The Natural Operators Similar to the Twisted Courant Bracket One

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Abstract. Given natural numbers \( m \geq 3 \) and \( p \geq 3 \), all \( \mathcal{M}_f^m \)-natural operators \( A_H \) sending \( p \)-forms \( H \in \Omega^p(M) \) on \( m \)-manifolds \( M \) into bilinear operators \( A_H : (\mathcal{X}(M) \oplus \Omega^1(M)) \times (\mathcal{X}(M) \oplus \Omega^1(M)) \to \mathcal{X}(M) \oplus \Omega^1(M) \) transforming pairs of couples of vector fields and 1-forms on \( M \) into couples of vector fields and 1-forms on \( M \) are founded. If \( m \geq 3 \) and \( p \geq 3 \), then that any (similar as above) \( \mathcal{M}_f^m \)-natural operator \( A \) which is defined only for closed \( p \)-forms \( H \) can be extended uniquely to the one \( A \) which is defined for all \( p \)-forms \( H \) is observed. If \( p = 3 \) and \( m \geq 3 \), all \( \mathcal{M}_f^m \)-natural operators \( A \) (as above) such that \( A_H \) satisfies the Leibniz rule for all closed 3-forms \( H \) on \( m \)-manifolds \( M \) are extracted. The twisted Courant bracket \([−, −]_H\) for all closed 3-forms \( H \) on \( m \)-manifolds \( M \) gives the most important example of such \( \mathcal{M}_f^m \)-natural operator \( A \).

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1. Introduction

The “doubled” tangent bundle \( T \oplus T^* \) over \( m \)-dimensional manifolds (\( m \)-manifolds) is full of interest because it has the natural inner product, and the Courant bracket, see [1]. Besides, generalized complex structures are defined on \( T \oplus T^* \), generalizing both (usual) complex and symplectic structures, see e.g. [3, 4].

In Sect. 2, the description from [2] of all \( \mathcal{M}_f^m \)-natural bilinear operators

\[
A : (\mathcal{X}(M) \oplus \Omega^1(M)) \times (\mathcal{X}(M) \oplus \Omega^1(M)) \to \mathcal{X}(M) \oplus \Omega^1(M),
\]

transforming pairs of couples of vector fields and 1-forms on \( m \)-manifolds \( M \) into couples of vector fields and 1-forms on \( M \) will be shortly cited. The most important example of such \( \mathcal{M}_f^m \)-natural bilinear operator \( A \) is given by the Courant bracket \([−, −]_H\) for all closed 3-forms \( H \) on \( m \)-manifolds \( M \) gives the most important example of such \( \mathcal{M}_f^m \)-natural operator \( A \).

This Courant bracket was used in [1] to define the concept of Dirac structures being hybrid of both symplectic and Poisson structures.
In Sect. 2 we also deduce that the “trivial” Lie algebroid $(TM \oplus T^*M, 0, 0)$ is the only $\mathcal{M}f_m$-natural Lie algebroid $(EM, [[-,-]], a)$ with $EM := TM \oplus T^*M$.

In Sect. 3, using essentially the results from [2], if $m \geq 3$ and $p \geq 3$, we find all $\mathcal{M}f_m$-natural operators $A$ sending $p$-forms $H \in \Omega^p(M)$ on $m$-manifolds $M$ into bilinear maps

$$A_H : (\mathcal{X}(M) \oplus \Omega^1(M)) \times (\mathcal{X}(M) \oplus \Omega^1(M)) \to \mathcal{X}(M) \oplus \Omega^1(M).$$

The most important example of such $A$ is given by the $H$-twisted Courant bracket $[-,-]_H$ for all $3$-forms $H$ on $m$-manifolds $M$, see Example 3.2. Properties of $[−,−]_H$ (as the Leibniz rule for closed $3$-forms $H$) were used in [7,8] to define the concept of exact Courant algebroid.

In Sect. 4, we observe that if $m \geq 3$ and $p \geq 3$, then any (similar as above) $\mathcal{M}f_m$-natural operator $A$ which is defined only for closed $p$-forms $H$ can be extended uniquely to the one $A$ which is defined for all $p$-forms $H$.

In Sect. 5, if $p = 3$ we extract all $\mathcal{M}f_m$-natural operators $A$ as above satisfying the Leibniz rule

$$A_H(\rho_1, A_H(\rho_2, \rho_3)) = A_H(A_H(\rho_1, \rho_2), \rho_3) + A_H(\rho_2, A_H(\rho_1, \rho_3)), $$

for any closed $H \in \Omega^3(M)$, $\rho_1, \rho_2, \rho_3 \in \mathcal{X}(M) \oplus \Omega^1(M)$ and $M \in \text{obj}(\mathcal{M}f_m)$.

From now on, $(x^i) (i = 1, \ldots, m)$ denote the usual coordinates on $\mathbb{R}^m$ and $\partial_i = \frac{\partial}{\partial x^i}$ are the canonical vector fields on $\mathbb{R}^m$.

All manifolds considered in this paper are assumed to be finite dimensional second countable Hausdorff without boundary and smooth (of class $C^\infty$). Maps between manifolds are assumed to be smooth (of class $C^\infty$)

2. The Natural Bilinear Operators Similar to the Courant Bracket

The general concept of natural operators can be found in the fundamental monograph [5]. In the paper, we need two particular cases of natural operators presented in Definitions 2.1 (below) and 3.1 (in the next section).

Let $\mathcal{M}f_m$ be the category of $m$-dimensional $C^\infty$ manifolds as objects and their immersions of class $C^\infty$ as morphisms ($\mathcal{M}f_m$-maps).

Definition 2.1. A natural (called also $\mathcal{M}f_m$-natural) operator $A$ sending pairs of couples of vector fields and 1-forms on $m$-manifolds $M$ into couples of vector fields and 1-forms on $M$ is a $\mathcal{M}f_m$-invariant family of operators (functions)

$$A : (\mathcal{X}(M) \oplus \Omega^1(M)) \times (\mathcal{X}(M) \oplus \Omega^1(M)) \to \mathcal{X}(M) \oplus \Omega^1(M),$$

for all $m$-manifolds $M$, where $\mathcal{X}(M) \oplus \Omega^1(M)$ is the vector space of couples $(X, \omega)$ of vector fields $X$ on $M$ and 1-forms $\omega$ on $M$. Such $\mathcal{M}f_m$-natural operator $A$ is called bilinear if $A$ is bilinear (i.e., $A(\rho^1, \cdot)$ and $A(\cdot, \rho^2)$ are linear (over the field $\mathbb{R}$ of real numbers) functions $\mathcal{X}(M) \oplus \Omega^1(M) \to \mathcal{X}(M) \oplus \Omega^1(M)$ for any fixed $\rho^1, \rho^2 \in \mathcal{X}(M) \oplus \Omega^1(M)$) for any $m$-manifold $M$. Such $\mathcal{M}f_m$-natural operator $A$ is called skew-symmetric if $A$ is skew-symmetric for any $m$-manifold $M$. 


The $\mathcal{M}_f^m$-invariance of $A$ means that if $(X^1 \oplus \omega^1, X^2 \oplus \omega^2)$ and $(\overline{X}^1 \oplus \overline{\omega}^1, \overline{X}^2 \oplus \overline{\omega}^2)$ are $\varphi$-related by an $\mathcal{M}_f^m$-map $\varphi : M \rightarrow \overline{M}$ (i.e., $\overline{X}^i \circ \varphi = T\varphi \circ X^i$ and $\overline{\omega}^i \circ \varphi = T^*\varphi \circ \omega^i$ for $i = 1, 2$), then so are $A(X^1 \oplus \omega^1, X^2 \oplus \omega^2)$ and $A(\overline{X}^1 \oplus \overline{\omega}^1, \overline{X}^2 \oplus \overline{\omega}^2)$.

The most important example of such $\mathcal{M}_f^m$-natural bilinear operator $A$ is given by the (skew-symmetric) Courant bracket $[-,-]^C$ for any $m$-manifold $M$.

