Propagation of nonlinear waves in disordered media

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Abstract

We study the propagation of stationary waves in disordered non-linear media described by the nonlinear Schrödinger equation and show that for given boundary conditions and a given coherent wave incident on the sample, the number of solutions of the equation increases exponentially with sample size. We also discuss the ballistic case, the sensitivity of the solutions to the change of external parameters, the similarity of this problem to the problem of spin glasses and time-dependent solutions.

I. INTRODUCTION

In this article we consider the propagation of a stationary coherent wave described by a field $\phi(r)$, in a nonlinear elastically scattering medium. Though we believe that our results are of a general character, for the sake of concreteness we consider the situation where the wave is described by a nonlinear Schrödinger equation

$$\left\{ -\frac{1}{2m} \frac{\partial^2}{\partial r^2} - \epsilon + u(r) + \beta n(r) \right\} \phi(r) = 0 \quad (1)$$

Here $n(r) = |\phi(r)|^2$ is the density, $m$ is the wave mass, $\epsilon$ is the wave energy, $\beta$ is a constant and $u(r)$ is a scattering potential, which is a random function of the coordinates. Eq.1
appears, for example, in the theory of electromagnetic waves propagating in nonlinear media [1], in the theory of hydrodynamic turbulence [2], and in the theory of turbulent plasma [3].

We will assume that \( u(\mathbf{r}) \) obeys the white noise statistics: \( \langle u(\mathbf{r}) \rangle = 0 , \langle u(\mathbf{r})u(\mathbf{r}_1) \rangle = \frac{\pi}{lm^2} \delta(\mathbf{r} - \mathbf{r}_1) \). Here brackets \( \langle \rangle \) correspond to averaging over realizations of \( u(\mathbf{r}) \), and \( l \) is the elastic mean free path \( (l \gg k^{-1} = (2\epsilon m)^{-\frac{1}{2}}) \). In the presence of the scattering potential \( u(\mathbf{r}) \), the spatial dependence of the \( n(\mathbf{r}) \) exhibits random sample-specific fluctuations, which are called speckles. In the case of elastically scattering diffusive linear media this problem was considered in [4–6]. Below we will be interested in the statistics of \( n(\mathbf{r}) \) in the case of nonlinear diffusive media. In particular, we will show that, for given boundary conditions, the number of solutions of Eq.1 increases exponentially with sample size. A brief summary of some of the results has been published in [7].

II. NONLINEAR SPECKLES.

Let us consider the case where a coherent wave \( \phi_0(\mathbf{r}) = \sqrt{n_0}e^{i\mathbf{k}\mathbf{r}} \) with momentum \( \mathbf{k} \) is incident on a disordered sample of size \( L \gg l \) (See the insert in Fig.1a).

The \( \mathbf{r} \)-dependence of the average density \( \langle n(\mathbf{r}) \rangle \) can be described by the diffusion equation,

\[
\langle \mathbf{j}(\mathbf{r}) \rangle = -D \frac{\partial \langle n(\mathbf{r}) \rangle}{\partial \mathbf{r}} \quad \text{div} \langle \mathbf{j}(\mathbf{r}) \rangle = 0
\]

which is equivalent the to calculation of the diagrams shown in Fig.2.a. Here \( \mathbf{j}(\mathbf{r}) \) is the current density. In the limit \( k^{-1} \ll l \ll L \) the expression for the diffusion coefficient \( D = \frac{lk}{3m} \) has a classical form. In the case of the geometry shown in Fig.1, the total flux through the sample is zero, and the average density \( \langle n(\mathbf{r}) \rangle = n_0 \) is spatially uniform.

The term \( \beta|\phi(\mathbf{r})|^2 \) in Eq.1 plays the role of an additional scattering potential. It will be shown that its contribution to the diffusion coefficient \( D \) can be neglected at small enough intensities of the incident beam, when
\[ |\beta n_0|^2 \ll \frac{ek}{lm}. \]  

At \( \beta > 0 \) the propagation of a uniform wave in an infinite medium described by Eq.1 is unstable at arbitrarily small \( n_0 \) due to the effect of nonlinear self focusing. The characteristic length at which the self-focusing takes place is of order \( r_{(sf)} \sim \frac{ek}{(\beta n_0)^2 m} \) [1]. Thus the inequality Eq.3 is equivalent to \( r_{(sf)} \gg l \). One can say that the regime Eq.3 corresponds to a system of randomly distributed weak concave and convex lenses.

