Grid homology and the unknotting number

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Abstract

We showed that the order of torsion homology classes in the grid homology of a knot is a lower bound for the unknotting number.

1 Introduction

The unknotting number $u(K)$ of a knot $K$ is the minimum number of crossing changes from $K$ to the unknot. It is one of the most elementary but intractable knot invariants. Upper bounds for the unknotting number are easy to obtain, since one can take some diagram and find a sequence of crossing changes giving the unknot. However, the lower bound for the unknotting number is usually hard to find since there is no method to determine which knot diagram will exhibit the actual unknotting number. The concordance invariant $|\tau(K)|\leq \frac{1}{2}|s(K)|$ from knot Floer homology [4] and Khovanov homology [8], respectively, are lower bounds of the slice genus $g_s(K)$. Since $g_s(K)\leq u(K)$, they are lower bounds for $u(K)$. Indeed, the bounds are sharp for torus knots.

In [1], Alishahi showed that the order of torsion homology classes in Bar-Natan deformation $H_{BN}(K)$ of Khovanov homology is lower bound for $u(K)$. More precisely, for $\alpha \in H_{BN}(K)$, let $t(\alpha)$ be the smallest $n$ such that $h^n(\cdot) = 0$, and define

$$t_{BN}(K) = \max\{ t(\alpha) \mid \alpha \text{ is a torsion element in } H_{BN}(K) \}$$

They proved that

$$t_{BN}(K) \leq u(K)$$

In [2], Alishahi and Eftekharz proved an analogous result:

$$t_{KF}(K) \leq u(K)$$

where $t_{KF}(K)$ is defined as above, with Bar-Natan homology replaced by knot Floer homology.

There are cases when these lower bounds are sharper than the concordance invariants $|\tau(K)|\leq \frac{1}{2}|s(K)|$, e.g. for knots of the form $-K\#K$, which is slice, therefore $\tau(-K\#K) = s(-K\#K) = 0$. 

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Our goal in this paper is to provide a similar lower bound for the unknotting number using grid homology. The grid homology of knots is a combinatorial counterpart of knot Floer homology in $S^3$. It was used to give a simple proof of the Milnor conjecture [10], to study transverse and Legendrian knots in $S^3$ [6].

There is one advantage of the grid-homology approach: Let $K_+, K_-$ be two knots which differ only at a crossing, where $K_+$ is positive and $K_-$ is negative. Denote by $H^*$ a knot homology theory (Bar-Natan homology, knot Floer homology, or grid homology) for knots. The core of proof for 2 and 3 is to construct homogeneous $F[h]$-module homomorphisms
\begin{align}
    f^+ : H^*(K_+) &\rightarrow H^*(K_-) \\
    f^- : H^*(K_-) &\rightarrow H^*(K_+) \tag{4}
\end{align}
such that $f^- \circ f^+ = f^+ \circ f^- = \text{multiplication by } h^m$ for some $m$. However, such maps already exist [1] and are constructed within the context of knot grid homology (without referring to tangle homology). Due to the isomorphism between knot Floer homology and grid homology, this gives a simple and self-contained proof for [3].

## 2 Lower bound from grid homology

### 2.1 Notions on grid homology

We reviewed some basic concepts and properties on grid homology.

A grid diagram $G$ is an $n \times n$ grid on the torus, with $n$ of these small squares marked with an $X$ and $n$ of them marked with an $O$. Furthermore, each row or column has exactly one $X$ and exactly one $Y$, and no square is marked with both an $X$ and $Y$. $n$ is called the grid number of $G$. Draw oriented segments connecting the $X$-marked squares to the $O$-marked squares in each column, then draw oriented segments connecting the $O$-marked squares to the $X$-marked squares in each row, with the convention that the vertical segments always cross above the horizontal ones. Then we get an oriented link $L$, and we call $G$ a grid diagram for $L$. Any link has a grid diagram representation.

A grid state for $G$ is an $n$-tuple of points $x = \{x_1, \ldots, x_n\}$ in the torus, so that each horizontal circle and vertical circles contains exactly one of the $x_i$. The (unblocked) grid complex $GC^-(G)$ for $G$ is a free $F[V_1, \ldots, V_n]$-module generated by the set $S(G)$ of grid states, where each $V_i$ corresponds to an $O$ on the diagram $G$. The differential of $GC^-(G)$ is defined by counting the rectangles not intersecting any $X$. There are 2 gradings on $GC^-(G)$, the Maslov grading $M$, and the Alexander grading $A$, satisfying
\begin{align}
    M(x) - M(y) &= 1 - 2#(r \cap \Omega) + 2#(x \cap Int(r)) \\
    A(x) - A(y) &= #(r \cap X) - #(r \cap \Omega) \tag{5}
\end{align}
where $r$ is a rectangle connecting $x$ to $y$.  

