Explicit $N = 2$ supersymmetry for higher-spin massless fields in $D = 4$ AdS superspace.

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Abstract

We develop two $N = 2$ superfield formulations of free equations of motion for the joint model of all $D = 4$ massless higher-superspin fields in generating form. The explicit $Osp(2|4)$ supersymmetry is achieved without exploiting the harmonic superspace, and with adding no auxiliary component fields to those of $N = 1$ superfields. The formulations are developed in two different $Osp(2|4)$ homogeneous superspaces which have a structure of a fibre bundle over the standard $D = 4$ AdS superspace, with dimensions $(7|4)$ and $(7|8)$. The $N = 2$ covariant derivatives in these spaces are expressed in terms of $N = 1$ ones which gives simple rules for component analysis.

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0 Introduction.

Not long ago, a universal formulation for the linearized dynamics of infinite system of higher spin massless fields on anti-de Sitter space has been suggested \[1\], where the gauge-invariant action is formulated in terms of two real and one complex unconstrained superfields on \(\tilde{M}^{7|4} = \text{super}AdS_4 \times H_3\), where \(\text{super}AdS_4\) is \(N = 1\) anti-de Sitter space and \(H_3\) is \(\mathbb{R}^{3,1}\) one-sheeted hyperboloid. The possibility to develop such a formulation arose due to special properties of \(N = 1\) AdS superfields along with the special choice of the spin spectrum including every \(N = 1\) superspin with multiplicity one. This spectrum structure coincides with that of the Vasiliev’s higher-spin interaction theory for \(D = 4\) massless higher spin fields \[2\], a significant breakthrough in the higher spin interaction problem. It is not known up to now, whether the equations of \[2\] are lagrangian or not.

The latter circumstance has urged some authors to try developing alternative approaches. In particular, many efforts were applied to construct different generating formulations, usually using auxiliary operators of creation and annihilation \[3\]. In this respect the formulation of \[1\] is distinguished as it is based on the commutative algebra and gives, in a sense, a theory, formulated in extended supermanifold. This property seems to be more suitable for describing nonlinear structures.

Another advantage of the approach undertaken in \[1, 4, 5\] is the manifest \(Osp(1|4)\) covariance. Although the equations of motion of \[2\] have clear gauge structure, the question of global symmetry of higher-spin fields has never been sufficiently studied. The hypothetical symmetry of the actions \[3\] under the Fradkin-Vasiliev superalgebra of higher spins \[1\] has never been observed even on the free level. Moreover, it is known that, in the MacDowell-Mansouri approach used in \[1\], even the usual global supersymmetry transformations of supergravity are very subtle. Contrary, in the generating superfield approach the algebra of transformations of \(N = 2\) supersymmetry closes off-shell \[1, 5\] and (as we prove here) without the breaking terms proportional to gauge transformations. Therefore, it is natural to set the question of a manifestly \(N = 2\)-covariant form of GKS action.

An additional motivation to develop such a formulation comes from the study of \(N = 2\) superfield theories for Poincaré super Yang-Mills and gravity in harmonic superspace \[3, 10\], which present themselves a new type of supergeometries which have no analogues in \(N = 1\) case. Here it is worth noting that in harmonic superspace infinitely many auxiliary or purely gauge component fields were introduced to achieve an explicit \(N = 2\) covariance. The model considered in the present paper also contains infinitely many component fields, because it describes all massless superspins. Nevertheless, for every superspin, the number of components is finite. Moreover, the formulation with explicit \(N = 2\) supersymmetry has neither auxiliary nor purely gauge components in addition to those of \(N = 1\) superfield version. Besides, to our knowledge, the \(N = 2\) superfield models in AdS superspace have not been considered before. The \(N = 2\) AdS supergeometry differs essentially from the flat one because \(osp(2|4)\) algebra does not possess the internal \(su(2)\) subalgebra (rotating supersym-
metry generators) crucial for the idea of harmonic superspace. Then, the known superfield formulations of $N = 2$ supergravity contain nonminimal formulation of $N = 1$ supergravity (see also the case of superspin $3/2$ in [8]). But the description of AdS superspace in the nonminimal formulation is unknown (see e.g. [11]). This can cause problems in a description of $N = 2$ AdS supergeometry in the framework of the conventional approach.

One may expect that the manifest $N = 2$ supersymmetry property may help to uncover the geometry of higher-spin gauge symmetries which is crucial for the non-linear action construction. That is why the superfield formulation developed in the present paper may be taken as a starting point to analyze the interaction problem and to construct a consistent nonlinear action.

In this paper, we construct a manifestly $N = 2$ covariant formulation for the equations of motion of GKS theory [1]. We construct the $N = 2$ superspaces $M^{7|4} = Osp(2|4)/\mathcal{H}$ (where $\mathcal{H}$ is the subgroup of $Osp(2|4)$ corresponding to the subalgebra $su(2,0|1,0) \subset osp(2|4)$) and $M^{7|8} = Osp(2|4)/SU(2) \oplus U(1)$ (differing by the number of odd coordinates) and present two equivalent forms of GKS equations written in terms of superfields on these superspaces.

The paper is organized as follows. In Section 1, we describe the $N = 2$ AdS superalgebra $osp(2|4)$ and its subalgebras $su(2,0|1,0)$ and $su(2) \oplus u(1)$ and exhibit the decomposition of $osp(2|4)$ w.r.t. these subalgebras. In Section 2, we recall the GKS theory and reformulate it by a twist which makes the $N = 2$ transformations being realized by local vector fields on $M^{7|4}$, we also observe that the $N = 2$ transformation laws may be improved by new terms, proportional to the equations of motion, what provides the fields transformations with a more transparent form. Remarkably, it appears that such transformations preserve just the GKS equations of motion and not the action. But it is this form of $N = 2$ transformations which is derived from the manifestly $N = 2$-covariant formalism introduced in the subsequent sections. In Section 3, the covariant derivatives on homogeneous spaces $M^{7|4} = Osp(1|4)/SU(2)$, $M^{7|4} = Osp(2|4)/\mathcal{H}$ and $M^{7|8} = Osp(2|4)/SU(2) \oplus U(1)$ are constructed in terms of covariant derivatives on $N = 1$ AdS superspace superAdS$_4 = Osp(1|4)/SO(3,1)$ and the so called ”small vielbein” field. It gives a possibility to study various $N = 2$ superfields and equations. In Section 4, the special ”strongly chiral” constrained superfields on $M^{7|8}$ are studied and their component content is exhibited in terms of $N = 1$ superfields on $M^{7|4}$ superfields. In Section 5, the GKS equations are equivalently reformulated in terms of two strongly chiral fields on $M^{7|8}$. This gives the anticipated manifestly $N = 2$ covariant formulation for the GKS equations of motion. In Section 6, we show that all the stuff may be reformulated once again in terms of superfields on $M^{7|4}$. The obtained equations of motion describe dynamics of superfields with indices, transforming under a local supergroup. In Conclusion, we discuss the results. We supply our paper with a few appendices carrying the information of technical character to be referred in the main text. Appendix A contains general facts about covariant derivatives on homogeneous spaces. The short Appendix B introduces our supertensor notation. Appendix C describes the $A(1|0)$ superalgebra and a brief study
of its finite-dimensional representations properties.

Let us describe our main notation. The letters $\alpha, \beta, \ldots \dot{\alpha}, \dot{\beta} \ldots$ describe $SL(2, C)$ spinor undotted and dotted indices. The letters $a, b, c, \ldots$ describe $su(2)$ spinor indices or $SO(3, 1)$ vector ones, depending on a situation. The letters $\xi, \eta, \ldots, \xi, \bar{\eta} \ldots$ describe the indices of two conjugated $A(1|0)$ (1|2)–dimensional representations.

We consider only Lorentz tensors symmetric in their undotted indices and separately in dotted ones. A tensor of type $(k, l)$ with $k$ undotted and $l$ dotted indices can be equivalently represented as $\psi(k, l) \equiv \psi_{(a_1 \ldots a_k)}(\dot{a}_1 \ldots \dot{a}_l) \equiv \psi_{(\alpha_1 \ldots \alpha_k)}(\dot{\alpha}_1 \ldots \dot{\alpha}_l)$. The indices, which are denoted by one and the same letter, should be symmetrized separately with respect to upper and lower indices; after the symmetrization, the maximal possible number of the upper and lower indices denoted by the same letter are to be contracted. In particular $\phi_{(\alpha k)}(\psi_{\alpha l}) \equiv \phi_{(\alpha_1 \ldots \alpha_k)}(\psi_{\alpha_{k+1} \ldots \alpha_{k+l}})$ and $\xi^a(\psi_{\alpha k}) \equiv \xi^a(\phi_{(\alpha_1 \ldots \alpha_{k+1})})$. The similar notation, with a proper account of grading, takes place in the case of supertensors, see Appendix B. The two-dimensional antisymmetric $\varepsilon$ tensors are defined such that $\varepsilon^{12} = -\varepsilon_{12} = 1$.

The notation $(P|Q)$ stands for the dimension of a superspace with $P$ even and $Q$ odd coordinates. $[A, B] = AB - (-)^{(A)(B)}BA$ stands for graded commutator while $[A, B]_- = AB - BA$ is an ordinary commutator.

1 $N = 2$ AdS superalgebra $osp(2|4)$ and its subalgebras.

In this section, we provide the information about the structure of $osp(2|4)$ superalgebra and its decompositions with respect to some subalgebras, this information is being used extensively below.

The $N = 2$ AdS superalgebra $osp(2|4)$ is defined via the following graded commutation relations

$$[D^i_A, D^j_B] = \delta^{ij} \Sigma_{AB} - i\delta^{ij}C_{AB}\Gamma$$  \hspace{1cm} (1.1)

$$[\Gamma, D^i_A] = -i\delta^{ij}\delta_{jk}D^k_A$$  \hspace{1cm} (1.2)

$$[\Sigma_{AB}, D^i_C] = C_{CA}D^i_B + C_{CB}D^i_A$$  \hspace{1cm} (1.3)

$$[\Sigma_{AB}, \Sigma_{CD}] = C_{CA}\Sigma_{BD} + C_{CB}\Sigma_{AD} + C_{DA}\Sigma_{BC} + C_{DB}\Sigma_{AC}$$  \hspace{1cm} (1.4)

$$[\Sigma_{AB}, \Gamma] = 0,$$  \hspace{1cm} (1.5)

Here $A, B, \ldots = 1, 2, 3, 4$ are $so(3, 2)$ spinor representation indices; $i, j, \ldots = 1, 2$ are $so(2)$ vector ones; Kronecker delta-symbol $\delta^{ij} = \delta_{ij}$ and antisymmetric tensor $\varepsilon^{ij} = -\varepsilon_{ij}$ are two-dimensional $SO(2)$-invariant tensors. The matrix $C_{AB}, C_{BA}, C^2 = -1$ is $so(3, 2)$ charge conjugation matrix, it’s explicit form is exhibited below (Eq. (1.1)–(1.9)). The $so(3, 2)$ generators $\Sigma_{AB}$ are symmetric in spinor indices, $\Sigma_{AB} = \Sigma_{BA}$. The generators $\Sigma_{AB}$ and $\Gamma$ are even while $D^i_A$ are odd.

We are interested in the decomposition of $osp(2|4)$ with respect to the maximal compact subalgebra of its even part $su(2) \oplus so(2) \oplus so(2)$, where $su(2) \oplus so(2)$ is the
maximal compact subalgebra of $so(3, 2)$ and one more $so(2)$ is associated to the $\Gamma$ generator. The decomposition is performed quite simply by employing the fact that the spinor representation of $so(3, 2)$ is decomposed into the direct sum $D(\frac{1}{2})_+ \oplus D(\frac{1}{2})_-$ w.r.t. $su(2) \oplus so(2)$ subalgebra, the "+", "−" subscripts denote the $so(2)$ weights. Therefore, the spinor generators $D^i_A$ are classified by their $su(2) \oplus so(2)$ indices and by additional $so(2)$ weight associated to the $\Gamma$ generator:

$$D^i_A = \{D^+_a, D^-_a, D^+_\pm a, D^-_\pm a\},$$

(1.6)

where the upper index labels the $\Gamma$-weight:

$$D^\pm_A = \frac{1}{\sqrt{2}}(D^1_A \pm iD^2_A)$$

(1.7)

and the lower indices correspond to $su(2) \oplus so(2)$ representation, $a, b, \ldots$ are the $su(2)$ spinor indices.

According to (1.1), the above classification induces the decomposition of the even generators into subsets

$$\Sigma_{++(ab)} = [D^+_a, D^+_b], \quad \Sigma_{--(ab)} = [D^-_a, D^-_b]$$

(1.8)

$$\Sigma_{+-ab} = [D^+_a, D^-_b] \equiv S_{(ab)} + \varepsilon_{ab}(\Gamma - \frac{1}{2}S)$$

(1.9)

$$\Sigma_{-+ab} = [D^-_a, D^+_b] \equiv S_{(ab)} + \varepsilon_{ab}(\Gamma + \frac{1}{2}S).$$

(1.10)

To establish the precise form of the rest $osp(2|4)$ commutation relations in this basis we need to know how the matrices $C_{AB}, \delta^{ij}, \varepsilon^{ij}$ look like. The answer is

$$C_{a+b} = C_{-a-b} = 0$$

(1.11)

$$C_{a-b} = C_{-a+b} = -\varepsilon_{ab}$$

(1.12)

$$\varepsilon^{+-} = -i; \quad \varepsilon^{++} = \varepsilon^{--} = 0$$

(1.13)

$$\delta^{+-} = 1; \quad \delta^{++} = \delta^{--} = 0,$$

(1.14)

and, therefore, the whole system of $osp(2|4)$ commutators consists of Eqs. (1.8–1.10) and

$$[S_{a(2)}, S_{b(2)}] = 2\varepsilon_{ab}S_{ab}$$

(1.15)

$$[S_{a(2)}, D^\pm_{\pm a}] = \varepsilon_{ab}D^\pm_{\pm a}, \quad [S_{a(2)}, D^\pm_{\mp a}] = \varepsilon_{ab}D^\pm_{\mp a}$$

(1.16)

$$[S, D^\pm_{\pm a}] = \pm D^\pm_{\pm a}; \quad [S, D^\pm_{\mp a}] = \pm D^\pm_{\mp a}$$

(1.17)

$$[\Gamma, D^\pm_{\pm a}] = \pm D^\pm_{\pm a}; \quad [\Gamma, D^\pm_{\mp a}] = \pm D^\pm_{\mp a}$$

(1.18)

$$[\Sigma_{++a(2)}, \Sigma_{--b(2)}] = 4\varepsilon_{ab}(S_{ab} - \frac{1}{2}\varepsilon_{ab}S)$$

(1.19)

$$[\Sigma_{\pm \pm a(2)}, D^+_b] = 2\varepsilon_{ab}D^+_b; \quad [\Sigma_{\pm \pm a(2)}, D^-_b] = 2\varepsilon_{ab}D^-_b.$$
The other commutators vanish. The $su(2) \oplus so(2) \oplus so(2)$ subalgebra is spanned by generators $S_{ab} \ (S_{ab} = S_{ba}), S, \Gamma$, where $S_{ab}$ are $su(2)$ generators and $S, \Gamma$ are $so(2)$ ones.

In what follows, we use the usual complex conjugation notations adopted in supersymmetry literature (see e.g. [12]): complex conjugation does not transpose operators but gives the additional minus sign for a product of odd operators:

$$(D_I D_J)^\dagger = (-)^{IJ} D_I^\dagger D_J^\dagger. $$

In this notation, the complex conjugation rules consistent with commutation relations $(1.8-1.20)$ look like

$$(D^\pm_a)^\dagger = -D^\mp_a, \quad (D^\mp_a)^\dagger = D^\pm_a. $$

This complex conjugation extracts $osp(2|4)$ superalgebra from its complexification.

Now it is seen that the complexification of superalgebra $osp(2|4)$ contains two $A(1|0)$ subalgebras (see Appendix C, Eq. (C.1)) spanned by

$$(1.24) \quad S_{ab}, \ T = \Gamma - \frac{1}{2} S, \ D_{+a} = D^{+a}, \ D_{-a} = D^{-a} $$

which intersect over $S_{ab}$ generators.

These subalgebras are invariant w.r.t. conjugation $(1.21, 1.23)$ that extracts their $su(2,0|1,0)$ real forms with the even part being isomorphic to $su(2) \oplus so(2)$. Also, they are connected to each other by $osp(2|4)$ automorphism $\Gamma' = -\Gamma, \Sigma'_{AB} = \Sigma_{AB}, D'^\pm_A = D^\mp_A$, therefore we can choose one of them without an information loss.

Let’s choose the subalgebra $(1.24)$ and study the decomposition of $osp(2|4)$ adjoint representation w.r.t. it. One finds a direct sum of graded subspaces: the first one is $su(2,0|1,0)$ itself

$$(1.26) \quad \mathcal{H}_0 = \left\{S_{ab}, \ \Gamma - \frac{1}{2} S, \ D^{+a}, \ D^{-a} \right\}, $$

the second and the third ones are mutually conjugated subspaces

$$(1.27) \quad \tilde{\mathcal{H}} = \left\{\Sigma_{++(ab)}, \ D_{+a} \right\} \quad \text{and} \quad \bar{\mathcal{H}} = \left\{\Sigma_{--(ab)}, \ D_{-a} \right\},$$

eventually, the forth selfconjugated subspace is $(1|0)$ dimensional:

$$(1.28) \quad \check{\mathcal{H}} = \left\{D \equiv \frac{1}{2}(\Gamma - S) \right\}. $$

6
In Appendix C, the finite-dimensional representations (fdr) of \( su(2, 0|1, 0) \) are studied to some depth and two conjugated \((1|2)\)-dimensional "minimal" representations \( \phi_\xi \) and \( \phi_\xi^\bar{\epsilon} \) are introduced (Eqs. (C.2-C.4)). Their tensor products are useful for the description of various fdr. In particular, the subspaces \( H_0, \tilde{H}, \bar{H}, \check{H} \) transform under \( su(2, 0|1, 0) \) according to

\[
H_0 = \mathcal{E}_\varsigma = \begin{pmatrix}
S_{ab} - \varepsilon_{ab} T & -D_{+a} \\
D_{-a} & 2T
\end{pmatrix}, \quad C^{\varsigma\varsigma}_\xi \mathcal{E}_\varsigma = 0,
\]

(1.29)

\[
\check{H} = D_\varsigma = \begin{pmatrix}
-\Sigma_{-a}^{--ab} & -D_{+a} \\
D_{-a} & 0
\end{pmatrix}, \quad D_\varsigma = -(-)^\varsigma D_{\varsigma^\epsilon},
\]

(1.30)

\[
\bar{H} = D_{\varsigma^\epsilon} = \begin{pmatrix}
-\Sigma^{++ab} & -D_{-a} \\
D_{+a} & 0
\end{pmatrix}, \quad D_{\varsigma^\epsilon} = -(-)^{\varsigma^\epsilon} D_{\varsigma^\epsilon},
\]

(1.31)

and for \( \check{H} \) one has the trivial representation.

The whole superalgebra \( osp(2|4) \) may be rewritten in \( su(2, 0|1, 0) \) covariant form (the "\( = \)" symbol stands for the ordinary equality "\( = \)" with a proper account of the sign factors in the r.h.s., see Appendix B):

\[
[\mathcal{E}_\varsigma, \mathcal{E}_\eta] \ [=] \ C_{\varsigma\eta} \mathcal{E}_\epsilon - C_{\epsilon\theta} \mathcal{E}_\eta \]

(1.32)

\[
[\mathcal{E}_\varsigma, D_{\eta}[2]] \ [=] \ 2C_{\varsigma\eta} D_{\epsilon[2]} + 2C_{\eta\varsigma} D_{\epsilon[2]} \]

(1.33)

\[
[\mathcal{E}_\varsigma, D_{\eta}[2]] \ [=] \ -2C_{\epsilon\eta} D_{\epsilon[2]} - 2C_{\epsilon\eta} D_{\varsigma\eta} \]

(1.34)

\[
[\mathcal{E}_\varsigma, D] = 0
\]

(1.35)

\[
[D_{\varsigma[2]}, D_{\eta[2]}] \ [=] \ -4C_{\epsilon\varsigma} \mathcal{E}_{\epsilon\varsigma} - 8C_{\epsilon\varsigma} C_{\varsigma\epsilon} D
\]

(1.36)

\[
[D, D_{\varsigma[2]}] = D_{\varsigma[2]}, \quad [D, D_{\varsigma\eta[2]}] = -D_{\varsigma[2]},
\]

(1.37)

where the indices denoted by the same letters are to be superantisymmetrized with sign factor \( (\mathcal{B}.10) \). The other commutators vanish, like \( [D_{\epsilon[2]}, D_{\varsigma[2]}] = 0 \).

