A Global Compact Result for a Fractional Elliptic Problem with Hardy Term and Critical Sobolev Non-linearity on the Whole Space

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Abstract

In this paper, we deal with a fractional elliptic equation with critical Sobolev non-linearity and Hardy term

\[
\begin{cases}
(-\Delta)^\alpha u - \mu \frac{u}{|x|^{2\alpha}} + a(x)u = |u|^{2^*-2}u + k(x)|u|^{q-2}u, \\
u \in H^\alpha(\mathbb{R}^N),
\end{cases}
\]

where \(2 < q < 2^*, 0 < \alpha < 1, N > 4\alpha, 2^* = 2N/(N - 2\alpha)\) is the critical Sobolev exponent, \(a(x), k(x) \in C(\mathbb{R}^N)\). Through a compactness analysis of the functional associated to (\star), we obtain the existence of positive solutions for (\star) under certain assumptions on \(a(x), k(x)\).

Key words. Fractional Laplacian, compactness, positive solution, unbounded domain, Hardy term, critical Sobolev nonlinearity.

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1 Introduction

We consider the following nonlinear elliptic equation:

\[
\begin{cases}
(-\Delta)^\alpha u - \mu \frac{u}{|x|^{2\alpha}} + a(x)u = |u|^{2^*-2}u + k(x)|u|^{q-2}u, \\
u \in H^\alpha(\mathbb{R}^N),
\end{cases}
\]

where \(2 < q < 2^*, 0 < \alpha < 1, N > 4\alpha, 2^* = 2N/(N - 2\alpha)\) is the critical Sobolev exponent, \(a(x), k(x) \in C(\mathbb{R}^N)\).
Recently the fractional Laplacian and more general nonlocal operators of elliptic type have been widely studied, both for their interesting theoretical structure and concrete applications in many fields such as optimization, finance, phase transitions, stratified materials, anomalous diffusion and so on (see [2, 5, 7, 8, 14, 21, 23, 24]). In particular, a lot of results have been accumulated for elliptic equations with critical nonlinearity related to (1.1). In [5], Dipierro et al. considered the critical problem with Hardy-Leray potential

\[ (-\Delta)^\alpha u - \mu \frac{u}{|x|^{2\alpha}} = |u|^{2^*-2}u, \quad x \in \mathbb{R}^N, \]

(1.2)

where \( \dot{H}^\alpha(\mathbb{R}^N) \) is defined in (1.6). They proved the existence, certain qualitative properties and asymptotic behavior of positive solutions to (1.2). Ghoussoub and Shakerian in [9] investigated the following double critical problem in \( \mathbb{R}^N \)

\[
\begin{cases}
(-\Delta)^\alpha u - \mu \frac{u}{|x|^{2\alpha}} = \left|u\right|^{2^*-2}u + \left|u\right|^{2^*-2}u, \quad x \in \mathbb{R}^N, \\
u > 0, \ u \in \dot{H}^\alpha(\mathbb{R}^N),
\end{cases}
\]

(1.3)

with \( \mu > 0, 0 < s < 2 \). Through the non-compactness analysis of the Palais-Smale sequence of (1.3), the existence of the solutions were obtained. The authors in [11] established a concentration-compactness result for a fractional Schrödinger equation with the subcritical nonlinearity \( f(x, u) \). Motivated by [5, 9, 11, 12, 27] we consider the existence of positive solutions for problem (1.1) in \( \mathbb{R}^N \). The main interest for this type of problems, in addition to the nonlocal fractional Laplacian is the presence of the singular potential \( \frac{1}{|x|^{2\alpha}} \) related to the fractional Hardy’s inequality. We recall the Hardy inequality ([5]),

\[ \Gamma_{N,\alpha} \left( \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2\alpha}} \, dx \right) \leq c_{N,\alpha} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\alpha}} \, dy, \forall u \in C_0^\infty(\mathbb{R}^N), \]

(1.4)

where

\[ \Gamma_{N,\alpha} = 2^{2\alpha} \Gamma^2 \left( \frac{N + 2\alpha}{4} \right) / \Gamma \left( \frac{N - 2\alpha}{4} \right), \quad c_{N,\alpha} = 2^{2\alpha - 1} \pi^{-\frac{N}{2}} \frac{\Gamma \left( \frac{N + 2\alpha}{2} \right)}{\Gamma \left( -\alpha \right)} \]  

(1.5)

The Sobolev embedding \( \dot{H}^\alpha(\mathbb{R}^N) \hookrightarrow L^2(|x|^{-2\alpha}, \mathbb{R}^N) \) is not compact, even locally, in any neighborhood of zero. As it is well known, the loss of the compactness of the embeddings is one of the main difficulties for elliptic problems with critical nonlinearities. Problem (1.1) has three factors, critical Sobolev term, Hardy term and unbounded domain which lead to the non-compactness of the embeddings. In [5] and [9], the authors can consider the solutions of critical problems in the homogeneous fractional Sobolev space \( \dot{H}^\alpha(\mathbb{R}^N) \), while we must deal with (1.1) in the nonhomogeneous fractional Sobolev space \( H^\alpha(\mathbb{R}^N) \) given the presence of low sub-critical terms in (1.1). This is why the methods in [5] and [9] can not be used directly to (1.1). As far as we know, the existence results for the fractional elliptic problems with a mixture of critical Sobolev terms, Hardy term and subcritical terms are relatively new. To
overcome the difficulties caused by the lack of compactness, we carry out a non-compactness analysis which can distinctly express all the parts which cause non-compactness. As a result, we are able to obtain the existence of nontrivial solutions of the elliptic problem with the critical nonlinear term on an unbounded domain by getting rid of these noncompact factors. To be more specific, for the Palais-Smale sequences of the variational functional corresponding to (1.1) we first establish a complete noncompact expression which includes all the blowing up bubbles caused by the critical Sobolev nonlinearity, the Hardy term and by the unbounded domain. Then we derive the existence of positive solutions for (1.1). Our methods are based on some techniques of \[4, 11, 13, 16, 19, 20, 25, 26\].

Before introducing our main results, we give some notations and assumptions.

**Notations and assumptions:**

Denote \(c\) and \(C\) as arbitrary constants which may change from line to line. Let 
\[B(x,r) = \mathbb{R}^N \setminus B(x,r)\]
\[B(x,r) C = B(x,r) \cap \mathbb{R}^N\]
\[\mathbb{R}^N \setminus B(x,r)\]

Let \(u \in L^2(\mathbb{R}^N)\), let the Fourier transform of \(u\) be
\[
\hat{u}(\xi) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-i\xi \cdot x} u(x) dx.
\]

We define the operator \((-\Delta)^\alpha u\) by the Fourier transform
\[
(-\Delta)^\alpha u(\xi) = |\xi|^{2\alpha} \hat{u}(\xi), \quad \forall u \in C^\infty_0(\mathbb{R}^N).
\]

Let \(\dot{H}^\alpha(\mathbb{R}^N)\) be the homogeneous fractional Sobolev space as the completion of \(C^\infty_0(\mathbb{R}^N)\) under the norm
\[
\|u\|_{\dot{H}^\alpha(\mathbb{R}^N)} = \|\xi|^{\alpha} \hat{u}\|_{L^2(\mathbb{R}^N)},
\]
and denote by \(H^\alpha(\mathbb{R}^N)\) the usual nonhomogeneous fractional Sobolev space with the norm
\[
\|u\|_{H^\alpha(\mathbb{R}^N)} = \|u\|_{L^2(\mathbb{R}^N)} + \|\xi|^{\alpha} \hat{u}\|_{L^2(\mathbb{R}^N)}.
\]

For \(0 < \alpha < 1\), a direct calculation (see e.g. \([14\), proposition 4.4\] or \([5\), Proposition 1.2\]) gives
\[
c_{N,\alpha} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2\alpha}} dx dy = \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2} u(x)|^2 dx = \|u\|_{H^\alpha(\mathbb{R}^N)}^2.
\]

where \(c_{N,\alpha} = 2^{2\alpha-1}\pi^{-N/2}\Gamma(N+2\alpha)/(\Gamma(-\alpha))\).

Let \(u^+ = \max\{u, 0\}\), \(u^- = u^+-u\). From the proof of \((2.15)\) in \([15]\), it follows
\[
\|u^+\|_{H^\alpha} \leq \|u\|_{H^\alpha}.
\]

We call \(u \neq 0\) in \(\mathbb{R}^N\) if the measure of the set \(\{x \in \mathbb{R}^N | u(x) \neq 0\}\) is positive.
Recall the definition of Morrey space. A measurable function $u : \mathbb{R}^N \to \mathbb{R}$ belongs to the Morrey space with $p \in [1, \infty)$ and $\nu \in (0, N]$, if and only if
\[
\|u\|_{L^p,\nu}(\mathbb{R}^N) = \sup_{r>0, \bar{x} \in \mathbb{R}^N} r^{-\nu} \int_{B(\bar{x}, r)} |u(x)|^p dx < \infty.
\]
By Hölder inequality, we can verify (refer to [14])
\[
L^{2^*}(\mathbb{R}^N) \hookrightarrow L^{p,\nu}(\mathbb{R}^N), \quad \text{for } 1 \leq p < 2^*, \quad (1.9)
\]
and
\[
L^{p_1,\nu}(\mathbb{R}^N) \hookrightarrow L^{p_1,\nu}(\mathbb{R}^N), \quad \text{for } 1 < p_1 < 2^*. \quad (1.10)
\]
Moreover, we have $L^{p,\nu}(\mathbb{R}^N) \hookrightarrow L^{1,\nu}(\mathbb{R}^N)$. 

Next we give the definition of the Palais-Smale sequence. Let $X$ be a Banach space, $\Phi \in C^1(X, \mathbb{R})$, $c \in \mathbb{R}$, we call $\{u_n\} \subset X$ is a Palais-Smale sequence of $\Phi$ if
\[
\Phi(u_n) \to c, \quad \Phi'(u_n) \to 0 \quad \text{as } n \to \infty. \quad (1.11)
\]
In this paper we assume that:

(a) $a(x) \in C(\mathbb{R}^N)$, $k(x) \in C(\mathbb{R}^N)$;

(b) $\lim_{|x| \to \infty} a(x) = \bar{a} > 0$, $\lim_{|x| \to \infty} k(x) = \bar{k} > 0$, $\inf_{x \in \mathbb{R}^N} a(x) = \hat{a} > 0$, $\inf_{x \in \mathbb{R}^N} k(x) = \hat{k} > 0$.

In the following, we assume that $a(x), k(x)$ always satisfy (a) and (b). The energy functional associated with (1.1) is for all $u \in H^\alpha(\mathbb{R}^N)$,
\[
I(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left(|(-\Delta)^{\alpha/2} u(x)|^2 - \mu \frac{u^2}{|x|^{2\alpha}} + a(x) |u(x)|^2 \right) dx
- \frac{1}{2^*} \int_{\mathbb{R}^N} (u^+(x))^{2^*} dx - \frac{1}{q} \int_{\mathbb{R}^N} k(x)(u^+(x))^q dx.
\]
Finally we present some problems associated to (1.1) as follows.

