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On minimum algebraic connectivity of graphs whose complements are bicyclic

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Abstract: The second smallest eigenvalue of the Laplacian matrix of a graph (network) is called its algebraic connectivity which is used to diagnose Alzheimer’s disease, distinguish the group differences, measure the robustness, construct multiplex model, synchronize the stability, analyze the diffusion processes and find the connectivity of the graphs (networks). A connected graph containing two or three cycles is called a bicyclic graph if its number of edges is equal to its number of vertices plus one. In this paper, firstly the unique graph with a minimum algebraic connectivity is characterized in the class of connected graphs whose complements are bicyclic with exactly three cycles. Then, we find the unique graph of minimum algebraic connectivity in the class of connected graphs $\Omega^c_n = \Omega^c_{1,n} \cup \Omega^c_{2,n}$, where $\Omega^c_{1,n}$ and $\Omega^c_{2,n}$ are classes of the connected graphs in which the complement of each graph of order $n$ is a bicyclic graph with exactly two and three cycles, respectively.

Keywords: Laplacian matrix, eigenvalues, algebraic connectivity

MSC: 15A18, 05C50, 05C40, 05D05

1 Introduction

Let $G = (V(G), E(G))$ be a graph having $V(G) = \{v_i : 1 \leq i \leq n\}$ and $E(G)$ as the sets of vertices and edges. The graph $G^c$ is complement of $G$ with same vertex-set and edge-set $E(G^c) = \{uv : u, v \in V(G), uv \notin E(G)\}$. The number of first neighbors of $v \in V(G)$ is called its degree denoted by $d(v)$. The adjacency matrix (A-matrix) of $G$ is $A(G) = [a_{ij}]_{n \times n}$ such that $a_{ij} = 1$ if $v_i$ is adjacent to $v_j$ and $a_{ij} = 0$ otherwise. By $D(G) = [d_i]_{n \times 1}$, we denote the degree matrix such that $d_i = d(v_i)$ and zero otherwise. The Laplacian matrix (L-matrix) of the graph $G$ is

$$L(G) = D(G) - A(G).$$

For $1 \leq i \leq n$, eigenvalues $\mu_i = \mu_i(G)$ and eigenvectors $Z_i = Z_i(G)$ of $L$-matrix ($L(G)$) are the L-eigenvalues and the L-eigenvectors of $G$. For $n$-dimensional column-vectors $Z_i \neq 0$, we have $L(G)Z_i = \mu_i Z_i$. Since $L(G)$ is real and symmetric therefore we have $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_{n-1} \geq \mu_n$, where $\mu_n = 0$ is a minimum L-eigenvalue and $\mu_{n-1}(G) = \alpha(G)$ is algebraic connectivity of $G$ that remains positive if and only if $G$ is connected. Moreover, eigenvectors corresponding to $\alpha(G)$ are called Fiedler vectors. For further study, we refer to [17].
The algebraic connectivity plays an important role in studies of communication and control theory to increase efficiency in air transportation system [8], measure connectivity, convergence speed & synchronization ability of the networks [9-11] generate or absorb the bipartition among the links [12] and construct multiplex model for the interconnected networks [13]. It is also used in brain networks to study the group differences and complex changes in Alzheimer’s disease, see [14].

A connected graph is called $k$-cyclic if $m = n - 1 + k$, where $n$ denotes the number of vertices, $m$ the number of edges and $k$ non-negative integers. In particular, if $k = 0, 1, 2$ or $3$, then $G$ is called a tree, unicyclic, bicyclic or tricyclic graph, respectively. Let $H'(n, 2)$ be a bicyclic graph with two cycles which is obtained from the star $K_{1,n-1}$ by the addition of two edges such that each edge joins two different pendant vertices. Similarly, $H(n, 2)$ is a bicyclic graph with three cycles obtained from the star $K_{1,n-1}$ by the addition of two edges in the pendant vertices such that both edges have one common end point. Let $\Omega_{1,n}$ and $\Omega_{2,n}$ be two classes of the bicyclic graphs with $n$ vertices having exactly two and three cycles other than $H'(n, 2)$ and $H(n, 2)$ respectively. Moreover, assume that $\Omega^1_{1,n}$ and $\Omega^2_{2,n}$ be the classes of the graphs whose complements are bicyclic with exactly two and three cycles respectively i.e $\Omega^1_{1,n} = \{G_{c} : |G_{c}| = n \text{ and } G \in \Omega_{1,n}\}$ and $\Omega^2_{2,n} = \{G_{c} : |G_{c}| = n \text{ and } G \in \Omega_{2,n}\}$. The condition to exclude $H'(n, 2)$ and $H(n, 2)$ from $\Omega_{1,n}$ and $\Omega_{2,n}$ respectively ensures that $\Omega^1_{1,n}$ and $\Omega^2_{2,n}$ are families of the connected graphs.

Many authors studied the algebraic connectivity for different families of graphs such as connected graphs with certain girth, lollipop graphs and caterpillar unicyclic graphs, see [15-17]. Moreover, the operation of complement in graphs has important role, especially when structures of the simple graphs become more complex than their complements. Recently, Jiang et al. [18], Li et al. [19] and Javaid et al. [20, 21] found the graphs with minimum algebraic connectivity among all the connected graphs whose complements are trees, unicyclic, and bicyclic with exactly two cycles. In this paper, firstly we characterize the unique graph with minimum algebraic connectivity in the class of connected graphs whose complements are bicyclic with three cycles. Then, we find the unique graph with minimum algebraic connectivity in the complete class of connected graphs whose complements are bicyclic with two or three cycles.

The rest of the paper is managed as; In Section 2, some basic definitions and results are given. Section 3 and Section 4 cover the main results. Section 5 includes the conclusion and some new directions of the problem.

2 Preliminaries

For any vector $Z \in R^n$, define a one-one map $\mu : V(G) \rightarrow Z$ such that $\mu(u) = Z_u$, where $Z_u$ is entry of $Z$ corresponding to $u \in V(G)$. Then, for $Z \neq 0$, we have

$$Z^T L(G) Z = \sum_{uv \in E(G)} (\mu(u) - \mu(v))^2 = \sum_{uv \in E(G)} (Z_u - Z_v)^2. \quad (1)$$

Moreover, if $\lambda$ is a L-eigenvalue of $G$ corresponding to $Z \neq 0$ then Laplacian eigenvalue equation (LE-equation) is

$$(d_G(v) - \lambda)Z_v = \sum_{u \in N_G(v)} Z_u \text{ for each vertex } v \in V(G), \quad (2)$$

where $N_G(v)$ is set of neighbors of $v \in V(G)$.

