Finite Time Blow-Up for Wave Equations with Strong Damping in an Exterior Domain

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Abstract. We consider the initial boundary value problem in exterior domain for strongly damped wave equations with power-type nonlinearity $|u|^p$. We will establish blow-up results under some conditions on the initial data and the exponent $p$, using the method of test function with an appropriate harmonic functions. We also study the existence of mild solution and its relation with the weak formulation.

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1. Introduction

This paper concerns the initial boundary value problem of the strongly damped wave equation in an exterior domain. Let $\Omega \subset \mathbb{R}^n$ be an exterior domain whose obstacle $\mathcal{O} \subset \mathbb{R}^n$ is bounded with smooth compact boundary $\partial \Omega$. We consider the initial boundary value problem

$$\begin{cases}
    u_{tt} - \Delta u - \Delta u_t = |u|^p, & t > 0, x \in \Omega, \\
    u(0, x) = u_0(x), & x \in \Omega, \\
    u_t(0, x) = u_1(x), & x \in \Omega, \\
    u = 0, & t > 0, x \in \partial \Omega,
\end{cases}$$

(1.1)

where the unknown function $u$ is real-valued, $n \geq 1$, and $p > 1$. Throughout this paper, we assume that

$$(u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega).$$

(1.2)

Without loss of generality, we assume that $0 \in \mathcal{O} \subset B(R)$, where $B(R) := \{x \in \mathbb{R}^n : |x| < R\}$ is a ball of radius $R$ centred at the origin. Moreover, we assume that
\[
\begin{cases}
1 < p < \infty, & \text{for } n = 1, 2, \\
1 < p \leq \frac{n}{n-2}, & \text{for } n \geq 3.
\end{cases}
\] (1.3)

For the simplicity of notations, \(\|\cdot\|_q\) and \(\|\cdot\|_{H^1_0}(1 \leq q \leq \infty)\) stand for the usual \(L^q(\Omega)-\text{norm}\) and \(H^1_0(\Omega)-\text{norm}\), respectively. For any confusion, we remember that \(\|u\|_{H^1_0} = \|u\|_2 + \|\nabla u\|_2\).

For the Cauchy problem for the semilinear wave equation with weak damping
\[
u_{tt} - \Delta u + u_t = |u|^p, \quad t > 0, \quad x \in \mathbb{R}^n,
\] (1.4)

Todorova and Yordanov [23] proved the global existence of solution for sufficiently small data in the energy space if \(p > 1 + 2/n\) and the blowing up of solution when \(p < 1 + 2/n\). Moreover, Zhang [25] succeeded in completing this study by proving that the critical case \(p = 1 + 2/n\) belongs to the blow-up existence of solution. The number \(p_c(n) = 1 + 2/n\) is known (due to Fujita [9]) as the critical exponent of corresponding heat equation \(-\Delta u + u_t = |u|^p\) since it divides \((1, \infty)\) into two subintervals so that the following take place: If \(p \in (1, p_c(n)]\), then solutions with non-negative (in some sense) initial values blow-up in finite time, while if \(p \in (p_c(n), \infty)\), then solutions with small (and sufficiently regular) initial values exist for all time.

For problem (1.4) in exterior domain
\[
u_{tt} - \Delta u + u_t = |u|^p, \quad t > 0, \quad x \in \Omega.
\]

Ikehata [12,13] proved that the solution exists globally when \(n = 2\) for \(p > p_c(n)\), and \(n = 3, 4, 5\) for \(1 + 4/(n + 2) < p \leq 1 + 2/(n - 2)\) under the assumption of the compactness of the initial data. Ono [22] obtained the same result of global existence of solution without compactness of the initial data by applying the result of Dan–Shibata [3] and cut-off method. Ogawa and Takeda [21] proved the non-existence of non-negative global weak solutions of (1.1) under the assumption that \(u_0 = 0, 1 < p < p_c(n)\), and the support of the initial data is compact. Recently, Fino et al. [5] proved the blow-up existence of solution for the critical exponent \(p = p_c(n)\) and in more general condition on the compactly supported initial data, while Lai and Yin [19] obtained the same result of [5] without the compactness of the initial data and by applying another method using a suitable Harmonic function.

For the Cauchy problem for the semilinear wave equation with strong damping
\[
u_{tt} - \Delta u - \Delta u_t = |u|^p, \quad t > 0, \quad x \in \mathbb{R}^n,
\] (1.5)

D’Abbicco–Reissig [2] tried to find the critical exponent; they proved that there exists a global solution for (1.5) when \(p > 1 + 3/(n - 1)\) \((n \geq 2)\) for sufficiently small initial data, while when \(1 < p \leq 1 + 2/n\) they obtained a blow-up existence of solution. Therefore, it still an open problem to find the critical exponent.

On the other hand, concerning the exterior domain of (1.5), namely problem (1.1), there exist few results. Ikehata and Inoue [15] proved the global existence of solution in two-dimensional case and just when \(p > 6\).
For more general equations in whole and exterior domain, we mentioned the recent papers of Ikehata et al. [1,16].

In this paper, we study the blow-up existence of solution for (1.1) as well as the existence of mild solution in a clear way.

First, the following local well-posedness result is needed.

**Proposition 1.1.** (Local existence)

Under the assumptions (1.2)–(1.3), there exists a maximal existence time $T_{\text{max}} > 0$ such that the problem (1.1) possesses a unique mild solution $u \in C([0,T_{\text{max}}), H_0^1(\Omega)) \cap C^1([0,T_{\text{max}}), L^2(\Omega))$, where $0 < T_{\text{max}} \leq \infty$. In addition,

$$
\begin{cases}
T_{\text{max}} = \infty, \\
T_{\text{max}} < \infty \text{ and } \|u(t, \cdot)\|_{H_0^1} + \|u_t(t, \cdot)\|_2 \to \infty \text{ as } t \to T_{\text{max}}.
\end{cases}
$$

(1.6)

For the proof of this proposition, we refer to Wakasugi [24] by an appropriate modification of the energy space and some estimations. For simplicity, we present all in details with all required modifications.

**Remark 1.2.** We say that $u$ is a global solution of (1.1) if $T_{\text{max}} = \infty$, while in the case of $T_{\text{max}} < \infty$, we say that $u$ blows up in finite time.

Our main result is the following:

**Theorem 1.3.** (Blow-up)

Assume that the initial data satisfy (1.2) such that $u_0, u_1 \phi_0 \in L^1(\Omega)$ and

$$
\int_{\Omega} u_1(x) \phi_0(x) \, dx > 0,
$$

where $\phi_0(x)$ is defined later (see Lemma 2.11, 2.12, 2.13). If

$$
\begin{cases}
1 < p \leq 1 + \alpha, & \text{for } n = 1, \\
1 < p < 3, & \text{for } n = 2, \\
1 < p \leq 1 + \frac{2}{n-1}, & \text{for } n \geq 3,
\end{cases}
$$

where $\alpha = \frac{1 + \sqrt{17}}{4}$ is the positive root of $2\alpha^2 - \alpha - 2 = 0$, then the solution of problem (1.1) blows up in finite time.

The proof of the blow-up results is based on the test function method introduced by [25], used by [6–8,20], and modified by [7].

This paper is organized as follows: in Sect. 2, we present several preliminaries. Section 3 is devoted to prove the local existence of mild solution (Proposition 1.1). Section 4 contains the proof of the blow-up theorem (Theorem 1.3).
2. Preliminaries

In this section, we give some preliminary properties that will be used in the proof of local existence and blow-up result. First, we start by

2.1. Linear Homogeneous Case

We consider the linear homogeneous equation

\[
\begin{cases}
  u_{tt} - \Delta u - \Delta u_t = 0, & t > 0, x \in \Omega, \\
  u(0, x) = u_0(x), & x \in \Omega, \\
  u_t(0, x) = u_1(x), & x \in \Omega, \\
  u = 0, & t > 0, x \in \partial\Omega.
\end{cases}
\] (2.1)

**Definition 2.1. (Strong solution)**

Let \((u_0, u_1) \in (H^2(\Omega) \cap H^1_0(\Omega))^2\). A function \(u\) is said to be a strong solution of (2.1) if

\[
u \in C^1([0, \infty); H^2(\Omega) \cap H^1_0(\Omega)) \cap C^2([0, \infty); L^2(\Omega)),
\]

and \(u\) has the initial data \(u(0) = u_0, u_t(0) = u_1\) and satisfies Eq. (2.1) in the sense of \(L^2(\Omega)\).

