Black Holes, Branes and
Superconformal Symmetry

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Abstract

The main focus of this lecture is on extended objects in $adS_{p+2} \times S^{d-p-2}$ bosonic backgrounds with unbroken supersymmetry. The backgrounds are argued to be exact, special consideration are given to the non-maximal supersymmetry case. The near horizon superspace construction is explained. The superconformal symmetry appears in the worldvolume actions as the superisometry of the near horizon superspace, like the superPoincaré symmetry of GS superstring and BST supermembrane in the flat superspace. The issues in gauge fixing of local kappa-symmetry are reviewed.

We describe the features of the gauge-fixed IIB superstring in $adS_{5} \times S^{5}$ background with RR 5-form. From a truncated boundary version of it we derive an analytic N=2 off shell harmonic superspace of Yang-Mills theory. The reality condition of the analytic subspace, which includes the antipodal map on the sphere, has a simple meaning of the symmetry of the string action in the curved space. The relevant issues of black holes and superconformal mechanics are addressed.

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During the last year there was some progress in establishing connections between exact solutions of the supergravity, near horizon geometry of black holes and branes, and quantum field theories with (super)conformal symmetries. The connection relies on the (super)isometries of the configurations which are products of anti de Sitter space and a sphere, $\text{adS} \ast S$ and which also are characterized by some charge since the configurations have a non-trivial form field. The (super)isometries of these exact configurations form a superconformal algebra. The purpose of this lecture is to discuss the set of connections between such configurations of space-time and (super)conformal theories.

This lecture is based mostly on my recent work with P. Claus, J. Kumar, A. van Proeyen, A. Rajaraman, J. Rahmfeld, P. Townsend, and A. Tseytlin and on numerous discussions with my collaborators. I will cover here some aspects of our work which relate the extended supersymmetric objects in space-time and the worldvolume actions with superconformal symmetry. In this lecture I rely on few other pedagogical lectures at this school where many aspects of M-theory, string theory and ADS/CFT Maldacena’s duality [1] were explained. The contributions to this proceedings by P. Claus and P. Termonia have an overlap with this lecture and may be useful to read in this context. The topics to be covered in the lecture are:

1. Black holes and branes as solutions of supergravity equations in space-time.

2. Is the supersymmetric $\text{adS}_{p+2} \times S^{d-p-2}$ geometry conformally flat?

3. Exactness of $\text{adS}_{p+2} \times S^{d-p-2}$ +form vacuum; special consideration for $\text{adS}_2 \times S^2 +2$-form near horizon black hole case with smaller supersymmetry.

4. Flat superspace background, its isometry and superPoincaré symmetry of the extended objects; near horizon superspace, its isometry and superconformal symmetry of the extended objects.

5. Choices of gauges for fixing $\kappa$-symmetry.

6. IIB Green-Schwarz superstring in $\text{adS}_5 \times S^5$ background with RR 5-form.
7. Analytic N=2 harmonic superspace from the quantization of the truncated GS string (superparticle) in the curved background of the AdS*S boundary. The role of the antipodal map on the sphere.

8. Black holes and superconformal mechanics of a particle approaching the black hole horizon.

An attempt will be made to describe the topics listed above in a relatively simple way, mostly to explain the new concepts, referring the reader to the published papers for the details.

1 Black Holes and Branes as solutions of supergravity equations in space-time.

Consider various supergravity theories in d-dimensions. Black holes ($p = 0$) and higher branes ($p > 0$) are solutions of supergravity field equations. The metric has the form

$$ ds^2_{brane} = H^{-2} dx^\mu \eta_{\mu\nu} dx^\nu + H^{2p+1} dy^m dy^m $$

Here $x^\mu$ are the $(p + 1)$ coordinates along the brane, $y^m$ are the remaining $(d - p - 1)$ coordinates of the space orthogonal to the brane, $y^m y^m \equiv r^2$ and $H$ is a harmonic function in $(d - p - 1)$-dimensional transverse space:

$$ H = 1 + \left( \frac{R}{r} \right)^{d-p-3} $$

Such metric has to be supplemented by some form-field $F \sim N \times \text{volume}$ so that the configuration has the maximal amount of unbroken supersymmetry. This leads to the interpretation of this solution as a set of $N$ parallel branes on top of each other. The number $N$ is proportional to $R$ in some power, where $R$ is the parameter in the harmonic function $H$. When the parameter $\omega$ picks up the ‘magic’ value [2]

$$ \omega = \frac{p + 1}{d - p - 3} , $$

the metric given above becomes a metric with the non-singular near horizon geometry at $r = 0$. The limiting metric at either very large $N$ when $R \to \infty$ or near the horizon at $r \to 0$
This metric is not yet in a form of the product space $adS_{p+2} \times S^{d-p-2}$ since here the cartesian coordinates of the $d$-dimensional target space are used and the split of $d$-dimensions into $p+1$ and $d-p-1$ is performed. To see that this metric is actually the $adS_{p+2} \times S^{d-p-2}$ one has to switch to spherical coordinates of the transverse space $d\vec{y}^2 = dr^2 + r^2 d^2 \Omega$, which gives

$$ds^2_{adS+S} = ds^2_{adS} + ds^2_S = \left(\frac{r}{R}\right)^{2\nu} dx^\mu \eta_{\mu\nu} dx^\nu + \left(\frac{R}{r}\right)^2 dr^2 + R^2 d^2 \Omega$$

Here the first two terms give the metric of $adS_{p+2}$ space $ds^2_{adS} = \left(\frac{r}{R}\right)^{2\nu} dx^\mu \eta_{\mu\nu} dx^\nu + \left(\frac{R}{r}\right)^2 dr^2$ and the third term gives the metric of $S^{d-p-2}$ sphere $ds^2_S = R^2 d^2 \Omega$. Now we have a split of the original $d$ dimensions into $p+2$ and $d-p-2$ coordinates of the product space.

The advantage of using cartesian coordinates is that the $R$-symmetry of the superconformal algebra, $SO(d-p-1)$ is manifest. In this coordinate system also the action of GS superstring in $adS_5 \times S^5$ background is the simplest.

The advantage of using the spherical coordinates is that the supercoset construction is simple.

It is important to realize that what in space-time is a coordinate which labels the points in space-time, in the worldvolume actions becomes a field, depending on the worldvolume coordinates. This means that the properties of coordinate system in space-time transfer into choice of the coordinate system in the space of fields on the worldvolume which may lead to various possibilities to develop a quantum theory.

The metric of $adS_{p+2} \times S^{d-p-2}$ with the form $F$ has an enhancement of unbroken supersymmetries comparative to the full brane metric. This means that the Killing spinor equation for the supersymmetry transformation rules of gravitino (and dilatino) has a solution with the maximal amount of the zero modes.

$$\delta \psi(g, F) = \nabla \epsilon + F \epsilon = 0 , \quad \epsilon \neq 0$$

2 Is the supersymmetric $adS_{p+2} \times S^{d-p-2}$ geometry conformally flat?

From questions I had during the school it become clear to me that there is a confusion with respect to the issue of conformal flatness of $adS_{p+2} \times S^{d-p-2}$ geometries. To enhance this
confusion and explain its source I will bring up here the private statement of S. Hawking and G. Horowitz who observed that this geometry is conformally flat for all cases contrary to the claim in [2]. The resolution of this controversy is the following. If the issue of supersymmetry is ignored, the metric of $adS_{p+2} \times S^{4-p-2}$ geometry can be taken in the form

$$ds^2_{adS*S} = ds^2_{adS} + ds^2_S = \left(\frac{r}{R}\right)^2 dx^\mu \eta_{\mu\nu} dx^\nu + \left(\frac{R}{r}\right)^2 dr^2 + R^2 d^2\Omega$$

We will show below that this metric is conformally flat. However if the metric $g$ with the form $F$ are required to solve the Killing spinor equation (8) in addition to solving the field equations, the choice of the parameter $\omega$ is not arbitrary and depends on $d$ and $p$ as shown in eq. (8). Thus for the supersymmetric solution given in the previous section only for

$$\omega = \frac{p + 1}{d - p - 3} = 1,$$

we have a conformally flat metric. For example, for D3 brane, $d = 10, p = 3$, a self-dual string $d = 6, p = 1$ and for black holes, $d = 4, p = 0$ the conformal flatness takes place even in supersymmetric case. However, for

$$\omega = \frac{p + 1}{d - p - 3} \neq 1,$$

in cases like M2 brane $d = 11, p = 2$, M5 brane $d = 11, p = 5$, for black holes $d = 5, p = 0$, for magnetic string $d = 5, p = 1$ the metric of the configuration with the unbroken supersymmetry is not conformally flat!

To understand it better let us perform a change of coordinates

$$\left(\frac{R}{r}\right) = z^\omega$$

The metric (8) of the supersymmetric solution becomes

$$ds^2_{adS*S} = \frac{1}{z^2} dx^\mu \eta_{\mu\nu} dx^\nu + (\omega R)^2 \frac{dz^2}{z^2} + R^2 d^2\Omega$$

Now we rescale $x = \omega R\bar{x}$ and the metric becomes

$$ds^2_{adS*S} = \left(\frac{\omega R}{z}\right)^2 \left[d\bar{x}^\mu \eta_{\mu\nu} d\bar{x}^\nu + dz^2\right] + R^2 d^2\Omega$$

One more step is required to combine the angles of the sphere with the radial direction $z$ into $d - p - 1$ coordinates $z^m$ so that $dz^2 = dz^2 + z^2 d^2\Omega$. This step is possible only for $\omega = 1$, the basic reason being the fact that $one\ can\ not\ rescale\ the\ angles\ of\ the\ sphere$ and therefore

$$\frac{(\omega R)^2}{z^2} dz^2 + z^2 R^2 d^2\Omega$$
In cases when $\omega = 1$ which also means that the dimension of the adS space equals the dimension of the sphere,

$$p + 2 = d - p - 2,$$

we have found that the metric $adS_{p+2} \times S^{p+2}$ of the supersymmetric configuration is conformally flat in the full target space $d$

$$ds^2_{adS_n \times S^n} = \frac{R^2}{z^2} [dx^2 + dz^2]$$

In case of $adS_5 \times S^5$, $adS_3 \times S^3$ and $adS_2 \times S^2$ the metric of the supersymmetric near horizon configuration is conformally flat in $d=10$ for D3 brane, in $d=6$ for the string and and in $d=4$ for black holes respectively.

