Fractons, geometrically

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We initiate a systematic study of fracton physics within the geometric framework of Double Field Theory. We ascribe the immobility and large degeneracy of the former to the non-Riemannian backgrounds of the latter, in terms of generalised geodesics and infinite-dimensional isometries. A doubled pure Yang–Mills or Maxwell theory reduces to an ordinary one coupled to a strain tensor of elasticity theory, and thus rather remarkably provides a unifying description of photons and phonons. Upon a general Double Field Theory background, which consists of Riemannian and non-Riemannian subspaces, the dual photon-phonon pair becomes fractonic over the non-Riemannian subspace. When the elasticity displacement vector condenses, minimally coupled charged particles acquire an effective mass even in the purely Riemannian case, yielding predictions for polaron physics and time crystals. Furthermore, the immobility of neutral particles along the non-Riemannian directions is lifted to a saturation velocity for charged particles. Utilising the differential geometry of Double Field Theory we also present curved spacetime extensions which exhibit general covariance.

I. Introduction

Fractons are novel quasiparticles with properties that challenge the conventional understanding of topological phases of matter \[1\,4\]. In modern condensed matter physics, it is generically expected that any lattice model with local interactions admits a well-defined continuum field theory limit in the far infrared regime. However, fractons defy this doctrine and appear to require ingenious or exotic field theories with some manner of UV/IR mixing \[5\,12\]. Fractons have further characteristic properties such as immobility, infinite ground-state degeneracy in the continuum limit, and higher moment conservation laws. We refer readers to \[12\,14\] for reviews and \[15\,48\] for further significant developments.

Recent advances have shown further that the immobility can be explained in terms of certain subsystem symmetries or conserved higher multipole moments. As symmetry has been a successful guiding principle in modern physics, the characteristics of fracton physics can be grasped through the underlying (though rather exotic) symmetry laws. For example, a charged particle with both monopole and dipole conservation in specific directions explains the immobility of the monopole in the corresponding subspace.

Parallel to the endeavours to find continuum field theory limits of all known fracton lattice models, it may be worthwhile to have a novel formalism which allows us to construct systematically and geometrically new types of quantum field theories featuring fractons.

In this paper, we launch a systematic top-down approach to fracton physics by employing the geometric framework of Double Field Theory (DFT), assuming the \(O(D,D)\) symmetry therein as the first principle. Historically, \(O(d,d;\mathbb{Z})\) was an ‘emerging’ discrete symmetry for string theory compactified on a torus background \(T^d\) \[49\]. However, from the modern DFT point of view, string theory itself ‘knows’ the \(O(D,D)=O(D,D;\mathbb{R})\) symmetry regardless of the chosen background, with \(D\) now denoting the full spacetime dimension. The theory is \textit{ab initio} ‘covariant’ (rather than invariant) under \(O(D,D)\) symmetry rotations. Only a specific individual background breaks it spontaneously, either fully or partially, such as \(O(D,D;\mathbb{R}) \to O(d,d;\mathbb{Z})\) upon the aforementioned toroidal compactification.

We shall demonstrate in the present paper that fracton physics may arise from such fully \(O(D,D)\) symmetric theories when the background is non-Riemannian, meaning that an invertible metric \(g_{\mu\nu}\) is not defined even locally. Analogous to General Relativity (GR) which describes physics on Riemannian geometries, the (stringy) gravitational theory for more general geometries, including both Riemannian and non-Riemannian ones, is DFT.

By embedding fracton physics into DFT, it becomes readily possible to further address fermionic extensions, supersymmetrisations, and curved spacetime generalisations, while likely maintaining consistency with string theory or quantum gravity.

DFT was originally conceived \[50\,53\] to make manifest the hidden symmetry of \(D\)-dimensional supergravity underlying the so-called ‘Buscher rule’ \[50\,52\]. In order to do so, the theory demands the coordinates to be formally doubled, \(x^A=\left(\tilde{x}_\mu, x^\mu\right)\), \(\partial_A=\left(\partial^\mu, \partial_\nu\right)\), and redefines the notion of general covariance: under infinitesimal doubled diffeomorphisms \(\delta x^A=\xi^A(x)\), a covariant tensor density

\[\delta x^A=\xi^A(x)+\partial_A\xi(x)\, x^A.\]
of weight \( w \) transforms through a generalised Lie derivative,

\[
\hat{\mathcal{L}}_\xi T_{A_1 \cdots A_n} = \xi^B \partial_B T_{A_1 \cdots A_n} + w \partial_B \xi^B T_{A_1 \cdots A_n} + \sum_{j=1}^n (\partial_A \xi^B - \partial_B \xi_A) T_{A_1 \cdots B \cdots A_n}.
\]  

(1)

Here \( A, B, \ldots, D + D \) are \( O(D, D) \) indices which are raised and lowered by an \( O(D, D) \) invariant metric,

\[
\mathcal{J}_{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]  

(2)

Closure of (1) requires imposing the so-called ‘section condition’,

\[
\partial_A \partial^A = 0,
\]  

(3)

which enforces that the contraction between any pair of derivatives should be trivial. Decomposing this as \( \partial_A \partial^A = \partial_{\mu} \partial^\mu + \partial^\mu \partial_{\mu} \), the condition is conveniently solved by switching off any tilde-coordinate dependence, \( \partial^\mu = 0 \). In this way, the theory is not truly doubled: rather, it packages various component fields into a unifying \( O(D, D) \) multiplet.

In DFT the ‘dilaton’ \( d \) and the ‘generalised metric’ \( \mathcal{H}_{AB} \) are the two fundamental variables that constitute the gravitational sector, in analogy with the Riemannian metric \( g_{\mu\nu} \) in GR. While the former exponentiates to a unit-weight scalar density \( e^{-2d} \), the latter satisfies its own defining properties,

\[
\mathcal{H}_{AB} = \mathcal{H}_{BA}, \quad \mathcal{H}_{AC} \mathcal{H}_{BD} \mathcal{J}_{CD} = \mathcal{J}_{AB}.
\]  

(4)

This implies that \( \det \mathcal{H}_{AB} = \pm 1 \), hence the generalised metric alone cannot produce any integral measure like \( \sqrt{g} \) in GR. Instead, combined with the \( O(D, D) \) invariant metric, it generates a pair of mutually orthogonal projectors, \( P_{AB} = \frac{1}{2} (J + \mathcal{H})_{AB} \) and \( \bar{P}_{AB} = \frac{1}{2} (J - \mathcal{H})_{AB} \), satisfying

\[
P_{AB} P_{BC} = P_{AC}, \quad P_{AB} \bar{P}_{BC} = \bar{P}_{AC}, \quad P_{AB} \bar{P}_{BC} = 0.
\]

(5)

Parallel to General Relativity (GR), DFT has its own Christoffel symbols \( \Gamma_{ABC} \), scalar/Ricci/Einstein curvatures, etc., all arising from \( \{ d, \mathcal{H}_{AB} \} \). Moreover, when coupled to extra ‘matter’ \( O(D, D) \) symmetrically, DFT satisfies “Einstein equations” [61],

\[
G_{AB} = \mathcal{H}_{AB}.
\]  

(6)

In the early days of the development of DFT, the generalised metric was simply assumed to be of the form

\[
\mathcal{H}_{AB} = \begin{pmatrix} \mathcal{H}_{\mu\nu} \mathcal{H}_{\kappa\lambda} \mathcal{H}_{\kappa\lambda} \\ \mathcal{H}_{\kappa\nu} \mathcal{H}_{\kappa\lambda} \mathcal{H}_{\kappa\lambda} \end{pmatrix} = \begin{pmatrix} g_{\mu\nu} - g_{\mu\rho} B_{\rho\lambda} \mathcal{H}_{\kappa\lambda} \\ B_{\kappa\rho} g_{\mu\nu} g_{\kappa\lambda} - B_{\kappa\rho} g_{\sigma\tau} B_{\sigma\lambda} \end{pmatrix}.
\]  

(7)

In this case, the \( D \)-dimensional diagonal blocks roughly correspond to the inverse metric \( g^{\mu\nu} \) and metric \( g_{\mu\nu} \), with additional components generated by a skew-symmetric tensor ‘\( B \)-field’, \( B_{\mu\nu} \). Together with the decomposition of the DFT volume element as \( e^{-2d} = e^{-2\phi} \sqrt{-\hat{g}} \), the resulting fields \( \{ g_{\mu\nu}, B_{\mu\nu}, \phi \} \) constitute the gravitational multiplet in the supergravity theory which arises as the low-energy effective description of the massless modes of the closed string propagating in Minkowskian spacetime.

However, this is not the most general parametrisation of the generalised metric that satisfies the two defining conditions [4]. Surprisingly, it turns out that DFT describes not only the Riemannian geometries given in (6) but also non-Riemannian ones where an invertible Riemannian metric cannot be defined even locally [77]. Namely, with respect to the section choice \( \partial^\mu = 0 \), the upper-indexed \( D \times D \) block matrix \( \mathcal{H}^{\mu\nu} \) can be degenerate. From the most general solutions to the conditions (4), all possible DFT geometries have been classified by two non-negative integers \( (n, \bar{n}) \), with \( \text{dim (ker } \mathcal{H}^{\mu\nu}) = n + \bar{n} \) [74]. Only those of type \( (0, 0) \) are Riemannian, while others are intrinsically non-Riemannian. In particular, the maximally non-Riemannian cases of \( (0, 0) \) or \( (0, D) \) correspond to \( \mathcal{H}_{AB} = \pm \mathcal{J}_{AB} \), and thus they are the two perfectly symmetric vacua of DFT, preserving the entire \( O(D, D) \) symmetry with no moduli [77]. Intriguingly then, the Riemannian spacetime (6) may arise after the spontaneous breaking of \( O(D, D) \) symmetry, which identifies \( g_{\mu\nu} \) and \( B_{\mu\nu} \) as the massless Nambu–Goldstone bosons [78]. Some intermediate types of non-Riemannian geometries, such as \( (1, 1), (D - 1, 0) \), etc., [76, 78, 80] have also been identified as non-relativistic or ultra-relativistic gravities and/or strings [81, 82]. In the present paper, we will establish further connections to fracton physics, for generic \( (n, \bar{n}) \neq (0, 0) \).

Splitting the coordinates into three parts,

\[
x^\mu = (x^a, y^i, \bar{y}^\bar{i}), \quad \partial_\mu = (\partial_a, \partial_i, \partial_{\bar{i}}),
\]  

(8)

where \( 1 \leq a \leq D - n - \bar{n}, \quad 1 \leq i \leq n, \quad 1 \leq \bar{i} \leq \bar{n} \), a flat \( (n, \bar{n}) \) background is given by constant \( d \) and, with a sub-dimensional (Minkowskian) metric \( \eta_{ab} \),

\[
\mathcal{H}^{\mu\nu} = \eta^{ab} \delta^i_\mu \delta^\bar{i}_\nu, \quad \mathcal{H}_{\mu\nu} = \delta^a_\mu \delta^b_\nu \eta_{ab}, \quad \mathcal{H}^{\mu\nu} = \delta^i_\mu \delta^{\bar{i}}_\nu - \delta^i_\nu \delta^{\bar{i}}_\mu.
\]  

(9)

while \( \mathcal{H}_{\mu\nu} = \mathcal{H}^{\nu\mu} \) and \( \Gamma_{ABC} = 0 \), hence \( \nabla_A = \partial_A \). Here ‘flat’ means simply being constant: unlike GR it appears that there is no four-indexed “Riemann curvature” in DFT [58, 60]. Nevertheless, any constant
background of \( \{d, \mathcal{H}_{AB}\} \) solves the vacuum Einstein equations \( G_{AB} = 0 \), hence in contrast to GR the constant flat geometries are not unique in DFT. The fact that \( \mathcal{H}^{\mu\nu} \) and \( \mathcal{H}_{\mu\nu} \) are degenerate \( D \times D \) matrices with \( n + \bar{n} \neq 0 \) characterises the non-Riemannianity.

