GENERALIZED DEHN FUNCTIONS II

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Abstract. For $G$ a group of type $F_q$, we establish the existence, finiteness, and uniqueness up to scaling of various $q$-dimensional isoperimetric profiles. We also show that these profiles all coincide for $q \geq 4$, and that significant overlap exists for $q = 3$. When $G$ has decidable word problem, this has mild consequences for the growth rates of these profiles. We also establish a metric analogue for highly connected Riemannian manifolds.

Introduction

In part I (see [5]) we examined the isoperimetric profiles $\Phi_{X,M}^{q}$ where $M$ is an orientable manifold of dimension $q \geq 2$ with nonempty boundary and $X$ is either a CW complex or a local Lipschitz neighborhood retract (LLNR). Briefly, if $f: \partial M \to X$ is a map with volume $v$ that extends to $M$, then $f$ extends to $g: M \to X$ with volume at most $\Phi_{X,M}^{q}(v)$, and $\Phi_{X,M}^{q}(v)$ is the smallest nonnegative extended real number with this property. In order for a map $g: M \to X$ to have a volume, it must be a map $(M, \partial M) \to (X^{(q)}, X^{(q-1)})$ (which in [5] is called quasi-cellular); there are several possible definitions in this case, as discussed in [5]. When $M = D^q$, these profiles are identical to the higher-dimensional Dehn functions defined in [1]. Similar profiles $\Phi_{X,q}^{q}$, for which we replace maps $\partial M \to X$ and $M \to X$ with $(q-1)$- and $q$-dimensional chains respectively, were also defined, both in [5] and earlier in [3] and [6].

Unlike the more familiar Dehn function of a complex (think $M = D^2$), the functions $\Phi_{X,M}^{q}$ and $\Phi_{X,q}^{q}$ (for $X$ a finite complex) are not dependent on $\pi_1(X)$ alone; the homotopy groups $\pi_2(X)$ through $\pi_{q-1}(X)$ are also relevant. Moreover, if the higher homotopy groups are nonzero, it is possible for $\Phi_{X,M}^{q}$ and $\Phi_{Y,M}^{q}$ to be different functions, and $\Phi_{X,M}^{q}$ may depend nontrivially on $M$ (this is not yet clear). The best theorem we have at present is the following: Given a continuous function $f: X \to Y$ where $f_*: \pi_t(X) \to \pi_t(Y)$ is an isomorphism for $1 \leq t < q$, the functions $\Phi_{X,M}^{q}$ and $\Phi_{Y,M}^{q}$ are quasi-equivalent; that is,

$$\Phi_{X,M}^{q}(v) \leq A \cdot \Phi_{Y,M}^{q}(Bv) + Cv + D$$

for some constants $A$, $B$, $C$, $D$, and vice versa. The chain versions $\Phi_{X,q}^{q}$ and $\Phi_{Y,q}^{q}$ are also quasi-equivalent. Moreover, if $X$ is a compact Riemannian manifold or compact Lipschitz neighborhood retract (CLNR) with a triangulation, the profile $\Phi_{X,M}^{q}$ may be interpreted as applying to cellular maps or to Lipschitz maps; the two interpretations are quasi-equivalent functions, and similarly for $\Phi_{X,q}^{q}$.

As in [1] (with $M = D^q$) and [2], given a finite CW complex or CLNR $X$ where $X$ is $(q-1)$-connected, one can define the isoperimetric profiles of $G = \pi_1(X)$

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to be those of $\tilde{X}$; that is, $\delta^M_G = \Phi^{G,M}_G = \Phi^{\tilde{X},M}$ and $FV^G_T = \Phi^{G,q}_G = \Phi^{\tilde{X},q}_G$. The above results ensure that $\Phi^{G,M}_G$ and $\Phi^{G,q}_G$ are well-defined up to quasi-equivalence. It remains to ask whether $\Phi^{G,M}_G = \Phi^{G,q}_G$, or whether $\Phi^{G,M}_G = \Phi^{G,N}_G$ for $N$ another $q$-dimensional connected orientable manifold with nonempty boundary.

Our major result is that $\Phi^{G,M}_G = \Phi^{G,q}_G$ almost everywhere for $q \geq 4$ and $M$ a $q$-dimensional manifold, and that $\Phi^{G,q}_G = \Phi^{G,N}_G$ for $M$ and $N$ manifolds of dimension $q \geq 3$ with $\partial M = \partial N$. More generally, $\Phi^{X,M}_X = \Phi^{X,q}_X$ almost everywhere and $\Phi^{X,M}_X = \Phi^{X,N}_X$ under these conditions, provided $X$ is $(q - 1)$-connected. In the cellular case, these equalities are exact; in the metric case, the second is exact, while the first may fail at discontinuities of the functions (of which there are at most countably many, since both functions are increasing). This generalizes a theorem in [12], which states that $\Phi^{G,M}_G \leq \Phi^{G,D}_D$ provided $q \geq 4$ and either $\partial M$ is connected or $\Phi^{G,D}_D$ is superadditive, and which can be proved by itself by the methods of lemma 1.

The proof uses, and is inspired by, a classic theorem of Brian White, proved in [12]: Let $X$ be a simply connected Riemannian manifold, let $M^q$ be an orientable connected manifold with boundary where $q \geq 3$, and let $f: M \to X$ be Lipschitz. Then the Plateau problem for Lipschitz maps $\gamma f \rel \partial M$ is equivalent to the Plateau problem for integral currents $T \sim f_\#([M])$.

Since we are looking at all functions which fill a given map $f: \partial M \to X$, and all currents $T$ where $\partial T = f_\#([M])$, this theorem is not quite sufficient. To finish the proof, we need for any integral current $T$ a function $f$ where $f_\#([M])$ is very close to $T$. We use the assumption that $X$ is $(q - 1)$-connected, the Hurewicz maps, and the strong approximation theorem to prove that such a function must exist. As in [12], we start with $X$ a CW complex (a mild generalization, as [12] uses simplicial complexes) and use this to prove the metric case.