**Example 2.2.** On the vector bundle $TM \oplus T^*M$ there exist canonical symmetric and skew-symmetric pairings

$$\langle X^1 \oplus \omega^1, X^2 \oplus \omega^2 \rangle_\pm = \frac{1}{2}(i_{X^2}\omega^1 \pm i_{X^1}\omega^2)$$

for any $X^1 \oplus \omega^1, X^2 \oplus \omega^2 \in \mathcal{X}(M) \oplus \Omega^1(M)$, where $i$ is the interior derivative. Further, the (skew-symmetric) Courant bracket is given by

$$[X^1 \oplus \omega^1, X^2 \oplus \omega^2]^C = [X^1, X^2] \oplus \left(\mathcal{L}_{X^2}\omega^1 - \mathcal{L}_{X^1}\omega^2 + d\langle X^1 \oplus \omega^1, X^2 \oplus \omega^2 \rangle_-\right)$$

for any $X^1 \oplus \omega^1, X^2 \oplus \omega^2 \in \mathcal{X}(M) \oplus \Omega^1(M)$, where $[ -, - ]$ is the usual bracket on vector fields, $\mathcal{L}$ is the Lie derivative and $d$ is the exterior derivative.

**Theorem 2.3** [2]. If $m \geq 2$, any $\mathcal{M}_f^m$-natural bilinear operator $A$ in the sense of Definition 2.1 is of the form

$$A(\rho^1, \rho^2) = a[X^1, X^2] \oplus \left(b_1\mathcal{L}_{X^2}\omega^1 + b_2\mathcal{L}_{X^1}\omega^2 + b_3 d\langle \rho^1, \rho^2 \rangle_+ + b_4 d\langle \rho^1, \rho^2 \rangle_-\right)$$

for (uniquely determined by $A$) real numbers $a, b_1, b_2, b_3, b_4$, where $\rho^i = X^i \oplus \omega^i \in \mathcal{X}(M) \oplus \Omega^1(M)$ for $i = 1, 2$ are arbitrary, and where $\langle -, - \rangle_+$ and $\langle -, - \rangle_-$ are as in Example 2.2.

**Corollary 2.4** [2]. If $m \geq 2$, any $\mathcal{M}_f^m$-natural skew-symmetric bilinear operator $A$ in the sense of Definition 2.1 is of the form

$$A(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = a[X^1, X^2] \oplus (b(\mathcal{L}_{X^1}\omega^2 - \mathcal{L}_{X^2}\omega^1) + cd\langle X^2 \oplus \omega^1, X^1 \oplus \omega^2 \rangle_-)$$

for (uniquely determined by $A$) real numbers $a, b, c$.

Roughly speaking, Corollary 2.4 says that if $m \geq 2$, then any $\mathcal{M}_f^m$-natural skew-symmetric bilinear operator $A$ in the sense of Definition 2.1 coincides with the one given by Courant bracket $[-,-]^C$ up to three real constants.

**Definition 2.5.** A $\mathcal{M}_f^m$-natural bilinear operator $A$ in the sense of Definition 2.1 satisfies the Leibniz rule if

$$A(\rho_1, A(\rho_2, \rho_3)) = A(A(\rho_1, \rho_2), \rho_3) + A(\rho_2, A(\rho_1, \rho_3))$$

for all $\rho_1, \rho_2, \rho_3 \in \mathcal{X}(M) \oplus \Omega^1(M)$ and all $m$-manifolds $M$.

Of course, in the case of skew-symmetric bilinear $A$ the Leibniz rule is equivalent to the Jacobi identity $\sum_{\text{cycl}(\rho_1, \rho_2, \rho_3)} A(\rho_1, A(\rho_2, \rho_3)) = 0$. Machines
Example 2.6. The (not skew-symmetric) Courant bracket given by

\[ [X^1 \oplus \omega^1, X^2 \oplus \omega^2]_0 \]

\[ := [X^1, X^2] \oplus (\mathcal{L}_{X^1} \omega^2 - i_{X^2} \omega^1), \]

where \( X^i \oplus \omega^i \in \mathcal{X}(M) \oplus \Omega^1(M) \), satisfies the Leibniz rule, see [7,8].

The Courant bracket \([-,-]_C\) from Example 2.2 does not satisfy the Leibniz rule.

Theorem 2.7 [2]. If \( m \geq 2 \), any \( \mathcal{M}_{f_m} \)-natural bilinear operator \( A \) in the sense of Definition 2.1 satisfying the Leibniz rule is one of the following ones:

\begin{align*}
A^{(1,a)}(\rho^1, \rho^2) &= a[X^1, X^2] \oplus 0, \\
A^{(2,a)}(\rho^1, \rho^2) &= a[X^1, X^2] \oplus (\mathcal{L}_{X^1} \omega^2 - i_{X^2} \omega^1), \\
A^{(3,a)}(\rho^1, \rho^2) &= [\pi \rho_1, \pi \rho_2], \\
A^{(4,a,0)}(\rho^1, \rho^2) &= a[X^1, X^2] \oplus (\mathcal{L}_{X^1} \omega^2 - i_{X^2} \omega^1),
\end{align*}

where \( a \) is an arbitrary real number, and where \( \rho^1 = X^1 \oplus \omega^1 \) and \( \rho^2 = X^2 \oplus \omega^2 \).

Corollary 2.8. If \( m \geq 2 \), the Courant bracket \([-,-]_0\) from Example 2.6 for \( m \)-manifolds \( M \) is the unique \( \mathcal{M}_{f_m} \)-natural bilinear operator \( A \) in the sense of Definition 2.1 satisfying the conditions:

(A1) \( A(\rho_1, A(\rho_2, \rho_3)) = A(\rho_1, A(\rho_2, \rho_3)) + A(\rho_2, A(\rho_1, \rho_3)), \)

(A2) \( \pi A(\rho_1, \rho_2) = [\pi \rho_1, \pi \rho_2], \)

(A3) \( A(\rho_1, \rho_1) = i_0 \delta (\rho_1, \rho_1)_+, \)

for all \( \rho_1, \rho_2, \rho_3 \in \mathcal{X}(M) \oplus \Omega^1(M) \) and all \( m \)-manifolds \( M \), where \( \langle -,- \rangle_+ \) is the pairing of Example 2.2, \( \pi : TM \oplus T^*M \to TM \) is the fibred projection given by \( \pi(v, \omega) = v \) and \( i_0 : T^*M \to TM \oplus T^*M \) is the fibred embedding \( i_0(\omega) = (0,0) \).

Consequently, if \( m \geq 2 \), then a \( \mathcal{M}_{f_m} \)-natural bilinear operator \( A \) in the sense of Definition 2.1 satisfying the conditions (A1)–(A3) satisfies the conditions:

(A4) \( \pi \rho_1 \langle \rho_2, \rho_3 \rangle_+ = \langle A(\rho_1, \rho_2), \rho_3 \rangle_+ + \langle \rho_2, A(\rho_1, \rho_3) \rangle_+, \)

(A5) \( A(\rho_1, f \rho_2) = \pi \rho_1 (f) \rho_2 + f A(\rho_1, \rho_2) \)

for all \( \rho_1, \rho_2 \in \mathcal{X}(M) \oplus \Omega^1(M) \), all \( f \in C^\infty(M) \) and all \( m \)-manifolds \( M \) (i.e., putting \( \llbracket -,- \rrbracket : A \to \ker \delta \llbracket -,- \rrbracket \) := \( E = (TM \oplus T^*M, \llbracket -,- \rrbracket, \langle -,- \rangle_+, \pi, i_0 \) in the sense of [8] for any \( m \)-manifold \( M \)).

Proof. By Theorem 2.7, the conditions (A1) and (A2) imply that \( A = A^{(1,1)} \) or \( A = A^{(2,1)} \) or \( A = A^{(3,1)} \) or \( A = A^{(4,1,0)} \). On the other hand if \( \rho_1 = X \oplus \omega, \) then \( i_0 \delta (\rho_1, \rho_1)_+ = 0 \oplus d_i X \omega \) and \( A^{(1,1)}(\rho_1, \rho_1) = 0 \oplus 0 \) and \( A^{(2,1)}(\rho_1, \rho_1) = 0 \oplus 0 \) and \( A^{(3,1)}(\rho_1, \rho_1) = 0 \oplus \mathcal{L}_X \omega \) and \( A^{(4,1,0)}(\rho_1, \rho_1) = 0 \oplus d_i X \omega \). Then \( A = A^{(4,1,0)}. \) \( \square \)

Corollary 2.9. If \( m \geq 2 \), any \( \mathcal{M}_{f_m} \)-natural Lie algebra brackets on \( \mathcal{X}(M) \oplus \Omega^1(M) \) (i.e., \( \mathcal{M}_{f_m} \)-natural skew-symmetric bilinear operator satisfying the
Lemma 2.10. \ni

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Theorem (Leibniz rule) is the constant multiple of the one of the following two Lie algebra brackets:

\[ [[X^1 \otimes \omega^1, X^2 \otimes \omega^2]]_1 = [X^1, X^2] \oplus 0, \]
\[ [[X^1 \otimes \omega^1, X^2 \otimes \omega^2]]_2 = [X^1, X^2] \oplus (\mathcal{L}_{X^1}\omega^2 - \mathcal{L}_{X^2}\omega^1). \]

At the end of this section we are going to describe completely all Lie algebroids \((TM \otimes T^*M, [[-,-]], a)\) which are invariant with respect to immersions \((\mathcal{M}f_m\text{-maps})\). The concept of Lie algebroids can be found in the fundamental book [6].

