Blow-up solutions of the intercritical inhomogeneous NLS equation: the non-radial case

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Abstract
In this paper we consider the inhomogeneous nonlinear Schrödinger (INLS) equation

\[
i \partial_t u + \Delta u + |x|^{-b}|u|^{2\sigma}u = 0, \quad x \in \mathbb{R}^N
\]

with \( N \geq 3 \). We focus on the intercritical case, where the scaling invariant Sobolev index \( s_c = \frac{N}{2} - \frac{2-b}{2\sigma} \) satisfies \( 0 < s_c < 1 \). In a previous work, for radial initial data in \( \dot{H}^{s_c} \cap \dot{H}^1 \), we prove the existence of blow-up solutions and also a lower bound for the blow-up rate. Here we extend these results to the non-radial case. We also prove an upper bound for the blow-up rate and a concentration result for general finite time blow-up solutions in \( H^1 \).

1 Introduction
In this work we consider the initial value problem (IVP) for the inhomogeneous nonlinear Schrödinger (INLS) equation

\[
\begin{cases}
i \partial_t u + \Delta u + |x|^{-b}|u|^{2\sigma}u = 0, & x \in \mathbb{R}^N, \ t > 0, \\
u(0) = u_0,
\end{cases}
\]

for \( N \geq 3 \) and some \( b, \sigma > 0 \). This model is a generalization of the classical nonlinear Schrödinger (NLS) equation and also has applications in laser beam propagation upon a nonlinear optical medium [22, 27].

We are mainly interested in the intercritical regime. To understand this terminology, we recall that if \( u(x, t) \) solves (1.1) so does \( u_\lambda(x, t) = \lambda^{\frac{4}{2\sigma}} u(\lambda x, \lambda^2 t) \) and also \( \|u_\lambda(0)\|_{\dot{H}^{s_c}} = \|u_0\|_{\dot{H}^{s_c}} \) where \( s_c = \frac{N}{2} - \frac{2-b}{2\sigma} \). The mass-supercritical and energy-subcritical regime is such

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that $0 < s_c < 1$ and we can reformulate this condition as

$$\frac{2-b}{N} < \sigma < \frac{2-b}{N-2}. \quad (1.2)$$

Over the last few years, the INLS equation has been the subject of a great deal of mathematical research. This is part of a recently growing interest in the global dynamics of NLS type equations lacking the usual symmetries. In the present case, the translation invariance is not present and there is a space-dependent singular coefficient in the nonlinearity. Several results concerning well-posedness theory, existence and concentration of blow-up solutions, stability of solitary waves and asymptotic behavior of global solutions have been recently obtained for the INLS model [1–9, 11–21, 23, 24, 26, 30, 31]. In particular, $H^1$-solutions have the mass and the energy conserved in time, more precisely, if $u(t)$ is a solution to (1.1) on some time interval $I \subset \mathbb{R}$, then for any $t \in I$

$$M[u(t)] = \int |u(x,t)|^2 \, dx = M[u_0] \quad (1.3)$$

and

$$E[u(t)] = \frac{1}{2} \int |\nabla u(x,t)|^2 \, dx - \frac{1}{2\sigma + 2} \int |x|^{-b} |u(x,t)|^{2\sigma+2} \, dx = E[u_0]. \quad (1.4)$$

The main goal of this paper is to remove the radial assumption in our previous work [7]. This is in the same vein as the recent papers by [3, 24], where the authors reported results for the INLS equation in the non-radial case that have remained out of reach so far for the NLS equation. Although the presence of the term $|x|^{-b}$ in the nonlinearity introduces several challenging technical difficulties in the study of well-posedness and scattering theories, its decay away from the origin provides a new tool in the study of the global dynamics of solutions that avoid the need for a radial assumption.

Our first result is related to finite time solutions in $H^1$. The existence of such solutions for the INLS equation (1.1) was obtained by the second author [17] in the virial space $\Sigma := \{ f \in H^1; |x| f \in L^2 \}$. This result was extended by [14] in radial setting and more recently by Ardila and the first author in [2, 3] for general initial data. In particular, the result in [3] says that if $\frac{2-b}{N} < \sigma < \min \left\{ \frac{2-b}{N-2}, \frac{2}{N} \right\}$, then blow-up occurs in finite time. Under the same restriction on $\sigma$, we prove an universal space-time upper bound on the blow-up rate.

**Theorem 1.1** Let $N \geq 3$, $0 < b < \min \left\{ \frac{N}{2}, 2 \right\}$ and $\frac{2-b}{N} < \sigma < \min \left\{ \frac{2-b}{N-2}, \frac{2}{N} \right\}$. Let $u_0 \in H^1$ and assume that the maximal time of existence $T^* > 0$ for the corresponding solution $u \in C([0, T^*): H^1)$ of (1.1) is finite. Define $\beta = \frac{2-\sigma N}{b}$, then there exists a universal constant $C = C(u_0, \sigma, b, N) > 0$ such that the following space-time upper bound holds

$$\int_t^{T^*} (T^* - \tau) \| \nabla u(\tau) \|^2_{L^2} \, d\tau \leq C (T^* - t)^{\frac{2\beta}{1+\beta}}, \quad (1.5)$$

for $t$ close enough to $T^*$.

This type of result was first proved by Merle et al. [29] for the NLS equation and then extended in [7] for the INLS equation, both in the radial case. Here we show that the radial assumption can be removed in the INLS setting. The new ingredient is a localized virial type inequality satisfied by the non-radial solutions of the INLS equation (see Lemma 2.1).
As a consequence of (1.5), we deduce an upper bound on the blow-up rate along a sequence of times. Indeed, there exists a sequence \( \{t_n\}_{n=1}^{\infty} \subset [0, T^*) \) with \( t_n \rightarrow T^* \) such that
\[
\|\nabla u(t_n)\|_{L^2} \leq \frac{C}{(T^* - t_n)^{\frac{1}{p}}}, \quad \text{as} \quad n \rightarrow +\infty.
\]

Finite time solutions in \( H^1 \) also enjoy other important properties. For instance, from the \( H^1 \) local Cauchy theory obtained by [23], if \( T^* < \infty \), then
\[
\|\nabla u(t)\|_{L^2} \rightarrow \infty, \quad \text{as} \quad t \rightarrow T^*.
\] (1.6)
Moreover, the recent work by [1] proved that these solutions obey the lower bound
\[
\|\nabla u(t)\|_{L^2} \geq \frac{c}{(T^* - t)^{\frac{1}{2}}},
\]
where \( V \) is a solution to elliptic equation
\[
\Delta V + |x|^{-b} |V|^{2\sigma} V - |V|^{\sigma_b} - 2 V = 0
\] (1.8)
with minimal \( L^{\sigma_b} \)-norm. Applying the last inequality to the energy conservation (1.4) we have
\[
E(u_0) \geq \frac{1}{2} \|\nabla u(t)\|_{L^2}^2 \left( 1 - \frac{\|u(t)\|_{L^{\sigma_b}}^{2\sigma}}{\|V\|_{L^{\sigma_b}}^{2\sigma}} \right),
\]
which implies that finite time solutions must satisfy
\[
\sup_{t \in [0, T^*)} \|u(t)\|_{L^{\sigma_b}} \geq \|V\|_{L^{\sigma_b}}.
\] In the next result, we prove that the \( L^{\sigma_b} \) norm of such solutions concentrates around the origin in the non-radial case.