The metric of the supersymmetric near horizon configuration $adS_4 \times S^7$ of the M2 brane and of $adS_7 \times S^4$ of the M5 brane in $d=11$ (and other cases with $p + 2 \neq d - p - 2$) is not conformally flat!

Having removed the confusion about the conformal flatness/non-flatness of the generic supersymmetric $adS_{p+2} \times S^{d-p-2}$ solution of classical supergravity equations, we may address another related controversial issue concerning the exactness of such configurations in the framework of the effective action of supergravities with all higher derivative terms present.

The existing lore relates the absence of quantum corrections to conformal flatness of the metric. Partially this is based on the proof presented by Banks and Green [3] that in $d=10$ IIB supergravity $(R_{abcd})^4$ terms do not correct the $adS_5 \times S^5$ metric since these terms actually depend on the Weyl tensor $C_{abcd}$ which vanishes for the conformally flat metric.

Now that we have clearly shown that for M2 and M5 brane there is no conformal flatness of the supersymmetric metric, we will explain how the exactness argument works and why in the particular case of $(R_{abcd})^4$ terms in IIB supergravity this more general argument is reduced to conformal flatness.

### 3 Exactness of $adS_{p+2} \times S^{d-p-2} + \text{form vacuum}$

We have argued so far that $adS \times S$ spaces with form-fields are solutions of classical equations of supergravities. Suppose that we have an effective action of supergravity where all possible terms with higher derivatives compatible with supersymmetries are added to the action. We know their structure from string theory or by supersymmetry arguments. One can study the problem how these terms will affect the classical solution [4].

3.1 Stability of pp-waves to quantum corrections in general covariant theories

It is instructive to remind here the situation with the exactness of the pp-waves in general relativity. Suppose that we have some general covariant theory where the action includes higher derivative terms/quantum corrections which are general covariant. Pp-wave geometries are space-times admitting a covariantly constant null vector field as shown by Brinkmann $\nabla_\mu l_\nu = 0, l^\mu l_\nu = 0$. For instance, for the class of d-dimensional pp-waves with metrics of the form $ds^2 = 2dudv + K(u, x^i)du^2 - dx^i dx^i$, the Riemann curvature is

$$R_{\mu\nu\rho\sigma} = -2l_\mu (\partial_\rho l_\sigma) l_\sigma$$.

(16)

The Ricci tensor vanishes if $K$ is a harmonic function in the transverse space: $R_{\mu\sigma} = -\frac{1}{2} (\partial_\rho l^\rho K) l_\mu l_\sigma$, $R = -\frac{1}{2} (\partial_\rho l^\rho K) l_\mu l^\mu = 0$ The curvature $R_{\mu\nu\rho\sigma}$ is therefore orthogonal to $l^\mu$ and to $\nabla^\mu$ in all its indices. Since $K$ is independent of $v$, the metric solves Einstein equations $G_{\mu\nu} = 0$ if $\partial_\mu^2 K = 0$. Possible corrections to field equations may come from higher dimension operators and depend on the curvature tensors and their covariant derivatives

$$G_{\mu\nu} = F_{\mu\nu}^{corr}(R_{\mu\nu\lambda\sigma}, D_\delta R_{\mu\nu\lambda\sigma}, \ldots)$$

(17)

Corrections to Einstein equations are quadratic or higher order in curvature tensors. Note that we do not consider the terms in the r.h.s of eq. (17) which vanish when classical field equation are satisfied. For pp-waves these terms are the Ricci tensor and Einstein curvature scalar and their covariant derivatives. We construct all possible higher order terms from the Riemann curvature and the covariant derivatives of it, which do not vanish for pp-wave solutions. This serves as an analog of the on-shell superfields which will be used in supersymmetric theories with maximal supersymmetry.

Now we may analyse all terms in (17) depending on Riemann curvature and the covariant derivatives of it. We find that there is no way to contract two or more of Riemann tensors which will form a two-component tensor to provide the r.h.s. of the Einstein equation coming from higher dimensions operators. Therefore all higher order corrections vanish for pp-waves solutions. They remain exact solutions of any higher order in derivatives general covariant theory. This includes supergravities and string theory with all possible sigma model and string loop corrections to the effective action, as long as these corrections respect general covariance. Note that supersymmetry played no role in establishing this non-renormalization theorem.
3.2 Maximal supersymmetry

The importance of having the maximal supersymmetry case when considering quantum corrections to the supersymmetric branes and black holes is in the fact that for a given dimensions the theory is unique, e.g. we have N=1, d=11 supergravity for M2 and M5 branes and N=2 d=10 IIB supergravity for D3 case [4]. All fields of these maximal supersymmetry supergravities are sitting in one multiplet, which includes the graviton, there is no coupling between different multiplets.

In maximally supersymmetric case of the near horizon M2, M5 and D3 branes the theories of d=11, d=10 supergravities can be described in the so-called on-shell superspace, i.e. in terms of superfields which satisfy the classical equations of motion. It is still possible to construct in each case the manifestly supersymmetric analog of eq. (17). The right hand side of this equation will depend on available superfields and their covariant derivatives.

The crucial part of the argument which in pp-wave case was the presence of the null Killing vector, here is the fact of the maximal unbroken supersymmetry of the relevant solutions. From this one can derive the characteristic property of the vacua of M theory and string theory: these vacua, \( adS_4 \times S^7 + 4 \)-form for the M2 brane, \( adS_7 \times S^4 + \) dual 4-form for the M5 brane and \( adS_5 \times S^5 + 5 \)-form for the D3 brane, can be defined completely by the covariantly constant superfields. In d=11 the basic superfield of Cremmer-Ferrara-Brink-Howe [6, 7] is \( W_{abcd}(X, \Theta) \) and for the near horizon configurations this superfield is covariantly independent \([4]\) on \( X \) and on \( \Theta \).

\[
D_a W_{abcd}(X, \Theta) = D_\alpha W_{abcd}(X, \Theta) = 0 \tag{18}
\]

In spherical coordinates of eq. (5) this superfield is actually \( X \)- and \( \Theta \)-independent:

\[
\frac{\partial}{\partial X^e} W_{abcd}(X, \Theta) = \frac{\partial}{\partial \Theta_\alpha} W_{abcd}(X, \Theta) = 0 \tag{19}
\]

and is given by the constant value of the form-field of this configuration. This is a generalization to the superspace of the fixed point behavior of the fields near the horizon, established in the usual space in \([8]\). In cartesian coordinates of eq. (4) the superfield is not constant but only covariantly constant. In case of IIB supergravity \([9]\) there are two superfields in Howe-West superspace \([0]\) of this theory, however, they are not independent as there is only one supermultiplet in this theory. One superfield starts with the dilatino, \( \Lambda_\alpha((X, \Theta) \) and was shown in \([4]\) to vanish for \( adS_5 \times S^5 + 5 \)-form vacuum. The second superfield, \( Z_{abcde}^+(X, \Theta) \) is supercovariantly constant. Here again in spherical coordinates of eq. (5) this superfield is
actually $X$- and $\Theta$-independent

$$\frac{\partial}{\partial X^k} Z_{\alpha}^{\mu}(X, \Theta) = \frac{\partial}{\partial \Theta^\alpha} Z_{\alpha}^{\mu}(X, \Theta) = 0. \quad (20)$$

and equal to the constant value of the RR 5-form.

Thus the correction to classical equations of motion which do not vanish on classical solutions which we discuss here may depend only on non-differentiated value of the superfields. To show that such contributions are absent, one has to observe that the bosonic equations of motion are given by some derivatives of the fermionic equations of motion since they come out as some higher components of the fermionic equations. The generic form of corrections to the fermionic equations inevitably has to carry a fermionic index. In our bosonic vacuum such index may come only from a fermionic derivatives on the superfields defined above. However such derivatives on the superfields of our vacua vanish. This accomplishes the chain of arguments about the exactness of the supersymmetric vacua of M-theory and string theory.

The $adS \times S^2$ vacua form a fixed point in the superspace, where the first derivatives on the superfields vanishes and the superfields take a fixed, non-vanishing value. In the string case it is a value of the RR 5-form, in M-theory it is the value of the 4-form and its dual. Note that for the trivial vacua, the flat superspace, all these superfields $W_{abcd}(X, \Theta)$, $\Lambda_{\alpha}(X, \Theta)$, and $Z_{\alpha}^{\mu}(X, \Theta)$ vanish everywhere. For generic supergravity they are functions of $(X, \Theta)$. The brane solutions interpolate between these two types of exact vacua, flat superspace and near horizon superspace, to be described below.

3.3 $adS_2 \times S^2$ +2-form near horizon black hole case with smaller supersymmetry

In case of smaller, non-maximal supersymmetry in a given dimension, the theory is not completely defined by dimension and the properties of the supergravity multiplet, including the graviton. Therefore the theories with non-maximal supersymmetry are not unique even before higher derivative terms are taken into account. For example, in $d=4$ $N=2$ supergravity there is a supergravity multiplet, which includes the graviton, and the matter multiplets without a graviton. These are vector multiplets, including gauge fields and hypermultiplets. Such theories require the information on the prepotential, a function which defines the coupling of the theory. The choice of such function is not unique and we will see below to which extent this affects the issue of exactness.
In [4] we looked at d=4 N=2 supergravity without vector or hypermultiplets (pure supergravity), as a toy model for d=11,10 theories with one multiplet. In such case there is only one vector field in the theory, belonging to the supergravity multiplet, no scalars, only one charge \( Q = Z = M \) and we have classically the Reissner-Nordstrom black hole. Near the horizon the metric tends to the Bertotti-Robinson adS_2 × S^2 and there is a covariantly constant 2-form. The only superfield of this theory, \( W_{ab} \) was shown in [12] to be covariantly constant due to enhancement of supersymmetry near the horizon. The argument about the absence of quantum corrections to supersymmetric Bertotti-Robinson configuration in pure d=4 N=2 supergravity was based on this fixed point behavior of the supergravity superfield

\[
\frac{\partial}{\partial X^e} W_{ab}^{ij}(X, \Theta) = \frac{\partial}{\partial \Theta^\alpha} W_{ab}^{ij}(X, \Theta) = 0.
\]

(21)
as in cases above.

