CONTINUOUS FAMILIES OF HAMILTONIAN TORUS ACTIONS

ANDRÉS VIÑA

Abstract. We determine conditions under which two Hamiltonian torus actions on a symplectic manifold $M$ are homotopic by a family of Hamiltonian torus actions, when $M$ is a toric manifold and when $M$ is a coadjoint orbit.

MSC 2000: 53D05, 57S05

1. Introduction

Let $(M,\omega)$ be a closed symplectic $2n$-manifold, and let $G$ be a compact Lie group. Given two Hamiltonian $G$-actions on $M$, it is a natural issue to analyze if one can be deformed in the another by a continuous family of Hamiltonian $G$-actions. In this paper we will consider this matter; that is, we will determine necessary conditions for two Hamiltonian $G$-actions on $M$ to be connected by a homotopy consisting of Hamiltonian $G$-actions.

The mathematical setting is the following: $\text{Ham}(M)$ will be the Hamiltonian group of $(M,\omega)$ [20], [21], and $\text{Hom}(G,\text{Ham}(M))$ will denote the space of all Lie group homomorphisms $\psi$ from $G$ to $\text{Ham}(M)$; that is, the space consisting of the Hamiltonian $G$-actions on $M$.

Given $\psi,\psi' \in \text{Hom}(G,\text{Ham}(M))$, we write $\psi \sim \psi'$ iff $\psi$ and $\psi'$ belong to the same connected component of $\text{Hom}(G,\text{Ham}(M))$. In other words, $\psi \sim \psi'$ iff there is a continuous family $\{\psi^s : G \to \text{Ham}(M)\}_{s \in [0,1]}$ of Lie group homomorphisms, such that $\psi^0 = \psi$ and $\psi^1 = \psi'$; that is, $\psi$ and $\psi'$ are homotopic by a family of group homomorphisms. The corresponding quotient space is denoted by $[G,\text{Ham}(M)]_{gh}$.

In this paper we will consider the following three general aspects:

1. Necessary conditions for the $\sim$-equivalence. It would be desirable to obtain necessary and sufficient conditions for the $\sim$-equivalence of two $G$-actions in the general case, but we will prove more modest results. To each Hamiltonian $G$-action $\psi$ we will associate a cohomology class $S_\psi$, such that the map $S_\sim$ is compatible with the relation $\sim$, and we will use the map $S_\sim$ to distinguish different elements of $[G,\text{Ham}(M)]_{gh}$ (see Theorem [1]). In some particular cases we will give direct proofs of the $\sim$-equivalence of two actions on which the map $S_\sim$ takes the same value.

2. Equivalence relations between toric actions. A $G$-action can be regarded as a representation of $G$ in the group $\text{Ham}(M)$, so we can consider another natural equivalence relation in the space $\text{Hom}(G,\text{Ham}(M))$: Two Hamiltonian $G$-actions on $M$, $\psi$ and $\psi'$, are said to be $r$-equivalent if there exist an element $h \in \text{Ham}(M)$, such that $h \circ \psi_g \circ h^{-1} = \psi'_g$ for all $g \in G$; that is, if they are equivalent as

Key words and phrases. Hamiltonian diffeomorphisms, toric actions, coadjoint orbits.

This work has been partially supported by Ministerio de Ciencia y Tecnología, grant MAT2003-09243-C02-01.

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representations of \(G\) in the group \(\text{Ham}(M)\). We will compare the \(r\)-equivalence and the relation \(\sim\) in the space of toric actions on \(M\). We will prove, among other properties, that the \(r\)-equivalence implies the \(\sim\)-equivalence, and that the equality of the moment polytopes of \(\psi\) and \(\psi'\) is a necessary condition for the \(\sim\)-equivalence of them (see Theorem \(3\)).

3. \(\sim\)-equivalence of \(U(1)\)-actions. When \(G = U(1)\), each \(\psi \in \text{Hom}(G, \text{Ham}(M))\) determines an element of \(\{\psi\} \in \pi_1(\text{Ham}(M))\). We will compare our relation \(\sim\) with the equivalence in \(\pi_1(\text{Ham}(M))\) in some particular cases for which the homotopy type of \(\text{Ham}(M)\) is known. In \([3]\) Gromov proved that \(\text{Ham}(\mathbb{CP}^2, \omega_{FS})\) is homotopy equivalent to the group \(PU(3)\), where \(\omega_{FS}\) is the Fubini-Study symplectic structure, and in \([1]\) Abreu and McDuff determined the homology of the group of symplectomorphisms of a rational ruled surface. In order to carry out the aforementioned comparison and also by its very interest, we will study the relation \(\sim\) between \(U(1)\)-actions in two cases:

a) When \(M\) is the total space of the fibre
\[
P(L_1 \oplus \cdots \oplus L_{n-1} \oplus \mathbb{C}) \to \mathbb{CP}^1,
\]
\(L_i\) being a holomorphic line bundle over \(\mathbb{CP}^1\).

b) When the manifold \(M\) is a coadjoint orbit; we look more closely at the case of \(U(1)\)-actions on coadjoint orbits of \(SU(n)\) diffeomorphic to Grassmann manifolds.

The comparison of the results of Gromov and Abreu-McDuff above mentioned and our results shows the non-injectivity of the obvious map
\[
[\psi] \in [U(1), \text{Ham}(M)]_{gh} \mapsto \{\psi\} \in \pi_1(\text{Ham}(M)),
\]
when \(M\) is a Hirzebruch surface and when \(M = \mathbb{CP}^2\) (see Remark below Theorem \(7\) the paragraph after Theorem \(8\)) and Remark below Theorem \(11\).

Statement of main results

1. Necessary conditions for the \(\sim\)-equivalence. An action \(\psi\) of \(G\) on \(M\) determines the fibre \(p : M_{\psi} := EG \times_G M \to BG\), with fibre \(M\), on the classifying space \(BG\) of the group \(G\). If \(\psi\) is Hamiltonian, the fibre \(M_{\psi}\) supports the coupling class \(c_\psi \in H^2(M_{\psi}, \mathbb{R})\) \([20], [10]\). Then fibre integration of \(\exp(c_\psi)\) defines an element \(S_\psi\) in the cohomology \(H(BG)\). We will prove that \(S_\psi = S_{\psi'}\), if the \(G\)-actions \(\psi\) and \(\psi'\) are \(\sim\)-equivalent. That is,

\[\text{Theorem 1.} \text{ Let } \psi \text{ and } \psi' \text{ be two Hamiltonian } G\text{-actions on } M, \text{ if } S_\psi \neq S_{\psi'}, \text{ then } [\psi] \neq [\psi'] \in [G, \text{Ham}(M)]_{gh}.\]

The cohomology class \(S_\psi\) can be calculated by the localization formula of the \(G\)-equivariant cohomology. This formula gives an expression for \(S_\psi\) which is a linear combination of exponentials functions. If \(G\) is the \(m\)-dimensional torus, the coefficients of this combination are rational functions of \(m\) variables \((u_1, \ldots, u_m)\). On the other hand, given real constants \(\gamma_{ij}\), the exponential functions \(F_i(u_1, \ldots, u_m) := \exp\left(\sum_j \gamma_{ij} u_j\right)\) are \(\mathbb{R}(u_1, \ldots, u_m)\)-linearly independent (see Lemma \([13]\)). This property will allow us to compare the values of the map \(S_\psi\) on different \(G\)-actions. The comparison of the values taken by this map, together with Theorem \(1\) will permit us to determine necessary conditions, with a simple geometric content, for two torus actions to be \(\sim\)-equivalent (see, for example, Theorem \(2\) and Theorem \(5\)).

Let us assume that \((M, \omega)\) is equipped with a Hamiltonian action of the Lie group \(G\). If \(X \in \mathfrak{g}\) defines a circle action \(\psi^X\) on \(M\), we say that \(X\) is of type
s when a connected component of the fixed point set of $\psi^X$ is a $2s$-dimensional submanifold of $M$, and there is no submanifold of greater dimension in the fixed point set. The set of elements of type $s$ will be denoted $\text{Type}(s)$. We will prove the following result, which gives a necessary condition for the $\sim$-equivalence whose geometric meaning is clear.

**Theorem 2.** If $X, Y \in \mathfrak{g}$ are of different type, then the corresponding circle actions $\psi^X$ and $\psi^Y$ are not homotopic by means of a family of $U(1)$-actions.

2. Equivalence relations between torus actions.

In Section 2 we will consider toric actions on $M$; that is, Hamiltonian effective actions of the $n$-dimensional torus $T := U(1)^n$ on $M$. A toric action $\psi$ defines a moment polytope $\Delta(\psi) = \Delta$ in $t^*$, and this polytope is uniquely determined by $\psi$ up to translation.

Let $\psi, \psi'$ be toric actions on $M$, and let $\Delta, \Delta'$ denote the moment polytopes with center of mass at 0 associated to $\psi$ and $\psi'$, respectively. By $\mu$ and $\mu'$ we denote the corresponding moment maps, such that $\text{im} \mu = \Delta$ and $\text{im} \mu' = \Delta'$. We state the following properties:

(i) $\Delta = \Delta'$.
(ii) $\Delta = \Delta'$ and there exists $h \in \text{Symp}_0(M)$ such that $h \circ \psi_g \circ h^{-1} = \psi'_g$ for all $g \in T$, where $\text{Symp}_0(M)$ is the connected component of the identity in the group $\text{Symp}(M)$ of symplectomorphisms of $M$.
(iii) $\psi$ and $\psi'$ are $r$-equivalent.
(iv) $S_\psi = S_{\psi'}$.
(v) $[\psi] = [\psi'] \in [T, \text{Ham}(M)]_{gh}$.

We will prove the following Theorem

**Theorem 3.** The above properties satisfy the implications described in the following diagram

$$
\begin{array}{ccc}
(iv) & \longrightarrow & (i) \\
\uparrow & & \uparrow \\
(v) & \longleftarrow & (iii) \longleftarrow (ii)
\end{array}
$$

From this theorem it follows, among other properties, that the maps $S_\cdot$ and $\Delta(\cdot)$ are equivalent tools for distinguishing $\sim$-inequivalent toric actions. The equality $\Delta = \Delta'$ is a very simple geometric necessary condition for the equality $[\psi] = [\psi']$, and a sufficient condition can be obtained by complementing that necessary condition as it is stated in (ii), but this complementary condition is not easy to check, in general.

Given $\varphi, \varphi' \in \text{Hom}(G, \text{Ham}(M))$, we write $\varphi \sim_{\tau \varphi} \varphi'$ if $\varphi$ and $\varphi'$ are $\sim$-equivalent “up to reparametrization” of $G$; that is, if there is an automorphism $v$ of $G$ such that $\varphi \circ v \sim \varphi'$. Following Karshon [15] we say that the Hamiltonian actions $\varphi$ and $\varphi'$ are *conjugate* if they are representations in $\text{Symp}(M)$ equivalent “up to reparametrization”; that is, if there is $v \in \text{Aut}(G)$ and $\xi \in \text{Symp}(M)$ such that $\xi \circ (\varphi \circ v)(g) \circ \xi^{-1} = \varphi'(g)$, for all $g \in G$. To complete the scheme we define the $r$-equivalence up to reparametrization: $\varphi$ and $\varphi'$ are $r$-equivalent up to reparametrization if there exist $v \in \text{Aut}(G)$ and $h \in \text{Ham}(M)$ such that $h \circ (\varphi(v(g)) \circ h^{-1} = \varphi'(g)$, for all $g \in G$. 
Some of the implications in Theorem 3 remain valid when the equivalence relations are replaced by the corresponding “up to reparametrization” relations. For \( \psi \) and \( \psi' \) toric actions on \( M \) we state the following properties:

(a) \( \psi \) and \( \psi' \) are \( r \)-equivalent up to reparametrization.
(b) \( \psi \sim_{rp} \psi' \).
(c) \( \psi \) and \( \psi' \) are conjugate.
(d) There exists \( V \in GL(n, \mathbb{Z}) \), such that \( V(\Delta) = \Delta' \).

The following Proposition is also proved in Section 2

**Proposition 4.** For the above properties the implications of the following diagram

\[
\begin{array}{ccc}
(a) & \rightarrow & (b) \\
\downarrow & & \downarrow \\
(c) & \leftrightarrow & (d)
\end{array}
\]

hold.

Let \( M \) be a toric manifold and \( \psi \) the effective action of \( T \) which endows \( M \) with the toric structure. If \( X \) belongs to the integer lattice of \( t \), it generates an \( S^1 \)-action \( \psi^X \). If the element \( X \) is of type \( s \), then there is an \( s \)-face \( f \) in the moment polytope \( \Delta \) such that \( \langle f, X \rangle = 0 \), but there is no an \( r \)-face orthogonal to \( X \) with \( r > s \). If \( X \) is of Type(0), the fixed point set for \( \psi^X \) is the set \( \{ \mu^{-1}(P_i) \mid i = 1, \ldots, s \} \), where the \( P_i \)'s are the vertices of the moment polytope \( \Delta \) and \( \mu \) the moment map.

