On Coreflexive Coalgebras and Comodules over Commutative Rings

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Abstract

In this note we study dual coalgebras of algebras over arbitrary (noetherian) commutative rings. We present and study a generalized notion of coreflexive comodules and use the results obtained for them to characterize the so called coreflexive coalgebras. Our approach in this note is an algebraically topological one.

Introduction

The concept of coreflexive coalgebras was studied, in the case of commutative base fields, by several authors. An algebraic approach was presented by E. Taft ([Taf72], [Taf77]), while a topological one was presented mainly by D. Radford ([HR74], [Rad73]) and studied by several authors (e.g. [Miy75], [Wit79]). In this note we present and study a generalized concept of coreflexive comodules and use it to characterize coreflexive coalgebras over commutative (noetherian) rings. In particular we generalize results from the papers mentioned above from the case of base fields to the case of arbitrary (noetherian) commutative ground rings.

Throughout this paper $R$ denotes a commutative ring with $1_R \neq 0_R$. We consider $R$ as a left and a right linear topological ring with the discrete topology. The category of $R$-(bi)modules will be denoted by $\mathcal{M}_R$. The unadorned $-\otimes -$ and Hom mean $-\otimes_R -$ and $\text{Hom}_R$ respectively. For an $R$-module $M$, an $R$-submodule $K \subset M$ will be called $N$-pure for some $R$-module $N$, if the canonical $R$-linear mapping $\iota_K \otimes \text{id}_N : K \otimes_R N \to M \otimes_R N$ is injective. We call $K \subset M$ pure (in the sense of Cohn), if it’s $N$-pure for every $R$-module $N$. For every $R$-module $L$, we denote with $L^*$ the algebraic dual $R$-module of all $R$-linear maps from $L$ to $R$.

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Let $S$ be a ring. We consider every left (right) $S$-module $K$ as a right (a left) module over $\text{End}(sK)^{\text{op}}$ ($\text{End}(K_S)$) and as a left (a right) module over $\text{Biend}(sK) := \text{End}(K_{\text{End}(sK)^{\text{op}}})$ ($\text{Biend}(K_S) := \text{End}(\text{End}(K_S))^{\text{op}}$), the ring of biendomorphisms of $K$ (e.g. [Wis88, 6.4]).

Let $A$ be an $R$-algebra and $M$ be an $A$-module. An $A$-submodule $N \subset M$ will be called $R$-cofinite, if $M/N$ is finitely generated in $\mathcal{M}_R$. The class of all $R$-cofinite $A$-submodules of $M$ is denoted with $\mathcal{K}_M$. We call $M$ cofinitely $R$-cogenerated, if $M/N$ is $R$-cogenerated for every $R$-cofinite $A$-submodule. With $\mathcal{K}_A$ we denote the class of all $R$-cofinite $A$-ideals and define

$$A^o := \{ f \in A^* | f(I) = 0 \text{ for some } R\text{-cofinite ideal } I \triangleleft A \}.$$ 

If $\mathcal{K}_A$ is a filter (e.g. $R$ is a noetherian ring), then $A^o \subset A^*$ is an $R$-submodule with equality, iff $R_A$ is finitely generated projective.

We assume the reader is familiar with the theory of Hopf Algebras. For any needed definitions or results the reader may refer to any of the classical books on the subject (e.g. [Swe69], [Abe80] and [Mon93]). For an $R$-coalgebra $(C, \Delta_C, \varepsilon_C)$ and an $R$-algebra $(A, \mu_A, \eta_A)$ we consider $\text{Hom}_R(C, A)$ as an $R$-algebra with multiplication the convolution product $(f \ast g)(c) := \sum f(c_1)g(c_2)$ and unity $\eta_A \circ \varepsilon_C$.

## 1 Preliminaries

In this section we present some definitions and lemmata.

**Definition 1.1.** Let $(C, \Delta_C, \varepsilon_C)$ be an $R$-coalgebra. We call an $R$-submodule $K \subset C$:

- an $R$-subcoalgebra, iff $K \subset C$ is pure and $\Delta_C(K) \subset K \otimes_R K$;
- a $C$-coideal, iff $K \subset \text{Ke}(\varepsilon_C)$ and

$$\Delta_C(K) \subset \text{Im}(\iota_K \otimes \text{id}_C) + \text{Im}(\text{id}_C \otimes \iota_K);$$

- a right $C$-coideal (resp. a left $C$-coideal, a $C$-bicoideal), if $K \subset C$ is $C$-pure and $\Delta_C(K) \subset K \otimes_R C$ (resp. $\Delta_C(K) \subset C \otimes_R K$, $\Delta_C(K) \subset (K \otimes_R C) \cap (C \otimes_R K)$).

### 1.2. Subgenerators

Let $A$ be an $R$-algebra and $K$ be a left $A$-module. We say a left $A$-module $N$ is $K$-subgenerated, if $N$ is isomorphic to a submodule of a $K$-generated left $A$-module (equivalently, if $N$ is kernel of a morphism between $K$-generated left $A$-modules). The full subcategory of $\mathcal{AM}$, whose objects are the $K$-subgenerated left $A$-modules is denoted by $\sigma[AK]$. In fact $\sigma[AK] \subset \mathcal{AM}$ is the smallest Grothendieck full subcategory that contains $K$. If $M$ is a left $A$-module, then

$$\text{Sp}(\sigma[AK], M) := \sum \{ f(N) \mid f \in \text{Hom}_{A-}(N, M), \ N \in \sigma[AK] \}$$

is the biggest $A$-submodule of $M$ that belongs to $\sigma[AK]$. The reader is referred to [Wis88] and [Wis96] for the well developed theory of categories of this type.
The linear weak topology

1.3. R-Pairings. An R-pairing \( P = (V,W) \) consists of R-modules \( V,W \) with an R-bilinear form
\[ \alpha : V \times W \to R, \quad (v,w) \mapsto <v,w>. \]
If the induced R-linear mapping \( \kappa_P : V \to W^* \) (resp. \( \chi_P : W \to V^* \)) is injective, then we call \( P \) left non-degenerating (resp. right non-degenerating). If both \( \kappa_P \) and \( \chi_P \) are injective, then we call \( P \) non-degenerating.

For R-pairings \( (V,W) \) and \( (V',W') \) a morphism \( (\xi, \theta) : (V',W') \to (V,W) \) consists of R-linear mappings \( \xi : V \to V' \) and \( \theta : W' \to W \), such that
\[ <\xi(v),w' > = <v,\theta(w')> \text{ for all } v \in V \text{ and } w' \in W'. \]

The R-pairings with the morphisms described above (and the usual composition of pairings) build a category which we denote with \( P \). If \( P = (V,W) \) is an R-pairing, \( V' \subset V \) is an R-submodule and \( W' \subset W \) is a (pure) R-submodule with \( < V',W' >= 0 \), then \( Q := (V/V',W') \) is an R-pairing, \( (\pi, \iota_K) : (V/V',W') \to (V,W) \) is a morphism in \( P \) and we call \( Q \subset P \) a (pure) R-subpairing.

Notation. Let \( P = (V,W) \) be an R-pairing. For subsets \( X \subset V \) resp. \( K \subset W \) set
\[ X^\perp := \{w \in W| <X,w> = 0\} \text{ resp. } K^\perp := \{v \in V| <v,K> = 0\}. \]
We say \( X \subset V \) (resp. \( K \subset W \)) is orthogonally closed w.r.t. \( P \), if \( X = X^{\perp\perp} \) (resp. \( K = K^{\perp\perp} \)). In case \( V = W^* \), then we set for every subset \( X \subset W^* \) (resp. \( K \subset W \)) \( \text{Ke}(X) = X^{\perp} \) (resp. \( \text{An}(K) =: K^{\perp} \)).

1.4. Let \( P = (V,W) \) be an R-pairing. Then the class of R-submodules of \( V \) :
\[ \mathcal{F}(0_V) := \{K^{\perp}| K \subset W \text{ is a finitely generated R-submodule}\} \]
is a filter basis consisting of R-submodule of \( V \) and induces on \( V \) a topology, the so called linear weak topology \( V[\mathfrak{W}_R(W)] \), such that \( (V,V[\mathfrak{W}_R(W)]) \) is a linear topological right R-module and \( \mathcal{F}(0_V) \) is a neighbourhood basis of \( 0_V \). In particular we call \( W^*[\mathfrak{W}_R(W)] \) the finite topology. The properties of this topology were studied by several authors in the case of commutative base fields (e.g. [Köt66], [KN63], [Rad73]). We refer mainly to the recent work of the author [Abu] for the case of arbitrary ground rings.
The $\alpha$-condition

In a joint work with J. Gómez-Torrecillas and J. Lobillo [AG-TL01] on the category of comodules of coalgebras over arbitrary commutative base rings, we presented the so called $\alpha$-condition. That condition has shown to be a natural assumption in the author’s study of duality theorems for Hopf algebras [Abu01]. We refer mainly to [Abu] for the properties of the such pairings over arbitrary ground rings.

1.5. $R$-Pairings. We say an $R$-pairing $P = (V, W)$ satisfies the $\alpha$-condition (or $P$ is an $\alpha$-pairing), if for every $R$-module $M$ the following map is injective

$$\alpha^P_M : M \otimes_R W \to \text{Hom}_R(V, M), \sum m_i \otimes w_i \mapsto v \mapsto \sum m_i < v, w_i >.$$  \hfill (1)

With $P^\alpha \subset P$ we denote the full subcategory of $R$-pairings satisfying the $\alpha$-condition. We call an $R$-pairing $P = (V, W)$ dense, if $\kappa_P(V) \subseteq W^*$ is dense (considering $W^*$ with the finite topology). It’s easy to see that $P^\alpha \subset P$ is closed under pure $R$-subpairings.

We say an $R$-module $W$ satisfies the $\alpha$-condition, if the $R$-pairing $(W^*, W)$ satisfies the $\alpha$-condition, i.e. for every $R$-module $M$ the canonical $R$-linear morphism $\alpha^W_M : M \otimes_R W \to \text{Hom}_R(W^*, M)$ in injective (equivalently, if $RW$ is locally projective in the sense of B. Zimmermann-Huisgen [ZH76]).

Remark 1.6. [Abu, Remark 2.2] Let $P = (V, W) \in P^\alpha$. Then $RW$ is $R$-cogenerated and flat. If $R$ is perfect, then $RW$ turns to be projective.

Notation. Let $W, W'$ be $R$-modules and consider for any $R$-submodules $X \subseteq W^*$ and $X' \subseteq W'^*$ the canonical $R$-linear mapping

$$\delta : X \otimes_R X' \to (W \otimes_R W')^*.$$  \hfill (2)

For $f \in X$ and $g \in X'$ set $f \otimes g = \delta(f \otimes g)$, i.e.

$$(f \otimes g)(\sum w_i \otimes w'_i) := \sum f(w_i)g(w'_i) \text{ for every } \sum w_i \otimes w'_i \in W \otimes_R W'.$$

2 Measuring $R$-pairings

2.1. For an $R$-coalgebra $C$ and an $R$-algebra $A$ we call an $R$-pairing $P = (A, C)$ a measuring $R$-pairing, if the induced mapping $\kappa_P : A \to C^*$ is an $R$-algebra morphism. In this case $C$ is an $A$-bimodule through the left and the right $A$-actions

$$a \rightarrow c := \sum c_1 < a, c_2 > \quad \text{and} \quad c \leftarrow a := \sum < a, c_1 > c_2 \text{ for all } a \in A, \ c \in C.$$  \hfill (2)

Let $(A, C)$ and $(B, D)$ be measuring $R$-pairings. We say a morphism of $R$-pairings $(\xi, \theta) : (B, D) \to (A, C)$ is a morphism of measuring $R$-pairings, if $\xi : A \to B$ is an $R$-algebra
morphism and \( \theta : D \to C \) is an \( R \)-coalgebra morphism. The category of measuring \( R \)-pairings and morphisms described above will be denoted by \( \mathcal{P}_m \). With \( \mathcal{P}^m \subset \mathcal{P}_m \) we denote the full subcategory of measuring \( R \)-pairings satisfying the \( \alpha \)-condition (we call these measuring \( \alpha \)-pairings). If \( P = (A, C) \) is a measuring \( R \)-pairing, \( D \subset C \) is an \( R \)-subcoalgebra and \( I \trianglelefteq A \) is an ideal with \( < I, D > = 0 \), then \( Q := (A/I, C) \) is a measuring \( R \)-pairing, \((\pi_I, \iota_D) : (A/I, D) \to (A, C)\) is a morphism in \( \mathcal{P}_m \) and we call \( Q \subset P \) a measuring \( R \)-subpairing. Since by convention an \( R \)-coalgebra is a pure \( R \)-submodule, it's easy to see that \( \mathcal{P}^m \subset \mathcal{P}_m \) is closed against measuring \( R \)-subpairings.

**Lemma 2.2.** Let \( P = (A, C), Q = (B, D) \in \mathcal{P}_m \) and \((\xi, \theta) : (B, D) \to (A, C)\) be a morphism of \( R \)-pairings.

1. Assume that \( P \otimes P := (A \otimes_R A, C \otimes_R C) \) is right non-degenerating (i.e. \( \chi := \chi_{P \otimes P} : C \otimes_R C \hookrightarrow (A \otimes_R A)^* \) is an embedding). If \( \xi \) is an \( R \)-algebra morphism, then \( \theta \) is an \( R \)-coalgebra morphism. If \( A \) is commutative, then \( C \) is cocommutative.

2. If \( Q \) is left non-degenerating (i.e. \( B \xleftarrow{Q^*} D^\ast \) is an embedding) and \( \theta \) is an \( R \)-coalgebra morphism, then \( \xi \) is an \( R \)-algebra morphism. If \( C \) is cocommutative and \( P \) is left non-degenerating (i.e. \( A \subseteq C^* \)), then \( A \) is commutative.

**Proof.**

1. If \( \xi \) is an \( R \)-algebra morphism, then we have for arbitrary \( d \in D, a, \tilde{a} \in A : \)

\[
\chi(\sum \theta(d)_1 \otimes \theta(d)_2)(a \otimes \tilde{a}) = \sum < a,\theta(d)_1 > < \tilde{a}, \theta(d)_2 > = < a\tilde{a}, \theta(d) > = < \xi(a\tilde{a}), d > = < \xi(a)\xi(\tilde{a}), d > = \sum < \xi(a), d_1 > < \xi(\tilde{a}), d_2 > = \sum < a, \theta(d_1) > < \tilde{a}, \theta(d_2) > = \chi(\sum \theta(d_1) \otimes \theta(d_2))(a \otimes \tilde{a}).
\]

By assumption \( \chi \) is injective and so \( \sum \theta(d)_1 \otimes \theta(d)_2 = \sum \theta(d_1) \otimes \theta(d_2) \) for every \( d \in D, \) i.e. \( \theta \) is an \( R \)-coalgebra morphism.

If \( A \) is commutative, then we have for all \( c \in C \) and \( a, \tilde{a} \in A : \)

\[
\chi(\sum c_1 \otimes c_2)(a \otimes \tilde{a}) = \sum < a, c_1 > < \tilde{a}, c_2 > = < a\tilde{a}, c > = < \tilde{a}a, c > = \sum < a, c_2 > < \tilde{a}, c_1 > = \chi(\sum c_2 \otimes c_1)(a \otimes \tilde{a}).
\]

By assumption \( \chi \) is injective and so \( \sum c_1 \otimes c_2 = \sum c_2 \otimes c_1 \) for every \( c \in C, \) i.e. \( C \) is cocommutative.

