Technical Report: The Policy Graph Improvement Algorithm

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Abstract—Optimizing a partially observable Markov decision process (POMDP) policy is challenging. The policy graph improvement (PGI) algorithm for POMDPs represents the policy as a fixed size policy graph and improves the policy monotonically. Due to the fixed policy size, computation time for each improvement iteration is known in advance. Moreover, the method allows for compact understandable policies. This report describes the technical details of the PGI [1] and particle based PGI [2] algorithms for POMDPs in a more accessible way than [1] or [2] allowing practitioners and students to understand and implement the algorithms.

I. POMDP

In a POMDP, the agent operates in a world defined by the current state $s$. However, the agent does not observe $s$ directly but makes indirect observations about the state of the world. At each time step $t$, the agent executes an action $a$, receives a reward $R(s,a)$, and the world transitions to a new state $s'$ with probability $P(s'|s,a)$. The agent makes then an observation $o$ with probability $P(o|s',a)$. In a POMDP, the agent can make optimal decisions based on the complete action-observation history or a probability distribution over states. The goal of the agent is to choose actions which maximize the expected total reward $E(\sum_{t=0}^{T-1} R(s(t),a(t))|\pi,b_0)$ over $T$ time steps, where $\pi$ is the policy and $b_0$ is the belief, the initial probability distribution over states.

II. POLICY GRAPH IMPROVEMENT (PGI) ALGORITHM

The PGI algorithm [1] Algorithm 1 improves the value, that is, the expected total reward over $T$ time steps, of a fixed size POMDP policy graph (and Dec-POMDP policy graphs) monotonically. The policy graph is an acyclic graph that consists of $T$ layers of nodes. A policy graph node executes an action and an edge, for each possible observation, defines the next node to transition to. Figure 1 shows an example of a policy graph. See the figure caption for a discussion on how the agent uses the policy graph for choosing actions.

PGI shares some properties with point based POMDP methods which apply value iteration with piece wise linear convex (PWLC) value functions represented as a set of alpha vectors [3]. Essentially, PGI contains an alpha vector at each graph node but it differs from standard PWLC methods in that it 1) restricts the number of alpha vectors at each time step, 2) restricts alpha vectors to specific time steps and allows only alpha vector backups from the next time step, 3) generates a completely new set of beliefs after changing the policy. PGI shares the high level idea of alternating between a forward and backward pass with methods such as differential dynamic programming (DDP) [https://en.wikipedia.org/wiki/Differential_dynamic_programming]

A. Notation

$t$ denotes current time step. $q$ denotes the index of a policy graph node. $t,q$ denotes graph node $q$ at time step $t$. $A$ is the set of actions and $a \in A$ is one action. $O$ denotes the set of observations and $o \in O$ denotes a single observation. $S$ is the set of states and $s \in S$ is a single state. $s'$ denotes the next time step state and $q'$ denotes the next time step policy graph node.

$P(o,s'|s,a)P(o'|s',a)$ is the joint transition and observation probability. $b_{t,q}(s)$ is the non-normalized belief at time step $t$ at policy graph node $q$. $b_t(s,a,o)$ denotes beliefs at all policy graph nodes.

$\pi$ denotes policy. $\pi$ consists of $P_t(a|q)$ and $P_t(q'|q,o)$. $P_t(a|q)$ denotes the probability to execute action $a$ at time step $t$ at policy graph node $q$. $P_t(q'|q,o)$ denotes probability to move to policy graph node $q'$ after observing $o$ in policy graph node $q$ at time step $t$. Note that in the PGI algorithm $P_t(a|q)$ and $P_t(q'|q,o)$ are deterministic. Because of the deterministic policy we often denote the action at time step $t$ and at node $q$ with $a_{t,q}$ and the best mapping from action $a$, observation $o$, current node $q$ to next node as $q_{t+1}(a,o,q)$. 

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B. PGI for POMDPs

Algorithm 1 defines PGI for POMDPs (without policy compression [1]). Figure 2 illustrates the forward pass which projects the initial belief, using the current policy, from the first time step to the last one. Figure 7 illustrates the dynamic programming back pass which optimizes the policy graph for the projected beliefs starting from the last layer and proceeding to the first one.

