We propose that the double scaling behavior of the unitary matrix models, and that of the complex matrix models, is related to type 0B and 0A fermionic string theories. The particular backgrounds involved correspond to \( \hat{c} < 1 \) matter coupled to super-Liouville theory. We examine in detail the \( \hat{c} = 0 \) or pure supergravity case, which is related to the double scaling limit around the Gross-Witten transition, and find that reversing the sign of the Liouville superpotential interchanges the 0A and 0B theories. We also find smooth transitions between weakly coupled string backgrounds with D-branes, and backgrounds with Ramond-Ramond fluxes only. Finally, we discuss matrix models with multicritical potentials that are conjectured to correspond to 0A/0B string theories based on \((2,4k)\) super-minimal models.
1. Introduction

Recent work on unstable D0-branes of two-dimensional bosonic string theory [1-7] has led to reinterpretation of the well-known large-$N$ matrix quantum mechanics formulation of this theory (for reviews, see [8-12]) as exact open/closed string duality. The open strings live on $N$ unstable D0-branes; the boundary state of such a D0-brane is a product of the ZZ boundary state [13] for the Liouville field, localized at large $\phi$, and of the Neumann boundary state for the time coordinate [3,4]. The dynamics of these open strings is governed by a gauged quantum mechanics of an $N \times N$ Hermitian matrix $M$ with an asymmetric (e.g. cubic) potential. This model is exactly solvable since the eigenvalues act as free fermions. In fact, these eigenvalues are the D0-branes. In the double scaling limit [14], the ground state of the 2-d bosonic string theory is constructed by filling one side of the inverted harmonic oscillator potential, $-\lambda^2/(2\alpha')$, with free fermions up to Fermi level $-\mu$ as measured from the top of the potential. Since $g_s \sim 1/\mu$, this state has a non-pertubative tunnelling instability.

While this matrix model formulation of the 2-d closed bosonic string has been known for quite some time [15], similar formulations of NSR strings have been a long-standing problem. In recent work [16,17] a solution of this problem was found for two-dimensional type 0 strings. Let us briefly summarize the logic that led to this solution. Consideration of unstable D0-branes of the type 0B theory indicates that the dynamics of open strings living on them is again governed by a gauged Hermitian matrix model, but now with a symmetric double-well potential. This led the authors of [16,17] to conjecture that the ground state of the closed 0B string theory corresponds to filling the potential $-\lambda^2/(4\alpha')$ symmetrically up to Fermi level $-\mu$. In the continuum formulation the parameter $\mu$ enters the superpotential of the super-Liouville theory as $\mu e^{\phi}$. This explains why in the 0B theory, unlike in the bosonic string, $\mu$ can have either sign. In fact, this theory has a symmetry under $\mu \rightarrow -\mu$ [8,12] which was called S-duality in [17]. For either sign of $\mu$ the fermions are divided symmetrically into two branches in phase space; hence, we may loosely call this a two-cut eigenvalue distribution.

A matrix model formulation of the 2-d type 0A closed string may be derived in a similar fashion. The 0A theory has charged D0-branes and anti D0-branes. The dynamics of open strings on $N + q$ D0-branes and $N$ anti D0-branes is described by $(N + q) \times N$ complex matrix quantum mechanics with $U(N + q) \times U(N)$ gauge symmetry. Just as the 0B model, this model is exactly solvable in terms of free fermions and is stable non-perturbatively [17].
In [16-23], these models were studied further. In particular, the matrix model formulations of the 2-d type 0 strings were subjected to a number of stringent checks vs. the continuum worldsheet formulation in terms of $\hat{c} = 1$ super-conformal field theory coupled to super-Liouville theory. In this paper we consider further extensions of these dualities to $\hat{c} < 1$ theories coupled to super-Liouville. If we turn on relevant operators in the $\hat{c} = 1$ theory, appropriately dressed by the Liouville field, then the theory undergoes gravitational RG flow to $\hat{c} < 1$ models coupled to the super-Liouville theory. Therefore, we expect such string theories to have matrix model duals closely related to the ones found for $\hat{c} = 1$. In this paper we indeed argue that type 0B theories are again dual to double-scaling limits of hermitian matrix models with two-cut eigenvalue distributions (or, equivalently, of the unitary matrix models), while type 0A theories are dual to complex matrix models.

For simplicity, we will restrict our discussion to the one-matrix models. Large $N$ unitary matrix models of this type were solved in [24-30] while the complex matrix models in [31-35]. The generic critical behavior of such models is that of the $\hat{c} = 0$ theory (pure supergravity), i.e. of type 0 strings in one dimension. The unitary matrix model below the Gross-Witten phase transition has a two-cut eigenvalue distribution and above the transition it has one cut. In the double scaling limit we conjecture that this model is dual to 0B string theory with $\mu > 0$ and $\mu < 0$ respectively. The complex matrix model has a phase transition where the eigenvalues reach the origin. We conjecture its double scaling limit around this phase transition to be the dual of 0A string theory. It was observed long ago that the double scaling limit of the generic complex matrix model is equivalent to the one of the generic unitary matrix model [31]. Indeed, for $\hat{c} = 0$ we find that a change in the sign of the left-moving fermion on the wordsheet sends the 0A theory at some value of $\mu$ to the 0B theory at $-\mu$.

By fine-tuning the potential in the matrix integrals, we are able to also describe a certain class of non-unitary $\hat{c} < 0$ SCFT’s coupled to super-Liouville theory. A worldsheet interpretation of these matrix models has been a longstanding puzzle. In fact, the idea that the two-cut hermitian matrix models are dual to SCFT’s coupled to supergravity was advanced in the early 90’s [36,24,28] but, as far as we know, was not tested thoroughly. We will present a number of consistency checks of this duality conjecture which rely on the interplay of the 0B and 0A models.

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1 Later in the paper we will sometimes use the phrase “two-cut hermitian matrix model” in referring to the model around this transition, either above or below the transition where it has one or two cuts.
In section 2 we discuss the unitary matrix models, reviewing and extending the existing literature on the subject. Section 3 is devoted to various aspects of the matrix model resolvent. It satisfies a loop equation which describes a Riemann surface. We demonstrate the discussion with the explicit solution of the simplest model at tree level. The Riemann surface also leads to a new insight into the FZZT \cite{37,38} and the ZZ \cite{13} branes in the theory. In sections 4 and 5 we present a detailed analysis of the simplest nontrivial theory − \( c = 0 \) theory – pure supergravity, and the first multicritical point. In section 6 we turn to discussion of the complex matrix model and its solutions. Section 7 presents a “spacetime” picture of these models. In section 8-10 we discuss the worldsheet interpretation of these theories. We analyze the R-R vertex operators and uncover interesting dependence on the sign of the cosmological constant, we discuss the properties of superminimal models coupled to supergravity and we explore the torus amplitude in these theories. In section 11 we present our conclusions and open questions for future research.

Several appendices provide more details for the interested reader. Some of these details are reviews of known results. In appendix A we discuss the comparison between the worldsheet and the matrix model results for the first multicritical point. In appendix B we mention some properties of superconformal minimal models and a restriction due to modular invariance on such models. Appendix C includes an assortment of results about the Zakharov-Shabat hierarchy of differential operators. In appendix D and appendix E we discuss various aspects of the complex matrix model.

\section{2. Unitary Matrix Models}

In this section we study unitary matrix models. The unitary one-matrix integrals have the form

\[ Z = \int dU \exp \left( -\frac{N}{\gamma} \text{Tr}V(U + U^\dagger) \right) , \quad (2.1) \]

where \( U \) is a unitary \( N \times N \) matrix. The simplest such model, with potential \( \sim \text{Tr}(U + U^\dagger) \), is obtained if one considers Wilson’s lattice action for a single plaquette. This one-plaquette model was originally solved in the large-\( N \) limit by Gross and Witten \cite{24}. For \( \gamma = \infty \) the eigenvalues \( e^{i\theta} \) are uniformly distributed on the circle parameterized by \( \theta \). As \( \gamma \) decreases, the eigenvalue distribution gets distorted: it starts decreasing in the region where the potential has a maximum. As \( \gamma \) is decreased below a critical value, \( \gamma_c \), a gap appears in the eigenvalue distribution: this is the third-order large \( N \) phase transition discovered in
The generic (and simplest) case is when the potential has a quadratic maximum. This is labelled as $k = 1$ in the classification of critical points. By further fine-tuning the potential, Periwal and Shevitz [25] found an infinite sequence of multi-critical points labelled by a positive integer $k$, and found their descriptions in the double-scaling limit in terms of the mKdV hierarchy of differential equations.

The double-scaling limit of the simplest unitary matrix model, $k = 1$, with potential $\text{Tr}(U + U^\dagger)$ is described by the Painlevé II equation [25]

$$2f'' - f^3 + xf = 0 ,$$

where $x \sim (\gamma_c - \gamma)N^{2/3}$. The free energy $F(x)$ is determined by $F'' = f(x)^2/4$. We will identify this model with pure 2-d supergravity where $x$ is proportional to the parameter $\mu$ in the super-Liouville interaction, and $-F(x)$ is the sum over surfaces.

More generally, the $k$-th critical point is described by a non-linear differential equation for $f(x)$ of order $2k$. The solution $f$ has the following large $x$ expansion:

$$f(x) = x^{1/(2k)} \left( 1 - \frac{2k + 1}{12k} x^{-(2k+1)/k} + \ldots \right) ,$$

from which it follows that

$$-F(x) = -\frac{x^{(2k+1)/k}}{4(2 + 1/k)(1 + 1/k)} - \frac{2k + 1}{24k} \ln x + \ldots .$$

In the double scaling limit the physics comes from the coalescence of two cuts. So, a double scaling limit of a hermitian matrix model around the point when two cuts meet will lead to the same free energy as the unitary matrix model. Important further steps in studying these models were made in [26,28,29] where both odd and even perturbations to the potential were considered. The complete model is described by two functions, $f(x)$ and $g(x)$, which in general satisfy coupled equations. An integer $m$ specifies the critical points studied in [28] (for even $m$ the relation to $k$ of [25] is $m = 2k$). In the $m = 2$ case the equations are

$$2f'' + f(g^2 - f^2) + xf = 0 ,$$

$$2g'' + g(g^2 - f^2) + xg = 0 ,$$

while the free energy is determined by

$$F'' = \frac{1}{4}[f(x)^2 - g(x)^2] .$$
More generally we will need two functions $F_l, G_l$ which are polynomials in $f, g$ and their derivatives. These polynomials are related to the Zakharov-Shabat hierarchy, which generalizes the mKdV hierarchy. They are defined by expanding the matrix resolvent

$$O = \langle x | J_3 R | x \rangle = \langle x | \frac{1}{D + Q - \zeta J_3} | x \rangle = \sum_{l=-1}^{\infty} (-J_1 F_l - iJ_2 G_l + J_3 H_l) \zeta^{-(l+1)} \quad (2.7)$$

where $J_i = \sigma_i / 2$ with $\sigma_i$ are the standard Pauli matrices and

$$D = \frac{d}{dx}, \quad Q = \begin{pmatrix} 0 & f + g \\ f - g & 0 \end{pmatrix}. \quad (2.8)$$

Our problem is invariant under a “boost” symmetry of $f$ and $g$. Therefore, it is natural to define

$$f \pm g = re^{\pm \beta}, \quad Q = \begin{pmatrix} 0 & f + g \\ f - g & 0 \end{pmatrix} = r \begin{pmatrix} 0 & e^\beta \\ e^{-\beta} & 0 \end{pmatrix}. \quad (2.9)$$

Similarly, we define a new “boosted” operator

$$\tilde{O} \equiv e^{-\beta J_3} O e^{\beta J_3} = \sum_{l=-1}^{\infty} (-J_1 R_l - iJ_2 \Theta_l + J_3 H_l) \zeta^{-(l+1)} \quad (2.10)$$

where $R_l$ and $\Theta_l$ are related to $F_l$ and $G_l$ by a boost. Using (2.10) and (2.7) we find

$$\tilde{O} = \frac{1}{D + rJ_1 + (\omega - \zeta) J_3}, \quad \text{with} \quad \omega = \beta'. \quad (2.11)$$

We see that undifferentiated $\beta$ does not appear in $\tilde{O}$, only $\omega$ and its derivatives appear. We can see from (2.11) that a constant shift in $\omega$ results in a shift in $\zeta$, which in turn produces a redefinition of the expansion where each $H_l$ gets mixed with lower $H_l$ terms.

The $2 \times 2$ matrix $O$ obeys the equation

$$[O, D + Q - \zeta J_3] = 0 \quad (2.12)$$

which leads to recursion relations for $F_l, G_l$ and $H_l$ in (2.7). So we can determine them all from the lowest ones $G_{-1} = F_{-1} = 0, H_{-1} = 1$. Equivalently, we can derive recursion relations for $R_l, \Theta_l$. It turns out that $r \Theta_l$ is always a total derivative. In fact, $r \Theta_l = -H'_l$. The recursion relations then become

$$R_{l+1} = \omega R_l - \left( \frac{H'_l}{r} \right)' + r H_l, \quad (2.13)$$

$$H'_{l+1} = \omega H'_l - r R'_l.$$
The first few terms are

\[ H_{-1} = 1, \quad R_{-1} = 0; \]
\[ H_0 = 0, \quad R_0 = r; \]
\[ H_1 = -r^2/2, \quad R_1 = \omega r; \]
\[ H_2 = -r^2\omega, \quad R_2 = -r^3/2 + r\omega^2 + r''; \]
\[ H_3 = \frac{3}{8}r^4 - \frac{3}{2}r^2\omega^2 + \frac{1}{2}rr^2 - rr'' , \]
\[ R_3 = -\frac{3}{2}r^3\omega + r\omega^3 + 3r\omega' + 3\omega r'' + r\omega''. \]  

(2.14)

These equations may also be obtained by “boosting” the recursion relations for \( F_l, G_l \) in

It is an interesting exercise to find the terms in \( H_l \) and \( R_l \) with no derivatives of \( r \) or \( \omega \).

In terms of the variables

\[ \rho^2 = r^2 + \omega^2, \quad \cos \varphi = \frac{\omega}{\rho}, \]  

(2.15)

we show in appendix C that

\[ H_l = -\rho^{l+1} \left[ \cos \varphi P_l(\cos \varphi) - P_{l+1}(\cos \varphi) \right] = -\rho^{l+1} \frac{\sin^2 \varphi}{l+1} P'_l(\cos \varphi), \]  

\[ R_l = \rho^{l+1} \sin \varphi P_l(\cos \varphi). \]  

(2.16)

The string equations of the matrix model may be stated as

\[ \sum_{l \geq 0} (l + 1)t_l G_l = 0 = \sum_{l \geq 0} (l + 1)t_l F_l \]  

(2.17)

where \( t_0 \sim x \). We also assume that \( t_l = 0 \) for \( l > m \). These equations follow from varying the action

\[ S = \int dx \sum_{l=0}^{m} t_l H_{l+1}. \]  

(2.18)

The \( l \)-th term correspond to a perturbation in the potential of the schematic form \( V \sim -t_l \nu^l Tr[M^{l+2}] \). As pointed out in [29], the terms with odd \( l \) (and real \( t_l \)) correspond to imaginary terms in the potential. Since we are doing an integral, there is nothing wrong

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2 Since \( \omega = \beta' \) this includes terms with derivatives in terms of the original variables.

3 This can be proven as in the KdV case. See [10] for a nice discussion of the KdV case.
with having these imaginary terms. In fact, they make the matrix integral more convergent, otherwise an odd term in the potential would be unbounded below if it ever dominates.\footnote{If we made \( t_m \) imaginary for odd \( m \) we would have a real potential. Then the string equation becomes real if we define \( \tilde{g} \to ig \). But in this case there is no real solution for \( f, \tilde{g} \).}

In all these models the free energy obeys \( F'' = -H_1/2 = r^2/4 \). The action (2.18) is invariant under \( x \)-independent shifts of \( \beta \). Therefore, the equation of motion of \( \beta \) is a total derivative and can be integrated by adding an integration constant \( q \). Alternatively, we can view \( \omega = \beta' \) as the independent variable in (2.18), and add to it \( q\omega \):

\[
S = \int dx (\sum_{l=0}^{m} t_l H_{l+1} + q\omega).
\] (2.19)

If we assume that this action can be used in a quantum theory such that \( e^{-S} \) is well defined, invariance under \( \beta \to \beta + 2\pi i \) leads to the conclusion that \( q \) must be quantized. In our case we are only solving the classical equations coming from this action, and it is not clear to us which problem the quantum theory is the answer for. Below we will discuss the physical interpretation of \( q \).

The action (2.19) leads to the equations of motion

\[
\frac{\delta}{\delta r(x)} \int dx (\sum_{l=0}^{m} t_l H_{l+1} + q\omega) = -\sum_{l=0}^{m} t_l (l+1) R_l = 0,
\]

\[
\frac{\delta}{\delta \omega(x)} \int dx (\sum_{l=0}^{m} t_l H_{l+1} + q\omega) = \sum_{l=0}^{m} t_l (l+1) H_l + q = 0.
\] (2.20)

Suppose we consider the \( m \)-th model, which has fixed \( t_m \). At first it seems that there are \( m \) parameters that we can vary: \( t_l \) with \( l = 0, \ldots, m-1 \). However the ability to shift \( \omega \) by a constant shows that we can set \( t_{m-1} = 0 \) at the expense of an analytic change of variables for the rest of the \( t_l \); i.e. the operator that couples to \( t_{m-1} \) is redundant (this is analogous to a similar operator in the bosonic string, which was discussed in \cite{39}). As a result, there are only \( m-1 \) operators. If we assign dimension minus one to \( x \sim t_0 \), then the operators have dimensions \( 1 - l/m \) for \( l = 0, \ldots, m-2 \). We will later match these operator dimensions with gravitational dimensions in (2,2\( m \)) super-minimal models coupled to super-Liouville theory.

If we set \( t_l = 0 \) for all odd \( l \) then there exists a solution with \( g = 0 \). We find that \( F_{2k}(f, g = 0) \) is the \( k \)-th member of the mKdV hierarchy derived for the unitary matrix
models in [25]. This corresponds to having an even potential and considering only even perturbations. For these even models $m = 2k$. We present examples of models with even $m$ in sections 4 and 5.

The simplest example of an odd model is the $m = 1$ theory. In this case (2.19) becomes

$$S = \int dx (-\frac{1}{2}r^2 \omega - \frac{1}{2}xr^2 + q\omega).$$

(2.21)

The equations of motion of this theory are

$$(\omega + x)r = 0$$

$$r^2 = 2q$$

(2.22)

and therefore $u = r^2/4 = q/2$. This “topological point” has been discussed in [28] and we now interpret it as associated with nonzero $q$. We will say more about the physical interpretation of $q$ below. Other examples of odd models are discussed in Appendix C.

3. Comments on the Matrix Model Resolvent

In this section we will analyze the matrix model in the standard large $N$ 't Hooft limit, i.e. in the planar approximation. Our main goals will be to identify the meaning of the parameter $q$ introduced above, as well as understand the relation between the FZZT [37,38] and ZZ [13] branes in string theory.

