On Correlation Function of High Moments of Large Wigner Random Matrices\textsuperscript{*†‡}

O. Khorunzhiy
Université de Versailles - Saint-Quentin, Versailles
FRANCE
e-mail: khorunjy@math.uvsq.fr

Abstract

We consider the Wigner ensemble of Hermitian $n$-dimensional random matrices $W_{ij}^{(n)} = w_{ij}/\sqrt{n}$ and study the asymptotic behavior of the expression

$$K_n(s',s'') = E\left\{\text{Tr} \left( W^{(n)} \right)^{2s'} \text{Tr} \left( W^{(n)} \right)^{2s''} \right\} - E\text{Tr} \left( W^{(n)} \right)^{2s'} E\text{Tr} \left( W^{(n)} \right)^{2s''}$$

in the limit $s', s'', n \to \infty$ such that $s' = s'_n$ and $s'' = s''_n$ are the values of the order $n^{2/3}$. Assuming that the random variables $\{w_{ij}\}$ are of the symmetric probability distribution such that its all moments $V_{2k}$ are of the sub-gaussian form, we prove that the limit of $K_n(s'_n, s''_n)$ exists and does not depend on the particular values of $V_{2k}$, $k \geq 2$.

The proof is based on the combination of the arguments by Ya. Sinai and A. Soshnikov with the detailed study of a moment analog of the Inverse Participation Ratio of the Gaussian Unitary Invariant Ensemble of random matrices (GUE).

Running title: Correlation Function of High Moments

1 Introduction

The famous semi-circle law is proved by E. Wigner in the fifties by the moment method \cite{13}. Since that, the moment method is widely used in the spectral theory of large random matrices. In particular, it has been employed in the studies of the asymptotic behavior of the spectral norm \cite{1, 4, 5}, where one considers the moments of the order that grows at the same time as the matrix dimension tends to infinity.

Later, the moment method has been developed to describe the properties of the eigenvalue distribution on the local scale \cite{11} that gives much more detailed information than that of the semi-circle law known as the global (or integral) one. In the subsequent paper \cite{12}, the so-called microscopic scale at the edge of the limiting spectra is reached in the frameworks of the proof of the universality conjecture of...
the properties of the local eigenvalue distribution. In [12], a spacious program of
the studies of the high moments of the Wigner ensemble as well as their correlation
functions (or cumulants) has been proposed and strong method has been developed.

In our previous two papers, first [8] and then [7], we followed this way and con-
sidered the asymptotic behavior of high moments of large Wigner random matrices.
In the present paper we continue the work in this area and pass to the studies of the
second correlation function (or the second cumulant) of these moments.

2 Main results

Let \( \{X_{ij}, Y_{ij}, 1 \leq i \leq j \} \) be family of jointly independent real random variables
determined on the same probability space and let \( E \) denote the mathematical expec-
tation with respect the corresponding probability measure. Assuming that
\[
E X_{ij} = E Y_{ij} = 0, \quad (2.1)
\]
and
\[
E X_{ij}^2 = (1 + \delta_{ij})/8 \quad \text{and} \quad E Y_{ij}^2 = (1 - \delta_{ij})/8 \quad (2.2)
\]
for all \( i \) and \( j \), where \( \delta_{ij} \) denotes the Kronecker \( \delta \)-function, we determine complex
random \( n \times n \) matrices \( W^{(n)} \) with the elements
\[
(W^{(n)})_{ij} = \frac{1}{\sqrt{n}} \begin{cases} X_{ij} + iY_{ij}, & \text{if } i \leq j, \\ X_{ji} - iY_{ji}, & \text{if } i > j, \end{cases}, \quad (2.3)
\]
and say that the family \( \{W^{(n)}\} \) represents the Wigner ensemble of Hermitian random
matrices [13]. Below we will omit the superscripts \( n \) when no confusion can arise.

We are interested in the asymptotic behavior of the covariance of the traces
\[
\text{Tr} (W^{(n)}_{2s}) = \sum_{i=1}^{n} (W^{2s})_{ii}
\]
in the limit when \( s \) and \( n \) tend to infinity. Our main result is as follows.

**Theorem 2.1.** Let random variables \( X_{ij} \) and \( Y_{ij} \) have the same symmetric
probability distribution such that, in addition to (2.2), the moments \( E(X_{ij})^{2k} = V_{2k} \)
exist for all \( k \geq 2 \) and \( V_{2k} \leq (ck)^k \) for some \( c > 0 \). Then there exists \( \chi_0 > 0 \) such
that the limit
\[
K(\chi', \chi'') = \lim_{n \to \infty} \left( E \left\{ \text{Tr} W^{2s'} \text{Tr} W^{2s''} \right\} - E \text{Tr} W^{2s'} E \text{Tr} W^{2s''} \right), \quad (2.4)
\]
where
\[
s' = s'_n = [(\chi'^2)1/3], \quad s'' = s''_n = [(\chi''^2)1/3], \quad (2.5)
\]
for any given positive \( \chi' \) and \( \chi'' \) less than \( \chi_0 \), exists and does not depend on the
particular values of \( V_{2k} \), \( k \geq 2 \).

The proof of Theorem 2.1 is performed consists of two parts. On the first step
we reduce the study of the covariance
\[
K_n(s', s'') = E \left\{ \text{Tr} W^{2s'} \text{Tr} W^{2s''} \right\} - E \text{Tr} W^{2s'} E \text{Tr} W^{2s''} \quad (2.6)
\]
to the study of a part of an expression that can be regarded as a moment analog of the Inverse Participation Ratio of the Wigner ensemble (2.3). We do this with the help of the arguments used in paper [11] combined with the results on the upper bound of the moments of the Wigner ensemble obtained in [8]. Let us note that the condition of the sub-gaussian moments $V_{2k} \leq (ck)^k$ imposed in Theorem 2.1 can be essentially relaxed [7]. We keep it here to make the main stress on the second part of the proof of Theorem 2.1.

The second stage contains a detailed study of the moment analog of a Green function version of the Inverse Participation Ratio considered in the case of the well-known Gaussian Unitary Invariant Ensemble (GUE). This ensemble represents a restriction of the Wigner ensemble (2.3) with (2.1) and (2.2) to the case of the joint Gaussian (normal) distribution of the random matrix elements [9]. On this stage we use the method of non-asymptotic estimates of the moments of GUE developed in [6].

Everywhere below, we will say for the sake of simplicity that $K_n(s', s'')$ (2.6) represents the correlation function of the moments of random matrices $W^{(n)}$.

### 3 Correlation function and moment analog of IPR

The study of the correlation functions of high moments of Wigner random matrices has been started in [11]. The reasoning of this paper is based on the natural representation of $K_n(s', s'')$ (2.6) as a weighted sum over the set of pairs of closed paths of $2s'$ and $2s''$ steps

$$T_{2s', 2s''}(n) = \left\{ (t_{2s'}, t_{2s''}) \right\}$$

where $I^{(1)} = (i_0^{(1)}, \ldots, i_{2s'-1}^{(1)}, i_0^{(1)})$, $I^{(2)} = (i_0^{(2)}, \ldots, i_{2s''-1}^{(2)}, i_0^{(2)})$, and $i_j \in \{1, 2, \ldots, n\}$ for $k = 1, 2$ and all $j$; namely

$$K_n(s', s'') = \sum_{I^{(1, 2)} \in T_{2s', 2s''}(n)} \Pi_n(I^{(1, 2)}),$$

(3.1)

where $\Pi_n$ is a weight of the pair $I^{(1, 2)}$ given by the mathematical expectation of the product of random variables $w_{ij}/\sqrt{n}$ determined by $I^{(1, 2)}$.

Clearly, a non-zero contribution to the right-hand side of (3.1) is given by a subset of path pairs such that $I^{(1)}$ and $I^{(2)}$ have at least one step in common. This means that there exist two instants of time $t'$ and $t''$ such that

$$\left\{ t_{i'}^{(1)} = t_{i'}^{(2)} \right\} \text{ and } \left\{ t_{i'+1}^{(1)} = t_{i'+1}^{(2)} \right\}$$

or

$$\left\{ i_{t'}^{(1)} = i_{t'+1}^{(2)} \right\} \text{ and } \left\{ i_{t'}^{(1)} = i_{t'+1}^{(2)} \right\}$$

(3.2)

(here and below we identify $i_{2s'}^{(1)}$ with $t_0^{(1)}$ and $i_{2s''}^{(2)}$ with $t_0^{(2)}$). In these two cases, we say that the step $(t', t'+1)$ of $I^{(1)}$ is passed by $I^{(2)}$ in the direct and the inverse orientations, respectively. In what follows, we consider $t'$ and $t''$ to be the first instants of time, i.e. the minimal ones, with the property (3.2) to hold. The number of times that the step $(t', t'+1)$ is seen in the path pair is referred as to the multiplicity $\mu$ of the step.

In paper [11], the main attention is paid to the subset $T_{2s', 2s''}(n)$ of path pairs such that $I^{(1)}$ and $I^{(2)}$ have at least one step of multiplicity $\mu = 2$ in common. This
kind of path pairs is referred to as to the simply correlated pairs [11]. Since $W^{(n)}$ are Hermitian, this step in common is passed by $I^{(2)}$ in the inverse orientation with respect to $I^{(1)}$.

By using the arguments of [11], one can prove that the sum over the set of simply correlated pairs

$$
\Sigma_n(s', s'') = \sum_{I^{(1,2)} \in \mathcal{I}^{(1,2)}} \Pi_n(I^{(1,2)})
$$

(3.3)

remains bounded in the limit (2.5) as $n \to \infty$ and that the contribution of the path pairs such that there exists at least one step of multiplicity $\mu \geq 4$ vanishes, no matter how many times this step is seen in $I^{(1)}$.