**Proposition 2.2.** For each \((u_0, u_1) \in (H^2(\Omega) \cap H^1_0(\Omega))^2\), there exists a unique strong solution \(u\) of problem (2.1) that satisfies the following energy estimates

\[
\int_\Omega (u^2(t, x) + |\nabla u(t, x)|^2) \, dx \leq \int_\Omega (u^2_1(x) + |\nabla u_0(x)|^2) \, dx,
\] (2.2)

\[
\|u(t)\|_{L^2} \leq \|u_0\|_{L^2} + T\| (u_1, \nabla u_0) \|_{L^2 \times L^2},
\] (2.3)

for any \(T > 0\), and all \(0 \leq t \leq T\).

**Proof.** The existence of the strong solution is done by Kobayashi–Pecher–Shibata [18, Theorem 3.1]. The energy estimates (2.2)–(2.3) can be deduced easily from Ikehata [14, Proposition 1.1]. To be self-contained, we prove them in details. We start by proving inequality (2.2). Multiply (2.1) by \(u_t\), and integrating over \(\Omega\), we have

\[
\int_\Omega u_{tt} u_t \, dx - \int_\Omega \Delta u u_t \, dx - \int_\Omega \Delta u_t u_t \, dx = 0.
\]

Using the boundary condition and the divergence theorem, we get

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega (u_t^2 + |\nabla u|^2) \, dx + \int_\Omega |\nabla u_t|^2 \, dx = 0.
\]

Integrating over \([0, t]\), we deduce that

\[
\frac{1}{2} \int_\Omega (u_t^2 + |\nabla u|^2) \, dx \leq \frac{1}{2} \int_\Omega (u_1^2 + |\nabla u_0|^2) \, dx + \int_0^t \int_\Omega |\nabla u_s|^2 \, dx \, ds
\]

\[
= \frac{1}{2} \int_\Omega (u_1^2 + |\nabla u_0|^2) \, dx;
\]
which implies the desired estimate. Next, we prove (2.3). Let \( T > 0 \), for all \( 0 \leq t \leq T \), we have
\[
u(t) = \int_0^t u_s(s) \, ds,
\]
then
\[
\|u(t)\|_{L^2} \leq \|u_0\|_{L^2} + \int_0^t \|u_s(s)\|_{L^2} \, ds
\]
\[
\leq \|u_0\|_{L^2} + \int_0^t \|(u_1, \nabla u_0)\|_{L^2 \times L^2} \, ds
\]
\[
\leq \|u_0\|_{L^2} + T \|(u_1, \nabla u_0)\|_{L^2 \times L^2},
\]
where we have used inequality (2.2). This completes the proof. \( \square \)

Let us denote by \( R(t) \) the operator which maps the initial data \((u_0, u_1) \in (H^2(\Omega) \cap H^1_0(\Omega))^2\) to the strong solution \( u(t) \in H^2(\Omega) \cap H^1_0(\Omega) \) at the time \( t \geq 0 \), i.e. the solution \( u \) of (2.1) is defined by \( u(t) = R(t)(u_0, u_1) \).

**Remark 2.3.** From Proposition 2.2, the operator \( R(t) \) can be extended uniquely such that \( R(t) : H^1_0(\Omega) \times L^2(\Omega) \rightarrow C([0, \infty), H^1_0(\Omega)) \cap C^1([0, \infty), L^2(\Omega)). \)

Indeed, for any fixed \( T > 0 \), due to the energy estimates (2.2)–(2.3), the following estimation
\[
\|R(t)(u_0, u_1)\|_{H^1_0} + \|\partial_t R(t)(u_0, u_1)\|_{L^2} \leq C(1 + T)\|(u_0, u_1)\|_{H^1_0 \times L^2}
\]
holds for all \( 0 \leq t \leq T \). It follows that the operator \( R(t) \) can be extended uniquely to an operator such that \( R(t) : H^1_0(\Omega) \times L^2(\Omega) \rightarrow C([0, T], H^1_0(\Omega)) \cap C^1([0, T], L^2(\Omega)). \) Since \( T \) is arbitrary, we conclude the desired extension.

### 2.2. Linear Inhomogeneous Case

Let us consider the linear inhomogeneous equation
\[
\begin{cases}
u_{tt} - \Delta u - \Delta u_t = F(t, x), & t > 0, x \in \Omega, \\
u(0, x) = u_0(x), & u_t(0, x) = u_1(x), & x \in \Omega, \\
u = 0, & t > 0, x \in \partial \Omega.
\end{cases}
\]

**Definition 2.4.** (Strong solution)

Let \((u_0, u_1) \in (H^2(\Omega) \cap H^1_0(\Omega))^2\) and \( F \in C([0, \infty); L^2(\Omega)) \). A function \( u \) is said to be a strong solution of (2.4) if \( u \in C^1([0, \infty); H^2(\Omega) \cap H^1_0(\Omega)) \cap C^2([0, \infty); L^2(\Omega)) \), and \( u \) has the initial data \( u(0) = u_0, u_t(0) = u_1 \) and satisfies the equation (2.4) in the sense of \( L^2(\Omega) \).

**Proposition 2.5.** [18, Theorem 3.1]

Let \((u_0, u_1) \in (H^2(\Omega) \cap H^1_0(\Omega))^2\) and \( F \in C^1([0, \infty); L^2(\Omega)) \), then problem (2.4) has a unique strong solution.
Definition 2.6. (Mild solution)
Let \((u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)\) and \(F \in C([0, \infty); L^2(\Omega))\). A function \(u\) is said to be a mild solution of (2.4) if
\[
u \in C([0, \infty); H_0^1(\Omega)) \cap C^1([0, \infty); L^2(\Omega)),
\]
and \(u\) has the initial data \(u(0) = u_0, \ u_t(0) = u_1\) and satisfies the integral equation
\[
u(t, x) = R(t)(u_0, u_1) + \int_0^t S(t - s)F(s, x) \, ds \tag{2.5}
\]
in the sense of \(H_0^1(\Omega)\), where \(S(t)g := R(t)(0, g)\) for all \(g \in L^2(\Omega)\).

Remark 2.7. It is easy to check that \(S(t)(u_0 + u_1) + \partial_t S(t)u_0\) is a strong solution of (2.1). It follows, by the uniqueness, that
\[
R(t)(u_0, u_1) = S(t)(u_0 + u_1) + \partial_t S(t)u_0.
\]

Definition 2.8. (Weak solution)
Let \(T > 0\), \(u_0, u_1 \in L^1_{loc}(\Omega)\) and \(F \in L^1((0, T); L^1_{loc}(\Omega))\). A function \(u\) is said to be a weak solution of (2.4) if
\[
u \in L^1((0, T); L^1_{loc}(\Omega)),
\]
and \(u\) satisfies the weak formulation
\[
\int_0^T \int_\Omega F(t, x) \varphi \, dx \, dt + \int_\Omega u_1(x) \varphi(0, x) \, dx - \int_\Omega u_0(x) \Delta \varphi(0, x) \, dx - \int_\Omega u_0(x) \varphi_t(0, x) \, dx
\]
\[
= \int_0^T \int_\Omega u_\varphi t \, dx \, dt + \int_0^T \int_\Omega u \Delta \varphi_t \, dx \, dt - \int_0^T \int_\Omega u \Delta \varphi \, dx \, dt, \tag{2.6}
\]
for all compactly supported test function \(\varphi \in C^2([0, T] \times \Omega)\) such that \(\varphi(\cdot, 0) = 0\) and \(\varphi_t(\cdot, 0) = 0\).

