On some quaternionic generalized slice regular functions

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Abstract

The quaternionic valued functions of a quaternionic variable, often referred to as slice regular functions has been studied extensively due to the large number of generalized results of the theory of one complex variable, see [1, 13, 15–18, 20, 26] and the references given there. Recently, several global properties of these functions has been found of the study of a differential operator, see [11, 19, 21, 24, 25]. Particularly, given a structural set $\psi$ the Borel-Pompieu formula induced by the operator $\psi G$ and its consequences in the slice regular function theory were studied in [24].

The aim of this paper is to present some global and local properties of a kind of quaternionic generalized slice regular functions. We shall see that the global properties are consequences of the study of the perturbed global-type operator:

$$\psi G_v[f] := \psi G[f] - \frac{x_\psi}{2}(x_\psi v + v x_\psi)f,$$

where $v$ is a quaternionic constant and $f$ is a quaternionic-valued continuously differentiable function with domain in $\mathbb{H}$ since our generalized slice regular function space coincides with $\text{Ker}^{\text{std}} G_v$ associated to an axially symmetric $s$-domain, where the $\psi_{\text{std}}$ is standard structural set.

Among the local properties studied in this work are the versions of Splitting Lemma and Representation Theorem that show us a deep relationship between this generalized slice regular function space with a complex generalized analytic function space on each slice.

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1 Introduction

The theory quaternionic slice regular functions has been studied extensively in the last 15 years, see [1, 13, 15–18, 20, 26] and the references given there.

The study of operator

\[ G := \| \mathbf{x} \|^2 \psi_0 \partial_0 + \mathbf{x} \sum_{k=1}^{3} x_k \partial_k. \]

has given us some global properties of the slice regular functions such as Cauchy-Riemann equations, Borel-Pompeiu, Cauchy formulas and Cauchy integral theorem and an analog of the chain role, see [11, 19, 21, 24, 25]. This operator can be associated to any structural set \( \psi \) as follows:

\[ \psi G := \| x_\psi \|^2 \psi_0 \partial_0 + x_\psi \sum_{k=1}^{3} x_k \partial_k, \]

see [24].

The aim of this paper is to present the Borel-Pompeiu and Cauchy integral formulas induced by

\[ \psi G_v[f] := \psi G[f] - \frac{x_\psi}{2} (x_\psi v + vx_\psi)f, \]

where \( v \) is a quaternionic constant and \( f \) is a continuously differentiable function. For the standard structural set \( \psi_{st} \) we shall see a conformal covariant property of \( \psi_{st} G_v \) and the consequences of all these facts in the function space \( \text{Ker} \psi_{st} G_v \) since we shall see that if \( \psi_{st} G_v \) is associated to an axially symmetric slice domain \( \Omega \subset \mathbb{H} \) then \( \text{Ker} \psi_{st} G_v \) if formed by continuously differential functions \( f : \Omega \rightarrow \mathbb{H} \) such that

\[ \frac{\partial f}{\partial z_i} |_{\Omega_i} + \frac{1}{4} (v - ivi)f |_{\Omega_i} = 0, \quad \text{on} \quad \Omega_i, \]

for all \( i \in \mathbb{S}^2 \), which are a kind of quaternionic generalized slice regular functions. What is more, these functions are given in terms of pairs of complex generalized analytic function associated to a complex Vekua-type problem.

In addition, this writing presents some local formulas of these kind of quaternionic generalized slice regular functions such as Splitting Lemma, Representation theorem, Cauchy’s formula, Identity principle, Liouville’s theorem and Morera’s theorem.
After this brief introduction, the structure of the paper is the following: Some preliminaries to the quaternionic slice regular function theory and to the operator $\psi G$ are given in Section 2. Our principal results are stated and proved in Section 3 which is divided in two subsection, the first one studies the function theory induced by the perturbed global-type operator $\psi G_\nu$, while the second one presents the global and local formulas of a kind of quaternionic generalized slice regular functions. Finally, Section 4 presents the conclusions of this paper and future works.

2 Preliminaries

The skew-field of quaternions $\mathbb{H}$ is formed by $q = \sum_{k=0}^{3} q_k e_k$ where $q_k \in \mathbb{R}$ for all $k$ and the quaternionic imaginary units satisfy: $e_0 = 1$, $e_1^2 = e_2^2 = e_3^2 = -1$, $e_1 e_2 = e_3$, $e_2 e_3 = e_1$, and $e_3 e_1 = e_2$. The real part of $q \in \mathbb{H}$ is $q_0$ and the vector part of $q$ is $q = q_1 e_1 + q_2 e_2 + q_3 e_3$ which is usually identified with $(q_1, q_2, q_3) \in \mathbb{R}^3$. The quaternionic conjugation and the quaternionic modulus of $q$ are $\bar{q} = q_0 - q$ and $\|q\| = \sqrt{\bar{q} q} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$, respectively.

On the other hand, given $q, r \in \mathbb{H}$ the inner product is given by

$$\langle q, r \rangle = \sum_{k=0}^{3} q_k r_k = \frac{1}{2} (\bar{q} r + \bar{r} q) = \frac{1}{2} (q \bar{r} + r \bar{q}).$$

The unit open ball in $\mathbb{H}$ is $\mathbb{B}^4(0, 1) = \{ q \in \mathbb{H} \mid \|q\| < 1 \}$ and the unit sphere $S^3 := \{ q \in \mathbb{H} \mid \|q\| = 1 \}$. The unit sphere in $\mathbb{R}^3$ is denoted by $S^2 := \{ q \in \mathbb{R}^3 \mid \|q\| = 1 \}$.

A quadruple of quaternions $\psi = \{ \psi_0, \psi_1, \psi_2, \psi_3 \}$ is called a structural set if $\langle \psi_k ; \psi_m \rangle = \delta_{k,m}$ for $k, m \in \{0, 1, 2, 3\}$. Therefore $\psi$ is a basis of $\mathbb{H}$ as real-linear space and then each quaternion is written by $x_\psi = \sum_{k=0}^{3} x_k \psi_k$ with $x_k \in \mathbb{R}$ for all $k$ and note that $x_{\psi_\text{st}} = x$. Denote $\overline{x_\psi} = \sum_{k=0}^{3} x_k \psi_k^*$ and given $x, y \in \mathbb{H}$ set $\langle x, y \rangle_\psi = \sum_{k=0}^{3} x_k y_k^*$, where $x_\psi = \sum_{k=0}^{3} x_k \psi_k$ and $x_\psi = \sum_{k=0}^{3} y_k \psi_k$. The orthonormality property of $\psi$ allows to see that

$$\langle x, y \rangle_\psi = \frac{1}{2} (\overline{x_\psi} y_\psi + \overline{y_\psi} x_\psi) = \frac{1}{2} (x_\psi \overline{y_\psi} + y_\psi \overline{x_\psi}).$$

The 4-tuple $\psi_\text{st} := \{ e_0, e_1, e_2, e_3 \}$ is called the canonical, or standard, structural set of $\mathbb{H}$. Then one sees that $\langle x, y \rangle_{\psi_\text{st}} = \langle x, y \rangle$ for all $x, y \in \mathbb{H}$, see, e.g. [27] and [28].

