Fast dual proximal gradient algorithms with rate $O(1/k^{1.5})$ for convex minimization

Donghwan Kim · Jeffrey A. Fessler

Abstract We consider minimizing the composite function that consists of a strongly convex function and a convex function. The fast dual proximal gradient (FDPG) method decreases the dual function with a rate $O(1/k^2)$, leading to a rate $O(1/k)$ for decreasing the primal function. We propose a generalized FDPG method that guarantees an $O(1/k^{1.5})$ rate for the dual proximal gradient norm decrease. By relating this to the primal function decrease, the proposed approach decreases the primal function with the improved $O(1/k^{1.5})$ rate.

Keywords Dual-based methods · Fast gradient methods · Convex optimization · Rate of Convergence

1 Introduction

This paper focuses on improving the rate of convergence of dual-based proximal gradient methods for minimizing the sum of two convex functions, where one is assumed to be strongly convex. The convergence analysis in this paper focuses on the rate of decrease of the dual proximal gradient norm, whereas the existing analysis in [1] focuses on the rate of decrease of the dual function.

This work is based on the alternating minimization algorithm by Tseng [2] that exploits the strong convexity. The method [2] is essentially equivalent to applying the proximal gradient method to the dual function, which is naturally named a dual proximal gradient (DPG) method in [1]. In [1,3], this alternating minimization algorithm (or DPG) is accelerated using the fast proximal gradient method (FPGM) in [4], widely popularized under the name FISTA. That fast DPG (FDPG) method decreases the dual function at rate $O(1/k^2)$ due to the acceleration of FISTA [4], where $k$ denotes the number of iterations; the FDPG method is effective for various applications such as total-variation-based image denoising problems [3,5] and model predictive control problems [6].

In the interest of the primal convergence analysis of DPG and FDPG methods, Beck and Teboulle [1] derived nonasymptotic convergence bounds for the decrease of the distance between the primal sequence and a primal solution, and for the primal function decrease of DPG and FDPG. In particular, the rate $O(1/k^2)$ for the dual function decrease of FDPG provided the rate $O(1/k)$ for both the primal distance and function decrease, which is superior to those rates of subgradient and DPG methods in [1].

In addition to analyzing the primal convergence analysis using the dual function decrease as in [1], Nesterov [7] pointed out that the dual gradient decrease is closely related to the primal function decrease for minimizing a strongly convex function with a linear equality constraint. He then suggested using an algorithm that decreases the dual gradient with a fast rate $O(1/k^{1.5})$, thus providing the same rate for the primal function decrease. That analysis was extended to a linear inequality constrained strongly convex problem in [8]. This paper further extends such analyses to strongly convex composite problems, by showing that the dual proximal gradient decrease is directly related to the primal function decrease.

This research was supported in part by NIH grant U01 EB018753.

Donghwan Kim · Jeffrey A. Fessler
Dept. of Electrical Engineering and Computer Science, University of Michigan, Ann Arbor, MI 48109 USA
E-mail: kimdongh@umich.edu, fessler@umich.edu
We recently proposed an accelerated proximal gradient method named the generalized FPGM (GFPGM) in [9] that has rate \( O(1/k^{1.5}) \) for decreasing the proximal gradient norm and that is computationally as efficient as FISTA. This paper proposes to incorporate that method with duality, leading to a generalized FDPG (GFDPG) method. We show that the proposed approach has the rate \( O(1/k^{1.5}) \) for decreasing the primal function, by extending the analysis in [7,8]. As a byproduct of our analysis, we prove an \( O(1/k) \) bound on the rate of decrease of the primal function, which is interestingly the same as that of the FDPG in [1].

Sometimes the function information such as the strong convexity parameter is unavailable or difficult to approximate, and the FDPG method (and FISTA) have a backtracking scheme [4] that circumvents that problem. By introducing such a backtracking scheme to GFPGM [9], we illustrate that the proposed GFDPG also satisfies an \( O(1/k^{1.5}) \) bound on the primal function decrease for such cases.

Section 2 presents the optimization problem of interest and its dual. Section 3 reviews the convergence analysis of FDPG in [1]. Section 4 analyzes the convergence rate of the primal function decrease using the dual proximal gradient norm convergence. Section 5 proposes using the accelerated proximal gradient method named GFPGM in [9] instead of FISTA to effectively tackle the dual problem, leading to an improved \( O(1/k^{1.5}) \) rate for the primal function decrease. Section 6 concludes.

2 Optimization problem and its dual

2.1 The problem

This paper considers the following composite convex problem:

\[
\text{x}_* := \text{argmin}_{x} \{ H(x) := f(x) + g(Ax) \}, \quad (P)
\]

where both \( f : \mathbb{R}^n \rightarrow (-\infty, +\infty) \) and \( g : \mathbb{R}^m \rightarrow (-\infty, +\infty) \) are proper, closed, and convex extended real-valued functions, while the function \( f \) is further assumed to be \( \sigma \)-strongly convex for \( \sigma > 0 \), and \( A \) is a \( m \times n \) matrix. Due to the strong convexity, problem (P) has a unique optimal solution \( x_* \).

