Conformally related metrics and Lagrangians and their physical interpretation in cosmology

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Conformally related metrics and Lagrangians are considered in the context of scalar-tensor gravity cosmology. After the discussion of the problem, we pose a lemma in which we show that the field equations of two conformally related Lagrangians are also conformally related if and only if the corresponding Hamiltonian vanishes. Then we prove that to every non-minimally coupled scalar field, we may associate a unique minimally coupled scalar field in a conformally related space with an appropriate potential. The latter result implies that the field equations of the non-minimally coupled scalar field are the same at the conformal level with the field equations of the minimally coupled scalar field. This fact is relevant in order to select physical variables among conformally equivalent systems. Finally, we find that the above propositions can be extended to a general Riemannian space of $n$-dimensions.

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1. INTRODUCTION

The detailed analysis of the current high quality cosmological data (Type Ia supernovae, cosmic microwave background, baryonic acoustic oscillations, etc), converge towards a new emerging “Cosmological Standard Model”. This cosmological model is spatially flat with a cosmic dark sector constituted by cold dark matter and some sort of dark energy, associated with large negative pressure, in order to explain the observed accelerating expansion of the universe (see [1–8] and references therein). Despite the mounting observational evidences on the existence of the dark energy component in the universe, its nature and fundamental origin remains an intriguing enigma challenging the very foundations of theoretical physics.

Indeed, during the last decade there has been an intense theoretical debate among cosmologists regarding the nature of this exotic “dark energy”. The absence of a fundamental physical theory, concerning the mechanism inducing the cosmic acceleration, has opened a window to a plethora of alternative cosmological scenarios. Most are based either on the existence of new fields in nature (dark energy) or in some modification of Einstein’s general relativity, with the present accelerating stage appearing as a sort of geometric effect (see [9–31] and references therein). Despite the mounting observational evidences on the existence of the dark energy component in the universe, its nature and fundamental origin remains an intriguing enigma challenging the very foundations of theoretical physics.

In order to investigate the dynamical properties of a particular “dark energy” model, we need to specify the covariant Einstein-Hilbert action of the model and find out the corresponding energy-momentum tensor. This methodology provides a mathematically consistent way to incorporate “dark energy” in cosmology. However, in the literature, there are many Lagrangians [31] which describe differently the physical features of the scalar field or the modified gravity [12]. Because of the large amount of “dark energy” models, it is essential to study them in a unified context in order to discriminate the true physical variables. From our viewpoint, this framework has to be at the level of geometry since the various Lagrangians which describe the nature of “dark energy” are embedded in the space-time. From a theoretical point of view, an easy way to study dynamics in a unified manner is to look for conformally related Lagrangians. In fact, the idea to use conformally related metrics and Lagrangians as a cosmological tool is not new. In particular, it has been proposed that the existence of conformally equivalent Lagrangians can be used in order to select viable cosmological models [32].

In general, the presence of scalar fields into the gravitational action can give rise to two classes of theories: minimally and non-minimally coupled Lagrangians. In the first case, the gravitational coupling is the standard Newton constant and the scalar field Lagrangian is simply added to the Ricci scalar. It can consist of (i) a kinetic term, (ii) a kinetic term and a self-interaction potential, or (iii) just an interaction potential. In the first case, the scalar field is nothing else but a cyclic variable and then it is related to a conserved quantity. The second case is the relevant one since, by a variational principle, it is possible to obtain a Klein-Gordon equation where the self-interacting potential $V(\phi)$ leads the dynamics. The third case means that the scalar field has no dynamics. When the coupling is the Newton constant, we are in the so-called Einstein frame.
In fact, as firstly pointed out by Brans and Dicke [33], a gravitational theory can be made more "Machian" by relaxing the hypothesis that the coupling is constant. They introduced a scalar field $\phi$ non-minimally coupled to the Ricci scalar $R$ and a kinetic term for such a scalar field into the gravitational action. The result was that the coupling was non-minimal and coordinate dependent. In other words, the gravitational interaction is assumed to change with distance and time. This approach can be generalized considering scalar-tensor theories of gravity where also a self-interaction potential comes into the game or more than one scalar field are taken into account. In general, any gravitational theory not simply linear in the Ricci scalar can be reduced to a scalar-tensor one. In the case of $f(R)$-gravity, it is straightforward to show that an O’Hanlon representation by scalar fields is possible [34]. In this case, a scalar field is non-minimally coupled to the Ricci scalar and a self-interacting potential is present while there is no kinetic term. Scalar field dynamics is guaranteed by the non-minimal coupling and the potential (see [12] and references therein). In other words, the further gravitational degrees of freedom, coming from the fact that $f(R) \neq R$, can be figured out as a scalar field. As soon as we are considering non-minimal couplings or higher-order terms in the Lagrangian, we are in the so-called Jordan frame, after Jordan who first introduced this notion [33, 34].

In this article we would like to address the following basic question: In the framework of the scalar field cosmology, is it possible to relate the available Lagrangians in a conformal way?

The structure of the paper is as follows. In section II, we discuss the issue of conformal transformations considering its historical development and connection with physical theories. From our point of view, this section is essential in order to fix the problem showing the urgency to discriminate physical variables among the conformally related models. The basic theoretical elements of the conformally related metric are presented in section III, where we also introduce the concept of conformally related Lagrangians and prove a lemma which shows that the field equations for two conformally related Lagrangian are also conformally related if the corresponding Hamiltonian vanishes. In section IV, we discuss the conformal equivalence of Lagrangians for scalar fields in a Riemannian space of dimension 4 (for an extension see appendix A). In particular we enunciate a theorem which proves that the field equations for a non-minimally coupled scalar field are the same at the conformal level with the field equations of the minimally coupled scalar field. The necessity to preserve Einstein’s equations in the context of Friedmann-Lemaître-Robertson-Walker (FRLW) space-time leads us to apply, in section V, the current general analysis to the scalar field (quintessence or phantom) flat FLRW cosmologies. Finally, we draw our main conclusions and discuss results in section VI.