The diffusion equation approximation completely neglects interference effects, which lead to the existence of speckles. To describe them one should solve Eq.1 before averaging over the realizations of \( u(r) \). It is convenient to expand the density

\[ \beta n(r) = \frac{D}{\sqrt{L}} \sum_{i=1}^{\infty} i^{1/3} \bar{u}_i n_i(r) \]

in a complete set of orthonormal eigenstates \( n_i(r) \) of the diffusion equation (\( \int dr n_i^2(r) = 1 \)):

\[ D\frac{\partial^2}{\partial r^2} n_i(r) = E_i n_i(r) \]  

where \( E_i \sim \frac{D}{2} i^{2/3} \) are the eigenvalues of Eq.5, and \( i = 1, 2... \) labels the eigenstates. We will show below why the expansion Eq.4 is convenient. We assume boundary conditions corresponding to zero current through a closed boundary, and \( n(r) = 0 \) at the open boundary.

Let us first substitute Eq.4 into the nonlinear term of Eq.1, and regard the \( \{\bar{u}_i\} = \bar{u}_1, \bar{u}_2, ... \) as independent parameters. This gives a linear equation for \( \phi(r) \)

\[ \left( -\frac{1}{2m} \frac{\partial^2}{\partial r^2} - \epsilon + u(r) + \beta \frac{D}{\sqrt{L}} \sum_{i=1}^{\infty} i^{1/3} \bar{u}_i n_i(r) \right) \phi(r) = 0 \]

Denoting a solution of Eq.6 at a given set of parameters \( \{\bar{u}_i\} \), as \( \phi(r, \{u(r)\}, \{\bar{u}_m\}) \) and constructing \( n(r, \{u(r)\}, \{\bar{u}_i\}) = |\phi(r, \{u(r)\}, \{\bar{u}_i\})|^2 \), we can write the self-consistency equations for \( \{\bar{u}_i\} \):

\[ \frac{i^{2/3} \bar{u}_i}{\gamma} = F_i(\{\bar{u}_i\}). \]  

Here
\[
\gamma = \left| \frac{3n_0 \beta}{2\epsilon} \right| \left( \frac{L}{l} \right)^{3/2}
\]

and

\[
F_i(\{\bar{u}_i\}) = \frac{k_i^{1/3} l^{1/2}}{n_0 L} \int d\mathbf{r} n(\mathbf{r}, \{u(\mathbf{r})\}, \{\bar{u}_i\}) n_i(\mathbf{r})
\]

are dimensionless random functions of \( \{\bar{u}_i\} \), the form of which depends on \( u(\mathbf{r}) \).

To investigate the properties of the solutions of Eq.7, we have to know the statistical properties of the random functions \( F_i(\{\bar{u}_i\}) \). We can infer these properties from values of different correlation functions of \( F_i(\{\bar{u}_i\}) \) obtained by averaging over realizations of \( u(\mathbf{r}) \). It is important that the statistical analysis of the random functions \( F_i(\{\bar{u}_i\}) \) is equivalent to the analysis of linear speckles, which has been done in [4–6]. To characterize \( F_i(\{\bar{u}_i\}) \) we calculate the following correlation functions

\[
\langle \delta F_i(\{\bar{u}_i\}) \delta F_j(\{\bar{u}_i\}) \rangle = \delta_{ij},
\]

\[
\langle [F_i(\{\bar{u}_i + \Delta \bar{u}_i\}) - F_i(\{\bar{u}_i\})]^2 \rangle \sim (\Delta \bar{u}_n)^2 \quad \text{at} \quad \Delta \bar{u}_i \sim 1.
\]

\[
\frac{\partial F_i}{\partial \bar{u}_r} \times \frac{\partial F_i}{\partial \bar{u}_s} \sim \frac{1}{((r/s)^{1/3} + (s/r)^{1/3})^2} \left( \frac{1}{|i - j| + |r - s|^{2/3}} \right)
\]

