Difference of convex algorithms for bilevel programs with applications in hyperparameter selection

Jane J. Ye 1 · Xiaoming Yuan 2 · Shangzhi Zeng 1 · Jin Zhang 3

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Abstract
In this paper, we present difference of convex algorithms for solving bilevel programs in which the upper level objective functions are difference of convex functions, and the lower level programs are fully convex. This nontrivial class of bilevel programs provides a powerful modelling framework for dealing with applications arising from hyperparameter selection in machine learning. Thanks to the full convexity of the lower level program, the value function of the lower level program turns out to be convex and hence the bilevel program can be reformulated as a difference of convex bilevel program. We propose two algorithms for solving the reformulated difference of convex program and show their convergence to stationary points under very mild assumptions. Finally we conduct numerical experiments to a bilevel model of support vector machine classification.

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Jin Zhang
zhangj9@sustech.edu.cn
Jane J. Ye
janeye@uvic.ca
Xiaoming Yuan
xmyuan@hku.hk
Shangzhi Zeng
zengshangzhi@uvic.ca

1 Department of Mathematics and Statistics, University of Victoria, Victoria, Canada
2 Department of Mathematics, The University of Hong Kong, Hong Kong SAR, China
3 Department of Mathematics, SUSTech International Center for Mathematics, Southern University of Science and Technology, National Center for Applied Mathematics Shenzhen, Peng Cheng Laboratory, Shenzhen, China
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1 Introduction

Bilevel programs are a class of hierarchical optimization problems which have constraints containing a lower-level optimization problem parameterized by upper-level variables. Bilevel programs capture a wide range of important applications in various fields including Stackelberg games and moral hazard problems in economics ([29, 41]), hyperparameter selection and meta learning in machine learning ([16, 21–23, 26, 27, 30, 31, 34]). More applications can be found in the monographs [3, 12, 15, 40], the survey on bilevel optimization [11, 14] and the references within.

In this paper, we develop some numerical algorithms for solving the following difference of convex (DC) bilevel program:

\[
\begin{align*}
\text{(DCBP)} \\
\min_{x \in \mathbb{R}^n, y \in \mathbb{R}^m} & \quad F(x, y) := F_1(x, y) - F_2(x, y) \\
\text{s.t.} & \quad x \in X, y \in S(x),
\end{align*}
\]

with \(S(x)\) being the set of optimal solutions of the lower level problem,

\[
\begin{align*}
\text{(P_x)} : \quad \min_{y \in Y} & \quad f(x, y) \\
\text{s.t.} & \quad g(x, y) \leq 0,
\end{align*}
\]

where \(X \subseteq \mathbb{R}^n\) and \(Y \subseteq \mathbb{R}^m\) are nonempty closed sets, \(g := (g_1, \ldots, g_l)\), all functions \(g_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, i = 1, \ldots, l\) are convex on an open convex set containing the set \(X \times Y\), and the functions \(F_1, F_2, f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}\) are convex on an open convex set containing the set

\[
C := \{(x, y) \in X \times Y : g(x, y) \leq 0\}.
\]

To ensure the bilevel program is well-defined, we assume that \(S(x) \neq \emptyset\) for all \(x \in X\). Moreover we assume that for all \(x\) in an open convex set \(O \supseteq X\), the feasible region for the lower level program \(\mathcal{F}(x) := \{y \in Y : g(x, y) \leq 0\}\) is nonempty and the lower level objective function \(f(x, y)\) is bounded below on \(\mathcal{F}(x)\).

Although the objective function in the DC bilevel program we consider must be a DC function, this setting is general enough to capture many cases of practical interests. In particular any lower \(C^2\) function (i.e., a function which can be locally written as a supremum of a family of \(C^2\) functions) and \(C^{1+}\) function (i.e., a differentiable function whose gradient is locally Lipschitz continuous) are DC functions and the class of DC functions is closed under many operations encountered frequently in optimization; see, e.g., [19, 44]. In the lower level program, we assume all functions
are fully convex, i.e., convex in both variables $x$ and $y$. However as pointed out by [24, Example 1 and Section 5], using some suitable reformulations one may turn a non-fully convex lower level program into a fully convex one. Also as demonstrated in this paper, the bilevel model for hyperparameter selection problem can be reformulated as a bilevel program where the lower level is fully convex.

Solving bilevel programs numerically is extremely challenging. It is known that even when all defining functions are linear, the computational complexity is already NP-hard [5]. If all defining functions are smooth and the lower level program is convex with respect to the lower level variable, the first order approach was popularly used to replace the lower level problem by its first order optimality condition and to solve the resulting problem as the mathematical program with equilibrium constraints (MPEC); see e.g. [1, 3, 14, 28, 36]. The first order approach may be problematic since it may not provide an equivalent reformulation to the original bilevel program if only local (not global) optimal solutions are considered; see [13]. Moreover even in the case of a fully convex lower level program, [24, Example 1] shows that it is still possible that a local optimal solution of the corresponding MPEC does not correspond to a local optimal solution of the original bilevel program. Recently some numerical algorithms have been introduced for solving bilevel programs where the lower level program is not necessarily convex in the lower level variable; see e.g., [25, 32, 33]. However these approaches have limitations in the numbers of variables in the bilevel program. In most of literature on numerical algorithms for solving bilevel programs, smoothness of all defining functions are assumed. In some special cases, non-smoothness can be dealt with by introducing auxiliary variables and constraints to reformulate a nonsmooth lower level program as a smooth constrained lower level program. But using such an approach the numbers of variables or constraints would increase.

Our research on the DC bilevel program is motivated by a number of important applications in model selection and hyperparameter learning. Recently in the statistical learning, the regularization parameters has been successfully used, e.g., in the least absolute shrinkage and selection operator (lasso) method for regression and support vector machines (SVMs) for classification. However the regularization parameters have to be set a priori and the choice of these parameters dramatically affects results on the model selection. The most commonly used method for selecting these parameters is the so-called $T$-fold cross validation. By $T$-fold cross validation, a data set $\Omega$ is randomly partitioned into $T$ pairwise disjoint subsets called the validation sets $\Omega_{val}^t$, $t = 1, \ldots, T$. For each fold $t = 1, \ldots, T$, a subset of $\Omega$ denoted by $\Omega_{trn}^t := \Omega \setminus \Omega_{val}^t$ is used for training and the validation set $\Omega_{val}^t$ is used for testing the result. Take the SVM problem for example, suppose the data set $\Omega = \{(a_j, b_j)\}_{j=1}^\ell$ where $a_j \in \mathbb{R}^n$, and the labels $b_j = \pm 1$ indicate the class membership. For each hyperparameters $\lambda > 0$, $\bar{w}$ and each fold $t = 1, \ldots, T$, the following SVM problem can be solved.

\[
(P_{\lambda, \bar{w}}^t) \quad \min_{\lambda \leq \bar{w} \leq \bar{w}} \left\{ \frac{\lambda}{2} \|\bar{w}\|^2 + \sum_{j \in \Omega_{trn}^t} \max(1 - b_j(a_j^T \bar{w} - c), 0) \right\}.
\]

The desirable hyperparameters $\lambda^*$ and $\bar{w}^*$ can be selected by minimizing some measure of validation accuracy over all folds.
such as

$$\Theta(\mathbf{w}^1, \ldots, \mathbf{w}^T, \mathbf{c}) := \frac{1}{T} \sum_{t=1}^{T} \frac{1}{\Omega_{\text{val}}^t} \sum_{j \in \Omega_{\text{val}}^t} \max(1 - b_j (\mathbf{a}_j^T \mathbf{w}^t_{\lambda, \bar{w}} - c_j^t, \bar{w}), 0),$$

where $|M|$ denotes the number of elements in set $M$ and $(\mathbf{w}^t_{\lambda, \bar{w}}, c_j^t, \bar{w})$ denotes a solution to the SVM problem $(P^t_{\lambda, \bar{w}})$. Here the cross validation error is based on the hinge loss function. Other possible functions that can be used for cross validation error can be found in [6, 21, 22]. In fact, the hyperparameter selection for SVM has been proposed as the following bilevel program with $T$ lower level programs by [21, 22]:

$$\begin{align*}
\min_{\lambda, \bar{w}, \mathbf{w}^1, \ldots, \mathbf{w}^T, \mathbf{c}} & \quad \Theta(\mathbf{w}^1, \ldots, \mathbf{w}^T, \mathbf{c}) \\
\text{s.t.} & \quad \lambda_{lb} \leq \lambda \leq \lambda_{ub}, \quad \bar{w}_{lb} \leq \bar{w} \leq \bar{w}_{ub}, \\
& \quad \text{for } t = 1, \ldots, T:
\end{align*}
$$

$$\begin{align*}
(\mathbf{w}^t, c_j^t) & \in \arg\min_{-\bar{w} \leq \mathbf{w} \leq \bar{w}} \{ \lambda \frac{1}{2} \|\mathbf{w}\|^2 + \sum_{j \in \Omega_{\text{trn}}^t} \max(1 - b_j (\mathbf{a}_j^T \mathbf{w} - c), 0) \},
\end{align*}$$

where $\mathbf{c} \in \mathbb{R}^T$ is the vector with $c_j^t$ as the $t$th component. Here $\lambda_{lb}, \lambda_{ub}$ are given positive numbers and $\bar{w}_{lb}, \bar{w}_{ub}$ are given vectors in $\mathbb{R}^n$. It is easy to see that by changing the variable $\lambda$ to $\mu := \frac{1}{\lambda}$ we can reformulate the above SV bilevel model selection equivalently as the following bilevel program with a single lower level program

$$\begin{align*}
\min_{\mu, \bar{w}, \mathbf{w}^1, \ldots, \mathbf{w}^T, \mathbf{c}} & \quad \Theta(\mathbf{w}^1, \ldots, \mathbf{w}^T, \mathbf{c}) \\
\text{s.t.} & \quad \lambda_{lb} \leq \mu \leq \lambda_{ub}, \quad \bar{w}_{lb} \leq \bar{w} \leq \bar{w}_{ub}, \\
& \quad (\mathbf{w}^1, \ldots, \mathbf{w}^T, \mathbf{c}) \in S(\mu, \bar{w}),
\end{align*}$$

( SVBP)

where $S(\mu, \bar{w})$ is the set of optimal solutions of the lower level problem

$$\begin{align*}
\min_{\mu, \bar{w}, \mathbf{w}^1, \ldots, \mathbf{w}^T, \mathbf{c}} & \quad \Theta(\mathbf{w}^1, \ldots, \mathbf{w}^T, \mathbf{c}) \\
\text{s.t.} & \quad \lambda_{lb} \leq \mu \leq \lambda_{ub}, \quad \bar{w}_{lb} \leq \bar{w} \leq \bar{w}_{ub}, \\
& \quad (\mathbf{w}^1, \ldots, \mathbf{w}^T, \mathbf{c}) \in S(\mu, \bar{w}),
\end{align*}$$

Moreover using the fact that a function in the form $\phi(x, \mu) = \|x\|^2 / \mu$ with $\mu > 0$ is convex as a perspective function [42, Example 3.18], the above bilevel program has a fully convex lower level program and all required assumptions hold; see the details in Sect. 4. The classical $T$-fold cross validation method for selecting hyperparameters usually implements a grid search: training $T$ models at each point of a discretized parameter space in order to find an approximate optimal parameter. This method has
many drawbacks and limitations. In particular its computational complexity scales exponentially with the number of hyperparameters and the number of grid points for each hyperparameter. Hence the grid search method is not practical for problem requiring several hyperparameters, including our SV bilevel model selection where the hyperparameters \((\mu, \bar{w})\) are \(n + 1\)imensional. To deal with limitations of grid search, introduced first in [6] in 2006, the bilevel program has been used to model hyperparameter selection problems in [6, 21–23, 30, 31].

The above fully convex transformation using the perspective function can be applied to some other model hyperparameter selection problems, for example, the \(T\)-fold cross validation Lasso problem.

This paper is motivated by an interesting fact that, under our problem setting, the value function of the lower level in (DCBP) defined by

\[
  v(x) := \inf_{y \in Y} \{ f(x, y) \text{ s.t. } g(x, y) \leq 0 \}
\]

is convex and locally Lipschitz continuous on \(X\). We take full advantage of this convexity and use the value function approach first proposed in [35] for a numerical purpose and further used to study optimality conditions in [48] to reformulate (DCBP) as the following DC program:

\[
  \begin{align*}
    (VP) \quad \min_{(x, y) \in C} & \quad F_1(x, y) - F_2(x, y) \\
    \text{s.t.} & \quad f(x, y) - v(x) \leq 0.
  \end{align*}
\]

Unfortunately, due to the value function constraint, (VP) violates the usual constraint qualification such as the nonsmooth Mangasarian Fromovitz constraint qualification (MFCQ) at each feasible point, see [48, Proposition 3.2] for the smooth case and Proposition 7 for the nonsmooth case. It is well-known that convergence of the difference of convex algorithm (DCA) is only guaranteed under constraint qualifications such as the extended MFCQ, which is MFCQ extended to infeasible points; see, e.g., [43]. To deal with this issue, we consider the following approximate bilevel program

\[
  \begin{align*}
    (VP)_\epsilon \quad \min_{(x, y) \in C} & \quad F_1(x, y) - F_2(x, y) \\
    \text{s.t.} & \quad f(x, y) - v(x) \leq \epsilon,
  \end{align*}
\]

for some \(\epsilon > 0\). Such a relaxation strategy has been used for example in [25] based with the reasoning that in numerical algorithms one usually obtain an inexact optimal solution anyway and the solutions of (VP), approximate a solution of the original bilevel program (VP) as \(\epsilon\) approaches zero. In this paper we will show that EMFCQ holds for problem \((VP)_\epsilon\) when \(\epsilon > 0\) automatically. Hence we propose to solve problem \((VP)_\epsilon\) with \(\epsilon \geq 0\). When \(\epsilon > 0\), the convergence of our algorithm to stationary points is guaranteed and when \(\epsilon = 0\), the convergence is not guaranteed but it could still converge if the penalty parameter sequence is bounded.