Problem (P) is general enough to model various applications; representative examples such as image denoising, projection onto the intersection of convex sets, and resource allocation problems are provided in [1] (see also [3,5,6]). Tackling such problems directly (in a primal domain) using algorithms such as subgradient methods suffer from relatively slow convergence rates [1]. The next subsection and Section 3 review the fast proximal gradient scheme combined with duality in [1,3] that exploits the properties of problem (P) and that converges faster than the subgradient methods [1].

2.2 The dual problem

Problem (P) has the following equivalent constrained form:

\[
\text{x}_* = \text{argmin}_{x} \min_{z} \left\{ \tilde{H}(x, z) := f(x) + g(z) : Ax - z = 0 \right\}, \quad (P')
\]

where \( H(x) = \tilde{H}(x, Ax) \). Problem (P') has the following dual problem:

\[
y_* \in \text{argmax}_{y} q(y), \quad (D)
\]

where the dual function is defined as [1]:

\[
q(y) := \min_{x,z} \left\{ \tilde{H}(x, z) - \langle y, Ax - z \rangle \right\} = -f^*(A^\top y) - g^*(-y) \quad (2.1)
\]

with dual variable vector \( y \in \mathbb{R}^m \). Let \( y_* \) denote an optimal (dual) solution of problem (D). The convex conjugates of \( f \) and \( g \) are defined as

\[
f^*(u) = \max_{x} \{ \langle u, x \rangle - f(x) \}, \quad \text{and} \quad g^*(y) = \max_{z} \{ \langle y, z \rangle - g(z) \}.
\]
To make problem (D) into an equivalent convex problem for convenience, \[1\] defines
\[
F(y) := f^*(A^\top y), \quad \text{and} \quad G(y) := g^*(-y),
\]
where \(F\) has a Lipschitz continuous gradient (due to the strong convexity of \(f\)) with a constant \(L_F := \frac{||A||^2}{\sigma}\) \[1, \text{Lemma 3.1}\], i.e., for any \(x, u \in \mathbb{R}^n\)
\[
||\nabla F(x) - \nabla F(u)|| \leq L_F ||x - u||.
\]
Then dual problem (D) is equivalent to the following:
\[
y_* \in \arg\min_y \{\tilde{q}(y) := F(y) + G(y)\}
\]
that consists of a smooth function \(F\) and a closed proper function \(G\). One can solve using proximal gradient methods. Note that \(\tilde{q}(y) = -q(y)\) by definition.

Even when solving the dual problem (D) (or (D')), we are eventually interested in analyzing the convergence rate of the primal sequence as in \[1\] and this paper. For a given dual variables vector \(y\), the corresponding primal variables vectors are defined as \((x(y), z(y)) \in \arg\min_{x,z} \left\{ \tilde{H}(x, z) - \langle y, Ax - z \rangle \right\}\), i.e.,
\[
x(y) := \arg\max_x \{ \langle A^\top y, x \rangle - f(x) \},
\]
\[
z(y) \in \arg\min_z \{ \langle y, z \rangle + g(z) \}.
\]
Then by definition, \(x_* = x(y_*)\) and these vectors satisfy
\[
\tilde{H}(x(y), z(y)) - q(y) = \langle y, Ax(y) - z(y) \rangle.
\]

Next, Section 3 reviews bounds on the convergence rate of the primal function decrease for the primal variable vector \(x(y)\) of dual-based proximal gradient methods using bounds on the dual function decrease \[1\]. In contrast, Sections 4 and 5 analyze the primal sequence using (2.6) and bounds on the dual proximal gradient decrease.

3 Fast dual-based proximal gradient methods

3.1 Dual-based proximal gradient methods

The proximal gradient method \[4\] for solving (D') has the following update at \(k\)th iteration for \(k \geq 1\) with given \(L_0\) and \(y_0\):\(^1\)
\[
y_k = p_{L_k}(y_{k-1})
\]
\[
:= \arg\min_y \left\{ Q_{L_k}(y, y_{k-1}) := F(y_{k-1}) + \langle y - y_{k-1}, \nabla F(y_{k-1}) \rangle + \frac{L_k}{2} ||y - y_{k-1}||^2 + G(y) \right\}
\]
\[
= \prox_{\frac{1}{L_k} G} \left( y_{k-1} - \frac{1}{L_k} \nabla F(y_{k-1}) \right),
\]
where \(L_k\) is chosen to satisfy \(L_{k-1} \leq L_k\) and \(\tilde{q}(p_{L_k}(y_{k-1})) \leq Q_{L_k}(p_{L_k}(y_{k-1}), y_{k-1})\), which guarantees descent because \(Q_{L_k}(p_{L_k}(y_{k-1}), y_{k-1}) \leq Q_{L_k}(y_{k-1}, y_{k-1}) = \tilde{q}(y_{k-1})\). Using the fixed constant \(L_k = L_F\) for all \(k\) can satisfy the condition on \(L_k\). However when \(L_F\) is unknown or cannot be easily approximated, a backtracking scheme in \[4\] can be adopted. This proximal gradient method decreases the (dual) function with rate \(O(1/k)\) \[4\].

