On exact solution for some integrable nonlinear equations of the Schrödinger type.

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Abstract

The outlook of a simple method to generate localized (soliton-like) potentials of time-dependent Schrödinger type equations is given. The conditions are discussed for the potentials to be real and nonsingular. For the derivative Schrödinger equation also is discussed its relation to the Ishimori-II model. Some peculiar soliton solutions of nonlinear Schrödinger type equations are given and discussed.

Four years ago a paper on exact solutions of a time-dependent Schrödinger equation with self consistent potentials was published in the journal, Particles and Nuclei [1]. In this work the method is developed to construct such solutions along with the nonlinear equations they obey.

Though the paper was of a survey character it contained a series of original results. Talking at International Conferences and in some European Centres I realized that these results were practically unknown to the audience. This makes me to look over the paper [1] and try to rewrite its main body adding some recent development in this direction. So the first part of the present paper reproduces in a more simple way (up to author’s opinion) the results of [1] related to studying the time-dependent linear Schrödinger Equation (TLSE)

\[ i\partial_t - \partial^2_x + U(x, t)\Psi(x, t, k) = 0. \] (1)

Only solution-type solutions will be discussed for the sake of simplicity. For finite zone solutions see [2,3]. And in the second part we give an outlook of how to extend this approach to include the equation (Derivate Schrödinger Equation, TDSE)

\[ [i\partial_t - \partial^2_x + iU(x, t)\partial_x]\Psi(x, t, k) = 0 \] (2)

which gets popular now due to its connection to some 2+1 dimensional models (for Ishimori-II and modified KP models, see, eg.[2]).
First of all we stress that there are two different levels of consideration: linear and nonlinear. On the first we shall find a special class of localized reflectionless (Bargman’s) potentials along with their wave-functions. They are defined, conditionally speaking, by certain “spectral data” (SD), namely, by a set of complex numbers \( \kappa_i \) \( i = 1, N \) and a complex valued \( N \times N \) matrix \( c_{ij} \) (normalization matrix). We give the conditions which \( \kappa_i \) and \( c_{ij} \) have to satisfy in order to the potential \( U \) to be real and nonsingular together with w.f. \( \Psi \). We also discuss degeneration of the solution with respect to the S.D. and give two possible representations of the w.f. viz., the polynomial one and the rational (pole-type) one. Asymptotic properties of the solutions allow us to judge of the structural units (bricks) which the solutions are constituted of.

On the second nonlinear level self-consistency conditions are found which relate the potential to the w.f. and its residues. Here boundary conditions for nonlinear fields play the crucial role and the fields are given as direct sums of the structural units.

Some simple concrete examples are discussed with special emphasis on peculiar bi-pole solutions.

1 Linear level.

1. We look for the soliton like solutions of eq. (1) via the plane wave Ansatz

\[
\Psi(x, t, k) = P(x, t, k)e^{ik(x+kt)} \equiv (k^n + a_{n-1}(x, t)k^{n-1} + \cdots + a_0(x, t))e^{ik(x+kt)}
\]

and by specifying the SD: poles \( \kappa_i, \ i = 1, N \) with nonzero imaginary parts and a \( N \times N \) matrix \( \alpha_{ij} \).

Generalizing the notion of the Beiker-Akhiezer function we assume the w.f. \( \Phi \) to satisfy the following \( N \) conditions

\[
\Psi(\bar{\kappa}_i) = -\sum_{j=1}^{N} b_{ij} \Psi(\kappa_j), \ i, j = 1, \cdots, N
\]

First proposition.

Given with (3) and (4) w.f. \( \Psi(k, x, t) \) defines the potential \( U(x, t) \) as follows

\[
U(x, t) = 2i\partial_x a_{N-1}(x, t)
\]

The proof is very easy, and can be found in [1] (and even earlier in [3] for the B-A. function).