2.1 On quaternionic slice regular functions

Let’s recall concepts and properties of the theory of quaternionic slice regular functions for achieving our goal.

Given $i \in S^2$ one sees that $i^2 = -1$ and so $\mathbb{C}(i) := \{ x + iy \mid x, y \in \mathbb{R} \} \cong \mathbb{C}$ as fields.
Let $\Omega \subset \mathbb{H}$ be a domain, i.e., $\Omega$ is an open and connected set, then $\Omega$ is called axially symmetric if $\Omega \cap \mathbb{R} \neq \emptyset$ and if $x + iy \in \Omega$, where $x, y \in \mathbb{R}$ and $i \in S^2$ then \{ $x + jy \mid j \in S^2$ \} $\subset \Omega$. Moreover, $\Omega$ is called slice domain, or s-domain, if $\Omega_i = \Omega \cap \mathbb{C}(i)$ is a domain in $\mathbb{C}(i)$ for all $i \in S^2$.

Let $\Omega \subset \mathbb{H}$ be an axially symmetric s-domain open set. A real differentiable function $f : \Omega \rightarrow \mathbb{H}$ is called quaternionic left slice regular function, or slice regular function, if

$$\frac{\partial f}{\partial \bar{z}_i} := \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) f \mid_{\Omega_i} = 0, \quad \text{on } \Omega_i = \Omega \cap \mathbb{C}(i),$$

for all $i \in S^2$ and its derivative, or Cullen’s derivative see [15], is $f' = \partial f \mid_{\Omega \cap \mathbb{C}(i)} = \frac{\partial}{\partial x} f \mid_{\Omega \cap \mathbb{C}(i)}$. By $SR(\Omega)$ we mean the quaternionic right-linear space of the slice regular functions defined on $\Omega$, see [7][12][13].

Splitting Lemma and Representation Theorem are two important results in the slice regular function theory. Let $\Omega \subset \mathbb{H}$ be an axially symmetric s-domain and $f \in SR(\Omega)$.

1. (Splitting Lemma) Given $i, j \in S$ orthogonal to each other there exist $F, G \in Hol(\Omega_i)$, holomorphic functions from $\Omega_i$ to $\mathbb{C}(i)$, such that $f|_{\Omega_i} = F + Cj$ on $\Omega_i$, see [13].

2. (Representation Formula) Given $q = x + i_qy \in \Omega$, where $x, y \in \mathbb{R}$ and $i_q \in S^2$, then

$$f(x + i_qy) = \frac{1}{2} (1 - i_q i) f(x + iy) + \frac{1}{2} (1 + i_q i) f(x - iy),$$

for all $i \in S^2$, see [7].

Moreover, given $i, j \in S$, orthogonal to each other, from the previous results one has the following operators: $Q_{ij} : SR(\Omega) \rightarrow Hol(\Omega_i) + Hol(\Omega_j)$ and $P_{ij} : Hol(\Omega_i) + Hol(\Omega_j)$ $\rightarrow SR(\Omega)$ defined by $Q_{ij}[f] = f \mid_{\Omega_i} = f_1 + f_2$ for all $f \in SR(\Omega)$, where $f_1, f_2 \in Hol(\Omega_i)$, and

$$P_{ij}[g](x + i_qy) = \frac{1}{2} [(1 + i_q i)g(x - y)i + (1 - i_q i)g(x + yi)], \quad \forall x + i_qy \in \Omega,$$

for all $g \in Hol(\Omega_i) + Hol(\Omega_j)$.

What is more, these operators satisfy that

$$P_{ij} \circ Q_{ij} = \mathcal{I}_{SR(\Omega)} \quad \text{and} \quad Q_{ij} \circ P_{ij} = \mathcal{I}_{Hol(\Omega_i) + Hol(\Omega_j)},$$

where $\mathcal{I}_{SR(\Omega)}$ and $\mathcal{I}_{Hol(\Omega_i) + Hol(\Omega_j)}$ are the identity operators in $SR(\Omega)$ and in $Hol(\Omega_i) + Hol(\Omega_j)$ respectively, see [20].

The development in quaternionic power series. Given $f, g \in SR(\mathbb{B}^4(0, 1))$ there exist sequences of quaternions $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ such that $f(q) = \sum_{n=0}^{\infty} q^n a_n$
and \( g(q) = \sum_{n=0}^{\infty} q^n b_n \) for all \( q \in \mathbb{B}^4(0,1) \) and their \(*\)-product is defined by \( f \ast g(q) = \sum_{n=0}^{\infty} q^n \sum_{k=0}^{n} a_k b_{n-k} \) for all \( q \in \mathbb{B}^4(0,1) \), see [7].

Let \( \Omega \subset \mathbb{H} \) be an axially symmetric s-domain. Identity Principle for the slice regular functions shows that if \( f \in \mathcal{SR}(\Omega) \) and if there exists \( i \in \mathbb{S}^2 \) such that \( Z_f \cap \Omega_i = \{ q \in \Omega \mid f(q) = 0 \} \cap \Omega_i \) has an accumulation point then \( f \equiv 0 \) on \( \Omega \), see [7]. Cauchy’s Integral Formula for slice regular functions shows that given \( f \in \mathcal{SR}(\Omega) , a \in \mathbb{R} \) and \( r > 0 \) such that \( q \in \Delta_q(a,r) = \{ x + y i \mid (x - a)^2 + y^2 \leq r^2 \} \subset \Omega \). one has that
\[
f(q) = \frac{1}{2\pi} \int_{\partial \Delta_q(a,r)} (\zeta - q)^{-1} d\zeta_i f(\zeta),
\]
where \( d\zeta_i := -d\zeta_i q \) and from Cauchy’s Integral Theorem one has that
\[
\int_{\Gamma} f d\zeta = 0,
\]
for all \( i \in \mathbb{S}^2 \) and for any closed, homotopic to a point and piecewise \( C^1 \) curve \( \Gamma \subset \Omega_i \), see [13].

Identity Principle for the slice regular functions. If \( f \in \mathcal{SR}(\Omega) \) and there exists \( i \in \mathbb{S}^2 \) such that \( Z_f \cap \Omega_i = \{ q \in \Omega \mid f(q) = 0 \} \cap \Omega_i \) has an accumulation point then \( f \equiv 0 \) on \( \Omega \), see [7]. Liouville’s Theorem for quaternionic slice regular functions. If \( f \in \mathcal{SR}(\mathbb{H}) \) is a bounded function then \( f \) is a quaternionic constant function,see [13].

Finally, Morera’s Theorem shows that if \( h : \Omega \to \mathbb{H} \) satisfies that
\[
\int_{\Gamma} h d\zeta = 0,
\]
for any closed, homotopic to a point and piecewise \( C^1 \) curve \( \Gamma \subset \Omega_i \) and for all \( i \in \mathbb{S}^2 \) then \( h \in \mathcal{SR}(\Omega) \), see [13].

