Equivariant $K$-theory of semi-infinite flag manifolds and Pieri-Chevalley formula

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Abstract

We propose a definition of equivariant (with respect to an Iwahori subgroup) $K$-theory of the formal power series model $Q_G$ of semi-infinite flag manifold and prove the Pieri-Chevalley formula, which describes the product, in the $K$-theory of $Q_G$, of the structure sheaf of a semi-infinite Schubert variety with a line bundle (associated to a dominant integral weight) over $Q_G$. In order to achieve this, we provide a number of fundamental results on $Q_G$ and its Schubert subvarieties including the Borel-Weil-Bott theory, whose special case is conjectured in [BF2]. One more ingredient of this paper besides the geometric results above is (a combinatorial version of) standard monomial theory for level-zero extremal weight modules over quantum affine algebras, which is described in terms of semi-infinite Lakshmibai-Seshadri paths. In fact, in our Pieri-Chevalley formula, the positivity of structure coefficients is proved by giving an explicit representation-theoretic meaning through semi-infinite Lakshmibai-Seshadri paths.

Key words and phrases: semi-infinite flag manifold, normality, $K$-theory, Pieri-Chevalley formula, standard monomial theory, semi-infinite Lakshmibai-Seshadri path.

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1 Introduction.

Let $G$ be a connected and simply-connected simple algebraic group over $\mathbb{C}$, and let $X$ be the flag variety of $G$. The torus-equivariant Grothendieck group $K_H(X)$ of $X$ affords rich structures from the perspective of geometry and representation theory. One of the highlights there is the positivity of the structure constants of the products among natural classes (called the Schubert classes; see Anderson-Griffeth-Miller \cite{AGM}, and Baldwin-Kumar \cite{BK}), which serves as a basis of its interaction with the eigenvalue problems \cite{Kl}. There is a variant of this theme (called the Pieri-Chevalley formula), namely the structure constants of the products between Schubert classes and (ample) line bundles in $K_H(X)$, which is also known to be positive by Mathieu \cite{Mat} and Brion \cite{B}.

Pittie and Ram \cite{PR} initiated a program to describe such a positive structure constant by relating them with the standard monomial theory (SMT for short). In particular, they gave an explicit meaning of each structure coefficient in the Pieri-Chevalley formula in terms of Lakshmibai-Seshadri paths (LS paths for short; see, e.g., \cite{Li}), which carries almost all information about simple $G$-modules. Their program is subsequently completed by Littelmann-Seshadri \cite{LiSe} and Lenart-Shimozono \cite{LeSh} (see also Lenart-Postnikov \cite{LeP}).

Peterson \cite{P} noticed that the quantum $K$-theory of $X$ should be intimately connected with the $K$-theory of the “affine version” of $X$ (see Lam-Shimozono \cite{LaSh} and Lam-Li-Mihalcea-Shimozono \cite{LLMS}). In view of Givental-Lee \cite{GL} and Braverman-Finkelberg \cite{BF1,BF2}, the quantum $K$-theory of $X$ can be defined through the space of quasi-maps, whose union forms a dense subset of the formal power series model $Q_G$ of semi-infinite flag manifolds (cf. Finkelberg-Mirković \cite{FM}).

Therefore, it is quite natural to make some rigorous sense of $K_H(Q_G)$ and provide the Pieri-Chevalley formula using SMT, which is compatible with the pictures provided by Pittie-Ram and Peterson. This is what we perform in this paper by affirming two new theories: 1) the Borel-Weil-Bott theory of $Q_G$ that enables us to define and calculate a version of $K_H(Q_G)$, and 2) the SMT of level-zero modules over quantum affine algebras. We remark that the level-zero modules over quantum affine algebras admit an interpretation through the geometry of affine Grassmannian (of Langlands dual type), which is the “affine version” of $X$ (see, e.g., Lenart-Naito-Sagaki-Schilling-Shimozono \cite{LNS2} Introduction).

In order to explain our results, theories, and ideas more precisely, we need some notation. Let $\mathfrak{g}$ denote the Lie algebra of $G$, and let $\mathfrak{g}_{af}$ denote the associated untwisted affine Lie algebra; we fix a Borel subgroup $B$ of $G$ and a maximal torus $H \subset B$, and set $N := [B, B]$. Let $W = N_G(H)/H$ be the Weyl group, which is generated by the simple reflections $s_i$, $i \in I$; $W$ can be thought of as acting on the dual space $\mathfrak{h}^*$ of the Cartan subalgebra $\mathfrak{h} := \text{Lie}(H)$. We set $W_{af} := W \ltimes Q^\vee$, with $Q^\vee := \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i^\vee \subset Q^\vee := \bigoplus_{i \in I} \mathbb{Z} \alpha_i^\vee$ (the coroot lattice). Let $\hat{P} = \bigoplus_{i \in I} \mathbb{Z} \varpi_i \subset \mathfrak{h}^*$ be the weight lattice generated by the fundamental weights $\varpi_i$, $i \in I$, and set $P^+ := \sum_{i \in I} \mathbb{Z}_{\geq 0} \varpi_i$.

For an algebraic group $E$ over $\mathbb{C}$, we denote by $E((z))$ and $E[[z]]$ the space of $\mathbb{C}((z))$-valued points and the space of $\mathbb{C}[[z]]$-valued points of $E$, respectively, viewed as an (ind-)scheme over $\mathbb{C}$. Let $\ev_0 : G[[z]] \to G$ be the evaluation map at $z = 0$, and set $\mathbf{I} := \ev_0^{-1}(B)$, an Iwahori subgroup of $G[[z]]$; we also set $\mathbf{I} := \mathbf{I} \rtimes \mathbb{C}^*$, the semi-direct product group, where the group $\mathbb{C}^*$ (of loop rotations) acts on $\mathbf{I}$ as the dilation on $z$. Now,
we define \( Q^\text{rat}_G := G((z))/HN((z)) \), which is a pure ind-scheme of infinite type. Then the set of \( I \)-orbits is in natural bijection with \( W_{af} \); let \( Q_G(x) \) denote the \( I \)-orbit closure corresponding to \( x \in W_{af} \). We define \( Q_G := Q_G(e) \).

Our first main result is the following:

**Theorem 1** \((\doteq \text{Theorem 4.26 and Corollary 4.27})\). For each \( x \in W_{af} \), the scheme \( Q_G(x) \) is normal. In addition, there is an explicit \( P^+\)-graded algebra \( R_G \) such that \( Q_G = \Proj R_G \); here our \( \Proj \) is the \( P^+\)-graded one.

We remark that Theorem 1 affirmatively answers [BF2, Conjecture 2.1] and relevant speculations therein. Also, as we see below, the scheme \( Q_G \) is far from being “compact” (cf. [Kat2, Theorem A] and [FGT, (7.1)]). In order to prove Theorem 1 naturally, we introduce a “semi-infinite” Bott-Samelson-Demazure-Hansen tower that yields a normal \( R_G \). From the construction, \( R_G \) contains the projective coordinate ring of \( Q_G \). Moreover, on the basis of the fact that \( R_G \) is generated by the primitive degree terms, a detailed comparison with the computation for the dense subset in [BF2] implies that the inclusion must be an isomorphism.

For each \( x \in W_{af} \) and \( \lambda \in P \), we have an associated \( G[[z]]\)-equivariant line bundle \( \mathcal{O}_{Q_G(x)}(\lambda) \) over \( Q_G(x) \). Also, for each \( x \in W \) and \( \lambda \in P^+ \), we have a Demazure submodule \( V^-_x(\lambda) \), in the sense of [Kas3], of the level-zero extremal weight module \( V(\lambda) \) (of extremal weight \( \lambda \)) over the quantum affine algebra \( U_q(\mathfrak{g}_{af}) \) associated to \( \mathfrak{g}_{af} \).

**Theorem 2** \((\doteq \text{Theorem 4.29})\). For each \( x \in W_{af} \) and \( \lambda \in P \), we have

\[
gch H^i(Q_G(x), \mathcal{O}_{Q_G(x)}(\lambda)) = \begin{cases} 
gch V^-_x(-w_0\lambda) & \text{if } i = 0 \text{ and } \lambda \in P^+, \\
0 & \text{otherwise,} \end{cases}
\]

where \( \text{gch} \) denotes the character taking values in \( \mathbb{Z}[[q^{-1}]] \)[\( P \)], and \( w_0 \in W \) is the longest element.

The higher cohomology vanishing part of Theorem 2 is based on the fact that the ring \( R_G \) is free over a polynomial ring with infinitely many variables (Theorem 4.28), which is also an interesting result in its own. We should mention that Theorem 2 have an ind-model counterpart in [BF2], but there are no implications between these and the two proofs are totally different.

**Proposition 3** \((\doteq \text{Proposition 5.1})\). For each \( x \in W_{af} \), every \( \tilde{I} \)-equivariant line bundle over the scheme \( Q_G(x) \) is isomorphic to some \( \mathcal{O}_{Q_G(x)}(\lambda) \) up to character twist.

Although \( R_G \) itself is highly infinite-dimensional (it is not even finitely generated), it admits a grading such that it is almost like an Artin algebra in a graded sense. Moreover, Proposition 3 supplies “graded indecomposable projectives” of \( R_G \). These two facts, combined with Theorem 2, assert that the category of \( \tilde{I} \)-equivariant sheaves on \( Q_G \) (and on \( Q^\text{rat}_G \)) behaves almost like the category of coherent sheaves on an affine scheme.

This series of observations enables us to define a reasonable variant of an equivariant \( K \)-group \( K'_I(Q_G) \) of \( Q_G \) (and \( K'_I(Q^\text{rat}_G) \) of \( Q^\text{rat}_G \)) with respect to \( \tilde{I} \); see Section 5 for details. They are rather involved, partly because we need to specify a class of formal power series that is large enough to afford the Pieri-Chevalley rule, and at the same time is small enough so that the Euler character map is injective. Nevertheless, we can prove that
\( K'_1(G) \) contains (the classes of) the sheaves \( \mathcal{O}_{Q_G(y)}(\lambda) \) for each \( \lambda \in P \) and relevant \( y \in W_{af} \). We also prove that our \( K'_1(G) \) is natural enough so that it admits a nil-DAHA action as an analog of Kostant-Kumar [KK] for \( Q'_G \) (see Section \( \circ \) for details).

Here we recall that in Ishii-Naito-Sagaki [INS] and Naito-Sagaki [NS3], the semi-infinite path model of the crystal basis of \( V_\infty(\lambda) \) is constructed for every \( \lambda \in P^+ \) and \( x \in W_{af} \); it is a specific subset of the set of “semi-infinite” LS paths \( B^\infty(x) \) of shape \( \lambda \) parametrizing the global crystal basis of \( V(\lambda) \). Note that it is endowed with three functions
\[
\iota, \kappa : B^\infty(x) \to W_{af} \quad \text{and} \quad \text{wt} : B^\infty(x) \to P \oplus \mathbb{Z}\delta,
\]
which are called the initial/final directions and the weight, respectively. We set
\[
B^\infty_{\iota, \kappa}(x) := \{ \eta \in B^\infty(x) \mid \kappa(\eta) \succeq x \}.
\]

In order to make use of the path model above to derive the Pieri-Chevalley formula for \( Q'_G \), we additionally need a combinatorial version of the semi-infinite SMT. This consists of the definition of the initial direction \( \iota(\eta, x) \in W_{af} \) of a semi-infinite LS path \( \eta \) with respect to \( x \) (based on the existence of the semi-infinite analog of the so-called Deodhar lift), and of the description of tensor product decomposition of crystals in terms of \( \iota(\bullet, x) \) (Theorem \( \text{[Strategy 3.1]} \) and Theorem \( \text{[Strategy 3.3]} \)). We remark that our \( \iota(\eta, x) \) is an analogue of the one in [LiSe, LeSh] in the setting of level-zero extremal weight modules over \( U_q(\mathfrak{g}_{af}) \). Using them, we obtain our Pieri-Chevalley formula:

**Theorem 4** (\( \equiv \) Theorem \( \text{[Strategy 5.10]} \)). For \( \lambda \in P^+ \) and \( x \in W_{af}^0 := W \times Q_{N^+,+} \), we have
\[
[\mathcal{O}_{Q_G(\lambda)} ] \cdot [\mathcal{O}_{Q_G(x)}] = \sum_{\eta \in B^\infty_{\iota, \kappa}(-w(\lambda))} e^{\text{fin}(\eta)} q^{\text{mul}(\eta)} \cdot [\mathcal{O}_{Q_G(\iota(\eta, x))}] \in K'_1(G),
\]
(1.1)

where \( \text{fin}(\eta) \in P \) and \( \text{mul}(\eta) \in \mathbb{Z} \) for \( \eta \in B^\infty_{\iota, \kappa}(-w(\lambda)) \) are defined by:
\[
\text{wt}(\eta) = \text{fin}(\eta) + \text{mul}(\eta)\delta.
\]

Once generalities on \( K'_1(G) \) and the semi-infinite SMT are given, our strategy for the proof of Theorem \( \text{[Strategy 4]} \) is along the line of [LiSe]. Namely, we compare the functionals
\[
P \ni \lambda \mapsto \sum_{i \geq 0} (-1)^i \text{gch} H^i(Q_G, \mathcal{E} \otimes_{Q_G} \mathcal{O}_{Q_G(\lambda)}) \in \mathbb{C}[P]((q^{-1}))
\]
where \( \mathcal{E} \) is taken from the both sides of (1.1).

This paper is organized as follows. In Section \( \text{[Strategy 2]} \) we fix our notation for untwisted affine Lie algebras, and then recall some basic facts about semi-infinite LS paths, extremal weight modules, and their Demazure submodules. In Section \( \text{[Strategy 3]} \) we state a combinatorial version of standard monomial theory for level-zero extremal weight modules, and also its refinement for Demazure submodules; the proofs of these results are given in Sections \( \text{[Strategy 7]} \) and \( \text{[Strategy 8]} \). In Section \( \text{[Strategy 4]} \) we first review the formal power series model \( Q_G \) of semi-infinite flag manifold, and then introduce a semi-infinite version of Bott-Samelson-Demazure-Hansen tower for \( Q_G \). Then, we study the cohomology spaces of line bundles over \( Q_G \), and prove the higher cohomology vanishing; also, we describe the spaces of global sections in terms of Demazure submodules of extremal weight modules. As an application, we prove the...
normality of the semi-infinite Schubert varieties $Q_G(x), x \in W_{af}^\geq 0$. In Section 5, after giving a definition of $I$-equivariant $K$-group $K_I(Q_G)$ of $Q_G$ (and $K_I(Q_{G}^{\text{rat}})$ of $Q_{G}^{\text{rat}}$), we establish the Pieri-Chevalley formula (Theorem 5.10) by combining our geometric results with the semi-infinite SMT. Also, in Section 6, we show that our $K$-group $K_I(Q_{G}^{\text{rat}})$ admits a natural nil-DAHA action. Appendices mainly contain some technical results concerning the semi-infinite Bruhat order; in particular, we prove the existence of analogs of Deodhar lifts for the semi-infinite Bruhat order.

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2 Algebraic setting.

2.1 Affine Lie algebras.

A graded vector space is a $\mathbb{Z}$-graded vector space over $\mathbb{C}$ all of whose homogeneous subspaces are finite-dimensional. Let $V = \bigoplus_{m \in \mathbb{Z}} V_m$ be a graded vector space with $V_m$ its subspace of degree $m$. We define

$$\text{gdim } V := \sum_{m \in \mathbb{Z}} (\dim V_m) q^m.$$ 

Also, we denote by $V^\vee$ (resp., $V^*$) the full (resp., restricted) dual of $V$; note that $V^* := \bigoplus_{m \in \mathbb{Z}} (V^*)_m$, with $(V^*)_m := (V_{-m})^*$. In addition, we set $\hat{V} := \prod_{m \in \mathbb{Z}} V_m$, which is a completion of $V$.

Let $G$ be a connected, simply-connected simple algebraic group over $\mathbb{C}$, and $B$ a Borel subgroup with unipotent radical $N$. We fix a maximal torus $H \subset B$, and take the opposite Borel subgroup $B^-$ of $G$ that contains $H$. In the following, for an (arbitrary) algebraic group $E$ over $\mathbb{C}$, we denote its Lie algebra $\text{Lie}(E)$ by the corresponding German letter $\mathfrak{g}$; in particular, we write $\mathfrak{g} = \text{Lie}(G), \mathfrak{b} = \text{Lie}(B), \mathfrak{n} = \text{Lie}(N)$, and $\mathfrak{h} = \text{Lie}(H)$. Thus, $\mathfrak{g}$ is a finite-dimensional simple Lie algebra over $\mathbb{C}$ with Cartan subalgebra $\mathfrak{h}$. Denote by $\{\alpha_i^\vee\}_{i \in I}$ and $\{\alpha_i\}_{i \in I}$ the set of simple coroots and simple roots of $\mathfrak{g}$, respectively, and set $Q := \bigoplus_{i \in I} \mathbb{Z}\alpha_i, Q^+ := \bigoplus_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$, and $Q^\vee := \bigoplus_{i \in I} \mathbb{Z}\alpha_i^\vee, Q^{\vee,+} := \bigoplus_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i^\vee$ for $\xi, \zeta \in Q^\vee$, we write $\xi \geq \zeta$ if $\xi - \zeta \in Q^{\vee,+}$. Let $\Delta$ and $\Delta^+$ be the set of roots and positive roots of $\mathfrak{g}$, respectively, with $\theta \in \Delta^+$ the highest root of $\mathfrak{g}$. For a root $\alpha \in \Delta$, we denote by $\alpha^\vee$ its dual root. We set $\rho := (1/2) \sum_{\alpha \in \Delta^+} \alpha$ and $\rho^\vee := (1/2) \sum_{\alpha \in \Delta^+} \alpha^\vee$. Also, let $\varpi_i, i \in I$, denote the fundamental weights for $\mathfrak{g}$, and set

$$P := \bigoplus_{i \in I} \mathbb{Z}\varpi_i, \quad P^+ := \sum_{i \in I} \mathbb{Z}_{\geq 0}\varpi_i. \quad (2.1)$$

Let $\mathfrak{g}_{af} = (\mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]) \oplus \mathbb{C}c \oplus \mathbb{C}d$ be the untwisted affine Lie algebra over $\mathbb{C}$ associated to $\mathfrak{g}$, where $c$ is the canonical central element, and $d$ is the scaling element (or the degree
operator), with Cartan subalgebra $\mathfrak{h}_{af} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$. We regard an element $\mu \in \mathfrak{h}^* := \text{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$ as an element of $\mathfrak{h}_{af}^*$ by setting $\langle \mu, c \rangle = \langle \mu, d \rangle = 0$, where $\langle \cdot, \cdot \rangle : \mathfrak{h}_{af}^* \times \mathfrak{h}_{af} \to \mathbb{C}$ is the canonical pairing of $\mathfrak{h}_{af}^* := \text{Hom}_{\mathbb{C}}(\mathfrak{h}_{af}, \mathbb{C})$ and $\mathfrak{h}_{af}$. Let $\{\alpha_i^v\}_{i \in I_{af}} \subset \mathfrak{h}_{af}^*$ and $\{\alpha_i\}_{i \in I_{af}} \subset \mathfrak{h}_{af}$ be the set of simple coroots and simple roots of $\mathfrak{g}_{af}$, respectively, where $I_{af} := I \cup \{0\}$; note that $\langle \alpha_i, c \rangle = 0$ and $\langle \alpha_i, d \rangle = \delta_{i0}$ for $i \in I_{af}$. Denote by $\delta \in \mathfrak{h}_{af}^*$ the null root of $\mathfrak{g}_{af}$; recall that $\alpha_0 = \delta - \theta$. Also, let $\Lambda_i \in \mathfrak{h}_{af}^*$, $i \in I_{af}$, denote the fundamental weights for $\mathfrak{g}_{af}$ such that $\langle \Lambda_i, d \rangle = 0$, and set

$$P_{af} := \left( \bigoplus_{i \in I_{af}} \mathbb{Z} \Lambda_i \right) \oplus \mathbb{Z} \delta \subset \mathfrak{h}^*, \quad P^0_{af} := \{ \mu \in P_{af} \mid \langle \mu, c \rangle = 0 \};$$

(2.2)

notice that $P^0_{af} = P \oplus \mathbb{Z} \delta$, and that

$$\langle \mu, \alpha_0^v \rangle = -\langle \mu, \theta^v \rangle \text{ for } \mu \in P^0_{af}. \quad (2.3)$$

Let $W := \langle s_i \mid i \in I \rangle$ and $W_{af} := \langle s_i \mid i \in I_{af} \rangle$ be the (finite) Weyl group of $\mathfrak{g}$ and the (affine) Weyl group of $\mathfrak{g}_{af}$, respectively, where $s_i$ is the simple reflection with respect to $\alpha_i$ for each $i \in I_{af}$, with length function $\ell : W_{af} \to \mathbb{Z}_{\geq 0}$, which gives the one on $W$ by restriction; we denote by $e \in W_{af}$ the identity element, and by $w_0 \in W$ the longest element. For each $\xi \in Q^\vee$, let $t_\xi \in W_{af}$ denote the translation in $\mathfrak{h}_{af}^*$ by $\xi$ (see [Kac Sect. 6.5]); for $\xi \in Q^\vee$, we have

$$t_\xi \mu = \mu - \langle \mu, \xi \rangle \delta \text{ if } \mu \in \mathfrak{h}_{af}^* \text{ satisfies } \langle \mu, c \rangle = 0. \quad (2.4)$$

Then, $\{t_\xi \mid \xi \in Q^\vee\}$ forms an abelian normal subgroup of $W_{af}$, in which $t_\xi t_\zeta = t_{\xi + \zeta}$ holds for $\xi, \zeta \in Q^\vee$. Moreover, we know from [Kac Proposition 6.5] that

$$W_{af} \cong W \ltimes \{t_\xi \mid \xi \in Q^\vee\} \cong W \ltimes Q^\vee;$$

we also set

$$W^\geq_{af} := \{wt_\xi \mid w \in W, \xi \in Q^{\vee,+}\} \subset W_{af}. \quad (2.5)$$

Denote by $\Delta_{af}$ the set of real roots of $\mathfrak{g}_{af}$, and by $\Delta^+_{af} \subset \Delta_{af}$ the set of positive real roots; we know from [Kac Proposition 6.3] that $\Delta_{af} = \{\alpha + n\delta \mid \alpha \in \Delta, n \in \mathbb{Z}\}$, and $\Delta^+_{af} = \Delta^+ \cup \{\alpha + n\delta \mid \alpha \in \Delta, n \in \mathbb{Z}_{\geq 0}\}$. For $\beta \in \Delta_{af}$, we denote by $\beta^v \in \mathfrak{h}_{af}$ its dual root, and $s_\beta \in W_{af}$ the corresponding reflection; if $\beta \in \Delta_{af}$ is of the form $\beta = \alpha + n\delta$ with $\alpha \in \Delta$ and $n \in \mathbb{Z}$, then $s_\beta = s_\alpha t_{n\alpha^v} \in W \ltimes Q^\vee$.

Finally, let $U_q(\mathfrak{g}_{af})$ denote the quantized universal enveloping algebra over $\mathbb{C}(q)$ associated to $\mathfrak{g}_{af}$, with $E_i$ and $F_i$, $i \in I_{af}$, the Chevalley generators corresponding to $\alpha_i$ and $-\alpha_i$, respectively. We denote by $U_q^-(\mathfrak{g}_{af})$ the negative part of $U_q(\mathfrak{g}_{af})$, that is, the $\mathbb{C}(q)$-subalgebra of $U_q(\mathfrak{g}_{af})$ generated by $F_i$, $i \in I_{af}$.

### 2.2 Parabolic semi-infinite Bruhat graph.

In this subsection, we fix a subset $J \subset I$. We set $Q_J := \bigoplus_{i \in J} \mathbb{Z} \alpha_i$, $Q^\vee_J := \bigoplus_{i \in J} \mathbb{Z} \alpha_i^v$, $Q^{\vee,+}_J := \sum_{i \in J} \mathbb{Z}_{\geq 0} \alpha_i^v$, $\Delta_J := \Delta \cap Q_J$, $\Delta^+_J := \Delta^+ \cap Q_J$, and $W_J := \langle s_i \mid i \in J \rangle$. Also, we denote by

$$[\cdot]_J : Q^\vee \to Q^\vee_J \quad \text{resp., } [\cdot]^J : Q^\vee \to Q^{\vee,+}_{I \setminus J}$$

(2.6)
we know from [BB, Sect. 2.4] that coset 2.3
Remark W
Let J define the projection from Proposition 2.4.
the set of minimal(-length) coset representatives for the cosets in W/W_J; we know from [BB Sect. 2.4] that

\[ W^J = \{ w \in W \mid w_\alpha \in \Delta^+ \text{ for all } \alpha \in \Delta_J^+ \} \]  

(2.7)

For w \in W, we denote by \( |w| = |w|^J \in W^J \) the minimal coset representative for the coset wW_J in W/W_J. Also, following [P] (see also [LaSh, Sect. 10]), we set

\[ (\Delta_J)_\alpha := \{ \alpha + n\delta \mid \alpha \in \Delta_J, n \in \mathbb{Z} \} \subset \Delta_{af}, \]  

(2.8)

\[ (\Delta_J)_+ := (\Delta_J)_\alpha \cap \Delta_{af}^+ = \Delta_J^+ \cup \{ \alpha + n\delta \mid \alpha \in \Delta_J, n \in \mathbb{Z}_{\geq 0} \}, \]  

(2.9)

\[ (W_J)_\alpha := W_J \times \{ t_\xi \mid \xi \in Q_J^\vee \} = \langle s_\beta \mid \beta \in (\Delta_J)_\alpha \rangle, \]  

(2.10)

\[ (W^J)_\alpha := \{ x \in W_{af} \mid x_\beta \in \Delta_{af}^+ \text{ for all } \beta \in (\Delta_J)_+ \}. \]  

(2.11)

The map \( \Pi^J : W_{af} \twoheadrightarrow (W^J)_\alpha, \ x \mapsto x_1, \)  

(2.12)

where \( x = x_1x_2 \) with \( x_1 \in (W^J)_\alpha \) and \( x_2 \in (W_J)_\alpha \).

Definition 2.1. Let \( x \in W_{af}, \) and write it as \( x = wt_\xi \) for \( w \in W \) and \( \xi \in Q^\vee \). We define the semi-infinite length \( \ell^\infty_\xi(x) \) of \( x \) by: \( \ell^\infty_\xi(x) = \ell(w) + 2(\rho, \xi) \).

Definition 2.2 ([Lu1], [Lu2]; see also [P]).

1. The (parabolic) semi-infinite Bruhat graph \( BG_{\infty}(W^J)_\alpha \) is the \( \Delta_{af}^+ \)-labeled, directed graph with vertex set \((W^J)_\alpha\) whose directed edges are of the following form:

\( x \xrightarrow{\beta} s_\beta x \) for \( x \in (W^J)_\alpha \) and \( \beta \in \Delta_{af}^+ \), where \( s_\beta x \in (W^J)_\alpha \) and \( \ell^\infty(s_\beta x) = \ell^\infty(x) + 1 \).

When \( J = \emptyset \), we write \( BG_{\infty}(W_{af}) \) for \( BG_{\infty}(W^\emptyset)_\alpha \).

2. The semi-infinite Bruhat order is a partial order \( \preceq \) on \((W^J)_\alpha\) defined as follows:

for \( x, y \in (W^J)_\alpha \), we write \( x \preceq y \) if there exists a directed path from \( x \) to \( y \) in \( BG_{\infty}(W^J)_\alpha \); we write \( x < y \) if \( x \preceq y \) and \( x \neq y \).

Remark 2.3. In the case \( J = \emptyset \), the semi-infinite Bruhat order on \( W_{af} \) is just the generic Bruhat order introduced in [Lu1]; see [INS Appendix A.3] for details. Also, for a general \( J \), the parabolic semi-infinite Bruhat order on \((W^J)_\alpha\) is nothing but the partial order on \( J \)-alcoves introduced in [Lu2] when we take a special point to be the origin.

In Appendix A we recall some of the basic properties of the semi-infinite Bruhat order.

For \( x \in (W^J)_\alpha \), let \( \text{Lift}(x) \) denote the set of lifts of \( x \) in \( W_{af} \) with respect to the map \( \Pi^J : W_{af} \twoheadrightarrow (W^J)_\alpha \), that is,

\[ \text{Lift}(x) := \{ x' \in W_{af} \mid \Pi^J(x') = x \}; \]  

(2.13)

for an explicit description of \( \text{Lift}(x) \), see Lemma [B.1] The following proposition will be proved in Appendix B.

Proposition 2.4. If \( x \in W_{af} \) and \( y \in (W^J)_\alpha \) satisfy the condition that \( y \succeq \Pi^J(x) \), then the set

\[ \text{Lift}_{\preceq x}(y) := \{ y' \in \text{Lift}(y) \mid y' \succeq x \} \]  

(2.14)

has the minimum element with respect to the semi-infinite Bruhat order on \( W_{af} \); we denote this element by \( \min \text{Lift}_{\preceq x}(y) \).
2.3 Semi-infinite Lakshmibai-Seshadri paths.

In this subsection, we fix $\lambda \in P^+ \subset P^0_{af}$ (see (2.1) and (2.2)), and set
\[ J := \{ i \in I \mid \langle \lambda, \alpha_i^\vee \rangle = 0 \} \subset I. \]  

(2.15)

**Definition 2.5.** For a rational number $0 < a < 1$, we define $BG_{\lambda,a}( (W^J)_{af} )$ to be the subgraph of $BG_{\lambda}( (W^J)_{af} )$ with the same vertex set but having only the edges of the form $x \xrightarrow{\beta} y$ with $a \langle x, \beta^\vee \rangle \in \mathbb{Z}$.

**Definition 2.6.** A semi-infinite Lakshmibai-Seshadri (LS for short) path of shape $\lambda$ is a pair
\[ \pi = (x ; a) = (x_1, \ldots, x_s ; a_0, a_1, \ldots, a_s), \quad s \geq 1, \]  

(2.16)

of a strictly decreasing sequence $x : x_1 \succ \cdots \succ x_s$ of elements in $(W^J)_{af}$ and an increasing sequence $a : 0 = a_0 < a_1 < \cdots < a_s = 1$ of rational numbers satisfying the condition that there exists a directed path from $x_{u+1}$ to $x_u$ in $BG_{\lambda,a_u}( (W^J)_{af} )$ for each $u = 1, 2, \ldots, s-1$. We denote by $B_{\lambda}(\lambda)$ the set of all semi-infinite LS paths of shape $\lambda$.

Following [LNS] Sect. 3.1 (see also [NS3], Sect. 2.4), we endow the set $B_{\lambda}(\lambda)$ with a crystal structure with weights in $P_{af}$ by the map $wt : B_{\lambda}(\lambda) \to P_{af}$ and the root operators $e_i, f_i, i \in I_{af}$; for details, see Appendix C. We denote by $B_{\lambda}(\lambda)$ the connected component of $B_{\lambda}(\lambda)$ containing $\pi_\lambda := (e ; 0, 1) \in B_{\lambda}(\lambda)$.

If $\pi \in B_{\lambda}(\lambda)$ is of the form (2.16), then we set
\[ \iota(\pi) := x_1 \in (W^J)_{af} \quad \text{(resp., } \kappa(\pi) := x_s \in (W^J)_{af}) \]  

(2.17)

we call $\iota(\pi)$ (resp., $\kappa(\pi)$) the initial (resp., final) direction of $\pi$. For $x \in W_{af}$, we set
\[ B_{\lambda, \geq x}(\lambda) := \{ \pi \in B_{\lambda}(\lambda) \mid \kappa(\pi) \succeq \Pi^J(x) \}. \]  

(2.18)

2.4 Extremal weight modules and their Demazure submodules.

In this subsection, we fix $\lambda \in P^+ \subset P^0_{af}$ (see (2.1) and (2.2)). Let $V(\lambda)$ denote the extremal weight module of extremal weight $\lambda$ over $U_q(\mathfrak{g}_{af})$, which is an integrable $U_q(\mathfrak{g}_{af})$-module generated by a single element $v_\lambda$ with the defining relation that $v_\lambda$ is an “extremal weight vector” of weight $\lambda$; recall from [Kas2], Sect. 3.1 and [Kas3], Sect. 2.6] that $v_\lambda$ is an extremal weight vector of weight $\lambda$ if and only if $(v_\lambda$ is a weight vector of weight $\lambda$ and) there exists a family $\{v_x\}_{x \in W_{af}}$ of weight vectors in $V(\lambda)$ such that $v_x = v_\lambda$, and such that for every $i \in I_{af}$ and $x \in W_{af}$ with $n := \langle x, \alpha_i^\vee \rangle \geq 0$ (resp., $\leq 0$), the equalities $E_i v_x = 0$ and $F_i^{(n)} v_x = v_{s_i x}$ (resp., $F_i^{(n)} v_x = 0$ and $E_i^{(n)} v_x = v_{s_i x}$) hold, where for $i \in I_{af}$ and $k \in \mathbb{Z}_{\geq 0}$, the $E_i^{(k)}$ and $F_i^{(k)}$ are the $k$-th divided powers of $E_i$ and $F_i$ respectively; note that the weight of $v_x$ is $x \lambda$. Also, for each $x \in W_{af}$, we define the Demazure submodule $V_x^{-}(\lambda)$ of $V(\lambda)$ by
\[ V_x^{-}(\lambda) := U_q^{-}(\mathfrak{g}_{af}) v_x. \]  

(2.19)

We know from [Kas1] Proposition 8.2.2] that $V(\lambda)$ has a crystal basis $B(\lambda)$ and the corresponding global basis $\{ G(b) \mid b \in B(\lambda) \}$; we denote by $u_\lambda$ the element of $B(\lambda)$ such that $G(u_\lambda) = v_\lambda$, and by $B_{0}(\lambda)$ the connected component of $B(\lambda)$ containing $u_\lambda$. Also, we
know from [Kas3, Sect. 2.8] (see also [NS3, Sect. 4.1]) that \( V^-_x(\lambda) \subset V(\lambda) \) is compatible with the global basis of \( V(\lambda) \), that is, there exists a subset \( B^-_x(\lambda) \) of the crystal basis \( B(\lambda) \) such that

\[
V^-_x(\lambda) = \bigoplus_{b \in B^-_x(\lambda)} \mathbb{C}(q)G(b) \subset V(\lambda) = \bigoplus_{b \in B(\lambda)} \mathbb{C}(q)G(b).
\]

(2.20)

Remark 2.7 ([NS3 Lemma 4.1.2]). For every \( x \in W_{af} \), we have \( V^-_x(\lambda) = V^-_{\Pi(x)}(\lambda) \) and \( B^-_x(\lambda) = B^-_{\Pi(x)}(\lambda) \).

We know the following from [NS Theorem 3.2.1] and [NS3 Theorem 4.2.1].

**Theorem 2.8.** There exists an isomorphism \( \Phi : B(\lambda) \cong \mathbb{B}^\infty(\lambda) \) of crystals such that \( \Phi(u_\lambda) = \pi_\lambda \) and such that \( \Phi(B^-_x(\lambda)) = \mathbb{B}^-_{\Pi(x)}(\lambda) \) for all \( x \in W_{af} \); in particular, we have \( \Phi(B_0(\lambda)) = \mathbb{B}^\infty_0(\lambda) \).

Let \( x \in W_{af} \). If \( x \) is of the form \( x = wt_\xi \) for some \( w \in W \) and \( \xi \in Q' \), then \( v_x \in V(\lambda) \) is a weight vector of weight \( x\lambda = w\lambda - \langle \lambda, \xi \rangle \delta \); note that \( w\lambda \in \lambda - Q^+ \). Also, for \( i \in I \) (resp., \( i = 0 \in I_{af} \)), the Chevalley generator \( F_i \) (resp., \( F_0 \)) of \( U_q(\mathfrak{g}_{af}) \) acts on \( V(\lambda) \) as a (linear) operator of weight \( -\alpha_i \in Q \) (resp., \( -\alpha_0 = \theta - \delta \in Q + \mathbb{Z}_{<0} \delta \)). Therefore, the Demazure submodule \( V^-_x(\lambda) = U^-_q(\mathfrak{g}_{af})v_x \) has the weight space decomposition of the form:

\[
V^-_x(\lambda) = \bigoplus_{k \in \mathbb{Z}} \left( \bigoplus_{\gamma \in Q} V^-_x(\lambda)_{\lambda + \gamma + k\delta} \right),
\]

where \( V^-_x(\lambda)_k = \{0\} \) for all \( k > -\langle \lambda, \xi \rangle \); in addition, by Theorem 2.8 together with the definition of the map \( \text{wt} : \mathbb{B}^\infty(\lambda) \to P_{af} \) (see [C3]), we see that if \( \xi \notin Q^+ \), then \( V^-_x(\lambda)_{\lambda + \gamma + k\delta} = \{0\} \) for all \( k \in \mathbb{Z} \), since \( W_{af} \lambda \subset \lambda - Q^+ + \mathbb{Z}\delta \) by the assumption that \( \lambda \in P^+ \). Here we claim that \( V^-_x(\lambda)_k \) is finite-dimensional for all \( k \in \mathbb{Z} \) with \( k \leq -\langle \lambda, \xi \rangle \); we show this assertion by descending induction on \( k \). Let \( U^-_q(\mathfrak{g}) \) denote the \( \mathbb{C}(q) \)-subalgebra of \( U^-_q(\mathfrak{g}_{af}) \) generated by \( F_i \), \( i \in I \). If \( k = -\langle \lambda, \xi \rangle \), then the assertion is obvious since \( V^-_x(\lambda)_{-\langle \lambda, \xi \rangle} = U^-_q(\mathfrak{g})v_x \) and \( V(\lambda) \) is an integrable \( U^-_q(\mathfrak{g}_{af}) \)-module. Assume that \( k < -\langle \lambda, \xi \rangle \). Observe that \( V^-_x(\lambda)_k \) is a \( U^-_q(\mathfrak{g}) \)-module generated by \( F_0V^-_x(\lambda)_{k+1} \). Because \( F_0V^-_x(\lambda)_{k+1} \) is finite-dimensional by our induction hypothesis, and \( V(\lambda) \) is an integrable \( U^-_q(\mathfrak{g}_{af}) \)-module, we deduce that \( V^-_x(\lambda)_k = U^-_q(\mathfrak{g})(F_0V^-_x(\lambda)_{k+1}) \) is also finite-dimensional, as desired.

