The deterministic-stochastic flow model

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1. The basic conceptions

Let us consider the movement of particles (vehicles) on multilane road fragment. Let $v$ be a velocity of regular movement determined by a number of particles called slow vehicles that differs sufficiently from zero. Let $d = d(v)$ be a dynamic distance, that includes one lane part of a road with the length that covers the length of vehicle and braking way (the size of discretization step by space coordinate) [1], fig. 1.

![Figure 1: Discretization](image)

The dependence $d(v)$ can be approximated by the quadratic relation

$$d(v) = c_0 + c_1 v + c_2 v^2$$

where $c_0$ is the length of vehicle base, $c_1$ is a coefficient which is connected with regard of driver’s reaction in case of unexpected traffic condition change, $c_2 v^2$ is an evaluation of braking way. For example let us consider Tanaka model, [2]

$$d(v) = 5.7 + 0.14v + 0.0022v^2,$$  (1)
If in the equation (1) the velocity is measured in m/sec, then 
$c_0 = 5.7 \text{ m}$, $c_1 = 0.14 \cdot 3.6 = 0.504 \text{ sec}$, $c_2 = 0.0022 \cdot (3.6)^2 = 0.0285 \text{ sec}^2/\text{m}$. In this case we have

$$d(v) = 5.7 + 0.504v + 0.0285v^2. \quad (2)$$

The coefficient $c_2$ depends on condition of the road covering. 
So, according [3], for the wet asphalt-concrete road covering the coefficient $c_2$ is in two times larger than for dry asphalt-concrete road covering, that is $d(v) = 5.7 + 0.504v + 0.057v^2$, and for a road covered with ice $d(v) = 5.7 + 0.504v + 0.165v^2$. Fig. 2 shows the dependence $d(v)$ for different conditions of road covering. The graph of the function $d(v)$ for different conditions of the road covering is shown on fig. 2.

![Graph showing dynamic distance for different road conditions](image)

**Figure 2:** Dynamic distance: $d_1(v)$ — dry asphalt; $d_2(v)$ — wet asphalt; $d_3(v)$ — road covered with ice

Thus the immediate location of a vehicle on the road can be shown by the cell field, which for simplicity is synchronized in
relation to lanes (fig. 3).

Let $T$ be magnitude of discrete time unit, that fixes flow state. If the ”snap” does not change then the flow is supposed to be steady. The ratio of the number of occupied cells to their total number is called the regularity, $r \in [0; 1]$. If $r = 1$ then we have the steady flow, an army column, which presents the movement of the column with the constant velocity $v = \text{const}$. It is a consequence of busy cells. If $r < 1$, then the individual transition of a vehicle to a front or neighbour cell diagonally is possible in a time unit $T$ (fig. 4).

These transitions are caused by several reasons but one of the most important is that drivers want to drive their cars with a higher velocity. Let $p = p(t)$ be a stochastic measure of individual transitions at time unit $t$ to the cell ahead when this cell is empty. Of course this value also depends on other characteristics of the flow and this is the subject of the further consideration.
Thus each car will do an attempt to move forward with probability $p(T)$ independent of the behavior of the other vehicles. If $r \approx 0$ then there is no obstacle for such transitions as a rule, and if $r$ is essentially greater than 0 then the considered measure depends on $r$. Let $p(r, t)$ be this measure. It is clear that $p(0, t) = p(T)$ and $p(r, t)$ is an non-increasing function $r \in [0; 1]$, $p(1, T) = 0$.

Let us evaluate the function $p(r, t)$. Suppose that in the neighbourhood of the considered cell the states of the three neighbour cells that follow ahead are independent (fig. 5).

![Manoeuvres from an inner cell](image)

Figure 5: Manoeuvres from an inner cell

Thus accurate to the coefficient $p(T)$ the probability $p_i(r; T)$ of a transition equals

$$R_3(r) = (1 - r) \cdot 1 + r(2(1 - r)^2 - (1 - r)^4) =$$

$$= (1 - r)(1 + 2r(1 - r) - r(1 - r)^3) =$$

$$= (1 - r)(1 + 2r - r^2 - r + 3r^2 - 3r^3 + r^4) =$$

$$= (1 - r)(1 + r + r^2 - 3r^3 + r^4) =$$

$$= 1 - 4r^3 + 4r^4 - r^5.$$

Similarly, for the case of inside lane on a multilane road (or on a two-lane road), we have

$$R_2(r) = 1 - 2r^2 + r^3.$$
At last for the movement on a lane (or for canalized movement) we obtain
\[ R_1(r) = 1 - r. \]
Thus
\[ p_1(r, T) \simeq p(T)(1 - r). \]
\[ p_2(r, T) \simeq p(T)(1 - 2r^2 + r^3); \]
\[ p_3(r, T) \simeq p(T)(1 - 4r^3 + 4r^4 - r^5), \]
(fig. 6).