**Theorem 1.2** Let \( \sigma_b = \frac{2N\sigma}{2-b} \) such that \( \dot{H}^{\sigma_b} \subset L^{\sigma_b} \). Assume \( N \geq 3 \), \( 0 < b < \min\left\{ \frac{N}{2}, 2 \right\} \) and \( \frac{2-b}{N} < \sigma < \min\left\{ \frac{2-b}{N-2}, \frac{2}{N} \right\} \). Given \( u_0 \in H^1 \) so that the maximal time of existence \( T^* > 0 \) for the corresponding solution \( u \in C([0, T^*) : H^1) \) of (1.1) is finite. Then there exist positive constants \( c_0 \) and \( c_1 \) depending only on \( N, \sigma \) and \( b \) such that
\[
\liminf_{t \rightarrow T^*} \int_{|x| \leq c_{u_0}} \|\nabla u(t)\|_{L^2}^{-2\cdot\sigma/N} |u(x, t)|^{\sigma_b} dx \geq c_0,
\] (1.9)
where \( c_{u_0} = c_1 \max \left\{ \|u_0\|_{L^2}^2, \|u_0\|_{L^2}^{2\sigma + 2 - \sigma_b} \right\} \).

Note that the concentration window size \( c_{u_0} \|\nabla u(t)\|_{L^2}^{-2\cdot\sigma/N} \) can be made arbitrarily small for times close to the finite maximal time of existence in view of (1.6) and (1.3). To prove this result we use a spatial and frequency localization also employed by [25] for the radial 3D cubic NLS equation (see also the work [5], where Campos and the first author treat the radial INLS setting).

Next we turn to the IVP (1.1) with initial data \( u_0 \in \dot{H}^{\sigma_b} \cap \dot{H}^1 \). Recently, in collaboration with Guzmán [8], we proved that the IVP (1.1) is locally well-posed in this space. In [7],
applying the techniques developed by [28] for the mass-supercritical NLS equation, we obtain the existence of radially symmetric solution non-positive energy solutions with finite maximal time of existence and the behavior of the $H^{s_c}$ norm. Here we consider a general initial data. Our main results in this direction are the following.

**Theorem 1.3** Let $N \geq 3$, $0 < b < \min\{\frac{N}{2}, 2\}$ and $\frac{2-b}{N} < \sigma < \min\{2-b, \frac{2}{N}\}$. If $u_0 \in \dot{H}^{s_c} \cap \dot{H}^1$ and $E(u_0) \leq 0$, then the maximal time of existence $T^* > 0$ of the corresponding solution $u(t)$ to (1.1) is finite.

**Theorem 1.4** Let $\sigma_c = \frac{2N\sigma}{2-b}$ such that $\dot{H}^{s_c} \subset L^{\sigma_c}$. Assume $N \geq 3$, $0 < b < \min\{\frac{N}{2}, 2\}$ and $\frac{2-b}{N} < \sigma < \min\{2-b, \frac{2}{N}\}$. Given $u_0 \in \dot{H}^{s_c} \cap \dot{H}^1$ so that the maximal time of existence $T^* > 0$ of the corresponding solution $u$ to (1.1) is finite and satisfies

$$\|\nabla u(t)\|_{L^2} \geq \frac{c}{(T^* - t)^{\frac{1-\sigma_c}{2}}},$$

for some constant $c = c(N, \sigma)$ and $t$ close enough to $T^*$. Then there exists $\gamma = \gamma(N, \sigma, b) > 0$ such that

$$c \|u(t)\|_{\dot{H}^{s_c}} \geq \|u(t)\|_{L^{\sigma_c}} \geq |\log(T^* - t)|^{\gamma}, \quad as \ t \to T^*.$$

The condition (1.10) is very natural and it is easily deduced if a local Cauchy theory in $\dot{H}^1$ is available (see [10] and also the Introduction in [28] for the argument in the NLS case). Moreover, if we additionally assume that $u_0 \in H^1 \subset \dot{H}^{s_c} \cap \dot{H}^1$, it is automatically satisfied as we mentioned before.

As a consequence of the previous theorem and the well-posedness theory in $\dot{H}^{s_c} \cap \dot{H}^1$ obtained in [8] we deduce that the $\dot{H}^{s_c}$-norm of the blow-up solution cannot be uniformly bounded.

**Corollary 1.5** Assume $N \geq 3$, $0 < b < \min\{\frac{N}{2}, 2\}$ and $\frac{2-b}{N} < \sigma < \min\{2-b, \frac{2}{N}\}$. Given $u_0 \in \dot{H}^{s_c} \cap \dot{H}^1$ so that the maximal time of existence $T^* > 0$ of the corresponding solution $u$ to (1.1) is finite, then

$$\limsup_{t \to T^*} \|u(t)\|_{\dot{H}^{s_c}} = +\infty.$$

Note that the previous result does not hold in the mass-critical case ($s_c = 0$), since the scaling invariant $L^2$-norm is preserved by the mass conservation (1.3).

It should be emphasized that all the above results are still unknown for the classical NLS equation in the non-radial case. In our proofs we need the restriction $\frac{2-b}{N} < \sigma < \min\{2-b, \frac{2}{N}\}$, which implies that the argument cannot be extended to the NLS setting since $b = 0$ in this case. Moreover, if $b > 4/N$, then $\frac{2-b}{N-2} < \frac{2}{N}$ and the result covers all the intercritical range (1.2). Finally, the assumption $N \geq 3$ can probably be relaxed and we assume this condition due to the local theory in $\dot{H}^{s_c} \cap \dot{H}^1$ obtained in [8].

This paper is organized as follows. In Sect. 2, we prove a virial type estimate and use it to prove Theorem 1.1. The proof of the concentration result Theorem 1.2 is also presented in this section. A non-radial interpolation estimate based on a Morrey–Campanato type semi-norm is presented in Sect. 3 (see Lemma 3.1). The non-radial interpolation estimate and the virial type estimate are the main tools to obtain the existence of blow-up solutions and also a lower bound for the blow-up rate.
2 Blow-up solutions in $H^1$

2.1 A virial type estimate

Let $\phi$ be a non-negative radial function $\phi \in C_0^\infty(\mathbb{R}^N)$, such that

$$\phi(x) = \begin{cases} |x|^2, & \text{if } |x| \leq 2 \\ 0, & \text{if } |x| \geq 4 \end{cases}$$

satisfying

$$\phi(x) \leq c|x|^2, \quad |\nabla \phi(x)|^2 \leq c\phi(x) \quad \text{and} \quad \partial_r^2 \phi(x) \leq 2, \quad \text{for all } x \in \mathbb{R}^N,$$

(2.1)

with $r = |x|$. Then, define $\phi_R(x) = R^{-2}\phi \left( \frac{x}{R} \right)$. Consider $u \in C([0, \tau_*], H^{s_c} \cap \dot{H}^1)$ a solution to (1.1). For any $R > 0$ and $t \in [0, \tau_\ast]$ define the function

$$z_R(t) = \int \phi_R |u(t)|^2 \, dx.$$ 

From direct computations (see, for instance, Proposition 7.2 in [19]), we obtain

$$z_R'(t) = 2\Im \int \nabla \phi_R \cdot \nabla u(t) \bar{u}(t) \, dx$$

and

$$z_R''(t) = 4\Re \sum_{j,k=1}^N \int \partial_j u(t) \partial_k \bar{u}(t) \partial_{jk}^2 \phi_R \, dx - \int |u(t)|^2 \Delta^2 \phi_R \, dx - \frac{2\sigma}{\sigma + 1} \int |x|^{-b} |u(t)|^{2\sigma + 2} \Delta \phi_R \, dx + \frac{2}{\sigma + 1} \int \nabla \left( |x|^{-b} \right) \cdot \nabla \phi_R |u(t)|^{2\sigma + 2} \, dx.$$  