Proposition 2.9. Let \((u_0, u_1) \in (H^2(\Omega) \cap H_0^1(\Omega))^2\) and \(F \in C([0, \infty); H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0, \infty); L^2(\Omega))\) and \(u\) be the strong solution of (2.4). Then \(u\) also is a mild solution, and satisfies the following energy estimates:
\[
\|(u_t, \nabla u)(t)\|_{L^2 \times L^2} \leq C\|(u_1, \nabla u_0)\|_{L^2 \times L^2} + C \int_0^t \|F(s, \cdot)\|_{L^2} \, ds, \tag{2.7}
\]
\[
\|u(t)\|_{L^2} \leq \|u_0\|_{L^2} + C \int_0^t \left(\|(u_1, \nabla u_0)\|_{L^2 \times L^2} + \int_0^s \|F(\tau, \cdot)\|_{L^2} \, d\tau\right) \, ds. \tag{2.8}
\]

Proof. Let
\[
\tilde{u}(t) := R(t)(u_0, u_1) + \int_0^t S(t - s)F(s, x) \, ds.
\]
As \((u_0, u_1) \in (H^2(\Omega) \cap H_0^1(\Omega))^2\) and \(F \in C([0, \infty); H^2(\Omega) \cap H_0^1(\Omega))\), using Proposition 2.2, we can easily conclude that \(\tilde{u}\) is a strong solution of (2.4). Therefore, by the uniqueness, we have \(u = \tilde{u}\).
Next, we prove estimate (2.7). Multiply (2.4) by $u_t$, integrating over $\Omega$, use the boundary condition and the divergence theorem, we obtain

$$\frac{1}{2} \int_{\Omega} (u_t^2(t) + |\nabla u|^2) \, dx + \int_{\Omega} |\nabla u_t|^2 \, dx = \int_{\Omega} F(t, x) u_t \, dx.$$ 

Integrating over $[0, t]$, and using Cauchy–Schwarz inequality, we deduce that

$$\frac{1}{2} \int_{\Omega} (u_t^2(t) + |\nabla u(t)|^2) \, dx \leq \frac{1}{2} \int_{\Omega} (u_1^2 + |\nabla u_0|^2) \, dx + \int_0^t \|F(s)\|_2 \|u_s(s)\|_2 \, ds \leq \frac{1}{2} \int_{\Omega} (u_1^2 + |\nabla u_0|^2) \, dx + \int_0^t \|F(s)\|_2 \left( \int_{\Omega} (u_s^2(s) + |\nabla u(s)|^2) \, dx \right)^{1/2} \, ds.$$

Therefore, by Gronwall’s Lemma (see, e.g. [24, Lemma 9.12]), we conclude that

$$\left( \int_{\Omega} (u_t^2(t) + |\nabla u(t)|^2) \, dx \right)^{1/2} \leq \left( \int_{\Omega} (u_1^2 + |\nabla u_0|^2) \, dx \right)^{1/2} + \int_0^t \|F(s)\|_2 \, ds.$$

In particular, we get (2.7). Finally, we prove (2.8). Let $T > 0$, for all $0 \leq t \leq T$, we have

$$u(t) = u_0 + \int_0^t u_s(s) \, ds,$$

then

$$\|u(t)\|_{L^2} \leq \|u_0\|_{L^2} + \int_0^t \|u_s(s)\|_{L^2} \, ds \leq \|u_0\|_{L^2} + C \int_0^t \left( \|(u_1, \nabla u_0)\|_{L^2 \times L^2} + \int_0^s \|F(\tau)\|_{L^2} \, d\tau \right) \, ds,$$

where we have used inequality (2.7). This completes the proof. \(\square\)

**Proposition 2.10.** Let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and $F \in C([0, \infty); L^2(\Omega))$. Then there exists a unique mild solution $u$ of (2.4). Moreover, the mild solution $u$ satisfies the estimates (2.7) and (2.8).

**Proof.** **Existence.** Let $T_0 > 0$ an arbitrary number. By the density argument, there exist sequences

$$\left\{(u_0^{(j)}, u_1^{(j)})\right\}_{j=1}^{\infty} \subseteq (H^2(\Omega) \cap H_0^1(\Omega))^2,$$

$$\left\{(F^{(j)})\right\}_{j=1}^{\infty} \subseteq C([0, T_0]; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0, T_0]; L^2(\Omega))$$

such that

$$\lim_{j \to \infty} (u_0^{(j)}, u_1^{(j)}) = (u_0, u_1) \text{ in } H_0^1(\Omega) \times L^2(\Omega),$$

$$\lim_{j \to \infty} F^{(j)} = F \text{ in } C([0, T_0]; L^2(\Omega)).$$

Using Proposition 2.5, let $u^{(j)}$ be the strong solution of the linear inhomogeneous equation (2.4) with the initial data $(u_0^{(j)}, u_1^{(j)})$ and the inhomogeneous
term $F^{(j)}(t, x)$. Then the difference $u^{(j)} - u^{(k)}$ is a strong solution of the initial value problem
\[
\begin{aligned}
&u_{tt} - \Delta u - \Delta u_t = F^{(j)}(t, x) - F^{(k)}(t, x), & t > 0, x \in \Omega, \\
&u(0, x) = u_0^{(j)}(x) - u_0^{(k)}(x), & x \in \Omega, \\
&u_t(0, x) = u_1^{(j)}(x) - u_1^{(k)}(x), & x \in \Omega, \\
&u = 0, & t > 0, & x \in \partial\Omega.
\end{aligned}
\]
Apply Proposition 2.9 to $u^{(j)} - u^{(k)}$, we have
\[
\begin{aligned}
&\|\partial_t (u^{(j)} - u^{(k)}), \nabla (u^{(j)} - u^{(k)}))\|_{L^2 \times L^2} \\
&\leq C \|\nabla (u_0^{(j)} - u_0^{(k)}), \nabla (u_1^{(j)} - u_1^{(k)}))\|_{L^2 \times L^2} \\
&+ CT_0 \sup_{s \in [0, T_0]} \|(F^{(j)} - F^{(k)})(s))\|_{L^2},
\end{aligned}
\]
and
\[
\begin{aligned}
&\|(u^{(j)} - u^{(k)})(t))\|_{L^2} \\
&\leq \|u_0^{(j)} - u_0^{(k)}\|_{L^2} + CT_0 \|\nabla (u_1^{(j)} - u_1^{(k)}), \nabla (u_0^{(j)} - u_0^{(k)}))\|_{L^2 \times L^2} \\
&+ CT_0^2 \sup_{s \in [0, T_0]} \|(F^{(j)} - F^{(k)})(s))\|_{L^2}.
\end{aligned}
\]
This shows that $\{u^{(j)}\}_{j=1}^\infty$ is a Cauchy sequence in the complete space $C([0, T_0]; H_0^1(\Omega)) \cap C^1([0, T_0]; L^2(\Omega))$. Therefore, we can define the limit
\[
\lim_{j \to \infty} u^{(j)} = u \in C([0, \infty); H_0^1(\Omega)) \cap C^1([0, \infty); L^2(\Omega)),
\]
(2.9)
since $T_0 > 0$ is arbitrary. Apply again Proposition 2.9 to $u^{(j)}$, it follows that $u^{(j)}$ satisfies the integral equation
\[
u^{(j)}(t, x) = R(t)(u_0^{(j)}, u_1^{(j)}) + \int_0^t S(t - s)F^{(j)}(s, x) \, ds.
\]
By Remark 2.3, $R(t)$ and $S(t)$ are extended uniquely to the operators defined on $H_0^1(\Omega) \times L^2(\Omega)$ and $L^2(\Omega)$, respectively. Letting $j \to \infty$, we get
\[
u(t, x) = R(t)(u_0, u_1) + \int_0^t S(t - s)F(s, x) \, ds,
\]
which indicates that $u$ is a mild solution of (2.4).

