On Supersymmetric Gauge Theories on $S^4 \times S^1$

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Abstract

We construct supersymmetric gauge theory on $S^4 \times S^1$. We find a consistent supersymmetry transformations which reduced to the 4D $N = 2$ supersymmetry transformation studied by Pestun by the dimensional reduction on $S^1$. We find there is no analogue of the usual Yang-Mills action except in the 4D limit. We also apply the localization technique to the partition function of the theories.
1 Introduction

The supersymmetric (SUSY) gauge theories on (Euclidean) curved spaces have been investigated very intensively recently. A motivation for studying these is the possibility of the exact computations of the partition function and some BPS operators. These can be done by the localization technique in field theory developed by [1] [2]. The work of [2], where the $N = 2$ SUSY gauge theory on $S^4$ was considered, have been generalized to other geometry [3]-[22]. One of the interesting applications of these is the computation of the $N^\frac{3}{2}$ scaling of the partition function of the ABJM model [23].

Especially, the applications of the localization technique to the 5D gauge theory on curved space will be important for studying the still mysterious M5-branes. Indeed, in the paper [15] the 5d Supersymmetric gauge theories on $S^5$ was constructed and some interesting results have been obtained for the theories [18] [19]. Since compactifications of the M5-branes give varieties of lower dimensional interesting theories, it will be important to extend the construction of the SUSY gauge theory on $S^5$ to other spaces for studying the M5-branes.

In this paper, we construct supersymmetric gauge theory on $S^4 \times S^1$. We find a consistent SUSY transformations which reduced to the 4d $N = 2$ SUSY transformation studied by Pestun [2] by the dimensional reduction on $S^1$. We find there is no analog of the usual Yang-Mills action which does not contain Lorentz violating constant, except in the 4D limit. It should be noted that we can not use the off-shell 5D supergravity [25] to construct a SUSY transformation and actions following [24] because there does not exist a field in the supergravity corresponding to the appropriate background field which appears in the Killing spinor equation. Thus we should think the theory has infinite coupling constant. We apply the localization technique to the partition function of the theory on $S^4 \times S^1$ following
and find the result is a simple extension of the corresponding partition function on $S^4$ with the contributions from Kaluza-Klein modes. Physical applications of the results in the paper, in particular to the M5-branes, are under investigations.

The organization of this paper is as follows: In section 2 we construct the Killing spinor and the consistent SUSY transformations of the vectormultiplets and hypermultiplets for the theory on $S^4 \times S^1$. In section 3, we see that an SUSY Yang-Mills action for the vectormultiplets on $S^4 \times S^1$ is difficult to construct. We can construct it only after taking the 4D limit by the dimensional reduction for the $S^1$. In section 4, we apply the localization technique to the partition function of the theory on $S^4 \times S^1$ and find the result is a simple extension of the corresponding partition function on $S^4$. We conclude with a short discussion in section 5.

2 SUSY transformations

In this section we will construct consistent SUSY transformations on $S^4 \times S^1$ by taking a simple ansatz on the Killing spinor.

2.1 Killing Spinor

We will construct SUSY gauge theories on $S^4 \times S^1$ following [15]. We will use the notations used in [15] in which the theory on $S^5$ was considered. The indices $m, n, \cdots$ runs from 1 to 5, on the other hand, $\mu, \nu, \cdots$ runs from 1 to 4. As in [15], we assume the following Killing Spinor equation:

$$\nabla_m \xi_I = \Gamma_m \tilde{\xi}_I,$$

(2.1)

where we define $\nabla_m$ as

$$\nabla_\mu \xi_I = D_\mu \xi_I,$$

$$\nabla_5 \xi_I = D_5 \xi_I + t_I^J \xi_J.$$

(2.2)

We further assume

$$\partial_5 \xi_I = 0,$$

(2.3)

for simplicity. The metric of $S^4 \times S^1$ is

$$ds^2_{S^4 \times S^1} = (dx^5)^2 + ds^2_{S^4},$$

$$ds^2_{S^4} = \ell^2 (d\theta^2 + \sin^2 \theta ds^2_{S^3}) = \frac{dr^2 + r^2 ds^2_{S^3}}{(1 + \frac{r^2}{4\ell^2})^2} = \frac{\sum dx_n^2}{(1 + \frac{r^2}{4\ell^2})^2},$$

(2.4)

where $r^2 = \sum_{n=1}^4 (x^n)^2$ and $e^a = f \delta^n_a dx^n$ and $f = (1 + \frac{r^2}{4\ell^2})^{-1}$. Here $x^5$ is a coordinate of $S^1$ with radius $R$, thus there is an identification $x^5 \sim x^5 + 2\pi R$. We can embed the $S^4$ in $R^5$ as $Y_1^2 + \cdots + Y_5^2 = \ell^2$. The relation between $x^n$ and $Y^n$ ($n = 1, \ldots, 4$) is $Y_n = \frac{x^n}{1 + \frac{r^2}{4\ell^2}}$. 

