Approximation and parameterized algorithms to find balanced connected partitions of graphs

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Abstract

Partitioning a connected graph into \( k \) vertex-disjoint connected subgraphs of similar (or given) orders is a classical problem that has been intensively investigated since late seventies. Given a connected graph \( G = (V, E) \) and a weight function \( w: V \to \mathbb{Q}_{\geq} \), a connected \( k \)-partition of \( G \) is a partition of \( V \) such that each class induces a connected subgraph. The balanced connected \( k \)-partition problem consists in finding a connected \( k \)-partition in which every class has roughly the same weight. To model this concept of balance, one may seek connected \( k \)-partitions that either maximize the weight of a lightest class (max-min BCP\(_k\)) or minimize the weight of a heaviest class (min-max BCP\(_k\)). Such problems are equivalent when \( k = 2 \), but they are different when \( k \geq 3 \). In this work, we propose a simple pseudo-polynomial \( \frac{1}{k^2} \)-approximation algorithm for min-max BCP\(_k\) which runs in time \( O(W|V||E|) \), where \( W = \sum_{v \in V} w(v) \). Based on this algorithm and using a scaling technique, we design a (polynomial) \( (\frac{1}{k^2} + \varepsilon) \)-approximation for the same problem with running-time \( O(|V|^3|E|/\varepsilon) \), for any fixed \( \varepsilon > 0 \). Additionally, we propose a fixed-parameter tractable algorithm based on integer linear programming for the unweighted max-min BCP\(_k\) parameterized by the size of a vertex cover.

Keywords: connected partition, approximation algorithms, fixed parameter tractable, parameterized algorithm

1 Introduction

The problem of partitioning a connected graph into a given number \( k \geq 2 \) of connected subgraphs with prescribed orders was first studied by Lovász\(^5\) and Györi\(^6\) in the late seventies. Let \([k]\) denote the set \( \{1, 2, \ldots, k\} \), for every integer \( k \geq 1 \). A connected \( k \)-partition of a connected graph \( G = (V, E) \) is a partition of \( V \) into classes \( \{V_i\}_{i=1}^k \) of nonempty subsets such that, for each \( i \in [k] \), the subgraph \( G[V_i] \) is connected, where \( G[V_i] \) denotes the subgraph of \( G \) induced by the vertices \( V_i \).

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Consider a pair \((G, w)\), where \(G = (V, E)\) is a connected graph and \(w : V \to \mathbb{Q}_+\) is a function that assigns positive weights to the vertices of \(G\). For each \(V' \subseteq V\), we define \(w(V') = \sum_{v \in V'} w(v)\). Furthermore, if \(G' = (V', E')\) is a subgraph of \(G\), we write \(w(G')\) instead of \(w(V')\). If \(P = \{V_i\}_{i \in [k]}\) is a connected \(k\)-partition of \(G\), then \(w^+(P)\) stands for \(\max_{i \in [k]} \{w(V_i)\}\), and \(w^-(P)\) stands for \(\min_{i \in [k]} \{w(V_i)\}\).

The concept of balance of the classes of a connected partition of a graph can be expressed in different ways. In this work, we consider two related variants whose objective functions express this concept.

**Problem 1.** **Min-Max Balanced Connected \(k\)-Partition (min-max \(BCP_k\))**

Instance: a connected graph \(G = (V, E)\), and a weight function \(w : V \to \mathbb{Q}_+\).

Find: a connected \(k\)-partition \(P\) of \(G\).

Goal: minimize \(w^+(P)\).

**Problem 2.** **Max-Min Balanced Connected \(k\)-Partition (max-min \(BCP_k\))**

Instance: a connected graph \(G = (V, E)\), and a weight function \(w : V \to \mathbb{Q}_+\).

Find: a connected \(k\)-partition \(P\) of \(G\).

Goal: maximize \(w^-(P)\).

We remark that \(\text{min-max} \text{ } BCP_2\) and \(\text{max-min} \text{ } BCP_2\) are equivalent, that is, for any instance, an optimal solution for the min-max version is also an optimal solution for the max-min version. If \(k > 2\) the corresponding optimal \(k\)-partitions may differ (the reader is referred to the examples given by Lucertini, Perl, and Simeone [20]).

Throughout this paper we assume that \(k \geq 2\). When \(k\) is in the name of the problem, we are considering that \(k\) is fixed. The problems in which \(k\) is part of the instance are denoted similarly but without specifying \(k\) in the name (e.g. \(\text{max-min} \text{ } BCP\)).

The unweighted (or cardinality) versions of these problems refer to the case in which all vertices have equal weight, which may be assumed to be 1. We denote the corresponding problems as \(\text{1-min-max} \text{ } BCP_k\) and \(\text{1-max-min} \text{ } BCP_k\)

In this paper, we show approximation algorithms for \(\text{min-max} \text{ } BCP_k\) and also mention approximation results for both \(\text{min-max} \text{ } BCP_k\) and \(\text{max-min} \text{ } BCP_k\). To make clear what we mean by an \(\alpha\)-approximation algorithm, we define this concept. We also define another closely related concept that will be used here.

Let \(\mathcal{A}\) be an algorithm for an optimization problem \(\Pi\). If \(I\) is an instance for \(\Pi\), we denote by \(\mathcal{A}(I)\) the value of the solution produced by \(\mathcal{A}\) for \(I\), and by \(\text{OPT}(I)\) the value of an optimal solution for \(I\). If \(\Pi\) is a minimization (resp. maximization) problem, and \(\mathcal{A}\) is a polynomial-time algorithm, we say that \(\mathcal{A}\) is an \(\alpha\)-approximation algorithm, for some \(\alpha \geq 1\), if \(\mathcal{A}(I) \leq \alpha \text{OPT}(I)\) (resp. \(\mathcal{A}(I) \geq 1/\alpha \text{OPT}(I)\)) for every instance \(I\) of \(\Pi\). We also say that \(\mathcal{A}\) is an approximation algorithm with ratio \(\alpha\). The approximation ratio \(\alpha\) need not be a constant: it can be a function \(\alpha(I)\) that depends on \(I\). So, whenever we refer to an approximation algorithm, we mean that it runs in polynomial time on the size of the instance.

When an approximation ratio \(\alpha\) can be guaranteed for an algorithm, but it may run in pseudo-polynomial time, we refer to it as a pseudo-polynomial \(\alpha\)-approximation. This is not a usual terminology, but it will be appropriate for our purposes.

Problems of finding balanced connected partitions can be used to model a rich collection of applications in logistics, image processing, database, operating systems, cluster analysis, education, robotics and biological networks [3, 19, 22, 24, 32].
1.1 Some known results

We first mention some hardness results to solve or to obtain certain approximate solutions for the $k$-connected partition problems. Dyer and Frieze [11] proved that $1$-$\text{max-min BCP}_k$ is $\mathcal{NP}$-hard on bipartite graphs. Furthermore, $1$-$\text{max-min BCP}_k$ have been shown by Chlebíková [9] to be $\mathcal{NP}$-hard to approximate within an absolute error guarantee of $n^{1-\varepsilon}$, for all $\varepsilon > 0$. For the weighted versions, Becker, Lari, Lucertini and Simeone [2] proved that $\text{max-min BCP}_2$ is $\mathcal{NP}$-hard on grid graphs. Wu [31] showed that $\text{max-min BCP}_k$ is $\mathcal{NP}$-hard on interval graphs for every $k$. Chataigner, Salgado, and Wakabayashi [6] showed that $\text{max-min BCP}_k$ is strongly $\mathcal{NP}$-hard, even on $k$-connected graphs. Hence, unless $\mathcal{P} = \mathcal{NP}$, $\text{max-min BCP}_k$ does not admit a fully polynomial-time approximation scheme (FPTAS). They also showed that when $k$ is part of the instance, $\text{min-max BCP}$ cannot be approximated within a ratio better than $6/5$.