Section 1 establishes the above argument in detail, allowing us in section 2 to define the isoperimetric profiles for certain groups $G$ and establish their equality. In a third section, we show that the profiles for a given group $G$ are finite everywhere, and computably bounded provided $G$ has solvable word problem.

Most of the concepts and conventions used herein were introduced in [5], and the acknowledgements there apply here as well.

1. The map vs. current equivalence

Let $X$ be a $(q - 1)$-connected CW complex where $q \geq 3$, and let $M$ be a $q$-dimensional compact orientable manifold with $\partial M \neq \emptyset$. We want to show that $\Phi^{X,M}_X \leq \Phi^{X,q}_X$, with equality in dimensions $q \geq 4$. To do this, we must show that chains in dimensions $q$ and $q - 1$ can be represented by functions $M^{q-1} \to X^{(q-1)}$ and $M^q \to X^q$, where $M$ is some compact manifold, possibly with boundary. The easiest cases are where $M = D^q$ or $M = S^{q-1}$; here we can interpret “volume” to mean “word length in $\pi_q(X^{(q)}, X^{(q-1)})$” or the same one dimension down. Since $X$ is highly connected, the Hurewicz maps are length-preserving isomorphisms, and this is sufficient. For more general $M$, we can triangulate $M$ and put all the volume on a single cell.

The metric case is similar, with one wrinkle. If $T$ is a polygonal current in $X$ of dimension $q$, then $T$ can be represented by a Lipschitz map $f: M \to X$ with the same volume (by the above logic applied to some highly connected simplicial complex which supports $T$). However, an integral current $T$ with volume $v$ can only
be approximated by a polygonal current, with volume at most $v + \epsilon$. Thus, if $\Phi_{\text{met}}^{X,M}$ has a jump discontinuity at $v$, then $\Phi_{\text{cut}}^{X,M}(v)$ lies between $\Phi_{\text{met}}^{X,M}(v)$ and the right limit. As $\Phi_{\text{met}}^{X,M}$ is an increasing function, this implies that $\Phi_{\text{met}}^{X,M} = \Phi_{\text{cut}}^{X,M}$ almost everywhere, and certainly the functions are quasi-equivalent.

To begin, consider the diagram

$$
\begin{array}{ccc}
\pi_q(X^{(q)}, X^{(q-1)}) & \xrightarrow{\varphi} & H_q(X^{(q)}, X^{(q-1)}) \\
\downarrow \varphi & & \downarrow \rho \\
\pi_{q-1}(X^{(q-1)}) & \xrightarrow{\varphi} & H_{q-1}(X^{(q-1)}) \\
\downarrow j & & \downarrow j_* \\
\pi_{q-1}(X^{(q-1)}, X^{(q-2)}) & \xrightarrow{\varphi} & H_{q-1}(X^{(q-1)}, X^{(q-2)})
\end{array}
$$

where $j: (X^{(q-1)}, s) \to (X^{(q-1)}, X^{(q-2)})$ is the inclusion, each $\varphi$ is a Hurewicz homomorphism, and each $\partial$ is the usual boundary map. By both parts of theorem 7.4.3, this diagram commutes. By the Hurewicz isomorphism theorem, all of the maps $\varphi$ are isomorphisms if $q \geq 4$, and the top and bottom maps preserve word length. If $q = 3$, then the top two $\varphi$ are still isomorphisms, the top $\varphi$ still preserves length, and the last $\varphi$ is a length-reducing epimorphism. The $j_*$ on the right is an injection, by the long exact sequence for $(X^{(q-1)}, X^{(q-2)})$ (certainly $H_{q-1}(X^{(q-2)}) = 0$), so the $j_*$ on the left is an injection as well for $q \geq 4$. (The left $j_*$ is also an injection if $q = 3$ by the exact sequence of homotopy groups, since $\pi_2(X^{(1)}) = 0$.) Note that the composition $j_* \circ \partial$ on the right is our previous notion of $\partial$ on chains; since $j_*$ is 1-1, we might ignore this distinction.

**Lemma 1.** Let $q \geq 3$, let $\tau$ be a triangulation of $M$, and let $X$ be a $(q-1)$-connected CW complex. Let $f: \partial M \to X$ be quasi-cellular, and let $T$ be a $q$-chain of $X$ where $\partial T = f_\tau(\partial M)$, Then there exists a $\tau$-cellular map $g: M \to X$ where $\partial g = f$, $g_\tau(\partial M) = T$, and $\text{Vol}_g = ||T||$.

**Proof.** Recall that $\text{Vol}_g$ is the sum, over all $q$-cells $\Delta$ of the triangulation $\tau$, of the word length of $[g \mid \Delta]$ in $\pi_q(X^{(q)}, X^{(q-1)})$. In fact, we will construct $g$ so that every $q$-cell except possibly one is sent to $X^{(q-1)}$. Let $G = (V,E)$ be the undirected graph where $V$ is the set of $q$-cells of $\tau$, and where $\{v,w\} \in E$ iff $v$ and $w$ share a $(q-1)$-face. $G$ is connected, so let $R$ be a spanning tree for $G$. We define a procedure which will define $g$ cell by cell on $M$, while removing cells from $R$. At any given stage, $g$ will be defined only on $\partial M$ and on those cells which are no longer in $R$. In particular, if $\{v,w\} \in R$ at any stage, then $g$ will not be defined on the interior of $v \cap w$, as this cell is neither a subset of $\partial M$ nor a face of any $q$-cell where $g$ is already defined.