From the Cauchy–Schwarz inequality and mass conservation (1.3), we have

$$z_R(t) \leq cR^2 \|u_0\|_{L^2}^2$$

and

$$z_R'(t) \leq cR \|\nabla u(t)\|_{L^2} \|u_0\|_{L^2}.$$  

On the other hand, since $\phi$ is radial, we have

$$z_R'(t) = 2 \Im \int \partial_r \phi_R \frac{x}{r} \cdot \nabla u(t) \bar{u}(t) \, dx$$

and

$$z_R''(t) = 4 \int \frac{\partial_r \phi_R}{r} |\nabla u(t)|^2 \, dx + 4 \int \left( \frac{\partial_r^2 \phi_R}{r^2} - \frac{\partial_r \phi_R}{r^3} \right) |x \cdot \nabla u(t)|^2 \, dx - \int |u(t)|^2 \Delta^2 \phi_R \, dx - \frac{2\sigma}{\sigma + 1} \int \left( N - 1 + \frac{b}{\sigma} \right) \frac{\partial_r \phi_R}{r} |x|^{-b} |u(t)|^{2\sigma + 2} \, dx,$$

(2.2)

where $\partial_r$ denotes the derivative with respect to $r = |x|$.

Consider the following functional

$$P[u(t)] = \int |\nabla u(t)|^2 \, dx - \frac{N\sigma + b}{2\sigma + 2} \int |x|^{-b} |u(t)|^{2\sigma + 2} \, dx,$$
then can rewrite $z_R''(t)$ in (2.2) as

$$z_R''(t) = 8P[u(t)] + K_1 + K_2 + K_3,$$

where

$$K_1 = 4 \int \left( \frac{\partial_r \phi_R}{r} - 2 \right) |\nabla u(t)|^2 dx + 4 \int \left( \frac{\partial_r^2 \phi_R}{r^2} - \frac{\partial_r \phi_R}{r^3} \right) |x \cdot \nabla u(t)|^2 dx,$$

$$K_2 = -\frac{2 \sigma}{\sigma + 1} \int \left[ \partial_r^2 \phi_R + \left( N - 1 + \frac{b}{\sigma} \right) \frac{\partial_r \phi_R}{r} - 2N - \frac{2b}{\sigma} \right] |x|^{-b} |u(t)|^{2\sigma + 2} dx,$$

$$K_3 = -\int |u(t)|^2 \Delta^2 \phi_R dx.$$

We claim that there exists $c > 0$ such that

$$z_R''(t) \leq 8P[u(t)] + \frac{c}{R^2} \int_{2R \leq |x| \leq 4R} |u(t)|^2 dx + \int_{|x| \geq 2R} |x|^{-b} |u(t)|^{2\sigma + 2} dx. \quad (2.3)$$

Indeed, we first show that $K_1 \leq 0$. To this end, we consider the following region in $\mathbb{R}^N$

$$\Omega = \left\{ x \in \mathbb{R}^N; \frac{\partial_r^2 \phi_R(x)}{r^2} - \frac{\partial_r \phi_R(x)}{r^3} \leq 0 \right\}.$$

Since $\partial_r^2 \phi_R \leq 2$ it follows that $\partial_r \phi_R(|x|) \leq 2|x|$, for all $x \in \mathbb{R}^N$. Now, splitting the integration and using the Cauchy-Schwartz inequality, we get

$$K_1 = 4 \int_\Omega \left( \frac{\partial_r \phi_R}{r} - 2 \right) |\nabla u(t)|^2 dx + 4 \int_\Omega \left( \frac{\partial_r^2 \phi_R}{r^2} - \frac{\partial_r \phi_R}{r^3} \right) |x \cdot \nabla u(t)|^2 dx$$

$$= 4 \int_{\mathbb{R}^N \setminus \Omega} \left( \frac{\partial_r \phi_R}{r} - 2 \right) |\nabla u(t)|^2 dx + 4 \int_{\mathbb{R}^N \setminus \Omega} \left( \frac{\partial_r^2 \phi_R}{r^2} - \frac{\partial_r \phi_R}{r^3} \right) |x \cdot \nabla u(t)|^2 dx$$

$$+ 4 \int_{\mathbb{R}^N \setminus \Omega} \left( \frac{\partial_r \phi_R}{r} - 2 \right) |\nabla u(t)|^2 dx + 4 \int_{\mathbb{R}^N \setminus \Omega} \left( \frac{\partial_r^2 \phi_R}{r^2} - \frac{\partial_r \phi_R}{r^3} \right) |x|^2 |\nabla u(t)|^2 dx$$

$$\leq 4 \int_{\mathbb{R}^N \setminus \Omega} \left( \frac{\partial_r \phi_R}{r} - 2 \right) |\nabla u(t)|^2 dx + 4 \int_{\mathbb{R}^N \setminus \Omega} \left( \frac{\partial_r^2 \phi_R}{r^2} - \frac{\partial_r \phi_R}{r^3} \right) |x|^2 |\nabla u(t)|^2 dx$$

$$= 4 \int_{\mathbb{R}^N \setminus \Omega} \left( \frac{\partial_r^2 \phi_R}{r^2} - 2 \right) |\nabla u(t)|^2 dx \leq 0. \quad (2.4)$$

where in the last inequality we have used the assumption (2.1).

To estimate $K_2$, first note that if $0 \leq |x| \leq 2R$, then

$$\partial_r \phi_R(x) = 2|x| = 2r, \quad \partial_r \phi_R(x) = 2,$$

which implies

$$\partial_r^2 \phi_R + \left( N - 1 + \frac{b}{\sigma} \right) \frac{\partial_r \phi_R}{r} - 2N - \frac{2b}{\sigma} = 0, \quad \text{for} \quad 0 \leq |x| \leq 2R.$$

Hence,

$$\text{supp} \left[ \partial_r^2 \phi_R + \left( N - 1 + \frac{b}{\sigma} \right) \frac{\partial_r \phi_R}{r} - 2N - \frac{2b}{\sigma} \right] \subset (2R, \infty).$$
and thus, there exists \( c > 0 \) such that
\[
K_2 \leq c \int_{|x| \geq R} |x|^{-b} |u(t)|^{2\sigma + 2} \, dx.
\] (2.5)
Moreover, again by definition of \( \phi_R \), we also conclude that
\[
K_3 \leq \frac{c}{R^2} \int_{2R \leq |x| \leq 4R} |u(t)|^2 \, dx.
\] (2.6)
Collecting (2.4), (2.5) and (2.6) we deduce (2.3).

Next, from the energy conservation (1.4), we can write
\[
P[u(t)] = -\sigma \varepsilon_c \| \nabla u(t) \|_{L^2}^2 + 2(\sigma \varepsilon_c + 1) E[u_0].
\]
Then, combining the last identity with the inequality (2.3), we deduce the following lemma.

**Lemma 2.1** Let \( N \geq 3 \), \( 0 < b < \min\{ \frac{N}{2}, 2 \} \) and \( \frac{2-b}{N} < \sigma < \min \left\{ \frac{2-b}{N}, \frac{2}{N} \right\} \). If \( u \in C([0, \tau_*] : \dot{H}^s_c \cap \dot{H}^1) \) is a solution to (1.1) with initial data \( u(0) = u_0 \), then there exists \( c > 0 \) depending only on \( N, \sigma, b \) such that for all \( R > 0 \) and \( t \in [0, \tau_*] \) we have
\[
8\sigma \varepsilon_c \| \nabla u(t) \|_{L^2}^2 + z''_R(t) - 16(\sigma \varepsilon_c + 1) E[u_0] \leq c \left( \frac{1}{R^2} \int_{2R \leq |x| \leq 4R} |u(t)|^2 \, dx + \int_{|x| \geq R} |x|^{-b} |u(t)|^{2\sigma + 2} \, dx \right).
\] (2.7)

### 2.2 Upper bound for the blow-up rate

As an application of Lemma 2.1 we obtain our first main result.

