Bell Transform and Teleportation-Based Quantum Computation

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Abstract
We propose the concepts of both the Bell transform and the teleportation operator in view of the standard description of quantum teleportation, and then with them make a detailed study on the fault-tolerant construction of single-qubit gates and two-qubit gates in teleportation-based quantum computation. The teleportation operator is a tensor product of the identity operator, the Bell transform and its inverse. Representative examples of the Bell transform can be recognized as parity-preserving gates, matchgates, magic gates, and the Yang–Baxter gates in the literature, all of which are Clifford gates and maximally entangling gates, so teleportation-based quantum computation can be reviewed as an extension of quantum Clifford gate computation or quantum matchgate computation or integrable quantum computation using the Yang–Baxter gates.

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1. Introduction

Quantum information and quantum computation [1, 2] is a newly developed research field in which information processing and computational tasks are accomplished by exploiting fundamental principles of quantum mechanics. Quantum algorithms [1, 2] are characteristic examples to show fundamental differences between classical information and computation and quantum information and computation. The quantum Fourier transform [1] plays the essential roles in Deutsch-Jozsa’s algorithm [3], Simon’s algorithm [3], Shor’s algorithm [3] and Kitaev’s phase estimation algorithm [4], see Jozsa’s paper [7]. In view of the quantum Fourier transform and its applications to quantum information and computation [1], we propose a new type of quantum transform called the Bell transform and apply it to the reformulation of both quantum teleportation [8, 9, 10, 11] and teleportation-based quantum computation [12, 13, 14, 15].

Quantum teleportation [8, 9, 10, 11] is an information protocol of transmitting an unknown qubit from Alice to Bob with the help of quantum entanglement and quantum measurement. Meanwhile, quantum teleportation is a quantum computation primitive [12, 13, 14, 15] exploited by universal quantum computation called teleportation-based quantum computation, which presents a simple and systematic approach to the construction of fault-tolerant quantum gates [12] in fault-tolerant quantum computation [1, 2, 16].

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The conceptual frameworks of both quantum teleportation [8, 9, 10, 11] and teleportation-based quantum computation [12, 13, 14, 15] have been already well set up years ago, and both have become widely used basic concepts in quantum information science [1, 2]. The motivation of this paper is to refine hidden algebraic structures in the standard description of quantum teleportation, which was initiated in [17], and then with them to reformulate the description of teleportation-based quantum computation. These algebraic structures [17] are so simple and surprisingly associated with several other topics in quantum computation [1, 2], so we hope that they are helpful for the further understanding on physics and mathematics underlying quantum teleportation and teleportation-based quantum computation.

We propose the concept of the teleportation operator [17] to characterize the quantum circuit of quantum teleportation. This operator is a tensor product in terms of the identity operator, the Bell transform and its inverse. We define the Bell transform as a unitary basis transformation matrix from the product basis to the Bell basis which consists of four Bell states (EPR pair states) [1, 2]. In this paper, these Bell states can be permuted with each other or differ by global phase factors respectively, so the Bell basis allows different forms. The Bell transform is capable of including examples in the literature which are respectively recognized as the Yang–Baxter gates [18, 19], magic gates proposed by Makhlin [20, 21], matchgates proposed by Valiant [22, 23, 24, 25, 26], and parity-preserving two-qubit gates [26].

The Yang–Baxter gates [18, 19] are nontrivial unitary solutions of the Yang–Baxter equation [27], and quantum computation using the Yang–Baxter gates is called integrable quantum computation [28], which is not the focus of the present paper, so interested readers are invited to refer to [18, 19, 28] for details. All the examples for the Bell transform in this paper are verified as Clifford gates [1, 16], some of which are also recognized as matchgates. Hence quantum computation using the Bell transform can be viewed as quantum Clifford gate computation associated with the Gottesman-Knill theorem [1, 16] and as quantum matchgate computation associated with the Valiant theorem [22, 23]. Note that quantum matchgate circuits and quantum Clifford circuits can be classically efficiently simulatable [29] as well as quantum teleportation can be efficiently simulated on a classical computer [1]. Therefore, the teleportation-based quantum computation [12, 13, 14, 15] can be reformulated as an extension of quantum computation using Clifford gates or quantum computation using matchgates or integrable quantum computation [28].

Compared with the previous research in the literature, we have done several things new in the paper. We propose the concept of the Bell transform which was originally called the Bell matrix only for a special solution of the Yang–Baxter equation [18, 19], so that it now includes two-qubit gates which do not satisfy the Yang–Baxter equation [27], meanwhile we make a comprehensive study on typical examples of the Bell transform from the viewpoint of quantum information and computation [1, 2]. We propose the concept of the teleportation operator instead of the braid teleportation in our previous research [17] in order to show non-topological features in the standard description of quantum teleportation. With the teleportation operator, we study the fault-tolerant construction of single-qubit gates and two-qubit gates in teleportation-based quantum computation, which can be regarded as an algebraic counterpart of our recent study [15] on a topological diagrammatical approach to teleportation-based quantum computation.

The plan of this paper is organized as follows. In Section 2, we present the definition of the Bell transform and explain it with four examples, including the $C_H$ gate as a tensor product of the $CNOT$ gate and the Hadamard gate, the Yang–Baxter gate $B$, and the magic gates $Q$ and $R$. We verify the $B$, $Q$ gates as matchgates and the $R$ gate as a parity-preserving non-matchgate, and verify these four gates as maximally entangling Clifford gates [30, 31, 32]. In Section 3, we
introduce the concept of the teleportation operator to characterize the quantum circuit of quantum teleportation, and then with it make a specific study on the fault-tolerant construction of single-qubit gates and two-qubit gates in teleportation-based quantum computation. In Section 4, we make concluding remarks and present an outlook on the future research. In Appendix A, we study exponential formulations of the Bell transforms $B, Q, R$ with associated two-qubit Hamiltonians.

2. Bell transform, matchgate and Clifford gate

In this section, we propose the concept of the Bell transform and regard it as a two-qubit gate in quantum computation [1, 2]. In Subsection 2.1, we define the Bell transform and explain it with the $C_1$ gate. In Subsection 2.2, we present the Yang–Baxter gate $B$ and the magic gates $Q$ and $R$ as examples for the Bell transform, and recognize the $B$ gate and the $Q$ gate as matchgates and the $R$ gate as a parity-preserving non-matchgate. In Subsection 2.3, we calculate the inverses of these four gates $C_H, B, Q, R$ and point out that the inverse of the Bell transform is not the Bell transform in general. In Subsection 2.4, we verify $C_H, B, Q, R$ and their inverses $C^{-1}_H, B^{-1}, Q^{-1}, R^{-1}$ as Clifford gates in two equivalent approaches. In Subsection 2.5, we calculate the entangling powers [3] of $C_H, B, Q, R$ and $C^{-1}_H, B^{-1}, Q^{-1}, R^{-1}$ to verify them as maximally entangling gates.