Proceed as follows: While $R$ contains more than one vertex, let $v$ be a leaf of $R$ and let $w$ be the unique $q$-cell where $\{v,w\} \in R$. Let $D = (\partial v) \setminus (v \cap w)^o$, which is homeomorphic to $D^{q-1}$. There is some subcomplex of $D$ on which $g$ is already defined (possibly empty). Since $X^{(q-1)}$ is $(q-2)$-connected, we may extend $g$ in some way to all of $D$. Finally, there is a retraction $r: v \to D$; define $g$ on $v$ as $g \circ r$. (Note that $g_\tau([v]) = g_\tau(\partial [v]) = 0$.) Now that $g$ is defined on $v$, remove the vertex $v$ and the edge $\{v,w\}$ from $R$. The remaining cells still form a connected, etc. manifold, at least in the PL category, so we may repeat.
When this procedure is finished, let $\Delta$ be the unique q-cell remaining in $R$, so that $g$ is defined on $M \setminus \Delta^q$. Since $g_0([v]) = 0$ for every other q-cell $v$, we have $g_0([\partial \Delta]) = \partial T$. Choose $h: (D^q, S^{q-1}) \to (X^{(q)}, X^{(q-1)})$ where $\varphi(h) = T$. Then there is a homotopy $H: \partial h \simeq f$. Attach $H$ to $h$ to obtain $g | \Delta$. \hfill $\Box$

**Theorem 1.** Let $q \geq 4$ and $\partial M \neq \emptyset$, and let $X$ be a $(q-1)$-connected CW complex. Then $\tilde{\Phi}_{\text{ch}}^{X,q} = \tilde{\Phi}_{\text{cell}}^{X,M}$. If $q = 3$, then $\tilde{\Phi}_{\text{cell}}^{X,M} \leq \tilde{\Phi}_{\text{ch}}^{X,q}$.

**Proof.** Let $q \geq 3$, let $f: \partial M \to X^{(q-1)}$ be a quasi-cellular function, and suppose $\text{Vol}_\tau f \leq n$. There is a triangulation $\tau$ of $M$ where $\text{Vol}_\tau f = \text{Vol}_\tau f$. Because $X^{(q)}$ is $(q-1)$-connected, there is some extension of $f$ to a $\tau$-cellular map $f': M \to X$. Thus the $(q-1)$-chain $S = f_1([\partial M])$ is the boundary of the q-chain $f_1([M])$. Let $T$ be a q-chain of least possible volume where $\partial T = S$. By lemma 1 there is a $\tau$-cellular map $g: M \to X$ where $\partial g = f$ and $\text{Vol}_\tau g = ||T|| \leq \tilde{\Phi}_{\text{ch}}^{X,q}(n)$. Generalizing over all $f$, $\tau$, and $n$, $\tilde{\Phi}_{\text{cell}}^{X,M} \leq \tilde{\Phi}_{\text{ch}}^{X,q}$.

To see the reverse inequality for $q \geq 4$, choose a triangulation $\tau$ on $M$ and a q-cell $\Delta$ with at least one face on the boundary on $M$; call this face $\delta$. Let $s$ be the map on $M$ which collapses all points of $M$, except those in $\Delta^q$ or $\delta^q$, to a single point $*$. $s$ is a map from $M$ to $D^q$ which is a diffeomorphism $\Delta^q \cong (D^q)^o$ and $\delta^q \cong (S^{q-1} \setminus \{\}$. Given a $(q-1)$-boundary $S = \partial T$ where $||S|| \leq n$, let $f': S^{q-1} \to X^{(q-1)}$ where $\varphi([f']) = S$ and $f'(\ast) = X^{(q-1)}$ (this is where we use $q \geq 4$). Let $f = f' \circ s: \partial M \to X^{(q-1)}$, so that $\text{Vol}_\tau f = ||S||$. $f'$ has a filling $g'$, by the Hurewicz isomorphism, so $f$ has a filling $g' \circ s$. Choose a filling $g$ for $f$ so that $\text{Vol}_\tau g$ is minimum; then $T = g_0([M])$ fills $S$ and

$$\text{FV}_{\text{ch}}(S) \leq ||T|| \leq \text{Vol}_\tau g = \text{FV}_{\text{cell}} f \leq \tilde{\Phi}_{\text{cell}}^{X,M}(n).$$

Generalizing over all $S$ and $n$, $\tilde{\Phi}_{\text{ch}}^{X,q} \leq \tilde{\Phi}_{\text{cell}}^{X,M}$. \hfill $\Box$

**Corollary 1.** For $q \geq 4$, $X$ as above, and $M$, $N$ manifolds with nonempty boundary, $\tilde{\Phi}_{\text{cell}}^{X,M} = \tilde{\Phi}_{\text{cell}}^{X,N}$. This still holds true for $q = 3$ provided $\partial M \cong \partial N$.

**Proof.** The case $q \geq 4$ follows directly from theorem 1, $\tilde{\Phi}_{\text{cell}}^{X,M} = \tilde{\Phi}_{\text{ch}}^{X,q} = \tilde{\Phi}_{\text{cell}}^{X,N}$. For $q = 3$, refer to the first paragraph of the above proof; by this reasoning, every quasi-cellular function $f: (\partial M = \partial N) \to X^{(q-1)}$ has a filling on both $M$ and $N$, and moreover $\text{FV}_{\text{cell}} ^{X,M} f = \text{FV}_{\text{ch}} f_1([\partial M]) = \text{FV}_{\text{cell}} f$. Thus $\tilde{\Phi}_{\text{cell}}^{X,M} = \tilde{\Phi}_{\text{cell}}^{X,N}$. \hfill $\Box$

Now we address the metric case. Let $X$ be an LLNR; that is, let $X \subseteq U \subseteq \mathbb{R}^N$ where $U$ is open in $\mathbb{R}^N$, and let $r: U \to X$ be a locally Lipschitz retraction.