2. Analogous to (1). \( \blacksquare \)
Lemma 2.5. Let $A$ be an $R$-algebra and $N$ be a left $A$-module (resp. a right $A$-module). For subsets $X,Y \subset N$ we set

$$(Y : X) := \{a \in A \mid aX \subset Y\} \quad \text{(resp.} \quad (Y : X) := \{a \in A \mid Xa \subset Y\}).$$

If $Y = \{0_N\}$, then we set also $\text{An}_A(X) := (0_N : X)$. If $N$ is an $A$-bimodule, then we set for every subset $X \subset N$:

$$\text{An}_A^l(X) := \{a \in A \mid aX = 0_N\} \quad \text{and} \quad \text{An}_A^r(X) := \{a \in A \mid Xa = 0_N\}.$$  

2.3. The $C$-adic topology. Let $(A, C) \in \mathcal{P}_m$ and consider $C$ as a left $A$-module with the left $A$-action “$-”$ in (2). Then the class of left $A$-ideals

$$\mathcal{B}_{C-}(0_A) := \{\text{An}_A^l(W) = (0 : W) \mid W = \{c_1, ..., c_k\} \subset C \text{ a finite subset}\}$$

is a neighbourhood basis of $0_A$ and induces on $A$ a topology, the so called left $C$-adic topology $\mathcal{T}_{C-}(A)$, so that $(A, \mathcal{T}_{C-}(A))$ is a left linear topological $R$-algebra (see [AW97], [Ber94]). A left $A$-ideal $I \trianglelefteq A$ is open w.r.t. $\mathcal{T}_{C-}(A)$, iff $A/I$ is $C$-subgenerated. If $\mathcal{T}$ is a left linear topology on $A$, then the category of discrete left $(A, \mathcal{T})$-modules is equal to the category of $C$-subgenerated left $A$-modules $\sigma_A[C]$, iff $\mathcal{T} = \mathcal{T}_{C-}(A)$. In particular we have for every left $A$-module $N$:

$$\text{Sp}(\sigma_A[C], N) = \{n \in N \mid \exists F = \{c_1, ..., c_k\} \subset C \text{ with } (0_C : F) \subset (0_N : n)\}.$$  

Analogously $C_A$ induces on $A$ a topology, the so called right $C$-adic topology $\mathcal{T}_{C-}(A)$, such that $(A, \mathcal{T}_{C-}(A))$ is a right linear topological $R$-algebra.

Rational modules

2.4. Let $P = (A, C)$ be a measuring $\alpha$-pairing. Let $M$ be a left $A$-module, $\rho_M : M \to \text{Hom}_R(A, M)$ be the canonical $A$-linear mapping and $\text{Rat}_C(A,M) := \rho_M^{-1}(M \otimes_R C)$. In case $\text{Rat}_C(A,M) = M$ we call $M$ a $C$-rational left $A$-module and define

$$\rho_M := (\alpha_M^P)^{-1} \circ \rho_M : M \to M \otimes_R C.$$ 

Analogously one defines the $C$-rational right $A$-modules. With $\text{Rat}_C(A, M_A) \subset M_A$ (resp. $\text{Rat}(M_A) \subset M_A$) we denote the full subcategory of $C$-rational left (resp. right) $A$-modules.

Lemma 2.5. ([Abu01, Lemma 2.2.7]) Let $P = (A, C)$ be a measuring $\alpha$-pairing. For every left $A$-module $M$ we have:

1. $\text{Rat}_C(A,M) \subseteq M$ is an $A$-submodule.

2. For every $A$-submodule $N \subset M$ we have $\text{Rat}_C(A,N) = N \cap \text{Rat}_C(A,M)$.
3. \( \text{Rat}^C(\text{Rat}^C(AM)) = \text{Rat}^C(AM) \).

4. For every \( L \in AM \) and \( f \in \text{Hom}_{AM}(M,L) \) we have \( f(\text{Rat}^C(AM)) \subseteq \text{Rat}^C(AL) \).

**Theorem 2.6.** ([Abu01, Lemmata 2.2.8, 2.2.9, Satz 2.2.16]) Let \( P = (A,C) \) be a measuring \( R \)-pairing. Then \( M^C \subseteq AM \) and \( CM \subseteq MA \) (not necessarily full subcategories). Moreover the following are equivalent:

1. \( P \) satisfies the \( \alpha \)-condition;
2. \( R^C \) is locally projective and \( \kappa_P(A) \subseteq C^* \) is dense.

If these equivalent conditions are satisfied, then \( M^C \subseteq AM \) and \( CM \subseteq MA \) are full subcategories and we have category isomorphisms

\[
M^C \cong \text{Rat}^C(AM) = \sigma[A^C] \quad \text{and} \quad CM \cong \text{Rat}(MA) = \sigma[C^A] \\
\cong \text{Rat}(C^*M) = \sigma[C^C] \quad \text{and} \quad \cong \text{Rat}(MC^*) = \sigma[C^A] \]

(3)

**Corollary 2.7.** Let \( Q = (B,C) \in P_m, \xi : A \to B \) be an \( R \)-algebra morphism and consider the induced measuring \( R \)-pairing \( P := (A,C) \). Then the following statements are equivalent:

(i) \( P \in P_m \);
(ii) \( Q \in P^a_m \) and \( \xi(A) \subseteq B \) is dense (w.r.t. the \( C \)-adic topology \( T_{C^*}(B) \));
(iii) \( C \) satisfies the \( \alpha \)-condition and \( \kappa_P(A) \subseteq C^* \) is dense.

If these equivalent conditions are satisfied, then we get category isomorphisms

\[
M^C \cong \text{Rat}^C(AM) = \sigma[A^C] \quad \text{and} \quad CM \cong \text{Rat}(MA) = \sigma[C^A] \\
\cong \text{Rat}(C^*M) = \sigma[C^C] \quad \text{and} \quad \cong \text{Rat}(MC^*) = \sigma[C^A] \]

(4)

2.8. Let \( (C, \Delta_C, \varepsilon_C) \) be an \( R \)-coalgebra and denote with \( \text{End}^C(C) \) (resp. \( C\text{End}(C) \)) the ring of all right (resp. left) \( C \)-colinear morphisms from \( C \) to \( C \) with the usual composition. For every right \( C \)-comodule \( M \) we have an isomorphism of \( R \)-modules

\[ \Psi : M^* \to \text{Hom}^C(M,C), \ h \mapsto [m \mapsto \sum f(m_{<0>})m_{<1>}] \]

(5)

with inverse \( g \mapsto \varepsilon_C \circ g \). Analogously \( N^* \cong C\text{Hom}(N,C) \) as \( R \)-modules for every left \( C \)-comodule \( N \). In particular \( C^* \cong \text{End}^C(C)^{op} \) and \( C^* \cong C\text{End}(C) \) as \( R \)-algebras.

If \( (A,C) \) is a measuring \( \alpha \)-pairing, then we have \( R \)-algebra isomorphisms

\[ \text{Biend}(AC) := \text{End}(C_{\text{End}(AC)^{op}}) \cong \text{End}(C_{\text{End}^C(C)^{op}}) \cong \text{End}(C^{C^*}) = C\text{End}(C) \cong C^* \]

and

\[ \text{Biend}(CA) := \text{End}(C_{\text{End}(CA)^{op}}) \cong \text{End}(C_{\text{End}(C)^{op}}) \cong \text{End}(C^{C^*})^{op} \cong \text{End}^C(C)^{op} \cong C^* \]

In particular, if \( R^C \) is locally projective, then \( \text{Biend}(C^*) \cong C^* \cong \text{Biend}(C^{C^*}) \) as \( R \)-algebras (i.e. \( C^* \) is faithfully balanced).
Corollary 2.9. Let $P = (A, C) \in \mathcal{P}_m^n$ and consider $A^*$ as an $A$-bimodule with the regular $A$-actions
\[(af)(\bar{a}) = f(\bar{a}a) \text{ and } (fa)(\bar{a}) = f(a\bar{a}).\] (6)

1. For every unitary left (right) $A$-submodule $D \subseteq A^*$ we have
\[\text{Rat}^C(AD) = C \cap D (\text{C-Rat}(DA) = C \cap D).\]
In particular $\text{Rat}^C(\alpha A^*) = C = \text{C-Rat}(A^*_\alpha)$.

2. If $D \subseteq A^*$ is an $A$-subbimodule, then $\alpha D$ is $C$-rational, iff $DA$ is $C$-rational.

3. Let $R$ be noetherian. If $\alpha A^\circ$ (equivalently $A^\circ_\alpha$) is $C$-rational, then $C = A^\circ$.

**Proof.**

1. Let $D \subseteq A^*$ be a left $A$-submodule. By Lemma 2.5 (2) $C \cap D$ is a $C$-rational left $A$-module, i.e. $C \cap D \subseteq \text{Rat}^C(AD)$. On the other hand, if $f \in \text{Rat}^C(AD)$ with \[\varrho_D(f) = \sum f_i \otimes c_i \in D \otimes_R C,\] then we have for every $a \in A$:
\[f(a) = (af)(1_A) = \sum f_i(1_A) < a, c_i >,\]
\[\text{i.e. } f = \sum f_i(1_A) c_i \in C. \text{ Hence } \text{Rat}^C(AD) = C \cap D. \text{ The corresponding result for right } A\text{-submodules } D \subseteq A^* \text{ follows by symmetry.}\]

2. Let $D \subseteq A^*$ be an $A$-subbimodule. Then by (1) $\text{Rat}^C(\alpha D) = C \cap D = \text{C-Rat}(DA)$.

3. If $R$ is noetherian, then $A^\circ \subseteq A^*$ is an $A$-subbimodule. Obviously $C \xrightarrow{\text{C}} A^\circ$ and it follows by assumption and (1) that $A^\circ = \text{Rat}^C(\alpha A^\circ) = C \cap A^\circ = C$. \[\square\]

An important rule by the study of the category of rational representations of measuring $\alpha$-pairings is played by the

2.10. Finiteness Theorem.

1. Let $P = (A, C)$ be a measuring $\alpha$-pairing. If $M \in \text{Rat}^C(A\mathcal{M})$ (resp. $M \in \text{C-Rat}(\mathcal{M}_A)$, $M \in \text{C-Rat}^C(\mathcal{M}_A)$), then there exists for every finite set $\{m_1, ..., m_k\} \subseteq M$ some $N \in \text{Rat}^C(A\mathcal{M})$ (resp. $N \in \text{C-Rat}(\mathcal{M}_A)$, $N \in \text{C-Rat}^C(\mathcal{M}_A)$), such that $N_R$ is finitely generated.

2. Let $C$ be a locally projective $R$-coalgebra. Then every finite subset of $C$ is contained in a right $C$-coideal (resp. a left $C$-coideal, a $C$-bicoideal), that is finitely generated in $\mathcal{M}_R$.

**Proof.**

1. Let $P = (A, C) \in \mathcal{P}_m^n$. Let $M \in \text{Rat}^C(A\mathcal{M})$ and $\{m_1, ..., m_k\} \subseteq M$. Then $Am_i \subseteq M$ is an $A$-submodule, hence a $C$-subcomodule. Moreover $m_i \in Am_i$ and so there exists a subset $\{(m_{ij}, c_{ij})\}_{j=1}^{n_i} \subseteq Am_i \times C$, such that $\varrho_M(m_i) = \sum_{j=1}^{n_i} m_{ij} \otimes c_{ij}$ for
Obviously \( N := \sum_{i=1}^{k} Am_i = \sum_{i=1}^{k} \sum_{j=1}^{n_i} Rm_{ij} \subset M \) is a \( C \)-subcomodule and contains \( \{m_1, ..., m_k\} \).

Using analogous arguments one can show the corresponding result for \( C \)-rational right \( A \)-modules and \( C \)-birational \( A \)-bimodules.

2. If \( C \) is a locally projective \( R \)-coalgebra, then \( (C^*, C) \in \mathcal{P}_m^\alpha \) and the result follows by (1).\( \blacksquare \)

The following result gives topological characterizations of the \( C \)-rational left \( A \)-modules and generalizes the corresponding result obtained by D. Radford [Rad73, 2.2] from the case of base fields to the case of arbitrary (artinian) commutative ground rings (see also [LR97, Proposition 1.4.4]).

**Proposition 2.11.** Let \( P = (A, C) \) be a measuring \( \alpha \)-pairing and consider \( A \) with the \( C \)-adic topology \( \mathcal{T}_{C,-}(A) = A[\Sigma_{ls}(C)] \). If \( M \) is a unitary left \( A \)-module, then for every \( m \in M \) the following statements are equivalent:

1. there exists a finite subset \( W = \{c_1, ..., c_k\} \subset C \), such that \( (0_C : W) \subset (0_M : m) \).
2. \( Am \) is \( C \)-subgenerated;
3. \( m \in \text{Rat}^C_{(A M)} \).
4. there exists a finitely generated \( R \)-submodule \( K \subset C \), such that \( K^\perp \subset (0_M : m) \).

If \( R \) is artinian, then “1-4” are moreover equivalent to:

5. \( (0_M : Am) \) contains an \( R \)-cofinite closed \( R \)-submodule of \( A \);
6. \( (0_M : Am) \) is an \( R \)-cofinite closed \( A \)-ideal;
7. \( (0_M : m) \) contains an \( R \)-cofinite closed \( A \)-ideal;
8. \( (0_M : m) \) is an \( R \)-cofinite closed left \( A \)-ideal.

**Proof.** (1) \( \Rightarrow \) (2) By assumption and 2.3 \( m \in N := \text{Sp}(\sigma_{[A C]} , M) \). Since \( Am \subset N \) is an \( A \)-submodule, it’s \( C \)-subgenerated.

(2) \( \Rightarrow \) (3) By assumption and Theorem 2.6 \( m \in Am \subset \text{Rat}^C_{(A M)} \).

(3) \( \Rightarrow \) (4) Let \( \varrho(m) = \sum_{i=1}^{k} m_i \otimes c_i \) and \( K := \sum_{i=1}^{k} Rc_i \subset C \). Then obviously \( K^\perp \subset (0_M : m) \).

(4) \( \Rightarrow \) (1) For every subset \( W \subset C \) we have \( (0_C : W) \subset W^\perp \).

Let \( R \) be artinian.
(3) ⇒ (5). By Theorem 2.6 Rat\(^C\)\((A,M)\) is a C-rational left A-module. Assume that 
\(g_M(m) = \sum_{i=1}^{k} m_i \otimes c_i \in \text{Rat}^C(A,M) \otimes_R C\), \(g_M(m_i) = \sum_{j=1}^{n_i} m_{ij} \otimes c_{ij}\) for \(i = 1, \ldots, k\) and set 
\(K := \sum_{i=1}^{k} \sum_{j=1}^{n_i} Rc_{ij}\). Then we have for every \(a \in K^\perp\) and arbitrary \(b \in A\):

\[a(bm) = a(\sum_{i=1}^{k} m_i < b, c_i>) = \sum_{i=1}^{k} \sum_{j=1}^{n_i} m_{ij} < a, c_{ij}> < b, c_i> = 0,\]

i.e. \(K^\perp \subseteq (0_M : Am)\). The \(R\)-module \(K\) is finitely generated and it follows from the 
embedding \(A/K^\perp \hookrightarrow K^*\), that \(K^\perp \subset A\) is an \(R\)-cofinite \(R\)-submodule. Moreover \(K^\perp\) is by 
[Abu, Lemma 1.7 (1)] closed.

If \(R\) is artinian, then the implications (5) ⇒ (6) ⇒ (7) ⇒ (8) ⇒ (4) follow from [Abu, 
Lemma 1.7 (4)].■

**Lemma 2.12.** Let \(C\) be an \(R\)-coalgebra and consider \(C^*\) with the finite topology. For every 
\(f \in C^*\) the \(R\)-linear mappings 
\(\xi^r_f : C^* \to C^*, \ g \mapsto g \ast f\) and \(\xi^l_f : C^* \to C^*, \ g \mapsto f \ast g\)

are continuous. If \(R\) is an injective cogenerator, then \(\xi^r_f\) and \(\xi^l_f\) are linearly closed (i.e. 
\(\xi^r_f(X) \subset C^*\) and \(\xi^l_f(X) \subset C^*\) are closed for every closed \(R\)-submodule \(X \subset C^*\)).

**Proof.** Consider for every \(f \in C^*\) the \(R\)-linear mappings 
\(\theta^r_f : C \to C, \ c \mapsto f \to c\) and \(\theta^l_f : C \to C, \ c \mapsto c \leftarrow f\).

