Observable adjustments in single-index models for regularized M-estimators with bounded p/n

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Abstract: We consider observations \((\mathbf{X}, y)\) from single index models with unknown link function, Gaussian covariates and a regularized M-estimator \(\hat{\beta}\) constructed from convex loss function and regularizer. In the regime where sample size n and dimension p are both increasing such that \(p/n\) has a finite limit, the behavior of the empirical distribution of \(\hat{\beta}\) and the predicted values \(X\hat{\beta}\) has been previously characterized in a number of models: The empirical distributions are known to converge to proximal operators of the loss and penalty in a related Gaussian sequence model, which captures the interplay between ratio \(\frac{p}{n}\), loss, regularization and the data generating process. This connection between \((\hat{\beta}, X\hat{\beta})\) and the corresponding proximal operators requires solving nonlinear system of equations that typically involve unobservable quantities such as the prior distribution on the index or the link function.

This paper develops a different theory to describe the empirical distribution of \(\hat{\beta}\) and \(X\hat{\beta}\). Approximations of \((\hat{\beta}, X\hat{\beta})\) in terms of proximal operators are provided that only involve observable adjustments. These proposed observable adjustments are data-driven, e.g., do not require prior knowledge of the index or the link function. These new adjustments yield confidence intervals for the index or the link function, as well as estimators of the correlation of \(\hat{\beta}\) with the index, enabling parameter tuning to maximize the correlation. The interplay between loss, regularization and the model is captured in a data-driven manner, without solving the nonlinear systems studied in previous works. The results are proved to hold both strongly convex regularizers and unregularized M-estimation.

1. Introduction

Consider iid observations \((\mathbf{x}_i, y_i)\) with Gaussian feature vectors \(\mathbf{x}_i \sim N(0, \Sigma)\), \(\Sigma \in \mathbb{R}^{p \times p}\) and response \(y_i\) valued in a set \(\mathcal{Y}\) following a single index model

\[ y_i = F(x_i^T \mathbf{w}, U_i) \tag{1.1} \]

where \(F : \mathbb{R}^2 \to \mathbb{R}\) is an unknown deterministic function, \(\mathbf{w} \in \mathbb{R}^p\) an unknown index, and \(U_i\) is a latent variable independent of \(\mathbf{x}_i\). The index \(\mathbf{w}\) is normalized with \(\text{Var}(x_i^T \mathbf{w}) = \|\Sigma^{1/2} \mathbf{w}\|^2 = 1\) by convention, since \(\|\Sigma^{1/2} \mathbf{w}\|\) can be otherwise absorbed into \(F(\cdot)\). The triples \((\mathbf{x}_i, U_i, y_i)\) are iid but only \((\mathbf{x}_i, y_i)\) are observed. Typical examples that we have in mind for \(F\) and \(U_i\) in (1.1) include

- **Linear regression**: \(F(v, u) = \|\Sigma^{1/2} \beta^* \| v + u\) for some \(\beta^* \in \mathbb{R}^p\), \(U_i \sim N(0, \sigma^2)\) and \(\mathbf{w} = \beta^*/\|\Sigma^{1/2} \beta^*\|\). Equivalently, \(y_i | x_i \sim N(x_i^T \beta^*, \sigma^2)\).
- **Logistic regression**: \(F(v, u) = 1\) if \(u \leq 1/(1 + e^{-\|\Sigma^{1/2} \beta^*\| v})\) and \(0\) otherwise for some \(\beta^* \in \mathbb{R}^p\). \(U_i \sim \text{Unif}[0, 1]\) and \(\mathbf{w} = \beta^*/\|\Sigma^{1/2} \beta^*\|\). Equivalently, \(y_i | x_i \sim \text{Bernoulli}(\rho'(x_i^T \beta^*))\) where \(\rho'(u) = 1/(1 + e^{-u})\) is the sigmoid function.
- **1-bit compressed sensing**: \(F(v, u) = \text{sign}(v)\) so that \(y_i = \text{sign}(x_i^T \mathbf{w})\), or 1-bit compressed sensing with \(\epsilon\)-contamination: \(F(v, u) = \text{usign}(v)\) for \(U_i \in \{\pm 1\}\) s.t. \(\mathbb{P}(U_i = -1) = \epsilon\).
- **Binomial logistic regression**: \(y_i | x_i \sim \text{Binomial}(q, \rho'(x_i^T \beta^*))\) for some integer \(q \geq 1\) and \(\rho'(u) = 1/(1 + e^{-u})\) the sigmoid function.

Throughout the paper, \(\hat{\beta}\) is a regularized M-estimator of the form

\[ \hat{\beta}(y, X) = \arg\min_{\beta \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^{n} \ell(y_i, (x_i^T \beta) + g(\beta)) \tag{1.2} \]
where $g : \mathbb{R}^p \to \mathbb{R}$ is a convex penalty function and for any $y_0 \in \mathcal{Y}$, the map $\ell_{y_0} : \mathbb{R} \to \mathbb{R}$, $t \mapsto \ell_{y_0}(t)$ is a convex loss function. For a fixed $y_0$, the derivatives of $\ell_{y_0}$ are denoted by $\ell'_y(t)$ and $\ell''_y(t)$ where these derivatives exist. If $y_i \in \{-1, 1\}$ or $y_i \in \{0, 1\}$ as in binary classification (e.g., with $y_i | x_i$ following a logistic regression or 1-bit compressed sensing model), a popular loss function is for instance the logistic loss $\ell_{y_i}(t) = \log(1 + e^{-t})$ if $y_i = 1$ and $\ell_{y_i}(t) = \log(1 + e^t)$ if $y_i \neq 1$. This paper focuses on the behavior of estimators of the form (1.2) in the single index model (1.1) when the dimensions and sample sizes are both large and of the same order, confidence intervals for individual components of the index $\mathbf{w}$, and estimation of performance metrics of interest such as the correlation between $\hat{\boldsymbol{\beta}}$ and the unknown index $\mathbf{w}$.

### 1.1. Prior works in asymptotic behavior of M-estimators

There is now a rich literature on the behavior of M-estimators in the regime where $p/n$ converges to a constant in linear models [2, 23, 37, 40, 41, 22, 29, 15, 33, 30, among others] and generalized models, including logistic regression [38, 34] and general teacher-student models [26]. To present typical results from this literature, assume in this section that $n/p \to +\infty$ with $n/p \to \delta$ for some constant $\delta > 0$ and isotropic covariance matrix $\Sigma = \frac{1}{p} I_p$.

Consider first a linear model with $y_i = x_i^T \beta^* + \epsilon_i$ for $\epsilon_i$ independent of $x_i$ and the unregularized estimator $\hat{\beta} = \arg\min_{b \in \mathbb{R}^p} \sum_{i=1}^n \ell(y_i - x_i^T b)$ for some convex loss $\ell : \mathbb{R} \to \mathbb{R}$. This corresponds to $g = 0$ and $\ell_{y_i}(u) = \ell(y_i - u)$ in (1.2). The works [23, 20, 25, 22] showed that the behavior of $\hat{\beta}$ is characterized by the system of two equations

$$
\begin{align*}
\delta^{-1} \lambda^2 &= \mathbb{E}[\|\prox_{\gamma \ell}(\epsilon_1 + \sigma Z) - \epsilon_1 - \sigma Z\|^2], \\
1 - \delta^{-1} &= \mathbb{E}[\|\prox_{\gamma \ell}(\epsilon_1 + \sigma Z)\|],
\end{align*}
$$

with two unknowns ($\sigma, \gamma$), where $Z \sim N(0, 1)$ is independent of $\epsilon_1$. In (1.3) and throughout the paper, for any convex function $f : \mathbb{R} \to \mathbb{R}$, the proximal operator $\prox(f)$ of $f$ is defined as $\prox(f)(u) = \arg\min_{u \in \mathbb{R}} (u - v)^2/2 + f(v)$, and we denote by $\prox(f)'$ its derivative. While [25] uses notation $(\tau, c)$ to reveal the connection with the larger systems (1.6) and (1.7) below. The solution $(\hat{\sigma}, \hat{\gamma})$ to (1.3) is such that $p^{-1}\|\hat{\beta} - \beta^*\|^2 \to \sigma^2$ in probability, and for each fixed component $j = 1, \ldots, p$ the convergence in distribution $\hat{\beta}_j - \beta^*_j \to^d N(0, \sigma^2)$ holds [23], so that the system (1.3) and its solution captures the asymptotic distribution of $\hat{\beta}$. Equipped with the system (1.3) and these results, [4] studied the optimal loss function $\ell(\cdot)$ that minimizes $\|\hat{\beta} - \beta^*\|^2$ for a given $\delta = \lim_{p \to \infty} \frac{p}{n}$ and given noise distribution for $(\epsilon_1, \ldots, \epsilon_n)$. In linear model with normally distributed noise, Bayati and Montanari [2] used Approximate Message Passing to establish a similar phenomenon for the Lasso $\hat{\beta} = \arg\min_{b \in \mathbb{R}^p} \|y - X\hat{\beta}\|^2/2 + \lambda\|b\|_1$, showing that the empirical distribution of $(\hat{\beta}_j)_{j=1,\ldots,p}$ is close in distribution to the empirical distribution of $(\eta(\hat{\beta}_j + \tau Z_j; \lambda \frac{Z_j}{E})_{j=1,\ldots,p}$ where $\eta(x; u) = \text{sign}(x)(|x| - u)_+$ is the soft-thresholding operator and $(\hat{\tau}, \hat{\beta})$ is solution to a nonlinear system of two equations of a similar nature as (1.3) [2, Theorem 1.5]; the works [30, 33, 16] provides explicit error bounds between functions of the Lasso $\hat{\beta}$ and their prediction from the nonlinear system of two equations. Inspired by early works from Stojnic [37], Thrampoulidis et al. [41] developed the Convex Gaussian Min-max Theorem and obtained analogous systems of equations to characterize the limit of $p^{-1}\|\hat{\beta} - \beta^*\|^2$ for a given loss-penalty pair in linear models, see for instance the system with four unknowns [41, Eq. (15)] for separable loss and penalty.

With the model (1.1), our focus in this paper is on single-index models where $\mathbb{E}[y_i | x_i]$ is topically not a linear function of $x_i$. To present some representative existing results of a similar nature as those in the previous paragraph, we now turn to logistic regression, a particular case of the single model (1.1). Consider iid observations from the logistic model

$$
y_i | x_i \sim \text{Bernoulli}(\rho'(x_i^T \beta^*)) \quad \text{for } \rho'(u) = 1/(1 + e^{-u}),
$$

and the M-estimator (1.2) with the logistic loss $\ell_{y_i}(u_i) = \rho(u_i) - y_i u_i$ for $\rho(u) = \log(1 + e^u)$. With no regularization, i.e., $g = 0$, $\hat{\beta}$ in (1.2) is the logistic Maximum Likelihood Estimate (MLE). Sur and Candès
describe the asymptotic distribution of the MLE as follows. For a sequence of logistic regression problems with \( n, p, \beta^* \) such that \( \| \beta^* \|^2/p \to \kappa^2 \) and \( n/p \to \delta > 0 \), the MLE exists with overwhelming probability if \( \delta > \min_{t \in \mathbb{R}} \int \left( f(z - tu) \right) \varphi(z)2\varphi'((kv)\varphi(v))dv \) [14] where \( \varphi(z) = \left( 2\pi \right)^{-1/2} \exp(-z^2/2) \) is the standard normal pdf. In this case, assuming additionally that the components of \( \beta^* \) are iid copies of a random variable \( \beta \), then for any Lipschitz function \( \phi : \mathbb{R}^2 \to \mathbb{R} \)

\[
\frac{1}{p} \sum_{j=1}^{p} \phi \left( \hat{\beta}_j - \bar{\alpha}\beta_j^*, \beta_j^* \right) \to^P \mathbb{E} \left[ \phi \left( \sigma Z, \beta \right) \right] \tag{1.5}
\]

where \( Z \sim N(0, 1) \) is independent of \( \beta \), the arrow \( \to^P \) denotes convergence in probability, and \((\bar{\alpha}, \sigma, \gamma)\) is solution of the nonlinear system of 3 equations

\[
\begin{align*}
\delta^{-1}\sigma^2 &= 2\mathbb{E} \left[ \rho'(-\kappa Z_1)(\gamma \rho'(\text{prox}[\gamma \rho](\kappa \alpha Z_1 + \sigma Z_2)))^2 \right], \\
0 &= 2\mathbb{E} \left[ \rho'(-\kappa Z_1)\gamma \rho'(\text{prox}[\gamma \rho](\kappa \alpha Z_1 + \sigma Z_2)))\right], \\
1 - \delta^{-1} &= 2\mathbb{E} \left[ \rho'(-\kappa Z_1)\text{prox}[\gamma \rho](\kappa \alpha Z_1 + \sigma Z_2)) \right].
\end{align*}
\tag{1.6}
\]

Above, \( \rho'(u) = 1/(1 + e^{-u}) \) is the sigmoid function as in (1.4), \( \rho''(u) = \rho'(u)(1 - \rho'(u)) \) its derivative, and inside the expectation in the third line \( \text{prox}[\gamma \rho](u) = 1/(1 + \gamma \rho(\text{prox}[\gamma \rho](u))) \).

Salehi et al. [34] extended such results to the M-estimator (1.2) constructed with the same logistic loss, \( \ell_{y_i}(u) = \rho(u_i) - y_i u_i \) for \( \rho(u) = \log(1 + e^u) \), and separable penalty function of the form \( g(b) = \frac{1}{2} \sum_{j=1}^{p} \tilde{f}(b_j) \) for some convex \( \tilde{f} : \mathbb{R} \to \mathbb{R} \). To describe the main result of [34], assume that the coefficients of \( \beta^* \) are iid copies of a random variables \( \beta \) with finite variance, and let \( (Z, Z_1, Z_2)^T \sim N(0, I_3) \) be independent of \( \beta \). Let \( \kappa^2 = \mathbb{E}[\beta^2] \) and consider the system of 6 equations

\[
\begin{align*}
\kappa^2 \alpha &= \mathbb{E} \left[ \beta \text{prox} [\kappa \alpha Z_1 + \sigma Z_2] \right], \\
\sqrt{\delta \gamma} &= \mathbb{E} \left[ Z \text{prox} [\kappa \alpha Z_1 + \sigma Z_2] \right], \\
\kappa^2 \alpha^2 + \sigma^2 &= \mathbb{E} \left\{ \left[ \text{prox} [\kappa \alpha Z_1 + \sigma Z_2] \right] \right\}^2, \\
r^2 \gamma^2 &= 2\mathbb{E} \left[ \rho'(-\kappa Z_1)(\kappa \alpha Z_1 + \sigma Z_2 - \text{prox}[\gamma \rho](\kappa \alpha Z_1 + \sigma Z_2))^2 \right], \\
-\theta \gamma &= 2\mathbb{E} \left[ \rho''(-\kappa Z_1)\text{prox}[\gamma \rho](\kappa \alpha Z_1 + \sigma Z_2) \right], \\
1 - \gamma/(\sigma \theta) &= 2\mathbb{E} \left[ \rho'(-\kappa Z_1)\text{prox}[\gamma \rho](\kappa \alpha Z_1 + \sigma Z_2) \right].
\end{align*}
\tag{1.7}
\]

with unknowns \((\alpha, \sigma, \gamma, \theta, \tau, r)\). As in the case of the MLE in [38] with (1.6), the system (1.8) captures the interplay between the logistic model (1.4), the penalty and the limit \( \delta \) of the ratio \( \alpha/n \). The main result of [34] states that if the nonlinear system (1.7) has a unique solution \((\bar{\alpha}, \bar{\sigma}, \bar{\gamma}, \bar{\theta}, \bar{\tau}, \bar{r})\) then for any locally Lipschitz function \( \Phi : \mathbb{R}^2 \to \mathbb{R} \)

\[
\frac{1}{p} \sum_{j=1}^{p} \Phi \left( \hat{\beta}_j, \beta_j^* \right) \to^P \mathbb{E} \left[ \Phi \left( \text{prox} [\bar{\sigma} \bar{\tau} \bar{f}(\cdot)], \bar{\sigma} \bar{\tau} (\bar{\theta} \bar{\beta}_j^* + \delta^{-1/2} \bar{r} Z_j) \right), \beta \right] \tag{1.8}
\]

as \( n, p \to \infty \) with \( n/p \to \delta \). An informal interpretation of (1.8) is the approximation

\[
\hat{\beta}_j \approx \text{prox} [\bar{\sigma} \bar{\tau} \bar{f}(\cdot), \bar{\sigma} \bar{\tau} (\bar{\theta} \bar{\beta}_j^* + \delta^{-1/2} \bar{r} Z_j)], \tag{1.9}
\]

where \( Z_j \sim N(0, 1) \), and (1.9) holds in an averaged sense over \( j = 1, \ldots, p \). This approximation means that in order to understand \( \hat{\beta} \), it is sufficient to understand the random vector with independent components \( \hat{b}_j = \text{prox} [\bar{\sigma} \bar{\tau} \bar{f}(\cdot), \bar{\sigma} \bar{\tau} y_j^{seq}] \) in the Gaussian sequence model \( y_j^{seq} \sim N(\bar{\theta} \bar{\beta}_j^*, \bar{\sigma}^2) \), \( j = 1, \ldots, p \). The works [24, 26] further extend these results to loss and penalty functions that need not be separable, provide a unified theory for the systems (1.3), (1.5) and (1.7), and describe the relationship of these results with predictions from the replica method in statistical physics.
The above system (1.7) may reduce to simpler forms for specific penalty functions. With Ridge penalty \( g(b) = \frac{\Delta}{2p} \|b\|^2 \), in the isotropic setting with covariance \( \Sigma = \frac{1}{p} I_p \), the system (1.7) reduces to

\[
\begin{cases}
\delta^{-1} \sigma^2 = 2E[\rho'(\kappa Z_1)(\kappa \alpha Z_1 + \sigma Z_2 - \text{prox}[\gamma \rho](\kappa \alpha Z_1 + \sigma Z_2))^2], \\
-\delta^{-1} \alpha = 2E[\rho'(-\kappa Z_1)\text{prox}[\gamma \rho](\kappa \alpha Z_1 + \sigma Z_2)], \\
1 - \delta^{-1} + \lambda \gamma = 2E[\rho'(-\kappa Z_1)\text{prox}[\gamma \rho](\kappa \alpha Z_1 + \sigma Z_2)],
\end{cases}
\]

(1.10)

see [34, Eq. (14), (16)]. By integration by parts, (1.10) is equivalent to (1.6) when \( \lambda = 0 \), and the convergence (1.8) with \( \Phi(u, v) = \phi(u - \alpha v, v) \) reduces to (1.5) due to the simple form of the proximal operator \( \text{prox}[f](x) = \frac{1}{1 + \alpha f(x)} \) where \( f(u) = \frac{1}{2} u^2 \).

As explained in [38, 34] among others, the systems (1.6), (1.7) and (1.10) combined with the asymptotic results (1.5) and (1.8) are powerful tools to analyze various characteristics of the M-estimator \( \hat{\beta} \), for instance the correlation \( p^{-1} \beta^T \beta^* \) with the true regression vector \( \beta^* \) by using \( \Phi(a, b) = ab \) in (1.8), giving \( p^{-1} \beta^T \beta^* \rightarrow^p \alpha \kappa^2 \) by the first line in (1.7). The Mean Squared Error (MSE) \( p^{-1} \|\hat{\beta} - \beta^*\|^2 \) is analogously characterized using \( \Phi(a, b) = (a - b)^2 \) in (1.8). If the penalty is of the form \( g = \lambda g_0 \) for some tuning parameter \( \lambda > 0 \), solving the above nonlinear systems for given \( \lambda, \delta \) provide curves of performance metrics of interest (e.g., correlation \( p^{-1} \beta^T \beta^* \) or MSE \( p^{-1} \|\hat{\beta} - \beta^*\|^2 \)) as a function of the tuning parameter and the limit \( \delta \) of \( n/p \) [34]. Plotting such curves by computing the solutions \((\tilde{\alpha}, \tilde{\sigma}, \gamma, \tilde{\theta}, \tilde{\tau}, \tilde{r})\) of (1.7) require the knowledge of the distribution \( \beta \) of the components of \( \beta^* \), or in the case of (1.6) and (1.10) the second moment \( \kappa^2 = E[\beta^2] \approx p^{-1}\|\beta^*\|^2 \). Solving the system also requires the knowledge of the single index model: If the Bernoulli parameter in (1.4) is of the form \( \tilde{\rho}(x_j^T \beta^*) \) for \( \tilde{\rho} : \mathbb{R} \rightarrow [0, 1] \) different than the sigmoid function \( \rho \), the systems (1.6) and (1.7) require an appropriate modification, and solving the new system requires the knowledge of \( \tilde{\rho} \) (see for instance the system in [26, Section B.8]).

In practice, \( \kappa^2 = E[1/p] \|\beta^*\|^2 \) and more generally the law of \( \beta \) are typically unknown. In this case the above results are not readily useful since \((\tilde{\alpha}, \tilde{\sigma}, \gamma, \tilde{\theta}, \tilde{\tau}, \tilde{r})\) cannot be computed only by looking at the data \((y, X)\). To illustrate this, given two loss-penalty pairs \((\ell, g)\) and \((\ell, \tilde{g})\), the practitioner may wish to pick the loss-penalty pair such that the resulting M-estimator in (1.2) has larger correlation with the true regression vector \( \beta^* \in \mathbb{R}^p \) in (1.4). With this application in mind, the above theory suggests to solve the system (1.7) for each loss-penalty pair and to pick the pair with the largest \( \tilde{\alpha} \) since \( \tilde{\alpha} \) is the limit of \( \beta^T \beta^* / \|\beta^*\|^2 \). This is feasible only if the law of \( \beta \) in (1.7) is known, or only if \( \kappa^2 \) is known in (1.5) and (1.10), as by rotational invariance the systems for the MLE and Ridge regularization only depend on the law of \( \beta \) through the second moment. The drawback that solving the above systems require typically unknown quantities is also present in linear models: (1.3) requires the knowledge of the noise distribution and the systems in (2, 41) additionally require the knowledge of the prior distribution \( \beta \) on the components of \( \beta^* \). Estimating scalar solutions of the above system is possible in certain cases, including in linear models [3, 30, 5, 16] [22, Proposition 2.7] and unregularized logistic regression [38, 45], but a general recipe in single index models is lacking.

For the MLE, (1.5)-(1.6) also provide confidence intervals for components \( \beta_j^* \) [38, 48]. Let \( z_{\alpha/2} > 0 \) be such that \( \mathbb{P}(\{N(0, 1) > z_{\alpha/2}\} = \alpha \). The result (1.5) applied to a smooth approximation of the indicator function \( \phi(a, b) = I_{\{a \leq \tilde{\sigma} z_{\alpha/2}\}} \) yields that approximately \((1 - \alpha)p \) covariates \( j = 1, ..., p \) are such that

\[
\beta_j^* \in \frac{1}{n}[\hat{\beta}_j - \tilde{\sigma} z_{\alpha/2}, \hat{\beta}_j + \tilde{\sigma} z_{\alpha/2}].
\]

This provides a confidence interval in an average sense. Zhao et al. [48] later proved that the same confidence interval is valid not only in this averaged sense, but also for a fixed, given component \( j = 1, ..., p \) under conditions on the amplitude of \( \beta_j^* \). As in the previous paragraph, such confidence interval can be constructed provided that \( \kappa^2 \) is known so that the solution \((\tilde{\alpha}, \tilde{\sigma}, \gamma)\) to the system (1.6) can be computed. This motivated the ProbeFrontier [38] and SLOE [45] procedures to compute approximations of \( \kappa^2 \) and of the solutions \((\tilde{\alpha}, \tilde{\sigma}, \gamma)\) of the system (1.6). These procedures [38, 45] to estimate \((\tilde{\alpha}, \tilde{\gamma})\) in (1.6) for the logistic MLE require the single-index model (1.1) to be well-specified (i.e., \( y_i | x_i^T \beta^* \) must follow an actual logistic model (1.4)): If the single-index model (1.1) for the conditional law \( y_i | x_i^T \beta^* \) deviates from the assumed model (1.4) then (1.6)-(1.7) must be modified to account for the actual generative model of the conditional law \( y_i | x_i^T \beta^* \), as in [26, Section B.8].
For regularized logistic regression with non-smooth penalty, even if the solution \((\hat{\beta}, \hat{\sigma}, \hat{\gamma}, \hat{\theta}, \hat{r}, \hat{v})\) were known, constructing confidence intervals for \(\beta_j^*\) from (1.8) alone is typically not possible since \(\text{prox}[\hat{\theta} f]\) is not injective (e.g., for the non-smooth penalty \(\hat{f}(u) = |u|\), the proximal of \(\hat{f}\) is the soft-thresholding operator which is not one-to-one). The lack of injectivity comes from the multi-valued nature of the subdifferential: \(x = \text{prox}[u f](z)\) holds if and only if \(\pm \frac{z - x}{\sqrt{\beta}} \in \partial f(x)\), but \((x, u)\) alone are not sufficient to recover \(z\) if the subdifferential \(\partial f(x)\) is multi-valued. Even if the value
\[
\text{prox}[\hat{\theta} f(\cdot)](\hat{\theta} \beta_j^* + \delta^{-1/2} \hat{r}Z_j)
\]
were known exactly (or approximately through \(\hat{\beta}_j\) as in (1.9)), recovering \(\beta_j^*\) from the value (1.11) still requires to choose a specific element of the subdifferential of \(\hat{f}\) at (1.11), and results such as (1.8)-(1.9) are not informative regarding which element of the subdifferential of \(\hat{f}\) at (1.11) should be used. Because of this difficulty, in general nonlinear models such as (1.4) with non-smooth penalty, confidence intervals for components of \(\beta^*\) are lacking.

1.2. Contributions

A peek at our results This paper develops a different theory to provide proximal approximations, confidence intervals as well as data-driven estimates of the bias of \(\hat{\beta}\) and of the correlation \(\hat{\beta}^T \Sigma \hat{w}\) with the index \(\hat{w}\) (here, correlation refers to the inner product \(\langle \hat{\beta}, \hat{w} \rangle\)). If \(\Sigma = \frac{1}{p} \hat{f}_p\) and \(g(b) = \frac{1}{b} \sum_{j=1}^p f(b_j)\) as in the previous subsection, a by-product of this paper is the approximation
\[
\hat{\beta}_j \approx \text{prox}[\hat{\theta} f(\cdot)](\pm w_j \frac{\hat{v}}{\hat{v}} + \frac{1}{\sqrt{\hat{v}}} \frac{\hat{r}}{\hat{v}} Z_j)
\]

for \(Z_j \sim N(0, 1)\) and \(\pm w_j\) the \(j\)-th component of the index \(\hat{w}\) in (1.1) up to an unidentifiable sign, where \((\hat{v}, \hat{t}, \hat{r})\) are observable scalars defined below in (3.8). The informal approximation (1.12) is made rigorous in Corollary 4.2. The approximation (1.12) mimics (1.9) with \(\beta^*\) replaced by the normalized index \(\hat{w}\), with the important difference that the adjustments \((\hat{v}, \hat{t}, \hat{r})\) of the previous display are observable from the data \((\hat{y}, \hat{X})\), while the deterministic adjustments \((\hat{\sigma}, \hat{\tau}, \hat{\bar{r}}, \hat{\bar{v}})\) in (1.9) requires the knowledge of \(\kappa^2\) and of the distribution of the components of \(\beta^*\) to solve the system (1.7). This means that the rich information contained in the system (1.7) and the limiting result (1.8) can be captured from the data \((\hat{y}, \hat{X})\) by computing \((\hat{v}, \hat{t}, \hat{r})\) in a data-driven fashion, bypassing solving (1.8) and the theory described in the previous subsection.

Main contributions A summary of the main contributions of the paper, with pointers to the main theorems in the next sections, is as follows.

- The present paper introduces the adjustments \((\hat{r}^2, \hat{t}^2, \hat{v})\) that appear in the proximal representation (1.12). For strongly convex penalty functions, Theorem 4.1 derivates an asymptotic normality result for debiased versions of \(\hat{\beta}_j\), which yields confidence intervals and hypothesis tests for the \(j\)-th component \(w_j\) of the index in the single index model (1.1), as well as proximal representations such as (1.12) involving the adjustments \((\hat{r}^2, \hat{t}^2, \hat{v})\) in Corollary 4.2. Similar results hold without a strongly convex penalty for unregularized M-estimators in Theorem 5.2.

- Additional adjustments \((\hat{\sigma}^2, \hat{a}^2)\) are introduced to estimate from the data \((\hat{X}, \hat{y})\) the correlation \(a_* = \hat{w}^T \Sigma \beta\). This is of practical interest, for instance to tune hyper-parameters in order to maximize the correlation with the index \(\hat{w}\) in the model (1.1). Bounds on the estimation error of \(\hat{a}^2 - a^2\) are derived in Theorem 4.4 below for strongly convex penalty and in Theorem 5.2 for unregularized M-estimation. The estimator \(\hat{a}^2\) takes the form
\[
\hat{a}^2 = \frac{\left(\frac{\lambda}{\hat{n}} \left| X \hat{\beta} - \hat{\gamma} \hat{\psi} \right|^2 + \frac{\lambda}{\hat{n}} \hat{\psi}^T X \hat{\beta} - \frac{\lambda}{\hat{n}} \left| \hat{\psi} \right|^2\right)^2}{\frac{1}{\hat{n}} \left| \Sigma^{-1/2} X^T \hat{\psi} \right|^2 + \frac{\lambda}{\hat{n}} \hat{\psi}^T X \hat{\beta} + \frac{\lambda}{\hat{n}} \left| X \hat{\beta} - \hat{\gamma} \hat{\psi} \right|^2 - \frac{\lambda}{\hat{n}} \left| \hat{\psi} \right|^2}.
\]
where \( \psi \in \mathbb{R}^n \) is the vector with components \( \psi_i = -\ell'_i(x_i^T \hat{\beta}) \) and \( (\hat{v}, \hat{\gamma}) \) are defined in (3.8) below from the derivatives of \( \hat{\beta}(y, X) \). The expression of \( \hat{\sigma}^2 \) above is notably involved, but also surprisingly accurate in simulations as shown in Section 6, in particular in Figure 1 for L1-regularized M-estimation in binomial logistic models.

- The most studied generalized linear model is logistic regression thanks to the seminal works [14, 38, 48] for the MLE, where the nonlinear system (1.6) describe the performance and distribution of \( \hat{\beta} \).
  
  Under the assumptions in these works with a logistic model with true coefficient \( \beta^* \) and \( n, p \to +\infty \) with a fixed ratio \( p/n \), asymptotic normality of \( \hat{\beta}_j \) for a fixed coordinate \( j \in [p] \) is granted provided \( \tau_j|\beta^*_j| = O(n^{-1/2}) \) where \( \tau_j^2 = (\Sigma^{-1})_{jj} \) [47, Theorem 3.1]. Under the same assumptions, in Section 5 we show that the techniques of the present paper allow to relax the requirement on the magnitude of \( |\beta^*_j| \) to \( \tau_j|\beta^*_j| = o(1) \) for asymptotic normality of \( \hat{\beta}_j \), so that asymptotic normality may fail for at most a finite number of covariates \( j \in [p] \).

The pivotal expressions enjoying approximate normality, as well as the estimates \( (\hat{\sigma}^2, \hat{\sigma}^2) \) developed in the paper, do not depend on a particular nonlinear function \( F \) in the model (1.1). Consequently, approximate normality (as well as the resuting confidence intervals), and the validity of \( (\hat{\sigma}^2, \hat{\sigma}^2) \) for estimating their targets, are maintained if the conditional law of \( y_i|x_i \) is replaced by another. This robustness to a particular model for \( y_i|x_i \) stands in contrast to procedures aimed at estimating the scalar solutions to the systems (1.6) or (1.7) as well as the resulting pivotal quantities and the validity of \( \hat{\sigma} \) are preserved if we change the law of \( y_i|x_i \) from a linear model to a classification model (logistic or 1-bit compressed sensing) or to a Poisson model.