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First, $D_\mu \xi_I = \Gamma_\mu \tilde{\xi}_I$ is solved [2] if we take

$$\xi_I = \frac{1}{\sqrt{1 + \frac{r^2}{4l^2}}} \left( \epsilon_I + \frac{x^i \Gamma_{5i}}{2l} \epsilon'_I \right), \quad (2.5)$$

$$\tilde{\xi}_I = \frac{1}{2l \sqrt{1 + \frac{r^2}{4l^2}}} \left( \epsilon'_I - \frac{x^i \Gamma_{5i}}{2l} \epsilon_I \right), \quad (2.6)$$

where $i, j = 1, \ldots, 4$ which are 4D flat indices and $\epsilon_I, \epsilon'_I$ are constants. For the $S^1$ direction, the condition $\nabla_{5} \xi_i = t_I^J \xi_J = \Gamma_5 \tilde{\xi}_I$ is written as

$$\tilde{\xi}_I = t_I^J \Gamma_5 \xi_J. \quad (2.7)$$

Assuming

$$t \Gamma_i = \pm \Gamma_i t, \quad (2.8)$$

we have

$$(te)_I = \frac{1}{2l} \Gamma_5 \epsilon'_I,$$

$$(te')_I = \pm \frac{1}{2l} \Gamma_5 \epsilon_I, \quad (2.9)$$

which implies

$$(t^2)^J_I = \pm \frac{1}{4l^2} \delta^J_I, \quad (2.10)$$

where we assume $[t, \Gamma_5] = 0$. Using (2.9), we rewrite (2.5) as

$$\xi_I = \frac{1}{\sqrt{1 + \frac{r^2}{4l^2}}} \left( \epsilon_I + x^i \Gamma_{5i} t_I^J \epsilon_J \right). \quad (2.11)$$

We will later see that the scaling transformation which enters in the SUSY algebra is $\rho \sim \epsilon^I J \xi_I t_J \eta_I - \epsilon^I J \eta_I \tilde{\xi}_J = \xi_I t_{I}^J \Gamma_5 \eta_J - \eta_I t_{I}^J \Gamma_5 \xi_J$ which should vanish for a non-scale invariant theory. For $t_I^J = 0$ which means $t_{I}^J$ is a symmetric tensor for the $SU(2)_R$, we find $\rho = 0$. 

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Therefore, our conditions essentially fix $t$ modulo the $SU(2)_R$ transformation:

$$t_I^J = \frac{1}{2l} (\sigma_3)_I^J,$$  \hspace{1cm} (2.14)

where

$$\xi_1 = \frac{1}{\sqrt{1 + \frac{4}{4l^2}}} \left( 1 + \frac{x^I \Gamma_I}{2l} \right) \psi_1,$$

and

$$\xi_2 = \frac{1}{\sqrt{1 + \frac{4}{4l^2}}} \left( 1 - \frac{x^I \Gamma_I}{2l} \right) \psi_2.$$  \hspace{1cm} (2.15)

This would correspond to a $SU(2)_R$ gauge field background in the supergravity, however, the term $\Gamma_m \tilde{\xi}^I = t_I^J \Gamma_m \delta \xi_J$ is absent in the SUSY transformation of the gravitino of the 5D supergravity \[25\]. Thus we can not use \[25\] to construct the SUSY invariant action.

### 2.2 SUSY transformations

Now we will construct the SUSY transformations on $S^4 \times S^1$ of the vector multiplets. First, following \[15\] we assume that the SUSY variation of fields on $S^4 \times S^1$ takes the form

$$\delta_\xi A_m = i \epsilon^{IJ} \xi_I \Gamma_m \lambda_J,$$

$$\delta_\xi \sigma = i \epsilon^{IJ} \xi_I \lambda_J,$$

$$\delta_\xi \lambda_I = -\frac{1}{2} \Gamma^{mn} \xi_I F_{mn} + \Gamma^m \xi_I D_m \sigma + \xi_J D_K e^{JK} + 2 \hat{\xi}_J \sigma,$$

$$\delta_\xi D_{IJ} = -i (\xi_I \Gamma^m \nabla_m \lambda_J + \xi_J \Gamma^m \nabla_m \lambda_I) + [\sigma, \xi_I \lambda_J + \xi_J \lambda_I] + i (\hat{\xi}_I \lambda_J + \hat{\xi}_J \lambda_I)$$

$$= -i (\xi_I \Gamma^m D_m \lambda_J + \xi_J \Gamma^m D_m \lambda_I) + [\sigma, \xi_I \lambda_J + \xi_J \lambda_I] + 2i \xi_K t_{IJ} \Gamma^K \lambda_K,$$  \hspace{1cm} (2.16)

where we have used

$$s^K_I t^J_K + t^K_I s^K_J = -s^K_L t^J_K \delta^J_I,$$  \hspace{1cm} (2.17)

\[2\] There is another choice:

$$t = i \frac{1}{2l} \sigma_3 \Gamma_5.$$  \hspace{1cm} (2.12)

where

$$\xi_1 = \frac{1}{\sqrt{1 + \frac{x^2}{4l^2}}} \left( 1 + \frac{x^I \Gamma_I}{2l} \right) \psi_1,$$

$$\xi_2 = \frac{1}{\sqrt{1 + \frac{x^2}{4l^2}}} \left( 1 - \frac{x^I \Gamma_I}{2l} \right) \psi_2.$$  \hspace{1cm} (2.13)

However, in this choice we find that it is difficult to construct a consistent SUSY transformation related to this. Thus, in this paper, we forget this possibility.
which is valid for an arbitrary symmetric tensor $s_{IJ}$.

Using

$$\Gamma^m \nabla_m \xi_I = \Gamma^m t^J_I \Gamma_5 \Gamma_m \xi_J = -\frac{3}{4 l^2} \xi_I, \quad (2.18)$$

and performing some computations, we can show that the commutator of the two SUSY generators is a sum of a translation ($v^m$), a gauge transformation ($\gamma + iv^m A_m$), a dilatation ($\rho$), an R-rotation ($R_{IJ}$) and a Lorentz rotation ($\Theta^{ab}$):

$$[\delta \xi, \delta \eta] A_m = -iv^m F_{nm} + D_m \gamma, \quad (2.19)$$

where $R_I^J = \epsilon^{JK} R_{JK}$ and

$$v^m = 2 \epsilon^{IJ} \xi_I \Gamma^m \eta_J, \quad \gamma = -2i \epsilon^{IJ} \xi_I \eta_J \sigma, \quad \rho = -2i \epsilon^{IJ} (\xi_I \eta_J - \eta_I \xi_J) = 0, \quad R_{IJ} = -3i(\xi_I \eta_J + \xi_J \eta_I - \eta_I \xi_J - \eta_J \xi_I) - 2i \epsilon^{KL} \xi_K \Gamma_5 t_{IJ} \eta_L, \quad (2.20)$$

We can see that $R_{IJ} = R_{IJ}$ and $R_{11} = R_{22} = 0$ which imply $R_{IJ} \sim t_{IJ}$. Therefore, the SUSY transformation which is inferred from the result of [13] is indeed consistent off-shell.

