Gravitational waves in Modified Gauss-Bonnet gravity

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Abstract

We study the gravitational waves in modified Gauss-Bonnet gravity. Applying the metric perturbation around a cosmological background, we obtain explicit expressions for the wave equations. It is shown that the speed of the traceless mode is equal to the speed of light. An additional massive scalar mode appears in the propagation of the gravitational waves. To find phenomena beyond the general relativity the scalar mode mass is calculated as a function of the background curvature in some typical models.

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I. INTRODUCTION

In 2015, LIGO first detected gravitational waves (GWs) from the merger of a binary black holes of around 36 and 29 solar masses, and its result is consistent with the prediction of General Relativity (GR) [1]. It indicates that GR is still correct under strong gravity. However, astrophysical observations provide phenomena which can not be explained in GR, like the accelerating expansion. It is considered that the cosmic expansion has two phases. The one is Inflation in the very early universe which is proposed to solve the horizon and flatness problems [2, 3]. The other is the current expansion of the universe from the observational consequence of type Ia supernovae [4], cosmic microwave background [5], baryon acoustic oscillations [6] and so on.

Modified gravity is one of candidates to induce the cosmic expansion at very early universe and the current period [7]. It is shown that the existence of the modified gravity which pass the inspections from the observations. The most popular model of the modified gravity is $F(R)$ gravity which replace Ricci scalar, $R$, in the Einstein-Hilbert action by an arbitrary function of Ricci scalar, $F(R)$. Since the gravitational degree of freedom in $F(R)$ gravity is three [8, 9], GWs in $F(R)$ gravity have, in addition to tensor modes propagation, a scalar mode propagation. The wave equations for $F(R)$ gravity in a cosmological background is given by [10]

$$[\Box - m_{F(R)}^2] \delta \Phi = 0, \quad m_{F(R)}^2 = \frac{1}{3} \left( \frac{F'(\bar{R})}{F''(\bar{R})} - \bar{R} \right),$$

(1)

where $\delta \Phi$ indicates the fluctuation of the scalar mode. The mass, $m_{F(R)}$, depends on the background curvature. The non-vanishing mass means that the speed of the scalar mode propagation is less than the light speed and it constrains the $F(R)$ gravity models [10–14]. The tensor modes in $F(R)$ gravity are massless and propagate with the light speed [9, 11]. It has been pointed out that the modified gravity changes the graviton amplitude and the propagation phase [15, 16].

$F(G)$ gravity is an alternative model of the modified gravity which is proposed in Ref. [17]. The topological Gauss-Bonnet (GB) invariant, $G$, which contains the contractions of Ricci and Riemann tensors is derived from string theory at high energy as a low energy effective action [18]. A scalar field coupled with the GB invariant is introduced in the Einstein-
GB gravity. In the Einstein-GB gravity the gravitational wave speed constrains the scalar coupling to the GB invariant [19]. In $F(\mathcal{G})$ gravity an arbitrary function of the GB invariant is included in the Einstein-Hilbert action and applied some cosmological problems [20, 21]. From the topological property of GB invariant, the equation of motion in $F(\mathcal{G})$ gravity only has 2nd derivative terms. It is shown the degrees of freedom in $F(\mathcal{G})$ gravity is also three [22], and the tensor modes are massless. The model can be generalized to $F(R, \mathcal{G})$ gravity in which the Lagrangian contains an arbitrary function with respect to $R$ and $\mathcal{G}$ [23, 24]. $F(R)$ and $F(\mathcal{G})$ gravity can be described as special cases of $F(R, \mathcal{G})$ gravity. The degrees of freedom in $F(R, \mathcal{G})$ is four. Remarkably, the degrees of freedom reduce to three in Friedmann-Lematre-Robertson-Walker (FLRW) background [25].

In this paper the scalar mode in $F(\mathcal{G})$ and $F(R, \mathcal{G})$ gravity are investigated in De Sitter background. The experiments to measure the polarizations of GWs have already been started in Advanced LIGO and Virgo detectors [26]. At the moment, it is hard to discern the polarizations of GWs but the additional detectors will work in the near future, KAGRA [27] and LIGO-India [28], LISA [29], DECIGO [30]. It is expected to test the models of $F(\mathcal{G})$ gravity and $F(R, \mathcal{G})$ in the precise polarization measurements.