Policy compression. Policy compression [1] recomputes policies for “redundant” nodes which get an identical policy as another policy graph node in the same layer/time step and, as such, are not useful. A new policy can be computed for the redundant node using a random belief.

1. \( \pi = \text{PGI}(b_0(s), \pi_0) \)

   **Input:** Initial belief \( b_0(s) \), initial policy \( \pi_0 = [P_0(a|q), P_0(q'|q, o)] \)

   **Output:** Optimized policy \( \pi \)

   while No convergence and time limit not exceeded do
   
   2. \( b = \text{ForwardPass}(b_0(s), \pi) \)
   
   3. \( \pi = \text{BackPass}(b) \)
   
   end

   **Algorithm 1:** Monotonic policy graph improvement (PGI) algorithm for POMDPs

1. \( b = \text{ForwardPass}(b_0(s), \pi) \)

2. \( b_{0,0}(s) = b_0(s) \)

3. for Time step \( t = 0 \) to \( T - 1 \) do
   
   4. foreach Policy graph node \( q' \) at layer \( t + 1 \) do
   
   5. \( b_{t+1,q'}(s') = \sum_{q,o,a,s} P(o, s'|s, a)P_t(a|q)P_t(q'|q, o)b_t,q(s) \)
   
   end

   **Algorithm 2:** PGI forward pass

C. Particle PGI

Particle PGI (PPGI) [2] can compute policies for POMDPs with very large state spaces by using a particle based approximation of the belief and by approximating the value function by sampling. Other POMDP methods which use a particle representation include DESPOT [4], POMCP [5], and MCVI [6]. One advantage of PPGI is a fixed size policy which is incrementally improved instead of growing the policy.

When using a particle based belief, the belief consists of a weighted set of particles: \( b(s) = \sum_{i=1}^{N} w_i \delta(s, s_i) \); \( \sum w_i = 1; 0 \leq w_i \leq 1 \), where \( w_i \) is the particle weight and \( \delta(s, s_i) \) is the delta function: \( \delta(s, s_i) = 1 \) only when \( s = s_i \), otherwise zero. Algorithm 7 shows how to perform an approximate Bayesian belief update using particles for a belief given an action and observation.

\( N_{t,q} \) denotes the number of particles in belief \( b_{t,q}(s) \). Algorithm 4 shows the particle based forward pass. Algorithm 5 shows the particle based backwards pass. Algorithm 8 follows [6] Algorithm 1], and, hence, the approximation error bounds in [6] apply.
Fig. 2. Illustration of the forward pass procedure. The top figure shows a summary of the procedure: project initial belief \(b_0(s)\) through the whole policy graph from left to right using the current policy. The following figures show an example of the initial belief and the projection steps.

\[
\begin{align*}
\bar{V}_1(s) &= \sum_{s', q} P(o', s' | s, a) b_0(s) V_{t+1}(s', q') \\
V_t(s) &= \max_q \sum_{s'} P(o, s' | s, a) V_{t+1}(s', q')
\end{align*}
\]

Fig. 3. Illustration of the dynamic programming back pass. The top figure shows a summary of the procedure: start from the last policy graph layer on the right and update the policy and value function at each policy graph layer going from right to left. At each node compute best action and for each observation a forward edge based on the belief at the node. The following figures show an example of the backward pass.
\begin{algorithm}
\SetAlgoLined
\STATE \textbf{b} = \texttt{ParticleForwardPass}(b_0(s), \pi)
\STATE \textbf{b}_{0,0}(s) = b_0(s)
\FOR{Time step $t = 0$ to $T-1$}
\STATE Set $b_{t+1, q'}(s')$ to an empty set for all $q'$
\FOR{Policy graph node $q$ at layer $t$}
\STATE // Sample state $s_t$, next state $s'_{t}$, and observation $o_t$:
\STATE $a = \pi(q)$
\STATE $s_t \sim b_{t, q}(s)$
\STATE $s'_{t} \sim P(s'_{t}|s_t, a)$
\STATE $o_t = P(o_s|s'_t, a)$
\STATE Add state $s'_{t}$ to $b_{t+1, q_{t+1}(a, o_t, q)}(s')$
\ENDFOR
\ENDFOR
\ENDFOR
\caption{PPGI forward pass}
\end{algorithm}

