Stochastic 3D Leray-α Model with Fractional Dissipation

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Abstract. In this paper, we establish the global well-posedness of stochastic 3D Leray-α model with general fractional dissipation driven by multiplicative noise. This model is the stochastic 3D Navier-Stokes equations regularized through a smoothing kernel of order \( θ_1 \) in the nonlinear term and a \( θ_2 \)-fractional Laplacian. In the case of \( θ_1 ≥ 0 \) and \( θ_2 > 0 \) with \( θ_1 + θ_2 ≥ \frac{5}{4} \), we prove the global existence and uniqueness of the strong solutions. The main results cover many existing works in the deterministic cases, and also generalize some known results of stochastic models such as stochastic hyperviscous Navier-Stokes equations and classical stochastic 3D Leray-α model as our special cases.

Keywords: Stochastic Leray-α model; fractional Laplacian; Stochastic Navier-Stokes equations; Well-posedness

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1 Introduction

The 3D Leray-α model of turbulence as a regularization of 3D Navier-Stokes equations was first introduced by Leray \textsuperscript{33} in order to prove the existence of solutions to the Navier-Stokes equation in \( \mathbb{R}^3 \). It has been studied in the following form (cf. \textsuperscript{13, 14, 35})

\[
\begin{align*}
    d\mathbf{u} - \nu \Delta \mathbf{u} - (\mathbf{v} \cdot \nabla) \mathbf{u} - \nabla \mathbf{p} \, dt &= f(\mathbf{u}) \, dt, \\
    \mathbf{u} &= \mathbf{v} - \alpha^2 \Delta \mathbf{v}, \\
    \nabla \cdot \mathbf{u} &= 0, \quad \nabla \cdot \mathbf{v} = 0,
\end{align*}
\]

where \( \mathbf{u} \) (and \( \mathbf{v} \)) are unknown fields, \( \nu > 0 \) is the viscosity constant, \( \alpha > 0 \) is a length-scale constant, \( \mathbf{p} \) denotes the pressure and \( f \) is an external force field acting on the fluid.

If \( \alpha \) approaches to zero, then (1.1) is reduced to the classical 3D Navier-Stokes equations for incompressible fluids. It is well-known that the uniqueness of global solutions of 3D Navier-Stokes equations is among the most challenging problems of contemporary mathematics. Many different types of modifications for 3D Navier-Stokes equations have been investigated in the literatures (see e.g. \textsuperscript{7, 13, 48} and the references therein). One would also like to discover whether there exists a noise perturbation such that the uniqueness (pathwise
or in law) holds for the stochastic Navier-Stokes equations (cf. \[25\]). However, the problem remains unsolved, in spite of considerable efforts.

In recent years, the fractional power of Laplacian has been paid a lot of attentions (see e.g. \[51 10 11 17 19 40 45 51 53 54\] and the references therein). One can regularize the fluid equation for the velocity by putting a fractional power of the Laplace operator. One interesting point is to check for the limiting case (critical case) whether one still have the global existence and uniqueness of solutions (see e.g. \[5, 11, 51, 53, 54\]) or not? The 3D Leray-\(\alpha\) model with fractional dissipation has been extensively studied in the following form

\[
\begin{aligned}
du + [\nu(-\Delta)^{\theta_2} u + (v \cdot \nabla) u + \nabla p] dt &= f(u) dt, \\
u &= v + \alpha^{2\theta_1} (-\Delta)^{\theta_1} v, \\
abla \cdot u &= 0, \quad \nabla \cdot v = 0,
\end{aligned}
\]

(1.2)

where \((-\Delta)^{\theta_i} (i = 1, 2)\) are fractional Laplace operators, the parameter \(\theta_1 \geq 0\) affects the strength of the non-linear term and \(\theta_2 \geq 0\) represents the degree of viscous dissipation. The idea of regularizing the equation by two terms was introduced by Olson and Titi in \[11\] for 3D Lagrangian averaged Navier-Stokes-\(\alpha\) model (LANS-\(\alpha\) model). The authors proposed the idea in \[41\] that a weaker nonlinearity and a stronger viscous dissipation could work together to yield the well-posedness of the system. In particular, when \(\theta_1 = 0\), the model \(1.2\) becomes the hyperviscous Navier-Stokes equations and it is well-known that this system has a unique global solution for \(\theta_2 \geq \frac{5}{4}\) (see e.g. \[34, 48, 51\]). When \(\theta_2 = 1\), Ali in \[2\] studied the global well-posedness of the critical Leray-\(\alpha\) model \((\theta_1 = \frac{4}{3})\) and also considered the convergence to a suitable solution of Navier-Stokes equations. Note that the uniqueness of global weak solutions to the model \(1.2\) for \(\theta_2 = 1\) and \(\theta_1 < \frac{4}{3}\) remains open with \(L^2\) initial data. Moreover, the Leray-\(\alpha\) models with more general dissipation terms have been studied in \[4, 42, 52\].

In the present paper, we will study the following stochastic 3D Leray-\(\alpha\) model with fractional dissipation on the 3D torus \(T^3 = [0, 2\pi]^3\) with periodic boundary conditions:

\[
\begin{aligned}
du + [\nu(-\Delta)^{\theta_2} u + (v \cdot \nabla) u + \nabla p] dt &= g(u) dW(t), \\
u &= v + \alpha^{2\theta_1} (-\Delta)^{\theta_1} v, \\
abla \cdot u &= 0, \quad \nabla \cdot v = 0,
\end{aligned}
\]

(1.3)

where \(W(t)\) is a cylindrical Wiener process in a separable Hilbert space. In order to emphasize the stochastic effects and for the simplicity of exposition, we do not include a deterministic force \(f\) in \(1.3\), but it’s easy to show that all results of this paper could be easily extended to this more general case.

To the best of our knowledge, except for some special cases, there is no result concerning stochastic 3D Leray-\(\alpha\) model with general fractional dissipation. In the case of \(\theta_1 = 0\), the stochastic fractional (or hyperviscous) Navier-Stokes equations have been intensively studied (see e.g. \[19, 24, 44, 46, 54\] and references within). Chueshov and Millet in \[16\] proved the well-posedness and large deviation principle of stochastic 3D Leray-\(\alpha\) model in the case of \(\theta_1 = \theta_2 = 1\) (see also \[22\]). The well-posedness and irreducibility of 3D Leray-\(\alpha\) model driven by Lévy noise have been studied in \[6, 24\]. The \(\alpha\)-approximation of stochastic Leray-\(\alpha\) model to the stochastic Navier–Stokes equations was established in \[8, 15\]. In addition, when the viscosity constant \(\nu = 0\), Barbato, Bessaih and Ferrario in \[9\] studied the 3D stochastic inviscid Leray-\(\alpha\) model. The purpose of this paper is to investigate the global existence and uniqueness of solutions to the equation \(1.3\) under certain assumptions on the parameters \(\theta_1\) and \(\theta_2\). More precisely, we prove that if \(\theta_1 \geq 0\) and \(\theta_2 > 0\) satisfy \(\theta_1 + \theta_2 \geq \frac{5}{4}\), the equation \(1.3\) has a unique (probabilistically) strong solution (see Theorems 3.1 and 4.1).
Comparing with the results in the deterministic case, Yamazaki in [52] obtained an unique global solution to the 3D Leray-α model with the logarithmical dissipation when \( \theta_2 \geq \frac{1}{2} \) and \( \theta_1 + \frac{\theta_2}{2} \geq \frac{5}{4} \); then it was improved by Pennington in [12] for \( \theta_2 > \frac{1}{2} \) and \( \theta_1 + \theta_2 \geq \frac{5}{4} \). Moreover, Barbato, Morandin and Romito in [1] proved the 3D Leray-α model with the logarithmical dissipation is well-posed for \( \theta_1 + \theta_2 \geq \frac{5}{4} \) with \( \theta_1 \) and \( \theta_2 \geq 0 \), this result mainly focused on the optimal condition on the correction to the dissipation and was obtained by analyzing energy dispersion over dyadic shells, which is a completely different approach with the one employed in this work. Another related work is about 3D regularized Boussinesq equations in [5], but their results are restricted to the case of \( \frac{1}{2} < \theta_2 < \frac{3}{2} \) and \( \theta_1 + \theta_2 = \frac{5}{4} \).

The proof of our main results is divided into two cases. If the viscous dissipation is strong enough, i.e. for \( \theta_2 > \frac{1}{2} \), we can apply the generalized variational approach (cf. [37, 39]) to get the well-posedness with initial data in \( H^0 \). The reason is that \( (-\Delta)^{\theta_2} \) is smoothing by 2\( \theta_2 \) derivatives, while the \((v \cdot \nabla)u\) has one derivative. Let \( H^s \) denote the Sobolev space of divergence free vector fields (see (2.1)). We shall consider the Geland triple \( H^{\theta_2} \subset H^0 \subset H^{-\theta_2} \), and get the existence and uniqueness of strong solutions when \( \theta_1 \geq 0 \) and \( \theta_2 > \frac{1}{2} \) with \( \theta_1 + \theta_2 \geq \frac{5}{4} \). In this case, we can check the coefficients satisfy the local monotonicity and coercivity conditions similarly as the second named author and Röckner did for various types of SPDEs in [37] (see also [36, 38, 40]). It is known that the 3D Navier-Stokes equations \((\theta_2 = 1, \theta_1 = 0)\) lie outside the framework in [39] (see Example 5.2.23 there), hence our result illustrates that if we regularize the 3D Navier-Stokes equations by putting a fractional dissipation term is not strong enough to control the non-linear term (i.e. \( -\Delta \)).

On the other hand, the case \( 0 < \theta_2 \leq \frac{1}{2} \) is much more difficult to handle. Since the dissipation term is not strong enough to control the non-linear term \((v \cdot \nabla)u\), the uniqueness of solutions with initial data in \( H^0 \) seems unavailable. To get the well-posedness for (1.3), we work in the phase space \( H^1 \) with the initial data in \( H^1 \). And we get the global existence of unique strong solution which is also continuous with respect to \( H^1 \) norm for the case \( \theta_1 \geq 0 \) and \( \theta_2 > 0 \) with \( \theta_1 + \theta_2 \geq \frac{5}{4} \). Note that the variational approach is not applicable to (1.3) in this case, so we use a different approach to get the well-posedness. Firstly, we get the existence of a unique local strong solution when \( \theta_1 \geq 0 \) and \( \theta_2 > 0 \) satisfying \( \theta_1 + \theta_2 > \frac{3}{4} \) based on the work of [44]. Then we prove that the solution is global for \( \theta_1 \geq 0 \) and \( \theta_2 > 0 \) with \( \theta_1 + \theta_2 \geq \frac{5}{4} \). In [28, 44] the authors show that, when the noise is linear multiplicative, the local solution is global with a high probability if the initial data is sufficiently small, or if the noise coefficient is sufficiently large. Different with the arguments in [28, 44], we consider the general multiplicative noise including but not limit to the linear case, and show that the local solution is global almost surely. The proof of global existence of solution is based on a stochastic version Gronwall’s lemma from [27] and some stopping time techniques. But it is clear that the proof of the main results is more involved than the arguments in [27, 44], as we need to maintain the balance of the mixed fractional dissipation terms in (1.3). Another main difficulty arising is to prove appropriate \( H^s \)-norm estimate with \( s > 0 \). Unlike the case \( s = 0 \), we do not have the cancellation property \( \int \Lambda^*(v \cdot \nabla u)\Lambda^*udx = 0 \) anymore. Inspired by the works [5, 17, 44, 45], we frequently use the commutator estimate to show that the \( H^1 \)-norm can be controlled. Due to the lack of the cancellation property, we are not able to show that the solution is in \( L^p(\Omega; L^{\infty}_{loc}([0, \infty); H^1)) \cap L^2(\Omega; L^{\infty}_{loc}([0, \infty); \mathbb{H}^{\theta_2+1})) \) for \( \theta_1 \geq 0 \) and \( \theta_2 > 0 \) with \( \theta_1 + \theta_2 \geq \frac{5}{4} \).