Now, we turn to existential or algorithmic results for the unweighted versions of the $k$-connected partition problems. When the input graph $G$ is $k$-connected, Győri [15], and Lovász [18] proved that one can always find a connected $k$-partition where each class has a prescribed number of vertices. It is not difficult to devise a polynomial-time algorithm to find a 2-connected partition in which the classes have prescribed sizes. However, the proof of the existence of the desired $k$-connected partition on any $k$-connected graph given by Győri [15] does not seem to yield a polynomial-time algorithm. (In the past, an algorithm for this problem was claimed to be polynomial, but such result has not been established yet.)

Polynomial-time algorithms have been designed for restricted cases of the result of Győri and Lovász. Suzuki, Takahashi and Nishizeki [29] devised a linear-time algorithm to find a connected 2-partition on a 2-connected graph. When the input graph is 3-connected, Suzuki et al. [30] presented a quadratic-time algorithm to compute a connected 3-partition. If $G$ is planar and 4-connected, Nakano, Rahman and Nishizeki [27] shows that a connected 4-partition can be found in linear-time.

More recently, Chen et al. [8] designed a $3/2$-approximation for $1$-$\text{min-max BCP}_3$ and $24/13$-approximation for $1$-$\text{min-max BCP}_4$. For $1$-$\text{min-max BCP}_k$, $k \geq 4$, they also provided a $k/2$-approximation.

Now, let us consider the weighted versions. For $\text{max-min BCP}_k$ (resp. $\text{min-max BCP}_k$), Perl and Schach [25] (resp. Becker, Schach, and Perl [4]) designed polynomial-time algorithm when the input graph is a tree. Also for trees, Frederickson [14] proposed linear-time algorithms for both $\text{max-min BCP}_k$ and $\text{min-max BCP}_k$. Polynomial-time algorithms were also derived for $\text{max-min BCP}_2$ on graphs with at most two cut-vertices [16,9]. For $\text{max-min BCP}_k$ on ladders, a polynomial-time algorithm was obtained by Becker et al. (2001).

In 1996, Chlebíková [9] designed a $4/3$-approximation algorithm for $\text{max-min BCP}_2$. More recently, Chen et al. [7] have shown that the algorithm obtained by Chlebíková has approximation ratio $5/4$ for $\text{min-max BCP}_2$. These authors also obtained approximation algorithms with ratio $3/2$ and $5/3$ for $\text{min-max BCP}_3$ and $\text{max-min BCP}_3$, respectively. In 2012, Wu [31] designed a FPTAS for $\text{max-min BCP}_2$ restricted to interval graphs.

In Table 1, we present some of the results we have mentioned. Other approximation algorithms have also been obtained, but they have slightly weaker ratios, or impose conditions on the input graph.

When $k$ is part of the input, approximation algorithms were proposed by Borndörfer, Elijazyfer and Schwartz [5]. For both $\text{max-min BCP}$ and $\text{min-max BCP}$, their algorithms has approximation ratio $\Delta$, where $\Delta$ is the maximum degree of an arbitrary spanning tree of the input graph $G$. Specifically for $\text{max-min BCP}$, their $\Delta$-approximation
only holds for instances in which the largest weight is at most $w(G)/(\Delta k)$.

The results we mentioned above were mostly concerned with polynomial-time algorithms, approximation algorithms and hardness results for the two variants of balanced connected $k$-partition problem.

For completeness, we mention that some mixed integer linear programming formulations were also proposed for these problems (see [23][25][26][32]).

1.2 Our contributions

We generalize the $k/2$-approximation algorithm for 1-MIN-MAX BCP$_k$, $k \geq 3$, designed by Chen et al. [8], and present an approximation algorithm for (the weighted version) MIN-MAX BCP$_k$, $k \geq 3$. We prove that it has basically the same approximation ratio: namely, $k/2 + \varepsilon$, for any arbitrarily small $\varepsilon > 0$. For that, we use a scaling technique to deal with weights that might be very large. When the weights assigned to the vertices of the input graph are bounded by a polynomial on the order of the graph, it achieves the ratio $k/2$. A 3/2-approximation algorithm for MIN-MAX BCP$_3$ was obtained by Chen et al. [7], but its analysis and implementation are slightly more complicated than the algorithm we show here.

We also prove that 1-MAX-MIN BCP$_k$ is fixed-parameter tractable when the parameter is the size of a vertex cover of the input graph. This algorithm is based on integer linear programming and it has a doubly exponential dependency on the size of a vertex cover. To the best of our knowledge, no other FPT algorithm for balanced connected partition problems is described in the literature. Despite the fact that the proposed algorithm is not practical, the strategy used to model connected partitions may be applicable to show that other problems involving connectivity constraints are fixed-parameter tractable when the parameter is the size of a vertex cover.

2 Approximation algorithm for min-max BCP$_k$

Chen et al. [8] devised an algorithm for 1-MIN-MAX BCP$_k$ with approximation ratio $k/2$. This algorithm iteratively applies two simple operations, namely PULL and MERGE, to reduce the size of the largest class. In what follows, we show how to generalize such operations for the weighted case to design a $(k/2 + \varepsilon)$-approximation for MIN-MAX BCP$_k$, for any $\varepsilon > 0$.

First we discuss the algorithm for the case $k = 3$, and then we show how to use the connected 3-partition returned by this algorithm to obtain a connected $k$-partition for any $k \geq 4$. 

| Author(s) | Approximation Ratio | Problem Version |
|-----------|---------------------|-----------------|
| Chen et al. [8] | $k/2$ | 1-MIN-MAX BCP$_k$, $k \geq 3$ |
| Chen et al. [8] | $24/13$ | 1-MIN-MAX BCP$_4$ |
| Chlebíková [9] | $5/4$ | MIN-MAX BCP$_2$ |
| Chen et al. [7] | $3/2$ | MIN-MAX BCP$_3$ |
| Chlebíková [9] | $4/3$ | MAX-MIN BCP$_2$ |
| Chen et al. [7] | $5/3$ | MAX-MIN BCP$_3$ |

Table 1: Approximation results for BCP$_k$. 


Throughout this section, $k$ is a positive integer, and $(G, w)$ denotes an instance of MIN-MAX BCP$_k$, where $G = (V, E)$ and $w: V \to \mathbb{Q}_{\geq 0}$. Also, when convenient, we denote by $W$ the sum of the weights of the vertices in $G$, that is, $W = w(G)$. In this section, we assume without loss of generality that $w$ is an integer-valued function (otherwise, we may simply multiply all weights by the least common multiple of the denominators). The following trivial fact is used to show the approximation ratio of the algorithms for MIN-MAX BCP$_k$ proposed here. When convenient, we denote by $OPT_k(I)$ the value of an optimal solution for an instance $I$ of MIN-MAX BCP$_k$.

**Fact 1.** Any optimal solution for an instance $I = (G, w)$ of MIN-MAX BCP$_k$ has value at least $w(G)/k$, that is, $OPT_k(I) \geq w(G)/k$.

For $k \geq 3$, let $\mathcal{G}_k$ be the class of connected graphs $G$ containing a cut-vertex $v$ such that $G - v$ has at least $k - 1$ components. We denote by $c(H)$ the number of components of a graph $H$. The next lemma provides a lower bound for the value of an optimal solution of MIN-MAX BCP$_k$ on instances $(G, w)$ with $G \in \mathcal{G}_k$.

**Lemma 1.** Let $I = (G, w)$ be an instance of MIN-MAX BCP$_k$ in which $G \in \mathcal{G}_k$, and $v$ is a cut-vertex of $G$ such that $c(G - v) = \ell \geq k - 1$. Let $\mathcal{C} = \{C_i\}_{i \in [\ell]}$ be the set of the components of $G - v$. Suppose further that $w(C_i) \leq w(C_{i+1})$ for every $i \in [\ell - 1]$. Then every connected $k$-partition $\mathcal{P}$ of $G$ satisfies

$$w^+(\mathcal{P}) \geq w(v) + \sum_{i \in [\ell - k + 1]} w(C_i).$$

In particular, $OPT_k(I) \geq w(v) + \sum_{i \in [\ell - k + 1]} w(C_i)$.