For the hypermultiplets, we also assume a SUSY transformation of the form given in [13]:

$$\delta q_I = -2i \xi_I \psi, \quad \delta \psi = \epsilon^{IJ} \Gamma^m \xi_I \nabla_m q_J + i \epsilon^{IJ} \xi_I \sigma q_J + 3 \epsilon^{IJ} \xi_I \tilde{q}_J + \epsilon^{IJ} \tilde{q}_I F_J + \epsilon^{IJ} \tilde{F}_I, \quad \delta F_I = 2 \tilde{\xi}_I (i \Gamma^m \nabla_m \psi + \sigma \tilde{\psi} + \epsilon^{KL} \lambda_K q_L). \quad (2.21)$$

The square of $\delta$ is

$$\delta^2 q_I = iv^m D_m q_I - i^\gamma q_I - R_I^J q_J, \quad \delta^2 \psi = iv^m D_m \psi - i^\gamma \psi - \frac{1}{4} \Theta^{ab} \Gamma^{ab} \psi, \quad \delta^2 F_I = iv^m D_m F_I - i^\gamma F_I + R_I^J F_J. \quad (2.22)$$
where

\[ v^m = \epsilon^{IJ} \xi_I \Gamma^m \xi_J , \]
\[ \gamma = -i \epsilon^{IJ} \xi_I \xi_J \sigma , \]
\[ R_{IJ} = 2i (e^{KL} \xi_K \Gamma^5 t_{IJ} \xi_L) , \]
\[ \Theta^{ab} = -2i \epsilon^{IJ} \dot{\xi}_I \Gamma^{ab} \xi_J , \]
\[ R'_{IJ} = -2i \dot{\epsilon}^{IJ} \Gamma^m \xi_I \Gamma^m \xi_J , \]

(2.23)

which is consistent with the one for the vectormultiplets. Therefore, the SUSY transformation for the hypermultiplets is consistent.

3 SUSY invariant action

In this section, we will try to construct SUSY invariant actions. We will see that our SUSY transformation corresponds to the 4D SUSY transformation of [2] by the dimensional reduction of \( S^1 \). We will drop the total divergent terms below for the notational convenience.

Now we will try to construct a SUSY invariant action for vectormultiplets on \( S^4 \times S^1 \).

We can show

\[
\delta \xi \left( \frac{1}{2} F_{mn} F^{mn} - D_m \sigma D^m \sigma - \frac{1}{2} D_{IJ} D^{IJ} + i \lambda_I \Gamma^m \nabla_m \lambda^I - \lambda_I [\sigma, \lambda^I] - i \lambda_I \lambda^I \right) = i \lambda_I [\Gamma_5 t^{IJ}, \Gamma^{mn}] \xi_J F_{mn} - 2i D_m \sigma (\xi_I \{ \Gamma_5 t^{IJ}, \Gamma^{mn} \} \lambda_J - 2i D^{IJ} (\xi_K \Gamma_5 t_{IJ} \lambda^K) + 2i \xi_I \lambda^I \sigma \text{Tr}(t^2) - 4i \xi_K \Gamma_5 t_K \Gamma_m \Gamma^j_I \Gamma_n \lambda^I \sigma + \text{(total divergence)},
\]

(3.1)

and then we find

\[
\delta \xi \left( \frac{1}{2} F^2 - (D \sigma)^2 - \frac{1}{2} D_{IJ} D^{IJ} + \lambda_I \Gamma^m D_m \lambda^I - \lambda_I [\sigma, \lambda^I] + 2 A_5 t_{IJ} D_{IJ} - t_{IJ} t^{IJ} (6(A_5)^2 - 4 \sigma^2) \right) = 4i t^{IJ} ((\xi_I \Gamma^m \lambda_J) \partial_5 A_m - (\xi_I \lambda_J) \partial_5 \sigma).
\]

(3.2)

This action is not gauge invariant nor SUSY invariant. However, by taking \( R \to 0 \), i.e. the dimensional reduction to \( S^4 \), both problems disappear. Thus, in this limit to the theory on \( S^4 \), we find the invariant action as

\[
L_{\text{vector}}^{S^4} = \frac{1}{2} F_{mn} F^{mn} - D_m \sigma D^m \sigma + i \lambda_I \Gamma^m D_m \lambda^I - \lambda_I [\sigma, \lambda^I] - \frac{1}{2} (D_{IJ} - 2 A_5 t_{IJ}) (D^{IJ} - 2 A_5 t^{IJ}) - 4 t_{IJ} t^{IJ} ((A_5)^2 - \sigma^2),
\]

(3.3)

where \( \partial_5 = 0 \), which is the usual SUSY Yang-Mills Lagrangian of the vector multiplet used in [2]. Note that the mass terms for the scalars (\( A_5, i \sigma \)) are same and the auxiliary field

\[ D'_{IJ} = D_{IJ} - 2 A_5 t_{IJ} \]

(3.4)
transforms as
\[
\delta \xi D'_{IJ} = -i(\xi_I \Gamma^m D_m \lambda_J + \xi_J \Gamma^m D_m \lambda_I) + [\sigma, \xi_I \lambda_J + \xi_J \lambda_I],
\] (3.5)
which is same as the one on \( R^4 \) and
\[
\delta \xi \lambda_I = -\frac{1}{2} \Gamma^{mn} \xi_I F_{mn} + \Gamma^m \xi_I D_m \sigma + \xi_J D'_{Kl} \epsilon^{JK} + 2 \left( \xi_I \sigma - \Gamma_5 \xi_I A_5 \right) .
\] (3.6)
These are consistent with the ones in [2].

