STRUCTURE CONSTANTS IN EQUIVARIANT ORIENTED COHOMOLOGY OF FLAG VARIETIES

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ABSTRACT. We introduce generalized Demazure operators for the equivariant oriented cohomology of the flag variety, which have specializations to various Demazure operators and Demazure-Lusztig operators in both equivariant cohomology and equivariant K-theory. In the context of the geometric basis of the equivariant oriented cohomology given by certain Bott-Samelson classes, we use these operators to obtain formulas for the structure constants arising in different bases. Specializing to divided difference operators and Demazure operators in singular cohomology and K-theory, we recover the formulas for structure constants of Schubert classes obtained in Goldin-Knutson [11]. Two specific specializations result in formulas for the structure constants for cohomological and K-theoretic stable bases as well; as a corollary we reproduce a formula for the structure constants of the Segre-Schwartz-MacPherson basis previously obtained by Su [21]. Our methods involve the study of the formal affine Demazure algebra, providing a purely algebraic proof of these results.

1. Introduction

Flag varieties $G/B$ are among the most studied varieties in topology and algebraic geometry. They have a cellular decomposition by Schubert cells, whose closures are called Schubert varieties. Schubert varieties are invariant under a torus action and, consequently, their torus-equivariant singular cohomology is spanned as a module by the Schubert classes.

Other classes associated to Schubert varieties in the equivariant singular cohomology $H^*_T$ and equivariant K-theory $K_T$ of the flag variety $G/B$ include Chern-Schwartz-MacPherson (CSM) classes and Motivic Chern (mC) classes, studied in [1, 2, 17, 19, 18, 20, 22]. These classes coincide with the corresponding stable bases of Maulik-Okounkov [16] for $H^*_T$ and $K_T$, of the Springer resolutions. Due to this fact, we always refer to the CSM classes as the cohomological stable basis, and to the mC classes as the K-theoretic stable basis. These classes behave like Schubert classes in their corresponding theories. Roughly speaking, Schubert classes in $H^*_T(G/B)$ and $K_T(G/B)$ are constructed by Demazure operators (also called divided difference operators), and elements of the stable bases are constructed by Demazure-Lusztig operators. All these operators generate various Hecke-type algebras.

Structure constants of Schubert classes are central objects in Schubert calculus, appearing in important questions of representation theory and combinatorics. In [11], the first author and Knutson obtain formulas for the structure constants in $H^*_T(G/B)$ and $K_T(G/B)$ using geometric properties of Bott-Samelson resolutions.
of Schubert varieties. They pull-back the Schubert classes to the equivariant cohomology (or equivariant K-theory) of Bott-Samelson variety, apply the cup product in this variety, then push-forward back to $G/B$. In [21], Su generalized this method to the so-called Segre-Schwartz-MacPherson (SSM) classes, a variant form of CSM classes.

We are interested in generalized cohomology theories, called oriented cohomology theories, defined by Levine and Morel [15]. These cohomologies are contravariant functors defined on the category of smooth projective varieties over a field $k$ of characteristic 0 to the category of commutative rings, such that for proper maps, there is a push-forward map on cohomology groups. Examples include Chow rings (singular cohomology), K-theory and algebraic cobordism. Chern classes are defined for each oriented cohomology theory $b$, and there is an associated formal group law $F$ defined over $R = b(pt)$. The machinery works equivariantly as well, resulting in a cohomology theory $b_T$ with an associated formal group law $F$ defined over $R = b_T(pt)$. For flag varieties, generalizing work of Kostant and Kumar [13, 14] on equivariant singular cohomology and equivariant K-theory of flag varieties, the ring $b_T(G/B)$ has a nice algebraic model, constructed in Hoffmann et al. in [12], and studied in [6, 7, 5] by Calmès, Zainoulline, and the second author. One can define the (formal) Demazure operators $X_\alpha$ associated to each simple root $\alpha$. These operators generate a non-commutative algebra, called the formal affine Demazure algebra $D_F$. It is a free left $b_T(pt)$-module isomorphic to $b_T(G/B)$, together with a dual basis $\{X^I_w \mid w \in W\}$. Indeed, for equivariant Chow group/singular cohomology/K-theory, $X^I_w$ coincides, up to various normalizations, to the Schubert class associated with $w$. Then $H^*_T(G/B)$ and $K_T(G/B)$ are achieved with the same module basis, and a restricted coefficient ring: a polynomial ring for $H^*_T(G/B)$ and Laurent polynomial ring for $K_T(G/B)$.

We notice that the product structure on $D_F^T$ is obtained by dualizing the coproduct structure of $D_F$. It follows that the structure constants of the basis $X^I_w$ may be deduced from the twisted Leibniz rule of the product $X_\beta w X_\gamma w \cdots X_\eta w$ for a reduced word $s_{\beta_1} \cdots s_{\beta_k}$ of $w \in W$. This is the main idea of the proof of Theorem 3.7, which implies the main result, Theorem 4.1. Specializing $b_T$ to equivariant singular cohomology and equivariant K-theory, we recover the formulas of the first author and Knutson in [11].

In the case of $H^*_T(G/B)$ and $K_T(G/B)$, replacing the Demazure operators $X_\alpha$ by the Demazure-Lusztig operators $T_\alpha$ and $\tau_\alpha$, one obtains the stable bases for $H^*_T(G/B)$ and $K_T(G/B)$, respectively. Both the cohomology stable basis and the K-theory stable basis can be described in an analogous fashion to the story for Schubert classes. That is, the Demazure-Lusztig operators generate a degenerate affine Hecke algebra (for equivariant cohomology) and an affine Hecke algebra (for equivariant K-theory). The dual elements to products of these operators are essentially the cohomological/K-theoretic stable bases, so their respective twisted
Leibniz rules result in a formula for the structure constants of stable bases. For instance, for cohomology, we recover the formula of Su [21] (see Remark 6.6).

To work with the Demazure operators $X_{\alpha}$ and Demazure-Lusztig operators $T_{\alpha}$ at the same time, we define a general operator $Z_{\alpha}$ (see §3) in a ring containing $D_F$, which can be specialized to $X_{\alpha}$ and $T_{\alpha}$. Our main results are Theorems 4.1 and 6.3, which state a formula for structure constants of the basis determined by $Z_{\alpha}$ and apply it to the cohomological stable basis.

The paper is organized as follows: In §2 we recall necessary notation introduced by the second author in [6, 7, 5]. We recall the definition of a Demazure element, the formal affine Demazure algebra, its dual, and relation with $h_T(G/B)$. In §3 we prove the twisted Leibniz rule for the operator $Z_{\alpha}$, which is used to derive the structure constants of the basis $Z_{I}^{*}$ determined by $Z_{\alpha}$. In §4 we specialize our result to Demazure operators in singular cohomology and K-theory, and recover the formulas in [11]. In §6 we specialize our result to Demazure-Lusztig operators in singular cohomology, which, as a by-product, recovers the formula due to Su in [21]. In §7, we consider Demazure-Lusztig operators in K-theory and obtain a formula for the structure constants of the K-theoretic stable basis. In §8, for equivariant oriented cohomology, we generalize some results of Kostant-Kumar ([13, Proposition 4.32], [14, Lemma 2.25]) by relating our formula for structure constants with a restriction formula of Schubert classes.

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guarantees that the elements $x_\alpha, \alpha \in \Lambda$ defined in $S$ below are non-zero-divisors. In particular, the Demazure operators $X_\alpha$ for simple roots $\alpha$ are well defined.

Let $G$ be a split semi-simple linear algebraic group with maximal torus $T$ and a Borel subgroup $B$. Let the associated root datum of $G$ be $\Sigma \hookrightarrow \Lambda^\vee$, so $\Lambda$ is the group of characters of $T$.

Let $h$ be an oriented cohomology theory of Levine and Morel. Roughly speaking, it is a contravariant functor from the category of smooth projective varieties to the category of commutative rings such that there is a push-forward map for any proper map. The Chern classes of vector bundles are defined. Associated to $\Lambda$, both can be extended to the torus equivariant setting. We assume the equivariant cohomology theory $h$ is Chern-complete over the point for $T$, that is, the ring $h_T(pt)$ is separated and complete with respect to the topology induced by the $\gamma$-filtration [5, Definition 2.2]. In particular, this includes the completed equivariant Chow ring, the completed equivariant K-theory and equivariant algebraic cobordism.

Let $S$ be the formal group algebra defined in [4]:

$$S = R[[\Lambda]]_F := \frac{R[[x_\lambda | \lambda \in \Lambda]]}{J_F},$$

where $J_F$ is the closure of the ideal generated by $x_0$ and $x_{\lambda+\mu} - F(x_\lambda, x_\mu)$, for all $\lambda, \mu \in \Lambda$. Indeed, if $\{t_1, ..., t_n\}$ is a basis of $\Lambda$, then $S$ is (non-canonically) isomorphic to $R[[x_{t_1}, ..., x_{t_n}]]$. According to [5, §3], $S \cong h_T(pt)$ with $x_\lambda$ corresponding to $c_1^h(L_\lambda)$ where $L_\lambda$ is the line bundle associated to $\lambda \in \Lambda$. Since $x_{-\lambda}$ is the formal inverse of $x_\lambda$, i.e. $F(x_\lambda, x_{-\lambda}) = 0$ in $S$, we may write

$$x_{-\lambda} = -x_\lambda + \text{higher degree terms} \in S.$$

Define $Q := S[\frac{1}{x_\lambda} | \alpha \in \Sigma]$. We will frequently need the special element of $Q$ given by $\kappa_\lambda := \frac{1}{x_\lambda} + \frac{1}{x_{-\lambda}}$. Note that $\kappa_\lambda$ actually belongs to $S$. Note also that the action of $W$ on $\Lambda$ induces an action of $W$ on $S$.

**Example 2.1.** *Two cases of the formal product appear widely in the literature [4, §2].*

1. If $F = F_a$ with $R = \mathbb{Z}$, then $h$ is the singular cohomology/Chow groups, and $S \cong \text{Sym}_2(\Lambda)^\wedge$ (with $x_\lambda \mapsto \lambda$) is the completion of the polynomial ring at the augmentation ideal. In this case $x_{-\lambda} = -x_\lambda$ and $\kappa_\lambda = 0$.

2. If $F = F_m$ with $R = \mathbb{Z}$, then $h$ is K-theory, and $S \cong \mathbb{Z}[\Lambda]^\wedge$ (with $x_\lambda \mapsto 1 - e^{-\lambda}$) is the completion of the Laurent polynomial ring at the augmentation ideal. In this case $x_{-\lambda} = \frac{x_\lambda}{x_\lambda - 1}$, and $\kappa_\lambda = 1$.

To obtain equivariant cohomology $H^*_\Gamma(X)$ and equivariant K-theory $K_\Gamma(X)$, we restrict the coefficient ring to $S^a = \text{Sym}[\Lambda]$ and $S^m = \mathbb{Z}[\Lambda]$, respectively.

### 2.1. The operator algebras $Q_W$ and $D_F$

This paper is concerned with various divided difference operators acting on $h_T(G/B)$, the equivariant cohomology of $G/B$. To create an algebraic framework for these operators, following [6, 7]
we localize $S$ at $\{x_\alpha\}$ to create an algebra out of this localization and the Weyl group, as follows.

Let $S$ be the ring described in (1), and let $Q := S[\frac{1}{x_\alpha} | \alpha \in \Sigma]$. Define $Q_W := Q \times R[W]$, as a left $Q$-module with basis $\{\delta_w\}, w \in W$.

We shall see that $Q_W$ acts on its dual space $Q_W^*$, which is identified with $Q \otimes_S \bigwedge^*_T(G/B)$, the cohomology of $G/B$ with inverted Chern classes.