### 2. WHAT IS THE PHYSICAL FRAME?

Some considerations are in order at this point. The conformal transformations from the Jordan to the Einstein frame are geometrical maps allowing to set many features of scalar-tensor gravity, $f(R)$ gravity and, in general, any modified theory of gravity. Taking into account both the Jordan and the Einstein conformal frames (infinite conformal frames can be chosen assuming a suitable conformal factor), the question is whether the frames are infinitely many physically equivalent or only mathematically related. In other words, the problem is whether the physical information contained in the theory is “preserved” or not by the conformal transformations. In other words, one has the metric $g_{ij}$ and its conformally related one $g_{ij}$; the question is what is the “physical metric”, i.e., the metric from which curvature, geometry, and physical effects have to be calculated and compared with experiments and observations [37].

More precisely, every Killing and homothetic vector is also a symmetry of the energy-momentum tensor. The latter is also a metric of spacetime because a "metric" can be considered any second order tensor. Therefore, up to the homothetic group, metric, energy-momentum tensor, and Ricci tensor have the same symmetries. This is not the case for the conformal group and then the requirement that the theory is invariant under conformal transformations is an additional assumption not related to the gauge invariance. Furthermore, we are concerned with Lagrangians which means that we assume the same kinematics on which these Lagrangians set up a dynamics. In general, kinematics has other symmetries with respect to dynamics and these symmetries are not related to those of the metric because no field equation relates kinematic quantities to metric. Kinematic symmetries are constrained only by the Ricci identity which gives the constraint and propagation equations.

The issue of “which is the physical frame” has been debated for a long time and it emerges as soon as some authors argue in favor of one frame against the other, and others support the idea that the two frames are physically equivalent. In the latter case, authors claim that the issue is a pseudo-problem. The final result is that there is a good deal of confusion in the literature.

Fierz was the first to pose the problem [38], but the main argument is due to Dicke, who discussed the conformal transformation for Brans-Dicke theory [40]. The point was that physics must be invariant under a rescaling of units and the conformal transformation is a local rescaling: units do not change rigidly over the entire spacetime manifold, but by amounts which are different at different spacetime points. From Dicke’s point of view, the two frames are equivalent provided that mass, length, time, and quantities derived from them scale with appropriate powers of the conformal factor in the Einstein frame [40].
From this point of view, it is not difficult to see why many authors consider the issue of which is the physical frame a pseudo-problem. In principle, it is difficult to object to this argument, but there are some difficulties.

Even though the above argument is clear in principle, its application to practical situations gives rise to problems. The assumption that the two conformal frames are just different representations of the same theory, similar to different gauges of a gauge theory, has to be checked explicitly by using the field equations of a given system. “Physical equivalence” is a vague concept because one can consider many different matter (or test) fields in curved spacetime and different types of physics, or different physical aspects of a problem. When checking explicitly the physical equivalence between the two frames, one has to specify which physical field, or physical process is considered and the equations describing it. The equivalence has to be shown explicitly, but there is no proof that holds in any situation (e.g. scalar fields, spinors, cosmology, black holes derived from the same theory). While physical equivalence can be proven for various physical aspects, no proof comprehensive of all physical fields and different physical applications exists.

It is important to stress that Dicke seems to mix the concept of dimensional units and the concept of measuring units. For example, spatial distance has the dimensional unit length $L$ but the measuring unit length, as we already know from Special Relativity, has to be defined in a relativistic inertial frame (RIF). After, by some rule, spatial lengths are compared in different RIFs. One of such rules is the Einstein one. It can be defined by means of light signals which are simultaneous in the RIF in which the measurement is taking place. This is the so called rest length (which, by assumption, coincides with the corresponding Euclidian length measured by a Newtonian inertial observer). This type of measurement has nothing to do with the concept of dimension which, for all RIFs, is the same i.e. the time $T$ (the second, in units, where $c = 1$, see e.g. $[39]$). Dicke is aware of that and then says: “Generally there may be more than one feasible way of establishing the equality of units at different spacetime points” $[40]$. It is essential to stress that such an approach is based on conventions which may lead to absurd results in real measurements. Geometrically it is a 1:1 map of two points defining a spacelike interval from the rest space of one observer at a spacetime point to a spacelike plane of another observer at another spacetime point. This definition it is not a coordinate transformation and there is no point or meaning to consider symmetry invariance with this transformation. Dicke says: “It is evident that the equations of motion of matter must be invariant under a general coordinate dependent transformation of units” $[40]$. This is a misunderstanding that can give rise to confusion. The method of comparing lengths at different spacetime points is a kinematic assumption while equations of motion give dynamics.

Furthermore, Dicke’s argument is purely classical. In cosmology, black hole physics, and quantum fields in curved space, the equivalence of conformal frames is not clear. At quantum level, this equivalence is not proven due to the lack of a definitive quantum gravity theory: in fact, when the metric $g_{ij}$ is quantized, inequivalent quantum theories can be found $[38, 41]$. One can consider the semiclassical regime in which gravity is classical and matter fields are quantized: again, one would expect the conformal frames to be inequivalent because the conformal transformations can be seen as Legendre transformations $[42]$, similar to the Legendre transformations of classical mechanics of point particles which switches from the canonical Lagrangian coordinates $q$ to the variables $\{q, p\}$ of the Hamiltonian formalism. Now, it is well known that Hamiltonians that are classically equivalent become inequivalent when quantized, producing different energy spectra and scattering amplitudes $[43, 44]$. However, the conformal equivalence between Jordan and Einstein frame seems to hold to some extent at the semi-classical level $[45]$. Again, only a particular kind of physics has been considered and one cannot make statements about all possible physical situations.