**Uniqueness** If two functions $u$ and $v$ satisfy the integral equation (2.5), then we immediately have $u = v$.

**Energy estimates** By Proposition 2.9, each strong solution $u^{(j)}$ constructed above satisfies the estimates (2.7)–(2.8) with $u_0^{(j)}, u_1^{(j)}, F^{(j)}$. By letting $j \to \infty$ and using (2.9), the same estimates hold for the mild solution $u$. \hfill \Box

### 2.3. Harmonic Functions

In this subsection, we give some harmonic function that will be used in the proof of Theorem 1.3.
Lemma 2.11. There exists a function $\phi_0(x) \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfying the following boundary value problem

$$\begin{cases} 
\Delta \phi_0(x) = 0, & \text{in } \Omega, \ n \geq 3, \\
\phi_0|_{\partial\Omega} = 0, \\
|x| \to \infty, & \phi_0(x) \to 1.
\end{cases} \tag{2.10}$$

Moreover, $\phi_0(x)$ satisfies:

- $0 < \phi_0(x) < 1$, for all $x \in \Omega$.
- $\phi_0(x) \geq C$, for all $|x| \gg 1$.
- $|\nabla \phi_0(x)| \leq \frac{C}{|x|^{n-2}}$, for all $|x| \gg 1$.

Proof. From [26, Lemma 2.2], there exists a regular solution $\phi_0$ of (2.10) such that $0 < \phi_0(x) < 1$, for all $x \in \Omega$.

The rest of this proof is inspired by [17]. On the other hand, since $\mathcal{O}$ is bounded and $0 \in \mathcal{O}$, there exists $r_2 > r_1 > 0$ such that $B_{r_1} \subseteq \mathcal{O} \subseteq B_{r_2}$, where $B_r$ stands for the open ball with center zero and radius $r$. Let $\phi_1(x)$ and $\phi_2(x)$, respectively, be the solution of (2.10) if $\Omega = \mathbb{R}^n \setminus B_{r_1}$ and if $\Omega = \mathbb{R}^n \setminus B_{r_2}$, we remember that $\phi_i(x) = 1 - \frac{|x|}{r_i}^{2-n}, i = 1, 2$. Consider $G = \phi_0 - \phi_2$, then $\Delta G = 0$ in $\mathbb{R}^n \setminus B_{r_2}$, and $G = \phi_0 \geq 0$ on $\partial B_{r_2}$. By the maximum principle we conclude that $G$ cannot have a negative minimum in $\mathbb{R}^n \setminus B_{r_2}$, so $\phi_0 \geq \phi_2$ in $\mathbb{R}^n \setminus B_{r_2}$. Similarly, we have $\phi_1(x) \geq \phi_0(x)$ in $\Omega$, so that $\phi_2 \leq \phi_0 \leq \phi_1$ in $\mathbb{R}^n \setminus B_{r_2}$. Hence,

$$|\phi_0(x) - 1| \leq C|x|^{2-n}, \quad \text{for } |x| \text{ large}, \tag{2.11}$$

and particularly, we obtain the second property.

To get the gradient estimate, we can apply [4, Theorem 7, p. 29], with $v(x) = \phi_0(x) - 1$, $|x_0|$ large and $r = |x_0|/2$, it gives

$$|\nabla \phi_0(x_0)| = |\nabla v(x_0)| \leq \frac{C}{r^{n+1}} \int_{B(x_0, r)} |v(y)| \, dy \leq \frac{C}{r^{n+1}} |x_0|^{2-n} r^n = C|x_0|^{1-n},$$

where we have used (2.11), this complete the proof. \qed

Similarly, we have the following:

Lemma 2.12. [10, Lemma 2.5] There exists a function $\phi_0(x) \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfying the following boundary value problem:

$$\begin{cases} 
\Delta \phi_0(x) = 0, & \text{in } \Omega, \ n = 2, \\
\phi_0|_{\partial\Omega} = 0, \\
|x| \to \infty, & \phi_0(x) \to +\infty, \text{ and } \phi_0(x) \text{ increase at the rate of } \ln(|x|).
\end{cases} \tag{2.12}$$

Moreover, $\phi_0(x)$ satisfies

- $0 < \phi_0(x) \leq C \ln(|x|)$, for all $x \in \Omega$.
- $\phi_0(x) \geq C$, for all $|x| \gg 1$.
- $|\nabla \phi_0(x)| \leq \frac{C}{|x|}$, for all $|x| \gg 1$. 

Lemma 2.13. [11, Lemma 2.2] There exists a function \( \phi_0(x) \in C^2([0, \infty)) \) satisfying the following boundary value problem:

\[
\begin{align*}
\Delta \phi_0(x) &= 0, \quad x > 0, \\
\phi_0|_{x=0} &= 0, \\
x \to \infty, \phi_0(x) &\to +\infty, \text{ and } \phi_0(x) \text{ increase at the rate of } x.
\end{align*}
\] (2.13)

Moreover, \( \phi_0(x) \) satisfies: there exist two positive constants \( C_1 \) and \( C_2 \) such that, for all \( x > 0 \), we have \( C_1 x \leq \phi_0(x) \leq C_2 x \). In fact, we can take \( \phi_0(x) = Cx \).

3. Local Existence

In this section, we prove the local existence of mild solution (Proposition 1.1). We start by giving the definition of the mild and weak solution of (1.1). For a nonlinear equation, it is not always true that the solution exists globally in time. Therefore, we consider solution defined on an interval \([0, T)\) for \( T > 0 \). When \( T < \infty \), such a solution is called local (mild or weak) solution, otherwise, it is called global (mild or weak) solution. Moreover, each global solution is local.

Definition 3.1. (Mild solution)

Let \( T > 0 \), and \((u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega)\). A function \( u \) is said to be a mild solution of (1.1) if

\[ u \in C([0, T); H^1_0(\Omega)) \cap C^1([0, T); L^2(\Omega)) \]

and \( u \) has the initial data \( u(0) = u_0, u_t(0) = u_1 \) and satisfies the integral equation

\[
u(t, x) = R(t)(u_0, u_1) + \int_0^t S(t - s)|u(s)|^p \, ds
\] (3.1)

in the sense of \( H^1_0(\Omega) \).

Definition 3.2. (Weak solution)

Let \( T > 0 \), and \( u_0, u_1 \in L^1_{loc}(\Omega) \). A function \( u \) is said to be a weak solution of (1.1) if

\[ u \in L^p_{loc}((0, T); L^p_{loc}(\Omega)) \]

and \( u \) satisfies the weak formulation

\[
\int_0^T \int_\Omega |u|^p \varphi \, dx \, dt + \int_\Omega u_1(x) \varphi(0, x) \, dx \\
- \int_\Omega u_0(x) \Delta \varphi(0, x) \, dx - \int_\Omega u_0(x) \varphi_t(0, x) \, dx \\
= \int_0^\infty \int_\Omega u \varphi_{tt} \, dx \, dt + \int_0^\infty \int_\Omega u \varphi_t \, dx \, dt - \int_0^\infty \int_\Omega u \Delta \varphi \, dx \, dt,
\] (3.2)

for any compactly supported test function \( \varphi \in C^2([0, T] \times \Omega) \) such that \( \varphi(\cdot, T) = 0 \) and \( \varphi_t(\cdot, T) = 0 \).

The following lemma is crucial for the proof of Theorem 1.3.
Lemma 3.3. \textbf{(Mild $\rightarrow$ Weak)}

Let $(u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega)$. Under assumption (1.3), if $u$ is a global mild solution of (1.1), then $u$ is a global weak solution of (1.1).