Here \( \delta F_i = F_i - \langle F_i \rangle \) and \( \langle F_i(\{\bar{u}_i\}) \rangle = \text{const} \) which is independent of \( \{\bar{u}_i\} \). We present the derivation of Eqs.10-12 in the next section. The simple form of Eqs.10-12 is a consequence of the choice of \( n_i(\mathbf{r}) \) in Eqs.4,5. Eq.10 indicates that mesoscopic fluctuations of different functions \( F_i \) are uncorrelated. According to Eq.12, the correlation of derivatives of \( F_i(\{\bar{u}_i\}) \) over different \( \bar{u}_r \) is small for \( |r - s| > 1 \). It will be shown in the next section that these facts are consequences of the choice of \( n_i(\mathbf{r}) \) in Eq.4 as eigenfunctions of Eq.5. The introduction of the coefficients \( l^{1/3} D/\sqrt{L} \) in Eq.4 ensures Eqs.10,11 appears as above.

Thus we arrive at the following picture: the \( F_i(\{\bar{u}_i\}) \) fluctuate randomly as functions of \( \{\bar{u}_i\} \) near their average, which is independent of \( \{\bar{u}_i\} \). According to Eq.11, the characteristic period of the fluctuations is of order one. The fluctuations both of different functions, and the same functions with respect to different \( \bar{u}_i \) are uncorrelated.
Using this information about the \( F_i(\{\bar{u}_i\}) \) we can estimate the number of solutions of Eq.7 (or Eq.1). If \( \gamma \ll 1 \), Eq.7 has a unique solution while for \( \gamma \gg 1 \) Eq.7 has many solutions. Let us consider the \( i^{th} \) equation in Eq.7 and fix all variables \( \bar{u}_j \) other than \( \bar{u}_i \). Then, at \( i^{2/3} \gamma^{-1} \gg 1 \) the equation has a unique solution, while at \( i^{2/3} \gamma^{-1} \ll 1 \) the number of the solutions is of order \( \gamma i^{-2/3} \). In Fig.1a we show a qualitative graphical solution of Eq.7, which corresponds to the intersection of two functions: \( F_i(\ldots, \bar{u}_i, \ldots) \) and \( \gamma^{-1} i^{2/3} \bar{u}_i \). The solutions are distributed in an interval of order \( \gamma i^{-2/3} \).

Therefore, to estimate number of solutions of Eqs.7 at \( \gamma \gg 1 \), we have to take into account only a subset of Eqs.7 with \( i < I = \gamma^{3/2} \). Since both the amplitude of fluctuations and the periods in the \( i^{th} \) direction of randomly rippled hypersurfaces \( F_i(\{\bar{u}_i\}) \) are of order unity, the number of solutions \( N \) of Eqs.1,7 is proportional to the volume of an \( I \)-dimensional hyperparallelepiped with sides of order \( \gamma i^{-2/3} \), \( i < I \). As a result we have

\[
N \sim \gamma^I \prod_{i=1}^{I} i^{-2/3} \sim \exp\left( \frac{2}{3} \gamma^{3/2} \right)
\]

(13)

Thus the number of the solutions \( N \) of Eq.1 increases exponentially with the sample size \( L \).

To illustrate Eq.13 we consider the case \( I = 2 \) (\( \gamma \sim 1 \)). Then Eqs.7 can be viewed as two surfaces \( z = F_1(\bar{u}_1, \bar{u}_2) \) and \( z = F_2(\bar{u}_1, \bar{u}_2) \), which are intersected by two planes \( z = \gamma^{-1} \bar{u}_1 \) and \( z = \gamma^{-1} i^{2/3} \bar{u}_2 \) respectively. A result of these intersections is two systems of lines in the plane \( \bar{u}_1, \bar{u}_2 \) shown in Fig.1b. The solid lines correspond to intersections between \( z = F_1(\bar{u}_1, \bar{u}_2) \) and \( z = \gamma^{-1} \bar{u}_1 \), and are located within a strip in the \( \bar{u}_1 \) direction of width of order \( \gamma \). The dashed lines correspond to the intersection of the surfaces \( z = F_2(\bar{u}_1, \bar{u}_2) \) and \( z = \gamma^{-1} i^{2/3} \bar{u}_2 \), and are located within a strip of width \( \gamma 2^{-2/3} \) in \( \bar{u}_2 \) direction. The intersections of solid and dashed lines in Fig.1b correspond to solutions of the system of Eqs.7. According to Eqs.10-12, the dashed and the solid lines in Fig.1b are uncorrelated. The typical distance between, say, solid lines is of order one. As a result, the number of solutions \( N_{I=2} \) of Eqs.7 in this case is of the order of the area of a parallelogram with sides \( \gamma \) and \( 2^{-2/3} \gamma \)

\[
N_{I=2} \sim \gamma \times 2^{-2/3} \gamma.
\]

