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Bond order via light-induced synthetic many-body interactions of ultracold atoms in optical lattices

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Abstract
We show how bond order emerges due to light mediated synthetic interactions in ultracold atoms in optical lattices in an optical cavity. This is a consequence of the competition between both short- and long-range interactions designed by choosing the optical geometry. Light induces effective many-body interactions that modify the landscape of quantum phases supported by the typical Bose–Hubbard model. Using exact diagonalization of small system sizes in one-dimension, we present the many-body quantum phases the system can support via the interplay between the density and bond (or matter–wave coherence) interactions. We find numerical evidence to support that dimer phases due to bond order are analogous to valence bond states. Different possibilities of light-induced atomic interactions are considered that go beyond the typical atomic system with dipolar and other intrinsic interactions. This will broaden the Hamiltonian toolbox available for quantum simulation of condensed matter physics via atomic systems.

1. Introduction

Ultracold gases loaded in optical lattices are an ideal tool for studying competing phases of quantum matter. Engineering the effective potential seen by the atoms using light beams allows to realise with optical lattices simple models of condensed matter, particle physics and even biological systems [1]. Moreover, the realisation of these models in experiments would aid in the development of applications towards quantum information processing and the development of novel quantum materials via quantum simulation [2]. Typically, one can realise effective Hamiltonians which contain short-range physical processes such as tunnelling between neighbour lattice sites and on-site interactions according to a prescribed lattice potential engineered by classical light fields, such as the Bose–Hubbard (BH) model. Long range interactions are experimentally challenging but accessible in principle via polar molecules [3, 4] or Rydberg atoms [5–7]. However, the nature of the interaction is fixed by the characteristics of its constituents. In addition, finite range interactions by other approaches of light–matter interactions [8–11], and extended BH models via dipolar interactions [12, 13] are also possible.

In contrast to the above, loading an optical lattice inside a cavity allows to engineer effective synthetic many-body interactions between light induced atomic modes with an arbitrary spatial profile [14–16]. These interactions are mediated by the light field and do not depend on the nature of the atoms considered, making them extremely tunable and suitable for realising quantum simulations of effective many-body long-range Hamiltonians. In principle fermionic, bosonic, molecular systems, etc can be studied. This allows to explore the interplay between additional non-conventional quantum many-body phases, besides from the typical superfluid (SF) and Mott-insulator (MI). It is now experimentally possible to access the regime where light–matter coupling is strong enough with cavity decay rates of MHz [17, 18] and kHz [19, 20], in the range to compete with typical short-range processes (tunnelling and on-site interactions). Moreover, bosonic ultracold atoms loaded in an optical lattice inside an optical cavity have been recently realised [17, 19], opening a new venue to analyse the interplay between competing orders of quantum matter by design. The light inside the cavity can be used to
control the formation of many-body phases of matter even in a single cavity mode by properly choosing the arrangement of the cavity, optical lattice and light pumped into the system [14, 15, 21, 22]. Additional freedom can be achieved by multimode cavities or multiple cavities extending the possibilities to condense into exotic quantum phases even further [16, 23–25]. Recent advances [26, 27, 28], will enable the experimental realisation of synthetic interactions by design with additional freedom in the near future. This can be realised as the cavity parameters (decay rates) and detunings with respect the cavity modes can be externally modified with respect to the atomic system. Moreover, the spatial profile of the cavity modes can be designed depending on the geometry of the coherent light beams pumped into the high-Q cavity [16].

In this article we present how different arrangements involving multiple probes and/or multiple light modes configurations, lead to the competition of atypical quantum many-body phases via synthetic interactions. In the past, single cavity competition between typical SF and MI phases [29, 30], density wave orders [14–16, 31–34], and disorder [35] has been addressed. In contrast to other works, here we consider the interplay with what we call ‘bond order’ and other orders in the system. Bond order is a form of self-organisation different from the usual type in 1D systems, of the Berezinskii–Kosterlitz–Thouless (BKT) type, can occur. In what follows, we analyse the system by using exact diagonalization for small number of sites in 1D. Our simulations are an indicative picture of the expected behaviour in a larger system. Additionally, we will study the behaviour related with the competition between bond-order and other orders present in the system. In particular, we find numerical evidence to support the analogy between valence bond states (VBS) [37, 38] and dimerised states that arise in the cavity system due to bond ordering via different mechanisms. Our results could be used as basis for the quantum simulation of analogous dimer states important in quantum magnetism [39]. Using classical optical lattices the AKLT Hamiltonian can be implemented in principle as the large interaction limit of the BH Hamiltonian [40]. Non-trivial entanglement properties have been found [41, 42], while bulk-boundary duality with entangled pair states occurs [43, 44] and spin glasses are also possible [45]. The states are potentially useful for measurement based quantum computation [46]. Other ultracold systems where the possibility of bond ordered states has been explored include dipolar gases with nearest neighbour and truncated finite range density interactions [47, 48], in the framework of the so-called extended BH models [4, 13]. Bond order can also occur via density dependent frustration of the hopping amplitudes with Raman assisted tunelling [49]. Moreover in condensed matter systems, bond order was first explored in low dimensions via extended Hubbard models with two species fermions [50–53].

Figure 1. Typical configurations with bond order $\mathcal{O}_B$ and density wave order $\mathcal{O}_{DW}$ of the ground state, circles denote sites $i$. We discuss the competition between these orders in the main text and their definition. (Top) For typical $\mathcal{O}_B \neq 0$, different ellipsoids correspond to pairs of nearest neighbour sites with the same matter field phase (dimers), colours denote different values of $\langle \hat{n}_i \hat{n}_{i+1} + \text{h.c.} \rangle = c_i$ with a constant $c_i > 0$, $n > i$. The thick lines in between ellipsoids have $\langle \hat{n}_{i+1} \hat{n}_{i+2} + \text{h.c.} \rangle = \tilde{c}_i$ with $\tilde{c}_i = c_i$ adjacent dimers. If $c_{i+1} = -c_i$ with $\tilde{c}_i = 0$ then maximal phase difference is established, if $\tilde{c}_i = 0$ and $c_i = 0$ maximal matterwave amplitude difference occurs. See main text on the structure of the one-body reduced density matrix. (Bottom) Different colours correspond to density values at sites $i$ for $\mathcal{O}_{DW} \neq 0$. For homogenous states, superfluid and Mott insulator there is no pattern in coherences or densities, $\tilde{c}_i = c_i \forall n$. The ground state is 4-fold degenerate for $\mathcal{O}_B$ and 2-fold degenerate for $\mathcal{O}_{DW}$.
In our treatment, we provide an alternative route to explore physics of this kind using cavity fields that relaxes the constraint on very strong on-site interaction \( U \), as cavity coupling can be tailored to simulate effective Hamiltonians with similar properties. In ultracold atoms in the fermionic version of our system, a closer analogy to investigate and simulate resonance valence bond (RVB) states important in high-\( T_c \) superconductivity [54], might be possible. Moreover, the competition between superconductivity and density wave orders is actively studied [55, 56], and light induced superconductivity is being researched [57–59]. We analyse the emergence and competition between SF, supersolid, insulating and dimerised quantum many-body phases of matter by means of the behaviour in their order parameters.