The remainder of this paper is organised as follows. In the next section [II] we present three key motivators for our proposal of studying fractons via DFT. In section [III] as elementary warm-up exercises, we consider a particle action and scalar field theory on the non-Riemannian \((n, \bar{n})\) constant background and verify their fractonic behaviours. We then turn to our main example, doubled pure Yang–Mills theory, in section [V]. We show that it reduces to ordinary Yang–Mills coupled to an elasticity theory of a non-Abelian strain tensor, and we spell out its infinite-dimensional Noether symmetries originating from the non-Riemannian isometries. We subsequently couple its Abelian version, i.e. doubled Maxwell theory, to charged particles in section [V] and study the resulting dynamics. In particular, we observe that the elasticity displacement vector, once condensed, changes the effective mass of the particle. In section [VI] we extend our results to curved \((n, \bar{n})\) backgrounds through the DFT formalism. We conclude with comments including a connection to polaron in section [VII] and we display some technical formulae in Appendix [A].

While our primary goal was to explore the fractonic nature of field theories on non-Riemannian DFT backgrounds, during the investigation of the doubled pure Yang–Mills theory, as well as Maxwell theory and its coupling to point particles, we uncovered some remarkable properties genuinely valid even for Riemannian backgrounds, or on Riemannian subspaces. One is that the doubled pure Yang–Mills theory \([30]\) reduces to an ordinary (undoubled) Yang–Mills theory coupled to a (non-Abelian) strain tensor theory \([11]\), such that its Abelian version provides a unifying description of photons and phonons. Furthermore, when minimally coupled to a point particle \([53]\), the particle will acquire an effective mass through the condensation of the displacement vector of phonons \([60]\), suggesting a potential application to polaron physics.

II. Three Key Motivators

The three key motivators for our proposal of studying fractons via DFT are A. geodesic immobility; B. infinite-dimensional isometries; and C. induced Noether currents. All of these assume non-Riemannian constant backgrounds \([8]\).

A. Geodesic Immobility

The geometric meaning of the section condition \([8]\) advocated in \([8]\) is that half of the doubled coordinates, e.g. \( \hat{x}_{\mu} \), are actually gauged as \( D x^4 = (d \hat{x}_{\mu} - a_{\mu}, dx^0) \). This enables us to define an \( O(D, D) \)-symmetric, doubled-diffeomorphism-invariant, proper length \([90]\) and consequently a particle action \([71, 72]\),

\[
S_{\text{particle}} = \int d\tau \frac{1}{2} e^{-1} D_{\tau} x^4 D_{\tau} x^4 \mathcal{H}_{AB} - \frac{1}{2} e m^2, \tag{9}
\]

where \( e \) (einbein) and \( a_{\mu} \) in \( D_{\tau} x^4 \) are auxiliary variables. After Gaussian integration of the \( a_{\mu} \)'s along the Riemannian directions, the above doubled particle action reduces upon the constant \((n, \bar{n})\) background \([8]\) to an undoubled one, \([76]\) (c.f. \([22]\)).

\[
S_{\text{particle}}^{(n, \bar{n})} = \int d\tau \frac{1}{2} e^{-1} x^a \dot{x}_a - \frac{1}{2} e m^2 + \Lambda_i \dot{y}^i + \bar{\Lambda}_i \dot{\bar{y}}^i. \tag{10}
\]

Here \( \Lambda_i \) and \( \bar{\Lambda}_i \) originate from the field redefinitions of the gauge components \( a_i \) and \( \bar{a}_i \), respectively, and crucially play the role of Lagrange multipliers, enforcing immobility along the non-Riemannian directions,

\[
\dot{y}^i = 0, \quad \dot{\bar{y}}^i = 0. \tag{11}
\]

Similarly, on a string worldsheet \([73, 74]\), \( y^i \) and \( \bar{y}^i \) become chiral and anti-chiral, respectively \([76]\).

B. Infinite-dimensional Isometries

Our second observation is that the isometry of the \((n, \bar{n}) \neq (0, 0)\) non-Riemannian constant background \([8]\) is infinite-dimensional \([53]\): the most general solution to the twofold Killing equations,

\[
\hat{\partial}_{\xi} \mathcal{H}_{AB} = 0, \quad \hat{\partial}_{\xi} e^{-2d} = 0, \tag{12}
\]

is, with \( \xi^A = (\lambda, \xi^a) \),

\[
\xi^a = w^a x^b + \zeta^a(y) + \bar{\zeta}^a(\bar{y}), \quad \lambda_a = \zeta_a(y) - \bar{\zeta}_a(\bar{y}),
\]

\[
\xi^i = \omega n \dot{y}^i + \zeta^i(y), \quad \lambda_i = \rho_i(y),
\]

\[
\bar{\xi}^i = -\omega n \dot{\bar{y}}^i + \bar{\zeta}^i(\bar{y}), \quad \bar{\lambda}_i = \bar{\rho}_i(\bar{y}). \tag{13}
\]

Here \( w_{ab} = -w_{ba} \) (Lorentz symmetry) and \( \omega \) are constants. All other parameters are arbitrary functions of the non-Riemannian coordinates \( y^i \) or \( \bar{y}^i \), as displayed in \([13]\). Furthermore, \( \zeta^i(y) \) and \( \bar{\zeta}^i(\bar{y}) \) should be divergence-less,

\[
\partial_i \zeta^i(y) = 0, \quad \bar{\partial}_{\bar{i}} \bar{\zeta}^i(\bar{y}) = 0, \tag{14}
\]

which ensures that \( \partial_i \zeta^i = 0 \), a requirement following from the Killing equation of the dilaton \( d \) \([12]\).

C. Induced Noether Currents

The third point of interest relates to the energy-momentum tensor in DFT \([61]\),

\[
T^{AB} = -e^{2d} \left[ 8 P^{[A} \tilde{C}^{\bar{B}]} D \frac{\delta S_{\text{Matter}}}{\delta \mathcal{H}_{CD}} + \frac{1}{2} \frac{\varepsilon^{AB}}{d^4} \frac{\delta S_{\text{Matter}}}{\delta d} \right]. \tag{15}
\]

This relates, for example, to the dilaton \( d \), and the induced Noether currents

\[
\mathcal{J}_{\text{induc}}^A = e^{-1} D_{\tau} D_{\tau} x^4 \mathcal{H}_{AB} D_{\mu} x^4 D_{\nu} x^4 \mathcal{H}_{CD} - \delta_{\text{Matter}} D_{\tau} x^4 \mathcal{H}_{AB} D_{\mu} x^4 D_{\nu} x^4 \mathcal{H}_{CD} \tag{16}
\]

for the doubled \( n, \bar{n} \)-background, \( \delta_{\text{Matter}} \) is the variation of the matter action with respect to the background fields \( \mathcal{H}_{AB} \).
By construction, for arbitrary $\xi^A$, it satisfies the off-shell relation
\[
\partial_A(e^{-2d}T^{AB}\xi^B) = e^{-2d}\xi^B\nabla_A T^{AB} + \frac{1}{2d}T^{AB}\xi^B e^{-2d} - \frac{1}{2}e^{-2d}(PTP)^{AB}\xi^B \mathcal{H}_{AB}.
\] (16)

Thus, for the constant background [8] with the Killing vector [13] we acquire an on-shell conserved current,
\[
J^\mu = T^\mu\xi^A = T^\mu\xi^\nu + T^{\mu\nu}\lambda_\nu, \quad \partial_\mu J^\nu = 0. \quad (17)
\]

Explicitly, as a collection of independent currents,
\[
J^\mu = (T^\mu_a + \eta_{ab}T^{\mu b})\zeta_a(y) + (T^\mu_a - \eta_{ab}T^{\mu b})\bar{\zeta}_a(\bar{y}) + \omega(\dot{\bar{y}}T^\mu_i\dot{\bar{y}}^j - nT^\mu_i\dot{\bar{y}}^j) + T^\mu_i\zeta_i(y) + T^\mu_i\bar{\zeta}_i(\bar{y}) + T^{\mu\nu}\rho_{\nu}(y) + T^{\mu\nu}\rho_{\nu}(\bar{y}) + T^\mu a\omega_{ab}x^b.
\] (18)

Evidently, power series expansions of the local parameters in the coordinates $y'$ and $\bar{y}'$ generate infinitely many higher multipole conservation laws. This includes dipole conservation laws generated by the parameter $\omega$ and other linear terms from $\{ \zeta^a(y), \zeta^a(\bar{y}), \bar{\zeta}^a(y), \bar{\zeta}^a(\bar{y}) \}$, modulo $so(n) \oplus so(\bar{n})$ rotations. In the special cases where $n = 1$ or $\bar{n} = 1$, the divergenceless condition [13] actually enforces $\zeta^a$ or $\bar{\zeta}^a$ simply to be constant, which implies the absence of all higher multipole conservation laws along the non-Riemannian directions. In particular, (1, 1) allows only dipole conservation, corresponding to the finite scale transformation
\[
y \rightarrow e^\omega y, \quad \bar{y} \rightarrow e^{-\omega}\bar{y},
\]
where the two non-Riemannian directions are inversely related. Note that the symmetry is still ‘supertranslational’ in the Riemannian directions for any $(n, \bar{n}) \neq (0, 0)$, as $\zeta^a(y)$ and $\bar{\zeta}^a(\bar{y})$ appearing in [13] are arbitrary functions.

Meanwhile, for the global translational symmetries generated by the constant terms in $\xi^\mu$ and $\lambda_\nu$, the conservation of the current [17] reduces to that of the energy-momentum tensors,
\[
\partial_\mu T^{\mu\nu} = 0, \quad (20)
\]

for the untide $x^\mu$-directions, and further, inequivalently,
\[
\partial_\mu T^{\mu\nu} = 0, \quad (21)
\]

for the tilde $\tilde{x}_\mu$-directions. The latter can be nontrivial even after switching off the tilde coordinates, i.e. setting $\tilde{\xi}^\mu = 0$ as our choice of section: as we shall see later, a scalar field theory has trivial $T^{\mu\nu}$ [27], whereas it is nontrivial for Yang–Mills theory [56].

The three points A, B, C imply that any (double field theorisable) field theory should feature the fractonic properties of higher multipole conservation [18] and a huge degeneracy of quantum states, as there are infinitely many conserved quantities. Intriguingly, the (1, 1) non-Riemanian background, corresponding to the non-relativistic string [81], allows only dipole conservation along the pair of non-Riemannian directions, $y, \bar{y}$ [19] which alludes to UV/IR mixing of these two directions. This property is comparable to known fracton field theory models [6, 7, 15, 16].