**Theorem 2.** Let $q \geq 3$, $\partial M \neq \emptyset$, $X$ a $(q-1)$-connected LLNR, $P$ a finite simplicial complex of dimension $q$, and $\psi: P \to X$ a 1-1 Lipschitz map. Let $T = \psi([P])$. Then there exists $f: M \to X$ where $f_1([M]) = T$ and $\text{Vol}_{\text{met}} f = \text{M}(T)$. If $q \geq 4$, one may choose $f$ so that $\text{Vol}_{\text{met}} \partial f = \text{M}(\partial T)$.

**Proof.** Construct $Q \supset P$ so that $Q$ is q-dimensional and $(q-1)$-connected. (For example, find a high-dimensional simplex which contains $P$ as a subcomplex and take its q-skeleton.) Extend $\psi$ to a continuous map $\psi': Q \to X$; this is possible because $X$ is $(q-1)$-connected. Mollify $\psi'$ outside $P$ to obtain a Lipschitz map $\psi'': Q \to U$, and let $\psi''' = r \circ \psi''$. Then $\psi'''$ is a Lipschitz map $Q \to X$ which extends $\psi$. For convenience, refer to $\psi'''$ as $\psi$. \hfill $\Box$
As in the proof of theorem 1, choose a triangulation $\tau$ of $M$, a cell $\Delta$ intersecting $\partial M$, and a collapsing map $s$: $(M, \partial M) \to (D^q, S^{q-1})$ which is a diffeomorphism on $\Delta^o$ and $(\Delta \cap \partial M)^o$. Choose $q: (D^q, S^{q-1}) \to (Q^q, Q^{q-1})$ where $q([g]) = [P]$. We may assume that $g$ covers each point in the interior of a $q$-cell of $P$ exactly once, that it does so smoothly, and that $g[D^q]$ does not intersect the interior of any other $q$-cell of $Q$; and similarly for $\partial g$ provided $q \geq 4$. Let $f = \psi \circ g \circ s$. Then

$$f_2([M]) = \psi_2(q_2([D^q])) = \psi_2([P]) = T.$$

Over every cell of $\tau$ except for $\Delta$, the volume form of $f^*(ds^2)$ disappears, and we calculate

$$\text{Vol}_{\text{met}} f = \text{Vol}_{\text{met}}(\psi \circ g) = \int_P \psi^*(ds^2) = M(T),$$

the last because $\psi$ is 1-1. A similar calculation shows that $\text{Vol}_{\text{met}} \partial f = \text{M}(\partial T)$ for $q \geq 4$.

A similar result holds for $q \geq 3$ and $\partial M = \emptyset$, provided $\partial T = 0$. (The only way this can happen is if $\partial P = \emptyset$ as well, which by exactness implies $P = \partial(P')$ for some $P' \in H_{q-1}(Q^{q-1}) \cong \pi_{q-1}(Q^{q-1})$.)

**Theorem 3.** Let $q \geq 4$ and let $M$ and $X$ be as in theorem 2. Then

$$\Phi^{X,M}_{\text{met}} \leq \Phi^{X,q}_{\text{cur}} \leq \Phi^{X,M}_{\text{met}}.$$  

In particular, $\Phi^{X,M}_{\text{met}} = \Phi^{X,q}_{\text{cur}}$ almost everywhere. If $q = 3$, then $\Phi^{X,M}_{\text{met}} \leq \Phi^{X,q}_{\text{cur}}$.

Recall that $\overline{f}$ is the upper envelope of $f$, where $f$ is any map from a topological space to $\mathbb{R}$.

**Proof.** [12 theorem 3] tells us that, given a function $f: M \to X$,

$$\inf \{ \text{Vol}_{\text{met}} g : g \simeq f \text{ rel } \partial M \} = \inf \{ \text{M}(T) : T = f_2([M]) \in \mathcal{B}_q(X) \}. $$

Let $q \geq 3$, $r \geq 0$, and $f \in C^{0,1}(\partial M, X)$, $\text{Vol}_{\text{met}} f \leq r$. Since $X$ is $(q-1)$-connected, $f$ is the boundary of some continuous map $h: M \to X$, and $h$ can be mollified and retracted to a Lipschitz map $M \to X$. Thus $S = f_2(\partial M)$ is the boundary of $h_2^*([M])$.

Let $R \in \mathcal{I}_q(X)$ where $\text{M}(R) < \text{FV}_{\text{cur}} S + \epsilon$. $R - h_2^*([M])$ is a $q$-cycle, which is a $q$-boundary up to an element of $H_q(X)$ (singular homology). But $H_q(X) \cong \pi_q(X)$, so by modifying $h$ in a disk, we may assume $R - h_2^*([M])$ is a boundary. Thus

$$\text{FV}_{\text{met}} f \leq \inf \{ \text{Vol}_{\text{met}} g : g \simeq h \text{ rel } \partial M \} = \inf \{ \text{M}(T) : T = h_2^*([M]) \in \mathcal{B}_q(X) \} \leq \text{M}(R) < \Phi^{X,q}_{\text{cur}}(r) + \epsilon.$$

Taking $\epsilon \to 0$ and generalizing over all $f$ and $r$, we have $\Phi^{X,M}_{\text{met}} \leq \Phi^{X,q}_{\text{cur}}$.

Conversely, for $q \geq 4$ let $S \in \mathcal{I}_{q-1}(X)$ where $\partial S = 0$ and $\text{M}(S) \leq r$. By the strong approximation theorem [4 lemma 4.2.19], $S$ is homologous by a current $R$ where $\text{M}(R) < \epsilon$ to a polyhedral current $S' = \psi_j([P]), \text{M}(S') \leq \text{M}(S) + \epsilon$. Note that $\partial S' = 0$ as well. By theorem 2 choose $f: \partial M \to X$ where $f_2(\partial M) = S'$ and $\text{Vol}_{\text{met}} f = \text{M}(S')$. $f$ extends to some $g: M \to X$; choose $g$ where $\text{Vol}_{\text{met}} g < \text{FV}_{\text{met}}(f) + \epsilon$. Then

$$\text{FV}_{\text{cur}} S < \text{M}(g_2([M])) + \epsilon < \text{FV}_{\text{met}}(f) + 2\epsilon \leq \Phi^{X,M}_{\text{met}}(r + \epsilon) + 2\epsilon;$$

taking $\epsilon \to 0$, $\text{FV}_{\text{cur}} S \leq \Phi^{X,q}_{\text{cur}}(r)$ (since $\Phi^{X,M}_{\text{met}}$ is increasing). Generalizing over $S$ and $r$, we have $\Phi^{X,q}_{\text{cur}} \leq \Phi^{X,M}_{\text{met}}$. 

