B-SPLINE CURVES AND SURFACES AS A MINIMIZATION OF QUADRATIC OPERATORS

SVETOSLAV I. NENOV

Abstract. The goal of this short note is to prove that every b-spline curve or surface (generated by uniform knots, without multiplicity) may be defined as minimum of positive quadratic operator.

1. Introduction

The b-spline curves and surfaces are an essential tool in many engineering software for design and visualization – for example ANSYS, RFEM 3D, etc. So, it is necessary to have in these applications many different methods to construct b-spline elements. Also see [11], [12], and references therein.

We will prove that any b-spline curve or surface minimizes positive quadratic operator: appropriate moving least-square error.

Let us mark that different approaches in moving least-squares method are used by Shepard – computer software SYMAP (Harvard Laboratory for Computer Graphics), Lancaster in 1979, and the works of D. Levin in 1999, see [8]. In [10] it has been shown that moving least-squares method is an adequate mathematical tool for determining diesel-fuel cetane-number (or cetane-index) from easily available physical properties of fuels.

In this section we will remind the definition of b-splines generated by control points in $\mathbb{R}^{d+1}$ and definition of moving least-squares approximation for a given data set $\{(x_i, f(x_i)) : x_i \in \mathbb{R}^d \} \subset \mathbb{R}^{d+1}$.

2. Preliminaries

2.1. b-Splines. Let $\{p_i \in \mathbb{R}^{d+1} : i = 0, \ldots, n\}$ be a set of $n + 1$ (control) points.

Let $r$ be an integer, $1 \leq r \leq n + 1$ (the order of spline).

We will use uniform knots, without multiplicity: $t_i = i, i = 0, \ldots, n + r$. 
Using Cox-de Boor recursion formula (see [3], [4]), let us define the following basis functions:

\[ B_{i,1}(t) = \begin{cases} 
1, & \text{if } t_i \leq t < t_{i+1}, \\
0, & \text{otherwise},
\end{cases} \]  

for \(0 \leq i \leq n + r - 1\); and

\[ B_{i,j}(t) = \frac{t - t_i}{t_{i+j-1} - t_i} B_{i,j-1}(t) + \frac{t_{i+j} - t}{t_{i+j} - t_{i+1}} B_{i+1,j-1}(t) \]

\[ = \frac{t - i}{j - 1} B_{i,j-1}(t) + \frac{i + j - t}{j - 1} B_{i+1,j-1}(t), \]  

for \(2 \leq j \leq r, 0 \leq i \leq n + r - j\).

The b-spline curve of order \(r\) is defined as a linear combination of control points \(p_i\):

\[ \gamma(t) = \sum_{i=0}^{n} B_{i,r}(t) p_i, \quad t \in [t_{r-1}, t_{n+1}]. \]  

2.2. Moving Least-Squares Method. Let:

1. \(\mathcal{D}\) be a bounded domain in \(\mathbb{R}^d\).
2. \(x_i \in \mathcal{D}, i = 0, \ldots, m; x_i \neq x_j, \text{if } i \neq j\).
3. \(f : \mathcal{D} \to \mathbb{R}\) be a continuous function.
4. \(p_i : \mathcal{D} \to \mathbb{R}\) be continuous functions, \(i = 1, \ldots, l\). The functions \(\{p_1, \ldots, p_l\}\) are linearly independent in \(\mathcal{D}\) and let \(\mathcal{P}_l\) be their linear span.
5. \(W : (0, \infty) \to (0, \infty)\) is a strictly positive functions.

Usually the basis in \(\mathcal{P}_l\) is constructed by monomials. For example:

\[ p_l(x) = x_1^{k_1} \ldots x_d^{k_d}, \text{ where } x = (x_1, \ldots, x_d), k_1, \ldots k_d \in \mathbb{N}, k_1 + \ldots + k_d \leq l - 1. \]  

In the case \(d = 1\), the standard basis is \(\{1, x, \ldots, x^{l-1}\}\).

