Instability in a Network Coevolving with a Particle System

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We study a coupled dynamics of a network and a particle system. Particles of density $\rho$ diffuse freely along edges, each of which is rewired at a rate given by a decreasing function of particle flux. We find that the coupled dynamics leads to an instability toward the formation of hubs and that the mean first passage time is determined by the network structure. For instance, the study on random networks separately. The aim of the present work is to investigate emerging structure of a network coevolving with a dynamical system.

For the past decade growing interests have been paid to complex networks. They are ubiquitous in nature and display intriguing properties which have not been observed in periodic lattices or random networks. The work of Ref. [1] triggered extensive and intensive studies on structure and dynamics of complex networks [2, 3, 4]. Dynamics and cooperative phenomena on networks have also attracted a lot of attention [5, 6]. Most studies so far have considered dynamics of networks or dynamics on networks separately. The aim of the present work is to investigate emerging structure of a network coevolving with a dynamical system.

Properties of dynamical systems or models for cooperative phenomena are strongly affected by underlying network structure. For instance, the study on random walks [7] shows that the density of diffusing particles at nodes is strictly proportional to the degree of nodes and that the mean first passage time is determined by the network structure through the so-called random walk centrality. Importance of underlying network structure is also shown in the study of critical phenomena [8], condensation [9], opinion dynamics [10], and so on.

Just as network structure affects dynamics on it, the former may also be influenced from the latter. The synaptic plasticity is an example of such phenomena [10]. In neural networks, bio-chemical signals are transmitted from neuron to neuron through synaptic links. At the same time, the strength of synapses can be enhanced or suppressed depending on synaptic activities. It is called the synaptic plasticity, which may result in deformation of neural networks.

When structure and dynamics are coupled, the interplay between them will drive a network to evolve in a self-organized way. It is challenging to study the emerging property of such a network. We will show that the interplay can lead to an instability toward the formation of hubs. There are a few recent works on coevolutionary dynamics of complex networks. Network dynamics combined with a game theoretical model was studied in Refs. [11, 12], and that combined with a voter model type opinion dynamics was studied in Refs. [13, 14]. However, the dynamic instability was not observed in those studies.

We study a minimal model which consists of a network and diffusing particles. A network is undirected and consists of $N$ nodes. Each edge $e = (i, j)$ between nodes $i$ and $j$ is assigned to a positive weight $w_e$. There are particles of density $\rho$ distributed over nodes. We adopt the following dynamic rule: (i) All particles hop to their neighboring nodes randomly and independently. (ii) If a particle hops from node $i$ to $j$, the weight of all edges attached to $j$ is increased by unity. (iii) After the hopping of all particles, each edge $e$ is rewired with the probability $1/w_e$. The weight of rewired edges is set to unity. The time is increased by unity after those processes.

The diffusion (i) mimics a transport taking place on a network. For simplicity, the particles are taken to be non-interacting. According to (ii), the weight $w_e$ of an edge $e = (i, j)$ established at time $t_e$ is given by

$$w_e(t) = 1 + \sum_{t'=t_e}^{t} (n_i(t') + n_j(t')).$$

Here, $n_i(t')$ denotes the number of particles visiting node $i$ at time $t'$. The more an edge contributes to a transport the more robust it is [15]. Less important edges are weeded out and replaced by new ones in the process (iii).

We start with a random network with $N$ nodes and mean degree $\langle k \rangle$ over which particles of density $\rho$ are distributed randomly. The weight of all edges are set to unity. Then we measure the degree $k_{\text{max}}$ of the node having the largest degree and the degree distribution $P_{\text{deg}}(k)$, which are averaged over $N_S$ samples. The mean degree is fixed to $\langle k \rangle = 4$ and $N_S = 10^3$ in numerical studies.

Figure 1 shows the numerical data for $k_{\text{max}}$ with $N = 1000$. One finds that $k_{\text{max}}$ increases in time exceeding the value $k_{\text{max}} = O(\ln N)$ which one would expect in random networks at all values of $\rho$ except 0.1. This suggests that there exists a dynamic instability toward the formation of hubs. Initially all edges have low weights...
and they are rewired randomly at a constant rate. Suppose that a node $i$ happens to be linked with more edges than others due to a statistical fluctuation. Then it will be visited by more particles since diffusing particles tend to be attracted toward higher degree nodes [7]. This will strengthen the edges emanating from $i$, and the node $i$ will have more chance to increase its degree. This feedback may be a possible mechanism for the instability. This idea will be elaborated in detail later.