We will prove the following Theorem that gives a simple geometric condition, under which two elements \( X, Y \) of type 0 define circle actions \( \psi^X \) and \( \psi^Y \) no homotopic by a family of \( U(1) \)-actions.

**Theorem 5.** Let \( M \) be a toric manifold and \( \Delta \) the moment polytope with center of mass at 0. If \( X, Y \) belong to the integer lattice of \( t \), they are in Type(0) and

\[
\{ \langle P_i, X \rangle : i = 1, \ldots, s \} \neq \{ \langle P_i, Y \rangle : i = 1, \ldots, s \},
\]

\( P_1, \ldots, P_s \) being the vertices of \( \Delta \). Then \( [\psi^X] \neq [\psi^Y] \in [U(1), \text{Ham}(M)]_{gh} \).

Proposition 21 in Section 2 is the version “up to reparametrization” of Theorem 5.

### 3. \( \sim \)-equivalence of \( U(1) \)-actions.

**S\( ^1 \)-actions on \( \mathbb{P}(L_1 \oplus \cdots \oplus L_{n-1} \oplus \mathbb{C}) \).**

If \( \Delta \) is a Delzant polytope in \( (\mathbb{R}^n)^* \), we denote by \( M_\Delta \) the symplectic toric manifold associated to \( \Delta \) \cite{5, 9}. We will study the relation \( \sim \) when \( M_\Delta \) is diffeomorphic to the total space of the fibration (1.1). In this case the result stated in Theorem 5 will be completed giving a sufficient condition for two elements in the integer lattice to define \( \sim \)-equivalent \( U(1) \)-actions.

Let us assume that \( M_\Delta \) is diffeomorphic to \( \mathbb{P}(L_1 \oplus \cdots \oplus L_{n-1} \oplus \mathbb{C}) \), where \( L_i \) is the holomorphic line bundle over \( CP^1 \) with first Chern number \(-a_i\). The \( S^1 \)-action defined by \( b \in \mathbb{Z}^n \) is denoted by \( \psi_b \). To study the relation \( \sim \) between the \( \psi_b \)'s we introduce the following definition:

Given \( b = (b_1, \ldots, b_n) \) and \( b' = (b'_1, \ldots, b'_n) \) two elements of the lattice \( \mathbb{Z}^n \). We write \( b \equiv b' \) iff either \( b = b' \), or

\[
b'_j = b_j + a_j b_n, \quad \text{for} \quad j = 1, \ldots, n-1 \quad \text{and} \quad b'_n = -b_n.
\]
We will prove the following Theorem, which gives the aforesaid sufficient condition for the \( \sim \)-equivalence between \( \psi_b \) and \( \psi_{b'} \).

**Theorem 6.** Let \( b \) and \( b' \) be two elements of the integer lattice \( \mathbb{Z}^n \). Let \( \psi_b, \psi_{b'} \) be the respective circle actions on the symplectic manifold \( M_\Delta \) diffeomorphic to \( \mathbb{P}(L_1 \oplus \cdots \oplus L_{n-1} \oplus \mathbb{C}) \). If \( b \equiv b' \), then \( \psi_b \sim \psi_{b'} \).

When the manifold \( M_\Delta \) is a Hirzebruch surface we will prove a stronger result than the one stated in Theorem 6. Let \( M_\Delta \) be the Hirzebruch surface \( \mathbb{P}(L_k \oplus \mathbb{C}) \) (where \( L_k \) is the holomorphic line bundle over \( \mathbb{C}P^1 \) with first Chern number \( k \)). If \( b = (b_1, b_2) \) is an element in \( \mathbb{Z}^2 \), set \( S_b = S_{\psi_b} \). We will prove the following Theorem.

**Theorem 7.** Let \( M_\Delta \) be the Hirzebruch surface \( \mathbb{P}(L_k \oplus \mathbb{C}) \), with \( k \neq 0 \), and \( b, b' \in \mathbb{Z}^2 \), then the following statements are equivalent

(a) \( b \equiv b' \).

(b) \( \psi_b \sim \psi_{b'} \).

(c) \( S_b = S_{b'} \).

**Remark.** Set \( c = (1, 0) \) and \( c' = (0, 1) \). When \( k \) is odd,

\[
\{ \psi_c \}, \{ \psi_{c'} \} \in \pi_1(\text{Ham}(\mathbb{P}(L_k \oplus \mathbb{C})))
\]

are elements of infinite order in this group (see Section 4 of [22]). On the other hand, \( \pi_1(\text{Ham}(\mathbb{P}(L_k \oplus \mathbb{C}))) = \mathbb{Z} \) (see [1]). Hence there exist \( m, n \in \mathbb{Z} \setminus \{0\} \) such that \( \{ \psi_{mc} \} = \{ \psi_{nc'} \} \). As the condition (a) of Theorem 7 does not hold for \( mc \) and \( nc' \), it follows that \( \psi_{mc} \) and \( \psi_{nc'} \) are not \( \sim \)-equivalent; that is, the map [172] is not injective for the Hirzebruch surface we are considering.

When \( k = 0 \) the manifold \( \mathbb{P}(L_k \oplus \mathbb{C}) \) is diffeomorphic to \( S^2 \times S^2 \). In this case let \( \tau \) and \( \sigma \) denote the total areas of respective the spheres in the product \( S^2 \times S^2 \). We will prove following result

**Theorem 8.** If \( b_1 b_2 \neq 0 \neq b_1' b_2' \) and \( \sigma / \tau \notin \mathbb{Q} \), then the following statements are equivalent

(i) \( |b_1'| = |b_1| \) and \( |b_2'| = |b_2| \).

(ii) \( S_b = S_{b'} \).

(iii) \( \psi_b \sim \psi_{b'} \).

When \( \sigma > 1 \) and \( \tau = 1 \),

\[
\pi_1(\text{Ham}(S^2 \times S^2)) = \mathbb{Z} / 2\mathbb{Z} \oplus \mathbb{Z} / 2\mathbb{Z} \oplus \mathbb{Z}.
\]

The first two summands of the right hand of (1.3) come from the action of \( SO(3) \times SO(3) \) on \( S^2 \times S^2 \). The generator of the third summand may be represented by the loop \( \{ R_t^u \}_t \), where

\[
R^u_t : (u, v) \in S^2 \times S^2 \mapsto (u, R^u_t(v)) \in S^2 \times S^2,
\]

\( R^u_t \) being the rotation of the fiber \( S^2 \), with angle \( 2\pi t \) and axis through \( u \) (see [11, 19]). Therefore \( \psi_c \), with \( c = (1, 0) \), is a representative of the generator of the first summand in the right hand of (1.3). Thus \( \{ \psi_c \} = \{ \psi_{mc} \} \in \pi_1(\text{Ham}(S^2 \times S^2)) \), for all \( m \in 2\mathbb{Z} + 1 \). But, according to Theorem 8 from

\[
[\psi_c] = [\psi_{mc}] = [U(1), \text{Ham}(S^2 \times S^2)]_{gh}
\]

it follows \( m = \pm 1 \), when \( \sigma \notin \mathbb{Q} \).
Although the results stated in Theorem 7 and in Theorem 8 the equality $S_b = S_{b'}$ is not equivalent to $[\psi_b] = [\psi_{b'}] \in [U(1), \text{Ham}(M)]_{gh}$, in general. As we said, $S_b = S_{b'}$ is a necessary condition for the equality of the classes $[\psi_b]$ and $[\psi_{b'}]$. But if $(M, \omega)$ possesses certain symmetries it can happen that $S_b = S_{b'}$ and $[\psi_b] \neq [\psi_{b'}]$ (see Remark in Subsection 3.2).

Circle actions on coadjoint orbits.

If $G$ is a compact connected Lie group and $\eta \in \mathfrak{g}^*$, the stabilizer of $\eta$ for the coadjoint action contains a maximal torus $H$. The integer elements of $\mathfrak{h}$ define Hamiltonian circle actions on the coadjoint orbit $O$ of $\eta$. We will consider the equivalence relation $\sim$ in the set of those $S^1$-actions on the orbit $O$. By $W$ we denote the Weyl group of the pair $(G, H)$. We will prove the following Theorem

**Theorem 9.** Let $X, X'$ be two elements of the integer lattice of $\mathfrak{h}$. If there exists $w \in W$ such that $w(X) = X'$, then the circle actions defined on $O$ by $X$ and $X'$ are $\sim$-equivalent.

A particular case is when $G = SU(n)$. $H$ its diagonal subgroup and $O$ is the orbit of an element whose stabilizer contains $H$. In this case we have the following Corollary

**Corollary 10.** If $X = (x_1, \ldots, x_n)$, $X = (x_1, \ldots, x_n) \in (2\pi i \mathbb{Z})^n$ and there exists a permutation $\tau$ of the set $\{1, \ldots, n\}$, such that $x'_j = x_{\tau(j)}$ for all $j$, then the circle actions defined by $X$ and $X'$ on the orbit $O$ of $SU(n)$ are $\sim$-equivalent.

When the coadjoint orbit is diffeomorphic to $\mathbb{C}P^{n-1}$ we will prove a stronger result.

**Theorem 11.** Let $O$ be a coadjoint orbit of $SU(n)$ diffeomorphic to $\mathbb{C}P^{n-1}$, and $X, X'$ diagonal elements of $\mathfrak{su}(n)$ such that $X, X' \in (2\pi i \mathbb{Z})^n$. If $\psi$ and $\psi'$ denote the circle actions on $O$ defined by $X$ and $X'$ respectively. Then the following are equivalent

(a) $[\psi] = [\psi'] \in [U(1), \text{Ham}(O)]_{gh}$.

(b) $S_\psi = S_{\psi'}$.

(c) There exists a permutation $\tau$ of the set $\{1, \ldots, n\}$, such that $x'_j = x_{\tau(j)}$ for $j = 1, \ldots, n$.

Remark. By the preceding Theorem, the elements $X \in (2\pi i \mathbb{Z})^3$ define an infinite family in $[U(1), \text{Ham}(\mathbb{C}P^2)]_{gh}$. But in contrast $\pi_1(\text{Ham}(\mathbb{C}P^2)) = 3$, since $\text{Ham}(\mathbb{C}P^2)$ has the homotopy type of $PU(3)$ [8]. Thus the map (1.2) is not injective when $M = \mathbb{C}P^2$.

To state a necessary condition for the equality $[\psi] = [\psi']$ when the coadjoint orbit of $SU(n)$ is the Grassmann manifold $G_k(\mathbb{C}^n)$ (of the $k$-subspaces in $\mathbb{C}^n$) we need to introduce new notations. By $C_k$ we denote the set of combinations of the set $\{1, \ldots, n\}$ taken $k$ at a time. If $X = (ia_1, \ldots, ia_n)$ and $\delta = \{j_1, \ldots, j_k\} \in C_k$ we write $a_\delta = \sum_{i=1}^k a_{j_i}$. We will prove the following result.

**Theorem 12.** Let $O$ be a coadjoint orbit of $SU(n)$ diffeomorphic to the Grassmann manifold $G_k(\mathbb{C}^n)$, and $X, X'$ diagonal regular elements of $\mathfrak{su}(n)$ such that $X, X' \in (2\pi i \mathbb{Z})^n$. If $\psi$ and $\psi'$ denote the circle actions on $G_k(\mathbb{C}^n)$ defined by $X$ and $X'$ respectively. Then the equality

$[\psi] = [\psi'] \in [U(1), \text{Ham}(O)]_{gh}$
implies that there exist a constant $\beta$ and a bijective map $f : C_k \rightarrow C_k$, such that $\alpha'_b = \alpha_{f(b)} + \beta$, for all $b \in C_k$.

Remarks. When $k = 1$ the necessary condition stated in Theorem 12 for the equality $[\psi] = [\psi']$ is precisely the condition (c) of Theorem 11 since in this case the constant $\beta = a'_b - a_{f(b)}$ vanishes.

On the other hand, the mentioned necessary condition given in Theorem 12 is independent of the symplectic structure. In Proposition 37 we give such a necessary condition when the orbit is diffeomorphic to the full flag manifold for $SU(n)$, which depends on the symplectic structure.

The paper is organized as follows. In Section 2 we introduce the cohomology class $S_\psi$ and we prove Theorem 3 and Proposition 4. Section 3 deals with the case when the manifold is the total space of the fibration (1.1); in particular Theorem 6 is proved. Subsection 3.1 is devoted to prove Theorem 7. The case $M = S^2 \times S^2$ is considered in Subsection 3.2.

In Section 4 we will concern with the relation \(\sim\) between circle actions in the coadjoint orbits of a Lie group $G$, defined by elements of a Cartan subalgebra of $G$. We will prove the general Theorem 9 and Theorem 11 and Theorem 12 relative to circle actions on coadjoint orbits of $SU(n)$.