**Proof of Theorem 1.1** Let \( R, \varepsilon > 0 \) real numbers to be chosen later. First, using interpolation and Sobolev embedding, or just the Gagliardo-Nirenberg inequality (see, for instance, Weinstein [32, inequality (1.2)]), we have
\[
\int_{|x| \geq R} |x|^{-b} |u(t)|^{2\sigma + 2} \, dx \leq \frac{1}{R^b} \| \nabla u(t) \|_{L^{2N/2}}^{\sigma N} \| u(t) \|_{L^2}^{2\sigma + 2 - \sigma N} \leq c \frac{1}{R^b} \| \nabla u(t) \|_{L^{2N/2}}^{\sigma N} \| u(t) \|_{L^2}^{2\sigma + 2 - \sigma N}.
\] (2.8)
Then, given \( \varepsilon > 0 \), by the Young inequality and mass conservation (1.3) we conclude, for some constant \( C_\varepsilon > 0 \), that
\[
\int_{|x| \geq R} |x|^{-b} |u(t)|^{2\sigma + 2} \, dx \leq \frac{c}{R^b} \| \nabla u \|_{L^2}^{\sigma N} \| u_0 \|_{L^2}^{2\sigma + 2 - \sigma N} \leq \varepsilon \| \nabla u \|_{L^2}^2 + C_\varepsilon \frac{\| u_0 \|_{L^2}^{2(2\sigma + 2 - \sigma N)}}{R^{\frac{2b}{2\sigma + 2 - \sigma N}}},
\] (2.9)
where we have used that \( \sigma < 2/N \).

1 This is the step where we explore the decaying factor in the nonlinearity instead of the radial assumption employed by Merle et al. [29, Theorem 1.1].
Combining the inequality (2.7), energy and mass conservation (1.4)–(1.3) and the inequality (2.9) we deduce that
\[
8\sigma_c \int |\nabla u(t)|^2 \, dx + z''(t) \leq C_{u_0} \left( 1 + \frac{1}{R^2} + \int_{|x| \geq R} |x|^{-b}|u(t)|^{2\sigma+2} \, dx \right)
\]
\[
\leq C_{u_0} \left( 1 + \frac{1}{R^2} + \varepsilon \int |\nabla u(t)|^2 \, dx + \frac{C_\varepsilon}{R^{2-\sigma N}} \right).
\]

Fix \( \varepsilon > 0 \) small enough such that \( 8\sigma_c - \varepsilon C_{u_0} > \sigma_c \). Therefore, there exists a universal constant \( C = C(u_0, \sigma, b, N) > 0 \) such that
\[
\sigma_c \int |\nabla u(t)|^2 \, dx + z''(t) \leq C \left( 1 + \frac{1}{R^2} + \frac{1}{R^{2-\sigma N}} \right).
\]
Now, since \( \frac{2b}{2-\sigma N} > 2 \) (or \( \sigma > \frac{2-b}{N} \)), if \( R \ll 1 \), then
\[
\sigma_c \int |\nabla u(t)|^2 \, dx + z''(t) \leq \frac{C}{R^{2-\sigma N}}.
\]

The rest of the argument is as in the proof of Theorem 1.1 in Merle et al. [29] (see also Theorem 1.4. in [7]). \( \square \)

### 2.3 \( L^\sigma \)-norm concentration

In this subsection we prove Theorem 1.2. We start by introducing some notation. Let \( \phi \in C_0^\infty(\mathbb{R}^N) \) be a positive radial cut-off solution such that
\[
\phi(x) = \left\{ \begin{array}{ll} 1, & \text{if } |x| \leq 1, \\
0, & \text{if } |x| \geq 2. \end{array} \right.
\]

For \( R(t) > 0 \), define the inner and outer spatial localizations of \( u(x, t) \) at radius \( R(t) \) as
\[
u_1(x, t) = \phi \left( \frac{x}{R(t)} \right) u(x, t) \quad \text{and} \quad u_2(x, t) = \left( 1 - \phi \left( \frac{x}{R(t)} \right) \right) u(x, t).
\]

Moreover, let \( \chi \in C_0^\infty(\mathbb{R}^N) \) be a radial function such that \( \chi(x) = 0 \) for \( |x| \geq 1 \) and \( \hat{\chi}(0) = 1 \). For \( \rho(t) \), define the inner and outer frequency localizations at radius \( \rho(t) \) of \( u_1(x, t) \) as
\[
\hat{u}_{1L}(\xi, t) = \hat{\chi} \left( \frac{\xi}{\rho(t)} \right) \hat{u}_1(\xi, t) \quad \text{and} \quad \hat{u}_{1H}(\xi, t) = \left( 1 - \hat{\chi} \left( \frac{\xi}{\rho(t)} \right) \right) \hat{u}_1(\xi, t).
\]

It is clear that
\[
\|\nabla u_{1L}(t)\|_{L^2} \leq c\|\nabla u_1(t)\|_{L^2} \quad \text{and} \quad \|\nabla u_{1H}(t)\|_{L^2} \leq c\|\nabla u_1(t)\|_{L^2}.
\]

**Proof of Theorem 1.2** First, from (1.6) and the energy conservation (1.4) we deduce
\[
\lim_{t \to T^*} \frac{\int |x|^{-b}|u(t)|^{2\sigma+2} \, dx}{\|\nabla u(t)\|^2_{L^2}} = \sigma + 1.
\]

Hence, for \( t \) close to \( T^* \), we have
\[
\|\nabla u(t)\|^2_{L^2} \leq \int |x|^{-b}|u(t)|^{2\sigma+2} \, dx \leq \int |x|^{-b}|u_{1L}(t)|^{2\sigma+2} \, dx + \int |x|^{-b}|u_{1H}(t)|^{2\sigma+2} \, dx + \int |x|^{-b}|u_2(t)|^{2\sigma+2} \, dx.
\]
where in the last line we have used that \( u = u_1 + u_2 \) with pairwise disjoint supports.

For a constant \( a_1 > 0 \), define

\[
R(t) = a_1 \frac{\max \{ \|u_0\|_{L^2}, \|u_0\|_{L^2}^{2b - \sigma \sigma N} \}}{\|\nabla u(t)\|_{L^2}^{2b - \sigma \sigma N}}. \tag{2.12}
\]

Applying (2.8), we obtain\(^2\)

\[
\int |x|^{-b} |u_2(t)|^{2b + 2} \, dx = \int |x|^{-b} \left( 1 - \phi \left( \frac{x}{R(t)} \right) \right) u(t)^{2b + 2} \, dx \\
\leq c \int_{|x| \geq R(t)} |x|^{-b} |u(t)|^{2b + 2} \, dx \\
\leq c^2 \frac{a_1}{b} \|\nabla u(t)\|_{L^2}^2 \\
\leq \frac{1}{4} \|\nabla u(t)\|_{L^2}^2. \tag{2.13}
\]

where we have chosen \( a_1 > 0 \) sufficiently large such that the last inequality holds.