2 Different forms of generating theory

Let us recollect the superfield formulations of free dynamics of higher spin fields in \( N = 1 \) AdS superspace \([3, 4]\). We denote \( 4 + 4 \) coordinates in AdS superspace by \( z \) so that \( \Phi(z) \) is a superfield in this superspace. All the necessary information about AdS superspace is contained in the superalgebra of covariant derivatives \( \mathcal{D}_A \):

\[
\mathcal{D}_A \equiv \{ \mathcal{D}_a \dot{\alpha}, \mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot{\alpha}} \equiv (\mathcal{D}_\alpha)^* \}^1,
\]

(2.1)

\( ^1 \)Here * means usual complex conjugation in Grassman algebra.
exhibited by the expressions\footnote{Remember, that brackets denote supercommutator. For example $[\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot{\alpha}}] = \mathcal{D}_\alpha \bar{\mathcal{D}}_{\dot{\alpha}} + \bar{\mathcal{D}}_{\dot{\alpha}} \mathcal{D}_\alpha$.}

$$
\begin{align*}
[\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot{\alpha}}] &= -2i\mathcal{D}^{\alpha\dot{\alpha}}, & [\mathcal{D}^{\alpha\dot{\alpha}}, \mathcal{D}_{\beta\dot{\beta}}] &= -2\bar{\mu}\epsilon_{\alpha\beta\dot{\alpha}\dot{\beta}} \bar{M}_{\alpha\dot{\alpha}} + \epsilon_{\dot{\alpha}\dot{\beta}M_{\alpha\beta}}, \\
[\mathcal{D}_\alpha, \mathcal{D}_\beta] &= -4\bar{\mu}M_{\alpha\beta}, & [\mathcal{D}_\alpha, \mathcal{D}_{\beta\dot{\beta}}] &= i\bar{\mu}\epsilon_{\alpha\beta\dot{\alpha}\dot{\beta}} \bar{\mathcal{D}}_{\beta}
\end{align*}
$$

and their complex conjugated counterparts. Here $M_{\alpha(2)}$ and $\bar{M}_{\dot{\alpha}(2)}$ are local Lorentz rotations, e.g. $M_{\alpha(2)}\psi_\beta = \epsilon_{\beta\gamma}\psi_\alpha$. More information about the geometry of AdS superspace can be found in \cite{1,4}. Here we need only the fact that this supermanifold is the coset space $Osp(1|4)/SO(3,1)$. The action of the $osp(1|4)$ algebra in AdS superspace can be expressed in terms of covariant derivatives (see Eq.(2.26) below).

In \cite{4,5}, there were found two formulations for each superspin: the ”transversal” and the ”longitudinal” ones. We will use here only the transversal formulation for half-integer superspin $s + 1/2$ and the longitudinal one for integer superspin $s$, denoting them as $S^\perp_{s+1/2}$ and $S^\parallel_s$. It is this choice that is necessary for constructing $N = 2$ supersymmetry transformations \cite{8} and generating formulation \cite{1}. The actions $S^\perp_{s+1/2}$ and $S^\parallel_s$ are expressed in terms of the following dynamical superfields:

$$
\begin{align*}
\mathcal{V}^\perp_{s+1/2} &= \{ H(s,s), \Gamma(s - 1, s - 1), \bar{\Gamma}(s - 1, s - 1) \}, & s &\geq 1 \\
\mathcal{V}^\parallel_s &= \{ H'(s - 1, s - 1), G(s,s), \bar{G}(s, s) \}, & s &\geq 1
\end{align*}
$$

where $H$ and $H'$ are arbitrary real superfields while $\Gamma$ and $G$ are constrained complex superfields, satisfying the following relations

$$
\mathcal{D}^\alpha \Gamma_{(s-1)\dot{\alpha}(s-1)} = 0, \iff (\mathcal{D}^2 - 2(s+1)\mu)\Gamma(s-1, s-1) = 0, \tag{2.5}
$$

$$
\mathcal{D}_{\dot{\alpha}} G_{(s)\dot{\alpha}(s)} = 0, \iff (\mathcal{D}^2 + 2s\mu)G(s,s) = 0 \tag{2.6}
$$

and called transversal linear and longitudinal linear superfields, respectively. These two types of superfields exhaust all possible off-shell constrained superfields with given index structure in AdS superspace. The gauge parameters are superfields of the same kind as the dynamical fields: the action $S^\perp_{s+1/2}$ is gauge invariant w.r.t. transformations with longitudinal linear parameter $g(s,s)$ and the action $S^\parallel_s$ is gauge invariant w.r.t. the ones with transversal linear parameter $\gamma(s - 1, s - 1)$.

To construct the generating formulation let us consider an $SO(3,1)$ four-vector, subjected to one-sheeted hyperboloid:

$$
q^\alpha q_{\alpha} = 1, \quad q^{\alpha\dot{\alpha}} q_{\alpha\dot{\alpha}} = -2.
$$

In \cite{1}, it is proposed to consider spin-tensors (2.3,2.4) as the coefficients arising in expansion of analytic functions in power series via the $q$-vector:

$$
\phi(q) = \sum_{s=0}^{\infty} \phi_{(s)\dot{\alpha}(s)} q^{\alpha(\dot{\alpha})s}, \quad q^{\alpha(\dot{\alpha})s} = q^{\alpha\dot{\alpha}} \cdots q^{\alpha\dot{\alpha}}. \tag{2.7}
$$
These functions can be considered as the elements of commutative algebra with the following composition law for basis elements:
\[
q_{\beta(k)\dot{\beta}(k)}q_{\alpha(k)\dot{\alpha}(k)} = \sum_{n=0}^{\min(k,l)} (-1)^n \frac{c^n_l c^n_k}{c^n_{l+k-n+1}} \varepsilon_{\beta\alpha} \varepsilon_{\dot{\beta}\dot{\alpha}} q_{\beta(k-n)\alpha(l-n)\dot{\beta}(k-n)\dot{\alpha}(l-n)}
\]
where \(c^n_m = \frac{s^m}{m!(s-m)!}\). It successfully turned out to be \([1]\) that the joint content of superfield sets \(\mathcal{V}_{s+1/2}^\perp\) and \(\mathcal{V}_s^\parallel\) for all values \(s \geq 1\), plus two scalar superfields describing superspins 0 and 1/2, is represented by the following functions:
\[
U(z,q) = \sum_{s=0}^{\infty} U_{\alpha(s)\dot{\alpha}(s)}q^{\alpha(s)\dot{\alpha}(s)} \quad U(s,s) = \Gamma(s,s) + G(s,s); \\
Y(z,q) = \sum_{s=0}^{\infty} Y_{\alpha(s)\dot{\alpha}(s)}q^{\alpha(s)\dot{\alpha}(s)} \quad Y(s,s) = H'(s,s) - H(s,s); \\
X(z,q) = \sum_{s=0}^{\infty} X_{\alpha(s)\dot{\alpha}(s)}q^{\alpha(s)\dot{\alpha}(s)} \quad X(s,s) = (s+1)\{H'(s,s) + H(s,s)\};
\]
(2.8)

The gauge parameters are described by the function
\[
\lambda(z,q) = \sum_{s=0}^{\infty} \lambda_{\alpha(s)\dot{\alpha}(s)}q^{\alpha(s)\dot{\alpha}(s)}, \quad \lambda(s,s) = \gamma(s,s) - g(s,s).
\]
(2.12)

The functions \((2.9)\) \((2.12)\) depend both on the vector \(q\) and on the coordinates of AdS superspace \(z\), we will call them superfields. Note that the dependence of \(U, Y, X\) and \(\varepsilon\) on their arguments is arbitrary (except for analyticity in \(q\) and reality of \(Y\) and \(X\)) though superfields \(G, \Gamma, g\) and \(\gamma\) are linear \((2.5)\) \((2.6)\). This is due to the fact that, in AdS superspace, the sum of transversal and longitudinal linear spin-tensor superfields gives an arbitrary superfield \([1]\) \([4]\).

To derive an action for these superfields, the Lorentz-invariant trace is introduced in \([1]\) to the algebra of analytic functions of vector \(q\):
\[
\text{tr} \phi = \phi(0,0) \implies \text{tr} (\phi \cdot \psi) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s+1} \phi^{\alpha(s)\dot{\alpha}(s)}\psi_{\alpha(s)\dot{\alpha}(s)};
\]
(2.13)

It is also convenient to use the following scalar operators:
\[
Q = q^{\alpha\dot{\alpha}} D_{\alpha} \tilde{D}_{\dot{\alpha}}, \quad \bar{Q} = -q^{\alpha\dot{\alpha}} \tilde{D}_{\alpha} D_{\dot{\alpha}} = Q^*, \\
\mathcal{P} = \frac{D^2}{2\mu} = \frac{1}{2\mu} D^\alpha D_{\alpha}, \quad \bar{\mathcal{P}} = \frac{\tilde{D}^2}{2\mu} = \mathcal{P}^*.
\]
(2.14)

These operators, quadratic in derivatives, satisfy the following relations due to \((2.3)\)
\[
Q\bar{P} = -(\mathcal{P} - 1)Q
\]
(2.15)
\[^3\text{In this section, the local \(so(3,1)\) rotations \(M_{\alpha(2)}\) and \(\bar{M}_{\dot{\alpha}(2)}\) from connection terms of covariant derivatives \(D_A\) do not act on the indices of vector \(q^{\alpha\dot{\alpha}}\).
\[ \mathcal{QP} = -(\mathcal{P} + \mathcal{P} - 2)\mathcal{Q} + (\mathcal{P} + 1)\mathcal{Q} \]  

(2.16)

It was noted in [1] that the real superfield \( Y \) is purely gauge and the gauge \( Y = 0 \) can be imposed and substituted into the action. Then the joint action of superfields of all superspins (including nongauge massless fields of low spins 0, 1/2) takes a simple form:

\[
S = \frac{1}{2}\text{tr} \int d^8z \, E^{-1} \{ -\mu \bar{\mu} X^2 + \frac{1}{2} X (QU + \bar{Q}U^*) - \frac{1}{2} U^* (\mathcal{P} + \mathcal{P} - 2) U \bar{U} - \frac{1}{2} U^* \bar{U} \mathcal{P} U \},
\]

(2.17)

where \( d^8z \, E^{-1} \) is AdS superinvariant measure. The action is invariant with respect to the following gauge transformations with real gauge parameter \( \varepsilon(z,q) \):

\[
\delta X = -i(\bar{P} - P) \varepsilon, \quad \delta U = i\bar{Q} \varepsilon, \quad \delta U^* = -iQ \varepsilon,
\]

(2.18)

This gauge invariance can be checked straightforwardly with the use of Eq.(2.16). It is remarkable that the corresponding calculation for a separate superspin [5] is much more complicated. \( N = 2 \) supersymmetry transformations of Ref.[8] can also be brought to generating form. For this purpose one should express them via the Killing scalar parameter \( t(z) \) \[1\]:

\[
t = \bar{t}, \quad (\mathcal{D}^2 - 4\bar{\mu})t = 0, \quad \mathcal{D}_a \bar{\mathcal{D}}_a t = 0,
\]

(2.19)

containing one spinor and one scalar constants which are exactly the parameters of transformations necessary to complete the manifestly realized \( N = 1 \) AdS superalgebra to the \( N = 2 \) one:

\[
\delta t Y = 0, \quad \delta t X = 2i[\bar{P}, t] \mathcal{R} U + \text{c.c.},
\]

\[
\delta t U = \frac{i}{2} \mu \bar{\mu} [\mathcal{P} + \mathcal{P}, t] \mathcal{R} X + \frac{i}{2} \mu \bar{\mu} (\mathcal{P} - 1)[\mathcal{P} + \mathcal{P}, t] \mathcal{R} Y + iq^{\alpha\dot{\alpha}} (\mathcal{D}_a t) \bar{\mathcal{D}}_{\dot{a}} \mathcal{R}(U + U^*).
\]

(2.20)

Here the new operation arises which inverts the odd powers of \( q^{\alpha\dot{\alpha}} \), \( \mathcal{R} \Phi(q) = \Phi(-q) \). Since the variation \( \delta t Y \) vanishes, the other variations \( \delta t X \) and \( \delta t U \) do not change in the gauge \( Y = 0 \).

Note that the actions \( S_{s+1/2}^\perp \) and \( S_s^\parallel \) for higher spin superfields contain the factor \((-1)^s\) to supply a positive Hamiltonian \[3\]. In generating formulation (2.17), this sign alternation arises due to relation \( q^{\alpha\dot{\alpha}} q_{\alpha\dot{\alpha}} = -2 \) what provides \((-1)^s\) in (2.13). Here we observe that there is another possible generating formulation in which superfields \( 2.3 \) are decomposed in functions on two-sheeted hyperboloid, parametrized by a four-vector \( r^{\alpha\dot{\alpha}} \), that can be chosen to be proportional to \( q \):

\[
r^{\alpha\dot{\alpha}} r_{\alpha\dot{\alpha}} = 2, \quad r^a r_a = -1, \quad q = ir.
\]

(2.21)

We will see that, in this formulation, the operation \( \mathcal{R} \) passes from the global symmetry transformations (2.20) to the action and the sign alternation still remains. It is
convenient here to define the new operation of complex conjugation † which acts in just the same way on the superfields not depending on \( q \) and \( r \) as the operation * used before, while their actions on \( q \) and \( r \) are opposite:
\[
\begin{align*}
q^* &= q, \quad r^* = -r, \\
q'^* &= -q, \quad r'^* = r.
\end{align*}
\]

The fact that * and † act in one and the same way on \( q \)-independent superfields means that if an action functional is real with respect to * then it is real with respect to † either.

The main goal of our reformulation \( q \rightarrow r \) is to choose a more adequate manifold from the global symmetry point of view. We will see that, in this new formulation, \( \text{Osp}(2|4) \) transformations are realized by local operators without \( R \) operation. Moreover, the new supermanifold parametrized by \((z, r)\) is the homogeneous superspace \( M^{7|4} = \text{Osp}(2|4)/H \) where the supergroup \( H \) is defined in the introduction and in subsect 3.2. The \( \text{Osp}(1|4) \)-covariant form of vector superfields that give the action of \( \text{osp}(2|4) \) superalgebra can be constructed with the use of Killing vector:
\begin{align*}
&k_{\alpha\dot{\alpha}}(z) = \bar{k}_{\dot{\alpha}\alpha}, \quad \mathcal{D}_{\alpha} k_{\alpha\dot{\alpha}} = 0, \quad \mathcal{D}^{\alpha}\bar{\mathcal{D}}^{\dot{\alpha}} k_{\alpha\dot{\alpha}} = 0, \quad (2.23)
\end{align*}
and Killing scalar \( t \) (2.19). It is convenient to introduce the following notation for all linearly independent derivatives of these Killing superfields:
\begin{align*}
&k_{\alpha} = \frac{i}{8}\bar{\mathcal{D}}^{\dot{\alpha}} k_{\alpha\dot{\alpha}}, \quad \bar{k}_{\dot{\alpha}} = (k_{\alpha})^\dagger, \\
&k_{\alpha(2)} = \mathcal{D}_{\alpha}k_{\alpha}, \quad \bar{k}_{\dot{\alpha}(2)} = (k_{\alpha(2)})^\dagger, \quad (2.24)
&t_{\alpha} = \frac{1}{2}\mathcal{D}_{\alpha}t, \quad \bar{t}_{\dot{\alpha}} = (t_{\alpha})^\dagger. \quad (2.25)
\end{align*}
Then the vector superfield giving the action of \( \text{osp}(1|4) \) superalgebra can be written as usual (see e. g. [11]),
\[
\mathcal{K} = -\frac{1}{2}k^{\alpha\dot{\alpha}}\mathcal{D}_{\alpha\dot{\alpha}} + k^{\alpha}\mathcal{D}_{\alpha} + \bar{k}^{\dot{\alpha}}\bar{\mathcal{D}}^{\dot{\alpha}} + k^{\alpha(2)}M_{\alpha(2)} + \bar{k}^{\dot{\alpha}(2)}\bar{M}_{\dot{\alpha}(2)} = \mathcal{K}^\dagger = \mathcal{K}^*. \quad (2.26)
\]
It appears that the rest vectors of \( \text{osp}(2|4) \) are parametrized by Killing scalar \( t \),
\[
\mathcal{M} = 2ir^{\alpha\dot{\alpha}}(t_{\alpha}\bar{D}_{\dot{\alpha}} + \bar{t}_{\dot{\alpha}}D_{\alpha}) + 2it\nabla, \quad \mathcal{M}^\dagger = \mathcal{M} = -\mathcal{M}^*, \quad (2.27)
\]
where \( \nabla \) is an imaginary vector field
\[
\nabla = -\frac{i}{4}(Q - \bar{Q}) = -\frac{i}{2}r^{\alpha\dot{\alpha}}\mathcal{D}_{\alpha\dot{\alpha}} \quad (2.28)
\]
It can be straightforwardly checked that a commutator of two operators \( \mathcal{M} \) closes on \( \mathcal{K} \),
\[
[\mathcal{M}(t), \mathcal{M}(t')] = -\mathcal{K}(k), \quad k_{\alpha\dot{\alpha}} = 16i(t_{\alpha}\bar{t}_{\dot{\alpha}} + \bar{t}_{\dot{\alpha}}t_{\alpha}). \quad (2.29)
\]
The conclusion that the coordinates \((z,r)\) parametrize superspace \(Osp(2|4)/\mathcal{H}\) will be proved by the construction of \(Osp(2|4)\)-covariant derivatives with the proper algebra (see Sec. [3]). Thus, vector superfields \(\mathcal{M}\) and \(\mathcal{K}\) give the action of \(osp(2|4)\) superalgebra.

Depending on \(r\) are new superfields \(Z = Z^\dagger\) and \(V\) whose contents coincide with the contents of \(X\) and \(U\), respectively.

\[
Z(z, r) = \sum_{s=0}^{\infty} r^{\alpha(s)\dot{\alpha}(s)} Z_{\alpha(s)\dot{\alpha}(s)}, \\
V(z, r) = \sum_{s=0}^{\infty} r^{\alpha(s)\dot{\alpha}(s)} V_{\alpha(s)\dot{\alpha}(s)},
\]

\[
Z(s, s) = (-1)^{[\lambda]} X(s, s), \\
V(s, s) = (-1)^{[\lambda]} U(s, s),
\]

where \([\lambda]\) denotes the integral part of \(\lambda\). Note that, even though \(q\) is connected to \(r\) by (2.21), and superfields \(Z, V, X, U\) have the identical content, they are not proportional. Therefore, the new formulation cannot be obtained by a trivial change like \(X \rightarrow \alpha Z, U \rightarrow \beta V, q \rightarrow ir\) in the action (2.17). Nevertheless, it is not difficult to check that every term in the action (2.17) can be equivalently rewritten in new formulation with the use of operation \(R\):

\[
S = \frac{1}{2} \text{tr} \int d^8z E^{-1} \left\{ -\mu \bar{\mu} Z R Z - \frac{i}{2} Z R (Q V + \bar{Q} V^\dagger) - \frac{1}{2} V^\dagger (P + \bar{P} - 2) RV - \frac{1}{2} V^\dagger P R V - \frac{1}{2} V^\dagger P R V^\dagger \right\},
\]

where 'tr' is the same operation as in (2.13): \(\text{tr}\Phi(r) = \text{tr}\Phi(-iq) \equiv \Phi(0, 0)\), \(Q\) and \(\bar{Q} = -Q^\dagger\) are the same operators (2.14) with \(ir\) being substituted for \(q\).