The representation (3) of the w.f. we call polynomial and now it’s of use to introduce the other

\[
\Psi = \frac{\Psi(k, x, t)}{\prod_{i=1}^{N}(k - \kappa_i)} \equiv \left\{ 1 + \sum_{j=1}^{N} \frac{r_j(x, t)}{k - \kappa_j} \right\} e^{ik(x+kt)}
\]
which naturally may be named the rational or pole-type representation. We need both of them in what follows.

In the pole type representation eq.(4) assumes the form

$$\Psi(x, t, \bar{\kappa}) = - \sum_{j=1}^{N} c_{ij} \Psi_j(x, t) \quad (7)$$

with

$$\Psi_j = \text{res}_{k=\kappa_j} \Psi(x, t, k) = \lim_{k \to \kappa_j} [(k - \kappa_j) \Psi(x, t, k)] \quad (8)$$

and

$$c_{ij} = b_{ij} \frac{R'(\kappa_j)}{R(\bar{\kappa}_j)} \quad (9)$$

$$R(k) = \prod_{j=1}^{N} (k - \kappa_j), \quad R' = \frac{d}{dk} R(k) \quad (10)$$

The SD $\kappa_i$ and $c_{ij}$ are still arbitrary (with to the above exception).

**Second proposition.**

In order to the potential $U(x, t)$ be real and non-singular for all real $(x, t)$ the following conditions are sufficient

1) The matrix $c_{ij}$ in (8) should be anti-Hermitian: $c = -c^+$.  
2) Let the poles $\kappa_i$ be arranged such that $\text{Im } \kappa_i > 0$ for $i = 1, \ldots, p$ and $\text{Im } \kappa_j < 0$ for $j = p+1, N$ then the Hermitian matrix $(i^{-1}c_{ij})$, $i, j = 1, p$ should be positive definite and the $((i^{-1}c_{ij})$, $i, j = p+1, N$ negative definite.

The proof is more sophisticated and also given in [1].

These conditions gives the first real limitation on the location of the poles related to the form of $c_{ij}$.

The following properties of the solution obtained have to be mentioned.

a) The solution is degenerate. In polynomial representation (3) both the w.f. $\Psi(x, t, k)$ and the potential $U(x, t)$ are $2^N$-fold degenerate with respect to changing the SD.

In the pole-type representation only the potential $U(x, t)$ is left $2^N$-fold degenerate, i.e. for $2^N$ definite but different sets of the SD we have the same potential $U(x, t)$.

Transformations from one set to another goes as follows [1]. Let the matrix $\alpha_{ij}$ be given in the block form

$$\begin{pmatrix} \alpha_+ \\ \gamma \\ \alpha_- \end{pmatrix}$$

$$a_{ij} = \begin{pmatrix} \alpha_+ & \beta \\ \gamma & \alpha_- \end{pmatrix}$$

3
where the square matrices $\alpha_+$ and $\alpha_-$ are $p \times p$ and $(N-p) \times (N-p)$ dimensional respectively (recall that $\text{Im } \kappa_i > 0$ for $i = 1, p$, and $\text{Im } \kappa_i < 0$ if $i = p+1, N$) and $\det \alpha_- \neq 0$, then the transformations are \{\kappa_i, b_{ij}\} \Rightarrow \{\kappa_i', b_{ij}'\}$ such that

$$
\kappa_i' = \begin{cases} 
\kappa_i & \text{for } i= \overline{1,p} \\
\tilde{\kappa}_i & \text{for } i= \overline{p+1,N}
\end{cases}
$$

and

$$(b_{ij}') = \begin{pmatrix} \alpha_+ - \beta \alpha_-^{-1} \gamma & \alpha_+^{-1} \gamma \\
\alpha_-^{-1} \gamma & \alpha_-^{-1} \end{pmatrix}
$$

b) One can consider the asymptotic behavior of the solution in $x$ and $t_i$ at arbitrary $N$.