Our findings will foster the study of competing orders in multicomponent optomechanical systems [60]. Moreover, the interplay of the quantum phases we study and their generalisation, may appear in hybrid system networks [61–64]. In connection to our work, non interacting fermions in cavity systems have been studied [65–67] and even chiral states have been found [68]. Towards quantum state engineering via measurement back-action, competition with other correlated quantum many-body states [69–79] and non-Hermitian dynamics [80–83] can occur. Moreover, feedback control [84–90] can also be explored in relation to the dynamical stabilisation of quantum many-body phases. As such, the behaviour of the emergent phases in the cavity system we will show, might aid towards the design of novel quantum materials with analogous properties. It follows, that the use of the mechanisms described here could be incorporated in the future development of real materials and composite devices in hybrid solid state systems [91], where both light and matter are in the quantum limit and quantum coherence can be exploited.

The article has the structure that follows. We introduce the general model of ultracold atoms in high-\( Q \) cavity(ies) where the atoms are in the regime of quantum degeneracy. We continue by stating the effective models we will consider due to synthetic interactions between light induced atomic spatial modes. Then, we define the different competing orders that can arise in the system. Next, we present our results via the phase diagrams of competing phases. Finally, we conclude our manuscript by summarising our findings.

2. The model

The system consists of atoms trapped in a cavity (single/multi mode) or many cavities with the cavity mode frequency(s) \( \omega_c \) and decay rate(s) \( \kappa_c \) [14–16, 22, 74, 92–94]. The atomic system is subject to additional light beam(s) pumped into the system in off-resonant light scattering. The off-resonant light scattering condition means that \( \Gamma \ll |\Delta_{pa}| \), where \( \Gamma \) is the spontaneous emission rate of the atoms, where \( \Delta_{pa} = \omega_p - \omega_a \) is the detuning between the light mode(s) frequency(ies) \( \omega_p \) and the atomic resonance frequency \( \omega_a \). The scattered light from the ultracold atoms in the OL is selected and coupling is enhanced by the optical cavity(ies), generating a quantum potential. The light pumped into the system has amplitude(s) \( \Omega_p \in \mathbb{C} \) (in units of the Rabi frequency). The pump–cavity detunings are \( \Delta_{pc} = \omega_p - \omega_c \). The light is pumped from the side of the main axis of the high-Q cavity(ies), at an angle not necessarily at 90° which allows for arbitrary control of the overlap of light induced spatial modes [16]. The cavity modes couple with the atoms via the effective coupling strengths \( g_{cp} = g_c \Omega_p/(2\Delta_{pa}) \), with \( g_c \) the light–matter coupling coefficient of the cavity. The light–matter Hamiltonian describing the system after the light-field has been adiabatically eliminated [15, 21] in the good cavity limit \( \kappa_c \ll \Delta_{pc} \) is:

\[
\mathcal{H}^{\text{eff}} = \mathcal{H}^{\text{b}} + \mathcal{H}^{\text{ad}}, \quad \text{where} \quad \mathcal{H}^{\text{b}} \quad \text{is the regular BH Hamiltonian [99, 100]},
\]

\[
\mathcal{H}^{\text{b}} = -U \sum_{i,j} (\hat{b}_i^\dagger \hat{b}_j + \text{h.c.} - \mu \sum_i \hat{n}_i - \frac{U}{2} \sum_i \hat{n}_i (\hat{n}_i - 1), \quad (1)
\]

with \( U \) the nearest neighbour tunnelling amplitude, \( U \) the on-site interaction and \( \mu \) the chemical potential. The operators \( \hat{b}_i^\dagger \) create (annihilate) bosonic atoms at site \( i \), the number operator of atoms per site is given by \( \hat{n}_i = \hat{b}_i^\dagger \hat{b}_i \). The on-site interaction and hopping amplitude terms are short-range local processes. The BH Hamiltonian contains the effective parameters forming the classical optical lattice [100]. The emergent effective light-induced interaction is [15, 16],

\[
\mathcal{H}^{\text{ad}} = \sum_{\phi, \phi'} \sum_{p,q} \left( \gamma_{\phi, \phi'}^{D, p} (c, p, q) \hat{N}_p \hat{N}_q + \gamma_{\phi, \phi'}^{B, p} (c, p, q) \hat{S}_p \hat{S}_q + \gamma_{\phi, \phi'}^{D, p} (c, p, q) (\hat{N}_p \hat{S}_q + \hat{S}_p \hat{N}_q) \right), \quad (2)
\]

where

\[
\gamma_{\phi, \phi'}^{\nu, p} (c, p, q) = \frac{|g_{c, \phi}^p|}{2} \left( \frac{\langle f_{\nu, \phi'}^{p} | f_{\nu, \phi}^{p} \rangle}{\Delta_{qc} + i \kappa_c} + \text{c.c.} \right), \quad (3)
\]

with \( \hat{g}_{pc} = g_c \Omega_p/(2\Delta_{pa}) \) where \( \{ \eta, \nu \} \in \{ D, B \} \). The sum over ‘\( p \)’ ans ‘\( q \)’ go over the number of pumps and ‘\( c \)’ goes over the cavity modes (for a multi-mode cavity/several cavities). The couplings \( f_{\nu, \phi}^{p} \in \mathbb{C} \), correspond to the possible values of \( f_{\nu, \phi}^{p} \) (Wannier overlap integrals) [14–16, 74, 92–94] for each mode of the cavity system through
the inter-site amplitudes, labelled \( R \), or through the site density, labelled \( D \). These can either be for a single mode cavity with one pump and one cavity or a multi-mode cavity, and even multiple cavities and multiple pumps. These coupling constants are given by,

\[
J_{ij}^R = \int w(\mathbf{x} - \mathbf{x}_i) u_{i\alpha}(\mathbf{x}) u_{j\alpha}(\mathbf{x}) w(\mathbf{x} - \mathbf{x}_j) d^nx,
\]

where \( i' \) and \( j' \) can be the same site for density coupling or be nearest neighbours for bond coupling (inter-site densities), where \( u_{i\alpha}(\mathbf{x}) \) are the cavity(ies) and pump(s) mode functions, typically travelling or standing waves. The \( w(\mathbf{x}) \) are the Wannier functions given by the classical optical lattice in the lowest band. The light induced ‘density’ \( \hat{N}_\varphi \) and ‘bond’ \( \hat{S}_\varphi \) mode operators are such that:

\[
\hat{N}_\varphi = \sum_{i \in \varphi} \hat{n}_i, \quad \text{and} \quad \hat{S}_\varphi = \sum_{(i,j) \in \varphi} (\hat{b}_i^{\dagger} \hat{b}_j + \hat{b}_j^{\dagger} \hat{b}_i).
\]