Quite recently some new results on the stability\(^2\) of the adS_2 × S^2 geometry with the 2-forms in presence of vector multiplets and \( R^2_{abcd} \) corrections were obtained [13]. One starts with the supergravity coupled to abelian vector multiplets, \( X^I \) are the scalar fields of the vector multiplets, and some chiral background field \( \hat{A} \). The coupling is encoded into a holomorphic function \( F(X^I, \hat{A}) \) which is homogeneous of degree two. In this theory the lowest components of the reduced chiral multiplet \( W_{ab}^{ij} \) related to Weyl multiplet is the tensor \( T_{ab}^{ij} \). The background chiral multiplet \( \hat{A} \) is identified with \( W^2 \) at some point. This allows to generate the curvature square terms \( R^2_{abcd} \) in the action in a supersymmetric way. In fact, one starts with the Lagrangian which has a superconformal symmetry, so that the action is of the form

\[
16\pi \mathcal{L} = -e^{-\kappa} R + \ldots
\]

(22)

where

\[
e^{-\kappa} = i \left[ \hat{X}^I F_I(X, \hat{A}) - \hat{F}_I(\hat{X}, \hat{A})X^I \right]
\]

(23)

If not for the dependence of the prepotential on the chiral field \( \hat{A} \), this would be a Kahler potential of the special geometry. The dots in the Lagrangian are for the action of the vector

\(^2\)The main purpose of [13] was to find the corrections to the black hole entropy in presence of \( R^2 \) terms. These corrections are due to Wald's redefinition of the black hole entropy in presence of \( R^2 \) terms and corrections to the prepotential which takes care of the second Chern class of the Calabi-Yau threefold. With these modifications the supergravity corrections to the entropy are found to be in agreement with microscopic calculations of the entropy by Maldacena, Strominger, Witten and Vafa [14]. An important step in establishing this result in [13] was the derivation of the adS_2 × S^2 solution in presence of \( R^2 \) terms in the action.
multiplets and couplings to the chiral multiplet $\hat{A}$. The central charge is defined as in the usual case of special geometry, however, the prepotential and all functions of it carry the additional dependence on $\hat{A}$ (on $R_{abcd}^2$)

$$Z = e^{K/2}(p^I F_I - q_I X^I)$$

(24)

The superconformal symmetry of the action (22) has been fixed by the choice of the gauge

$$e^{-K} = i \left[ \bar{X}^I F_I(X, \hat{A}) - \bar{F}_I(\bar{X}, \hat{A})X^I \right] = 1$$

(25)

Note that the presence of $R_{abcd}^2$ terms does affect the choice of the gauge. In this gauge we have a usual Poincaré supergravity theory with supersymmetry, without conformal symmetry.

From full supersymmetry at the horizon, in the presence of the $R^2$ terms, it was found that for nonzero 2-forms (i.e. nonzero electric-magnetic charges) the spacetime remains the Bertotti-Robinson one: $adS_2 \times S^2$. Furthermore it was established that $X^I, F_I, \hat{A}$ and $T_{ab}^{ij}$ are constant. At this stage it has not yet been shown that there is fixed-point behaviour. But assuming that the values of the moduli are determined by the charges, one can invoke symplectic invariance and uniquely determine the relevant equations for the moduli. The metric then equals

$$ds^2 = ds_{adS}^2 + ds_S^2 = -r^2 \frac{dt^2}{|Z|^2} + \frac{|Z|^2}{r^2} dr^2 + |Z|^2 d^2 \Omega$$

(26)

where the central charge defining the size of the adS throat and the radius of the sphere is related to the 2-form as follows

$$T_{01}^{ij} = -2 \epsilon^{ij} \bar{Z}^{-1}.$$ 

(27)

Here the central charge in the chosen gauge at the fixed point is given by

$$Z = (p^I F_I(X_{fix}, \hat{A}_{fix}) - q_I X^I_{fix}).$$

(28)

Substituting these results into the entropy formula that includes Wald’s modification, one then establishes agreement with the microscopic entropy as determined in [14]. Thus one can conclude that for this particular theory of N=2, d=4 supergravity with vector multiplets and $R^2$ terms the $adS_2 \times S^2$ geometry with the 2-forms defining the size of the radius is a solution.
3.4 Comment on exactness versus conformal flatness

In the generic case of M-theory as well as in string theory vacua we have not used the conformal flatness of the metric to prove the stability of the classical solution. In fact, near horizon metric of supersymmetric M2 and M5 branes is strictly not conformally flat, as \( \omega = \frac{p+1}{d-p-3} \) is equal to 1/2 and 2, respectively. Still in D3 case \( \omega = 1 \) and the metric is conformally flat. Moreover, the argument in [3] about \( R_{abcd}^4 \) terms not affecting the \( \text{ads}_5 \times S^5 \) configuration is based completely on conformal flatness of the near horizon geometry of the D3 brane. The resolution of this puzzle is the following.

The maximal amount of 32 unbroken supersymmetries is valid in M-theory as well as in string theory. The integrability condition of eq. (6) in both cases reads

\[
\delta \check{\nabla} [a \psi_b] = \check{\nabla} [a \check{\nabla} b] = 0 \quad (29)
\]

When translated into the superfield language this allows to prove that some higher in \( \Theta \) component of the superfield \( W_{abcd} \) in M-theory or \( Z_{+abcde} \) in string theory, vanishes. In M-theory we get for the \( \Theta^2 \) component of the superfield \( W_{abcd} \) (see [3] for details) the following combination of the Riemann curvatures \( R_{rsmn} \) and 4-form \( F_{tuvw} \)

\[
W'' \sim \frac{1}{8} R_{rsmn} \gamma^{mn} + \frac{1}{2} \left[ T_r^{tuvw} , T_s^{xyzp} \right] F_{tuvw} F_{xyzp} + T_s^{tuvw} D_r F_{tuvw} \quad (30)
\]

Here \( T_r^{tuvw} \) is some combination of \( \gamma \)-matrices. On the near horizon supersymmetric M2 and M5 brane solutions this expression vanishes, i.e. the term linear in curvature is compensated by terms quadratic in form-fields. The terms with the covariant derivative of the form-field vanish for our vacua independently of the other terms in this equation.

\[
\left[ \frac{1}{8} R_{rsmn} \gamma^{mn} + \frac{1}{2} \left[ T_r^{tuvw} , T_s^{xyzp} \right] F_{tuvw} F_{xyzp} \right]_{\text{vac}} = 0 \quad (31)
\]

\[
T_s^{tuvw} D_r F_{tuvw} \big|_{\text{vac}} = 0 \quad (32)
\]

For IIB string theory the second component of the superfield \( Z_{+abcde} \) is also given by some combination of the curvature \( R_{abcd} \) and of the 5-form fields of the type [4]

\[
(Z_{+abcde})'' \sim \frac{1}{4} \left( \sigma^{cd} \right)^\gamma_{\delta} R_{abcd} - T_a^\gamma T_b^\delta + T_a^\gamma T_b^\delta - D_a T_b^\gamma \quad (33)
\]

Here the torsion tensor \( T_b^\gamma \) is a function of the RR 5-form field. On the near horizon supersymmetric D3 brane solutions this expression vanishes, i.e. the term linear in curvature
Table 1: Supergravity brane solutions with $adS_{p+2} \times S^{d-p-2}$ and $(p+2)$ form.

is compensated by terms quadratic in form-fields
\[ \left[ \frac{1}{4} (\sigma^{cd})_{\gamma}^\delta R_{abcd} - T_{a(c}^\gamma T_{b)\delta} + T_{a\gamma}^\gamma T_{b\delta} \right]_{\text{vac}} = \left[ \frac{1}{4} (\sigma^{cd})_{\gamma}^\delta C_{abcd} \right]_{\text{vac}} = 0 . \] (34)

The terms with the covariant derivative of the form-field vanish independently.
\[ [T_{[s}^{\ell uvw} D_r] F_{\ell uvw}]_{\text{vac}} = 0 . \] (35)

The important difference with the M-theory case is that the combination of curvature and forms in eq. (34) on shell forms exactly the Weyl tensor! The bilinear combination of forms provides the difference between the Riemann tensor and Weyl tensor. Weyl tensor vanishes for $adS_5 \times S^5$ supersymmetric solution and this is a particular form of the proof of the fact that the superfield $Z_{abcde}^+$ is $\Theta$-independent. This particular form of the argument does not work in M-theory, however the fact that a combination of curvature and forms vanishes still works! Thus the unbroken supersymmetry which in all cases provides the $\Theta$-independence of the superfield is the fundamental reason for exactness. In string case this manifests itself via conformal flatness.

4 Flat superspace and near horizon superspace, symmetries of extended objects

The worldvolume actions of Green-Schwarz superstring and Bergshoeff-Sezgin-Townsend M2 supermembrane are known in the generic background of supergravity. The coordinates $Z = (X, \Theta)$ of the target (super)-space become functions of the world-volume coordinates of the brane $\sigma^\mu$
\[ Z^A(\sigma) = (X(\sigma), \Theta(\sigma)) \] (36)
The worldvolume Lagrangians depend on the pullback of the geometric objects, vielbeins and forms, in the target superspace to the worldvolume

$$\mathcal{L}[E_\mu \hat{\lambda} = \partial_\mu Z^\Lambda E_\Lambda \hat{\lambda}(Z), A_{\mu_0...\mu_p} = \partial_{\mu_0} Z^{\Lambda_0} \cdots \partial_{\mu_p} Z^{\Lambda_p} A_{\Lambda_0...\Lambda_p}(Z)]$$

(37)

Thus if we know the supervielbein form

$$E_\Lambda \hat{\lambda}(Z)$$

and the $p + 1$ form

$$A_{\Lambda_0...\Lambda_p}(Z)$$

in the superspace for any supergravity theory, one can use this information to construct the worldvolume actions in any background.