Let us turn to \(N = 2\) supersymmetry transformations. It is shown in Ref. [8] that their algebra closes off-shell. Nevertheless, this algebra is broken by the terms proportional to gauge transformations (see [1] for the generating form of this algebra). Here we observe that it can be closed eventually by adding certain gauge transformation to (2.20). The structure of the global symmetry becomes more clear after splitting \(V\) into its real and imaginary parts (with respect to \(\dagger\)) \(V = \gamma + i\rho\). Then the action (2.32) reads

\[
S = \frac{1}{2} \text{tr} \int d^8z E^{-1} \left\{ -\mu \bar{\mu} Z R Z - 2i Z R \nabla \rho + \rho R \rho + \frac{i}{2} Z R (Q + \bar{Q}) \gamma + i\gamma R (P - \bar{P}) \rho - \gamma (P + \bar{P} - 1) R \gamma \right\}.
\]

Gauge transformations (2.18) convert into

\[
\delta Z = -i(P - \bar{P}) \varepsilon, \\
\delta \rho = -i(Q + \bar{Q}) \varepsilon, \\
\delta \gamma = -2i \nabla \varepsilon,
\]

From (2.33,2.34) it is seen that \(Z\) and \(\rho\) are similar fields because they are both auxiliary (either \(Z\) or \(\rho\), not both, can be excluded by the equations of motion), and
enter the action in an analogous way. One can show that the superfield $\gamma$ is purely gauge. Really, using (2.21) and (2.8) one can derive gauge transformations for the components $\gamma_{s,s}$ of the superfield $\gamma$:

$$
\delta\gamma_{\alpha(s)\dot{\alpha}(s)} = a_s \mathcal{D}_{\alpha\dot{\alpha}} \varepsilon_{\alpha(s-1)\dot{\alpha}(s-1)} + b_s \mathcal{D}^{\alpha\dot{\alpha}} \varepsilon_{\alpha(s+1)\dot{\alpha}(s+1)}
$$

where $b_s \neq 0$ and $a_s$ are numerical factors. Using the freedom in the choice of $\varepsilon_{\alpha(s+1)\dot{\alpha}(s+1)}$ we can put $\gamma_{\alpha(s)\dot{\alpha}(s)} = 0$. This gauge imposes the differential restriction $\nabla \varepsilon = 0$ on the parameter and therefore cannot be inserted in action.

To obtain the $N = 2$ supersymmetry transformations with completely closed algebra one should add to transformations (2.20) the gauge variation (2.34) with the parameter $\varepsilon = -2t\gamma$, rewritten in terms of superfields $Z$, $\rho$ and $\gamma$. The result is

$$
\begin{align*}
\tilde{\delta}_t \gamma &= \mathcal{M}\gamma, \\
\tilde{\delta}_t Z &= -np + it(\mathcal{P} - \mathcal{\bar{P}})\gamma, \\
\tilde{\delta}_t \rho &= \mu\bar{\mu}n Z + \frac{i}{2}t(\mathcal{Q} + \mathcal{\bar{Q}})\gamma, \\
n &= [\mathcal{P} + \mathcal{\bar{P}}, t] = \mathcal{N} + 4t, \quad \mu\bar{\mu} = 2(\mu t^{\alpha} \mathcal{D}_\alpha + \bar{\mu} \bar{t}^\dot{\alpha} \bar{\mathcal{D}}^\dot{\alpha})
\end{align*}
$$

Here $\mathcal{M}$ is a vector superfield of $osp(2|4)$ superalgebra (2.27) and $\mathcal{N}$ is another vector superfield whose commutator has the opposite, as compared to (2.29), sign.

$$
[N(t), N(t')] = \frac{1}{\mu\bar{\mu}} K = [n(t), n(t')].
$$

Note that $n$-dependent terms in (2.35) may be represented with the use of matrix $J$:

$$
\begin{pmatrix} 
\tilde{\delta} Z \\
\tilde{\delta} \rho 
\end{pmatrix} = Jn \begin{pmatrix} 
Z \\
\rho 
\end{pmatrix} + \ldots, \quad J = \begin{pmatrix} 
0 & -1 \\
\mu\bar{\mu} & 0 
\end{pmatrix}, \quad J^2 = -\mu\bar{\mu},
$$

and the dots denote $\gamma$-dependent terms. This provides the correct sign in the commutator:

$$
[\tilde{\delta}_t, \tilde{\delta}_t'] = -K,
$$

where $K$ is the Killing operator (2.26) with vector parameter $k_{\alpha\dot{\alpha}}$ defined in (2.29). The difference in sign with $[1]$ is conditioned by a different definition of commutator of variations $[\tilde{\delta}_t, \tilde{\delta}_t']$.

Let us emphasize that the superfield $\gamma$ transforms via itself by the supervector operator $\mathcal{M}$. The whole representation of the superalgebra $osp(2|4)$ on superfields $\gamma$, $Z$ and $\rho$ has the opposite, as compared to (2.29), sign.

Let us write them down

$$
\begin{align*}
\delta S/\delta Z &= -2\mu\bar{\mu} \mathcal{R}Z - 2i \mathcal{R} \nabla \rho + \frac{i}{2} \mathcal{R}(\mathcal{Q} + \mathcal{\bar{Q}})\gamma = 0, \\
\delta S/\delta \rho &= 2\mathcal{R}\rho - 2i \mathcal{R} \nabla Z + i \mathcal{R}(\mathcal{P} - \mathcal{\bar{P}})\gamma = 0, \\
\delta S/\delta \gamma &= \frac{i}{2} \mathcal{R}(\mathcal{Q} + \mathcal{\bar{Q}})Z + i \mathcal{R}(\mathcal{P} - \mathcal{\bar{P}})\rho - 2(\mathcal{P} + \mathcal{\bar{P}} - 1)\mathcal{R}\gamma = 0.
\end{align*}
$$
Then we can define new variations $\bar{\delta}_t$ with the same algebra:

\[
\begin{align*}
\bar{\delta}_t \gamma &= \bar{\delta}_t = M \gamma, \\
\bar{\delta}_t Z &= \bar{\delta}_t Z - t R \delta S / \delta \rho = - (N + 6t) \rho + 2it \nabla Z, \\
\bar{\delta}_t \rho &= \bar{\delta}_t \rho - t R \delta S / \delta Z = \mu \bar{\mu} (N + 6t) Z + 2it \nabla \rho.
\end{align*}
\]  

(2.42)

Unfortunately, the new transformations leave invariant only the equations of motion (2.39–2.41) and not the action (2.32). The reason is that to maintain the invariance of the action, the expression $\bar{\delta} - \tilde{\delta}$ to transformations should be a composition of an antisymmetric matrix with the action variations. However, in (2.42) a symmetric matrix is used.

3 Structure of supercovariant derivatives.

Now we begin the description of the formalism with explicit $Osp(2|4)$ symmetry. To develop it we need the $Osp(2|4)$-covariant derivatives and their expressions in terms of $Osp(1|4)$-covariant operators used previously.

In this section, the following sequence of derivatives (which starts from ordinary $Osp(1|4)/SO(3,1)$ derivatives (1) defined in $N = 1$ AdS superspace) is constructed: $Osp(1|4)/SU(2)$, $Osp(2|4)/H$ and $Osp(2|4)/(SU(2) \oplus U(1)) \equiv Osp(2|4)/U(2)$. Two latter sets of $Osp(2|4)$-covariant derivatives will be used in the explicitly $N = 2$ invariant formulation of higher-spin equations. We will construct every set in this sequence in terms of the preceding one. In this way, all the expressions for covariant derivatives may be expressed in terms of $Osp(1|4)/SO(3,1)$ ones, that provides us with the possibility to make a bridge between various $N = 2$ covariant equations and those of $N = 1$ GKS theory.

Let us sketch our technology. Recall that, according to general facts collected in App.A, the derivatives covariant with respect to a (super)group $G$ are standardly determined for every coset space $G/H$ where $H$ is the local subgroup. In particular, given a supergroup $G$ of dimension $(P|Q)$ and a supergroup $H$ of dimension $(p|q)$, there exist $(P - p|Q - q)$ covariant derivatives for $G/H$ coset space, and the rest $(p|q)$ generators span a local $H$ supergroup.

We will start with $(4|4)$ $Osp(1|4)/SO(3,1)$ covariant derivatives in $N = 1$ AdS space $M^{4|4}$. Then we will observe that the extension of $M^{4|4}$ by a constrained Lorentz four vector $\gamma^{\alpha \dot{\alpha}}$ (being the essence of GKS formulation) can be considered as the $(7|4)$-dimensional homogeneous space $Osp(1|4)/SU(2)$, and build corresponding $(7|4)$ covariant derivatives. To this aim, we introduce a "small vielbein field" (3.13, 3.14) and construct the $\bar{M}^{7|4}$ derivatives via $M^{4|4}$ ones and the small vielbein (subsection 3.1). Further, the $Osp(2|4)/H$ superspace has the same dimension $(7|4)$ as the superspace $Osp(1|4)/SU(2)$, and the numbers of covariant derivatives in these spaces are equal. In fact, these new $N = 2$ covariant derivatives in a certain basis are given by $N = 1$ ones, this is shown in subsection 3.2. It will remain to build the $M^{7|8}$ superspace.
lent to (2.2):

The superalgebra $\text{osp}_M$ terms of covariant derivatives (2.2) of superspace $\text{osp}$ form the superalgebra derivatives in this superspace $\bar{\text{Osp}}$, for $(\Sigma, \Xi)$, which differs from two previous ones by the number of odd coordinates. We will not introduce these new odd coordinates $\theta$ explicitly, instead, we describe an $M^{\ell|8}$ superfield by its $M^{\ell|4}$ proections, associated with $\theta = 0$ components of its covariant derivatives.

Before starting the superspace building process let us make a remark. Note that the supergroup $\mathcal{H}$ serves as a local supegroup for $\text{Osp}(2|4)/\mathcal{H}$-covariant derivatives. The corresponding superfields in the coset space carry the indices which transform under the supergroup (we call superindices the indices of such a kind). Therefore these indices are graded, they have even and odd parts. For example, one has the superindex $\xi$ (3.2): $\phi_\xi = (\phi_+a, \phi_+\pm)$. The corresponding superfield has the odd part $\phi_+a$ and the even part $\phi_+\pm$ (and each of them has even and odd components after an expansion in power series by odd coordinates). As far as we know superfields with superindices have not yet been used in physically interesting models. In Sect. 3.3, we present the models with these superfields.

3.1 $\text{Osp}(1|4)/SU(2)$-covariant derivatives.

Consider the supermanifold $\bar{M}^{\ell|4}$ parametrized by coordinates $(z, r)$ defined in Sect.2. The dynamical superfields of the theory under consideration are $r$-analytic functions in this supermanifold, (2.30,2.31). The supergroup $\text{Osp}(1|4)$ acts on $\bar{M}^{\ell|4}$ transitively. Indeed, it acts transitively on the supermanifold $M^{\ell|4}$ parametrized by $z$, then, the stability subgroup for $z$ – the Lorentz group $SL(2,C)$ – acts transitively on a two-sheeted hyperboloid parametrized by $r$. The stability subgroup of the point $(z, r) \in M^{\ell|4}$ is $SU(2)$. According to App. A, there exist $\text{Osp}(1|4)/SU(2)$ covariant derivatives in this superspace $M^{\ell|4}$. In this subsection, we find their expressions in terms of covariant derivatives (2.2) of superspace $M^{4|4}$. Our guide is the requirement for $\text{Osp}(1|4)/SU(2)$ covariant derivatives, together with local generators of $su(2)$, to form the superalgebra $\text{osp}(1|4)$. It is convenient to introduce $su(2)$-covariant basis in this superalgebra which has the following structure

$$S_{a(2)}, \Sigma_I = \{\Sigma_{\pm\pm}, \Sigma_{\pm}, \Sigma\}.$$  

The superalgebra $\text{osp}(1|4)$ is defined by the following commutation relations equivalent to (2.2):

$$[S_{a(2)}, S_{b(2)}] = 2\varepsilon_{ab}S_{ab}, \quad (3.2)$$
$$[S_{a(2)}, \Sigma_{\pm\pm}(b)] = 2\varepsilon_{ab}\Sigma_{\pm\pm}, \quad [S_{a(2)}, \Sigma_\pm(b)] = \varepsilon_{ab}\Sigma_\pm, \quad (3.3)$$
$$[\Sigma, \Sigma_{\pm\pm}(a(2))] = \pm\Sigma_{\pm\pm}(a(2)), \quad [\Sigma, \Sigma_\pm(a)] = \pm\frac{1}{2}\Sigma_\pm, \quad (3.4)$$
$$[\Sigma_{++}(a(2)), \Sigma_{--}(b)] = 4\varepsilon_{ab}(S_{ab} - \varepsilon_{ab}\Sigma), \quad (3.5)$$
$$[\Sigma_{\pm\pm}(a(2)), \Sigma_{\mp}(b)] = 2\varepsilon_{ab}\Sigma_{\pm\pm}, \quad (3.6)$$
$$\{\Sigma_{\pm\pm}, \Sigma_{\pm\pm}\} = -\Sigma_{\pm\pm}, \quad \{\Sigma_{\pm\pm}, \Sigma_{\pm}\} = -S_{ab} \pm \varepsilon_{ab}\Sigma, \quad (3.7)$$

with reality conditions

$$\left(S_{a(2)}\right)^\dagger = S^a(2) \quad (3.8)$$
$$\left(\Sigma_{\pm\pm}\right)^\dagger = -S^a_{\pm\pm}, \quad (\Sigma_\pm)^\dagger = \mp\Sigma^a_\mp, \quad \Sigma^\dagger = -\Sigma \quad (3.9)$$

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Commutation relations \((3.2)\), together with reality conditions \((3.8)\), define the \(su(2)\) algebra. The rules \((3.8,3.9)\) reflect the fact that the complex conjugate of the fundamental representation of \(su(2)\) is equivalent to its contragredient representation.

Let us now discuss the connection between the considered basis \(\Sigma_{I}, S_{a(2)}\) and the basis \(D_{A, M_{\alpha(2)}, \bar{M}_{\dot{\alpha}(2)} (2.2)}\) used in the previous section. To express the former via the latter we need to pass to \(SU(2)\)-covariant indices. Note that conjugation rules for generators \(D_{a, M_{\alpha(2)}, \bar{M}_{\dot{\alpha}(2)} (2)}\) are as follows:

\[
(M_{a(2)})^\dagger = \bar{M}_{\dot{\alpha}}, \quad (D_{a\beta})^\dagger = D_{\dot{\alpha} \dot{\beta}}, \quad (D_{a})^\dagger = \bar{D}_{\dot{\alpha}}\tag{3.10}\]

That is why if \(D_{a}\) is transformed under the fundamental representation of \(SU(2)\), then \(\bar{D}_{\dot{\alpha}}\) is transformed under the contragredient one. Therefore we should define

\[
M_{(ab)} = M_{(a\beta)}, \quad \bar{M}^{(ab)} = \bar{M}_{(a\dot{\beta})}, \quad D_{a,b} = D_{a\dot{\beta}}, \quad D_{a} = D_{a\dot{\beta}}, \quad D_{a} = -\bar{D}_{\dot{\alpha}}\tag{3.11}\]

here, in every equality, \(a\) and \(b\) take the same values as \(\alpha\) and \(\beta\) (or \(\dot{\alpha}\) and \(\dot{\beta}\)), respectively. Now, the commutation relations \((3.2-3.7)\) are connected with \((2.2)\) by the expressions

\[
\Sigma_{\pm a(2)} = D_{a,a} \pm i(\bar{M}_{a(2)} - M_{a(2)}), \quad S_{a(2)} = -(M_{a(2)} + \bar{M}_{a(2)}), \quad \Sigma = -\frac{i}{2} D_{a,a}\tag{3.12}\]

\[
\Sigma_{+a} = \frac{j}{2}(D_{a} + \bar{D}_{a}), \quad \Sigma_{-a} = -\frac{j}{2}(D_{a} - \bar{D}_{a}),
\]

where \(j = e^{i\pi/2}\).

The covariant derivatives (we denote them \(\Sigma_{I}\)) possessing the required algebra and reality conditions should depend both on \(z\) and \(r\). To find their expressions, first of all we introduce a ‘small vielbein’ \(e_{a\alpha}(z, r)\) and its conjugated \((e_{a\alpha})^\dagger = e_{a\dot{\alpha}}\) which convert Lorentz two-component spinor indices \(\alpha\) and \(\dot{\alpha}\) into two-component spinor \(SU(2)\)-index \(a\). We subject this vielbein to the following conditions invariant both with respect to \(SL(2, C)\) and \(SU(2)\). First, the \(2 \times 2\)-matrix \(ie_{a\alpha}\) is an element of the \(SL(2, C)\) group:

\[
\varepsilon_{ab} = -e_{a\alpha} e_{b\beta} \varepsilon_{\alpha\beta}.\tag{3.13}\]

Second, it is convenient to restrict partially the remaining freedom by

\[
e^{\alpha\alpha} e_{a} \dot{\alpha} = r^{\alpha\dot{\alpha}},\tag{3.14}\]

(remember that the vielbein \(e\) is a function of \(z\) and \(r\)). One can see that the latter condition \((3.14)\) is consistent with \(r^{\alpha\dot{\alpha}} r_{\alpha\dot{\alpha}} = 2\) and \((r_{\dot{\alpha}\dot{\alpha}})^\dagger = r_{\alpha\dot{\alpha}}\). It implies

\[
e_{a} \alpha r_{\dot{\alpha}} = e_{a} \dot{\alpha}, \quad e_{a} \dot{\alpha} r_{\alpha} = -e_{a} \alpha.
\]

\(^4\) As usual, we raise and lower indices \(\alpha, \dot{\alpha}\) and \(a\) of the small vielbein with help of antisymmetric symbols \(\varepsilon_{\alpha\dot{\beta}}, \varepsilon_{\dot{\alpha}\dot{\beta}}, \varepsilon_{ab}\) and so on. For example, \(e_{a} \alpha = \varepsilon_{a\beta} e_{b} \beta\).
Let us remind that the Lorentz generators $M_{\alpha(2)}$, $\tilde{M}_{\dot{\alpha}(2)}$ enter the connection parts of covariant derivatives $\mathcal{D}_A$ in all formulas of Sect. 2. These generators do not act on the indices of vector $r^{\alpha\dot{\alpha}}$. Their action on a scalar superfield $\Phi(r)$ is equivalent to the action of the following operators

$$\tilde{M}_{\alpha(2)} = \frac{1}{2} r^{\alpha\dot{\alpha}} \partial_{\alpha\dot{\alpha}}, \quad \tilde{M}_{\dot{\alpha}(2)} = \frac{1}{2} r^{\alpha\dot{\alpha}} \partial_{\dot{\alpha}a}, \quad \partial_{\dot{\alpha}a} \equiv \frac{\partial}{\partial r^{\alpha\dot{\alpha}}}$$

that preserve the constraint $r^2 = -1$ and therefore are (complex) vector fields on the hyperboloid. Let us introduce operators $\hat{M}_{\alpha(2)}$, $\hat{\tilde{M}}_{\dot{\alpha}(2)}$ that act on all indices $\alpha$, $\dot{\alpha}$ and include the differential operator:

$$\hat{M}_{\alpha(2)} e^\beta_\alpha \equiv \tilde{M}_{\alpha(2)} e^\beta_\alpha + \varepsilon_{\alpha \beta} e^a_\alpha,$$

$$\hat{M}_{\alpha(2)} r_{\beta\dot{\beta}} \equiv \tilde{M}_{\alpha(2)} r_{\beta\dot{\beta}} + \varepsilon_{\beta \dot{\beta}} r^{\alpha\dot{\alpha}} = 0,$$

$$\hat{M}_{\alpha(2)} \Phi(r) \equiv \tilde{M}_{\alpha(2)} \Phi(r)$$

and conjugated expressions for $\hat{\tilde{M}}_{\dot{\alpha}(2)}$. Then define two sets of modified covariant derivatives $\mathcal{D}_A$, substituting either $\hat{M}_{\alpha(2)}$ or $\hat{\tilde{M}}_{\dot{\alpha}(2)}$ instead of $M_{\alpha(2)}$ in the connection terms:

$$\tilde{\mathcal{D}}_A \equiv \mathcal{D}_A|_{M \rightarrow \tilde{M}}, \quad \hat{\mathcal{D}}_A \equiv \mathcal{D}_A|_{M \rightarrow \hat{M}}.$$ 

Then we have the following equations for an arbitrary $SU(2)$ spinor superfield $\Phi_a(z, r)$:

$$\tilde{\mathcal{D}}_A \Phi_a(z, r) = \mathcal{D}_A \Phi_a(z, r),$$

$$\hat{\mathcal{D}}_A \hat{\mathcal{D}}_B \Phi_a(z, r) = \mathcal{D}_A \mathcal{D}_B \Phi_a(z, r).$$

Now we choose (7|4) independent vector fields:

$$\nabla_a^{(2)} \equiv e^\alpha_a e^\dot{\alpha}_a \tilde{\mathcal{D}}_{\alpha\dot{\alpha}} = - (\nabla_a^{(2)})^\dagger, \quad \nabla \equiv - \frac{i}{2} e^\alpha_\alpha \tilde{\mathcal{D}}_{\alpha\dot{\alpha}} = - \nabla^\dagger, $$

$$\partial_a^{(2)} \equiv - i e^\alpha_a e^\dot{\alpha}_a \partial_{\alpha\dot{\alpha}} = - 2 i e^\alpha_a e^\dot{\alpha}_a \tilde{M}_{\alpha(2)} = (\partial_a^{(2)})^\dagger, $$

$$\nabla_a \equiv e^\alpha_a \tilde{\mathcal{D}}_\alpha, \quad \nabla_a \equiv e^\dot{\alpha}_a \tilde{\mathcal{D}}_{\dot{\alpha}} = (\nabla_a)^\dagger$$

It is also convenient to define the modified operators $\hat{\nabla}$:

$$\hat{\nabla}_A \equiv \mathcal{D}_A|_{M \rightarrow \hat{M}}, \quad \hat{\nabla}_A \Phi_a(z, r) = \mathcal{D}_A \Phi_a(z, r), \quad \hat{\nabla}_A \nabla_B \Phi_a(z, r) = \mathcal{D}_A \mathcal{D}_B \Phi_a(z, r)$$

Note their useful properties:

$$\partial_a^{(2)} r^{\alpha\dot{\alpha}} = 2 i e^\alpha_a e^\dot{\alpha}_a, \quad \hat{\nabla}_A r^{\alpha\dot{\alpha}} = 0$$

In the analogous equation for $\nabla_A$, the connection parts of covariant derivatives act on $r^{\alpha\dot{\alpha}}$ and do not give zero. Making use of these equations, all independent first order derivatives of the small vielbein prove to be described by the expressions

$$v_{a(2)b(2)} \equiv \frac{1}{2} \{(\partial_a^{(2)} e^\beta_b) e_{b\alpha} + (\partial_a^{(2)} e^\dot{\beta}_b) e_{b\dot{\alpha}}\},$$

$$\partial_a^{(2)} e^\beta_b \equiv - v_{a(2)b(2)} e^{ba} - i \varepsilon_{ab} e^\alpha_a,$$
To derive the algebra of the vector fields \( \partial_{a(2)} \), \( \nabla_A \) it is sufficient to calculate the algebra of the modified operators \( \partial_{a(2)} \), \( \nabla_A \) since, due to (3.19), these algebras coincide modulo curvature terms with local rotations \( M_{a(2)} \). The latter algebra can be derived straightforwardly using the commutation relations for \( \mathcal{D}_A \) (given by (2.2)) along with the substitution \( \mathcal{D}_A \to \mathcal{D}_A^\dagger \), \( M \to \bar{M} \). In so doing one can deduce that the supercommutators of vector fields \( \partial_{a(2)} \), \( \nabla_A \) close on themselves with coefficients of non-holonomicity being proportional to tensor superfields \( v \), \( u \), \( w \), and \( y \).