Some of the techniques used below to derive results of the form (1.12) have been used previously to derive asymptotic normality results in linear models for penalized least-squares estimators [8, 9] and robust/regularized M-estimators [6, 10]. The present paper builds upon these techniques to tackle single-index models. While the factor \( \hat{\tau}/\hat{v} \) of the Gaussian part \( Z \) in (1.12) is reminiscent of the construction of the scalar solutions to the systems (1.6) or (1.7) by assuming a specific generalized linear model for \( y_i|x_i \) (cf. [45, 35] where such procedures are developed). This phenomenon is illustrated in Table 2 where simulations show that approximate normality of pivotal quantities and the validity of \( \hat{\sigma} \) are preserved if we change the law of \( y_i|x_i \) from a linear model to a classification model (logistic or 1-bit compressed sensing) or to a Poisson model.

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### 1.3. Organization

Section 2 states our working assumptions. Section 3 obtains formula for the derivatives of \( \hat{\beta} \) with respect to \( X \) and defines the observable adjustments (e.g., \( \hat{\tau}, \hat{v}, \hat{\tau}^2 \) in (1.12)) and related notation that will be used throughout the paper. Section 4 states the main results in the paper regarding confidence intervals, proximal mapping representations for \( \hat{\beta} \) and for the predicted values \( X \hat{\beta} \), and estimation of the correlation of \( \hat{\beta} \) with the index. Section 5 develops similar results for unregularized M-estimation. Section 6 presents examples and simulations. The proofs are delayed to the appendix.

### 1.4. Notation

For an event \( E \), denote by \( I_E \) or \( I\{E\} \) its indicator function. Vectors are denoted by bold lowercase letters and matrices by bold uppercase. The euclidean norm of a vector \( v \) is denoted by \( \|v\| \), the operator norm (largest singular value) of a matrix \( M \) by \( \|M\|_{op} \) and its Frobenius norm by \( \|M\|_F \). If \( S \) is a symmetric matrix, \( \lambda_{\min}(S) \) is its smallest eigenvalue. If \( S, T \) are two symmetric matrices, we write \( S \preceq T \) if and only if \( T - S \) is positive semi-definite. The identity matrix of size \( p \times p \) is \( I_p \). Convergence in distribution is denoted by \( \overset{d}{\rightarrow} \) and convergence in probability by \( \overset{p}{\rightarrow} \).
2. Assumptions

We now come back to the single index model (1.1) with unknown function $F$. That is, we assume from now on

$$y_i = F(x_i^T w, U_i), \quad U_i \text{ independent of } x_i, \quad \text{Var}[x_i^T w] = w^T \Sigma w = 1.$$ 

Each of our results will require a subset of the following assumptions. Let $\mathcal{Y} \subset \mathbb{R}$ be the set of allowed values for $y_i$, i.e., a set such that $\mathbb{P}(y_i \in \mathcal{Y}) = 1$. For instance, $\mathcal{Y} = \mathbb{R}$ for continuous response regression and $\mathcal{Y} = \{\pm 1\}$ or $\mathcal{Y} = \{0, 1\}$ in binary classification.

**Assumption A.** Let $\delta > 0, \kappa \geq 1$ be constants independent of $n, p$. Consider the model (1.1) for some unknown nonrandom $w \in \mathbb{R}^p$ with $	ext{Var}[x_i^T w] = 1$. Assume that $\frac{1}{2\kappa} \leq \frac{p}{n} \leq \frac{1}{\delta}$, that $\mathbf{X}$ has iid $N(0, \Sigma)$ rows for $\Sigma$ with $\|\Sigma\|_{\text{op}} \|\Sigma^{-1}\|_{\text{op}} \leq \kappa$, that $\ell_{y_0}(\cdot), g(\cdot)$ are convex with $\ell_{y_0}(\cdot)$ continuously differentiable and $\ell_{y_0}'(\cdot)$ $1$-Lipschitz for any fixed $y_0 \in \mathcal{Y}$.

**Assumption B.** For all $b, \hat{b} \in \mathbb{R}^p$ we have $g(0) \leq g(b)$ and for some constant $\tau > 0$ independent of $n, p$,

$$(b - \hat{b})^T (d - \hat{d}) \geq \tau \|\Sigma^{1/2} (b - \hat{b})\|^2, \quad \forall d \in \partial g(g), \hat{d} \in \partial g(\hat{b}). \quad (2.1)$$

The loss satisfies $\mathbb{P}(\ell_{y_0}'(0) = 0) = 0$ where $y_i$ satisfies the model (1.1).

In (2.1) above, $\partial g(b) = \{d \in \mathbb{R}^p : \nabla b, g(b) \geq g(b) + d^T (b - b)\}$ denotes the subdifferential of $g$ at $b \in \mathbb{R}^p$. The assumption that $\ell_{y_0}'(0) = 0$ happens with probability 0 is mild: For instance it is always satisfied for the logistic loss or the probit negative log-likelihood. The goal if this assumption is to ensure

$$\mathbb{P}(\sum_{i=1}^n \ell_{y_i}'(x_i^T \hat{\beta})^2 > 0) = 1. \quad (2.2)$$

Indeed, in the event $\sum_{i=1}^n \ell_{y_i}'(x_i^T \hat{\beta})^2 = 0$, it must be that $0 \in \partial g(\hat{\beta})$ so that $\hat{\beta}$ is a minimizer of $g$, so that $\hat{\beta} = 0$ since $0$ is the only minimizer of $g$ by strong convexity. This implies $\sum_{i=1}^n \ell_{y_i}'(0)^2 = 0$ which has probability 0. Thus, (2.2) holds under Assumption B; this allows us to divide by $\sum_{i=1}^n \ell_{y_i}'(x_i^T \hat{\beta})^2$ with probability one.

Alternatively to strong convexity of the penalty in Assumption B, we will derive results for unregularized M-estimation when $p < n$ under the following assumptions.

**Assumption C.** Assume $\frac{1}{2\kappa} \leq \frac{p}{n} \leq \frac{1}{\delta} < 1$ and that the penalty is $g = 0$.

**Assumption D.** Assume that $\ell_{y_0}$ is twice continuously differentiable for all $y_0 \in \mathcal{Y}$, that $\sup_{y \in \mathbb{R}} \max_{y_0 \in \mathcal{Y}} |\ell_{y_0}'(u)| \leq 1$, and that the maps $u \mapsto \min_{y \in \mathcal{Y}} (\ell_{y_0}'(u))^2$ and $u \mapsto \min_{y \in \mathcal{Y}} \ell_{y_0}''(u)$ are both continuous and valued in the open interval $(0, +\infty)$. Here, the loss $\ell$ (as a function $\mathcal{Y} \times \mathbb{R} \to \mathbb{R}$) is assumed independent of $n, p$.

3. Derivatives and observable adjustments

Let us define some notation used throughout. Denote by $\partial g(b) \subset \mathbb{R}^p$ the subdifferential of a convex function $g : \mathbb{R}^p \to \mathbb{R}$ at a point $b \in \mathbb{R}^p$. The KKT conditions of the minimization problem (1.2) with convex penalty $g$ and convex loss $\ell_{y_i}$ read

$$X^T \hat{\psi}(y, X) = n \partial g(\hat{\beta}(y, X)) \quad (3.1)$$

where $y \in \mathbb{R}^n$ is the response vector with components $y_1, \ldots, y_n$, $X$ is the design matrix with rows $x_1^T, \ldots, x_n^T$, and $\hat{\psi}(y, X) \in \mathbb{R}^n$ has components $\psi_i(y, X) = -\ell_{y_i}'(x_i^T \hat{\beta}(y, X))$. The minus sign here is used so that, in the special case of linear models with the square loss $\ell_{y_i}(u) = (y_i - u)^2/2$, the usual $i$-th residual is $\psi_i(y, X) = y_i - x_i^T \hat{\beta}$. Similarly to $\hat{\beta}(y, X)$ in (1.2), we view $\hat{\psi}(y, X)$ as a function of $(y, X)$, i.e., $\hat{\psi} : \mathbb{R}^n \times \mathbb{R}^{n \times p} \to \mathbb{R}^n$. We let $\ell'$ act componentwise and denote by $\ell_{y_i}'(u) \in \mathbb{R}^n$ the vector with components $\ell_{y_i}'(u_i)$ for every $u, y \in \mathbb{R}^n$, so that

$$\hat{\psi}(y, X) = -\ell_{y_i}'(X \hat{\beta}(y, X)). \quad (3.2)$$
Similarly, \( \ell'' \) acts componentwise on vectors in \( \mathbb{R}^n \) so that \( \ell''(u) \) has components \( \ell''(u) = \ell''(u_i) \) for \( y, u \in \mathbb{R}^n \) and each \( i = 1, \ldots, n \). This notation highlights that derivatives will always be with respect to \( u \) for a fixed value of \( y \). If the context is clear, we write simply \( \hat{\beta} \) for \( \hat{\beta}(y, X) \) and \( \hat{\psi} \) for \( \hat{\psi}(y, X) \); in this case the functions \( \beta, \psi \) and their derivatives are implicitly taken at the observed data \( (y, X) \).

The observable adjustments \((\hat{t}, \hat{v})\) in (1.12) are constructed from the derivatives of \( \beta \) with respect to \( X \). The following proposition provides the structure of these derivatives, extending the corresponding result for linear models [6, Theorem 1] to the present setting.

**Proposition 3.1.** Assume that \( \ell''_y \) is 1-Lipschitz for all \( y \in \mathcal{Y} \) and that (2.1) holds for some some \( \tau > 0 \) and positive definite \( \Sigma \in \mathbb{R}^{p \times p} \). Then for any fixed \( y \in \mathcal{Y} \) the function \( \hat{\beta}(y, \cdot) \) is differentiable almost everywhere in \( \mathbb{R}^{n \times p} \) with derivatives

\[
(\partial/\partial x_{ij}) \hat{\beta}(y, X) = \hat{A}(e_j \hat{v}_i - X^T D e_i \hat{\beta}_j)
\]

where \( D \overset{\text{def}}{=} \text{diag}(\ell''(X \hat{\beta})) \) and some matrix \( \hat{A} \in \mathbb{R}^{p \times p} \) with

\[
\|\Sigma^{1/2} \hat{A} \Sigma^{1/2}\|_{op} \leq (n\tau)^{-1} \tag{3.3}
\]

and such that the following holds: If \( DX \hat{\beta} \neq \hat{\psi} \) we have

\[
0 \leq \hat{d} \hat{r} \leq n \quad \text{where} \quad \hat{d} \overset{\text{def}}{=} \text{Tr}[X \hat{A} X^T D], \quad \hat{r} \overset{\text{def}}{=} \text{Tr}[V] \leq \text{Tr}[D], \quad \text{where} \quad V \overset{\text{def}}{=} D - DX \hat{A} X^T D,
\]

where \( \hat{c} = \frac{1}{n\tau} \|D^{1/2} X \Sigma^{-1/2}\|_{op}^2 \), while if \( DX \hat{\beta} = \hat{\psi} \) we have the slightly weaker

\[
-\hat{c} \leq \hat{d} \hat{r} \leq n + \hat{c}, \\
-4\hat{c} + \text{Tr}[D]/(1 + \hat{c}) \leq \text{Tr}[V] \leq \text{Tr}[D] + 4\hat{c}. \tag{3.5, 3.6}
\]

The proof is given in Appendix A. If \( \hat{\psi} \neq DX \hat{\beta} \), which we expect to happen in practice, (3.5)-(3.6) become the simpler bounds in (3.4). This shows that \( \text{Tr}[D] = \sum_{i=1}^n \ell''_{yi}(x_i^T \hat{\beta}) \) and \( \text{Tr}[V] \) are of the same order, since \( \hat{c} \) is of constant order by (C.19) below. It is unclear why the special case \( \hat{\psi} = DX \hat{\beta} \) needs to be handled separately, and we believe that it is an artefact of the proof. If \( \hat{\psi} = DX \hat{\beta} \) we still obtain the slightly weaker (3.5)-(3.6), which is sufficient for our purpose.

By Proposition 3.1, since \( \hat{\psi} \) in (3.2) is given by \( \hat{\psi} = -\ell''_{yi}(X \hat{\beta}) \) we find by the chain rule \((\partial/\partial x_{ij}) \hat{\beta} = D[-X \hat{A} e_j \hat{v}_i - (I_n - \hat{A} X^T D)e_i \hat{\beta}_j] \). Summarizing the derivatives of both \( \hat{\psi} \) and \( \hat{\beta} \) side by side, for all \( i \in [n], j \in [p] \),

\[
(\partial/\partial x_{ij}) \hat{\beta}(y, X) = \hat{A} e_j \hat{v}_i - \hat{A} X^T D e_i \hat{\beta}_j, \\
(\partial/\partial x_{ij}) \hat{\psi}(y, X) = -DX \hat{A} e_j \hat{v}_i - V e_i \hat{\beta}_j. \tag{3.7}
\]

Equipped with these derivatives of \( \hat{\beta} \), our main result requires the following scalar adjustments to correct the bias and scaling of \( \hat{\beta} \) for estimation and confidence interval about components of the index \( w \). Define the scalars \((\hat{v}, \hat{r}, \hat{t}, \hat{a}, \hat{a}^2)\) by

\[
\begin{align*}
\hat{v} & \overset{\text{def}}{=} \frac{1}{n} \text{Tr}[V], \\
\hat{r} & \overset{\text{def}}{=} \left( \frac{1}{n} \|\hat{\psi}\|^2 \right)^{1/2}, \\
\hat{t}^2 & \overset{\text{def}}{=} \frac{1}{n} \|\Sigma^{-1/2} X^T \hat{\psi}\|^2 + \frac{2\hat{d}}{n} \|\hat{\psi}\| X \hat{\beta} + \frac{\hat{c}}{n} \|X \hat{\beta} - \hat{\gamma} \hat{\psi}\|^2 - \frac{\hat{c}}{n} \hat{r}^2, \\
\hat{a} & \overset{\text{def}}{=} \hat{t}^2 \left( \frac{\hat{c}}{n} \|X \hat{\beta} - \hat{\gamma} \hat{\psi}\|^2 + \frac{1}{n} \|\hat{\psi}\| X \hat{\beta} - \hat{\gamma} \hat{r}^2 \right)^2, \\
\hat{a}^2 & \overset{\text{def}}{=} \frac{1}{n} \|X \hat{\beta} - \hat{\gamma} \hat{\psi}\|^2 - \hat{a}^2.
\end{align*}
\tag{3.8}
\]
Next, for each covariate index \( j \in [p] \), define the de-biased estimate \( \hat{\beta}_j^{(d)} \) by
\[
\hat{\beta}_j^{(d)} \equiv \hat{\beta}_j + \text{Tr}[V^{-1}e_j^T \Sigma^{-1}X^T \hat{\psi}] = \hat{\beta}_j + \hat{v}_j e_j^T (n\Sigma)^{-1}X^T \hat{\psi}
\] (3.9)
where \( e_j \) is the \( j \)-th canonical basis vector in \( \mathbb{R}^p \). Finally, let \( \Omega_{jj} \equiv (\Sigma^{-1})_{jj} \) be the \( j \)-th diagonal element of \( \Sigma^{-1} \).

Alternatively to the adjustments \( (\hat{\gamma}, \hat{\iota}, \hat{\alpha}^2, \hat{\sigma}^2) \) in (3.8), the quantities \( \frac{1}{n}\|X\hat{\beta} - \hat{\gamma}\hat{\psi}\|^2 \) and \( \hat{\gamma} \) in the expressions for \( (\hat{\iota}, \hat{\alpha}^2, \hat{\sigma}^2) \) in (3.8) may be replaced by \( \|\Sigma^{1/2}\hat{\beta}\|^2 \) and \( \gamma_* = \text{Tr}[\Sigma \hat{\Delta}] \) thanks to the approximations \( \frac{1}{n}\|X\hat{\beta} - \hat{\gamma}\hat{\psi}\|^2 \approx \|\Sigma^{1/2}\hat{\beta}\|^2 \) and \( \hat{\gamma} \approx \gamma_* \) justified in Theorem 4.4 below. This gives the alternative estimates
\[
\begin{align*}
\hat{\iota}^2 &= \|\Sigma^{-1/2}X^T \hat{\psi}/n + \hat{v}\Sigma^{1/2}\hat{\beta}\|^2 - (p/n)^2, \\
\hat{\alpha}^2 &= \hat{t}^{-2}(\hat{v}\|\Sigma^{1/2}\hat{\beta}\|^2 + \hat{\psi}^T X \hat{\beta}/n - \gamma_* t^2), \\
\hat{\sigma}^2 &= \|\Sigma^{1/2}\hat{\beta}\|^2 - \hat{\alpha}^2.
\end{align*}
\] (3.10)

The expressions \( (\hat{\iota}^2, \hat{\alpha}^2, \hat{\sigma}^2) \) in (3.8) are preferred when the covariance \( \Sigma \) is unknown, as \( \Sigma \) only occurs in (3.8) in the expression of \( \hat{t}^2 \). For unregularized \( M \)-estimation (penalty \( g = 0 \)), the term \( \Sigma^{-1/2}X^T \hat{\psi} \) in \( \hat{t} \) is equal to 0 by the optimality conditions of the optimization problem (1.2), so that \( (\hat{\iota}, \hat{v}, \hat{\gamma}, \hat{\iota}^2, \hat{\alpha}^2, \hat{\sigma}^2) \) are all computable without any knowledge of the covariance \( \Sigma \). The special form of \( (\hat{\iota}, \hat{v}, \hat{\gamma}, \hat{\iota}^2, \hat{\alpha}^2, \hat{\sigma}^2, \hat{d}_f) \) in unregularized \( M \)-estimation is detailed in Section 5.

4. Main results under strong convexity

Throughout, \( C_1, C_2, \ldots \) denotes absolute constants, and \( C_3(\delta), C_4(\delta, \tau), \ldots \) denote constants that depend only on \( \delta \) and only on \( (\delta, \tau) \), respectively.

4.1. Confidence intervals for individual components of the index

Theorem 4.1. Let Assumptions A and B be fulfilled. Then for all \( j = 1, \ldots, p \), there exists \( Z_j \sim N(0,1) \) such that
\[
\frac{1}{p} \sum_{j=1}^{p} \mathbb{E} \left[ \left( \frac{\sqrt{n}}{\hat{\Omega}_{jj}^{1/2}} \left( \frac{\hat{v}}{\hat{r}} \hat{\beta}_j^{(d)} - \frac{\hat{t}}{\hat{r}} w_j \right) - Z_j \right)^2 \right] \leq \frac{C_5(\delta, \tau, \kappa)}{\sqrt{p}}
\] (4.1)
where \( \pm \) denotes the sign of the unknown scalar \( t_* \equiv w^T(\text{Tr}[V]\Sigma \hat{\beta} + X^T \hat{\psi})/n \), and \( \hat{t} = \max(0, \hat{t}^2)^{1/2} \).

If additionally Assumption D holds then for some event \( E \) with \( \mathbb{P}(E) \to 1 \), we have \( \max\{\frac{1}{\sqrt{n}}, \hat{r}\} I_E \leq C_6(\delta, \tau, \ell) \) almost surely and
\[
\frac{1}{p} \sum_{j=1}^{p} \frac{1}{\hat{\Omega}_{jj}^{1/2}} \mathbb{E} \left[ I_E \left( \sqrt{n} \left( \hat{\beta}_j^{(d)} - \frac{\hat{t}}{\hat{r}} w_j \right) - \hat{\Omega}_{jj}^{1/2} Z_j \frac{\hat{r}^2}{\hat{v}} \right)^2 \right] \leq \frac{C_7(\delta, \tau, \kappa, \ell)}{\sqrt{p}}.
\] (4.2)

In the single index model (1.1), the sign of \( w \) is not identifiable as replacing \( w \) with \( -w \) and \( F \) with \( (a, u) \mapsto F(-a, u) \) in (1.1) leaves \( y \) unchanged. Consequently, for each covariate \( j = 1, \ldots, p \) we focus on confidence intervals for \( w_j \) up to an unidentifiable sign denoted by the random variable \( \pm \) in the above displays. We now describe the confidence intervals that stem from Theorem 4.1. By (4.1), for any \( \varepsilon > 0 \) there exist at most \( \sqrt{p}C(\delta, \tau, \kappa)/\varepsilon \) covariates \( j \in [p] \) such that \( W_2(N(0, 1), \sqrt{p}_{jj}^{-1/2} \mathbb{E} \left( \hat{v} \hat{\beta}_j^{(d)} - \frac{\hat{t}}{\hat{r}} w_j \right) ) > \varepsilon \) where \( W_2 \) is the Wasserstein distance. This justifies the approximation
\[
\sqrt{n} \left( \hat{v} \hat{\beta}_j^{(d)} - \frac{\hat{t}}{\hat{r}} w_j \right) \approx \hat{r} \hat{\Omega}_{jj}^{1/2} Z_j
\] (4.3)
for most coordinates. The quantile-quantile plots in Tables 2 and 3 illustrate this normal approximation. Since convergence in 2-Wasserstein distance implies weak convergence, for all \( j \in [p] \setminus J_p \) for some set \( J_p \) with cardinality \(|J_p| \leq \sqrt{p}C\delta, \tau, \kappa, \alpha, \epsilon\) we thus have

\[
\sup_{j \in [p] \setminus J_p} \left| \mathbb{P} \left( \frac{\hat{\beta}(d)}{\sqrt{n}} \leq w_j \right) - \mathbb{P} \left( \frac{\hat{\beta}(d)}{\sqrt{n}} \leq \epsilon \right) \right| \leq \epsilon
\]

where \( z_{\alpha/2} \) is the standard normal quantile defined by \( \mathbb{P}(|N(0,1)| > z_{\alpha/2}) = \alpha \). This provides a confidence interval for the \( j \)-th component of \( w_j \), the index \( w \), to the unknown sign \( \pm = \text{sign}(t_*) \). By the same argument and taking \( \epsilon = \epsilon_{n,p} \) depending on \( n, p \) and converging to 0, for instance \( \epsilon_{n,p} = 1/\log n \), we obtain

\[
\sup_{j \in [p] \setminus J_{n,p}} \left| \mathbb{P} \left( \frac{\hat{\beta}(d)}{\sqrt{n}} \leq w_j \right) - \mathbb{P} \left( \frac{\hat{\beta}(d)}{\sqrt{n}} \leq \epsilon \right) \right| \rightarrow 0
\]

for some set \( J_{n,p} \subset [p] \) with cardinality at most \( \sqrt{p}C\delta, \tau, \kappa, \epsilon)/\epsilon_{n,p} \).

Unpacking the proof of Theorem 4.1 reveals that (4.1) is a consequence of

\[
\frac{1}{p} \sum_{j=1}^{p} \mathbb{E} \left( \left( \frac{\sqrt{n}}{\Omega_{jj}^{1/2}} \left( \frac{\hat{\beta}(d)}{\sqrt{n}} - t_* w_j \right) - Z_j \right)^2 \right) \leq \frac{C_{10}(\delta, \tau, \kappa)}{p},
\]

\[
\mathbb{E} \left[ \frac{1}{\sqrt{\tilde{t}^2 - \tilde{t}^2}} \right] \leq \frac{C_{11}(\delta, \tau)}{\sqrt{p}}.
\]

The first line is of the same form as (4.1) with \( \pm \tilde{t} \) replaced by \( t_* \) and a right-hand side of order \( 1/p \), much smaller than the right-hand side of (4.1). The right-hand side of order \( 1/\sqrt{p} \) in (4.1) is paid due to the approximation \( \tilde{t}^2 \approx \tilde{t}^2 \) in (4.5), which features an error term of order \( 1/\sqrt{p} \). The approximation (4.5) is only used in the terms of (4.1) for which \( w_j \neq 0 \), so that if \( N = \{ j = 1, \ldots, p : w_j = 0 \} \) denotes the set of null covariates, (4.4) implies the bound

\[
\frac{1}{p} \sum_{j \in N} \mathbb{E} \left( \left( \frac{\sqrt{n}}{\Omega_{jj}^{1/2}} \left( \frac{\hat{\beta}(d)}{\sqrt{n}} - Z_j \right)^2 \right) \leq \frac{C_{12}(\delta, \tau, \kappa)}{p}
\]

with a smaller right-hand side of order \( \frac{1}{p} \). To perform an hypothesis test of

\[
H_0 : w_j = 0 \quad \text{against} \quad H_1 : |w_j| > 0
\]

at level \( \alpha \in (0,1) \), the test that rejects \( H_0 \) if

\[
\frac{\sqrt{n}||\hat{\beta}(d)||}{\tilde{t}} > z_{\alpha/2} \Omega_{jj}^{1/2}
\]

has type I error in \([\alpha - \epsilon, \alpha + \epsilon] \) for all components \( j \in [p] \setminus J^0_p \) where \(|J^0_p| \leq C_{13}(\delta, \tau, \kappa) \), i.e., the desired type I error holds for all null covariates except a finite number at most.

### 4.2. Proximal mapping representation for \( \hat{\beta}_j \)

For isotropic designs with \( \Sigma = \frac{1}{p} I_p \), the de-biased estimate (3.9) reduces in vector form to \( \hat{\beta}(d) = \hat{\beta} + \hat{v}^{-1} X^T \hat{\phi} \) with \( X^T \hat{\phi} \in n \partial_g(\hat{\beta}) \), where \( \partial_g(\hat{\beta}) \) is the subdifferential of the penalty \( g \) at \( \hat{\beta} \). By the definition of the proximal operator in \( \mathbb{R}^p \), we thus have for any \( b \in \mathbb{R}^p \)

\[
\hat{\beta} + \hat{v}^{-1} X^T \hat{\phi} = b \quad \text{iff} \quad \hat{\beta} = \text{prox}[n/\hat{v}g](b).
\]

That is, in this isotropic setting, Theorem 4.1 lets us express \( \hat{\beta} \) as a proximal operator of the penalty function \( g \) scaled by \( n/\hat{v} \). The following result makes such proximal approximation precise in the case of separable penalty.
Corollary 4.2. Let Assumptions A and B be fulfilled and set $\Sigma = \frac{1}{n} I_p$. Assume that the penalty $g$ is separable, of the form $g(b) = \frac{1}{n} \sum_{j=1}^{p} g_j(b_j)$ for convex functions $g_j : \mathbb{R} \to \mathbb{R}$. Then
\[
\frac{1}{p} \sum_{j=1}^{p} \mathbb{E} \left[ \frac{\hat{\tau}^2}{\tau^2} \hat{\beta}_j - \text{prox} \left[ \frac{1}{\hat{v}} g_j \left( \hat{r} \hat{v} Z_j + \frac{\hat{t}}{\hat{v}} w_j \right) \right] \right] \leq \frac{C_{14}(\delta, \tau, \kappa)}{\sqrt{p}} \tag{4.8}
\]
where $Z_j \sim N(0,1)$ for each $j \in [p]$ and $\pm = \text{sign}(t_j)$ as in Theorem 4.1.

Proof. By the KKT conditions, with $\Sigma = \frac{1}{n} I_p$ and separable penalty $g$, $\hat{\beta}_j(\delta) \in \hat{\beta}_j + \hat{v}^{-1}\partial g_j(\hat{\beta}_j)$. Since $\Omega_{ij} = n$, Theorem 4.1 gives that $\hat{\beta}_j + \hat{v}^{-1}\partial g_j(\hat{\beta}_j) \ni \xi Z_j + (\pm \hat{t}) w_j + \frac{\xi}{\hat{v}} \text{Rem}_j$ where $\partial g_j(\hat{\beta}_j)$ is the subdifferential of $g_j$ at $\hat{\beta}_j$ and $\text{Rem}_j$ are such that $\frac{1}{p} \sum_{j=1}^{p} \mathbb{E}[\text{Rem}_j^2] \leq C_{15}(\delta, \tau, \kappa)/\sqrt{p}$. Equivalently, $\hat{\beta}_j = \text{prox}(\hat{v}^{-1} g_j)((\xi Z_j + (\pm \hat{t}) w_j) + \frac{\xi}{\hat{v}} \text{Rem}_j)$ by definition of the proximal operator. Since $x \mapsto \text{prox}[h](x)$ is 1-Lipschitz for any convex, proper lower semi-continuous function $h$, the left-hand side of Corollary 4.2 is bounded from above by $\frac{1}{p} \sum_{j=1}^{p} \mathbb{E}[(\text{Rem}_j)^2]$.

For $\Sigma = \frac{1}{n} I_p$, Corollary 4.2 provides the proximal approximation
\[
\hat{\beta}_j \approx \text{prox} \left[ \frac{1}{\hat{v}} g_j \left( \hat{r} \hat{v} Z_j + \frac{\hat{t}}{\hat{v}} w_j \right) \right] \tag{4.9}
\]
for $\hat{\beta}_j$, in an averaged sense over $j \in [p]$. If $\Sigma = \frac{1}{p} I_p$ is used (instead of $\Sigma = \frac{1}{n} I_p$ in Corollary 4.2) and the penalty is $g(b) = \frac{1}{p} \sum_{j=1}^{p} \hat{f}(b_j)$, the same argument yields (1.12), which is analogous to (1.9) from [34, 24] with the important difference that the adjustments $(\hat{r}, \hat{v}, \hat{t}^2)$ are observable and computed from the data, while the deterministic adjustments in (1.9) are not observable. We note in passing that Theorem 4.1 is more informative than Corollary 4.2 because the subgradient is explicit: If $\Sigma = \frac{1}{n} I_p$ and $g(b) = \frac{1}{n} \sum_{j=1}^{p} g_j(b_j)$, Theorem 4.1 provides
\[
\hat{\beta}_j + \frac{1}{\hat{v}} e_j^T X^T \hat{\psi} \approx \pm w_j \hat{t} \hat{v} + Z_j \hat{\beta}_j \quad \text{with} \quad e_j^T X^T \hat{\psi} \in \partial g_j(\hat{\beta}_j).
\]
On the other hand, the information that $e_j^T X^T \hat{\psi}$ is the subgradient of $g_j$ at $\hat{\beta}_j$ appearing in the KKT conditions of the proximal operator is not visible from results such as (4.9).

In Corollary 4.2 the index $w$ is nonrandom. If $g_j = g_0$ for all $j \in [p]$ and some function $g_0 : \mathbb{R} \to \mathbb{R}$, and if $w$ is random independent of $X$ with exchangeable entries then
\[
\max_{j=1,...,p} \mathbb{E} \left[ \frac{\hat{\tau}^2}{\tau^2} \hat{\beta}_j - \text{prox} \left[ \frac{1}{\hat{v}} g_0 \left( \hat{r} \hat{v} Z_j + \frac{\hat{t}}{\hat{v}} w_j \right) \right] \right] \leq \frac{C_{16}(\delta, \tau, \kappa)}{\sqrt{p}}.
\]
Indeed, by exchangeability the expectation inside the maximum is the same for all $j = 1,...,p$, so that the maximum over $[p]$ is equal to the average over $[p]$. The previous display is thus a consequence of Corollary 4.2 conditioned on $w$, followed by integrating with respect to the probability measure of $w$.

Previous studies on generalized linear models such as Salehi et al. [34], Loureiro et al. [27] discussed in the introduction derived proximal representations such as (1.8)-(1.9) in logistic and single-index models, although the connection between M-estimator $\tilde{\beta}$ and the scalar $\tilde{\sigma} \tilde{r}$ in $\text{prox}[[\tilde{\sigma} \tilde{r} f]]$ of (1.8)-(1.9) has remained unclear. Results such as (4.8)-(4.9) shed light on this connection, showing that $1/\hat{v}$ plays the role of the deterministic $\sigma \tilde{r}$ appearing in (1.8)-(1.9). In other words, the multiplicative coefficient of $\tilde{f}$ inside the proximal operator in (1.5)-(1.9) has a simple expression, $1/\hat{v}$, in terms of the derivatives of $\hat{\beta}(y, X)$ (cf. (3.4) and (3.8) for the definition of $\hat{v}$). Such connection was previously only established in linear models, see [22, Lemma 3.1 and discussion following Proposition 2.7] for unregularized M-estimation, [16, Theorem 9] for the Lasso and [10] for penalized robust estimators. The following subsection studies proximal representations of the predicted values $x_i^T \hat{\beta}$.
4.3. Proximal mapping representation for predicted values

The same techniques as Theorem 4.1 provide a proximal representation for the predicted value $x_i^T \hat{\beta}$ for a fixed $i \in [n]$.