i) In the simplest case $N = 1$, $\kappa = \alpha + i \beta$ the potential assumes soliton like form

$$
U = -2\beta^2 \cosh^{-2} \beta (x - x_0 + 2\alpha t),
$$

and w.f. is

$$
\Psi = \left[ 1 + \frac{i \beta}{k - \kappa} (1 + \text{th} \beta(x - x_0 + 2\alpha t)) \right] e^{i(k(x+kt))}
$$

$$
2i\beta c = -e^{2\beta x_0}
$$

For $N > 1$, all $\text{Im } \kappa_i > 0$ and $\text{Re } \kappa_i \neq \text{Re } \kappa_j$ if $i \neq j$ the potential asymptotically decays in a direct sum of the potentials (14). Hence such $N = 1$ potentials can be regarded as simple construction bricks for complexes with $N > 1$. Below we call them the solibricks.

ii) The next fundamental construction units are strings:

$N > 1$ $\text{Re } \kappa_i = \text{Re } \kappa_j$ which are periodic or quasiperiodic in time solutions [4] sometimes named the breathers. Both types of solutions are well-known.

iii) Now we proceed to a new type of construction units the bions. They are defined by an of diagonal matrix $c$, e.g. in the $N = 2$ case

$$
c = \begin{pmatrix} 0 & 1 \\
-1 & 0
\end{pmatrix}
$$

and are given by [5]:

$$
\Psi = \left(1 + \frac{c_3 \cos(q\xi + \Omega x + \theta_3) + c_4 e^{\beta \xi}}{c_2 \cosh(\beta\xi + \theta_2) + c_1 \cos(q\xi + \Omega x + \theta_1)} \right) e^{i(k(\xi + k'x))}
$$

with

$$
c_1 = -\left| \frac{c}{\tilde{\kappa}_{12}} \right|, \quad e^{\theta_2} = \left| \frac{\tilde{\kappa}_{12}}{c\tilde{\kappa}_{11}} \right| \sqrt{\frac{1}{\tilde{\kappa}_{11}\tilde{\kappa}_{22}}} \quad \text{4}
$$
The parameters defining solution (17) are

\[ \kappa_j = \alpha_j + i\beta_j \]

\[ q = \alpha_2 - \alpha_1, \quad \Omega = \omega - qv, \quad k' = k - v, \quad \beta = \beta_1 + \beta_2, \]

\[ \omega = \alpha_2^2 - \alpha_1^2 + \beta_1^2 - \beta_2^2, \quad \xi = t + vx, \]

\[ v = \frac{2\alpha_1\beta_1 + \alpha_2\beta_2}{\beta_1 + \beta_2} \]

\[ \tilde{\kappa}_{ij} = \tilde{\kappa}_i - \kappa_j, \quad \kappa_{ij} = \kappa_i - \kappa_j \]

and \( \alpha_i \) give velocities of the constituents, \( \beta_i \) their “masses”.

These solutions also are periodic or quasiperiodic, and we call them the bions. The nature of breathers and bions is strongly different, especially from the physical point of view (it will be evident on the nonlinear level). Though a breather and a bion are formed with two solibricks and both quasiperiodic in time the first can be easily destroyed to produce the constituents, for the second such a process is strongly forbidden. The constituents of the bion behave like quarks in a meson. Therefore we found so far three types of structural units: solibricks, breathers and bions (next are possible), and the potential asymptotically assumes the form

\[ U(x, t)_{t \to \infty} = \sum \text{solibricks} + \sum \text{breathers} + \sum \text{bions} + \cdots \] (18)

These results are directly applicable to construct soliton-like solutions of the Davey-Stewartson-I equation. First attempts to find such solutions were done in [6] and [7] and the dromions were born. In [8] some new results based on the approach proposed were obtained. In this sense one can consider the above technique as a constructive way to produce solutions in the scope of 2+1 dimensional KP and DS-I models.