Now, from the Gagliardo–Nirenberg inequality (1.7) and the Sobolev embedding \( \dot{H}^{\sigma_c} \subset L^{\sigma_c} \), we get

\[
\int |x|^{-b} |u_1H(t)|^{2b + 2} \, dx \leq c \|\nabla u_1H(t)\|_{L^2}^2 \|u_1H(t)\|_{L^{2 \sigma_c}}^{2 \sigma_c} \leq c \|\nabla u_1H(t)\|_{L^2}^2 \|u_1H(t)\|_{H^{\sigma_c}}^{2 \sigma_c} \\
= c \|\nabla u_1H(t)\|_{L^2}^2 \|\xi|^{\sigma_c} \left( 1 - \tilde{\chi} \left( \frac{\xi}{\rho(t)} \right) \right) \tilde{u}_1(t) \|_{L^2}^{2 \sigma_c} \tag{2.14}
\]

We claim that

\[
|\xi|^{\sigma_c} \left( 1 - \tilde{\chi} \left( \frac{\xi}{\rho(t)} \right) \right) \leq c \frac{|\xi|}{\rho(t)^{1 - \sigma_c}}. \tag{2.15}
\]

Indeed, by the mean value theorem

\[
|1 - \tilde{\chi}(\xi)| = |\tilde{\chi}(0) - \tilde{\chi}(\xi)| \leq c \min\{1, |\xi|\}.
\]

Therefore, for \( |\xi| \leq \rho(t) \), we have

\[
|\xi|^{\sigma_c} \left| 1 - \tilde{\chi} \left( \frac{\xi}{\rho(t)} \right) \right| \leq c |\xi|^{\sigma_c} \frac{|\xi|}{\rho(t)} \leq c \frac{|\xi|}{\rho(t)^{1 - \sigma_c}}
\]

and if \( |\xi| \geq \rho(t) \), then

\[
|\xi|^{\sigma_c} \left| 1 - \tilde{\chi} \left( \frac{\xi}{\rho(t)} \right) \right| \leq c |\xi|^{\sigma_c} = |\xi|^{\sigma_c} \frac{|\xi|}{|\xi|} \leq \frac{|\xi|}{\rho(t)^{1 - \sigma_c}}.
\]

\(^2\) In [25, Theorem 1.2] the authors used in this part the radial Gagliardo-Nirenberg estimate

\[
\|f\|_{L^4(|x| \geq R)} \leq \frac{c}{R^2} \|\nabla f\|_{L^2(|x| \geq R)} \|f\|_{L^2(|x| \geq R)}^3,
\]

and hence, they need the radial restriction. Here, we use the decay of \( |x|^{-b} \) away from the origin to obtain the desired estimate in the general case.
Moreover, since \( \sigma > \frac{2-b}{N} \) and \( \| u(t) \|_{L^2} > 1 \) for \( t \) close enough to \( T^* \), from the definition of \( R(t) \) (2.12) we get
\[
\| \nabla u_1(t) \|_{L^2} \leq \frac{c}{R(t)} \| u_0 \|_{L^2} + c \| \nabla u(t) \|_{L^2}
\leq c \left( \| \nabla u(t) \|_{L^2}^{\frac{2-\sigma N}{N}} + \| \nabla u(t) \|_{L^2} \right)
\leq c \| \nabla u(t) \|_{L^2}.
\] (2.16)

For a constant \( a_2 > 0 \), let
\[
\rho(t) = a_2 \| \nabla u(t) \|_{L^2}^{\frac{1}{1-\sigma}}.
\]
Collecting the estimates (2.10), (2.14), (2.15), (2.16) and choosing \( a_2 > 0 \) sufficiently large we have
\[
\int |x|^{-b} |u_{1H}(t)|^{2\sigma+2} \, dx \leq c \| \nabla u_1(t) \|_{L^2}^2 \left| \frac{|\xi|}{\rho(t)^{1-\sigma}} \hat{u}_1(t) \right|^{2\sigma} \leq c \frac{\| \nabla u(t) \|_{L^2}^{2\sigma+2}}{\rho(t)^{2\sigma(1-\sigma)}} \leq \frac{1}{4} \| \nabla u(t) \|_{L^2}^2.
\] (2.17)

Therefore, in view of the inequality (2.11), (2.13) and (2.17), we deduce
\[
\| \nabla u(t) \|_{L^2}^2 \leq c \int |x|^{-b} |u_{1L}(t)|^{2\sigma+2} \, dx.
\] (2.18)

Next, using again the Gagliardo–Nirenberg inequality (1.7) and the estimates (2.10), (2.16) and (2.18), we obtain
\[
\int |x|^{-b} |u_{1L}(t)|^{2\sigma+2} \, dx \leq c \| \nabla u_{1L}(t) \|_{L^2}^2 \| u_{1L}(t) \|_{L^{2\sigma}}^{2\sigma}
\leq c \| u_{1L}(t) \|_{L^{2\sigma}}^{2\sigma} \int |x|^{-b} |u_{1L}(t)|^{2\sigma+2} \, dx.
\]

Moreover, Young’s convolution inequality yields
\[
\| u_{1L}(t) \|_{L^{2\sigma}} = \| \rho^N \chi(\rho \cdot) * u_1(t) \|_{L^{2\sigma}} \leq c \| u_1(t) \|_{L^{2\sigma}}.
\]

From the last two inequalities it follows that \( \| u_1(t) \|_{L^{2\sigma}} \) is bounded from below by a universal constant \( c > 0 \) independent of the initial data \( u_0 \) and time \( t \). Finally, by definition of \( u_1(t) \), we get
\[
c^{\sigma_c} \leq \| u_1(t) \|_{L^{2\sigma_c}}^\sigma_c = \left\| \phi \left( \frac{x}{R(t)} \right) u(t) \right\|_{L^{2\sigma_c}}^\sigma_c \leq \int_{|x| \leq 2R(t)} |u(t)|^{2\sigma_c} \, dx = \| u(t) \|_{L^{2\sigma_c}}^{\sigma_c} \| \chi_{|\{x| \leq 2R(t)\}} \|_{L^{\sigma_c}} ,
\]

which, by the definition of \( R(t) \) (2.12), implies (1.9) and complete the proof of Theorem 1.2.

As a consequence of the above argument, if we further assume that \( \| u_1(t) \|_{L^{2\sigma_c}} \) is not bounded above, then there exist a sequence \( t_n \to T^* \) such that
\[
\int_{|x| \leq c_{k_0}} \| \nabla u(t_n) \|_{L^2}^{2-2\sigma N} |u(x, t_n)|^{\sigma_c} \, dx \to +\infty.
\]

On the other hand, if \( \| u_1(t) \|_{L^{2\sigma_c}} \) is bounded, then it may be possible to prove that the concentration window shrinks at a different rate following the strategy of [25, Theorem 1.2].

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We have decided not to explore this for the moment, since our main purpose here is to show how to remove the radial assumption and still obtain a concentration result around the origin for the INLS equation.

3 Blow-up solutions in \( \dot{H}^{s_c} \cap \dot{H}^1 \)

3.1 Non-radial Gagliardo–Nirenberg inequality

We first recall the following scaling invariant Morrey–Campanato type semi-norm used in [28] (see also [7])

\[
\rho(u, R) = \sup_{R' \geq R} \frac{1}{(R')^{2s_c}} \int_{R' \leq |x| \leq 2R'} |u|^2 \, dx. \tag{3.1}
\]

It is easy to see that \( \rho(u, R) \) is non-increasing in \( R > 0 \). Moreover, by Holder’s inequality, there exists a universal constant \( c > 0 \) such that for all \( u \in L^{2\sigma_c} \) and \( R > 0 \)

\[
\frac{1}{R^{2s_c}} \int_{|x| \leq R} |u|^2 \, dx \leq c \|u\|^{2}_{2\sigma_c}
\]

and

\[
\lim_{R \to +\infty} \frac{1}{R^{2s_c}} \int_{|x| \leq R} |u|^2 \, dx = 0
\]

(see Merle and Raphäel in [28, Lemma 1] and also [7, Lemma 2.1]). We should point out that no radial symmetry is needed to obtain these estimates. Next, we prove a crucial interpolation inequality for general functions in \( \dot{H}^{s_c} \cap \dot{H}^1 \).