**Theorem 4.3.** Let Assumptions A and B be fulfilled. Define $a_i = w^T \Sigma \hat{\beta}$, $\sigma^2 = \|\Sigma^{1/2} \hat{\beta}\|^2 - a_i^2$ and $\gamma = \text{Tr}[\Sigma \hat{A}]$. Then

$$\max_{i=1,\ldots,n} E \left[ \left| x_i^T \hat{\beta} - \text{prox}[\gamma, \ell_{y_i}(\cdot)](a_i U_i + \sigma Z_i) \right|^2 \right] \leq C_1 \tau(\delta, \tau) / n \quad (4.10)$$

where $U_i = x_i^T w$ and $Z_i$ are independent $N(0,1)$ random variables.

Inequality (4.10) justifies the approximation $x_i^T \hat{\beta} \approx \text{prox}[\gamma, \ell_{y_i}(\cdot)](a_i U_i + \sigma Z_i)$, or equivalently by definition of the proximal operator,

$$x_i^T \hat{\beta} + \gamma \ell_{y_i}(x_i^T \hat{\beta}) \approx a_i U_i + \sigma Z_i.$$

This provides a clear description of the predicted value $x_i^T \hat{\beta}$, although $(a_i^2, \sigma^2)$ is not observable. The topic of the next subsection is the estimation of these quantities by $(\hat{a}^2, \hat{\sigma}^2)$.

4.4. Correlation estimation

Recall the notation $a_i = w^T \Sigma \hat{\beta}$, $\sigma_i^2 = \|\Sigma^{1/2} \hat{\beta}\|^2 - a_i^2$ and $\gamma_i = \text{Tr}[\Sigma \hat{A}]$ given in Theorem 4.3, and $t_i$ given in Theorem 4.1. While the adjustments ($\hat{r}, \hat{v}, \hat{t}$) for the proximal representation $(4.8)$ are observable from the data, the quantities $(a_i, \sigma_i)$ in (4.10) are not. The star subscript in $(a_i, \sigma_i)$ is meant to emphasize that they are not observable. Estimation of the quantity $a_i$ is of interest in itself: An estimate $\hat{a}^2$ with $\hat{a}^2 \approx a_i^2$ would allow the Statistician to estimate the correlation $a_i / \|\Sigma^{1/2} \hat{\beta}\|$ up to a sign, or other performance metrics for $\hat{\beta}$. The following result shows that $(a_i^2, \sigma_i^2, \gamma_i, t_i^2)$ can be estimated by $(\hat{a}^2, \hat{\sigma}^2, \hat{\gamma}, \hat{t}^2)$.

**Theorem 4.4.** Let Assumptions A and B hold. Let $(\hat{r}, \hat{v}, \hat{\gamma}, \hat{\bar{a}}^2, \hat{\sigma}^2, \hat{\gamma}, \hat{t}^2)$ be as in (3.8), and let $(t_i, a_i, \sigma_i)$ be as in Theorems 4.1 and 4.3. Then

$$E \left[ \hat{\gamma} - \gamma \right] \leq C_1 \tau(\delta, \tau) n^{-1/2}, \quad (4.11)$$

$$E \left[ \hat{r}^2 - t_i^2 \right] \leq C_2 \tau(\delta, \tau) n^{-1/2}, \quad (4.12)$$

$$E \left[ \frac{1}{n} \left\| X \hat{\beta} - \hat{\gamma} \hat{\phi} \right\|^2 - \|\Sigma^{1/2} \hat{\beta}\|^2 \right] \leq C_3 \tau(\delta, \tau) n^{-1/2}, \quad (4.13)$$

$$E \left[ \frac{1}{n} \left( \|\hat{a}^2 - a_i^2\| + \|\hat{\sigma}^2 - \sigma_i^2\| \right) \right] \leq C_4 \tau(\delta, \tau) n^{-1/2}. \quad (4.14)$$

If additionally Assumption D holds, then there exists an event $E$ with $P(E) \to 1$ such that

$$E \left[ I_E \left( \|\hat{\gamma} - \gamma \| + \|\hat{r}^2 - t_i^2 \| + \frac{1}{n} \| X \hat{\beta} - \hat{\gamma} \hat{\phi} \|^2 - \|\Sigma^{1/2} \hat{\beta}\|^2 \right) \right] \leq C_5 \tau(\delta, \tau) n^{-1/2}. \quad (4.15)$$

Theorem 4.4 justifies the approximations $\hat{\gamma} \approx \gamma$, $\hat{r}^2 \approx t_i^2$, $\hat{a}^2 \approx a_i^2$ and $\hat{\sigma}^2 \approx \sigma_i^2$, so that the quantities $(\gamma_i, a_i^2, \sigma_i^2)$ appearing in the proximal representation for $x_i^T \hat{\beta}$ in Theorem 4.3 are estimable by $(\hat{\gamma}, \hat{a}^2, \hat{\sigma}^2)$. We focus on estimation of $a_i^2$ here instead of $a_i$, because the sign of $a_i = w^T \Sigma \hat{\beta}$ is unidentifiable in the single index model (1.1) as any sign change can be absorbed into $F(\cdot)$. The accuracy of $\hat{a}^2$ for estimation $a_i^2$ is confirmed in simulations (Section 6) in Tables 2 and 3 and Figure 1.

5. Main results for unregularized $M$-estimation

Unregularized $M$-estimation refers to the special case of penalty $g$ in (1.2) being identically 0. In this case, (1.2) includes Maximum Likelihood Estimator if the negative log-likelihood is used for the loss functions $\ell_{y_i}(\cdot)$ in (1.2). Our first task is to justify the derivative formula (3.7) and to determine $\hat{A}$ without the strong convexity assumption made in Proposition 3.1. The following lemma follows from the implicit function theorem and is proved in Appendix D.
Lemma 5.1. Let Assumptions C and D be fulfilled so that the penalty is \( g = 0 \). Let \( y \in \mathcal{Y}^n \) and \( \hat{X} \in \mathbb{R}^{n \times p} \) be fixed. If a minimizer \( \hat{\beta} \) exists at \((y, \hat{X})\) and \( X^T \hat{X} \) is invertible, then there exists a neighborhood of \( \hat{X} \) such that \( \hat{\beta}(y, X) \) exists in this neighborhood, the map \( X \mapsto \hat{\beta}(y, X) \) restricted to this neighborhood is continuously differentiable, and (3.7) holds with \( \hat{A} = \left( \sum_{i=1}^{n} x_i x_i^T \hat{\beta} \right)^{-1} \).

For unregularized \( M \)-estimation with \( p < n \), the optimality conditions of the optimization problem (1.2) read \( X^T \hat{\psi} = 0 \). With \( X^T \hat{\psi} = 0 \) and the explicit expression \( \hat{A} = \left( \sum_{i=1}^{n} x_i x_i^T \hat{\beta} \right)^{-1} \) from the previous lemma, the adjustments \((\hat{d}\hat{f}, \hat{g}, \hat{a}, \hat{\delta})\) in (3.8) reduce to the simpler forms

\[
\hat{d}\hat{f} = p, \quad \hat{g} = \frac{p}{\hat{v}}, \quad \hat{a}^2 = \frac{\|X\hat{\beta}\|^2}{n} - \frac{p}{n} \left( 1 - \frac{p}{n} \right) \hat{v}^2, \quad \hat{\delta}^2 = \frac{p}{n} \left( \frac{\hat{r}}{\hat{v}} \right)^2.
\]

Here, the fact that \( \hat{d}\hat{f} = p \) justifies the notation \( \hat{d}\hat{f} \) for the quantity defined in Proposition 3.1: In unregularized \( M \)-estimation \( \hat{d}\hat{f} \) is the number of parameters, or degrees of freedom of the estimator. A similar justification for the notation \( \hat{d}\hat{f} \) is observed for \( L1 \) regularized \( M \)-estimation as discussed in Section 6.4 below.

The low-dimensional systems of equations (e.g., (1.3), (1.6)) characterizing the behavior of unregularized \( M \)-estimation were obtained in linear models [23, 20, 22], in logistic regression models [38, 48], other generalized linear models [35], single index models [26] and Gaussian mixture models [28, 27]. These results state that if the low-dimensional system corresponding to the loss and the generative model for \( y_i \) given \( x_i \) admits a unique solution, then this solution characterizes the limit in probability of \( \| \Sigma^{1/2} \hat{\beta} \|^2 \) as well as the limit in probability of the correlation \( a_* = w^T \Sigma \hat{\beta} \) with the index \( w \), and limits of averages with respect to test functions \( \phi \) as in (1.5). In logistic regression, the existence and unicity of solutions to the low-dimensional system (1.6) is provably related to the existence of the minimizer \( \hat{\beta} \) and the linear separability of the data [14, 38]. If \( \| \Sigma^{1/2} \hat{\beta} \|^2 \) and \( a_* \) admits non-zero limits in probability as in these references, then \( \mathbb{P}(\frac{1}{K} \leq |a_*|, \frac{1}{n} \|X\hat{\beta}\|^2 \leq K) \) for some constant \( K > 0 \). The aforementioned results thus justify the following assumption that will be used in the theorems of this section.

Assumption E. Let \( K > 0 \) be constant. Assume that Assumptions A, C and D hold so that the penalty is \( g = 0 \). Assume that with probability approaching one as \( n, p \to \infty \), the estimator \( \hat{\beta} \) exists, is bounded and has non-vanishing correlation with the index \( w \) in the sense that \( \mathbb{P}(\text{minimizer } \hat{\beta} \text{ in (1.2) exists and } \frac{1}{n} \|X\hat{\beta}\|^2 \leq K \text{ and } \frac{1}{K} \leq |a_*|) \to 1 \).

The next result involves the square root of \( \Omega_{jj} - w_j^2 \). This quantity is always non-negative since it equals \( e_j^T \Sigma^{-1} e_j - e_j^T w w^T e_j = e_j^T \Sigma^{-1/2} (I_p - v v^T) \Sigma^{-1/2} e_j \) for \( v = \Sigma^{1/2} w \) and \( I_p - v v^T \) is positive semi-definite thanks to \( \|v\| = 1 \).

Theorem 5.2. Let Assumptions A, C and D be fulfilled so that the penalty is \( g = 0 \). Let \( K > 0 \) and define the event \( E = \{ \text{the minimizer } \hat{\beta} \text{ in (1.2) exists and } \frac{1}{n} \|X\hat{\beta}\|^2 \leq K \} \). For each \( j = 1, ..., p \) such that \( w_j^2 \neq \Omega_{jj} \), there exists a standard normal \( Z_j \sim N(0,1) \) satisfying

\[
\mathbb{E}[I_E \left( \frac{n}{\Omega_{jj} - w_j^2} \right)^{1/2} \left( \hat{\beta}_j - a_* w_j \right)^2] \leq C_{24}(\delta, K, \ell) \frac{1}{p}
\]

for \( a_* = \beta^T \Sigma w \) and the observables \((\hat{g}, \hat{v}, \hat{\delta}, \hat{a})\) in (5.1). For \( a^2_* = \|\Sigma^{1/2} \hat{\beta}\|^2 - a^2_* \), it holds

\[
\mathbb{E}[I_E (|a^2_* - \hat{a}^2| + |a^2_* - \hat{a}^2|)] \leq C_{24}(\delta, K, \ell) / \sqrt{p}
\]

and \( \mathbb{E}[I_E \max(\hat{r}, \frac{1}{\hat{r}}, \hat{v}, \frac{1}{\hat{v}})] \leq C_{25}(K, \delta, \ell) \). If additionally Assumption E holds, then \( |\hat{a} - a_*| = O_p(n^{-1/2}) \) for \( \hat{a} = \max(0, a^2_* - \hat{a}^2)^{1/2} \) where \( O_p(\cdot) \) hides constants depending only on \((\delta, K, \ell)\).

Theorem 5.2 is proved in Appendix D.5. In logistic regression where \( \hat{\beta} \) is the MLE and a well-specified logistic model is assumed for \( y \) as in [14, 38], the data is separable and \( \hat{\beta} \) does not exist with positive probability. The event that \( \hat{\beta} \) does exist has positive probability, although exponentially small, on the side of the phase transition studied in [14] where the data is not separable with high-probability.
This makes the indicator function \( I_E \) (of an event \( E \) on which the MLE exists) unavoidable inside the expectations in (5.2)-(5.3), since we cannot make any statement about \( \hat{\beta} \) in the event that it does not exist.

When \( \mathbb{P}(E) \to 1 \) as in Assumption E and \( \Omega_{jj} > w_j^2 \), (5.2) implies the approximation

\[
\sqrt{\frac{n}{\Omega_{jj} - w_j^2}} (\hat{\beta}_j - a_* w_j) - Z_j \frac{\varepsilon}{\sigma} = \sqrt{\frac{n}{\Omega_{jj} - w_j^2}} (\hat{\beta}_j - a_* w_j) - Z_j \bar{a} \sqrt{\frac{n}{p}} \sim O_p(\frac{1}{p})
\]

(5.4)

with \( Z_j \sim N(0, 1) \) having pivotal distribution, and the remainder \( O_p(1/p) \) converges to 0 in probability.

In a logistic model where \( \hat{\beta} \) is the logistic MLE, the stochastic representation result in [48, Lemma 2.1] yields the normal approximation \( \sqrt{\mathbb{P}(\Omega_{jj} - w_j^2)^{-1/2} (\hat{\beta}_j - a_* w_j) / \sigma_*} \to_d N(0, 1) \) which is comparable to the previous display with \( \bar{a} \) replaced by \( \sigma_* \); the advantage of the asymptotic normality involving \( \bar{a} \) is that the asymptotic variance is already estimated.

It remains to explain how the unknown \( a_* \) in (5.4) can be estimated. One important consequence of Theorem 5.2 resides in the rate of convergence for \( |a_*^2 - \hat{a}^2| \) in (5.3) and for \( |\hat{a} - a_*| \) under Assumption E.

From (5.4), in order to relate \( \hat{\beta}_j \) to the unknown parameter \( w_j \) of interest through data-driven quantities, one needs to replace the unknown correlation \( a_* = \beta^T \Sigma w \) by some known scalar \( a \), leading to the approximation

\[
\sqrt{\frac{n}{\Omega_{jj} - w_j^2}} (\hat{\beta}_j - a w_j) - Z_j \frac{\varepsilon}{\sigma} = O_p(\frac{1}{p}) + \frac{|w_j|}{\sqrt{\Omega_{jj} - w_j^2}} \sqrt{n} |a - a_*|
\]

(5.5)

In order to obtain an asymptotic normality result from the previous display, the scalar \( a \) needs to estimate \( a_* \) fast enough to ensure that the right-hand side converges to 0 in probability; similar observations were already made in [48, Section 3.2.2]. Zhao et al. [48] further conjectured that \( |\hat{a} - a_*| = O_p(n^{-1/2}) \) holds if \( \hat{a} \) is the deterministic limit in probability of \( a_* = \hat{\beta}^T \Sigma w \), (that is, \( a_* \to_d \hat{a} \) obtained by solving the nonlinear system of three unknowns (1.6) studied in [38] under the assumptions in [38, 48]. Instead of considering deterministic scalars \( a \) in (5.5), our proposal is to use \( a = \pm \hat{a} \) for \( a = \max(0, \hat{a}^2)^{1/2} \) and \( \pm = \text{sign}(a_*) \), where \( \hat{a}^2 \) is the observable adjustment defined in (5.1). The last part of Theorem 5.2 shows that \( \hat{a} \) is a \( n^{-1/2} \)-consistent estimate of \( |a_*| \). On the other hand, as discussed in [48, 3.2.2] and [45, Remark 1], obtaining \( O_p(n^{-1/2}) \) error bounds on \( |\hat{a} - a_*| \) appear out of reach of current techniques.

Developing \( n^{-1/2} \)-consistent estimates of \( |a_*| \) is important because (5.4) and (5.5) with \( a = \hat{a} \) imply

\[
\sqrt{\frac{n}{\Omega_{jj} - w_j^2}} (\hat{\beta}_j - \pm \hat{w}_j - Z_j \frac{\varepsilon}{\sigma}) = O_p(1) + O_p(\frac{|w_j|}{\sqrt{\Omega_{jj} - w_j^2}}(n^{-1/2}))(5.6)
\]

if \( |\hat{a} - a_*| = O_p(n^{-1/2}) \) as in the discussion of the previous paragraph. Then the right-hand side converges to 0 in probability for any covariate \( j \in [p] \) such that \( w_j^2 / \Omega_{jj} \to 0 \). Since max\( \{\bar{v}, \bar{r}, \bar{v} / \bar{r}\} \) is \( O_p(1) \) by Theorem 5.2, \( w_j^2 / \Omega_{jj} \to 0 \) implies the normal approximation

\[
\sqrt{\frac{n}{\Omega_{jj} - w_j^2}} (\hat{w} / \bar{r}) (\hat{\beta}_j - \pm \hat{w}_j) \to_d N(0, 1) \quad \text{where } \pm = \text{sign}(a_*), \ \hat{a} = \max(0, \hat{a}^2),
\]

(5.7)

which relates \( \hat{\beta}_j \) and the unknown parameter of interest \( w_j \) to the pivotal standard normal distribution via the observable adjustments \( \hat{w}, \bar{r} \). We summarize this discussion in the next corollary.

**Corollary 5.3.** Let Assumption E be fulfilled. For any sequence \( j = j_n \), if

\[
w_j^2 / \Omega_{jj} \to 0 \quad \text{as} \quad n, p \to +\infty,
\]

(5.8)

then the normal approximation (5.7) holds.

**Proof of Corollary 5.3.** Since \( \mathbb{P}(E) \to 1 \) for the event \( E \) in Theorem 5.2 under Assumption E, the bounds

\[
|\hat{a} - a_*| = |\hat{a} - \pm a_*| = O_p(n^{-1/2}) \quad \text{and max}\{\bar{v}, \bar{r}, \bar{v} / \bar{r}\} = O_p(1)
\]

hold by the last part of Theorem 5.2. Consequently the right-hand side in (5.6) converges to 0 in probability if (5.8) holds.

If the condition number \( \kappa \) of \( \Sigma \) is bounded from above by \( \kappa \) we have \( 1 / \kappa \leq \Omega_{jj} / \|w\|^2 \leq \kappa \) thanks to

\[
\frac{1}{\kappa} \|w\|^2 \leq \frac{w_j^2}{\Omega_{jj}} \leq \kappa \|w\|^2.
\]
As long as the condition number $\kappa$ is a constant independent of $n, p$, the quantity $w_j^2/\Omega_{jj} \to 0$ appearing in the right-hand side of (5.2), (5.4) and of (5.6) is of the same order as $w_j^2/\|w\|^2$, and the condition $w_j^2/\Omega_{jj} \to 0$ is equivalent to $w_j^2$ being negligible compared to the full squared norm $\|w\|^2$ of the index.

In other words, in typical cases where the unregularized M-estimator $\hat{\beta}$ exists, is bounded and has non-vanishing correlation $a_r$ with high-probability as in Assumption E, if the condition number of $\Sigma$ is bounded and $w_j^2/\|w\|^2 \to 0$ then the normal approximation (5.7) holds.

In a well-specified binary logistic regression model where $\hat{\beta}$ is the MLE and $\beta^*$ is the true logistic regression vector normalized such that $\mathbb{E}(\|x_i^T \beta^*\|^2) = (\beta^*)^T \Sigma \beta^*$ is a fixed constant $\gamma^2$, [48, Theorem 3.1] established asymptotic normality of $\hat{\beta}_j$ under the assumption that $\sqrt{n} t_j \beta_j^* = O(1)$ where $t_j^2 = (\epsilon_j^T \Sigma^{-1} \epsilon_j)^{-1}$ in the notation of [48]. Corollary 5.3 obtains asymptotic normality results under the relaxed assumption (5.8), which is equivalent to $t_j \beta_j^* \to 0$ as $n, p \to +\infty$ since $t_j \beta_j^* = \gamma w_j/\sqrt{\Omega_{jj}}$. In other words, the amplitude of $t_j \beta_j^*$ for which asymptotic normality provably holds in Corollary 5.3 is almost $\sqrt{n}$ greater than allowed in previous studies.

### 5.1. Non-separable loss function

We use this section to explain the presence of the indicator function $I_E$ in Theorem 5.2 and to explain the argument behind its proof. Throughout this subsection, assume $p < n$ and $X^T X$ positive definite, which holds with probability one under Assumptions A and C. The situation where $\hat{\beta}$ does not exist stems from the lack of coercivity of the loss function: in the minimization problem $\min_{\beta \in \mathbb{R}^p} \sum_{i=1}^n \ell_{y_i}(x_i^T \beta)$, one can find a sequence $(b(i))_{i \geq 1}$ with $\|X b(i)\| \to +\infty$ as $t \to +\infty$ such that $\ell_{y_i}(x_i^T b(i)) \to \inf_{\beta \in \mathbb{R}^p} \sum_{i=1}^n \ell_{y_i}(x_i^T \beta)$, although the infimum is not attained at any $b \in \mathbb{R}^p$. This is for instance the case in logistic regression when the data is separable (see, e.g., [14] and the references therein).

This situation can be avoided by modifying the loss function to make it coercive. We prove Theorem 5.2 for loss functions satisfying Assumption D by introducing the modified optimization problem

$$
\min_{b \in \mathbb{R}^p} \sum_{i=1}^n \ell_{y_i}(x_i^T b) + nH \left( \frac{1}{2} \left( \frac{1}{n} \|X b\|^2 - K \right) \right)
$$

where $K > 0$ is a fixed constant and $H : \mathbb{R} \to \mathbb{R}$ is a convex smooth function with derivative $h = H'$ satisfying $h(t) = 0$ for $t < 0$, $h(t) = 1$ for $t > 1$. An example of such function $H$ is given in Theorem 5.5 below. This modified optimization maintains desirable properties of $\hat{\beta}$ if $\beta$ exists and satisfies $\frac{1}{n} \|X \beta\|^2 \leq K$, since in this case $\hat{\beta}$ is also a minimizer of (5.9). In other words, the event $E$ of Theorem 5.2, $\hat{\beta}$ is also a minimizer of (5.9) so we may as well study the optimization problem (5.9) to bound from above the expectations in (5.2)-(5.3). The second term in (5.9) could also be useful in practice if $\min_{b \in \mathbb{R}^p} \sum_{i=1}^n \ell_{y_i}(x_i^T b)$ is not attained, as the coercivity of the second term in (5.9) ensures that a minimizer always exists. The modified objective function in (5.9) is not a separable function of $(x_i^T b)_{i \notin [n]}$.

**Theorem 5.4.** Assume that $1 < \delta \leq n/p \leq 26$, that $X$ has iid $N(0, \Sigma)$ rows and $w \in \mathbb{R}^p$ satisfies $w^T \Sigma w = 1$. Let $U$ be a latent random variable independent of $X$ and assume that $\mathcal{L} : \mathbb{R}^n \to \mathbb{R}$ is a random loss function of the form $\mathcal{L}(v) = F(v, U, X w)$ for all $v \in \mathbb{R}^n$ for some deterministic measurable function $F$. Assume that with probability one with respect to $(U, X w)$, $\mathcal{L}$ is convex, coercive, twice differentiable with positive definite Hessian everywhere. Let $\hat{\beta} = \arg\min_{b \in \mathbb{R}^p} \mathcal{L}(X b)$ and assume that $\mathbb{P}(\|\nabla \mathcal{L}(X \hat{\beta})\| \leq \sqrt{n}) = 1$. Extend the notation $D, \hat{A}, V$ to this non-separable setting by

$$
\hat{\psi}(y, X) = -\nabla \mathcal{L}(X \hat{\beta}) \in \mathbb{R}^n, \quad D = \nabla^2 \mathcal{L}(X \hat{\beta}) \in \mathbb{R}^{n \times n}, \quad \hat{A} = (X^T D X)^{-1} \in \mathbb{R}^{p \times p}
$$

and $V = D - DX \hat{A} X^T D$. Define $\hat{\rho} = \|\hat{\psi}\|^2/n$ as well as $\hat{\rho} = \text{Tr}[V]/n$. Let $(\hat{\sigma}^2, \hat{\sigma}_2^2)$ be as in (5.1) and define $\tilde{a}_r = \hat{\beta}_r^T \Sigma w$ as well as $\sigma^2 = \|\Sigma^{1/2} \beta^*\|^2 - \tilde{a}^2_r$. Then

$$
\mathbb{E}[\hat{\rho}^2 - \sigma^2] + \mathbb{E}[\hat{\rho}^2 - \sigma^2] + \mathbb{E}[\hat{\rho}^2 - \sigma^2] + \mathbb{E}[(\sqrt{n} \hat{\psi}^T u - \hat{\rho} Z)^2]^{1/2} \leq \frac{C_2(\delta)}{\sqrt{n}} \left[ \right. \left. (1 \vee \|n^{1/2} \Sigma a \|_{op}) \vee \frac{\sqrt{n}}{D \vee \|\Sigma^{1/2} \beta\| \vee \|n^{-1/2} X \Sigma^{-1/2}\|_{op}) \right]^8
$$
for some $Z \sim N(0,1)$ for any deterministic $u \in \mathbb{R}^p$ with $\|\Sigma^{-1/2}u\| = 1$ such that $w^T u = 0$.

Theorem 5.4 is proved in Appendix D.2. To obtain a desired $n^{-1/2}$ upper bound on the first line of (5.10), we only need to show that the expectation on the right-hand side of (5.10) is bounded from above by a constant, which is done for the loss (5.9) in the next result.

**Theorem 5.5.** Let Assumptions A, C and D be fulfilled so that the penalty is $g = 0$. Let $K > 0$. Define $h(t) = 0$ for $t < 0$, $h(t) = 1$ for $t > 1$ and $h(t) = 3t^2 - 2t^3$ for $t \in [0,1]$, set $H(t) = \int_0^t h(u) du$ and note that $H$ is convex, twice continuously differentiable and nondecreasing. For any $v \in \mathbb{R}^n$, define

$$
\mathcal{L}(v) = \frac{1}{1 + \sqrt{K} + 2} \left[ \sum_{i=1}^n \ell_{y_i}(v_i) + nH\left(\frac{1}{n}\|v\|^2 - K\right)\right].
$$

Then the minimizer $\hat{b} = \arg\min_{b \in \mathbb{R}^p} \mathcal{L}(Xb)$ satisfies $\mathbb{P}(\|\nabla\mathcal{L}(X\hat{b})\| \leq \sqrt{n}) = 1$, the assumptions of Theorem 5.4 are satisfied and the right-hand side of (5.10) is bounded from above by $C_{27}(\delta, K, t)n^{-1/2}$. In the event $E$ of Theorem 5.2, the minimizer $\hat{\beta} = \arg\min_{b \in \mathbb{R}^p} \sum_{i=1}^n \ell_{y_i}(x_i^T b)$ is equal to $\hat{b}$ and $\max\{\hat{r}_1, \frac{1}{\sqrt{n}}, \hat{v}, \frac{1}{\sqrt{n}}\} \leq C_{28}(K, \delta, t)$.

Theorem 5.5 is proved in Appendix D.4. Theorem 5.2 is finally obtained as a consequence of Theorems 5.4 and 5.5 thanks to $E[I_k | \cdot |] \leq E[|\cdot |]$ and using that $\hat{\beta}$ from Theorem 5.2 equals $\hat{b}$ from Theorem 5.5 in $E$, where $E$ is the event in Theorem 5.2.

6. Examples and simulations

6.1. Linear models: Square loss and Huber loss

It is first instructive to specialize Theorem 4.1 and the adjustments (3.8) to loss functions usually used in linear models. For the square loss $\ell_{y_i}(u_i) = \frac{1}{2}(y_i - u_i)^2$ and convex penalty $g$ in (1.2) we have $\ell_{y_0}(u) = 1$ for all $y_0 \in \mathbb{R}$ and

$$
\hat{v} = 1 - \frac{\hat{d}_f}{n}, \quad \hat{\gamma} = \frac{\hat{d}_f}{n - \hat{d}_f} = \frac{1}{\hat{v}} - 1, \quad \hat{r}^2 = \frac{\|y - X\hat{\beta}\|^2}{n}
$$

(6.1)

where $\hat{d}_f$ is defined in (3.4), and the quantity $\hat{d}_f$ equals $\text{Tr}[\frac{\partial}{\partial y} X\hat{\beta}(y, X)]$ thanks to [6, Theorem 2.1] which relates the matrix $\hat{A}$ of Proposition 3.1 to the derivatives with respect to $y$. Thus, $\hat{d}_f$ is the usual notion of degrees-of-freedom of the estimator $\hat{\beta}$ in linear models as introduced in Stein [36], and $\hat{v} = 1 - \hat{d}_f/n$ captures the difference between sample size and degrees-of-freedom. On the other hand, $n\hat{r}^2$ is the usual residual sum of squares.

If the Huber loss $H(u) = \int_0^{\|u\|} \min(1,v)dv$ is used and $\hat{\beta}$ is the estimate (1.2) with $\ell_{y_i}(x_i^T b) = H(y_i - x_i^T b)$ and convex penalty $g$, then

$$
\hat{v} = \frac{n - \hat{d}_f}{n}, \quad \hat{\gamma} = \frac{\hat{d}_f}{n - \hat{d}_f}, \quad \hat{r}^2 = \frac{\|H'(y - X\hat{\beta})\|^2}{n}
$$

(6.2)

where $\hat{I} \equiv \{i = 1, \ldots, n : |y_i - x_i^T \hat{\beta}| \leq 1\}$ and $\hat{n} = |\hat{I}|$ denotes the number of residuals that fall within the interval $[-1, 1]$ where the loss $H(\cdot)$ is quadratic. The estimate $\hat{r}^2$ is the averaged of the squared residuals clipped to $[-1,1]$, since here the derivative $H'$ of the Huber loss is $H'(u) = \max(-1, \min(1,u))$. The integer $\hat{n} = |\hat{I}|$ represents the effective sample size, since observations $i \notin \hat{I}$ do not participate in the fit of (1.2) in the sense that $\frac{\partial}{\partial y_i} \hat{\beta}(y, X) = 0$ for $i \notin \hat{I}$ [6]. Similarly to the square loss case, $n\hat{v} = \hat{n} - \hat{d}_f$ captures the effective sample size left after subtracting the degrees-of-freedom of the estimator $\hat{\beta}$.

For the square loss, both $(n - \hat{d}_f)$ and $\|y - X\hat{\beta}\|^2$ are expected to appear in the confidence interval about $\beta^*_\gamma$ for regularized least-squares [9], while $(\hat{n} - \hat{d}_f)$ and $\|H'(y - X\hat{\beta})\|^2$ are expected to appear in confidence intervals about $\beta^*_\gamma$ for the Huber loss [10]. In Theorem 4.1 on the other hand, the confidence interval is about the component $w_j$ of the normalized index $w$. This is where $\hat{I}$ enters the picture: the role of $\hat{I}$ is to bring the index $w$ (which is normalized with $\|\Sigma^{1/2}w\| = 1$) on the same scale as $\hat{v}\hat{\beta}$. The following proposition makes this precise.
Proposition 6.1. Let Assumptions A and B be fulfilled and additionally assume a linear model where observations \( y_i = x_i^T \beta^* + \varepsilon_i \) are iid with additive noise \( \varepsilon_i \) independent of \( x_i \). Set \( w = \beta^* / ||\Sigma^{1/2} \beta^*|| \), and assume that \( ||\Sigma^{1/2} \beta^*|| \) equals a constant independent of \( n, p \). Then
\[
\hat{t} = \hat{v} ||\Sigma^{1/2} \beta^*|| + n^{-1/2} O_P(1) [\hat{r} + ||\Sigma^{1/2} \beta^*||].
\]

Above, \( O_P(1) \) denotes a random variable \( W \) such that for any \( \eta > 0 \) there exists a constant \( K \) depending on \( (\eta, \delta, \tau, ||\Sigma^{1/2} \beta^*||) \) only such that \( \mathbb{P}(|W| > K) \leq \eta \).