Using DCA approach, at each iterate point \((x^k, y^k)\), one linearizes the concave part of the function, i.e., the functions \(F_2(x, y), v(x)\) by using an element of the subdifferentials.
ferentials $\partial F_2(x^k, y^k)$, $\partial v(x^k)$ and solve a resulting convex subproblem. The value function is an implicit function. How do we obtain an element of the subdifferential $\partial v(x^k)$? At current iterate $x^k$, assuming we can solve the lower level problem $(\mathcal{P}_{x^k})$ with a global minimizer $\tilde{y}^k$ and a corresponding Karush–Kuhn–Tucker (KKT) multiplier denoted by $\gamma^k$. Suppose that the following partial derivative formula holds:

$$\partial f(x, y) = \partial_x f(x, y) \times \partial_y f(x, y), \quad \partial g_i(x, y) = \partial_x g_i(x, y) \times \partial_y g_i(x, y) \quad (1)$$

at $(x, y) = (x^k, \tilde{y}^k)$. Then since by convex analysis

$$\partial_x f(x^k, \tilde{y}^k) + \sum_{i=1}^{l} \gamma_i^k \partial_x g_i(x^k, \tilde{y}^k) \subseteq \partial v(x^k),$$

we can select an element of $\partial v(x^k)$ from the set

$$\partial_x f(x^k, \tilde{y}^k) + \sum_{i=1}^{l} \gamma_i^k \partial_x g_i(x^k, \tilde{y}^k)$$

and use it to linearize the value function. We then solve the resulting convex subproblem approximately to obtain a new iterate $(x^{k+1}, y^{k+1})$. Thanks to recent developments in large-scale convex programming, using this approach we can deal with a large scale DC bilevel program.

Now we summarize our contributions as follows.

– We propose two new algorithms for solving DC program. These algorithms have modified the classical DCA in two ways. First, we add a proximal term in each convex subproblem so the the objective function is strongly convex and at each iterate point, only an approximate solution for the convex subproblem is solved. Second, our penalty parameter update is simpler.

– We have laid down all theoretical foundations from convex analysis that are required for our algorithms to work. In particular we have demonstrated that under the minimal assumptions that we specify for problem (DCBP), the value function is convex and locally Lipschitz on set $X$ automatically.

– Using the two new algorithms for solving DC program, we propose two corresponding algorithms to solve problem (DCBP). Our algorithms hold under very mild and natural assumptions. In particular we allow all defining functions to be nonsmooth and we do not require any constraint qualification to hold for the lower level program. The main assumptions we need are only the partial derivative formula (1) which holds under many practical situations (see Proposition 1 for sufficient conditions) and the existence of a KKT multiplier for the lower level program under each iteration.

– Taking advantage of large scale convex programming, our algorithm can handle high dimensional hyperparameter selection problems. To test effectiveness of our algorithm, we have tested it in the SV bilevel model selection (SVBP). Our results compare favourably with the MPEC approach [21–23].
This paper is organized as follows. In Sect. 2 we propose two modified DCAs and study their convergence to stationary points for a class of general DC programs. In Sect. 3, we derive explicit conditions for the bilevel program under which the algorithms introduced in Sect. 3 can be applied. Numerical experiments on the SV bilevel model selection is conducted on Sect. 4. Section 5 concludes the paper.

2 Modified inexact proximal DC algorithms

In order to solve the (relaxed) value function reformulation of problem DCBP, in this section we propose numerical algorithms to solve the following difference of convex program:

\[(DC) \quad \min_{z \in \Sigma} f_0(z) := g_0(z) - h_0(z) \text{ s.t. } f_1(z) := g_1(z) - h_1(z) \leq 0,\]

where \(\Sigma\) is a closed convex subset of \(\mathbb{R}^d\) and \(g_0(z), h_0(z), g_1(z), h_1(z) : \Sigma \rightarrow \mathbb{R}\) are convex functions. Although the results in this section can be generalized to the case where there are more than one inequality in a straightforward manner, to simplify the notation and concentrate on the main idea we assume there is only one inequality constraint in problem (DC). Our algorithms are modifications of the classical DCA (see [43]). Recently, [37] studied problem (DC) where \(h_1\) is a maximum of finitely many smooth convex functions and proposed an algorithm for finding B-stationary points of it.

Before introducing our algorithms and conduct the convergence analysis, we recall some notations from convex analysis and variational analysis. Let \(\varphi(x) : \mathbb{R}^n \rightarrow [\pm \infty]\) be a convex function, and let \(\bar{x}\) be a point where \(\varphi\) be finite. The subdifferential of \(\varphi\) at \(\bar{x}\) is a closed convex set defined by

\[\partial \varphi(\bar{x}) := \{ \xi \in \mathbb{R}^n | \varphi(x) \geq \varphi(\bar{x}) + \langle \xi, x - \bar{x} \rangle, \forall x \},\]

and a subgradient is an element of the subdifferential. For a function \(\varphi : \mathbb{R}^n \times \mathbb{R}^m \rightarrow [\pm \infty]\), we denote the partial subdifferential of \(\varphi\) with respect to \(x\) and \(y\) by \(\partial_x \varphi(x, y)\) and \(\partial_y \varphi(x, y)\) respectively. Let \(\Sigma\) be a convex subset in \(\mathbb{R}^n\) and \(\bar{x} \in \Sigma\). The normal cone to \(\Sigma\) at \(\bar{x}\) is denoted by \(N_{\Sigma}(\bar{x})\). Let \(\delta_{\Sigma}(x)\) denote the indicator function of set \(\Sigma\) at \(x\). The following partial subdifferentiation rule will be useful.

**Proposition 1** (Partial subdifferentiation) Let \(\varphi : \mathbb{R}^n \times \mathbb{R}^m \rightarrow [\pm \infty]\) be a convex function and let \((\bar{x}, \bar{y})\) be a point where \(\varphi\) be finite. Then

\[\partial \varphi(\bar{x}, \bar{y}) \subseteq \partial_x \varphi(\bar{x}, \bar{y}) \times \partial_y \varphi(\bar{x}, \bar{y}).\]  

(2)

The inclusion (2) becomes an equality under one of the following conditions.

(a) For every \(\xi \in \partial_x \varphi(\bar{x}, \bar{y})\), it holds that \(\varphi(x, y) - \varphi(\bar{x}, \bar{y}) \geq \langle \xi, x - \bar{x} \rangle, \forall (x, y) \in \mathbb{R}^n \times \mathbb{R}^m\).
(b) $\varphi(x, y) = \varphi_1(x) + \varphi_2(y)$.

(c) For any $\epsilon > 0$, there is $\delta > 0$ such that

\[
either \quad \partial_x \varphi(\bar{x}, \bar{y}) \subseteq \partial_x \varphi(x, y) + \epsilon \mathbb{B}_{\mathbb{R}^n} \quad \forall y \in B(\bar{y}; \delta) \tag{3}
either \quad \partial_y \varphi(\bar{x}, \bar{y}) \subseteq \partial_y \varphi(x, y) + \epsilon \mathbb{B}_{\mathbb{R}^n} \quad \forall x \in B(\bar{x}; \delta), \tag{4}
\]

where $B(\bar{x}; \delta)$ denotes the open ball centered at $\bar{x}$ with radius equal to $\delta$ and $\mathbb{B}_{\mathbb{R}^n}$ denotes the open unit ball centered at the origin in $\mathbb{R}^n$.

(d) $\varphi(x, y)$ is continuously differentiable respect to one of the variables $x$ or $y$ at $(\bar{x}, \bar{y})$.

Moreover (b) $\implies$ (a), (d) $\implies$ (c) $\implies$ (a).

**Proof** The inclusion (2) and its reverse under (a) follow directly from definitions of the convex subdifferential and the partial subdifferential.

When $\varphi(x, y) = \varphi_1(x) + \varphi_2(y)$, we have that $\partial \varphi(x, y) = \partial \varphi(x) \times \{0\} + \{0\} \times \partial \varphi(y)$. Hence obviously (b) implies (a). The implication of (d) to (c) is obvious.

Now suppose that (3) holds. Let $\xi \in \partial_x \varphi(\bar{x}, \bar{y})$. Then according to (3), for any $\epsilon > 0$, there is $\delta > 0$ such that $\xi = \eta + \epsilon e$, where $e \in \mathbb{B}_{\mathbb{R}^n}$, and

\[
\langle \xi, x - \bar{x} \rangle \leq \varphi(x, y) - \varphi(\bar{x}, \bar{y}) + \epsilon \|x - \bar{x}\| \quad \forall y \in B(\bar{y}; \delta).
\]

Thanks to the convexity of $\varphi$, using the proof technique of [10, Corollary 2.6 (c)], we can easily show that (a) holds. The proof for the case where (4) holds is similar and thus omitted. $\square$

Next, we first brief some solution quality characterizations for problem (DC).

**Definition 1** Let $\bar{z}$ be a feasible solution of problem (DC). We say that $\bar{z}$ is a stationary/KKT point of problem (DC) if there exists a multiplier $\lambda \geq 0$ such that

\[
0 \in \partial g_0(\bar{z}) - \partial h_0(\bar{z}) + \lambda (\partial g_1(\bar{z}) - \partial h_1(\bar{z})) + N_\Sigma(\bar{z}),
\]

\[
(g_1(\bar{z}) - h_1(\bar{z}))\lambda = 0.
\]

**Definition 2** Let $\bar{z}$ be a feasible point of problem (DC). We say that the nonzero abnormal multiplier constraint qualification (NNAMCQ) holds at $\bar{z}$ for problem (DC) if either $f_1(\bar{z}) < 0$ or $f_1(\bar{z}) = 0$ but

\[
0 \notin \partial g_1(\bar{z}) - \partial h_1(\bar{z}) + N_\Sigma(\bar{z}). \tag{5}
\]

Let $\bar{z} \in \Sigma$, we say that the extended no nonzero abnormal multiplier constraint qualification (ENNAMCQ) holds at $\bar{z}$ for problem (DC) if either $f_1(\bar{z}) < 0$ or $f_1(\bar{z}) \geq 0$ but (5) holds.

Note that NNAMCQ (ENNAMCQ) is equivalent to MFCQ (EMFCQ) respectively; see e.g., [20].
Denote by $\partial^c \varphi(x)$ the Clarke generalized gradient \cite{9} of a locally Lipschitz function $\varphi$ at $x$. The following optimality condition follows from the nonsmooth multiplier rule in terms of Clarke generalized gradients (see e.g. \cite{9,20}) and the fact that for two convex functions $g, h$ which are Lipschitz around point $\tilde{z}$, we have $\partial^c (g(\tilde{z}) - h(\tilde{z})) \subseteq \partial^c g(\tilde{z}) - \partial^c h(\tilde{z}) = \partial g(\tilde{z}) - \partial h(\tilde{z})$.

**Proposition 2** Let $\tilde{z}$ be a local solution of problem (DC). If NNAMCQ holds at $\tilde{z}$ and all functions $g_0, g_1, h_0, h_1$ are Lipschitz around point $\tilde{z}$, then $\tilde{z}$ is a KKT point of problem (DC).

### 2.1 Inexact proximal DCA with simplified penalty parameter update

In this subsection we propose an algorithm called inexact proximal DCA to solve problem (DC) and show its convergence to stationary points.

By using the main idea of DCA which linearizes the concave part of the DC structure, we propose a sequential convex programming scheme as follows. Given a current iterate $z^k \in \Sigma$ with $k = 0, 1, \ldots$, we select a subdifferential $\xi_i^k \in \partial h_i(z^k)$, for $i = 0, 1$. Then we solve the following subproblem approximately and select $z^{k+1}$ as an approximate minimizer:

$$
\min_{z \in \Sigma} \tilde{\varphi}_k(z) := g_0(z) - h_0(z^k) - \langle \xi_0^k, z - z^k \rangle + \beta_k \max \left\{ g_1(z) - h_1(z^k) - \langle \xi_1^k, z - z^k \rangle, 0 \right\} + \rho \|z - z^k\|^2, \quad (6)
$$

where $\rho$ is a given positive constant and $\beta_k$ represents the adaptive penalty parameter. Our scheme is similar to that of DCA2 in \cite{43} but different in that the subproblem (6) has a strongly convex objective function, the subproblem is only solved approximately, and a simpler penalty parameter update is used. We propose the following two inexact conditions for choosing $z^{k+1}$ as an approximate solution to (6):

$$
\text{dist}(0, \partial \tilde{\varphi}_k(z^{k+1}) + N_\Sigma(z^{k+1})) \leq \zeta_k, \quad \text{for some } \zeta_k \geq 0 \text{ satisfying } \sum_{k=0}^{\infty} \zeta_k^2 < \infty, \quad (7)
$$

and

$$
\text{dist}(0, \partial \tilde{\varphi}_k(z^{k+1}) + N_\Sigma(z^{k+1})) \leq \frac{\sqrt{2}}{2} \rho \|z^k - z^{k-1}\|, \quad (8)
$$

where $\text{dist}(x, M)$ denotes the distance from a point $x$ to set $M$.

Using above constructions, we are ready to propose the inexact proximal DCA (iP-DCA) in Algorithm 1.

In DCA2 of \cite{43}, the subproblem (6) was solved as a constrained optimization problem and a Lagrange multiplier is used to update the penalty parameter. Since our penalty parameter update rule does not involve any multipliers, it is easier to implement. In the rest of this section we show that the proposed algorithm converges.
Algorithm 1 iP-DCA

1: Take an initial point $z^0 \in \Sigma$; $\delta \beta > 0$; an initial penalty parameter $\beta_0 > 0$, $tol > 0$.
2: for $k = 0, 1, \ldots$ do
   1. Compute $\xi_i^k \in \partial h_i(z^k)$, $i = 0, 1$.
   2. Obtain an inexact solution $z^{k+1}$ of (6) satisfying (7) or (8).
   3. Stopping test. Compute $t_{k+1}^k := \max\{g_1(z^{k+1})\!-\!h_1(z^k)\!-\!(\xi_1^k \cdot z^{k+1} \!-\! z^k), 0\}$. Stop if $\max\{\|z^{k+1} \!-\! z^k\|, t_{k+1}^k\} < tol$.
   4. Penalty parameter update. Set $\beta_{k+1} = \begin{cases} \beta_k + \delta \beta, & \text{if } \max\{\beta_k, 1/t_{k+1}^k\} < \|z^{k+1} \!-\! z^k\|^{-1}, \\ \beta_k, & \text{otherwise.} \end{cases}$
   5. Set $k := k + 1$.
3: end for

Let us start with the following lemma which provides a sufficient decrease of the merit function of (DC) defined by

$$\varphi_k(z) := g_0(z) - h_0(z) + \beta_k \max\{g_1(z) - h_1(z), 0\}.$$  \(\varphi_k(z) := g_0(z) - h_0(z) + \beta_k \max\{g_1(z) - h_1(z), 0\}.\)

Lemma 1 Let \(\{z^k\}\) be a sequence of iterates generated by iP-DCA as defined in Algorithm 1. If the inexact criterion (7) or (8) is applied, then $z^k$ satisfies

$$\varphi_k(z^k) \geq \varphi_k(z^{k+1}) + \frac{\rho}{2} \|z^{k+1} - z^k\|^2 - \frac{1}{2\rho} \xi_k^2,$$

or $$\varphi_k(z^k) \geq \varphi_k(z^{k+1}) + \frac{\rho}{2} \|z^{k+1} - z^k\|^2 - \frac{\rho}{4} \|z^k - z^{k-1}\|^2,$$

where $\xi_k \geq 0$ satisfying $\sum_{k=0}^{\infty} \xi_k^2 < \infty$ respectively.