Following [5], [6], [7], [8], we use the following definition. The moving least-squares approximation of order \(l\) at a fixed point \(x\) is the value of \(p^*(x)\), where \(p^* \in \mathcal{P}_l\) is minimizing the least-squares error

\[ \sum_{i=1}^{m} W(\|x - x_i\|)(p(x) - f(x_i))^2 \]  

among all \(p \in \mathcal{P}_l\).

The approximation is “local” if weight function \(W(s)\) is fast decreasing as \(s\) tends to infinity. Interpolation is achieved if \(W(0) = \infty\). We
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define additional function \( w: [0, \infty) \rightarrow [0, \infty) \), such that:

\[
  w(s) = \begin{cases} 
    \frac{1}{W(s)}, & \text{if } (s > 0) \text{ or } (s = 0 \text{ and } W(0) < \infty), \\
    0, & \text{if } (s = 0 \text{ and } W(0) = \infty). 
  \end{cases}
\]

Some examples of \( W(s) \) and \( w(s) \), \( s \geq 0 \):

\[
  W(s) = e^{-a^2s^2} \quad \text{exp-weight},
  \quad W(s) = s^{-a^2} \quad \text{Shepard weights},
  \quad w(s) = s^2e^{-a^2s^2} \quad \text{McLain weight},
  \quad w(s) = e^{a^2s^2} - 1 \quad \text{see Levin’s works}.
\]

Here and below: the superscript \(^t\) denotes transpose of matrix; \( I \) is the identity matrix.

Let us introduce the matrices:

\[
  E = \begin{pmatrix}
    p_1(x_1) & p_2(x_1) & \cdots & p_l(x_1) \\
    p_1(x_2) & p_2(x_2) & \cdots & p_l(x_2) \\
    \vdots & \vdots & \ddots & \vdots \\
    p_1(x_m) & p_2(x_m) & \cdots & p_l(x_m)
  \end{pmatrix},
  \quad a = \begin{pmatrix}
    a_1 \\
    a_2 \\
    \vdots \\
    a_m
  \end{pmatrix},
  \quad c = \begin{pmatrix}
    p_1(x) \\
    p_2(x) \\
    \vdots \\
    p_l(x)
  \end{pmatrix}
\]

\[
  D = 2 \begin{pmatrix}
    w(\|x - x_1\|) & 0 & \cdots & 0 \\
    0 & w(\|x - x_2\|) & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & w(\|x - x_m\|)
  \end{pmatrix}.
\]

Through the article, we assume the following conditions (H1):

(H1.1) \( 1 \in \mathcal{P}_l \).
(H1.2) \( 1 \leq l \leq m \).
(H1.3) \( \text{rank}(E^t) = l \).
(H1.4) \( w \) is a smooth function.

**Theorem 2.1** (see [6]). Let the conditions (H1) hold true.

Then:

1. The matrix \( E^tD^{-1}E \) is non-singular.
2. The approximation defined by the moving least-squares method is

\[
  \hat{L}(f) = \sum_{i=1}^{m} a_if(x_i),
\]

where

\[
  a = A_0c \quad \text{and} \quad A_0 = D^{-1}E\left(E^tD^{-1}E\right)^{-1}.
\]
(3) If \( w(0) = 0 \), then the approximation is interpolatory.

3. B-Spline Curve as a Minimum of Moving Least-Squares Error

Using the definitions and notations introduced in Section 2, our goal is to prove the following theorem.

**Theorem 3.1.** Let:

1. \( d = 1, n, r \in \mathbb{Z}_+, r \leq n + 1, f : [0, n + r] \rightarrow \mathbb{R} \) be a continuous function.
2. \( p_i = (i, f(i)), i = 0, \ldots, n + r \).
3. Let \( \gamma(t) = \left( \begin{array}{c} \gamma_1(t) \\ \gamma_2(t) \end{array} \right) \) be the b-spline of order \( r \) and knot vector \( \{ t_i = i : i = 0, \ldots, n + r \} \).

Then \( \gamma_1(t) = t - 2, \; t \in [0, n + r] \) and there exists a weight function \( W \), such that

\[
\gamma_2(x) = \hat{L}(f)(x), \quad x \in [r - 1, n + 1],
\]

where \( \hat{L}(f)(x) \) is the approximation defined by the moving least-squares method for the data \( \{ p_i : i = 0, \ldots, n + r \} \).