Consider the hermitian matrix model with potential $V(M)$. We will be interested in two closely related operators. The macroscopic loop (the FZZT brane)

$$W(z) = -\frac{1}{N} \text{Tr} \log(M - z) = \lim_{\epsilon \to 0} \left( \int_{\epsilon}^{\infty} \frac{dl}{l} \frac{1}{N} \text{Tr} e^{l(z - M)} + \log \epsilon \right)$$

(3.1)

($\epsilon$ is a UV cutoff) and the resolvent

$$R(z) = \frac{\partial W(z)}{\partial z} = \frac{1}{N} \text{Tr} \frac{1}{M - z}.$$ 

(3.2)

Without causing confusion we will denote by $W$ and $R$ both the matrix model operators and their expectation values in the large $N$ theory. We will later think of $R(z)dz = dW(z)$ as a one-form. It is clear from the expression for $W(z)$ that it can have an additive ambiguity of $\frac{2\pi ik}{N}$ with integer $k$. If this is the case, the one-form $R(z)dz$ is not exact.
Using the invariance of the matrix integral under $\delta M = \frac{1}{M-z}$ we derive the loop equation

$$\left\langle \text{Tr} \frac{1}{M-z} \text{Tr} \frac{1}{M-z} \right\rangle + N \left\langle \text{Tr} \frac{V'(M)}{M-z} \right\rangle = 0 \quad (3.3)$$

The last term can be written as $NV'(z)\left\langle \text{Tr} \frac{1}{M-z} \right\rangle$ plus a polynomial of degree $n-2$ if the degree of $V$ is $n$. In the large $N$ limit the first term factorizes and we find the loop equation for the resolvent

$$R(z)^2 + V'(z)R(z) - \frac{1}{4}f(z) = 0, \quad (3.4)$$

where $f(z)$ is a polynomial of degree $n-2$. The solution of (3.4) is

$$2R(z) = -V'(z) \pm \sqrt{V'(z)^2 + f(z)}. \quad (3.5)$$

The cuts in this expression mean that it is a function on a Riemann surface which is a two-fold cover of the complex plane. For each value of the parameter $z$ there are two points on the Riemann surface

$$y^2 = V'(z)^2 + f(z). \quad (3.6)$$

We will denote them by $P_{\pm}(z)$. They differ in the sign of $y$, $y(P_+(z)) = -y(P_-(z)) = \sqrt{V'(z)^2 + f(z)}$. The asymptotic behavior as $z \to \infty$ should be $R(P_+(z)) \to -\frac{1}{2}$ in the upper sheet. This determines

$$2R(P_{\pm}(z)) = \pm \sqrt{V'(z)^2 + f(z)} - V'(z). \quad (3.7)$$

In the planar limit the eigenvalues form cuts. Their density $\rho(\lambda) = \frac{1}{N} \sum_i \delta(\lambda - \lambda_i)$ is supported only on the cuts and it can be used to compute expectation values of operators, e.g.

$$R(z) = \frac{1}{N} \left\langle \text{Tr} \frac{1}{M-z} \right\rangle = \int d\lambda \frac{\rho(\lambda)}{\lambda - z} \quad \text{for } z \text{ not on the cuts} \quad (3.8)$$

from which we find

$$R(x + i\epsilon) + R(x - i\epsilon) = 2\mathcal{P} \int d\lambda \frac{\rho(\lambda)}{\lambda - x} \quad \text{for } x \text{ on the cuts,} \quad (3.9)$$

where $\mathcal{P}\int$ denotes the principal part.

Using the expression for the integral in the planar limit we define the effective potential of a probe eigenvalue at $z$ away from the cuts

$$V_{\text{eff}}(z) = V(z) - 2\int d\lambda \rho(\lambda) \log(z - \lambda) = V(z) + 2W(z). \quad (3.10)$$
The force on a probe eigenvalue away from the cuts is

$$F_{\text{eff}}(z) = -V_{\text{eff}}'(z) - 2W'(z) = -V'(z) - 2R(z)$$

$$= -R(P_+(z)) + R(P_-(z)) = -y(z). \quad (3.11)$$

The force on an eigenvalue on the cut, $-\frac{1}{2}\left(V_{\text{eff}}'(x+i\epsilon) + V_{\text{eff}}'(x-i\epsilon)\right)$ vanishes, as can be verified using (3.4) and (3.7)

$$V'(x) + 2\mathcal{P} \int d\lambda \frac{\rho(\lambda)}{\lambda - x} = V'(x) + R(x + i\epsilon) + R(x - i\epsilon) = 0. \quad (3.12)$$

We can also analytically continue the FZZT brane (3.4) to the second sheet and distinguish between $W(P_+(z))$ and $W(P_-(z))$. Since the analytic continuation of an analytic function is unique, this generalizes the analytic continuation discussed in [40,2].

This discussion makes it clear that the force $-y(z)dz$ or $Rdz$ are one forms on the Riemann surface, while $W$ is the corresponding potential. We can calculate the periods of the force around the $a$-cycles of the Riemann surface

$$\frac{1}{2\pi i} \oint_{C_i} Rdz = -\frac{N_i}{N} \quad (3.13)$$

where $C_i$ is a contour around the cut $i$ and $N_i$ is the number of eigenvalues in that cut. If there is only one cut, $N_1 = N$. More generally, the residue of $R$ at $z \to \infty$ is $-1$. This discussion makes it clear that $W(z)$ of (3.1) indeed has additive ambiguities of $2\pi ik/N$ with integer $k$.

Similarly, we can study the periods of $Rdz$ around the $b$-cycles of the Riemann surface (3.4). Assume for simplicity that there is only one such cycle; i.e. there are only two cuts. It is a contour which connects the two cuts in the upper sheet and then connects them also in the lower sheet. The period is

$$\hat{q} = \frac{q}{N} \equiv \frac{1}{4\pi i} \oint_b ydz = \frac{1}{2\pi i} \oint_b Rdz. \quad (3.14)$$

This means that $W$ has an additive monodromy of $2\pi iq/N$. We will later identify it with the flux or the number of D-branes. This discussion suggest that $q$ must be quantized. It does not prove it because this discussion applies only in the large $N$ limit with finite $\hat{q} = q/N$ which is arbitrary.
3.1. Solution of the simplest model in the planar limit

Now we will consider the simplest double-well model with $V'(z) = \frac{1}{g}(z^3 - z)$ which leads to the $k = 1$ critical behavior. The second order polynomial $f(z)$ is determined by two parameters:

$$f(z) = -\frac{4}{g}z^2 + f_1z + f_0,$$

where the first term is fixed by the condition that $R(z) \to -1/z$ for large $z$. We impose that the sixth order polynomial $y^2(z)$ has one double zero at $-i\omega$ ($\omega$ can in principle be any complex number; we will later take it to to be real) and four simple zeros. This gives one relation between the remaining two parameters and leads to the curve

$$y^2 = \frac{1}{g^2}(z + i\omega)^2(z^4 - 2i\omega z^3 - z^2(2 + 3\omega^2) + 4i\omega z(1 + \omega^2) + A),$$

with

$$A = (1 - 4g + 6\omega^2 - 5\omega^4)$$

Turning on the remaining parameter would spread the zero of $y$ into a branch cut.

The scaling limit is obtained when two of the simple zeros of $y^2$ go to infinity. Equivalently, we may scale $z$ and $\omega$ to zero at fixed ratio, while neglecting higher powers of $z$ and $\omega$. It is clear from the curve (3.16) that $A$ should scale like $z^2$. Therefore, the critical limit of $g$ is $g = 1/4$. So, we parametrize $g = \frac{1}{4} - x/2$, with $x$ scaling as $\omega$. In this limit the curve (3.16) becomes

$$g_c^2 y^2 = -2(z + i\omega)^2((z - i\omega)^2 - 4u)$$

with $g_c = 1/4$ and

$$4u = x + 2\omega^2$$

It is straightforward to compute the period of this $y$ around the cut between the simple zeros $z_{\pm} = i\omega \pm 2\sqrt{u}$

$$\frac{1}{4\pi i} \int ydz = 8\sqrt{2}\omega u = \hat{q} = \frac{q}{N}.$$

It turns out that the second derivative of the free energy is $\partial_x^2 F = 8u$. Up to a rescaling of $u, \omega$ and $x$, these are the same as the equations that follow from (2.20) for the simplest case, $t_2 = 1/3, t_0 = x/2$ and $q$ nonzero. We will discuss these equations in more detail in section 4.

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5 The rescaling is $x_{\text{there}} = 2x_{\text{here}}, u_{\text{there}} = 2u_{\text{here}}$ and $w_{\text{there}} = \sqrt{2}w_{\text{here}}$. 

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Now we solve the two equations (3.19) (3.20). One can identify \( u \) with the second derivative of the free energy. We will discuss first the case of \( q = 0 \) and then \( q \neq 0 \). For \( \hat{q} = 0 \), either \( u = 0 \) or \( \omega = 0 \). With \( \omega = 0 \) (3.10) has a double zero at the origin and two simple zeros at \( z_\pm = \pm 2\sqrt{u} \). It is natural to run the cuts in \( R \) from these two zeros to infinity. We take \( x = 4u \) to be positive and identify this with the positive \( x \) phase of the two cut model. The zero of \( y \) at the origin shows that an eigenvalue at that position feels no force. In string theory such an eigenvalue corresponds to an unstable ZZ brane. The potential has a maximum at this point.

With \( u = 0 \) (3.16) has double zeros at \( \pm i\omega \). Here we take \( \omega \) to be real to describe the negative \( x = -2\omega^2 \) phase of the two/one cut model. The ends of the other cut of this model were scaled to infinity and are not visible in this limit. The effective potential does not have stationary points for real \( z_0 \). Therefore it is not clear whether the system has unstable ZZ branes. However, one can place some eigenvalues at \( \pm i\omega \) where the effective potential is stationary. We will see that this is equivalent to taking \( q \neq 0 \) and will be interpreted in the string theory as a charged ZZ brane. As we will soon see, branes at \( i\omega \), along the positive imaginary axis, are charged D-branes and branes at \( -i\omega \), along the negative imaginary axis, are charged anti-D-branes.

Now let us turn to the case that \( q \neq 0 \). We take \( q \to \infty \) with finite \( \hat{q} = \frac{q}{N} \) so that its effects are visible in the planar limit.

Since the period around the \( b \)-cycle of the Riemann surface is nonzero, this cycle cannot collapse, and the theory cannot have a phase transition.

For large positive \( x \) we have \( u \approx \frac{x}{4} \) and \( \omega = \frac{x}{4u} \approx \frac{\hat{q}}{x} \). Therefore the two simple zeros at \( z_\pm = i\omega \pm 2\sqrt{u} \approx i\frac{\hat{q}}{x} \pm \sqrt{x} \) are approximately on the real axis and are far separated. In this limit it is natural to draw the cuts from these points to infinity.

As \( x \) becomes smaller and negative the two zeros move more into the complex plane. They move to the upper (lower) half plane for \( \hat{q} \) positive (negative). Let \( \hat{q} \) be positive. As \( x \to -\infty \) they approach \( i\omega \pm 2\sqrt{u} \approx i\sqrt{-\frac{\hat{q}}{2}} \pm \sqrt{-2x} \), i.e. they move up along the positive imaginary axis and approach each other. Here it is more natural to connect the simple zeros of \( y^2 \) by a cut rather than running the cuts to infinity. This amounts to performing a modular transformation on the Riemann surface. In this configuration the period around this cycle appears to measure the number of D-branes there.

This gives us a clear picture of why the phase transition at \( x = 0 \) is smoothed out and how the flux (period around the \( b \)-cycle) is continuously connected to D-branes (period around the \( a \) cycle).
Finally, we would like to suggest that the contour integral (3.20) of the force around the \( b \) cycle can be interpreted as an imaginary energy difference between the two Fermi surfaces. In other words, \( l = iq \) is the difference in fermi energies between the two sides. This puts the parameters \( \hat{q} \) and \( \mu \) on somewhat similar footing and might suggest a deeper interpretation of these phenomena.

### 3.2. The relation between FZZT and ZZ branes

In this subsection, we summarize and slightly extend the discussion in [2,40]. As we have argued above, the operator (3.1) should be interpreted as the insertion of an FZZT brane. It was observed in the CFT analysis that one can analytically continue the formulae for FZZT branes. In other words, the branes are labelled by a parameter \( s \), and two different values of \( s \) could give the same boundary cosmological constant \( \mu_B \). We will interpret this as the two sheets of the Riemann surface, in other words we can define the operators \( W(P_+(z)) \) and \( W(P_-(z)) \). It is of interest to find whether there are points \( z_0 \) away from the cuts where \( F_{\text{eff}}(z_0) = 0 \). A probe eigenvalue located at these points will remain stationary, even though it could be unstable. At these points \( y = 0 \) and \( R(P_+(z_0)) = R(P_-(z_0)) \). In fact we suggest that the operator

\[
Z = W(P_+(z_0)) - W(P_-(z_0))
\]

creates a ZZ brane at \( z_0 \). In the classical limit of string theory, the ZZ brane is infinitely heavy and it makes sense to think of it as an operator, a deformation of the theory. Similarly, the above analysis in terms of Riemann surfaces is also valid in the classical limit. Note that since \( y \) is zero at \( z_0 \) we can think of \( z_0 \) as the position of an infinitesimally short branch cut. Then (3.21) corresponds to integrating \( R(z) \) through the cycle that goes through this infinitesimal cut and the main cut. In fact, the formula (3.21) was inspired by the formulae in [2] which express the boundary state of the ZZ brane as a difference between boundary states of FZZT branes.

Let us consider, for example, the bosonic string. In that case the FZZT brane parameter \( s \) is given by

\[
\cosh \pi bs = \mu_B / \sqrt{\tilde{\mu}}
\]

where \( \tilde{\mu} \) is, up to an unimportant factor, the bulk cosmological constant, and \( \mu_B \) is the boundary cosmological constant. Then the ZZ brane boundary state can be written as\( ^6 \)

\[
|D\rangle_{ZZ} = |D(s_+)\rangle_{FZZT} - |D(s_+)\rangle_{FZZT}, \quad s_\pm = i(\frac{1}{b} \pm b)
\]

\( ^6 \) We concentrate on the \((1,1)\) ZZ brane.
Note that $\mu_B(s_+) = \mu_B(s_-)$. One can compute the $\mu$ and $\mu_B$ dependence of the FZZT branes for the bosonic string with $c = 0$. We do this by computing the disk one point function for the insertion of a bulk cosmological constant, which equals the derivative of the disk with respect to the bulk cosmological constant: $\partial_\mu Z_{FZZT} = \langle V_b \rangle$. We compute this using the formulas in [37,38]. Using (3.22) we express the answer in terms of the boundary cosmological constant to find that

$$\partial_{\mu_B} D \sim (2\mu_B - \sqrt{\mu})\sqrt{\mu_B + \sqrt{\mu}}$$

which is the expected form of the singular part of the resolvent for the matrix model corresponding to bosonic $c = 0$. This also gives the force on an eigenvalue. We see that the force vanishes if $\mu_B^0 = \sqrt{\mu}/2$. One can check that this is indeed the value of $\mu_B$ that appears for the FZZT in (3.23). Furthermore, as $s$ varies continuously between $s_+$ and $s_-$, the value of $\mu_B$ goes from $\mu_B^0$ through the cut that starts at $\mu_B = -\sqrt{\mu}$ and back to $\mu_B^0$.

Now let us consider the superstring theory. There are two types of FZZT branes distinguished by the sign in the supercharge boundary condition. We associate the operator (3.1) to the FZZT brane of 0B theory with $\eta = -1$ (in the notations of [17]) with $\mu_B = iz$. For $\hat{c} = 0$ and $\mu > 0$, one finds that the disk expectation value for the $\eta = -1$ brane obeys [41]

$$\partial_\mu D_- \sim \sqrt{\mu} + \mu_B^2.$$  \hfill (3.25)

This implies

$$\partial_{\mu_B} D_- \sim \mu_B\sqrt{\mu} + \mu_B^2$$

which is the expected answer; i.e. it is the same as the resolvent of the two cut model. On the other hand for the $\eta = +1$ FZZT brane we find

$$D_+ \sim \mu_B\bar{\mu}$$

which is analytic in $\mu_B$.

We expect that at negative $\mu$ the expectation value of the $\eta = -1$ brane will be given by (3.27). This is as expected of the resolvent of the two cut model for negative $\mu$. 

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4. The $m = 2$ case – pure supergravity

In this section we consider in detail the simplest even critical point of the unitary matrix model, which corresponds to $m = 2$ or $k = 1$. As discussed above, we remove $R_1$ away by a shift of $\omega$, and find the following equations:

$$R_2 + \frac{1}{2} x R_0 = 0, \quad r \Theta_2 = 0. \quad (4.1)$$

The equation $r \Theta_2 = -H'_2 = 0$ is a total derivative, which is integrated to $H_2 = -q$. Throughout most of the paper we choose the integration constant $q$ to be real. Since $H_2 = -r^2 \omega$, $q$ has a simple interpretation as the boost eigenvalue (the “rapidity”). Solving for $\omega$ and inserting it into (4.1), we find

$$r'' - \frac{1}{2} r^3 + \frac{1}{2} x r + \frac{q^2}{r^3} = 0. \quad (4.2)$$

It follows from the Lagrangian

$$\frac{1}{2} (r')^2 + \frac{1}{8} r^4 - \frac{1}{4} x r^2 + \frac{q^2}{2r^2}. \quad (4.3)$$

In [30] the two-cut hermitian matrix model was studied using different conventions. In (2.9) $g$ was taken to be imaginary; therefore, $\beta = i \theta$ where $\theta$ is a conventionally defined angle with periodicity $2 \pi$, see (2.9). Therefore,

$$r'' - \frac{r^3}{2} + \frac{x r}{2} - r(\theta')^2 = 0. \quad (4.4)$$

This equation of motion follows from the Lagrangian

$$\frac{(r')^2}{2} + \frac{r^2(\theta')^2}{2} + \frac{r^4}{8} - \frac{x r^2}{4}. \quad (4.5)$$

In Euclidean space the standard way to impose the constraint on the angular momentum is to add $iq \theta'$ to the Lagrangian (the factor $i$ is due to the presence of a single derivative). This method reproduces (4.2).

Instead, in [30] the “angular momentum” $r^2 \theta'$ was taken to be real. This corresponds to imaginary $q$ in (4.2): $q = i l / 2$, so that

$$r'' - \frac{1}{2} r^3 + \frac{1}{2} x r - \frac{l^2}{4r^3} = 0. \quad (4.6)$$
The real parameter $l$ corresponds to shifting the left “Fermi level” relative to the right one, i.e. to having different number of eigenvalues in the two wells. This can be seen by computing the eigenvalue distribution from the leading order solution (neglecting the $f''$ term). Note that the equation (4.6) comes from the (Euclidean) Lagrangian (4.3). We see that with $q = il/2$ this action is not bounded below. This implies that we cannot find well behaved non-perturbative solutions. More precisely, if we look at solutions of $V'(r) = 0$, we see that, for very large $x$, $r \sim \sqrt{x}$. As we decrease $x$ we find that at a critical value of $x$ the solution becomes complex. We can focus on this critical region by taking $l \to \infty$ limit and defining the scaled variables

$$r = l^{1/3} + l^{-1/15}u, \quad x = \frac{3}{2}l^{2/3} + l^{-2/15}y.$$  \hspace{0.5cm} (4.7)

Inserting this into (4.6), taking the $l \to \infty$ limit, and rescaling $y$ and $u \to \tilde{u}$ by numerical factors, we find the Painlevé I equation, which is well-known to describe the double-scaling limit of a one-cut Hermitian matrix model [42-44]:

$$\frac{1}{3} \tilde{u}'' - \tilde{u}^2 + y = 0, \quad \text{and} \quad \partial_y^2 F = \frac{\tilde{u}}{2}.$$  \hspace{0.5cm} (4.8)

Thus, the large $l$ limit corresponds to removing the eigenvalues from one side of the potential, and filling the other side near the top where we recover the single cut critical behavior.

