On the Classifying Space for Commutativity in $U(3)$

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Abstract

For a Lie group $G$ let $B_{\text{com}}G$ be the classifying space for commutativity first introduced in [4]. Let $E_{\text{com}}G$ be the total space of the principal $G$-bundle associated to $B_{\text{com}}G$. In this article we present a computation of the cohomology of $E_{\text{com}}U(3)$ using the spectral sequence associated to a homotopy colimit. As a part of our computation we will also compute the integral cohomology of $U(3)/N(T)$ and $(U(3)/T) \times_{\Sigma_3} (U(3)/T)$ where $T$ is a maximal torus of $U(3)$ with normalizer $N(T)$.

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1 Introduction

For a topological group $G$ with non-degenerate identity element $1_G$, let $C_n(G)$ be the set of $n$-tuples $(g_1, \ldots, g_n)$ such that $g_i g_j = g_j g_i$ for all $1 \leq i, j \leq n$, topologized as a subspace of $G^n$. We can identify $C_n(G)$ with the space of homomorphisms Hom($\mathbb{Z}^n, G$). The spaces \{Hom($\mathbb{Z}^n, G$)\}_{n \geq 0} can be assembled into a simplicial subspace Hom($\mathbb{Z}^*, G$) of $NG$. Define $B_{\text{com}}G := |\text{Hom}(\mathbb{Z}^*, G)|$.

In [4] it has been shown that $B_{\text{com}}G$ is a classifying space for principal $G$-bundles whose transition functions are transitonally commutative. The pullback of the universal bundle $EG \to BG$ along $i : B_{\text{com}}G \to BG$ is denoted as $E_{\text{com}}G$. The bundle $E_{\text{com}}G \to B_{\text{com}}G$ has a transitonally commutative structure. The space $E_{\text{com}}G$ can also be described simplicially. We will mostly study the space $E_{\text{com}}G$ in this article. Note that the fibration above
splits after looping and there is a homotopy equivalence \( G \times \Omega E_{\text{com}} G \simeq \Omega B_{\text{com}} G \) (see [4]).

Most of the computations regarding \( B_{\text{com}} G \) and \( E_{\text{com}} G \) has been centered around compact Lie groups. This is mainly because in ([I], Theorem 3.1), it was shown that for a real or complex reductive algebraic group \( G \) with maximal compact subgroup \( K \) the inclusion map \( K \subset G \) induces homotopy equivalences \( B_{\text{com}} K \simeq B_{\text{com}} G \) and \( E_{\text{com}} K \simeq E_{\text{com}} G \). When the spaces \( \{\text{Hom}(\mathbb{Z}^n, G)\}_{n \geq 0} \) are not path-connected, we have spaces \( B_{\text{com}} G_3 \) and \( E_{\text{com}} G_1 \), arising from the connected component of identity in \( \{\text{Hom}(\mathbb{Z}^n, G)\}_{n \geq 0} \). When \( \{\text{Hom}(\mathbb{Z}^n, G)\} \) are path-connected for all \( n \geq 0 \) the constructions \( B_{\text{com}} G_1 \) and \( E_{\text{com}} G_1 \) agree with \( B_{\text{com}} G \) and \( E_{\text{com}} G \) respectively.

In [I], the spaces \( B_{\text{com}} G_3 \) and \( E_{\text{com}} G_3 \) are described as a homotopy colimit over a poset \( S(n) \), where \( n = \text{Rank}(G) - \text{Rank}(Z(G)) \). The poset \( S(n) \) consists of all non-empty subsets of \( \{0, 1, \ldots, n\} \), with order being reverse inclusion. By ([I], Theorem 6.3) and ([I], Theorem 6.5)

\[
B_{\text{com}} G_1 \simeq \text{hocolim}_{i \in S(n)} F_G(i) \tag{1}
\]

\[
E_{\text{com}} G_1 \simeq \text{hocolim}_{i \in S(n)} H_G(i) \tag{2}
\]

where \( F_G, H_G : S(n) \to \text{Top} \) are functors defined later. We are interested in finding the integral cohomology of \( B_{\text{com}} G_1 \) and \( E_{\text{com}} G_1 \) for \( G = U(n) \) and \( SU(n) \). The characteristic classes for transitionally commutative complex vector bundles of rank \( n \) over a space \( X \) are the pullback of cohomology classes in \( H^*(B_{\text{com}} U(n); \mathbb{Z}) \) along the classifying map \( X \to BU(n) \), which factors through \( B_{\text{com}} U(n) \) as \( X \to B_{\text{com}} U(n) \hookrightarrow BU(n) \). Knowing \( H^*(B_{\text{com}} U(n); \mathbb{Z}) \) helps us better understand transitionally commutative complex vector bundles.

The cohomology of \( B_{\text{com}} G \) and \( E_{\text{com}} G \) for \( G = SU(2), U(2) \) were computed in [I] and [5]. In this article we will focus on the rank 3 case. In section 5.1 we give an additive description of the mod 2 cohomology of \( E_{\text{com}} U(3) \).

**Theorem 5.5.** The \( \mathbb{F}_2 \)-cohomology of \( E_{\text{com}} U(3) \) is given by

\[
H^k(E_{\text{com}} U(3); \mathbb{F}_2) = \begin{cases} 
\mathbb{F}_2; & k = 0, 4, 5, 7, 9, 11, 12, 14, 15 \\
(\mathbb{F}_2)^2; & k = 8, 10 \\
(\mathbb{F}_2)^3; & k = 6 \\
0; & k = \text{otherwise}
\end{cases}
\]

We will use the spectral sequence associated to a homotopy colimit for this computation. In section 5.2 we show that \( H^*(E_{\text{com}} U(3)) \) can only have 2 and 3 torsion. Finally using the previous result along with the Universal Coefficient Theorem we get the integral homology and cohomology in Theorem 5.9. In section 5.3 we describe the relations between the generators of \( H^*(E_{\text{com}} U(3); \mathbb{Q}) \). This allows us to write down the integral cohomology ring of \( E_{\text{com}} U(3) \) as follows:

**Theorem 5.12.** The integral cohomology ring of \( E_{\text{com}} U(3) \) is given by

\[
H^*(E_{\text{com}} U(3); \mathbb{Z}) \cong \mathbb{Z}[\gamma_4, \gamma_6, \tilde{\gamma}_6, \gamma_{12}, t_6, t_8, t_{10}, t_{11}, t_{15}] / \left(2t_4, \gamma_4^2, \gamma_6^2, \tilde{\gamma}_6^2, t_4 \tilde{t}, t_4 \gamma, t_4 \tilde{\gamma}_6, 3 \gamma_6 \tilde{\gamma}_6 - 2 \gamma_{12}\right)
\]
where \( i, j \) runs through all possible values and \( \deg(t_i) = i \).

## 2 Description of the Spaces \( \mathcal{H}_{U(3)}(i) \)

**Definition 2.1.** Let \( G \) be a Lie group. Define \( \mathcal{T}(G) \) to be the poset whose objects are closed subspaces \( S \subset G \), which are intersection of a collection of maximal tori in \( G \). The partial order in \( \mathcal{T}(G) \) is given by inclusion.

The topology on the set \( \mathcal{T}(G) \) can be described as follows. Let \( \mathcal{C}(G) \) be the set of all closed subspaces of \( G \). Suppose \( \mathcal{U} = \{U_1, \ldots, U_k\} \) is a finite collection of open sets in \( G \). Define \( \mathcal{C}(G, \mathcal{U}) \) as the set of elements \( A \in \mathcal{C}(G) \) such that \( A \subseteq \bigcup_{i=1}^k U_i \) and \( A \cap U_i \neq \emptyset \) for all \( 1 \leq i \leq k \). The sets \( \mathcal{C}(G, \mathcal{U}) \) forms a basis for a topology in \( \mathcal{C}(G) \) called the finite topology. We give \( \mathcal{T}(G) \) the subspace topology via the inclusion \( \mathcal{T}(G) \subseteq \mathcal{C}(G) \).

Let \( \mathfrak{g} \) and \( \mathfrak{t} \) be the Lie algebras of \( G \) and a maximal torus \( T \subset G \) respectively. Also, let \( \Phi \) be a root system associated to \( (\mathfrak{g}, \mathfrak{t}) \) with \( \Phi^+ \) the subset of positive roots. Define for each \( \alpha \in \Phi^+ \)

\[
\mathfrak{t}_\alpha = \{X \in \mathfrak{t}|\alpha(X) \in 2\pi i \mathbb{Z}\}
\]

Let \( I = \{\alpha_1, \ldots, \alpha_k\} \) be a set of positive roots and

\[
\mathfrak{t}_I = \cap_{i=1}^k \mathfrak{t}_{\alpha_i} \quad \text{and} \quad T_I := \exp(\mathfrak{t}_I).
\]

We also set \( \mathfrak{t}_\emptyset := \mathfrak{t} \) and hence \( T_\emptyset = \exp(\mathfrak{t}) = T \). We take a set of simple roots \( \Delta = \{\alpha_1, \ldots, \alpha_l\} \) for the root system \( \Phi \). Denote the Weyl group of \( G \) by \( W \), which is a reflection subgroup generated by the reflections \( s_\alpha, \alpha \in \Delta \). For a subset \( I \subseteq \Delta \), define \( W_I \) to be the subgroup of \( W \) generated by the reflections \( s_\alpha, \alpha \in I \). For \( I, J \subseteq \Delta, \ W_I \) and \( W_J \) are conjugate if and only if \( I = wJ \) for some \( w \in W \). When this happens we say that \( I \) and \( J \) are in the same Coxeter class and denote it as \( I \sim_W J \). The relation \( \sim_W \) is an equivalence relation on the subsets of \( \Delta \) and the set of equivalence class is denoted as \( \mathcal{E}_W \). The following theorem is from ([1], Theorem 5.4).

**Theorem 2.2.** Let \( G \) be a compact connected Lie group. Fix a set of simple roots \( \Delta \). Then any element \( S \in \mathcal{T}(G) \) is conjugated to some \( T_I \) for \( I \subseteq \Delta \). Moreover, there is a \( G \)-equivariant homeomorphism \( \mathcal{T}(G) \cong \bigsqcup_{I \in \mathcal{E}_W} \mathbb{G}/N_G(T_I) \)

For a compact connected Lie group \( G \), let \( n = \text{rank}(G) - \text{rank}(Z) \), where \( Z := Z(G) \) is the center of \( G \). Define \( \rho : \mathcal{T}(G) \longrightarrow \mathbb{N} \cup \{0\} \) by \( \rho(S) := \text{rank}(S) - \text{rank}(Z) \). We see that for all \( S \in \mathcal{T}(G), \ 0 \leq \rho(S) \leq n \). Also \( \rho \) is strictly increasing and constant on each connected component of \( \mathcal{T}(G) \cong \bigsqcup_{I \in \mathcal{E}_W} \mathbb{G}/N_G(T_I) \).