Proof Since $z^{k+1}$ is an approximation solution to problem (6) with inexact criterion (7) or (8), there exists a vector $e_k$ such that $e_k \in \partial \bar{\varphi}_k(z^{k+1}) + N_\Sigma(z^{k+1}) \subseteq \partial (\bar{\varphi}_k + \delta \Sigma)(z^{k+1})$ and

$$\|e_k\| \leq \xi_k \text{ or } \|e_k\| \leq \sqrt{2}\frac{\rho}{2} \|z^k - z^{k-1}\|, \quad (9)$$

respectively. As $\bar{\varphi}_k$ is strongly convex with modulus $\rho$, $\Sigma$ is a closed convex set and $z^k \in \Sigma$, we have

$$\bar{\varphi}_k(z^k) \geq \bar{\varphi}_k(z^{k+1}) + \langle e_k, z^{k+1} - z^k \rangle + \frac{\rho}{2} \|z^{k+1} - z^k\|^2$$

$$\geq \bar{\varphi}_k(z^{k+1}) - \frac{1}{2\rho} \|e_k\|^2 - \frac{\rho}{2} \|z^{k+1} - z^k\|^2 + \frac{\rho}{2} \|z^{k+1} - z^k\|^2$$

$$= \bar{\varphi}_k(z^{k+1}) - \frac{1}{2\rho} \|e_k\|^2. \quad (10)$$

\(\varphi_k(z) := g_0(z) - h_0(z) + \beta_k \max\{g_1(z) - h_1(z), 0\}.\)
Next, by the convexity of \( h_i(z) \) and \( \xi_i^k \in \partial h_i(z^k) \), \( i = 0, 1 \), there holds that
\[
h_i(z^{k+1}) \geq h_i(z^k) + \langle \xi_i^k, z^{k+1} - z^k \rangle, \quad i = 0, 1,
\]
and thus \( \varphi_k(z^{k+1}) \geq \varphi_k(z^k) + \frac{\rho}{2} \| z^{k+1} - z^k \|^2 \). Combined with (10), we have
\[
\varphi_k(z^k) = \varphi_k(z^k) \geq \varphi_k(z^{k+1}) - \frac{1}{2\rho} \| e_k \|^2 \geq \varphi_k(z^{k+1}) - \frac{1}{2\rho} \| z^{k+1} - z^k \|^2.
\]
The conclusion follows immediately from (9). \( \square \)

The following theorem is the main result of this section. It proves that any accumulation point of iP-DCA is a KKT point as long as the penalty parameter sequence \( \{\beta_k\} \) is bounded.

**Theorem 1** Suppose \( f_0 \) is bounded below on \( \Sigma \) and the sequences \( \{z^k\} \) and \( \{\beta_k\} \) generated by iP-DCA are bounded. Moreover suppose functions \( g_0, g_1, h_1, h_0 \) are locally Lipschitz on set \( \Sigma \). Then every accumulation point of \( \{z^k\} \) is a KKT point of problem (DC).

**Proof** Since \( \{\beta_k\} \) is bounded, there exists some iteration index \( k_0 \) such that \( \beta_k = \beta_{k_0}, \quad \forall k \geq k_0 \), and thus \( \varphi_k(z) = \varphi_{k_0}(z) \) for all \( k \geq k_0 \). As \( f_0 \) is bounded below, \( \varphi_k(z) \) is bounded below for all \( k \geq k_0 \). Then, by the inequality (9) and (9) obtained in Lemma 1, we have
\[
\sum_{k=1}^{\infty} \| z^{k+1} - z^k \|^2 < +\infty, \quad \lim_{k \to \infty} \| z^{k+1} - z^k \| = 0,
\]
and thus \( \beta_k < \| z^{k+1} - z^k \|^{-1} \) always holds when \( k \) is large enough. According to the update strategy of \( \beta_k \) in iP-DCA, there exists some iteration index \( k_1 \) such that
\[
t^{k+1} := \max\{g_1(z^{k+1}) - h_1(z^k) - \langle \xi^k, z^{k+1} - z^k \rangle, 0\} \leq \| z^{k+1} - z^k \| \quad \forall k \geq k_1,
\]
and thus \( t^k \to 0 \). Since \( z^{k+1} \) is an approximate solution to problem (6) and inexact criterion (7) or (8) holds, there exists a vector \( e_k \) such that \( e_k \in \partial \tilde{\varphi}_k(z^{k+1}) + \mathcal{N}_\Sigma(z^{k+1}) \) and (9) holds. According to the sum rule (see, e.g., [38, Theorem 23.8] [9, Corollary 1 to Theorem 2.9.8]) and the subdifferential calculus rules for the pointwise maximum (see, e.g., [9, Proposition 2.3.12]), there exist \( \tilde{\lambda}^{k+1} \in [0, 1] \) and \( \eta_i^{k+1} \in \partial g_i(z^{k+1})(i = 0, 1) \) such that
\[
e_k \in \eta_0^{k+1} - \xi_0^k + \beta_k \tilde{\lambda}^{k+1}(\eta_1^{k+1} - \xi_1^k) + \rho(z^{k+1} - z^k) + \mathcal{N}_\Sigma(z^{k+1}), \quad (11)
g_1(z^{k+1}) - h_1(z^k) - \langle \xi^k, z^{k+1} - z^k \rangle \leq t^{k+1}, \quad (12)
\tilde{\lambda}^{k+1}(g_1(z^{k+1}) - h_1(z^k) - \langle \xi^k, z^{k+1} - z^k \rangle - t^{k+1}) = 0, \quad (13)
t^{k+1}(1 - \tilde{\lambda}^{k+1}) = 0, \quad t^{k+1} \geq 0. \quad (14)
Since \( \{ \beta_k \hat{\lambda}^{k+1} \} \) is bounded, we may suppose that \( \hat{z} \) and \( \hat{\lambda} \) are accumulation points of \( \{ z^k \} \) and \( \{ \beta_k \hat{\lambda}^{k+1} \} \) respectively. Taking subsequences if necessary, without loss of generality we may assume that \( z^k \to \hat{z} \in \Sigma \) and \( \beta_k \hat{\lambda}^{k+1} \to \hat{\lambda} \). Now passing onto the limit as \( k \to \infty \) in (11)–(13), as \( e_k \to 0 \) from \( \xi_k \to 0 \) in (7) or \( \| z^{k+1} - z^k \| \to 0 \) in (8) and \( t^k \to 0 \), since \( g_i(z), h_i(z), i = 0, 1 \) are locally Lipschitz continuous at \( \hat{z} \), \( \partial g_i(z), \partial h_i(z), i = 0, 1 \) and \( N_\Sigma(z) \) are outer semicontinuous, we obtain that \( \hat{z} \) is a KKT solution of problem (DC).

Notice that the boundedness of the penalty parameters is needed for an accumulation point to be a KKT point. The following proposition provides a sufficient condition for the boundedness of the penalty parameters sequence \( \{ \beta_k \} \).

**Proposition 3** Suppose that the iterate sequence \( \{ z^k \} \) generated by iP-DCA is bounded. Moreover suppose functions \( g_0, g_1, h_1, h_0 \) are Lipschitz around at any accumulation point of \( \{ z^k \} \). Assume that ENNAMCQ holds at any accumulation points of the sequence \( \{ z^k \} \). Then the sequence \( \{ \beta_k \} \) must be bounded.

**Proof** The proof is inspired by [43, Theorem 3.1]. To the contrary, suppose that \( \beta_k \to +\infty \) as \( k \to \infty \). Then there exist infinitely many indices \( j \) such that

\[
\beta_k j < \| z^{k+1}_j - z^{k}_j \|^{-1} \quad \text{and} \quad t^{k+1}_j > \| z^{k+1}_j - z^{k}_j \|,
\]

and thus

\[
\lim_{j \to \infty} z^{k+1}_j - z^{k}_j = 0, \quad t^{k+1}_j > 0, \quad \forall j.
\]

From (14), since \( t^{k+1}_j > 0 \) for all \( j \), we have \( \beta_k \hat{\lambda}^{k+1}_j = 1 \) for all \( j \) and thus \( \lambda^{k+1}_j := \beta_k \hat{\lambda}^{k+1}_j \to +\infty \) as \( j \to \infty \). Taking a further subsequence, if necessary, we can assume that \( z^{k_j}_j \to \hat{z} \in \Sigma \) as \( j \to \infty \). If \( g_1(\hat{z}) - h_1(\hat{z}) < 0 \), then as \( g_1, h_1 \) are locally Lipschitz continuous at \( \hat{z} \), \( \{ \xi^{k_j}_j \} \) is bounded, and \( \lim_{j \to \infty} \| z^{k+1}_j - z^{k}_j \| = 0 \), when \( j \) is sufficiently large, one has \( g_1(z^{k+1}_j) - h_1(z^{k}_j) - \langle \xi^{k_j}_j, z^{k+1}_j - z^{k}_j \rangle < 0 \), which contradicts the assumption \( t^{k+1}_j := \max \{ g_1(z^{k+1}_j) - h_1(z^{k}_j) - \langle \xi^{k_j}_j, z^{k+1}_j - z^{k}_j \rangle, 0 \} > 0 \) for all \( j \). Thus, \( g_1(\hat{z}) - h_1(\hat{z}) \geq 0 \).

From (11), we have

\[
e_k_j \in \partial g_0(z^{k+1}_j) - \partial h_0(z^{k}_j) + \lambda^{k+1}_j \partial g_1(z^{k+1}_j) - \lambda^{k+1}_j \partial h_1(z^{k}_j)
+ \rho(z^{k+1}_j - z^{k}_j) + N_\Sigma(z^{k+1}_j),
\]

where \( \lambda^{k+1}_j := \beta_k \hat{\lambda}^{k+1}_j \). Dividing both sides of this equality by \( \lambda^{k+1}_j \), and passing onto the limit as \( j \to \infty \), we have \( 0 \in \partial g_1(\hat{z}) - \partial h_1(\hat{z}) + N_\Sigma(\hat{z}) \), which contradicts ENNAMCQ.

**2.2 Inexact proximal linearized DCA with simplified penalty parameter update**

Recall that iP-DCA defined in Algorithm 1 requires minimization of a strongly convex subproblem (6). In this subsection, we assume that \( g_1 \) is \( L \)-smooth which means that
\( \nabla g_1(z) \) is Lipschitz continuous with constant \( L \) on \( \Sigma \). This setting motivates a very simple linearization approach inspired by the idea behind the proximal gradient method (see [4] and the references therein). Specifically, we linearize both the concave part and the convex smooth part of the DC structure. Such a linearization approach makes subproblems easier to solve compared to iP-DCA. Given a current iterate \( z^k \in \Sigma \) with \( k = 0, 1, \ldots \), we select a subgradient \( \xi_i^k \in \partial h_i(z^k) \), for \( i = 0, 1 \). Then we solve the following subproblem approximately.

\[
\min_{z \in \Sigma} \hat{\phi}_k(z) := g_0(z) - h_0(z^k) - (\xi_0^k, z - z^k) + \frac{\rho_k}{2} \| z - z^k \|^2 \\
+ \beta_k \max\{g_1(z^k) + (\nabla g_1(z^k), z - z^k) - h_1(z^k) - (\xi_1^k, z - z^k), 0\},
\]

(15)

where \( \rho_k \) and \( \beta_k \) are the adaptive proximal and penalty parameters respectively. Choose \( z^{k+1} \) as an approximate minimizer of the convex subproblem (15) satisfying one of the following two inexact criteria

\[
\text{dist}(0, \partial \hat{\phi}_k(z^{k+1}) + N_{\Sigma}(z^{k+1})) \leq \zeta_k, \quad \text{for some } \zeta_k \text{ satisfying } \sum_{k=0}^{\infty} \zeta_k^2 < \infty,
\]

(16)

and

\[
\text{dist} \left( 0, \partial \hat{\phi}_k(z^{k+1}) + N_{\Sigma}(z^{k+1}) \right) \leq \frac{\sqrt{2}}{2} \sigma \| z^k - z^{k-1} \|.
\]

(17)

This yields the inexact proximal linearized DCA (iPL-DCA), whose exact description is given in Algorithm 2.

Recall that the merit function of (DC) is defined by \( \phi_k(z) := g_0(z) - h_0(z) + \beta_k \max\{g_1(z) - h_1(z), 0\} \). Similar to Lemma 1, we first give following sufficiently decrease result of iPL-DCA.

**Lemma 2** Let \( \{z^k\} \) be the sequence of iterates generated by iPL-DCA as defined in Algorithm 2. If the inexact criterion (16) or (17) is applied, then \( z^k \) satisfies

\[
\phi_k(z^k) \geq \phi_k(z^{k+1}) + \frac{\sigma}{2} \| z^{k+1} - z^k \|^2 - \frac{1}{2\sigma} \zeta_k^2,
\]

\[
\phi_k(z^k) \geq \phi_k(z^{k+1}) + \frac{\sigma}{2} \| z^{k+1} - z^k \|^2 - \frac{\sigma}{4} \| z^k - z^{k-1} \|^2,
\]

respectively.