On the other hand, for real $q$ the equation (4.2) has a smooth solution. We analyze these solutions below.

4.1. Solving the “$q$-deformed” equation

Let us examine the solutions of (4.2) in more detail. For large positive $x$ we find

$$r(x) = x^{1/2} + (4q^2 - 1) \left( \frac{1}{4x^{5/2}} - \frac{-73 + 36q^2}{32x^{11/2}} + \frac{10657 - 7048 q^2 + 1040 q^4}{128 x^{17/2}} \right)$$

$$- \frac{-13912277 + 10303996 q^2 - 2156528 q^4 + 144704 q^6}{2048x^{23/2}} + O(x^{-29/2}) \right) \right),$$

$$u(x) = r^2(x)/4 = \frac{x}{4} + \left( q^2 - \frac{1}{4} \right) \left[ \frac{1}{2x^2} + \left( q^2 - \frac{9}{4} \right) \left( -\frac{2}{x^5} + \frac{14 \left( q^2 - \frac{21}{4} \right)}{x^8} \right) \right]$$

$$- \frac{5 \left( -29 + 4q^2 \right) \left( -83 + 12q^2 \right)}{2x^{11}} + O \left( \frac{1}{x^{14}} \right), \hspace{0.5cm} (4.9)$$

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and for large negative $x$,

$$
\begin{align*}
    r(x) &= \frac{2^{1/4} \sqrt{|q|}}{|x|^{1/4}} \left[ 1 - \frac{|q| \sqrt{2}}{4|x|^{3/2}} + \frac{5 + 18q^2}{32|x|^3} - \frac{|q| (107 + 110q^2) \sqrt{2}}{128|x|^{9/2}} \\
    & \quad + \frac{2285 + 13572q^2 + 6188q^4}{2048|x|^6} + O(|x|^{-15/2}) \right] \\
    u(x) &= r^2(x)/4 = \frac{|q| \sqrt{2}}{4|x|^{1/2}} - \frac{q^2}{4|x|^2} + \frac{5 |q| (1 + 4q^2) \sqrt{2}}{64|x|^{7/2}} - \frac{q^2 (7 + 8q^2)}{8|x|^5} \\
    & \quad + \frac{11 |q| (105 + 664q^2 + 336q^4) \sqrt{2}}{2048|x|^{13/2}} + O(|x|^{-8}).
\end{align*}
$$

(4.10)

The fact that there are terms in the free energy non-analytic in $q$ at $q = 0$ suggests that in the dual string theory there is no R-R vertex operator that corresponds to turning on $q$ continuously: had there been a standard vertex operator which couples to $q$, its $n$-point functions at $q = 0$ would have been non-singular. This suggests that $q$ is quantized.

It is possible to argue that the asymptotic expansion (4.10) matches onto the expansion (4.9) as follows. The differential equation (4.2) comes from the action

$$
S \sim \int dx \left[ \frac{1}{2} r'^2 + \frac{1}{8} (r'^2 - x)^2 + \frac{1}{2} q^2 \right] = \int dx \left[ \frac{1}{2} r'^2 + V(r^2) \right].
$$

(4.11)

This action is bounded below. We can find an approximate variational solution by neglecting the derivative term and minimizing the potential for each $x$ independently. This gives a continuous function of $x$. For $q > 0$ the function is smooth; for $q = 0$ it has a discontinuous first derivative at $x = 0$. Including the second derivative term will lead to some changes, especially near $x \sim 0$, but a solution will exist since it is clear that the action has a minimum.

Since we started with a well-defined and convergent integral (2.1), we should end up with a finite and real answer for the free energy $F$. Therefore, it is natural to expect that the differential equation has a unique real and smooth solution. Indeed, the argument above shows this. It is important that this is the case both for zero and for nonzero $q$. Note that, in order to select the appropriate solution, it is important to set boundary conditions both at $x \to \pm \infty$.

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7 It is amusing to also present two exact solutions of our equations which do not satisfy our boundary conditions. For $q = \pm \frac{1}{2}$ the problem is solved with $r = \sqrt{x}$, $u = x/4$ and for $q = \pm \frac{3}{2}$ it is solved with $r = \sqrt{x + \frac{1}{2x^2}}$, $u = \frac{x}{4} + \frac{1}{x^2}$. 

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In the worldsheet interpretation of these theories we associate the term of order $|x|^{1-3(h+b/2)}$ in $u$ with worldsheets having $h$ handles and $b$ boundaries. The $q$ dependence in the expressions (4.10)(4.9) is consistent with this scaling and with a factor of $q^b$ arising from a surface with $b$ boundaries. This explains why the coefficient of $x^{1-3n/2}$ is a polynomial in $q$ of degree $n$.

Note also that in terms of the original rank of the matrix, $N$, before taking the double scaling limit for positive $x$ we have an expansion with only even powers of $1/N$ while for negative $x$ we have an expansion with both odd and even powers.

4.2. A large $q$ limit

An interesting limit to consider is $q \to \infty$, $x \to \pm \infty$ with $t = q^{-2/3}x$ fixed. This is the 't Hooft limit with $t^{-3/2}$ being the 't Hooft coupling. After we define $s = q^{-1/3}r$, equation (4.2) becomes

$$\frac{2}{q^2} s^3 \partial_t^2 s - s^6 + ts^4 + 2 = 0. \tag{4.12}$$

In the large $q$ limit the first term is negligible, and we end up with a cubic equation for $v(t) = s^2$

$$v^3 - tv^2 = 2. \tag{4.13}$$

The solution of this equation leads to a free energy $F = q^2 f(t)$ where $v(t)/4 = \partial_t^2 f(t)$. This is exactly the expected behavior in the large $q$ limit when thought of as a large $N$ limit.

For generic $t$ (4.13) has three solutions. Only one of them is real for all $t$ (this is easy to see for $t \approx 0$)

$$v(t) = \frac{1}{3} \left[ t + \left( t^3 + 27 - 3\sqrt{81 + 6t^3} \right)^{\frac{1}{3}} + \left( t^3 + 27 + 3\sqrt{81 + 6t^3} \right)^{\frac{1}{3}} \right] \tag{4.14}$$

(here the branches of the two cube roots should be handled with care). For $t > -\frac{3}{\sqrt{21/3}}$ the arguments of the square roots are positive and $v(t)$ is real. It is clear from the form of (4.14) that the half integer powers of $t$ cancel when we do the expansion for $t \to \infty$. For $t < -\frac{3}{\sqrt{21/3}}$ the arguments of the square roots are negative and the second and third terms in (4.14) are complex, but $v(t)$ is real. As we move in $t$ and the argument of the cubic root moves in the complex plane and it is important to keep track of the branch.

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8 Interpretation of this matrix model in terms of world sheets with boundaries was proposed already in [33].

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of the cubic root. When we expand for very large $t \to -\infty$ we are are then saying that $(t^3 + \cdots)^{1/3} = e^{\pm i 2\pi/3 t} + \cdots$ for the second or third term in (4.14). This implies that there is a term where the two square roots add, which leads to the half integer powers of $t$. The expansions of $v(t)$ for large negative and large positive $t$

$$v(t) = \begin{cases} \frac{\sqrt{2}}{|t|^{3/2}} - \frac{1}{|t|^2} + \frac{5\sqrt{2}}{4|t|^{5/2}} - \frac{4}{|t|^6} + \frac{231\sqrt{2}}{32|t|^8} + O\left(\frac{1}{t^8}\right), & t < 0, \\ \frac{2}{t^2} - \frac{8}{t^6} + \frac{56}{t^8} - \frac{480}{t^{10}} + O\left(\frac{1}{t^{14}}\right), & t > 0, \end{cases}$$

(4.15)

reproduce the highest powers of $q$ in the expansions $[4.10]$ $[4.9]$ after remembering that $u = v/4$. It is interesting that one smooth function (4.14) captures the limiting behavior of large $q$.

Equation (4.13) is obtained from (4.12) by neglecting the derivative term. In the picture of a particle with coordinate $s$ (or $r$) moving in time $t$ (or $x$) we presented above, this is a limit where we neglect the acceleration term and keep only the potential term. The particle is forced to stay at the stationary points of the potential. This suggests that we can take (4.14), or more precisely $s(t) = \sqrt{v(t)}$, as the starting point of a systematic expansion of the solution of (4.12) in powers of $1/q^2$, even though usually such an expansion with the highest derivative term is singular.

We interpret this limit as an ’t Hooft limit where only spherical topologies, perhaps with boundaries, survive. For $q = 0$ the theory exhibits the Gross-Witten transition [24] at $x = 0$. Namely $F'' = x/4$ for $x > 0$ and $F'' = 0$ for $x < 0$. It is known that this transition can be smoothed by the genus expansion with $q = 0$. Now we see that it can also be smoothed by the expansion in the ’t Hooft parameter for infinite $q$.

An interpretation of this result is the following. For negative $t$ we have D-branes and the power of $|t|^{-3/2}$ is the number of boundaries in the spherical worldsheet. For positive $t$ the D-branes are replaced by flux; therefore, we have spherical worldsheets with insertion of RR fields. Each RR field comes with a power of $t^{-3/2}$ but their number must be even, and hence the expansion in powers of $t^{-3}$. This is the RR field discussed in section 7. Note that the power of $t$ agrees with the KPZ scaling of this operator. It is extremely interesting that the theory exhibits, both for finite $q$ and infinite $q$, a smooth transition between the two domains of positive and negative $x$. This is reminiscent of geometric transitions [47-49]. The fact that the theory is smooth at $t = 0$, and the strong coupling singularity is smoothed by the RR flux, may have implications for QCD-like theories which arise from the conifold with D-branes [48-49].
5. The $m = 4$ theory

Here we briefly discuss the $m = 4$ theory which has some new phenomena compared to the $m = 2$ case. In particular, for $x < 0$ we find that the free energy is discontinuous with respect to turning on the parameter $q$. This happens due to the fact that for $q = 0$ and negative $x$ there exist three different solutions: the trivial one where $r = 0$, and two non-trivial $Z_2$ symmetry breaking solutions with the sphere free energy scaling as $|x|^{5/2}$ (they are related by the $Z_2$ transformation $\omega \to -\omega$). For $q = 0$ the trivial symmetric solution matches to the positive $x$ solution. For non-vanishing $q$ the positive $x$ solution matches one of the non-trivial symmetry breaking solutions. It is clear that the structure of solutions gets even richer with increasing $m$.

The basic equations for $m = 4$ are

$$R_4 - \frac{3xr(x)}{8} = 0, \quad H_4 + q = 0 \quad (5.1)$$

where $R_4$, $H_4$ are given in appendix C. For $x > 0$ and $q = 0$ there is a solution with $\omega = 0$, and the free energy is given in (2.4). Deformation of this solution by $q$ is straightforward, and we find

$$r(x) = x^{1/4} + \frac{2}{3} \left( \frac{4q^2}{3} - \frac{5}{16} \right) x^{-9/4} + \frac{1}{3} \left( \frac{297}{128} + \frac{125q^2}{9} - \frac{608q^4}{81} \right) x^{-19/4} + O(x^{-29/4})$$

$$\omega(x) = -\frac{2q}{3x} + \frac{2q}{3} \left( \frac{80q^2}{27} - \frac{5}{4} \right) x^{-7/2} + O(x^{-6})$$

$$\frac{d^2F}{dx^2} = \frac{r^2(x)}{4} = \sqrt{x} + \frac{64q^2 - 15}{144} x^{-2} + \left( -\frac{1757}{2304} + \frac{245q^2}{54} - \frac{560q^4}{243} \right) x^{-9/2} + O(x^{-7}) \quad (5.2)$$

The free energy contains even powers of $q$ only, so we identify turning on $q$ in the unitary matrix model with turning on R-R flux $\sim q$ in the dual string theory.

Now let us consider $x < 0$. If $q = 0$ then there is an obvious trivial solution where $r(x) = 0$. We have not been able to find a real deformation of this solution produced by $q$. For such a deformation one expects $\omega$ to behave as $A|x|^{1/4}$, while $r \sim |x|^{-3/8}$. However, (5.1) require that $A^4 < 0$; therefore, the solution is complex.

There are two other, less obvious, solutions where both $r(x)$ and $\omega(x)$ are non-vanishing. One of these solutions has the following asymptotic expansion for $x \to -\infty$:

$$r(x) = (2|x|/7)^{1/4} - 2^{3/4} \cdot 7^{1/4} \cdot \frac{5}{48|x|^{9/4}} - (7/2)^{3/4} \cdot \frac{319}{256|x|^{19/4}} + O(|x|^{-29/4})$$

$$\omega(x) = -\frac{\sqrt{3}}{2} (2|x|/7)^{1/4} - \frac{5}{32\sqrt{3}|x|^{9/4}} - (7/2)^{3/4} \cdot \frac{1111}{1024\sqrt{3}|x|^{19/4}} + O(|x|^{-29/4}) \quad (5.3)$$

$$\frac{r^2(x)}{4} = \frac{|x|^{1/2}}{2\sqrt{14}} - \frac{5}{48|x|^2} - (7/2)^{1/2} \cdot \frac{2821}{4608|x|^{9/2}} + O(|x|^{-7}),$$

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and the other is related to it by $\omega \rightarrow -\omega$. Since these new solutions break the $Z_2$ symmetry spontaneously, in the string interpretation they must involve background RR fields. This can be shown explicitly by computing a nonzero one point function of these RR fields using the matrix model. The RR fields which condense in the broken symmetry background are present in the 0B theory but not in the 0A theory.\footnote{In the next section we will discuss a description of the 0A theory by a complex matrix model, and we will not find any new solutions exhibiting the symmetry breaking phenomenon.} We also trace the fact that the $m = 2$ unitary matrix model does not exhibit symmetry breaking to the absence of continuous R-R parameters in this model. As $m$ increases, so does the number of R-R fields that can condense and produce different solutions.

Both solutions described above possess real extensions to non-vanishing $q$. Thus, we can expand in small $q$ around the broken symmetry solutions, but not around the symmetric solution. This means that it is impossible to vary $q$ in a continuous fashion around the symmetric solution. A non-analyticity in $q$ was already observed for $m = 2$, but for $m = 4$ it becomes a more dramatic phenomenon related to the symmetry breaking. Just as for $m = 2$, in the full nonperturbative $m = 4$ theory there is no vertex operator which changes $q$ in the symmetric background. This suggests that $q$ is quantized.

The requirement of matching an $x \rightarrow -\infty$ expansion to the solution (5.2) for large positive $x$ appears to pick one of the two broken symmetry solutions for $q > 0$, and the other for $q < 0$. The asymptotic expansion of the $q > 0$ solution for $x \rightarrow -\infty$ is

$$r(x) = \frac{(2|x|/7)^{1/4} + q}{\sqrt{3}|x|} - 2^{3/4} \cdot 7^{1/4} \cdot \frac{15 - 52q^2}{144|x|^{9/4}} + q\sqrt{42} \frac{165 - 584q^2}{864|x|^{7/2}} + O(|x|^{-19/4})$$

$$\omega(x) = -\frac{\sqrt{3}}{2} \frac{(2|x|/7)^{1/4} + 2q}{3|x|} - \frac{5(7/2)^{1/4} + 1 + 32q^2/3}{32\sqrt{3}} \frac{1}{|x|^{9/4}} + 25q\sqrt{14} \frac{27 - 32q^2}{1296|x|^{7/2}} + O(|x|^{-19/4})$$

$$\frac{y^2(x)}{4} = \frac{|x|^{1/2}}{2\sqrt{14}} + \frac{q}{\sqrt{3} \cdot 7^{1/4} \cdot (2|x|)^{3/4} + \frac{64q^2 - 15}{144|x|^2} - 5(7/2)^{1/4} \frac{32q^2 - 9}{96\sqrt{3}|x|^{13/4}} + O(|x|^{-18/4}) \quad (5.4)$$

Remarkably, the torus term in the free energy, $\frac{15 - 64q^2}{144} \ln |x|$, is the same as for $x > 0$ (see (5.2)). Perhaps this means that the type 0B string theory dual to the $m = 4$ model has a symmetry under a change of sign of $x$.

To obtain the solution for $q < 0$ we act on the above with the transformation $q \rightarrow -q$, $\omega \rightarrow -\omega$. The free energy has the same structure as in the $k = 2$ complex matrix model with positive $x$ that will be presented in the next section, although the coefficients are different. The key question is whether the expansion (5.4) for large negative $x$ matches...
onto the large positive $x$ expansion in (5.2). We believe that this is the case: a clear argument in favor of this can be given in the limit of large $q$.

To take such a limit, we assume $q > 0$ and define $\omega = q^{1/5}h$, $x = q^{4/5}t$, $\tau^2 = q^{2/5}v$. Then the derivative terms in (5.1) are suppressed, and we find

$$
\frac{3}{8}v^2 - 3vh^2 + h^4 - \frac{3}{8}t = 0, \\
vh(-\frac{3}{2}v + 2h^2) = 1.
$$

(5.5)

From the second equation we solve for $v$

$$
v_{\pm} = \frac{2}{3}h^2 \pm \sqrt{\left(\frac{2}{3}h^2\right)^2 - \frac{2}{3}h}.
$$

(5.6)

From the expansions (5.2), (5.4) we see that $h$ is negative. For $h < 0$ we can only choose the solution $v_+$.

Now we substitute this into the first equation in (5.5), move the square roots to one side of the equal sign, and then square the equation to remove the square root. After all this we find the equation

$$
-12 - 864h^5 + 448h^{10} - 36ht - 96h^6t - 27h^2t^2 = 0.
$$

(5.7)

In order to analyze it, we define the variable

$$
y = ht.
$$

(5.8)

Then the equation becomes quadratic for $h^5$

$$
-12 - 864h^5 + 448h^{10} - 36y - 96h^5y - 27y^2 = 0.
$$

(5.9)

Once we find $h$ as a function of $y$ we can find $t = y/h$. The solutions of this equation are

$$
h_5^{\pm} = \frac{54 + 6y \pm 5\sqrt{3}\sqrt{28 + 3(y+2)^2}}{56}.
$$

(5.10)

The solution with the plus sign, $h_5^+$, is always positive and nonzero as a function of $y$. As we remarked above, we are not interested in solutions with positive $h$. The solution $h_-$ is non-positive, and it is zero only for

$$
y_0 = -2/3.
$$

(5.11)
As a function of $y$, $h_-$ has a maximum at this point.

As $y \to \pm \infty$ we find that $h_5^\pm \sim c_\pm |y|^{1/5}$, where $c_\pm$ is a numerical constant. This implies that $t$ diverges as $y^{4/5}$ and that $h \sim |t|^{1/4}$. This agrees with the expansion (5.4) which applies for negative $t$. If we are interested in a solution where $h$ decreases as $t \to \infty$ it is clear that we should look at the region near $y_0 = -2/3$. This forces us to choose the solution $h_5^-$. As we change $t$ continuously, we will stay on the $h_-$ branch of (5.10) since the two branches never cross. For $y$ close to $y_0$ we find that $h \sim y_0/t$, which agrees with the expansion (5.2) for large positive $t$.