Of course, the anchor \(a : TM \oplus T^*M \to TM\) for all \(m\)-manifolds \(M\) must be \(\mathcal{M}f_m\)-natural transformation \(\text{i.e.}, Tf \circ a = a \circ (Tf \oplus T^*f)\) for any \(\mathcal{M}f_m\)-map \(f : M \to M^1\) and fibre linear. By Corollary 2.9, \([[,-,-]] = \mu[[[-,-]]_1 or [[-,-]] = [\mu[[[-,-]]_2 for some \(\mu \in \mathbb{R}\).

Lemma 2.10. Any \(\mathcal{M}f_m\)-natural transformation \(a : TM \oplus T^*M \to TM\) which is fibre linear is the constant multiple of the fibre projection \(\pi : TM \oplus T^*M \to TM\).

Proof. Clearly, \(a\) is determined by the values \(< \eta, a_x(v, \omega) > \in \mathbb{R}\) for all \(\omega, \eta \in T^*_xM, v \in T_xM, x \in M, M \in \text{Obj}(\mathcal{M}f_m)\). By the standard chart arguments, we may assume \(M = \mathbb{R}^m, x = 0, \eta = d_0x^n\). We can write \(< d_0x^1, a_0(v, \omega) > = \sum_{\alpha} \alpha_i v^i + \sum_{\beta} \beta^j \omega_j\), where \(v^i\) are the coordinates of \(v\) and \(\omega_j\) are the coordinates of \(\omega\), and where \(\alpha_i\) and \(\beta^j\) are the real numbers determined by \(a_0\). Then using the invariance of \(a_0\) with respect to the maps \((\tau^1x^1, ..., \tau^m x^m)\) for \(\tau^1 > 0, ..., \tau^m > 0\) we deduce that \(\alpha_2 = \cdots = \alpha_m = 0\) and \(\beta_1 = \cdots = \beta_m = 0\). Then the vector space of all \(a\) in question is at most 1-dimensional. Thus the dimension argument completes the proof.

\[ a = k\pi \]

So, \(a = k\pi\) for some real number \(k\). It must be \(a([[X^1 \oplus 0, X^2 \oplus 0]]) = [a(X^1 \oplus 0), a(X^2 \oplus 0)]\) for any vector fields \(X^1\) and \(X^2\) on \(M\). This gives the condition \(k\mu[1, X^2] = k^2[X^1, X^2]\). Then \(k\mu = k^2\), and then \((k = 0\) and \(\mu = k\)) or \((k \neq 0\) and \(\mu = k\)). Consider two cases:

1. \([[,-,-]] = \mu [[-,-]]_1. Let \(\rho^1 = X^1 \oplus \omega^1\) and \(\rho^2 = X^2 \oplus \omega^2\). It must be \([[\rho^1, f\rho^2]] = a(\rho^1(f)\rho^2 + f[[\rho^1, \rho^2]]). \) Considering the \(\Omega^1(M)\)-parts of both sides of this equality we get \(0 = kX^1(f)\omega^2 + 0\) for any vector fields \(X^1, X^2\) on \(M\) any map \(f : M \to \mathbb{R}\) and any \(\omega^1, \omega^2 \in \Omega^1(M)\). Then \(k = 0\). Then considering the \(\mathcal{X}(M)\)-parts we get \(\mu[X^1, fX^2] = f\mu[X^1, X^2]\). Then \(\mu X^1(f)X^2 = 0\) for all vector fields \(X^1\) and \(X^2\) on \(M\) and all maps \(f : M \to \mathbb{R}\), i.e., \(\mu = 0\).

2. \([[,-,-]] = \mu [[-,-]]_2. Let \(\rho^1 = 0 \oplus \omega^1\) and \(\rho^2 = X^2 \oplus 0\). It must be \([[\rho^1, f\rho^2]] = a(\rho^1(f)\rho^2 + f[[\rho^1, \rho^2]]). \) Considering the \(\Omega^1(M)\)-parts of both sides of this equality we get \(-\mu\mathcal{L}_{fX^2}\omega^1 = -\mu f\mathcal{L}_{X^2}\omega^1\). Then \(\mu = 0\) or \(d_i fX^2\omega^1 + i_{dX^2} f\omega^1 = f d_i X^2\omega^1 + f i_{X^2} \omega^1\). Putting \(\omega^1 = dg\) we get \(\mu = 0\) or \(d_i fX^2\omega^1 = f d_i X^2\omega^1\). Then \(\mu = 0\) or \(d_i X^2 g = f d_i X^2 g\). Then \(\mu = 0\) or \(X^2 g \odot f = 0\) for any \(X^2, g, f\) in question. Putting \(X^2 = \frac{\partial}{\partial x}r\) and \(f = g = x^1\) we get \(\mu = 0\) or \(dx^1 = 0\). Then \(\mu = 0\), and then \(k = \mu = 0\).

On the other hand one can directly show that \((TM \oplus T^*M, 0 [[-,-]]_1, 0 \pi)\) is a Lie algebroid. Thus we have
Proposition 2.11. If $m \geq 2$, $(TM \otimes T^*M, 0, 0)$ is the only invariant with respect to $\mathcal{M}f_m$-maps Lie algebroid $(EM, [[-,-]], a)$ with $EM = TM \oplus T^*M$.

3. The Natural Operators Similar to the Twisted Courant Bracket

Definition 3.1. A $\mathcal{M}f_m$-natural operator $A$ sending $p$-forms $H \in \Omega^p(M)$ on $m$-manifolds $M$ into bilinear operators

$$A_H : (\mathcal{X}(M) \oplus \Omega^1(M)) \times (\mathcal{X}(M) \oplus \Omega^1(M)) \to \mathcal{X}(M) \oplus \Omega^1(M),$$

is a $\mathcal{M}f_m$-invariant family of regular operators (functions)

$$A : \Omega^p(M) \to \text{Lin}_2((\mathcal{X}(M) \oplus \Omega^1(M)) \times (\mathcal{X}(M) \oplus \Omega^1(M)), \mathcal{X}(M) \oplus \Omega^1(M))$$

for all $m$-manifolds $M$, where $\text{Lin}_2(U \times V, W)$ denotes the vector space of all bilinear (over $\mathbb{R}$) functions $U \times V \to W$ for any real vector spaces $U, V, W$.

The $\mathcal{M}f_m$-invariance of $A$ means that if $H^1 \in \Omega^p(M)$ and $H^2 \in \Omega^p(M)$ are $\varphi$-related and $(X^1 \oplus \omega^1, X^2 \oplus \omega^2)$ and $(\overline{X}^1 \oplus \overline{\omega}^1, \overline{X}^2 \oplus \overline{\omega}^2)$ are $\varphi$-related by an $\mathcal{M}f_m$-map $\varphi : M \to \overline{M}$, then so are $A_{H^1}(X^1 \oplus \omega^1, X^2 \oplus \omega^2)$ and $A_{H^2}(\overline{X}^1 \oplus \overline{\omega}^1, \overline{X}^2 \oplus \overline{\omega}^2)$.

The regularity of $A$ means that it transforms smoothly parametrized families $(H_t, X^i_t \oplus \omega^i_t, X^j_t \oplus \omega^j_t)$ into smoothly parametrized families $A_{H_t}(X^i_t \oplus \omega^i_t, X^j_t \oplus \omega^j_t)$.