Acknowledgements. The relation \(\sim\) has been introduced in [23]. I thank to a referee of paper [23] for suggesting me to study the relation between the space $[G, \text{Ham}(M)]_{gh}$ and the $r$-equivalence of $G$-actions introduced above. I thank the referee for his useful comments.

2. The cohomology class $S_\psi$

Here we will denote by $H$ the Hamiltonian group $\text{Ham}(M, \omega)$. From the universal principal bundle $EH \rightarrow BH$ of the group $H$, we construct the universal bundle with fibre $M$

$$
\begin{array}{ccc}
M & \longrightarrow & M_H := EH \times_H M \\
\downarrow \pi_H & & \downarrow \pi_H \\
BH & \longrightarrow & BH
\end{array}
$$

There exists a unique class $c \in H^2(M_H, \mathbb{R})$ called the coupling class (see [14]) satisfying:

a) $\pi_{H*}(c^{n+1}) = 0$, where $\pi_{H*}$ is the fibre integration.

b) $c$ extends the fiberwise class $[\omega]$.

Let $G$ be a compact Lie group. We assume that $M$ is equipped with the Hamiltonian $G$-action given by the group homomorphism $\psi : G \rightarrow \text{Ham}(M, \omega)$. This homomorphism induces a map $\Psi : BG \rightarrow BH$, between the corresponding classifying spaces. The pullback $\Psi^{-1}(M_H)$ of $M_H$ by $\Psi$ is a bundle over $BG$ which can be identified with $p : M_\psi = EG \times_G M \rightarrow BG$. Thus we have the following commutative diagram

$$
\begin{array}{ccc}
M_\psi & \xrightarrow{p} & M_H \\
\downarrow \pi_H & & \downarrow \pi_H \\
BG & \xrightarrow{\psi} & BH
\end{array}
$$
By \( c_\psi \) we denote the pullback of \( c \) by \( \hat{\psi} \); that is, \( c_\psi = \hat{\psi}^*(c) \). The class \( c_\psi \) is the coupling class of the \( \hat{F} \)-action.

If \( \psi \) and \( \psi' \) are homotopic by means of a family of Hamiltonian \( G \)-actions, then the bundles \( \Psi'^{-1}(M_H) \) and \( \Psi^{-1}(M_H) \) are isomorphic and the isomorphism from \( M_\psi \) to \( M_{\psi'} \) applies \( c_\psi \) to \( c_{\psi'} \).

Since \( c_\psi \in H^2(M_\psi) = H^2_c(M) \) we can integrate \( e^{c_\psi} \) along the fibre and we obtain \( S_\psi := p_*(e^{c_\psi}) \) in the cohomology of \( H(BG) \). The above arguments prove Theorem 1.

Strictly speaking \( e^{c_\psi} \) is not an equivariant de Rham cohomology class. For instance, when \( G \) is the \( n \)-torus \( T \), \( e^{c_\psi} \) belongs to \( \hat{H}_T(M) \) [3], where \( \hat{H}_T(M) \) is

\[
\hat{H}_T(M) = H_T(M) \otimes_{\mathbb{R}[u_1, \ldots, u_n]} \mathbb{R}[[u_1, \ldots, u_n]],
\]

\( \mathbb{R}[[u_1, \ldots, u_n]] \) being the ring of formal power series in the variables \( u_1, \ldots, u_n \).

By the localization theorem [3] we have, in a suitable localization of \( \hat{H}(BT) = \mathbb{R}[[u_1, \ldots, u_n]] \),

\[
S_\psi = \sum_F p_F^* \left( \frac{e^{c_\psi}}{E(N_F)} \right),
\]

where \( F \) varies in the set of connected components of the fixed point set of the \( T \)-action, \( p_F^* : \hat{H}_T(F) \to \hat{H}(BT) \) is the fibre integration on \( F \), and \( E(N_F) \) is the equivariant Euler class of the normal bundle \( N_F \) to \( F \) in \( M \).

Let us assume that \( M \) is endowed with a Hamiltonian action of the torus \( T = U(1)^n \). We say that a moment map \( \mu : M \to \mathfrak{t}^* \) for the \( T \)-action is the normalized moment map if \( \int_M \mu(X) \omega^n = 0 \) for all \( X \in \mathfrak{t} \). If \( \Delta \) denotes the moment polytope of \( \mu \), that is \( \Delta = \text{Im}(\mu) \), then

\[
\int_M \mu \omega^n = \text{Cm}(\Delta) \int_M \omega^n,
\]

where \( \text{Cm}(\Delta) \) is the center of mass of \( \Delta \). It follows that \( \text{Cm}(\Delta) = 0 \), iff \( \mu \) is normalized.

If \( \mu \) is a normalized moment for the \( T \)-action \( \psi \), then \( \omega - \mu \) is a \( T \)-equivariant closed 2-form [13] which represents the coupling class \( c_\psi \), because it satisfies the properties which determine the coupling class uniquely.

Let \( M \) be a toric manifold and \( \psi \) the corresponding effective action of \( T \). We denote by \( \Delta \) the respective moment polytope with center of mass at 0. By \( u_1, \ldots, u_n \) we denote the basis of \( \mathfrak{t}^* \) dual of \( e_1, \ldots, e_n \in \mathfrak{t} = \mathbb{R}^n \). The image by moment map \( \mu \) of the fixed points \( q_1, \ldots, q_s \) of the \( T \)-action \( \psi \) are the vertices of \( \Delta \) [2], [11]. The equivariant Euler class \( E(N_{q_i}) \) of the normal bundle to \( q_i \) in \( M \) is

\[
E(N_{q_i}) = \prod_{j=1}^n l_{ij},
\]

where the \( l_{ij} \) are the weights of isotropy representation of \( T \) at the tangent space \( T_{q_i} \) (see [13]). So \( E(N_{q_i}) \) is a homogeneous degree \( n \) polynomial in the variables \( u_1, \ldots, u_n \).

On the other hand \( \mu(q_i) = \sum_j \beta_{ij} u_j \), where \( (\beta_{ij})_j \) are the coordinates of the vertex \( \mu(q_i) \). Let \( f \) be the polynomial \( \prod_i \prod_j l_{ij} \). Denoting by \( [S_\psi]_f \) the class of \( S_\psi \)
in the localization \( \hat{H}(BT)_f \) (see [3]) it follows from (2.1)

\[
(2.2) \quad [S_\psi]_f = \sum_{i=1}^{n} e^{-\sum_j \beta_{ij} u_j} / \prod_j l_{ij} \in \hat{H}(BT)_f. 
\]

So we can write

\[
(2.3) \quad [S_\psi]_f = \frac{A}{f} \in \hat{H}(BT)_f, 
\]

where \( A \) is a linear combination of elements of

\[ E := \{ E_i := \exp(-\sum_j \beta_{ij} u_j) \}_{i}. \]

with coefficients in the polynomial ring \( \mathbb{R}[u_1, \ldots, u_n] \).

Let \( \psi' \) be another effective action of \( T \) on \( M \), and \( f' \) be the polynomial defined by the equivariant Euler classes of the normal bundles of the fixed points. So

\[
(2.4) \quad [S_{\psi'}]_{f'} = \frac{C}{f'} \in \hat{H}(BT)_{f'}, 
\]

In order to compare \([S_\psi]_f\) and \([S_{\psi'}]_{f'}\) we introduce the extension \( A \) of \( \hat{H}(BT) \)

\[ A := \mathbb{R}[[u_1, \ldots, u_n]] \otimes_{\mathbb{R}[u_1, \ldots, u_n]} \mathbb{R}(u_1, \ldots, u_n), \]

where \( \mathbb{R}(u_1, \ldots, u_n) \), is the field of rational functions in the variables \( u_1, \ldots, u_n \).

We have the following obvious Lemma

**Lemma 13.** Suppose that \( S_\psi = S_{\psi'} \in \hat{H}(BT) \). If \([S_\psi]_f = \frac{A}{f} \in \hat{H}(BT)_f \) and \([S_{\psi'}]_{f'} = \frac{C}{f'} \in \hat{H}(BT)_{f'} \), with \( p, q \in \mathbb{Z} \), then there exists a rational function \( g \in \mathbb{R}(u_1, \ldots, u_n) \) such that \( A = Cg \) in \( A \).

**Proof.** \( g = f^p/(f')^q \) satisfies the required condition. \( \square \)

Let \( \{\gamma_{ij} | i = 1, \ldots, r; j = 1, \ldots, m\} \) be a family of real numbers. We put

\[ F_i(u_1, \ldots, u_m) = \exp \left( \sum_j \gamma_{ij} u_j \right). \]

We will prove that the set of exponential functions \( \{F_i\} \in A \) is \( \mathbb{R}(u_1, \ldots, u_m) \)-independent.

**Lemma 14.** If \( \{Q_i(u_1, \ldots, u_m)\}_{i=1, \ldots, r} \) is a set of rational functions, such that \( \sum_i Q_i F_i = 0 \) and \( F_i \neq F_k \) for \( i \neq k \), then \( Q_i = 0 \) for \( i = 1, \ldots, r \).

**Proof.** We define the linear functions \( h_i(u_1, \ldots, u_m) = \sum_j \gamma_{ij} u_j \). We write \( \gamma_i = (\sum_j \gamma_{ij}^2)^{1/2} \). Let us assume that \( |\gamma_i| \geq |\gamma_{ij}| \) for \( i = 1, \ldots, r \). Then \( |\text{grad}(h_i)| \leq |\text{grad}(h_1)| \) and moreover \( \text{grad}(h_i) \neq \text{grad}(h_1) \), for \( i \neq 1 \) by the assumption. Hence the vector \( v := (\gamma_{11}, \ldots, \gamma_{1m}) \in \mathbb{R}^m \) defines a direction such that

\[
(2.5) \quad \lim_{\lambda \to +\infty} \frac{F_i(\lambda v)}{F_1(\lambda v)} = 0, 
\]

for \( i \neq 1 \).

If \( Q_1 \) would be nonzero

\[
(2.6) \quad 1 = - \sum_{i \neq 1} \frac{Q_i}{Q_1} \frac{F_i}{F_1}. 
\]
If we take the limit along \( \lambda \nu \) as \( \lambda \to +\infty \) in (2.6), we arrive to a contradiction, by the exponential decay (2.5). Hence the initial linear combination reduces to
\[
\sum_{i>1} Q_i F_i = 0.
\]
By repeating the argument it follows \( Q_i = 0 \) for all \( i \).

Sometimes we will use particular cases of Lemma 13, for example, the \( \mathbb{R}[u, u^{-1}] \)-independence of a family of exponential functions \( \{e^{\gamma u}\}_i \), with \( \gamma_i \neq \gamma_j \) for \( i \neq j \).

**Proposition 15.** Let \( \psi \) and \( \psi' \) be toric actions on \( M \) and \( \Delta \) and \( \Delta' \) be the respective moment polytopes with center of mass at \( 0 \). Then \( \Delta \neq \Delta' \) iff \( S_\psi \neq S_{\psi'} \).

**Proof.** Let us suppose that \( \Delta \neq \Delta' \). According to (2.2), the element \( A \) in (2.3) is a linear combination \( A = \sum_i g_i E_i \), with \( 0 \neq g_i \in \mathbb{R}[u_1, \ldots, u_n] \). Similarly, \( C \) in (2.4) is a linear combination of elements of
\[
\mathcal{E}' := \{ E_i' := \exp(-\sum_j \beta_{ij}' u_j) \}_i.
\]
As the moment polytopes are not equal, there is a vertex in \( \Delta \) which does not belong to \( \Delta' \). So there is an element, say \( E_k \), in \( \mathcal{E} \) which is not in \( \mathcal{E}' \).

If there were a rational function \( g \in \mathbb{R}(u_1, \ldots, u_n) \) such that \( A = Cg \) in \( A \), then \( 0 = A - Cg \) is a linear combination in \( A \) with coefficients in \( \mathbb{R}(u_1, \ldots, u_n) \), in which \( E_k \) appears once with the coefficient \( g_k \neq 0 \). That is,
\[
(2.7) \quad 0 = A - gC = g_k E_k + \sum_{G_r \in \mathcal{E}''} \tilde{g}_r G_r,
\]
where \( \mathcal{E}'' = (\mathcal{E} \cup \mathcal{E}') \setminus \{ E_k \} \). By the \( \mathbb{R}(u_1, \ldots, u_n) \)-linearity of the set \( \mathcal{E}'' \cup \{ E_k \} \), it follows from (2.7) that \( g_k = 0 \). This contradiction shows that there is no a rational function \( g \) such that \( A = Cg \). It follows from Lemma 13 that \( S_\psi \neq S_{\psi'} \).

On the other hand, if \( \Delta = \Delta' \), then trivially \( \beta_{ij} = \beta_{ij}' \). Moreover the equality of the polytopes implies the equality of the weights of the respective isotropy representations at the fixed points [11]. Hence, by (2.2), \( [S_\psi] |_{f} = [S_{\psi'}] |_{f} \). As the polynomial \( f \) and \( f' \) are equal, it turns out that \( S_\psi = S_{\psi'} \).

