A family of solutions to the inverse problem in gravitation: building a theory around a metric

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A method is presented to construct a particular, non-minimally coupled scalar-tensor theory such that a given metric is an exact vacuum solution in that theory. In contrast to the standard approach in studies of gravitational dynamics, where one begins with an action and then solves the equations of motion, this approach allows for an explicit theory to be built around some pre-specified geometry. Starting from some parameterized black hole geometry with generic, non-Kerr hairs, it is shown how an overarching family of theories can be designed to fit the metric exactly.

INTRODUCTION

In the study of Lagrangian field theory and the calculus of variations, one typically begins with an action functional and then investigates the dynamics of the associated theory. In many cases however, the inverse problem is also of fundamental interest: starting from a particular field configuration, can one find an invariant Lagrangian density whose equations of motion admit that field as an exact solution? Owing to the complexity of the differential equations involved, which are typically non-linear in realistic problems, finding such a Lagrangian, let alone all Lagrangians, can be a challenging task. This is especially true in studies of gravitation (e.g., \[1\,2\]), where the action is built from geometric invariants which depend on the tensorial metric field in complicated ways. Even conceptually simple cases like the \(f(R)\) theories \[3\,4\], which involve only some function, \(f\), of the Ricci scalar curvature, \(R\), admit rich configuration spaces \[5\,8\].

In the context of tests of general relativity (GR) from observations of compact objects, two main techniques are employed. One approach (sometimes called ‘top-down’) involves picking a particular theory of gravity and comparing the solutions obtained within that theory with a suitable GR counterpart (e.g., \[9\,10\]). In this way, the predictions of a given theory are challenged directly using experimental data. Top-down methods are however limited because exact solutions describing realistic compact objects within a given theory are often not known, and it can be impractical to test multiple theories simultaneously using a given framework. The other approach (‘bottom-up’) involves a phenomenological parameterization of the spacetime that incorporates generic deformations of the GR counterpart \[11\,14\]. However, even if the deviation parameters of the parameterized metric can be constrained, bottom-up approaches do not necessarily guide one towards the ‘true’ theory of gravity. Moreover, backreaction effects cannot be self-consistently accounted for when a metric is considered independently of a parent theory \[15\,16\]. A unification of these two approaches, which would remedy the above shortcomings, boils down to requiring a solution to the inverse problem: given a metric (reconstructed from astrophysical data), can one find a (theoretically robust) theory of gravity that supports the solution exactly?

In this article we show how one can construct such a theory. In particular, a new class of mixed scalar-\(f(R)\) theories are presented which involve a function \(f\) whose argument includes the scalar curvature and both potential and kinetic terms of a scalar field in a precise way, and reduces to a number of well-known cases in some limits. We show that there are large families of functions \(f\) such that, for a particular scalar-field configuration, a given metric is an exact solution to the equations of motion. While the presented theory provides only one particular (not necessarily physically-motivated) example of a covariant action that can be tailored to a given metric, having an explicit construction on hand helps toward finding a general solution to the inverse problem. The approach has the benefit that gravitational perturbations of a given spacetime can be studied self-consistently in bottom-up approaches, without the ad hoc use of the Teukolsky or related equations that, strictly speaking, only apply to the dynamics of GR and not its modified variants.

Except where needed for clarification, natural units with \(c = G = 1\) are adopted throughout.

A MIXED SCALAR-\(f(R)\) GRAVITY

Consider the theory governed by the action

\[
A = \kappa \int d^4x \sqrt{-g} \left[ F(\phi) R + V(\phi) - \omega(\phi) \nabla_\alpha \phi \nabla^\alpha \phi \right],
\]

where \(\kappa = (16\pi G)^{-1}\), \(G\) is Newton’s (bare) constant, \(R \equiv R_{\mu\nu} g^{\mu\nu}\) is the scalar curvature for metric tensor \(g\), and \(F, V, \omega\) are potential functions of the scalar field \(\phi\). When the function \(f\) is linear in its argument \(X\), where \(X \equiv F(\phi) R + V(\phi) - \omega(\phi) \nabla_\alpha \phi \nabla^\alpha \phi\), the theory described by the action \[1\], which resembles a curvature-coupled modification of 'k-field’ theory \[17\,18\], reduces to standard scalar-tensor theory in the Jordan frame \[19\,21\]. The \(f(R)\) theory of gravity is also recovered for constant scalar field \(5\). In any case, matter fields can be included in the usual way.