We can construct a Yang-Mills action on \( S^4 \times S^1 \) which is invariant under a SUSY generator, but it is the SUSY exact action which will be used as a localization computation and it depends on constant tensors. It is difficult to construct the SUSY Yang-Mills action on \( S^4 \times S^1 \) which reduces to the standard SUSY Yang-Mills action on \( S^4 \) by the dimensional reduction. This is partly because the dimensional reduction gives a massless scalar in the vector multiplet, but there is the mass term for all the scalar fields in the SUSY Yang-Mills action on \( S^4 \) by the conformal mapping from \( R^4 \). To resolve this, we probably need to modify the ansatz for the Killing spinor although we will not try this in the paper. We can instead think that the theory has infinite gauge coupling constant although this would cause some divergences even for the BPS protected properties.

Now we will consider the hypermultiplets. By explicit computations, we find that the following one is a SUSY invariant Lagrangian on \( S^4 \times S^1 \):
\[
L_{\text{hyper}} = \epsilon^{IJ}(D_m \bar{q}_I D^n q_J - \bar{q}_I \sigma^2 q_J) - 2(i \bar{\psi} \Gamma^m D_m \psi + \bar{\psi} \sigma \psi)
- i \bar{q}_I D^{IJ} q_J - 4 \epsilon^{IJ} \bar{\psi} \lambda_I q_J - \epsilon^{I' J'} \bar{F}_{I'J'} F_{J'}
- 2 \epsilon^{IJ} \bar{q}_I D_5 q_J - 8 \epsilon^{IJ} \bar{q}_I q_J .
\] (3.7)
Taking the 4D limit, \( R \to 0 \), we have the SUSY invariant Lagrangian on \( S^4 \):
\[
L_{\text{hyper}}^{S^4} = \epsilon^{IJ}(D_\mu \bar{q}_I D^\mu q_J + \bar{q}_I (A_5)^2 q_J - \bar{q}_I \sigma^2 q_J) - 2(i \bar{\psi} \Gamma^\mu D_\mu \psi + \bar{\psi} \Gamma_5 A_5 \psi + \bar{\psi} \sigma \psi)
- i \bar{q}_I D^{IJ} q_J - 4 \epsilon^{IJ} \bar{\psi} \lambda_I q_J - \epsilon^{I' J'} \bar{F}_{I'J'} F_{J'}
- 8 \epsilon^{IJ} \bar{q}_I q_J .
\] (3.8)
This Lagrangian and the SUSY transformation are different from the ones of [2]. In [2] the action and the SUSY transformation contain quartic terms of the scalars in the hypermultiplets however, in ours they are quadratic. The Lagrangian (3.8) will correspond to the round sphere limit of the Lagrangian in [22]. The hypermultiplet Lagrangian in [2] is expected to be related to ours by a field redefinition.

A mass term for the hypermultiplets is also introduced by giving a VEV to the vectormultiplets which does not break the SUSY. Here, we can take \( \langle A_5 \rangle = m, \sigma = 0 \) and \( \langle D_{IJ} \rangle = 0 \) in the Lagrangian (3.7) or (3.8). Then, a collection of the \( m \) dependent terms is the mass term.
4 Localization

In this section, we apply the localization technique to the theory on $S^4 \times S^1$ following [2]. We take $\xi_I$ as Grassmann-even spinors such that $\delta \xi$ is a fermionic transformation.

First, we will compute the bilinear of the Killing spinors which will be used for the localization technique. For

$$t_I^J = \frac{1}{2l}(\sigma_3)_I^J,$$  \hspace{1cm} (4.1)

the explicit form of the Killing spinor is

$$\xi_1 = \frac{1}{\sqrt{1 + \frac{r^2}{4l^2}}} \left(1 + \frac{x^i \Gamma_i}{2l} \Gamma_5\right) \psi_1,$$

$$\xi_2 = \frac{1}{\sqrt{1 + \frac{r^2}{4l^2}}} \left(1 - \frac{x^i \Gamma_i}{2l} \Gamma_5\right) \psi_2,$$

$$\tilde{\xi}_1 = \frac{1}{2l} \Gamma_5 \xi_1, \quad \tilde{\xi}_2 = -\frac{1}{2l} \Gamma_5 \xi_2,$$  \hspace{1cm} (4.2)

where $\psi_1, \psi_2$ are constant spinors. Note that we can not impose the $SU(2)$ majorana condition $\xi^\dagger_I = \epsilon^{IJ} \xi_J^T C$ for this. Instead, we can impose a “twisted” $SU(2)$ majorana condition

$$\xi^\dagger_I = \epsilon^{IJ} \xi_J^T C \Gamma_5,$$  \hspace{1cm} (4.3)

by imposing $\psi^\dagger_I = \epsilon^{IJ} \psi_J^T C \Gamma_5$.