The sums go over illuminated sites \( N_{i\varphi} \) and nearest neighbour pairs \((i, j)\) that belong to the light-induced atomic spatial mode \( \varphi \). As it has been shown [16] the coupling constants can be designed with great freedom by choosing the angle of incident light with respect to the classical optical lattice plane and the cavity axis. The spatial structure of light is useful as a natural basis to define these atomic modes, as the coupling coefficients \( J_{ij}^R \) can periodically repeat in space [14–16, 74, 75, 92, 94]. The atoms that belong to a particular light-induced atomic mode scatter light with the same phase. Thus, one can use the distribution of values of \( J_{ij}^R \) to define the light induced spatial atomic modes. As the pump and cavity modes are external to the internal structure of the system (the BH model), they provide a large set of independently tuneable parameters. This allows to tailor the effective light-induced atomic mode interaction with an arbitrary spatial profile. By addressing the density via the couplings \( J_{ij}^R \) one can generate multi-component density orders. Density wave orders correspond to different groups of atoms for each light induced atomic mode. In the case of \( J_{ij}^R \) one can generate dimer, trimers, tetramers, etc which will form as a consequence of the pattern induced to the MW coherences or ‘bonds’. These two kinds of orders will compete in addition with the SF order in the system and the Mott insulating phase of the BH model as we will see. The current effective model disregards additional density dependent Wannier functions modified dynamically by light, which are difficult to calculate self-consistently. However the proper redefinition and self-consistent determination of these functions won’t alter the essential structure of the effective Hamiltonian. This will only renormalise the effective coupling strengths and parameters of the BH model. Thus our results are applicable in a frame of reference with this renormalized parameters. In addition, coupling between cavity modes has not been included, as these processes have much smaller amplitudes compared to the pump modes.

### 3. Effective Hamiltonians

In contrast to previous works, here we focus in the large effective light–matter interaction where quantum fluctuations cannot be accounted for in mean-field theory regarding bond order. Moreover, we will consider the interplay between density coupling and bond order in the strong-coupling limit. To do this we will analyse the following Hamiltonian corresponding to a single cavity and a single pump, where the incident light illuminating from the side has been designed to scatter through the bonds and densities as a staggered following Hamiltonian corresponding to a single cavity and a single pump, where the incident light illuminating from the side has been designed to scatter through the bonds and densities as a staggered self-consistent determination of these functions won’t alter the essential structure of the effective HH model. The current effective model disregards additional density dependent Wannier functions modified dynamically by light, which are difficult to calculate self-consistently. However the proper redefinition and self-consistent determination of these functions won’t alter the essential structure of the effective Hamiltonian. This will only renormalise the effective coupling strengths and parameters of the BH model. Thus our results are applicable in a frame of reference with this renormalized parameters. In addition, coupling between cavity modes has not been included, as these processes have much smaller amplitudes compared to the pump modes.

\[
\mathcal{H}_{\text{eff}} = \mathcal{H}^{0} + \frac{g_{\text{eff}}}{N_{\varphi}} \sum_{i=0}^{N_{\varphi} - 1} (b_{i}^{\dagger} b_{i+1}^{\dagger} + \text{h.c.}) + J_{D}(\hat{B}^{\dagger} \hat{D}^{\dagger} + \hat{D} \hat{B}),
\]

the effective interaction strength is \( g_{\text{eff}} \), which depends on amplitude of the light pumped into the system and the light detunnings, equation (3). The bond \( \hat{B} \) and density operators \( \hat{D} \) are:

\[
\hat{B} = \sum_{i=0}^{N_{\varphi} - 1} (b_{i}^{\dagger} b_{i+1}^{\dagger} + \text{h.c.}) \quad \text{and} \quad \hat{D} = \sum_{i=0}^{N_{\varphi} - 1} (b_{i}^{\dagger} b_{i}).
\]

For ultracold atoms in an optical cavity in the adiabatic limit, cavity decay rates are of the order of MHz. The effective interaction strength, \( g_{\text{eff}} \), can be typically be made of the same order of magnitude or larger than on-site interactions, \( |g_{\text{eff}}| \gtrsim U \sim \omega_{0} \sim \omega_{D} \sim \text{kHz} \) [17], with \( E_{R} \) the recoil energy. Note that the ratio \( \omega_{0}/U \) can be tuned via the classical optical lattice depth and/or Feshbach resonances [100]. Essentially the sign of the light induced interaction can be chosen via the cavity-pump detunning \( \Delta \), and the amplitude by the pump strength \( \Omega_{p} \) [14–16]. In addition, without loss of generality, \( \{J_{0}, J_{D}\} \in [0, 1] \). Depending on the lattice depth of the classical optical lattice, e.g. the BH Wannier functions and the choice of illumination, the magnitude of the \( J_{0,D} \) coupling constants can be tuned using real Wannier functions [92]. Beyond a gaussian ansatz this gives \( J_{0} \approx 0 \) depending on the lattice depth of the classical optical lattice. Typically for a lattice depth of \( 5E_{R} \) (where the sing band approximation is valid) then \( J_{0} \approx 0.05 \text{[sin}(\Sigma \phi - \phi_{0})/2\text{[sin}(\Sigma \phi + \phi_{0})/2]\text{] while } J_{D} \approx 0.8\text{[cos}(\Sigma \phi - \phi_{D})/2\text{[cos}(\Sigma \phi + \phi_{D})/2]\text{] with } \Delta \phi = \phi_{1} - \phi_{0} \text{ and } \Sigma \phi = \phi_{1} + \phi_{0}. \Delta \phi (\Sigma \phi) \text{ is the} \)
difference (sum) of phases between two crossed standing waves (with phases \( \phi_{0i} \)) pumped from the side at 90° with respect to the classical optical lattice potential. The beams are arranged such that \( k_{m1} = 0 \) and \( k_{m2} = \pi/a \), with \( a \) the lattice spacing (typically \( a = \lambda/2 \) for a standing wave in the classical optical lattice). Therefore, the ratio between the value of the two contributions can be adjusted arbitrarily, e.g. for \( \beta/R = 0.25 \) we have \( \Delta \phi \approx 0.844\pi \) with \( \Sigma \phi = 0 \) for simplicity. Thus, any ratio between the coefficients \( \beta \) and \( \gamma \) is possible and can be modified in addition by changing the classical optical lattice depth, maximal bond coupling is achieved by \( \Delta \phi = \pi/2 \) while maximal density coupling is achieved by \( \Delta \phi = 0 \) with \( \Sigma \phi = 0 \). By increasing the depth of the classical optical lattice the coefficient \( \beta \) becomes smaller and it is basically negligible for a lattice depth of 15 \( ER \).