Proof. Let $u$ be a global mild solution of (1.1), $T_0 > 0$, and let $\varphi \in C^2([0, T_0] \times \Omega)$ be a compactly supported function such that $\varphi(\cdot, T_0) = 0$ and $\varphi_t(\cdot, T_0) = 0$. For $f(u) := |u|^p$, it follows from Gagliardo–Nirenberg inequality, under the assumption (1.3), that

$$\|f(u)\|_2 = \|u\|_{2p}^p \leq C\|\nabla u\|_{2p}^p \|u\|_{L^2}^{1-\sigma^p} \leq C\|u\|_{L^\infty([0, T_0]; H^1_0(\Omega))}^p,$$

where $\sigma = n(p - 1)/(2p) \in [0, 1]$. This inequality shows that $f(u) \in C([0, T_0]; L^2)$. Due to the density argument, there exist sequences

$$\left\{(u_0^{(j)}, u_1^{(j)})\right\}_{j=1}^\infty \subseteq (H^2(\Omega) \cap H^1_0(\Omega))^2,$$

$$\left\{F^{(j)}\right\}_{j=1}^\infty \subseteq C([0, T_0]; H^2(\Omega) \cap H^1_0(\Omega)) \cap C^1([0, T_0]; L^2(\Omega))$$

such that

$$\lim_{j \to \infty} (u_0^{(j)}, u_1^{(j)}) = (u_0, u_1) \text{ in } H^1_0(\Omega) \times L^2(\Omega),$$

$$\lim_{j \to \infty} F^{(j)} = f(u) \text{ in } C([0, T_0]; L^2(\Omega)).$$

Using Proposition 2.5, let $u^{(j)}$ be the strong solution of the linear inhomogeneous equation (2.4) with the initial data $(u_0^{(j)}, u_1^{(j)})$ and the inhomogeneous term $F^{(j)}(t, x)$. Apply Proposition 2.9 to $u^{(j)}$, and using the fact that $u$ is a mild solution of (1.1), we have

$$u^{(j)}(t) - u(t) = R(t)(u_0^{(j)} - u_0, u_1^{(j)} - u_1) + \int_0^t S(t-s)(F^{(j)}(s) - f(u(s))) \, ds$$

and hence, using Proposition 2.2, we get

$$\|(u^{(j)} - u)(t)\|_{L^2} \leq \|R(t)(u_0^{(j)} - u_0, u_1^{(j)} - u_1)\|_{L^2} + \int_0^t \|R(t-s)(0, F^{(j)}(s) - f(u(s)))\|_{L^2} \, ds$$

$$\leq C\|u_0^{(j)} - u_0\|_{L^2} + T_0\|\nabla (u_0^{(j)} - u_0, u_1^{(j)} - u_1)\|_{L^2 \times L^2}$$

$$+ \int_0^t T_0\|F^{(j)}(s) - f(u(s))\|_{L^2} \, ds$$

$$\leq C(1+ T_0)\|u_0^{(j)} - u_0, u_1^{(j)} - u_1\|_{H^1_0 \times L^2} + T_0^2 \sup_{s \in [0, T_0]} \|F^{(j)}(s) - f(u(s))\|_{L^2},$$

which implies, by letting $j \to \infty$, that

$$u^{(j)} \longrightarrow u \text{ in } C([0, T_0]; L^2(\Omega)).$$

Moreover, as $u^{(j)}$ is a strong solution of (2.4), $u^{(j)}$ is also a weak solution of (2.4), that is,
Theorem 1.1. We will use the Banach fixed-point theorem. Let $T > 0$, $M > 0$, $X(T) := C([0, T]; H^1_0(\Omega) \cap C^1([0, T]; L^2(\Omega)))$ and

$$B_M(T) = \{v \in X(T); \|v\|_{X(T)} \leq 2M\},$$

where $\|v\|_{X(T)} := \sup_{t \in [0, T]}(\|v_t\|_2 + \|\nabla v\|_2 + \|v\|_2)$. As above, by Gagliardo–Nirenberg inequality, we have

$$v \in B_R(T) \implies f(v) = |v|^p \in C([0, T]; L^2(\Omega)),$$

which allow us, using Proposition 2.10, to define a mapping $\Phi : B_M(T) \to X(T)$ such that $u(t, x) = \Phi(v)(t, x)$ is the unique mild solution to the linear inhomogeneous equation

$$u_{tt} - \Delta u - \Delta u_t = f(v), \quad t > 0, x \in \Omega,$$

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \Omega,$$

$$u = 0, \quad t > 0, x \in \partial \Omega;$$

moreover,

\begin{align*}
\|(u_t, \nabla u)(t)\|_{L^2 \times L^2} &\leq C\|(u_1, \nabla u_0)\|_{L^2 \times L^2} + C \int_0^t \|f(v(s))\|_{L^2} \, ds, \quad (3.3) \\
\|u(t)\|_{L^2} &\leq \|u_0\|_{L^2} + C \int_0^t \left(\|(u_1, \nabla u_0)\|_{L^2 \times L^2} + \int_0^s \|f(v(\tau))\|_{L^2} \, d\tau\right) \, ds. \quad (3.4)
\end{align*}

$\Phi : B_M(T) \to B_M(T)$. Let $v \in B_M(T)$, and $u = \Phi(v)$. By the inequalities (3.3)–(3.4) and the Sobolev imbedding theorem $H^1_0(\Omega) \hookrightarrow L^{2p}(\Omega)$, one has

\begin{align*}
\|(u_t, \nabla u)(t)\|_{L^2 \times L^2} &\leq C\|(u_1, \nabla u_0)\|_{L^2 \times L^2} + C \int_0^t \|v(s)\|_{L^2} \, ds \\
&\leq C (1 + T) I_0 + C \int_0^t \|v(s)\|_{H^1_0}^p \, ds \\
&\leq C I_0 + C \int_0^t \|v\|_{X(T)}^p \, ds \\
&\leq C I_0 + C 2^p M^p T,
\end{align*}
and
\[ \|u(t)\|_{L^2} \leq \|u_0\|_{L^2} + C \int_0^t \left( \|(u_1, \nabla u_0)\|_{L^2} + \int_0^s \|f(v(\tau))\|_{L^2} \, d\tau \right) \, ds \]
\[ \leq C I_0 + C \int_0^t \int_0^s \|f(v(\tau))\|_{L^2} \, d\tau \, ds \]
\[ \leq C I_0 + C \int_0^t \int_0^s \|v(\tau)\|_{H^1_0}^p \, d\tau \, ds \]
\[ \leq C I_0 + C \int_0^t \int_0^s \|v\|_{X(T)}^p \, d\tau \, ds \]
\[ \leq C I_0 + C 2^p M^p T^2, \]
where \( I_0 := \|u_0\|_{H^1} + \|u_1\|_2 \), and \( T \ll 1 \). Therefore
\[ \|u\|_{X(T)} \leq C I_0 + C 2^p M^p T. \]
We take \( M \) satisfying
\[ M \geq C I_0 \]
and then choose \( T \) sufficiently small so that
\[ C 2^p M^p - 1 \leq 1, \]
it follows that \( \|u\|_{X(T)} \leq 2M \) and this proves that \( \Phi \) is a mapping from \( B_M(T) \) to \( B_M(T) \).