We impose a product on $Q_W$ by

$$(p\delta_w)(p'\delta_{w'}) = pw(p')\delta_{ww'},$$

for all $p, p' \in Q$, and $w, w' \in W$,

using the natural $W$ action on $Q$ induced from that on $\Lambda$ and extending linearly. Note that $Q$ identified with $Q\delta_e$ is a subring of $Q_W$ under this product, where $e \in W$ denotes the identity element of $W$. We routinely abuse notation and write $\delta_\alpha$ for $\delta_{s_\alpha}$, and use $1 = \delta_e$ to denote the identity element of $Q_W$. The ring $Q_W$ acts on $Q$ by

$$p\delta_w \cdot p' = pw(p'),$$

for all $p, p' \in Q$.

The action of $Q_W$ on $Q$ induces a coproduct structure on $Q_W$ as follows. Let

$$\eta = \sum_{w \in W} q_w \delta_w \in Q_W.$$ Then

$$\eta \cdot (pq) = \sum_{w} q_w w(pq) = \sum_{w} q_w w(p)w(q) = \sum_{w} q_w (\delta_w \cdot p)(\delta_w \cdot q).$$

This action factors through the coproduct $\Delta : Q_W \to Q_W \otimes_Q Q_W$

$$\Delta(\eta) = \sum_{w} q_w \Delta(\delta_w) = \sum_{w} q_w \delta_w \otimes \delta_w.$$ In other words, the coproduct structure on $Q_W$ is induced from the $Q_W$-action on $Q$.

For any simple root $\alpha$ we, define the Demazure element $X_\alpha$ and the push-pull element $Y_\alpha$ in $Q_W$:

$$X_\alpha = \frac{1}{x_\alpha} (1 - \delta_\alpha) \quad \text{and} \quad Y_\alpha = \frac{1}{x_{-\alpha}} + \frac{1}{x_\alpha} \delta_\alpha.$$

We observe the relationship $Y_\alpha = \kappa_\alpha - X_\alpha$. In particular, if $F = F_\alpha$ (resp. $F = F_m$), then $Y_\alpha = -X_\alpha$ (resp. $Y_\alpha = 1 - X_\alpha$).

The way $Q_W$ acts on $Q$ implies that $X_\alpha$ acts in a fashion similar to the Demazure operator defined in [8] (and there denoted $D_\alpha$). In particular, $X_\alpha \cdot S \subset S$ and, for any $r \in R, X_\alpha \cdot r = 0$ and $\delta_\alpha \cdot r = s_\alpha(r) = r$.

Let $D_F$ be the $R$-subalgebra of $Q_W$

$$D_F = \langle S, X_{\alpha_1}, \ldots, X_{\alpha_m} \rangle$$
generated by $S$ and the elements $X_\alpha \in Q_W$ for simple roots $\alpha$, and call it the formal affine Demazure algebra. It is also generated by $S$ and $\{Y_\alpha : \alpha \text{ simple}\}$. As a left $S$ module, $D_F$ is a also free with basis $\{X_{I_w}\}_{w \in W}$, or with basis $\{Y_{I_w}\}_{w \in W}$; see [6, Proposition 7.7].

Let $w = s_{i_1} \cdots s_{i_k}$ be a reduced word decomposition and $I_w = (i_1, \ldots, i_k)$ the corresponding sequence of reflections. Define

$$X_{I_w} = X_{\alpha_{i_1}} \cdots X_{\alpha_{i_k}} \quad \text{and} \quad Y_{I_w} = Y_{\alpha_{i_1}} \cdots Y_{\alpha_{i_k}}.$$ In particular, $X_{(i)} = X_{\alpha_i}$ and $Y_{(i)} = Y_{\alpha_i}$, though we eliminate parentheses when there is no confusion. We write $X_e := 1 \in Q_W$ to indicate $X_I$ when $I$ is the empty sequence.
The Demazure and push-pull elements have the following properties:

**Lemma 2.2.** [24, Proposition 3.2] Let \( \alpha \) and \( \beta \) be simple roots. The following identities hold in \( Q_W \):

1. \( X_\alpha^2 = \kappa_\alpha X_\alpha \), \( Y_\alpha^2 = \kappa_\alpha Y_\alpha \).
2. \( X_\alpha p = s_\alpha(p)X_\alpha + X_\alpha \cdot p \), \( p \in Q \).
3. If \((s_\alpha s_\beta)^2 = e\), then \( X_\alpha X_\beta = X_\beta X_\alpha \).
4. If \((s_\alpha s_\beta)^3 = e\), then \( X_\beta X_\alpha X_\beta - X_\alpha X_\beta X_\alpha = \kappa_{\alpha\beta} X_\alpha - \kappa_{\beta\alpha} X_\beta \), where
   \[
   \kappa_{\alpha\beta} = \frac{1}{x_{\alpha + \beta}} - \frac{1}{x_{\alpha + \beta x_{-\alpha}}} - \frac{1}{x_{\alpha x_{\beta}}},
   \]

   Furthermore, \( \kappa_{\alpha\beta} \in S \) by [12, Lemma 6.7].
5. Suppose \( s_\alpha s_\beta \) has order \( m \) with \( m = 4 \) or \( 6 \), and \( I_w \) is a choice of reduced word for \( w \in W \). Then
   \[
   \frac{X_\alpha X_{\beta} X_{\alpha} \cdots - X_{\beta} X_{\alpha} X_{\beta} \cdots}{m} = \sum_{v \in W} c_{I_v} X_{I_v},
   \]
   where \( c_{I_v} = 0 \) if \( v \not\subseteq s_\alpha s_\beta \cdots \). Moreover, \( c_{I_v} = 0 \) if \( \ell(v) = m - 1 \) or \( v = e \).

Lemma 2.2.(4)-(5) imply that the operators \( X_\alpha \) (and similarly \( Y_\alpha \)) do not satisfy braid relations for general \( F \). For \( F = F_n \) or \( F = F_m \), they do; in these cases, the coefficients \( \kappa_{\alpha\beta} \) and \( c_{I_v} \) all vanish. In general, \( X_{I_w} \) and \( Y_{I_w} \) depend on the choice of \( I_w \) due to this failure of braid relations.

For the purposes of this paper, we fix a reduced sequence \( I_w \) of \( w \) for each \( w \in W \). While the specific coefficients and calculations regarding \( X_{I_w} \) and \( Y_{I_w} \) depend on this choice, statements regarding bases and ring phenomena do not.

By construction, \( \{\delta_v : v \in W\} \) form a basis of \( Q_W \) as a module over \( Q \). In [6], and extended in [7], the second author proves that \( \{X_{I_v} : v \in W\} \) and \( \{Y_{I_v} : v \in W\} \) also form bases of \( Q_W \) as a module over \( Q \), and that the change of basis matrix from \( \{X_{I_v}\} \) (or from \( \{Y_{I_v}\} \)) to \( \{\delta_v\} \) consists of elements of \( S \). In particular, \( \{\delta_v\} \) are elements of \( D_F \). The lower-triangularity of the change of bases matrices is expressed in the following lemma.

**Lemma 2.3.** [7, Lemma 3.2, Lemma 3.3] For each \( v \in W \), choose a reduced decomposition of \( v \) and let \( I_v \) be its corresponding sequence. There exist elements \( a_{I_{w,v}}^X \in Q \) for \( v \in W \), and \( b_{w,I_v}^X \in S \) such that

\[
X_{I_w} = \sum_{v \leq w} a_{I_{w,v}}^X \delta_v, \text{ and } \delta_w = \sum_{v \leq w} b_{w,I_v}^X X_{I_v}.
\]

Similarly, there exist \( a_{I_{w,v}}^Y \in Q \) and \( b_{w,I_v}^Y \in S \) such that

\[
Y_{I_w} = \sum_{v \leq w} a_{I_{w,v}}^Y \delta_v, \text{ and } \delta_w = \sum_{v \leq w} b_{w,I_v}^Y Y_{I_v}.
\]

Notice that nonzero coefficients \( b_{w,I_v}^X \) are elements of \( S \) with \( v \leq w \).

**Example 2.4.** Consider the root datum \( A_2 \), with

\[
W = \{e, s_1, s_2, s_1s_2, s_2s_1, w_0\},
\]
where \( w_0 \) is the longest element and \( s_i \) is the reflection corresponding to \( \alpha_i \) for \( i = 1, 2 \). We fix the reduced sequence \( I_{w_0} = (1, 2, 1) \) for \( w_0 \). For simplicity, let \( \alpha_{13} = \alpha_1 + \alpha_2 \). By direct computation,

\[
\begin{align*}
\delta_1 &= X_e, & \delta_{s_1 s_2} &= 1 - x_1X_{(1)} - x_{\alpha_{13}}X_{(1)} + x_{\alpha_1 x\alpha_{13}}X_{(1, 2)}, \\
\delta_1 &= 1 - x_{\alpha_1}X_{(1)}, & \delta_{s_2 s_1} &= 1 - x_{\alpha_2}X_{(2)} - x_{\alpha_{13}}X_{(1)} + x_{\alpha_1 x\alpha_{13}}X_{(2, 1)}, \\
\delta_2 &= 1 - x_{\alpha_2}X_{(2)}, & \delta_{w_0} &= 1 - x_{\alpha_{13}}X_{(2)} - (x_{\alpha_1} + x_{\alpha_2} - \kappa_{\alpha_1 x\alpha_{13}})X_{(1)} \\
&& & + x_{\alpha_1}x_{\alpha_{13}}X_{(1, 2)} - x_{\alpha_1 x\alpha_{13}}X_{I_{w_0}}.
\end{align*}
\]

2.2. The dual operator algebras. The dual \( Q^*_W \) module

\[ Q^*_W = \text{Hom}_G(QW, Q) \cong \text{Hom}(W, Q), \]

contains a natural basis \( \{ f_w \}_{w \in W} \) dual to \( \{ \delta_w \}_{w \in W} \), defined by

\[ \langle f_w, \delta_v \rangle = \begin{cases} 1 & \text{if } w = v; \\ 0 & \text{otherwise.} \end{cases} \]

One may think of \( Q^*_W \) as the \( T \)-equivariant oriented cohomology of \( W \) with the trivial \( T \) action, tensored with \( Q \). In particular,

\[ Q^*_W = Q \otimes_S \mathfrak{h}_T(W) = Q \otimes_S \mathfrak{h}_T(G/B). \]

The module \( Q^*_W \) forms a ring with product \( f_w f_v = 1 \) if an only if \( w = v \), and 0 otherwise, extended linearly to all elements of \( Q^*_W \), and unity \( 1 = \sum_{w \in W} f_w \). This product structure is equivalent to the one induced from the coproduct structure (see §1 below).

The ring \( QW \) acts on \( Q^*_W \) by

\[ \langle z \cdot f, z' \rangle = \langle f, z' z \rangle, \quad \text{for all } z, z' \in QW, \ f \in Q^*_W. \]

In the bases \( \{ \delta_w \} \) of \( QW \) and \( \{ f_w \} \) of \( Q^*_W \), the action has explicit formulation

\[ p \delta_w \cdot (q f_v) = q v w^{-1} (p) f_{v w^{-1}}, \quad \text{for all } p, q \in Q. \]

Denote

\[ \text{pt}_w = \left( \prod_{\alpha < 0} x_{\alpha} \right) \cdot f_w = w \left( \prod_{\alpha < 0} x_{\alpha} \right) f_w \in Q^*_W. \]

Let \( D^*_F := \text{Hom}_S(D_F, S) \subset Q^*_W \) be the dual \( S \)-module to \( D_F \). It is proved in [7, Lemma 10.3] that \( \text{pt}_w \in D^*_F \). Let

\[ \zeta^X_{I_{w}} = X_{I_{w}, w}, \quad \text{and} \]

\[ \zeta^Y_{I_{w}} = Y_{I_{w}, w}. \]

Then \( \{ \zeta^X_{I_{w}} \} \) forms a basis of \( D^*_F \) over \( S \), as does \( \{ \zeta^Y_{I_{w}} \} \).

Finally, let \( \{ X^*_I \} \) (respectively \( \{ Y^*_I \} \)) be the bases dual to \( \{ X_I \} \) (resp. \( \{ Y_I \} \)) in \( D^*_F \), which are also \( Q \)-basis of \( Q^*_W \).