Consider first the flat superspace. There are no form fields,

$$A_{\Lambda_0...\Lambda_p}(Z) = 0$$

(38)

The supervielbein forms are simple

$$E^\alpha = d\Theta^\alpha \quad E^a = dx^a - \bar{\Theta}\Gamma^a d\Theta$$

(39)

The isometries of the flat superspace

$$\delta \Theta = \epsilon \quad \delta x^a = \epsilon \Gamma^a \Theta$$

(40)

form the super-Poincaré algebra. This is the reason why the GS superstring and BST-supermembrane are ‘manifestly supersymmetric’. For example, GS classical superstring action depends on the pullback to the world-sheet with coordinates $\sigma^\mu$ of the manifestly supersymmetric vielbein forms of the flat target superspace:

$$E_\mu^\alpha \equiv \partial_\mu \Theta^\alpha \quad E^a_\mu = \partial_\mu x^a - \bar{\Theta}\Gamma^a \partial_\mu \Theta$$

(41)

Under the superspace isometries these objects are invariant

$$\delta_{\text{isom}} E^\alpha_\mu = 0 \quad \delta_{\text{isom}} E^a_\mu = 0$$

(42)

and therefore the choice of the background provides the global symmetry of the GS and BST actions.
One would like to construct the string and the M2 and M5 and Dp brane actions not only in the flat superspace but also in the background of the supersymmetric branes. The most interesting case would be the IIB string interaction with RR 5-form of the D3 brane.

To construct the supersymmetric worldvolume actions in any background other than the flat superspace some time ago was looking like an impossible task. The point is that the vielbeins of M-theory and IIB string theory depend on 32 fermionic coordinates $\Theta$ and therefore they look like

$$E(X, \Theta)_{\Lambda}^{\bar{\Lambda}} = (E_0(X))_{\Lambda}^{\bar{\Lambda}} + (\Theta E_1(x))_{\Lambda}^{\bar{\Lambda}} + \ldots + (\Theta^{32} E_{32}(x))_{\Lambda}^{\bar{\Lambda}}$$

For any particular background one would be able to find such long superfield depending on 32 fermionic coordinates $\Theta$ but one may not expect to get any closed form of it, in general. A beautiful exception from this rule is the superspace generalization of the the near horizon bosonic background of M2, M5 and D3 branes (and other cases in the Table 1), suggested in [15]. The supercoset construction was developed for the IIB superstring and D3 brane in [16] and with the use of the closed form of the near horizon superspace [15] these actions have been presented in the supersymmetric $adS_5 \times S^5$ with RR form background in a closed form.

One may either use the supercoset construction $G/H$ or equivalently, use the supergravity theory to find the near horizon superspace which at $\Theta = 0$ is a bosonic near horizon M2, M5, D3 brane, etc. One starts with the superalgebra $G$, which for each case is shown in the Table 1. The supercoset construction $G/H$ consists of solving the set of Maurer-Cartan equations

$$\mathcal{D}^2 = 0$$

where

$$\mathcal{D} \equiv d + L^A B_A + L^a F_a$$

Here $B$ and $F$ are bosonic and fermionic generators of the superconformal algebra $G$. The solution of MC equations for fermionic 1-forms takes the following form: there is a term linear in $\Theta$ and higher order corrections enter via a multiple commutators of fermionic generators:

$$F_{\alpha} L^\alpha = F_{\alpha} D\Theta^\alpha + [F_{\alpha} \Theta^\alpha [F_\beta \Theta^\beta, F_\gamma D\Theta^\gamma] + \cdots \) (45)$$

Here the fermionic generator $F$ consists of supersymmetry $Q$ and special supersymmetry $S$ and $D$ is the value of the operator $\mathcal{D}$ at $\Theta = 0$ and $L_0^A$ is the bosonic Cartan form at $\Theta = 0$. 

\(^3\text{See the contribution of P. Claus to these Proceedings.}\)
The solution for Cartan forms can be written in a closed form as

\[ L^\alpha = \left( \frac{\sinh \mathcal{M}}{\mathcal{M}} D\Theta \right)^\alpha, \quad L^A = L^A_0 + 2\Theta^\alpha f^A_{\alpha\beta} \left( \frac{\sinh^2 \mathcal{M}/2}{\mathcal{M}^2} D\Theta \right)^\beta, \] (46)

where the matrix \(\mathcal{M}\) is quadratic in \(\Theta\) and depends on the structure constants of the superconformal algebra

\[(\mathcal{M}^2)^\alpha_\beta = f^\alpha_{A\gamma} \Theta^\gamma \Theta^\delta f^A_{\delta\beta}.\] (47)

The superisometries of this background have been found recently in a closed form in [17] and they are given by the transformations of near horizon superspace coordinates \(Z\)

\[ \delta_{\text{ads*S}} Z = \delta_{\text{ads*S}}(Z) \] (48)

and the compensating stability \(H\)-group transformations. These isometries form a superconformal algebra. Therefore the actions of the extended objects in this background have a superconformal symmetry since the pullback to the worldvolume \(\partial_\mu Z^A E^A_\lambda(Z)\) of the spacetime forms are invariant under the isometries.

The M2 supermembrane classical action in \(\text{adS}_4 \times S^7\) and \(\text{adS}_7 \times S^4\) backgrounds of the near horizon M2 and M5 branes has been constructed in the near horizon superspace in [18].

5 Issues in gauge-fixing of \(\kappa\)-symmetry

The supersymmetric actions of extended objects have local worldvolume \(\kappa\)-symmetry in generic background. Therefore 1/2 of the 32-component spinor \(\Theta\) are unphysical and one have to get rid of them. The standard procedure consists of gauge-fixing of this symmetry, by choosing an algebraic constraint on \(\Theta\) with non-propagating ghosts (if the constraint includes the worldvolume derivatives, one has to consider the ghosts action since in this case the ghost are propagating fields). In the near horizon superspace one can consider at least 3 possibilities.

- Light-cone gauge, \(\Gamma^+ \Theta = 0\) or \(\Gamma^- \Theta = 0\).

The first possibility is to consider the same gauge which has been used for the quantization of the GS superstring in the flat superspace [19]. This gauge requires the \(P^+\) components of the momenta to be non-vanishing since the kinetic term for the remaining fermions looks as \(\dot{\Theta} \Gamma^- P^+ \partial \Theta\) and one has to be able to divide on \(P^+\). This works well even for massless states for which \(P^+ P^- + (P^i)^2 = 0\). The problem in the near
horizon space is that in the light-cone gauge the values of the vielbein forms do not seem to simplify and each of these superfields still goes all the way till $\Theta^{16}$. This may not necessarily be a major problem, but still one may try to do different things. Note that by choosing $\Gamma^{-}\Theta = 0$ gauge we would not change anything in proper notation, it is an equivalent gauge.

- $Q, S$ class of gauges for the near horizon background\textsuperscript{4}. In the context of the superconformal algebra there is a natural split of the fermions into

$$F = \begin{pmatrix} Q_{+}^{1/2} \\ S_{-}^{-1/2} \end{pmatrix}$$

where the supersymmetry generator $Q$ has a conformal weight $+1/2$ and the special supersymmetry generator $S$ has a conformal weight $-1/2$. The coordinates also can be split in analogous way:

$$F\Theta = Q\Theta_Q + S\Theta_S$$

Let us now consider these two inequivalent possibilities. The basic reason why these two gauges are inequivalent is due to the triangular nature of the supervielbein in the Killing spinor gauge in the superspace. In these class of gauges $\Theta$ are considered to be the functions of $X$ of the form $\Theta(X) = K(X)\theta$, or in split form:

$$\begin{pmatrix} \Theta_Q(X) \\ \Theta_S(X) \end{pmatrix} = \begin{pmatrix} K_Q^+(X) & K_Q^-(X) \\ 0 & K_S^-(X) \end{pmatrix} \begin{pmatrix} \theta_+ \\ \theta_- \end{pmatrix}$$

and $\theta$ are $X$-independent coordinates. In such case one can show \cite{15} that

$$D\Theta = K(X)d\theta$$

- $Q$-gauge, $\theta_- = 0$. \textsuperscript{20}.

This gauge gives the remarkable simplification of superspace vielbeins. Note that in this gauge

$$\Theta_Q = K_Q^+\theta_+ \quad (D\Theta)_Q = K_Q^+d\theta_+$$
$$\Theta_S = 0 \quad (D\Theta)_S = 0$$

\textsuperscript{4}In \cite{20} we called these gauges Killing (anti-Killing) spinor gauge for fixing $\kappa$-symmetry, since the Killing spinors play an important role here. On the other hand there is a choice of the gauge in the superspace which was also given a name of a Killing spinor gauge versus Wess-Zumino gauge \cite{15}. In Killing spinor gauge in superspace the spinor-spinor component of the vielbein at vanishing $\Theta$ is taken from the the Killing spinors of the bosonic space \cite{21}. To avoid misunderstanding we will refer to the relevant gauge in the superspace as to ‘Killing spinor gauge’ and to the gauges for fixing $\kappa$-symmetry, as to $Q$ or $S$ gauge.
Therefore at $\theta_\perp = 0$

\begin{align}
\mathcal{M}^{2n}\Theta &= 0, \quad \text{at } n = 1, 2, \ldots \quad (55) \\
\mathcal{M}^{2n}D\Theta &= 0, \quad \text{at } n = 1, 2, \ldots \quad (56)
\end{align}

The vielbein forms are reduced to the following expressions

\begin{equation}
L_Q = (D\Theta)_Q, \quad L_S = 0, \quad L^A = L^A_0 + 2\Theta^a_\alpha f^A_{\alpha\beta}(D\Theta)_Q^{\beta}, \quad (57)
\end{equation}

i.e. the vielbeins are quadratic in $\Theta_Q$ at most, like in the flat superspace. Therefore the actions in the $Q$-gauge are no more complicated than those in the flat superspace, what concerns the fermions. One has to specify the conditions when such gauge is admissible and we will give examples of this for the GS string in $adS_5 \times S^5$ in [23, 24]. In case we consider the action for the extended object in its own near horizon background, e.g. the D3 brane in $adS_5 \times S^5$, a special requirement has to be imposed to make the $Q$-gauge admissible. This requirement is that the momenta in directions transverse to the brane are not vanishing.

- $S$-gauge, $\Theta_Q = 0$.