A function can differ from its upper envelope only at points of discontinuity; since \( \Phi_{\text{met}}^{X,M} \) is increasing, there are at most countably many of these (see [9]), so \( \Phi_{\text{met}}^{X,M} = \Phi_{\text{met}}^{X,q} \) almost everywhere. \( \square \)

**Corollary 2.** For \( q \geq 4 \), \( X \) as in theorem 2 and \( M \) and \( N \) manifolds with nonempty boundary, \( \Phi_{\text{met}}^{X,M} = \Phi_{\text{met}}^{X,N} \) almost everywhere. Provided \( \partial M \cong \partial N \), the conclusion holds for \( q \geq 3 \) and with exact equality.

**Proof.** The first follows from theorem 3. For the second, follow the reasoning in the first paragraph of the preceding proof. Every \( f \in C^{0,1}(\partial M, X) \) has a filling, and \( FV_{\text{met}}^M f = FV_{\text{cur}} f(\partial M) = FV_{\text{met}}^N f \). Therefore \( \Phi_{\text{met}}^{X,M} = \Phi_{\text{met}}^{X,N} \). \( \square \)

The case \( q = 3 \) deserves some attention. For simplicity, assume that we only consider \( M \) where \( \partial M \) is connected; thus \( \partial M = \Sigma_g \) is the surface of genus \( g \) for some \( g \geq 0 \). By Corollary 2 only \( \partial M \) is relevant, so assume \( M = \Gamma_g \) is the solid torus of genus \( g \). If \( g \leq h \), then every map \( \Gamma_g \to X \) can be composed with a collapsing map \( \Gamma_h \to \Gamma_g \) to produce a new map with equal volume and filling volume. Thus \( \Phi_{\text{cell}}^{X,\Gamma_g} \leq \Phi_{\text{cell}}^{X,\Gamma_h} \) and \( \Phi_{\text{cur}}^{X,\Gamma_g} \leq \Phi_{\text{cur}}^{X,\Gamma_h} \) almost everywhere. As every 2-chain or polygonal 2-current \( T \) can be represented by a map \( f: \Sigma_g \to X \) for some \( g \), we have \( \Phi_{\text{ch}}^{X,3} = \lim_{g \to 0} \Phi_{\text{cell}}^{X,\Gamma_g} \) almost everywhere and \( \Phi_{\text{cur}}^{X,3} = \lim_{g \to 0} \Phi_{\text{met}}^{X,\Gamma_g} \) almost everywhere.

As we will note later, there are spaces \( X \) for which \( \Phi_{\text{cell}}^{X,1} \neq \Phi_{\text{cell}}^{X,1} \). Such a separation between \( \Phi_{\text{cell}}^{X,\Gamma_g} \) and \( \Phi_{\text{cell}}^{X,\Gamma_h} \) for \( 1 \leq g < h \), or between \( \Phi_{\text{cell}}^{X,\Gamma_g} \) and \( \Phi_{\text{ch}}^{X,3} \), is not yet known.

### 2. Applications to geometric group theory

We say that a group \( G \) is of type \( F_q \) if there is a CW complex \( X = K(G,1) \) where \( X(q) \) is finite. Equivalently, \( G \) is of type \( F_q \) if there is a finite CW-complex \( Y \) where \( Y \) is \((q-1)\)-connected and \( \pi_1(Y) = G \). \( Y \) may be taken as \( q \)-dimensional; alternatively, \( Y \) may be a compact manifold.

For example, every group is of type \( F_0 \). A group is of type \( F_1 \) if it is finitely generated, and of type \( F_2 \) if it is finitely presented. Every type \( F_q \) is a strict subtype of \( F_{q-1} \); this follows in the case \( q = 3 \) by the results in [11].

In this case, where \( X = K(G,1) \) has finite \( q \)-skeleton, we may take the functions \( \Phi_{\text{ch}}^{X,q} \) and \( \Phi_{\text{cell}}^{X,M} \) (for \( M \) a \( q \)-manifold) as invariants of \( G \). For example, \( \Phi_{\text{cell}}^{X,D^2} \) is the classical Dehn function, while \( \Phi_{\text{cell}}^{X,D^3} \) is the higher-order Dehn function \( \delta_{q-1} \) studied in [4]. As in these cases (and for similar reason), if we change \( X \) to a different \( K(G,1) \), we obtain a quasi-equivalent function. By the results of the last section, many of these functions coincide.

**Lemma 2.** Let \( q \geq 2 \), and let \( G \) be a group of type \( F_q \). Let \( X \) and \( Y \) be CW complexes which are \( K(G,1) \)'s and where \( X(q) \) and \( Y(q) \) are both finite. Then \( \Phi_{\text{ch}}^{X,q} \approx \Phi_{\text{ch}}^{Y,q} \). Also \( \Phi_{\text{cell}}^{X,M} \approx \Phi_{\text{cell}}^{Y,M} \) for \( M q \)-dimensional and \( \partial M \neq \emptyset \).