Example 3.2. The most important example of $\mathcal{M}f_m$-natural operator in the sense of Definition 3.1 for $p = 3$ is given by the $H$-twisted Courant bracket

$$[X^1 \oplus \omega^1, X^2 \oplus \omega^2]_H := [X^1, X^2] \oplus (L_{X^1} \omega^2 - i_{X^2} d\omega^1 + i_{X^1} i_{X^2} H)$$

for all 3-forms $H \in \Omega^3(M)$ and all $m$-manifolds $M$. We call this $\mathcal{M}f_m$-natural operator the twisted Courant bracket $\mathcal{M}f_m$-natural operator.

Example 3.3. The operator given by $[-,-]_{dH}$ for all $H \in \Omega^2(M)$ and all $m$-manifolds $M$ is a $\mathcal{M}f_m$-natural operator in the sense of Definition 3.1 for $p = 2$.

The main result of this section is the following

Theorem 3.4. Assume $m \geq 3$. Then we have:

1. Any $\mathcal{M}f_m$-natural operator $A$ in the sense of Definition 3.1 for $p = 2$ such that $A_H = A_{H + dH}$ for any $H \in \Omega^2(M)$ and any $H^1 \in \Omega^1(M)$ is of the form

$$A_H(\rho^1, \rho^2) = a[X^1, X^2]$$

$$\oplus \left( b_1 L_{X^2} \omega^1 + b_2 L_{X^1} \omega^2 + b_3 d\langle \rho^1, \rho^2 \rangle_+ + b_4 d\langle \rho^1, \rho^2 \rangle_- + ci_{X^1} i_{X^2} dH \right),$$

for (uniquely determined by $A$) reals $a, b_1, ..., c$, where 2-forms $H \in \Omega^2(M)$, pairs $\rho^i = X^i \oplus \omega^i \in \mathcal{X}(M) \oplus \Omega^1(M)$ for $i = 1, 2$ and $m$-manifolds $M$ are arbitrary.
2. Any \( Mf_m \)-natural operator (not necessarily satisfying \( A_H = A_{H + dH^1} \)) in the sense of Definition 3.1 for \( p = 3 \) is of the form

\[
A_H(\rho^1, \rho^2) = a[X^1, X^2] \oplus \left( b_1\mathcal{L}X_2 \omega^1 + b_2\mathcal{L}X_1 \omega^2 + b_3d(\rho^1, \rho^2)_+ + b_4d(\rho^1, \rho^2)_- + c_i X_i X_2 H \right),
\]

for \( \rho \in \mathcal{A} \), where 3-forms \( H \in \Omega^3(M) \), pairs \( \rho^i = X^i \oplus \omega^i \in \mathcal{X}(M) \oplus \Omega^1(M) \) for \( i = 1, 2 \) and manifolds \( M \) are arbitrary.

3. If \( p \geq 4 \), any \( Mf_m \)-natural operator (not necessarily satisfying \( A_H = A_{H + dH^1} \)) in the sense of Definition 3.1 is of the form

\[
A_H(\rho^1, \rho^2) = a[X^1, X^2] \oplus \left( b_1\mathcal{L}X_2 \omega^1 + b_2\mathcal{L}X_1 \omega^2 + b_3d(\rho^1, \rho^2)_+ + b_4d(\rho^1, \rho^2)_- \right)
\]

for \( \rho \in \mathcal{A} \), where \( p \)-forms \( H \in \Omega^p(M) \), pairs \( \rho^i = X^i \oplus \omega^i \in \mathcal{X}(M) \oplus \Omega^1(M) \) for \( i = 1, 2 \) and manifolds \( M \) are arbitrary.

**Proof.** Clearly, \( A_0 \), where 0 is the zero \( p \)-form, can be treated as the bilinear operator in the sense of Definition 2.1. Then \( A_0 \) is described in Theorem 2.3. So we can replace \( A \) by \( A - A_0 \). In other words, we have assumption \( A_0 = 0 \).

By the invariance, \( A \) is determined by the values \( A_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2)|_0 \) for all \( H \in \Omega^p(R^m), X^i \oplus \omega^i \in \mathcal{X}(R^m) \oplus \Omega^1(R^m) \). Put

\[
A_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2)|_0 = \left( A_{H}^{1}(X^1 \oplus \omega^1, X^2 \oplus \omega^2)|_0, A_{H}^{2}(X^1 \oplus \omega^1, X^2 \oplus \omega^2)|_0 \right),
\]

where \( A_{H}^{1}(\ldots)|_0 \in T_0 R^m \) and \( A_{H}^{2}(\ldots)|_0 \in T^* R^m \). Then \( A \) is determined by

\[
\langle A_{H}^{1}(X^1 \oplus \omega^1, X^2 \oplus \omega^2)|_0, \eta \rangle \in \mathbb{R} \quad \text{and} \quad \langle A_{H}^{2}(X^1 \oplus \omega^1, X^2 \oplus \omega^2)|_0, \mu \rangle \in \mathbb{R}
\]

for all \( H \in \Omega^p(R^m), X^i \oplus \omega^i \in \mathcal{X}(R^m) \oplus \Omega^1(R^m), \eta \in T_0^* R^m, \mu \in T_0 R^m, i = 1, 2 \).

By the non-linear Peetre theorem, see [5], \( A \) is of finite order. It means that there is a finite number \( r \) such that from \( (j^i_x H = j^i_x \mathcal{H}, j^i_x(\rho^i) = j^i_x(\rho^i)), i = 1, 2 \) it follows \( A_H(\rho^1, \rho^2)|_x = A_{\mathcal{H}}(\rho^1, \rho^2)|_x \). So, we may assume that \( H, X^1, X^2, \omega^1, \omega^2 \) are polynomials of degree not more than \( r \).

Using the invariance of \( A \) with respect to the homotheties and the bilinearity of \( A_H \) (for given \( H \)) we obtain homogeneity condition

\[
\left\langle A_{(\frac{1}{t} \text{id})_*}^{1} H(t(\frac{1}{t} \text{id})_* X^1 \oplus t(\frac{1}{t} \text{id})_* \omega^1, t(\frac{1}{t} \text{id})_* X^2 \oplus t(\frac{1}{t} \text{id})_* \omega^2)|_0, \eta \right\rangle \quad = \quad t\left\langle A_{H}^{1}(X^1 \oplus \omega^1, X^2 \oplus \omega^2)|_0, \eta \right\rangle.
\]

Then, by the homogeneous function theorem, since \( A \) is of finite order and regular and \( A_0 = 0 \) and \( p \geq 2 \), we have \( \left\langle A_{H}^{1}(X^1 \oplus \omega^1, X^2 \oplus \omega^2)|_0, \eta \right\rangle = 0. \)
Using the same arguments we get homogeneity condition

\[
\left\langle A^2_{\left(\frac{1}{t}id\right)_*} H \left( t \left( \frac{1}{t}id \right)_* X^1 \oplus t \left( \frac{1}{t}id \right)_* \omega^1, t \left( \frac{1}{t}id \right)_* X^2 \right) \right\rangle_{\omega^1, \mu} = t^3 \left\langle A^2_H (X^1 \oplus \omega^1, X^2 \oplus \omega^2)_{\omega^1, \mu} \right\rangle.
\]