The limit equation of (1.1) involving subcritical and critical terms is
\[
\begin{cases}
(-\Delta)^\alpha u + \bar{a} u = \bar{k} |u|^{q-2} u + |u|^{2^*-2} u, \\
u \in H^\alpha(\mathbb{R}^N),
\end{cases} \quad (1.12)
\]
and its corresponding variational functional is
\[
I^\infty(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left(|(-\Delta)^{\alpha/2} u(x)|^2 + \bar{a} |u(x)|^2 \right) dx
- \frac{1}{2^*} \int_{\mathbb{R}^N} (u^+(x))^{2^*} dx, \quad u \in H^\alpha(\mathbb{R}^N).
\]
The limit equation of (1.1) involving the Hardy term and critical Sobolev nonlinearity is
\[
\begin{aligned}
&\left\{ (-\Delta)^{\alpha} u - \mu \frac{u}{|x|^{2\alpha}} = |u|^{2^*-2} u, \\
&u \in \dot{H}^\alpha(\mathbb{R}^N),
\right.
\end{aligned}
\tag{1.13}
\]
and the corresponding variational functional is
\[
I_{\mu}(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left| (-\Delta)^{\alpha/2} u(x) \right|^2 - \mu \frac{u^2}{|x|^{2\alpha}} \, dx - \frac{1}{2^*} \int_{\mathbb{R}^N} (u^+(x))^{2^*} \, dx, \quad u \in \dot{H}^\alpha(\mathbb{R}^N).
\]
The limit equation of (1.1) involving critical Sobolev nonlinearity is
\[
\begin{aligned}
&\left\{ (-\Delta)^{\alpha} u = |u|^{2^*-2} u, \\
&u \in \dot{H}^\alpha(\mathbb{R}^N),
\right.
\end{aligned}
\tag{1.14}
\]
and the corresponding variational functional is
\[
I_0(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left| (-\Delta)^{\alpha/2} u(x) \right|^2 \, dx - \frac{1}{2^*} \int_{\mathbb{R}^N} (u^+(x))^{2^*} \, dx, \quad u \in \dot{H}^\alpha(\mathbb{R}^N).
\]
Define
\[
S_{\alpha,\mu} = \inf_{u \in H^\alpha(\mathbb{R}^N) \setminus \{0\}} \frac{c_{N,\alpha} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\alpha}} \, dx \, dy - \mu \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2\alpha}} \, dx}{(\int_{\mathbb{R}^N} |u(x)|^{2^*} \, dx)^{2/2^*}}, \tag{1.15}
\]
the Euler equation associated to (1.15) is (1.13). In particular it has been showed in Theorem 1.2 of [5] that for any positive solution $U_{\mu}(x) \in H^\alpha(\mathbb{R}^N)$ of (1.13), there exist two positive constants $c, C$ such that
\[
\frac{c}{(|x|^{1-\eta_{\mu}}(1 + |x|^{2\eta_{\mu}}))^{N-2\alpha}} \leq U_{\mu}(x) \leq \frac{C}{(|x|^{1-\eta_{\mu}}(1 + |x|^{2\eta_{\mu}}))^{N-2\alpha}}, \quad \text{in } \mathbb{R}^N \setminus \{0\} \tag{1.16}
\]
where
\[
\eta_{\mu} = 1 - \frac{2\alpha_{\mu}}{N-2\alpha}, \tag{1.17}
\]
and $\alpha_{\mu} \in (0, \frac{N-2\alpha}{2})$ is a suitable parameter whose explicit value will be determined as the unique solution to the following equation
\[
\varphi_{s,N}(\alpha_{\mu}) = 2^{2\alpha} \frac{\Gamma(\frac{\alpha_{\mu} + 2\alpha}{2})\Gamma(\frac{N-\alpha_{\mu}}{2})}{\Gamma(\frac{N-\alpha_{\mu} - 2\alpha}{2})\Gamma(\frac{\alpha_{\mu}}{2})} = \mu, \tag{1.18}
\]
and $\varphi_{\alpha,N}$ is strictly increasing. That is
\[
\alpha_{\mu} = \varphi_{\alpha,N}^{-1}(\mu).
All the positive solutions of (1.13) are of the form

$$U_{\mu}^{\varepsilon}(x) := \varepsilon^{2\alpha-N/2} U_{\mu}(x/\varepsilon).$$  \hspace{1cm} (1.19)

In particular, for $\mu = 0$, it follows that (refer to [6])

$$U_{0}(x) = \frac{C}{1 + |x|^{N-2\alpha}},$$  \hspace{1cm} (1.20)

where $C > 0$ is a constant. These solutions $U_{\mu}^{\varepsilon,y} := \varepsilon^{2\alpha-N/2} U_{\mu}(x-y/\varepsilon)$ are also minimizers for the quotient

$$S_{\alpha} = \inf_{u \in H^{\alpha}(\mathbb{R}^{N})\setminus\{0\}} \frac{\int_{\mathbb{R}^{N}} |(-\Delta)^{\alpha/2} u(x)|^{2} dx}{\left(\int_{\mathbb{R}^{N}} |u(x)|^{2^{*}} dx\right)^{2/2^{*}}}.$$  

Define

$$D_{\mu} = \int_{\mathbb{R}^{N}} \left(\frac{1}{2}|(-\Delta)^{\alpha/2} U_{\mu}(x)|^{2} - \frac{1}{2} \frac{U_{\mu}(x)^{2}}{|x|^{\alpha}} - \frac{1}{2} |U_{\mu}(x)|^{2^{*}}\right) dx = \frac{\alpha}{N} S_{\alpha}^{2\alpha},$$  \hspace{1cm} (1.21)

$$D_{0} = \int_{\mathbb{R}^{N}} \left(\frac{1}{2}|(-\Delta)^{\alpha/2} U_{0}|^{2} - \frac{1}{2} |U_{0}|^{2}\right) dx = \frac{\alpha}{N} S_{\alpha}^{2\alpha},$$  \hspace{1cm} (1.22)

$$N = \{u \in H^{\alpha}(\mathbb{R}^{N}) \setminus \{0\} \mid \int_{\mathbb{R}^{N}} \left(|(-\Delta)^{\alpha/2} u(x)|^{2} + \tilde{k} |u(x)|^{2} - \tilde{k} (u(x))^{q}\right) dx = 0\},$$  \hspace{1cm} (1.23)

$$J^{\infty} = \inf_{u \in N} I^{\infty}(u).$$  \hspace{1cm} (1.24)

It is known that $N \neq \emptyset$ since problem (1.12) has at least one positive solution if $N > 2\alpha$ (see Theorem 1.3 in [28] for $2 < q < 2^{*}$ and $\tilde{k} > \lambda^{*}(\lambda^{*} > 0$ is a positive constant defined in [28]).

The main result of our paper is as follows:

**Theorem 1.1.** Suppose $a(x), k(x)$ satisfy (a) and (b), $\tilde{k} > \lambda^{*}, 2 < q < 2^{*}, 0 < \alpha < 1, N > 4\alpha, 0 < \mu < \phi_{\alpha,N}(\frac{N-4\alpha}{2})$. Assume that $\{u_{n}\}$ is a positive Palais-Smale sequence of $I$ at level $d \geq 0$, then there exist sequences $\{y_{n}^{k}\} \subset \mathbb{R}^{N}$ ($1 \leq k \leq l_{1}$), $\{R_{n}^{j}\} \subset \mathbb{R}^{+}$ ($1 \leq j \leq l_{2}$), $\{x_{i}\} \subset \mathbb{R}^{+}, x_{i}^{n} \subset \mathbb{R}^{N}$ ($1 \leq i \leq l_{3}$) and $u_{k} \in H^{\alpha}(\mathbb{R}^{N})$ ($1 \leq k \leq l_{1}(l_{1},l_{2},l_{3} \in \mathbb{N})$, $u \in H^{\alpha}(\mathbb{R}^{N})$, such that up to a subsequence:

$$d = I(u) + \sum_{j=1}^{l_{1}} I^{\infty}(u_{k}) + l_{2} D_{\mu} + l_{3} D_{0};$$

$$\|u_{n} - u - \sum_{k=1}^{l_{1}} u_{k}(x - y_{n}^{k}) - \sum_{j=1}^{l_{2}} U_{j}^{i} - \sum_{i=1}^{l_{3}} U_{0}^{R_{n},x_{i}^{n}}\|_{H^{\alpha}(\mathbb{R}^{N})} = o(1) \text{ as } n \to \infty,$$  \hspace{1cm} (1.25)
where \( u \) and \( u_k (1 \leq k \leq l_1) \) satisfy
\[
I'(u) = 0, \quad I^{\infty'}(u_k) = 0
\]
and
\[
|y^k_n| \to \infty (1 \leq k \leq l_1), \quad R^j_n \to 0 (1 \leq j \leq l_2), \quad \bar{R}^i_n \to 0, \quad \frac{|x^i_n|}{\bar{R}^i_n} \to \infty (1 \leq i \leq l_3), \quad \text{as } n \to \infty.
\]
In particular, if \( u \neq 0 \), then \( u \) is a weakly solution of (1.1). Note that the corresponding sum in (1.25) will be treated as zero if \( l_i = 0 (i = 1, 2, 3) \).

**Remarks:**

1) Similar as Corollary 3.3 in [19], one can show that any Palais-Smale sequence for \( I \) at a level which is not of the form \( m_1D_0 + m_2D_\mu + m_3J^\infty \), \( m_1, m_2 \in \mathbb{N} \cup \{0\} \), gives rise to a non-trivial weak solution of equation (1.1).

2) In our non-compactness analysis, we prove that the blowing up positive Palais-Smale sequences can bear exactly three kinds of bubbles. Up to harmless constants, they are either of the form
\[
U_{\mu}^{R_n}(x), \quad |R_n| \to 0 \text{ as } n \to \infty,
\]
or
\[
U_0^{R_n, x_n}(x), \quad |\bar{R}_n| \to 0, \quad \frac{|x_n|}{\bar{R}_n} \to \infty \text{ as } n \to \infty,
\]
or
\[
u(x - y_n) \in H^\alpha(\mathbb{R}^N), \quad |y_n| \to \infty, \quad \text{as } n \to \infty,
\]
where \( \nu \) is the solution of (1.12). For any Palais-Smale sequence \( u_n \) for \( I \), ruling out the above two bubbles yields the existence of a non-trivial weak solution of equation (1.1).

Using the compactness results and the Mountain Pass Theorem [1] we prove the following existence result.

**Theorem 1.2.** Assume that \( 2 < q < 2^* \), \( 0 < \alpha < 1 \), \( N > 4\alpha \), \( 0 < \mu < \phi_{\alpha,N}(\frac{N-4\alpha}{2}) \). If \( a(x), k(x) \) satisfy (a), (b) and
\[
\bar{a} \geq a(x), \quad k(x) \geq \bar{k} > 0, \quad k(x) \neq \bar{k} \quad (1.26)
\]
Then (1.1) has a nontrivial solution \( u \in H^\alpha(\mathbb{R}^N) \) which satisfies
\[
I(u) < \min\{\frac{\alpha}{N} \int_{\mathbb{R}^N} |\nabla u|^q, J^\infty\}.
\]

This paper is organized as follows. In Section 2, we prove Theorem 1.1 by carefully analyzing the features of a positive Palais-Smale sequence for \( I \). Theorem 1.2 is proved in Section 3 by applying Theorem 1.1 and the Mountain Pass Theorem. Finally we put some preliminaries in the last section as an appendix.
2 Non-compactness analysis

In this section, we prove Theorem 1.1 by using the Concentration-Compactness Principle and a delicate analysis of the Palais-Smale sequences of $I$. Firstly we give the following Lemmas.

