ON CENTRAL FUBINI-LIKE NUMBERS AND POLYNOMIALS

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Abstract. We introduce the central Fubini-like numbers and polynomials using Rota approach. Several identities and properties are established as generating functions, recurrences, explicit formulas, parity, asymptotics and determinantal representation.

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1. INTRODUCTION

We start by giving some definitions that will be used throughout this paper. For $n \geq 1$, the falling factorial denoted $x^\underline{n}$ is defined by

\[ x^\underline{n} = x(x-1)\cdots(x-n+1), \]

and the central factorial $x^{[n]}$, see [4, 9], is defined by

\[ x^{[n]} = x(x+n/2-1)(x+n/2-2)\cdots(x-n/2+1). \]

We use the convention, $x^\underline{0} = x^{[0]} = 1$.

It is well-known that, for all non-negative integers $n$ and $k$ ($k \leq n$), Stirling numbers of the second kind are defined as the coefficients $S(n,k)$ in the expansion

\[ x^n = \sum_{k=0}^{n} S(n,k)x^\underline{k}. \tag{1.1} \]

Riordan, in his book [15], shows that, for all non-negative integers $n$ and $k$ ($k \leq n$), the central factorial numbers of the second kind are the coefficients $T(n,k)$ in the expansion

\[ x^n = \sum_{k=0}^{n} T(n,k)x^{[k]}. \tag{1.2} \]

In combinatorics, the number of ways to partition a set of $n$ elements into $k$ nonempty subsets are counted by Stirling numbers $S(n,k)$, and the central factorial numbers $T(2n, 2n-2k)$ count the number of ways to place $k$ rooks on a 3D-triangle board of size $(n-1)$, see [11].

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The coefficients $S(n,k)$ and $T(n,k)$ satisfy, respectively, the triangular recurrences

$$S(n,k) = kS(n-1,k) + S(n-1,k-1) \quad (1 \leq k \leq n) \quad (1.3)$$

and

$$T(n,k) = \left(\frac{k}{2}\right)^2 T(n-2,k) + T(n-2,k-2) \quad (2 \leq k \leq n), \quad (1.4)$$

where $S(n,k) = T(n,k) = 0$ for $k > n$, $S(0,0) = T(0,0) = T(1,1) = 1$ and $T(1,0) = 0$.

$S(n,k)$ and $T(n,k)$ admit also the explicit expressions

$$S(n,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n, \quad (1.5)$$

and

$$T(n,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \left(\frac{k}{2} - j\right)^n. \quad (1.6)$$

**TABLE 1.** The first few values of $S(n,k)$.

| $n \setminus k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|-----------------|---|---|---|---|---|---|---|
| 0               | 1 |
| 1               | 0 | 1 |
| 2               | 0 | 1 | 1 |
| 3               | 0 | 1 | 3 | 1 |
| 4               | 0 | 1 | 7 | 6 | 1 |
| 5               | 0 | 1 | 15| 25| 10| 1 |
| 6               | 0 | 1 | 31| 90| 65| 15| 1 |

**TABLE 2.** The first few values of $T(n,k)$.

| $n \setminus k$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  |
|-----------------|----|----|----|----|----|----|----|
| 0               | 1  |
| 1               | 0  | 1  |
| 2               | 0  | 0  | 1  |
| 3               | 3  | 0  | $\frac{1}{4}$ | 0 | 1  |
| 4               | 4  | 0  | 1  | 0  | 1  |
| 5               | 5  | 0  | $\frac{1}{16}$ | 0 | $\frac{5}{8}$ | 0 | 1  |
| 6               | 6  | 0  | 0  | 1  | 0  | 5  | 0  | 1  |
The usual difference operator $\Delta$, the shift operator $E^a$ and the central difference operator $\delta$ are given respectively by
\[
\Delta f(x) = f(x + 1) - f(x),
\]
\[
E^a f(x) = f(x + a)
\]
and
\[
\delta f(x) = f(x + 1/2) - f(x - 1/2).
\]
Riordan, [15], mentioned that the central factorial operator $\delta$ satisfies the following property
\[
\delta f_n(x) = n f_{n-1}(x), \quad (1.7)
\]
where $(f_n(x))_{n \geq 0}$ is a sequence of polynomials with $f_0(x) = 1$.

