WEAK SOLUTIONS OF THE HAMILTON-JACOBI EQUATION FOR TIME PERIODIC LAGRANGIANS

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Abstract. In this work we prove the existence of Fathi’s weak KAM solutions for periodic Lagrangians and give a construction of all of them.

0. Introduction and statement of results

Let $M$ be a closed connected manifold, $TM$ its tangent bundle. Let $L : TM \times \mathbb{R} \to \mathbb{R}$ be a $C^\infty$ Lagrangian. We will assume for the Lagrangian the hypothesis of Mather’s seminal paper [9]. The Lagrangian $L$ should be:

1. Convex. The Lagrangian $L$ restricted to $T_xM$, in linear coordinates should have positive definite Hessian.

2. Superlinear. For some Riemannian metric we have

$$\lim_{|v| \to \infty} \frac{L(x, v, t)}{|v|} = \infty,$$

uniformly on $x$ and $t$.

3. Periodic. The Lagrangian should be periodic in time, i.e.

$$L(x, v, t + 1) = L(x, v, t),$$

for all $x, v, t$.

4. Complete. The Euler Lagrange flow associated to the Lagrangian should be complete.

Let $\mathcal{M}(L)$ be the set of probabilities on the Borel $\sigma$-algebra of $TM \times S^1$ that have compact support and are invariant under the Euler-Lagrange flow $\phi_t$.

The action of $\mu \in \mathcal{M}(L)$ is defined by

$$A_L(\mu) = \int L \, d\mu.$$

Mather defined the function $\alpha : H^1(M, \mathbb{R}) \to \mathbb{R}$ as

$$\alpha([\omega]) = -\min \left\{ \int (L - \omega) \, d\mu : \mu \in \mathcal{M}(L) \right\}.$$

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For any \( k \in \mathbb{R} \) define the \((L + k)\)-action of an absolutely continuous curve \( \gamma : [a, b] \to M \) as
\[
A_{L+k}(\gamma) = \int_a^b (L + k)(\gamma(\tau), \dot{\gamma}(\tau), \tau) d\tau
\]

For \( t \in \mathbb{R} \) we denote by \([t]\) the corresponding point in \( S^1 \). For any pair of points \((x, [s]), (y, [t])\) on \( M \times S^1 \) and \( n \) a non negative integer, define \( C((x, [s]), (y, [t]); n) \) as the set of absolutely continuous curves \( \gamma : [a, b] \to M \) with \( \gamma(a) = x \) and \( \gamma(b) = y \) such that \([a] = [s] \) and \([b] = [t] \), and the integer part of \( b - a \) is \( n \).

Let \( \Phi^n_k \) be the real function defined on \( M \times S^1 \times M \times S^1 \) as
\[
\Phi^n_k((x, [s]), (y, [t])) = \min_{\gamma \in C((x, [s]), (y, [t]); n)} \{ A_{L+k}(\gamma) \}.
\]
so that \( \Phi^n_k = \Phi^n_0 + kn \).

Then the action functional is defined by
\[
\Phi_k = \inf_n \Phi^n_k,
\]
and the Extended Peierls barrier by
\[
h_k = \liminf_{n \to \infty} \Phi^n_k.
\]
Thus \( \Phi_k \leq h_k \).

A curve \( \gamma : [a, b] \to M \) will be called closed if \( \gamma(a) = \gamma(b) \) and \( b - a \) is an integer.

In analogy to the autonomous case [8], [2], there is a critical value \( c(L) \) given by the following proposition:

1. **Proposition.**

   (1) If \( k < c(L) \), then \( \Phi_k((x, [s]), (y, [t])) = -\infty \), for all \((x, [s]), (y, [t])\) on \( M \times S^1 \)

   (2) \( c(L) = \min\{k : \int_\gamma L + k \geq 0 \text{ for all closed curves } \gamma \} \)

   (3) If \( k \geq c(L) \), then \( \Phi_k((x, [s]), (y, [t])) > -\infty \) for all \((x, [s]), (y, [t])\) on \( M \times S^1 \).

   (4) In terms of Mather’s \( \alpha \) function we have

   (2) \[ c(L) = -\min\left\{ \int Ld\mu : \mu \text{ is an invariant probability} \right\} \]

   (3) \[ = \alpha(0) \]

   Invariant probabilities realizing the infimum above are called minimizing measures.
From now on, set \( c = c(L) \).

In contrast with the autonomous case, the action potential \( \Phi_c \) may fail to be continuous and to satisfy the triangle inequality. However, for the extended Peierls barrier we shall prove the following:

2. Proposition.

(1) If \( k < c \), \( h_k \equiv -\infty \).

(2) If \( k > c \), \( h_k \equiv \infty \).

(3) \( h_c \) is finite.

(4) \( h_c((x,[s]),(z,[\tau])) \leq h_c((x,[s]),(y,[t])) + \Phi_c((y,[t]),(z,[\tau])) \).

(5) \( h_c \) is Lipschitz.