Then we have for every \(g \in C^*\) and \(c \in C\):

\[\xi^r_f(g)(c) = (g \ast f)(c) = \sum g(c_1)f(c_2) = g(f \to c) = g(\theta^r_f(c)) = ((\theta^r_f)^* g)(c).\]

So \(\xi^r_f = (\theta^r_f)^*\) and analogously \(\xi^l_f = (\theta^l_f)^*\). The result follows then by [Abu, Proposition 1.10].■

If \(P = (A,C) \in \mathcal{P}^\alpha_m\), then the Grothendieck category \(\text{Rat}^C(A,M) \simeq \sigma_{[A,C]}\) is in 
general not closed under extensions:

**Example 2.13.** ([Rad73, Page 520]) Let \(R\) be a base field, \(V\) be an infinite dimensional 
vector space over \(R\) and consider the \(R\)-coalgebra \(C := R \oplus V\) (with \(\Delta(v) = 1 \otimes v + v \otimes 1\) 
and \(\varepsilon(v) = 1_R\)). Let \(I \subset V^*\) be vector subspace that is not closed, and consider the exact 
sequence of \(C^*\)-modules 
\[0 \to V^*/I \to C^*/I \to C^*/V^* \to 0.\]

Then \(V^*/I\) and \(C^*/V^*\) are \(C\)-rational, while \(C^*/I\) is not.
Lemma 2.14. ([Swe69, Lemma 6.1.1, Corollary 6.1.2]) Let $I \triangleleft A$ be an ideal.

1. Let $M$ be a finitely generated left (right) $A$-module. If $A I (I A)$ is finitely generated, then also $IM \subset M$ ($MI \subset M$) is a finitely generated $A$-submodule. If $I \subset A$ is $R$-cofinite, then $IM \subset M$ ($MI \subset M$) is an $R$-cofinite $A$-submodule.

2. If $A I (I A)$ is finitely generated, then $A I^n (I^n A)$ is finitely generated for every $n \geq 1$. If moreover $I \subset A$ is $R$-cofinite, then $I^n \subset A$ is $R$-cofinite.

The following result generalizes [Rad73, 2.5] from the case of base fields to the case of arbitrary commutative QF rings.

Proposition 2.15. Let $R$ be a QF Ring, $C$ be a projective $R$-coalgebra and consider an exact sequence of left $C^*$-modules

$$0 \rightarrow N \xrightarrow{i} M \xrightarrow{\pi} L \rightarrow 0.$$ 

If $N, L \in \text{Rat}^C(C^*M)$ and $C^*(0 : l)$ is finitely generated for every $l \in L$, then $M$ is $C$-rational.

Proof. Let $m \in M$ and $\{f_1, ..., f_k\}$ be a generating system of $C^*(0_L : \pi(m))$. By assumption $\pi(m)$ is $C$-rational and so there exist by Proposition 2.11 $R$-cofinite closed $A$-ideals $J_i \subset (0_N : f_i m)$ for $i = 1, ..., k$. So we have for the closed $R$-cofinite $A$-ideal $J := \bigcap_{i=1}^k J_i \triangleleft C^*$:

$$J(0_L : \pi(m)) \ni m = (J \ast f_1 + ... + J \ast f_k) \rightarrow m = 0,$$

i.e. $J(0_L : \pi(m)) \subseteq (0_M : m)$. By Lemmata 2.12, 2.14 and [Abu, Proposition 1.10 (3.d)] $J(0_L : \pi(m)) = \sum_{i=1}^k J \ast f_i$ is $R$-cofinite and closed. It follows then by [Abu, Lemma 1.7 (4)] that $(0_M : m) \triangleleft_l C^*$ is $R$-cofinite and closed, hence $m \in \text{Rat}^C(C^*M)$ by Proposition 2.11.

Definition 2.16. An $R$-algebra $A$ is called nearly left noetherian (resp. nearly right noetherian, nearly noetherian), if every $R$-cofinite left (resp. right, two-sided) $A$-ideal is finitely generated in $A \mathcal{M}$ (resp. in $\mathcal{M}_A$, in $A \mathcal{M}_A$).

As a corollary of Theorem 2.6 and Proposition 2.15 we get

Corollary 2.17. Let $R$ be a QF Ring and $C$ be a projective $R$-coalgebra. If $C^*$ is nearly left noetherian (resp. nearly right noetherian), then $\mathcal{M}^C \simeq \text{Rat}^C(C^*C) = \sigma[C^*C]$ (resp. $C^* \mathcal{M} \simeq ^C\text{Rat}(\mathcal{M}_{C^*}) = \sigma[C^*]$) is closed under extensions.
Duality relations between substructures

As an application of our results in this section and our observations about the linear weak topology [Abu] we generalize known results on the duality relations between substructures of a coalgebra and substructures of its dual algebra from the case of base fields (e.g. [Swe69], [Abe80] and [DNR01, 1.5.29]) to the case of measuring \( \alpha \)-pairings over arbitrary commutative rings.

As a consequence of Theorem 2.6 and [Abu, Theorem 1.8] we get

**Proposition 2.18.** Let \( P = (A, C) \in \mathcal{P}_m \).

1. Let \( K \subset C \) be an \( R \)-submodule.
   
   If \( K \) is a right (a left) \( C \)-coideal, then \( K^\perp = \text{An}_A^r(K) \) is a right (a left) \( A \)-ideal;
   
   If \( K \) is a \( C \)-bicoideal, then \( K^\perp = \text{An}_A^r(K) \cap \text{An}_A^l(K) \) is a two-sided \( A \)-ideal.

2. Let \( P \in \mathcal{P}_m^\alpha \).
   
   (a) For every \( R \)-submodule \( I \subset A \) we have:
   
   If \( I \subset A \) is a right (a left) ideal, then \( I^\perp \subset C \) is a right (a left) coideal;
   
   If \( I \triangleleft A \) is a two-sided ideal (and \( I^\perp \subset C \) is pure), then \( I^\perp \subset C \) is a bicoideal (an \( R \)-subcoalgebra).

   (b) Let \( R \) be an injective cogenerator. For a closed \( R \)-submodule \( I \subset A \) we have:
   
   \( I \) is a right (a left) ideal, iff \( I^\perp \subset C \) is a right (a left) coideal.
   
   \( I \) is a two-sided ideal (and \( I^\perp \subset C \) is pure), iff \( I^\perp \subset C \) is a bicoideal (an \( R \)-subcoalgebra).

**Lemma 2.19.**

1. If \( P = (A, C) \) is a measuring \( R \)-pairing and \( K \subset C \) is a coideal, then \( K^\perp \subset A \) is an \( R \)-subalgebra with unity \( 1_A \).

2. Let \( R \) be a QF Ring, \( C \) be a projective \( R \)-coalgebra and \( A \subset C^* \) be an \( R \)-subalgebra (with \( \varepsilon_C \in A \)). If \( \text{Ke}(A) \subset C \) is pure, then \( \Delta_C(\text{Ke}(A)) \subset \text{Ke}(A) \otimes_R C + C \otimes_R \text{Ke}(A) \) (\( \text{Ke}(A) \subset C \) is a \( C \)-coideal).

**Proof.**

1. Obvious.

2. Let \( A \subset C^* \) be an \( R \)-subalgebra and consider the canonical \( R \)-linear mappings
   
   \[ \kappa : A \otimes_R A \to (C \otimes_R C)^* \quad \text{and} \quad \chi : C \otimes_R C \to (A \otimes_R A)^*. \]

   If \( \text{Ke}(A) \subset C \) is pure, then it follows form [Abu, Proposition 1.10 (3.c), Corollary 2.9] that
   
   \[ \text{Ke}(A) = \text{Ke}(\Delta_C^*(\kappa(A \otimes_R A))) = \Delta_C^{-1}(\text{Ke}(\kappa(A \otimes_R A))) \]

   \[ (7) \]

   i.e. \( \Delta_C(\text{Ke}(A)) \subset \text{Ke}(A) \otimes_R C + C \otimes_R \text{Ke}(A) \). If moreover \( \varepsilon_C \in A \), then \( \varepsilon_C(\text{Ke}(A)) = 0 \), i.e. \( \text{Ke}(A) \subset C \) is a \( C \)-coideal. ■
As a consequence of Propositions 2.18, 2.19 and [Abu, Theorem 1.8] we get

**Corollary 2.20.** Let \( R \) be an injective cogenerator and \( C \) a locally projective \( R \)-coalgebra. If we denote with \( \mathcal{C} \) the class of all \( R \)-submodules of \( C \) and with \( \mathcal{H} \) the class of all \( R \)-submodules of \( C^* \), then

\[
\text{An}(\cdot) : \mathcal{C} \to \mathcal{H} \text{ and } \text{Ke}(\cdot) : \mathcal{H} \to \mathcal{C}
\]

induce bijections

\[
\begin{align*}
\{K \subset C \text{ a right } C\text{-coideal}\} & \leftrightarrow \{I \triangleleft_r C^* \text{ a closed right } A\text{-ideal}\} \\
\{K \subset C \text{ a left } C\text{-coideal}\} & \leftrightarrow \{I \triangleleft_l C^* \text{ a closed left } A\text{-ideal}\} \\
\{K \subset C \text{ a } C\text{-bicoideal}\} & \leftrightarrow \{I \triangleleft C^* \text{ a closed two-sided ideal}\}, \\
\{K \subset C \text{ an } R\text{-subcoalgebra}\} & \leftrightarrow \{I \triangleleft C^* \text{ a closed two-sided ideal, } \text{Ke}(I) \subset C \text{ pure}\}.
\end{align*}
\]

If \( R \) is moreover a QF ring, then (8) induces a bijection

\[
\{K \subset C \text{ a pure } C\text{-coideal}\} \leftrightarrow \{A \subset C^* \text{ a closed } R\text{-subalgebra, } \varepsilon_C \in A, \text{Ke}(A) \subset C \text{ pure}\}.
\]

3 Dual coalgebras

Every \( R \)-coalgebra \((C, \Delta_C, \varepsilon_C)\) has a dual \( R \)-algebra, namely \( C^* \) with multiplication the convolution product

\[
\star : C^* \otimes_R C^* \xrightarrow{\delta} (C \otimes_R C)^* \xrightarrow{\Delta^*_C} C^*,
\]

where \( \delta \) is the canonical \( R \)-linear mapping, and with unity element \( \varepsilon_C \). If \((A, \mu_A, \eta_A)\) is an \( R \)-algebra that is finitely generated projective as an \( R \)-module, then \( A^* \) becomes an \( R \)-coalgebra with comultiplication given by

\[
\mu_A^\circ : A^* \xrightarrow{\mu_A^\circ} (A \otimes_R A)^* \xrightarrow{\delta^{-1}} A^* \otimes_R A^*,
\]

where \( \delta : A^* \otimes_R A^* \to (A \otimes_R A)^* \) is the canonical isomorphism, and with counity \( \eta_A^* : A^* \to R \). If \( A \) is not finitely generated projective, then \( \delta \) in not surjective anymore (and not even injective over arbitrary ground rings), hence \( \mu_A^\circ \) is not well defined and \( \mu_A \) incudes on \( A^* \) no \( R \)-coalgebra structure. However, if \( R \) is base field and we consider the \( R \)-algebra \( A \) with the cofinite topology \( \text{Cf}(A) \) (see 3.20) and \( R \) with the discrete topology, then the character module \( A^\circ \) of all continuous \( R \)-linear mappings from \( A \) to \( R \) is an \( R \)-coalgebra ([Swe69]). That result was generalized in [CN90] to the case of Dedekind domains and in [AG-TW00] to the case of arbitrary noetherian (hereditary) commutative rings.

In this section we consider coalgebra structures on the character module of an algebra, considered with a linear topology induced from a filter basis consisting of cofinite ideals over an arbitrary (noetherian) ring.
3.1. Let $A$ be an $R$-algebra and $\mathcal{B}$ be a filter basis consisting of $R$-cofinite $A$-ideals. Then $\mathcal{B}$ induces on $A$ a left linear topology $\mathcal{F}(\mathcal{B})$, such that $(A, \mathcal{F}(\mathcal{B}))$ is a left linear topological $R$-algebra and $\mathcal{B}$ is a neighbourhood basis of $0_A$. With

$$A^*_B := \{ f \in A^* \mid \exists I \in \mathcal{B}, \text{ such that } f(I) = 0 \} = \lim_{\rightarrow} B(A/I)^*$$

we denote the character module of all continuous $R$-linear mappings from $A$ to $R$ (where $R$ is considered as usual with the discrete topology). The completion of $A$ w.r.t. $\mathcal{B}$ is denoted with

$$\widehat{A}_B := \lim_{\leftarrow} \{ A/I \mid I \in \mathcal{B} \}.$$ 

If $A^*_B$ is an $R$-coalgebra, then we call $A^*_B$ the continuous dual $R$-coalgebra of $A$ w.r.t. $\mathcal{B}$.

Analogously $\mathcal{B}$ induces on $A$ a right linear topology, such that $A$ is a right linear topological $R$-algebra and $\mathcal{B}$ is a neighbourhood basis of $0_A$.

Remark 3.2. (Compare [CG-RTvO01, Proposition 3.1]) Let $R$ be noetherian and $A$ be an $R$-algebra. Let $I$ be an $R$-cofinite left $A$-ideal, say $A/I = \sum_{i=1}^k R(a_i + I)$, and consider the two-sided $A$-ideal

$$J := \bigcap_{i=1}^k (I : a_i) = (I : A) \subset (I : 1_A) = I.$$ 

Then

$$\varphi_I : A \to \text{End}_R(A/I), \ a \mapsto [b + I \mapsto ab + I]$$

is an $R$-algebra morphism with $\text{Ke}(\varphi_I) = J$, i.e. $J$ is an $R$-cofinite $A$-ideal.

Analogously one can show that every $R$-cofinite right $A$-ideal contains an $R$-cofinite two-sided $A$-ideal.  

The following result extends [AG-TW00, 1.11] and [AG-TL01, Remark 2.14]:

**Theorem 3.3.** Let $R$ be noetherian and $A$ an $R$-algebra. If $C \subseteq A^A$ is an $A$-subbimodule and $P := (A, C)$, then the following statements are equivalent:

1. $RC$ is locally projective and $\kappa_P(A) \subseteq C^* \text{id dense}$.

2. $RC$ satisfies the $\alpha$-condition and $\kappa_P(A) \subseteq C^*$ is dense;

3. $(A, C)$ is an $\alpha$-pairing;

4. $C^A \subseteq R^A$ is pure (in the sense of Cohn);

5. $C$ is an $R$-coalgebra and $(A, C) \in \mathcal{P}_{\alpha}^m$.

If $R$ is a QF Ring, then “1-4” are moreover equivalent to

6. $RC$ is projective.
Proof. The equivalences (1) \(\Leftrightarrow\) (2) and (3) \(\Leftrightarrow\) (4) follow from [Abu, Lemma 2.13, Proposition 2.6 (3)].

(2) \(\Rightarrow\) (3) follows from [Abu, Proposition 2.4 (2)].

(4) \(\Rightarrow\) (5) If \(C \subset R^A\) is pure, then \(C\) is by [AG-TW00, 1.11] an \(R\)-coalgebra. It follows moreover for all \(f \in C\) and arbitrary \(a, \tilde{a} \in A\) that

\[
\kappa_P(a) = \sum f_1(a) f_2(\tilde{a}) = (\kappa_P(a)) \kappa_P(\tilde{a})(\Delta(f)) = (\kappa_P(a) \kappa_P(\tilde{a}))(f)
\]

and

\[
\kappa_P(1_A)(f) = f(1_A) = \varepsilon_C(f) \text{ for all } f \in C.
\]

So \(\kappa_P : A \rightarrow C^\ast\) is an \(R\)-algebra morphism, i.e. \(P \in \mathcal{P}_m\). By [Abu, Proposition 2.6] \(P\) satisfies the \(\alpha\)-condition, hence \(P \in \mathcal{P}_m^\alpha\).

(5) \(\Rightarrow\) (2) follows from Theorem 2.6.

Let \(R\) be a QF ring.

(2) \(\Rightarrow\) (6) follows from Remark 1.6.