This paper is organized as follow. In Sec. II, we briefly review the equation of motion in $F(\mathcal{G})$ gravity. Using the perturbations around cosmological background, we obtain the wave equations of the trace modes and scalar mode. In Sec. III, we calculate the wave equation of the scalar mode in $F(R, \mathcal{G})$ gravity. In Sec. IV, the scalar mode mass is calculated in some typical models of $F(R)$ and $F(\mathcal{G})$ gravity. In Sec. V, conclusions are given and we discuss the possibility to detect the scalar mode propagation in the future experiment.

II. GWS IN $F(\mathcal{G})$ GRAVITY

A. $F(\mathcal{G})$ gravity

$F(\mathcal{G})$ gravity is defined by the action [17]

$$S_{GB} = \int d^4x \sqrt{-g} \left( \frac{\sqrt{\mathcal{G}}}{2} R + F(\mathcal{G}) + L_{\text{matter}} \right),$$

(2)
where $M_{\text{pl}}$ denotes Planck mass and $\mathcal{G}$ is the GB invariant, $\mathcal{G} = R^2 - 4R_{\mu\nu}R_{\mu\nu} + R^\lambda_{\mu\rho\nu\sigma}R_{\lambda\mu\rho\nu\sigma}$. A key difference between $F(\mathcal{G})$ and $F(R)$ gravity comes from contractions of Ricci and Riemann tensors. The GB invariant has topological property known as Chern-Gauss-Bonnet theorem [31],

$$\int d^4x \sqrt{-g}\mathcal{G} = 8\pi^2\chi(M),$$

where $\chi(M)$ is the Euler characteristic. The variation of the action (2) with respect to the metric tensor yields the equation of motion (EoM). In the vacuum the contribution from the matter Lagrangian, $\mathcal{L}_{\text{matter}}$, is dropped and EoM is found to be

$$\frac{M_{\text{pl}}^2}{2} \left( -R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R \right) + \frac{1}{2}g_{\mu\nu}F(\mathcal{G}) - \frac{1}{2}g_{\mu\nu}\mathcal{G}F'(\mathcal{G})$$

$$- 2g_{\mu\nu}\Box F'(\mathcal{G}) + 2R_{\mu\rho}\nabla_\rho F'(\mathcal{G}) + 4R_{\mu\nu}\Box F'(\mathcal{G})$$

$$+ 4g_{\mu\nu}R^\rho_{\sigma\rho}\nabla_\sigma F'(\mathcal{G}) - 4R^\rho_{\nu\sigma\rho}\nabla_\nu F'(\mathcal{G}) - 4R^\rho_{\mu\nu\sigma}\nabla_\rho F'(\mathcal{G}) - 4R^\rho_{\mu\nu}\nabla_\rho F'(\mathcal{G}) - 4R^\rho_{\mu\nu}\nabla_\rho F'(\mathcal{G})$$

$$= 0. \quad (3)$$

Eq. (3) does not contain the first derivative of the curvature, $R$, and the Gauss-Bonnet invariant, $\mathcal{G}$. As is known, the Einstein equations of GR are obtained for Gauss-Bonnet gravity, $F(\mathcal{G}) = \mathcal{G}$.

**B. Wave equations of $F(\mathcal{G})$ gravity**

To find the wave equation of $F(\mathcal{G})$ gravity in De Sitter background we employ the metric perturbations around the background,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \quad (4)$$

where $\bar{g}_{\mu\nu}$ shows the background metric. The Gauss-Bonnet invariant is perturbatively expanded as

$$\mathcal{G} \simeq \bar{\mathcal{G}} + \delta\mathcal{G} + \mathcal{O}(h^2). \quad (5)$$
Under the De Sitter background the Riemann tensor and Ricci tensor can be written by the background metric, $\tilde{g}_{\mu\nu}$, and the background scalar curvature, $\tilde{R}$,

$$\tilde{R}_{\mu\rho\nu\sigma} = \frac{\tilde{g}_{\mu\nu}\tilde{g}_{\rho\sigma} - \tilde{g}_{\mu\sigma}\tilde{g}_{\nu\rho}}{12} \tilde{R},$$

(6)

$$\tilde{R}_{\mu\nu} = \frac{\tilde{g}_{\mu\nu}}{4} \tilde{R}.$$  

(7)