### 2.2 On Global operator \( \psi G \)

From now on, given an structural set \( \psi \) we shall suppose that \( \psi_0 = \pm 1 \) because the real part of the Cauchy-Riemann operator \( \bar{\partial}_i \) is \( \frac{1}{2} \frac{\partial}{\partial x_0} \) for all \( i \in \mathbb{S}^2 \). The operators \( \psi G \) and \( \psi G_r \) were introduced in [21] as follows:

\[
\psi G[f] := \|\bar{x}_\psi\| \psi_0 \partial_0 f + \bar{x}_\psi \sum_{k=1}^{3} x_k \partial_k f,
\]
\[
\psi G_r[f] := \|\bar{x}_\psi\| \psi_0 \partial_0 f + \sum_{k=1}^{3} x_k (\partial_k f) \bar{x}_\psi,
\]
for any continuously differentiable $\mathbb{H}$-valued function $f$. It is easily verified that 
$\text{Ker}(\psi G)$ is a quaternionic right-linear space and $\text{Ker}(\psi G_r)$ is a quaternionic left-linear space. For short denote $G = \psi_{st} G$.

The Borel-Pompeiu- and the Cauchy-type formulas induced by $\psi G$ and $\psi G_r$ were proved in [24].

The global quaternionic Borel-Pompeiu-type formula. Let $\Omega \subset \mathbb{H}$ be a domain such that $\partial \Omega$ is a 3-dimensional compact smooth surface and $\overline{\Omega} \subset \mathbb{H} \setminus \mathbb{R}$. Then

$$
\int_{\partial \Omega} \| \tau_\psi \|^2 [g(\tau) \nu_\psi A_\psi(x, \tau) - A_\psi(\tau, x) \nu_\psi f(\tau)]
+ \int_{\Omega} [B_\psi(y, x) f(y) - g(y) C_\psi(x, y)] dy
+ 2 \int_{\Omega} [A_\psi(y, x) \psi G[f](y) - \psi G_r [g](y) A_\psi(x, y)] dy
= \begin{cases} f(x) + g(x), & x \in \Omega, \\ 0, & x \in \mathbb{H} \setminus \overline{\Omega}, \end{cases}
$$

(2.3)

for all $f, g \in C^1(\overline{\Omega}, \mathbb{H})$, where the reproducing functions are given by

$$
A_\psi(x, t) = \frac{1}{2\pi^2} \frac{x_\psi(t_\psi - x_\psi) t_\psi}{\|x_\psi\|^2 \|t_\psi - x_\psi\|^4 \|t_\psi\|^2},
$$

$$
B_\psi(t, x) = \frac{1}{\pi^2} \frac{x_\psi}{\|x_\psi\|^2} \left[ t_\psi + 3x_\psi - 4t_\psi \frac{t_\psi - x_\psi}{\|t_\psi - x_\psi\|^4} + \frac{4(\tau_\psi - x_\psi)(x_0 - t_\psi) t_\psi - \langle t, x \rangle_\psi}{\|t_\psi - x_\psi\|^6} \right],
$$

$$
C_\psi(x, t) = \frac{1}{\pi^2} \frac{t_\psi + 3x_\psi - 4t_\psi}{\|x_\psi - t_\psi\|^4} + \frac{4[(x_0 - t_\psi) t_\psi - \langle t, x \rangle_\psi](\tau_\psi - x_\psi)}{\|t_\psi - x_\psi\|^6} \frac{x_\psi}{\|x_\psi\|^2},
$$

where $\tau_\psi, x_\psi \in \mathbb{H} \setminus \mathbb{R}$ such that $x_\psi = x_0 \psi_0 + \sum_{k=1}^3 x_k \psi_k = x_0 \psi_0 + x_\psi$ and $x_\psi \neq \tau_\psi$.

For abbreviation denote $A = A_{\psi_{st}}$, $B = B_{\psi_{st}}$ and $C = C_{\psi_{st}}$. The differential form

$$
\nu^\psi_x = 2\psi_0 d\hat{x}_0 + 2 \frac{x_\psi}{\|x_\psi\|^2} \sum_{k=1}^3 x_k d\hat{x}_k = \sigma^\psi_x - \frac{x_\psi}{\|x_\psi\|^2} \frac{x_\psi}{\|x_\psi\|^2},
$$

where $\sigma^\psi_x$ is the quaternionic differential form of the 3 dimensional volume in $\mathbb{R}^4$, see [24].

The following notation is used all over the paper:

- Given two domains $\Xi, \Omega \subset \mathbb{R}^4$, let $\alpha : \Xi \to \Omega$ be a one-to-one correspondence; if $f$ belongs to a function space on $\Omega$ then denote $W_\alpha : f \mapsto f \circ \alpha$.  

6
If β is a \( \mathbb{H} \)-valued function then denote \( \beta M : f \mapsto \beta f \), \( M \beta : f \mapsto f \beta \) on the same function space.

Particularly, if \( \Omega \subset \mathbb{H} \) is an axially symmetric slice domain in \([25]\) was proved that \( \text{Ker}(G) \cap C^1(\Omega, \mathbb{H}) = \text{SR}(\Omega) \), where \( G = \psi G \).

Given the quaternionic Möbius transformation
\[
T(x) = (ax + b)(cx + d)^{-1},
\]
where \( a, b, c, d \in \mathbb{H} \) satisfy that \( a\bar{c}, (b - ac^{-1}d)\bar{c}, d(b - ac^{-1}d) \in \mathbb{R} \) then the conformal covariant property of \( G \):
\[
G \circ (A_T M \circ W_T) = (B_T M \circ W_T) \circ G
\]
on \( C^1(T^{-1}(\Omega), \mathbb{H}) \) where \( A_T(x) = \bar{c}, B_T(y) = ||c||b - ac^{-1}d||\bar{c}(y - ac^{-1})^{-2} \) and \( y = T(x) \) was given in \([25]\).

3 Main results

3.1 On a quaternionic perturbed global-type operator

Let \( \psi \) be a structural set consider
\[
\psi G_v[f] := \psi G[f] - \frac{x_\psi}{2}(x_\psi v + v x_\psi)f,
\]
\[
\psi G_{r, v}[f] := \psi G_r[f] - f \frac{x_\psi}{2}(x_\psi v + v x_\psi),
\]
where \( v \) is a quaternionic constant and \( f \) is a continuously differentiable function on a domain in \( \mathbb{H} \).

**Theorem 3.1.** (Quaternionic Borel-Pompeiu-type formula induced by \( \psi G_v \) and \( \psi G_{r, v} \)) Let \( \Omega \subset \mathbb{H} \) be a domain such that \( \partial \Omega \) is a 3-dimensional compact smooth surface and \( \overline{\Omega} \subset \mathbb{H} \setminus \mathbb{R} \). Then
\[
\int_{\partial \Omega} ||\tau_\psi||^2 [g(\tau)u_\psi A_\psi(x, \tau, u) - A_\psi(\tau, x, v)u_\psi f(\tau)]
+ \int_{\Omega} [B_\psi(y, x, v)f(y) - g(y)C_\psi(x, y, u)]dy
+ 2\int_{\Omega} [A_\psi(y, x, v)\psi G_v[f](y) - \psi G_{r, u}[g](y)A_\psi(x, y, u)]dy
= \begin{cases} f(x) + g(x), & x \in \Omega, \\ 0, & x \in \mathbb{H} \setminus \overline{\Omega}, \end{cases}
\]
for all \( f, g \in C^1(\overline{\Omega}, \mathbb{H}) \), where

\[
A_\psi(x, t, u) = \frac{1}{2\pi^2} e^{(t-x, u)_\psi} x_\psi (t_\psi - x_\psi) t_\psi,
\]

\[
B_\psi(t, x, v) = \frac{1}{\pi^2} e^{(t-x, v)_\psi} x_\psi \left[ \frac{t_\psi + 3t_\psi - 4t_\psi}{||t_\psi - x_\psi||^4} + \frac{4(t_\psi - x_\psi)(x_0 - t_\psi) t_\psi - (t, x)_\psi}{||t_\psi - x_\psi||^6} \right],
\]

\[
C_\psi(x, t, u) = \frac{1}{\pi^2} \left[ \frac{t_\psi + 3t_\psi - 4t_\psi}{||x_\psi - t_\psi||^4} + \frac{4[(x_0 - t_\psi) t_\psi - (t, x)_\psi](t_\psi - x_\psi)}{||t_\psi - x_\psi||^6} \right] e^{(t-x, u)_\psi} x_\psi.
\]