\begin{algorithm}
\SetAlgoLined
\STATE \texttt{ParticleBackPass}(b)
\FOR{Time step $t = T$ to 0}
\FOR{Policy graph node $q$ at layer $t$}
\FOR{Action $a$ do}
\STATE $R_a = 0$
\STATE $V_{a, o, q'} = 0$ for all $o$, $q'$
\STATE $V_{a, o} = 0$ for all $o$
\FOR{$i = 1$ to $N$}
\STATE // Sample state $s_t$, next state $s'_{t}$, and observation $o_t$:
\STATE $s_t \sim b_{t, q}(s)$
\STATE $s'_{t} \sim P(s'_{t}|s_t, a)$
\STATE $o_t = P(o_s|s'_t, a)$
\STATE // Update immediate reward
\STATE $R_a = R_a + R(s_t, a)$
\FOR{Next node $q'$ do}
\STATE // Simulate future value of $s'_{t}$
\STATE $V_{a, o, q'} = V_{a, o, q'} + V_{a, o, q'}$
\STATE Simulate $(s'_{t}, t+1, q', \pi)$
\ENDFOR
\ENDFOR
\ENDFOR
\ENDFOR
\STATE // Next layer node $q_{t+1}(a, o, q)$ for each $a$, $o$ combination:
\STATE $q_{t+1}(a, o, q) = \text{argmax}_{q'} V_{a, o, q'}$
\STATE $V_{a, o} = V_{a, o, q_{t+1}(a, o, q)}$
\STATE $V_a = (R_a + \sum_o V_{a, o}) / N$
\STATE // Optimized action for node at $t, q$:
\STATE $a_{t, q} = \text{argmax}_a V_a$
\ENDFOR
\ENDFOR
\STATE \texttt{ParticleBeliefUpdate}(b, a, o)
\STATE Set $b(s')$ to an empty set
\FOR{$i = 1$ to $|b(s)|$}
\STATE $w_i, s_i = b_i(s)$
\STATE $s'_{i} \sim P(s'_{i}|s_i, a)$
\STATE $w'_i = w_i P(o_s|s'_i, a)$
\STATE Add $w'_i, s'_i$ to $b(s')$
\ENDFOR
\STATE Normalize $b(s')$
\caption{PPGI dynamic programming back pass}
\end{algorithm}

\begin{algorithm}
\SetAlgoLined
\STATE $V = \texttt{Simulate}(s_t, t, q, \pi)$
\STATE $V = R(s_t, a_t, q)$
\FOR{Time step $t$ to $T-1$}
\STATE $s'_{t} \sim P(s'_{t}|s_t, a_t, q)$
\STATE $o_t = P(o_s|s'_t, a_t, q)$
\STATE $q' = q_{t+1}(a_t, q, o_t)$
\STATE $V = V + R(s'_{t}, a_{t+1}, q')$
\ENDFOR
\STATE // Move policy node and state to next time step:
\STATE $q = q', s = s'_i$
\STATE \textbf{Algorithm 5:} PPGI dynamic programming back pass
\end{algorithm}

\begin{algorithm}
\SetAlgoLined
\STATE $b(s') = \texttt{ParticleBeliefUpdate}(b(s), a, o)$
\STATE Set $b(s')$ to an empty set
\FOR{$t = 1$ to $|b(s)|$}
\STATE $w_i, s_i = b_i(s)$
\STATE $s'_{i} \sim P(s'_{i}|s_i, a)$
\STATE $w'_i = w_i P(o_s|s'_i, a)$
\STATE Add $w'_i, s'_i$ to $b(s')$
\ENDFOR
\STATE Normalize $b(s')$
\caption{Particle belief update}
\end{algorithm}

\begin{thebibliography}{10}
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