We stress that all of these are direct consequences of the underlying constant non-Riemannian background. In the following we verify these properties explicitly for several examples, such as particles, scalar fields, doubled Yang–Mills theory, and a strain-Maxwell theory minimally coupled to charged particles. The advantage of embedding fracton physics into DFT is that generalisations to curved geometries, supersymmetry à la [63], and consistent string backgrounds are readily available, by setting $D = 10$ or 26 and $n = \bar{n}$ [92].

### III. Particle and Scalar Field

The doubled energy-momentum tensor of the point particle [19] was obtained in [61] from the variation of the covariant particle action [6] following the prescription [13],
\[
T^{\mu\nu} = 0, \quad T^\mu_\nu(x) = \int d\tau \dot{x}^\mu(\tau)p_\nu\delta^D(x - x(\tau)),
\]

where the delta function is defined for the untide coordinates $x^\mu - x^\mu(\tau)$, and $p_\mu = (e^{-1}\dot{x}_a, \Lambda_\xi, \bar{\Lambda}_\xi)$ is the conjugate momentum of $x^\mu$, of which all components are constant on-shell. Thus, conservation indeed holds,
\[
\partial_\mu T^{\mu\nu} = -\int d\tau p_\nu \frac{d}{d\tau} \delta^D(x - x(\tau)) = 0. \quad (23)
\]

Further, from the on-shell relations
\[
T^a_\nu \eta^{cb} = T^b_\nu \eta^{ca}, \quad T^\mu_\nu = 0 = T^\nu_\mu, \quad (24)
\]

the conservation of the current [13] readily follows. The corresponding Noether symmetries of the reduced particle action [10] inherited from the doubled particle action [9] read, with [13],
\[
\delta \xi^a = \xi^a, \quad \delta y^i = \xi^i, \quad \delta \bar{y}^i = \bar{\xi}^i, \quad \delta e = 0, \quad \delta \Lambda_i = -e\dot{x}_a \partial_\nu \xi_a(y) - \omega n \Lambda_i - \Lambda_j \partial_\xi \xi_j(y), \quad \delta \bar{\Lambda}_i = -e\dot{x}_a \partial_\nu \bar{\xi}_a(\bar{y}) + \omega n \bar{\Lambda}_i - \bar{\Lambda}_j \partial_\bar{\xi} \bar{\xi}_j(\bar{y}). \quad (25)
\]

As a target-spacetime counterpart to the particle action, we turn to a scalar field theory with Lagrangian (density) $e^{-2d}L_\Phi$, c.f. [6, 6],
\[
L_\Phi = -\frac{1}{2}H^{AB} \partial_A \Phi \partial_B \Phi - V(\Phi) = -\frac{1}{2} \eta^{ab} \partial_a \Phi \partial_b \Phi - V(\Phi). \quad (26)
\]
The doubled Yang–Mills Lagrangian takes the form
\[ L_{YM} = \text{Tr}[P^{AC}P^{BD}F_{AB}F_{CD}] + \delta_{\mu} \partial^{\alpha} \Phi \Phi + \delta_{\nu} L_{\Phi}, \]
which is conserved on-shell as
\[ \partial_{\mu} T_{\mu}^\nu = [\partial_{\mu} \partial^{\alpha} \Phi - \nabla^{\alpha} (\Phi)] \partial_{\nu} \Phi = 0. \]

The infinite-dimensional Noether symmetries for the Killing vector [13] are given simply by \( \delta \Phi = \xi^{\mu} \partial_{\mu} \Phi \). In particular, when the scalar theory is free with a Lagrangian \( L_{\Phi} = \frac{1}{2} \phi^{\mu} \partial_{\mu} \phi - m^{2} \phi^{2} \) which vanishes on-shell, its energy-momentum tensor also satisfies (28).

The doubled pure Yang–Mills field strength \( (\Phi) \), the equations of motion \((\Phi) = 0\) and \((\Phi) = 0\) are read, respectively,
\[
\begin{aligned}
P_{A}^{B} &= \frac{1}{2} \left( \delta_{A}^{B} + \mathcal{H}_{AB} \right) = \frac{1}{2} \left( \delta_{\mu}^{\nu} + Z_{\nu}^{\mu} \right. \\
& \hfill \left. \begin{array}{ll}
H_{\mu \nu}^{\rho} & K_{\rho \nu} \\
K_{\rho \nu} & \delta_{\mu}^{\rho} + Z_{\mu}^{\rho}
\end{array} \right),
\end{aligned}
\]
\[
\begin{aligned}
\tilde{P}_{A}^{B} &= \frac{1}{2} \left( \delta_{A}^{B} - \mathcal{H}_{AB} \right) = \frac{1}{2} \left( \delta_{\mu}^{\nu} - Z_{\nu}^{\mu} \right. \\
& \hfill \left. \begin{array}{ll}
\delta_{\mu}^{\rho} & - K_{\rho \nu} \\
- K_{\rho \nu} & \delta_{\mu}^{\rho} + Z_{\mu}^{\rho}
\end{array} \right),
\end{aligned}
\]
\[
\mathcal{F}_{AB} = 2 \partial_{[A} \nabla_{B]} - i [\nabla_{A}, \nabla_{B}] = \left( -i[\phi^{\mu}, \phi^{\nu}] - D_{\rho} \phi^{\mu} \phi^{\nu}, f_{\rho \sigma} \right),
\]
where \( D_{\mu} = \partial_{\mu} - i[A_{\mu}, \cdot] \) and
\[
f_{\mu \nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} - i[A_{\mu}, A_{\nu}] \]
is the field strength of ordinary undoubled Yang–Mills theory. From these ingredients we obtain the fully covariant field strength,
\[
(P \tilde{P})_{AB} = \left( (P \tilde{P})^{\mu \rho} (P \tilde{P})^{\rho \nu} \right),
\]
\[
(P \tilde{P})^{\mu \nu} = -\frac{1}{4} \left[ H_{\mu \nu}^{\rho} (\delta_{\rho}^{\alpha} + Z_{\rho}^{\alpha}) (\delta_{\alpha}^{\lambda} - Z_{\alpha}^{\lambda}) \right],
\]
\[
(P \tilde{P})^{\mu \rho} = \frac{1}{4} \left[ H_{\mu \rho}^{\sigma} (\delta_{\sigma}^{\lambda} + Z_{\sigma}^{\lambda}) (\delta_{\lambda}^{\nu} - Z_{\lambda}^{\nu}) \right],
\]
\[
(P \tilde{P})_{\rho \nu} = -\frac{1}{4} \left[ (\delta_{\rho}^{\alpha} + Z_{\rho}^{\alpha}) (\delta_{\alpha}^{\lambda} - Z_{\alpha}^{\lambda}) \right].
\]

We now compute the Lagrangian explicitly on the constant \((n, \bar{n})\) non-Riemannian background [8], which we denote using \( \{ H_{\mu \nu}^{\rho}, K_{\mu \rho}, Z_{\mu \nu} \} \) as
\[
H_{\mu \nu}^{\rho} = \eta^{\rho \delta} \delta^{\mu}_{\rho} \delta^{\nu}_{\delta} = \mathcal{H}_{\mu \nu}^{\rho},
K_{\mu \rho} = \delta^{\mu}_{\delta} \delta^{\rho}_{\sigma} \delta^{\nu}_{\sigma} = \mathcal{H}_{\mu \rho},
Z_{\mu \nu}^{\rho} = \delta^{\rho}_{\mu} \delta^{\nu}_{\rho} - \delta^{\mu}_{\rho} \delta^{\nu}_{\rho} = \mathcal{H}_{\mu \nu}^{\rho}.
\]

Parametrising the doubled vector as
\[
\nabla_{A} = (\phi^{\mu}, A_{\mu}), \]
the projectors and the semi-covariant Yang–Mills field strength [29] read, respectively,
\[
P_{A}^{B} = \frac{1}{2} (\delta_{A}^{B} + \mathcal{H}_{AB}) = \frac{1}{2} \left( \delta_{\mu}^{\nu} + Z_{\nu}^{\mu} \right. \\
& \hfill \left. \begin{array}{ll}
H_{\mu \nu}^{\rho} + Z_{\nu}^{\rho} & H_{\mu \nu}^{\rho} \\
H_{\mu \nu}^{\rho} & K_{\rho \nu} + \delta_{\mu}^{\rho} + Z_{\mu}^{\rho}
\end{array} \right),
\]
\[
\tilde{P}_{A}^{B} = \frac{1}{2} (\delta_{A}^{B} - \mathcal{H}_{AB}) = \frac{1}{2} \left( \delta_{\mu}^{\nu} - Z_{\nu}^{\mu} \right. \\
& \hfill \left. \begin{array}{ll}
\delta_{\mu}^{\rho} & - K_{\rho \nu} - \delta_{\mu}^{\rho} - Z_{\mu}^{\rho}
\end{array} \right),
\]
where \( D_{\mu} = \partial_{\mu} - i[A_{\mu}, \cdot] \) and
\[
f_{\mu \nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} - i[A_{\mu}, A_{\nu}] \]
is the field strength of ordinary undoubled Yang–Mills theory. From these ingredients we obtain the fully covariant field strength,
one arrives at the undoubled Lagrangian,

\[ L^{(n,\bar{n})}_{YM} = \text{Tr} \left[ -\frac{1}{4} (f_{ab} + i[\phi_a, \phi_b]) (f^{ab} + i[\phi^a, \phi^b]) - \frac{1}{4} u_{ab} u^{ab} - f_{a\dot{a}} D^a \phi^a + f_a^i D^a \phi^i \right] - 2D_i \phi^i D\phi^i - 2i \phi_i [\phi^i, \phi^j] \]

(41)

where we have defined a symmetric tensor

\[ u_{ab} = D_a \phi_b + D_b \phi_a . \]

(42)

Identifying \( \phi^\mu \) as the displacement vector in elasticity theory, \( u_{ab} \) corresponds to a strain tensor which now interacts with the undoubled Yang–Mills. The symmetric strain tensor originates essentially from the projection

\[ 4 (PFP)_{ab} = f_{ab} + i[\phi_a, \phi_b] - u_{ab} = \partial_a A_b^+ - \partial_b A_a^+ - i[A_a^+, A_b^-] . \]

(43)

Since \( A_\mu \) and \( \phi^\mu \) are dual to each other à la Buscher \cite{Buscher1987}, so are their (Abelian) elementary quanta, photon and phonon (c.f. \cite{Sak1974}).