**Proof** Since \( z^{k+1} \) is an approximation solution to problem (15) with inexact criterion (16) or (17), there exists a vector \( e_k \) such that \( e_k \in \partial \hat{\phi}_k(z^{k+1}) + N_{\Sigma}(z^{k+1}) \subseteq \partial (\hat{\phi}_k + \delta_{\Sigma})(z^{k+1}) \) and

\[
\| e_k \| \leq \zeta_k \text{ or } \| e_k \| \leq \frac{\sqrt{2}}{2} \sigma \| z^k - z^{k-1} \|, \quad (18)
\]
Algorithm 2 iPL-DCA

1: Take an initial point $z^0 \in \Sigma, \delta_\beta > 0, \sigma > 0$, an initial penalty parameter $\beta_0 > 0$, an initial regularizer parameter $\rho_0 = \frac{1}{2} \beta_0 L + \sigma, tol > 0$.

2: for $k = 0, 1, \ldots$ do
   1. Compute $\xi^k_i \in \partial h_i(z^k), i = 0, 1$.
   2. Obtain an inexact solution $z^{k+1}$ of (15) satisfying (16) or (17).
   3. Stopping test. Compute $t_{k+1} := \max \{ g_1(z^k) + \langle \nabla g_1(z^k), z^{k+1} - z^k \rangle - h_1(z^k) - (\xi^k_i, z^{k+1} - z^k), 0 \}$. Stop if $\max\{\|z^{k+1} - z^k\|, t_{k+1}\} < tol$.
   4. Penalty parameter update. Set $\beta_{k+1} = \begin{cases} \beta_k + \delta_\beta, & \text{if } \max\{\beta_k, 1/t_{k+1}\} < \|z^{k+1} - z^k\|^{-1}, \\ \beta_k, & \text{otherwise.} \end{cases}$
   $\rho_{k+1} = \frac{1}{2} \beta_{k+1} L + \sigma$.
   5. Set $k := k + 1$.
3: end for

respectively. As $\hat{\phi}_k$ is strongly convex with modulus $\rho_k$ and $\Sigma$ is a closed convex set, we have

$$
\hat{\phi}_k(z^k) \geq \hat{\phi}_k(z^{k+1}) + \langle e_k, z^{k+1} - z^k \rangle + \frac{\beta_k}{2} \|z^{k+1} - z^k\|^2 \\
\geq \hat{\phi}_k(z^{k+1}) - \frac{1}{2\sigma} \|e_k\|^2 - \frac{\sigma}{2} \|z^{k+1} - z^k\|^2 + \frac{\rho_k}{2} \|z^{k+1} - z^k\|^2 \\
= \hat{\phi}_k(z^{k+1}) - \frac{1}{2\sigma} \|e_k\|^2 + \frac{\rho_k - \sigma}{2} \|z^{k+1} - z^k\|^2. \quad (19)
$$

Next, by the convexity of $h_i(z)$ and $\xi^k_i \in \partial h_i(z^k), i = 0, 1$, we have

$$
h_i(z^{k+1}) \geq h_i(z^k) + \langle \xi^k_i, z^{k+1} - z^k \rangle, \quad i = 0, 1.
$$

And since $g_1$ is $L$-smooth, we have

$$
g_1(z^{k+1}) \leq g_1(z^k) + \langle \nabla g_1(z^k), z - z^k \rangle + \frac{L}{2} \|z^{k+1} - z^k\|^2.
$$

Thus, we have

$$
\hat{\phi}_k(z^{k+1}) \geq \phi_k(z^{k+1}) + \frac{\rho_k - \beta_k L}{2} \|z^{k+1} - z^k\|^2.
$$
Combined with (19), we have

\[
\varphi_k(z^k) = \hat{\varphi}_k(z^k) \geq \hat{\varphi}_k(z^{k+1}) - \frac{1}{2\sigma} \|e_k\|^2 + \frac{\beta_k - \sigma}{2} \|z^{k+1} - z^k\|^2
\]

\[
\geq \varphi_k(z^{k+1}) - \frac{1}{2\sigma} \|e_k\|^2 + \frac{2\rho_k - \beta_k L - \sigma}{2} \|z^{k+1} - z^k\|^2
\]

\[
\geq \varphi_k(z^{k+1}) - \frac{1}{2\sigma} \|e_k\|^2 + \frac{\sigma}{2} \|z^{k+1} - z^k\|^2.
\]

Then the conclusion follows immediately from (18). \qed

Similar to Theorem 1 and Proposition 3, by Lemma 2, the following convergence results of iPL-DCA can be derived easily. The proofs are purely technical and thus omitted.

**Theorem 2** Suppose \( f_0 \) is bounded below and the sequences \( \{z^k\} \) and \( \{\beta_k\} \) generated by iPL-DCA are bounded, functions \( g_0, h_1, h_0 \) are locally Lipschitz on set \( \Sigma \). Then every accumulation point of \( \{z^k\} \) is a KKT point for problem (DC).

**Proposition 4** Suppose the sequence \( \{z^k\} \) generated by iPL-DCA is bounded, functions \( g_0, h_1, h_0 \) are Lipschitz around at any accumulation point of \( \{z^k\} \), and ENNAMCQ holds at any accumulation points of the sequence \( \{z^k\} \). Then the sequence \( \{\beta_k\} \) is bounded.

**Remark 1** In fact, if \( g_0 \) is further assumed to be differentiable and \( \nabla g_0 \) is Lipschitz continuous, we can also linearize \( g_0 \) in iPL-DCA. The proof of convergence is similar.

### 3 DC algorithms for solving DCBP

In this section we will show how to solve problem (DCBP) numerically. It is obvious that problem (VP)\( _\epsilon \) is problem (DC) with

\[
z := (x, y), \quad f_0(x, y) := F_1(x, y) - F_2(x, y), \quad f_1(x, y) := f(x, y) - v(x) - \epsilon, \quad \Sigma = C.
\]

According to [38, Theorem 10.4], since \( F_1(x, y), F_2(x, y), f(x, y) \) are Lipschitz continuous near every point on an open convex set containing \( C \) and hence Lipschitz continuous near every point on \( C \). However our problem (VP)\( _\epsilon \) involves the value function which is an extended-value function \( v(x) : X \to [-\infty, \infty] \) defined by

\[
v(x) := \inf_{y \in Y} \{ f(x, y) \text{ s.t. } g(x, y) \leq 0 \},
\]

with the convention of \( v(x) = +\infty \) if the feasible region \( F(x) \) is empty. To apply the proposed DC algorithms, we need to answer the following questions.

(a) Is the value function convex and locally Lipschitz on the convex set \( X \) and how to obtain one element from \( \partial v(x^k) \) in terms of problem data?
(b) Will the constraint qualification ENNAMCQ hold at any accumulation point of the iterate sequence?

We now give answers to these questions in the next two subsections.

### 3.1 Lipschitz continuity and the subdifferential of the value function

Thanks to the full convex structure of the lower level problem in (DCBP), the value function turns out to be convex and Lipschitz continuous under our problem setting as shown below.

**Lemma 3** The value function $v(x) : X \rightarrow \mathbb{R}$ is convex and Lipschitz continuous around any point in set $X$. Given $\bar{x} \in X$ and $\bar{y} \in S(\bar{x})$, we have

$$\partial v(\bar{x}) = \{ \xi \in \mathbb{R}^n : (\xi, 0) \in \partial \phi(\bar{x}, \bar{y}) \}, \quad (20)$$

where $\phi(x, y) := f(x, y) + \delta_D(x, y)$, $D := \{(x, y) \in \mathcal{O} \times Y : g(x, y) \leq 0\}$, with $\mathcal{O}$ being the open set defined in the introduction.

**Proof** First we extend the definition of the value function from any element $x \in X$ to the whole space $\mathbb{R}^n$ as follows:

$$v(x) := \inf_{y \in \mathbb{R}^m} \phi(x, y), \quad \forall x \in \mathbb{R}^n.$$ 

It follows that $v(x) = +\infty$ for $x \notin \mathcal{O}$. In our problem setting, $f$ is fully convex on an open convex set containing the convex set $C$ and hence we can assume without loss of generality that $f$ is fully convex on the convex set $D$. Therefore the extended-valued function $\phi(x, y) : \mathbb{R}^n \times \mathbb{R}^m \rightarrow [-\infty, \infty]$ is convex.

The convexity of the value function $v(x) = \inf_{y \in \mathbb{R}^m} \phi(x, y)$ then follows from [39, Theorem 1]. Hence the value function restricted on set $X$ is convex. Next, according to [39, Theorem 24], we have the Eq. (20). By assumption stated in the introduction of the paper, the feasible region of the lower level program $F(x) := \{ y \in Y : g(x, y) \leq 0 \} \neq \emptyset$ and $v(x) \neq -\infty$ for all $x$ in the open set $\mathcal{O}$.

Hence $v(x) : \mathbb{R}^n \rightarrow [-\infty, \infty]$ is proper convex. Since $\text{dom}v := \{ x : v(x) < +\infty \} = \{ x : F(x) \neq \emptyset \} \supseteq \mathcal{O} \supseteq X$, we have $X \subseteq \text{int}(\text{dom}v)$. The result on Lipschitz continuity of the value function follows from [38, Theorem 10.4]. \qed

By using some sensitivity analysis techniques, a subgradient of the value function $v(x)$ can be expressed in terms of Lagrangian multipliers. In particular, given $\bar{y} \in S(\bar{x})$, we denote the set of KKT multipliers of the lower-level problem $(P_{\bar{x}})$ by

$$K T(\bar{x}, \bar{y})$$

$$:= \left\{ \gamma \in \mathbb{R}_+^l \left| 0 \in \partial_y f(\bar{x}, \bar{y}) + \sum_{i=1}^l \gamma_i \partial_y g_i(\bar{x}, \bar{y}) + N_Y(\bar{y}), \quad \sum_{i=1}^l \gamma_i g_i(\bar{x}, \bar{y}) = 0 \right. \right\}. \quad (21)$$
Theorem 3 Let $\bar{x} \in X$ and $\bar{y} \in S(\bar{x})$. Then
\[
\partial v(\bar{x}) \supseteq \left\{ \xi | (\xi, 0) \in \partial f(\bar{x}, \bar{y}) + \sum_{i=1}^{l} \gamma_i \partial g_i(\bar{x}, \bar{y}) + \{0\} \times N_Y(\bar{y}), \right. \\
\left. \gamma \in \mathbb{R}^l, \gamma \geq 0, \sum_{i=1}^{l} \gamma_i g_i(\bar{x}, \bar{y}) = 0 \right\}, \tag{21}
\]
and the equality holds in (21) provided that
\[
N_E(\bar{x}, \bar{y}) = \left\{ \sum_{i=1}^{l} \gamma_i \partial g_i(\bar{x}, \bar{y}) + \{0\} \times N_Y(\bar{y}) | \gamma \geq 0, \sum_{i=1}^{l} \gamma_i g_i(\bar{x}, \bar{y}) = 0 \right\}, \tag{22}
\]
where $E := \{(x, y) \in \mathbb{R}^n \times Y : g(x, y) \leq 0\}$. Moreover if the partial derivative formula holds
\[
\partial f(\bar{x}, \bar{y}) = \partial_x f(\bar{x}, \bar{y}) \times \partial_y f(\bar{x}, \bar{y}), \quad \partial g_i(\bar{x}, \bar{y}) = \partial_x g_i(\bar{x}, \bar{y}) \times \partial_y g_i(\bar{x}, \bar{y}) \tag{23}
\]
then
\[
\bigcup_{\gamma \in KT(\bar{x}, \bar{y})} \left( \partial_x f(\bar{x}, \bar{y}) + \sum_{i=1}^{l} \gamma_i \partial_x g_i(\bar{x}, \bar{y}) \right) \subseteq \partial v(\bar{x}), \tag{24}
\]
and the equality in (24) holds provided that (22) holds.

Proof Let $\phi_E(x, y) := f(x, y) + \delta_E(x, y) = f(x, y) + \delta_Y(y) + \sum_{i=1}^{l} \delta_{C_i}(x, y)$ with $C_i := \{(x, y) | g_i(x, y) \leq 0\}$. Then by the sum rule (see, e.g., [38, Theorem 23.8] [9, Corollary 1 to Theorem 2.9.8]) and the fact that $N_E = \partial \delta_E$, we have
\[
\partial f(\bar{x}, \bar{y}) + \{0\} \times N_Y(\bar{y}) + \sum_{i=1}^{l} N_{C_i}(\bar{x}, \bar{y}) \subseteq \partial \phi_E(\bar{x}, \bar{y}). \tag{25}
\]
When $g_i(\bar{x}, \bar{y}) < 0$, we have $(\bar{x}, \bar{y}) \in \text{int} C_i$ and hence $\gamma_i \partial g_i(\bar{x}, \bar{y}) = 0 \in N_{C_i}(\bar{x}, \bar{y})$. Otherwise if $g_i(\bar{x}, \bar{y}) = 0$, by definition of subdifferential and the normal cone we can show that for any $\gamma_i \geq 0$, $\gamma_i \partial g_i(\bar{x}, \bar{y}) \subseteq N_{C_i}(\bar{x}, \bar{y})$. Hence together with (25), we have
\[
\left\{ \partial f(\bar{x}, \bar{y}) + \sum_{i=1}^{l} \gamma_i \partial g_i(\bar{x}, \bar{y}) + \{0\} \times N_Y(\bar{y}) | \gamma \in \mathbb{R}^l, \gamma \geq 0, \sum_{i=1}^{l} \gamma_i g_i(\bar{x}, \bar{y}) = 0 \right\} \subseteq \partial \phi_E(\bar{x}, \bar{y}).
\]
Since \( \partial \phi_E(\bar{x}, \bar{y}) = \partial \phi(\bar{x}, \bar{y}) \) where \( \phi(x, y) = f(x, y) + \delta_D(x, y) \) with \( D := \{(x, y) \in \mathcal{O} \times Y \mid g(x, y) \leq 0\} \), it follows from Lemma 3 that

\[
\partial v(\bar{x}) = \{\xi \mid (\xi, 0) \in \partial \phi(\bar{x}, \bar{y})\} = \{\xi \mid (\xi, 0) \in \partial \phi_E(\bar{x}, \bar{y})\}
\]

\[
\supseteq \left\{\xi \mid (\xi, 0) \in \partial f(\bar{x}, \bar{y}) + \sum_{i=1}^{l} \gamma_i \partial g_i(\bar{x}, \bar{y}) + \{0\} \times \mathcal{N}_Y(\bar{y}), \right. \\
\gamma \in \mathbb{R}^l, \gamma \geq 0, \sum_{i=1}^{l} \gamma_i g_i(\bar{x}, \bar{y}) = 0 \right\}.
\]

Hence (21) holds. Since \( f \) is Lipschitz continuous at \((\bar{x}, \bar{y})\), by the sum rule (see, e.g., [38, Theorem 23.8] [9, Corollary 1 to Theorem 2.9.8]), we have \( \partial \phi_E(\bar{x}, \bar{y}) = \partial f(\bar{x}, \bar{y}) + \mathcal{N}_E(\bar{x}, \bar{y}) \). Hence if (22) holds, then

\[
\partial v(\bar{x}) = \{\xi \mid (\xi, 0) \in \partial \phi(\bar{x}, \bar{y})\} = \{\xi \mid (\xi, 0) \in \partial \phi_E(\bar{x}, \bar{y})\}
\]

\[
= \{\xi \mid (\xi, 0) \in \partial f(\bar{x}, \bar{y}) + \mathcal{N}_E(\bar{x}, \bar{y})\}
\]

\[
= \left\{\xi \mid (\xi, 0) \in \partial f(\bar{x}, \bar{y}) + \sum_{i=1}^{l} \gamma_i \partial g_i(\bar{x}, \bar{y}) + \{0\} \times \mathcal{N}_Y(\bar{y}), \right. \\
\gamma \geq 0, \sum_{i=1}^{l} \gamma_i g_i(\bar{x}, \bar{y}) = 0 \right\}.
\]

This shows that the equality holds in (21).