**Proof.** Let \( \varphi: \pi_1(X,*) \to \pi_1(Y,*) \) be an isomorphism. By [7] Theorem 1B.9, there is a continuous function \( f: X \to Y \) where \( f_* = \varphi \). The conclusions follow from [5] theorem 2]. \( \square \)

**Definition 1.** Let \( q \geq 2 \). Let \( G \) be a group of type \( F_q \), and let \( X \) be a \( K(G,1) \) where \( X(q) \) is finite. The **chain isoperimetric profile of \( G \)** in dimension \( q \) is that of
Likewise, for $M$ a $q$-manifold with $\partial M \neq \emptyset$, the \textit{cellular isoperimetric profile} of $G$ for $M$ is that of $\tilde{X}$;

$$\Phi_{\text{ch}}^{G,M} := \Phi_{\text{ch}}^{\tilde{X},M}.$$  

For example, if $G$ is finitely presented, the function $\Phi^{G,D^2}_{\text{ch}}$ is defined, and is in fact the usual Dehn function of $G$. As one would expect, this definition is an abuse of language, since $\Phi_{\text{ch}}^{G,q}$ and $\Phi_{\text{cell}}^{G,M}$ are defined only up to quasi-equivalence. It still makes sense to ask whether these functions are linear, or polynomial, or exponential, or computably bounded, or everywhere finite.

\textbf{Theorem 4.} Let $G$ be a group of type $F_q$, where $q \geq 4$. Then for any $q$-manifold $M$ with $\partial M \neq \emptyset$, $\Phi_{\text{cell}}^{G,M} \simeq \Phi_{\text{ch}}^{G,q}$. If $q = 3$, then $\Phi_{\text{cell}}^{G,M} \ll \Phi_{\text{ch}}^{G,q}$, and if $M$ and $N$ are two 3-manifolds with $\partial M = \partial N \neq \emptyset$, then $\Phi_{\text{cell}}^{G,M} \simeq \Phi_{\text{cell}}^{G,N}$.

\textbf{Proof.} Apply theorem 1 and corollary 2 to any $K(G,1)$ with finite $q$-skeleton. \hfill \Box

\textbf{Theorem 5.} Let $X$ be a closed Riemannian manifold where $\pi_1(X) = G$ and $\tilde{X}$ is $(q-1)$-connected. Then $\Phi_{\text{ch}}^{\tilde{X},q} \simeq \Phi_{\text{ch}}^{G,q}$, and $\Phi_{\text{cell}}^{\tilde{X},M} \simeq \Phi_{\text{cell}}^{G,M}$ for any $M^q$.

\textbf{Proof.} Choose a triangulation $\tau$ on $X$. There is a $K(G,1)$ whose $q$-skeleton is $\tau^{(q)}$; one adds $(q+1)$-cells to kill $\pi_q$, etc. Thus

$$\Phi_{\text{ch}}^{\tilde{X},q} \simeq \Phi_{\text{ch}}^{\tau,q} \simeq \Phi_{\text{ch}}^{G,q} \text{ and } \Phi_{\text{cell}}^{\tilde{X},M} \simeq \Phi_{\text{cell}}^{\tau,M} \simeq \Phi_{\text{cell}}^{G,M}. \hfill \Box$$

3. Finiteness and computability

One of the classic results on Dehn functions of finitely presented groups $G$ is that $\delta_G$ is recursive, or even subrecursive, iff the word problem on $G$ is solvable. The proof from right to left is fairly simple: Fix $X = K(G,1)$ with finite 2-skeleton. Up to translation by $G$, there are only finitely many loops in $\tilde{X}$ whose length is at most $n$, for any fixed $n$. Of these, use the word problem solution to determine which are contractible, then search all the disk maps into $\tilde{X}$ (up to translation) until one is found for each contractible loop.

The situation for generalized Dehn functions is not so clear-cut. For example, a result by Papasoglu (see [8]) states that the “second Dehn function” $\Phi_{\text{cell}}^{G,D^2}$ is subrecursive for all groups $G$ of type $F_3$, while an unpublished result by Young (see [13]) shows that there is a group $G$, which is of type $F_4$ for all $q$, where $\Phi_{\text{cell}}^{G,D^1}$ is not subrecursive, nor is $\Phi_{\text{cell}}^{G,q-1}$. In fact, the second Dehn function is effectively the only generalized Dehn function which is subrecursive for all qualifying $G$.

Nevertheless, results in one direction is possible.

\textbf{Theorem 6.} Let $q \geq 2$ and let $G$ be a group of type $F_q$. Then $\Phi_{\text{ch}}^{G,q}(n)$ is finite for all $n$. If $G$ has solvable word problem, then $\Phi_{\text{ch}}^{G,q}$ is recursive.

The proof is fairly similar to that for the classical Dehn function, with the wrinkle that the class of chains with volume at most $n$ is not generally finite, even up to translation by $G$. We observe, however, that each chain can be decomposed into connected components, and the class of small connected chains can be exhausted.
The equivalent of $\Phi_{\text{ch}}^{G,q}$ where only connected boundaries are considered is therefore computable. If the components of a cycle are far enough apart, then the most efficient way to fill the cycle is to fill each component separately. This allows us to compute $\Phi_{\text{ch}}^{G,q}$ in terms of the specialized version.

**Theorem 7.** Let $q \geq 2$, and let $G$ be a group of type $F_q$, and let $M$ be a $q$-dimensional manifold with nonempty boundary. Then $\Phi_{\text{cell}}^{G,M}(n)$ is finite for all $n$. If $G$ has solvable word problem, then $\Phi_{\text{cell}}^{G,M}$ is subrecursive.

For $q \geq 3$, the [recursive] bound on $\Phi_{\text{cell}}^{G,M}$ is precisely $\Phi_{\text{ch}}^{G,q}$; for $q = 2$, the bound may be computed in terms of the classical Dehn function.

We need some preliminaries. Let $X$ be a CW-complex. For any $t$ and for any $t$-chains $A$, $B$, we say $B$ is a *subchain* of $A$ if $\|A\| = \|B\| + \|A - B\|$. Equivalently, for ever $t$-cell $\sigma$ of $X$, let $n_{A,\sigma}$ and $n_{B,\sigma}$ be the coefficients of $\sigma$ in the expansions of $A$ and $B$ respectively. Then $B$ is a subchain of $A$ iff $0 \leq n_{B,\sigma} \leq n_{A,\sigma}$ or $n_{A,\sigma} \leq n_{B,\sigma} \leq 0$ for all $\sigma$. A given chain $A$ has finitely many subchains.