From Eqs. (6) and (7), the background Gauss-Bonnet invariant is given by

$$\tilde{\mathcal{G}} = \frac{\tilde{R}^2}{6}.$$  

(8)

In the leading order of the metric perturbations Eq. (3) is simplifies to

$$\frac{M_{pl}^2}{2} \tilde{R} + 2F(\tilde{\mathcal{G}}) - 2\tilde{\mathcal{G}} F'(\tilde{\mathcal{G}}) = 0.$$  

Next we calculate the equation of motion up to the next-to-leading order of the metric perturbations. The Riemann, Ricci tensors and Ricci scalar are expanded as

$$R_{\mu\rho\nu\sigma} \simeq \tilde{R}_{\mu\rho\nu\sigma} + \delta R_{\mu\rho\nu\sigma} + \mathcal{O}(h^2),$$

(9)

$$R_{\mu\nu} \simeq \tilde{R}_{\mu\nu} + \delta R_{\mu\nu} + \mathcal{O}(h^2),$$

(10)

$$R \simeq \tilde{R} + \delta R + \mathcal{O}(h^2).$$

(11)

The perturbative terms can be expressed in terms of $h_{\mu\nu}[32]$

$$\delta R_{\mu\nu} = -\frac{1}{2}(\Box h_{\mu\nu} + \nabla_\mu \nabla_\nu h - \nabla_\mu \nabla_\lambda h_{\lambda\nu} - \nabla_\nu \nabla_\lambda h_{\lambda\mu} - \frac{2}{3} \tilde{R} h_{\mu\nu} + \frac{\tilde{R}}{6} \tilde{g}_{\mu\nu} h),$$

(12)

$$\delta R = -\Box h + \nabla^\mu \nabla_\mu h_{\mu\nu} - \frac{\tilde{R}}{4} h,$$

(13)

where $\nabla_\mu$ represents the covariant derivative in De Sitter metrics and $\Box$ is D’Alembert operator, $\Box \equiv \tilde{g}^{\mu\nu} \nabla_\mu \nabla_\nu$. The perturbations of Gauss-Bonnet invariant is found to be

$$\delta \mathcal{G} = \frac{\tilde{R}}{3} \left(-2\tilde{R}^{\mu\nu} h_{\mu\nu} + \tilde{g}^{\mu\nu} \tilde{g}^{\alpha\beta} \delta R_{\alpha\mu\beta\nu} \right) = \frac{\tilde{R}}{3} \delta R,$$

(14)

where we use the following expression of the perturbations of Ricci scalar,

$$\delta R = -\tilde{R}^{\mu\nu} h_{\mu\nu} + \tilde{g}^{\mu\nu} \delta R_{\mu\nu} = -2\tilde{R}^{\mu\nu} h_{\mu\nu} + \tilde{g}^{\mu\nu} \tilde{g}^{\alpha\beta} \delta R_{\alpha\mu\beta\nu}.$$  

(15)
\( F(\mathcal{G}) \) and \( F'(\mathcal{G}) \) are expanded as

\[
F(\mathcal{G}) \simeq F(\tilde{\mathcal{G}}) + F'(\tilde{\mathcal{G}})\delta\mathcal{G},
\]

\[
F'(\mathcal{G}) \simeq F'(\tilde{\mathcal{G}}) + F''(\tilde{\mathcal{G}})\delta\mathcal{G}.
\]

Therefore the equation of motion (3) reduces to

\[
- \frac{M_{\text{pl}}^2}{2} \left[ \delta R_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \delta R - \frac{1}{4} \bar{R} h_{\mu\nu} \right] + \frac{1}{3} \left[ \bar{g}_{\mu\nu} \Box - \nabla_\mu \nabla_\nu + \bar{g}_{\mu\nu} \frac{\bar{R}}{4} \right] F''(\tilde{\mathcal{G}}) \delta\mathcal{G} = 0.
\]

(18)

To find a wave equation for the physical degrees of freedom we take the following gauge conditions,

\[
\nabla^\mu \bar{h}_{\mu\nu} = 0, \quad \bar{h}_{0i} = 0 \quad (i = 1, 2, 3),
\]

(19)

where \( \bar{h}_{\mu\nu} \) is defined as

\[
\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} h.
\]