By construction from \cite{Bekaert2005}, the Lagrangian \cite{Bekaert2005} enjoys ‘supertranslational’ Noether symmetries given \emph{a priori} by \( \delta_\xi A_\mu = \bar{L}_\xi \Lambda_\mu \) with the Killing vector \cite{Bekaert2005}, which reduce in terms of the ordinary Lie derivative to

\[ \delta A_\mu = \bar{L}_\xi A_\mu + 2 \partial_\mu \lambda_\rho [\phi^\rho] , \quad \delta \phi^\mu = \bar{L}_\xi \phi^\mu . \]

(44)

In particular, under the transformations \cite{Bekaert2005} with the Killing vector \cite{Bekaert2005}, \( (PFP)_{AB} \) transforms covariantly as

\[ \delta (PFP)^\mu\nu = \bar{L}_\xi (PFP)^\mu\nu , \]

\[ \delta (PFP)^\mu_\nu = \bar{L}_\xi (PFP)^\mu_\nu + 2 \partial_\mu \lambda_\rho (PFP)^\rho\nu , \]

\[ \delta (PFP)^\mu_\nu = \bar{L}_\xi (PFP)^\mu_\nu + 2 \partial_\nu \lambda_\rho (PFP)^\rho\mu , \]

\[ \delta (PFP)^\mu\nu = \bar{L}_\xi (PFP)^\mu\nu + 2 \partial_\nu \lambda_\rho (PFP)^\rho\mu + 2 \partial_\mu \lambda_\rho (PFP)^\rho\nu , \]

(45)

from which the invariance of the action, or \cite{Bekaert2005}, follows straightforwardly,

\[ \delta L_{YM} = \xi^\mu \partial_\mu L_{YM} = L_{\xi^\mu L_{YM}} \].

(46)

Note that although \cite{Bekaert2005} can be verified directly by brute force, it can be understood simply as a natural consequence of the \( O(D,D) \)-symmetric general covariance of the doubled Yang–Mills field strength \cite{Bekaert2005}, encoded via \cite{Bekaert2005}.

The equations of motion \( D_A (PFP)^{AB} = 0 \) \cite{Bekaert2005} are, exhaustively,

\[ 0 = D_b (F^b_{\dot{a}} + i[\phi^b, \phi_{\dot{a}}]) - D_\mu D_i \phi^i + D_a^+ D_{\dot{a}} \phi_{\dot{a}} + i[\phi_b, u_{\dot{a}}] + 2i[\phi^i, f_{\dot{a}i}] - 2i[\phi^i, f^i_{\dot{a}}] , \]

\[ 0 = D_a D^a \varphi^i + 4i[\phi^i, D_i \phi^j] - 2i[\phi^i, D_i \psi^j] , \]

\[ 0 = D_a^+ D^a \varphi^i - 4i[\phi^i, D_i \psi^j] + 2i[\phi^i, D_i \psi^j] , \]

\[ 0 = D_{b\dot{a}} D_{\dot{a}} \psi^i + 4i[\phi^i, f^i_{\dot{a}}] + 2i[\phi^i, f^i_{\dot{a}}] , \]

\[ 0 = D_{a\dot{a}} - f_{\dot{a}i} D_i \psi^i + 4i[\phi^i, f_{\dot{a}i}] , \]

\[ 0 = D_{a\dot{a}} - f_{\dot{a}i} D_i \psi^i + 4i[\phi^i, f_{\dot{a}i}] . \]

(47)

We directly verified the conservation \( \partial_{\mu} \psi^\mu = 0 \) \cite{Bekaert2005} by checking, first of all, the conservation of the doubled energy-momentum tensor itself (up to the Bianchi identity, the on-shell equations \cite{Bekaert2005}, and the section condition),

\[ \partial_{\mu} T^B_{\mu A} = \text{Tr} \left[ 6 (PFP)^{BC} D_{A}(F_{BC}) - 4 F_{AC} D_B (PFP)^{[BC]} \right] + 4 \partial_c Tr \left[ \partial C (PFP)^{[BC]}(PFP)^{[BA]} - \partial C D^B (PFP)^{[BA]} \right] = 0 , \]

(48)

and secondly, the vanishing of each of

\[ \{ T^{ij} , T^{ij} , T^{ia} + \eta^{ab} T^{ib} , T^{ia} - \eta^{ab} T^{ib} , T^{ii} + T^{ii} \} \]

as consequences of the orthogonal projections performed in \cite{Bekaert2005}, with \( T^{ij} = \delta^{ij} L_{YM} \) and \( T^{ij} = \delta^{ij} L_{YM} \), c.f. \cite{Bekaert2005}.

Unlike the previous particle and scalar field theory cases, both \( T^{\mu\nu} \) and \( T^{\mu\nu} \) turn out to be nontrivial in Yang–Mills theory: see Appendix \ref{appendixA} for the full expression, or the upcoming eq. \cite{Bekaert2005} for the simple \((0,0)\) Abelian case.

As already mentioned, it is a rather unexpected and remarkable result of the DFT formalism that the doubled Yang–Mills theory produces a unifying description \cite{Bekaert2005} of ordinary Yang–Mills theory and a non-Abelian elasticity theory. We emphasize that this holds true even upon genuine Riemannian backgrounds, \emph{i.e.} \((n,\bar{n}) = (0,0)\). The Abelian reduction of \cite{Bekaert2005} on a flat Minkowskian spacetime is simply the sum of Maxwell and strain tensor theories describing free photons and phonons, respectively,

\[ L^{(0,0)}_{\text{Maxwell–Strain}} = -\frac{1}{4} f_{ab} f^{ab} - \frac{1}{4} u_{ab} u^{ab} . \]

(49)

The corresponding energy-momentum tensors are

\[ T^{ab} = 2 \partial_c \varphi^a f^{bc} + \partial_c (f^{bc} \varphi^c) , \]

\[ T^a_b = f^{ac} f_{bc} + \partial^a \varphi^c \partial_b \varphi_c - \partial_c \varphi^a \partial^b \varphi_c + \partial_c (\varphi^a u^b_b) - \frac{1}{4} \delta^a_b (f_{cd} f^{cd} + u_{cd} u^{cd}) . \]

(50)

While clearly \( T^{ab} \neq T^a_c \eta^{cb} \), these satisfy the off-shell
relations
\[
\partial_a T^{ab} = 2\partial_\epsilon (\partial_a f^{ab} \phi^c),
\]
\[
\partial_a T^b = f_{bd} \partial_c \epsilon^{fed} - \frac{2}{3} f^{ed} \partial_b f_{cd} + \partial_b \phi^d \partial_c u^d + \partial_d (\phi^d \partial_c u^c),
\]
and thus become conserved on-shell due to the equations of motion, \(\partial_a f^{ab} = 0 = \partial_b u^{ab}\).

Intriguingly, with the decomposition of the displacement vector into temporal and spatial components, \(\phi^a = (\phi^t, \phi^s)\) where \(a = 1, 2, \ldots, D-1\), if we truncate the temporal component by setting \(\phi^t = 0\), the strain tensor part in (50) becomes
\[
-\frac{1}{2} u_{ab} u^{ab} = \frac{1}{2} \dot{\phi}^s \dot{\phi}^s - \frac{1}{2} u_{ab} u^{ab},
\]
where the dots denote time derivatives \(\frac{d}{dt}\). This is precisely the elasticity theory considered in (26), shown therein to be dual to a \(U(1)\) symmetric tensor gauge theory. The topological defects of the former map onto charges of the latter, with disclinations and dislocations corresponding to fractons and dipoles, respectively. Our result is in some sense (orthogonally) complementary to (24), as we are principally focusing on the fracton physics that arises geometrically from non-Riemannian backgrounds. Nevertheless, we remark that both non-Riemannian spaces and topological defects are, after all, singular configurations from a conventional perspective.

V. Doubled Maxwell Coupled to Charged Particles

We now consider the doubled Maxwell theory (30) minimally coupled to particles (9) with charge \(q\).

\[
\sum_\alpha \int dt \left( \frac{1}{2} e^{-1} D_\tau x^A D_\tau x^B H_{AB} - \frac{1}{2} \epsilon m^2 - q D_\tau x^A V_A \right),
\]
where \(\alpha\) denotes a (negligible) particle index. On the constant \((n, \bar{n})\) background (53), the single-particle Lagrangian becomes
\[
L_q = \frac{1}{2} e^{-1} D_\tau x^A D_\tau x^A H_{AB} - \frac{1}{2} \epsilon m^2 - q D_\tau x^A V_A
\]
\[
= \frac{1}{2} e^{-1} \dot{x}_a \dot{x}_a - \frac{1}{2}\epsilon (m^2 + q^2 \phi^a \phi_a) - q \dot{x}_\mu A_\mu
\]
\[
+ \frac{1}{2} e^{-1} \left( \dot{x}_a - q \phi_a \right) \left( \dot{x}_b - q \phi_b \right) \eta^{ab}
\]
\[
+ \left( e^{-1} \dot{y}_i - q \phi^i \right) \left( \dot{y}_j - q \phi^j \right) + \tilde{\alpha}_i \left( \dot{y}_i + q \phi^i \right) + \tilde{\alpha}_i \left( \dot{y}_i + q \phi^i \right).
\]
The corresponding action generalises (10) to
\[
S_q = \int \frac{1}{2} e^{-1} \dot{x}_a \dot{x}_a - \frac{1}{2}\epsilon (m^2 + q^2 \phi^a \phi_a) - q \dot{x}_\mu A_\mu
\]
\[
+ \tilde{\alpha}_i (\dot{y}_i - q \phi^i) + \tilde{\alpha}_i (\dot{y}_i + q \phi^i),
\]
where we have integrated out the auxiliary variables \(a_a\) thereby setting the on-shell value \(D_\tau \dot{x}_a = e q \phi_a\), and we have identified \(\Lambda_i = e^{-1} D_\tau \dot{y}_i\) and \(\tilde{\alpha}_i = -e^{-1} D_\tau \tilde{y}_i\). This action then couples the particle to a strain-Maxwell theory, i.e. the Abelian reduction of (11),
\[
L_0 = -\frac{1}{4} f_{ab} f^{ab} - \frac{1}{2} u_{ab} u^{ab} - f_{a1} \partial^a \phi^1 + f_{a1} \partial^a \phi^1 - 2 \partial_1 \phi^1 \partial_1 \phi^1.
\]
The Hamiltonian action for (55) is
\[
S_H = \int dt \ p_i \dot{\phi}^i - e H, \quad H = \frac{1}{2} (p_i + q A_i) (p_i + q A_i) + \frac{1}{2} (m^2 + q^2 \phi^a \phi_a) + q (p_i + q A_i) \phi^1 = q (\bar{p}_i + q A_i) \phi^1.
\]
Integrating out the \(p_i\)’s from (57), one recovers (55) with the identification \(p_i = \Lambda_i - q A_i\), \(\bar{p}_i = \Lambda_i - q A_i\).

Clearly, from (55) or (57), the immobility is lifted by the displacement vector to a ‘saturation velocity’,
\[
\dot{y}_i = e q \phi^i, \quad \dot{\bar{y}}_i = -e q \phi^i.
\]
Further, the Hamiltonian constraint \(H = 0\) (57) leads to a modified dispersion relation,
\[
k_a k^a + m^2 + q^2 \phi^a \phi_a + 2q k_i \phi^i - 2q k_i \phi^i = 0,
\]
in which one may identify an effective mass (c.f. (52) for an interpretation in terms of a Higgs mechanism),
\[
m_{\text{eff}}^2 = m^2 + q^2 \phi^a \phi_a.
\]
We stress that while (58) is a consequence of the non-Riemannian geometry, (60) holds on the Riemannian subspace.