The numerical data in Fig. 1 also suggest that there is a dynamic phase transition at $\rho = \rho_c \approx 0.6$. The threshold will be estimated from a scaling theory which will be presented later. When $\rho$ is small (see Fig. 1a)), $k_{\text{max}}$ remains almost constant up to a certain time scale $\tau$. Then it grows ballistically as $k_{\text{max}} \sim t$ until it reaches the limiting value $k_{\text{max}} \approx N$. We will call $\tau$ the instability time.

More detailed information is obtained from the degree distribution presented in Fig. 2a). It follows the Poisson distribution for $t \ll \tau$, which indicates that all nodes are statistically equivalent and edges are being rewired randomly. At $t \approx \tau$, a hub emerges spontaneously developing a peak in the degree distribution. The hub grows until it is connected to almost all other nodes. Finally there is an isolated peak in the degree distribution and the network becomes star-like.

The system exhibits distinct behaviors when $\rho$ is large (see Fig. 1b)). The instability sets in immediately and then $k_{\text{max}}$ increases sublinearly in time, whose time dependence has not been characterized yet. The numerical data in Fig. 2b) show that the degree distribution remains continuous and keeps broadening. These behaviors allow us to interpret that hubs emerge simultaneously and compete with each others to grow into larger ones. During the growth, the degree distribution can be fitted into the power-law form as $P_{\text{deg}}(k) \sim k^{-\gamma}$ with $\gamma \approx 2.0$. The power-law degree distribution persists for a long time, but is not a stationary one. The numerical data show that there appears a dip in the intermediate $k$ regime. It suggests that a single hub will dominate and the network will become star-like eventually, which could not be observed numerically up to $t = \mathcal{O}(10^7)$ though.

We present a phenomenological theory that explains the mechanism for the instability. On a non-evolving complex networks, a diffusing particle relaxes quickly to the stationary state in which the visiting frequency to a node is strictly proportional to its degree [7]. Using this property, we assume that the diffusing particles remain in the quasi-stationary state to a given network at each moment. The quasi-stationarity assumption allows us to make the approximation $n_i(t) \approx \rho k_i(t) / \langle k \rangle$ in Eq. (1), with which we can eliminate the particles degrees of freedom.

In order to describe the onset of the instability, it suffices to consider an effective dynamics of a single node $I$ and its degree $K$. Before the onset, all edges in the network are rewired randomly at a constant rate. So we can assume that $K$ is increased ($K \to K + 1$) at each time step with a suitable choice of time unit. The weight $w_\alpha$ of each edge $\alpha = 1, \ldots, K$ is set to unity when it is attached to $I$, and then increased by the amount of $\Delta w_\alpha = \lambda K(t)$ at time step $t$ according to the quasi-stationarity assumption. The constant factor $\lambda$ should be an increasing function of $\rho$, whose explicit form is not necessary. So, the weight of an edge $\alpha$ having been attached to $I$ since time $t_\alpha$, is given by

$$w_\alpha(t) = 1 + \lambda \sum_{t' = t_\alpha}^{t} K(t').$$ (2)

The degree $K$ decreases when an edge $\alpha$ is rewired with the probability $1/w_\alpha$. Combining those processes, we can write down the rate equation for the time evolution of the mean value of the degree:

$$\Delta K \equiv K(t+1) - K(t) = f_{\text{in}} - f_{\text{out}},$$ (3)

where the incoming flux is given by $f_{\text{in}} = 1$ and the outgoing flux is given by $f_{\text{out}} = \sum_{\alpha = 1}^{K(t)} 1/w_\alpha(t)$. Since the

![FIG. 1: Time evolution of $k_{\text{max}}$ in networks with $N = 10^3$.](image1.png)

![FIG. 2: Time evolution of the degree distribution of the networks with $N = 10^3$ and with (a) $\rho = 0.2$ and (b) $\rho = 1.0$. The solid curves represent the Poisson distribution $P_{\text{deg}}(k) = e^{-\langle k \rangle} \langle k \rangle^k / k!$ with $\langle k \rangle = 4$. The dashed line in (b) has the slope $-2$.](image2.png)
However, dynamic features may be different depending on the value of $\lambda$: (i) When $\lambda > \lambda_c \equiv (2q_c)^2 \simeq 2.03$, $\Delta K = f_{in} - f_{out} > 0$ for all values of $K$. Hence the queue size $K(t)$ grows immediately and asymptotically linearly in time. (ii) When $\lambda < \lambda_c$, there may be a dynamic barrier in an interval $K_1 < K < K_2$ where $\Delta K < 0$. In that case the queue can be trapped to an attractor at $K(t) = K_1$. It, however, cannot stay there permanently because the queue can escape from the barrier due to a statistical fluctuation in a characteristic time scale $\tau$. For $t > \tau$, the queue size $K(t)$ will grow linearly in time asymptotically.