Then, if \( p = 2 \), by the homogeneous function theorem and the bilinearity of \( A_H \) and the assumptions \( A_0 = 0 \) and \( A_H = A_{H+1}H \), the value \( \left\langle A^2_H (X^1 \oplus \omega^1, X^2 \oplus \omega^2)_{\omega^1, \mu} \right\rangle \) depends quadrilinearly on \( X^1_{\omega^1}, X^2_{\omega^1}, j^0 \) and \( \mu \), only. By \( m \geq 3 \) and the regularity of \( A \), we may assume that \( X^1_{\omega^1}, X^2_{\omega^1} \) and \( \mu \) are linearly independent. Then by the invariance we may assume \( X^1_{\omega^1} = \partial_1_{\omega^1}, X^2_{\omega^1} = \partial_2_{\omega^1} \) and \( \mu = \partial_3_{\omega^1} \). Then \( A \) is determined by the values \( \left\langle A^2_{x_{1,i_1}x_{2,i_2}\lambda d_{x_{1,i_3}}} (\partial_1 \oplus 0, \partial_2 \oplus 0), \partial_3_{\omega^1} \right\rangle \) for all \( i_1 = 1, \ldots, m \) and \( i_2, i_3 \) with \( 1 \leq i_2 < i_3 \leq m \). Then using the invariance of \( A \) with respect to \( \tau^i \) for \( \tau^i > 0 \) we deduce that only \( v := \left\langle A^2_{x_{1,i_1}x_{2,i_2}\lambda d_{x_{1,i_3}}} (\partial_1 \oplus 0, \partial_2 \oplus 0), \partial_3_{\omega^1} \right\rangle, \ w := \left\langle A^2_{x_{1,i_1}x_{2,i_2}\lambda d_{x_{1,i_3}}} (\partial_1 \oplus 0, \partial_2 \oplus 0), \partial_3_{\omega^1} \right\rangle, \ z := \left\langle A^2_{x_{1,i_1}x_{2,i_2}\lambda d_{x_{1,i_3}}} (\partial_1 \oplus 0, \partial_2 \oplus 0), \partial_3_{\omega^1} \right\rangle \) may be not-zero. But \( x^1dx^1 \vee dx^2 = -x^2dx^1 \vee dx^3 + d(...) \). So, \( v = -w \). Similarly, \( v = -z \). Therefore the vector space of all \( A \) in question with \( A_0 = 0 \) and \( A_H = A_{H+1}H \) is at most one-dimensional. The part (1) of the theorem is complete. If \( p = 3 \), then (by almost the same arguments as for \( p = 2 \)) \( A \) is determined by the values \( \left\langle A^2_{x_{1,i_1}x_{2,i_2}\lambda d_{x_{1,i_3}}} (\partial_1 \oplus 0, \partial_2 \oplus 0), \partial_3_{\omega^1} \right\rangle \in \mathbb{R} \) for all \( i_1, i_2, i_3 \) with \( 1 \leq i_1 < i_2 < i_3 \leq m \). Then using the invariance with respect to \( (\tau^1x^1, \ldots, \tau^m x^m) \) for \( \tau^i > 0 \) we deduce that only the value \( \left\langle A^2_{x_{1,i_1}x_{2,i_2}\lambda d_{x_{1,i_3}}} (\partial_1 \oplus 0, \partial_2 \oplus 0), \partial_3_{\omega^1} \right\rangle \in \mathbb{R} \) may be not-zero. Therefore the vector space of all \( A \) in question with \( A_0 = 0 \) is one-dimensional (generated by the natural operator \( 0 \oplus i_{X^1}i_{X^2}H \)).

If \( p \geq 4 \), then (similarly as for \( p = 2 \)) \( \left\langle A^2_H (X^1 \oplus \omega^1, X^2 \oplus \omega^2)_{\omega^1, \mu} \right\rangle = 0 \).

Theorem 3.4 is complete. \( \square \)

**Corollary 3.5.** If \( m \geq 3 \), any \( \mathcal{M}f_m \)-natural operator \( A \) in the sense of Definition 3.1 for \( p = 3 \) such that \( A_H \) is skew-symmetric for any \( H \in \Omega^2(M) \) and any \( m \)-manifold \( M \) is of the form

\[
A_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = a[X^1, X^2] \oplus \left( b(L_{X^1}\omega^1 - L_{X^2}\omega^1) + cd \right) (X^2 \oplus \omega^1, X^1 \oplus \omega^2) + ei_{X^1}i_{X^2}H,
\]

for (uniquely determined by \( A \)) real numbers \( a, b, c, e \).

Roughly speaking, Corollary 3.5 says that any \( \mathcal{M}f_m \)-natural operator \( A \) in the sense of Definition 3.1 such that \( A_H \) is skew-symmetric for any \( H \in \Omega^2(M) \) and any \( m \)-manifold \( M \) coincides with the “skew-symmetrization” of the twisted Courant bracket \( \mathcal{M}f_m \)-natural operator up to four real constants \( a, b, c, e \).

**Corollary 3.6.** If \( m \geq 3 \), then the twisted Courant bracket \( \mathcal{M}f_m \)-natural operator from Example 3.2 is the unique \( \mathcal{M}f_m \)-natural operator \( A \) in the sense of Definition 3.1 for \( p = 3 \) satisfying the following properties:
Corollary 4.2. We have

\[
\text{(B1) } A_0(\rho_1, \rho_2) = [\rho_1, \rho_2]_0,
\]

\[
\text{(B2) } A_H(X \oplus 0, Y \oplus 0) = [X, Y] \oplus i_X i_Y H
\]

for all closed \( H \in \Omega^3_{cl}(M) \), all \( \rho_1, \rho_2, X \oplus 0, Y \oplus 0 \in \mathcal{X}(M) \oplus \Omega^1(M) \) and all \( m \)-manifolds \( M \), where \([\cdot, \cdot]_0\) is the \( \mathcal{M}_f\)-natural bilinear operator given by the (not skew-symmetric) Courant bracket as in Example 2.6.

Proof. Clearly, the twisted Courant bracket \( \mathcal{M}_f\)-natural operator satisfies (B1) and (B2). Consider \( A \) in question satisfying (B1) and (B2). Then by Theorem 3.4, there exist uniquely determined reals \( a, b_1, \ldots, c \) such that for all \( H \in \Omega^3(M) \) and \( m \)-manifolds \( M \)

\[
A_H(\rho^1, \rho^2) = a[X^1, X^2]
\]

\[
\oplus \left( b_1 L_{X^2} \omega^1 + b_2 L_{X^1} \omega^2 + b_3 d \langle \rho^1, \rho^2 \rangle_+ + b_4 d \langle \rho^1, \rho^2 \rangle_- + ci_{X^1} i_{X^2} H \right),
\]

where \( \rho^i = X^i \oplus \omega^i \in \mathcal{X}(M) \oplus \Omega^1(M) \) are arbitrary. Putting \( \omega^1 = \omega^2 = 0 \) we get \( A_H(\rho^1, \rho^2) = a[X^1, X^2] \oplus ci_{X^1} i_{X^2} H \). Then condition (B2) implies \( c = 1 \). Putting \( H = 0 \) we get

\[
A_0(\rho^1, \rho^2) = a[X^1, X^2] \oplus \left( b_1 L_{X^2} \omega^1 + b_2 L_{X^1} \omega^2 + b_3 d \langle \rho^1, \rho^2 \rangle_+ + b_4 d \langle \rho^1, \rho^2 \rangle_- \right)
\]

for all \( \rho^i = X^i \oplus \omega^i \in \mathcal{X}(M) \oplus \Omega^1(M) \) and all \( m \)-manifolds \( M \). But \( A_0 \) is a \( \mathcal{M}_f\)-natural bilinear operator in the sense of Definition 2.1. Then \( a, b_1, b_2, b_3, b_4 \) are uniquely determined because of Theorem 2.3. Then \( a, b_1, \ldots, c \) are uniquely determined. So, \( A \) is uniquely determined by conditions (B1) and (B2). \( \square \)

4. The Natural Operators Similar to the Twisted Courant Bracket and Defined for Closed \( p \)-Forms Only

In the previous section, we considered \( \mathcal{M}_f\)-natural operators \( A \) which are defined for all \( p \)-forms \( H \). In this section, we observe what happens if \( A \) are defined for closed \( p \)-forms \( H \), only. We start with the following

Definition 4.1. A \( \mathcal{M}_f\)-natural operator \( A \) sending closed \( p \)-forms \( H \in \Omega^p_{cl}(M) \) on \( m \)-manifolds \( M \) into bilinear operators

\( A_H : (\mathcal{X}(M) \oplus \Omega^1(M)) \times (\mathcal{X}(M) \oplus \Omega^1(M)) \rightarrow \mathcal{X}(M) \oplus \Omega^1(M) \),

is a \( \mathcal{M}_f\)-invariant family of regular operators (functions)

\( A : \Omega^p_{cl}(M) \rightarrow \text{Lin}_2((\mathcal{X}(M) \oplus \Omega^1(M)) \times (\mathcal{X}(M) \oplus \Omega^1(M)), \mathcal{X}(M) \oplus \Omega^1(M)) \),

for all \( m \)-manifolds \( M \).

We have the following corollary of Theorem 3.4.