**Proof.** We will prove the theorem for the cubic splines, i.e. \( r = 4 \). The proof for the different orders is similar.

From conditions (2) and (3), if we set \( x_i = i \), then \( t_i = x_i, i = 0, \ldots, n \) and hence \( \gamma_1(t) = t \).

The b-spline curve of order 4, defined using knots \( \{ t_i = i : i = 0, \ldots, n + r \} \), is

\[
\gamma_2(t) = \sum_{i=0}^{n} B_{i,4}(t) f(x_i), \quad x \in [t_{r-1}, t_{n+1}] \equiv [3, n + 1].
\]

The following properties of b-spline basis functions \( B_{i,j}(t) \) are well known:
Table 1. Cox-de Boor recursion algorithm

(BS-0) By the direct calculation (see formulas (1), (2), and the schema illustrated in Table 1, r = 4):

\[ B_{i,4}(t) = \begin{cases} 
0, & \text{if } t < i, \\
\frac{1}{6} (t - i)^3, & \text{if } i \leq t < i + 1, \\
-\frac{2}{3} (t - i - 1)^3 + \frac{1}{6} (t - i)^3, & \text{if } i + 1 \leq t < i + 2, \\
(t - i - 2)^3 - \frac{2}{3} (t - i - 1)^3 & \\
+ \frac{1}{6} (t - i)^3, & \text{if } i + 2 \leq t < i + 3, \\
-\frac{2}{3} (t - i - 3)^3 + (t - i - 2)^3 & \\
-\frac{3}{3} (t - i - 1)^3 + \frac{1}{6} (t - i)^3, & \text{if } i + 3 \leq t < i + 4, \\
0, & \text{if } i + 4 \leq t. 
\]

(BS-1) If \( t \in (i, i + j) \), then \( B_{i,j}(t) > 0 \).

(BS-2) If \( t \in [0, i] \cup [i + j, n + j] \), then \( B_{i,j}(t) = 0 \).

(BS-3) \( \sum_{i=0}^{n} B_{i,j}(t) = 1 \), for any \( t \in (j - 1, n + 1) \).

(BS-4) \( B_{i,j}(t) \) has \( C^{j-2} \) continuity at each knot.

(BS-5) By the simple substitutions in the formulas in (BS-0):

\[ B_{i,j}(t + i) = B_{k,j}(t + k), \quad t \in (0, j), \]

\[ B_{i,j}(t - i) = B_{k,j}(t - k), \quad t \in (j, n + 1). \]
Figure 1. The graphics of $W(x)$, $x \in (-2.5, 2.5)$ and $w(x)$, $x \in (-1, 1)$

$$B_{i-2,j}(t) = B_{i,j}(t + 2), \quad t \in (i - 2, i + j - 2).$$

Let $t$ be a fixed point in the interval $(3, n + 1)$ and $i_0$ be an integer such that $3 \leq i_0 < t < i_0 + 1 \leq n + 1$. Then

$$\gamma_2(t) = \sum_{i=0}^{n} B_{i,4}(t)f(i) = \sum_{i=i_0-3}^{i_0} B_{i,4}(t)f(i).$$ (7)

because if $i = 1, \ldots, i_0 - 4, i_0 + 1, \ldots, n$, then $B_{i,4}(x) = 0$, see (BS-1) and (BS-2).

On the other hand, let us consider the moving least-squares problem for the given data $\{(i, f(i)) : i = 0, \ldots, n + 4\}$. Let us set $l = 1$, and

$$W(x) = \begin{cases} 
0, & \text{if } x < -2, \\
\frac{1}{6}(x + 2)^3, & \text{if } -2 \leq x < -1, \\
-\frac{1}{2}x^3 - x^2 + \frac{2}{3}, & \text{if } -1 \leq x < 0, \\
\frac{1}{2}x^3 - x^2 + \frac{2}{3}, & \text{if } 0 \leq x < 1, \\
-\frac{1}{6}(x - 2)^3, & \text{if } 1 \leq x < 2, \\
0, & \text{if } 2 \leq x,
\end{cases}$$

see Figure 1.