We say $B$ is a *component* of $A$ if $B$ is a subchain of $A$ and $\partial B$ is a subchain of $\partial A$, and that a $t$-chain $A$ is *connected* if its only components are 0 and $A$. By induction on $\|A\|$, every chain $A$ can be expressed as a sum of connected components $A = B_1 + \cdots + B_n$. Given such an expansion, we have $\|A\| = \|B_1\| + \cdots + \|B_n\|$ and $\|\partial A\| = \|\partial B_1\| + \cdots + \|\partial B_n\|$.

**Proof of theorem 7.** Let $X$ be a $K(G,1)$ with finite $q$-skeleton, and let $\tilde{X}$ be its universal cover. Assume $G$ is finite, so that $\tilde{X}^{(q)}$ is finite. For $0 \leq t \leq q$ and $v \geq 0$, we can determine the set of all elements $T \in C_t(\tilde{X})$ where $\|T\| \leq v$. Also we can determine whether a given $T \in C_t(\tilde{X})$ is a cycle, hence a boundary. Thus, we have the following algorithm for $\Phi_{\text{ch}}^{G,q}(n)$: determine all the $(q - 1)$-chains $S$ which are boundaries; for each $N$ starting with 0 and every $q$-chain $T$ where $\|T\| = N$, flag $\partial T$; stop when every $S$ is flagged and output $N$.

Now assume $G$ is infinite. Each cell of $X$ is covered by a collection of cells of $\tilde{X}$ in 1-1 correspondence with the elements of $G$, and the collection is invariant under the natural $G$-action. For each $t \leq q$, let $\Sigma_t$ be a set containing exactly one $t$-cell of $\tilde{X}$ which covers any given $t$-cell of $X$; thus $\Sigma_t$ is finite.

Up to $G$-action, there are finitely many connected $t$-chains of any fixed volume $n$. For $n = 0$, this is obvious. For $n \geq 1$, suppose $A$ is a connected $t$-chain and $\|A\| = n$. We generate chains $B_1, B_2, \ldots, B_n = A$, each a subchain of the next, where $\|B_k\| = k$, as follows: After translation by some element of $G$, $A$ must contain $\pm \sigma$ where $\sigma \in \Sigma_t$. Let $B_1 = \pm \sigma$. Suppose $B_k$ has been chosen for some $k < n$. Since $A$ is connected, $B_k$ is not a component of $A$, so there is some subchain $C$ of $A - B_k$ where $\|C\| = 1$ and $\|\partial(B_k + C)\| < \|\partial B_k + \partial C\|$. Let $B_{k+1} = B_k + C$. There are only finitely many possibilities for $B_1$; and for any chain $B$, there are only finitely many choices for $B_{k+1}$. Thus there are only finitely many possible $B_n = A$, which is what we wanted.

Let $\Psi(n) = \max FV_{\text{ch}}(A)$, where the maximum is taken over connected $(q-1)$-cycles $A$ with $\|A\| \leq n$. Let

$$\Phi(n) = \max_{\text{partitions } P} \sum_{k \in P} \Psi(k),$$
where $P$ is interpreted as a multiset. $\Psi(n)$ and $\Phi(n)$ are finite for all $n$.

We claim that $\Phi_{ch_q}^X(n) = \Phi(n)$ for all $n$. To see $\Phi_{ch_q}^X(n) \leq \Phi(n)$, let $A$ be a $(q-1)$-cycle with $\|A\| \leq n$. Let $A = B_1 + \cdots + B_m$ be a sum of connected components. Each $B_i$ is a cycle; let $C_i \in C_q(\tilde{X})$ with minimum volume where $\partial C_i = B_i$. The multiset $\{\|B_1\|, \ldots, \|B_m\|\}$ is a partition of $\|A\|$, so

$$FV_{ch}(A) \leq \sum_{i=1}^m \|C_i\| = \sum_{i=1}^m FV_{ch}(B_i) \leq \sum_{i=1}^m \Psi(\|B_i\|) \leq \Phi(\|A\|) \leq \Phi(n).$$

Take the supremum over all $A$ to see $\Phi_{ch_q}^X(n) \leq \Phi(n)$. In particular, $\Phi_{ch_q}^X(n)$ is finite for all $n$.

To see $\Phi(n) \leq \Phi_{ch_q}^X(n)$, let $B_1, \ldots, B_m$ be a finite list of connected $(q-1)$-cycles where $\sum_i \|B_i\| \leq n$. Choose translates $B_i' = g_i B_i$ as follows: Let $g_1 = e$ and $B_1' = B_1$. For $i = j + 1$, note that there are finitely many connected $q$-chains $C$ where $\partial C$ shares a cell with any of $B_1', \ldots, B'_j$ and where $\|C\| \leq \Phi_{ch_q}^X(n)$. Since $G$ is infinite, there must be some $h \in G$ where $h B_i = B_i$ does not share a cell with $\partial C$ for any such $C$. Take $g_i$ to be some such $h$. Note that $\|B_i'\| = \|B_i\|$, $\partial B_i' = 0$, and $FV_{ch}(B_i') = FV_{ch}(B_i)$ for all $i$.