The classes \( X^*_I \) for each \( v \in W \) are determined by duality. Under the dual pairing,

\[ \langle X^*_I, \delta_w \rangle = \langle X^*_I, \sum_{u \in W} b^X_{w, I_u} X_{I_u} \rangle = b^X_{w, I_v}. \]
Set $X_w^* = \sum_{u \in W} m_{I_w,u} f_u$, which implies
\[
\langle X_w^*, \delta_w \rangle = \langle \sum_{u \in W} m_{I_w,u} f_u, \delta_w \rangle = m_{I_w,w},
\]
and thus $X_w^* = \sum_{w \in W} b_{w,I_w} f_w$.

**Example 2.5.** Consider the root datum $A_2$, with $W = \{e, s_1, s_2, s_1 s_2, s_2 s_1, w_0\}$. Fix the reduced sequence $w_0 = s_1 s_2 s_1$. The calculations from Example 2.4 imply
\[
\begin{align*}
X_e^* &= 1 = \sum_{w \in W} f_w, & X_{(1,2)}^* &= x_{\alpha_1} x_{\alpha_3} (f_{s_1 s_2} + f_{w_0}) \\
X_{(1)}^* &= -x_{\alpha_1} (f_{s_1} + f_{s_2 s_1}) - x_{\alpha_1} s_{s_2 s_1} - y f_{w_0}, & X_{(2,1)}^* &= x_{\alpha_2} x_{\alpha_3} (f_{s_2 s_1} + f_{w_0}) \\
X_{(2)}^* &= -x_{\alpha_2} (f_{s_2} + f_{s_2 s_1}) - x_{\alpha_3} (f_{s_1 s_2} + f_{w_0}), & X_{I_w^0}^* &= -x_{\alpha_1} x_{\alpha_2} x_{\alpha_3} f_{w_0},
\end{align*}
\]
where $y = x_{\alpha_1} + x_{\alpha_2} - \kappa_{\alpha_1} x_{\alpha_1} x_{\alpha_2}$. In case $F = F_a$ or $F_m$, we have $y = x_{\alpha_1}$.

The following proposition explains the relationship between the algebraic construction above and equivariant oriented cohomology of $G/B$.

For each reduced sequence $I_w$, let $X_w \to G/B$ denote the Bott-Samelson resolution. The push-forward in $\mathfrak{h}_T$ of the fundamental class along this resolution is called the Bott-Samelson class of $I_w$, which we denote by $\eta_{I_w}$. Define a map
\[
\Phi : D_F^\ast \to \mathfrak{h}_T(G/B)
\]
given by $\Phi(\zeta_I^Y) = \eta_{I_w}$ and $\Phi(1) = [G/B]$, the fundamental class of $G/B$, and extended as a module over $S$.

**Proposition 2.6.** The isomorphism $\Phi$ satisfies the following properties:

1. [5, Theorem 8.2, Lemma 8.8] The map $\Phi$ is a functorial isomorphism.
2. [7, Theorem 14.7] The basis $\{\Phi(X_{I_w}^*) : w \in W\}$ (resp. $\{\Phi(Y_{I_w}^*) : w \in W\}$) is dual to $\Phi(\zeta_I^Y)$ (resp. $\Phi(\zeta_I^Y)$) via the nondegenerate dual pairing on $\mathfrak{h}_T(G/B)$ given by multiplying and pushing forward to a point.
3. [5, Corollary 6.4] Let $i_w : wB \to G/B$ be the inclusion of the $T$-fixed point corresponding to $w \in W$, and $(i_w)_* : \mathfrak{h}_T(wB) \to \mathfrak{h}_T(G/B)$ be the pushforward map. Then $\Phi(pt_w) = (i_w)_*(1)$.
4. There is a commutative diagram
\[
\begin{array}{ccc}
D_F^\ast & \xrightarrow{\cong} & \mathfrak{h}_T(G/B) \\
\oplus_{w \in W} i_w & \xrightarrow{\cong} & Q \otimes_S \mathfrak{h}_T(W),
\end{array}
\]
where the top horizontal map is the embedding of the $S$-module into the $Q$-module $Q_w$.

By specializing the formal group law to $F_a$ or $F_m$, respectively, and restricting $S$ to $R[A]/J_F$, we obtain a map $\Phi^H : D_F^\ast \to H_T^2(G/B)$ or $\Phi^K : D_F^\ast \to K_T(G/B)$ to the equivariant cohomology or equivariant K-theory. The map remains an isomorphism over the corresponding module. From now on we will not distinguish between $D_F^\ast$ and $\mathfrak{h}_T(G/B)$. 


Example 2.7. Let $X(w) = BwB/B$ be the Schubert variety and $Y(w) = B^{-w}B/B$ be the opposite Schubert variety. For $H^*_T(G/B)$ (with $F = F_\alpha$) or $K_T(G/B)$ (with $F = F_m$), we write $w$ for $I_w$ since $X_{I_w}$ and $Y_{I_w}$ are independent of the reduced sequence.

1. [11, §1.2] For $H^*_T(G/B)$, $\zeta^X_w = [X(w)]$, and $\zeta^Y_w = (-1)^{l(w)}[X(w)]$, where each homology class is identified with its dual cohomology class. Then $Y^*_w = [Y(w)]$ and similarly $X^*_w = (-1)^{l(w)}[Y(w)]$.

2. [3, §3] For $K_T(G/B)$, $\zeta^Y_w = [O_{X(w)}]$ is the class of the structure sheaf of $X(w)$, $Y^*_w = [O_{Y(w)}(-\partial Y(w))], \zeta^X_w = (-1)^{l(w)}[O_{X(w)}(-\partial X(w))], and \ X^*_w = (-1)^{l(w)}[O_{Y(w)}]$.

3. Generalized Demazure operators and the generalized Leibniz rule

In this section, we generalize the operators $X_{I_w}$ and $Y_{I_w}$ on $H^*_T(G/B)$ to a more general class of elements of $Q_W$, and prove the generalized Leibniz rule for $D_F$ acting on $Q$. We use this result to compute the coproduct structure in $Q_W$, and then the product structure in $Q^*_W$.

Let $\{a_\alpha, b_\alpha \in Q : \alpha \in \Sigma\}$ be a set of elements with the property that, for all $w \in W$,

$$w(a_\alpha) = a_{w(\alpha)}, \quad w(b_\alpha) = b_{w(\alpha)}, \text{ and } b_\alpha \text{ are all invertible in } Q.$$ 

For any simple root $\alpha$, define operators $Z_\alpha \in Q_W$ by

$$Z_\alpha = a_\alpha + b_\alpha \delta_\alpha.$$ 

Clearly $X_\alpha$ and $Y_\alpha$ result from $Z_\alpha$ as special cases of $a_\alpha$ and $b_\alpha$. For any sequence $I = (i_1, \ldots, i_k)$, define $Z_I \in Q_W$ by

$$Z_I = Z_{\alpha_{i_1}} Z_{\alpha_{i_2}} \cdots Z_{\alpha_{i_k}}.$$ 

We call $Z_I$ generalized Demazure operators.

As before, we choose a reduced word expression $I_w$ for each $v \in W$.

Lemma 3.1. The set of generalized Demazure operators $\{Z_{I_w}\}$ forms a basis of $Q_W$ as a module over $Q$.

Proof. This follows from the fact that $b_\alpha \in Q$ is invertible for all simple roots $\alpha$ (hence, for all roots $\alpha$). \qed

Remark 3.2. Note that $Z_\alpha \in D_F$ if and only it satisfies the residue condition [23, Definition 3.7]. If this is satisfied, then $Z_{I_w} \in D$ and equivalently, $Z^*_w \in D_F$. Moreover, $Z_{I_w}$ forms a basis of $D_F$ if and only if $\frac{1}{\alpha} \in S$ for all $\alpha$. For example, this holds for $X_\alpha, Y_\alpha$, but fails for $T_\alpha$ considered in Section 6 and 7. This is precisely why the stable basis is only a basis after localization.

Lemma 3.3. For any sequence $J$, define coefficients $c_{J, I_w} \in Q$ by

$$Z_J = \sum_{w \in W} c_{J, I_w} Z_{I_w},$$

Then $c_{J, I_w} = 0$ unless $w \leq \prod J$. 

where the operators $I_{\Leibniz}$ coefficients $C_{J,I}$ may be expressed as a $Q$-linear combination of $\delta_v$ for $v \leq \prod J$.

For any $v \in W$ and reduced sequence $I_v = (i_1, \ldots, i_k)$, let $\gamma_j = \alpha_{i_j}$ for $j = 1, \ldots, k$. The coefficient of $\delta_v$ in $Z_{I_v}$ is

$$b_{\gamma_1} s_{\gamma_1} (b_{\gamma_2}) s_{\gamma_2} (b_{\gamma_3}) \cdots s_{\gamma_k-1} (b_{\gamma_k}).$$

In particular, since $b_{\gamma_j}$ is invertible, so is $w(b_{\gamma_j})$ for any Weyl group element $w$, and thus the coefficient of $\delta_v$ in $Z_{I_v}$ is nonzero.

Let $A = \{ w \in W : c_{J,I_w} \neq 0 \text{ and } w \not\leq \prod J \}$, and assume $A$ is nonempty. Pick $v \in A$ to be a maximal element of $A$ in the Bruhat order. By support considerations, the only terms contributing to the coefficient of $\delta_v$ in (4) is $c_{J,I_w, Z_{I_w}}$. Since the coefficient of $\delta_v$ in $Z_{I_w}$ is a unit, we conclude $c_{J,I_w} = 0$, contrary to assumption. □

The structure constants $c_{J,I_w}$ reflect geometric properties in some special cases (see Section 5). When $Z_\alpha = X_\alpha$ for all $\alpha$ or $Z_\alpha = Y_\alpha$ for all $\alpha$, and $F = F_\alpha$, the coefficients in the sum (4) vanish unless $J$ is a reduced word for $w$, in which case $c_{J,I_w} = 1$; this reflects the property that the pushforward map in homology sends the orientation class $[BS_J]$ to the Schubert variety $X(w)$ when $J$ is a reduced word for $w$. When $Z_\alpha = X_\alpha$ for all $\alpha$ or $Z_\alpha = Y_\alpha$ for all $\alpha$, and $F = F_\alpha$, coefficients vanish except when the Demazure product of $J$ is $w$, which occurs exactly once and results in $c_{J,I_w} = 1$. In this case, the K-theoretic pushforward of $[O_{BS_J}]$ is the structure sheaf of $X(w)$ when $w = \prod J$. More generally, $Z_J$ is an (equivariant) operator whose dual has support only on those fixed points in the Schubert variety $X(w)$, where $w = \prod J$.

We have the following lemma describing the action of $Z_\alpha$ on a product.

**Lemma 3.4.** For a simple root $\alpha$, and $p, q \in Q$, we have

$$Z_\alpha \cdot (pq) = \frac{a_\alpha (a_\alpha + b_\alpha)}{b_\alpha} pq - \frac{a_\alpha}{b_\alpha} [(Z_\alpha \cdot p) q + p (Z_\alpha \cdot q)] + \frac{1}{b_\alpha} (Z_\alpha \cdot p) (Z_\alpha \cdot q).$$

**Proof.** One just has to plug in $Z_\alpha = a_\alpha + b_\alpha \delta_s$, and use the definition of the action $\delta_s \cdot p = s_a (p)$. A comparison of both sides yields the identity. □

The coefficients occurring in Lemma 3.4 may be generalized to the case of the action of $Z_I$ on a product $pq$.

