Knudsen number, ideal hydrodynamic limit for elliptic flow and QGP viscosity in 
$\sqrt{s_{NN}}=62$ and 200 GeV Cu+Cu/Au+Au collisions

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Taking into account of entropy generation during evolution of a viscous fluid, we have estimated inverse Knudsen number, ideal hydrodynamic limit for elliptic flow and QGP viscosity to entropy ratio in $\sqrt{s_{NN}}=62$ and 200 GeV Cu+Cu/Au+Au collisions. Viscosity to entropy ratio is estimated as $\eta/s = 0.17 \pm 0.10 \pm 0.20$, the first error is statistical, the second one is systematic. In a central Au+Au collision, inverse Knudsen number is $\approx 2.80 \pm 1.63$, which presumably small for complete equilibration. In peripheral collisions it is even less. Ideal hydrodynamic limit for elliptic flow is $\sim 40\%$ more than the experimental flow in a central collision.

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Recent experiments at RHIC produced convincing evidences that a collective matter is created in Au+Au collisions [1,4]. The evidences come mainly from observing finite elliptic flow in non-central collisions, which is now regarded as a definitive signature of collective effect [5,6]. Elliptic flow measure the azimuthal correlation of produced particles with respect to the reaction plane. It is also best understood in a collective model like hydrodynamics [6]. In a non-central collision, the reaction zone is spatially asymmetric. Differential pressure gradient convert the spatial asymmetry in to momentum asymmetry. Ideal hydrodynamics has been quite successful explaining a large part of the experimental elliptic flow [6]. However it is now realized that the experimentally measured scaling of integrated $v_2$ with multiplicity or with collision centrality is not in agreement with ideal hydrodynamics. While ideal hydrodynamics predicts approximate scaling [8], in experiments scaling is violated [9, 10]. As discussed in [11], violation of the scaling can be understood only from explicit transport calculations. From Monte-Carlo simulation of transport equations $K_0$ was estimated, $K_0 = 0.70 \pm 0.03$ [12]. In [11] it was argued that inverse of the Knudsen number $K^{-1}$ can be determined from the experimental data, as it is proportional to $\frac{1}{S \ dy}$, where $\frac{dN}{dy}$ is the total multiplicity density and $S$ is a measure of the transverse area of the collision zone,

\begin{equation}
\frac{1}{K} = c_s \sigma \frac{1}{S} \frac{dN}{dy}
\end{equation}

In Eq\[3\] $c_s$ is the speed of sound of the medium and $\sigma$ is the inter-particle cross section. Eq\[2\] and\[3\] connect two experimental observables, elliptic flow and particle multiplicity and can be used to determine unknown quantities e.g. ideal hydrodynamic limit of elliptic flow ($\frac{v_2}{\epsilon}$)$_{ih}$, the combination of parameters $K_0 \sigma c_s$. However, there is a serious flaw in the derivation of Eq\[3\] and the conclusions derived from it could be misleading. Eq\[3\] was obtained with the assumption that the total particle number is conserved throughout the evolution [11]. The assumption is justified in an isentropic expansion, i.e. one dimensional evolution of ideal fluid, when entropy density ($s$) times the proper time ($\tau$) is a constant. Under such condition, $\frac{1}{S} \frac{dN}{dy} \times s \tau \approx n \tau$ [13]. However, in a viscous evolution, entropy is generated and initial and final state entropy are not same and the assumption is clearly violated. Only in systems with very

1 is the Knudsen regime where hydrodynamics become inapplicable. The simple formula,

\begin{equation}
\left(\frac{v_2}{\epsilon}\right)_{ih} = \left(\frac{v_2}{\epsilon}\right)_{\text{ex}} K^{-1} + K_0 \tau.
\end{equation}