Moreover, we consider the previously unexplored scenario where via multiple cavities or multiple pumps one can perform the quantum simulation of the following light induced atomic mode interactions [16]:

- Bessel type potential, \( V_{\phi_{m}} = j_0(\pi (x_m - 1)) \), see figure 2(a).
- Morse type potential, \( V_{\phi_{m}} = [1 - e^{-(m-1)^2}]^2 - 1/2 \), see figure 2(b).

The function \( j_0 \) is the zero order spherical Bessel function of the first kind and \( \phi_{m} \) with \( m \in [1, R] \) with \( R \) light-induced spatial atomic modes. \( x_m - 1 \) are the locations of the maxima and minima of \( j_0(y) \) with \( y \in [0, 3] \). The maximum amplitude of the interactions has been chosen such that, \( \max(V_{\phi_{m}}) \sim 1 \), for simplicity. Morse type potentials are a typical phenomenological tool to model effective molecular systems. The Bessel type potential we consider, shows an example of the flexibility of the construction with respect to the degree of control that can be achieved via the synthetic light-induced atomic mode interactions. The relationship with the pump and cavity coupling via inverse discrete Fourier transforms can be found in general in [16]. Certainly other types of potentials can be tailored with great flexibility depending on the quantum many-body system we would like to simulate. For many cavity modes (multiple cavities/ multimode cavity) we have,

\[
\mathcal{H}_{\text{eff}} = \mathcal{H}^0 + \frac{g_{\text{eff}}}{N_c} \sum_{\phi, \phi'} V_{\phi - \phi'} \hat{N}_{\phi'} \hat{N}_{\phi},
\]

where the interaction depends on the mode distance \( |\phi - \phi'| \). This kind of effective many-body interaction is physically motivated to account for finite range effective interacting potentials. For many pumps in a single mode cavity, we have,

\[
\mathcal{H}_{\text{eff}} = \mathcal{H}^0 + \frac{g_{\text{eff}}}{N_c} \sum_{\phi, \phi'} V_{\phi - \phi'} \hat{N}_{\phi} \hat{N}_{\phi'},
\]

where the interaction depends on the position between light-induced atomic modes. This is potentially useful for the simulation of biological systems and other hybrid networks [61–64]. Here in contrast to the many cavity mode case, the interaction is position dependent and corresponds to the interaction between different branches, channels or nodes in the network. Without loss of generality, we will consider the case of \( R = 4 \) light-induced atomic modes, such that \( \phi_{m} \) with \( m \in [1, 4] \) for simplicity. The correspondence rules between light induced
modes \( \nu_{\text{ph}} \) and lattice sites \( i \) is shown in figure 2(c) for 1D lattice with \( N_z = 8 \). In principle, the number of pump modes can be arbitrarily increased by shining the light at different angles with respect to the cavity axis in combination with beam splitters. We call our interactions synthetic, as they are artificially designed by the choice of the spatial profile of the light pumped into the system and the cavity modes [116]. The properties of the light pumped into the system and the cavity parameters are external to the intrinsic properties of the atoms (\( J_0 \) and \( U \)) and easily tuned in the range of the atomic processes of the order of the recoil energy.

### 4. Order parameters

Bond order occurs whenever dimerised structures appear in the ground state of the Hamiltonian. These bosonic dimerised structures, akin to VBS [37], appear in the particular case where the structure of light–matter coupling alternates sign in the inter-site amplitudes or bond between two neighbouring sites [14]. Concretely, the ground state of the system is such that in order to maximise light scattering the inter-site coherences self-organise to minimise the energy. This can be extracted from the ground state configuration via a function of the operator \( \hat{B} \). In exact diagonalisation, if \( \langle \hat{B} \rangle = 0 \), it inherently implies a degenerate quantum superposition (Schrödinger 'cat-state'), if \( \langle \hat{B} \rangle \neq 0 \). Thus, a useful order parameter regarding bond-order can be defined as,

\[
O^2_B = \frac{\langle \hat{B}^2 \rangle}{N_z^2},
\]

akin to a staggered magnetisation, a bond order structure factor [49]. In the case when \( O_B = 0 \) there is imbalance between matter wave coherences in the ground state. This is a manifestation of a broken time reversal symmetry in the ground state. This is not necessarily coexisting with broken translational invariance, e. g. a ground state with density wave order. It is worth noting that this order parameter will signal bond-order whenever we are not in the ground state. This is not necessarily coexisting with broken translational invariance, e. g. a ground state between matter wave coherences in the ground state. This is a manifestation of a broken time reversal symmetry.

In large density wave insulator with maximal imbalance we have:

\[
\langle \hat{B} \rangle^2 = (\rho + 1) \rho / N_z, \text{ with } \rho \in \mathbb{Z}^+, \text{ which in large } N_z \text{ limit vanishes.}
\]

For a density wave insulator with maximal imbalance we have:

\[
\langle \hat{B} \rangle^2 = 2 \rho / N_z.
\]

In order to tell the difference between a MI and a bond ordered state we will use the fact that the on-site minimise the energy. This can be extracted from the ground state configuration via a function of the operator \( \hat{B} \). In exact diagonalisation, if \( \langle \hat{B} \rangle = 0 \), it inherently implies a degenerate quantum superposition (Schrödinger 'cat-state'), if \( \langle \hat{B} \rangle \neq 0 \). Thus, a useful order parameter regarding bond-order can be defined as,

\[
O^2_B = \frac{\langle \hat{B}^2 \rangle}{N_z^2},
\]

Maximal phase difference in the coherences is stablished when:

\[
\langle \hat{b}_{i+1} \hat{b}_{i+1}^\dagger \rangle = (\hat{b}_{i+1} \hat{b}_{i+1}^\dagger + \text{h.c.}).
\]

Maximal matterwave coherence amplitude difference is stablished when \( \langle \hat{b}_{i+1} \hat{b}_{i+1}^\dagger \rangle = 0 \) and \( \langle \hat{b}_{i+1} \hat{b}_{i+1}^\dagger + \text{h.c.} \rangle = 0 \). The typical MW coherence patterns found with bond order in the one-body reduced density matrix of sites \( (i,j) \) are:

- **Partial MW amplitude imbalance:**

\[
\begin{pmatrix}
\rho_0 & c & c & c & c & c & c \\
0 & \rho_1 & c & c & c & c & c \\
0 & 0 & \rho_2 & c & c & c & c \\
0 & 0 & 0 & \rho_3 & c & c & c \\
0 & 0 & 0 & 0 & \rho_4 & c & c \\
0 & 0 & 0 & 0 & 0 & \rho_5 & c \\
0 & 0 & 0 & 0 & 0 & 0 & \rho_6
\end{pmatrix}
\]

- **Maximal MW amplitude imbalance:**

\[
\begin{pmatrix}
\rho_0 & c & c & c & c & c & c \\
0 & \rho_1 & c & c & c & c & c \\
0 & 0 & \rho_2 & c & c & c & c \\
0 & 0 & 0 & \rho_3 & c & c & c \\
0 & 0 & 0 & 0 & \rho_4 & c & c \\
0 & 0 & 0 & 0 & 0 & \rho_5 & c \\
0 & 0 & 0 & 0 & 0 & 0 & \rho_6
\end{pmatrix}
\]
Maximal MW phase difference imbalance:

$$
\begin{pmatrix}
\rho_0 & 0 & -\tilde{\epsilon} & 0 & c & 0 & -\tilde{\epsilon} & 0 \\
0 & \rho_1 & 0 & -\tilde{\epsilon} & 0 & c & 0 & -\tilde{\epsilon} \\
-\tilde{\epsilon} & 0 & \rho_2 & 0 & -\tilde{\epsilon} & 0 & c & 0 \\
0 & -\tilde{\epsilon} & 0 & \rho_3 & 0 & -\tilde{\epsilon} & 0 & c \\
c & 0 & -\tilde{\epsilon} & 0 & \rho_4 & 0 & -\tilde{\epsilon} & 0 \\
0 & c & 0 & -\tilde{\epsilon} & 0 & \rho_5 & 0 & -\tilde{\epsilon} \\
-\tilde{\epsilon} & 0 & c & 0 & -\tilde{\epsilon} & 0 & \rho_6 & 0 \\
0 & -\tilde{\epsilon} & 0 & c & 0 & -\tilde{\epsilon} & 0 & \rho_7
\end{pmatrix}
$$

for $\tilde{\epsilon}$ and $c$ positive real constants, where each entry in the matrix corresponds to $\langle \hat{b}_i^\dagger \hat{b}_j \rangle$ for $(i, j) \in 0, \ldots, N_s - 1$. Thus, distant MW amplitudes are correlated. In a perfect SF ($U = 0$, e.g. $g_{\text{eff}} = 0$), $c = \tilde{\epsilon} = \rho_1 = \rho$. Deep in the MI ($t_0 = 0$, e.g. $g_{\text{eff}} = 0$) $\rho_1 = \rho \in \mathbb{Z}^+$, $c = \tilde{\epsilon} = 0$. A pictorial representation is given in figure 1. On the other hand the matrix representing the product of nearest neighbour coherences can be constructed, a typical structure for bond-ordered states with maximal phase difference between MW is the following:

$$
\begin{pmatrix}
\alpha & -\beta & \lambda & -\lambda & \bar{\lambda} & -\bar{\lambda} & \lambda & -\lambda \\
-\beta & \alpha & -\beta & \lambda & -\lambda & \bar{\lambda} & -\bar{\lambda} & \lambda \\
\lambda & -\beta & \alpha & -\beta & \lambda & -\lambda & \bar{\lambda} & -\bar{\lambda} \\
-\lambda & \bar{\lambda} & -\beta & \alpha & -\beta & \lambda & -\lambda & \bar{\lambda} \\
\bar{\lambda} & -\lambda & \bar{\lambda} & -\beta & \alpha & -\beta & \lambda & -\lambda \\
\lambda & -\bar{\lambda} & \bar{\lambda} & -\beta & \alpha & -\beta & \lambda & -\lambda \\
-\lambda & \lambda & -\lambda & \bar{\lambda} & -\bar{\lambda} & \lambda & -\lambda & \bar{\lambda} \\
-\lambda & -\lambda & \lambda & -\lambda & \bar{\lambda} & -\bar{\lambda} & \lambda & -\lambda
\end{pmatrix}
$$

where $\alpha, \beta, \lambda, \bar{\lambda}$ are positive real constants. Here each entry corresponds to the product of elements $\langle \hat{s}_n \hat{s}_m \rangle$ with $\hat{s}_n = (\hat{b}_n^\dagger \hat{b}_{n+1} + \text{h.c.})$. The alternating character of the sign of its elements is characteristic of bond ordered states, e.g. for SF and SS all elements are positive. Note that $O^2_{\text{DW}} = (1/N^2_s) \sum_{n,m} (-1)^{n+m} \langle \hat{s}_n \hat{s}_m \rangle$ (where we have used periodic boundary conditions, e.g. $\hat{s}_N = \hat{b}_0$). In the large $N_s$ limit we have for a bond ordered state:

$$O^2_{\text{DW}} \approx \lambda$$

where $\lambda \approx -\bar{\lambda} > 0$. In terms of the above matrix elements, deep in the MI or DW insulators we have $\lambda = \bar{\lambda} = \beta \approx 0$, thus $O^2_{\text{DW}} / N_s \approx \alpha / N_s$ for $N_s \gg 1$.

Moreover, the above order will compete and coexist with density wave order, typically given by the structure factor [95],

$$O^2_{\text{DW}} = \frac{1}{N^2_s} \sum_{i,j} (-1)^{i-j} \langle \hat{n}_i \hat{n}_j \rangle \equiv \frac{\langle \hat{N}_j^2 \rangle}{N^2_s}.
$$

(11)

DW order breaks translation invariance in the ground state and signals a $\mathbb{Z}_2$ symmetry between odd and even sites (Schrödinger ‘cat state’) in the density configurations. Deep in a density wave insulator the bond order parameter is: $O^2_{\text{DW}} = 2\rho / N_s$ while $O^2_{\text{DW}} = \rho^2$ is maximal.

We use the condensate fraction as an estimator of the SF fraction in the system,

$$f_{\text{SF}} \approx f_c = \frac{1}{2N_s} \sum_{i=0}^{N_s-1} \langle \hat{b}_i^\dagger \hat{b}_{i+1} + \text{h.c.} \rangle.
$$

(12)

Alternatively one could use the difference in energy with respect to a phase twist [98]. For an ideal SF, $f_{\text{SF}} = \rho$.