The equations of motion of the photon \(A_\mu\) are the following generalised Maxwell equations,
\[
\partial_b f^{ba} - \partial^a \partial_i \phi^i + \partial^a \partial_i \phi^i = J^a, \quad \partial_b \partial^a \phi^i = J^i, \quad -\partial_b \partial_i \phi^i = \tilde{J}^i.
\]

Here \(J^\mu\) is the usual electric current,
\[
J^\mu(x) = \sum_\alpha \int d\tau \ q \phi_a^\alpha (\tau) \delta^D(x - x_a(\tau)),
\]
which is identically conserved in the manner of (24), ensuring the consistency of (11) à la Maxwell in 1865.

Meanwhile, the phonon \(\phi^\mu\) has equations of motion
\[
\partial_\tau \phi^a + \partial_\tau \partial_\mu \phi^i + \partial_\tau \partial_\mu \phi^i = \tilde{J}_a, \quad \partial_\tau \partial_\mu \phi^i + 2\partial_\tau \partial_\mu \phi^i + \partial_\tau f^b_i = \tilde{J}_i, \quad \partial_\tau \partial_\mu \phi^i + 2\partial_\tau \partial_\mu \phi^i - \partial_\tau f^b_i = \tilde{J}_i,
\]
for which we introduce a (by no means conserved) dual ‘pseudo-current’,
\[
\tilde{J}_\mu(x) = \sum_\alpha \int d\tau \ q \phi_a^\alpha (\tau) \delta^D(x - x_a(\tau)).
\]
On shell, in terms of $\varphi_a$ and the conjugate momenta $p_i, \tilde{p}_i$ of $y^i, \tilde{y}^i$, we can express the dual pseudo-current without explicitly invoking the tilde coordinates,

$$\hat{J}_a(x) = \sum_\alpha \int d\tau \, e q^2 \varphi_a \delta^D(x - x_\alpha(\tau)),$$

$$\hat{J}_i(x) = \sum_\alpha \int d\tau \, e q(p_{\alpha i} + q A_i) \delta^D(x - x_\alpha(\tau)),$$

$$\hat{J}_\tilde{i}(x) = \sum_\alpha \int d\tau \, (-e q)(\tilde{p}_{\alpha i} + q A_i) \delta^D(x - x_\alpha(\tau)).$$  \hspace{1cm} (65)

Evidently, (61) and (63) generalise the usual Maxwell equations. In particular, in the absence of sources, they describe electromagnetic-strain waves that propagate exclusively through the Riemannian subspace, and thus are fractonic,

$$\partial_\alpha \partial^\alpha f_{\mu\nu} = \partial_\alpha \partial^\alpha \varphi^i = \partial_\alpha \partial^\alpha \varphi^a = \partial_\alpha \partial^\alpha [\partial_\alpha \varphi_\beta] = 0.$$  \hspace{1cm} (66)

The first term vanishes on-shell via the Bianchi identity as $\partial_\alpha \partial^\alpha f_{\mu\nu} = -2 \partial_\alpha \partial^\alpha f_{\nu\mu} = 0$, while the final equality, which pertains to the ‘rotation tensor’ $\partial_\alpha \varphi_\beta$, follows from the more general relation $\partial_\alpha \partial^\alpha \varphi_\beta + \partial_\alpha \partial_\beta \varphi^\alpha = 0$.

The consistent matching between the particle action, (49) or (50), and the scalar field theory (26) generalises to the interacting theory of a charged particle (53) and a charged complex scalar field. The Lagrangian of the complex scalar field in the fundamental representation is, with $D_\alpha \Phi = (\partial_\alpha + i q V_\alpha) \Phi$ and $D_\mu \Phi = (\partial_\mu + i q A_\mu) \Phi$,

$$L_{\text{VE}} = -\frac{1}{4} \mathcal{H}^{AB} D_\alpha \Phi D_\mu \Phi^\dagger - \frac{1}{2} m^2 \Phi \Phi^\dagger$$

$$= -\frac{1}{2} D_\alpha \Phi D^\alpha \Phi - \frac{1}{2} (m^2 + q^2 \varphi_a \varphi^a) \Phi \Phi^\dagger$$

$$- i \frac{2}{q^2} (\Phi \tilde{D}_\alpha \Phi - \Phi^\dagger \tilde{D}_\alpha \Phi)$$

$$+ i \frac{2}{q^2} (\Phi \tilde{D}_\alpha \Phi - \Phi^\dagger \tilde{D}_\alpha \Phi).$$  \hspace{1cm} (67)

Remarkably, the resulting equation of motion agrees with the Hamiltonian constraint of the charged particle (57), including the mass enhancement,

$$\left[ -D_\alpha D^\alpha + m^2 + q^2 \varphi_a \varphi^a - i q \{ D_\alpha, \varphi^i \} + i q \{ D_\alpha, \varphi^i \} \right] \Phi = 0,$$  \hspace{1cm} (68)

while it further produces a symmetric ‘prescription’ for the Hamiltonian constraint at the quantum level,

$$\{ p_i, \varphi^i \} = p_i \varphi^i + \varphi^i p_i,$$

$$\{ \tilde{p}_i, \tilde{\varphi}^i \} = \tilde{p}_i \tilde{\varphi}^i + \tilde{\varphi}^i \tilde{p}_i,$$

after identifying $-i D_\mu = -i \partial_\mu + q A_\mu$, with $p_\mu + q A_\mu$.

**VI. On Curved Non-Riemannian Backgrounds**

Owing to the geometric $O(D, D)$ formalism applicable to non-Riemannian geometries that has been developed in the literature [76, 95], all results in the previous sections can be readily generalised to curved backgrounds. In the context of DFT this also includes the possibility of a non-trivial $B$-field and dilaton. Here we present such curved results in full generality: in doing so it is necessary to define many covariant quantities while carefully distinguishing upper and lower $D$-dimensional curved indices, $\mu, \nu$. Though this may at first glance seem over-elaborate, we remind and warn the reader that raising and lowering indices is generically not possible in the absence of an invertible metric.

Ref. [17] and two recent works [96, 97] already considered fracton physics on curved backgrounds using sub-Riemannian or Carollian/Aristotelian geometries, which in the present framework would correspond to the $(1, 0)$ or $(D - 1, 0)$ non-Riemannian geometries. Such scenarios, not being organised under the generalised metric, break $O(D, D)$ symmetry. The consequences of this remain to be seen. Here we merely present our $O(D, D)$ symmetric, curved spacetime extension of the previous particle, scalar field, and Yang–Mills theories.

On general curved backgrounds, it is convenient to factorise out the $B$-field contribution via an $O(D, D)$ transformation,

$$B^A_B = \begin{pmatrix} \delta^\mu_\sigma & 0 \\ B_{\mu\sigma} & \delta^\nu_\tau \end{pmatrix}, \quad B^A_C B^C_B J_{CD} = J_{AB}. \hspace{1cm} (69)$$

The $(O(D, D))$-covariant generalised metric on a general curved background then takes the form [70]

$$\mathcal{H}_{AB} = B_A^C B_B^D \mathcal{H}_{CD}, \quad \mathcal{H}_{AB} = \begin{pmatrix} H^{\mu\nu} & Z^{\lambda\mu} \\ Z_{\kappa\nu} & K_{\kappa\lambda} \end{pmatrix}. \hspace{1cm} (70)$$

The vectors $X_i^j, \tilde{X}_i^j$ and $Y_i^\mu, \tilde{Y}_i^\mu$ belong to the kernels of $H^{\mu\nu}$ and $K_{\mu\nu}$, respectively,

$$H^{\mu\nu} X_i^j = 0 = H^{\mu\nu} \tilde{X}_i^j, \quad K_{\mu\nu} Y_i^\mu = 0 = K_{\mu\nu} \tilde{Y}_i^\mu,$$  \hspace{1cm} (72)

and correspond to the $n + \bar{n}$ non-Riemannian directions. These objects satisfy a completeness relation,

$$H^{\mu\nu} K_{\rho\nu} + Y_i^\mu X_i^\rho = \tilde{Y}_i^\mu \tilde{X}_i^\rho = \delta^{\mu\rho}. \hspace{1cm} (73)$$

From (72), (73), and the linear independence of the null eigenvectors, it follows that

$$X_i^\mu Y_j^\mu = \delta^i_j, \quad \tilde{X}_i^\mu \tilde{Y}_j^\mu = \delta^i_j, \quad X_i^\mu \tilde{Y}_j^\mu = 0 = \tilde{X}_i^\mu Y_j^\mu,$$

$$\langle HKH \rangle^{\mu\nu} = H^{\mu\nu}, \quad (HKH)_{\mu\nu} = K_{\mu\nu}. \hspace{1cm} (74)$$

Further from (69) and (70), a similar factorisation of the projectors holds,

$$P_{AB} = B_A^C B_B^D \tilde{P}_{CD}, \quad \tilde{P}_{AB} = B_A^C B_B^D \tilde{P}_{CD}. \hspace{1cm} (75)$$
where $\hat{P}_{AB} = \frac{1}{2}(J + \hat{H})_{AB}$ and $\tilde{P}_{AB} = \frac{1}{2}(J - \hat{H})_{AB}$.

Remarkably, the doubled metric $\mathcal{H}_{AB}$ is invariant under $\text{GL}(n) \times \text{GL}(\bar{n})$ local rotations, which act on the unbarred $i,j,\ldots$ and on the barred $\bar{i},\bar{j},\ldots$ indices, and further under the ‘Milne-shift’ symmetry—generalising the ‘Galilean boost’ in the Newtonian gravity literature [20, 21]—which acts with arbitrary local parameters, $V_{\bar{\mu}}$ and $V_{\bar{\mu}}$, as

$$
\delta_{\bar{m}}H^{\mu\nu} = 0, \quad \delta_{\bar{m}}X_{\mu} = 0, \quad \delta_{\bar{m}}\bar{X}_{\mu} = 0,
$$

$$
\delta_{\bar{m}}Y_{\nu}^{\mu} = H^{\mu\nu}V_{\bar{\nu}}, \quad \delta_{\bar{m}}\bar{Y}_{\nu}^{\mu} = H^{\mu\nu}\bar{V}_{\bar{\nu}},
$$

$$
\delta_{\bar{m}}K_{\mu\nu} = -2X_{\nu}^{(\mu}K_{\nu)}\rho H^{\rho\sigma}V_{\bar{\sigma}} - 2\bar{X}_{\nu}^{(\mu}\bar{K}_{\nu)}\rho H^{\rho\sigma}\bar{V}_{\bar{\sigma}},
$$

$$
\delta_{\bar{m}}B_{\mu\nu} = -2X_{\nu}^{(\mu}V_{\bar{\nu}} + 2\bar{X}_{\nu}^{(\mu}\bar{V}_{\bar{\nu}}
$$

$$
+ 2X_{\nu}^{\mu}X_{\sigma}^{(\nu\bar{\mu})}(Y^{\sigma}PV_{\bar{\rho}} + \bar{Y}^{\sigma}\bar{V}_{\bar{\rho}}).
$$

(76)

In fact, both local symmetries are part of the local Lorentz symmetries in DFT and should not be broken.

First let us briefly comment on the scalar field case. Upon the generic $(\nu, \bar{\nu})$ curved background above, with the choice of section $\partial^\nu = 0$, the scalar field kinetic term reduces to

$$
-\frac{1}{2}e^{-2d}H^{AB}\partial_\Phi B_{\partial\Phi} \Phi = -\frac{1}{2}e^{-2d}H_{\nu\rho}\partial_\nu \Phi \partial_\rho \Phi,
$$

which obviously generalises [20] into a covariant form.

