Research Article
On Genocchi Numbers and Polynomials

Seog-Hoon Rim, Kyoung Ho Park, and Eun Jung Moon

Department of Mathematics, Kyungpook National University, Taegu 702-701, South Korea

Correspondence should be addressed to Kyoung Ho Park, sagamath@yahoo.co.kr

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The main purpose of this paper is to study the distribution of Genocchi polynomials. Finally, we construct the Genocchi zeta function which interpolates Genocchi polynomials at negative integers.

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1. Introduction

Let \( p \) be a fixed odd prime number. Throughout this paper, \( \mathbb{Z}_p \), \( \mathbb{Q}_p \), \( \mathbb{C} \), and \( \mathbb{C}_p \) will, respectively, denote the ring of \( p \)-adic rational integers, the field of \( p \)-adic rational numbers, the complex number field, and the completion of the algebraic closure of \( \mathbb{Q}_p \). Let \( \nu_p \) be the normalized exponential valuation of \( \mathbb{C}_p \) with \( |p|_p = p^{-\nu_p(p)} = 1/p \). When one talks about \( q \)-extension, \( q \) is variously considered as an indeterminate, a complex, \( q \in \mathbb{C} \), or a \( p \)-adic number, \( q \in \mathbb{C}_p \). If \( q \in \mathbb{C} \), one normally assumes \( |q| < 1 \). If \( q \in \mathbb{C}_p \), then we assume \( |q - 1|_p < 1 \).

The ordinary Genocchi polynomials are defined as the generating function:

\[
F(t, x) = \frac{2t}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}, \quad |t| < \pi.
\]  

(1.1)

For a fixed positive integer \( d \) with \( (p, d) = 1 \), set

\[
X = X_d = \lim_{N} \frac{\mathbb{Z}}{d\mathbb{Z} N}, \quad X_1 = \mathbb{Z}_p,
\]

\[
X^* = \bigcup_{0 < a < dp, \ (a, p) = 1} (a + dp\mathbb{Z}_p),
\]

(1.2)

\[
a + dp^N \mathbb{Z}_p = \{ x \in X \mid x \equiv a \mod dp^N \}
\]
2. Genocchi numbers and polynomials

The Genocchi numbers are defined as

\[ G_n = 0, \quad (G + 1)^n + G_n = \begin{cases} 2 & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \]  
(2.1)

where \( G^n \) is replaced by \( G_n \), symbolically. The Genocchi polynomials are also defined as

\[ G_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} G_k. \]  
(2.2)

By (2.1), we note that \( G_1 = 1, \ G_2 = -1, \ G_3 = 0, \ G_4 = 1, \ldots, G_{2k+1} = 0, \) and \( G_{2k} \in \mathbb{Z} \) \((k = 1, 2, \ldots)\). The fermionic \( p \)-adic invariant integral on \( \mathbb{Z}_p \) is defined as

\[ I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \to \infty} \frac{1}{[p^N]_{-1}} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \]  
(2.3)

Let \( f_1(x) \) be translation with \( f_1(x) = f(x + 1) \). Then we have the following integral equation. Note that \( I_{-1}(f_1) + I_{-1}(f) = 2f(0) \). From (2.3), we can derive

\[ I_{-1}(f) = \int_{\mathbb{Z}_p} e^{xt} d\mu_{-1}(x) = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!} = t \sum_{n=0}^{\infty} \frac{G_{n+1} t^n}{n+1 n!}, \]  
(2.4)
Thus, we have
\[ \int_{\mathbb{Z}_p} e^{(x+y)t} \, d\mu_{-1}(y) = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} = t \sum_{n=0}^{\infty} \frac{G_{n+1}(x)}{n+1} \frac{t^n}{n!}. \quad (2.5) \]

Thus, we obtain
\[ \int_{\mathbb{Z}_p} x^n \, d\mu_{-1}(x) = \frac{G_{n+1}}{n+1}, \quad \int_{\mathbb{Z}_p} (x+y)^n \, d\mu_{-1}(y) = \frac{G_{n+1}(x)}{n+1}. \quad (2.6) \]

For \( n \in \mathbb{N} \), we have
\[ \int_{\mathbb{Z}_p} f(x+n) \, d\mu_{-1}(x) = (-1)^n \int_{\mathbb{Z}_p} f(x) \, d\mu_{-1}(x) + 2 \sum_{\ell=0}^{n-1} (-1)^{n-1+\ell} f(\ell), \quad \text{see } [1-27]. \quad (2.7) \]