(6) \(\Rightarrow\) (2) If \(R C\) is projective, then \(C\) satisfies the \(\alpha\)-condition by [Abu, Proposition 2.14 (3)]. Consider the \(R\)-submodule \(\kappa_P(A) \subset C^\ast\). By [Abu, Theorem 1.8 (1)] we have

\[
\kappa_P(A) := \text{AnKe}(\kappa_P(A)) = \text{An}(A^\perp) = \text{An}(0_C) = C^\ast,
\]

i.e. \(\kappa_P(A) \subset C^\ast\) is dense. }

Definition 3.4. An \(R\)-algebra \(A\) is said to satisfy the \(\alpha\)-condition, if the class \(K_A\) of all \(R\)-cofinite \(A\)-ideals is a filter and the induced \(R\)-pairing \((A, A^\circ)\) satisfies the \(\alpha\)-condition (in case \(R\) is noetherian, this is equivalent to the purity of \(A^\circ \subset R^A\)). An \(R\)-coalgebra \(C\) is said to satisfy the \(\alpha\)-condition or to be an \(\alpha\)-coalgebra, if the \(R\)-pairing \((C^\ast, C)\) satisfies the \(\alpha\)-condition (equivalently, if \(R C\) is locally projective). With \(\text{Alg}_R^\alpha \subseteq \text{Alg}_R\) resp. \(\text{Cog}_R^\alpha \subseteq \text{Cog}_R\) we denote the full subcategory of \(\alpha\)-algebras resp. \(\alpha\)-coalgebras.

Remark 3.5. Let \(R\) be noetherian and \(A\) be an \(\alpha\)-algebra. Then there is obviously a 1-1 correspondence

\[
\{P = (A, C) \mid P \in \mathcal{P}_m^\alpha\} \longleftrightarrow \{C \mid C \subseteq A^\circ \text{ is an } R\text{-subcoalgebra}\}.
\]

Lemma 3.6. 1. If \(C, D\) are \(R\)-coalgebras and \(\theta : D \rightarrow C\) is an \(R\)-coalgebra morphism, then \(\theta^* : C^\ast \rightarrow D^\ast\) is an \(R\)-algebra morphism and

\[
(\theta^*, \theta) : (D^\ast, D) \rightarrow (C^\ast, C)
\]

is a morphism in \(\mathcal{P}_m\).

2. Let \(R\) be noetherian, \(A, B\) be \(\alpha\)-algebras and \(\xi : A \rightarrow B\) be an \(R\)-algebra morphism. Then we have a morphism in \(\mathcal{P}_m^\alpha\)

\[
(\xi, \xi^\circ) : (B, B^\circ) \rightarrow (A, A^\circ).
\]

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Proof. 1. Trivial.

2. If $f \in B^\circ$, then there exists an $R$-cofinite $B$-ideal $I \triangleleft B$, such that $f \in (B/I)^\ast$. By assumption $R$ is noetherian and so $\xi^{-1}(I) \subset A$ is an $R$-cofinite $A$-ideal, i.e. $\xi^\circ(f) \in A^\circ$ and we get a morphism of $R$-pairings

$$(\xi, \xi^\circ) : (B, B^\circ) \to (A, A^\circ).$$

By assumption $\xi$ is an $R$-algebra morphism. Moreover the canonical $R$-linear mapping $A^\circ \otimes_R A^\circ \to (A \otimes_R A)^\ast$ is by [Abu, Corollary 2.8 (1)] an embedding, hence $\xi^\circ : B^\circ \to A^\circ$ is an $R$-coalgebra morphism by Lemma 2.2 (1). \hfill \blacksquare

Lemma 3.7. Let $R$ be noetherian, $B$ an $\alpha$-algebra and consider the $\alpha$-pairing $(B, B^\circ)$. If $A \subset B$ is an $\alpha$-subalgebra with $1_B \in A$, then $A^\perp := \text{An}(A) \cap B^\circ$ is a $B^\circ$-coideal.

Proof. The embedding $\iota_A : A \hookrightarrow B$ is an $R$-algebra morphism and so $\iota_A^\circ : B^\circ \to A^\circ$ is by Lemma 2.2 (1) an $R$-coalgebra morphism. Hence $A^\perp := \text{Ke}(\iota_A^\circ) \subset B^\circ$ is a $B^\circ$-coideal. \hfill \blacksquare

The following result follows directly from Propositions 2.18, 2.19, Lemma 3.6 and [Abu, Theorem 1.8]:

Corollary 3.8. Let $R$ be a QF Ring, $A$ be an $\alpha$-algebra, $P := (A, A^\circ)$ and consider $A$ with the linear weak topology $A[\mathfrak{F}_A(A^\circ)]$. Let $I \subset A$ be a closed $R$-submodule and set $I^\perp := \text{An}(I) \cap A^\circ$. Then $I$ is a right (a left) $A$-ideal, iff $I^\perp$ is a right (a left) $A^\circ$-coideal. Moreover $I \subset A$ is a two-sided $A$-ideal (and $I^\perp \subset A^\circ$ is pure), iff $I^\perp \subset A^\circ$ is an $A^\circ$-bicoideal (an $R$-subcoalgebra).

The convolution coalgebra

Dual to the convolution algebra, D. Radford presented in [Rad73] the so called convolution coalgebra in the case of base fields. Over arbitrary noetherian ground rings the following version of his definition makes sense:

3.9. Let $R$ be noetherian. If $C$ is an $R$-coalgebra and $A$ is an $\alpha$-algebra, then we call $A \star C := A^\circ \otimes_R C$ the convolution coalgebra of $A$ and $C$. In the special case $C = R$ we have $A \star R \simeq A^\circ$.

The following result generalizes results of D. Radford [Rad73] on the convolution coalgebra from the case of base fields to the case of arbitrary noetherian ground rings:

3.10. Let $R$ be noetherian, $C$ be a locally projective $R$-coalgebra and $A$ be an $\alpha$-algebra. It’s easy to see then that $P := (A \otimes_R C^\ast, A \star C)$ is a measuring $R$-pairing, which satisfies the $\alpha$-condition by [Abu, Lemma 2.8]. By [Rad73] the following mappings are $R$-algebra morphisms:

$$\beta : \text{Hom}_R(C, A) \to (A^\circ \otimes_R C)^\ast, \quad f \mapsto [h \otimes c \mapsto h(f(c))].$$

$$\gamma : A \otimes_R C^\ast \to \text{Hom}_R(C, A), \quad a \otimes g \mapsto [c \mapsto g(c)a].$$
By Corollary 2.7 (Hom$_R(C, A), A \star C) \in P^m_A$, $\gamma(A \otimes_R C^*) \subset$ Hom$_R(C, A)$ is dense (w.r.t. the left $C$-adic topology) and we get category isomorphisms

\[
\mathcal{M}^{A \star C} \simeq \text{Rat}^{A \star C}(A \otimes_R C^*) = \sigma_{A \otimes_R C^*}(A \star C) \\
\simeq \text{Rat}^{A \star C}(A \cdot C^*) = \sigma_{A \cdot C^*}(A \star C) \\
\simeq \text{Rat}^{A \star C}(\text{Hom}_R(C, A)) = \sigma_{\text{Hom}_R(C, A)}(A \star C).
\]

**Proposition 3.11.** If $R$ is noetherian, then we have bifunctors

\[- \star : \text{Alg}^\alpha_R \times \text{Cog} \to \text{Cog}_R \quad \text{and} \quad -\star - : \text{Alg}^\alpha_R \times \text{Cog}^\alpha_R \to \text{Cog}^\alpha_R. \quad (11)\]

**Proof.** Let $A \in \text{Alg}^\alpha_R$. Then $A^\circ$ is by Theorem 3.3 a locally projective $R$-coalgebra (i.e. an $\alpha$-coalgebra). If $C$ is a (locally projective) $R$-coalgebra, then $A \star C := A^\circ \otimes_R C$ is a (locally projective) $R$-coalgebra by [Abu, Lemma 2.8]). Analog to [Par73] one can see that (11) are bifunctors. ■

**Continuous Dual Coalgebras**

**Definition 3.12.** Let $A$ be an $R$-algebra, $\mathcal{K}_A$ be the class of all $R$-cofinite $A$-ideals and

\[\mathcal{E}_A := \{I \triangleleft A \mid A/I \text{ is finitely generated projective}\}.\]

For every subclass $\mathcal{F} \subseteq \mathcal{K}_A$ set

\[A^\circ_\mathcal{F} := \{f \in A^* \mid f(I) = 0 \text{ for some } I \in \mathcal{F}\}.\]

1. We call a filter $\mathcal{F} = \{I_\lambda\}_\Lambda$ consisting of $R$-cofinite $A$-ideals:

   - an $\alpha$-filter, if the $R$-pairing $(A, A^\circ_\mathcal{F})$ satisfies the $\alpha$-condition;
   - cofinitary, if $\mathcal{F} \cap \mathcal{E}_A$ is a filter basis of $\mathcal{F}$;
   - cofinitely $R$-cogenerated, if $A/I$ is $R$-cogenerated for every $I \in \mathcal{F}$.

2. We call $A$:

   - an $\alpha$-algebra, if $\mathcal{K}_A$ is an $\alpha$-filter;
   - cofinitary, if $\mathcal{K}_A$ is a cofinitary filter;
   - cofinitely $R$-cogenerated, if $A/I$ is $R$-cogenerated for every $I \in \mathcal{K}_A$.

**Definition 3.13.** ([Tak81]) An $R$-coalgebra $C$ is called infinitesimal flat, if $C = \varinjlim C_\lambda$ for a directed system of finitely generated projective $R$-subcoalgebras $\{C_\lambda\}_\Lambda$.

**Proposition 3.14.** Let $A$ be an $R$-algebra, $\mathcal{F}$ be a filter consisting of $R$-cofinite $A$-ideals, $P := (A, A^\circ_{\mathcal{F}})$ and consider $A$ as a left (a right) linear topological $R$-algebra with the induced topology $\mathfrak{F}(\mathcal{F})$. 17
1. Assume $\mathcal{F}$ to be cofinitely $R$-cogenerated. Then $\mathfrak{T}(\mathcal{F})$ is Hausdorff, iff $\kappa_P : A \to A_\mathcal{F}^*$ is an embedding.

2. Assume $R$ to be noetherian and $\mathcal{F}$ to be an $\alpha$-filter. Then $A_\mathcal{F}^*$ is an $\alpha$-coalgebra, $(A, A_\mathcal{F}^*) \in \mathcal{P}_m^\alpha$ and $\kappa_P(A) \subset A_\mathcal{F}^*$ is dense (w.r.t. the finite topology).

3. If $A/I$ is $R$-reflexive for every $I \in \mathcal{F}$ (e.g. $R$ is an injective cogenerator), then $\hat{A} \simeq A_\mathcal{F}^*$ as left (right) linear topological $R$-modules.

Proof. 1. By assumption $A/I$ is $R$-cogenerated for every $I \in \mathcal{F}$, hence

$$\overline{U_A} = \bigcap_{I \in \mathcal{F}} I = \bigcap_{I \in \mathcal{F}} \text{KeAn}(I) = \text{Ke}(\sum_{I \in \mathcal{F}} \text{An}(I)) = \text{Ke}(A_\mathcal{F}^*) = \text{Ke}(\kappa_P).$$

2. Every $I \in \mathcal{F}$ is a two-sided $A$-ideal and so $A_\mathcal{F}^* \subset A^*$ is an $A$-subbimodule. The results follows then from Theorem 3.3.

3. If $A/I$ is $R$-reflexive for every $I \in \mathcal{F}$, then we have isomorphisms of topological $R$-modules

$$\hat{A} = \lim_{\leftarrow} A/I \simeq \lim_{\leftarrow} (A/I)^* \simeq (\lim_{\rightarrow} (A/I)^*)^* = (A_\mathcal{F}^*)^*.$$

If $R$ is an injective cogenerator, then all finitely generated $R$-modules are $R$-reflexive (e.g. [Wis88, 48.13]) and we are done.

The following result extends observations in [Lar98] (resp. [AG-TL01]) on cofinitary $R$-algebras over Dedekind domains (resp. noetherian rings) to the case of cofinitary filters for algebras over arbitrary commutative base rings:

**Proposition 3.15.** Let $A$ be an $R$-algebra, $\mathcal{F}$ be a filter consisting of $R$-cofinite $A$-ideals, $P := (A, A_\mathcal{F}^*)$ and consider $A$ as a left (a right) linear topological $R$-algebra with the induced left (right) linear topology $\mathfrak{T}(\mathcal{F})$. If $\mathcal{F}$ is cofinitary, then

1. $\mathfrak{T}(\mathcal{F})$ is Hausdorff, iff $\kappa_P : A \to A_\mathcal{F}^*$ is an embedding.

2. $A_\mathcal{F}^*$ is an infinitesimal flat $\alpha$-coalgebra, $P \in \mathcal{P}_m^\alpha$ and $\kappa_P(A) \subset A_\mathcal{F}^*$ is dense.

3. $\hat{A} \simeq A_\mathcal{F}^*$ as left (right) linear topological $R$-algebras.

Proof. 1. For every $I \in \mathcal{E}_A$ the $R$-module $A/I$ is in particular $R$-cogenerated and the result follows from Proposition 3.14 (1).

2. For $I, J \in \mathcal{F} \cap \mathcal{E}_A$ set $I \leq J$, if $I \supset J$ and consider the canonical $R$-algebra epimorphism $\pi_{I,J} : A/J \to A/I$. Then

$$\{(A/I)^*, \pi_{I,J}^* : (A/I)^* \to (A/J)^* \}$$
is a directed system of finitely generated projective \( R \)-coalgebras with \( R \)-coalgebra morphisms \( \pi_{i,j} : (A/I)^* \rightarrow (A/J)^* \). Then \( A^\circ_\mathcal{F} = A^\circ_\mathcal{F} \otimes_{\mathcal{E}_A} \mathcal{E}_A \simeq \lim_{\rightarrow_{\mathcal{F} \otimes_{\mathcal{E}_A}}} (A/I)^* \) is an infinitesimal flat \( R \)-coalgebra.

Let \( M \) be an arbitrary \( R \)-module. If \( \sum_{i=1}^{k} m_i \otimes g_i \in \text{Ke}(\alpha_M^P) \), then there exists \( I \in \mathcal{F} \cap \mathcal{E}_A \), such that \( \{g_1, ..., g_n\} \subset \text{An}(I) \). If \( \{(a_I + I, f_i)\}_{i=1}^{k} \) is a dual basis for \((A/I)^*\), then

\[
\sum_{i=1}^{n} m_i \otimes g_i = \sum_{i=1}^{n} m_i \otimes (\sum_{l=1}^{k} g_i(a_l + I)f_l) = \sum_{i=1}^{n} m_i \otimes (\sum_{l=1}^{k} g_i(a_l)f_l) = \sum_{l=1}^{k} (\sum_{i=1}^{n} g_i(a_l)m_i) \otimes f_l = 0.
\]

Obviously the canonical \( R \)-linear mapping \( \kappa_P : A \rightarrow A^\circ_\mathcal{F} \) is an \( R \)-algebra morphism, i.e. \( P \) is a measuring \( \alpha \)-pairing. The density of \( \kappa_P(A) \subset A^\circ_\mathcal{F} \) follows then by Theorem 3.3.