(20)

It should be noted that the traceless of the metric perturbation is also imposed as the gauge condition in GR. A discrepancy between GR and \( F(\mathcal{G}) \) gravity is found by dividing the metric perturbation \( h_{\mu\nu} \) into the traceless and scalar parts,

\[
h_{\mu\nu} = h^T_{\mu\nu} + \frac{h}{4} \bar{g}_{\mu\nu},
\]

(21)

where the traceless part, \( h^T_{\mu\nu} \), holds

\[
g^{\bar{\mu}\bar{\nu}} h^T_{\bar{\mu}\bar{\nu}} = 0, \quad h^T_{0i} = 0 \quad (i = 1, 2, 3).
\]

(22)

Substituting Eq. (21) into Eq. (18), we obtain

\[
\frac{M_{\text{pl}}^2}{4} \left[ \Box h^T_{\mu\nu} - \bar{R} h^T_{\mu\nu} \right] - \bar{R} \left[ \nabla_\mu \nabla_\nu - \frac{m_{\text{F}(\mathcal{G})}^2}{4} \bar{g}_{\mu\nu} \right] F''(\tilde{\mathcal{G}}) \delta\mathcal{G} = 0,
\]

(23)

where \( m_{\text{F}(\mathcal{G})}^2 \) is defined by

\[
m_{\text{F}(\mathcal{G})}^2 = \frac{3M_{\text{pl}}^2/2}{F''(\tilde{\mathcal{G}})R^2} - \frac{1}{3} \bar{R}.
\]

(24)

The wave equation for the scalar mode fluctuation is obtained by contractions of Eq. (23) by \( \bar{g}^{\mu\nu} \),

\[
[\Box - m_{\text{F}(\mathcal{G})}^2] \delta\Phi = 0.
\]

(25)
where we identify $F''(\tilde{G})\delta G$ with the scalar mode fluctuation, $\delta \Phi$,

$$F''(\tilde{G})\delta G \equiv \delta \Phi.$$  

The background curvature dependence of the scalar mode mass is given by Eq. (24).

The rest part of Eq. (23) corresponds to the wave equation for the tensor modes,

$$\Box h_{\mu\nu} - \frac{\tilde{R}}{6} h_{\mu\nu} = 0.$$  

(26)

It is more convenient to normalize the traceless parts as $h_{\mu\nu}^T = a^2 e_{\mu\nu}$. Then the wave equations for the tensor modes are written as

$$\left(-\partial_0^2 - 3H\partial_0 + \sum_k \frac{\partial_0^2}{a^2}\right) e_{ij} = 0,$$  

(27)

where $H$ is Hubble rate defined as $H \equiv \dot{a}/a$. To derive Eq. (27) we use $\tilde{R} = 12H^2$.

Thus we conclude that a massive scalar mode appears as an additional degree of freedom and tensor modes propagate with the speed of light under the De Sitter background in $F(G)$ gravity. It should be noticed that the perturbation of Gauss-Bonnet invariant (14) vanishes and only the tensor modes propagate in a flat spacetime [33].

III. GWS IN $F(R, G)$ GRAVITY

Here we consider $F(R, G)$ gravity, a more general class of theories of the modified gravity. The action of $F(R, G)$ gravity is defined as an integral of a function of Ricci scalar $R$ and Gauss-Bonnet invariant $G$ [24],

$$S_{RG} = \int d^4x \sqrt{-g} \left( \frac{M_{pl}^2}{2} F(R, G) + \mathcal{L}_{\text{matter}} \right).$$  

(28)

$F(R)$ and $F(G)$ gravity are regarded as a special cases of $F(R, G)$ gravity. The EoM is obtained by taking the variation of this action with respect to the metric tensor,

$$R_{\mu\nu} F(R, G) + \frac{1}{2} g_{\mu\nu} \mathcal{G}(R, G) - \frac{1}{2} g_{\mu\nu} F(R, G)$$

$$+ [g_{\mu\nu} \Box - \nabla_{\mu} \nabla_{\nu}] F(R, G)$$

$$+ [2g_{\mu\nu} R - 2R \nabla_{\mu} \nabla_{\nu} - 4R_{\mu\nu} \Box - 4g_{\mu\nu} R^{\rho\sigma} \nabla_{\rho} \nabla_{\sigma}] F(R, G) = 0.$$  