By (2.6) and (2.7), if we take \( f(x) = x^k (k \in \mathbb{Z}^+) \), we easily see that
\[ \int_{\mathbb{Z}_p} (x+n)^k \, d\mu_{-1}(x) - \int_{\mathbb{Z}_p} x^k \, d\mu_{-1}(x) = 2 \sum_{\ell=0}^{n-1} (-1)^{\ell-1} \ell^k \quad \text{if } n \equiv 0 \pmod{2}. \quad (2.8) \]

Thus, we have
\[ \frac{G_{k+1}(n)}{k+1} - \frac{G_{k+1}}{k+1} = 2 \sum_{\ell=0}^{n-1} (-1)^{\ell-1} \ell^k \quad \text{if } n \equiv 0 \pmod{2}. \quad (2.9) \]

If \( n \equiv 1 \pmod{2} \), then we know that
\[ \int_{\mathbb{Z}_p} (x+n)^k \, d\mu_{-1}(x) + \int_{\mathbb{Z}_p} x^k \, d\mu_{-1}(x) = 2 \sum_{\ell=0}^{n-1} (-1)^{\ell} \ell^k \quad \text{if } n \equiv 1 \pmod{2}. \quad (2.10) \]

Thus, we get
\[ \frac{G_{k+1}(n)}{k+1} + \frac{G_{k+1}}{k+1} = 2 \sum_{\ell=0}^{n-1} (-1)^{\ell} \ell^k, \quad \text{see } [1-30]. \quad (2.11) \]

Let \( \chi \) be the Dirichlet character with conductor \( d \in \mathbb{N} \), with \( d \equiv 1 \pmod{2} \). Then, we consider the generalized Genocchi numbers attached to \( \chi \) as follows:
\[ \frac{G_{n+1,\chi}}{n+1} = \int_X \chi(x) x^n \, d\mu_{-1}(x), \quad G_{0,\chi} = 0, \quad (2.12) \]

where \( n \in \mathbb{Z}_+ \). From (2.7) and (2.12), we note that
\[ t \int_X e^{\xi t} \chi(x) \, d\mu_{-1}(x) = \sum_{\ell=0}^{d-1} (-1)^{\ell} \chi(\ell) e^{\xi \ell} \ell! + t = \sum_{n=0}^{\infty} \frac{G_{n,\chi} t^n}{n!}. \quad (2.13) \]

By (2.12) and (2.13), it is not difficult to show that
\[ \frac{G_{n+1,\chi}}{n+1} = \int_X \chi(x) x^n \, d\mu_{-1}(x) \]
\[ = d^n \sum_{a=0}^{d-1} \chi(a)(-1)^a \int_{\mathbb{Z}_p} \left( \frac{a}{d} + x \right)^n \, d\mu_{-1}(x) \quad (2.14) \]
\[ = d^n \sum_{a=0}^{d-1} \chi(a)(-1)^a \frac{G_{n+1}(a/d)}{n+1}, \]
\[ \int_X (x+y)^n \, d\mu_{-1}(y) = d^n \sum_{a=0}^{d-1} (-1)^a \int_{\mathbb{Z}_p} \left( \frac{x+a}{d} + y \right)^n \, d\mu_{-1}(y). \quad (2.15) \]
By (2.6) and (2.15), we obtain the following theorem.

**Theorem 2.1.** Let \( d \in \mathbb{N} \) with \( d \equiv 1(\text{mod } 2) \), and let \( \chi \) be the Dirichlet character with conductor \( d \). Then, one has

\[
\frac{G_{n+1}(x)}{n+1} = d^n \sum_{a=0}^{d-1} (-1)^a \frac{G_{n+1}((x + a)/d)}{n+1} \quad \text{(distribution relation for Genocchi polynomials),}
\]

(2.16)

\[
\frac{G_{n+1,\chi}}{n+1} = d^n \sum_{a=0}^{d-1} \chi(a) (-1)^a \frac{G_{n+1}(a/d)}{n+1}.
\]

(2.17)

### 3. Genocchi zeta function

Let \( F(t, x) \) be the generating function of \( G_k(x) \) in complex plane as follows:

\[
F(t, x) = \frac{2te^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!},
\]

(3.1)

\[
= t \sum_{n=1}^{\infty} G_{n+1}(x) \frac{t^n}{n+1}, \quad |t| < \pi.
\]

Then, we show that

\[
F(t, x) = 2t \sum_{n=0}^{\infty} (-1)^n e^{(n+1)x}.
\]

(3.2)

By (3.1) and (3.2), we easily see that

\[
G_k(x) = \left. \frac{d^k F(t, x)}{dt^k} \right|_{t=0} = 2k \sum_{n=0}^{\infty} (-1)^n (n + x)^{k-1}.
\]

(3.3)

Therefore, we obtain the following proposition.