We can also express $\delta$ by means of both $\Delta$ and $E^a$, see [9, 15], as follows:
\[
\delta f(x) = \Delta E^{-1/2} f(x). \quad (1.8)
\]
For more details about difference operators, we refer the reader to [9].

2. Central Fubini-Like Numbers and Polynomials

In 1975, Tanny [17], introduced the Fubini polynomials (or ordered Bell polynomials) $F_n(x)$ by applying a linear transformation $L$ defined as
\[
L(x^n) := n! x^n.
\]
The polynomials $F_n(x)$ are given by
\[
F_n(x) := \sum_{k=0}^{n} k! S(n,k) x^k, \quad (2.1)
\]
according to,
\[
F_n(x) := L(x^n) = L \left( \sum_{k=0}^{n} S(n,k) x^k \right) = \sum_{k=0}^{n} S(n,k) L(x^k) = \sum_{k=0}^{n} k! S(n,k) x^k.
\]
Putting $x = 1$ in (2) we get
\[
F_n := F_n(1) = \sum_{k=0}^{n} k! S(n,k), \quad (2.2)
\]
which is the $n$-th Fubini number.

The Fubini polynomial $F_n(x)$ has the exponential generating function given by, see [17],
\[
\sum_{n=0}^{\infty} F_n(x) \frac{t^n}{n!} = \frac{1}{1 - x(e^t - 1)}. \quad (2.3)
\]
For more details concerning Fubini numbers and polynomials, see [3, 6, 8, 12, 17, 18, 20] and papers cited therein.

Now, we introduce the linear transformation $Z$ as follows.
**Definition 1.** For \( n \geq 0 \), we define the transformation
\[
Z(x^n) = n!x^n.
\] (2.4)

Then, we have
\[
Z(x^n) = Z \left( \sum_{k=0}^{n} T(n,k)x^k \right) = \sum_{k=0}^{n} T(n,k)Z(x^k) = \sum_{k=0}^{n} k!T(n,k)x^k.
\] (2.5)

And due to Formula (1.6), we are now able to introduce the main notion of the present paper.

**Definition 2.** The \( n \)-th central Fubini-like polynomial is given by
\[
C_n(x) := \sum_{k=0}^{n} k!T(n,k)x^k.
\] (2.6)

Setting \( x = 1 \), we obtain the central Fubini-like numbers,
\[
C_n = C_n(1) := \sum_{k=0}^{n} k!T(n,k).
\] (2.7)

The first central polynomials \( C_n(x) \) are given in Table 3.

| \( n \) | \( C_{2n}(x) \) | \( 2^{2n}C_{2n+1}(x) \) |
|---|---|---|
| 0 | 1 | x |
| 1 | \( 2x^2 \) | \( x + 24x^4 \) |
| 2 | \( 2x^2 + 24x^4 \) | \( x + 240x^3 + 1920x^5 \) |
| 3 | \( 2x^2 + 120x^4 + 720x^6 \) | \( x + 2184x^3 + 67200x^5 + 322560x^7 \) |
| 4 | \( 2x^2 + 504x^4 + 10080x^6 + 40320x^8 \) | \( x + 19680x^3 + 1854720x^5 + 27095040x^7 + 92897280x^9 \) |

**Table 3.** First value of \( C_n(x) \).

The first few central Fubini-like numbers are
\[
\left( C_{2n} \right)_{n \geq 0} : 1, 2, 26, 842, 50906, 4946282, 704888186, 138502957322, \ldots
\]

\[
\left( 2^{2n}C_{2n+1} \right)_{n \geq 0} : 1, 25, 2161, 391945, 121866721, 57890223865, 38999338931281, \ldots
\]

2.1. **Exponential generating function**

We begin by establishing the exponential generating function of the central Fubini-like polynomials.

