GUARANTEED COST CONTROL OF DISCRETE-TIME SWITCHED SATURATED SYSTEMS

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Abstract. The problem of guaranteed cost control is investigated for a class of discrete-time saturated switched systems. The purpose is to design the switched law and state feedback control law such that the closed-loop system is asymptotically stable and the upper-bound of the cost function is minimized. Based on the multiple Lyapunov functions approach, some sufficient conditions for the existence of guaranteed cost controllers are obtained. Furthermore, a convex optimization problem with linear matrix inequalities (LMI) constraints is formulated to determine the minimum upper-bound of the cost function. Finally, a numerical example is given to demonstrate the effectiveness of the proposed method.

1. Introduction. The study on switched systems has attracted much attention for its theoretical and practical importance in recent years [11, 23, 17, 24]. Basically, switched systems are an important class of hybrid systems, which consist of a family of continuous-time or discrete-time subsystems, together with a switching law specifying which subsystem is active along the system trajectory during a certain interval of time. Many important real-world systems and processes can be modeled as switched systems, such as chemical processes, power systems and many other fields. It is pointed out in [11] that stability has been a major focus in studying switched systems. Considerably many analytical methodologies and techniques have been used for the study of switched systems [4, 5, 2].

On the other hand, actuator saturation is inevitable in almost all practical systems due to the physical limitations of actuators and safety constraints. It is well recognized that actuator saturation is a source of degradation of system performance, occurrence of limit cycle, more than one equilibrium states with different stability behavior, and even system instability caused by some large perturbation. For the practical significance and the theoretical challenges, a great deal of attention has been focused on the analysis and control synthesis for non-switched systems with input saturation for a long time [26, 27, 6, 25]. Many methods have been developed to deal with actuator saturation [7, 8, 9].

Also it is important to study switched systems subject to actuator saturation and a series of meaningful results are obtained [21, 1, 15, 13, 20, 14]. By utilizing the
switched Lyapunov function approach, the stabilization problem was investigated for a class of discrete-time switched systems with input saturations in [1]. Based on the minimum dwell time method, [15] studied the problem of stabilization of switched linear systems subject to actuator saturation. In [13], the multiple Lyapunov functions method was used to design a switching scheme for the stability of switched linear systems in the presence of actuator saturation. For a class of uncertainty switched linear systems with saturating actuators, the robust stabilization problem was addressed by applying the multiple Lyapunov functions method in [20]. In order to further enlarge the domain of attraction of a linear system subject to actuator saturation, [14] proposed the idea of switching among multiple anti-windup gains. All the results mentioned above are about the stability or stabilization problem for switched saturated systems. However, when controlling a real plant, it is also desirable to design a control system which guarantees not only stability but also an adequate level of performance index [19, 3, 10, 16, 18, 12]. One way to address the performance problem is the so called guaranteed cost control approach which was firstly presented in [19]. Then, many significant results on the GCC problem have been proposed [3]. To the best of the authors’ awareness, nearly no results on the problem of GCC have been reported for switched systems with input saturation in the existing literature, which is indeed our motivation.

2. Problem statement and preliminaries. We consider the following class of discrete time switched linear systems subject to actuator saturation:

\[
\begin{align*}
\{ & x(k+1) = A_\sigma x(k) + B_\sigma \operatorname{sat}(u(k)), \\
& x(0) = x_0.
\end{align*}
\]  

(1)

where \( k \in \mathbb{Z}^+ \), \( x(k) \in \mathbb{R}^n \) is the state vector and \( u(k) \in \mathbb{R}^m \) is the control input vector. The function \( \operatorname{sat}: \mathbb{R}^m \rightarrow \mathbb{R}^m \) is the vector valued standard saturation function defined as

\[
\operatorname{sat}(u) = \begin{bmatrix} \operatorname{sat}(u_1), & \cdots, & \operatorname{sat}(u_m) \end{bmatrix}^T,
\]

\[
\operatorname{sat}(u_j) = \text{sign}(u_j) \min\{1, |u_j|\},
\]

\( j \in Q_m = \{1, \cdots, m\} \).

The function \( \sigma(k) \) is a switching law which takes its values in the finite set \( I_N = \{1, \cdots, N\} \); \( \sigma(k) = i \) means that the \( i \)-th subsystem is active. \( A_i, B_i \) are constant matrices of appropriate dimensions that describe the set of nominal systems.

The cost function associated with switched system (1) is given by

\[
J = \sum_{k=0}^{\infty} \left[ x^T(k) Q x(k) + (\operatorname{sat}(u(k)))^T R \operatorname{sat}(u(k)) \right],
\]

(2)

where \( Q \) and \( R \) are given positive-definite weighted matrices.