3. For every \( I \in \mathcal{F} \cap \mathcal{E}_A \) the \( R \)-module \( A/I \) is finitely generated projective, hence \((A/I)^*\) is an \( R \)-coalgebra and \((A/I)^{**} \simeq A/I\) as \( R \)-algebras. So we have an isomorphisms of topological \( R \)-algebras

\[
\hat{A} = \lim_{\rightarrow_{\mathcal{F} \otimes_{\mathcal{E}_A}}} A/I \simeq \lim_{\rightarrow_{\mathcal{F} \otimes_{\mathcal{E}_A}}} (A/I)^{**} \simeq (\lim_{\rightarrow_{\mathcal{F} \otimes_{\mathcal{E}_A}}} (A/I)^* \simeq (\lim_{\rightarrow_{\mathcal{F} \otimes_{\mathcal{E}_A}}} (A/I)^*)^* = (A^\circ_\mathcal{F})^* \].
\]

As a consequence of Propositions 3.14, 3.15 and Theorem 2.6 we get

**Corollary 3.16.** Let \( A \) be an \( R \)-algebra and \( \mathcal{F} \) be a filter consisting of \( R \)-cofinite \( A \)-ideals. If \( R \) is noetherian and \( \mathcal{F} \) is an \( \alpha \)-filter, or \( \mathcal{F} \) is cofinitary, then we have isomorphisms of categories

\[
\mathcal{M}^{A^\circ_\mathcal{F}} \simeq \text{Rat}^{A^\circ_\mathcal{F}}(A) \Rightarrow \sigma[\mathcal{M}^{A^\circ_\mathcal{F}}] \quad \text{\&} \quad \mathcal{M}^{A^\circ_\mathcal{F}} \simeq \text{Rat}^{A^\circ_\mathcal{F}}(\mathcal{M}_A) \Rightarrow \sigma[\mathcal{M}^{A^\circ_\mathcal{F}}].
\]

3.17. Let \( A, B \) be \( R \)-algebras, \( \mathcal{F}_A, \mathcal{F}_B \) be filter bases consisting of \( R \)-cofinite \( A \)-ideals, \( B \)-ideals respectively and

\[
\mathcal{F}_A \times \mathcal{F}_B := \{ \text{Im}(\iota_I \otimes id_B) + \text{Im}(id_A \otimes \iota_J) \mid I \in \mathcal{F}_A, J \in \mathcal{F}_B \}. \quad (12)
\]

Obviously \( \mathcal{F}_A \times \mathcal{F}_B \) is a filter basis consisting of \( R \)-cofinite \( A \otimes_R B \)-ideals and induces so a linear topology \( \mathcal{F}(\mathcal{F}_A \times \mathcal{F}_B) \) on \( A \otimes_R B \), such that \((A \otimes_R B, \mathcal{F}(\mathcal{F}_A \times \mathcal{F}_B))\) is a linear topological \( R \)-algebra and \( \mathcal{F}_A \times \mathcal{F}_B \) is a neighbourhood basis of \( 0_{A \otimes_R B} \).

One can generalize [AG-TL01, Proposition 4.9, Theorem 4.10] to the following more general result:
Theorem 3.18. Let $A, B$ be $R$-algebras, $\mathfrak{F}_A, \mathfrak{F}_B$ be filters consisting of $R$-cofinite $A$-deals, $B$-ideals respectively and consider the canonical $R$-linear mapping $\delta : A^* \otimes_R B^* \to (A \otimes_R B)^*$.

1. If $\mathfrak{F}_A$ and $\mathfrak{F}_B$ are cofinitary, then the filter of $A \otimes R B$-ideals generated by $\mathfrak{F}_A \times \mathfrak{F}_B$ is cofinitary and $(A \otimes_R B)^\circ_{\mathfrak{F}_A \times \mathfrak{F}_B}$ is an $R$-coalgebra. If $R$ is noetherian, then $\delta$ induces an $R$-coalgebra isomorphism

$$A^\circ_{\mathfrak{F}_A} \otimes_R B^\circ_{\mathfrak{F}_B} \simeq (A \otimes_R B)^\circ_{\mathfrak{F}_A \times \mathfrak{F}_B}.$$

2. Let $R$ be noetherian. If $\mathfrak{F}_A$ is an $\alpha$-filter and $\mathfrak{F}_B$ is cofinitary, then the filter generated by $\mathfrak{F}_A \times \mathfrak{F}_B$ is an $\alpha$-filter, $(A \otimes_R B)^\circ_{\mathfrak{F}_A \times \mathfrak{F}_B}$ is an $R$-coalgebra and $\delta$ induces an $R$-coalgebra isomorphism

$$A^\circ_{\mathfrak{F}_A} \otimes_R B^\circ_{\mathfrak{F}_B} \simeq (A \otimes_R B)^\circ_{\mathfrak{F}_A \times \mathfrak{F}_B}.$$

Theorem 3.19. Let $R$ be hereditary and noetherian.

1. All $R$-algebras satisfy the $\alpha$-condition, i.e. $\text{Alg}_R^\alpha = \text{Alg}_R$.

2. There is a duality between $\text{Alg}_R$ and $\text{Cog}_R$ through the right-adjoint contravariant functors

$$(-)^* : \text{Cog}_R \to \text{Alg}_R, \ (-)^\circ : \text{Alg}_R \to \text{Cog}_R.$$

Proof. 1. Let $A$ be an arbitrary $R$-algebra. By [AG-TW00, Proposition 2.11] $A^\circ \subset R^A$ is pure and so $(A, A^\circ)$ is an $\alpha$-pairing by [Abu, Proposition 2.6].

2. For every $R$-algebra $A$ the canonical mapping $\lambda_A : A \to A^*$ is an $R$-algebra morphism and for every $R$-coalgebra $C$ the canonical mapping $\Phi_C : C \to C^{**}$ is an $R$-coalgebra morphism (compare Lemma 3.6). Moreover for every $A \in \text{Alg}_R$ and every $C \in \text{Cog}_R$

$$\Upsilon_{A,C} : \text{Alg}_R(A, C^*) \to \text{Cog}_R(C, A^\circ), \ \xi \mapsto \xi^\circ \circ \Phi_C$$

is an isomorphism with inverse

$$\Psi_{A,C} : \text{Cog}_R(C, A^\circ) \to \text{Alg}_R(A, C^*), \ \theta \mapsto \theta^* \circ \lambda_A.$$

It’s easy to see that $\Upsilon_{A,C}$ and $\Psi_{A,C}$ are functorial in $A$ and $C$.\qed

Locally finite modules

3.20. The cofinite topology. Let $R$ be noetherian. For every $R$-algebra $A$, the class $\mathcal{K}_A$ of all $R$-cofinite two-sided $A$-ideals is obviously a filter and induces on $A$ a left (a right) linear topology, the so called left (right) cofinite topology $\text{Cf}(A)$, such that $\mathcal{K}_A$ is
Consider $A$ with the left linear topology $Cf(A)$. Let $M$ be a left $A$-module and consider the filter $K_M$ of all $R$-cofinite $A$-submodules of $M$. Let $L \subseteq M$ be an $R$-cofinite $A$-submodule and consider the $R$-linear mapping

$$\varphi_L : A \to \text{End}_R(M/L), \ a \mapsto [m + L \mapsto am + L].$$

Then $A/Ke(\varphi_L) \hookrightarrow \text{End}_R(M/L)$ and so

$$I_L := Ke(\varphi_L) = \{a \in A | aM \subseteq L\}$$

is an $R$-cofinite two-sided $A$-ideal. If $m \in M$ is arbitrary, then $I_L := (L : M) \subseteq (L : m)$, hence $(L : m)$ is open w.r.t. to the left cofinite topology $Cf(A)$. So $M$ becomes a topology, the so called cofinite topology $Cf(M)$, such that $(M, Cf(M))$ is a linear topological left $(A, Cf(A))$-module and $K_M$ is a neighbourhood basis of $0_M$.

Considering $A$ with the right cofinite topology $Cf(A)$ it turns out that for every right $A$-module $M$, the filter of $R$-cofinite right $A$-submodules of $M$ induces on $M$ a topology, the cofinite topology $Cf(M)$, such that $(M, Cf(M))$ is a linearly topological right $(A, Cf(A))$-module.

**3.21.** Let $R$ be noetherian and $A$ be an $R$-algebra. A left $A$-module $M$ is called locally finite, if $Am$ is finitely generated for every $m \in M$. With $\text{Loc}(AM) \subseteq AM$ we denote the full subcategory of locally finite left $A$-modules. For every left $A$-module $M$ it follows that $\text{Loc}(M) \subseteq M$ is an $A$-submodule (since the ground ring $R$ is noetherian) and we get a preradical

$$\text{Loc}(-) : AM \to AM, \ M \mapsto \{m \in M | Am \text{ is finitely generated in } M\}$$

with pretorsion class the full subcategory $\text{Loc}(AM) \subseteq AM$ of locally finite left $A$-modules.

Analogously one defines the preradical $\text{Loc}(-) : MA \to MA$ with pretorsion class the full subcategory of locally finite right $A$-modules $\text{Loc}(MA) \subseteq MA$.

**Lemma 3.22.** Let $R$ be noetherian and $A$ be an $R$-algebra. For every right $A$-module $M$ we have

$$M^* := \{f \in M^* | f(MI) = 0 \text{ for some } R\text{-cofinite (right) } A\text{-ideal } I \subseteq A\}$$

$$= \{f \in M^* | Af \text{ is finitely generated in } M\} \quad (= \text{Loc}(AM^*)) \quad (13)$$

$$= \{f \in M^* | f(L) = 0 \text{ for some } R\text{-cofinite right } A\text{-submodule } L \subseteq M\}.$$

**Proof.** Let $f \in M^*$ with $f(MI) = 0$ for an $R$-cofinite right $A$-ideal $I$. If $\{a_1 + I, ..., a_k + I\}$ is a generating system for $A/I$ over $R$, then $\{a_1 f, ..., a_k f\}$ is a generating system for $Af$ over $R$, i.e. $f \in \text{Loc}(AM^*)$.

Let $f \in \text{Loc}(AM^*)$ and assume that $Af = \sum_{i=1}^k Rf_i$ with $\{f_1, ..., f_k\} \subseteq M^*$. Then $L := \text{Ke}(Af) = \bigcap_{i=1}^k \text{Ke}(f_i) \subseteq M$ is a right $A$-submodule and moreover $M/L \hookrightarrow \bigoplus_{i=1}^k M/\text{Ke}(f_i)$, i.e. $L \subseteq M$ is an $R$-cofinite $A$-submodule.
Let \( f \in (M/L)^* \cong \text{An}(L) \) for some \( R \)-cofinite \( A \)-submodule \( L \subseteq M \). Then \( I_L := (L : M) \) is an \( R \)-cofinite \textit{two-sided} \( A \)-ideal (compare 3.20) and moreover \( f(MI_L) \subseteq f(L) = 0 \), i.e. \( f \in M^\circ. \square \)

It's well known that for an \( R \)-algebra \( A \) over a base field \( R \), the category of right (left) \( A^\circ \)-comodules and the category of locally finite left (right) \( A \)-modules coincide (e.g. [Abe80], [Rad73]). Over arbitrary commutative rings we have

**Proposition 3.23.** Let \( R \) be noetherian and \( A \) be an \( R \)-algebra.

1. Every \( A^\circ \)-subgenerated left (right) \( A \)-module is locally finite.

2. If \( A \) is cofinitely \( R \)-cogenerated, then \( \sigma[A^\alpha A^\circ] = \text{Loc}(A\mathcal{M}) \) and \( \sigma[A^\circ_A] = \text{Loc}(\mathcal{M}_A) \).

3. If \( A \) is an (cofinitely \( R \)-cogenerated) \( \alpha \)-algebra, then we have category isomorphisms

\[
\mathcal{M}^{A^\circ} \cong \text{Rat}^{A^\circ}(A\mathcal{M}) = \sigma[A^\alpha A^\circ] \quad \& \quad A^\circ\mathcal{M} \cong A^\circ\text{Rat}(\mathcal{M}_A) = \sigma[A^\alpha_A].
\]

If \( A \) is moreover cofinitely \( R \)-cogenerated, then

\[
\mathcal{M}^{A^\circ} \cong \text{Loc}(A\mathcal{M}) \text{ and } A^\circ\mathcal{M} \cong \text{Loc}(\mathcal{M}_A).
\]

**Proof.**

1. Let \( M \in \sigma[A^\alpha A^\circ] \). Then there exists for every \( m \in M \) a finite subset \( W = \{f_1, \ldots, f_k\} \subset A^\circ \), such that \((0_{A^\circ} : W) \subset (0_M : m)\). Choose for every \( i = 1, \ldots, k \) an \( R \)-cofinite \( A \)-ideal \( J_i \subset Ke(f_i) \) and consider the \( R \)-cofinite \( A \)-ideal \( J := \bigcap_{i=1}^k J_i \). If \( a \in J \), then for every \( \tilde{a} \in A \) and \( i = 1, \ldots, k \) we have \( (a \rightarrow f_i)(\tilde{a}) = f_i(\tilde{a}a) = 0 \). Consequently \( J \subset (0_{A^\circ} : W) \subset (0_M : m) \) and so \( Am \cong A/(0_M : m) \) is finitely generated in \( \mathcal{M}_R \). Hence \( A\mathcal{M} \) is locally finite.

2. By (1) \( \sigma[A^\alpha A^\circ] \subset \text{Loc}(A\mathcal{M}) \). Assume now that \( A \) is cofinitely \( R \)-cogenerated. Let \( N \) be a locally finite left \( A \)-module. For every \( n \in N \) the \( R \)-module \( A/(0_N : n) \cong An \) is finitely generated and so there exists by Remark 3.2 an \( R \)-cofinite \( A \)-ideal \( I \subset (0_N : n) \). By assumption \( A/I \) is \( R \)-cogenerated and so \( I = \text{KeAn}(I) \) (e.g. [Wis88, 28.1]). If \( \text{An}(I) \cong (A/I)^* = \sum_{i=1}^k \text{Rg}_i \) and \( W := \{g_1, \ldots, g_k\} \), then it follows for every \( a \in (0_{A^\circ} : W) \) that \( g_i(a) = (a \rightarrow g_i)(1_A) = 0 \). So \((0_{A^\circ} : W) \subset \text{KeAn}(I) = I \subset (0_N : n) \), i.e. \( _A\mathcal{N} \) is \( A^\circ \)-subgenerated.

3. The category isomorphisms follow from Theorem 2.6 (and (2)). \( \square \)

4 **Dual comodules**

In this section we discuss for every \((A, C) \in \mathcal{P}_m^\alpha \) the \textit{duality} between the category of right (left) \( A \)-modules and the category the right (left) \( C \)-comodules.
4.1. Let \( P = (A, C) \in \mathcal{P}_m \). By Theorem 2.6 \( \mathcal{M}^C \subset \mathcal{M} \) is a subcategory and so we have a contravariant functor
\[
(-)^* : \mathcal{M}^C \to \mathcal{M}_A, \quad (N, \varrho_N) \mapsto (N^*, \rho_{N^*}),
\]
where
\[
\rho_{N^*} : N^* \to \text{Hom}_R(A, N^*), \quad f \mapsto [a \mapsto [n \mapsto \sum f(n_{<0>}) < a, n_{<1>} >]].
\]
(15)
If moreover \( P \) satisfies the \( \alpha \)-condition, then we get by Theorem 2.6 the contravariant functor
\[
(-)^r : \mathcal{M}_A \to \mathcal{M}^C, \quad M \mapsto M^r := \text{Rat}^C(A M^*).
\]
(17)

4.2. ([Wis2000]) Let \( R_C \) be a flat \( R \)-coalgebra, \( N \) be a left \( C \)-comodule and consider the \( R \)-linear mapping
\[
\gamma : N^* \to \text{Hom}_R(N, C), \quad f \mapsto [n \mapsto \sum f(n_{<-1>}) n_{<0>}].
\]
(16)
If \( R \) is finitely presented, then by \( N^* \otimes_R C \simeq \text{Hom}_R(N, C) \) (e.g. [Wis96, 15.7]) and \( N^* \) becomes a structure of a right \( C \)-comodule through
\[
\varrho_{M^r} : N^* \overset{\gamma}{\to} \text{Hom}_R(N, C) \simeq N^* \otimes_R C.
\]
(17)

Theorem 4.3. For every \((A, C) \in \mathcal{P}_m^C\) there is a duality between the category of right \( C \)-comodules and the category of right \( A \)-modules through the right adjoint contravariant functors
\[
(-)^* : \mathcal{M}^C \to \mathcal{M}_A \quad \text{and} \quad (-)^r : \mathcal{M}_A \to \mathcal{M}^C.
\]
Proof. For every right \( C \)-comodule \( N \) the canonical mapping \( \Phi_N : N \to N^{**} \) is \( A \)-linear, we have by Lemma 2.5 (4) \( \Phi_N(N) \subseteq N^{**r} \) and so \( \Phi_N : N \to N^{**r} \) is \( C \)-colinear by Theorem 2.6. On the other hand, for every right \( A \)-module \( M \) the canonical mapping \( \lambda_M : M \to M^{*r} \) is \( A \)-linear. It’s easy to see then that we have functorial homomorphisms (in \( M \in \mathcal{M}_A \) and \( N \in \mathcal{M}^C \))
\[
\Upsilon_{N,M} : \text{Hom}_A(M, N^*) \to \text{Hom}^C(N, M^r), \quad f \mapsto f^r \circ \Phi_N,
\]
\[
\Psi_{N,M} : \text{Hom}^C(N, M^r) \to \text{Hom}_A(M, N^*), \quad g \mapsto g^* \circ \lambda_M.
\]
Moreover \( \Upsilon_{N,M} \) is bijective with inverse \( \Psi_{N,M} \). □
Notation. For every $R$-algebra $A$ denote with $\mathcal{M}_A^f$ (or $A\mathcal{M}^f$) the category of finitely generated right (left) $A$-modules.