Now suppose that (23) holds. Then for any \( \gamma \in \mathbb{R}^l, \gamma \geq 0, \sum_{i=1}^{l} \gamma_i g_i(\bar{x}, \bar{y}) = 0 \), by the sum rule (see, e.g., [38, Theorem 23.8] [9, Corollary 1 to Theorem 2.9.8]) we have

\[
\partial \phi_E(\bar{x}, \bar{y})
\]

\[
\supseteq \partial f(\bar{x}, \bar{y}) + \sum_{i=1}^{l} \gamma_i \partial g_i(\bar{x}, \bar{y}) + \{0\} \times \mathcal{N}_Y(\bar{y})
\]

\[
= \left\{\partial_x f(\bar{x}, \bar{y}) + \sum_{i=1}^{l} \gamma_i \partial_x g_i(\bar{x}, \bar{y}) \right\} \times \left\{\partial_y f(\bar{x}, \bar{y}) + \sum_{i=1}^{l} \gamma_i \partial_y g_i(\bar{x}, \bar{y}) + \mathcal{N}_Y(\bar{y}) \right\}
\]

Combining with (21), we obtain (24).

Similarly when (22) holds, the equality holds in (24). \(\square\)

By the description in (24), \( \partial v(\bar{x}) \) can be calculated as long as (22) is satisfied. We claim that (22) is a mild condition. In fact, by convexity, there always holds the inclusion

\[
\mathcal{N}_E(\bar{x}, \bar{y}) \supseteq \left\{\sum_{i=1}^{l} \gamma_i g_i(\bar{x}, \bar{y}) + \{0\} \times \mathcal{N}_Y(\bar{y}) : \gamma \geq 0, \sum_{i=1}^{l} \gamma_i g_i(\bar{x}, \bar{y}) = 0 \right\}.
\]
By virtue of [18, Theorem 4.1], the reverse inclusion also follows under standard constraint qualifications. Some sufficient conditions for (22) are thus summarized in the following proposition.

**Proposition 5** Equation (22) holds provided that the set-valued map

\[ \Psi(\alpha) := \{ (x, y) \in \mathbb{R}^n \times Y : g(x, y) + \alpha \leq 0 \} \]

is calm at \((0, \bar{x}, \bar{y})\), i.e., there exist \(\kappa, \delta > 0\) such that

\[ \text{dist}_E(x, y) \leq \kappa \| \max\{g(x, y), 0\} \| \forall (x, y) \in B_{\delta}(\bar{x}, \bar{y}) \cap E; \]

in particular if one of the following conditions:

(a) The linear constraint qualification holds: \(g(x, y)\) is an affine mapping of \((x, y)\) and \(Y\) is convex polyhedral.

(b) The Slater condition holds: there exists a point \((x_0, y_0) \in \mathbb{R}^n \times Y\) such that \(g(x_0, y_0) < 0\).

**Proof** By virtue of [18, Theorem 4.1], the reverse inclusion

\[ N_E(\bar{x}, \bar{y}) \subseteq \left\{ \sum_{i=1}^{l} \gamma_i \partial g_i(\bar{x}, \bar{y}) + \{0\} \times N_Y(\bar{y}) : \gamma \geq 0, \sum_{i=1}^{l} \gamma_i g_i(\bar{x}, \bar{y}) = 0 \right\} \]

holds provided that the system \(y \in Y, g(x, y) \leq 0\) is calm at \((0, \bar{x}, \bar{y})\). It is well-known that (a) or (b) is a sufficient condition for calmness.

3.2 Motivations for studying the approximate bilevel program

There are three motivations to consider the approximate program \((VP_\epsilon)\). First, as shown in [25], the solutions of \((VP_\epsilon)\) approximate a true solution of the original bilevel program (DCBP) as \(\epsilon\) approaches zero. Second, the proximity from a local minimizer of \((VP_\epsilon)\) to the solution set of (DCBP) can be controlled by adjusting the value of \(\epsilon\). The third motivation is that the approximate program \((VP_\epsilon)\) when \(\epsilon > 0\) would satisfy the required constraint qualification automatically. We present the second motivation in the following proposition.

**Proposition 6** Suppose \(\mathcal{S}^*\), the solution set of problem (VP), is compact and the value function \(v(x)\) is continuous. Then for any \(\delta > 0\), there exists \(\bar{\epsilon} > 0\) such that for any \(\epsilon \in (0, \bar{\epsilon}]\), there exists \((x_\epsilon, y_\epsilon)\) which is a local minimum of \(\epsilon\)-approximation problem \((VP_\epsilon)\) with \(\text{dist}((x_\epsilon, y_\epsilon), \mathcal{S}^*) < \delta\).

**Proof** To the contrary, assume that there exist \(\delta > 0\) and sequence \(\{\epsilon_k\}\) with \(\epsilon_k \downarrow 0\) as \(k \to \infty\) such that there does not exist \((x, y)\) being a local minimum of \(\epsilon_k\)-approximation problem \((VP_{\epsilon_k})\) satisfying \(\text{dist}((x_k, y_k), \mathcal{S}^*) < \delta\) for all \(k\). Consider a
point \((\hat{x}^k, \hat{y}^k)\), which is a global minimum to the following problem

\[
\min_{(x, y) \in C} F(x, y)
\]

\[
\text{s.t. } f(x, y) - v(x) \leq \epsilon, \quad \text{dist}((x, y), S^*) \leq \delta.
\]

Then by assumption, it holds that \(\text{dist}((\hat{x}^k, \hat{y}^k), \Sigma^*) = \delta\) and \(F(\hat{x}^k, \hat{y}^k) \leq F^*\), where \(F^*\) is the optimal value of problem (VP). As \(S^*\) is compact, sequence \(\{(\hat{x}^k, \hat{y}^k)\}\) is bounded and we can assume without lost of generality that \((\hat{x}^k, \hat{y}^k) \to (\bar{x}, \bar{y})\) as \(k \to \infty\). Since the value function \(v\) is continuous, by taking \(k \to \infty\) in \((\hat{x}^k, \hat{y}^k) \in C\) and \(f(\hat{x}^k, \hat{y}^k) - v(\hat{x}^k) \leq \epsilon^k\), we obtain the feasibility of the limit point \((\bar{x}, \bar{y})\) for problem (VP). Next, by taking \(k \to \infty\) in \(\text{dist}((\hat{x}^k, \hat{y}^k), S^*) = \delta\) and \(F(\hat{x}^k, \hat{y}^k) \leq F^*\), we obtain that \(\text{dist}((\bar{x}, \bar{y}), S^*) = \delta\) and \(F(\bar{x}, \bar{y}) \leq F^*\), a contradiction. \(\square\)

Before we clarify the third motivation, we define the concepts of NNAMCQ for problem (VP) and ENNAMCQ for problem \((\text{VP})_\epsilon\) where \(\epsilon \geq 0\).

**Definition 3** Let \((\bar{x}, \bar{y})\) be a feasible solution to problem (VP). We say that NNAMCQ holds at \((\bar{x}, \bar{y})\) for problem (VP) if

\[
0 \notin \partial f(\bar{x}, \bar{y}) - \partial v(\bar{x}) \times \{0\} + \mathcal{N}_C(\bar{x}, \bar{y}). \tag{26}
\]

Let \((\bar{x}, \bar{y}) \in C\), we say that ENNAMCQ holds at \((\bar{x}, \bar{y})\) for problem \((\text{VP})_\epsilon\) if either \(f(\bar{x}, \bar{y}) - v(\bar{x}) < \epsilon\) or \(f(\bar{x}, \bar{y}) - v(\bar{x}) \geq \epsilon\) but (26) holds.

**Proposition 7** Let \((\bar{x}, \bar{y})\) be a feasible solution to problem (VP). Suppose that \(v(x)\) is Lipschitz continuous around \(\bar{x}\). Then NNAMCQ never holds at \((\bar{x}, \bar{y})\).

**Proof** By definition of the value function, we can never have \(f(\bar{x}, \bar{y}) - v(\bar{x}) < 0\) and we always have \(f(\bar{x}, \bar{y}) - v(\bar{x}) = 0\). But since \((\bar{x}, \bar{y})\) is a feasible solution to problem (VP), it is easy to see that \((\bar{x}, \bar{y})\) must be a solution to the following problem

\[
\min_{(x, y) \in C} \{f(x, y) - v(x)\}.
\]

But by the optimality condition we must have

\[
0 \in \partial f(\bar{x}, \bar{y}) - \partial v(\bar{x}) \times \{0\} + \mathcal{N}_C(\bar{x}, \bar{y}).
\]

This means that (26) would never hold. \(\square\)

Although the NNAMCQ never hold for (VP), for \(\epsilon > 0\), ENNAMCQ is a standard constraint qualification for \((\text{VP})_\epsilon\); see e.g. [25, Proposition 8]. Moreover thanks to the model structures, it holds automatically for \((\text{VP})_\epsilon\) if \(\epsilon > 0\). Hence, according to the DC theories established in the preceding section, powerful DCA can be employed to solve \((\text{VP})_\epsilon\).
Proposition 8 For any \((\bar{x}, \bar{y}) \in C\), problem \((VP)_\epsilon\) with \(\epsilon > 0\) satisfies ENNAMCQ at \((\bar{x}, \bar{y})\).

Proof If \(f(x, y) - v(x) < \epsilon\) holds, then by definition, ENNAMCQ holds at \((\bar{x}, \bar{y})\). Now suppose that \(f(\bar{x}, \bar{y}) - v(\bar{x}) \geq \epsilon\) and ENNAMCQ does not hold, i.e.,

\[
0 \in \partial f(\bar{x}, \bar{y}) - \partial v(\bar{x}) \times \{0\} + \mathcal{N}_C(\bar{x}, \bar{y}).
\]

It follows from the partial subdifferentiation formula (2) that

\[
0 \in \left[ \partial_x f(\bar{x}, \bar{y}) - \partial_y f(\bar{x}, \bar{y}) \right] + \mathcal{N}_C(\bar{x}, \bar{y}). \tag{27}
\]

By (2) we have

\[
\mathcal{N}_C(\bar{x}, \bar{y}) = \partial \delta_C(\bar{x}, \bar{y}) \subseteq \partial_x \delta_C(\bar{x}, \bar{y}) \times \partial_y \delta_C(\bar{x}, \bar{y}) \subseteq \mathbb{R}^n \times \mathcal{N}_{C(\bar{x})}(\bar{y}),
\]

where \(C(\bar{x}) := \{y \in Y \mid g_i(\bar{x}, y) \leq 0, i = 1, \ldots, l\}\). Thus, it follows from (27) that

\[
0 \in \partial_y f(\bar{x}, \bar{y}) + \mathcal{N}_{C(\bar{x})}(\bar{y}),
\]

which further implies that \(\bar{y} \in S(\bar{x})\). However, an obvious contradiction to the assumption that \(f(\bar{x}, \bar{y}) - v(\bar{x}) \geq \epsilon\) occurs and thus the desired conclusion follows immediately.

By using the definition of KKT points for (DC) in Definition 1, under the assumption that the value function is locally Lipschitz continuous, we define KKT points for problem \((VP)_\epsilon\).

Definition 4 We say a point \((\bar{x}, \bar{y})\) is a KKT point of problem \((VP)_\epsilon\) with \(\epsilon \geq 0\) if there exists \(\lambda \geq 0\) such that

\[
\begin{align*}
0 & \in \partial F_1(\bar{x}, \bar{y}) - \partial F_2(\bar{x}, \bar{y}) + \lambda \partial f(\bar{x}, \bar{y}) - \lambda \partial v(\bar{x}) \times \{0\} + \mathcal{N}_C(\bar{x}, \bar{y}), \\
f(\bar{x}, \bar{y}) - v(\bar{x}) - \epsilon & \leq 0, \quad \lambda (f(\bar{x}, \bar{y}) - v(\bar{x}) - \epsilon) = 0.
\end{align*}
\]

By virtue of Proposition 8 and Theorem 1, we have the following necessary optimality condition. Since the issue of constraint qualifications for problem \((VP)_\epsilon\) is complicated and it is not the main concern in this paper, we refer the reader to discussions on this topic in [2, 47].