\( \square \)

**Theorem 16.** Let \( \psi \) and \( \psi' \) be toric actions on \( M \). Let \( \Delta \) and \( \Delta' \) be the respective moment polytopes with center of mass at \( 0 \). If \( \Delta \neq \Delta' \), then \([\psi] \neq [\psi'] \in [T, \text{Ham}(M)]_{gh}\).

**Proof.** It is a consequence of Proposition 13 and Theorem 11.

\( \square \)

**Proposition 17.** If \( \psi \) and \( \psi' \) are \( r \)-equivalent Hamiltonian \( G \)-actions on \( M \), then \([\psi] = [\psi'] \in [G, \text{Ham}(M)]_{gh}\).

**Proof.** There exists \( h \in \text{Ham}(M) \) such that \( h \circ \psi_g \circ h^{-1} = \psi'_g \), for all \( g \). Let \( h_s \) be a path in \( \text{Ham}(M) \) from \( Id \) to \( h \), then \( \{ h_s \circ \psi \circ h_s^{-1} \}_s \) defines a homotopy of \( G \)-actions between \( \psi \) and \( \psi' \).

\( \square \)

**Proof of Theorem 3**

Properties (iv) and (i) are equivalent by Proposition 13. 

(v) “implies” (iv) by Theorem 11.

(iii) \( \implies \) (v) is Proposition 17.

If \( h \in \text{Ham}(M) \) satisfies \( \psi'_g = h \circ \psi_g \circ h^{-1} \), for all \( g \in T \), then the corresponding normalized moment maps satisfy \( \mu' = \mu \circ h \). So \( \Delta = \text{im} \mu = \text{im} \mu' = \Delta' \); that is, (iii) implies (ii).
Since $M$ is a toric manifold $\text{Ham}(M) = \text{Symp}_0(M)$; thus (ii) implies (iii).

**Lemma 18.** Let $\psi$ be a toric action on $M$ and $\Delta$ its moment polytope with center of mass at $0$. If $v$ is an automorphism of $T$, then there exists $V \in \text{GL}(n, \mathbb{Z})$ such that $V(\Delta)$ is the moment polytope with center of mass at $0$ associated to the action $\psi \circ v$.

**Proof.** Let $v^*$ be the automorphism of $t^*$ induced by $v$. If $\mu$ is the normalized moment map of $\psi$, then $v^* \circ \mu$ is the corresponding moment map of $\psi \circ v$. So $V = v^*$ in the identification $t^* = (\mathbb{R}^n)^*$.

**Proof of Proposition 4.**

(a) “implies” (b). It is consequence of Proposition 17.

(b) “implies” (d). By definition of $\sim_{rp}$, there exists an automorphism $v$ of $T$ such that $\psi' \sim \psi \circ v$. By the implication (v) $\implies$ (i) of Theorem 3, $\Delta' = \Delta(\psi \circ v)$. The implication follows from Lemma 18.

(c) “implies” (d). If $\psi'$ and $\psi$ are conjugate then $\xi \circ (\psi \circ v)(g) \circ \xi^{-1} = \psi'(g)$, with $\xi$ a symplectomorphism. By the proof of Lemma 18, $\mu' = v^* \circ \mu \circ \xi$. So $\Delta' = \nu^*(\Delta)$.

(d) “implies” (c). Let us assume that $V(\Delta) = \Delta'$ with $V \in \text{GL}(n, \mathbb{Z})$. $V$ induces a group automorphism of $T$, which is denoted by $v$, such that $V^* = V$. So the actions $\psi \circ v$ and $\psi'$ have the same moment polytope. By Delzant’s Theorem (Th. 2.1, [3]) there is a symplectomorphism $\xi$ such that $\xi \circ (\psi \circ v)(g) \circ \xi^{-1} = \psi'(g)$; that is, $\psi$ and $\psi'$ are conjugate.

Let us assume that $M$ is a symplectic manifold equipped with a Hamiltonian $G$-action, where $G$ is a Lie group. Let $X$ be an element of $\mathfrak{g}$ of Type (s) which determine a circle action $\psi^X$ on $M$, and let $F$ be a $2s$-dimensional submanifold of $M$, connected component of the fixed point set of the circle action $\psi^X$. The normal bundle $N_F$ to $F$ in $M$ decomposes in direct sum of $n - s$ $U(1)$-equivariant line bundles

$$N_F = \bigoplus_{j=1}^{n-s} N_j.$$ 

On $N_j$, $U(1)$ acts with a weight $p_k u \neq 0$, where $u$ we denote the vector in $\mathfrak{u}(1)^*$ dual of the standard basis of $\mathfrak{u}(1)$. Then the $U(1)$-equivariant Euler class of $N_F$ is

$$E(N_F) = \prod_{k=1}^{n-s} (c_1(N_k) + p_k u) = u^{n-s} \prod_k p_k \left(1 + \frac{c_1(N_k)}{p_k u}\right).$$

So

$$(E(N_F))^{-1} = u^{s-n} \left(\alpha_0 + \frac{\alpha_1}{u} + \frac{\alpha_2}{u^2} + \ldots\right),$$

$\alpha_r$ being a 2r-form on $F$ and $\alpha_0 \neq 0$. That is, $(E(N_F))^{-1}$ is a Laurent polynomial of degree $s - n$ with coefficients even dimensional forms on $F$.

On the other hand, if $f$ is a normalized Hamiltonian for the $U(1)$-action $\psi^X$, then $\omega - u f$ is a $U(1)$-equivariant closed 2-form which represents the coupling class $c_{\psi^X}$ [10]. Hence the contribution of $F$ to $S_{\psi^X}$ in (2.1) is

$$\frac{e^{\alpha u}}{u^{n-s}} \int_F e^\omega \left(\alpha_0 + \frac{\alpha_1}{u} + \ldots\right).$$
where \(-a\) is the value of \(f\) on \(F\). That is,
\[
e^{au} \frac{u^{-s}}{A_0u + A_1 + \cdots + \frac{A_{m-1}}{u^{m-2}}},
\]
with \(A_0 \neq 0\).

Let \(F'\) be another component of the fixed point set. As \(\text{codim } F' \geq \text{codim } F \geq 2\), there is a curve \(\sigma\) in \(M\) which joins \(F\) and \(F'\) and such that it does not meet other fixed point. So the only stationary points of the function \(f \circ \sigma\) are the end points; that is, \(f \circ \sigma\) is strictly monotone, and thus \(f(F) \neq f(F')\). Therefore the contribution of \(F'\) to \(S_{\psi X}\) will be of the form \(e^{au}Q'(u)\), with \(a \neq a'\) and \(Q'(u)\) a Laurent polynomial of degree \((1/2)\dim F' - n\). We note that the contributions of \(F\) and \(F'\) can not be grouped together in the form \(e^{au}Q(u)\). So we have the following Lemma

**Lemma 19.** If \(X\) is of Type \((s)\) and \(F_1, \ldots, F_c\) are the 2s-dimensional components of the fixed point set of the circle action \(\psi^X\), then
\[
S_{\psi X} = \sum_{i=1}^c e^{ai}Q_i(u) + \sum_{r} e^{br}Q_r(u),
\]
with \(a_i \neq a_j\) for \(i \neq j\), \(Q_i(u)\) a Laurent polynomial of degree \(s - n\) and \(Q_r(u)\) a Laurent polynomial of degree less than \(s - n\).

**Proof of Theorem 2.** If \(X\) and \(Y\) are of different Type, say Type of \(X = s\) greater than Type of \(Y\), then in the expression for \(S_{\psi X}\) given in Lemma 19 appears an exponential function with coefficient a Laurent polynomial of degree \(s - n\), but all the Laurent polynomials in \(S_{\psi X}\) have degree less than \(s - n\). The Theorem follows from the aforesaid \(\mathbb{R}[u, u^{-1}]\)-linear independence of the exponential functions \(\{e^{au}\}\) together with Theorem 1. 

Let \(M\) be a toric manifold. To avoid trivial cases we assume that \(\dim M > 2\). By \(\psi\) we denote the corresponding Hamiltonian effective \(T\)-action and by \(\mu\) a moment map. In terms of the basis \(\{u_j\}\) we write \(\mu = \sum_j \mu_j u_j\). Given \(X = \sum_j k_j e_j\) in the integer lattice of \(t\), then \(\mu^X = \langle \mu, X \rangle = \sum_j k_j \mu_j\) is a moment map for the \(U(1)\)-action \(\psi^X\) defined as the composition of \(\psi\) with \(z \in U(1) \rightarrow (z^{k_j}) \in T\).

The following Proposition gives equivalent conditions to the property of being of type 0.

**Proposition 20.** Let \(X\) be a vector in the integer lattice of \(t\), the following are equivalent.

(a) The set of fixed points for the \(U(1)\)-action \(\psi^X\) is equal to the set of fixed points of the \(T\)-action \(\psi\).

(b) \(\langle v, X \rangle \neq 0\), for all \(v\) parallel to an edge of the moment polytope \(\Delta\).

(c) \(X \notin g_a = \text{Lie}(G_a),\) for all \(a,\) with \(G_a\) proper isotropy subgroup of the \(T\)-action \(\psi\).

**Proof.**

(a) \(\implies\) (b). Let \(e\) be an edge of moment polytope \(\Delta\). \(e\) is of the form \(p + tv\), with \(t\) in an interval of \(\mathbb{R}\), \(p\) a vertex of \(e\) and \(v \in t^*\). If \(\langle v, X \rangle = 0\), then \(\mu^X\) takes on \(\mu^{-1}(e)\) the value \(\langle p + tv, X \rangle = (p, X)\). Thus the infinitely many points of \(\mu^{-1}(e)\) are fixed by \(\psi^X\). But the fixed point set of the \(T\)-action is \(\{q_1, \ldots, q_s\}\), the set of inverse images by \(\mu\) of the vertices of \(\Delta\).
If $X \in \mathfrak{g}_b$, we may assume that $\mathfrak{g}_b$ is maximal; that is, $G_b$ is the stabilizer group of the points of $N := \mu^{-1}(e)$, with $e$ an edge of $\Delta$. So the induced vector field $X_M$ on $M$ vanishes on the submanifold $N$, and $d\mu^X = 0$ on this set. The values of the function $\mu^X$ on $N$ are of the form $(p + tv, X)$, with $t$ varying in an interval of $\mathbb{R}$, and where $p$ is a vertex of $e$ and $v$ is parallel to $e$. It follows from the vanishing of $d\mu_X|_N$ that $(v, X) = 0$.

(\gamma) \implies (\alpha). If $X$ satisfies (\gamma) then the fixed point set for the $U(1)$-action $\psi_X$ is $\{q_1, \ldots, q_s\}$ (see Corollary 10.9.1 in [14]).

If the equivalent properties stated in Proposition 20 hold, the fixed point set for $\psi_X$ is $\{q_1, \ldots, q_s\}$. The $U(1)$-equivariant Euler class $E(N_{q_i})$ is a degree $n$ monomial in a variable $u$, that we denote by $l_i$. If the moment map $\mu$ is normalized, then $\mu^X$ is the normalized Hamiltonian of the $S^1$-action $\psi_X$. Therefore

$$[S_{\psi_X}] = \sum_{i=1}^{s} \frac{e^{-\beta_i u}}{l_i} \in \hat{H}(BU(1))u,$$

with $\beta_i = (P_i, X)$, and $P_i$ the vertex of $\Delta$ image of $q_i$ by $\mu$. The proof of Theorem 5 is similar to the one given for Proposition 15.

**Proof of Theorem 5.** We put $\mathcal{E} = \{E_i = e^{-\beta_i u}\}$ and $\mathcal{E}' = \{E'_i = e^{-\beta'_i u}\}$, with $\beta'_i = \langle \mu(q_i), Y \rangle$. We write $[S_{\psi_X}] = A/u^n$ and $[S_{\psi_Y}] = C/u^n$, where $A, C \in \mathbb{R}[u]$ are linear combinations

$$A = \sum_{i=1}^{s} a_i E_i, \quad C = \sum_{i=1}^{s} c_i E'_i; \quad a_i \neq 0 \neq c_i, \quad \text{for all } i.$$

By hypothesis, there is an $E_k$ that does not belong to $\mathcal{E}'$. Then, by the $\mathbb{R}$-linear independence of the exponential functions, $A \neq C$. Thus $S_{\psi_X} \neq S_{\psi_Y}$, and by Theorem 1 $[\psi_X] \neq [\psi_Y] \in [U(1), \text{Ham}(M)]_{gh}$.

**Proposition 21.** Let $X, Y$ be elements in the integer lattice of $t$ that satisfy the conditions of Proposition 20. If

$$\lambda(P_i, X) : i = 1, \ldots, s \neq \{\langle P_i, Y \rangle : i = 1, \ldots, s\},$$

for all $0 \neq \lambda \in \mathbb{Z}$, where the $P_i$’s are the vertices of the moment polytope with center at $0$. Then $\psi_X$ and $\psi_Y$ are $S^1$-actions $\sim_{\text{rp}}$-inequivalent.