(14)
Now let us estimate the corrections to the diffusion coefficients originating from scattering from the potential \( \beta n(r) \). In the case \(|r - r'| < l\) we have \((D = 3)\) [6]

\[
\langle n(r)n(r') \rangle = \frac{n_0^2 m^2}{k^2 |r - r'|^2},
\]

(15)

independently of the values of the parameters \( \{\bar{u}_i\} \). In the Born approximation, the nonlinear mean free path corresponding to scattering from the potential \( \beta n(r) \) is:

\[
l_{(nl)} = \frac{\epsilon k^2}{(\beta n_0)^2 m}.
\]

(16)

Thus the criterion Eq.3 is equivalent to \( l_{(nl)} = r_{(sf)} \gg l \).

In the opposite limit \( l_{(nl)} \ll l \), the scattering mean free path is determined by the scattering from the potential \( \beta n(r) \), and one can neglect the random potential \( u(r) \) in Eq.1. In this case one can estimate the number of solutions of Eq.1 by substituting \( l_{(nl)} \) instead of \( l \) into Eqs.8, or by substituting \( \gamma_{(nl)} \) into Eq.13 instead of \( \gamma \), where

\[
\gamma_{(nl)} = 3 \left( \frac{n_0 \beta}{\epsilon} \right)^4 \left( \frac{Lm}{\epsilon k} \right)^{3/2}.
\]

(17)

In the conclusion of this section we would like to discuss the condition \( \gamma > 1 \) for the existence of multiple solutions of Eq.1. In the absence of the nonlinear term \( \beta n(r) \) the solution of Eq.1 corresponds to particles traveling along diffusion trajectories. In the presence of the nonlinear term \( \beta n(r) \), the probability amplitude for traveling along a diffusive trajectory acquires an additional phase of order \( \delta \chi_{(nl)} = \frac{\beta k}{2} \int ds \delta n(r) \sim 1 \), where the integration is taken along a diffusive trajectory. To estimate the value of the additional phase let us calculate the integral

\[
\left\langle \left( \delta \chi_{(nl)} \right)^2 \right\rangle = \left( \frac{\beta k}{2} \right)^2 \int ds ds' \langle \delta n(r) \delta n(r') \rangle
\]

(18)

The correlation function of densities is given by Eq.15 and the estimate for Eq.18 is \((\beta k)^2 L^3 / \ell^2\). Thus the criterion \( \gamma > 1 \) corresponds to \( \langle(\delta \chi_{(nl)})^2 \rangle > 1 \). Another interpretation is that at \( \gamma > 1 \) the sensitivity of the solutions of Eq.1 to a change of \( u(r) \) increases significantly as compared to its single particle value [7].

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III. CALCULATION OF CORRELATION FUNCTIONS OF EQUATIONS 11-13

Now we turn to the calculation of correlation functions Eqs.10-12. According to the definition Eq.9 we get

\[
\langle F_i(\{\bar{u}_i\}) F_j(\{\bar{u}_i + \Delta u_i\}) \rangle = \frac{i^{1/3} j^{1/3} L^2}{2 n_0^2} \int d\mathbf{r} d\mathbf{r}' n_i(r) n_j(\mathbf{r}') \langle n(\mathbf{r}, \{\bar{u}_i\}) n(\mathbf{r}_1, \{\bar{u}_i + \Delta u_i\}) \rangle \tag{19}
\]