Now, we define the graded character \( \text{gch} V^-_x(\lambda) \) of \( V^-_x(\lambda) \) to be

\[
\text{gch} V^-_x(\lambda) := \sum_{k \in \mathbb{Z}} \left( \sum_{\gamma \in Q} \dim(V^-_x(\lambda)_{\lambda + \gamma + k\delta}) e^{\lambda + \gamma} \right) q^k;
\]

(2.21)

observe that

\[
\text{gch} V^-_x(\lambda) \in \left( \mathbb{Z}[e^\nu]_{\nu \in P} \right) \llbracket q^{-1} \rrbracket q^{-\langle \lambda, \xi \rangle}.
\]

(2.22)

For \( \gamma \in Q \) and \( k \in \mathbb{Z} \), we set \( \text{fin}(\lambda + \gamma + k\delta) := \lambda + \gamma + P \) and \( \text{nul}(\lambda + \gamma + k\delta) := k \in \mathbb{Z} \). Then, by Theorem 2.8 we have

\[
\text{gch} V^-_x(\lambda) = \sum_{\pi \in \mathbb{B}^\infty^-_{\Pi(x)}(\lambda)} e^{\text{fin}(\lambda + \gamma + k\delta)} q^{\text{nul}(\lambda + \gamma + k\delta)}.
\]

(2.23)
3 Combinatorial standard monomial theory for semi-infinite LS paths.

In this section, we fix $\lambda, \mu \in P^+ \subset P^0_{af}$ (see (2.1) and (2.2)), and set

$$ J := \{ i \in I \mid \langle \lambda, \alpha_i^\vee \rangle = 0 \}, \quad K := \{ i \in I \mid \langle \mu, \alpha_i^\vee \rangle = 0 \}. $$

3.1 Standard paths.

We consider the following condition (SP) on $\pi \otimes \eta \in B^\pm(\lambda) \otimes B^\pm(\mu)$:

$$ \begin{cases} 
\text{there exist } x, y \in W_{af} \text{ such that } x \succeq y \text{ in } W_{af}, \\
\text{and such that } \Pi^j(x) = \kappa(\pi), \Pi^K(y) = \iota(\eta); 
\end{cases} \quad (\text{SP}) $$

we set

$$ S^\pm(\lambda + \mu) := \{ \pi \otimes \eta \in B^\pm(\lambda) \otimes B^\pm(\mu) \mid \pi \otimes \eta \text{ satisfies condition (SP)} \}. $$

Theorem 3.1. The set $S^\pm(\lambda + \mu) \cup \{0\}$ is stable under the action of the Kashiwara (or, root) operators $e_i, f_i, i \in I_{af}$, on $B^\pm(\lambda) \otimes B^\pm(\mu)$; in particular, $S^\pm(\lambda + \mu)$ is a crystal with weights in $P_{af}$. Moreover, $S^\pm(\lambda + \mu)$ is isomorphic as a crystal to $B^\pm(\lambda + \mu)$.

We will give a proof of Theorem 3.1 in Section 7.

3.2 Defining chains.

Definition 3.2. Let $\pi = (x_1, \ldots, x_s; a) \in B^\pm(\lambda)$ and $\eta = (y_1, \ldots, y_p; b) \in B^\pm(\mu)$. A defining chain for $\pi \otimes \eta$ is a sequence $x'_1, \ldots, x'_s, y'_1, \ldots, y'_p$ of elements in $W_{af}$ satisfying the condition:

$$ \begin{cases} 
x'_1 \succeq \cdots \succeq x'_s \succeq y'_1 \succeq \cdots \succeq y'_p \quad \text{in } W_{af}; \\
\Pi^j(x'_u) = x_u \quad \text{for } 1 \leq u \leq s; \\
\Pi^K(y'_q) = y_q \quad \text{for } 1 \leq q \leq p; 
\end{cases} \quad (\text{DC}) $$

we call $x'_1$ (resp., $y'_p$) the initial element (resp., the final element) of this defining chain.

Proposition 3.3. Let $\pi \in B^\pm(\lambda)$ and $\eta \in B^\pm(\mu)$. Then, $\pi \otimes \eta \in S^\pm(\lambda + \mu)$ if and only if there exists a defining chain for $\pi \otimes \eta \in B^\pm(\lambda) \otimes B^\pm(\mu)$.

We will give a proof of Proposition 3.3 in Section 8.1.

Now, let $\eta = (y_1, \ldots, y_p; b) \in B^\pm(\mu)$. For each $x \in W_{af}$ such that $\kappa(\eta) = y_p \succeq \Pi^K(x)$, we define a specific lift $\iota(\eta, x) \in W_{af}$ of $\iota(\eta) = y_1 \in (W^K)_{af}$ as follows. Since $y_p \succeq \Pi^K(x)$ by the assumption, it follows from Proposition 2.4 that $\text{Lift}_{\succeq x}(y_p)$ has the minimum element $\text{min Lift}_{\succeq x}(y_p) =: \varrho_p$. Similarly, since $y_{p-1} \succeq y_p = \Pi^K(\varrho_p)$, it follows again from Proposition 2.4 that $\text{Lift}_{\succeq \varrho_p}(y_{p-1})$ has the minimum element $\text{min Lift}_{\succeq \varrho_p}(y_{p-1}) =: \varrho_{p-1}$. Continuing in this way, we obtain $\varrho_p, \varrho_{p-1}, \ldots, \varrho_1$. Namely, these elements are defined by the following recursive procedure (from $p$ to $1$):

$$ \begin{cases} 
\varrho_p := \text{min Lift}_{\succeq x}(y_p), \\
\varrho_q := \text{min Lift}_{\succeq \varrho_{q+1}}(y_q) \quad \text{for } 1 \leq q \leq p - 1. 
\end{cases} \quad (3.1) $$

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Finally, we set
\[ \iota(\eta, x) := \overline{y}; \tag{3.2} \]
this element is sometimes called the initial direction of \( \eta \) with respect to \( x \).

**Proposition 3.4.** Let \( \pi \in B^\infty_\mu(\lambda) \) and \( \eta \in B^\infty_\mu(\lambda) \). Then, \( \pi \otimes \eta \in S^\infty(\lambda + \mu) \) (or equivalently, there exists a defining chain for \( \pi \otimes \eta \in B^\infty_\mu(\lambda) \otimes B^\infty_\mu(\mu) \) by Proposition 3.3) if and only if \( \kappa(\pi) \geq \Pi^J(\iota(\eta, x)) \) for some \( x \in W_{af} \) such that \( \kappa(\eta) \geq \Pi^K(x) \).

We will give a proof of Proposition 3.4 in Section 8.2.

### 3.3 Demazure crystals in terms of standard paths.

We set \( \mathcal{S} := \{ i \in I \mid \langle \lambda + \mu, \alpha_i^\vee \rangle = 0 \} = J \cap K \). For each \( x \in W_{af} \), we define \( \mathcal{S}^\infty_{\leq x}(\lambda + \mu) \subset S^\infty(\lambda + \mu) \) to be the image of \( B^\infty_{\leq x}(\lambda + \mu) = \{ \psi \in B^\infty_{\leq x}(\lambda + \mu) \mid \kappa(\psi) \geq \Pi^S(x) \} \) under the isomorphism \( B^\infty_{\leq x}(\lambda + \mu) \cong S^\infty(\lambda + \mu) \) in Theorem 3.1.

**Theorem 3.5.** Let \( x \in W_{af} \). For \( \pi \otimes \eta \in B^\infty_\mu(\lambda) \otimes B^\infty_\mu(\mu) \), the following conditions \( (D1), (D2), \) and \( (D3) \) are equivalent:

\[ \pi \otimes \eta \in \mathcal{S}^\infty_{\leq x}(\lambda + \mu); \tag{D1} \]

\[ \begin{cases} 
\text{there exists a defining chain for } \pi \otimes \eta \text{ whose final element, say } y, \\
\text{satisfies the condition that } \Pi^S(y) \geq \Pi^S(x); \\
\kappa(\eta) \geq \Pi^K(x) \text{ and } \kappa(\pi) \geq \Pi^J(\iota(\eta, x)). 
\end{cases} \tag{D2} \]

Therefore, we have
\[ \mathcal{S}^\infty_{\leq x}(\lambda + \mu) = \{ \pi \otimes \eta \mid \eta \in B^\infty_{\leq x}(\mu) \text{ and } \pi \in B^\infty_{\leq \iota(\eta, x)}(\lambda) \}, \]
and hence (see (2.23))
\[ \operatorname{gch} V^-_x(\lambda + \mu) = \sum_{\eta \in B^\infty_{\leq x}(\mu)} e^{\operatorname{fin}(\operatorname{wt}(\eta))} q^{\operatorname{null}(\operatorname{wt}(\eta))} \sum_{\pi \in B^\infty_{\leq \iota(\eta, x)}(\lambda)} e^{\operatorname{fin}(\operatorname{wt}(\pi))} q^{\operatorname{null}(\operatorname{wt}(\pi))}. \tag{3.3} \]

We will give a proof of Theorem 3.5 in Section 9.

### 4 Semi-infinite Schubert varieties and their resolutions.

#### 4.1 Geometric setting.

An (algebraic) variety is an integral separated scheme of finite type over \( \mathbb{C} \). Also, a pro-affine space is a product of finitely many copies of \( \text{Spec} \mathbb{C}[x_m \mid m \geq 0] \), equipped with a truncation morphism \( \text{Spec} \mathbb{C}[x_m \mid m \geq 0] \rightarrow \text{Spec} \mathbb{C}[x_m \mid 0 \leq m \leq n] \) for \( n \gg 0 \); by a
morphism of pro-affine spaces, we mean a morphism of schemes that is also continuous with respect to the topology induced by the truncation morphisms (this topology itself is irrelevant to the Zariski topology). For a $\mathbb{C}$-vector space $V$, we set $\mathbb{P}(V) := (V \setminus \{0\})/\mathbb{C}^\times$. We usually regard $\mathbb{P}(V)$ as an algebraic variety over $\mathbb{C}$ when $\dim V < \infty$, or as an ind/pro-scheme when $\dim V = \infty$ in accordance with the topology of $V$.

For an algebraic group $E$, let $E[[z]]$, $E((z))$, and $E[z]$ denote the set of $\mathbb{C}[[z]]$-valued points, $\mathbb{C}((z))$-valued points, and $\mathbb{C}[z]$-valued points of $E$, respectively; the corresponding Lie algebras are denoted by $\mathfrak{e}[[z]]$, $\mathfrak{e}((z))$, and $\mathfrak{e}[z]$, respectively, with $E$ replaced by its German letter $\mathfrak{e} = \text{Lie}(E)$. Also, we denote by $R(E)$ the representation ring of $E$.

Recall that $G$ is a connected, simply-connected simple algebraic group over $\mathbb{C}$; concerning the Lie algebra $\mathfrak{g} = \text{Lie}(G)$ and its untwisted affinization $\mathfrak{g}_{af}$, we use the notation of Section 2.

We have an evaluation map $\text{ev}_0 : G[[z]] \to G$ at $z = 0$. Let $I := \text{ev}_0^{-1}(B)$ be an Iwahori subgroup of $G[[z]]$. Also, for each $i \in I_{af}$, we have a minimal parahoric subgroup $I \subset I(i) \subset G[[z]]$ corresponding to $\alpha_i$, so that $I(i)/I \cong \mathbb{P}^1$. Note that both $G[[z]]$ and $I$ admit an action of $G_m$ obtained by the scalar dilation on $z$; we denote the resulting semi-direct product groups by $\tilde{G}[[z]]$ and $\tilde{I}$, respectively. The (finite) Weyl group $W$ of $\mathfrak{g}$ is isomorphic to $N_G(H)/H$, and $Q^\vee$ is isomorphic to $H((z))/H[[z]]$, both of which fit in the following commutative diagram involving the (affine) Weyl group $W_{af}$ of $\mathfrak{g}_{af}$:

$$
\begin{array}{cccccc}
0 & \longrightarrow & Q^\vee & \longrightarrow & W_{af} & \longrightarrow & W & \longrightarrow & e \\
& & \cong & & \cong & & \cong & & \cong \\
& & H((z))/H[[z]] & \longrightarrow & N_G((z))(H)/H[[z]] & \longrightarrow & N_G((z))((z))/H((z)))) \cong N_G(H)/H, \\
\end{array}
$$

where the first row is exact, and the rightmost isomorphism in the second row holds since $N_G((z))(H) \cong N_G((z))(H((z)))) \cong (N_G(H))(z))$. In particular, we have a lift $\hat{w} \in N_G((z))(H)$ for each $w \in W_{af}$. Now, for $x \in W_{af}$ and $i \in I_{af}$, we set

$$
\begin{align*}
s_i \ast x := \begin{cases} x & \text{if } s_i x < x, \\
s_i x & \text{if } s_i x > x, 
\end{cases}
\end{align*}
$$

(4.1)

where we denote by $>$ the ordinary Bruhat order on $W_{af}$. Then, the set $W_{af}$ becomes a monoid, which we denote by $W_{af}$, under the product $\ast$; this monoid is also obtained as a subset of the generic Hecke algebra associated to $(W_{af}, I_{af})$ by setting $a_i = 1$ and $b_i = 0$ for $i \in I_{af}$ in [Hu2], Sect. 7.1, Theorem].

### 4.2 Semi-infinite flag manifolds.

Here we review two models of semi-infinite flag manifold associated to $G$, for which the basic references are [FM] and [FFKM].

Let $L(\lambda)$ denote the (finite-dimensional) irreducible highest weight $\mathfrak{g}$-module of highest weight $\lambda \in P^+$. Recall that for each $\lambda, \mu \in P^+$, we have a canonical embedding of irreducible highest weight $\mathfrak{g}$-modules (and hence of $G$-modules) up to scalars:

$$
L(\lambda + \mu) \hookrightarrow L(\lambda) \otimes_{\mathbb{C}} L(\mu).
$$

(4.2)
The embedding (4.2) induces an embedding
\[ L(\lambda + \mu) \otimes R \hookrightarrow (L(\lambda) \otimes L(\mu)) \otimes R \cong (L(\lambda) \otimes R) \otimes R (L(\mu) \otimes R) \] (4.3)
for every commutative, associative \( \mathbb{C} \)-algebra \( R \).

**Theorem 4.1** ([BG 1.1.2]). Let \( \mathbb{K} \) be a field containing \( \mathbb{C} \). The set of collections \( \{ \ell^\lambda \}_{\lambda \in P^+} \) of one-dimensional \( \mathbb{K} \)-vector subspaces \( \ell^\lambda \) in \( L(\lambda) \otimes \mathbb{K} \) such that \( \ell^\lambda \otimes \mathbb{K} = \ell^{\lambda + \mu} \) for every \( \lambda, \mu \in P^+ \) (under the embedding (4.3)) is in bijection with the set of closed \( \mathbb{K} \)-points of \( G/B \).

For a \( g \)-module \( V \), we set \( V[[z]] := V \otimes \mathbb{C}[[z]] \) and \( V((z)) := V \otimes \mathbb{C}((z)) \).

**Definition 4.2.** Consider a collection \( L = \{ L^\lambda \}_{\lambda \in P^+} \) of one-dimensional \( \mathbb{C} \)-vector subspaces \( L^\lambda \) in \( L(\lambda)[[z]] = L(\lambda) \otimes \mathbb{C}[[z]] \) (resp., \( L(\lambda)((z)) = L(\lambda) \otimes \mathbb{C}((z)) \)). The datum \( L \) is called a formal (resp., rational) Drinfeld-Plücker (DP for short) datum if for every \( \lambda, \mu \in P^+ \) (under the embedding (4.3)) is determined uniquely by a collection \( \{ \ell^\lambda \}_{\lambda \in P^+} \) of one-dimensional \( \mathbb{K} \)-vector subspaces \( \ell^\lambda \) in \( L(\lambda) \) such that \( \ell^\lambda \otimes \mathbb{K} = \ell^{\lambda + \mu} \) for every \( \lambda, \mu \in P^+ \) (under the embedding (4.3)) is in bijection with the set of closed \( \mathbb{K} \)-points of \( G/B \).

Let \( Q_G \) (resp., \( Q_G^\text{rat} \)) denote the set of formal (resp., rational) DP data.

**Remark 4.3.** By the compatibility condition (4.3), a DP datum \( \{ L^\lambda \}_{\lambda \in P^+} \) is determined completely by a collection \( \{ u^i \}_{i \in I} \) of nonzero elements \( u^i \in L^{\varpi_i} \) for \( i \in I \). We call this collection \( \{ u^i \}_{i \in I} \) DP coordinates.

Let \( L = \{ L^\lambda \}_{\lambda \in P^+} \in Q_G \). We define \( \deg L^\lambda \) to be the degree of a nonzero element in \( L^\lambda \), viewed as an \( L(\lambda) \)-valued formal power series (if it is bounded). For each \( \xi \in Q^{\lambda, +} \), a DP datum of degree \( \xi \) is a formal DP datum \( L = \{ L^\lambda \}_{\lambda \in P^+} \) such that \( \deg L^\lambda \leq \langle \lambda, \xi \rangle \) for all \( \lambda \in P^+ \). For each \( \xi \in Q^{\lambda, +} \), let \( \Omega_G(\xi) \) denote the set of formal DP data of degree \( \xi \). Here we note that if \( \xi, \zeta \in Q^{\lambda, +} \) satisfy \( \xi \leq \zeta \), i.e., \( \zeta - \xi \in Q^{\lambda, +} \), then \( \Omega_G(\xi) \subset \Omega_G(\zeta) \).

We set \( \Omega_G := \bigcap_{\xi \in Q^{\lambda, +}} \Omega_G(\xi) \); observe that \( \Omega_G(\xi) \supset \Omega_G \subset Q_G \) for each \( \xi \in Q^{\lambda, +} \).

Also, we remark that \( Q_G \) has a natural \( G[[z]] \)-action, and that its subsets \( \Omega_G(\xi), \xi \in Q^{\lambda, +} \), and \( \Omega_G \) are stable under the action of \( G \) on \( Q_G \).

**Lemma 4.4.** We have an embedding
\[ Q_G \ni \{ L^\lambda \}_{\lambda \in P^+} \mapsto \{ [L^{\varpi_i}] \}_{i \in I} \in \prod_{i \in I} \mathbb{P}(L(\varpi_i)[[z]]), \] (4.5)
which gives the set \( Q_G \) a (reduced) structure of an infinite type closed subscheme of \( \prod_{i \in I} \mathbb{P}(L(\varpi_i)[[z]]) \). In particular, \( Q_G \) is separated.

**Proof.** Because a DP datum \( \{ L^\lambda \}_{\lambda \in P^+} \) is determined uniquely by \( \{ L^{\varpi_i} \}_{i \in I} \) (see Remark 4.3), the map above is injective. Also, condition (4.4) is equivalent to saying that the image of \( L^\lambda \otimes \mathbb{C} L^\mu \) lies in the \( \mathbb{C} \)-vector subspace \( L(\lambda + \mu)[[z]] \subset (L(\lambda) \otimes \mathbb{C} L(\mu))[[z]] \) for all \( \lambda, \mu \in P^+ \). This condition imposes infinitely many equations on \( \prod_{i \in I} \mathbb{P}(L(\varpi_i)[[z]]) \) that define \( Q_G \) as its closed subscheme. Since each \( \mathbb{P}(L(\varpi_i)[[z]]) \) is separated, so is \( Q_G \). This proves the lemma. \( \square \)
Theorem 4.5 ([FM]). The set of $G[[z]]$-orbits in $Q_G$ is labeled by $Q^{\vee,+}$. The codimension of the $G[[z]]$-orbit corresponding to $\xi \in Q^{\vee,+}$ is equal to $2(\rho, \xi)$. 

Corollary 4.6. The set of $I$-orbits in $Q_G$ is in bijection with the set $W_{af}^{\geq 0} = \{ \text{wt}_\xi \mid w \in W, \xi \in Q^{\vee,+} \}$.

Proof. Apply (the consequence of) the Bruhat decomposition $G[[z]] = \bigsqcup_{w \in W} IwI$ to each $G[[z]]$-orbit in $Q_G$ described in Theorem 4.5. □

For each $x = \text{wt}_\xi \in W_{af}^{\geq 0}$, we denote by $O(x)$ the $I$-orbit of $Q_G$ that contains a unique $(H \times G_m)$-fixed point corresponding to $\{ z^{(w_\xi, \xi)} \}_{i \in I}$ (see Lemma 4.4), where for each $\lambda \in P^+$ and $w \in W$, we take and fix a nonzero vector $v_{w\lambda}$ of weight $w\lambda$ in $L(\lambda)$; note that the codimension of $O(x) \subset \bar{O}(e)$ is given by $\ell(x)$. We set $Q_G(x) := \bar{O}(x) \subset Q_G$. For $x, y \in W_{af}^{\geq 0}$, we have

$$Q_G(x) \subset Q_G(y) \iff x \succeq y \ (\text{[FFKM Sect. 5.1]})$$

$$\iff xw_0 \preceq yw_0 \ (\text{see [P Lecture 13]}).$$

Also, we have $Q_G(e) = Q_G$ by inspection; in fact, $e \in W_{af}^{\geq 0}$ is the minimum element in the semi-infinite Bruhat order restricted to $W_{af}^{\geq 0}$.

For $\xi \in Q^{\vee,+}$ and $x \in W_{af}^{\geq 0}$, we set $\Omega_G(x, \xi) := \Omega_G(\xi) \cap Q_G(x)$, and for $x \in W_{af}^{\geq 0}$, we set $\Omega_G(x) := \bigcup_{\xi \in Q^{\vee,+}} \Omega_G(x, \xi)$.

For each $\xi \in Q^{\vee,+}$, we have an embedding

$$i_\xi : Q_G \ni \{ L^\lambda \}_{\lambda \in P^+} = \{ u^\lambda \}_{\lambda \in P^+} \mapsto \{ z^{(\lambda, \xi)} u^\lambda \}_{\lambda \in P^+} \in Q_G.$$ 

Thus we have its direct limit $Q_G^{\text{str}} \cong \lim_{\xi \to Q_G}$, on which we have an action of $G((z))$. By its construction, the embedding $i_\xi$ is $G[[z]]$-equivariant, and sends the $I$-orbit $O(x)$ to $O(x t_\xi)$.

Now, by Lemma 4.4, we have a $G[[z]]$-equivariant line bundle $O_{Q_G}(w_i)$ obtained by the pullback of the $i$-th $O(1)$ through (4.5). For $\lambda = \sum_{i \in I} m_i w_i \in P$, we set $O_{Q_G}(\lambda) := \bigotimes_{i \in I} O_{Q_G}(w_i)^{\otimes m_i}$ (as a tensor product of line bundles). Also, for each $x \in W_{af}^{\geq 0}$, we have the restriction $O_{Q_G(x)}(\lambda)$ obtained through (4.5), which is $\tilde{I}$-equivariant. Similarly, we have $(B \times G_m)$-equivariant line bundles $O_{\Omega_G(x, \xi)}(\lambda)$ and $O_{\Omega_G(x)}(\lambda)$ by further pullbacks (the latter is $G[z]$-equivariant whenever $x = t_\xi$ for some $\xi \in Q^{\vee,+}$); we set

$$H^n(\Omega_G(x), O_{\Omega_G(x, \xi)}(\lambda)) := \lim_{\xi \to \Omega_G(x)} H^n(\Omega_G(x, \xi), O_{\Omega_G(x, \xi)}(\lambda)) \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \quad (4.6)$$

Let $\lambda \in P^+$. As explained in [Kat1] Theorem 1.6, the restricted dual of the Demazure submodule $V^\vee(\xi) \subset V(\xi)$ (see (2.19)) of the extremal weight $U_q(g_{af})$-module $V(\xi)$ of extremal weight $-w_\lambda$ gives rise to an integrable $g[z]$-module (by taking the classical limit $q \to 1$), called the global Weyl module; we denote it by $W(\lambda)$. Here we note that global Weyl modules carry natural gradings arising from the dilation of the $z$-variable.

Theorem 4.7 ([BF2 Proposition 5.1]). For $\lambda \in P$, we have

$$H^0(Q_G, O_{Q_G}(\lambda)) \cong \begin{cases} W^*(\lambda) & \text{if } \lambda \in P^+, \\ \{0\} & \text{otherwise}, \end{cases}$$

where $Q_G$ denotes the open dense $G[[z]]$-orbit in $Q_G$. 

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Lemma 4.10. \(\in \xi wt\) In this subsection, we construct two kind of (pro-)schemes of infinite type, which can be thought of as “resolutions” of \(Q\) antidominant. Then we see from Lemma 4.11 that \(\ell(wt) = -\ell(\tilde{\mathcal{F}}(wt))\) for all \(w \in W\).

Proof. The proof is exactly the same as that of [Kat1, Corollary 2.7].

4.3 Bott-Samelson-Demazure-Hansen towers.

In this subsection, we construct two kind of (pro-)schemes of infinite type, which can be thought of as “resolutions” of \(Q_G(x)\) for \(x \in W_{af}^\geq\), and study their properties.

Lemma 4.11 ([Mac2, (2.4.1)]). If \(\xi \in Q^\vee\) is a strictly antidominant coweight, i.e., \(\langle \alpha_i, \xi \rangle < 0\) for all \(i \in I\), then \(\ell(t_\xi) = -2\langle \rho, \xi \rangle\), and \(\ell(wt_\xi) = \ell(t_\xi) - \ell(w)\) for all \(w \in W\); hence we have \(\ell(wt_\xi) = -\ell(\tilde{\mathcal{F}}(wt_\xi))\) for all \(w \in W\).

Lemma 4.12.

1. \(\ell(\tilde{\mathcal{F}}(xt_{-2m\rho})) + \ell(t_{2m\rho}) = \ell(xt)\) for all \(x \in W_{af}\) and \(m \in \mathbb{Z}\).
2. There exists \(m_0 \geq 0\) such that \(-\ell(\tilde{\mathcal{F}}(xt_{-2m\rho})) = \ell(xt_{-2m\rho})\) for all \(m \geq m_0\).

Proof. Part (1) is obvious from the definition of \(\ell(\tilde{\mathcal{F}}(\cdot))\). For part (2), write \(x = wt_\xi \in W_{af}\) for some \(w \in W\) and \(\xi \in Q^\vee\), and take \(m_0 \geq 0\) such that \(\xi - 2m_0\rho\) is strictly antidominant. Then we see from Lemma 4.11 that \(-\ell(\tilde{\mathcal{F}}(xt_{-2m\rho})) = \ell(xt_{-2m\rho})\) for all \(m \geq m_0\). This proves the lemma.

Remark 4.13. Keep the setting of Lemma 4.12. We have

\[\ell(xt_{-2(m_0 + m)\rho}) = \ell(xt_{-2m\rho}) + m\ell(t_{-2\rho}) = \ell(xt_{-2m\rho}) + \ell(t_{-2\rho})\]

for all \(m \geq 0\).

In what follows, we fix \(x \in W_{af}^\geq\) unless stated otherwise. For this \(x\), we take \(m_0 \geq 0\) as in Lemma 4.12(2), and fix reduced expressions

\[xt_{-2m\rho} = s_{i_1} s_{i_2} \cdots s_{i_{\ell}} \quad \text{and} \quad t_{-2\rho} = s_{i_{\ell}} s_{i_{\ell-1}} \cdots s_{i_2},\]

where \(i_1, \ldots, i_{\ell}, i_{\ell}', i_{\ell}'', \ldots, i_{\ell}\) \(\in I_{af}\), with \(\ell = \ell(xt_{-2m\rho})\) and \(\ell = \ell(t_{-2\rho})\). We concatenate these sequences periodically to obtain an infinite sequence

\[i = (i_1, i_2, i_3, \ldots, i_{\ell}, i_{\ell}', i_{\ell}'', \ldots, i_{\ell}', i_{\ell}', i_{\ell}', i_{\ell}', i_{\ell}', i_{\ell}', i_{\ell}', \ldots) \in I_{af}^\infty,\]

for \(xt_{-2m\rho}\) for \(t_{-2\rho}\) for \(t_{-2\rho}\)
Lemma 4.15. Let \( m \in \mathbb{Z}_{\geq 0} \) and write it as: \( i = (i_1, i_2, \ldots) \in I_{af}^\infty \); remark that \( s_i s_2 \cdots s_k \) is reduced for all \( k \geq 0 \). For \( k \in \mathbb{Z}_{\geq 0} \), we set \( i_k := (i_1, i_2, \ldots, i_k) \).

Let \( k \in \mathbb{Z}_{\geq 0} \), and let \( j = (i_{j_1}, \ldots, i_{j_h}) \) be a subsequence of \( i_k \), where \( 1 \leq j_1 < \cdots < j_h \leq k \). We set \( \sigma(j) = (k(i(j)) := \{1, 2, \ldots, k\} \setminus \{j_1, \ldots, j_h\} \). We identify a subsequence \( j \) of \( i_k \) with a subsequence \( j' \) of \( k' \) if and only if \( \sigma(k) \) (as subsets of \( \mathbb{Z}_{\geq 0} \)); namely, if \( k' \geq k \), then \( j = (i_{j_1}, \ldots, i_{j_h}) \subset i_k \) and \( j' = (i_{j_1'}, \ldots, i_{j_h'}) \subset i_{k'} \) are identified. Thus, we identify a subsequence \( j \) of \( i_k \) with a subsequence of \( i \) by taking the limit in \( \lim_{k \to \infty} i_k = i \).

Let \( k \in \mathbb{Z}_{\geq 0} \), and let \( j = (i_{j_1}, i_{j_2}, \ldots, i_{j_h}) \) be a subsequence of \( i_k \). We set \( x(j; k) := s_{i_{j_1}} s_{i_{j_2}} \cdots s_{i_{j_h}} \in W_{af} \).

**Remark 4.14.** Let \( k' \in \mathbb{Z}_{\geq 0} \) be such that \( k' \geq k \). Because the sequence \( j \) above is identified with the subsequence \((i_{j_1}, \ldots, i_{j_h}, i_{k+1}, \ldots, i_{k'})\) of \( i_{k'} \), we have

\[
x(j; k') := s_{i_{j_1}} \cdots s_{i_{j_h}} s_{i_{k+1}} \cdots s_{i_{k'}} \in W_{af}.
\]

**Lemma 4.15.** Let \( m \in \mathbb{Z}_{\geq 0} \), and let \( j \) be a subsequence of \( i_{r+m'} \). Then, there exists \( m_1 \geq m \) such that \( x(j; l' + m' \ell) = x(j; l' + m'' \ell) \cdot t_{-2(m''-m')\rho'} \) for every \( m' \geq m'' \geq m_1 \). In particular, the element \( x(j) := x(j; l' + m' \ell) \cdot t_{2(m' + m_0)\rho'} \in W_{af} \) does not depend on the choice of \( m' \geq m_1 \).

**Proof.** We first note that

\[
x(j; l' + m' \ell) = x(j; l' + m'' \ell) \cdot (s_{i_{r2}'} s_{i_{r3}'} \cdots s_{i_{r_k}'}) \cdots (s_{i_{r1}'} s_{i_{r2}'} \cdots s_{i_{r_k}'})
\]

for \( (m' - m'') \) times, corresponding to \( t_{-2\rho'} \) and \( t_{-2\rho'} \).

Since \( y \cdot s_i = ys_i \) if and only if \( \ell(ys_i) = \ell(y) + 1 \) for \( y \in W_{af} \) and \( i \in I_{af} \), it suffices to show that there exists \( m_1 \geq m \) such that

\[
\ell(x(j; l' + m'' \ell) \cdot t_{-2n\rho'}) = \ell(x(j; l' + m' \ell)) + \ell(t_{-2n\rho'})
\]

for all \( n > 0 \) and \( m'' \geq m_1 \). Let \( k \in \mathbb{Z}_{\geq 0} \) be such that \( k \geq m \). Since \( x(j; l' + k\ell) = x(j; l' + m\ell) \cdot t_{-2(k-m)\rho'} \) and \( \ell(t_{-2(k-m)\rho'}) = (k - m)\ell \), we see that

\[
\ell(x(j; l' + k\ell)) \geq (k - m)\ell;
\]

note that \( \ell(y \cdot y') \geq \max\{\ell(y), \ell(y')\} \) for \( y, y' \in W_{af} \), as is verified by induction. Now, for each integer \( k \geq m \), we set

\[
d_k := \ell(t_{-2\rho'}) - \ell(x(j; l' + (k + 1)\ell)) - \ell(x(j; l' + k\ell))
\]

observe that \( d_k \in \mathbb{Z}_{\geq 0} \). Also, for \( k \geq m \), we have

\[
(k - m)\ell = \ell(x(j; l' + k\ell)) - \ell(x(j; l' + m\ell)) + \sum_{k' = m}^{k-1} d_{k'}.
\]

If \( d_k > 0 \) for infinitely many \( k \geq m \), then \( (k - m)\ell > \ell(x(j; l' + k\ell)) \) for \( k \gg m \), which contradicts (4.10). Hence we deduce that \( d_k > 0 \) only for finitely many \( k \geq m \). Thus, if we set \( m_1 := \max\{k \geq m \mid d_k > 0\} \), then (4.9) holds. This proves the lemma.
Lemma 4.16. For each $y, y' \in W_{af}$ such that $y \preceq y'$, there exists $m_2 \in \mathbb{Z}_{\geq 0}$ such that $yt_{-2m\rho^\vee} \geq y't_{-2m\rho^\vee}$ in the ordinary Bruhat order on $W_{af}$ for all $m \geq m_2$.

Proof. It suffices to prove the assertion in the case that $y \overset{\beta}{\rightarrow} s_\beta y = y'$ for some $\beta \in \Delta^+_w$. Here we see from [INS Corollary 4.2.2] that $\beta$ is either of the following forms:

(i) $\beta = \alpha$ with $\alpha \in \Delta^+$;
(ii) $\beta = \alpha + \delta$ with $-\alpha \in \Delta^+$.

Moreover, if $y = wt_\xi$ with $w \in W$ and $\xi \in Q^\vee$, then $\gamma := w^{-1} \alpha \in \Delta^+$ in both cases above. Also, it follows from [INS Proposition A.1.2] that

\[ \ell(w) = \begin{cases} \ell(w_\gamma) - 1 & \text{in case (i)}, \\ \ell(w_\gamma) - 1 + 2\langle \rho, \gamma^\vee \rangle & \text{in case (ii)}. \end{cases} \quad (4.11) \]

If we set $\zeta := \xi - 2m\rho^\vee$ for $m \in \mathbb{Z}$, then $yt_{-2m\rho^\vee} = wt_\xi - 2m\rho^\vee = wt_\zeta$, and

\[ s_\beta(yt_{-2m\rho^\vee}) = s_{\alpha+k\delta}wt_\zeta, \quad \text{where } k = 0 \text{ in case (i), and } k = 1 \text{ in case (ii)}, \]
\[ = wt_\zeta s_{t_{-w^{\alpha+k\delta}}} = (yt_{-2m\rho^\vee})s_{\gamma+n\delta}, \quad \text{with } n := k + \langle \gamma, \zeta \rangle. \quad (4.12) \]

Therefore, in case (i) (resp., case (ii)), we deduce from [LNS Proposition 5.1 (1) (resp., (2)) with $v = e$], together with equalities (4.11) and (4.12) that $yt_{-2m\rho^\vee} = s_\beta yt_{-2m\rho^\vee} = (yt_{-2m\rho^\vee})s_{\gamma+n\delta} \geq yt_{-2m\rho^\vee}$ for all $m \gg 0$. This proves the lemma. $\square$

Lemma 4.17. For each $y \in W_{af}$ such that $y \preceq x$, there exist $k \in \mathbb{Z}_{\geq 0}$ and a subsequence $j$ of $i_k$ such that $y = x(j)$.

Proof. By Lemma 4.16 there exists $m \gg 0$ such that the assertion of Lemma 4.12 (2) holds for $m_0 + m$ instead of $m_0$ (see also Remark 4.13), and such that

\[ yt_{-2(m_0+m)\rho^\vee} < xt_{-2(m_0+m)\rho^\vee} = xt_{-2m\rho^\vee} \cdot t_{-2m\rho^\vee}; \]

note that $i_{r+m_\ell}$ gives a reduced expression for $xt_{-2(m_0+m)\rho^\vee}$. By the Subword Property (see, e.g., [BB Theorem 2.2.2]), $yt_{-2(m_0+m)\rho^\vee}$ is obtained as a subexpression of the reduced expression for $xt_{-2(m_0+m)\rho^\vee}$ corresponding to $i_{r+m_\ell}$. Namely, there exists a subsequence $j = (i_{j_1}, \ldots, i_{j_r})$ of $i_{r+m_\ell}$ such that

\[ yt_{-2(m_0+m)\rho^\vee} = s_{j_1} \cdots s_{j_r} = s_{j_1} \cdots * s_{j_r} = x(j, \ell' + m_\ell). \]

Let us take $m_1 \gg m$ as in Lemma 4.15. Then, we see by Remark 4.13 that for $m' \geq m_1$,

\[ x(j, \ell' + m_\ell) = x(j, \ell' + m_\ell) \cdot t_{-2(m' - m)\rho^\vee} = x(j, \ell' + m_\ell) \cdot t_{-2(m' - m)\rho^\vee} = yt_{-2(m'+m_0)\rho^\vee}. \]

Thus, we obtain $y = x(j)$, as desired. This proves the lemma. $\square$

For each $i \in I_{af}$ and $y \in W_{af}$, we define a map

\[ q_{i,y} : I(i) \times^1 Q_G(y) \to Q_G^{rat}, \quad (p, L) \mapsto pL. \quad (4.13) \]

Lemma 4.18. Let $y \in W_{af}^{\geq 0}$, and $i \in I_{af}$. If $s_i y \succeq y$, then the map $q_{i,y}$ induces a $\mathbb{P}^1$-fibration $I(i) \times^1 Q_G(y) \to Q_G(y)$, which we also denote by $q_{i,y}$. If $e \preceq s_i y \preceq y$, then the map $q_{i,y}$ induces a birational map $I(i) \times^1 Q_G(y) \to Q_G(s_i y)$, which we also denote by $q_{i,y}$. In both cases, the map $q_{i,y}$ is proper.
Proof. If $s_iy \geq y$, then the action of $I(i)$ stabilizes $O(y) \cup O(s_iy) \subset Q_G(y)$. Hence taking the closure in $Q_G$ implies that $Q_G(y)$ admits an $I(i)$-action. Therefore, the assertions hold in this case since $I(i)/I \cong \mathbb{P}^1$.