**Lemma 2.1.** Let $0 < \alpha < N/2$, $\{u_n\} \subset \dot{H}^\alpha(\mathbb{R}^N)$ be a bounded sequence such that

\[
\inf_{n \in \mathbb{N}} \int_{\mathbb{R}^N} (u_n^+(x))^2\,dx \geq c > 0. \tag{2.1}
\]

Then, up to subsequence, there exist two sequences $\{r_n\} \subset \mathbb{R}^+$ and $\{x_n\} \subset \mathbb{R}^N$ such that

\[
\bar{u}_n \rightharpoonup w \text{ in } \dot{H}^\alpha(\mathbb{R}^N) \text{ with } w \not\equiv 0, \tag{2.2}
\]

where

\[
\bar{u}_n(x) = \begin{cases} 
\frac{N-2\alpha}{r_n^2} u_n(r_n x) & \text{when } \frac{x_n}{r_n} \text{ is bounded,} \\
\frac{2}{2N-2\alpha} u_n(r_n x + x_n) & \text{when } |\frac{x_n}{r_n}| \to \infty.
\end{cases} \tag{2.3}
\]

**Proof.** By Theorem 1 in [16],

\[
\left( \int_{\mathbb{R}^N} |u_n(x)|^2\,dx \right)^{1/2^*} \leq C \|u_n\|_{\dot{H}^\alpha(\mathbb{R}^N)}^{\theta} \|u_n\|_{L^{2,N-2\alpha}(\mathbb{R}^N)}^{1-\theta}, \tag{2.4}
\]

where $\frac{N-2\alpha}{N} \leq \theta < 1$.

Then there exists a constant $c > 0$ such that

\[
\|u_n\|_{L^{2,N-2\alpha}(\mathbb{R}^N)}^2 = \sup_{\bar{x} \in \mathbb{R}^N, R \in \mathbb{R}^+} R^{-2\alpha} \int_{B(\bar{x}, R)} |u_n(x)|^2\,dx \geq c > 0. \tag{2.5}
\]

From (2.5), we may find $r_n > 0$ and $x_n \in \mathbb{R}^N$ such that for $n$ large enough,

\[
r_n^{-2\alpha} \int_{B(x_n,r_n)} |u_n(x)|^2\,dx \geq \|u_n\|_{L^{2,N-2\alpha}(\mathbb{R}^N)}^2 - \frac{c}{2^* n} \geq c/2 > 0. \tag{2.6}
\]

Denote

\[
\bar{u}_n(x) = \begin{cases} 
\frac{N-2\alpha}{r_n^2} u_n(r_n x) & \text{when } \frac{x_n}{r_n} \text{ is bounded,} \\
\frac{2}{2N-2\alpha} u_n(r_n x + x_n) & \text{when } |\frac{x_n}{r_n}| \to \infty.
\end{cases} \tag{2.7}
\]

Since $\{u_n\}$ is bounded in $\dot{H}^\alpha(\mathbb{R}^N)$, from the scaling and translation invariance of $\dot{H}^\alpha(\mathbb{R}^N)$, then $\{\bar{u}_n\}$ is bounded in $\dot{H}^\alpha(\mathbb{R}^N)$, therefore, up to a subsequence (still denoted by $\bar{u}_n$),

\[
\bar{u}_n \rightharpoonup w \text{ in } \dot{H}^\alpha(\mathbb{R}^N) \text{ and } \bar{u}_n \to w \text{ in } L^2_{\text{loc}}(\mathbb{R}^N), \text{ as } n \to \infty.
\]
If $\frac{x_n}{r_n}$ is bounded, there exists a $\tilde{R} > 1$ such that $B(\frac{x_n}{r_n}, 1) \subset B(0, \tilde{R})$, then
\[ c/2 < \int_{B(\frac{x_n}{r_n}, 1)} |\bar{u}_n(x)|^2 dx \leq \int_{B(0, \tilde{R})} |\bar{u}_n(x)|^2 dx \to \int_{B(0, \tilde{R})} |w(x)|^2 dx. \] (2.8)

If $|\frac{x_n}{r_n}| \to \infty$, then
\[ c/2 < \int_{B(0, 1)} |\bar{u}_n(x)|^2 dx \leq \int_{B(0, \tilde{R})} |\bar{u}_n(x)|^2 dx \to \int_{B(0, \tilde{R})} |w(x)|^2 dx \] (2.9)

where $\tilde{R} > 1$. Obviously we have $w \not\equiv 0$. From (2.8) and (2.9), Lemma 2.1 is complete. \(\square\)

**Lemma 2.2.** Assume $N > 4\alpha, 2 < q < 2^*, 0 < \alpha < 1$. Let \(\{v_n\} \subset H^\alpha(\mathbb{R}^N)\) be a Palais-Smale sequence of \(I\) at level $d_1$ and $v_n \to 0$ in $H^\alpha(\mathbb{R}^N)$ as $n \to \infty$. If there exists two sequence \(\{r_n\} \subset \mathbb{R}^+\) and \(\{x_n\} \subset \mathbb{R}^N\) with $r_n \to 0$, $|\frac{x_n}{r_n}| \to \infty$ as $n \to \infty$ such that $\bar{v}_n(x) := \frac{r_n^2}{r_n^2} v_n(r_n x + x_n)$ converges weakly in $\tilde{H}^\alpha(\mathbb{R}^N)$ and almost everywhere to some $v_0 \in \tilde{H}^\alpha(\mathbb{R}^N)$ as $n \to \infty$ with $v_0 \not\equiv 0$, then $v_0$ solves (1.14), the sequence $z_n(x) := v_n(x) - v_0(\frac{x-x_n}{r_n})r_n^2 \to 0$ in $H^\alpha(\mathbb{R}^N)$ and $z_n(x)$ is a Palais-Smale sequence of $I$ at level $d_1 - I_0(v_0)$.

**Proof.** First, we prove that $v_0$ solves (1.14) and $I(z_n) = I(v_n) - I_0(v_0)$. Fix a ball $B(0, r)$ and a test function $\phi \in C_0^\infty(B(0, r))$. Since

$$\bar{v}_n \to v_0 \text{ in } \tilde{H}^\alpha(\mathbb{R}^N),$$

it implies

\[
\langle \phi, I_0'(v_0) \rangle = \langle \phi, I_0'(\bar{v}_n) \rangle + o(1)
\]

\[
= \langle \phi, I_0'(\bar{v}_n) \rangle + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(\bar{v}_n(x) - \bar{v}_n(y))(\phi(x) - \phi(y))}{|x - y|^{N+2\alpha}} dx dy
\]

\[
= c_{N, \alpha} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(\bar{v}_n(x) - \bar{v}_n(y))(\phi(x) - \phi(y))}{|x - y|^{N+2\alpha}} dx dy - \mu \int_{\mathbb{R}^N} \frac{\bar{v}_n(x)\phi(x)}{|x + \frac{x_n}{r_n}|^{2\alpha}} dx - \int_{\mathbb{R}^N} (\bar{v}_n^+(x))^{2^*-1} \phi(x) dx
\]

\[
+ r_n^{2\alpha} \int_{\mathbb{R}^N} a(r_n x + x_n)\phi(x)\bar{v}_n(x) dx - r_n^{N - N - 2\alpha} \int_{\mathbb{R}^N} k(r_n x + x_n)\phi(x)(\bar{v}_n^+(x))^{q-1} dx + o(1)
\]

\[
= c_{N, \alpha} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(v_n(x) - v_n(y))(\phi(x) - \phi(y))}{|x - y|^{N+2\alpha}} dx dy - \mu \int_{\mathbb{R}^N} \frac{v_n(x)\phi(x)}{|x|^{2\alpha}} dx - \int_{\mathbb{R}^N} (v_n^+(x))^{2^*-1} \phi(x) dx
\]

\[
+ \int_{\mathbb{R}^N} a(x)\phi_n(x)v_n(x) dx - \int_{\mathbb{R}^N} k(x)\phi_n(x)(v_n^+(x))^{q-1} dx + o(1) = o(1) \text{ as } n \to \infty,
\] (2.10)

where $\phi_n = r_n^{-\frac{N-2\alpha}{2}} \phi(\frac{x-x_n}{r_n})$. The last equality in (2.10) holds since

\[
\int_{\mathbb{R}^N} |\phi_n(x)|^2 dx = r_n^{2\alpha} \int_{\mathbb{R}^N} |\phi(x)|^2 dx = o(1),
\]
there exists a sequence \( \{ \alpha \} \) and a test function \( \phi \) of Lemma 4.6 in the Appendix, we can prove that

\[
z \quad \text{and} \quad v \quad \text{converges weakly in} \quad \dot{H}^\alpha(\mathbb{R}^N).
\]

First, we prove that

\[
\text{Proof.} \quad \int_{\mathbb{R}^N} |v_0(x)|^p dx \leq c \int_{\mathbb{R}^N} \frac{1}{(1 + |x|^N)^p} dx \leq c, \quad \forall \ p \geq 2 \quad (2.11)
\]

which implies that \( v_0 \in L^2(\mathbb{R}^N) \). Let

\[
z_n(x) = v_n(x) - r_n \frac{x - x_n}{r_n} \in H^\alpha(\mathbb{R}^N).
\]

From (2.11), \( v_0 \in L^p(\mathbb{R}^N) \) for all \( p \in [2, 2^*) \). Then it follows

\[
\int_{\mathbb{R}^N} \left| v_0(x) - r_n \frac{x - x_n}{r_n} \right|^2 \leq c \int_{\mathbb{R}^N} \left( 1 + |x|^N \right)^{-2} \leq c, \quad \forall \ n \to \infty, \quad \forall \ 2 \leq p < 2^*, \quad (2.12)
\]

Thus \( z_n \to 0 \) in \( H^\alpha(\mathbb{R}^N) \) as \( n \to \infty \). Now we prove that \( \{z_n\} \) is a Palais-Smale sequence of \( I \) at level \( d_1 - I_0(v_0) \). By the Brézis-Lieb Lemma and the weak convergence, similar to Lemma 4.6 in the Appendix, we can prove that

\[
I(z_n) = I(v_n) - I_0(v_0),
\]

\[
\langle I'(z_n), \phi \rangle = o(1)
\]

as \( n \to \infty \). It completes the proof.

**Lemma 2.3.** Assume \( N > 4\alpha, 2 < q < 2^*, 0 < \alpha < 1, 0 < \mu < \phi_{\alpha,N}(\frac{N-4\alpha}{2}) \). Let \( \{v_n\} \subset H^\alpha(\mathbb{R}^N) \) be a Palais-Smale sequence of \( I \) at level \( d_1 \) and \( v_n \to 0 \) in \( H^\alpha(\mathbb{R}^N) \) as \( n \to \infty \). If there exists a sequence \( \{r_n\} \subset \mathbb{R}^+ \), with \( r_n \to 0 \) as \( n \to \infty \) such that \( \bar{v}_n(r_n) := r_n \frac{v_n(x)}{r_n} \) converges weakly in \( H^\alpha(\mathbb{R}^N) \) and almost everywhere to some \( v_0 \in H^\alpha(\mathbb{R}^N) \) as \( n \to \infty \) with \( v_0 \neq 0 \), then \( v_0 \) solves (1.13), the sequence \( z_n(x) := v_n(x) - v_0(x) = o(\frac{r_n}{x}) \) \( \to 0 \) in \( \dot{H}^\alpha(\mathbb{R}^N) \) and \( z_n(x) \) is a Palais-Smale sequence of \( I \) at level \( d_1 - I_\mu(v_0) \).