It is important to point out a very basic argument among particle physicists that relies on the equivalence theorem of Lagrangian field theory. It states that the $S$-matrix is invariant under local (nonlinear) field redefinitions $[46, 47]$. Since the conformal transformation is, essentially, a field redefinition, it would seem that quantum physics is invariant under the change of the conformal field. However, the field theory in which the equivalence theorem is derived applies to gravity only in the perturbative regime in which the fields deviate slightly from the Minkowski space-time. In this regime, tree level quantities can be calculated in any conformal frame with same result, but in the non-perturbative regime field theory and the equivalence theorem do not apply.

Unfortunately, the scaling of units in the Einstein frame often produces results that either do not make sense or are incorrect in the Jordan frame, reinforcing the opposite view that the two frames are not equivalent. While Dicke’s explanation is very appealing and several claims supporting the view that the two frames are inequivalent turned out to be incorrect because they simply neglected the scaling of units in the Einstein frame, one should not forget that Dicke’s argument is not inclusive of all areas of physics and it is better to check explicitly that the physics of a certain field does not depend on the conformal representation and not make sweeping statements. Certain points have been raised in the literature which either constitute a problem for Dicke’s view, or, at least, indicate that this viewpoint cannot be applied blindly, including the following ones.

For example, massive particles follow time-like geodesics in the Jordan frame, while they deviate from geodesic motion in the Einstein frame due to a force proportional to the gradient of the conformal scalar field $[48]$. Hence, the Weak Equivalence Principle is satisfied only in the Jordan frame but not in the Einstein frame due to the coupling
of the scalar field to the ordinary matter. Since the Equivalence Principle is the foundation of any relativistic theory of gravity, this aspect is important and there are two ways to consider it. One can ask for the two conformal frames are equivalent also with respect to the Equivalence Principle. This means that Equivalence Principle is formulated in a way that depends on the conformal frame representation. Then, a representation-independent formulation must be sought for. However, up to now, no definite result exists in this direction. On the other hand, we can ask for the violation of the Weak Equivalence Principle in the Einstein frame by saying that the "physical equivalence" of the two frames must be precisely defined and this concept cannot be used blindly. In fact the Equivalence Principle holds only in one frame but not in the other. This fact could be used as an argument against the physical equivalence of the frames. However, the fact that the Equivalence Principle holds in a given frame and not in all frames means that it is not a covariant requirement but a kinematic one. In other words, the Equivalence Principle could not be sufficient to discriminate among conformally related frames.

In the scalar-tensor theories of gravity, the energy conditions are easily violated in the Jordan frame, but they are satisfied in the Einstein frame. This fact does not eliminate singularities and then the two frames remain equivalent with respect to the singularities and not with respect to the energy conditions. This difficulty arises because part of the matter sector of the theory, in the Einstein frame, comes from the conformal factor; in other words, the conformal transformation mixes matter and geometric degrees of freedom, which is the source of many interpretational problems. Thus, even if the theory turns out to be independent of the conformal representation, its interpretation is not.

There are results in cosmology in which the universe accelerates in one frame but not in the other. From the pragmatic point of view of an astronomer attempting to fit observational data (for example, type Ia supernovae data to a model of the present acceleration of the universe), the two frames certainly do not appear to be "physically equivalent".

To approach correctly the problem of physical equivalence under conformal transformations, one has to compare physics in different conformal frames at the level of the Lagrangian, of the field equations, and of their solutions. This comparison may not always be easy but, in certain cases, it is extremely useful to discriminate between frames. It has been adopted, for example, in Ref. to compare cosmological models in the Einstein and the Jordan frame. Specifically, it has been shown that solutions of $f(R)$ and scalar-tensor gravity cannot be assumed to be physically equivalent to those in the Einstein frame when matter fields are given by generalized Equations of State.

In these and in other situations, one must specify precisely what "physical equivalence" means. In certain situations physical equivalence is demonstrated simply by taking into account the coupling of the scalar field to matter and the varying units in the Einstein frame, but in other cases the physical equivalence is not obvious and it does not seem to hold. At the very least, this equivalence, if it is valid at all, must be defined in precise terms and discussed in ways that are far from obvious. For this reason, it would be too simplistic to dismiss the issue of the conformal frame as a pseudo-problem that has been solved for all physical situations of interest.

### 3. Conformally Related Metrics and Lagrangians

Taking in mind the above discussion, we want to seek for geometrical structures that are conformally invariant.

Our aim is to compare cosmological models coming from scalar-tensor gravity in order to select conformal quantities in view of a possible physical meaning. This is a very delicate issue that has to be discussed in details. On one hand, an invariant quantity, (i.e. a quantity that remains the same under conformal transformation) should have a physical meaning. However, in cosmology, this statement does not necessarily hold since the problem of equivalence between the two frames is not well posed and it may happen that one of them has to be taken as the physical one in a particular case and then such invariant quantity would not have a physical meaning. This situation often happen if cosmological solutions fit data and then related quantities are assumed as "physical". On the other hand, if a scalar field describes an actual particle in a given frame, then its properties (e.g. its mass and couplings to other fields) would change in the conformally-related metric. This fact does not mean that its properties have no physical meaning. In other words, the identification of conformally invariant physical quantities is a very difficult task if it is not based on first principles.

With these considerations in mind, let us start with defining some geometrical structures that will be useful in the discussion.