Now we are in a position to construct the operators

\[
\mathcal{L}_{a(2)} = -\partial_{a(2)} - v_{a(2)}^{b(2)} S_{b(2)}, \\
\mathcal{D}_{a(2)} = \nabla_{a(2)} + u_{a(2)}^{b(2)} S_{b(2)}, \\
\hat{\Sigma} = \nabla + w^{b(2)} S_{b(2)}, \\
\mathcal{D}_a = i\nabla_a + iy_a^{b(2)} S_{b(2)},
\]

acting on sections of \( su(2) \) fiber bundle over \( \bar{M}^7 \), with \( S_{a(2)} \) representing local \( su(2) \) generators. In fact, these expressions already define \( Osp(1|4)/SU(2) \)-covariant derivatives, but for our purpose it is more convenient to make a linear substitution

\[
\hat{\Sigma}_{\pm a(2)} = \mathcal{D}_{a(2)} \pm \mathcal{L}_{a(2)}, \\
\hat{\Sigma}_{a} = \frac{i}{2}(\mathcal{D}_a + \mathcal{D}_a), \\
\hat{\Sigma}_{-a} = -\frac{i}{2}(\mathcal{D}_a - \mathcal{D}_a).
\]

Using the algebra of vector fields (3.18) one can show that the introduced operators \( \hat{\Sigma}_I \), along with \( S_{ab} \) local rotations, obey the superalgebra \( osp(1|4) \) (3.4–3.7). They also satisfy the reality conditions (3.9). The obtained results exhibit a general situation, briefly discussed in App. A. Suppose that we have two homogeneous manifolds \( G/L \) and \( G/H \) for the same group \( G \), and \( H \) is a subgroup of \( L, H \subset L \subset G \). Then there exists a general expression for \( G/H \)-derivatives via \( G/L \)-ones. The small vielbein field can be viewed as an object which parametrizes different embeddings \( L/H \subset L \).

\subsection*{3.2 Osp(2|4)-covariant derivatives in \( \bar{M}^7 \).}

Here we construct other covariant derivatives in the same supermanifold \( \bar{M}^7 \), where the derivatives \( \hat{\Sigma}_I \) were defined previously. The point is that, according to Sect.[\ref{sect:super}], this supermanifold can be considered as the homogeneous manifold \( Osp(2|4)/\mathcal{H} \) for \( N = 2 \ AdS \) supergroup \( Osp(2|4) \), the corresponding vector fields spanning \( osp(2|4) \) superalgebra are given by Eqs. (2.26, 2.27). Here, as it is mentioned in the Introduction, \( \mathcal{H} \) denotes the (super)subgroup of the supergroup \( Osp(2|4) \), associated with superalgebra \( su(2, 0|1, 0) \subset osp(2|4) \). Note that the bosonic part of \( su(2, 0|1, 0) \) coincides with \( u(2) = so(3) \oplus so(2) \). The algebra of the corresponding \( Osp(2|4) \)-covariant derivatives and local rotations is \( osp(2|4) \) superalgebra. We denote these \( Osp(2|4)/\mathcal{H} \)
derivatives by the same letter as the corresponding generators in (1.8–1.10,1.15–1.20) with the additional emphasis \( \dot{\cdot} \). The local rotations are denoted identically with the generators of \( A(1|0) \):

\[
\text{local rotations} \quad S_{a(2)}, \quad T = \Gamma - S/2, \quad D_{+a}^+, \quad D_{-a}^- \quad D = \Gamma/2 - S/2
\]

(3.30)

The commutation relations in this basis can be read off (1.8–1.10,1.15–1.20) after switching from \( \Gamma, S \) to \( T, \bar{D} \). To derive the \( Osp(2|4) \)-covariant derivatives (3.30) in terms of derivatives \( \Sigma_I \) constructed in the previous subsection, it is convenient to pass to a new basis. Let us recall the odd generators \( D_{+a}^+, D_{-a}^- \) related to \( D_{\mp a} \) and \( D_{\mp a} \) by the rules (1.7). Introduce the notation

\[
\tilde{\Sigma}_{\mp a} = \mp \sqrt{2}(\tilde{D}_{\mp a} - \tilde{D}_{\mp a}^-), \quad \tilde{\Sigma} = T - 2\bar{D}.
\]

(3.31)

Strictly speaking, \( \tilde{\Sigma}_{\pm a} \) and \( \tilde{\Sigma} \) are no longer pure covariant derivatives for \( Osp(2|4)/H \) but the combinations of covariant derivatives and local \( H \) rotations. Therefore, the supercommutator of \( \tilde{\Sigma}_{\pm a} \) with local rotations \( D_{\mp a} \) is not a linear combination of bosonic covariant derivatives and contains explicitly local rotations \( S_{a(2)} \). Nevertheless, it is possible to consider the following basis of operators (the set of generators \( D_{\mp a}^+, D_{\mp a}^- \) forms the basis in the odd subspace of superalgebra \( osp(2|4) \)):

\[
\mathcal{H} - \text{local rotations} \quad S_{a(2)}, \quad T, \quad D_{\pm a}^+, \quad \Sigma_{\pm a}, \quad \tilde{\Sigma}
\]

(3.32)

(3.33)

Now our task reduces to finding the operators \( \Sigma_I \). The basis (3.32,3.33) for \( osp(2|4) \) was chosen so that generators \( \Sigma_I, S_{a(2)} \) span \( osp(1|4) \) subalgebra, which can be derived from Eqs.(1.8–1.10,1.15–1.20). This means that the commutators of operators \( \Sigma_I \) close on \( \Sigma_I \) and \( S_{a(2)} \) and coincide with those of derivatives \( \Sigma_I \) explicitly constructed in the previous subsection. That is why we can identify the desired combinations (3.33) with the available operators (3.28–3.29). Thus we do not need to 'construct' any new objects to built \( Osp(2|4)/H \) covariant derivatives. In this reasoning, we have used the fact that the whole superalgebra \( osp(2|4) \) can be spanned by two intersecting subalgebras: \( osp(1|4) \) with basis \( \Sigma_I, S_{a(2)} \) and subalgebra of local rotations \( S_{a(2)}, D_{\pm a}^+, T \). The action of \( osp(1|4) \) was defined in the subsection 3.1 while the local rotations act by definition on the indices of covariant derivatives. Of course, we can go back to the basis (3.30) that appears to be more convenient in what follows:

\[
\bar{D}_{\mp a}^\pm = \pm \sqrt{2}\Sigma_{\mp a} + D_{\mp a}^\mp,
\]

(3.34)

\[
\bar{D} = -\Sigma/2 + T/2.
\]

(3.35)

Let us overview the subsection results. Even covariant derivatives \( \Sigma_{\pm a(2)} \) are given by (3.28). Odd derivatives \( \bar{D}_{\mp a}^\pm \) and scalar derivative \( \bar{D} \) are given by (3.34), where \( \Sigma_{\pm a}, \tilde{\Sigma} \).
are $Osp(1|4)$-covariant derivatives (3.26, 3.29) and $D_{\pm a}$, $T$ are local rotations. Let us stress that here only a particular solution is found for $Osp(2|4)/\mathcal{H}$-derivatives. That is why it is not $\mathcal{H}$-invariant and the connections, corresponding to local generators $D_{\pm a}$ and $T$ are fixed constants. To obtain a general solution, one has to perform a general local $\mathcal{H}$-rotation of the constructed covariant derivatives. Nevertheless, this particular solution is convenient for the component analysis of the superfield models considered in the following sections.

3.3 $Osp(2|4)$-covariant derivatives in $M^{7|8}$.

Here we study $Osp(2|4)/U(2)$ derivatives and their relation to the derivatives constructed in the previous section. We denote the set of these derivatives by the same letters as the corresponding generators of $osp(2|4)$ superalgebra:

\[
\text{local rotations} \quad S_{a(2)}, T, D_{\pm a} \\
\text{covariant derivatives} \quad D_I = \{\Sigma_{\pm a(2)}, D_{\mp a}, D\} \text{ and } D_a = \{D_{\pm a}\}. \quad (3.36)
\]

First, let us consider the coset superspace $\mathcal{H}/U(2)$, parametrized by four odd coordinates $\theta$. Let us call $D_{\pm a}$ the corresponding odd covariant derivatives with local rotations $S_{a(2)}$, $T$. The algebra of covariant derivatives coincides with the (real form of) $A(1|0)$ superalgebra written out in Eq.(3.33). Note that every superfield $\Phi$ in this supermanifold is described by $2^4 = 16$ coefficients of the decomposition by power series in odd coordinates $\theta$. Since the covariant derivatives $D_{\pm a}$ are independent the following projections can be chosen for these coefficients:

\[
\Phi|, \quad D_{\pm a}\Phi|, \quad D_{b}D_{\pm a}\Phi| \equiv (D_{\pm})^2\Phi|, \quad [D_{+a}, D_{-b}]\Phi|, \\
D_{\mp b}(D_{\pm})^2\Phi|, \quad (D_{\mp})^2(D_{\pm})^2\Phi| \quad (3.37)
\]

where $\Phi|$ denotes the null component $\Phi(\theta = 0)$.

Now let us discuss the relation of $Osp(2|4)/U(2)$ covariant derivatives in $M^{7|8}$ to the odd derivatives for $\mathcal{H}/SU(2)$ and $Osp(2|4)/\mathcal{H}$ derivatives constructed in the previous subsections. Here it is convenient to introduce a shorthand notations. $D_I = \{\Sigma_{\pm a(2)}, D_{\mp a}, \tilde{D}\}$ are covariant derivatives from (3.28, 3.33, 3.35); $l_a = \{D_{\pm a}\}$ are just introduced odd derivatives in $\mathcal{H}/U(2)$, while $\mathcal{L}_a = \{D_{\pm a}\}$ are odd local rotations from (3.34); $S_a = \{S_{a(2)}, T\}$ are bosonic generators of local algebra $U(2)$; at last $x = \{z,r\}$ are coordinates of the superspace $M^{7|4}$. Then the derivatives $\tilde{D}_I$ and $l_a$ have the following structure:

\[
\tilde{D}_I = e_I + \omega_I^a(x)\mathcal{L}_a + \omega_I^a(x)S_a \\
l_a = e_a + \omega_a^a(\theta)S_a \quad (3.39)
\]

where $e$‘s are vielbein vector superfields and $\omega$‘s are connections\footnote{It follows from (3.34) that $\omega_I^a$ are constants in that particular basis for $Osp(2|4)/\mathcal{H}$ derivatives.}. The derivatives $\tilde{D}_I$ satisfy the superalgebra $osp(2|4)$, while $l_a$ form the $A(1|0)$ superalgebra

\[
[\tilde{D}_I, \tilde{D}_J] = f_{IK}^J \tilde{D}_K + f_{IJ}^a \mathcal{L}_a + f_{IJ}^a S_a, \\
[l_a, l_b] = f_{ab}^c l_c + f_{a}^b S_a \quad (3.41)
\]
where the structure constants $f$’s can be found in (1.8–1.10,1.15–1.20). Let us seek for $Osp(2\mid4)/U(2)$ covariant derivatives $D_I$, $D_\alpha$ in the following form:

\[ D_I = E^J_I(x, \theta) e_J + E^\alpha_I(x, \theta) l_\alpha + \Omega^\alpha_I(x, \theta) S_a, \]  
(3.42)

\[ D_\alpha = l_\alpha, \]  
(3.43)

\[ E^J_I(x, 0) = \delta^J_I, \quad E^\alpha_I(x, 0) = \omega^\alpha_I, \quad E^a_I(x, 0) = \omega^a_I. \]  
(3.44)

Eqs.(3.44) give the initial conditions for the vielbeins $E_I$ and the connection $\Omega_I$. Then we have to impose the following equations which define the dependence of $D_I$ on $\theta$’s:

\[ [l_\alpha, D_I] = f^J_{\alpha I} D_J. \]  
(3.45)

Calculating the supercommutator and using the relations $[e_\alpha, D_I] = 0$ and $[l_\alpha, S_a] = f^\beta_{\alpha a} l_\beta$, we obtain differential equations for $E_I$, $\Omega_I$ of the following form:

\[ e_\alpha E^J_I(x, \theta) + \ldots = 0, \quad e_\alpha E^\beta_I(x, \theta) + \ldots = 0, \quad e_\alpha \Omega^a_I(x, \theta) + \ldots = 0 \]  
(3.46)

with dots standing for terms without derivatives. This system of differential equations is self-consistent as it is checked using Jacobi identities for the constants $f$, Eqs.(3.43) and the identity

\[ [l_\alpha, [l_\beta, D_I]] + [l_\beta, [l_\alpha, D_I]] + [[l_\alpha, l_\beta], D_I] = 0. \]

The self-consistency of the equations (3.46) and the linear independence of the vector fields $e_\alpha$ assures the existence and uniqueness of a solution for $E_I$ and $\Omega_I$ with the initial conditions (3.44). To ensure that this solution gives $Osp(2\mid4)/\mathcal{H}$ derivatives we have to check the supercommutator $[D_I, D_J]$. Due to linear independence of operators $D_I$, $l_\alpha$ and $S_a$ in every point, the supercommutator $[D_I, D_J]$ can be expanded in this basis, with some function coefficients $f(x, \theta)$:

\[ [D_I, D_J] = f^K_{IJ}(x, \theta) D_K + f^\alpha_{IJ}(x, \theta) l_\alpha + f^a_{IJ}(x, \theta) S_a \]  
(3.47)

Then $f(x, 0) = f$ – the structure constants of the superalgebra $osp(2\mid4)$. Really, the operator $l_\alpha$ in Eq.(3.47) (and also inside operators $D_I$ (3.42)) can be replaced by the local generators $\mathcal{L}_\alpha$ with the same algebra and action on $D_I$ (3.45). The null component of the obtained operator $D^{l_{2\mathcal{L}}}_I$ is equal to $\tilde{D}_I$, whence the null component of the equation (3.47) converts into (3.40) that provides the equation $f(x, 0) = f = \text{const}$. Then, taking the commutator of $l_a$ with the equation (3.47), we get the differential equations on $\theta$-dependence of $f(x, \theta)$. With the use of (3.41,3.45) these equations can be reduced to form

\[ e_\alpha f(x, \theta) + \ldots = 0, \]  
(3.48)

the dots are standing for terms without derivatives. These equations have the unique solution which is constant, $f(x, \theta) = f$. So we have proved that the new covariant
derivatives $D_I = \{\Sigma_{\pm \pm a(2)}, \ D_{\mp a}, \ D\}$ and $D_a = \{D_{\pm a}\}$ satisfy the superalgebra $osp(2|4)$.

A general superfield in $Osp(2|4)/U(2)$ is represented locally by a set of functions $\Phi_{pa(k)}$, carrying a finite-dimensional representation of $u(2)$ algebra with $a(k)$ representing basis of $su(2)$ rank-$k$ symmetric spinor representation, and $p/2$ being the $u(1)$ T-weight. In this paper only the case $k = 0$ appears.

Analogously to the superfields in superspace $\mathcal{H}/U(2)$, all the content of an arbitrary superfield $\Phi_p(z,r,\theta)$ in superspace $M^{7|8}$ is given by the following component superfields in $\bar{\mathcal{M}}^{7|4}$:

\[
\Phi_p|, \quad D_{\pm a}\Phi_p|, \quad D_{\mp a}^\pm D_{\pm a}\Phi_p| \equiv (D_{\pm a}^\pm)^2\Phi_p|, \quad [D_{+a}, D_{-b}]\Phi_p|,
\]

\[
D_{\mp b}(D_{\pm a}^\pm)^2\Phi_p|, \quad (D_{\pm a}^\pm)^2(D_{\pm a}^\pm)^2\Phi_p|.
\] (3.49)

The initial conditions (3.44), together with Eqs.(3.34,3.35) and the fact that $\omega_I^\alpha$ in (3.42) are constant superfields gives us the following component rules:

\[
(S_{\pm \pm a(2)}\Phi_p)| = \hat{S}_{\pm \pm a(2)}(\Phi_p)|,
\] (3.50)

\[
(D_{\mp a}^\pm \Phi_p)| = \pm \sqrt{2}S_{\mp a}(\Phi_p)| + (D_{\pm a}^\pm \Phi_p)|,
\] (3.51)

\[
(D\Phi_p)| = -\frac{1}{2}\hat{S}(\Phi_p)| + \frac{1}{2}(T\Phi_p)|
\] (3.52)

where in (3.51) the second term in r.h.s. is another independent component of the superfield $\Phi_p$ while $T\Phi_p \equiv p/2\Phi_p$ in the second term of (3.52) is the local rotation.

4 The chiral superfields on $M^{7|8}$.

We introduce the main building blocks of the $N = 2$ covariant equations to be constructed, the strongly chiral fields on $M^{7|8}$.

A one-component superfield on $M^{7|8}$ is called weakly chiral if it satisfies the constraint

\[
E_a\Phi_p = 0,
\] (4.1)

where $E_a$ is one piece from the four odd covariant derivatives pieces on $M^{7|8}$:

\[
E_a = D_{+a}^+ \text{ or } D_{-a}^- \text{ or } D_{+a}^- \text{ or } D_{-a}^+.
\] (4.2)

The strongly chiral fields are defined to be weakly chiral w.r.t. two special pieces from Eq.(4.2). Namely, the strongly chiral field of a first kind is defined by the constraints

\[
D_{+a}^+\gamma_p = D_{-a}^-\gamma_p = 0 \Rightarrow p = 0.
\] (4.3)

It is clear from the commutation relations (1.9) that in this case a strongly chiral field should carry zero T-weight.
The strongly chiral field of a second kind is defined by the constraints in which the two covariant derivatives pieces have equal Γ-weights and, therefore, commute:

\[ D^+_{+a} \Phi_p = D^+_{-a} \Phi_p = 0 \]  

(4.4)

or

\[ D^-_{+a} \Phi_p = D^-_{-a} \Phi_p = 0 \]  

(4.5)

In these cases, there are no restrictions on the T-weight. Note that the superfield \( γ \) can be real, \( γ = γ^\dagger \), while \( Φ_p \) is necessarily complex, as the complex conjugation “\( \dagger \)” maps the covariant derivatives \( D^+_{+a}, D^+_{-a} \) to \( D^-_{-a}, D^-_{+a} \) (Eq. (1.23)).