If $e \leq s_iy \leq y$, then $q_{i,y}^{-1}(O(s_iy))$ contains $I_i \times I O(y)$. Here we deduce from Lemmas A.2 and A.4 that if $y' \in W_{af}$ satisfies $y' > y$, then $s_iy' > s_iy$; in particular, we have $\ell(\tilde{x}) < \ell(\tilde{x})$. Hence we have $I_i \times I O(y) = q_{i,y}^{-1}(O(s_iy))$. In addition, the unipotent one-parameter subgroup of $I$ corresponding to $\alpha_i$ gives an isomorphism $A^1 \times I O(y) \cong O(s_iy)$. Therefore, $q_{i,y}$ is birational. Also, by applying the same observation for $s_iy$ instead of $y$, we deduce that $q_{i,y}$ is obtained as the restriction of the $\mathbb{P}^1$-fibration $q_{i,s_iy}$ to a closed subscheme. Hence $q_{i,y}$ defines a proper map. This proves the lemma. □

For each $k \in \mathbb{Z}_{\geq 0}$, we set $x(k) := s_{i_k} \cdots s_{i_1} x$. We claim that

$$\ell(\tilde{x})(x(k + 1)) = \ell(\tilde{x})(x(k)) + 1 \quad \text{for all } k \geq 0. \quad (4.14)$$

Indeed, since $x(k+1) = s_{i_k} x(k)$ for $k \geq 0$, we see by (A.5) that $\ell(\tilde{x})(x(k+1)) = \ell(\tilde{x})(x(k)) + 1$ for each $k \geq 0$. Therefore, it suffices to show that

$$\ell(\tilde{x})(x((m - m_0)\ell + \ell')) = \ell(\tilde{x})(x(0)) + (m - m_0)\ell + \ell' \quad \text{for all } m \geq m_0; \quad (4.15)$$

note that $x(0) = x$. We see by the definition that $x((m - m_0)\ell + \ell') = (x_{-2m_0\rho'})^{-1} x = t_{2m_0\rho'}$. Hence we compute:

$$\ell(\tilde{x})(x((m - m_0)\ell + \ell')) = \ell(\tilde{x})(t_{2m_0\rho'}) = 2\langle \rho, 2m_0\rho' \rangle = -m\ell(\tilde{x})(t_{-2\rho'}) = m\ell \quad \text{by Lemma 4.11} \quad (4.16)$$

Also, by Lemma 4.12(1), we have $\ell(\tilde{x})(x_{-2m_0\rho'} + \ell(\tilde{x})(t_{2m_0\rho'}) = \ell(\tilde{x})(x)$. Here we deduce that $\ell(\tilde{x})(x_{-2m_0\rho'}) = -\ell(x_{-2m_0\rho'}) = -\ell'$ by Lemma 4.12(2), and that $\ell(\tilde{x})(t_{2m_0\rho'}) = m_0 \ell$. Hence we obtain $\ell(\tilde{x})(x) = -\ell' + m_0 \ell$. Combining this equality with (4.16) shows (4.15), as desired.

Now, we set

$$Q_G(i_k) := I(i_1) \times I(i_2) \times I(i_3) \times I(i_k) \times I O(x(k)). \quad (4.17)$$

We also define its ambient space

$$Q_G^*(i_k) := I(i_1) \times I(i_2) \times I(i_3) \times I(i_k) \times I Q_G(x(k)).$$

Since $s_{i_k} x(k) = x(k - 1)$ and $x(k) \geq x(k - 1)$ for each $k \geq 1$ (see (4.14)), we have an $\tilde{I}$-equivariant embedding $O(x(k - 1)) \hookrightarrow I(i_k) \times I O(x(k))$ by the latter case of Lemma 4.18 and hence an $\tilde{I}$-equivariant embedding $Q_G(i_{k-1}) \hookrightarrow Q_G(i_k)$ for each $k \geq 1$. By infinite repetition of these embeddings, we obtain a scheme of infinite type

$$Q_G(i) := \lim_{\longrightarrow} Q_G(i_k) = \bigcup_{k \geq 0} Q_G(i_k), \quad (4.18)$$

endowed with an $\tilde{I}$-action. Also, the multiplication of components yields an $\tilde{I}$-equivariant morphism

$$m : Q_G(i) \rightarrow Q_G(x).$$
Similarly, we have $m_k : Q_G^{#}(i_k) \to Q_G(x)$ for each $k \geq 0$. The natural inclusion $Q_G(i_k) \hookrightarrow Q_G^{#}(i_k)$ yields a map $Q_G(i) \to Q_G^{#}(i_k)$ by taking the product of factors at position $> k$ from the left in (4.17). Namely, we collect the maps

$$Q_G(i_{k'}) \ni (p_1, \ldots, p_{k'}, L) \mapsto (p_1, \ldots, p_k, p_{k+1} \cdots p_{k'} L) \in Q_G^{#}(i_k)$$

for $k' > k$ through (4.18), where $p_j \in I(i_j), 1 \leq j \leq k'$, and $L \in O(x(k'))$; here each closed point of $Q_G(i_{k'})$ is an equivalence class with respect to the $I^k$-action, and the map above respects equivalence classes. This yields a factorization of $m$ through arbitrary $m_k$ in such a way that

$$Q_G(i) \to Q_G^{#}(i_{k'}) \to Q_G^{#}(i_k) \to Q_G(x)$$

(4.19)

for each $k' \geq k$. This also yields an inclusion

$$Q_G(i) \hookrightarrow Q_G^{#}(i) := \lim_{\longrightarrow k} Q_G^{#}(i_k),$$

which fits into the following commutative diagram of $\tilde{I}$-equivariant morphisms for $k < k'$:

$$\begin{array}{ccc}
Q_G(i) & \longrightarrow & Q_G^{#}(i) \\
m \downarrow & & \downarrow \\
Q_G(x) & \leftarrow & Q_G^{#}(i_k) \leftarrow Q_G^{#}(i_{k'}). \\
\end{array}$$

Lemma 4.19. The scheme $Q_G(i)$ is separated and normal.

Proof. The scheme $Q_G(i)$ is an inclusive union of countably many open subschemes each of which is isomorphic to a pro-affine space bundle over a finite successive $P^1$-fibrations. Since each of such a space is separated, we deduce the desired separatedness.

Also, by its construction, each $Q_G(i_k)$ is a union of pro-affine spaces labeled by subsequences of $i_k$ (so that $Q_G(i_k)$ is covered by a total of $2^k$-copies of open cover consisting of pro-affine spaces). Thus, $Q_G(i)$ is a union of countably many pro-affine spaces. Since all of these pieces are normal, we deduce the desired normality. This proves the lemma. □

For each subsequence $j \subset i_k$, we obtain an $I$-stable pro-affine space $O(j) \subset Q_G(i_k)$ by replacing $I(i_j), 1 \leq j \leq k$, with $I_{s_j}I$ (resp., $I$) if $i_j \in j$ (resp., $i_j \notin j$) in (4.17); we refer to $O(j)$ as the stratum corresponding to a subsequence $j$ of $i_k$.

Lemma 4.20.

(1) For each $k \in \mathbb{Z}_{\geq 1}$, it holds that $Q_G(i_k) = \bigsqcup_{j \subset i_k} O(j)$.

(2) Let $k \in \mathbb{Z}_{\geq 1}$, and let $j$ and $j'$ be subsequences of $i_k$. Then, $O(j) \subset \overline{O(j')}$ in $Q_G(i_k)$ if and only if $j \subset j' \subset i_k$.

Proof. The proofs of the assertions are straightforward by the definitions. □

If we regard $O(j)$ as a locally closed subscheme of $Q_G(i)$ via (4.18) for $j \subset i_k$, then its images in $Q_G^{#}(i_k)$ and $Q_G^{#}(i_{k'})$ for $k < k'$ are isomorphic through (4.19). Therefore, we can freely replace $m$ with $m_k$ when we analyze a single stratum in $Q_G(i)$. We set $Q_G(j) := \overline{O(j)}$, where the closure is taken in $Q_G(i)$.  

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Lemma 4.21. We have $Q_G(i) = \bigsqcup_{j} O(j)$, where $j$ runs over all subsequences of $i$ such that $|\sigma(j)| < \infty$.

Proof. Since $Q_G(i) = \bigsqcup_{k \geq 1} Q_G(i_k)$, the assertion is a consequence of Lemma 4.20 and the consideration just above.

Lemma 4.22. The map $m$ is surjective.

Proof. By Lemma 4.17, each fiber of $m$ along an $(H \times \mathbb{G}_m)$-fixed point is nonempty. Because applying the $I$-action to the set of $(H \times \mathbb{G}_m)$-fixed points exhausts $Q_G(x)$ by Corollary 4.6, we conclude that $m$ is surjective, as required. This proves the lemma.

Lemma 4.23. The map $m_k$ is $\tilde{I}$-equivariant, birational, and proper for each $k \geq 1$.

Proof. Since $\ell \tilde{\tau}(x(k)) - \ell \tilde{\tau}(x) = k$ for each $k \geq 1$, repeated application of Lemma 4.18 shows that $m_k$ is birational and proper for each $k \geq 1$; this is because $m_k$ is obtained as the base change of the composite of morphisms of type $q_{ij}, x(j)$ for $1 \leq j \leq k$. This proves the lemma.

The map $m$ is also $\tilde{I}$-equivariant and birational since the embedding $Q_G(i) \subset Q_G^\#(i)$ is open.

4.4 Cohomology of line bundles over $Q_G$.

In this subsection, keeping the setting of the previous subsection, we also assume that $x = e$, the identity element; in this case, we can (and do) take the $m_0$ (in Lemma 4.12(2)) to be 0, so that we have $\ell' = 0$ (see (4.7)). In particular, we have $Q_G(x) = Q_G(e) = Q_G$.

For each $\lambda \in P$, the $\tilde{I}$-equivariant line bundle $O_{Q_G(x(k))}(\lambda)$ induces a line bundle over $Q_G(i_k)$ for each $k \in \mathbb{Z}_{\geq 0}$.

Proposition 4.24. For each $\lambda \in P$, we have an isomorphism

$$H^0(Q_G(i), O_{Q_G(i)}(\lambda)) \cong W(\lambda)^*$$

of $\tilde{I}$-modules. In particular, the left-hand side carries a natural structure of graded $g[z]$-module.

Proof. We adopt the notation of the previous subsection, with $x = e$, $m_0 = 0$, and $\ell' = 0$. Also, since $\ell(t_{-2p^\nu}) = \ell(w_o) + \ell(w_o t_{-2p^\nu})$, we can rearrange the reduced expression $(i_{t_1}', \ldots, i_{t_\ell}') = (i_1, \ldots, i_\ell)$ for $t_{-2p^\nu}$, if necessary, in such a way that the first $\ell(w_o)$-entries $(i_1, \ldots, i_{\ell(w_o)})$ give rise to a reduced expression for $w_o$.

The map $m$ factors as $m' \circ m''$, where $m''$ is the map obtained by collapsing the first $\ell(w_o)$-factors in $Q_G(i)$ as:

$$Q_G(i) = I(i_1) \times I(i_2) \times \cdots \times I(i_{\ell(w_o)}) \times I(1) X \to G[[z]] \times X,$$  \hspace{1cm} (4.20)

with $X$ a certain scheme admitting an $\tilde{I}$-action, and $m'$ is the natural map $G[[z]] \times X \to Q_G$ induced by the action. In particular, we deduce that $m_*^* O_{Q_G(i)}(\lambda)$ admits a $G[[z]]$-action, and hence that the space of global sections of $m_*^* O_{Q_G(i)}(\lambda)$ is a $g[z]$-module.

Now, for each $m \in \mathbb{Z}_{\geq 0}$, we have $Q_G(i_{\ell(w_o)+\ell m}) \subset Q_G(i)$. Also, for each $k \in \mathbb{Z}_{\geq 0}$, we have an inclusion $Q_G(i_k) \hookrightarrow Q_G(i_{k+\ell})$ by sending the stratum $O(j)$ corresponding to
Theorem 4.25 (Kneser-Platonov; see, e.g., [Gil, Proof of Theorem 5.8]). The subset $G[z] \subset G[[z]]$ is dense.
we conclude the assertion.

Theorem 4.26. We have $m_*\mathcal{O}_{Q_G(i)} \cong \mathcal{O}_{Q_G}$. In particular, the scheme $Q_G$ is normal.

Proof. We (can) employ the same reduced expression for $t_{-2\rho^\vee}$ as in the proof of Proposition 4.24, recall the last sentence of the proof. The pullback defines a map $m^*\mathcal{O}_{Q_G} \to \mathcal{O}_{Q_G(i)}$, whose adjunction in turn yields $\mathcal{O}_{Q_G} \to m_*\mathcal{O}_{Q_G(i)}$. From this, by twisting by $\mathcal{O}_{Q_G}(\lambda)$ for some $\lambda \in P^+$, we obtain the following short exact sequence:

$$0 \to \mathcal{O}_{Q_G}(\lambda) \to m_*\mathcal{O}_{Q_G(i)}(\lambda) \to \text{Coker} \to 0,$$

from which we deduce a $\mathfrak{g}[z]$-module inclusion

$$H^0(Q_G, \mathcal{O}_{Q_G}(\lambda)) \hookrightarrow H^0(Q_G, m_*\mathcal{O}_{Q_G(i)}(\lambda)) \cong H^0(Q_G(i), \mathcal{O}_{Q_G(i)}(\lambda)),$$

by taking their global sections. The rightmost one is isomorphic to $W(\lambda)^*$ by Proposition 4.23. In particular, we have algebra homomorphisms:

$$\bigoplus_{\lambda \in P^+} \Gamma(Q_G, \mathcal{O}_{Q_G}(\lambda)) \subset \bigoplus_{\lambda \in P^+} \Gamma(Q_G, m_*\mathcal{O}_{Q_G(i)}(\lambda))$$

$$\cong \bigoplus_{\lambda \in P^+} \Gamma(Q_G(i), \mathcal{O}_{Q_G(i)}(\lambda))$$

$$\cong \bigoplus_{\lambda \in P^+} W(\lambda)^*;$$

let us denote the leftmost one by $R'_G$ and the rightmost one by $R_G$. Since $Q_G(i)$ is normal, we deduce that $R_G$ is normal when localized with respect to homogeneous elements in $W(\varpi_i)^*$ for each $i \in I$. For the same reason, $R_G$ is an integral domain. The ring structure of $R_G$ is induced by the unique (up to scalar) $\mathfrak{g}[z]$-module map

$$W(\lambda + \mu) \longrightarrow W(\lambda) \otimes \mathbb{C} W(\mu), \quad \lambda, \mu \in P^+, \quad \text{(4.23)}$$

of degree zero. In view of [Kat1, Proof of Theorem 3.3], we deduce that the multiplication map $W(\lambda)^* \otimes W(\mu)^* \to W(\lambda + \mu)^*$ is surjective since (4.23) is injective. Therefore, $R_G$ is a normal ring generated by terms of primitive degree (see, e.g., [Ha, Chap. II, Exerc. 5.14]; cf. [Kat1, Proof of Theorem 3.3]).

From the above, it suffices to prove $R'_G = R_G$. For this purpose, it is enough to prove that the associated graded ring of the projective coordinate ring $R'_G$ of $Q_G$, which is arising from its structure of a closed subscheme of $\prod_{i \in I} \mathbb{P}(L(\varpi_i)[[z]])$, contains $R_G$ (see, e.g., [EGAII, Sect. 2.6] for convention). Recall that the projective coordinate ring of $\mathbb{P}(L(\varpi_i)[[z]])$ is $\bigoplus_{n \geq 0} S^n(L(\varpi_i)[z]^*)$, where $S^nV$ denotes the n-th symmetric power of a vector space $V$. Thanks to the surjectivity of multiplication map of $R_G$, it is further reduced to seeing that for each $i \in I$, every element of the part $W(\varpi_i)^*$ of degree $\varpi_i$,
in \( R_G \) is written as the quotient of an element of \( \prod_{i \in I} S^{(\lambda, \alpha_i^\gamma)}(L(\varpi_i)[z]^*) \) by some power of monomials in elements of \( L(\varpi_j)[z]^* \subset W(\varpi_j)^* \), \( j \in I \) (note that this condition is particularly apparent in types \( A \) and \( C \) since \( W(\varpi_i) = L(\varpi_i)[z] \) for each \( i \in I \).

By [BF2] Proof of Theorem 3.1, each \( \Omega_G(\xi) \), \( \xi \in Q^{\vee,+} \), is a projective variety with rational singularities. By the Serre vanishing theorem [EGAIII Théorème 2.2.1] applied to the ideal sheaf that defines \( \Omega_G(\xi) \) (inside the product of finite-dimensional projective spaces in \( \mathbb{P}(L(\varpi_i)[z]) \) obtained by bounding the degree; cf. [KatI (2.1)]), the restriction map

\[
\bigotimes_{i \in I} H^0(\mathbb{P}(L(\varpi_i)[z]), \mathcal{O}(\lambda, \alpha_i^\gamma)) \to H^0(\Omega_G(\xi), \mathcal{O}_{\Omega_G(\xi)}(\lambda))
\]  

(4.24)
is surjective for sufficiently large \( \lambda \in P^+ \), where we have used the fact that

\[
\bigotimes_{i \in I} H^0(\mathbb{P}(L(\varpi_i)[z]), \mathcal{F}_i) \cong H^0\left( \prod_{i \in I} \mathbb{P}(L(\varpi_i)[z]), \bigotimes_{i \in I} \mathcal{F}_i \right)
\]

holds for vector bundles \( \mathcal{F}_i \) of finite rank on \( \mathbb{P}(L(\varpi_i)[z]) \).

**Claim 1.** For a given degree \( n \in \mathbb{Z}_{\leq 0} \), we can choose \( \lambda \in P^+ \) and \( \xi \in Q^{\vee,+} \) sufficiently large in such a way that for every \( m \in \mathbb{Z}_{> 0} \), the restriction map

\[
W(m\lambda)^* \subset W(m\lambda)^{\vee} = H^0(\Omega_G, \mathcal{O}_{\Omega_G}(m\lambda)) \to H^0(\Omega_G(\xi), \mathcal{O}_{\Omega_G(\xi)}(m\lambda))
\]
is injective at degree greater than or equal to \( n \).

**Proof of Claim 1.** Let us denote by \( W(\varpi_i)_{\geq n}^* \subset W(\varpi_i)^* \) and \( W(\varpi_i)_{\leq -n} \subset W(\varpi_i) \) the direct sum of the homogeneous components of \( W(\varpi_i)^* \) of degree greater or equal to \( n \), and the the direct sum of the homogeneous components of \( W(\varpi_i) \) of degree less than or equal to \( -n \), respectively. Also, let \( R^n_G \) be the subring of \( R_G \) generated by the \( W(\varpi_i)_{\geq n}^*, i \in I \); every homogeneous component of \( R_G \) of degree greater than or equal to \( n \) is contained in \( R^n_G \) by the surjectivity of multiplication map and the fact that \( W(\lambda)^* \) is concentrated in nonpositive degrees. The value of a section of a line bundle over \( \text{Proj} R_G \) (our \( \text{Proj} \) here is the \( P^+ \)-graded \( \text{proj} \)), by which we mean that the \( H \)-quotient of the subset of the affine spectrum in \( \prod_{i \in I} (W(\varpi_i) \setminus \{0\}) \) arising from \( R^n_G \) at a point is determined completely by its image under the projection

\[
\text{pr} : \prod_{i \in I} \mathbb{P}(\overline{W(\varpi_i)}) \setminus Z \to \prod_{i \in I} \mathbb{P}(W(\varpi_i)_{\leq -n})
\]

induced by the \( g[z] \)-module surjection \( W(\varpi_i) \to \overline{W(\varpi_i)_{\leq -n}} \), where \( Z \) denotes the loci in which \( \text{pr} \) is not well-defined; for the notation \( W(\varpi_i) \), see Section 2.1. Thanks to Theorem 4.25 we deduce that

\[
\text{pr}(\Omega_G \setminus Z) = \text{pr}(Q_G \setminus Z)
\]
as the set of closed points. Since the restriction of \( \text{pr} \) to \( \Omega_G(\xi) \setminus Z \) for each \( \xi \in Q^{\vee,+} \) is a morphism of Noetherian schemes, it follows that \( \text{pr}(\Omega_G(\xi) \setminus Z) \) is a constructible subset of \( \prod_{i \in I} \mathbb{P}(W(\varpi_i)_{\leq -n}) \). Moreover, the irreducibility of \( \Omega_G(\xi) \) forces \( \text{pr}(\Omega_G(\xi) \setminus Z) \) to be irreducible. Therefore, the equality

\[
\text{pr}(\Omega_G \setminus Z) = \bigcup_{\xi \in Q^{\vee,+}} \text{pr}(\Omega_G(\xi) \setminus Z)
\]

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implies that there exists some $\xi \in Q^{V_i^+}$ such that

$$\text{pr}(\Omega_G(\xi) \setminus Z) \subset \text{pr}(Q_G \setminus Z)$$

(4.25)

is Zariski dense, since $\prod_{i \in I} \mathbb{P}(W(\varpi_i)_{\leq -n})$ is a Noetherian scheme.

Thanks to [BF2, Proposition 5.1] (cf. Theorem 4.8), we can find $\lambda$ (by replacing $\xi$ with a larger one if necessary) such that the assertion holds for $m = 1$. Now we assume the contrary to deduce the assertion for $m > 1$. Then, we have an additional equation on $\text{pr}(Q_G \setminus Z) \subset \prod_{i \in I} \mathbb{P}(W(\varpi_i)_{\leq -n})$ vanishing along $\Omega_G(\xi)$ by (taking sum of) the multiplication of $W(\lambda)^*_Z$. However, this is impossible in view of (4.25) since $\text{Proj} R_G$ is reduced (and hence $R_G^0$ is integral). Thus we have proved Claim 1.

We return to the proof of Theorem 4.26. We fix $n \in \mathbb{Z}_{\leq 0}$ and $\beta \in Q^{V_i^+}$ such that Claim 1 holds. By replacing $\lambda$ if necessary to guarantee the surjectivity of the restriction map (4.24) with keeping the situation of Claim 1, we deduce that all the maps

$$\bigotimes_{i \in I} H^0(\mathbb{P}(L(\varpi_i)[z]), \mathcal{O}(\{\lambda, \alpha_i^\gamma\})) \longrightarrow H^0(Q_G, \mathcal{O}_{Q_G}(\lambda))$$

$$\bigotimes_{i \in I} S^{(\lambda, \alpha_i^\gamma)}(L(\varpi_i)[z]^*) \longrightarrow W(\lambda)^*$$

(4.26)

are surjective at degree $n$ from the commutativity of the diagram and the surjectivity of the bottom horizontal map. For a degree $n$ element $f \in W(\varpi_i)^*$ and degree zero element $h_j \in L(\varpi_j)^* \subset W(\varpi_j)^*$ for each $j \in I$, we can choose sufficiently large integers $N_i$, $i \in I$, such that

$$f \cdot \prod_{j \in I} h_j^{N_j} \in S^{1+N_i}(L(\varpi_i)[z]^*) \cdot \prod_{j \in I, j \neq i} S^{N_j}(L(\varpi_j)[z]^*) \subset R_G$$

as the corresponding claim is true after sending to the bottom line of (4.26). This forces $W(\varpi_i)^*$ to be contained in the part of degree $\varpi_i$ of the associated graded ring of $R_G^0$, as required. This completes the proof of the theorem.

\textbf{Corollary 4.27.} The projective coordinate ring $R_G$ of $Q_G$ arising from the embedding by means of the DP-coordinates is isomorphic to $\bigoplus_{\lambda \in P^+} W(\lambda)^*$.

\textbf{Theorem 4.28.} The projective coordinate ring $R_G$ of $Q_G$ in Corollary 4.27 is free over the polynomial algebra $A_G$ given by the lowest weight components with respect to the $H$-action.

\textbf{Proof.} During this proof, we denote by $v_\lambda \in W(\lambda)$ the unique $h_{af}$-eigenvector of weight $\lambda$ (which is determined up to a scalar, and is the specialization of the corresponding vector of $V(\lambda)$ through $q \to 1$).

For each $\lambda \in P^+$, we set

$$\mathbb{C}[A(\lambda)] := \bigotimes_{i \in I} S^{(\lambda, \alpha_i^\gamma)}(\mathbb{C}[z]) \cong \bigotimes_{i \in I} \mathbb{C}[z_{(i)}^{(i)}, \ldots, z_{(\lambda, \alpha_i^\gamma)}]^{S_{(\lambda, \alpha_i^\gamma)}}.$$
By the results [FL, N], due to Fourier-Littelmann and Naoi, we know that \( W(\lambda) \) is a free module over \( \mathbb{C}[\mathbb{A}^{(\lambda)}] \), and the \( \lambda \)-isotypical component of \( W(\lambda) \) is free of rank one. Here we define the polynomial algebra \( A_G \) by collecting the \((-\lambda)\)-isotypical component of \( W(\lambda)^* \) for all \( \lambda \in P^+ \). It follows that the ring \( A_G \) is of the form

\[
\bigotimes_{i \in I} S^* \mathbb{C}[z]^* \cong \bigoplus_{\lambda \in P^+} \bigotimes_{i \in I} S(\lambda, \alpha_i^\vee) \mathbb{C}[z]^* \cong \bigoplus_{\lambda \in P^+} \mathbb{C}[\mathbb{A}^{(\lambda)}]^*.
\]

Let \( \psi \in W(\mu)^* \) and \( \xi \in \mathbb{C}[\mathbb{A}^{(\lambda-\mu)}]^* \subset A_G \), where \( \lambda, \mu, \lambda - \mu \in P^+ \), and assume that both of them are homogeneous with respect to \( P \)-weights and degrees. Then, we find a product of \( h_{af}\)-eigen PBW basis element \( F_1 \in U(n^-[z]) \) and monomials \( f_1, f_2 \in U(\mathfrak{h}[z]z) \cong S^*(\mathfrak{h}[z]z) \) such that \( \psi(F_1 f_1 v_{\mu}) \neq 0 \) and \( \xi(f_2 v_{(\lambda-\mu)}) \neq 0 \) by the Poincaré-Birkhoff-Witt theorem. It follows that \( (\psi \cdot \xi)(F_1 f_1 f_2 v_{\lambda}) \neq 0 \), since we need to collect the terms \( F_1 m_1 v_{\mu} \otimes m_2 v_{\lambda-\mu} \), with \( m_1, m_2 = f_1 f_2 \), in the tensor product through the embedding \( W(\lambda) \subset W(\mu) \otimes W(\lambda - \mu) \). This means that \( 0 \neq \psi \cdot \xi \in W(\lambda)^* \); in particular, the ring \( R_G \) is torsion-free as an \( A_G \)-module.

Now, let us fix \( i_0 \in I \) and \( \lambda \in P^+ \) so that \( \langle \lambda, \alpha_{i_0}^\vee \rangle = 0 \), and set \( \lambda_m := \lambda + m \varpi_{i_0} \) for each \( m \in \mathbb{Z}_{\geq 0} \). We also set \( A_{G_{i_0}} := S^* \mathbb{C}[z]^* \subset A_G \) for the fixed \( i_0 \). For each \( m, l \in \mathbb{Z}_{\geq 0} \) such that \( m \geq l \), we denote by \( W(\lambda; m, l) \) the space \( (\mathbb{C}[\mathbb{A}^{(\varpi_{i_0})}]^* \cdot W(\lambda_m - l)^*)^* \), which is a \( \mathfrak{h}[z] \)-submodule of \( W(\lambda_m) \). From this description, we have an inclusion

\[
W(\lambda; m, l) \subset W(\lambda; m, l + 1)
\]

when \( l > 0 \). In particular, \( W(\lambda; m, l)^* \) is a quotient of \( W(\lambda; m, l + 1)^* \).

By repeated use of (the dual of) the surjectivity of the multiplication map of \( R_G \), we have an embedding:

\[
\Phi : W(\lambda_m) \hookrightarrow W(\varpi_{i_0})^\otimes m \otimes W(\lambda).
\]

(4.27)

For \( 0 \leq l \leq m \), let \( \mathbb{W}(\lambda; m, l) \) denote the linear span of pure tensors

\[
\left( \bigotimes_{j=1}^{m} v_{i_0,j} \right) \otimes v \in W(\varpi_{i_0})^\otimes m \otimes W(\lambda)
\]

(4.28)

of \( \mathfrak{h}_{af}\)-eigenvectors in which at most \( l \)-elements of \( \{v_{i_0,j}\}_{j=1}^{m} \) is of the form \( z^e v_{\varpi_{i_0}} \) for some \( e \in \mathbb{Z}_{\geq 0} \). If we denote by \( W(\lambda; m, l)' \) the preimage of \( \mathbb{W}(\lambda; m, l) \) through \( \Phi \), then we have

\[
W(\lambda; m, l - 1)' \subset W(\lambda; m, l)'
\]

whenever \( l > 0 \). By construction, \( \{W(\lambda; m, l)\}_{0 \leq l \leq m} \) forms a \( \mathbb{Z} \)-graded increasing filtration whose associated graded modules

\[
\text{gr}_l W(\lambda_m) := W(\lambda; m, l)' / W(\lambda; m, l - 1)', \quad l \in \mathbb{Z}_{\geq 0},
\]

stratify \( W(\lambda_m) \).

Here, \( W(\varpi_{i_0})^\otimes m \otimes W(\lambda) \) admits a graded decomposition coming from the number of elements in \( \{v_{i_0,j}\}_{j=1}^{m} \) of the form \( z^e v_{\varpi_{i_0}} \) for \( e \in \mathbb{Z}_{\geq 0} \) through (4.28) and (4.27). It follows that the space \( \mathbb{W}(\lambda; m, l) \) is the annihilator of the subspace

\[
\sum_{w \in S_m} \sum_{a=0}^{m} w(\mathbb{C}[\mathbb{A}^{(\varpi_{i_0})}]^\otimes a \otimes W(\varpi_{i_0})^\otimes (m-a)^* \otimes W(\lambda)^*) \subset (W(\varpi_{i_0})^\otimes m)^* \otimes W(\lambda)^*,
\]

(4.29)
where $\mathfrak{S}_m$ permutes the tensor factors of $W(\varpi_i)^\otimes m$. Pulling back by $\Phi$, we deduce that $W(\lambda; m, l)\dagger$ is the annihilator of the space (1.29) in $W(\lambda_m)$ through the embedding (1.27). Therefore, $W(\lambda; m, l)\dagger \subset W(\lambda_m)$ is exactly the annihilator of $\mathbb{C}[A^{(\varpi_i)}]^*$. $W(\lambda_{m-l})^* \subset W(\lambda_m)$. It follows that we have a canonical isomorphism

$$\text{gr}_l W(\lambda_m) \cong W(\lambda; m, l)/W(\lambda; m, l + 1).$$

If we define a subquotient $M(\lambda; n)$ of $R_G$ for each $n \in \mathbb{Z}_{\geq 0}$ by

$$M(\lambda; n) := \bigoplus_{l \geq 0} M(\lambda; n)_l, \quad M(\lambda; n)_l = (\text{gr}_l W(\lambda_{n+l}))^*,$$

then $M(\lambda; n)$ admits an $A_{C_0}^{(\lambda)}$-action.

From the construction of $M(\lambda; n)_l$ through $\{W(\lambda; m, l)\}_m,l \geq 0$, we deduce that $M(\lambda; n)$ is generated by $M(\lambda; n)_0$ as an $A_{C_0}^{(\lambda)}$-module. Also, from the construction of $M(\lambda; n)_l$ through $\{W(\lambda; m, l)\}_m,l \geq 0$, we deduce that the dual of the multiplication map is the natural map

$$\mathbb{C}[A^{(\varpi_i)}] \otimes \text{gr}_0 W(\lambda_n) \to \text{gr}_0 W(\lambda_{n+l})$$

of $\mathbb{C}[A^{(\lambda_{n+l})}]$-modules. The $(\mathbb{C}[A^{(\varpi_i)}], \mathfrak{b}[z])$-modules $\text{gr}_l W(\lambda_m), 0 \leq l \leq m$, stratify $W(\lambda_m)$. In addition, the maximality of $W(\lambda; m, l)\dagger$ guarantees that each $\text{gr}_l W(\lambda_m)$ is torsion-free as a $\mathbb{C}[A^{(\lambda_m)}]$-module.

For each $\lambda \in P^+$, we can regard $W(\lambda)$ as a module corresponding to a vector bundle (or a free sheaf) $W(\lambda)$ over $A^{(\lambda)}$, where $A^{(\lambda)}$ denotes $\text{Spec} \mathbb{C}[A^{(\lambda)}]$; its fiber is known to be the tensor product of local Weyl modules $W(\mu, x)$, where $\mu \in P^+$ and $x \in \mathbb{C}$ runs over the configurations of points determined by a point of $A^{(\lambda)}$ (see, e.g., [Kat1], Theorem 1.4]).

The spaces $\text{gr}_l W(\lambda_m), 0 \leq l \leq m$, give torsion-free sheaves $\mathcal{W}_l(\lambda_m)$ on $A^{(\lambda_m)}$ that stratify $W(\lambda_m)$. Hence a section of $\mathcal{W}_l(\lambda_m)$ is an equivalence class of the set of sections $A^{(\lambda_m)} \to W(\lambda_m)$ whose specialization to a general point gives an element of the tensor product of local Weyl modules

$$v = \sum_{k} \bigotimes_{i,j} v_{i,j}^{(k)} \in \bigotimes_{i \in I} \bigotimes_{j = 1} W(\varpi_i, x_{i,j})$$

(4.30)

such that exactly $l$-elements in $\{v_{i_0,j}^{(k)}\}_{j = 1}^m$ are highest weight vectors, and the other vectors do not have a highest weight component for each $k$.

Since every two points in $\{x_{i,j}\}_{i,j}$ are generically distinct, each pure tensor in (4.30) divides $\{1, 2, \ldots, m\}$ into two subsets $S_1$ and $S_2$, with $\# S_1 = l$, so that $\{x_{i_0,j}\}_{j \in S_1}$ carries a highest weight vector (i.e., $v_{i_0,j}^{(k)} = z^\epsilon v_{\varpi_i}$ for some $e \in \mathbb{Z}_{\geq 0}$) and $\{x_{i_0,j}\}_{j \in S_2}$ carries a vector lying in non-highest-weight components (as a $\mathfrak{b}$-module). The coordinates $\{x_{i_0,j}\}_{j \in S_1}$ gives rise to the action of $\mathbb{C}[A^{(\varpi_i)}]$ on $\text{gr}_l W(\lambda_m)$, while the coordinates $\{x_{i_0,j}\}_{j \in S_2}$ gives rise to a $\mathbb{C}[A^{(\lambda_{m-l})}]$-module structure on $\text{gr}_l W(\lambda_m)$, and these two module structures are (mutually commutative and) distinct. It follows that every pair of elements of $\mathbb{C}[A^{(\varpi_i)}]$ and $\text{gr}_0 W(\lambda_{m-l})$ appears as a section in $\mathcal{W}_l(\lambda_m)$ after a generic localization. This particularly gives us the $\mathfrak{S}_m$-action on $\mathbb{C}[A^{(\varpi_i)}] \otimes \mathbb{C}[A^{(\lambda_{m-l})}]$ and $\text{gr}_l W(\lambda_m)$ that changes the order of the highest weight vectors and the one of non-highest weight vectors (or mixes up $S_1$ and $S_2$). Therefore, $\text{gr}_l W(\lambda_m)$ itself is torsion-free as a $\mathbb{C}[A^{(\varpi_i)}] \otimes \mathbb{C}[A^{(\lambda_{m-l})}]$-module.
Because $M(\lambda; n)$ is generated by $M(\lambda; n)_0$ as an $A^n_G$-module, we have an injective map
\[ \eta : \text{gr}_i W(\lambda_m) \hookrightarrow \mathbb{C}[\hat{A}^{(l \omega_u)}] \otimes \text{gr}_0 W(\lambda_{m-l}) \]
of $\mathbb{C}[\hat{A}^{(l \omega_u)}] \otimes \mathbb{C}[A^{(\omega_u)}]$-modules, which is an isomorphism after a localization to some Zariski open subset of $\hat{A}^{(l \omega_u)} \times A^{(\omega_u)}$.

Since $\text{gr}_i W(\lambda_m)$, $0 \leq l \leq m$, stratifies $W(\lambda_m)$, we deduce that the $M(\lambda; n)$’s, with $\lambda$ varying, give a stratification of the $A^n_G$-module $R_G$. Hence, in order to prove that $R_G$ is free over $A^n_G$, it suffices to verify that each $M(\lambda; n)$ is a free $A^n_G$-module. By construction, the image of $\eta$ contains $\mathbb{C} \otimes \text{gr}_0 W(\lambda_{m-l})$, which gives a $\mathbb{C}[\hat{A}^{(l \omega_u)}]$-module generator of $\text{gr}_i W(\lambda_m)$. Therefore, the map $\eta$ must be an isomorphism. As a consequence, we conclude that
\[ \mathbb{C}[\hat{A}^{(l \omega_u)}]^* \otimes M(\lambda; n)_0 \cong M(\lambda; n)_l \]
through the multiplication map. In other words, $M(\lambda; n)$ is a free $A^n_G$-module.