**Definition 3.5.** For each simple root $\alpha$, let $Z_\alpha = a_\alpha + b_\alpha \delta_s$ with $a_\alpha, b_\alpha \in Q$ and $b_\alpha$ invertible. Let $I = (i_1, \ldots, i_k)$ be a sequence of indices of simple roots, with $\gamma_j := \alpha_{i_j}$ corresponding to the $j$th entry of $I$. For $E, F \subset \{1, \ldots, k\}$, define the **Leibniz coefficients** $C^I_{E,F} \in Q$ by

$$C^I_{E,F} = (B_1^E B_2^Z \cdots B_k^Z) \cdot 1,$$

where the operators $B_j^Z \in Q_W$ are given by

$$B_j^Z = \begin{cases} 
\frac{1}{b_{\gamma_j}} \delta_{\gamma_j}, & \text{if } j \in E \cap F, \\
\frac{a_{\gamma_j}}{b_{\gamma_j}} \delta_{\gamma_j}, & \text{if } j \in E \text{ or } F, \text{ but not both}, \\
a_{\gamma_j} + \frac{a_{\gamma_j}^2}{b_{\gamma_j}} \delta_{\gamma_j}, & \text{if } j \notin E \cup F.
\end{cases}$$

10 REBECCA GOLDIN, CHANGLONG ZHONG
Similarly, if \( Z = Y \) indicate the push-pull operators,
\[
B^Y_j = \begin{cases} 
\frac{x_{ij}}{x_{ij} - \gamma_j}, & \text{if } j \in E \cap F, \\
\frac{x_{ij}}{x_{ij} - \gamma_j}, & \text{if } j \in E \text{ or } F, \text{ but not both,} \\
\frac{1}{x_{ij}} + \frac{x_{ij}}{(x_{ij} - \gamma_j)}, & \text{if } j \notin E \cup F.
\end{cases}
\]

Now we prove the main technical result of this paper, generalizing [6, Lemma 4.8].

**Theorem 3.7 (Generalized Leibniz Rule).** Let \( Z_I \) be a generalized Demazure operator for \( I = (i_1, \ldots, i_k) \), and let \( \gamma_j = \alpha_{i_j} \) denote the \( j \)th simple root in the list. Then for any \( p, q \in Q \),
\[
Z_I \cdot (pq) = \sum_{E,F \subset \{1, \ldots, k\}} C^I_{E,F}(Z_E \cdot p)(Z_F \cdot q),
\]
where \( C^I_{E,F} \) are the Leibniz coefficients defined in (5).

**Proof.** For any simple root \( \alpha \), observe the following two identities:
\[
\begin{align*}
(7) \quad a_\alpha (1 - \delta_\alpha) + \frac{a_\alpha (a_\alpha + b_\alpha)}{b_\alpha} \delta_\alpha &= a_\alpha + \frac{a_\alpha^2}{b_\alpha} \delta_\alpha = \frac{a_\alpha}{b_\alpha} Z_\alpha \delta_\alpha, \\
(8) \quad Z_\alpha \cdot (pq) &= a_\alpha (p - s_\alpha(p))q + s_\alpha(p) (Z_\alpha \cdot q).
\end{align*}
\]

Now assume it holds for all \( I \) with \( \ell(I) < k \), and let \( I = (i_1, \ldots, i_k) \). Let \( J = (i_2, \ldots, i_k) \) and let \( \alpha = \alpha_{i_1} \). We have
\[
Z_I \cdot (pq) = (Z_\alpha Z_J) \cdot (pq) = Z_\alpha \cdot (Z_J \cdot (pq))
\]
\[
= Z_\alpha \cdot \left[ \sum_{E,F \subset \{2, \ldots, k\}} C^J_{E,F}(Z_E \cdot p)(Z_F \cdot q) \right]
\]
\[
= \sum_{E,F \subset \{2, \ldots, k\}} a_\alpha \left[ C^J_{E,F} - s_\alpha(C^J_{E,F}) \right] (Z_E \cdot p)(Z_F \cdot q)
\]
\[
+ \sum_{E,F \subset \{2, \ldots, k\}} s_\alpha(C^J_{E,F}) Z_J \cdot [(Z_E \cdot p)(Z_F \cdot q)] \quad \text{by Equation (8)}
\]
\[
= \sum_{E,F \subset \{2, \ldots, k\}} a_\alpha \left[ C^J_{E,F} - s_\alpha(C^J_{E,F}) \right] (Z_E \cdot p)(Z_F \cdot q)
\]
\[
+ \sum_{E,F \subset \{2, \ldots, k\}} s_\alpha(C^J_{E,F}) \frac{a_\alpha (a_\alpha + b_\alpha)}{b_\alpha} (Z_E \cdot p)(Z_F \cdot q)
\]
Comparing the coefficients with $B_E^I \cdot (C^I_{E,F})$ from (6), we see that they coincide. The proof then follows by induction. \hfill \Box

The following corollary follows immediately. We see in Section 8 that the Leibniz coefficients $C^I_{E,[k]}$ arise as factors in summands of specific structure constants in Schubert calculus, justifying the name. Here $[k] = \{1, 2, \ldots, k\}$.

**Corollary 3.8.** [*Generalized Billey’s Formula*] Let $I = (i_1, \ldots, i_k)$ be a sequence of indices of simple roots, and denote $m_j = s_i s_i \cdots s_{i_{j-1}}(a_i)$ and $n_j = s_i s_i \cdots s_{i_{j-1}}(b_i)$.

For $E \subset [k]$, we have

$$C^I_{E,[k]} = C^I_{E,[k]} = (-1)^{k-|E|} \prod_{j \in [k] \setminus E} m_j \prod_{j \in [k]} n_j^{-1}.$$ 

As a consequence of Theorem 3.7, [6, Proposition 9.5] and the coproduct defined in Equation (2), we obtain the following theorem.

**Theorem 3.9.** Let $Z_\alpha = a_\alpha + b_\alpha \delta_\alpha \in Q_W$ with $b_\alpha$ invertible, then for any $I = (i_1, \ldots, i_k)$, we have

$$\Delta(Z_I) = \sum_{E,F \subset [k]} C^I_{E,F} Z_E \otimes Z_F,$$

where $C^I_{E,F}$ are defined in Definition 3.5.

We specialize Theorem 3.7 to the elements $X_I$ and $Y_I$. For any index $j$, the operators $B_j^X$ and $B_j^Y$ preserve $S$ under the action of $Q_W$ on $Q$, and thus $B_j^X, B_j^Y \in D_E$ (see [6, Remark 7.8]). The first statement in the next corollary is the result [6, Proposition 9.5].

**Corollary 3.10.** For the Demazure elements $X_\alpha$, and $I = (i_1, \ldots, i_k)$, we have

$$X_I \cdot (pq) = \sum_{E,F \subset [k]} A^I_{E,F}(X_E \cdot p)(X_F \cdot q),$$

where $A^I_{E,F} = (B^X E \cdot \cdots B^X F \cdot 1)$ with $B^X E \in D_E$ defined in Example 3.6. Similarly, for the push-pull elements $Y_\alpha$, and $I = (i_1, \ldots, i_k)$, we have

$$Y_I \cdot (pq) = \sum_{E,F \subset [k]} B^I_{E,F}(Y_E \cdot p)(Y_F \cdot q),$$

where $B^I_{E,F} = (B^Y E \cdot \cdots B^Y F \cdot 1)$, and $B^Y E \in D_E$ is defined in Example 3.6.
4. The structure constants of equivariant oriented cohomology of flag varieties

In this section we prove the main result, i.e., the formulas of structure constants of $Z^*_w$ in $\mathfrak{h}^T(G/B)$, with resulting formulas for the structure constants of $X^*_w$ and of $Y^*_w$.

Let $\{Z^*_w\}$ be the basis of $Q_W$ (as a module over $Q$) dual to the basis $\{Z_w\}$ of $Q_W$ introduced in Section 3.

**Theorem 4.1.** For any $u, v \in W$, the product $Z^*_w Z^*_v$ is given by

$$Z^*_w Z^*_v = \sum_{w \in \mathcal{W}} c^I_{I_w, I_v} Z^*_w,$$

where

$$c^I_{I_w, I_v} = \sum_{E,F \in \Delta(w)} C^I_{E,F,I_w} C_{F,I_v} \in Q,$$

$C^I_{E,F,I_w} \in Q$ are the Leibniz coefficients given in Definition 3.5. As before, the $Q$ elements $c_{E,I_w}$ and $c_{F,I_v}$ are defined as constants appearing in the expansion

$$(9) \quad Z_f = \sum_{w \in W} c_{E,I_w} Z_w.$$

**Example 4.2.** Consider the $A_3$-case. Consider $I_w = (2,3,1,2,1), I_v = (1,2,3,2,1)$, then $c^I_{I_w, I_v} = 0$ unless $w = w_0$ is the longest element. Fix $I_{w_0} = (1,2,3,1,2,1)$, in which case we have

$$C^I_{I_{w_0}(2,3,4,5,6), (1,2,3,5,6)} = B_1^2 B_2^2 B_3^2 B_4^2 B_5^2 B_6^2 \cdot 1$$

and $c_{(2,3,4,5,6), I_w} = c_{(1,2,3,5,6), I_v} = 1$. Therefore,

$$Z^*_w \cdot Z^*_v = \frac{a_{\alpha_1} a_{\alpha_2}}{b_{\alpha_1} b_{\alpha_2 + \alpha_3}} Z^*_w.$$

**Proof of Theorem 4.1.** The coproduct structure $\Delta$ on $Q_W$ (Equation (2)) naturally induces a product on $Q_W$. For all $f, g \in Q_W$ and $\sum_{w \in W} q_w \delta_w \in Q_W$,

$$\langle f, g, \sum_{w \in W} q_w \delta_w \rangle = \langle f \otimes g, \Delta(\sum_{w \in W} q_w \delta_w) \rangle$$

$$= \langle f \otimes g, \sum_{w \in W} q_w \delta_w \otimes \delta_w \rangle$$

$$= \sum_{w \in W} q_w \langle f, \delta_w \rangle \langle g, \delta_w \rangle.$$

Note that this product corresponds to the product on $Q^*_W$ introduced at the beginning of Section 2.2 since

$$\langle f, g, \sum_{w \in W} q_w \delta_w \rangle = \begin{cases} \langle f, \sum_{w \in W} q_w \delta_w \rangle = \sum_{w} q_w \langle f, \delta_w \rangle, & \text{if } u = v; \\ 0, & \text{otherwise,} \end{cases}$$

$$= \begin{cases} q_u, & \text{if } u = v; \\ 0, & \text{otherwise.} \end{cases}$$
From Theorem 3.9 we have
\[
\Delta(Z_{I_w}) = \sum_{E,F \subseteq [\ell(w)]} C_{E,F}^{I_w} Z_E \otimes Z_F
\]
\[
= \sum_{E,F \subseteq [\ell(w)]} C_{E,F}^{I_w} \left[ \left( \sum_{u \in W} c_{E,I_u} Z_{I_u} \right) \otimes \left( \sum_{v \in W} c_{F,I_v} Z_{I_v} \right) \right]
\]
\[
= \sum_{u,v \in W} C_{E,F}^{I_w} c_{E,I_u} c_{F,I_v} Z_{I_u} \otimes Z_{I_v}
\]
\[
= \sum_{u,v \in W} c_{I_w,I_u,I_v} Z_{I_u} \otimes Z_{I_v}.
\]

Finally we obtain the coefficient by calculating the pairing:
\[
\langle Z_{I_w}^*, Z_{I_u}^*, Z_{I_v}^* \rangle = \langle Z_{I_w} \otimes Z_{I_v}^*, \Delta(Z_{I_w}) \rangle = c_{I_w,I_u,I_v}.
\]

Let \( I_w|E \) be the subsequence obtained from restricting \( I_w \) to \( E \). Since \( w = \prod I_w \geq \prod (I_w|E) \) for any \( E \subseteq [\ell(w)] \), by Lemma 3.3, \( c_{I_w,I_u,I_v} = 0 \) unless \( u \leq w \) and \( v \leq w \).

The coproduct structure on the left \( Q \)-module \( Q_W \) restricts to a coproduct structure on the left \( S \)-module \( D_F \) [6, Theorem 9.2]. Consequently, the embedding \( D_F \subset Q_W^* \) is an embedding of subrings. So the structure constants of the \( S \)-bases \( \{ X_{I_w}^* \} \) and \( \{ Y_{I_w}^* \} \) in \( D_F \) are precisely those of the \( Q \)-bases \( \{ X_{I_w}^* \} \) and \( \{ Y_{I_w}^* \} \) in \( Q_W^* \).