Proof. Apply formula (2.3) to functions \( e^{(\cdot, y)_\psi} f \) and \( e^{(\cdot, u)_\psi} g \) to get the following:

\[
\int_{\partial\Omega} \|ar{\tau}_\psi\|^2 \left[ e^{(\tau, u)_\psi} g(\tau) \nu_\psi A_\psi(x, \tau) - A_\psi(\tau, x) \nu_\psi e^{(\tau, v)_\psi} f(\tau) \right] + \int_{\Omega} \left[ B_\psi(y, x) e^{(y, v)_\psi} f(y) - e^{(y, u)_\psi} g(y) C_\psi(x, y) \right] dy
\]

\[
+ 2 \int_{\Omega} \left[ A_\psi(y, x) e^{(y, v)_\psi} G[e^{(y, u)_\psi} f](y) - e^{(y, u)_\psi} g(y) A_\psi(x, y) \right] dy \]

\[
= \left\{ \begin{array}{ll}
e^{(x, v)_\psi} f(x) + e^{(x, u)_\psi} g(x), & x \in \Omega, \\
0, & x \in \mathbb{H} \setminus \overline{\Omega}, \end{array} \right.
\]

for all \( f, g \in C^1(\overline{\Omega}, \mathbb{H}) \). From direct computations one sees that

\[
\psi G[e^{(x, v)_\psi} f](x) = e^{(x, v)_\psi} \left( \psi G[f](x) - \frac{x_\psi}{2} (x_\psi v + v x_\psi) f(x) \right)
\]

\[
= e^{(x, v)_\psi} \psi G[f](x),
\]

\[
\psi G_r[e^{(x, v)_\psi} f](x) = e^{(x, v)_\psi} \left( \psi G_r[f](x) - f(x) \frac{x_\psi}{2} (x_\psi v + v x_\psi) \right)
\]

\[
= e^{(x, v)_\psi} \psi G_{r,v}[f](x),
\]

where we denote \( v = v_\psi \) since it is a quaternionic constant. Therefore,

\[
\int_{\partial\Omega} \|ar{\tau}_\psi\|^2 \left[ g(\tau) \nu_\psi e^{(\tau, u)_\psi} A_\psi(x, \tau) - e^{(\tau, v)_\psi} A_\psi(\tau, x) \nu_\psi f(\tau) \right] + \int_{\Omega} \left[ e^{(y, v)_\psi} B_\psi(y, x) f(y) - g(y) e^{(y, u)_\psi} C_\psi(x, y) \right] dy
\]

\[
+ 2 \int_{\Omega} \left[ e^{(y, v)_\psi} A_\psi(y, x) e^{(y, v)_\psi} G_r[f](y) - e^{(y, u)_\psi} g(y) e^{(y, u)_\psi} A_\psi(x, y) \right] dy \]

\[
= \left\{ \begin{array}{ll}
e^{(x, v)_\psi} f(x) + e^{(x, v)_\psi} g(x), & x \in \Omega, \\
0, & x \in \mathbb{H} \setminus \overline{\Omega}, \end{array} \right.
\]

(3.1)
for all \(f, g \in C^1(\mathbb{H}, \mathbb{H})\). Then consider \(g = 0\) and multiply the obtained identity by the factor \(e^{(t-x,u)\psi}\) on both sides. Repeat the reasoning for \(f = 0\) and \(e^{(t-x,u)\psi}\) to obtain two expression. Finally, add these two expressions and note that
\[
A_{\psi}(x, t, u) = e^{(t-x,u)\psi} A_{\psi}(x, t), \quad B_{\psi}(t, x, v) = e^{(t-x,v)\psi} B_{\psi}(t, x), \quad C_{\psi}(x, t, u) = e^{(t-x,u)\psi} C_{\psi}(x, t).
\]

**Corollary 3.2.** (Quaternionic Cauchy-type formula induced by \(\psi G_v\) and \(\psi G_r,v\)) Let \(\Omega \subset \mathbb{H}\) be a domain such that \(\partial \Omega\) is a 3-dimensional compact smooth surface and \(\overline{\Omega} \subset \mathbb{H}\). Given \(f, g \in C^1(\mathbb{H}, \mathbb{H})\) such that \(f \in \text{Ker}^\psi G_v\) and \(g \in \text{Ker}^\psi G_u\) then
\[
\int_{\partial \Omega} \|\overline{\tau}\|^2 [g(\tau)\nu^\psi_{\tau} A_{\psi}(x, \tau, u) - A_{\psi}(\tau, x, v)\nu^\psi_{\tau} f(\tau)]
+ \int_{\Omega} [B_{\psi}(y, x, v) f(y) - g(y) C_{\psi}(x, y, u)] dy
= \begin{cases} f(x) + g(x), & x \in \Omega, \\ 0, & x \in \mathbb{H} \setminus \Omega, \end{cases}
\]
for all \(f, g \in C^1(\overline{\Omega}, \mathbb{H})\).

**Proof.** It’s follows from Proposition 3.1.

We shall see a conformal covariant property of \(G_v\).

**Proposition 3.3.** Conformal covariant-type property of \(G_u\). Let \(\Omega \subset \mathbb{H}\) be a domain and \(u, v \in \mathbb{H}\). Given the quaternionic Möbius transformation \(T : T^{-1}(\Omega) \rightarrow \Omega\) given by (2.4). Denote \(y = T(x)\). Then
\[
G_u \circ e^{<v>} AT M \circ W_T = E_T M \circ W_T \circ G_{G_T}
\]
on \(C^1(\Omega, \mathbb{H})\), where \(G_{\Gamma_T} := G - \Gamma_T M\),
\[
\Gamma_T(y) = \frac{\|\ell\|^2}{2\|c\|^3} \left[ (\ell c - m)^{-1} - p \right] [(\ell c - m)^{-1} - p] (u + v) + (u + v)( (\ell c - m)^{-1} - p ) \bar{e},
\]
\[
E_T(y) = \frac{\|c\|^2}{\|\ell\|^2} e^{<T^{-1}(y),v>} \bar{e}(y - ac^{-1})^{-2},
\]
where \(\ell = (b - ac^{-1}d)^{-1}, m = \ell a\) and \(p = c^{-1}d\).