Because the above argument is consistent with the filtrations and their associated graded modules arising from a different choice of $\omega_u \in \mathbb{I}$, we can vary $\omega_u \in \mathbb{I}$ and construct the associated graded modules inductively on a fixed total order on $\mathbb{I}$. This gives a stratification of $R_G$ that is free over $A_G$. Hence the ring $R_G$ itself is free over $A_G$. This completes the proof of the theorem. 

**Theorem 4.29.** For each $\lambda \in P$, we have
\[ \text{gch} H^n(Q_G, \mathcal{O}_{Q_G}(\lambda)) = \begin{cases} \text{gch} V^- w\lambda & \text{if } \lambda \in P^+ \text{ and } n = 0, \\ 0 & \text{otherwise.} \end{cases} \]

**Proof.** We know that $Q_G$ is a closed subscheme of $\prod_{i \in I} \mathbb{P}(L(\omega_i))$ by (4.5). Therefore, we have a countable set $\Omega$ of $I$-tuples of $(H \times \mathbb{G}_m)$-eigenvectors of $\bigcup_{i \in I} L(\omega_i)$, one for each $i \in I$, so that it induces an affine open cover $U := \{\mathcal{U}_S\}_{S \subseteq \Omega}$ of $Q_G$ (where $\mathcal{U}_S := \{f \neq 0 \mid f \in S\}$) that is closed under intersection.

Now, the maps $L(\omega_i)[z] \setminus \{0\} \rightarrow \mathbb{P}(L(\omega_i)[z])$, $i \in I$, induce a (right) $H$-fibration $\tilde{Q}_G$ that defines an open scheme of $\prod_{i \in I} L(\omega_i)[z]$, which corresponds to specifying a nonzero vector $u^{\omega_i}$ instead of a one-dimensional $\mathbb{C}$-vector subspace $L^{\omega_i} \ni u^{\omega_i}$ in the definition of DP data; its closure $\bar{Q}_G$, which corresponds to allowing $u^{\omega_i} = 0$ in a DP datum, is an affine subscheme of $\prod_{i \in I} L(\omega_i)[z]$ of infinite type. We set $Z := \bar{Q}_G \setminus \tilde{Q}_G$, which is a closed subscheme of $Q_G$. Also, the pullback $\tilde{U}_S$ of $\mathcal{U}_S$ to $\tilde{Q}_G$ defines an affine open subset of $\tilde{Q}_G$. By the finiteness of the defining functions, $\tilde{U}_S \hookrightarrow \tilde{Q}_G$ is quasi-compact by [EGAII Proposition 1.1.10]. For each finite subset $S \subseteq \Omega$, we set $\tilde{U}^S := \{\tilde{U}_T\}_{T \subseteq S}$, which is again a collection of affine subschemes that is closed under intersections, and $\tilde{U}^S := \bigcup_{T \subseteq S} \tilde{U}_T$.

In addition, we set $\tilde{Z}_S := \tilde{Q}_G \setminus \tilde{U}^S$.

Let us denote the natural projection $\tilde{Q}_G \rightarrow Q_G$ by $\pi$. Since $Q_G$ is a (right) free quotient of $\tilde{Q}_G$ by $H$, we deduce that $\mathcal{O}_{Q_G}(\lambda) = (\pi_* \mathcal{O}_{\tilde{Q}_G})^{(H, \lambda)}$, where $(\pi_* \mathcal{O}_{\tilde{Q}_G})^{(H, \lambda)}$ denotes the $\lambda$-isotypical component with respect to the right $H$-action. Because discarding open sets (in such a way that the remaining ones are closed under intersection) in the Čech complex defines a projective system of complexes satisfying the Mittag-Leffler condition, [EGAIII Proposition 13.2.3] yields an isomorphism
\[ H^n(\tilde{Q}_G, \mathcal{O}_{\tilde{Q}_G}) \cong \lim_{\rightarrow S} H^n(\tilde{U}^S, \mathcal{O}_{\tilde{Q}_G}) \quad \text{for each } n \in \mathbb{Z}_{\geq 0}. \]
We have
\[ H^n(Q_G, \mathcal{O}_{Q_G}(\lambda)) \cong H^n(\hat{Q}_G, \mathcal{O}_{\hat{Q}_G})^{(H, \lambda)} \quad \text{for } n \in \mathbb{Z}_{\geq 0}, \tag{4.32} \]

since the right \( H \)-action on \( \hat{Q}_G \) is free, and it induces a semi-simple action on the level of \( \check{\text{C}} \)ech complex.

The long exact sequence of local cohomologies (see [SGAI III, Exposé I, Corollaire 2.9]) yields:
\[ \cdots \to H^n_{Z^S}(\hat{Q}_G, \mathcal{O}_{\hat{Q}_G}) \to H^n(\hat{Q}_G, \mathcal{O}_{\hat{Q}_G}) \to H^n(\hat{U}^S, \mathcal{O}_{\hat{Q}_G}) \to H^{n+1}_{Z^S}(\hat{Q}_G, \mathcal{O}_{\hat{Q}_G}) \to \cdots. \tag{4.33} \]

Since \( \hat{Q}_G \) is affine, this induces
\[ H^n(\hat{U}^S, \mathcal{O}_{\hat{Q}_G}) \cong H^{n+1}_{Z^S}(\hat{Q}_G, \mathcal{O}_{\hat{Q}_G}) \quad \text{for } n \geq 1 \tag{4.34} \]
by [EGAIII, Théorème 1.3.1]. Here the quasi-compactness of the embedding \( \hat{U}^S \hookrightarrow \hat{Q}_G \), together with [SGAI Exposé II, Proposition 5], implies that
\[ H^n_{Z^S}(\hat{Q}_G, \mathcal{O}_{\hat{Q}_G}) \cong H^n(K_S(\mathbb{C}[\hat{Q}_G])), \tag{4.35} \]
where \( K_S(\mathbb{C}[\hat{Q}_G]) \) denotes the (cohomological) \( \mathbb{C}[\hat{Q}_G] \)-Koszul complex defined through \( S \subset \Omega \) (see [EGAIII (1.1.2)]).

In view of (4.33), the comparison of (4.31) and (4.35) via (4.34) yields an isomorphism
\[ H^n_{Z}(\hat{Q}_G, \mathcal{O}_{\hat{Q}_G}) \cong \varprojlim_{S} H^n_{Z^S}(\hat{Q}_G, \mathcal{O}_{\hat{Q}_G}) \cong \varprojlim_{S} H^n_{Z^S}(\hat{Q}_G, \mathcal{O}_{\hat{Q}_G}). \tag{4.36} \]
We know from [EGAIII, Proposition 1.1.4] that the Koszul complex \( K_S(\mathbb{C}[\hat{Q}_G]) \) has trivial cohomology at degree \( < n \) if \( S \) contains a regular sequence of length \( n \). Here we see from Corollary 4.27 that \( \mathbb{C}[Q_G] = \bigoplus_{\lambda \in P^+} W(\lambda)^{\ast} \). Also, by Theorem 4.28 we can rearrange \( \Omega \) if necessary in such a way that for each \( i \in I \), the set of the \( i \)-th components of the elements in \( \Omega \) contain a regular sequence of arbitrary length. Then, we deduce from (4.36) that
\[ H^n_{Z}(\hat{Q}_G, \mathcal{O}_{\hat{Q}_G}) = \{0\} \quad \text{for all } n \in \mathbb{Z}_{\geq 0}. \]

Therefore, (4.33) and the affinity of \( \hat{Q}_G \) imply that
\[ H^n(\hat{Q}_G, \mathcal{O}_{\hat{Q}_G}) = H^n(\hat{Q}_G, \mathcal{O}_{\hat{Q}_G}) = \{0\} \quad \text{for all } n > 0. \]
In view of (4.32) and Theorem 4.4, we conclude the assertion of the theorem. \( \square \)

**Corollary 4.30.** For each \( x \in W^{\geq 0}_{a^+} \), the scheme \( Q_G(x) \) is normal. Moreover, for each \( \lambda \in P \) and \( x \in W^{\geq 0}_{a^+} \), we have
\[ \text{gch } H^n(Q_G(x), \mathcal{O}_{Q_G(x)}(\lambda)) = \begin{cases} \text{gch } V_x^-(\omega_0^\ast \lambda) & \text{if } \lambda \in P^+ \text{ and } n = 0, \\ 0 & \text{otherwise}. \end{cases} \tag{4.37} \]

In particular, we have
\[ \text{gch } H^0(Q_G(x), \mathcal{O}_{Q_G(x)}(\lambda)) \in (\mathbb{Z}((q^{-1})))[P] \subset (\mathbb{Z}[P])((q^{-1})). \tag{4.38} \]
Proof. Once we know the normality of \( Q_G \) and the cohomology vanishing result in Theorem 4.29, the same argument as in [Kat1, Theorem 4.7] (see Theorem 4.8) yields all the assertions except for the last one.

The last assertion on the character estimate follows from a result about extremal weight modules ([Kas2, Corollary 5.2]) and the fact that \( U_q^- (\mathfrak{g}) \) is concentrated on subspaces of \( q \)-degree \( \leq 0 \).

**Corollary 4.31.** For an arbitrary \( x \in W_{\text{af}}^0 \), we take \( i \) as in (4.8). Then we have \( m_* O_{Q_G(i)} \cong O_{Q_G(x)} \).

**Proof.** We adopt the notation of Section 4.3. The map \( m \) factors as the composite of the map \( m \) for \( t_{2m \rho} \) and a successive composite of the \( d_{i, x(k)} \) for \( 1 \leq k \leq \ell' \). The case \( x = t_\xi \) for \( \xi \in Q^{\vee, +} \) is clear since \( Q_G(t_\xi) \cong Q_G \) (through \( t_\xi \)). Therefore, it suffices to show that \( (q \xi)_*, O_{I(i)} \times Q_G(x) \cong O_{Q_G(s_i x)} \) for each \( x \in W_{\text{af}}^0 \) and \( i \in I_{\text{af}} \) such that \( s_i x < x \). This assertion itself follows from Corollary 4.30 and [Kat1, Theorem 4.7] (in view of Lemma 4.10). Hence we obtain the assertion of the corollary.

5 \( K \)-theory of semi-infinite flag manifolds.

We keep the notation and setting of Section 4.3.

**Proposition 5.1.** Every \( \tilde{I} \)-equivariant locally free sheaf of rank one (i.e., line bundle) on \( Q_G(x) \) is of the form \( \chi \otimes_C O_{Q_G(x)}(\lambda) \) for some \( \lambda \in P \) and an \( \tilde{I} \)-character \( \chi \).

**Proof.** For \( x = e \), the boundary of the open \( G[[z]] \)-orbit \( O \) in \( Q_G \) is of codimension at least two, and the open \( G[[z]] \)-orbit \( O \) has a structure of pro-affine bundle over \( G/B \). In particular, an \( \tilde{I} \)-equivariant line bundle over \( O \) is the pullback of a \( B \times \mathbb{G}_m \)-equivariant line bundle over \( G/B \). Because every line bundle over \( G/B \) carries a unique \( G \)-equivariant structure by [KKV, Sect. 3.3], and \( B \)-equivariant structures of the trivial line bundle \( O_{G/B} \) are in bijection with \( P \) (since \( H^0(G/B, O_{G/B}) = \mathbb{C} \)), we deduce that every \( B \)-equivariant line bundle over \( G/B \) is an \( H \)-character twist of a \( G \)-equivariant line bundle, which is obtained as the restriction of some \( O_{Q_G}(\lambda) \). Consequently, the assertion follows for \( x = e \).

Now, for \( y_1, y_2 \in W_{\text{af}}^0 \) such that \( y_1 \prec y_2 \), the restriction map transfers an \( \tilde{I} \)-equivariant line bundle over \( Q_G(y_1) \) to an \( \tilde{I} \)-equivariant line bundle over \( Q_G(y_2) \). Also, for an arbitrary \( x \in W_{\text{af}}^0 \), we can find \( \xi \in Q^{\vee, +} \) such that

\[
Q_G(x t_\xi) \subset Q_G(t_\xi) \subset Q_G(x) \subset Q_G(e) = Q_G
\]

by (the proof of) Lemma 4.17 since \( x = x(j) \leq t_{2m \rho} \) for \( m \gg 0 \). Because we have \( Q_G(t_\xi) \cong Q_G \) as schemes with an \( \tilde{I} \)-action, we conclude that a nonisomorphic pair of \( \tilde{I} \)-equivariant line bundles over \( Q_G \) restricts to a nonisomorphic pair of \( \tilde{I} \)-equivariant line bundles over \( Q_G(t_\xi) \). Since \( Q_G(x t_\xi) \cong Q_G(x) \), the same is true for line bundles over \( Q_G(x) \). Therefore, by means of (5.1), we deduce the assertion of the proposition for an arbitrary \( x \in W_{\text{af}}^0 \) from the case \( x = e \). This proves the proposition.

The following is an immediate consequence of Corollary 4.30 and Proposition 5.1.

**Corollary 5.2.** For each \( \tilde{I} \)-equivariant line bundle \( \mathcal{L} \) over \( Q_G(x) \), we have

\[
H^n(Q_G(x), \mathcal{L}) = \{ 0 \} \quad \text{for all } n > 0.
\]
Lemma 5.3. For each $\bar{I}$-equivariant quasi-coherent sheaf $E$ on $Q_G$ such that

$$\Gamma(Q_G, E(\lambda)) = \{0\}$$

for all $\lambda \in P$, where $E(\lambda) = E \otimes_{\mathcal{O}_{Q_G}} \mathcal{O}_{Q_G}(\lambda)$, we have $E = \{0\}$.

Proof. By the quasi-coherence and $\bar{I}$-equivariance, every nonzero section of $E$ has an $\bar{I}$-stable support, which must be a union of $I$-orbits. In addition, it defines a regular section on a complement of finitely many hyperplanes having poles of finite order around the boundary points. Therefore, Lemma 4.10 implies the desired result. \qed

Theorem 5.4. Let $E$ be an $\bar{I}$-equivariant quasi-coherent $\mathcal{O}_{Q_G}$-module satisfying the following conditions:

$$\text{gdim} \Gamma(Q_G, E(\lambda)) \in \mathbb{Z}[q]$$

for every $\lambda \in P$; \hspace{1cm} (5.2)

there exists $\lambda_0 \in P$ such that $\Gamma(Q_G, E(\lambda)) = \{0\}$

for all $\lambda \in P$ with $\langle \lambda, \alpha_i^\vee \rangle < \langle \lambda_0, \alpha_i^\vee \rangle$ for some $i \in I$. \hspace{1cm} (5.3)

Then, we have a resolution $\cdots \to P^2_E \to P^1_E \to P^0_E \to E \to 0$ of $\bar{I}$-equivariant $\mathcal{O}_{Q_G}$-modules such that

1. $\text{gdim} \Gamma(Q_G, P^k_E(\lambda)) \in \mathbb{Z}[q]$ for every $k \geq 0$ and $\lambda \in P$;

2. for each $k \geq 0$, the $\bar{I}$-equivariant $\mathcal{O}_{Q_G}$-module $P^k_E$ is a direct sum of line bundles (if we forget the $I$-module structure);

3. for each $m \in \mathbb{Z}$ and $\lambda \in P$, the number of direct summands $P^k_E(\lambda)$ of $\bigoplus_{k \geq 0} P^k_E(\lambda)$ contributing to the homogeneous subspace of degree $m$ of $\Gamma(Q_G, \bigoplus_{k \geq 0} P^k_E(\lambda))$ is finite.

Moreover, we have $H^n(Q_G, E) = \{0\}$ for all $n > 0$.

Proof. Let

$$R_G := \bigoplus_{\lambda \in P^+} W(\lambda)^* = \bigoplus_{\lambda \in P^+} H^0(Q_G, \mathcal{O}_{Q_G}(\lambda))$$

be the projective coordinate ring. Thanks to Lemma 5.3, the sheaf $E$ is determined by the $R_G$-module

$$M(E) := \bigoplus_{\lambda \in P} H^0(Q_G, E(\lambda)).$$

Because $M(E)$ is nonpositively graded and each homogeneous subspace with respect to the $(P \times \mathbb{Z})$-grading is finite-dimensional, we obtain a surjection $P^0_E \to M(E)$, where $P^0_E$ is a direct sum of $(P \times \mathbb{Z})$-graded projective $R_G$-modules tensored with $\bar{I}$-modules; indeed, we can construct the desired maps inductively by starting with $\lambda = \lambda_0 \in P$, and then by adding the $w_i$, $i \in I$, repeatedly, by means of the projectivity of $R_G$. Since a (graded) projective $R_G$-module is obtained from $R_G$ by a grading shift and an $\bar{I}$-module twist, we deduce that $P^0_E \cong M(P^0_E)$ as $(P \times \mathbb{Z})$-graded $R_G$-modules for a certain direct sum $P^0_E$ of $\bar{I}$-equivariant line bundles (with some twist of the $\bar{I}$-equivariant structure). Here the surjectivity of $P^0_E \to M(E)$ of $R_G$-modules implies that $P^0_E \to E$ is also surjective. Also,
by our character estimate, we deduce that \( \text{gdim} \Gamma(Q_G, \mathcal{P}^0_{\mathcal{E}}(\lambda))^* \in \mathbb{Z}[q] \). Now, let \( \Xi(\mathcal{E}) \) be the set of those pairs \((\lambda, m) \in P \times \mathbb{Z}\) for which

\[
\bigoplus_{\mu \in \lambda - P^+} \left( \bigoplus_{n > m} \Gamma(Q_G, \mathcal{E}(\mu))_n \right) \bigoplus \bigoplus_{\mu \in \lambda - P^+, \mu \neq \lambda} \Gamma(Q_G, \mathcal{E}(\mu))_m = \{0\}.
\]

Then, we can rearrange \( \mathcal{P}^0_{\mathcal{E}} \), if necessary, to assume that \( \text{gdim} \ker(\Gamma(Q_G, \mathcal{P}^0_{\mathcal{E}}(\lambda)) \rightarrow \Gamma(Q_G, \mathcal{E}(\lambda)))^* \in \mathbb{Z}[q] \) for all \( \lambda \in P \); (5.4)

\[
\Gamma(Q_G, \mathcal{P}^0_{\mathcal{E}}(\lambda))^* = \{0\} \text{ for all } \lambda \in P \text{ with } \langle \lambda, \alpha_i^\vee \rangle < \langle \lambda_0, \alpha_i^\vee \rangle \text{ for some } i \in I; \quad (5.5)
\]

\[
\ker(\Gamma(Q_G, \mathcal{P}^0_{\mathcal{E}}(\lambda)) \rightarrow \Gamma(Q_G, \mathcal{E}(\lambda)))_m = \{0\} \text{ for every } (\lambda, m) \in \Xi(\mathcal{E}). \quad (5.6)
\]

Thanks to (5.4) and (5.5), we can replace \( \mathcal{E} \) with \( \ker(\mathcal{P}_{\mathcal{E}} \rightarrow \mathcal{P}_{\mathcal{E}}^{-1}) \) repeatedly (with the convention \( \mathcal{P}_{\mathcal{E}}^{-1} = \mathcal{E} \)) to apply the procedure above in order to obtain \( \mathcal{P}_{\mathcal{E}}^{k+1} \) for each \( k \geq 0 \). This yields an \( \tilde{I} \)-equivariant resolution

\[
\cdots \rightarrow \mathcal{P}^2_{\mathcal{E}} \rightarrow \mathcal{P}^1_{\mathcal{E}} \rightarrow \mathcal{P}^0_{\mathcal{E}} \rightarrow \mathcal{E} \rightarrow 0, \quad (5.7)
\]

in which each \( \mathcal{P}^k_{\mathcal{E}} \) is a direct sum of \( \tilde{I} \)-equivariant line bundles (with some twist by \( \tilde{I} \)-modules). By the construction, we have

\[
\emptyset = \bigcap_{k \geq 0} \Xi(\ker d_k) \subset P \times \mathbb{Z},
\]

and hence the resolution (5.7) satisfies the first two of the requirements. Also, taking into account (5.5) and (5.6), we see that the resolution (5.7) satisfies the third one of the requirements.

Finally, by applying Corollary 5.2, we conclude the desired cohomology vanishing. This completes the proof of the theorem.

\[\square\]

**Corollary 5.5.** Keep the setting of Theorem 5.4. We have

\[
\sum_{k \geq 0} \text{gdim} \Gamma(Q_G, \mathcal{P}^k_{\mathcal{E}}(\lambda))^* \in \mathbb{Z}[q] \text{ for all } \lambda \in P,
\]

and an (unambiguously defined) equality

\[
\text{gdim} \Gamma(Q_G, \mathcal{E}(\lambda))^* = \sum_{k \geq 0} (-1)^k \text{gdim} \Gamma(Q_G, \mathcal{P}^k_{\mathcal{E}}(\lambda))^* \in \mathbb{Z}[q] \text{ for all } \lambda \in P.
\]

**Proof.** By Theorem 5.4(3), there are only finitely many terms \( \mathcal{P}^k_{\mathcal{E}}(\lambda) \) contributing to each homogeneous subspace of a fixed \( q \)-degree of \( \Gamma(Q_G, \mathcal{E}(\lambda))^* \). Therefore, the projective resolution afforded in Theorem 5.4 implies the desired result. This proves the corollary.

\[\square\]

We say that a condition depending on \( \lambda \in P \) holds for \( \lambda \gg 0 \) if there exists \( \gamma \in P \) and the condition holds for every \( \lambda \in \gamma + P^+ \).
For an element \( f \in (\mathbb{Z}[P])[q^{-1}] \), we define \( |f| \in (\mathbb{Z}_{\geq 0}[P])[q^{-1}] \) as follows:

\[
f = \sum_{n \in \mathbb{Z}_{\leq 0}} \left( \sum_{\nu \in P} c_{\nu,n} e^{\nu} \right) q^n \quad \Rightarrow \quad |f| := \sum_{n \in \mathbb{Z}_{\leq 0}} \left( \sum_{\nu \in P} |c_{\nu,n}| e^{\nu} \right) q^n.
\]

We now define \( K'_1(\mathcal{O}_G) \) to be the following set of formal infinite sums, modulo equivalence relation \( \sim \):

\[
\left\{ f = \sum_{\lambda \in P} f_{\lambda} \cdot [\mathcal{O}_Q(\lambda)] \left| f_{\lambda} \in (\mathbb{Z}[P])[q^{-1}], \lambda \in P, \text{ satisfy condition } (\#) \right\} / \sim
\]

where condition \( (\#) \) is given by:

\[
\sum_{\lambda \in P} |f_{\lambda}| \cdot \text{gch} H^0(\mathcal{O}_Q, \mathcal{O}_Q(\lambda + \mu)) \in (\mathbb{Z}_{\geq 0}[P])[q^{-1}] \quad \text{for every } \mu \in P, \quad (\#)
\]

and the equivalence relation \( \sim \) is:

\[
f \sim 0 \iff \sum_{\lambda \in P} f_{\lambda} \cdot \text{gch} H^0(\mathcal{O}_Q, \mathcal{O}_Q(\lambda + \mu)) = 0 \quad \text{if } \mu \gg 0. \quad (\sim)
\]

By construction, \( K'_1(\mathcal{O}_G) \) is topologically spanned by classes of I-equivalent line bundles. Hence we deduce that the following map is well-defined:

\[
\text{Pic}^I \mathcal{O}_Q \times K'_1(\mathcal{O}_G) \to K'_1(\mathcal{O}_G), \quad \left( \mathcal{L}, f = \sum_{\lambda \in P} f_{\lambda} [\mathcal{O}(\lambda)] \right) \mapsto [\mathcal{L}] \cdot f = \sum_{\lambda \in P} f_{\lambda} [\mathcal{L} \otimes \mathcal{O}(\lambda)].
\]

By Corollary 3.4, each \( \mathcal{E} \) from Theorem 3.3 satisfies

\[
[\mathcal{E}] := \sum_{k \geq 0} (-1)^k [P^k_\mathcal{E}] \in K'_1(\mathcal{O}_G).
\]

In particular, thanks to Corollary 4.30, we have \([\mathcal{O}_Q(x)(\lambda)] \in K'_1(\mathcal{O}_G)\) for every \( x \in W^0_{\text{af}} \) and \( \lambda \in P \).

Let \( \text{Fun}_{(\mathbb{C}[P])[q^{-1}]} P \) denote the space of \((\mathbb{C}[P])[q^{-1}]\)-valued functions on \( P \), and let \( \text{Fun}_{(\mathbb{C}[P])[q^{-1}]}^f P \subset \text{Fun}_{(\mathbb{C}[P])[q^{-1}]} P \) be the subset consisting of those functions that are zero on \( \gamma + P^+ \) for some \( \gamma \in P \). Then we form a \((\mathbb{C}[P])[q^{-1}]\)-module quotient

\[
\text{Fun}_{(\mathbb{C}[P])[q^{-1}]}^{\text{ess}} P := \text{Fun}_{(\mathbb{C}[P])[q^{-1}]} P / \text{Fun}_{(\mathbb{C}[P])[q^{-1}]}^f P.
\]

For each \( \mu \in P \), we regard the assignment

\[
P \ni \lambda \mapsto \begin{cases} \text{gch } V_{\mathcal{E}}(-w_o(\lambda + \mu)) & \text{if } \lambda + \mu \in P^+, \\ 0 & \text{otherwise,} \end{cases}
\]

as an element of \( \text{Fun}_{(\mathbb{C}[P])[q^{-1}]} P \), which we denote by \( \Psi([\mathcal{O}_Q(\mu)]) \). Passing to the quotient, we obtain a map \( \Psi : \{ [\mathcal{O}_Q(\mu)] \}_{\mu \in P} \ni [\mathcal{O}_Q(\mu)] \mapsto \Psi([\mathcal{O}_Q(\mu)]) \in \text{Fun}_{(\mathbb{C}[P])[q^{-1}]}^{\text{ess}} P \).
**Theorem 5.6.** The map $\Psi$ extends to an injective $(\mathbb{Z}[P])[\![q^{-1}]\!]$-linear map $\Psi : K'_1(\mathcal{Q}_G) \to \text{Fun}_{\text{ess}}(\mathbb{C}(P))[\![q^{-1}]\!] P$.

**Proof.** We assume the contrary to deduce a contradiction. Let $C \in K'_1(\mathcal{Q}_G)$, and expand the $C$ as:

$$C = \sum_{n \in \mathbb{Z}_{\leq 0}, \nu, \mu \in P} c_{n,\nu,\mu} q^n e^\nu \cdot [\mathcal{O}_{\mathcal{Q}_G}(\mu)], \quad \text{with } c_{n,\nu,\mu} \in \mathbb{Z},$$

inside $K'_1(\mathcal{Q}_G)$. We have $\Psi(C) = 0$ if and only if there exists $\gamma \in P$ such that

$$\sum_{n \in \mathbb{Z}_{\leq 0}, \nu, \mu \in P} c_{n,\nu,\mu} q^n e^\nu \cdot \text{gch} V_e^-(\gamma - w_0(\lambda + \mu)) = 0 \quad \text{for } \mu \in \gamma + P^+.$$

This is exactly the condition $C \sim 0$. Hence the map $\Psi$ defines an injective map. It is $(\mathbb{Z}[P])[\![q^{-1}]\!]$-linear by construction.

For countably many elements $C_p, p \geq 0$, in $K'_1(\mathcal{Q}_G)$ that represent the classes of $\tilde{I}$-equivariant quasi-coherent sheaves, we expand them as:

$$C_p = \sum_{\lambda \in P} a_\lambda(C_p) [\mathcal{O}_{\mathcal{Q}_G}(\lambda)], \quad \text{with } a_\lambda(C_p) \in (\mathbb{Z}[P])[\![q^{-1}]\!]$$

by using the procedure of Theorem 5.4; we say that the sum $\sum_{p \geq 0} C_p$ converges absolutely to an element of $K'_1(\mathcal{Q}_G)$ if there exists some $\lambda_0 \in P$ (uniformly for all $p \geq 0$) such that $a_\lambda(C_p) = 0$ for all $\lambda \in P$ with $\langle \lambda, \alpha_i^\vee \rangle < \langle \lambda_0, \alpha_i^\vee \rangle$ for some $i \in I$, and if the number of those $(\lambda, p) \in P \times \mathbb{Z}_{\geq 0}$ for which $a_\lambda(C_p)$ has a nonzero term of $q$-degree $m$ is finite for each $m \in \mathbb{Z}$. It is straightforward to see that $\sum_{p \geq 0} C_p$ defines an element of $K'_1(\mathcal{Q}_G)$, which does not depend on the order of the $C_p$’s.

**Remark 5.7.** Since the coefficients for $K'_1(\mathcal{Q}_G)$ are in $\mathbb{Z}$, the sum $\sum_{p \geq 0} C_p$ must “diverge” or “oscillate” when it does not converge absolutely.

**Proposition 5.8.** Let $f_y \in (\mathbb{Z}[P])[\![q^{-1}]\!], y \in W_{\text{af}}^{\geq 0}$. Then the formal sum

$$\sum_{y \in W_{\text{af}}^{\geq 0}} f_y \cdot [\mathcal{O}_{\mathcal{Q}_G}(y)]$$

(5.8)

converges absolutely to an element of $K'_1(\mathcal{Q}_G)$ if and only if $\sum_{y \in W_{\text{af}}^{\geq 0}} |f_y| \in (\mathbb{Z}_{\geq 0}[P])[\![q^{-1}]\!]$. Moreover, in this case, the equation

$$\sum_{y \in W_{\text{af}}^{\geq 0}} f_y \cdot [\mathcal{O}_{\mathcal{Q}_G}(y)] = 0$$

implies $f_y = 0$ for all $y \in W_{\text{af}}^{\geq 0}$.

**Proof.** First, we remark that $[\mathcal{O}_{\mathcal{Q}_G}(y)] \in K'_1(\mathcal{Q}_G)$ for each $y \in W_{\text{af}}^{\geq 0}$ by Corollary 4.30 and Theorem 5.4. More precisely, by means of the cohomology vanishing:

$$H^*(\mathcal{Q}_G; \mathcal{O}_{\mathcal{Q}_G(y)}(\mu)) = \{0\} \quad \text{if } \mu \notin P^+,$$
we can take $\lambda_0 = 0$ in Theorem 5.6 by setting $E = \mathcal{O}_{Q_G(y)}$. In addition, we have $H^0(Q_G, \mathcal{O}_{Q_G(y)}) = \mathbb{C}$. Hence the construction in Theorem 5.6 implies that

$$[\mathcal{O}_{Q_G(y)}] = [\mathcal{O}_{Q_G}] + \sum_{\lambda \in P^+} a_y(\lambda)[\mathcal{O}_{Q_G}(\lambda)] \in K'_1(Q_G) \tag{5.9}$$

for some $a_y(\lambda) \in (\mathbb{Z}[P])[[q^{-1}]]$. Therefore, the coefficient of $[\mathcal{O}_{Q_G}]$ in (5.8) must be the sum $\sum_{y \in W_{af}^G} f_y$, which unambiguously defines an element of $(\mathbb{Z}[P])[[q^{-1}]]$ if and only if the coefficient ($\in \mathbb{Z}$) of each $q^n$, $n \in \mathbb{Z}_{\leq 0}$, in the sum $\sum_{y \in W_{af}^G} f_y$ converges absolutely (see Remark 5.7). This proves the first assertion.

We prove the second assertion. Let us assume the contrary to deduce a contradiction. Let $S$ be the set of those $y \in W_{af}^G$ for which $f_y \neq 0$; denote by $n_0$ the maximal $q$-degree of all $f_y$, $y \in S$. Also, let $S_1$ be the set of those $y \in S$ for which $\tau(y) \neq \tau(y')$ for any $y' \in S$, where for $y \in W_{af}^G$ of the form $y = wt_e$ with $w \in W$ and $\xi \in Q^{\vee +}$, we set $\tau(y) := \xi$; since a polynomial ring (of finite variables) is Noetherian, we deduce that $|S_1| < \infty$. We choose and fix $y_0 \in S_1$ such that

$$P^\# := \{ \lambda \in P^+ \mid \langle \lambda, \tau(y_0) \rangle < \langle \lambda, \tau(y) \rangle \text{ for all } y \in S \text{ with } y \neq y_0 \}$$

is Zariski dense in $\mathfrak{h}^*$. Let $n_1$ denote the maximal $q$-degree of those $f_y$, $y \in S_1$, for which $\tau(y) = \tau(y_0)$. Then, the subset

$$P^{\# \#} := \{ \lambda \in P^+ \mid \langle \lambda, \tau(y_0) \rangle < \langle \lambda, \tau(y) \rangle + n_0 - n_1 \text{ for all } y \in S \text{ with } y \neq y_0 \}$$

of $P^\#$ is still Zariski dense in $\mathfrak{h}^*$.

For each $\lambda \in P^{\# \#}$, the coefficient of the part of degree $(n_1 - \langle \lambda, \tau(y_0) \rangle)_{S}$ of

$$\Psi \left( \sum_{y \in W_{af}^G} f_y \cdot [\mathcal{O}_{Q_G(y)}] \right)(\lambda)$$

is equal to

$$\sum_{w \in W} f_{w, \tau(y_0)}^{(n_1)} \cdot \text{ch} L_w^\#(-w_0 \lambda),$$

where $f_{y}^{(n_1)} \in \mathbb{Z}[P]$ is the part of degree $n_1$ of $f_y$; here, for $w \in W$ and $\mu \in P^+$, $L_w^\#(\mu) := U(\mathfrak{h}^*) L(\mu)_{w_0}$ denotes the (opposite) Demazure submodule of $L(\mu)$. This defines a $\mathbb{Z}[P]$-valued function of $\lambda \in P^{\# \#}$; note that the above is a finite sum. Here we have the equality $\text{ch} L_w^\#(-w_0 \lambda)^* = D_{w_0}(e^\lambda)$ in terms of the Demazure operator $D_{w_0}$ for each $w \in W$ (see [Kum, Theorem 8.2.9]); recall that the Demazure operator $D_i = D_{s_i}$, $i \in I$, is defined by $D_i(e^\mu) := (e^{\mu_i} - e^{-\mu_i})/(1 - e^{-\alpha_i})$ for $\mu \in P$. Also, we know by [MacI, pp. 28–29] that the operators $D_w$, $w \in W$, form a set of $\mathbb{Z}[P]$-linearly independent $\mathbb{Z}$-linear operators acting on $\mathbb{Z}[P]$. Therefore, we obtain $f_{w, \tau(y_0)}^{(n_1)} = 0$ for all $w \in W$. This is a contradiction, and hence we cannot take the $S_1$ above from the beginning. Thus, we conclude the desired result. This completes the proof of the proposition. \(\square\)

**Corollary 5.9** (of Theorem 5.6). Let $x \in W_{af}^G$ and $\lambda \in P^+$. Consider a collection $f_y(\lambda) \in (\mathbb{Z}[P])[[q^{-1}]]$, $y \in W_{af}^G$, such that $\sum_{y \in W_{af}^G} f_y(\lambda) \cdot [\mathcal{O}_{Q_G(y)}]$ converges absolutely in $K'_1(Q_G)$. Then,

$$[\mathcal{O}_{Q_G(x)}(\lambda)] = \sum_{y \in W_{af}^G} f_y(\lambda) \cdot [\mathcal{O}_{Q_G(y)}] \tag{5.10}$$
if and only if
\[ gch V_x^-(−w_0(λ + µ)) = \sum_{y \in W_{af}^≥} f_y(λ) \cdot gch V_y^-(−w_0µ) \quad \text{for } µ \gg 0. \] (5.11)

Proof. We have an expansion
\[ [O_{QG}(x)](λ) = \sum_{y \in W_{af}^≥} f_y(λ) \cdot [O_{QG}(y)] \]
inside $K'_1(Q_G)$. From this, by twisting by the line bundle $O(µ)$ for $µ \in P^+$, we obtain
\[ [O_{QG}(x)(λ + µ)] = \sum_{y \in W_{af}^≥} f_y(λ) \cdot [O_{QG}(y)(µ)]. \]

By Corollary 4.30, this equation in turn implies (5.11), which proves the “only if” part of the assertion.

We now assume (5.11). Then we have
\[ \Psi([O_{QG}(x)(λ)]) = \sum_{y \in W_{af}^≥} f_y(λ) \cdot \Psi([O_{QG}(y)]) \]
by Corollary 4.30. Therefore, by Theorem 5.6, we deduce that both sides of (5.10) represent the same class in $K'_1(Q_G)$. Thus, we have proved the “if” part of the assertion. This proves the corollary.

Theorem 5.10 (Pieri-Chevalley formula for semi-infinite flag manifolds). For each $λ \in P^+$ and $x \in W_{af}^≥$, there holds the equality
\[ [O_{QG}(λ)] \cdot [O_{QG}(x)] = \sum_{η \in B_{≥x}^∞(−w_0λ)} e^{\text{fin}(\text{wt}(η))} q^{\text{nul}(\text{wt}(η))} \cdot [O_{QG}(ι(η, x))] \]
in $K'_1(Q_G)$.

Proof. By Theorem 3.5, we have
\[ gch V_x^-(−w_0(λ + µ)) = \sum_{η \in B_{≥x}^∞(−w_0λ)} e^{\text{fin}(\text{wt}(η))} q^{\text{nul}(\text{wt}(η))} \cdot gch V_{ι(η, x)}^-(−w_0µ) \]
for each $µ \in P^+$. Taking into account the fact that the LHS is zero if $λ + µ \notin P^+$, and the RHS is zero if $µ \notin P^+$, we conclude the above equation for $µ \gg 0$. Here we see from Section 2.4 that $\text{nul}(\text{wt}(η)) \in Z_{≤0}$ for each $η \in B_{≥x}^∞(−w_0λ)$. Also, we deduce from (2.22) that for each $m \in Z_{≤0}$, there exist only finitely many $η \in B_{≥x}^∞(−w_0λ)$ such that $\text{nul}(\text{wt}(η)) \geq m$. Because $\text{gdim} H^0(Q_G, O_{QG}(ι(η, x)))(µ)) \in Z[[q^{-1}]]$ by Corollary 4.30, we deduce that
\[ \sum_{η \in B_{≥x}^∞(−w_0λ)} e^{\text{fin}(\text{wt}(η))} q^{\text{nul}(\text{wt}(η))} \cdot [O_{QG}(ι(η, x))] \in K'_1(Q_G). \]
From this, by applying Corollary 5.9, we conclude the desired result. This proves the theorem.
6 nil-DAHA action on $K^1_G(Q^\text{rat})$.