Let’s study the \( M^7|4 \) component content of strongly chiral fields along the previous section lines. It is clear that the strongly chiral field of the first kind possess the unique unconstrained component \( γ(z, r) = γ_0 \), the rest components like \( (D^+_{+a} γ_0) \) are zero by virtue of constraints (4.3).

Technically, the situation for the chiral fields of a second kind is more involved. Consider the constraint (4.4) and introduce the \( M^7|4 \) components of the field \( Φ_p \):

\[ a = Φ_p |, \quad ψ_a = (D^-_{-a} Φ_p) |, \quad b = ((D^-)^2 Φ_p) |. \]  

(4.6)

The rest components like \( (D^+_{+a} Φ_p) | \) etc. are zero by virtue of the first constraint from (4.4). We have to analyze how the remaining constraint \( Y^-_{-a} ≡ D^-_{-a} Φ_p = 0 \) looks like in terms of the \( a, ψ_a, b \) fields by studying the components of this equation. There are four groups of components (since \( D^+_b Y^+_a = 0 \)):

\[ I : \quad Y^+_a | = ψ_a + \sqrt{2} \Sigma^-_{-a} a = 0 \]  

(4.7)

\[ II : \quad D^-_{-a} Y^+_a | = \dot{Σ}_{-a} a - \sqrt{2} \Sigma^-_{-a} ψ_a = 0 \]  

(4.8)

\[ III : \quad D^-_{-a} Y^-_{-a} | = b + \sqrt{2} \Sigma^a ψ_a = 0 \]  

(4.9)

\[ IV : \quad ((D^-)^2 Y^+_a) | = 2 \dot{Σ}_{-a} ψ_a + \sqrt{2} \Sigma^-_{-a} b = 0. \]  

(4.10)

Here we have used formulas (3.50–3.52) and \( Osp(1|4) \) covariant derivatives \( \dot{Σ} \) in supermanifold \( M^7|4 \). The first and the third groups show that \( ψ_a \) and \( b \) are expressed via the unconstrained field \( a \). The second and the fourth groups prove to be the consequences of the first and the third ones and do not lead to any restriction on \( a \). Thus, the chiral field of the second kind \( Φ_p \) is equivalent to the unconstrained \( M^7|4 \)-superfield, being expressed by the first projection \( a = Φ_p | \).

As far as the component content of either of the considered strongly chiral superfields reduces to one unconstrained field in \( M^7|4 \), they can be chosen on the role of dynamical variables in physical models. Though both these components are similar superfields in \( M^7|4 \), their \( N = 2 \) supersymmetry transformations differ. One can show that for \( γ = γ_0 \) the transformations of the second supersymmetry (2.19), parametrized by Killing scalar \( t \), are realized by the operator \( \mathcal{M} \) (2.27). This means that the \( osp(2|4) \) representation carried by \( γ \) is an ordinary regular representation on...
the functions in homogeneous space. However, for \( a = \Phi_p \), the second supersymmetry is given by the operator \( \mathcal{N} \) whose action on the real and imaginary parts of \( a \) is similar to the action on the superfields \( Z \) and \( \rho \) in (2.42).

The strongly chiral fields of a third kind are defined to obey the constraints

\[
D_{-a}^+ \Phi_p = D_{+a}^- \Phi_p = 0 \Rightarrow \bar{T} \Phi_p \equiv (3T - 4D) \Phi_p = 0 \quad (4.11)
\]

One may show that such a field possesses just one constrained component on \( \bar{M}^{|4} \):
\( \phi = \Phi_p \), \( (3p + 4\Sigma)\phi = 0 \), and all higher components are expressed via \( \phi \).

5 Equations of motion for higher spin superfields with manifest \( N = 2 \) supersymmetry.

In this section, we present explicitly \( N = 2 \) supersymmetric equations equivalent to (2.33–2.41). They are the dynamical equations for higher spin fields formulated in superspace \( M^{7|8} = OSP(2|4)/U(2) \). According to App. A, in this superspace there exist the covariant derivatives \( \Sigma_{\pm \pm a}(2) \), \( D_{\pm a} \), \( D_{\pm a} \) and \( D \) that satisfy the \( osp(2|4) \) superalgebra (1.8–1.10, 1.15–1.20) together with local rotations \( S_{a(2)} \) and \( T \). The explicit expressions for these derivatives in terms of \( OSP(1|4) \)-covariant objects are derived in Sect. 3, and used below in the component analysis.

Let us begin to formulate the theory. The superfields on \( M^{7|8} \) are characterized by their \( su(2) \) indices and \( T \)-weights. Dynamical variables are strongly chiral superfields on \( M^{7|8} \): real \( \gamma = \gamma^\dagger \), complex \( \Phi_6 \equiv \Phi_{++} \) and its conjugated \( \Phi_- = (\Phi_{++})^\dagger \)

subjected to the constraints

\[
\begin{align*}
D_{+a}^+ \gamma &= D_{-a}^- \gamma = 0, & T \gamma &\equiv 0, \\
D_{+a}^+ \Phi_{++} &= D_{-a}^- \Phi_{++} = 0, & T \Phi_{++} &\equiv 3\Phi_{++}, \\
D_{-a}^- \Phi_{--} &= D_{+a}^+ \Phi_{--} = 0, & T \Phi_{--} &\equiv -3\Phi_{--}. 
\end{align*}
\]

Equivalently, \( \gamma \) is a strongly chiral superfield of a first kind and \( \Phi_{++} \) is of a second kind one (see Sec. 4). The r.h.s. of the last equations are not actual constraints in \( M^{7|8} \), they just exhibit the \( T \)-weights of the superfields introduced. The peculiar notation \( \Phi_{++} \) for the \( \Phi_6 \) superfield will be justified in a special on-shell gauge (5.12) \( \gamma = 0 \), where \( \Gamma \Phi_{++} = 2\Phi_{++}, S \Phi_{++} = -2\Phi_{++}, \) thus this field has definite weights w.r.t. both \( \Gamma \) and \( S \) generators like generators \( D_{\pm a} \) etc. do, see Eqs.(1.17,1.18).

The higher-spin dynamics is described by two (one real and one complex) equations of motion for these superfields:

\[
\begin{align*}
D_{-a}^- D_{+a}^+ \Phi_{++} + D_{+a}^+ D_{-a}^- \Phi_{--} &= 8i([D_{-a}^- , D_{+a}^+] - 2)\gamma, \\
i(D - 2)\Phi_{++} &= D_{+a}^+ D_{-a}^- \gamma.
\end{align*}
\]

These equations have gauge symmetry with real first kind strongly chiral gauge parameter \( \varepsilon \),

\[
\begin{align*}
\delta \Phi_{++} &= D_{-a}^- D_{+a}^+ \varepsilon, & \delta \Phi_{--} &= -D_{+a}^+ D_{-a}^- \varepsilon,
\end{align*}
\]

24
\[ \delta \gamma = iD\varepsilon, \quad D^\dagger = -D, \]  
\[ D^+_a\varepsilon = D^-_a\varepsilon = 0, \quad T\varepsilon \equiv 0 \]  
(5.5)

The invariance of the second equation follows from \([D, D^\pm]\) and of the first one – from the relation

\[ [D^+_a D^-_{-a}, D^+_{-a} D^-_a]\varepsilon = -8([D^+_a, D^-_{-a}] - 2)D\varepsilon, \]  
(5.6)

which is valid for scalars with zero weight w.r.t. generator \(T\). The null \(\tilde{M}^{7|4}\) component of \(\varepsilon\) is nothing but the gauge parameter of GKS formulation (2.18).

Using the previous section results one concludes that the component content of superfields (5.1) is characterized by one real and one complex scalar in \(\tilde{M}^{7|4}\). This means that neither auxiliary nor pure gauge degrees of freedom were added to the dynamical content of the model (2.33). It can be considered as the reflection of the fact that though the \(N = 2\) invariance (2.42) of the equations of motion (2.39–2.41) is implicit, its algebra closes off-shell and without gauge additions (2.38). One can see that the manifest \(N = 2\) supersymmetry (realized on superfields (5.1) in terms of a canonical left action of a group on its homogeneous space), gives the laws (2.42) on the component level. This canonical action in \(M^{7|8}\) is given by a Killing operator analogous to operator \(K\) (2.26) in \(\tilde{M}^{7|4}\), and its component form can be derived with help of rules of component analysis formulated in the end of subsection 3.3. It agrees with the fact that the components \(Z, \rho\) and \(\gamma\) from (2.33) decompose in two \(N = 2\) superfields \(\Phi^{++}\) and \(\gamma\) (5.1). The coincidence of this explicit symmetry with (2.42) means that it does not leave the action invariant (see Sect.2). That is why though the proposed equations of motion are Lagrangian and the corresponding action is \(N = 2\)-supersymmetric, it cannot be expressed in explicitly \(Osp(2|4)\)-invariant form via the superfields \(\Phi^{++}\) and \(\gamma\). Nevertheless, there still remains the question if there exists a formulation with an explicit realization of the supersymmetry \(\tilde{\delta}_t\) (2.35). In such a formulation, the action would have to possess the manifestly invariant form.

Now we investigate the component form of the equations (5.2,5.3). To begin with, consider the null components of these equations. It was shown in Sect.4 that the superfield \(\gamma\) contains only one independent real component (which we also denote \(\gamma\)) and the superfield \(\Phi^{++}\) has only one independent complex component \(a\).

\[ \gamma(z, r) = \gamma(z, r, \theta)| \equiv \gamma(z, r, \theta = 0), \]  
\[ a = \Phi^{++}|, \quad D^-_a \Phi^{++}| \equiv \psi_a = -\sqrt{2}\tilde{\Sigma}_a a, \]  
\[ (D^-)^2\Phi^{++}| \equiv b = -\sqrt{2}\tilde{\Sigma}_a^\dagger \psi_a, \]  
(5.7,5.8)

with \(\tilde{\Sigma}_a\) defined in (3.29).

Now we are in a position to consider \(\theta = 0\) components of the equations (5.2,5.3). To calculate them one should, first of all, pass to \(Osp(1|4)\)-covariant derivatives \(\tilde{\Sigma}_t\) by the formulas (3.50–3.52) and definitions (5.7,5.8) for components. Second, one should use the expressions (3.24–3.29) for these derivatives. At last, one has to pass
to $SL(2, C)$ covariant indices by (3.18). Then all the dependence on the small vielbein and its derivatives (3.21–3.23) disappear and one is rest with ordinary $N = 1$ AdS superspace covariant equations on $\tilde{M}^{7|4}$. In so doing we come to the following two equations:

\[-(i(\mathcal{P} - \bar{\mathcal{P}}) + \frac{1}{2}(\mathcal{Q} + \bar{\mathcal{Q}}))a - c.c. = 16i(\mathcal{P} + \bar{\mathcal{P}} - 1)\gamma, \quad (5.9)\]
\[-\frac{i}{2}(\nabla + 1)a = \{i(\mathcal{P} - \bar{\mathcal{P}}) - \frac{1}{2}(\mathcal{Q} + \bar{\mathcal{Q}})\} \gamma. \quad (5.10)\]

These equations convert into (2.39–2.41) after the identification $a = -4(Z + i\rho)$. The real equation (5.9) turns into (2.41) while the complex equation (5.10) gives the sum (2.39) + $i$ (2.40). So, it is proved that the dynamical equations (5.2, 5.3) for superfields (5.1) lead to the original higher spin GKS equations (2.39–2.41).

Let us discuss the higher components of equations of motion (5.2, 5.3) and prove that they do not imply any new information. First note that the second equation (5.3) is strongly chiral, it satisfies identically the same constraints the superfield $\Phi^{++}$ does. A strongly chiral superfield of this type has only one independent component – its null component (see the previous section). Hence the equation (5.10) in superspace $\tilde{M}^{7|4}$ is equivalent to the superfield equation (5.3).

The situation with the first equation (5.2) is a bit more complicated. It is convenient to consider this equation in the gauge $\gamma = 0$ which admissibility is shown after Eq.(2.34). Let us introduce a weakly chiral superfield $B = D^a_+ D^-a \Phi^{+-}$, $D^-a B \equiv 0$. Then, the equations of motion (5.2, 5.3) in this gauge take the form

\[E \equiv B + B^\dagger = 0, \quad (5.11)\]
\[(D - 2)\Phi^{+-} = 0 \quad (5.12)\]

To ascertain that all the components of the equation (5.11) are derivatives of the null one we use Eq.(5.12) which is already shown to be equivalent to (5.10) with $\gamma = 0$. Thereby (5.3) follows from the couple of equations (5.3, 5.10), not from the only Eq.(5.9). Using the algebra (1.8–1.10, 1.15–1.20) of covariant derivatives the following relation can be derived straightforwardly

\[D^+_a B = D^+_a D^-b D^-b \Phi^{+-} = 8D^+_{-a}(2 - D)\Phi^{+-}\]

Thus we come to the additional constraint $D^-a B = 0$ on superfield $B$ (note that $D^-a B = D^+a B = 0$ implies $DB = 0$ as it follows from (1.10)). So, if Eq.(5.12) is satisfied, the superfield $E$ obeys the constraints

\[D^-a E = D^+_a E = 0 \quad (5.13)\]

Therefore, $E$ is a third kind strongly chiral superfield which possesses only one independent component $E$ (see Sec. 4), this completes the proof.
6 Picture changing and manifestly $N = 2$ supersymmetric equations on $M^{7|4}$.

In this section, we show that the whole system of equations and constraints on $M^{7|8}$ may be equivalently reformulated as a system of equations and constraints on $M^{7|4} = Osp(2|4)/H$. This framework seems to be quite natural since, from the $N = 1$ viewpoint, $M^{7|4}$ is just that manifold the original manifestly $N = 1$ supersymmetric theory was defined.

Calculations of this section do not deal with $N = 1$ component expressions, and $N = 2$ covariance is kept manifest. It is convenient therefore to introduce a universal notation: given an $osp(2|4)$ coset space, both the covariant derivatives and local rotations are denoted by the same letter $D$ (except for local rotations $S_{su(2)}$ and $T$) differing by the indices they carry according to their place in $osp(2|4)$ superalgebra. In what follows, we use two equivalent notations for components of $Osp(2|4)$ generators $D_{\epsilon \zeta}$, $D_{\bar{\epsilon} \bar{\zeta}}$ and $D_{\epsilon \bar{\zeta}}$. First, we use the index notation introduced in App. C: $\epsilon = (a, z)$, $\bar{\epsilon} = (\bar{a}, \bar{z})$, etc., where $a$, $\bar{a}$ are $su(2)$-indices and $z$, $\bar{z}$ denote the single bosonic value of superindices $\epsilon$, $\bar{\epsilon}$ respectively. Then, we also use the notation explicitly indicating $T$-weights. The following expressions include all these notations

$$D_{\epsilon \zeta} = \begin{pmatrix} D_{ab} & D_{az} \\ D_{zb} & D_{zz} \end{pmatrix} = \begin{pmatrix} S_{ab} - \varepsilon_{ab}T & -D_{+a} \\ D_{-b} & 2T \end{pmatrix}, \quad C^{\bar{\epsilon} \bar{\zeta}} D_{\epsilon \zeta} = 0, \quad \text{(6.1)}$$

$$D_{\bar{\epsilon} \bar{\zeta}} = \begin{pmatrix} D_{ab} & D_{az} \\ D_{zb} & D_{zz} \end{pmatrix} = \begin{pmatrix} D_{2ab} & -D_{3a} \\ D_{3b} & 0 \end{pmatrix}, \quad D_{\epsilon \zeta} = -(-)^{\epsilon \zeta} D_{\epsilon \zeta}, \quad \text{(6.2)}$$

$$D_{\bar{\epsilon} \bar{\zeta}} = \begin{pmatrix} D_{ab} & D_{az} \\ D_{zb} & D_{zz} \end{pmatrix} = \begin{pmatrix} D_{-2ab} & -D_{-3a} \\ D_{-3b} & 0 \end{pmatrix}, \quad D_{\bar{\epsilon} \bar{\zeta}} = -(-)^{\bar{\epsilon} \bar{\zeta}} D_{\bar{\epsilon} \bar{\zeta}}, \quad \text{(6.3)}$$

The rest piece of $osp(2|4)$, the scalar derivative is always denoted $D$. Similar notations are used for superfields, for example, a supersymmetric tensor superfield $\Phi_{\epsilon \xi z \tilde{\zeta}}$ contains three $SU(2)$-irreducible component fields. At last, the covariant derivatives in superspaces $M^{7|8}$ and $M^{7|4}$ are denoted identically; the manifold, on which a derivative lives, is specified by the superfield this derivative acts.

To start with we observe that there exist special constrained superfields on $M^{7|8}$ which are equivalent to the unconstrained $M^{7|4}$ ones. Indeed, consider a superfield $\Phi^A(z, r, \theta, \phi)$ on $M^{7|8}$, which takes its values in an $su(2,0|1,0)$ fdr (or in $A(1|0)$ fdr, which is the complexification of $su(2,0|1,0)$), the index $A$ just labels this fdr’s basis. Let $D_i$ be $su(2,0|1,0)$ generators, $\hat{D}_i$ be their image in the covariant derivatives algebra on $M^{7|8}$ and $(D_i)^A_B$ be their form in the $su(2,0|1,0)$ fdr. Impose the constraints

$$\hat{D}_i \Phi^A = (D_i)^A_B \Phi^B. \quad \text{(6.4)}$$

The even part of these constraints is satisfied identically as for $S_{ab}, T$ generators the equations (6.3) are just the definition of how these generators enter the whole covariant derivatives algebra (see App. A). The constraints may be solved in a standard
way, (at least locally on $M^{7|8}$) in terms of local fields on $M^{7|4}$ which is identified with the subspace $\theta = 0$ (see Sect. 3). A general solution reads

$$\Phi^A(z, r, \theta_{\pm a}) = \exp(\theta^+ a D_{+a} + \theta^- a D_{-a})^A B \Phi^B(z, r, 0)$$  \hspace{1cm} (6.5)

It establishes a bijection between the $M^{7|8}$ constrained fields of the form (6.4-6.5) and the unconstrained local tensor fields $\Phi^B(z, r, 0) \equiv \phi^B(z, r)$ on $M^{7|4}$ transforming in the same $su(2, 0|1, 0)$ fdr as the original $M^{7|8}$ ones. If the representation labeled by $A$ is irreducible, then, for every fixed value of $A$, the component fields $\Phi^A|, D_{\pm a} \Phi^A|\ldots$ are obviously in one-to-one correspondence with the set of all $\phi^B(z, r)$ components; the pictures arising for different $A$ are connected by an $A(1|0)$ transformation.

In order not to be too abstract it is pertinent to introduce now an example and then illustrate all the ideas on it, the generalization to an arbitrary fdr is straightforward. Consider the $A(1|0)$—representation $(2|2)_p, p \not\equiv \pm 1$ (see Appendix C, Eqs. (C.10- C.11)[1]), so the field’s $\Phi_A$ components are $(p, p \pm 1$ labels the $T$-weight and $a$ is $su(2)$ spinor index)

$$\Phi_A = (\Phi_{pa} , \Phi_{p-1} , \Phi_{p+1}),$$

while the constraints (6.4) read (compare to (C.11))

$$D_{+ b} \begin{pmatrix} \Phi_{p a} \\ \Phi_{p+1} \\ \Phi_{p-1} \end{pmatrix} = \begin{pmatrix} \varepsilon_{ba} \Phi_{p+1} \\ 0 \\ 1 \frac{1}{2} \Phi_{pb} \end{pmatrix}; \quad D_{- b} \begin{pmatrix} \Phi_{p a} \\ \Phi_{p+1} \\ \Phi_{p-1} \end{pmatrix} = \begin{pmatrix} \varepsilon_{ba} \Phi_{p-1} \\ 0 \frac{1}{2} + p \Phi_{pb} \end{pmatrix}. \hspace{1cm} (6.6)$$

We see that all the content of superfields $\Phi_A$ reduces to the component fields of superfield $\Phi_{p+1}$ because the part of relations (6.4) leads to

$$\Phi_{p a} = 2(1 + p)^{-1} D_{-a} \Phi_{p+1} ; \quad \Phi_{p-1} = (D_{-})^2 \Phi_{p+1},$$  \hspace{1cm} (6.7)

while the rest constraints imply

$$D_{+ a} \Phi_{p+1} = 0. \hspace{1cm} (6.8)$$

So, one can equivalently describe any $H$-tensor superfield on $M^{7|4}$ in two ways: first, as a cross section of the $H$-bundle on $M^{7|4}$, second, as a constrained superfield on $M^{7|8}$. We shall refer to the first way as to $M^{7|4}$-picture and to the second one as to $M^{7|8}$-picture. For instance, it is seen that the weakly chiral (w.r.t. $D_{+ a}$—derivative) one-component superfield $\Phi_{p+1}$ on $M^{7|8}$ is equivalent to the $\phi_i$ unconstrained superfield on $M^{1|4}, p \not\equiv \pm 1$, or they form the two pictures of the same $osp(2|4)$ superalgebra representation.