\(^1\) The Moreau proximal map \[10\] of a proper closed and convex function \(h : \mathbb{R}^m \to (-\infty, \infty]\) in (3.1) is defined as \(\prox_h(w) = \arg\min_{y \in \mathbb{R}^m} \{h(y) + \frac{1}{2} ||y - w||^2\}\).
The proximal gradient update $p_{L_k}(y_{k-1})$ in (3.1) has an equivalent efficient update in terms of the original functions $f$ and $g$ as follows [1, Lemma 3.2]:

\[
\begin{align*}
    u_k &= x(y_{k-1}), \\
    v_k &= \text{prox}_{L_k,g}(Au_k - L_k y_{k-1}), \\
    y_k &= y_{k-1} - \frac{1}{L_k}(Au_k - v_k),
\end{align*}
\]

which exactly matches the update of the alternating minimization algorithm in [2]. The advantage of this alternating minimization algorithm over the augmented Lagrangian-based methods [11] for solving (P) (or (P')) is that the method can exploit separability of $f$ in the update step (3.2).

The next section reviews FDPG [1,3], the accelerated version of DPG using FISTA [4].

3.2 FDPG method and its convergence analysis

In [1,3], DPG is accelerated using FISTA [4] with negligible extra computation per iteration as shown below, which is named FDPG.

| The FDPG Method with backtracking |
|-----------------------------------|
| **Input:** Take $L_0$, $y_0 = w_0$, $t_0 = 1$. |
| **Step $k$. ($k \geq 1$)** |
| Choose $L_k$ s.t. $L_{k-1} \leq L_k$, and |
| $q(p_{L_k}(w_{k-1})) \leq Q_{L_k}(p_{L_k}(w_{k-1}), w_{k-1})$. |
| $y_k = p_{L_k}(w_{k-1})$ |
| $t_k = \frac{1+\sqrt{1+4t_{k-1}^2}}{2}$ |
| $w_k = y_k + \frac{t_{k-1}}{t_k}(y_k - y_{k-1})$ |

This FDPG has the following bound on the dual function decrease with rate $O(1/k^2)$ [4, Theorem 4.4], i.e.,

\[ q(y_*) - q(y_k) = \tilde{q}(y_k) - \tilde{q}(y_*) \leq \frac{2L_k||y_0 - y_*||^2}{t_{k-1}^2} \leq \frac{2L_k||y_0 - y_*||^2}{(k+1)^2}. \]  (3.5)

This rate is superior to the rate $O(1/k)$ for the dual function decrease of DPG [4, Theorem 3.1].

In [1], it is shown that the rate $O(1/k^2)$ of the dual function decrease in (3.5) provides the $O(1/k)$ bound on the convergence of the primal distance and function decrease. In particular, with the following assumption:

**Assumption 1** The function $H$ is subdifferentiable for all $x \in \mathbb{R}^n$, and its subgradients are bounded as

\[ \gamma_H := \max_{x \in \mathbb{R}^n} \max_{d \in \partial H(x)} ||d|| < \infty, \]

the corresponding primal sequence $\{x(y_k)\}$ of FDPG defined by (2.4) decreases the primal function with rate $O(1/k)$ [1, Theorem 4.3], i.e.,

\[ H(x(y_k)) - H(x_*) \leq 2\gamma_H \sqrt{\frac{L_k ||y_0 - y_*||}{\sigma (k+1)}}. \]  (3.6)

In addition, the proof of [1, Theorem 4.3] for (3.6) implies the following $O(1/\sqrt{k})$ bound for the primal function decrease of DPG.

---

2 [1] defines the closed and convex feasibility set $\mathcal{X} = \{x \in \mathbb{R}^n : x \in \text{dom}(f), Ax \in \text{dom}(g)\}$ and assumes $\gamma_H := \max_{x \in \mathcal{X}} \max_{d \in \partial H(x)} ||d|| < \infty$, whereas this paper uses $\mathcal{X} = \mathbb{R}^n$. 
Theorem 3.1 Let \( \{y_k\} \) be the sequence generated by DPG. Then for any \( k \geq 1 \) and with Assumption 1, the corresponding primal sequence \( \{x(y_k)\} \) defined by (2.4) satisfies

\[
H(x(y_k)) - H(x_*) \leq \gamma H \sqrt{\frac{L_k}{\sigma} ||y_0 - y_*||}.
\]  (3.7)

Proof This can be easily proven using [4, Theorem 3.1] that shows the \( O(1/k) \) rate for the dual function decrease of DPG, and using the proof of [1, Theorems 4.1 and 4.3]. \( \square \)

Both the bounds (3.6) and (3.7) resulting from the bound on the dual function decrease of FDPG and DPG respectively seem to suggest that the primal function decrease of FDPG is faster than that of DPG. However, the next section improves on (3.7) by deriving an \( O(1/k) \) bound on the primal function decrease for DPG, which is the same rate as that of FDPG in (3.6). This new analysis in Section 4 uses a bound on the dual proximal gradient norm decrease with an assumption that is weaker than Assumption 1 to analyze the primal function decrease.