Proposition 6.1 justifies the approximation \( \hat{t} \approx \hat{v} ||\Sigma^{1/2} \beta^*|| \), and combined with (4.3), \( \sqrt{n} \hat{v}(\hat{\beta}^{(d)} - \beta^*_1) \approx \hat{r} \Omega^{1/2}_{jj} Z_j \) with \( Z_j \sim N(0,1) \). This recovers the asymptotic normality result proved for regularized least-squares in [9] and for some robust loss functions in [10]. Proposition 6.1 illustrates that \( \hat{t} \) brings the normalized index \( w \) on the same scale as \( \hat{v} \beta^* \) and \( \hat{v}\hat{\beta}^{(d)} \) in this linear model setting.

We note in passing that Proposition 6.1 justifies the use of \( \hat{t}/\hat{v} \) to estimate the signal strength \( ||\Sigma^{1/2} \beta^*|| \), when an initial M-estimator \( \hat{\beta} \) is provided to estimate to high-dimensional parameter \( \beta^* \). For \( \hat{\beta} = 0 \) (which can be seen as a special case of (1.2) with penalty satisfying \( g(0) = 0 \) and \( g(b) = +\infty \) for \( b \neq 0 \)), the quantity \( \hat{t}^2/\hat{v}^2 \) reduces to the estimator of the signal strength in [19]. Table 1 reports experiments demonstrating the accuracy of Proposition 6.1. The approximate normality of \( \hat{t}/\hat{v} - ||\Sigma^{1/2} \beta^*|| \) observed in the QQ-plot of Table 1 is not currently proved theoretically—a proof of this observed normality remains an open problem.

| Estimate | Signal strength |
|----------|-----------------|
| \( \hat{t}/\hat{v} \) | \( ||\Sigma^{1/2} \beta^*|| \) |
| 2.071 ± 0.122 (average ± std) | 2.000 |

![Histogram and QQ-plot of \( \hat{t}/\hat{v} \) and \( ||\Sigma^{1/2} \beta^*|| \).](image)

**Table 1**

**Estimate \( \hat{v}/\hat{v} \) of the signal strength \( ||\Sigma^{1/2} \beta^*|| \) in a linear model \( y_i = x_i^T \beta^* + \varepsilon_i \) independent of \( x_i \sim N(0, \Sigma) \) with \( \Sigma = I_p \), with \( n = 1500 \), \( p = 1501 \) and \( ||\beta^*||_0 = 200 \) with all non-zero coordinates equal to the same value. The M-estimator \( \hat{\beta} \) is chosen with \( \ell_{0.1}(u) = H(u - y_i) \) for the Huber loss \( H(u) = \int_0^u \min(1, v) dv \) and Elastic-Net penalty \( g(b) = n^{-1/2} ||b||_1 + 0.05 ||b||_2^2 \). Average, standard deviation, histogram and QQ-plot of \( \hat{t}/\hat{v} \) were computed over 100 independent realizations of the dataset. Over the 100 repetitions, \( \hat{\beta} \) has False Negatives 47.4 ± 7.3, False Positives 271.36 ± 13.4 and True Positives 152.6 ± 7.3.

In summary, for the square loss and Huber loss,

- \( \hat{r}^2 \) is a generalization of the residual sum of squares,
- \( \hat{v} \) is the difference of an effective sample size minus the degrees-of-freedom of \( \hat{\beta} \), and
- \( \hat{t}/\hat{v} \) brings the normalized index \( w \) on the same scale as \( \hat{\beta} \).
6.2. Least-squares with nonlinear response

Let \( p/n \leq \gamma < 1 \). We now focus on the square loss, \( \ell_{\text{sq}}(u) = (y_i - u)^2 \) with no penalty (\( g = 0 \) in (1.2)), so that \( \hat{\beta} \) is the least-squares estimate \( \hat{\beta} = (X^TX)^{-1}X^Ty \). We emphasize that here, \( y \) does not follow a linear model: \( y_i \) is allowed to depend non-linearly on \( x_i^T w \) as in the single index model (1.1). In this setting, \( \hat{\sigma}^2 = \frac{1}{n} ||y - X\hat{\beta}||^2 \) is the residual sum of squares as in (6.1), \( \hat{\nu} = 1 - \frac{\nu}{\nu+1} \) since the degrees-of-freedom \( \hat{df} \) equals \( p \), and \( (\hat{a}, \hat{t}) \) defined in (5.1) satisfy

\[
\frac{\hat{t}^2}{\hat{\nu}^2} = \frac{\hat{a}^2}{\hat{\nu}^2} = \frac{||X\hat{\beta}||^2}{||X\beta - y||^2} - \frac{p}{n-p}.
\]

(6.3)

Assuming \( p/n \leq \gamma < 1 \), Theorem 5.2 yields the approximation

\[
\frac{n-p}{2} \Omega_{jj}^{1/2} \left[ \frac{\hat{\beta}_j}{\sqrt{n}} - \frac{\pm w_j}{\sqrt{n}} \left( \frac{||X\hat{\beta}||^2}{||X\beta - y||^2} - \frac{p}{n-p} \right)^{1/2} \right] \approx N(0,1)
\]

(6.4)

where \( \pm \) is the sign of \( w^T \Sigma \hat{\beta} \). If \( \Sigma \) is unknown, the quantity \( \Omega_{jj} \) is linked to the noise variance in the linear model of regressing \( Xe_i \) onto \( X_{-j} \): \( \Omega_{jj} \) can be estimated using

\[
\Omega_{jj} \left[ \frac{I_n - X_{-j}(X_{-j}^TX_{-j})^{-1}X_{-j}^T}{X_{-j}} \right] e_i^2 \sim \chi^2_{n-p+1}
\]

where \( X_{-j} \in \mathbb{R}^{n \times (p-1)} \) has the \( j \)-th column removed. While (6.4) does not formally follow from Theorem 5.2 because the square loss fails to satisfy Assumption D, the argument of the proof of Theorem 5.2 only requires minor modifications to obtain (5.2) and (6.4) for the square loss (thanks to \( \ell_{\text{sq}}' = 1 \), the proof for the square loss is actually much simpler than for the logistic loss and other loss functions covered by Theorem 5.2). Alternatively, (6.4) follows from Theorem 5.4 with \( L(v) = \frac{1}{2} v - \frac{1}{2} \frac{p}{n} y^2 \).

For fixed \( p \) and \( n \to +\infty \), asymptotic normality and confidence intervals for the least-squares and penalized least-squares is studied in [46, 32, 31]. With \( \Sigma = I_p \) to simplify comparison, asymptotic normality in [13] concerns the random variable \( \hat{\beta} - \mu w \) and the estimation bounds in [46, 32, 31] bounds the estimation error of \( ||\hat{\beta} - \mu w|| \) where \( \mu = \mathbb{E}_{\gamma \sim N(0,1)}[gF(g, U_i)] \) where \( F \) is the function defining the single index model in (1.1). The scaled vector \( \mu w \) appears here because it is the minimizer of the population minimization problem \( \min_{b \in \mathbb{R}^p} \mathbb{E}[||x_i^T b - y_i||^2] \). The constant \( \mu \) is typically unknown. A major difference with these previous results featuring the unknown \( \mu \) is that the multiplicative coefficient of \( w_j \) in (6.4) is an estimate from the data.

We illustrate the normal approximation (6.4) in Table 2 with \( n = 3000, p = 2400 \) and four different models: linear, logistic, Poisson, and 1-Bit compressed sensing with a 20% probability of flipped bits (\( \mathbb{P}(u_i = -1) = 0.2 = 1 - \mathbb{P}(u_i = 1) \) with \( u_i \) independent of \( x_i \)).

6.3. Ridge regularized M-estimation

Consider now an isotropic design with \( \Sigma = \frac{1}{p} I_p \) and the Ridge penalty \( g(b) = \lambda ||b||^2 / (2p) \) as for the nonlinear system (1.10) of [34]. The optimality conditions of the optimization problem (1.2) are

\[
X^T \hat{\psi} = (n/p)\lambda \hat{\beta}
\]

(6.5)

so that the terms \( \hat{\psi}^T X \hat{\beta} / n \) and \( ||\Sigma^{-1/2}X^T \hat{\psi}||^2 / n^2 \) in (3.8) and (3.10) reduce to

\[
\hat{\psi}^T X \hat{\beta} / n = \lambda ||\hat{\beta}||^2 / p, \quad ||\Sigma^{-1/2}X^T \hat{\psi}||^2 / n^2 = \lambda^2 ||\hat{\beta}||^2 / p.
\]

(6.6)

Computing explicitly \( \partial / \partial x_{ij} \hat{\beta}(y, X) \) in Proposition 3.1 for this Ridge penalty yields \( \hat{A} = (X^TDX + \lambda^2 I_p)^{-1} \). This implies that \( \gamma_{\text{ridge}} \) satisfies

\[
n\lambda \gamma_{\text{ridge}} + \hat{df} = Tr[\lambda \frac{\hat{A}}{\hat{\gamma}} x + X^TDX \hat{A}] = Tr[I_p] = p
\]
by definition of $\hat{d}$ in Proposition 3.1. Next, (4.11) provides $\hat{d}f/n = \hat{v} + O_p(n^{-1/2})$ so that $(\lambda + \hat{v})\gamma_* = \frac{\hat{v}^2}{n} + O_p(n^{-1/2})$. This justifies replacing $\gamma_*$ by $\frac{\hat{v}^2}{n} (\lambda + \hat{v})^{-1}$ in the expression of $\hat{a}^2$, which provides the approximations

$$
\begin{align*}
\hat{a}^2 &\approx \frac{1}{p} \|\hat{\beta}\|^2 - \left(\frac{p}{n}\right)\hat{v}^2, \\
\hat{a}^2 &\approx \frac{1}{p} \|\hat{\beta}\|^2 - \hat{a}^2 \approx \frac{p}{n} \left(\frac{p}{p + \lambda}\right)^2 \approx \frac{n}{p} \gamma_*^2 \hat{v}^2.
\end{align*}
$$

In this setting, the normal approximation (4.3) from Theorem 4.1 becomes

$$
\frac{(n/p)^{1/2}}{\hat{v}} \left[ (\hat{v} + \lambda)\hat{\beta}_j - \mp \hat{t} w_j \right] \approx Z_j \quad \text{with} \quad Z_j \sim N(0, 1).
$$

We illustrate these results for Ridge regularized M-estimation with the simulation study in Table 3 with the logistic loss in the logistic model.

Because the simple structure of the matrix $\hat{A}$ and the KKT conditions (6.5), for isotropic design the estimation error of $\sigma_*^2 = (w^T \Sigma \hat{\beta})^2$ and of $\sigma_*^2 = \|\Sigma^{1/2} \hat{\beta}\|^2 - a_*^2$ can be slightly improved compared to the estimation error in (4.14), and the proof is significantly simpler. With $\Sigma = \frac{1}{p} I_p$ as in the present subsection, we have $\gamma_* = \frac{1}{p} \text{Tr}[\hat{A}]$, $a_* = \frac{1}{p} w^T \hat{\beta}$ and $\sigma_* = \frac{1}{p} \|\hat{\beta}\|^2 - a_*^2$.

**Proposition 6.2.** Let Assumption A be fulfilled with $\Sigma = \frac{1}{p} I_p$. Set $g(b) = \lambda_0 \|b\|_2^2/p$ as the penalty in (1.2). If additionally $\sup_{y \in \mathcal{Y}} |\hat{t}'_{y\theta}| \leq 1$ then

$$
\mathbb{E}\left[ \left( \sigma_*^2 - \frac{\gamma_* \hat{v}^2}{\lambda + \hat{v}} \right)^2 \right] \leq \frac{C_{29}(\delta, \lambda)}{p}, \quad \mathbb{E}\left[ \left( a_*^2 - \left(\frac{1}{p} \|\hat{\beta}\|^2 - \frac{\gamma_* \hat{v}^2}{\lambda + \hat{v}} \right) \right)^2 \right] \leq \frac{C_{30}(\delta, \lambda)}{p}.
$$

The proof of Theorem 4.4 for general penalty functions encompasses (6.9) up to multiplicative factor $\hat{v}^2\hat{v}^2/\hat{v}^2$ in (4.14). The bound (6.9) avoids such multiplicative factor. We give a short proof of (6.9) in Appendix E, in order to present a concise overview of some of the techniques behind the proof of more general result in Theorem 4.4.

### 6.4. L1 regularized M-estimation

The last example concerns L1 regularized M-estimation, with penalty $g(b) = \lambda_{n,p} \|b\|_1$ where $\lambda_{n,p} > 0$ is a tuning parameter. Throughout Section 6.4, let $\mathcal{S} = \{j \in [p] : \hat{\beta}_j = 0\}$ be the set of active covariates...
For each \( \lambda \) and the probabilistic inequalities used in the proof of (4.14). We verify in Figure 1 the validity of this

| \( \lambda \) | 0.01 | 0.10 | 1.00 |
|-----------------|---|---|---|
| \( \frac{1}{n} \| \hat{X} \hat{\beta} - \hat{\psi} \|^{2} - \frac{\pi / n}{(1 + \lambda)^{2}} r^{2} \) | 0.621942±0.160021 | 0.168159±0.043360 | 0.016483±0.004789 |
| \( \frac{1}{n} \| \hat{\beta} \|^{2} - \frac{\pi / n}{(1 + \lambda)^{2}} r^{2} \) | 0.630087±0.167536 | 0.170862±0.039237 | 0.016765±0.003354 |
| \( \alpha^{2} = \frac{1}{p} \frac{(w_{j}^{T} \hat{\beta})^{2}}{\| w_{j} \|^{2}} \) | 0.610240±0.039087 | 0.164714±0.009765 | 0.016184±0.000914 |

(6.8) for \( j : w_{j} = 0 \)

| \( \frac{p}{n} \| \hat{\beta} \|^{2} \) | 8.540738±0.265670 | 2.105677±0.047351 | 0.185784±0.001738 |
| \( \sigma^{2} = \frac{1}{p} \left\| \left( I - \frac{w_{j}^{T} w_{j}}{p} \right) \hat{\beta} \right\|^{2} \) | 8.560584±0.110858 | 2.111824±0.014522 | 0.186364±0.001680 |

Table 3

In the logistic model \( y_{i} \in\{0, 1\} \), \( P(y_{i} = 1) = 1/(1 + \exp(-a_{j}^{T} w_{j})) \) with \( \hat{x}_{i} \sim N(0, \Sigma) \) and isotropic covariance \( \Sigma = \frac{1}{2} I_{p} \), the M-estimator \( \hat{\beta} \) is constructed with logistic loss \( \ell_{y_{i}}(u) = \log(1 + e^{-u}) - y_{i}u \) and Ridge penalty \( g(b) = \lambda \| b \|^{2} / p \).

Dimension and sample size are \((n, p) = (5000, 10000)\) and the tuning parameters \( \lambda \in \{0.01, 0.1, 1.0\} \) are used. For \( \alpha^{2} \), its estimates \( \frac{1}{n} \| \hat{X} \hat{\beta} - \hat{\psi} \|^{2} - \frac{\pi / n}{(1 + \lambda)^{2}} r^{2} \) and \( \frac{1}{p} \| \hat{\beta} \|^{2} - \frac{\pi / n}{(1 + \lambda)^{2}} r^{2} \), as well as \( \sigma^{2} \) and its estimate \( \frac{p}{n} \| \hat{\beta} \|^{2} \), we report the average standard deviation over 50 independent repetitions of the dataset. The index \( w \) has \( s = 100 \) entries equal to \( \sqrt{p/s} \) and \( p - s \) entries equal to 0. Middle rows show standard normal QQ-plots of \( \{ r^{-1}(n/p)^{1/2}[(\hat{y} + \gamma_{i})\hat{\beta} + \pm \hat{w}_{j}]_{j=1,...,p} \} \) (as in (6.8)) for the null covariates \( j \in [p] : w_{j} = 0 \) collected over the 50 repetitions (the QQ-plot thus featuring 50\( p \) points), and for the non-nulls \( j \in [p] : w_{j} \neq 0 \) (the QQ-plot thus featuring 50\( s \) points).

of the corresponding L1 regularized M-estimator in (1.2). Then, by now well-understood arguments for the Lasso (cf. [43, 42][7, Proposition 3.10][5, Proposition 2.4]), the KKT conditions of (1.2) hold strictly with probability one, and the formulae (3.7) hold true with \( \hat{A} \) symmetric and diagonal by block with \( \hat{A}_{S,S} = (X_{S}^{T}DX_{S})^{-1} \) and \( \hat{A}_{S^{c},S^{c}} = 0 \). This implies that \( df \) defined in (3.4) is simply

\[
\hat{df} = \text{Tr}[\hat{A}X^{T}DX] = \text{Tr}[\hat{A}_{S,S}(\hat{A}_{S,S})^{-1}] = |\hat{S}|
\]

for almost every \( X \) if the diagonal matrix \( D \) has at least \( |\hat{S}| \) positive entries. This motivates the notation \( \hat{df} \) for the effective degrees-of-freedom of \( \hat{\beta} \), following tradition on the Lasso in linear regression [49, 43] in the context of Stein’s Unbiased Risk Estimate [36].

In previous sections, the algebraic nature of the loss and/or the penalty allowed some major simplifications in the expression of \( \hat{\alpha}^{2}, \hat{I}^{2} \) and other adjustments in (3.8), see for instance (6.3) for the square loss and (6.7) for Ridge regularized estimates under isotropic design. In this case of L1-regularized M-estimation, no such simplification other than \( \hat{df} = |\hat{S}| \) is available and the estimator \( \hat{a}^{2} \) in (3.8) of the squared correlation \( (\hat{\beta}^{T} \Sigma w)^{2} \) has expression

\[
\hat{a}^{2} = \frac{1}{n^{2}} \left\| \Sigma^{-1/2} X^{T} \hat{\psi} \right\|^{2} + \frac{2 \hat{u}^{T} \hat{\beta}}{n} \left\| X^{T} \hat{\beta} - \hat{\psi} \right\|^{2} - \frac{\hat{u}^{T}}{n} r^{2}.
\]

(6.10)

We could not think of any intuition to suggest the validity this expression, except from the algebra and the probabilistic inequalities used in the proof of (4.14). We verify in Figure 1 the validity of this estimate under the following simulation setting. Let \( \Sigma = I_{p} \), \( n = 1000 \) and \( p \in \{500, 1000, 2000, 2500\} \). For each \( q \in \{1, 2, 4, 6, 8\} \), the response \( y_{i} \) is generated from the model \( y_{i} | x_{i} \sim \text{Binomial}(q, \rho'(x_{i}^{T} \beta^{*})) \) where \( \rho'(t) = 1/(1 + e^{-t}) \) is the sigmoid. The unknown vector \( \beta^{*} \) has \( \frac{q}{20} \) nonzero coefficients and is
which denominator of the normalized correlation to avoid numerical instability for large tuning parameters for $X$ dataset (averages (as well as error bands depicting standard errors times 1.96), over 50 independent copy of the $X$ dataset). Figure 1 reports averages (as well as error bands depicting standard errors times 1.96), over 50 independent copy of the $X$ dataset. Figure 1 shows that (6.10) is accurate across different dimensions, different tuning parameter $\lambda$ and different binomial parameter $q$. Estimation inaccuracies start to appear at small tuning parameter $\lambda$ and for the largest values of the dimension $p$.

Fig 1. Left: Correlation $a_* = w^T \Sigma \hat{\beta}$ and its estimate $\sqrt{\hat{a}^2}$ for $\hat{a}^2$ in (6.10) for $n = 1000$, in the binomial model $y_i | x_i \sim Binomial(q, \rho'(x_i^T \beta^*))$ and the L1-penalized M-estimator (6.11). The precise simulation setting is described in Section 6.4. The x-axis represents the tuning parameter $\lambda$ in (6.11).

deterministically generated as follows: first generate $\beta^0$ with the first $\frac{p}{20}$ coefficients being equispaced in $[0.5, 4]$ and with remaining coefficients equal to 0, second set $\beta^* = 1.1 \beta^0 / \| \Sigma^{1/2} \beta^0 \|$. For each dataset, we compute the regularized M-estimate

$$\hat{\beta} = \arg\min_{b \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^{n} \left[ q \log(1 + e^{x_i^T b}) - y_i x_i^T b \right] + \frac{\lambda}{\sqrt{n}} \| b \|_1$$

for each tuning parameter $\lambda$ in a logarithmic grid of cardinality 20 from 0.3 to 3.0. The x-axis represents the tuning parameter $\lambda$ in (6.11).

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SUPPLEMENT

Appendix A: Derivatives

Proposition 3.1. Assume that $\ell''_y$ is $1$-Lipschitz for all $y_0 \in \mathcal{Y}$ and that (2.1) holds for some some $\tau > 0$ and positive definite $\Sigma \in \mathbb{R}^{p \times p}$. Then for any fixed $y \in \mathcal{Y}^n$ the function $\hat{\beta}(y, \cdot)$ is differentiable almost everywhere in $\mathbb{R}^{n \times p}$ with derivatives

\[ (\partial/\partial x_{ij}) \hat{\beta}(y, X) = \hat{A}(e_j \hat{\psi}_i - X^T D e_i \hat{\beta}_j) \]

where $D \overset{\text{def}}{=} \text{diag}(\ell''_y(X \hat{\beta}))$ and some matrix $\hat{A} \in \mathbb{R}^{p \times p}$ with

\[ \|\Sigma^{1/2} \hat{A}\Sigma^{1/2}\|_{\text{op}} \leq (nt)^{-1} \]

and such that the following holds: If $D X \hat{\beta} \neq \hat{\psi}$ we have

\[ 0 \leq \hat{d} \leq n \quad \text{where } \hat{d} \overset{\text{def}}{=} \text{Tr}[X \hat{A} X^T D], \]

\[ \text{Tr}[D]/(1 + \hat{c}) \leq \text{Tr}[V] \leq \text{Tr}[D] \quad \text{where } V \overset{\text{def}}{=} D - D X \hat{A} X^T D, \]

where $\hat{c} = \frac{1}{n^2} \|D^{1/2} X \Sigma^{-1/2}\|_{\text{op}}^2$, while if $D X \hat{\beta} = \hat{\psi}$ we have the slightly weaker

\[ -\hat{c} \leq \hat{d} \leq n + \hat{c}, \]

\[ -4\hat{c} + \text{Tr}[D]/(1 + \hat{c}) \leq \text{Tr}[V] \leq \text{Tr}[D] + 4\hat{c}. \]

Proof of Proposition 3.1. Throughout, let $y \in \mathbb{R}^n$ be fixed. Let $X, \tilde{X} \in \mathbb{R}^{n \times p}$ with corresponding minimizers $\hat{\beta} = \hat{\beta}(y, X), \tilde{\beta} = \hat{\beta}(y, \tilde{X})$, for the same response vector $y \in \mathbb{R}^n$. Let also $\tilde{\psi} = \tilde{\psi}(y, \tilde{X})$ be the counterpart of $\hat{\psi} = \hat{\psi}(y, X)$ for $\tilde{X}$. The KKT conditions read $X^T \tilde{\psi} \in n \partial g(\tilde{\beta})$ and $X^T \hat{\psi} \in n \partial g(\hat{\beta})$. Multiplying by $\hat{\beta} - \tilde{\beta}$ and taking the difference we find

\[ n(\hat{\beta} - \tilde{\beta})^T (\partial g(\hat{\beta}) - \partial g(\tilde{\beta})) + (X \hat{\beta} - \tilde{X} \tilde{\beta})^T (\ell''_y(X \hat{\beta}) - \ell''_y(\tilde{X} \tilde{\beta})) \]

\[ = (\hat{\beta} - \tilde{\beta})^T [X^T \hat{\psi} - \tilde{X}^T \tilde{\psi}] + (X \hat{\beta} - \tilde{X} \tilde{\beta})^T (\hat{\psi} - \tilde{\psi}) \]

\[ = (\hat{\beta} - \tilde{\beta})^T (X - \tilde{X})^T \hat{\psi} + (X \hat{\beta} - \tilde{X} \tilde{\beta})^T (\hat{\psi} - \tilde{\psi}) \]

Note that by convexity of $g$ and of $\ell$, the two terms in the first line are non-negative. If $g$ is strongly convex ($\tau > 0$ from Assumption B), the first term on the first line is bounded from below as follows: $\mu \|\hat{\beta} - \tilde{\beta}\|^2 \leq n (\hat{\beta} - \tilde{\beta})^T (\partial g(\hat{\beta}) - \partial g(\tilde{\beta}))$ for some constant $\mu > 0$ (e.g., $\mu = \phi_{\text{min}}(\Sigma) \tau n$ works). For the second term in the first line of (A.1),

\[ \|\hat{\psi} - \tilde{\psi}\|^2 = \|\ell''_y(X \hat{\beta}) - \ell''_y(\tilde{X} \tilde{\beta})\|^2 \leq (X \hat{\beta} - \tilde{X} \tilde{\beta})^T (\ell''_y(X \hat{\beta}) - \ell''_y(\tilde{X} \tilde{\beta})) \]

since $\ell''_y$ is increasing and 1-Lipschitz for all $y_0 \in \mathbb{R}$. Using $(\hat{\beta} - \tilde{\beta})^T (X - \tilde{X})^T \hat{\psi} + \tilde{\beta}^T (\tilde{X} - X)^T (\hat{\psi} - \tilde{\psi}) \leq \|X - \tilde{X}\|_{\text{op}} (\|\hat{\psi}\| + \|\tilde{\beta} - \tilde{\beta}\| + \|\hat{\beta} - \tilde{\beta}\|)$ we find

\[ \mu \|\hat{\beta} - \tilde{\beta}\|^2 + \|\hat{\psi} - \tilde{\psi}\|^2 \leq \|\hat{\beta} - \tilde{\beta}\| \|X - \tilde{X}\|_{\text{op}} \|\hat{\psi}\| + \|\hat{\beta} - \tilde{\beta}\| \|\hat{\psi} - \tilde{\psi}\|. \]

Note that $X \mapsto \hat{\beta}(y, X)$ is continuous as $\hat{\beta}(y, X)$ is the unique minimizer, by strong convexity, of the continuous objective function $L_y(X, b) = \frac{1}{n} \sum_{i=1}^n \ell_y(x_i^T b) + g(b)$. By continuity $\sup_{X \in K} \|\hat{\psi}(y, X)\| + \|\hat{\beta}(y, X)\|$ is bounded for any compact $K$. This proves that $X \mapsto \hat{\beta}(y, X)$ is Lipschitz in any compact and differentiable almost everywhere in $\mathbb{R}^{n \times p}$ by Rademacher’s theorem.

Recall that $y$ is fixed throughout this proof. Assume that $X \mapsto \hat{\beta}(y, X)$ is differentiable at $X$. This means that for some linear map $\hat{\beta}_X : \mathbb{R}^{n \times p} \rightarrow \mathbb{R}^p$ we have for $\hat{X} \in \mathbb{R}^{n \times p}$

\[ \hat{\beta}(y, X + \hat{X}) - \hat{\beta}(y, X) = \hat{\beta}_X(\hat{X}) + o(\|\hat{X}\|_F) \]
and \( \frac{\partial}{\partial x_i} \hat{\beta}(y, X) = \hat{\beta}_X(e_i, e_j^T) \). In the following we consider a fixed direction \( \hat{X} \in \mathbb{R}^{n \times p} \) and write \( \hat{\beta} \) for the directional derivative in direction \( \hat{X} \), that is, \( \hat{\beta} = \frac{d}{dt} \hat{\beta}(y, X + t\hat{X})\big|_{t=0} \). This is equivalent to \( \hat{\beta} = \hat{\beta}_X(\hat{X}) \) with the notation \( \hat{\beta}_X(\cdot) \) of the previous display; we write \( \hat{\beta} \) for brevity. Then we have \( \frac{d}{dt}(X + t\hat{X})\hat{\beta}(y, X + t\hat{X})\big|_{t=0} = X\hat{\beta} + \hat{X}\hat{\beta} \) and \( \frac{d}{dt}(\hat{\psi}(y, X + t\hat{X}))\big|_{t=0} = -D(X\hat{\beta} + \hat{X}\hat{\beta}) \) by the chain rule. By bounding \( n(\hat{\beta} - \beta)^T(\partial g(\hat{\beta}) - \partial g(\beta)) \) from below by \( n\tau \| \Sigma^{1/2}(\hat{\beta} - \beta) \|^2 \) in (A.1) for \( \hat{X} = X + t\hat{X} \) and \( \beta = \beta(y, X + t\hat{X}) \), dividing by \( t^2 \) and taking the limit as \( t \to 0 \), we find
\[
\tau n\| \Sigma^{1/2}\beta \|^2 + (X\beta + \hat{X}\beta)^T D(X\beta + \hat{X}\beta) \leq \beta^T \hat{\psi} + \hat{\beta}^T \hat{\psi}^T D(X\beta + \hat{X}\beta).
\]
Equivalently, noting that the terms \( (X\hat{\beta})^T D(X\hat{\beta} + \hat{X}\hat{\beta}) \) cancel out,
\[
\tau n\| \Sigma^{1/2}\beta \|^2 + \| D^{1/2} X\beta \|^2 \leq \beta^T \mathcal{L}(X) \text{ with } \mathcal{L}(X) = \hat{\psi}^T X - X^T D X \hat{\beta}.
\] (A.3)

The matrix \( n\tau \Sigma + X^T D X \) is positive definite thanks to \( \tau > 0 \). Thus \( \hat{\beta} = 0 \) for every direction \( \hat{X} \) such that \( \hat{X}^T \hat{\psi} - X^T D \hat{X} \hat{\beta} = 0 \). We have established the inclusion of kernel of linear mappings \( \mathbb{R}^{n \times p} \to \mathbb{R}^p \)
\[
\ker(\hat{X} \to \mathcal{L}(\hat{X})) \subseteq \ker(\hat{X} \to \hat{\beta}).
\]

This implies the existence\(^1\) of \( \hat{A} \in \mathbb{R}^{p \times p} \) with \( \ker(\hat{A})^\perp \cap \text{Range}(\mathcal{L}(\cdot)) = \{ \mathcal{L}(\hat{X}), \hat{X} \in \mathbb{R}^{n \times p} \} \) such that \( \beta = \hat{A}\mathcal{L}(\hat{X}) \). The choice \( \hat{X} = e_i e_j^T \) for canonical basis vectors \( e_i \in \mathbb{R}^n, e_j \in \mathbb{R}^p \) gives the desired formula for \( (\partial/\partial x_{ij}) \hat{\beta}(y, X) \).

Inequality (A.3) implies that for all \( u \in \text{Range}(\mathcal{L}(\cdot)) \supset \ker(\hat{A})^\perp \),
\[
\tau n\| \Sigma^{1/2}\hat{A}^T u \|^2 + \| D^{1/2} X \hat{A} u \|^2 \leq u^T \hat{A}^T u.
\] (A.4)

Let \( v \) with \( \| v \| = 1 \) such that \( \| \Sigma^{1/2} \hat{A} \Sigma^{1/2} \|_{op} = \| \Sigma^{1/2} \hat{A} \Sigma^{1/2} \| \) and \( v \in \ker(\hat{A})^\perp \). Then \( u = \Sigma^{1/2} v \) satisfies \( \| \Sigma^{1/2} \hat{A} \Sigma^{1/2} \|_{op} = \| \Sigma^{1/2} \hat{A} u \| \) and \( u \in \ker(\hat{A})^\perp \) with \( \| \Sigma^{-1/2} u \| = 1 \), so that the previous display gives
\[
\tau n\| \Sigma^{1/2} \hat{A} \Sigma^{1/2} \|^2_{op} \leq u^T \hat{A}^T u \leq \| \Sigma^{1/2} \hat{A} \Sigma^{1/2} \|^2_{op}
\]
which proves (3.3).