2.1. The Bell transform

A single-qubit Hilbert space [1, 2] is a two-dimensional Hilbert space $\mathcal{H}_2$, and a two-qubit Hilbert space is a four dimensional Hilbert space $\mathcal{H}_2 \otimes \mathcal{H}_2$. The orthonormal basis of $\mathcal{H}_2$ is chosen as the eigenvectors $|0\rangle$ and $|1\rangle$ of the Pauli matrix $Z$ with $Z|0\rangle = |0\rangle$ and $Z|1\rangle = -|1\rangle$. The Pauli matrices $X$ and $Z$ with the identity matrix $I_2$ have the conventional form

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

with the Pauli matrix $Y = ZX$. The product basis of $\mathcal{H}_2 \otimes \mathcal{H}_2$ is denoted by $|i\rangle \otimes |j\rangle$ or $|ij\rangle$, with $i, j = 0, 1$, which are eigenvectors of the parity-bit operator $Z \otimes Z$ with $Z \otimes Z|i, j\rangle = (-1)^{i+j}|i, j\rangle$ where the addition $i + j$ is the binary addition modulo 2 and $i + j$ represents the parity-bit of the state $|i, j\rangle$. Obviously, $|00\rangle$ and $|11\rangle$ are even-parity states, while $|01\rangle$ and $|10\rangle$ are odd-parity states.

The Bell states $|\psi(i, j)\rangle$ [15] are maximally entangled bipartite pure states widely used in quantum information and computation, denoted by

$$|\psi(i, j)\rangle = (I_2 \otimes X'Z')|\psi(0, 0)\rangle,$$

with $|\psi(0, 0)\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and $i, j = 0, 1$, which are simultaneous eigenvectors of the parity-bit operator $Z \otimes Z$ and the phase-bit operator $X \otimes X$ with

$$Z \otimes Z|\psi(i, j)\rangle = (-1)^{i+j}|\psi(i, j)\rangle,$$
$$X \otimes X|\psi(i, j)\rangle = (-1)^{j}|\psi(i, j)\rangle,$$

with the parity-bit $i$ and the phase-bit $j$. The Bell states $|\psi(i, j)\rangle$ present an orthonormal basis of the two-qubit Hilbert space $\mathcal{H}_2 \otimes \mathcal{H}_2$, which is called the Bell basis or maximally entangling basis [1, 2].

3
The Bell transform is defined as a unitary basis transformation matrix from the product basis $|i, j\rangle$ to the Bell basis $e^{i\alpha_{kl}}|\psi(k, l)\rangle$ with the global phase factor $e^{i\alpha_{kl}}$ depending on $k$ and $l$ (which are functions of $i, j$, i.e., $k(i, j)$ and $l(i, j)$), so it is a one-by-one mapping between $|i, j\rangle$ and $e^{i\alpha_{kl}}|\psi(k, l)\rangle$ given by

$$e^{i\alpha_{kl}}|\psi(k, l)\rangle = B_{ell}|i, j\rangle$$

where the notation $B_{ell}$ denotes the Bell transform.

Our first example on the $B_{ell}$ transform is the $C_H$ gate defined by

$$|\psi(j, i)\rangle = C_H|i, j\rangle$$

which has the matrix form

$$C_H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}.$$  

With the Hadamard gate $H$ and the $CNOT$ gate \[1, 2\] given by

$$H = \frac{1}{\sqrt{2}}(X + Z), \quad CNOT = |0\rangle\langle 0| \otimes \mathbb{I}_2 + |1\rangle\langle 1| \otimes X,$$

the Bell transform $C_H$ can be formulated as

$$C_H = CNOT \cdot (H \otimes \mathbb{I}_2)$$

with the diagrammatical quantum circuit given by

$$C_H = \begin{array}{c} H \\
\downarrow \\
\end{array}$$

which has been explicitly exploited in the standard description of quantum teleportation \[1, 12\].

2.2. Parity-preserving gate, matchgate and magic gate

With the definition of the Bell transform \[4\], it allows different forms so that it is capable of including various of examples in the literature. We study the Bell transform which is a matchgate or its extension, a parity-preserving gate, because quantum computation of both matchgates and parity-preserving gates have been well studied in \[22, 23, 24, 25, 26\].

The notation on the parity-preserving gate \[23, 26\] refers to our research on integrable quantum computation \[28\], and it has the form

$$G(A_G, B_G) = \begin{pmatrix} \omega_1 & 0 & 0 & \omega_7 \\ 0 & \omega_5 & \omega_3 & 0 \\ 0 & \omega_4 & \omega_6 & 0 \\ \omega_8 & 0 & 0 & \omega_2 \end{pmatrix},$$

with two $SU(2)$ matrices $A_G$ and $B_G$ given by

$$A_G = \begin{pmatrix} \omega_1 & \omega_7 \\ \omega_8 & \omega_2 \end{pmatrix}, \quad B_G = \begin{pmatrix} \omega_5 & \omega_3 \\ \omega_4 & \omega_6 \end{pmatrix}.$$
The parity-preserving gate $G(A_G, B_G)$ \[10\] has very good algebraic properties,

$$
G(A_G, B_G) = G(A_G, B_G^\dagger),
G(A_G, B_G)G(C_G, D_G) = G(A_GC_G, B_GD_G)
$$

(12)

with $\dagger$ denotes the Hermitian conjugation. Note that the $G(A_G, B_G)$ matrix is called the parity-preserving gate because it commutes with the parity-bit operator $Z \otimes Z$ due to $Z \otimes Z = G(\mathbb{1}_2, -\mathbb{1}_2)$.

When the determinants of the two $SU(2)$ matrices $A_G, B_G$ are equal, $det(A_G) = det(B_G)$, the parity-preserving gate is a matchgate \[23, 26\]. When we call a gate as a parity-preserving gate, we usually mean it is a parity-preserving non-matchgate. Quantum computation of matchgates can be classically simulated \[22\], and it plays important roles in the research topic \[29\] of distinguishing classical computation with quantum computation. A matchgate with single-qubit gates \[23\] are capable of performing universal quantum computation, while a matchgate with a parity-preserving gate \[26\] can do too.