It is clear that, as $t$ decreases from infinity, $y$ departs from $y_0$. Since $h_5^-$ is always negative, in order to have negative values of $t$ we need to have positive values of $y$. We see then that the relevant region is $y > y_0$. When $y \to y_0$ we have $t \to \infty$; when $y = 0$ we have $t = 0$, and when $y \to +\infty$ we have $t \to -\infty$. For large positive $y$, $h_5^- = -9y/56$, which implies

$$h_4^- = -\frac{9}{56}t, \quad v = \frac{4}{3}h_2^2 = \left(\frac{2}{t}|t|\right)^{1/2}.$$  \hspace{1cm} (5.12)

This agrees with the limiting form of the solution for $t \to -\infty$, (5.4). Expanding the solution of (5.10),(5.8) further for large negative $t$, we find

$$h_- = -\frac{\sqrt{3}}{2}\left(\frac{2|t|}{7}\right)^{1/4} + \frac{2}{3|t|} - \frac{5(7/2)^{1/4}}{3\sqrt{3}|t|^{9/4}} - \frac{50\sqrt{14}}{81|t|^{7/2}} + O(|t|^{-19/4}).$$  \hspace{1cm} (5.13)

For large positive $t$ we find the expansion

$$h_- = -\frac{2}{3t} + \frac{160}{81t^{7/2}} + O(t^{-6}).$$  \hspace{1cm} (5.14)

The coefficients in these expansions agree with the leading powers for large $q$ in the expansions of $\omega(x)$ in (5.4),(5.2). This shows that an appropriately chosen solution of (5.3) indeed interpolates between the expansion (5.2) and (5.4) in the limit of large $q$. This also suggests that there is a smooth interpolating solution for finite $q$. Presumably, this could be checked through numerical work.

6. Complex matrix models

The complex matrix models are based on an $(N + q) \times N$ complex matrix $M$ with partition function $Z$, free energy $F$ and potential $V$ \cite{31,33}

$$Z = e^{-F} = \int dMdM^\dagger e^{-\frac{N}{2} \text{Tr} V(MM^\dagger)},$$

$$V(MM^\dagger) = \sum_{j=1}^p g_j (MM^\dagger)^j.$$  \hspace{1cm} (6.1)
Using the $U(N + q) \times U(N)$ symmetry we can bring $M$ to the form $M_{ij} = \lambda_i \delta_{ij}$ with $\lambda_i \geq 0$. Then (6.1) becomes

$$Z = e^{-F} = \prod_{i=1}^{N} \int_{0}^{\infty} d\lambda_i \lambda_i^{1+2q} e^{-\frac{N}{\tau} V(\lambda_i^2) \Delta(\lambda^2)^2} \sim \prod_{i=1}^{N} \int_{0}^{\infty} dy_i y_i^{q} e^{-\frac{N}{\tau} V(y_i) \Delta(y)^2} ,$$

(6.2)

$\Delta(y) = \prod_{i>j} (y_i - y_j)$.

Consider first the large $N$ limit with a fixed $\gamma$ – not the double scaling limit. The second form of the integral in (6.2) in terms of $y$ is similar to the standard hermitian matrix model with two exceptions: the factor of $y_i^q$ in the measure which can be written as a contribution to the potential of the form $q \sum_i \ln y_i$, and the fact that the domain of $y_i$ is restricted to be positive. Let us now examine the first form of the integral in (6.2). We replace $\lambda_i^{1+2q}$ with $|\lambda_i|^{1+2q}$, and extend the range of the integral over each $\lambda_i$ to $(-\infty, +\infty)$ (this adds an inessential constant to $F$ which is independent of the parameters in $V$). Introducing the eigenvalue density $\rho(\lambda) d\lambda$ the effective potential is

$$V_{eff} = \int_{0}^{\infty} d\lambda \rho(\lambda) \left( V(\lambda^2) - \frac{2q+1}{N} \ln |\lambda| \right) = \int_{0}^{\infty} d\lambda d\lambda' \rho(\lambda) \rho(\lambda') \ln |\lambda^2 - \lambda'^2|$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} d\lambda \tilde{\rho}(\lambda) \left( V(\lambda^2) - \frac{2q+1}{N} \ln |\lambda| \right) = \frac{1}{2} \int_{-\infty}^{\infty} d\lambda d\lambda' \tilde{\rho}(\lambda) \tilde{\rho}(\lambda') \ln |\lambda - \lambda'|$$

(6.3)

where we have neglected terms of order $\frac{1}{N}$, we used the fact that $V$ is invariant under $\lambda \to -\lambda$ and we defined $\tilde{\rho} = \rho(\lambda) + \rho(-\lambda)$ with $\rho(\lambda < 0) = 0$. In appendix E we derive the loop equation for this model and we analyze the solution.

For $q \ll N$ (6.3) is exactly like the standard hermitian matrix model with an even potential (up to an inessential over all factor of 2). We conclude that in this case the large $N$ limit of correlation functions are identical to those of even operators in the standard hermitian model. Furthermore, if $V(\lambda^2)$ and $N$ are such that the complex matrix model has one cut that does not reach $\lambda = 0$, then the corresponding hermitian model would have two cuts. If the cut in the complex matrix model reaches $\lambda = 0$ then the hermitian matrix model has just one cut. The important lesson is that the planar correlation functions are the same in the two models. This will be important for us since we will interpret these two models as the 0A/0B version of certain string backgrounds.
For $q \sim N$ (6.3) is like the standard hermitian matrix model with an additive logarithmic potential. This is the Penner model. We will not explore it further but will assume that $q$ is finite when $N \to \infty$.

Consider now the $\frac{1}{N}$ corrections and the double scaling limit. We should distinguish three classes of critical behavior

1. A cut of eigenvalues ends at generic $\lambda$. This critical behavior is described by the standard KdV hierarchy of the single cut hermitian matrix model.

2. Ends of two different cuts meet at generic $\lambda$. This critical behavior is described by the mKdV hierarchy because it is the same as in the two cut hermitian matrix model or as in the unitary matrix model.

3. A cut ends at $\lambda \approx 0$. In the leading order in $\frac{1}{N}$ (with fixed finite $q$) this critical behavior is the same as in two cut models. However, because of the measure factor $|\lambda_i|^{1+2q}$, the $\frac{1}{N}$ corrections are different than in those models.

Since only the third kind of behavior is new, let us focus on it. The string equation is a differential equation for

$$u = 2\partial_z^2 F(z) \tag{6.4}$$

where $z$ is a scaled version of $N$. The string equation in this case \[31-34\] is

$$uR^2[u] - \frac{1}{2}R[u]\partial_z^2 R[u] + \frac{1}{4}(\partial_z R[u])^2 = q^2$$

$$R[u] = \sum_{l=1}^{\infty} (l + \frac{1}{2})t_lQ_l[u] - z \tag{6.5}$$

where $Q_l[u]$ are the Gel’fand-Dikii differential polynomials

$$Q_0[u] = \frac{1}{2}$$

$$Q_1[u] = -\frac{1}{4}u$$

$$Q_2[u] = -\frac{1}{16}(\partial_z^2 u - 3u^2)$$

$$Q_3[u] = -\frac{1}{64}(\partial_z^4 u - 5(\partial_z u)^2 - 10u\partial_z^2 u + 10u^3) \tag{6.6}$$

and $R[u]$ is the string equation of the hermitian matrix model with couplings $t_l$. Usually the sum over $l$ in $R$ is finite going up to $k$. Then we normalize the highest $t_l$ such that the $R = u^k + \ldots$.

\[10\] For the simplest case, we provide a derivation of this equation in Appendix D.
For \( q = 0 \) equation (6.3) is satisfied by a solution of the differential equation \( \mathcal{R} = 0 \). This is exactly the equation of the standard hermitian one-matrix model in its one cut phase. The fact that we find this equation can be easily understood by examining the expression for the integral (6.2) in terms of \( y \). For \( q = 0 \) this is exactly the same integral as in the one-matrix model, except that the integrals over the eigenvalues are only over positive \( y \). As long as the support of the eigenvalue distribution is away from \( y = 0 \), the effect of the restricted range of integration is nonperturbative. Indeed, for \( z > 0 \), it is possible to check that to all orders in the \( 1/z \) expansion (6.3) leads to \( \mathcal{R} = 0 \). Clearly, this is not the case for \( z \) negative when the effect of the restricted range of integration is important. In fact, the nonperturbative corrections to the expansion in \( 1/z \) for \( z \) positive are also sensitive to the bounded range of \( y \). Therefore, these nonperturbative effects are captured by (6.3) but not by \( \mathcal{R} = 0 \).

6.1. The \( k = 1 \) case

For a general \( q \), the string equation of the first nontrivial case \( k = 1 \) is

\[
\begin{align*}
\mathcal{R}^2[u] &- \frac{1}{2} \mathcal{R}[u] \partial^2_z \mathcal{R}[u] + \frac{1}{4} (\partial_z \mathcal{R}[u])^2 - q^2 = 0 \\
\mathcal{R}[u] & = u - z.
\end{align*}
\]

(6.7)

Using the substitution of (31),

\[
u(z) = f(z)^2 + z
\]

(6.8)

the string equation becomes

\[
\partial^2_z f - f^3 - zf + \frac{q^2}{f^3} = 0.
\]

(6.9)

After rescaling \( f = 2^{-1/6}r \) and \( z = 2^{-1/3}x \) it becomes the equation we found in the two-cut matrix model, (4.2), but with the sign of \( x \) reversed [31]. Note also that the equations (6.4), (6.8) for the free energy become, upon the rescaling,

\[
\frac{\partial^2 F}{\partial x^2} = \frac{1}{4} r^2(x),
\]

(6.10)

up to a non-universal term \( \sim x \). This is consistent with the normalization of the free energy in (2.6).
We conclude that the complex matrix model with positive \( z \) is the same as the two cut theory with negative \( x \) and vice versa, i.e. \( z = -x \). Therefore, asymptotic expansions of the solution (4.9), (4.10) apply to the complex matrix model as well. The complex matrix model with parameter \( q \) describes, for positive \( x \), type 0A string theory in the background of \( q \) D0-branes. This agrees with the fact that, in (4.10), the maximum power of \( q \) in each term in the free energy corresponds to the number of boundaries in a spherical worldsheet. Our interpretation of the complex matrix model at negative \( x \), described by the expansion (4.9), is in terms of closed string in presence of R-R flux.

6.2. The \( k = 2 \) case

In the \( k = 2 \) case some new phenomena appear compared with \( k = 1 \). First, let us consider \( q = 0 \). For \( z > 0 \), the \( k = 1 \) complex matrix model is topological, but the \( k = 2 \) complex matrix model has a non-trivial genus expansion. Indeed, for \( k = 2 \), the equation is

\[
u R^2 - R R''/2 + (R')^2/4 = 0, \quad R = u^2 - u''/3 - z. \tag{6.11}
\]

Clearly, it is solved by any solution of \( R = 0 \), which for \( k = 2 \) is the Painlevé I equation. Around \( z \to \infty \) the solution of this equation has the asymptotic expansion

\[
u(z) = \sqrt{z} - \frac{1}{24 z^2} - \frac{49}{1152 z^{9/2}} + O(z^{-7}), \tag{6.12}
\]

but it cannot be continued in a smooth way to negative \( z \).

Luckily, there are solutions of (6.11) which do not satisfy \( R = 0 \). Neglecting the derivatives of \( u \) we see that we can solve (6.11) for negative \( z \) without satisfying \( R = 0 \) by \( u = 0 \). Expanding around it we find

\[
u(z) = -\frac{1}{4 z^2} - \frac{225}{32 z^7} - \frac{6906075}{1024 z^{12}} + O(z^{-17}). \tag{6.13}
\]

The authors of [50] showed numerically that the two asymptotic expansions (6.12) and (6.13) are connected through a real and smooth solution of (6.11).

Generalizing (6.11) to include \( q \), we find (6.14). Now the asymptotic expansions of the solution are, for positive \( z \),

\[
u(z) = \sqrt{z} + \frac{|q|}{2 z^2} - \frac{1 + 6 q^2}{24 z^2} + O(z^{-13/4}), \tag{6.14}
\]

\[\text{11} \quad \text{The boundary conditions are } u \to z (1/m) (0) \text{ for } z \to +\infty (0). \text{ We also demand that } R \text{ is positive and goes to zero as } z \to +\infty. \text{ See appendix D for further explanation.}\]
and for negative $x$,

$$u(z) = (q^2 - \frac{1}{4}) \left( \frac{1}{z^2} + \frac{2(q^2 - \frac{9}{4})(q^2 - \frac{25}{4})}{z^7} + O(z^{-12}) \right). \quad (6.15)$$

The $q \to 0$ limit of this solution is (6.13). Thus, in the $k = 2$ complex matrix model we do not find a discontinuity at $q = 0$ that is present in the $m = 4$ unitary matrix model even though the two models have the same critical exponents. The discontinuity of the unitary matrix model is related to the $Z_2$ symmetry breaking. From the string point of view, the RR fields that condense in the $0B$ string are absent from the spectrum of the $0A$ string dual to the complex matrix model.

We identify this theory with the $0A$ version of the $(2, 8)$ superminimal model for which the KPZ scaling of the $h$ handle and $b$ boundary surface is $z^{-5(h+b/2-1)/2-2}$. The expressions above are consistent with this identification, with the power of $q$ being the number of boundaries or insertions of the RR ground state operator.

As for $k = 1$, on one side we have boundaries – D-branes; on the other side we have no boundaries but only fluxes, hence only even powers of $q$ appear.

7. String theory in one dimension

Here we consider the type 0 theory in one target space dimension. This theory is also known as pure supergravity or $\hat{c} = 0$ noncritical string theory. Using standard Liouville conventions we have $Q = \sqrt{\frac{3}{\sqrt{2}}}$ and $\gamma = \frac{1}{\sqrt{2}}$. Therefore, the contribution of worldsheets with $h$ handles and $b$ boundaries scale like $\mu^{3(2-2h-b)/2}$.

The theory has a single closed-string NS operator, the worldsheet cosmological constant. In the $(-1, -1)$ picture it is $e^{\gamma \phi}$. For $\mu = 0$, using free field theory, we find two candidate R-R operators. In the $(-1/2, -1/2)$ picture they are given by $V_\pm = \sigma^\pm e^{Q \phi/2}$, where $\sigma^\pm$ are the two spin fields of the super-Liouville theory. The worldsheet $(-1)^F$ projection in the $0A$ theory leaves only $V_+$, and in $0B$ it leaves only $V_-$. For nonzero $\mu$ their wave functions are determined by solving the minisuperspace Schrodinger equations, as in [17]. Because of the behavior at $\phi \to \infty$, only one of the two operators is acceptable. Our conventions are such that for $\mu < 0$ only $V_+$ is acceptable, and for $\mu > 0$ only $V_-$ is

$^{12}$ It is amusing to point out three simple, but unphysical solutions of (1.5). For $q = 1/2$ it is solved by $u = 0$, for $q = 3/2$ it is solved by $u = 2/z^2$ and for $q = 5/2$ by $u = 6/z^2$. More generally, it is easy to show that the solution can have double poles with residues 2 or 6.
acceptable. This means that in the 0A theory there is one R-R operator $V_+$ for $\mu < 0$ and no R-R operator for $\mu > 0$. In the 0B theory the situation is reversed: there is $V_-$ for $\mu > 0$ and no R-R operator for $\mu < 0$.

More generally, reversing the sign of $\mu$ can be undone by transforming the worldsheet Liouville fermions as $\psi \rightarrow -\psi$ with $\overline{\psi} \rightarrow \overline{\psi}$. Since these are the only fermions in the problem, this has the effect of changing the sign of the worldsheet $(-1)^F$ in the R-R sector. Therefore, it amounts to reversing the projection in this sector and interchanging 0A and 0B. We conclude that the 0A theory with $\mu$ is the same as the 0B theory with $-\mu$.

Unlike the situation in $\hat{c} = 1$ where both theories are invariant under $\mu \rightarrow -\mu$, here they are interchanged.

All this can be summarized by a simple "spacetime" picture. Recalling the situation in the $\hat{c} = 1$ theory, and reducing it to one dimension, we find only one field. In the 0B theory it is $C$; in the 0A theory it a component of a gauge field $A_t$ in what was originally the Euclidean time direction. Since 0A and 0B are related by $\mu \rightarrow -\mu$ we can focus, without loss of generality on 0B. The leading order term in the action for $C$ is

$$\int d\phi \frac{1}{2} e^{2T(\phi)} (\partial_\phi C)^2$$

(7.1)

where $T(\phi) = \mu e^{\gamma\phi}$ is the tachyon field. The equation of motion is solved by

$$C(\phi) = \int^{\phi} e^{-2T(\phi')} d\phi'.$$

(7.2)

For $\mu > 0$ this solution is well behaved as $\phi \rightarrow \infty$, and leads to a "fluctuating degree of freedom." Since it is linearly divergent as $\phi \rightarrow -\infty$, it is not a normalizable mode and corresponds to a physical vertex operator $V_-$. For $\mu < 0$ this solution is badly behaved for $\phi \rightarrow \infty$ and therefore should be discarded.

We can introduce background charge $q$ at infinity which leads to background field

$$C(\phi) = iq \int^{\phi} e^{-2T(\phi')} d\phi'.$$

(7.3)

---

13 This fact is familiar from the study of the Ising model. Changing the sign of the fermion mass (moving from one side of the transition to the other), the order and the disorder operators change roles. This means that we change the sign of the $(-1)^F$ projection in the R-R sector.

14 We put the word spacetime in quotation marks because we have only space parametrized by $\phi$.

15 There is also a RR one form $A_\phi$. 

---
This can be done by adding the surface term

$$iq \int d\phi^' \partial_{\phi^'} C(\phi^') = iq (C(\infty) - C(-\infty))$$

(7.4)

to (7.1). (We view the action (7.1) as a Euclidean action and therefore we put an \(i\) in front of the topological term \(\partial_{\phi} C\). This leads to imaginary classical solution for \(C(\phi)\) as is common in classical solutions of D-instantons.) For \(\mu > 0\), integrating out \(C(\phi)\) we find \(\frac{\mu^2}{2} \log(\Lambda/|\mu|)\) where \(\Lambda\) is a cutoff on \(\phi\).

For \(\mu < 0\), the solution of (7.2) is badly behaved. In this case it seems possible to add charged ZZ branes that will absorb the flux of \(C\) and somehow lead to a finite answer. These are charged \(D(-1)\) branes and their number should be precisely equal to the flux. Therefore, from the string theory point of view it seems that \(q\) should be quantized.

More generally, it is natural to assume that \(q\) arises from the charges which exist in the theory. These are ZZ branes, which are localized near \(\phi = \infty\). Therefore, \(q\) should be quantized. With integer \(q\) the surface term (7.4) does not change if \(C(\infty) \rightarrow C(\infty) + 2\pi\), as expected from the periodicity of \(C\).

\(q\) appears as a \(\theta\) term, but unlike similar terms, the theory is not periodic in \(q\) because the standard process of pair creation which can screen its charge does not take place because the D-branes are locked at infinity. These D-branes look like instantons. However, unlike with ordinary instantons, we do not sum over \(q\) but keep it fixed. There are two reasons for that. First, the standard argument which forces us to sum over \(q\) relies on creating an instanton anti-instanton pair and separating them. This argument does not apply here because the D-branes are forced to be at one point in our “spacetime.” Second, backgrounds with different values of \(q\) differ by infinite action.

8. R-R physical vertex operators

In this section we classify the R-R vertex operators, which distinguish the 0A theory from the 0B theory.