**Proof.** It is important to comment that the differentiation variable of \(G_u\) is \(x\) and
the variable of $G$ is $y$. Given $f \in C^1(\Omega, \mathbb{R})$ from (3.1) and (2.5) one obtains that

$$G_u[e^{<x,v>\Lambda_T}M \circ W_T[f](x)] = G[e^{<x,v>\Lambda_T}M \circ W_T[f](x) - \frac{x}{2}(\mathbf{x}u + u\mathbf{x})e^{<x,v>\Lambda_T}M \circ W_T[f](x)]$$

$$= e^{<x,v>} \{ G^{\Lambda_T}M \circ W_T[f](x) - \frac{x}{2}(\mathbf{x}v + v\mathbf{x})^{\Lambda_T}M \circ W_T[f](x) \}$$

$$= e^{<x,v>} \{ G^{\Lambda_T}M \circ W_T[f] - \frac{x}{2}(\mathbf{x}(u + v) + (u + v)\mathbf{x})^{\Lambda_T}M \circ W_T[f] \}$$

Due to the properties of the coefficients of $T$ one sees that the mapping $y \to \ell y c$ preserves $\mathbb{R}^3$ and so $x = [\ell yc - m]^{-1} - p$. Therefore

$$G_u[e^{<x,v>\Lambda_T}M \circ W_T[f](x)] = e^{<x,v>} \{ (B_T \circ W_T) \circ G[f](y) - \frac{1}{2}[([\ell yc - m]^{-1} - p) \{ (\ell yc - m)^{-1} - p \}(u + v) + (u + v)((\ell yc - m)^{-1} - p) \}$$

From definitions of $\Lambda_T$ and $B_T$ one has that

$$G_u[e^{<x,v>\Lambda_T}M \circ W_T[f](x)] = e^{<x,v>} \left\{ \| c \| \| \ell \| c(y - ac^{-1})^{-2}W_T \circ G[f] - \frac{1}{2}[([\ell yc - m]^{-1} - p) \{ (\ell yc - m)^{-1} - p \}(u + v) + (u + v)((\ell yc - m)^{-1} - p) \} eW_T[f] \right\}$$

Therefore,

$$G_u[e^{<x,v>\Lambda_T}M \circ W_T[f](x)] = e^{<x,v>} \frac{\| c \|}{\| \ell \|} \| c \| c(y - ac^{-1})^{-2} \left\{ W_T \circ G[f] - \frac{\| \ell \|((y - ac^{-1})^{2}c}{2\| c \|^3} \right\} \{ (\ell yc - m)^{-1} - p \} \{ (\ell yc - m)^{-1} - p \}(u + v) + (u + v)((\ell yc - m)^{-1} - p) \} eW_T[f] \}$$

and according to definitions of $\Gamma_T$ and $E_T$ one concludes the main identity.

**Remark 3.4.** The previous proposition explains the behavior of the new perturbed operators $G + \Gamma_T M$ in terms of our perturbed operators.
The conformal covariant-type property of \( G_u \) is a quaternionic version of the chain rule in complex analysis and it implies that \( f \in \text{Ker} G_{\Gamma^T} \cap C^1(\Omega, \mathbb{H}) \) if and only if \( e^{\langle v, -u \rangle A_T} M \circ W_T[f] \in \text{Ker} G_u C^1(T^{-1}(\Omega), \mathbb{H}) \). What is more, the previous property allows us to establish a bijective quaternionic-right linear mapping from \( \text{Ker} G_{\Gamma^T} \cap C^1(\Omega, \mathbb{H}) \) to \( \text{Ker} G_u \cap C^1(T^{-1}(\Omega), \mathbb{H}) \).

Choosing \( v = -u \) one obtains that \( G_u \circ e^{\langle v, -u \rangle A_T} M \circ W_T = e^{\langle v, -u \rangle (B_T M \circ W_T) \circ G} \), on \( C^1(\Omega, \mathbb{H}) \), i.e., the perturbation is eliminated.

The computations shown above can be repeated on any structural set \( \psi \) to obtain a conformal covariant-type property of \( \psi G_v \). But our interest to study some generalized slice regular functions; that is why we only will consider \( \psi \) in the following sentences.

Note that the computations presented in this subsection extend the results obtained in [24, 25] since \( \psi G_0 = \psi G \) and \( \psi G_{r,0} = \psi G_r \). In addition, papers [5, 6] present several results about some quaternionic operators that can be used to study \( \psi G_r \) and \( \psi G_{r,v} \).

3.2 On some quaternionic generalized slice regular functions

**Definition 3.5.** Let \( \Omega \subset \mathbb{H} \) be an axially symmetric s-domain and given \( v \in \mathbb{H} \). If \( f \in C^1(\Omega, \mathbb{H}) \) satisfies

\[
\frac{\partial f}{\partial \bar{z}_i} + \frac{1}{4}(v - ivi)f |_{\Omega_i} = 0, \quad \text{on } \Omega_i,
\]

for all \( i \in S^2 \) then \( f \) is called \( v \)-slice regular functions, that is a kind of quaternionic generalized slice regular function. The quaternionic right-linear space formed by these functions is denoted by \( \mathcal{SR}_v(\Omega) \).

**Definition 3.6.** Let \( \Omega \subset \mathbb{H} \) be an axially symmetric s-domain. Given \( v \in \mathbb{H} \) and \( i \in S^2 \) define

\[
M_v(\Omega_i) = \{ h \in C^1(\Omega_i, \mathbb{C}(i)) \mid \frac{\partial h}{\partial \bar{z}_i} + \frac{1}{4}(v - ivi)h = 0, \quad \text{on } \Omega_i \}.
\]

Note that if \( v = v_1 + v_2 j \), where \( v_1, v_2 \in \mathbb{C}(i) \) with \( j \in S^2 \) orthogonal to \( i \), then \( v - ivi = 2v_1 \). Therefore the identity

\[
\frac{\partial h}{\partial \bar{z}_i} + \frac{1}{4}(v - ivi)h = 0, \quad \text{on } \Omega_i
\]

is equivalent to

\[
\frac{\partial h}{\partial \bar{z}_i} + \frac{1}{2} v_1 h = 0, \quad \text{on } \Omega_i.
\]
In complex analysis it is well-known that $M_v(\Omega_i)$ is a generalized analytic function space associated to a Vekua-type problem and several applications in which appears the bi-dimensional Helmholtz equation, see [2, 29, 30]. The function space $M_v(\Omega_i)$ is considered as a generalization of the analytic function space since it is related with the Cauchy-Riemann operator perturbed by the multiplication operator by the complex number. Analogously, $SR_v(\Omega)$ is considered a quaternionic generalized slice regular function space.

Definition 3.5 and $\text{Ker} G_v$ are related in next result.

**Proposition 3.8.** Let $\Omega \subset \mathbb{H}$ be an axially symmetric s-domain and $v \in \mathbb{H}$ then $SR_v(\Omega) = \text{Ker} G_v \cap C^1(\Omega, \mathbb{H})$.