Recall that the $N = 2$ covariant derivatives on $M^{7|8}$ constitute, along with $su(2, 0|1, 0)$-inner rotations, the full $osp(2|4)$-superalgebra which decomposes into direct sum of four graded subspaces w.r.t. $su(2, 0|1, 0)$ (1.28-1.28). Therefore, for the covariant derivative set the constraints (6.4) are satisfied identically.

In this section, we use extensively the results and notation of Appendix C.
The main result of this section is the observation that all equations and constraints of Sect. 5 constitute the $M^{7|8}$-picture of some $M^{7|4}$ tensor equations. First note that, as is obvious from the above analysis, the constrained fields $\Phi_6 \equiv \Phi_+^+$ and $\gamma$ (5.4) may be viewed as $M^{7|8}$-picture of $M^{7|4}$ tensor fields $\phi_i$, and $\gamma$, (the strongly chiral field $\gamma$ corresponds to the trivial $(1|0)$-representation). Recalling the formula (C.10) from the App. C we represent the $\phi_i$ field as the supersymmetrized product of three $\xi$ fdr's: $\phi_{\xi(3)}$, with the highest weight component being identified with $\Phi_6$:

$$
\Phi_6| = \phi_6 = \phi_{zzz}, \quad \Phi_{5a}| = \phi_{5a} = \phi_{azz}, \quad \Phi_4| = \phi_4 = \varepsilon^{ba} \phi_{abz} \quad (6.9)
$$

The $\Phi_6, \Phi_{5a}, \Phi_4$ superfields are assumed to obey the equations (6.6–6.8) with $p = 5$. It is clear from the above discussion that, in $M^{7|4}$ picture, the counterparts of constraints $D_+^a \Phi_6 = 0$ are absent. At the same time, the constraint

$$
D_3^a \Phi_6 = 0 \quad (6.10)
$$

turns into a tensor equation

$$
D_{\xi_1(\xi_2} \phi_{\xi_3\xi_4)} = 0. \quad (6.11)
$$

The round brackets denote supersymmetrization, as usual. As covariant derivatives $D_{\xi_1\xi_2}$ are superantisymmetric in its indices, the last tensor equation is characterized by the Young tableaux of the form

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+ +
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To prove the relation (6.11), it is sufficient to observe that the first $M^{7|4}$ projection of (6.10) presents the highest-weight component of (6.11) and the next projections, associated with all $D_{\pm a}$ derivatives of (6.11), are identified with the rest components of the tensor equation (6.11). It is important that one should not break the $H$-covariant structures by passing to the $N = 1$ components as in Sections 3 and 4. Indeed, consider the basis in $osp(2|4)$ superalgebra introduced in (3.32,3.33). Then it follows from Eqs. (3.50–3.52) that for combinations $\Sigma_I$ (3.33), the relation holds:

$$
(\Sigma_I \Phi^A)| = \tilde{\Sigma}_I(\Phi^A) \quad (6.12)
$$

Due to the constraints (6.4) the same equation is valid for all generators $D_{\xi\zeta} \in osp(2|4)$:

$$
(D_{\xi\zeta} \Phi^A)| = D_{\xi\zeta}(\Phi^A) \quad (6.13)
$$

where in r.h.s. $D_{\xi\zeta}$ acts as the local rotation on index $A$. So, these equations are satisfied for every generator of $osp(2|4)$ and give the $osp(2|4)$-covariant way to pass to $M^{7|4}$-components of superfields in $M^{7|8}$. For example, one has

$$
(D_{3a} \Phi_6)| = D_{3a}(\Phi_6)| = D_{z a} \phi_{zzz} = 24 D_{z(a} \phi_{zz)} \quad (6.14)
$$
for the first $M^{7,4}$ projection, or

$$ (D_{-b}D_{3a}\Phi_6) = $$

$$ D_{-b}D_{3a}(\Phi_6) = |D_{-b}D_{3a}\phi_6 - D_{3a}D_{-a}\phi_6 = \Sigma_{++ba}\phi_6 - 3D_{3a}\phi_{5b} = $$

$$ (6.15) $$

$$ = D_{-a}D_{2a}\phi_{zzz} = D_{ba}\phi_{zzz} - 3D_{za}\phi_{bzz} = 2D_{a(b}\phi_{zzz}) $$

for the "$D_{-b}$"-projection, and so on. This proves the Eq. (6.11).

To find the picture changed $M^{7,4}$ reformulation of two equations (5.2, 5.3) introduce supersymmetric third-rank tensor operators (for the definition of invariant seven-rank $\tilde{C}$-tensors, see App.C, Eqs.(C.24-C.28))

$$ \Delta_{\epsilon_1\epsilon_2\epsilon_3} = D_{\epsilon_1\eta_1} D_{\epsilon_2\eta_2} \tilde{C}_{\epsilon_1\epsilon_2\eta_1\eta_2\epsilon_3} $$

$$ \Delta_{\epsilon_1\epsilon_2\epsilon_3} = D_{\epsilon_1\eta_1} D_{\epsilon_2\eta_2} \tilde{C}_{\epsilon_1\epsilon_2\eta_1\eta_2\epsilon_3}, $$

possessing the components

$$ \Delta_{zzz} = -4(D_3)^2 $$

$$ \Delta_{zzz} = -4(D_{-3})^2 $$

$$ \Delta_{zzc} = -8(D_3 D_{2a}) $$

$$ \Delta_{zzc} = -8(D_{-3} D_{-2a}) $$

$$ \Delta_{zcd} = -2\varepsilon_{cd} (D_{2b} D_{2ba}) $$

$$ \Delta_{zcd} = -2\varepsilon_{cd} (D_{-2b} D_{-2ba}) $$

Also, we need another operator of the form

$$ \Delta_{\epsilon\epsilon} = D_{\epsilon\eta} D_{\eta\epsilon} + (-)^{\epsilon\eta} D_{\epsilon\eta} D_{\eta\epsilon} $$

with the components

$$ \Delta_{z\bar{z}} = -(D_3^a D_{-3a} + D_{-3}^a D_{3a}) \equiv -[D_3 D_{-3}] $$

$$ \Delta_{a\bar{z}} = -2D_{2a}^b D_{-3b} $$

$$ \Delta_{z\bar{a}} = -2D_{-2a}^b D_{3b} $$

$$ \Delta_{a\bar{b}} = D_{2a}^c D_{-2b} - D_{-2a}^c D_{2b} + D_{-3a} D_{3b} - D_{3a} D_{-3b} $$

Then, the $M^{7,4}$ picture of equations (5.2, 5.3) looks as follows (the "[=]") symbol means” = " with an account of a sign factor (App. B)):

$$ -\frac{1}{4} \Delta_{(3)\gamma} = i(D - 2)\phi_{(3)} $$

$$ \frac{1}{4} \left( \Delta_{(3)\phi_{(3)}} - (-)^{(3)\epsilon(3)} \Delta_{(3)\phi_{(3)}} \right) = -8i (C_{\epsilon\epsilon} C_{\epsilon\epsilon} \Delta_{\epsilon\epsilon} - 2C_{\epsilon\epsilon} C_{\epsilon\epsilon} C_{\epsilon\epsilon}) \gamma $$

(6.20)

Indeed, the "zzz"-component of the first equation and the "zzz, zzz" component of the second equation are

$$ -\frac{1}{4} \Delta_{zzz} \gamma = i(D - 2)\phi_{zzz} $$

$$ \frac{1}{4} \left( \Delta_{zzz}\phi_{zzz} - \Delta_{zzz}\phi_{zzz} \right) = 8i(\Delta_{zz} - 2)\gamma $$

(6.21)
which, after accounting the component form of the operators (6.17, 6.19) are seen to co-
incide with $M^7|4$-projection of Eq. (5.2,5.3) upon the identification $\Phi_6 \equiv \Phi^{++}, \Phi_{-6} = -\Phi^{+-}$. It remains to check that the rest components of Eq.(6.20) are in one to one
correspondence with all components of $M^7|8$ equations (5.2, 5.3). We have performed
this check explicitly and have found they match indeed.

The $M^7|4$-picture of gauge invariance reads
\[
\delta \phi_\epsilon(3) = -\frac{1}{4} \Delta_\epsilon(3) \varepsilon, \quad \delta \gamma = iD\varepsilon
\]  
(6.22)

where $\varepsilon$ is a real scalar field presenting the unique $M^7|4$ projection of the strongly
chiral field of a first kind $\varepsilon$ on $M^7|8$.

This is easy to prove since the ”zzz” component of the first equation gives null
component of the gauge transformation law (5.4) and the rest components coincide
with $\theta = 0$ projections of the $D_{\pm a}$-derivatives of this law; the only component of the
second equation is identical to the only null component of (2.3). On the other hand,
one may check the gauge invariance in $M^7|4$-picture directly, by using the covariant
derivatives algebra (1.32–1.37).

Thus, we have observed that the higher spin equations (5.2,5.3) are well defined
on $M^7|4$ and derived such a picture. It is characterized by two $M^7|4$-superfields $\phi_\epsilon(3)$
and $\gamma$ satisfying the equations (6.11,6.20), which turn out to be gauge invariant w.r.t.
transformations (6.22).

A few remarks are in order. Recall that the fdrs $\xi(3)\bar{\xi}(3)$ and $\xi(2)\bar{\xi}(2)$ have equal
dimension (see App.C), therefore there is no information loss if one takes the first
trace of the second equation (6.14) by an invariant second rank tensor $C^{\xi\bar{\xi}}$ (C.6 -
C.8):
\[
\frac{1}{4} \left( \Delta_\epsilon(2) \bar{\xi}_\epsilon(2) - (\epsilon(2))^\epsilon(2) \Delta_\epsilon(2) \bar{\xi}_\epsilon(2) \right) \left[ = \right]
\]
\[
\left[ = \right] -8i \left( \frac{1}{6} C_{\epsilon\epsilon} \Delta_\epsilon + \frac{1}{9} C_{\epsilon\epsilon} C_{\epsilon\epsilon} \Delta_\bar{\epsilon} + \frac{1}{3} C_{\epsilon\epsilon} C_{\epsilon\epsilon} \right) \gamma
\]
(6.23)

Now we apply the analysis of $\xi(2)\bar{\xi}(2)$ representation from Appendix C (C.33 - C.44)
to the l.h.s. and the r.h.s. of the last equation.

Obviously, the r.h.s $\in H_{\text{trace}} \subset H^7|8$ where $H^7|8$ is a subspace of double-traceless
tensors and $H_{\text{trace}}$ is defined by (C.38). Therefore, the l.h.s. obeys this condition
either. The relation l.h.s $\in H^7|8$ leads to the covariant $\gamma$-independent consequence of
(6.23):
\[
\frac{1}{4} \left( \Delta_\epsilon(3) \phi_\epsilon(3) - \Delta^\epsilon(3) \phi_\epsilon(3) \right) = 0
\]  
(6.24)

Further, $H_{\text{trace}}$ is a general solution of the constraints (C.42, C.44) so one gets
\[
0 \left[ = \right] C_{\epsilon(2); \bar{\epsilon}} \bar{\xi}(2) (l.h.s.)(\xi(2)(\bar{\xi}(2)
\]  
\[
0 \left[ = \right] C_{\bar{\epsilon}(2); \bar{\epsilon}} \xi(2) (l.h.s.)(\xi(2)(\bar{\xi}(2)
\]  
(6.25)
i.e. two additional $\gamma$-independent consequences of Eq. (6.23). Here 'l.h.s.' means left hand side of the Eq. (6.23).

In the $M^{7|8}$-picture, these consequences are expressed via the constraint

$$
(D_{\pm})^2 (r.h.s.) = (D_{\pm})^2 (8i(\Delta_{zz} - 2)) \gamma = 0, \quad (6.26)
$$

Therefore, the l.h.s. should obey these equations either:

$$
(D_{\pm})^2 \frac{1}{4} (\Delta_{zzzz} \Phi_{-6} - \Delta_{zzzz} \Phi_{-6}) = 0 \quad (6.27)
$$

The $M^{7|4}$-projections of the $D_{\pm}$-derivatives of the last equation give rise to tensor equations (6.25), which include only $\phi_{e(3)}$ and not $\gamma$. All these additional equations are gauge-invariant as the equations (6.11, 6.20) are.

7 Conclusion.

Let us summarize the results. We succeeded in constructing two manifestly $N=2$ covariant formulations for the equations of motion of GKS theory [4], which describes free massless fields of all superspins in $D = 4$ AdS space in terms of a few scalar fields on $M^{7|4}$. The first formulation is achieved through employing the constrained (strongly chiral) superfields in $N = 2$ superspace $M^{7|8} = Osp(2|4)/U(2)$ while the second one uses a smaller superspace $M^{7|4} = Osp(2|4)/SU(2,0|1,0)$ which, from $N = 1$ viewpoint, is just the same manifold the original GKS theory is defined. The obtained equations have a neat form and, in a sense, look simpler than $N = 1$ ones.

Unfortunately, all of this does not give a possibility to construct an $N = 2$ manifestly invariant form of the action, since the laws of global $N = 2$ transformations, which follow from the superfield formalism developed in the paper, differ from those leaving the action invariant by terms proportional to the equation of motion multiplied by a symmetric matrix.

However, there still exist a lot of possibilities to improve the situation. Here we discuss how one can try to construct an explicitly $N = 2$ invariant action. As far as the action is not invariant w.r.t. transformations $\delta (2.42)$, one should make explicit the supersymmetry $\tilde{\delta} (2.33)$. These transformations have the structure of a semidirect sum and an invariant subspace extracted by the equation $\gamma = 0$ exists. Note that the algebra of global transformations $\tilde{\delta}|_{\gamma=0}$ is determined by the algebra of vector fields $\mathcal{N} (2.30)$; the addition of scalar $t$ with an arbitrary factor does not change the algebra. Thus there is a one-parameter set of multiplets, each multiplet consists of two real superfields $(Z, \rho)$ transforming through each other by the laws differing from $\tilde{\delta}|_{\gamma=0}$ by the addition of scalar $t$ with an arbitrary factor.

On the other hand, it is possible to construct different versions of $M^{7|8}$ as $M_{(a,b)}^{7|8} = Osp(2|4)/SU(2) \oplus U(1)_{(a,b)}$ where the $su(2)$ generators are the same as for $M^{7|8}$ while the $U(1)_{(a,b)}$ generator is taken as $a\Gamma + bS$ (in the present paper we dealt with $a =
$1, b = -\frac{1}{2}$). Then one has to study diverse chiral superfields on these manifolds just analogously to the procedure of Sec. (4) and find those reducing to unconstrained superfields on $\tilde{M}^{7|4}$. The existence of the superfields of such a kind is hinted by the observed evidence of one-parametric family of nonequivalent $Osp(2|4)$ multiplets. It is probably that, at some specific value of $(a, b)$, the chiral superfields on $M^{7|8}_{(a, b)}$ may be found which reproduce the semidirect type global supersymmetry laws $\tilde{\delta}$ for its $\tilde{M}^{7|4}$ components. Then there should exist a manifestly supersymmetric action.

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APPENDIX A. Covariant derivatives on a coset space.
Let us consider a group $G$, its subgroup $H$ and the homogeneous space $M = G/H$. The group $G$ can be treated as a principle fibre bundle $(G, M, H)$ with the base space $M$ and the structure group = standard fibre $H$. This is the fibre bundle of the group in its cosets. All the covariant derivatives used in the present paper arise in a construction of this type. Here we give general formulas and argue the existence of covariant derivatives.

Let $X_a, a = 1, \ldots, \dim H$ be a basis in Lie algebra $L_H$ of subgroup $H$. The first supposition we have to do is that the set of vectors $X_a$ can be complemented by vectors $X_\alpha, \alpha = 1, \ldots, \dim G - \dim H$ to give the basis $\{X_a, X_\alpha\}$ in Lie algebra $L_G$ of $G$ so that the commutation relations are partially graded:

\[
\begin{align*}
[X_a, X_b] &= C^c_{ab} X_c, \quad (L_H \text{ is subalgebra,}) \quad (A.1) \\
[X_a, X_\alpha] &= C^\beta_{a\alpha} X_\beta, \quad C^b_{a\alpha} = 0, \quad (A.2) \\
[X_\alpha, X_\beta] &= C^\gamma_{\alpha\beta} X_\gamma + C^{a\beta}_{\alpha\alpha} X_a. \quad (A.3)
\end{align*}
\]

Note that this choice of vectors $X_\alpha$ can always be done if $L_G$ is a semisimple algebra and $L_H$ do not contain null vectors. Then $X_\alpha$ can be chosen in to belong to orthogonal complement of $L_H$. For the factor-spaces considered in this work the choice with (A.2) is always possible. But if we chose the (Abelian) subalgebra spanned by $D_{\pm a}^\pm$ in the superalgebra $osp(2|4)$ (1.8–1.10, 1.15–1.20), a choice with (A.2) would be impossible. This could cause problems in attempts to construct covariant derivatives.

Now we turn to the principle bundle $(G, M, H)$. Let $\pi : G \to M$ be canonical projection

\[
\forall g \in G, \quad \forall h \in H \quad \pi(gh) = gh \equiv m \in M \quad (A.4)
\]

where left coset $gH$ is to be treated as a point of manifold $M$. A fibre bundle is defined by trivializations for every open domain $U$ from an open covering of base
space $M$. Trivialization is a one-to-one map $\psi$ satisfying

$$\psi : U \times H \to \pi^{-1}(U) \subset G, \quad \pi \circ \psi(m, h) = m.$$  

In our case this trivialization reduces to a smooth function $g(m)$ which maps $U \to G$ and satisfies $\pi(g(m)) = m$. Then $\psi(m, h) = g(m)h$.