Assume that $Z \in R^n$ is a unit vector and perpendicular to all-ones vector, then by Courant-Fisher theorem [3], we have

$$a(G) \leq Z^T L(G) Z, \quad (3)$$

where $a(G)$ achieves the upper bound if $Z$ is a Fiedler vector. If $J$ is all-ones matrix, $I$ is identity matrix and $L(G^*)$ is L-matrix of the complement of $G$, then for any vector $Z \in R^n$

$$Z^T L(G^*) Z = Z^T (nI - J) Z - Z^T L(G) Z. \quad (4)$$
Suppose that $C_4$, $C_5$ and $C_6$ are cycles of length 4, 5 and 6 respectively. Now, some graphs are defined which are used in the main results.

Let $H_1$ and $H_2$ be two bicyclic graphs with exactly three cycles which are obtained by joining any single non adjacent pair of vertices with an edge in $C_4$ and $C_5$, respectively. The bicyclic graphs with exactly three cycles $H_3$ is obtained from $C_6$ by joining a pair of vertices with an edge such that $H_3$ consists of a outer cycle of length 6 and two inner cycles both of lengths 4. If we insert a vertex in an edge which is incident on two vertices of degree 3 in $H_4$, then we obtain a bicyclic graph with three cycles $H_4$ such that its all the cycles (one outer and two inner) are of lengths 4.

![Figure 1: Bicycle graphs with three cycles $G_1(m_1, m_2)$, $G_2(m_1, m_2)$, $G_3(m_1, m_2)$, $G_4(m_1, m_2)$, and $G_5(m_1, m_2)$.](image)

Let $G_1(m_1, m_2)$ be obtained by attaching $m_1$ and $m_2$ pendant vertices with two adjacent vertices of degree 3 and 2 in $H_1$ respectively and $G_2(m_1, m_2)$ is obtained by attaching $m_1$ and $m_2$ pendant vertices with both the vertices of degree 3 in $H_1$. If we identify $H_1$ by a vertex of degree 2 with an edge of length one and attach $m_2$ and $m_1$ pendant vertices with a pendant vertex and a vertex adjacent to it respectively, then we obtain $G_3(m_1, m_2)$. Similarly, $G_4(m_1, m_2)$ is obtained when we identify $H_1$ by a vertex of degree 3 with an edge of length one and attach $m_2$ and $m_1$ pendant vertices with a pendant vertex and a vertex adjacent to it, respectively (see Figure 1).

By attaching $m_1$ and $m_2$ pendant vertices with two adjacent vertices of degrees 2 in $H_2$, $G_5(m_1, m_2)$ is obtained and if we attach $m_1$ and $m_2$ pendant vertices with vertices of degree 3 in $H_2$, we obtain $G_6(m_1, m_2)$. The graph $G_6(m_1, m_2)$ is obtained by joining $m_1$ and $m_2$ pendant vertices with vertices of degree 3 in $H_1$. Finally, we obtain $G_7(m_1, m_2)$ from $H_4$ by attaching $m_1$ and $m_2$ pendant vertices with two adjacent vertices of degree 2 and 3 in $H_4$ such both are on the outer cycle of $H_4$ (see Figure 2).

Moreover, $G_8(m_1, m_2)$ is a bicyclic graph with exactly two cycles which is obtained by attaching $m_2$ pendant vertices with a vertex of degree 2 of the graph $H(m_1 + 5, 2)$ (see Figure 3). Now, we state some results which are frequently used in main results.

**Lemma 2.1.** [3] Let $G$ be a simple graph. Then $\alpha(G) \leq \delta(G)$, where $\delta(G) = \min\{d_G(v), v \in V(G)\)$. 
Lemma 2.2. [18] If \( Z_i \) for \( 1 \leq i \leq n \) is a non-increasing sequence, then, for any \( 1 \leq i, j \leq n, (Z_i - Z_j)^2 \leq \max((Z_i - Z_1)^2, (Z_i - Z_n)^2) \leq (Z_1 - Z_n)^2 \).

Theorem 2.3. [21] Let \( n, m_1 \) and \( m_2 \) be any positive integers such that \( m_1 \geq m_2 \geq 1, n \geq 11 \) and \( n = m_1 + m_2 + 5 \), then for any bicyclic graph with exactly two cycles \( G \in \Omega_{1,n} \),

\[
a(G'_1(n - 6, 1)^c) \leq a(G'_1(m_1, m_2)^c) \leq a(G^c),
\]

where equalities hold if and only if \( G'_1(n - 6, 1)^c \cong G'_1(m_1, m_2) \cong G \).

![Figure 2: Bicycle graphs with three cycles \( G_6(m_1, m_2), G_7(m_1, m_2), G_8(m_1, m_2), \) and \( G_9(m_1, m_2) \).](image)

![Figure 3: Bicycle graphs with two cycles \( G'_1(m_1, m_2) \).](image)

### 3 Computational results of minimum algebraic connectivity

The computational results of the algebraic connectivity are presented in this section.

**Lemma 3.1.** Let \( m_1, m_2 \) and \( n \) be positive integers, such that \( m_1 \geq m_2 \geq 1, n \geq 11 \) and \( m_1 + m_2 + 5 = n \). Then

\[
a(G_1(n - 6, 1)^c) < \ldots < a(G_1(m_1 + 1, m_2 - 1)^c) < a(G_1(m_1, m_2)^c).
\]

**Proof.** Let \( G_1(m_1, m_2) \) be a graph with labeled vertices as shown in Figure 1 and \( Z \) be a unit Fiedler vector of \( G_1(m_1, m_2)^c \). By Lemma 2.1 and LE-equation (3), \( a(G_1(m_1, m_2)^c) \neq d_{G_1(m_1, m_2)^c}(v) + 1 \) for any
\( v \in V(G_1(m_1, m_2)^c) \) and all the pendant vertices attached with same vertex have same values given by \( Z \). Therefore, \( Z_i := Z_v \), for \( 1 \leq i \leq 7 \) and we have the following equations for \( a = a(G_1(m_1, m_2)^c) \),

\[
(m_1 + m_2 + 3 - a)Z_1 = (m_1 - 1)Z_1 + Z_2 + Z_4 + Z_5 + Z_6 + m_2Z_7,
\]

\[
(m_1 + m_2 + 3 - a)Z_2 = m_1Z_1 + Z_4 + Z_5 + Z_6 + m_2Z_7,
\]

\[
(m_2 - a)Z_3 = m_2Z_7,
\]

\[
(m_1 + m_2 + 2 - a)Z_4 = m_1Z_1 + Z_2 + Z_4 + m_2Z_7,
\]

\[
(m_1 + m_2 + 2 - a)Z_5 = m_1Z_1 + Z_2 + Z_4 + m_2Z_7,
\]

\[
(m_1 + 2 - a)Z_6 = m_1Z_1 + Z_2 + Z_6,
\]

\[
(m_1 + m_2 + 3 - a)Z_7 = m_1Z_1 + Z_2 + Z_3 + Z_4 + Z_5 + (m_2 - 1)Z_7.
\]