The Yang–Baxter gate $B$ is a solution of the Yang–Baxter equation \[27\] and has been recently recognized as a two-qubit quantum gate \[18, 19\]. It has a compact form as a basis unitary transformation from the product basis to the Bell basis,

$$
|\psi(i + j, j + 1)\rangle = (-1)^{(i+j)(j+1)}|\psi(i, j)\rangle
$$

(13)

with the multiplication $(i + j) \cdot (j + 1)$ defined as the logical AND operation between $i + j$ and $j + 1$. The matrix form of the Yang–Baxter gate $B$ is given by

$$
B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}
$$

(14)

which is the matchgate $B = G(A_B, A_B^{-1})$ with the $SU(2)$ matrix $A_B = e^{iY}$. The Yang–Baxter gate $B$ is a real orthogonal matrix leading to its inverse and transpose given by $B^T = B^{-1} = G(A_B^{-1}, A_B)$ which is also the Bell transform and the matchgate. Quantum computation of the Yang–Baxter gate $B$ can be therefore viewed as an interesting example for quantum matchgate computation \[22, 23, 24, 25, 26\].

The magic gates $Q$ and $R$ respectively exploited by Makhlin \[20\] and Fujii \[21\] are the Bell transform, and with them, tensor products of two single-qubit gates, $SU(2) \otimes SU(2)$, can be proved to be isomorphic to the special orthogonal group $SO(4)$. The magic gate $Q$ \[20\] given by

$$
|\psi(i + j, i)\rangle = (-\sqrt{-1})^i |\psi(i, j)\rangle
$$

(15)

with the imaginary unit $\sqrt{-1}$ has the matrix form

$$
Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & i \\ 0 & i & 1 & 0 \\ 0 & i & -1 & 0 \\ 1 & 0 & 0 & -i \end{pmatrix}
$$

(16)

and it is the matchgate $Q = G(A_Q, B_Q)$ with two single-qubit gates $A_Q$ and $B_Q$ given by

$$
A_Q = HS, \quad B_Q = iA_QZ
$$

(17)
where the Hadamard gate $H$, the Pauli-$Z$ gate, and the phase gate $S$ given by

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

are exploited and obviously $S^2 = Z$ and $S^\dagger = S^3$. Note that the $\pi/8$ gate or $T$ gate \[1\] is a square root of the phase gate $S$, i.e., $S = T^2$.

The magic gate $R$ \[21\] given by

$$|\psi(i + j, i)\rangle = (\sqrt{-1})^i (\sqrt{-1})^j |ij\rangle$$

has the matrix form

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -i \\ 0 & -i & -1 & 0 \\ 0 & -i & 1 & 0 \\ 1 & 0 & 0 & i \end{pmatrix}.$$  \[(20)\]

It is the parity-preserving gate $R = G(A_R, B_R)$ with single-qubit gates

$$A_R = -i B_Q, \quad B_R = -B_Q.$$  \[(21)\]

which give rise to $R = Q \cdot G(Z, -I_2)$. Obviously $\det(A_R) \neq \det(B_R)$, so the $R$ gate is a non-matchgate.

In Appendix A, we study exponential formulations of the Bell transforms $B, Q, R$ with associated Hamiltonians, based on which we can show the essential difference between the matchgates $B, Q$ and the non-matchgate $R$ from the viewpoint of universal quantum computation \[1, 2\].

2.3. The inverse of the Bell transform

In general, the inverse of the Bell transform \[4\] is not the Bell transform. For example, the inverse of the Bell transform $C_H$ \[6\] denoted by $C_H^{-1}$ has the form

$$C_H^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix}.$$  \[(22)\]

which gives rise to $C_H^{-1}|00\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)$, so $C_H^{-1}$ is not the Bell transform. The inverse of the magic gate $Q$ \[16\] has the form

$$Q^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -i & -i & 0 \\ 0 & 1 & -1 & 0 \\ -i & 0 & 0 & i \end{pmatrix}.$$  \[(23)\]

which leads to a local unitary transformation of Bell states when acting on product states, for instance, $Q^{-1}|11\rangle = (I_2 \otimes S)|\psi(1, 1)\rangle$. Similarly, the inverse of the magic gate $R$ \[20\] given by

$$R^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & i & i & 0 \\ 0 & -1 & 1 & 0 \\ i & 0 & 0 & -i \end{pmatrix}.$$  \[(24)\]
has $R^{-1}|00\rangle = Q^{-1}|11\rangle$. So both $Q^{-1}$ and $R^{-1}$ are not the Bell transform, and $Q^{-1}$ is a matchgate and $R^{-1}$ is a parity-preserving non-matchgate. Occasionally, the inverse of the Yang–Baxter gate $B$ \(^{(14)}\) given by

$$B^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 0 & 0 & -1 \\
0 & 1 & 1 & 0 \\
0 & -1 & 1 & 0 \\
1 & 0 & 0 & 1
\end{pmatrix} \quad (25)$$

is still both the Bell transform and the Yang–Baxter gate\(^{[18, 19]}\).

According to our reformulation of the standard description of quantum teleportation in Section 3, the Bell transform \(^{(4)}\) is explained as the creation operator of Bell states acting on product states and the inverse of the Bell transform is associated with Bell measurements, so the inverse of the Bell transform can not be the Bell transform from the theoretical point of view.

2.4. Clifford gate and Bell transform

We recognize the Bell transforms $C_H, B, Q$ and $R$ and their inverses $C_H^{-1}, B^{-1}, Q^{-1}$ and $R^{-1}$ as Clifford gates \(^{[1, 16]}\). We focus on the Yang–Baxter gate $B$ \(^{(14)}\) in this subsection to explain basic concepts of Clifford gates, and present the results on other Bell transforms in Table 1 and Table 2.