In this paper, we consider the following linear state feedback control laws:

\[
u(k) = F_i x(k), \quad i \in I_N,
\]

(3)

where \( F_i \) are to be designed. Then the closed-loop system is

\[
\begin{align*}
\{ & x(k+1) = A_\sigma x(k) + B_\sigma \operatorname{sat}(F_i x(k)), \\
& x(0) = x_0.
\end{align*}
\]

(4)

Definition 2.1. For switched system (1), if there exist a state feedback control law \( u^* \) for each subsystem and a positive scalar \( J^* \) under the switched law \( \sigma \) such that the closed-loop system (4) is asymptotically stable and the value of the cost
Theorem 3.1. Consider the closed-loop system (4). If there exist positive definite matrices $P_i$, matrices $F_i$, and $H_i$ and a set of scalars $\beta_{ir} \geq 0$ such that

$$[A_i + B_i(D_s F_i + D_s^{-} H_i)]^T P_i [A_i + B_i(D_s F_i + D_s^{-} H_i)]$$

$$+ \sum_{r=1, r\neq i}^{N} \beta_{ir} (P_r - P_i) - P_i + Q$$

$$(D_s F_i + D_s^{-} H_i)^T R (D_s F_i + D_s^{-} H_i) < 0, \quad \forall (i, r) \in I_N \times I_N,$$

$$\Omega(P_i, \beta) \cap \Phi_i \subset L(H_i), \forall i \in I_N.$$

and

$$x_0^T P_i x_0 \leq \beta, \forall i \in I_N,$$

Then under the switched law

$$\sigma = \arg \min \{x^T(k)P_i x(k), i \in I_N\}.$$
where \( \Phi_i = \{x(k) \in \mathbb{R}^n : x^T(k)(P_r - P_{\ast})x(k) \geq 0, \forall r \in I_N, r \neq i\} \), the switched system (4) is asymptotically stable at the origin with controllers such that the closed-loop system (4) satisfies the guaranteed cost control.

Proof. By virtue of Lemma 2.2, for every \( x(k) \in \Omega(P_i, \beta) \cap \Phi_i \subset L(H_i) \),

\[
\text{sat}(F_i x(k)) \in \text{co} \{D_x F_i x(k) + D_\sigma H_i x(k), s \in Q\}.
\]

It follows that

\[
A_i x(k) + B_i \text{sat}(F_i x(k)) \in \text{co} \{A_i x(k) + B_i (D_x F_i x(k) + D_\sigma H_i x(k)), s \in Q\}.
\]

In view of the switching law (12), for \( \forall x(k) \in \Omega(P_i, \beta) \cap \Phi_i \subset L(H_i) \), the \( i \)-th subsystem is active.

Choose multiple Lyapunov functions candidate for the system (4) as

\[
V(x(k)) = V_{\sigma(k)}(x(k)) = x^T(k)P_{\sigma(k)} x(k).
\]

Then, when \( \sigma = i \), for \( \forall x(k) \in \Omega(P_i, \beta) \cap \Phi_i \subset L(H_i) \), it follows:

\[
\Delta V(x(k)) + x^T(k)Q x(k) + (\text{sat}(u(k)))^T R s \text{at}(u(k))
= x^T(k + 1)P_x x(k + 1) - x^T(k)P_x x(k) + x^T(k)Q x(k)
+ \left( \sum_{s=1}^{m} \eta_s (D_s F_i + D_\sigma H_i) x(k) \right)^T R \left( \sum_{s=1}^{m} \eta_s (D_s F_i + D_\sigma H_i) x(k) \right)
\leq \max_{s \in Q} x^T(k) [A_i + B_i (D_s F_i + D_\sigma H_i)]^T P_i [A_i + B_i]
\times (D_s F_i + D_\sigma H_i) x(k) - x^T(k)P_x x(k)
+ x^T(k)Q x(k) + x^T(k)(D_s F_i + D_\sigma H_i)^T R (D_s F_i + D_\sigma H_i) x(k)).
\]

Then, in view of the switched law (12), we obtain

\[
\Delta V(x(k)) + x^T(k)Q x(k)
+ (\text{sat}(u(k)))^T R \text{sat}(u(k)) < 0,
\]

since \( Q \) and \( R \) are given positive-definite matrices, it holds that

\[
\Delta V(x(k)) < 0,
\]

which indicates that under the switched law (12), the closed-loop switched system (4) is asymptotically stable at the origin with \( \cup_{i=1}^{N} (\Omega(P_i, \beta) \cap \Phi_i) \) contained in the domain of attraction.