**Definition 6.1** (cf. [CF, Sect. 1.2]). The nil-DAHA $\mathcal{H}$ (of adjoint type) is the unital $\mathbb{Z}[q^{\pm 1}]$-algebra generated by $T_i$, $i \in I_{af}$, and $e(\nu)$, $\nu \in P$, subject to the following relations:

\[
\begin{cases}
T_i(T_i + 1) = 0 & \text{for each } i \in I_{af}; \\
\text{if } s_is_j \cdots = s_js_i \cdots, \text{ then } T_iT_j \cdots = T_jT_i \cdots & \text{for each } i, j \in I_{af}; \\
e(\nu_1)e(\nu_2) = e(\nu_1 + \nu_2) \text{ and } e(0) = 1 & \text{for each } \nu_1, \nu_2 \in P; \\
T_i e(\nu) - e(s_i \nu) T_i = \frac{e(s_i \nu) - e(\nu)}{1 - e(\alpha_i)} & \text{for each } \nu \in P \text{ and } i \in I_{af}.
\end{cases}
\]

(6.1)

We define $\mathcal{H}$ to be the $\mathbb{Z}[q^{-1}]$-subalgebra of $\mathcal{H}$ generated by $T_i$, $i \in I$, and $e(\nu)$, $\nu \in P$.

**Proposition 6.2.** The assignment

\[
e(\omega_i) : [\mathcal{C}_{\lambda} \otimes_{\mathcal{C}} \mathcal{O}_{G}(\mu)] \mapsto [\mathcal{C}_{-\omega_i + \lambda} \otimes_{\mathcal{C}} \mathcal{O}_{G}(\mu)],
\]

\[
T_i : [\mathcal{C}_{\lambda} \otimes_{\mathcal{C}} \mathcal{O}_{G}(\mu)] \mapsto \frac{e(-\lambda) - e(-s_i \lambda + \alpha_i)}{1 - e(\alpha_i)}[\mathcal{O}_{G}(\mu)],
\]

for each $i \in I$ and $\lambda, \mu \in P$, equips $K^1_G(Q_G)$ with an action of the subalgebra $\mathcal{H}$ of $\mathcal{H}$ through the identifications:

\[
q \mapsto q^{-1}, \quad T_i \mapsto T_i - 1 \quad \text{for } i \in I, \quad e(\nu) \mapsto e(\nu) \quad \text{for } \nu \in P.
\]

**Proof.** By the construction, $K^1_G(Q_G)$ contains a dense subset isomorphic to $(\mathbb{Z}[P])[q^{-1}] \otimes_{\mathbb{Z}} K_G(G/B)$ (see Proposition 5.1). Also, we have a surjection $(\mathbb{Z}[P])[q^{-1}] \otimes_{\mathbb{Z}} K_G(G/B) \rightarrow \mathbb{Z}[q^{-1}] \otimes_{\mathbb{Z}} K_B(G/B)$; see, e.g., [KK, (3.17)]. Here, for each $i \in I$, the action of $T_i$ is identical to the action of the Demazure operator $D_i = D_{s_i}$, and the action of $e(\omega_i)$ corresponds to the twist by the $B$-character $-\omega_i$; these define an $\mathcal{H}$-action on $K_G(G/B)$ by [KK, Sect. 3]. Notice that both of the actions of $e(\cdot)$ and $T_i$, $i \in I$, are neutral with respect to tensoring with $\mathcal{O}_{G/B}(\lambda)$ for each $\lambda \in P$, and that they also commute with the $G_m$-twist corresponding to $q^{-1}$. Therefore, the $\mathcal{H}$-action on $K_G(G/B) \cong (\mathbb{Z}[P])[q^{-1}][\mathcal{O}_{G/B}]$ induces an $\mathcal{H}$-action on $(\mathbb{Z}[P])[q^{-1}] \otimes_{\mathbb{Z}} K_G(G/B)$ through

\[
(\mathbb{Z}[P])[q^{-1}] \otimes_{\mathbb{Z}} K_G(G/B) \cong (\mathbb{Z}[P])[q^{-1}] \otimes_{\mathbb{Z}} K_B(G/B) \cong \bigoplus_{\lambda \in P} (\mathbb{Z}[P])[q^{-1}][\mathcal{O}_{G/B}(\lambda)],
\]

where the second factor of the leftmost one is responsible for the factors $\{[\mathcal{O}_{G/B}(\lambda)]\}_{\lambda \in P}$. Finally, we complete $(\mathbb{Z}[P])[q^{-1}] \otimes_{\mathbb{Z}} K_B(G/B)$ to obtain the desired assertion. This proves the proposition. \hfill \Box

**Corollary 6.3.** The $\mathcal{H}$-action in Proposition 6.2 is induced by the $I$-character twists and the convolution action of the structure sheaves through $q_{i,e}$ for $i \in I$ (see (4.13)).
Proof. The assertion holds for the actions of $\mathcal{H}$ on $K_B(G/B)$ and $\mathbb{Z}[q^{-1}] \otimes \mathbb{Z} K_B(G/B)$ by [KK Sect. 3]. Also, by Lemma 4.18 for each $i \in I$, the map $q_{i,e}$ is a $\mathbb{P}^1$-fibration, and hence
\[ \mathbb{R}^k(q_{i,e})_* \mathcal{O}_{\mathbb{I}(i)_1^1} \mathcal{Q}_G \cong \begin{cases} \mathcal{O}_{\mathcal{Q}_G} & \text{if } k = 0, \\ \{0\} & \text{if } k \neq 0. \end{cases} \]

This implies that the convolution action of $\mathcal{I}(i)/\mathcal{I}$, $i \in I$, on $\mathcal{Q}_G$ fixes the classes of $[\mathcal{O}_{\mathcal{Q}_G}(\lambda)]$ for each $\lambda \in P$ by the projection formula. Taking into account the fact that the twist of $R(\mathcal{I}) \cong R(B \times \mathbb{G}_m)$ has an effect through the fiber of $q_{i,e}$, we conclude that $T_i$, $i \in I$, is identical to the convolution action induced by $q_{i,e}$ through the inclusion $\mathbb{Z}[q^{-1}] \otimes \mathbb{Z} K_B(G/B) \subset K'_I(\mathcal{Q}_G)$. This proves the corollary. 

For each $\xi \in Q^{V,+}$, the natural inclusion map $q_{\xi} : \mathcal{Q}_G \hookrightarrow \mathcal{Q}_G$ induces an inclusion $(q_{\xi})_* : K'_I(\mathcal{Q}_G) \hookrightarrow K'_I(\mathcal{Q}_G)$ of $(\mathbb{Z}[P])[[q^{-1}]]$-modules such that $(q_{\xi})_*[\mathcal{O}_{\mathcal{Q}_G(x)}(\lambda)] = [\mathcal{O}_{\mathcal{Q}_G(x\xi)}(\lambda)]$ for each $x \in W_{af}$. We define
\[ K'_I(\mathcal{Q}_G^{rat}) := \mathbb{Z}((q^{-1})) \otimes_{\mathbb{Z}[q^{-1}]} \lim_K K'_I(\mathcal{Q}_G). \]

Theorem 6.4 ([BF4]). The assignment
\[ e(\varpi_i) : [C_\lambda \otimes \mathbb{C} \mathcal{O}_{\mathcal{Q}_G(t_\xi)}(\mu)] \mapsto [C_{-\varpi_i + \lambda} \otimes \mathbb{C} \mathcal{O}_{\mathcal{Q}_G(t_\xi)}(\mu)] \quad \text{for } i \in I, \]
\[ T_i : [C_\lambda \otimes \mathbb{C} \mathcal{O}_{\mathcal{Q}_G(t_\xi)}(\mu)] \mapsto \begin{cases} \frac{e(-\lambda) - e(-s_0\lambda)}{1 - e(\alpha_i)}[\mathcal{O}_{\mathcal{Q}_G(t_\xi)}(\mu)] + e(-s_0\lambda)[\mathcal{O}_{\mathcal{Q}_G(s_0t_\xi)}(\mu)] & \text{for } i = 0, \\ \frac{e(-\lambda) - e(-s_i\lambda + \alpha_i)}{1 - e(\alpha_i)}[\mathcal{O}_{\mathcal{Q}_G(t_\xi)}(\mu)] & \text{for } i \neq 0, \end{cases} \]

where $\xi \in Q^{V,+}$, and $\lambda, \mu \in P$, equips $K'_I(\mathcal{Q}_G^{rat})$ with an action of $\mathcal{H}$ through the identifications:
\[ q \mapsto q^{-1}, \quad T_i \mapsto T_i - 1 \quad \text{for } i \in I_{af}, \quad e(\nu) \mapsto e(\nu) \quad \text{for } \nu \in P. \]

Proof. Thanks to [KK Sect. 3] (and Lemma 4.18), for each $i \in I$, the action of $T_i$ is induced by the pushforward of an $\overline{\mathcal{I}}$-equivariant inflated sheaf through $q_{i,e}$ (see Section 4.3), and the action of $e(\varpi_i)$ is induced by an $\overline{\mathcal{I}}$-character twist. Because these geometric counterparts commute with the pullback through $q_{\xi}$ for each $\xi \in Q^{V,+}$, our formulas define an action of $\mathcal{H}$ on $K'_I(\mathcal{Q}_G^{rat})$ induced by Proposition 6.2.

Now, we have
\[ \frac{fg - e^{-\alpha_0}s_0(fg)}{1 - e^{-\alpha_0}} = \frac{f - s_0(f)}{1 - e^{-\alpha_0}} g + s_0(f) \frac{g - e^{-\alpha_0}s_0(g)}{1 - e^{-\alpha_0}} \quad \text{for } f, g \in (\mathbb{C}[P])((q^{-1})). \] (6.2)

Let $p_{0,t_\xi} : \mathcal{I}(0) \times \mathcal{Q}_G(t_\xi) \to \mathbb{P}^1$ be the inflation of the structure map of $\mathcal{Q}_G(t_\xi)$, and let $\mathcal{E}(W)$ denote the vector bundle over $\mathbb{P}^1 \cong \mathcal{I}(0)/\mathcal{I}$ associated to an $\overline{\mathcal{I}}$-module $W$. Then, by taking into account equation (6.2), Corollary 4.30 and Theorem 5.4, and [Kat1 Corollary 4.8], we deduce that for each $\lambda, \mu \in P$, $\nu \in P$, and $\xi \in Q^{V,+}$ such that $s_0t_\xi \in W_{af}^\geq 0$,
\[ \sum_{m,n \geq 0} (-1)^{m+n} gch^m \left( \mathcal{Q}_G^{rat}, \mathbb{R}^{n}(p_{0,t_\xi})_* \left( \mathcal{C}_{-\nu} \otimes \mathcal{O}_{\mathcal{Q}_G(t_\xi)}(\lambda) \right) \otimes \mathcal{O}_{\mathcal{Q}_G}(\mu) \right) \]
where the first and fourth equalities follow by the Leray spectral sequence. In particular, the term (6.3) represents the image under \( \Psi \) of the convolution of \([C_{-\nu} \otimes \mathcal{O}_{\hat{Q}_G(t_{\xi})}(\lambda)]\) with respect to \( q_{0,t_{\xi}} \). Therefore, from the injectivity of \( \Psi \), we conclude that \( T_0 \) is induced by the pushforward of an \( \hat{I} \)-equivariant inflated sheaf through \( q_{0,t_{\xi}} \) for some \( \xi \in Q^{\nu,+} \).

From the above, we deduce that the actions \( T_i, i \in I_{af}, \) and \( e(\nu), \nu \in P, \) generate the convolution action of Schubert cells and the \( \widetilde{f} \)-character twists of the (thin) affine flag manifold \( G(\mathbf{z})/I \) on \( \hat{Q}_G^{\text{at}} \) (or rather, on \( Q_G \)). In particular, the \( T_i, i \in I_{af}, \) generate the nil-Hecke algebra of affine type by [KK] Sect. 3. Therefore, their commutation relations with \( e(\varpi_i), i \in I, \) imply that the \( T_i, i \in I_{af}, \) and the \( e(\varpi_i), i \in I, \) satisfy the relations for \( \mathcal{H} \) (see also [BF4] Sect. 3.4); we remark that their convention differs from ours by the twist by the Serre duality and line bundle twist [BF4, Sects. 3.1 and 3.21].

Finally, we complete the proof by observing that \( T_0 \) preserves \( K_i(Q_G^{\text{at}}) \) by inspection.

\[ \square \]

7 Proof of Theorem 3.1

7.1 Affine Weyl group action.

Let \( B \) be a regular crystal in the sense of [Kas2, Sect. 2.2] (or, a normal crystal in the sense of [HK, p. 389]); for example, \( \mathbb{B}^{\mathfrak{h}}(\lambda) \) for \( \lambda \in P^+ \) is a regular crystal by Theorem 2.3 and hence so is \( \mathbb{B}^{\mathfrak{h}}(\lambda) \otimes \mathbb{B}^{\mathfrak{h}}(\mu) \) for \( \lambda, \mu \in P^+ \). Then we know from [Kas1, Sect. 7] that the affine Weyl group \( W_{af} \) acts on \( B \) as follows: for \( b \in B \) and \( i \in I_{af}, \)

\[
s_i \cdot b := \begin{cases} f_i^n b & \text{if } n := (\text{wt}(b), \alpha_i^\vee) \geq 0, \\ e_i^{-n} b & \text{if } n := (\text{wt}(b), \alpha_i^\vee) \leq 0. \end{cases}
\]

(7.1)

Also, for \( b \in B \) and \( i \in I_{af}, \) we define \( e_i^{\max} b = e_i^{\varepsilon_i(b)} b \) and \( f_i^{\max} b = f_i^{\varphi_i(b)} b, \) where \( \varepsilon_i(b) := \max\{ n \geq 0 \mid e_i^n b \neq 0 \} \) and \( \varphi_i(b) := \max\{ n \geq 0 \mid f_i^n b \neq 0 \} \); note that if \( b \in B \) satisfies \( e_i b = 0 \) (resp., \( f_i b = 0 \)), i.e., \( \varepsilon_i(b) = 0 \) (resp., \( \varphi_i(b) = 0 \)), then \( f_i^{\max} b = s_i \cdot b \) (resp., \( e_i^{\max} b = s_i \cdot b \)).
7.2 Connected components of $\mathbb{B}_\phi^\infty(\lambda)$.

Let $\lambda \in P^+$, and write it as $\lambda = \sum_{i \in I} m_i \omega_i$, with $m_i \in \mathbb{Z}_{\geq 0}$; note that $J = \{ i \in I \mid \langle \lambda, \alpha_i^\vee \rangle = 0 \} = \{ i \in I \mid m_i = 0 \}$. We define $\text{Par}(\lambda)$ to be the set of $I$-tuples of partitions $\rho = (\rho^{(i)})_{i \in I}$ such that $\rho^{(i)}$ is a partition of length (strictly) less than $m_i$ for each $i \in I$; a partition of length less than 0 (or 1) is understood to be the empty partition $\emptyset$. Also, for $\rho = (\rho^{(i)})_{i \in I} \in \text{Par}(\lambda)$, we set $|\rho| := \sum_{i \in I} |\rho^{(i)}|$, where for a partition $\rho = (\rho_1 \geq \rho_2 \geq \cdots \geq \rho_m)$, we set $|\rho| := \rho_1 + \cdots + \rho_m$. We endow the set $\text{Par}(\lambda)$ with a crystal structure as follows: for $\rho \in \text{Par}(\lambda)$ and $i \in I_\text{af}$, 

$$e_i \rho = f_i \rho := 0, \quad \varepsilon_i(\rho) = \varphi_i(\rho) := -\infty, \quad \text{wt}(\rho) := -|\rho|\delta.$$

We recall from [INS] Sect. 7 the relation between $\text{Par}(\lambda)$ and the set $\text{Conn}(\mathbb{B}_\phi^\infty(\lambda))$ of connected components of $\mathbb{B}_\phi^\infty(\lambda)$. We set $\text{Turn}(\lambda) := \{ k/m_i \mid i \in I \setminus J \text{ and } 0 \leq k \leq m_i \}$. By [INS] Proposition 7.1.2, each connected component of $\mathbb{B}_\phi^\infty(\lambda)$ contains a unique element of the form:

$$(\Pi^J(t\xi_1), \ldots, \Pi^J(t\xi_{s-1}), e; a_0, a_1, \ldots, a_s), \tag{7.2}$$

where $s \geq 1$, $\xi_1, \ldots, \xi_{s-1} \in \mathbb{Q}_{\lambda \setminus J}$ such that $\xi_1 > \cdots > \xi_{s-1} > 0 =: \xi_s$, and $a_u \in \text{Turn}(\lambda)$ for all $0 \leq u < s$. For each element of the form (7.2) (or equivalently, each connected component of $\mathbb{B}_\phi^\infty(\lambda)$), we define an element $\rho = (\rho^{(i)})_{i \in I} \in \text{Par}(\lambda)$ as follows. First, let $i \in I \setminus J$; note that $m_i \geq 1$. For each $1 \leq k \leq m_i$, take $0 \leq u \leq s$ such that $a_u$ is contained in the interval $((k-1)/m_i, k/m_i]$. Then we define $\rho^{(i)}$ to be $\langle \omega_i, \xi_u \rangle$, the coefficient of $\alpha_i^\vee$ in $\xi_u$; we know from (the proof of) [INS] Proposition 7.2.1 that $\rho^{(i)}$ does not depend on the choice of $u$ above. Since $\xi_1 > \cdots > \xi_{s-1} > 0 = \xi_s$, we see that $\rho^{(1)} \geq \cdots \geq \rho^{(i-1)} \geq \rho^{(i)} = 0$. Thus, for each $i \in I \setminus J$, we obtain a partition of length less than $m_i$. For $i \in J$, we set $\rho^{(i)} := \emptyset$. Thus we obtain an element $\rho = (\rho^{(i)})_{i \in I} \in \text{Par}(\lambda)$, and hence a map from $\text{Conn}(\mathbb{B}_\phi^\infty(\lambda))$ to $\text{Par}(\lambda)$. Moreover, we know from [INS] Proposition 7.2.1 that this map is bijective; we denote by $\pi_{\rho} \in \mathbb{B}_\phi^\infty(\lambda)$ the element of the form (7.2) corresponding to $\rho \in \text{Par}(\lambda)$ under this bijection.

**Remark 7.1.** Let $\rho = (\rho^{(i)})_{i \in I} \in \text{Par}(\lambda)$, with $\rho^{(i)} = (\rho^{(i)} \geq \cdots)$ for $i \in I$; note that $\rho^{(1)} = 0$ if $\rho^{(i)} = 0$. It follows from the definition that

$$\iota(\pi_{\rho}) = \Pi^J(t\xi_1), \quad \text{where} \quad \xi_1 = \sum_{i \in I} \rho^{(i)} \alpha_i^\vee \in \mathbb{Q}_{\lambda \setminus J}. \tag{7.3}$$

For $\rho \in \text{Par}(\lambda)$, we denote by $\mathbb{B}_\rho^\infty(\lambda)$ the connected component of $\mathbb{B}_\phi^\infty(\lambda)$ containing $\pi_{\rho}$. Also, we denote by $\mathbb{B}_0^\infty(\lambda)$ the connected component of $\mathbb{B}_\phi^\infty(\lambda)$ containing $\pi_\lambda = (e; 0, 1)$; note that $\pi_\lambda = \pi_{\rho}$ for $\rho = (0)_{i \in I}$. We know from [INS] Proposition 3.2.4 (and its proof) that for each $\rho \in \text{Par}(\lambda)$, there exists an isomorphism $\mathbb{B}_\rho^\infty(\lambda) \cong \{ \rho \} \otimes \mathbb{B}_0^\infty(\lambda)$ of crystals, which maps $\pi_{\rho}$ to $\rho \otimes \pi_\lambda$. Hence we have

$$\mathbb{B}_\phi^\infty(\lambda) = \bigsqcup_{\rho \in \text{Par}(\lambda)} \mathbb{B}_\rho^\infty(\lambda) \cong \bigsqcup_{\rho \in \text{Par}(\lambda)} \{ \rho \} \otimes \mathbb{B}_0^\infty(\lambda) \quad \text{as crystals.} \quad \tag{7.4}$$

The following lemma is shown by induction on the (ordinary) length $\ell(x)$ of $x$; for part (1), see also [NS3] Remark 3.5.2.
Lemma 7.2.

(1) Let \( \lambda \in P^+ \). If \( \pi \in \mathbb{B}^{\oplus}_N(\lambda) \) is of the form (7.2), then for \( x \in W_{af} \),
\[
x \cdot \pi = (\Pi^I(x_{t_1}^\xi), \ldots, \Pi^I(x_{t_{s-1}^\xi}), \Pi^I(x); a_0, a_1, \ldots, a_s).
\] (7.5)

(2) Let \( \lambda, \mu \in P^+ \). Let \( \rho \in \text{Par}(\lambda), \chi \in \text{Par}(\mu) \), and \( \xi, \zeta \in Q' \). Then, for \( x \in W_{af} \),
\[
x \cdot ((t_{\xi} \cdot \pi_\rho) \otimes (t_{\xi} \cdot \pi_\chi)) = (x_{t_\xi} \cdot \pi_\rho) \otimes (x_{t_\xi} \cdot \pi_\chi).
\] (7.6)

Let \( \xi \in Q' \). It follows from Lemma A.5(3) that if \( \pi = (x_1, \ldots, x_s; a) \in \mathbb{B}^{\oplus}_N(\lambda) \), then
\[
T_\xi \pi := (\Pi^I(x_1t_\xi), \ldots, \Pi^I(x_st_\xi); a) \in \mathbb{B}^{\oplus}_N(\lambda);
\] (7.7)

the map \( T_\xi : \mathbb{B}^{\oplus}_N(\lambda) \to \mathbb{B}^{\oplus}_N(\lambda) \) is clearly bijective, with \( T_\xi^{-1} = T_{-\xi} \). We can verify by the definitions that
\[
\begin{aligned}
&T_{\xi}e_i \pi = e_iT_{\xi} \pi, \quad T_{\xi}f_i \pi = f_iT_{\xi} \pi \quad \text{for } \pi \in \mathbb{B}^{\oplus}_N(\lambda) \text{ and } i \in I_{af},
&T_{\xi} \varepsilon_i(\pi) = \varepsilon_i(\pi), \quad \varphi_i(T_{\xi} \pi) = \varphi_i(\pi) \quad \text{for } \pi \in \mathbb{B}^{\oplus}_N(\lambda) \text{ and } i \in I_{af},
\end{aligned}
\] (7.8)

where \( T_\xi 0 \) is understood to be \( 0 \).

Remark 7.3. Let \( \rho \in \text{Par}(\lambda) \), and assume that \( \pi_\rho \) is of the form (7.2). For \( \xi \in Q' \), we see from (7.5) and (7.7) that
\[
T_\xi \pi_\rho = (\Pi^I(x_1t_\xi), \ldots, \Pi^I(x_st_\xi); a_0, a_1, \ldots, a_s) = t_\xi \cdot \pi_\rho,
\]

which implies that \( T_\xi \pi_\rho \in \mathbb{B}_\rho(\lambda) \). Therefore, it follows from (7.8) that \( T_\xi(\mathbb{B}_\rho(\lambda)) = \mathbb{B}_\rho^{\oplus}(\lambda) \).

7.3 Quantum Lakshmibai-Seshadri paths.

Let \( \lambda \in P^+ \). Let \( \text{cl} : \mathbb{R} \otimes_{\mathbb{Z}} P_{af} \to (\mathbb{R} \otimes_{\mathbb{Z}} P_{af})/\mathbb{R}a \) denote the canonical projection. For an element \( \pi = (x_1, \ldots, x_s; a_0, a_1, \ldots, a_s) \in \mathbb{B}^{\oplus}_N(\lambda) \), we define a piecewise-linear, continuous map \( \text{cl}(\pi) : [0, 1] \to (\mathbb{R} \otimes_{\mathbb{Z}} P_{af})/\mathbb{R}a \) by \( (\text{cl}(\pi))(t) := \text{cl}(\pi(t)) \) for \( t \in [0, 1] \) (for \( \pi \), see (C.2)). As explained in [NS3, Sect. 6.2], the set \( \{ \text{cl}(\pi) \mid \pi \in \mathbb{B}^{\oplus}_N(\lambda) \} \) is identical to the set \( \mathbb{B}(\lambda)_{cl} \) of all “projected (by cl)” LS paths of shape \( \lambda \), introduced in [NS1 (3.4)] and [NS2, page 117] (see also [LNS2, Sect. 2.2]). Also, by [LNS3, Theorem 3.3], \( \mathbb{B}(\lambda)_{cl} \) is identical to the set QLS(\( \lambda \)) of all quantum LS paths of shape \( \lambda \), introduced in [LNS3, Sect. 3.2]. We can endow the set \( \mathbb{B}(\lambda)_{cl} = \text{QLS}(\lambda) \) with a crystal structure with weights in \( \text{cl}(P_{af}) \) in such a way that
\[
\begin{aligned}
&\varepsilon_i \text{cl}(\pi) = \text{cl}(\varepsilon_i \pi), \quad f_i \text{cl}(\pi) = \text{cl}(f_i \pi) \quad \text{for } \pi \in \mathbb{B}^{\oplus}_N(\lambda) \text{ and } i \in I_{af},
&\text{wt}(\text{cl}(\pi)) = \text{cl}(\text{wt}(\pi)) \quad \text{for } \pi \in \mathbb{B}^{\oplus}_N(\lambda),
&\varepsilon_i(\text{cl}(\pi)) = \varepsilon_i(\pi), \quad \varphi_i(\text{cl}(\pi)) = \varphi_i(\pi) \quad \text{for } \pi \in \mathbb{B}^{\oplus}_N(\lambda) \text{ and } i \in I_{af},
\end{aligned}
\] (7.9)

where we understand that \( \text{cl}(0) = 0 \). The next theorem follows from [NS1, Proposition 3.23 and Theorem 3.2].
Theorem 7.4.

(1) For every \( \lambda \in P^+ \), the crystal \( \text{QLS}(\lambda) = \mathcal{B}(\lambda)_{\text{cl}} \) is connected.

(2) For every \( \lambda, \mu \in P^+ \), there exists an isomorphism \( \text{QLS}(\lambda) \otimes \text{QLS}(\mu) \cong \text{QLS}(\lambda + \mu) \) of crystals. In particular, \( \text{QLS}(\lambda) \otimes \text{QLS}(\mu) \) is connected.

Lemma 7.5. Let \( \lambda, \mu \in P^+ \), and set \( J := \{ i \in I \mid \langle \lambda, \alpha_i^\vee \rangle = 0 \} \), \( K := \{ i \in I \mid \langle \mu, \alpha_i^\vee \rangle = 0 \} \). Each connected component of \( \mathcal{B}^\mathfrak{g}(\lambda) \otimes \mathcal{B}^\mathfrak{g}(\mu) \) contains an element of the form: \( (t_{\xi} \cdot \pi_\rho) \otimes \pi_\chi \) for some \( \xi \in Q_\Lambda^\vee(J\cup K) \), \( \rho \in \text{Par}(\lambda) \), and \( \chi \in \text{Par}(\mu) \).

Proof. Let \( \pi \in \mathcal{B}^\mathfrak{g}(\lambda) \) and \( \eta \in \mathcal{B}^\mathfrak{g}(\mu) \). By Theorem 7.4(2), there exists a monomial \( X \) in root operators on \( \text{QLS}(\lambda) \otimes \text{QLS}(\mu) \) such that \( X(\text{cl}(\pi) \otimes \text{cl}(\eta)) = \text{cl}(\pi_\lambda) \otimes \text{cl}(\pi_\mu) \); recall that \( \pi_\lambda = (e; 0, 1) \in \mathcal{B}^\mathfrak{g}(\lambda) \) and \( \pi_\mu = (e; 0, 1) \in \mathcal{B}^\mathfrak{g}(\mu) \). It follows from (7.3) and the tensor product rule for crystals that \( X(\pi \otimes \eta) \) is of the form \( \pi_1 \otimes \eta_1 \) for some \( \pi_1 \in \mathcal{B}^\mathfrak{g}(\lambda) \) such that \( \text{cl}(\pi_1) = \text{cl}(\pi_\lambda) \) and \( \eta_1 \in \mathcal{B}^\mathfrak{g}(\mu) \) such that \( \text{cl}(\eta_1) = \text{cl}(\pi_\mu) \). Here, we see from [NS3 Lemma 6.2.2] that

\[
\begin{align*}
\{ \pi \in \mathcal{B}^\mathfrak{g}(\lambda) \mid \text{cl}(\pi) = \text{cl}(\pi_\lambda) \} & = \{ t_{\xi} \cdot \pi_\rho \mid \rho \in \text{Par}(\lambda), \xi \in Q^\vee \}, \\
\{ \eta \in \mathcal{B}^\mathfrak{g}(\mu) \mid \text{cl}(\eta) = \text{cl}(\pi_\mu) \} & = \{ t_{\zeta} \cdot \pi_\chi \mid \chi \in \text{Par}(\mu), \zeta \in Q^\vee \}.
\end{align*}
\]

Therefore, \( X(\pi \otimes \eta) = (t_{\xi_1} \cdot \pi_\rho) \otimes (t_{\zeta_1} \cdot \pi_\chi) \) for some \( \rho \in \text{Par}(\lambda), \xi_1 \in Q^\vee \) and \( \chi \in \text{Par}(\mu), \zeta_1 \in Q^\vee \). Also, by (7.6), we have \( t_{-\zeta_1} \cdot ((t_{\xi_1} \cdot \pi_\rho) \otimes (t_{\zeta_1} \cdot \pi_\chi)) = (t_{\xi_1-\zeta_1} \cdot \pi_\rho) \otimes \pi_\chi \); we deduce from (7.5) and (A.3) that \( t_{\xi_1-\zeta_1} \cdot \pi_\rho = t_{\xi_2} \cdot \pi_\rho \), with \( \xi_2 = [\xi_1 - \zeta_1]^J \), where \( [\cdot]^J : Q^\vee \to Q^\vee_{\Lambda,J} \) is the projection in (2.6). We set \( \gamma := \xi_2 \bigcup K \) such that \( [\cdot]_K : Q^\vee \to Q^\vee_K \) is the投影 defined as in (2.6); we deduce from (7.5) and (A.3) that \( t_{-\gamma} \cdot \pi_\chi = \pi_\chi \). In addition, we set \( \xi := \xi_2 - \gamma \); notice that \( \xi \in Q^\vee_{\Lambda,J,K} \). Summarizing the above, we have

\[
(t_{-\gamma} t_{-\zeta_1}) \cdot X(\pi \otimes \eta) = t_{-\gamma} t_{-\zeta_1} \cdot ((t_{\xi_1} \cdot \pi_\rho) \otimes (t_{\zeta_1} \cdot \pi_\chi)) = t_{-\gamma} \cdot ((t_{\xi_2} \cdot \pi_\rho) \otimes \pi_\chi).
\]

Because the action of the affine Weyl group \( W_\text{af} \) on \( \mathcal{B}^\mathfrak{g}(\lambda) \otimes \mathcal{B}^\mathfrak{g}(\mu) \) is defined by means of root operators (see (7.1)), we conclude that \( \pi \otimes \eta \) and \( (t_{\xi} \cdot \pi_\rho) \otimes \pi_\chi \) above are in the same connected component of \( \mathcal{B}^\mathfrak{g}(\lambda) \otimes \mathcal{B}^\mathfrak{g}(\mu) \). This proves the lemma.

\[\square\]

7.4 Proof of Theorem 3.1.

Recall that \( \lambda, \mu \in P^+ \), and \( J = \{ i \in I \mid \langle \lambda, \alpha_i^\vee \rangle = 0 \}, K = \{ i \in I \mid \langle \mu, \alpha_i^\vee \rangle = 0 \} \).

Proposition 7.6. The set \( \mathcal{S}^\mathfrak{g}(\lambda+\mu) \cup \{0\} \) is stable under the action of the root operators \( e_i, f_i; i \in I_{\text{af}} \), on \( \mathcal{B}^\mathfrak{g}(\lambda) \otimes \mathcal{B}^\mathfrak{g}(\mu) \).

Proof. We give a proof of the assertion only for \( e_i, i \in I_{\text{af}} \); the proof for \( f_i, i \in I_{\text{af}} \), is similar. Let \( \pi \otimes \eta \in \mathcal{S}^\mathfrak{g}(\lambda+\mu) \), and \( i \in I_{\text{af}} \). We may assume that \( e_i(\pi \otimes \eta) \neq 0 \). Then it follows from the tensor product rule for crystals that

\[
e_i(\pi \otimes \eta) = \begin{cases} (e_i\pi) \otimes \eta & \text{if } \varphi_i(\pi) \geq \varepsilon_i(\eta), \\
\pi \otimes (e_i\eta) & \text{if } \varphi_i(\pi) < \varepsilon_i(\eta); \end{cases}
\]
recall from Remark C.2 that
\[ \varepsilon_i(\eta) = -m_i^\eta \quad \text{and} \quad \varphi_i(\pi) = H_i^\pi(1) - m_i^\pi. \] (7.11)

Let \( x, y \in W_{af} \) be such that \( x \geq y \) and \( \Pi^j(x) = \kappa(\pi), \Pi^K(y) = \iota(\eta) \) (see (SP)); we write \( x \) and \( y \) as:
\[
\begin{aligned}
x &= \kappa(\pi)x_1 \quad \text{with } x_1 \in (W_J)_{af}, \\
y &= \iota(\eta)y_1 \quad \text{with } y_1 \in (W_K)_{af}.
\end{aligned}
\] (7.12)

**Case 1.** Assume that \( \varphi_i(\pi) \geq \varepsilon_i(\eta) \), i.e., \( e_i(\pi \otimes \eta) = (e_i \pi) \otimes \eta \). Note that \( \kappa(e_i \pi) \) is equal either to \( \kappa(\pi) \) or to \( s_i \kappa(\pi) \) by the definition of the root operator \( e_i \). If \( \kappa(e_i \pi) = \kappa(\pi) \), then there is nothing to prove. Hence we may assume that \( \kappa(e_i \pi) = s_i \kappa(\pi) \). Then we deduce from the definition of the root operator \( e_i \) that the point \( t_1 = \min\{t \in [0, 1] \mid H_i^\pi(t) = m_i^\pi\} \) is equal to 1, and hence
\[ H_i^\pi(1) = m_i^\pi \quad \text{and} \quad \langle \kappa(\pi)\lambda, \alpha_i^\vee \rangle < 0. \] (7.13)

By (7.11), the equality in (7.13), and our assumption that \( \varphi_i(\pi) \geq \varepsilon_i(\eta) \), we see that \( m_i^\eta \geq 0 \), and hence \( m_i^\eta = 0 \); in particular, we obtain \( \langle \iota(\eta)\mu, \alpha_i^\vee \rangle \geq 0 \). In addition, it follows from (7.13) that \( \kappa(\pi)\alpha_i \in -\left(\Delta^+ \setminus \Delta_j^+\right) + Z\delta \). Since \( x_1 \in (W_J)_{af} \) by (7.12), we have
\[ x^{-1}_1 \alpha_i = x^{-1}_1 \kappa(\pi)^{-1}\alpha_i \in -\left(\Delta^+ \setminus \Delta_j^+\right) + Z\delta \subset -\Delta^+ + Z\delta. \] (7.14)

Also, since \( s_i \kappa(\pi) = \kappa(e_i \pi) \in (W^J)_{af} \) and \( x_1 \in (W_J)_{af} \), we have
\[ \Pi^j(s_i x) = \Pi^j(s_i \kappa(\pi)x_1) = s_i \kappa(\pi) = \kappa(e_i \pi). \]

If \( y^{-1}_1 \alpha_i \in \Delta^+ + Z\delta \), then we see from Lemma A.4(2) (applied to the case \( J = \emptyset \)) and (7.14) that \( s_i x \geq y \) in \( W_{af} \). Therefore, \( s_i \), \( y \in W_{af} \) satisfy condition (SP) for \( e_i(\pi \otimes \eta) = (e_i \pi) \otimes \eta \). If \( y^{-1}_1 \alpha_i \in -\Delta^+ + Z\delta \), then we see from Lemma A.4(3) (applied to the case \( J = \emptyset \)) and (7.14) that \( s_i x \geq s_i y \) in \( W_{af} \). We now claim that \( \Pi^K(s_i y) = \iota(\eta) \). Indeed, since \( \langle \iota(\eta)\mu, \alpha_i^\vee \rangle \geq 0 \) as seen above, we have \( \iota(\eta)^{-1}\alpha_i \in \left(\Delta^+ \cup \left(-\Delta_j^+\right)\right) + Z\delta \).

In addition, since \( y_1 \in (W_K)_{af} \), we deduce that \( y^{-1}_1 \alpha_i = y_1^{-1} \iota(\eta)^{-1}\alpha_i \) is contained in \( \left(\Delta^+ \cup \left(-\Delta_j^+\right)\right) + Z\delta \). However, since \( y^{-1}_1 \alpha_i \in -\Delta^+ + Z\delta \) by our assumption, we have \( y^{-1}_1 \alpha_i \in -\Delta_j^+ + Z\delta \), which implies that \( s_i y^{-1}_1 \alpha_i \in (W_K)_{af} \). Therefore, we obtain \( \Pi^K(s_i y) = \Pi^K(s_i y^{-1}_1 \alpha_i) = \Pi^K(y) = \iota(\eta) \), as desired. Thus, \( s_i x, s_i y \in W_{af} \) satisfy condition (SP) for \( e_i(\pi \otimes \eta) = (e_i \pi) \otimes \eta \).