**Proof.** First, we prove that \( v_0 \) solves (1.13) and \( I(z_n) = I(v_n) - I_\mu(v_0) \). Fix a ball \( B(0, r) \) and a test function \( \phi \in C^\infty_0(B(0, r)) \). Since

\[
\bar{v}_n \to v_0 \quad \text{in} \quad \dot{H}^\alpha(\mathbb{R}^N), \quad (2.13)
\]
it implies
\[ \langle \phi, I'_\mu(v_0) \rangle + o(1) = \langle \phi, I'_\mu(\bar{v}_n) \rangle \]

\[ = c_{N, \alpha} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(\bar{v}_n(x) - \bar{v}_n(y))(\phi(x) - \phi(y))}{|x - y|^{N+2\alpha}} \, dx \, dy - \mu \int_{\mathbb{R}^N} \frac{\bar{v}_n(x) \phi(x)}{|x|^{2\alpha}} \, dx - \int_{\mathbb{R}^N} (\bar{v}_n^+(x))^{2\alpha-1} \phi(x) \, dx \]

\[ = c_{N, \alpha} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(\bar{v}_n(x) - \bar{v}_n(y))(\phi(x) - \phi(y))}{|x - y|^{N+2\alpha}} \, dx \, dy - \mu \int_{\mathbb{R}^N} \frac{\bar{v}_n(x) \phi(x)}{|x|^{2\alpha}} \, dx - \int_{\mathbb{R}^N} (\bar{v}_n^+(x))^{2\alpha-1} \phi(x) \, dx \]

\[ + r_n^{2\alpha} \int_{\mathbb{R}^N} a(r_n x) \phi(x) \bar{v}_n(x) \, dx - r_n^{N-2\alpha} \int_{\mathbb{R}^N} k(r_n x) \phi(x)(\bar{v}_n^+(x))^{q-1} \, dx + o(1) \]

\[ = c_{N, \alpha} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(v_n(x) - v_n(y))(\phi_n(x) - \phi_n(y))}{|x - y|^{N+2\alpha}} \, dx \, dy - \mu \int_{\mathbb{R}^N} \frac{v_n(x) \phi_n(x)}{|x|^{2\alpha}} \, dx - \int_{\mathbb{R}^N} (v_n^+(x))^{2\alpha-1} \phi_n(x) \, dx \]

\[ + \int_{\mathbb{R}^N} a(x) \phi_n(x) v_n(x) \, dx - \int_{\mathbb{R}^N} k(x) \phi_n(x)(v_n^+(x))^{q-1} \, dx + o(1) = o(1) \text{ as } n \to \infty, \]

(2.14)

where \( \phi_n = r_n^{-\frac{N-2\alpha}{2}} \phi(\frac{x}{r_n}) \). The last equality in (2.14) holds since

\[ \int_{\mathbb{R}^N} |\phi_n(x)|^2 \, dx = r_n^{2\alpha} \int_{\mathbb{R}^N} |\phi(x)|^2 \, dx = o(1), \]

\[ \|\phi\|_{H^\alpha(\mathbb{R}^N)} = \|\phi_n\|_{H^\alpha(\mathbb{R}^N)} = \|\phi_n\|_{H^\alpha(\mathbb{R}^N)} + o(1), \text{ as } n \to \infty. \]

Thus \( v_0 \) is a nontrivial critical point of \( I_\mu \). Noting the fact \( N > 4\alpha, \mu < \phi_{\alpha,N}(\frac{N-4\alpha}{2}) \) and \( \phi_{\alpha,N} \) is a strictly increasing, it follows

\[ \eta_\mu > \frac{2\alpha}{N - 2\alpha}, (1 + \eta_\mu)(N - 2\alpha)p \geq \frac{(1 + \eta_\mu)(N - 2\alpha)}{2} \]

\[ \geq (1 + \eta_\mu)(N - 2\alpha) > N, \forall \ p \geq 2 \]

then by Lemma [4.5] and (1.16), it follows

\[ \int_{\mathbb{R}^N} |v_0(x)|^p \, dx \leq c, \forall \ p \geq 2 \]

(2.15)

which implies that \( v_0 \in L^2(\mathbb{R}^N) \). Let

\[ z_n(x) = v_n(x) - r_n^{\frac{2\alpha-N}{2}} v_0\left(\frac{x}{r_n}\right) \in H^\alpha(\mathbb{R}^N). \]

From (2.15) and \( v_0 \in L^p(\mathbb{R}^N) \) for all \( p \in [2, 2^*], \) it follows

\[ \int_{\mathbb{R}^N} \left| v_0\left(\frac{x - x_n}{r_n}\right) r_n^{\frac{2\alpha-N}{2}} \right|^p \, dx = r_n^{\frac{N-p(N-2\alpha)}{2}} \|v_0\|_{L^p(\mathbb{R}^N)}^p \to 0, \text{ as } n \to \infty, \text{ for all } 2 \leq p < 2^*, \]

(2.16)
Thus \( z_n \to 0 \) in \( H^\alpha(\mathbb{R}^N) \) as \( n \to \infty \). Now we prove that \( \{z_n\} \) is a Palais-Smale sequence of \( I \) at level \( d_1 - I_\mu(v_0) \). By the Brézis-Lieb Lemma and the weak convergence, similar to Lemma 4.6 in the Appendix, we can prove that

\[
I(z_n) = I(v_n) - I_\mu(v_0),
\]

\[
\langle I'(z_n), \phi \rangle = o(1)
\]
as \( n \to \infty \). It completes the proof. \( \square \)

**Proof of Theorem 1.1.** By Lemma 4.3 in the appendix, we can assume that \( \{u_n\} \) is bounded. Up to a subsequence, let \( n \to \infty \), we assume that

\[
\begin{align*}
    u_n &\rightharpoonup u \text{ in } H^\alpha(\mathbb{R}^N), \quad (2.17) \\
    u_n &\to u \text{ in } L^p_{\text{loc}}(\mathbb{R}^N) \text{ for } 2 \leq p < 2^*, \quad (2.18) \\
    u_n &\to u \text{ a.e. in } \mathbb{R}^N. \quad (2.19)
\end{align*}
\]

Denote \( v_n(x) = u_n(x) - u(x) \), then \( \{v_n\} \) is a Palais-Smale sequence of \( I \) and \( v_n \to 0 \) in \( H^\alpha(\mathbb{R}^N) \) and

\[
\begin{align*}
    v_n &\to 0 \text{ in } H^\alpha(\mathbb{R}^N), \quad (2.20) \\
    v_n &\to 0 \text{ in } L^p_{\text{loc}}(\mathbb{R}^N) \text{ for } 2 \leq p < 2^*, \quad (2.21) \\
    v_n &\to 0 \text{ a.e. in } \mathbb{R}^N. \quad (2.22)
\end{align*}
\]

Then by Lemma 4.6 we know that

\[
\begin{align*}
    I(v_n) &= I(u_n) - I(u) + o(1), \text{ as } n \to \infty; \quad (2.23) \\
    I'(v_n) &= o(1), \text{ as } n \to \infty, \quad (2.24) \\
    \|v_n\|_{H^\alpha(\mathbb{R}^N)} &= \|u_n\|_{H^\alpha(\mathbb{R}^N)} - \|u\|_{H^\alpha(\mathbb{R}^N)} + o(1), \text{ as } n \to \infty. \quad (2.25)
\end{align*}
\]

Without loss of generality, we may assume that

\[
\|v_n\|_{H^\alpha(\mathbb{R}^N)}^2 \to l > 0 \text{ as } n \to \infty.
\]

In fact if \( l = 0 \), Theorem 1.1 is proved for \( l_1 = 0, l_2 = 0, l_3 = 0 \).

**Step 1:** getting rid of the blowing up bubbles caused by unbounded domains.

Suppose there exists a constant \( 0 < \delta < \infty \) such that

\[
(\int_{\mathbb{R}^N} (v_n^+(x))^q dx)^{\frac{q}{q}} \geq \delta > 0, \text{ for } 2 < q < 2^*. \quad (2.26)
\]

By interpolation inequality, it follows

\[
\|v_n\|_{L^q} \leq \|v_n\|_{L^{2^*}}^{\lambda} \|v_n\|_{L^2}^{1-\lambda}, \text{ for } 2 < q < 2^*
\]
where $0 < \lambda < 1$. Thus there exists a $\tilde{\delta} > 0$ such that
\[ \|v_n\|_{L^2}^2 \geq \tilde{\delta} > 0. \]
By Lemma 4.1, there exists a subsequence still denoted by \{v_n\}, such that one of the following two cases occurs.

i) Vanish occurs.
\[ \forall 0 < R < \infty, \sup_{y \in \mathbb{R}^N} \int_{B(y, R)} |v_n(x)|^2 dx \to 0 \text{ as } n \to \infty. \]

By Lemma 4.2 (1.7) and Sobolev inequality, it follows
\[ \int_{\mathbb{R}^N} (v_n^+(x))^q dx \to 0 \text{ as } n \to \infty, \forall 2 < q < 2^*, \]
which contradicts (2.20).

ii) Nonvanish occurs.
There exist $\beta > 0$, $0 < \bar{R} < \infty$, \{y_n\} $\subset \mathbb{R}^N$ such that
\[ \lim \inf_{n \to \infty} \int_{y_n + B_{\bar{R}}} |v_n(x)|^2 dx \geq \beta > 0. \]  
(2.27)

We claim that $|y_n| \to \infty$ as $n \to \infty$. Otherwise, if there exists a constant $M > 0$ such that $|y_n| \leq M$, then we can choose a $R_2 > 0$ large enough such that
\[ \int_{y_n + B_{R_2}} |v_n(x)|^2 dx \leq \|v_n\|_{L^2(B(0,R_2))}^2 \to 0 \text{ as } n \to \infty, \]  
(2.28)
which contradicts (2.27).

To proceed, we first construct the Palais-Smale sequences of $I^\infty$.

Denote $\bar{v}_n(x) = v_n(x + y_n)$. Since $\|\bar{v}_n\|_{H^\alpha(\mathbb{R}^N)} = \|v_n\|_{H^\alpha(\mathbb{R}^N)} \leq c$, without loss of generality, we assume that as $n \to \infty$,
\[ \bar{v}_n \to v_0 \text{ in } H^\alpha(\mathbb{R}^N), \]
\[ \bar{v}_n \to v_0 \text{ in } L^p_{loc}(\mathbb{R}^N), \text{ for any } 1 < p < 2^*. \]  
(2.29)

Then $\forall \phi \in C_0^\infty(B(0,r))$ as $n \to \infty$,
\[ \int_{\mathbb{R}^N} \frac{\bar{v}_n^+(x)\phi(x)}{|x + y_n|^{2\alpha}} dx = \int_{B(0,r)} \frac{\bar{v}_n^+(x)\phi(x)}{|x + y_n|^{2\alpha}} dx \]
\[ \leq \frac{2}{|y_n|^{2\alpha}} \int_{\mathbb{R}^N} |\bar{v}_n(x)\phi| dx + o(1) \]
\[ \leq \frac{2}{|y_n|^{2\alpha}} \|\phi\|_{H^\alpha} \|\bar{v}_n\|_{H^\alpha} + o(1) \]
\[ \leq \frac{c}{|y_n|^{2\alpha}} + o(1) = o(1). \]  
(2.30)
Similarly we have
\[ \int_{\mathbb{R}^N} \frac{(\bar{v}_n^+(x))^2}{|x + y_n|^{2\alpha}} dx = o(1) \text{ as } n \to \infty. \] (2.31)

Since \( v_n \to 0 \) in \( H^\alpha(\mathbb{R}^N) \) and \( \lim_{n \to \infty} a(x + y_n) = \bar{a} \), we have as \( n \to \infty \),
\[ o(1) = \int_{\mathbb{R}^N} a(x)v_n(x)\phi_n(x)dx = \int_{\mathbb{R}^N} \bar{a}v_n(x)\phi(x)dx + \int_{\mathbb{R}^N} \left[a(x + y_n) - \bar{a}\right]v_n(x)\phi(x)dx \]
and
\[ |\int_{\mathbb{R}^N} \left[a(x + y_n) - \bar{a}\right]v_n(x)\phi(x)dx| \leq c\left(\int_{\mathbb{R}^N} |a(x + y_n) - \bar{a}|^2\phi(x)^2dx\right)^{1/2} = o(1), \]
that is,
\[ \int_{\mathbb{R}^N} \bar{a}v_n(x)\phi(x)dx = o(1) = \int_{\mathbb{R}^N} a(x)v_n(x)\phi_n(x)dx \text{ as } n \to \infty. \] (2.32)

Similarly we have
\[ \int_{\mathbb{R}^N} k(x)(v_n^+(x))^q\phi_n(x)dx = \int_{\mathbb{R}^N} \tilde{k}(\bar{v}_n^+(x))^{q-1}\phi(x)dx = o(1) \text{ as } n \to \infty. \] (2.33)

Recall that \( v_n \) is a Palais-Smale sequence of \( I \), by (2.29) and (2.31)-(2.33) we have
\[ o(1) = \langle I'(v_n), \phi_n \rangle = \langle I^\infty(\bar{v}_n), \phi \rangle + o(1) = \langle I^\infty(v_0), \phi \rangle + o(1), \text{ as } n \to \infty. \] (2.34)

This shows that \( v_0 \) is a weak solution of (1.12).