Let \( n = \text{rank}(G) - \text{rank}(Z) \geq 0 \) and \( S(n) \) be the poset of all non-empty subsets of \( \{0, 1, \ldots, n\} \), with the order being reverse inclusion of sets. Elements of \( S(n) \) are of the form \( \mathbf{i} = \{i_0, \ldots, i_k\} \), where \( 0 \leq i_0 < i_1 < \cdots < i_k \leq n \). We define the functors \( \mathcal{F}_G \) and \( \mathcal{H}_G \) associated to a compact connected Lie group \( G \) as follows. Let \( \mathbf{i} = \{i_0, \ldots, i_k\} \in S(n) \) and define

\[
\mathcal{H}_G(\mathbf{i}) := \{(S_0, \ldots, S_k, b)|S_0 \subset \cdots \subset S_k \in \mathcal{T}(G), \ \rho(S_r) = i_r \text{ for } 0 \leq r \leq k \text{ and } b \in G/S_0\}.
\]
Note that \( H_G(i) \subset T(G)^{k+1} \times G/Z(G) \). We give \( H_G(i) \) the subspace topology. If \( j \subset i \), then the natural projection map induce continuous functions \( p_{ij} : H_G(i) \rightarrow H_G(j) \). Hence, \( H_G : S(n) \rightarrow \text{Top} \) defines a functor. Fix \( i = \{ i_0, \ldots, i_k \} \in S(n) \). The space \( H_G(i) \) can be described explicitly in the following way. Let \( S_i = \{ S_{i_0}, \ldots, S_{i_k} \} \) and \( S'_i = \{ S'_{i_0}, \ldots, S'_{i_k} \} \) be two chains in \( T(G) \) such that \( \rho(S_{i_l}) = \rho(S'_{i_l}) = i_l \) for \( 0 \leq l \leq k \). Then we define the equivalence relation on the chains in \( T(G) \) by setting \( S_i \sim S'_i \) if and only if there is some \( g \in G \) such that \( gS_{i_l}g^{-1} = S'_{i_l} \), that is \( gS_{i_l}g^{-1} = S'_{i_l} \) for all \( 0 \leq l \leq k \). Denote the set of equivalence classes by \( E(i) \). Then we have a continuous map

\[
\mu_{S_i} : G \times G/S_0 \rightarrow H_G(i)
\]

\[
(g, a) \mapsto (gS_i g^{-1}, gag^{-1})
\]

Let \( N_G(S_i) \) be the subgroup of \( G \) consisting of all \( g \in G \) such that \( gS_{i_l}g^{-1} = S_{i_l} \). We refer to \( N_G(S_i) \) as the normaliser of \( S_i \) in \( G \). The groups \( N_G(S_i) \) acts on \( G \) as \( n \cdot g = gn^{-1} \) and on \( G/S_0 \) by conjugation. The map \( \mu_{S_i} \) is invariant under the diagonal action of \( N_G(S_i) \) on \( G \times G/S_0 \). If we vary \( S_i \) through all the equivalence classes in \( E(i) \) we get a continuous map

\[
\mu_i := \bigsqcup_{S_i \in E(i)} \mu_{S_i} : \bigsqcup_{S_i \in E(i)} G \times_{N_G(S_i)} G/S_0 \rightarrow H_G(i).
\]

This maps is bijective and \( \mu_i^{-1} \) is continuous. So, we have a homeomorphism which describes \( H_G(i) \) just as the one in section 6 of \( \text{[1]} \) describing \( F_G(i) \):

\[
H_G(i) \cong \bigsqcup_{S_i \in E(i)} G \times_{N_G(S_i)} G/S_{i_0}.
\]

We are mostly interested in the case when \( G = U(n) \) or \( SU(n) \). When \( G = U(n) \) or \( SU(n) \), \( B_{com}G \) and \( E_{com}G \) are path-connected as \( \text{Hom}(\mathbb{Z}^n, G) \) is path-connected. Moreover \( E_{com}G \) is 3-connected (\( \text{[1]} \), Proposition 3.3). The cases \( G = U(2) \) and \( SU(2) \) were computed in \( \text{[5]} \). We look at the case \( G = U(3) \) in detail. We will use the spectral sequence associated to a homotopy colimit (\( \text{[9]} \), Theorem 18.3) to compute the cohomology of \( E_{com}U(3) \). By Theorem 5.8 the integral cohomology of \( E_{com}U(3) \) can only have 2 and 3-torsion. We will work with \( \mathbb{Z}/2 \) and \( \mathbb{Z}/3 \) coefficients to make the computations a bit easier.

One can interpret the functor \( H_G \) from the poset \( S(2) \) as a diagram of spaces. The homotopy colimit over the poset is same as the homotopy colimit of the associated diagram. We get the following diagram for the poset \( S(2) = \{ (0), (1), (2), (0, 1), (0, 2), (1, 2), (0, 1, 2) \} \):
We are omitting the set notation while referring to the elements of the poset. So, \( \mathcal{H}_G(i, j) \) should be interpreted as \( \mathcal{H}_G(\{i, j\}) \). The homotopy colimit of the above diagram can also be interpreted as a homotopy pushout cube as follows:

\[
\begin{array}{ccc}
\mathcal{H}_G(1, 2) & \longrightarrow & \mathcal{H}_G(2) \\
\mathcal{H}_G(0, 1, 2) & \longrightarrow & \mathcal{H}_G(0, 2) \\
\mathcal{H}_G(1) & \longrightarrow & E_{\text{com}} U(3) \\
\mathcal{H}_G(0, 1) & \longrightarrow & \mathcal{H}_G(0) \\
\end{array}
\]

Before going to the computation of the homotopy colimit, we take a closer look at the spaces \( \mathcal{H}_{U(3)}(i) \) and their cohomology.

Let \( T_d, 1 \leq d \leq n \) be a torus of dimension \( d \) in \( G = U(3) \) of the form

\[
\{\text{diag}(\lambda_1, \ldots, \lambda_1, \ldots, \lambda_d, \ldots, \lambda_d)|\lambda_i \in S^1\}.
\]

\( T_1 \) is the centre and \( T_n \) is the maximal torus. Note that all \( d \) dimensional tori of the form \( T_d \) for \( 1 \leq d \leq n \) in \( U(n) \) are conjugate. Let \( i = \{i_0, \ldots, i_k\} \in S(n-1) \). Then \( \mathcal{E}(i) \) consists of equivalence classes of chains \( \{S_{i_0}, \ldots, S_{i_k}\} \), with \( \rho(S_{i_l}) = i_l \) and the equivalence relation being \( S_i \sim S'_i \) if and only if there is some \( g \in G \) such that \( gS_i g^{-1} = S'_i \), that is \( g S_{i_l} g^{-1} = S'_{i_l} \) for all \( 0 \leq l \leq k \). By Theorem 2.2, \( S_{i_l} \) is conjugated to some \( T_I \) for \( I \subseteq \Delta \), where \( \Delta \) is the set of all simple roots. Replacing \( S_i \) with a suitable conjugate we can assume that \( S_i \) is of the form \( S_{i_0} \subset S_{i_1} \subset \cdots \subset S_{i_k} \subset T_n \). In [1] it was shown that any chain \( S_i = \{S_{i_0} \subset \cdots \subset S_{i_k}\} \) is conjugated to a chain of the form \( \{T_{i_0} \subset \cdots \subset T_{i_k}\} \), where \( I_l \subseteq \Delta \). It turns out that \( S_{i_l} \) is conjugated to some \( T_{i_{l+1}} \), the \( (i_l + 1) \)-dimensional tori in \( U(n) \) described above. Hence, any chain \( S_i = \{S_{i_0} \subset \cdots \subset S_{i_k}\} \) is conjugated to a chain of the form \( \{T_{i_0+1} \subset \cdots \subset T_{i_{k+1}}\} \). So, \( \mathcal{E}(i) \) consists of a single equivalence class represented by \( \{T_{i_0+1} \subset \cdots \subset T_{i_{k+1}}\} \).

If \( d_1 \leq d_2 \), then \( N_{U(n)}(T_{d_2}) \subseteq N_{U(n)}(T_{d_1}) \). So, the normalizer of \( S_i \) is thus given by \( N_{U(n)}(S_i) = N_{U(n)}(S_{i_k}) = N_{U(n)}(T_{i_{k+1}}) \).

Let, \( T = T_3 \) be the maximal torus, \( T_2 \) be the 2-torus \( \{\text{diag}(\lambda_1, \lambda_1, \lambda_2)|\lambda_i \in S^1\} \) and \( Z \) be the center in \( U(3) \). By Theorem 2.2

\[
\mathcal{T}(U(3)) = \{e\} \sqcup U(3)/N(T) \sqcup U(3)/N(T).
\]

Also, by (3)

\[
\mathcal{H}_{U(3)}(i) \cong \bigsqcup_{S_i \in \mathcal{E}(i)} G \times_{N_{U(3)}(S_i)} U(3)/S_{i_0}.
\]
We can now describe all the pieces $\mathcal{H}_{U(3)}(i)$ for $i \in S(2)$. From now on we will denote $\mathcal{H}_{U(3)}$ by $\mathcal{H}$ and $N_{U(3)}$ by $N$. We have

$$
\begin{align*}
\mathcal{H}(0) &= U(3)/Z = PU(3) \\
\mathcal{H}(1) &= U(3) \times_{N(T_2)} U(3)/T_2 \\
\mathcal{H}(2) &= U(3) \times_{N(T)} U(3)/T = U(3)/T \times_{\Sigma_3} U(3)/T \\
\mathcal{H}(0, 1) &= U(3)/N(T_2) \times PU(3) \\
\mathcal{H}(0, 2) &= U(3)/N(T) \times PU(3) \\
\mathcal{H}(1, 2) &= U(3) \times_{N(T)} U(3)/T_2 = U(3)/T \times_{\Sigma_3} U(3)/T_2 \\
\mathcal{H}(0, 1, 2) &= U(3)/N(T) \times PU(3)
\end{align*}
$$

In the next couple of sections we will compute cohomology of some of these spaces. One particular space of interest is $U(3)/N(T)$. Even though it is known for all $n$ that $U(n)/N(T)$ is rationally acyclic, barely anything is known about their integral cohomology except that it can have $p$-torsion for those $p$ dividing the order of the Weyl group. Before going into the cohomology computations we state a general result relating $E_{com} U(n)$ and $E_{com} SU(n)$.

**Proposition 2.3.** There are homotopy equivalences $E_{com} U(n) \simeq E_{com} SU(n)$ for all $n \geq 2$.

**Proof.** To see this we need the following lemma from [12]:

**Lemma 2.4.** ([12], Lemma 1.2.8) If $\tilde{G} \to G$ is a covering homomorphism of compact connected Lie groups, then the following diagram is a homotopy pullback diagram:

$$
\begin{array}{ccc}
B_{com} \tilde{G}_1 & \to & B\tilde{G} \\
\downarrow & & \downarrow \\
B_{com} G_1 & \to & BG
\end{array}
$$

We have a covering homomorphism $S^1 \times SU(n) \to U(n)$ given by $(\lambda, g) \mapsto \lambda g$. Applying the above lemma gives us the following homotopy pullback square:

$$
\begin{array}{ccc}
BS^1 \times B_{com} SU(n) & \to & BS^1 \times BSU(n) \\
\downarrow & & \downarrow \\
B_{com} U(n) & \to & BU(n)
\end{array}
$$

The homotopy fibers of the horizontal maps are homotopy equivalent and hence

$$E_{com} U(n) \simeq E S^1 \times E_{com} SU(n) \simeq E_{com} SU(n).$$

[12]
3 Cohomology of $U(3)/N(T)$

We want to compute the cohomology of $U(3)/N(T)$. As the Weyl group $W = N(T)/T$ for both $U(3)$ and $SU(3)$ is isomorphic to $\Sigma_3$, the symmetric group of 3 letters and $U(3)/T \cong SU(3)/T$, we have $U(3)/N(T) \cong SU(3)/N(T)$. The normal subgroup $A_3 \leq \Sigma_3$ is isomorphic to $\mathbb{Z}/3$ and is generated by the 3-cycle $(123)$. We have a short exact sequence

$$0 \to A_3 \xrightarrow{i} \Sigma_3 \xrightarrow{\text{sign}} \mathbb{Z}/2 = \{1, -1\}$$

which splits and hence $\Sigma_3$ can be written as a semidirect product $A_3 \rtimes \mathbb{Z}/2$.

Given a finite group $\pi$ and a fibration $\pi \to X \to X/\pi$, we have a spectral sequence, called the Cartan-Leray spectral sequence. The $E_2^{p,q}$ term is given by $H^p(\pi; H^q(X; \mathbb{Z}))$, the group cohomology of $\pi$ with coefficients in the singular cohomology of $X$. Note that we have used $H$ to denote both cohomologies, but cohomology of all finite groups are to be considered as group cohomology. Also, the action of $\pi$ on $H^q(X; \mathbb{Z})$ could be non-trivial as will be in most of our cases. The spectral sequence converges to $H^*(X; \mathbb{Z})$.