**Theorem 1.** The polynomials \( C_n(x) \) have the following exponential generating function
\[
G(x; t) := \sum_{n=0}^{\infty} C_n(x) \frac{t^n}{n!} = \frac{1}{1 - 2x \sinh(t/2)}.
\] (2.8)
Proof. We have
\[
\sum_{n=0}^{\infty} \mathcal{C}_n(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} k! T(n,k) x^k \frac{t^n}{n!} = \sum_{k=0}^{\infty} k! x^k \sum_{n=k}^{\infty} T(n,k) \frac{t^n}{n!},
\]
from \cite[p. 214]{15}, we have
\[
\sum_{n=0}^{\infty} T(n,k) \frac{t^n}{n!} = \frac{1}{k!} (2 \sinh(t/2))^k,
\]
therefore
\[
\sum_{n=0}^{\infty} \mathcal{C}_n(x) \frac{t^n}{n!} = \sum_{k=0}^{\infty} (2 \sinh(t/2))^k x^k = \frac{1}{1 - 2x \sinh(t/2)}.
\]
\[ \square \]

**Corollary 1.** The sequence \((\mathcal{C}_n)_{n \geq 0}\) has the following exponential generating function
\[
\sum_{n=0}^{n} \mathcal{C}_n(x) \frac{t^n}{n!} = \frac{1}{1 - 2 \sinh(t/2)}.
\]

\[ (2.9) \]

### 2.2. Explicit representations

In this subsection we propose some explicit formulas for the central Fubini-like polynomials, we start by the derivative representation.

**Proposition 1.** The polynomials \((\mathcal{C}_n(x))_{n \geq 0}\) correspond to the higher derivative expression
\[
\mathcal{C}_n(x) = \sum_{k=0}^{\infty} \frac{\partial^n}{\partial t^n} \left( 2x \sinh \left( \frac{t}{2} \right) \right)^k \bigg|_{t=0}.
\]

**Proof.** Let
\[
\left. \frac{\partial^n}{\partial t^n} \left( \sum_{m=0}^{\infty} \mathcal{C}_m(x) \frac{t^m}{m!} \right) \right|_{t=0} = \sum_{m=n}^{\infty} \mathcal{C}_m(x) \frac{t^{m-n}}{(m-n)!} \bigg|_{t=0} = \sum_{m=0}^{\infty} \mathcal{C}_{m+n}(x) \frac{t^m}{m!} \bigg|_{t=0} = \mathcal{C}_n(x).
\]

Thus from Theorem 1 we get the result. \[ \square \]

From Formula (1.6), it is clear that the following proposition holds.

**Proposition 2.** The central Fubini-like polynomials satisfy the following explicit formula
\[
\mathcal{C}_n(x) = \sum_{k=0}^{n} x^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k/2 - j)^n.
\]

**Proof.** It suffices to replace \(T(n,k)\) in Equation (2.6) by its explicit formula (Equation (1.6)),
\[
\mathcal{C}_n(x) = \sum_{k=0}^{n} k! T(n,k) x^k = \sum_{k=0}^{n} x^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k/2 - j)^n.
\]
Theorem 2. For non-negative $n$, the following explicit representation holds true.

$$C_n(x) = x^{n-1} \sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j} \left( -\frac{1}{2} \right)^{k-j} \mathcal{E}_j(x) = x \sum_{j=0}^{n-1} \binom{n}{j} \delta[0^{n-j}] \mathcal{E}_j(x),$$

(2.10)

where $\delta[0^{n-j}] = (1/2)^{n-j} - (-1/2)^{n-j}$.

The proof will depend on Lemma 1, Lemma 2 and Relation (1.8).

Lemma 1. For all polynomials $p_n(x)$ the following relation holds true.

$$Z(p_n(x)) = xZ(\delta p_n(x)).$$

Proof. We have

$$Z(x^n) = n! x^n = x(n-1)! x^{n-1} = xZ(n! x^{n-1}) = xZ(\delta x^n),$$

as any polynomial can be written as sums of central factorial $x^n$. Thus, we have the result. □

Lemma 2 (Tanny [17]). For all polynomials $p_n(x)$ we have

$$\Delta p_n(x) = \sum_{k=0}^{n-1} \binom{n}{k} p_k(x).$$

(2.11)

Now we give the proof of Theorem 2,

Proof of Theorem 2. Using Lemma 1, Lemma 2 and setting $p_n(x) = x^n$, we get

$$Z(x^n) = xZ(\delta x^n) = xZ(\Delta E^{-1/2} x^n) = xZ(\Delta x^{n-1}) = xZ(\delta x^n),$$

as any polynomial can be written as sums of central factorial $x^n$. Thus, we have the result. □

