On the efficient representation of the half-space impedance Green’s function for the Helmholtz equation

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October 3, 2011

Abstract

A classical problem in acoustic scattering concerns the evaluation of the Green’s function for the Helmholtz equation subject to impedance boundary conditions on a half-space. The two principal approaches used for representing this Green’s function are the Sommerfeld integral and the (closely related) method of complex images. The former is extremely efficient when the source is at some distance from the half-space boundary, but involves an unwieldy range of integration as the source gets closer and closer. Complex image-based methods, on the other hand, can be quite efficient when the source is close to the boundary, but they do not easily permit the use of the superposition principle, since the selection of complex image locations depends on both the source and the target. We have developed a new, hybrid representation which uses a finite number of real images (dependent only on the source location) coupled with a rapidly converging Sommerfeld-like integral. Our method applies in both two and three dimensions and its efficiency is illustrated with numerical examples.

1 Introduction

A number of problems in acoustics (and electromagnetics) involve the solution of the Helmholtz equation

\[(\Delta + k^2)u(x) = f(x),\]

in the half-plane \(P = \{(x, y) \in \mathbb{R}^2; y > 0\}\) or the half-space \(S = \{(x, y, z) \in \mathbb{R}^3; z > 0\}\), subject to suitable boundary and radiation conditions. In acoustics, the Helmholtz coefficient \(k\) is given by \(k = \frac{\omega}{c}\), where \(\omega\) is the governing frequency (assuming time-harmonic motion) and \(c\) is the sound speed. In the present paper, we assume \(k\) is constant throughout the region of interest. For concreteness, suppose that we wish to compute the scattered field due to a unit strength point source located at \(x_0\) in the presence of a “sound-hard” obstacle over an infinite half-space subject to impedance boundary conditions (Fig. 1).

We let the total field \(u_{\text{tot}} = u_{\text{in}} + u\), where \(u_{\text{in}}\) denotes the (known) incoming field due to the point source and \(u\) denotes the scattered field. On a sound-hard obstacle \(\Omega\) with boundary \(\Gamma\), the total field must satisfy homogeneous Neumann boundary conditions. Since the scattered field involves no sources outside \(\Omega\), it must satisfy the homogeneous Helmholtz equation

\[(\Delta + k^2)u(x) = 0\]

for \(x \in P \setminus \Omega\) in 2D, and for \(x \in S \setminus \Omega\) in 3D. On the obstacle boundary \(\Gamma\), we have

\[\frac{\partial u}{\partial n} = -\frac{\partial u_{\text{in}}}{\partial n}.\]

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Finally, on the interface, we assume the impedance condition takes the form:
\[
\frac{\partial u^{\text{tot}}}{\partial n} - i\alpha u^{\text{tot}} = 0.
\] (1.3)

There is a substantial literature on impedance problems and we mention only a few relevant papers which discuss the computation of the corresponding Green’s function \([1, 2, 9, 11, 6, 12]\). We assume that \(\text{Im}(\alpha) \leq 0\) to ensure that the lower half-space be energy-absorbing \([2, 3]\).

A simple ansatz for solving this scattering problem is to represent the total field as
\[
u^{\text{tot}}(x) = \int_{\Gamma} G_{\text{imp}}(x, y) \sigma(y) \, dy + G_{\text{imp}}(x, x_0),
\]
where \(G_{\text{imp}}(x, y)\) is the Green’s function for the half space with homogeneous impedance boundary conditions. Imposing the Neumann conditions on \(\Gamma\) yields an integral equation of the form
\[
-\frac{1}{2} \sigma(x) + \int_{\Gamma} \frac{\partial G_{\text{imp}}}{\partial n}(x, y) \sigma(y) \, dy = -\frac{\partial G_{\text{imp}}}{\partial n}(x, S)
\]
for \(x \in \Gamma\), where \(\frac{\partial}{\partial n}\) is the outward normal derivative from the obstacle and the integral is interpreted in the principal value sense. This integral equation is a Fredholm equation of the second kind, although it has spurious resonances for a discrete set of frequencies \(k\). More robust representations are well-known (see \([3]\)). In this short note, we restrict our attention to the calculation of the impedance Green’s function \(G_{\text{imp}}(x, y)\) itself.