We now bound \( \text{Tr}[V] \) from below. For any \( v \in \mathbb{R}^n \), we would like to find some \( \hat{X} \) with \( \mathcal{L}(\hat{X}) = X^T D v \).

Consider \( \hat{X} \) of the form \( S D X \) for symmetric \( S \in \mathbb{R}^{n \times n} \). Then \( \mathcal{L}(\hat{X}) = X^T D S (\hat{\psi} - D \hat{X} \hat{\beta}) \), since for any two vectors \( a, b \) of the same dimension, there exists a symmetric \( S \) such that \( S a = b \), if \( \hat{\psi} - D X \hat{\beta} \neq 0 \) then we can always find some \( \hat{X} \) such that \( X^T D v = \mathcal{L}(\hat{X}) \) and (A.4) with \( u = \mathcal{L}(\hat{X}) \) yields
\[
\tau n\| \Sigma^{1/2} \hat{A} X^T D v \|^2 + \| D^{1/2} X \hat{A} X^T D v \|^2 \leq v^T D X \hat{A} X^T D v.
\]
The LHS is further lower bounded by \( (\tau n\| DX \Sigma^{-1/2} \|_{op}^2 + 1)\| D^{1/2} X \hat{A} X^T D v \|^2 \). Since for any \( v' \in \mathbb{R}^n \) we can find \( v \in \mathbb{R}^n \) such that \( D^{1/2} v' = D v \), this shows that
\[
(\tau n\| D^{1/2} X \Sigma^{-1/2} \|_{op}^2 + 1)\| M v' \|^2 \leq v'^T M v' \leq \| M \|_{op} \| v' \|^2 \quad \text{for } M = D^{1/2} X \hat{A} X^T D^{1/2}.
\]
This proves \( \| M \|_{op} \leq (\tau n\| DX \Sigma^{-1/2} \|_{op}^2 + 1)^{-1} \). Thus
\[
\text{Tr}[V] = \text{Tr}[D] - \text{Tr}[D^{1/2} M D^{1/2}] = \text{Tr}[D] - \text{Tr}[D^{1/2} M^* M D^{1/2}] \geq \text{Tr}[D][1 - (\tau n\| D^{1/2} X \Sigma^{-1/2} \|_{op}^2 + 1)^{-1}]
\]
for \( M^* = \frac{1}{2}(M + M^T) \) the symmetric part. We have established the desired lower bound on \( \text{Tr}[V] \) in the case \( \hat{\psi} - D X \hat{\beta} \neq 0 \). The previous displays also show that \( M^* \) is psd, so that \( \text{Tr}[V] \leq \text{Tr}[D] \leq n \) and \( 0 \leq \hat{d} \leq \text{Tr}[M^*] \leq n \). This proves (3.4) if \( \hat{\psi} \neq D X \hat{\beta} \).

The situation is more delicate if \( \hat{\psi} - D X \hat{\beta} = 0 \). In this case, \( \mathcal{L}(\hat{X}) = (X^T D X - X^T D \hat{X}) \hat{\beta} \). If \( \hat{\beta} = 0 \) then \( \ker(\hat{A})^\perp \cap \text{Range}(\mathcal{L}) = \{ 0 \} \) implies that \( \hat{A} = 0 \) and all stated results (3.3)-(3.4) hold trivially. Now assume \( \hat{\beta} \neq 0 \) and Denote by \( \dagger \) the pseudo-inverse. Define the subspace \( V = \{ v' \in \)

\(^1\)Indeed, if ker \( B \subseteq \ker C \) and \( B \) has SVD \( B = \sum_i u_i s_i v_i^T \) then \( A = \sum_i C v_i u_i^T / s_i \) is such that \( C = AB \).
$\mathbb{R}^n : \hat{\psi}^T (D^{1/2})^i v' = 0$. For any $v' \in \mathbb{V}$, let $v = (D^{1/2})^i v'$ so that $Dv = D^{1/2}v'$ and \(\hat{\psi}^T v = 0\). Set $X = -v \hat{\beta}' \| \hat{\beta} \|^{-2}$ so that $\mathcal{L}(X) = X^T D^{1/2} v'$. By (A.4) and $M = D^{1/2} X \hat{X} \hat{X}^T D^{1/2}$ we find \((\tau n)^{1/2} \| \mathcal{L} D^{1/2} X \hat{X} \hat{X}^T D^{1/2} \|_{op} + 1 \| M v' \| \| v' \| \leq u^T M v'\). If $P_V \in \mathbb{R}^{n \times n}$ is the orthogonal projector onto $\mathbb{V}$ with rank at least $n - 1$, this proves that the symmetric matrix $P_V M^* P_V$ is positive semi-definite with eigenvalues at most \((\tau n)^{1/2} \| D^{1/2} X \Sigma^{-1/2} \|_{op}^2 + 1 \)^{-1}, so that

$$0 - \| M^* \|_{op} \leq \hat{d} = \text{Tr}[M^*] \leq (n - 1) + \| M^* \|_{op}.$$ For $V$ we find

$$\text{Tr}[V] - \text{Tr}[D] + \text{Tr}[D^{1/2} P_V M^* P_V D^{1/2}] = \text{Tr}[D^{1/2}(-M^* + P_V M^* P_V) D^{1/2}].$$

The matrix $M^* - P_V M^* P_V$ is rank at most 2 and operator at most $2 \| M^* \|_{op}$. Thus the absolute value of the RHS is at most $4 \| M^* \|_{op}$. Since $0_{n \times n} \leq P_V M^* P_V \leq (\tau n)^{1/2} \| D^{1/2} X \Sigma^{-1/2} \|_{op}^2 + 1 \)^{-1} I_n$, we find the following upper and lower bounds on $\text{Tr}[V]$:

$$\text{Tr}[V] \geq \text{Tr}[D] (1 - (\tau n)^{1/2} \| D^{1/2} X \Sigma^{-1/2} \|_{op}^2 + 1) - 4 \| M^* \|_{op} = \frac{1}{1 - \tau n} \text{Tr}[D] - 4 \| M^* \|_{op},$$

$$\text{Tr}[V] \leq \text{Tr}[D] + 4 \| M^* \|_{op}.$$

We conclude with $\| M^* \|_{op} \leq \| M \|_{op} \leq \hat{c} = (\tau n)^{-1} \| D^{1/2} X \Sigma^{-1/2} \|_{op}^2$ thanks to (3.3).\hfill $\square$

Appendix B: Probabilistic tools

**Lemma B.1** (Variant of [9]). Let $z \sim N(0, I_n)$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ be weakly differentiable with $\mathbb{E}[\| f(z) \|] < +\infty$. Then $Z = \| f(z) \|^{-1} z^T \mathbb{E}[f(z)] \sim N(0, I_n)$ is such that

$$\mathbb{E}\left[\left( z^T f(z) - \sum_{i=1}^n \frac{\partial f_i}{\partial z_i}(z) - \| f(z) \| Z \right)^2 \right] \leq 15 \mathbb{E}\left[ \sum_{i=1}^n \left( \frac{\partial f_i}{\partial z_i}(z) \right)^2 \right].$$

**Proof.** Variants of the following argument were developed in [9, 10]. A short proof is provided for completeness, and because the exact statement in Lemma B.1 slightly differ from previous results. Let $g(z) = f(z) - \mathbb{E}[f(z)]$ and $W = \| f(z) \| - \mathbb{E}[\| f(z) \|]$. Then the square root of the left-hand side is $\mathbb{E}[z^T g(z) - \sum_{i=1}^n \frac{\partial g_i}{\partial z_i}(z) - W Z]^{1/2}$ which is smaller than $\sqrt{E_1 + \sqrt{E_2}}$ by the triangle inequality, where $E_1 = \mathbb{E}[(z^T g(z) - \sum_{i=1}^n \frac{\partial g_i}{\partial z_i}(z))^2]$ and $E_2 = \mathbb{E}[Z^2 W^2]$. For $E_1$, by [7] applied to $g$ we find

$$E_1 = \mathbb{E}\left[\| f(z) - \mathbb{E}[f(z)] \| ^2 + \sum_{i=1}^n \sum_{l=1}^n \frac{\partial f_i}{\partial z_l}(z) \frac{\partial f_l}{\partial z_i}(z) \right] \leq 2 \mathbb{E} \sum_{i=1}^n \| \frac{\partial f}{\partial z_i} \| ^2$$

by the Gaussian Poincaré inequality [12, Theorem 3.20] for the first term and the Cauchy-Schwarz inequality for the second. For $E_2$, by the triangle inequality $E_2 \leq \mathbb{E}[Z^2 \| g(z) \| ^2]$. Write $Z = \sum_{i=1}^n \sigma_i z_i$ for some $\sigma_i \geq 0$ with $\sum_{i=1}^n \sigma_i^2 = 1$. By Stein’s formula,

$$\mathbb{E}\left[ Z^2 \| g(z) \| ^2 \right] = \sum_{i=1}^n \mathbb{E}\left[ \sigma_i z_i Z \| g(z) \| ^2 \right] = \sum_{i=1}^n \left\{ \sigma_i^2 \mathbb{E}[\| g(z) \| ^2] + \sigma_i \mathbb{E}[Z \frac{\partial}{\partial z_i}(\| g(z) \| ^2)] \right\}$$

$$= \mathbb{E}[\| g(z) \| ^2] + 2 \sum_{i=1}^n \sum_{l=1}^n \mathbb{E}\left[ \sigma_i Z g_l(z) \frac{\partial g_l}{\partial z_i}(z) \right] \leq RHS + 2 \left( \sum_{i=1}^n \sum_{l=1}^n \mathbb{E}\left[ \sigma_i^2 (Z g_l(z))^2 \right] \right)^{1/2} (RHS)^{1/2}$$

where $RHS = \sum_{i=1}^n \sum_{l=1}^n \mathbb{E}[(\frac{\partial g_l}{\partial z_i}(z))^2] \leq \mathbb{E}[\| g(z) \| ^2] \leq RHS$ by the Gaussian Poincaré inequality for the first term and the Cauchy-Schwarz inequality for the second term. By completing the square, \(\mathbb{E}[Z^2 \| g(z) \| ^2]^{1/2} - (RHS)^{1/2} \leq 2RHS\) and $E_2 \leq \mathbb{E}[Z^2 \| g(z) \| ^2] \leq (1 + \sqrt{2})^2 RHS$. Hence $\sqrt{E_1 + \sqrt{E_2}} \leq (\sqrt{2} + 1 + \sqrt{2})(RHS)^{1/2}$.\hfill $\square$
Corollary B.2. Let \( X \in \mathbb{R}^{n \times p} \) with iid \( N(0, \Sigma) \) rows with invertible \( \Sigma \).

(i) If \( a \in \mathbb{R}^p \) and \( h : \mathbb{R}^{n \times p} \rightarrow \mathbb{R} \) is weakly differentiable then for some \( Z \sim N(0,1) \),

\[
\mathbb{E}_0 \left[ \left\| \mathbf{a}^T \Sigma^{-1} X^T h(X) - \sum_{i=1}^n \sum_{k=1}^p \frac{\partial h_i}{\partial x_{ik}}(X) - \| \Sigma^{-1/2} a \| Z h(X) \right\| \right]^2 \leq 15 \mathbb{E}_0 \sum_{i=1}^n \left\| \sum_{k=1}^p a_k \frac{\partial h}{\partial x_{ik}}(X) \right\|^2
\]

where \( \mathbb{E}_0 \) is the conditional expectation given \( X(I_p - \Sigma^{-1} \mathbf{a} \mathbf{a}^T) \).

(ii) If \( h : \mathbb{R}^n \times \mathbb{R}^{n \times p} \rightarrow \mathbb{R} \) is such that \( h(y, \cdot) \) is weakly differentiable for all \( y \), if \( w \) is such that \( \text{Var}[w^T x] = w^T \Sigma w = 1 \) and \( (Xw, y) \) is independent of \( X \Sigma^{-1} (I_p - \Sigma w w^T) \) as in the single index model (1.1), then for \( P = I_p - \Sigma w w^T \) we have

\[
\frac{1}{2} \sum_{j=1}^p \mathbb{E} \left[ \left\| e_j^T P^T \Sigma^{-1} X^T h(y, X) - \sum_{i=1}^n \sum_{k=1}^p P_{kj} \frac{\partial h_i}{\partial x_{ik}}(y, X) - Z_j \Omega_j^{1/2} \| h(y, X) \| \right\|^2 \right] \leq 15 \mathbb{E} \sum_{i=1}^n \left\| \sum_{k=1}^p \frac{\partial h}{\partial x_{ik}}(y, X) e_k^T P \right\|^2 + \sum_{j=1}^p \mathbb{E} \left[ Z_j^2 \| h(y, X) \|^2 \right] w_j^2
\]

where \( \Omega_j = e_j^T \Sigma^{-1} e_j \) and \( \mathbb{E}[\cdot] \) is either the conditional expectation given \( (y, Xw) \) or the unconditional expectation.

Proof. Without loss of generality, assume that \( \mathbf{a}^T \Sigma^{-1} \mathbf{a} = 1 \). Following the notation and conditioning technique in [8, 9], \( z = X \Sigma^{-1} \mathbf{a} \) is independent of \( X(I_p - \Sigma^{-1} \mathbf{a} \mathbf{a}^T) \) so that the conditional distribution of \( z \) given \( X(I_p - \Sigma^{-1} \mathbf{a} \mathbf{a}^T) \) is \( N(0, I_n) \). The proof is completed by application of Lemma B.1 to \( z \) conditionally on \( X(I_p - \Sigma^{-1} \mathbf{a} \mathbf{a}^T) \) with \( f(z) = h(z \mathbf{a}^T + X(I_p - \Sigma^{-1} \mathbf{a} \mathbf{a}^T)) \).

(ii) By application of the first part of the theorem to \( a = a^{(j)} = P e_j \) for each \( j \in [p] \) we obtain

\[
\mathbb{E} \sum_{j=1}^p \left[ \left( \sum_{i=1}^n \sum_{k=1}^p a_{k}^{(j)} \frac{\partial h_i}{\partial x_{ik}} \right) - \hat{\Omega}_j^{1/2} Z_j \| h \| ^2 \right] \leq 15 \mathbb{E} \sum_{j=1}^p \sum_{k=1}^n \left\| \sum_{k=1}^p a_k^{(j)} \frac{\partial h}{\partial x_{ik}} \right\|^2
\]

where \( \hat{\Omega}_j^{1/2} = \| \Sigma^{-1/2} a^{(j)} \| \). Next, using \( \frac{1}{2}(u + v) \leq u^2 + v^2 \) we find

\[
(B.1) \leq 15 \mathbb{E} \sum_{j=1}^p \sum_{k=1}^n \left\| \sum_{k=1}^p a_k^{(j)} \frac{\partial h}{\partial x_{ik}} \right\|^2 + \sum_{j=1}^p \mathbb{E} \left[ Z_j^2 \| h \|^2 \right] \left( \hat{\Omega}_j^{1/2} - \Omega_j^{1/2} \right)^2
\]

where we omit the arguments of \( h \) and its derivatives for brevity. Here the second term appears due to \( \Omega_j^{1/2} \) in the term \( Z_j \| h \| \) in (B.1) instead of \( \hat{\Omega}_j^{1/2} \). To bound from above the right-hand side of the previous display by (B.2), we use for the first term \( a_k^{(j)} = e_k^T a^{(j)} = e_k^T P e_j \), while for the second term

\[
| \hat{\Omega}_j^{1/2} - \Omega_j^{1/2} | = \| \Sigma^{-1/2} P e_j \| - \| \Sigma^{-1/2} e_j \| \leq \| \Sigma^{-1/2} (I_p - P) e_j \|
\]

by the triangle inequality and \( \Sigma^{-1/2} (I_p - P) e_j = \Sigma^{1/2} w w_j \) has squared euclidean norm equal to \( w_j^2 \) thanks to \( \| \Sigma^{1/2} w \| = 1 \).

Appendix C: Proofs of Section 4

C.1. Proofs: Approximate normality and proximal representation for \( \hat{\beta}_j \)

Theorem 4.1. Let Assumptions A and B be fulfilled. Then for all \( j = 1, \ldots, p \), there exists \( Z_j \sim N(0,1) \) such that

\[
\frac{1}{p} \sum_{j=1}^p \mathbb{E} \left[ \left( \frac{\sqrt{n}}{\Omega_j^{1/2}} \left( \hat{\beta}_j^{(d)} - \frac{z_j}{r} w_j \right) - Z_j \right)^2 \right] \leq \frac{C_{31}(\delta, \tau, \kappa)}{\sqrt{p}}
\]
where $\pm$ denotes the sign of the unknown scalar $t_* := w^T(\text{Tr}[V]\Sigma\hat{\beta} + X^T\hat{\psi})/n$, and $\hat{\ell} = \max(0, \ell^2)^{1/2}$. If additionally Assumption D holds then for some event $E$ with $\mathbb{P}(E) \to 1$, we have $\max\{\frac{\hat{\ell}}{\hat{\ell}^*}, \hat{\ell}\} I_E \leq C_{32}(\delta, \tau, \ell)$ almost surely and

$$
\frac{1}{p} \sum_{j=1}^{p} \frac{1}{\Omega_{jj}} \mathbb{E} \left[ I_E \left( \sqrt{n} \left( \hat{\beta}_{(j)} - \frac{-\ell}{\hat{\ell}} w_j, \right) \right) - \Omega_{jj}^{1/2} Z_{jj} \right]^2 \leq \frac{C_{33}(\delta, \tau, \kappa, \ell)}{\sqrt{p}}. \tag{4.2}
$$

**Proof of Theorem 4.1.** By the product rule and (3.7),

$$
\frac{\partial}{\partial x_{ik}} \left[ \frac{\hat{\psi}}{\|\psi\|} \right] = \frac{\|\psi\|^{-1}}{\|\psi\|^2} \left( I_n - \frac{\hat{\psi} \hat{\psi}^T}{\|\psi\|^2} \right) \frac{\partial}{\partial x_{ik}} \hat{\psi} = \|\psi\|^{-1} \left( I_n - \frac{\hat{\psi} \hat{\psi}^T}{\|\psi\|^2} \right) \left[ -DX \hat{A} \hat{e}_k \hat{\psi}_i - V e_i \hat{\beta}_k \right].
$$

Let $P = I_p - \Sigma \omega w^T$ as in Corollary B.2. Define for each $j = 1, \ldots, p$,

$$
\text{Rem}_j \overset{\text{def}}{=} \frac{\text{Tr}[V \hat{e}_j \hat{\beta}^T \hat{\beta}]}{\|\psi\|} + \sum_{i=1}^{n} \sum_{k=1}^{p} P_{kj} \frac{\partial}{\partial x_{ik}} \left( \hat{\psi}_i \right) = \frac{(e_i^T P^T \hat{\beta}) \hat{\psi}_i V}{\|\psi\|^3}.
$$

Notice that $\sum_{j=1}^{p} \text{Rem}_j \leq \|P^T \hat{\beta}\| \|V\|_{op}^2 \|\psi\|^2$ which will be used below to show that $\text{Rem}_j$ is negligible. We apply the second part of Corollary B.2 to $h(X) = \hat{\psi}/\|\hat{\psi}\|$. With the notation

$$
U_j \overset{\text{def}}{=} \frac{e_i^T P^T (\Sigma - 1)^{X^T \hat{\psi} + \hat{\beta}^T \text{Tr}[V])}{\Omega_{jj}^{1/2} \|\hat{\psi}\|} - Z_j
$$

for each $j = 1, \ldots, p$ for brevity where $Z_j$ is given by Corollary B.2, we find

$$
\sum_{j=1}^{p} \frac{\Omega_{jj}}{2} \mathbb{E} \left[ (U_j - \text{Rem} \|\hat{\psi}\|)^2 \right] \leq 15 \mathbb{E} \sum_{i=1}^{n} \frac{1}{\|\psi\|^2} \left( I_n - \frac{\hat{\psi} \hat{\psi}^T}{\|\psi\|^2} \right) \sum_{k=1}^{p} \frac{\partial \hat{\psi}}{\partial x_{ik}} e_k^T P \|w\|^2 + \|w\|^2
$$

by (B.1)-(B.2)

$$
= 15 \mathbb{E} \sum_{i=1}^{n} \frac{1}{\|\psi\|^2} \left( I_n - \frac{\hat{\psi} \hat{\psi}^T}{\|\psi\|^2} \right) \left( DX \hat{A} \hat{e}_i \hat{\psi}_i + V e_i \hat{\beta}_i \right) P \|w\|^2 + \|w\|^2
$$

using (3.7)

$$
\leq 30 \mathbb{E} \left[ \|DX \hat{A} P\|^2_{op} + 30 \|P^T \hat{\beta}\| \|\psi\|^2 \|\hat{\psi}\|^2 + \|w\|^2, \right.
$$

thanks to $(a + b)^2 \leq 2a^2 + 2b^2$ and $\sum_{i=1}^{n} \|M_{ei}\|^2 = \|M\|^2$ for the last inequality. We further use $\min_{k=1,\ldots,p} \Omega_{kk} \leq \Omega_{jj}$ and $\Omega_{jj} U_j^2 \leq \Omega_{jj} (U_j - \text{Rem} \|\hat{\psi}\|)^2 + \text{Rem}_j^2$ to lower bound the first line, so that

$$
\text{Rem}_* \overset{\text{def}}{=} \sum_{j=1}^{p} \mathbb{E} \left[ \left( e_j^T P^T (\Sigma - 1)^{X^T \hat{\psi} + \hat{\beta}^T \text{Tr}[V]) \right) \frac{\Omega_{jj}}{\|\hat{\psi}\|} - Z_j \right]^2 = \sum_{j=1}^{p} \mathbb{E} \left[ U_j^2 \right] \tag{C.1}
$$

$$
\leq C_{34} \left( \sum_{j=1}^{p} \mathbb{E} \left[ \left( DX \hat{A} \right)^2 \right] + \mathbb{E} \left[ \|V\|^2 \|\hat{\beta}\|^2 + \|w\|^2 \right] \right). \tag{C.2}
$$

For the middle term, we use $\|P^T \hat{\beta}\|^2 \leq \|P^T (\Sigma - 1/2)^{2 \|\cdot\|^2} (\Sigma - 1/2)\|_F^2$. By (2.1) and since $\hat{\beta} \in \text{argmin}_{\theta \in \mathbb{R}^p} g(b)$ we have have $0 \in \partial g(0)$ and $\beta^T X^T \psi = n \hat{\beta}^T \partial g(\hat{\beta}) = n (\hat{\beta} - 0)^T (\partial g(\hat{\beta}) - 0) \geq n \tau \|\Sigma^{1/2} \hat{\beta}\|^2$ so that

$$
n \|\Sigma^{1/2} \hat{\beta}\|^2 \|\hat{\psi}\|^2 \leq n^{-1/2} X \Sigma^{-1/2} \|\hat{\beta}\|_{op}^2. \tag{C.3}
$$

For the first term in (C.2), we use $\|DX \hat{A}\|^2 \|P\|_{op} \leq \|DX \hat{A} \Sigma^{1/2} \|_{op}^2 \|\Sigma^{-1/2} P\|_{op}^2$. Next, the matrices inside the Frobenius norms $\|DX \hat{A} \Sigma^{-1/2}\|_F$ and $\|V\|_F$ have rank at most $n$. We use $\|\cdot\|_F \leq \text{rank}(\cdot) \|\cdot\|_{op}$ and we bound the operator norms using $\|\Sigma^{1/2} \hat{A} \Sigma^{1/2}\|_{op} \leq (n \tau)^{-1}$ by (3.3), $\|D\|_{op} \leq 1$ since $\ell_{\gamma}$ is assumed 1-Lipschitz, and $\|V\|_{op} \leq \|D\|_{op} + \|DX \Sigma^{-1/2}\|_{op} (n \tau)^{-1}$ by definition of $V$ in (3.4). This implies that (C.2) is bounded from above by

$$
C_{35} \left( \min_{j=1,\ldots,p} \Omega_{jj} \right)^{-1} \left( \mathbb{E} \left[ \frac{X \Sigma^{-1/2}}{n^{1/2}} \right] \frac{\|\Sigma^{-1/2} P\|_{op}^2}{\tau^2} + \mathbb{E} \left[ 1 + \frac{X \Sigma^{-1/2}}{n^{1/2}} \right] \|\Sigma^{-1/2} P\|_{op} \|P^T (\Sigma - 1/2)^{2 \|\cdot\|^2} (\Sigma - 1/2)\|_{op} \|\hat{\beta}\|_{op}^2 + \|w\|^2 \right). \tag{C.3}
$$
We then use \( \max_{j=1,\ldots,p} \Omega_{jj}^{-1} \leq \| \Sigma \|_{op} \) as well as \( \| w \|_2 \leq \| \Sigma^{-1/2} w \|_2 \) thanks to \( \| \Sigma^{1/2} w \| = 1 \). Furthermore, 
\[
\| \Sigma^{-1/2} P \|_{op} = \| \Sigma^{-1/2} P \Sigma^{1/2} \|_{op} \leq \| \Sigma^{-1/2} \|_{op}^2 \text{ since } \Sigma^{-1/2} P \Sigma^{1/2} \text{ is an orthogonal projection.}
\]
Combined with \( \mathbb{E}[\| n^{-1/2} \mathbf{X} \Sigma^{-1/2} \|_{op}^2] \leq C_3(\delta) \) due to \( p/n \leq \delta^{-1} \) (cf. (C.19) below) we have proved that
\[
\text{Rem}_* = (C.1) = \mathbb{E} \sum_{j=1}^p \left( \frac{\sqrt{\Omega_{jj}}}{n} \frac{\hat{v}_{jj}(\hat{\theta}) - t_j w_j}{\hat{t}} - Z_j \right)^2 \leq C_37(\delta) \kappa \left( \frac{1}{\hat{t}^2} + \frac{1}{\hat{t}^2 + 1} \right).
\]
Finally, using \( (a + b)^2 \leq 2a^2 + 2b^2 \) we have with \( \hat{t} = \max(0, \hat{t}^2)^{1/2} \)
\[
\mathbb{E} \sum_{j=1}^p \left( \frac{\sqrt{\Omega_{jj}}}{n} \frac{\hat{v}_{jj}(\hat{\theta}) - t_j w_j}{\hat{t}} - Z_j \right)^2 \leq 2\text{Rem}_* + 2\mathbb{E} \left[ n \left| \frac{\hat{t}}{\hat{t}^2} - t_j \right| \right] \sum_{j=1}^p \frac{u_j^2}{\Omega_{jj}}.
\]
Since \( \text{Var}[\mathbf{x}^T w] = 1 \) and \( \max_{j=1,\ldots,n} (\Omega_{jj}^{-1}) \leq \| \Sigma \|_{op} \) we find \( \sum_{j=1}^p u_j^2 / \Omega_{jj} \leq \| \Sigma \|_{op} \| w \|_2^2 \leq \kappa \). Since \( \pm \) is the sign of \( t_j \), \( |t_j| \leq \| t_j \| \leq \| \hat{t} \| \leq \| \hat{t} \| \), the basic inequality
\[
|t_j - \hat{t}^2| = |t_j| - \hat{t}^2| \leq |t_j^2 - \hat{t}^2| = |t_j - \hat{t}|^2 \leq \| t_j^2 - \hat{t}^2 \|
\]
and inequality \( \mathbb{E}[|t_j^2 - \hat{t}^2|/\hat{t}^2] \leq C_38(\delta, \tau) \sqrt{p} \) from Theorem 4.4 completes the proof of (4.1). To prove (4.2) from (4.1), we use the event and the lower bound on \( \hat{t}^2/\hat{t}^2 \) established in (C.25) below.

\section*{C.2. Rotational invariance, change of variable}

In the next proofs, the following change of variable will be useful to transform the correlated design problem to an isotropic one such that the index is concentrated on the first component. With this in mind, let \( Q \in O(p) \) be any rotation such that \( Q^T Q = Q Q^T = I_p \) and \( \theta^* \equiv Q \Sigma^{1/2} w = e_1 \) is the first canonical basis vector in \( \mathbb{R}^p \). Define
\[
G = \mathbf{X} \Sigma^{-1/2} Q^T, \quad \hat{\theta}(y, G) = Q \Sigma^{1/2} \hat{\beta} = \arg\min_{\theta \in \mathbb{R}^p} -\frac{1}{n} \sum_{i=1}^n \ell_{gi}(g_i^T \theta) + h(\theta) \tag{C.4}
\]
where \( h(\theta) = g(\Sigma^{-1/2} Q^T \theta) \) is convex and where \( g_i = G_i^T e_1 \) for each \( i = 1, \ldots, n \) are the rows of \( G \). Then \( G \) has iid \( N(0, 1) \) entries, \( \hat{\beta} = G \hat{\theta}, \ h(\hat{\theta}) = g(\hat{\beta}), \ (\theta - \hat{\theta})^T (\partial h(\theta) - \partial h(\hat{\theta})) \geq \| \theta - \hat{\theta} \|^2 \) for all \( \theta, \hat{\theta} \in \mathbb{R}^p \) thanks to (2.1). Since \( Q \Sigma^{1/2} w = e_1 \) is the first canonical basis vector in \( \mathbb{R}^p \), the matrix \( G(I_p - e_1 e_1^T) \) is independent of \( Ge_1 = G e_1 \) and thus \( G(I_p - e_1 e_1^T) \) is independent of \( y \). Furthermore, by the chain rule we can deduce the derivatives of \( \hat{\theta} \) with respect to the entries of \( G \) for a fixed \( y \) from the derivatives (3.7) of \( \hat{\beta}, \hat{\psi} \) with respect to the entries of \( X \):

\[
\frac{\partial \hat{\theta}}{\partial g_{ij}} = A e_j \hat{\psi}_i - A G^T D e_i \hat{\theta}_j, \quad \frac{\partial \hat{\psi}}{\partial g_{ij}} = -D G A e_j \hat{\psi}_i - \partial \hat{\psi}_j \tag{C.5}
\]
where \( A = Q \Sigma^{1/2} A \Sigma^{1/2} Q^T \), while the quantities
\[
D = \text{diag}(\ell''(X \hat{\beta})), \quad V = D - DX \hat{X}^T D = D - DG A^T D, \quad \hat{\psi} = -\ell''(X \hat{\beta}),
\]
\[
\hat{\theta} = \text{Tr} [X \hat{X}^T D], \quad \text{Tr}[GAG^T D], \quad \hat{\beta} = \text{Tr} [X \hat{X}^T D], \quad \hat{\psi} = -\ell''(G \hat{\theta}) \tag{C.6}
\]
are unmodified by the change of variable. The bound (3.3) then reads \( \| A \|_{op} \leq 1/(n \tau) \).

Finally, let us rewrite \( (\hat{t}^2, \hat{a}^2, \hat{d}^2) \) in (3.8) after the change of variable:

\[
\left\{ \begin{array}{l}
\hat{t}^2 = \frac{1}{n} \| G^T \hat{\psi} \|^2 + \frac{n}{\hat{r}^2} \| G \hat{\theta} - \hat{\gamma} \hat{\psi} \|^2 - \frac{n}{\hat{r}^2} \hat{r}^2 \\
\hat{a}^2 = \frac{1}{n} \| G^T \hat{\psi} \|^2 + \text{Tr}[V \| \hat{\theta} \|^2 + \hat{\theta}^T (\frac{1}{n} \| G \hat{\theta} - \hat{\gamma} \hat{\psi} \|^2 - \| \hat{\theta} \|^2) - \frac{n}{\hat{r}^2} \text{Tr}[\hat{X}^T D],
\end{array} \right.
\]
\[
\hat{a}^2 = \frac{1}{n} \| G \hat{\theta} - \hat{\gamma} \hat{\psi} \|^2 - \hat{d}^2. \tag{C.7}
\]
C.3. **Proofs: \((\hat{\gamma}, \hat{r}^2, \hat{a}_\gamma^2, \hat{\sigma}^2)\) estimate \((\gamma_\ast, t_\ast^2, a_\ast^2, \sigma_\ast^2)\).**

Before reading the following arguments, we recommend to first go through Proposition 6.2 and its short proof. The techniques there are used in a simpler and more restricted setting than the general setting of the present section, but are still representative of the arguments below.