Then for any $i = 0, \ldots, n$, we have (see also (BS-5)):

$$W(|x|) = W(x) = B_{i,4}(x + i + 2) = B_{i-2,4}(x + i), \quad x \in [-2, 2].$$

Hence $W(|x - i|) = B_{i-4,4}(x)$, $x \in [i - 2, i + 2]$. 
The least-squares error is (see also conditions (H1): \( p_1(x) = 1 \), so \( p(x) = p \) has to be a constant)
\[
\sum_{i=1}^{n+r} W(x - i) (p - f(i))^2 = \sum_{i=i_0-1}^{i_0+2} W(x - i) (p - f(i))^2 ,
\]
because \( W(x - i) > 0 \), iff \( i_0 - 1 \leq i \leq i_0 + 2 \).

It is not hard to compute
\[
E = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad c = (1),
\]
\[
D = 2\begin{pmatrix} w(|x - (i_0 - 1)|) & 0 & 0 & 0 \\
0 & w(|x - i_0|) & 0 & 0 \\
0 & 0 & w(|x - (i_0 + 1)|) & 0 \\
0 & 0 & 0 & w(|x - (i_0 + 2)|) \end{pmatrix},
\]
where \( w(x) = \frac{1}{W(x)} \), and
\[
E^t D^{-1} E = \frac{1}{2} \left( \frac{1}{w(|x - (i_0 - 1)|)} + \frac{1}{w(|x - i_0|)} + \frac{1}{w(|x - (i_0 + 1)|)} + \frac{1}{w(|x - (i_0 + 2)|)} \right)
\]
\[
= \frac{1}{2} (B_{i_0-3,4}(x) + B_{i_0-2,4}(x) + B_{i_0-1,4}(x) + B_{i_0,4}(x))
\]
\[
= \frac{1}{2}, \quad \text{because } x \in [i_0, i_0 + 1],
\]
\[
a = D^{-1} E (E^t D^{-1} E)^{-1} c = \begin{pmatrix} B_{i_0-3,4}(x) \\
B_{i_0-2,4}(x) \\
B_{i_0-1,4}(x) \\
B_{i_0,4}(x) \end{pmatrix}.
\]

Hence, by Theorem 2.1, we have
\[
\hat{L}(f)(x) = \sum_{i=1}^{4} a_i f(x_i) = \sum_{i=i_0-3}^{i_0} B_{i,4}(x)f(i),
\]
i.e. we received b-spline (7). \( \square \)
Remark 1. Using the method, illustrated in the proof of Theorem 3.1, it is not difficult to generalize the result in whole interval:

\[ \gamma_2(x) = \sum_{i=0}^{n} W(x-i)f(i), \quad x \in [r-1,n]. \]

See Example 4.1 in Section 4.

Remark 2. Using mentioned Subsection 1.2, Levin’s approach (i.e. working with weight-function \( w(x) \), such that \( w(x_i) = 0 \)) in moving least-squares method, it is not difficult to receive interpolation. Let

\[ \tilde{W}(x) = W(x) + \delta, \quad x \in \mathbb{R}, \]

where \( \delta > 0 \). Then \( \tilde{W}(x) \geq \delta > 0 \), for any \( x \in \mathbb{R} \) and \( \max\{\tilde{W}(x) : x \in \mathbb{R}\} = \tilde{W}(0) = \frac{2}{3} + \delta \). Let moreover

\[ \tilde{w}(x) = \frac{1}{W(x) + \delta} - \frac{3}{2 + 3\delta}, \quad x \in \mathbb{R}. \]

Then: \( \tilde{w}(x) > 0 \), for any \( x \in \mathbb{R} \setminus \{0\} \) and \( \min\{\tilde{w}(x) : x \in \mathbb{R}\} = \tilde{w}(0) = 0 \), see Figure 2.

In this case the method used in the proof of Theorem 2.1 produces interpolation.

See Example 4.2 in Section 4.