Specializing Theorem 4.1 to the \( X \)-operators, we have
\[
X_{I_u}^* X_{I_v}^* = \sum_{w \geq u, v \geq u} \partial_{I_u,I_v}^{I_w} X_{I_w}^*.
\]
with
\[
(10) \quad \partial_{I_u,I_v}^{I_w} = \sum_{E,F \subseteq [\ell(w)]} A_{E,F}^{I_w} c_{E,I_u} c_{F,I_v},
\]
where \( c_{I,I_u} \) are the coefficients that occur in the expansion \( X_I = \sum_v c_{I,I_u} X_{I_u} \). It follows from [6, Theorem 9.2 and Proposition 7.7] that \( A_{E,F}^{I_w} \in S \), that \( c_{I,I_u} \in S \), so \( \partial_{I_u,I_v}^{I_w} \in S \). Similarly, specializing to the \( Y \)-operators, the structure constants for \( Y_{I_w}^* \) are denoted by \( \partial_{I_u,I_v}^{I_w} \) and can be expressed as
\[
Y_{I_u}^* Y_{I_v}^* = \sum_{E,F \subseteq [\ell(w)]} B_{E,F}^{I_w} c_{E,I_u} c_{F,I_v},
\]
where now the coefficients \( c_{I,I_u} \) are those appearing in the expansion of \( Y_I \). As before, \( B_{E,F}^{I_w} \in S \) and \( c_{I,I_u} \in S \), so \( \partial_{I_u,I_v}^{I_w} \in S \). In \S 5 we show that these coefficients simplify in the case that \( F = F_m \) or \( F = F_m^* \), resulting in Theorem 1 from [11]. It is worth noting that the formula (10) can be used to prove the Leray-Hirsch Theorem for flag varieties (see [9]).

**Example 4.3.** Assume the root datum is of type \( A_1 \), then \( W = \{ e, s_1 \} \). We calculate the basis change explicitly:
\[
X_e^* = f_e + f_{s_1}, \quad X_{(1)}^* = -x_{\alpha_1} f_{s_1}.
\]
and then we may obtain the products directly:

\[X_e^* X_e^* = X_e^*, \quad X_e^* X_{(1)} = X_{(1)}^*, \quad X_{(1)}^* X_{(1)} = -x_\alpha X_{(1)}^*\]

and note that it agrees with Theorem 4.1 with \(Z = X\).

**Example 4.4.** Consider the root datum \(A_2\), with \(W = \{e, s_1, s_2, s_1s_2, s_2s_1, w_0\}\). For the longest element \(w_0\), we fix the reduced sequence \(I_{w_0} = s_1s_2s_1\).

We use the calculation in Example 2.5, and the product structure on \(Q_w^*\) to obtain the multiplication table for \(X_{I_{w_0}}^*\). Recall that \(f_u f_v = 1\) if \(u = v\) and 0 otherwise, and that \(X_e^* = f_e + s_1 + f_{s_2} + s_1 s_2 + s_2 s_1 + f_{w_0}\). If \(X_w = \sum_u a_u f_u\), we have

\[X_w^* X_e^* = \left( \sum_u a_u f_u \right) \left( \sum_v f_v \right) = \sum_u a_u f_u = X_w^*\]

for all \(w \in W\). Similarly,

\[X_{I_{w_0}}^* X_{(1,2)}^* = (-x_{\alpha_1} x_{\alpha_2} x_{\alpha_13} f_{w_0}) (x_{\alpha_1} x_{\alpha_13} (f_{s_1 s_2} + f_{w_0})) = -x_{\alpha_1} x_{\alpha_2} x_{\alpha_13}^2 f_{w_0}\]

\[= x_{\alpha_1} x_{\alpha_13} X_{I_{w_0}}^*.\]

The other products are as follows: Here \(y\) was defined in Example 2.5.

\[
\begin{align*}
X_{I_{w_0}}^* X_{I_{w_0}}^* &= -x_1 x_2 x_{\alpha_13} X_{I_{w_0}}^* \\
X_{(1)}^* X_{(2)}^* &= X_{(1,2)}^* + X_{(2,1)}^* + x_1 X_{I_{w_0}}^* \\
X_{(2)}^* X_{(2)}^* &= \frac{x_{\alpha_13} - x_2}{x_1 - x_2} X_{(1,2)}^* - x_2 X_{(2)}^* \\
X_{(2,1)}^* X_{(2)}^* &= \frac{x_{\alpha_13} - x_2}{x_1 - x_2} X_{I_{w_0}}^* - x_2 X_{(2,1)}^* \\
X_{(1)}^* X_{(1)}^* &= -x_1 X_{(1)}^* + \frac{x_{\alpha_13} - x_1}{x_1 - x_2} X_{(2,1)}^* + \frac{y^2 - y}{x_1 x_2 x_{\alpha_13}} X_{I_{w_0}}^* \\
X_{(2,1)}^* X_{(1)}^* &= x_1 x_{\alpha_13} X_{(1,2)}^* \\
X_{(2)}^* X_{(1)}^* &= x_2 x_{\alpha_13} X_{I_{w_0}}^* \\
X_{(1)}^* X_{(2)}^* &= -x_2 x_{\alpha_13} X_{I_{w_0}}^* \\
X_{(2,1)}^* X_{(1)}^* &= -x_2 x_{\alpha_13} X_{I_{w_0}}^* \\
X_{(1)}^* X_{(2)}^* &= -y X_{I_{w_0}}^* \\
X_{(2,1)}^* X_{(2)}^* &= -y X_{I_{w_0}}^*. \\
\end{align*}
\]

One can check that the above coefficients \(a_{I_{w_0}}^{I_v}\) agree with the formula (10). Note that when computing \(a_{I_{w_0}}^{I_v}\), one needs to compute the following coefficients:

\[
A_{I_{w_0}}^{I_{w_0}}^{(3),\{3\}}, A_{I_{w_0}}^{I_{w_0}}^{I_{w_0} \in \{3\},\{3\}}, A_{I_{w_0}}^{I_{w_0}}^{\{3\},\{3\}}, A_{I_{w_0}}^{I_{w_0}}^{\{13\},\{13\}}. 
\]

As an application, we consider the case of a partial flag variety. Let \(K\) be a subset of \([n]\). Let \(P_K\) be the standard parabolic subgroup, \(W_K < W\) the corresponding subgroup, and \(W_{\mathbb{K}} \subseteq W\) be the set of minimal length representatives of \(W/W_K\). We say a set of reduced sequences \(I_w\) is \(K\)-compatible if for each \(w = w', u \in W_K, v \in W_K\), we have \(I_w = I_u \cup I_v\), i.e., \(I_w\) is the concatenation of \(I_u\) with \(I_v\).

**Theorem 4.5.** Suppose the set \(\{I_w\}\) is \(K\)-compatible. Then for any \(v, u \in W^K\), we have

\[X_{I_u}^* X_{I_v}^* = \sum_{w \in W^K, w \geq v} a_{I_{w}}^{I_{w}} X_{I_{w}}^*.\]

**Proof.** It follows from [7, Corollary 8.4] that \(X_{I_u}^*, u \in W^K\) is a basis of \((Q_W^*)^{W_K}\). Moreover, from Lemma 4.3 of loc.it., we know \(\delta_w \bullet (ff') = (\delta_w \bullet f)(\delta_w \bullet f')\). Therefore, \(X_{I_u}^* X_{I_v}^* \in (Q_W^*)^{W_K}\), so is a linear combination of \(X_{I_{w}}^*, w \in W^K\). \(\square\)
Geometrically, under the assumption of this theorem, it follows from [7, Corollary 8.4] that \( \{X^*_w\}_{w \in W^K} \) is a basis of \((D^*_F)^{W^K} \cong H_T(G/P_K)\). So the product \( X^*_u X^*_v, u, v \in W^K \) is a linear combination of \( X^*_w, w \in W^K \).

**Corollary 4.6.** Let \( F = F_a \) or \( F_m \), and suppose \( u \in W \) satisfies that \( u \in W^K \) for some \( K \) and \( u \) is the longest element in \( W^K \). Then for any \( v \in W^K \), \( a^w_{u,v} = 0 \) for any \( w \in W \), unless \( w = u \).

**Proof.** In these cases, the braid relations are satisfied, so the structure constants do not depend on the choice of reduced sequences. In other words, fixing \( u \) and \( K \), we can assume we have chosen \( K \)-compatible reduced sequences. Then Theorem 4.5 applies, which implies that for any \( v \in W^K, w \in W \), we have \( c^I_{w,v} = 0 \) unless \( w \in W^K \) and \( w \geq u \). Since \( u \) is maximal in \( W^K \), so \( w = u \). \( \square \)

5. **Structure constants in singular cohomology and K-theory**

We restrict our attention to \( H^*_F(G/B) \) and \( K_T(G/B) \) to recover formulas in [11] of structure constants of Schubert classes for singular cohomology \((F = F_a)\) and K-theory \((F = F_m)\). We first simplify the coefficients \( c^X_{I, I_w} \) and \( c^Y_{I, I_w} \) in these two cases. Recall that, when the formal group law is \( F = F_a \) or \( F = F_m \), the braid relations are satisfied for \( Z_\alpha = X_\alpha \) and \( Z_\alpha = Y_\alpha \). We consider the equivariant oriented cohomology together with either the additive or multiplicative formal group law, and restrict the coefficient ring to \( S^n \) or \( S^m \).

**Lemma 5.1.** Let \( J \) be a word in the Weyl group. As in Lemma 3.3, define coefficients \( c_{J, I_w} \) by

\[
Z_J = \sum_{w \in W} c_{J, I_w} Z_{I_w}.
\]

(1) Let \( F = F_a \). If \( Z_\alpha = X_\alpha \) or \( Z_\alpha = Y_\alpha \), then

\[
c_{J, I_w} = \begin{cases} 
1, & \text{if } J \text{ is a reduced word for } w; \\
0, & \text{else.}
\end{cases}
\]

(2) Let \( F = F_m \). If \( Z_\alpha = X_\alpha \) or \( Z_\alpha = Y_\alpha \), then

\[
c_{J, I_w} = \begin{cases} 
1, & \text{if } w = \prod J; \\
0, & \text{else.}
\end{cases}
\]

**Proof.** When \( F = F_a \) or \( F = F_m \), it is well-known that the braid relations are satisfied. We write \( c_{J, w} \) for the coefficient \( c_{J, I_w} \). When \( F = F_a \), \( Z_\alpha^2 = 0 \), so if \( J \) is not reduced, \( Z_J = 0 \). If \( J \) is reduced and \( \prod J = w \), then \( Z_J = Z_w \), so \( c_{J, w} = 1 \) and \( c_{J, v} = 0 \) for \( v \neq w \).