Corollary 2. The central Fubini-like numbers satisfy

$$C_n = \sum_{j=0}^{n-1} \binom{n}{j} \delta[0^{n-j}] \mathcal{E}_j.$$

(2.12)

Now we give an explicit formula connecting the central Fubini-like polynomials with Stirling numbers of the second kind $S(n,k)$. 
Theorem 3. The central Fubini-like polynomials \( C_n(x) \) satisfy
\[
C_n(x) = \sum_{k=0}^{n} k! x^k \sum_{j=0}^{n} \binom{n}{j} \left( \frac{-k}{2} \right)^j S(n-j,k).
\] (2.13)

Proof. From Theorem 1, we have
\[
\sum_{n=0}^{\infty} C_n(x) \frac{t^n}{n} = \frac{1}{1-2x \sinh(t/2)}.
\]
Using the exponential form of \( 2x \sinh(t/2) \) we get
\[
\sum_{n=0}^{\infty} C_n(x) \frac{t^n}{n} = \frac{1}{1-x e^{(-t/2)}(e^t-1)} = \sum_{k=0}^{\infty} x^k e^{(-kt/2)}(e^t-1)^k.
\]
It is also known that
\[
\sum_{n=0}^{\infty} S(n,k) \frac{t^n}{n!} = \frac{(e^t-1)^k}{k!}.
\]
Therefore
\[
\sum_{n=0}^{\infty} C_n(x) \frac{t^n}{n} = \sum_{k=0}^{\infty} x^k! \sum_{j=0}^{n} \binom{n}{j} \left( \frac{-k}{2} \right)^j \sum_{n=0}^{\infty} S(n,k) \frac{t^n}{n!}.
\]
Then Cauchy’s product implies the identity. \( \square \)

Corollary 3. The central Fubini-like numbers \( C_n \) satisfy
\[
C_n = \sum_{k=0}^{n} k! \sum_{j=0}^{n} \binom{n}{j} \left( \frac{-k}{2} \right)^j S(n-j,k).
\] (2.14)

2.3. Umbral representation

Umbral (or Blissard or symbolic) calculus originated as a method for discovering and proving combinatorial identities in which subscripts are treated as powers. Bell in [1] gave a postulational bases of this calculus. In this section we use the following property given by Riordan [16]. As specified by the author in [16], "A sequence \( a_0, a_1, ... \) may be replaced by \( a_0^0, a_1^1, ... \) with the exponents are treated as powers during all formal operations, and only restored as indexes when operations are completed". Then when we have
\[
a_n = \sum_{k=0}^{n} \binom{n}{k} b_k c_{n-k}
\]
we can write it as
\[
a_n = (b + c)^n,
\]
where \( b^n = b_n \) and \( c^n = c_n \). We note that \( b^0 \) and \( c^0 \) is not necessary equal to 1.

In the following theorem we use the umbral notation \( \mathcal{C}_k(x) \equiv C^k(x) \) and \( \mathcal{C}_k \equiv C^k \).

Theorem 4. Let \( n \) be a non-negative integer, for all real \( x \) we have
\[
C_n(x) = x \left[ (C(x) + 1/2)^n - (C(x) - 1/2)^n \right].
\]
Proof. From Theorem 2 and using the umbral notation, a simple calculation gives the umbral representation result.

Corollary 4. For non-negative integer \( n \), we have

\[
\mathcal{C}_n = (\mathcal{C} + 1/2)^n - (\mathcal{C} - 1/2)^n.
\]

2.4. Parity

A function \( f(x) \) is said to be even when \( f(x) = f(-x) \) for all \( x \) and it is said to be odd when \( f(x) = -f(-x) \).

Theorem 5. For all non-negative \( n \) and real variable \( x \) we have

\[
\mathcal{C}_n(x) = (-1)^n \mathcal{C}_n(-x).
\]

Proof. Using the fact that the function \( f : t \mapsto \sinh(t) \) is odd, this gives \( G(x; t) = G(-x; -t) \), then comparing the coefficients of \( t^n/n! \) in \( G(x; t) \) and \( G(-x; -t) \) the theorem follows.

Corollary 5. The polynomials \( \mathcal{C}_n(x) \) are odd if and only if \( n \) is odd.