Indeed, regarding a simply correlated pair $(I^{(1)}, I^{(2)})$, one can remove the steps $(t^1, t^1 + 1)$ and $(t^{(2)}, t^{(2)} + 1)$ and consider the collection of remaining steps as a closed path of $2s' + 2s'' - 2$ steps

$$I_{2s' + 2s'' - 2} = (t_0^{(1)}, \ldots, t_{i'}^{(1)}, t_{i'+2}^{(2)}, \ldots, t_0^{(2)}, \ldots, t_{i''}^{(2)}, \ldots, t_{i'+2}^{(2)}, \ldots, t_0^{(1)})$$

such that

$$\Pi(I_{2s' + 2s'' - 2}) = \frac{1}{4n} \Pi(I_{2s' + 2s'' - 2}).$$

(3.4)

It is natural to say that the two-steps reduction procedure $(I^{(1)}, I^{(2)}) \to I_{2s' + 2s'' - 2}$ is performed here.

To get a non-zero weight $\Pi(I_{2s' + 2s'' - 2})$, one has to consider $I_{2s' + 2s'' - 2}$ to be an even closed path [11]. Since

$$\sum_{I_{2s' + 2s'' - 2}} \Pi(I_{2s' + 2s'' - 2}) = E \left\{ \text{Tr} \left( W^{(n)} \right)^{2s' + 2s'' - 2} \right\},$$

one can use the result about the universal upper bound for the averaged moments $nM_{2s' + 2s'' - 2}^{(n)} = E \left\{ \text{Tr} \left( W^{(n)} \right)^{2s' + 2s'' - 2} \right\}$ [12] and prove the existence of their limit. Let us stress that the proof of this estimate given in [12] should be completed and modified (see [7] or [8]). To show that $\Sigma_n(s', s'')$ remains bounded, it suffices to estimate the number of simply correlated path pairs $I^{(1,2)}$ that can be reconstructed from an even closed path $I_{2s' + 2s'' - 2}$. Omitting the details, one can say that this number is proved in [11] to be proportional to $(2s' + 2s'' - 2)^{3/2}$ that is compensated by the factor $1/(4n)$ of (3.4) and this completes the argument.

As for the path pairs that are not simply correlated ones, it is claimed in [11] and subsequent papers that the study of the corresponding sums can be also trivially reduced to the study of $M_{2s' + 2s'' - 2}^{(n)}$. Indeed, using the arguments described above, it is not to hard to show that the contribution to the right-hand side of (3.1) that comes from the path pairs such that $I^{(1)}$ and $I^{(2)}$ have at least one step of multiplicity $\mu \geq 6$ in common, vanishes in the limit (2.5) as $n \to \infty$. The same concerns the path pairs that have one step of multiplicity $\mu = 4$ in common and have some other step in common passed 4 times or more.

However, it remains one more case of non-simply correlated pairs that seems to be not so easy to treat. Consider a path pair such that $I^{(1)}$ and $I^{(2)}$ have a step
of multiplicity 4 in common and do not have any other common step. Then the two-steps reduction procedure described above leads to a path $I_{2s'+2s''-2}$ that, in particular, can be free from the steps of multiplicity $\mu \geq 4$. Then it is not clear how to prove the fact of vanishing contribution of the corresponding sum $\hat{\Sigma}_n(s', s'')$ by

the use of the arguments of [11] only. Up to our knowledge, no rigorous study of this question has been published or reported. In the present paper we give a proof based on the method of [6].

If the common step is passed by $I^{(1)}$ twice in opposite directions, then we get a sub-sum of $\Sigma_n(s', s'')$ given by expression

$$\hat{\Sigma}_n^{(1)}(s', s'') = \frac{V_1}{n^2} \sum_{u,v} \sum_{i_1,i_2} E \left\{ \left[ W_{i_1i_2}^{2s'-2-u} W_{i_2i_1}^{u} \right]^* \right\} E \left\{ \left[ W_{i_1i_2}^{2s''-2-v} W_{i_2i_1}^{v} \right]^* \right\}. \quad (3.5)$$

Here the symbol $[\cdot]^*$ indicates the fact that the corresponding product does not contain the steps of multiplicity 4 or greater. Therefore we can bound the right-hand side of the previous inequality by the expression

$$R_n(s', s'') = \frac{V_4}{n^2} \sum_{u,v} \sum_{i_1,i_2} E_{\text{GUE}} \left\{ A_{i_1i_1}^{2s'-u} A_{i_2i_2}^{u} \right\} E_{\text{GUE}} \left\{ A_{i_1i_1}^{2s''-v} A_{i_2i_2}^{v} \right\}, \quad (3.6)$$

where $A = A^{(n)}$ are the random matrices of the Gaussian Unitary Invariant Ensemble (GUE) and $E_{\text{GUE}}$ denotes the corresponding mathematical expectation [9]. Certainly, we assume that the matrix elements $A^{(n)}$ have the mean zero and the variance $(4n)^{-1}$ (cf. (2.2)). We used here the fact that all terms of the right-hand side of (3.6) are positive.

If the common step is passed by $I^{(1)}$ twice in the same direction, we get a sub-sum

$$\hat{\Sigma}_n^{(2)}(s', s'') = \frac{V_4}{n^2} \sum_{u,v} \sum_{i_1,i_2} E \left\{ \left[ W_{i_1i_2}^{2s'-2-u} W_{i_2i_1}^{u} \right]^* \right\} E \left\{ \left[ W_{i_1i_2}^{2s''-2-v} W_{i_2i_1}^{v} \right]^* \right\}$$

bounded by

$$S_n(s', s'') = \frac{V_4}{n^2} \sum_{u,v} \sum_{i_1,i_2} E_{\text{GUE}} \left\{ A_{i_1i_2}^{2s'-u} A_{i_1i_2}^{u} \right\} E_{\text{GUE}} \left\{ A_{i_1i_2}^{2s''-v} A_{i_1i_2}^{v} \right\}, \quad (3.7)$$

In the proof, we will see that $S_n(s', s'') = o(R_n(s', s''))$ in the limit (2.5) as $n \to \infty$. If the common step is passed by $I^{(1)}$ one or three times, the corresponding sub-sums are zero.

Let us point out that the expression (3.6) resembles the expression

$$\sum_{x=1}^{n} E \left\{ G_{xx}(z_1) G_{xx}(z_2) \right\} \quad (3.8)$$

with $G(z) = (A - zI)^{-1}$, $z_1 = \lambda + i\eta$, $z_2 = \lambda - i\eta$, $\eta > 0$ known in theoretical physics as a version of the Inverse Participation Ratio. This quantity that reflects the (de)localization properties of the eigenvectors of random operators naturally appears
also in the studies of the spectral properties of random matrices [2, 10]. Recently, it has been studied in the frameworks of the proof of the universality of the bulk spectral distribution of Wigner random matrices [3] on the microscopic scale $1/n$.

It is natural to consider (3.6) as a natural analog of (3.8). This become even clearer when one considers the diagonal part of (3.6) (see Section 5). Let us note that in the studies of the edge of the spectrum of Wigner ensemble, the asymptotic regime (2.5) corresponds to the microscopic scale $n^{-2/3}$.

4 Estimates of moments of GUE

Let us consider the Gaussian Unitary Invariant Ensemble of random matrices (GUE) that is given by a family of random Hermitian matrices with elements

$$
\left( A^{(n)} \right)_{xy} = \frac{1}{\sqrt{n}} a_{xy}, \quad x, y = 1, \ldots, n
$$

(4.1)
such that the law of $A^{(n)}$ has a density proportional to $\exp\{-2n\text{Tr} \left( A^{(n)} \right)^2\}$ [9]. Then (2.1) and (2.2) hold. For $p$ non-negative integer, we denote

$$
L_p^{(n)} = \frac{1}{n} \sum_{x=1}^{n} (A^{(n)})^p_{xx} = \frac{1}{n} \text{Tr} \left( A^{(n)} \right)^p,
$$

(4.2)
and consider the averaged moments $M_p^{(n)} = \mathbf{E} L_p^{(n)}$, where $\mathbf{E}$ denotes the mathematical expectation with respect to the probability measure generated by $\{A^{(n)}\}$. We also denote $U_p^{(n)}(x) = \mathbf{E}(A^{(n)})^p_{xx}$. Everywhere below, we will omit the superscripts $n$ when no confusion can arise.

We represent $R_n(s', s'')$ (3.6) as a sum of four terms

$$
R_n(s', s'') = \frac{V_4}{n^2} \sum_{k=1}^{4} R_n^{(k)} (s', s''),
$$

(4.3)
such that

$$
R_n^{(1)} (s', s'') = \sum_{x,y=1}^{n} \left( \sum_{\alpha_1 + \beta_1 = 2s'} U_{\alpha_1}(x) U_{\beta_1}(y) \right) \left( \sum_{\alpha_2 + \beta_2 = 2s''} U_{\alpha_2}(x) U_{\beta_2}(y) \right),
$$

$$
R_n^{(2)} (s', s'') = \sum_{x,y=1}^{n} \left( \sum_{\alpha_1 + \beta_1 = 2s'} U_{\alpha_1}(x) U_{\beta_1}(y) \right) \left( \sum_{\alpha_2 + \beta_2 = 2s''} \mathbf{E} \left\{ (A_{xx}^{\alpha_2})^o (A_{yy}^{\beta_2})^o \right\} \right),
$$

$$
R_n^{(3)} (s', s'') = \sum_{x,y=1}^{n} \left( \sum_{\alpha_1 + \beta_1 = 2s'} \mathbf{E} \left\{ (A_{xx}^{\alpha_1})^o (A_{yy}^{\beta_1})^o \right\} \right) \left( \sum_{\alpha_2 + \beta_2 = 2s''} U_{\alpha_2}(x) U_{\beta_2}(y) \right),
$$

and

$$
R_n^{(4)} (s', s'') = \sum_{x,y=1}^{n} \left( \sum_{\alpha_1 + \beta_1 = 2s'} \mathbf{E} \left\{ (A_{xx}^{\alpha_1})^o (A_{yy}^{\beta_1})^o \right\} \right) \left( \sum_{\alpha_2 + \beta_2 = 2s''} \mathbf{E} \left\{ (A_{xx}^{\alpha_2})^o (A_{yy}^{\beta_2})^o \right\} \right),
$$

(4.4)
where we denoted by \( X^\circ \) the centered random variable \( X^\circ = X - \mathbb{E}X \).