Theorem 4 Let \((\bar{x}, \bar{y})\) be a local optimal solution to problem \((VP)_\epsilon\) with \(\epsilon \geq 0\). Suppose either \(\epsilon > 0\) or \(\epsilon = 0\) and a constraint qualification holds. Then \((\bar{x}, \bar{y})\) is a KKT point of problem \((VP)_\epsilon\).
3.3 Inexact proximal DCA for solving (VP)\(_{\epsilon}\)

In this subsection we implement the proposed DC algorithms in Sect. 2 to solve (VP)\(_{\epsilon}\). To proceed, let us describe iP-DCA to solve (VP)\(_{\epsilon}\). Given a current iterate \((x^k, y^k)\) for each \(k = 0, 1, \ldots\), solving the lower level problem parameterized by \(x^k\)

\[
\min_{y \in \mathcal{Y}} f(x^k, y), \text{ s.t. } g(x^k, y) \leq 0
\]

leads to a solution \(\tilde{y}^k \in S(x^k)\) and a corresponding KKT multiplier \(\gamma^k \in KT(x^k, \tilde{y}^k)\). Select

\[
\xi^k_0 \in \partial F_2(x^k, y^k), \quad \xi^k_1 \in \partial_x F(x^k, \tilde{y}^k) + \sum_{i=1}^l y^k_i \partial_i g_i(x^k, \tilde{y}^k). \tag{28}
\]

Note that according to (24) in Theorem 3, if the partial derivative formula holds, then \(\xi^k_1 \in \partial_x f(x^k, \tilde{y}^k) + \sum_{i=1}^l y^k_i \partial_i g_i(x^k, \tilde{y}^k)\) is an element of the subdifferential \(\partial v(x^k)\). Compute \((x^{k+1}, y^{k+1})\) as an approximate minimizer of the strongly convex subproblem for (VP)\(_{\epsilon}\) given by

\[
\min_{(x, y) \in \mathcal{C}} F_1(x, y) - \langle \xi^k_0, (x, y) \rangle + \frac{\rho}{2} \| (x, y) - (x^k, y^k) \|^2 + \beta_k \max \{ f(x, y) - f(x^k, \tilde{y}^k) - \langle \xi^k_1, x - x^k \rangle - \epsilon, 0 \}, \tag{29}
\]

satisfying one of the two inexact criteria. Under the assumption that \(KT(x^k, y)\) is nonempty for all \(y \in S(x^k)\), the description of iP-DCA on (VP)\(_{\epsilon}\) with \(\epsilon \geq 0\) now follows:

**Algorithm 3** iP-DCA for solving (VP)\(_{\epsilon}\)

1. Take an initial point \((x^0, y^0) \in X \times Y; \delta_\beta > 0;\) an initial penalty parameter \(\beta_0 > 0, tol > 0\).
2. for \(k = 0, 1, \ldots\) do
   1. Solve the lower level problem \(P_{x^k}\) defined in (3.3) and obtain \(\tilde{y}^k \in S(x^k)\) and \(y^k \in KT(x^k, \tilde{y}^k)\).
   2. Compute \(\xi^k_i, i = 0, 1\) according to (28).
   3. Obtain an inexact solution \((x^{k+1}, y^{k+1})\) of (29).
   4. Stopping test. Compute \(\ell^{k+1} = \max \{ f(x^{k+1}, y^{k+1}) - f(x^k, \tilde{y}^k) - \langle \xi^k_1, x^{k+1} - x^k \rangle - \epsilon, 0 \}\). Stop if \(\max \{ \| (x^{k+1}, y^{k+1}) - (x^k, y^k) \|, \ell^{k+1} \} < tol\).
   5. Penalty parameter update. Set
      \[
      \beta_{k+1} = \begin{cases} 
      \beta_k + \delta_\beta, & \text{if } \max \{ \beta_k, 1/\ell^{k+1} \} < \| (x^{k+1}, y^{k+1}) - (x^k, y^k) \|^{-1}, \\
      \beta_k, & \text{otherwise}. 
      \end{cases}
      \]
3. Set \(k := k + 1\).
Thanks to Proposition 8, when $\epsilon > 0$, problem $(VP)_\epsilon$ satisfies ENNAMCQ automatically. Moreover, since the partial subgradient formula (23) holds, according to (24) in Theorem 3, the selection criteria in (28) implies that $\xi^k_1 \in \partial v(x^k)$. Hence the convergence of iP-DCA for solving $(VP)_\epsilon$ (Algorithm 3) follows from Theorem 1 and Proposition 3.

**Theorem 5** Assume that $F$ is bounded below on $C$. Let $\{(x^k, y^k)\}$ be an iterate sequence generated by Algorithm 3. Moreover assume that the partial subgradient formula (1) holds at every iterate point $(x^k, y^k)$ and $KT(x^k, y) \neq \emptyset$ for all $y \in S(x^k)$. Suppose that either $\epsilon > 0$ or $\epsilon = 0$ and the penalty sequence $\{\beta_k\}$ is bounded. Then any accumulation point of $\{(x^k, y^k)\}$ is a KKT point of problem $(VP)_\epsilon$.

We now assume that the lower level objective $f$ is differentiable and $\nabla f$ is Lipschitz continuous with modulus $L_f$ on set $C$. Given a current iterate $(x^k, y^k)$, the next iterate $(x^{k+1}, y^{k+1})$ can be returned as an approximate minimizer of subproblem (29) with linearized $f$ given by

$$
\min_{(x, y) \in C} F_1(x, y) - \langle \xi_0^k, (x, y) \rangle + \frac{\rho_k}{2} \|(x, y) - (x^k, y^k)\|^2 \\
+ \beta_k \max\{f(x^k, y^k) + \langle \nabla f(x^k, y^k), (x, y) - (x^k, y^k) \rangle - f(x^k, y^k) - \langle \xi_1^k, x - x^k \rangle - \epsilon, 0\}.
$$

(30)

Under the assumption that $KT(x^k, y) \neq \emptyset$ for all $y \in S(x^k)$, the iterate scheme of iPPL-DCA on $(VP)_\epsilon$ with $\epsilon \geq 0$ thus reads as:

**Algorithm 4** iP-DCA for solving $(VP)_\epsilon$

1: Take an initial point $(x^0, y^0) \in X \times Y$; $\delta_{\beta}, \sigma > 0$; $\beta_0 > 0$; $\rho_0 = \frac{1}{2} \beta_0 L_f + \sigma$; $tol > 0$.

2: for $k = 0, 1, \ldots$ do

1. Solve the lower level problem $P_{x^k}$ defined in (3.3) and obtain $y^k \in S(x^k)$ and $y^k \in KT(x^k, y^k)$.

2. Compute $\xi_i^k$, $i = 0, 1$ according to (28).

3. Obtain an inexact solution $(x^{k+1}, y^{k+1})$ of (30).

3. Stopping test. Compute $\delta^{k+1} = \max\{f(x^k, y^k) + \langle \nabla f(x^k, y^k), (x^{k+1}, y^{k+1}) - (x^k, y^k) \rangle - f(x^k, y^k) - \langle \xi_1^k, x^{k+1} - x^k \rangle - \epsilon, 0\}$. Stop if $\max\|\{(x^{k+1}, y^{k+1}) - (x^k, y^k)\|_1, \delta^{k+1} < tol$.

4. Penalty parameter update. Set

$$
\beta_{k+1} = \begin{cases} 
\beta_k + \delta_{\beta}, & \text{if } \max\{\beta_k, 1/\delta^{k+1}\} < \|x^{k+1}, y^{k+1} - (x^k, y^k)\|^{-1}, \\
\beta_k, & \text{otherwise}.
\end{cases}
$$

$$
\rho_{k+1} = \frac{1}{2} \beta_{k+1} L_f + \sigma.
$$

5. Set $k := k + 1$.

3: end for

The convergence of iP-DCA follows from Theorem 2 and Proposition 4 directly.
Theorem 6 Assume that $F$ is bounded below on $C$, $f$ is $L_f$ smooth on $C$. Moreover assume that the partial subgradient formula (1) holds at every iterate point $(x^k, y^k)$ and $KT(x^k, y) \neq \emptyset$ for all $y \in S(x^k)$. Suppose that either $\epsilon > 0$ or $\epsilon = 0$ and the penalty parameter sequence $\{\beta_k\}$ is bounded. Then any accumulation point of $(x^k, y^k)$ is a KKT point of problem (VP)$_e$.

4 Numerical experiments on SV bilevel model selection

In this section, we will conduct numerical experiments on the SV bilevel model selection problem (SVBP). Extensive numerical experiments of iP-DCA on multiple hyperparameter selection models (e.g., elastic net, sparse group lasso, low-rank matrix completion and etc) are presented in [17].

Let $x := (\mu, \bar{w}) \in \mathbb{R}^{n+1}$, $y := (w^1, \ldots, w^T, c) \in \mathbb{R}^{(n+1)T}$, $X = [\frac{1}{x_{lb}}, \frac{1}{x_{ub}}] \times [\bar{w}_{lb}, \bar{w}_{ub}]$, $Y = \mathbb{R}^{(n+1)T}$,

$$f(x, y) = \sum_{t=1}^{T} \left( \frac{\|w^t\|^2}{2\mu} + \max_{j \in \Omega_{trn}^t}(1 - b_j(a_j^T w^t - c_t), 0) \right),$$

and

$$g(x, y) = \begin{pmatrix} g_1(x, y) \\ \vdots \\ g_T(x, y) \end{pmatrix} \text{ with } g_t(x, y) = \begin{pmatrix} -\bar{w} - w^t \\ w^t - \bar{w} \end{pmatrix}, t = 1, \ldots, T.$$

Obviously $F$, $f$ and $g$ are all convex functions defined an open set containing $X \times Y$, and problem (SVBP) can be regarded as a special case of the DC bilevel program (DCBP). Both $F$ and $f$ are bounded below on $X \times Y$. When $\bar{w}_{lb} \geq \bar{w}_{lb} > 0$, $\mathcal{F}(x) \neq \emptyset$ for an open set containing $X$. And since $b_j \in \{-1, 1\}$, $f(x, y)$ is coercive and continuous with respect to lower-level variable $y$ for any given $x$ in an open set containing $X$, thus $S(x) \neq \emptyset$ for all $x$ in an open set containing $X$. The function $g$ is smooth and $f$ is a sum of a smooth function and a function which is independent of variable $x$. Hence by Proposition 1, the partial differential formula (1) holds at each point $(x, y)$. Since the lower level constraints are affine, KKT conditions holds at any $y \in S(x)$ for any $x \in X$. Therefore, all conditions required by the convergence results of iP-DCA in Theorem 5 are satisfied.

We now describe how to calculate the main objects that are required in iP-DCA on problem (SVBP). At the current iterate $x^k := (\mu^k, \bar{w}^k)$, solve $(P_{\mu^k,\bar{w}^k})$, the lower level problem parameterized by $\mu^k$, $\bar{w}^k$ and obtain a solution $\bar{y}^k := (\bar{w}^1, \ldots, \bar{w}^T, \bar{c}) \in S(x^k)$ and a corresponding KKT multiplier

$$(\gamma_{1,1}^k, \ldots, \gamma_{1,T}^k, \gamma_{2,1}^k, \ldots, \gamma_{2,T}^k) \in KT(x^k, \bar{y}^k).$$
where $\gamma_{1,t}^k$ and $\gamma_{2,t}^k$ are multipliers corresponding to constraints $-\tilde{w}^k - w^t \leq 0$, $t = 1, \ldots, T$ and $w^t - \tilde{w}^k \leq 0$, $t = 1, \ldots, T$, respectively. Since $F(x,y)$ is convex, we have $\xi_0^k = 0$. Since $f(x,y)$ and $g(x,y)$ are both smooth in variable $x$, $\xi_1^k$ can be calculated by

$$
\xi_1^k = \left( -\sum_{t=1}^T \frac{\|w^t\|^2}{2\mu} \gamma_{1,t}^k - \sum_{t=1}^T \gamma_{2,t}^k \right) = \nabla_x f(x^k, \tilde{y}^k) + \sum_{t=1}^T \nabla_x g_t(x^k, \tilde{y}^k) \gamma_t^k \subseteq \partial v(x^k),
$$

where $\gamma_t^k := (\gamma_{1,t}^k, \gamma_{2,t}^k)$. With these objects calculated, we can then carry out the rest of steps in Algorithm 3.

Although the SV bilevel model selection is a nonsmooth bilevel program, by using auxiliary variables, the problem can be reformulated as a smooth bilevel program with a convex lower level program for which MPEC approach and the iPL-DCA are both applicable. This approach has been taken in [22] in which some nonlinear program solver has been used to solve the resulting MPEC. Because the smooth lower level objective in the reformulated bilevel program consists of $\sum_{t=1}^T \frac{\|w^t\|^2}{2\mu}$, and the Lipschitz constant $L_f$ of the gradient of $\sum_{t=1}^T \frac{\|w^t\|^2}{2\mu}$ with respect to variables $(w^1, \ldots, w^T)$ and $\mu$ will be extremely large when $\mu$ is optimized over the interval with small values. According to the update rule for the regularizer parameter $\rho_k$ in the iPL-DCA, $\rho_k$, as the coefficient of the regularizer terms in the subproblem during each iteration, is linear w.r.t. $L_f$ and will be extremely large. For this reason, the iPL-DCA is not a good choice for this problem. In next subsection, we will compare our proposed algorithms with the MPEC approach considered in [22]. In numerical experiments, we will follow the suggestions given in [22] to replace the complementarity constraints with the relaxed complementarity constraints. As claimed by [22], such approach can facilitate an early termination of cross-validation and ease the difficulty of dealing the complementarity constraints for nonlinear program solver.