Transform \((A_1)\) into a matrix equation \((M - aI)Z = 0\), where \( Z = (Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7)^T \) and

\[
M = \begin{bmatrix}
m_2 + 4 & -1 & 0 & -1 & -1 & -1 & -m_2 \\
-m_1 & m_1 + m_2 + 3 & 0 & -1 & -1 & -1 & -m_2 \\
0 & 0 & m_2 & 0 & 0 & 0 & -m_2 \\
-m_1 & -1 & 0 & m_1 + m_2 + 2 & 0 & -1 & -m_2 \\
-m_1 & -1 & 0 & -1 & m_1 + m_2 + 2 & 0 & -m_2 \\
-m_1 & -1 & 0 & -1 & 0 & m_1 + 2 & 0 \\
-m_1 & -1 & -1 & -1 & -1 & 0 & m_1 + 4
\end{bmatrix}
\]

Let \( g(\lambda; m_1, m_2) := det(\lambda I - M) \), then we have

\[
g(\lambda; m_1, m_2) = \lambda(-4 - m_1 - m_2 + \lambda)(-3 - m_1 - m_2 + \lambda)(15m_2 + 23m_1m_2 + 9m_1^2m_2 + m_1^3m_2 + 8m_1^2 + 10m_1m_2^2 + 2m_1^3m_2^2 + m_1^4m_2 + m_1^5m_2 - 16\lambda - 24m_1^2 - 9m_1^3\lambda - m_1^4\lambda - 32m_1^2\lambda - 28m_1m_2^2\lambda - 5m_1^3m_2\lambda - 11m_2^2\lambda - 5m_1^2m_2^2\lambda - m_1^3m_2\lambda + 24\lambda^2 + 18m_1^2\lambda^2 + 3m_1^3\lambda^2 + 19m_2\lambda^2 + 7m_1m_2\lambda^2 + 3m_2^2\lambda^2 - 9\lambda^3 - 3m_1\lambda^3 - 3m_2\lambda^3 + \lambda^6).
\]

Since \( g(0; m_1, m_2) = 0 = g(a; m_1, m_2) \), thus \( a \) is the second smallest root of \( g(\lambda; m_1, m_2) \). Observe that

\[
g(\lambda; m_1 + 1, m_2 - 1) - g(\lambda; m_1, m_2) = -\lambda(2 + m_1 - m_2)(3 + m_1 + m_2 - \lambda)\frac{(3 + m_1 + m_2 - \lambda)}{(5 + m_1 + m_2 - \lambda)} = -\lambda(1 + m_1 - m_2)(3 + m_1 + m_2 - \lambda)\frac{(3 + m_1 + m_2 - \lambda)}{(5 + m_1 + m_2 - \lambda)}(n - \lambda).
\]

By Lemma 2.1, \( a = a(G_1(m_1, m_2)^c) < n \). Then by \( m_1 \geq m_2, a > 0 \) and \( g(a; m_1, m_2) = 0 \), we have \( g_1(a; m_1 + 1, m_2 - 1) = -\lambda(2 + m_1 - m_2)(3 + m_1 + m_2 - a)^2(n - \lambda < 0 \). This shows that \( a(G_1(m_1 + 1, m_2 - 1)^c) < a(G_1(m_1, m_2)^c) \). Similarly, we can prove \( a(G_1(m_1 + 2, m_2 - 2)^c) < a(G_1(m_1 + 1, m_2 - 1)^c) \). Consequently, \( a(G_1(n - 6, 1)^c) < ... < a(G_1(m_1 + 1, m_2 - 1)^c) < a(G_1(m_1, m_2)^c) \).

Lemma 3.2. Let \( m_1, m_2 \) be positive integers, such that \( m_1 \geq m_2 \geq 1, n \geq 11, m_1 + m_2 + 5 = n \) and \( 2 \leq i \leq 9 \). Then \( a(G_1(m_1, m_2)^c \leq a(G_1(m_1, m_2)^c) \), where equality holds if \( G_1(m_1, m_2) \cong G_1(m_1, m_2) \).

Proof. Using Lemma 2.1 and (3) (as in Lemma 3.1), we find the following polynomials of the graphs \( G_1(m_1, m_2)^c \) for \( 2 \leq i \leq 9 \).

\[
g_2(a; m_1 + 1, m_2 - 1) = -(m_1 + m_2 - a - 1)(14 + 20m_1 + 8m_1^2 + m_1^3 + 10m_2 + 6m_1m_2 - m_1m_2^2 - m_1^2m_2 - 15\lambda - 11m_1a - 2m_1^2a - m_1^3a + 3m_2^2a - m_2^3a),
\]

\[
g_3(a; m_1 + 1, m_2 - 1) = -(m_1 + m_2 - a - 1)(25 - 25m_1 + 9m_1^2 - m_1^3 - 11m_2 - 6m_1m_2 - m_1m_2^2 + 3m_2^2 + m_1m_2^3 + m_2^3 - 50a - 97m_1a - 57m_1^2a - 13m_1^3a - m_1^4a - 69m_1a - 60m_1m_2a - 19m_1^2m_2a - 2m_1^3m_2a - 3m_2^2a + m_1m_2^3a + 7m_2^3a + 2m_1m_2^3a + m_1^3a + 86a^2 + 88m_1a^2 + 29m_1^2a^2 + 3m_2a^2 + 32m_2^2a^2 + 18m_1m_2a^2 + 3m_2^3a^2 - 11m_2^3a^2 - 18m_1m_2^3a^2).
On MAC of graphs...

\[ 3m_1 m_2^2 a^2 - 3m_1^2 a^2 - 30a^3 - 19m_1 a^3 - 3m_1^2 a^3 + m_2 a^3 + 3m_1^2 a^3 + 3a^4 + m_1 a^4 - m_2 a^4, \]

\[ g_4(a; m_1 + 1, m_2 - 1) = -a(4 + m_1 - m_2)(1 + m_1 + m_2 - a)(3 + m_1 + m_2 - a)(4 + m_1 + m_2 - a)(5 + m_1 + m_2 - a), \]

\[ g_5(a, m_1 + 1, m_2 - 1) = -a(-4 - m_1 - m_2 + a)(-50 - 86m_1 - 50m_1^2 - 12m_1^2 - 10m_1^3 - 54m_2 - 53m_1 m_2 - 18m_2^2 m_2 - 2m_1 m_2 - 3m_2^2 + 6m_2 + 2m_1 m_2^2 + m_1^2 + 71a + 77m_1 a + 27m_1 a + 3m_1 a + 30m_2 a + 18m_1 + m_2 + 3m_1 m_2 a - 9m_2 a - 3m_1 m_2 a - 3m_2^2 a - 27a^2 - 18m_1 a^2 - 3m_2 a^2 - 3a^3 + m_1^3 a^3 - m_2^2 a^3), \]