Let \( H(x,p,t) \) be the Hamiltonian associated to the Lagrangian;

\[
H : T^* M \times \mathbb{R} \rightarrow \mathbb{R}
\]

(4) \[
H(x,p,t) = \max_{v \in T_x M} pv - L(x,v,t)
\]

In [4] the critical value or \( \alpha(0) \) for the autonomous case is characterized as

\[
c(L) = \inf_{f \in C^\infty(M,\mathbb{R})} \sup_{x \in M} H(x,d_x f)
\]

\[
= \inf\{k \in \mathbb{R} : \text{there exists } f \in C^\infty(M,\mathbb{R}) \text{ such that } H(df) < k\},
\]

This can be restated in physical terms, by saying that \( c(L) \) is either the infimum of the values of \( k \in \mathbb{R} \) for which there is an exact Lagrangian graph with energy less than \( k \), or the infimum of the values of \( k \in \mathbb{R} \) for which there exist smooth solutions of the Hamilton-Jacobi inequality \( H(df) < k \).

The second interpretation has a natural generalization. We will prove in section 2 the following:

3. Theorem. The critical value, \( c(L) \) or \( \alpha(0) \) is characterized as the infimum of \( k \) such that there exists a subsolution \( f : M \times S^1 \rightarrow \mathbb{R} \) of the Hamilton Jacobi equation,

\[
d_t f + H(x,d_x f,t) \leq k.
\]

We can recover the previous interpretation by using the autonomous Hamiltonian \( \mathbb{H}(x,p,t,e) = H(x,p,t) + e \) on \( T^*(M \times S^1) \). Then \( df = (d_x f, d_t f) \) is an exact Lagrangian graph and \( c(L) = \inf_u \sup_{(x,t)} \mathbb{H}(d(x,t) u) \). The results in [4] can not be directly applied to this case because the Hamiltonian \( \mathbb{H} \) does not come from a Lagrangian.

The other values of Mather’s alpha function can be similarly characterized by recalling that \( \alpha([\omega]) = c(L - \omega) \) and that the Hamiltonian of \( L - \omega \) is \( (x,p,t) \mapsto H(x,p + \omega,t) \).

In corollary 14 we observe that differentiable solutions may only exist when \( k = c(L) \).
4. Definition. Following Fathi [6] we say that \( u : M \times S^1 \rightarrow \mathbb{R} \) is a forward weak KAM solution if

1. \( u \) is \( L + c \) dominated, i.e.
\[
 u(y, [t]) - u(x, [s]) \leq \Phi_c((x, [s]), (y, [t])).
\]

We use the notation \( u < L + c \).

2. For every \((x, [s]) \in M \times \mathbb{R}\) there exists a curve \( \gamma : (s, \infty) \rightarrow M \) such that
\[
 u(\gamma(t), [t]) - u(x, [s]) = A_{L+c}(\gamma|_{[s,t]}),
\]
in that case we say that \( \gamma \) realizes \( u \).

Similarly \( u : M \times S^1 \rightarrow \mathbb{R} \) is a backward weak KAM solution if it is dominated and for every \((x, [s]) \in M \times S^1\) there exists a curve \( \gamma : (-\infty, s) \rightarrow M \) such that
\[
 u(x, [s]) - u(\gamma(t), [t]) = A_{L+c}(\gamma|_{[t,s]}).
\]

Let \( S^- \) (resp. \( S^+ \)) be the set of backward (resp. forward) weak KAM solutions.

A point \((x, v, [s]) \in TM \times S^1\) is a positive (resp. negative) semistatic point if the solution \( \gamma = \gamma(x, v, s) \) of the Euler-Lagrange equation with initial conditions \((x, v, [s])\), satisfies for all \( t \)
\[
 A_{L+c}(\gamma|_{[s,t]}) = \Phi_c((x, [s]), (\gamma(t), [t]));
\]
(resp. \( A_{L+c}(\gamma|_{[t,s]}) = -\Phi_c((\gamma(t), [t]), (x, [s])) \) for all \( t \)).

A point \((x, v, [s]) \in TM \times S^1\) is a static point if it is positive semistatic and
\[
 A_{L+c}(\gamma|_{[s,t]}) = -\Phi_c((\gamma(t), [t]), (x, [s])).
\]

It turns out that if a point is static then its whole orbit under the Euler-Lagrange flow is static.

We denote by \( \Sigma^+ \) the set of positive semistatic points.

For a forward weak KAM solution \( u \) we define its forward basin as
\[
 \Gamma^+_0(u) = \{(x, v, [s]) \in \Sigma^+ : \quad u(\gamma(x, v, s)(t), [t]) - u(x, [s]) = \Phi_c((x, [s]), (\gamma(x, v, s)(t), [t])) \forall t > s \};
\]
and define its cut locus by \( \pi(\Gamma^+_0(u) \setminus \Gamma^+(u)) \), where \( \pi : TM \times S^1 \rightarrow M \times S^1 \) is the projection,
\[
 \Gamma^+(u) = \bigcup_{t > 0} \phi_t(\Gamma^+_0(u)),
\]
and \( \phi_t \) is the Euler-Lagrange flow. It is easy to see that the sets \( \Sigma^+ \) and \( \Gamma^+_0(u) \) are positively invariant and so \( \Gamma^+(u) \subset \Gamma^+_0(u) \). Similarly, define the backward basins \( \Gamma^-_0(u), \Gamma^-(u) \) for \( u \in S^- \).

The relevance of weak KAM solutions is that they have several properties, including those given by the following theorem.

5. Theorem. If \( u : M \times S^1 \rightarrow \mathbb{R} \) is a weak KAM solution then

1. \( u \) is Lipschitz and satisfies the Hamilton Jacobi equation
\[
 H(x, dxu, t) + d_tu = c
\]
at any point of differentiability. Moreover, \( d_x u \) and \( \dot{\gamma} \) are Legendre conjugate.