4. Some Examples. Case of Interpolation

Example 4.1. Consider the following example of control points:

\[ \Xi_0 = \{(x_i, f(x_i)) : x_i = i, \]
Figure 3. The plots of data-set $\Xi_0$ and cubic b-spline in $[1,9]$

Figure 4. The plots of data-set $\Xi_0$ (red dots), cubic b-spline in $[1,9]$ (blue curve) and moving least square approximation in $[2,10]$ (cyan bold curve)

$$f(x) = e^{-x^2} + 3e^{-(x-4)^2} + 1.7e^{-(x-8)^2}, \ i = 0, \ldots, 10.$$

Here $n = 10$, $r = 4$, knots: $t_i = i$, $i = 1, \ldots, 14$. The control points and cubic b-spline $\gamma(x)$ are illustrated on Figure 3. Here we used standard Maple expression `BSplineCurve`, see [9].

Using Remark 1 it is easy to construct the b-spline curve in whole interval $[2,10]$ – see Figure 4.

Let us apply Theorem 3.1 in the interval $(5,6)$, for example. We have $W(x) = B_{5-2.4}(x + 5 + 2)$ and moreover

$W(x) > 0, \ i = 4, 5, 6, 7; \ W(x) = 0, \ i = 0, 1, 2, 3, 8, 9, 10.$
Following the proof of theorem: let \( w(s) = \frac{1}{W(s)} \). Let \( x \in (5, 6) \) be a fixed point, then:
\[
E^t D^{-1} E = \frac{1}{2} \left( \frac{1}{w(|x - 4|)} + \frac{1}{w(|x - 5|)} + \frac{1}{w(|x - 6|)} + \frac{1}{w(|x - 7|)} \right) = \frac{1}{2},
\]
see (BS-3). Moreover

\[
a = D^{-1} E \left( E^t D^{-1} E \right)^{-1} c = D^{-1} E 2 = \begin{pmatrix} B_{2,4}(x) \\ B_{3,4}(x) \\ B_{4,4}(x) \\ B_{5,4}(x) \end{pmatrix}.
\]

So, we receive the classical b-spline formula in the interval (5, 6):
\[
\gamma_2(x) = \sum_{i=2}^{5} W(x - i) f(i) = \sum_{i=2}^{5} B_{i,4}(x) f(i).
\]

The b-splines in the intervals [5, 6], [6, 7], and [7, 8] are plotted on Figure 5, respectively.

**Example 4.2.** To construct the interpolation in the interval \([x_0, x_{10}]\), following Remark 2 let us set \( \delta = 0.1 \), for example. Then
\[
\tilde{w}(x) = \frac{1}{W(x) + 0.1} - \frac{3}{5}, \quad x \in \mathbb{R}.
\]

Applying the moving least-squares method (i.e. applying Remark 2 and Theorem 2.1 at each point \( x = l/100, l = 1, \ldots, 1000 \)) we received the function presented in Figure 6.
Figure 6. The plots of data (red dots), b-spline (blue), and interpolation (cyan, bold)

5. B-SPLINE SURFACES

Let \( \{p_{ij} \in \mathbb{R}^3 : i, j = 0, \ldots, n\} \) be a set of \((n+1)^2\) control points. Let \( r \) be an integer, \( 1 \leq r \leq n+1 \) (the order of spline).

We will use again the uniform knots, without multiplicity: \( u_i = v_j = i, i = 0, \ldots, n+r \). Then, the corresponding b-spline surface of order \( r \) is given by

\[
r(u, v) = \sum_{i=0}^{n} \sum_{j=0}^{n} B_{i,r}(u)B_{j,r}(v)p_{ij}.
\]

Arguments similar to the proof of Theorem 3.1 yield to the following result.

Theorem 5.1. Let:
(1) \( d = 2, n, r \in \mathbb{Z}_+, r \leq n + 1, f : [0, n+r] \times [0, n+r] \to \mathbb{R} \) be a continuous function.
(2) \( p_{ij} = (i, j, f(i, j)), i = 0, \ldots, n+r \).
(3) Let \( \gamma(u, v) = \left( \begin{array}{c} u \\ v \\ \gamma_3(u,v) \end{array} \right) \) be the b-spline of order \( r \) and knots \( \{(u_i, v_j) = (i, j) : i, j = 0, \ldots, n+r\} \).