The set of all tensor products of Pauli matrices acting on $n$ qubits with phase factor $\pm 1, \pm i$ is called the Pauli group $\mathcal{P}_n$. Clifford gates are defined in two equivalent approaches. They are unitary quantum gates preserving tensor products of Pauli matrices under conjugation, and they can be represented as tensor products of the Hadamard gate $H$, the phase gate $S$ and the $CNOT$ gate. Note that quantum computation of Clifford gates can be efficiently simulated on a classical computer in view of the Gottesman-Knill theorem \(^{[1, 16]}\). Clifford gates with the $\pi/8$ gate \(^{[13]}\) are capable of performing universal quantum computation \(^{[1, 2]}\).

| Operation | Input | Output |
|-----------|-------|--------|
| $C_H$     | $X_1$ | $Z_1$  |
|           | $X_2$ | $X_1X_2$ |
|           | $Z_1$ | $X_1Z_2$ |
|           | $Z_2$ | $Z_1Z_2$ |
| $B$       | $X_1$ | $X_1$  |
|           | $X_2$ | $X_1Z_2$ |
|           | $Z_1$ | $-Y_1Y_2$ |
|           | $Z_2$ | $Z_1X_2$ |
| $Q$       | $X_1$ | $Z_1X_2$ |
|           | $X_2$ | $-\sqrt{-1}Y_1Z_2$ |
|           | $Z_1$ | $X_1X_2$ |
|           | $Z_2$ | $Y_1Y_2$ |
| $R$       | $X_1$ | $X_1Z_2$ |
|           | $X_2$ | $-\sqrt{-1}Y_1Y_2$ |
|           | $Z_1$ | $X_1X_2$ |
|           | $Z_2$ | $Y_1Y_2$ |

Table 1: Transformation properties of elements of the Pauli group under conjugation by the Bell transforms $B_{\text{ell}} = C_H, B, Q, R$. 

---

"Operation" | "Input" | "Output"
---|---|---
$C_H$ | $X_1$ | $Z_1$
| $X_2$ | $X_1X_2$
| $Z_1$ | $X_1Z_2$
| $Z_2$ | $Z_1Z_2$

$B$ | $X_1$ | $X_1$
| $X_2$ | $X_1Z_2$
| $Z_1$ | $-Y_1Y_2$
| $Z_2$ | $Z_1X_2$

$Q$ | $X_1$ | $Z_1X_2$
| $X_2$ | $-\sqrt{-1}Y_1Z_2$
| $Z_1$ | $X_1X_2$
| $Z_2$ | $Y_1Y_2$

$R$ | $X_1$ | $X_1Z_2$
| $X_2$ | $-\sqrt{-1}Y_1Y_2$
| $Z_1$ | $X_1X_2$
| $Z_2$ | $Y_1Y_2$
Table 2: Transformation properties of elements of the Pauli group under conjugation by the inverse of the Bell transforms $B_{\text{ell}}^{-1} = C_H^{-1}, B^{-1}, Q^{-1}, R^{-1}$.

| Operation | Input | Output |
|-----------|-------|--------|
| $C_H^{-1}$ | $X_1$ | $Z \cdot X_2$ |
|           | $X_2$ | $X_2$ |
|           | $Z_1$ | $X_1$ |
|           | $Z_2$ | $X_1 \cdot Z_2$ |
| $B^{-1}$  | $X_1$ | $X_1$ |
|           | $X_2$ | $-X_1 \cdot Z_2$ |
|           | $Z_1$ | $Y_1 \cdot Y_2$ |
|           | $Z_2$ | $X_1 \cdot X_2$ |
| $Q^{-1}$  | $X_1$ | $\sqrt{\gamma} \cdot Z_1 \cdot Y_2$ |
|           | $X_2$ | $\sqrt{\gamma} \cdot Y_2$ |
|           | $Z_1$ | $\sqrt{\gamma} \cdot X_1 \cdot Y_2$ |
|           | $Z_2$ | $\sqrt{\gamma} \cdot Y_1 \cdot X_2$ |
| $R^{-1}$  | $X_1$ | $-\sqrt{\gamma} \cdot Y_1$ |
|           | $X_2$ | $-\sqrt{\gamma} \cdot Y_2$ |
|           | $Z_1$ | $-\sqrt{\gamma} \cdot Y_1 \cdot X_2$ |
|           | $Z_2$ | $-\sqrt{\gamma} \cdot Y_1 \cdot Y_2$ |

The Pauli group $\mathcal{P}_2$ on two qubits is generated by $X_1 = X \otimes I_2$, $X_2 = I_2 \otimes X$, $Z_1 = Z \otimes I_2$, $Z_2 = I_2 \otimes Z$ which are transformed under conjugation by the Yang–Baxter gate $B$ (14) in the way

$$BX_1B^\dagger = X_1, \quad BX_2B^\dagger = X_1 \cdot Z_2,$$

$$BZ_1B^\dagger = -Y_1 \cdot Y_2, \quad BZ_2B^\dagger = -X_1 \cdot X_2,$$

(26)

so the Yang–Baxter gate $B$ is a Clifford gate preserving the Pauli group under conjugation. See Table 1 for transformation properties of $X_1$, $X_2$, $Z_1$ and $Z_2$ under conjugation by the Bell transforms $B_{\text{ell}} = C_H, B, Q, R$, and see Table 2 for transformation properties of $X_1$, $X_2$, $Z_1$ and $Z_2$ under conjugation by the inverse of the Bell transforms $B_{\text{ell}}^{-1} = C_H^{-1}, B^{-1}, Q^{-1}, R^{-1}$.

The $\text{CNOT}_{ij}$ gate is the controlled-NOT gate with qubit $i$ as the control and qubit $j$ as the target. Note that the $\text{CZ}$ gate has the form of $\text{CZ}_{12} = (H \otimes I_2) \text{CNOT}_{21}(H \otimes I_2)$ and $\text{CZ}_{12} = \text{CZ}_{21}$, so the $\text{CZ}$ gate is a Clifford gate. With the research work [25] by Ramelow et al., the parity-preserving gate $G = G(A_G, B_G)$ (10) has been reformulated as

$$G(A_G, B_G) = \text{CNOT}_{12} \text{CU}_{21} (A_G \otimes I_2) \text{CNOT}_{12}$$

(27)

where the controlled-$U$ gate $\text{CU}_{21}$ given by

$$\text{CU}_{21} = I_2 \otimes |0\rangle\langle 0| + B_G A_G^{-1} \otimes |1\rangle\langle 1|$$

(28)

can be further decomposed as tensor products of $\text{CNOT}$ gates and single-qubit gates, see Nilesen and Chuang’s description on controlled operations [1]. As an example, the Yang–Baxter gate $B$ has the form

$$B = \text{CNOT}_{12} \text{CZ}_{21}(Z \cdot H \otimes I_2) \text{CNOT}_{12},$$

(29)

or equivalently

$$B = \text{CNOT}_{12}(Z \cdot H \otimes I_2) \text{CNOT}_{21} \text{CZ}_{21} \text{CNOT}_{12},$$

(30)
which has another more simplified form

\[ B = \text{CNOT}_{21}(\mathbb{I}_2 \otimes Z H)\text{CNOT}_{21}, \]  

(31)

so the Yang–Baxter gate \( B \) is verified again as a Clifford gate. For the other Bell transforms \( B_{\text{ell}} = C_H, Q, R \), we have the decomposition such as \((31)\) respectively given by

\[
C_H = \text{CNOT}_{12}(H \otimes \mathbb{I}_2), \\
Q = \text{CNOT}_{12}(H S \otimes S)CZ_{21}\text{CNOT}_{12}, \\
R = \text{CNOT}_{12}(H S^\dagger \otimes S^\dagger)\text{CNOT}_{12},
\]

(32)

which naturally give rise to the decomposition of the inverse of the Bell transforms \( B_{\text{ell}}^{-1} = C_H^{-1}, Q^{-1}, R^{-1} \).