**Proof.** Suppose the Proposition were false. Then there were an automorphism $v$ of $U(1)$ such that $\psi_X \circ v \sim \psi_Y$. But $\psi_X \circ v = \psi_Z$, with $Z = v_e(X)$; that is, $Z = \lambda X$, with $0 \neq \lambda \in \mathbb{Z}$. Since $\langle P_i, Z \rangle = \lambda \langle P_i, X \rangle$, it follows from (2.9) together with Theorem 5 that $[\psi_Z] \neq [\psi_Y]$; that is, $\psi_X \circ v$ is not $\sim$-equivalent to $\psi_Y$. This contradicts our supposition.

**3. Circle actions in $\mathbb{P}(L_1 \oplus \cdots \oplus L_{n-1} \oplus \mathbb{C})$.**

We denote by $e_1, \ldots, e_n$ the standard basis of $\mathbb{R}^n$. Given $a_1, \ldots, a_{n-1} \in \mathbb{Z}$, we define the following vectors of $\mathbb{R}^n$:

$$v_j = -e_j, \quad j = 1, \ldots, n, \quad v_{n+1} = \sum_{j=1}^{n-1} e_j, \quad v_{n+2} = e_n - \sum_{i=1}^{n-1} a_i e_i.$$
Let \( \{k_i\}_{i=1,...,n+2} \) be a family of real numbers with \( k_j = 0 \), for \( j = 1, \ldots, n, k_{n+1} = \sigma > 0, k_{n+2} = \tau > 0 \) satisfying \( \tau + a_j \sigma > 0 \) for all \( i \), then we consider the following Delzant polytope \( \Delta \) in \( (\mathbb{R}^n)^* \)

\[
\Delta = \bigcap_{i=1}^{n+2} \{ x \in (\mathbb{R}^n)^* : \langle x, v_i \rangle \leq k_i \}.
\]

When \( n = 3 \) the polytope \( \Delta \) is represented in the Figure 1.

![Figure 1. The polytope \( \Delta \) when \( n = 3 \)](image)

The symplectic manifold \( M_\Delta \) associated to \( \Delta \) is

\[
M_\Delta = \{(z_i) \in \mathbb{C}^{n+2} : \sum_j |z_j|^2 = \sigma / \pi, -\sum_{j=1}^{n-1} a_j |z_j|^2 + |z_n|^2 + |z_{n+2}|^2 = \tau / \pi \} / \simeq,
\]

in the sum \( \sum_j' \) the index \( j \) runs in the set \( \{1, \ldots, n-1, n+1\} \). The equivalence relation \( \simeq \) is defined by \( (z_j') \simeq (z_j) \) iff there exist \( \epsilon, \lambda \in U(1) \) such that

\[
\begin{aligned}
& z_j' = \lambda e^{-a_j} z_j, \; j = 1, \ldots, n-1; \\
& z_k' = \epsilon z_k, \; k = n, n+2; \\
& z_{n+1}' = \lambda z_{n+1}.
\end{aligned}
\]

The action of \( U(1)^n \) on \( M_\Delta \) given by

\[
(\lambda_j) \cdot [z] = [\lambda_1 z_1, \ldots, \lambda_n z_n, z_{n+1}, z_{n+2}]
\]

defines a structure of toric manifold on \( M_\Delta \). Moreover the map \( \mu \) on \( M_\Delta \) defined by \( \mu([z]) = \pi(|z_1|^2, \ldots, |z_n|^2) \) is the corresponding moment map with \( \text{im}\mu = \Delta \).

The manifold \( M_\Delta \) is the total space of the fibration \( \mathbb{P}(L_1 \oplus \cdots \oplus L_{n-1} \oplus \mathbb{C}) \to \mathbb{C}P^1 \), where \( L_i \) is the holomorphic line bundle over \( \mathbb{C}P^1 \) with first Chern number \(-a_i\).

Given \( [z] \in M_\Delta \), we write \( w = (z_n, z_{n+2}) \). The element in \( \mathbb{P}(L_1 \oplus \cdots \oplus L_{n-1} \oplus \mathbb{C}) \) which corresponds to \( [z] \) is the following one in the fibre over \( [w] = [z_n : z_{n+2}] \in \mathbb{C}P^1 \)

\[
([z_n]^{a_1} z_1 : (z_{n+2})^{a_1} z_1 : \cdots : (z_n)^{a_{n-1}} z_{n-1} : (z_{n+2})^{a_{n-1}} z_{n-1} : z_{n+1}]
\]

We will write \( ([w], [w^{a_j} z_j : z_{n+1}]) \) for the element in fibre bundle associated to \( [z] \in M_\Delta \).

An element \( b = (b_1, \ldots, b_n) \) in the integer lattice \( \mathbb{Z}^n \) determines a Hamiltonian circle action \( \psi_b \) on \( M_\Delta \) through the action \((3.1)\). In the notation introduced above

\[
\psi_b(t)[z] = ([w_1], ([w_1]^{a_j} z_j : z_{n+1}] ),
\]
where $\epsilon_k = \exp(2\pi i b_k t)$ and $w_t = (z_n \epsilon_n, z_{n+2})$.

**Proof of Theorem** Let $b' = (b'_1, \ldots, b'_n)$ with
\[
b'_j = b_j + a_j b_n, \quad \text{for } j = 1, \ldots, n - 1, \quad \text{and } b'_n = -b_n,
\]
then
\[
\psi_{b'}(t)[z] = \left( [w'_i], [(w'_i)^{a_i} z_j' : z_{n+1}^+] \right),
\]
with $w'_i = (z_n e'_n, z_{n+2})$ and $\epsilon'_k = \exp(2\pi i b'_k t)$. So
\[
\psi_{b'}(t)[z] = \left( [u_i], [(u_i)^{a_i} z_j : z_{n+1}^+] \right),
\]
where $u_t = (z_n, z_{n+2} \epsilon_n)$.

Let $\{\hat{n}_s : s \in [0, 1]\}$ be a continuous curve of unitary vectors in $\mathbb{R}^3$, with $\hat{n}_0 = e_3$ and $\hat{n}_1 = -e_3$. We denote by $\xi^s(t)$ the rotation of $S^2 = \mathbb{C}P^1$ around the axis defined by the unitary vector $\hat{n}_s$ and angle $2\pi b_n t$. Then $\xi^0(t)[z_n : z_{n+2}] = [w_t]$ and $\xi^1(t)[z_n : z_{n+2}] = [u_t]$. We write $[w_{s, t}] := \xi^s(t)[z_n : z_{n+2}]$.

For each $s \in [0, 1]$ we define
\[
\psi^s(t)[z] = \left( [w_{s, t}], [(w_{s, t})^{a_i} z_j : z_{n+1}^+] \right).
\]
Thus $\psi^0 = \psi_b$ and $\psi^1 = \psi_{b'}$. Since $\xi^s(t + t') = \xi^s(t) \circ \xi^s(t')$, the family $\{\psi^s\}$ is a homotopy between $\psi_b$ and $\psi_{b'}$, consisting of circle actions. □

### 3.1. Hirzebruch surfaces

Here we will consider the manifold $\mathbb{P}(L_1 \oplus \cdots \oplus L_{n-1} \oplus \mathbb{C})$ when $n = 2$. In this case $\Delta$ is the trapezoid in $(\mathbb{R}^2)^*$ with vertices $P_1(0, 0)$, $P_2(0, \tau)$, $P_3(\sigma, 0)$, $P_4(\sigma, \tau + a\sigma)$ and $M_{\Delta}$ is a Hirzebruch surface $\mathbb{P}(L_k \oplus \mathbb{C})$, where $L_k$ is the holomorphic line bundle over $\mathbb{C}P^1$ with first Chern number $k := -a$. More explicitly
\[
M_{\Delta} = \{ (z_j) \in \mathbb{C}^4 : |z_1|^2 + |z_3|^2 = \sigma/\pi, -a|z_1|^2 + |z_2|^2 = \tau/\pi \}/ \simeq,
\]
with
\[
(z_1, z_2, z_3, z_4) \simeq (\epsilon_1 \epsilon_2^a z_1, \epsilon_2 z_2, \epsilon_1 z_3, \epsilon_2 z_4),
\]
$\epsilon_i$ being an element of $U(1)$.

The map $[z_j] \mapsto ([z_2 : z_4], [z_2^a z_3 : z_4^a z_3 : z_1])$ gives a representation of $M$ as a submanifold of $\mathbb{C}P^1 \times \mathbb{C}P^2$.

An element $b$ in the lattice $\mathbb{Z}^2$ determines a Hamiltonian circle action $\psi_b$ on $M_{\Delta}$. The following Proposition is a particular case of Theorem [3.2]

**Proposition 22.** If $b = (b_1, b_2)$ in the integer lattice $\mathbb{Z}^2$ and $b' = (b_1 - kb_2, -b_2)$, then $\psi_b \sim \psi_{b'}$.

Now we assume that $k = -a \neq 0$. The case $k = 0$ will be considered in Subsection [3.2]. In order to calculate $S_b := S_{\psi_b}$ it is necessary to take into account the Type of $b$. The element $b \neq 0$ is of Type(0) when $\langle f, b \rangle \neq 0$, for all edge $f$ of $\Delta$. And it is of Type (1) if there is an edge $f$ such that $\langle f, b \rangle = 0$.

Values of $S_{\psi}$ on Type(0)

If $b = (b_1, b_2) \in \mathbb{Z}^2$ is of type 0, then $b_1 \neq 0 \neq b_2 \neq 0 \neq b_0 := (b_1 + ab_2)$, and the fixed points of the $S^1$-action $\psi_b$ are the inverse images by the moment map $\mu$ of the vertices of $\Delta$. If $P$ is a vertex, the $U(1)$-equivariant Euler class of the normal bundle of $\mu^{-1}(P)$ in $M$ is the product of the weights of the corresponding isotropy representation. On the other hand, the edges meeting at $P$ can be expressed as $\{P + t\rho_i : t \in [0, c_i]\}$, where $\rho_i \in (\mathbb{Z}^2)^*$ and $\rho_1, \rho_2$ is a basis of $(\mathbb{Z}^2)^*$. The aforementioned weights are precisely the numbers $\langle \rho_i, b \rangle$, with $i = 1, 2$. 


The normalized Hamiltonian function for the $S^1$-action $\psi_b$ is $\langle \mu, b \rangle - \langle Cm(\Delta), b \rangle$, where $Cm(\Delta)$ is the center of mass of $\Delta$. Hence

$$S_b = \frac{e^{qu}}{u^2} \sum_{j=1}^{4} \frac{e^{-\langle P_j, b \rangle u}}{\prod_{i=1}^{2} \langle \rho_{ji}, b \rangle},$$

with $\{\rho_{ji}\}_{i=1, 2}$ is the basis of $(\mathbb{Z}^2)^*$ formed with the aforementioned vectors which give the directions of the edges meeting at $P_j$, and $q := \langle Cm(\Delta), b \rangle$.

On $U_1 := \{[z] \in M : z_3 \neq 0 \neq z_4\}$ we can consider the complex coordinates $w = z_2/z_4$ and $w' = z_1/(z_3z_4^*)$. Then

$$\psi_b(w, w') = (e^{2\pi i b_1 t}w, e^{2\pi i b_4 t}w').$$

The point $\mu^{-1}(P_1)$ has coordinates $w = 0 = w'$, so the isotropy representation at this point has weights $b_1$ and $b_2$; in fact these values are the products by $b$ of the direction vectors of edges meeting at $P_1$. The remaining terms in (3.3) can be obtained in a similar way, or directly from the vectors $\rho_j$. So (3.3) can be written

$$u^2 S_b = e^{qu}(A - Be^{-\tau b_2 u} - Ae^{-\sigma b_1 u} + Be^{-(\tau b_2 + \sigma b_3)u})$$

where $A := (b_1b_2)^{-1}$ and $B := (b_0b_2)^{-1}$. Note that $A \neq B$; otherwise $b_0$ would be equal to $b_1$, but this is impossible in Type(0).