- \( \Phi \) is a contraction. Let \( v, \overline{v} \in B_M(T) \), \( u := \Phi(v) \), \( \overline{u} := \Phi(\overline{v}) \) and \( w := u - \overline{u} \). By Proposition 2.10, \( w \) is the unique mild solution to the linear inhomogeneous equation
\[
\begin{cases}
  w_{tt} - \Delta w - \Delta w_t = f(v) - f(\overline{v}), & t > 0, x \in \Omega, \\
  w(0, x) = 0, \quad w_t(0, x) = 0, & x \in \Omega, \\
  w = 0, \quad t > 0, & x \in \partial \Omega,
\end{cases}
\]
and the following energy estimates hold
\[ \|(w_t, \nabla w)(t)\|_{L^2} \leq C \int_0^t \|f(v(s)) - f(\overline{v}(s))\|_{L^2} \, ds, \]
\[ \|w(t)\|_{L^2} \leq C \int_0^t \int_0^s \|f(v(\tau)) - f(\overline{v}(\tau))\|_{L^2} \, d\tau \, ds. \]
Therefore,
\[ \|(w_t, \nabla w)(t)\|_{L^2} \]
\[ \leq C \int_0^t \|f(v(s)) - f(\overline{v}(s))\|_{L^2} \, ds \]
\[ \leq C \int_0^t \left( \|v(s)|^{p-1}\|_{2p/(p-1)} + \|\overline{v}(s)|^{p-1}\|_{2p/(p-1)} \right) \|v(s) - \overline{v}(s)\|_{2p} \, ds \]
\[ = C \int_0^t \left( \|v(s)|^{p-1} + \|\overline{v}(s)|^{p-1} \right) \|v(s) - \overline{v}(s)\|_{2p} \, ds \]
\[ C \int_0^t \left( \|v(s)\|_{H^1_0}^{p-1} + \|\overline{v}(s)\|_{H^1_0}^{p-1} \right) \|v(s) - \overline{v}(s)\|_{H^1_0} \, ds \leq C \int_0^t \left( \|v\|_{X(T)}^{p-1} + \|\overline{v}\|_{X(T)}^{p-1} \right) \|v - \overline{v}\|_{X(T)} \, ds \leq C 2^p M^{p-1} T \|v - \overline{v}\|_{X(T)} \leq \frac{1}{4} \|v - \overline{v}\|_{X(T)}, \]

thanks to Hölder’s inequality, the fact that \( H^1_0(\Omega) \hookrightarrow L^{2p}(\Omega) \), and the following well-known estimation:

\[ \|v\|^p - |\overline{v}|^p \leq C(p) \|v - \overline{v}\|(\|v|^{p-1} + |\overline{v}|^{p-1}); \quad (3.5) \]

\( T \) is chosen such that

\[ C 2^p M^{p-1} T \leq \frac{1}{4}. \]

Similarly,

\[ \|w(t)\|_{L^2} \leq C \int_0^t \int_0^t \|f(v(\tau)) - f(\overline{v}(\tau))\|_{L^2} \, d\tau \, ds \leq C 2^p M^{p-1} T^2 \|v - \overline{v}\|_{X(T)} \leq \frac{1}{4} \|v - \overline{v}\|_{X(T)}. \]

We conclude that

\[ \|w(t)\|_{X(T)} \leq \frac{1}{2} \|v - \overline{v}\|_{X(T)}. \]

This implies that \( \Phi \) is a contraction mapping. Then by the Banach fixed point theorem, there exists a unique mild solution \( u \in X(T) \) to problem (1.1).

Moreover, by uniqueness, there exists a maximal interval \([0, T_{\text{max}})\), where

\[ T_{\text{max}} := \sup \{ T > 0 ; \text{there exist a mild solution } u \in X(T) \text{ to (1.1)} \} \leq +\infty. \]

Finally, if the lifespan \( T_{\text{max}} \) is finite, then the energy of the solution blows up at \( T_{\text{max}} \):

\[ \lim_{t \to T_{\text{max}}} (\|u(t)\|_{H^1_0} + \|u_t(t)\|_2) = \infty. \]

Indeed, if

\[ \lim_{t \to T_{\text{max}}} (\|u(t)\|_{H^1_0} + \|u_t(t)\|_2) =: L < \infty, \]

then there exists a time sequence \( \{t_m\}_{m \geq 0} \) tending to \( T_{\text{max}} \) as \( m \to \infty \) and such that

\[ \sup_{m \in \mathbb{N}} (\|u(t_m)\|_{H^1_0} + \|u_t(t_m)\|_2) \leq L + 1. \]

The argument before shows that there exists \( T(L + 1) > 0 \) such that the solution \( u(t) \) can be extended on the interval \([t_m, t_m + T(L + 1)]\) for any \( m \).

By taking \( m \) sufficiently large so that \( t_m \geq T_{\text{max}} - (1/2)T(L + 1) \), the solution
$u(t)$ can be extended on $[T_{\text{max}}, T_{\text{max}} + (1/2)T(L + 1)]$. This contradicts the definition of $T_{\text{max}}$. Thus, we complete the proof. \hfill \square

4. Blow-Up

This section is devoted to prove the blow-up result, namely Theorem 1.3.

**Proof of Theorem 1.3.** The idea of the proof is to use the variational formulation of the weak solution by choosing the appropriate test function. Note that the harmonic functions in Lemmas 2.11, 2.12 and 2.13 play a crucial role in the exterior domain, because of their good behaviour and vanishing on the boundary $\partial \Omega$. We argue by contradiction assuming that $u$ is not a blow-up solution of (1.1). Using Proposition 1.1 and Lemma 3.3, we have

$$
\int_0^T \int_{\Omega} |u|^p \varphi \, dx \, dt + \int_{\Omega} u_1(x)\varphi(0, x) \, dx
- \int_{\Omega} u_0(x)\Delta \varphi(0, x) \, dx - \int_{\Omega} u_0(x)\varphi_t(0, x) \, dx
= \int_0^T \int_{\Omega} u \varphi_{tt} \, dx \, dt + \int_0^T \int_{\Omega} u \Delta \varphi_t \, dx \, dt - \int_0^T \int_{\Omega} u \Delta \varphi \, dx \, dt, \quad (4.1)
$$

for all $T > 0$ and all compactly supported function $\varphi \in C^2([0, T] \times \Omega)$ such that $\varphi(\cdot, T) = 0$ and $\varphi_t(\cdot, T) = 0$. Take $\varphi(x, t) = \phi_0(x)\varphi_T^\ell(x)\eta_T^k(t)$, where $\phi_0$ is the harmonic function introduced in Lemmas 2.11, 2.12 and 2.13, $\eta_T(t) := \eta(T, T)$, $\ell, k \gg 1$, and $\eta(\cdot) \in C^\infty(\mathbb{R}_+)$ is a cut-off non-increasing function such that

$$
\eta(t) := \begin{cases} 
1 & \text{if } 0 \leq t \leq 1/2 \\
0 & \text{if } t \geq 1,
\end{cases}
$$

$0 \leq \eta(t) \leq 1$ and $|\eta'(t)| \leq C$ for some $C > 0$ and all $t > 0$; and $\varphi_T(x) = \Phi(|x|/T)$ with the following smooth, non-increasing cut-off function

$$
\Phi(r) := \begin{cases} 
1 & \text{if } 0 \leq r \leq 1 \\
0 & \text{if } r \geq 2,
\end{cases}
$$

such that $0 \leq \Phi(r) \leq 1$, $|\Phi'(r)| \leq C/r$ and $|\Phi''(r)| \leq C/r^2$. We obtain

$$
\int_0^T \int_{\Omega_1} |u|^p \varphi \, dx \, dt + \int_{\Omega_1} u_1(x)\phi_0(x)\varphi_T^\ell(x) \, dx
- \int_{\Omega_1} u_0(x)\Delta [\phi_0(x)\varphi_T^\ell(x)] \, dx
= \int_0^T \int_{\Omega_1} u \phi_0(x)\varphi_T^\ell(x)(\eta_T^k(t))'' \, dx \, dt + \int_0^T \int_{\Omega_1} u \Delta [\phi_0(x)\varphi_T^\ell(x)](\eta_T^k(t))' \, dx \, dt
- \int_0^T \int_{\Omega_1} u \Delta [\phi_0(x)\varphi_T^\ell(x)]\eta_T^k(t) \, dx \, dt
=: I_1 + I_2 + I_3, \quad (4.2)
$$

where $\Omega_1 := \{x \in \Omega; |x| \leq 2T\}$. At this stage, we have to distinguish three cases:
Case 1: $n \geq 3$. To estimate the right-hand side of (4.2), we introduce the term $\varphi^{1/p} \varphi^{-1/p}$ in $I_1$, and we use Young’s inequality to obtain

$$I_1 \leq \int_0^T \int_{\Omega_1} |u| \varphi^{1/p} \varphi^{-1/p} \phi_0(x) \varphi_T^\ell(x) \left[|\eta_T^k(t)|''\right] \, dx \, dt \leq \frac{1}{6} \int_0^T \int_{\Omega_1} |u|^p \varphi \, dx \, dt + C \int_0^T \int_{\Omega_1} \varphi^{-p'/p} \phi_0^\ell'(x) \varphi_T^{\ell p'}(x) \left[|\eta_T^k(t)|''\right] \, dx \, dt \leq \frac{1}{6} \int_0^T \int_{\Omega_1} |u|^p \varphi \, dx \, dt + C \int_0^T \int_{\Omega_1} \phi_0(x) \varphi_T^\ell(x) \eta_T(t)^{(k-2)p'} \left[|\eta_T'(t)|''\right] \, dx \, dt $$

$$+ C \int_0^T \int_{\Omega_1} \phi_0(x) \varphi_T^\ell(x) \eta_T(t)^{(k-1)p'} \left[|\eta_T''(t)|''\right] \, dx \, dt. \quad (4.3)$$