In the approximation Eq.3, the value of \( \langle n(\mathbf{r}, \{\bar{u}_i\}) n(\mathbf{r}_1, \{\bar{u}_i + \Delta u_i\}) \rangle \) is independent of \( \{\bar{u}_i\} \) and has the same form as in the linear case [4,6]. Therefore, one can use the standard diagram technique for averaging over \( u(\mathbf{r}) \) [11]. The diagrams describing the correlation functions \( \langle n(\mathbf{r}, \{\bar{u}_i\}) n(\mathbf{r}', \{\bar{u}_i + \Delta u_i\}) \rangle \) are shown in Fig.2b,c. Alternatively, one can solve the Langevin equation [4,6], valid at \( |\mathbf{r} - \mathbf{r}'| \gg l \)

\[
div \delta \mathbf{j}(\mathbf{r}) = 0 \tag{20}
\]

\[
\delta \mathbf{j}(\mathbf{r}) = -D \frac{\partial}{\partial \mathbf{r}} \delta n(\mathbf{r}) + J^L(\mathbf{r}, \{u(\mathbf{r}), \{\bar{u}_i\}\}) \tag{21}
\]

The correlation function of random Langevin forces \( J^L(\mathbf{r}) \) is given by the diagram shown in Fig.2b. It can be written as

\[
\begin{align*}
\langle J^L_i(\mathbf{r}, \{u(\mathbf{r}), \{\bar{u}_i\}\}) & J^L_j(\mathbf{r}', \{u(\mathbf{r}), \{\bar{u}_i + \Delta u_i\}\}) \rangle = \\
&= \frac{2\pi l}{3m^2} \frac{1}{|\phi(\mathbf{r}, \{\bar{u}_i\}) \phi^*(\mathbf{r}, \{\bar{u}_i + \Delta u_i\})|^2} \delta_{ij} \delta(\mathbf{r} - \mathbf{r}') \tag{22}
\end{align*}
\]

The expression for \( \langle \phi(\mathbf{r}, \{\bar{u}_i\}) \phi^*(\mathbf{r}, \{\bar{u}_i + \Delta u_i\}) \rangle \) is given by the ladder diagrams shown in Fig.2b, where the inner Green function correspond to \( \{\bar{u}_i\} \), while the outer Green functions correspond to \( \{\bar{u}_m + \Delta \bar{u}_m\} \). Eq.22 is a generalization of corresponding equations introduced in [4,6]. Namely, in the case \( \Delta \bar{u}_i = 0 \) one has to substitute \( \langle n \rangle^2 = n_0^2 \) in Eq.22 instead of \( |\langle \phi(\mathbf{r}, \{\bar{u}_i\}) \phi^*(\mathbf{r}, \{\bar{u}_i + \Delta u_i\}) \rangle|^2 \).

Using Eqs.20-22 we get

\[
\begin{align*}
\langle n(\mathbf{r}, \{\bar{u}_i\}) n(\mathbf{r}', \{\bar{u}_i + \Delta u_i\}) \rangle &= \\
&= \frac{2\pi l}{3m^2} \int d\mathbf{r_1} \frac{d\Pi(\mathbf{r}, \mathbf{r}_1)}{d\mathbf{r}_1} \frac{d\Pi(\mathbf{r}', \mathbf{r}_1)}{d\mathbf{r}_1} |\langle \phi(\mathbf{r}, \{\bar{u}_i\}) \phi^*(\mathbf{r}, \{\bar{u}_i + \Delta u_i\}) \rangle|^2. \tag{23}
\end{align*}
\]
\[ \Pi(r, r') = \sum_l \frac{n_l(r)n_l(r')}{E_l} \tag{24} \]

Here \( \Pi(r, r') \) is the Green’s function of Eq.5.

At \( \Delta \bar{u}_i = 0 \), doing the integral over \( r_1 \) in Eq.23 by parts, using Eq.22 and taking into account the orthogonality of the functions \( n_i(r) \), we get Eq.10. The diagrams in Fig.2.b for \( \langle \phi(r, \{ \bar{u}_i \})\phi^*(r, \{ \bar{u}_i + \Delta \bar{u}_i \}) \rangle \) are equivalent to the equation [9,10]

\[ \left( D \frac{\partial^2}{\partial^2 r} + i \frac{D}{\sqrt{L}} \sum_i i^{1/3} \Delta \bar{u}_i n_i(r) \right) \langle \phi(r, \{ \bar{u}_i \})\phi^*(r, \{ \bar{u}_i + \Delta \bar{u}_i \}) \rangle = 0 \tag{25} \]

The existence of the term proportional to \( \Delta \bar{u}_i \) in Eq.25 reflects the fact that the inner lines in the diagrams shown in Fig.2b correspond to the Green’s functions in the scattering potential characterized by \( \{ \bar{u}_i \} \) while the outer lines correspond to those of \( \{ \bar{u}_i + \Delta \bar{u}_i \} \). Note that as long as Eq.3 holds Eq.25 is independent of \( \{ \bar{u}_i \} \).