**Proof.** Set $f \in \text{Ker} G_v \cap C^1(\Omega, \mathbb{H})$ or equivalently, from (3.1) one sees that $e^{(\cdot, v)} f \in \text{Ker} G \cap C^1(\Omega, \mathbb{H})$. From the global characterization of the quaternionic slice regular functions presented in Subsection 5 of [11] one sees that

$$\frac{\partial e^{(\cdot, v)} f}{\partial \bar{z}_i} |_{\Omega_i} = 0, \quad \text{on } \Omega_i,$$

for all $i \in S^2$. One easily sees that $\langle z, v \rangle = \langle z, v_1 \rangle$ for all $z \in \Omega_i$ where $v = v_1 + v_2$ with $v_1, v_2 \in \mathbb{C}(i)$ and $j \in S^2$ is orthogonal to $i$. From direct computations one gets that

$$\frac{\partial e^{(\cdot, v_1)} f}{\partial \bar{z}_i} |_{\Omega_i} = e^{(\cdot, v_1)} \left[ \frac{\partial f}{\partial \bar{z}_i} |_{\Omega_i} + \frac{1}{2} v_1 f |_{\Omega_i} \right],$$

or equivalently

$$\frac{\partial e^{(\cdot, v)} f}{\partial \bar{z}_i} |_{\Omega_i} = e^{(\cdot, v)} \left[ \frac{\partial f}{\partial \bar{z}_i} |_{\Omega_i} + \frac{1}{4} (v - iv) f |_{\Omega_i} \right],$$

on $\Omega_i$ for all $i \in S^2$. Recall that $2v_1 = v - iv$.

Therefore $f \in \text{Ker} G_v \cap C^1(\Omega, \mathbb{H})$ iff

$$\frac{\partial f}{\partial \bar{z}_i} |_{\Omega_i} + \frac{1}{4} (v - iv) f |_{\Omega_i} = 0,$$

on $\Omega_i$ for all $i \in S^2$. \qed

**Corollary 3.9.** (The global quaternionic Cauchy-type formula for $SR_v(\Omega)$) Let $\Omega \subset \mathbb{H}$ be an axially symmetric s-domain such that $\partial \Omega$ is a 3-dimensional compact smooth surface and $\overline{\Omega} \subset \mathbb{H} \setminus \mathbb{R}$. Given $f \in SR_v(\Omega)$ then

$$- \int_{\partial \Omega} ||\bar{\tau}_\psi||^2 A_\psi(\tau, x, v) v_\psi f(\tau) + \int_\Omega B_\psi(y, x, v) f(y) dy$$

$$f(x), \quad x \in \Omega,$$

$$0, \quad x \in \mathbb{H} \setminus \overline{\Omega},$$
Proof. Direct consequence of Proposition 3.8 and Corollary 3.2.

The quaternionic Möbius $T$ given in (2.4) preserves axially symmetric s-domains, i.e., $\Omega \subset \mathbb{H}$ and $T^{-1}(\Omega)$ are axially symmetric s-domains simultaneously, see [25].

**Corollary 3.10.** Let $\Omega \subset \mathbb{H}$ be an axially symmetric s-domain and set $v \in \mathbb{H}$. Given $T: \mathbb{C}(\Omega, \mathbb{H})$ and $f \in C^1(\Omega, \mathbb{H})$. Then $e^{(r,v)M \circ W_T[f]} \in \mathcal{SR}_v(T^{-1}(\Omega))$ if and only if $G_{r,v}[f] = 0$ on $\Omega$.

Proof. Its follows from Proposition 3.8 and Proposition 3.3.

Now we shall see several local properties of $\mathcal{SR}_v(\Omega)$.

**Proposition 3.11.** (Splitting Lemma and Representation Theorem for $\mathcal{SR}_v(\Omega)$). Let $\Omega \subset \mathbb{H}$ be an axially symmetric s-domain and $v \in \mathbb{H}$. For $f \in \mathcal{SR}_v(\Omega)$ one has the following:

1. Let $i, j \in \mathbb{S}$ be orthogonal to each other there exist $F, G \in M_v(\Omega_i)$ such that $f|_{\Omega_i} = F + Gj$ on $\Omega_i$.

2. Given $q = x + i_qy \in \Omega$, where $x, y \in \mathbb{R}$ and $i_q \in \mathbb{S}^2$, one has that

$$f(x + i_qy) = \frac{1}{2} \left\{ e^{(1-i_q)y,v}(1 - i_qi) f(x + iy) + e^{(1-i_q)y,v}(1 + i_qi) f(x - iy) \right\},$$

for all $i \in \mathbb{S}^2$.

Proof. First of all, let us recall that $v - ivi \in \mathbb{C}(i)$ for all $i \in \mathbb{S}^2$.

1. Set $f = f_1 + f_2j$, where $f_1, f_2 \in C^1(\Omega, \mathbb{C}(i))$, then

$$\frac{\partial(f_1 + f_2j)}{\partial z} + \frac{1}{4}(v - ivi)(f_1 + f_2j) = 0 \text{ on } \Omega_i.$$

Therefore,

$$\frac{\partial f_1}{\partial z} + \frac{1}{4}(v - ivi)f_1 = 0 \text{, and } \frac{\partial f_2}{\partial z} + \frac{1}{4}(v - ivi)f_2 = 0 \text{ on } \Omega_i,$$

i.e., $f_1, f_2 \in M_v(\Omega_i)$.

2. From the proof of Proposition 3.8 one gets that $q \mapsto e^{<q,v>}f(q) \in \mathcal{SR}(\Omega)$ and from Representation Theorem for the slice regular functions one has that

$$e^{<x+i_qy,v>}f(x + i_qy) = \frac{1}{2} \left\{ e^{<x+iy,v>}(1 - i_qi)f(x + iy) + e^{<x-iy,v>}(1 + i_qi)f(x - iy) \right\},$$

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or equivalently

\[
f(x + iy) = \frac{1}{2} \left\{ e^{-i(x+iy)}(1 - iy)f(x + iy) + e^{-i(x+iy)}(1 + iy)f(x - iy) \right\}.
\]

\[\square\]

**Remark 3.12.** The previous proposition shows a deep relationship between the \(v\)-slice regular-type functions with a kind of complex generalize analytic functions that are solutions of some complex Vekua problems.

Let \(i, j \in \mathbb{S}\) be orthogonal to each other. Proposition 3.11 defines the operators \(Q_{i,j}^v: \mathcal{SR}_v(\Omega) \rightarrow M_v(\Omega_1) + M_v(\Omega_4)j\) and \(P_{i,j}^v: M_v(\Omega_1) + M_v(\Omega_4)j \rightarrow \mathcal{SR}_v(\Omega)\) given by

\[
Q_{i,j}^v[f] = f \mid_{\Omega_1} = f_1 + f_2j, \quad \forall f \in \mathcal{SR}_v(\Omega),
\]

where \(f_1, f_2 \in M_v(\Omega_1)\) and

\[
P_{i,j}^v[f](x + iy) = \frac{1}{2} \left\{ e^{i(x+iy)}(1 - iy)f(x + iy) + e^{i(x+iy)}(1 + iy)f(x - iy) \right\},
\]

for all \(x + iy \in \Omega\) and for all \(g \in M_v(\Omega_1) + M_v(\Omega_4)j\). What is more,

\[
P_{i,j}^v \circ Q_{i,j}^v = I_{\mathcal{SR}_v(\Omega)} \quad \text{and} \quad Q_{i,j}^v \circ P_{i,j}^v = I_{M_v(\Omega_1) + M_v(\Omega_4)j},
\]

where \(I_{\mathcal{SR}_v(\Omega)}\) and \(I_{M_v(\Omega_1) + M_v(\Omega_4)j}\) are the identity operators in \(\mathcal{SR}_v(\Omega)\) and in \(M_v(\Omega_1) + M_v(\Omega_4)j\), respectively.