![Diagram](image)

**Figure 6:** Evaluation \( R_i(r) \) of dependence \( p_i(r; T)/p(T) \)

In common case it can be considered that \( p(r; T) \) is a continuous function on \( r \); (the addition of one busy cell cannot have any essential influence upon the mean velocity).

At last in common case the considered function can be also depend on \( v \). Really, every driver has his own knowledge of velocity. Therefore the additive component to expected velocity \( (v_1 > v) \) is compensated by individual attempts. For example according to the next scheme we have \( v + d(v)p(r, T, v)/T = v_1. \)
2. DST-flows (Deterministic-stochastic traffic)

The regular velocity can be also considered as the determinate component \( v_{\text{det}} \) of the flow. Beside the collective motive of the particle behavior in the flow each particle also has its own intensions. Hence the flow can be presented as a composition of total (common, socialist) and private (liberal, individual) behavior. Let us consider a flow which consists of the particles with identical strategies of behavior (homogeneous dst-flow,). Then the flow velocity is the sum of the deterministic and stochastic components

\[
v_{\text{dst}} = v_{\text{det}} + v_{\text{st}}
\]

(3)

where the stochastic transitions are independent and distributed equally. In this relation average meaning of individual transition is

\[
\bar{v}_{\text{st}} = p(r, T, v) \frac{d(v)}{T}.
\]

Hence the average value and the dispersion of the velocity is accordingly equal to

\[
\bar{v} = v + p(r, T, v) \frac{d(v)}{T},
\]

(4)

\[
\bar{v}^2 = p(r, T, v)(1 - p(r, T, v))(\frac{d(v)}{T})^2.
\]

Let \( \rho \) be the density of traffic flow per lane. Then

\[
\bar{\rho} = \frac{r}{d(v)}.
\]

(5)

Hence we have intensity per lane

\[
\bar{q} \simeq \rho \times \bar{v} = \frac{r}{d(v)}(v + p(r, T, v)\frac{d(v)}{T}) = \frac{rv}{d(v)} + \frac{rp(r, T, v)}{T}.
\]

(6)
The function (6) depends on three variables. As

$$p_i(r, T, v) \simeq p(0, T, v)R_i(r)$$

we have ($i$ is number of lanes)

$$\bar{q}_i = \frac{rv}{d(v)} + \frac{rp(0, T, v)R_i(r)}{T}.$$ 

Suppose $T \to 0$. Reassume that

$$\frac{p(0, T, v)}{T} \to p(v).$$

(7)

Then

$$\bar{v}_{st}(i) = p(0, T, v)R_i(r)\frac{d(v)}{T} = p(v)R_i(r)d(v)$$

and the intensity is equal to

$$\bar{q}_i(v, p(v), r) = \frac{rv}{d(v)} + rp(v)R_i(r).$$

(8)

According to (5) we receive

$$\bar{q}_i(\rho, v, p(v), \rho) = \rho v + \rho d(v)p(v)R_i(\rho d(v)).$$

(9)

Equation (9) generalizes the classical relation (main diagram) which is obtained for $p(v) = 0$.

In this case

$$\bar{v}_i = v + p(v)R_i(r)d(v).$$

3. The single-lane traffic "Regularity-Velocity-Rate"

Let us consider the case of one lane, $n = 1$. We have $\bar{v}_{st} = p(v)(1 - r)d(v)$ and the equation (6) for small $T$ can be written as

$$\bar{q}_1 \simeq \frac{rv}{d(v)} + p(v)r(1 - r) = v\rho + p(v)d(v)\rho(1 - d(v)\rho),$$
where $\rho d(v) = r \leq 1$. For $\rho d(v) > 1$ the dynamic distance is not regarded and for this reason $v$ diminishes.

As $d(v) = \frac{r}{\rho} = c_0 + c_1 v + c_2 v^2$ we have

$$\bar{q}_1 \simeq \bar{q}_1(r, v, p) = \frac{rv}{c_0 + c_1 v + c_2 v^2} + pr(1 - r), \quad (10)$$

where coefficients $c_0, c_1, c_2$ are assigned as in equation (1); $p = p(v)$ 1/sec.

The function $\bar{q}_1$ is defined on the rectangular $0 < r < 1$, $0 < v < v_{max}$. Suppose $p \equiv 1$. Let us represent the graph of the intensity function (fig.7).