**Case 2.** Assume that \( \varphi_i(\pi) < \varepsilon_i(\eta) \), i.e., \( e_i(\pi \otimes \eta) = \pi \otimes (e_i \eta) \). If \( \iota(e_i \eta) = \iota(\eta) \), then there is nothing to prove. Hence we may assume that \( \iota(e_i \eta) = s_i \iota(\eta) \). Then we deduce from the definition of the root operator \( e_i \) that \( \langle \iota(\eta)\mu, \alpha_i^\vee \rangle < 0 \); observe that with notation in (C.5) and (C.6), \( t_0 = 0 \), and \( H_i^\eta(t) \) is strictly decreasing on \([t_0, t_1] = [0, t_1] \). Thus we obtain \( \iota(\eta)^{-1}\alpha_i \in -\left(\Delta^+ \setminus \Delta_j^+\right) + Z\delta \). In addition, since \( y_1 \in (W_K)_{af} \) (see (7.12)), we deduce that
\[ y^{-1}_1 \alpha_i = y_1^{-1} \iota(\eta)^{-1}\alpha_i \in -\left(\Delta^+ \setminus \Delta_j^+\right) + Z\delta \subset -\Delta^+ + Z\delta, \]
which implies that \( y \succ s_i y \) by Lemma A.2. Since \( x \geq y \), we get \( x \succ s_i y \). Also, since \( s_i \iota(\eta) = \iota(e_i \eta) \in (W^K)_{af} \) and \( y_1 \in (W_K)_{af} \), we have
\[ \Pi^K(s_i y) = \Pi^K(s_i \iota(\eta)y_1) = s_i \iota(\eta) = \iota(e_i \eta). \]
Thus, $x, s, y \in W_{af}$ satisfy condition (SP) for $e_i(\pi \otimes \eta) = \pi \otimes (e_i\eta)$.

This completes the proof of the proposition. □

By this proposition, $S_\infty^\mp(\lambda + \mu)$ is a subcrystal of $B_\infty^\mp(\lambda) \otimes B_\infty^\mp(\mu)$. Hence our remaining task is to prove that $S_\infty^\mp(\lambda + \mu) \cong B_\infty^\mp(\lambda + \mu)$ as crystals. Recall from Lemma 7.5 that each connected component of $B_\infty^\mp(\lambda) \otimes B_\infty^\mp(\mu)$ contains an element of the form: $(t_\xi \cdot \pi_\rho) \otimes \pi_\chi$ for some $\xi \in Q_{I,(J \cup K)}^\vee$, $\rho \in \text{Par}(\lambda)$, and $\chi \in \text{Par}(\mu)$.

**Proposition 7.7.** Let $\rho = (\rho^{(i)}) \in \text{Par}(\lambda)$, $\chi = (\chi^{(i)}) \in \text{Par}(\mu)$, and $\xi = \sum c_i \alpha_i^\gamma \in Q_{I,(J \cup K)}^\vee$. Then,

\[
(t_\xi \cdot \pi_\rho) \otimes \pi_\chi \in S_\infty^\mp(\lambda + \mu) \tag{7.15}
\]

if and only if

\[
c_i \geq \chi_1^{(i)} \quad \text{for all} \quad i \in I \setminus (J \cup K). \tag{7.16}
\]

**Proof.** We first show the “only if” part; assume that (7.15) holds. We see that

\[
\kappa(t_\xi \cdot \pi_\rho) = \Pi^J(t_\xi) \quad \text{by (7.5)};
\]

\[
\iota(\pi_\chi) = \Pi^K(t_\zeta_1), \quad \text{with} \quad \zeta_1 = \sum_{i \in I} \chi_1^{(i)} \alpha_i^\gamma \in Q_{I \setminus K}^\vee \quad \text{by (7.3)}.
\]

Since (7.15) holds, there exist $x, y \in W_{af}$ such that $x \succeq y$ in $W_{af}$, and such that

\[
\begin{cases}
\Pi^J(x) = \kappa(t_\xi \cdot \pi_\rho) = \Pi^J(t_\xi), \\
\Pi^K(y) = \iota(\pi_\chi) = \Pi^K(t_\zeta_1);
\end{cases}
\]

we write $x$ and $y$ as:

\[
\begin{cases}
x = \Pi^J(t_\xi)x_1 \quad \text{with} \quad x_1 \in (W_J)_{af}, \\
y = \Pi^K(t_\zeta_1)y_1 \quad \text{with} \quad y_1 \in (W_K)_{af}.
\end{cases}
\]

Because $x \in W_{af}$ is a lift of $\Pi^J(t_\xi) \in (W^J)_{af}$, it follows from Lemma B.1 that $x = vt_{\xi + \gamma}$ for some $v \in W_J$ and $\gamma \in Q_J^\vee$. Similarly, we have $y = vt_{\zeta_1 + \gamma'}$ for some $v' \in W_K$ and $\gamma' \in Q_K^\vee$. Because $x \succeq y$ in $W_{af}$, we deduce from Lemma A.3(1) (applied to the case $J = \emptyset$) that $\xi + \gamma \succeq \zeta_1 + \gamma'$, which implies (7.16) since $\gamma, \gamma' \in Q_{I \setminus K}^\vee$.

We next show the “if” part; assume that (7.16) holds. Recall that

\[
\kappa(t_\xi \cdot \pi_\rho) = \Pi^J(t_\xi) \quad \text{by (7.5)};
\]

\[
\iota(\pi_\chi) = \Pi^K(t_\zeta_1), \quad \text{with} \quad \zeta_1 = \sum_{i \in I} \chi_1^{(i)} \alpha_i^\gamma \in Q_{I \setminus K}^\vee \quad \text{by (7.3)}.
\]

We set $\gamma := \sum_{i \in J \setminus K} \chi_1^{(i)} \alpha_i^\gamma \in Q_J^\vee$. Then it follows from (7.16) that $\xi + \gamma \succeq \zeta_1$ since $I \setminus K = (I \setminus (J \cup K)) \cup (J \setminus K)$. Hence we deduce from Lemma A.3(2) that $x := t_{\xi + \gamma} \succeq t_{\zeta_1} =: y$ in $W_{af}$. It is obvious that $\Pi^K(y) = \iota(\pi_\chi)$. Also, since $\gamma \in Q_J^\vee$, we see from (A.3) that $\Pi^J(x) = \Pi^J(t_\xi) = \kappa(t_\xi \cdot \pi_\rho)$. Thus, $x$ and $y$ satisfy condition (SP) for $(t_\xi \cdot \pi_\rho) \otimes \pi_\chi$, which implies (7.15). This proves the proposition. □
Proposition 7.8. Each connected component of \( \mathbb{S}^{\lambda}(\lambda + \mu) \) contains a unique element of the form: \((t_{\xi} \cdot \pi_{\rho}) \otimes \pi_{\chi}\) for some \(\rho \in \text{Par}(\lambda), \chi \in \text{Par}(\mu), \) and \(\xi \in Q^{J}_{\lambda(J\cup K)}\) satisfying condition (7.16) in Proposition 7.7. Therefore, there exists a one-to-one correspondence between the set \(\text{Conn}(\mathbb{S}^{\lambda}(\lambda + \mu))\) of connected components of \(\mathbb{S}^{\lambda}(\lambda + \mu)\) and the set of triples \((\rho, \chi, \xi) \in \text{Par}(\lambda) \times \text{Par}(\mu) \times Q^{J}_{\lambda(J\cup K)}\) satisfying condition (7.16) in Proposition 7.7.

Proof. The “existence” part follows from Lemma 7.8 and Proposition 7.7. Hence it suffices to prove the “uniqueness” part. Let \((\rho, \chi, \xi)\) and \((\rho', \chi', \xi')\) be elements in \(\text{Par}(\lambda) \times \text{Par}(\mu) \times Q^{J}_{\lambda(J\cup K)}\) satisfying condition (7.16) in Proposition 7.7 and suppose that \((t_{\xi} \cdot \pi_{\rho}) \otimes \pi_{\chi}\) and \((t_{\xi'} \cdot \pi_{\rho'}) \otimes \pi_{\chi'}\) are contained in the same connected component of \(\mathbb{S}^{\lambda}(\lambda + \mu)\). Then there exists a monomial \(X\) in root operators such that \(X((t_{\xi} \cdot \pi_{\rho}) \otimes \pi_{\chi}) = (t_{\xi'} \cdot \pi_{\rho'}) \otimes \pi_{\chi'}\). By the tensor product rule for crystals, we see that \(X((t_{\xi} \cdot \pi_{\rho}) \otimes \pi_{\chi}) = X_{1}(t_{\xi} \cdot \pi_{\rho}) \otimes X_{2}(t_{\xi} \cdot \pi_{\rho}')\) for some monomials \(X_{1}, X_{2}\) in root operators. Then we have \(X_{1}(t_{\xi} \cdot \pi_{\rho}) = t_{\xi'} \cdot \pi_{\rho'}\), which implies that \(t_{\xi} \cdot \pi_{\rho}\) and \(t_{\xi'} \cdot \pi_{\rho'}\) are contained in the same connected component of \(\mathbb{S}^{\lambda}(\lambda)\), and hence so are \(\pi_{\rho}\) and \(\pi_{\rho'}\). Therefore, by the uniqueness of an element of the form (7.2) in a connected component of \(\mathbb{S}^{\lambda}(\lambda)\) (see Section 7.2), we deduce that \(\rho = \rho'\). Similarly, we obtain \(\chi = \chi'\). Suppose, for a contradiction, that \(\xi \neq \xi'\); we may assume that for some \(k \in I \setminus (J \cup K)\), the coefficient of \(\alpha_{k}^{\vee}\) in \(\xi\) is greater than that in \(\xi'\), i.e., the coefficient of \(\alpha_{k}^{\vee}\) in \(\xi' - \xi\) is a negative integer. Because \((t_{\xi} \cdot \pi_{\rho}) \otimes \pi_{\chi}\) and \((t_{\xi'} \cdot \pi_{\rho'}) \otimes \pi_{\chi'}\) are contained in the same connected component, there exists a monomial \(Y\) in root operators such that \(Y((t_{\xi} \cdot \pi_{\rho}) \otimes \pi_{\chi}) = (t_{\xi'} \cdot \pi_{\rho}) \otimes \pi_{\chi} = (t_{\xi} \cdot \pi_{\rho}) \otimes \pi_{\chi} = (t_{\xi} \cdot \pi_{\rho}) \otimes \pi_{\chi}'\). Here, the same argument as in the proof of [NS] Lemma 7.1.4 (or, as in the proof of [NS] Proposition 7.1.2) shows that
\[
Y^{N}(t_{\xi} \cdot \pi_{\rho}) \otimes \pi_{\chi} = (t_{\xi+N(\xi'-\xi)} \cdot \pi_{\rho}) \otimes \pi_{\chi} \quad \text{for all } N \in \mathbb{Z}_{\geq 1}.
\]
Since this element is contained in \(\mathbb{S}^{\lambda}(\lambda + \mu)\) for all \(N \in \mathbb{Z}_{\geq 1}\) by Proposition 7.6, it follows from Proposition 7.7 that the coefficient of \(\alpha_{k}^{\vee}\) in \(\xi + N(\xi' - \xi)\) is greater than or equal to \(\chi_{1}^{(k)}\) for all \(N \geq 1\). This contradicts the fact that the coefficient of \(\alpha_{k}^{\vee}\) in \(\xi' - \xi\) is a negative integer. This proves the proposition. \(\square\)

Now, we write \(\lambda\) and \(\mu\) as: \(\lambda = \sum_{i \in I} m_{i} \varpi_{i}\) and \(\mu = \sum_{i \in I} n_{i} \varpi_{i}\). For \((\rho, \chi, \xi) \in \text{Par}(\lambda) \times \text{Par}(\mu) \times Q^{J}_{\lambda(J\cup K)}\) satisfying (7.16), define \(\omega = (\omega^{(i)})_{i \in I} \in \text{Par}(\lambda + \mu)\) as follows.

Write \(\rho, \chi,\) and \(\xi\) as:
\[
\rho = (\rho^{(i)})_{i \in I}, \quad \text{with } \rho^{(i)} = (\rho^{(i)}_{1} \geq \cdots \geq \rho^{(i)}_{m_{i} - 1} \geq 0) \quad \text{for } i \in I,
\]
\[
\chi = (\chi^{(i)})_{i \in I}, \quad \text{with } \chi^{(i)} = (\chi^{(i)}_{1} \geq \cdots \geq \chi^{(i)}_{n_{i} - 1} \geq 0) \quad \text{for } i \in I,
\]
\[
\xi = \sum_{i \in I \setminus (J \cup K)} c_{i} \chi^{(i)}; \quad \text{recall that } c_{i} \geq \chi^{(i)}_{1} \quad \text{for all } i \in I \setminus (J \cup K).
\]

Let \(i \in I\).

- If \(i \in J \cap K\) (note that \(m_{i} = n_{i} = 0\)), we set \(\omega^{(i)} := \emptyset\);
- if \(i \in J \setminus K\) (note that \(m_{i} = 0\)), we set \(\omega^{(i)} := \chi^{(i)}\), which is a partition of length less than \(n_{i} = 0 + m_{i} = m_{i} + n_{i}\);
- if \(i \in K \setminus J\) (note that \(n_{i} = 0\)), we set \(\omega^{(i)} := \rho^{(i)}\), which is a partition of length less than \(m_{i} = m_{i} + 0 = m_{i} + n_{i}\);

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It follows from Proposition 7.8 and the bijectivity of $\Theta$ that $B$ isomorphic, as a crystal, to $\rho$ where the last map sends $(\rho, \chi, \xi) \in \text{Par}(\lambda + \mu) \times \text{Par}(\mu) \times Q_{\lambda + \mu}^\vee$ satisfying (7.15) to the set $\text{Par}(\lambda + \mu)$; we can easily deduce that the map $\Theta$ is bijective. Also, by direct calculation, we have

$$\begin{align*}
\text{wt} \left( t_\xi \cdot \pi_\rho \otimes \pi_\chi \right) & = \text{wt} (t_\xi \cdot \pi_\rho) + \text{wt} (\pi_\chi) = \text{wt} (\lambda - |\rho| + \mu - |\chi|) \\
& = (\lambda + \mu) - (|\rho| + (\lambda, \xi) + |\chi|) \delta \\
& = (\lambda + \mu) - \sum_{\ell \in \lambda \setminus (J \cup K)} m_i c_i + |\chi| \delta \\
& = (\lambda + \mu) - |\Theta(\rho, \chi, \xi)\delta|.
\end{align*}$$

We claim that the connected component of $S_{\rho}^\vee (\lambda + \mu)$ containing $(t_\xi \cdot \pi_\rho) \otimes \pi_\chi$ is isomorphic, as a crystal, to $\{\Theta(\rho, \chi, \xi)\} \otimes S_0^\vee (\lambda + \mu)$, where $S_0^\vee (\lambda + \mu)$ denotes the connected component of $S_{\rho}^\vee (\lambda + \mu)$ containing $\pi_\lambda \otimes \pi_\mu = (\varepsilon ; 0, 1) \otimes (\varepsilon ; 0, 1) \in B_{\rho}^\vee (\lambda) \otimes B_{\rho}^\vee (\mu)$. Indeed, let us consider the composite of the following bijections:

$$B_{\rho}^\vee (\lambda) \otimes B_{\rho}^\vee (\mu) \xrightarrow{T_{\rho} \otimes \text{id}} B_{\rho}^\vee (\lambda) \otimes B_{\rho}^\vee (\mu) \quad \text{(see (7.8) and Remark 7.3)}$$

$$\xrightarrow{\sim} (\{\rho\} \otimes B_0^\vee (\lambda)) \otimes (\{\chi\} \otimes B_0^\vee (\mu)) \quad \text{(see the comment preceding (7.4))}$$

$$\xrightarrow{\sim} (\{\rho\} \otimes \{\chi\}) \otimes (B_0^\vee (\lambda) \otimes B_0^\vee (\mu)) \quad \text{(by the tensor product rule for crystals)}$$

$$\xrightarrow{\sim} \{\Theta(\rho, \chi, \xi)\} \otimes (B_0^\vee (\lambda) \otimes B_0^\vee (\mu)), $$

where the last map sends $(\rho \otimes \chi) \otimes (\pi \otimes \eta)$ to $\Theta(\rho, \chi, \xi) \otimes (\pi \otimes \eta)$ for each $\pi \in B_0^\vee (\lambda)$ and $\eta \in B_0^\vee (\mu)$. We deduce by (7.8) and the tensor product rule for crystals that the composite of these bijections is an isomorphism of crystals, which sends $(t_\xi \cdot \pi_\rho) \otimes \pi_\chi$ to $\Theta(\rho, \chi, \xi) \otimes (\pi_\lambda \otimes \pi_\mu)$. Therefore, the connected component of $S_{\rho}^\vee (\lambda + \mu)$ containing $(t_\xi \cdot \pi_\rho) \otimes \pi_\chi$ is mapped to $\{\Theta(\rho, \chi, \xi)\} \otimes S_0^\vee (\lambda + \mu)$ under this isomorphism of crystals. It follows from Proposition 7.8 and the bijectivity of $\Theta$ that

$$S_{\rho}^\vee (\lambda + \mu) \cong \bigsqcup_{\omega \in \text{Par}(\lambda + \mu)} \{\omega\} \otimes S_0^\vee (\lambda + \mu). \quad (7.17)$$

**Proposition 7.9.** As crystals, $S_0^\vee (\lambda + \mu) \cong B_0^\vee (\lambda + \mu)$.

**Proof.** Write $\lambda$ and $\mu$ as: $\lambda = \sum_{i \in I} m_i \omega_i$ with $m_i \in \mathbb{Z}_{\geq 0}$, and $\mu = \sum_{i \in I} n_i \omega_i$ with $n_i \in \mathbb{Z}_{\geq 0}$, respectively. We know from [Kas2, Conjecture 13.1 (iii)], which is proved in [BN, Remark 4.17], that there exists an isomorphism $B(\lambda + \mu) \xrightarrow{\sim} \bigotimes_{i \in I} B((m_i + n_i) \omega_i)$
of crystals, which maps $u_{\lambda+\mu}$ to $\bigotimes_{i \in I} u_{(m_i+n_i)\varpi_i}$; the restriction of this isomorphism to $B_0(\lambda+\mu) \subset B(\lambda+\mu)$ gives an embedding $B_0(\lambda+\mu) \hookrightarrow \bigotimes_{i \in I} B_0((m_i+n_i)\varpi_i)$ of crystals. Also, we know from [Kas2, Conjecture 13.2 (iii)], which is proved in [BN, Remark 4.17], that for each $i \in I$, there exists an embedding $B_0((m_i+n_i)\varpi_i) \hookrightarrow B(\varpi_i)^{\otimes(m_i+n_i)}$ of crystals, which maps $u_{(m_i+n_i)\varpi_i}$ to $u_{\varpi_i}^{\otimes(m_i+n_i)}$; recall from [Kas2, Proposition 5.4] that $B(\varpi_i)$ is connected. Thus we obtain an embedding

$$B_0(\lambda+\mu) \hookrightarrow \bigotimes_{i \in I} B_0((m_i+n_i)\varpi_i) \hookrightarrow \bigotimes_{i \in I} B(\varpi_i)^{\otimes(m_i+n_i)}$$

of crystals, which maps $u_{\lambda+\mu}$ to $\bigotimes_{i \in I} u_{\varpi_i}^{\otimes(m_i+n_i)}$. Here, we recall from [Kas2, Sect. 10] that for each $j, k \in I$, there exists an isomorphism $B(\varpi_j) \otimes B(\varpi_k) \cong B(\varpi_j) \otimes B(\varpi_k)$ of crystals, which maps $u_{\varpi_j} \otimes u_{\varpi_k}$ to $u_{\varpi_k} \otimes u_{\varpi_j}$. Hence we obtain an isomorphism of crystals

$$\bigotimes_{i \in I} B(\varpi_i)^{\otimes(m_i+n_i)} \cong \left( \bigotimes_{i \in I} B(\varpi_i)^{\otimes m_i} \right) \otimes \left( \bigotimes_{i \in I} B(\varpi_i)^{\otimes n_i} \right) =: B,$$

which maps $\bigotimes_{i \in I} u_{\varpi_i}^{\otimes(m_i+n_i)}$ to $\left( \bigotimes_{i \in I} u_{\varpi_i}^{\otimes m_i} \right) \otimes \left( \bigotimes_{i \in I} u_{\varpi_i}^{\otimes n_i} \right) =: b$. From these, we obtain an embedding $B_0(\lambda+\mu) \hookrightarrow B$ of crystals, which maps $u_{\lambda+\mu}$ to $b$. Similarly, we obtain an embedding $B_0(\lambda) \otimes B_0(\mu) \hookrightarrow B$ of crystals, which maps $u_{\lambda} \otimes u_{\mu}$ to $b$. Consequently, there exists an isomorphism of crystals from $B_0(\lambda+\mu)$ to the connected component (denoted by $S_0(\lambda+\mu)$) of $B_0(\lambda) \otimes B_0(\mu)$ containing $u_{\lambda} \otimes u_{\mu}$, which maps $u_{\lambda+\mu}$ to $u_{\lambda} \otimes u_{\mu}$. Now, by Theorem 2.28, we have an isomorphism $B_0(\lambda+\mu) \cong B_0^\otimes(\lambda+\mu)$ of crystals, which maps $u_{\lambda+\mu}$ to $\pi_{\lambda+\mu}$. In addition, we have an isomorphism $B_0(\lambda) \otimes B_0(\mu) \cong B_0^\otimes(\lambda) \otimes B_0^\otimes(\mu)$ of crystals, which maps $u_{\lambda} \otimes u_{\mu}$ to $\pi_{\lambda} \otimes \pi_{\mu}$; by restriction, we obtain an isomorphism of crystals from $S_0(\lambda+\mu)$ to $S_0^\otimes(\lambda+\mu)$. Summarizing, we obtain the following isomorphism of crystals:

$$B_0^\otimes(\lambda+\mu) \cong B_0(\lambda+\mu) \cong S_0(\lambda+\mu) \cong S_0^\otimes(\lambda+\mu),$$

$$\pi_{\lambda+\mu} \mapsto u_{\lambda+\mu} \mapsto u_{\lambda} \otimes u_{\mu} \mapsto \pi_{\lambda} \otimes \pi_{\mu}.$$

This proves the proposition. □

By using (7.17), Proposition 7.9 and (7.4) (with $\lambda$ replaced by $\lambda+\mu$), we conclude that

$$S_0^\otimes(\lambda+\mu) \cong \bigsqcup_{\omega \in \text{Par}(\lambda+\mu)} \{\omega\} \otimes S_0^\otimes(\lambda+\mu) \cong \bigsqcup_{\omega \in \text{Par}(\lambda+\mu)} \{\omega\} \otimes B_0^\otimes(\lambda+\mu) \cong B_0^\otimes(\lambda+\mu) \cong S_0^\otimes(\lambda+\mu),$$

as crystals. This completes the proof of Theorem 3.1.

**Corollary 7.10.** For each $\omega \in \text{Par}(\lambda+\mu)$, the element $\pi_{\omega} \in B_0^\otimes(\lambda+\mu)$ is mapped to $(t_\xi \cdot \rho \cdot \pi_{\chi}) \otimes \pi_{\gamma} \in S_0^\otimes(\lambda+\mu)$ for some $\xi \in Q_1^{\text{Y}(J,J,K)}$ and $\rho \in \text{Par}(\lambda)$, $\chi \in \text{Par}(\mu)$ satisfying (7.16) under the isomorphism $B_0^\otimes(\lambda+\mu) \cong S_0^\otimes(\lambda+\mu)$ of crystals in Theorem 3.1.
8 Proof of Propositions 3.3 and 3.4

Recall that \( \lambda, \mu \in P^+ \), and that \( J = \{ i \in I \mid \langle \lambda, \alpha_i^\vee \rangle = 0 \} \), \( K = \{ i \in I \mid \langle \mu, \alpha_i^\vee \rangle = 0 \} \).

8.1 Proof of Proposition 3.3

The “if” part is obvious from the definition of defining chains and condition (SP). Let us prove the “only if” part. Assume that \( \pi \otimes \eta \in S^\infty(\lambda + \mu) \), and write \( \pi \) and \( \eta \) as: \( \pi = (x_1, \ldots, x_s; a) \in B^\infty(\lambda) \) and \( \eta = (y_1, \ldots, y_p; b) \in B^\infty(\mu) \), respectively. It follows from (SP) that there exist \( x'_1, y'_1 \in W_{af} \) such that \( x'_s \geq y'_1 \in W_{af} \), and such that \( \Pi^J(x'_s) = x_s, \Pi^K(y'_1) = y_1 \); we write \( x'_s = x_s z_1 \) for some \( z_1 \in (W_J)_{af} \), and \( y'_1 = y_1 z_2 \) for some \( z_2 \in (W_K)_{af} \). Now we set

\[
\begin{align*}
x'_u &:= x_u z_1 \quad \text{for } 1 \leq u \leq s, \\
y'_q &:= y_q z_2 \quad \text{for } 1 \leq q \leq p.
\end{align*}
\]

Because \( x_1 \geq x_2 \geq \cdots \geq x_s \) in \( (W^J)_{af} \) by the definition of semi-infinite LS paths, it follows from Lemma A.7 that \( x'_1 \geq x'_2 \geq \cdots \geq x'_s \) in \( W_{af} \). Similarly, we see that \( y'_1 \geq y'_2 \geq \cdots \geq y'_p \) in \( W_{af} \). Combining these inequalities with the inequality \( x'_s \geq y'_1 \), we obtain \( x'_1 \geq \cdots \geq x'_s \geq y'_1 \geq \cdots \geq y'_p \) in \( W_{af} \). Since \( \Pi^J(x'_u) = x_u \) for all \( 1 \leq u \leq s \), and \( \Pi^J(y'_q) = y_q \) for all \( 1 \leq q \leq p \), the sequence \( x'_1, \ldots, x'_s, y'_1, \ldots, y'_p \) is a defining chain for \( \pi \otimes \eta \). This completes the proof of Proposition 3.3.

8.2 Proof of Proposition 3.4

We write \( \pi \) and \( \eta \) as: \( \pi = (x_1, \ldots, x_s; a) \in B^\infty(\lambda) \) and \( \eta = (y_1, \ldots, y_p; b) \in B^\infty(\mu) \), respectively. First, we prove the “only if” part. Take an (arbitrary) defining chain \( x'_1, \ldots, x'_s, y'_1, \ldots, y'_p \in W_{af} \) for \( \pi \otimes \eta \). It suffices to show the following claim.

Claim 1. Let \( x \in W_{af} \) be such that \( y'_p \succeq x \); note that \( \kappa(\eta) = y_p = \Pi^K(y'_p) \succeq \Pi^K(x) \) by Lemma A.3. Then, \( \kappa(\pi) \succeq \Pi^J(\iota(\eta, x)) \).

Proof of Claim 1. For the given \( x \in W_{af} \), we define \( \tilde{y}_p, \tilde{y}_{p-1}, \ldots, \tilde{y}_1 = \iota(\eta, x) \) as in (8.1). We show by descending induction on \( q \) that

\[
y'_q \succeq \tilde{y}_q \quad \text{for all } 1 \leq q \leq p. \tag{8.1}
\]

Because \( y'_p \succeq x \) and \( \Pi^K(y'_p) = y_p \), it follows that \( y'_p \in \Lift_{\geq x}(y_p) \), and hence \( y'_q \succeq \min \Lift_{\geq x}(y_p) = \tilde{y}_p \). Assume that \( q < p \). Since \( y'_q \succeq y'_{q+1} \) by the definition of defining chains, and since \( y'_{q+1} \succeq \tilde{y}_{q+1} \) by our induction hypothesis, we obtain \( y'_q \succeq \tilde{y}_{q+1} \). In addition, we have \( \Pi^K(y'_q) = y_q \). From these, we deduce that \( y'_q \in \Lift_{\geq \tilde{y}_{q+1}}(y_q) \), and hence \( y'_q \succeq \min \Lift_{\geq \tilde{y}_{q+1}}(y_q) = \tilde{y}_q \). Thus we have shown (8.1). Hence we have \( x'_s \succeq y'_1 \succeq \tilde{y}_1 = \iota(\eta, x) \) by the assumption. Therefore, it follows from Lemma A.8 that

\[
\kappa(\pi) = x_s = \Pi^J(x'_s) \succeq \Pi^J(\iota(\eta, x)). \tag{8.2}
\]

This proves Claim 1.
Next, we prove the "if" part. We define \( \tilde{y}_p, \tilde{y}_{p-1}, \ldots, \tilde{y}_1 = \iota(\eta, x) \) as in (3.1). By the definitions, we have

\[
\begin{aligned}
\iota(\eta, x) &= \tilde{y}_1 \geq \tilde{y}_2 \geq \cdots \geq \tilde{y}_p \geq x, \\
\Pi^K(\tilde{y}_q) &= y_q \quad \text{for } 1 \leq q \leq p.
\end{aligned}
\]  

(8.3)

Write \( \iota(\eta, x) \in W_{af} \) as: \( \iota(\eta, x) = \Pi^J(\iota(\eta, x))z \) with \( z \in (W_J)_{af} \). Since \( \kappa(\pi) \geq \Pi^J(\iota(\eta, x)) \) by the assumption, we deduce from Lemma A.7 that \( x'_s := \kappa(\pi)z \geq \Pi^J(\iota(\eta, x))z = \iota(\eta, x) = \tilde{y}_1 \). Similarly, if we set \( x'_u := x_uz \) for \( 1 \leq u \leq s \), then we have

\[
\begin{aligned}
&\begin{cases} 
  x'_1 \geq x'_2 \geq \cdots \geq x'_s (\geq \iota(\eta, x) = \tilde{y}_1), \\
  \Pi^J(x'_u) = x_u & \text{for } 1 \leq u \leq s.
\end{cases}
\end{aligned}
\]  

(8.4)

Concatenating the sequences in (8.3) and (8.4), we obtain a defining chain

\[
\begin{aligned}
x'_1 \geq x'_2 \geq \cdots \geq x'_s \geq \tilde{y}_1 \geq \tilde{y}_2 \geq \cdots \geq \tilde{y}_p
\end{aligned}
\]  

(8.5)

for \( \pi \otimes \eta \in B^\Sigma(\lambda) \otimes B^\Sigma(\mu) \). This completes the proof of Proposition 3.4.

9 Proof of Theorem 3.5.

Recall that \( \lambda, \mu \in P^+ \), and that \( J = \{ i \in I \mid \langle \lambda, \alpha^+_i \rangle = 0 \}, K = \{ i \in I \mid \langle \mu, \alpha^+_i \rangle = 0 \}, \) and \( S = \{ i \in I \mid \langle \lambda + \mu , \alpha^+_i \rangle = 0 \} = J \cap K \).

9.1 Proof of (D2) ⇔ (D3).

We prove the implication (D2) ⇒ (D3). Let \( x'_1, \ldots, x'_s, y'_1, \ldots, y'_p =: y \) be a defining chain for \( \pi \otimes \eta \) such that \( \Pi^S(y) \geq \Pi^S(x) \). Write \( x = \Pi^S(x)z \) for some \( z \in (W_S)_{af} \); note that \( (W_S)_{af} \subset (W_J)_{af} \cap (W_K)_{af} \). We deduce from Lemma A.7 that

\[
\Pi^S(x'_1)z, \ldots, \Pi^S(x'_s)z, \Pi^S(y'_1)z, \ldots, \Pi^S(y'_p) = \Pi^S(y)z
\]

is also a defining chain for \( \pi \otimes \eta \) such that \( \Pi^S(\Pi^S(y)z) = \Pi^S(y) \geq \Pi^S(x) \). Hence we may assume from the beginning that \( y \geq x \). We deduce from Lemma A.8 that \( \kappa(\eta) = \Pi^K(y) \geq \Pi^K(x) \). Also, the inequality \( \kappa(\pi) \geq \Pi^J(\iota(\eta, x)) \) was shown in Claim 1 in the proof of Proposition 3.4.

The implication (D3) ⇒ (D2) follows from the fact that the defining chain (8.5) for \( \pi \otimes \eta \) satisfies the desired condition in (D2).

9.2 Proof of (D1) ⇔ (D2).

Let \( D_{\geq x}^\Sigma(\lambda + \mu) \) denote the set of elements in \( B^\Sigma(\lambda) \otimes B^\Sigma(\mu) \) satisfying condition (D2) (or equivalently, condition (D3)); by Proposition 3.3 we see that \( D_{\geq x}^\Sigma(\lambda + \mu) \subset S^\Sigma(\lambda + \mu) \).

Lemma 9.1 (cf. [NS3] Lemma 5.3.1 and Proposition 5.3.2)).

1. The set \( D_{\geq x}^\Sigma(\lambda + \mu) \cup \{0\} \) is stable under the action of the root operator \( f_i \) for all \( i \in I_{af} \).
(2) The set $\mathbb{D}_{\geq x}^\omega (\lambda + \mu) \cup \{0\}$ is stable under the action of the root operator $e_i$ for those $i \in I_{af}$ such that $\langle x(\lambda + \mu), \alpha_i^\vee \rangle \geq 0$.

(3) Let $i \in I_{af}$ be such that $\langle x(\lambda + \mu), \alpha_i^\vee \rangle \geq 0$. Then,

$$\mathbb{D}_{\geq x}^\omega (\lambda + \mu) = \{ e_i^n (\pi \otimes \eta) \mid \pi \otimes \eta \in \mathbb{D}_{\geq s_{ix}}^\omega (\lambda + \mu), n \geq 0 \} \setminus \{0\}. \quad (9.1)$$

Proof. (1) Let $\pi \otimes \eta \in \mathbb{D}_{\geq x}^\omega (\lambda + \mu)$, and let $i \in I_{af}$; we may assume that $f_i(\pi \otimes \eta) \neq 0$. We give a proof only for the case that $f_i(\pi \otimes \eta) = \pi \otimes f_i \eta$; the proof for the case that $f_i(\pi \otimes \eta) = f_i \pi \otimes \eta$ is similar. We write $\pi$ and $\eta$ as: $\pi = (x_1, \ldots, x_s; a) \in \mathbb{H}^\omega (\lambda)$ and $\eta = (y_1, \ldots, y_p; b) \in \mathbb{H}^\omega (\mu)$, respectively. Let $x'_1 \geq \cdots \geq x'_s \geq y'_1 \geq \cdots \geq y'_p$ be a defining chain for $\pi \otimes \eta$ such that $\Pi^S (y'_p) \geq \Pi^S (x)$. Take $0 \leq t_0 < t_1 \leq 1$ as in (C.7) (with $\pi$ replaced by $\eta$); note that $H_i^n (t)$ is strictly increasing on the interval $[t_0, t_1]$. We see from (C.8) that $f_i \eta$ is of the form:

$$f_i \eta := (y_1, \ldots, y_k, s_i y_{k+1}, \ldots, s_i y_m, s_i y_{m+1}, y_{m+1}, \ldots, y_p; b')$$

for some $0 \leq k \leq m \leq p - 1$ and some increasing sequence $b'$ of rational numbers in $[0, 1]$. Here, since $H_i^n (t)$ is strictly increasing on the interval $[t_0, t_1]$, it follows that $\langle y_k \mu, \alpha_i^\vee \rangle > 0$, and hence that $y_n^{-1} \alpha_i \in (\Delta^+ \setminus \Delta^+_K) + Z \delta$ for all $k + 1 \leq n \leq m + 1$. Hence we deduce that

$$\langle y'_n \rangle = \langle y_n \rangle = \langle y_n \rangle = \langle y_n \rangle \in (\Delta^+ \setminus \Delta^+_K) + Z \delta \subset \Delta^+ + Z \delta \text{ for all } k + 1 \leq n \leq m + 1 \quad (9.2)$$

since $y'_n = y_n z_n$ for some $z_n \in (W_K)_{af}$. Therefore, it follows from Lemma A.4.3 (applied to the case $J = \emptyset$) that $s_i y_{k+1} \geq \cdots \geq s_i y_{m+1}$. Also, we see from (A.5) that $s_i y'_{m+1} \geq y'_{m+1}$. Thus we obtain

$$s_i y'_{k+1} \geq \cdots \geq s_i y'_{m+1} \geq y'_{m+1} \geq \cdots \geq y'_p; \quad (9.3)$$

note that $\Pi^K (s_i y'_n) = s_i y_n$ for all $k + 1 \leq n \leq m + 1$ by Lemma A.2 since $s_i y_n \in (W_K)_{af}$. If $t_1 \neq 1$, then $\kappa (f_i \eta) = \kappa (\eta) = y_p$, and the final element of the sequence $s_i y_p$ which satisfies $\Pi^S (y'_p) \geq \Pi^S (x)$ by our assumption. If $t_1 = 1$, then $m + 1 = p$, $\kappa (f_i \eta) = s_i y_p$, and the final element of the sequence $s_i y_p$ which satisfies $\Pi^S (y'_p) \geq \Pi^S (x)$ by our assumption. As shown above, we deduce from Lemma A.8 that $\Pi^S (s_i y'_p) = \Pi^S (s_i y'_{m+1}) \geq \Pi^S (y'_p) \geq \Pi^S (x)$. In what follows, we will give a defining chain for $f_i (\pi \otimes \eta) = \pi \otimes (f_i \eta)$ in which the sequence $s_i y_p$ lies at the tail.