Let us denote $U(3)/T$ by $\mathcal{M}$. Then $U(3)/N(T) = \mathcal{M}/\Sigma_3 = (\mathcal{M}/A_3)/(\mathbb{Z}/2)$ as (4) splits. So, we have fibrations

$$A_3 \to \mathcal{M} \to \mathcal{M}/A_3$$

and

$$\mathbb{Z}/2 \to \mathcal{M}/A_3 \to \mathcal{M}/\Sigma_3$$

The Cartan-Leray spectral sequences associated to these fibrations are

$$H^p(A_3; H^q(\mathcal{M}; \mathbb{Z})) \Longrightarrow H^{p+q}(\mathcal{M}/A_3; \mathbb{Z})$$

$$H^p(\mathbb{Z}/2; H^q(\mathcal{M}/A_3; \mathbb{Z})) \Longrightarrow H^{p+q}(\mathcal{M}/\Sigma_3; \mathbb{Z})$$

Our strategy will be to carry out the computation in two steps. First using the spectral sequence (7) and then (8). Recall that $\mathcal{M} = U(3)/T = Fl(3)$ is the complete flag variety. The integral cohomology of $Fl(3)$ is given by

$$H^*(\mathcal{M}; \mathbb{Z}) = \mathbb{Z}[x_1, x_2, x_3]/(e_1, e_2, e_3),$$

where $x_i \in H^2(\mathcal{M}; \mathbb{Z})$ and $e_i$ is the $i$-th elementary symmetric polynomial in the variables $x_i$'s. The proof of this result can be found in [11]. Moreover $\{x_1^{i_1} \cdot x_2^{i_2} \cdot x_3^{i_3} | i_j \leq 3 - j \}$ forms a basis for $H^*(\mathcal{M}; \mathbb{Z})$ and hence

$$H^0(\mathcal{M}; \mathbb{Z}) = \mathbb{Z}$$

$$H^2(\mathcal{M}; \mathbb{Z}) = \mathbb{Z}[x_1] \oplus \mathbb{Z}[x_2] \oplus \mathbb{Z}[x_3] =: \mathcal{A}$$

$$H^4(\mathcal{M}; \mathbb{Z}) = \mathbb{Z}[x_1 x_2] \oplus \mathbb{Z}[x_2 x_3] \oplus \mathbb{Z}[x_3 x_1] \oplus \mathbb{Z}[x_1 x_2 + x_2 x_3 + x_3 x_1] =: \overline{\mathcal{A}}$$

$$H^6(\mathcal{M}; \mathbb{Z}) = \mathbb{Z}[x_1^2 x_2]$$
$\Sigma_3$ acts on $H^*(M; \mathbb{Z})$ by permuting the $x_i$’s. In terms of representations of $\Sigma_3$, $H^0(M; \mathbb{Z})$ is the trivial representation, $H^2(M; \mathbb{Z})$ is the 2-dimensional standard representation, $H^4(M; \mathbb{Z})$ its dual and $H^6(M; \mathbb{Z})$ is the sign representation. Let us denote the $\mathbb{Z}$-module $H^2(M; \mathbb{Z})$ by $A$ and $H^4(M; \mathbb{Z})$ by $\overline{A}$. By definition

$$H^p(A_3; A) := \text{Ext}^p_{\mathbb{Z}[A_3]}(\mathbb{Z}; A)$$

We identify $\mathbb{Z}[A_3]$ with $\mathbb{Z}[\mathbb{Z}/3] \cong \mathbb{Z}[x]/(x^3 - 1)$. Let $\sigma = (123) \in A_3$. Note that $A$ is a $\mathbb{Z}[A_3] = \mathbb{Z}[x]/(x^3 - 1)$-module via $x \cdot x_i = x_{\sigma(i)}$. Consider the following projective resolution:

$$\cdots \xrightarrow{(x^2 + x + 1)} \mathbb{Z}[x] \xrightarrow{(x - 1)} \mathbb{Z}[x] \xrightarrow{(x^2 + x + 1)} \mathbb{Z}[x] \xrightarrow{(x - 1)} \mathbb{Z}[x] \xrightarrow{x + 1} \mathbb{Z} \quad (10)$$

Taking $\text{Hom}_{\mathbb{Z}[A_3]}(-; A)$ we see that $\text{Ext}^p_{\mathbb{Z}[A_3]}(\mathbb{Z}; A)$ is given by the $p$-th cohomology of the following chain complex:

$$A \xrightarrow{\phi} A \xrightarrow{0} A \xrightarrow{\phi} A \xrightarrow{0} A \xrightarrow{\phi} \cdots$$

where the map $\phi$ is given by $x_i \mapsto (x_{\sigma(i)} - x_i)$. We can identify $A$ as a free $\mathbb{Z}$-module of rank 2 with generators $x_1$ and $x_2$. Under this identification, $\phi$ can be represented as the following integer matrix:

$$\phi = \begin{pmatrix} -1 & 1 \\ -1 & -2 \end{pmatrix}$$

It is now easy to see that $\phi$ is injective and hence $\text{Ext}^p_{\mathbb{Z}[A_3]}(\mathbb{Z}; A) = 0$ for all $k \geq 0$. For $p = 2k + 1$, $\text{Ext}^p_{\mathbb{Z}[A_3]}(\mathbb{Z}; A) = A/\text{im}(\phi) \cong \mathbb{Z}/3$. So, we have

$$H^p(A_3; A) = \begin{cases} \mathbb{Z}/3 & \text{if } p \text{ is odd} \\ 0 & \text{if } p \text{ is even} \end{cases}$$

Similar computation yields similar results if we replace $A$ with $\overline{A}$. The action of $\sigma$ on the generator of $H^0(M; \mathbb{Z})$ is trivial

$$\sigma \cdot x_1^2 x_2 = x_2 x_3 = -x_2^2 x_1 = x_1^2 x_2.$$ 

So, $H^6(M; \mathbb{Z})$ is the trivial $\mathbb{Z}[A_3]$-module. Recall the group cohomology of $A_3 \cong \mathbb{Z}/3$ with coefficients in the trivial module is given by

$$H^p(A_3; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } p = 0 \\ \mathbb{Z}/3 & \text{if } p \text{ is even and } p > 0 \\ 0 & \text{if } p \text{ is odd} \end{cases}$$

Finally we can write down the spectral sequence (17). From our computations we have

$$E_2^{p,q} = H^p(A_3; H^q(M; \mathbb{Z})) = \begin{cases} \mathbb{Z} & \text{if } (p = 0, q = 0) \text{ or } (p = 0, q = 6) \\ \mathbb{Z}/3 & \text{if } (p = \equiv 1 \mod 2, q = 2, 4) \text{ or } (p = \equiv 0 \mod 2, p > 0, q = 0, 6) \\ 0 & \text{otherwise} \end{cases}$$
As \( d_2 \equiv 0 \), \( E^{p,q}_3 = E^{p,q}_2 \). Note that \( \mathcal{M}/A_3 \) is a 6-dimensional orientable closed manifold as \( H^6(\mathcal{M}/A_3; \mathbb{Z}) \cong \mathbb{Z} \). Also \( \pi_1(\mathcal{M}/A_3) \cong \mathbb{Z}/3 \) and hence \( H_1(\mathcal{M}/A_3; \mathbb{Z}) \cong \mathbb{Z}/3 \). Poincaré duality tells us that \( H^5(\mathcal{M}/A_3; \mathbb{Z}) \cong \mathbb{Z}/3 \). So, \( d_3 : E^{1,2}_3 \rightarrow E^{4,0}_3 \) or \( d_5 : E^{1,4}_5 \rightarrow E^{6,0}_5 \) must be non-zero. It turns out that \( d_3 \) is non-zero.

**Lemma 3.1.** The differential \( d_3 : E^{1,2}_3 \rightarrow E^{4,0}_3 \) in the above spectral sequence is non-zero.

We will prove this result in the next section. Right now we proceed with assuming that this result is true. All possible non-zero \( d_3 \)'s in the \( E_3 \)-page are isomorphism except \( d_3 : E^{6,6}_3 = \mathbb{Z} \rightarrow E^{3,4}_3 = \mathbb{Z}/3 \), which is surjective. All higher differentials are zero making \( E^{p,q}_4 = E^{p,q}_2 \) and the cohomology of \( \mathcal{M}/A_3 \) would be the following:

\[
H^k(\mathcal{M}/A_3; \mathbb{Z}) = \begin{cases} 
\mathbb{Z} & \text{if } k = 0, 6 \\
\mathbb{Z}/3 & \text{if } k = 2, 5 \\
0 & \text{otherwise}
\end{cases}
\]

(11)

We look the spectral sequence \([\mathbb{S}]\), specifying on the action of \( \mathbb{Z}/2 < \Sigma_3 \) on \( H^*(\mathcal{M}/A_3; \mathbb{Z}) \). Let us denote the transposition (12) \( \in \Sigma_3 \) by \( \tau \). Then \( \{1, \tau\} \) can be thought of as the \( \mathbb{Z}/2 \) inside \( \Sigma_3 \). \( H^0(\mathcal{M}/A_3; \mathbb{Z}) = \mathbb{Z} \) is the trivial \( \mathbb{Z}/2 \) module.

On the other hand \( H^6(\mathcal{M}/A_3; \mathbb{Z}) = E^{0,6}_\infty \). Now in both the above cases \( E^{0,6}_\infty \) is the kernel of the surjective map \( \mathbb{Z} \rightarrow \mathbb{Z}/3 \). That means the map \( \mathcal{M} \rightarrow \mathcal{M}/A_3 \) induces on cohomology the map

\[
H^6(\mathcal{M}/A_3; \mathbb{Z}) \xrightarrow{\cdot 3} H^6(\mathcal{M}; \mathbb{Z}) = H^6(\mathcal{M}; \mathbb{Z})^A_3 \cong H^0(\mathcal{A}_3; \mathbb{H}^6(\mathcal{M}; \mathbb{Z}))
\]

So, \( H^6(\mathcal{M}/A_3; \mathbb{Z}) \) is embedded as a submodule of the sign representation, so it is also the sign representation. So, \( \tau \) acts on \( H^6(\mathcal{M}/A_3; \mathbb{Z}) \) as multiplication by \(-1\). To distinguish \( H^6(\mathcal{M}/A_3; \mathbb{Z}) \) from the trivial rank one \( \mathbb{Z} \)-module, we will denote it as \( \mathbb{Z}_4 \).
To see the action on \(H^2(A_3; \mathbb{Z})\) recall a non-zero element \(\alpha \in H^2(A_3; \mathbb{Z})\) is a class of a non-split extension of \(A_3\) by \(\mathbb{Z}\)

\[
0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \xrightarrow{f} A_3 \rightarrow 0
\]

with \(f(1) = \sigma\). Any element \(\delta \in \Sigma_3\) acts on \(H^2(A_3, \mathbb{Z})\) as follows. Given any \(\delta \in \Sigma_3\) we define \(c_\delta : A_3 \rightarrow A_3\) by \(c_\delta(\eta) = \delta \eta \delta^{-1}\). Note as \(A_3\) is normal in \(\Sigma_3\), the map \(c_\delta\) is well defined and for a transposition \(c_\delta = c_\delta^{-1}\). We have an induced map on cohomology \(c_\delta^* : H^2(A_3; \mathbb{Z}) \rightarrow H^2(A_3; \mathbb{Z})\). We define \(\tau \cdot \alpha = (c_\tau^*)^{-1}(\alpha)\), where \((c_\tau^*)^{-1}(\alpha)\) is the class of the non-split extension

\[
0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \xrightarrow{c_\delta \circ f} A_3 \rightarrow 0
\]

with \(c_\tau \circ f(1) = c_\tau(\sigma) = \tau \sigma \tau^{-1} = \sigma^2\). So, the transposition \(\tau\) acts as \(-1\) on \(H^2(A_3; \mathbb{Z}) \cong \mathbb{Z}/3\).