4 Rate of convergence of the primal function

4.1 Preliminaries

This section presents two Lemmas that are the ingredients for relating the dual proximal gradient norm \( ||p_L(y) - y|| \) to the primal-dual gap \( H(x(p_L(y))) - q(p_L(y)) \). This in turn determines the rate of the decrease of the primal function \( H(x(p_L(y))) - H(x_*) \), because \( H(x_*) = H(x_*, Ax_*) \geq q(y_*) \geq q(p_L(y)) \).

Lemma 4.1 For any \( y, w \in \mathbb{R}^m \), the following inequality holds:

\[
||x(y) - x(w)|| \leq \frac{||A||}{\sigma} ||y - w||.
\]  (4.1)

Proof Since \( f \) is \( \sigma \)-strongly convex, for any \( x, u \in \mathbb{R}^n \) we have

\[
\sigma||x - u|| \leq ||f'(x) - f'(u)||,
\]

where \( f'(x) \in \partial f(x) \). Then, using \( A^T y \in \partial f(x(y)) \) that follows from the optimality condition of (2.4), we have

\[
\sigma||x(y) - x(w)|| \leq ||A^T(y - w)|| \leq ||A|| \cdot ||y - w||.
\]

\( \square \)

Lemma 4.2 For any \( y \in \mathbb{R}^m \) and \( L > 0 \), the following equality holds:

\[
Ax(y) - z(p_L(y)) + L(p_L(y) - y) = 0.
\]  (4.2)

Proof We show that the following vector \( \bar{z} \):

\[
\bar{z} := L(p_L(y) - y) + Ax(y)
\]  (4.3)

corresponds to \( z(p_L(y)) \).

Using (3.2), (3.3) and (3.4), we have

\[
\bar{z} = \text{prox}_{Lg} (Ax(y) - Ly)
\]

\[
= \arg\min_z \left\{ Lg(z) + \frac{1}{2} ||z - (Ax(y) - Ly)||^2 \right\}.
\]  (4.4)

The optimality condition of (4.4) implies that there exists \( g' (\bar{z}) \in \partial g(\bar{z}) \) such that \( Lg'(\bar{z}) + \bar{z} - Ax(y) + Ly = 0 \) that is equivalent to

\[
g'(\bar{z}) + p_L(y) = 0
\]  (4.5)

using (4.3). This condition (4.5) holds for \( \bar{z} = z(p_L(y)) \) based on the optimality condition of

\[
z(p_L(y)) \in \arg\min_z \{ q_L(y), z + g(z) \}
\]

in (2.5), which concludes the proof. \( \square \)
4.2 Relating the dual proximal gradient norm to the primal-dual gap

Based on Lemmas 4.1 and 4.2, the following Lemma analyzes the convergence bound for the primal-dual gap decrease $\tilde{H}(x(p_L(y)), z(p_L(y))) - q(p_L(y))$ of $(P').$

**Lemma 4.3** For any $y \in \mathbb{R}^m$, $L > 0$ and the corresponding primal vectors defined by (2.4) and (2.5), the following inequality holds:

$$\tilde{H}(x(p_L(y)), z(p_L(y))) - q(p_L(y)) \leq (L + L_F) \| p_L(y) \| \cdot \| p_L(y) - y \|. \quad (4.6)$$

**Proof** We have

$$\tilde{H}(x(p_L(y)), z(p_L(y))) - q(p_L(y)) = (p_L(y), Ax(p_L(y)) - z(p_L(y)))$$

$$= (p_L(y), Ax(y) - z(p_L(y)) + A(x(p_L(y)) - x(y)))$$

$$\leq \| p_L(y) \| (\| Ax(y) - z(p_L(y)) \| + \| A \| \cdot \| x(p_L(y)) - x(y) \|)$$

$$\leq \| p_L(y) \| \left( L(p_L(y) - y) \| + \frac{\| A \|^2}{\sigma} \| p_L(y) - y \| \right)$$

$$\leq \| p_L(y) \| (L + L_F) \| p_L(y) - y \|, \quad (4.7)$$

where the first equality uses (2.6), the first inequality uses the Cauchy-Schwartz and triangle inequalities, and the second inequality uses Lemmas 4.1 and 4.2. $\square$

Lemma 4.3 shows that the primal-dual gap decrease of $(P')$ depends on the decrease of the dual proximal update $\| p_L(y) - y \|$. However, we are more interested in the primal-dual gap decrease of $(P)$ than of $(P')$. Towards that end, we introduce the following assumption that is weaker than Assumption 1.

**Assumption 2** The function $g$ is subdifferentiable for all $z \in \mathbb{R}^m$, and its subgradients are bounded as

$$\gamma_g := \max_{z \in \mathbb{R}^m} \max_{d \in \partial g(z)} \| d \| < \infty.$$

We next analyze the convergence bound for the primal-dual gap $H(x(p_L(y))) - q(p_L(y))$ of $(P)$ using Assumption 2, which is one of the main contributions of this paper.