**Proof.** Consider a connected $k$-partition $\mathcal{P}$ of $G$, and let $V^*$ be the class in $\mathcal{P}$ that contains $v$. Let $q^* := \{|C \in \mathcal{C}: V(C) \subseteq V^*\}$ and $q := \{|C \in \mathcal{C}: V(C) \not\subseteq V^*\}$.

Note that each class in $\mathcal{P} \setminus \{V^*\}$ is either a component of $G - v$ or it is a set properly contained in a unique component of $G - v$. Hence, $q^* + q = \ell$ and $q \leq k - 1$. Therefore, $q^* = \ell - q \geq \ell - k + 1$. Since $w(C_1) \leq w(C_2) \leq \ldots \leq w(C_\ell)$, we conclude that $w(V^*) \geq w(v) + \sum_{i \in [\ell - k + 1]} w(C_i)$, and therefore, $w^+(\mathcal{P}) \geq w(V^*)$. Clearly, it holds that $OPT_k(I) \geq w(v) + \sum_{i \in [\ell - k + 1]} w(C_i)$. \hfill \Box

We now present an algorithm for MIN-MAX BCP$_3$ that generalizes the algorithm proposed by Chen et al. [8] for the unweighted version of this problem. We adopt basically the same notation used by these authors to refer to the basic operations which are the core of the algorithm.

The strategy used in the algorithm is to start with an arbitrary connected 3-partition and improve it by applying successively (while it is possible) the operations MERGE and PULL, defined in what follows.

We say that a connected 3-partition $\{V_1, V_2, V_3\}$ of $G$ is ordered if $w(V_1) \leq w(V_2) \leq w(V_3)$. The input for PULL and MERGE is an ordered connected 3-partition $\{V_1, V_2, V_3\}$. As these operations may be applied several times, a reordering of the classes is performed at the end, if necessary. In this context, we say that an ordered 3-partition $\mathcal{P} = \{V_1, V_2, V_3\}$ is better than an ordered 3-partition $\mathcal{Q} = \{X_1, X_2, X_3\}$ if $w(V_3) < w(X_3)$.

We say that two classes $V_i$ and $V_j$ are adjacent if there is an edge in $G$ joining these classes. For $X \subset V$, we denote by $N(X)$ the set of vertices in $G$ that are adjacent to a vertex of $X$.
For simplicity, we say that such a set \( U \) is pull-admissible w.r.t. \( G \), if \( U \) is a proper subset of \( V(G) \) and \( w(U) < w(V(G) \setminus U) \). Note that a depth-first search suffices to check the pre-conditions. Moreover, a connected 2-partition of \( G[V_3] \) can be easily obtained from any spanning tree of this graph. Hence MERGE can be executed in \( O(|V| + |E|) \).

**Pull(\( P, U, i \))**

- Input: an ordered connected 3-partition \( P = \{V_1, V_2, V_3\} \) of \( G \), a nonempty subset \( U \) of vertices, and \( i \in \{1, 2\} \).
- Pre-conditions: (a) \( w(V_3') > w(V_3) \); (b) \( U \subseteq V_3 \), \( G[V_i \cup U] \) and \( G[V_3 \setminus U] \) are connected; (c) \( w(V_i \cup U) < w(V_3) \).
- Output: a connected 3-partition \( \{V_j, V_i \cup U, V_3 \setminus U\} \) where \( j \in \{1, 2\} \setminus \{i\} \).

Reorder the classes if necessary, and return an ordered partition.

Note that Pull(\( P, U, i \)) improves the input partition \( P = \{V_1, V_2, V_3\} \), since \( w(V_3 \setminus U) < w(V_3') \), \( w(V_j) < w(V_3) \) and \( w(V_j \cup U) < w(V_3) \). Moreover, it is only executed when a set \( U \) satisfying the pre-conditions is given. Thus, this operation can be executed in \( O(|V|) \) time. We next discuss the time complexity to find such a set \( U \subseteq V_3 \), if it exists.

For simplicity, we say that such a set \( U \) is pull-admissible (w.r.t. \( i \)). Observe that if \( V_3 \) contains a pull-admissible subset, then \( V_3 \) has at least 2 vertices.

**Algorithm 1 PullCheck**

**Input:** An ordered connected 3-partition \( P = \{V_1, V_2, V_3\} \) of \( (G, w) \), and \( i \in \{1, 2\} \).

**Output:** Either a set \( U \subset V_3 \) that is pull-admissible w.r.t. \( i \), or the empty set \( \emptyset \).

1: **procedure** PullCheck(\( P, i \))
2:     for \( v \in N(V_i) \cap V_3 \) do
3:         Let \( C = \{C_1, \ldots, C_\ell\} \) be the components of \( G[V_3 - v] \) with \( w(C_1) \leq \ldots \leq w(C_\ell) \).
4:         if \( w(V_i) + w(v) + \sum_{j\in[\ell-1]} w(C_j) < w(V_3) \) then
5:             return \( U \), where \( U = \{v\} \cup \bigcup_{j\in[\ell-1]} V(C_j) \) # \( U \) is pull-admissible.
6:     return \( \emptyset \)

**Lemma 2.** If there is a pull-admissible set, then Algorithm 1 finds one. Moreover, on input \( (G, w) \), where \( G = (V, E) \), it runs in \( O(|V|(|V| + |E|)) \) time.

**Proof.** Let \( T \subset V_3 \) be a pull-admissible set w.r.t. \( i \). We may assume that \( G[T] \) is connected, otherwise, each of its components is pull-admissible and we can consider any of them. Let \( v \in T \) be a vertex adjacent to \( V_i \). Clearly, \( v \) will be checked at line 2. Let \( C = \{C_1, \ldots, C_\ell\} \) be the components of \( G[V_3 - v] \) with \( w(C_1) \leq \ldots \leq w(C_\ell) \). (Note that \( \ell \geq 1 \), as \( T \) is a proper subset of \( V_3 \).) Since \( G[V_3 \setminus T] \) and \( G[T] \) are connected, we conclude that \( G[T] \) must contain \( \ell - 1 \) components of \( C \). Moreover, precisely one of the components in \( C \)
is not contained in \( G[T] \) (but \( T \) may contain part of it). (To see this, consider the block structure of \( G[V_3] \) and analyse when \( \ell = 1 \) and \( \ell \geq 2 \).) It follows from this observation that the set \( U = \{ v \} \cup \bigcup_{l \in [\ell-1]} V(C_l) \) is such that \( w(U) \leq w(T) \). As \( T \) is pull-admissible, it holds that \( w(V_i) + w(U) \leq w(V_i) + w(T) < w(V_3) \). Hence, at line 3 the set \( U \), which is pull-admissible, will be returned by Algorithm 1.

Since the connected components of \( G[V_3 - v] \) can be computed in time \( \mathcal{O}(|V| + |E|) \), Algorithm 1 runs in \( \mathcal{O}(|V|(|V| + |E|)) \) time.

Lemma 3. Algorithm 2 on input \((G, w)\) of \textsc{min-max bcp3} finds a connected 3-partition of \( G \) in \( \mathcal{O}(w(G)|V||E|) \) time.

Proof. The algorithm starts with an arbitrary connected 3-partition of \( G \), and only modifies the current partition when a \textsc{merge} or \textsc{pull} operation is performed. As both operations are performed only when the corresponding pre-conditions are satisfied, they yield connected 3-partitions of \( G \), and the algorithm is correct.

Each time a \textsc{merge} or a \textsc{pull} operation is executed, the weight of the heaviest class decreases. Thus, at most \( w(G) \) calls of such operations are performed by the algorithm. Recall that both \textsc{merge} and \textsc{pull} operations take \( \mathcal{O}(|V| + |E|) \) time. Moreover, by Lemma 2 the procedure \textsc{pullCheck} has time complexity \( \mathcal{O}(|V||E|) \) \((G \text{ is connected, so } |E| \geq |V| - 1) \). It follows from the above remarks that Algorithm 2 has time complexity \( \mathcal{O}(w(G)|V||E|) \).