At commensurate fillings, in addition to a MI the system can present hidden string order, the string order parameter is given by,

$$O_S = \lim_{|\tilde{\epsilon}| \to \infty} \langle \hat{b}_k \hat{b}_l e^{i\tilde{\epsilon} \sum_{j<k,j<l} \hat{b}_j \hat{b}_j^\dagger} \rangle
$$

(13)

with $\hat{b}_k = \hat{n}_k - \rho$ and $\rho$ the average density per-site (the filling factor). In order to distinguish the MI and string ordered states, it is necessary to define the parity order parameter,

$$O_p = \lim_{|\tilde{\epsilon}| \to \infty} \langle e^{i\tilde{\epsilon} \sum_{j<k,j<l} \hat{b}_j} \rangle.
$$

(14)

In combination with the other order parameters, the string and parity order parameters allow to distinguish the emergence of a Haldane insulator (HI) [38, 95, 96]. While $O_{\text{DW}} = 0$, $O_p = 0$, and $f_{\text{SF}} = 0$, if $O_S > 0$, with a gapped spectrum, the system is a HI. If the spectrum is gapless ($f_{\text{SF}} = 0$) or gapped (insulator, $f_{\text{SF}} = 0$) and
order parameters is eliminated by renormalising with respect to the perfect SF value. In this section we will analyse the results from simulations performed using the effective Hamiltonian.

5.1. Bond order versus density wave order

5. Results

At integer filling \( \rho = 1 \), the simplest case to understand is when there is only density coupling \( f_{d0} \neq 0 \) and \( I_B = 0 \), as the density wave instability forms for negative \( g_{eff} \). The system in addition to SF and MI states is able to support the emergence of DW insulator and SS phases, see figures 3 (a) and (d). The SF and MI exist for \( g_{eff} I_0 > -U \) while the transition point shifts to higher values of \( t_0/U \) as \( g_{eff} I_0 \) increases. For \( g_{eff} I_0 < 0 \), compared to the system without cavity light. This occurs as on-site fluctuations are enhanced because light scatters minimally being a quantum optical lattice effect, not recoverable by simple mean-field analysis as corrections must be included [14]. Once \( g_{eff} I_0 < -U \), the system has a discontinuous transition to the DW insulator state. While increasing the effective tunnelling, the system goes from DW through SS to SF.

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Table 1. Relation between order parameters and quantum many-body phases (QP). The criteria to distinguish in our finite size simulations has been relaxed to define DW, SS with \( \mathcal{O}_{DW} > \mathcal{O}_b \) and SFD with \( \mathcal{O}_D > \mathcal{O}_{DW} \). The discussion on several finite size effects is in the main text.

| QP      | \( \mathcal{O}_b \) | \( \mathcal{O}_{DW} \) | \( \mathcal{O}_D \) | \( \Delta(\theta_i) \) | \( f_{DW} \) |
|---------|---------------------|---------------------|---------------------|---------------------|---------------------|
| SF      | 0                   | 0                   | 0                   | 0                   | 0                   |
| SS      | 0                   | =0                  | 0                   | =0                  | 0                   |
| SSD     | =0                  | =0                  | =0                  | =0                  | 0                   |
| SFD     | =0                  | 0                   | =0                  | =0                  | =0                  |
| MI      | 0                   | =0                  | 0                   | 0                   | 0                   |
| BI      | =0                  | 0                   | =0                  | =0                  | 0                   |
| BI + DW | =0                  | =0                  | =0                  | =0                  | 0                   |

\( \mathcal{O}_b > 0, \mathcal{O}_p = 0 \), then we have a type of VBS, a dimerised phase. For filling \( \rho = 1, \theta = \pi \), otherwise \( \theta \) needs to be determined with the help of additional methods [97].

If the system is in the SF state with coexisting bond order \( \mathcal{O}_b = 0 \), it is in the SF dimer phase (SFD). The system has matter wave coherence patterns but is homogeneous in the density. The typical difference between the order in the ground state for either \( \mathcal{O}_b = \sqrt{|\mathcal{O}_b|^2} \) or \( \mathcal{O}_{DW} = \sqrt{|\mathcal{O}_{DW}|} \) is shown in figure 1. Note that whenever \( \mathcal{O}_{DW} > \mathcal{O}_b \) the system will be in a DW phase either an insulator if the SF component is zero or a supersolid phase (SS) if \( f_{SB} \neq 0 \). If \( \mathcal{O}_{DW} < \mathcal{O}_b \) and with SF component different from zero the system will be in a supersolid dimer phase (SSD). In the SSD, the system has density variation and matter wave coherence pattern with finite SF fraction. Whenever the system is in the SS, SSD or SFD phases, the spectrum is gapless as there is a finite SF component in the system. In addition, it can occur that the system is in a bond insulator (BI) phase, where \( \mathcal{O}_{DW} = 0 \) and \( \mathcal{O}_b > 0 \) and the SF component is zero at incommensurate fillings. Here the system has phase pattern but there is no SF fraction or DW order and it is not a MI. In BI, dimerised structures form the ground state and it is homogenous in space. It can also happen that, we have a coexistence SF phase where, \( 0 < \mathcal{O}_{DW} \leq \mathcal{O}_b \), while the SF fraction is completely suppressed, this is a BI + DW insulating phase. This phase is a dimer insulator akin to VBS with density imbalance between components. For incommensurate filling \( \rho = 1 \) the BI + DW phase presents \( \mathcal{O}_b = 0 \) and \( \mathcal{O}_p = 0 \). Thus bond ordered phases can be gapped (BI) and gapless (SFD, SSD). SSD and SFD phases are bear similarities of RVB states [54], being bosonic gapless ground states with dimerised structures. On the other hand, BI states are similar to VBS, being gapped.

As finite size effects are considerable for small number of sites in the order parameters, to circumvent this problem, we have used the fact that in the large \( t_0/U \) limit the system will tend to be a perfect SF. Thus, all other order parameters besides \( f_{SB} \) should approach zero. Therefore, the finite size spurious contribution in other order parameters is eliminated by renormalising with respect to the perfect SF value \( f_{SB} \) in the limit \( t_0/U \gg 0 \). We subtract the SF fraction profile multiplied by the large \( t_0/U \) limit off-set due to finite size. Besides from this, some intermediate phases found will be harder to observe as the number of sites increases, concretely: SFD, SSD and SS which appear as the system moves from insulating states when \( t_0/U = 0 \) to the ideal SF in the limit \( t_0/U \gg 1 \).

In table 1, we summarise the quantum many-body phases of the system and the relation with the order parameters defined.

In what follows, we will constrain our discussion on the half-filled \( \rho = 1/2 \), and integer filling \( \rho = 1 \) cases, while considering simulations for \( N_s = 8 \) and renormalized order parameters as previously explained.