Proof. Using Theorem 5, it suffices to replace \( n \) by \( 2k + 1 \) (resp. \( 2k \)) and establish the property.

2.5. Recurrences and derivatives of higher order

Now we are interested to derive some recurrences for \( \mathcal{C}_n(x) \) in terms of their derivatives.

First, we deal with a recurrence of second order.

Theorem 6. For \( n \geq 2 \), the polynomials \( \mathcal{C}_n(x) \) satisfy the following recurrence relation

\[
\mathcal{C}_n(x) = 2x^2 \mathcal{C}_{n-2}(x) + \left( \frac{x^2}{4} + 4x^3 \right) \mathcal{C}'_{n-2}(x) + \left( \frac{x^2}{4} + x^4 \right) \mathcal{C}''_{n-2}(x).
\]

Here \( \mathcal{C}_n'(x) \) and \( \mathcal{C}_n''(x) \) are respectively the first and second derivative of \( \mathcal{C}_n(x) \).

Proof. From Equation (1.4) we have

\[
\mathcal{C}_n(x) = \sum_{k=0}^{n} k!T(n,k)x^k
= \sum_{k=2}^{n} k!T(n-2,k-2)x^k + \frac{1}{4} \sum_{k=0}^{n} k^2k!T(n-2,k)x^k
= \sum_{k=0}^{n} (k+2)T(n-2,k)x^{k+2} + \frac{x}{4} \left( \sum_{k=0}^{n} kk!T(n-2,k)x^k \right)'.
\]
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\[ x^2 \left( x^2 \sum_{k=0}^{n} k! T(n-2,k)x^k \right)'' + \frac{x}{4} \left( \frac{x}{4} \sum_{k=0}^{n} k! T(n-2,k)x^k \right) \]

\[ = x^2 \left( x^2 T_{n-2}(x) \right)'' + \frac{x}{4} \left( x T_{n-2}(x) \right)'' \]

\[ = 2x^2 T_{n-2}(x) + \left( \frac{x}{4} + 4x^3 \right) T_{n-2}(x) + \left( \frac{x^2}{4} + x^4 \right) T_n''(x), \]

this concludes the proof. □

In the next theorem we give a recurrence formula for the \( r \)-th derivative of \( C_n(x) \).

**Proposition 3.** The \( r \)-th derivative of \( G(x;t) \), defined in (2.8), is given by

\[ \frac{\partial^r}{\partial x^r} G(x;t) = \frac{r!}{x^r} G(x;t)(G(x;t) - 1)' \]

**Proof.** Induction on \( r \) implies the equality. □

**Theorem 7.** Let \( C_n^{(r)}(x) \) be the \( r \)-th derivative of \( C_n(x) \). Then \( C_n^{(r)}(x) \) is given by

\[ C_n^{(r)}(x) = \frac{r!}{x^r} \sum_{k=0}^{r} \left( \begin{array}{c} r \\ k \end{array} \right) (-1)^{r-k} \sum_{j_0+j_1+\cdots+j_k=n} \left( \begin{array}{c} n \\ j_0, j_1, \ldots, j_k \end{array} \right) C_{j_0}(x) C_{j_1}(x) \cdots C_{j_k}(x). \]

**Proof.** Using Proposition 3, by applying Cauchy product and comparing the coefficients of \( t^n/n! \), we get the result. □

**Corollary 6.** The following equality holds for any real \( x \):

\[ x C_n'(x) = \sum_{k=0}^{n-1} \left( \begin{array}{c} n \\ k \end{array} \right) C_k(x) C_{n-k}(x). \]

**Proof.** Setting \( r = 1 \) in Proposition 3, we get the first derivative of \( G(x;t) \) as

\[ \frac{\partial}{\partial x} G(x;t) = \frac{2 \sinh \left( \frac{t}{2} \right)}{(1 - 2x \sinh \left( \frac{t}{2} \right))^2} = \frac{G(x;t)}{x} \left( x(G(x;t) - 1) \right), \]

\[ x \frac{\partial}{\partial x} G(x;t) = G(x;t)^2 - G(x;t), \]

\[ x \sum_{n=0}^{\infty} C_n'(x) \frac{t^n}{n!} = \left( \sum_{n=0}^{\infty} C_n(x) \frac{t^n}{n!} \right)^2 - \sum_{n=0}^{\infty} C_n(x) \frac{t^n}{n!}, \]

then applying the Cauchy product in the right hand side and comparing the coefficients of \( t^n/n! \) we get the result. □
2.6. Integral representation

Integral representation is a fundamental property in analytic combinatorics. The central Fubini-like polynomials can be represented as well.