However, we prove that the solution is in \( L^p(\Omega; L^{\infty}_{loc}([0, \infty); H^1)) \cap L^2(\Omega; L^{\infty}_{loc}([0, \infty); \mathbb{H}^{\theta_2+1})) \) in the subcritical case (i.e. \( \theta_1 \geq 0, \theta_2 > 0, \theta_1 + \theta_2 > \frac{5}{4} \)).

The main results of this paper illustrate how the non-linearity and viscous dissipation can be balanced with each other to yield the well-posedness of stochastic 3D Leray-α model.
In particular, if we take $\theta_1 = 0$ and $\theta_2 = 1$ in (1.3), the corresponding stochastic 3D Navier-Stokes equations lie in the region of the local well-posedness and outside the region of the global well-posedness of our main results. Moreover, our global well-posedness results are applicable for the case $\theta_2 = 1$, $\theta_1 = \frac{1}{4}$ and the case $\theta_1 = 0$, $\theta_2 \geq \frac{5}{4}$, which are corresponding to the the stochastic critical Leray-$\alpha$ model (see [2] for deterministic case) and hyperviscous Navier-Stokes equations. Hence our results cover and generalize some corresponding results in [2, 19, 22, 24, 27, 44]. And we believe that the methods presented in this paper are also useful for tackling other types of SPDEs with fractional Laplacian.

We should mention that there also exist many works concerning other types of SPDE with fractional Laplacian such as stochastic fractional Burgers equation, stochastic quasi-geostrophic equation, stochastic fractional Euler equations, stochastic fractional reaction-diffusion equation and stochastic fractional Boussinesq equations (see e.g. [9, 12, 17, 19, 29, 30, 40, 44, 45] and the references therein).

The paper is organized as follows. In Section 2 we introduce some notations and preliminaries. In Section 3 we prove the well-posedness for equation (1.3) with initial data in $H^0$ when $\theta_1 \geq 0, \theta_2 > \frac{1}{2}$ and $\theta_1 + \theta_2 \geq \frac{5}{4}$. In Section 4, we establish the well-posedness for equation (1.3) with initial data in $H^1$ when $\theta_1 \geq 0, \theta_2 > 0$ and $\theta_1 + \theta_2 \geq \frac{5}{4}$. At the end of this Section, we also show that there exist the finite moments of $H^1$ norm of the solution at any given deterministic time $t$ in the subcritical case (i.e. $\theta_1 \geq 0, \theta_2 > 0, \theta_1 + \theta_2 > \frac{5}{4}$).

## 2 Preliminaries

In this section, we introduce some notations and preliminaries which are commonly used in the analysis of fluid equations.

We denote by $L^p = L^p(T^3)^3$ the usual Lebesgue space over $T^3$ with the norm $\| \cdot \|_{L^p}$. As usual in the periodic setting, we can restrict ourself to deal with initial data with vanishing spatial average; then the solutions will enjoy the same property at any fixed time $t > 0$.

Since we work with periodic boundary condition, we can expand the velocity in Fourier series as

$$u(x) = \sum_{k \in \mathbb{Z}^3_0} \hat{u}_k e^{ik \cdot x},$$

where $\mathbb{Z}^3_0 = \mathbb{Z}^3 \setminus \{0\}$ and $\hat{u}_k^*$ denotes the complex conjugate of $\hat{u}_k$.

For $s \in \mathbb{R}$, the Sobolev spaces $H^s(T^3)^3$ can be represented as

$$H^s = \left\{ u(x) = \sum_{k \in \mathbb{Z}^3_0} \hat{u}_k e^{ik \cdot x} : \hat{u}_k = \hat{u}_k^*, \| u \|_s^2 < \infty \right\},$$

where

$$\| u \|_s^2 = \sum_{k \in \mathbb{Z}^3_0} |k|^{2s} |\hat{u}_k|^2.$$ 

In the Fourier space, the divergence free condition can be formulated as

$$\hat{u}_k \cdot k = 0 \quad \text{for every} \; k.$$ 

Define the divergence free Sobolev space by

$$H^s : = \{ u \in H^s : \hat{u}_k \cdot k = 0 \; \text{for every} \; k \}.$$  

(2.1)
which is a Hilbert space with scalar product
\[ \langle u, v \rangle_{\mathbb{H}^s} = \sum_{k \in \mathbb{Z}_0^3} |k|^{2s} \hat{u}_k \cdot \hat{v}_k. \]

Following the standard notation, we denote the norm in space \( \mathbb{H}^0 \) by \( \|u\|_{L^2} \) and inner product \( \langle u, v \rangle = \sum_k \hat{u}_k \cdot \hat{v}_k \). Thus, \( \mathbb{H}^0 \) is the Hilbert space of \( L^2 \)-integrable functions on \( \mathbb{T}^3 \) taking values in \( \mathbb{R}^3 \) which are divergence free and have zero mean. For simplicity, we also identify the continuous dual space of \( \mathbb{H}^s \) as \( \mathbb{H}^{-s} \) with the dual action of \( \mathbb{H}^{-s} \) on \( \mathbb{H}^s \) by the same notation \( \langle u, v \rangle = \sum_k \hat{u}_k \cdot \hat{v}_k \). In particular, we denote the norm of \( \mathbb{H}^1 \) by \( \|u\| \).

The nonlocal operator \( \Lambda^s \) is defined as
\[ \Lambda^s u := \sum_{k \in \mathbb{Z}_0^3} |k|^{-s} \hat{u}_k e^{ik\cdot x}, \]
hence \( \Lambda^2 = -\Delta \). Note that \( \Lambda^s \) maps \( H^r \) onto \( H^{r-s} \) and
\[ \|u\|_{L^2}^2 = \sum_{k \in \mathbb{Z}_0^3} |k|^{2s} |\hat{u}_k|^2 = \|\Lambda^s u\|_{L^2}^2. \]

Denote by \( P_\sigma \) the Leray-Helmholtz projection from \( H^3 \) to \( \mathbb{H}^0 \). It’s well-known that the operators \( P_\sigma \) and \( \Lambda^s \) are commutative. Note that the space \( H^{s+\epsilon} \) is compactly embedded in \( H^s \) (resp. \( \mathbb{H}^{s+\epsilon} \) is compactly embedded in \( \mathbb{H}^s \)) for any \( \epsilon > 0 \), moreover we have the following Sobolev embedding theorem (see e.g. [5] [17]).

**Lemma 2.1** If \( 0 \leq s < \frac{3}{2} \) and \( \frac{1}{p} + \frac{s}{2} = \frac{1}{q} \), then \( H^s \subset L^p \). Moreover, there is a constant \( C = C(s, p) > 0 \) such that
\[ \|f\|_{L^p} \leq C \|\Lambda^s f\|_{L^q}. \]

If \( s = \frac{3}{2} \), then for any finite \( p \),
\[ \|f\|_{L^p} \leq C \|f\|_s \]
and if \( s > \frac{3}{2} \), then
\[ \|f\|_{L^\infty} \leq C \|f\|_s. \]

Define the bilinear operator \( B : \mathbb{H}^1 \times \mathbb{H}^1 \rightarrow \mathbb{H}^{-1} \) by
\[ \langle B(u, v), w \rangle := \int_{\mathbb{T}^3} ((u \cdot \nabla)v) \cdot w \, dx, \]
i.e. \( B(u, v) = P_\sigma ((u \cdot \nabla)v) \) for smooth vectors \( u \) and \( v \).

We list some well-known properties of the bilinear operator \( B \) below (see e.g. [19] [5]).

**Lemma 2.2** For any \( u, v, w \in \mathbb{H}^1 \), we have
\[ \langle B(u, v), w \rangle = -\langle B(u, w), v \rangle, \quad \langle B(u, v), v \rangle = 0. \] (2.2)
(2.2) holds more generally for any \( u, v, w \) giving a meaning to the trilinear forms, as stated precisely in the following:
\[ \langle B(u, v), w \rangle \leq C \|u\|_{m_1} \|v\|_{m_2+1} \|w\|_{m_3}, \] (2.3)
with the nonnegative parameters fulfilling
\[ m_1 + m_2 + m_3 \geq \frac{3}{2} \quad \text{if } m_i \neq \frac{3}{2} \text{ for any } i \]
or
\[ m_1 + m_2 + m_3 > \frac{3}{2} \quad \text{if } m_i = \frac{3}{2} \text{ for some } i. \]
Put \( G = (I + \alpha^{2\theta_1} \Lambda^{2\theta_1})^{-1} \), then \( G \) is a linear operator. For \( u = v + \alpha^{2\theta_1} \Lambda^{2\theta_1} v \), we have \( v = Gu \). Denote \( B(u) := B(Gu, u) \). By applying \( P_\sigma \) to Eq. (1.3) we remove the pressure term and reformulate it as the following abstract stochastic evolution equation:

\[
\begin{aligned}
  &du(t) + \nu \Lambda^{2\theta_2} u(t)dt + B(u(t))dt = g(u(t))dW(t), \\
  &u(0) = u_0,
\end{aligned}
\]

(2.4)

where \( W(t) \) is a cylindrical Wiener process in a separable Hilbert space \( U \) w.r.t. a complete filtered probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \).

The regularization effect of the nonlocal operator involved in the relation between \( Gu \) and \( u \) is described by the following lemma (see [2, Lemma 2.2]).

**Lemma 2.3** Let \( 0 \leq \beta \leq 2\theta_1 \), \( s \in \mathbb{R} \) and \( u \in H^s \), then \( Gu \in H^{s+\beta} \) and there exists a constant \( C = C_{\alpha, \beta} > 0 \) such that

\[ ||Gu||_{s+\beta} \leq C ||u||_s. \]

Define the commutator

\[ [\Lambda^s, f]g = \Lambda^s(fg) - f\Lambda^s g. \]

The following commutator estimate is very important for later use (see [5, 31]).

**Lemma 2.4** (Commutator estimate) Suppose that \( s > 0, p, p_2, p_3 \in (1, \infty) \) and \( p_1, p_4 \in (1, \infty] \) satisfy

\[
\frac{1}{p} \geq \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{p} \geq \frac{1}{p_3} + \frac{1}{p_4}. \tag{2.5}
\]

Then we have

\[ ||[\Lambda^s, f]g||_{L^p} \leq C \left( ||\nabla f||_{L^{p_1}} ||\Lambda^{s-1}g||_{L^{p_2}} + ||\Lambda^s f||_{L^{p_3}} ||g||_{L^{p_4}} \right). \]

We recall the following important product estimate (see, e.g. [5, 45]).

**Lemma 2.5** Suppose that \( s > 0, p, p_2, p_3 \in (1, \infty), p_1, p_4 \in (1, \infty] \) satisfy (2.5). We have

\[ ||\Lambda^s(fg)||_{L^p} \leq C \left( ||f||_{L^{p_1}} ||\Lambda^s g||_{L^{p_2}} + ||\Lambda^s f||_{L^{p_3}} ||g||_{L^{p_4}} \right). \]

For any Hilbert space \( K \), we use \( (L_2(U, K), || \cdot ||_{L_2(U, K)}) \) to denote the space of all Hilbert-Schmidt operators from \( U \) to \( K \). In this paper we use \( C \) to denote some generic constant which may change from line to line.

## 3 Main results for \( \theta_2 > \frac{1}{2} \) with initial data in \( H^0 \)

In this section, we will prove the existence and uniqueness of Eq. (2.4) with initial data in \( H^0 \) when \( \theta_2 > \frac{1}{2} \). To this end, we first impose the following assumptions on \( g \).