A vector field $X^a$ is a Conformal Killing Vector (hereafter CKV) of the metric $g_{ij}$ if there exists a function $\psi (x^k)$ so that:

$$\mathcal{L}_x g_{ij} = 2\psi (x^k) g_{ij}$$

where $\mathcal{L}_x$ is the Lie derivative. In case that $\psi_x = 0$ i.e. $\psi =$ constant, the vector $X$ is called homothetic (hereafter HV) while if $\psi = 0$ then the vector $X$ is a Killing vector (hereafter KV). In this context, two metrics $g_{ij}, \bar{g}_{ij}$ are
said to be conformal or conformally related if there exists a function \( N^2 (x^k) \) so that \( \bar{g}_{ij} = N^2 (x^k) g_{ij} \). From the mathematical point of view the CKVs form the so called conformal Lagrangians of the metric. The conformal algebra contains two closed sub-algebras the Homothetic algebra and the Killing algebra. Interestingly the above algebras are related as follows:

\[
KV_x \subseteq HV_x \subseteq CKV_x .
\]  

(2)

The dimension of the conformal algebra of an \( n \)-dimensional metric \( (n > 2) \) of constant curvature equals \( \frac{(n+1)(n+2)}{2} \), the dimension of the Killing algebra \( \frac{n(n+1)}{2} \) and that of the homothetic algebra \( \frac{n(n+1)}{2} + 1 \). Note that two conformally related metrics have the same conformal algebra [53], however not the same subalgebras. Indeed if \( X \) is a CKV for the metric \( g_{ij} \) i.e. \( \mathcal{L}_X g_{ij} = 2\psi (x^k) g_{ij} \) then for the metric \( \bar{g}_{ij} \) the vector \( X \) is again a CKV with conformal factor \( \bar{\psi} (x^k) \), that is:

\[
\mathcal{L}_X \bar{g}_{ij} = 2\bar{\psi} (x^k) \bar{g}_{ij}
\]  

(3)

where the conformal factors \( \psi (x^k), \bar{\psi} (x^k) \) are related as follows:

\[
\bar{\psi} (x^k) = \psi (x^k) + \mathcal{L}_X (\ln N) .
\]  

(4)

The Ricci scalars of the conformally related metrics \( g_{ij}, \bar{g}_{ij} \) are related as follows [54]:

\[
\bar{R} = N^{-2} R - 2(n - 1)N^{-3} \Delta_2 N - (n - 1)(n - 4)\Delta_1 N
\]  

(5)

where (note that \( \Delta_2 N \) contains the covariant derivative whereas \( \Delta_1 N \) the partial derivative):

\[
\Delta_1 N = g_{ij} N^{ij} N^j \]

(6)

\[
\Delta_2 N = \bar{g}_{ij} N^{ij} .
\]  

(7)

From the above discussion it becomes clear that all two dimensional spaces are Einstein Spaces (i.e. \( R_{ab} = \frac{R}{2} g_{ab} \)) and conformally flat. The metric of a two dimensional space can be written in the generic form:

\[
ds^2 = N^2 (x, y) \left( \varepsilon dx^2 + dy^2 \right) , \quad \varepsilon = \pm 1 .
\]  

(8)

3.1. Conformal Lagrangians

Due to the fact that almost every dynamical system is described by a corresponding Lagrangian, below we study generically, as much as possible, the problem of the conformal Lagrangians and then we apply the current ideas to the scalar field cosmology. To begin with, consider the Lagrangian of a particle moving under the action of a potential \( V(x^k) \) in a Riemannian space with metric \( g_{ij} \)

\[
L = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j - V (x^k) , \quad \dot{x}^i = \frac{dx^i}{dt}
\]  

(9)

where \( t \) is a path parameter. The equations of motion follow from the action

\[
S = \int dx dt L (x^k, \dot{x}^k) = \int dx dt \left[ \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j - V (x^k) \right].
\]  

(10)

Changing the variables in Eq.(10) from \( t \) to \( \tau \) via the relation:

\[
d\tau = N^2 (x^i) dt
\]  

(11)

the action is given by

\[
S = \int dx \frac{d\tau}{N^2 (x^k)} \left[ \frac{1}{2} g_{ij} N^4 (x^k) x'^i x'^j - V (x^k) \right]
\]  

(12)

where \( x'^i = \frac{dx^i}{d\tau} \). Obviously, the Lagrangian in the new coordinate system \((\tau, x^i)\) becomes:

\[
\bar{L} (x^k, x'^k) = \frac{1}{2} N^2 (x^k) g_{ij} x'^i x'^j - \frac{V (x^k)}{N^2 (x^k)}
\]  

(13)
Now if we consider a conformal transformation of the metric \( g_{ij} = N^2 (x^k) g_{ij} \) and a new potential function \( \bar{V} (x^k) = \frac{V (x^k)}{N^2 (x^k)} \) then the new Lagrangian \( \bar{L} (x^k, x'^k) \) takes the following form:

\[
\bar{L} (x^k, x'^k) = \frac{1}{2} \bar{g}_{ij} x'^i x'^j - \bar{V} (x^k)
\tag{14}
\]

implying that Eq. (14) is of the same form as the Lagrangian \( L \) in Eq. (9). From now on the Lagrangian \( L (x^k, \dot{x}^k) \) of Eq. (9) and the Lagrangian \( \bar{L} (x^k, x'^k) \) of Eq. (14) will be called conformal. In this framework, the action remains the same i.e. it is invariant under the change of parameter, the equations of motion in the new variables \((\tau, x^i)\) will be the same with the equations of motion for the Lagrangian \( L \) in the original coordinates \((t, x^i)\).

It has been shown [55] that the Noether symmetries of the conformal Lagrangians (9)-(14) are elements of the common conformal algebra of the metrics \( g_{ij}, \bar{g}_{ij} \). As we have remarked the conformal algebra of the metrics \( g_{ij}, \bar{g}_{ij} \) (as a set) is the same however their closed subgroups of HVs and KV's are in general different. Now, we formulate and prove the following proposition:

**Lemma:** The Euler-Lagrange equations for two conformal Lagrangians transform covariantly under the conformal transformation relating the Lagrangians iff the Hamiltonian vanishes.