The last cohomology group we consider is \(H^1(A_3; \mathcal{A})\). As the action of \(\mathbb{Z}/2\) on \(H^1(A_3; \mathcal{A})\) and \(H^1(A_3; \mathcal{A})\) are similar, we work with the latter. Using the standard cochain complex as in chapter III of \([\mathcal{M}/A_3, \mathcal{A}]\) is the set of all maps \(f : A_3 \rightarrow \mathcal{A}\) with \(Z^1(A_3, \mathcal{A})\) consists of those maps that satisfy \(f(\delta \eta) = f(\delta) + \delta \cdot f(\eta)\) and \(B^1(A_3, \mathcal{A})\) consists of all maps of the form \(d_\eta(\delta) = \delta \cdot q - q\) for a fixed element \(q \in \mathcal{A}\). One can easily verify that \(B^1(A_3, \mathcal{A})\) is generated by \(\{d_{x_1}, d_{x_2}, d_{x_3}\}\) as an abelian group. Let \(f_i \in Z^1(A_3, \mathcal{A})\) be such that \(f_i(\sigma) = x_i\). By definition \(f_i(\sigma^2) = f_i(\sigma) + \sigma \cdot f_i(\sigma) = x_i + x_{\sigma(i)}\). Under the map \(c_\sigma^*\), the cohomology class \(f_i + B^1(A_3, \mathcal{A})\) is sent to \(f_i \circ c_\tau + B^1(A_3, \mathcal{A})\). Now \(f_i \circ c_\tau(\sigma) = f_i(\sigma^2) = x_i + x_{\sigma(i)}\). Also we have \(f_i(\sigma^2) - d_{x_i}(\sigma) = (x_i + x_{\sigma(i)}) - (x_{\sigma(i)} - x_i) = 2x_i = 2f_i(\sigma)\). Hence, we have \(f_i \circ c_\tau + B^1(A_3, \mathcal{A}) = 2f_i + B^1(A_3, \mathcal{A})\). It is quite clear that \(f_i = f_{\sigma(i)}\) in \(H^1(A_3; \mathbb{Z})\) as \(f_i + d_{x_i} = f_{\sigma(i)}\). Also, the cohomology class \([f_i]\) generates \(H^1(A_3; \mathbb{Z}) \cong \mathbb{Z}/3\). Hence, \(\tau \cdot [f_i] = [f_i \circ c_\tau] = 2[f_i]\) and we conclude that \(\tau\) acts by \(-1\) on \(H^1(A_3; \mathcal{A})\).

So, we have that \(\mathbb{Z}/2\) acts on the \(\mathbb{Z}/3\) factors of \(H^*(\mathcal{M}/A_3; \mathbb{Z})\) non-trivially. Let us denote those modules as \((\mathbb{Z}/3)_t\). We get \(H^p(\mathbb{Z}/2; (\mathbb{Z}/3)_t)\) as the \(p\)-th cohomology of the following chain complex:

\[
\mathbb{Z}/3 \cong \mathbb{Z}/3 \rightarrow 0 \rightarrow \mathbb{Z}/3 \cong \mathbb{Z}/3 \rightarrow 0 \rightarrow \mathbb{Z}/3 \cong \cdots
\]

As the above chain complex is exact, \(H^p(\mathbb{Z}/2; (\mathbb{Z}/3)_t) = 0\) for all \(p \geq 0\).

Recall the group cohomology of \(\mathbb{Z}/2\):

\[
H^p(\mathbb{Z}/2; \mathbb{Z}) = \begin{cases} 
\mathbb{Z} & \text{if } p = 0 \\
\mathbb{Z}/2 & \text{if } p > 0 \text{ is even} \\
0 & \text{if } p \text{ is odd}
\end{cases}
\]

\[
H^p(\mathbb{Z}/2; \mathbb{Z}_t) = \begin{cases} 
\mathbb{Z}/2 & \text{if } p \text{ is odd} \\
0 & \text{if } p \text{ is even}
\end{cases}
\]

By definition

\[
E_2^{p,q} = H^p(\mathbb{Z}/2, H^q(\mathcal{M}/A_3; \mathbb{Z}))
\]

and the spectral sequence is as follows:
As the only non-zero rows in the spectral sequence are 0 and 6, the only possible non-zero differential is \( d_7 \). As \( U(3)/N(T) \) is a 6-dimensional complex, the differential \( d_7 \) must be non-zero. All other higher differentials are zero and hence \( E_{8,q}^{p,q} = E_{8,q}^{p,q} \). Therefore,

\[
H^k(U(3)/N(T); \mathbb{Z}) = H^k(\mathcal{M}/\Sigma_3; \mathbb{Z}) = \begin{cases} 
\mathbb{Z} & \text{if } k = 0 \\
\mathbb{Z}/2 & \text{if } k = 2, 4, 6 \\
0 & \text{otherwise}
\end{cases}
\]

From the product structure in \( E_2 \)-page we can easily determine the product on the \( E_\infty \)-page.

**Theorem 3.2.** The cohomology ring of \( U(3)/N(T) \) is given by

\[
H^*(U(3)/N(T); \mathbb{Z}) = \mathbb{Z}[t]/(2t, t^4); \quad |t| = 2.
\]

Notice the absence of 3-torsion in the integral cohomology of \( U(3)/N(T) \) is mainly due to the non-trivial action of \( \mathbb{Z}/2 \) on \( A_3 < \Sigma_3 \). In general the integral cohomology of \( U(n)/N(T) \) can have \( p \)-torsion for those \( p \) dividing \( n! \). The absence of 3-torsion in the integral cohomology of \( U(3)/N(T) \) is particularly intriguing and we are interested in finding out more details about the general case.

## 4 Cohomology of \( (U(3)/T) \times_{\Sigma_3} (U(3)/T) \) and \( U(3)/T_2 \)

We will follow the same strategy for this computation as well. As before \( \mathcal{M} := U(3)/T \). Consider the fibration \( \Sigma_3 \rightarrow \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M} \times_{\Sigma_3} \mathcal{M} \). The Weyl group \( \Sigma_3 \) acts on \( \mathcal{M} \) by conjugation and hence induces a diagonal action on \( \mathcal{M} \times \mathcal{M} \). By definition \( \mathcal{M} \times_{\Sigma_3} \mathcal{M} = (\mathcal{M} \times \mathcal{M})/\Sigma_3 \). As \( \square \) splits, \( \mathcal{M} \times_{\Sigma_3} \mathcal{M} = (\mathcal{M} \times_{A_3} \mathcal{M})/(\mathbb{Z}/2) \). We have fibrations:
An $A_3 \to \mathcal{M} \times \mathcal{M} \to \mathcal{M} \times_{A_3} \mathcal{M}$

and

$\mathbb{Z}/2 \to \mathcal{M} \times A_3 \mathcal{M} \to \mathcal{M} \times_{\Sigma_3} \mathcal{M}$

The Cartan-Leray spectral sequence associated to these fibrations are

$$H^p(A_3; H^q(\mathcal{M} \times \mathcal{M}; \mathbb{Z}) \Rightarrow H^{p+q}(\mathcal{M} \times_{A_3} \mathcal{M}; \mathbb{Z}) \tag{12}$$

$$H^p(\mathbb{Z}/2; H^q(\mathcal{M} \times A_3 \mathcal{M}; \mathbb{Z}) \Rightarrow H^{p+q}(\mathcal{M} \times_{\Sigma_3} \mathcal{M}; \mathbb{Z}) \tag{13}$$

Before going ahead with the spectral sequence computations we describe the cohomology of $\mathcal{M} \times \mathcal{M}$ explicitly.

$$H^0(\mathcal{M} \times \mathcal{M}; \mathbb{Z}) = \mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}$$

$$H^2(\mathcal{M} \times \mathcal{M}; \mathbb{Z}) = (\mathcal{A} \otimes \mathbb{Z}) \oplus (\mathbb{Z} \otimes \mathcal{A}) = \mathcal{A} \oplus \mathcal{A}$$

$$H^4(\mathcal{M} \times \mathcal{M}; \mathbb{Z}) = (\overline{\mathcal{A}} \otimes \mathbb{Z}) \oplus (\mathbb{Z} \otimes \overline{\mathcal{A}}) \oplus (\mathcal{A} \otimes \mathcal{A}) = \overline{\mathcal{A}} \oplus \overline{\mathcal{A}} \oplus \mathbb{Z}[\mathbb{Z}/3] \oplus \mathbb{Z}_t$$

$$H^6(\mathcal{M} \times \mathcal{M}; \mathbb{Z}) = (\mathbb{Z}_t \otimes \mathbb{Z}_t) \oplus (\mathbb{Z} \otimes \mathbb{Z}_t) \oplus (\overline{\mathcal{A}} \otimes \overline{\mathcal{A}}) = \mathbb{Z}_t^2 \oplus \mathbb{Z}[\mathbb{Z}/3] \oplus \mathbb{Z}_t$$

$$H^8(\mathcal{M} \times \mathcal{M}; \mathbb{Z}) = (\mathcal{A} \otimes \mathbb{Z}_t) \oplus (\mathbb{Z}_t \otimes \mathcal{A}) \oplus (\overline{\mathcal{A}} \otimes \overline{\mathcal{A}}) = \mathcal{A} \oplus \mathcal{A} \oplus \mathbb{Z}[\mathbb{Z}/3] \oplus \mathbb{Z}_t$$

$$H^{10}(\mathcal{M} \times \mathcal{M}; \mathbb{Z}) = (\overline{\mathcal{A}} \otimes \mathbb{Z}_t) \oplus (\mathbb{Z}_t \otimes \overline{\mathcal{A}}) = \overline{\mathcal{A}} \oplus \overline{\mathcal{A}}$$

$$H^{12}(\mathcal{M} \times \mathcal{M}; \mathbb{Z}) = \mathbb{Z}_t \otimes \mathbb{Z}_t = \mathbb{Z}$$

In the above description we view the cohomology groups as $A_3$-modules. As $A_3$-modules $\mathcal{A} \otimes \mathcal{A} \cong \mathbb{Z}[A_3] \oplus \mathbb{Z}$. Note that the transposition $\pi \in \Sigma_3$ acts by $-1$ on the $\mathbb{Z}$-summand. To remember this we denote the $A_3$-module $\mathcal{A} \otimes \mathcal{A}$ as $\mathbb{Z}[A_3] \oplus \mathbb{Z}_t$. Recall

$$H^k(G; \oplus_{i=1}^l A_i) = \text{Ext}^k_{\mathbb{Z}[G]}(\mathbb{Z}, \oplus_{i=1}^l A_i) \cong \oplus_{i=1}^l \text{Ext}^k_{\mathbb{Z}[G]}(\mathbb{Z}, A_i) = \oplus_{i=1}^l H^k(G; A_i).$$

and

$$H^p(A_3; \mathbb{Z}[\mathbb{Z}/3]) = \begin{cases} \mathbb{Z} & \text{if } p = 0 \\ 0 & \text{otherwise} \end{cases}$$

The $E_{2}^{p,q}$ term for the spectral sequence (12) is as follows:

$$E_{2}^{p,q} = H^p(A_3; H^q(\mathcal{M} \times \mathcal{M}; \mathbb{Z})) = \begin{cases} \mathbb{Z} & \text{if } (p = 0, q = 0, 12) \\ \mathbb{Z} \oplus \mathbb{Z}_t & \text{if } (p = 0, q = 4, 8) \\ \mathbb{Z}_t^2 \oplus \mathbb{Z}_t^4 & \text{if } (p = 0, q = 6) \\ \mathbb{Z}/3 & \text{if } (p \equiv 0 \text{mod } 2, q = 0, 4, 8, 12) \\ (\mathbb{Z}/3)^2 & \text{if } (p \equiv 1 \text{mod } 2, q = 2, 4, 8, 10) \\ (\mathbb{Z}/3)^4 & \text{if } (p \equiv 0 \text{mod } 2, q = 6) \end{cases}$$

Note that $d_2 \equiv 0$ and hence $E_{2}^{p,q} = E_3^{p,q}$. Note that $H^{12}(\mathcal{M} \times \mathcal{M}; \mathbb{Z})$ is the trivial rank one $\Sigma_3$-representation and hence $H^{12}(\mathcal{M} \times_{\Sigma_3} \mathcal{M}; \mathbb{Z}) \cong \mathbb{Z}$. Therefore $\mathcal{M} \times_{\Sigma_3} \mathcal{M}$ is an Orientable 12-dimensional manifold. So, either $d_3$ or $d_5$ must be non-zero in the spectral sequence (12).
Lemma 4.1. The differential \( d_3 : E_3^{1,2} \rightarrow E_3^{4,0} \) in the above spectral sequence is non-zero.