### C.3.1. Notation and deterministic preliminary

Consider the change of variable and notation defined in Appendix C.2. Next, define \(P_1^\perp = I_p - e_1e_1^T = I_p - \theta^*\theta^{*T}\) as well as

\[
\hat{t}^2 \overset{\text{def}}{=} \|G^T\hat{\psi}/n + \hat{v}\|^2 - \frac{p}{n}r^2
\]

which will be proved to be close to \(\hat{t}^2\) and \(\hat{t}_n^2 = [e_1^T(G^T\hat{\psi} + \text{Tr}[V\hat{\theta}])^2 / n^2].\) Consider \(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_5, \Gamma_6, \Gamma_7\)

defined by

\[
\begin{align*}
-\hat{r}^2 + \hat{t}^2 &= \frac{p}{n} - \frac{1}{n\tau} \|P_1^\perp (G^T\hat{\psi}/n + \hat{v}\theta)\|^2 = \Gamma_1, \\
\|P_1^\perp (G^T\hat{\psi}/n + \hat{v}\theta)\|^2 + \hat{v}\theta^T G P_1^\perp \hat{\theta} + \hat{v}\theta - \frac{p}{n} r^2 &= \Gamma_2, \\
-\hat{v}^2 r^2 - \hat{v}\theta^T G P_1^\perp \hat{\theta} + \hat{v}\theta - \frac{p}{n} r^2 &= \Gamma_3, \\
\|GP_1^\perp \hat{\theta}\|^2 = \frac{\sigma_\ast^2}{\tau^2} - \frac{\hat{v}\theta^T G P_1^\perp \hat{\theta}}{\tau^2} - \frac{\hat{v}\theta - \frac{p}{n} r^2}{\tau^2} &= \Gamma_4, \\
\frac{1}{n}\theta^T (GP_1^\perp \hat{\theta} - \gamma \hat{\psi}) = \frac{\hat{v}\theta (GP_1^\perp \hat{\theta} - \gamma \hat{\psi})}{\tau^2} &= \Gamma_5, \\
\frac{1}{n}\theta^T (GP_1^\perp \hat{\theta} - \gamma \hat{\psi})^2 = \frac{\hat{v}\theta (GP_1^\perp \hat{\theta} - \gamma \hat{\psi})^2}{\tau^2} &= \Gamma_6, \\
\hat{v}\theta^T (GP_1^\perp \hat{\theta} - \gamma \hat{\psi}) &= \Gamma_7.
\end{align*}
\]

The quantities \(\Gamma_1, \Gamma_2, \Gamma_4\) will be proved to be of order \(n^{-1/2}\) in Lemma C.3 below, so that each right-hand side is of negligible order. Before proving that the \(\Gamma_1, \Gamma_2, \Gamma_4\) are at most of order \(n^{-1/2}\) under our working assumptions, which follows from techniques already developed in the linear model \([6]\), we explain how well-chosen weighted combinations of the above quantities provide the desired relationships \((4.11)-(4.14)\). We have

\[
\hat{v}(\gamma - \gamma_\ast) = \Gamma_1 \hat{v} + \Gamma_2 + \Gamma_3.
\]

If the previous display is of negligible order, then \(\hat{v}\gamma \approx \hat{v}\gamma_\ast: This means that after multiplication by \(\hat{v}\), we can replace \(\gamma_\ast\) by \(\hat{\gamma}\) in \((C.9)\) without significantly enlarging the right-hand sides in the three equations involving \(\Gamma_1, \Gamma_5, \Gamma_6\). With this in mind, define \(\Gamma_1, \Gamma_5, \Gamma_6\) by

\[
\begin{align*}
\hat{v}(\frac{\hat{v}\theta^T GP_1^\perp \hat{\theta}}{\tau^2} + \frac{\sigma_\ast^2}{\tau^2} - \hat{\gamma}) &= \hat{v}\theta (\hat{\gamma} - \hat{\gamma}_\ast) \overset{\text{def}}{=} \Gamma_1, \\
\hat{v}(\frac{\hat{v}\theta^T GP_1^\perp \hat{\theta}}{\tau^2} - \sigma_\ast^2 - \hat{\gamma} + \frac{\hat{v}\theta^T GP_1^\perp \hat{\theta}}{\tau^2} + \hat{v}\theta - \frac{p}{n} r^2) &= \hat{v}\theta (\hat{\gamma} - \hat{\gamma}_\ast) \overset{\text{def}}{=} \Gamma_5, \\
\hat{v}\theta (\hat{\gamma} - \hat{\gamma}_\ast) &= \hat{v}\theta (\hat{\gamma} - \hat{\gamma}_\ast) \overset{\text{def}}{=} \Gamma_6.
\end{align*}
\]

Here \(\Gamma_1, \Gamma_5, \Gamma_6\) are the analogous of \(\Gamma_1^\perp, \Gamma_5^\perp, \Gamma_6^\perp\) after multiplication by \(\hat{v}\) and replacing \(\gamma_\ast\) by \(\hat{\gamma}\) in the left-hand side of \((C.9)\). Completing the square, we find using \(\Gamma_5, \Gamma_6\) and \(\hat{\gamma}^2 \hat{r}^2 = \|\hat{\gamma} \hat{\psi}\|^2 / n\)

\[
\hat{v}(\|GP_1^\perp \hat{\theta} - \gamma \hat{\psi}\|^2 - \frac{\sigma_\ast^2}{\tau^2}) = \frac{1}{n}(\Gamma_5 - \hat{\gamma} \Gamma_1).
\]

For the following, recall that \(\hat{a}_\ast^2 = \hat{\theta}^2, \sigma_\ast^2 = \|P_1^\perp \hat{\theta}\|^2\) and \(\|\hat{\theta}\|^2 = \hat{a}_\ast^2 + \sigma_\ast^2\). Expanding now the square with \(G\hat{\theta} - \hat{\gamma} \hat{\psi} = (GP_1^\perp \hat{\theta} - \gamma \hat{\psi}) + GP_1^\perp \hat{\theta},\)

\[
\hat{v}(\|GP_1^\perp \hat{\theta} - \gamma \hat{\psi}\|^2 - \frac{\sigma_\ast^2 + \hat{a}_\ast^2}{\tau^2}) = \hat{v}(\sigma_\ast^2 + \hat{a}_\ast^2) = \hat{v}\|\hat{\theta}\|^2 \text{ when the right-hand side of the previous display is of negligible order.}
\]
We now focus on $\hat{t}^2$ in (C.8), $t^2$ in (3.8), and $t_* = \frac{e^T(G^*\hat{\psi} + Tr|\hat{\theta}|)}{n} = (G\hat{e})^T\hat{\psi} + \hat{v}a_*$ as defined in Theorem 4.1. We have by the second line in (C.7) and the definition of $\Gamma_8$,

$$\frac{\hat{t}^2 - t^2}{\hat{t}^2} = -\hat{v}\Gamma_8, \quad \frac{t_*^2 - t^2}{t_*^2} = \Gamma_2 - \hat{v}\Gamma_8. \quad (C.12)$$

This will justify the approximation $\hat{t}^2 \approx t_*^2$ in (4.12) when the right-hand sides are negligible. Using $G\hat{\theta} = GP_1^*\hat{\theta} + GE_1\hat{\theta}$ and $a_* = \hat{\theta}_1$ we find

$$\hat{\psi}_T = t_*a_* = \hat{\psi}_T GP_1^*\hat{\theta} + \psi_Ge_1\hat{\theta}_1 - a_*(\frac{\hat{\psi}_T(Ge_1)}{n} + \hat{v}a_*) = \hat{\psi}_T GP_1^*\hat{\theta} - \hat{v}a_*$$

so that by definition of $\Gamma_1$,

$$\hat{v}^2[\hat{\psi}_T GP_1^*\hat{\theta} - \hat{\psi}^2GP_1^*\hat{\theta} - 2\hat{\psi}\hat{v}a_* - \hat{\psi}^2a_*/n] = \hat{v}^2(\|\hat{\psi}_T GP_1^*\hat{\theta} - \hat{\psi}^2GP_1^*\hat{\theta} - 2\hat{\psi}\hat{v}a_* - \hat{\psi}^2a_*/n)$$

with $\Gamma_9 \overset{def}{=} \hat{v}\Gamma_8 + \Gamma_1$. If the right-hand side is small, this means that the approximation $\hat{\psi}_T GP_1^*\hat{\theta} + \hat{\psi}^2GP_1^*\hat{\theta} - \hat{\psi}\hat{v}a_* - \hat{\psi}^2a_*/n$ holds and one can estimate the product $\hat{\psi}_T GP_1^*\hat{\theta}$ by the left-hand side of the approximation.

To find an estimate for $a_*^2 = \hat{a}_1^2$, let $W = \hat{\psi}_T GP_1^*\hat{\theta} + \hat{\psi}^2GP_1^*\hat{\theta} - \hat{\psi}\hat{v}a_* - \hat{\psi}^2a_*/n$ so that $\hat{v}(W - \hat{a}_*^2) = \Gamma_9$. Expanding the square $(\hat{v}W)^2 = (\hat{v}\hat{a}_1^2 + \Gamma_9)^2$ and noticing that $W^2 = \frac{\hat{a}_1^2}{\hat{v}^2}$ by definition of $(\hat{v}, \hat{a}_1^2, \hat{a}_*^2)$, we obtain

$$\hat{v}^2\frac{\hat{a}_1^2}{\hat{v}^2} = \hat{v}^2\frac{\hat{a}_1^2}{\hat{v}^2} + 2\hat{v}\hat{a}_1^2 + \hat{v}^2\frac{\hat{a}_1^2}{\hat{v}^2} + \Gamma_9 + \Gamma_9^2$$

$$\hat{v}^2\frac{\hat{a}_1^2}{\hat{v}^2} = \hat{v}^2\frac{\hat{a}_1^2}{\hat{v}^2} + 2\hat{v}\hat{a}_1^2 + \Gamma_9 + \Gamma_9^2$$

by subtracting $\hat{v}^2\frac{\hat{a}_1^2}{\hat{v}^2}$ on both sides for the last line. This will justify the approximation $\hat{a}_1^2 \approx a_*^2$ when $\Gamma_2, \Gamma_8, \Gamma_9$ have negligible order and $\hat{v}, \hat{v}, \hat{r}, a_*$ all have constant order.

### 3.3.2. All $\Gamma_i, \Gamma_i^*$ are of order at most $n^{-1/2}$

Controlling the terms $\text{Rem}_i$ relies on the two following probabilistic propositions developed for analysing M-estimators in linear models.

**Proposition C.1.** [Theorem 7.2 in [5]] Let $Z \in \mathbb{R}^{K \times Q}$ be a matrix with iid $N(0,1)$ entries. Let $u : \mathbb{R}^{K \times Q} \to \mathbb{R}^K$ be weakly differentiable such that $\|u(Z)\| \leq 1$ almost surely. Then, with $e_q$ the $q$-th canonical basis vector in $\mathbb{R}^Q$,

$$\mathbb{E} \left[ \|Q\|u(Z)\|^2 - \sum_{q=1}^{Q} \left( e_q^T Z^T u(Z) - \sum_{k=1}^{K} \frac{\partial u_q}{\partial z_{kq}}(Z) \right)^2 \right] \leq C_{39} \left( \sqrt{Q}(1 + \Xi^{1/2}) + \Xi \right) \quad (C.14)$$

where $\Xi = \mathbb{E} \sum_{k=1}^{K} \sum_{q=1}^{Q} \|\frac{\partial u_q}{\partial z_{kq}}(Z)\|^2$.

**Proposition C.2.** [Proposition 6.5 in [5]] Let $Z \in \mathbb{R}^{K \times Q}$ be a matrix with iid $N(0,1)$ entries. Let $f : \mathbb{R}^{K \times Q} \to \mathbb{R}^Q$, $u : \mathbb{R}^{K \times Q} \to \mathbb{R}^K$ be weakly differentiable. Then, omitting the dependence on $Z$ in $u(Z), f(Z)$ and their derivatives,

$$\mathbb{E} \left[ \left( u^T Z f - \sum_{k=1}^{K} \sum_{q=1}^{Q} \frac{\partial f}{\partial z_{kq}}u^T \right)^2 \right] \leq \mathbb{E} \left[ \|u\|^2 \|f\|^2 + \sum_{k=1}^{K} \sum_{q=1}^{Q} \left( \frac{\partial f}{\partial z_{kq}}u^T \right)^2 \right].$$

**Lemma C.3.** Let Assumptions $A$ and $B$ be fulfilled. Then $\mathbb{E}[\Gamma_1^2 + \Gamma_3^2 + \Gamma_5^2 + \Gamma_6^2 + \Gamma_7^2] \leq C_{40}(\delta, \tau)/\sqrt{n}$ and $\mathbb{E}[\|\Gamma_2\|^2] \leq C_{41}(\delta, \tau)/\sqrt{n}$. 
Let us recall some bounds that will be useful throughout the proof:

\[ \|A\|_{op} \leq (nr)^{-1} \quad \text{by (3.3)}, \]
\[ \|V\|_{op} = \|D - DGAGD\|_{op} \leq 1 + \|n^{-1/2}G\|_{op}^2 \quad \text{by (C.15) and def. of } V, \]
\[ \|\hat{\psi}\| = \|\hat{\psi}\|_{n} \leq n(1 + \|n^{-1/2}G\|_{op}^2 \tau) \quad \text{by (C.16) or (3.6)}, \]
\[ n\|\partial^2 \|_{\psi}\|_{n} = \|\hat{\psi}\|^2 \leq \|n^{-1/2}G\|_{op}^2 \tau^2 \quad \text{by (C.3)}, \]
\[ \mathbb{E}[\|n^{-1/2}G\|_{op}^2 c] \leq C_{32}(\delta, c) \]

for any absolute constant \( c \geq 1 \). Here the last line follows, for instance, from [44, Corollary 7.3.3] or [18, Theorem II.13]. As we explain next, the bound on \( \mathbb{E}[\|\Gamma_2\|] \) follows from Proposition C.1 while the bounds \( \mathbb{E}[\Gamma_1^2], \mathbb{E}[\Gamma_3^2], \mathbb{E}[\Gamma_4^2], \mathbb{E}[\Gamma_5^2] \) are consequences of Proposition C.2.

**Proof of Lemma C.3.** Proof of \( \mathbb{E}[\|\Gamma_2\|] \leq C_{43}(\delta, \tau)/n. \) By the Cauchy-Schwarz inequality, with \( \|Ge_1\|_2 \sim \chi_n^2 \) we have \( \mathbb{E}[\|\Gamma_2\|^2] \leq \mathbb{E}[(\chi_n^2 - n)^{1/2}E[\delta^2 \hat{\theta}^2/\tau^2]^{1/2}/n^2. \) Next, \( \mathbb{E}[(\chi_n^2 - n)^{1/4} \leq C_{44}\sqrt{n} \) by concentration properties of the \( \chi_n^2 \) distribution, and \( \mathbb{E}[\|\Gamma_2\|^2] \leq C_{43}(\delta, \tau)/n \) is obtained by combining (C.17)-(C.19).

**Proof of \( \mathbb{E}[\|\Gamma_2\|] \leq C_{44}(\delta, \tau/n^2. \)** We apply Proposition C.1 with respect to the Gaussian matrix \( G \) with the first column removed. By construction, since \( y \) is independent of the submatrix of \( G \) made of columns indexed in \( \{2, ..., p\} \), we are in a position to apply Proposition C.1 conditionally on \((y, Ge_1)\) where \( e_1 \) is the first canonical basis vector in \( \mathbb{R}^p \). With \( \Xi = \mathbb{E}\sum_{j=2}^p \sum_{i=1}^n \|\partial g_{ij}\|^2 \), the choice \( u = \hat{\psi}/\|\hat{\psi}\| \) in Proposition C.1 yields

\[ \mathbb{E}[p - 1 - \sum_{j=2}^p (e_j^T G T u - \sum_{i=1}^n \|\partial g_{ij}\|^2) \leq C_{47}(1 + \sqrt{\Xi})\sqrt{p} + C_{48}\Xi. \]

By (C.5) we have \((\partial/\partial g_{ij})(\hat{\psi}/\|\hat{\psi}\|) = (I_n - \hat{\psi} \hat{\psi}^T/\|\hat{\psi}\|^2)(-DGAe_j \hat{\psi} - Ve_j \hat{\theta}_j)\) by the chain rule, hence

\[ \sum_{i=1}^n \|\partial g_{ij}\|^2 = \sum_{i=1}^n \|\partial g_{ij}\|^2 \left( \hat{\psi} - \|\hat{\psi}\| \right) = -\text{Tr}(I_n - \hat{\psi} \hat{\psi}^T/\|\hat{\psi}\|^2)V \hat{\theta}_j \]

in the left-hand side of (C.20). In the right-hand side of (C.20), we bound \( \Xi \) using

\[ \sum_{i=1}^n \sum_{j=2}^p \left( \|\partial g_{ij}\|^2 \right)^{1/2} \leq \frac{1}{\|\hat{\psi}\|^2} \sum_{i=1}^n \sum_{j=2}^p \left( \|\partial \hat{\psi}\|_{g_{ij}} \right)^2 \]
\[ \leq 2 \sum_{i=1}^n \sum_{j=2}^p \left( \|DGAE_j\|_{\hat{\psi}^2} + \|V e_j\|_{\hat{\theta}^2} \right) \]
\[ \leq 2(\|DGA\|_{F}^2 + n\|V\|_{op}^2 \|\hat{\theta}\|_{2}^2 /\|\hat{\psi}\|^2). \]

Using inequalities (C.16)-(C.18) we find

\[ \Xi \leq 2E[\|G\|_{op}(\sqrt{n})^2] + 2E[(1 + \|n^{-1/2}G\|_{op}^2 /\tau)^2 \|n^{-1/2}G\|_{op}^2 /\tau^2] \leq C_{49}(\delta, \tau). \]

Thanks to (C.19) this yields

\[ \mathbb{E}[p - 1 - \|G^T \hat{\psi} + \text{Tr}(\hat{\psi} \hat{\theta})\|^2 /\|\hat{\psi}\|^2 + (e_j^T G^T \hat{\psi} + \text{Tr}(\hat{\psi} e_j^T \hat{\psi})\|^2 /\|\hat{\psi}\|^2)] \leq C_{50}(\delta, \tau) \sqrt{p} \]

where \( \hat{V} = (I_n - \hat{\psi} \hat{\psi}^T /\|\hat{\psi}\|^2) \). We have \( e_j^T \hat{\theta} = \hat{\theta}^T \theta^* = w^T \Sigma \hat{\beta}, Ge_1 = G \theta^* = Xw \) and \( \|G^T \hat{\psi} + \text{Tr}(\hat{\psi} \hat{\theta})\| = \|\Sigma^{-1/2}X^T \hat{\psi} + \text{Tr}(\hat{\psi} \Sigma^{-1/2} \hat{\theta})\| \) since \( Q \in O(p) \) is a rotation. Note that \( p - 1 \) can be replaced by \( p \) in the left-hand side by changing the right-hand side constant if necessary. It remains to show that we can
replace $\text{Tr}[\tilde{V}]$ by $\text{Tr}[V]$ in (C.23). Thanks to $||a||^2 - ||b||^2 = (a-b)^T(a+b)$ we have with $P_1^\perp = I_p - e_1e_1^T$ and $\text{Tr}[V - \tilde{V}] = \tilde{V}^TV\tilde{\psi}/||\tilde{\psi}||^2$

$$\frac{||P_1^\perp(GT\tilde{\psi} + \text{Tr}[V]\hat{\theta})||^2 - ||P_1^\perp(GT\tilde{\psi} + \text{Tr}[V]\hat{\theta})||^2}{||\tilde{\psi}||^2} = \frac{\tilde{V}^TV\tilde{\psi}\hat{\theta}P_1^\perp(2GT\tilde{\psi} + \text{Tr}[V + \tilde{V}]\hat{\theta})}{||\tilde{\psi}||^2}$$

which is smaller in absolute value than $2||V||_{op}(||G||_{op}||\tilde{\theta}||)/||\tilde{\psi}|| + n||\tilde{\theta}||^2/||\tilde{\psi}||^2$. Thanks to the bounds (C.16)-(C.18) and $\mathbb{E}[n^{-1/2}G||_{op}] \leq C_{51}(\delta)$, by the triangle inequality $\text{Tr}[V]$ in (C.23) can be replaced by $\text{Tr}[V]$. We have thus established

$$n\mathbb{E}\left[\frac{\tilde{V}^2 - \tilde{\theta}^2}{\tilde{\theta}^2}\right] = \mathbb{E}\left[p - \frac{||GT\tilde{\psi} + \text{Tr}[V]\hat{\theta}||^2}{||\tilde{\psi}||^2} + \frac{e_1^TGT\tilde{\psi} + \text{Tr}[V]e_1^T\hat{\theta}}{||\tilde{\psi}||^2}\right] \leq C_{52}(\delta, \tau)\sqrt{p}$$

as well as $\mathbb{E}[\Gamma_2^2] \leq C_{53}(\delta, \tau)n^{-1/2}$.

**Proof of $\mathbb{E}[\Gamma_1^2] \leq C_{54}(\delta, \tau)/n$.** Let $K = n, Q = p-1$ and let $Z \in \mathbb{R}^{K \times Q}$ be the matrix $G$ with the first column removed. Then $Z$ is independent of $(y, Ge_1)$ and Proposition C.2 is applicable conditionally on $(y, Ge_1)$. Chose $f = (\tilde{\theta}/||\tilde{\psi}||, 1, \ldots, p$ valued in $\mathbb{R}^{p-1}$ and $u = Ge_1$ valued in $\mathbb{R}^n$. Here, $u$ has zero derivatives with respect to $Z$. Using the derivatives in (C.5),

$$u^Tzf - K \sum_{k=1}^{Q} \sum_{q=1}^{P} \frac{\partial(f_{kq})}{\partial z_{qj}} = (Ge_1)^TGP_1^\perp\tilde{\theta} - \sum_{i=1}^{n} \sum_{j=2}^{p} G_{i1} \left( \frac{1}{||\tilde{\psi}||} \frac{\partial \theta_j}{\partial g_{ij}} + \theta_j \frac{\partial}{\partial g_{ij}} \left( \frac{1}{||\tilde{\psi}||} \right) \right)$$

$$= ||\tilde{\psi}||^{-1}(Ge_1)^T(GP_1^\perp\tilde{\theta} - \text{Tr}[A]) + \text{Rem}_6 + \text{Rem}_6'$$

where the square bracket on the last line equals $\sqrt{n}\Gamma_6^\perp \frac{\tilde{\theta}}{\tilde{\theta}}$ and

$$\text{Rem}_6 = \frac{(Ge_1)^T\text{Tr}[A]}{||\tilde{\psi}||} - \sum_{i=1}^{n} \sum_{j=2}^{p} \frac{\partial \theta_j}{\partial g_{ij}}$$

$$\text{Rem}_6' = \frac{(Ge_1)^T\text{Tr}[A]}{||\tilde{\psi}||} - \sum_{i=1}^{n} \sum_{j=2}^{p} \frac{\partial \theta_j}{\partial g_{ij}}$$

thanks to $\text{Tr}[A] = A_{11} + \sum_{j=2}^{p} A_{jj}$ for $\text{Rem}_6$. Here, $\text{Rem}_6'$ comes from differentiatation of $||\tilde{\psi}||^{-1}$ thanks to $\frac{\partial}{\partial g_{ij}} (||\tilde{\psi}||^{-1}) = -\frac{\partial}{\partial g_{ij}} ||\tilde{\psi}||^{-3} \tilde{\psi}^T \frac{\partial \tilde{\psi}}{\partial g_{ij}}$. Now, $\text{Rem}_6$ and $\text{Rem}_6'$ both have second moment bounded by $C_{55}(\delta, \tau)$ thanks to (C.18), (C.15) and (C.19). In the right-hand side of Proposition C.2, $||u||^2/|f|^2$ has expectation smaller than $C_{56}(\delta, \tau)$ again thanks to (C.18) and (C.19). The derivative term in the right-hand side of Proposition C.2 is bounded by $C_{57}(\delta, \tau)$ by explicitly computing the derivatives using (C.5) and using again (C.15)-(C.19).

Bounds on $\Gamma_1^*, \Gamma_3, \Gamma_6^*$ are obtained similarly by the following applications of Proposition C.2. As the precise calculations using (C.15)-(C.19) follow the same arguments as for $\Gamma_2, \Gamma_6^*$ above, we omit some details.

**Proof of $\mathbb{E}[\Gamma_1^2] \leq C_{58}(\delta, \tau)/n$.** The bound on $\Gamma_1^*$ is obtained similarly using Proposition C.2 with the same $Z \in \mathbb{R}^{n \times (p-1)}$ (that is, $Z$ is the matrix $G$ with the first column removed), this time with $f$ valued in $\mathbb{R}^{p-1}$ with components $f_j = \theta_j/||\tilde{\psi}||$ for each $j = 2, \ldots, p$ and $u = \tilde{\psi}/||\tilde{\psi}||$. The key algebra is that using the derivatives (C.5),

$$\sum_{i=1}^{n} \tilde{\psi}_i \sum_{j=2}^{p} \frac{\partial \theta_j}{\partial g_{ij}} = \left( \sum_{j=2}^{p} A_{jj} \right) ||\tilde{\psi}||^2 - \tilde{\theta}^T P_1^\perp AG^T \tilde{\psi} = (\gamma_* - A_{11})n\sigma^2 - \tilde{\theta}^T P_1^\perp AG^T \tilde{\psi},$$

$$\sum_{j=2}^{p} \frac{\partial \theta_j}{\partial g_{ij}} = -\tilde{\psi}^T DGAP_1^\perp \tilde{\theta} - \text{Tr}[V] ||P_1^\perp \tilde{\theta}||^2 = -\tilde{\psi}^T DGAP_1^\perp \tilde{\theta} - n\tilde{\sigma}^2.$$
so that \( u^T Zf - \sum_{ij} \frac{\partial}{\partial z_{ij}}(u_i f_j) \) appearing in the left-hand side of Proposition C.2 equals
\[
\left[ \frac{\hat{\psi}^T G P_n^+ \hat{\theta}}{n \tau^2} + \hat{\psi} \frac{\sigma_n^2}{\tau^2} - \gamma_* \right] + \left[ A_{11} + \frac{\hat{\psi}^T D G A \hat{\psi} + \hat{\theta}^T P_n^+ AG^T D \hat{\psi}}{\|\hat{\psi}\|^2} \right] - \frac{n}{n} \sum_{i=1}^{p} \frac{\partial}{\partial g_{ij}} \|\hat{\psi}\|^2 \]
where the first square bracket is exactly \( \Gamma^*_1 \) and the second moment of the second bracket is smaller than \( C_{59}(\delta, \tau)/n \) using (C.15)-(C.19). Similarly to \( \Gamma_6^* \), the derivative term in the right-hand side of Proposition C.2 is bounded by \( C_{60}(\delta, \tau) \) by explicitly computing the derivatives using (C.5) and using again (C.15)-(C.19).

**Proof of Theorem 4.4.** The bound on \( \Gamma_3 \) is obtained similarly using Proposition C.2 with the same \( Z \), this time with \( f \) valued in \( \mathbb{R}^{p-1} \) with components \( f_j = \frac{e_j^T G^T \hat{\psi}}{\|\hat{\psi}\|} \) for each \( j = 2, \ldots, p \) and \( u = n^{-1} \|\hat{\psi}\| \). The key algebra is that \( u^T Zf - \sum_{ij} \frac{\partial}{\partial z_{ij}}(u_i f_j) \) appearing in the left-hand side of Proposition C.2 is then equal to
\[
\left[ \frac{\|GP^+ \hat{\psi}\|^2}{n \tau^2} - \frac{\sigma_n^2}{\tau^2} - \gamma_* \frac{\hat{\psi}^T G P_n^+ \hat{\theta}}{n \tau^2} + \frac{\hat{\psi}}{n \tau^2} \right] + \left[ A_{11} \frac{\hat{\psi}^T D G A \hat{\psi} + \hat{\theta}^T P_n^+ AG^T D \hat{\psi} - (AG^T DG)_{ij} \|P_n^+ \hat{\theta}\|^2}{\|\hat{\psi}\|^2} \right] - \frac{n}{n} \sum_{i=1}^{p} \frac{\partial}{\partial g_{ij}} \|\hat{\psi}\|^2 \]
with \( \hat{\psi}/n = \text{Tr}[G^T D G A]/n = \hat{\gamma} \) so that the first square bracket is exactly \( \Gamma_3 \) and the second bracket is negligible using (C.15)-(C.19).