**Lemma 4.4** With Assumption 2, for any $y \in \mathbb{R}^m$, $L > 0$ and the corresponding primal vector defined by (2.4), the following primal-dual gap inequality holds:

$$H(x(p_L(y))) - q(p_L(y)) \leq (L + L_F) (\| p_L(y) \| + \gamma_g) \| p_L(y) - y \|. \quad (4.8)$$

**Proof** We have

$$H(x(p_L(y))) - q(p_L(y)) = f(x(p_L(y))) + g(Ax(p_L(y))) - q(p_L(y))$$

$$\leq f(x(p_L(y))) + g(z(p_L(y)))$$

$$- \langle g'(Ax(p_L(y))), z(p_L(y)) - Ax(p_L(y)) \rangle - q(p_L(y))$$

$$= \tilde{H}(x(p_L(y)), z(p_L(y))) - q(p_L(y))$$

$$- \langle g'(Ax(p_L(y))), z(p_L(y)) - Ax(p_L(y)) \rangle$$

$$= (p_L(y) + g'(Ax(y)) - z(p_L(y))$$

$$\| Ax(p_L(y)) - z(p_L(y)) \| + \gamma_g) \| p_L(y) - y \|.$$
4.3 New convergence analysis of the DPG and FDPG method

Both DPG and FDPG have the following bound on the (dual) proximal gradient norm [9, Theorem 1 and Equation (5.1)]:

\[ \| p_{L_k}(y_k) - y_k \| \leq \frac{2\| y_0 - y^* \|}{k}, \]  

(4.9)

for any \( L_k \) that satisfies \( \bar{q}(p_{L_k}(y_k)) \leq Q_{L_k}(p_{L_k}(y_k), y_k) \). (Inequality (4.9) simplifies for DPG by using \( L_k = L_{k+1} \) and \( y_{k+1} = p_{L_k}(y_k) \).) This inequality leads to new bounds on the primal-dual gap decrease of DPG and FDPG using Lemma 4.4 as shown next. 4

**Theorem 4.1** Let \( \{ y_k \} \) be the sequence generated by either DPG or FDPG. Then with Assumption 2, the corresponding primal sequence defined by (2.4) satisfies

\[ H(x(p_{L_k}(y_k))) - q(p_{L_k}(y_k)) \leq (L_k' + LF)(\| y_0 - y^* \| + \| y^* \| + \gamma_d) \frac{2\| y_0 - y^* \|}{k}, \]  

(4.10)

for any \( L_k' \) that satisfies \( \bar{q}(p_{L_k}(y_k)) \leq Q_{L_k'}(p_{L_k}(y_k), y_k) \).

**Proof** [4, Equation (3.6)] and Lemma 5.1 in Section 5 imply that the sequence \( \{ y_k \} \) of both DPG and FDPG satisfy

\[ \| p_L(y_k) \| \leq \| p_L(y_k) - y^* \| + \| y^* \| \leq \| y_0 - y^* \| + \| y^* \|, \]  

(4.11)

where the first inequality uses the triangle inequality. Inserting (4.9) and (4.11) in Lemma 4.4 concludes the proof.

To accelerate the rate of the primal function decrease, the next section proposes to replace FISTA with GFPGM [9] because it decreases the proximal gradient norm with rate \( O(1/k^{1.5}) \).

5 Generalized FDPG with rate \( O(1/k^{1.5}) \)

The following generalized FDPG (GFDPG) is an extension of GFPGM (with fixed \( L_k \)) in [9] that can adopt a backtracking scheme based on [4].

| The GFDPG method with backtracking |
|-----------------------------------|
| **Input.** Take \( L_0, y_0 = w_0, t_0 = T_0 \in (0, 1] \). |
| **Step.** \( k (k \geq 1) \) |
| Choose \( L_k \) s.t. \( L_{k-1} \leq L_k \), and |
| \( \bar{q}(p_{L_k}(w_{k-1})) \leq Q_{L_k}(p_{L_k}(w_{k-1}), w_{k-1}) \). |
| \( y_k = p_{L_k}(w_{k-1}) \) |
| Choose \( t_k \) s.t. \( t_k > 0 \) and \( t_k^2 \leq T_k := \sum_{i=0}^{k-1} t_i \). |
| \( w_k = y_k + \frac{(T_{k-1} - t_{k-1})t_k}{t_{k-1}-t_k}(y_k - y_{k-1}) + \frac{(t_{k-1} - T_{k-1})t_k}{t_{k-1}-t_k}(y_k - w_{k-1}) \). |

This GFDPG has the following bounds on the dual function decrease and dual proximal gradient norm decrease that extend [9, Theorems 3 and 4] for the GFDPG (GFPGM) with fixed \( L_k \). Note that the GFDPG and (5.1) reduce to FDPG and (3.5) respectively when one chooses \( t_k^2 = T_k \) for all \( k \).

3 This bound is tight up to a constant for DPG [12]. However, it is unknown whether or not FDPG (FISTA) has a bound for the proximal gradient norm decrease that is better than the rate \( O(1/k) \), which is an interesting open question considering that the Nesterov’s fast gradient method [13] (equivalent to FISTA for unconstrained smooth convex problems) decreases the gradient norm with rate \( O(1/k^{1.5}) \) in [12].