We claim that \( v_0 \neq 0 \). From (2.26), we may assume that there exists a sequence \( \{y_n\} \)
satisfying (2.27) and
\[ \int_{B(y_n, R)} (v_n^+(x))^q dx = b + o(1) > 0, \text{ as } n \to \infty, \] (2.35)
where \( b > 0 \) is a constant. If \( v_0 \equiv 0 \), we have
\[ \int_{B(0, R)} (\bar{v}_n^+(x))^q dx = \int_{B(y_n, R)} (v_n^+(x))^q dx = o(1) \text{ as } n \to \infty \text{ for } 0 < R < \infty \]
which contradicts (2.35).

Denote \( z_n(x) = v_n(x) - v_0(x - y_n) \). Since
\[
I(v_n) = \frac{1}{2} \int_{\mathbb{R}^N} \left|(-\Delta)^{\alpha/2} v_n(x)\right|^2 + a(x)|v_n(x)|^2 - \mu \frac{|v_n(x)|^2}{|x|^{2\alpha}} dx \\
- \frac{1}{2^*} \int_{\mathbb{R}^N} (v_n^+(x))^2 dx - \frac{1}{q} \int_{\mathbb{R}^N} k(x)(v_n^+(x))^q dx \\
= \frac{1}{2} \int_{\mathbb{R}^N} \left|(-\Delta)^{\alpha/2} \bar{v}_n(x)\right|^2 + a(x + y_n)|\bar{v}_n(x)|^2 - \mu \frac{|\bar{v}_n(x)|^2}{|x + y_n|^{2\alpha}} dx \\
- \frac{1}{2^*} \int_{\mathbb{R}^N} (\bar{v}_n^+(x))^2 dx \\
- \frac{1}{q} \int_{\mathbb{R}^N} k(x)(\bar{v}_n^+(x))^q dx
\]

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\[-\frac{1}{q} \int_{\mathbb{R}^N} k(x + y_n)(\tilde{v}^+_n(x))^q \, dx \]
\[= \frac{1}{2} \int_{\mathbb{R}^N} \left( |(-\Delta)^{\alpha/2} \tilde{v}_n(x)|^2 + \tilde{a}|\tilde{v}_n(x)|^2 \right) \, dx - \frac{1}{q} \int_{\mathbb{R}^N} \tilde{k}(\tilde{v}^+_n(x))^q \, dx - \frac{1}{2^*} \int_{\mathbb{R}^N} (\tilde{v}^+_n(x))^{2^*} \, dx + o(1) \]
\[= I^\infty(\tilde{v}_n) + o(1), \]
where the last equality but one is a result of (2.31), therefore, as \( n \to \infty \),
\[\|z_n\|_{H^\alpha(\mathbb{R}^N)} = \|\tilde{v}_n\|_{H^\alpha(\mathbb{R}^N)} - \|v_0\|_{H^\alpha(\mathbb{R}^N)} + o(1) \leq \|v_n\|_{H^\alpha(\mathbb{R}^N)} - \|v_0\|_{H^\alpha(\mathbb{R}^N)} + o(1), \quad (2.36)\]
\[I(z_n) = I^\infty(\tilde{v}_n) - I^\infty(v_0) + o(1) = I(v_n) - I^\infty(v_0) + o(1). \quad (2.37)\]
Hence \( z_n \to 0 \) in \( H^\alpha(\mathbb{R}^N) \) as \( n \to \infty \), and \( z_n \) is a Palais-Smale sequence of \( I \). From (4.7) in Lemma 4.1 it follows \( \|v_0\|_{H^\alpha} = 0 \), that is \( v_0 \equiv 0 \) a.e. in \( \mathbb{R}^N \). Then by Brezis-Lieb Lemma and (4.7), there exists a constant \( c > 0 \) such that
\[\int_{\mathbb{R}^N} (z^+_n(x))^q \, dx = \int_{\mathbb{R}^N} (v^+_n(x))^q \, dx - \int_{\mathbb{R}^N} (v^+_0(x))^q \, dx + o(1) \leq \int_{\mathbb{R}^N} (v^+_n(x))^q \, dx - c \quad (2.38)\]
where the last inequality follows from the fact \( v_0 \not\equiv 0 \). If \( \|z_n\|_{L^q(\mathbb{R}^N)} \to \delta_2 > 0 \) as \( n \to \infty \), from (2.38) and the boundedness of \( \|v_n\|_{L^q} \), then one can repeat Step 1 for finite times \( (l_1) \) times. Thus we obtain a new Palais-Smale sequence of \( I \), without loss of generality still denoted by \( v_n \), such that
\[d = I(u) + I(v_n) + \sum_{k=1}^{l_1} I^\infty(u_k) + o(1), \quad (2.39)\]
\[v_n(x) = u_n(x) - u(x) - \sum_{k=1}^{l_1} u_k(x - y^k_n), \text{ with } y^k_n \to \infty, \quad (2.40)\]
\[\|v^+_n\|_{L^q(\mathbb{R}^N)} \to 0 \quad (2.41)\]
as \( n \to \infty \).

**Step 2:** Getting rid of the blowing up bubbles caused by the critical terms.

Suppose there exists \( 0 < \delta < \infty \) such that
\[\inf_{n \in \mathbb{N}} \int_{\mathbb{R}^N} (v^+_n(x))^{2^*} \, dx \geq \delta > 0, \text{ for some } 0 < R < \infty. \quad (2.42)\]

It follows from Lemma 2.1 that there exist two sequences \( \{r_n\} \subset \mathbb{R}^+ \) and \( \{x_n\} \subset \mathbb{R}^N \), such that
\[\tilde{v}_n(x) \to v_0 \not\equiv 0 \text{ in } \dot{H}^\alpha(\mathbb{R}^N), \quad (2.43)\]
where
\[\tilde{v}_n(x) = \begin{cases} \frac{N-2\alpha}{2} v_n(r_n x) & \text{when } \frac{x_n}{r_n} \text{ is bounded,} \\ \frac{N-2\alpha}{2} v_n(r_n x + x_n) & \text{when } |\frac{x_n}{r_n}| \to \infty. \end{cases} \quad (2.44)\]
Now we claim that $r_n \to 0$ as $n \to \infty$. In fact there exists a $R_1 > 0$ such that
\[
\int_{B(0,R_1)} |v_0(x)|^p \, dx = \delta_1 > 0, \quad \text{for } 2 \leq p < 2^*.
\] (2.45)

From the Sobolev compact embedding, \(2.48\), \(2.43\) and \(2.45\), we have that for all $r > 0$,
\[
v_n \to 0 \text{ in } L^p(B(0, r)) \text{ for all } 2 \leq p < 2^*,
\]
\[
\bar{v}_n \to v_0 \text{ in } L^p(B(0, r)) \text{ for all } 2 \leq p < 2^*,
\]
\[
0 \neq \|v_0\|^2_{L^2(B(0, R_1))} + o(1)
\]
\[
= \int_{B(0, R_1)} |\bar{v}_n(x)|^2 \, dx
\]
\[
= \begin{cases} 
  r_n^{-2\alpha} \int_{B(0, r_n R_1)} |v_n(x)|^2 \, dx, & \text{if } \frac{\bar{v}_n}{r_n} \text{ is bounded}, \\
  r_n^{-2\alpha} \int_{B(x_n, r_n R_1)} |v_n(x)|^2 \, dx & \text{if } |\frac{\bar{v}_n}{r_n}| \to \infty.
\end{cases}
\] (2.46)

If $|\frac{\bar{v}_n}{r_n}| \to \infty$, then there exists a constant $\tilde{c}$ such that
\[
0 < \tilde{c} < r_n^{-2\alpha} \int_{B(x_n, r_n R_1)} |v_n(x)|^2 \, dx
\]
\[
\leq cr_n^{-2\alpha} \left( \int_{B(x_n, r_n R_1)} |v_n(x)|^q \, dx \right)^{2/q} (w_N(r_n R_1)^N)^{1-\frac{2}{q}}
\]
\[
\leq cr_n^{N(1-\frac{2}{q})-2\alpha} \left( \int_{\mathbb{R}^N} |v_n(x)|^q \, dx \right)^{2/q}
\] (2.47)

Then from \(2.41\) and \(2.47\) and the fact $q < 2^*$, it follows that $r_n \to 0$. Similarly, if $\frac{\bar{v}_n}{r_n}$ is bounded, we also have that $r_n \to 0$.

For the case that $\frac{\bar{v}_n}{r_n}$ is bounded and $\bar{v}_n(x) = r_n^{\frac{N-2\alpha}{2}} v_n(r_n x)$, define $z_n(x) = v_n(x) - v_0(\frac{x}{r_n}) r_n^{\frac{2\alpha-N}{2}}$. It follows from Lemma 2.2 that \(\{z_n\}\) is a Palais-Smale sequence of $I$ satisfying
\[
I(z_n) = I(v_n) - I_\mu(v_0) + o(1), \quad \text{as } n \to \infty.
\] (2.48)

and $z_n \to 0$ in $H^\alpha(\mathbb{R}^N)$. Since $v_0$ satisfies \(1.13\), from Lemma 4.5 \(1.19\) and \(1.21\) there exists $\varepsilon_1 > 0$ such that
\[
v_0(x) = \varepsilon_1^{\frac{2\alpha-N}{2}} U_\mu(\frac{x}{\varepsilon_1}), \quad I_\mu(v_0) = D_\mu.
\] (2.49)

Let $R_n^1 = r_n \varepsilon_1$, from \(2.49\), it follows
\[
\begin{align*}
  \frac{2\alpha-N}{2} v_0(\frac{x}{r_n}) = (R_n^1)^{2\alpha-N} U_\mu(\frac{x}{R_n^1}) = U_\mu(R_n^1, x),
\end{align*}
\] (2.50)

with $R_n^1 \to 0$. Then from \(2.23\) it follows
\[
\begin{align*}
  z_n(x) &= v_n(x) - U_\mu(R_n^1, x), \\
  I(z_n) &= I(v_n) - D_\mu + o(1).
\end{align*}
\] (2.51)
with $R_n^1 \to 0$. From (4.7), we have that $z_n \geq 0$, a.e. in $\mathbb{R}^N$. From Lemma 4.7, let $a = v_n, b = U_{\mu}^{R_n^1}$, it follows
\[ \int_{\mathbb{R}^N} (z_n^+(x))^2 \, dx = \int_{\Omega} (z_n(x))^2 \, dx \]
\[ \leq \int_{\Omega} (v_n(x))^2 - (U_{\mu}^{R_n^1}(x))^2 \, dx \]
\[ = \int_{\Omega} (v_n^+(x))^2 \, dx - C \]
\[ \leq \int_{\mathbb{R}^N} (v_n^+(x))^2 \, dx - C \] (2.52)
where $\Omega = \{ x \mid z_n(x) \geq 0 \} \cap \mathbb{R}^N$.