Algorithms for the computation of \(G_{\text{imp}}(x, y)\) date back to the classical work of Sommerfeld, Weyl, and Van der Pol \([10, 13, 12]\), who developed both what is now referred to as the Sommerfeld integral and what is now referred to as the method of complex images. For a more recent treatment of this problem, see \([7, 4, 5, 9, 11]\).

The main contribution of the present work is the observation that there is an overlooked solution to the impedance problem using a hybrid representation involving a finite number of real images which guarantee rapid convergence of a residual Sommerfeld-type integral (sections \([4]\) and \([5]\)). Our approach is somewhat related to that of Cai and Yu \([1]\), which also separates near and far field contributions, but uses an asymptotic method for the near field.

2 Spectral representation of the Green’s function

For \(k \in \mathbb{C}\) with non-negative imaginary part, the solution to
\[
(\Delta + k^2)G(x) = -\delta(x_0),
\] (2.1)
in a homogeneous medium is referred to as the free-space Green’s function for the Helmholtz equation, where \( \delta(x_0) \) represents the usual delta function at \( x_0 \). It is well known that the Green’s functions are

\[
G(x, x_0) = \begin{cases} 
-\frac{i}{4} H_0(k|x - x_0|) & \text{in 2D,} \\
\frac{e^{ik|x-x_0|}}{4\pi|x-x_0|} & \text{in 3D.}
\end{cases}
\]

A continuous spectral representation of the Green’s functions can be obtained by taking the Fourier transform of equation (2.1). In three dimensions, the Green’s function can be written as

\[
G(x, x_0) = \frac{1}{8\pi^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} -\frac{e^{i(k_0(x-x_0)+\lambda_0(y-y_0)+\lambda_0(z-z_0))}}{\lambda_0^2 + \lambda_0^2 + \lambda_0^2 - k^2} \, d\lambda_x \, d\lambda_y \, d\lambda_z. \tag{2.2}
\]

Switching to polar coordinates and evaluating the integral in \( \lambda_2 \) via contour deformation yields the well-known plane-wave formula (which itself is sometimes called the Sommerfeld integral):

\[
G(x, x_0) = \frac{1}{4\pi} \int_0^\infty \frac{e^{-\sqrt{\rho^2-k^2}|z-z_0|}}{\sqrt{\rho^2-k^2}} J_0(\rho r) \, \rho \, d\rho, \tag{2.3}
\]

where \( r = \sqrt{(x-x_0)^2 + (y-y_0)^2} \).

Due to the central role of this formula in scattering theory, there has been much energy devoted to its numerical integration. We do not give a comprehensive review of the various schemes available, which are largely based on contour deformation, because they are mainly concerned with the square-root singularity in the denominator. Our method will address the range of integration required in the Fourier integral parameter \( \rho \).

### 3 The impedance problem

Impedance (or Robin) boundary conditions are used in acoustic scattering to describe surfaces which are neither sound-hard (Neumann) or sound-soft (Dirichlet). They are typically derived from the limiting behavior of a high-contrast layered medium (where the sound speed is vastly different in the lower half-space).

If we represent the scattered response to a single point source located at \( x_0 \) in a form similar to equation (2.3), then it suffices to match Fourier modes between the scattered and incoming solution using equation (1.3). We analyze only the 3D case in detail. For this, let the incoming field be given by a point source at \( x_0 = (x_0, y_0, z_0) \) located in the upper half-space. The incoming field below \( x_0 \) can be written as

\[
u^{\text{in}}(x) = G(x, x_0) = \frac{1}{4\pi} \int_0^\infty \frac{e^{i\sqrt{\rho^2-k^2}|z-z_0|}}{\sqrt{\rho^2-k^2}} J_0(\rho r) \, \rho \, d\rho, \tag{3.1}
\]

where \( r \) is given by (2.3). If we assume a similar spectral representation for the scattered field \( u \) with unknown density \( \sigma \),