Now we regard $\xi_I$ as Grassmann even spinors and will compute

$$s \equiv \epsilon^{IJ} \xi_I \xi_J,$$

$$v^m \equiv \epsilon^{IJ} \xi_I \Gamma^m \xi_J,$$

$$w^{mn}_{IJ} \equiv \xi_I \Gamma^{mn} \xi_J,$$  \hspace{1cm} (4.4)

which appear in $(\delta \xi)^2$. We can show that

$$D_m s = 2\epsilon^{IK} w_{m5IK},$$

$$D_m v_n = 2\epsilon^{IK} \xi_I \Gamma_{nm5} \xi_K = \epsilon_{mn\mu5} w^{\mu}_{m5IK},$$

$$D_n w_{mlIJ} = -t_{IJ} (\epsilon_{nml5\mu} v^\mu + s(\delta_{5l}g_{nm} - \delta_{5m}g_{nl})).$$  \hspace{1cm} (4.5)

These implies that $\partial_5 s = \partial_5 v_m = \partial_m v_5 = 0$ and

$$D_m v_n + D_n v_m = 0,$$  \hspace{1cm} (4.6)

i.e. $v^m$ is a Killing vector of $S^4 \times S^1$. 

There are choices for the constant spinors $\psi_I$. In this paper we choose
\[ \Gamma_5 \psi_2 = -\psi_2, \quad \Gamma^{12} \psi_2 = \Gamma^{34} \psi_2 = i \psi_2, \quad (4.7) \]
because this corresponds to the Killing spinor used in $S^4$ case \[2\] as we will see later. Other choices may be different from this and interesting to be studied although we will concentrate this choice in the paper. We also normalize the $\psi_i$ as
\[ 2 \psi_i^T C \psi_2 = -1, \quad (4.8) \]
for the convenience. Note that this choice is consistent with the twisted $SU(2)$ majorana condition. Then, we obtain explicitly
\[ s = 2 \frac{1}{1 + \frac{x^2}{a^2}} \psi_i^T C (1 + \Gamma_5 \frac{x^I \Gamma^I}{2l} \Gamma_5) \psi_2 = - \frac{1 - \frac{x^2}{a^2}}{1 + \frac{x^2}{a^2}} = - \cos \theta \]
\[ v^5 = 1 \]
\[ v^\mu \frac{\partial}{\partial x^\mu} = i \frac{1}{l} \left( x^1 \partial_2 - x^2 \partial_1 + x^3 \partial_4 - x^4 \partial_3 \right) = i \frac{1}{l} \left( Y_1 \frac{\partial}{\partial Y_2} - Y_2 \frac{\partial}{\partial Y_1} + Y_3 \frac{\partial}{\partial Y_4} - Y_4 \frac{\partial}{\partial Y_3} \right). \quad (4.9) \]
Note that $v^\mu$ is pure imaginary, but $v^5 = 1$. Of course, we can multiply a phase factor to the $\psi_I$. Then, the majorana condition also has an extra phase and all bilinears of the Killing spinors are multiplied by a same phase factor. This can make $v^\mu$ real although we will not do so.

Let us first concentrate on the vector multiplets. We take the regulator Lagrangian as
\[ \delta \xi V = \text{tr} \left[ \left( \delta \xi^{\dagger} \lambda \right) \lambda \right]. \quad \text{Here we recall} \]
\[ \delta \xi I = - \frac{1}{2} \Gamma^{nm} \xi I F_{mn} + \Gamma^m \xi I D_m \sigma + \xi J D^J I + 2 \xi I \sigma. \quad (4.10) \]
We define
\[ \delta \xi I = \frac{1}{2} \xi I \Gamma^{mn} F_{mn} - \xi I \Gamma^m D_m \sigma + \xi J D^J I - 2 \xi I \Gamma_5 t^I K \sigma, \quad (4.11) \]
where we take $\xi_I$ as a twisted $SU(2)$ majorana spinor in order for manifest positive definiteness and the condition, $\delta \xi \left( \int_{S^4 \times S^1} \delta V \right) = 0$. Note that path-integral contour should be $\sigma^+ = - \sigma$ and $D^I_{IJ} = - D_I^J$ as in \[15\].

Now we can show that
\[ \delta \xi I \delta \xi I |_{\text{bos}} = \frac{1}{2} F_{mn} F^{mn} - D_m \sigma D^m \sigma - \frac{1}{2} D_{IJ} D_I^J + 2 t^I t^I \sigma^2 \]
\[ - \frac{1}{4} \epsilon^{\mu \nu \rho \sigma} F_{\mu \nu} F_{\rho \sigma} - \frac{1}{2} \epsilon^{\mu \nu \rho} (D_\rho v_\sigma) F_{\mu \nu} \sigma \]
\[ = F_{\mu 5} F^{\mu 5} + \frac{1 + s}{2} \left( F_+ + \frac{\sigma}{1 + s} (dv)_- \right)^2 + \frac{1 - s}{2} \left( F_- - \frac{\sigma}{1 - s} (dv)_+ \right)^2 \]
\[ - D_m \sigma D^m \sigma - \frac{1}{2} D_{IJ} D_I^J, \quad (4.12) \]
where

\[ 2(F_{\pm})_{\mu\nu} \equiv F_{\mu\nu} \pm \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}, \]  

(4.13)

and \((d\nu)_{\mu\nu} = D_{\mu} v_{\nu}\) which is an antisymmetric tensor by (4.5). Here we have used

\[ (D_{\mu} v_{\nu})^2 + \frac{s}{2} \epsilon^{\mu\nu\alpha\beta} (D_{\mu} v_{\nu})(D_{\alpha} v_{\beta}) = 4 t_{IJ} t^{IJ} (1 - s^2), \]  

(4.14)

and other identities follows from the Fierz identities which are summarized in the Appendix.