Solving Eq.25 by perturbation theory with respect to \( \Delta \bar{u}_i \) we get

\[ \langle \phi(r, \{ \bar{u}_i \})\phi^*(r, \{ \bar{u}_i + \Delta \bar{u}_i \}) \rangle = n_0 \left( 1 + i \frac{D}{\sqrt{L}} \sum_i i^{1/3} n_i(r) \Delta \bar{u}_i - \frac{D^2}{L} \int d\mathbf{r}' \Pi(r, \mathbf{r}') \sum_i \frac{i^{2/3}(\Delta \bar{u}_i n_i(r'))^2}{E_i^2} + \ldots \right). \tag{26} \]

We can neglect the second term in brackets in Eq.26 because it is of order \( (i^{-1/3} \Delta \bar{u}_i) \) (and its contribution to Eq.22 is of order \( i^{-2/3}(\Delta \bar{u}_i)^2 \)), while the contribution to Eq.22 from the third term is \( (\Delta \bar{u}_i)^2 \). To get the latter estimate we took into account that \( \Pi(r, \mathbf{r}') \sim (DL)^{-1} \) at \( |r - r'| \sim L \). Substituting Eq.26 into Eq.19,22,23, we get Eqs.11,12.

### IV. DISCUSSION

The estimate Eq.14 was made for the case of a typical realization of the scattering potential. On the other hand, even at \( \gamma < 1 \), there are rare realizations of \( u(r) \), which correspond to several solutions of Eq.1.

The results presented above hold for arbitrary sign of \( \beta \). This is quite different from the situation in the pure case \( (u(r) = 0) \) [1] where at \( \beta > 0 \) self-focusing takes place.
At \( \gamma \gg 1 \), the solutions of Eq.1 exhibit exponentially large sensitivity to changes of parameters of the system \([7]\). Consider, for example, the case where the incident angle \( \theta \) of the wave is changing, and suppose that a solution of Eq.1 follows this change adiabatically. Then an exponentially small change

\[ \Delta \theta \sim \exp\left(-\frac{2}{3} \gamma^{3/2}\right) \]  

(27)

will lead to disappearance of the solution, and the system will jump to another solution.

Similar phenomenon may occur in the system of interacting electrons is disordered metals: it can be unstable with respect to the creation of random magnetic moments. This would correspond to Finkelshtain’s scenario \([12]\). However, in this case in order to get a self-consistency equation, which would be an analog of Eq.7, we have to integrate over electron energies, which decreases the amplitude of mesoscopic fluctuations. As a result, the situation with many solutions may occur only in the D=2 case and the characteristic spatial scale will be of the order of the electron localization length in the linear problem. Thus the problem of interacting electrons in disordered metals remains unsolved.

We would like to mention a similarity of the problem considered above to the problem of spin glasses. To illustrate this point let us consider a model in which the coefficient \( \beta(r) \) in Eq.1 is nonzero only at points \( r = r_{\alpha}, \alpha = 1, 2,... \)

\[ \beta(r) = \beta_0 \sum_{\alpha} U(r - r_{\alpha}), \]  

(28)

where \( U(|r|) \) is a short range function decaying on characteristic distance \( R < 1/k \), and of a maximum height \( U_0 \), and \( r_i \) are randomly distributed in space with given density. Then Eq.1 can be rewritten only in terms of values of \( \phi_{\alpha} = \phi(r = r_{\alpha}) \)

\[ \phi_{\alpha} = \beta_0 U_0 R^2 \sum_{\beta} G(r_{\alpha}, r_{\beta}) |\phi_{\beta}|^2 \phi_{\beta}, \]  