**Theorem 8.** The polynomials \( C_n(x) \) satisfy

\[
C_n(x) = \frac{2n!}{\pi} \text{Im} \int_0^\pi \frac{\sin(n\theta)}{1 - 2x \sinh(e^{i\theta}/2)} \, d\theta.
\]

**Proof.** We will use here the known identity, see [5],

\[
k^n = \frac{2n!}{\pi} \text{Im} \int_0^\pi \exp(k e^{i\theta}) \sin(n\theta) \, d\theta.
\]

We have

\[
C_n(x) = \sum_{k=0}^\infty k!T(n,k)x^k
\]

\[
= \sum_{k=0}^\infty x^k \sum_{j=0}^k (-1)^j \binom{k}{j} \left( \frac{k}{2} - j \right)^n
\]

\[
= \sum_{k=0}^\infty x^k \sum_{j=0}^k (-1)^j \binom{k}{j} \frac{2n!}{\pi} \text{Im} \int_0^\pi \exp\left( \frac{k}{2} - j \right) e^{i\theta} \sin(n\theta) \, d\theta
\]

\[
= \frac{2n!}{\pi} \text{Im} \int_0^\pi \sin(n\theta) \sum_{k=0}^\infty x^k \exp\left( -\frac{k}{2} e^{i\theta} \right) \left( \exp(e^{i\theta}) - 1 \right)^k \, d\theta
\]

\[
= \frac{2n!}{\pi} \text{Im} \int_0^\pi \frac{\sin(n\theta)}{1 - 2x \sinh(e^{i\theta}/2)} \, d\theta.
\]

\( \square \)

2.7. Determinantal representation

Several papers have been published on determinantal representations of many sequences as Bernoulli numbers, Euler numbers, ordered Bell numbers (or Fubini numbers), etc.

Komatsu and Ramírez in a recent paper gives the following theorem.

**Theorem 9** (Komatsu & Ramírez [10]). Let \((R(j))_{j \geq 0}\) be a sequence, and let \(\alpha_n\) be defined by the following determinantal expression for all \(n \geq 1:\)

\[
\alpha_n = \begin{vmatrix}
R(1) & 1 \\
R(2) & R(1) \\
\vdots & \vdots & \ddots & 1 \\
R(n-1) & R(n-2) & \cdots & R(1) & 1 \\
R(n) & R(n-1) & \cdots & R(2) & R(1)
\end{vmatrix}. \tag{2.15}
\]
Then we have

\[ \alpha_n = \sum_{j=1}^{n} (-1)^{j-1} R(j) \alpha_{n-j} \quad (n \geq 1). \quad (2.16) \]

We set \( \alpha_0 = 1 \).

By applying the previous theorem we get

**Theorem 10.** For \( n \geq 1 \), we have

\[ C_n(x) \, n! = \left| \begin{array}{cccc}
R(1) & 1 \\
R(2) & R(1) \\
\vdots & \vdots & \ddots & 1 \\
R(n-1) & R(n-2) & \cdots & R(1) & 1 \\
R(n) & R(n-1) & \cdots & R(2) & R(1)
\end{array} \right|, \quad (2.17) \]

where

\[ R(j) = x^{(-1)^{j-1}} j! \delta[0^j] = x^{(-1)^{j-1}} j! \left( \left( \frac{1}{2} \right)^j - \left( -\frac{1}{2} \right)^j \right). \]

**Proof.** From Theorem 2 we have,

\[ C_n(x) = x \sum_{j=0}^{n-1} \binom{n}{j} \delta[0^{n-j}] C_j(x) = x \sum_{j=1}^{n} \binom{n}{j} \delta[0^j] C_{n-j}(x) \]

\[ \frac{C_n(x)}{n!} = \sum_{j=1}^{n} \frac{x}{j!} \delta[0^j] \frac{C_{n-j}(x)}{(n-j)!}. \]

It suffices to set \( \alpha_n = \frac{C_n(x)}{n!} \) and \( R(j) = x^{(-1)^{j-1}} j! \delta[0^j] \) to get the result. \( \square \)

**Remark 1.** The function \( R(j) = 0 \) for \( j \) even.

Using Remark 1, we establish the following binomial convolution for the polynomials \( C_n(x) \).