Consider a “matter” SCFT with central charge \(\hat{c}\) and an R-R primary operator with dimension \(\Delta > \frac{\hat{c}}{16}\). Before performing any GSO projection there are two such operators corresponding to the states \(|\pm\rangle_M\), where the sign denotes \((-1)^F\) with \(F\) the worldsheet fermion number. It is convenient to consider the sum and the difference of the zero modes
of the left and right moving matter supercharges, \( G_M = \frac{1}{\sqrt{2}}(G_{M,\text{Left}} + iG_{M,\text{Right}}) \) and \( \bar{G}_M = \frac{1}{\sqrt{2}}(G_{M,\text{Left}} - iG_{M,\text{Right}}) \). They act of these two states as

\[
G_M|\rangle = \sqrt{\Delta - \frac{\hat{c}}{16}}|\rangle \\
G_M|\rangle = 0 \\
\bar{G}_M|\rangle = \sqrt{\Delta - \frac{\hat{c}}{16}}|\rangle \\
\bar{G}_M|\rangle = 0.
\]

(8.1)

There can also be other representations in which \( G_M|\rangle_M = \bar{G}_M|\rangle_M = 0 \), but \( G_M|\rangle_M \) and \( \bar{G}_M|\rangle_M \) do not vanish. Such representations are not present in the superminimal models. But they do exist in more generic systems. For example, they exist in \( \hat{c} = 1 \) and in the flat space critical theory, where we can think of the “matter part” as the superconformal field theory of nine free superfields. We will focus on models where all the representations are of the type (8.1).

We couple this system to Liouville with central charge \( \hat{c}_L = 10 - \hat{c} \). Physical vertex operators in the R-R sector have ghosts. In the \((-\frac{1}{2}, -\frac{1}{2})\) picture the total dimension of the matter and Liouville is \( \frac{5}{8} \). Therefore, the dimension of the Liouville operator \( \Delta_L \) needed to dress the matter operators of dimension \( \Delta \) satisfies \( \Delta_L - \frac{\hat{c}_L}{16} = - (\Delta - \frac{\hat{c}}{16}) \). We denote the two R-R operators with these dimensions as \( |\pm\rangle_L \). (Since \( \Delta_L - \frac{\hat{c}_L}{16} < 0 \), these are operators and not normalizable states \([51]\); nevertheless, we will use the state notation.)

The action of the zero modes of the Liouville supercharges \( G_L \) and \( \bar{G}_L \) (which are again linear combinations of left and right moving supercharges) on them are

\[
G_L|\rangle_L = i\sqrt{\Delta - \frac{\hat{c}}{16}}|\rangle_L \\
G_L|\rangle_L = 0 \\
\bar{G}_L|\rangle_L = i\sqrt{\Delta - \frac{\hat{c}}{16}}|\rangle_L \\
\bar{G}_L|\rangle_L = 0.
\]

(8.2)

The total supercharges\(^{16}\) are \( G = G_M + G_L \) and \( \bar{G} = \bar{G}_M + \bar{G}_L \) of (8.1) and (8.2). In the 0A theory the candidate operators in the \((-\frac{1}{2}, -\frac{1}{2})\) picture have \((-1)^F = 1 \), and

\(^{16}\) Actually, in order to obey the proper anticommutation relations we need a cocycle. The proper expression is \( G = G_M \times 1_L + (-1)^{F_M} \times G_L \) where \((-1)^{F_M} \) is the matter fermion number. We have a similar expression for \( \bar{G} \). In order not to clutter the equations, we will suppress the cocycle.
therefore the allowed operators are $|+\rangle_M|+\rangle_L$ and $|−\rangle_M|−\rangle_L$. Imposing that they are annihilated by the total supercharges we find that there are no such physical states.

In the 0B theory the candidate operators in the $(-\frac{1}{2}, -\frac{1}{2})$ picture have $(-1)^F = -1$, and therefore the allowed operators are $|+\rangle_M|−\rangle_L$ and $|−\rangle_M|+\rangle_L$. Imposing that they are annihilated by $G = G_M + G_L$ and $\bar{G} = \bar{G}_M + \bar{G}_L$ of (8.1) and (8.2) we find one physical operator

$$|+\rangle_M|−\rangle_L + i|−\rangle_M|+\rangle_L.$$

(8.3)

As we mentioned above, in theories with $\hat{c} > 1$, there can be matter operators which satisfy (8.1) with the interchange of $+$ ↔ $−$. For such operators, the above conclusions about the spectra in 0A and 0B are interchanged.

Consider now the R-R ground states which exists in those SCFT which do not break supersymmetry. It has $\Delta = \frac{\hat{c}}{16}$. Here there is only one state $|+\rangle_M$ satisfying

$$G|+\rangle_M = \bar{G}|+\rangle_M = 0,$$

(8.4)

which we take to have even fermion number by convention. In some examples like the $\hat{c} = 1$ theory there are two such ground states $|\pm\rangle_M$ and the discussion below is modified appropriately.

In the free field description of Liouville there are two states with $\Delta_L = \frac{\hat{c}_L}{16}$ which satisfy

$$G_L|\pm\rangle_L = \bar{G}_L|\pm\rangle_L = 0.$$  

(8.5)

For nonzero $\mu$ only one of them is well behaved as $\phi \to \infty$ [17]. For $\mu < 0$ it is $|+\rangle_L$ and for $\mu > 0$ it is $|−\rangle_L$. Imposing the GSO projection we find the physical operators. The 0A theory has an R-R ground state operator (in the $(-\frac{1}{2}, -\frac{1}{2})$ picture)

$$|+\rangle_M|+\rangle_L \quad \text{for} \quad \mu < 0 \quad \text{in} \quad 0A$$

(8.6)

and no such operator for $\mu > 0$, while the 0B theory has no R-R ground state operator for $\mu < 0$, and

$$|+\rangle_M|−\rangle_L \quad \text{for} \quad \mu > 0 \quad \text{in} \quad 0B.$$  

(8.7)

The spectrum of NS operators in these theories is independent of the sign of $\mu$, and it is the same in the 0A and 0B theories. Let us summarize the spectrum of R-R operators. In the 0A theory there are no R-R operators for $\mu > 0$ and for $\mu < 0$ only the R-R ground state is present. In the 0B theory with $\mu > 0$ there is a physical vertex operator for each
R-R primary. For $\mu < 0$ the spectrum is the same except that the R-R ground state is absent.

One interesting special case is $\hat{c} = 0$. Here the general picture above is valid, but there are no R-R operators other than the ground state. Therefore, the spectrum of the 0A theory with one sign of $\mu$ is the same as the spectrum of the 0B theory with the opposite sign of $\mu$. This is consistent with our general claim that $\mu \to -\mu$ exchanges 0A and 0B in this case.

Another interesting case is $\hat{c} = 1$. We can also apply the above construction, keeping in mind that there will be two types of matter operators, the ones in (8.1) and similar ones with plus and minus interchanged. There are two R-R ground states in the matter theory with opposite fermion numbers. For every sign of $\mu$ one of them leads to a physical vertex operator in the theory coupled to gravity. This is consistent with the fact that these theories are invariant under $\mu \to -\mu$ [17].

In nonunitary theories the lowest dimension matter operator is typically not the identity operator. Then, a generic perturbation of the SCFT is given by this operator $x \int d^2 \theta O_{\text{min}}$, and upon coupling to Liouville, this operator is dressed. We assume that in the conclusions (8.6)(8.7) about the existence of the R-R ground state, we simply have to replace $\mu$ by the coefficient $x$ in this case.

The R-R ground state operators (8.6) and (8.7) represent fluxes. If these fluxes can only be induced by D-branes, they are quantized and cannot be changed in a continuous fashion. Then, these operators can appear as vertex operators in the perturbative string theory, and they can appear with quantized coefficients in the worldsheet theory. But they cannot exist as standard vertex operators in the complete nonperturbative theory.

9. Superminimal Models Coupled to Supergravity

In this section we review some basic facts about $(A_{p-1}, A_{p'-1})$ superconformal minimal models and their coupling to super-Liouville theory.

The central charges of the superminimal models are

$$\hat{c} = 1 - \frac{2(p - p')^2}{pp'}.$$  

The operators are labelled by two positive integers $j$ and $j'$ subject to

$$1 \leq j' \leq p' - 1, \quad 1 \leq j \leq p - 1, \quad jp' \geq j'p.$$  

(9.1)
Their dimensions are
\[ h_{jj'} = \frac{(jp' - j'p)^2 - (p - p')^2}{8pp'} + \frac{1 - (-1)^{j-j'}}{32}. \] (9.2)
Operators with even \( j' - j \) are NS operators and those with odd \( j' - j \) are R operators. The R operator with \( jp' = j'p \) is the R ground state. It has \( h = \frac{\hat{c}}{16} \).

The super-minimal models are characterized by two positive integers \( p' \) and \( p \) subject to restrictions \( p' > p ; p' - p = 0 \mod 2 \); if both are odd, they are coprime, and if both are even, then \( p/2 \) and \( p'/2 \) are coprime. There is also a standard restriction that if \( p \) and \( p' \) are even then \( (p-p')/2 \) is odd [52] (see Appendix B). For models with odd \( p, p' \) there is no R state with \( h = \frac{\hat{c}}{16} \); hence these models break supersymmetry. For models with even \( p, p' \) there is such a state, but we will show that upon coupling to super-Liouville theory, it does not give rise to a local operator.

When coupled to super-Liouville we need the total \( \hat{c} = 10 \). This fixes
\[ Q = \sqrt{\frac{9 - \hat{c}}{2}} = \frac{p + p'}{\sqrt{pp'}}. \] (9.3)
The operator labelled by \((j, j')\) is dressed by \( e^{\beta_{jj'}\phi} \) with exponent
\[ \beta_{jj'} = \frac{p + p' - jp' + j'p}{2\sqrt{pp'}}. \] (9.4)
In the super-Liouville action we may include the coupling to the lowest dimension operator in the NS sector of the matter theory, \( O_{\min} \),
\[ S(\Phi) = \frac{1}{4\pi} \int d^2z d^2\theta [D_\theta \Phi \bar{D}_\theta \bar{\Phi} + xO_{\min}e^{\beta_{\min}\Phi}], \] (9.5)
In models with odd \( p, p' \) the lowest dimension operator is in the R sector, and then we find \( \beta_{\min} = \frac{p+p'-1}{2\sqrt{pp'}} \). Since models with odd \((p, p')\) break supersymmetry, one might suspect that their coupling to supergravity leads to theories which are equivalent to some bosonic minimal models coupled to gravity. Indeed, there is evidence that they are equivalent to bosonic \((p, p')\) minimal models coupled to gravity, with the dressed lowest dimension operator turned on. In these bosonic models, \( Q = \frac{\sqrt{2}(p+p')}{\sqrt{pp'}} \) and \( \beta_{\min} = \frac{p+p'-1}{\sqrt{2pp'}}, \) so that \( \frac{Q}{\beta_{\min}} = \frac{2(p+p')}{p+p'-1} \). This is the same as in odd \((p, p')\) superminimal models coupled to super-Liouville and perturbed by the dressed lowest dimension operator from the R sector. Thus, the scaling of the partition function is the same. So are the gravitational dimensions of dressed operators.
Consider now the series of theories \((p = 2, p' = 2m)\) with \(m = 1, 2, \ldots\). The standard restriction of \([52]\) (see Appendix B) requires that \(m\) is even. If we denote \(m = 2k\), then the resulting \((2, 4k)\) theories, when coupled to super-Liouville, will match in the 0B case with the critical points of the two-cut matrix models (these critical points are also labelled by positive integer \(k\) and belong to the mKdV hierarchy found in \([25]\)).

Theories with odd \(m\) can be obtained by starting with theories with higher even \(m\) and flowing down by turning on a \(Z_2\) odd operator. In terms of the worldsheet description this is a superconformal field theory coupled to supergravity which is perturbed by a R-R operator. Such theories are not expected to be massive field theories coupled to Liouville. The reason is that the R-R vertex operator involves the ghosts and it is no longer true that the ghosts and the matter fields are decoupled. This is the origin of the difficulty of describing strings in background R-R fields in the RNS formalism. However, in the matrix model there is no such difficulty and the ZS hierarchy allows us to describe such backgrounds. In particular, we find new critical points, the ones with odd \(m\), by turning on such R-R operators. For a discussion of two-cut matrix models with odd \(m > 1\), see Appendix C.

The super-Liouville theory is characterized by

\[
Q_m = \frac{m + 1}{\sqrt{m}}.
\]

(9.6)

The operators are labelled by \((j = 1, j' = 1, 2, \ldots, m - 1)\). The operators with odd \(j'\) are from the NS sector, and those with even \(j'\) are from the R sector. The Liouville dressings of these operators are determined by

\[
\beta_{j'} = \frac{j' + 1}{2\sqrt{m}}.
\]

(9.7)

The lowest dimension operator is the operator with \(j' = m - 1\) and

\[
\beta_{\text{min}} = \beta_{m-1} = \frac{\sqrt{m}}{2}.
\]

(9.8)

Since this is the most relevant operator, we will turn it on as the generic perturbation with coefficient \(x\). For even \(m\) it is an NS operator, while for odd \(m\) it is an R operator.\(^{18}\) We removed from the list the operator labelled by \(j' = m\). For the SCFT with even \(m\) this is the R ground state which was discussed in the previous section.
Therefore, for even $m$ the theory has $Z_2$ symmetry under which all the R fields are odd, while for odd $m$ this symmetry is broken.

Standard KPZ/DDK scaling shows that the correlation functions on a surface with $h$ handles scale like

$$\langle \prod_i O_{j_i'} \rangle_h \sim x^{Q_j (h-1) + \sum_i \beta j_i'} = x^{(-2 - \frac{2}{m}) (h-1) - \sum_i j_i' + 1}.$$  

This matches the discussion of operator dimensions that we discussed after (2.20). The enumeration of operators, the scaling in (9.9) and the $Z_2$ assignments agree with that in the double cut (unitary) matrix model [25-28] with the string action

$$S = \int dx \sum_{l=0}^m t_l H_{l+1}; \quad t_m = 1, \quad t_0 \sim x.$$  

Here $t_{m-1}$ is redundant and can be shifted away by an appropriate redefinition of the couplings. As in [39], it can be interpreted as the boundary cosmological constant which is important only when worldsheet boundaries are present. More precisely, it is the boundary cosmological constant in presence of the FZZT branes [37,38]. $t_l$ is the coefficient of the operator labelled by $j' = m - l - 1$ in the CFT. For example, $t_{l=0} = x$ is the coefficient of the lowest dimension operator labelled by $j' = m - 1$, $t_{l=m-2}$ is the cosmological constant (the dressed identity operator) labelled by $j' = 1$.

It is simple to repeat this for the 0A theory. The complex matrix model is a $Z_2$ orbifold of the two cut model. It has the same spectrum of $Z_2$ invariant operators as its parent two cut model and the $Z_2$ odd operators are absent. The same conclusion about the spectrum applies to the 0A theory which is an orbifold of the 0B theory. Therefore, the 0A theory has the same spectrum as the complex matrix model.

10. The Torus Path Integral

In this section we compare the matrix model results for 0B and 0A theories with the genus-1 path integral for supergravity coupled to the $(A_{p-1},A_{p'}-1)$ superconformal minimal models. For even spin structures this was calculated in [33].

As in the bosonic case, we perform the integral over $\phi_0$ first, and it contributes a volume factor $V_L = -(\ln |x|)/\beta_{\text{min}}$. Integrating over the rest of the modes, we find

$$\frac{Z^{(S)}_{\text{even}}}{V_L} = \frac{1}{4\pi \sqrt{2}} \int d^2 \tau \tau^{-3/2} \sum_{(r,s)} |D_{r,s}|^{-2} Z^m_{r,s}(\tau, \bar{\tau})$$  

(10.1)
where $Z_{m,r,s}(\tau, \bar{\tau})$ is the matter partition function in the spin structure $(r, s)$. One can represent the partition functions of the superconformal minimal models as linear combinations of the partition functions of a compactified scalar superfield \[52\]. For the supersymmetric $(A_{p-1}, A_{p'-1})$ series, in each even spin structure there are relations \[52\]

\[
Z_{p,p'} = \frac{1}{2} \tilde{Z}(\mathcal{R}/\sqrt{\alpha'} = \sqrt{p/p'}) - \frac{1}{2} Z(\mathcal{R}/\sqrt{\alpha'} = \sqrt{p'/p'}) , \quad p, p' \text{ odd} \\
Z_{p,p'} = \frac{1}{2} \tilde{Z}(\mathcal{R}/\sqrt{\alpha'} = \sqrt{p'/p'}) - \frac{1}{2} Z(\mathcal{R}/\sqrt{\alpha'} = \sqrt{p/p'}) , \quad p, p' \text{ even} ,
\]

where $\tilde{Z}$ is the partition function of the “circle” $\hat{c} = 1$ theory, and $Z$ is the partition function of the super-affine theory. Substituting these expressions into eq. (10.1), and performing the integrals, we find the sum of the contributions of the $(-,-)$, $(-,+)$ and $(+,-)$ spin structures, each weighted with the factor $1/2$:

\[
Z_{\text{even}} = -\frac{1}{16} \frac{(p-1)(p'-1)}{(p+p'-1)} \ln |x| , \quad p, p' \text{ odd} \\
Z_{\text{even}} = -\frac{1}{16} \frac{(p-1)(p'-1)+1}{(p+p'-2)} \ln |x| , \quad p, p' \text{ even} .
\]

Note that the answers depend on $\ln |x|$. Our discussion in section 8 about the spectrum of these theories shows that the even spin structures are independent of the sign of $x$. This sign is important in the odd spin structure.

The continuum calculation of the odd spin structure is more difficult and is beyond the scope of this paper. However, even without calculating it, we can carry out an interesting consistency check of our 0B and 0A conjectures. Since the odd spin structure contributes with opposite signs in the 0A and 0B theories, we can check that the average of their torus partition sums agrees with (10.3). Furthermore, since the even spin structures are independent of the sign of $x$, we should find

\[
\frac{1}{2} (Z_A(x) + Z_B(x)) = Z_{\text{even}}(|x|) .
\]

\[19\] In models with odd $(p, p')$, $x$ is the coefficient of the dressed lowest dimension Ramond operator. Note that the sum over even spin structures is the same as for bosonic $(p, p')$ models coupled to gravity, up to overall coefficient (in the bosonic case the coefficient is $1/24$ \[54\]). It is possible that inclusion of the odd spin structure will give exact agreement with the bosonic theory. This would provide further evidence that the odd $(p, p')$ superminimal models coupled to super-Liouville are equivalent to $(p, p')$ minimal models coupled to Liouville.
Let us consider theories with \( p = 2, p' = 4k \) where the 1-matrix models discussed in sections 2-5 and in section 6 describe 0B and 0A theories respectively. In this case the worldsheet computations lead to

\[
Z_{\text{even}} = -\frac{1}{16} \ln |x| ,
\]

(10.5)

independent of \( k \). Let us compare this with the matrix models. We start with the complex matrix models discussed in section 5, changing the notation \( z \rightarrow x \). If we set \( q = 0 \) in (6.3), then the perturbative solution for \( x > 0 \) is obtained simply by setting \( R(x) = 0 \). This is exactly the set of KdV equations that describe one-cut Hermitian matrix model, except the result is divided by 2 to eliminate the doubling of the free energy that appears for symmetric potentials: this is the origin of the factor 2 in (10.4). It follows that the torus path integral in the \( k \)-th multicritical complex matrix model for \( x > 0 \) is the same as in gravity coupled to the \((2, 2k - 1)\) minimal model which was calculated in [54]. Substituting \( p = 2, p' = 2k - 1 \) into the result of [54], we find

\[
-\frac{(p - 1)(p' - 1)}{24(p + p' - 1)} \ln x = -\frac{k - 1}{24k} \ln x.
\]

(10.6)

Of course, this result can be obtained also by using the KdV equations \( R(x) = 0 \) and integrating (6.4).