(2) Graph property: \( \pi: \Gamma^+(u) \to M \times S^1 \) is injective and its inverse is Lipschitz.

(3) \( u \) is differentiable on \( \pi(\Gamma^+(u)) \).

Observe that since a weak KAM solution \( u \) is Lipschitz, by Rademacher’s theorem it is differentiable Lebesgue almost everywhere.

Define the Aubry set \( A \) as
\[
A := \{ (x, [t]) \in M \times S^1 | h_c((x, [t]), (x, [t])) = 0 \}.
\]

We define an equivalence relation on \( A \) by \( (x, [s]) \sim (y, [t]) \) if and only if
\[
\Phi_c((x, [s]), (y, [t])) + \Phi_c((y, [t]), (x, [s])) = 0.
\]

The equivalence classes of this relation are called static classes.

Let \( \mathcal{A} \) be the set of static classes. For each static class \( \Gamma \in \mathcal{A} \) choose a point \( (p, [s]) \in \Gamma \) and let \( \mathcal{A} \) be the set of such points.

6. **Remark.** Observe that by item 4 of proposition 2 if \( (p, [\tau]) \in \mathcal{A} \) then
\[
h_c((p, [\tau]), (x, [t])) = \Phi_c((p, [\tau]), (x, [t])).
\]

7. **Theorem.** The map \( \{ f : \mathcal{A} \to \mathbb{R} | f \text{ dominated } \} \to S^- \)
\[
f \mapsto u_f(x, [t]) = \min_{(p, [s]) \in \mathcal{A}} f(p, [s]) + h_c((p, [s]), (x, [t])),
\]
and the map \( \{ f : \mathcal{A} \to \mathbb{R} | f \text{ dominated } \} \to S^+ \)
\[
f \mapsto v_f(x, [t]) = \max_{(p, [s]) \in \mathcal{A}} f(p, [s]) - h_c((x, [t]), (p, [s])),
\]
are bijections.

1. **The Peierls barrier**

We will be using the following lemma due to Mather [9]. We say that an absolutely continuous curve \( \gamma: [a, b] \to M \) is a minimizer if \( A_L(\gamma) \leq A_L(\eta) \) for any absolutely continuous curve \( \eta: [a, b] \to M \) with \( \eta(a) = \gamma(a) \) and \( \eta(b) = \gamma(b) \). It turns out that a minimizer is a solution of the Euler-Lagrange equation \( \frac{d}{dt} L_v = L_x \).

8. **Lemma.** There is \( A > 0 \) such that if \( b - a \geq 1 \) and \( \gamma: [a, b] \to M \) is a minimizer, then \( |\dot{\gamma}(t)| \leq A \) for \( t \in [a, b] \).

The proof of most of Propositions 1 and 2 follow standard arguments. We only give the proof of the Lipschitz continuity of \( h_c \).

9. **Lemma.** Given \( (z, [\sigma]) \in M \times S^1 \) define
\[
u(x, [t]) := h_c((z, [\sigma]; x, [t]), \quad \nu(x, [t]) := -h_c((x, [t]; z, [\sigma]).
\]
Then \( u \in S^- \) and \( v \in S^+ \).
Proof: By item 4 of proposition\textsuperscript{2} \( h(z, [\sigma]), (x, [t]) \leq h_c((z, [\sigma]), (y, [s]))+\Phi_c((y, [s]), (x, [t])) \) for all \((y, [s]), (x, [t]) \in M \times S^1 \). Thus \( u < L + c \).

Given \((x, [t]) \in M \times S^1 \) choose sequences \( n_k \to +\infty \), \( n_k \in \mathbb{Z} \) and \((x, v_k) \in T_x M \) such that
\[
h_c((z, [\sigma]), (x, [t])) = \lim_k A_{L+c}(\gamma_k|_{\sigma-n_k,t}),
\]
where \( \gamma_k(s) = \pi \varphi_{s-t}(x, v_k, t) \) is the solution of the Euler-Lagrange equation such that \((\gamma_k(t), \dot{\gamma}_k(t)) = (x, v_k) \). By lemma \textsuperscript{8} the norm \( \|v_k\| \) is uniformly bounded. Choose a convergent subsequence \( v_k \to w \). Let \( \eta(s) := \pi \varphi_{s-t}(x, w, t) \), then for any fixed \( s < 0 \),
\[
h_c((z, [\sigma]), (x, [t])) \leq h_c((z, [\sigma]), (\eta(s), [s])) + A_{L+c}(\eta|_{s,t})
\]
\[
= \lim_k h_c((z, [\sigma]), (\gamma_k(s), [s])) + A_{L+c}(\gamma_k|_{s,t})
\]
\[
\leq \lim_k A_{L+c}(\gamma_k|_{\sigma-n_k,s}) + A_{L+c}(\gamma_k|_{s,t})
\]
\[
= h_c((z, [\sigma]), (x, [t])).
\]
So that \( u(x, [t]) - u(\eta(s), [s]) = A_{L+c}(\eta|_{s,t}) \) for all \( s < 0 \).

10. \textbf{Lemma.} If \( u : M \times S^1 \to \mathbb{R} \) is a weak KAM solution (i.e. \( u \in S^+ \cup S^- \)) then it is Lipschitz. Moreover the Lipschitz constant does not depend on \( u \).