**Lemma 3.1** [Non-radial Gagliardo–Nirenberg inequality] Suppose that \( N \geq 3 \), \( 0 < b < 2 \), \( \frac{2-b}{N} < \sigma < \min\{\frac{2}{N}, \frac{2-b}{N-2}\} \). Then, for all \( \eta > 0 \), there exists a constant \( C_\eta > 0 \) such that for all \( u \in \dot{H}^{s_c} \cap \dot{H}^1 \) the following inequality holds

\[
\int_{|x| \geq R} |x|^{-b} |u|^{2\sigma+2} \, dx \leq \eta \|\nabla u\|_{L^2}^2 + \frac{C_\eta}{R^{2(1-s_c)}} [\rho(u, R)]^{\frac{2\sigma+2-\sigma N}{2-\sigma N}}. \tag{3.2}
\]

**Proof** Since \( \rho(u, R) \) is non-increasing in \( R > 0 \), given \( j \in \mathbb{N} \), we first note that \( \rho(u, 2^j R) \leq \rho(u, R) \). Now, for each \( j \in \mathbb{N} \), let

\[
C_j = \{x \in \mathbb{R}; \ 2^j R \leq |x| \leq 2^{j+1} R\}.
\]

By interpolation and Sobolev embedding, we first have\(^3\)

\[
\int_{C_j} |x|^{-b} |u|^{2\sigma+2} \, dx \leq c \frac{1}{(2^j R)^b} \|u\|_{L^{2\sigma_c}(C_j)}^{\sigma N} \|u\|_{L^2(C_j)}^{2\sigma+2-\sigma N} \\
\leq c \|\nabla u\|_{L^2}^{\sigma N} \frac{1}{(2^j R)^{(1-s_c)(2-\sigma N)}} [\rho(u, 2^j R)]^{\sigma+1-\frac{2\sigma N}{2-\sigma N}},
\]

where in the last inequality we have also used the definition (3.1) and the fact that

\[
b - s_c(2\sigma + 2 - \sigma N) = (1 - s_c)(2 - \sigma N) > 0.
\]

\(^3\) As in the proof of Theorems 1.1 and 1.2, this is the step where we use the decaying factor in the nonlinearity to replace the radial assumption.
Let θ ∈ (0, (1 − sc)(2 − σ N)). Given ˜η > 0, by Young’s inequality, there exists a constant ˜C ˜η > 0 such that

\[ \int_{C_j} |x|^{-b} |u|^{2\sigma + 2} \, dx \leq \frac{\tilde{\eta}}{(2j)^{\frac{2\sigma}{2\sigma - N}}} \|\nabla u\|_{L^2}^2 + \frac{\tilde{C}_{\tilde{\eta}}}{(2j R)^{2(1 - sc)(2j) - 2}} \left( \rho(u, 2j R) \right)^{\frac{2\sigma + 2 - \sigma N}{2 - \sigma N}}. \]  

(3.3)

Therefore, from (3.3), we deduce that

\[ \int_{|x| \geq R} |x|^{-b} |u(x)|^{2\sigma + 2} \, dx = \sum_{j=0}^{\infty} \int_{C_j} |x|^{-b} |u|^{2\sigma + 2} \, dx \]

\[ \leq \tilde{C} \left( \sum_{j=0}^{\infty} \frac{1}{(2j)^{\frac{2\sigma}{2\sigma - N}}} \right) \|\nabla u\|_{L^2}^2 \]

\[ + \frac{\tilde{C}_{\tilde{\eta}}}{R^{2(1 - sc)}} \left( \sum_{j=0}^{\infty} \frac{1}{2^{(2j R^{2 - \sigma N - \theta})} j} \right) \left[ \rho(u, R) \right]^{\frac{2\sigma + 2 - \sigma N}{2 - \sigma N}}, \]

which completes the proof of the lemma since the above two series are summable. □

### 3.2 Existence of blow-up solutions and lower bound for the blow-up rate

We now prove some uniform estimates that are the main ingredients in the proof of Theorems 1.3–1.4 and Corollary 1.5. The technique is very similar to the one used in the proof of [7, Propositions 4.1–4.2] and was inspired by the work of [28], these papers treat the radial setting. Here, we show that Lemmas 2.1–3.1 allow us to consider general solutions of the INLS equation (1.1).

**Proposition 3.2** Let N ≥ 3, 0 < b < min{N, 2}, \( \frac{2 - b}{N - 2} < \sigma < \min \left\{ \frac{2 - b}{N - 2}, \frac{2}{N} \right\} \) and \( v_0 \in \dot{H}^{sc} \cap \dot{H}^1 \) such that the corresponding solution \( v \in C([0, \tau_\ast]: \dot{H}^{sc} \cap \dot{H}^1) \) to (1.1) satisfies

\[ \tau_\ast^{1 - sc} \max \{ E[v_0], 0 \} < 1 \]  

(3.4)

and

\[ M_0 := \frac{4\|v_0\|_{L^{\sigma c}}}{\|V\|_{L^{\sigma c}}} \geq 2, \]  

(3.5)

where \( V \) is a solution to elliptic equation (1.8) with minimal \( L^{\sigma c} \)-norm. Then, there exist universal constants \( C_1, \alpha_1, \alpha_2 > 0 \) depending only on N, σ and b such that, for all \( \tau_0 \in [0, \tau_\ast] \), the following uniform estimates hold

\[ \rho(v(\tau_0), M_0^{\alpha_1} \sqrt{\tau_0}) \leq C_1 M_0^2 \]  

(3.6)

and

\[ \int_0^{\tau_0} (\tau_0 - \tau) \|\nabla v(\tau)\|_{L^2}^2 \, d\tau \leq M_0^{\alpha_2} \tau_0^{1 + sc}. \]  

(3.7)
Proof For all $A > 0$ and $\tau_0 \in [0, \tau_*]$, let $R = A\sqrt{\tau_0}$ and $M_\infty$ defined by

$$M^2_\infty(A, \tau_0) = \max_{\tau \in [0, \tau_0]} \rho(v(\tau), A\sqrt{\tau}).$$

Then using the non-radial Gagliardo–Nirenberg inequality (3.2) and following the proof of Lemma 4.3 in [7] we obtain the existence of a universal constant $c > 0$ such that

$$8\sigma sc \int_0^{\tau_0} (\tau_0 - \tau) \|\nabla v(\tau)\|^2_{L^2} d\tau$$

$$\leq c\tau_0^{1+sc} \left[ A^{2(1+sc)} \|v_0\|^2_{L^\infty} + \frac{[M^2_\infty(A, \tau_0)]^{2\sigma+2-\sigma N}}{A^{2(2-\sigma N)}} + M^2_\infty(A, \tau_0) \right]$$

$$+ 2\tau_0 \left[ z'_R(0) + 8\tau_0(\sigma sc + 1) E[v_0] \right]$$

(3.8)

and

$$\frac{1}{R^{2sc}} \int_{|x| \leq 2R} |v(\tau_0)|^2 dx \leq c \|v_0\|^2_{L^\infty} + \frac{c}{A^4} \left[ [M^2_\infty(A, \tau_0)]^{2\sigma+2-\sigma N} + M^2_\infty(A, \tau_0) \right]$$

$$+ \frac{2}{\tau_0^{sc}} A^{2(1+sc)} \left[ z'_R(0) + 8\tau_0(\sigma sc + 1) E[v_0] \right].$$

(3.9)