The equations of motion for the theory \[1\] are found via the Euler-Lagrange equations, and are qualitatively similar to those of \(f(R)\) gravity. Variation of \[1\] with respect to the metric yields

\[
0 = F(\phi) f'(X) R_{\mu\nu} - \frac{f(X)}{2} g_{\mu\nu} + g_{\mu\nu} \Box [F(\phi) f'(X)]
- \nabla_\mu \nabla_\nu [F(\phi) f'(X)] - \omega(\phi) f'(X) \nabla_\mu \phi \nabla_\nu \phi,
\]

end of text
while variation with respect to $\phi$ gives
\begin{equation}
0 = f'(X) \left[ 2\omega(\phi)\Box \phi + \frac{d\omega(\phi)}{d\phi} \nabla_\alpha \phi \nabla^\alpha \phi + R \frac{dF(\phi)}{d\phi} + \frac{dV(\phi)}{d\phi} \right] + 2\omega(\phi) \nabla_\alpha \phi \nabla^\alpha f'(X),
\end{equation}
in vacuo.

In general, several conditions are imposed on scalar-tensor dynamics to ensure a well-defined theory. In the case of linear $f$, demanding that the graviton carries a positive energy and that the scalar field carries a non-negative kinetic energy requires that $F(\phi) > 0$ and $2F(\phi)\omega(\phi) + 3\left| \frac{dF(\phi)}{d\phi} \right|^2 \geq 0$, respectively [23]. In the case of Brans-Dicke theory, which consists of the choices $F(\phi) = \phi$ and $\omega(\phi) \propto \phi^{-1}$ [23], the aforementioned conditions are satisfied automatically for linear $f$ (see also Ref. [18]). For any $f$, however, an appealing feature of the theory described by (1) is that energy-momentum is conserved identically. Employing the Bianchi identities $\nabla_\mu \left( R_{\mu\nu} - \frac{2}{3} g_{\mu\nu} R \right) = 0$ and $(\Box \nabla_\mu - \nabla_\mu \Box) Z = R_{\mu\nu} \nabla^\mu Z$, the first of which is familiar from GR while the second is valid for any function $Z$ [24], some extensive though not particularly difficult algebra shows that applying a contravariant divergence to the right-hand side of (4) produces a sequence of terms which vanish identically when equation (5) is used. As such, for the non-vacuum case where a stress-energy tensor $T_{\mu\nu}$ occupies the left-hand side of (2), geometric identities give $\nabla^\mu T_{\mu\nu} = 0$ exactly, as in the pure $f(R)$ and scalar-tensor cases [25].

**CONSTRUCTING A SOLUTION TO THE INVERSE PROBLEM**

In the case of pure $f(R)$ gravity, families of functions $f$ can be constructed such that any metric $g$ with constant scalar curvature, $R_0$, can be admitted as an exact solution. For example, if $f$ has a critical point at $R_0$ and also happens to vanish there (i.e., $f = 0$ is a local extremum at the point $R_0$), the equations of motion are necessarily satisfied for any metric $g$ which has $R = R_0$. One such theory in this class is the Starobinsky-like quadratic theory with $f(R) = (R - R_0)^2$ [26], for example. Therefore, in the case of constant-scalar-curvature (though not necessarily Einstein) spacetimes, certain $f(R)$ theories are already examples of solutions to the inverse problem [4].

A similar but more extensive phenomenon to that described above can occur in the generalized theories associated with the action [1]. If the scalar field is tuned in such a way that the Ricci curvature is counterbalanced in some precise way, the function $f$ within (1) can be chosen to vanish at a realizable local extremum. As in the case of $f(R)$ gravity, this implies that, given any reasonable metric $g$, there exists a family of mixed scalar-$f(R)$ theories admitting that particular $g$ as an exact solution. To see this explicitly suppose that, for a given $g$, the scalar field $\phi$ solves the equation $\nabla_\mu X = 0$, where again $X \equiv F(\phi)R + V(\phi) - \omega(\phi)\nabla_\alpha \phi \nabla^\alpha \phi$ is the argument of the function $f$ within (1). This implies that $X$ is constant, $X = X_0$, for this particular metric and scalar field combination. If the function $f$ has a critical point at $X_0$ and also vanishes there, then the field equations are necessarily satisfied for this combination of $g$ and $\phi$, as each term within (2) and (3) can be seen to vanish. This means that, provided the scalar field can be chosen such that $\nabla_\mu X = 0$ for a given $g$, there exists a function $f$ [e.g., $f(X) = (X - X_0)^2$ for some $X_0$] for which that particular $g$ is an exact, vacuum solution to the theory governed by (1). In fact, there are infinitely many such functions. If we consider only those $f$ that are analytic, then the most general such $f$ can be represented as a power series, viz. $f(X) = \sum_{k=2}^{\infty} a_k (X - X_0)^k$ for arbitrary coefficients $a_k$. Allowing for non-analytic $f$ further widens the class of suitable functions (see the example given in the next section).