**Proposition 3.13.** If \(f \in \mathcal{SR}_v(\mathbb{B}^4(0,1))\) then there exists a sequence of quaternions \((a_n)_{n=0}^\infty\) such that

\[
f(q) = \sum_{n=0}^\infty \sum_{k=0}^n q^{-k}(vq + q\tilde{v})^k \frac{(-1)^k a_{n-k}}{2^k k!}, \quad \forall q \in \mathbb{B}^4(0,1).
\]

Particularly,

\[
f(q) = \sum_{n=0}^\infty q^n \sum_{k=0}^n \frac{(-1)^k c_{v,k}(q)a_{n-k}}{2^k k!}, \quad \forall q \in \mathbb{B}^4(0,1) \setminus \{0\},
\]

where \(c_{v,k}(q) = \left( \frac{\tilde{q}}{|q|} \cdot \frac{\bar{q}}{|q|} \right) + \tilde{v}^k\).

**Proof.** As \(f \in \mathcal{SR}_v(\mathbb{B}^4(0,1))\) from Proposition 3.8 and equation (3.1) one sees that the mapping \(q \mapsto e^{(q,v)} f(q)\) belongs to \(\mathcal{SR}(\mathbb{B}^4(0,1))\) then there exists a sequence of quaternions \((a_n)_{n=0}^\infty\) such that

\[
f(q) = e^{-(q,v)} \sum_{n=0}^\infty q^n a_n = \sum_{n=0}^\infty \frac{(-1)^n}{n!} (q,v)^n \sum_{n=0}^\infty q^n a_n,
\]

\(14\)
or equivalently,
\[ f(q) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} q^{n-k} (v\bar{q} + q\bar{v})^k (-1)^k \frac{a_{n-k}}{2^k k!}. \]

Particularly, if \( q \neq 0 \) then
\[ f(q) = \sum_{n=0}^{\infty} q^n \sum_{k=0}^{n} (-1)^k c_{v,k}(q) a_{n-k} \frac{2^k k!}{2^k k!}, \]
where
\[ c_{v,k}(q) = (\frac{\bar{q}}{\|q\|} v + q\bar{v})^k. \]

Remark 3.14. Using the * product of slice regular functions we can establish the multiplication in \( \mathcal{SR}_v(\mathbb{B}^4(0,1)) \) as follows:
\[ f * v g(q) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} q^{n-k} (v\bar{q} + q\bar{v})^k (-1)^k a_{n-k} \frac{2^k k!}{2^k k!}, \]
for all \( q \in \mathbb{B}^4(0,1) \), where \( f, g \in \mathcal{SR}_v(\mathbb{B}^4(0,1)) \) and
\[ f(q) = \sum_{n=0}^{\infty} q^n \sum_{k=0}^{n} (-1)^k c_{v,k}(q) a_{n-k} \frac{2^k k!}{2^k k!}, \]
\[ g(q) = \sum_{n=0}^{\infty} q^n \sum_{k=0}^{n} (-1)^k b_{n-k} \frac{2^k k!}{2^k k!}, \]
for all \( q \in \mathbb{B}^4(0,1) \). Note that \( f * v g = f * g \) for \( f, g \in \mathcal{SR}_0(\mathbb{B}^4(0,1)) \)

Proposition 3.15. (Cauchy’s Integral Formula for \( \mathcal{SR}_v(\Omega) \)). Let \( \Omega \subset \mathbb{H} \) be an axially symmetric s-domain. Given \( f \in \mathcal{SR}_v(\Omega), a \in \mathbb{R} \) and \( r > 0 \) such that \( q \in \Delta_q(a,r) \subset \Omega \) then
\[ f(q) = \frac{1}{2\pi} \int_{\partial\Delta_q(a,r)} e^{(\zeta - q,v)} (\zeta - q)^{-1} d\zeta \xi f(\zeta), \]
where \( d\zeta \xi := -d\zeta \).

Proof. As the mapping \( q \mapsto e^{(q,v)} f(q) \) belongs to \( \mathcal{SR}(\Omega) \) then (2.1) implies
\[ e^{(q,v)} f(q) = \frac{1}{2\pi} \int_{\partial\Delta_q(a,r)} (\zeta - q)^{-1} d\zeta \xi e^{(\zeta,v)} f(\zeta), \]
or equivalently
\[ f(q) = \frac{1}{2\pi} \int_{\partial\Delta_q(a,r)} e^{(\zeta - q,v)} (\zeta - q)^{-1} d\zeta \xi f(\zeta), \]
where \( d\zeta \xi := -d\zeta \).
Corollary 3.16. *(Cauchy’s Integral Theorem for \( \mathcal{SR}_v(\Omega) \)).* If \( f \in \mathcal{SR}_v(\Omega) \) then
\[
\int_{\Gamma} f e^{\langle \zeta, v \rangle} d\zeta = 0,
\]
for all \( i \in \mathbb{S}^2 \) and for any closed, homotopic to a point and piecewise \( C^1 \) curve \( \Gamma \subset \Omega_i \).

**Proof.** As \( q \mapsto e^{\langle q, v \rangle} f(q) \) belongs to \( \mathcal{SR}(\Omega) \) from (2.2) one has
\[
\int_{\Gamma} e^{\langle \zeta, v \rangle} f(\zeta) d\zeta = 0,
\]
for all \( i \in \mathbb{S}^2 \) and for any closed, homotopic to a point and piecewise \( C^1 \) curve \( \Gamma \subset \Omega_i \).

We finish this paper by showing the Identity Principle, Liouville’s Theorem and Morera’s Theorem for \( \mathcal{SR}_v(\Omega) \).

**Proposition 3.17.** Let \( \Omega \subset \mathbb{H} \) be an axially symmetric s-domain.

1. **Identity Principle for \( \mathcal{SR}_v(\Omega) \).** If \( f \in \mathcal{SR}_v(\Omega) \) and there exists \( i \in \mathbb{S}^2 \) such that \( Z_f \cap \Omega_i = \{ q \in \Omega \mid f(q) = 0 \} \cap \Omega_i \) has an accumulation point then \( f = 0 \) on \( \Omega \).

2. **Liouville’s Theorem for \( \mathcal{SR}_v(\Omega) \).** If \( f \in \mathcal{SR}_v(\mathbb{H}) \) and there exists \( M > 0 \) such that
\[
\| f(q) \| \leq M e^{-\langle q, v \rangle}, \quad \forall q \in \mathbb{H}.
\]
Then there exists \( k \in \mathbb{H} \) such that \( f(q) = e^{-\langle q, v \rangle} k \) for all \( q \in \mathbb{H} \).