(29)
where we write
\[ F_R(R, G) \equiv \frac{\partial F(R, G)}{\partial R}, \quad F_G(R, G) \equiv \frac{\partial F(R, G)}{\partial G}. \] (30)

It is known that the degrees of freedom in \( F(R, G) \) gravity decrease from four to three under FLRW and De Sitter background [25]. Following the procedure developed in the previous section, we find that the gravitational wave is composed by the massless tensor modes and a massive scalar mode. To achieve the scalar mode mass we evaluate the first order perturbations of \( F_R(R, G) \) and \( F_G(R, G) \) around the De Sitter background,
\[ F_R(R, G) \simeq F_R(\tilde{R}, \tilde{G}) + F_{RR}(\tilde{R}, \tilde{G}) \delta R + F_{RG}(\tilde{R}, \tilde{G}) \delta G, \] (31)
\[ F_G(R, G) \simeq F_G(\tilde{R}, \tilde{G}) + F_{GR}(\tilde{R}, \tilde{G}) \delta R + F_{GG}(\tilde{R}, \tilde{G}) \delta G. \] (32)

Using the relation \( \delta G = \tilde{R} \delta R / 3 \), Eqs. (31) and (32) are rewritten as
\[ F_R(R, G) \simeq F_R(\tilde{R}, \tilde{G}) + \left( 1 + \frac{\tilde{R}}{3} F_{RG}(\tilde{R}, \tilde{G}) \right) \frac{\delta \Phi}{F_{RR}(\tilde{R}, \tilde{G})}, \] (33)
\[ F_G(R, G) \simeq F_G(\tilde{R}, \tilde{G}) + \left( F_{RG}(\tilde{R}, \tilde{G}) + \frac{\tilde{R}}{3} F_{GG}(\tilde{R}, \tilde{G}) \right) \frac{\delta \Phi}{F_{RR}(\tilde{R}, \tilde{G})}, \] (34)

where we identify \( F_{RR} \delta R \) with the scalar mode fluctuation \( \delta \Phi \).

The wave equation for the scalar model is obtained by the contraction of Eq. (29) with \( \tilde{g}^{\mu \nu} \),
\[ [\Box - m_{F(R,G)}^2] \delta \Phi = 0, \quad m_{F(R,G)}^2 = \frac{F_R - F_{RR} \tilde{R} - \frac{2 \tilde{R}^2}{3} F_{RG} - \frac{\tilde{R}^3}{9} F_{GG}}{3F_{RR} + 2 \tilde{R} F_{RG} + \frac{\tilde{R}^2}{3} F_{GG}}. \] (35)

The scalar mode mass depends on the background curvature through Eq. (35). It becomes a generalization from the mass function for \( F(R) \) and \( F(G) \) gravity. For \( G = 0 \) Eq. (35) reduces to the wave equation of \( F(R) \) gravity (1). Eq. (35) coincides with Eq. (25) for a special case \( F(R, G) = R + F(G)/M_{pl}^2 \).

IV. SCALAR MODE MASS

A. Exponential model

De Sitter metric satisfies the Friedmann equation if the energy density of the universe is asymptotically constant. It gives the simplest background to exhibit the accelerating
expansion of the universe. The exponential model of $F(R)$ gravity is introduced to turn on the cosmological constant for a strong curvature $R > R_0$ and suppress it at the flat limit, $R \to 0$ [34]. The model is defined by

$$F(R) = R - 2\Lambda(1 - e^{-R/R_0}).$$

(36)

To describe the current accelerating expansion of the universe $R_0$ is fixed smaller than a typical curvature of the universe and $\Lambda$ is at the present scale of the cosmological constant, $(10^{-33}\text{eV})^2$. From the formula (1) the mass for the scalar mode propagation is estimated as

$$m_{F(R)}^2 = \frac{1}{3} \left( \frac{R_0^2}{2\Lambda} e^\tilde{R}/R_0 - R_0 - \tilde{R} \right),$$

(37)

A similar model is studied in the modified Gauss-Bonnet gravity to switch on the cosmological constant term for $G > G_0$ [35],

$$F(G) = -\frac{M_{\text{pl}}^2}{2} 2\Lambda e^{\tilde{G}/G_0}.$$  

(38)

Substituting Eq. (38) into Eq. (24), we compute the mass of the scalar mode propagation,

$$m_{F(G)}^2 = \frac{G_0^2}{4G \Lambda} e^{\tilde{G}/G_0} - \frac{1}{3} \tilde{R}.$$  

(39)

The mass is sensitive to the model parameters $R_0$ and $G_0$, since the dominant contributions in Eq. (37) and Eq. (39) come from the exponential terms.