In the present section we study the values \( U_{2s}(x) \). We do this mostly in the frameworks of the method of recurrent non-asymptotic estimates developed in [6] with respect to the moments \( M_{2s}^{(n)} \) of GUE. The statement we prove generalizes the results of [6] and can be formulated as follows.

**Theorem 4.1.** Given any constant \( h > 1/16 \), there exists \( 0 < \kappa < 12 - 3/(4h) \) such that the estimate

\[
\sup_{x=1,\ldots,n} U_{2s}(x) \leq \left( 1 + h \frac{s(s^2 - 1)}{n^2} \right) m_s
\]  

holds for all values of integer positive \( s \) and \( n \) satisfying condition \( s^3/n^2 \leq \kappa \), where \( m_s \) are the moments of the corresponding to (4.1) semi-circle distribution [13]

\[
m_s = \frac{1}{2^{2s}(s+1)} \binom{2s}{s} = \frac{(2s)!}{2^{2s} s!(s+1)!}, \quad s = 0, 1, 2, \ldots
\]

Regarding the generating function \( \varphi(\tau) = \sum_{s=0}^{\infty} m_s \tau^s \), one can rewrite (4.4) in the form

\[
\sup_{x=1,\ldots,n} U_{2s}(x) \leq \left[ \varphi(\tau) + \frac{h\tau^2}{n^2 (1 - \tau)^{3/2}} \right]_s,
\]

where \( [f(\tau)]_s = f_s \) denotes the coefficients of the corresponding generating function \( f(\tau) = \sum_{s\geq0} f_s \tau^s \).

The proof of Theorem 4.1 is based on the analysis of recurrent relations for \( U_{2s}(x) \) and related variables

\[
D_{2s}^{(r)}(x) = \sum_{\alpha_1+\ldots+\alpha_r=2s} |\mathbb{E} \{(A_{\alpha_1})_{xx}^\circ L_{\alpha_2}^\circ \cdots L_{\alpha_r}^\circ \}|, \quad \alpha_i \geq 1.
\]

In fact, we need to consider more general than \( U_{2s}(x) \) variable

\[
U_{2s}(x, y) = \mathbb{E}(A_{2s})_{xy} = \sum_{t=1}^{n} \mathbb{E} \{A_{xt} (A_{2s-1})_{ty} \}.
\]

Applying to the last mathematical expectation the integration by parts formula (see Section 7 for the details), we get equality

\[
\sum_{t=1}^{n} \mathbb{E} \{A_{xt} (A_{2s-1})_{ty} \} = \frac{1}{4^n} \sum_{t=1}^{n} \sum_{j=0}^{2s-2} \mathbb{E} \{(A^j)_{yt} (A_{2s-2-j})_{xy} \} = \frac{1}{4} \sum_{j=0}^{s-1} M_{2j} U_{2s-2-2j}(x, y) + \sum_{\alpha_1+\alpha_2=2s-2} \mathbb{E} \{(A_{\alpha_1})_{xy}^\circ L_{\alpha_2}^\circ \}.
\]  

Introducing variable

\[
D_{2s}^{(r)}(x, y) = \sum_{\alpha_1+\ldots+\alpha_r=2s} |\mathbb{E} \{(A_{\alpha_1})_{xy}^\circ L_{\alpha_2}^\circ \cdots L_{\alpha_r}^\circ \}|,
\]

\[
\text{Suppose that the coefficients of the corresponding generating function}
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\text{one can rewrite (4.4) in the form}
\]

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\sup_{x=1,\ldots,n} U_{2s}(x) \leq \left[ \varphi(\tau) + \frac{h\tau^2}{n^2 (1 - \tau)^{3/2}} \right]_s,
\]

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\text{where } [f(\tau)]_s = f_s \text{ denotes the coefficients of the corresponding generating function}
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Applying to the last mathematical expectation the integration by parts formula (see Section 7 for the details), we get equality

\[
\sum_{t=1}^{n} \mathbb{E} \{A_{xt} (A_{2s-1})_{ty} \} = \frac{1}{4^n} \sum_{t=1}^{n} \sum_{j=0}^{2s-2} \mathbb{E} \{(A^j)_{yt} (A_{2s-2-j})_{xy} \} = \frac{1}{4} \sum_{j=0}^{s-1} M_{2j} U_{2s-2-2j}(x, y) + \sum_{\alpha_1+\alpha_2=2s-2} \mathbb{E} \{(A_{\alpha_1})_{xy}^\circ L_{\alpha_2}^\circ \}.
\]  

Introducing variable

\[
D_{2s}^{(r)}(x, y) = \sum_{\alpha_1+\ldots+\alpha_r=2s} |\mathbb{E} \{(A_{\alpha_1})_{xy}^\circ L_{\alpha_2}^\circ \cdots L_{\alpha_r}^\circ \}|,
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\sup_{x=1,\ldots,n} U_{2s}(x) \leq \left[ \varphi(\tau) + \frac{h\tau^2}{n^2 (1 - \tau)^{3/2}} \right]_s,
\]

\[
\text{where } [f(\tau)]_s = f_s \text{ denotes the coefficients of the corresponding generating function}
\]

\[
f(\tau) = \sum_{s\geq0} f_s \tau^s.
\]
we deduce from (4.7) the main inequality
\[ U_{2s}(x, y) \leq \frac{1}{4} (M \ast U(x, y))_{2s-2} + D_{2s-2}^{(2)}(x, y), \quad s \geq 1 \] (4.8)
with the initial condition \( U_0(x, y) = \delta_{xy}, \ D_0^{(2)} = 0. \)

Regarding the expectation
\[
E \left\{ (A^{|x|})_{xy} \left[ L_{a_2}^o \cdots L_{a_r}^o \right]^o \right\} = \sum_{t=1}^{n} E \left\{ A_{xt} (A^{a_1})_{ty} \left[ L_{a_2}^o \cdots L_{a_r}^o \right]^o \right\}
\]
and using again the integration by parts formula, we get equality
\[
E \left\{ (A^{|x|})_{xy} \left[ L_{a_2}^o \cdots L_{a_r}^o \right]^o \right\} = \frac{1}{4} \sum_{j=0}^{\alpha_1-2} E \left\{ L_j (A^{a_1-2-j})_{xy} \left[ L_{a_2}^o \cdots L_{a_r}^o \right]^o \right\}
\]
and
\[
+ \frac{1}{4n^2} \sum_{i=2}^{r} \alpha_i E \left\{ (A^{a_1+a_i-2})_{xy} L_{a_2}^o \cdots L_{a_{i-1}}^o \cdots L_{a_r}^o \right\}
\]
Taking into account identity
\[ E(XYZ^o) = E(X)E(Y^oZ) + E(Y)E(X^oZ) + E(X^oY^oZ) - E(X^oY^o)E(Z) \] (4.9)
and performing elementary transformations (see [6] for details), we get the second main inequality
\[
D_{2s}^{(r)}(x, y) \leq \frac{1}{4} (M \ast D^{(r)}(x, y))_{2s-2} + \frac{1}{4} (U(x, y) \ast D^{(r)}(x, y))_{2s-2}
\]
\[
+ \frac{1}{4} D_{2s-2}^{(r+1)}(x, y) + \frac{1}{4} \left( D^{(2)}(x, y) \ast D^{(r-1)} \right)_{2s-2}
\]
\[
+ \frac{r-1}{4n^2} \left( U''(x, y) \ast D^{(r-2)} \right)_{2s-2} + \frac{s(2s-1)(r-1)}{4n^2} D_{2s-2}^{(r-1)}(x, y), \] (4.10)
where we denoted
\[ U''_{2s} = \frac{(2s+2)(2s+1)}{2} U_{2s}(x, y) \]
and \( D_{2s}^{(r)} = n^{-1} \sum_{x=1}^{n} D_{2s}^{(r)}(x, x). \) The initial condition for (4.10) is given by the obvious equality \( D_{2s}^{(r)}(x, x) = \delta_{xy}(4n^2)^{-1}. \) Also we accept that \( D_{2s}^{(r)} = 0 \) whenever \( r > 2s \) and that \( D_{2s}^{(0)} = 0 \) and \( D_{2s}^{(0)} = \delta_{s,0}. \)

Consider a triangular domain of integers
\[ \Delta = \{(s, r) : \ s \geq 1, \ 2 \leq r \leq 2s \} \]
and the numbers \( U_s(x, y), \ \mathcal{D}_{2s}^{(r)}(x, y) \) determined in \( \Delta \) by the following system of recurrent relations
\[ U_s(x, y) = \frac{1}{4} (U \ast U(x, y))_{s-1} + \frac{1}{4} D_{s-1}^{(2)}(x, y), \] (4.11)
and

\[ D_s^{(r)}(x,y) \leq \frac{1}{4} \left( U \ast D_s^{(r)}(x,y) \right)_{s-1} + \frac{1}{4} \left( U(x,y) \ast D_s^{(r)} \right)_{s-1} + \frac{1}{4} D_{s-1}^{(r+1)}(x,y) + \frac{1}{4} \left( D^{(2)}(x,y) \ast D_s^{(r-1)} \right)_{s-1} \]

\[ + \frac{r - 1}{4n^2} \left( U''(x,y) \ast D_s^{(r-2)} \right)_{s-1} + \frac{s^2 r}{2n^2} D_{s-1}^{(r-1)}(x,y), \]

where

\[ U''(x,y) = \frac{(2s+2)(2s+1)}{2} U_s(x,y), \quad U_s = \frac{1}{n} \sum_{t=1}^{n} U_s(t,t). \]

\[ D_s^{(r)} = n^{-1} \sum_{t=1}^{n} D_s^{(r)}(t,t), \]

and the initial conditions are given by equalities

\[ U_0(x,y) = \delta_{xy} \quad \text{and} \quad D_1^{(2)}(x,y) = \frac{\delta_{xy}}{4n^2}. \]

The main technical result of the present section is as follows.