**Proof:** Consider the Lagrangian \( L = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j - V (x^k) \) whose Euler-Lagrange equations are:

\[
\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k + V^i = 0
\tag{15}
\]

where \( \Gamma^i_{jk} \) are the Christoffel symbols. The corresponding Hamiltonian is given by

\[
E = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j + V (x^k).
\tag{16}
\]

For the conformally related Lagrangian \( \bar{L} (x^k, x'^k) = \left( \frac{1}{2} N^2 (x^k) g_{ij} x'^i x'^j - \frac{V (x^k)}{N^2 (x^k)} \right) \) where \( N_j \neq 0 \) the resulting Euler Lagrange equations are

\[
x''^i + \bar{\Gamma}^i_{jk} x'^j x'^k + \frac{1}{N^4} V^i - \frac{2 V}{N^3} N^i = 0
\tag{17}
\]

where

\[
\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + (\ln N)_j \delta^i_k + (\ln N)_j \delta^i_k - (\ln N)^i g_{jk}
\tag{18}
\]

and the corresponding Hamiltonian is

\[
\bar{E} = \frac{1}{2} N^2 (x^k) g_{ij} \dot{x}^i \dot{x}^j + \frac{V (x^k)}{N^2 (x^k)}.
\tag{19}
\]

In order to show that the two equations of motion are conformally related we start from Eq. (17) and apply the conformal transformation

\[
x'^i = \frac{dx^i}{d\tau} = \frac{dx^i}{dt} \frac{dt}{d\tau} = \dot{x}^i \frac{1}{N^2}
\]

\[
x''^i = \ddot{x}^i \frac{1}{N^4} - 2 \dot{x}^i \dot{x}^j (\ln N)_j \frac{1}{N^4}
\]

Replacing in Eq. (17) we find:

\[
\ddot{x}^i \frac{1}{N^4} - 2 \dot{x}^i \dot{x}^j (\ln N)_j \frac{1}{N^4} + \frac{1}{N^4} \bar{\Gamma}^i_{jk} \dot{x}^j \dot{x}^k + \frac{1}{N^4} V^i - \frac{2 V}{N^3} N^i = 0.
\]

\[\text{References:}\] The Noether symmetries of the conformal Lagrangians (9)-(14) are elements of the common conformal algebra of the metrics \( g_{ij}, \bar{g}_{ij} \). A clear definition of the Noether symmetries can be founds in [55-58] (for applications to cosmology see [59-61] and references therein).
Replacing $\hat{\Gamma}_{jk}^i$ from Eq. (18) we have

$$\ddot{x}^i - 2\dot{x}^i \dot{\ln N} - \Gamma_{jk}^i \dot{x}^j \dot{x}^k + 2(\ln N)_{,j} \dot{x}^j \dot{x}^i - (\ln N)^i_{,j} g_{jk} \dot{x}^j \dot{x}^k + V = 2V(\ln N)^i_{,j} = 0$$

from which follows

$$\ddot{x}^i + \Gamma_{jk}^i \dot{x}^j + V - (\ln N)^i_{,j} (g_{jk} \dot{x}^j + 2V) = 0.$$ 

Obviously, the above Euler-Lagrange equations coincide with Eqs. (15) if and only if $(g_{jk} \dot{x}^j + 2V) = 0$, which implies that the Hamiltonian of Eq. (16) vanishes. The steps are reversible hence the inverse is also true.

The physical meaning of such a result is that systems with vanishing energy are conformally invariant at the level of equations of motion.

4. CONFORMALLY EQUIVALENT LAGRANGIANS IN THE SCALAR FIELD COSMOLOGY

Let us discuss now the conformal equivalence of Lagrangians for scalar fields in a general Riemannian space of four dimensions (see also [32]). The field equations in the scalar-tensor cosmology can be derived from two different variational principles. In the first case we consider a scalar field $\phi$ which is minimally coupled to gravity and the equations of motion follow from the action

$$S_M = \int d\tau dx^3 \sqrt{-g} \left[ R + \frac{\varepsilon}{2} g_{ij} \dot{\phi}^i \dot{\phi}^j - V(\phi) \right]$$ (20)

where $\varepsilon = \pm 1$ defines quintessence or phantom field cosmology respectively.

In the second case we assume a scalar field $\psi$ (different from the minimally coupled scalar field $\phi$) which interacts with the gravitational field (non minimal coupling) and the corresponding action is given by

$$S_{NM} = \int d\tau dx^3 \sqrt{-\bar{g}} \left[ F(\psi) \bar{R} + \frac{\varepsilon}{2} \bar{g}^{ij} \dot{\psi}_i \dot{\psi}_j - \bar{V}(\psi) \right]$$ (21)

where $F(\psi)$ is the coupling function between the gravitational and the scalar field $\psi$ respectively. Below we pose the following proposition.

**Theorem:** The field equations for a non minimally coupled scalar field $\psi$ with Lagrangian $\bar{L}(\tau, x^k, \dot{x}^k)$ and coupling function $F(\psi)$ in the gravitational field $\bar{g}_{ij}$ are the same with the field equations of the minimally coupled scalar field $\Psi$ for a conformal Lagrangian $L(\tau, x^k, \dot{x}^k)$ in the conformal metric $g_{ij} = N^{-2} \bar{g}_{ij}$ where the conformal function is $N = \frac{1}{\sqrt{-2F(\psi)}}$ with $F(\psi) < 0$. The inverse is also true, that is, to a minimally coupled scalar field it can be associated a unique non minimally coupled scalar field in a conformal metric and with a different potential function.