Evaluating the scalar mode mass for a fixed background curvature as $R_0$ or $G_0$ varies, we find the lower bound of the mass at

$$\tilde{R} = R_0 \left( 2 + W \left( -\frac{2\Lambda}{\epsilon^2 R_0} \right) \right),$$

$$\tilde{G} = 2G_0,$$

where $W(x)$ is the Lambert W function. In Fig. 1, we plot the lower bound of the scalar mode mass-squared as a function of the background curvature. The possible regions are shown by colored areas for each model. The modified term reduces to a cosmological constant at the large curvature limit, $F(R) \to -2\Lambda$, $F(G) \to -M_{\text{pl}}^2\Lambda$. The scalar mode obtains heavy mass and decouples from low energy phenomena. It should be noticed that the mass-squared
FIG. 1: Behavior of mass-squared of the scalar mode in the exponential models of $F(R)$ and $F(G)$ gravity. The lower limit of mass-squared is plotted by solid lines as a function of the background curvature. Colored areas show the possible regions in each models.

becomes negative and the tachyonic mode appears for a small curvature (See the enlarged plot in Fig. 1). Since the modified term vanishes at the limit, $R \rightarrow 0$, the scalar mode disappears.

If we set the model parameters $R_0$ and $G_0$ near the cosmological constant scale, $\tilde{R} \sim R_0 \sim \Lambda$, the scalar mode acquires the mass at the scale, $m^2 \sim \Lambda \sim (10^{-33} \text{ eV})^2$. The correlation length of the scalar mode is so long that there is a chance to detect the extra mode beyond GR. As is shown in Fig. 1, similar behavior is observed in each model and the mass ratio is of order unity, $m^2_{F(G)}/m^2_{F(R)} \sim \mathcal{O}(1)$. Thus we conclude that it is hard to test the difference between $F(R)$ and $F(G)$ gravity by the observation of GWs.

B. Power-law model

Another familiar model of $F(R)$ gravity is the power-law model [36–41],

$$\mathcal{L} = \frac{M^2_{\text{Pl}}}{2} R + \left| \frac{R}{R_0} \right|^{\alpha - 1} \times |R| \quad (\alpha > 1),$$

(40)
FIG. 2: Behavior of mass-squared of the scalar mode in the power-law models of $F(R)$ gravity for $r_0 = 1$[eV$^2$], $\tilde{R} = 10^{-66}, 10^{-46}, 10^{-26}, 10^{-6}, 10^{14}$[eV$^2$].

where $r_0$ is a constant parameter with mass dimension dim[$r_0$] = eV$^2$. It is considered that the higher order term of $R$ induces the accelerated expansion at inflation era. The term decreases more rapidly than Einstein-Hilbert term, then the universe exit from the era. The modified term gives a negligible contribution for a small curvature and the model is able to pass all the constraints on the solar system. From Eq. (1), the mass-squared of the scalar mode is given by

$$m^2_{F(R)} = \frac{\tilde{R}}{3\alpha(\alpha - 1)} \left[ \left( \frac{\tilde{R}}{r_0} \right)^{1 - \alpha} + \alpha(2 - \alpha) \right].$$  \hspace{1cm} (41)

The model is extended to the modified Gauss-Bonnet gravity [42–44],

$$\mathcal{L} = \frac{M^2_{\text{pl}}}{2} R + \left| \frac{\mathcal{G}}{g_0} \right|^{\beta - 1} \times |\mathcal{G}| \ (\beta > 1),$$  \hspace{1cm} (42)

where $g_0$ is a constant parameter with the mass dimension, dim[$g_0$] = eV$^4$. From Eq. (24) we obtain the mass-squared of the scalar mode under the De Sitter background,

$$m^2_{F(\mathcal{G})} = \frac{M^2_{\text{pl}}}{\beta(\beta - 1)} \left| \frac{\mathcal{G}}{g_0} \right|^{1 - \beta} - \frac{1}{3}\tilde{R}.$$  \hspace{1cm} (43)