\[ g_6(a, m_1 + 1, m_2 - 1) = (16 + 56m_1 + 53m_1^2 + 18m_1^3 + 2m_1^4 + 56m_2 + 113m_1 m_2 + 67m_2^2 + 15m_1 m_2^2 + m_1^2 m_2 + 53m_2 + 67m_1 m_2 + 26m_1^2 m_2 + 3m_1^2 m_2^2 + 18m_2 + 15m_1 m_2^2 + 3m_1^3 m_2^2 + 2m_1^2 m_2 + m_1^3 m_2^2 - 171a - 353m_1 a - 260m_2 a - 90m_1 m_2^2 - 15m_2^2 m_2 - 27m_1 a - 378m_1 m_2 a - 186m_2 m_2 a - 39m_1 m_2^2 a - 3m_2^2 m_2 a - 105m_1 a - 88m_2 a - 24m_2^2 m_2 a - 2m_1 m_2 a + 8m_2 a + 9m_1 m_2 a - 2m_1^2 m_2 a + 9m_2 a + 3m_1 m_2 a + 3m_2 a + 265a + 394a^2 + 210m_2^2 a + 48m_1^2 a^2 + 4m_1^2 a + 239m_2 a^2 + 231m_2 a^2 + 75m_1 m_2 a^2 + 8m_1 m_2 a^2 + 14m_2 a^2 + 3m_1 m_2 a^2 - 24m_2 a^2 - 8m_1 m_2 a^2 - 4m_2 a^2 - 152a - 158m_1 a - 54m_2 a^2 - 6m_1 a^2 - 6m_1 m_2 a + 18m_2 a^2 + 6m_1 m_2 a^2 + 36a^4 + 24m_1 a^4 + 4m_1^2 a^4 - 4m_2 a^4 - 3a^5 - m_1 a^5 + m_2 a^5), \]

\[ g_7(a, m_1 + 1, m_2 - 1) = -a(2 + m_1 + m_2)(-5 - m_1 - m_2 + a)(-4 - m_1 - m_2 + a)(-3 - m_1 - m_2 + a)(-1 - m_1 - m_2 + a), \]

\[ g_8(a, m_1 + 1, m_2 - 1) = -a(1 + m_1 - m_2)(2 + m_1 + m_2 - a)(3 + m_1 + m_2 - a)(4 + m_1 + m_2 - a)(5 + m_1 + m_2 - a), \]

\[ g_9(a, m_1 + 1, m_2 - 1) = (2 + m_1 + m_2 - a)(4 + m_1 + m_2 - a)(-8m_1 - 6m_1^2 - m_1^3 + 8m_2 - m_1 m_2 + 6m_2^2 + m_1 m_2^2 + m_2^3 + 4a + 32m_1 a + 32m_2 a + 10m_1 a + 30m_2 a + 10m_1 m_2 a + 2m_1 m_2 a - 32m_2 a - 10m_1 a - 2m_1 m_2 a - m_2 a^2 - 3m_1 a^2 - 20m_2 a^2 + 3m_1^2 a^2 - 3m_2 m_2 a^2 + 2m_2 a^2 + 3m_1 m_2 a^2 + 3m_2 a^2 + 10m_1 a^3 + 3m_1^2 a^2 - 10m_2 a^2 - 3m_2^2 a - m_1 a^4 + m_2 a^4). \]

Now, consider

\[ g_1(a, m_1 + 1, m_2 - 1) - g_2(a, m_1 + 1, m_2 - 1) = a(1 + a - 2m_2)(-5 - a - m_1 - m_2)(-4 + a + m_1 - m_2) a^2 > 0. \]

Thus,

\[ a(G_1(m_1 + 1, m_2 - 1))^2 < a(G_2(m_1 + 1, m_2 - 1))^2. \]

\[ g_1(a, m_1 + 1, m_2 - 1) - g_3(a, m_1 + 1, m_2 - 1) = 2a(1 + a + 2m_2)(-5 + a - m_1 - m_2)(-4 + a - m_1 - m_2) a^3 > 0. \]

Thus,

\[ a(G_1(m_1 + 1, m_2 - 1))^3 < a(G_3(m_1 + 1, m_2 - 1))^3. \]

\[ g_1(a, m_1 + 1, m_2 - 1) - g_4(a, m_1 + 1, m_2 - 1) = a(4 + a - m_1 - m_2)(40 - 7a - 5a^2 + a^3 + 37m_1 - 6am_1 - a^2 m_1 + 11m_1^2 - am_1^2 + m_1^2 - 21m_2 + 25am_1^2 - 5a^2 m_2 - 9m_1 m_2 + 6am_1 m_2 - m_2 m_2 - 20m_2^2 + 7am_2^2 - 5m_1 m_2^2 - 3m_2^2) > 0. \]

Thus,

\[ a(G_1(m_1 + 1, m_2 - 1))^4 < a(G_4(m_1 + 1, m_2 - 1))^4. \]

\[ g_1(a, m_1 + 1, m_2 - 1) - g_5(a, m_1 + 1, m_2 - 1) = (a + 4 - a - m_1 - m_2)(40 - 7a - 5a^2 + a^3 + 37m_1 - 6am_1 - a^2 m_1 + 11m_1^2 - am_1^2 + m_1^2 - 21m_2 + 25am_1^2 - 5a^2 m_2 - 9m_1 m_2 + 6am_1 m_2 - m_2 m_2 - 20m_2^2 + 7am_2^2 - 5m_1 m_2^2 - 3m_2^2) > 0. \]

Thus,

\[ a(G_1(m_1 + 1, m_2 - 1))^5 < a(G_5(m_1 + 1, m_2 - 1))^5. \]

\[ g_1(a, m_1 + 1, m_2 - 1) - g_6(a, m_1 + 1, m_2 - 1) = (a + 4 - a - m_1 - m_2)(40 - 7a - 5a^2 + a^3 + 37m_1 - 6am_1 - a^2 m_1 + 11m_1^2 - am_1^2 + m_1^2 - 21m_2 + 25am_1^2 - 5a^2 m_2 - 9m_1 m_2 + 6am_1 m_2 - m_2 m_2 - 20m_2^2 + 7am_2^2 - 5m_1 m_2^2 - 3m_2^2) > 0. \]

Thus,

\[ a(G_1(m_1 + 1, m_2 - 1))^6 < a(G_6(m_1 + 1, m_2 - 1))^6. \]
Thus,

\[
a(G_1(m_1 + 1, m_2 - 1)^c) < a(G_6(m_1 + 1, m_2 - 1)^c).
\]

(9)

g_1(a, m_1 + 1, m_2 - 1) - g_7(a, m_1 + 1, m_2 - 1) = 2a(-5+a-m_1-m_2)(-3+a-m_1-m_2)(2+m_1-m_2) > 0.

Thus,

\[
a(G_1(m_1 + 1, m_2 - 1)^c) < a(G_7(m_1 + 1, m_2 - 1)^c).
\]

(10)

g_1(a, m_1 + 1, m_2 - 1) - g_6(a, m_1 + 1, m_2 - 1) = -a(-4+a-2m_1)(-5+a-m_1-m_2)(-3+a-m_1-m_2) < 0.