**Hypothesis (3.1)** Suppose that \( g \) is measurable mapping from \( H^{\theta_2} \) to \( L_2(U, H^0) \) and satisfies the following conditions:

(i) There exists a constant \( C > 0 \) such that, for any \( u \in H^{\theta_2} \),

\[ ||g(u)||_{L_2(U, H^0)}^2 \leq C(1 + ||u||_{L_2}^2). \]
(ii) There exists a constant $C > 0$ such that, for any $u, v \in \mathbb{H}^{\theta_2}$,
\[
\|g(u) - g(v)\|_{L^2(U; H^0)}^2 \leq \nu \|u - v\|_{\theta_2}^2 + C(1 + \rho(v))\|u - v\|_{L^2}^2,
\]
where $\rho: \mathbb{H}^{\theta_2} \to [0, +\infty)$ is a measurable and locally bounded function in $\mathbb{H}^{\theta_2}$ such that
\[
\rho(v) \leq C(1 + \|v\|_{\theta_2}^2)(1 + \|v\|_{L^2}^2).
\]

Next lemma plays an essential role and the proof can be found from [31, Lemma 4.2].

**Lemma 3.1** Suppose that $\theta_1 \geq 0$ and $\theta_2 > \frac{1}{2}$ with $\theta_1 + \theta_2 \geq \frac{5}{4}$, then $B: \mathbb{H}^{\theta_1+\theta_2} \times \mathbb{H}^{\theta_2} \to \mathbb{H}^{-\theta_2}$ is well defined. Moreover, for $u \in \mathbb{H}^{\theta_1+\theta_2}$ and $v, w \in \mathbb{H}^{\theta_2}$,
\[
|\langle B(u, v), w \rangle| \leq C (\|u\|_{\theta_1+\theta_2} \|w\|_{L^2} + \|u\|_{\theta_1} \|w\|_{\theta_2}) \|v\|_{\theta_2}.
\]

**Definition 3.1** A continuous adapted $\mathbb{H}^0$-valued process $\{u(t)\}_{t \in [0, T]}$ is called a (strong) solution of (2.4), if for its $dt \otimes \mathbb{P}$-equivalent class $\bar{u}$, we have $\bar{u} \in L^2([0, T] \times \Omega; \mathbb{H}^{\theta_2})$ and $\mathbb{P}$-a.s.,
\[
u \int_0^t \Lambda^{\theta_2} \bar{u}(s) ds + \int_0^t B(\bar{u}(s)) ds = u_0 + \int_0^t g(\bar{u}(s)) dW(s), \quad t \in [0, T].
\]

The first main result of this paper is given in the next statement.

**Theorem 3.1** Suppose that $\theta_1 \geq 0$, $\theta_2 > \frac{1}{2}$ with $\theta_1 + \theta_2 \geq \frac{5}{4}$ and the Hypothesis (3.1) hold. We have that, for any $u_0 \in L^p(\Omega; \mathbb{H}^0)$ with $p \geq 4$, (2.4) has a unique strong solution $\{u(t)\}_{t \in [0, T]}$, which satisfies
\[
\mathbb{E} \left( \sup_{t \in [0, T]} \|u(t)\|_{L^2}^p + \int_0^T \|u(t)\|_{\theta_2}^2 dt \right) < \infty.
\]

Moreover, the solution $\{u(t)\}_{t \in [0, T]}$ is a Markov process.

**Proof** Now we consider the following Geland triple
\[
\mathbb{H}^{\theta_2} \subset \mathbb{H}^0 \subset \mathbb{H}^{-\theta_2}.
\]

We first note that the following mappings
\[
\Lambda^{\theta_2}: \mathbb{H}^{\theta_2} \to \mathbb{H}^{-\theta_2}, \quad B: \mathbb{H}^{\theta_2} \to \mathbb{H}^{-\theta_2}
\]
are well defined. In particular, by Lemma 2.2, we have
\[
\langle B(Gv, u), w \rangle = -\langle B(Gv, w), u \rangle, \quad \langle B(Gv, u), u \rangle = 0, \quad \text{for } v, u, w \in \mathbb{H}^{\theta_2}.
\]

Let $F: \mathbb{H}^{\theta_2} \to \mathbb{H}^{-\theta_2}$ be defined by
\[
F(u) := -\nu \Lambda^{\theta_2} u - B(u), \quad \text{for all } u \in \mathbb{H}^{\theta_2}.
\]

We only need to verify that all conditions of Theorem 5.1.3 in [39] hold for (2.4).

(1) Since $B$ is bilinear map, the hemicontinuity of $F$ is obvious.

(2) Note that $\langle B(Gu, u), u \rangle = 0$, it is easy to verify the following coercivity condition:
\[
2\langle F(u), u \rangle + \|g(u)\|_{L^2(U; H^0)}^2 \leq -\nu \|u\|_{\theta_2}^2 + C(1 + \|u\|_{L^2}^2).
\]
(3) By the Young’s inequality, Lemma 2.3 and Lemma 3.1, for \( u_1, u_2 \in H^{\theta_2} \) we have

\[
\langle B(u_1) - B(u_2), u_1 - u_2 \rangle = \langle B(G(u_1 - u_2), u_2), u_1 - u_2 \rangle \\
\leq C (\|G(u_1 - u_2)\|_{2\theta_1 + \theta_2} \|u_1 - u_2\|_{L^2} + \|G(u_1 - u_2)\|_{2\theta_1} \|u_1 - u_2\|_{\theta_2}) \|u_2\|_{\theta_2} \\
\leq C \|u_1 - u_2\|_{\theta_2} \|u_1 - u_2\|_{L^2} \|u_2\|_{\theta_2} \\
\leq \frac{\nu}{4} \|u_1 - u_2\|_{\theta_2}^2 + C_\nu \|u_2\|_{\theta_2} \|u_1 - u_2\|_{L^2}.
\]

(3.1)

Then by Hypothesis (3.1) we can deduce that

\[
2\langle F(u_1) - F(u_2), u_1 - u_2 \rangle + \|g(u_1) - g(u_2)\|_{L^2(U;H^0)}^2 \\
\leq C (1 + \rho(u_2) + \|u_2\|_{\theta_2}^2) \|u_1 - u_2\|_{L^2},
\]

hence the local monotonicity condition holds.

(4) By Lemma 2.2 and Lemma 3.1, for any \( u, w \in H^{\theta_2} \)

\[
\langle B(u), w \rangle = \langle B(Gu, w), u \rangle \leq C (\|Gu\|_{2\theta_1 + \theta_2} \|u\|_{L^2} + \|Gu\|_{2\theta_1} \|u\|_{\theta_2}) \|w\|_{\theta_2},
\]

from which and Lemma 2.3 we have the following growth condition:

\[
\|F(u)\|_{\theta_2} \leq C \|u\|_{\theta_2} (1 + \|u\|_{L^2}^2).
\]

Therefore, all conclusions follow from Theorem 5.1.3 in [39]. The proof is completed. □

4 Main results for \( \theta_2 > 0 \) with initial data in \( H^1 \)

In this section, we show the existence and uniqueness of solutions to Eq. (2.4) for \( \theta_2 > 0 \) with initial data in \( H^1 \). Here we first prove that Eq. (2.4) is local well-posedness and then show that the local solution is global. Firstly, we make the following assumptions of \( \theta_2 \) such that Eq. (2.4) has a unique local strong solution.

**Hypothesis (4.1)** Suppose that \( g \) is measurable mapping from \( H^0 \) to \( L_2(U, H^0) \) and it satisfies the following conditions:

(i) For all \( s \in [1, 2] \), \( g \) is an operator from \( H^s \) to \( L_2(U, H^s) \) and there exists a locally bounded function \( \rho_1 \) on \( \mathbb{R} \) such that for all \( u \in H^s \)

\[
\|\Lambda^s g(u)\|_{L_2(U;H^0)} \leq \rho_1(\|u\|_s)(1 + \|u\|_s).
\]

(ii) There exist locally bounded functions \( \rho_2 \) and \( \rho_3 \) on \( \mathbb{R} \) such that for all \( u, v \in H^1 \)

\[
\|g(u) - g(v)\|_{L_2(U;H^0)} \leq (\rho_2(\|u\|) + \rho_3(\|v\|)) \|u - v\|_{L^2}.
\]

In order to prove the global well-posedness, we also need to impose some further assumptions on \( g \).

**Hypothesis (4.2)** There exists a constant \( C \) such that

\[
\|g(u)\|^2_{L_2(U;H^0)} \leq C(1 + \|u\|^2_{L^2}), \quad \text{for all } u \in H^{\theta_2}
\]

and

\[
\|\Lambda g(u)\|^2_{L_2(U;H^0)} \leq C(1 + \|u\|^2), \quad \text{for all } u \in H^{\theta_2+1}.
\]
Remark 4.1 The assumptions of $g$ in Hypotheses (4.1) and (4.2) may be shown to cover a wider class of examples, including but not limited to the classic cases of additive and linear multiplicative noise.

We recall the following notions of local, maximal and global solutions of Eq. \ref{eq:2.4}.

**Definition 4.1** Fix a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t, W)$.

(i) A local strong solution of \ref{eq:2.4} is a pair $(u, \tau)$, where $\tau$ is an $\mathcal{F}_t$-stopping time and $(u(t))_{t \geq 0}$ is a predictable $\mathbb{H}^1$-valued process such that $u(\cdot \wedge \tau) \in L^2(\Omega; L^2_{loc}([0, \infty); \mathbb{H}^{d+1}))$, $u(\cdot \wedge \tau) \in C([0, \infty); \mathbb{H}^1)$ $\mathbb{P}$-a.s., and for every $t \geq 0$, $\xi \in \cap_{i=1}^{\infty} \mathbb{H}^i$, $\mathbb{P}$-a.s.,

$$
\langle u(t \wedge \tau), \xi \rangle + \int_0^{t \wedge \tau} \langle \nu \Lambda^{(2d)} u + B(u), \xi \rangle ds = \langle u_0, \xi \rangle + \int_0^{t \wedge \tau} \langle g(u) dW, \xi \rangle. \tag{4.1}
$$

(ii) We say that local pathwise uniqueness holds if given any pair $(u^1, \tau^1)$ and $(u^2, \tau^2)$ of local strong solutions of \ref{eq:2.4} with the same initial condition, the following holds:

$$
\mathbb{P}\left\{ u^1(t) = u^2(t); \forall t \in [0, \tau^1 \wedge \tau^2] \right\} = 1.
$$

(iii) A maximal strong solution of \ref{eq:2.4} is a pair $(u^R, \tau_R)_{R \in \mathbb{N}}$ such that for each $R \in \mathbb{N}$, the pair $(u^R, \tau_R)$ is a local strong solution, $\tau_R$ is increasing such that $\xi := \lim_{R \to \infty} \tau_R > 0$, $\mathbb{P}$-a.s. and

$$
\sup_{t \in [0, \tau_R]} \|u^R(t)\| \geq R, \quad \mathbb{P}$-a.s. on the set $\{\xi < \infty\}. \tag{4.2}
$$

(iv) If the local pathwise uniqueness holds, then $\xi$ does not depend on the sequences $(u^R)_{R \in \mathbb{N}}$, $(\tau_R)_{R \in \mathbb{N}}$. In this case we denote the maximal solution by $(u, (\tau_R)_{R \in \mathbb{N}}, \xi)$, and we say a maximal strong solution $(u, (\tau_R)_{R \in \mathbb{N}}, \xi)$ is global if $\xi = \infty$ $\mathbb{P}$-a.s.

Now we formulate the main result in this section concerning the global well-posedness of Eq. \ref{eq:2.4}.

**Theorem 4.1** Suppose that $\theta_1 \geq 0, \theta_2 > 0$ with $\theta_1 + \theta_2 \geq \frac{5}{4}$ and the Hypotheses (4.1), (4.2) hold. Then for any $u_0 \in L^2(\Omega; \mathbb{H}^1)$ \ref{eq:2.4} has a unique global strong solution. More precisely, there exists a unique predictable $\mathbb{H}^1$-valued process $\{u(t)\}_{t \geq 0}$ such that $u \in L^2_{loc}([0, \infty); \mathbb{H}^{d+1}) \cap C([0, \infty); \mathbb{H}^1)$ $\mathbb{P}$-a.s., and for every $t \geq 0$, $\xi \in \cap_{i=1}^{\infty} \mathbb{H}^i$

$$
\langle u(t), \xi \rangle + \int_0^t \langle \nu \Lambda^{(2d)} u + B(u), \xi \rangle ds = \langle u_0, \xi \rangle + \int_0^t \langle g(u) dW, \xi \rangle \quad \mathbb{P}$-a.s. \tag{4.3}
$$

The proof of Theorem 4.1 will be given in the section 4.2.
4.1 Local existence and uniqueness

In this section, we establish the existence of local solution and maximal solution for Eq. (2.4) with initial data in $L^2(\Omega; H^1)$. The proof is based on the results in [44], which have been applied for various types of SPDEs with fractional dissipation.