**Lemma 4.1.** The family of numbers \( \{U_s(x,y), D_s^{(r)}(x,y), (s,r) \in \Delta\} \) exists, is uniquely determined by (4.11) and (4.12) and is such that

\[ \sup_{x,y=1,...,n} U_{2s}^{(n)}(x,y) \leq \delta_{xy} U_s^{(n)} \quad \text{and} \quad \sup_{x,y=1,...,n} D_{2s}^{(r)}(x,y) \leq D_s^{(r)}. \]  

(4.13)

For any \( h > 1/12 \), there exists \( \kappa > 0 \) such that for all \( 1 \leq s \leq s_0 \), with \( s_0^2 \leq \kappa n^2 \) the following inequality holds

\[ U_s \leq \left[ \varphi(\tau) + \frac{h}{n^2} \frac{\tau^2}{(1-\tau)^{5/2}} \right]_s; \]  

(4.14)

moreover, there exists \( C, 1/24 < C < \min\{2h/3, 24\} \) such that inequalities

\[ D_s^{(r)} \leq \begin{cases} C(3r')! n^{-2r'} \left[ \tau(1-\tau)^{-2r'} \right]_s, & \text{if } r = 2r' \\ C(3r' + 3)! n^{-2r'-2} \left[ \tau(1-\tau)^{-(4r'+5)/2} \right]_s, & \text{if } r = 2r' + 1 \end{cases} \]

(4.15)

hold for all \( r \) and \( s \) such that \( s + 2r + 5 \leq s_0 \).

The proof of Lemma 4.1 repeats almost literally the proof of Lemmas 2.1, 2.2 and 2.3 of [6], so we omit the computations and explain only the key points of the method. One more reason for this is that the main ingredients of this method will be used also in the next section, where we present the principal computations that are similar for those of the proof of Lemma 4.1.

First of all, it follows from relations (4.11) and (4.12) that \( U_{2s}(x,y) = 0 \) for \( x \neq y \). Then we can consider the diagonal terms only \( U_{2s}(x,x) \). Moreover, we can introduce auxiliary numbers \( \bar{U} \) and \( \bar{D} \) that serve as the upper bounds for \( \sup_x U(x,x) \) and \( \sup_x D(x,x) \) and verify relations of the form (4.11) and (4.12) and therefore satisfy (4.14) and (4.15).
Next, assuming that $D^{(2)}$ is of the order $o(1)$, it is easy to deduce from (4.11) that the leading contribution to $\bar{U}^{(n)}_s$ is given by $[\varphi(\tau)]_s$. This generating function is determined as a solution of the quadratic equation

$$\varphi(\tau) = 1 + \tau \varphi^2(\tau)/4$$

(4.16)

and therefore is given by equality

$$\frac{\tau \varphi(\tau)}{2} = 1 - \sqrt{1 - \tau}.$$  

(4.17)

Assuming that $\bar{D}^{(3)}_s = o(D^{(2)}_s)$, it is not to hard to get from (4.15) with the help of (4.17) and the identity (see also relation (7.4) of Section 7)

$$(s + 1)(2s + 1)m_s = \left[ \frac{1}{(1 - \tau)^{3/2}} \right]_s$$

(4.18)

that

$$\bar{D}^{(2)}_s = \frac{C'}{n^2} \frac{1}{(1 - \tau)^2} (1 + o(1)).$$

Finally, regarding (4.17) and (4.18), it is natural to consider the third derivative $\varphi'''(\tau)$ as a function that determines the next-in-order corrections for $\bar{U}_s$ and therefore to find the optimal the form of (4.14).

Now let us return to the relation (4.11) written down for $\bar{U}_s$. Accepting the estimate (4.14) to hold, we see that the first term of the right-hand side of (4.11) can be rewritten as

$$\frac{1}{4} \left( \bar{U} \ast \bar{U} \right)_{s-1} = \left[ \frac{\tau}{4} \left( \varphi(\tau) + \frac{h\tau^2}{n^2(1 - \tau)^{3/2}} \right) \right]_s$$

$$= \left[ \varphi(\tau) - 1 + \frac{h\tau^2(1 - \sqrt{1 - \tau})}{n^2(1 - \tau)^{5/2}} \right]_s + \frac{h^2}{4n^2} \left[ \frac{\tau^5}{(1 - \tau)^5} \right]_s.$$ 

The negative part of the last expression given by $\Phi_n(\tau) = h\tau^2(1 - \tau)^{-2} n^{-2}$ is exactly what we need to compensate the contribution $C'(1 - \tau^2)^{-2} n^{-2}$ to the right-hand side of (4.11) that comes from the estimate of $\bar{D}^{(2)}_{s-1}$. It remains to control the last term $h^2[\tau^5(1 - \tau)^{-5}]_s(4n^4)^{-1}$ no to be greater than $[\Phi(\tau)_n]_s$. This is true provided the positive ratio $\kappa = \sup s^3/n^2$ is sufficiently small.

Finally, accepting that $\bar{D}^{(5)}_s = o(D^{(4)}_s)$, we get from (4.12) that

$$\bar{D}^{(4)}_s = \frac{C''}{n^3} \left[ \frac{1}{(1 - \tau)^3} \right]_s (1 + o(1)).$$

This leads to the conclusion that the value $n^{-4}[(1 - \tau)^{-9/2}]_s$ can be used to estimate $\bar{D}^{(3)}_s$. The form of (4.15) for general $r$ is dictated by (4.12) and by the detailed analysis of the expressions and constants involved into the computations.
5 Correlation function terms

In the present section we study variables

\[ P^{(r)}_{2s}(x,y) = \sum_{\alpha + \beta + \gamma_1 + \ldots + \gamma_{r-2} = 2s} |\mathbb{E} \left\{ (A^\alpha)^\circ_{xx} (A^\beta)^\circ_{yy} L^\circ_{\gamma_1} \cdots L^\circ_{\gamma_{r-2}} \right\}|, \quad x \neq y \]

\[ Q^{(r)}_{2s}(x,y) = \sum_{\alpha + \beta + \gamma_1 + \ldots + \gamma_{r-2} = 2s} |\mathbb{E} \left\{ (A^\alpha)^\circ_{xy} (A^\beta)^\circ_{yx} L^\circ_{\gamma_1} \cdots L^\circ_{\gamma_{r-2}} \right\}|, \quad x \neq y \]

and

\[ T^{(r)}_{2s}(x) = \sum_{\alpha + \beta + \gamma_1 + \ldots + \gamma_{r-2} = 2s} |\mathbb{E} \left\{ (A^\alpha)^\circ_{xx} (A^\beta)^\circ_{xx} L^\circ_{\gamma_1} \cdots L^\circ_{\gamma_{r-2}} \right\}|, \]

that we refer to as the non-crossing, crossing and diagonal terms, respectively.

**Theorem 5.1.** Under conditions of Lemma 4.1, there exists \( \chi < \min\{\kappa, 2^{-6}\} \) such that inequalities

\[
\sup_{x,y=1,\ldots,n} P^{(r)}_{s}(x,y) \leq \begin{cases} 
\frac{C(3r'+1)!}{n^{2r'+2}} \left[ \frac{\tau}{(1-r)^{2r'+3/2}} \right]_s, & \text{if } r = 2r'; \\
\frac{C(3r'+4)!}{n^{2r'+2}} \left[ \frac{\tau}{(1-r)^{2r'+5/2}} \right]_s, & \text{if } r = 2r' + 1,
\end{cases} \tag{5.1}
\]

\[
\sup_{x,y=1,\ldots,n} Q^{(r)}_{s}(x,y) \leq \begin{cases} 
\frac{C(3r')!}{n^{2r'+2}} \left[ \frac{\tau}{(1-r)^{2r'+3/2}} \right]_s, & \text{if } r = 2r'; \\
\frac{C(3r'+3)!}{n^{2r'+2}} \left[ \frac{\tau}{(1-r)^{2r'+5/2}} \right]_s, & \text{if } r = 2r' + 1,
\end{cases} \tag{5.2}
\]

and

\[
\sup_{x=1,\ldots,n} T^{(r)}_{s}(x) \leq \begin{cases} 
\frac{C(3r')!}{n^{2r'-1}} \left[ \frac{\tau}{(1-r)^{2r'-3/2}} \right]_s, & \text{if } r = 2r'; \\
\frac{C(3r'+3)!}{n^{2r'-1}} \left[ \frac{\tau}{(1-r)^{2r'-5/2}} \right]_s, & \text{if } r = 2r' + 1,
\end{cases} \tag{5.3}
\]

hold for all \( s \) and \( r \) such that \( s + 2r + 5 \leq s_0 \) with \( s_0^3 \leq \chi n^2 \).