For the case that $|x_n^r| \to \infty$ and $\bar{v}_n(x) = \varepsilon_1^{N-2a} v_n(x)$, define $z_n(x) = v_n(x) - v_0(x^r_n)\varepsilon_1^{\frac{N}{2}}$. It follows from Lemma 2.3 that $\{ z_n \}$ is a Palais-Smale sequence of $I$ satisfying
\[ I(z_n) = I(v_n) - I_0(v_0) + o(1), \quad \text{as } n \to \infty. \] (2.53)
and $z_n \to 0$ in $H^\alpha(\mathbb{R}^N)$. Since $v_0$ satisfies (1.14), from Lemma 4.5, (1.19) and (1.22) there exists $\varepsilon_1 > 0$ such that
\[ v_0(x) = \varepsilon_1^{\frac{2a-N}{2}} U_0(\frac{x-y}{\varepsilon_1}), \quad I_0(v_0) = D_0. \] (2.54)
Let $\bar{R}_n^1 = r_n\varepsilon_1$ and $x_n^1 = y + \varepsilon_1 x_n$, from (2.54), it follows
\[ r_n^{\frac{2a-N}{2}} v_0\left(\frac{x}{r_n}\right) = (\bar{R}_n^1)^{\frac{2a-N}{2}} U_0\left(\frac{x-x_n^1}{\bar{R}_n^1}\right) = U_0, \] (2.55)
with $\bar{R}_n^1 \to 0$. Then from (2.23) it follows
\[ z_n(x) = v_n(x) - U_0^{R_n^1,x_n^1}(x), \]
\[ I(z_n) = I(v_n) - D_0 + o(1) = I(u_n) \] (2.56)
with $\bar{R}_n^1 \to 0$. Similar to (2.52), it follows
\[ \int_{\mathbb{R}^N} (z_n^+(x))^2 \, dx \leq \int_{\mathbb{R}^N} (v_n^+(x))^2 \, dx - C \] (2.57)
If still there exists a $\bar{\delta} > 0$, such that
\[ \int_{\mathbb{R}^N} (z_n^+(x))^2 \, dx \geq \bar{\delta} > 0, \]
then repeat the previous argument. From (2.52) and the fact
\[ \int_{\mathbb{R}^N} (z_n^+(x))^2 \, dx \leq \| v_n \|^2_{H^\alpha} \leq c, \]
we deduce that the iteration must stop after finite times. That is to see, there exist non-negative constants $l_2, l_3$ and a new Palais-Smale sequence of $I$, (without loss of generality) denoted by $\{v_n\}$, such that as $n \to \infty$,

$$d = I(v_n) + I(u) + \sum_{k=1}^{l_1} I^\infty(u_k) + l_2 D_\mu + l_3 D_0,$$

with $R_n^i \to 0$, $\bar{R}_n^i \to 0$ and $|x_n^i/\bar{R}_n^i| \to \infty$.

$$\int_{\mathbb{R}^N} (v_n^+)^{2^*} dx = o(1), \quad \|v_n\|_{L^q(\mathbb{R}^N)} \to 0 \quad \text{(2.59)}$$

and

$$v_n \rightharpoonup 0 \text{ in } H^\alpha(\mathbb{R}^N). \quad \text{(2.60)}$$

Then from the fact $<I'(v_n), v_n> = o(1)$, it follows

$$\|v_n\|^2_{H^\alpha(\mathbb{R}^N)} \leq c \int_{\mathbb{R}^N} \left( |(-\Delta)^{\alpha/2} v_n(x)|^2 + a(x)|v_n(x)|^2 - \mu \frac{(v_n^+(x))^2}{|x|^{2\alpha}} \right) dx$$

$$= c \left( \int_{\mathbb{R}^N} k(x)(v_n^+(x))^q dx + \int_{\mathbb{R}^N} (v_n^+(x))^{2^*} dx \right) \to 0 \quad \text{(2.61)}$$

as $n \to \infty$. From (2.61), it gives that

$$I(v_n) = o(1). \quad \text{(2.62)}$$

From (2.58)-(2.62), the proof of Theorem 1.1 is complete.

### 3 Proof of Theorem 1.2

Now we are ready to prove Theorem 1.2 by Mountain Pass Theorem [1] and Theorem 1.1.

**Proof of Theorem 1.2.** From

$$I(tu) = \frac{t^2}{2} \left[ \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2} u(x)|^2 dx - \mu \int_{\mathbb{R}^N} \frac{(u^+(x))^2}{|x|^{2\alpha}} dx + \int_{\mathbb{R}^N} a(x)|u(x)|^2 dx \right]$$

$$- \frac{|t|^2}{2^*} \int_{\mathbb{R}^N} (u^+(x))^{2^*} dx - \frac{|t|^q}{q} \int_{\mathbb{R}^N} k(x)(u^+(x))^q dx,$$

we deduce that for a fixed $u \not\equiv 0$ in $H^\alpha(\mathbb{R}^N)$, $I(tu) \to -\infty$ if $t \to \infty$. Since

$$\int_{\mathbb{R}^N} (u^+(x))^q dx \leq C \|u\|^q_{H^\alpha(\mathbb{R}^N)} , \quad \int_{\mathbb{R}^N} (u^+(x))^{2^*} dx \leq C \|u\|^{2^*}_{H^\alpha(\mathbb{R}^N)},$$
we have
\[ I(u) \geq c\|u\|_{H^\alpha(\mathbb{R}^N)}^2 - C(\|u\|_{H^\alpha(\mathbb{R}^N)}^q + \|u\|_{H^\alpha(\mathbb{R}^N)}^{2^*}). \]

Hence, there exists \( r_0 > 0 \) small such that \( I(u)|_{\partial B(0,r_0)} \geq \rho > 0 \) for \( q, 2^* > 2 \).

As a consequence, \( I(u) \) satisfies the geometry structure of Mountain-Pass Theorem. Now define
\[ c^* = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} I(\gamma(t)), \]
where \( \Gamma = \{ \gamma \in C([0,1], H^\alpha(\mathbb{R}^N)) : \gamma(0) = 0, \gamma(1) = \psi_0 \in H^\alpha(\mathbb{R}^N) \} \) with \( I(t\psi_0) \leq 0 \) for all \( t \geq 1 \).

To complete the proof of Theorem 1.2, we need to verify that \( I(u) \) satisfies the local Palais-Smale conditions. According to Remarks 1), we only need to verify that
\[ c^* < \min\{ \frac{\alpha}{N}S_{\alpha,\mu}^{\frac{N}{2}}, \frac{\alpha}{N}S_{\alpha,\mu}^{\frac{N}{2}}, J^\infty \} = \min\{ \frac{\alpha}{N}S_{\alpha,\mu}^{\frac{N}{2}}, J^\infty \}. \] (3.1)

Set \( v_\varepsilon(x) = \frac{U_\mu^\varepsilon(x)}{\langle \int_{\mathbb{R}^N} |U_\mu^\varepsilon(x)|^{2^*} \rangle^{1/2^*}}. \) We claim
\[ \max_{t > 0} I(tv_\varepsilon) < \frac{\alpha}{N}S_{\alpha,\mu}. \] (3.2)
In fact, from (1.20) it is easy to calculate the following estimates
\[ \|v_\varepsilon\|_{H^\alpha(\mathbb{R}^N)}^2 = S_{\alpha,\mu}, \] (3.3)
\[ \int_{\mathbb{R}^N} (v_\varepsilon(x))^2 dx = c\varepsilon^{2\alpha}\|U_\mu\|_{L^2}^2 = O(\varepsilon^{2\alpha}), \text{ for } N > 4\alpha, \mu < \phi_{\alpha,N}(\frac{N - 4\alpha}{2}), \] (3.4)
\[ \int_{\mathbb{R}^N} (v_\varepsilon(x))^q dx = O(\varepsilon^{\frac{(2\alpha - N)q}{2}} + N). \] (3.5)
Since \( 2^* > q > 2 \) we have
\[ O(\varepsilon^{2\alpha}) = o(\varepsilon^{\frac{(2\alpha - N)q}{2}} + N). \] (3.6)
Denote \( t_\varepsilon \) the attaining point of \( \max_{t > 0} I(tv_\varepsilon) \), similar to the proof of Lemma 3.5 in [3] we can prove that \( t_\varepsilon \) is uniformly bounded. In fact, we consider the function
\[ h(t) = I(tv_\varepsilon) \]
\[ = \frac{t^2}{2}\|(-\Delta)^{\alpha/2}v_\varepsilon\|_{L^2(\mathbb{R}^N)}^2 - \frac{\mu}{2^*} \int_{\mathbb{R}^N} \frac{v_\varepsilon^2}{|x|^{2\alpha}} dx + \int_{\mathbb{R}^N} a(x)(v_\varepsilon(x))^2 dx \]
\[ - \frac{t^2}{2^*} \int_{\mathbb{R}^N} (v_\varepsilon(x))^{2^*} dx - \frac{t^q}{q} \int_{\mathbb{R}^N} (k(x)v_\varepsilon(x))^q dx \]
\[ \geq \frac{ct^2}{2}\|v_\varepsilon\|_{H^\alpha(\mathbb{R}^N)}^2 - \frac{Ct^2}{2^*}\|v_\varepsilon\|_{H^\alpha(\mathbb{R}^N)}^{2^*} - \frac{Ct^q}{q}\|v_\varepsilon\|_{H^\alpha(\mathbb{R}^N)}^q, \] (3.7)
Since $\lim_{t \to +\infty} h(t) = -\infty$ and $h(t) > 0$ when $t$ is closed to 0, then $\max_{t>0} h(t)$ is attained for $t_\varepsilon > 0$. From the fact $\int_{\mathbb{R}^N} (v_\varepsilon(x))^2 dx = 1$, it follows
\[
h'(t_\varepsilon) = t_\varepsilon \left( \|(-\Delta)^{\alpha/2} v_\varepsilon \|^2_{L^2(\mathbb{R}^N)} - \mu \int_{\mathbb{R}^N} \frac{v_\varepsilon^2}{|x|^{2\alpha}} dx + \int_{\mathbb{R}^N} a(x)(v_\varepsilon(x))^2 dx \right)
\]
\[- t^{2*}_\varepsilon - t^{q-1}_\varepsilon \int_{\mathbb{R}^N} k(x)(v_\varepsilon(x))^q dx = 0. \tag{3.8}
\]
Since $k(x) > 0$, from (3.3) and (3.4) for $\varepsilon$ sufficiently small, we have
\[
t^{2*}_\varepsilon \leq \|(-\Delta)^{\alpha/2} v_\varepsilon \|^2_{L^2(\mathbb{R}^N)} - \mu \int_{\mathbb{R}^N} \frac{v_\varepsilon^2}{|x|^{2\alpha}} dx + \int_{\mathbb{R}^N} a(x)(v_\varepsilon(x))^2 dx < 2S_{\alpha,\mu}. \tag{3.9}
\]
Then
\[
\|(-\Delta)^{\alpha/2} v_\varepsilon \|^2_{L^2(\mathbb{R}^N)} - \mu \int_{\mathbb{R}^N} \frac{v_\varepsilon^2}{|x|^{2\alpha}} dx + \int_{\mathbb{R}^N} a(x)(v_\varepsilon(x))^2 dx
\]
\[
= t^{2*}_\varepsilon + t^{q-2}_\varepsilon \int_{\mathbb{R}^N} k(x)(v_\varepsilon(x))^q dx \leq t^{2*}_\varepsilon + (2S_{\alpha,\mu})^{\frac{q-2}{2}} \int_{\mathbb{R}^N} k(x)(v_\varepsilon(x))^q dx. \tag{3.10}
\]
Choosing $\varepsilon > 0$ small enough, by (3.3)- (3.5), there exists a constant $\alpha_1 > 0$ such that $t_\varepsilon > \alpha_1 > 0$. Combining this with (3.9), it implies that $t_\varepsilon$ is bounded for $\varepsilon > 0$ small enough.