**Theorem 4.2** Suppose that $\theta_1 \geq 0$, $\theta_2 > 0$ with $\theta_1 + \theta_2 > \frac{3}{4}$ and the Hypothesis (4.1) hold. Then for any $u_0 \in L^2(\Omega; H^1)$, we have local pathwise uniqueness strong solution for (2.4), and there exists a maximal strong solution $(u_0, (\tau_R)_{R \in \mathbb{N}}, \xi)$ of (2.4).

**Remark 4.2** In particular, Theorem 4.2 is applicable for the case $\theta_1 = 0$ and $\theta_2 > \frac{3}{4}$. Hence our results generalize the results of stochastic 3D Navier-Stokes equations ($\theta_1 = 0$ and $\theta_2 = 1$) in [27].

**Remark 4.3** If $(u_0, (\tau_R)_{R \in \mathbb{N}})$ is a local strong solution for (2.4), then we have

$$
\mathbb{E} \left( \sup_{0 \leq t \leq \tau_R} \|u(t)\|^2 + \int_0^{\tau_R} \|u(t)\|^2_{\theta_2+1} dt \right) < \infty.
$$

(4.4)

However, we are not able to show that (4.4) holds when replacing the stopping time $\tau_R$ by any fixed (deterministic) $T > 0$. This is the case even in the case $\theta_1 \geq 0, \theta_2 > 0$ and $\theta_1 + \theta_2 \geq \frac{3}{4}$ where we can prove the existence of global strong solution.

In the section 4.3, we will prove that (4.4) also holds for any fixed (deterministic) $T > 0$ in the subcritical case, i.e. $\theta_1 \geq 0$ and $\theta_2 > 0$ with $\theta_1 + \theta_2 > \frac{5}{4}$.

**Remark 4.4** We remark that the proof can be spited in two cases, namely the case $0 < \theta_2 \leq 1$ and the case $\theta_2 > 1$. For $0 < \theta_2 \leq 1$, we can use the result in [44]. While, the framework is not adapted to $\theta_2 > 1$. We shall give a direct proof for this case.

Before the proof of Theorem 4.2, we introduce the following space for later use. Let $K$ be a separable space, given $p > 1$, $\kappa \in (0, 1)$, let $W^{\kappa,p}([0, T]; K)$ be the Sobolev space of all $u \in L^p([0, T]; K)$ such that

$$
\int_0^T \int_0^T \frac{|u(t) - u(s)|_K^p}{|t-s|^{1+\kappa p}} dt ds < \infty,
$$

endowed with the norm

$$
\|u\|_{W^{\kappa,p}([0, T]; K)} := \int_0^T |u|_K^p dt + \int_0^T \int_0^T \frac{|u(t) - u(s)|_K^p}{|t-s|^{1+\kappa p}} dt ds.
$$

For the case $\kappa = 1$, we take $W^{1,p}([0, T]; K) := \{u \in L^p([0, T]; K); \frac{du}{dt} \in L^p([0, T]; K)\}$ with the norm

$$
\|u\|_{W^{1,p}([0, T]; K)} := \int_0^T |u|_K^p + \left|\frac{du}{dt}\right|_K^p dt.
$$

Note that for $\kappa \in (0, 1), W^{1,p}([0, T]; K) \subset W^{\kappa,p}([0, T]; K)$ and there exists a constant $C > 0$ such that $\|u\|_{W^{\kappa,p}([0, T]; K)} \leq C \|u\|_{W^{1,p}([0, T]; K)}$.

**Proof of Theorem 4.2:**

**Case 1:** (The case $\theta_1 \geq 0$, $0 < \theta_2 \leq 1$ and $\theta_1 + \theta_2 > \frac{3}{4}$)

For this case, we shall apply Theorem 3.2 of [44] to (2.4). Thanks to the assumptions for $g$, it is sufficient to check the conditions (b.1)-(b.3) in [44] for $B(u)$.
(1) For $s \in [1, 2]$, since $\langle Gu \cdot \nabla \Lambda^s u, \Lambda^s u \rangle = 0$, by the Hölder’s inequality we get that for $u \in H^s$, 

$$
-\langle B(u), \Lambda^{2s} u \rangle \leq \left| \langle \Lambda^s ((Gu \cdot \nabla)u), \Lambda^s u \rangle - \langle (Gu \cdot \nabla)\Lambda^s u, \Lambda^s u \rangle \right|
= |\langle [\Lambda^s, Gu] \cdot \nabla u, \Lambda^s u \rangle| 
\leq C \| [\Lambda^s, Gu] \cdot \nabla u \|_{L^p} \| \Lambda^s u \|_{L^q}.
$$

(4.5)

Here we choose

$$
\frac{1}{p} = \frac{2}{2} + \frac{\delta}{3}, \quad \frac{1}{q} = \frac{2}{2} - \frac{\delta}{3},
$$

with $\delta = \delta(\theta_2) \in (0, \theta_2)$ such that $2\theta_1 + \theta_2 + \delta \geq \frac{\delta}{2}$.

To bound the first term of (4.5), we make use of the commutator estimate and get that

$$
\| [\Lambda^s, Gu] \cdot \nabla u \|_{L^p} \leq C \| \Lambda Gu \|_{L^{p_1}} \| \Lambda^s u \|_{L^{p_2}} + \| Au \|_{L^{p_3}} \| \Lambda^s Gu \|_{L^{p_4}}.
$$

Here we choose

$$
\frac{1}{p_1} = \frac{\theta_2 + \delta}{3}, \quad \frac{1}{p_2} = \frac{2}{2} - \frac{\theta_2}{3}, \quad p_3 = 2, \quad \frac{1}{p_4} = \frac{\delta}{3}.
$$

(4.6)

$\theta_2 \in (0, 1)$ implies $p_1 \in \left(\frac{3}{2}, \infty\right)$, $p_2 \in (2, 3]$ and $p_4 \in (3, \infty)$.

Since $\theta_1 + \theta_2 > \frac{3}{2}$ and $2\theta_1 + \theta_2 + \delta \geq \frac{3}{2}$, by Lemma 2.1 and Lemma 2.3, we obtain

$$
\| \Lambda Gu \|_{L^{p_1}} \leq C \| Gu \|_{\frac{3}{2} - \theta_2 - \delta} \leq C \| u \|,
$$

$$
\| \Lambda^s u \|_{L^{p_2}} \leq C \| u \|_{\theta_2 + s},
$$

$$
\| \Lambda^s Gu \|_{L^{p_4}} \leq C \| Gu \|_{\frac{3}{2} - \delta + s} \leq C \| u \|_{\theta_2 + s},
$$

$$
\| \Lambda^s u \|_{L^q} \leq C \| u \|_{\delta + s}.
$$

Therefore, putting the above estimates all together and by the interpolation inequality as well as the Young’s inequality, it leads to

$$
-\langle B(u), \Lambda^{2s} u \rangle \leq C \| u \| \| u \|_{\delta + s} + \| u \|_{\theta_2 + s} \leq \varepsilon \| u \|_{\theta_2 + s}^2 + C \| u \|^{2 + \frac{2\theta_2 + s - 1}{\theta_2 - \frac{3}{2}}}. 
$$

Thus, the coercivity condition (b.1) in [44] is satisfied.

(2) Lemma 2.1 and Lemma 2.5 imply that, for $u \in H^{2+\theta_2}$,

$$
\| B(u) \|_{L^2} \leq C \| u \| \| u \|_{\theta_2 + 1}
$$

and

$$
\| \Lambda B(u) \|_{L^2} \leq C \| Gu \|_{L^{p_1}} \| \Lambda^2 u \|_{L^{p_2}} + \| \Lambda Gu \|_{L^{p_3}} \| \Lambda u \|_{L^{p_4}}.
$$

(4.7)

Now we choose $\frac{1}{p_1} = \frac{\theta_2}{3}$, $\frac{1}{p_2} = \frac{1}{2} - \frac{\theta_2}{3}$, so by the Sobolev embeddings we get

$$
\| Gu \|_{L^{p_1}} \leq C \| Gu \|_{\frac{3}{2} - \theta_2} \leq C \| u \|,
$$

$$
\| \Lambda^2 u \|_{L^{p_2}} \leq C \| u \|_{\theta_2 + 2}.
$$

For the last two terms in the right side of (4.7), if $0 < \theta_2 \leq \frac{1}{2}$, we take $p_3 = 3$, $p_4 = 6$ and the assumption $\theta_1 + \theta_2 > \frac{3}{2}$ gives $\theta_1 \geq \frac{1}{4}$. So, we can get $\| \Lambda Gu \|_{L^{p_3}} \leq C \| Gu \|_{\frac{3}{2}} \leq C \| u \|$ and
\[ \| \Lambda u \|_{L^p} \leq C \| u \|_2 \leq C \| u \|_{\theta_2 + 2}. \]

On the other hand, if \( \frac{1}{2} < \theta_2 \leq 1 \), we choose \( p_3 = 2 \), \( p_4 = \infty \), Lemma 2.1 implies that \( \| \Lambda u \|_{L^\infty} \leq C \| u \|_{\theta_2 + 2} \).

Thus

\[ \| \Lambda B(u) \|_{L^2} \leq C \| u \|_{\theta_2 + 2}, \]

which shows that the growth condition (b.2) in \([44]\) is proved.

(3) For the local monotonicity condition (b.3) in \([44]\), since \( 2\theta_1 + \delta + \theta_2 \geq \frac{3}{2} \), Lemma 2.2 (with \( m_1 = 2\theta_1 + \delta, m_2 = 0, m_3 = \theta_2 \)) yields

\[ \langle (B(u_1) - B(u_2)), u_1 - u_2 \rangle = \langle B(G(u_1 - u_2), u_2), u_1 - u_2 \rangle \]
\[ \leq C \| G(u_1 - u_2) \|_{2\theta_1 + \delta} \| u_2 \| \| u_1 - u_2 \|_{\theta_2} \]
\[ \leq C \| u_1 - u_2 \|_{L^2}^{\frac{\theta_1 - \delta}{\theta_2}} \| u_2 \| \| u_1 - u_2 \|_{\theta_2}^{\frac{\theta_1 + \delta}{\theta_2}} \]
\[ \leq \varepsilon \| u_1 - u_2 \|_{\theta_2}^2 + C \varepsilon \| u_2 \|^{2\theta_1 - \delta} \| u_1 - u_2 \|_{L^2}^2, \]

(4.8)

where we use Lemma 2.3 in the second inequality, the interpolation inequality in the third inequality and the Young’s inequality in the last inequality.

Hence all conclusions follow from Theorem 3.2 in \([44]\). The proof is completed in this case.