Thus,

\[
a(G_1(m_1 + 1, m_2 - 1)^c) < a(G_6(m_1 + 1, m_2 - 1)^c).
\]

(11)

g_1(a, m_1 + 1, m_2 - 1) - g_5(a, m_1 + 1, m_2 - 1) = (8a-6a^2+2a^3-2a^4-16m_1+199am_1-178a^8m_1+68a^9m_1-13a^9m_1+5a^9m_1-20m_1^2+163am_1^2-131a^2m_1^3+39a^3m_1^3-4a^4m_1^3-8m_1^4+66am_1^4-39a^2m_1^4+6a^3m_1^4-13am_1^4+4a^4m_1^4+16m_1^5-31am_1^5+86a^2m_1^5-56a^3m_1^5+13a^4m_1^5-a_5^5m_1^5+46am_1^5-12a^6m_1^5-8m_1^6+70am_1^6-39a^2m_1^6+6a^3m_1^6-2m_1^6+26am_1^6-8a^2m_1^6+3am_1^6+20m_1^7-117am_1^7+119a^2m_1^7+39a^3m_1^7+4a^4m_1^7+8m_1^8+58am_1^8-39a^2m_1^8+6a^3m_1^8+2am_1^8+8m_1^9-62am_1^9+39a^2m_1^9+6a^3m_1^9+2m_1^9-26am_1^9+8a^2m_1^9-2am_1^9+6m_1^10-13am_1^10+4a^2m_1^10-3am_1^10-6m_1^11) > 0.

Thus,

\[
a(G_1(m_1 + 1, m_2 - 1)^c) < a(G_5(m_1 + 1, m_2 - 1)^c).
\]

(12)

From (5) to (12), we have \(a(G_1(m_1 + 1, m_2 - 1)^c) < a(G_1(m_1 + 1, m_2 - 1)^c)\). Consequently, for \(2 \leq i \leq 9\), \(a(G_1(m_1, m_2)^c) \leq a(G_1(m_1, m_2)^c)\), where equality holds if \(G_1(m_1, m_2) \cong G_1(m_1, m_2)\).

Lemma 3.3. Let \(m_1\) and \(m_2\) and \(n\) be positive integers, such that \(m_1 \geq m_2 \geq 1\), \(n \geq 11\), and \(m_1 + m_2 + 5 = n\). Then \(a(G_1(m_1, m_2)^c) < a(G_1(m_1, m_2)^c)\).

Proof. Let \(Z = \{Z_v\}\) be a unit Fiedler vector of the graph \(G_1(m_1, m_2)^c\) for \(1 \leq i \leq n\). After deleting the edge \(v_4v_5\) and adding \(v_4v_5\) in \(G_1(m_1, m_2)^c\), we obtain \(G_1(m_1, m_2)\) (see Figure 1 and Figure 3, for particular values of \(m_1 = 5\) and \(m_2 = 1\) see Figure 4). Now, by (2) and Lemma 2.2, we have

\[
Z^T L(G_1(m_1, m_2)) Z = \sum_{v_i \in E(G_1(m_1, m_2))} (Z_{v_i} - Z_v)^2 = \sum_{v_i \in E(G_1(m_1, m_2))} (Z_{v_i} - Z_v)^2 = Z^T L(G_1(m_1, m_2)) Z.
\]

Using (5), we obtain \(a(G_1(m_1, m_2)^c) = Z^T L(G_1(m_1, m_2)^c) Z = Z^T (n-I) Z - Z^T L(G_1(m_1, m_2)) Z > Z^T (n-I) Z - Z^T L(G_1(m_1, m_2)) Z > a(G_1(m_1, m_2)^c)\). Consequently, \(a(G_1(m_1, m_2)^c) < a(G_1(m_1, m_2)^c)\).
4 Characterization

This section consists on the main results of the paper.

**Theorem 4.1.** Let \( n, m_1 \) and \( m_2 \) be any positive integers such that \( m_1 \geq m_2 \geq 1 \), \( n \geq 11 \) and \( n = m_1 + m_2 + 5 \), then for any bicyclic graph with three cycles \( G \in \Omega_{2,n} \),

\[
a(G(m_1, m_2)) \leq a(G),
\]

where \( 1 \leq i \leq 9 \).

**Proof.** Let \( G \) be a bicyclic graph with three cycles \( C_1(l_1), C_2(l_2) \) and \( C_3(l_3) \) with lengths \( l_1 \geq 3, l_2 \geq 3 \) and \( l_3 \geq 4 \), respectively. The cycles \( C_1(l_1) \) and \( C_2(l_2) \) are inner cycles with at least one common edge and \( C_3(l_3) \) is an outer cycle such that \( l_3 = l_1 + l_2 - 2k \), where \( k \) are common edges in \( C_1(l_1) \) and \( C_2(l_2) \). Let \( Z \) be a unit Fiedler vector of \( G \). Then, we have a sequence \( \{V_n\} \) such that

\[
Z_{v_1} \geq Z_{v_2} \geq \ldots \geq Z_{v_n}.
\]

For \( d_G(v_1, v_n) > 1 \), we can assume the path \( v_1Gv_n = v_1w_1 \ldots v_2v_n \). In the path \( v_1Gv_n \), \( d_G(v_1, v_n) = 2 \) if \( w_1 = w_2 \). Add the edge \( v_1w_1 \) and delete \( v_1w_1 \) or \( w_2v_n \) such that the resulting bicyclic graph \( G_a \) is not \( H(n, 2) \). Then by (2) and Lemma 2.2, we have

\[
Z^T L(G)Z = \sum_{v_i, v_j \in E(G)} (Z_{v_i} - Z_{v_j})^2 \leq \sum_{v_i, v_j \in E(G_a)} (Z_{v_i} - Z_{v_j})^2 = Z^T L(G_a)Z,
\]

where \( G_a \) is a bicyclic graph with three cycles \( C_1(l'_1), C_2(l'_2) \) and \( C_3(l'_3) \) having some trees attached with the vertices of one or both the cycles \( C_1(l'_1) \) and \( C_2(l'_2) \). The lengths \( l'_1, l'_2 \) and \( l'_3 \) may or may not different from \( l_1, l_2 \) and \( l_3 \) respectively. Most importantly, we note that \( d_{G_a}(v_1, v_n) = 1 \). If \( G_a \notin \{G_i : 1 \leq i \leq 9\} \), then we have the following three cases for \( G_a \):

**Case a.** Both the vertices \( v_1 \) and \( v_n \) are cycle vertices. In this case, we discuss further four possibilities (1) both the vertices are on exactly one inner cycle, (2) one of \( v_1 \) and \( v_n \) is a common vertex of both the inner cycles, (3) both the vertices \( v_1 \) and \( v_n \) are common vertices of the inner cycles, and (4) each inner cycle contains exactly one of \( v_1 \) and \( v_n \).