### 4.1 Numerical tests

All the numerical experiments are implemented on a laptop with Intel(R) Core(TM) i7-9750H CPU@ 2.60 GHz and 32.00 GB memory. All the codes are implemented on MATLAB 2019b. The subproblems in iP-DCA are all convex optimization problems and we apply the Matlab software package SDPT3 [45, 46] with default setting to solve them. MPEC problem is solved by using `fmincon` in Matlab optimization toolbox with setting ’Algorithm’ being ’interior – point’, ’MaxIterations’ being 200 and ’MaxFunctionEvaluations’ being $10^6$. MPEC approach is implemented with low and strict tolerance by setting ’OptimalityTolerance’, ’ConstraintTolerance’, ’StepTolerance’ being $tol = 10^{-2}$, and $10^{-6}$. As `fmincon` needs extremely long time to solve large dimension MPEC problems, we first use small size datasets to conduct the numerical comparison between iP-DCA and MPEC approach. We test here three real datasets “australian_scale”, “breast-cancer_scale” and “diabetes_scale” downloaded...
from the SVMLib repository [8]. Each dataset is randomly split into a training set \( \Omega \) with \(|\Omega| = \ell_{\text{train}}\) data pairs, which is used in the cross-validation bilevel model and a hold-out test set \( \mathcal{N} \) with \(|\mathcal{N}| = \ell_{\text{test}}\) data pairs. We give the descriptions of datasets in Table 1. For each dataset, we use a three-fold cross-validation in the SV bilevel model selection problem (SVBP), i.e. \( T = 3 \), and that each training fold consists of two-thirds of the total training data and validation fold consists of one-third of the total training data. We repeat the experiments 20 times for each dataset. The values of parameters in SV bilevel model selection (SVBP) are set as: \( \lambda_{\text{lb}} = 10^{-4} \), \( \lambda_{\text{ub}} = 10^{4} \), and \( \bar{w}_{\text{ub}} = 10^{-6} \). For our approach, we test three different values of relaxation parameter \( \epsilon \in \{0, 10^{-2}, 10^{-4}\} \) in (VP)_\( \epsilon \). And the value of relaxation parameter of the relaxed complementarity constraints in MPEC is set to be \( 10^{-6} \). The initial points for both iP-DCA and MPEC approach are chosen as \( \hat{\lambda} = 1 \), \( \bar{w} = 0 \), where \( \mathbf{e} \) denotes the vector whose elements are all equal to 1, and the values of other variables are all equal to 0. These settings are used for all experiments. Parameters in iP-DCA are set as \( \beta_{0} = 1 \), \( \rho = 10^{-2} \) and \( \delta_{\text{P}} = 5 \). And we terminate iP-DCA when \( t^{k+1} < 10^{-4} \) and \( \| (x^{k+1}, y^{k+1}) - (x^{k}, y^{k}) \| (1 + \| (x^{k}, y^{k}) \|) < \text{tol} \).

For each experiment, after we obtain the hyperparameters \( \hat{\mu} \) and \( \hat{w} \) from implementing our proposed algorithm and MPEC approach for the SV bilevel model selection, we calculate their corresponding cross-validation error (CV error) and test error for comparing the performances of these two methods. For calculating the CV error, we put \( \hat{\mu} \) and \( \hat{w} \) back to the lower level problem in problem (SVBP) to get the corresponding lower level solution \( (\hat{w}^{1}, \ldots, \hat{w}^{T}, \hat{c}) \) and calculate the corresponding cross-validation error \( \Theta(\hat{w}^{1}, \ldots, \hat{w}^{T}, \hat{c}) \). Next, as in [22] we implement a post-processing procedure to calculate the generalization error on the hold-out data for each instance. In particular as suggested in [22], since only two thirds of the data in \( \Omega \) was used in each fold while in testing we use all the training data from \( \Omega \), we should solve the following support vector classification problem with \( \frac{3}{2} \hat{\lambda} = \frac{3}{2\hat{\mu}} \) and \( \hat{w} \) as hyperparameter

\[
\min_{\hat{w}} \frac{1}{2} \| \mathbf{w} \|^2 + \sum_{j \in \Omega} \max(1 - b_j (a_j^T \mathbf{w} - c), 0)
\]

to obtain the final classifier \( (\hat{w}, \hat{c}) \). Then the test (hold-out) error rate is calculated as:

\[
\text{Test error} = \frac{1}{\ell_{\text{test}}} \sum_{i \in \mathcal{N}} \frac{1}{2} \text{sign}(a_i^T \hat{w} - \hat{c}) - b_i,\]

where \( \text{sign}(x) \) denote the sign function. Note that for each \((a_i, b_i)\) in the test set \( \mathcal{N} \), \( |\text{sign}(a_i^T \hat{w} - \hat{c}) - b_i| \) is either equal to zero or 2 and hence the test error is the average misclassification by the final classifier. The achieved numerical results averaged over 20 repetitions for each dataset are reported in Table 2.

We compare the computational performance of iP-DCA with different values of \( \epsilon \) and \text{tol}, i.e., \( \epsilon = 0, 10^{-4}, 10^{-2} \) and \text{tol} = \( 10^{-2}, 10^{-3} \) and the MPEC approach with different values of \text{tol}, i.e., \( \text{tol} = 10^{-2}, 10^{-6} \). Observe from Table 2 that different

\[1\] http://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/.
Table 1  Description of datasets used

| Dataset         | $\epsilon_{\text{train}}$ | $\epsilon_{\text{test}}$ | $n$ | $T$  |
|-----------------|-----------------------------|---------------------------|-----|------|
| australian_scale| 345                         | 345                       | 14  | 3    |
| breast-cancer_scale | 339                     | 344                       | 10  | 3    |
| diabetes_scale  | 384                         | 384                       | 8   | 3    |
| mushrooms       | 4062                        | 4062                      | 112 | 3    |
| phishing        | 5526                        | 5529                      | 68  | 3    |

Table 2  Numerical results comparing iP-DCA and MPEC approach

| Dataset         | Method                               | CV error     | Test error    | Time(sec) |
|-----------------|--------------------------------------|--------------|---------------|-----------|
| australian_scale| iP-DCA($\epsilon = 0, tol = 10^{-2}$)| 0.28 ± 0.03  | 0.15 ± 0.01   | 70.0 ± 116.9 |
|                 | iP-DCA($\epsilon = 0, tol = 10^{-3}$)| 0.28 ± 0.03  | 0.15 ± 0.01   | 75.7 ± 123.8 |
|                 | iP-DCA($\epsilon = 10^{-2}, tol = 10^{-2}$)| 0.28 ± 0.03  | 0.15 ± 0.01   | 10.1 ± 5.1 |
|                 | iP-DCA($\epsilon = 10^{-2}, tol = 10^{-3}$)| 0.28 ± 0.03  | 0.15 ± 0.01   | 119.8 ± 55.8 |
|                 | iP-DCA($\epsilon = 10^{-4}, tol = 10^{-2}$)| 0.28 ± 0.03  | 0.15 ± 0.01   | 58.5 ± 108.8 |
|                 | iP-DCA($\epsilon = 10^{-4}, tol = 10^{-3}$)| 0.28 ± 0.03  | 0.15 ± 0.01   | 118.6 ± 123.6 |
|                 | MPEC approach($tol = 10^{-2}$)       | 0.28 ± 0.03   | 0.15 ± 0.01   | 130.0 ± 82.4 |
|                 | MPEC approach($tol = 10^{-6}$)       | 0.28 ± 0.03   | 0.15 ± 0.01   | 391.3 ± 226.9 |
| breast-cancer_scale | iP-DCA($\epsilon = 0, tol = 10^{-2}$)| 0.07 ± 0.01   | 0.03 ± 0.01   | 26.2 ± 19.5 |
|                 | iP-DCA($\epsilon = 0, tol = 10^{-3}$)| 0.06 ± 0.01   | 0.03 ± 0.01   | 69.6 ± 52.3 |
|                 | iP-DCA($\epsilon = 10^{-2}, tol = 10^{-2}$)| 0.07 ± 0.01   | 0.03 ± 0.01   | 15.6 ± 4.5 |
|                 | iP-DCA($\epsilon = 10^{-2}, tol = 10^{-3}$)| 0.07 ± 0.01   | 0.03 ± 0.01   | 106.5 ± 44.8 |
|                 | iP-DCA($\epsilon = 10^{-4}, tol = 10^{-2}$)| 0.07 ± 0.01   | 0.03 ± 0.01   | 17.5 ± 6.3 |
|                 | iP-DCA($\epsilon = 10^{-4}, tol = 10^{-3}$)| 0.06 ± 0.01   | 0.03 ± 0.01   | 72.5 ± 55.0 |
|                 | MPEC approach($tol = 10^{-2}$)       | 0.09 ± 0.01   | 0.03 ± 0.01   | 54.3 ± 13.7 |
|                 | MPEC approach($tol = 10^{-6}$)       | 0.09 ± 0.01   | 0.03 ± 0.01   | 226.0 ± 108.4 |
| diabetes_scale  | iP-DCA($\epsilon = 0, tol = 10^{-2}$)| 0.55 ± 0.02   | 0.24 ± 0.02   | 14.5 ± 30.3 |
|                 | iP-DCA($\epsilon = 0, tol = 10^{-3}$)| 0.55 ± 0.02   | 0.24 ± 0.02   | 24.2 ± 44.8 |
|                 | iP-DCA($\epsilon = 10^{-2}, tol = 10^{-2}$)| 0.56 ± 0.02   | 0.24 ± 0.02   | 3.1 ± 0.4 |
|                 | iP-DCA($\epsilon = 10^{-2}, tol = 10^{-3}$)| 0.56 ± 0.02   | 0.24 ± 0.02   | 65.7 ± 33.6 |
|                 | iP-DCA($\epsilon = 10^{-4}, tol = 10^{-2}$)| 0.55 ± 0.02   | 0.24 ± 0.02   | 18.2 ± 27.1 |
|                 | iP-DCA($\epsilon = 10^{-4}, tol = 10^{-3}$)| 0.55 ± 0.02   | 0.24 ± 0.02   | 45.1 ± 53.8 |
|                 | MPEC approach($tol = 10^{-2}$)       | 0.59 ± 0.02   | 0.26 ± 0.02   | 52.1 ± 42.2 |
|                 | MPEC approach($tol = 10^{-6}$)       | 0.59 ± 0.02   | 0.26 ± 0.02   | 326.8 ± 312.0 |

values of $\epsilon$ and $tol$ do not influence the cross-validation error and test error obtained by iP-DCA a lot. The case $\epsilon = 0$ takes more time to achieve desired tolerance on some data sets. This may be because that, as we have shown in Propositions 7 and 8, when $\epsilon = 0$, NNAMCQ never hold for the problem (VP), the ENNAMCQ always holds for the problem (VP)$_\epsilon$ when $\epsilon > 0$. As a consequence, the problem (VP) is more ill-
conditioned compared to the problem \((VP)_{\epsilon}\) with \(\epsilon > 0\), then iP-DCA performs better on the problem \((VP)_{\epsilon}\) with \(\epsilon > 0\). Compared with MPEC approach, our proposed iP-DCA achieves a smaller cross-validation error, which is exactly the value of upper level objective of the bilevel problem (SVBP), on datasets “breast-cancer_scale” and “diabetes_scale”. Furthermore, the time taken by our proposed iP-DCA is shorter than MPEC approach. The test errors of our proposed iP-DCA and MPEC approach are similar and iP-DCA achieves a smaller test error than MPEC approach on dataset “diabetes_scale”. Moreover both iP-DCA and MPEC approach can obtain a relatively good solution without requiring a small \(tol\).

Next, we are going to test our proposed iP-DCA on two large scale datasets “mushrooms” and “phishing” downloaded from the SVMLib repository. The descriptions of datasets are given in Table 1. We set \(tol = 10^{-2}\) for these tests. The numerical results averaged over 20 repetitions for each data set are reported in Table 3. It can be observed from Table 3 that different values of \(\epsilon\) and \(tol\) do not influence the cross-validation error obtained by iP-DCA much but the case \(\epsilon = 0\) takes more time to achieve desired tolerance on some data sets. And iP-DCA can obtain a satisfactory solution within an acceptable time on large scale problems.

### 4.2 Further numerical tests

In the follow-up paper [17], experiments on the SV bilevel model selection problem (SVBP) and comparisions with the state-of-the-art approaches in machine learning community, including the grid search, the random search and the tree-structured Parzen estimator Bayesian approach (TPE) [7] have been conducted. In this subsection we summarize these results.

All the numerical experiments are implemented on a computer with Intel(R) Core(TM) i9-9900K CPU @ 3.60GHz and 16.00 GB memory. All the codes are implemented in Python and are available at [https://github.com/SUSTech-Optimization/VF-iDCA](https://github.com/SUSTech-Optimization/VF-iDCA). Six real datasets “liver-disorders_scale”, “diabetes_scale”, “breast-cancer_scale”, “sonar”, “a1a”, “w1a” collected from the SVMLib repository are tested. For each dataset, the SV bilevel model selection problem (SVBP) with three-fold and six-fold cross-validations, i.e. \(T = 3, 6\) are solved respectively. Each dataset is randomly split in the same way as in Sect. 4.1. The experiments are repeated

| Dataset | Method | CV error | Test error | Time (sec) |
|---------|--------|----------|------------|------------|
| mushrooms | iP-DCA(\(\epsilon = 0\) | 6.36e-04 ± 5.94e-04 | 0 ± 0 | 334.3 ± 346.1 |
| | iP-DCA(\(\epsilon = 10^{-2}\) | 1.53e-03 ± 3.85e-03 | 3.57e-04 ± 1.34e-03 | 109.3 ± 35.2 |
| | iP-DCA(\(\epsilon = 10^{-4}\) | 6.38e-04 ± 6.08e-04 | 0 ± 0 | 162.9 ± 27.4 |
| phishing | iP-DCA(\(\epsilon = 0\) | 0.29 ± 0.00 | 0.09 ± 0.00 | 357.9 ± 95.2 |
| | iP-DCA(\(\epsilon = 10^{-2}\) | 0.29 ± 0.00 | 0.09 ± 0.00 | 222.1 ± 18.9 |
| | iP-DCA(\(\epsilon = 10^{-4}\) | 0.29 ± 0.00 | 0.09 ± 0.00 | 215.4 ± 46.5 |
30 times for each dataset. The values of parameters in SV bilevel model selection (SVBP) are set as: $\lambda_{lb} = 10^{-4}$, $\lambda_{ub} = 10^4$, $\bar{w}_{lb} = 10^{-6}$ and $\bar{w}_{ub} = 10$.

For the implementation of iP-DCA, unlike in Sect. 4.1, in which iP-DCA is applied to solve problem (SVBP) directly, for the numerical experiments in this part, the hyperparameter decoupling technique is applied to reformulate the problem (SVBP) into a DC bilevel program (DCBP) (see [17] for details) before applying iP-DCA. The strongly convex subproblem in iP-DCA at each iteration is solved by using the CVXPY package. Parameters in iP-DCA are set as $\epsilon = 0$, $\beta_0 = 1$ and $\delta_\beta = 5$. And iP-DCA is terminated when $\max\{(\|x^{k+1} - y^{k+1}\|/(1 + \|x^k - y^k\|)), t^{k+1}\} < tol$. The computational performance of iP-DCA is compared using two different values of $tol$, i.e., $tol = 10^{-1}, 10^{-2}$.