On the other hand, using Lemma 2.11 with all properties of $\phi_0$, $T \gg 1$, and Young’s inequality, we conclude that

$$I_2 \leq C \int_0^T \int_{\Omega_1} |u| \varphi_T^{\ell -1}(x) |\nabla \phi_0(x)| |\nabla \varphi_T(x)| |\partial_t (\eta_T^k(t))| \, dx \, dt $$

$$+ C \int_0^T \int_{\Omega_1} |u| \varphi_T^{\ell -2}(x) \phi_0(x) |\nabla \varphi_T(x)|^2 |\partial_t (\eta_T^k(t))| \, dx \, dt $$

$$+ C \int_0^T \int_{\Omega_1} |u| \varphi_T^{\ell -1}(x) \phi_0(x) |\Delta \varphi_T(x)| |\partial_t (\eta_T^k(t))| \, dx \, dt $$

$$= C \int_0^T \int_{\Omega_1} |u| \varphi^{1/p} \varphi^{-1/p} \varphi_T^{\ell -1}(x) |\nabla \phi_0(x)| |\nabla \varphi_T(x)| |\partial_t (\eta_T^k(t))| \, dx \, dt $$

$$+ C \int_0^T \int_{\Omega_1} |u| \varphi^{1/p} \varphi^{-1/p} \varphi_T^{\ell -2}(x) \phi_0(x) |\nabla \varphi_T(x)|^2 |\partial_t (\eta_T^k(t))| \, dx \, dt $$

$$+ C \int_0^T \int_{\Omega_1} |u| \varphi^{1/p} \varphi^{-1/p} \varphi_T^{\ell -1}(x) \phi_0(x) |\Delta \varphi_T(x)| |\partial_t (\eta_T^k(t))| \, dx \, dt $$

$$\leq \frac{1}{6} \int_0^T \int_{\Omega_1} |u|^p \varphi \, dx \, dt $$

$$+ C \int_0^T \int_{\Omega_1^2} \varphi^{\ell -p'}(x) \eta_T^{k-p'}(t) |\nabla \phi_0(x)|^{p'} |\nabla \varphi_T(x)|^{p'} |\partial_t (\eta_T(t))|^{p'} \, dx \, dt $$

$$+ C \int_0^T \int_{\Omega_1^2} \varphi^{\ell -2p'}(x) \eta_T^{k-2p'}(t) |\nabla \varphi_T(x)|^{2p'} |\partial_t (\eta_T(t))|^{p'} \, dx \, dt $$

$$+ C \int_0^T \int_{\Omega_1^2} \varphi^{\ell -p'}(x) \eta_T^{k-p'}(t) |\Delta \varphi_T(x)|^{p'} |\partial_t (\eta_T(t))|^{p'} \, dx \, dt, \quad (4.4)$$

where $\Omega_1^2 := \{x \in \Omega; \ T \leq |x| \leq 2T\}$. Similarly,

$$I_3 \leq \frac{1}{6} \int_0^T \int_{\Omega_1} |u|^p \varphi \, dx \, dt $$

$$+ C \int_0^T \int_{\Omega_1^2} \varphi^{\ell -p'}(x) \eta_T^{k}(t) |\nabla \phi_0(x)|^{p'} |\nabla \varphi_T(x)|^{p'} \, dx \, dt$$
Now, we have to distinguish two subcases. Using (4.3)–(4.5), it follows from (4.2) that

\[
\int_{\Omega_1} u_1(x)\phi_0(x)\phi_T^k(x) \, dx - \int_{\Omega_1} u_0(x)\Delta [\phi_0(x)\phi_T^k(x)] \, dx \\
\leq \frac{1}{2} \int_0^T \int_{\Omega_1} |u|^p \varphi \, dx \, dt \\
+ \int_{\Omega_1} u_1(x)\phi_0(x)\phi_T^k(x) \, dx - \int_{\Omega_1} u_0(x)\Delta [\phi_0(x)\phi_T^k(x)] \, dx \\
\leq C \int_0^T \int_{\Omega_1} \phi_0(x)\phi_T^k(x)\eta_T(t)^{(k-2)p'} |\partial_t \eta_T(t)|^{2p'} \, dx \, dt \\
+ C \int_0^T \int_{\Omega_1} \phi_0(x)\phi_T^k(x)\eta_T(t)^{(k-1)p'} |\partial_t^2 \eta_T(t)|^{p'} \, dx \, dt \\
+ C \int_0^T \int_{\Omega_1} \varphi_T^{\ell-p'}(x)\eta_T^{k-p'}(t)|\nabla \phi_0(x)|^{p'} |\nabla \phi_T(x)|^{p'} |\partial_t (\eta_T(t))|^{p'} \, dx \, dt \\
+ C \int_0^T \int_{\Omega_1} \varphi_T^{\ell-p'}(x)\eta_T^{k-p'}(t)|\Delta \phi_T(x)|^{p'} |\partial_t (\eta_T(t))|^{p'} \, dx \, dt \\
+ C \int_0^T \int_{\Omega_1} \varphi_T^{\ell-p'}(x)\eta_T^{k}(t)|\nabla \phi_0(x)|^{p'} |\nabla \phi_T(x)|^{p'} \, dx \, dt \\
+ C \int_0^T \int_{\Omega_1} \varphi_T^{\ell-p'}(x)\eta_T^{k}(t)|\Delta \phi_T(x)|^{p'} \, dx \, dt. \tag{4.6}
\]

Using (4.3)–(4.5), it follows from (4.2) that

\[
\int_{\Omega_1} u_1(x)\phi_0(x)\phi_T^k(x) \, dx - \int_{\Omega_1} u_0(x)\Delta [\phi_0(x)\phi_T^k(x)] \, dx \\
\leq \frac{1}{2} \int_0^T \int_{\Omega_1} |u|^p \varphi \, dx \, dt \\
+ \int_{\Omega_1} u_1(x)\phi_0(x)\phi_T^k(x) \, dx - \int_{\Omega_1} u_0(x)\Delta [\phi_0(x)\phi_T^k(x)] \, dx \\
\leq C \int_0^T \int_{\Omega_1} \phi_0(x)\phi_T^k(x)\eta_T(t)^{(k-2)p'} |\partial_t \eta_T(t)|^{2p'} \, dx \, dt \\
+ C \int_0^T \int_{\Omega_1} \phi_0(x)\phi_T^k(x)\eta_T(t)^{(k-1)p'} |\partial_t^2 \eta_T(t)|^{p'} \, dx \, dt \\
+ C \int_0^T \int_{\Omega_1} \varphi_T^{\ell-p'}(x)\eta_T^{k-p'}(t)|\nabla \phi_0(x)|^{p'} |\nabla \phi_T(x)|^{p'} |\partial_t (\eta_T(t))|^{p'} \, dx \, dt \\
+ C \int_0^T \int_{\Omega_1} \varphi_T^{\ell-p'}(x)\eta_T^{k-p'}(t)|\Delta \phi_T(x)|^{p'} |\partial_t (\eta_T(t))|^{p'} \, dx \, dt \\
+ C \int_0^T \int_{\Omega_1} \varphi_T^{\ell-p'}(x)\eta_T^{k-p'}(t)||\nabla \phi_0(x)|^{p'} |\nabla \phi_T(x)|^{p'} \, dx \, dt \\
+ C \int_0^T \int_{\Omega_1} \varphi_T^{\ell-p'}(x)\eta_T^{k}(t)|\Delta \phi_T(x)|^{p'} \, dx \, dt. \tag{4.6}
\]

Now, we have to distinguish two subcases.