Now, let $\varepsilon > 0$ a fixed small enough real number to be chosen later and define

$$G_\varepsilon = M^\frac{1}{2}_0 \quad \text{and} \quad A_\varepsilon = \left( \frac{\varepsilon G_\varepsilon}{M_0^2} \right)^{\frac{1}{2(1+sc)}},$$

(3.10)

where $M_0$ is given in (3.5). Consider the following estimates

$$\int_0^{\tau_0} (\tau_0 - \tau) \|\nabla v(\tau)\|^2_{L^2} d\tau \leq G_\varepsilon \tau_0^{1+sc}$$

(3.11)

and

$$M^2_\infty(A_\varepsilon, \tau_0) \leq \frac{2M^2_0}{\varepsilon}.$$  

(3.12)

We define

$$S_\varepsilon = \{ \tau \in [0, \tau_*] ; \ (3.11) \text{ and } (3.12) \text{ hold for all } \tau_0 \in [0, \tau] \}$$

(3.13)

and

$$\tau_1 = \max_{\tau \in [0, \tau_1]} S_\varepsilon.$$

Note that $S_\varepsilon \neq \emptyset$, for $\varepsilon > 0$ sufficiently small, since $v \in C ([0, \tau_*] : \dot{H}^{sc} \cap \dot{H}^1)$ and also by the definition of $M_0$ in (3.5) and the Sobolev embedding $\dot{H}^{sc} \subset L^{sc}$.

The goal is to show that $\tau_1 = \tau_*$ and therefore inequalities (3.11) and (3.12) hold at the maximal time $\tau_*$, which clearly imply (3.6) and (3.7), in view of definition (3.10). Indeed,
from (3.10) and (3.12) it is easy to see that
\[
\frac{[M_\infty^2(A_\varepsilon, \tau_0)]^{2\sigma+2-\sigma N}}{A_\varepsilon^{2(1-s_c)}} + \frac{M_\infty^2(A_\varepsilon, \tau_0)}{A_\varepsilon^{2(1-s_c)}} \\
\leq \left(\frac{2M_0^2}{\varepsilon}\right)^{\frac{1-s_c}{1+s_c}} \leq \frac{2M_0^2}{\varepsilon}
\]
\[
\leq \frac{1}{M_0^2} \left(\frac{M_0^2}{\varepsilon}\right)^{\frac{2\sigma+2-\sigma N}{2-\sigma N}} \leq \frac{1}{10^s}
\]
for \(\varepsilon > 0\) small enough, where we have used the assumption (3.5). Combining the last inequality with (3.8) and using again definition (3.10) and assumption (3.5), we have, for \(R = A_\varepsilon \sqrt{\tau_0}\), that
\[
\int_0^{\tau_0} (\tau_0 - \tau) \|\nabla v(\tau)\|_{L^2}^2 \, d\tau \\
\leq c\tau_0^{1+s_c} \left[M_0^2 A_\varepsilon^{2(1+s_c)} + \frac{[M_\infty^2(A_\varepsilon, \tau_0)]^{2\sigma+2-\sigma N}}{A_\varepsilon^{2(1-s_c)}} + \frac{M_\infty^2(A_\varepsilon, \tau_0)}{A_\varepsilon^{2(1-s_c)}}\right] \\
+ 2c\tau_0 \left[|z_R'(0)| + 8\tau_0(\sigma s_c + 1)E[v_0]\right] \\
\leq c\tau_0^{1+s_c} \left[\varepsilon G_\varepsilon + \frac{1}{10}\right] + 2c\tau_0 \left[|z_R'(0)| + 8\tau_0(\sigma s_c + 1)E[v_0]\right] \\
\leq G_\varepsilon \tau_0^{1+s_c} \left[\frac{c}{10} + \frac{2c}{G_\varepsilon \tau_0} \left[|z_R'(0)| + 8\tau_0(\sigma s_c + 1)E[v_0]\right]\right] \\
\leq G_\varepsilon \tau_0^{1+s_c} \left[\frac{1}{10} + \frac{2c}{G_\varepsilon \tau_0} \left[|z_R'(0)| + 8\tau_0(\sigma s_c + 1)E[v_0]\right]\right],
\]
for \(\varepsilon > 0\) small enough.

Next, from Lemma 4.4 in [7]\(^4\), there exists \(\varepsilon_0 > 0\) small enough and \(c > 0\) a universal constant such that, for all \(\tau_0 \in [0, \tau_1], 0 < \varepsilon \leq \varepsilon_0, A \geq A_\varepsilon\) and \(R = A_\varepsilon \sqrt{\tau_0}\), the following inequality holds
\[
|z_R'(0)| + 8\tau_0(\sigma s_c + 1)E[v_0] \leq c\frac{M_0^2 A_\varepsilon^{2(1+s_c)}}{\varepsilon^{\frac{1+s_c}{1-s_c}}} \tau_0^{s_c}. \tag{3.14}
\]
Therefore, from estimate (3.14) with \(A = A_\varepsilon\), for all \(\tau_0 \in [0, \tau_1]\) and since \(R = A_\varepsilon \sqrt{\tau_0}\), we get
\[
\int_0^{\tau_0} (\tau_0 - \tau) \|\nabla v(\tau)\|_{L^2}^2 \, d\tau \\
= G_\varepsilon \tau_0^{1+s_c} \left[\frac{1}{10} + \frac{c}{G_\varepsilon} \frac{M_0^2 A_\varepsilon^{2(1+s_c)}}{\varepsilon^{\frac{1+s_c}{1-s_c}}} \tau_0^{s_c}\right] \\
\leq G_\varepsilon \tau_0^{1+s_c} \left[\frac{1}{10} + c\varepsilon^{\frac{s_c}{1-s_c}}\right] \\
\leq G_\varepsilon \frac{1}{2} \tau_0^{1+s_c}. \tag{3.15}
\]

\(^4\) In the proof of [7, Lemma 4.4] does not require a radial assumption.
Moreover, let $A \geq A_\varepsilon$ and $R = A\sqrt{\tau_0}$. First, since $\rho$ is non-increasing in $R$, then for all $\tau_0 \in [0, \tau_1]$ we obtain
\[
M_\infty^2(A, \tau_0) \leq M_\infty^2(A_\varepsilon, \tau_0) \leq \frac{2M_0^2}{\varepsilon},
\]
where in the last inequality we have used (3.12). Combining (3.9) with (3.5), the last inequality and (3.14) we deduce
\[
\frac{1}{R^{2\varepsilon}} \int_{R \leq |x| \leq 2R} |v(\tau_0)|^2 \, dx \leq cM_0^2 + \frac{c}{A^4} \left[ M_\infty^2(A, \tau_0) \right]^{\frac{2\sigma + 2 - \sigma N}{4 - \sigma N}} + \frac{2M_0^2}{\varepsilon} + \frac{2M_0^2}{\varepsilon^{1+\varepsilon}} \left( \frac{G_0}{M_0^2} \right)^{\frac{2}{1+\varepsilon}}.
\]

\[
\leq cM_0^2 + c \left( \frac{2M_0^2}{\varepsilon} \right)^{\frac{2\sigma + 2 - \sigma N}{4 - \sigma N}} + \frac{2M_0^2}{\varepsilon^{1+\varepsilon}} \left( \frac{G_0}{M_0^2} \right)^{\frac{2}{1+\varepsilon}} + cM_0^2 + cM_0^2 \left( \frac{M_0^2}{\varepsilon} \right)^{\frac{2\sigma + 2 - \sigma N}{4 - \sigma N}} + \frac{2M_0^2}{\varepsilon^{1+\varepsilon}} \left( \frac{G_0}{M_0^2} \right)^{\frac{2}{1+\varepsilon}}
\]
\[
= \frac{M_0^2}{\varepsilon} \left[ c + \frac{c}{M_0^2} \left( \frac{M_0^2}{\varepsilon} \right)^{\frac{2\sigma + 2 - \sigma N}{4 - \sigma N}} + \frac{2}{\varepsilon^{1+\varepsilon}} \left( \frac{G_0}{M_0^2} \right)^{\frac{2}{1+\varepsilon}} \right]
\]
\[
< \frac{M_0^2}{\varepsilon},
\]
for $\varepsilon > 0$ small enough.