Now we turn to the doubled Yang–Mills theory on curved backgrounds. We first factorise the doubled gauge potential, similarly to [70], as

$$
\mathcal{V}_A = B_A^B \mathcal{V}_B = \begin{pmatrix} \varphi^\mu \\ A_{\nu} + B_{\nu\rho}\varphi^\rho \end{pmatrix}, \quad \mathcal{V}_A = \begin{pmatrix} \varphi^\mu \\ A_{\nu} \end{pmatrix}.
$$

(78)

Like the doubled metric, the doubled vector potential $\mathcal{V}_A$ should be invariant under the DFT local Lorentz symmetries, and thus the Milne-shift transformations of the component fields are

$$
\delta_{\bar{m}}\varphi^\mu = 0, \quad \delta_{\bar{m}}A_{\nu} = -\left(\delta_{\bar{m}}B_{\mu\nu}\right)\varphi^\nu.
$$

(79)

The semi-covariant Yang–Mills field strength is

$$
F_{AB} = \nabla_A \mathcal{V}_B - \nabla_B \mathcal{V}_A - i [\mathcal{V}_A, \mathcal{V}_B] = 2\partial_{[A}\mathcal{V}_{B]} + 2\Gamma_{[AB]}C^C \mathcal{V}_C - i [\mathcal{V}_A, \mathcal{V}_B],
$$

(80)

where $\Gamma_{CAB}$ are the the DFT Christoffel symbols 58. From the torsionless property $\Gamma_{[ABC]} = 0$, it follows that $\Gamma_{[AB]C} = -\frac{1}{2}\Gamma_{CAB}$. Since the fully covariant field strength is $(PF\tilde{P})_{AB}$, we only need the projection

$$
P_M A^P N^B \Gamma_{CAB} = (P\partial_C \tilde{P} \bar{P})_{MN} = \frac{1}{2} (P\partial_C \mathcal{H} \bar{P})_{MN}.
$$

(81)

Using [59], [70], [75], and [81] in [80], the fully covariant field strength is given by

$$
(\mathcal{B}^{-1}P)_A^C (\mathcal{B}^{-1}\tilde{P})_B^D \mathcal{F}_{CD} = \hat{P}_A^C \hat{P}_B^D \left(2\partial_{[C} \mathcal{V}_{D]} - i[\mathcal{V}_C, \mathcal{V}_D] - \frac{1}{2} \hat{\nabla}^E \partial_E \mathcal{H}_{CD} \right) + 3\hat{\nabla}^E \partial_E (\mathcal{B} \mathcal{C} D)
$$

(82)

This is the curved generalisation of [87]. After a lengthy calculation we may obtain the explicit components,

$$
(\mathcal{B}^{-1}P \mathcal{F}(\mathcal{B}^{-1}\tilde{P}))_{\mu\nu} = -\frac{1}{2} \mathcal{F}^{\mu\nu}(\delta^\mu_{\nu} + Z_{\nu}^\mu)(\delta_{\nu}^\mu - Z_{\nu}^\nu),
$$

$$
(\mathcal{B}^{-1}P \mathcal{F}(\mathcal{B}^{-1}\tilde{P}))_{\mu\rho} = \frac{1}{2} \left[ H^{\mu\kappa}(\delta^\kappa_{\rho} - Z^\kappa_{\rho})\hat{f}_{\kappa\lambda} - \Upsilon_{\sigma\mu} \right],
$$

$$
(\mathcal{B}^{-1}P \mathcal{F}(\mathcal{B}^{-1}\tilde{P}))_{\nu\rho} = -\frac{1}{2} \left[ (\delta^\rho_{\mu} + Z^\rho_{\mu})H^\nu\lambda\hat{f}_{\nu\lambda} - \tilde{\Upsilon}_{\nu\rho} \right],
$$

$$
(\mathcal{B}^{-1}P \mathcal{F}(\mathcal{B}^{-1}\tilde{P}))_{\rho\sigma} = \frac{1}{2} \left[ \delta_{\rho\sigma}^{\nu} + Z^{\nu}_{\rho}\nu Z^{\nu}_{\sigma} \right] \hat{f}_{\nu\lambda},
$$

(83)

where the constituent tensors are now defined as

$$
\hat{f}^{\mu\nu} = H^{\mu\rho}H^\nu\sigma (f_{\rho\sigma} + H_{\rho\sigma})\varphi^\nu + i[\varphi^\mu, \varphi^\nu] - \bar{\omega}_{\mu\nu},
$$

$$
\hat{f}_{\mu\nu} = f_{\mu\nu} + iK_{\mu\rho}K_{\nu\sigma}\varphi^\rho\varphi^\sigma + \varphi^\rho H_{\rho\mu\nu} - u_{\mu\nu},
$$

$$
\Upsilon_{\mu\nu} = 2K_{\mu\rho}(\mathcal{D}^\rho\varphi^\nu + i[\varphi^\rho, \varphi^\nu])X^{\nu}_{\rho} + 4\bar{X}^{\nu}_{\rho} \mathcal{D}^\rho\varphi^\nu X^{\nu}_{\rho} + 4\bar{X}^{\nu}_{\rho} \mathcal{D}^\rho\varphi^\nu \bar{X}^{\nu}_{\rho}.
$$

(84)

Further explanation is in order. Every term in [54] is covariant under (ordinary undoubled) diffeomorphisms and Yang–Mills gauge symmetry, and also invariant under $\text{GL}(n) \times \text{GL}(\bar{n})$ local rotations, but not under the Milne shift, only the whole set of components of $(PF\tilde{P})_{AB}$ is so.

Specifically, $f_{\mu\nu}$ is the usual Yang–Mills field strength [70], while $\bar{\omega}_{\mu\nu}$ is $\bar{\omega}_{\mu\nu}$, $u_{\mu\nu}$ are the strain tensors for the vector field $\varphi^\nu$, carrying upper or lower symmetric indices, which can be expressed as symmetrisations of appropriate covariant derivatives,

$$
\bar{\omega}_{\mu\nu} = \mathcal{D}^\mu \varphi^\nu + \mathcal{D}^\nu \varphi^\mu, \quad u_{\mu\nu} = \mathcal{D}_\mu \varphi^\nu + \mathcal{D}_\nu \varphi^\mu,
$$

(85)

where, with $\Omega_{\mu\nu}$ to be explained later [71],

$$
\mathcal{D}^\mu \varphi^\nu = H^{\mu\rho} (\partial_\rho \varphi^\nu - i[A_\rho, \varphi^\nu]) + \Omega_{\mu\nu} \varphi^\rho,
$$

$$
\mathcal{D}_\mu \varphi^\nu = (\partial_\mu \varphi^\rho - i[A_\mu, \varphi^\rho]) K^\rho_{\nu}
$$

+ \frac{1}{2} (\partial_\rho K_{\mu\nu} + \partial_\nu K_{\mu\rho} - \partial_\mu K_{\nu\rho}) \varphi^\rho.
$$

(86)

Further, we have defined various covariant derivatives which are nontrivial only in genuine non-Riemannian
By construction, this action is invariant under diffeomorphisms, local $GL(n) \times GL(\bar{n})$ rotations, and Yang–Mills gauge transformations. However, without the contractions as in (87) and (88), the bare derivatives

\[
D_i \phi^\mu = Y_i^\rho (\partial_\rho \phi^\mu - i [A_\rho, \phi^\mu]) - \phi^\rho \partial_\rho Y_i^\mu,
\]

\[
\bar{D}_i \phi^\mu = \bar{Y}_i^\rho (\partial_\rho \phi^\mu - i [A_\rho, \phi^\mu]) - \phi^\rho \partial_\rho \bar{Y}_i^\mu,
\]

are anomalous under local $GL(n) \times GL(\bar{n})$ rotations due to the final terms containing derivatives of $Y_i^\mu$ and $\bar{Y}_i^\mu$.

While most of these covariant derivatives are novel, the upper-indexed covariant derivative,

\[
\mathcal{D}^\mu = H^{\mu \rho} \partial_\rho - i H^{\mu \rho} [A_\rho, _] + \Omega^\mu,
\]

was proposed in [93]. It can act on an arbitrary (undoubled) tensor, as it is equipped with a generalised Christoffel connection,

\[
\Omega^{\mu \nu \lambda} = - \frac{1}{2} \partial_\lambda H^{\mu \nu} - H^{[\mu \rho} \partial_{\nu \sigma]} X_\lambda^\rho - H^{[\mu \rho} \partial_{\nu \sigma]} \bar{X}_\lambda^\rho
\]

\[
-2 H^{\rho [\nu} \partial_{\nu \sigma]} X_\lambda^\rho \bar{Y}_i^\mu \bigg( Y_i^\gamma X_\lambda^\gamma - \bar{Y}_i^\gamma \bar{X}_\lambda^\gamma \bigg).
\]

Note that according to (87), only the symmetric part, $\Omega^{(\mu \nu) \lambda} = - \frac{1}{2} \partial_\lambda H^{\mu \nu}$, contributes to the strain tensor $\bar{n}^{\mu \nu}$.

The Lagrangian for the doubled Yang–Mills theory on general curved and non-Riemannian backgrounds,

\[
L_{YM} = 2 \text{Tr} \left[ \left( P \bar{F} F \right)^{\mu \nu} (P \bar{F} F)_{\mu \nu} + (P \bar{F} F)^{\mu \nu} (P \bar{F} F)^{\mu \nu} \right],
\]

can be obtained analogously to the flat case from the components in (83). The full result is

\[
L_{YM} = \text{Tr} \left[ \begin{pmatrix}
- \frac{1}{2} H^{\mu \nu} H^{\sigma \rho} \left( f_{\mu \nu} + \phi^\kappa H_{\kappa \rho \sigma} \right) \left( f_{\mu \nu} + \phi^\lambda H_{\lambda \rho \sigma} \right) - \frac{1}{2} H^{\mu \nu} H^{\sigma \rho} u_{\mu \nu} u_{\rho \sigma} + \frac{1}{2} K_{\mu \nu} [\phi^\rho, [\phi^\sigma, [\phi^\mu, \phi^\nu], \phi^\sigma]] \\
- \left( D^\mu Y_i^\nu - \bar{D}^\mu \phi^\nu Y_i^\mu + i \frac{1}{2} [\phi^\rho, [\phi^\sigma, \phi^\nu]] (1 + Z) X_i^\rho \right) \right] - K_{\mu \nu} D_i \phi^\mu \left( D^{\nu \rho} \phi^\nu - i [\phi^\nu, \phi^\rho] X_i^\rho \right) - 2 D_i \phi^\mu \bar{X}_i^\rho \bar{D}_i \phi^\nu X_i^\nu
\]

as well as some projection properties,

\[
H^{\mu \rho} H^{\sigma \nu} f_{\mu \nu} = (H K)^{\rho \nu} (H K)^{\nu \sigma} f_{\mu \nu},
\]

\[
H^{\mu \rho} u_{\mu \nu} u_{\rho \sigma} = (H K)^{\rho \nu} (H K)^{\nu \sigma} \bar{u}^{\rho \sigma},
\]

\[
H^{\mu \rho} u_{\mu \nu} Y_i^\nu = (H K)^{\rho \nu} D_i \phi^\nu,
\]

\[
H^{\mu \rho} u_{\rho \nu} \bar{Y}_i^\nu = (H K)^{\mu \nu} \bar{D}_i \phi^\nu,
\]

and

\[
K_{\mu \nu} \bar{u}^{\rho \sigma} X_i^\rho = K_{\mu \nu} D_\rho \phi^\nu, \quad K_{\mu \nu} \bar{u}^{\rho \sigma} \bar{X}_i^\rho = K_{\mu \nu} \bar{D}_\rho \phi^\nu.
\]

\[
\Upsilon_\mu ^\nu K_{\nu \rho} = 0 = \bar{\Upsilon}_\mu ^\nu K_{\nu \rho}, \quad Y_i^\rho u_{\mu \nu} Y_i^\nu = 0 = X_i^\rho \bar{u}^{\mu \nu} X_i^\nu.
\]

Upon taking the flat-spacetime limit with vanishing $B$-field, the connection $\Omega$ vanishes and the covariant derivatives $\mathcal{D}, \bar{D}$ all reduce to the Yang–Mills covariant derivative, $D = \partial - i [A, _]$. Thus, the field
The second and third lines are quadratic in the auxiliary constraints, while from the Lagrangian \( \mathcal{L}_q \) we recover \( \mathcal{L} \).