When \( F = F_m \), we have \( Z_\alpha^2 = Z_\alpha \) and thus \( Z_J = Z_w \) where \( w := \prod J \). It follows that \( c_{J, w} = 1 \) and \( c_{J, v} = 0 \) for \( v \neq w \). \( \square \)

**Example 5.2.** For \( H^*(G/B) \) and \( F = F_a \), as described in Example 2.7 and Proposition 2.7, the element \( c^X_w \) in \( D^*_F \) corresponds under a natural isomorphism

\[
D^*_F \rightarrow H_T(G/B)
\]

to the equivariant cohomology class Poincaré dual to \( [X(w)] \), where \([X(w)]\) is the homology class of the Schubert variety. Furthermore, the first Chern classes of the corresponding line bundles are \( x_\alpha = \alpha \) for all simple roots \( \alpha \).
For each \( w \in W \), fix a reduced sequence \( I_w \). From the specialization of Theorem 4.1, we have defining relations
\[
Y_u^* Y_v^* = \sum_{w \geq u, w \geq v} B^I_{u,w} Y_w^*
\]
for \( B^I_{u,w} \). Then
\[
B^I_{u,w} = \sum_{E,F \subset [\ell(w)]} B^I_{E,F} c^Y_{E,F,u} c^Y_{E,F,w} \text{ by Theorem 4.1},
\]
\[
= \sum_{E,F \text{ reduced for } u,v} B^I_{E,F}, \text{ by Lemma 5.1(1)}
\]
where the second sum is over \( E,F \) whose corresponding products of reflections are reduced and equal to \( u,v \) respectively. Recall that
\[
B^I_{E,F} = (B^Y_{1} B^Y_{2} \cdots B^Y_{\ell(w)}) \cdot 1,
\]
with
\[
B^Y_{j} = \begin{cases} 
 x_{\beta_j} \delta_{\beta_j}, & \text{if } j \in E \cap F, \\
 \delta_{\beta_j}, & \text{if } j \in E \text{ or } F, \text{ but not both,} \\
 Y_{\beta_j}, & \text{if } j \notin E \cup F.
\end{cases}
\]
with \( \beta_j = \alpha_{i_j} \).

The coefficients \( B^I_{u,w} \) coincide with the structure constants \( c^w_{u,v} \) in [11, Theorem 1]. Note that in this case, \( Y_w^* \), so \( c^w_{u,v} = (-1)^{\ell(w)} c_{u,v} \), and thus \( X_w^* = (-1)^{\ell(w)} Y_w^* \).

Therefore,
\[
a^I_{u,w} = (-1)^{\ell(w) + \ell(u) + \ell(v)} B^I_{u,w}.
\]

Example 5.3. For \( K_T(G/B) \) (and \( F = F_m \)), we have \( x = 1 - e^\alpha \). The action of \( X_{\alpha} \) (resp. \( Y_{\alpha} \)) on \( K_T(pt) \) corresponds to the action of the ordinary (resp. isobaric) Demazure operator in [11].

Fixing a reduced sequence \( I_w \) for each \( w \), we have
\[
X_u^* X_v^* = \sum_{w \geq u, w \geq v} a^I_{u,w} X_w^* = \sum_{w \geq v, w \geq u} \sum_{E,F} A^I_{E,F} Y_w^*
\]
where by Lemma 5.1(2), the second sum is over all \( E,F \subset [\ell(w)] \) such that \( \prod E = u \) and \( \prod F = v \). Here, we have
\[
B^X_{j} = \begin{cases} 
 -(1 - e^{-\beta_j}) \delta_{\beta_j}, & \text{if } j \in E \cap F, \\
 \delta_{\beta_j}, & \text{if } j \in E \text{ or } F, \text{ but not both,} \\
 X_{\beta_j}, & \text{if } j \notin E \cup F,
\end{cases}
\]
where \( \beta_j = \alpha_{i_j} \).

The classes \( \{\xi_w : w \in W\} \) in [11] are defined as the dual basis to \( [\mathcal{O}_{X(w)}(-\partial X(w))] \) under the pairing obtained by taking the equivariant cap product and pushing forward to a point. Each \( \xi_w \) coincides with the Poincaré dual class to \( [\mathcal{O}_{Y(w)}] \). In Example 2.7 we note that \( X_w^* = (-1)^{\ell(w)} [\mathcal{O}_{Y(w)}] \), and thus \( \xi_w = (-1)^{\ell(w)} X_w^* \).

Therefore,
\[
\xi_u \xi_v = (-1)^{\ell(u) + \ell(v)} \sum_{w \geq u,v} a^I_{u,w} \xi_w.
\]
It follows that the coefficients \( (-1)^{\ell(u) + \ell(v) + \ell(w)} a^I_{u,w} \) coincide with \( a^w_{uv} \) in [11], as is clear from the formula.
Let \( \xi_w = w \in W \) defined in [11] satisfy \( \xi_w = Y_w^* \), a similar argument implies that \( \xi_{w, h}^* \) coincide with the structure constants \( \hat{a}_{u, v} \) defined in [11].

**Example 5.4.** Let \( F = F_4 \). Consider the \( A_2 \) case. If \( I_w = s_1 s_2 s_1, u = s_1, v = s_1 s_2 \), then

\[
\delta_{s_1 s_2 s_1}^{s_1 s_2 s_1} = B_{\{1\}, \{1\}, \{2\}} + B_{\{3\}, \{1\}, \{2\}} = \left( \begin{array}{ccc}
\alpha_1 & \delta_1 & Y_1 \\
\delta_1 & \delta_2 & \delta_1 \\
Y_1 & \delta_1 & \delta_1
\end{array} \right) \cdot 1 = 0 + 1 = 1.
\]

Similarly,

\[
\delta_{s_1 s_2 s_1}^{s_1 s_2 s_1} = B_{\{1\}, \{2\}, \{3\}} + B_{\{3\}, \{2\}, \{3\}} = \left( \begin{array}{ccc}
\delta_1 & \delta_2 & \delta_1 \\
\alpha_2 & \delta_2 & \delta_1 \\
\alpha_2 & \delta_2 & \delta_1
\end{array} \right) \cdot 1 = 1 - 1 = 0.
\]

For the \( A_3 \) case, one can also compute

\[
\delta_{s_2 s_1 s_2 s_1}^{s_2 s_1 s_2 s_1} = B_{\{2\}, \{3\}, \{2\}, \{3\}} + B_{\{2\}, \{3\}, \{1\}, \{2\}} + B_{\{2\}, \{3\}, \{2\}, \{1\}} = \left( \begin{array}{ccc}
\delta_1 & \alpha_2 & \delta_1 \\
\delta_3 & \delta_3 & \delta_1 \\
\delta_3 & \delta_3 & \alpha_2
\end{array} \right) \cdot 1
\]

\[
= (\alpha_1 + \alpha_2 + \alpha_3).
\]

**Example 5.5.** Let \( F = F_4 \). Consider the \( A_3 \) case, with \( I_w = s_1 s_2 s_3 s_1 s_2, u = s_2 s_3 s_2, v = s_1 s_2 s_1 \). We have

\[
\delta_{s_2 s_3 s_2 s_1}^{s_2 s_3 s_2 s_1} = A_{\{2\}, \{3\}, \{2\}, \{1\}} + A_{\{2\}, \{3\}, \{1\}, \{2\}} + A_{\{2\}, \{3\}, \{2\}, \{1\}} = \left( \begin{array}{ccc}
\delta_1 & -x_2 & \delta_1 \\
x_1 & \delta_2 & \delta_1 \\
-\delta_1 & -x_2 & \delta_1
\end{array} \right) \cdot 1
\]

\[
= -x_1 + x_1 + x_2 + x_2 + x_1 + x_1 = x_2 - x_1 + 2x_2 + x_1
\]

\[
= x_2 - x_1 + 2x_2 + x_1
\]

6. **Structure constants of cohomological stable bases**

In this section, we let \( F = F_4 \) and \( R = R^a = \mathbb{Z}[h] \). We recall the definition of the cohomological stable basis of Maulik-Okounkov, and generalize Su’s formula of structure constants for Segre-Schwartz-MacPherson classes (Theorem 6.3). We use the twisted group algebra language for singular cohomology, whose \( K \)-theory version was given in [22]. As the framework and proofs are very similar to earlier sections, we will only review essential properties. Some of the notation introduced below is restricted to this section only.

Let \( R^a = \mathbb{Z}[h] \), \( S^a = \text{Sym}_{R^a}(A) \) and \( Q^a = \text{Frac}(S^a) \). Define

\[
Q^a_W = Q^a \times_{R^a} R^a[W]
\]

with \( Q^a \)-basis \( \delta_w, w \in W \). For simplicity we introduce the following notation:

\[
\hat{\alpha} = h - \alpha, \quad \alpha_{\alpha'\alpha} = \prod_{\alpha > 0} \alpha, \quad \hat{\alpha}_{\alpha'\alpha} = \prod_{\alpha > 0} (h - \alpha).
\]

Finally, for any simple root \( \alpha \), define an operator associated to this root by

\[
T_{\alpha} = -h \frac{1}{\alpha} (1 - \delta_\alpha) - \delta_\alpha = -h + \hat{\alpha} \delta_\alpha \in Q^a_W.
\]
By direct computation, the set \( \{ T_\alpha \}_{\alpha \in \{ \alpha_1, \ldots, \alpha_n \}} \) satisfies the braid relations, and \( T_\alpha^2 = 1 \). Indeed, the algebra generated by \( \{ T_\alpha \} \) is called the degenerate (or graded) Hecke algebra. Note that \( T_\alpha \) is a special case of \( Z_\alpha \), occurring over \( R = R^a \).

For any sequence \( I = (i_1, \ldots, i_\ell) \) (not necessarily reduced), we define the Demazure-Lusztig operator

\[
T_I = T_{\alpha_{i_1}} \cdots T_{\alpha_{i_\ell}}
\]

in cohomology to be the product of the operators indicated in the list \( I \). It follows from the relations that, if \( I \) and \( I' \) are two sequences with \( w := \prod I = \prod I' \), then \( T_I = T_I' \), and we denote it \( T_w \). The set \( \{ T_w \mid w \in W \} \) is a basis of \( Q_W^a \).

Let \( (Q_W^a)^* \) be the \( Q^a \)-dual of \( Q_W^a \), and let \( \{ T_w^* \} \subseteq (Q_W^a)^* \) be the dual basis. Denote the basis of \( (Q_W^a)^* \) dual to \( \{ \delta_w \in Q_W^a \} \) by \( \{ f_w \} \), as in §2. The identity of the ring \( (Q_W^a)^* \) is denoted by \( 1 = \sum_{w \in W} f_w \). The ring \( Q_W^a \) acts on \( (Q_W^a)^* \) via the

\[
\bullet \cdot (z \bullet q^* , z') = (q^* , z' z) \quad \text{for } z , z' \in Q_W^a , q^* \in (Q_W^a)^* .
\]

It induces a \( W \)-action on \( (Q_W^a)^* \) via the embedding \( W \subseteq Q_W^a \). Let \( ((Q_W^a)^*)^W \) denote the Weyl-invariant subgroup of \( (Q_W^a)^* \).

In this section only, denote by \( \hat{Y} \in Q_W^a \) the element

\[
\sum_{w \in W} \delta_w \frac{1}{\alpha \cdot \alpha_{w_0}} = \sum_{w \in W} \delta_w \prod_{\alpha > 0} (h - \alpha) .
\]

The map \( \hat{Y} \bullet : (Q_W^a)^* \to ((Q_W^a)^*)^W = Q^a \cdot 1 \) is the algebraic analogue of the composition of the map

\[
Q^a \otimes_{S^a} H_G^* \to Q^a \otimes_{S^a} H_B^* \to Q^a \otimes_{S^a} H_{\mathfrak{T} \times \mathfrak{C}}^* \to Q^a \otimes_{S^a} H_{\mathfrak{T} \times \mathfrak{C}}^* ,
\]

where the last map is the equivariant pushforward of cohomology class \( G/B \) to a point on the second term. The proofs in [7, Lemma 7.1] and [22, Lemma 5.1] easily extend to show that, for any \( f, g \in (Q_W^a)^* \),

\[
\hat{Y} \bullet ((T_\alpha \bullet f) \cdot g) = \hat{Y} \bullet (f \cdot (T_\alpha \bullet g)) .
\]

**Definition 6.1.** We define two bases of \( (Q_W^a)^* \) as a module over \( Q^a \). Let

\[
\begin{align*}
\text{stab}_w^+ &= T_{w^{-1}} \bullet (\alpha_{w_0} f_w) , \\
\text{stab}_w^- &= (-1)^{\ell(w_0)} T_{w^{-1} w_0} \bullet (\alpha_{w_0} f_{w_0}) .
\end{align*}
\]

Then \( \{ \text{stab}_w^+ : w \in W \} \) and \( \{ \text{stab}_w^- : w \in W \} \) each form a basis for \( (Q_W^a)^* \) as a module over \( Q^a \). We call these bases the **cohomological stable bases**. See [20] for more details.