For the same values of parameters the dependence of the individual velocity on $r$ and $v$ is shown on fig. 8.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig7.png}
\caption{Function $\bar{q}_1(r, v, 1)$ vehicle/sec}
\end{figure}

It can be seen from fig. 7 that the largest flow rate is reached when the values of basic variables $r$ and $v$ are near by $(0.8;15)$. In some neighborhood of point of maximum the trajectories of
intensity level lines are close of and are included to the considered set of values. Thus the flow intensity can be still invariant when the parameters of the flow change.

Comparing the behavior of graphs of the intensity and the velocity (fig. 7–8) we note that the conflict of the collective and individual purposes occurs. Just, if the velocity remains constant then the intensity changes, as the level lines intersect.

In common case we have

$$\frac{\partial q_1}{\partial r} = \frac{v}{c_0 + c_1 v + c_2 v^2} + p - 2rp = 0,$$

$$\frac{\partial q_1}{\partial v} = \frac{r}{c_0 + c_1 v + c_2 v^2} - \frac{rv(c_1 + c_2 v)}{(c_0 + c_1 v + c_2 v^2)^2} = 0,$$

i.e. $c_0 + c_1 v + c - 2v^2 = v(c_1 + 2c_2 v)$ and $c_2 v^2 = c_0$. Thus

$$v^* = \sqrt{\frac{c_0}{c_2}},$$
\[ r^* = \frac{1}{2} + \frac{1}{2p c_0 + c_1 v^* + c_2 v^{*2}}. \]

4. Stability of the traffic and scattering of the fundamental diagram

Figure 9: Experimental data in field "density–intensity", [4]
Figure 10: Experimental data in field "density–velocity" with different intervals of averaging, the upper fragment is taken from [4], the below fragment is taken from [6]
The considered model allows to explain the appearance on the fundamental diagram of scattering domains of density where different values of intensity can correspond to the same value of density. The existence of scattered domains has been discovered on the basis of the experimental results, described in [4,5], fig. 9–11.

If the flow density \( \rho < \rho_{\text{max}}/2 \) then the movement is stable and the traffic can be described well by a hydrodynamic model. In this case the regular movement \( r = 1 \) is a single stable state. The effect of random disturbances are short-lived and the traffic return fast to the previous regime.

For sufficiently large densities, that are out of the range of the stability of the traffic, the appearances of disturbances result in that regularity of traffic has values \( r < 1 \) because of reaction of drivers on changes of the situation that gives a change of the dynamic dimension. The density becomes a random value. Dispersion of the density becomes not equal to zero. The changes of rate \( r/d(v) \) result in that values of densities, which have been found in the experiments, has dispersion near the average value when the intensity of the traffic does not change. Thus the same value of the intensity, obtained in the experiments, can correspond to different values of densities and therefore scattered domains on the fundamental diagram appear.
5. Multilane motion "Regularity – velocity – intensity"

Similarly, for the case of two lanes

\[
\bar{q}_2 = \bar{q}_2(r, v, p) = \frac{rv}{c_0 + c_1v + c_2v^2} + pr(1 - 2r^2 + r^3). \tag{11}
\]

we have the graphic dependences for the intensity and the velocity (fig. 12, 13).
Figure 12: Function $\bar{q}_2(r, v, 1)$ (vehicle/sec)

![Diagram](image1)

Figure 13. Function $\bar{v}_2(r, v, 1)$ (m/sec)

![Diagram](image2)
Finally, for three lanes

$$\bar{q}_3 = \bar{q}_3(r, v, p) = \frac{rv}{c_0 + c_1v + c_2v^2} + pr(1 - 4r^3 + 4r^4 - r^5) \quad (12)$$

we obtain the dependences shown on fig. 14, 15.

Figure 14. $\bar{q}_3(r, v, 1)$ (vehicle/sec)

Figure 15. $\bar{v}_3(r, v, 1)$ (m/sec)
6. Collective and individual for the flows with a constant density

For fixed density $\rho_0 = r/d(v)$ from (12) we obtain

$$\bar{q}_1 = \bar{q}_1(\rho, v, p) = v\rho + pd(v)\rho(1 - d(v)\rho) = v\rho + p(c_0 + c_1v + c_2v^2)\rho(1 - \rho(c_0 + c_1v + c_2v^2)) = d^{-1}(r/v)\rho + pr(1 - r).$$  \hspace{1cm} (13)

The inequality $r < 1$ is equivalent to inequality $\rho d(v) < 1$, i.e.

$$c_0 + c_1v + c_2v^2 < \frac{1}{\rho},$$

$$v < -\frac{c_1}{2c_2} + \sqrt{\left(\frac{c_1}{2c_2}\right)^2 - \frac{c_0}{c_2} + \frac{1}{\rho}}.$$  \hspace{1cm} (14)