5. Results

5.1. Bond order versus density wave order

In this section we will analyse the results from simulations performed using the effective Hamiltonian (6).

At integer filling \( \rho = 1 \), the simplest case to understand is when there is only density coupling \( f_{d0} = 0 \) and \( I_B = 0 \), as the density wave instability forms for negative \( g_{eff} \). The system in addition to SF and MI states is able to support the emergence of DW insulator and SS phases, see figures 3 (a) and (d). The SF and MI exist for \( g_{eff} I_0 > -U \) while the transition point shifts to higher values of \( t_0/U \) as \( g_{eff} I_0 \) increases. For \( g_{eff} I_0 < 0 \), compared to the system without cavity light. This occurs as on-site fluctuations are enhanced because light scatters minimally being a quantum optical lattice effect, not recoverable by simple mean-field analysis as corrections must be included [14]. Once \( g_{eff} I_0 < -U \), the system has a discontinuous transition to the DW insulator state. While increasing the effective tunnelling, the system goes from DW through SS to SF.
smoothly. This can be seen in the behaviour of $f_{SB}$ and $O_{DW}$, figure 3(g). The insulating character of the MI is confirmed by the absence of string order parameter $O_{S} = 0$ away from the DW order phases, while having $O_{P} = 0$, figure 3(d), while the onsite fluctuations are also minimal figure 3(j).

When only dimer coupling occurs ($J_{D} = 0$, $J_{B} = 0$), the system for strong on-site interactions $t_{0}/U \ll 1$ supports DW for $g_{\text{eff}} I_{B}^{\perp} > 0$ and for $g_{\text{eff}} I_{B}^{\perp} < 0$ BI + DW states. For $g_{\text{eff}} I_{B}^{\perp} > 0$, the system evolves from DW to SF via an intermediate SFD phase as tunnelling increases. As $g_{\text{eff}} I_{B}^{\perp} < 0$ the system goes from the BI + DW state the SF phase as $t_{0}/U$ increases rather sharply, figure 3(h). Complementarily, figure 3(e), $O_{S}$ is different from zero as SFD and BI + DW phases emerge. We have that $O_{S} > O_{DW} = 0$ and $f_{SB} = 0$, thus a BI + DW, a coexisting bond Insulating with density wave insulating phase. This state is different from a SSD state as there is no SF component (the state is gapped), and it is neither a MI, neither a HI. Summarising, the transitions from the insulating phases towards the SF state are sharp for $g_{\text{eff}} I_{B}^{\perp} > 0$ and continuous for $g_{\text{eff}} I_{B}^{\perp} < 0$. The $O_{P}$ smoothly decreases to zero as DW is approached for $g_{\text{eff}} > 0$, while there is sharp change as $g_{\text{eff}} < 0$ in the BI + DW phase, where $O_{P} = 0$. We consider this as numerical evidence to support that the BI + DW phase is analogous to a gapped VBS state.
The analogy between dimer states in the cavity system and VBS states can be traced back to the relationship between spin operators and bosonic operators via the Schwinger mapping [101]. Thus, the bond operators with alternating sign coupling in the bosonic system induce an analogous staggered field interaction. However, typically in spin systems interactions are of local character. In contrast to this, our interactions are global but not trivial, as they are structured. This produces a similar mechanism for the formation of an anti-ferromagnetic like state. However, as we are considering soft-core bosons the analogy is not complete to spins. The dimers in the system are similar to the typical spin singlets of the original VBS [38]. The typical hardcore bosonic representation of the BH Hamiltonian (limit \( U \to \infty \)) via Matsubara–Matsuda mapping [102] \((\hat{b}_i^\dagger \rightarrow S_i^+, \hat{b}_i \rightarrow S_i^- \) and \( \hat{a}_i \rightarrow S_i^z + 1/2 \)) is:

\[
\mathcal{H}_{\text{HC}}^{\text{bf}} \approx -\frac{t_0}{2} \sum_i (S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+) - \mu \sum_i S_i^z
\]

which is an anisotropic Heisenberg model [101] while

\[
\mathcal{H}_{\text{HC}}^{\text{sd}} \approx \frac{g_{\text{eff}}}{N} \left( \sum_i \left[ \frac{J_b}{2} (-1)^i (S_i^z S_{i+1}^- - S_i^- S_{i+1}^z) + J_D (-1)^i \hat{S}_i^z \right] \right)^2
\]

Now making the above contributions isotropic, only keeping the \( J_b \) terms \((J_D = 0)\) and at fixed particle number we have:

\[
\mathcal{H}_{\text{HC}}^{\text{eff}} \approx -\frac{t_0}{2} \sum_i (S_i \cdot S_{i+1}) + \frac{g_{\text{eff}} J_b}{4N} \left( \sum_i (-1)^i (S_i \cdot S_{i+1}) \right)^2
\]

where \( S = (S^x, S^y, S^z) \). On the other hand the AKLT Hamiltonian [37] is:

\[
\mathcal{H}_{\text{AKLT}}^{\text{eff}} = \alpha \sum_i (S_i \cdot S_{i+1}) - \alpha \beta \sum_i (S_i \cdot S_{i+1})^2
\]

Therefore, one can see (17) is a global relative to (18) from which our Hamiltonian is an anisotropic relative without the hard core constraint formally. As such, some similarities might be expected between their ground states.

When density coupling and bond coupling act simultaneously \((J_b \approx 0 \text{ and } J_D \approx 0, \text{ with } J_b/J_D \approx 0.25)\), the situation interpolates in between the above two limits. However the additional bond-density coupling terms have strong effect even for small \( J_b \). Bond ordering can take over the system behaviour instead of DW order, see figures 3(g), (l), (i) and (l). Interestingly, for large on-site interactions \((\text{small } t_0/U)\) and \( g_{\text{eff}} J_D < 0 \), we find that a state with both DW and bond order occurs while being insulating, the DW does not destroy bond order. When we increase the effective tunnelling for \( g_{\text{eff}} J_D \ll 0 \), the system smoothly transitions from the BI + DW state to the SF via a mixture of SSD and SS phases. In contrast, when \( g_{\text{eff}} > 0 \) and \( t_0/U = 0 \) the system is a DW insulating state. The system transitions smoothly from this state one increases \( t_0/U \) to the SF state via an intermediate SFD phase. In general, bond order takes over with DW order as the ratio \( J_b/J_D \) increases for strong on-site interactions while smoothly transitioning to the SF state as \( t_0/U \) increases via SSD and SS \((g_{\text{eff}} J_D < 0)\) and SF dimer phases SFD \((g_{\text{eff}} J_D > 0)\). In the current parameter range explored there is no indication of a HI phase for \( \alpha = 1 \). The phase diagram of the above cases is shown in figure 4.

