very simple for the moments of the distribution because $G(z)$ and $\ln G(z)$ are the generating functions of $F_q$ and $K_q$, correspondingly, according to (3), (4). Using this fact, one gets the solution which looks like

$$
H_q = \frac{K_q}{F_q} = \frac{\gamma_0^2 [1 - 2h_1 q \gamma + h_2 (q^2 \gamma^2 + q \gamma')]}{q^2 \gamma^2 + q \gamma'},
$$

where the anomalous dimension $\gamma$ is related to $\gamma_0$ by

$$
\gamma = \gamma_0 - \frac{1}{2} h_1 \gamma_0^2 + \frac{1}{8} (4h_2 - h_1^2) \gamma_0^3 + O(\gamma_0^4).
$$

The formula (11) shows how the ratio $H_q$ behaves in different approximations. In double-log approximation, where $h_1 = h_2 = 0$, it monotonously decreases as $q^{-2}$ that corresponds to the negative binomial law with its parameter $k = 2$ i.e. to very wide distribution. Let us note that it owes to the singular part of the kernel and is absent on more regular theories like $\lambda \phi^3$.

In modified-log approximation ($h_2 = 0$) it acquires a negative minimum at

$$
q_{\min} = \frac{1}{h_1 \gamma_0} + \frac{1}{2} + O(\gamma_0) \approx 5
$$

and approaches the abscissa axis from below asymptotically at large ranks $q$. In the next approximation given by (11) it preserves the minimum location but approaches a positive constant crossing the abscissa axis. In ever
higher orders it reveals the quasi-oscillatory behavior about this axis. This prediction of the minimum at \( q \approx 5 \) and subsequent specific oscillations is the main theoretical outcome.

It is interesting to note that the equation (5) can be solved exactly in the case of fixed coupling constant \( \beta \). All the above qualitative features are noticeable here as well.

While the above results are valid for gluon distributions in gluon jets (and pertain to QCD with quarks taken into account \([1]\)), the similar qualitative features characterize the multiplicity distributions of hadrons in high energy reactions initiated by various particles. The numerous demonstration of it can be found in the review paper \([1]\).

The multiplicity distributions can be measured not only in the total phase space (as has been discussed above for very large phase-space volumes) but in any part of it. For the homogeneous distribution of particles within the volume, the average multiplicity is proportional to the volume and decreases for small volumes but the fluctuations increase. The most interesting problem here is the law governing the growth of fluctuations and its possible departure from a purely statistical behavior related to the decrease of the average multiplicity. Such a variation has to be connected with the dynamics of the interactions. In particular, it has been proposed to look for the power-law behavior of the factorial moments for small rapidity intervals \( \delta y \)

\[
F_q \propto (\delta y)^{-\phi(q)} \quad (\phi(q) > 0) \quad (\delta y \to 0),
\]

inspired by the idea of intermittency in turbulence. In the case of statistical fluctuations with purely Poissonian behavior, the intermittency indices \( \phi(q) \) are identically equal to zero.

Experimental data on various processes in a wide energy range support this idea, and QCD provides a good basis for its explanation as a result of parton showers.

At moderately small rapidity windows, one can get in the double-log approximation the power-law behavior with

\[
\phi(q) = D(q - 1) - \frac{q^2 - 1}{q} \gamma_0.
\]

The running property of QCD coupling constant is not important in that region. This property becomes noticeable at ever smaller windows, where
(e.g., at \(q=2\)) \(\ln \delta y_0 / \delta y > \alpha_s^{-1}\), and leads to smaller numerical values of \(\phi(q)\) compared to (15). The general trends in this region decline somewhat from the simple power law (14) due to logarithmic corrections. Qualitatively, these predictions correspond to experimental findings at relatively small ranks \(q\) where the steep increase in the region of \(\delta y > 1\) on the log-log plot of the dependence (14) is replaced by slower one at smaller intervals \(\delta y\). The transition point between the two regimes depends on the rank in qualitative agreement with QCD predictions also. Namely, the transition happens at smaller bins for higher ranks. These findings can be interpreted as an indication on fractal structure of particle distributions within the available phase space. When interpreted in terms of fluctuations, they show that the fluctuations become stronger in small phase-space regions in a definite power-like manner and, surely, exceed trivial statistical fluctuations.

Let us turn now to the \(q\)-behavior of moments at small bins. The phenomenon of the oscillations of cumulants discussed above reveals itself here as well if one goes beyond the double-log approximation of (15). In terms of factorial moments, it means the non-monotonous behavior of the intermittency indices as functions of \(q\). (Compare it to the steady increase with \(q\) at \(q > 1\) given by (15).) It gives rise to the negative values of \(K_q\) and \(H_q\).

The fate of the first minimum can be easily guessed from the formula (13). For large enough virtualities (i.e. small \(\gamma_0\)), the minimum location moves to higher values of rank \(q\) for jets with larger virtuality \(Q\) since the QCD coupling constant is running as \(\ln^{-1} Q\). Therefore, the predicted shift of the minimum is

\[
q_{\text{min}} \propto \ln^{1/2} Q. \tag{16}
\]

It follows that \(q_{\text{min}}\) moves to higher ranks at higher energies because more massive jets become available. Another corollary is that it should shift to smaller values of \(q\) for smaller bins at fixed energy.

While former statement finds some support in experiment, the second one does not look to be true. On the contrary, the minimum appears at higher ranks for smaller bins. There is no solution of this problem yet but it should be ascribed to the higher-order effects. Actually, one can guess that the higher order terms shown as \(O(\gamma_0)\) in (13) become so important at small bins that they overpower the weak \(Q\)-dependence of \(\gamma_0^{-1}\) in the first term of (13). It is important to stress here that at large rapidity intervals the modified leading-log term with \(h_2\) does not influence the value of \(q_{\text{min}}\), and
only increases the value of $H_q$ by $h_2\gamma_0^2$. Thus, the next-to-leading corrections should be in charge of the additional shift of $q_{min}$, and, therefore, small bins help us look into higher orders of QCD.

There is another fascinating feature of multiplicity distributions – it happens that zeros of the truncated (sum in $n$ runs up to $n = N_{max}$) generating function form a spectacular pattern in the complex plane of the variable $z$. Namely, they seem to lie close to a single circle. At enlarged values of $N_{max}$ they move closer to the real axis pinching it at some positive value of $z$.

No QCD interpretation of the fact exists because it is hard to exploit the finite cut-off in analytic calculations. The interest to it stems from the analogy to the locations of zeros of the grand canonical partition function as described by Lee and Yang who related them to possible phase transitions in statistical mechanics. In that case, $z$ variable plays the role of fugacity, and pinching of the real axis implies existence of two phases in the system considered.

In particle physics, it shows up the location of the singularity of the generating function i.e. the number of zeros of truncated generating functions increases and they tend to move to the singularity point when $N_{max} \to \infty$. Since it happens to lie close to the origin, it drastically influences the behavior of moments (see (3), (4)), and, therefore, determines the shape of the distribution. The study of the singularities is at the very early stage now, and one can only say that the singularity is positioned closer to the origin in nucleus-nucleus collisions and it is farthest in $e^+e^-$ that appeals to our intuitive guess.

To conclude, I would like to stress that, once again, QCD demonstrates its power in predicting new features of particle distributions when dealing with parton distributions.

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