Proof. We have already seen that either \( d_3 \) or \( d_5 \) must be non-zero. If \( d_3 \) is zero, then \( d_5 \) must be non-zero and all higher differentials must be zero. In that case we see that \( H^4(M \times A_3; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/3 \). In the previous section we proved that the action of \( \mathbb{Z}/2 < \Sigma_3 \) on \( H^2(A_3; \mathbb{Z}) = \mathbb{Z}/3 \) is by \(-1\) and hence on \( H^4(A_3; \mathbb{Z}) = \mathbb{Z}/3 \) is by \((-1)^2 = 1\). So, \( \mathbb{Z}/2 \) acts on the \( \mathbb{Z}/3 \) summand \( H^4(M \times A_3; \mathbb{Z}) \) by 1. In the spectral sequence (13) \( E_2^{0,4} \) has a \( \mathbb{Z}/3 \) summand and therefore so does \( H^4(M \times \Sigma_3; \mathbb{Z}) \). By Poincaré duality, \( H^0(M \times \Sigma_3; \mathbb{Z}) \) should have a \( \mathbb{Z}/3 \) summand. As \( \mathbb{Z}/2 \) acts on \( H^9(M \times A_3; \mathbb{Z}) \cong \mathbb{Z}/3 \) by \(-1\), \( E_2^{0,9} \) in the spectral sequence (13) does not have any \( \mathbb{Z}/3 \) summand. Recall \( H^p(\mathbb{Z}/2; \mathbb{Z}/3) = 0 \) for all \( p > 0 \). So, only possible \( \mathbb{Z}/3 \) summands in the spectral sequence (13) are in \( E_2^{0,q} \). This shows that it is impossible to have a \( \mathbb{Z}/3 \) summand in \( H^9(M \times \Sigma_3; \mathbb{Z}) \), which is a contradiction to our conclusion in the previous paragraph. So, \( d_5 \) must be zero and \( d_3 : E_3^{1,2} \rightarrow E_3^{4,0} \) must be non-zero. 

Now we can prove Lemma 3.1 using the previous lemma.
Proof of Lemma 3.1. We compare the spectral sequence (12) and (7). Denote the latter by $E^*_1$ and the former by $E^*$. The map between spectral sequences commutes with the differentials. We have the following commutative diagram:

\[
\begin{array}{ccc}
E_1^{1,2} = (\mathbb{Z}/3)^2 & \xrightarrow{d_3} & E_3^{4,0} = \mathbb{Z}/3 \\
\downarrow & & \downarrow \\
E_3^{1,2} = \mathbb{Z}/3 & \xrightarrow{d_3'} & E_3^{4,0} = \mathbb{Z}/3
\end{array}
\]

By Lemma 4.1, $d_3'$ is non-zero. Hence $d_3$ is also non-zero. ■

Coming back to our current computation, one can write down $H^*(\mathcal{M} \times_{A_3} \mathcal{M}; \mathbb{Z})$ from the spectral sequence. But we will not do the computation. Recall from previous section that the action of $\mathbb{Z}/2$ on both $H^2(A_3; \mathbb{Z}) \cong \mathbb{Z}/3$ and $H^1(A_3; \mathcal{A}) \cong \mathbb{Z}/3$ is non-trivial, i.e. by $-1$. So, in our next spectral sequence all $H^*(\mathbb{Z}/2; H^1(A_3; \mathcal{A}))$ and $H^*(\mathbb{Z}/2; H^2(A_3; \mathbb{Z}))$ are zero. We only need the “free” part of the cohomology of $\mathcal{M} \times_{A_3} \mathcal{M}$ for our next spectral sequence calculation. We write

\[
H^*(\mathcal{M} \times_{A_3} \mathcal{M}; \mathbb{Z}) = H^*_{\text{free}}(\mathcal{M} \times_{A_3} \mathcal{M}; \mathbb{Z}) \oplus H^*_{\text{tor}}(\mathcal{M} \times_{A_3} \mathcal{M}; \mathbb{Z})
\]

where

\[
H^k_{\text{free}}(\mathcal{M} \times_{A_3} \mathcal{M}; \mathbb{Z}) = \begin{cases} 
\mathbb{Z} & \text{if } k = 0, 12 \\
\mathbb{Z} \oplus \mathbb{Z}_t & \text{if } k = 4, 8 \\
\mathbb{Z}^2 \oplus \mathbb{Z}_t^4 & \text{if } k = 6
\end{cases}
\]

We have from our previous discussions $\mathbb{Z}/2 = \{1, \tau\} < \Sigma_3$ acts as $-1$ on $\mathbb{Z}_t$ and trivially on $\mathbb{Z}$. Finally the $E_2$-page of the spectral sequence (13) is as follows:

\[
E_2^{p,q} = H^p(\mathbb{Z}/2; H^q(\mathcal{M} \times_{A_3} \mathcal{M}; \mathbb{Z})) = \begin{cases} 
\mathbb{Z} & \text{if } p = 0 \text{ and } q = 0, 4, 8, 12 \\
\mathbb{Z}^2 & \text{if } (p, q) = (0, 6) \\
\mathbb{Z}/2 & \text{if } p \equiv 0 \text{mod} 2, p > 0 \text{ and } q = 0, 12 \\
\mathbb{Z}/2 & \text{if } p > 0 \text{ and } q = 4, 8 \\
(\mathbb{Z}/2)^2 & \text{if } p \equiv 0 \text{mod} 2, p > 0 \text{ and } q = 6 \\
(\mathbb{Z}/2)^4 & \text{if } p \equiv 1 \text{mod} 2 \text{ and } q = 6 \\
0 & \text{otherwise}
\end{cases}
\]

As $\mathcal{M} \times_{\Sigma_3} \mathcal{M}$ is a finite dimensional complex, we must have some non-zero differentials. The only way to avoid the $E_\infty$-page to have infinite cohomology is to have $d_3$ non-zero. All the maps in the differential $d_3$ are the obvious ones. Also, $d_4 = d_5 = d_6 = 0$. The $E_7$-page is as follows:
For all \( i \geq 0 \), \( d_7 : E_7^{2i,12} \rightarrow E_7^{2i+7,6} \) is given by inclusion in the first factor and \( d_7 : E_7^{2i+1,6} \rightarrow E_7^{2i+8,0} \) is given by projection of the second factor so that \( d_7^2 = 0 \). All higher differentials are zero and hence \( E_{\infty}^{p,q} = E_8^{p,q} \). All possible extensions of \( \mathbb{Z}^{2} \) by \( \mathbb{Z}^{2} \) are given by \( \mathbb{Z}/2 \oplus \mathbb{Z}/2 \) as there is no 4-torsion in \( H^*(\mathcal{M} \times \Sigma_3 \mathcal{M}; \mathbb{Z}) \). So, we can simply write \( H^k(\mathcal{M} \times \Sigma_3 \mathcal{M}; \mathbb{Z}) = \oplus_{p+q=k} E_8^{p,q} \). Hence we have the following theorem:

**Theorem 4.2.** The cohomology of \( \mathcal{M} \times \Sigma_3 \mathcal{M} \) is given by

\[
H^k(\mathcal{M} \times \Sigma_3 \mathcal{M}; \mathbb{Z}) = \begin{cases} 
\mathbb{Z} & \text{if } k = 0, 12 \\
\mathbb{Z}/2 & \text{if } k = 2, 5, 9, 11 \\
\mathbb{Z} \oplus \mathbb{Z}/2 & \text{if } k = 4, 8 \\
(\mathbb{Z}/2)^2 & \text{if } k = 7 \\
\mathbb{Z}^2 \oplus (\mathbb{Z}/2)^2 & \text{if } k = 6 \\
0 & \text{otherwise}
\end{cases}
\]

The rational cohomology of \( \mathcal{M} \times \Sigma_3 \mathcal{M} \) is same as the rational cohomology of \( E_{com} U(3) \). From Corollary 8.2 of [1], we have \( P_{E_{com} U(3)}(t) = 1 + t^4 + 2t^6 + t^8 + t^{12} \), which matches with the Poincaré polynomial \( P_{\mathcal{M} \times \Sigma_3 \mathcal{M}}(t) \) according to our calculation.
Next, we will compute the cohomology of $U(3)/T_2$. First recall from (Corollary 4.3, [10]), the cohomology of $PU(3)$ is given by

$$H^*(PU(3); \mathbb{Z}) = \Lambda[\rho_3, \rho_5] \otimes \mathbb{F}_3[\omega]/(\omega^3)$$ (14)

with $|\rho_3| = 3$, $|\rho_5| = 5$ and $|\omega| = 2$. We will proceed similarly as in section 4 of [10] to compute $H^*(U(3)/T_2)$. We have the covering homomorphism $f : U(3) \rightarrow U(3)/T_2$ and the fiber sequence $U(3) \rightarrow U(3)/T_2 \rightarrow (\mathbb{C}P^\infty)^2$. Since $U(3)/T_2 \rightarrow (\mathbb{C}P^\infty)^2$ is a principal $U(3)$ bundle, there is a set of generators of $H^*(U(3); \mathbb{Z})$ which transgresses into Chern classes of the associated complex bundle $\eta$ over $(\mathbb{C}P^\infty)^2$.