(29)

\[ G(r_{\alpha}, r_{\beta}) \sim \frac{\exp(ik|r_{\alpha} - r_{\beta}| + i\delta(r_{\alpha}, r_{\beta}))}{|r_{\alpha} - r_{\beta}|^{1/2}} \quad |r_{\alpha} - r_{\beta}| \gg l \]  

(30)

where \( G(r_{\alpha}, r_{\beta}) \) is the Green function of the linear Schrödinger equation, and the phase \( \delta(r_{\alpha}, r_{\beta}) \) is a random quantity at \( |r_{\alpha} - r_{\beta}| \gg l \). The major difference between Eq.30 and
the spin glass problem is that in the former case one is interested in the minimum of the
free energy, while in the case of Eq.30 the boundary conditions are given. Thus there are no
thermodynamic criteria on how to choose between multiple solutions of the stationary Eq.1.
One of the possibilities is that the state of the system is determined by its history.

We may question how many of these stationary states are stable. The time-dependent
non-linear Schrödinger equation may be obtained from Eq.1 by substituting \( \epsilon \) for \( i\partial_t \). Equivalently, we can write equations for \( \bar{u}_i(t) \). We still assume the same stationary boundary
conditions. In the absence of a complete solution we present a qualitative picture.

The characteristic time of change of the \( i^{th} \) harmonic \( \bar{u}_i \) is \( \tau_i \sim E_i^{-1} \). Thus we have

\[
\tau_i (1 + \gamma g_i(\{\bar{u}_i\})) \partial_t \bar{u}_i(t) = \gamma i^{-\frac{2}{3}} F(\{\bar{u}_i\}) - \bar{u}_i
\]

(31)

Here \( g_i(\{\bar{u}_i\}) \sim 1 \) is a dimensionless random function. The statistical properties of the
function \( g_i(\{\bar{u}_i\}) \) are roughly the same as those for \( F_i(\{\bar{u}_i\}) \). Namely, the amplitude and
sign randomly oscillate with a characteristic periods of order one. Strictly speaking, Eq.31
holds if \( |\tau_i \partial_t \bar{u}_i| \ll 1 \) and the characteristic time of establishing stationary distributions \( n(r) \)
at given \( \bar{u}_i(t) \) is much shorter than the characteristic time of change of \( \bar{u}_i(t) \). In other words,
Eqs.31 represent a sort of hydrodynamics. This takes place, for example, near the critical
instability points (see, for example, the point "a" on Fig1.a.). Generally, \( \tau_i \partial_t \bar{u}_i(t) \sim 1 \), so
we have to keep not only the first but also several higher time derivatives in Eqs.31. We
believe, however, that the model Eqs.31 captures the qualitative features of the system’s
dynamics correctly even in this case.

Linearizing Eqs.31 near the stationary solutions, we arrive at the conclusion that the
fraction of solutions of Eq.7 which are stable is of order \( 2^{-I} \). Thus, at \( \gamma \gg 1 \) the number of
stable stationary solutions is still exponentially large.

In principle, Eqs.31 can also have nonstationary solutions as \( t \to \infty \). Obviously, the
characteristic amplitudes of the solutions are given by Eq.7, \( |\delta \bar{u}_i(t)| < \gamma i^{-2/3} \). A complete
investigation of \( \{\bar{u}_i(t)\} \) is a complicated and still unsolved problem. For example, for \( I = 1 \)
and \( t \to \infty \) only stable stationary solutions are relevant. For \( I = 2 \), and \( t \to \infty \), depending
on the properties of the realizations of the random sample specific functions $F_1(\tilde{u}_1, \tilde{u}_2)$ and $F_2(\tilde{u}_1, \tilde{u}_2)$, one can, additionally, have periodic, quasiperiodic, and chaotic in time solutions. If $I = 3$, one can also, have strange (or stochastic) attractors as solutions of three differential equations.