For \( x < 0 \) we return to (6.3) and set \( q = 0 \). It is easy to see that the asymptotic solution of the equations is \( u \approx -\frac{1}{4x} \) for all \( k \) (note the agreement with our \( k = 1, 2 \) expressions). Using (6.4) we learn that

\[
Z_A = \begin{cases} 
-\frac{k - 1}{24k} \ln x & x > 0 \\
0 & x < 0
\end{cases}
\]

(10.7)

In the multicritical two-cut models (2.4), we find that the torus path integral for 0B theories is

\[
Z_B = \begin{cases} 
-\frac{2k + 1}{24k} \ln x & x > 0 \\
0 & x < 0
\end{cases}
\]

(10.8)

The vanishing result for \( x < 0 \) follows from the fact that, for \( q = 0 \), the relevant solution for large negative \( x \) is the trivial one, with \( r = 0 \) up to non-perturbative corrections.

Using (10.7), (10.8)

\[
Z_{\text{even}} = \frac{1}{2} (Z_A(x) + Z_B(x)) = -\frac{1}{16} \ln |x|
\]

(10.9)
in agreement with the worldsheet value (10.5). As expected, it is independent of the sign of \(x\). Using the matrix model results we get a prediction for the odd spin structure

\[
Z_{\text{odd}} = \frac{1}{2} (Z_A(x) - Z_B(x)) = \begin{cases} 
\frac{k+2}{48k} \ln x & x > 0 \\
-\frac{1}{16} \ln |x| & x < 0
\end{cases},
\]

which does depend on the sign of \(x\). It will be interesting to check this with an explicit continuum calculation.

11. Conclusions and Future Directions

In this paper we studied two types of matrix models:

1. The unitary matrix model (or, equivalently, the two-cut Hermitian matrix model). In the double scaling limit it is described by the differential equations (2.20)

\[
\frac{\delta}{\delta r(x)} \int dx (\sum_{l=0}^{m} t_l H_{l+1} + q \omega) = -\sum_{l=0}^{m} t_l (l + 1) R_l = 0
\]

\[
\frac{\delta}{\delta \omega(x)} \int dx (\sum_{l=0}^{m} t_l H_{l+1} + q \omega) = \sum_{l=0}^{m} t_l (l + 1) H_l + q = 0.
\]

We identify this model with type 0B string theory in a background characterized by \(t_l\) and \(q\).

2. The complex matrix model, which is described in the double-scaling limit by the differential equation (6.5)

\[
u R^2[u] = \frac{1}{2} R[u] \partial_z^2 R[u] + \frac{1}{4} (\partial_z R[u])^2 = q^2
\]

\[R[u] = \sum_{l=1}^{\infty} (l + \frac{1}{2}) t_l Q_l[u] - z.
\]

We identify this model with type 0A string theory in a background characterized by \(t_l\) and \(q\).

Unlike in the standard Hermitian matrix model, the potentials of these models are bounded from below, and we expand around their global minima. Correspondingly, the differential equations (11.1)(11.2) have smooth and real solutions.

We pointed out that, when a superconformal field theory is coupled to super-Liouville theory to make a superstring theory, there are in general four independent weak coupling
limits. First, we have a two-fold ambiguity in the sign of the odd spin structures. This leads to the type 0A and type 0B theories. Second, we can change the sign of the cosmological constant $\mu \rightarrow -\mu$, or more generally, the sign of the coefficient of the lowest dimension operator $x \rightarrow -x$. In the simplest case of pure supergravity ($\hat{c} = 0$), these theories are identical pairwise: 0A with $\mu$ is equivalent to 0B with $-\mu$. This is not the case, however, for more general theories.

The recent advances in non-critical string theory have been based on the idea that the matrix model is a theory of a large number of ZZ D-branes [1-7, 16, 17]. The identification of the FZZT branes with the integral of the matrix model resolvent leads to another insight. The boundary cosmological constant on this brane can be analytically continued to take values on a Riemann surface which is a double cover of the complex plane. The cuts in the complex plane represent the eigenvalues; the discontinuity of the resolvent (the derivative of the FZZT brane with respect to the boundary cosmological constant) is their density. This works well for one of the FZZT branes of the type 0 theory. It would be nice to identify the other FZZT brane in the matrix model.

One novelty of the matrix models is that they allow us to analyze theories with background Ramond-Ramond fields. This is an important topic that we have only started to analyze in detail. Our preliminary investigation has already led to the following observations:

1. Even without turning on the background R-R couplings $t_l$, the 0B theory has solutions, e.g. (5.3), which break the $Z_2$ symmetry that acts as $-1$ on all the R-R fields. It would be nice to clarify the nature of these solutions in more detail.

2. By turning on background R-R fields, the $Z_2$ odd $t_l$ couplings, we find new theories. They involve nontrivial couplings of fields and ghosts, but their description in the matrix model is as easy as the even $t_l$ flows. Among them we find scaling solutions which are described by the odd $m$ equations in the ZS hierarchy. Unlike the even $m$ scaling solutions, these are not superconformal matter field theories coupled to supergravity.

3. The parameter $q$ is a particular $Z_2$ odd coupling. In the 0B matrix model it appears as an integration constant. In the 0A matrix model it can be introduced by considering a model of rectangular matrices. Alternatively, it can be introduced by changing

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20 On the matrix model side, they seem related to performing the matrix integral along other contours.
the measure in the complex matrix model of square matrices by adding a factor of \((\det MM^\dagger)^{|q|}\) to the measure. In some cases \(q\) appears to be related to adding D-branes to the system. This is particularly clear in the complex matrix model with rectangular matrices. From this point of view it is natural to assume that \(q\) must be quantized. However, this conclusion might be misleading. If \(q\) is introduced by changing the measure of the 0A model, or if it is introduced as an integration constant in the equations of the 0B model, we see no reason why it should be quantized. It would be nice to find a clear argument which determines whether \(q\) should be quantized or not. If \(q\) is not quantized, we have seen some physical amplitudes which are not analytic in \(q\) around \(q = 0\). Therefore \(q\) cannot vary in a continuous fashion and its change is not described by a standard vertex operator.

4. It is common that for one sign of \(x\) the parameter \(q\) appears with boundaries on the worldsheets and can be interpreted as associated with D-branes, while for the opposite sign of \(x\) only even powers of \(q\) appear, and it can be interpreted as background R-R flux. This difference in the behavior as \(x\) changes sign is consistent with the behavior of the profile of the R-R field strength as a function of the Liouville field \(\phi\). What is surprising is that the system smoothly interpolates between the behavior at positive \(x\) and at negative \(x\). Such a smooth interpolation between D-branes and fluxes is reminiscent of geometric transitions [45-49]. Our solvable models provide simple, tractable and explicit examples of this phenomenon.

5. By examining the planar limit with \(x\) positive and \(x\) negative one often finds a phase transition associated with nonanalytic behavior of the free energy \(F\) [24]. We have seen that the finite \(x\) (higher genus) corrections smooth out these transitions. Alternatively, we can keep \(|x|\) large, i.e. continue to focus only on spherical worldsheet topologies, but smooth out the transition by turning on \(q\). In other words, the transition is smoothed out either by including worldsheet handles, or by including worldsheet boundaries or nontrivial R-R backgrounds. These latter cases provide particularly simple examples of the interpolation from D-branes to flux. They can be seen with spherical worldsheets and are described by polynomial equations rather than differential equations. In the 0B theory we have interpreted \(q\) as a certain period around a cycle in a Riemann surface. The transition associated with the collapse of this cycle is prevented by a nonzero period.

We found many situations in which amplitudes vanish without a simple worldsheet or spacetime explanation. For example, the perturbative expansion of \(F\) with negative
and $q = 0$ vanishes for some of our solutions of the 0B theory; see e.g. (4.10) but not (5.4). Also, the even genus amplitudes in the expansion (6.13) of the 0A theory vanish, but the odd genus amplitudes are nonzero. Perhaps these vanishing amplitudes reflect a deep structure of these theories.

Interesting insights into the bosonic string counterpart of these systems has been gained by interpreting them as topological theories [55]. It is possible that a similar topological structure underlies our examples and our results. For a possible starting point for investigating this question, see [28].

An obvious generalization of the dualities we studied is provided by multi-matrix model versions of them. It is likely that they can provide realizations of all $(p,p')$ super-minimal models coupled to supergravity. We expect these systems to exhibit a rich structure which generalizes the phenomena seen in this paper and in the analysis of the bosonic noncritical string.

We have related the FZZT brane of the 0B theory with $\eta = -1$ with the resolvent of the two cut matrix model. The FZZT brane with $\eta = 1$ related to the resolvent of the complex matrix model. It would be nice to have a description of the two branes within the same theory. It would also be nice to have a clearer description of the relation of the ZZ branes to the FZZT branes.

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Appendix A. A simple comparison – the $m = 4$ model

In this appendix we compute some simple tree level correlation functions of RR vertex operators using the matrix model results.
The pure supergravity example \((m = 2)\) discussed above is particularly simple in that the model has only one coupling constant which corresponds to turning on the super-Liouville superpotential \(e^\phi\) from the NS sector. As we discussed in the previous subsection, the R ground state is not present as a standard deformation of the theory. Another would be R-operator is redundant and can be shifted away.

For this reason let us consider the next even critical point, \(m = 4\) which we expect to correspond to the \((2,8)\) super-minimal model coupled to gravity. In this model the effective Lagrangian is given by

\[
L = xH_1 + eH_2 + \tilde{x}H_3 - \frac{8}{15}H_5. \tag{A.1}
\]

Now there are 2 NS operators, corresponding to coupling constants \(x\) and \(\tilde{x}\), and one RR operator corresponding to \(e\). The coupling \(x\) corresponds to the dressed \(j = 1, j' = 3\) operator of negative dimension, while the coupling \(\tilde{x}\) to the dressed identity. The coupling \(e\) corresponds to dressed R-R operator \(O_R\) with \(j = 1, j' = 2\). We will set \(\tilde{x} = 0\) and calculate the expansion of the free energy in powers of \(e\).

To perform the sphere calculation we neglect derivatives of \(r\) and \(\omega\). From the KPZ scaling we expect that \(\langle O_R O_R \rangle \sim x \ln x\), but we do not find such a term. It follows that the two-point function vanishes, or is given by a non-universal term \(\sim x\). However, the four point function of \(O_R\) scales as \(x^{-1/2}\), in agreement with the KPZ scaling.

### Appendix B. Super-minimal models with even \(p\) and \(p'\)

The super-minimal models are characterized by two positive integers \(p'\) and \(p\) subject to: \(p' > p\), \(p' - p = 0\) mod 2, if both are odd, they are coprime, and if both are even, then \(p/2\) and \(p'/2\) are coprime. There is often also a restriction that if \(p\) and \(p'\) are even then \((p - p')/2\) is odd. The purpose of this appendix is to review parts of the discussion of [52] emphasizing why this requirement is needed. Along the way we will also review some useful facts about the super-minimal models.

The central charge of the super-minimal model labelled by \((p, p')\) is

\[
\hat{c} = 1 - 2\frac{(p - p')^2}{pp'}. \tag{B.1}
\]

The operators are labelled by two positive integers \(m\) and \(m'\) subject to

\[
1 \leq m' \leq p' - 1, \quad 1 \leq m \leq p - 1, \quad mp' \geq m'p. \tag{B.2}
\]
Their dimensions are

$$h_{mm'} = \frac{(mp' - m'p)^2 - (p - p')^2}{8pp'} + \frac{1 - (-1)^{m-m'}}{32}. \tag{B.3}$$

Operators with even \(m' - m\) are NS operators and those with odd \(m' - m\) are R operators. The R operator with \(mp' = m'p\) is the R ground state. It has \(h = \hat{c}/16\).

For even \(p\) and \(p'\) and odd \((p - p')/2\) the R ground state is the operator labelled by \((m = \frac{n}{2}, m' = \frac{n'}{2})\). For even \(p\) and \(p'\) and even \((p - p')/2\) the operator \((m = \frac{n}{2}, m' = \frac{n'}{2})\) is in the NS sector and it has \(h = -\frac{(p - p')^2}{8pp'} = -\frac{\hat{c} - 1}{16}\). This means that the effective central charge of the theory \([51,56]\), which is given in terms of the lowest dimension operator is \(\hat{c}_{eff} = \hat{c} - 16h_{min} = 1\), and therefore the density of states of such a theory is as in the \(\hat{c} = 1\) theory. We conclude that the theory must have an infinite number of super-Virasoro primaries, and it cannot be a super-minimal model. Since this argument depends on the modular invariance of the partition function, let us examine it in more detail.

The superconformal characters in the different sectors are \([52]\)

$$\chi_{\lambda}^{NS}(\tau) = \text{Tr}_{\lambda} q^{L_0 - \frac{c}{16}} = \chi_{\hat{c}=1,h=0}(\tau) [K_\lambda(\tau) - \tilde{K}_\lambda(\tau)]$$

$$\chi_{\lambda}^{\bar{NS}}(\tau) = \text{Tr}_{\lambda} q^{L_0 - \frac{c}{16}} (-1)^F = \chi_{\lambda}^{NS}(\tau + 1) = \chi_{\hat{c}=1,h=0}(\tau) [K_\lambda(\tau + 1) - \tilde{K}_\lambda(\tau + 1)]$$

$$\chi_{\lambda}^{R}(\tau) = \text{Tr}_{\lambda} q^{L_0 - \frac{c}{16}} = \chi_{\hat{c}=1,h=\frac{1}{16}}(\tau) [K_\lambda(\tau) - \tilde{K}_\lambda(\tau)]. \tag{B.4}$$

Here the traces are in the representation labelled by \(\lambda = mp' - m'p\) and we use \(\chi^{NS}\) or \(\chi^{R}\) depending on whether \(m' - m\) is even (NS) or odd (R). In \((B.4)\) we use the notation \(\hat{\lambda} = mp' + m'p\). \(\chi_{\hat{c}=1,h}\) are the characters in \(\hat{c} = 1\) in the same spin structure and

$$K_\lambda = \sum_{n=-\infty}^{\infty} q^{\frac{2pp'n}{8pp'} + \frac{\hat{c}^2}{8pp'}} \tag{B.5}$$

(we moved a factor of \(\eta(\tau)\) from \(K\) to the factor that multiplies it relative to \([52]\)).

As a simple consistency checks examine the leading behavior of the different characters as \(q = e^{2\pi i \tau} \to 0\). Since by \((B.2)\) \(0 \leq \lambda < pp'\), the leading term in the sum in \((B.5)\) is with \(n = 0\), and therefore

$$\chi_{\lambda}^{NS}(\tau) \to q^{0 - \frac{c}{16}} + \frac{\lambda^2}{8pp'} = q^{h - \frac{c}{16}}$$

$$\chi_{\lambda}^{\bar{NS}}(\tau) \to q^{0 - \frac{c}{16}} + \frac{\lambda^2}{8pp'} = q^{h - \frac{c}{16}}$$

$$\chi_{\lambda}^{R}(\tau) \to q^{\frac{c}{16} - \frac{c}{16}} + \frac{\lambda^2}{8pp'} = q^{h - \frac{c}{16}}. \tag{B.6}$$
Under $\tau \to \tau + 1$ each term in the sum (B.5) is multiplied by $e^{\pi i (pp'n^2 + (mp' - m'p)n + \lambda^2 / 4pq')}$.

If $p$ and $p'$ are even, all the terms have the same phase and $K$ transforms by a phase. If $p$ and $p'$ are odd, and $m - m'$ is odd (R representations) again all the terms have the same phase and $K$ transforms by a phase. Finally for $p$ and $p'$ odd and $m - m'$ even (NS representations) the different terms in $K$ transform with the same phase up to $\pm 1$. Now let us compare the phase of $K_\lambda$ and $K_{\tilde{\lambda}}$. Since

$$\frac{(2pp'n + \lambda)^2 - (2pp'n + \tilde{\lambda})^2}{8pp'} = -nm'p - \frac{mm'}{2},$$

for R representations (since $mm'$ is even) $K_\lambda$ and $K_{\tilde{\lambda}}$ transform with the same phase. For NS representations the phase of the terms in $K_\lambda$ can differ by a minus sign relative to the phase of the terms in $K_{\tilde{\lambda}}$. This is precisely the behavior expected from the characters because the $L_0$ value of the different states in the representation differ by integer or half integer in the NS representations and they differ by an integer in the R representations.

We conclude that under $\tau \to \tau + 1$ the characters transform up to an overall phase as $\chi^{\text{NS}}_\lambda \leftrightarrow \chi^{\text{NS}}_{\tilde{\lambda}}$ and $\chi^R_\lambda$ are invariant.

Now let us consider the behavior under $\tau \to -\frac{1}{\tau}$. We use the Poisson resummation formula

$$\sum_n e^{-\pi an^2 + 2\pi ibn} = \frac{1}{\sqrt{a}} \sum_m e^{-\pi (m - b)^2 / a}$$

(B.7)

to write

$$K_\lambda(\tau') = -\frac{1}{\tau} = \sum_n e^{\frac{2\pi i (2pp'n + \lambda)^2}{8pp'} / \tau} = \sqrt{\frac{\tau}{4pp'}} \sum_n q_n^2 \sum_n e^{2\pi i n\lambda / 2pp'}. \quad (B.8)$$

Consider for example the theory with $(p = 2, p' = 8)$. It has two NS representations: the identity with $\lambda = 6, \tilde{\lambda} = 10$ and another representation with $\lambda = 2, \tilde{\lambda} = 14$. Using

$$\chi^{\text{NS}}_6(\tau) = \chi^{\text{NS}}_{\tilde{\lambda}=1, h=0}(\tau) [K_6(\tau) - K_{10}(\tau)] = \chi^{\text{NS}}_{\tilde{\lambda}=1, h=0}(\tau) \left[ \sum_n q_n^2 \frac{(16n+3)^2}{32} - \sum_n q_n^2 \frac{(16n+5)^2}{32} \right]$$

$$\chi^{\text{NS}}_2(\tau) = \chi^{\text{NS}}_{\tilde{\lambda}=1, h=0}(\tau) [K_2(\tau) - K_{14}(\tau)] = \chi^{\text{NS}}_{\tilde{\lambda}=1, h=0}(\tau) \left[ \sum_n q_n^2 \frac{(16n+1)^2}{32} - \sum_n q_n^2 \frac{(16n+7)^2}{32} \right]$$

(B.9)
we easily find

\[
\chi_6^{NS}(\frac{-1}{\tau}) = \chi_{\tilde{c}=1, h=0}(\frac{-1}{\tau}) \left[ K_6(-\frac{1}{\tau}) - K_{10}(-\frac{1}{\tau}) \right] \\
= \sqrt{\frac{\tau}{16\tau}} \chi_{\tilde{c}=1, h=0}(\frac{-1}{\tau}) \sum_n \frac{n^2}{q^{1/2}} \left[ e^{\frac{2\pi in3}{16}} - e^{\frac{2\pi in5}{16}} \right] \\
= \sqrt{\frac{\tau}{4l}} \chi_{\tilde{c}=1, h=0}(\frac{-1}{\tau}) \sum_n \frac{n^2}{q^{1/2}} \left[ \sqrt{2 - \sqrt{2}}(q^{\frac{(16k+1)^2}{32}} - q^{\frac{(16k+7)^2}{32}}) \\
- \sqrt{2 + \sqrt{2}}(q^{\frac{(16k+3)^2}{32}} - q^{\frac{(16k+5)^2}{32}}) \right] \\
\]

\[
\chi_2^{NS}(\frac{-1}{\tau}) = \chi_{\tilde{c}=1, h=0}(\frac{-1}{\tau}) \left[ K_2(-\frac{1}{\tau}) - K_{14}(-\frac{1}{\tau}) \right] \\
= \sqrt{\frac{\tau}{16\tau}} \chi_{\tilde{c}=1, h=0}(\frac{-1}{\tau}) \sum_n \frac{n^2}{q^{1/2}} \left[ e^{\frac{2\pi in7}{16}} - e^{\frac{2\pi in3}{16}} \right] \\
= \sqrt{\frac{\tau}{4l}} \chi_{\tilde{c}=1, h=0}(\frac{-1}{\tau}) \sum_k \frac{n^2}{q^{1/2}} \left[ \sqrt{2 + \sqrt{2}}(q^{\frac{(16k+1)^2}{32}} - q^{\frac{(16k+7)^2}{32}}) \\
+ \sqrt{2 - \sqrt{2}}(q^{\frac{(16k+3)^2}{32}} - q^{\frac{(16k+5)^2}{32}}) \right].
\]

(B.10)

Therefore, they can be expressed as linear combinations of \( \chi_6^{NS}(\tau) \) and \( \chi_2^{NS}(\tau) \).