Finally, as a curved spacetime generalisation of \( \mathcal{L}_q \), we present the full particle action minimally coupled to the doubled Maxwell vector potential on a generic \((n, \bar{n})\) curved background,

\[
\mathcal{L} = \frac{1}{2} e^{-1} \mathcal{D}_x x^A \mathcal{D}_x x^B \mathcal{H}_{AB} - \frac{1}{2} e m^2 - q \mathcal{D}_x x^A \mathcal{Y}_A
\]

\[
= \frac{1}{2} e^{-1} \dot{x} \dot{x} K_{\mu \nu} - \frac{1}{2} e \left( m^2 + q^2 \varphi^\mu \varphi^\nu K_{\mu \nu} \right) - q \dot{\varphi} A_{\mu} + \frac{1}{2} e^{-1} \left( \dot{x}_\mu - a_{\mu} - B_{\mu \nu} \dot{x}^\nu - e q K_{\mu \nu} \varphi^\rho \right) H^{\mu \nu}
\]

\[
\times \left( \dot{x}_\nu - a_{\nu} - B_{\nu \sigma} \dot{x}^\sigma - e q K_{\nu \sigma} \varphi^\rho \right)
\]

\[
+ X_{\mu} (e^{-1} \dot{x}^{\mu} - q \varphi^\mu) \left( \dot{x}_\nu - a_{\nu} - B_{\nu \mu} \dot{x}^\mu \right) Y_i^\nu
\]

\[
- \dot{X}^\mu_{\mu} (e^{-1} \dot{x}^{\mu} + q \varphi^\mu) \left( \dot{x}_\nu - a_{\nu} - B_{\nu \mu} \dot{x}^\mu \right) Y_i^\nu .
\]

(97)

The first line on the right-hand side of the second equality is essentially the usual (undoubled) action for a charged ‘relativistic’ particle in the Riemannian subspace, with effective mass generated by the displacement vector field,

\[
m_{\text{eff}}^2 = m^2 + q^2 \varphi^\mu \varphi^\nu K_{\mu \nu} .
\]

(98)

The second and third lines are quadratic in the auxiliary variable \( a_{\mu} \) on the Riemannian subspace and hence are to be negligibly integrated out, resulting in

\[
H^{\mu \nu} \left( \dot{x}_\nu - a_{\nu} - B_{\nu \mu} \dot{x}^\mu - e q K_{\nu \mu} \varphi^\rho \right) = 0 .
\]

(99)

The last two lines are linear in \( a_{\mu} \) and hence impose constraints,

\[
X^i_{\mu} (e^{-1} \dot{x}^{\mu} - q \varphi^\mu) = 0 , \quad \bar{X}^i_{\mu} (e^{-1} \dot{x}^{\mu} + q \varphi^\mu) = 0 ,
\]

(100)

which can be unified into a single expression,

\[
e^{-1} \dot{x}^\rho Z_{\rho}^{\mu} = q \varphi^\rho (1 - KH)_{\mu} .
\]

(101)

It is worthwhile to note the conjugate momenta of \( x^\mu \),

\[
p_{\mu} = e^{-1} \left( K_{\mu \nu} + B_{\mu \rho} Z^\rho \right) \dot{x}^\nu - q A_{\mu} - q B_{\mu \nu} (Y_i x_i + \tilde{Y}_i \bar{x}_i) \varphi^\nu + e^{-1} (B_{\mu \rho} H^{\nu \sigma} + Z_\rho^{\nu \sigma}) \left( \dot{x}_\nu - a_{\nu} - B_{\nu \sigma} \dot{x}^\sigma - e q K_{\nu \sigma} \varphi^\rho \right) ,
\]

(102)

and some on-shell values for the tilde coordinates,

\[
e^{-1} (D_x \dot{x}_\mu - B_{\mu \nu} \dot{x}^\nu) = Z_\mu (p_{\mu} + q A_{\nu}) + q K_{\mu \nu} \varphi^\nu ,
\]

(103)

where actually only the momenta along the non-Riemannian directions, \( p_{\mu}, \bar{p}_{\mu} \), are relevant.

Clearly, (103) generalises the saturation velocity (58) as well as the immobility constraint (11) of the constant background (53) to the case of a curved background. Specifically, for a neutral particle of \( q = 0 \), from (103) we obtain the vanishing of the velocity along the (curved non-Riemannian) \( X^i_\mu \) and \( \bar{X}^i_\mu \) directions,

\[
X^i_{\mu} \dot{x}^\mu = 0 , \quad \bar{X}^i_{\mu} \dot{x}^\mu = 0 .
\]

(104)

Taking the \( \tau \)-derivative of these gives expressions that may be viewed as ‘non-Riemannian geodesic equations’,

\[
X^i_{\mu} \dot{x}^\mu + \partial_{(\mu} X^i_{\nu)} \dot{x}^\nu = 0 , \quad \bar{X}^i_{\mu} \dot{x}^\mu + \partial_{(\mu} \bar{X}^i_{\nu)} \dot{x}^\nu = 0 ,
\]

(105)

For genuine curved backgrounds where \( \partial_{(\mu} X^i_{\nu)} \) or \( \partial_{(\mu} \bar{X}^i_{\nu)} \) are nontrivial, this indicates that the immobility of a fracton is rather non-trivial. Related to this, it is worthwhile to note that the first curved non-Riemannian DFT background reported in [73] was shown in [82] to admit only a finite number of isometries, implying the absence of higher-moment conservations. Further investigation with more examples is desired.

VII. Discussion

Existing field theoretical intuition on fractons is largely based on dipole conservation for charged particles. Contradistinctly, in our scheme the immobility is universal regardless of charge, since it originates from the ‘geodesic’ particle action (9). Accordingly, our current (107) contains the energy-momentum tensor rather than a charge density of any sort.

Analysis on a spinor field is also possible, following [61, 64]. We merely comment that, since DFT vielbeins square to projectors like \( V^A \bar{V}^B = P_{AB} \), on the flat Background (8) the doubled Dirac equation

\[
\gamma \psi = V^A \gamma^\alpha \partial_A \psi = 0
\]

(106)

gives

\[
0 = (\gamma)^2 \psi = P^{AB} \partial_A \partial_B \psi - \frac{1}{2} \mathcal{H}^{AB} \partial_A \partial_B \psi = \frac{1}{2} \partial_A \partial^A \psi .
\]

Thus, the massless spinor is also fractonic, like (66). Dualisation of the full strain-Maxwell model including the non-Riemannian sector (56) and the connection to D-branes, following [26, 24, 91, 100], and also deserve further study.

The charged particle action (55, 97) is of interest even upon a genuine Riemannian/Minkowskian (0, 0) flat background,

\[
S_{\gamma}^{(0,0)} = \int d \tau \frac{1}{2} e^{-1} \dot{x}^i \dot{x}_i - \frac{1}{2} e (m^2 + q^2 \varphi^\mu \varphi^\mu) - q \dot{x}^\mu A_\mu .
\]

(107)

Minimally coupled to the doubled vector potential \( V_A \), this particle action naturally interacts with the Maxwell vector potential of photons \( A_\mu \), and further with the elasticity displacement vector of phonons \( \varphi^\mu \), satisfying, from
This set of equations may provide a novel effective description of polarons \[101\]–\[104\]. Deformations of a periodic potential of a crystal lattice are described by phonons, or the displacements of atoms from their equilibrium positions. Electrons moving inside the crystal interact with the displacements, which is known as electron-phonon coupling. Such electrons with the accompanying deformation are called polarons. They move freely across the crystal, but with increased effective mass. This polaron picture essentially agrees with \[107\] and \[108\] above. The charged particles can correspond to both atomic nuclei and electrons. From \[108\], the lattice structure of the atomic nuclei naturally sets the dual periodic potential of a crystal lattice are described by the effective mass \[107\] from the condensation of the displacement vector \(\varphi^a\). We recall the effective mass formula \[60\] and expand the square root,

\[
m_{\text{eff.}} = \sqrt{m^2 + q^2 \varphi^a \varphi_a} = m \left[ 1 + \frac{1}{3} \left( \frac{\varphi^a \varphi_a}{m} \right)^2 + \cdots \right].
\]

We compare this with a well-known formula for the effective mass of a polaron obtained from estimating its self-energy \[105, 106\],

\[
m_{\text{known}} \simeq m \left[ 1 + \frac{1}{3}(\alpha_{e-ph})^2 + 0.0236(\alpha_{e-ph})^2 \right],
\]

where \(\alpha_{e-ph}\) is a dimensionless electron-phonon coupling constant. From the leading-order terms in the two formulae, we identify \(2 \varphi^a \varphi_a = \frac{1}{3} \alpha_{e-ph}\) which in turn gives, from \[109\], \(m_{\text{eff.}}/m \simeq 1 + \frac{1}{3}(\alpha_{e-ph})^2\), and hence differs from \[110\] at sub-leading order. We call for experimental verification.

For the Riemannian subspace we have mostly envisaged a Minkowskian signature \(\ddot{g}\), such that time can flow without suffering from immobility and that the effective mass \[60\] is not necessarily bigger than the true mass. Intriguingly then, in the case of time crystals \[107\], where \(\varphi^a \varphi_a\) would be time-like or negative, our formula seems to predict that the effective mass of polarons in time crystals should become smaller. Note that a time crystal is a quantum system of particles for which the ground state is characterised by repetitive periodic motion of the particles.

On the other hand, if we were to choose the Euclidean signature \(\eta_{ab} = \delta_{ab}\) for the Riemannian subspace and let \((n, \bar{n}) = (1, 1)\), thereby including two non-Riemannian directions, one temporal and the other spatial \(74, 75, 78–80, 83, 85, 108, 109\), the corresponding fracton physics would match that of the non-relativistic string \[81, 82, 110\] and Newton–Cartan gravities \[57, 111, 117\]. Eq. \[68\] then further implies that time therein can start to flow if the timelike displacement vector condenses, setting

\[
\dot{t} = eq\varphi^a.
\]

It would be of utmost interest if any of the non-Riemannian geometries underlying the modified Maxwell equations \[61, 63\] are realised in Nature. Some well-known singularities in GR \[118–120\] have recently been identified as regular non-Riemannian geometries \[121\]. Approaching them, geodesics indeed become immobile. Extra dimensions, if any, might be non-Riemannian \[76\] and therefore fractonic.