\[
u(x) = \int_0^\infty \frac{e^{-i\sqrt{\rho^2-k^2}|z-z_0|}}{\sqrt{\rho^2-k^2}} \delta(\rho) J_0(\rho r) \, \rho \, d\rho, \tag{3.2}
\]

then on the surface \( z = 0 \) the impedance boundary condition (1.3) turns into a simple condition on \( \delta(\rho) \) for each \( \rho \):

\[
\frac{1}{4\pi} e^{-\sqrt{\rho^2-k^2}z_0} - \delta(\rho) - i\alpha \left( \frac{1}{4\pi} e^{-\sqrt{\rho^2-k^2}z_0} + \delta(\rho) \sqrt{\rho^2-k^2} \right) = 0. \tag{3.3}
\]

Solving for the density \( \sigma \), we have

\[
\sigma(\rho) = \frac{1}{4\pi} e^{-\sqrt{\rho^2-k^2}z_0} \left( \frac{\sqrt{\rho^2-k^2} - i\alpha}{\sqrt{\rho^2-k^2} + i\alpha} \right), \tag{3.4}
\]
Therefore, the scattered field $u$ can be evaluated via

$$ u(x) = \frac{1}{4\pi} \int_0^\infty \frac{e^{-\sqrt{\rho^2-k^2}(z+z_0)}}{\sqrt{\rho^2-k^2}} \left( \frac{\sqrt{\rho^2-k^2} - i\alpha}{\sqrt{\rho^2-k^2} + i\alpha} \right) J_0(\rho r) \rho d\rho. $$

Representation 3.5, also called the Sommerfeld integral, can be used to efficiently evaluate the scattered field if either $z$ or $z_0$ are $O(1)$, since the integrand is exponentially decaying. However, if $z+z_0 \sim O(h) \ll 1$, then the size of the integration interval must be chose to be $O(h^{-1})$, which can be unreasonably large when, for example, a scatterer is near the interface. Figure 2 shows the rate of decay of the density $\hat{\sigma}$ for varying values of $z_0$ evaluated at $z = 0$.

Since straightforward use of the Sommerfeld integral is clearly not robust for sources and targets near the half-space boundary, there have been many attempts at developing more efficient schemes. For example, Cai and Yu added a mollifier to the Sommerfeld representation and expanded the remaining high frequency components asymptotically \[1]. More common is the use of the method of complex images, which we turn to next.

4 The method of images

Using image sources or charges to impose a given homogeneous boundary condition is a well-known technique in classical mathematical physics and potential theory (see \[8] for a thorough discussion). When solving the half-space problem with homogeneous Dirichlet boundary conditions, the response to a point source located at $(x_0, y_0, z_0)$ is exactly the field generated by a point source of equal and opposite strength located at $(x_0, y_0, -z_0)$. Similarly, for the homogeneous Neumann problem, the response to a point source located at $(x_0, y_0, z_0)$ is exactly the field generated by a point source of equal strength located at $(x_0, y_0, -z_0)$. Unfortunately, there is no single image source that can be used in the case of impedance boundary conditions. However, it is possible to develop an explicit representation of the impedance Green’s function using an infinite ray of images, starting at the reflection of the source point $(x_0, y_0, -z_0)$ and continuing vertically down (see \[11] for a historical overview).

Under this assumption, the scattered field $u$ takes the form:

$$ u(x) = \int_0^\infty G(x, x_0 - 2z_0\hat{z} - \xi\hat{z})\eta(\xi)d\xi, $$

Figure 2: For $k = 1$ and $\alpha = 2$, the absolute value of the density $\hat{\sigma}$ is shown for various values of $z_0$.\[2]
where \( \hat{z} = (0, 0, 1) \) is the unit normal vector in the \( z \) direction. Using the Sommerfeld representation for the Green’s function \( G \), we then write the scattered field \( u \) as

\[
u(x) = \frac{1}{4\pi} \int_0^\infty \int_0^\infty \frac{e^{-\sqrt{\rho^2 - k^2}(z + z_0 + \xi)}}{\sqrt{\rho^2 - k^2}} J_0(\rho r) \eta(\xi) \rho d\rho d\xi. \tag{4.2}\]