As a remark, the proposal of the concept of the Bell transforms \((4)\) or its inverse in this paper is mainly based on the reformulation of the standard description of quantum teleportation (see Section 3). Since quantum teleportation is a well-known example for quantum Clif- ford gate computation \([1, 16]\), the Bell transforms \( B_{\text{ell}} = C_H, B, Q, R \) and their inverses \( B_{\text{ell}}^{-1} = C_H^{-1}, B^{-1}, Q^{-1}, R^{-1} \) can be assumed as Clifford gates in principle.

2.5. Entangling power of the Bell transform

The product states are separable states and the Bell states are maximally entangled states \([1, 2]\), so it is reasonable to require the Bell transform and its inverse as a maximally entangling two-qubit gate in any entanglement measurement theory \([30, 31]\). We calculate the entangling powers \([32]\) of both the Bell transforms \( C_H, B, Q, R \) and their inverses to support this statement.

With the magic gates \( Q \) or \( R \) \([20, 21]\), two-qubit entangling gates in the special unitary group \( SU(4) \) can be characterized by the homogenous space \( SU(4)/SO(4) \otimes SO(4) \), namely, two-qubit entangling gates are locally equivalent when they are associated with single-qubit transformations. So any two-qubit gate \( U \) \([34]\) can be characterized by three non-local parameters \((a, b, c)\) which give rise to a locally equivalent two-qubit gate \( e^{i(aX + bY + cZ)} \). The entangling power \( e_p(U) \) \([26]\) of the two-qubit gate \( U \) has the form

\[ e_p(U) = 1 - \cos^2 2a \cos^2 2b \cos^2 2c - \sin^2 2a \sin^2 2b \sin^2 2c \]

with the maximum 1, which is useful enough in our purpose.

The non-local parameters \((a, b, c)\) of the Bell transform \( C_H \) \((6)\) and its inverse \( C_H^{-1} \) are the same as those of the CNOT gate which can be calculated as \((\frac{\pi}{4}, 0, 0)\). After some algebra, those of the Yang–Baxter gate \( B \) \((14)\) and its inverse \( B^{-1} \) are found to also be \((\frac{\pi}{4}, 0, 0)\).

The magic gate \( Q \) \((16)\) and its inverse \( Q^{-1} \) are locally equivalent to the Bell transform \( B' \) given by

\[ B' = e^{-i\frac{\pi}{2}Y \otimes Y} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \]

(33)

which informs that \( B' \) has non-local parameters \((\frac{\pi}{4}, 0, 0)\), with \( Q = B'(Z \otimes S) \) or \( Q^{-1} = (\mathbb{I}_2 \otimes S^\dagger)B'(Z \otimes \mathbb{I}_2) \), so \( Q, Q^{-1} \) and \( B' \) have the same non-local parameters.
Figure 1: Quantum circuit for quantum teleportation as a diagrammatic representation of the teleportation equation (40) in terms of the teleportation operator (38). The diagram is read from the left to the right. The single lines denote qubits and the double lines denote classical bits. The box $B_{ell}$ or $B_{ell}^{-1}$ denotes a two-qubit gate which is either the Bell transform $B_{ell}$ or its inverse $B_{ell}^{-1}$. The box $U_{ij}^{-1}V_{kl}^{-1}$ denotes local unitary correction operators.

Table 3: Local unitary operators $U_{ij}$ and $V_{kl}$ with $i, j = 0, 1$ and $k, l = 0, 1$ in the teleportation equation (39) for the Bell transforms $B_{ell} = C_H, B, Q, R$. Here $\sqrt{-1}$ denotes the imaginary unit and $i + j$ is the binary addition modulo 2. The local unitary operators $U_{ij}$ and $V_{kl}$ for $C_H$ and $Q$ have the form of products of the Pauli matrix $X$ and the Pauli matrix $Z$, while for $B$ and $R$ have of the Pauli matrix $Z$ and the Pauli matrix $X$.

\[
\begin{array}{ccc}
B_{ell} & U_{ij} & V_{kl} \\
C_H & X/Z & XZ^2 \\
Q & (-\sqrt{-1})X^{i+j}Z^2 & (-\sqrt{-1})X^{i+j}Z^2 \\
B & Z^{i+j}X^{i+j} & Z^{i+j}X^{i+j} \\
R & (\sqrt{-1})Z^{i+j}X^{i+j} & (\sqrt{-1})Z^{i+j}X^{i+j}
\end{array}
\]

Hence the entangling powers of the Bell transforms $C_H, B, Q, R$ and their inverses can be calculated exactly as 1, so they are maximally entangling gates.

3. Teleportation operator and teleportation-based quantum computation

In this section, we propose the concept of the teleportation operator and then with it reformulate the standard description of both quantum teleportation and teleportation-based quantum computation. In Subsection 3.1, we make a brief sketch on quantum teleportation using the Bell transform $C_H$. In Subsection 3.2, we define the teleportation operator and apply it to the algebraic description of quantum circuit of quantum teleportation. In Subsection 3.3, we make a thorough study on the fault-tolerant construction of single-qubit gates and two-qubit gates in teleportation-based quantum computation.

3.1. Bell transform and quantum teleportation

Quantum teleportation [8, 9, 10, 11] is an information protocol with which an unknown qubit is sent from Alice to Bob by successfully performing the operations including state preparation,
Figure 2: Fault-tolerant construction of the single-qubit gate $U$ in teleportation-based quantum computation as a diagrammatical representation of the teleportation equation (45) in terms of the teleportation operator (38).
Table 5: Local unitary operators $R_{ij}$ and $S_{kl}$ with $i, j = 0, 1$ and $k, l = 0, 1$ in the teleportation equation (35) for the Bell transforms $B_{ell} = C_H, B, Q, R$ with the single-qubit gate $U$ as the $\pi/8$ gate $T$. The single-qubit gate $W = (X - iY)/\sqrt{2}$ is a Clifford gate.

the teleportation operator given by

$$(B_{ell}^{-1} \otimes \mathbb{I}_2)(\mathbb{I}_2 \otimes B_{ell})$$

or given by

$$(\mathbb{I}_2 \otimes B_{ell}^{-1})(B_{ell} \otimes \mathbb{I}_2)$$

in terms of the Bell transform $B_{ell}$, its inverse $B_{ell}^{-1}$ and the identity operator $\mathbb{I}_2$. For examples, the $B_{ell}$ transform in this paper can be the $C_H$ gate (6), the Yang–Baxter gate $B$ (14), the matchgate $Q$ (16) and the parity-preserving gate $R$ (20).