2. Consider now the equation (2)

\[ (i\partial_t - \partial_x^2 \pm iU\partial_x)\Psi(x, t, k) \equiv L\Psi(x, t, k) = 0 \] (19)

Adopting the above procedure to (2) we construct its solutions through the Ansatz slightly different from (3)

\[ \Psi(x, t, k) = Q_N exp[i(k(x + kt))] \] (20)
with

\[ Q_N = a_N k^N + \cdots + a_1 k + 1 \]  \hspace{1cm} (21)

Specify again \( N \) poles \( \kappa_i \) and complex \( N \times N \) constant matrix \( b_{ij} \) we can prove the first proposition [9] for the w.f. \( \Psi \) (18) satisfying (4) but now

\[ U(x, t) = 2i \partial_x \ln a_N \]  \hspace{1cm} (22)

One can introduce again the pole-type representation

\[ \hat{\Psi} = a_N \left\{ 1 + \sum_{j=1}^{N} \frac{r_j(x, t)}{k - \kappa_j} \right\} e^{i(k+kt)} \]  \hspace{1cm} (23)

to obtain exactly the equations (7) for the function

\[ \Psi = a_N^{-1} \hat{\Psi} \]  \hspace{1cm} (24)

and then

\[ a_N = \frac{(-1)^N}{\prod_{j=1}^{N} \kappa_j} \frac{1}{1 - \sum_{j=1}^{N} \frac{r_j}{\kappa_j}} \]  \hspace{1cm} (25)

Denoting \( \Psi_0 = \Psi(x, t, k = 0) = 1 - \sum_{j=1}^{N} \frac{r_j}{\kappa_j} \) we have

\[ a_N = \prod_{j=1}^{N} \left( \frac{-1}{\kappa_j} \right) \frac{1}{\Psi_0} \]  \hspace{1cm} (26)

and

\[ U = -2i \partial_x \ln \Psi_0 \]  \hspace{1cm} (27)

\[ \hat{\Psi} = \prod_{j=1}^{N} \left( \frac{-1}{\kappa_j} \right) \frac{1}{\Psi_0} \Psi(x, t, k) \]  \hspace{1cm} (28)

From (25) it follows that \( U = \bar{U} \) when

\[ \left| 1 - \sum_{j=1}^{N} \frac{r_j}{\kappa_j} \right| = \text{const} \]  \hspace{1cm} (29)

General conditions are still unknown which the SD should satisfy in order to obtain a real and non-singular potential \( U(x, t) \).

So we give a simplest \( N = 1 \) example and find the connection of the such obtained one soliton solution with that of the Ishimori-II equation.

In the one soliton case system (4) gives

\[ a = \left( -\frac{1}{\kappa} \right) \frac{1 + b e^{i(\theta - \bar{\theta})}}{1 + b \kappa e^{i(\theta - \bar{\theta})}} \]  \hspace{1cm} (30)
The potential \( U \) is real when \(|a| = \text{const} \) or
\[
|a| = \text{const}
\] or \((b = b_1 + ib_2)\)
\[
b_1\beta + b_2\alpha = 0
\] (32)

By differentiating \( \ln a \) (30) we arrive at
\[
V = \frac{8|\alpha|\beta^2 \text{sgn}\alpha}{4\alpha^2 \cosh^2 \eta + \beta^2 e^{-2\eta}}, \quad b_1 = e^{-2\beta x_0} > 0
\] (33)
\[
U = -\frac{8|\alpha|\beta^2 \text{sgn}\beta}{4\alpha^2 \sinh^2 \eta + \beta^2 e^{-2\eta}}, \quad b_1 = -e^{-2\beta x_0} < 0
\] (34)
\[
\eta = \beta(x + 2\alpha \tau + x_0)
\] (35)
also we have
\[
\Psi = \left(1 - \frac{k_1 + be^{-2(\eta - \beta x_0)}}{\bar{k}_1 + be^{-2(\eta - \beta x_0)}}\right) e^{ik(x+\tau)}
\] (36)

If one sets \( k = \bar{k} \) then
\[
\Psi = \frac{2i}{2\alpha \cosh \eta + i\beta e^{-\eta}} \sqrt{b_1 \beta} e^{i\theta}, \quad \theta = i\alpha x + i(\alpha^2 - \beta^2)t
\] (37)
Fixing \( k = \kappa \) we come to the same expression up to the constant factor.