It is clear that when Algorithm 2 halts and returns a partition \( \mathcal{P} \), one of the two cases occurs: (a) either the loop condition in line 2 failed, and in this case, \( \mathcal{P} \) has value \( w^{+}(\mathcal{P}) \leq w(G)/2 \), or (b) neither \textsc{merge} nor \textsc{pull} operations could be performed (and \( w^{+}(\mathcal{P}) > w(G)/2 \)). In what follows, we prove that in case (b) the input graph has a particular “star-like” structure which allows us to conclude that the solution produced by the algorithm is optimal.

Lemma 4. Let \( \mathcal{P} = \{ V_1, V_2, V_3 \} \) be an ordered connected 3-partition produced by Algorithm 2 and let \( G_i = G[V_i] \), for \( i = 1, 2, 3 \). If \( |V_3| \geq 2 \) and \( w(V_3) > w(G)/2 \), the following hold:

(i) \( w(V_1) < w(G)/4 \), and \( V_1 \) and \( V_2 \) are not adjacent; and

\[ \]
(ii) there exists \( u \in V_3 \) such that \( u \) is a cut-vertex of \( G \), \( \{G_1, G_2\} \subseteq C \), \( w(C) \leq w(V_1) \leq w(V_2) \) for each \( C \subseteq C \setminus \{G_1, G_2\} \), where \( C \) is the set of components of \( G - u \). Moreover, if \( |C| = 3 \) then \( w(u) > w(G)/4 \).

Proof. Since \( w(V_3) > w(G)/2 \), the algorithm terminated after executing line [9]. This implies that neither MERGE nor PULL operation can be performed on \( P \). Particularly, it follows that \( V_1 \) and \( V_2 \) are not adjacent. Moreover, note that \( w(V_1) + w(V_2) < w(G)/2 \), again because \( w(V_3) > w(G)/2 \). Hence, it holds that \( w(V_1) < w(G)/4 \), since \( w(V_1) \leq w(V_2) \). This proves [1].

Since \( G \) is connected, and \( V_1 \) and \( V_2 \) are not adjacent, there exists \( w \in E(G) \) such that \( u \in V_3 \) and \( v \in V_1 \). Let \( C' \) be the set of components of \( G_2 - u \). Note that \( C' \neq \emptyset \) because \( |V_3| > 1 \), and consider a component \( C \in C' \). Let us define \( S = \{u\} \cup \bigcup_{C' \in C \setminus \{C\}} C' \). It is clear that \( G[C] \) and \( G[S] \) are connected subgraphs of \( G \). Since it is not possible to perform PULL(\( P, S, 1 \)), it holds that \( w(S) + w(C) \leq w(S) + w(V_1) \). Therefore, we have that \( w(C) \leq w(V_1) \leq w(V_2) \).

Suppose to the contrary that \( V_1 \) is adjacent to \( C \). Thus the partition \( \{V_1 \cup V(C), V_2, V_3 \setminus V(C)\} \) is a connected 3-partition of \( G \). Since \( w(C) + w(V_1) \leq 2w(V_1) < w(G)/2 \), Algorithm [2] could perform PULL(\( P, V(C), 1 \)), a contradiction. Similarly, \( V_2 \) is not adjacent to \( C \), otherwise the algorithm could execute PULL(\( P, V(C), 2 \)) because \( w(C) + w(V_2) \leq w(V_1) + w(V_2) < w(G)/2 < w(V_2) \). By claim [1], \( V_1 \) and \( V_2 \) are not adjacent, and thus we have \( N(V_1) \cap V_3 = N(V_2) \cap V_3 = \{u\} \). Therefore, \( u \) is a cut-vertex of \( G \) and \( C = \{G_1, G_2\} \subseteq C' \). This concludes the proof of [ii].

Theorem 5. Algorithm [3] is a pseudo-polynomial \( \frac{3}{4} \)-approximation for MIN-MAX BCP\(_3\) which runs in \( O(w(G)|V||E|) \) time on an instance \((G, w)\), where \( G = (V, E) \).

Proof. Let \( P = \{V_1, V_2, V_3\} \) be an ordered 3-partition of \( G \), returned by the algorithm; and let \( G_i = G[V_i] \), for \( i = 1, 2, 3 \). By Lemma [8], \( P \) is indeed a connected 3-partition of \( G \) and it can be computed in time \( O(w(G)|V||E|) \). If \( w(V_3) \leq w(G)/2 \), then it follows directly from Fact [1] that \( w^+(P) = w(V_3) \leq \frac{3}{4} \text{OPT}_3(G, w) \).

Suppose now that \( w(V_3) > w(G)/2 \). If \( V_3 \) is a singleton \( \{u\} \), then \( w(u) \leq \text{OPT}_3(G, w) \) and \( P \) is optimal. Otherwise, the algorithm terminated because neither MERGE nor PULL operation can be performed on \( P \). By Lemma [11], there exists \( u \in V_3 \) such that \( u \) is a cut-vertex of \( G \), \( \{G_1, G_2\} \subseteq C \), and \( w(C) \leq w(V_1) \leq w(V_2) \) for each \( C \subseteq C \setminus \{G_1, G_2\} \), where \( C \) is the set of components of \( G - u \). By Lemma [1], we have

\[
w^+(P) = w(V_3) = w(u) + \sum_{C \subseteq C \setminus \{G_1, G_2\}} w(C) \leq \text{OPT}_3(G, w).
\]

Therefore, in this case the partition \( P \) produced by the algorithm is an optimal solution for the instance \((G, w)\) of MIN-MAX BCP\(_3\). \( \square \)

In what follows, we show how to extend the result obtained for MIN-MAX BCP\(_3\) to obtain results for MIN-MAX BCP\(_k\), for all \( k \geq 4 \). For simplicity, we say that a vertex \( u \) satisfying condition [ii] of Lemma [1] is a star-center. Moreover, when \( u \) is a star-center, we name the \( \ell \) components of \( G - u \) as \( C = \{C_1, C_2, \ldots, C_\ell\} \), where \( C_\ell = G[V_2], C_{\ell - 1} = G[V_1] \) and \( w(C_i) \leq w(C_{i+1}) \) for all \( i \in \{\ell - 1\} \).
Theorem 6. For each integer $k \geq 3$, Algorithm 4 is a pseudo-polynomial $\frac{k}{2}$-approximation for the problem min-max BCP$_k$ that runs in $O(w(G) |V| |E|)$ time on an instance $(G, w)$, where $G = (V, E)$.

Proof. We first note that Algorithm 3 is correct, since it iteratively removes $q$ non-cut vertices from non-trivial classes of $P$ and create new singleton classes. As $|V| \geq k + q$, there always exists $U \in P$ satisfying the conditions in line 3. Furthermore, note that if $P$ and $P'$ are the input and output of Algorithm 3 respectively, then it holds that $w^+(P') \leq w^+(P)$.

Now we turn to Algorithm 4. We may assume that $k \geq 4$, since for $k = 3$ the result follows from Theorem 5. Let $\{V_1, V_2, V_3\}$ be the ordered connected 3-partition produced in line 2, and let $G_i = G[V_i^+]$, for $i = 1, 2, 3$.

If the condition in line 3 is satisfied, then since Algorithm 3 is correct, the partition $P'$ (in line 4) is a connected $k$-partition of $G$. Clearly if $V_3$ is a singleton $\{u\}$, $P'$ is optimal, since $w(u) \leq \text{OPT}_k(G, w)$. Moreover, when $w^+(P) \leq w(G)/2$, it holds that $w^+(P') \leq w(G)/2 \leq (k/2) \text{OPT}_k(G, w)$, where the last inequality is justified by Fact 1.

Suppose now that $w(V_3) > w(G)/2$ and $|V_3| \geq 2$. By Lemma III, there exists a star-center $u \in V_3$. Let $C = \{C_i\}_{i \in [\ell]}$ be the components of $G - u$. We now consider two cases according to the values of $k$ and $\ell$.  