Lemma 4.4. Let $R$ be noetherian. For every $(A, C) \in \mathcal{P}_m^a$ there is a duality between $\text{Rat}^C(\mathcal{A}\mathcal{M}^f)$ and $\text{cRat}(\mathcal{M}_A^f)$ through the right-adjoint contravariant functors
\[
(-)^* : \text{cRat}(\mathcal{M}_A^f) \rightarrow \text{Rat}^C(\mathcal{A}\mathcal{M}^f) \quad \text{and} \quad (-)^* : \text{Rat}^C(\mathcal{A}\mathcal{M}^f) \rightarrow \text{cRat}(\mathcal{M}_A^f).
\]

Proof. Let $M \in \text{cRat}(\mathcal{M}_A^f)$ $(M \in \text{Rat}^C(\mathcal{A}\mathcal{M}^f))$. By [Abu01, Folgerung 2.2.24] every finitely generated $C$-rational left $A$-module is finitely generated over $R$, hence $M_R$ is finitely generated and so $A M^* (M_A^*)$ is finitely generated. By assumption $R$ is noetherian and so $M_R$ is finitely presented. Consequently $M^*$ is by 4.2 a $C$-rational left (right) $A$-module. The claimed duality follows then from Theorem 4.3.■

4.5. If $C$ is a locally projective $R$-coalgebra, then we get by Theorem 4.3 right-adjoint contravariant functors
\[
(-)^* : \mathcal{M}_C \rightarrow \mathcal{M}_{C^*}, \quad N \mapsto N^*,
\]
\[
(-)^\Box : \mathcal{M}_{C^*} \rightarrow \mathcal{M}_C, \quad M \mapsto M^\Box := \text{Rat}^C(C^* M^*).
\]

Lemma 4.6. Let $R$ be an injective cogenerator and $C$ be a locally projective $R$-coalgebra. If $M$ a right $C^*$-module, $L \subset M$ is a $C^*$-submodule and $M^\Box \subset M^*$ is dense, then $L^\Box \subset L^*$ is dense.

Proof. By Lemma 2.5 (4) $\iota_L^*(M^\Box) \subset L^\Box$ and it follows from [Abu, Proposition 1.10 (3.b)] that $\iota_L^*(M^\Box) = \iota_L^*(M^*) = \iota_L^*(M^*) = L^*$.■

M. Takeuchi [Tak74] studied the category of locally finite modules of a commutative algebra over a base field. In what follows we transfer some results obtained by him to the category $\text{Rat}^C(\mathcal{A}\mathcal{M})$ corresponding to a measuring $\alpha$-pairing $P = (A, C) \in \mathcal{P}_m^a$ with $A$ a commutative algebra over an arbitrary commutative ground ring.

Proposition 4.7. Let $P = (A, C) \in \mathcal{P}_m^a$ with $A$ commutative and denote with $\mathcal{M}_A^f \subset \mathcal{A}\mathcal{M}$ the full subcategory of finitely generated $A$-modules. Then we have an isomorphism of functors
\[
\text{Hom}_A(-, C) \simeq (-)^\pi : \mathcal{M}_A^f \rightarrow \mathcal{M}_C.
\]

Proof. Step 1. Let $M \in \mathcal{M}_A^f$ be arbitrary and consider $\text{Hom}_A(M, C)$ with the canonical $A$-module structure induced by $M_A$. For arbitrary $f \in \text{Hom}_A(M, C)$ the $A$-subbimodule $N := f(M) \subset C$ is by Theorem 2.6 a $C$-bicoideal. Moreover $N_A$ is finitely generated and so finitely generated in $\mathcal{M}_R$ (since $\text{Rat}^C(\mathcal{A}\mathcal{M})$ is by [Abu01, Folgerung 2.2.24] $(A, R)$-finite). Assume that $N = \sum_{i=1}^l R c_i$, with $\Delta(c_i) = \sum_{j=1}^k c_{ij} \otimes c_{ij}$ for every $i = 1, \ldots, k$ and set
\[
K := \sum_{i=1}^l \sum_{j=1}^k R c_{ij} \quad \text{Then} \quad K^\perp \subset \langle 0 : f \rangle \quad \text{and so} \quad f \quad \text{is by Proposition 2.11} \quad C\text{-rational. By our choice} \quad f \in \text{Hom}_A(M, C) \quad \text{is arbitrary, i.e.} \quad \text{Hom}_A(M, C) \in \text{Rat}^C(\mathcal{A}\mathcal{M}).
\]
Step 2. \((-)^r \simeq \text{Hom}_A(-, C)\).

Let \(N \in \mathcal{M}^C\), \(M \in \mathcal{M}_A\) and consider the \(C\)-comodule \(\text{Hom}_A(M, C)\). The result follows then from the functorial isomorphisms:

\[
\begin{align*}
\text{Hom}^C(N, \text{Hom}_A(M, C)) &= \text{Hom}_A(N, \text{Hom}_A(M, C)) \quad \text{(Theorem 2.6)} \\
&\simeq \text{Hom}_A(M, \text{Hom}_A(N, C)) \\
&\simeq \text{Hom}_A(M, \text{Hom}^C(N, C)) \quad \text{(Theorem 2.6)} \\
&\simeq \text{Hom}_A(M, N^*) \quad \text{(5)} \\
&\simeq \text{Hom}^C(N, M^r) \quad \text{(Theorem 4.3)}.
\end{align*}
\]

As a consequence of Proposition 4.7 we get

Corollary 4.8.

1. Let \(R\) be noetherian, \(A\) be an \(\alpha\)-algebra and consider the functor

\((-)^0 = \text{Rat}^{A^0}(-) \circ (-)^* : \mathcal{M}_A \to \mathcal{M}^{A^0}, \ M \mapsto M^0 := \text{Rat}^{A^0}(A M^r)\)

If \(A\) is commutative, then we have a functorial isomorphism

\[
\text{Hom}_A(-, A^0) \simeq (-)^0 : \mathcal{M}_A^f \to \mathcal{M}^{A^0}.
\]

2. If \(C\) is a cocommutative locally projective \(R\)-coalgebra, then we have a functorial isomorphism

\[
\text{Hom}_{C^*}(-, C) \simeq (-)^\square : \mathcal{M}^f_{C^*} \to \mathcal{M}^C.
\]

Corollary 4.9. Let \(P = (A, C) \in \mathcal{P}^\alpha_m\), where \(A\) is commutative and noetherian. If \((-)^r : \mathcal{M}_A^f \to \sigma_A[C]\) is exact, then \(C\) is a injective \(A\)-module.

Proof. By Baer’s criteria (e.g. [Wis88, 16.4]) it’s enough to show that \(C\) is \(A\)-injective. Let \(I\) be an \(A\)-ideal. Then \(I_A\) is finitely generated and by assumption the following set mapping is surjective

\[
A^r \xrightarrow{\ell^r} I^r \to 0.
\]

By Proposition 4.7 \(\text{Hom}_A(-, C) \simeq (-)^r\) and so

\[
\text{Hom}_A(A, C) \xrightarrow{(\bar{\iota}, C)} \text{Hom}_A(I, C) \to 0
\]

is a surjective set mapping, i.e. \(C\) is \(A\)-injective and we are done.

Continuous dual comodules

In what follows we consider the dual comodules of modules of an \(\alpha\)-algebra over an arbitrary noetherian base ring. These were considered in the case of base fields by several authors (e.g. [Tak74], [Liu94], [GK97] and [Lu98]) and in the case of Dedekind domains by R. Larson [Lar98].

4.10. Let \(R\) be noetherian, \(A\) be an \(R\)-algebra, \(\mathfrak{F}\) be a filter consisting of \(R\)-cofinite \(A\)-ideals and consider \(A\) with the right induced linear topology \(\Sigma(\mathfrak{F})\).
1. If $\mathfrak{F}$ is an $\alpha$-filter basis, then by Proposition 3.14 (2) $A_\mathfrak{F}^\circ$ is an $R$-coalgebra and $(A, A_\mathfrak{F}) \in \mathcal{P}_m^\alpha$. By Theorem 4.3 we get right-adjoint contravariant functors

$(-)^*: \mathcal{M}^{A_\mathfrak{F}} \to \mathcal{M}_A$, $M \mapsto M^*$,

$(-)_0^\mathfrak{F}: \mathcal{M}_A \to \mathcal{M}^{A_\mathfrak{F}}$, $M \mapsto M_0^\mathfrak{F} := \text{Rat}^{A_\mathfrak{F}}(A M^*)$.

For every $M \in \mathcal{M}_A$ we call $M_0^\mathfrak{F}$ the dual comodule of $M$ w.r.t. $\mathfrak{F}$. If $A$ is an $\alpha$-algebra, then we call $M^0 := \text{Rat}^{A_\mathfrak{F}}(A M^*)$ the dual comodule of $M$.

2. For every right $A$-module $M$ we call

$M_\mathfrak{F}^\circ := \{f \in M^* | f(M I) = 0 \text{ for some } I \in \mathfrak{F}\} = \lim_{\mathfrak{F}}(M/MI)^*$

the continuous dual module of $M$ w.r.t. $\mathfrak{F}$. If $A_\mathfrak{F}^\circ$ is an $R$-coalgebra and $M_\mathfrak{F}^\circ$ is a right $A_\mathfrak{F}^\circ$-comodule, then we call $M_\mathfrak{F}^\circ$ the continuous dual comodule of $M$ w.r.t. $\mathfrak{F}$.

**Notation.** Let $R$ be noetherian, $A$ be an ($\alpha$)-algebra and $M, N$ be right $A$-modules. For every $A$-linear mapping $\gamma: M \to N$ we denote with $\gamma^*: N^0 \to M^0$ ($\gamma^0: N^0 \to M^0$) the restriction of $\gamma^*$ on $N^0$ (on $N^0$).

The following result generalizes the corresponding one [DNR01, Corollary 2.2.16] stated for the canonical pairing $(C^*, C)$ over a base field to an arbitrary measuring $\alpha$-pairing $(A, C)$ over an arbitrary noetherian ground ring:

**Proposition 4.11.** Let $P = (A, C) \in \mathcal{P}_m^\alpha$, $N \in C\text{Rat}(\mathcal{M}_A) \text{ and consider for every } f \in N^*$ the $R$-linear mapping

$$\theta_f: N \to C, \quad \theta_f(n) = \sum n_{<i>} f(n_{<i>}).$$

If $R$ is noetherian, then:

$$\text{Rat}^C( A N^*) = \text{Sp}(\sigma[A C], A N^*) := \{\text{Im}(g) | g \in \text{Hom}_A(U, N^*), U \in \sigma[A C]\} = \{f \in N^* | A f \text{ is finitely generated}\} (= \text{Loc}(A N^*)) = \{f \in N^* | \exists \text{ an } R\text{-cofinite (right) ideal } I \subset A \text{ with } f(N I) = 0\} = \{f \in N^* | \exists \text{ an } R\text{-cofinite } A\text{-submodule } L \subset N \text{ with } f(L) = 0\} = \{f \in N^* | \exists \text{ an } R\text{-cofinite } C\text{-subcomodule } L \subset N \text{ with } f(L) = 0\} = \{f \in N^* | \theta_f(N) \subset C \text{ is a finitely generated } R\text{-submodule}\}.$$

**Proof.** The equality $\text{Rat}^C( A N^*) = \text{Sp}(\sigma[A C], A N^*)$ follows from 2.3 and Theorem 2.6. Obviously $\text{Rat}^C( A N^*) \subseteq \text{Loc}(A N^*)$.

By Theorem 2.6 and Lemma 3.22 $f \in \text{Loc}(A N^*) \iff f(N I) = 0$ for an $R$-cofinite (right) ideal $I \subset A \iff f(L) = 0$ for an $R$-cofinite right $A$-submodule $L \subset N \iff f(L) = 0$ for an $R$-cofinite left $C$-subcomodule $L \subset N$.

Let $f \in N^*$ with $f(L) = 0$ for an $R$-cofinite left $C$-subcomodule $L \subset N$. Analogous to 5.2 below, $\theta_f: N \to C$ is $C$-colinear. Notice that $\theta_f(L) = 0$ and so there exists a $C$-colinear morphism $\overline{\theta}_f: N/L \to C$, such that $\overline{\theta}_f \circ \pi_L = \theta_f$. Consequently $\theta_f(N) = \overline{\theta}_f(N/L)$ is finitely generated in $\mathcal{M}_R.$
To every \( f \in N^* \) there corresponds the left \( C \)-coideal \( \theta_f(N) \subset C \). If \( \theta_f(N) \) is finitely generated in \( \mathcal{M}_R \), then \((\theta_f(N))^*\) is a right \( C \)-comodule by 4.2 and we have for every \( n \in N \):

\[
\varepsilon(\theta_f(n)) = \varepsilon(\sum f(n_{<0>})n_{<-1>}) = f(\sum \varepsilon(n_{<-1>})n_{<0>}) = f(n),
\]

i.e. \( f \in (\theta_f(N))^* \subset \text{Rat}^C(A N^*) \).

As a special case of Proposition 4.11 we get

**Corollary 4.12.** Let \( R \) be noetherian. For every locally projective \( R \)-coalgebra \( C \) we have

\[
\text{Rat}^C(C^*, C^*^*) = \text{Sp}(\sigma_{[C^*, C^*^*]}):= \sum \{ \text{Hom}_{C^*^*}(U, C^*), U \in \sigma_{[C^*, C^*^*]} \}
\]

\[
= \{ f \in C^*^* \mid C^* \ast f \text{ is finitely generated in } \mathcal{M}_R \}
\]

\[
= \{ f \in C^*^* \mid \exists \text{ an } R \text{-cofinite (right) ideal } I \subset C^* \text{ with } f(CI) = 0 \}
\]

\[
= \{ f \in C^*^* \mid \exists \text{ an } R \text{-cofinite right } C^*\text{-submodule } K \subset C \text{ with } f(K) = 0 \}
\]

\[
= \{ f \in C^*^* \mid \exists \text{ an } R \text{-cofinite left } C \text{-coideal } K \subset C \text{ with } f(K) = 0 \}
\]

\[
= \{ f \in C^*^* \mid f \rightarrow C \subset C \text{ is a finitely generated } R\text{-submodule} \}.
\]

**4.13. Cofree comodules.** A right \( C \)-comodule \((M, \varrho_M)\) is called **cofree**, if there exists an \( R \)-module \( K \), such that \((M, \varrho_M) \simeq (K \otimes_R C, id_K \otimes \Delta_C)\) as right \( C \)-comodules. Note that if \( K = R^A \), a free \( R \)-module, then \( M \simeq R^A \otimes_R C \simeq C^A \) as right \( C \)-comodules (in fact, this is the reason for the terminology **cofree**).

**Lemma 4.14.** Let \( R \) be noetherian and \( A \) be a cofinitary \( R \)-algebra. Let \( M \) be an \( R \)-module and consider the right \( A \)-module \( N := M \otimes_R A \). Then \( N^o \simeq M^* \otimes_R A^* \) as \( A^* \)-comodules (i.e. \( N^o \) is a cofree right \( A^* \)-comodule).