The expression (3.4) will be written

$$u^2 S_b = A e^{\alpha_1 u} - Be^{\alpha_2 u} - Ae^{\alpha_3 u} + Be^{\alpha_4 u},$$

where $\alpha_1 = q$, $\alpha_2 = q - \tau b_2$, $\alpha_3 = q - \sigma b_1$, $\alpha_4 = q - (\tau b_2 + \sigma b_3)$. If $\alpha_2 \neq \alpha_3$ and $\alpha_1 \neq \alpha_4$, there are four different exponential functions in (3.5). By contrast, if $\alpha_2 = \alpha_3$ and $\alpha_1 = \alpha_4$ there are only two different exponentials in (3.5). Therefore we will distinguish three subtypes in Type(0).

$$\begin{align*}
\text{Type(0)}(\alpha) & \quad \text{iff } \alpha_2 \neq \alpha_3, \alpha_1 \neq \alpha_4 \\
\text{Type(0)}(\beta) & \quad \text{iff either } \alpha_2 = \alpha_3, \alpha_1 \neq \alpha_4, \text{ or } \alpha_2 \neq \alpha_3, \alpha_1 = \alpha_4 \\
\text{Type(0)}(\gamma) & \quad \text{iff } \alpha_2 = \alpha_3, \alpha_1 = \alpha_4
\end{align*}$$

It follows from the Lemma [14] the following Proposition

**Proposition 23.** If $b$ and $b'$ are in Type(0) and belong to different subtype, then $S_b \neq S_{b'}$.

Given $b, b' \in \text{Subtype}(0)\alpha$, if $S_b = S_{b'}$, then by Lemma [14] it follows from (3.5)

$$u^2 S_b = \{A e^{\alpha_1 u}, -Be^{\alpha_2 u}, -Ae^{\alpha_3 u}, Be^{\alpha_4 u}\} = \{A' e^{\alpha'_1 u}, -B' e^{\alpha'_2 u}, -A' e^{\alpha'_3 u}, B' e^{\alpha'_4 u}\}.$$  

This set equality gives, in principle, the following eight possibilities

$$\begin{align*}
\{i\} & \quad A = A' & (a) & B = B' \\
\{ii\} & \quad A = B' & (a) & B = A' \\
\{iii\} & \quad A = -A' & (a) & B = B' \\
\{iv\} & \quad A = -B' & (a) & B = A'
\end{align*}$$

If $b, b' \in \text{Subtype}(0)\alpha$
Where the corresponding exponential function which multiplies each constant has been deleted in the equalities; for example, we have written \( B = -A' \) instead of \( B e^{A' u} = -A' e^{A' u} \).

**Proposition 24.** If \( b \) and \( b' \) are in Subtype(0)\( \alpha \) and \( S_b = S_{b'} \), then \( b \equiv b' \).

**Proof.** In the case (i)(a), \( A = A' \), \( B = B' \), \( \alpha_1 = \alpha'_1 \), \( \alpha_4 = \alpha'_4 \). And by \( (3.6) \)
\[-Be^{\alpha_2 u} = -B' e^{\alpha_2 u}, \quad -Ae^{\alpha_3 u} = -A'e^{\alpha_3 u} .\]
From \( \alpha_j = \alpha'_j \) for all \( j \), we deduce \( b = b' \).

In the case (i)(b), \(-Ae^{\alpha_3 u} = -A'e^{\alpha_3 u} \). So \( \alpha_3 = \alpha'_3 \) and of course \( \alpha_1 = \alpha'_1 \). Thus \( b_1 = b'_1 \). From \( A = A' \) it follows \( b'_2 = b_2 \); that is, \( b = b' \). But this is contradictory with \( B = -B' \).

It is also easy to check that the cases (ii)(a), (ii)(b), (iii)(a), (iii)(b) and (iv)(a) do not occur. In the case (iv)(b) it is straightforward to deduce \( b'_1 = b_1 - kb_2 \) and \( b'_2 = -b_2 \).

Next we study the map \( S_- \) on Subtype(0)\( \beta \). If \( b \) belongs to Subtype(0)\( \beta \), then either
\[ u^2 S_b = Ae^{\alpha_1 u} - (A + B)e^{\alpha_2 u} + Be^{\alpha_4 u}, \quad \text{when } \alpha_2 = \alpha_3 \]
or
\[ u^2 S_b = (A + B)e^{\alpha_1 u} - Ae^{\alpha_2 u} - Be^{\alpha_3 u}, \quad \text{when } \alpha_1 = \alpha_4 .\]

We will consider the partition \( P \) of Subtype(0)\( \beta \) formed by two subsets: The one consisting of all elements for which \( \alpha_2 = \alpha_3 \), and the other one its complement subset. It is not hard to prove the following Proposition

**Proposition 25.** Given \( b, b' \) \( \in \) Subtype(0)\( \beta \) with \( S_b = S_{b'} \), if they belong to the same subset of the partition \( P \), then \( b' = b \). If they belong to different subsets, then \( b' \equiv b \).

It is easy to prove

**Proposition 26.** The map \( S_- \) is injective on Subtype(0)\( \gamma \).

Propositions 24, 25 and 26 can be put together

**Proposition 27.** If \( b, b' \) \( \in \) Type(0) and \( S_b = S_{b'} \), then \( b \equiv b' \).

**Values of \( S_- \) on Type(1)**

If \( b \) belongs to Type(1) one of \( b_j \)'s vanishes. We will distinguish three possibilities: \( b_2 = 0 \), \( b_1 = 0 \) and \( b = (kb_2, b_2) \), that is \( b_0 = 0 \).

We first consider the case \( b_2 = 0 \). Now \( \psi_b[z] = [e^{2\pi ib_1 t} z_1, z_2, z_3] \). So the fixed point set is \( \{ [z] \in M : z_1 z_3 = 0 \} \). On \( U_2 = \{ [z] \in M : z_2 \not= 0 \not= z_3 \} \) we consider the complex coordinates \( v = z_4/z_2 \) and \( v' = z_4/(z_3 z_2^k) \). So \( \frac{\partial}{\partial v} \) is a section of the normal bundle \( N_Q \) to \( Q = \{ [z] : z_1 = 0 \} \) on \( Q \cap U_2 \). On the other hand, \( \frac{\partial}{\partial v'} \) is a section of \( N_Q \) on \( Q \cap U_1 \). As
\[ \frac{\partial}{\partial w'} = \left( \frac{z_4}{z_2} \right)^k \frac{\partial}{\partial v'} , \]
identifying \( Q \) with \( \mathbb{C}P^1 \) it turns out that \( N_Q \) is \( O(k) \).

Since the coordinate \( w' \) is transformed by \( \psi_b(t) \) in \( \exp(2\pi ib_1 t)w' \), the \( U(1) \)-equivariant Euler class \( E(N_Q) \) of the normal bundle to \( N_Q \) is \( c_1(O(k)) + b_1 u \).
Analogously, if we denote by $Q' := \{ [z] : z_3 = 0 \}$, then $E(N_{Q'}) = c_1(O(-k)) - b_1u$. As $\omega(Q) = \tau$ and $\omega(Q') = \tau - k\sigma =: \lambda$, it follows from the localization formula

$$
S_b = e^{gu} \left( \frac{1}{b_1u} (\tau - \frac{k}{b_1u}) - \frac{e^{-\sigma b_1 u}}{b_1u} (\lambda - \frac{k}{b_1u}) \right),
$$

where $q := \langle \text{Cm}(\Delta), b \rangle$.

Next we consider the case when $b = (0, b_2)$, with $b_2 \neq 0$. Now the fixed point set is $\{ [z] : z_2 = 0 \} \cup \{ P_2, P_4 \}$. The vector field $\frac{\partial}{\partial w}$ is a global section to the normal bundle to $\{ [z] : z_2 = 0 \}$, so its $U(1)$-equivariant Euler class is $b_2u$. The equivariant Euler classes of the normal bundles $N_{P_2}$ and $N_{P_4}$ are $k(b_2u)^2$ and $-k(b_2u)^2$ respectively. As $\omega(\{ [z] : z_2 = 0 \}) = \sigma$, it follows

$$
S_b = e^{gu} \left( \frac{e^{-\tau b_2 u}}{k(b_2u)^2} - \frac{e^{-\lambda b_2 u}}{k(b_2u)^2} + \frac{\sigma}{b_2u} \right).
$$

Finally, let us suppose that $b = (kb_2, b_2)$. The corresponding fixed point set is $\{ [z] : z_4 = 0 \} \cup \{ P_1, P_3 \}$. The equivariant Euler class of the normal bundles to $\{ [z] : z_4 = 0 \}$, $P_1$ and $P_3$ are $-b_2u$, $k(b_2u)^2$ and $-k(b_2u)^2$ respectively. So

$$
S_b = e^{gu} \left( \frac{1}{k(b_2u)^2} - \frac{e^{-\sigma kb_2 u}}{k(b_2u)^2} - \frac{\sigma e^{-\tau b_2 u}}{b_2u} \right).
$$

To compare the expressions $S_b$ for different elements $b$, we classify the elements of Type(0) according to the following scheme of subtypes

| Type(1) | $b \in \text{Subtype(1)}\alpha$ iff $b_2 = 0$ |
|---------|-----------------------------------------------|
|         | $b \in \text{Subtype(1)}\beta$ iff $b_2 \neq 0$ |

From (3.7), (3.8) and (3.9) together with Lemma 1.4 it follows the following Proposition

**Proposition 28.** If $b, b'$ belong to Type(1) but to different subtype, then $S_b \neq S_{b'}$.

It follows from Theorem 1 together with Theorem 2 Proposition 28 and Proposition 29 the following Theorem

**Theorem 29.** Let $M_\Delta$ be a Hirzebruch surface non diffeomorphic to $S^2 \times S^2$. Let $b$ and $b'$ be nonzero elements of $\mathbb{Z}^2$. If they belong to different subtype then $\psi_b \sim \psi_{b'}$.

Next we study whether the map $b \mapsto S_b$ is injective restricted to each of the different subtypes of Type(1).

If $b = (b_1, 0)$ belongs to Subtype(1)$\alpha$, in the expression of $u^2S_b$ obtained from (3.7) there are two different exponentials

$$
u^2 S_b = A(u)e^{gu} + B(u)e^{\lambda u},$$

where $A(u)$ and $B(u)$ are the following polynomials $A(u) = (b_1)^{-1}\tau u - k(b_1)^{-2}$, $B(u) = -(b_1)^{-1}\lambda u + k(b_1)^{-2}$. If $b'$ belongs to Subtype (1)$\alpha$ and $S_b = S_{b'}$, then either $A(u) = A'(u)$ and $B(u) = B'(u)$ or $A = B'$ and $B = A'$. In the first case $b_1' = b_1$. In the second one implies $-(b_1)^2 = (b_1')^2$. Hence we have the following Proposition

**Proposition 30.** $S_-$ is injective on Subtype(1)$\alpha$. 
In Subtype(1)\(\beta\) one can distinguish the case in which \(b\) is of the form \((0, c)\) and when it is of the form \((kc, c)\). As in the preceding Proposition, if \(b = (0, b_2)\), \(b' = (0, b'_2)\) and \(S_b = S_{b'}\), then \(b = b'\). Analogously when \(b = (kc, c)\) and \(b' = (kc', c')\).

Finally, given \(b = (0, b_2)\) and \(b' = (b_1, b'_2) = (kc, c)\), in the expressions for \(u^2S_b\) and \(u^2S_{b'}\) obtained from (3.8) and (3.9) there are three different exponentials:

\[
\begin{align*}
u^2S_b &= Ae^{(q - \tau b_2)u} + Be^{(q - \lambda b_2)u} + C(u)e^{\sigma u}, \\
u^2S_{b'} &= A'e^{q' - \lambda b'_2} + B'e^{(q' - \sigma k b'_2)u} + C'(u)e^{(q' - \tau b'_2)u},
\end{align*}
\]

with \(C(u) = b_2^{-1}\sigma\) and \(C'(u) = -(b'_2)^{-1}\sigma\). The equality \(S_b = S_{b'}\) implies \(C(u) = C'(u)\); that is \(b'_2 = -b_2\). In other words \(b \equiv b'\). We have proved the Proposition.

**Proposition 31.** If \(b\) and \(b'\) belong to Subtype(1)\(\beta\) with \(S_b = S_{b'}\), then \(b \equiv b'\).

The preceding Propositions can be summarized in the following one.

**Proposition 32.** If \(b\) and \(b'\) are elements of type 1 with \(S_b = S_{b'}\), then \(b \equiv b'\).

**Proof of Theorem 7.** Theorem 7 is consequence of Proposition 32, Proposition 3, Proposition 27, and Theorem 9. \(\square\)

### 3.2. The manifold \(S^2 \times S^2\).

Here we study the relation \(\psi_b \sim \psi_{b'}\) when \(M\) is the symplectic manifold associated to the polytope \(\Delta\) in the case \(a = 0\). That is, \(M\) is diffeomorphic to \(S^2 \times S^2\). Now \(b_0 = b_1\) and \(Cm(\Delta) = (\sigma/2, \tau/2)\), so the corresponding discussion is simpler than the one of case \(a \neq 0\).

If \(b_1 \neq 0 \neq b_2\), the expression (3.4) (which is also valid when \(k = 0\)) reduces to

\[
S_b = \frac{4}{b_1b_2u^2} \left( \sin \left( \frac{\sigma b_1u}{2} \right) \sin \left( \frac{\tau b_2u}{2} \right) \right)
\]

When \(b = (b_1, 0)\) the expression for \(S_b\) can be obtained from (3.7)

\[
S_b = \frac{2\tau}{b_1u} \sinh \left( \frac{\sigma b_1u}{2} \right)
\]

Analogously, if \(b = (0, b_2)\), then

\[
S_b = \frac{2\sigma}{b_2u} \sinh \left( \frac{\tau b_2u}{2} \right)
\]

Using Lemma 14 it is straightforward to deduce from (3.10) the following Proposition.