With the teleportation operator (38), the teleportation equation has a generalized formalism

$$(B_{ell}^{-1} \otimes \mathbb{I}_2)(\mathbb{I}_2 \otimes B_{ell})|\phi\rangle|kl\rangle = \frac{1}{2} \sum_{i, j = 0}^{1} |ij\rangle V_{kl} U_{ij}|\phi\rangle$$

where $k, l = 0, 1$ and $V_{kl}$ and $U_{ij}$ are local unitary operators as products of Pauli matrices acting on the single-qubit. The quantum circuit diagram in Figure 1 denotes the quantum circuit of quantum teleportation, as a diagrammatical representation of the teleportation equation (40) in terms of the teleportation operator (38), in which Bob performs the local unitary operation $U_{ij} V_{kl}^{-1}$ on his qubit to obtain the quantum state $|\phi\rangle$. See Table 5 for local unitary operators $U_{ij}$ and $V_{kl}$ for the Bell transforms $B_{ell} = C_H, B, Q, R$.

Besides the form (40), the teleportation equation has the other form

$$(\mathbb{I}_2 \otimes B_{ell}^{-1})(B_{ell} \otimes \mathbb{I}_2)|kl\rangle|\phi\rangle = \frac{1}{2} \sum_{i, j = 0}^{1} V_{kl}^{T} U_{ij}^{T} |ij\rangle |\phi\rangle$$

with the teleportation operator (39) and with the symbol $T$ denoting the matrix transpose, which is to be exploited in the fault-tolerant construction of two-qubit gates in teleportation-based quantum computation, see Figure 3 in Subsection 3.3.

3.3. Fault-tolerant construction of single-qubit gates and two-qubit gates in teleportation-based quantum computation

Teleportation-based quantum computation has been well studied in both algebraic and topological approach in [12, 13, 14, 15]. Here we focus on the fault-tolerant construction of single-qubit gates and two-qubit entangling gates using the Bell transform formulation (40) or (41) of quantum teleportation. Note that any entangling two-qubit gate with single-qubit gates [35, 36, 37] can perform universal quantum computation [1, 2].
Figure 3: Fault-tolerant construction of the two-qubit gate $CU$ in teleportation-based quantum computation as a diagrammatical representation of the teleportation equation (49).

Table 6: Local unitary operators $Q$ and $P$ in the teleportation equation (49), with the teleportation operator (38) in terms of the Bell transforms $B$ and $R$, for two-qubit gates $CU$ including the $CNOT$ gate, $CZ$ gate, the Bell transforms $B_{ij} = C_{ij}$, $B$, $Q$, $R$, and their inverses. The local unitary operators $Q$ and $P$ have the form as products of the Pauli matrices $X$, $Z$, and $Y$; with respective indices $a, b, c, d$ in Table 5 and respective phase factors $E_Q$ and $E_P$ in Table 5.

Table 7: Local unitary operators $Q$ and $P$ in the teleportation equation (49), with the teleportation operator (38) in terms of the Bell transforms $B$ and $R$, for two-qubit gates $CU$ including the $CNOT$ gate, $CZ$ gate, the Bell transforms $B_{ij} = C_{ij}$, $B$, $Q$, $R$, and their inverses. Indices $a, b, c, d$ are shown in Table 5. Phase factors $E_Q$ and $E_P$ are in Table 5.
with given by the prepared quantum state. These two successive operations lead to the teleportation equation

\[ (B_{\ell}^{-1} \otimes I_2)(I_2 \otimes B_{\ell})(I_2 \otimes B_{\ell})(\psi) |kl\rangle = \frac{1}{2} \sum_{i,j=0}^{1} i\overline{j}S_{ik}R_{ij}U|\phi\rangle \]  

Table 8: The indices \( a, b, c, d \) for the Bell transforms \( B_{\ell} = C_{H}, Q \) in Table 6 and the Bell transforms \( B_{\ell} = B, R \) in Table 7. All the index addition + is the binary addition.

|     | \( a \) | \( b \) | \( c \) | \( d \) |
|-----|-------|-------|-------|-------|
| \( C_{H} \) | \( j_1 + l_1 \) | \( i_1 + k_1 \) | \( i_2 + k_2 \) | \( j_2 + l_2 \) |
| \( B \) | \( j_1 + l_1 \) | \( i_1 + j_1 + k_1 + l_1 \) | \( i_2 + j_2 + k_2 + l_2 \) | \( j_2 + l_2 \) |
| \( Q \) | \( i_1 + j_1 + k_1 + l_1 \) | \( i_2 + k_2 \) | \( i_2 + j_2 + k_2 + l_2 \) | \( i_2 + k_2 \) |
| \( R \) | \( i_1 + k_1 \) | \( i_1 + j_1 + k_1 + l_1 \) | \( i_2 + j_2 + k_2 + l_2 \) | \( i_2 + k_2 \) |

Table 9: The phase factors \( E_Q \) and \( E_P \) for the Bell transforms \( B_{\ell} = C_{H}, Q \) in Table 6 and the Bell transforms \( B_{\ell} = B, R \) in Table 7. All the index addition + is the binary addition and all the index multiplication \( \cdot \) is the logical AND operation.

|     | \( E_Q \) | \( E_P \) |
|-----|-------|-------|
| \( C_{H} \) | \((-1)^{i_1 i_2} (-1)^{j_1 j_2} \) | \((-1)^{i_1 i_2} (-1)^{j_1 j_2} \) |
| \( B \) | \((-1)^{i_1 i_2} (-1)^{j_1 j_2} \) | \((-1)^{i_1 i_2} (-1)^{j_1 j_2} \) |
| \( Q \) | \((-1)^{i_1 i_2} (-1)^{j_1 j_2} \) | \((-1)^{i_1 i_2} (-1)^{j_1 j_2} \) |
| \( R \) | \((-1)^{i_1 i_2} (-1)^{j_1 j_2} \) | \((-1)^{i_1 i_2} (-1)^{j_1 j_2} \) |

The \( CNOT \) gate, the \( CZ \) gate, the Bell transforms \( C_{H}, B, Q, R \) and their inverses \( C_{H}^{-1}, B^{-1}, Q^{-1}, R^{-1} \), all of them are good candidates for an entangling two-qubit gate with the entangling power 1 [26, 32]. Single-qubit gates can be generated by the Hadamard gate \( H \) and \( \pi/8 \) gate [33]. Note that the \( \pi/8 \) gate given by

\[
T = \begin{pmatrix}
1 & 0 \\
0 & e^{i\pi/4}
\end{pmatrix}
\]  

is not a Clifford gate, since transformations of elements of the Pauli group \( \mathcal{P}_2 \) under conjugation by the \( \pi/8 \) gate have the form

\[
TXT^4 = W, \quad TZZT = Z
\]

in which \( W = XZ^2 \) is a Clifford gate [16, 16].