Let us contrast this with a putative theory with \((p = 2, p' = 6)\). It has two NS representations: the identity with \( \lambda = 4, \tilde{\lambda} = 8 \) and another representation with \( \lambda = 0, \tilde{\lambda} = 12 \). Now

\[
\chi_4^{NS}(\tau) = \chi_{\tilde{c}=1, h=0}(\tau) [K_4(\tau) - K_8(\tau)] = \chi_{\tilde{c}=1, h=0}(\tau) \left[ \sum_n \frac{(6n+1)^2}{6} - \sum_n \frac{(6n+2)^2}{6} \right] \\
\chi_0^{NS}(\tau) = \chi_{\tilde{c}=1, h=0}(\tau) [K_0(\tau) - K_{12}(\tau)] = \chi_{\tilde{c}=1, h=0}(\tau) \left[ \sum_n \frac{(6n)^2}{6} - \sum_n \frac{(6n+3)^2}{6} \right]
\]

(B.11)
and their modular transforms are

\[
\chi_{NS}^{4}(-\frac{1}{\tau}) = \chi_{\tilde{c}=1, h=0}^{NS}(-\frac{1}{\tau}) \left[ K_{4}(-\frac{1}{\tau}) - K_{8}(-\frac{1}{\tau}) \right] \\
= \sqrt{\frac{\tau}{12i}} \chi_{\tilde{c}=1, h=0}^{NS}(-\frac{1}{\tau}) \sum_{n} q^{\frac{n^2}{24}} \left[ e^{\frac{2\pi in}{6}} - e^{\frac{2\pi in}{3}} \right] \\
= \sqrt{\frac{\tau}{3i}} \chi_{\tilde{c}=1, h=0}^{NS}(-\frac{1}{\tau}) \sum_{n} q^{\frac{(6k+1)^2}{24} - q^{\frac{(6k+3)^2}{24}}} \\
= \sqrt{\frac{\tau}{12i}} \chi_{\tilde{c}=1, h=0}^{NS}(-\frac{1}{\tau}) \sum_{n} q^{\frac{n^2}{24}} \left[ 1 - (-1)^n \right] \\
= \sqrt{\frac{\tau}{3i}} \chi_{\tilde{c}=1, h=0}^{NS}(-\frac{1}{\tau}) \sum_{n} q^{\frac{2n^2}{24}}.
\]

These are not linear combinations of characters of the theory. Therefore, the theory with \((p = 2, p' = 6)\) is not modular invariant. More generally, when \(p\) and \(p'\) are even we must assume that \((p - p')/2\) is odd [52].

**Appendix C. Some properties of the ZS hierarchy**

In this Appendix we derive the form of the Zakharov-Shabat operators \(R_m, H_m\) which is relevant to solving string theory on a sphere. We also review how the mKdV hierarchy is related to the equation (6.5) for the matrix model.

We first recall that, in terms of \(H_m\) and \(R_m\), the recursion relations are given by (2.13). It is interesting to solve the recursion relations (2.13) on the sphere, i.e. by dropping the derivative terms and using the ansatz

\[
R_m = \sum_{l=0}^{[m/2]} a_l^m r^{2l+1} \omega^{m-2l} \\
H_m = \sum_{l=0}^{[(m-1)/2]} b_l^m r^{2l+2} \omega^{m-2l-1}
\]

(C.1)

When we insert this into the recursion relations (2.13) we can drop the term involving derivatives of \(H\) in the recursion relation for \(R_{m+1}\). Equating coefficients on both sides we
find the recursion relations for the coefficients:
\[ a_{l+1}^{m} = a_{l}^{m} + b_{l}^{m-1} , \]
\[ b_{l+1}^{m} = b_{l}^{m} - \frac{(2l + 1)}{(2l + 2)} a_{l}^{m} , \]
\[ (m - 2l)b_{l+1}^{m} = -(m - 2l)a_{l}^{m} + (m - 2l - 1)b_{l}^{m} . \]
Note that the last two equations have to be compatible with each other. This implies the equation
\[ b_{l}^{m} = -\frac{m - 2l}{2l + 2} a_{l}^{m} \]
This relation can be stated as
\[ \partial_{r} H_{m} = -\partial_{\omega} R_{m} \]
which is precisely the integrability condition for the second equation in (2.13), when we drop derivative terms in \( H_{n} \) and \( R_{m} \). Note also that once we set \( a_{0}^{0} = 1 \) (C.2) implies that \( a_{0}^{m} = 1 \). By demanding that the right hand sides of the first two lines in (C.2) obey (C.3) we find that
\[ a_{l}^{m} = (-1)^{l} \frac{m!}{2^{l!}l!(m - 2l)!} \]
By defining new variables \( \rho \) and \( \varphi \) through
\[ \rho^{2} = r^{2} + \omega^{2} , \quad \cos \varphi = \frac{\omega}{\rho} , \quad \sin \varphi = \frac{r}{\rho} , \]
we can see from (C.5) that \( R_{m} \) and \( H_{m} \) can be written in terms of Legendre Polynomials
\[ H_{m} = -\rho^{m+1}[\cos \varphi P_{m}(\cos \varphi) - P_{m+1}(\cos \varphi)] = -\rho^{m+1} \frac{\sin^{2} \varphi}{m + 1} P'_{m}(\cos \varphi) , \]
\[ R_{m} = \rho^{m+1} \sin \varphi P_{m}(\cos \varphi) , \]
which is our main result.
This relation can be derived more directly by looking at equation (2.7). In the limit that \( r \) and \( \omega \) are independent of \( x \) (i.e. commute with \( \frac{d}{dx} \)), we have the equation
\[ \tilde{O} = \int \frac{dp}{2\pi i p + rJ_{1} + (\omega - \zeta)J_{3}} = \int \frac{dp}{2\pi p^{2} + |\tilde{v}|^{2}/4} = \frac{\tilde{v}}{\tilde{v}} J \]
We see that
\[ v_{1} = \frac{\rho}{|\zeta|} \sin \varphi \frac{1}{(1 - 2 \cos \varphi \xi + \xi^{2})^{1/2}} = \frac{\zeta}{|\zeta|} \sum_{l=0}^{\infty} \zeta^{l-1} \rho^{l+1} \sin \varphi P_{l}(\cos \varphi) , \]
\[ v_{3} = \frac{\rho}{|\zeta|} \frac{(\cos \varphi - \frac{\xi}{\rho})}{(1 - 2 \cos \varphi \xi + \xi^{2})^{1/2}} = \frac{\zeta}{|\zeta|} \sum_{l=-1}^{\infty} \zeta^{l-1} \rho^{l+1}[\cos \varphi P_{l}(\cos \varphi) - P_{l+1}(\cos \varphi)] . \]
For $\zeta < 0$ these equations imply (C.7). Note that in this limit the quantities $\Theta_l$ in (2.10) are zero since they have no term without derivatives.

Note that (2.7) implies that the resolvent, $\text{Tr}[\frac{1}{M-z}]$, is proportional to $\int_x^\infty dx' \text{Tr}[J_3 \tilde{O}(x')]$ with $z = i\zeta$. In the planar limit we can use (C.9) to compute it. We find that the ends of the cuts are at $z = i\zeta = i\rho e^{\pm i\phi} = \pm r + i\omega$.

Let us first consider even $m$ and $q = 0$. The equations on the sphere are

$$\rho \sin \varphi [-\alpha_m x + \rho^m P_m(\cos \varphi)] = 0,$$
$$\rho^{m+1} \sin^2 \varphi P_m'(\cos \varphi) = 0,$$

with $(-1)^{m/2} \alpha_m > 0$. First note that $\sin \varphi = 0$ is a trivial solution of the equations with vanishing $r$ and vanishing free energy. Aside from this trivial solution, the second equation implies that $P_m'(\cos \varphi) = 0$. For $x > 0$ there is always a solution with $\cos \varphi = 0$: this is the symmetric solution with vanishing $\omega$. The number of possible solutions actually grows with $m$. For $m = 2$ the solutions with $\cos \varphi = 0, 1$ are the only ones. For $m = 4$ and $x < 0$ we also find solutions with $\cos \varphi = \pm \sqrt{1/7}$; these are the broken symmetry solutions discussed in section 4 (see (5.4)). For $m = 6$ and $x < 0$ we again find two broken symmetry solutions with $\cos \varphi = \pm \sqrt{15 + 2\sqrt{15}/\sqrt{33}}$; for $x > 0$, in addition to the standard solution with $\cos \varphi = 0$, there are also non-trivial solutions with $\cos \varphi = \pm \sqrt{15 - 2\sqrt{15}/\sqrt{33}}$. In general, the equation $P_m'(\cos \varphi) = 0$ admits $m - 2$ non-trivial broken symmetry solutions; some of them are compatible with $x < 0$ and others with $x > 0$. For all the nontrivial solutions, $\rho$, $r$, and $\omega$ scale as $|x|^{1/m}$ for large $|x|$.

If $m$ is odd then let us choose $\alpha_m > 0$. For $x > 0$ we look for solutions of the second equation in (C.10) with $P_m(\cos \varphi) > 0$ (the solutions with $x \to -x$ are obtained by $\cos \varphi \to -\cos \varphi$). The number of solutions for a given sign of $x$ is $(m - 1)/2$. After we include derivative terms in the string equations, the sphere solutions receive higher genus corrections.

Consider, for instance, $m = 3$, which is the first non-topological “odd” critical point. Here the equations are $R_3 = 5xr/8; H_3 = 0$ (see (2.14)). We find the following solution as $x \to \infty$:

$$r(x) = x^{1/3} - \frac{2}{9} x^{-7/3} - \frac{1162}{729} x^{-5} + \mathcal{O}(x^{-23/3})$$,
$$\omega(x) = -\frac{1}{2} x^{1/3} - \frac{2}{27} x^{-7/3} - \frac{128}{243} x^{-5} + \mathcal{O}(x^{-23/3})$$,
$$\frac{d^2 F}{dx^2} = \frac{r^2(x)}{4} = \frac{1}{4} x^{2/3} - \frac{1}{9} x^{-2} - \frac{572}{729} x^{-14/3} + \mathcal{O}(x^{-22/3})$$.

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Remarkably, for $m$ large enough that there are multiple solutions, we find that the one loop partition function is independent of the choice of solution (except for the trivial solution $r = 0$ where it vanishes). The results we find are consistent with the general formula $F_{\text{torus}} = \frac{m+1}{12m} \ln x$.

Finally, we extend (2.14) by presenting a few more terms in the Zakharov-Shabat hierarchy, generated via the recursion relations:

\[
R_4 = \frac{3r^5}{8} - 3r^3 \omega^2 + r\omega^4 - \frac{5r^2 r''}{2} + 12r' \omega' + 3r\omega'^2 - \frac{5r^2 r'''}{2} + 6\omega^2 r'' + 4r\omega r'' + r^{(4)}
\]

\[-H_4 = \frac{-3r^4 \omega}{2} + 2r^2 \omega^3 - 2r \omega r'' + 4r\omega r'' + r^2 \omega'' \]

\[
R_5 = \frac{15r^5 \omega}{8} - 5r^3 \omega^3 + r\omega^5 - \frac{25r\omega r^2}{2} - \frac{25r^2 r' \omega'}{2} + 30\omega^2 r' \omega' + 15r \omega r^2 - \frac{25r^2 \omega r''}{2} + 10\omega^3 r'' - \frac{5r^3 \omega''}{2} + 10r \omega^2 r'' + 10r'' r'' + 10r' r^{(3)} + \)

\[
5r' \omega^{(3)} + 5r \omega^{(4)} + r^{(4)} \]

\[-H_5 = \frac{5r^6}{16} - \frac{15r^4 \omega^2}{4} + \frac{5r^2 \omega^4}{2} - \frac{5r^2 r'^2}{4} - 5\omega^2 r'^2 + 10r \omega r' \omega' + \frac{5r^2 \omega^2}{2} - \]

\[
\frac{5r^3 \omega''}{2} + 10r \omega^2 r'' + \frac{r''^2}{2} + 5r^2 \omega r'' - r' r^{(3)} + r r^{(4)} \]

(C.12)

**Appendix D. Double scaling limit of the complex matrix model**

In this appendix we derive (6.5) for the simplest case, $k = 1$, where it reduces to (6.7). This is a review of the discussion in [31], which is further expanded in [32].

We start with the complex matrix model (6.1), which reduces to the integral (6.2)

\[
Z = \prod_{i=1}^{N} \int_{0}^{\infty} dy_i y_i q e^{-\frac{N}{2} V(y_i)} \Delta(y)^2 . \quad \text{(D.1)}
\]

We define orthogonal polynomials with respect to the measure

\[
\int d\mu P_n P_m = \int_{0}^{\infty} dy y q e^{-\frac{N}{2} V} P_n P_m = \delta_{n,m} h_n , \quad P_n = y^n + \cdots \quad \text{(D.2)}
\]

**D.1. $q = 0$**

Let us first set $q = 0$. We derive recursion relations in the usual way by writing

\[
y P_n = P_{n+1} + s_n P_n + r_n P_{n-1}, \quad r_n = h_n / h_{n-1} \quad \text{(D.3)}
\]
and then writing
\[ h_{n-1}^{-1}\langle n|V'|n \rangle = \frac{\gamma}{N} \frac{P_n(0)^2}{h_n} e^{-\frac{N}{\gamma}V(0)} = \Omega_n, \quad \text{(D.4)} \]
\[ h_{n-1}^{-1}\langle n-1|V'|n \rangle = \frac{\gamma}{N} + \frac{\gamma}{N} \frac{P_{n-1}(0)P_n(0)}{h_{n-1}} e^{-\frac{N}{\gamma}V(0)} \equiv x_n + \tilde{\Omega}_n, \]
where \( \langle m|n \rangle = h_n \delta_{mn} \), and \( x_n = \frac{\gamma n}{N} \). \( \Omega \) and \( \tilde{\Omega} \) are defined in terms of the values of the polynomials at zero
\[ \Omega_n \equiv \frac{\gamma}{N} \frac{P_n(0)}{h_n} e^{-\frac{N}{\gamma}V(0)} , \quad \tilde{\Omega}_n \equiv \frac{\gamma}{N} \frac{P_{n-1}(0)P_n(0)}{h_{n-1}} e^{-\frac{N}{\gamma}V(0)}. \quad \text{(D.5)} \]
They satisfy
\[ \tilde{\Omega}_n + \tilde{\Omega}_{n+1} = -s_n \Omega_n, \]
\[ r_n \Omega_n \Omega_{n-1} = \tilde{\Omega}_n^2. \quad \text{(D.6)} \]

The simplest potential, which arises for the \( k = 1 \) model, is \( V(y) = -y + y^2/2 \) and then we find that (D.4) are
\[ -1 + s_n = \Omega_n, \quad r_n - x_n = \tilde{\Omega}_n. \quad \text{(D.7)} \]

We can eliminate \( s_n \) from this equation and substitute in (D.6) to end up with
\[ r_n - x_n = \tilde{\Omega}_n, \]
\[ \tilde{\Omega}_n + \tilde{\Omega}_{n+1} = -(\Omega_n + 1)\Omega_n, \quad \text{(D.8)} \]
\[ r_n \Omega_n \Omega_{n-1} = \tilde{\Omega}_n^2. \]

When the cut is far from \( y = 0 \), the polynomials are small at the origin; therefore, we find \( \Omega = \tilde{\Omega}_n = 0 \) and recover the standard hermitian matrix model equations.

One can then combine these equations with (D.4) to obtain the string equations. In the simplest model this works as follows. Let us start by considering the planar limit. The equations (D.6) and (D.7) (or equivalently (D.8)) have three solutions on the sphere:

1. \( \Omega = \tilde{\Omega} = 0, s = 1, r = x \).
2. \( \Omega = \frac{1}{3}(-2 + \sqrt{1 + 12x}), \tilde{\Omega} = \frac{1}{18}(1 - 12x + \sqrt{1 + 12x}), s = \frac{1}{3}(1 + \sqrt{1 + 12x}) \) and \( r = \frac{x^2}{4} = \frac{1}{18}(1 + 6x + \sqrt{1 + 12x}) \).
3. \( \Omega = \frac{1}{3}(-2 - \sqrt{1 + 12x}), \tilde{\Omega} = \frac{1}{18}(1 - 12x - \sqrt{1 + 12x}), s = \frac{1}{3}(1 - \sqrt{1 + 12x}) \) and \( r = \frac{x^2}{4} = \frac{1}{18}(1 + 6x - \sqrt{1 + 12x}) \). At this point we need more physical input. It arises from (D.5) and the interpretation of \( \Omega \) in terms of \( P_n(0) \). First, it is clear that
\(\Omega \geq 0\). This rules out the third solution and allows the second solution only for \(x \geq \frac{1}{4}\).

Second, it is clear that we cannot have \(P_n(0) = 0\) for all \(n\). Therefore we cannot take the first solution for all \(x\). We conclude that for \(0 < x \leq \frac{1}{4}\) we should take the first solution and for \(\frac{1}{4} \leq x \leq 1\) the second solution. In terms of the eigenvalues, for \(x < \frac{1}{4}\) they stay away from the origin so that \(\Omega = 0\); but for \(\frac{1}{4} \leq x \leq 1\) they reach the origin so that \(\Omega \neq 0\).