Let $h^i_j$ be matrices of a representation of subgroup $H$ in a vector space $V$ in some basis:

$$[hv]^i = h^i_j v^j,$$

where $v^i$ are components of a vector $v \in V$. It is known that there is the unique vector bundle $(E, H, V, H)$ associated with principle bundle $(G, M, H)$ and the chosen representation of subgroup $H$ in a standard fibre $V$. Let $\pi_E$ be canonical projection in $E$ which maps the fibre $V_m$ over the point $m$ to $m$: $\pi_E(V_m) = m$. Sections $s(m)$ in the vector bundle can be described by vector functions $v : G \to V$ constrained by

$$h^i_j v^j(gh) = v^i(g) \quad \forall g \in G, \quad h \in H. \quad (A.5)$$

The bases in fibres $V_m = \pi_E^{-1}(m)$ are defined by the trivialization $g(m)$. The components of the section corresponding to the function $v(g)$ \((A.6)\) in these bases look like

$$s^i(m) = v^i(g(m)). \quad (A.6)$$

The left action of the group $G$ on its homogeneous manifold is defined as

$$\forall f \in G \quad D_f : M \to M, \quad D_f(gH) = fgH, \quad D_f m = \pi(fg(m)).$$

This action induces the regular representation of the group $G$ in the space of scalar functions $\phi$ in the manifold $M$:

$$T_f \phi(m) = \phi(D_{f^{-1}}m).$$

Analogously the regular representation of the group $G$ in the space of sections in the vector bundle $E$ can be defined. It is easier to do in terms of vector functions $v$ \((A.5,A.6)\):

$$T_f v(g) = v(f^{-1}g). \quad (A.7)$$

It is evident that new function $T_f v$ also satisfies the equation \((A.5)\) and thus defines some section. We can write this action in the infinitesimal form: for every vector $X \in L_G$

$$T_X v(g) = lv(g) \quad (A.8)$$

here $l$ is right-invariant vector field corresponding to $X$,

$$l(g) = Xg \equiv dR_g X, \quad R_g g' = g' g \quad (A.9)$$

where $dR_g$ is the differential of the right shift $R_g$ (a right-invariant vector field generates left action and therefore denoted '$l$').
Let us denote \( l_a, l_\alpha \) right-invariant and \( r_a, r_\alpha \) left-invariant vector fields corresponding to basis vector fields \( X_a, X_\alpha \):

\[
\begin{align*}
  l_a(g) &= X_a g, & l_\alpha(g) &= X_\alpha g, & Xg &\equiv dB_g X \\
  r_a(g) &= gX_a, & r_\alpha(g) &= gX_\alpha, & gX &\equiv dB_g X
\end{align*}
\]  

(A.10)

where \( L_g g' = gg' \). It is convenient to introduce also dual basis of 1-forms

\[
\begin{align*}
  l^\alpha \cdot l_\beta &= \delta^\alpha_\beta, & l^a \cdot l_b &= \delta^a_b, & l^\alpha \cdot l_b = l^a \cdot l_\beta = 0
\end{align*}
\]

and the same for \( r^\alpha, r^a \) via \( r_\alpha, r_a \). Dots stand for the contraction of vector with 1-form. Then, in every trivialization, the connection in the principle bundle is the Lie algebra valued 1-form in the base space being equal to

\[
A = (g^* l^a) X_a
\]  

(A.11)

where \( g^* l^a \) is the pull-back of the 1-form \( l^a \) from point \( g(m) \) to point \( m \). The exterior covariant derivative is equal to

\[
\mathcal{D} = d + (g^* l^a) X_a.
\]  

(A.12)

To obtain the usual covariant derivative the vielbein vector field \( e_\alpha \) and its inverse 1-form have to be introduced in the base space:

\[
\begin{align*}
  e^\alpha &= g^* r^\alpha, & e_\alpha &= d\pi(r_\alpha(g(m))),
\end{align*}
\]

(A.13)

here \( d\pi \) is the differential of the canonical projection. One can show that \( e^\alpha \cdot e_\beta = \delta^\alpha_\beta \). Then the covariant derivatives are

\[
\mathcal{D}_\alpha = e_\alpha \cdot \mathcal{D} = e_\alpha + A_\alpha^a X_a, \quad A_\alpha^a = e_\alpha \cdot (g^* r^a).
\]  

(A.14)

Consider a section \( s \) in a vector bundle with a representation \( T \) of group \( H \) on a fibre. Then \( \mathcal{D}_\alpha s \) is also a section in the vector bundle associated with the same principle bundle and with the representation \( T \otimes \text{Ad}_\perp \) in a fibre. Here \( \text{Ad}_\perp \) denotes the restriction of adjoint representation of subgroup \( H \) to the linear envelope of vectors \( X_\alpha \). Further, one can show that

\[
\mathcal{D}_\alpha s^i(m) = r_\alpha v^i(g(m)),
\]  

(A.15)

the action of covariant derivative on the section \( s \) equals to the action of left-invariant vector field on the function \( v^i(g) \) (A.6). That is why derivatives \( \mathcal{D}_\alpha \) commute with left infinitesimal action (A.8) of the algebra \( L_G \) on sections. In this sense these derivatives are covariant. The action of local rotations \( M_a \) on a section can be generated by the action of vector fields \( r_a \) on the corresponding function \( v(g) \) in the group as follows from infinitesimal form of Eq.(A.5). Hence the algebra of covariant derivatives and
local rotations coincides with the algebra of left-invariant vector fields, that is with algebra $L_G$:

\[
\begin{align*}
[M_a, M_b] &= C^c_{ab} M_c, \\
[M_a, D_\alpha] &= C^\beta_{a\alpha} D_\beta, \\
[D_\alpha, D_\beta] &= C^\gamma_{\alpha\beta} D_\gamma + C^a_{a\alpha} M_a.
\end{align*}
\] (A.16, A.17, A.18)

Let us emphasize that the covariant derivatives $D_\alpha$ were expressed for general case in terms of geometrical objects (left and right invariant vector fields, projections and trivialization map $g(m)$) that are available for any group $G$ and its subgroup $H$ provided in (A.2) $C_{a\alpha}^b = 0$. If we have two subgroups $H \subset L \subset G$, then three factors-spaces arise: $G/H$, $G/L$ and $L/H$. It can be shown in general case that covariant derivatives in space $G/H$ can be expressed in terms of derivatives in space $L/H$, vector fields in $L$, projection $\pi : L \rightarrow L/H$ and trivialization $l : L/H \rightarrow L$. For short, we do not consider this in detail. In subsection 3.a, we find these expressions for $G/H$ covariant derivatives in particular case $G = Osp(1|4)$, $L = SL(2, C)$, $H = SU(2)$ from the requirement that they satisfy the proper algebra. All the necessary geometrical objects constructed from vector fields, projection and trivialization in this case reduce to the ’small vielbeins’ (3.13), their tensor products and derivatives.

**APPENDIX B. The supertensor notation.**

Given a basis in a finite-dimensional supervector space $H$, a supervector is characterized by a set of components $\eta_{\mu}$, with the Grassmann parity being assigned by the rule $\varepsilon(\eta_{\mu}) = \varepsilon(\mu) \equiv \mu$, where $\varepsilon(\mu) = 0, 1 \mod 2$ is the grading function of this basis. Equivalently, one has

\[ \eta_{\mu_1} \eta_{\mu_2} = (-)^{\mu_1 \mu_2} \eta_{\mu_2} \eta_{\mu_1} \] (B.1)

Let

\[ \eta^\prime_{\mu} = G_{\mu}^{\nu} \eta_{\nu} ; \varepsilon(G_{\mu}^{\nu}) = \varepsilon(\mu) + \varepsilon(\nu) \equiv \mu + \nu \] (B.2)

be a representation of a supergroup $G$. The contragredient representation is defined by the rule

\[ \chi^\mu = \chi^\nu (G^{-1})^\nu_{\mu}, \] (B.3)

where

\[ (G^{-1})^\rho_{\mu} G^{\mu}_{\nu} = \delta^\rho_{\nu} \] (B.4)

and $\delta^\rho_{\nu}$ is an ordinary $\delta$-symbol.

An invariant contraction, therefore, takes the form

\[ \chi^\nu \eta_\nu = (-)^{\nu} \eta_\nu \chi^\nu = Str \eta_\nu \chi^\nu \] (B.5)

A supertensor $A_{\mu_1 \mu_2 \mu_3 \mu_4 \ldots}$ is defined to be transformed, by definition, as a product of supervectors and supercovectors being multiplied in the same order the indices $\mu_1 \mu_2 \nu_1 \mu_3 \nu_2$ are:

\[ A_{\mu_1 \mu_2 \mu_3 \nu_2 \ldots} = \eta_{\mu_1} \eta_{\mu_2} \chi_{\mu_3}^{\nu_1} \eta_{\nu_3} \chi_{\mu_2}^{\nu_2} \ldots \] (B.6)
For example,
\[ D^{\nu \mu} = (-)^{\nu' + \nu + \mu' + \mu} (G^{-1})^{\nu}_{\nu'} G^{\mu'}_{\mu} D^{\nu'}_{\mu'}, \quad (B.7) \]

To perform a covariant contraction (i.e., a contraction with a supertensor result) over one upper and one lower index one have to transfer them to each other by indices permutation and multiply the whole expression by the corresponding sign factor according to the Eq. (B.1) and then contract them by the rule (B.5). For example, one can make the contractions like

\[ B_{\mu_1 \mu_2 \nu_2 \ldots} = A_{\mu_1 \mu_2 \nu \nu_2 \ldots} \]
\[ D_{\mu_1 \nu_1 \mu_3 \ldots} = (-)^{\nu(\nu_1 + \mu_3 + 1)} A_{\mu_1 \nu_1 \mu_3 \nu \ldots} \quad (B.8) \]
\[ B_{\mu_1 \ldots \nu} = (-)^{\mu} B_{\mu_1 \mu \nu \ldots} = (-)^{\mu} A_{\mu_1 \mu \nu \nu \ldots} \]

Two nonzero tensors may be equal only if their indices orders are equivalent. Since we will work with tensor expressions in the main text, the indices order in the right hand side of some equation is unambiguously determined by the one of the left hand side. With this in mind we equivalently represent the formulas like (B.7) as follows

\[ D^{\nu \mu} \equiv (G^{-1})^{\nu}_{\nu'} G^{\mu'}_{\mu} D^{\nu'}_{\mu'} \quad (B.9) \]

The notation "\( \equiv \)" means that each term in the right hand side should be multiplied by a sign factor arising when the indices to be contracted are transferred to their partners and the rest indices are put to the same order as in the left hand side. It’s easy to see that this prescription determines the sign factors unambiguously.

The supersymmetrization and superantisymmetrization are defined standardly as follows,

\[ A_{(\mu \nu)} = \frac{1}{2} (A_{\mu \nu} + (-)^{\mu \nu} A_{\nu \mu}) \]
\[ A_{[\mu \nu]} = \frac{1}{2} (A_{\mu \nu} - (-)^{\mu \nu} A_{\nu \mu}), \quad (B.10) \]

the generalization to the multiple number of indices is straightforward. We also employ the condensed notation \( A_{\mu(n)} \) and \( B_{\mu[n]} \) for supersymmetric and superantisymmetric tensors of rank \( n \).

The standard notation

\[ (-)^{\epsilon(k)\mu(l)} \equiv (-)^{[k]\mu[l]} \equiv (-)^{(\epsilon_1 + \epsilon_2 + \ldots + \epsilon_k)(\mu_1 + \mu_2 + \ldots + \mu_l)} \quad (B.11) \]

is assumed.

In the main text, one meets the supertensors transforming in the tensor products of two nonequivalent representations. All the rules of this appendix are assumed to be employed, the only point is that the signs factors arising from a permutation of nonequivalent indices should be also taken into account.
APPENDIX C. The $A(1|0)$ superalgebra and properties of its finite dimensional representations.

The $A(1|0)$ superalgebra is defined via the following graded commutation relations:

\[
[S_{a(2)}, S_{b(2)}] = 2\varepsilon_{ab} S_{ab}
\]

\[
[S_{a(2)}, D_{\pm b}] = \varepsilon_{ab} D_{\pm a}
\]

\[
[T, D_{\pm a}] = \pm \frac{1}{2} D_{\pm a}
\]

\[
[D_{+a}, D_{-b}] = S_{ab} + \varepsilon_{ab} T ; \quad S_{ab} = S_{ba}
\]

The other commutators vanish. Among the real forms of this superalgebra is $su(2,0|1,0)$ superalgebra with the even part being isomorphic to $su(2) \oplus so(2)$, this real form is explored in the main text.

The irreducible finite dimensional representations (fdrs) of $A(1|0)$ were classified by Tierry-Mieg [14]. For our purposes it is useful, however, to develop an independent treatment, including non completely reducible fdrs. We do not pursue an exhaustive classification and just describe the fdrs relevant for the constructions of the paper and their close relatives, along with the information appropriate for our exposition in the main text.

First, let’s find the representations with smallest dimensions. Since the even generators of $A(1|0)$ are expressed via the odd ones (see Eq.(C.1)), a fdr is trivial if the odd generators act trivially. Therefore, there are no nontrivial $(1|0)$, $(0|1)$, $(1|1)$ representations. For the dimension $(1|2)$, one finds exactly two inequivalent fdrs: $(1|2) \equiv \xi$ and $(1|2) \equiv \bar{\xi}$, they are both irreducible. We denote the elements of corresponding supervector spaces by $\phi_\xi$ and $\phi_{\bar{\xi}}$. The decomposition of $\phi_\xi, \phi_{\bar{\xi}}$ representations w.r.t. even subalgebra $su(2) \oplus so(2)$ and the action of the odd generators in such a basis looks as follows

\[
\phi_\xi = \begin{pmatrix} \phi_{+a} \\ \phi_{++} \end{pmatrix} ; \quad \phi_{\bar{\xi}} = \begin{pmatrix} \phi_{-a} \\ \phi_{--} \end{pmatrix}
\]

\[
D_{+b} \begin{pmatrix} \phi_{+a} \\ \phi_{++} \end{pmatrix} = \begin{pmatrix} \varepsilon_{ba} \phi_{++} \\ 0 \end{pmatrix} ; \quad D_{+b} \begin{pmatrix} \phi_{-a} \\ \phi_{--} \end{pmatrix} = \begin{pmatrix} 0 \\ \phi_{-b} \end{pmatrix}
\]

\[
D_{-b} \begin{pmatrix} \phi_{+a} \\ \phi_{++} \end{pmatrix} = \begin{pmatrix} 0 \\ \phi_{+b} \end{pmatrix} ; \quad D_{-b} \begin{pmatrix} \phi_{-a} \\ \phi_{--} \end{pmatrix} = \begin{pmatrix} \varepsilon_{ba} \phi_{--} \\ 0 \end{pmatrix}
\]

Here the pluses and minuses label the $T$-weights, namely,

\[
T\phi_{\pm a} = \frac{1}{2} \phi_{\pm a} ; \quad T\phi_{\pm} = \pm \phi_{\pm}
\]

It is easy to see by direct analysis of commutation relations (C.1) that these $T$-weights combinations are the only ones allowing to construct a nontrivial $(1|2)$ dimensional
A(1|0) representation. In what follows, we adopt a simpler notation for the indices running the fdr’s basis: $\xi = (z, a)$; $\bar{\xi} = (\bar{z}, \bar{a})$, where $z, \bar{z}$ label one-dimensional even subspaces.

Two representations, $\xi$ and $\bar{\xi}$, are conjugated in the sense that they are mapped to each other by an outer $A(1|0)$-automorphism of the form

$$T' = -T, \quad S'_{ab} = S_{ab}, \quad D'_{\pm a} = D_{\mp a}$$

(C.5)

As a consequence, there exist the second rank invariant tensors $C_{\epsilon \xi}$ and $C_{\bar{\epsilon} \bar{\xi}}$ (see App. B for our supertensor notation):

$$C_{\epsilon \xi} = \begin{pmatrix} -\varepsilon_{ab} & 0 \\ 0 & 1 \end{pmatrix} \quad ; \quad C_{\bar{\epsilon} \bar{\xi}} = (\varepsilon)_{\xi \bar{\xi}} = \begin{pmatrix} -\varepsilon_{0} & 0 \\ 0 & 1 \end{pmatrix},$$

(C.6)

as well as their inverse tensors $C^{\epsilon \xi}$ and $C^{\bar{\epsilon} \bar{\xi}}$:

$$C^{\epsilon \xi} = \begin{pmatrix} \varepsilon_{ab} & 0 \\ 0 & 1 \end{pmatrix} \quad ; \quad C^{\bar{\epsilon} \bar{\xi}} = (\varepsilon)_{\xi \bar{\xi}} = \begin{pmatrix} \varepsilon_{0} & 0 \\ 0 & 1 \end{pmatrix},$$

(C.7)

satisfying the relations

$$C^{\epsilon \xi} C^{\bar{\eta} \bar{\xi}} = (-)^{\xi} \delta^{\epsilon \bar{\eta}} \quad ; \quad (-)^{\xi} C_{\xi \xi} C^{\bar{\eta} \bar{\xi}} = \delta_{\epsilon \bar{\eta}}$$

$$C^{\epsilon \xi} C^{\xi \xi} = C^{\epsilon \xi} - C_{\xi \xi} = -1$$

$$\varepsilon(C^{\epsilon \xi}) = \varepsilon(\xi) + \varepsilon(\bar{\xi}) = 0,$$

(C.8)

where $\delta^{\epsilon \bar{\eta}}$ and $\delta_{\xi \xi}$ are the ordinary $\delta$-symbols. These tensors may be used to raise and lower the $\xi$ and $\bar{\xi}$ indices by the rule:

$$\phi^{\xi} = C^{\epsilon \xi} \phi_{\xi} \quad ; \quad \phi^{\bar{\xi}} = C^{\epsilon \xi} \phi_{\xi};$$

$$\phi_{\xi} = \phi^{\epsilon} C^{\epsilon \xi} \quad ; \quad \phi_{\bar{\xi}} = \phi^{\epsilon} C^{\xi \xi}$$

(C.9)

Moreover, these invariant supermatrices enable one to obtain tensor form of expressions (C.2, C.4) for the action of the generators of $A(1|0)$ on $\phi_{\xi}$ and $\phi_{\bar{\xi}}$:

$$\mathcal{E}_{\epsilon \xi} \phi_{\xi} \quad = \quad C_{\epsilon \xi} \phi_{\xi} + \phi_{\xi} C_{\xi \xi},$$

$$\mathcal{E}_{\epsilon \xi} \phi_{\xi} \quad [=] \quad (C_{\epsilon \xi} \phi_{\xi} + C_{\xi \xi} \phi_{\xi}).$$

To discuss tensor products of $\xi$, $\bar{\xi}$, it’s worth working out (2|2) dimensional representations. Obviously, their decomposition w.r.t. even subalgebra should contain two $su(2)$ scalars in the even sector and one spinor in the odd sector, assigned with corresponding $T$-weights (otherwise the (2|2) is a direct sum of fdrs with lower dimensions).

Direct analysis of commutation relations (C.1) shows that there is the unique series of (2|2) fdrs, which we denote $(2|2)_{p}$, parametrized by a complex number $p \in \mathbb{C}$. 

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Denote the corresponding index by $i_p$. The decomposition w.r.t even subalgebra looks as

$$\phi_{i_p} = \left( \begin{array} {c} \phi_{pa} \\ \phi_{p+1} \\ \phi_{p-1} \end{array} \right); \quad T \left( \begin{array} {c} \phi_{pa} \\ \phi_{p+1} \\ \phi_{p-1} \end{array} \right) = \left( \begin{array} {c} \frac{p+1}{2}\phi_{p+1} \\ \frac{p-1}{2}\phi_{p+1} \end{array} \right), \quad (C.10)$$

while the odd generators action is set by the formulas

$$D_+ \left( \begin{array} {c} \phi_{pa} \\ \phi_{p+1} \\ \phi_{p-1} \end{array} \right) = \left( \begin{array} {c} \epsilon_{ba}\phi_{p+1} \\ 0 \\ \frac{1-p}{2}\phi_{pb} \end{array} \right); \quad D_- \left( \begin{array} {c} \phi_{pa} \\ \phi_{p+1} \\ \phi_{p-1} \end{array} \right) = \left( \begin{array} {c} \epsilon_{ba}\phi_{p+1} \\ \frac{1+p}{2}\phi_{pb} \\ 0 \end{array} \right) \quad (C.11)$$

The automorphism (C.5) maps the $\left(2\right)_{2\rho}$ and $\left(2\right)_{-2\rho}$ fdrs onto each other. The corresponding second rank invariant tensor $C_{ipj\rho}$ has the form

$$C_{ipj\rho} = \left( \begin{array} {ccc} -\epsilon_{ab} & 0 & 0 \\ 0 & 0 & \frac{1+p}{2} \\ 0 & \frac{1-p}{2} & 0 \end{array} \right). \quad (C.12)$$

The $\left(2\right)_{2\rho}$ fdrs are irreducible for any $p$ except the values $p = \pm 1$. In those cases, the fdrs $\left(2\right)_{2\rho}$ and $\left(2\right)_{-2\rho}$ have the structure of semidirect product $\left(2\right)_{2\rho} = (1|0) \rightarrow \xi$, $\left(2\right)_{-2\rho} = (1|0) \rightarrow \bar{\xi}$ with invariant subspaces carrying the $\left(1\right)_{2}$ dimensional fdrs and the trivial representation being realized in the factor space.

Obviously, the $T$-weights structure appearing in the decomposition (C.10) is the only possible one for $\left(2\right)_{2\rho}$ dimensiona fdrs. The representations $\left(2\right)_{2\rho}$ exhaust all $\left(2\right)$ fdrs in the case $p \neq \pm 1$. In the last two cases, one finds two additional representations contragredient to $i_1$ and $i_{-1}$ : $i_1$ and $i_{-1}$. Their structure is as follows:

$$\psi_{i_1} = \left( \begin{array} {c} \psi_{-a} \\ \psi_{-2} \\ \psi_{0} \end{array} \right); \quad \psi_{i_{-1}} = \left( \begin{array} {c} \psi_{+a} \\ \psi_{2} \\ \psi_{0} \end{array} \right) \quad (C.13)$$

the odd generators action is

$$D_+ \left( \begin{array} {c} \psi_{+a} \\ \psi_{2} \\ \psi_{0} \end{array} \right) = \left( \begin{array} {c} \epsilon_{ba}\psi_{2} \\ 0 \\ \psi_{+b} \end{array} \right); \quad D_- \left( \begin{array} {c} \psi_{+a} \\ \psi_{2} \\ \psi_{0} \end{array} \right) = \left( \begin{array} {c} 0 \\ \psi_{+b} \\ 0 \end{array} \right) \quad (C.14)$$

$$D_- \left( \begin{array} {c} \psi_{-a} \\ \psi_{-2} \\ \psi_{0} \end{array} \right) = \left( \begin{array} {c} \epsilon_{ba}\psi_{-2} \\ 0 \\ \psi_{-b} \end{array} \right); \quad D_+ \left( \begin{array} {c} \psi_{-a} \\ \psi_{-2} \\ \psi_{0} \end{array} \right) = \left( \begin{array} {c} 0 \\ \psi_{-b} \\ 0 \end{array} \right) \quad (C.15)$$

These two representations, along with the direct sums $\xi \oplus (1|0); \bar{\xi} \oplus (1|0); (1|0) \oplus (0|1) \oplus (0|1)$ complete the list of all $\left(2\right)$ dimensional fdrs.