4 We have a primal-dual gap bound at the point \( p_L(x(y_k)) \) of FDPG in (4.10) rather than at the point \( x(y_k) \) in (3.6), since we only know a proximal gradient norm bound at \( p_L(x(y_k)) \) in (4.9).
**Theorem 5.1** Let \( \{y_k, w_k\} \) be the sequence generated by GFDPG. Then for any \( k \geq 1 \),
\[
q(y_\ast) - q(y_k) = \bar{q}(y_k) - \bar{q}(y_\ast)
\]
and
\[
\min \left\{ \left\| y_i - w_{i-1} \right\|, \left\| p_{L_k^\prime}(y_k) - y_k \right\| \right\} \leq \frac{L_k \left\| y_0 - y_\ast \right\|}{2T_k - 1},
\]
(5.1)
for any \( L_k^\prime \) that satisfies \( \bar{q}(p_{L_k^\prime}(y_k)) \leq Q_{L_k^\prime}(p_{L_k^\prime}(y_k), y_k) \), where \( y_i = p_{L_k}(w_{i-1}) \).

**Proof** See Appendix 7.1. \( \square \)

A specific version of GFDPG requires selecting the parameters \( t_k \). We consider the choice \( t_k = \frac{k + a}{a} \) for any \( a > 2 \) that leads to the following Corollary that provides an \( O(1/k^{1.5}) \) bound on the proximal gradient norm decrease using [9, Corollary 2].

**Corollary 5.1** Let \( \{y_k, w_k\} \) be the sequence generated by GFDPG with \( t_k = \frac{k + a}{a} \) for any \( a > 2 \). Then for any \( k \geq 1 \),
\[
\min \left\{ \left\| y_i - w_{i-1} \right\|, \left\| p_{L_k^\prime}(y_k) - y_k \right\| \right\} \leq \frac{a \sqrt{6}}{\sqrt{a} - 2} \frac{\left\| y_0 - y_\ast \right\|}{k^{1.5}}
\]
(5.3)
for any \( L_k^\prime \) that satisfies \( \bar{q}(p_{L_k^\prime}(y_k)) \leq Q_{L_k^\prime}(p_{L_k^\prime}(y_k), y_k) \), where \( y_i = p_{L_k}(w_{i-1}) \).

The following Lemma shows that the sequence \( \{y_k, w_k\} \) of GFDPG is bounded.

**Lemma 5.1** Let \( \{y_k, w_k\} \) be the sequence generated by GFDPG. Then for any \( k \geq 1 \),
\[
\max \left\{ \left\| p_{L_k^\prime}(y_k) \right\|, \left\| y_k \right\|, \left\| w_k \right\| \right\} \leq \left\| y_0 - y_\ast \right\| + \left\| y_\ast \right\|,
\]
(5.4)
for any \( L_k^\prime \) that satisfies \( \bar{q}(p_{L_k^\prime}(y_k)) \leq Q_{L_k^\prime}(p_{L_k^\prime}(y_k), y_k) \).

**Proof** See Appendix 7.2. \( \square \)

Inserting Corollary 5.1 and Lemma 5.1 to Lemmas 4.3 and 4.4 leads to the following Theorem that bounds the primal-dual gap decrease of \((P')\) and \((P)\) respectively for GFDPG with \( t_k = \frac{k + a}{a} \) for any \( a > 2 \).

**Theorem 5.2** Let \( \{y_k, w_k\} \) be the sequence generated by GFDPG with \( t_k = \frac{k + a}{a} \) for any \( a > 2 \). Then the corresponding primal sequence defined by (2.4) satisfies
\[
\min \left\{ \left\{ \bar{H}(x(y_i), x(y_i)) - q(y_i) \right\}_{i=1}^k, H(x(p_{L_k^\prime}(y_k)), x(p_{L_k^\prime}(y_k))) - q(p_{L_k^\prime}(y_k)) \right\}
\]
\[
\leq \frac{a \sqrt{6}}{\sqrt{a} - 2} (L_k + L_F) \left( \left\| y_0 - y_\ast \right\| + \left\| y_\ast \right\| \right) \frac{\left\| y_0 - y_\ast \right\|}{k^{1.5}},
\]
and with Assumption 2 the sequence satisfies
\[
\min \left\{ \left\{ H(x(y_i)) - q(y_i) \right\}_{i=1}^k, H(x(p_{L_k^\prime}(y_k))) - q(p_{L_k^\prime}(y_k)) \right\}
\]
\[
\leq \frac{a \sqrt{6}}{\sqrt{a} - 2} (L_k + L_F) \left( \left\| y_0 - y_\ast \right\| + \left\| y_\ast \right\| + \gamma \right) \frac{\left\| y_0 - y_\ast \right\|}{k^{1.5}},
\]
for any \( L_k^\prime \) that satisfies \( \bar{q}(p_{L_k^\prime}(y_k)) \leq Q_{L_k^\prime}(p_{L_k^\prime}(y_k), y_k) \).