Then there exists a weight function \( W(x, y) \), such that

\[
\gamma_3(x, y) = \hat{L}(f)(x, y), \quad (x, y) \in [r-1, n] \times [r-1, n].
\]

Example 5.1. As an example, consider the following data

\[
\Xi_0 = \{(x_i, y_j, f(x_i)) : x_i = i, \ y_j = j, \\
f(x_i, y_j) = e^{-x_i^2} + 3e^{-(y-1)^2} + e^{-(x-6)^2-(y-6)^2}, \ i = 0, \ldots, 10\}.
\]
The set of control points $p_{ij} = \left( x_i, y_j, f(x_i, y_j) \gamma_2(t) \right)$ and function $f(x, y)$ are plotted in Figure 7.

Using the formula

$$\hat{L}(f)(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{n} p_{ij} W(x - i)W(x - j).$$

we receive the plot of b-spline surface $\gamma_3(u, v)$ in $[1, 9] \times [1, 9]$, see Figure 8.
If we need the plot of b-spline only in the segment $[i_0, i_0+1] \times [j_0, j_0+1]$, then

$$\hat{L}(f)(x, y) = \sum_{i=i_0-3}^{i_0} \sum_{j=j_0-3}^{j_0} p_{ij} W(x - i) W(y - j),$$

see Figure 9.

**References**

[1] Gary D. Knott, *Interpolating Cubic Splines*, Springer Science & Business Media (2000), 244 pages.

[2] N.M. Patrikalakis, T. Maekawa, W. Cho, [http://web.mit.edu/hyperbook/Patrikalakis-Maekawa-Cho/](http://web.mit.edu/hyperbook/Patrikalakis-Maekawa-Cho/)

[3] E.T.Y. Lee, A simplified b-spline computation routine, *Computing*, Springer-Verlag, **29**, No. 4 (1982), 365–371, [doi: 10.1007/BF02246763](https://doi.org/10.1007/BF02246763).

[4] E.T.Y. Lee, Comments on some b-spline algorithms, *Computing*, Springer-Verlag, **36**, No. 3 (1986), 229–238, [doi: 10.1007/BF02240069](https://doi.org/10.1007/BF02240069).

[5] Marc Alexa, Johannes Behr, Daniel Cohen-Or, Shachar Fleishman, David Levin, Claudio T. Silva, Point-Set Surfaces, [http://www.math.tau.ac.il/~levin/](http://www.math.tau.ac.il/~levin/)

[6] D. Levin, The approximation power of moving least-squares, [http://www.math.tau.ac.il/~levin/](http://www.math.tau.ac.il/~levin/)
[7] D. Levin, Mesh-independent surface interpolation, http://www.math.tau.ac.il/~levin/

[8] D. Levin, Stable integration rules with scattered integration points, Journal of Computational and Applied Mathematics, 112 (1999), 181-187, http://www.math.tau.ac.il/~levin/

[9] Maple 2015 Help System, http://www.maplesoft.com/

[10] Dicho Stratiev, Ivaylo Marinov, Rosen Dinkov, Ivelina Shishkova, Ilian Velkov, Ilshat Sharafutdinov, Svetoslav Nenov, Tsvetelin Tsvetkov, Sotir Sotirov, Magdalena Mitkova, Nikolay Rudnev, Opportunity to improve diesel-fuel cetane-number prediction from easily available physical properties and application of the least-squares method and artificial neural networks, Energy & Fuels, 29, No. 3 (2015).

[11] Arne Laksøa, Børre Bang, Lubomir T. Dechevsky, Geometric modelling with beta-function b-splines, I: Parametric curves, International Journal of Pure and Applied Mathematics, 65, No. 3 (2010), 339-360.

[12] Arne Laksøa, Børre Bang, Lubomir T. Dechevsky, Geometric modelling with beta-function b-splines, II: Tensor-product parametric surfaces, International Journal of Pure and Applied Mathematics, 65, No. 3 (2010), 361-380.

Department of Mathematics, University of Chemical Technology and Metallurgy, Sofia 1756, Bulgaria

E-mail address: nenov@uctm.edu