**Proposition 33.** If \(b_1b_2 \neq 0 \neq b'_1b'_2\), then \(S_b = S_{b'}\) iff one of the following statements are true:

(i) \(|b_1| = |b'_1|\) and \(|b_2| = |b'_2|\).

(ii) \(\sigma b_1 = \pm \tau b'_2\) and \(\tau b_2 = \pm \sigma b'_1\).

(iii) \(\sigma b_1 = \pm \tau b'_2\) and \(\tau b_2 = \mp \sigma b'_1\).

**Proof of Theorem 8.** In this case

\[
\psi_b(t)[z] = \left( [e^{2\pi ib_1t}z_2 : z_4], [z_3 : e^{2\pi ib_1t}z_1] \right),
\]

where \([z] = ([z_2 : z_4], [z_3 : z_1]) \in S^2 \times S^2\). On the other hand, if \(\sigma/\tau \not\in \mathbb{Q}\), the possibilities (ii) and (iii) in the preceding Proposition do not occur. If, for example, \(b'_1 = -b_1\) and \(b'_2 = b_2\), let \(\xi^s\) be the rotation of angle \(2\pi b_1t\) around the vector \(\hat{n}_s\) introduced in the Proof of Theorem 6. We set

\[
\psi^s(z) = \left( [e^{2\pi ib_1t}z_2 : z_4], \xi^s[z_3 : z_1] \right).
\]
Then $\psi^*$ is a homotopy between $\psi_b$ and $\psi_{b'}$ consisting of $U(1)$-actions. That is, $|b_1| = |b_2|$ and $|b_3| = |b_2|$ implies $\psi_b \sim \psi_{b'}$. So Theorem 8 follows from Proposition 33 together with Theorem 1.

Remark. In general $S_b = S_{b'}$ does not imply $[\psi_b] = [\psi_{b'}] \in [U(1), \text{Ham}(M)]$. For instance, if $\tau = \sigma, \ b = (1, 0)$ and $b' = (0, 1)$, it follows from (3.11) and (3.12) that $S_b = S_{b'}$. Gromov proved that $\text{Ham}(S^2 \times S^2)$ deformation retracts to the group $SO(3) \times SO(3)$ [8]. So $\pi_1(\text{Ham}(S^2 \times S^2)) = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\psi_b$ and $\psi_{b'}$ are the generators of the respective groups in the product. Hence $\psi_b$ and $\psi_{b'}$ are not homotopic, and a fortiori $\psi_b \not\sim \psi_{b'}$.

4. $S^1$-ACTIONS IN COADJOURT ORBITS.

Let $G$ be a compact connected Lie group and $\eta \in \mathfrak{g}^*$. We consider the coadjoint orbit $\mathcal{O}$ of the element $\eta$, endowed with the standard symplectic structure $\omega$ (see [17]). A moment map $\mu_G : \mathcal{O} \to \mathfrak{g}^*$ for the action of $G$ on $\mathcal{O}$ is the opposite of the inclusion. The manifold $\mathcal{O}$ can be identified with the quotient $G/G_\eta$, where $G_\eta$ is the subgroup of isotropy of $\eta$. Let $H$ be a Cartan subgroup of $G$ contained in $G_\eta$ ([12], page 166). We have the decomposition of $\mathfrak{g}_C$ as direct sum of root spaces

$$\mathfrak{g}_C = \mathfrak{h}_C \oplus \bigoplus_{\alpha \in \mathcal{R}} \mathfrak{g}_\alpha,$$

where $\mathcal{R}$ is the set of roots determined by $H$. We will denote by $\check{\alpha} \in \mathfrak{h}$ the coroot associated to $\alpha \in \mathcal{R}$ [7]. Let $\mathfrak{p}$ be the parabolic subalgebra

$$\mathfrak{p} = \mathfrak{h}_C \oplus \bigoplus_{\eta(i\check{\alpha}) \leq 0} \mathfrak{g}_\alpha.$$

If $P$ is the parabolic subgroup of $G_C$ generated by $\mathfrak{p}$, then $G_C/P$ is a complexification of $\mathcal{O}$ and

$$T^1,0_\mathcal{O} = \bigoplus_{\alpha \in \mathcal{R}^+} \mathfrak{g}_\alpha,$$

where $\mathcal{R}^+ = \{\alpha \in \mathcal{R} : \langle \eta, i\check{\alpha} \rangle > 0\}$.

Given $X \in \mathfrak{g}$ we consider the map $h := \langle \mu_G, X \rangle : \mathcal{O} \to \mathbb{R}$, and define the constant $\kappa \in \mathbb{R}$ by the relation

$$\kappa \int_{\mathcal{O}} \omega^n = \int_{\mathcal{O}} h \omega^n.$$

The function $h - \kappa$ is the normalized Hamiltonian function for the isotopy in $\mathcal{O}$ generated by $X$ through the $G$-action.

If $X$ belongs to the integer lattice of $\mathfrak{h}$, then $X$ determines a $U(1)$-action $\psi$ on $\mathcal{O}$. By $u$ we denote a coordinate in $\mathfrak{u}(1)$, then closed $U(1)$-equivariant 2-form $\omega - (\mu - \kappa)u$ is a representative of the coupling class $c_\psi$ [16]. Thus

$$S_\psi = e^{\kappa u} \int_{\mathcal{O}} e^{-\mu u + \omega} \in \hat{H}(BU(1)).$$

In order to apply the localization formula we need to determine the fixed point set of the $S^1$-action $\psi$. Let us assume that $X$ is a regular element [6] of the lattice of $\mathfrak{h}$. We denote by $W_\eta$ the stabilizer subgroup of $\eta$ in the Weyl group $W$ of $(G, H)$. It is easy to see that the fixed point set of the action $\psi$ is $\{w \cdot \eta : w \in W/W_\eta\}$ (see [4] page 231).
Given $w \in W$, the $U(1)$-equivariant Euler class of the normal bundle to the point $w \cdot \eta$ in $\mathcal{O}$ can be calculated as follows: From (4.1) it follows
\[
T_{w\eta}^{1,0}\mathcal{O} = \bigoplus_{\alpha \in \mathcal{R}^+} \mathfrak{g}_{w \cdot \alpha}.
\]
Moreover $[Y, Z] = (w \cdot \alpha)(Y)Z$ for any $Z \in \mathfrak{g}_{w \cdot \alpha}$ and all $Y \in \mathfrak{h}$. So the representation of $U(1)$ on $\mathfrak{g}_{w \cdot \alpha}$ induced by $\psi$ therefore has the weight $-i(w \cdot \alpha) \in u(1)^*$, and the weights of the isotropy representation at the point $w \cdot \eta \in \mathcal{O}$ are $-i(w \cdot \alpha)$, with $\alpha \in \mathcal{R}^+$. That is, the $U(1)$-equivariant Euler class satisfies
\[
E(N_{w\eta})(Y) = \prod_{\alpha \in \mathcal{R}^+} -i(w \cdot \alpha)(Y),
\]
for any $Y = tX$ with $t \in \mathbb{R}$. Thus $E(N_{w\eta})$ is a monomial in $u$
\[
E(N_{w\eta}) = \prod_{\alpha \in \mathcal{R}^+} A_{\alpha}^wu \in H(BU(1)),
\]
with $A_{\alpha}^w$ constant.

Consequently, if $X$ is a regular element of the integer lattice of $\mathfrak{h}$, in the localization $H(BU(1))_u$ one has
\[
S_\psi = e^{\kappa u} \sum_{w \in W/W_\eta} e^{i(w^{-1} \cdot X)u} \prod_{\alpha \in \mathcal{R}^+} A_{\alpha}^wu,
\]
since $\mu(w \cdot \eta) = -\langle w \cdot \eta, X \rangle = -\eta(w^{-1} \cdot X)$.

**Theorem 34.** Given $X, X'$ regular elements of the integer lattice of $\mathfrak{h}$, let $\psi$ and $\psi'$ be the circle actions on $\mathcal{O}$ generated by $X$ and $X'$, respectively. If there is no a translation in $\mathbb{R}$ which applies the set of real numbers $\{\eta(w \cdot X) : w \in W/W_\eta\}$ in $\{\eta(w \cdot X') : w \in W/W_\eta\}$, then $\psi$ and $\psi'$ are not homotopically equivalent by means of a homotopy consisting of circle actions

**Proof.** The class $S_\psi$ is obtained from (4.4) by substituting $\kappa, X$ and $A_{\alpha}^w$ by the corresponding $\kappa', X'$ and $A_{\alpha}^{w'}$. By the hypothesis the sets of functions
\[
\mathcal{E} := \{e^{(\kappa + i(w^{-1} \cdot X))u} \mid w \in W/W_\eta\} \quad \text{and} \quad \mathcal{E}' := \{e^{(\kappa' + i(w^{-1} \cdot X'))u} \mid w \in W/W_\eta\}
\]
are distinct. Hence $\mathcal{E}$ is not contained in the $\mathbb{R}[u]$-submodule of $\mathbb{R}[u]$ generated by $\mathcal{E}'$. Reasoning as in the proof of Theorem 35 we conclude that $S_\psi \neq S_{\psi'}$, and the theorem follows from Theorem 34.

**Proposition 35.** Let $\psi$ and $\psi'$ be the circle actions on the coadjoint orbit $\mathcal{O}$ generated by the regular elements $X$ and $X'$ of the integer lattice of $\mathfrak{h}$, respectively. If there is no an affine transformation
\[
B : x \in \mathbb{R} \rightarrow \lambda x + a \in \mathbb{R}, \quad \text{with} \ 0 \neq \lambda \in \mathbb{Z} \text{ and } a \in \mathbb{R},
\]
such that
\[
B(\{\eta(w \cdot X) : w \in W/W_\eta\}) = \{\eta(w \cdot X') : w \in W/W_\eta\},
\]
then $\psi$ and $\psi'$ are $\sim_{rp}$-inequivalent.
Proof. If $\psi$ and $\psi'$ were $\sim_T$-equivalent, there would be an automorphism $v$ of $U(1)$ with $\psi' \sim v \circ \psi$. Moreover $\psi \circ v$ is the $S^1$-action generated by $Z = cx$, with $c$ a nonzero integer. By Theorem 4.3 there must be a translation $T_h$ in $\mathbb{R}$ which applies $\{\eta(w \cdot Z) = c \eta(w \cdot X) : w \in W/W_\eta\}$ in $\{\eta(w \cdot X') : w \in W/W_\eta\}$. Hence the affine transformation $B: x \in \mathbb{R} \mapsto cx + b \in \mathbb{R}$ contradicts the hypothesis of the Proposition.

\[\Box\]

Proof of Theorem 4.9 We consider the coadjoint orbit $O = G/Q$ of $G$, where $Q$ is a subgroup of $G$ which contains a maximal torus $H$ of $G$. The element $X$ in the integer lattice of $\mathfrak{h}$ generates the circle action

$$\psi_t: gQ \in G/Q \mapsto (e^{tx}g)Q \in G/Q.$$ 

We denote by $\psi'$ the $S^1$-action defined by $X'$. Let $g_1$ be an element of the normalizer $N(H)$ of $H$ in $G$, such that $g_1 H \subseteq N(H)/H = W$ corresponds to the element $w$. And let $g: s \in [0,1] \mapsto g(s) \in G$ be a curve with $g(0) = 1$ and $g(1) = g_1$. We write $X_s := \text{Ad}g(s)X$, then $X_1 = w(X) = X'$. For $s,t \in [0,1]$ we put

$$\psi_t^s: gQ \in G/Q \mapsto (e^{tx}s)gQ \in G/Q.$$ 

As $e^{tx}s = g(s)e^{Xs}(g(s))^{-1} = 1$, for each $s$ the family $\{\psi_t^s\}_t$ is a circle action on the orbit $O = G/Q$. Moreover

$$\psi_1^s(gQ) = (e^{tx}s)gQ = (e^{tx}s')gQ = \psi_t'(gQ),$$

and $\psi_0^s(gQ) = \psi_t(gQ)$. Hence $\{\psi^s\}$ is a homotopy consisting of $U(1)$-actions between $\psi$ and $\psi'$.

\[\Box\]

By $G_k(\mathbb{C}^n)$ we denote the Grassmann manifold consisting of the $k$-subspaces in $\mathbb{C}^n$. If $M$ is a coadjoint orbit of $SU(n)$ diffeomorphic to $G_k(\mathbb{C}^n)$, it is the orbit of an element $\eta \in \mathfrak{su}(n)^*$ such that for $C \in \mathfrak{su}(n)$

$$\eta(C) = \sum_{j=1}^{k} C_{jj} + \sum_{j=k+1}^{n} C_{jj},$$

with $\alpha_1, \alpha_2 \in i \mathbb{R}$ and $\alpha_1 \neq \alpha_2$. That is, $\eta(C) = -ir \sum_{j=1}^{k} C_{jj}$, with $r \in \mathbb{R} \setminus \{0\}$.