---

**Algorithm 3 GetSingletons**

**Input:** A connected graph $G = (V, E)$, a connected $k'$-partition $P$ of $G$, and an integer $q \geq 0$ such that $k' + q \leq |V|$

**Output:** A connected $(k' + q)$-partition of $G$

1: **procedure** GetSingletons($G, w, q, P$)
2: while $q > 0$
3: Let $U \in P$ be such that $|U| \geq 2$ and $u$ be a non-cut vertex of $G[U]$.
4: $P \leftarrow (P \setminus \{U\}) \cup (\{\{u\}\} \cup \{U \setminus \{u\}\})$
5: $q \leftarrow q - 1$
6: return $P$

**Algorithm 4 Min-Max-BCP$_k$ ($k \geq 3$)**

**Input:** An instance $(G = (V, E), w)$ of min-max BCP$_k$, $3 \leq k \leq |V|$

**Output:** A connected $k$-partition of $G$

**Routines:** Min-Max-BCP3, GetSingletons

1: **procedure** Min-Max-BCP$_k$(G, w)
2: $P \leftarrow \text{Min-Max-BCP3}(G, w)$ \# $P = \{V_1, V_2, V_3\}$
3: if $w^+(P) \leq \frac{w(G)}{2}$ or $|V_3| = 1$ then
4: $P' \leftarrow \text{GetSingletons}(G, w, k - 3, P)$
5: else
6: Let $u$ be the star-center and let $C = \{C_i\}_{i \in [\ell]}$ be the components of $G - u$.
7: if $\ell \geq k - 1$ then
8: Let $t = \ell - k + 1$ and $V' = \bigcup_{i \in [\ell]} V(C_i) \cup \{u\}$.
9: $P' \leftarrow \{V', V(C_{t+1}), \ldots, V(C_{\ell-1}), V(C_{\ell})\}$
10: else
11: $P \leftarrow \{\{u\}\} \cup \{C_i\}_{i \in [\ell]}$
12: $P' \leftarrow \text{GetSingletons}(G, w, k - 1 - \ell, P)$
13: return $P'$
If \( \ell \geq k - 1 \), then in any connected \( k \)-partition of \( G \) the class containing \( u \) must also contain \( \ell = \ell - k + 1 \) components of \( G - u \). The class \( V' \) defined in line 8 consists of the union of \( u \) and the \( t \) lightest such components. Clearly, \( P' := \{ V', V(C_{t+1}), \ldots, V(C_l), V(C_t) \} \) is a connected \( k \)-partition of \( G \), and \( w^+(P') = \max\{ w(V'), w(V_2) \} \) (recall that \( C_t = G[V_2] \)). If \( w^+(P') = w(V') \), it follows from Lemma \( \ref{lem:lightest-components} \) that \( P' \) is an optimal connected \( k \)-partition of \( G \). Otherwise, \( w^+(P') = w(V_2) \leq w(G)/2 \leq (k/2) \OPT_k(G, w) \).

If \( \ell \leq k - 2 \), starting with the connected \( (\ell + 1) \)-partition \( P = \{ \{ u \}, V(C_1), \ldots, V(C_\ell) \} \) (as defined in line 11), using Algorithm \( \ref{alg:lightest-components} \) we obtain a connected \( k \)-partition \( P' \) of \( G \). Clearly, \( w^+(P') \leq w(V_2) \leq w(G)/2 \), and so \( w^+(P') \leq (k/2) \OPT_k(G, w) \).

Finally, observe that non-cut vertices can be obtained by removing leaves of any spanning tree of the graph. Thus, Algorithm \( \ref{alg:lightest-components} \) has time complexity \( O(|V||E|) \). Since Algorithm \( \ref{alg:naive} \) (in line 2) is a pseudo-polynomial algorithm for MIN-MAX BCP\(_3\) that runs in \( O(w(G)|V||E|) \) time (cf. Theorem \( \ref{thm:naive} \)), we conclude that Algorithm \( \ref{alg:approx} \) is a pseudo-polynomial \( \frac{k}{2} \)-approximation for MIN-MAX BCP\(_k\) that runs in \( O(w(G)|V||E|) \) time.

The algorithm given by Theorem \( \ref{thm:approx} \) is a (polynomial) \( \frac{k}{2} \)-approximation if the weights assigned to the vertices are bounded by a polynomial on the order of the graph. In case the weights assigned to the vertices are arbitrary, it is possible to apply a scaling technique and use the previous algorithm as a subroutine to obtain a polynomial algorithm for MIN-MAX BCP\(_k\) with approximation ratio \( (\frac{k}{2} + \varepsilon) \), for any fixed \( \varepsilon > 0 \).

We prove next a more general result, concerning any pseudo-polynomial \( \alpha \)-approximation algorithm for MIN-MAX BCP\(_k\) whose running time depends on the value of the weights.

**Algorithm 5** \( \varepsilon \)-MIN-MAX-BCP\(_k\) \( (k \geq 3) \)

**Input:** An instance \( (G = (V, E), w) \) of MIN-MAX BCP\(_k\), \( 3 \leq k \leq |V| \)

**Output:** A connected \( k \)-partition of \( G \)

**Routine:** a pseudo-polynomial \( \alpha \)-approximation algorithm \( A \) for MIN-MAX BCP\(_k\)

1: \( \theta \leftarrow \max_{v \in V} w(v) \)
2: \( \lambda \leftarrow \frac{\theta}{|V|} \)
3: \( \text{for } v \in V \text{ do} \)
4: \( \quad \hat{w}(v) \leftarrow \left\lfloor \frac{w(v)}{\lambda} \right\rfloor \)
5: \( \quad \text{end for} \)
6: \( \mathcal{P} \leftarrow A(G, \hat{w}) \)
7: \( \text{return } \mathcal{P} \)

**Theorem 7.** Let \( k \geq 3 \) be an integer, and let \( I = (G = (V, E), w) \) be an instance of MIN-MAX BCP\(_k\). If there is a pseudo-polynomial \( \alpha \)-approximation algorithm \( A \) for MIN-MAX BCP\(_k\) that runs in \( O(w(G)^c|V||E|) \) time for some constant \( c \), then Algorithm \( \ref{alg:approx} \) is an \( \alpha(1 + \varepsilon) \)-approximation for MIN-MAX BCP\(_k\) that runs in \( O(|V|^{2c+1}|E|/\varepsilon^c) \) time.

**Proof.** Let \( I = (G = (V, E), w) \) be an instance of MIN-MAX BCP\(_k\), and let \( \mathcal{P}^* \) (resp. \( \mathcal{P} \)) be an optimal solution (resp. a solution produced by Algorithm \( \ref{alg:approx} \)) on input \( I \). Denote by \( V^*_k \) and \( V_k \) the heaviest classes in \( \mathcal{P}^* \) and \( \mathcal{P} \), respectively. First, note that \( \mathcal{P}^* \) is a feasible solution for the instance \( (G, \hat{w}) \), and so \( \hat{w}^+(\mathcal{P}^*) = \sum_{v \in V^*_k} \hat{w}(v) \geq \OPT_k(G, \hat{w}) \).

Moreover, \( \hat{w}^+(\mathcal{P}) = \sum_{v \in V_k} \hat{w}(v) \leq \alpha \OPT_k(G, \hat{w}) \) since \( A \) is an \( \alpha \)-approximation. It is clear from line \( \ref{alg:approx} \) that \( w(v)/\lambda \leq \hat{w}(v) \leq w(v)/\lambda + 1 \) for every \( v \in V \). Hence, the following
sequence of inequalities hold:

\[
    w^+(\mathcal{P}) = \sum_{v \in V_k} w(v) \leq \lambda \sum_{v \in V_k} \hat{w}(v) \leq \lambda \alpha \text{OPT}_k(G, \hat{w}) \\
    \leq \lambda \alpha \sum_{v \in V_k^*} \hat{w}(v) \leq \lambda \alpha \sum_{v \in V_k^*} \left( \frac{w(v)}{\lambda} + 1 \right) \\
    = \alpha \text{OPT}_k(G, w) + \lambda \alpha |V_k^*|.
\]

Since \( \lambda = \varepsilon \theta / |V| \) (see line 3) and \( \theta \leq \text{OPT}_k(G, w) \), it follows from inequality (1) that

\[
    w^+(\mathcal{P}) \leq \alpha \text{OPT}_k(G, w) + \alpha \varepsilon \theta \leq \alpha(1 + \varepsilon) \text{OPT}_k(G, w).
\]

The running-time of Algorithm 3 is clearly dominated by the running-time of \( \mathcal{A} \) on input \((G, \hat{w})\) in line 6 which takes time \( O(\hat{w}(G)^{\varepsilon}|V||E|) \). It follows from the scaling in line 5 that \( \hat{w}(v) \leq w(v)/\lambda + 1 \leq |V|/\varepsilon + 1 \) for every \( v \in V \). Therefore, \( \hat{w}(G) \leq |V|^2/\varepsilon + |V| \), and thus, the algorithm runs in \( O(|V|^{2\varepsilon + 1}|E|/\varepsilon^\varepsilon) \) time.