**Case 2:** (The case \( \theta_1 \geq 0 \) and \( \theta_2 > 1 \))

We first establish the existence of weak solutions to the following equation:

\[
\begin{cases}
    du + \nu \Lambda^{2\theta_2} u dt + \chi_R(\|u\|) B(u) dt = \chi_R(\|u\|) g(u) dW, \\
    u(0) = u_0,
\end{cases}
\]

(4.9)

where \( R > 0 \) is a fixed constant, and \( \chi_R : [0, \infty) \to [0, 1] \) is a \( C^\infty \) smooth function such that

\[ \chi_R(x) = \begin{cases} 
    1 & \text{for } x \leq R, \\
    0 & \text{for } x > 2R.
\end{cases} \]

We denote by \( P_n \) the projection operator onto \( \mathbb{H}_n := \text{span}\{e^{ikx} : |k| \leq n\} \). Consider the Galerkin approximation \( u_n \) of \((4.9)\) as

\[
\begin{cases}
    du_n + \nu \Lambda^{2\theta_2} u_n dt + \chi_R(\|u_n\|) P_n B(u_n) dt = \chi_R(\|u_n\|) P_n g(u_n) dW, \\
    u_n(0) = P_n u_0.
\end{cases}
\]

(4.10)

Then by the theory of SDE in finite-dimension space (see, e.g. \([39]\)), there exists a unique global solution to \((4.10)\). According to Itô’s formula, we obtain

\[
d\|u_n\|^2 + 2\nu \|u_n\|_{\theta_2 + 1}^2 dt \leq -2\chi_R(\|u_n\|) \langle B(u_n), \Lambda^2 u_n \rangle dt \\
+ \chi_R(\|u_n\|) \| \Lambda g(u_n) \|_{L^2(L^2(\mathbb{H}_n)))}^2 dt \\
+ 2\chi_R(\|u_n\|) \langle \Lambda u_n, \chi_R(\|u_n\|) \Lambda g(u_n) dW \rangle.
\]

(4.11)

By the commutator lemma, one may conclude that

\[
\langle B(u_n), \Lambda^2 u_n \rangle \leq |\langle [\Lambda, G u_n] \cdot \nabla u^n, \Lambda u_n \rangle| \\
\leq C \| [\Lambda, G u_n] \cdot \nabla u^n \|_{L^1} \| u_n \|_{L^3} \\
\leq C \| \nabla G u_n \|_{L^6} \| u_n \|_{L^2} + \| G u_n \|_{L^6} \| \nabla u \|_{L^2} \| u_n \|_{L^3} \\
\leq C \| u_n \|_{L^2} \| u_n \|_{L^2} \| u_n \|_{\theta_2}^2 \\
\leq \varepsilon \| u_n \|_{\theta_2 + 1}^2 + C \varepsilon \| u_n \|_6^6,
\]

(4.12)
where we use the Hölder inequality in the second inequality, the Sobolev embedding inequality and Lemma 2.3 in the fourth inequality, the interpolation inequality as well as the Young’s inequality in the last inequality.

Then, (4.11), (4.12) and the Hypothesis (4.1) imply
\[
d\|u_n\|^2 + 2\nu\|u_n\|^2_{\dot{H}^{2} + 1}dt \leq \left[ C\chi_R(\|u_n\|)\|u_n\|^2 + 2\varepsilon\|u_n\|^2_{\dot{H}^{2} + 1} + C \right]dt
+ 2\chi_R(\|u_n\|)(\Lambda u_n, \chi_R(\|u_n\|)\Lambda g(n)dw) .
\]

By the BDG’s inequality, it follows that
\[
\mathbb{E} \sup_{t \in [0,T]} \|u_n\|^2 + \mathbb{E} \int_0^T \|u_n\|^2_{\dot{H}^{2} + 1}dt
\leq C\mathbb{E}\|u_0\|^2 + CT + C\mathbb{E}\left(\int_0^T \|u_n\|^2_2 \chi_R(\|u_n\|)\|\Lambda g(n)\|^2_{L^2(U,\mathbb{P}^\varepsilon)}dt\right)^{1/2}
\leq C_T ,
\]
where $C_T$ is a constant independent of $n$.

Now, we prove that the family $\mathcal{L}(u_n)_{n \in \mathbb{N}}$ is tight in $C([0,T];\mathbb{H}^{1-\theta_2})$. Here, $\mathcal{L}(u_n)$ means the law of $u_n$. By (4.13), for each $t \in [0,T]$, $\mathcal{L}(u_n(t))$ is tight on $\mathbb{H}^{1-\theta_2}$. Then according to Aldous’s criterion in [H], it suffices to check that for all stopping times $\tau_n \leq T$ and $\eta_n \to 0$,
\[
\lim_n \mathbb{E}\|u_n(\tau_n + \eta_n) - u_n(\tau_n)\|_1 = 0 .
\]

Note that
\[
u u_n(\tau_n + \eta_n) - u_n(\tau_n) = -\int_{\tau_n}^{\tau_n + \eta_n} (\nu \Lambda^{2\theta_2} u_n + \chi_R(\|u_n\|))P_n B(u_n)dt
+ \int_{\tau_n}^{\tau_n + \eta_n} \chi_R(\|u_n\|)P_n g(u_n)dw .
\]

By (4.13), we get that for large $n$
\[
\mathbb{E}\left\|\int_{\tau_n}^{\tau_n + \eta_n} \nu \Lambda^{2\theta_2} u_n dt \right\|_{1-\theta_2} \leq C\eta_n^{1/2} \left(\mathbb{E}\left(\int_0^{T+1} \|u_n\|^2_{\dot{H}^{2} + 1}dt\right)\right)^{1/2} \to 0, \text{ as } \eta_n \to 0 .
\]

Thanks to Lemmas 2.2 and 2.3, it is easy to get
\[
\|B(u_n)\|_{1-\theta_2} \leq C\|B(u_n)\|_{L^2} \leq C\|u_n\|\|u_n\|_{\dot{H}^{2} + 1} .
\]

From (4.16) we infer that
\[
\mathbb{E}\left\|\int_{\tau_n}^{\tau_n + \eta_n} \chi_R(\|u_n\|)P_n B(u_n)dt \right\|_{1-\theta_2} \leq C\mathbb{E}\int_{\tau_n}^{\tau_n + \eta_n} \|u_n\|_{\dot{H}^{2} + 1}dt
\leq C\eta_n^{1/2} \left(\mathbb{E}\left(\int_0^{T+1} \|u_n\|^2_{\dot{H}^{2} + 1}dt\right)\right)^{1/2} \to 0, \text{ as } \eta_n \to 0 .
\]

Similarly, we obtain by Hypothesis (4.1)
\[
\mathbb{E}\left\|\int_{\tau_n}^{\tau_n + \eta_n} \chi_R(\|u_n\|)P_n g(u_n)dw \right\|_{1-\theta_2}^2 \leq C\mathbb{E}\int_{\tau_n}^{\tau_n + \eta_n} \chi_R(\|u_n\|)\|\Lambda g(u_n)\|^2_{L^2(U,\mathbb{P}^\varepsilon)}dt
\leq C\eta_n^{1/2} \left(\mathbb{E}\left(\int_0^{T+1} \|u_n\|^2_{\dot{H}^{2} + 1}dt\right)\right)^{1/2} \to 0, \text{ as } \eta_n \to 0 .
\]
Thus, (4.11) follows, which implies the tightness of \( L(u_n)_{n \in \mathbb{N}} \) in \( C([0, T]; \mathbb{H}^{1-\theta_2}) \).

We also make use of a variation of the BDG’s inequality (see, Lemma 2.1 in [26]) and get that for \( \kappa \in [0, 1/2) \),

\[
E \left\| \int_0^t \chi_R(||u_n||)g(u_n) dW \right\|_{W^{1,2}([0,T];\mathbb{H}^{1-\theta_2})}^2 dt \leq C E \int_0^T \chi_R(||u_n||)g(u_n)^2_{L^2(U,\mathbb{H}^0)} dt
\]

\[
\leq C E \int_0^T \chi_R(||u_n||)\Lambda g(u_n)^2_{L^2(U,\mathbb{H}^0)} dt
\]

\[
\leq C_{1,T}. \tag{4.17}
\]

By (4.16), we conclude

\[
E \left\| u_n(t) - \int_0^t \chi_R(||u_n||)g(u_n) dW \right\|_{W^{1,2}([0,T];\mathbb{H}^{1-\theta_2})}^2
\]

\[
\leq C E \|u_0\|_{L^2}^2 + C E \int_0^T (\|\Lambda^{\theta_2} u_n\|_{1-\theta_2}^2 + \chi_R(||u_n||)\|B(u_n)\|_{L^2} dt
\]

\[
\leq C E \|u_0\|_{L^2}^2 + C E \int_0^T \|u_n\|_{\theta_2+1} dt
\]

\[
\leq C_{2,T}. \tag{4.18}
\]

(4.13), (4.17) and (4.18) imply that the laws \( L(u_n)_{n \in \mathbb{N}} \) are bounded in probability in

\( L^2([0, T]; \mathbb{H}^{\theta_2+1}) \cap W^{1,2}([0, T]; \mathbb{H}^{1-\theta_2}) \).

Thus, by [26] Theorem 2.1 we get that \( L(u_n)_{n \in \mathbb{N}} \) is tight in \( L^2([0, T]; \mathbb{H}^1) \). Therefore, there exists a subsequence, still denoted by \( u_n \), such that \( L(u_n) \) converges weakly in \( L^2([0, T]; \mathbb{H}^1) \cap C([0, T]; \mathbb{H}^{1-\theta_2}) \). The Skorohod’s embedding theorem yields that there exists a stochastic basis \((\Omega^1, \mathcal{F}^1, \{\mathcal{F}^1_t\}_{t \geq 0}, \mathbb{P}^1)\) and \( L^2(0, T; \mathbb{H}^1) \cap C([0, T]; \mathbb{H}^{1-\theta_2})\)-valued random variables \( u^1 \) and \( u^1_n, n \geq 1 \) on it, such that \( u^1_n \) and \( u_n \) have the same law and \( u^1_n \to u^1 \) in \( L^2([0, T]; \mathbb{H}^1) \cap C([0, T]; \mathbb{H}^{1-\theta_2}) \), \( \mathbb{P}^1\)-a.s. For \( u^1_n \) we also have (4.13). This and the Fatou’s lemma imply

\[ u^1 \in L^2(\Omega^1, L^2([0, T]; \mathbb{H}^{\theta_2+1}) \cap L^\infty([0, T]; \mathbb{H}^1)). \]

For \( n \geq 1 \), define the \( \mathbb{H}_n\)-valued process

\[ M^1_n(t) := u^1_n(t) - P_n u_0 + \int_0^t (\nu \Lambda^{\theta_2} u^1_n + \chi_R(||u^1_n||)P_n B(u^1_n)) \, ds. \]

Then \( M^1_n(t) \) is a square integrable martingale with respect to the filtration \( \{\mathcal{F}^1_t\}_t = \sigma\{u^1_n(s), s \leq t\} \) with quadratic variation process

\[ \langle M^1_n \rangle_t = \int_0^t \chi_R(||u^1_n||)^2 P_n g(u^1_n(s)) g(u^1_n(s))^* P_n ds. \]