Let us represent the evaluations of dependence (13) of intensity on velocity for the fixed density. The question is how the rate changes if the flow velocity changes abruptly during a small period of time and the density is constant. If the case of one lane is considered then for $\rho_0 = 0.01$ vehicle/m we have the dependence shown in fig. 16. As for dependences, shown on fig. 17–19, this dependence corresponds to the case of dry asphalt (eq. (2)): $c_0 = 5.7$ m/sec; $c_1 = 0.504$ sec; $c_2 = 0.0285$ sec/m. Range of velocities below 25 m/sec, i.e 90 km/h, is considered.
Figure 16. $\bar{q}_1(v, p)$ (vehicle/sec), $\rho = 0.01$ (vehicle/m)

Figure 17. Dependence $\bar{q}_1(v, p)$ (vehicle/sec), $\rho = 0.05$ (vehicle/m)

The dependence for $\rho_0 = 0.05$ is shown on fig. 14. On fig.17 there is the evident unstability in the neighbourhood of the right lower angle. For the case of the three-lane road we have dependences shown on fig.18-19.
Figure 18. $\bar{q}_3(v,p)$ (vehicle/sec), $\rho = 0.01$ (vehicle/m)

Figure 19. $\bar{q}_3(v,p)$ (vehicle/sec), $\rho = 0.05$ (vehicle/m)
7. Influence of "blue lights" upon the flow rate

The particles of the two types characterized by of the different functions of dynamic distances \( d_1(v) \) and \( d_2(v) \) are considered. Suppose \( d_2(v) < d_1(v) \) for each allowed \( v \). If \( v \) is the regular velocity of the unmixed flow with zero individual velocity of the unmixed flow, then by mixing between two large particles small particle would emerge (percolation). Hence the flow velocity can be evaluated as \( v_1(v) = d_1^{-1}(d_1(v)/(2)) \). Let us suppose still that the number of such particles is rather small. That the flow intensity in the new conditions is

\[
\rho v_1(v) + \rho \frac{p(d(v)/2, T)}{T}(d(v)/2),
\]

where the function \( p(r, T) \) is defined as the function \( p(r, T) \) introduced in section 1; dynamic distance \( d(v) \) is calculated according equation (2).

The value \( v_1(v) \), which satisfies equation \( d(v_1(v)) = \frac{d(v)}{2} \) that is equation

\[
c_0 + c_1 v_1(v) + c_2 v_1^2(v) = \frac{c_0 + c_1 v + c_2 v^2}{2},
\]

exists only in the case

\[
c_1 v + c_2 v^2 \geq c_0. \tag{15}
\]

Suppose also

\[
1 - \rho d(v)/2 > 0,
\]

i.e.

\[
c_0 + c_1 v + c_2 v^2 < \frac{2}{\rho},
\]

\[
v < -\frac{c_1}{2c_2} + \sqrt{\left(\frac{c_1}{2c_2}\right)^2 - \frac{c_0}{c_2} + \frac{2}{c_2\rho}}.
\]
Thus the equalities
\[-\frac{c_1}{2c_2} + \sqrt{\left(\frac{c_1}{2c_2}\right)^2 + \frac{c_0}{c_2}} < v < -\frac{c_1}{2c_2} + \sqrt{\left(\frac{c_1}{2c_2}\right)^2 - \frac{c_0}{c_2} + \frac{2}{c_2\rho}}.\]

are true.

Figure 20. Dependence $Q(\rho, v)$

Suppose that
\[p(v)d(v)/2 = v - v_1(v).\]
Hence
\[q_i(\rho, v) = \rho\left(v_1(v) + (v - v_1(v))R_i(\rho d(v)/2)\right)\].

For example, if $i = 1$ we have
\[q_1(\rho, v) = \rho(v_1(v) + (v - v_1(v))(1 - \rho d_1(v)/2))\].

As
\[v_1(v) = -\frac{c_1}{2c_2} + \sqrt{\left(\frac{c_1}{2c_2}\right)^2 - \left(\frac{c_0}{2c_2} - \frac{c_1}{2c_2}v - \frac{v^2}{2}\right)},\]
we obtain the ratio of intensity of disturbed and undisturbed conditions

\[ Q(\rho, v) = \frac{v_1(v) + (v - v_1(v))(1 - \rho d_1(v)/2)}{v}. \]

Relative variation of flow rate of slow particles in case of small number of fast special vehicles is shown on fig. 20. We still suppose that \( c_0 = 5.7 \text{ m/sec}; c_1 = 0.504 \text{ sec}; c_2 = 0.0285 \text{ sec/m}. \)

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