proposed in [11] give qualitatively correct behavior of the experimental elliptic flow. In the limit of small Knudsen number, experimental flow approaches the ideal hydrodynamic limit ($\frac{v_2}{\epsilon}$)$_{ih}$ with a correction linear in $K$. In the other extreme limit of large $K$, flow is proportional to the Knudsen number $K$. In Eq\[2\] $K_0^{-1}$ is a number of the order of unity, whose precise value can be determined only from explicit transport calculations. From Monte-Carlo simulation of transport equations $K_0$ was estimated, $K_0 = 0.70 \pm 0.03$ [12]. In [11] it was argued that inverse of the Knudsen number $K^{-1}$ can be determined from the experimental data, as it is proportional to $\frac{1}{S \ dy}$, where $\frac{dN}{dy}$ is the total multiplicity density and $S$ is a measure of the transverse area of the collision zone,
small viscosity, the assumption may be approximately
valid, but not in systems where sufficient entropy is gen-
erated. Explicit numerical simulations indicate that in
Au+Au collisions, entropy generation can be substan-
tial, e.g. \( \sim 20\%, 30\% \) and \( 50\% \) in fluid evolution with
viscosity to entropy ratio, \( \eta/s=0.08, 0.12 \) and 0.16 [14].
One may argue that unlike in explicit numerical simu-
lation of viscous hydrodynamics, Knudsen number ansatz
does not require the entire dynamic range of evolution.
Time of validity of Knudsen number ansatz is \( \tau < \frac{R}{c_s} \)
[11]. For \( c_s = \sqrt{1/3} \), and characteristic size \( R \approx 1-2 \) fm,
the time scale is \( \tau < 1.7-3.5 \) fm. Though the time scale is
small compared to the entire dynamic range of evolution,
it is large enough for significant entropy generation.
In explicit simulation of viscous hydrodynamics, entropy
generation is fast and most of the entropy is generated
with first 2-4 fm of evolution [15]. Thus even though
Knudsen number ansatz does not require full dynamic
range of evolution, in the time scale for validity of Knud-
sen ansatz most of the entropy will be generated.

In the present paper accounting for the entropy gen-
eration in viscous evolution, we generalise Eq.(3). As in
[11], we also consider one dimensional Bjorken longitu-
dinal expansion. If \( \eta \) and \( \zeta \) is the shear and bulk viscosity
coefficients, for Bjorken flow, energy-momentum conser-
vation equation, \( \partial_\mu T^{\mu \nu} = 0 \) reduces to [16[17],

\[
\frac{d(sT)}{d\tau} = \frac{1}{\tau T} \left( \frac{4}{3} \eta + \zeta \right) \tag{4}
\]

where \( s \) is the entropy density and \( \tau \) is the proper time.
In general bulk viscosity is much less than the shear vis-
cosity, however, near the phase transition region bulk
viscosity can be large [18][19]. In the following we ne-
glect the bulk viscosity. We further assume that shear
viscosity is proportional to cube of the temperature. For

\( \eta \propto T^3 \), Eq.(11) can be analytically integrated between
time \( \tau_i \) and \( \tau_f \). For \( \tau_f \gg \tau_i \), the final state entropy can be
written as [16][17],

\[
\tau_f s_f \approx \tau_i s_i \left[ 1 + \frac{2}{3 \tau_i T_i} \left( \frac{\eta}{s} \right) \right]^3 \tag{5}
\]

In Eq.(5) \( T_i \) is the temperature at the time scale \( \tau_i \).
Equating final state entropy density \( s_f \) with particle mul-
tiplicity per unit transverse area \( (S) \), \( \tau_f s_f \approx 3.6 \frac{1}{3} \frac{dN}{dy} \)
[13], we obtain the particle density at the time scale \( \tau_i \)
as,

\[
n_i \approx \frac{1}{K} \left[ \frac{1}{\tau_i} \frac{dN}{dy} \right] \left[ 1 + \frac{2}{3 \tau_i T_i} \left( \frac{\eta}{s} \right) \right]^{-3} \tag{6}
\]

One immediately observes that neglect of entropy gen-
eration during evolution will over estimate \( K^{-1} \), by the
factor \( \left[ 1 + \frac{2}{3 \tau_i T_i} \left( \frac{\eta}{s} \right) \right] \). As it will be shown below, ex-
perimental data indicate that the factor could be large,
\( \sim 2-7 \).

Inserting Eq.(7) in Eq.(2) eccentricity scaled elliptic flow
now can be related to observed particle multiplicity as,

\[
\left( \frac{v_2}{c} \right)_{\text{ex}} = \left( \frac{v_2}{c} \right)_{\text{th}} \left[ \frac{1}{K_0 \sigma c_s} \right] \left[ 1 + \frac{2}{3 \tau_i T_i} \left( \frac{\eta}{s} \right) \right] \tag{8}
\]

Experimental data on elliptic flow and particle mul-
tiplicity can be fitted with Eq.(8) to obtain estimates of
the hydrodynamic limit of elliptic flow \( \left( \frac{v_2}{c} \right)_{\text{th}} \) and vis-
cosity to entropy ratio in unit of initial time and tem-
perature \( \left( \frac{1}{\tau_i T_i} \right) \). Eq.(8) also involve the quantity \( K_0 \sigma \),
\( K_0 \) and \( \sigma \) and \( c_s \) are known approximately. For example,
inter parton cross section is expected to be small, \( \sigma=3-4 \) mb.
From transport calculations, \( K_0 \) was estimated as
\( K_0 = 0.7 \pm 0.3 \). The speed of sound of QGP medium is
expected to be \( c_s \approx \sqrt{1/3} \). In the following, we fix \( \sigma=3 \) mb,
\( K_0=0.7 \) and \( c_s = \sqrt{1/3} \).