For \( s \leq t \in [0, T] \), bounded continuous functions \( \phi \) on \( L^2([0, s]; \mathbb{H}^1) \cap C([0, s]; \mathbb{H}^{1-\theta_2}) \) and \( v \in \cap_{t=1}^\infty \mathbb{H}^\theta \), we have

\[ E^1(\langle M^1_n(t) - M^1_n(s), v \rangle \phi(u^1_n|_{[0,s]}) = 0 \tag{4.19} \]

and

\[ E^1 \left[ (\langle M^1_n(t), v \rangle - \langle M^1_n(s), v \rangle)^2 - \int_s^t \chi_R(||u^1_n||)^2 ||g(u^1_n)^* P_n v||^2 dt \right] \phi(u^1_n|_{[0,s]}) = 0. \tag{4.20} \]
In order to take the limit in (4.19) as $n \to \infty$, we estimate
\[
\int_0^t |\langle \chi_R(\|u_n^1\|)P_n B(u_n^1), v\rangle - \langle \chi_R(\|u^1\|)B(u^1), v\rangle|\,ds \\
\leq \int_0^t |\langle \chi_R(\|u_n^1\|)(B(u_n^1) - B(u^1)), P_n v\rangle|\,ds + \int_0^t |\langle \chi_R(\|u_n^1\|)B(u^1), (I - P_n)v\rangle|\,ds \\
+ \int_0^t |\langle \chi_R(\|u_n^1\|) - \chi_R(\|u^1\|)\rangle B(u^1), v\rangle|\,ds \\
=: I_1 + I_2 + I_3. \tag{4.21}
\]
For $I_1$, by the bilinearity of $B$, we have
\[B(u_n^1) - B(u^1) = B(Gu_n^1 - Gu^1, u_n^1) + B(Gu^1, u_n^1 - u^1),\]
which together with Lemma 2.2 implies that
\[I_1 \leq C\|v\| \int_0^t (\|u_n^1\| + \|u^1\|)\|u_n^1 - u^1\|\,ds \\
\leq C\|v\| \left( \int_0^t \|u_n^1\|^2 + \|u^1\|^2 \,ds \right)^{1/2} \left( \int_0^t \|u_n^1 - u^1\|^2 \,ds \right)^{1/2} \\
\to 0, \text{ for } \mathbb{P}^1\text{-a.e. } \omega \in \Omega^1. \tag{4.22}
\]
For $I_2$, we have
\[I_2 \leq C\|(I - P_n)v\| \int_0^t \|u^1\|^2 \,ds \to 0, \text{ for } \mathbb{P}^1\text{-a.e. } \omega \in \Omega^1. \tag{4.23}
\]
By the dominated convergence theorem, we can get the last term $I_3 \to 0$, for almost every $\omega \in \Omega^1$. Thus, combining (4.21)–(4.23), we infer that for almost every $\omega \in \Omega^1$
\[\int_0^t \langle \chi_R(\|u_n^1\|)P_n B(u_n^1), v\rangle \to \int_0^t \langle \chi_R(\|u^1\|)B(u^1), v\rangle ds. \tag{4.24}\]
Applying the BDG’s inequality, we get that for any $p \geq 2$
\[\sup_n \mathbb{E}^1|\langle M_n^1(t), v\rangle|^{2p} \leq C \sup_n \mathbb{E}^1 \left( \int_0^t \chi_R(\|u_n^1\|)^2\|g(u_n^1)^*P_n v\|_0^2 \,dr \right)^p < \infty. \tag{4.25}\]
Note that the convergence for the linear term is direct, by (4.24) and (4.25) we can get that for $v \in \cap_{i=1}^\infty \mathbb{H}^i$,
\[\lim_{n \to \infty} \mathbb{E}^1|\langle M_n^1(t) - M^1(s), v\rangle| = 0\]
and
\[\lim_{n \to \infty} \mathbb{E}^1|\langle M_n^1(t), v\rangle|^2 = 0,\]
where
\[M^1(t) := u^1(t) - u^1(0) + \int_0^t (\nu A^2 u^1 + \chi_R(\|u^1\|)B(u^1)) \,ds.\]
Taking the limit in (4.19) and (4.20), we derive that for $s \leq t \in [0, T]$, bounded continuous functions $\phi$ on $L^2([0, s]; \mathbb{H}^1) \cap C([0, s]; \mathbb{H}^{1-\theta_2})$ and $v \in \cap_{i=1}^\infty \mathbb{H}^i$, we have
\[\mathbb{E}^1(\langle M^1(t) - M^1(s), v\rangle)\phi(u^1|_{[0, s]}) = 0\]

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and
\[
\mathbb{E}^1 \left( \langle M^1(t), v \rangle^2 - \langle M^1(s), v \rangle^2 - \int_s^t \chi_R(||u^1||)^2 ||g(u^1)\ast v||^2 \eta^2 dW \right) \phi(u^1|_{[0,s]}) = 0.
\]

Thus, according to the martingale representation theorem (cf. [18, Theorem 8.2]), there exists a stochastic basis \((\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{F}_t\}_{t \geq 0}, \tilde{P})\), a cylindrical Wiener process \(\tilde{W}\) and an \(\tilde{\mathcal{F}}_t\)-adapted process \(\tilde{u}\) with path in \(L^2([0, T]; \mathbb{H}^1) \cap C([0, T]; \mathbb{H}^{1-\theta_2})\) such that \(\tilde{u}\) satisfies (4.19) with \(W\) replaced by \(\tilde{W}\) and \(\tilde{u}_0\) has the same distribution as \(u_0\). For \(\tilde{u}\) we also have (4.13), hence, it follows that
\[
\tilde{u} \in L^2(\tilde{\Omega}, L^2([0, T]; \mathbb{H}^{\theta_2+1}) \cap L^\infty([0, T]; \mathbb{H}^1)).
\]

Now we want to show that \(\tilde{u} \in C([0, T]; \mathbb{H}^1)\) a.s., which is needed in order to justify the following stopping time (4.30) is well defined. To this end we define
\[
dz + \nu \lambda^{2g_2} z = \chi_R(\|\tilde{u}\|) g(\tilde{u}) d\tilde{W}, \quad z(0) = 0.
\]

Since \(\chi_R(\|\tilde{u}\|) g(\tilde{u}) \in L^2(\tilde{\Omega}, L^2([0, T], L^2(U, \mathbb{H}^1)))\), we have
\[
z \in L^2(\tilde{\Omega}, C([0, T]; \mathbb{H}^1)) \cap L^2(\tilde{\Omega}, L^2([0, T]; \mathbb{H}^{1+\theta_2})).
\]

Take \(\tilde{v} = \tilde{u} - z\). Subtracting (4.27) from (4.10), then we see that \(\tilde{v}\) solves
\[
\begin{cases}
\frac{d\tilde{v}}{dt} + \nu \lambda^{2g_2} \tilde{v} + \chi_R(\|\tilde{u} + z\|) B(\tilde{u} + z) = 0, \\
\tilde{v}(0) = \tilde{u}_0,
\end{cases}
\]

which is a (pathwise) deterministic PDE.

Due to (4.20) and (4.28), we infer that \(\tilde{v} \in L^2(\tilde{\Omega}, L^2([0, T]; \mathbb{H}^{\theta_2+1}) \cap L^\infty([0, T]; \mathbb{H}^1))\). Hence, we have
\[
\lambda^{2g_2} \tilde{v}, \chi_R(\|\tilde{u} + z\|) B(\tilde{u} + z) \in L^2(\tilde{\Omega}, L^2([0, T]; \mathbb{H}^{1-\theta_2})).
\]

We conclude with (4.29) that
\[
\frac{d\tilde{v}}{dt} \in L^2(\tilde{\Omega}, L^2([0, T]; \mathbb{H}^{1-\theta_2})), \quad \tilde{v} \in L^2(\tilde{\Omega}, L^2([0, T]; \mathbb{H}^{\theta_2+1})).
\]

Applying the strong continuity result (see [49, Chap3, Lemma 1.2] or [5, Lemma 6]), we infer that \(\tilde{v} \in C([0, T]; \mathbb{H}^1)\), a.s. This and (4.28) imply that \(\tilde{u} \in C([0, T]; \mathbb{H}^1)\) a.s.

Define the stopping time
\[
\tau_R := \inf \{ t \geq 0; \|\tilde{u}(t)\| \geq R \}.
\]

Then \((\tilde{u}, \tau_R)\) is a local weak solution of (2.41) such that \(\tilde{u}(\cdot \wedge \tau_R) \in C([0, \infty); \mathbb{H}^1)\) a.s., and \(\tilde{u}(\cdot \wedge \tau_R) \in L^2(\tilde{\Omega}, L^2_{loc}([0, \infty), \mathbb{H}^{\theta_2+1})).\)

To complete the proof of the case, it remains to prove the pathwise uniqueness and apply the Yamada-Watanable theorem (cf [32, Thorem 3.14]). These technical details are similar to the arguments in [44], so we omit further details.

Now the proof of Theorem 4.2 is completed.
4.2 Global existence

In the deterministic case \((g \equiv 0)\), the global solutions of Leray-regularized equations with fractional dissipation have been intensively investigated (cf. \([2, 4, 5, 11, 12, 52]\)). In particular, Barbato, Morandin and Romito in \([4]\) showed the existence of a smooth global solution to the 3D Leray-\(\alpha\) model with the logarithmical dissipation when \(\theta_1 + \theta_2 \geq \frac{5}{4}\) (with \(\theta_1, \theta_2 \geq 0\)). The result in \([4]\) was obtained by analyzing energy dispersion over dyadic shells, which is a completely different approach with the one employed in this work.

The existence of the noise perturbation makes the problem more interesting and challenging. In this section, inspired by \([21, 27]\), we show that the strong solution of (2.4) is global when \(\theta_1 \geq 0, \theta_2 > 0\) and \(\theta_1 + \theta_2 \geq \frac{5}{4}\). By Theorem 4.2, let \((u, (\tau_R)_{R \in \mathbb{N}}, \xi)\) be a maximal strong solution, if \(\xi(\omega) < \infty\) for \(\omega \in \Omega\), then the \(H^1\) norm of the solution must blow up at this maximal time as expressed by (1.2). Next lemma establishes some estimates for the solution of (2.4), which do not depend on \(\tau_R\) and are useful for the proof of global existence.

**Lemma 4.1** Under the assumptions of Theorem 4.1, if \((u, (\tau_R)_{R \in \mathbb{N}}, \xi)\) is a maximal strong solution of (2.4), then for any \(t > 0\)

\[
\mathbb{E} \left( \sup_{0 \leq s \leq \tau_R \wedge t} \|u(s)\|_{L^2}^2 + \int_0^{\xi \wedge t} \|u(s)\|_{L^2}^2 ds \right) < \infty. \tag{4.31}
\]

**Proof** Taking Itô’s formula, we obtain that, for any \(t > 0, s \in [0, \tau_R \wedge t]\)

\[
\|u(s)\|_{L^2}^2 = \|u_0\|_{L^2}^2 - 2 \int_0^s \left( \nu \Lambda^{2\theta_2} u(r) + B(u(r), u(r)) \right) dr
+ \int_0^s \|g(u(r))\|_{L_2(H^0)}^2 dr + 2 \int_0^s \langle g(u(r)) dW(r), u(r) \rangle. \tag{4.32}
\]

By the Hypothesis (4.2) and the fact that \(\langle B(u), u \rangle = 0\), we have

\[
\|u(s)\|_{L^2}^2 + \int_0^s \|u(r)\|_{L^2}^2 dr
\leq C + \|u_0\|_{L^2}^2 + C \int_0^s \|u(r)\|_{L^2}^2 dr + 2 \int_0^s \langle g(u(r))dW(r), u(r) \rangle. \tag{4.33}
\]

The BDG’s inequality and the Young’s inequality yield

\[
\mathbb{E} \sup_{s \in [0, \tau_R \wedge t]} \left| \int_0^s \langle g(u(r))dW(r), u(r) \rangle \right|
\leq 3 \mathbb{E} \left( \int_0^{\tau_R \wedge t} \|u(r)\|_{L^2}^2 \|g(u(r))\|_{L_2(H^0)}^2 dr \right)^{1/2}
\leq 3 \mathbb{E} \left( \sup_{s \in [0, \tau_R \wedge t]} \|u(s)\|_{L^2}^2 \cdot C \int_0^{\tau_R \wedge t} (1 + \|u(s)\|_{L^2}^2) ds \right)^{1/2}
\leq \varepsilon \mathbb{E} \sup_{s \in [0, \tau_R \wedge t]} \|u(s)\|_{L^2}^2 + C \varepsilon \mathbb{E} \int_0^{\tau_R \wedge t} (1 + \|u(s)\|_{L^2}^2) ds, \tag{4.34}
\]

where \(\varepsilon > 0\) is a small constant.

By (4.33) and (4.34), one deduces

\[
\mathbb{E} \sup_{s \in [0, \tau_R \wedge t]} \|u(s)\|_{L^2}^2 + \mathbb{E} \int_0^{\tau_R \wedge t} \|u(s)\|_{L^2}^2 ds
\leq C + C \mathbb{E} \|u_0\|_{L^2}^2 + \mathbb{E} \int_0^{\tau_R \wedge t} \|u(s)\|_{L^2}^2 ds. \tag{4.35}
\]
The Gronwall’s lemma implies that
\[ \mathbb{E} \sup_{0 \leq s \leq T} \| u(s) \|_{L^2}^2 + \mathbb{E} \int_0^T \| u(s) \|_{\hat{H}_2}^2 ds \leq C(1 + \mathbb{E} \| u_0 \|_{L^2}^2), \]  
(4.36)

where \( C = C(u_0, T) \) is a constant independent of \( R \).