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Multiplication by $V_i$ for different $i$ are chain homotopic. The (unblocked) grid homology of $G$, denoted $\text{GH}^-(G)$, is the homology of $GC^-(G)$, viewed as a bigraded module over $\mathbb{F}[U]$, where the action by $U$ is induced by multiplication by any $V_i$.

We will use the following property on grid homology of knots:

**Proposition 1.** ([7], Proposition 6.1.1) Let $K_+, K_-$ be two knots which differ only at a single crossing, where $K_+$ is positive and $K_-$ is negative. Then there are $\mathbb{F}[U]$-module maps

$$
\begin{align*}
C_- : \text{GH}^-(K_+) &\longrightarrow \text{GH}^-(K_-) \\
C_+ : \text{GH}^-(K_-) &\longrightarrow \text{GH}^-(K_+)
\end{align*}
$$

which are homogeneous of degree $(0,0)$, $(-2,-1)$ respectively, and the composition $C_- \circ C_+, C_+ \circ C_-$ are equal to multiplication by $U$.

Let $K_1, K_2$ be two knots and $I$ is a sequence of crossing changes which modifies $K_1$ to $K_2$. The **Gordian distance** $u(K_1, K_2)$ between $K_1$ and $K_2$ is the least length of such sequence $I$. In particular, $u(K) = u(K, U)$, where $U$ is the unknot.

### 2.2 The torsion bound

The grid homology of a knot has rank 1, i.e. $\text{GH}^-(K) \cong \mathbb{F}[U] \oplus T(K)$, where $T(K)$ is the torsion part of $\text{GH}^-(K)$ as an $\mathbb{F}[U]$-module. Indeed, we have the following theorem on the structure of knot grid homology:

**Theorem 1.** For each knot $K$, there are integers $n_1, \ldots, n_k$ such that

$$
\text{GH}^-(K) \cong \mathbb{F}[U] \oplus \bigoplus_{i=1}^k \mathbb{F}[U]/U^{n_i}
$$

as $\mathbb{F}[U]$-modules.

For any element $\alpha \in T(K)$, define the order $\text{ord}(\alpha)$ of $\alpha$ to be the minimal integer $m$ such that $U^m \cdot \alpha = 0$. By Theorem 1, $\text{ord}(\alpha)$ is well-defined.

**Definition 1.** Define the torsion order of $K$ to be

$$
t(K) := \max\{\text{ord}(\alpha) \mid \alpha \in T(K)\}
$$

i.e. $t(K)$ is the minimal number $n$ such that multiplication by $U^n$ is trivial on $T(K)$.

**Theorem 2.** For any two knots $K, K'$,

$$
|t(K) - t(K')| \leq u(K, K')
$$

In particular, $t(K) \leq u(K)$. 

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**Proof.** Let $K_+, K_-$ be two knots which differ only at a crossing point, where is positive and $K_-$ is negative. We will show that

$$|t(K_+) - t(K_-)| \leq 1 \quad (10)$$

Let $C_+, C_-$ be the homomorphisms given in Proposition[1]. For $\alpha \in T(K_+)$, we have $C_-(\alpha) \in T(C_-)$, and it follows from $C_+ \circ C_-(\alpha) = U \cdot \alpha$ that

$$\text{ord}(U \cdot \alpha) \leq \text{ord}(C_-(\alpha)) \leq \text{ord}(\alpha) \quad (11)$$

Then, $\text{ord}(\alpha) \leq \text{ord}(C_-(\alpha)) + 1$, and so $t(K_+) \leq t(K_-) + 1$. Similarly, we can prove $t(K_-) \leq t(K_+) + 1$, and therefore $|t(K_+) - t(K_-)| \leq 1$.

Choose an oriented diagram for $K$ so that after switching $u(K, K')$ crossings we get a diagram for $K'$. For any $i = 1, \ldots, u(K, K')$, let $K_i$ be the diagram obtained from $K$ by switching the first $i$ crossings of them. Using the property proved above we see that $|t(K_i) - t(K_{i-1})| \leq 1$. Therefore $|t(K) - t(K')| \leq u(K, K')$.

The **alternation number** $\text{alt}(K)$ of a knot $K$ is the minimum Gordian distance between $K$ and an alternating knot. When $K$ is an alternating knot, the grid homology of $K$ is of the form $GH^-(K) \cong \bigoplus_{i \in I} \mathbb{F} \oplus \mathbb{F}[U]$, where $I = \emptyset$ if and only if $K$ is the unknot. Therefore $t(K) = 1$ if $K$ is not the unknot.

**Corollary 1.** The alternation number $\text{alt}(K)$ of a knot $K$ satisfies

$$\text{alt}(K) \geq t(K) - 1 \quad (12)$$

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