Proof: Assume that \( u \in S^- \), the case \( u \in S^+ \) is similar. Let \((x, [t_0]), (y, [s_0]) \in M \times S^1 \) be nearby points with \(|s_0 - t_0| < \frac{1}{4} \). Let \( \gamma : [0, 1] \to M \) be a length minimizing geodesic joining \( x \) to \( y \) and let \( \tau(r) = t_0 + r(s_0 - t_0), r \in [0, 1] \). Fix \( \delta > 1 \) and let \( z : [t_0 - \delta, t_0] \to M \) be such that
\[
u(x, [t_0]) = u(z(t), [t]) + \int_{t_0}^{t_0} L(z, \dot{z}) + c \, dt \quad \text{for all } t_0 - \delta < t \leq t_0.
\]
For \( r \in [0, 1] \), let \( \eta(r, t), t \in [t_0 - \delta, \tau(r)] \), be a locally minimizing solution of (E-L) such that \( \eta(r, t_0 - \delta) = z(t_0 - \delta) \) and \( \eta(r, \tau(r)) = \gamma(r) \).

Then
\[
u(\gamma(r), [\tau(r)]) \leq u(z(t_0 - \delta), [t_0 - \delta]) + \int_{t_0 - \delta}^{\tau(r)} L(\eta, \frac{\partial \eta}{\partial t}, t) + c \, dt.
\]
with equality for \( r = 0 \). Substracting the equality \textsuperscript{5} at \( r = 0 \), we get that
\[
u(\gamma(r), [\tau(r)]) - u(x, [t_0]) \leq \int_{t_0 - \delta}^{\tau(r)} (L + c) \, dt - A_{L+c}(z|_{t_0 - \delta, t_0}).
\]
Observe that this formula holds either for $s_0 \leq t_0$ or $t_0 \leq s_0$. As we shall see below, formula (6) implies that $u(y, s) - u(x, t) \leq K \left( |s-t| + d(x, y) \right)$ for some fixed $K > 0$. Then changing the roles of $s$ and $t$ we get that $u$ is Lipschitz.

Indeed, differentiating the right hand side and integrating by parts, we have

$$
\frac{d}{dr} \int_{t_0}^{t_0 + \tau(r)} L(\eta, \frac{\partial \eta}{\partial t}, t) + c \ dt =
$$

$$
= \left[ L(\eta, \frac{\partial \eta}{\partial t}, t) \right]_{(r,\tau(r))} + c \left( s_0 - t_0 \right) + \int_{t_0}^{\tau(r)} L_x \frac{\partial \eta}{\partial r} + L_v \frac{\partial^2 \eta}{\partial t \partial r} \ dr
$$

$$
= \left[ L(\eta, \frac{\partial \eta}{\partial t}, t) \right]_{(r,\tau(r))} + c \left( s_0 - t_0 \right) + \frac{\partial L}{\partial v} \left( \eta, \frac{\partial \eta}{\partial t}, t \right) \left|_{(r,\tau(r))} \right| \frac{\partial \eta}{\partial r} + \left|_{(r,\tau(r))} \right|
$$

Observe that since $u$ is dominated the realizing curve $z$ must be a minimizer. By lemma 8, $\|\dot{z}\|$ is uniformly bounded. By the continuity of the solutions of (E-L) with respect to initial values, $\left| \frac{\partial \eta}{\partial t} \right|$ is uniformly bounded. Hence there is a uniform constant $K > 0$ (independent of $z(t), x, y, [s_0], [t_0], u$) such that

$$
\left| L(\eta, \frac{\partial \eta}{\partial t}, t) + c \right| \leq K \quad \text{and} \quad \left\| \frac{\partial L}{\partial v} (\eta, \frac{\partial \eta}{\partial t}, t) \right\| < K.
$$

Since $\frac{\partial \eta}{\partial r} \left|_{(r,\tau(r))} \right| = \dot{\gamma}(r)$, we get that

$$
\frac{d}{dr} \int_{t_0 - \delta}^{\tau(r)} \left[ L + c \right] \leq K |s_0 - t_0| + K \left\| \dot{\gamma} \right\|.
$$

The value of the right hand side of (6) is 0 at $r = 0$. Integrating this inequality,

$$
u(y, [s_0]) - u(x, [t_0]) \leq K \left[ |s_0 - t_0| + d(x, y) \right].
$$

Interchanging the roles of $(x, [t_0])$ and $(y, [s_0])$ we obtain that the function $u$ is Lipschitz.

Combining lemmas 9 and 10 we get that the functions $f, g : M \times S^1 \rightarrow \mathbb{R}$, $f(y, [t]) = h_\epsilon((x, [s]), (y, [t]))$ and $g(x, [s]) = h_\epsilon((x, [s]), (y, [t]))$ are Lipschitz. This implies that $h_\epsilon$ is Lipschitz.

2. Subsolutions of the Hamilton Jacobi equation

Following the same ideas as in [4], one obtains

11. Lemma. If $k$ is a real number such that there exists a function $f$ in $C^1(M \times S^1)$ subsolution of the Hamilton Jacobi equation

$$
H(x, d_x f) + d_i f \leq k
$$

Then $k \geq c(L)$. 

Lemma. Let \( k \geq c(L) \). If \( f : M \times S^1 \to \mathbb{R} \) is differentiable at \((x, [t]) \in M \times S^1\) and satisfies
\[
f(y, [t_2]) - f(x, [t_1]) \leq \Phi_k(x, [t_1], y, [t_2])
\]
for all \( y \) in a neighbourhood of \( x \), then \( H(x, d_x f) + d_t f \leq k \).