We prove Theorem 5.1 by using a modification of the recurrent relations method developed in [6]. To derive these relations, let us consider expression

\[ H(\alpha, \beta, \Gamma_{r-2}) = \mathbb{E} \left\{ (A^\alpha)_{ab} (A^\beta)_{cd} L^\circ_{\gamma_1} \cdots L^\circ_{\gamma_{r-2}} \right\} \]

\[ = \sum_{t=1}^{n} \mathbb{E} \left\{ A_{at} (A^\alpha_{-1})_{tb} (A^\beta)_{cd} L^\circ_{\gamma_1} \cdots L^\circ_{\gamma_{r-2}} \right\} \]

and apply to the last mathematical the integration by parts formula (7.1). We get equality

\[ H(\alpha, \beta, \Gamma_{r-2}) = \frac{1}{4} \sum_{j=0}^{\alpha-2} \sum_{k=0}^{\beta-2} \mathbb{E} \left\{ (A^\alpha_{2-j})_{ab} (A^\beta)_{cd} L^\circ_{\gamma_1} \cdots L^\circ_{\gamma_{r-2}} \right\} \]

and

\[ H(\alpha, \beta, \Gamma_{r-2}) = \frac{1}{4} \sum_{j=0}^{\alpha-2} \sum_{k=0}^{\beta-2} \mathbb{E} \left\{ (A^\alpha_{2-j})_{ab} (A^\beta)_{cd} L^\circ_{\gamma_1} \cdots L^\circ_{\gamma_{r-2}} \right\} \]

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\[ T. \text{The structure of these relations resembles very much the one of (4.12) and the} \]

\[ \begin{align*}
  &+ \frac{1}{4n} \sum_{j=0}^{\beta-1} \mathbf{E} \left\{ (A^\gamma)_{cd} (A^{\alpha+\beta-2-j})_{cb} L_{\gamma_1}^o \ldots L_{\gamma_{r-2}}^o \right\} \\
  &+ \frac{1}{4n^2} \sum_{l=1}^{r-2} \mu_l \mathbf{E} \left\{ (A^{\alpha+\gamma-l})_{ab} (A^\beta)_{cd} L_{\gamma_1}^o \ldots L_{\gamma_{r-1}}^o \right\}.
\end{align*} \]

Using (4.9), we obtain that \( H(\alpha, \beta, \Gamma_{r-2}) \) can be represented as a sum of ten terms,

\[ \mathbf{E} \left\{ (A^\alpha)_{ab} (A^\beta)_{cd} L_{\gamma_1}^o \ldots L_{\gamma_{r-2}}^o \right\} = \sum_{k=1}^{10} J^{(k)}(\alpha, \beta, \Gamma_{r-2}), \quad (5.4) \]

where

\[ J^{(1)} = \frac{1}{4} \sum_{j=0}^{\alpha-2} M_j \mathbf{E} \left\{ (A^{\alpha-2-j})_{ab} (A^\beta)_{cd} L_{\gamma_1}^o \ldots L_{\gamma_{r-2}}^o \right\}; \]

\[ J^{(2)} = \frac{1}{4} \sum_{j=0}^{\alpha-2} U_{\alpha-j} (a, b) \mathbf{E} \left\{ (A^\beta)_{cd} L_{\gamma_1}^o \ldots L_{\gamma_{r-2}}^o \right\}; \]

\[ J^{(3)} = \frac{1}{4} \sum_{j=0}^{\alpha-2} \mathbf{E} \left\{ (A^{\alpha-2-j})_{ab} (A^\beta)_{cd} L_j L_{\gamma_1}^o \ldots L_{\gamma_{r-2}}^o \right\}; \]

\[ J^{(4)} = -\frac{1}{4} \sum_{j=0}^{\alpha-2} \mathbf{E} \left\{ (A^{\alpha-2-j})_{ab} (A^\beta)_{cd} \right\} \mathbf{E} \left\{ L_j L_{\gamma_1}^o \ldots L_{\gamma_{r-2}}^o \right\}; \]

\[ J^{(5)} = \frac{1}{4n} \sum_{j=0}^{\beta-1} U_j (a, d) U_{\alpha+j} (c, b) \mathbf{E} \left\{ L_{\gamma_1}^o \ldots L_{\gamma_{r-2}}^o \right\}; \]

\[ J^{(6)} = \frac{1}{4n} \sum_{j=0}^{\beta-1} U_{\alpha+j} (c, b) \mathbf{E} \left\{ (A^\gamma)_{cd} L_{\gamma_1}^o \ldots L_{\gamma_{r-2}}^o \right\}; \]

\[ J^{(7)} = \frac{1}{4n} \sum_{j=0}^{\beta-1} U_j (a, d) \mathbf{E} \left\{ (A^{\alpha+j-2})_{cb} L_{\gamma_1}^o \ldots L_{\gamma_{r-2}}^o \right\}; \]

\[ J^{(8)} = \frac{1}{4n} \sum_{j=0}^{\beta-1} \mathbf{E} \left\{ (A^\gamma)_{cd} (A^{\alpha+j-2})_{cb} L_{\gamma_1}^o \ldots L_{\gamma_{r-2}}^o \right\}; \]

\[ J^{(9)} = \frac{1}{4n^2} \sum_{l=1}^{r-2} \gamma_l U_{\alpha+l} (a, b) \mathbf{E} \left\{ (A^\beta)_{cd} L_{\gamma_1}^o \ldots L_{\gamma_{r-1}}^o \right\}, \] and

\[ J^{(10)} = \frac{1}{4n^2} \sum_{l=1}^{r-2} \gamma_l \mathbf{E} \left\{ (A^{\alpha+l-2})_{ab} (A^\beta)_{cd} L_{\gamma_1}^o \ldots L_{\gamma_{r-1}}^o \right\}. \]

Basing on (5.4), we derive a system of recurrent relations for the terms \( P, Q \) and \( T \). The structure of these relations resembles very much the one of (4.12) and the
triangular scheme of recurrent estimates can be applied to them. In general words, we assume the estimates of the form (5.1), (5.2) and (5.3) to be true for all the terms of the right-hand sides of the relations we get, and prove that the terms of the left-hand sides also obey the corresponding estimates. More detailed discussion of the triangular scheme of the proof of the recurrent relations can be found in [6].

5.1 Relations and estimates for \( P_{2s}^{(r)}(x, y) \)

Taking into account that \( U_{2s}(x, y) = 0 \) when \( x \neq y \) and denoting \( U_{2s}(x, x) = U_{2s}(x) \), we get from (5.4) the following system of relations for \( P_{2s}^{(r)}(x, y) \):

\[
P_{2s}^{(r)}(x, y) \leq \frac{1}{4} \left( M * P^{(r)}(x, y) \right)_{2s-2} + \frac{1}{4} \left( U(x) * D^{(r)}(y) \right)_{2s-2} + \frac{1}{4} P_{2s-2}^{(r+1)}(x, y) + \frac{1}{4} \left( P^{(2)}(x, y) * D^{(r-1)} \right)_{2s-2} + \frac{s}{2n} Q_{2s-2}^{(r)}(x, y)
\]

\[
+ \frac{r - 2}{4n^2} \left( U''(x) * D^{(r-2)} \right)_{2s-2} + \frac{r - 2}{4n^2} \frac{(2s + 2)(2s + 1)}{2} P_{2s-2}^{(r-1)}(x, y) \tag{5.5}
\]

with the initial condition \( P_{2s}^{(r)}(x, y) = 0 \).

Let us consider the case of \( P_{2s}^{(2r)} \). Regarding the estimate of \( U_{2s}(x) \) (4.6) and taking into account formula (7.5) of Section 7, we can write that

\[
U_{2s}''(x) \leq \left[ \frac{1}{(1 - \tau)^{3/2}} + \frac{18h}{n^2} \frac{\tau^2}{(1 - \tau)^{9/2}} \right]_{s}. \tag{5.6}
\]

Using these estimates and substituting (5.1) and (5.2) to the right-hand side of (5.5), we get the ten terms that present as the following sum;

\[
P_{2s}^{(2r)}(x, y) \leq \sum_{k=1}^{6} \mathcal{P}_{2s}^{(2r;k)},
\]

where

\[
\mathcal{P}_{2s}^{(2r;1)} = \frac{C(3r + 1)!}{4n^{2r}} \left[ \frac{\tau^2 \varphi(\tau)}{(1 - \tau)^{2r}} \right]_{s}
\]

\[
= \frac{1}{2} \frac{C(3r + 1)!}{n^{2r}} \left[ \frac{\tau}{(1 - \tau)^{2r}} \right]_{s} - \frac{1}{2} \frac{C(3r + 1)!}{n^{2r}} \left[ \frac{\tau}{(1 - \tau)^{2r-1/2}} \right]_{s};
\]

\[
\mathcal{P}_{2s}^{(2r;2)} = \frac{C(3r)!}{4n^{2r}} \left[ \frac{\tau^2 \varphi(\tau)}{(1 - \tau)^{2r}} \right]_{s}
\]

\[
= \frac{1}{2} \frac{C(3r)!}{n^{2r}} \left[ \frac{\tau}{(1 - \tau)^{2r}} \right]_{s} - \frac{1}{2} \frac{C(3r)!}{n^{2r}} \left[ \frac{\tau}{(1 - \tau)^{2r-1/2}} \right]_{s};
\]

\[
\mathcal{P}_{2s}^{(2r;3)} = \left( \frac{hC(3r + 1)!}{4n^{2r+2}} + \frac{hC(3r)!}{4n^{2r+2}} + \frac{C(3r + 4)!}{4n^{2r+2}} \right) \left[ \frac{\tau^4}{(1 - \tau)^{2r+5/2}} \right]_{s};
\]

\[
= \frac{1}{2} \frac{hC(3r + 1)!}{n^{2r+2}} \left[ \frac{\tau}{(1 - \tau)^{2r+2}} \right]_{s} - \frac{1}{2} \frac{hC(3r)!}{n^{2r+2}} \left[ \frac{\tau}{(1 - \tau)^{2r+1/2}} \right]_{s};
\]
\[
P^{(2r;4)}_{2s} = \left( \frac{C^24!(3r)!}{4n^{2r+2}} + \frac{hC(2r-2)(3r-3)!}{4n^{2r+2}} \right) \left[ \frac{\tau^3}{(1-\tau)^{2r+5/2}} \right]_s;
\]
\[
P^{(2r;5)}_{2s} = \frac{sC(3r)!}{2n^{2r}} \left[ \frac{\tau}{(1-\tau)^{2r-3/2}} \right] + \frac{C(2r-2)(3r-3)!}{4n^{2r}} \left[ \frac{\tau^4}{(1-\tau)^{2r-1/2}} \right]_s;
\]
and
\[
P^{(2r;6)}_{2s} = \frac{C(2r-2)(3r+1)!}{4n^{2r+2}} \frac{(2s+2)(2s+1)}{2} \left[ \frac{\tau}{(1-\tau)^{2r+1/2}} \right]_s.
\]

Regarding the sum of \( P^{(2r;1)}_{2s} \) and \( P^{(2r;2)}_{2s} \), we can write that
\[
P^{(2r;1)}_{2s} + P^{(2r;2)}_{2s} = \frac{C(3r+1)!}{n^{2r}} \left[ \frac{\tau}{(1-\tau)^{2r}} \right]_s - \mathcal{X}_0 - \mathcal{Y}_0,
\]
where
\[
\mathcal{X}_0 = \frac{3Cr(3r)!}{2n^{2r}} \left[ \frac{\tau}{(1-\tau)^{2r}} \right]_s \quad \text{and} \quad \mathcal{Y}_0 = \frac{C(3r+2)(3r)!}{2n^{2r}} \left[ \frac{\tau}{(1-\tau)^{2r-1/2}} \right]_s.
\]
The first term of the right-hand side of (5.7) reproduces the expression that we use as the estimate of \( B^{(2r)}_{2s} \). Therefore all that we need is to check that the sum of the remaining four terms \( P \) is less than the sum \( \mathcal{X}_0 + \mathcal{Y}_0 \). In fact, we are going to compare the sum of these four terms with \( \mathcal{Y}_0 \).