Hence, for $\varepsilon > 0$ small,
\[
\max_{t>0} I(t v_\varepsilon) = I(t_\varepsilon v_\varepsilon)
\]
\[
\leq \max_{t>0} \left\{ t^2 \int_{\mathbb{R}^N} \|(-\Delta)^{\alpha/2} v_\varepsilon(x) \|^2 dx - \mu \int_{\mathbb{R}^N} \frac{v_\varepsilon^2}{|x|^{2\alpha}} dx - \frac{t^{2*}_\varepsilon}{2*} \int_{\mathbb{R}^N} (v_\varepsilon(x))^2 dx \right\}
\]
\[- O(\varepsilon^{2(\alpha-N)/2} + N) + O(\varepsilon^{2\alpha}),
\]
\[
< \frac{\alpha}{N} S_{\alpha,\mu} \frac{N}{\varepsilon} (\text{by (3.6)}).
\]
This completes the proof of (3.2). By the definition of $c^*$, we have $c^* < \frac{\alpha}{N} S_{\alpha,\mu} \frac{N}{\varepsilon}$.

Next we verify
\[
c^* < J^\infty. \tag{3.11}
\]
Let $\{u_0\}$ be the minimizer of $J^\infty$, $I^\infty(u_0) = J^\infty$ and
\[
\int_{\mathbb{R}^N} \left( \|(-\Delta)^{\alpha/2} u_0(x) \|^2 + a |u_0(x)|^2 \right) dx = \int_{\mathbb{R}^N} \bar{k}(u_0^+(x))^q dx + \int_{\mathbb{R}^N} (u_0^+(x))^{2*} dx.
\]
Let
\[
g(t) = I^\infty(tu_0) = \frac{1}{2} t^2 \int_{\mathbb{R}^N} \left( \|(-\Delta)^{\alpha/2} u_0(x) \|^2 + a |u_0(x)|^2 \right) dx
\]
\[- \frac{t^q}{q} \int_{\mathbb{R}^N} \bar{k}(u_0^+(x))^q dx - \frac{t^{2*}}{2*} \int_{\mathbb{R}^N} (u_0^+(x))^{2*} dx,
\]
20
\begin{equation}
\frac{d}{dt} g(t) = t \int_{\mathbb{R}^N} \left( |(-\Delta)^{\alpha/2} u_0(x)|^2 + a|u_0(x)|^2 \right) dx - t^{\alpha - 1} \int_{\mathbb{R}^N} k(u_0^+(x))^q dx - t^{2\alpha - 1} \int_{\mathbb{R}^N} (u_0^+(x))^{2^*} dx.
\end{equation}

Thus \( g'(t) \geq 0 \) if \( t \in (0, 1) \); \( g'(t) \leq 0 \) if \( t \geq 1 \). Then

\begin{equation}
g(1) = I^\infty(u_0) = \max_l I^\infty(u);
\end{equation}

where \( l = \{tu_0, t \geq 0, u_0 \text{ fixed} \} \).

Since there exists a \( t_0 > 0 \) such that \( \sup_{t \geq 0} I(tu_0) = I(t_0u_0) \), from (3.12) and the assumptions of \( a(x) \) and \( k(x) \), we have

\( J^\infty = I^\infty(u_0) \geq I^\infty(t_0u_0) > I(t_0u_0) = \sup_{t \geq 0} I(tu_0) \).

It proves (3.11). By (3.2) and (3.11) we have (3.1). Then the proof is completed.

4 Appendix

In this appendix, we give some lemmas and detailed proofs for the convenience of the reader.

**Lemma 4.1.** (Lemma 2.1, [22]) Let \( \{\rho_n\}_{n \geq 1} \) be a sequence in \( L^1(\mathbb{R}^N) \) satisfying

\begin{equation}
\rho_n \geq 0 \text{ on } \mathbb{R}^N, \quad \lim_{n \to \infty} \int_{\mathbb{R}^N} \rho_n(x) dx = \lambda > 0,
\end{equation}

where \( \lambda > 0 \) is fixed. Then there exists a subsequence \( \{\rho_{n_k}\} \) satisfying one of the following two possibilities:

(i) (Vanishing):

\begin{equation}
\lim_{k \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B(y,R)} \rho_{n_k}(x) dx = 0, \text{ for all } R < +\infty.
\end{equation}

(ii) (Nonvanishing): \( \exists \alpha > 0, R < +\infty \) and \( \{y_k\} \subset \mathbb{R}^N \) such that

\begin{equation}
\lim_{k \to +\infty} \int_{y_k + BR} \rho_{n_k}(x) dx \geq \alpha > 0.
\end{equation}

**Lemma 4.2.** (Lemma 2.2, [17]) If \( \{u_n\} \) is bounded in \( H^\alpha(\mathbb{R}^N) \) and for some \( R > 0 \), we have

\begin{equation}
\sup_{y \in \mathbb{R}^N} \int_{B(y,R)} |u_n(x)|^2 dx \to 0 \text{ as } n \to \infty.
\end{equation}

Then \( u_n \to 0 \) in \( L^q(\mathbb{R}^N) \), for \( 2 < q < \frac{2N}{N-2\alpha} \).
Lemma 4.3. Let \( \{u_n\} \) be a Palais-Smale sequence of \( I \) at level \( d \in \mathbb{R} \). Then \( d \geq 0 \) and \( \{u_n\} \subset H^\alpha(\mathbb{R}^N) \) is bounded. Moreover, every Palais-Smale sequence for \( I \) at a level zero converges strongly to zero.

Proof. Since \( a(x) \geq 0 \), \( \bar{a} > 0 \) and \( \inf_{x \in \mathbb{R}^N} a(x) = \hat{a} > 0 \), we have

\[
\|u_n\|^2_{H^\alpha(\mathbb{R}^N)} + \int_{\mathbb{R}^N} a(x)|u_n(x)|^2dx \geq c\|u_n\|^2_{H^\alpha(\mathbb{R}^N)},
\]

and hence for \( 2 < q < 2^* \)

\[
d + 1 + o(\|u_n\|) \geq I(u_n) - \frac{1}{q}(I'(u_n), u_n)
\]

\[
= \left(\frac{1}{2} - \frac{1}{q}\right) \int_{\mathbb{R}^N} \left(\{(-\Delta)^{\alpha/2}u_n(x)|^2 - \mu \frac{|u_n(x)|^2}{|x|^{2\alpha}} + a(x)|u_n(x)|^2\right)dx
\]

\[
+ \left(\frac{1}{q} - \frac{1}{2^*}\right) \int_{\mathbb{R}^N} (u_n^+(x))^2dx
\]

\[
\geq C\|u_n\|^2_{H^\alpha(\mathbb{R}^N)}.
\]

It follows that \( \{u_n\} \) is bounded in \( H^\alpha(\mathbb{R}^N) \) for \( 2 < q < 2^* \). Since

\[
d = \lim_{n \to \infty} I(u_n) - \frac{1}{q}(I'(u_n), u_n) \geq C \limsup_{n \to \infty} \|u_n\|^2_{H^\alpha(\mathbb{R}^N)},
\]

we have \( d \geq 0 \). Suppose now that \( d = 0 \), we obtain from the above inequality that

\[
\lim_{n \to \infty} \|u_n\|_{H^\alpha(\mathbb{R}^N)} = 0.
\]

\[\square\]

Lemma 4.4. Let \( \{u_n\} \) be a Palais-Smale sequence of \( I \) at level \( d \in \mathbb{R} \) and \( u_n^+ = \max\{u_n, 0\} \). Then \( \{u_n^+\} \) is also a Palais-Smale sequence of \( I \) at level \( d \).

Proof. By the definition of \( I \) we have that as \( n \to \infty \)

\[
I(u_n) = \frac{1}{2} \int_{\mathbb{R}^N} \left(\{(-\Delta)^{\alpha/2}u_n(x)|^2 - \mu \frac{|u_n(x)|^2}{|x|^{2\alpha}} + a(x)|u_n(x)|^2\right)dx
\]

\[
- \frac{1}{2^*} \int_{\mathbb{R}^N} (u_n^+(x))^2dx - \frac{1}{q} \int_{\mathbb{R}^N} k(x)(u_n^+(x))^qdx \to d,
\]

and

\[
< I'(u_n), \phi >
\]

\[
= C_{N, \alpha} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u_n(x) - u_n(y))(\phi(x) - \phi(y))}{|x - y|^{N+2\alpha}}dxdy + \int_{\mathbb{R}^N} a(x)u_n(x)\phi(x)dx - \mu \int_{\mathbb{R}^N} \frac{u_n(x)\phi(x)}{|x|^{2\alpha}}dx
\]

\[
- \int_{\mathbb{R}^N} (u_n^+(x))^{2^*-1}\phi(x)dx - \int_{\mathbb{R}^N} k(x)(u_n^+(x))^{q-1}\phi(x)dx \to 0, \text{ for all } \phi \in H^\alpha(\mathbb{R}^N).
\]
Taking $\phi = -u_n^- = \min\{u_n, 0\}$, from the fact
\[ u_n(x) = u_n^+(x) - u_n^-(x), \quad u_n^+(x)u_n^-(x) = 0, \tag{4.5} \]
we have
\[ o(1) = \langle I'(u_n), -u_n^- \rangle = -C_{N,\alpha} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u_n(x) - u_n(y))(u_n^-(x) - u_n^-(y))}{|x - y|^{N+2\alpha}} \, dx \, dy + \mu \int_{\mathbb{R}^N} u_n(x)u_n^-(x) \, dx \]
\[ - \int_{\mathbb{R}^N} a(x)u_n(x)u_n^- \, dx + \int_{\mathbb{R}^N} (u_n^+(x))^2u_n^- \, dx + \int_{\mathbb{R}^N} k(x)(u_n^+(x))^{q-1}u_n^- \, dx \]
\[ = C_{N,\alpha} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u_n^-(x) - u_n^-(y))^2}{|x - y|^{N+2\alpha}} \, dx \, dy - \mu \int_{\mathbb{R}^N} \frac{(u_n^-(x))^2}{|x|^{2\alpha}} \, dx \]
\[ + C_{N,\alpha} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u_n^-(x)u_n^-(y) + u_n^+(y)u_n^-(x)}{|x - y|^{N+2\alpha}} \, dx \, dy + \int_{\mathbb{R}^N} a(x)(u_n^-)^2 \, dx \]
\[ \geq c \|u_n^-\|^2_{H^\alpha}, \tag{4.6} \]
from (4.6) and the fact $u_n^+(x) \geq 0$, $u_n^-(x) \geq 0$, $a(x) > 0$, then
\[ \|u_n^\|_{H^\alpha} \to 0, \tag{4.7} \]
and
\[ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{2(u_n^+(x) - u_n^+(y))(u_n^-(x) - u_n^-(y))}{|x - y|^{N+2\alpha}} \, dx \, dy \to 0. \tag{4.8} \]
Then from (4.5) and (4.7)-(4.8), we have
\[ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u_n(x) - u_n(y))^2}{|x - y|^{N+2\alpha}} \, dx \, dy \]
\[ = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u_n^+(x) - u_n^+(y))^2 + (u_n(x) - u_n(y))^2 - 2(u_n^+(x) - u_n^+(y))(u_n^-(x) - u_n^-(y))}{|x - y|^{N+2\alpha}} \, dx \, dy \]
\[ = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u_n^+(x) - u_n^+(y))^2}{|x - y|^{N+2\alpha}} \, dx \, dy + o(1). \tag{4.9} \]
That is
\[ \|u_n\|_{H^\alpha} = \|u_n^\|_{H^\alpha} + o(1). \tag{4.10} \]
Thus
\[ \lim_{n \to \infty} I(u_n^+) = \lim_{n \to \infty} I(u_n) = d \]
and
\[ I'(u_n^+, \phi) = I'(u_n, \phi) \to 0 \]
as $n \to \infty$. This complete the proof.