For the implementation of the grid search and the random search, the searches are run over two-dimension hyperparameter $\theta = (\theta_1, \theta_2)$ on $\{−4, −3, \ldots, 3, 4\} \times \{−6, −5, \ldots, 1, 2\}$ and with setting $\mu = 10^{\theta_1}$ and $\bar{w} = (10^{\theta_2}, \ldots, 10^{\theta_2})^T$ in the problem (SVBP). The subproblems in the grid search and the random search are all solved by using the CVXPY package. And for the implementation of TPE, the hyperparameter $\log_{10}(\mu)$ in $[−4, 4]$, and the hyperparameter $\log_{10}(\bar{w}_i)$ in $[−6, 2]$ are searched, respectively. Because TPE will be extremely slow when the dimension of the hyperparameters is too high, the maximum number of iteration of TBE is set to be 10. And the TPE method is also tested on the simplified model with a two-dimension hyperparameter, that has the same setting as the one solved by the search methods. This method is denoted by “TPE2”. For this simplified “TPE2”, the maximum number of iterations of the TPE method are set to be 100. The TPE method is implemented using the code collected from https://github.com/hyperopt/hyperopt and its subproblem is solved by using the CVXPY package. In the implementation of all the methods, the CVXPY package is set with using the open source solvers ECOS and SCS.

The achieved numerical results of the three-fold and the six-fold SV bilevel model selection problem averaged over 30 repetitions for each dataset are reported in Tables 4 and 5, respectively. The cross-validation error (CV error) and the test error are calculated in the same way as in Sect. 4.1. Observe from Tables 4 and 5 that compared with the state of the art approaches, including the grid search method, the random search method and the TPE, our proposed iP-DCA shows superiority by achieving a smaller cross-validation error and also a smaller test error. Furthermore, the time spent by our proposed iP-DCA is shorter than other approaches on most of the datasets. It can be also observed that on all the datasets except “breast-cancer_scale”, different values of $tol$ do not influence the cross-validation error and test error obtained by iP-DCA a lot. In view of the test error, iP-DCA can always obtain a relatively good solution without requiring a tight tolerance. This suggests to set a moderate algorithmic tolerance for iP-DCA when we apply it on practical problems for obtaining a satisfactory solution with shorter computation time.
Table 4 Numerical results of three-fold SV bilevel model selection problem on datasets “liver-disorders_scale”, “diabetes_scale”, “breast-cancer_scale”, “sonar”, “a1a”, “w1a”

| Dataset           | Method         | CV error       | Test error     | Time(sec) |
|-------------------|----------------|----------------|----------------|-----------|
| liver – disorders_scale | iP-DCA(tol = 10^{-1}) | 0.53 ± 0.07    | 0.27 ± 0.03    | 0.09 ± 0.02 |
| \( \ell_{train} = 72 \) | iP-DCA(tol = 10^{-2}) | 0.53 ± 0.09    | 0.28 ± 0.05    | 0.20 ± 0.06 |
| \( \ell_{test} = 73 \) | Grid Search    | 0.64 ± 0.08    | 0.34 ± 0.06    | 0.53 ± 0.01 |
| \( n = 5 \)      | Random Search  | 0.58 ± 0.06    | 0.32 ± 0.05    | 0.56 ± 0.03 |
|                   | TPE            | 0.65 ± 0.07    | 0.34 ± 0.05    | 0.37 ± 0.29 |
|                   | TPE2           | 0.61 ± 0.07    | 0.33 ± 0.06    | 2.88 ± 1.16 |
|                   | Grid Search    | 0.55 ± 0.03    | 0.33 ± 0.05    | 1.70 ± 0.11 |
| diabetes_scale    | iP-DCA(tol = 10^{-1}) | 0.48 ± 0.02    | 0.23 ± 0.01    | 0.18 ± 0.02 |
| \( \ell_{train} = 384 \) | iP-DCA(tol = 10^{-2}) | 0.48 ± 0.02    | 0.23 ± 0.01    | 0.28 ± 0.03 |
| \( \ell_{test} = 384 \) | Random Search  | 0.56 ± 0.04    | 0.30 ± 0.06    | 1.83 ± 0.09 |
| \( n = 8 \)      | TPE            | 0.55 ± 0.03    | 0.29 ± 0.05    | 6.64 ± 4.30 |
|                   | TPE2           | 0.54 ± 0.03    | 0.32 ± 0.06    | 18.67 ± 7.84 |
| breast – cancer_scale | iP-DCA(tol = 10^{-1}) | 0.09 ± 0.01    | 0.04 ± 0.01    | 0.14 ± 0.01 |
| \( \ell_{train} = 388 \) | iP-DCA(tol = 10^{-2}) | 0.05 ± 0.01    | 0.03 ± 0.01    | 1.12 ± 0.59 |
| \( \ell_{test} = 345 \) | Grid Search    | 0.08 ± 0.01    | 0.12 ± 0.06    | 1.63 ± 0.04 |
| \( n = 10 \)     | Random Search  | 0.09 ± 0.01    | 0.08 ± 0.09    | 1.80 ± 0.03 |
|                   | TPE            | 0.09 ± 0.01    | 0.10 ± 0.11    | 9.14 ± 4.55 |
|                   | TPE2           | 0.07 ± 0.01    | 0.09 ± 0.10    | 14.72 ± 6.02 |
| sonar             | iP-DCA(tol = 10^{-1}) | 0.03 ± 0.02    | 0.24 ± 0.04    | 0.48 ± 0.09 |
| \( \ell_{train} = 102 \) | iP-DCA(tol = 10^{-2}) | 0.00 ± 0.00    | 0.24 ± 0.04    | 2.22 ± 1.50 |
| \( \ell_{test} = 106 \) | Grid Search    | 0.58 ± 0.08    | 0.40 ± 0.12    | 3.19 ± 0.10 |
| \( n = 60 \)     | Random Search  | 0.54 ± 0.06    | 0.34 ± 0.10    | 3.23 ± 0.06 |
|                   | TPE            | 0.64 ± 0.10    | 0.41 ± 0.12    | 40.77 ± 7.12 |
|                   | TPE2           | 0.57 ± 0.08    | 0.37 ± 0.13    | 18.47 ± 6.84 |
| a1a               | iP-DCA(tol = 10^{-1}) | 0.27 ± 0.02    | 0.17 ± 0.01    | 1.10 ± 0.07 |
| \( \ell_{train} = 801 \) | iP-DCA(tol = 10^{-2}) | 0.27 ± 0.02    | 0.18 ± 0.01    | 10.17 ± 5.47 |
| \( \ell_{test} = 804 \) | Grid Search    | 0.41 ± 0.02    | 0.24 ± 0.02    | 8.04 ± 0.15 |
| \( n = 123 \)    | Random Search  | 0.41 ± 0.02    | 0.22 ± 0.03    | 8.62 ± 0.30 |
|                   | TPE            | 0.42 ± 0.03    | 0.23 ± 0.03    | 176.59 ± 17.38 |
|                   | TPE2           | 0.41 ± 0.02    | 0.24 ± 0.02    | 65.51 ± 16.24 |
| w1a               | iP-DCA(tol = 10^{-1}) | 0.01 ± 0.00    | 0.02 ± 0.00    | 4.87 ± 0.51 |
| \( \ell_{train} = 1236 \) | iP-DCA(tol = 10^{-2}) | 0.01 ± 0.00    | 0.02 ± 0.00    | 27.49 ± 7.31 |
| \( \ell_{test} = 1241 \) | Grid Search    | 0.06 ± 0.01    | 0.03 ± 0.00    | 20.21 ± 0.82 |
| \( n = 300 \)    | Random Search  | 0.06 ± 0.01    | 0.03 ± 0.00    | 20.44 ± 1.10 |
|                   | TPE            | 0.06 ± 0.01    | 0.03 ± 0.00    | 299.62 ± 78.72 |
|                   | TPE2           | 0.06 ± 0.01    | 0.03 ± 0.00    | 86.10 ± 28.19 |
| Dataset                              | Method                  | CV error      | Test error     | Time(sec)  |
|-------------------------------------|-------------------------|---------------|----------------|------------|
| liver-disorders_scale, \(\ell_{\text{train}} = 72\), \(\ell_{\text{test}} = 73\), \(n = 5\) | iP-DCA(\(tol = 10^{-1}\)) | 0.41 ± 0.08   | 0.27 ± 0.04   | 0.18 ± 0.03 |
|                                     | iP-DCA(\(tol = 10^{-2}\)) | 0.41 ± 0.08   | 0.27 ± 0.05   | 0.33 ± 0.14 |
|                                     | Grid Search             | 0.63 ± 0.08   | 0.33 ± 0.07   | 0.78 ± 0.02 |
|                                     | Random Search           | 0.62 ± 0.07   | 0.31 ± 0.05   | 0.79 ± 0.04 |
|                                     | TPE                     | 0.63 ± 0.08   | 0.34 ± 0.05   | 1.06 ± 1.04 |
|                                     | TPE2                    | 0.62 ± 0.07   | 0.32 ± 0.06   | 6.88 ± 4.15 |
| diabetes_scale, \(\ell_{\text{train}} = 384\), \(\ell_{\text{test}} = 384\), \(n = 8\) | iP-DCA(\(tol = 10^{-1}\)) | 0.43 ± 0.02   | 0.23 ± 0.01   | 0.35 ± 0.01 |
|                                     | iP-DCA(\(tol = 10^{-2}\)) | 0.43 ± 0.02   | 0.23 ± 0.01   | 0.56 ± 0.08 |
|                                     | Grid Search             | 0.55 ± 0.03   | 0.32 ± 0.05   | 3.18 ± 0.14 |
|                                     | Random Search           | 0.56 ± 0.03   | 0.31 ± 0.05   | 3.63 ± 0.21 |
|                                     | TPE                     | 0.55 ± 0.03   | 0.27 ± 0.06   | 29.52 ± 13.13 |
|                                     | TPE2                    | 0.55 ± 0.03   | 0.33 ± 0.05   | 51.85 ± 20.49 |
| breast-cancer_scale, \(\ell_{\text{train}} = 388\), \(\ell_{\text{test}} = 345\), \(n = 10\) | iP-DCA(\(tol = 10^{-1}\)) | 0.08 ± 0.01   | 0.04 ± 0.01   | 0.29 ± 0.08 |
|                                     | iP-DCA(\(tol = 10^{-2}\)) | 0.03 ± 0.01   | 0.03 ± 0.01   | 2.01 ± 0.17 |
|                                     | Grid Search             | 0.08 ± 0.02   | 0.15 ± 0.06   | 3.38 ± 0.25 |
|                                     | Random Search           | 0.08 ± 0.02   | 0.07 ± 0.08   | 3.92 ± 0.29 |
|                                     | TPE                     | 0.09 ± 0.01   | 0.11 ± 0.13   | 25.96 ± 12.95 |
|                                     | TPE2                    | 0.07 ± 0.02   | 0.08 ± 0.09   | 38.69 ± 16.27 |
| sonar, \(\ell_{\text{train}} = 102\), \(\ell_{\text{test}} = 106\), \(n = 60\) | iP-DCA(\(tol = 10^{-1}\)) | 0.00 ± 0.00   | 0.23 ± 0.04   | 0.92 ± 0.02 |
|                                     | iP-DCA(\(tol = 10^{-2}\)) | 0.00 ± 0.00   | 0.23 ± 0.04   | 0.92 ± 0.02 |
|                                     | Grid Search             | 0.59 ± 0.08   | 0.39 ± 0.11   | 6.57 ± 0.32 |
|                                     | Random Search           | 0.54 ± 0.06   | 0.32 ± 0.08   | 6.44 ± 0.28 |
|                                     | TPE                     | 0.60 ± 0.07   | 0.39 ± 0.12   | 97.65 ± 31.37 |
|                                     | TPE2                    | 0.57 ± 0.08   | 0.36 ± 0.12   | 58.19 ± 29.60 |
| a1a, \(\ell_{\text{train}} = 801\), \(\ell_{\text{test}} = 804\), \(n = 123\) | iP-DCA(\(tol = 10^{-1}\)) | 0.19 ± 0.01   | 0.17 ± 0.01   | 4.22 ± 0.37 |
|                                     | iP-DCA(\(tol = 10^{-2}\)) | 0.18 ± 0.02   | 0.17 ± 0.01   | 63.01 ± 186.14 |
|                                     | Grid Search             | 0.40 ± 0.02   | 0.25 ± 0.02   | 17.60 ± 0.36 |
|                                     | Random Search           | 0.40 ± 0.02   | 0.21 ± 0.03   | 18.59 ± 0.42 |
|                                     | TPE                     | 0.41 ± 0.03   | 0.23 ± 0.03   | 312.63 ± 60.60 |
|                                     | TPE2                    | 0.40 ± 0.02   | 0.24 ± 0.02   | 161.68 ± 42.67 |
| w1a, \(\ell_{\text{train}} = 1236\), \(\ell_{\text{test}} = 1241\), \(n = 300\) | iP-DCA(\(tol = 10^{-1}\)) | 0.01 ± 0.00   | 0.02 ± 0.00   | 26.74 ± 3.67 |
|                                     | iP-DCA(\(tol = 10^{-2}\)) | 0.01 ± 0.00   | 0.02 ± 0.00   | 97.50 ± 35.99 |
|                                     | Grid Search             | 0.05 ± 0.00   | 0.03 ± 0.00   | 44.29 ± 1.39 |
|                                     | Random Search           | 0.05 ± 0.00   | 0.03 ± 0.00   | 61.80 ± 2.91 |
|                                     | TPE                     | 0.05 ± 0.01   | 0.03 ± 0.00   | 703.72 ± 82.75 |
|                                     | TPE2                    | 0.05 ± 0.00   | 0.03 ± 0.00   | 190.04 ± 39.00 |
5 Concluding remarks

Motivated by hyperparameter selection problems, in this paper we develop two DCA type algorithms for solving the DC bilevel program. Our numerical experiments on the SV bilevel model selection show that our approach is promising. Due to the space limit, we are not able to present more studies for more complicated models in hyperparameter selection problems. We hope to study these problems in our future work. Note that all of our results except the result on the constraint qualification ENNAMCQ in Proposition 8 can be applied to the case where there are also some extra upper level constraints $G(x, y) := (G_1(x, y), \ldots, G_k(x, y)) \leq 0$ as long as each function $G_i(x, y)$ is a difference of convex function. In this case, the corresponding approximate bilevel program has an extra DC constraint $G(x, y) \leq 0$. Although the constraint qualification ENNAMCQ no longer holds automatically, it is reasonable to impose ENNAMCQ for the corresponding approximate bilevel program to hold.

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