The saddle points of (4.12) are

\[ D_{m\sigma} = 0, \quad D_{IJ} = 0, \quad F_{\mu5} = 0, \]  

(4.15)

and for \(s \neq \pm 1\)

\[ (F_{\mu\nu})_{\pm} \mp \frac{s}{2(1 \mp s)} (D_{\mu} v_{\nu})_{\pm} = 0. \]  

(4.16)

This implies \(D_5 F_{\mu\nu} = 0\) and

\[ \sigma d \left( \frac{s}{1 - s^2} d\nu + \frac{1}{1 - s^2} \ast d\nu \right) = 0, \]  

(4.17)

from the Bianchi identity. We can check that this implies

\[ \sigma = 0, \quad \text{and then} \quad F_{\mu\nu} = 0. \]  

(4.18)

Only the Wilson line along \(x^5\), i.e. the constant part of \(A_5 = a\) in a gauge choice, remains as a moduli for \(s^2 \neq 1\). Note that \(a\) is taken in the Cartan subalgebra. The Wilson loop

\[ P e^{i \int_{\theta_0}^{2\pi R} d^5 A_5} = e^{2\pi i Ra} \]  

(4.19)

is invariant under \(a \rightarrow a + \frac{1}{\pi} H_i\) where \(\{H_i\}\) is any basis of the Cartan algebra such that the inner product with any weight in any representation is an integer. This means that \(a\) is an periodic variable with the above identification. Except this periodicity, the saddle points for the theory on \(S^4 \times S^1\) are same as the ones for the theory on the \(S^4\).

Next, we consider the localization of the hypermultiplets. The SUSY transformation for the fermion in the hypermultiplets is

\[ \delta \psi = \epsilon^{IJ} \Gamma^m \xi_I D_m q_J + i \epsilon^{IJ} \xi_I \sigma q_J + \epsilon^{IJ'} \xi_I' F_{J'} - 2t^{IJ} \Gamma_5 \xi_I q_J, \]  

(4.20)

For positivity of the action of the hypermultiplets, we have assumed that \(F\) is “pure imaginary” and \(q\) is “real”. With the rotation of the contours for \(\sigma, D_{IJ}, F_{J'}\), we find

\[ (\delta \psi)^{\dagger} = \xi_I C \Gamma_5 \Gamma^m D_m q_I \Omega + i \xi_I C \Gamma_5 q_J^{\dagger} \Omega \sigma - \xi_I' C \Gamma_5 F_{J'} \Omega + 2t^{IJ} \xi_I C q_J \Omega. \]  

(4.21)
The regulator Lagrangian for the localization will be \( \delta V_{\text{hyper}} \) where
\[
V_{\text{hyper}} = (\delta \psi) \psi .
\] (4.22)

Then, the bosonic part of the regulator Lagrangian is \( \delta V_{\text{hyper}}|_{\text{bos}} = (\delta \psi) \delta \psi \) which becomes
\[
\delta V_{\text{hyper}}|_{\text{bos}} = \frac{1}{2} \epsilon^{IJ} D_m \bar{q}_I D_m q_J + w_{IJ}^{5mn} D_m \bar{q}^I D_n q^J - \frac{1}{2} \epsilon^{I} J^{J'} \bar{F}_I F_{J'} - \frac{1}{2} \epsilon^{IJ} \bar{q}_I \sigma^2 q_J
\]
\[-t^{IJ} \epsilon^{KL} \bar{q}_K q_L + 2i w_{IJ}^{5\mu} q^I \sigma D_\mu q^J - 2i u^{IJ} \bar{q}_I D_\mu q_J - 2i s^{IJ} \bar{q}_I \sigma q_J ,\] (4.23)

where
\[
w_{IJ}^{5mn} = \xi_I \Gamma^{mn} \xi_J .
\] (4.24)

In order to derive the saddle point of this, we note there are following two inequalities:
\[
0 \leq |\epsilon^{IJ} \Gamma^m \xi_I D_m q_J|^2 = \xi_I \mathrm{CT}_5 \Gamma^m D_m q_I \Omega \epsilon^{IJ} \Gamma^m \xi_I D_m q_J
\]
\[= \frac{1}{2} \epsilon^{IJ} D_m \bar{q}_I D_m q_J + w_{IJ}^{5mn} D_m \bar{q}^I D_n q^J ,
\] (4.25)

and \( 0 \leq \frac{1}{2} \epsilon^{IJ} D_m \bar{q}_I D_m q_J \). Using
\[
w_{IJ}^{5\mu} \epsilon^{IJ} \bar{q}^I D_\mu q^J = \frac{1}{2} w_{IJ}^{5\mu} \bar{q}^I F_\mu q^J - 3t_{IJ} q^I D_\mu q^J v^\mu ,
\] (4.26)

which is valid up to a total divergence term, we can show
\[
\delta V_{\text{hyper}}|_{\text{bos}} = \frac{1}{6} \epsilon^{IJ} D_m \bar{q}_I D_m q_J + \frac{5}{3} \left( \frac{1}{2} \epsilon^{IJ} D_m \bar{q}_I D_m q_J + w_{IJ}^{5mn} D_m \bar{q}^I D_n q^J \right) - \frac{1}{2} \epsilon^{I} J^{J'} \bar{F}_I F_{J'}
\]
\[-t^{IJ} \epsilon^{KL} \bar{q}_K q_L + \frac{5}{3} t_{IJ} \epsilon^{IJ} \bar{q}_I D_\mu q_J - \frac{1}{2} \epsilon^{IJ} \bar{q}_I \sigma q_J ,\] (4.27)

at the saddle points of the vectormultiplets. Because this is written as a sum of positive
definite terms, we conclude
\[
q^I = 0 , \quad F^I = 0 ,
\] (4.28)

for the hypermultiplets at the saddle points. Thus, both of the saddle points of vectormulti-
plets and hypermultiplets essentially coincide with the one for \( S^4 \) \cite{2} \cite{22}.