**Proof.** If \( N \simeq M \otimes_R A \) as right \( A \)-modules, then there are isomorphisms in \( A \mathcal{M} \):

\[
N^o := \lim_{\rightarrow} \{ ((M \otimes_R A)/(M \otimes_R A)I)^* \mid I \in \mathcal{K}_A \}
\]

\[
= \lim_{\rightarrow} \{ ((M \otimes_R A)/(M \otimes_R A)\tilde{I})^* \mid \tilde{I} \in \mathcal{E}_A \} \quad (A \text{ is cofinitary})
\]

\[
= \lim_{\rightarrow} \{ ((M \otimes_R A)/(M \otimes_R \tilde{I}))^* \mid \tilde{I} \in \mathcal{E}_A \}
\]

\[
\simeq \lim_{\rightarrow} \{ (M^* \otimes_R (A/\tilde{I}))^* \mid \tilde{I} \in \mathcal{E}_A \}
\]

\[
\simeq \lim_{\rightarrow} \{ M^* \otimes_R (A/\tilde{I})^* \mid \tilde{I} \in \mathcal{E}_A \}
\]

\[
\simeq M^* \otimes_R A^o \quad (A/\tilde{I} \text{ is f.g. projective in } \mathcal{M}_R);
\]

\[
\simeq M^* \otimes_R \lim_{\rightarrow} \{ (A/\tilde{I})^* \mid \tilde{I} \in \mathcal{E}_A \}
\]

In contradiction with [Wit79, Corollary 2] the following example shows that for an arbitrary \( R \)-algebra \( A \) the preradical \( \text{Loc}(-) : A \mathcal{M} \rightarrow \text{Loc}(A \mathcal{M}) \) is in general not a torsion radical:

**Counter Example 4.15.** (Compare [Mon93, Seite 155]) Let \( R \) be a field and consider the Hopf \( R \)-algebra \( H := R[x_1, x_2, \ldots, x_n, \ldots] \), with the usual multiplication, the usual unit \( \epsilon \) and the comultiplication, counit and antipode defined on the generators through

\[
\Delta(x_i) = 1 \otimes x_i + x_i \otimes 1, \quad \varepsilon(x_i) = 0, \quad S(x^i) := (-1)^i x^i.
\]
If we consider $H$ with the left cofinite topology, then $(H, \text{Cf}(H))$ is a left linear topological $R$-algebra with preradical $\text{Loc}(-) : H\mathcal{M} \to H\mathcal{M}$ and pretorsion class $\text{Loc}(H\mathcal{M})$ (see 3.20 and Proposition 3.23). If we consider the $H$-ideal $\omega := \text{Ke}(\varepsilon_H)$, then $H/\omega \simeq R$ while $\dim(H/\omega^2) = \infty$, i.e. $\omega^2 \notin \mathcal{K}_H$. So Cf$(H)$ is not a Gabriel-topology and consequently Loc$(H\mathcal{M})$ is not closed against extensions (see [Ste75, Chapter VI, Theorem 5.1, Lemma 5.3]).

5 Coreflexive comodules

In [Taf72], [Taf77] E. Taft developed an algebraic aspect to the study of coreflexive coalgebras over base field (i.e. coalgebras $C$ with $C \simeq C^{**}$). Independently, R. Heyneman and D. Radford [Rad73], [HR74] studied the coreflexive coalgebras with the help of the finite topology on $C^*$. In this section we present and study for every $(A, C) \in \mathcal{P}_m$ over an arbitrary noetherian ring the notions of reflexive $A$-modules and coreflexive $C$-comodules. We get algebraic as well as topological characterizations for the (co)reflexive (co)modules. Our results will be applied then to the study of (co)reflexive (co)algebras, where we generalize also results from the papers mentioned above and from [Wit79].

$(A, C)$-pairings

In the case of base fields, D. Radford [Rad73] presented for every measuring $R$-pairing $P = (A, C)$ the so called right (left) $P$-pairings. In what follows we consider duality relations for such pairings.

5.1. Let $P = (A, C) \in \mathcal{P}_m$. A pairing of $R$-modules $Q = (M, N)$ is called a right (a left) $P$-pairing, if $M$ is a right (a left) $A$-module, $N$ is a right (a left) $C$-comodule and the induced mapping $\kappa_Q : M \to N^*$ is $A$-linear. By $Q^r_P \subset \mathcal{P}$ ($Q^l_P \subset \mathcal{P}$) we denote the subcategory of right (left) $P$-pairings with morphisms described as follows: if $(M, N)$, $(M', N')$ are right (left) $P$-pairings, then a morphism of $R$-pairings

$$(\xi, \theta) : (M', N') \to (M, N)$$

is a morphism in $Q^r_P$ (resp. in $Q^l_P$), if $\xi : M \to M'$ is $A$-Linear and $\theta : N' \to N$ is $C$-colinear.

A $P$-bi-pairing is an $R$-pairing $(M, N)$, where $M$ is an $A$-bimodule, $N$ is a $C$-bicomodule and $\kappa_Q : M \to N^*$ is $A$-bilinear. With $Q_P$ we denote the category of $P$-bi-pairings with morphisms described as follows: if $(M, N)$, $(M', N')$ are $P$-bi-pairings, then a morphism of $R$-pairings

$$(\xi, \theta) : (M', N') \to (M, N)$$

is a morphism in $Q_P$, if $\xi : M \to M'$ is $A$-bilinear and $\theta : N' \to N$ is $C$-bicolinear. In particular every measuring $R$-pairing $P$ is itself a $P$-bi-pairing.
5.2. Let $P = (A, C) \in \mathcal{P}_m$, $Q = (M, N) \in \mathcal{Q}_p^r$ and define for every $m \in M$:

$$
\begin{align*}
\xi_m : & A \to M, \ a \mapsto ma \quad \text{for all } a \in A, \\
\theta_m : & N \to C, \ n \mapsto \sum < m, n_{<0}> > n_{<1>} \quad \text{for all } n \in N.
\end{align*}
$$

Then we have for all $a \in A$ and $n \in N$:

$$
< \xi_m(a), n > = < ma, n > = < m, an > = \sum < m, n_{<0}> > n_{<1>} = < a, n_{<1}> > = < a, \theta_m(n) > .
$$

Obviously $\xi_m : A \to M$ is $A$-linear. Moreover it follows for all $n \in N$ and $a \in A$ that

$$
\alpha_N^P(\sum \theta_m(n_1) \otimes \theta_m(n_2))(a) = \sum \theta_m(n_1) < a, \theta_m(n_2) >
$$

$$
= a \rightarrow \theta_m(n)
$$

$$
= \sum < m, n_{<0}> > (a \rightarrow n_{<1>})
$$

$$
= \sum < m, n_{<0}> > n_{<1>1} < a, n_{<1>2} >
$$

$$
= \sum < m, n_{<0}> < n_{<0>_{<1}>} > n_{<1>2} < a, n_{<1>1} >
$$

$$
= \alpha_N^P(\sum \theta_m(n_{<0>}) \otimes n_{<1>})(a).
$$

If $\alpha_N^P : N \otimes_R C \to \text{Hom}_R(A, N)$ is injective, then

$$
\sum \theta_m(n_1) \otimes \theta_m(n_2) = \sum \theta_m(n_{<0>}) \otimes n_{<1>} \quad \text{for every } n \in N,
$$

i.e. $\theta_m : N \to C$ is $C$-colinear and

$$(\xi_m, \theta_m) : (M, N) \to (A, C)$$

is a morphism in $\mathcal{Q}_p^r$.

**Notation.** Let $P = (A, C) \in \mathcal{P}_m$ and $Q = (M, N) \in \mathcal{Q}_p^r$. For $R$-submodules $L \subset M$, $K \subset N$ we set

$$
K^\perp := \{ m \in M | < m, K > = 0 \}, \quad \text{Ann}_M(K) := \{ m \in M | \theta_m(k) = 0 \ \forall \ k \in K \},
$$

$$
L^\perp := \{ n \in N | < L, n > = 0 \}, \quad \text{Ann}_N(L) := \{ n \in N | \theta_m(n) = 0 \ \forall \ m \in L \}.
$$

As a consequence of Theorem 2.6 one can easily derive the following result:

**Lemma 5.3.** Let $P = (A, C) \in \mathcal{P}_m$ and $Q = (M, N) \in \mathcal{Q}_p^r$ (resp. $Q \in \mathcal{Q}_p^l$, $Q \in \mathcal{Q}_p$).

1. Every right $C$-subcomodule (resp. left $C$-subcomodule, $C$-subbicomodule) $K \subset N$ is a left $A$-submodule (resp. a right $A$-submodule, an $A$-subbimodule) and $K^\perp \subset M$ is a right $A$-submodule (resp. a left $A$-submodule, an $A$-subbimodule).

2. Let $(A, C) \in \mathcal{P}_m^a$. If $L \subset M$ a right $A$-submodule (resp. a left $A$-submodule, an $A$-subbimodule), then $L^\perp \subset N$ is a right $C$-subcomodule (resp. a left $C$-subcomodule, a $C$-subbicomodule).
The topology $\mathfrak{T}_N(M)$

Let $P = (A, C)$ be a measuring $R$-pairing and consider $A$ as a right linear topological $R$-algebra with the right $C$-adic topology $\mathcal{T}_C(A)$. For every $Q = (M, N) \in \mathcal{Q}_P$ we present on $M$ a topology $\mathfrak{T}_N(M)$, such that $(M, \mathfrak{T}_N(M))$ is a linear topological $(A, \mathcal{T}_C(A))$-module.

5.4. Let $P = (A, C) \in \mathcal{P}_m$, $Q = (M, N) \in \mathcal{Q}_P$ and consider $C$ with the canonical right $A$-module structure and $A$ as a right linear topological $R$-algebra with the right $C$-adic topology $\mathcal{T}_C(A)$ (compare 2.3). If $K \subseteq N$ is an $R$-submodule and $m \in \mathcal{A}_M(K)$, then we have for arbitrary $n \in K$ and $a \in A$:

$$\theta_{ma}(n) := \sum < ma, n_{<0}> > n_{<1>}$$

$$= \sum < m, an_{<0}> > n_{<1>}$$

$$= \sum < m, n_{<0}, <a, n_{<0} <1}> > n_{<1>1}$$

$$= \sum < m, n_{<0}, > < a, n_{<0} <1}> > n_{<1,2>2}$$

$$= \{(\alpha_l \circ \Delta^\cop_c)((\sum < m, n_{<0}, > n_{<1}>))(a)$$

$$= \{(\alpha_l \circ \Delta^\cop_c)((\theta_m(n)))(a) = 0,$$

i.e. $\mathcal{A}_M(K) \subseteq M$ is an $A$-submodule. Let $K = \sum_{i=1}^l Rn_i \subseteq N$ be an arbitrary finitely generated $R$-submodule with $\rho_N(n_i) = \sum_{j=1}^{l_i} n_{ij} \otimes c_{ij}$ for $i = 1, ..., l$ and set $W := \sum_{i=1}^l \sum_{j=1}^{l_i} Rc_{ij}$. Let $m \in M$ be arbitrary. If $a \in \mathcal{A}_A(W)$, then for $i = 1, ..., l$:

$$\theta_{ma}(n_i) = \sum_{i=1}^l \sum_{j=1}^{l_i} < ma, n_{ij} > c_{ij}$$

$$= \sum_{i=1}^l \sum_{j=1}^{l_i} < m, an_{ij} > c_{ij}$$

$$= \sum_{i=1}^l \sum_{j=1}^{l_i} \sum_{n_{ij}} < m, n_{ij} <0> > < a, n_{ij} <1>> c_{ij}$$

$$= \sum_{i=1}^l \sum_{j=1}^{l_i} \sum_{c_{ij}} < m, n_{ij} > < a, c_{ij}1 > c_{ij2}$$

$$= \sum_{i=1}^l \sum_{j=1}^{l_i} < m, n_{ij} > (c_{ij} \leftarrow a) = 0,$$

i.e. $(\mathcal{A}_M(K) : m) \supset \mathcal{A}_A(W)$ and so it’s open w.r.t. the $C$-adic topology $\mathcal{T}_C(A)$. So

$$\mathcal{B}(0_M) := \{\mathcal{A}_M(K) | K \subseteq N \text{ is a finitely generated } R\text{-submodule}\}$$

is neighbourhood basis of $0_M$ consisting of $A$-submodules of $M$ and $M$ becomes a topology $\mathfrak{T}_N(M)$, such that $(M, \mathfrak{T}_N(M))$ is a linear topological right $(A, \mathcal{T}_C(A))$-module.
5.5. Let \( P = (A, C) \in \mathcal{P}_m \) and \( Q = (M, N) \in \mathcal{Q}_P \). Then

\[
\mathcal{F}(0_M) := \{ K^\perp | K \subset N \text{ is a finitely generated } R\text{-submodule} \}
\]

is a filter basis consisting of \( R\text{-submodules of } M \) and induces on \( M \) the \textit{linear weak topology} \( M[\Sigma_{ls}(N)] \), such that \( (M, M[\Sigma_{ls}(N)]) \) is a linear topological \( R\text{-module} \) and \( \mathcal{F}(0_M) \) is a neighbourhood basis of 0

5.6. Let \( R \) be noetherian, \( P = (A, C) \in \mathcal{P}_m \) and consider \( C^* \) with the \textit{right cofinite topology} \( \text{Cf}(C^*) \) (see 3.20). The \( R\)-algebra morphism \( \kappa_P : A \to C^* \) induces on \( A \) a linear topology \( \kappa_P\text{-Cf}(A) \) with neighbourhood basis of 0:

\[
\mathcal{B}_{\kappa_P}(0_A) := \{ \kappa_P^{-1}(J) | J \subset C^* \text{ is an } R\text{-cofinite two-sided ideal} \}.
\]

By definition \( \kappa_P\text{-Cf}(A) \) the \textit{finest} linear topology \( \mathfrak{T} \) on \( A \), such that \( (A, \mathfrak{T}) \) is a linear topological \( R\text{-algebra} \) and \( \kappa_P : A \to C^* \) is continuous.

Let \( Q = (M, N) \in \mathcal{Q}_P \) and consider \( N_A^* \) with the cofinite topology \( \text{Cf}(N^*) \). The \( A\)-linear mapping \( \kappa_Q : M \to N^* \) induces on \( M \) a topology \( \kappa_Q\text{-Cf}(M) \) with neighbourhood basis of 0

\[
\mathcal{B}_{\kappa_Q}(0_M) := \{ \kappa_Q^{-1}(L) | L \subset N^* \text{ is an } R\text{-cofinite } A\text{-module} \}.
\]

Clearly \( \kappa_Q\text{-Cf}(M) \) is a linear topological right \( \text{Cf}(A)\)-module and is the \textit{finest} topology \( \mathfrak{T} \) on \( M \), such that \( (M, \mathfrak{T}) \) is a linear topological right \( (A, \text{Cf}(A))\)-module and \( \kappa_Q : M \to N^* \) is continuous.

**Lemma 5.7.** Let \( P = (A, C) \in \mathcal{P}_m \) and \( Q = (M, N) \in \mathcal{Q}_P \).

1. The linear weak topology \( M[\Sigma_{ls}(N)] \) and the topology \( \mathfrak{T}_N(M) \) coincide. So \( M \), considered with the linear weak topology, is a linear topological right \( (A, \mathfrak{T}_C(A))\)-module.

2. If \( R \) is noetherian and \( P \) satisfies the \( \alpha \)-condition, then

\[
M[\Sigma_{ls}(N)] \leq \kappa_Q\text{-Cf}(M) \leq \text{Cf}(M). \tag{18}
\]

**Proof.**

1. Let \( U \subset M \) be a neighbourhood of 0\textit{M} w.r.t. \( M[\Sigma_{ls}(N)] \). Then there exists a finitely generated \( R\text{-submodule } K \subset N \), such that \( K^\perp \subset U \). If \( m \in \text{An}_M(K) \), then we have for arbitrary \( n \in K : \)

\[
<m, n> = \sum n_{<0>} \varepsilon(n_{<1>}) = \varepsilon(\sum m, n_{<0>}) > n_{<1>} = \varepsilon(\theta_m(n)) = 0.
\]

So \( \text{An}_M(K) \subset K^\perp \subset U \), i.e. \( U \) is a neighbourhood of 0\textit{M} w.r.t. \( \mathfrak{T}_N(M) \).

On the other hand, let \( U \subset M \) be a neighbourhood of 0\textit{M} w.r.t. \( \mathfrak{T}_N(M) \). Then there exists a finitely generated \( R\text{-submodule } K = \sum_{i=1}^l Rn_i \subset N \), such that \( \text{An}_M(K) \subset U \).