It is immediate from the definition that \( \text{stab}_w^+ \) has support on \( \{ f_v : v \leq w \} \) and \( \text{stab}_w^- \) has support on \( \{ f_v : v \geq w \} \).

The following lemma is the analogue of Theorem 5.7 and Lemma 5.6 in [22]. The first identity was due to Maulik-Okounkov originally.

**Lemma 6.2.** We have

\[
\hat{Y} \bullet [\text{stab}_w^+ \cdot \text{stab}_w^-] = (-1)^{\ell(w_0)} \delta_{w_0} 1 , \quad \hat{Y} \bullet [\text{stab}_w^+ \cdot \alpha_{w_0} T_w^*] = \delta_{w_0} 1 .
\]

Define structure constants \( \ell_{u,v} \in Q^a \) by the equation

\[
\text{stab}_w^- \cdot \text{stab}_v^- = \sum_{w \in W} \ell_{u,v} \text{stab}_u^-.
\]
We now present the main result about the stable basis \( \{\text{stab}^-\} \).

**Theorem 6.3.** The classes \( \text{stab}^- \) and the coefficients \( \xi^w_{u,v} \) satisfy the following properties:

1. We have \( \text{stab}^- = (-1)^{f(wo)} \hat{\alpha}_{wo} T^*_u \).

2. For each \( w \in W \), fix a reduced sequence \( I_w \). Then

\[
\xi^w_{u,v} = \sum_{E,F \subset \{I(w)\}} \widehat{c}^2_{w0} t^I_{E,F},
\]

where \( t^I_{E,F} = (B^I_1 B^I_2 \cdots B^I_k) \cdot 1 \) with

\[
B^I_j = \begin{cases} 
\frac{\alpha_{ij} h}{\alpha_{ij}}, & \text{if } j \in E \cap F, \\
\frac{\alpha_{ij} h}{\alpha_{ij}} + \frac{\alpha_{ij}^2 h^2}{\alpha_{ij}^2}, & \text{if } j \notin E \cup F, \\
\alpha_{ij} \frac{h}{\alpha_{ij}} + \frac{h^2}{\alpha_{ij}^2} \delta_{ij}, & \text{if } j \in E \text{ and not both.}
\end{cases}
\]

**Proof.** (1). This follows from Lemma 6.2 above.

(2). For each \( w \in W \), fix a reduced decomposition. We have

\[
\text{stab}^- \cdot \text{stab}^- = (-1)^{f(wo)} \hat{\alpha}_{wo} T^*_u \cdot (-1)^{f(wo)} \hat{\alpha}_{wo} T^*_v = \widehat{c}^2_{w0} T^*_u \cdot T^*_v.
\]

Therefore, it suffices to consider the structure constants for \( T^*_u \). But the elements \( T_u \) are an instantiation of \( Z^* \) with the coefficient ring \( R^* \), with \( a_{ij} = -h/\alpha_{ij} \) and \( b_{ij} = \hat{\alpha}_{ij}/\alpha_{ij} \). Thus Theorem 4.1 indicates how to multiply the corresponding dual elements, resulting in \( B^I_j \) defined as above. \( \square \)

When \( h = -1 \), the Demazure Lusztig operator \( T_\alpha \) specializes to the operator considered by Su in [21], allowing us to recover his formula for the structure constants from Theorem 6.3.

**Example 6.4.** Consider the \( A_2 \)-case. If \( I_w = s_1 s_2 s_1, u = v = s_1 \), then

\[
\xi^{s_1 s_2 s_1}_{s_1, s_1} = \hat{\alpha}_1 \hat{\alpha}_2 \hat{\alpha}_3 (t^{121}_{(1),(1)} + t^{121}_{(3),(1)} + t^{121}_{(1),(1)} + t^{121}_{(3),(3)}) \cdot 1
\]

\[
= \hat{\alpha}_1 \hat{\alpha}_2 \hat{\alpha}_3 \begin{pmatrix} \frac{h}{\alpha_1} \delta_1 & -\frac{h}{\alpha_2} + \frac{h^2}{\alpha_2^2} \delta_2 & \frac{h}{\alpha_1} \delta_1 \\ \frac{h}{\alpha_2} \delta_1 & -\frac{h}{\alpha_2} + \frac{h^2}{\alpha_2^2} \delta_2 & \frac{h}{\alpha_2} \delta_1 \\ -\frac{h}{\alpha_1} + \frac{h^2}{\alpha_1^2} \delta_1 & -\frac{h}{\alpha_2} + \frac{h^2}{\alpha_2^2} \delta_2 & -\frac{h}{\alpha_1} + \frac{h^2}{\alpha_1^2} \delta_1 \end{pmatrix} \cdot 1 = h^2(h + \alpha_1).
\]

If \( I_w = s_1 s_2 s_1, u = s_1, v = s_1 s_2 \), then

\[
\xi^{s_1 s_2 s_1}_{s_1, s_1} = \hat{\alpha}_1 \hat{\alpha}_2 \hat{\alpha}_3 (t^{121}_{(1),(1)} + t^{121}_{(3),(1,2)}) \cdot 1 = h^2(h + \alpha_1).
\]

Similarly, for \( v' = s_2 s_1 \), we have

\[
\xi^{s_1 s_2 s_1}_{s_1, s_2 s_1} = \hat{\alpha}_1 \hat{\alpha}_2 \hat{\alpha}_3 (t^{121}_{(1),(2,3)} + t^{121}_{(3),(2,3)}) \cdot 1 = h^2(h + \alpha_1).
\]
Consider the Example 6.5. Consider the $A_3$ case. For $I_w = s_1 s_2 s_3 s_4 s_2$, $u = s_2 s_3 s_2$, $v = s_1 s_2 s_1$, with $\alpha_{ij} = \alpha_i + \cdots + \alpha_{j-1}$ for $1 \leq i < j \leq 4$, we have

$$t_{s_1 s_2 s_3 s_4 s_2} = \hat{t}_{u_0} (s_{1,2,3,5}, \{1,2,4\} + t_{s_2 s_3 s_2, \{2,3,5\}, \{2,4,5\}})$$

$$= \hat{a}_{u_0} \left( -\frac{h}{\alpha_1} \delta_1 + \frac{\alpha_2 \delta_2}{\alpha_2} \alpha_2 \delta_2 \frac{h}{\alpha_3} \delta_3 \frac{h}{\alpha_4} \delta_1 \frac{h}{\alpha_2} \delta_2 \right) \cdot 1$$

$$= \hat{a}_{u_0} \left( \begin{array}{c}
\hat{a}_{12312} = \hat{a}_{u_0} (t_{12312}, \{1\} + t_{12312}, \{2,3,5\}, \{4\}) \\
\hat{a}_{12312} = \hat{a}_{u_2} (t_{12312}, \{2\} + t_{12312}, \{2,3,5\}, \{5\}) \\
\end{array} \right)$$

$$= \hat{a}_{u_0} (\alpha_2 + 2\delta_3).$$

Remark 6.6. In [21, Theorem 1.1], the authors find a formula for the structure constants of $\sigma^*_w \in (Q^m_W)^*$, where

$$\sigma_i = \frac{1 + \alpha_i}{\alpha_i} \delta_i - \frac{1}{\alpha_i} \in Q^m_W.$$ 

This is equal to our $-T_0$ with $\hbar = -1$.

7. Structure Constants for K-theoretic Stable Bases

In this section, we give a formula of the structure constants of the K-theory stable basis. Similar to our strategy in §6, we use the twisted group algebra method. This method was introduced by Su, Zhao and the second author in [22]; we only recall the definitions below. Here we use $F = F_m$ and $R = R^m = \mathbb{Z}[q^{1/2}, q^{-1/2}]$.

Let $S^m = R^m[\Lambda]$. We use the following notation in this section:

$$x_{\pm, 1} = 1 - e^\mp \alpha, \quad \hat{x}_{\alpha} = 1 - q e^{-\alpha}, \quad \hat{x}_w = \prod_{\alpha > 0, w^{-1} \alpha < 0} \hat{x}_\alpha, \quad q_w = q^{\ell(w)}.$$ 

Let $Q^m = \text{Frac}(S^m)$ and apply the twisted group algebra construction to obtain the module

$$Q^m_W = Q^m \times_{R^m} R^m[W].$$

Define the operator $\tau^-_\alpha$ by

$$\tau^-_\alpha = \frac{q - 1}{1 - e^{\alpha}} + \frac{1 - q e^{-\alpha}}{1 - e^{\alpha}} \delta_\alpha \in Q^m_W.$$ 

Observe that $\tau^-_\alpha$ is a special case of $Z_\alpha$ when $Q = Q^m$.

A simple calculation shows that $(\tau^-_\alpha)^2 = (q - 1) \tau^-_\alpha + q$, and that $\{\tau^-_\alpha\}$ satisfies the braid relations. It follows that the K-theoretic Demazure-Lusztig operator $\tau^-_w$, given by the product

$$\tau^-_w = \tau^-_{\alpha_1} \tau^-_{\alpha_2} \cdots \tau^-_{\alpha_i},$$

is independent of choice of reduced word $s_{i_1} s_{i_2} \cdots s_{i_t}$ for $w$. The set $\{\tau^-_w, w \in \Lambda\}$ is a $Q^m$-basis of $Q^m_W$. 
For each not-necessarily reduced sequence $I = (i_1, \ldots, i_k)$, let $\tau_I^-$ be the concatenation $\tau_{\alpha_{i_1}}^- \cdots \tau_{\alpha_{i_k}}^-$, and define the structure constants $c_{I,w}^- \in \mathbb{R}^m$ by the equations

$$
\tau_I^- = \sum_{w \in W} c_{I,w}^- \tau_w^-.
$$

**Lemma 7.1.** The coefficients $c_{I,w}^- \in \mathbb{R}^m$ in (11) satisfy the following:

1. For all $w \in W$ and sequences $I$, $c_{I,w}^- = 0$ unless $w \leq \prod I$.
2. If $I$ is reduced, then

$$
c_{I,w}^- = \begin{cases} 
0 & \text{if } w \neq \prod I \\
1 & \text{if } w = \prod I.
\end{cases}
$$

**Proof.** Statement (1) follows from the quadratic relation $(\tau_{\alpha_i}^-)^2 = (q - 1)\tau_{\alpha_i}^- + q$.

Statement (2) follows from the braid relations satisfied by the $\tau_{\alpha_i}^-$. \hfill \square

The analogous statement to Theorem 3.7 is the following proposition.