This constraints on spinors was considered in detail in [22] with respect to quantization of the D3 brane. Using our set up we can show that at $\Theta_Q = 0$

\begin{align}
\mathcal{M}^{2n}\Theta &= 0, \quad \text{at } n = 1, 2, \ldots \quad (58) \\
\mathcal{M}^{2n}D\Theta \neq 0, \quad \mathcal{M}^{2n}D\Theta = 0, \quad \text{at } n = 2, 3, \ldots \quad (59)
\end{align}

Thus in this gauge the fermionic vielbeins have terms $\Theta^3$ and the bosonic ones have up to $\Theta^4$. The advantage of this gauge that one can consider the actions of the extended objects in its own near horizon background without requiring the non-vanishing momenta in transverse directions to the brane.

In conclusion, the fixing of $\kappa$-symmetry in supersymmetric $adS \ast S$ backgrounds has been studied in $Q$-gauge, $S$-gauge and light-cone gauge. The vielbeins depend on up to $\Theta^2$, $\Theta^4$ and $\Theta^{16}$, in these gauges, respectively.

An alternative procedure is available for $adS \ast S$ spaces, it was called in [25, 26, 22] an a priori gauge-fixing. It is based on the Supersolvable subalgebra of the superconformal algebra.
6 IIB Green-Schwarz superstring in $adS_5 \times S^5$ and RR-form background

Maldacena’s conjecture about the duality between the IIB superstring in $adS_5 \times S^5$ and RR-form and N=4 supersymmetric Yang-Mills theory is based on the fact that both theories have the same symmetry forming the $SU(2,2|4)$ superalgebra.

The classical superstring action in this background was constructed recently [16, 15] in the background whose coordinates $Z = (X, \Theta)$ form an $SU(2,2|4)$ supercoset space.

\[
S = -\frac{1}{2} \int d^2 \sigma \left( \sqrt{-g} g^{ij} L_i \dot{L}_j + 4 i \epsilon^{i j} \int_0^1 ds \ L_i \ K^{IJ} \Theta^I \Gamma J L_\mu \right). \tag{60}
\]

The coupling to RR-form $F$ is included into a term of the form $\partial X \partial X \Theta \Theta F$. Here $K^{IJ} \equiv \text{diag}(1, -1)$, $I, J = 1, 2$ and $\hat{a} = (a, a') = (0, ..., 4, 5, ..., 9)$. The invariant 1-forms $L^I = L^I_{s=1}$, $\hat{L}^a = L^a_{s=1}$ are given by

\[
L^I_s = \left( \frac{\sinh(sM)D\Theta}{M} \right)^I, \quad \hat{L}^a_s = \epsilon^a_{\hat{m}}(X)dx^{\hat{m}} - 4 i \Theta^I \Gamma J \left( \frac{\sinh^2 \left( \frac{1}{2} sM \right) D\Theta}{M^2} \right)^I, \tag{61}
\]

where $X^{\hat{m}}$ and $\Theta^I$ are the bosonic and fermionic superstring coordinates and

\[
(M^2)^I = \epsilon^{IJ} \left( -\gamma^a \Theta^I \Theta^L \gamma^a + \gamma^{a'} \Theta^I \Theta^L \gamma^{a'} \right) + \frac{1}{2} \epsilon^{KL} \left( \gamma^{a'b'} \Theta^I \Theta^L \gamma^{a'b'} - \gamma^{a'b} \Theta^I \Theta^L \gamma^{a'b'} \right), \tag{62}
\]

\[
(D\Theta)^I = \left[ d + \frac{1}{4} (\omega^{a'b'} \gamma_{ab} + \omega^{a'b} \gamma_{a'b'}) \right] \Theta^I - \frac{1}{2} \epsilon^{IJ} (e^a \gamma_a + ie^{a'} \gamma_{a'}) \Theta^J. \tag{63}
\]

The Dirac matrices are split in the ‘$5+5$’ way, $\Gamma a = \gamma^a \times 1 \times \sigma_1$, $\Gamma a' = 1 \times \gamma^{a'} \times \sigma_2$, where $\sigma_k$ are Pauli matrices (see [16] for details on notation). This classical action has two type of symmetries.

- Global $SU(2,2|4)$ symmetries.

  The global symmetries are the near horizon superspace isometries found in [17] which form the $SU(2,2|4)$ superalgebra. As explained above, these symmetries act on the coordinates of the superspace: $\delta_{adS_5} Z$. The isometries are functions of $Z$ and of the global parameters of the $SU(2,2|4)$ superalgebra. Under these transformations the classical string action is invariant. One may expect the spectrum of states to have the superconformal symmetry $SU(2,2|4)$.

- Local symmetries
The action is invariant under local symmetries, *reparametrization and κ-symmetry*, whose parameters depend on the worldsheet coordinates \( \sigma \). The action in \((60)\) is a particular example of the IIB superstring action in a generic background of supergravity \([27]\) where the local symmetries symmetries are given. These local symmetries have to be gauge-fixed.

The important property of \( κ \)-symmetry in the curved background of supergravity is that the background has to be on shell. This means in particular that any brane action known in generic IIB supergravity superspace can be coupled consistently to the near horizon superspace of the D3 brane. For example, one can couple the GS IIB string, D1, D3 and D5 brane to \( \text{adS}_5 \times S^5 \) superspace with RR form.

By gauge-fixing \( κ \)-symmetry one can reduce the number of fermions by 1/2 to match the number of physical bosonic and fermionic degrees of freedom. The gauge-fixing of \( κ \)-symmetry was performed in \([23]\) developing the proposal \([20]\) and the action was found which has terms at most quartic in fermions. The special \( κ \)-symmetry gauge using the projector parallel to D3-brane directions allowed to substantially reduce the power of fermionic terms in the action.

Let us review the \( κ \)-symmetry gauge fixing of this action performed in \([23]\). We shall use the ‘D3-brane adapted’ or ‘4+6’ bosonic coordinates \( X^m = (x^p, y^t) \) in which the \( \text{AdS}_5 \times S^5 \) metric is split into the parts parallel and transverse to the D3-brane directions (we take the radius parameter to be \( R = 1 \))

\[
d s^2 = y^2 d x^p d x^p + \frac{1}{y^2} d y^t d y^t , \quad y^2 \equiv y^t y^t ,
\]

where \( p = 0, \ldots, 3, \ t = 4, \ldots, 9 \). In what follows the contractions of the indices \( p \) is understood with Minkowski metric and indices \( t \) – with Euclidean metric. The \( κ \)-symmetry gauge is fixed using the ‘parallel to D3-brane’ \( Γ \)-matrix projector

\[
\Theta^I_\pm = 0 , \quad \Theta^I_\pm \equiv \mathcal{P}^I_\pm \Theta^J , \quad \mathcal{P}^I_\pm = \frac{1}{2} \left( \delta^{IJ} \pm \Gamma_{0123} \epsilon^{IJ} \right) , \quad \mathcal{P}_+ \mathcal{P}_- = 0 .
\]

In ‘5+5’ coordinates \( (x^a = (x^p, x^4 = y) \) and \( ξ^a \) coordinates on \( S^5 \) one finds that \( (\Gamma_{0123} = i \gamma_4 \times 1 \times 1 , \ \omega^{a}_{b} = \epsilon^{a}_{b}) \)

\[
(D\Theta)^I = \left[ \delta^{IJ} (d + \frac{1}{4} \omega^{a}_{b} \gamma_{a b}) + \frac{1}{2} \epsilon^{IJ} (\epsilon^{a}_{b} \gamma_{a b} - i \epsilon^{4}_{a} \gamma_4) + \frac{1}{2} \epsilon^{\rho}_{b} \gamma_{\rho} \gamma_4 \mathcal{P}^{IJ}_- \right] \Theta^J .
\]

\(^{5}\)A similar action was found in \([28]\) using supersolvable (Ssolv) algebra approach \([25]\). In \([28]\) and \([26]\) different choices of bosonic (and fermionic) coordinates were used: Cartesian and horospherical in \([23]\) and projective coordinates on \( S^5 \) in \([23]\). A change of variables has been found in \([28]\) which brings both actions to the same form.
Using that the $S^5$ part of the covariant derivative satisfies $D^I_5 \equiv \delta^{IJ}(d + \frac{1}{2} \omega^{a'b'} \gamma_{a'b'}) + \frac{1}{2} \epsilon^{IJ} e^{a'} \gamma_{a'} = (\Lambda d\Lambda^{-1})^{IJ}$, $(D_5)^2 = 0$, where the spinor matrix $\Lambda^{IJ} = \Lambda^{IJ}(\xi)$ is a function of the $S^5$ coordinates, one finds that $(D\Theta_+)^I$ can be written as

$$D\Theta_+ = (d - \frac{1}{2} d \log y + \Lambda d\Lambda^{-1})\Theta_+ = y^{1/2} \Lambda \ d (y^{-1/2} \Lambda^{-1} \Theta_+) \, .$$

Eq. (67) suggests to make the change of the fermionic variable $\Theta \to \theta$

$$\Theta_+^I = y^{1/2} \Lambda_{IJ}(\xi) \ \theta_+^J \, , \quad \mathcal{P}_{-}^{IJ} \theta_+^J = 0 \, , \quad D\Theta_+^I = y^{1/2} \Lambda d\theta_+^I \, . \quad (68)$$

If we further transform from the coordinates $(y, \xi^a)$ to the 6-d Cartesian coordinates $y'$ in (64), $y' = \frac{y}{\sqrt{1 + \xi^2}}(1, \xi^a)$, that would effectively absorb the matrix $\Lambda$ into an $SO(6)$ spinor rotation. In the Cartesian coordinates $y^I = y^I y'$ the 6-d part of the covariant derivative has the form $D^I_6 = \delta^{IJ}(d + \frac{1}{2} \gamma_{st} d'y^t d'y^s) + \frac{1}{2} \epsilon^{IJ} \gamma_{tt}(d'y^t + d'y^s \log y)$. This simplification is suggested by the form of the Killing spinors in $AdS_5 \times S^5$ space viewed as the near-horizon D3-brane background. In particular, writing the 10-d covariant derivative (including the Lorentz connection and 5-form terms) in the ‘4+6’ coordinates in (64) one learns that when acting on the constrained spinor $\Theta_+$ it becomes simply $D\Theta_+^I = y^{1/2} d\theta_+^I$, $\theta_+^I \equiv y^{-1/2} \Theta_+^I$.