Proposition. For any \( k > c(L) \) there exists \( f \in C^\infty(M \times S^1, \mathbb{R}) \) such that
\[
H(x, d_x f, t) + d_t f < k.
\]

We give a proof of the following fact

Corollary. If \( u \) is a \( C^{1+\text{Lip}} \) global solution of the Hamilton-Jacobi equation \( u_t + H(x, u_x, t) = k \), then \( k = c(L) \) and \( u \) is a weak KAM solution in \( S^- \cap S^+ \).

Proof: By lemma 11, \( k \geq c(L) \). Let \( L_t(x, v) = L(v)(x, v, t) \) be the conjugate moment associated to \( L \) and let \( \xi(x, t) \) be the vector field defined by
\[
\xi(x, t) = L_t^{-1}(u_x) \in T_x M.
\]
Then the vector field \( (\xi, 1) \) in \( M \times S^1 \) is Lipschitz. Let \( \rho_t \) be the flow of \( (\xi, 1) \) in \( M \times S^1 \). From the Hamilton-Jacobi equation we get that
\[
d_{(x, [t])} u \cdot (v, 1) = u_x(x, t) \cdot v + u_t \cdot 1 \leq L(x, v, [t]) + k.
\]
and that
\[
d_{(x, [t])} u \cdot (\xi(x, t), 1) = L(x, \xi(x, t), t) + k \quad \text{for all } (x, [t]) \in M \times S^1.
\]
Integrating equation (7) along absolutely continuous curves \((\gamma(t), [t]) \) in \( M \times S^1 \) from \((x, [s]) \) to \((y, [t]) \), we get that
\[
u(y, [t]) - u(x, [s]) \leq \inf_{\gamma} \int_{\gamma} (L + k) = \Phi_k((x, [s]), (y, [t])).
\]
So that \( u \prec L + k \).

Also, integrating equation (8), we get that the orbits of \( \rho_t \) realize \( u \) in the sense of the definition of a weak KAM solution. In particular, the orbits of \( \rho \) are global minimizers of the \( (L + k) \)-action, and thus they are solutions of the Euler-Lagrange equation.

It remains to prove that \( k = c(L) \). Let \( \nu \) be an invariant Borel probability for \( \rho_t \) and let \( \mu \) be its lift to \( TM \times S^1 \) using the vectorfield \( \xi \). Then \( \mu \) is an invariant probability of the Lagrangian flow and, by equation (8),
\[
\int (L + k) \, d\mu = \int d\nu \, d\mu = 0.
\]
This implies that \( k \leq c(L) \). Thus \( k = c(L) \) and also \( \mu \) is a minimizing measure. \( \square \)
3. Weak KAM solutions

Proof of theorem 5:

We first prove item 1. By lemma 10 we have that \( u \) is Lipschitz and hence it is differentiable almost everywhere. Let \((x, [t])\) be a point of differentiability, then by lemma 12 we have

\[
H(x, d_x u, t) + u_t \leq c.
\]

Moreover let \( \gamma : [t, \infty) \to M \) be such that

\[
u(\gamma(s), [s]) - u(x, [t]) = A_{L+c}(\gamma|_{[t,s]}),
\]

\[
\lim_{s \to t} \frac{u(\gamma(s), [s]) - u(x, [t])}{s - t} = \lim_{s \to t} \frac{1}{s - t} \int_{t}^{s} (L + c)(\gamma(s), \dot{\gamma}(s), [s])ds,
\]

so

\[
d_x u(x, [t]) \dot{\gamma} + d_t u(x, [t]) = L(x, \dot{\gamma}, t) + c.
\]

Therefore

\[
c = d_x u \dot{\gamma} - L + d_t u \leq H(x, d_x u, t) + d_t u \leq c.
\]

So \( u \) is a solution of the Hamilton Jacobi Equation and \( d_x u \) and \( \dot{\gamma} \) are related by the Legendre transformation of \( L \).

\[\Box\]

Proof of the Graph Property:

We need the following lemma due to Mather, a proof of which can be found in [9].

15. Lemma. Given \( A > 0 \) there exists \( K > 0 \) \( \varepsilon_1 > 0 \) and \( \delta > 0 \) with the following property: if \( |v_i| < A \), \((p_i, v_i, [t_i]) \in TM \times S^1 \), \( i = 1, 2 \) satisfy \( d((p_1, [t_1]), (p_2, [t_2])) < \delta \) and \( d((p_1, v_1, [t_1]), (p_2, v_2, [t_2])) \geq K^{-1}d((p_1, [t_1]), (p_2, [t_2])) \) then, if \( a \in \mathbb{R} \) and \( x_i : \mathbb{R} \to M \), \( i = 1, 2 \), are the solutions of \( L \) with \( x_i(t_i) = p_i \), \( \dot{x}_i(p_i) = v_i \), there exist solutions \( \gamma_i : [t_i - \varepsilon, t_i + \varepsilon] \to M \) of \( L \) with \( 0 < \varepsilon < \varepsilon_1 \), satisfying

\[
\gamma_1(t_1 - \varepsilon) = x_1(t_1 - \varepsilon), \quad \gamma_1(t_1 + \varepsilon) = x_2(t_2 + \varepsilon),
\]

\[
\gamma_2(t_2 - \varepsilon) = x_2(t_2 - \varepsilon), \quad \gamma_2(t_2 + \varepsilon) = x_1(t_1 + \varepsilon),
\]

\[
S_L(x_1|_{[t_1 - \varepsilon, t_1 + \varepsilon]}) + S_L(x_2|_{[t_1 - \varepsilon, t_1 + \varepsilon]}) > S_L(\gamma_1) + S_L(\gamma_2)
\]