In fault-tolerant quantum computation [1, 2, 16], the fault-tolerant construction of Clifford gates including Pauli gates can be realized in a systematic approach, so the fault-tolerant construction of non-Clifford gates such as the \( \pi/8 \) gate becomes a problem of how to introduce a set of Clifford gates to play the role of these non-Clifford gates, which motivates the original research of teleportation-based quantum computation [12]. The fault-tolerant construction of single-qubit gates and two-qubit gates using quantum teleportation [12] has a rather simpler topological diagrammatical interpretation [15] which presents the guideline on our fault-tolerant construction of single-qubit and two-qubit gates using the teleportation operator [3].

To perform a single-qubit gate \( U \) on the unknown quantum state \( |\psi\rangle \), Alice prepares the quantum state given by

\[
|\phi\rangle \otimes |\psi_L(k, l)\rangle,
\]

with \( |\psi_L(k, l)\rangle = (I_2 \otimes U)B_{\ell}|k\rangle \), and then applies the Bell measurement denoted by \( B_{\ell}^{-1} \otimes I_2 \) to the prepared quantum state. These two successive operations lead to the teleportation equation given by

\[
(B_{\ell}^{-1} \otimes I_2)(I_2 \otimes B_{\ell})(I_2 \otimes B_{\ell})(I_2 \otimes U)|k\rangle = \frac{1}{2} \sum_{i,j=0}^{1} i\overline{j}S_{ik}R_{ij}U|\phi\rangle
\]
with $S_{kl}R_{ij} = U(V_{kl}U_{ij})U^\dagger$. When Bob gets the classical two-bit $(i,j)$ information from Alice, he performs the local unitary correction operator $R_{ij}^{-1}S_{kl}^{-1}$ on his qubit to obtain the quantum state $U|\phi\rangle$. See Figure 2 for the quantum circuit associated with the teleportation equation (45).

Note that $V_{kl}U_{ij}$ (see Table 3) is a Pauli operator. As the single-qubit gate $U$ is the Hadamard gate $H$, the $S_{kl}R_{ij}$ (see Table 4) is still a Pauli operator. As $U$ is the $\pi/8$ gate, the $S_{kl}R_{ij}$ (see Table 5) is a Clifford gate. Hence, the fault-tolerant procedure of performing the $\pi/8$ gate consists of two-steps [12]: The first step is to fault tolerantly prepare the state $|\psi_{CU}(k,l)\rangle$ with $U = \pi/8$, and the second step is to fault-tolerantly perform the associated Clifford gate $R_{ij}^{-1}S_{kl}^{-1}$.

The fault-tolerant construction of a two-qubit gate $CU$ depends on the fault-tolerant construction of the quantum state given by $|\psi_{CU}\rangle = (I \otimes CU \otimes I)(B_{eU} \otimes B_{eU})|k_1l_1\rangle \otimes |k_2l_2\rangle$ (46) which is a four-qubit state. This state together with an unknown two-qubit product state $|\alpha\beta\rangle$ has the form given by $|\alpha\rangle \otimes |\psi_{CU}\rangle \otimes |\beta\rangle$ (47) which is the prepared quantum state for use. Applying the joint Bell measurement given by $B_{eU}^{-1} \otimes I_2 \otimes I_2 \otimes B_{eU}^{-1}$ (48) to the prepared quantum state (47) gives rise to the teleportation equation

$$
(B_{eU}^{-1} \otimes I_2 \otimes I_2 \otimes B_{eU}^{-1})|\alpha\rangle \otimes |\psi_{CU}\rangle \otimes |\beta\rangle
$$

$$
= \frac{1}{4} \sum_{i_1,j_1=0}^{1} \sum_{i_2,j_2=0}^{1} (I_2 \otimes I_2 \otimes Q \otimes P \otimes I_2 \otimes I_2)|i_1j_1\rangle \otimes CU|\alpha\beta\rangle \otimes |i_2j_2\rangle
$$

(49)
with $Q \otimes P$ defined by $Q \otimes P = CU(V_{k_1l_1}U_{i_1j_1} \otimes V_{k_2l_2}^T U_{i_2j_1}^T)CU^\dagger$ (50) where obviously both the teleportation equations (44) and (41) have been exploited, so that the unitary correction operator is given by $Q^\dagger \otimes P^\dagger$. See Figure 3 for the diagrammatical representation associated with the teleportation equation (49), and see Table 6, Table 7, Table 8, and Table 9 for local unitary operators $Q$ and $P$ (50).

4. Concluding remarks

Together with our recent study [15] on the topological diagrammatic approach to teleportation-based quantum computation [12, 13, 14], this paper represents a further conceptual development of our previous research [17] in which both topological and algebraic structures in the standard description of quantum teleportation [8, 9, 10, 11] have been presented in detail, especially, the concept of the braid teleportation has been made very clear.

In this paper, we propose the concept of the Bell transform [49] to include various examples in the literature for the unitary basis transformation from the product basis to the Bell basis [11, 2] such as the $C_H$ gate [6], the Yang–Baxter gate $B$ [14], the magic matrices $Q$ [16] and $R$ [20]. From the viewpoint of quantum computation [11, 2], we make a comprehensive study on the properties of these four Bell transforms. We show that the Yang–Baxter gate $B$ and the magic gate $Q$ are matchgates [22, 23, 24, 25, 26] and that the magic gate $R$ is a parity-preserving non-matchgate.
and also verify these four two-qubit gates as both Clifford gates \[1, 16\] and maximally entangling gates \[32, 26\]. For three parity-preserving gates \(B, Q\) and \(R\), furthermore, we study associated Hamiltonians for a possible physical realization.