Finally, in view of the regularity of $v$ and the definition of $S_\varepsilon$ in (3.13), the inequalities (3.15) and (3.16) imply that $\tau_1 = \tau_*$ and the proof of Proposition 3.2 is completed. \hfill $\Box$

Next, we prove a lower bound on the $L^2$ norm of the initial data around the origin, assuming an additional restriction on the energy.

**Proposition 3.3** Let $0 < b < 2$, $\frac{2-b}{N} < \sigma < \min \left\{ \frac{2-b}{N-2}, \frac{2}{N} \right\}$ and $v \in C ([0, \tau_*] : \dot{H}_\varepsilon \cap \dot{H}^1)$ a solution to (1.1) with initial data $v_0 \in \dot{H}_\varepsilon \cap \dot{H}^1$ such that (3.4) and (3.5) of Proposition 3.2 hold. Let
\[
\tau_0 \in \left[ 0, \frac{\tau_*}{2} \right].
\]

Define $\lambda_v(\tau) = \| \nabla v(\tau) \|_{L^2}^{-\frac{1}{1+\varepsilon}}$ and assume that
\[
E[v_0] \leq \frac{\| \nabla v(\tau_0) \|_{L^2}^2}{4} = \frac{1}{4A^{2(1-\varepsilon)}(\tau_0)}.
\]

Then, there exist universal constants $C_2, \alpha_3 > 0$ depending only on $N, \sigma$ and $b$ such that if
\[
F_* = \frac{\sqrt{\tau_0}}{\lambda_v(\tau_0)} \quad \text{and} \quad D_* = M_0^{\alpha_3} \max [1, F_*^{\frac{1}{1+\varepsilon}}],
\]
then
\[
\frac{1}{\lambda_v(\tau_0)} \int_{|x| \leq D_* \lambda_v(\tau_0)} |v_0|^2 \, dx \geq C_2.
\]
**Proof** The argument is quite similar to the one in [7, Proposition 4.2] and we only sketch the details. Given \( v(\tau_0) \) we define

\[
    w(x) = \frac{\lambda^{2-b}}{2\pi^2} (v(\lambda v(\tau_0) x, \tau_0),
\]

where \( \lambda v(\tau_0) = \| \nabla v(\tau_0) \|_{L^2}^{-\frac{1}{2}} \). By a straightforward calculation and assumption (3.18), we have

\[
    \| \nabla w \|_{L^2}^2 = 1 \quad \text{and} \quad E[w] \leq \frac{1}{4}.
\]

Thus, by the definition of the energy (1.4), we get

\[
    \int |x|^{-b} |w|^{2\sigma+2} \, dx \geq \frac{\sigma+1}{2}.
\]

Recalling the definition of \( F_\ast \) in (3.19), for \( J \geq M_0^{\alpha_1} F_\ast \), the estimate (3.6) obtained in Proposition 3.2 and the definition of the semi-norm \( (3.1) \) imply

\[
    \rho(w, J) \leq \rho(v(\tau_0), M_0^{\alpha_1} \sqrt{\tau_0}) \leq C M_0^2.
\]

On the other hand, for \( J \geq C M_0^{2(2\sigma+2-\sigma N)/(4-\sigma N)} \) and all \( \eta > 0 \), from estimate (3.2) we get

\[
    \int |x|^{-b} |f|^{2\sigma+2} \, dx \leq \eta \| \nabla f \|_{L^2}^{2\sigma+2} + \frac{C_\eta}{J^{2(1-\sigma_c)}} [\rho(w, J)]^{2(2\sigma+2-\sigma N)/(4-\sigma N)}
\]

\[
    \leq \eta + \frac{C_\eta}{C^{2(1-\sigma_c)} M_0^{2(2\sigma+2-\sigma N)/(4-\sigma N)}} [\rho(w, J)]^{2(2\sigma+2-\sigma N)/(4-\sigma N)}.
\]

Now, taking

\[
    J_\ast = C_\ast \max[M_0^{\alpha_1} F_\ast, M_0^{2(1-\sigma_c)/(4-\sigma N)}],
\]

for \( C_\ast > 1 \) large enough, the previous two inequalities (with \( \eta > 0 \) small enough) and (3.21) yield

\[
    \int_{|x| \leq J_\ast} |x|^{-b} |w|^{2\sigma+2} \, dx \geq \frac{\sigma+1}{4}.
\]

Next, we apply the Gagliardo–Nirenberg type inequality

\[
    \int |x|^{-b} |f|^{2\sigma+2} \, dx \leq c \| \nabla f \|_{L^2}^{2\sigma+2} \| f \|_{L^2}^{2\sigma(1-\sigma_c)}
\]

(see [17, Theorem 1.2]) to deduce a lower bound on the \( L^2 \) norm of \( v(\tau_0) \) around the origin. Indeed, let \( \varphi \in C_0^\infty(\mathbb{R}^N) \) be a cut-off function such that

\[
    \varphi(x) = \begin{cases} 
    1, & |x| \leq 1 \\
    0, & |x| \geq 2
    \end{cases} \quad \text{and} \quad \varphi_A(x) = \varphi \left( \frac{x}{A} \right).
\]

It is easy to see that

\[
    \| \nabla (\varphi_A w) \|_{L^2} = \| \nabla \varphi_A w + \varphi_A \nabla w \|_{L^2} \leq \| \nabla \varphi_A w \|_{L^2} + \| \varphi_A \nabla w \|_{L^2} \leq c(\| w \|_{L^2(|x| \leq 2J_\ast)} + 1),
\]
where we have used (3.20) in the last inequality. Combining (3.22), (3.23) and the previous inequality, we deduce
\[
\frac{\sigma + 1}{4} \leq \int_{|x| \leq J_*} |x|^{-b} |w|^{2\sigma + 2} \, dx \leq \int |x|^{-b} |\varphi_{J_*} w|^{2\sigma + 2} \, dx
\]
\[
\leq c \|\nabla (\varphi_{J_*} w)\|_{L^2}^{2\sigma + 2} \|\varphi_{J_*} w\|_{L^2}^{2\sigma (1-s_c)} \leq c \left( \|w\|_{L^2(|x| \leq 2J_*)}^{2\sigma + 2} + \|w\|_{L^2(|x| \leq 2J_*)}^{2\sigma (1-s_c)} \right).
\]
Therefore, there exists a constant $c_3 > 0$ such that
\[
c_3 \leq \int_{|x| \leq 2J_*} |w|^2 \, dx = \frac{1}{\lambda_{c_3}^2 (\tau_0)} \int_{|x| \leq 2J_* \lambda_{c_3} (\tau_0)} |v(\tau_0)|^2 \, dx.
\]
Finally, the end of the proof of [7, Proposition 4.2] implies that the above estimate is also verified at the initial time, where we use the assumption (3.17), and complete the proof of Proposition 3.3. \hfill \Box

Once Propositions 3.2 and 3.3 are established, the proofs of Theorem 1.3–1.4 and Corollary 1.5 follow from exactly the same arguments as those of in [7, Theorem 1.1–1.2 and Corollary 1.3] and hence need not be repeated here.

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