Recall from section 2 that $T_2$ is the 2-torus

$$\{\text{diag}(\lambda_1, \lambda_1, \lambda_2) | \lambda_i \in S^1\} \cong S^1 \times S^1.$$ 

The inclusion $T_2 \hookrightarrow U(3)$ can be factored by $S^1 \times S^1 \xrightarrow{\Delta} S^1 \times S^1 \times S^1 \xrightarrow{i} U(3)$, where $\Delta(\lambda_1, \lambda_2) = (\lambda_1, \lambda_1, \lambda_2)$. These homomorphisms of groups give us the following maps of classifying spaces

$$BS^1 \times BS^1 \xrightarrow{\Delta'} BS^1 \times BS^1 \times BS^1 \xrightarrow{i'} BU(3).$$

Now $i' : (\mathbb{C}P^\infty)^3 \rightarrow BU(3)$ induces the bundle $\pi_*(\xi) \oplus \pi_2^*(\xi) \oplus \pi_3^*(\xi)$, where $\xi$ is the canonical line bundle over $\mathbb{C}P^\infty$ and $\pi_j : (\mathbb{C}P^\infty)^3 \rightarrow \mathbb{C}P^\infty$ is the projection of the $j$-th factor. The total Chern class $c(\xi)$ is given by $1 + \alpha$ for $\alpha \in H^2(\mathbb{C}P^\infty; \mathbb{Z})$. By Whitney Product Formula

$$c(\pi_*(\xi) \oplus \pi_2^*(\xi) \oplus \pi_3^*(\xi)) = (1 + \alpha_1)(1 + \alpha_2)(1 + \alpha_3),$$

where $\alpha_j = \pi_j^*\alpha$. The induced map $\Delta' : (\mathbb{C}P^\infty)_1 \times (\mathbb{C}P^\infty)_2 \rightarrow (\mathbb{C}P^\infty)^3$ is given by $\Delta'(b_1, b_2) = (b_1, b_1, b_2)$. Note that we have $(\Delta')^*(\alpha_1) = \beta_1$, $(\Delta')^*(\alpha_2) = \beta_1$, $(\Delta')^*(\alpha_3) = \beta_2$, where $\beta_j \in H^2((\mathbb{C}P^\infty)_j; \mathbb{Z})$. The map $i' \circ \Delta'$ induces the bundle $\eta$ over $(\mathbb{C}P^\infty)^2$. Hence,

$$c(\eta) = (\Delta')^*((1 + \alpha_1)(1 + \alpha_2)(1 + \alpha_3)) = (1 + \beta_1)^2(1 + \beta_2) = 1 + 2\beta_1 + 2\beta_2 + \beta_1^2 + 2\beta_1\beta_2 + \beta_2^2.$$

So, $c_1(\eta) = 2\beta_1 + \beta_2, c_2(\eta) = \beta_1^2 + 2\beta_1\beta_2, c_3(\eta) = \beta_1^2\beta_2$. This determines the transgressions of the generators of $H^*(U(3); \mathbb{Z}) \cong \Lambda[z_1, z_3, z_5], |z_j| = j$. Following the result in Theorem 4.1 of [10], we see that $z_j$ transgresses to $c_j(\eta)$ in the spectral sequence associated to the fiber sequence $U(3) \rightarrow U(3)/T_2 \rightarrow (\mathbb{C}P^\infty)^2$. As a consequence we have the following result:

**Theorem 4.3.** The integral cohomology of $U(3)/T_2$ is given by $H^*(U(3)/T_2; \mathbb{Z}) = \Lambda[y_2, y_5], |y_j| = j$.

**Proof.** In the above spectral sequence the $E_2$-page is given by

$$H^*(U(3)) \otimes H^*((\mathbb{C}P^\infty)^2) = \Lambda[z_1, z_3, z_5] \otimes \mathbb{Z}[\beta_1, \beta_2].$$

It follows from the above discussion that $z_1$ transgresses to $2\beta_1 + \beta_2$ and hence we have $d_2(z_1) = 2\beta_1 + \beta_2$. This determines the differential $d_2$ on the $E_2$-page from the product
structure. Note that $E_3^{2,0} \cong (\mathbb{Z}/\beta_1 \oplus \mathbb{Z}/\beta_2)/(2\beta_1 + \beta_2) \cong \mathbb{Z}/\beta_1$.

Therefore, the $E_3$ page is given by $\Lambda[z_3, z_5] \otimes \mathbb{Z}[\beta_1]$. As $d_3 \equiv 0$, $E_3 = E_4$. The differential $d_4 : E_4^{0,3} = \mathbb{Z}(z_3) \to E_4^{1,0} = \mathbb{Z}(\beta_1^2)$ is given by $d_4(z_3) = \beta_1^2$. This determines $d_4$ on the $E_4$-page. All higher differentials are zero and hence $E_5 = E_{\infty}$. If we set $y_2 := \beta_1$ and $y_5 := z_5$ then the result follows from the induced cup product structure on the $E_{\infty}$-page.

5 Cohomology of $E_{\text{com}}U(3)$

We want to compute the cohomology of the homotopy colimit (2) over the poset $S(2)$. The following theorem is from [9]:

**Theorem 5.1.** Let $D : I \to \text{Top}$ be a diagram and let $E_\ast$ be a homology theory. Write $E_k(D)$ for the diagram $I \to \text{Ab}$ given by $i \mapsto E_k(D_i)$, and $E^k(D)$ for the diagram $I^{\text{op}} \to \text{Ab}$.

(a) There is a spectral sequence $E_2^{p,q} = H_p(I; E_q(D))$ converging to $E^{p+q}(\text{hocolim}D)$. The differentials have the form $d^r : E_r^{p,q} \to E_r^{p+r,q-r}$. (b) There is a spectral sequence $E_2^{p,q} = H^p(I^{\text{op}}; E^q(D))$ converging to $E^{p+q}(\text{hocolim}D)$. The differentials have the form $d^r : E_r^{p,q} \to E_r^{p+r,q-r+1}$.

We are interested in the cohomological version, hence we want to define $H^p(I^{\text{op}}; E^qD)$. The following definition is also due to [9].

**Definition 5.2.** Let $A$ be an abelian category, $I$ be a small category and $F : I \to A$ be a functor. We write the cosimplicial replacement for $F$:

$$\prod_i F(i) \Longrightarrow \prod_{i_0 \to i_1} F(i_1) \Longrightarrow \prod_{i_0 \to i_1 \to i_2} F(i_2) \Longrightarrow \cdots$$

This is a cosimplicial object over $A$. Take the alternating sum of the coface maps, which gives a co-chain complex over $A$. We define $H^p(I; F)$ to be the $p$-th cohomology of this cochain complex.

Note that $H^0(I; F)$ (equalizer of the first two arrows) is simply the ordinary limit $\lim F$. So, one can think of $H^p(I; F)$ as higher limit functors and write $H^p(I; F) = \lim^p F$. In our case the abelian category $A$ will be the category of abelian groups $\text{Ab}$, $I$ will be $S(2)$ and $F$ will be $H^k(\mathcal{H}; R)$, where $R = \mathbb{F}_3$ or $\mathbb{F}_2$. So, we have the functor $H^k(\mathcal{H}; R) : S(2)^{\text{op}} \to \text{Ab}$, given by $i \mapsto H^k(\mathcal{H}(i); R)$ and the corresponding diagram:

$$
\begin{array}{ccc}
H^k(\mathcal{H}(0)) & \xrightarrow{d} & H^k(\mathcal{H}(0,1)) \\
& & \uparrow \\
H^k(\mathcal{H}(0,2)) & \xrightarrow{d} & H^k(\mathcal{H}(0,1,2)) \\
& & \uparrow \\
H^k(\mathcal{H}(2)) & \xrightarrow{d} & H^k(\mathcal{H}(1,2)) \\
\end{array}
$$
To compute the higher limits, we want to write the cosimplicial replacement of the above diagram. By definition 12.1 of [13] the cosimplicial replacement of $H^k(\mathcal{H}; R)$ is given by

$$
\prod_{i \in S(2)} H^k(\mathcal{H}(i); R) \longrightarrow \prod_{i_0 \rightarrow i_1} H^k(\mathcal{H}(i_1); R) \longrightarrow \prod_{i_0 \rightarrow i_1 \rightarrow i_2} H^k(\mathcal{H}(i_2); R) \longrightarrow 0 \quad (15)
$$

such that the projection of the coface map $d^j : \prod_i H^k(\mathcal{H}(i); R) \longrightarrow \prod_{i_0 \rightarrow i_1} H^k(\mathcal{H}(i_1); R)$ onto the factor $H^k(\mathcal{H}(i_1); R)$ indexed by $(i_0 \rightarrow i_1)$ is the composition of a projection from the product with

(a) the identity map from the factor $H^k(\mathcal{H}(i_1); R)$ when $j = 0$,
(b) the map $H^k(\mathcal{H}(i_0); R) \longrightarrow H^k(\mathcal{H}(i_1); R)$ when $j = 1$.

The projection of the coface map $d^j : \prod_{i_0 \rightarrow i_1} H^k(\mathcal{H}(i_1); R) \longrightarrow \prod_{i_0 \rightarrow i_1 \rightarrow i_2} H^k(\mathcal{H}(i_2); R)$ onto the factor $H^k(\mathcal{H}(i_2); R)$ indexed by $(i_0 \rightarrow i_1 \rightarrow i_2)$ is the composition of a projection from the product with

(a) the identity map from the factor $H^k(\mathcal{H}(i_2); R)$ indexed by $i_1 \rightarrow i_2$ when $j = 0$,
(b) the identity map from the factor $H^k(\mathcal{H}(i_2); R)$ indexed by $i_0 \rightarrow i_2$ when $j = 1$,
(c) the map $H^k(\mathcal{H}(i_1); R) \longrightarrow H^k(\mathcal{H}(i_2); R)$ indexed by $i_0 \rightarrow i_1$ when $j = 2$.

By Definition 5.2 taking the alternate sum of the coface maps $d^j$ makes $\prod$ a cochain complex.

### 5.1 $\mathbb{F}_2$ and $\mathbb{F}_3$-Cohomology

**Lemma 5.3.** Let $R$ be a commutative ring. Then $\lim^2 H^k(\mathcal{H}; R) = H^2(S(2)^{op}; H^k(\mathcal{H}; R)) = 0$.

**Proof.** From the description of the pieces $\mathcal{H}(i)$ earlier, we see that $\mathcal{H}(0, 2) = \mathcal{H}(0, 1, 2)$. Consider the map $d^0 - d^1 + d^2 : \prod_{i_0 \rightarrow i_1} H^k(\mathcal{H}(i_1); R) \longrightarrow \prod_{i_0 \rightarrow i_1 \rightarrow i_2} H^k(\mathcal{H}(i_2); R)$. Note that in $S(2)^{op}$ all 3-chains end with $(0, 1, 2)$ and hence $i_2 = (0, 1, 2)$ is constant in all terms in the product $\prod_{i_0 \rightarrow i_1 \rightarrow i_2} H^k(\mathcal{H}(i_2); R)$. So, we can write the product as $\prod_{i_0 \rightarrow i_1 \rightarrow i_2} H^k(\mathcal{H}(0, 1, 2); R)$. Let any $(a_1, \ldots, a_6) \in \prod_{i_0 \rightarrow i_1 \rightarrow i_2} H^k(\mathcal{H}(0, 1, 2); R)$ be given. We will pick an element $b$ in $\prod_{i_0 \rightarrow i_1} H^k(\mathcal{H}(i_1); R)$ in the following way:

The entries of $b$ which are indexed by $(0) \rightarrow (0, 1, 2), (2) \rightarrow (0, 1, 2), (0, 1) \rightarrow (0, 1, 2), (1, 2) \rightarrow (0, 1, 2)$ are $a_3 - a_1, a_4 - a_6, a_3, a_4 \in H^k(\mathcal{H}(0, 1, 2); R)$ respectively. The entries which are indexed by $(0) \rightarrow (0, 2), (2) \rightarrow (0, 1, 2)$ are $a_2 + a_3 - a_1, a_5 + a_4 - a_6 \in H^k(\mathcal{H}(0, 2); R) = H^k(\mathcal{H}(0, 1, 2); R)$ respectively and all other entries in $b$ are 0.

One can easily verify from our description of the cochain maps that $d^0 - d^1 + d^2$ maps $b \in \prod_{i_0 \rightarrow i_1} H^k(\mathcal{H}(i_1); R)$ onto $(a_1, \ldots, a_6) \in \prod_{i_0 \rightarrow i_1 \rightarrow i_2} H^k(\mathcal{H}(i_2); R)$. This shows that $d_0 - d_1 + d_2$ is surjective and hence $H^2(S(2)^{op}; H^k(\mathcal{H}; R)) = 0$. 

We first deal with the case when $R = \mathbb{F}_3$. Due to the above lemma we only need to worry about $\lim^0$ and $\lim^1$. From Corollary 7.4 of [11] we have an isomorphism

$$
H^*(E_{con}U(3); \mathbb{Q}) \cong H^*(U(3)/T \times_{\Sigma_3} U(3)/T; \mathbb{Q}) = (H^*(U(3)/T; \mathbb{Q}) \otimes H^*(U(3)/T; \mathbb{Q}))^{\Sigma_3}.
$$
We get a similar result when we work with \( \mathbb{F}_3 \) coefficient as stated in the following theorem:

**Theorem 5.4.** We have an isomorphism \( H^*(E_{com}U(3); \mathbb{F}_3) \cong H^*(U(3)/T \times \Sigma_3 U(3)/T; \mathbb{F}_3) \).