In short, the main result of this article is that, for any given metric $g$, if

i) a scalar field $\phi$ can be chosen such that $X = X_0$ for some constant $X_0$, and

ii) the function $f$ satisfies $f(X_0) = f'(X_0) = 0$,

then $g$ is a solution to the field equations (2) and (3) for the gravitational action (1).

It is important to note that we do not comment here on the physical viability or otherwise of such theories. Indeed, further analysis, beyond the scope of this article, is required to determine whether there exists members of the class constructed above that can accommodate existing (and upcoming) astrophysical experiments. For example, there may be no such $f$ which simultaneously satisfies the above and passes Solar system [29] and/or strong-field [30] tests, even with screening mechanisms [31][32]. A thorough investigation of these considerations will be conducted elsewhere.

**AN EXAMPLE: PARAMETERIZED BLACK HOLE GEOMETRIES**

Various techniques based on electromagnetic [33, 34] and gravitational-wave [35-36] observations allow one to, with varying degrees of precision, identify the local spacetime geometry surrounding a monitored (usually compact) object. However, especially in the gravitational case, these tests inherently presuppose a particular set of field equations. Radiation of any sort saps energy from the system, and backreaction

\footnote{These observations have the implication that given any metric $g$, the conformal metric $e^{2\varphi}g$ for conformal factor $\varphi$ is a solution to some $f(R)$ theory provided that the factor $\varphi$ is chosen such that the conformal scalar curvature is constant; mathematically, this requires the existence of a solution to the Yamabe problem on the spacetime manifold under consideration [27][28]. As such, practically any conceivable null-cone structure can arise in some $f(R)$ theory, because a metric conformally related to any given metric can be admitted as an exact solution.}

cannot be self-consistently accounted for without some over-arching equations of motion. Backreaction effects are negligible in many cases of course, though those tests which involve oscillations or violent outbursts of compact objects may be sensitive to the particulars of the gravitational action. Many metric reconstruction techniques, which use some parameterized scheme in lieu of an exact theory, are therefore limited in their validity to some degree. The construction given in the previous section essentially allows one to build a theory around a given metric, which allows for a potential resolution to this problem.

In this section, we show how one may tailor a particular theory of gravity to a given family of parameterized black holes, such as those considered in Refs. [11-14]. For demonstration purposes, we consider a simple generalization of the Kerr metric whose line element, in Boyer-Lindquist coordinates \((t, r, \theta, \varphi)\), reads

\[ ds^2 = \frac{a^2 \sin^2 \theta - \Delta}{\Sigma} dt^2 - \frac{2a \sin^2 \theta (a^2 + r^2 - \Delta)}{\Sigma} dt d\varphi \\
+ \frac{\Delta}{\Sigma} dr^2 + \frac{(a^2 + r^2)^2 - a^2 \sin^2 \theta \Delta}{\csc^2 \theta \Sigma} d\varphi^2, \]

where \(\Delta = r^2 - 2Mr + a^2 + \epsilon M^3/r\) and \(\Sigma = r^2 + a^2 \cos^2 \theta\). In expression (4), \(M\) and \(a\) denote the mass and spin of the black hole, respectively, while \(\epsilon\) is a dimensionless deformation parameter that is to be constrained by observations. The metric (4) admits an (outer) event horizon at the largest positive root of \(\Delta = 0\), which occurs near the Kerr value for sufficiently small \(\epsilon\), viz. \(r \approx M + \sqrt{M^2 - a^2 + \mathcal{O}(\epsilon)}\).

The geometry described by (4) represents a generalization of the Kerr metric with several desirable properties. Most notably, 1) the metric is asymptotically flat, 2) many post-Newtonian constraints are automatically satisfied due to the absence of quadratic terms in the static limit \(a = 0\), and 3) the metric coefficients are algebraically simple, so that astrophysical tests involving electromagnetic data analysis are numerically easy to handle. The metric (4) is a member of those considered in Ref. [14], for instance.