**Proof of Theorem 4.4.** The bound on \( \Gamma_5 \) is obtained similarly using Proposition C.2 with the same \( Z \), this time with \( f \) valued in \( \mathbb{R}^{p-1} \) with components \( f_j = \hat{\theta}_j/\|\hat{\psi}\| \) for each \( j = 2, \ldots, p \) and \( u = GP^+ \hat{\theta}/\|\hat{\psi}\| \). The key algebra is that \( u^T Zf - \sum_{ij} \frac{\partial}{\partial z_{ij}}(u_i f_j) \) appearing in the left-hand side of Proposition C.2 is then equal to
\[
\left[ \frac{\|GP^+ \hat{\psi}\|^2}{n \tau^2} - \frac{\sigma_n^2}{\tau^2} - \gamma_* \frac{\hat{\psi}^T G P_n^+ \hat{\theta}}{n \tau^2} + \frac{\hat{\psi}}{n \tau^2} \right] + \left[ A_{11} \frac{\hat{\psi}^T D G A \hat{\psi} + \hat{\theta}^T P_n^+ AG^T D \hat{\psi} - (AG^T DG)_{ij} \|P_n^+ \hat{\theta}\|^2}{\|\hat{\psi}\|^2} \right] - \frac{n}{n} \sum_{i=1}^{p} \frac{\partial}{\partial g_{ij}} \|\hat{\psi}\|^2 \]
The first bracket is exactly \( \Gamma_5^* \) and the second bracket is negligible using (C.15)-(C.19).

\[ \square \]

**Theorem 4.4.** Let Assumptions A and B hold. Let \( (\hat{\gamma}, \hat{\psi}, \hat{\gamma}, \hat{\psi}, \hat{\gamma}, \hat{\psi}) \) be as in (3.8), and let \( (t_*, a_*, \sigma_*) \) be as in Theorems 4.1 and 4.3. Then
\[
E[|\hat{\psi} - \gamma_*|] \leq C_{63}(\delta, \tau)n^{-1/2}, \tag{4.11}
E[\|\hat{\psi} - t_*^2\|] \leq C_{64}(\delta, \tau)n^{-1/2}, \tag{4.12}
E[\|\hat{\psi}^2 - \gamma^2\|] \leq C_{65}(\delta, \tau)n^{-1/2}, \tag{4.13}
E[\|\hat{\psi}^2 - \sigma^2_\psi\|] \leq C_{66}(\delta, \tau)n^{-1/2}. \tag{4.14}
\]
If additionally Assumption D holds, then there exists an event \( E \) with \( \mathbb{P}(E) \to 1 \) such that
\[
E[I_E \left( |\hat{\gamma} - \gamma_*| + \|\hat{\psi}^2 - t_*^2\| + \|\hat{\psi} - \gamma^2\|^2 - \|\Sigma^{1/2} \hat{\theta}\|^2 \right) ] \leq C_{67}(\delta, \tau, \ell)n^{-1/2}. \tag{4.15}
\]

**Proof of Theorem 4.4.** Using (C.15) and (C.16)-(C.18), we have almost surely
\[
\max \left\{ \|\hat{\gamma} - \gamma_*\|, |\hat{\psi}|, |\hat{\psi}|, \frac{\sigma_n^2}{\tau^2}, \frac{\|GP^+ \epsilon\|^2}{n \tau^2}, \frac{\|\hat{\psi}^2 - \gamma^2\|^2}{n \tau^2}, \frac{\ell^2 - t_*^2}{\tau^2} \right\} \leq C_{68}(\delta, \tau)(1 + \|\varepsilon\|_{op})^c \tag{C.24}
\]
for some numerical constant \(c \geq 1\). The event \(\tilde{E} = \{\|n^{-1/2}G\|_{op} \leq 2 + \delta^{-1/2}\}\) has exponentially large probability, \(P(\tilde{E}) \leq e^{-n/2}\), by [18, Theorem II.13]. Let

\[
\text{Rem} = \left| \hat{\delta}(\hat{\gamma} - \gamma_\ast) \right| + \frac{1}{2} \left| \tilde{t}^2 - \tilde{t}^2 \right| + \frac{1}{2} \left| \frac{1}{n} \right| \lVert X\hat{\beta} - \hat{\gamma} \hat{\psi} \rVert^2 - \left| \hat{\theta} \right|^2 \right| + \frac{\tilde{t}^2}{n} \lVert \tilde{a}^2 - a_\ast^2 \rVert.
\]

Using (C.10), (C.11), (C.12) and (C.13) for the first term and the Cauchy-Schwarz inequality for the second, we find

\[
\mathbb{E}\text{Rem} \leq \mathbb{E}[I_{\tilde{E}}\text{Rem}] + \mathbb{E}[I_{\tilde{E}}\text{Rem}] \leq C_{\text{90}}(\gamma, \tau)\mathbb{E}[\hat{\Gamma}] + P(\tilde{E})^{1/2}\mathbb{E}[\text{Rem}^2]^{1/2}
\]

where \(\hat{\Gamma} = \max\{\lVert \Gamma_1^\ast \rVert, \lVert \Gamma_2^\ast \rVert, \lVert \Gamma_3^\ast \rVert, \lVert \Gamma_4^\ast \rVert, \lVert \Gamma_5^\ast \rVert, \lVert \Gamma_7^\ast \rVert\}\). Lemma C.3 shows that \(\mathbb{E}[\hat{\Gamma}] \leq C_{\text{70}}(\delta, \tau)n^{-1/2}\). By (C.24) and (C.19) we have \(\mathbb{E}\text{Rem}^2 \leq C_{\text{71}}(\delta, \tau)\) so that the exponential small probability of \(\tilde{E}\) completes the proof of (4.11)-(4.14).

Under the additional Assumption D, we have \(\hat{r}^2 = \|\hat{\psi}\|^2/n \leq 1\) hence by (C.18) and in \(\tilde{E}\), \(\|G\hat{\beta}\|^2/n \leq C_{\text{72}}(\delta, \tau)\). Hence with \(u_i = e_i^T G\hat{\beta}\), there exists at least \(n/2\) indices \(i \in [n]\) such that \(|u_i| \leq K\) for some constant \(K = C_{\text{73}}(\delta, \tau)\). By Assumption D, continuity and compactness, there exists deterministic constants \(c_\ast, m_\ast > 0\) depending only on \(K\) and the loss \(\ell\) such that \(\inf_{y_i \in [r; \lambda; \lambda]} \min_{u_i \in [r; \lambda; \lambda]} \ell'_{y_i}(u_i) \geq c_\ast\) and \(\inf_{y_i \in [r; \lambda; \lambda]} \min_{u_i \in [r; \lambda; \lambda]} \ell''_{y_i}(u_i)^2 \geq m_\ast\). Since at least \(n/2\) components \(u_i\) are such that \(|u_i| \leq K\), this implies \(\hat{r}^2 = \|\hat{\psi}\|^2/n \geq m_\ast/2\) and \(\text{Tr}[D] = \sum_{i=1}^n \ell''_{y_i}(u_i) \geq (n/2)c_\ast\). The lower bound in (3.6) then yields \(\tilde{E} \supseteq \text{Tr}[\hat{V}] \geq C_{\text{75}}(\delta, \tau) \text{Tr}[D] - C_{\text{76}}(\delta, \tau) \geq nC_{\text{77}}(\delta, \tau, c_\ast) = nC_{\text{78}}(\delta, \tau)\) for \(n \geq C_{\text{79}}(\delta, \tau)\). We have proved that

\[
in the event \(\tilde{E} = \{\|n^{-1/2}G\|_{op} \leq 2 + \delta^{-1/2}\}\), \(\max\{\hat{r}^2, \|\hat{\psi}\|/\sqrt{n}\} \leq C_{\text{80}}(\delta, \tau)\)
\]

and the proof of (4.15) is complete.

\(\square\)

### C.4. Proofs: Proximal representation for predicted values

**Theorem 4.3.** Let Assumptions A and B be fulfilled. Define \(a_\ast = w^T \Sigma \hat{\beta}\), \(\sigma_\ast^2 = \|\Sigma^{1/2} \hat{\beta}\|^2 - a_\ast^2\) and \(\gamma_\ast = \text{Tr}[\Sigma \hat{A}]\). Then

\[
\max_{i=1,...,n} \mathbb{E}\left[ \frac{1}{n^2} \left| x_i^T \hat{\beta} - \text{pro}x[\gamma_\ast \ell_{y_i}(\cdot)](a_\ast U_i + \sigma_\ast Z_i) \right|^2 \right] \leq \frac{C_{\text{81}}(\delta, \tau)}{n}
\]

where \(U_i = x_i^T w\) and \(Z_i\) are independent \(N(0, 1)\) random variables.

**Proof of Theorem 4.3.** Consider the change of variable and notation defined in Appendix C.2. Then (4.10) holds if and only if

\[
\mathbb{E}[\hat{r}^{-2} (g_i^T \hat{\theta} - \text{pro}x[\text{Tr}[A] \ell_{y_i}(\cdot)](a_\ast U_i + \sigma_\ast Z_i))^2] \leq C_{\text{80}}(\delta, \tau, \kappa)/n
\]

for independent standard normals \(U_i, Z_i\) where \(U_i\) is the \((i, 1)\) element of the matrix \(G\). Furthermore the quantities in (3.8) can be expressed in terms of \(\hat{\theta}, G\); we thus work with \(G, \hat{\theta}\) and its derivatives instead of \(\hat{\beta}\). Next, we have the decomposition

\[
x_i^T \hat{\beta} = g_i^T \hat{\theta} = a_\ast U_i + \hat{\theta}_i^T f
\]

where we recall \(a_\ast \overset{\text{def}}{=} e_i^T \hat{\beta} = w^T \Sigma \hat{\beta}\), the standard normal \(z_i \sim N(0, I_{p-1})\) as \(z_i = (y_i)_{k=2,...,p}\) and \(f \in \mathbb{R}^{p-1}\) as \(f = \left(\frac{y_i}{y_i}\right)_{k=2,...,p}\). We apply Lemma B.1 conditionally on \((y_i, (I_n - e_i e_i^T)G, G_{i1})\). Since \(z\) is independent of \((y_i, (I_n - e_i e_i^T)G, G_{i1})\), the expectations in Lemma B.1 are simply integrals with respect to the Gaussian measure of \(z\). Rewriting Lemma B.1 with the notation of the present context yields

\[
\mathbb{E}\left[ \left( z_i^T f - \frac{\sigma_\ast}{\sqrt{n}} Z_i - \sum_{k=2}^p \frac{\partial}{\partial g_{ik}} \left( \frac{\hat{\theta}_i^T}{\|\hat{\psi}\|/\sqrt{n}} \right) \right) \right] \leq C_{83} \mathbb{E} \left[ \left( \frac{\partial}{\partial g_{ik}} \left( \frac{\hat{\theta}_i^T}{\|\hat{\psi}\|/\sqrt{n}} \right) \right) \right]\]

(C.26)
where $\sigma_* = \|P_1^T \hat{\theta}\|$ for $P_1^T \overset{\text{def}}{=} (I_p - e_1 e_1^T)$. We focus first on the sum in the left-hand side. By the product rule

$$
\frac{\partial}{\partial g_{ik}} \left( \frac{\hat{\theta}}{\psi} \right) = \frac{1}{\|\psi\|} \left[ \frac{\partial}{\partial g_{ik}} \hat{\theta} \right] - \frac{\hat{\psi}}{\|\psi\|^3} \left[ \frac{\partial}{\partial g_{ik}} \hat{\psi} \right] \hat{\theta},
$$

(C.27)

the derivatives (C.5), the definition of $\gamma_*$, $\hat{r}$ and $\sum_{k=2}^p \sigma_{\psi_{ik}}^2$, $\frac{\|\psi\|}{\|\psi\|^3}$, $\frac{n}{r} \sum_{k=2}^p \frac{\sqrt{n} \hat{\psi}_{ik}}{\|\psi\|} - \gamma_* \frac{\hat{\psi}_{il}}{\hat{r}} = - \frac{A_{11} \hat{\psi}_{il}}{\hat{r}} - \sum_{k=2}^p \frac{\sqrt{n} \hat{\psi}_{ik}}{\|\psi\|^3} \frac{\hat{\psi}_{il}}{\hat{r}}$

$$
\begin{align*}
 &\quad - \frac{A_{11} \hat{\psi}_{il}}{\hat{r}} + \sum_{k=2}^p \frac{\sqrt{n}}{\|\psi\|^3} \hat{\psi}_{il} \left( \text{DGA}_1 \hat{\theta} \hat{\psi}_{il} + V e_1 \sigma_*^2 \right), \\
 &\quad \text{and the third term in the right-hand side, by symmetry, satisfies} \\
 &\quad E[\sigma_*^4 \|\psi\|^{-6} (V^T \hat{\psi})^2] = E[C_8 \|\psi\|^{-6} \|V^T \hat{\psi}\|] \leq E[C_8 \|\psi\|^{-4} \|V\|_o^2].
\end{align*}
$$

The bounds (C.15), (C.16) and (C.18) thus show that $E[(C.28)^2] \leq C_{85}(\delta, \tau, \kappa)/n$. It remains to bound from above the right-hand side of (C.26). By exchangeability of $i = 1, ..., n$ and the product rule (C.27),

$$
E \sum_{k=2}^p \left( \frac{\partial}{\partial g_{ik}} \left( \frac{\hat{\theta}}{\|\psi\|/\sqrt{n}} \right) \right)^2 = \frac{1}{n} \sum_{i=1}^n E \sum_{k=2}^p \left( \frac{\partial}{\partial g_{ik}} \left( \frac{\hat{\theta}}{\|\psi\|/\sqrt{n}} \right) \right)^2 (\text{by exchangeability})
$$

$$
\leq \frac{1}{n} \sum_{i=1}^n \left[ \frac{\sqrt{n} \hat{\theta} \hat{\psi}^T \hat{\theta}}{\|\psi\|^3} \|\hat{\psi}\|^3 \|\hat{\theta}\|^2 \right]^2 (\text{chain rule})
$$

$$
\leq \frac{1}{n} \sum_{i=1}^n \left[ \frac{\sqrt{n} \hat{\theta} \hat{\psi}^T \hat{\theta}}{\|\psi\|^3} \|\hat{\psi}\|^3 \|\hat{\theta}\|^2 \right]^2 (\text{C.29})
$$

by $(a + b)^2 \leq 2a^2 + 2b^2$. For the first term, using the explicit derivatives in (C.5) and again $(a + b)^2 \leq 2a^2 + 2b^2$,

$$
\frac{1}{n} \sum_{i=1}^n \sum_{k=2}^p \frac{1}{\|\psi\|^2} \left( \frac{\partial}{\partial g_{ik}} \|\hat{\theta}\|^2 \right) \leq \frac{\|A\|^2}{\|\hat{\psi}\|^2} + \frac{\|AGD\|^2}{\|\hat{\psi}\|^2}. \\
$$

(C.30)

while $\sum_{i=1}^n \sum_{k=2}^p \frac{\|\psi\|^2}{\|\psi\|^3} \|\hat{\psi}\|^2$ is already bounded in (C.22). The bounds (C.15), (C.16) and (C.18) completes the proof of $(C.26)^2 \leq (C.29)^2 \leq C_{85}(\delta, \tau, \kappa)/n$. 

Appendix D: Proof in the unregularized case with $p < n$

D.1. Derivatives

Lemma 5.1 is restated before its proof for convenience.

Lemma 5.1. Let Assumptions C and D be fulfilled so that the penalty is $g = 0$. Let $y \in \mathcal{Y}$ and $X \in \mathbb{R}^{n \times p}$ be fixed. If a minimizer $\hat{\beta}$ exists at $(y, X)$ and $X^T X$ is invertible, then there exists a neighborhood of $X$ such that $\hat{\beta}(y, X)$ exists in this neighborhood, the map $X \mapsto \hat{\beta}(y, X)$ restricted to this neighborhood is continuously differentiable, and (3.7) holds with $A = \left( \sum_{i=1}^n x_i \epsilon_i y_i (x_i^T \hat{\beta} x_i)^{-1} \right)$. 

\begin{flushright}
$\Box$
\end{flushright}
Proof of Lemma 5.1. If a minimizer \( \hat{\beta} \) of (1.2) exists and \( X^T X \) is invertible, the optimality conditions read \( \varphi(\hat{X}, \hat{\beta}) = 0 \) for \( \varphi(\hat{X}, b) = \sum_{i=1}^{n} x_i^T \ell(y_i | x_i^T \beta) \) (for the derivatives in this paragraph, \( y \) is considered a constant). Then the Jacobian \( \frac{d}{db} \varphi \in \mathbb{R}^{n \times p} \) is the symmetric matrix \( \sum_{i=1}^{n} x_i \ell''(y_i | x_i^T \beta) x_i^T \) which is continuous and positive semi-definite in the open set \( \{ X : \det(X^T X) > 0 \} \) since \( \ell''_w > 0 \). By the implicit function theorem, there exists a continuously differentiable function \( b : \mathbb{R}^{n \times p} \to \mathbb{R}^p \) in a neighborhood of \( X \) such that \( \varphi(X, b(X)) = 0 \) in this neighborhood. In other words, in this neighborhood, \( b(X) \) is a solution of (1.2) and \( X \to \hat{\beta}(y, X) \) is continuously differentiable. By differentiation of \( \sum_{i=1}^{n} x_i \ell''(x_i^T \hat{\beta}) = 0 \), we obtain (3.7) with \( \hat{A} = \left( \sum_{i=1}^{n} x_i \ell''(x_i^T \beta) x_i^T \right)^{-1} \).

D.2. Non-separable losses

Theorem 5.4 is restated before its proof for convenience.

Theorem 5.4. Assume that \( 1 < \delta \leq n/p \leq 2\delta \), that \( X \) has iid \( N(0, \Sigma) \) rows and \( w \in \mathbb{R}^p \) satisfies \( w^T \Sigma w = 1 \). Let \( U \) be a latent random variable independent of \( X \) and assume that \( L : \mathbb{R}^n \to \mathbb{R} \) is a random loss function of the form \( L(v) = F(v, U, Xw) \) for all \( v \in \mathbb{R}^n \) for some deterministic measurable function \( F \). Assume that with probability one with respect to \((U, Xw), L \) is convex, coercive, twice differentiable with positive definite Hessian everywhere. Let \( \hat{\beta} = \arg\min_{b \in \mathbb{R}^p} L(Xb) \) and assume that \( \mathbb{P}(\|\nabla L(X\hat{\beta})\| \leq \sqrt{n}) = 1 \). Extend the notation \( D, \hat{A}, V \) to this non-separable setting by

\[
\hat{\psi}(y, X) = -\nabla L(X\hat{\beta}) \in \mathbb{R}^n, \quad D = \nabla^2 L(X\hat{\beta}) \in \mathbb{R}^{n \times n}, \quad \hat{A} = (X^T DX)^{-1} \in \mathbb{R}^{p \times p}
\]

and \( V = D - DX \hat{A} X^T D \). Define \( \hat{r}^2 = \|\hat{\psi}\|^2/n \) as well as \( \hat{v} = \text{Tr}(V)/n \). Let \((\hat{a}^2, \hat{\sigma}^2)\) be as in (5.1) and define \( a_* = \hat{\beta}^T \Sigma \hat{a}^2 \) as well as \( \sigma_*^2 = \|\Sigma^{1/2} \hat{a}\|^2 - \hat{a}_*^2 \). Then

\[
\mathbb{E}[\hat{v}^2 - \sigma_*^2] + \mathbb{E}[\hat{v}^2 - \hat{a}_*^2] + \mathbb{E}[(\sqrt{n} \hat{v}^T D \hat{u} - \hat{r} Z)^2]^{1/2} \leq \frac{C_{\text{sc}(\delta)}}{\sqrt{n}} \mathbb{E}[(1 \vee \|n^{1/2} \hat{A} \Sigma^{1/2}\|_{op} \vee \|D\|_F \vee \|\Sigma^{1/2} \hat{\beta}\| \vee \|n^{-1/2} X^T X \Sigma^{-1/2}\|_{op})^8]
\]

for some \( Z \sim N(0, 1) \) for any deterministic \( u \in \mathbb{R}^p \) with \( \|\Sigma^{-1/2} u\| = 1 \) such that \( w^T u = 0 \).

Proof of Theorem 5.4. By the same argument as the proof of Lemma 5.1, the partial derivatives are given by (3.7). Consider the change of variable in Appendix C.2, so that \( G \in \mathbb{R}^{n \times p} \) has iid \( N(0, 1) \) entries, and \( \hat{G} = X \hat{\beta} \) for \( \hat{\beta} = \arg\min_{\beta \in \mathbb{R}^p} \sum_{i=1}^{n} L(G\beta) \). Then

\[
D = \nabla^2 L(G\hat{\beta}), \quad \hat{\psi} = -\nabla L(G\hat{\beta}), \quad \hat{A} = (G^T DG)^{-1}, \quad \hat{V} = D - DGAG^T D
\]

and \( \hat{\theta}_j = a_* = \hat{\beta}^T \Sigma \hat{a}^2 \) as well as \( \sigma_*^2 = \|P_1^\perp \hat{a}\|^2 \) for \( P_1^\perp = I_p - e_1 e_1^T \). After the change of variable, the formula (C.5) for the derivatives holds. Applying Proposition 5.1 conditionally on \((y, Ge_1)\) with \( K = n, Q = p-1 \) to the matrix \( Z \in \mathbb{R}^{n \times (p-1)} \) made of the last \( p-1 \) columns of \( G \) and with \( u(Z) = \psi/\sqrt{n} \), we obtain

\[
\frac{1}{n} \mathbb{E}[\|P_1^\perp \hat{a}\|^2 - \sum_{j=2}^{p} (e_j^T G^T \hat{\psi} - \sum_{i=1}^{n} \frac{\partial \hat{\psi}_i}{\partial g_{ij}})^2] \leq C_{\text{sc}}(\sqrt{n}(1 + \Xi^{1/2} + \Xi))
\]

with \( \Xi = \sum_{j=2}^{p} \sum_{i=1}^{n} \frac{1}{n} \|\frac{\partial}{\partial g_{ij}} \hat{\psi}\|^2 \) as defined in Proposition 5.1. Since \( G^T \hat{\psi} = 0_p \) by the KKT conditions of \( \hat{\theta} \), the first term inside the square in the left-hand side is 0. For the second term inside the square in the left-hand side, \(- \sum_{i=1}^{n} \frac{\partial \hat{\psi}_i}{\partial g_{ij}} = \text{Tr}(V) \hat{\theta}_j \) + \( \hat{u}^T D G A e_j \). With the notation \( c \in \mathbb{R}^p, c_j \equiv \hat{\psi}^T D G A e_j \), by expanding the square \( (\text{Tr}(V) \hat{\theta}_j + c_j)^2 \) for each \( j = 2, ..., p \) using the triangle inequality, and dividing both sides by \( n \) gives

\[
\frac{1}{n} \mathbb{E}[|P_1^\perp \hat{a}|^2 - \text{Tr}(V)^2 \|P_1^\perp \hat{\theta}\|^2] \leq \frac{1}{n} \mathbb{E}[(2 \text{Tr}(V) \hat{\theta}^T P_1^\perp c + \|P_1^\perp c\|^2 - \|\hat{\psi}\|^2] + \frac{1}{n} C_{\text{sc}}(\sqrt{n}(1 + \Xi^{1/2} + \Xi)).
\]
The left-hand side reads \(\mathbb{E} |\hat{p}^2 - n\hat{\theta}^2\hat{\sigma}_\theta^2| = \mathbb{E} |n\hat{\theta}^2(\hat{\sigma}^2 - \sigma^2_\theta)|\) thanks to \(\hat{\sigma}^2 = \frac{p}{n} - \hat{\sigma}^2\). We now bound from above \(|\hat{a}^2 - \hat{a}^2|\). We start with

\[
\|G\hat{\theta}\|^2 = \|GP_\hat{a}^1\hat{\theta} + Ge_1\hat{\theta}_1\|^2 = \hat{\theta}^T P_\hat{a}^1G^T(GP_\hat{a}^1\hat{\theta} + 2Ge_1\hat{\theta}_1) + \|Ge_1\|^2\hat{\theta}_1^2.
\]

We wish to study the first term in the right-hand side. Applying Proposition C.2 conditionally on \((\mathbf{y}, Ge_1)\) to \(K = n, Q = p - 1\), \(\mathbf{Z}\) being the last \(p - 1\) columns of \(\mathbf{G}\), \(\mathbf{u}\) equal to the last \(p - 1\) components of \(\hat{\theta}\) and \(f = G[I_p + e_1e_1^T]\hat{\theta}\), we find \(\|GP_\hat{a}^1\hat{\theta} + 2Ge_1\hat{\theta}_1\|^2 = f^T\mathbf{Z}f\)

\[
\mathbb{E}\left[(\|GP_\hat{a}^1\hat{\theta}\| - \|Ge_1\|\hat{\theta}_1^2 - \sum_{i=1}^p f_i \sum_{j=2}^p \frac{\partial \hat{\theta}_j}{\partial g_{ij}} - \sum_{i=1}^p \hat{\theta}_j \sum_{j=2}^p e_i G(I_p + e_1e_1^T) \frac{\partial \hat{\theta}}{\partial g_{ij}}) \right] \leq \mathbb{E} \sum_{j=2}^p \left\| \frac{\partial (f^T \hat{\theta})}{\partial g_{ij}} \right\|^2.
\]

Applying the product rule for \(\psi\) after observing that \(\|\hat{\theta}\| - \hat{\theta}^T P_\hat{a}^1AG^T D f = -\hat{\theta}^T P_\hat{a}^1AG^T D f = -\|P_\hat{a}^1\hat{\theta}\|^2\)

because \(f^T\hat{\psi} = 0\) thanks to \(G^T\hat{\psi} = 0\), which are KKT conditions for the second equality, and using \(AG^T DG = I_p\), for the third equality. Now let us focus on the second term appearing from the product rule:

\[
\sum_{j=2}^p \hat{\theta}_j \sum_{j=2}^p e_i G(I_p + e_1e_1^T) \frac{\partial \hat{\theta}}{\partial g_{ij}}.
\]

For the right-most double sum, using \((\partial/\partial g_{ij})\hat{\theta} = A(e_1 e_i G - e_1 G)\) in (C.5), the term stemming from \(Ae_j\hat{\psi}\) equals 0 thanks to \(\sum_{i=1}^n \hat{e}_i G = 0\) by the KKT conditions. This gives that the previous display equals \((n - \hat{p})\|P_\hat{a}^1\hat{\theta}\|^2\) where \(\hat{p} \equiv \mathbb{E}\left[\mathbb{E}(G[I_p + e_1e_1^T]AG^T D)\right] = p + 1\) since \(AG^T DG = I_p\). Combining the above identities yields

\[
\mathbb{E}\left[(\|GP_\hat{a}^1\hat{\theta}\| - \|Ge_1\|^2\hat{\theta}_1^2 - (n - p - 2)\|P_\hat{a}^1\hat{\theta}\|^2) \right] \leq \mathbb{E} \sum_{i=1}^p \left\| \frac{\partial (f^T \hat{\theta})}{\partial g_{ij}} \right\|^2.
\]

We apply Jensen’s inequality \(\mathbb{E} |\cdot| \leq \mathbb{E}[|\cdot|^2]^{1/2}\) to the left-hand side and replace \(\|Ge_1\|^2\hat{\theta}_1^2\) by \(n\hat{\theta}_1^2\), incurring the error term \(\|Ge_1\|^2 - n\hat{\theta}_1^2\) by the triangle inequality. This yields

\[
\frac{1}{n} \mathbb{E}|\hat{\theta}^2| - \mathbb{E}(\|P_\hat{a}^1\hat{\theta}\|^2) \leq \left| \mathbb{E}\left(\frac{\|Ge_1\|^2}{n} - 1\right) \right| + \mathbb{E}\left(\sum_{i=1}^p \sum_{j=2}^p \left\| \frac{\partial (f^T \hat{\theta})}{\partial g_{ij}} \right\|^2 \right)^{1/2}
\]

\[
\leq \left( \frac{2}{n} \mathbb{E}[\hat{\theta}_1^4] \right)^{1/2} + \mathbb{E}\left(\sum_{i=1}^p \sum_{j=2}^p \left\| \frac{\partial (f^T \hat{\theta})}{\partial g_{ij}} \right\|^2 \right)^{1/2}\]

\[
(D.2)
\]

thanks to the Cauchy-Schwarz inequality and \(\mathbb{E}[\hat{\theta}_1^4] = 2n\) for the second inequality. Finally, for any vector \(a^2, a^3, \ldots, a^p\) with unit norm orthogonal to \(e_1\), consider an orthonormal basis \(a^2, a^3, \ldots, a^p\) of \(\{0\} \times \mathbb{R}^{p-1}\).

By application of Lemma B.1 to \(n = G\hat{a}^k\) conditionally on \((U, G[I_p - a^k(a^k)^T])\) and \(f\) in Lemma B.1 equal to \(\hat{\psi}\), we have that for random variables \(Z_k \sim N(0,1)\),

\[
\sum_{k=2}^p \mathbb{E}\left[\left(\sum_{i=1}^n a^k_j \frac{\partial \hat{\psi}_k}{\partial g_{ik}} - \|\hat{\psi}\|Z_k\right)^2\right] \leq 15 \mathbb{E}\left(\sum_{k=2}^p \sum_{i=1}^n a^k_j \frac{\partial \hat{\psi}_k}{\partial g_{ik}}\right)^2
\]

after observing that \(z^TF = 0\). Furthermore, in the left-hand side \(\sum_{i=1}^n \sum_{j=2}^p a^k_j \frac{\partial \hat{\psi}_k}{\partial g_{ij}} = -\mathbb{E}(V^T \hat{\theta}^T a^k - \hat{\psi}^T DG A a^k)\).