**Proposition 7.2 (K-Stable Leibniz Rule).** If $I = (i_1, \ldots, i_k)$, we have

$$
\tau_I^- \cdot (pq) = \sum_{E,F \subseteq [k]} P_{E,F}^I(\tau_{I|E}^- \cdot p)(\tau_{I|F}^- \cdot q), \quad p, q \in Q.
$$

where $P_{E,F}^I = (B_1^- B_2^- \cdots B_k^-) \cdot 1$ with $B_j^- \in Q_W^m$ defined by

$$
B_j^- = \begin{cases} 
\frac{1 - e^{-\alpha_j}}{1 - qe^{-\alpha_j}} \delta_j, & \text{if } j \in E \cap F, \\
\frac{1 - qe^{-\alpha_j}}{1 - qe^{-\alpha_j}} \delta_j, & \text{if } j \in E \text{ or } F, \text{ but not both,} \\
\frac{q - 1}{1 - qe^{-\alpha_j}} \tau_{\alpha_j} \delta_j, & \text{if } j \not\in E \cup F.
\end{cases}
$$

Similar to §6, we take the dual $(Q_W^m)^*$, and $Q_W^m$ acts on $(Q_W^m)^*$ via the $\cdot$-action. Indeed, we have

$$(Q_W^m)^* \cong Q^m \otimes_{S_m} K_{C^* \times T}(G/B) \cong Q^m \otimes_{S_m} K_{C^* \times T}(T^*G/B).$$

**Definition 7.3.** [22, Definition 5.3, Theorem 5.4] The K-theoretic stable basis elements are defined by

$$
\text{stab}_{w}^-= q_{w_0}q_{w}^{-1/2}(\tau_{w_0, w})^{-1} \cdot (\prod_{\alpha > 0} (1 - e^\alpha) f_{w_0}) \in (Q_W^m)^*.
$$

Moreover, by [22, Theorem 5.4, Theorem 6.5], we have

$$
\text{stab}_{w}^- = q_{w}^{1/2} q_{w_0} (\tau_{w_0, w})^-.
$$

The following theorem gives a formula for the structure constants of the K-theory stable basis:

**Theorem 7.4.** Let $\{\text{stab}_{w}^- | w \in W\}$ denote the K-theory stable basis of $(Q_W^m)^*$. Define coefficients $p_{u,v}^w \in Q^m$ by the equation

$$
\text{stab}_u^- \cdot \text{stab}_v^- = \sum_{w \geq u, v} p_{u,v}^w \text{stab}_w^-.
$$

Then

$$
p_{u,v}^w = q_{w_0}^{\frac{1}{2}(|\ell(u) + \ell(v) - \ell(w)|)} q_{w_0} \sum_{E,F} \sum_{E,F} c_{I,w}^- c_{I,w}^- c_{I,w}^-.
$$
where the sum is over all $E,F \subset [\ell(w)]$ such that $\prod (I_w|_E) \geq u$ and $\prod (I_w|_F) \geq v$, and coefficients $c_{I_w|_F,v}^u$ are given in Lemma 7.1

Proof. The proof follows a similar argument as that of Theorem 6.3.

\[ \square \]

Remark 7.5. Due to the quadratic relation $(\tau_\alpha^-)^2 = (q-1)\tau_\alpha^- + q$, it is difficult to express the sum in terms of formulas in Section 5 and Section 6. Indeed, this is also the reason why it is difficult to express the restriction formula of $\text{stab}_w$ in $[22]$ in terms of an AJ-S-Billey-Graham-Willems type formula.

8. THE RESTRICTION FORMULA

In this section we relate the structure constants of $Z_\alpha^w$ with its restriction coefficients. This generalizes such relations in cohomology and K-theory due to Kostant and Kumar in [13, Proposition 4.32] and [14, Lemma 2.25].

Let $Z_\alpha^w$ be given in Definition 3.5. Following Lemma 2.3, we obtain coefficients $b_{u,I_w}^Z \in \mathbb{Q}$ using the defining relations

$$\delta_u = \sum_{w \in W} b_{u,I_w}^Z Z_{I_w},$$

Then $Z_{w}^* = \sum_{u} b_{u,I_w}^Z f_u$, i.e., $Z_{w}^*(\delta_u) = b_{u,I_w}^Z$. We call $b_{u,I_w}^Z$ the restriction coefficients of $Z_{w}^*$.

**Theorem 8.1.** For any $w \in W$, define the matrix $p_w^Z$ with $p_w^Z(u,v) = \epsilon_{I_w,I_u}^I$, the matrix $b^Z$ with $b^Z(u,v) = b_{u,I_w}^Z$, and the matrix $b^Z$ with $b_w^Z(u,v) = \delta_w b_{u,I_w}^Z$. Then

$$p_w^Z = b^Z \cdot b_w^Z \cdot (b^Z)^{-1}.$$

Proof. We have

$$ (p_w^Z \cdot b^Z)(u,v) = \sum_{z \in W} p_w^Z(u,z)b^Z(z,v) = \sum_{z \in W} \epsilon_{I_w,I_u}^I b_{u,z}^Z Z_{I_u}^* Z_{I_z}^*(\delta_u) = Z_{I_w}^*(\delta_u) \cdot Z_{I_u}^*(\delta_u) = b_{u,I_w}^Z Z_{I_u}^*(\delta_u) = \sum_{z \in W} b_w^Z(u,z)\delta_z v b_{z,I_w}^Z = \sum_{z \in W} b^Z(u,z) b_w^Z(z,v) = (b^Z \cdot b_w^Z)(u,v).$$

\[ \square \]

**Corollary 8.2.** For any $v,w \in W$, we have

$$\epsilon_{I_w,I_v}^I = b_{v,I_w}^Z.$$

In particular, $\epsilon_{I_w,I_v}^I$ does not depend on the choice of $I_v$.

Proof. Denote $Z_{I_v} = \sum_{u \leq v} a_{I_v,u}^Z \delta_u$. Then the matrix $a^Z$ with $a^Z(u,v) = a_{I_v,u}^Z$ is the inverse of $b^Z$. Theorem 8.1 implies that

$$\epsilon_{I_w,I_v}^I = p_w^Z(v,v) = \sum_{z_1,z_2 \in W} b^Z(v,z_1) b_{w}^Z(z_1,z_2) a^Z(z_2,v)$$
can also be proved with the following argument: since

\[ Z_{I_w}^*(\delta_u) = b_{u,I_w}^Z = 0 \] unless \( u \geq w \),

\[ (Z_{I_u}^* \cdot Z_{I_v}^*)^*(\delta_u) = \left( \sum_{u \geq w, w \geq v} c_{I_u,I_v}^W Z_{I_u}^*(\delta_u) \right) = c_{I_u,I_v}^W b_{u,I_v}^Z \]

On the other hand,

\[ (Z_{I_u}^* \cdot Z_{I_v}^*)^*(\delta_u) = Z_{I_u}^*(\delta_u) Z_{I_v}^*(\delta_u) = b_{u,I_u}^Z b_{v,I_v}^Z. \]

Therefore, \( c_{I_u,I_v}^W = b_{u,I_v}^Z \).

\begin{remark}
Corollary 8.2 can also be proved with the following argument: since

\[ Z_{I_w}^*(\delta_u) = b_{u,I_w}^Z = 0 \] unless \( u \geq w \),

\[ (Z_{I_u}^* \cdot Z_{I_v}^*)^*(\delta_u) = \left( \sum_{u \geq w, w \geq v} c_{I_u,I_v}^W Z_{I_u}^*(\delta_u) \right) = c_{I_u,I_v}^W b_{u,I_v}^Z \]

On the other hand,

\[ (Z_{I_u}^* \cdot Z_{I_v}^*)^*(\delta_u) = Z_{I_u}^*(\delta_u) Z_{I_v}^*(\delta_u) = b_{u,I_u}^Z b_{v,I_v}^Z. \]

Therefore, \( c_{I_u,I_v}^W = b_{u,I_v}^Z \).
\end{remark}

\begin{remark}
As mentioned in [11], specializing Corollary 8.2 and Examples 5.2 and 5.3 to singular cohomology or K-theory, and \( Z_\alpha \) to the \( X_\alpha \) and \( Y_\alpha \)-operators, one recovers the AJS/Billey formula and Graham-Willems formula of restriction coefficients of Schubert classes, which are obtained by using root polynomials.
\end{remark}

\begin{example}
Consider the \( A_2 \)-case with \( w = s_1, v = s_1s_2s_1 \). We compute \( b_{v,w}^X = X_{I_w}^*(\delta_v) \). For \( A_{1,3}^I \), we only need to consider the following three:

\[ A_{1,3}^I \{1\} = -x_1, \quad A_{1,3}^I \{3\} = -x_2, \quad A_{1,3}^I \{1,3\} = x_1x_2. \]

On the other hand, \( c_{1,3}^X_{v,w} = 1 \) when \( E = \{1\}, \{3\} \), and \( X_1X_1 = \kappa_1X_1 \). So \( c_{1,3}^X_{v,w} = 1 \). Therefore,

\[ b_{w,I_w}^X = -x_1 - x_2 + \kappa_1x_1x_2. \]

In particular, if \( F = F_a \), then \( b_{w,I_w}^X = -x_1 - x_2 \), and if \( F = F_m \), then \( b_{w,I_w}^X = -x_1 - x_2 + x_1x_2 = x_{a_1+a_2} \).
\end{example}

\begin{example}
Let \( w = s_1s_2, v = s_1s_2s_3s_1s_2 \). Let us compute \( b_{v,w}^X = X_{I_w}^*(\delta_v) \). We write \( X_{i_1j_1} \cdots X_{i_kj_k} \), for \( X_1X_2X_3 \cdots X_{i_kj_k} = x_{i_kj_k} = x_{i_kj_k}, \) and \( \kappa_\pm = \kappa_{\pm, \pm} = \kappa_{\pm, \pm} = \kappa_{\pm, \pm} = \kappa_{\pm, \pm} \). To compute \( A_{1,2}^I \), we only need to consider

\[ A_{1,2}^I \{1,2\} = x_1x_2, \quad A_{1,2}^I \{1,5\} = x_1x_2. \]

On the other hand, \( c_{1,2}^X_{v,w} = 1 \) when \( E = \{1,2\}, \{1,5\}, \{4,5\} \). Concerning \( X_{I_w}^\ell_{\{1,2,5\}} \) is \( X_{122} \), since

\[ X_1X_2X_2 = X_1\kappa_2X_2 = s_1(\kappa_2)X_1 + \Delta_1(\kappa_2)X_2 = \kappa_1+2X_1+2 + \Delta_1(\kappa_2)X_2, \]

so

\[ c_{1,2}^X_{v,w} = \kappa_1+2. \]
\end{example}
For $X_{I_w(1,4,5)} = X_{112}$, from $X_1X_1X_2 = \kappa_1 X_1X_2$, we get
\[ c^X_{I_w(1,4,5),I_w} = \kappa_1. \]
Lastly, for $X_{I_w(1,2,4,5)} = X_{1122}$, from Lemma 2.2 we know
\[ X_{1212} = X_1(X_{121} + \kappa_12X_1 - \kappa_21X_2) = \kappa_1 X_{121} + X_1 \kappa_2X_1 - \kappa_1 \kappa_2X_2 \]
\[ = \kappa_1 X_{121} + s_1(\kappa_12)X_1^2 + \Delta_1(\kappa_21)X_1 - s_1(\kappa_21)X_2 - \Delta_1(\kappa_21)X_2 \]
\[ = \kappa_1 X_{121} + \kappa_{1,12} s_1X_1 + \Delta_1(\kappa_11)X_1 - \kappa_{1,2} - 1X_12 - \Delta_1(\kappa_21)X_2, \]
so
\[ c^X_{I_w(1,2,4,5),I_w} = s_1(\kappa_21) = -\kappa_{1,2} - 1. \]
Therefore,
\[ b^X_{s_1, s_2, s_1, s_2} = A^L_{[6], \{1,2\}} + A^L_{[6], \{1,5\}} + A^L_{[6], \{4,5\}} \]
\[ + A^L_{[6], \{1,2,5\}} \kappa_{1,2} + A^L_{[6], \{1,4,5\}} \kappa_1 + A^L_{[6], \{1,2,4,5\}} (-\kappa_{1,2} - 1) \]
\[ = x_1 x_1 x_2 + x_1 x_2 x_3 + x_2 x_2 x_3 - x_1 x_1 x_2 x_2 + 3 \kappa_1 x_1 + x_1 x_2 x_2 x_3 x_1 + x_1 x_1 x_2 x_2 x_3 x_1 + 2 \kappa_1 - 1 \]
\[ = x_1 x_1 x_2 + x_1 x_2 x_3 + x_2 x_2 x_3 - x_2 x_3 x_1 + x_1 + x_2 + x_1 x_2 \]}
In particular, if $F = F_a$, then
\[ b^X_{s_1, s_2, s_1, s_2} = \alpha_1 (\alpha_1 + \alpha_2) + \alpha_1 (\alpha_2 + \alpha_3) + \alpha_2 (\alpha_2 + \alpha_3). \]
If $F = F_m$, then
\[ b^X_{s_1, s_2, s_1, s_2} = x_1 x_1 x_2 + x_1 x_2 x_3 + x_2 x_2 x_3 - x_1 x_2 x_3 (x_1 + x_2). \]
These agree with the result computed by using root polynomials.

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