It should be pointed out that a general form for the w.f. of time dependent eq.(2) is that of (36) with arbitrary complex number \( k \), so the solution is a five real parameter \((\alpha, \beta, b_1, k_1, k_2)\) function.

3. We now utilize the solutions found in order to obtain such for the Ishimori-II model
\[
\vec{S}_t(x, y, t) + \vec{S} \wedge (\vec{S}_{xx} + \vec{S}_{yy}) + \varphi_x \vec{S}_y + \varphi_y \vec{S}_x = 0
\] (38)
\[
\varphi_{xx} - \varphi_{yy} + 2\vec{S}(\vec{S}_x \wedge \vec{S}_y) = 0
\] (39)
where \( \vec{S} = (S_x, S_y, S_z) \), \( \vec{S}^2 = 1 \) and \( \varphi(x, t) \) is a real function.

To treat eqs.(38), (39) we shall use the results of work [2] where solutions of the problem was reduced to solutions of two linear equations of type (2), namely
\[
iX_t(\xi, t) + \frac{1}{2}X_{\xi\xi} + iU_2(\xi, t)X_\xi = 0
\] (40)
\[
iY_t(\eta, t) + \frac{1}{2}Y_{\eta\eta} - iU_1(\eta, t)Y_\eta = 0
\] (41)
with \( \xi = \frac{1}{2}(x + y), \ \eta = \frac{1}{2}(y - x) \) and real potentials: \( U_i = \overline{U_i} \).
The special class of solutions of (36) related to degenerate spectral data (factorized) are given by the formulas [2]: (N=1)

\[ S_x + iS_y = 2 \frac{XY}{|1 - AB|^2} (1 + \bar{A}B), \quad S_- = \bar{S}_+ \] (42)

\[ S_3 = - \left( 1 + 2 \frac{(A + \bar{A})(B + \bar{B})}{|1 - AB|^2} \right), \] (43)

\[ \varphi(\xi, \eta, t) = 2 i \ln(\det \Delta) + 2 \partial_\xi^{-1} U_2(\xi, t) + 2 \partial_\eta^{-1} U_1(\eta, t) \] (44)

\[ A = \int_{-\infty}^{\eta} d_y \bar{Y}(y, t) \partial_y Y(y, t) \] (45)

\[ B = - \int_{-\infty}^{\xi} d_x X(x, y) \partial_x \bar{X}(x, t) \] (46)

\[ \Delta = \frac{1 - \bar{A}B}{1 + A\bar{B}} \] (47)

one can easily see from (40), (41) that

\[ Y(x, t) = \bar{X}(x, -t) \]

Consider the case \( b_1 > 0, \ k = \kappa \). Then

\[ X = \frac{e^{i\alpha_1 x + i(\beta_1^2 - \alpha_1^2) t}}{2 \alpha_1 \cosh z_1 + i \beta_1 e^{-z_1}}, \quad z_1 = \beta_1 (x - 2 \alpha_1 t + x_0) \] (48)

\[ Y = \bar{X}(y, -t) = \frac{e^{-i\alpha_2 y + i(\beta_2^2 - \alpha_2^2) t}}{2 \alpha_2 \cosh z_2 - i \beta_2 e^{-z_2}}, \quad z_2 = \beta_2 (y + 2 \alpha_2 t + y_0) \] (49)

\[ A = \frac{1}{2} \frac{1 - i \frac{\alpha_2}{\beta_2} (1 + e^{2z_2})}{4 \alpha_2^2 \cosh^2 z_2 + \beta_2^2 e^{-2z_2}} \] (50)

\[ B = - \frac{1}{2} \frac{1 - i \frac{\alpha_1}{\beta_1} (1 + e^{2z_1})}{4 \alpha_1^2 \cosh^2 z_1 + \beta_1^2 e^{-2z_1}} \] (51)

The solution (42)–(51) is a one soliton solution which moves with the velocity \( \vec{v} = (2\alpha_1, -2\alpha_2) \). One can proceed to the moving coordinate frame to obtain the solution at rest.