Corollary 4.2. Assume \( m \geq 3 \). Then we have:

1. If \( p = 3 \), any \( \mathcal{M}_f\)-natural operator in the sense of Definition 4.1 is of the form

\[
A_H(\rho^1, \rho^2) = a[X^1, X^2]
\]

\[
\oplus (b_1 L_{X^2} \omega^1 + b_2 L_{X^1} \omega^2 + b_3 d \langle \rho^1, \rho^2 \rangle_+ + b_4 d \langle \rho^1, \rho^2 \rangle_- + ci_{X^1} i_{X^2} H),
\]
for uniquely determined by A reals $a, b_1, \ldots, c$, where closed 3-forms $H \in \Omega^3_{cl}(M)$, pairs $\rho^i = X^i \oplus \omega^i \in \mathcal{X}(M) \oplus \Omega^1(M)$ for $i = 1, 2$ and $m$-manifolds $M$ are arbitrary.

2. If $p \geq 4$, any $\mathcal{M}f_m$-natural operator in the sense of Definition 4.1 is of the form

$$A_H(\rho^1, \rho^2) = a[X^1, X^2]$$

$$\oplus \left( b_1 \mathcal{L}_{X^2}\omega^1 + b_2 \mathcal{L}_{X^1}\omega^2 + b_3 d\langle \rho^1, \rho^2 \rangle_+ + b_4 d\langle \rho^1, \rho^2 \rangle_- \right)$$

for uniquely determined by A reals $a, b_1, \ldots, b_4$, where closed $p$-forms $H \in \Omega^p_{cl}(M)$, pairs $\rho^i = X^i \oplus \omega^i$ for $i = 1, 2$ and $m$-manifolds $M$ are arbitrary.

**Proof.** Let $A$ be a $\mathcal{M}f_m$-natural operator in the sense of Definition 4.1 for $p$. Define a $\mathcal{M}f_m$-natural operator $A^1$ in the sense of Definition 3.1 for $p-1$ by $A^1_H = A_{d\tilde{H}}$. Then $A^1_{H+dH_1} = A^1_{\tilde{H}}$ for any $\tilde{H} \in \Omega^{p-1}(M)$ and $H_1 \in \Omega^{p-2}(M)$.

If $p = 3$, then by Theorem 3.4, $A^1$ is of the form

$$A^1_H(\rho^1, \rho^2) = a[X^1, X^2]$$

$$\oplus \left( b_1 \mathcal{L}_{X^2}\omega^1 + b_2 \mathcal{L}_{X^1}\omega^2 + b_3 d\langle \rho^1, \rho^2 \rangle_+ + b_4 d\langle \rho^1, \rho^2 \rangle_- + c_i X^1 i X^2 d\tilde{H} \right)$$

for uniquely determined reals $a, b_1, \ldots, c$ and all $\tilde{H} \in \Omega^2(M)$, where $\rho^i = X^i \oplus \omega^i$ for $i = 1, 2$. Then

$$A_H(\rho^1, \rho^2) = a[X^1, X^2]$$

$$\oplus \left( b_1 \mathcal{L}_{X^2}\omega^1 + b_2 \mathcal{L}_{X^1}\omega^2 + b_3 d\langle \rho^1, \rho^2 \rangle_+ + b_4 d\langle \rho^1, \rho^2 \rangle_- + c_i X^1 i X^2 H \right)$$

for all exact 3-forms $H$, where $\rho^i = X^i \oplus \omega^i$ for $i = 1, 2$. But by the locality of $A$ and the Poincare lemma we may replace the phrase “all exact 3-forms” by “all closed 3-forms”.

If $p \geq 4$, then by Theorem 3.4, $A^1$ is of the form

$$A^1_H(\rho^1, \rho^2) = a[X^1, X^2]$$

$$\oplus \left( b_1 \mathcal{L}_{X^2}\omega^1 + b_2 \mathcal{L}_{X^1}\omega^2 + b_3 d\langle \rho^1, \rho^2 \rangle_+ + b_4 d\langle \rho^1, \rho^2 \rangle_- + c_i X^1 i X^2 d\tilde{H} \right)$$

for uniquely determined reals $a, b_1, \ldots, c$ (with arbitrary $c$ if $p = 4$ and with $c = 0$ if $p \geq 5$) and all $\tilde{H} \in \Omega^{p-1}(M)$, where $\rho^i = X^i \oplus \omega^i$ for $i = 1, 2$. The condition $A^1_H = A^1_{H+dH_1}$ implies $c_i X^1 i X^2 dH_1 = 0$ for any $H_1 \in \Omega^{p-2}(M)$. If $p = 4$, putting $X^2 = \partial_1$, $X^2 = \partial_2$ and $H_1 = x^1 dx^2 \land dx^3$, we get $c(-dx^3) = 0$, i.e., $c = 0$. If $p \geq 5$, then $c = 0$, see above. Next, we proceed similarly as in the case $p = 3$. □

The above corollary and Theorem 3.4 imply

**Theorem 4.3.** If $m \geq 3$ and $p \geq 3$ then any $\mathcal{M}f_m$-natural operator in the sense of Definition 4.1 can be extended uniquely to a $\mathcal{M}f_m$-natural operator in the sense of Definition 3.1.

Roughly speaking, if $m \geq 3$ and $p \geq 3$, then any $\mathcal{M}f_m$-natural operator in the sense of Definition 4.1 can be treated as the $\mathcal{M}f_m$-natural operator in the sense of Definition 3.1, and vice-versa.
5. The Natural Operators Similar to the Twisted Courant Bracket and Satisfying the Leibniz Rule for Closed 3-Forms

**Definition 5.1.** A $\mathcal{M}_m$-natural operator $A$ in the sense of Definition 3.1 (or equivalently in the sense of Definition 4.1) satisfies the Leibniz rule for closed $p$-forms if

$$A_H(\rho_1, A_H(\rho_2, \rho_3)) = A_H(A_H(\rho_1, \rho_2), \rho_3) + A_H(\rho_2, A_H(\rho_1, \rho_3))$$

for all $\rho_1, \rho_2, \rho_3 \in \mathcal{X}(M) \oplus \Omega^1(M)$, all closed $p$-forms $H \in \Omega^p_{\text{cl}}(M)$ and all $m$-manifolds $M$.

**Example 5.2.** The twisted Courant bracket $\mathcal{M}_m$-natural operator presented in Example 3.2 satisfies the Leibniz rule for closed 3-forms, see [3,8].

**Theorem 5.3.** If $m \geq 3$, any $\mathcal{M}_m$-natural operator $A$ in the sense of Definition 3.1 (or equivalently of Definition 4.1) for $p = 3$ satisfying the Leibniz rule for closed 3-forms is one of the $\mathcal{M}_m$-natural operators:

$$
\begin{align*}
A_H^{(1,a)}(\rho_1, \rho_2) &= a[X^1, X^2] \oplus 0, \\
A_H^{(2,a)}(\rho_1, \rho_2) &= a[X^1, X^2] \oplus (a(L_{X^1}\omega^2 - L_{X^2}\omega^1)), \\
A_H^{(3,a)}(\rho_1, \rho_2) &= a[X^1, X^2] \oplus (a(L_{X^1}\omega^2), \\
A_H^{(4,a,e)}(\rho_1, \rho_2) &= a[X^1, X^2] \oplus (a(L_{X^1}\omega^2 - i_{X^2}d\omega^1) + ei_{X^1}i_{X^2}H),
\end{align*}
$$

where $\rho_1 = X^1 \oplus \omega^1$ and $\rho_2 = X^2 \oplus \omega^2$, and $a$ and $e$ are arbitrary real numbers.