As a result, $\mathcal{M}^2 D\Theta_+ = 0$ and the fermionic sector of the action reduces only to terms quadratic and quartic in $\theta_+^I$. Using $\mathcal{P}_{-}^{IJ} \theta_+^J = 0$ to eliminate $\theta_+^2$ in favour of

$$\theta_+^1 \equiv \vartheta \quad (69)$$

one finds that the $\kappa$-symmetry gauge-fixed string action in $AdS_5 \times S^5$ background (60) expressed in terms of the bosonic coordinates $X^{\hat{m}} = (x^p, y^I)$ and the single $D = 10$ Majorana-Weyl spinor $\vartheta$ takes the following simple form

$$S = -\frac{1}{2} \int d^2 \sigma \left[ \sqrt{-g} \, g^{ij} \left( y^2 (\partial_i x^p - 2 i \bar{\partial} \Gamma^p \partial_i \vartheta)(\partial_j x^p - 2 i \bar{\partial} \Gamma^p \partial_j \vartheta) + \frac{1}{y^2} \partial_i y^I \partial_j y^I \right) 
+ 4 i \epsilon^{ij} \partial_i y^I \bar{\partial} \Gamma^I \partial_j \vartheta \right] \, .$$

(70)

The $\Theta \Theta \partial X \partial X$ terms representing the coupling to the RR background present in the original action [6] in eq. (60) are now ‘hidden’ in the $\bar{\partial} \partial \vartheta \partial X$ terms because of the redefinition made in (68). We may also put the action in the first-order form by introducing the ‘momenta’ (Lagrange multipliers) $P_p^i$ in the D3 brane directions 0, 1, 2, 3:

$$S = -\frac{1}{2} \int d^2 \sigma \left[ \sqrt{-g} g^{ij} \left( -\frac{1}{y^2} P_p^i P_p^j + 2 P_p^i (\partial_j x^p - 2 i \bar{\partial} \Gamma^p \partial_j \vartheta) + \frac{1}{y^2} \partial_i y^I \partial_j y^I \right) 
+ 4 i \epsilon^{ij} \partial_i y^I \bar{\partial} \Gamma^I \partial_j \vartheta \right] \, .$$

(71)
One can use the conformal gauge $g^{ij} = \eta^{ij} f(\sigma)$ to fix the reparametrization symmetry which leads to the standard $b, c$ ghosts. The gauge-fixed action becomes

$$S = -\frac{1}{2} \int d^2 \sigma \left[ \eta^{ij} \left( -\frac{1}{y^2} P^p_i P^p_j + 2 P^p_i (\partial_j x^p - 2 i \bar{\vartheta} \Gamma^p \partial_j \vartheta) + \frac{1}{y^2} \partial_i \vartheta' \partial_j \vartheta' \right) \\
+ 4 i e^{ij} \partial_i \vartheta' \bar{\vartheta} \Gamma^i \partial_j \vartheta \right] + S_{\text{ghosts}}(b, c).$$

(72)

In order to achieve an understanding of the $\kappa$-symmetry gauge choice in (65) it is useful to study the issue of invertibility of the fermionic kinetic operator in the actions (70), (71). In particular, we shall consider the flat space case obtained by omitting (or just treating as constant) the $y^2$ and $1/y^2$ factors in the action (72). In general, the constraints coming from the equation of motion for the 2-d metric can be written in terms of the vielbein components of the ‘momentum’ $\Pi^p_i$ defined by the $g_{ij}$-dependent part of the action which does not include the WZ term $(z, \bar{z} = \sigma \pm \tau$, $\sigma^0 \equiv \tau$, $\sigma^1 \equiv \sigma$)

$$\Pi_z \cdot \Pi_z \equiv \Pi^p_z \Pi^p_z + \Pi^i_z \Pi^i_z = 0, \quad \Pi_{\bar{z}} \cdot \Pi_{\bar{z}} \equiv \Pi^p_{\bar{z}} \Pi^p_{\bar{z}} + \Pi^i_{\bar{z}} \Pi^i_{\bar{z}} = 0$$

(73)

Dots stand for the fermionic terms in the constraints and as above the indices $p$ are contracted with 4-d Minkowski metric and the indices $t$ – with 6-d Euclidean metric. In the case of the action $\Pi^p_i = y (\partial_i x^p - 2 i \bar{\vartheta} \Gamma^p \partial_j \vartheta)$, $\Pi^i_t = y^{-1} \partial_i y^t$.

Before $\kappa$-symmetry gauge fixing the quadratic fermionic terms in the flat-space GS action are

$$\bar{\vartheta}^1 (\Pi \cdot \Gamma) z \partial_z \Theta \equiv \bar{\Theta}^1 A_1 \Theta^1, \quad \bar{\vartheta}^2 (\Pi \cdot \Gamma) z \partial_z \Theta^2 \equiv \bar{\Theta}^2 A_2 \Theta^2.$$

(74)

On the classical equations and constraints we get $(A_1)^2 = (A_2)^2 = 0$, i.e. the fermionic operator is degenerate for any classical string background. As we shall see below, after the $\kappa$-symmetry gauge fixing as in (65) the degeneracy is removed provided the background is constrained in a certain way. In case of the gauge-fixed action (70) the background has to be a BPS one so that the gauge is admissible. Let us look at this in more details.

The quadratic term in the fermionic part of the gauge-fixed action (70) is (we omit the fermionic terms in $\Pi$)

$$\bar{\vartheta} y \left[ (\Pi \cdot \Gamma) z \partial_z + (\Pi * \Gamma) z \partial_z \right] \vartheta \equiv \bar{\vartheta} A \vartheta,$$

(75)

where we introduced the notation

$$\Pi^p \Gamma^p + \Pi^i \Gamma^i = (\Pi \cdot \Gamma), \quad \Pi^p \Gamma^p - \Pi^i \Gamma^i = (\Pi * \Gamma).$$

(76)
Using the equations of motion for \( X^m \) and the constraints (73), the square of the kinetic operator \( A \) can be written as

\[
A^2 = y^2 \left[ (\Pi \cdot \Gamma) z (\Pi \star \Gamma) z + (\Pi \star \Gamma) z (\Pi \cdot \Gamma) z \right] \partial_z \partial_{z} + ... \\
= y^2 \left[ \Pi^p \Pi^p - \Pi^p \Pi^p \right] \partial_z \partial_{z} + ... ,
\]

(77)

where dots stand for lower-derivative \( \partial y \) dependent terms which are absent in the flat space limit (\( y = \text{const} \)). In flat space \( A^2 \) is invertible even on massless (\( M^2_{10} = 0 \)) 10-d string states with \( (\Pi \cdot \Pi)_r = 0 \) and \( (\Pi \cdot \Pi)_\sigma = 0 \) if the \( X^m \) background is a BPS one,

\[
(\Pi^p \Pi^p)_r = -M^2_4 , \quad (\Pi^p \Pi^p)_\sigma = Z^2 , \quad M^2_4 = Z^2 , \quad (78)
\]

\[
A^2 = -2y^2M^2_4 \partial_z \partial_{z} = -2y^2Z^2 \partial_z \partial_{z} .
\]

(79)

To conclude, we have shown that choosing the ‘D3-brane’ or ‘4-d space-time’ adapted \( \kappa \)-symmetry gauge in the \( AdS_5 \times S^5 \) superstring action one obtains an action in which the fermionic term is quartic. The ‘4+6’ Cartesian parametrization of the 10-d space leads to a substantial simplification of the fermionic sector of the theory. This should hopefully allow one to make progress towards extracting more non-trivial information about the \( AdS_5 \times S^5 \) string theory and thus about its dual \([1]\) – \( N=4 \) super Yang-Mills theory.

7 Superparticle at the boundary of \( AdS_5 \times S^5 \); analytic harmonic superspace of N=2 super Yang-Mills theory

The string action on \( AdS_5 \times S^5 \) is not quadratic and therefore as different from the flat superspace background it is not clear how to construct the quantum theory. In curved background one can not have expected to find a simple quadratic action. However one may try to find some suitable variables in which the theory will become more useful. We will try to make the fermionic action quadratic in a way relevant to super-Yang-Mills theory. We also would like to use some guide from the Yang-Mills theory. This guide is an analytic harmonic superspace\(^6\) where the N=2 d=4 Yang-Mills theory can be formulated off-shell \([30]\).

\(^6\)Recently the limit to the boundary of the \( adS_5 \times S^5 \) superspace was used in \([29]\) to derive the N=2 harmonic superspace. Here we perform a closely related study where we in addition derive the analytic subspace and get the fermionic action quadratic.
As an example of such a possibility we will consider here a particular approximation to the full IIB string theory on $\text{AdS}_5 \times S^5$, starting with the gauge-fixed string action in the form (72). The approximation consists of

- **Boundary limit** $|y| \to \infty$

  To take this limit we rewrite the action in spherical coordinates on $S^5$ which include the radius of the sphere, $|y|$ and 5 angles $\phi_1, \ldots, \phi_5$. Only the angular part of the action $(\partial_i y/y)^2$ given by $\partial_i \phi^{m'} \partial^i \phi^{n'} g_{m'n'}(\phi)$ survives at the boundary.

  The WZ term in the form when the derivative hit the fermions has a term proportional to the $|y|$. To provide the existence of the limit is sufficient to require that $\vartheta$ depend only on $\tau$ and do not depend on $\sigma$. This suggests also the next approximation:

- **Dimensional reduction of the gauge-fixed string action.**

  Here we extend the independence of fermionic variables from one of the world sheet directions, suggested by the boundary limit, to other fields, coordinates of the 4-dimensional boundary of $\text{AdS}_5$, and angles of the sphere.