**Proof:** We first start with the action of Eq. (21). Let $g_{ij}$ be the conformally related metric (this is not a coordinate transformation):

$$g_{ij} = N^{-2} \bar{g}_{ij}.$$ 

Then the action provided by Eq. (21) becomes $2$:

$$S_{NM} = \int d\tau dx^3 N^4 \sqrt{-\bar{g}} \left[ F(\psi) \bar{R} + \frac{\varepsilon}{2} N^{-2} \bar{g}^{ij} \ddot{\psi}_i \ddot{\psi}_j - \bar{V}(\psi) \right].$$

Inserting the Ricci scalar $\bar{R}$ (using $n = 4$) from Eq. (13) into the latter equation we find:

2 For a $4 \times 4$ matrix namely, $A = (a_{ij})$ we have

$$\det A = \varepsilon^{ijkl} a_{ij} a_{kl}$$

hence

$$\bar{g} = \varepsilon^{ijkl} \bar{g}_{ij} \bar{g}_{kl} = N^4 g.$$
\[ S_{NM} = \int d\tau d^3x N^4 \sqrt{-g} \left[ F(\psi) N^{-2} R - 6 F(\psi) N^{-3} \Delta_2 N + \frac{\varepsilon}{2} N^{-2} \Delta_1 \psi - \bar{V}(\psi) \right]. \] \tag{22}

Now we can define the conformal function \(N\) in terms of the coupling function \(F(\psi)\) [where \(F(\psi) < 0\)]:

\[ N = \frac{1}{\sqrt{-2F(\psi)}}. \] \tag{23}

with

\[ N_i^j = \frac{F_{\psi \psi i}^{\psi j}}{(-2F)^{\frac{1}{2}}}. \] \tag{24}

Using Eqs. \(23\) and \(24\) the first term of the integral in Eq. \(22\) becomes:

\[ \int d\tau d^3x \sqrt{-g} \frac{F(\psi)}{2} N^2 R = \int d\tau d^3x \sqrt{-g} \left( \frac{R}{2} \right). \]

On the other hand the second term in Eq. \(22\) gives, after integration by parts:

\[ \int d\tau d^3x \sqrt{-g} \left[ -6 F(\psi) N \Delta_2 N \right] = \int d\tau d^3x \sqrt{-g} \left[ -6 \frac{F}{\sqrt{-2F}} N_{i}^{j} g^{ij} \right] \]
\[ = \int d\tau d^3x \sqrt{-g} \left[ -6 \frac{F}{\sqrt{-2F}} \left( \sqrt{-g} g^{ij} N_{k} \right)_{,j} \right] \]
\[ = \int d\tau d^3x \sqrt{-g} \left[ -6 \frac{F}{\sqrt{-2F}} \left( \sqrt{-g} g^{ij} N_{k} \right)_{,j} \right] \]
\[ = \int d\tau d^3x \sqrt{-g} \left[ 3 \frac{F_{\psi}^{\psi i} N_{i}^{j} g^{ij}}{\sqrt{-2F}} \right] \]
\[ = \int d\tau d^3x \sqrt{-g} \left[ 3 \frac{F_{\psi}^{\psi i} N_{i}^{j} g^{ij}}{\sqrt{-2F}} \right]. \] \tag{25}

The third term provides:

\[ \frac{\varepsilon}{2} N^2 \Delta_1 \psi = \frac{\varepsilon}{4F} \psi_i^{\psi \psi j} g^{ij}. \]

Finally, collecting all terms and inserting them into Eq. \(22\), the action is written as

\[ S_{NM} = \int d\tau d^3x \sqrt{-g} \left[ \frac{R}{2} + \frac{3F_{\psi}^{\psi i} N_{i}^{j} g^{ij}}{4F^2} - \frac{\varepsilon}{4F} \psi_i^{\psi \psi j} g^{ij} - \bar{V}(\psi) \right] \]
\[ = \int d\tau d^3x \sqrt{-g} \left[ \frac{R}{2} + \frac{\varepsilon}{2} \left( \frac{3F_{\psi}^{\psi i} N_{i}^{j} g^{ij}}{2F^2} \right) - \frac{\bar{V}(\psi)}{4F^2} \right]. \] \tag{26}

Interestingly, introducing the scalar field \(\Psi\) with the requirement:

\[ d\Psi = \sqrt{\left( \frac{3F_{\psi}^{\psi i} N_{i}^{j} g^{ij}}{2F^2} \right)} d\psi \] \tag{27}

the action of Eq. \(26\) can be written as follows

\[ S_{NM} = \int d\tau d^3x \sqrt{-g} \left[ \frac{R}{2} + \frac{\varepsilon}{2} \Psi_i^{\Psi \Psi j} g^{ij} - \frac{\bar{V}(\Psi)}{4F^2} \right]. \] \tag{28}

We conclude that the scalar field \(\Psi\) is minimally coupled (modulus a constant) to the gravitational field. In other words, we find that to every non-minimally coupled scalar field, we may associate a unique minimally coupled scalar field in a conformally related space with an appropriate potential. All considerations are reversible, hence the result is reversible. Finally, we would like to remark that the above theorem can be extended to general Riemannian spaces of \(n\)–dimensions (see appendix A).
5. CONFORMAL LAGRANGIANS IN FLRW COSMOLOGY

In this section we consider a spatially flat \((K = 0)\) FLRW spacetime\(^3\) whose metric is

\[
ds^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j
\]

(29)

where \(\delta_{ij}\) is the 3-space metric in Cartesian coordinates. The Lagrangian of a scalar field \(\phi\) minimally coupled to gravity in this coordinate system \((a, \phi)\) is

\[
L_M = -3a \dot{a}^2 + \frac{\varepsilon}{2} a^2 \dot{\phi}^2 - a^3 V(\phi).
\]

(30)

On the other hand, the Lagrangian of the non minimally coupled scalar field \(\psi\) in the coordinate system \((a, \psi)\) is given by

\[
L_{NM} = 6F(\psi) a \dot{a}^2 + 6F_\psi(\psi) a^2 \dot{\psi} + \frac{\varepsilon}{2} a^3 \dot{\psi}^2 + a^3 V(\psi).
\]

(31)

where \(F(\psi) < 0\) is the coupling function. The Hamiltonian of the above Lagrangian is

\[
E = 6F(\psi) a \dot{a}^2 + 6F_\psi(\psi) a^2 \dot{\psi} + \frac{\varepsilon}{2} a^3 \dot{\psi}^2 + a^3 V(\psi).
\]