- **Case (i):** $1 < p < 1 + \frac{2}{n-1}$. By Lemma 2.11, we have $|\nabla \phi_0(x)| \leq \frac{C}{|x|^n} \leq \frac{C}{T} \leq \frac{T}{n} \leq \frac{C}{T}$ in $\Omega_1^T$, therefore, using the change of variables: $y = T^{-1}x$, $s = T^{-1}t$, we get from (4.6) that

\[
\int_{\Omega_1} u_1(x)\phi_0(x)\phi_T^k(x) \, dx - \int_{\Omega_1} u_0(x)\Delta [\phi_0(x)\phi_T^k(x)] \, dx \\
\leq C T^{-2p'+1+n} + C T^{-3p'+1+n} \\
\leq C T^{-2p'+1+n}, \tag{4.7}
\]
where $C$ is independent of $T$. As $p < 1 + \frac{2}{n-1} \iff -2p' + 1 + n < 0$, it follows, by letting $T \to \infty$ that

$$\lim_{T \to \infty} \int_{\Omega_1} u_1(x) \phi_0(x) \varphi_T^\ell(x) \, dx - \lim_{T \to \infty} \int_{\Omega_2^T} u_0(x) \Delta[\phi_0(x) \varphi_T^\ell(x)] \, dx \leq 0.$$  

On the other hand, as $u_0, u_1, \varphi_0 \in L^1(\Omega)$ and $\Delta[\phi_0(x) \varphi_T^\ell(x)] \leq C/T^2$ in $\Omega_1^T$, it follows by Lebesgue’s dominated convergence theorem that

$$0 < \int_{\Omega} u_1(x) \phi_0(x) \, dx \leq 0;$$

contradiction.

- Case (ii): $p = 1 + \frac{2}{n-1}$. From (4.6) in Case 1 and the fact that $p = 1 + \frac{2}{n-1}$, there exists a positive constant $D$ independent of $T$ such that

$$\int_0^T \int_{\Omega_1} |u|^p \varphi \, dx \, dt \leq D, \quad \text{for all } T > 0,$$

which implies that

$$\int_{T/2}^T \int_{\Omega_1} |u|^p \varphi \, dx \, dt, \int_{T/2}^T \int_{\Omega_1^T} |u|^p \varphi \, dx \, dt, \int_0^T \int_{\Omega_1^T} |u|^p \varphi \, dx \, dt \to 0 \text{ as } T \to \infty. \tag{4.8}$$

On the other hand, we use Hölder’s inequality instead of Young’s one in $I_1$, $I_2$, and $I_3$, together with the same change of variables, we get

$$I_1 \leq \left( \int_{T/2}^T \int_{\Omega_1} |u|^p \varphi \, dx \, dt \right)^{1/p} \leq \left( C \int_0^T \int_{\Omega_1} \varphi_0(x) \varphi_T^\ell(x) \left[ \eta_T(t)^{(k-2)p'} |\eta_T(t)|^{2p'} + \eta_T(t)^{(k-1)p'} |\eta_T''(t)|^{p'} \right] \, dx \, dt \right)^{1/p'} \leq C \left( \int_{T/2}^T \int_{\Omega_1} |u|^p \varphi \, dx \, dt \right)^{1/p}, \tag{4.9}$$
	hanks{to the fact that $p = 1 + \frac{2}{n-1}$. Similarly,

$$I_2 \leq C \left( \int_{T/2}^T \int_{\Omega_1^T} |u|^p \varphi \, dx \, dt \right)^{1/p}, \tag{4.10}$$

and

$$I_3 \leq C \left( \int_0^T \int_{\Omega_1^T} |u|^p \varphi \, dx \, dt \right)^{1/p}. \tag{4.11}$$

Finally, using (4.9)–(4.11), it follows from (4.2) that

$$\int_{\Omega_1} u_1(x) \phi_0(x) \varphi_T^\ell(x) \, dx - \int_{\Omega_2^T} u_0(x) \Delta[\phi_0(x) \varphi_T^\ell(x)] \, dx.$$
In this case, we have a blow-up result just in the sub-critical case \( 1 < p < 1 + \frac{2}{n-1} = 3 \). By repeating the same calculation in the Case of \( n \geq 3 \) and using Lemma 2.12 instead of Lemma 2.11 (noted that the big difference is the fact that \( \phi_0(x) \leq C \ln(|x|) \)), we easily conclude that

\[
I_1 \leq \frac{1}{6} \int_0^T \int_{\Omega_1} |u|^p \, \varphi \, dx \, dt + C \ln(T) \, T^{-2p'+3},
\]

\[
I_2 \leq \frac{1}{6} \int_0^T \int_{\Omega_1} |u|^p \, \varphi \, dx \, dt + C \, T^{-3p'+3} + C \ln(T) \, T^{-3p'+3},
\]

and

\[
I_3 \leq \frac{1}{6} \int_0^T \int_{\Omega_1} |u|^p \, \varphi \, dx \, dt + C \, T^{-2p'+3} + C \ln(T) \, T^{-2p'+3}.
\]

This implies that

\[
\int_{\Omega_1} u_1(x)\phi_0(x)\varphi_T(t) \, dx - \int_{\Omega_1^T} u_0(x)\Delta[\phi_0(x)\varphi_T(t)] \, dx \leq C \ln(T) \, T^{-2p'+3}
\]

\[
\leq C \, T^{-p'+3/2},
\]

where we have used, e.g., the fact that \( \ln(T) \leq C \, T^{p'-3/2} \), thanks to \( p < 3 \). By letting \( T \) goes to infinity and using \( p < 3 \), we obtain the desired contradiction.

- Case 3: \( n = 1 \). In this case, we replace the space test function by \( \varphi_T(x) = \Phi(|x|/T^\alpha) \). If \( p < 1 + \alpha \), repeating the same calculation as in the Case of \( n \geq 3 \) and using Lemma 2.13 instead of Lemma 2.11, we easily get

\[
I_1 \leq \frac{1}{6} \int_0^T \int_{\Omega_1} |u|^p \, \varphi \, dx \, dt + C \, T^{-2p'+2\alpha+1},
\]

\[
I_2 \leq \frac{1}{6} \int_0^T \int_{\Omega_1} |u|^p \, \varphi \, dx \, dt + C \, T^{-(\alpha+1)(p'-1)},
\]

and

\[
I_3 \leq \frac{1}{6} \int_0^T \int_{\Omega_1} |u|^p \, \varphi \, dx \, dt + C \, T^{-\alpha p'+\alpha+1},
\]

where we have used the change of variables: \( y = T^{-\alpha} x, \ s = T^{-1} t \). Therefore, as \( p < 1 + \alpha < \alpha/2(\alpha - 1) \) implies \( -2p' + 2\alpha + 1 < -\alpha p' + \alpha + 1 \), we get from (4.6) that

\[
\int_{\Omega_1} u_1(x)\phi_0(x)\varphi_T(t) \, dx - \int_{\Omega_1^T} u_0(x)\Delta[\phi_0(x)\varphi_T(t)] \, dx \leq C \, T^{-\alpha p' + \alpha + 1},
\]

which leads to a contradiction by letting \( T \to \infty \).
For the critical case $p = 1 + \alpha$, we get the contradiction by applying a similar calculation as in the case (ii) above by taking into account the support of $\nabla \varphi_T$, $\Delta \varphi_T$, and $\partial_t \eta_T$.

This completes the proof of Theorem 1.3. \qed

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