In order to compute the 1-loop determinant for the regulator Lagrangian, we need to
first fix the gauge. This can be done following \cite{2}. However, our SUSY transformation for
\( S^4 \times S^1 \) is closer to the one in \cite{22}. Thus, it is more convenient to closely follow \cite{22}. As in
\cite{22}, we introduce the BRST transformation \( Q_B \) for the field in the vectormultiplets as the
usual one with ghost field \( c \) and define \( Q_B c = ic c + a_0 \) where \( a_0 \) is constant. We also define
\[
Q c = i \Phi \equiv i (-s \sigma + v^m A_m) ,
\] (4.29)
where $Q = \delta \xi$. We need to introduce other ghosts and their transformation rules:

$$
Q_B a_0 = Q a_0 = 0, \quad Q_B \bar{c} = B, \quad Q \bar{c} = 0, \quad Q_B B = i[a_0, \bar{c}], \quad Q_B = iv^m \partial_m \bar{c},
$$

$$
Q_B \bar{a}_0 = \bar{c}_0, \quad Q \bar{a}_0 = 0, \quad Q_B \bar{c}_0 = i[a_0, \bar{a}_0], \quad Q \bar{c}_0 = 0,
$$

$$
Q_B B_0 = c_0, \quad Q_B = i[a_0, B_0], \quad Q c_0 = 0.
$$

For the gauge fixing, we introduce $\hat{Q} = Q + Q_B$ and take the regulator Lagrangian as

$$
\hat{Q}(V + V_{GF}),
$$

where

$$
V_{GF} = \text{tr}(\bar{c}G + \bar{c}B_0 + c\bar{a}_0),
$$

$$
G = iD_m A^m + i\mathcal{L}_v(\Phi - A_5) = iD_\mu A^\mu + i\mathcal{L}_{v^\prime}(\Phi - A_5) + i\partial_5 \Phi,
$$

and $v^\prime$ is the vector on $S^4$ which is obtained from $v$ by the projection. This gauge fixing function is taken to be slightly different from the one used in [22], in order to fix the gauge symmetry related to $x^5$ direction. The saddle points of the vectormultiplets are unchanged and other bosonic field vanish at saddle points except $a_0 = A_5$ which is from $\hat{Q} c = 0$.

We introduce

$$
\Psi = Q \sigma = i\xi I \lambda^I, \quad \Psi_\mu = QA_\mu = i\xi I \Gamma_\mu \lambda^I, \quad \Xi_{IJ} = \xi I \Gamma_5 \lambda_J + \xi_J \Gamma_5 \lambda_I,
$$

which means

$$
\lambda^I = -i\Gamma_5 \xi I \Psi - i\Gamma^\mu_5 \xi I \Psi_\mu + \xi^J \Xi_{IJ}.
$$

Then, the fields are classified by boson-fermion and $\hat{Q}$-doublet:

$$
X = (\sigma, A_\mu, \bar{a}_0, B_0), \quad \Xi = (\Xi_{IJ}, \bar{c}, c)
$$

$$
\hat{Q} X = (\Psi, \Psi_\mu + D_\mu c, \bar{c}_0, c_0),
$$

$$
\hat{Q} \Xi = (-w_{IJ, \rho \sigma} \epsilon^{\mu \nu \rho \sigma} F_{\mu \nu} + 2w_{IJ} D_\mu \sigma - D_{IJ} - 2s t_{IJ, \sigma, B, a_0 - \Phi})
$$

(4.35)

where we have neglected higher order terms except ones including moduli $a_0$ and $A_5$. In terms of these, we can rewrite $V + V_{GF}$ and then we define $D_{ab}$ by

$$
V + V_{GF} = (\hat{Q} X, \Xi) \left( \begin{array}{cc} D_{00} & D_{01} \\ D_{10} & D_{11} \end{array} \right) \left( \begin{array}{c} X \\ \hat{Q} \Xi \end{array} \right).
$$

(4.36)

From this equation, we find

$$
\Xi D_{10} X + \Xi D_{11} \hat{Q} \Xi = \left( \frac{1}{2} w_{IJ, \rho \sigma} \epsilon^{\mu \nu \rho \sigma} F_{\mu \nu} - w_{IJ} D_\mu \sigma + \frac{1}{2} D_{IJ} + t_{IJ, \sigma} \right) \Xi^{IJ}
$$

$$
+ \frac{i}{4} (-v_\nu F^{\nu \mu} + v^5 F^{5 \mu} - s D^\mu \sigma + 2w_{IJ} t_{IJ, \sigma}) D_\mu c + \bar{c} G + \bar{c} B_0 + c a_0,
$$

(4.37)
This $D_{10}$ is different from the one for the round $S^4$ in [22] only by the terms $\partial_5 A_\mu$. Then, the principal symbol of $D_{10}$ is modified. However, if we Fourier-expand $S^1$-direction and think that the theory is on $S^4$ with the Kaluza-Klein towers, we can see that the principal symbol of $D_{10}$ is same as the one for $S^4$ because $\partial_5$ is regarded as a constant, not a differential. Therefore we can apply the index theorem for the transversally elliptic operator to each Kaluza-Klein (KK) momentum. The $Q^2 = (\delta_5)^2$ is modified by replacing $-i[a_0, *]$ to $D_5(*) = \partial_5(*) - i[a_0, *]$ except for $B_0, \bar{a}_0$ which have zero mode only. Therefore, the 1-loop determinant is just given by the product of the one for the vectormultiplets on $S^4$ for the KK tower with replacement of $a_0 \cdot \alpha$ to $n \frac{R}{4}$ where $\alpha$ is the root of the gauge group and the $n$ is the integer KK momentum.