The toy model action is a gauge-fixed dimensionally reduced to one dimension string action in the limit to the boundary of $\text{AdS}_5 \times S^5$. It is plausible that one can get this action by considering a superparticle in $\text{AdS}_5 \times S^5$ background and gauge fixing the local symmetries of the superparticle action and taking the limit to the boundary. The action is

$$S_{\text{superparticle}} = \int d\tau \left[ P_p (\partial x^p - 2i \bar{\vartheta} \Gamma^p \partial \vartheta) + \frac{1}{2} \partial \phi^{m'} \partial \phi^{n'} g_{m'n'}(\phi) + S_{\text{ghosts}}(b,c) \right]$$

$$= S_{\text{boundary}}^{\text{adS}} + S_{\text{sphere}} + S_{\text{ghosts}} . \quad (80)$$

The part of the action coming from the $S^5$ part of the geometry is written here in terms of the independent variables, angles of the $S^5$. Alternatively one can use from the beginning the action in the form

$$S_{S^5} = \frac{1}{2} \int d\tau L^a(\phi) L^{a'}(\phi) = \frac{tr}{c} \int d\tau u u^{-1} \partial u u^{-1} \partial u = \frac{tr}{c} \int d\tau \partial u u^{-1} \partial u . \quad (81)$$

where $L^a, a = 1, 2, 3, 4, 5$ are the Cartan form on the sphere $SO(6)/SO(5)$ and $u(\phi)$ are the coset representative on the sphere, spherical harmonics taking values in $SU(4)$ algebra. 

23
The first term in the action $S_{\text{boundary}}^{\text{adS}}$ has a manifest d=4, N=4 global supersymmetry. We may rewrite it using the decomposition of one d=10 Majorana-Weyl spinor into 4 d=4 two-component spinors $\theta_{\alpha i}, \alpha = 1, 2, i = 1, 2, 3, 4$ and their complex conjugate, $\bar{\theta}^{i}_{\alpha}$. 

$$S_{\text{boundary}}^{\text{adS}} = \int d\tau P_{p} (\partial x^{p} - 2i \bar{\theta}^{\alpha} \Gamma^{p} \partial \bar{\theta}) = \int d\tau P_{p} (\partial x^{p} + i \partial \theta^{i} \sigma^{p} \bar{\theta}_{i} - i \theta^{i} \sigma^{p} \partial \bar{\theta}_{i}). \quad (82)$$

The total action (80) 

$$S_{\text{toy}} = \int d\tau \left[ P_{p} (\partial x^{p} + i \partial \theta^{i} \sigma^{p} \bar{\theta}_{i} - i \theta^{i} \sigma^{p} \partial \bar{\theta}_{i}) + L_{\text{sphere}} [u(\phi)] + S_{\text{ghosts}} (b, c) \right] \quad (83)$$

has superconformal symmetry which follows from the isometry of the background and is non-linearly realized after the gauge-fixing. The global $Q$-supersymmetry of this action is manifest. It is given by the following transformations 

$$\delta \theta_{\alpha i} = \epsilon_{\alpha i}, \quad \delta \bar{\theta}^{i}_{\alpha} = \bar{\epsilon}^{i}_{\alpha}, \quad \delta x^{p} = i(\epsilon^{i} \sigma^{p} \bar{\theta}_{i} - \theta^{i} \sigma^{p} \bar{\epsilon}_{i}), \quad \delta u(\phi) = 0 \quad (84)$$

These are precisely the N=4 supersymmetry transformations of global supersymmetry in the central basis [30]. The fermionic part of the action is cubic in fields and the action depends on the 16-component spinor.

We would like to make the fermionic part of this action quadratic in fields. It is instructive to truncate the action to N=2 supersymmetric one. For this we have to take 

$$\theta_{\alpha 3} = \theta_{\alpha 4} = 0 \quad \phi_{3} = \phi_{4} = \phi_{5} = 0 \quad (85)$$

Thus we cut $S^{5}$ down to $S^{2}$ and keep only 1/2 of the fermions. The action of the sphere variables can be written in two possible ways: either in terms of the angles on the sphere or in terms of spherical harmonics.

$$S_{S^{2}} = \frac{1}{2} \int d\tau [ (\partial \phi_{1})^{2} + (\sin \phi_{1} \partial \phi_{2})^{2} ] = tr \int d\tau \partial u^{-1} \partial u. \quad (86)$$

Here one can take harmonics on $S^{2}$ in the form suggested in [31]:

$$u = \begin{pmatrix} u_{-}^{1} & u_{+}^{1} \\ u_{-}^{2} & u_{+}^{2} \end{pmatrix} = \begin{pmatrix} \cos \frac{\phi_{1}}{2} e^{-i \frac{\phi_{2}}{2}} & i \sin \frac{\phi_{1}}{2} e^{-i \frac{\phi_{2}}{2}} \\ i \sin \frac{\phi_{1}}{2} e^{i \frac{\phi_{2}}{2}} & \cos \frac{\phi_{1}}{2} e^{i \frac{\phi_{2}}{2}} \end{pmatrix} \quad (87)$$

Our notation here as in [3].
Consider the toy model of the string action, the superparticle action at the truncated boundary of the near horizon D3 brane:

\[
S_{\text{toy}} = \int d\tau \left[ P^p (\partial x^p + i \partial \theta^i \sigma^p \bar{\theta}_i - i \theta^i \sigma^p \partial \bar{\theta}_i) + \frac{1}{2} \int d\tau [(\partial \phi_1)^2 + (\sin \phi_1 \partial \phi_2)^2] + S_{\text{ghosts}} \right]
\]  

(88)

The symmetries of this action are the symmetries of the \(N=2\) harmonic superspace in the central basis which consists of \((Z^M, u^\pm(\phi))\) where \(Z^M = (x^p, \theta_{\alpha i}, \bar{\theta}_i^\dot{\alpha})\).

The supersymmetries are shown above. There are two possibilities to formulate the reality condition of this action which precisely fit the known two possibilities to formulate the reality condition of harmonic superspace.

1. **Standard hermitean conjugation.** \(P^p\) and \(x^p\) and angles on the sphere \(\phi_1\) and \(\phi_2\) are real and the chiral spinors \(\theta_{\alpha i}\) and \(\bar{\theta}_i^\dot{\alpha}\) are conjugate of each other.

   \[
P^*_p = P^p, \quad x^*_p = x^p, \quad \phi^*_1,2 = \phi_{1,2}, \quad \bar{\theta}^*_i = \bar{\theta}_i^\dot{\alpha}
\]

(89)

The action is hermitean conjugate, \(\overline{S} = S\).

2. **Hermitean conjugation+antipodal map on the sphere.** In the central basis this second reality condition is the symmetry of the action and it means that as before that \(P^p\) and \(x^p\) are real, the chiral spinors \(\theta_{\alpha i}\) and \(\bar{\theta}_i^\dot{\alpha}\) are conjugate of each other. The new feature is the antipodal map on \(S^2\) which means that each point on the sphere is projected to the antipodal one by the following shift of the angles:

   \[
   \phi^*_1 = \pi - \phi_1, \quad \phi^*_2 = \pi + \phi_2
   \]

(90)

In particular the North pole of the sphere become a South pole under the antipodal map. The other variables are neutral under the antipodal map and therefore for them we have, as before

\[
\overline{P}^*_p = P^p, \quad \overline{x}^*_p = x^p, \quad \overline{\theta}^*_i = \bar{\theta}_i^\dot{\alpha}
\]

(91)

The action is hermitean+antipodal map conjugate, \(\overline{\overline{S}} = S\).

It is of particular importance that it is the second reality condition under hermitean+antipodal map conjugation that is the property of the unconstrained analytic superfields in \(N=2\) analytic superspace. This is quite different from the usual chiral superspace of \(N=1\) supersymmetry where the chiral superfields in the chiral basis are complex.

Now that we have found this reality condition under hermitean+antipodal map conjugation to be a symmetry of the toy model string action \(\overline{\overline{S}}\), we may ask if the knowledge of
the analytic superspace describing the Yang-Mills theory can help us to make the fermionic part of the action quadratic in fields. The answer is positive. As suggested by variables in the analytic basis \[30\] \((Z^{M}_{A}, u_{\pm}^{\pm})\) where \(Z^{M}_{A} = (x^{p}_{A}, \theta^{+}, \bar{\theta}^{+}, \theta^{-}, \bar{\theta}^{-})\), we have to perform the following change of the variables in our action:

\[
\begin{align*}
\theta_{\alpha}^{i} &= u^{i+} \theta_{\alpha}^{-} - u^{i-} \theta_{\alpha}^{+}, & \theta_{\alpha}^{+} &= \theta_{\alpha}^{i} u_{i}^{+}, & \theta_{\alpha}^{-} &= \theta_{\alpha}^{i} u_{i}^{-}
\end{align*}
\]

and

\[
\begin{align*}
x_{A}^{p} &= x^{p} - 2i \theta^{i} (\sigma^{p} \bar{\theta}^{j}) u_{i}^{+} u_{j}^{-}
\end{align*}
\]

The action becomes

\[
S_{\text{anal}} = \int d\tau \left[ P_{p} (\partial x_{A}^{p} + 2i \partial \theta^{+} \sigma^{p} \bar{\theta}^{-} + 2i \theta^{-} \sigma^{p} \partial \bar{\theta}^{+}) + L(\phi_{1}, \phi_{2}, \theta) + S_{\text{ghosts}} \right]
\]

The terms in \(L(\phi, \theta)\) in addition to the action depending only on angles of the sphere has now also some terms with derivatives on angles which depend on fermions \(\theta\). We will focus here on the first part of the action which has derivatives on fermions. The nice thing happened: the derivatives hit only \(\theta^{+}, \bar{\theta}^{+}\) and not \(\theta^{-}, \bar{\theta}^{-}\). We may introduce the new variables now:

\[
\begin{align*}
\Pi^{i} &= 2i \theta^{i} P_{p} \sigma^{p}, & \bar{\Pi}^{i} &= 2i P_{p} \sigma^{p} \bar{\theta}^{-}
\end{align*}
\]

The part of the action which comes from the boundary of \(\text{adS}\) after truncation becomes a free quadratic action which depends only on \(\theta^{+}\) and \(\bar{\theta}^{+}\) and their canonical conjugate variables, which are related to \(\theta^{-}\) and \(\bar{\theta}^{-}\).

\[
S_{\text{anal}}^{\text{adS}} = \int d\tau \left[ P_{p} \partial x_{A}^{p} + \partial \theta^{+} \bar{\Pi}^{i} + \Pi^{i} \partial \bar{\theta}^{+} \right]
\]

Our action is still supersymmetric but the supersymmetry is realized on the smaller set of coordinates related to the analytic subspace which includes \((\zeta^{M}, u_{i}^{\pm})\) where \(\zeta^{M} = (x^{p}_{A}, \theta^{+}, \bar{\theta}^{+})\).