**Theorem 11.** For \( n \geq 0 \) we have

\[ C_{n+1}(x) = x \sum_{k=0}^{[n/2]} 4^{-k} \binom{n+1}{2k+1} C_{n-2k}(x). \quad (2.18) \]

**Proof.** From Remark 1 and using Formula (2.16) with \( \alpha_n = C_n(x)/n! \) and \( R(j) = x^{(-1)^{j-1}} j! \left( \left( \frac{1}{2} \right)^j - \left( -\frac{1}{2} \right)^j \right) \) we get the result. \( \square \)

**Remark 2.** Formula (2.18) is better than result of Theorem 2 from a computational point of view.
2.8. Asymptotic result with respect to $C_n$

Find an asymptotic behaviour of a sequence $(a_n)_{n \geq 0}$ means to find a second function depending on $n$ simple than the expression of $a_n$ which gives a good approximation to the values of $a_n$ when $n$ is large.

In this subsection, we are interested to obtaining the asymptotic behaviour of the central Fubini-like numbers.

Let $(a_n)_{n \geq 0}$ be a sequence of non-negative real numbers, the asymptotic behaviour $a_n$ is closely tied to the poles in $G(z)$, where $G(z)$ is the generating function of $a_n$,

$$G(z) = \sum_{n=0}^{\infty} a_n z^n.$$ 

Wilf, in his book [19] and Flajolet et al. in [7] gave a method to determine the asymptotic behaviour $a_n$ which can be summarized in the following steps:

1. Find the poles $z_0, z_1, \ldots, z_s$ in $G(z)$.
2. Calculate the principal parts $P(G(z), z_i)$ at the dominant singularities $z_i$ (which have the smallest modulus $R$) as

$$P(G(z), z_i) = \frac{\text{Res}(G(z), z_i)}{(z - z_i)},$$

where $\text{Res}(G(z), z_i)$ is the residue of $G(z)$ at the pole $z_i$.
3. Set $H(z) = \sum_{i=0}^{s} P(G(z), z_i)$ then write $H(z)$ as the expansion below,

$$H(z) = \sum_{n=0}^{\infty} b_n z^n.$$

4. The sequence $(b_n)_{n=0}$ is the asymptotic behaviour of $a_n$ when $n$ is big enough,

$$a_n \sim b_n + O\left(\frac{1}{R'} + \varepsilon\right)^n, \quad n \rightarrow \infty,$$

where $R'$ is the next smallest modulus of the poles.

For more details about singularities analysis method we refer to [7].

Remark 3. Poles $z_0, z_1, \ldots, z_s$ are considered as simple poles (has a multiplicity equal to 1).

Analytic methods of determining the asymptotic behavior of a sequence $(a_n)_n$ are widely discussed on [2, 7, 13, 14, 19].

**Theorem 12.** The asymptotic behaviour of the $C_n$ is given by

$$C_n \sim \frac{n!}{2^n \sqrt{5} \log^{n+1} (\phi)} + O((0.15732 + \varepsilon)^n), \quad n \rightarrow \infty$$

where $\phi$ is the Golden ratio.

**Proof.** Applying the previous steps in the generating function $G(z) = \frac{1}{1 - 2 \sinh(z/2)}$ gives
The poles of $G(z)$ are

$$z_0 = -2\log \left( \frac{1 + \sqrt{5}}{2} \right) + 2i\pi + 4i\pi k$$

and

$$z_1 = 2\log \left( \frac{1 + \sqrt{5}}{2} \right) + 4i\pi k,$$

with $k \in \mathbb{Z}$.