In terms of the Bell transform \[4\], we propose the teleportation operator \[38\] or \[39\] instead of the braid teleportation \[17\] to both refine the standard description of quantum teleportation \[8, 9, 10, 11\] and study the fault-tolerant construction of single-qubit gates and two-qubit gates in teleportation-based quantum computation \[12, 13, 14, 15\]. Our research is meaningful mainly due to the following aspects. First, with the teleportation operator \[38\] or \[39\], both quantum teleportation and teleportation-based computation obtain a very concrete and well-organized algebraic formulation which is called the teleportation equation \[17\]. Second, we show that teleportation-based computation can be regarded as a kind of platform on which quantum Clifford gate computation \[1, 16\], quantum matchgate computation \[22, 23, 24, 25, 26\], and integrable quantum computation \[28\] can be set up. Third, we expect that the Bell transform can play the important role in topics other than quantum teleportation since the Bell states are widely used in quantum information and computation \[12\].

We make an outlook on future research related to key topics of this paper. About the Bell transform, one can construct high-dimensional Bell transforms to include examples in the literature, for example, the Bell transform as a unitary basis transformation from the product basis to the GHZ basis \[38, 21\]. One can also study the Bell transform as a function of parameters, for example, the Yang–Baxter gate \(B(x)\) \[19, 38, 39\] depending on the spectral parameter \(x\). About quantum teleportation, we only focus on the teleportation of one qubit via maximally entangling two-qubit pure states \[8, 9, 10, 11\], so one can apply the Bell transform and the teleportation operator to multi-qubit teleportation \[40, 41, 42\] or quantum teleportation via non-maximally entangling resources \[43, 44, 45\]. About universal quantum computation, one can study topological and algebraic aspects in the one-way quantum computation \[46, 47\] in terms of the Bell transform, which together with teleportation-based quantum computation \[12, 13, 14, 15\] present representative examples for measurement-based quantum computation. About quantum algorithms, in view of quantum algorithms \[3, 4, 5, 6, 7\] based on the quantum Fourier transform \[1, 49\], it is reasonable for us to expect some new quantum algorithms based on the Bell transform.

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Appendix A. Exponential formulations of the Bell transforms \(B, Q, R\) with associated Hamiltonians

In this paper, we study the application of four maximally entangling Clifford gates \(C_H \[6\], B \[14\], Q \[16\], R \[20\) in teleportation-based quantum computation \[12, 13, 14, 15\], but we do not discuss how to prepare them in experiments. The \(C_H\) gate can be easily performed as a tensor product of the CNOT gate and Hadamard gate \(H\) which are the most popular quantum gates realized in experiments \[1\], so we only present the exponential formulations of three parity-preserving gates \[26\] \(B, Q\) and \(R\) with associated Hamiltonians.

With the Hamiltonian \(H_B = iX \otimes Y\), the Yang–Baxter gate \(B \[14\) has the form

\[
B = e^{-iH_B|_\pi/4},
\]

(A.1)
where \( t \) denotes the evolutional time.

The magic gate \( Q \) (16) has the exponential form with the global phase \( e^{i3\pi/4} \) given by

\[
Q = e^{\frac{i\pi}{2}} e^{-\frac{i}{2}Y \otimes X} e^{-\frac{i}{2} (Z \otimes \mathbb{1}_2 + \mathbb{1}_2 \otimes Z)}, \tag{A.2}
\]

which gives rise to a time-dependent Hamiltonian,

\[
H_Q(t) = \theta(t - \frac{\pi}{4}) (-iY \otimes X) + \theta(t - \frac{\pi}{4}) (2Z \otimes 1 + 1 \otimes Z) + \theta(t - \frac{\pi}{4}) (-iY \otimes X) \tag{A.3}
\]

with the step functions \( \theta(t - \frac{\pi}{4}) \) and \( \theta(t - \frac{\pi}{4}) \) and \( 0 \leq t \leq \pi/2 \). Equivalently, the magic gate \( Q \) has the other exponential formulation

\[
Q = e^{-i\frac{\pi}{4}} e^{-i\frac{\pi}{4} (Z \otimes Z + X \otimes X)} e^{-\frac{i}{4} (Z \otimes Z)} \tag{A.4}
\]

with the associated Hamiltonian given by

\[
H_Q(t) = \theta(t - \frac{\pi}{4}) (-iY \otimes X) + \theta(t - \frac{\pi}{4}) (2Z \otimes X + X \otimes Z) \tag{A.5}
\]

with \( t \in [0, \pi/2] \).

The magic gate \( R \) (20) has the exponential form given by

\[
R = e^{-i\frac{\pi}{4} (Z \otimes Z)} e^{-i\frac{\pi}{4} (Z \otimes Z)} \tag{A.6}
\]

with a time-dependent Hamiltonian given by

\[
H_R(t) = \theta(t - \frac{\pi}{4}) (Z \otimes Z) + \theta(t - \frac{\pi}{4}) (-iY \otimes X - Z \otimes Z) \tag{A.7}
\]

and equivalently has the other exponential formulation

\[
R = e^{-i\frac{\pi}{4} (Z \otimes Z)} e^{-\frac{i}{4} (Z \otimes Z)} \tag{A.8}
\]

with the associated Hamiltonian

\[
H_R(t) = \theta(t - \frac{\pi}{4}) (-iY \otimes X) + \theta(t - \frac{\pi}{4}) (X \otimes X - Z \otimes Z) \tag{A.9}
\]

with \( t \in [0, \pi/2] \).

Note that two-qubit matchgates \( 23 \) are generated by Hamiltonians as linear combinations of \( Z \otimes \mathbb{1}_2, \mathbb{1}_2 \otimes Z, X \otimes X, Y \otimes Y, X \otimes Y \) and \( Y \otimes X \). In the above Hamiltonians, only the Hamiltonians of the magic gate \( R \) has an exceptional term \( Z \otimes Z \), so the \( B \) gate and \( Q \) gate are matchgates and the \( R \) gate is a non-matchgate. Quantum computation with matchgates \( B \) or \( Q \) can be efficiently simulated on a classical computer, whereas quantum computation with the parity-preserving gate \( R \) can boost universal quantum computation mainly due to the computational power of the term \( \exp^{i\pi/4 (Z \otimes Z)} \), see [26, 48]. Similarly, we can derive the exponential formulations of the inverses of the Bell transforms \( B^{-1}, Q^{-1}, R^{-1} \) with associated Hamiltonians, for example, \( B^{-1} = -e^{-iH_B}\big|_{t=3\pi/4} \).
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