We now prove the graph property. Let \((p_1, v_1, [t_1]), (p_2, v_2, [t_2]) \in \Gamma^+(u)\) and suppose that \( K d((v_1, [t_1]), (v_2, [t_2])) > d((p_1, [t_1]), (p_2, [t_2])) \), where \( K \) is from lemma 15 and the \( A \) that we input on lemma 15 is from lemma 8. Let \( y_i^+ = x_i(t_i + \varepsilon), i = 1, 2 \), and \( y_i^- = x_i(t_i - \varepsilon) \) for \( \varepsilon \) small, then

\[
u(y_i^+, [t_1 + \varepsilon]) - u(y_i^-, [t_1 - \varepsilon]) = \Phi_c((y_i^-, [t_1 - \varepsilon]), (y_i^+, [t_1 + \varepsilon]))
\]

\[
u(y_2^+, [t_2 + \varepsilon]) - u(y_2^-, [t_2 - \varepsilon]) = \Phi_c((y_2^-,[t_2 - \varepsilon]), (y_2^+, [t_2 + \varepsilon]))
\]
Then using that \( u < L + c \) and lemma 15, we get that

\[
(11) \quad u(y_2^+, [t_2 + \varepsilon]) - u(y_1^-, [t_1 - \varepsilon]) + u(y_1^+([t_1 + \varepsilon]) - u(y_2^-, [t_2 - \varepsilon]) \\
\leq S_{L+c}(\gamma_1) + S_{L+c}(\gamma_2) \\
< S_{L+c}(x_1|_{[t_1 - \varepsilon, t_2 + \varepsilon]} + S_{L+c}(x_2|_{[t_2 - \varepsilon, t_1 + \varepsilon]}) \\
= \Phi_c((y_1^-, [t_1 - \varepsilon]), (y_1^+([t_1 + \varepsilon]) + \Phi_c((y_2^-, [t_2 - \varepsilon]), (y_2^+([t_2 + \varepsilon]).
\]

Which is a contradiction with the sum of (9) and (10). \(\square\)

**Proof of item 3:**

Let \((x, [s]) \in \pi \Gamma^+(u)\), let \((\sigma(\tau), [\alpha(\tau)])\) be a curve on \(M \times S^1\) with \(\sigma(0) = x, \alpha(0) = s\). Let \(\gamma_s\) be the curve such that

\[
u(\gamma_s(t), [t]) - u(\gamma_s(s - \delta), [s - \delta]) = A_{L+c}(\gamma_s|_{[s-\delta, t]}).
\]

Since \((x, [s])\) is in \(\pi \Gamma^+(u)\) we can make a backwards variation \((\gamma_\tau)\) of the solution \(\gamma_s\). That is, \(\gamma_\tau : [s - \delta, \alpha(\tau)] \rightarrow M\) is a solution of the Euler-Lagrange equation joining the points \(p = \gamma_s(s - \delta)\) and \(\sigma(\tau)\).

Since \(u\) is dominated we have

\[
u(\sigma(\tau), [\alpha(\tau)]) - u(x, [s]) = u(\sigma(\tau), [\alpha(\tau)]) - u(p, [s - \delta]) - (u(x, [s]) - u(p, [s - \delta])) \\
\leq A_{L+c}(\gamma_\tau|_{[s - \delta, \alpha(\tau)]}) - A_{L+c}(\gamma_s|_{[s - \delta, s]}) \\
= A_{L+c}(\gamma_\tau|_{[s - \delta, s]}) - A_{L+c}(\gamma_s|_{[s - \delta, s]}) + A_{L+c}(\gamma_{\tau|_{[s, \alpha(\tau)]}})
\]

Dividing by \(\tau - s\) and taking limits as \(\tau\) tends to \(s\) and using the fact that \(\gamma_\tau\) is a solution of the Euler-Lagrange equation, we obtain

\[
\limsup_{\tau \rightarrow s} \frac{u(\sigma(\tau), [\alpha(\tau)]) - u(x, [s])}{\tau - s} \leq L_v(\dot{\gamma}_s, s) \cdot \sigma'(0) + L + c(\dot{\gamma}_s, s)\alpha'(0)
\]

Similarly we can make a forward variation to get

\[
\liminf_{\tau \rightarrow s} \frac{u(\sigma(\tau), [\alpha(\tau)]) - u(x, [s])}{\tau - s} \geq L_v(\dot{\gamma}_s, s) \cdot \sigma'(0) + L + c(\dot{\gamma}_s, s)\alpha'(0)
\] \(\square\)

**Proof of theorem 7:**

Let \(u \in \mathcal{S}^+\), since \(u\) is dominated, then

\[
u(x, [t]) \leq \min_{(y, [r])} u(y, [r]) + \Phi_c((y, [r]), (x, [t])).
\]