**Remark 5.1** When one selects the total number of iterations \( N \) in advance, one can decrease the proximal gradient norm faster than the bound (5.3). It is found in [9] that the following choice
\[
t_k = \begin{cases} 
1, & k = 0, \\
\frac{1 + \sqrt{1 + 4k^2}}{2}, & k = 1, \ldots, \left\lfloor \frac{N}{2} \right\rfloor - 1, \\
\frac{N - k + 1}{2}, & \text{otherwise},
\end{cases}
\]
(5.5)
for GFPGM (and thus GFDPG) provides the best known proximal gradient norm bound.

**Remark 5.2** Other accelerated proximal gradient methods such as [14,15] that have \( O(1/k^{1.5}) \) bounds for decreasing the proximal gradient norm could be considered instead of using GFPGM for GFDPG, but their bounds are larger than those of GFPGM [9].
6 Conclusions

We provided a new analysis of the primal function decrease of the dual-based proximal gradient methods using the convergence analysis of the dual proximal gradient norm. As a consequence, we showed that using proximal gradient methods that decrease the proximal gradient norm with rate $O(1/k^{1.5})$ leads to the same fast rate for the primal function (and the primal-dual gap) decrease, improving on the previously best known rate $O(1/k)$.

7 Appendix

7.1 Proof of Theorem 5.1

Proof This proof uses the fact that the sequence $\{w_k\}$ of GFDPG is equivalent to the following [9, Proposition 2]:

$$w_k = \frac{T_{k-1}}{T_k} y_k + \frac{t_k}{T_k} s_k,$$

where $s_k := s_{k-1} + t_{k-1} (y_k - w_{k-1})$. This proof also uses the following two inequalities [4, Lemma 2.3]:

$$\tilde{q}(y_{k+1}) - \tilde{q}(y_k) \leq -\frac{L_{k+1}}{2} ||y_{k+1} - w_{k+1}||^2 - L_{k+1} \langle w_{k+1} - y_k, y_{k+1} - w_k \rangle,$$

and the following equality:

$$||s_{k+1} - y_\ast||^2 = ||s_k + t_k (y_{k+1} - w_k) - y_\ast||^2$$

$$= ||s_k - y_\ast||^2 + 2t_k \langle s_k - y_\ast, y_{k+1} - w_k \rangle + t_k^2 ||y_{k+1} - w_k||^2.$$

Using the above, we have

$$t_0 (\tilde{q}(y_1) - \tilde{q}(y_\ast))$$

$$\leq -\frac{L_{k+1}}{2} ||y_1 - w_0||^2 - L_{1+0} \langle w_0 - y_\ast, y_1 - w_0 \rangle$$

$$= -\frac{L_{k+1}}{2} (T_0 - t_0^2) ||y_1 - w_0||^2 + \frac{L_1}{2} \left(||s_0 - y_\ast||^2 - ||s_1 - y_\ast||^2\right),$$

and for $k \geq 1$, we have

$$T_{k-1} (\tilde{q}(y_{k+1}) - \tilde{q}(y_k)) + t_k (\tilde{q}(y_{k+1}) - \tilde{q}(y_\ast))$$

$$\leq -\frac{L_{k+1} T_k}{2} ||y_{k+1} - w_k||^2 - L_{k+1} \langle T_k w_k - T_{k-1} y_k - t_k y_\ast, y_{k+1} - w_k \rangle$$

$$= -\frac{L_{k+1} T_k}{2} ||y_{k+1} - w_k||^2 - L_{k+1} t_k \langle s_k - y_\ast, y_{k+1} - w_k \rangle$$

$$= -\frac{L_{k+1}}{2} \left(T_k - t_k^2\right) ||y_{k+1} - w_k||^2 + \frac{L_{k+1}}{2} \left(||s_k - y_\ast||^2 - ||s_{k+1} - y_\ast||^2\right),$$

which becomes

$$\frac{1}{2} \left(T_k - t_k^2\right) ||y_{k+1} - w_k||^2 + \frac{T_k}{L_{k+1}} (\tilde{q}(y_{k+1}) - \tilde{q}(y_\ast))$$

$$\leq \frac{T_{k-1}}{L_{k+1}} (\tilde{q}(y_k) - \tilde{q}(y_\ast)) + \frac{1}{2} \left(||s_k - y_\ast||^2 - ||s_{k+1} - y_\ast||^2\right)$$

$$\leq \frac{T_k}{L_k} (\tilde{q}(y_k) - \tilde{q}(y_\ast)) + \frac{1}{2} \left(||s_k - y_\ast||^2 - ||s_{k+1} - y_\ast||^2\right),$$

where the last inequality uses $L_k \leq L_{k+1}$.
Using a telescoping sum of (7.2) and (7.3), we have
\[
\sum_{i=0}^{k-1} \frac{1}{2} (T_i - t_i^2) \| y_{i+1} - w_i \|^2 + \frac{T_{k-1}}{L_k} (\tilde{q}(y_k) - \tilde{q}(y_*)) \
\leq \frac{1}{2} (\| s_0 - y_* \|^2 - \| s_k - y_* \|^2),
\] (7.4)
which implies (5.1).