Let $A = B_1' + \cdots + B_m'$, so that $\partial A = 0$ and $\|A\| \leq n$. Let $C$ be a $q$-chain where $\partial C = A$ and $\|C\| = FV_{ch}(A)$. If $C' \neq 0$ is a connected component of $C$, then $\partial C'$ shares cells with $B_i'$ for a unique $i$, since $\|C'\| \leq \|C\| \leq \Phi_{ch_q}^X(n)$ and we chose the $B_i'$ so that no such connected $q$-chain exists with boundary cells from two distinct $B_i'$. Conversely, any boundary cells of $\partial C'$ must come from $\partial C = A = \sum_i B_i'$, and if $\partial C' = 0$, then $C - C'$ is a filling cycle for $A$ of smaller volume, which is impossible. Thus, given a sum of connected components $C = C_1 + \cdots + C_N$, each $C_j$ is associated with a unique $B_i'$, and $C$ is therefore a sum over $i$ of fillings for $B_i'$. Hence

$$\sum_{i=1}^m FV_{ch}(B_i') = FV_{ch}(A) \leq \Phi_{ch_q}^X(n).$$

Taking the supremum over all finite lists $B_1, \ldots, B_m$, we have $\Phi(n) \leq \Phi_{ch_q}^X(n)$, as desired.

Finally, suppose $G$ is infinite with solvable word problem (let $Y$ be a finite set of generators). We first show that $\Psi$ is computable. If $w \in F(Y)$ is a word and $\sigma \in \Sigma_1$ for $t \leq q$, then $(w, \sigma)$ represents the $t$-cell $\overline{w}\sigma$. Every $t$-cell is represented by some pair, and $(w, \sigma)$ and $(w', \sigma')$ represent the same cell iff $\sigma = \sigma'$ and $w^{-1}w'$ represents the trivial element of $G$. Chains $A \in C_q(\tilde{X})$ can be represented as linear combinations of pairs. From a representation for $A$, one can calculate $\|A\|$; also, one can calculate a representation for $\partial A$. Also, one can determine all the subchains of $A$, and one can decide whether a chain $A$ is connected: for each subchain $B$ where $B \neq 0$ and $B \neq A$, determine whether $\partial B$ is a subchain of $\partial A$. $A$ is connected iff this is never the case.

The proof that $\Psi$ is computable is similar to the case with $G$ finite. Generate representations of all of the $(q-1)$-chains $A$ with $\|A\| \leq n$, up to $G$-action, by starting with cells in $\Sigma_{q-1}$ and adding cells which share boundary components; there are finitely many such chains. Of these chains, take the subset consisting of connected cycles. For each connected cycle $A$, generate the $q$-chains $B$ where $\partial B$ and $A$ share at least one cell, in order of increasing volume, stopping when a chain
B with ∂B = A is found; remember the volume of B. The maximum such volume, across all A, is Ψ(n).

Given an algorithm for computing Ψ, computing Φ = ΦX,q is easy. □

Proof of theorem 7 For q ≥ 3 the theorem is an easy consequence of theorems 1 and 2. Now consider q = 2. As noted previously, the classical Dehn function of G, here ΦG,D2, is finite for all inputs, and that it is computably bounded (even computable) iff G has solvable word problem. For arbitrary M2, we have ∂M a disjoint union of k copies of S1, which implies

$$\Phi_{\text{ch}}^{G,M}(n) \leq \max_{n_1 + \cdots + n_k = n} \sum_{i=1}^{k} \Phi_{\text{cell}}^{G,D^2}(n_i).$$

Similarly, if A is a 1-cycle, it can be represented as A1 + · · · + Ak where each Ai is the image of a copy of S1; hence

$$\Phi_{\text{ch}}^{G,2}(n) \leq \max_{\text{a partition of } n} \sum_{k \in P} \Phi_{\text{cell}}^{G,D^2}(k).$$

4. Further questions

The relationship between Φch^{G,2} and the various Φcell^{G,M} for dim M = 2 is not yet clear. Also we do not yet know whether Φcell^{G,M} is always recursive for dim M ≤ 3 and G decidable, rather than subrecursive. Finally, there is no obvious converse to the results of section 3 that is, if G does not have decidable word problem, it is unclear what restrictions this places on Φch^{G,q} or Φcell^{G,M}.

References

[1] J. M. Alonso, X. Wang, and S. J. Pride. Higher-dimensional isoperimetric (or Dehn) functions of groups. J. Group Theory, 2:81–112, 1999.
[2] Noel Brady, Martin R. Bridson, Max Forester, and Krishnan Shankar. Snowflake groups, Perron-Frobenius eigenvalues, and isoperimetric spectra. Geometry & Topology, 13:141–187, 2009, arXiv:math/0608155v2.
[3] David Bernard Alper Epstein, James W. Cannon, Derek F. Holt, S. V. F. Levy, M. S. Paterson, and William P. Thurston. Word Processing in Groups. A. K. Peters Ltd., 1992.
[4] Herbert Federer. Geometric Measure Theory. Classics in Mathematics. Springer-Verlag, 1969.
[5] Chad Groft. Generalized Dehn functions I. January 2009, 0901.2303. 19 pages, to appear.
[6] Mikhail Gromov. Geometric Group Theory, vol. 2: Asymptotic Invariants of Infinite Groups, volume 182 of London Mathematical Society Lecture Note Series. Cambridge University Press, 1993.
[7] Allen Hatcher. Algebraic Topology. Cambridge University Press, 2002. http://www.math.cornell.edu/~hatcher/AT/ATpage.html.
[8] Panos Papasoglu. Isodiametric and isoperimetric inequalities for complexes and groups. Journal of the London Mathematical Society, 62:97–106, 2000, http://users.uoa.gr/~ppapazog/research/isodiametric.pdf.
[9] Halsey L. Royden, Jr. Real Analysis. Prentice-Hall, third edition, 1988.
[10] Edwin H. Spanier. Algebraic Topology. Springer-Verlag, 1966.
[11] John Stallings. A finitely presented group whose 3-dimensional integral homology is not finitely generated. American Journal of Mathematics, 85:541–543, 1963.
[12] Brian White. Mappings that minimize area in their homotopy classes. Journal of Differential Geometry, 20:433–446, 1984.
[13] Robert Young. A note on higher-order filling functions. 2008, 0805.0584v2.