Now, consider the double scaling limit around the transition point \(x = 1/4\),

\[
r = \frac{1}{4} - \epsilon \hat{u}, \quad x = \frac{1}{4} - \epsilon \tilde{z}, \quad \Omega = \epsilon \hat{\Omega}, \quad \tilde{\Omega} = -\epsilon \tilde{\hat{R}}, \quad N = \gamma \epsilon^{-3/2}.
\]

Equations (D.8) become

\[
\tilde{\hat{R}} = \hat{u} - \tilde{z}, \nonumber
\]

\[
2\tilde{\hat{R}} - \epsilon^{1/2} \tilde{\hat{R}}' + \frac{\epsilon}{2} \tilde{\hat{R}}'' + O(\epsilon^{3/2}) = \hat{\Omega} + \epsilon \hat{\Omega}^2, \quad \text{(D.10)}
\]

\[
\left(\frac{1}{4} - \epsilon \hat{u}\right)\hat{\Omega}(\hat{\Omega} + \epsilon^{1/2} \hat{\Omega}' + \frac{\epsilon}{2} \hat{\Omega}'') = \tilde{\hat{R}}^2 + O(\epsilon^{3/2}),
\]

where derivatives are evaluated with respect to \(\tilde{z}\). Solving the second equation for \(\hat{\Omega}\), and substituting into the last one, we find

\[
\frac{1}{2} \tilde{\hat{R}} \tilde{\hat{R}}'' - \frac{1}{4} (\tilde{\hat{R}}')^2 - 4 \tilde{\hat{R}}^2 (\tilde{\hat{R}} + \hat{u}) = 0. \quad \text{(D.11)}
\]

Substituting \(\hat{u} = \tilde{u} + \tilde{z}/2\) we end up with

\[
8\tilde{u} \tilde{\hat{R}}^2 - \frac{1}{2} \tilde{\hat{R}} \tilde{\hat{R}}'' + \frac{1}{4} (\tilde{\hat{R}}')^2 = 0, \nonumber
\]

\[
\tilde{\hat{R}} = \tilde{u} - \tilde{z}/2, \quad \text{(D.12)}
\]

\[
F'' = 4\tilde{u}. \nonumber
\]

### D.2. \(q > 0\)

For \(q > 0\), we start again with the left-hand sides in (D.4) and integrate by parts. The boundary term vanishes, but there is an extra term where the derivative acts on the factor of \(\lambda^q\). We again call them \(\Omega_n, \tilde{\Omega}_n\):

\[
\frac{N}{\gamma q} \Omega_n = h_n^{-1} \int \frac{1}{y} P_n(y)^2 = h_n^{-1} \int \frac{1}{y} (P_n(y) - P_n(0)) P_n(y) + h_n^{-1} P_n(0) \int \frac{1}{y} P_n(y)
\]

\[
= h_n^{-1} P_n(0)^2 \int \frac{1}{y} + P_n(0) P'_{n}(0) h_0 / h_n, \nonumber
\]

\[
\frac{N}{\gamma q} \tilde{\Omega}_n = h_n^{-1} P_n(0) P_{n-1}(0) \int \frac{1}{y} + P_{n-1}(0) P'_n(0) h_0 / h_{n-1}
\]

\[
= h_n^{-1} P_n(0) P_{n-1}(0) \int \frac{1}{y} + P_n(0) P'_{n-1}(0) h_0 / h_{n-1} + 1. \quad \text{(D.13)}
\]

52
Since the measure includes a factor of $y^q$, all these integrals converge. Note that in the formal limit $q \to 0$ we recover the values for $\Omega$ and $\tilde{\Omega}$ of (D.3). Using (D.13) (and the two forms of $\tilde{\Omega}$), we find the generalization of (D.6)

$$\tilde{\Omega}_n + \tilde{\Omega}_{n+1} = -s_n \Omega_n + \frac{\gamma q}{N},$$

$$r_n \Omega_n \Omega_{n-1} = \tilde{\Omega}_n^2 - \frac{q \gamma}{N} \tilde{\Omega}_n.$$  

(D.14)

The corrections due to $q$ are of order $1/N$; therefore, they do not contribute in the planar limit. However, they have to be kept in the double scaling limit. Repeating the derivation of the differential equation (D.11) we find

$$16 \tilde{R}^3 + 16 \tilde{R}^2 \dot{\tilde{u}} + (\tilde{R}')^2 - 2 \tilde{R} \tilde{R}'' = q^2,$$

$$\tilde{R} = \tilde{u} - \tilde{z}.$$  

(D.15)

Substituting $\dot{\tilde{u}} = \tilde{u} + \tilde{z}/2$ we end up with

$$32 \tilde{R}^2 \tilde{u} + (\tilde{R}')^2 - 2 \tilde{R} \tilde{R}'' = q^2,$$

$$\tilde{R} = \tilde{u} - \tilde{z}/2,$$

$$F'' = 4\tilde{u}.$$  

(D.16)

Equations (6.7), (6.4) may be brought to this form by defining

$$u = 2^{5/3} \tilde{u}, \quad z = 2^{2/3} \tilde{z},$$

(D.17)

from which it follows that $R = 2^{5/3} \tilde{R}$. No rescaling of $q$ is needed. This completes the derivation for $k = 1$.

Note that (D.13) also implies that $\Omega \geq 0$, and that it approaches zero as $\tilde{z} \to +\infty$. The equations (D.6) imply that in the scaling limit, where $s_n$ is close to one, $\tilde{\Omega}$ is negative. Then (D.3) implies that $\tilde{R}$ is positive and that $\tilde{R} \to 0$ as $\tilde{z} \to +\infty$. It can also be seen from the second sphere solution above that $\tilde{R} \sim -\tilde{z}/2$ for $\tilde{z} \to -\infty$, which is indeed a property that we get from (6.5) once we impose that $u \to 0$ as $\tilde{z} \to -\infty$.

D.3. The Miura transformation

In this subsection we consider the ZS hierarchy for $g = 0 = \omega$, so that $f = r$. In this case only the $F_l$ in (2.7) with $l$ even, $l = 2k$, are nonzero. These are the terms in the mKdV hierarchy. There is an interesting relation between these $F_{2k}(f, g = 0) = R_{2k}(r, \omega = 0)$
and the Gelfand-Dikii Polynomials $Q_k(u)$ \[5\]. The functions $F_{2k}$ can be defined in terms of matrix elements of the operator

\[
O = \frac{1}{(d_x + f J_1 - \zeta J_3)} = \frac{1}{(d_x + f J_1 - \zeta J_3)(d_x - f J_1 - \zeta J_3)} = (d_x - f J_1 - \zeta J_3) \frac{1}{d_x^2 - f' J_1 - \frac{f^2}{4} - \zeta^2/4}
\]

We can derive a similar relation where we write the numerator on the right. Taking the $\langle x | \cdots | x \rangle$ matrix elements of these operators, extracting the piece proportional to $1$, which should vanish, and the piece proportional to $J_1$, which should equal $-\sum F_{2k} \zeta^{-2k-1}$, we find that

\[
F_{2k} = 2^{2k+1} (d_x + f) Q_k(u-) = -2^{2k+1} (d_x - f) Q_k(u+)
\]

where

\[
u_\pm = \frac{f^2}{4} \pm \frac{f'}{2}
\]

These relations can be viewed as arising from supersymmetric quantum mechanics, where the operator $O$ is the resolvent of the supercharge, and the operator $d_x^2 + u$ is the Hamiltonian.

Let us introduce a derivative $\mathcal{D}_+ = d_x + f$ and assume that $u$ in (6.5) can be written in terms of $f$ through (D.20) as $u = u_-$. Then it is possible to show that (6.5) becomes

\[
q^2 = \frac{f}{2} \mathcal{R} \mathcal{D}_+ \mathcal{R} - \frac{1}{2} \mathcal{R} \mathcal{D}_+^2 \mathcal{R} + \frac{1}{4} (\mathcal{D}_+ \mathcal{R})^2 = -\frac{1}{2} \mathcal{R} d_x (\mathcal{D}_+ \mathcal{R}) + \frac{1}{4} (\mathcal{D}_+ \mathcal{R})^2
\]

Acting with $\mathcal{D}_+$ on the second line in (6.5) we get

\[
\mathcal{D}_+ \mathcal{R} = \sum_{k \geq 0} (2k + 1) \tilde{t}_{2l} F_{2l} - 1, \quad \tilde{t}_{2k} = 2^{-2k-2} t_k
\]

Note that in (6.5) we are implicitly saying that $t_0 = -4x$. This translates into $\tilde{t}_0 = -x$. This $x$ dependence of $t_0$ is the source of the $-1$ in the right hand side of (D.22). These relations were found in [29,33].

In conclusion suppose that we find a solution $f$ that solves (D.22) with $\mathcal{D}_+ \mathcal{R} = 2q$ with $q \geq 0$. In other words, a solution of the equation

\[
2q + 1 = \sum_{k \geq 0} (2k + 1) \tilde{t}_{2k} F_{2k}
\]
Then we will also solve (D.21). In [27] it was argued that smooth solutions exist for the mKdV equation with a zero left hand side. Since (D.23) follows from a lagrangian that is bounded below[21], if the highest nonzero \( \tilde{t}_{2k} \) has the right sign, then it is easy to see that a smooth solution should exist that interpolates between \( f \sim x^{1/2m} \) for large \( x \) and \( f \sim -(2q+1)/x \) for large negative \( x \). Note that if \( 2q+1 = 0 \), then \( f \to -f \) is a symmetry of the problem. On the other hand if \( 2q+1 \) is nonzero the minimum of the action at large \( x \) with positive \( f \) is the one with lowest energy.

This solution is such that \( u \) obeys the physically relevant boundary conditions \( u \to x^{1/m} \) for \( x \to +\infty \) and \( u \to 0 \) for \( x \to -\infty \). We should also impose that \( R \geq 0 \) and that it goes to zero for large \( x \), as we showed above. By integrating the equation \( D_+ R = 2q \) we find that

\[
R = 2q e^{-\int_{x_0}^{x} f} \int_{-\infty}^{x} dy e^{\int_{x_0}^{y} f}
\]

(D.24)

For \( q \geq 0 \) and with \( f \) obeying the above boundary conditions we see that \( R \) has the requisite properties. For \( q = 0 \) it was proven in [28], that a unique solution of (6.5) exists.

We can study the concrete example of \( k = 1 \) in order to see how this works [33,34]. Defining \( u = u_- \) in (D.20), we find

\[
2q + 1 = -\frac{1}{2} f'' + \frac{1}{4} f^3 - xf
\]

(D.25)

We see that this equation comes from an action bounded below and that the lowest minimum at large \( x \) has \( f \sim x^{1/2} \).

Note that if we had imposed instead \( D_+ R = -2|q| \) we would also solve (D.21), but \( R \) would not obey the right conditions.

This relation between (6.5) and the equation (D.23), can also be understood as follows. When we are dealing with a hermitian two-cut model with a symmetric potential, we can set \( g = 0 \) and we can consider symmetric and anti-symmetric polynomials independently. Namely, the free energy can be expressed as a sum \( F = F_+ + F_- \) where \( F_\pm \) represents the contribution to the integral from even and odd polynomials. Then we have that

\[
F''_\pm = \frac{f^2}{8} \mp \frac{f'}{4},
\]

where \( f \) obeys the same equation for both cases. Of course, \( F'' = f^2/4 \) is the equation for the free energy for the two cut model. In the hermitian model we can add a logarithmic potential of the form \( M \log |\lambda| \) [59], which introduces a factor of \( |\lambda|^M \) in

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21 We have checked up to \( k = 3 \) that all the terms with derivatives in \( \int H_{2k+1}(f,g = 0) \) are positive definite. We think it is true in general.
the measure that defines the orthogonal polynomials. The resulting equation for \( f \) is the same as in (2.20) but with a constant proportional to \( M \) added to the right hand side (remember that we have set \( g = 0 \)). Let us consider now the complex matrix model, and look at the first form of the integral in (6.2). Then we can take the even polynomials \( P_{2l} \) of the hermitian matrix model defined with the measure \( d\mu = \lambda^{1+2q} e^{-N V(\lambda^2)} \). These polynomials are functions of \( \lambda^2 \) and we can rewrite the Vandermonde determinant that appears in the first line of (6.2) in terms of them. This will lead to an expression for the partition function of the complex matrix model in terms of the norms of the even polynomials of a two-cut model with a logarithmic potential with coefficient \( M = 1 + 2q \). This last problem is precisely the one solved by the equation (D.25), with free energy given by \( F'' = f^2/8 - f'/4 \).

**Appendix E. The loop equation of the complex matrix model**

For \( M \) a complex rectangular \( N \times (N + q) \) matrix, with \( q \) positive, there are two closely related resolvents

\[
R = \frac{1}{N} \text{Tr} \frac{1}{M^\dagger M - z}, \quad \tilde{R} = \frac{1}{N} \text{Tr} \frac{1}{MM^\dagger - z} = R - \frac{q}{Nz}. \tag{E.1}
\]

Let us derive their loop equations. We compute

\[
\int dMdM^\dagger \text{Tr} \left( \frac{\partial}{\partial M^\dagger} \frac{1}{M^\dagger M - z} \right) e^{-N \text{Tr} V(M^\dagger M)} = 0 \tag{E.2}
\]

which vanishes because it is a total derivative.\(^{23}\) This reflects the invariance of the integral under the change of variables \( \delta M = M - \frac{1}{M^\dagger M - z} \). It is straightforward to calculate (E.2)

\[
\begin{align*}
- N \left< \text{Tr} \frac{M^\dagger M V'(M^\dagger M)}{M^\dagger M - z} \right> + (N + q) \left< \text{Tr} \frac{1}{M^\dagger M - z} \right> - \left< \text{Tr} \frac{M^\dagger M}{M^\dagger M - z} \text{Tr} \frac{1}{M^\dagger M - z} \right> \\
= - N \left< \text{Tr} \frac{M^\dagger M V'(M^\dagger M) - z V'(z)}{M^\dagger M - z} \right> - Nz V'(z) \left< \text{Tr} \frac{1}{M^\dagger M - z} \right> \\
+ q \left< \text{Tr} \frac{1}{M^\dagger M - z} \right> - z \left< \text{Tr} \frac{1}{(M^\dagger M - z)^2} \right> \\
= - zN^2 \left( \left< R(z)^2 \right> + \left( V'(z) - \frac{q}{Nz} \right) \left< R(z) \right> - \frac{f(z)}{4z} \right)
\end{align*}\]

\(^{22}\) Normalizations in [59] are different, \( f_{\text{Minahan}} = f_{\text{here}}/2 \).

\(^{23}\) Compared to (1.1), we absorb \( 1/\gamma \) into the definition of \( V(M^\dagger M) \).
with \( f(z) = -\frac{4}{N} \left\langle \operatorname{Tr} \frac{M^†M V'(M^†M) - z V'(z)}{M^†M - z} \right\rangle \) a polynomial of the same degree as \( V'(z) \).

In the large \( N \) limit the term \( \langle R(z)^2 \rangle \) factorizes. We do not neglect the term proportional to \( \hat{q} = \frac{q}{N} \) because we allow for the possibility that \( q \) is going to infinity with finite \( \hat{q} \). We derive the loop equation

\[
R(z)^2 + \left( V'(z) - \frac{\hat{q}}{z} \right) R(z) - \frac{f(z)}{4z} = 0 \tag{E.4}
\]

where we denoted the expectation value of the operator \( R \) by \( \bar{R} \). As a check, we can substitute \( R = \tilde{R} + \frac{\hat{q}}{z} \) and derive the loop equation for \( \tilde{R} \)

\[
\tilde{R}(z)^2 + \left( V'(z) + \frac{\hat{q}}{z} \right) \tilde{R}(z) - \frac{f(z) - 4\hat{q}V'(z)}{4z} = 0 \tag{E.5}
\]

i.e. exactly the same as (E.4) but with \( R \to \tilde{R}, q \to -\hat{q} \), and the transformation of \( f(z) \) which follows from \( M^†M \to MM^† \).

The solution of (E.4) is

\[
2R(z) = - \left( V'(z) + \frac{\hat{q}}{z} \right) \pm \sqrt{\left( V'(z) + \frac{\hat{q}}{z} \right)^2 + \frac{f(z)}{z}} \tag{E.6}
\]

We see that the parameter \( z \) in \( R(z) \) takes values in a two fold cover of the complex plane which is the Riemann surface

\[
y^2 = \left( V'(z) + \frac{\hat{q}}{z} \right)^2 + \frac{f(z)}{z} \tag{E.7}
\]

or equivalently in terms of \( \hat{y} = yz \)

\[
\hat{y}^2 = (zV'(z) + \hat{q})^2 + zf(z). \tag{E.8}
\]

As \( z \to \infty \) on the upper sheet, we have \( R \to -\frac{1}{z} \). This determines

\[
2R(P_\pm(z)) = - \left( V'(z) + \frac{\hat{q}}{z} \right) \pm \sqrt{\left( V'(z) + \frac{\hat{q}}{z} \right)^2 + \frac{f(z)}{z}} \tag{E.9}
\]

where \( P_\pm(z) \) denote the points on the upper and lower sheets which correspond to \( z \). Finally, note that \( R \) has a pole on the lower sheet \( R \approx -\frac{\hat{q}}{P_-(z)} \). The other resolvent has a pole in the upper sheet \( \tilde{R} \approx \frac{\hat{q}}{P_+(z)} \).
Let us consider the simplest model with $V(z) = \frac{1}{4}(-z + z^2/2)$. The polynomial $\hat{y}^2$ in (E.8) is of fourth order, describing a genus one surface. The polynomial $f(z)$ is $f = -4z/\gamma + f_0$. If we are interested in a one-cut model we impose that the polynomial has a double zero. This determines the constant $f_0$. The most general such polynomial is

$$\hat{y}^2 = \frac{1}{\gamma^2}(z - a)^2 [z^2 + 2z(a - 1) + (1 - 4a + 3a^2 - 4\gamma - 2\gamma\hat{q})]$$

(E.10)

where $a$ obeys the equation

$$4a^3 - 3a^4 + \gamma^2\hat{q}^2 + a^2(4\gamma - 1 + 2\gamma\hat{q}) = 0 .$$

(E.11)

We are interested in the double scaling limit, where one of the ends of the cut approaches $z \sim 0$. Let us first set $\hat{q} = 0$. From (E.11) we see that $a = 0$ is always a solution and leads to a cut extending from $1 - \sqrt{4\gamma}$ to $1 + \sqrt{4\gamma}$. This solution makes sense as long as $4\gamma < 1$, otherwise the cut would extend to negative values of $z$ which are not allowed physically.

Thus, for $4\gamma > 1$ the solution must have $a \neq 0$. We write $4\gamma = 1 - \delta$ and take the limit of small negative $\delta$ and small $a$. From (E.11) with $\hat{q} = 0$, we see that we get a solution with $4a = \delta$. For $\hat{q}$ nonzero we see that we need to scale $\hat{q}$ as $|\delta|^{3/2}$. Then we find that (E.11) simplifies to

$$4a^3 + \frac{\hat{q}^2}{16} - \delta a^2 = 0 ,$$

(E.12)

and the second derivative of the free energy is proportional to the position $z$ of the cut closest to the origin, which is given by $u = -4a + \delta$, in the scaling limit. After we scale $-a \sim \hat{q}^{2/3}v$, and $-\delta \sim \hat{q}^{2/3}t$, then (E.12) reduces to (4.13).

To summarize, for $\hat{q} = 0$ we have a phase transition between the regime with $a = 0$ and $a \neq 0$. In one phase $\hat{y}^2$ has a double zero at $z = 0$, and in the other phase the double zero is at negative $z$ while the cut reaches $z = 0$. For $\hat{q} \neq 0$ there is no such transition, and the cut does not reach $z = 0$.

Let us now discuss the branes of 0A theory. The FZZT branes with $\eta = \pm 1$ in the 0A are the same as the FZZT branes with $\eta = \mp 1$ in the 0B theory (at $\hat{c} = 0$). We think that the resolvent (E.1) corresponds to the FZZT brane with $\eta = -1^{\text{24}}$ and $\mu_B^2 = -z$. Indeed we see that the disk diagram has the expected one cut when expressed in terms of $z$ (3.20). On the other hand, for negative $\mu$, this brane will have an expectation value similar to the FZZT brane with $\eta = 1$ at positive $\mu$ (3.27), which in terms of $z$ has a cut at the origin as expected.

\textsuperscript{24} Or the FZZT with $\eta = +1$ in 0B.
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