**Proposition 3.1.** For \( k \in \mathbb{N} \), one has

\[
\frac{G_k(x)}{k} = 2 \sum_{n=0}^{\infty} (-1)^n (n + x)^{k-1}.
\]

(3.4)

From Proposition 3.1, we can derive the Genocchi zeta function which interpolates Genocchi polynomials at negative integers.

For \( s \in \mathbb{C} \), we define the Hurwitz-type Genocchi zeta function as follows.

**Definition 3.2.** For \( s \in \mathbb{C} \),

\[
\zeta_{G}(s, x) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n + x)^s}, \quad \zeta_{G}(s) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}.
\]

(3.5)
Theorem 3.3. For \( k \in \mathbb{N} \), one has
\[
\zeta_C(1 - k, x) = \frac{G_k(x)}{k}, \quad \zeta_C(1 - k) = \frac{G_k}{k}.
\] (3.6)

Let \( \chi \) be the Dirichlet character with conductor \( d \in \mathbb{N} \), with \( d \equiv 1 \pmod{2} \), and let \( F_\chi(t) \) be the generating function in \( \mathbb{C} \) of \( G_{n,\chi} \). Then, we have
\[
F_\chi(t) = 2 \frac{\sum_{a=0}^{d-1} (-1)^a \chi(a) e^a}{e^{dt} + 1} - t \sum_{n=0}^{\infty} G_{n,\chi} \frac{t^n}{n!} \quad |t| < \frac{\pi}{d}. \tag{3.7}
\]

From (3.7), we derive
\[
F_\chi(t) = \sum_{n=0}^{\infty} G_{n,\chi} \frac{t^n}{n!} = \sum_{n=1}^{\infty} G_{n,\chi} \frac{t^n}{n!}
\]
\[
= t \sum_{n=0}^{\infty} \frac{G_{n+1,\chi} t^n}{n+1 n!}
\]
\[
= t \sum_{n=0}^{\infty} \left( \sum_{a=0}^{d-1} d^n \sum_{a=0}^{d-1} \chi(a)(-1)^a \frac{G_{n+1}(a/d)}{n+1} \right) \frac{t^n}{n!}
\]
\[
= \sum_{a=0}^{d-1} \chi(a)(-1)^a \left( dt \sum_{n=0}^{\infty} G_{n+1}(a/d) \frac{d^n t^n}{n!} \right). \tag{3.8}
\]

By (3.1), (3.2), and (3.8), we easily see that
\[
F_\chi(t) = \sum_{a=0}^{d-1} \chi(a)(-1)^a \left( 2t \sum_{k=0}^{\infty} (-1)^k e^{(k+a/d)dt} \right)
\]
\[
= 2t \sum_{k=0}^{\infty} \sum_{a=0}^{d-1} \chi(a + dk)(-1)^a e^{(dk+a)t}
\]
\[
= 2t \sum_{n=0}^{\infty} \chi(n)(-1)^n e^{nt}
\]
\[
= 2t \sum_{n=1}^{\infty} \chi(n)(-1)^n e^{nt}. \tag{3.9}
\]

From (3.9), we can derive
\[
G_{k,\chi} = \frac{d^k}{dt^k} F_\chi(t) \bigg|_{t=0} = k \left( 2 \sum_{n=1}^{\infty} \chi(n)(-1)^n n^{k-1} \right). \tag{3.10}
\]

Thus, we have
\[
\frac{G_{k,\chi}}{k} = 2 \sum_{k=0}^{\infty} \chi(n)(-1)^n n^{k-1}. \tag{3.11}
\]
Now, we consider the Dirichlet-type Genocchi $\ell$-function in complex plane as follows. For $s \in \mathbb{C}$, define

$$\ell_{\ell, \ell}(s) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n \chi(n)}{n^s}. \tag{3.12}$$

By (3.11) and (3.12), we obtain the following theorem.

**Theorem 3.4.** Let $\chi$ be the Dirichlet character with conductor $d \in \mathbb{N}$, with $d \equiv 1 \pmod{2}$, and let $k \in \mathbb{Z}^+$. Then, one has

$$\ell_{\ell, \ell}(1 - k) = \frac{G_{k, \ell}}{k}. \tag{3.13}$$

**Remark 3.5.** In [1], we can observe the value of Genocchi zeta function at positive integers as follows:

$$\zeta_{\ell}(2n) = \frac{(-1)^{n-1} \pi^{2n}(2 - 4^n)}{2(2n)! (1 - 4^n)} G_{2n}, \quad \text{cf. [1].} \tag{3.14}$$

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