(1) We assume without loss of generality that both the vertices \( v_1 \) and \( v_n \) are on the cycle \( C_1(l'_1) \). Since, for \( l'_1 \geq 4 \) and \( l'_2 = 3 \) the cycles \( C_1(l'_1) \) and \( C_2(l'_2) \) have two common vertices, therefore, we can assume \( C_1(l'_1) = v_1w_1w_2w_3 \ldots w_{m-1}w_mv_1 \), where \( m = l'_1 - 2 \) and two vertices other than \( v_1 \) and \( v_n \) are also of \( C_2(l'_2) \).

(2) Suppose that \( w_{m-1} \) and \( w_m \) are common vertices of the inner cycles. If \( (Z_{w_{m-1}} - Z_{v_1})^2 \geq (Z_{w_{m-1}} - Z_{v_n})^2 \), we delete the edge \( w_{m-2}w_{m-1} \) and add \( w_{m-1}v_1 \) (as (b) is obtained from (a) in Figure 5). The resulting graph \( G_a \) is a bicyclic graph with two inner cycles \( C_1(l'_1) \) and \( C_2(l'_2) \), and an outer cycle \( C_3(l'_3) \) such that \( l'_1 = 3 = l'_2 \), \( l'_3 = 4 \), some trees are attached on \( v_1 \) in \( C_1(l'_1) \) and some trees are attached on \( v_n \) which is non cycle. Thus, \( G_{aa} \) is a bicyclic graph \( G_3(m_1, m_2) \) which is obtained when we identify \( B_1 \) by a vertex of degree 2 with end point say \( v_1 \) of an edge \( v_1v_n \) having some trees on \( v_1 \) and \( v_n \) (see Figure 1 with \( v_1v_n = v_2v_3 \)).

If \( (Z_{w_{m-1}} - Z_{v_1})^2 < (Z_{w_{m-1}} - Z_{v_n})^2 \), we delete the edge \( w_{m-2}w_{m-1} \) and add \( w_{m-1}v_n \) (as (c) is obtained from (a) in Figure 5). The resulting graph \( G_{aa} \) is a bicyclic graph with two inner cycles \( C_1(l'_1) \) and \( C_2(l'_2) \), and an outer cycle \( C_3(l'_3) \) such that \( l'_1 = 4, l'_2 = 3, l'_3 = 5 \) and some trees are attached on both \( v_1 \) and \( v_n \) in \( C_1(l'_1) \). Thus, \( G_{aa} \) is a bicycle graph which is infect \( B_2 \) with some trees on the two adjacent vertices of degree 2 i.e \( G_{aa}(m_1, m_2) \) (see Figure 2). If we proceed from the other side of the path, then for \( (Z_{w_2} - Z_{v_1})^2 \geq (Z_{w_2} - Z_{v_n})^2 \), we delete
the edge $w_1w_2$ and add the edge $w_2v_1$ otherwise, we delete $w_1w_2$ and add $w_2v_n$. Thus, the resulting graph $G_{aa}$ is a bicyclic graph with three cycles such that the lengths of the inner cycles are $l'_1 = 2$ and $l'_2 = 3$ or $l'_1 = 1$ and $l'_2 = 3$. Now, repeat the process for the vertex $w_3$ and continue up to the vertex $w_{m-1}$. Thus, we obtain the same graphs $G_3(m_1, m_2)$ and $G_6(m_1, m_2)$.

(ii) Suppose that $w_l$ and $w_{l+1}$ are common vertices of the inner cycles, where $2 \leq i \leq m - 2$. If $(Z_{w_i} - Z_{v_i})^2 \geq (Z_{w_i} - Z_{v_{i-1}})^2$, we delete $w_{i+1}$ and add $w_{i}v_{i+1}$. Now, if $(Z_{w_i} - Z_{v_i})^2 \geq (Z_{w_{i-1}} - Z_{v_{i-1}})^2$, we delete $w_{i+1}w_{i+2}$ and add $w_{i+1}v_{i+2}$, otherwise delete $w_{i+1}w_{i+2}$ and add $w_{i+1}v_{n}$. Thus, the resulting graphs are $G_3(m_1, m_2)$ or $G_6(m_1, m_2)$, respectively.

If $(Z_{w_i} - Z_{v_i})^2 < (Z_{w_i} - Z_{v_{i-1}})^2$, we delete $w_{i}w_{i+1}$ and add $w_{i}v_{i+1}$. Now, if $(Z_{w_{i-1}} - Z_{v_{i-1}})^2 \geq (Z_{w_{i-1}} - Z_{v_{i}})^2$, we delete $w_{i+1}w_{i+2}$ and add $w_{i+1}v_{1}$, otherwise delete $w_{i+1}w_{i+2}$ and add $w_{i+1}v_{n}$. Thus, the resulting graphs are $G_6(m_1, m_2)$ or $G_3(m_1, m_2)$, respectively.

(iii) Suppose that $w_1$ and $w_2$ are common vertices of the inner cycles, then we repeat (i) and the obtained graphs are same as there.

If in (1(i)-(iii)), $l'_2 \geq 4$, then we can assume $C_2(l'_2) = u_1u_2\ldots u_{l'_2}w_{i+1}$, where $w_l$ and $w_{l+1}$ are two common vertices of the cycles $C_1(l'_1)$ and $C_2(l'_2)$ for $1 \leq i \leq m - 1$ and $l = l'_2 - 2$. By the use of (1(i)-(iii)), we have $C_1(l'_1)$ with $l'_1 = 3$, some trees attached on $v_1 \in C_1(l'_1)$ and some trees attached on $v_n$ (pendant vertex) or with $l'_1 = 4$ and some trees attached on $v_1$ and $v_n$ (both are in $C_1(l'_1)$). Now, for $C_2(l'_2)$, if $(Z_{u_{l'_2}} - Z_{v_{l'_2}})^2 \geq (Z_{u_{l'_2}} - Z_{v_{l'_2}})^2$, delete the edge $u_1u_2$ and add the edge $u_1v_{1}$, otherwise delete $u_1u_2$ and add $u_1v_{n}$. Thus, the resulting graphs are $G_4(m_1, m_2)$ and $G_5(m_1, m_2)$ or $G_5(m_1, m_2)$ and $G_7(m_1, m_2)$, (see Figure 1 and Figure 2) respectively. Moreover, $l'_1 = 3 = l'_2$ is not possible as both the vertices $v_1$ and $v_n$ can not appear on only $C_1(l'_1)$.

(2) Without loss of generality suppose that $v_n$ is on the cycle $C_1(l'_1)$ and $v_1$ is a common vertex of the inner cycles. Assume that $l'_1 \geq 4$, $l'_2 = 3$ and $C_1(l'_1) = v_1v_nw_1w_2\ldots w_{m}v_1$, where $w_m$ is also a common vertex of
the inner cycles and $m = l'_2 - 2$.