3. **Morera’s Theorem for \( \mathcal{SR}_v(\Omega) \).** Suppose that \( h \in C(\Omega, \mathbb{H}) \) satisfies
\[
\int_{\Gamma} h(\zeta) e^{-\langle \zeta, v \rangle} d\zeta = 0,
\]
for any closed, homotopic to a point and piecewise \( C^1 \) curve \( \Gamma \subset \Omega_i \) and for all \( i \in \mathbb{S}^2 \). Then \( h \in \mathcal{SR}_v(\Omega) \).

**Proof.**

1. The zero set of the slice regular function \( q \mapsto e^{\langle q, v \rangle} f(q) \) has an accumulation point and from identity Principle for the slice regular functions one gets that \( e^{\langle q, v \rangle} f(q) = 0 \) for all \( q \in \Omega \). Therefore \( f(q) = 0 \) for all \( q \in \Omega \).

2. The slice regular functions \( q \mapsto e^{\langle q, v \rangle} f(q) \) is a bounded function on \( \mathbb{H} \) and Liouville’s Theorem for slice regular functions implies that \( q \mapsto e^{\langle q, v \rangle} f(q) \) is a constant function.
3. As \( h : \Omega \to \mathbb{H} \) satisfies

\[
\int_{\Gamma} he^{\langle \zeta, v \rangle} d\zeta = 0,
\]

for any closed, homotopic to a point and piecewise \( C^1 \) curve \( \Gamma \subset \Omega_i \) and for all \( i \in S^2 \). Then \( q \mapsto e^{\langle q, v \rangle h(q)} \) satisfies the Morera’s Theorem for slice regular functions and \( h \in \mathcal{SR}_v(\Omega) \).

\[ \square \]

References

[1] Colombo, F. Gentili, G., Sabadini, I., Struppa, D., Extension results for slice regular functions of a quaternionic variable. Advances in Mathematics 222 1793-1808 (2009).

[2] Bers, L., Theory of pseudo-analytic functions. Institute of Mathematics and Mechanics, New York University (1953).

[3] Brackx, F., Delanghe, R., Sommen F., Clifford Analysis, Pitman Research notes, London, 76 1982.

[4] Castillo-Villalba, M. P., Colombo, F., Gantner, J., González-Cervantes, J. O., Bloch, Besov and Dirichlet Spaces of Slice Hyperholomorphic Functions Complex Anal. Oper. Theory DOI 10.1007/s11785-014-0380-4

[5] Colombo, F., Gantner, J., Quaternionic closed operators, fractional powers and fractional diffusion processes. Operator Theory: Advances and Applications, Birkhäuser-Springer 274 (2019).

[6] Colombo, F., Gantner, J., Kimsey, D. P., Spectral theory on the S-spectrum for quaternionic operators. Operator Theory: Advances and Applications, Birkhäuser-Springer 270 (2018).

[7] Colombo, F., Gentili, G., Sabadini, I., Struppa, D.C., Extension results for slice regular functions of a quaternionic variable, Adv. Math., 222 1793-1808 (2009)

[8] Colombo, F., González-Cervantes, J. O., Luna-Elizarrarás, M. E., Sabadini, I., Shapiro, M., On two approaches to the Bergman theory for slice regular functions, Springer INdAM Series 1, 39-54 (2013).

[9] Colombo, F., González-Cervantes, J. O., Sabadini, I., The Bergman-Sce transform for slice monogenic functions, Math. Meth. Appl. Sci., 34 1896-1909 (2011).
[10] Colombo, F., González-Cervantes, J. O., Sabadini, I., *On slice biregular functions and isomorphisms of Bergman spaces*, Compl. Var. Ell. Equa., 57 825-839 (2012).

[11] Colombo, F., González-Cervantes, J. O., Sabadini, I., *A non constant coefficients differential operator associated to slice monogenic functions*. Transactions of the American Mathematical Society 365 303-318 (2013).

[12] Colombo, F., González-Cervantes, J. O., Sabadini, I., *The C-property for slice regular functions and applications to the Bergman space*, Compl. Var. Ell. Equa., 58 1355-1372 (2013).

[13] Colombo, F., Sabadini, I., Struppa, D.C., *Noncommutative Functional Calculus. Theory and Applications of Slice Hyperholomorphic Functions*, Birkhauser, 2011.

[14] Gantner, J., González-Cervantes, J. O., Janssens, T., *BMO- and VMO-spaces of slice hyperholomorphic functions*, Mathematische Nachrichten, 2259-2279 (2017), https://doi.org/10.1002/mana.201600379

[15] Gentili, G., Struppa, D. C., *A new approach to Cullen-regular functions of a quaternionic variable* C. R. Acad. Sci. Paris, Ser. I 342 741–744 (2006).

[16] Gentili, G., Struppa, D.C. *A new theory of regular functions of a quaternionic variable*. Adv. Math. 216 279-301 (2007).

[17] Gentili, G., Stoppato, C., Struppa, D.C., *Regular functions of a quaternionic variable*, Springer Monographs in Mathematics, Springer, Berlin-Heidelberg (2013).

[18] Ghiloni, R., Perotti, A., *Volume Cauchy formulas for slice functions on real associative *-algebras*, Complex Variables and Elliptic Equations 58 1701-1714 (2013).

[19] Ghiloni, R., Perotti, A., *Global differential equations for slice regular functions*. Mathematische Nachrichten, 287(5-6), 561-573 (2014).

[20] Ghiloni, R., Perotti, A., Recupero V., *Noncommutative Cauchy integral formula*, Complex Analysis and Operator Theory, 11 289-306 (2017).

[21] González-Cervantes, J. O., *On Cauchy Integral Theorem for Quaternionic Slice Regular Functions*. Complex Analysis and Operator Theory, 13 6 2527-2539 (2019).

[22] González-Cervantes, J. O. *A Fiber Bundle over the Quaternionic Slice Regular Functions*. Adv. Appl. Clifford Algebras 31, 55 (2021). DOI: 10.1007/s00006-021-01158-z
[23] González-Cervantes, J. O. *Quaternionic slice regular functions with some sphere bundles*. Complex Variables and Elliptic Equations, (2021) DOI: 10.1080/17476933.2021.1971658

[24] González-Cervantes J. O., González-Campos, D., *The global Borel-Pompeiu-type formula for quaternionic slice regular functions*. Complex Variables and Elliptic Equations 66(5): 1-10 (2021), DOI:10.1080/17476933.2020.1738410.

[25] González-Cervantes, J. O., González-Campos D., *On the conformal mappings and the Global operator G*. Advances in Applied Clifford Algebras 31 1 (2021), DOI:10.1007/s00006-020-01103-6

[26] González-Cervantes, J. O., Sabadini, I., *On some splitting properties of slice regular functions*, Compl. Var. Ell. Equa. , 62 1393-1409 (2017).

[27] Shapiro, M., Vasilevski, N. L., *Quaternionic ψ-monogenic functions, singular operators and boundary value problems. I. ψ-Hyperholomorphy function theory*. Compl. Var. Theory Appl. 27 17-46 (1995).

[28] Sudbery, A., Quaternionic analysis, Math. Proc. Phil. Soc. 85 199-225 (1979).

[29] Spröessig, W., *On generalized Vekua type problems*, Article in Advances in Applied Clifford Algebras, February 2001

[30] Vekua, I.N., *Generalized analytic functions*. Nauka, Moscow (Russian) (1959).