Case 1. Assume that the set $\{ 1 \leq n \leq k \mid \langle y_n \mu, \alpha_i^\vee \rangle = 0 \}$ is nonempty, and let $k_0$ be the maximum element of this set. Because the function $H_i^n (t)$ attains its minimum value $m^n_i$ at $t = t_0$, it follows that $\langle y_{k_0} \mu, \alpha_i^\vee \rangle = \langle y_{k_0-1} \mu, \alpha_i^\vee \rangle = \cdots = \langle y_{k_0+1} \mu, \alpha_i^\vee \rangle = 0$, and $\langle y_{k_0} \mu, \alpha_i^\vee \rangle < 0$, which implies that $y_n^{-1} \alpha_i \in \Delta_K + Z \delta$ for all $k_0 + 1 \leq n \leq k$, and $y_n^{-1} \alpha_i \in \Delta_K + Z \delta$. Hence we deduce that $\langle y_n \rangle^{-1} \alpha_i \in \Delta_K + Z \delta$ for all $k_0 + 1 \leq n \leq k$, and $\langle y_n \rangle^{-1} \alpha_i \in \Delta_K + Z \delta$. Therefore, there exists $k_0 \leq k_1 \leq k$ such that $\langle y_n \rangle^{-1} \alpha_i \in \Delta_K + Z \delta$ for all $k_1 + 1 \leq n \leq k$, and such that $\langle y_n \rangle^{-1} \alpha_i \in \Delta_K + Z \delta$; recall from (9.2) that $\langle y'_{k+1} \rangle^{-1} \alpha_i \in \Delta^+ + Z \delta$. Hence, in this case, we deduce from Lemma A.4.1 and (3) that $y'_{k+1} \geq \cdots \geq s_i y'_{k+1} \geq \cdots \geq s_i y_{k+1} \geq \cdots \geq s_i y_{m+1} \geq y_{m+1} \geq \cdots \geq y'_p \quad (9.4)$

for $f_i (\pi \otimes \eta) = \pi \otimes (f_i \eta)$.
Case 2. Assume that the set \( \{ 1 \leq n \leq k \mid \langle y_n, \alpha_i^\gamma \rangle \neq 0 \} \) is empty, i.e., \( \langle y_n, \alpha_i^\gamma \rangle = 0 \) for all \( 1 \leq n \leq k \); note that \( (y_n')^{-1}\alpha_i \in \Delta_K + Z\delta \) for all \( 1 \leq n \leq k \). If there exists \( 1 \leq k_1 \leq k \) such that \( (y_n')^{-1}\alpha_i \in \Delta^+_K + Z\delta \) for all \( k_1 + 1 \leq n \leq k \), and \( (y_{k_1}')^{-1}\alpha_i \in -\Delta^+_K + Z\delta \), then we obtain a defining chain of the form (9.4) for \( f_i(\pi \otimes \eta) = \pi \otimes (f_i \eta) \) in exactly the same way as in Case 1. Hence we may assume that \( (y_n')^{-1}\alpha_i \in \Delta^+_K + Z\delta \) for all \( 1 \leq n \leq k \). It follows from Lemma A.4(3) and (9.2) that

\[
 s_iy_1' \geq \cdots \geq s_iy'_{k_1} \geq s_iy_{k_1+1} \geq \cdots \geq s_iy_{m+1} \geq s_iy_{m+1} \geq \cdots \geq y_p;
\]

note that by Remark A.3 \( \Pi^K(s_iy'_n) = \Pi^K(y_n) = y_n \) for all \( 1 \leq n \leq k \). Now, we define \( u_0 \) to be the maximum element of the set \( \{ 1 \leq u \leq s \mid \langle x_u, \alpha_i^\gamma \rangle \neq 0 \} \cup \{ 0 \} \). We claim that if \( u_0 \geq 1 \), then \( \langle x_{s}, \alpha_i^\gamma \rangle = \langle x_{s-1}, \alpha_i^\gamma \rangle = \cdots = \langle x_{u_0+1}, \alpha_i^\gamma \rangle = 0 \), and that if \( u_0 \geq 1 \), then \( \langle x_{u_0}, \alpha_i^\gamma \rangle < 0 \); this would imply that \( x_u^{-1}\alpha_i \in \Delta^-_J + Z\delta \) for all \( u_0 + 1 \leq u \leq s \), and that if \( u_0 \geq 1 \), then \( x_u^{-1}\alpha_i \in -\Delta^+_K + Z\delta \). Indeed, since \( \langle y_n, \alpha_i^\gamma \rangle = 0 \) for all \( 1 \leq n \leq k \) by our assumption, we see that \( H^\pi(t) \) is identically zero on the interval \([0, t_1]\), and hence \( m''_i = 0 \), from which it follows that \( \varepsilon_i(\eta) = -\eta_i = 0 \) by Remark C.2. Here we recall that \( f_i(\pi \otimes \eta) = \pi \otimes f_i \eta \) (if and only if) \( \varphi_i(\pi) \leq \varepsilon_i(\eta) \) by the tensor product rule for crystals. Hence we see that \( \varphi_i(\pi) = H^\pi(1) - \eta_i = 0 \) by Remark C.2. Since \( \langle x_{s}, \alpha_i^\gamma \rangle = \langle x_{s-1}, \alpha_i^\gamma \rangle = \cdots = \langle x_{u_0+1}, \alpha_i^\gamma \rangle = 0 \) by our assumption, we obtain \( \langle x_{u_0}, \alpha_i^\gamma \rangle < 0 \) if \( u_0 \geq 1 \), as desired. Therefore, by the same argument as in Case 1, we get \( 0 \leq u_0 \leq u_1 \leq s \) such that \( \langle x_u', \alpha_i^\gamma \rangle \in \Delta^+_J + Z\delta \) for all \( u_1 + 1 \leq u \leq s \), and such that \( \langle x_u', \alpha_i^\gamma \rangle \in -\Delta^+_J + Z\delta \) if \( u_1 \geq 1 \); recall that \( \langle y'_1, \alpha_i^\gamma \rangle \in \Delta^+_K + Z\delta \). Also we note that by Remark A.3 \( \Pi^J(s_iy'_1) = \Pi^J(x_u') = x_u \) for all \( u_1 + 1 \leq i \leq s \). In this case, by Lemma A.4(1) and (3), together with (9.5), we obtain a defining chain

\[
x_1' \geq \cdots \geq x_{u_1}' \geq s_1x_{u_1+1}' \geq \cdots \geq s_1y_1' \geq \cdots \geq s_1y_{m+1}' \geq s_1y_{m+1}' \geq \cdots \geq y_p'
\]

for \( f_i(\pi \otimes \eta) = \pi \otimes (f_i \eta) \). This proves part (1).

(2) Let \( \pi \otimes \eta \in D_{\mathcal{L}}^x(\lambda + \mu) \), and let \( i \in I_{af} \) be such that \( \langle x(\lambda + \mu), \alpha_i^\gamma \rangle \geq 0 \); we may assume that \( e_i(\pi \otimes \eta) \neq 0 \). Since \( \pi \otimes \eta \in D_{\mathcal{L}}^x(\lambda + \mu) \), there exists a defining chain for \( \pi \otimes \eta \) whose final element, say \( y \in W_{af} \), satisfies \( \Pi^S(y) \geq \Pi^S(x) \). We can show the following claims by arguments similar to those in part (1).

Claim 1.

(i) If \( e_i(\pi \otimes \eta) = e_i(\pi \otimes \eta) \), or if \( e_i(\pi \otimes \eta) = \pi \otimes (e_i \eta) \) and \( \kappa(e_i \eta) = \kappa(\eta) \), then there exists a defining chain for \( e_i(\pi \otimes \eta) \) such that its final element is \( y \).

(ii) If \( e_i(\pi \otimes \eta) = \pi \otimes (e_i \eta) \) and \( \kappa(e_i \eta) = s_i \kappa(\eta) \), then there exists a defining chain for \( e_i(\pi \otimes \eta) \) whose final element is \( s_iy \).

In case (i) of Claim 1, it is obvious that \( e_i(\pi \otimes \eta) \notin D_{\mathcal{L}}^x(\lambda + \mu) \). In case (ii) of Claim 1, we see by the definition of the root operator \( e_i \) that with notation in (C.5) and (C.6), \( t_1 = 1 \), and the function \( H^\pi_i(t) \) is strictly decreasing on \([t_0, t_1] = [t_0, 1]) \). Hence we have \( \langle \kappa(\eta) \mu, \alpha_i^\gamma \rangle < 0 \), which implies that \( \kappa(\eta)^{-1}\alpha_i \in -\Delta^+_J + Z\delta \). Since \( \Pi^K(\Pi^S(y)) = \Pi^K(y) = \kappa(\eta) \), we see that \( \Pi^S(y) = \Pi^K(y) = \kappa(\eta) \), which implies that \( \Pi^S(y)(\lambda + \mu), \alpha_i^\gamma \rangle < 0 \). Also, it follows from Lemma A.3 that \( s_i \Pi^S(x) \in (W^S)_{af} \) and \( \Pi^S(s_iy) = s_i \Pi^S(y) \). Here, by the assumption, we have \( \Pi^S(x)(\lambda + \mu), \alpha_i^\gamma \rangle = \langle x(\lambda + \mu), \alpha_i^\gamma \rangle \geq 0 \). Therefore, we deduce from Lemma A.4(2), together with \( \Pi^S(y) \geq \Pi^S(y) \)
\( \Pi^s(x) \), that \( \Pi^s(s_iy) = s_i \Pi^s(y) \geq \Pi^s(x) \). Thus, we conclude that \( c_i(\pi \otimes \eta) \in \mathbb{D}_{\geq \lambda}^\infty(\lambda + \mu) \). This proves part (2).

(3) If \( \langle x(\lambda + \mu), \alpha_i^\gamma \rangle = 0 \), then \( \Pi^s(s_i x) = \Pi^s(x) \) by Remark A.3 and hence \( \mathbb{D}_{\geq s_i x}^\infty(\lambda + \mu) = \mathbb{D}_{\geq x}^\infty(\lambda + \mu) \). Hence the assertion is obvious from part (2).

Assume that \( \langle x(\lambda + \mu), \alpha_i^\gamma \rangle > 0 \). Then we see from Lemma A.2 that \( \Pi^s(s_i x) = s_i \Pi^s(x) \in (W^s)_{af} \) and \( \Pi^s(s_i x) \geq \Pi^s(x) \), which implies that \( \mathbb{D}_{\geq s_i x}^\infty(\lambda + \mu) \supseteq \mathbb{D}_{\geq x}^\infty(\lambda + \mu) \). Therefore, by part (2), we obtain the inclusion \( \supseteq \) in \[ (9.11) \]. In order to show the opposite inclusion \( \subseteq \) in \[ (9.11) \], it suffices to show that \( f_{i_1}^{\text{max}}(\pi \otimes \eta) \in \mathbb{D}_{\geq s_i x}^\infty(\lambda + \mu) \) for all \( \pi \otimes \eta \in \mathbb{D}_{\geq s_i x}^\infty(\lambda + \mu) \). In view of part (1), this assertion itself follows from the following claim.

**Claim 2.** Let \( \pi \otimes \eta \in \mathbb{D}_{\geq x}^\infty(\lambda + \mu) \). If \( f_{i_1}(\pi \otimes \eta) = 0 \), i.e., \( \varphi_i(\pi \otimes \eta) = 0 \), then \( \pi \otimes \eta \in \mathbb{D}_{\geq s_i x}^\infty(\lambda + \mu) \).

**Proof of Claim 2.** We write \( \pi \) and \( \eta \) as: \( \pi = (x_1, \ldots, x_s; a) \) and \( \eta = (y_1, \ldots, y_p; b) \), respectively. Let \( x'_1 \geq \cdots \geq x'_s \geq y'_1 \geq \cdots \geq y'_p \) be a defining chain for \( \pi \otimes \eta \) such that \( \Pi^s(y'_p) \geq \Pi^s(x) \). We see from Lemma A.8 that \( \Pi^s(x'_1) \geq \cdots \geq \Pi^s(x'_s) \geq \Pi^s(y'_1) \geq \cdots \geq \Pi^s(y'_p) \) is also a defining chain for \( \pi \otimes \eta \) satisfying \( \Pi^s(\Pi^s(y'_p)) = \Pi^s(\Pi^s(y'_p)) = \Pi^s(\Pi^s(x)) \). Hence we may assume from the beginning that \( x'_1, \ldots, x'_s, y'_1, \ldots, y'_p \in (W^s)_{af} \).

Assume first that the set
\[
\{ 1 \leq q \leq p \mid \langle y'_q(\lambda + \mu), \alpha_i^\gamma \rangle \geq 0 \}
\]
is nonempty. Let \( q_1 \) be the maximum element of this set; notice that \( \langle y'_q(\lambda + \mu), \alpha_i^\gamma \rangle > 0 \) and \( \langle y_q(\lambda + \mu), \alpha_i^\gamma \rangle = 0 \) for all \( q \neq q_1 \). Note that \( f_i(\pi \otimes \eta) = 0 \) implies \( f_i \eta = 0 \) by the tensor product rule for crystals, and hence \( H^i(1) - m^i = 0 \). From this it follows that \( \langle y'_q(\lambda + \mu), \alpha_i^\gamma \rangle = 0 \), and hence \( \langle y'_q, \alpha_i^\gamma \rangle \geq 0 \) by the definition of \( q_1 \), which implies that \( \langle y'_q, \alpha_i^\gamma \rangle \leq 0 \). Therefore, we see from Lemma A.4(1) and (3) that
\[
x'_1 \geq \cdots \geq x'_s \geq y'_1 \geq \cdots \geq y'_q \geq s_iy_{q_1+1} \geq \cdots \geq s_iy_p;
\]

note that \( \Pi^K(s_i y'_q) = \Pi^K(y'_q) \) for all \( q_1 < q \leq p \) since \( \langle y'_q(\lambda + \mu), \alpha_i^\gamma \rangle = 0 \). Thus the sequence \[ (9.7) \] is also a defining chain for \( \pi \otimes \eta \). If \( q_1 = p \), then the final element of \[ (9.7) \] is \( y'_p \), and
\[
\langle y'_p(\lambda + \mu), \alpha_i^\gamma \rangle \leq 0.
\]
Hence it follows from Lemma A.4(1) that \( y'_p \geq s_i \Pi^s(x) = \Pi^s(s_i x) \). If \( q_1 < p \), then the final element of \[ (9.7) \] is \( s_i y'_p \), and \( \langle y'_p(\lambda + \mu), \alpha_i^\gamma \rangle > 0 \). This implies that \( s_i y'_p \leq (W^s)_{af} \) by Lemma A.2 and that \( s_i y'_p \geq s_i \Pi^s(x) = \Pi^s(s_i x) \) by Lemma A.4(3).

Hence we conclude that \( \pi \otimes \eta \in \mathbb{D}_{\geq \lambda}^\infty(\lambda + \mu) \).

Assume next that the set \[ (9.6) \] is empty, that is, \( \langle y'_q, \alpha_i^\gamma \rangle \leq 0 \) for all \( 1 \leq q \leq p \); notice that \( \langle y'_q(\lambda + \mu), \alpha_i^\gamma \rangle > 0 \) for all \( 1 \leq q \leq p \). Also, since \( \langle y'_q, \alpha_i^\gamma \rangle = \langle y'_q, \alpha_i^\gamma \rangle = 0 \) for all \( 1 \leq q \leq p \), we have \( H^i(t) = 0 \) for all \( t \in [0,1] \), and hence \( \varepsilon_i(\eta) = 0 \). Since \( f_i(\pi \otimes \eta) = 0 \) by the assumption, we obtain \( f_i \pi = 0 \) by the tensor product rule for crystals. Let \( u_1 \) be the maximum element of the set \( \{ 1 \leq u \leq s \mid \langle x'_u, \alpha_i^\gamma \rangle \geq 0 \} \). Then we have \( \langle x'_u(\lambda + \mu), \alpha_i^\gamma \rangle > 0 \) and \( \langle x'_u, \alpha_i^\gamma \rangle = \langle x, \alpha_i^\gamma \rangle = 0 \) for all \( u_1 < u \leq s \). In addition, we can show by the same
argument as above that if \( u_1 \geq 1 \), then \( x'_{u_1} (\lambda + \mu), \alpha_i^{\gamma} \leq 0 \). Therefore, it follows from Lemma A.4(1) and (3) that

\[
x_1' \geq \cdots \geq x'_{u_1} \geq s_i x_{u_1 + 1} \geq \cdots \geq s_i x'_s \geq s_i y'_1 \geq \cdots \geq s_i y'_s.
\] (9.8)

In the same way as for (9.7), we can verify that the sequence (9.8) is a defining chain for \( \pi \otimes \eta \) satisfying the condition in (D2). This proves Claim 2.

This completes the proof of Lemma 9.1. \( \blacksquare \)

**Corollary 9.2** (cf. [NS3, Corollary 5.3.3]). Let \( x \in W_{af} \), and \( i \in I_{af} \). For every \( \pi \otimes \eta \in \mathbb{D}^\infty_{x, \leq x}(\lambda + \mu) \), we have \( f_i^{\max}(\pi \otimes \eta) \in \mathbb{D}^\infty_{x}(\lambda + \mu) \).

**Proof.** If \( \langle x(\lambda + \mu), \alpha_i^{\gamma} \rangle \geq 0 \), then the assertion follows from the proof of Lemma 9.1(3). If \( \langle x(\lambda + \mu), \alpha_i^{\gamma} \rangle < 0 \), then we have \( \Pi^S(s_i x) = s_i \Pi^S(x) \leq \Pi^S(x) \) by Lemma A.2, and hence \( \mathbb{D}^\infty_{x, \leq x}(\lambda + \mu) \supset \mathbb{D}^\infty_{x}(\lambda + \mu) \). Therefore, the assertion follows from Lemma 9.1(1). This proves the corollary. \( \blacksquare \)

**Lemma 9.3.** Let \( x, y \in W_{af} \).

1. If \( \Pi^S(y) \geq \Pi^S(x) \) in \( (W^S)_{af} \), then \( y \cdot (t_\xi \cdot \pi_\rho \otimes \pi_\chi) \in \mathbb{D}^\infty_{x, \leq x}(\lambda + \mu) \) for all \( (\rho, \chi, \xi) \in \Par(\lambda) \times \Par(\mu) \times \mathcal{Q}^\nu_{i,(J \cup K)} \) satisfying (7.16).

2. If \( y \cdot (t_\xi \cdot \pi_\rho \otimes \pi_\chi) \in \mathbb{D}^\infty_{x, \leq x}(\lambda + \mu) \) for some \( (\rho, \chi, \xi) \in \Par(\lambda) \times \Par(\mu) \times \mathcal{Q}^\nu_{i,(J \cup K)} \) satisfying (7.16), then \( \Pi^S(y) \geq \Pi^S(x) \).

**Proof.** (1) By the definitions (see (7.2)), \( \pi_\rho \in \mathcal{B}^\infty_{x, \leq x}(\lambda) \) and \( \pi_\chi \in \mathbb{B}^\infty(\mu) \) are of the form:

\[
\begin{align*}
\pi_\rho &= (\Pi^J(t_{\xi_1}), \ldots, \Pi^J(t_{\xi_{s-1}}), e; a), \\
\pi_\chi &= (\Pi^K(t_{\zeta_1}), \ldots, \Pi^K(t_{\zeta_{p-1}}), e; b)
\end{align*}
\] (9.9)

for some \( \xi_1, \ldots, \xi_{s-1} \in Q^\nu_{1,J} \) such that \( \xi_1 > \cdots > \xi_{s-1} > 0 \), and \( \zeta_1, \ldots, \zeta_{p-1} \in Q^\nu_{i,K} \) such that \( \zeta_1 > \cdots > \zeta_{p-1} > 0 \), respectively. Also, recall from (7.3) that

\[
y \cdot (t_\xi \cdot \pi_\rho) = (\Pi^J(t_{\xi_1+\xi}), \ldots, \Pi^J(t_{\xi_{s-1}+\xi}), \Pi^J(t_\xi); a),
\] (9.10)

and from Lemma 7.2 that

\[
y \cdot (t_\xi \cdot \pi_\rho) = (\Pi^J(yt_{\xi_1+\xi}), \ldots, \Pi^J(yt_{\xi_{s-1}+\xi}), \Pi^J(yt_\xi); a),
\] (9.11)

with

\[
y \cdot (t_\xi \cdot \pi_\rho) = (y \cdot (t_\xi \cdot \pi_\rho)) \otimes (y \cdot \pi_\chi),
\]

for some \( \xi_1, \ldots, \zeta_{p-1} \in Q^\nu_{i,J} \) such that \( \xi_1 > \cdots > \zeta_{p-1} > 0 \), and \( \xi_1, \ldots, \zeta_{p-1} \in Q^\nu_{i,K} \) such that \( \xi_1 > \cdots > \zeta_{p-1} > 0 \), respectively.

Now, if \( \chi = (\chi^{(i)})_{i \in I} \in \Par(\mu) \), with \( \chi^{(i)} = (\chi_i^{(i)} \geq \chi_2^{(i)} \geq \cdots) \) for \( i \in I \), then we have \( \zeta_1 = \sum_{i \in I} \chi_i^{(i)} \alpha_i^{\gamma} \) by (7.3); we set \( \gamma := \sum_{i \in J \cup K} \chi_i^{(i)} \alpha_i^{\gamma} \in Q^\nu J \). Since \( (\rho, \chi, \xi) \) satisfies (7.16), and \( \chi_i^{(i)} = 0 \) for all \( i \in K \), we deduce that \( \xi + \gamma \geq \zeta_1 \), and hence that

\[
\xi_1 + \xi + \gamma > \cdots > \xi_{s-1} + \xi + \gamma > \xi + \gamma \geq \zeta_1 > \cdots > \zeta_{p-1} > 0.
\]

Therefore, it follows from Lemma A.5(2) (applied to the case \( J = \emptyset \)) that

\[
yt_{\xi_1+\xi+\gamma} > \cdots > yt_{\xi_{s-1}+\xi+\gamma} > yt_{\xi+\gamma} \geq yt_\xi \geq \cdots > yt_{\zeta_{p-1}} \geq y = yt_0 \quad \text{in } W_{af};
\] (9.12)
note that by Lemma A.1, \( \Pi^J(yt_{\xi_u+\zeta}) = \Pi^J(yt_{\xi_u+\zeta}) \) for all \( 1 \leq u \leq s \) since \( \gamma \in Q^J \).
Hence the sequence (9.12) is a defining chain for \( y \cdot (t_{\xi} \cdot \pi_{\rho} \otimes \pi_{\chi}) \).
Since \( \Pi^S(y) \geq \Pi^S(x) \) in \((W^S)_{af}\) by the assumption, we conclude that \( y \cdot (t_{\xi} \cdot \pi_{\rho} \otimes \pi_{\chi}) \in \mathbb{D}^K_{\geq}(\lambda + \mu) \). This proves part (1).

(2) We divide the proof into several steps.

**Step 1.** Assume that \( x = t_{\zeta'} \) for some \( \zeta' \in Q^J \), and \( y = t_{\xi'} \) for some \( \xi' \in Q^J \); in this case, in order to prove that \( \Pi^S(y) \geq \Pi^S(x) \) in \((W^S)_{af}\), it suffices to show that \( \lceil \xi' \rceil^S \geq \lceil \zeta' \rceil^S \) (see Lemma A.5(2)). In the same way as for (9.11), we obtain

\[
\begin{align*}
y \cdot (t_{\xi} \cdot \pi_{\rho} \otimes \pi_{\chi}) &= (y \cdot (t_{\xi} \cdot \pi_{\rho})) \otimes (y \cdot \pi_{\chi}), \\
y \cdot (t_{\xi} \cdot \pi_{\rho}) &= (\Pi^J(t_{\xi_1+\xi+\zeta}), \ldots, \Pi^J(t_{\xi_{s-1}+\xi+\zeta}), \Pi^J(t_{\xi+\zeta}); \mathbf{a}), \\
y \cdot \pi_{\chi} &= (\Pi^K(t_{\zeta_1+\xi'}), \ldots, \Pi^K(t_{\zeta_{s-1}+\xi'}), \Pi^K(t_{\xi+\zeta}); \mathbf{b}).
\end{align*}
\]

Here, since \( y \cdot (t_{\xi} \cdot \pi_{\rho} \otimes \pi_{\chi}) \in \mathbb{D}^K_{\geq}(\lambda + \mu) \) satisfies condition (D2) (or equivalently, condition (D3); see Section 9.2), we have

\[
\kappa(y \cdot \pi_{\chi}) \geq \Pi^K(x) \quad \text{and} \quad \kappa(y \cdot (t_{\xi} \cdot \pi_{\rho}))) \geq \Pi^J(\iota(y \cdot \pi_{\chi}, x)).
\]

We deduce from the first inequality in (9.14) that \( \Pi^K(t_{\xi'}) = \kappa(y \cdot \pi_{\chi}) \geq \Pi^K(x) = \Pi^K(t_{\zeta'}) \), which implies that \( \lceil \xi' \rceil^K \geq \lceil \zeta' \rceil^K \) by Lemma A.5(2). Since \( I \setminus S = (I \setminus K) \cup (K \setminus S) \), it remains to show that \( \lceil \xi \rceil^K_{S} \geq \lceil \zeta \rceil^K_{S} \). We define \( \tilde{y}_p, \tilde{y}_{p-1}, \ldots, \tilde{y}_1 \) by the recursive procedure (3.1), that is,

\[
\begin{align*}
\tilde{y}_p &:= \min \text{Lift}_{\geq \chi}(\Pi^K(t_{\zeta'})), \quad \text{with} \quad x = t_{\zeta'}, \\
\tilde{y}_{p-1} &:= \min \text{Lift}_{\geq \tilde{y}_p}(\Pi^K(t_{\zeta_{p-1}+\zeta'})), \\
& \vdots \\
\tilde{y}_1 &:= \min \text{Lift}_{\geq \tilde{y}_2}(\Pi^K(t_{\zeta_1+\xi'})) = \iota(y \cdot \pi_{\chi}, x).
\end{align*}
\]

**Claim 1.** The elements \( \tilde{y}_q, 1 \leq q \leq p, \) are of the form: \( \tilde{y}_q = t_{\zeta_q+\xi'+\gamma_q} \) for some \( \gamma_q \in Q^J_K \), where we set \( \zeta_p := 0 \).

**Proof of Claim 1.** We show the assertion by descending induction on \( 1 \leq q \leq p \). Assume that \( q = p \). We see from Lemma B.1 that \( \tilde{y}_p = z_p t_{\xi' + \gamma_p} \) for some \( z_p \in W_K \) and \( \gamma_p \in Q^J_K \). Since \( z_p t_{\xi' + \gamma_p} = \tilde{y}_p \geq x = t_{\zeta'} \), it follows from Lemma A.5(1) and (2) that \( t_{\xi' + \gamma_p} \geq t_{\zeta'} = x \) in \( W_{af} \). Also, we have \( t_{\xi' + \gamma_p} \in \text{Lift}(\Pi^K(t_{\zeta'})) \) by Lemma B.1 Combining these, we obtain \( t_{\xi' + \gamma_p} \in \text{Lift}(\Pi^K(t_{\zeta'})) \). Since \( \tilde{y}_p = z_p t_{\xi' + \gamma_p} \geq t_{\xi' + \gamma_p} \) in \( W_{af} \) by Remark A.6, we deduce that \( \tilde{y}_p = t_{\xi' + \gamma_p} \) by the minimality of \( \tilde{y}_p \).

Assume that \( q < p \); by our induction hypothesis, we have \( \tilde{y}_{q+1} = t_{\zeta_{q+1}+\xi'+\gamma_{q+1}} \) for some \( \gamma_{q+1} \in Q^J_K \). Also, we see from Lemma B.1 that \( \tilde{y}_q = z_q t_{\xi_q+\xi'+\gamma_q} \) for some \( z_q \in W_K \) and \( \gamma_q \in Q^J_K \). Now, the same argument as above shows that

\[
\tilde{y}_q = z_q t_{\xi_q+\xi'+\gamma_q} \geq t_{\xi_q+\xi'+\gamma_q} \geq t_{\xi_{q+1}+\xi'+\gamma_{q+1}} = \tilde{y}_{q+1},
\]

and that \( t_{\xi_q+\xi'+\gamma_q} \in \text{Lift}(\Pi^K(t_{\xi_q+\xi'})) \). Hence we obtain \( \tilde{y}_q = t_{\xi_q+\xi'+\gamma_q} \) by the minimality of \( \tilde{y}_q \). This proves Claim 1.
Because \( \tilde{y}_1 \geq \cdots \geq \tilde{y}_p \geq x \) in \( W_{af} \), it follows from Lemma A.3(2) and Claim I that
\[
\zeta_1 + \xi^t + \gamma_1 \geq \zeta_2 + \xi^t + \gamma_2 \geq \cdots \geq \zeta_p + \xi^t + \gamma_p \geq \zeta^t;
\]
in particular, we have
\[
[\zeta_1 + \xi^t + \gamma_1]_{K \cup S} \geq [\zeta^t]_{K \cup S}. \quad (9.15)
\]
Also, we see by the second inequality in (9.14) and Claim I that
\[
\Pi^t(t_{\zeta^t + \gamma_1}) = \kappa(y \cdot (t_{\zeta^t} \cdot \pi_{\rho})) \geq \Pi^t(\nu(y \cdot \pi_{\chi}, x)) = \Pi^t(t_{\zeta_1 + \xi^t + \gamma_1}),
\]
which implies that \([\xi + \xi^t]^t \geq [\zeta_1 + \xi^t + \gamma_1]^t\) by Lemma A.5(2); in particular, we have \([\xi + \xi^t]_{K \cup S} \geq [\zeta_1 + \xi^t + \gamma_1]_{K \cup S}\). Here, since \( \xi \in Q^\vee_{\ell(J, K)} \), we have \([\xi + \xi^t]_{K \cup S} = [\zeta^t]_{K \cup S}\).
Therefore, we deduce that
\[
[\zeta^t]_{K \cup S} \geq [\zeta_1 + \xi^t + \gamma_1]_{K \cup S}. \quad (9.16)
\]
Combining (9.15) and (9.16), we obtain \([\zeta^t]_{K \cup S} \geq [\zeta^t]_{K \cup S}\), as desired.

**Step 2.** Assume that \( x = t_{\zeta^t} \) for some \( \zeta^t \in Q^\vee \), and write \( y \in W_{af} \) as \( y = vt_{\zeta^t} \) for some \( v \in W \) and \( \xi^t \in Q^\vee \). Let us show the assertion by induction on \( \ell(v) \). If \( \ell(v) = 0 \), i.e., \( v = e \), then the assertion follows from Step 1. Assume that \( \ell(v) > 0 \). We take \( i \in I \) such that \( \ell(s_i v) = \ell(v) - 1 \); note that \( y^{-1} \alpha_i \in -\Delta^+ + \mathbb{Z} \delta \). Since \( \langle y \lambda, \alpha_i^\vee \rangle \leq 0 \) and \( \langle y \mu, \alpha_i^\vee \rangle \leq 0 \), we see by the definition of the root operator \( f_i \) and (9.11) that \( f_i(y \cdot (t_{\zeta^t} \cdot \pi_{\rho} \otimes \pi_{\chi})) = 0 \), and hence that
\[
e_i^{\text{max}}(y \cdot (t_{\zeta^t} \cdot \pi_{\rho} \otimes \pi_{\chi})) = (s_i y) \cdot (t_{\zeta^t} \cdot \pi_{\rho} \otimes \pi_{\chi}). \quad (9.17)
\]
Since \( x = t_{\zeta^t} \), we have \( \langle x(\lambda + \mu), \alpha_i^\vee \rangle = \langle \lambda + \mu, \alpha_i^\vee \rangle \geq 0 \) Therefore, by Lemma 9.1(2), together with (9.17), we obtain \( (s_i y) \cdot (t_{\zeta^t} \cdot \pi_{\rho} \otimes \pi_{\chi}) \in \mathbb{D}_{\geq x}^\vee (\lambda + \mu) \). Hence, by our induction hypothesis, we have \( \Pi^S(s_i y) \geq \Pi^S(x) \). Here we recall that \( \langle y(\lambda + \mu), \alpha_i^\vee \rangle \leq 0 \) since \( y^{-1} \alpha_i \in -\Delta^+ + \mathbb{Z} \delta \). If \( \langle y(\lambda + \mu), \alpha_i^\vee \rangle < 0 \), then \( \Pi^S(y) \geq s_i \Pi^S(y) = \Pi^S(s_i y) \geq \Pi^S(x) \) by Lemma A.2 and Remark A.3. If \( \langle y(\lambda + \mu), \alpha_i^\vee \rangle = 0 \), then \( \Pi^S(y) = \Pi^S(s_i y) \geq \Pi^S(x) \) by Remark A.3. In both cases, we obtain \( \Pi^S(y) \geq \Pi^S(x) \), as desired.

**Step 3.** Let \( x, y \in W_{af} \). We see from [AKK] that there exist \( i_1, \ldots, i_N \in I_{af} \) such that \( \langle s_{i_{n-1}} \cdots s_{i_1} x(\lambda + \mu), \alpha_i^\vee \rangle \geq 0 \) for all \( 1 \leq n \leq N \), and such that \( s_{i_{N}} \cdots s_{i_1} x = t_{\zeta^t} \) for some \( \zeta^t \in Q^\vee \). Let us show the assertion by induction on \( N \). If \( N = 0 \), i.e., \( x = t_{\zeta^t} \), then the assertion follows from Step 2. Assume that \( N > 0 \); for simplicity of notation, we set \( i := i_1 \). It follows from Corollary A.2 that
\[
f_i^{\text{max}}(y \cdot (t_{\zeta^t} \cdot \pi_{\rho} \otimes \pi_{\chi})) \in \mathbb{D}_{\geq s_i x}^\vee (\lambda + \mu). \quad (9.18)
\]
**Case 3.1.** Assume that \( \langle y(\lambda + \mu), \alpha_i^\vee \rangle \leq 0 \); note that \( \langle y \lambda, \alpha_i^\vee \rangle \leq 0 \) and \( \langle y \mu, \alpha_i^\vee \rangle \leq 0 \). We see by the definition of the root operator \( f_i \) and (9.11) that \( f_i^{\text{max}}(y \cdot (t_{\zeta^t} \cdot \pi_{\rho} \otimes \pi_{\chi})) = y \cdot (t_{\zeta^t} \cdot \pi_{\rho} \otimes \pi_{\chi}) \). Hence, by our induction hypothesis, we have \( \Pi^S(y) \geq \Pi^S(s_i x) \). Here we recall that \( \langle x(\lambda + \mu), \alpha_i^\vee \rangle \geq 0 \). If \( \langle x(\lambda + \mu), \alpha_i^\vee \rangle = 0 \), then we have \( \Pi^S(s_i x) = \Pi^S(x) \) by Remark A.3. If \( \langle x(\lambda + \mu), \alpha_i^\vee \rangle > 0 \), then it follows from Lemma A.2 that \( \Pi^S(s_i x) = s_i \Pi^S(x) \geq \Pi^S(x) \). In both cases, we obtain \( \Pi^S(y) \geq \Pi^S(x) \), as desired.
Case 3.2. Assume that \( \langle y(\lambda + \mu), \alpha_i^\vee \rangle > 0 \); note that \( \langle y\lambda, \alpha_i^\vee \rangle \geq 0 \) and \( \langle y\mu, \alpha_i^\vee \rangle \geq 0 \). We see by the definition of the root operator \( e_i \) and \((9.11)\) that \( e_i(y \cdot (t_\xi \cdot \pi_\rho \otimes \pi_\chi)) = 0 \), and hence

\[
 f_i^{\max}(y \cdot (t_\xi \cdot \pi_\rho \otimes \pi_\chi)) = (s_iy) \cdot (t_\xi \cdot \pi_\rho \otimes \pi_\chi).
\]

Hence, by our induction hypothesis, we have \( \Pi^S(s_iy) \succeq \Pi^S(s_i\lambda) \). As in Case 3.1, we see that \( \Pi^S(s_iy) \succeq \Pi^S(s_i\lambda) \), and hence \( \Pi^S(s_iy) \succeq \Pi^S(s_i\lambda) \). Also, since \( \langle y(\lambda + \mu), \alpha_i^\vee \rangle > 0 \), it follows from Lemma \ref{lem:A.2} that \( \Pi^S(s_iy) = s_i\Pi^S(y) \), and hence from Lemma \ref{lem:A.4}(2) that \( \Pi^S(y) \succeq \Pi^S(s_iy) \).

This proves part (2), and completes the proof of Lemma \ref{lem:9.3}. \( \square \)

Now, the equivalence \((D1) \Leftrightarrow (D2)\) follows from the next lemma.

**Lemma 9.4.** Let \( \psi \in B^\infty_\geq \lambda + \mu \), and assume that \( \psi \) is mapped to \( \pi \otimes \eta \in S^\infty_\leq \lambda + \mu \) under the isomorphism \( B^\infty_\geq \lambda + \mu \cong S^\infty_\leq \lambda + \mu \) in Theorem \ref{thm:3.1}. Let \( x \in W_{af} \).

1. If \( \psi \in B^\infty_{\geq x} \lambda + \mu \), then \( \pi \otimes \eta \in D^\infty_{\geq x} \lambda + \mu \).
2. If \( \pi \otimes \eta \in D^\infty_{\geq x} \lambda + \mu \), then \( \psi \in B^\infty_{\geq x} \lambda + \mu \).

**Proof.** By \cite[Lemma 5.4.1]{NS3}, there exist \( i_1, i_2, \ldots, i_N \in I_{af} \) satisfying the conditions that

\[
\left\{ \begin{align*}
\langle s_{i_{n-1}}s_{i_{n-2}} \cdots s_{i_2}s_{i_1}x(\lambda + \mu), \alpha_i^\vee \rangle & \geq 0 \quad \text{for all } 1 \leq n \leq N, \\
n^{\max}_N f^{\max}_N \cdots f^{\max}_{i_2} f^{\max}_{i_1} \psi = t_{\xi'} \cdot \pi_\omega & \quad \text{for some } \xi' \in Q^\vee \text{ and } \omega \in \operatorname{Par}(\lambda + \mu).
\end{align*} \right.
\]

We prove part (1) by induction on \( N \). Assume that \( N = 0 \), i.e., \( \psi = t_{\xi'} \cdot \pi_\omega \); recall from \eqref{eq:7.3} that \( \kappa(\psi) = \Pi^S(t_{\xi'}) \). Since \( \psi \in B^\infty_{\geq x} \lambda + \mu \) by the assumption, we have

\[
\Pi^S(t_{\xi'}) = \kappa(\psi) \succeq \Pi^S(x).
\] (9.19)

By Corollary \ref{cor:7.10}, \( \psi = t_{\xi'} \cdot \pi_\omega \) is mapped to \( t_{\xi'} \cdot (t_\xi \cdot \pi_\rho \otimes \pi_\chi) \), which is \( \pi \otimes \eta \), for some \( \xi \in Q^\vee \setminus \{J \cup K\} \) and \( \rho \in \operatorname{Par}(\lambda), \chi \in \operatorname{Par}(\mu) \) satisfying \eqref{eq:7.16} under the isomorphism \( B^\infty_\geq \lambda + \mu \cong S^\infty_\leq \lambda + \mu \) of crystals in Theorem \ref{thm:3.1}. Therefore, we deduce from Lemma \ref{lem:9.3}(1), together with \eqref{eq:9.19}, that \( \pi \otimes \eta \in D^\infty_{\geq x} \lambda + \mu \).