Using identity (7.3) (see Section 7), we can write that for any integer \( k \)
\[
\left[ \frac{\tau^{1+k}}{(1-\tau)^{2r+5/2}} \right]_s \leq \left[ \frac{\tau}{(1-\tau)^{2r+5/2}} \right]_s
\]
\[
= \frac{(2s + 4r - 3)(2s + 4r - 1)(2s + 4r + 1)}{(4r - 1)(4r + 3)} \left[ \frac{\tau}{(1-\tau)^{2r-1/2}} \right]_s.
\]
Similar computations show that
\[
\left[ \frac{\tau^4}{(1-\tau)^{2r-3/2}} \right]_s \leq \left[ \frac{\tau}{(1-\tau)^{2r-3/2}} \right]_s = \frac{4r-3}{2s+4r-5} \left[ \frac{\tau}{(1-\tau)^{2r-1/2}} \right]_s.
\]
Finally, we get equality
\[
\left[ \frac{\tau}{(1-\tau)^{2r+1/2}} \right]_s = \frac{2s+4r-3}{4r-1} \left[ \frac{\tau}{(1-\tau)^{2r-1/2}} \right]_s.
\]
Using these three relations and taking into account that \( s + 2r + 1 \leq s_0 = \chi n^{2/3} \), we can write that
\[
\frac{1}{\mathcal{Y}_0} \sum_{k=3}^{6} P^{(2r;k)}_{2s} \leq \frac{16h\chi}{(4r-1)^3} + 4\chi \frac{(3r+1)(3r+3)(3r+4)}{(4r-1)(4r+1)(4r+3)} + \frac{192C\chi}{(3r+2)(4r-1)(4r+3)}
\]
\[
+ \frac{(4r-3)s}{(3r+2)(2s+4r-5)} + \frac{r-1}{(3r-2)(3r-1)(3r+2)} + \frac{18h\chi}{(4r-1)(4r+1)(4r+3)}.
\]
Taking into account that \( r \geq 1 \), we see that the right-hand side of the last inequality is strictly less than \( 5/6 \) when \( h \) and \( \chi \) verify conditions of Theorem 5.1. The estimate (5.1) is proved for \( P_{2s}^{(2r)} \).

Let us consider relation (5.5) for \( P_{2s}^{(2r+1)} \). Taking into account estimates (4.6) and (5.6) and using expression (4.17), we get an inequality for \( P_{2s}^{(2r+1)} \) that counts ten terms that we present as the following sum

\[
P_{2s}^{(2r+1)}(x, y) \leq \sum_{k=1}^{6} P_{2s}^{(2r+1,k)},
\]

where

\[
P_{2s}^{(2r+1;1)} = \frac{C(3r + 4)!}{4n^{2r+2}} \left[ \frac{\tau^2 \phi(\tau)}{(1 - \tau)^{2r+5/2}} \right]_s,
\]

\[
P_{2s}^{(2r+1;2)} = \frac{C(3r + 3)!}{2n^{2r+2}} \left[ \frac{\tau^2 \phi(\tau)}{(1 - \tau)^{2r+5/2}} \right]_s - \frac{C(3r + 4)!}{2n^{2r+2}} \left[ \frac{\tau}{(1 - \tau)^{2r+2}} \right]_s;
\]

\[
P_{2s}^{(2r+1;3)} = \left( \frac{9Ch(3r + 4)!}{2n^{2r+4}} + \frac{9Ch(3r + 3)!}{2n^{2r+4}} + \frac{9hC(2r - 1)(3r)!}{2n^{2r+4}} \right) \left[ \frac{\tau^4}{(1 - \tau)^{2r+5}} \right]_s;
\]

\[
P_{2s}^{(2r+1;4)} = \left( \frac{C(3r + 4)!}{4n^{2r+2}} + \frac{C^24!(3r)!}{4n^{2r+2}} \right) \left[ \frac{\tau}{(1 - \tau)^{2r+2}} \right]_s;
\]

\[
P_{2s}^{(2r+1;5)} = \frac{sC(3r + 3)!}{2n^{2r+2}} \left[ \frac{\tau}{(1 - \tau)^{2r+1}} \right]_s + \frac{C(2r - 1)(3r)!}{4n^{2r+2}} \left[ \frac{\tau^2}{(1 - \tau)^{2r+2}} \right]_s;
\]

and

\[
P_{2s}^{(2r+1;6)} = \frac{C(2r - 1)(3r + 1)!}{2n^{2r+4}} \left( \frac{2s + 2)(2s + 1)}{2} \right) \left[ \frac{\tau}{(1 - \tau)^{2r}} \right]_s.
\]

Regarding the sum of two first terms, we can write that

\[
P_{2s}^{(2r+1;1)} + P_{2s}^{(2r+1;2)} = \frac{C(3r + 4)!}{2n^{2r+2}} \left[ \frac{\tau}{(1 - \tau)^{2r+5/2}} \right]_s - \mathcal{X}_1 - \mathcal{Y}_1,
\]

where

\[
\mathcal{X}_1 = \frac{C(3r + 5)(3r + 3)!}{2n^{2r+2}} \left[ \frac{\tau}{(1 - \tau)^{2r+2}} \right]_s
\]

and

\[
\mathcal{Y}_1 = \frac{C(3r + 3)(3r + 3)!}{2n^{2r+2}} \left[ \frac{\tau}{(1 - \tau)^{2r+5/2}} \right]_s.
\]
Using identity (7.3) of Section 7, it is not hard to show that for any integer \( s \geq 1 \)
\[
\left[ \frac{\tau}{(1 - \tau)^{2r+5/2}} \right]_s \geq \left[ \frac{\tau}{(1 - \tau)^{2r+2}} \right]_s.
\]
Then we can write that
\[
X_1 + Y_1 \leq \frac{C(3r + 3)(3r + 3)!}{n^{2r+2}} \left[ \frac{\tau}{(1 - \tau)^{2r+2}} \right]_s = \tilde{X}_1.
\]
Now it remains to show that the sum \( \sum_{k=3}^{10} P_{2s}^{(2r+1;k)} \) does not exceed \( \tilde{X}_1 \). To do this, we will use (7.6) and its consequences given by the following three relations;
\[
\left[ \frac{\tau^4}{(1 - \tau)^{2r+5}} \right]_s \leq \frac{s^3}{(2r + 2)(2r + 3)(2r + 4)} \left[ \frac{\tau}{(1 - \tau)^{2r+2}} \right]_s, \tag{5.11}
\]
\[
\left[ \frac{\tau}{(1 - \tau)^{2r+1}} \right]_s = \frac{2r + 1}{s + 2r} \left[ \frac{\tau}{(1 - \tau)^{2r+2}} \right]_s, \tag{5.12}
\]
and
\[
\left[ \frac{\tau}{(1 - \tau)^{2r}} \right]_s = \frac{2r(2r + 1)}{(s + 2r - 1)(s + 2r)} \left[ \frac{\tau}{(1 - \tau)^{2r+2}} \right]_s. \tag{5.13}
\]
Using these three relations, we can write that
\[
\frac{1}{X_1} \sum_{k=3}^{6} P_{2s}^{(2r+1;k)} \leq \frac{9h\chi(3r + 5)}{4(3r + 3)(2r + 2)(2r + 3)(2r + 4)}
\]
\[
+ \frac{3r + 4}{4(3r + 5)} + \frac{3C}{(3r + 1)(3r + 2)(3r + 3)^2} + \frac{2r + 1}{2(3r + 5)} + \frac{r}{(3r + 1)(3r + 2)(3r + 3)^2}
\]
\[
+ \frac{9h\chi(2r - 1)}{2(3r + 1)(3r + 2)(3r + 3)^2} + \frac{r(2r - 1)(2r + 1)}{(3r + 2)(3r + 3)^2}.
\]
It is clear that under conditions of Theorem 5.1 the right-hand side of the last inequality is strictly less than 1 for any \( r \geq 1 \). The estimate (5.1) is proved for the variable \( P_{2s}^{(2r+1)}(x, y) \).