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**Appendix**

**A. Doubled Yang–Mills Energy-Momentum Tensor on a Constant Flat Background**

In this Appendix, we write down the explicit components of the energy-momentum tensor \[32\] for the doubled Yang–Mills theory \[30\]. In terms of undoubled spacetime indices, the relevant pieces appearing in the on-shell

\[
\partial_e f^{ca} = J^a, \quad J^a(x) = \sum \alpha \int \delta \phi^a_c(x) \delta^D(x - x_a(\tau)),
\]

\[
\partial_e u^a = \tilde{T}_a, \quad \tilde{T}_a(x) = \sum \alpha \int \delta \phi^2 \varphi_a \delta^D(x - x_a(\tau)).
\]

\[\text{(108)}\]

flow without suffering from immobility and that the effective mass \[60\] is not necessarily bigger than the true mass. Intriguingly then, in the case of time crystals \[107\], where \(\varphi^a \varphi_a\) would be time-like or negative, our formula seems to predict that the effective mass of polarons in time crystals should become smaller. Note that a time crystal is a quantum system of particles for which the ground state is characterised by repetitive periodic motion of the particles.

On the other hand, if we were to choose the Euclidean signature \(\eta_{ab} = \delta_{ab}\) for the Riemannian subspace and let \((n, \bar{n}) = (1, 1)\), thereby including two non-Riemannian directions, one temporal and the other spatial \(74, 75, 78–80, 83, 85, 108, 109\), the corresponding fracton physics would match that of the non-relativistic string \[81, 82, 110\] and Newton–Cartan gravities \[57, 111, 117\]. Eq. \[68\] then further implies that time therein can start to flow if the timelike displacement vector condenses, setting

\[
\dot{t} = eq\varphi^a.
\]

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conserved current \([17]\) are

\[
T^{\mu \nu} = \nabla \left[ (FHF)^{\mu} \rho H_{\rho \nu} - H^{\mu \nu} (FHF)_{\rho \nu} + (FHF)^{\mu} Z_\rho \nu - Z_\rho (FHF)^{\mu} \right] - 2 \partial_\rho \nabla \left[ \varphi^\lambda (PFP + \bar{PFP})^{\mu \nu} \right],
\]

\[
T^{\mu \nu} = \nabla \left[ (FHF)^{\mu} \rho H_{\rho \nu} - H^{\mu \nu} (FHF)_{\rho \nu} + (FHF)^{\mu} Z_\rho \nu - Z_\rho (FHF)^{\mu} \right] - 2 \partial_\rho \nabla \left[ \varphi^\lambda (PFP + \bar{PFP})^{\mu \nu} \right] + \delta^{\mu \nu}_\rho \left. \right| \right|_{YM},
\]

To evaluate these we need the explicit expressions,

\[
(FHF)^{\mu \nu} = -D_\rho \varphi^\mu H^\rho \sigma D_\sigma \varphi^\nu + i Z_\rho \varphi^\rho D_\sigma \varphi^\nu - [\varphi^\mu, \varphi^\nu] D_\sigma \varphi^\rho - [\varphi^\mu, \varphi^\nu] K_{\rho \sigma} \varphi^\rho \varphi^\nu,
\]

\[
(FHF)^{\nu} = -D_\rho \varphi^\rho H^\rho \sigma D_\sigma \varphi^\nu + i \varphi^\alpha \varphi^\nu \left( K_{\rho \sigma} D_\rho \varphi^\rho \sigma D_\sigma \varphi^\nu + Z_\rho \sigma D_\rho \varphi^\rho D_\sigma \varphi^\nu \right),
\]

\[
(FHF)^{\mu \nu} = f_\mu \varphi^\mu H^\rho \sigma D_\sigma \varphi^\nu - i \left( D_\mu \varphi^\nu K_{\rho \sigma} + f_\mu \varphi^\nu \right) \left[ \varphi^\rho, \varphi^\nu \right] + D_\mu \varphi^\nu Z_\rho \sigma D_\sigma \varphi^\nu,
\]

\[
(FHF)^{\mu \nu} = f_\mu \varphi^\mu H^\rho \sigma D_\sigma \varphi^\nu - D_\mu \varphi^\rho D_\nu \varphi^\sigma K_{\rho \sigma} + Z^\mu \rho D_\mu \varphi^\rho D_\nu \varphi^\sigma,
\]

as well as

\[
-2 (PFP + \bar{PFP})^{\mu \nu} = H^{\mu \nu} + i \varphi^\alpha D_\mu \varphi^\nu - i \left( D_\mu \varphi^\nu K_{\rho \sigma} + f_\mu \varphi^\nu \right) \left[ \varphi^\rho, \varphi^\nu \right] + Z_\rho \sigma K_{\rho \sigma} + \left( H^{\mu \nu} f_\rho - Z_\rho \sigma D_\sigma \varphi^\rho \right) Z^\nu \sigma.
\]

Substituting into \([A1]\), we acquire all the components of the doubled Yang–Mills energy-momentum tensor,

\[
T_{a b} = \nabla \left[ f^{a c} f_{b c} + D_\rho \varphi^\rho D_\sigma \varphi^\nu - D_\sigma \varphi^\rho D_\nu \varphi^\sigma + [\varphi^\rho, \varphi^\sigma] \left[ \varphi^\rho, \varphi^\sigma \right] + \partial_\lambda \left( \varphi^\lambda u^a_b \right) - 2 Z_\rho \sigma \left( f^{a \rho} f^{\sigma} \right) \right] + \delta_{a b} \left| \right| \right|_{YM},
\]

\[
T_{a i} = \nabla \left[ \left( f^{a c} + D_\rho \varphi^\rho \right) f_{b c} + D_\sigma \varphi^\rho D_\nu \varphi^\sigma - \partial_\lambda \left( \varphi^\lambda f_{a b} \right) + Z_\rho \sigma \left( f^{a \rho} D_\sigma \varphi^\nu + D^\sigma \varphi^\rho \right) \right],
\]

\[
T_{a i} = \nabla \left[ \left( f^{a c} - D_\rho \varphi^\rho \right) f_{b c} + D_\sigma \varphi^\rho D_\nu \varphi^\sigma - \partial_\lambda \left( \varphi^\lambda f_{a b} \right) + Z_\rho \sigma \left( f^{a \rho} D_\sigma \varphi^\nu + D^\sigma \varphi^\rho \right) \right],
\]

\[
T_{a i} = \nabla \left[ \left. - D^\rho \varphi^\rho D_\sigma \varphi^\nu + \partial_\lambda \left( \varphi^\lambda D_\nu \varphi^\sigma \right) - Z_\rho \sigma \left( D_\sigma \varphi^\nu D_\rho \varphi^\sigma \right) + i \left[ \varphi^\rho, \varphi^\sigma \right] f_{a b} \right) \right],
\]

\[
T_{a i} = \nabla \left[ \left. - D^\rho \varphi^\rho D_\sigma \varphi^\nu + \partial_\lambda \left( \varphi^\lambda D_\nu \varphi^\sigma \right) + Z_\rho \sigma \left( D_\sigma \varphi^\nu D_\rho \varphi^\sigma \right) + i \left[ \varphi^\rho, \varphi^\sigma \right] f_{a b} \right) \right],
\]

\[
T_{a i} = \nabla \left[ \left. - D^\rho \varphi^\rho D_\sigma \varphi^\nu - i \left[ \varphi^\rho, \varphi^\sigma \right] D_\nu \varphi^\sigma + \partial_\lambda \left( \varphi^\lambda D_\nu \varphi^\sigma \right) - Z_\rho \sigma \left( D_\sigma \varphi^\nu D_\rho \varphi^\sigma \right) - i \left[ \varphi^\rho, \varphi^\sigma \right] f_{a b} \right) \right],
\]

\[
T_{a i} = \nabla \left[ \left. - D^\rho \varphi^\rho D_\sigma \varphi^\nu - i \left[ \varphi^\rho, \varphi^\sigma \right] D_\nu \varphi^\sigma + \partial_\lambda \left( \varphi^\lambda D_\nu \varphi^\sigma \right) + Z_\rho \sigma \left( D_\sigma \varphi^\nu D_\rho \varphi^\sigma \right) + i \left[ \varphi^\rho, \varphi^\sigma \right] f_{a b} \right) \right],
\]

\[
T^{a b} = \nabla \left[ \left. 2 D_\rho \varphi^\rho D^\rho f_{a b} - 2 i \left[ \varphi^\rho, \varphi^\sigma \right] D_\rho f_{b c} + \partial_\lambda \left( \varphi^\lambda f_{a b} \right) + Z_\rho \sigma \left( D_\sigma \varphi^\nu D_\rho \varphi^\sigma - i \left[ \varphi^\rho, \varphi^\sigma \right] f_{a b} \right) \right) \right],
\]

\[
T^{a i} = \nabla \left[ \left. - \left( f^{a c} + D_\rho \varphi^\rho \right) D_\nu \varphi^\sigma - i \left[ \varphi^\rho, \varphi^\sigma \right] D^\nu \varphi^\sigma + \partial_\lambda \left( \varphi^\lambda D^\nu \varphi^\sigma \right) - Z_\rho \sigma \left( D^\nu \varphi^\sigma D_\rho \varphi^\sigma \right) - i \left[ \varphi^\rho, \varphi^\sigma \right] f_{a b} \right) \right],
\]

\[
T^{a i} = \nabla \left[ \left. - \left( f^{a c} - D_\rho \varphi^\rho \right) D_\nu \varphi^\sigma - i \left[ \varphi^\rho, \varphi^\sigma \right] D^\nu \varphi^\sigma + \partial_\lambda \left( \varphi^\lambda D^\nu \varphi^\sigma \right) + Z_\rho \sigma \left( D^\nu \varphi^\sigma D_\rho \varphi^\sigma \right) - i \left[ \varphi^\rho, \varphi^\sigma \right] f_{a b} \right) \right],
\]

\[
T^{a i} = \nabla \left[ \left. 2 D_\rho \varphi^\rho D^\rho f_{a b} - \left[ \varphi^\rho, \varphi^\sigma \right] D_\nu \varphi^\sigma + \partial_\lambda \left( \varphi^\lambda f_{a b} \right) + 2 i Z_\rho \sigma \left( D_\sigma \varphi^\nu D_\rho \varphi^\sigma - i \left[ \varphi^\rho, \varphi^\sigma \right] f_{a b} \right) \right) \right],
\]

and, from the projection properties,

\[
T^{i j} = 0, \quad T^{i j} = 0, \quad T^{j i} = {\delta^{i j}}_L Y_{YM}, \quad T^{i j} = {\delta^{i j}}_L Y_{YM}, \quad T^{i a} = \eta^{i a} T^a_b, \quad T^{a i} = \eta^{a i} T^i_a, \quad T^{i i} = -T^{i i}, \quad T^{a c} \eta^{c b} = T^b_c \eta^{a c}.
\]

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