In the following, we will show that the closed-loop system (4) satisfies the upper bound of the cost function. According to the inequality (15), we have

\[
J < - \sum_{k=0}^{\infty} [V(x(k + 1) - V(x(k))]
= -[V(x(\infty)) - V(x(0))],
\]

Due to \( x(\infty) \geq 0 \), it is easy to see that

\[
J < V(x(0)) = x_0^T P_0 x_0 \leq \beta.
\]

The proof of Theorem 1 is complete. \( \square \)

It should be noted that the above sufficient condition in Theorem 3.1 is not the LMI-based condition, which cannot be solved easily. Then, on the basis of the Theorem 3.1, we propose an LMI approach for designing the state feedback controllers such that the closed-loop system (4) satisfies the guaranteed cost control.
Theorem 3.2. For the resulting system (4), suppose that there exist positive definite matrices $X_i$, matrices $M_i$, $N_i$ and a set of scalars $\beta_i > 0$ and $\delta_i$ satisfying

$$
- X_i - \sum_{r=1, r \neq i}^N \beta_i X_i \begin{bmatrix} * & * & * & * & * & * \\ A_iX_i + B_i(D_iM_i + D_i^+ N_i) & -X_i & * & * & * & * \\ D_iM_i + D_i^+ N_i & 0 & -\beta R^{-1} & * & * & * \\ X_i & 0 & 0 & -\beta Q^{-1} & * & * \\ X_i & 0 & 0 & 0 & -\beta_i^{-1} X_i & * \\ X_i & 0 & 0 & 0 & 0 & -\beta_i^{-1} X_N \end{bmatrix} < 0, 
$$

(19)

$$
X_i + \sum_{r=1, r \neq i}^N \delta_i X_i \begin{bmatrix} N_i & \varepsilon & * & * & * \\ X_i & 0 & \delta^{-1}_i X_1 & * & * \\ X_i & 0 & 0 & * & \cdots \\ X_i & 0 & 0 & 0 & \delta^{-1}_i X_N \end{bmatrix} \geq 0, 
$$

(20)

and

$$
\begin{bmatrix} 1 & x_0^T \\ * & X_i \end{bmatrix} \geq 0, \forall i \in I_N.
$$

(21)

Then, under the switched law

$$
\sigma(k) = \arg\min\{x^T(k)x_i^{-1}x(k), i \in I_N\},
$$

(22)

the set $\bigcup_{i=1}^N (\Omega(X_i, 1) \cap \Phi_i)$ is inside the domain of attraction of the system (4). Furthermore, $u(k) = M_i X_i^{-1} x(k)$ the guaranteed cost controllers for the system (4) and the corresponding performance index is

$$
J < \beta,
$$

(23)

where $N_i^j$ are the $j$-th row of matrices $N_i$, $X_i = \beta P_i^{-1}$, $M_i = F_iX_i$, $N_i = H_iX_i$.

Proof. Pre- and post-multiplying both sides of inequality (9) by $\beta_i^2 P_i^{-1}$, we obtain

$$
\begin{aligned}
\beta_i P_i^{-1}[A_i + B_i(D_i F_i + D_i^+ H_i)]^T \beta_i^{-1} P_i[A_i + B_i(D_i F_i + D_i^+ H_i)] & \beta_i P_i^{-1} \\
+ \sum_{r=1, r \neq i}^N \beta_i \beta_i^{-1} - \beta_i P_i^{-1} & + \beta_i^{-1} Q_i \beta_i P_i^{-1} \\
+ \beta_i^{-1} (D_i F_i + D_i^+ H_i)^T \beta_i^{-1} R(D_i F_i + D_i^+ H_i) \beta_i P_i^{-1} & < 0,
\end{aligned}
$$

(24)

Let $X_i = \beta P_i^{-1}$, $M_i = F_iX_i$, $N_i = H_iX_i$. Then, by virtue of Schur’s complements, it is easy to see that the inequality (24) can be equivalently transformed to the inequality (19).

Then, by applying a similar method as used in changing (9) into (19), the condition (10), the inequality (11) and the switched law (12) are equivalent to the inequality (20), the inequality (21) and (22) respectively. Thus, the proof of Theorem 3.2 is complete.

Theorem 3.2 provides a sufficient condition for the solution to the guaranteed cost control problem with the performance index $J < \beta$. However, the cost upper bound relies on feasible solution of LMI and any feasible solution of the set of LMI
provides a cost upper bound. Therefore, our objective is to design the guaranteed cost controllers in order to minimize the upper bound of the cost function. The problem can be formulated as the following optimization problem:

\[
\inf_{X_i, M_i, N_i, \beta_{ir}, \delta_{ir}} \beta,
\]

s.t. (a) inequality (19), \( \forall i \in I_N \), (b) inequality (20), \( \forall i \in I_N, \forall j \in Q_M \), (c) inequality (21), \( \forall i \in I_N \).

(25)

Solving these new optimization problems (25), we can obtain the minimum cost upper bound \( \beta^* \) and then compute the guaranteed cost controllers \( u^*(k) = M_iX_i^{-1}x(k) \). Thus, for the system (4), the problem of guaranteed cost control is solved too.