**Proof.** All the cohomologies in this proof are cohomology with \( \mathbb{F}_3 \) coefficient unless mentioned otherwise. From the description of the \( \mathcal{H}(i) \)'s in Section 3, we see that with mod 3 coefficients

\[
H^*(\mathcal{H}(0)) = H^*(\mathcal{H}(0, 1)) = H^*(\mathcal{H}(0, 2)) = H^*(\mathcal{H}(0, 1, 2)) = H^*(PU(3))
\]

\[
H^*(\mathcal{H}(1)) = H^*(\mathcal{H}(1, 2)) = H^*(U(3)/T_2)
\]

Fix any \( k \in \mathbb{N} \). We know that \( k \)-th cohomology group of \( PU(3) \), \( U(3)/T_2 \) and \( U(3)/T \times \Sigma_3 U(3)/T \) with \( \mathbb{F}_3 \) coefficient is free of finite rank. Let us take \( H^k(PU(3)) = (\mathbb{F}_3)^n \), \( H^k(U(3)/T_2) = (\mathbb{F}_3)^m \) and \( H^k(U(3)/T \times \Sigma_3 U(3)/T) = (\mathbb{F}_3)^l \), where \( n, m, l \in \mathbb{N} \cup \{0\} \). Then the cochain complex \([15]\) is of the form:

\[
(\mathbb{F}_3)^{4n} \oplus (\mathbb{F}_3)^{2m} \oplus (\mathbb{F}_3)^l \xrightarrow{(d^0 - d^1)} (\mathbb{F}_3)^{4m} \oplus (\mathbb{F}_3)^{2n} \oplus (\mathbb{F}_3)^{6n} \rightarrow (\mathbb{F}_3)^{6n}
\]

From previous lemma, the third map is surjective and hence by first isomorphism theorem

\[
\ker(d^0 - d^1) = (\mathbb{F}_3)^{4n} \oplus (\mathbb{F}_3)^{2m}.
\]

One can think of \( \ker(d^0 - d^1) \) as an equalizer of \( d^0 \) and \( d^1 \). By definition of \( d^0 \) and \( d^1 \), we see that the equalizer always has \( l \) copies of \( \mathbb{F}_3 \) for different values of \( m, n, l \). Hence, \( \ker(d^0 - d^1) \cong (\mathbb{F}_3)^l \). This forces \( \text{im}(d^0 - d^1) \cong (\mathbb{F}_3)^{4n} \oplus (\mathbb{F}_3)^{2m} \) by first isomorphism theorem.

So, we have \( \lim^1 H^k(\mathcal{H}; \mathbb{F}_3) = 0 \) and \( \lim^0 H^k(\mathcal{H}; \mathbb{F}_3) = (\mathbb{F}_3)^l = H^k(U(3)/T \times \Sigma_3 U(3)/T; \mathbb{F}_3) \).

As both \( \lim^1 \) and \( \lim^2 \) are 0, the spectral sequence in Theorem 5.1(b) has non-zero terms in only \( E^{0,q}_2 \) column and collapses at the \( E_2 \)-page. Hence,

\[
H^k(E_{com}U(3); \mathbb{F}_3) \cong \lim^0 H^k(\mathcal{H}; \mathbb{F}_3) = H^k(U(3)/T \times \Sigma_3 U(3)/T; \mathbb{F}_3).
\]

The above theorem tells us \( H^*(E_{com}U(3); \mathbb{Z}) \) has no 3-torsion as \( H^*(U(3)/T \times \Sigma_3 U(3)/T; \mathbb{Z}) \) has no 3-torsion. It remains to compute \( H^*(E_{com}U(3); \mathbb{F}_2) \), which will give us a very good idea about \( H^*(E_{com}U(3); \mathbb{Z}) \).

With \( \mathbb{F}_2 \) coefficients, we can describe the \( E_2 \)-page of the spectral sequence in Theorem 5.1(b). Only finitely many terms in the spectral sequence are non-zero. Explicitly from computation of \( \lim^0 \) and \( \lim^1 \), we have

\[
E^{0,q}_2 = \lim^0 H^q(\mathcal{H}; \mathbb{F}_2) \cong \mathbb{F}_2; \quad \text{for } q = 0, 4, 5, 7
\]

\[
E^{1,q}_2 = \lim^1 H^q(\mathcal{H}; \mathbb{F}_2) \cong \mathbb{F}_2; \quad \text{for } q = 8, 10, 11, 13, 14
\]

\[
E^{0,6}_2 = \lim^0 H^6(\mathcal{H}; \mathbb{F}_2) \cong (\mathbb{F}_2)^3
\]

\[
E^{0,8}_2 = \lim^0 H^8(\mathcal{H}; \mathbb{F}_2) \cong (\mathbb{F}_2)^2
\]

\[
E^{1,9}_2 = \lim^1 H^9(\mathcal{H}; \mathbb{F}_2) \cong (\mathbb{F}_2)^2
\]

\[
E^{2,q}_2 = \lim^0 H^q(\mathcal{H}; \mathbb{F}_2) = 0; \quad \text{otherwise}
\]

We see immediately that there is no non-trivial differential on the \( E_2 \)-page. Hence the spectral sequence collapses at the \( E_2 \)-page and \( E_{\infty}^{0,*} = E_2^{0,*} \). This gives the following theorem:
Theorem 5.5. The $\mathbb{F}_2$-cohomology of $E_{com}U(3)$ is given by

$$H^k(E_{com}U(3); \mathbb{F}_2) = \begin{cases} 
\mathbb{F}_2; & k = 0, 4, 5, 7, 9, 11, 12, 14, 15 \\
(\mathbb{F}_2)^2; & k = 8, 10 \\
(\mathbb{F}_2)^3; & k = 6 \\
0; & k = \text{otherwise}
\end{cases}$$

5.2 Integral Cohomology

To compute the integral cohomology of $E_{com}U(3)$, we first show that $H^*(E_{com}U(3); \mathbb{Z})$ can have only 2 and 3-torsion. This follows from a much more general statement that the integral cohomology of $E_{com}G$ can have only $p$-torsion for those $p$ dividing the order of the Weyl group of $G$. To prove this we need the following two lemmas. We will denote the order of a group $K$ by $|K|$.

Lemma 5.6. Fix a compact connected Lie group $G$. Let $T \subset G$ be a maximal torus and $W$ be the Weyl group. Then we have isomorphisms

$$H^*(G/T \times_W BT; \mathbb{F}) \cong H^*(B_{com}G_1; \mathbb{F}) \quad \text{and} \quad H^*(G/T \times_W G/T; \mathbb{F}) \cong H^*(E_{com}G_2; \mathbb{F})$$

whenever $\gcd(\text{char}(\mathbb{F}), |W|) = 1$.

Proof. Consider the map

$$\varphi_n : G \times T^n \longrightarrow \text{Hom}(\mathbb{Z}^n, G)_1, \quad (g, t_1, \ldots, t_n) \longmapsto (gt_1g^{-1}, \ldots, gt_ng^{-1})$$

There is a natural action of the normalizer $N(T)$ on $G \times T^n$ by

$$\eta \cdot (g, t_1, \ldots, t_n) = (g\eta^{-1}, \eta t_1\eta^{-1}, \ldots, \eta t_n\eta^{-1}).$$

Note that $\varphi_n(\eta \cdot (g, t_1, \ldots, t_n)) = \varphi_n(g, t_1, \ldots, t_n)$, i.e. $\varphi_n$ is invariant under the action of the normalizer. As a result we have a continuous map

$$\varphi_n : G \times N(T) T^n = G/T \times_W T^n \longrightarrow \text{Hom}(\mathbb{Z}, G)_1$$

Here $W$ acts diagonally on $T^n$. Any $n$-tuple $(g_1, \ldots, g_n)$ such that $g_i \in G$ belongs to $\text{Hom}(\mathbb{Z}, G)_1$ if and only if there is a maximal torus in $G$ that contains $g_1, \ldots, g_n$. As, all the maximal tori in $G$ are conjugated, we have that $\varphi_n$ is surjective. By ([6], Lemma 3.2) it follows that fibers of $\varphi_n$ have trivial cohomology with $\mathbb{F}$ coefficients. Hence by Vietoris-Begle mapping theorem, $\varphi_n$ induces an isomorphism in cohomology:

$$\varphi_n^* : H^*(\text{Hom}(\mathbb{Z}, G)_1; \mathbb{F}) \xrightarrow{\cong} H^*(G/T \times_W T^n; \mathbb{F}).$$

The collection $\{\varphi_n\}_{n \geq 0}$ defines a simplicial map and by passing to the geometric realization, we get a continuous map

$$\varphi : G/T \times_W BT \longrightarrow B_{com}G_1.$$
Here $G/T$ is seen as a constant simplicial space. In the same way as in [4, Theorem 6.1] we conclude that $\varphi$ induces an isomorphism in cohomology with $\mathbb{F}$ coefficients and thus we obtain an isomorphism

$$\varphi^* : H^*(B_{\text{com}}G_1; \mathbb{F}) \xrightarrow{\cong} H^*(G/T \times_W BT; \mathbb{F}).$$

Also in the same way as in ([1], Corollary 7.4), we have an isomorphism

$$\varphi^* : H^*(E_{\text{com}}G_1; \mathbb{F}) \xrightarrow{\cong} H^*(G/T \times_W G/T; \mathbb{F}).$$

### Lemma 5.7.

Let $G$ be a compact connected Lie group, $T \subset G$ be a maximal torus and $W$ be the corresponding Weyl group. Then $H^*(G/T \times_W BT; \mathbb{Z})$ and $H^*(G/T \times_W G/T; \mathbb{Z})$ can only have $p$-torsion for those $p$ dividing $|W|$. 

**Proof.** Consider the covering space $\pi : G/T \times BT \rightarrow G/T \times_W BT$ and the transfer homomorphism $\tau^* : H^*(G/T \times BT; R) \rightarrow H^*(G/T \times_W BT; R)$, where $R$ is a commutative ring. The composition $\tau^* \circ \pi^*$ is the multiplication by $|W|$ map in the $R$-module $H^k(G/T \times_W BT; R)$. Let $q$ be a prime such that $\gcd(q, |W|) = 1$. Set $R = \mathbb{Z}_{(q)}$ the localization of the integers at $q$. Since $\gcd(q, |W|) = 1$, $|W|$ is invertible in $\mathbb{Z}_{(q)}$ and multiplication by $|W|$ in $H^k(G/T \times_W BT; \mathbb{Z}_{(q)})$ is an isomorphism. Therefore $\tau^* \circ \pi^*$ is an isomorphism and $\pi^* : H^k(G/T \times_W BT; \mathbb{Z}_{(q)}) \rightarrow H^k(G/T \times BT; \mathbb{Z}_{(q)})$ is injective. As $H^*(G/T \times BT; \mathbb{Z})$ is torsion free, $H^*(G/T \times_W BT; \mathbb{Z})$ has no $q$ torsion, otherwise $\pi^*$ would not be injective. Therefore $H^*(G/T \times_W BT; \mathbb{Z})$ can only have $p$-torsion for those $p$ dividing $|W|$. The argument for $H^*(G/T \times_W G/T; \mathbb{Z})$ is similar.

### Theorem 5.8.

Let $G$ be a compact connected Lie group, $T \subset G$ be a maximal torus and $W$ be the corresponding Weyl group. Then $H^*(B_{\text{com}}G_2; \mathbb{Z})$ and $H^*(E_{\text{com}}G_1; \mathbb{Z})$ can only have $p$-torsion for those $p$ dividing $|W|$. 