**Corollary 8.** For each integer \( k \geq 3 \) and \( \varepsilon' > 0 \), there is a \((k/2 + \varepsilon')\)-approximation for \text{MIN-MAX BCP}_k that runs in \( O(|V|^\varepsilon'|E|/\varepsilon') \) time on an input \((G, \hat{w})\).

**Proof.** The result follows from Theorem 6 by taking Algorithm 3 with \( \varepsilon = \varepsilon'/k/2 \) and Algorithm 2 as the routine \( \mathcal{A} \) it requires. The approximation ratio \( k/2 \) of Algorithm 3 is guaranteed by Theorem 6.

An algorithm analogous to Algorithm 3 can be designed for \text{MAX-MIN BCP}_k. In this case, change line 2 to \( \theta \leftarrow \min_{v \in V} w(v) \), change line 5 to \( \hat{w}(v) \leftarrow \left\lceil \frac{w(v)}{\lambda} \right\rceil \), and consider a routine that is a pseudo-polynomial \( \alpha \)-approximation for \text{MAX-MIN BCP}_k. Then, a theorem similar to Theorem 7 can be obtained for \text{MAX-MIN BCP}_k.

## 3 Parameterized max-min BCP

This section is devoted to design a fixed-parameter tractable (FPT) algorithm based on integer linear programming for the max-min version of the unweighted balanced connected partition problem when parameterized by the vertex cover. In this problem, we are given an unweighted graph \( G \), a positive integer \( k \), and a vertex cover \( X \) of \( G \). The objective is to find a connected \( k \)-partition of \( G \) that maximizes the size of the smallest class. Let us consider a fixed instance \((G, k)\) of \text{MAX-MIN-BCP} and a vertex cover \( X \) of \( G \).

Let us denote by \( I \) the stable set \( V(G) \setminus X \). Recall that we assume \( k \leq |V(G)| = |X| + |I| \). If \( k > |X| \), then there are at least \( k - |X| \) classes of size exactly 1 contained in \( I \), and so an optimal solution (which has value equal to 1) can be easily computed. If \( |X| = 1 \), then \( G \) is a star, and so it is trivial to compute an optimal solution. From now on, we assume that \( k \leq |X| \) and \( |X| \geq 2 \).

Before presenting the details of the proposed algorithm, we prove a simple lemma that guarantees the existence of an optimal solution in which each class intersects the given vertex cover \( X \).

**Lemma 9.** Let \((G, k)\) be an instance of \text{MAX-MIN-BCP} and let \( X \) be a vertex cover of \( G \). Then, there exists an optimal connected \( k \)-partition \( \{V_i\}_{i \in [k]} \) of \( G \) such that \( V_i \cap X \neq \emptyset \) for all \( i \in [k] \).
Proof. Suppose to the contrary that no such partition exists. Let \(\{V'_i\}_{i \in [k]}\) be a connected \(k\)-partition of \(G\) with the smallest number of classes contained in \(I\), and let \(V'_j = \{v\} \subseteq I\), for some \(j \in [k]\), be one of these classes. Since \(k \leq |X|\), there exists \(\ell \in [k] \setminus \{j\}\) such that \(|V'_\ell \cap X| \geq 2\). One may easily find a partition \(\{V'_{\ell,1}, V'_{\ell,2}\}\) of \(V'_\ell\) such that, for \(i \in \{1, 2\}\), \(G[V'_{\ell,i}]\) is connected and \(V'_{\ell,i} \cap X \neq \emptyset\). If \(N(v) \cap V'_\ell \neq \emptyset\), then assume without loss of generality that \(N(v) \cap V'_\ell \neq \emptyset\) and \(|N(v) \cap V'_\ell| \neq 0\). In this case, there is a connected \(k\)-partition \(\{V_i\}_{i \in [k]}\) of \(G\) such that \(V_j = V'_{j,1} \cup \{v\}\), \(V_t = V'_{t,2}\) and \(V_i = V'_i\) for every \(i \in [k] \setminus \{j, \ell\}\). If \(N(v) \cap V'_\ell = \emptyset\), then there exists \(t \in [k] \setminus \{j, \ell\}\) such that \(N(v) \cap V'_t \neq \emptyset\) since \(G\) is connected. Clearly, such a class intersects \(X\), that is, \(V'_t \cap X \neq \emptyset\). Thus, there is a connected \(k\)-partition \(\{V_i\}_{i \in [k]}\) of \(G\) such that \(V_j = V'_{j,1}\), \(V_t = V'_{t,2}\), \(V_i = V'_i \cup \{v\}\) and \(V_i = V'_i\) for every \(i \in [k] \setminus \{j, \ell, t\}\). In both cases, the partition has a smaller number of classes contained in \(I\) than \(\{V'_i\}_{i \in [k]}\), a contradiction to the choice of this partition.

We next use hypergraphs to model the constraints of our formulation for \textsc{max-min-bcp}. A hyperpath of length \(m\) between two vertices \(u\) and \(v\) in a hypergraph \(H\) is a set of hyperedges \(\{e_1, \ldots, e_m\} \subseteq E(H)\) such that \(u \in e_1, v \in e_m, e_i \cap e_{i+1} \neq \emptyset\) for each \(i \in \{1, \ldots, m - 1\}\). A set of hyperedges \(F \subseteq E(H)\) is a \((u, v)\)-cut if there is no hyperpath between \(u\) and \(v\) in \(H - F\).

For each \(S \subseteq X\), we define \(I(S) = \{v \in I : N(v) = S\}\). Let \(u, v \in X\) be a pair of non-adjacent vertices in \(G\), and let \(\Gamma_X(u, v)\) be the set of all separators of \(u\) and \(v\) in \(G[X]\). Consider a separator \(Z \in \Gamma_X(u, v)\), and denote by \(C(Z)\) the set of components of \(G[X - Z]\). Let \(H_Z\) denote the hypergraph with vertices \(C(Z)\) such that, for each \(S \subseteq X\) with \(I(S) \neq \emptyset\), there is a hyperedge \(\{C \in \mathcal{C}(Z) : S \cap V(C) \neq \emptyset\}\) in \(H_Z\). We denote by \(\Lambda_Z(u, v)\) the set of all \((C_u, C_v)\)-cuts in \(H_Z\), where \(C_u\) and \(C_v\) are the components of \(G[X - Z]\) containing \(u\) and \(v\), respectively.

For each \(v \in X\) and \(i \in [k]\), there is a binary variable \(x_{v,i}\) that equals 1 if and only if \(v\) belongs to the \(i\)-th class of the partition. Moreover, for every \(S \subseteq X\) and \(i \in [k]\), there is an integer variable \(y_{S,i}\) that equals the amount of vertices in \(I(S)\) that are assigned to the \(i\)-th class. The intuition for the meaning of the \(y\)-variables is that all vertices in \(I(S)\), for a fixed \(S \subseteq X\), play essentially the same role in a connected partition. Hence, the formulation does not need to have decision variables associated with the vertices in \(I(S)\), and so it only has an integer variable to count the number of these vertices that are chosen to be in each class. The idea of using integer variables to count indistinguishable vertices in a stable set was used before by Fellows et al. [12] for the \textsc{imbalance} problem.