### 5.2 Relations and estimates for \( Q_{2s}^{(r)}(x, y) \)

Regarding (5.4) with \( a = d = x \) and \( b = c = y \) and taking into account equality \( U_{2s}(x, y) = 0 \) for \( x \neq y \), we obtain the following recurrence;
\[
Q_{2s}^{(r)}(x, y) \leq \frac{1}{4} \left( M * Q^{(r)}(x, y) \right)_{2s-2} + \frac{1}{4} Q_{2s-2}^{(r+1)}(x, y)
\]
\[
+ \frac{1}{4} \left( Q_{2s-2}^{(2)}(x, y) * D^{(r-1)} \right)_{2s-2} + \frac{1}{4n} \left( U'(x) * U(y) * D^{(r-2)} \right)_{2s-2}
\]
\[
+ \frac{1}{4n} \left( U'(y) * D^{(r-1)}(x) \right)_{2s-2} + \frac{s}{2n} \left( U(x) * D^{(r-1)}(y) \right)_{2s-2}
\]
+ \frac{s}{2n} P^{(r-2)}_{2s-2}(x, y) + \frac{(r-2)(2s+1)}{4n^2} \cdot \frac{(2s+2)(2s+1)}{2} Q^{(r-1)}_{2s-2}(x, y), \quad (5.14)

where we denoted

\[ U_2'(x) = (2s + 2)U_2. \]

Let us consider the case of \( Q^{(2r)}_{2s} \). Regarding (4.6) and using formulas (7.4), we obtain that

\[ U_2'(x) \leq 2 \left[ \frac{1}{\sqrt{1 - \tau}} + \frac{h \tau^2}{n^2 (1 - \tau)^{3/2}} \right] . \quad (5.15) \]

Using this estimate and substituting (5.1) and (5.2) to the right-hand side of (5.14), we get twelve terms that we regroup in the sum of six terms as follows;

\[ Q^{(2r)}_{2s}(x, y) \leq \sum_{k=1}^{6} Q^{(2r,k)}_{2s}, \quad (5.16) \]

where

\[ Q^{(2r:1)}_{2s} = \frac{C(3r)!}{4n^{2r-1}} \left[ \frac{\tau^2 \varphi(\tau)}{(1 - \tau)^{2r-3/2}} \right]_{s}; \]

\[ Q^{(2r:2)}_{2s} = \frac{C(3r)!}{2n^{2r-1}} \left[ \frac{\tau}{(1 - \tau)^{2r-3/2}} \right]_{s} - \frac{C(3r)!}{2n^{2r-1}} \left[ \frac{\tau}{(1 - \tau)^{2r-2}} \right]_{s}; \]

\[ Q^{(2r:3)}_{2s} = \frac{C(3r)!}{2n^{2r+1}} \left( \frac{h(3r)!}{(1 - \tau)^{2r+3/2}} \right) + (s + 1)(2s - 1) \left[ \frac{\tau}{(1 - \tau)^{2r-1}} \right]_{s}; \]

\[ Q^{(2r:4)}_{2s} = \frac{C h}{2n^{2r+3}} \left( h(3r - 3)! \left[ \frac{\tau^2}{(1 - \tau)^{2r+4}} \right]_{s} + s(3r)! \left[ \frac{\tau^4}{(1 - \tau)^{2r+3}} \right]_{s} \right); \]

\[ Q^{(2r:5)}_{2s} = \frac{C(3r - 3)!}{2n^{2r-1}} \left[ \frac{\tau^4}{(1 - \tau)^{2r-3/2}} \right]_{s}, \]

and

\[ Q^{(2r:6)}_{2s} = \frac{C}{2n^{2r+1}} \left( s(3r)! \left[ \frac{\tau^2}{(1 - \tau)^{2r+1/2}} \right]_{s} + h(3r - 3)! \left[ \frac{\tau^4}{(1 - \tau)^{2r+3/2}} \right]_{s} \right). \]

Regarding \( Q^{(2r:1)}_{2s} \), we can write that

\[ Q^{(2r:1)}_{2s} = \frac{C(3r)!}{n^{2r-1}} \left[ \frac{\tau}{(1 - \tau)^{2r-3/2}} \right]_{s} - \mathcal{X}_2 - \mathcal{Y}_2, \quad (5.17) \]

where

\[ \mathcal{X}_2 = \frac{C(3r)!}{2n^{2r-1}} \left[ \frac{\tau}{(1 - \tau)^{2r-2}} \right]_{s}, \quad \mathcal{Y}_2 = -\frac{C(3r)!}{2n^{2r-1}} \left[ \frac{\tau}{(1 - \tau)^{2r-3/2}} \right]_{s}. \]
We are going to show that the sum $\sum_{k=2, 3, 4} Q_{2s}^{(2r; k)}$ is strictly less than $X_2$ and that the sum $Q_{2s}^{(2r; 5)} + Q_{2s}^{(2r; 6)}$ is strictly less than $Y_2$. To do this, we will use the following consequences of formulas (7.3) and (7.6):

\[
\left[ \frac{\tau}{(1 - \tau)^{2r-2+q}} \right]_s = \frac{(s + 2r - 3)(s + 2r - 2) \cdots (s + 2r - 4 + q)}{(2r - 1)2r \cdots (2r - 2 + q)} \left[ \frac{\tau}{(1 - \tau)^{2r-2}} \right]_s,
\]

for $q = 1, 2, 3, 5, 6$ and

\[
\left[ \frac{\tau}{(1 - \tau)^{2r+1/2}} \right]_s = \frac{(2s + 4r - 5)(2s + 4r - 3)}{(4r - 3)(4r - 1)} \left[ \frac{\tau}{(1 - \tau)^{2r-3/2}} \right]_s
\]

(5.18)

and

\[
\left[ \frac{\tau}{(1 - \tau)^{2r+3/2}} \right]_s = \frac{(2s + 4r - 5)(2s + 4r - 3)(2s + 4r - 1)}{(4r - 3)(4r - 1)(4r + 1)} \left[ \frac{\tau}{(1 - \tau)^{2r-3/2}} \right]_s
\]

(5.19)

Using (5.17); we conclude that

\[
\frac{1}{X_2} \sum_{k=2, 3, 4} Q_{2s}^{(2r; k)} \leq \frac{\chi}{2r(2r - 1)(2r - 3)} + \frac{\chi^2}{2r(2r - 1)} + \frac{\chi}{2r(2r - 1)2r} + \frac{\chi^2}{2r(2r - 1)2r^2} + \frac{\chi^3}{2r(2r - 1)2r^3} + \frac{\chi^4}{2r(2r - 1)2r^4} + \frac{\chi^5}{2r(2r - 1)2r^5} + \frac{\chi^6}{2r(2r - 1)2r^6}
\]

(5.20)

where the right-hand side is strictly less than 1 for any $s, r \geq 1$ and $s_0^3/n^2 \leq \chi$. Also we have inequality

\[
\frac{1}{Y_2} (Q_{2s}^{(2r; 5)} + Q_{2s}^{(2r; 6)}) \leq \frac{1}{3r(3r - 1)(3r - 2)} + \frac{\chi}{(4r - 1)(4r - 3)} + \frac{h\chi}{(4r - 1)(4r + 1)(3r - 1)(3r - 2)}
\]

(5.21)

where the right-hand side is strictly less than 1 under conditions of Theorem 5.1.

Let consider $Q_{2s}^{(2r+1)}(x, y)$. We have from (5.14) that

\[
Q_{2s}^{(2r+1)}(x, y) \leq \sum_{k=1}^{6} Q_{2s}^{(2r+1; k)},
\]

(5.22)

where

\[
Q_{2s}^{(2r+1; 1)} = \frac{C(3r + 3)!}{4n^{2r+1}} \left[ \frac{\tau^2 \varphi(\tau)}{(1 - \tau)^{2r+1}} \right]_s
\]

\[
= \frac{C(3r + 3)!}{2n^{2r+1}} \left[ \frac{\tau}{(1 - \tau)^{2r+1}} \right]_s - \frac{C(3r + 3)!}{2n^{2r+1}} \left[ \frac{\tau}{(1 - \tau)^{2r+1/2}} \right]_s
\]

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Now it is sufficient to show that
\[\sum Q^{(2r+1;2)}_{2s} = \frac{C}{4n^{2r+1}} ((3r + 3)! + 6C(3r)!/n) \left( \frac{\tau}{(1 - \tau)^{2r+1/2}} \right)_s + C(3r)! \left( \frac{(2r - 1)(s + 1)(2s + 1)}{4n^{2r+1}} \right) \left( \frac{\tau}{(1 - \tau)^{2r-3/2}} \right)_s;\]
\[Q^{(2r+1;3)}_{2s} = \frac{sC}{2n^{2r+3}} (h(3r)! + (3r + 3)! \left( \frac{\tau}{(1 - \tau)^{2r+5/2}} \right)_s;\]
\[Q^{(2r+1;4)}_{2s} = \frac{hC}{2n^{2r+3}} (C(3r+3)! + (3r)! \left( \frac{\tau}{(1 - \tau)^{2r+7/2}} \right)_s + \frac{h^2 C(3r)!}{2n^{2r+4}} \left( \frac{\tau}{(1 - \tau)^{2r+13/2}} \right)_s;\]
and
\[Q^{(2r+1;5)}_{2s} = \frac{C(3r)!}{2n^{2r+1}} \left( s \left( \frac{\tau}{(1 - \tau)^{2r+1}} \right)_s + 2 \left( \frac{\tau}{(1 - \tau)^{2r+1}} \right)_s + \frac{h}{n^2} \left( \frac{\tau}{(1 - \tau)^{2r+4}} \right)_s.\]

Regarding \(Q^{(2r+1;1)}_{2s}\), we can write that
\[Q^{(2r+1;1)}_{2s} = \frac{C(3r + 3)!}{n^{2r+1}} \left( \frac{\tau}{(1 - \tau)^{2r+1}} \right)_s - X_3 - Y_3, \tag{5.23}\]
where
\[X_3 = \frac{C(3r + 3)!}{2n^{2r+1}} \left( \frac{\tau}{(1 - \tau)^{2r+1}} \right)_s, \quad Y_3 = \frac{C(3r + 3)!}{2n^{2r+1}} \left( \frac{\tau}{(1 - \tau)^{2r+1/2}} \right)_s;\]

Now it is sufficient to show that \(\sum_{k=2,3,4} Q^{(2r+1;k)}_{2s}\) is strictly less than \(Y_3\) and that \(Q^{(2r+1;5)}_{2s}\) is strictly less than \(X_3\).

It follows from (7.6) that
\[\left( \frac{\tau}{(1 - \tau)^{2r-3/2}} \right)_s = \frac{(4r - 3)(4r - 1)}{(2s + 4r - 3)(2s + 4r - 1)} \left( \frac{\tau}{(1 - \tau)^{2r+1/2}} \right)_s;\]
and that
\[\left( \frac{\tau}{(1 - \tau)^{2r+1/2+q}} \right)_s \leq \frac{(2s_0)^q}{(4r)^q} \left( \frac{\tau}{(1 - \tau)^{2r+1/2}} \right)_s.\]

Then we easily get inequality
\[\frac{1}{X_3} \sum_{k=2,3,4} Q^{(2r+1;k)}_{2s} \leq \frac{1}{2} + \frac{8}{27} + \frac{h\chi(1+C)}{16} + \frac{h^2\chi^3}{4^9}. \tag{5.24}\]
Clearly, the right-hand side of it is less than 1 under conditions of Theorem 5.1.