By setting $k = 0$, the dominant singularity is $z_1 = 2\log (\phi)$ (the modulus $R = 0.96$), then,

$$P(G(z), z_1) = -\frac{2}{\sqrt{5}(z - 2\log(\phi))}.$$

Set $H(z) = -\frac{2}{\sqrt{5}(z - 2\log(\phi))}$, if we write $H(z)$ as the expansion we get

$$H(z) = \sum_{n=0}^{\infty} \frac{1}{2^n \sqrt{5} \log^{n+1}(\phi)} z^n.$$

The next smallest modulus of the poles $R' = 6.356\ldots$, then the asymptotic behaviour of $C_n$ when $n$ is big enough is,

$$C_n \sim \frac{n!}{2^n \sqrt{5} \log^{n+1}(\phi)} + O\left( (0.15732 + \epsilon)^n \right), \quad n \to \infty.$$  

\[\square\]

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**REFERENCES**

[1] E. T. Bell, “Postulational bases for the umbral calculus,” *American Journal of Mathematics*, vol. 62, no. 1, pp. 717–724, 1940, doi: 10.2307/2371481.

[2] E. A. Bender, “Asymptotic methods in enumeration,” *SIAM review*, vol. 16, no. 4, pp. 485–515, 1974, doi: 10.1137/1016082.

[3] K. N. Boyadzhiev, “A series transformation formula and related polynomials,” *International Journal of Mathematics and Mathematical Sciences*, vol. 2005, no. 23, pp. 3849–3866, 2005, doi: 10.1155/IJMMS.2005.3849.

[4] P. L. Butzer, K. Schmidt, E. Stark, and L. Vogt, “Central factorial numbers; their main properties and some applications.” *Numerical Functional Analysis and Optimization*, vol. 10, no. 5-6, pp. 419–488, 1989, doi: 10.1080/01630568908816313.

[5] D. Callan, “Cesaro’s integral formula for the Bell numbers (corrected).” *arXiv preprint arXiv:0708.3301*, 2007.

[6] A. Dil and V. Kurt, “Investigating geometric and exponential polynomials with Euler-Seidel matrices,” *J. Integer Seq*, vol. 14, no. 4, 2011.

[7] P. Flajolet and R. Sedgewick, *Analytic combinatorics*. Cambridge University Press, 2009.

[8] O. A. Gross, “Preferential arrangements,” *The American Mathematical Monthly*, vol. 69, no. 1, pp. 4–8, 1962, doi: 10.1080/00029890.1962.11989826.
[9] C. Jordan and K. Jordán, *Calculus of finite differences*. American Mathematical Soc., 1965, vol. 33.

[10] T. Komatsu and J. L. Ramírez, “Some determinants involving incomplete Fubini numbers,” *An. Stiint. Univ. Ovidius* Constanta Ser. Mat. 26, no. 3, 2018, doi: 10.2478/auom-2018-0038.

[11] N. Krzywonos and F. Alayont, “Rook polynomials in three and higher dimensions,” *Involve*, vol. 6, no. 1, pp. 35–52, 2013, doi: 10.2140/involve.2013.6.35.

[12] I. Mező, “Periodicity of the last digits of some combinatorial sequences,” *J. Integer Seq*, vol. 17, pp. 1–18, 2014.

[13] A. M. Odlyzko, “Asymptotic enumeration methods,” *Handbook of combinatorics*, vol. 2, no. 1063, p. 1229, 1995.

[14] J. Plotkin and J. Rosenthal, “Some asymptotic methods in combinatorics,” *Journal of the Australian Mathematical Society*, vol. 28, no. 4, pp. 452–460, 1979, doi: 10.1017/S1446788700012593.

[15] J. Riordan, *Combinatorial identities*. Wiley New York, 1968, vol. 6.

[16] J. Riordan, *Introduction to combinatorial analysis*. Courier Corporation, 2012.

[17] S. M. Tanny, “On some numbers related to the Bell numbers,” *Canadian Mathematical Bulletin*, vol. 17, no. 5, pp. 733–738, 1975, doi: 10.4153/CMB-1974-132-8.

[18] W. A. Whitworth, *Choice and chance: with 1000 exercises*. D. Bell and Company;[etc., etc.], 1901.

[19] H. S. Wilf, *generatingfunctionology*. AK Peters/CRC Press, 2005.

[20] D. Zeitlin, “Remarks on a formula for preferential arrangements,” *The American Mathematical Monthly*, vol. 70, no. 2, pp. 183–187, 1963, doi: 10.2307/2312890.

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