**Proof.** Let $A$ be a $\mathcal{M}_m$-natural operator in the sense of Definition 3.1 for $p = 3$ such that $A_H$ satisfies the Leibniz rule for any closed $H \in \Omega^3_{\text{cl}}(M)$. By Theorem 3.4, $A$ is of the form

$$
A_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = a[X^1, X^2] \\
\oplus (b_1 L_{X^2}\omega^1 + b_2 L_{X^1}\omega^2 + c_1 d_i X^2\omega^1 + c_2 d_i X^1\omega^2 + e i_{X^1}i_{X^2}H),
$$

for (uniquely determined by $A$) real numbers $a, b_1, b_2, c_1, c_2, e$. Then for any $X^1, X^2, X^3 \in \mathcal{X}(M)$ and $\omega^1, \omega^2, \omega^3 \in \Omega^1(M)$ we have

$$
\begin{align*}
A_H(X^1 \oplus \omega^1, A_H(X^2 \oplus \omega^2, X^3 \oplus \omega^3)) &= a^2[X^1, [X^2, X^3]] \oplus \Omega, \\
A_H(A_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2), X^3 \oplus \omega^3) &= a^2[[X^1, X^2], X^3] \oplus \Theta, \\
A_H(X^2 \oplus \omega^2, A_H(X^1 \oplus \omega^1, X^3 \oplus \omega^3)) &= a^2[X^2, [X^1, X^3]] \oplus \mathcal{T},
\end{align*}
$$
where
\[
\Omega = b_1\mathcal{L}_{a[X^2,X^3]}\omega^1 + c_1 di a[X^2,X^3]\omega^1 + ei X^1 i a[X^2,X^3]H
\]
\[
+ b_2 L_{X^1}(b_1 L_{X^2}\omega^2 + b_2 L_{X^2}\omega^3 + c_1 di X^3\omega^2 + c_2 di X^2\omega^3 + ei X^2 i X^3 H)
\]
\[
+ c_2 di X^1(b_1 L_{X^3}\omega^2 + b_2 L_{X^2}\omega^3 + c_1 di X^3\omega^2 + c_2 di X^2\omega^3 + ei X^2 i X^3 H),
\]
\[
\Theta = b_2 L_{a[X^1,X^2]}\omega^3 + c_2 di a[X^1,X^2]\omega^3 + ei a[X^1,X^2]i X^3 H
\]
\[
+ b_1 L_{X^2}(b_1 L_{X^2}\omega^1 + b_2 L_{X^1}\omega^2 + c_1 di X^3\omega^1 + c_2 di X^1\omega^2 + e X^1 i X^2 H)
\]
\[
+ c_1 di X^3(b_1 L_{X^2}\omega^1 + b_2 L_{X^1}\omega^2 + c_1 di X^3\omega^1 + c_2 di X^1\omega^2 + e X^1 i X^2 H),
\]
\[
T = b_1 L_{a[X^1,X^3]}\omega^3 + c_1 di a[X^1,X^3]\omega^3 + ei X^2 i a[X^1,X^3]H
\]
\[
+ b_2 L_{X^2}(b_1 L_{X^3}\omega^1 + b_2 L_{X^1}\omega^3 + c_1 di X^3\omega^1 + c_2 di X^1\omega^3 + e X^1 i X^3 H)
\]
\[
+ c_2 di X^2(b_1 L_{X^3}\omega^1 + b_2 L_{X^1}\omega^3 + c_1 di X^3\omega^1 + c_2 di X^1\omega^3 + e X^1 i X^3 H).
\]
The Leibniz rule of $A_H$ is equivalent to $\Omega = \Theta + T$.

Putting $H = 0$, we are in the situation of Theorem 2.7. Then by Theorem 2.7 (i.e., by Theorem 3.2 in [2]) we get $(b_1, b_2, c_1, c_2) = (0, 0, 0, 0)$ or $(b_1, b_2, c_1, c_2) = (a, 0, 0, 0)$ or $(b_1, b_2, c_1, c_2) = (−a, a, 0, 0)$ or $(b_1, b_2, c_1, c_2) = (−a, a, 0, 0)$. More, $A_0$ for such $(b_1, b_2, c_1, c_2)$ satisfies the Leibniz rule.

Therefore (as $c_2 = 0$) the Leibniz rule of $A_H$ is equivalent to the equality
\[
eai X^1 i a[X^2,X^3]H + b_2 e L_{X^1} i X^2 i X^3 H
\]
\[
eai X^1 i a[X^2,X^3]H + b_2 e L_{X^1} i X^2 i X^3 H + c_1 e di X^3 i X^1 i X^2 H
\]
\[
eai X^1 i a[X^2,X^3]H + b_2 e L_{X^1} i X^2 i X^3 H.
\]

If $(b_1, b_2, c_1, c_2) = (0, 0, 0, 0)$, the above equality is equivalent to
\[
eai X^1 i a[X^2,X^3]H = eai [X^1,X^2] i X^3 H + eai X^2 i [X^1,X^3] H.
\]

Putting $X^1 = \partial_1$, $X^2 = \partial_2$ and $X^3 = \partial_3$ we have $[X^2, X^3] = 0$, $[X^1, X^2] = 0$ and $[X^1, X^3] = 0$ and then $0 = eai \partial_1 i \partial_2 H$ for any closed $H$ (for example $H = dx^1 \wedge dx^2 \wedge dx^3$). Consequently $a = 0$ or $a = 0$.

If $(b_1, b_2, c_1, c_2) = (0, a, 0, 0)$, the above equality is equivalent to
\[
eai X^1 i a[X^2,X^3]H + a L_{X^1} i X^2 i X^3 H
\]
\[
eai X^1 i a[X^2,X^3]H + a L_{X^1} i X^2 i X^3 H + a L_{X^2} i X^1 i X^3 H.
\]

Putting $X^1 = \partial_1$, $X^2 = \partial_2$ and $X^3 = \partial_3$ and $H = x^2 dx^1 \wedge dx^2 \wedge dx^3$ (it is closed) we have $[X^2, X^3] = 0$, $[X^1, X^2] = 0$, $[X^1, X^3] = 0$, $L_{X^2} i X^1 i X^3 H = L_{\partial_2} x^2 dx^2 = dx^2$ and $L_{X^1} i X^2 i X^3 H = L_{\partial_1} (−x^2 dx^1) = 0$. Then $e ad x^2 = 0$. So, $a = 0$ or $e = 0$.

If $(b_1, b_2, c_1, c_2) = (−a, a, 0, 0)$, the above equality is equivalent to
\[
eai X^1 i a[X^2,X^3]H + a L_{X^1} i X^2 i X^3 H
\]
\[
eai X^1 i a[X^2,X^3]H + a L_{X^1} i X^2 i X^3 H + a L_{X^1} i X^2 i X^3 H.
\]

Putting $X^1 = \partial_1$, $X^2 = \partial_2$ and $X^3 = \partial_3$ and $H = x^2 dx^1 \wedge dx^2 \wedge dx^3$ we have (see above) $[X^2, X^3] = 0$, $[X^1, X^2] = 0$, $[X^1, X^3] = 0$, $L_{X^2} i X^1 i X^3 H = dx^2$, $L_{X^1} i X^2 i X^3 H = 0$ and $L_{X^3} i X^1 i X^2 H = L_{\partial_3} (−x^2 dx^3) = 0$. Then $e ad x^2 = 0$. So, $a = 0$ or $e = 0$. 


If \((b_1, b_2, c_1, c_2) = (-a, a, a, 0)\), the above equality is equivalent to
\[
e a \sum \left\{ i_{X_1} i_{[X_2, X_3]} H + \mathcal{L}_{X_1} i_{X_2} i_{X_3} H \right\} = e a d i_{X_1} i_{X_2} i_{X_3} H,
\]
where \(\sum\) is the cyclic sum \(\sum_{cyclic}(X_1, X_2, X_3)\). Then \(e\) is arbitrary real number because from \(dH = 0\) it follows
\[
\sum \left\{ i_{X_1} i_{[X_2, X_3]} H + \mathcal{L}_{X_1} i_{X_2} i_{X_3} H \right\} = di_{X_1} i_{X_2} i_{X_3} H.
\]
Indeed, using \(dH = 0\) and \(i_{[X_1, X_4]} = \mathcal{L}_{X_1} i_{X^4} - i_{X^4} \mathcal{L}_{X_1}\) and the well-known formula expressing \(dH(X^1, X^2, X^3, X^4)\), we have
\[
\sum \left\{ i_{X_4} i_{X_1} i_{[X_2, X_3]} H + i_{X_4} \mathcal{L}_{X_1} i_{X_2} i_{X_3} H \right\}
= \sum \left\{ i_{X_4} i_{X_1} i_{[X_2, X_3]} H + \mathcal{L}_{X_1} i_{X_4} i_{X_2} i_{X_3} H - i_{[X_1, X_4]} i_{X_2} i_{X_3} H \right\}
= 6 \sum \{ H([X^2, X^3], X^1, X^4) + X^1 H(X^3, X^2, X^4)
- H(X^3, X^2, [X^1, X^4])\}
= -24dH(X^1, X^2, X^3, X^4) + 6X^1 H(X^3, X^2, X^1) = i_{X^4} di_{X_1} i_{X_2} i_{X_3} H.
\]
Summing up, given a real number \(a \neq 0\) we have \((b_1, b_2, c_1, c_2, e) = (0, 0, 0, 0, 0)\) or \((b_1, b_2, c_1, c_2, e) = (-a, a, 0, 0, 0)\) or \((b_1, b_2, c_1, c_2, e) = (-a, a, a, 0, e)\), where \(e\) may be arbitrary real number. If \(a = 0\) we have \((b_1, b_2, c_1, c_2, e) = (0, 0, 0, 0, e)\), where \(e\) may be arbitrary. Theorem 5.3 is complete. \(\square\)