Thus (4.31) can be proved by the monotone convergence theorem as \( R \to \infty \). The proof is completed. \( \square \)

**Proof of Theorem 4.1:**

For any \( K, M > 0 \) we define the following stopping times
\[ \rho_M := \inf_{t \geq 0} \left\{ \sup_{s \in [0, t]} \| u(s) \|^2 + \int_0^t \| u(s) \|_{\hat{H}_2}^2 ds \geq M \right\} \wedge \xi \]
and
\[ \gamma_K := \inf_{t \geq 0} \left\{ \int_0^{t \wedge \xi} \| u(s) \|^2_{\hat{H}_2} ds \geq K \right\}. \]

Here we take \( \inf \emptyset = \infty \). It is clear that \((u, \rho_M)\) is a local strong solution.

Applying \( \Lambda \) to equation (2.4) and taking Itô’s formula, for any \( t > 0, s \in [0, \gamma_K \wedge \rho_M \wedge t] \), we have
\[ \| u(s) \|^2 + 2\nu \int_0^s \| u(r) \|^2_{\hat{H}_2} dr = \| u_0 \|^2 - 2 \int_0^s \langle \Lambda B(u(r)), \Lambda u(r) \rangle dr + \int_0^s \| \Lambda g(u(r)) \|^2_{L^2} dr \]
\[ + 2 \int_0^s \langle \Lambda g(u(r)) dW(r), \Lambda u(r) \rangle. \]
(4.37)

For any stopping times \( 0 \leq \tau_a \leq \tau_b \leq \gamma_K \wedge \rho_M \wedge t \), taking a supremum over the interval \([\tau_a, \tau_b]\) and then taking expectation with respect to the resulting expression, we deduce that
\[ \mathbb{E} \left( \sup_{s \in [\tau_a, \tau_b]} \| u(s) \|^2 + \nu \int_{\tau_a}^{\tau_b} \| u(r) \|^2_{\hat{H}_2} dr \right) \]
\[ \leq C_0 \mathbb{E} \left( \| u(\tau_a) \|^2 + \int_{\tau_a}^{\tau_b} (| \langle \Lambda B(u(r)), \Lambda u(r) \rangle | + (1 + \| u(r) \|^2)) dr \right) \]
\[ + C_0 \mathbb{E} \sup_{s \in [\tau_a, \tau_b]} \left| \int_{\tau_a}^{s} \langle \Lambda g(u(r)) dW(r), \Lambda u(r) \rangle \right|, \]
(4.38)

where we use the Hypothesis (4.2), and \( C_0 \) is a constant independent of \( \tau_a \) and \( \tau_b \).

Similar as the proof of Theorem 4.2, by the commutator lemma, we have
\[ | \langle \Lambda B(u), \Lambda u \rangle | \leq C(\| \Lambda Gu \|_{L^p_1}, \| \Lambda u \|_{L^p_2}) \| \Lambda u \|_{L^2}, \]
where \( p_1 \) and \( p_2 \) are given by
\[ \left\{ \begin{array}{ll}
\frac{1}{p_1} = \frac{\theta_2}{3}, & \frac{1}{p_2} = \frac{1}{2} - \frac{\theta_2}{3} \quad \text{if } 0 < \theta_2 < \frac{3}{2}, \\
p_1 = 6, & p_2 = 3 \quad \text{if } \theta_2 \geq \frac{3}{2}.
\end{array} \right. \]
Thanks to the fact \( \theta_1 + \theta_2 \geq \frac{3}{4} \), by Lemmas 2.1 and 2.3, it is not difficult to get that for \( 0 < \theta_2 < \frac{3}{2} \),

\[
\| \Lambda u \|_{L^p} \leq C \| u \|_{\theta_2+1}, \quad \| \Lambda G u \|_{L^p} \leq C \| G u \|_{\frac{3}{2}-\theta_2} \leq C \| u \|_{\theta_2}.
\]

Similarly, for \( \theta_2 \geq \frac{3}{2} \) we also have

\[
\| \Lambda u \|_{L^p} \leq C \| u \|_{\frac{3}{2}} \leq C \| u \|_{\theta_2}, \quad \| \Lambda G u \|_{L^p} \leq C \| G u \|_2 \leq C \| u \|_{\theta_2+1}.
\]

Therefore, putting the above estimates all together and applying the Young’s inequality, it leads to

\[
\| \langle AB(u), \Lambda u \rangle \| \leq C \| u \|_{\theta_2} \| u \|_{\theta_2+1} \leq \varepsilon \| u \|_{\theta_2+1}^2 + C \varepsilon \| u \|_{\theta_2}^2 \| u \|^2, \tag{4.39}
\]

where \( \varepsilon > 0 \) is a small constant.

By the BDG’s inequality and Young’s inequality, we have

\[
E \sup_{s \in [\tau_0, \tau_b]} \left| \int_{\tau_0}^s \langle A g(u(r)) dW(r), \Lambda u(r) \rangle \right| \\
\leq 3E \left( \int_{\tau_0}^{\tau_b} \| u(s) \|^2 \| A g(u(s)) \|_{L^2(\mathbb{R}^d)}^2 ds \right)^{1/2} \\
\leq 3E \left( \sup_{s \in [\tau_0, \tau_b]} \| u(s) \|^2 \cdot C \int_{\tau_0}^{\tau_b} (1 + \| u(s) \|^2) ds \right)^{1/2} \\
\leq \varepsilon E \sup_{s \in [\tau_0, \tau_b]} \| u(s) \|^2 + C \varepsilon E \int_{\tau_0}^{\tau_b} (1 + \| u(s) \|^2) ds. \tag{4.40}
\]

Combining the estimates (4.38)-(4.40), we conclude that

\[
E \left( \sup_{s \in [\tau_0, \tau_b]} \| u(s) \|^2 + \int_{\tau_0}^{\tau_b} \| u(r) \|_{\theta_2+1}^2 dr \right) \\
\leq C E \left( \| u(\tau_0) \|^2 + \int_{\tau_0}^{\tau_b} (1 + \| u(r) \|_{\theta_2}^2 \| u(r) \|^2 dr \right), \tag{4.41}
\]

where \( C \) is independent of \( \tau_0 \) and \( \tau_b \). Now, we can apply the stochastic version Gronwall’s lemma from [27, Lemma 5.3], which is recalled in the appendix as Lemma 5.1. Note that by definition of \( \gamma_k \),

\[
\int_0^{\gamma_k} \| u \|_{\theta_2}^2 ds \leq K, \quad \text{a.s.}
\]

Taking \( X := \| u \|^2 \), \( Y := \| u \|_{\theta_2+1}^2 \), \( R := 1 + \| u \|_{\theta_2}^2 \), \( Z := 0 \) and \( \tau := \gamma_K \wedge \rho_M \wedge t \) in Lemma 5.1, we therefore get that

\[
E \left( \sup_{0 \leq s \leq \gamma_K \wedge \rho_M \wedge t} \| u(s) \|^2 + \int_0^{\gamma_K \wedge \rho_M \wedge t} \| u(s) \|_{\theta_2+1}^2 ds \right) \leq C_{c_0,t,K} (1 + E \| u_0 \|^2), \tag{4.42}
\]
where $C_{c_0,t,K}$ is a constant independent of $M$. Then we have that, for any $t > 0$
\[
\mathbb{P}(\rho_M \leq t) \leq \mathbb{P} \left( \{ \rho_M \leq t \} \cap \{ \gamma_K > t \} \right) + \mathbb{P}(\gamma_K \leq t) \\
= \mathbb{P} \left( \left\{ \sup_{0 \leq s \leq \rho_M \wedge t} \|u(s)\|^2 + \int_{0}^{\rho_M \wedge t} \|u(s)\|_{\partial_2 + 1}^2 ds \geq M \right\} \cap \{ \gamma_K > t \} \right) + \mathbb{P}(\gamma_K \leq t) \\
\leq \mathbb{P} \left( \sup_{0 \leq s \leq \gamma_K \wedge \rho_M \wedge t} \|u(s)\|^2 + \int_{0}^{\gamma_K \wedge \rho_M \wedge t} \|u(s)\|_{\partial_2 + 1}^2 ds \geq M \right) + \mathbb{P}(\gamma_K \leq t) \\
\leq \frac{1}{M} \mathbb{E} \left( \sup_{0 \leq s \leq \gamma_K \wedge \rho_M \wedge t} \|u(s)\|^2 + \int_{0}^{\gamma_K \wedge \rho_M \wedge t} \|u(s)\|_{\partial_2 + 1}^2 ds \right) + \mathbb{P}(\gamma_K \leq t) \\
\leq \frac{C_{c_0,t,K}(1 + \mathbb{E}\|u_0\|^2)}{M} + \mathbb{P}(\gamma_K \leq t).
\]
Thus, for any fixed $K > 0$,
\[
\lim_{M \to \infty} \mathbb{P}(\rho_M \leq t) \leq \mathbb{P}(\gamma_K \leq t).
\]
By Lemma 4.1 we obtain that
\[
\mathbb{P}(\gamma_K \leq t) \leq \mathbb{P} \left( \int_{0}^{t \wedge \xi} \|u(s)\|_{\partial_2}^2 ds \geq K \right) \leq \frac{1}{K} \mathbb{E} \left( \int_{0}^{t \wedge \xi} \|u(s)\|_{\partial_2}^2 ds \right),
\]
which goes to zero as $K \to \infty$. Hence, we get that for any $t > 0$,
\[
\lim_{M \to \infty} \mathbb{P}(\rho_M \leq t) = 0.
\]
Let $M \to \infty$, by the monotone convergence theorem, we can get that for any $t > 0$,
\[
\mathbb{E} \left( \sup_{0 \leq s \leq \gamma_K \wedge \rho_M \wedge t} \|u(s)\|^2 + \int_{0}^{\gamma_K \wedge \rho_M \wedge t} \|u(s)\|_{\partial_2 + 1}^2 ds \right) \leq C_{c_0,t,K}(1 + \mathbb{E}\|u_0\|^2). \tag{4.43}
\]
We now want to show that for any $K > 0$, $\gamma_K \leq \xi$ a.s. Suppose $\mathbb{P}(\gamma_K > \xi) > 0$. Denoted by $\mathbb{Q}^+$ the set of negative rational numbers, we have $\{ \gamma_K > \xi \} = \bigcup_{t \in \mathbb{Q}^+} \{ \gamma_K \wedge t > \xi \}$. Hence there is a $t_0 \in \mathbb{Q}^+$ such that $\mathbb{P}(\gamma_K \wedge t_0 > \xi) > 0$. By the definition of $\xi$ (see (4.2)), we would infer that
\[
\sup_{0 \leq s \leq \gamma_K \wedge t_0} \|u(s)\|^2 + \int_{0}^{\gamma_K \wedge t_0} \|u(s)\|_{\partial_2 + 1}^2 ds \geq \sup_{0 \leq s \leq \xi} \|u(s)\|^2 = \infty
\]
on $\{ \gamma_K \wedge t_0 > \xi \}$. Hence, we have a contradiction with (4.43). This means that $\gamma_K \leq \xi$ a.s. for any $K > 0$.
According to Lemma 4.1, it is obvious that
\[
\lim_{K \to \infty} \gamma_K = \infty \quad \text{a.s.}
\]
Consequently, $\xi = \infty$ a.s., and the solution $u$ is global in the sense of definition 4.1. We complete the proof of Theorem 4.1. \hfill \Box
4.3 Moments of the solution

As a continuation of Remark 4.3, we establish some moment estimates of the solution for (2.4) in this section, which do not depend on any stopping times.