If $(Z_{w_{m-1}} - Z_{v_1})^2 > (Z_{w_{m-1}} - Z_{v_2})^2$, we delete the edge $w_{m-2}w_{m-1}$ and add the edge $w_{m-1}v_1$. The resulting graph $G_{aa}$ is a bicyclic graph with two inner cycles $C_1(l'_1)$ and $C_2(l'_2)$, and an outer cycle $C_3(l'_3)$ such that $l'_1 = 3 = l'_2$, $l'_3 = 4$, some trees are attached on $v_1$ which is a common vertex of both the inner cycles and some trees are attached on $v_2$ in $C_2(l'_2)$. Thus, $G_{aa}$ is a bicycle graph $G_6(m_1, m_2)$ which is obtained when we identify $B_1$ by a vertex of degree 3 with an end point $v_1$ of an edge $v_1v_n$ having some trees on $v_1$ and $v_n$ (see Figure 1 with $v_1v_n = v_2v_n$).

If $(Z_{w_{m-1}} - Z_{v_1})^2 < (Z_{w_{m-2}} - Z_{v_2})^2$, we delete the edge $w_{m-2}w_{m-1}$ and add the edge $w_{m-1}v_2$. The resulting graph $G_{aa}$ is a bicyclic graph with two inner cycles $C_1(l'_1)$ and $C_2(l'_2)$, and an outer cycle $C_3(l'_3)$ such that $l'_1 = 4$, $l'_3 = 3$, $l'_3 = 5$, some trees are attached on $v_1$ which is a common vertex of both the inner cycles and some trees are attached on $v_2$ in $C_2(l'_2)$. Thus, $G_{aa}$ is a bicycle graph $G_5(m_1, m_2)$ (see Figure 1) which is infect $B_2$ with some trees which are attached on two adjacent vertices of degree 2 and 3 in $C_5 \subseteq B_2$.

If we proceed from the other side of the path, then for $(Z_{w_{m-1}} - Z_{v_1})^2 > (Z_{w_{m-2}} - Z_{v_2})^2$, we delete the edge $w_1w_2$ and add the edge $w_2v_1$, otherwise, we delete the edge $w_1w_2$ and add the edge $w_2v_2$. Thus, the resulting graph $H_{aa}$ is a bicyclic graph with three cycles such that the lengths of the inner cycles are $l'_2 - 2$ and $l'_3 = 3$ or $l'_1 - 1$ and $l'_2 = 3$. Now, repeat the process for the vertex $v_3$ and continue up to the vertex $w_{m-1}$. Thus, we obtain the same graphs $G_6(m_1, m_2)$ and $G_5(m_1, m_2)$.

If in (2), $l'_2 \geq 4$, then we can assume $C_2(l'_2) = u_1u_2u_3...u_{l'_2}w_1v_1u_1$, where $w_1$ and $v_1$ are two common vertices of the cycles $C_1(l'_1)$ and $C_2(l'_2)$ for $1 \leq i \leq m$ and $l = l'_2 - 2$. By the use of (2), we have $C_1(l'_1)$ with $l'_1 = 3$, some trees attached on $v_1 \in C_1(l'_1)$ and some trees attached on $v_n$ (pendant vertex) or with $l'_1 = 4$ and some trees attached on $v_1$ and $v_n$ (both are in $C_1(l'_1)$). Now, for $C_2(l'_2)$, if $(Z_{w_{m-1}} - Z_{v_1})^2 > (Z_{w_1} - Z_{v_1})^2$, delete the edge $u_{l'_1-1}u_{l'_1}$ and add the edge $u_{l'_1}v_1$, otherwise delete the edge $u_{l'_1-1}u_{l'_1}$ and add the edge $w_1v_1$. Thus, the resulting graphs $G_{aa}$ are $G_6(m_1, m_2)$ and $G_5(m_1, m_2)$ or $G_5(m_1, m_2)$ and $G_7(m_1, m_2)$, respectively. Moreover, if $l'_1 = 3 = l'_2$, then we obtain $G_2(m_1, m_2)$.

(3) Suppose that $v_1$ and $v_n$ both are common vertices of the inner cycles. Assume that $l'_1 \geq 4$, $l'_2 = 3$ and $C_1(l'_1) = v_1v_nw_1w_2w_3...w_{m-1}v_1$, where $m = l'_1 - 2$.

If $(Z_{w_{m-1}} - Z_{v_1})^2 > (Z_{w_{m-1}} - Z_{v_2})^2$, we delete the edge $w_{m-2}w_{m-1}$ and add the edge $w_{m-1}v_1$. The resulting graph $G_{aa}$ is a bicyclic graph with two inner cycles $C_1(l'_1)$ and $C_2(l'_2)$, and an outer cycle $C_3(l'_3)$ such that $l'_1 = 4$, $l'_2 = 3$, $l'_3 = 5$, some trees are attached on $v_1$ and $v_2$ which are common vertices of the inner cycles. Thus, $G_{aa}$ is a bicycle graph $G_6(m_1, m_2)$ (see Figure 2) which is obtained from $B_2$ by attaching some trees on both the vertices of degree 3.

If $(Z_{w_{m-1}} - Z_{v_1})^2 < (Z_{w_{m-2}} - Z_{v_2})^2$, we delete the edge $w_{m-2}w_{m-1}$ and add the edge $w_{m-1}v_1$. Then, check for $w_{m-2}$, if $(Z_{w_{m-2}} - Z_{v_1})^2 > (Z_{w_{m-2}} - Z_{v_2})^2$, delete the edge $w_{m-3}w_{m-2}$ and add the edge $w_{m-2}v_n$. The resulting graph is $H_B(m_1, m_2)$. If $(Z_{w_{m-2}} - Z_{v_1})^2 > (Z_{w_{m-2}} - Z_{v_2})^2$, delete $w_{m-2}w_{m-1}$ and add $w_{m-2}v_1$. Repeat this process until we reach on the vertex $w_2$. If $(Z_{w_2} - Z_{v_1})^2 > (Z_{w_2} - Z_{v_2})^2$, delete the edge $w_2w_1$ and add the edge $w_2v_n$, otherwise delete $w_2w_1$ and add $w_2v_1$. The resulting graph is $G_6(m_1, m_2)$.

If in (3), $l'_2 \geq 4$, then we can assume $C_2(l'_2) = u_1u_2u_3...u_{l'_2}v_nv_1u_1$. By the use of (3), we have $C_1(l'_1)$ with $l'_1 = 4$ and some trees attached on $v_1$ and $v_n$ (both are common vertices). Now, again repeat (3) for $C_2(l'_2)$ and we obtain $G_6(m_1, m_2)$. If $l'_1 = 3 = l'_2$, then we obtain $G_7(m_1, m_2)$.