For $\lambda < \lambda_c$, we can estimate the time scale $\tau$ roughly. As a crude approximation, we regard Eq. (4) as an equality so that the result $\tau'$ obtained thus will provide a lower bound for $\tau$. The queue size increases by unity at each time step if no packet escapes from the queue. It happens with the probability $P_{\text{esc}}(K, \lambda) = \prod_{K=1}^{K_2}(1 - 1/w_{s\alpha})$. For large $K$ it is approximated as $P_{\text{esc}} \sim \exp(- \sum_{\alpha=1}^{K_2} 1/w_{s\alpha}) = \exp(-f_{out}(K, \lambda))$. Thus, we can estimate the probability to overcome the dynamic barrier at $K_1 < K < K_2$ as $P_{\text{esc}}(\lambda) = \prod_{K=K_1}^{K_2}\exp(-f_{out}(K, \lambda))$ and the time scale as $\tau' = 1/P_{\text{esc}}(\lambda)$. Using Eqs. (7) and (8), we obtain that

$$\tau' \sim \exp\left(-\frac{a(ln \lambda)^2}{\lambda}\right)$$

with a constant $a$.

Using the knowledge from the effective single node dynamics, one can understand the dynamic property of the original model. We first consider the small $\rho$ case (corresponding to the case with $\lambda < \lambda_c$). Initially all nodes are trapped into the dynamic barrier. That is to say, all edges are rewired randomly and the degree of all nodes is fluctuating around the mean value $\langle k \rangle$. In the meanwhile a certain node may escape from the barrier in the instability time $\tau$ acquiring more and more edges. Once it happens, the number of particles available to all other nodes decreases, which leaves them into a deeper barrier. Consequently, the network will become star-like eventually. This is consistent with the numerical observation presented in Figs. 1 and 2. A rough estimate of the instability time $\tau$ is given by Eq. (8) with $\lambda$ replaced by $\rho$. It increases very rapidly as $\rho$ decreases. It explains the reason why we could not observe the instability at $\rho = 0.1$ numerically.

When $\rho$ is large (corresponding to the case with $\lambda > \lambda_c$), the single node picture predicts that the degrees of all nodes increase simultaneously since there is no dynamic barrier hindering growth. However, the simultaneous growth will give rise to competition among nodes. One cannot apply the independent single node picture any more to the network dynamics.

The quasi-stationarity condition for diffusing particles is still acceptable since the edge rewiring dynamics becomes slower under the competition. So, the weight $w_e$ of an edge $e$ will increase linearly in time as $w_e \simeq c\rho t$ with
numerical data for $P_{\text{deg}}(k)$ at several values of $\rho$ and $t$ with fixed $t^{1/\rho} = 2^{15}$. (b) $k_{\text{max}}$ versus $t^{1/\rho}$.

![Graphs showing $P_{\text{deg}}(k)$ and $k_{\text{max}}$ for different values of $\rho$.]

(4) $P_{\text{deg}}(k)$ for networks of $N = 10^3$ nodes at several values of $\rho$ and $t$ with fixed $t^{1/\rho} = 2^{15}$. (b) $k_{\text{max}}$ versus $t^{1/\rho}$.

FIG. 4: (a) $P_{\text{deg}}(k)$ for networks of $N = 10^3$ nodes at several values of $\rho$ and $t$ with fixed $t^{1/\rho} = 2^{15}$. (b) $k_{\text{max}}$ versus $t^{1/\rho}$.

We consider the rewiring dynamic of a network which coevolves with diffusing particles. Our study reveals that the feedback between dynamics and structure can give rise to a dynamic instability toward the formation of hubs. This may be one of the origins for the broad degree distribution observed in real-world networks. We have presented the analytic theory explaining the mechanism for the instability. We have also presented the scaling theory with which one can understand the dynamic scaling behaviors and the dynamic phase transition of the model.

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[15] One may adopt an alternative protocol where the weight of an edge increases only when a particle moves through it. We found that this dynamics does not show an interesting feature.
[16] Since the weight increment is determined by the degree of nodes at both ends of an edge, the form $\Delta w_e = \lambda (K(t) + \langle k \rangle )$ would be more realistic. We, however, ignore the constant term for simplicity because it does not modify the conclusion.
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