Using (7.6), we can write that
\[\frac{1}{X_3} Q^{(2r+1;5)}_{2s} \leq \frac{2rs}{(s + 2r)(3r)^3} + \frac{2}{(3r)^3} + \frac{h\chi}{(2r)^3}. \tag{5.25}\]
The right-hand side of this inequality is obviously less than 1 under conditions of Theorem 5.1. Inequality (5.2) is proved.
5.3 Relations and estimates for $T_{2s}^{(r)}(x)$

Using (5.4) with $a = b = c = d = x$, we get the following recurrence for the diagonal term $T_{2s}^{(r)}(x)$:

$$
T_{2s}^{(r)}(x) \leq \frac{1}{4} \left( M \ast T^{(r)}(x) \right)_{2s-2} + \frac{1}{4} \left( U(x) \ast D^{(r)}(x) \right)_{2s-2} + \frac{1}{4} T_{2s-2}^{(r+1)}(x) \\
+ \frac{1}{4} \left( T_{2s-2}^{(r)}(x) \ast D^{(r-1)} \right)_{2s-2} + \frac{1}{4n} \left( U'(x) \ast U(x) \ast D^{(r-2)} \right)_{2s-2} \\
+ \frac{s}{n} \left( U(x) \ast D^{(r-1)}(x) \right)_{2s-2} + \frac{s}{2n} T_{2s-2}^{(r)}(x) \\
+ \frac{r - 2}{4n^2} \left( U'' \ast D^{(r-2)} \right)_{2s-2} + \frac{r - 2}{4n^2} \cdot \frac{(2s + 2)(2s + 1)}{2} T_{2s-2}^{(r-1)}(x, y),
$$

(5.26)

The right-hand side of (5.26) contains more terms than those of the relations for the non-crossing terms $P_{2s}^{(r)}$ and the crossing term $Q_{2s}^{(r)}$. Therefore the total number of terms to consider raises up to 16. However, the estimates repeat in the most part the estimates performed to prove (5.2). This is because the second and the eighth terms of the right-hand side of (5.21) that are absent in (5.14) but present in (5.5) are of the order smaller than the leading terms of the right-hand side of (5.21). That is why the diagonal term $T_{2s}^{(r)}(x)$ is bounded by the same expression (5.3) as the non-crossing term $Q_{2s}^{(r)}(x)$.

We do not present the detailed proof of (5.3) because it is very similar to that of the proof of (5.2) and uses the same formulas of Section 7. Indeed, when estimating $T_{2s}^{(2r)}$, we conclude from relation (5.25) and expressions (5.3) that the leading term and the negative part $-\chi^4 - \psi_4$ are given by the corresponding terms of the right-hand side of relation (5.17). Then it is not hard to see that the "extra" terms coming from the right-hand side of (5.26) with respect to (5.14) add the terms

$$
\frac{\chi}{24} + \frac{h^2 \chi^2}{3r(3r - 1)(3r - 2)} \quad \text{and} \quad \frac{h \chi}{3} + \frac{h \chi^3}{12(3r - 1)(3r - 2)}
$$

to the right-hand sides of (5.20) and (5.21), respectively. Certainly, this does not alter much the result of the sum that is still strictly less than 1 under conditions of Theorem 5.1.

Regarding $T_{2s}^{(2r+1)}$, we see that the leading term and the negative contributions $\chi_5$ and $\psi_5$ are exactly the same as the corresponding terms of the right-hand side of (5.23). The "extra" terms of (5.26) produce then the additional terms

$$
\frac{4\chi}{(4r + 1)(4r + 3)} \quad \text{and} \quad \frac{\chi^3}{32} + \frac{\chi}{8(3r)^3} + \frac{3h \chi^3}{8(3r)^3}
$$

for the right-hand sides of (5.24) and (5.25), respectively. Certainly, this does not alter the result of sums that are strictly less than 1 under conditions of Theorem 5.1.
6 Proof of Theorem 2.1

Regarding variable $R_n(s', s'')$ (4.3), we apply the result (4.6) of Theorem 4.1 to the factors of the first term $R_n^{(1)}(s', s'')$ and write that

$$R_n^{(1)}(s', s'') \leq n^2 \left[ (\varphi(\tau) + h\tau^2 n^{-2}(1 - \tau)^{-5/2})^2 \right]_{\bar{s}'} \left[ (\varphi(\tau) + h\tau^2 n^{-2}(1 - \tau)^{-5/2})^2 \right]_{\bar{s}'},$$

where $\bar{s} = s - 1$. Repeating computations of (4.19), we get inequality

$$R_n^{(1)}(s', s'') \leq 16n^2 \left[ \varphi(\tau) + h\tau^2 n^{-2}(1 - \tau)^{-5/2} \right]_{\bar{s}'} \left[ \varphi(\tau) + h\tau^2 n^{-2}(1 - \tau)^{-5/2} \right]_{\bar{s}''}$$

$$\leq 16n^2 m_{s'} m_{s''} \left( 1 + \frac{h(\bar{s}')^3}{n^2} \right) \left( 1 + \frac{h(\bar{s}'')^3}{n^2} \right).$$

It follows from the expression for $m_\tau$ (4.5) that $m_\tau = (\pi \bar{s}^{-3})^{-1/2}(1 + o(1))$ as $s \to \infty$. Then in the limit (2.5) we have the bound

$$\limsup_{n \to \infty} R_n^{(1)}(s', s'') \leq 16 \frac{(1 + h\chi')(1 + h\chi'')}{\pi \sqrt{\chi' \chi''}}. \quad (6.1)$$

Let us pass to the last term of the sum (4.3). Using the results of Theorem 5.1, we can write that

$$R_n^{(4)}(s', s'') \leq \sum_{x,y=1}^{n} P_{2s'-2}(x, y) P_{2s''-2}(x, y) + \sum_{x=1}^{n} T_{2s'-2}(x) T_{2s''-2}(x)$$

$$\leq n^2 \left( \frac{24s'C}{n^2} \right) \left( \frac{24s''C}{n^2} \right) + n \left( \frac{6s'C}{n} m_{s'-1} \right) \left( \frac{6s'C}{n} m_{s''-1} \right).$$

Taking into account the asymptotic expression for $m_\tau$, we get in the limit (2.5)

$$R_n^{(4)}(s', s'') \leq \left( \frac{24C}{n^{2/3}} \right)^2 (\chi' \chi'')^{1/3}(1 + o(1)) + \frac{36C^2}{n^{5/3} (\chi' \chi'')^{1/6}}(1 + o(1)). \quad (6.2)$$

Similar computations show that

$$R_n^{(2)}(s', s'') + R_n^{(3)}(s', s'') \leq \frac{96C}{n^{1/3}} \left( \frac{\chi''(1 + h\chi'')}{(\chi')^{1/2}} \right) + \frac{96C}{n^{1/3}} \left( \frac{(1 + h\chi')^{1/3}}{(\chi''^{1/2})} \right). \quad (6.3)$$

Remembering the factor $V_4/n^2$ of (4.3), we see that $\bar{S}_n(s', s'')$ (3.3) vanishes in the limit (2.5) as $n \to \infty$.

Let us consider the variable $S_n(s', s'')$ (3.7) and its representation in four terms similar to (4.3). It follows from the results of Section 4 that the terms $S_n^{(k)}(s', s'')$, $k = 1, 2, 3$ that contain factors $U_\alpha(x, y)U_\beta(x, y)$ are equal to zero. Then only the term

$$S_n^{(4)}(s', s'') = \sum_{x,y=1}^{n} \sum_{\alpha_1 + \beta_1 = 2s'} E \left\{ (A_{x,y}^{(\alpha_1)})^\circ (A_{x,y}^{(\beta_1)})^\circ \right\} \left( \sum_{\alpha_2 + \beta_2 = 2s''} E \left\{ (A_{x,y}^{(\alpha_2)})^\circ (A_{x,y}^{(\beta_2)})^\circ \right\} \right),$$

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gives a non-zero contribution to $S_n(s', s'')$. Repeating computations of Subsection 5.2, it is not hard to show that $S_n^{(4)}(s', s'') = o(1)$ in the limit (2.5) as $n \to \infty$. This completes the proof of Theorem 2.1.

7 Auxiliary relations

For completeness, let us refer to some of the equalities and identities of [6] that we use in the present paper.

The first equality is a consequence of the integration by parts formula applied to the normal (Gaussian) random variable $\xi \sim N(0, v^2)$ that is

$$E\xi f(\xi) = v^2 E f'(\xi),$$

where $f(x)$ is a non-random function such that corresponding mathematical expectations exist. Then for the random matrix $A$ from GUE we get, in particular, relation

$$E_{xy} (A^k)_{uv} = \frac{1}{4n} \sum_{j=0}^{k-1} E \{ (A^j)_{uy} (A^{k-1-j})_{xv} \}. \quad (7.1)$$

The second identity relates the generating functions $(1 - \tau)^{-r - 1/2}$ with the moments $m_s$. It is not hard to obtain that

$$\left[ \frac{1}{(1 - \tau)^{r+1/2}} \right]_s = r \left( \frac{2^r + 2s}{s+1} \right) m_s, \quad (7.2)$$

or in equivalent form,

$$\left[ \frac{1}{(1 - \tau)^{r+1/2}} \right]_s = \frac{1}{2^{2s} s!} \frac{(2r + 2s)!}{(r+s)!} \frac{r!}{(2r)!}. \quad (7.3)$$

Two particular cases are important:

$$\frac{(2s+1)(2s+2)}{2} m_s = \left[ \frac{1}{(1 - \tau)^{3/2}} \right]_s \quad (7.4)$$

and

$$\frac{(2s+1)(2s+2)(2s+3)}{3!} m_s = \left[ \frac{1}{(1 - \tau)^{5/2}} \right]_s. \quad (7.5)$$

We also use the equality

$$\left[ \frac{1}{(1 - \tau)^{k+1}} \right]_s = \frac{(s+1)(s+2)\cdots(s+k)}{k!}. \quad (7.6)$$

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