**Figure 4.** Panel (a) shows the system with \( J_b = 0 \) and \( J_D = 0 \), bond order is not supported only DW (insulator and SS), MI and SF. Panel (b) shows \( J_b = 0 \) and \( J_D > 0 \), DW and BI insulators are supported. SFD phase exists as intermediate phase between DW and SF for \( g_{\text{eff}} J_D > 0 \). Panel (c) shows \( J_b = 0 \) and \( J_D = 0 \) with \( J_b/J_D = 0.23 \), the intermediate SS in (a) turns into SSD and SS. The SSD has partial matter wave coherence imbalance between dimers. In (b) and (c) BI has maximal phase difference between dimers. The intermediate SS (a), SFD (b), and SSD/SS (c) phases shrink as the number of lattice sites \( N_c \) increases, SF takes over. Note that for \( g_{\text{eff}} = 0 \), the system only has MI and SF, the thin black region is not visible in (b) and (c). Parameters in all panels are the same as in figure 3.
In this section we will study the results from simulations of effective Hamiltonians for many pumps and either Bessel or Morse-like potentials. There are slight changes in critical points but overall the phase diagram can be summarised qualitatively in the phase diagram for Bessel-like interactions with many cavities, figure 7. For light-induced many-body effective interaction strengths such that $g_{\text{eff}} > g_c$, with $g_c \approx -2.25 U$ the system is in the MI for small $t_0/U$. The critical effective tunnelling...

**5.2. Bond order and synthetic potentials**

In this section we will study the results from simulations of effective Hamiltonians (8) and (9). In the case where synthetic interactions via bond coupling are considered, the situation is qualitatively the same for filling $\rho = 1$ and either for many pumps (8) or many cavities (9) and either Bessel or Morse-like potentials. There are slight changes in critical points but overall the phase diagram can be summarised qualitatively in the phase diagram for Bessel-like interactions with many cavities, figure 7. For light-induced many-body effective interaction strengths such that $g_{\text{eff}} > g_c$, with $g_c \approx -2.25 U$ the system is in the MI for small $t_0/U$. The critical effective tunnelling...
In this paper we have shown the interplay between bond ordered states (BIs, supersolid dimers and SF dimers), density wave ordered states (supersolid and density wave insulators), SF and Mott insulators due to strong cavity induced interactions in 1D, via exact diagonalization. We have shown numerical evidence to support the

6. Conclusions

In this paper we have shown the interplay between bond ordered states (BIs, supersolid dimers and SF dimers), density wave ordered states (supersolid and density wave insulators), SF and Mott insulators due to strong cavity induced interactions in 1D, via exact diagonalization. We have shown numerical evidence to support the
identification of bond ordered states with valence bond states, via the calculation of string and parity order parameters in states with bond addressing. We have investigated the suppression and stabilisation of density wave ordered phases and their competition with bond ordered phases due to different choices of synthetic light-induced many-body matter interactions. We have found that using multiple cavity modes and multiple pump modes and the single cavity single pump case but with a shifted critical $g_{\text{eff}}$ for the emergence of DW order. Parameters are: $N_s = 8$ with $\rho = 1$ and $R = 4$ light-induced modes.

**Figure 7.** (a) Order parameters at integer filling $\rho = 1$ for synthetic Bessel interaction with many cavity modes, as a function of the tunnelling $t_{0}/U$ and $g_{\text{eff}}/U$. $f_{\text{SF}}$ superfluid fraction (yellow), $O_{B}$ bond order parameter (green), $O_{DW}$ density wave order parameter (blue). (b) Phase diagram, the green region corresponds to SS. These are qualitatively similar for multiple cavity modes, multiple pumps and the single cavity single pump case but with a shifted critical $g_{\text{eff}}$ for the emergence of DW order. Parameters are: $N_s = 8$ with $\rho = 1$ and $R = 4$ light-induced modes.

**Figure 8.** Phase diagrams at half filling $\rho = 1/2$ for synthetic interactions via cavity modes. (a) Order parameters at integer filling $\rho = 1$ for synthetic Bessel interaction with many cavity modes, as a function of the tunnelling $t_{0}/U$ and $g_{\text{eff}}/U$. $f_{\text{SF}}$ superfluid fraction (yellow), $O_{B}$ bond order parameter (green), $O_{DW}$ density wave order parameter (blue). (b) Phase diagram, the green region corresponds to SS. These are qualitatively similar for multiple cavity modes, multiple pumps and the single cavity single pump case but with a shifted critical $g_{\text{eff}}$ for the emergence of DW order. Parameters are: $N_s = 8$ with $\rho = 1/2$. 

We have investigated the suppression and stabilisation of density wave ordered phases and their competition with bond ordered phases due to different choices of synthetic light-induced many-body matter interactions. We have found that using multiple cavity modes and multiple pump modes, one can modify the behaviour of the supported quantum many-body phases in the system. We have found that one can induce bond ordering even by density addressing. In general, the interplay of the BH model with the cavity induced interaction can change the nature of the quantum phase transitions that appear in the system. These can be either discontinuous or continuous depending on the design of the spatial profile of the interactions, modifying the typical scenario of the SF–MI transition (BKT type in 1D). We have shown the typical quantum many-body phases the system can support and how do these compete as relevant experimental parameters are changed and different geometries of light are chosen: tunnelling, on-site interaction and the effective light induced interaction strength. The interplay between different order parameters demonstrates the connection between the designed light induced interactions and the supported quantum many-body phases in the system. This provides a rich landscape of phases to explore experimentally with intriguing properties. Our results support the possibility to use synthetic many-body matter interactions for the quantum simulation of analogous strongly correlated states related to quantum magnetism from condensed matter. Our findings also suggest, the possibility to use them in the study of the fermionic variant of our system to perform quantum
simulations of other interesting states such as, RVB states, which are relevant in mechanisms related to the on set of Hi-temperature superconductivity in real materials.

Beyond ultracold atoms in cavities, our results can extended to other arrays of naturally occurring, synthetic, hybrid systems or solid state devices with quantum degrees of freedom with strong light–matter coupling in low dimensions. The results in principle can be applied to systems of fermions, spins, molecules (including biological ones) [103], atoms in multiple cavities [104], ions [105] and semiconductor [106] or superconducting qubits [107]. The setup we study might aid in the design of novel quantum materials, where the concepts that we describe can be exploited for the design of properties of real materials and composite devices in solid state systems with strong light–matter coupling. This opens a new chapter on what can be achieved by quantum simulation via atomic systems.

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