For the hypermultiplets, the auxiliary fields can be integrated out trivially. The $D_{10}$ is modified only by adding term like $D_0q$ which is not a leading term in the symbol. Thus, for the hypermultiplets the 1-loop determinant is obtained from the one for the theory on $S^4$ as for the vectormultiplet.

Finally, the instanton contribution at $s = \pm 1$, i.e. at the north and south pole of $S^4$, is expected to be the Nekrasov’s partition function $Z_{\text{inst}}(a_0, \epsilon_1 = \frac{1}{7}, \epsilon_2 = \frac{1}{7}, \tau = 0, \beta = R)$ for 5D space. Therefore, by rescaling $a_0 \rightarrow a_0/l$ our final expression for the partition function on $S^4 \times S^1$ obtained from [2] [22] is

$$Z_{S^4 \times S^1} = \int \frac{da_0}{Z_{\text{inst}}} \prod_{k \in \mathbb{Z}} \frac{\Upsilon(i a_0 \cdot \alpha + i \frac{Q}{R} k) \Upsilon(-i a_0 \cdot \alpha - i \frac{Q}{R} k)}{\prod_{\rho \in R} \Upsilon(i a_0 \cdot \rho + \frac{Q}{2} + i \frac{Q}{R} k)}$$

(4.38)

where $R$ is the representation of the hypermultiplets under the gauge group, $\rho$ is a weight of $R$ and $Q = b + \frac{1}{b}$ with $b = 1$. Here, according to [22], we used the function $\Upsilon(x) = \prod_{n_1, n_2 \geq 0} (n_1 + n_2 + x)(n_1 + n_2 + 2 - x)$ to express the regularized infinite product.

The integration variables $a_0$ is periodic, i.e. $a_0 \sim a_0 + \frac{l}{R} H_i$, and the integrand of (4.38) is indeed periodic under this. Taking $R \rightarrow 0$ limit, (4.38) reduces to the partition function for the theory on $S^4$ [2] [22] times a numerical factor. We have introduced the mass term which affects the SUSY transformation by $\langle A_5 \rangle$. This also enters the expression of the partition function with some shift of the mass [26].

5 Conclusion

In this paper we have constructed supersymmetric gauge theory on $S^4 \times S^1$. We have found there is no analogue of the usual Yang-Mills action except in the 4D limit. It should be noted that We have applied the localization technique to the partition function of the theory on $S^4 \times S^1$ following [21] [22] and find the result is a simple extension of the corresponding partition function on $S^4$ with the contributions from Kaluza-Klein modes.

To extend our work to the ellipsoid [22] will be straightforward. This will give the result with $b \neq 1$. In this paper, we only computed the partition function. Of course, it is
interesting to compute the Wilson loop, the ’t Hooft loop \cite{27} and other operators. (The Wilson loop operators in the maximally SUSY Yang-Mills theory was considered in \cite{28})

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Note added:

As this article neared completion, we became aware of the very interesting preprint \cite{29} where the 5d superconformal index on $S^4 \times S^1$ was calculated and the enhancement of global symmetry was checked. The part of their result concerning the SUSY transformation for the vectormultiplets and the localization computations presumably coincide with ours.

A  Formula for bilinears of Killing spinors

We can show some relations between the bilinears of Killing spinors \cite{15}. Here we present them in the form valid for any 5d space. First,

$$\Gamma_m \xi_I \cdot v^m = s \xi_I ,$$

which implies

$$v_m v^m = s^2 .$$

Others including $w_{IJ}^{mn}$ are

\begin{align*}
0 &= -w_{mm IJ} w_{KL}^{mn} + s^2 (\epsilon_{IJ} \epsilon_{KL} + 2 \epsilon_{IL} \epsilon_{JK}) , \\
0 &= 2s (\epsilon_{IJ} \epsilon_{KL} + 2 \epsilon_{IL} \epsilon_{JK}) v^p + 2 v_m (\epsilon_{IJ} w_{KL}^{pm} - \epsilon_{KL} w_{IJ}^{pm}) \\
&\quad + \epsilon^{pqmnq} w_{mn IJ} w_{qr KL} , \\
0 &= s (8 \epsilon_{JK} w_{LJ}^{pq} - 2 \epsilon_{JI} w_{LK}^{pq} - 2 \epsilon_{LK} w_{IJ}^{pq}) \\
&\quad + \epsilon_{JI} v_m \epsilon^{pqmrs} w_{rs KL} + \epsilon_{JK} v_m \epsilon^{pqmrs} w_{rs IJ} \\
&\quad - 4 (w_{IJ}^{nm} w_{KL}^p - w_{IJ}^{np} w_{KL}^{nm}) .
\end{align*}
These implies

\[ 0 = v_m w_{KL}^{pm} , \quad (A.6) \]
\[ 0 = 2 w_{pq}^{IJ} + v_m \epsilon^{pqrst} w_{rsIJ} , \quad (A.7) \]
\[ 0 = w_{KL}^{mn} I I w_{mn,J} + 3 s^2 \epsilon_{JK} , \quad (A.8) \]
\[ 0 = 6 \epsilon_{JK} s w^P - \epsilon^{PMNQR} w_{MN} w_{QRK} , \quad (A.9) \]
\[ 0 = -2 s w_{pq}^{JK} + \epsilon^{IL} (w_{IJ} w_{nKL} + w_{IK} w_{nLJ}) , \quad (A.10) \]
\[ \epsilon^{IL} (w_{IJ} w_{KLM} + w_{IJ} w_{KLM}) = 3^2 \epsilon_{IK} (s^2 g^{pq} - v^p v^q) , \quad (A.11) \]
\[ \epsilon^{JK} w_{KL}^{pm} w_{nJL} = s w_{pq}^{IJ} + \frac{3}{4} (s^2 g^{pq} - v^p v^q) \epsilon_{IJ} . \quad (A.12) \]

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