In order for this representation for \( u \) to satisfy the impedance boundary conditions at \( z = 0 \), it is necessary that the density \( \eta \) satisfy the impedance boundary condition for all \( \rho \),

\[
e^{-\sqrt{\rho^2 - k^2}z_0} + \int_0^\infty \frac{e^{-\sqrt{\rho^2 - k^2}(z_0 + \xi)}}{\sqrt{\rho^2 - k^2}} \eta(\xi) d\xi + \imath \alpha e^{-\sqrt{\rho^2 - k^2}z_0} - \imath \alpha \int_0^\infty e^{-\sqrt{\rho^2 - k^2}(z_0 + \xi)} \eta(\xi) d\xi = 0, \tag{4.3}\]

which, after some algebra, reduces to a condition on the Laplace transform of \( \eta \),

\[
\int_0^\infty e^{-\sqrt{\rho^2 - k^2}z_0} \eta(\xi) d\xi = \frac{\sqrt{\rho^2 - k^2} - \imath \alpha}{\sqrt{\rho^2 - k^2} + \imath \alpha} = 1 - 2 \imath \alpha \frac{1}{\sqrt{\rho^2 - k^2} + \imath \alpha}. \tag{4.4}\]

An alternative method for deriving the previous formula is to equate the representation (4.2) with the Sommerfeld response formula (3.5). In any case, all we now require is the two well-known Laplace transform identities:

\[
\mathcal{L}[e^{-\beta t}](s) = \frac{1}{s + \beta}, \tag{4.5}\]

\[
\mathcal{L}[\delta(t - t_0)](s) = e^{-st},
\]

where

\[
\mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t) dt.
\]

Using these formulas, we solve equation (4.4) for the density \( \eta \), which is given by

\[
\eta(\xi) = \delta(\xi) - 2 \imath \alpha e^{-\imath \alpha \xi}. \tag{4.6}\]

The full image formula for the scattered field \( u \) is then

\[
u(x) = G(x, x_0 - 2z_0 \hat{z}) - 2 \imath \alpha \int_0^\infty G(x, x_0 - 2z_0 \hat{z} - \xi \hat{z}) e^{-\imath \alpha \xi} d\xi. \tag{4.7}\]

Note that, if \( \alpha \) is real, the density \( \eta(\xi) \) oscillates throughout its range, so that the decay in the integrand comes only from the \( 1/\xi \) decay in the Green’s function \( G \). The standard solution to performing numerical integration over the infinite interval is to complexify the coordinate \( \xi \) (see [11,12]). This is usually done by simply replacing \( \xi \) with \( -i\xi \) in formula (4.1), yielding

\[
u(x) = G(x, x_0 - 2z_0 \hat{z}) - 2 \imath \alpha \int_0^\infty G(x, x_0 - 2z_0 \hat{z} + i\xi \hat{z}) e^{-\alpha \xi} d\xi. \tag{4.8}\]

Instead of an oscillatory density, the complex image contour has enforced exponential decay. The difficulty with the formula is that the behavior of the integrand is rather complicated since it involves evaluating the Green’s function at a complex argument as \( \xi \) varies. Figure 3 illustrates the norm of the real part of the Green’s function at two distinct target locations with the same \( z \) value.

### 5 A hybrid approach

Rather than using either the Sommerfeld integral or the complex image formula, it is worth reconsidering the real image formula, separating it into two parts, a near-field component and a far-field component:

\[
u(x) = G(x, x_0 - 2z_0 \hat{z}) - 2 \imath \alpha \left( \int_0^C G(x, x_0 - 2z_0 \hat{z} - \xi \hat{z}) e^{-\imath \alpha \xi} d\xi + \int_C^\infty G(x, x_0 - 2z_0 \hat{z} - \xi \hat{z}) e^{-\imath \alpha \xi} d\xi \right), \tag{5.1}\]
where $C$ is a parameter of our choosing. From equations (3.5) and (5.1), it is straightforward to show that
\[
\int_C G(x, x_0 - 2z_0 \hat{z} - \xi \hat{z}) e^{-i\alpha \xi} d\xi = \frac{1}{4\pi} \int_0^\infty \frac{e^{-\sqrt{\rho^2-k^2}(z+z_0)} e^{-\sqrt{\rho^2-k^2+i\alpha}C}}{\sqrt{\rho^2-k^2}} J_0(\rho r) d\rho. \tag{5.2}
\]
This is a Sommerfeld-like formula, which for $C \sim O(1)$, decays exponentially fast once $\rho \gtrsim |k|$, independent of $z$ and $z_0$. Therefore, the full impedance Green’s function can be written as
\[
G_{imp}(x, x_0) = G(x, x_0) + G(x, x_0 - 2z_0 \hat{z} - \xi \hat{z}) e^{-i\alpha \xi} d\xi
- \frac{i\alpha}{2\pi} \int_0^\infty \frac{e^{-\sqrt{\rho^2-k^2}(y+y_0)} e^{-\sqrt{\rho^2-k^2+i\alpha}C}}{\sqrt{\rho^2-k^2}} d\lambda, \tag{5.3}
\]
In Fig. 4 we plot the integrand on the right-hand side of (5.2) for various values of $z_0$ at the interface $z = 0$.

6 Conclusions

We have derived a new formula for the half-space Helmholtz Green’s function with impedance boundary conditions in three dimensions. The representation (5.3) consists of a short segment of real images and a rapidly converging Sommerfeld integral that accounts for the far field. Unlike the method of complex images, it permits straightforward use of the principle of superposition. The impedance Green’s function is easily evaluated to full double precision accuracy, independent of the location of the source and target, using only $O(k + \alpha)$ operations and suitable quadratures.

Although we provided a detailed derivation only for the three dimensional case, the results are nearly identical in two dimensions. In two dimensions, repeating the analysis above yields:
\[
G_{imp}(x, x_0) = G(x, x_0) + G(x, x_0 - 2y_0 \hat{y} - \xi \hat{y}) e^{-i\alpha \xi} d\xi
- \frac{i\alpha}{2\pi} \int_{-\infty}^\infty \frac{e^{-\sqrt{\lambda^2-k^2}(y+y_0)} e^{-\sqrt{\lambda^2-k^2+i\alpha}C}}{\sqrt{\lambda^2-k^2}} d\lambda. \tag{6.1}
\]
where the free space Green’s function is now $G(x, x_0) = -\frac{i}{4} H_0(k|\mathbf{x} - \mathbf{x}_0|)$.

An obvious extension of our approach is to the evaluation of the layered medium Green’s function, where each layer has a distinct Helmholtz coefficient and continuity conditions are imposed across the layers. Unfortunately, unlike in the impedance case, an explicit image structure is not available. We are currently investigating a semi-numerical method that appears promising. Analogous problems appear in electromagnetic scattering, and we expect that the hybrid approach discussed here will apply with minor modifications.

Finally, the reader may have noted that formulas (5.1) and (5.3) make use of a discrete image at $(x_0, y_0, -z_0)$. As $\alpha \to 0$, the impedance condition becomes a sound-hard condition and the discrete image is all that remains, as one would expect. As $|\alpha| \to \infty$, the limit of the impedance condition is a Dirichlet condition, while the image formula diverges. It should yield a simple image source at $(x_0, y_0, -z_0)$ with the opposite sign. A complementary image formula can be obtained with the leading discrete image having a negative sign by manipulating the Laplace transform. Rather than (4.4), we could write

$$
\int_0^\infty e^{-\sqrt{\rho^2-k^2}\xi} \eta(\xi) d\xi = \frac{\sqrt{\rho^2-k^2} - i\alpha}{\sqrt{\rho^2-k^2} + i\alpha} = -1 + \frac{2\sqrt{\rho^2-k^2}}{\sqrt{\rho^2-k^2} + i\alpha},
$$

and derive a different image formula. In applications $\alpha$ is typically $O(1)$, so we prefer the formula here.

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