Assume now that \( \varrho_N(n_i) = \sum_{j=1}^{l_i} n_{ij} \otimes c_{ij} \) and set \( W := \sum_{i=1}^l \sum_{j=1}^{l_i} Rn_{ij} \). Then \( W^\perp \subset \text{An}_M(K) \subset U \), i.e. \( U \) is a neighbourhood of 0\textit{M} w.r.t. \( M[\Sigma_{ls}(N)] \). Consequently \( M[\Sigma_{ls}(N)] = \mathfrak{T}_N(M) \).
2. Let $R$ be noetherian and $P \in \mathcal{P}_m^\alpha$. Let $U \subset M$ be a neighbourhood of $0_M$ w.r.t. $M[\Sigma_{ls}(N)]$, i.e. there exists a finitely generated $R$-submodule $K \subset N$, such that $K^\perp \subseteq U$. By assumption $P \in \mathcal{P}_m^\alpha$ and so there exists by the Finiteness Theorem 2.10 a left $A$-submodule $\tilde{K} \subset N$, such that $K \subseteq \tilde{K}$ and $\tilde{K}_R$ is finitely generated. Moreover $N^*/\text{An}(\tilde{K}) \hookrightarrow \tilde{K}^*$, i.e. $\text{An}(\tilde{K}) \subset N^*$ is an $R$-cofinite right $A$-submodule and $\kappa_Q^{-1}(\text{An}(\tilde{K})) := \tilde{K}^\perp \subseteq K^\perp \subset U$, i.e. $U$ is a neighbourhood of $0_M$ w.r.t. $\kappa_Q$-$\text{Cf}(M)$.

Let $U \subset M$ be a neighbourhood of $0_M$ w.r.t. $\kappa_Q$-$\text{Cf}(M)$, i.e. there exists an $R$-cofinite $A$-submodule $L \subset N^*$, such that $\kappa_Q^{-1}(L) \subseteq U$. Then $M/\kappa_Q^{-1}(L) \hookrightarrow N^*/L$, and so $\kappa_Q^{-1}(L) \subset M$ is an $R$-cofinite $A$-submodule. Consequently $U$ is a neighbourhood of $0_M$ w.r.t. $\text{Cf}(M)$.

**Definition 5.8.** Let $P = (A, C) \in \mathcal{P}_m$ and $Q = (M, N) \in \mathcal{Q}_p$.

1. If $P \in \mathcal{P}_m^\alpha$, then we call $Q$ weakly coreflexive, if $N = M^*$.

2. If $R$ is noetherian, then we call $Q$ coreflexive, if $M[\Sigma_{ls}(N)] = \text{Cf}(M)$.

3. We call $Q$ proper (resp. weakly reflexive, reflexive), if $\kappa_Q : M \rightarrow N^*$ is injective (resp. surjective, bijective).

**Definition 5.9.**

1. Let $C$ be an $R$-coalgebra and $N$ be a right $C$-comodule.

   (a) If $rC$ is locally projective, then we call $N$ weakly coreflexive, if $N = N^{\square}$.

   (b) If $R$ is noetherian, then we call $N$ coreflexive, if $N^*[\Sigma_{ls}(N)] = \text{Cf}(N^*)$.

2. Let $R$ be noetherian and $A$ be an $R$-algebra. We call a right $A$-module $M$ proper (resp. weakly reflexive, reflexive), if the canonical $A$-linear mapping $\lambda_M : M \rightarrow M^{\diamond}$ is injective (resp. surjective, bijective).

**Remarks 5.10.**

1. Consider the ground ring $R$ as a trivial $R$-bialgebra. Then $R^* \simeq R$, $\mathcal{M}_R \simeq \mathcal{M}^R$ and for every $R$-(co-)module $N$ we have $N^{**} = \text{Rat}^R(N^{**}) = \text{Loc}(R^*, N^{**})$. So $N$ is (co)reflexive, iff $N$ is reflexive in the usual sense, i.e. if the canonical $R$-linear mapping $\Phi_N : N \rightarrow N^{**}$ is bijective.

2. For every $P = (A, C) \in \mathcal{P}_m^\alpha$ we have $C = A^r$ (by Corollary 2.9 (1)) and so $P \in \mathcal{Q}_p$ is weakly coreflexive.

**Proposition 5.11.** Let $R$ be noetherian and $A$ be an $R$-algebra.

1. If $A$ is proper (i.e. the canonical mapping $\lambda_A : A \rightarrow A^{\diamond}$ is injective), then $\text{Cf}(A)$ is Hausdorff.

2. Let $A$ be cofinitely $R$-cogenerated. Then $A$ is proper, iff $\text{Cf}(A)$ is Hausdorff.

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3. If $R$ is a QF ring, then

$$A \text{ is proper } \iff \text{Cf}(A) \text{ is Hausdorff } \iff A^0 \subset A^* \text{ is dense.}$$

**Proof.**

1. Obviously $0_A := \bigcap_{\mathcal{K}_A} I \subset \text{Ke}(\lambda_A)$ and the result follows.

2. Assume $\text{Cf}(A)$ to be Hausdorff. If $A$ is not proper, then there exists some $0 \neq \tilde{a} \in A$, such that $f(\tilde{a}) = 0$ for every $f \in A^0$. If $I \triangleleft A$ is an arbitrary $R$-cofinite $A$-ideal, then $\tilde{a} \in \text{KeAn}(I) = I$ (compare [Wis88, 28.1]) and so $\bigcap_{\mathcal{K}_A} I \neq 0$ (contradiction).

3. By [Abu, Theorem 1.8 (1)] we have

$$\overline{A^0} = \text{AnKe}(A^0) = \text{An}(\text{Ke}(\sum_{I \in \mathcal{K}_A} \text{An}(I))) = \text{An}(\bigcap_{I \in \mathcal{K}_A} \text{KeAn}(I)) = \text{An}(\bigcap_{I \in \mathcal{K}_A} I).$$

So $A^0 \subset A^*$ is dense, iff $\bigcap_{\mathcal{K}_A} I = 0$. $\blacksquare$

**Lemma 5.12.** (Krull’s Theorem) Let $A$ be a commutative noetherian ring. For every finitely generated $A$-module $M$ and every $A$-ideal $I \triangleleft A$ we have

$$\bigcap_{k=0}^{\infty} MI^{k+1} = \{m \in M \mid \exists b \in I, \text{ such that } m(1_A - b) = 0\}.$$ 

The following result was obtained in [Swe69, 6.1.3] for commutative affine algebras over base fields:

**Lemma 5.13.** Let $R$ be a QF ring and $A$ be a commutative noetherian $R$-algebra. If every maximal $A$-ideal is $R$-cofinite, then $A^0 \subset A^*$ is dense.

**Proof.** Let $0 \neq a \in A$ be arbitrary and consider the $A$-ideal $J := (0 : a)$. Let $m \triangleleft A$ be a maximal $A$-ideal, such that $J \subset m$. Since $A$ is noetherian, $m_A$ is finitely generated. If $a \in \bigcap_{k=0}^{\infty} m^{k+1}$, then there exists by Krull’s Theorem some $b \in m$, such that $a(1_A - b) = 0$ and so $1_A \in m$ (contradiction). So there exists $k \geq 0$, such that $a \notin m^{k+1}$. By assumption $m \subset A$ is $R$-cofinite and it follows then from Lemma 2.14 that $m^{k+1} \subseteq A$ is $R$-cofinite, i.e. $a \notin \bigcap_{\mathcal{K}_A} I$. Since $0 \neq a \in A$ is arbitrary by our choice, it follows that $\bigcap_{\mathcal{K}(A)} I = 0$, i.e. $A$ is proper and consequently $A^0 \subset A^*$ is dense by Proposition 5.11. $\blacksquare$

Analog to the proof of Proposition 5.11 we get

**Proposition 5.14.** Let $R$ be noetherian, $A$ be an $R$-algebra and $M$ be a right $A$-module.

1. If $M$ is proper, then $\text{Cf}(M)$ is Hausdorff.

2. Let $M$ be cofinitely $R$-cogenerated. Then $\text{Cf}(M)$ is Hausdorff, iff $M$ is proper.
3. If $R$ is a QF ring, then

$$M \text{ is proper} \iff \text{Cf}(M) \text{ is Hausdorff} \iff M^\circ \subset M^* \text{ is dense}.$$ 

**Theorem 5.15.** Let $R$ be noetherian, $P = (A, C) \in P_m^\alpha$ and $Q = (M, N) \in Q_P$.

1. If $Q$ is coreflexive, then $M^r = M^\circ$.

2. Let $M$ be cofinitely $R$-cogenerated.

   (a) If $N \cong M^\circ$, then $Q$ is coreflexive.

   (b) Let $Q$ be weakly coreflexive. Then $Q$ is coreflexive, iff $N \cong M^\circ$.

**Proof.**  1. Assume $Q$ to be coreflexive and consider $A$ and $M$ with the linear weak topology $A[\mathfrak{T}_{ls}(C)]$, $M[\mathfrak{T}_{ls}(N)]$ respectively. Let $f \in M^*$ with $f(L) = 0$ for an $R$-cofinite $A$-submodule $L \subset M$, say $M/L = \sum_{i=1}^k R(m_i + L)$. By assumption $M[\mathfrak{T}_{ls}(N)] = \text{Cf}(M)$ and so $L$ is open w.r.t. $M[\mathfrak{T}_{ls}(N)]$. By [Abu, Corollary 1.9] $\xi_{m_i} : A \to M$ is continuous and so there exist finitely generated $R$-submodules $Z_1, ..., Z_k \subseteq C$, such that $Z_i \subseteq \xi_{m_i}^{-1}(L)$. Consequently $(\sum_{i=1}^k Z_i)^{-1} = \bigcap_{i=1}^k Z_i^\perp \subseteq (0_{M^*} : f)$, i.e. $f \in M^r$ (by Proposition 2.11). Obviously $M^r \subset M^\circ$ and the result follows.

2. Let $M$ be cofinitely $R$-cogenerated.

   (a) Assume that $N \cong M^\circ$. Let $L \subset M$ be an $R$-cofinite $A$-submodule with \{f_1, ..., f_k\} a generating system of $\text{An}(L) \cong (M/L)^*$. Then there exists by assumption \{n_1, ..., n_k\} $\subset N$, such that $\chi_Q(n_i) = f_i$. By [Wis88, 28.1] we have then

   $$\left(\sum_{i=1}^k Rn_i\right)^{\perp} = \bigcap_{i=1}^k \text{Ke}(f_i) = \text{Ke}(\sum_{i=1}^k Rf_i) = \text{KeAn}(L) = L,$$

   i.e. $L$ is open w.r.t. $M[\mathfrak{T}_{ls}(N)]$. Consequently $\text{Cf}(M) \subset M[\mathfrak{T}_{ls}(N)]$. By Lemma 5.7 (2) $M[\mathfrak{T}_{ls}(N)] \subset \text{Cf}(M)$ and so $M[\mathfrak{T}_{ls}(N)] = \text{Cf}(M)$, i.e. $Q$ is coreflexive.

   (b) The result follows from (1) and (a). \[\Box\]

**Corollary 5.16.** Let $R$ be noetherian and $C$ be a locally projective $R$-coalgebra.

1. If $N$ is coreflexive, then $N^{*\square} = N^{*\circ}$.

2. Let $N^*$ be cofinitely $R$-cogenerated.

   (a) If $N \cong N^{*\circ}$, then $N$ is coreflexive.

   (b) Let $N$ be weakly coreflexive. Then $N$ is coreflexive, iff $N \cong N^{*\circ}$. 

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Theorem 5.17. Let $R$ be noetherian and $P = (A, C) \in \mathcal{P}_m^\alpha$.

1. If $P$ is coreflexive, then $C = A^\circ$.

2. Assume $R$ to be artinian. Then $P$ is coreflexive, iff all $R$-cofinite $A$-ideals are closed w.r.t. $A[\Sigma_{ls}(C)] = T_{-C}(A)$.

3. If $A$ is cofinitely $R$-cogenerated, then the following statements are equivalent:
   (i) $P$ is coreflexive;
   (ii) $C = A^\circ$.
   (iii) every locally finite left $A$-module is $C$-rational, i.e. $\text{Loc}(AM) = \sigma[A^\circ C]$.

Proof. 1. By Corollary 2.9 (1) $C = A^\circ$ and so the result follows from Theorem 5.15 (1).

2. Let $R$ be artinian. By [Abu, Lemma 1.7 (4)] every $R$-cofinite closed $A$-ideal is open and the result follows.

3. (i) $\iff$ (ii) follows from Theorem 5.15 (3).
   
   (ii) $\implies$ (iii) By assumption and Proposition 3.23 (2) $\text{Loc}(AM) = \sigma[AA^\circ] = \sigma[A^\circ C]$.
   
   (iii) $\implies$ (ii) Assume all locally finite left $A$-modules to be $C$-rational. Then in particular $AA^\circ$ is $C$-rational and it follows from Corollary 2.9 (2) that $C = A^\circ$. $\blacksquare$

Corollary 5.18. Let $R$ be noetherian and $C$ be a locally projective $R$-coalgebra.

1. If $C$ is coreflexive, then the canonical $R$-linear mapping $\phi_C : C \to C^{**}$ induces an isomorphism $C^{\phi_C} \cong C^{**}$.

2. Let $R$ be artinian. Then $C$ is coreflexive, iff all $R$-cofinite $C^*$-ideals are closed w.r.t. the finite topology.

3. If $C^*$ is cofinitely $R$-cogenerated, then the following statements are equivalent:
   (i) $C$ is coreflexive;
   (ii) $C \simeq C^{**}$;
   (iii) every locally finite left $C^*$-module is $C$-rational.

As a consequence of Lemma 3.22 and Theorem 5.17 (3) get we

Proposition 5.19. Let $R$ be noetherian. If $A$ is a cofinitely $R$-cogenerated $\alpha$-algebra and $M$ is a right $A$-module with structure map $\phi_M : M \otimes_R A \to M$, then for every $f \in M^*$ the following statements are equivalent:

1. $f \in M^\circ$. 35
2. $\phi^*_M(f) \in M^\circ \otimes_R A^\circ$.

3. $\phi^*_M(f) \in M^\circ \otimes_R A^\ast$.

4. $Af$ is finitely generated in $M_R$.

5. $f(MI) = 0$ for an $R$-cofinite (right) $A$-ideal.

6. $f(L) = 0$ for an $R$-cofinite right $A$-submodule $L \subset M$.

Analog to [Taf72] we get

Corollary 5.20. Let $R$ be a QF ring.

1. A projective $R$-coalgebra $C$ is coreflexive, iff $C^\ast$ is a reflexive $R$-algebra.

2. Let $A$ be an $\alpha$-algebra. If $A$ is weakly reflexive, then $A^\circ$ is a coreflexive $R$-coalgebra.

Example 5.21. ([Lin77, Example 5]) Let $R$ be a field and consider the Hopf $R$-algebra $(H, \mu, \eta, \Delta, \varepsilon, S)$ with countable basis $\{h_0, h_1, h_2, \ldots\}$ and

\[
\begin{align*}
\mu(h_n \otimes h_k) &:= \binom{n+k}{n}h_{n+k}, & \Delta(h_n) &:= \sum_{i+j=n} h_i \otimes h_j, & S(h_n) &:= (-1)^n h_n. \\
\eta(1_R) &:= h_0, & \varepsilon(h_n) &:= \delta_{0,n}.
\end{align*}
\]

1. $H^\ast \simeq R[[x]]$ is a principal ideal domain and

\[\mathcal{M}^H \simeq \text{Rat}^H(H^\ast, \mathcal{M}) = \{M \in H^\ast \mathcal{M} | M \text{ is a torsion module}\}.\]

So $\text{Rat}^H(-)$ is a radical and $\text{Rat}^H(H^\ast, \mathcal{M})$ is closed under extensions.

2. $H^\Box := \text{Rat}^H(H^\ast H^\ast) = 0$.

3. There exists no finite dimensional nonzero projective right $H$-comodules.

4. $H \simeq H^{\circ\circ}$, i.e. $H$ is a coreflexive $R$-coalgebra.

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