Further, let us show that the supersymmetric tensor product of $n$, $(n \neq 1)$ $\xi$ representations is isomorphic to $\left(2\right)_{2\rho_{2n-1}}$ and analogously for $\bar{\xi}$ (the round brackets denote the supersymmetrization):

$$(\xi_1 \ldots \xi_n) = (2\rho)_{2\rho_{2n-1}}; \quad (\bar{\xi}_1 \ldots \bar{\xi}_n) = (2\rho)_{-(2\rho-1)}. \quad (C.16)$$
Indeed, consider tensor $\phi_{(\xi_1...\xi_n)}$. It’s independent nonzero components are

$$
\begin{align*}
\phi_{z,z,...,z} & \equiv \phi_{2n} \\
\phi_{a,z,...,z} & \equiv \phi_{2n-1a} \\
\phi_{a,b,z,...,z} & \equiv \varepsilon_{ab}\phi_{2n-2}
\end{align*}
$$

As the component content of $\phi_{(\xi_1...\xi_n)}$ is identical to that of $(2|2)_{2n-1}$ and there is no another $(2|2)$ dimensional representation with such $T$-weight structure, these representations are isomorphic. Of course, this assertion may be proved by explicit calculation either.

The next important fact we would like to deliver deals with the supersymmetrized product of two $(2|2)_p$ fdrs.

$$( (2|2)_p \otimes (2|2)_p ) = (2|2)_{2p-1} \oplus (2|2)_{2p+1} \ , p \neq 0$$

It is proved by direct calculation, so we do not dwell on details here. Making use the Eq. (C.16), one can rewrite (C.18) in the case $p \in \mathbb{Z}_+$ in the manner (the external round brackets in the l.h.s. stand for the supersymmetrization w.r.t. two groups of indices, $\xi$ and $\eta$, and not the complete supersymmetrization over all indices)

$$((\xi_1...\xi_n) \otimes (\eta_1...\eta_n)) = (\epsilon_1...\epsilon_{2n-1}) \oplus (\epsilon_1...\epsilon_{2n}) ; n \neq 1.$$  

Equivalently, given two supersymmetric rank-$n$ tensors, their supersymmetric tensor product is decomposed as follows

$$
\phi_{\xi_1...\xi_n} \psi_{\eta_1...\eta_n} = (-)^{\xi(n)\eta(n)} \phi_{\eta_1...\eta_n} \psi_{\xi_1...\xi_n} = \phi_{(\eta_1...\eta_n) \psi_{\xi_1...\xi_n}} + C_{\xi_1...\xi_n;\eta_1...\eta_n}^{\epsilon_1...\epsilon_{2n-1}} \chi_{\epsilon_{2n-1}...\epsilon_1} ; n \neq 1.
$$

We see that there exist two subspaces in the supersymmetrized product of two supersymmetric tensors: one is the complete supersymmetrization over all indices, and $(\epsilon_1...\epsilon_{2n-1})$ is something new. Moreover, we see that there exists an invariant $4n-1$-rank tensor $C_{\xi(n);\eta(n)}^{\epsilon(2n-1)}$.

It follows from (C.18) that $C_{\xi(n);\eta(n)}^{\epsilon(2n-1)}$ has the properties

$$
C_{\xi(n);\eta(n)}^{\epsilon(2n-1)} = (-)^{\xi(n)\eta(n)} C_{\eta(n);\xi(n)}^{\epsilon(2n-1)}
$$

$$
C_{\xi_1...\xi_n;\eta_1...\eta_n}^{\epsilon(2n-1)} = 0
$$

$$
C^{\epsilon\xi} C_{\xi_1...\xi_n;\eta_1...\eta_{n-1}\xi;\epsilon_2...\epsilon_{2n-1}} = 0
$$

$$
\varepsilon (C_{\xi(n);\eta(n)}^{\epsilon(2n-1)}) = \xi(n) + \eta(n) + \epsilon(2n - 1) = 0.
$$

Obviously, the higher rank $C$-tensors may be expressed via the lower rank ones in the manner

$$
C_{\xi(n+1);\eta(n+1)}^{\epsilon(2n+1-1)} [\varepsilon] C_{\xi(n);\eta(n)}^{\epsilon(2n-1)} \delta_{\xi} \delta_{\eta} \epsilon.
$$
The lowest rank tensor in this sequence is \( C_{(\xi_1\xi_2):\eta_1\eta_2):\xi_1\xi_2\xi_3} \). Explicitly, it has the components

\[
C_{ab;cd;\bar{e}\bar{f}} = \varepsilon_{ab}\varepsilon_{cd}\varepsilon_{ef} \quad C_{ab;cz;\bar{e}\bar{f}} = \varepsilon_{ab}\varepsilon_{cf}
\]

\[
C_{az;bz;\bar{e}\bar{f}} = \varepsilon_{ab} \quad C_{ab;az;\bar{e}\bar{f}} = -2\varepsilon_{ab} \tag{C.23}
\]

and those obtained from (C.23) by indices permutations, the other components equal zero. The invariance of tensor (C.23) is checked by direct calculation.

Introduce the next important tensor \( \tilde{C}_{[\xi_1\xi_2]:\eta_1\eta_2):\xi_1\xi_2\xi_3} \) which is superantisymmetric in the first two index pairs and supersymmetric in the last three-index group:

\[
\tilde{C}_{[\xi_1\xi_2]:\eta_1\eta_2):\xi_1\xi_2\xi_3} = (-)^{(\xi_1+\xi_2)n} C_{\eta_1\eta_2;\xi_1\xi_2;\xi_1\xi_2\xi_3} - \]
\[
\tilde{C}_{[\xi_1\xi_2]:\eta_1\eta_2):\xi_1\xi_2\xi_3} = (-)^{(\xi_1+\xi_2)n} C_{\eta_1\eta_2;\xi_1\xi_2;\xi_1\xi_2\xi_3}.
\]

The inverse transformation reads

\[
C_{(\xi_1\xi_2):\eta_1\eta_2):\xi_1\xi_2\xi_3} = -\frac{1}{3} \left[ (-)^{(\xi_1+\xi_2)n} \tilde{C}_{(\eta_1\eta_2):[\xi_1\xi_2]:\xi_1\xi_2\xi_3} + \right.
\]
\[
+ (-)^{(\xi_1+\xi_2)n} \tilde{C}_{(\eta_1\eta_2):[\xi_1\xi_2]:\xi_1\xi_2\xi_3} \right)
\]

The \( \tilde{C} \) tensor is also supersymmetric w.r.t. permutation of two first groups of indices, and traceless:

\[
\tilde{C}_{[\xi_1\xi_2]:\eta_1\eta_2):\xi_1\xi_2\xi_3} = (-)^{(\xi_1+\xi_2)(\eta_1+\eta_2)} \tilde{C}_{[\eta_1\eta_2]:[\xi_1\xi_2]:\xi_1\xi_2\xi_3}
\]

\[
C^{\alpha\xi} \tilde{C}_{[\xi_1\xi_2]:[\eta_1\xi]:\xi_1\xi_2\xi_3} = 0,
\]

and satisfies the property

\[
\tilde{C}_{[\xi_1\eta_1\eta_2\eta_3):\xi_1\xi_2\xi_3} = 0,
\]

The component content of the \( \tilde{C} \) tensor is

\[
\tilde{C}_{ab;cd;\bar{e}\bar{f}} = (\varepsilon_{ca}\varepsilon_{bd} + \varepsilon_{cb}\varepsilon_{ad})\varepsilon_{ef} \quad \tilde{C}_{ab;cz;\bar{e}\bar{f}} = \varepsilon_{ca}\varepsilon_{bf} + \varepsilon_{cb}\varepsilon_{af}
\]

\[
\tilde{C}_{az;bz;\bar{e}\bar{f}} = \varepsilon_{ab} \quad \tilde{C}_{ab;az;\bar{e}\bar{f}} = 0 \tag{C.28}
\]

The formulas (C.21, C.26, C.27) just indicate that, for \( C \) and \( \tilde{C} \), two first groups of indices are characterized by the Young tableaux of the form

\[
\begin{array}{c|c|c|c} \\
\end{array}
\]

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The completely analogous formulas take place for conjugated representations, therefore, we also have the conjugated tensors \( \bar{C}_\xi(\bar{\xi}_1 \bar{\xi}_2; \bar{\eta}_1 \bar{\eta}_2; \bar{\epsilon}_1 \bar{\epsilon}_2 \bar{\epsilon}_3) \), \( \overline{\overline{C}}_\xi(\bar{\xi}_1 \bar{\xi}_2; \bar{\eta}_1 \bar{\eta}_2; \bar{\epsilon}_1 \bar{\epsilon}_2 \bar{\epsilon}_3) \); just with indices \( \xi \) being substituted for \( \bar{\xi} \). In what follows, we will not write the bar over the \( C \)-letters as it is clear from the index structure which tensor is used.

The tensor \( \tilde{C} \) is relevant for the component description of the equality

\[
(\xi[2] \otimes \xi[2]) = \epsilon[4] \oplus \epsilon(3),
\]

which is proved by direct analysis. We see that there are two irreducible subspaces in the supersymmetric tensor product of two second rank superantisymmetric tensors: the first is obtained by complete superantisymmetrization over all indices, the second one is extracted with the \( \tilde{C} \) help by the rule

\[
r_{\epsilon(3)} = \left( f^{\bar{\eta}_2 \bar{\eta}_1} g^{\xi \bar{\xi}_1} - (-)^{\epsilon[2] \bar{\eta}[2]} g^{\xi \bar{\xi}_1} f^{\bar{\eta}_2 \bar{\eta}_1} \right) \tilde{C}_{\xi_1 \xi_2; \bar{\eta}_1 \bar{\eta}_2; \epsilon(3)}. \tag{C.30}
\]

The last formula plays an important role in the main text (see Eqs.(6.16)).

Let us eventually describe the structure of \( f_{\epsilon(p)} = \xi(p) \otimes \bar{\xi}(p) \). The cases \( p = 1 \) and \( p \neq 1 \) are different. For \( p = 1 \), one finds

\[
\tilde{\xi} \tilde{\xi} = \xi \xi \oplus (1|0)
\]

(C.31)

where \( \tilde{\xi} \tilde{\xi} \) stands for the traceless part of \( \xi \xi \) fdr – the adjoint fdr of \( A(1|0) \) (see Eq.(1.29)), and \( (1|0) \) is the trivial fdr. This assertion is easy to prove as every element of the \( \xi \xi \) fdr possesses the unique decomposition

\[
\phi_{\xi \xi} = \circ \phi_{\xi \xi} - C_{\xi \xi} \phi_{\tilde{\xi} \tilde{\xi}};
\]

\[
\circ \phi_{\tilde{\xi} \tilde{\xi}} = 0.
\]

(C.32)

It appears that all \( I_p, p \geq 2 \) are isomorphic: \( \forall p \geq 2 \ I_p = F \) and have the structure

\[
F = (1|0) \rightarrow \left( \xi \oplus \tilde{\xi} \oplus \circ \tilde{\xi} \right) \rightarrow (1|0)
\]

(C.33)

Let’s prove this statement. Exhibit the component content of the \( \phi_{\epsilon(2)\epsilon(2)} \) supertensor:

\[
\phi_{\epsilon(2)\epsilon(2)} = \left\{ \phi, \pi, \chi, \phi_+a, \phi_--a, \pi_+a, \pi_-a, \phi_2, \phi_2-a, \pi_a(2) \right\}
\]

(C.34)

The isomorphism \( I_p = F \ \forall p \neq \pm 1 \) is established easily: one observes that dimensions of all \( I_p \) fdrs are \( (8|8) \) and equal to each other, on the other hand, there exist an invariant map \( I_{p+1} \) on \( I_p \) which may be written in tensor notation as follows:

\[
\phi_{\epsilon(p)\epsilon(p)} = C \tilde{\xi} \phi_{\epsilon(p)\epsilon\tilde{\xi}(p)}.
\]

(C.35)

So we may take \( p = 2 \) without the loss of generality and represent a general element of \( F \) representation space by \( \phi_{\epsilon(2)\epsilon(2)} \). There exist the invariant subspace of double-traceless tensors:

\[
H^{7|8} : \left\{ \phi_{\epsilon(2)\epsilon(2)} \in H^{7|8} | \phi_{\circ \tilde{\xi} \tilde{\xi}} = 0 \right\}
\]

(C.36)
The factor space $F/H^{7|8}$ carries (1|0) dimensional (trivial) fdr, so one observes that $F = (1|0) \rightarrow H^{7|8}$.

Further, there exist the invariant subspace (the "$\equiv$" symbol stands for the ordinary equality with an account of a sign factor, see App.B)

$$H^{1|0} \subset H^{7|8} : \{ \phi_{\ell(2)\ell(2)} \in H^{1|0} | \phi_{\ell\ell\ell} \equiv C_{\ell\ell\ell} \}$$

(C.37)

the inclusion $H^{1|0} \subset H^{7|8}$ is clear as the double-$C^{\ell\ell}$ trace in $H^{1|0}$ is proportional to the supertrace of the identity operator in the $F$ space, which is equal zero as $F$ is (8|8)-dimensional. Therefore, one gets the structure $H^{7|8} = (6|8) \rightarrow H^{1|0}$, where (6|8) is some (6|8)-dimensional fdr. Consider two subspaces:

$$H_{\text{trace}} \subset H^{7|8} : \{ \phi_{\ell(2)\ell(2)} \in H_{\text{trace}} | \phi_{\ell\ell\ell} \equiv C_{\ell\ell\ell} \}$$

$$H_{\text{traceless}} \subset H^{7|8} : \{ \phi_{\ell(2)\ell(2)} \in H_{\text{traceless}} | \phi_{\ell\ell\ell} = 0 \}$$

(C.38)

These two subspaces, $H_{\text{trace}}$ and $H_{\text{traceless}}$, overlap only via the $H^{1|0}$ since

$$\phi_{\ell(2)\ell(2)} \in H_{\text{trace}} \Rightarrow \phi_{\ell\ell\ell} = C_{\ell\ell\ell},$$

$$\phi_{\ell(2)\ell(2)} \in H_{\text{traceless}} \Rightarrow \phi_{\ell\ell\ell} = C_{\ell\ell\ell} \Rightarrow \phi_{\ell(2)\ell(2)} \in H^{1|0}$$

(C.39)

It is easy to see that the component form of $H_{\text{trace}}$ and $H_{\text{traceless}}$ is

$$H_{\text{trace}} = \{ \chi, \pi, \pi_{+a}, \pi_{-a}; \pi_{a(2)} \}$$

$$H_{\text{traceless}} = \{ \chi, \phi, \phi_{+a}; \phi_{-a}, \phi_2, \phi_{-2} \}$$

(C.40)

and their dimensions are (5|4) and (3|4), correspondingly. As they have (1|0)-dimensional overlap (parametrized by $\chi$ in the last formula), one has $H^{7|8} = H_{\text{trace}} + H_{\text{traceless}}$ and $H^{7|8}/H^{1|0} = (H_{\text{trace}}/H^{1|0}) \oplus (H_{\text{traceless}}/H^{1|0})$ in the factor space. The representation $H_{\text{trace}}/H^{1|0}$ is isomorphic to $\zeta \zeta$ as $\xi \xi = \zeta \zeta \oplus (1|0)$ (see Eqs.(C.31, C.32)) while $H_{\text{traceless}}/H^{1|0}$ is some (2|4)-dimensional fdr with component content exhibited by $\phi$'s in the formula (C.40). Obviously, it is decomposed into direct sum of two (1|2)-fdrs with components $\{ \phi_{+a}, \phi_2 \}$ and $\{ \phi_{-a}, \phi_{-2} \}$, which are identified with two unique representations with such dimension (see Eq.(C.2)) : $H_{\text{traceless}}/H^{1|0} = \zeta \oplus \zeta$.

Therefore, we come to the overall result of our analysis in the form of the Eq.(C.33). The fact that the decomposition (C.33) is not a direct sum is checked by explicit component calculation.

Let’s make a few remarks.

One may wonder about the explicit tensor form of the $H_{\text{traceless}}$ subspace. It is easy to prove that every element of $H_{\text{traceless}}$ may be written in the form

$$\phi_{\ell(2)\ell(2)} \equiv C_{\ell(2)\zeta(2);\ell(2)\zeta} \phi_{\ell\ell\ell(2)} + C_{\ell(2)\zeta(2);\ell(2)\zeta} \phi_{\ell\ell\ell(2)}$$

(C.41)
the invariant seven-rank $C$-tensors play an essential role here. The last formula presents a general solution to the tracelessness constraint, with the first and the second terms in the r.h.s. parametrizing the preimages (w.r.t. factorization by $H^{10}$) of $\xi$ and $\xi$ fdrs, $H_\xi$ and $H_\xi$, correspondingly.

One may impose another constraint in $H^{78}$:

$$0 \equiv C_{\eta(2)} \bar{\eta}(2) \bar{\eta}(2) \phi_{\eta(2)}\bar{\eta}(2)$$

(C.42)

It’s now easy to derive that its general solution is

$$\phi_{\eta(2)}\bar{\eta}(2) \in H_\xi + H_{trace}.$$  

(C.43)

Quite analogously,

$$0 \equiv C_{\eta(2)} \bar{\eta}(2) \bar{\eta}(2) \phi_{\eta(2)}\bar{\eta}(2) \Rightarrow$$

$$\Rightarrow \phi_{\eta(2)}\bar{\eta}(2) \in H_\xi + H_{trace}.$$  

(C.44)

The joint effect of constraints (C.42) and (C.44) is $H_{trace}$ subspace.

References

[1] S.J. Gates, S.M. Kuzenko and A.G. Sibiryakov, Phys. Lett. B 394 (1997) 343, hep-th/9611143

[2] M.A. Vasiliev, Phys. Lett. B243 (1990) 378; Class. Quantum Grav. 8 (1991) 1387; Phys. Lett. B285 (1992) 225.

[3] S. Ouvry and J. Stern, Phys. Lett. B 177 (1986) 335; M. Bellon and S. Ouvry, Phys. Lett. B 187 (1987) 93; Phys. Lett. B 193 (1987) 67; A.K.H. Bengtsson, Class. Quant. Grav. 5 (1988) 437; A. Pashnev and M. Tsulaya, Mod. Phys. Lett. A12 (1997) 861, hep-th/9703011.

[4] S.M. Kuzenko, V.V. Postnikov and A.G. Sibiryakov, JETP Lett. 57 (1993) 534; S.M. Kuzenko and A.G. Sibiryakov, JETP Lett. 57 (1993) 539.

[5] S.M. Kuzenko and A.G. Sibiryakov, Phys. At. Nucl. 57 (1994) 1257.

[6] E.S. Fradkin and M.A. Vasiliev, Phys. Lett. B189 (1992) 89; Nucl. Phys. B291 (1987) 141.

[7] E.S. Fradkin and M.A. Vasiliev, Int. J. Mod. Phys. A3 (1988) 2983.

[8] S.J. Gates, S.M. Kuzenko and A.G. Sibiryakov, Phys. Lett. B 412 (1997) 59, hep-th/9609141

[9] A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetski and E. Sokatchev, Class. Quant. Grav. 1 (1984) 469.
[10] A. Galperin, Nguyen Ahn Ky and E. Sokatchev, Class. Quant. Grav. 4 (1987) 1235; A. Galperin, E. Ivanov, V. Ogievetski and E. Sokatchev, Class. Quant. Grav. 4 (1987) 1255;

[11] I.L. Buchbinder and S.M. Kuzenko, *Ideas and Methods of Supersymmetry and Supergravity*. Institute of Physics Publishing, Bristol and Philadelphia, 1995.

[12] J. Wess and J. Bagger, *Supersymmetry and Supergravity*. Cambridge Univ. Press, Cambridge, 1983.

[13] S.J. Gates, M.T. Grisaru, M. Roček and W. Siegel, *Superspace*. Benjamin–Cummings, Reading, MA, 1983.

[14] B.W. Keck, J. Phys. A8 (1975) 1819; B. Zumino, Nucl. Phys. B127 (1977) 189; E.A. Ivanov and A.S. Sorin, J. Phys. A13 (1980) 1159.

[15] J. Thierry-Mieg, (1983). Table des representations irreducibles des superalgebres de Lie $SU(m|n), SU(n|n)/U(1), Osp(m,n), D(2|1; \alpha), G(3), F(4)$ (unpublished).