In Figs. 2 and 3, we show the behavior of the scalar mode mass as a function of the exponent $\alpha$ or $\beta$. If the background values are smaller than the constant parameters, $\tilde{R} < r_0$
FIG. 3: Behavior of mass-squared of the scalar mode in the power-law models of $F(G)$ gravity for $g_0 = 1 [eV^4]$, $\tilde{G} = 10^{-66}, 10^{-46}, 10^{-26}, 10^{-6}, 10^{14} [eV^4]$.

or $\tilde{G} < g_0$, we observe a minimum of the scalar mode mass at

$$\alpha = 1 - \frac{1}{2 \log \tilde{R}/r_0},$$

$$\beta = 1 - \frac{1}{\log \tilde{G}/g_0}.$$  

The explicit expressions of the mass-squared at the minimum are given in Table. I. For a large background, $\tilde{R} > r_0$ or $\tilde{G} > g_0$, the mass-squared decreases asymptotically to a negative value $-\frac{\tilde{R}}{3}$, as the exponent $\alpha$ or $\beta$ increases. Thus the power-low models have a ghost mode.

In Fig. 2 we observe that the mass-squared has a fixed value, $r_0/6$, which is independent of the background curvature for $\alpha = 2$. In Fig. 3 no fixed point appears. Therefore we find

$$m^2_{F(R)} = -\frac{2}{3}(e^{1/2} - 1)\tilde{R}\log\left(\frac{\tilde{R}}{r_0}\right)$$

$$m^2_{F(G)} = -\frac{2}{3}M_{pl}^2\log\left(\frac{\tilde{G}}{g_0}\right) - \frac{\tilde{R}}{3}$$

| Condition | Expression |
|-----------|------------|
| $\tilde{R}, \tilde{G} < r_0, g_0$ | $\alpha, \beta \to \infty$ |
| $\tilde{R}, \tilde{G} > r_0, g_0$ | $\alpha, \beta \to \infty$ |

TABLE I: Mass-squared $m^2_{F(R)}$ and $m^2_{F(G)}$. 

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some differences on the scalar mode mass in the power-law models of $F(R)$ and $F(G)$ gravity.

V. CONCLUSION

We have investigated the GWs in modified Gauss-Bonnet gravity which is introduced as a generalization of GR. $F(G)$ and $F(R,G)$ gravity are defined as a family of theories with an arbitrary function of the Gauss-Bonnet invariant in the Lagrangian.

In the theories GWs have extra degrees of freedom and propagate as massless tensor modes and a massive scalar mode under the De Sitter background. Evaluating the wave equation in $F(G)$ and $F(R,G)$ gravity, we find formulae to calculate the scalar mode mass. In Table. II we summarize the features of GWs in the theories under consideration. We have considered two types of realistic models which can pass all the constraints on the solar system. The exponential models are studied at the current cosmological constant scale, $\Lambda \sim (10^{-33}\text{eV})^2$. In the models the mass-squared of the scalar mode is of order of the cosmological constant, $O((10^{-33}\text{eV})^2)$. The mass is small enough to observe the signal of the scalar mode. Similar behavior is observed in the exponential models of $F(R)$ and $F(G)$ gravity. On the other hand some different properties are found in the power-law models. The scalar mode mass is fixed independent of the background curvature in $R^2$ gravity.

The present work is restricted mostly to the analysis under the cosmological background, the longest section that GWs propagate. We are interested in including the contribution from the spherical symmetric background [45] and matter effect [46]. The chameleon mechanism

| Theory     | Physical modes     | Scalar mode mass |
|------------|--------------------|------------------|
| GR         | tensor modes       | -                |
| $F(R)$     | tensor + scalar modes | Eq. 1           |
| $F(G)$     | tensor + scalar modes | Eq. 24          |
| $F(R,G)$   | tensor + scalar mode | Eq. 35          |

TABLE II: GWs in GR, $F(R)$, $F(G)$ and $F(R,G)$ gravity
might affect GWs propagation in the solar system [47]. It is also interesting to apply the result to the analysis of GWs in $F(T)$ gravity [48] and several modified theories [49]. We will continue our work further and hope to report on these problems.

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