In this respect we would like to mention papers [13–15], where an attempt to describe temporal nonlinear speckles was done. We believe that these results are incorrect. To estimate the instability threshold it is sufficient to expand the nonlinear Langevin equations in powers of $\beta$ (see diagrams shown in Fig.2 in [7], or in Fig.4 in [14]). This is why the authors of [13–15] were able to reproduce the instability criterion $\gamma > 1$ obtained in [7]. Strictly speaking, this approach holds only in the case when $\gamma \ll 1$. For $\gamma > 1$ this approach describes the system incorrectly. In the absence of solid mathematical procedure the authors of [13–15] made assumptions about the nature of solutions of Eq.1 beyond the instability point $\gamma > 1$. Namely, they made assumptions that at $t \to \infty$ the function $n(r, t)$ must exhibit oscillations in time and they assumed some form of time correlations of $n(r, t)$. Both assumptions are incorrect. This can be seen, for example, from the fact that their approach cannot reproduce the existence of an exponentially big (for $\gamma \gg 1$) number of multiple stationary solutions of Eqs.7, which are singular points of Eqs.31.

The simplest situation where the deficiency of the approach of [13–15] is most evident takes place near the first instability point $\gamma = \gamma_c \sim 1$. This is schematically shown by the dashed line in Fig.1a. The critical value $\gamma_c$ is sample specific. In this case, say, the first equation ($i = 1$) in the system of Eqs.7 has three solutions. Two of them are close to each other (see the point “a” in Fig.1a.). Let us start with a discussion of the system dynamics near this point. We would like to stress that in this case $(\tau_1 \partial_i \tilde{u}_i(t))/\tilde{u}_i \ll 1$ and our analysis is rigorous. This is exactly the regime considered in [15]. Since $\partial_i \tilde{u}_i/\tilde{u}_i \sim E_i \gg (\gamma - \gamma_c)E_1$ for $i > 1$, we can neglect the time derivatives in all equations of the system Eqs.31, except the one with $i = 1$. Moreover, since this point of instability is a rare one, all equations in Eqs.7 with $i > 1$ have unique solutions. Thus, the system’s dynamics is described by just one first order differential equation for $\tilde{u}_1$. Near the point ”a” in Fig.1a the functions
\( g_i(\{\bar{u}_i\}) \) and \( F_{i>1}(\{\bar{u}_i\}) \) change slowly. Thus, one stationary solution is stable and the other is unstable. The solution indicated by the letter ”b” in Fig.1a is also stable because it is related adiabatically to the unique stable solution for \( \gamma < \gamma_c \). At \( t \to \infty \) the system approaches one of the stable stationary solutions, described by Eqs.7 [7]. Thus the assumption made in [15] about existence of oscillating in time solutions near the critical point is wrong.

Far from the critical point, when \( I \gg 1 \) and the number of relevant equations of the system of Eqs.31 is larger than one, depending on initial conditions and the form of random functions \( F_i(\{\bar{u}_i\}) \), Eqs.31 can have periodic, chaotic solutions and strange attractors, in addition to stationary points. The fractions of the phase space which at \( t \to \infty \) are attracted to these types of motion are currently not known. In any case, the solutions of the time-dependent nonlinear Schrödinger equation, or Eqs.31, have a very complicated non-Gaussian character, which is very different from that assumed in [13–15].

Finally, we would like to mention that in reality, in the case of very large \( N \), the time dependent fluctuations of external sources become important. They are not described by Eq.1 and the problem remains unsolved.

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FIGURES

FIG. 1. a) Graphical solution of Eq.7. The wavy line corresponds to $F_i(...\bar{u}_i,...)$ and the straight lines corresponds to $\gamma^{-1}i^{2/3}\bar{u}_i$ at different values of $\gamma$. The dashed line illustrates the critical instability point, when a solution of Eq.7 becomes nonunique. b) The solid lines correspond to the intersection $F_1(\bar{u}_1, \bar{u}_2)$ and $\gamma^{-1}\bar{u}_1$, while the dashed lines correspond to the intersection of $F_2(\bar{u}_1, \bar{u}_2)$ and $\gamma^{-1}2^{2/3}\bar{u}_2$. 
FIG. 2. a) Diagrams describing \( \langle n(r) \rangle \). Solid lines correspond to Green functions of Eq.1 with \( \beta = 0 \). Dashed lines correspond to \( \pi \delta(r - r')/lm^2 \). b) and c) Diagrams describing Eq.24. The inner solid lines describe the Green functions which correspond to \( \{\bar{u}_i\} \), while the outer solid lines correspond to \( \{\bar{u}_i + \Delta\bar{u}_i\} \).