## 2 Nonlinear level

Here we give a short outlook of the results in the scope of (1). The main problem is to find a relation connecting the potential \( U(x, t) \) and the w.f. \( \Psi(x, t, k) \) or its
residues $\Psi_i(x, t)$. Since by construction we deal with meromorphic functions it is naturally of use to apply the residue theorem and calculate residues sum.

This prompts the form of a rational function $E(k)$ using which one can find the self consistency condition via calculating the residues of the function

$$\Omega = E(k)\Psi(x, t, k)\Psi(x, t, k)$$

(52)

If we specify $E$ as the polynomials:

1) $E_1 = k$

(53)

2) $E_2 = k^2 + ak$

(54)

3) $E_3 = k^3 + 2bk^2 + 2dk$

(55)

we arrive at the following relations respectively

1) $U = -2F(x, t)$

(56)

2) $U_t + aU_x = -2\partial_x F(x, t)$

(57)

3) $(\partial_t^2 - \frac{1}{3}\partial_x^4)U + 2(U^2)_{xx} + \frac{8}{3}bU_{xt} + \frac{8}{3}dU_{xx} = -\frac{8}{3}\partial_x^2 F(x, t)$

(58)

where $F(x, t)$ is the quadratic form

$$F(x, t) = \sum_{ij}^N \bar{\Psi}_i E_{ij} \Psi_j$$

(59)

in which

$$E_{ij} = (E(\bar{\kappa}_j) - E(\kappa_j))c_{ij}$$

(60)

is the Hermitian matrix.

This matrix along with the set of the poles $\kappa_i$ completely define solutions of the nonlinear equation

$$(i\partial_t - \partial_x^2 + U(x, t))\Psi_i(x, t) = 0$$

(61)

with corresponding self consistency relations (56), (57) or (58).

In eq.(61) nonlinear fields are just the residues of the w.f. $\Psi(x, t, k)$ and they possess “right” asymptotic behaviors at $x \rightarrow \pm \infty$ at certain sets of the SD (see proposition 2). In these cases

$$\Psi_i(x \rightarrow \pm \infty) \rightarrow 0$$

(62)

and we have the nonlinear problem with Trivial Boundary Conditions (TBC). For other sets of the SD $\Psi_i$ infinitely grow and usually are not interesting from the physical point of view (at least for homogeneous systems).

In order to extend our treatment to the case of nontrivial (so-called Condensate Boundary Conditions)(CBC)

$$|\Psi_i(x, t)| \rightarrow const$$

(63)
one has to consider instead of (53)–(55) the following function $E = \tilde{E}$ with
\[
\tilde{E} = \sum_{i=1}^{n} \frac{\varepsilon_i b_i^2}{k - k_i} + E, \quad i = 1, 2, 3
\]
and $\varepsilon_i = \pm 1$, $b_i$ and $k_i$ are arbitrary real numbers.

Calculating the residues of $\Omega$ (see (52)) we found equations (56)–(58) but $F(x, t)$ is now
\[
\tilde{F}(x, t) = \sum_{ij} \bar{\Psi}_i E_{ij} \Psi_j + \sum_{m=1}^{n} \varepsilon_m (|\Phi_m|^2 - b_m^2)
\]
and nonlinear fields
\[
\Phi_i(x, t) = b_i \Psi(x, t, k = k_i)
\]
are the w.f. at fixed points $k = k_i$.

From asymptotic behavior at $x \to \pm \infty$ it follows (see [1]) that $\Psi(t, k_i x \to \pm \infty) = 1$ and as a result we have CBC (63) for the nonlinear fields $\Phi_i(x, t)$.