We further move \(-\hat{\psi}^T DG A a^k\) to the right-hand side using the triangle inequality to find

\[
\sum_{k=2}^p \mathbb{E}\left[(-\mathbb{E}(V^T \hat{\theta}^T a^k - \|\hat{\psi}\|Z_k)\right)^2\right] \leq 30 \mathbb{E}\left(\sum_{k=2}^p \sum_{i=1}^n a^k_j \frac{\partial \hat{\psi}_k}{\partial g_{ik}}\right)^2 + 2 \mathbb{E}(\psi^T DG A a^k)\]
Using $\sum_{k=2}^{p} a^k (a^k)^T = P_1^1$ for the second term in the right-hand side, and
\[
\sum_{k=2}^{p} \left\| \sum_{j=2}^{p} \left\| M e_j e_j^T a^k \right\| \right\|^2 = \sum_{k=2,j,j \geq 2} \text{Tr}[M^T M e_j e_j^T a^k (a^k)^T e_j e_j^T] = \text{Tr}[M^T M P_1^1]
\]
for the matrix $M$ with columns $M e_j = \frac{\partial \hat{\phi}}{\partial g_j}$, again thanks to $\sum_{k=2}^{p} a^k (a^k)^T = P_1^1$, we find that the right-hand side equals $30 \mathbb{E} \sum_{i=1}^{n} \sum_{j=2}^{p} \left\| \frac{\partial \hat{\phi}}{\partial g_{ij}} \right\|^2 + 2 \mathbb{E} \left\| AGD \hat{\phi} \right\|^2$. By symmetry and rotational invariance, in the summands of $\sum_{k=2}^{p}$ in the left-hand side, the expectations $\mathbb{E}[(-\text{Tr}[V] \hat{\theta}^T a^k - \| \hat{\phi} \| Z_k)^2]$ are all equal to other so, that for any $a \in \{0\} \times \mathbb{R}^{p-1}$ with $\|a\| = 1$,
\[
\frac{p-1}{n^{2}} \mathbb{E} \left[ (-\text{Tr}[V] a^T \hat{\theta} - \| \hat{\phi} \| Z)^2 \right] \leq 30 \mathbb{E} \sum_{j=2}^{p} \sum_{i=1}^{n} \left\| \frac{\partial \hat{\psi}}{\partial g_{ij}} \right\|^2 + \frac{2}{n^{2}} \mathbb{E} \left\| AG^T D \hat{\psi} \right\|^2
\]
for some $Z \sim N(0,1)$. Note that to obtain (D.3), we divided both sides by $n^2$. It remains to bound from above the right-hand sides in (D.1), (D.2), (D.3). We will use inequality
\[
\left\| DGA \right\|_F^2 \leq \left\| D^{1/2} \right\|_F^2 \left\| A \right\|_{op} = \text{Tr}[D] \left\| A \right\|_{op},
\]
which follows from writing $DGA = D^{1/2} (D^{1/2} GA^{1/2}) A^{1/2}$ with the matrix $D^{1/2} GA^{1/2}$ inside the parenthesis having all singular values equal to one since $A^{-1} = G^T D G$. We will also use $\left\| P_1^1 \right\|_{op} \leq 1$ and $0_{n \times n} \leq V \leq D$ in the sense of psd matrices which implies $\text{Tr}[V] \leq \text{Tr}[D]$ and $\left\| V \right\|_F^2 \leq \left\| D \right\|_F^2$. We first focus on the quantity $\Xi = \frac{1}{n} \mathbb{E} \sum_{j=2}^{p} \sum_{i=1}^{n} \left\| \frac{\partial \hat{\psi}}{\partial g_{ij}} \right\|^2$ appearing in the right-hand side of (D.1) and (D.3). By (C.5),
\[
\begin{align*}
\frac{1}{2} \Xi & \leq \frac{1}{n} \mathbb{E} \left[ \left\| \hat{\psi} \right\|^2 \left\| DGA \right\|_F^2 + \left\| V \right\|_F^2 \left\| P_1^1 \right\|_{op} \right] \quad \text{by} \quad \frac{1}{2} \left\| \cdot \right\|^2 + \frac{1}{2} \left\| \cdot \right\|^2 \\
& \leq \mathbb{E} \left[ \left\| DG \right\|_F \left\| A \right\|_{op} + \frac{1}{n} \left\| D \right\|_F \left\| P_1^1 \right\|_{op} \right] \quad \text{using} \quad \left\| \hat{\psi} \right\|^2 \leq n \\
& \leq \mathbb{E} \left[ \left\| DG \right\|_F \left\| A \right\|_{op} + \left\| D \right\|_F \left\| P_1^1 \right\|_{op} \right] \quad \text{using} \quad \text{Tr}[D] \leq \sqrt{n} \left\| D \right\|_F
\end{align*}
\]
We are left with only the quantities $\left\| \hat{\theta} \right\|^2$, $\frac{1}{n} \left\| D \right\|_F^2$, and $\left\| n A \right\|_{op}$ that appear in the right-hand side of (5.10). We now bound from above the terms in the right-hand side of (D.1), using the loose bound $\left\| DGA \right\|_{op} \leq \left\| DGA \right\|_F$ and (D.4):
\[
\begin{align*}
\hat{\Xi} & \overset{\text{def}}{=} \frac{1}{n^{2}} \mathbb{E} \left[ 2 \text{Tr}[V] \hat{\theta}^T P_1^1 c + \left\| P_1^1 c \right\|^2 \right] \\
& \leq \frac{1}{n^{2}} \mathbb{E} \left[ 2 \text{Tr}[V] \left\| \hat{\psi} \right\|^2 \left\| DGA \right\|_{op} + \left\| V \right\|_F \left\| DGA \right\|_{op} \right] \quad \text{since} \quad c \overset{\text{def}}{=} AG^T D \hat{\psi} \quad \text{in} \quad (D.1) \\
& \leq \mathbb{E} \left[ \frac{2}{n} \text{Tr}[D] \left\| A \right\|_{op} \left\| \hat{\theta} \right\| + \frac{1}{n} \text{Tr}[D] \left\| A \right\|_{op} \right] \quad \text{by} \quad \text{Tr}[V] \leq \text{Tr}[D], (D.4), \left\| \hat{\psi} \right\|^2 \leq n \\
& \leq 2 \left[ \frac{1}{n} \left\| D \right\|_F \right] \left\| n A \right\|_{op} \quad \text{using} \quad \text{Tr}[D] \leq \sqrt{n} \left\| D \right\|_F
\end{align*}
\]
Since (D.1) $\leq C_{91}(n^{-1/2}(\hat{\Xi} + \Xi^{1/2}) + \left\| n A \right\|_{op})$, we have proved that $\left\| \Xi \right\|^{1/2}$ and (D.1) are bounded from above by the right-hand side of (5.10) as desired.
It remains to bound (D.2), again using the derivatives in (C.5) to differentiate $\hat{\theta} f^T$ in (D.2). Since $f = G(I_p + e_1 e_1^T) \hat{\theta}$ and $(I_p + e_1 e_1^T)$ has operator norm at most 2, denoting $\hat{\theta}$ for $\frac{\partial \hat{\theta}}{\partial g_{ij}}$ for brevity, if $j \geq 2$,
\[
\left\| \partial \hat{\theta} f^T \right\| = \left\| (\partial \hat{\theta}) \hat{\theta}^T (I_p + e_1 e_1^T) G^T + \hat{\theta} \partial_{j} e_1 e_1^T + \hat{\theta} (\partial \hat{\theta})^T (I_p + e_1 e_1^T) G^T \right\| \\
\leq 8 \left\| \partial \hat{\theta} \right\| \left\| \hat{\theta} \right\| \left\| G \right\|_{op} + \left\| \hat{\theta} \right\| \left\| \partial_{j} \right\| \\
\leq 8 \left\| A e_j \right\| \left\| v_i \right\| \left\| \hat{\theta} \right\| \left\| G \right\|_{op} + 8 \left\| A G^T D e_i \right\| \left\| \hat{\theta} \right\| \left\| G \right\|_{op} + \left\| \hat{\theta} \right\| \left\| \partial_{j} \right\|
\]
thanks to (C.5) for the last inequality. Taking the square, taking the double sum ∑_{i=1}^{n} ∑_{j=2}^{p} \|M|_{ij}\|^2 = \|M\|^2_F for any matrix M we find

\[ \sum_{i=1}^{n} \sum_{j=2}^{p} \left( \frac{\partial (\hat{\theta} f^T)}{\partial g_{ij}} \right) \leq C_{90} \left[ \|\hat{\psi}\|^2 F |G|_{op}^2 \hat{\theta}^2 + \|AG^T D\|^2 F |G|_{op}^2 \hat{\theta}^4 + n \|\hat{\theta}\|^4 \right]. \]

We use \( \|\hat{\psi}\|^2 \leq n \) for the first term and \( \|AG^T D\|^2 F \leq \text{Tr}[D]|A|_{op} \) by (D.4) for the second. We have proved that (D.2) is bounded from above by

\[ \mathbb{E}[(\frac{1}{n} + \frac{1}{\sqrt{n}} |G|_{op}^2 \text{Tr}[D]|A|_{op}) \|\hat{\theta}\|^{4+1/2} + \mathbb{E}[\|AG^T D\|^2 F |G|_{op}^2 \|\hat{\theta}\|^2/n]^{1/2}. \]

We complete the proof using \( 0 \leq \text{Tr}[D] \leq \sqrt{n}|D|_F \) and \( |A|_F^2 < p |A|_{op}^2 \), so that (D.1), (D.2) and (D.3) are all bounded from above by the second line in (5.10).

**D.3. Lemmas using the density of the smallest eigenvalue of Wishart matrices**

**Lemma D.1.** If \( \frac{p}{n} \leq \delta^{-1} < (1 - \alpha) \) for constants \( \delta, \alpha > 0 \) and \( G \in \mathbb{R}^{n \times p} \) has iid \( N(0,1) \) entries then for \( P_i = \sum_{t \in I} e_t e_t^T \) we have

\[ \mathbb{P} \left[ \min_{I \subseteq [n], |I| = \lfloor (1 - \alpha) n \rfloor} \lambda_{\min} \left( \frac{G^T P_i G}{n} \right) < c_0 \right] \leq \sum_{I \subseteq [n], |I| = \lfloor (1 - \alpha) n \rfloor} \mathbb{P} \left[ \lambda_{\min} \left( \frac{1}{n} \frac{G^T P_i G}{n} \right) < c_0 \right] \to 0 \]

as \( n, p \to +\infty \) for some constant \( c_0 = c_0(\delta, \alpha) > 0 \) depending on \( (\delta, \alpha) \) only.

**Proof.** Variants of the following argument were used previously to study restricted isometry properties in [11] as a consequence of results from [21]. By [17, proof of Lemma 4.1] (with \( \frac{2}{\pi} \) there equal to \( t > 0 \) here), if \( N = |I| \)

\[ \mathbb{P} \left( \lambda_{\min} \left( \frac{G^T P_i G}{N} \right) \leq t^2 \right) \leq \frac{(tN)^{N-p+1}}{\Gamma(N-p+1)} \leq \frac{etN}{N-p+1} \frac{1}{(2\pi(N-p+1))^{1/2}} \]

where the second inequality follows from the lower bound on \( \Gamma(N-p+1) \) given for instance in [17, p10, Proof of Theorem 4.5]. Taking the union bound over \( \binom{n}{N} \) possible sets \( I \subseteq [n] \), it is sufficient to show that for a small enough constant \( t > 0 \),

\[ \binom{n}{N} \left( \frac{etN}{N-p+1} \right)^{N-p+1}(2\pi(N-p+1))^{-1/2} \]

converges to 0. First, \( \binom{n}{N} \leq \exp(n \frac{N}{\log(c\frac{2}{\pi})}) \) by a standard bound on binominal coefficient, so that \( \binom{n}{N} \leq \exp(n \log(c(1-\alpha)^{-1})) \leq \exp((N-p+1)C(\delta, \alpha)) \) where \( C(\delta, \alpha) \) depends on \( \delta, \alpha \) only. This proves that the previous display converges to 0 exponentially fast in \( n \) if \( t = c_0(\delta, \alpha) \) is a small enough constant depending only on \( \delta, \alpha \).

**Lemma D.2.** If \( \frac{p}{n} \leq \delta^{-1} < (1 - \alpha) \) for constants \( \delta, \alpha > 0 \) and \( G \in \mathbb{R}^{n \times p} \) has iid \( N(0,1) \) entries then for \( P_i = \sum_{t \in I} e_t e_t^T \) we have for any \( k \geq 1 \)

\[ \mathbb{E} \left[ \max_{I \subseteq [n], |I| = \lfloor (1 - \alpha) n \rfloor} \lambda_{\min} \left( \frac{G^T P_i G}{n} \right)^{-k} \right] \leq C_{91}(\delta, \alpha, k) \]

for a constant depending on \( \delta, \alpha, k \) only.

**Proof.** Let \( N = \lfloor (1-\alpha)n \rfloor, W_I = \lambda_{\min}(\frac{1}{n} G^T P_I G) \) and \( u_0 > 0 \) a constant to be specified. Using \( \mathbb{E}[W] = \int_0^\infty P(W > u)du \leq u_0 + \int_{u_0}^\infty P(W > u)du \) for any non-negative \( W \),

\[ \mathbb{E} \left[ \max_{I \subseteq [n], |I| = N} W_I^{-k} \right] \leq u_0 + \int_{u_0}^\infty \mathbb{P} \left[ \max_{I \subseteq [n], |I| = N} W_I^{-k} > u \right]du \leq u_0 + \int_{u_0}^\infty \binom{n}{N} \mathbb{P}(W_I^{-k} > u)du \]
by the union bound for the second term. The integrand is thus smaller than (D.5) for $t = u^{-1/(2k)}$. Integrating with respect to $u$ over $[u_0, \infty)$ gives the upper bound
\[
u_0 + \binom{n}{N} e^{\frac{N-p+1}{N-p+1}} \sum_{i=1}^{n} \mathbb{E}^v \left( \frac{1}{n} \left( \left\| \mathbf{y}_i \right\|^2 - K \right) \right).
\]

The previous display is bounded from above by $C_{92}(\delta, \alpha, k)$ if $u_0$ is a large enough constant depending only on $\delta, \alpha, k$ by the same argument as that used after (D.5).

**D.4. Adding a coercive term to the M-estimation objective**

Theorem 5.5 is restated here before its proof for convenience.

**Theorem 5.5.** Let Assumptions A, C and D be fulfilled so that the penalty is $g = 0$. Let $K > 0$. Define $h(t) = 0$ for $t < 0$, $h(t) = 1$ for $t > 1$ and $h(t) = 3t^2 - 2t^3$ for $t \in [0, 1]$, set $H(t) = \int_0^t h(u) du$ and note that $H$ is convex, twice continuously differentiable and nondecreasing. For any $v \in \mathbb{R}^n$, define
\[
\mathcal{L}(v) = \frac{1}{1 + \sqrt{K + 2}} \left( \sum_{i=1}^{n} \mathbb{E}^v \left( \frac{1}{n} \left( \left\| \mathbf{y}_i \right\|^2 - K \right) \right) \right).
\]

Then the minimizer $\hat{\mathbf{b}} = \arg\min_{b \in \mathbb{R}^p} \mathcal{L}(\mathbf{X}^T \mathbf{b})$ satisfies $\mathbb{P}(\left\| \mathcal{G} \mathbf{L}(\hat{\mathbf{b}}) \right\| \leq \sqrt{n}) = 1$, the assumptions of Theorem 5.4 are satisfied and the right-hand side of (5.10) is bounded from above by $C_{94}(\delta, K, \ell)n^{-1/2}$. In the event $E$ of Theorem 5.2, the minimizer $\hat{\mathbf{b}} = \arg\min_{b \in \mathbb{R}^p} \sum_{i=1}^{n} \mathbb{E}^v \left( \mathbf{y}_i \right) \mathbf{x}_i^T \mathbf{b} = \min_{b \in \mathbb{R}^p} \sum_{i=1}^{n} \mathbb{E}^v \left( \mathbf{y}_i \right) \mathbf{x}_i^T \mathbf{b}$ is equal to $\hat{\mathbf{b}}$ and $\max\{\hat{\alpha}, \frac{1}{\hat{\ell}_1}, \frac{1}{\hat{K}_1}, \frac{1}{\hat{\nu}}\} \leq C_{94}(K, \delta, \ell)$.

**Proof of Theorem 5.5.** Note that $\mathcal{L}$ is coercive thanks to the second term, so that $\hat{\mathbf{b}}$ always exists. Let $\hat{u} = \frac{1}{n}\left( \frac{1}{n} \left\| \mathbf{X} \hat{\mathbf{b}} \right\|^2 - K \right)$. The KKT conditions of $\hat{\mathbf{b}}$ read $\mathbf{X}^T (\ell''_y(\mathbf{X} \hat{\mathbf{b}})) + h(\hat{u}) \mathbf{X} \hat{\mathbf{b}} = 0$. Note that by construction, $\hat{u} < 1$ if and only if $\frac{1}{n} \left\| \mathbf{X} \hat{\mathbf{b}} \right\|^2 < K + 2$ if and only if $h(\hat{u}) < 1$. If $\frac{1}{n} \left\| \mathbf{X} \hat{\mathbf{b}} \right\|^2 > K + 2$ then $h(\hat{u}) = 1$ and the KKT conditions imply $\mathbf{X} \hat{\mathbf{b}} = -\mathbf{X}(\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \ell''_y(\mathbf{X} \hat{\mathbf{b}})$ so that $\left\| \mathbf{X} \hat{\mathbf{b}} \right\|^2 \leq \left\| \ell''_y(\mathbf{X} \hat{\mathbf{b}}) \right\|^2 \leq n$ because $|\ell''_y(|)\| \leq 1$ by Assumption D. This contradicts $\frac{1}{n} \left\| \mathbf{X} \hat{\mathbf{b}} \right\|^2 > K + 2$ so that $\frac{1}{n} \left\| \mathbf{X} \hat{\mathbf{b}} \right\|^2 \leq K + 2$ always holds. We also have
\[
\frac{\partial}{\partial \nu} \mathcal{L}(v) = \frac{1}{(1 + \sqrt{K + 2})} \left[ \ell''_y(v_i) + h\left( \frac{1}{n} \left\| \mathbf{y}_i \right\|^2 - K \right) \right] v_i
\]
\[
\left\| \nabla \mathcal{L}(v) \right\| \leq \frac{1}{(1 + \sqrt{K + 2})} \left[ \frac{1}{n} \left\| \mathbf{y}_i \right\|^2 \right].
\]

Since $\left\| \mathbf{X} \hat{\mathbf{b}} \right\|^2 \leq n(K + 2)$, it follows that $\left\| \nabla \mathcal{L}(\mathbf{X} \hat{\mathbf{b}}) \right\| \leq \sqrt{n}$ always holds. Let $\alpha > 0$ be a constant such that $\delta^{-1} \leq 1 - \alpha$ (for instance, $\alpha = 1 - \delta^{-1/2}$). Since $\frac{1}{n} \left\| \mathbf{X} \hat{\mathbf{b}} \right\|^2 \leq K + 2$, by Markov’s inequality there exists a random set $I \subset [n]$ of size $((1 - \alpha)n)$ such that $\max_{i \in I} (\mathbf{x}_i^T \hat{\mathbf{b}})^2 \leq (K + 2)/\alpha$. Let $P_I = \sum_{i \in I} \mathbf{e}_i \mathbf{e}_i^T$. Since the Hessian of a convex function is positive, we have
\[
\nabla^2 \mathcal{L}(\mathbf{X} \hat{\mathbf{b}}) \succeq \min_{i \in I} \ell''_y(\mathbf{x}_i^T \hat{\mathbf{b}}) \mathbf{e}_i \mathbf{e}_i^T \preceq \min_{i \in I} \ell''_y(\mathbf{x}_i^T \hat{\mathbf{b}}) P_I \succeq P_I c(\alpha, K, \ell)
\]
(5.8)
in the sense of the psd order, where $c(\alpha, K, \ell) = \min_{v \in \mathbb{R}^p} \min_{u \in \mathbb{R}^n: \left\| u \right\| \leq \sqrt{K + 2}/\sqrt{\alpha}} \ell''_y(u)$ which is a positive constant by Assumption D. If $\hat{\mathbf{A}} = (\mathbf{X}^T \nabla^2 \mathcal{L}(\mathbf{X} \hat{\mathbf{b}}) \mathbf{X})^{-1}$,
\[
\left\| n \Sigma^{1/2} \hat{\mathbf{A}} \Sigma^{1/2} \right\|_{op} \leq \frac{1}{\lambda_{\min}(\Sigma^{-(1/2)} \mathbf{X}^T \mathbf{X} \Sigma^{-(1/2)})} \leq \frac{c(\alpha, K, \ell)^{-1}}{\lambda_{\min}(\Sigma^{-(1/2)} P_I X \Sigma^{-(1/2)})}.
\]

The idea behind (D.8) is to ensure strong convexity of the loss $\mathcal{L}$ when restricted to an L2 ball of any radius, with the strong convexity constant depending on the radius; such local strong convexity has been used before, e.g., [39, Lemma 4] or [1]. We now combine this local strong convexity with the moment
bounds in Lemma D.2: By (D.6), any finite moment of the above display is bounded from above by $C_05(\alpha, \delta, K, \ell)$. Finally,

\[
\nabla^2 \mathcal{L}(v) = \text{diag} \{(e''_{\varphi}(v_i))_{i \in [n]} + h (\frac{1}{n} \|v\|^2 - K)\}I_n + \frac{1}{n} h' (\frac{1}{n} \|v\|^2 - K)\|v\|^T
\]

\[
\|\nabla^2 \mathcal{L}(\hat{\theta})\|_{op} \leq \max_{y \in \mathbb{Y}} \sup_{\|y\| \leq \sqrt{K+2}/\sqrt{\alpha}} \ell''_y(u) + \sup |h'(K + 2)|.
\]

We have proved that the right-hand side in (5.10) is bounded from above by $C_{96}(\delta, K, \ell)$.

We now show that $\hat{r} \equiv \hat{r} \in \mathbb{Y}$ for $\mathbb{V} = D - DX^T \hat{A} XD$ for $\mathbb{V} = \nabla^2 \mathcal{L}(\hat{\theta})$. Using $\mathbb{V} \leq D$ in the sense of the psd order gives $\hat{v} \leq \|D\|_{op}$ so that $\hat{v} \leq C_{98}(\ell, \delta, K)$ thanks to (D.10). It remains to show that $\hat{v}^{-1}$ is bounded from above. Since $P \equiv D^{1/2} X^T \hat{A} XD^{1/2}$ is an orthogonal projector of rank $p$, by the interlacing eigenvalue theorem, $\text{Tr}[D(I_{n} - P)] \geq \sum_{i=n-p}^{n-1} \lambda_i(D)$ where $(\lambda_i(D))_{i=1,\ldots,n-p}$ are the $n-p$ smallest eigenvalues of $D$. Let $J \subset [n]$ of size $|J| = n - p$ be the set of indices of these $n-p$ smallest eigenvalues. Then $\sum_{i \in J} (x_i^T \hat{b})^2 \leq n(K + 2)$ so at least $(1 - \alpha)(n - p)$ indices $i \in J$ satisfy $(x_i^T \hat{b})^2 \leq \frac{1}{n} (K + 2)/\alpha$. This implies that at least $(1 - \alpha)(n - p)$ eigenvalues of $D$ are at least larger than $\min_{y \in \mathbb{Y}} \min_{u \in \mathbb{R} : |u| \leq \sqrt{\delta^{-1}(K+2)/\alpha}} \ell''_y(u)$ and $\text{Tr}[\mathbb{V}] \geq C_{99}(\alpha, \delta, K)(n - p)$ must hold.

\[\square\]

D.5. Proof of Theorem 5.2

The result is restated for convenience.

Theorem 5.2. Let Assumptions A, C and D be fulfilled so that the penalty is $g = 0$. Let $K > 0$ and define the event $E = \{\text{the minimizer } \hat{\beta} \text{ in (1.2) exists and } \frac{1}{n} \|X \hat{\beta}\|^2 \leq K\}$ for each $j = 1, \ldots, p$ such that $w_j^2 \neq \Omega_{jj}$, there exists a standard normal $Z_j \sim N(0,1)$ satisfying

\[
E\left[ I_E \left( \frac{n}{\Omega_{jj} - w_j^2} \right)^{1/2} (\hat{\beta}_j - a_* w_j)^2 \right] \leq C_{100}(\delta, K, \ell) \frac{1}{p}
\]

for $a_* = \hat{\beta}^T \Sigma w$ and the observables $(\hat{\gamma}, \hat{\nu}, \hat{r}, \hat{\sigma}^2)$ in (5.1). For $\sigma_*^2 = \|\Sigma^{1/2} \hat{\beta}\|^2 - a_*^2$, it holds

\[
E\left[ I_E \left( \|\Sigma^{1/2} \hat{\beta} - a^2\| + \|\Sigma^{1/2} \hat{\beta}^2 - a_*^2\| \right) \right] \leq C_{101}(\delta, K, \ell)/\sqrt{p}
\]

and $E[I_E \max\{\hat{r}, \frac{1}{p}, \hat{\gamma}, \frac{1}{p}\}] \leq C_{102}(K, \delta, \ell)$. If additionally Assumption E holds, then $|\hat{a} - a_*| = O_p(n^{-1/2})$ for $\hat{a} = \max(0, \hat{\sigma}^2)^{1/2}$ where $O_p(\cdot)$ hides constants depending only on $(\delta, K, \ell)$.

Proof of Theorem 5.2. By Theorem 5.5 we know that $\hat{b} = \hat{\beta}$ and $\hat{\nu}^{-1} \leq C_{103}(K, \ell, \delta)$ in the event $E$ from Theorem 5.2, and that the right-hand side of (5.10) is bounded by $C_{104}(K, \ell, \delta, \sqrt{n})$. This the upper bound on $E[\hat{\nu}^2 \hat{\sigma}^2 - \sigma_*^2] + E[\hat{\nu}^2 \hat{\sigma}^2 - a_*^2]$ implies the desired upper bound on $E[I_E |\sigma^2 - \sigma_*^2]|$ and $E[I_E |\hat{a}^2 - a_*^2]|$ stated in (5.3).

Let $P = I_p - \Sigma w w^T$. Finally, (5.2) is a direct consequence of the upper bound (5.10) on $E[\sqrt{\hat{v}}^T \hat{\beta}^T u - \hat{r} Z^2]^{1/2}$ by taking $u = \frac{P e_j}{\Sigma^{-1/2} P e_j}$ since we have $w^T P e_j = w_j - w^T \Sigma w w_j = 0$ so that $u^T w = 0$ on the one hand, and on the other

\[
\hat{\beta}^T P e_j = \hat{\beta}_j - a_* w_j, \quad \|\Sigma^{-1/2} P e_j\|^2 = e_j^T P \Sigma^{-1} P e_j = \Omega_{jj} - w_j^2
\]

so that $\hat{\beta}^T u = (\Omega_{jj} - w_j^2)^{-1/2}(\hat{\beta}_j - a_* w_j)$ as desired in (5.2). \[\square\]

Appendix E: Proof in the case of Ridge regularization

The theorem is restated for convenience.
Proposition 6.2. Let Assumption A be fulfilled with \( \Sigma = \frac{1}{p}I_p \). Set \( g(b) = \lambda\|b\|^2_2/p \) as the penalty in (1.2). If additionally \( \| b_{y_0}^{\prime} \| \leq 1 \) then
\[
\mathbb{E} \left[ (\hat{\sigma}_n^2 - \frac{\gamma i^2}{\lambda + \hat{v}})^2 \right] \leq \frac{C_{105}(\delta, \lambda)}{p}, \quad \mathbb{E} \left[ (\hat{\sigma}_n^2 - \left( \frac{1}{p}\|\hat{\beta}\|^2 - \frac{\gamma i^2}{\lambda + \hat{v}} \right))^2 \right] \leq \frac{C_{106}(\delta, \lambda)}{p}. \tag{6.9}
\]

Proof of Proposition 6.2. By rotational invariance and without loss of generality, assume that \( w = p^{1/2}e_1 \) where \( e_1 \) is the first canonical basis vector in \( \mathbb{R}^p \). Let \( P = I_p - \frac{ww^T}{\|w\|^2} = I_p - e_1e_1^T \). Proposition C.2 applied to the Gaussian matrix \( (p^{1/2}X_{ij})_{i\in[n], j=2,..., p} \), \( u = \hat{\psi} \) and \( f = (\hat{\beta}_j/\sqrt{p})_{j=2,..., p} \) gives
\[
\mathbb{E} \left[ (\hat{\psi}^T XP\hat{\beta} - \sum_{j=2}^p \sum_{i=1}^n \frac{1}{p} (\hat{\beta}_j \hat{\psi}_i))^2 \right] \leq \mathbb{E} \left[ \|\hat{\psi}\|^2 \left( \frac{P\hat{\beta}}{p} \right)^2 + \sum_{j=2}^p \sum_{i=1}^n \frac{1}{p^2} \left\| \frac{\partial (\hat{\psi}^T P)}{\partial x_{ij}} \right\|^2 \right].
\]

In the left-hand side, \( \hat{\psi}^T XP\hat{\beta} = \frac{\lambda_n}{p}\|P\hat{\beta}\|^2 \) by (6.5). By (3.7) we also have
\[
\sum_{j=2}^p \sum_{i=1}^n \frac{\partial \hat{\beta}_j}{\partial x_{ij}} = \left( \sum_{j=2}^p \hat{A}_{ij} \right) \|\hat{\psi}\|^2 - \hat{\beta}^T \hat{P} \hat{A} X^T D \hat{\psi},
\]
Therefore,
\[
\sum_{j=2}^p \sum_{i=1}^n \frac{\partial \hat{\psi}_i}{\partial x_{ij}} = -\hat{\psi}^T DX \hat{A} \hat{P} \hat{\beta} - \text{Tr}[V]\|P\hat{\beta}\|^2.
\]
Since \( \sum_{j=2}^p \hat{A}_{ij} = \text{Tr}[P \hat{A}] = \text{Tr}[\hat{A}] - \hat{A}_{11} \), moving the terms involving \( \hat{A}_{11} \) and \( \hat{\psi}^T DX \hat{A} \hat{P} \hat{\beta} \) to the right-hand side, this proves
\[
n^2 \mathbb{E} \left[ (\lambda + \hat{v}) \hat{\sigma}_n^2 - \gamma i^2 \right] = \mathbb{E} \left[ (\lambda n + \text{Tr}[V])\|P\hat{\beta}\|^2 - \frac{\text{Tr}[\hat{A}]}{p}\|\hat{\psi}\|^2 \right] \leq C_{107} \mathbb{E} \left[ \|\hat{\psi}\|^2 \left( \frac{\hat{A}_{11}}{p^2}\|\hat{\psi}\|^2 + \left( \frac{1}{p} + \frac{\|DX \hat{A}\|_F}{p^2} \right)\|P\hat{\beta}\|^2 \right) + \sum_{j=2}^p \sum_{i=1}^n \frac{1}{p^2} \left\| \frac{\partial (\hat{\psi}^T P)}{\partial x_{ij}} \right\|^2 \right].
\]

We further bound the last line using \( \|\hat{\beta}\| \leq \frac{n}{p}\|X\|_F\|\hat{\psi}\| \) by (6.6), \( \|D\|_F \leq 1 \) by Assumption A, inequality \( \|\hat{\psi}\|^2 \leq n \) granted by the additional assumption \( \max_{y_0} \|t_{y_0}^{\prime}\| \leq 1 \), inequality \( \|\hat{A}\|_F \leq \frac{p}{\|X\|_F} \) by (3.7) for the rightmost term, and \( \mathbb{E}[\|X\|_F] \leq C_{108}(\delta, c) \) for any numerical constant \( c > 0 \) by (19) below. The conclusion (6.9) is then obtained by dividing by \( n^2(\lambda + \hat{v}) \). \( \square \)

Appendix F: Proof: \( \hat{\ell}/\hat{\psi} \) estimates the signal strength in linear models

Proposition 6.1 is restated before its proof for convenience.

Proposition 6.1. Let Assumptions A and B be fulfilled and additionally assume a linear model where observations \( y_i = \mathbf{x}_i^T \beta^* + \epsilon_i \) are iid with additive noise \( \epsilon_i \) independent of \( \mathbf{x}_i \). Set \( w = \beta^*/\|\Sigma^{1/2}\beta^*\| \), and assume that \( \|\Sigma^{1/2}\beta^*\| \) equals a constant independent of \( n, p \). Then
\[
\hat{\ell} = \hat{v}\|\Sigma^{1/2}\beta^*\| + n^{-1/2}O_p(1) \left[ \hat{v} + \|\Sigma^{1/2}\beta^*\| \right].
\]

Above, \( O_p(1) \) denotes a random variable \( W \) such that for any \( \eta > 0 \) there exists a constant \( K \) depending on \( (\eta, \delta, \tau, \|\Sigma^{1/2}\beta^*\|) \) only such that \( P(|W| > K) \leq \eta \).

Proof of Proposition 6.1. Recall that \( t_{\epsilon} = \mathbf{x}_i^T (\mathbf{X}^T \hat{\psi}/n + \hat{v}\Sigma\hat{\beta}) \). It is easier to work with the change of variable of Appendix C.2, so that after the change of variable \( t_{\epsilon} = e_1^T (G^T \hat{\psi} + \text{Tr}[V\hat{\theta}])/n \) with \( e_1 \in \mathbb{R}^p \) the first canonical basis vector. Let also \( \Theta \) be the true regression vector after change of variable, that is, \( \Theta \in \mathbb{R}^p \) such that \( y = G\Theta + \epsilon \) (i.e., \( \Theta = e_1\|\Sigma^{1/2}\beta^*\| \)). We apply Lemma B.1 to \( z = G\mathbf{e}_1 \sim N(0, I_n) \) conditionally on \( G(I_p - e_1e_1^T) \) (i.e., conditionally on the last \( p-1 \) columns of \( G \)), and to the function
\[ f(z) = \frac{\hat{\psi}}{\left(\|\psi\|^2 + n\Theta_1^2\right)^{1/2}}. \] Here, we have to take into account that \( y \) is not independent of \( Ge_1 \), and to take into account the derivatives of \( y \) with respect to \( Ge_1 \) conditionally on the last \( p - 1 \) columns of \( G \) and the noise \( \varepsilon \). This gives

\[
\frac{\partial f}{\partial z_i} = \left( I_n - \frac{\hat{\psi}\hat{\psi}^T}{\text{denom}} \right) \frac{1}{\text{denom}} \left( \frac{\partial}{\partial y_{i1}} + \Theta_1 \frac{\partial}{\partial y_i} \right) \hat{\psi}
\]

\[
= \left( I_n - \frac{\hat{\psi}\hat{\psi}^T}{\text{denom}} \right) \frac{1}{\text{denom}} \left[ -DGAe_1\hat{\psi}_1 + Ve_i(\Theta_1 - \hat{\theta}_1) \right]
\]

where \( \text{denom} = (\|\hat{\psi}\|^2 + n\Theta_2^2)^{1/2} \) for brevity, and the second line follows from [6, Theorem 2.1]. The matrix in the first parenthesis has operator norm at most 1, so that using \((a+b)^2/2 \leq a^2 + b^2\) gives

\[
\frac{1}{2} \sum_{i=1}^n \left\| \frac{\partial f}{\partial z_i} \right\|^2 \leq \|DGAe_1\|^2 \frac{\|\hat{\psi}\|^2}{\text{denom}^2} + \|V\|^2 \frac{(\hat{\theta}_1 - \Theta_1)^2}{\text{denom}^2}.
\]

By (C.16), (C.18) and (C.19), the expectation of the previous display is at most \( C_{109}(\delta, \tau)/n \). We also have that \( \left| \sum_{i=1}^n \frac{\partial f}{\partial z_i} + \frac{\text{Tr}[V(\hat{\theta}_1 - \Theta_1)]}{\text{denom}} \right| \leq C_{110}(\tau, \delta)O_p(n^{-1/2}) \) again thanks to (C.16), (C.18) and (C.19). We conclude that, omitting constants depending only on \( \tau, \delta \),

\[
|t_* - \hat{t}| \leq O_p(n^{-3/2})\text{denom} \leq O_p(n^{-1})(\hat{\tau} + \|\Sigma^{1/2}\beta^*\|).
\]

The bound \( |\hat{t} - t_*| \leq O_p(n^{-1/2})\hat{\tau} \) is provided by (4.12) and completes the proof. \( \Box \)