(32)

We construct a conformal Lagrangian which corresponds to a minimally coupled scalar field. To do that we introduce the following transformation (see \([56, 57]\)):

\[
A(t) = \sqrt{-2F(t)} a(t).
\]

(33)

Then the Lagrangian (31) takes the form:

\[
L_{NM} = \frac{1}{\sqrt{-2F}} \left[ -3AA'^2 + \frac{\varepsilon}{2} \left( \frac{3\varepsilon F_\psi^2 - F}{2F^2} \right) A^3 \dot{\psi}^2 \right]
\]

\[
- \frac{A^3}{(-2F)^{\frac{3}{2}}} V(\psi).
\]

(34)

It is interesting to mention here that the cross term \(\dot{a} \dot{\psi}\) disappears from Eq.(34). Utilizing simultaneously Eq.(27) and the conformal transformation

\[
d\tau = \sqrt{-2F(\psi)} dt
\]

(35)

we find, after some algebra, that Eq.(34) can be written as

\[
L_M(A, A', \Psi, \Psi') = -3AA'^2 + \frac{\varepsilon}{2} A^3 \Psi'^2 - A^3 \bar{V}(\Psi)
\]

(36)

where

\[
\bar{V}(\Psi) = \frac{A^3}{(-2F)^{\frac{3}{2}}} V(\Psi).
\]

(37)

Notice that the prime denotes derivative with respect to the conformal time \(\tau\).

Evidently, the functional form of the Lagrangian (36) has the general form of Eq.(30) proving our assessment.

Furthermore, considering in the new coordinates \((\tau, x)\) the metric

\[
d\bar{s}^2 = -d\tau^2 + A^2(\tau) \delta_{ij} dx^i dx^j
\]

(38)

we find that the term \(3AA'^2\) equals the Ricci scalar \(\bar{R}\) of the conformally flat metric \(d\bar{s}^2\). In other words, the Lagrangian (30) can be seen as the Lagrangian of a scalar field \(\Psi\) of potential \(\bar{V}(\Psi)\) which is minimally coupled to the gravitational
field $\tilde{g}_{ij}$ in the space with metric $d\tilde{s}^2$. Replacing the quantity $A(\tau)$ and the coordinate $\tau$ from Eq. 63 and Eq. 65 respectively, we obtain:

$$d\tilde{s}^2 = \sqrt{-2F} \left[ -dt^2 + a^2(t) \delta_{ij} dx^i dx^j \right] = \sqrt{-2F} ds^2$$

(39)

that is, the metric $d\tilde{s}^2$ is conformally related to the metric $ds^2$ with conformal function $\sqrt{-2F}$. This means that the non-minimally coupled scalar field in the gravitational field $ds^2$ is equivalent to a minimally coupled scalar field - with appropriate potential defined in terms of the coupling function - in the gravitational field $d\tilde{s}^2$. For the benefit of the reader, we would like to stress that the above geometrical/dynamical result is reversible in the sense that a minimally coupled scalar field $\phi$ in a metric $ds^2$ can be seen as a non-minimally coupled scalar field $\psi$ in the flat FRWL space in which the Eq. 61 is equivalent to the minimally coupled scalar field $\Psi = \Psi(\phi)$ in the conformally related metric $d\tilde{s}^2$, where the conformal function is defined in terms of the coupling function. Equivalently the Lagrangians $L_M$ and $L_{NM}$ are conformally related. Finally, we want to stress that the result of the previous lemma is automatically recovered since the Hamiltonian 32 is equal to zero being the $\{0,0\}$ Einstein equation of the system (see also 55).

6. DISCUSSION AND CONCLUSIONS

In this article we have investigated conformally related metrics and Lagrangians in the context of scalar-tensor cosmology. The aim is to select which is the frame where conformally related solutions have an immediate physical meaning. As discussed in section II, no final statement is available for the problem if solutions have to be interpreted either in the Jordan frame or in the Einstein frame since the physical equivalence can be questioned according to several issues (quantum vs classical measurements, energy conditions, choice of physical units, etc.). Due to this situation, it is too simplistic to consider the problem of conformal frames just a pseudo-problem since we are facing only a mathematical equivalence.

Clearly, it has to be addressed at three levels: i) Lagrangians (or in general, effective actions); ii) field equations; iii) solutions. Actually, the last issue means also the choice of a set of observables where the interpretation of solutions is evident. To this goal, seeking for dynamical quantities invariant under conformal transformations is a fundamental issue. However, such quantities have to be related to geometry and possibly to be conserved like Noether symmetries.

With this target in mind, we have firstly proved a lemma which shows that the field equations of two conformally related Lagrangians are also conformally related if the corresponding Hamiltonian vanishes. This fact is extremely relevant being the Hamiltonian the energy constraint of a given mechanical system and, in particular, it constitutes a non-holonomic constraint for dynamical systems describing cosmological models. It is the $\{0,0\}$ Einstein equation of the system.

Secondly, we have found that to every non-minimally coupled scalar field, we can associate a unique minimally coupled scalar field in a conformally related space with an appropriate potential. The existence of such a connection can be used in order to study the dynamical properties of the various cosmological models, since the field equations of a non-minimally coupled scalar field are the same, at the conformal level, of the field equations of the minimally coupled scalar field. The above propositions can be extended to general Riemannian spaces in $n$-dimensions.

It is worth stressing that the above results are in agreement with the so-called Bicknell’s Theorem which states that a general non-linear $f(R)$ Lagrangian is equivalent to a minimally coupled scalar field with a general potential in the Einstein frame. In Ref. 62, this result is achieved in the case of $R^2$-gravity. In 63, the result is generalized to any analytic $f(R)$-gravity. We’d like to point out that in a recent paper 52, based on the Noether symmetry approach, we have studied the issue of physical solutions in $f(R)$ gravity models and scalar field dark energy models. Starting from these results, it is possible to identify the Noether symmetries, the physical solutions and the corresponding conformal properties of the scalar tensor theories (including $f(T)$ gravity). Such an analysis is in progress.