In general case one can consider a $(n + m)$ component vector field
\[
\varphi = \begin{pmatrix}
\Psi_1 \\
\vdots \\
\Psi_n \\
\Phi_1 \\
\vdots \\
\Phi_m
\end{pmatrix}
\]
satisfying the equation
\[
\left[ i\partial_t - \partial_x^2 + U(x, t) \right] \varphi = 0
\]
with self consistency conditions (56), (57) or (58) and (65).

It is clear that in the case of pure condensate fields the form $F(x, t) = \sum_{ij} \bar{\Psi}_i E_{ij} \Psi_j$ must vanish for every nonzero solitrons $\Psi_i$. This puts extra (nonlinear) restrictions on the SD, namely, on the location of the poles. Consider for example a well-known case of the scalar NSE
\[
(i\partial_t - \partial_x^2 + \varepsilon (|\Phi|^2 - b^2) \Phi = 0
\]
with $N = 1$ and CBC (63).

In this case we have
\[
\tilde{E}(\bar{\kappa}_1) - \tilde{E}(\kappa_1) = 0
\]
or
\[
(\bar{\kappa}_1 - \kappa_1)(\varepsilon b^2 |\bar{\kappa}_1 - \kappa_1|^2 - 1) = 0
\]

One can easily see from (71) that the equation
\[
\varepsilon \frac{b^2}{|\bar{\kappa}_1 - \kappa_1|^2} = 1
\]
has a solution when $\varepsilon = 1$, i.e. only in the case of the repulsive NSE, moreover poles allowed are on the circle $|\kappa_1 - k_1|^2 = 1/b^2$. In framework of the attractive NSE ($\varepsilon = -1$) a one–pole condensate solution is absent.

As the second example, we consider a new two–pole solution namely bions (17). Then instead of (71) we have

$$\tilde{E}(\bar{\kappa}_1) - E(\kappa_2) = 0 \quad \text{or} \quad (\bar{\kappa}_1 - \kappa_2) \left( \varepsilon \frac{b^2}{(\bar{\kappa}_1 - k_1)(\kappa_2 - k_1)} - 1 \right) = 0$$

(73)

The first solution $\bar{\kappa}_1 = \kappa_2$ is just the well–known Zakharov–Shabat breather (string solution), so we discuss solutions of the equation

$$\kappa_2 = k_1 + \varepsilon \frac{b^2}{|\kappa_1 - k_1|^2}(\kappa_1 - k_1)$$

(74)

From (74) it follows that the condition $Im \kappa_2 \cdot Im \kappa_1 < 0$ needed for nonsingularity of the solution (Proposition 2) is valid when $\varepsilon < 0$, i.e. in the scope of the attractive NSE.

So the qualitative conclusion is as follows:

In the system with condensate (plane–wave boundary conditions) one–pole solutions (kink) exist for the repulsive type interaction and absent otherwise, meanwhile two–pole solutions (bions) appear in the scope of the attractive NSE as the elementary nonlinear excitation and consist of the invisible (like quarks) constituents, the solibricks.

Finally we note that in the scope of the above technique vector nonlinear Schrödinger equations with nondiagonal potentials [9] can be treated, too. For example, the following simplest system

$$i\phi_{it} - \phi_{ixx} + U(x,t)\phi_i = 0 \quad i = 1, 2$$

(75)

$$U(x,t) = \bar{\phi}_1\phi_2 + \phi_1\bar{\phi}_2$$

has the solution

$$\phi_i = \frac{A_ie^{i(\beta_1(x + v_1t) + b_1)} + B_ie^{i(\beta_2(x + v_2t) + a_1)}}{B_1\chi_1(x + v^+t) + h_1} + B_2e^{i(\beta_2(x + v_2t) + a_2)}$$

(76)

with the trivial boundary conditions

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \bigg|_{x \to \pm\infty} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and the coefficients similar to those defined in (17).

This paper is dedicated to the kind memory of Professor M.K.Polivanov talks with whom always were interesting and often instructive.
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