Let \(\gamma : -\infty, t] \rightarrow M\) be such that for all \(s \leq t\),

\[
u(x, [t]) - u(\gamma(s), [s]) = A_{L+c}(\gamma|_{[s, t]}).
\]
Then $\gamma(s)$ is semistatic and the minimum in (12) is realized at every point $(\gamma(s), [s])$ with $s < t$. Choose a convergent sequence $(\gamma(s_n), [s_n]) \to (p, [\tau]) \in \mathbb{M} \times S^1$, with $s_n \to -\infty$. Then by lemma 16 below, $(p, [\tau])$ is in the Pierls set. Therefore, using the continuity of $\Phi_c$ at $(p, [\tau])$ (see lemma 17 below) and (12), we have that

\[
\begin{align*}
    u(x, [t]) &= u(p, [\tau]) + \Phi_c((p, [\tau]), (x, [t])) \\
    &= \min_{(q, [\sigma]) \in \mathcal{A}} u(q, [\sigma]) + \Phi_c((q, [\sigma]), (x, [t])).
\end{align*}
\]

(13)

We show now that it is enough to choose one point on each static class to achieve the minimum on (13). Suppose that $(p, [\tau])$ and $(q, [\sigma])$ are in the same static class. Then

\[
\Phi_c((q, [\sigma]), (x, [t])) \leq \Phi_c((q, [\sigma]), (p, [\tau])) + \Phi_c((p, [\tau]), (x, [t])) \leq \Phi_c((q, [\sigma]), (p, [\tau])) + \Phi_c((q, [\sigma]), (x, [t])) = \Phi_c((q, [\sigma]), (x, [t])).
\]

So that $\Phi_c((q, [\sigma]), (x, [t])) = \Phi_c((q, [\sigma]), (p, [\tau])) + \Phi_c((p, [\tau]), (x, [t]))$. Moreover,

\[
\begin{align*}
    u(p, [\tau]) &\leq u(q, [\sigma]) + \Phi_c((q, [\sigma], p, [\tau])) \\
    &\leq u(p, [\tau]) + \Phi_c((p, [\tau]), (q, [\sigma])) + \Phi_c((q, [\sigma]), (p, [\tau])) \\
    &= u(p, [\tau]).
\end{align*}
\]

So that $u(q, [\sigma]) + \Phi_c((q, [\sigma]), (p, [\tau])) = u(p, [\tau])$. Thus

\[
\begin{align*}
    u(q, [\sigma]) + \Phi_c((q, [\sigma]), (x, [t])) &= u(q, [\sigma]) + \Phi_c((q, [\sigma]), (p, [\tau])) + \Phi_c((p, [\tau]), (x, [t])) \\
    &= u(p, [\tau]) + \Phi_c((p, [\tau]), (x, [t])).
\end{align*}
\]

So that $u = u_f$, with $f = u|_A$.

Observe that by definition, if $f : A \to \mathbb{R}$ is dominated, then $u_f|_A \equiv f$. This implies that the map $\{ f \text{ dominated} \} \mapsto u_f$ is injective.

Finally, it remains to prove that if $f : A \to \mathbb{R}$ then $u_f \in S^-$. This follows from lemma 9 and lemma 18 below.

\[\text{16. Lemma. If } \gamma : [-\infty, t_0] \to \mathbb{M} \text{ is semistatic and } s_n \to -\infty \text{ is such that } \lim_n (\gamma(s_n), [s_n]) = (p, [\tau]) \text{ exists. Then } (p, [\tau]) \text{ is in the Aubry set.}
\]

\[\text{Proof: Let } \varepsilon > 0 \text{ be small. Chose } n_0 > 0 \text{ such that for } n > n_0, \text{ we have}
|s_n - \tau \mod 1| < \frac{\varepsilon}{2}, \quad d(\gamma(s_n), p) < \frac{\varepsilon}{2}.
\]

Let $\lambda_n^{-} : [\tau, s_n + \varepsilon \mod 1] \to \mathbb{M}$ be a minimizer with $\lambda_n^{-}(\tau) = p, \lambda_n^{-}(s_n + \varepsilon \mod 1) = \gamma(s_n + \varepsilon)$. By lemma 8, $\|\gamma\|$ is uniformly bounded. By the same argument, using the first variation formula, as in proposition 2c,

\[
A_L(\lambda_n) \leq K_1 \left[ d(\gamma(s_n), p) + |s_n + \varepsilon - \tau \mod 1| \right] \leq 3\varepsilon K_1.
\]
Let $\lambda_n^+: [s_n - \varepsilon \mod 1, \tau] \to M$ be a minimizer with $\lambda_n^+(s_n - \varepsilon) = \gamma(s_n - \varepsilon)$, $\lambda_n^+(\tau) = p$. Similarly,

$$A_{L+c}(\lambda_n^+) \leq 3 \varepsilon K_1.$$ 

We have that

$$h_c((p, [\tau]), (p, [\tau])) \leq \liminf_{N \to \infty} A_{L+c}(\lambda_N^+) + A_{L+c}(\gamma|_{[s_N + \varepsilon, s_n - \varepsilon]}) + A_{L+c}(\lambda_n^+)$$

(14)

$$\leq 6 \varepsilon K_1 + \liminf_{N \to \infty} A_{L+c}(\gamma|_{[s_N + \varepsilon, s_n - \varepsilon]}).$$

Adding the action of $\gamma$ on the intervals with endpoints $s_N - \varepsilon < s_N + \varepsilon < s_n - \varepsilon < s_n + \varepsilon$ and using that $\gamma$ is semistatic on $[s_N - \varepsilon, s_n + \varepsilon]$, we have that

$$A_{L+c}(\gamma|_{[s_N + \varepsilon, s_n - \varepsilon]}) = \Phi_c((\gamma(s_N - \varepsilon), s_N - \varepsilon), (\gamma(s_n + \varepsilon), s_n + \varepsilon))$$