Let us define \(\eta = 2^{|X|}\), that is, \(\eta\) is the number of subsets of \(X\). Let \(\mathcal{B}(G, X, k)\) be the set of vectors in \(\mathbb{R}^{|(X| + \eta)k}\) that satisfy the following inequalities [2–8].
∑_{v \in X} x_{v,i} + \sum_{S \subseteq X} y_{S,i} \leq \sum_{v \in X} x_{v,i+1} + \sum_{S \subseteq X} y_{S,i+1} \quad \forall i \in [k-1], \tag{2}

\sum_{i \in [k]} x_{v,i} = 1 \quad \forall v \in X, \tag{3}

x_{u,i} + x_{v,i} - \sum_{z \in Z} x_{z,i} - \sum_{S \in F} y_{S,i} \leq 1 \quad \forall uv \notin E(G[X]), Z \in \Gamma_X(u,v), F \in \Lambda_Z(u,v), i \in [k], \tag{4}

y_{S,i} \leq |I(S)| \left( \sum_{v \in S} x_{v,i} \right) \quad \forall S \subseteq X, i \in [k], \tag{5}

\sum_{i \in [k]} y_{S,i} = |I(S)| \quad \forall S \subseteq X, \tag{6}

x_{v,i} \in \{0,1\} \quad \forall v \in X \text{ and } i \in [k], \tag{7}

y_{S,i} \in \mathbb{Z}_{\geq 0} \quad \forall S \subseteq X \text{ and } i \in [k]. \tag{8}

Inequalities \((2)\) establish a non-decreasing ordering of the classes according to their sizes. Inequalities \((3)\) and \((6)\) guarantee that every vertex of the graph belongs to exactly one class (i.e. the classes define a partition). Due to Lemma \((9)\) we may consider only partitions such that each of its classes intersects \(X\). Thus, whenever a vertex in the stable set \(I\) is chosen to belong to some class, at least one of its neighbors in \(X\) has to be in the same class. This explains the meaning of inequalities \((5)\). Inequalities \((4)\) guarantee that each class of the partition induces a connected subgraph. This will be more clear in the proof of the following proposition.

**Lemma 10.** Let \(G\) be a connected graph, let \(k \geq 2\) be an integer, and let \(X\) be a vertex cover of \(G\). The problem \textsc{max-min-BCP} on instance \((G,k)\) is equivalent to

\[
\max \left\{ \sum_{v \in X} x_{v,1} + \sum_{S \subseteq X} y_{S,1} : (x,y) \in B(G,X,k) \right\}.
\]

**Proof.** Let \(\{V_i\}_{i \in [k]}\) be a connected \(k\)-partition of \(G\) such that \(V_i \cap X \neq \emptyset\) for every \(i \in [k]\). Suppose further that the classes in this partition are ordered so that \(|V_i| \leq |V_{i+1}|\) for all \(i \in [k-1]\). From \(\{V_i\}_{i \in [k]}\), we construct a vector \((\bar{x},\bar{y}) \in \mathbb{R}^{|X|k} \times \mathbb{R}^{k}\) such that its non-null entries are precisely defined as follows. For each \(i \in [k]\), we set \(\bar{x}_{v,i} = 1\) for every \(v \in X \cap V_i\), and \(\bar{y}_{S,i} = |I(S) \cap V_i|\) for every \(S \subseteq X\).

We next show that \((\bar{x},\bar{y})\) satisfies inequalities \((2)-(5)\). It easily follows from the construction of the vector and from the hypothesis on \(\{V_i\}_{i \in [k]}\) that inequalities \((2),(3),(4),(7)\), and \((5)\) hold for \((\bar{x},\bar{y})\). Moreover, since \(V_i \cap X \neq \emptyset\) for every \(i \in [k]\), inequalities \((5)\) hold for \((\bar{x},\bar{y})\). It remains to prove that inequalities \((1)\) are satisfied.
Consider an integer $i \in [k]$ such that there is a pair of non-adjacent vertices $u, v \in X \cap V_i$. Let $Z \subseteq X \setminus \{u, v\}$ be a separator of $u$ and $v$ in $G[X]$. Since $G[V_i]$ is connected, there exists a path $P$ in this graph with endpoints $u$ and $v$ such that either $V(P) \cap Z \neq \emptyset$ or $V(P) \cap I \neq \emptyset$. Suppose that $V(P) \cap Z = \emptyset$, otherwise inequalities (4) for $Z$ are clearly satisfied by $(\bar{x}, \bar{y})$. Hence there is a hyperpath in $H_Z$ linking $C_u$ and $C_v$, where $C_u$ and $C_v$ are the components of $G[X - Z]$ containing $u$ and $v$, respectively. As a consequence, for each cut $F$ separating $C_u$ and $C_v$ in the hypergraph $H_Z$, there exists a vertex $z \in V(P) \cap I$ such that $N(z) \in F$, and so $\bar{y}_{N(z), i} \geq 1$. Therefore, inequalities (4) are satisfied by $(\bar{x}, \bar{y})$.

Let $(\bar{x}, \bar{y}) \in B(G, X, k)$. Consider a subset $S \subseteq X$. It follows from inequality (3) for $S$ that there is a partition $\{I_i(S)\}_{i \in [k]}$ of $I(S)$ such that $|I_i(S)| = \bar{y}_{S,i}$. We remark that some classes in this partition may be empty. For each $i \in [k]$, let us define $V_i = (\bigcup_{S \subseteq X} I_i(S)) \cup \{v \in X : \bar{x}_{v,i} = 1\}$. One may easily verify that $|V_i| = \sum_{v \in X} \bar{x}_{v,i} + \sum_{S \subseteq X} \bar{y}_{S,i}$. Observe now that $I(S) \cap I(S') = \emptyset$ for all $S, S'$ subsets of $X$ with $S \neq S'$, and thus $\{I_i(S)\}_{S \subseteq X}$ is a partition of $I$ with possibly some empty classes. Furthermore, inequalities (3) guarantee that each vertex in $X$ belongs to exactly one class $V_i$, for some $i \in [k]$. Therefore, $\{V_i\}_{i \in [k]}$ is a partition of the vertices in $G$. Due to inequalities (2), it also holds that $|V_i| \leq |V_{i+1}|$ for all $i \in [k - 1]$. We shall prove that $G[V_i]$ is connected for each $i \in [k]$.

Suppose to the contrary there exists $i \in [k]$ such that $G[V_i]$ is not connected. Because of inequalities (3), every component of $G[V_i]$ has to intersect $X$. Let us define $Z = X \setminus V_i$, and let $H_Z$ be the hypergraph of the components of $G[X - Z]$ as defined earlier. It follows that, for each hyperedge $S$ of $H_Z$, no vertex in $I(S)$ belongs to $V_i$. Hence, for every pair of vertices $u, v \in V_i \cap X$ belonging to distinct components of $G[V_i]$, it holds that

$$
\bar{x}_{u,i} + \bar{x}_{v,i} - \sum_{z \in Z} \bar{x}_{z,i} - \sum_{S \in E(H_Z)} \bar{y}_{S,i} = 2 > 1.
$$

This is a contradiction to the fact that $(\bar{x}, \bar{y})$ satisfies inequalities (4). As a consequence, we conclude that $G[V_i]$ is connected for each $i \in [k]$. Therefore $\{V_i\}_{i \in [k]}$ is a connected $k$-partition of $G$.

Finally, it follows from Lemma 5 that the proposed integer linear program has an optimal solution that corresponds to an optimal connected $k$-partition of $G$. As a consequence, it is equivalent to solving the MAX-MIN-BCP problem on instance $(G, k)$. \hfill \Box

The main tool to design fixed-parameter tractable algorithms using integer linear programming (ILP) is a theorem due to Lenstra [17] which shows that checking the feasibility of an ILP problem with a fixed number of variables can be solved in polynomial time. The time and space complexity of Lenstra’s algorithm were later improved by Kannan [16], and Frank and Tardos [13]. In this work, we consider the following optimization version of their results.

In the INTEGER LINEAR PROGRAMMING problem, we are given as input a matrix $A \in \mathbb{Z}^{p \times q}$, vectors $b \in \mathbb{Z}^p$ and $c \in \mathbb{Z}^q$. The objective is to find a vector $x \in \mathbb{Z}^q$ that satisfies all inequalities (i.e. $Ax \leq b$), and maximizes $c^T x$. Let us denote by $L$ the size of the binary representation of an input $(A, b, c)$ of the problem.