In general, the Noether symmetries play an important role in physics because they can be used to simplify a given system of differential equations as well as to determine the integrability of the system. The latter will provide the necessary platform in order to solve the equations of motion analytically and thus to obtain the evolution of the physical quantities. In cosmology, such a method is extremely relevant in order to compare cosmographic parameters, such as scale factor, Hubble expansion rate, deceleration parameter, density parameters with observations 55, 59, 61.

Appendix A: Generalization of the Theorem to $n$-dimensions

In this appendix we generalize the theorem of section III to a Riemannian space of dimension $n$. Briefly, we consider the non-minimally coupled scalar field $\psi$ whose field equations are obtained from the action:
\[ S_{NM} = \int dx^n N^n \sqrt{-g} \left[ F(\psi) \bar{R} + \frac{\varepsilon}{2} N^{-2} g^{ij} \psi_i \psi_j - \bar{V}(\psi) \right] \]
\[ = \int dx^n \sqrt{-g} \left[ F(\psi) N^{n-2} R - 2(n-1) F(\psi) N^{n-3} \Delta_1 N + \frac{\varepsilon}{2} N^{-2} g^{ij} \psi_i \psi_j - N^n \bar{V}(\psi) \right] \]  
(A1)

where in order to derive the last equality we have used Eq.(5). Note that we can define the function \( N(x^i) \) in terms of the coupling function \( F(\psi) \) by the requirement:

\[ N^{n-2} = -\frac{1}{2F} \quad F = -\frac{N^{2-n}}{2} \]

which also implies that

\[ N \frac{1}{(-2F)^{\frac{n-2}{2}}} \rightarrow N^{-1} = (-2F)^{\frac{n-2}{2}} \]  
(A2)

\[ N^{-1} g^{-1} = -\frac{F^2}{(n-2)^2 F^2} \psi_i \psi_j. \]  
(A3)

We start now to treat the terms of the action in Eq.(A1). In particular, the first term gives:

\[ \int dx^n \sqrt{-g} \left( F(\psi) N^{n-2} R \right) = \int dx^n \sqrt{-g} \left( \frac{R}{2} \right). \]  
(A4)

If we utilize Eqs.(A2) and (A3) then the second (integrating by parts) and the third terms of the general action are

\[ \int dx^n \sqrt{-g} \left[ -2(n-1) F(\psi) N^{n-3} N_i j g^{ij} \right] = \int dx^n \sqrt{-g} \left[ (n-1) N^{n-3} N_i j g^{ij} \right] \]
\[ = \int dx^n \sqrt{-g} \left[ (n-1) N^{-1} \frac{1}{\sqrt{-g}} (\sqrt{-g} g^{ij} N_k) j \right] \]
\[ = \int dx^n \sqrt{-g} \left[ (n-1) N^{-1} N_i j g^{ij} \right] \]
\[ = \int dx^n \sqrt{-g} \left[ -(n-1) \left( N^{-1} \right) j N_i g^{ij} \right] \]
\[ = \int dx^n \sqrt{-g} \left[ \frac{(n-1) F^2}{(n-2)^2 F^2} \psi_i \psi_j g^{ij} \right]. \]  
(A5)

\[ \int dx^n \sqrt{-g} \left[ F(\psi) N^{n}(n-1)(n-4) \Delta_1 N \right] = \int dx^n \sqrt{-g} \left( -\frac{N^{2-n}}{2} N^{n}(n-1)(n-4) \Delta_1 N \right) \]
\[ = \int dx^n \sqrt{-g} \left[ -\frac{1}{2} N^{2}(n-1)(n-4) \Delta_1 N \right] \]
\[ = \int dx^n \sqrt{-g} \left[ -\frac{1}{2} \frac{(n-1)(n-4)}{(n-2)^2 \frac{F^2}{(-2F)^{\frac{n-2}{2}}} \psi_i \psi_j g^{ij} \right]. \]  
(A6)

To this end the final term gives:

\[ \int dx^n \sqrt{-g} \left( \frac{\varepsilon}{2} N^{n-2} g^{ij} \psi_i \psi_j \right) = \int dx^n \sqrt{-g} \left( -\frac{\varepsilon}{4} \frac{1}{F^2} g^{ij} \psi_i \psi_j \right). \]  
(A7)

Now we change the variable \( \psi \) to \( \Psi \) as follows
\[ d\Psi = \left[ \frac{2\varepsilon(n-1) F_\psi^2}{(n-2)^2 F^2} - \frac{\varepsilon(n-1)(n-4)}{(n-2)^2} \frac{F_\psi^2}{(-2F)^{2-n} F^2} - \frac{1}{2F} \right]^{\frac{1}{2}} d\psi. \]  

(A8)

Collecting the results of the above terms namely Eqs. [A4], [A5], [A6], [A7] and [A8] we find after some non-trivial algebra that the general action of Eq. [A11] is written in terms of \( \psi \) as follows

\[ S_{NM} = \int d^nx \sqrt{-g} \left[ -\frac{R}{2} + \frac{\varepsilon}{2} \Psi_{,i} \Psi_{,j} g^{ij} - \frac{V(\Psi)}{(-2F)^{2-n}} \right]. \]

We would like to remind the reader that the new scalar field \( \Psi \) is minimally coupled to the gravitational field \( g_{ij} \) and that the potential of \( \Psi \) is given by \( \frac{V(\Psi)}{(-2F)^{2-n}} \). Notice that for \( n = 4 \) the above expressions boil down to those of section III as they should. The above proof agrees with that provided by Keiser [64] however in our work we have used a different methodology which is simple and transparent.

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