Assume that \( N > 0 \). For simplicity of notation, we set \( i_1 := i \); note that \( \langle x(\lambda + \mu), \alpha_i^\vee \rangle \geq 0 \). We see from \cite[Corollary 5.3.3]{NS3} that \( f_i^{\max} \psi \in B^\infty_{\geq s_i x} \lambda + \mu \). By our induction hypothesis, we have \( f_i^{\max}(\pi \otimes \eta) \in D^\infty_{\geq s_i x} \lambda + \mu \). Since \( \pi \otimes \eta = e_i^{x} f_i^{\max}(\pi \otimes \eta) \) for some \( k \geq 0 \), we deduce from Lemma \ref{lem:9.1}(3) that \( \pi \otimes \eta \in D^\infty_{\geq x} \lambda + \mu \). This proves part (1).

We can prove part (2) similarly, using Lemma \ref{lem:9.3}(2) instead of Lemma \ref{lem:9.3}(1). \( \square \)

**Appendices.**

**A Basic properties of the semi-infinite Bruhat order.**

We fix \( J \subseteq I \) and \( \lambda \in P^+ \cap P^0_{af} \) (see \eqref{eq:2.1} and \eqref{eq:2.2}) such that \( \{ i \in I \mid \langle \lambda, \alpha_i^\vee \rangle = 0 \} = J \).
Lemma A.1 ([INS Lemmas 2.3.3 and 2.3.5]).

1. It holds that
\[
\begin{align*}
\Pi^J(w) &= [w] & \text{for all } w \in W; \\
\Pi^J(xt_\xi) &= \Pi^J(x)\Pi^J(t_\xi) & \text{for all } x \in W_{af} \text{ and } \xi \in Q^\vee;
\end{align*}
\]
\[\text{in particular,} \quad (W^J)_{af} = \{ w\Pi^J(t_\xi) \mid w \in W^J, \xi \in Q^\vee \}. \quad (A.1)\]

2. For each \(\xi \in Q^\vee\), the element \(\Pi^J(t_\xi)\) is of the form: \(\Pi^J(t_\xi) = z_\xi t_{\xi + \phi_J(\xi)}\) for (a unique) \(z_\xi \in W_J\) and \(\phi_J(\xi) \in Q^\vee_J\).

3. For \(\xi, \zeta \in Q^\vee\),
\[\Pi^J(t_\xi) = \Pi^J(t_\zeta) \iff \xi - \zeta \in Q^\vee_J. \quad (A.3)\]

Lemma A.2 ([INS Remark 4.1.3]). Let \(x \in (W^J)_{af}\), and \(i \in I_{af}\). Then,
\[s_i x \in (W^J)_{af} \iff \langle x, \alpha_i^J \rangle \neq 0 \iff x^{-1}\alpha_i \in (\Delta^+ \setminus \Delta_J) + Z\delta. \quad (A.4)\]
Moreover, in this case,
\[
\begin{align*}
x \xrightarrow{a_i} s_i x & \iff \langle x, \alpha_i^J \rangle > 0 \iff x^{-1}\alpha_i \in (\Delta^+ \setminus \Delta_J^+) + Z\delta, \\
s_i x \xrightarrow{a_i} x & \iff \langle x, \alpha_i^J \rangle < 0 \iff x^{-1}\alpha_i \in -(\Delta^+ \setminus \Delta_J^+) + Z\delta.
\end{align*}
\]

Remark A.3. Keep the setting of Lemma A.2. If \(x^{-1}\alpha_i \in \Delta_J + Z\delta\), i.e., \(\langle x, \alpha_i^J \rangle = 0\), then \(\Pi^J(s_i x) = x\).

Lemma A.4 ([INS Lemma 2.3.6]). Let \(x, y \in (W^J)_{af}\) be such that \(x \preceq y\), and let \(i \in I_{af}\).

1. If \(\langle x, \alpha_i^J \rangle > 0\) and \(\langle y, \alpha_i^J \rangle \leq 0\), then \(s_i x \preceq y\).

2. If \(\langle x, \alpha_i^J \rangle \geq 0\) and \(\langle y, \alpha_i^J \rangle < 0\), then \(x \preceq s_i y\).

3. If \(\langle x, \alpha_i^J \rangle > 0\) and \(\langle y, \alpha_i^J \rangle > 0\), or if \(\langle x, \alpha_i^J \rangle < 0\) and \(\langle y, \alpha_i^J \rangle < 0\), then \(s_i x \preceq s_i y\).

Lemma A.5 ([INS Lemmas 4.3.3–4.3.5]).

1. Let \(w, v \in W^J\), and \(\xi, \zeta \in Q^\vee\). If \(w\Pi^J(t_\xi) \succeq v\Pi^J(t_\zeta)\), then \([\xi]^J \succeq [\zeta]^J\), where \([\cdot]^J : Q^\vee \to Q^\vee_{\Delta^+_J}\) is the projection in (2.6).

2. Let \(w \in W^J\), and \(\xi, \zeta \in Q^\vee\). Then, \(w\Pi^J(t_\xi) \succeq w\Pi^J(t_\zeta)\) if and only if \([\xi]^J \succeq [\zeta]^J\).

3. Let \(x, y \in (W^J)_{af}\) and \(\beta \in \Delta^+_J\) be such that \(x \xrightarrow{\beta} y\) in \(BG^\Delta(W^J)_{af}\). Then, \(\Pi^J(xt_\xi) \xrightarrow{\beta} \Pi^J(yt_\xi)\) in \(BG^\Delta((W^J)_{af})\) for all \(\xi \in Q^\vee\). Therefore, if \(x \succeq y\), then \(\Pi^J(xt_\xi) \succeq \Pi^J(yt_\xi)\) for all \(\xi \in Q^\vee\).

Remark A.6. Let \(w \in W\). Since \(w \succeq e\) in the ordinary Bruhat order on \(W\), we see that \(w \succeq e\) in the semi-infinite Bruhat order on \(W_{af}\). Hence it follows from Lemma A.5(3) that \(wt_\xi \succeq t_\xi\) for all \(\xi \in Q^\vee\).
Lemma A.7. Let \( x, y \in (W^J)_{af} \) be such that \( x \preceq y \) in \((W^J)_{af} \). Then, \( xz \preceq yz \) in \( W_{af} \) for all \( z \in (W^J)_{af} \).

Proof. Let \( z \in (W^J)_{af} \). We know from [P] (see also [A] Theorem 3.3) that \( \ell \bar{\pi}(xz) = \ell \bar{\pi}(x) + \ell \bar{\pi}(z) \) and \( \ell \bar{\pi}(yz) = \ell \bar{\pi}(y) + \ell \bar{\pi}(z) \). Also, we may assume that \( x \xrightarrow{\beta} y \) in \( \text{BG} \bar{\pi}((W^J)_{af}) \) for some \( \beta \in \Delta_{af}^+ \); by the definition of \( \text{BG} \bar{\pi}((W^J)_{af}) \), we have \( y = s_\beta x \), with \( \ell \bar{\pi}(y) = \ell \bar{\pi}(x) + 1 \). Therefore, \( yz = s_\beta xz \), and

\[
\ell \bar{\pi}(yz) = \ell \bar{\pi}(y) + \ell \bar{\pi}(z) = \ell \bar{\pi}(x) + 1 + \ell \bar{\pi}(z) = \ell \bar{\pi}(xz) + 1.
\]

Thus, we obtain \( xz \xrightarrow{\beta} yz \) in \( \text{BG} \bar{\pi}(W_{af}) \), as desired.

\( \Box \)

Lemma A.8 ([INS] Lemma 6.1.1]). If \( x, y \in W_{af} \) satisfy \( x \succeq y \), then \( \Pi^J(x) \succeq \Pi^J(y) \).

B Proof of Proposition 2.4.

Lemma B.1. Let \( x \in (W^J)_{af} \), and write it as: \( x = w\Pi^J(t_\xi) \in (W^J)_{af} \) for some \( w \in W^J \) and \( \xi \in Q^\vee \) (see (A.2)). Then, \( \text{Lift}(x) = \{ w't_{\xi+\gamma} \mid w' \in wW_J, \gamma \in Q^\vee_J \} \).

Proof. We set \( L := \{ w't_{\xi+\gamma} \mid w' \in wW_J, \gamma \in Q^\vee_J \} \). We first prove that \( L \subseteq \text{Lift}(x) \).

Let \( w't_{\xi+\gamma} \in L \). Then we see from (A.1) and (A.3) that \( \Pi^J(w't_{\xi+\gamma}) = \Pi^J(w')\Pi^J(t_{\xi+\gamma}) = [w']\Pi(t_\xi) = w\Pi(t_\xi) = x \). This proves the inclusion \( L \subseteq \text{Lift}(x) \).

We next show that \( L \supseteq \text{Lift}(x) \). Each element of \( \text{Lift}(x) \) is of the form \( xz \) for some \( z \in (W^J)_{af} = W_J \times Q^\vee_J \); we write \( z \) as \( z = v_1t_{\gamma_1} \) for some \( v_1 \in W_J \) and \( \gamma_1 \in Q^\vee_J \). Since \( \Pi^J(t_{\xi+\gamma}) = v_2t_{\xi+\gamma_2} \) for some \( v_2 \in W_J \) and \( \gamma_2 \in Q^\vee_J \), we have \( xz = (w\Pi^J(t_\xi))(v_1t_{\gamma_2}) = w(v_2t_{\xi+\gamma_2})v_1t_{\gamma_1} \), which is of the form \( wv't_{\xi+\gamma} \) for some \( v \in W_J \) and \( \gamma \in Q^\vee_J \). Thus, this element is contained in \( L \). This proves the opposite inclusion, and hence the lemma.

\( \Box \)

Now, we give a proof of Proposition 2.4. If \( J = I \), then the assertion is obvious. Hence we may assume that \( J \nsubseteq I \).

Step 1. Assume that \( x = t_\xi \) for some \( \xi \in Q^\vee \) and \( y = \Pi^J(t_\xi) \) for some \( \zeta \in Q^\vee \); since \( \Pi^J(t_\xi) = w \Pi^J(t_\xi) = \Pi^J(t_\xi) \) by the assumption, it follows from Lemma A.5(1) that \( [\xi]^J \succeq [\xi]^J \), where \( [\cdot]^J : Q^\vee \to Q^\vee_J \) is the projection in (2.6). We set \( \gamma := [\zeta - \xi]^J \in Q^\vee_J \), where \( [\cdot]^J : Q^\vee \to Q^\vee_J \) is the projection in (2.6); note that \( [\zeta - \xi]^J = [\xi]^J \). We claim that \( t_{\zeta - \gamma} \) is the minimum element of \( \text{Lift}_{\geq x}(y) \). It is clear by Lemma B.1 that \( t_{\zeta - \gamma} \in \text{Lift}(y) \).

In addition, since \( [\zeta - \gamma]^J \succeq [\zeta - \gamma]^J \) and \( [\zeta - \gamma]^J = [\xi]^J \), we have \( \zeta - \gamma \preceq \xi \), and hence \( t_{\zeta - \gamma} \preceq t_\xi = x \) by Lemma A.5(2). Thus, \( t_{\zeta - \gamma} \in \text{Lift}_{\geq x}(y) \). Now, by Lemma B.1 each element \( y' \in \text{Lift}_{\geq x}(y) \) is of the form \( y' = v't_{\zeta - \gamma} \) for some \( v' \in W_J \) and \( \gamma' \in Q^\vee_J \). Since \( v't_{\zeta - \gamma} = y' \succeq x = t_\xi \) in \( W_{af} \) by the assumption, we deduce from Lemma A.5(1) that \( \zeta - \gamma' \succeq \xi \); in particular, \( [\zeta - \gamma']^J \succeq [\xi]^J \). Here, since \( \gamma = [\zeta - \xi]^J \) by the definition, we have \( [\zeta - \gamma']^J \succeq [\xi]^J \). Also, since \( \gamma, \gamma' \in Q^\vee_J \), we have \( [\zeta - \gamma']^J = [\zeta - \gamma]^J \). Combining these, we obtain \( \zeta - \gamma' \preceq \zeta - \gamma \). Therefore, by Remark A.6 and Lemma A.5(2), \( y' = v't_{\zeta - \gamma} \succeq t_{\zeta - \gamma} \preceq t_{\zeta - \gamma} \). Thus, \( t_{\zeta - \gamma} \) is the minimum element of \( \text{Lift}_{\geq x}(y) \).

\( \Box \)

In the following, we fix \( \Lambda \in P^+ \) and \( \lambda \in P^+ \) such that \( \{ i \in I \mid \langle \Lambda, \alpha_i^\vee \rangle = 0 \} = \emptyset \) and \( \{ i \in I \mid \langle \lambda, \alpha_i^\vee \rangle = 0 \} = J \). Note that \( \langle \Lambda, \beta^\vee \rangle \neq 0 \) for all \( \beta \in \Delta_{af}^+ \).

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Step 2. Let $x \in W_{af}$, and assume that $y = \Pi^J(t_\xi)$ for some $\xi \in Q^\vee$. We deduce from [AkK] that there exist $i_1, \ldots, i_N \in I_{af}$ such that $\langle s_{i_n} \cdots s_{i_1} x, \lambda, \alpha_i \rangle > 0$ for all $1 \leq n \leq N$, and such that $s_{i_N} \cdots s_{i_1} x = t_\xi$ for some $\xi \in Q^\vee$. We show the assertion of the proposition by induction on $N$. If $N = 0$, then the assertion follows from Step 1. Assume that $N \geq 1$; for simplicity of notation, we set $i := i_1 \in I_{af}$. Since

$$\langle x, \alpha_i \rangle > 0$$

by the assumption, it follows that $x^{-1} \alpha_i \in \Delta^+$, and hence $\langle \Pi^J(x) \lambda, \alpha_i \rangle = \langle x, \alpha_i \rangle > 0$. Also, by Lemma A.2 and Remark A.3,

$$\Pi^J(s_i x) = \begin{cases} s_i \Pi^J(x) & \text{if } \langle x, \alpha_i \rangle > 0, \\ \Pi^J(x) & \text{if } \langle x, \alpha_i \rangle = 0. \end{cases} \quad \text{(B.2)}$$

Case 2.1. Assume that $i \in I \setminus J$, and hence $\langle y, \alpha_i \rangle = \langle \lambda, \alpha_i \rangle > 0$; note that $s_i y \in (W^J)_{af}$ by Lemma A.2. We first claim that $s_i y \geq \Pi^J(s_i x)$. Indeed, if $\langle \Pi^J(x) \lambda, \alpha_i \rangle = \langle x, \alpha_i \rangle > 0$, then it follows from Lemma A.4 (3), (B.2), and the assumption $y \geq \Pi^J(x)$ that $s_i y \geq s_i \Pi^J(x) = \Pi^J(s_i x)$. If $\langle \Pi^J(x) \lambda, \alpha_i \rangle = \langle x, \alpha_i \rangle = 0$, then it follows from Lemma A.2 (B.2) and the assumption $y \geq \Pi^J(x)$ that $s_i y \geq \Pi^J(x)$ that $s_i y \geq \Pi^J(x) = \Pi^J(s_i x)$. In both cases, we obtain $s_i y \geq \Pi^J(s_i x)$, as desired. Hence, by our induction hypothesis,

$$\text{Lift}_{s_i} (s_i y) = \{ y'' \in W_{af} \mid \Pi^J(y'') = s_i y \text{ and } y'' \geq s_i x \}$$

has the minimum element $y''_{\text{min}}$. We next claim that $s_i y''_{\text{min}} \in \text{Lift}_{s_i} (y)$. Indeed, since $\langle y''_{\text{min}}, \alpha_i \rangle = \langle y, \alpha_i \rangle = \langle \lambda, \alpha_i \rangle > 0$ by our assumption, we see that $(y''_{\text{min}})^{-1} \alpha_i \in \Delta^+ \oplus \mathbb{Z} \delta$, and hence $\langle y^\prime, \alpha_i \rangle > 0$. Also, since $\langle x, \alpha_i \rangle > 0$ by (B.1), it follows from Lemma A.3 (3) that $s_i y' \geq s_i x$, which implies that $s_i y' \in \text{Lift}_{s_i} (s_i y)$. Therefore, we obtain $s_i y' \geq y''_{\text{min}}$. Since $\langle y''_{\text{min}}, \alpha_i \rangle > 0$ as seen above, and $\langle s_i y', \alpha_i \rangle < 0$, we deduce from Lemma A.3 (3) that $y' \geq s_i y''_{\text{min}}$. This shows (B.3).

Case 2.2. Assume that $i \in J$, and hence $\langle y, \alpha_i \rangle = \langle \lambda, \alpha_i \rangle = 0$. We first claim that $y \geq \Pi^J(s_i x)$. Indeed, if $\langle \Pi^J(x) \lambda, \alpha_i \rangle = \langle x, \alpha_i \rangle > 0$, then it follows from Lemma A.4 (1), (B.2), and the assumption $y \geq \Pi^J(x)$ that $y \geq s_i \Pi^J(x) = \Pi^J(s_i x)$. If $\langle \Pi^J(x) \lambda, \alpha_i \rangle = \langle x, \alpha_i \rangle = 0$, then $y \geq \Pi^J(x) = \Pi^J(s_i x)$ by (B.2). In both cases, we obtain $y \geq \Pi^J(s_i x)$, as desired. Hence, by our induction hypothesis,

$$\text{Lift}_{s_i} (y) = \{ y'' \in W_{af} \mid \Pi^J(y'') = y \text{ and } y'' \geq s_i x \}$$

has the minimum element $y''_{\text{min}}$. We set

$$y'_{\text{min}} := \begin{cases} y''_{\text{min}} & \text{if } \langle y''_{\text{min}}, \alpha_i \rangle > 0, \\ s_i y''_{\text{min}} & \text{if } \langle y''_{\text{min}}, \alpha_i \rangle < 0; \end{cases} \quad \text{(B.3)}$$

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remark that $y'_m \leq y''_m$ by Lemma [A.2]. First, we show that $y'_m \in \text{Lift}_{z,x}(y)$. Since $y''_m \in \text{Lift}(y)$ and $i \in J$, it follows from Remark [A.3] that $y''_m \in \text{Lift}(y)$. Also, since $\langle x\Lambda, \alpha_i^\vee \rangle > 0$, we see by Lemma [A.2] that $s_i x \succeq x$. Hence we have $y'_m = y''_m \succeq s_i x \succeq x$ if $\langle y''_m \Lambda, \alpha_i^\vee \rangle > 0$. If $\langle y''_m \Lambda, \alpha_i^\vee \rangle < 0$, then we deduce from Lemma [A.4](3) that $y'_m = s_i y''_m \succeq s_i(s_i x) = x$ since $y''_m \succeq s_i x$. Thus, in both cases, we obtain $y'_m \in \text{Lift}_{z,x}(y)$, as desired. Next, we show that

$$y'_m \text{ is the minimal element of } \text{Lift}_{z,x}(y). \quad (B.4)$$

Let $y' \in \text{Lift}_{z,x}(y)$. If $\langle y' \Lambda, \alpha_i^\vee \rangle < 0$, then it follows from Lemma [A.4](1) that $y' \succeq s_i x$, and hence $y' \in \text{Lift}_{z,s_i x}(y)$. This implies that $y' \succeq y'_m$. If $\langle y''_m \Lambda, \alpha_i^\vee \rangle > 0$, then we have $y' \succeq y''_m = y'_m$ by the definition. If $\langle y''_m \Lambda, \alpha_i^\vee \rangle < 0$, then we see from Lemma [A.2] that $y' \succeq y'_m \succeq s_i y''_m = y'_m$. Assume now that $\langle y' \Lambda, \alpha_i^\vee \rangle > 0$. Since $\langle x\Lambda, \alpha_i^\vee \rangle > 0$, it follows from Lemma [A.4](3) that $s_i y' \succeq s_i x$. In addition, since $i \in J$ and $y' \in \text{Lift}(y)$, we see from Remark [A.3] that $\Pi^J(s_i y') = y$. Hence we obtain $s_i y' \in \text{Lift}_{z,s_i x}(y)$, so that $s_i y' \succeq y'_m$; note that $\langle s_i y' \Lambda, \alpha_i^\vee \rangle < 0$ by our assumption. If $\langle y''_m \Lambda, \alpha_i^\vee \rangle < 0$, then we deduce from Lemma [A.4](2) that $y' \succeq y''_m = y'_m$. If $\langle y''_m \Lambda, \alpha_i^\vee \rangle < 0$, then we deduce from Lemma [A.4](3) that $y' \succeq y''_m = y'_m$. Thus, in all cases, we have shown that $y' \succeq y'_m$, as desired.

**Case 2.3.** Assume that $i = 0$. In this case, we have $\langle y \lambda, \alpha_0^\vee \rangle = \langle \lambda, \alpha_0^\vee \rangle < 0$ since $\alpha_0 = -\theta + \delta$, where $\theta \in \Delta^+$ is the highest root. By the same argument as that at the beginning of Case 2.2, we see that $y \succeq \Pi^J(s_0 x)$. Hence, by the induction hypothesis,

$$\text{Lift}_{z,s_0 x}(y) = \{ y'' \in W_{af} \mid \Pi^J(y'') = y \text{ and } y'' \succeq s_0 x \}$$

has the minimum element $y''_m$. Since $\langle x\Lambda, \alpha_0^\vee \rangle > 0$ by (B.1), it follows from Lemma [A.2] that $s_0 x \succeq x$, which implies that $y''_m \in \text{Lift}_{z,x}(y)$. Here we claim that

$$y''_m \text{ is the minimal element of } \text{Lift}_{z,x}(y).$$

Let $y' \in \text{Lift}_{z,x}(y)$. Then we have $\langle y' \Lambda, \alpha_0^\vee \rangle < 0$. Indeed, we deduce from Lemma [B.1] that $y' = z t_{\zeta+\gamma}$ for some $z \in W_J$ and $\gamma \in Q^\vee_J$. Since $z \in W_J$ and $\theta \in \Delta^+ \setminus \Delta^+_J$ (recall that $J \subseteq I$), we see that $z^{-1} \theta \in \Delta^+ \setminus \Delta^+_J$, and hence that $\langle y' \Lambda, \alpha_0^\vee \rangle = \langle \Lambda, -z^{-1} \theta \gamma \rangle < 0$. Since $\langle x\Lambda, \alpha_0^\vee \rangle > 0$ by (B.1), it follows from Lemma [A.4](1) that $y' \succeq s_0 x$, and hence $y' \in \text{Lift}_{z,s_0 x}(y)$. This shows that $y' \succeq y''_m$. \hfill \qed

**Step 3.** Let $x \in W_{af}$, $y \in (W^J)_{af}$, and write $y$ as $y = v \Pi^J(t_{\zeta})$ for some $v \in W^J$ and $\zeta \in Q^\vee$. We show the assertion by induction on $\ell(v)$. If $\ell(v) = 0$, then the assertion follows from Step 2. Assume that $\ell(v) \geq 1$, and take $i \in I$ such that $\langle v \lambda, \alpha_i^\vee \rangle < 0$; note that in this case, $v^{-1} \alpha_i \in -(\Delta^+ \setminus \Delta^+_J)$, and $s_i v \in W^J$ (see, for example, Lemmas 5.8 and 5.9), and hence that $s_i y \in (W^J)_{af}$. Also, for all $y' \in \text{Lift}(y)$, we have $\langle y', \alpha_i^\vee \rangle = \langle y \lambda, \alpha_i^\vee \rangle = \langle v \lambda, \alpha_i^\vee \rangle < 0$, which implies that

$$\langle y' \Lambda, \alpha_i^\vee \rangle < 0. \quad (B.5)$$
Case 3.1. Assume that \( \langle xA, \alpha \rangle > 0 \); note that \( \langle \Pi^I(x), \alpha_i \rangle = \langle x \lambda, \alpha_i \rangle \geq 0 \). Since \( \langle y \lambda, \alpha \rangle = \langle v \lambda, \alpha \rangle < 0 \), it follows from Lemma A.4 (2) that \( s_i y \geq \Pi^I(x) \). Hence, by our induction hypothesis,

\[
\Lift_{\geq x}(s_i y) = \{ y'' \in \Waf \mid \Pi^I(y'') = s_i y \text{ and } y'' \succeq x \}
\]

has the minimum element \( y''_{\min} \). Since \( \langle y_{\min} \lambda, \alpha \rangle = \langle s_i y \lambda, \alpha \rangle > 0 \), it follows that \( (y''_{\min})^{-1} \alpha_i \in \Delta^+ \), and hence \( \langle y''_{\min} \lambda, \alpha \rangle > 0 \). This implies that \( s_i y''_{\min} \succeq y''_{\min} = x \) by Lemma A.2. In addition, we see by Lemma A.2 that \( \Pi^I(s_i y''_{\min}) = s_i \Pi^I(y''_{\min}) = s_i (s_i y) = y \). Therefore, we conclude that \( s_i y''_{\min} \in \Lift_{\geq x}(y) \). Here we claim that

\( s_i y''_{\min} \) is the minimum element of \( \Lift_{\geq x}(y) \). \hfill (B.6)

Let \( y' \in \Lift_{\geq x}(y) \). Since \( \langle xA, \alpha \rangle > 0 \) by the assumption, and \( \langle y \lambda, \alpha \rangle < 0 \) by (B.5), we deduce from Lemma A.4 (2) that \( s_i y' \succeq x \). In addition, we see by Lemma A.2 that \( \Pi^I(s_i y') = s_i \Pi^I(y') = s_i y \), which implies that \( s_i y' \in \Lift_{\geq x}(s_i y) \), and hence \( s_i y' \succeq y''_{\min} \). Because \( \langle y''_{\min} \lambda, \alpha \rangle > 0 \) and \( \langle s_i y' \lambda, \alpha \rangle > 0 \), it follows from Lemma A.4 (3) that \( y' \succeq s_i y''_{\min} \). This shows (B.6).

Case 3.2. Assume that \( \langle xA, \alpha \rangle < 0 \); note that \( \langle y \lambda, \alpha \rangle \leq 0 \). Since \( \langle y \lambda, \alpha \rangle < 0 \), it follows from Lemma A.4 (2) and (3), together with Lemma A.2 and Remark A.3, that \( s_i y \geq \Pi^I(s_i x) \). Hence, by our induction hypothesis,

\[
\Lift_{\geq x}(s_i y) = \{ y'' \in \Waf \mid \Pi^I(y'') = s_i y \text{ and } y'' \succeq s_i x \}
\]

has the minimum element \( y''_{\min} \); as in Case 3.1, we obtain \( \langle y''_{\min} \lambda, \alpha \rangle > 0 \). Since \( \langle s_i x \lambda, \alpha \rangle > 0 \), we deduce from Lemma A.4 (3) that \( s_i y''_{\min} \succeq x \). In addition, we see by Lemma A.2 that \( \Pi^I(s_i y''_{\min}) = s_i \Pi^I(y''_{\min}) = s_i (s_i y) = y \), which implies that \( s_i y''_{\min} \in \Lift_{\geq x}(y) \). Here we claim that

\( s_i y''_{\min} \) is the minimum element of \( \Lift_{\geq x}(y) \). \hfill (B.7)

Let \( y' \in \Lift_{\geq x}(y) \). Since \( \langle y \lambda, \alpha \rangle = \langle y \lambda, \alpha \rangle \leq 0 \), it follows that \( \langle y' \lambda, \alpha \rangle < 0 \). Also, since \( \langle xA, \alpha \rangle < 0 \) by the assumption, we deduce from Lemma A.4 (3) that \( s_i y' \succeq s_i x \). In addition, we see by Lemma A.2 that \( \Pi^I(s_i y') = s_i \Pi^I(y') = s_i y \), which implies that \( s_i y' \in \Lift_{\geq x}(s_i y) \), and hence \( s_i y' \succeq y''_{\min} \). Because \( \langle y''_{\min} \lambda, \alpha \rangle > 0 \) and \( \langle s_i y' \lambda, \alpha \rangle > 0 \), it follows from Lemma A.4 (3) that \( y' \succeq s_i y''_{\min} \). This shows (B.7).

This completes the proof of Proposition 2.4.

C Crystal structure on \( \mathbb{B}_\infty^\infty(\lambda) \).

We fix \( \lambda \in P^+ \subset P^0_{af} \) (see (2.1) and (2.2)). Let

\[
\pi = (x ; a) = (x_1, \ldots, x_s ; a_0, a_1, \ldots, a_s) \in \mathbb{B}_\infty(\lambda). \hfill (C.1)
\]

Define \( \pi : [0, 1] \to \mathbb{R} \otimes_{\mathbb{Z}} P_{af} \) to be the piecewise-linear, continuous map whose “direction vector” on the interval \([a_{u-1}, a_u]\) is \( x_u \lambda \in P_{af} \) for each \( 1 \leq u \leq s \), that is,

\[
\pi(t) := \sum_{k=1}^{u-1} (a_k - a_{k-1}) x_k \lambda + (t - a_{u-1}) x_u \lambda \quad \text{for } t \in [a_{u-1}, a_u], \ 1 \leq u \leq s. \hfill (C.2)
\]
We know from [INS Proposition 3.1.3] that $\pi$ is an (ordinary) LS path of shape $\lambda$, introduced in [Li Sect. 4]. We set

$$\text{wt}(\pi) := \pi(1) = \sum_{u=1}^{s} (a_u - a_{u-1}) x_u \lambda \in P_{af}.$$  \hfill (C.3)

Now, we define root operators $e_i$, $f_i$, $i \in I_{af}$. Set

$$\begin{align*}
H_i^\pi(t) &:= \langle \pi(t), \alpha_i^\vee \rangle \quad \text{for } t \in [0, 1], \\
m_i^\pi &:= \min \{ H_i^\pi(t) \mid t \in [0, 1] \}.
\end{align*}$$  \hfill (C.4)

As explained in [NS3 Remark 2.4.3], all local minima of the function $H_i^\pi(t)$, $t \in [0, 1]$, are integers; in particular, the minimum value $m_i^\pi$ is a nonpositive integer (recall that $\pi(0) = 0$, and hence $H_i^\pi(0) = 0$).

We define $e_i \pi$ as follows. If $m_i^\pi = 0$, then we set $e_i \pi := 0$, where 0 is an additional element not contained in any crystal. If $m_i^\pi \leq -1$, then we set

$$\begin{align*}
t_1 &:= \min \{ t \in [0, 1] \mid H_i^\pi(t) = m_i^\pi \}, \\
t_0 &:= \max \{ t \in [0, t_1] \mid H_i^\pi(t) = m_i^\pi + 1 \};
\end{align*}$$  \hfill (C.5)

notice that $H_i^\pi(t)$ is strictly decreasing on the interval $[t_0, t_1]$. Let $1 \leq p \leq q \leq s$ be such that $a_{p-1} \leq t_0 < a_p$ and $t_1 = a_q$. Then we define $e_i \pi$ to be

$$e_i \pi := (x_1, \ldots, x_p, s_i x_p, s_i x_{p+1}, \ldots, s_i x_q, x_{q+1}, \ldots, x_s;$$
$$a_0, \ldots, a_{p-1}, t_0, a_p, \ldots, a_q = t_1, \ldots, a_s);$$

if $t_0 = a_{p-1}$, then we drop $x_p$ and $a_{p-1}$, and if $s_i x_q = x_{q+1}$, then we drop $x_{q+1}$ and $a_q = t_1$.

Similarly, we define $f_i \pi$ as follows. Note that $H_i^\pi(1) - m_i^\pi$ is a nonnegative integer. If $H_i^\pi(1) - m_i^\pi = 0$, then we set $f_i \pi := 0$. If $H_i^\pi(1) - m_i^\pi \geq 1$, then we set

$$\begin{align*}
t_0 &:= \max \{ t \in [0, 1] \mid H_i^\pi(t) = m_i^\pi \}, \\
t_1 &:= \min \{ t \in [t_0, 1] \mid H_i^\pi(t) = m_i^\pi + 1 \};
\end{align*}$$  \hfill (C.7)

notice that $H_i^\pi(t)$ is strictly increasing on the interval $[t_0, t_1]$. Let $0 \leq p \leq q \leq s - 1$ be such that $t_0 = a_p$ and $a_q < t_1 \leq a_{q+1}$. Then we define $f_i \pi$ to be

$$f_i \pi := (x_1, \ldots, x_p, s_i x_{p+1}, \ldots, s_i x_q, s_i x_{q+1}, x_{q+1}, \ldots, x_s;$$
$$a_0, \ldots, a_p = t_0, \ldots, a_q, t_1, a_{q+1}, \ldots, a_s);$$

if $t_1 = a_{q+1}$, then we drop $x_{q+1}$ and $a_{q+1}$, and if $x_p = s_i x_{p+1}$, then we drop $x_p$ and $a_p = t_0$.

In addition, we set $e_i 0 = f_i 0 := 0$ for all $i \in I_{af}$.

**Theorem C.1** (see [INS Theorem 3.1.5]).

1. The set $\mathbb{P}^\pi(\lambda) \cup \{0\}$ is stable under the action of the root operators $e_i$ and $f_i$, $i \in I_{af}$.
(2) For each \( \pi \in B_\geq^\infty (\lambda) \) and \( i \in I_{af} \), we set \[
\begin{aligned}
\varepsilon_i(\pi) &:= \max\{ n \geq 0 \mid e^n_i \pi \neq 0 \}, \\
\varphi_i(\pi) &:= \max\{ n \geq 0 \mid f^n_i \pi \neq 0 \}.
\end{aligned}
\]

Then, the set \( B_\geq^\infty (\lambda) \), equipped with the maps \( \text{wt}, e_i, f_i, i \in I_{af} \), and \( \varepsilon_i, \varphi_i, i \in I_{af} \), defined above, is a crystal with weights in \( P_{af} \).

Remark C.2. Let \( \pi \in B_\geq^\infty (\lambda) \), and \( i \in I_{af} \). If \( e_i \pi \neq 0 \), then we deduce from the definition of the root operator \( e_i \) that \( m_i^{e_i \pi} = m_i^\pi + 1 \). Hence it follows that \( \varepsilon_i(\pi) = -m_i^\pi \). Similarly, we have \( \varphi_i(\pi) = H_i^\pi (1) - m_i^\pi \).

## D A formula for graded characters of Demazure sub-modules.

**Proposition D.1.** For each \( x \in W_{af} \) and \( \xi \in Q^\vee \), there holds the equality \[
gch V_{xt_\xi}^-(\lambda) = q^{-\langle \lambda, \xi \rangle} \text{gch} V_x^- (\lambda). \tag{D.1}
\]

**Proof.** Let \( \pi = (x_1, \ldots, x_s ; a) \in B_\geq^\infty (\lambda) \). We see that \[
\begin{aligned}
\pi \in B_\geq^\infty (\lambda) &\Rightarrow x_s \geq \Pi^I (x) = \Pi^I (x_st_\xi) \geq \Pi^I (\Pi^I (x)t_\xi) \quad \text{by Lemma A.5 (3)} \\
&\Rightarrow \Pi^I (x_st_\xi) \geq \Pi^I (xt_\xi) \quad \text{by (A.1)} \\
&\Rightarrow T_\xi(\pi) \in B_\geq^\infty (\lambda);
\end{aligned}
\]
for the definition of \( T_\xi \), see (7.8). From this, we conclude that \( T_\xi (B_\geq^\infty _{\geq xt_\xi} (\lambda)) \subset B_\geq^\infty _{\geq xt_\xi} (\lambda) \). Replacing \( x \) by \( xt_\xi \), and \( T_\xi \) by \( T_{-\xi} \), we obtain \( T_{-\xi} (B_\geq^\infty _{\geq xt_\xi} (\lambda)) \subset B_\geq^\infty _{\leq x} (\lambda) \), and hence \( B_\geq^\infty _{\geq xt_\xi} (\lambda) \subset T_\xi (B_\geq^\infty _{\leq x} (\lambda)) \). Combining these, we conclude that \( T_\xi (B_\geq^\infty _{\leq x} (\lambda)) = B_\geq^\infty _{\geq xt_\xi} (\lambda) \). Therefore, using (2.23), we compute:

\[
\begin{aligned}
gch V_{xt_\xi}^- (\lambda) &= \sum_{\pi \in B_\geq^\infty _{\geq xt_\xi} (\lambda)} e^{\text{fin}(\text{wt}(\pi))} q^{\text{mul}(\text{wt}(\pi))} \\
&= \sum_{\pi \in B_\geq^\infty _{\leq x} (\lambda)} e^{\text{fin}(\text{wt}(\pi)) - \langle \lambda, \xi \rangle} q^{\text{mul}(\text{wt}(\pi)) - \langle \lambda, \xi \rangle} \quad \text{by (7.8)} \\
&= \sum_{\pi \in B_\geq^\infty _{\leq x} (\lambda)} e^{\text{fin}(\text{wt}(\pi)) - \langle \lambda, \xi \rangle} q^{\text{mul}(\text{wt}(\pi))} - \langle \lambda, \xi \rangle = q^{-\langle \lambda, \xi \rangle} \text{gch} V_x^- (\lambda).
\end{aligned}
\]

This proves the proposition. \( \square \)
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