(4) Suppose that $v_1$ is on $C_1(l'_1)$ and $v_n$ is on $C_2(l'_2)$, where $l'_1$, $l'_2 \geq 3$. We note that $d_{G_6}(v_1, v_n) \geq 2$, which is not possible.
Case b. One of $v_1$, $v_n$ is a cycle vertex.

We assume that $v_1$ is a cycle vertex and $v_n$ is non cycle vertex without loss of generality. In this case, for $v_1$, we have three possibilities (1) $v_1$ is on $C_1(l'_1)$, (2) $v_1$ is a common vertex of both the inner cycles and (3) $v_1$ is on $C_2(l'_2)$.

(1) If $v_1$ is only on $C_1(l'_1)$, then for $l'_1 \geq 4$ and $l'_2 = 3$, we have $C_1(l'_1) = v_1w_1w_2w_3...w_{l'_1 - 1}w_{l'_1}$. Assume $w_1$ and $w_{l'_1 - 1}$ are common vertices of the inner cycles, where $1 \leq i \leq m - 1$. Now, we repeat (Case a (1)) and obtain $G_3(m_1, m_2)$ or $G_4(m_1, m_2)$. If $l'_2 \geq 4$ then again by (Case a (1)), the resulting graphs are $G_3(m_1, m_2)$ and $G_5(m_1, m_2)$ or $G_6(m_1, m_2)$ and $G_7(m_1, m_2)$. Moreover, if $l'_1 = 3 = 1$, then the resulting graph is $G_3(m_1, m_2)$.

(2) If $v_1$ is a common vertex of the inner cycles. Assume that $w_m$ is another common vertex such that $l'_1 \geq 4$, $l'_2 = 3$, $C_1(l'_1) = v_1v_nw_1w_2w_3...w_m$ and $m = l'_1 - 1$. Now, by (Case a (2)), we obtain $G_4(m_1, m_2)$ or $G_5(m_1, m_2)$. If $l'_2 \geq 4$ then again by (Case a (2)), the resulting graphs are $G_6(m_1, m_2)$ and $G_7(m_1, m_2)$. Moreover, if $l'_1 = 3 = l'_2$, then the resulting graph is $G_6(m_1, m_2)$. (3) If $v_1$ is only on $C_2(l'_2)$, then follow (Case b (1)).

Case c. Both $v_1$ and $v_n$ are non cycle vertices.

Suppose that $u$ and $v$ are common vertices of the inner cycles such that $C_1(l'_1) = uvw_1w_2w_3...v_m$ and $C_2(l'_2) = uvu_1u_2u_3...u_l$, where $l'_1 \geq 4$, $m = l'_1 - 2$ and $l = l'_2 - 2$. Assume that there is a path $P$ containing the vertices $v_1$ and $v_n$ has one end point either $u$ or $v$. If $i$ is on $P$ and $w_1$ is adjacent to $u$ in $C_1(l'_1)$ such that $(Z_{w_1} - Z_{v_n})^2 \geq (Z_{w_1} - Z_{v_n})^2$, then we delete $w_1$ and add $w_1v_1$, otherwise delete $w_1$ and add $w_1v_n$. Then the resulting bicyclic graph $G_{aa}$ is in Case a or Case b.

Then by equation (2) and Lemma (3), we have

$$Z^TL(G_{aa})Z = \sum_{v_1v_j \in E(G_{aa})} (Z_{v_1} - Z_{v_j})^2 \leq \sum_{v_1v_j \in E(G_{aa})} (Z_{v_1} - Z_{v_j})^2 = Z^TL(G_{aa})Z,$$

(14)

If $G_{aa} \notin \{G_1: 1 \leq i \leq 9\}$ and there exists a pendant vertex $v$, whose neighbor $a$ is neither $v_1$ nor $v_n$, satisfying $(Z_{v} - Z_{v_n})^2 \geq (Z_{v} - Z_{v_n})^2$, then delete $av$ and add $vv_1$; otherwise delete $av$ and add $vv_n$. Repeat this rearranging until the resulting graph

$G_{aaa} \in \{G_1: 1 \leq i \leq 9\}$. Then by equation (2) and Lemma (2), we have

$$Z^TL(G_{aaa})Z = \sum_{v_1v_j \in E(G_{aaa})} (Z_{v_1} - Z_{v_j})^2 \leq \sum_{v_1v_j \in E(G_{aaa})} (Z_{v_1} - Z_{v_j})^2 = Z^TL(G_{aaa})Z$$

(15)

By (13) – (15), we have

$$a(G^c) = Z^TL(G^c) = Z^T(n - J)Z - Z^TL(G)Z \geq Z^T(n - J)Z - Z^TL(G_0)Z \geq Z^T(n - J)Z - Z^TL(G_{aaa})Z \geq Z^T(n - J)Z - Z^TL(G_{aaa})Z \geq a(G_{aaa}).$$

Hence we have $a(G^c) \geq a(G_{aaa})$. Consequently, $a(G(m_1, m_2)^c) \leq a(G^c)$ and equality holds if and only if $G(m_1, m_2) \cong G$, where $G \in \Omega_{2,n}$ and $1 \leq i \leq 9$.

Theorem 4.2. Let $n$, $m_1$ and $m_2$ be any positive integers such that $m_1 \geq m_2 \geq 1$, $n \geq 11$ and $n = m_1 + m_2 + 5$, then for any bicyclic graph with three cycles $G \in \Omega_{2,n}$,

$$a(G(n - 6, 1)^c) \leq a(G(m_1, m_2)^c) \leq a(G^c),$$

where equalities hold if and only if $G(n - 6, 1)^c \cong G(m_1, m_2) \cong G$.

Proof. The proof of this theorem follows from Lemma 3.1, Lemma 3.2 and Theorem 4.1.
Theorem 4.3. Let $n$, $m_1$ and $m_2$ be any positive integers such that $m_1 \geq m_2 \geq 1$, $n \geq 11$ and $n = m_1 + m_2 + 5$, then for any bicyclic graph with two or three cycles $G \in \Omega_n = \Omega_{1,n} \cup \Omega_{2,n}$,

$$a(G_2(n-6, 1)^c) \leq a(G_1(m_1, m_2)^c) \leq a(G^c),$$

where equalities hold if and only if $G_1(n-6, 1)^c \cong G_1(m_1, m_2)^c \cong G$.

Proof. The proof of this theorem follows from Theorem 2.3, Lemma 3.3 and Theorem 4.2.

5 Conclusions

In this paper, we have characterized the unique graph in the class of connected graphs whose complements are bicyclic having exactly three cycles with respect to the second least Laplacian eigenvalue (algebraic connectivity) of the Laplacian matrix. Mainly, we found the unique graph with minimum algebraic connectivity in the complete class of connected graphs whose complements are bicyclic with two or three cycles. The problem is still open to discuss the algebraic connectivity of the other families of the connected graphs whose complements are $k$-cyclic graphs for $k \geq 3$ (tricyclic, tetracyclic and so on.)

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