The off-shell Yang-Mills theory is described by the analytic superfields which in the analytic basis depend only on the coordinates of the subspace. The analytic subspace is real under hermitean+ antipodal map conjugation. Our action in the analytic basis inherits the reality condition which includes the antipodal map on the sphere from the original action in the central basis.

As a final remark in this section we would like to stress that the action of the superparticle on \(\text{AdS}_{5} \times S^{5}\) space upon truncation to \(N=2\) supersymmetric part offers new possibilities to link the string theory with Yang-Mills theory via analytic subspace of the harmonic superspace. One can hope to develop the analogous methods for the untruncated string theory and simplify the structure of the theory.
8 Black holes and conformal mechanics

This section is based on recent work in [32, 33, 34]. One of the deep issues in black hole physics is the existence of the horizon which prevents the standard quantum mechanical treatment of this system. On the other hand there is an issue in the conformal mechanics model of [35] known as the absence of the ground state with $E = 0$. The Hamiltonian of [35] is

$$ H = \frac{p^2}{2m} + \frac{g}{2x^2}. \quad (97) $$

In the black hole interpretation of the model, the classical analog of an eigenstate of $H$ is an orbit of a timelike Killing vector field $k$, equal to $\partial/\partial t$ in the region outside the horizon, and the energy is then the value of $k^2$. The absence of a ground state of $H$ at $E = 0$ can now be interpreted as due to the fact that the orbit of $k$ with $k^2 = 0$ is a null geodesic generator of the event horizon, which is not covered by the static coordinates adapted to $\partial_t$. The procedure used in [35] to cure this problem was to choose a different combination of conserved charges as the Hamiltonian. This corresponds to a different choice of time, one for which the worldlines of static particles pass through the black hole horizon instead of remaining in the exterior spacetime.

Therefore one can believe that the study of conformal quantum mechanics has potential applications to the quantum mechanics of black holes. As one can see from the Table 1. in Sec. 3, the near horizon geometry of the supersymmetric d=4 black holes with electric and/or magnetic charge, is the simplest example of $adS_{p+2} \times S^{d-4-2}$ with the 2-form configuration with $p = 0, d = 4$ relevant for the superconformal symmetry of the particle mechanics.

A surprising connection between black holes and superconformal mechanics models of Akulov and Pashnev and of Fubini and Rabinovici [36] was found in [32]. The main new observation is that one recovers the supersymmetric conformal mechanics of [36] in the limit of the large black hole mass $M \to \infty$ but one also finds a generalization of these superconformal models for the black holes with the arbitrary mass $M$.

We start from the extreme RN metric in isotropic coordinates

$$ ds^2 = - \left(1 + \frac{M}{\rho}\right)^{-2} dt^2 + \left(1 + \frac{M}{\rho}\right)^2 [d\rho^2 + \rho^2 d\Omega^2], \quad (98) $$

where $d\Omega^2 = d\theta^2 + \sin^2 \theta \, d\varphi^2$ is the $SO(3)$-invariant metric on $S^2$, and $M$ is the black hole...
mass, in units for which $G = 1$. The near-horizon geometry is therefore

$$ds^2 = - \left( \frac{\rho}{M} \right)^2 dt^2 + \left( \frac{M}{\rho} \right)^2 d\rho^2 + M^2 d\Omega^2,$$

(99)

which is the Bertotti-Robinson (BR) metric. It can be characterized as the $SO(1, 2) \times SO(3)$ invariant conformally-flat metric on $adS_2 \times S^2$. The parameter $M$ may now be interpreted as the $S^2$ radius (which is also proportional to the radius of curvature of the $adS_2$ factor).

A test particle in this near-horizon geometry provides a model of conformal mechanics in which the $SO(1, 2)$ isometry of the background spacetime is realized as a one-dimensional conformal symmetry. If the particle’s mass $m$ equals the absolute value of its charge $q$ then this is just the $p = 0$ case of the construction of [3].

In horospherical coordinates $(t, \phi = \rho/M)$ for $adS_2$, the 4-metric and Maxwell 1-form of the BR solution of Maxwell-Einstein theory are

$$ds^2 = -\phi^2 dt^2 + \frac{M^2}{\phi^2} d\phi^2 + M^2 d\Omega^2,$$

$$A = \phi dt.$$

(100)

The metric is singular at $\phi = 0$, but this is just a coordinate singularity and $\phi = 0$ is actually a non-singular degenerate Killing horizon of the timelike Killing vector field $\partial/\partial t$. We now define a new radial coordinate $r$ by

$$\phi = (2M/r)^2.$$

(101)

The BR metric is then

$$ds^2 = -(2M/r)^4 dt^2 + (2M/r)^2 dr^2 + M^2 d\Omega^2.$$

(102)

Note that the Killing horizon in these coordinates is now at $r = \infty$.

The (static-gauge) Hamiltonian of a particle of mass $m$ and charge $q$ in this background is $H = -p_0$ where $p_0$ solves the mass-shell constraint $(p - qA)^2 + m^2 = 0$. This yields

$$H = (2M/r)^2 [\sqrt{m^2 + (r^2 p_r^2 + 4L^2)/4M^2} - q],$$

(103)

where $L^2 = p_\theta^2 + \sin^{-2} \theta p_\phi^2$, which becomes minus the Laplacian upon quantization (with eigenvalues $\ell(\ell + 1)$ for integer $\ell$). We can rewrite this Hamiltonian as

$$H = \frac{p_r^2}{2f} + \frac{mg}{2r^2 f},$$

(104)
where
\[
f = \frac{1}{2} \left[ \sqrt{m^2 + (r^2 p_r^2 + 4L^2)} / 4M^2 + q \right] ,
\] (105)
and
\[
g = 4M^2(m^2 - q^2)/m + 4L^2/m .
\] (106)
This Hamiltonian defines a new model of conformal mechanics [32]. The full set of generators of the conformal group are
\[
H = \frac{1}{2f} p_r^2 + \frac{g}{2r^2 f} , \quad K = -\frac{1}{2} f r^2 , \quad D = \frac{1}{2} r p_r ,
\] (107)
where \( K \) generates conformal boosts and \( D \) generates dilatations. It may be verified that the Poisson brackets of these generators close to the algebra of \( Sl(2,R) \).

To make contact with previous work on this subject, we restrict to angular quantum number \( \ell \) and consider the limit
\[
M \to \infty , \quad (m - q) \to 0 ,
\] (108)
with \( M^2(m - q) \) kept fixed. In this limit \( f \to m \), so
\[
H = \frac{p_r^2}{2m} + \frac{g}{2r^2} ,
\] (109)
with
\[
g = 8M^2(m - q) + 4\ell(\ell + 1)/m .
\] (110)
This is the conformal mechanics of [35]. For obvious reasons we shall refer to this as ‘non-relativistic’ conformal mechanics; the ‘non-relativistic’ limit can be thought of as a limit of large black hole mass. When \( \ell = 0 \) an ‘ultra-extreme’ \( m < q \) particle corresponds to negative \( g \) and the particle falls to \( r = 0 \), i.e. it is repelled to \( \phi = \infty \). On the other hand, a ‘sub-extreme’ \( m > q \) particle is pushed to \( r = \infty \), which corresponds to it falling through the black hole horizon at \( \phi = 0 \). The force vanishes (again when \( \ell = 0 \)) for an ‘extreme’ \( m = q \) particle, this being a reflection of the exact cancellation of gravitational attraction and electrostatic repulsion in this case. A static extreme particle of zero angular momentum follows an orbit of \( \partial / \partial t \), and remains outside the black hole horizon.

A discussion of supersymmetric versions of the conformal mechanics related to black holes can be found in [32, 33, 34] and this field will require more studies. For example the superconformal mechanics of the particle approaching the black hole horizon which is
expected to have a full $SU(1,1|2)$ superconformal algebra, has not been constructed so far but it will be constructed in the near future.

An interesting aspect of the relation between black holes and integrable Calogero models has been studied in [34]. The model has the Hamiltonian

$$H = \sum_i \frac{p_i^2}{2m} + \sum_{i<j} \frac{\lambda^2}{(q_i - q_j)^2}.$$  \hspace{1cm} (111)

The case of 2-particle Calogero model was considered in detail. It has been found that to have an agreement with the black hole hamiltonian (in case of large black hole mass) one has to find a way to constrain the particle orbital angular momentum to even integers so that from Calogero model the relevant coupling constant equals $4l(l + 1)$ as in (110) at $m = q = 1$. This can be achieved by requiring the identification of the antipodal points\footnote{This antipodal map on $S^2$ does not seem to be related to the one described in the context of reality of the analytic subspace of N=2 harmonic superspace. It is interesting, however that in both cases this operation is particular for the sphere which is part of curved geometry under consideration. Antipodal map has no analog in the flat space.} on $S^2$ and opens the possibility to consider Calogero models with $q = q_1 - q_2$ both positive and negative, which may give some insights about the exterior and also interior of the black hole horizon.

It would be very interesting to understand the quantum mechanical features of the new conformal and superconformal models in [32] before the limit to the large black hole mass is taken. In this limit the curvature of the $adS_2 \times S^2$ space vanishes and one may loose the important properties of the curved space. However when the mass of the black hole is not very large, we may find interesting quantum mechanical properties starting from the classical superconformal mechanics described above.

9 Concluding Comments

One of the purpose of this lecture was to explain the new concepts and new approaches to strong gravity in the framework of supersymmetry. We are trying to understand new issues in superstring theory, supergravity and superconformal field theories. Our current interest in various aspects of $adS \times S$ supersymmetric geometry is based on the fact that this near horizon geometry of D3, M2, M5 branes has 32 unbroken supersymmetry and is exact and the isometries of the relevant superspace form the superconformal algebra. Therefore there is some hope that we are learning new connections between algebraic and geometric
concepts which may survive in the fundamental theory unifying quantum gravity with other interactions.
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