**Theorem 4.3** Under the assumptions of Theorem 4.1, and suppose $u_0 \in L^p(\Omega, \mathbb{H}^0)$ for some $p \geq 2$. Then for any $T > 0$ we have

\[
E \left( \sup_{0 \leq t \leq T} \|u(t)\|^p_{L^2} + \int_0^T \|u(t)\|^{p-2}_2 \|u(t)\|_2^2 dt \right) \leq C_T(1 + E\|u_0\|_{L^2}^p). \tag{4.44}
\]

**Proof** The result can be obtained by using similar arguments as in the proof of Lemma 4.1. \qed

**Theorem 4.4** Suppose that $\theta_1 \geq 0$, $\theta_2 > 0$ with $\theta_1 + \theta_2 > \frac{5}{4}$ and $u_0 \in L^{mp}(\Omega, \mathbb{H}^0) \cap L^p(\Omega, \mathbb{H}^1)$ for some $p \geq 2$ with a constant $m = m(\theta_2) > 0$ given by

\[
\begin{align*}
m & = 1 + \frac{1 + \frac{\theta_2}{2}}{2(\theta_1 + \theta_2 - \frac{1}{4})}, \quad \text{if} \quad \theta_1 + \frac{\theta_2}{2} < \frac{5}{4}, \\
m & = 2 + \frac{1}{\theta_2}, \quad \text{if} \quad \theta_1 + \frac{\theta_2}{2} \geq \frac{5}{4}.
\end{align*}
\]

If $g$ satisfies the Hypotheses (4.1) and (4.2), then for any $T > 0$ we have

\[
E \left( \sup_{0 \leq t \leq T} \|u(t)\|^p + \int_0^T \|u(t)\|^{p-2}_2 \|u(t)\|_2^2 dt \right) \leq C_{p,T}(1 + E\|u_0\|^p + C E\|u_0\|_{L^2}^mp). \tag{4.45}
\]

**Remark 4.5** By the standard Krylov-Bogolyubov argument, an immediate corollary of Theorem 4.4 is the existence of an invariant measure for the associate transition semigroup. The topic of ergodicity for this model will be investigated in a separate paper.

**Proof** By Theorem 4.1, there exists a unique strong global solution for (2.4). Applying Itô’s formula, we obtain that, for any $T > 0, t \in [0, T]$

\[
\|u(t)\|^p + pu \int_0^t \|u(s)\|^{p-2}_2 \|u(s)\|_{\theta_2+1}^2 ds = \|u_0\|^p - p \int_0^t \|u(s)\|^{p-2}_2 (\Lambda B(u(s)), \Lambda u(s)) ds + \frac{p}{2} \int_0^t \|u(s)\|^{p-2}_2 \|\Lambda g(u(s))\|_{L^2(U^1, \mathbb{H}^0)}^2 ds + \frac{p(p-2)}{2} \int_0^t \|u(s)\|^{p-4}_2 \|\Lambda g(u(s))\|^2 \Lambda u(s) ds + p \int_0^t \|u(s)\|^{p-2}_2 \|\Lambda g(u(s))\|dW(s), \Lambda u(s)). \tag{4.46}
\]

To bound the second term at the right hand side of (4.46), we split it into two cases, namely, the case $\theta_2 \leq 1$ and the case $\theta_2 > 1$. For the case $\theta_2 \leq 1$, thanks to the commutator estimate, it directly yields

\[
\|\langle \Lambda B(u), \Lambda u \rangle \| = \|\langle [\Lambda, Gu] \cdot \nabla u^n, \Lambda u \rangle \| \leq C \| [\Lambda, Gu] \cdot \nabla u \|_{L^\infty} \| \Lambda u \|_{L^\infty} \leq C (\| \Lambda Gu \|_{L^P_1} \| \Lambda u \|_{L^P_2} + \| \Lambda u \|_{L^P_3} \| \Lambda Gu \|_{L^P_4}) \| \Lambda u \|_{L^{\frac{q}{2+\frac{1}{4}}}}. \]

\]

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By the H"older's inequality and Hypothesis (4.2), we also get

\[ 1 = \frac{1}{p_1} + \frac{1}{p_4} = \frac{\delta_0 + \theta_2}{3}, \quad \frac{1}{p_2} = \frac{1}{p_3} = \frac{1}{2} - \frac{\delta_0}{3}, \]

where \( \delta_0 = \delta_0(\theta_2) := \{ \frac{3}{2} - 2\theta_1 - \theta_2 \} \vee 0. \)

The assumptions \( 0 < \theta_2 \leq 1, \theta_1 \geq 0 \) and \( \theta_1 + \theta_2 > \frac{5}{4} \) imply that \( \delta_0 \in [0, \theta_2) \), \( p_1, p_4 \in \left( \frac{3}{2}, \infty \right) \) and \( p_2, p_3 \in [2, 6) \). By Lemmas 2.1 and 2.3, we have

\[
\| \Lambda Gu \|_{L^{p_1}} \leq C \| Gu \|_{L^{\frac{2}{\theta_2} - \delta_0}} \leq C \| u \|_{L^2},
\]

\[
\| \Lambda u \|_{L^{p_2}} \leq C \| u \|_{\delta_0 + 1},
\]

\[
\| \Lambda u \|_{L^{\frac{6}{3-2\theta_2}}} \leq C \| u \|_{\theta_2 + 1}.
\]

From the above estimates, we thus obtain

\[
\| \left\langle \Lambda B(u), \Lambda u \right\rangle \| \leq C \| u \|_{L^2} \| u \|_{\delta_0 + 1} \| u \|_{\theta_2 + 1}.
\]

For the case \( \theta_2 > 1 \), the Lemma 2.2 would suffice our purpose. Actually, since \( 2\theta_1 + \delta_0 + \theta_2 > \frac{5}{2} \), applying Lemma 2.2 (with \( m_1 = 2\theta_1, m_2 = \delta_0, m_3 = \theta_2 - 1 \)), it leads to

\[
\| \left\langle \Lambda B(u), \Lambda u \right\rangle \| \leq C \| u \|_{L^2} \| u \|_{\delta_0 + 1} \| u \|_{\theta_2 + 1}.
\]

Therefore, by the Young’s inequality and interpolation inequality, one may conclude that

\[
|p\| u \|^{p-2} \| \left\langle \Lambda B(u), \Lambda u \right\rangle \| \leq C \| u \|^{p-2} \| u \|_{L^2} \| u \|_{\delta_0 + 1} \| u \|_{\theta_2 + 1} \| u \|_{\theta_2 + 1} \| u \|_{\theta_2 + 1} \| u \|_{\theta_2 + 1} \leq C \| u \|^{p-2} \| u \|_{\theta_2 + 1}^2 + C \| u \|^2 \| u \|_{\theta_2 + 1} \| u \|^2 \| u \|_{\theta_2 + 1} \| u \|^2 \leq C \| u \|^{p-2} \| u \|^2 + C \| u \|_{L^2} \| u \|^p. \quad (4.47)
\]

By the Hölder’s inequality and Hypothesis (4.2), we also get

\[
\frac{p}{2} \| u \|^{p-2} \| \Lambda g(u) \|_{L^2(U, H^p)}^2 \leq C \| u \|^{p-2} \| \Lambda g(u) \|_{L^2(U, H^p)}^2 \leq C(1 + \| u \|^p). \quad (4.48)
\]

From (4.46)-(4.48), we obtain

\[
\| u(t) \|^p + \int_0^t \| u^n(s) \|^{p-2} \| u(s) \|_{\theta_2 + 1}^2 ds \\
\leq \| u_0 \|^p + CT + C \int_0^t \| u(s) \|^p ds + C \int_0^t \| u(s) \|_{L^\infty}^{mp} ds \\
+ \int_0^t p \| u(s) \|^{p-2} \| \Lambda g(u(s)) \| dW(s), \quad \Lambda u(s). \quad (4.49)
\]

For any \( R_1 > 0 \), define the stopping time

\[
\tau_{R_1} := \inf \{ t \in [0, T] : \| u(t) \| > R_1 \} \wedge T.
\]

As a result of Theorem 4.1, we have

\[
\lim_{R_1 \to \infty} \tau_{R_1} = T, \quad \mathbb{P} - a.s.
\]

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By the BDG’s inequality we have
\[
\mathbb{E} \sup_{r \in [0, t \wedge T_1]} \left| \int_0^r p\|u(s)\|^{p-2} \langle \Lambda g(u(s)) dW(s), \Lambda u(s) \rangle \right| \\
\leq 3 \mathbb{E} \left( \int_0^{t \wedge T_1} p\|u(s)\|^{2p-2} \|\Lambda g(u(s))\|^2_{L^2(U; H_0)} ds \right)^{1/2} \\
\leq 3 \mathbb{E} \left( \sup_{s \in [0, t \wedge T_1]} \|u(s)\|^{2p-2} C \int_0^{t \wedge T_1} (1 + \|u(s)\|^2) ds \right)^{1/2} \\
\leq 3 \varepsilon \sup_{s \in [0, t \wedge T_1]} \|u(s)\|^p + C_\varepsilon \left( \int_0^{t \wedge T_1} (1 + \|u(s)\|^2) ds \right)^{p/2} \\
\leq 3\varepsilon \sup_{s \in [0, t \wedge T_1]} \|u(s)\|^p + 3(2T)^{p/2-1}C_\varepsilon \mathbb{E} \int_0^{t \wedge T_1} (1 + \|u(s)\|^p) ds, \quad (4.50)
\]
where \(\varepsilon > 0\) is a small constant and \(C_\varepsilon\) comes from the Young’s inequality.

(4.44), (4.49) and (4.50) yield that, for \(t \in [0, T]\),
\[
\mathbb{E} \sup_{0 \leq t \leq \tau} \|u(t)\|^p + \mathbb{E} \int_0^{T_1} \|u(t)\|^{p-2} \|u(t)\|_{L^2}^{2+1} dt \\
\leq \mathbb{E} \|u_0\|^p + C \mathbb{E} \|u_0\|^p_{L^2} + CT + C \mathbb{E} \int_0^{T_1} \|u(t)\|^p dt \\
\leq C(1 + \mathbb{E} \|u_0\|^p + \mathbb{E} \|u_0\|^p_{L^2} + C \mathbb{E} \int_0^{T_1} \|u(t)\|^p dt,
\]
where \(C = C_{T,P}\) is a constant independent of \(R_1\).

Thus, (4.45) follows from the classic Gronwall’s lemma and the monotone convergence theorem. \(\square\)

5 Appendix

We recall the following stochastic Gronwall’s lemma (cf. [27, Lemma 5.3]) which is used in the proof of our results.

Lemma 5.1 Fix \(T > 0\). Assume that \(X, Y, Z, R : [0, T] \times \Omega \to \mathbb{R}\) are real-valued, non-negative stochastic process. Let \(\tau < T\) be a stopping time such that
\[
\mathbb{E} \int_0^\tau (RX + Z) ds < \infty.
\]
Assume, moreover, that for some fixed constant \(\kappa\),
\[
\int_0^\tau R ds < \kappa, \quad \text{a.s.}
\]
Suppose that for all stopping times \(0 \leq \tau_a < \tau_b \leq \tau\)
\[
\mathbb{E} \left( \sup_{t \in [\tau_a, \tau_b]} X + \int_{\tau_a}^{\tau_b} Y ds \right) \leq c_0 \mathbb{E} \left( X(\tau_a) + \int_{\tau_a}^{\tau_b} (RX + Z) ds \right), \quad (5.1)
\]
where \(c_0 = c_0(T, P)\) is a constant independent of \(R_1\).
where $c_0$ is a constant independent of the choice of $\tau_a, \tau_b$. Then

$$
\mathbb{E} \left( \sup_{t \in [0, \tau]} X + \int_0^\tau Y \, ds \right) \leq c \mathbb{E} \left( X(0) + \int_0^\tau Z \, ds \right),
$$

(5.2)

where $c = c_{c_0, T, \kappa}$.

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