**Lemma 4.5.** All nontrivial critical points of $I_\mu$ are the positive solutions of (1.14).
Proof. Let \( u \neq 0 \) and \( u \in H^\alpha(\mathbb{R}^N) \) be a nontrivial critical point of \( I_\mu \). First, arguing as in the proof of Lemma 4.4 (similar to (4.6) and (4.7)), we can obtain that \( \|u^-\|_{H^\alpha} = 0 \) which gives that
\[
u \geq 0 \text{ a.e. in } \mathbb{R}^N.
\]
(4.11)

Then for any \( x_0 \in \mathbb{R}^N \),
\[
(-\Delta)^\alpha u = \mu \frac{u}{|x|^{2\alpha}} + |u|^{2^* - 2}u \geq 0, \text{ a.e. in } B(x_0, 1), \int_{\mathbb{R}^N} \frac{|u(x)|}{1 + |x|^{N + 2\alpha}} dx \leq c\|u\|_{L^2} \leq c, \tag{4.12}
\]
from Proposition 2.2.6 in [22], we have \( u \) is lower semicontinuous in \( B(x_0, 1/2) \). Combining this with (4.11), it follows \( u(x_0) \geq 0 \). Then \( u(x) \geq 0 \) pointwise in \( \mathbb{R}^N \).

Next we claim that \( u > 0 \) in \( \mathbb{R}^N \). Otherwise there exist \( x_1 \in \mathbb{R}^N \) such that \( u(x_1) = 0 \).

Since \( u \) is lower semicontinuous in \( \overline{B(x_1, 1/2)} \), from Proposition 2.2.8 in [22], it follows \( u \equiv 0 \) in \( \mathbb{R}^N \). This contradicts the assumption \( u \) is nontrivial.

Let \( \{u_n\} \) be a Palais-Smale sequence at level \( d \). Up to a subsequence, we assume that
\[
u_n \rightharpoonup u \text{ in } H^\alpha(\mathbb{R}^N) \text{ as } n \to \infty.
\]

Obviously, we have \( I'(u) = 0 \). Let \( v_n(x) = u_n(x) - u(x) \), as \( n \to \infty \),
\[
v_n \to 0 \text{ in } H^\alpha(\mathbb{R}^N),
\]
\[
v_n \to 0 \text{ in } L^q_{\text{loc}}(\mathbb{R}^N) \text{ for all } 2 < q < 2^*,
\]
\[
v_n \to 0, \text{ a.e. in } \mathbb{R}^N.
\]
(4.15)

As a consequence, we have the following Lemma.

**Lemma 4.6.** \( \{v_n\} \) is a Palais-Smale sequence for \( I \) at level \( d_0 = d - I(u) \).

**Proof.** For \( \phi(x) \in C^\infty_0(\mathbb{R}^N) \), there exists a \( B(0, r) \) such that \( \text{supp } \phi \subset B(0, r) \). Then as \( n \to \infty \),
\[
\left| \int_{\mathbb{R}^N} k(x)(v_n^+(x))^{q-1}\phi(x)dx \right| \leq c \int_{B(0, r)} (v_n^+(x))^{q-1}\phi(x)dx = o(1), \tag{4.16}
\]
\[
\left| \int_{\mathbb{R}^N} \frac{v_n^+(x)\phi(x)}{|x|^{2\alpha}} dx \right| = \int_{|x|\leq r} \frac{v_n^+(x)\phi(x)}{|x|^{2\alpha}} dx \leq c \int_{|x|\leq r} \frac{v_n^+(x)}{|x|^{2\alpha}} dx \tag{4.17}
\]
\[
= \left( \int_{|x|\leq r} |v_n|^q dx \right)^{\frac{1}{q}} \left( \int_{|x|\leq r} \frac{1}{|x|^{2\alpha q}} dx \right)^{1 - \frac{1}{q}} = o(1).
\]

where \( \frac{N}{N - 2\alpha} < \tilde{q} < 2^* \).
By (4.16), (4.17) and (4.18), we have $\langle \phi, I'(v_n) \rangle = o(1)$ as $n \to \infty$. Then similar to (4.17), we have
\[ \|v_n^-\|_{H^s} \to 0, \|u^-\|_{H^s} = 0. \] (4.18)
By Sobolev inequality, (4.17) and (4.18) it follows
\[ \|u_n\|_{L^q} = \|u_n^+\|_{L^q} + o(1), \|v_n\|_{L^q} = \|v_n^+\|_{L^q} + o(1), \|u\|_{L^q} = \|u^+\|_{L^q}. \]
Then by the Brézis-Lieb Lemma in [1] as $n \to \infty$, we have
\[ \int_{\mathbb{R}^N} (v_n^+(x))^q \, dx = \int_{\mathbb{R}^N} (u_n^+(x))^q \, dx - \int_{\mathbb{R}^N} (u^+(x))^q \, dx + o(1) \text{ for all } 2 \leq q \leq 2^*. \] (4.19)
Similarly
\[ \int_{\mathbb{R}^N} (z_n^+(x))^{2^*} \, dx = \int_{\mathbb{R}^N} (u_n^+(x))^{2^*} \, dx - \int_{\mathbb{R}^N} (u^+(x))^{2^*} \, dx + o(1). \] (4.20)
Then similar to (4.19)-(4.21), it follows $I(v_n) = I(u_n) - I(u) + o(1) = d - I(u) + o(1)$. \hfill \Box

**Lemma 4.7.** Assume $t \geq b > 0$ and $q > 1$, then
\[ t^q - (t - b)^q \geq b^q. \]

**Proof.** Let $f(t) = t^q - (t - b)^q$, it follows
\[ f'(t) = qt^{q-1} - q(t - b)^{q-1} > 0 \text{ for } t \geq b > 0, q > 1. \]
Then $f(t) = t^q - (t - b)^q \geq f(b) = b^q$. \hfill \Box

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References

[1] H. Brézis, L. Nirenberg. Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. *Communications on Pure and Applied Mathematics* **36**(1983), 437-477.

[2] C. Bucur. Valdinoci E. *Nonlocal diffusion and applications* (Springer, 2016).

[3] Y. Deng, Z. Guo, G. Wang. Nodal solutions for p-Laplace equations with critical growth. *Nonlinear Analysis* **54** (2003), 1121-1151.

[4] Y. Deng, L. Jin, S. Peng. A Robin boundary problem with Hardy potential and critical nonlinearities. *Journal d’Analyse Mathématique* **104**(2008), 125-154.

[5] S. Dipierro, L. Montoro, I. Peral, et al. Qualitative properties of positive solutions to nonlocal critical problems involving the Hardy-Leray potential. *Calculus of Variations and Partial Differential Equations* **55**(2016), 1-29.

[6] A. Cotioli. Best constants for Sobolev inequalities for higher order fractional derivatives. Journal of Mathematical Analysis and Applications, 2004, 295(1):225-236.

[7] A. Fiscella, G. M. Bisci, R. Servadei. Bifurcation and multiplicity results for critical non-local fractional Laplacian problems. *Bulletin Des Sciences Mathématiques* **140**(2015), 14-35.

[8] P. Felmer, A. Quaas, J. Tan. Positive solutions of the nonlinear Schrödinger equation with the fractional Laplacian. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics* **142**(2012), 1237-1262.

[9] N. Ghoussoub, S. Shakerian. Borderline variational problems involving fractional Laplacians and critical singularities. *Advanced Nonlinear Studies* **15**(2015), 527-555.

[10] L. Y. Jin, Y. B. Deng. A global compact result for a semilinear elliptic problem with Hardy potential and critical non-linearities on $\mathbb{R}^N$. *Science China Mathematics* **53**(2010), 385-400.

[11] J. M. Do Ó and D. Ferraz. Concentration-compactness at the mountain pass level for nonlocal Schrödinger equations, *arXiv preprint arXiv* (2017), 1610.04724v3.

[12] L.Y. Jin, S. M. Fang. Existence of solutions for a fractional elliptic problem with critical Sobolev-Hardy nonlinearities in $\mathbb{R}^N$. *Electronic Journal of Differential Equations*, 2018, 12, 123.

[13] P. L. Lions. The concentration-compactness principle in the calculus of variations. The limit case. II. *Annales De L Institut Henri Poincare Non Linear Analysis* **1**(1985), 45-121.
[14] E. Di Nezza, G. Palatucci, E. Valdinoci. Hitchhiker’s guide to the fractional Sobolev spaces. *Bulletin Des Sciences Mathématiques* **136**(2011), 521-573.

[15] F. Patricio, A. Q. J. Tan. Positive solutions of the nonlinear Schrödinger equation with the fractional Laplacian. *Proceedings of the Royal Society of Edinburgh* **142**(2012), 1237-1262.

[16] G. Palatucci, A. Pisante. Improved Sobolev embeddings, profile decomposition, and concentration-compactness for fractional Sobolev spaces. *Calculus of Variations and Partial Differential Equations* **50**(2014), 799-829.

[17] Felmer, Patricio, A. Q. J. Tan. Positive solutions of the nonlinear Schrödinger equation with the fractional Laplacian. *Proceedings of the Royal Society of Edinburgh* **142**(2012), 1237-1262.

[18] S. Secchi. Ground state solutions for nonlinear fractional Schrödinger equations in $\mathbb{R}^N$. *Journal of Mathematical Physics* **54**(2013), 056108-305.

[19] D. Smets. Nonlinear Schrödinger equations with Hardy potential and critical nonlinearities. *Transactions of the American Mathematical Society* **357**(2005), 2909-2938.

[20] M. Struwe. A global compactness result for elliptic boundary value problems involving limiting nonlinearities. *Mathematische Zeitschrift* **187**(1984), 511-517.

[21] R. Servadei, E. Valdinoci. The Brezis-Nirenberg result for the fractional Laplacian. *Transactions of the American Mathematical Society* **367**(2015), 67-102.

[22] L. Silvestre, Regularity of the obstacle problem for a fractional power of the Laplace operator. *Communications on Pure and Applied Mathematics* **60**(2007), 67-112.

[23] R. Servadei, E. Valdinoci. Fractional Laplacian equations with critical Sobolev exponent. *Revista Matemática Complutense* **28**(2015), 1-22.

[24] X. Shang, J. Zhang, Y. Yang. Positive solutions of nonhomogeneous fractional Laplacian problem with critical exponent. *Communications on Pure and Applied Analysis* **13**(2014), 567-584.

[25] X. Zhu, D. Cao. The concentration-compactness principle in nonlinear elliptic equations. *Acta Mathematica Scientia* **9**(1989), 307-323.

[26] J. Yang. Fractional Sobolev-Hardy inequality in $\mathbb{R}^N$. *Nonlinear Analysis* **119**(2015), 179-185.

[27] X. Wang, J. Yang. Singular critical elliptic problems with fractional Laplacian. *Electronic Journal of Differential Equations* **297**(2015), 1-12.
[28] X. Zhang, B. Zhang, M. Xiang. Ground states for fractional Schrodinger equations involving a critical nonlinearity. Advances in Nonlinear Analysis, 2016, 5(3):293-314.

[29] B. Barrios, E. Colorado, A. D. Pablo, et al. On some critical problems for the fractional Laplacian operator. Journal of Differential Equations, 2012, 252(11):6133-6162.

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