The condition \( \tilde{q}(p_{L_k}^\prime (y_k)) \leq Q_{L_k}^\prime (p_{L_k}^\prime (y_k), y_k) \) implies the following inequality [16, Theorem 1]:
\[
\frac{L_k^\prime}{2} \| p_{L_k}^\prime (y_k) - y_k \|^2 \leq \tilde{q}(y_k) - \tilde{q}(p_{L_k}^\prime (y_k)) \leq \tilde{q}(y_k) - \tilde{q}(y_*),
\]
and inserting this into (7.4) leads to (5.2). Note that using \( 0 \leq \tilde{q}(y_k) - \tilde{q}(p_{L_k}^\prime (y_k)) \) instead leads to
\[
\min\{ \| y_i - w_{i-1} \| \}_{i=1}^k \leq \frac{\| y_0 - y_* \|}{\sqrt{\sum_{i=0}^{k-1} (T_i - t_i^2)}},
\] (7.5)
which does not require computing \( L_k \) and \( p_{L_k}^\prime (y_k) \) unlike (5.2), but (7.5) has an upper bound that is looser than (5.2).

7.2 Proof of Lemma 5.1

Proof We have
\[
\| p_L (y_{k+1}) - y_* \| \leq \| y_{k+1} - y_* \| \leq \| w_k - y_* \| \\
\leq \frac{T_{k-1}}{T_k} \| y_k - y_* \| + \frac{t_k}{T_k} \| s_k - y_* \| \\
\leq \frac{T_{k-1}}{T_k} \| y_k - y_* \| + \frac{t_k}{T_k} \| y_0 - y_* \| \\
\leq \max\{ \| y_k - y_* \|, \| y_0 - y_* \| \},
\] (7.6)
where the first and second inequalities use [4, Equation (3.6)], the third inequality uses (7.1), \( T_k = T_{k-1} + t_k \), and a triangle inequality, the fourth inequality uses (7.4), and the last inequality uses convexity. The inequality (7.6) implies
\[
\max\{ \| p_L (y_k) - y_* \|, \| y_k - y_* \|, \| w_k - y_* \| \} \leq \| y_0 - y_* \|
\]
for any \( k \geq 1 \), and thus inequality (5.4) follows from a triangle inequality.

References

1. A. Beck, M. Teboulle, A fast dual proximal gradient algorithm for convex minimization and applications, Operations Research Letters 42 (1) (2014) 1–6.
2. P. Tseng, Applications of a splitting algorithm to decomposition in convex programming and variational inequalities, SIAM J. Cont. Opt. 29 (1) (1991) 119–38.
3. T. Goldstein, B. O’Donoghue, S. Setzer, R. Baraniuk, Fast alternating direction optimization methods, SIAM J. Imaging Sci. 2 (3) (2014) 1588–623.
4. A. Beck, M. Teboulle, A fast iterative shrinkage/thresholding algorithm for linear inverse problems, SIAM J. Imaging Sci. 2 (1) (2009) 183–202.
5. A. Beck, M. Teboulle, Fast gradient-based algorithms for constrained total variation image denoising and deblurring problems, IEEE Trans. Im. Proc. 18 (11) (2009) 2419–34.
6. Y. Pu, M. N. Zeilinger, C. N. Jones, Fast alternating minimization algorithm for model predictive control, in: Proc. 19th World Congress of the International Federation of Automatic Control, 2014, pp. 11980–6.
7. Y. Nesterov, How to make the gradients small, Optima 88 (2012) 10–11.
8. I. Necoara, A. Patrascu, Iteration complexity analysis of dual first order methods for conic convex programming, Optimization Methods and Software 31 (3) (2016) 645–78.
9. D. Kim, J. A. Peleser, Another look at the “Fast Iterative Shrinkage/Thresholding Algorithm (FISTA)”, arxiv 1608.03861 (2016).
10. J. J. Moreau, Proximité et dualité dans un espace hilbertien, Bulletin de la Société Mathématique de France 93 (1965) 273–99.
11. R. Glowinski, P. L. Tallec, Augmented Lagrangian and operator-splitting methods in nonlinear mechanics, Soc. Indus. Appl. Math., 1989.
12. D. Kim, J. A. Fessler, Generalizing the optimized gradient method for smooth convex minimization, arxiv 1607.06764 (2016).
13. Y. Nesterov, A method for unconstrained convex minimization problem with the rate of convergence $O(1/k^2)$, Dokl. Akad. Nauk. USSR 269 (3) (1983) 543–7.
14. S. Ghadimi, G. Lan, Accelerated gradient methods for nonconvex nonlinear and stochastic programming, Mathematical Programming 156 (1) (2016) 59–99.
15. R. D. C. Monteiro, B. F. Svaiter, An accelerated hybrid proximal extragradient method for convex optimization and its implications to second-order methods, SIAM J. Optim. 23 (2) (2013) 1092–1125.
16. Y. Nesterov, Gradient methods for minimizing composite functions, Mathematical Programming 140 (1) (2013) 125–61.