**Corollary 5.4.** If \(m \geq 3\), then the twisted Courant bracket \(\mathcal{M}f_m\)-natural operator from Example 3.2 is the unique \(\mathcal{M}f_m\)-natural operator \(A\) in the sense of Definition 3.1 for \(p = 3\) satisfying the following conditions:

\[(C1)\quad A_H(\rho_1, A_H(\rho_2, \rho_3)) = A_H(A_H(\rho_1, \rho_2), \rho_3) + A_H(\rho_2, A_H(\rho_1, \rho_3)),\]

\[(C2)\quad A_H(X \oplus 0, Y \oplus 0) = [X, Y] \oplus i_{X^1 Y} H\]

for all \(\rho_1, \rho_2, \rho_3, X \oplus 0, Y \oplus 0 \in X(M) \oplus \Omega^1(M), \) all closed \(H \in \Omega^3_M\) and all \(m\)-manifolds \(M\).

**Proof.** Indeed, the condition \((C1)\) and Theorem 5.3 imply that \(A = A^{(1, a)}\) or \(A = A^{(2, a)}\) or \(A = A^{(3, a)}\) or \(A = A^{(4, a, e)}\) for some real numbers \(a\) and \(e\). Then \((C2)\) implies that \(A = A^{(4, a, e)}\) and \(a = 1\) and \(e = 1\) because \(A^{(1, a)}_H(X \oplus 0, Y \oplus 0) = a[X, Y] \oplus 0\) and \(A^{(2, a)}_H(X \oplus 0, Y \oplus 0) = a[X, Y] \oplus 0\) and \(A^{(3, a)}_H(X \oplus 0, Y \oplus 0) = a[X, Y] \oplus 0\) and \(A^{(4, a, e)}_H(X \oplus 0, Y \oplus 0) = a[X, Y] \oplus e i_{X^1 Y} H.\) \(\square\)

**Corollary 5.5.** If \(m \geq 3\), any \(\mathcal{M}f_m\)-natural operator \(A\) in the sense of Definition 3.1 for \(p = 3\) such that \(A_H\) is a Lie algebra bracket (i.e., it is skew-symmetric, bilinear and satisfying the Leibniz rule) for all closed 3-forms \(H \in \Omega^3_M\) and all \(m\)-manifolds \(M\) is one of the \(\mathcal{M}f_m\)-natural operators:

\[
A^{(1, a)}_H(\rho_1, \rho_2) = a[X^1, X^2] \oplus 0,
A^{(2, a)}_H(\rho^1, \rho^2) = a[X^1, X^2] \oplus (a(L_{X^1} \omega^2 - L_{X^2} \omega^1)),
A^{(4, a, e)}_H(\rho^1, \rho^2) = 0 \oplus e i_{X^1} i_{X^2} H,
\]

where \(\rho^1 = X^1 \oplus \omega^1\) and \(\rho^2 = X^2 \oplus \omega^2\), and \(a\) and \(e\) are arbitrary real numbers.
Corollary 5.6. If \( m \geq 3 \), any \( \mathcal{M}f_m \)-natural operator \( A \) in the sense of Definition 3.1 for \( p = 3 \) satisfying the Leibniz rule for all 3-forms \( H \) (or for all closed 3-forms and at least one non-closed 3-form) is one of the \( \mathcal{M}f_m \)-natural operators:

\[
\begin{align*}
A_H^{(1,a)}(\rho_1, \rho_2) &= a[X^1, X^2] \oplus 0, \\
A_H^{(2,a)}(\rho_1, \rho_2) &= a[X^1, X^2] \oplus (a(\mathcal{L}_{X^1} \omega^2 - \mathcal{L}_{X^2} \omega^1)), \\
A_H^{(3,a)}(\rho_1, \rho_2) &= a[X^1, X^2] \oplus (a \mathcal{L}_{X^1} \omega^2), \\
A_H^{(4,a,0)}(\rho_1, \rho_2) &= a[X^1, X^2] \oplus (a(\mathcal{L}_{X^1} \omega^2 - i_{X^2} d\omega^1)), \\
A_H^{(4,0,e)}(\rho_1, \rho_2) &= 0 \oplus e i_{X^1} i_{X^2} H,
\end{align*}
\]

where \( \rho_1 = X^1 \oplus \omega^1 \) and \( \rho_2 = X^2 \oplus \omega^2 \), and \( a \) and \( e \) are arbitrary real numbers.

Proof. It follows from Theorem 5.3. \( \square \)

Remark 5.7. It is well-known that given closed 3-form \( H \in \Omega^3_{cl}(M) \) on a \( m \)-manifold \( M \), the twisted Courant bracket \( [\cdot, \cdot]_H : \mathcal{X}(M) \oplus \Omega^1(M) \times \mathcal{X}(M) \oplus \Omega^1(M) \rightarrow \mathcal{X}(M) \oplus \Omega^1(M) \) is bilinear and satisfies the properties (A1)–(A5) from Corollary 2.8 for all \( \rho_1, \rho_2, \rho_3 \in \mathcal{X}(M) \oplus \Omega^1(M) \) and all \( f \in C^\infty(M) \), see [3,8], but \( [\cdot, \cdot]_H \neq [\cdot, \cdot]_0 \) if \( H \neq 0 \). Is it a contradiction with the uniqueness from Corollary 2.8? No, it is not. Indeed, \( [\cdot, \cdot]_H \) is not extendable to a \( \mathcal{M}f_m \)-natural bilinear operator in the sense of Definition 2.1 because it is invariant only with respect to \( \mathcal{M}f_m \)-maps \( \varphi : M \rightarrow M \) preserving \( H \), in fact.

Remark 5.8. By Corollary 5.5, given a closed 3-form \( H \) on \( M \), the skew-symmetric bracket \( [[X^1 \oplus \omega^1, X^2 \oplus \omega^2]](H) := 0 \oplus i_{X^1} i_{X^2} H \) satisfies the Leibniz rule. One can easily directly verify that \( (TM \oplus T^* M, e[[\cdot, \cdot]](H), 0\pi) \) for arbitrary fixed \( e \in \mathbb{R} \) and closed 3-form \( H \) is a Lie algebroid canonically depending on \( H \). So, if we have a closed 3-form \( H \) on a \( m \)-manifold \( M \), we can construct canonical (in \( H \)) Lie algebroids \( (EM, [[\cdot, \cdot]](H), a(H)) \) with \( EM = TM \oplus T^* M \) different than the one from Proposition 2.11.

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References

[1] Courant, T.: Dirac manifolds. Trans. Am. Math. Soc. 319(631), 631–661 (1990)

[2] Doupovec, M., Kurek, J., Mikulski, W.M.: The natural brackets on couples of vector fields and 1-forms. Turk. J. Math. 42(2), 1853–1862 (2018)

[3] Gualtieri, M.: Generalized complex geometry. Ann. Math. 174(1), 75–123 (2011)

[4] Hitchin, N.: Generalized Calabi–Yau manifolds. Q. J. Math. 54(3), 281–308 (2003)

[5] Kolář, I., Michor, P.W., Slovák, J.: Natural Operations in Differential Geometry. Springer, Berlin (1993)

[6] Mackenzie, K.C.H.: General Theory of Lie Groupoids and Lie Algebroids. London Math. Soc., Lecture Note 213. Cambridge University Press, Cambridge (2005)

[7] Liu, Z.J., Weinstein, A., Xu, P.: Main triples for Lie bialgebroids. J. Differ. Geom. 45, 547–574 (1997)

[8] Ševera, P., Weinstein, A.: Poisson geometry with a 3-form background. Progr. Teoret. Phys. Suppl., 145–154 (2001)

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