Now we suppose that $k = 1$, then the coadjoint orbit $O := O_\eta = SU(n)/U(n-1) = \mathbb{C}P^{n-1}$. The coset $gU(n-1) \subseteq SU(n)/U(n-1)$ determines the line in $\mathbb{C}^n$ defined by the first column of $g$. This is the identification $O \simeq \mathbb{C}P^{n-1}$ we will use.

We write $\mathcal{R}$ for the set of roots determined by $H$, the diagonal subgroup of $SU(n)$. The stabilizer of $\eta$ in the Weyl group $W = N(H)/H$ consists of all the permutations $\sigma$ of $\{1, \ldots, n\}$ with $\sigma(1) = 1$. Denoting by $\sigma_1$ the permutation identity and by $\sigma_j$ the transposition $(1,j)$, with $j = 2, \ldots, n$, then $\{\sigma_1, \ldots, \sigma_n\}$ is a set of representatives of the classes in $W/W_\eta$.

Given $X \in \mathfrak{su}(n)$, the Hamiltonian isotopy generated by $X$ has the map $h - \kappa$ as normalized Hamiltonian function, $\kappa$ being a constant and $h$

$$h : \nu \in O \mapsto -\langle \nu, X \rangle \in \mathbb{R}.$$

If $X = i(a_1, \ldots, a_n)$ with $(a_j) \in (2\pi \mathbb{Z})^n$, then $X$ generates a circle action $\psi$ on $O$. Furthermore, if $a_i \neq a_j$ for $i \neq j$, then the fixed point set for the action $\psi$ is $\{\sigma_k \cdot \eta\}_{k=1,\ldots,n}$.
Denoting by \( \alpha_{ij} \) the root defined by \( \alpha_{ij}(B) = b_i - b_j \), for \( B = \text{diag}(b_k) \), then the set of positive roots is \( R^+ = \{ \alpha_{12}, \ldots, \alpha_{1n} \} \), assumed \( r > 0 \). For \( k \neq 1 \)

\[
(-i \sigma_k \cdot \alpha_{1s})(X) = \begin{cases} 
  a_k - a_s, & \text{if } s \neq k \\
  a_k - a_1, & \text{if } s = k
\end{cases}.
\]

And \( (-i \sigma_1 \cdot \alpha_{1s})(X) = a_1 - a_s \). By (4.2) we obtain for the equivariant Euler class of normal bundle \( N_k \) to \( \sigma_k \cdot \eta \)

\[
E(N_k) = \prod_{s \neq k} (a_k - a_s)u^{n-1}.
\]

Since \( h(\sigma_k \cdot \eta) = -\eta(\sigma_k^{-1} \cdot X) = -r a_k \), it follows from (4.7) together with (4.1)

\[
S_\psi = e^{\kappa u} u^{n-1} \sum_{k=1}^n \frac{e^{r a_k u}}{\prod_{s \neq k} (a_k - a_s)}.
\]

**Proposition 36.** Let \( X = (x_1, \ldots, x_n) \), \( X' = (x'_1, \ldots, x'_n) \) be regular elements of the integer lattice of \( \eta \), and \( \psi, \psi' \) respective the circle actions on the coadjoint orbit \( O \) of \( SU(n) \) diffeomorphic to \( \mathbb{C}P^{n-1} \). Then the following statements are equivalent:

(a) \( S_\psi = S_{\psi'} \).
(b) There is a permutation \( \tau \) of \( \{1, \ldots, n\} \) such that \( x'_j = x_{\tau(j)} \), for all \( j \).
(c) \( \psi \sim \psi' \).

**Proof.** (a) \( \Rightarrow \) (b). If we write \( X' = i(a'_1, \ldots, a'_n) \) the expression for \( S_{\psi'} \) can be obtained from (4.8) by substituting the \( a_1 \)'s by the \( a'_1 \)'s and \( \kappa \) by \( \kappa' \). From the equality \( S_{\psi} = S_{\psi'} \) together with the \( \mathbb{R}[u, u^{-1}] \)-linear independence of the exponentials, it follows that there is a permutation \( \tau \) of \( \{1, \ldots, n\} \) such that \( \kappa' + r a'_j = \kappa + r a_{\tau(j)} \). Thus there is a constant \( c \) such that \( a'_j - a_{\tau(j)} = c \) for all \( j = 1, \ldots, n \). As \( \sum_j a'_j = \sum_j a_1 = 0 \), it turns out \( c = 0 \); hence (b) holds.

(b) \( \Rightarrow \) (c) by Corollary 11. The equivalence of (a), (b) and (c) follows from Theorem 1.

Theorem 11 is the generalization of Proposition 36 when the hypothesis about the regularity of \( X \) and \( X' \) is deleted.

**Proof of Theorem 11**

If

\[
-iX = (a_1, \ldots, a_n, \ldots, a_1, \ldots) \equiv (a_1, \ldots, a_n) \in (2\pi \mathbb{Z})^n
\]

with \( a, b, \ldots \) different numbers, then a connected component of the fixed point set for the circle action \( \psi \) determined by \( X \) is \( F = \{ [z_1 : \cdots : z_n : 0 : \cdots : 0] \} \subset \mathbb{C}P^{n-1} = O \). The contribution of the component \( F \) to \( S_\psi \) can be calculated as in the paragraphs before Lemma 13. That is, \( N_F \) can be decomposed in a direct sum of \( U(1) \)-equivariant line bundles \( \oplus_{j=m+1}^n N_j \), and

\[
E(N_F) = \prod_{j=m+1}^n \left( c_1(N_j) + (a - a_j)u \right).
\]

So we have as in (2.8)

\[
(E(N_F))^{-1} = \frac{1}{u^{n-m}} (a_0 + \alpha_1 u + \ldots).
\]
The aforementioned contribution to \( S_\psi \) is
\[
e^{(n+r)a}Q_a(u),
\]
\( Q_a(u) \) being a Laurent polynomial of degree \( m-n \). Thus expression for \( S_\psi \) is
\begin{equation}
S_\psi = e^{(n+r)a}Q_a(u) + \cdots + e^{(n+rd)a}Q_d(u).
\end{equation}

Let
\begin{equation}
- iX' = (a', \ldots, a', e', \ldots, e') \equiv (a'_1, \ldots, a'_n) \in (2\pi \mathbb{Z})^n.
\end{equation}
The equality \( S_\psi = S_{\psi'} \) implies the following set equality \( \{a + \beta, b + \beta, \ldots, d + \beta\} = \{a', \ldots, e'\} \), with \( \beta \) constant. Furthermore, if for example \( a' = b + \beta \) then degree \( Q_{a'} = \text{degree} \ Q_b \). That is, the number of \( b' \)’s in \( \{a', \ldots, e'\} \) must be equal to the number of \( a' \)’s in \( \{a, \ldots, e\} \). Hence there is a permutation \( \tau \) of \( \{1, \ldots, n\} \), such that \( a_i + \beta = a'_\tau(i) \) and if \( a_i = a_j \) then \( a'_\tau(i) = a'_\tau(j) \). As \( \sum_j a_j = 0 = \sum_j a'_j \), then \( \beta = 0 \).

So the theorem is consequence of Corollary 10 and Theorem 1. □

**Proof of Theorem**

Now we must consider the case \( 1 < k < n \) in \( 4.3 \). In this case, if the permutation \( \sigma \in W_\eta \) then \( \sigma \) applies the subset \( \{1, \ldots, k\} \) onto itself (and of course \( \{k+1, \ldots, n\} \) onto itself). The stabilizer \( W_\eta \) has \( k!(n-k)! \) elements. A set of representatives of the elements of \( W/W_\eta \) is the set \( C_k \) of combinations of \( \{1, \ldots, n\} \) with \( n-k \) elements. If \( \mathfrak{d} \in C_k \), \( \sigma_\mathfrak{d} \) will denote the corresponding element in the quotient \( W/W_\eta \).

Let \( X = \text{diag}(ia_1, \ldots, ia_n) \in (2\pi i \mathbb{Z})^n \) be a regular element, and \( \psi \) is the corresponding circle action on the Grassmannian \( G_k(\mathbb{C}^n) \). If \( \mathfrak{c} = \{1, \ldots, n\} \setminus \{j_1, \ldots, j_k\} \), the value of the Hamiltonian \( 4.6 \) on \( \mathfrak{c} \cdot \eta \) is
\[
h(\sigma_{\mathfrak{d}} \cdot \eta) = -r \sum_{i=1}^k a_{j_i}.
\]
We write \( a_{\mathfrak{d}} = \sum_{i=1}^k a_{j_i} \), when \( \mathfrak{d} = \{i_1, \ldots, i_k\} \in C_k \). Then
\begin{equation}
S_\psi = e^{nu} \sum_{\mathfrak{c} \subseteq C_k} \sum_{p_\mathfrak{c} \in \mathbb{Z}} \frac{e^{\kappa_{\mathfrak{c}}u}}{p_\mathfrak{c}},
\end{equation}
where \( p_\mathfrak{c} \in \mathbb{Z} \) and \( r \) is a real number.

If \( X' \) is another regular element and \( S_\psi = S_{\psi'} \), it follows from \( 4.12 \) that there is a bijective map \( f : C_k \rightarrow C_k \) such that
\[
\kappa + \alpha_{f(\mathfrak{c})} = \kappa' + \alpha_{\mathfrak{c}},
\]
for all \( \mathfrak{c} \in C_k \). □

If \( \mathfrak{o} \) is a coadjoint orbit of \( SU(n) \) diffeomorphic to the full flag manifold, it is the orbit of an element \( \eta \in \mathfrak{su}(n)^* \), such that for any \( C \in \mathfrak{su}(n) \)
\begin{equation}
\eta(C) = -i \sum_{j=1}^{n-1} r_j C_{jj},
\end{equation}
with \( r_i \neq r_j \) for \( i \neq j \). We will assume that \( r_1 > r_2 > \cdots > r_{n-1} \). So \( \mathcal{R}^+ = \{\alpha_{ij} : i < j\} \).
Given $X \in \mathfrak{su}(n)$ with $-iX = (a_1, \ldots, a_n) \in (2\pi \mathbb{Z})^n$ and $a_i \neq a_j$ for $i \neq j$, the fixed point set for the $S^1$-action $\psi$ defined by $X$ on $\mathcal{O}$ is $\{ \sigma \cdot \eta : \sigma \in S_n \}$, where $S_n$ is the symmetric group. The $U(1)$-equivariant Euler class

$$E(N_{\sigma \cdot \eta}) = u^m \epsilon(\sigma) \prod_{r < s} (a_r - a_s),$$

$\epsilon(\sigma)$ being the signature of $\sigma$, and $m = n(n - 1)/2$. As

$$-\eta(\sigma \cdot X) = \sum_{j=1}^{n-1} r_j a_{\sigma(j)},$$

we obtain

$$S_\psi = \frac{e^{cu}}{u^m \prod_{r < s} (a_r - a_s)} \sum_{\sigma \in S_n} \epsilon(\sigma) \exp \left( u \sum_{j=1}^{n-1} r_j a_{\sigma(j)} \right).$$

If $X'$ is another regular element with $-iX' = (a'_1, \ldots, a'_n) \in (2\pi \mathbb{Z})^n$ and $S_\psi = S_{\psi'}$, it follows from (4.14) together with Lemma 14 that there exist a constant $\beta$ and a bijective map $f$ from $S_n$ into itself, such that for all $\sigma \in S_n$

$$\sum_{j=1}^{n-1} r_j a'_{\sigma(j)} = \sum_{j=1}^{n-1} r_j a_{f\sigma(j)} + \beta. \quad (4.15)$$

We have proved the following Proposition

**Proposition 37.** Let $\mathcal{O}$ be the coadjoint orbit of the element $\eta \in \mathfrak{su}(n)^*$ given by (4.13), and let $\psi$ and $\psi'$ be the $U(1)$-actions defined by the regular elements $(ia_1, \ldots, ia_n)$ and $(ia'_1, \ldots, ia'_n)$ respectively. If $\psi \sim \psi'$, then there exist a constant $\beta$ and a bijective map $f$ from $S_n$ into itself, such that (4.15) holds for all $\sigma \in S_n$.

**Remark.** In particular, if there exist $\tau \in S_n$, such that $a'_j = a_{\tau(j)}$ for all $j$; that is, the hypothesis of Corollary 10 are satisfied, then we can construct the map $f : \sigma \in S_n \rightarrow \tau \cdot \sigma \in S_n$. Thus $a'_{\sigma(k)} = a_{f\sigma(k)}$ and equation (4.15) holds with $\beta = 0$. That is, Proposition 37 is consistent with Corollary 10.

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Departamento de Física. Universidad de Oviedo. Avda Calvo Sotelo. 33007 Oviedo. Spain.
E-mail address: vina@uniovi.es