We next present the maximization version of the theorem showed in Cygan et al. [10] on the existence of an FPT algorithm for INTEGER LINEAR PROGRAMMING parameterized by the number of variables.

**Theorem 11** (Cygan et al. [10]). An INTEGER LINEAR PROGRAMMING instance of size $L$ with $q$ variables can be solved using $O(q^{4.54} + o(q)) \cdot (L + \log M_x) \log(M_x M_z)$ arithmetic operations and space polynomial in $L + \log M_x$, where $M_x$ is an upper bound on the
absolute value a variable can take in a solution, and \( M_c \) is the largest absolute value of a coefficient in the vector \( c \).

The previous theorem is now used to show that the max-min unweighted version of the balanced connected partition problem admits an algorithm that runs in time doubly exponential in the size of a vertex cover of the input graph.

**Theorem 12.** The problem max-min-BCP, parameterized by the size of a vertex cover of the input graph, is fixed-parameter tractable.

**Proof.** Consider an instance \((G, k)\) of max-min-BCP, and a vertex cover \( X \) of \( G \). It follows from Lemma 10 that \( \max \left\{ \sum_{v \in X} x_{v,1} + \sum_{S \subseteq X} y_{S,1} : (x, y) \in \mathcal{B}(G, X, k) \right\} \) is equivalent to solving instance \((G, k)\). Observe now that the size of this integer linear program (ILP) is \( 2^{O(|X|)} \log |G| \). By Theorem 11, this ILP problem can be solved in time \( 2^{2^{O(|X|)}} |G|^{O(1)} \). Therefore max-min-BCP is fixed parameter-tractable when parameterized by the size of a vertex cover of the input graph. \( \square \)

**References**

[1] P. Alimonti and T. Calamoneri. On the complexity of the max balance problem. In Argentinian Workshop on Theoretical Computer Science (WAIT’99), pages 133–138, 1999.

[2] R. I. Becker, I. Lari, M. Lucertini, and B. Simeone. Max-min partitioning of grid graphs into connected components. Networks, 32(2):115–125, 1998.

[3] R. I. Becker and Y. Perl. Shifting algorithms for tree partitioning with general weighting functions. Journal of Algorithms, 4(2):101–120, 1983.

[4] R. I. Becker, S. R. Schach, and Y. Perl. A shifting algorithm for min-max tree partitioning. J. ACM, 29(1):58–67, 1982.

[5] R. Borndörfer, Z. Elijazyfer, and S. Schwartz. Approximating balanced graph partitions. Technical Report 19-25, ZIB, Takustr. 7, 14195 Berlin, 2019.

[6] F. Chataigner, L. R. B. Salgado, and Y. Wakabayashi. Approximation and inapproximability results on balanced connected partitions of graphs. Discrete Mathematics & Theoretical Computer Science, Vol. 9 no. 1, 2007.

[7] G. Chen, Y. Chen, Z.-Z. Chen, G. Lin, T. Liu, and A. Zhang. Approximation algorithms for the maximally balanced connected graph tripartition problem. J. Comb. Optim., pages 1–21, 2020.

[8] Y. Chen, Z.-Z. Chen, G. Lin, Y. Xu, and A. Zhang. Approximation algorithms for maximally balanced connected graph partition. In International Conference on Combinatorial Optimization and Applications, pages 130–141. Springer, 2019.

[9] J. Chlebíková. Approximating the maximally balanced connected partition problem in graphs. Information Processing Letters, 60(5):225–230, 1996.

[10] M. Cygan, F. V. Fomin, L. Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk, and S. Saurabh. Miscellaneous, pages 129–150. Springer International Publishing, Cham, 2015.
[11] M. Dyer and A. Frieze. On the complexity of partitioning graphs into connected subgraphs. *Discrete Applied Mathematics*, 10(2):139–153, 1985.

[12] M. R. Fellows, D. Lokshtanov, N. Misra, F. A. Rosamond, and S. Saurabh. Graph layout problems parameterized by vertex cover. In *Proceedings of the 19th International Symposium on Algorithms and Computation*, ISAAC ’08, page 294–305, Berlin, Heidelberg, 2008. Springer-Verlag.

[13] A. Frank and É. Tardos. An application of simultaneous diophantine approximation in combinatorial optimization. *Combinatorica*, 7(1):49–65, 1987.

[14] G. N. Frederickson. Optimal algorithms for tree partitioning. In *Proceedings of the Second Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA ’91, page 168–177, USA, 1991. Society for Industrial and Applied Mathematics.

[15] E. Györi. On division of graph to connected subgraphs. In *Combinatorics (Proc. Fifth Hungarian Colloq., Keszthely, 1976)*, volume 18 of *Colloq. Math. Soc. János Bolyai*, pages 485–494, 1978.

[16] R. Kannan. Minkowski’s convex body theorem and integer programming. *Mathematics of Operations Research*, 12(3):415–440, 1987.

[17] H. W. Lenstra. Integer programming with a fixed number of variables. *Mathematics of Operations Research*, 8(4):538–548, 1983.

[18] L. Lovász. A homology theory for spanning tress of a graph. *Acta Mathematica Academiae Scientiarum Hungarica*, 30:241–251, 1977.

[19] M. Lucertini, Y. Perl, and B. Simeone. *Image enhancement by path partitioning*, pages 12–22. Springer Berlin Heidelberg, 1989.

[20] M. Lucertini, Y. Perl, and B. Simeone. Most uniform path partitioning and its use in image processing. *Discrete Applied Mathematics*, 42(2):227–256, 1993.

[21] M. Maravalle, B. Simeone, and R. Naldini. Clustering on trees. *Computational Statistics & Data Analysis*, 24(2):217–234, 1997.

[22] D. Matić and M. Božić. Maximally balanced connected partition problem in graphs: application in education. *The Teaching of Mathematics*, (29):121–132, 2012.

[23] D. Matić. A mixed integer linear programming model and variable neighborhood search for maximally balanced connected partition problem. *Applied Mathematics and Computation*, 237:85–97, 2014.

[24] D. Matić and M. Grbić. Partitioning weighted metabolic networks into maximally balanced connected partitions. In *2020 19th International Symposium INFOTEH-JAHORINA (INFOTEH)*, pages 1–6, 2020.

[25] F. K. Miyazawa, P. F. Moura, M. J. Ota, and Y. Wakabayashi. Partitioning a graph into balanced connected classes: Formulations, separation and experiments. *European Journal of Operational Research*, 293(3):826–836, 2021.

[26] F. K. Miyazawa, P. F. S. Moura, M. J. Ota, and Y. Wakabayashi. Cut and flow formulations for the balanced connected k-partition problem. In M. Baïou, B. Gendron, O. Günlük, and A. R. Mahjoub, editors, *Combinatorial Optimization*, pages 128–139. Springer International Publishing, 2020.
[27] S. Nakano, M. Rahman, and T. Nishizeki. A linear-time algorithm for four-partitioning four-connected planar graphs. *Information Processing Letters*, 62(6):315–322, 1997.

[28] Y. Perl and S. R. Schach. Max-min tree partitioning. *J. ACM*, 28(1):5–15, 1981.

[29] H. Suzuki, N. Takahashi, and T. Nishizeki. A linear algorithm for bipartition of biconnected graphs. *Information Processing Letters*, 33(5):227–231, 1990.

[30] H. Suzuki, N. Takahashi, T. Nishizeki, H. Miyano, and S. Ueno. An algorithm for tripartitioning 3-connected graphs. *Journal of Information Processing Society of Japan*, 31(5):584–592, 1990.

[31] B. Y. Wu. Fully polynomial-time approximation schemes for the max-min connected partition problem on interval graphs. *Discrete Mathematics, Algorithms and Applications*, 04(01):1250005, 2012.

[32] X. Zhou, H. Wang, B. Ding, T. Hu, and S. Shang. Balanced connected task allocations for multi-robot systems: An exact flow-based integer program and an approximate tree-based genetic algorithm. *Expert Systems with Applications*, 116:10–20, 2019.