The mixed scalar-\(f(R)\) theory described by the action \(f\) with \(f(X) = X^{1+\delta}\) for any \(\delta > 0\) admits the metric (4) as an exact solution, provided that the scalar field \(\phi\) solves the kinematic constraint equation

\[ 0 = F(\phi) R + V(\phi) - \omega(\phi) \nabla_\alpha \phi \nabla^\alpha \phi. \]

(5)

Note that this function \(f\) is not analytic, though \(f = 0\) is a critical point at \(X = 0\) for \(\delta > 0\). We make the Brans-Dicke [23] choices \(\omega(\phi) = \phi^{-1}\) and \(F(\phi) = \phi\) with vanishing potential, \(V = 0\), for simplicity, though more complicated examples can be readily designed. For the metric (4) we find \(R = -2M^3 \epsilon \left[ r^3 \left( r^2 + a^2 \cos^2 \theta \right) \right]\), and equation (5) reduces to the remarkably simple form

\[ 0 = 2M^3 \epsilon \frac{\phi(r)}{r} + r^3 \Delta(r) \left( \frac{d\phi(r)}{dr} \right)^2. \]

(6)

In general, there exists a well-behaved solution for \(\phi\) to the constraint equation (6) for a wide-range of \(\epsilon\). Figure 1 shows numerical solutions to (6) for \(M = 1\) and \(a = 0.9\) subject to the boundary condition \(\lim_{r \to \infty} \phi(r) = 1\), which forces \(\phi\) to asymptote towards the Newtonian value at large radii. Figure 1 illustrates that the scalar hair induced by the non-Kerr parameter \(\epsilon\) is rather short-ranged, as \(\phi \approx 1\) to within 1% already for \(r > 30\) for all values considered. This particular field \(\phi\), required to have the metric (4) be an exact solution to the theory (1) for \(f(X) = X^{1+\delta}\), therefore appears to be well behaved and physically reasonable. For vanishing \(\epsilon\) we find that \(\phi\) is everywhere constant, as expected, since the Kerr metric is Ricci-flat and equation (6) simply reduces to \(d\phi/dr = 0\).

**DISCUSSION**

In this article, a method is presented to build a covariant, Lagrangian theory of gravity around a pre-specified spacetime metric; in other words, a particular solution to the inverse problem in gravitation is found. Given some metric \(g\), we show that a function \(f\) and scalar field \(\phi\) can often be found (so long as a solution to the kinematic constraint equation \(\nabla_\mu X = 0\) exists) such that \(g\) is an exact solution to the mixed scalar-\(f(R)\) theory governed by the action (1). For the particular case of \(f(X) = X^{1+\delta}\) for \(\delta > 0\), we found that a parametrically-deformed Kerr metric (4) (cf. Ref. [14]) is an exact solution to the field equations (2) and (3), provided that the scalar field satisfies the kinematic constraint (5). Solutions to equation (6) are shown in Fig. 1 for a variety of parameters.

In all cases considered, the scalar field \(\phi\) is short ranged, well behaved, and asymptotes to the Newtonian value \(\phi_\infty = 1\), as expected of physical black hole geometries.

One of the major benefits of the construction detailed herein
is that gravitational perturbations of a given spacetime can be studied self-consistently. Given a solution to the equations of motion \([2, 3]\) [such as \([4]\), for instance], a perturbation, encapsulated by the Eulerian scheme \(g \rightarrow g + \delta g\) and \(\phi \rightarrow \phi + \delta \phi\), can be introduced to deduce stability \([29]\) and characterize any resulting gravitational radiation (e.g., \([16, 35, 36]\)). At least in the case of spherical symmetry, the equations describing a perturbation around a particular background, a l’a Bardeen-Press \([40]\) or Regge-Wheeler-Zerilli \([41, 42]\), within the theory \([1]\) can be derived without too much difficulty. Nevertheless, despite a number of attractive features, important questions remain about whether or not the theories considered herein are compatible with astrophysical constraints \([29, 30]\). A thorough investigation will be conducted elsewhere.

Some philosophical curiosities arise by noting that the approach presented here involves the construction of vacuum solutions. Since the seed metric could arise as a matter-filled solution in GR (for example), this implies that vacuum gravitational fields in the theory governed by expression \([1]\) can imitate the gravitational fields of material bodies in a different theory. In this way, the gravitational field within and surrounding a star, for instance, could be mimicked by that of a vacuum object in the theory \([1]\). This raises the interesting possibility of “gravitational doppelgängers”. Some examples of this phenomenon are already familiar from the literature; for instance, it is known that the electrovacuum Kerr-Newman metric arises as a pure vacuum solution in some modified theories of gravity \([10, 43]\).

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