**Proof.** Follows immediately from Lemma 5.6 and 5.7.

From ([1], Corollary 8.2) we have the Poincaré series for $E_{\text{com}}U(3)$ as following

$$P_{E_{\text{com}}U(3)}(t) = 1 + t^4 + 2t^6 + t^8 + t^{12}.$$ 

From Theorem 5.8 we know that $H^*(E_{\text{com}}U(3); \mathbb{Z})$ can only have 2 and 3 torsion. Again by Theorem 5.4 $H^*(E_{\text{com}}U(3); \mathbb{Z})$ does not have any 3-torsion. So, $H^*(E_{\text{com}}U(3); \mathbb{Z})$ only has 2-torsion. We compare the the Poincaré series with $H^*(E_{\text{com}}U(3); \mathbb{F}_2)$ from Theorem 5.3 to identify the $\mathbb{F}_2$ factors that are in the image of the map $H^*(E_{\text{com}}U(3); \mathbb{Z}) \rightarrow H^*(E_{\text{com}}U(3); \mathbb{F}_2)$. Finally using the Universal Coefficient Theorem we have the following result.

### Proposition 5.9.

The integral homology of $E_{\text{com}}U(3)$ is given by

$$H^k(E_{\text{com}}U(3); \mathbb{Z}) = \begin{cases} \mathbb{Z}; & k = 0, 4, 8, 12 \\ \mathbb{Z}^2; & k = 6 \\ \mathbb{F}_2; & k = 5, 7, 9, 10, 14 \\ 0; & \text{otherwise} \end{cases}$$
and consequently the integral cohomology of \( E_{\text{com}} U(3) \) is given by

\[
H^k(E_{\text{com}} U(3); \mathbb{Z}) = \begin{cases} 
\mathbb{Z}; & k = 0, 4, 12 \\
\mathbb{Z}^2 \oplus \mathbb{F}_2; & k = 6 \\
\mathbb{Z} \oplus \mathbb{F}_2; & k = 8 \\
\mathbb{F}_2; & k = 10, 11, 15 \\
0; & \text{otherwise}
\end{cases}
\]

### 5.3 Cohomology Ring

Let us first review the rational cohomology of \( E_{\text{com}} U(3) \). All cohomologies bellow are with rational coefficients unless otherwise specified. From Corollary 7.4 of [1] we have an isomorphism

\[
H^*(E_{\text{com}} U(3)) \cong H^*(U(3)/T \times_{\Sigma_3} U(3)/T) = (H^*(U(3)/T) \otimes H^*(U(3)/T))^\Sigma_3. \tag{16}
\]

Recall \( H^*(U(3)/T) \cong \mathbb{Q}[x_1, x_2, x_3]/(e_1, e_2, e_3) \), where \( e_i \) is the \( i \)-the elementary symmetric polynomial in the \( x_i \)'s. Consider the averaging operator

\[
\rho : H^*(U(3)/T) \otimes H^*(U(3)/T) \longrightarrow (H^*(U(3)/T) \otimes H^*(U(3)/T))^\Sigma_3
\]

\[
\rho(f(x, y)) = \frac{1}{6} \sum_{\omega \in \Sigma_3} f(\omega x, \omega y).
\]

For each \( \omega \in \Sigma_3 \) the diagonal descent monomial is defined as follows:

\[
f_\omega := \prod_{\omega^{-1}(i) > \omega^{-1}(i+1)} (x_1 \cdots x_i) \otimes \prod_{\omega(j) > \omega(j+1)} (y_{\omega(j)} \cdots y_{\omega(j)}) \tag{17}
\]

The collection \( \{ \rho(f_\omega) \}_{\omega \in \Sigma_3} \) forms a free basis of \( (H^*(U(3)/T) \otimes H^*(U(3)/T))^\Sigma_3 \) as a \( \mathbb{Q} \)-module. So, we obtain the following basis of \( (H^*(U(3)/T) \otimes H^*(U(3)/T))^\Sigma_3 \):

\[
\mathcal{B} = \{1, \rho(x_1 \otimes y_2), \rho(x_1 \otimes x_2 \otimes y_3), \rho(x_1 x_2 \otimes y_3), \rho(x_1 x_2 \otimes x_2 x_3 y_2), \rho(x_1 x_2 \otimes y_2 y_3), \rho(x_1^2 x_2 \otimes y_3^2 y_2)\}
\]

It follows that images of \( \rho(f_\omega) \) for \( \omega \in \Sigma_3 \) form a basis of \( H^*(E_{\text{com}} U(3)) \). We will also denote the image as \( \rho(f_\omega) \) for now. Also note that the degree of \( \rho(f_\omega) \) is given by \( \deg(\rho(f_\omega)) = 2(\text{maj}(\omega) + \text{maj}(\omega^{-1})) \) where

\[
\text{maj}(\omega) := \sum_{\omega(i) > \omega(i+1)} i.
\]

**Theorem 5.10.** We have the following relations between the classes in \( \mathcal{B} \):

1. \( 2\rho(x_1 \otimes y_2)^2 = -\rho(x_1 x_2 \otimes y_2 y_3) \),
2. \( 3\rho(x_1 \otimes y_2 y_3) \cdot \rho(x_1 x_2 \otimes y_3) = 2\rho(x_1^2 x_2 \otimes y_3^2 y_2) \),
3. All other products are zero.
Proof. We will show the relations in $(H^*(U(3)/T) \otimes H^*(U(3)/T))^\Sigma_3$, which is enough due to the isomorphism in Eq [10]. First note that in $H^*(U(3)/T)$ we have the following relation between the generators: $x_i^2 = x_j x_k$, $i \neq j \neq k$. This follows from combining the relations $e_1 = e_2 = 0$. Using the definition of $\rho$ given above we check that

$$
\rho(x_1 \otimes y_2) = -\frac{1}{6}(x_1 \otimes y_1 + x_2 \otimes y_2 + x_3 \otimes y_3)
$$

$$
\rho(x_1 x_2 \otimes y_2 y_3) = -\frac{1}{6}(x_1 x_2 \otimes y_1 y_2 + x_2 x_3 \otimes y_2 y_3 + x_1 x_3 \otimes y_1 y_3)
$$

Hence,

$$
2 \rho(x_1 \otimes y_2)^2 = \frac{1}{18}(x_1^2 \otimes y_1^2 + x_2^2 \otimes y_2^2 + x_3^2 \otimes y_3^2 + 2x_1 x_2 \otimes y_1 y_2 + 2x_2 x_3 \otimes y_2 y_3 + 2x_1 x_3 \otimes y_1 y_3)
$$

$$
= \frac{1}{6}(x_1 x_2 \otimes y_1 y_2 + x_2 x_3 \otimes y_2 y_3 + x_1 x_3 \otimes y_1 y_3) \quad \text{(using } x_i^2 = x_j x_k)\n$$

$$
= -\rho(x_1 x_2 \otimes y_2 y_3)
$$

The other relations also follow directly by writing down the corresponding $\rho(f_\omega)$ and using the relations $e_1 = e_2 = e_3 = 0$. ■

Corollary 5.11. Let us denote image of $\rho(x_1 \otimes y_2)$ as $\gamma_4$, $\rho(x_1 \otimes y_2 y_3)$ as $\gamma_6$, $\rho(x_1 x_2 \otimes y_3)$ as $\tilde{\gamma}_6$ in $H^*(E_{com}U(3); \mathbb{Q})$. Then

$$
H^*(E_{com}U(3); \mathbb{Q}) \cong \mathbb{Q}[\gamma_4, \gamma_6, \tilde{\gamma}_6]/(\gamma_4^2, \gamma_6^2, \tilde{\gamma}_6, \gamma_4 \gamma_6, \gamma_4 \tilde{\gamma}_6).
$$

Further we denote the image of $\rho(x_1 x_2 \otimes y_2 y_3)$ as $\gamma_8$ and $\rho(x_1^2 x_2 \otimes y_2^2 y_3)$ as $\gamma_{12}$ in $H^*(E_{com}U(3); \mathbb{Z})$. Also, we use the same notation to denote the lifts of these classes in $H^*(E_{com}U(3); \mathbb{Z})$ under the map $H^*(E_{com}U(3); \mathbb{Z}) \rightarrow H^*(E_{com}U(3); \mathbb{Q})$. The relations in Theorem 5.10 translates to $2\gamma_4 = -\gamma_8$ and $3\gamma_6 = 2\gamma_{12}$. So, $H_{com}^8(E_{com}U(3); \mathbb{Z}) = \mathbb{Z}$ is generated by $\gamma_4^2$ and $H_{com}^{12}(E_{com}U(3); \mathbb{Z}) = \mathbb{Z}$ is generated by $\gamma_12 - \gamma_6 \tilde{\gamma}_6$. The mod 2 classes in $H^*(E_{com}U(3); \mathbb{Z})$ have no non-trivial product. Hence, we have the following result:

Theorem 5.12. The integral cohomology ring of $E_{com}U(3)$ is given by

$$
H^*(E_{com}U(3); \mathbb{Z}) \cong \mathbb{Z}[\gamma_4, \gamma_6, \gamma_8, \gamma_6, \gamma_12, t_8, t_10, t_{11}, t_{12}, t_6, t_8, t_{10}, t_{11}, t_{12}] / (2t_i, \gamma_4^2, \gamma_6^2, \gamma_8^2, t_it_j, t_it_j, t_it_j, t_it_j, t_it_j, 3\gamma_6 \gamma_6 - 2\gamma_{12})
$$

where $i, j$ runs through all possible values and $\text{deg}(t_i) = i$.

We have a fiber sequence $E_{com}U(3) \rightarrow B_{com}U(3) \longrightarrow BU(3)$, which could be used to extract some information about the integral cohomology of $B_{com}U(3)$. We have the following immediate result.

Proposition 5.13. We have an isomorphism $H^*(B_{com}U(3); \mathbb{Z}/3) \cong H^*(U(3)/T \times \Sigma_3 BT; \mathbb{Z}/3)$

Proof. We have the following diagram and hence a map between fiber sequences

$$
U(3)/T \times \Sigma_3 U(3)/T \rightarrow U(3)/T \times \Sigma_3 BT \rightarrow BU(3)
$$

$$
E_{com}U(3) \rightarrow B_{com}U(3) \rightarrow BU(3)
$$

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By Theorem 5.4, we have $H^*(E_{com}U(3); \mathbb{Z}/3) \cong H^*(U(3)/T \times_{\Sigma_3} U(3)/T; \mathbb{Z}/3)$, induced by the map $U(3)/T \times_{\Sigma_3} U(3)/T \to E_{com}U(3)$. Hence, by comparing the spectral sequences associated to the fiber sequences in the above diagram, we have the desired isomorphism. □

A similar result also holds for $B_{com}SU(3)$. The above proposition tells us $H^*(B_{com}U(3); \mathbb{Z})$ has no 3-torsion, which is expected from the absence of 3-torsion in $H^*(E_{com}U(3); \mathbb{Z})$. We do not know the significance or reason of this result. We believe it is closely related to the absence of 3-torsion in $H^*(U(3)/N(T); \mathbb{Z})$. For a general result about the kind of torsion appearing in $H^*(B_{com}U(n); \mathbb{Z})$ for $n > 3$ we need more insight into the problem of computation of $H^*(U(n)/N(T); \mathbb{Z})$ for $n > 3$.

One can write down the mod 2 Serre spectral sequence $(E^*, d)$ associated to the fiber sequence $E_{com}U(3) \to B_{com}U(3) \to BU(3)$. We see that $d_2 : E_2^{0,5} \to E_2^{2,4}$ is non zero sending $t_5 \in E_2^{0,5}$ to $\overline{c}_1$, where $\overline{c}_1$ is the image of the first Chern class in mod 2 cohomology. This can be seen by comparing this Serre spectral spectral sequence with the similar one for $U(2)$. But we do not know anything about other possible non-zero differentials. On the other hand we see that for $SU(3)$ in the corresponding spectral sequence $d_2 \equiv 0$, but we do not fully understand the higher differentials. We expect $B_{com}SU(3)$ and $BSU(3) \times E_{com}SU(3)$ to have the same cohomology as was the case for $SU(2)$.

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