**Remark 1.** If the scalar parameters \( \beta_{ir}, \delta_{ir} \) are given in advance, (19) and (20) are formulated and solved as LMI constraints optimization problems.

4. Numerical example. In the section, in order to illustrate the validity of the results in section 3, we consider the following discrete-time switched linear system with input saturation

\[
\begin{cases}
  x(k+1) = A_\sigma x(k) + B_\sigma \text{sat}(F_i x(k)), \\
  x(0) = x_0,
\end{cases}
\]

where \( \sigma(k) \in I_2 = \{1, 2\} \),

\[
A_1 = \begin{bmatrix} -0.9 & 0.4 & 0.2 \\ 0 & 1.7 & -0.1 \\ 0 & 0 & -1.2 \end{bmatrix},
A_2 = \begin{bmatrix} 2 & 0 & 0 \\ -0.5 & -0.8 & 0 \\ 0.2 & 0.3 & -1.4 \end{bmatrix},
B_1 = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix},
B_2 = \begin{bmatrix} 0 & -0.1 \\ 0 & 0.09 \\ 6 & 1.2 \end{bmatrix},
x_0 = \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix},
Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},
R = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.6 \end{bmatrix},
\]

Now, we will design the switched law and the guaranteed cost controllers to stabilize the resulting system (26) subject to actuator saturation with the minimized the upper bound of the cost function. Let \( \beta_1 = \beta_2 = 10, \delta_1 = \delta_2 = 3 \). Then, solving the optimization problem (25) results in

\( \beta^* = 17.5725 \),

\[
X_1 = \begin{bmatrix} 35.3681 & 4.0247 & 3.8197 \\ 4.0247 & 17.3528 & 6.1576 \\ 3.8197 & 6.1576 & 8.6475 \end{bmatrix},
X_2 = \begin{bmatrix} 42.3245 & 8.3284 & 6.9743 \\ 8.3284 & 27.6843 & 4.6536 \\ 6.9743 & 4.6536 & 18.0227 \end{bmatrix},
\]

\[
M_1 = \begin{bmatrix} 12.6447 & -6.3478 & 7.0431 \\ -5.2483 & 15.2946 & -3.1862 \end{bmatrix},
M_2 = \begin{bmatrix} -14.3562 & 8.6475 & 11.0279 \\ 9.1580 & -12.8659 & 4.1934 \end{bmatrix},
N_1 = \begin{bmatrix} 4.8752 & 2.5738 & -6.2835 \\ -3.3719 & 5.0674 & 1.5160 \end{bmatrix},
N_2 = \begin{bmatrix} -6.8013 & 4.2752 & -3.1347 \\ 1.2879 & 5.9461 & -2.1457 \end{bmatrix},
\]

and the corresponding guaranteed cost controller gain matrices are

\[
F_1 = \begin{bmatrix} 0.3182 & -0.9082 & 1.3206 \\ -0.1672 & 1.3712 & -1.2710 \end{bmatrix},
F_2 = \begin{bmatrix} -0.5275 & 0.3490 & 0.7259 \\ 0.2886 & -0.5979 & 0.2753 \end{bmatrix}.
\]
To compare with the existing literature, we apply the method in [22] to the system and find that all the optimisation problems have no solutions. This is because the problem of guaranteed cost control is required to be solvable for every subsystem in [22]. However, it is easy to verify that in the example, the guaranteed cost control problem for each subsystem is not solvable.

5. Conclusions. In this paper, we have studied the guaranteed cost control problem for a class of discrete-time switched linear systems with saturating actuators by using the multiple Lyapunov functions approach. Some sufficient conditions for satisfying simultaneously the stabilization and the cost performance index of the closed-loop system are derived. Furthermore, the state feedback controllers and the switching law which minimize the upper-bound of the cost function are presented by solving a convex optimization problem with a set of LMI constraints.

Compared with the existing results for switched systems with actuator saturation, the results in this paper have three distinct features. First of all, the guaranteed cost control problem are addressed for the switched systems with saturating actuator, while most existing works considered only the problem of stability; second, the multiple Lyapunov functions method is used to study the guaranteed cost control problem of switched systems subject to actuator saturation for the first time, while in most existing literature, the problem has been investigated by using the switched Lyapunov function method; thirdly, by using the multiple Lyapunov functions method, no solvability of the control problem for subsystems is required, while solvability of the problem for each subsystem is needed when using the arbitrary switching method.

In addition, the chattering phenomenon generated by state-dependent switching laws will bring a lot of trouble, which will lead to fatal damage for actual engineering systems. Therefore, for switching systems with actuator saturation, how to choose switching laws to avoid chattering phenomenon is an interesting problem that deserves further study.

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