(15)

$$- A_{L+c}(\gamma|_{[s_N - \varepsilon, s_N + \varepsilon]}) - A_{L+c}(\gamma|_{[s_n - \varepsilon, s_n + \varepsilon]}).$$

Comparing $\Phi_c$ with the action of a minimal length geodesic, parameterized by the small interval $I = [s_N - \varepsilon \mod 1, s_n + \varepsilon \mod 1]$ of length $\varepsilon \leq \ell(I) \leq 3\varepsilon$, with speed $\leq \frac{1}{\varepsilon} d(\gamma(s_N - \varepsilon), \gamma(s_n + \varepsilon)) \leq \frac{1}{\varepsilon} \left[ \varepsilon \| \gamma' \| + d(\gamma(s_N), \gamma(s_n)) + \varepsilon \| \gamma' \| \right]$

$$\leq 2 \| \gamma' \| + 1;$$

we have that

$$\Phi_c((\gamma(s_N - \varepsilon), s_N - \varepsilon), (\gamma(s_n + \varepsilon), s_n + \varepsilon)) \leq \ell(I) \left[ \max_{|v| \leq 2 \| \gamma' \| + 1} L + c \right] \leq 3 \varepsilon K_2.$$

The two actions in (15) are bounded by $2(2\varepsilon \cdot K_2)$. Thus, from (15),

$$A_{L+c}(\gamma|_{[s_N + \varepsilon, s_n - \varepsilon]}) \leq 7 \varepsilon K_2.$$

From (14),

$$0 \leq h_c((p, [\tau]), (p, [\tau])) \leq 6 \varepsilon K_1 + 7 \varepsilon K_2.$$

Now let $\varepsilon \to 0$. 

17. Lemma. If $\lim_n(y_n, [s_n]) = (p, [\tau]) \in \mathcal{A}$ then for all $(x, [t]) \in M \times S^1$,

$$\lim_n \Phi_c((y_n, [s_n]), (x, [t])) = \Phi_c((p, [\tau]), (x, [t])).$$

**Proof:** Recall that by remark 6 $h_c((p, [\tau]), (x, [t])) = \Phi_c((p, [\tau]), (x, [t]))$. By item 4 of proposition 2

$$\Phi_c((p, [\tau]), (x, [t])) = h_c((p, [\tau]), (x, [t]))$$

(16)

$$\leq h_c((p, [\tau]), (y_n, [s_n])) + \Phi_c((y_n, [s_n]), (x, [t]))$$

$$\leq h_c((p, [\tau]), (y_n, [s_n])) + h_c((y_n, [s_n]), (x, [t]))$$

(17)

$$\leq h_c((p, [\tau]), (y_n, [s_n])) + h_c((y_n, [s_n]), (p, [\tau])) + \Phi_c((p, [\tau]), (x, [t]))$$

Using that $h_c$ is continuous, taking $\lim_n$ on inequalities (16) and (17), we get that $\lim_n \Phi_c((y_n, [s_n]), (x, [t])) = \Phi_c((p, [\tau]), (x, [t]))$. 

18. Lemma.

If \( U \subset S^\circ \), let \( u(x, [t]) := \inf_{u \in U} u(x, [t]) \) then either \( u \equiv -\infty \) or \( u \in S^\circ \).

If \( V \subset S^+ \), let \( v(x, [t]) := \sup_{v \in V} v(x, [t]) \) then either \( v \equiv +\infty \) or \( v \in S^+ \).

**Proof:** Since \( u \prec L + c \) for all \( u \in U \), for all \( (x, [s]), (y, [t]) \in M \times S^1 \),

\[
\min_{u \in U} u(y, [t]) \leq u(x, [s]) + \Phi_c((x, [s]), (y, [t])), \quad \text{for all } u \in U,
\]

\[
\min_{u \in U} u(y, [t]) = u(y, [t]) \leq u(x, [s]) + \Phi_c((x, [s]), (y, [t])), \quad \text{for all } u \in U,
\]

(18) \[
\min_{u \in U} u(y, [t]) \leq u(x, [s]) + \Phi_c((x, [s]), (y, [t])).
\]

Now fix \( (x, [t]) \in M \times S^1 \) and fix a sequence \( u_k \in U \) such that \( u(x, [t]) = \lim_k u_k(x, [t]) \). Let \( (x, v_k, [t]) \in \Gamma^-(u_k) \). By lemma 8 \( \|v_k\| \) is uniformly bounded. We can assume that \( v_k \to w \). Let \( \gamma_{v_k}(s) := \pi \varphi_{s-t}(x, v_k, t) \) and \( \gamma_w(s) := \pi \varphi_{s-t}(x, w, t) \).

Then

\[
u_k(x, t) = u_k(\gamma_{v_k}(s), [s]) + A_{L+c}(\gamma_{v_k}|_{[s,t]}), \quad \text{for all } s < t,\]

Since \( \gamma_{v_k} \xrightarrow{C^1} \gamma_w \) uniformly on bounded intervals, using that by lemma 10 all the \( u_k \)'s have the same Lipschitz constant, taking the lim inf on \( k \) we get that

(19) \[
\min_{u \in U} u(x, t) \geq u(\gamma_w(s), [s]) + A_{L+c}(\gamma_w|_{[s,t]}), \quad \text{for all } s < t,
\]

The domination condition (18) implies that (19) is an equality. \( \Box \)

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