In Fig.1 PHOBOS [20][22] data for the centrality de-
pendence of (participant) eccentricity scaled charged par-
ticles elliptic flow, in Au+Au and Cu+Cu collisions at
\( \sqrt{s_{NN}}=62 \) and 200 GeV are shown. PHOBOS data
have large error bars and within the error data do not
show any system size or energy dependence. All the
four data sets are fitted together with Eq. In Fig.1 shows the fit. Data are well explained, value is also small, 0.1. However, and (1/Ti)ιh and (1/Ti)2ιh can be determined only with large uncertainties, (1/Ti)ιh = 0.33 ± 0.12, (1/Ti)2ιh = 0.83 ± 0.51. Uncertainty in (1/Ti)ιh and (1/Ti)2ιh would be reduced with better quality data. While definitive conclusions cannot be made due to large uncertainty in (1/Ti)ιh, the central value suggests that even in central Au+Au collisions, ideal hydrodynamic limit is not reached. In central/mid-central collisions, experimental flow is only ~ 60% of the ideal fluid limit. In peripheral collisions, it is even less. For comparison purpose, in Fig.1 we have shown the fit to the data in the ideal fluid approximation, (1/Ti)2ιh = 0. In the ideal fluid approximation ideal hydrodynamic limit for elliptic flow is estimated as, (1/Ti)ιh = 0.19 ± 0.04. However, description to the data is comparatively poor, 0.13 is ~ 10 times larger than that obtained in viscous evolution.

With the estimate of (1/Ti)2ιh, we can compute inverse Knudsen number from Eq. In table.1 we have listed K−1 for different centrality ranges of Au+Au collisions. Particle multiplicities are taken from the PHOBOS data [20,22]. For comparison, we have also listed the values in the ideal fluid approximation. K−1 decreases by a factor of 2-7 when entropy generation is accounted for. In the ideal fluid approximation, K−1 ≈ 8 in a central collision. It is reduced to ~2.8 if entropy generation during evolution is accounted for. While it is debatable whether K−1 ≈ 8 can lead to complete equilibration, it is unlikely that complete thermalization will be achieved with K−1 ≈ 2.8. In more peripheral collisions, equilibration is certainly incomplete.

QGP viscosity is an important parameter characterizing QGP medium. String theory based models (ADS/CFT) give a lower bound on viscosity to entropy ratio of any matter, 4πη/s ≥ 1 [23]. In [24], from experimental data, a phenomenological upper bound was conjectured, 4πη/s < 5. We have obtained viscosity to entropy ratio in unit of initial time and temperature. It can be converted to more comprehensible viscosity to entropy ratio if the initial time and temperature scale is known. Recently, in [14] STAR data on φ mesons multiplicity, mean pT and integrated v2 were analyzed in ideal and viscous fluid dynamics. At the initial time T=0.6 fm, ideal or viscous fluid was initialized to reproduce experimental φ meson multiplicity. Viscous fluid requires less initial temperature than an ideal fluid. Results of the analysis are shown in Fig.2 where η/s as a func-

TABLE I: Inverse Knudsen number (K−1), in ideal and viscous fluid, as a function of collision centrality in Au+Au collision. Also listed are charged particles multiplicity per unit transverse area (dNch/dη) from PHOBOS experiment [20,22] and characteristic size R of the system. Inter parton cross section is assumed to be σ= 3 mb, speed of sound cs = √1/3.

| Centrality(%) | dNch/dη(fm−2) | R (1/Ti)ιh | 0.83 ± 0.51 | (1/Ti)2ιh | 0 |
|--------------|---------------|------------|----------------|-----------------| -
| 0-3          | 30.46         | 2.01       | 2.80 ± 1.63    | 7.91            | -
| 3-6          | 28.84         | 1.93       | 2.65 ± 1.55    | 7.49            | -
| 6-10         | 27.06         | 1.84       | 2.49 ± 1.45    | 7.03            | -
| 10-15        | 24.95         | 1.75       | 2.29 ± 1.33    | 6.48            | -
| 15-20        | 23.17         | 1.64       | 2.13 ± 1.24    | 6.02            | -
| 20-25        | 21.39         | 1.55       | 1.97 ± 1.15    | 5.56            | -
| 25-30        | 19.44         | 1.45       | 1.79 ± 1.04    | 5.05            | -
| 30-35        | 17.50         | 1.38       | 1.61 ± 0.94    | 4.55            | -
| 35-40        | 14.26         | 1.30       | 1.31 ± 0.76    | 3.70            | -
| 40-45        | 11.83         | 1.23       | 1.09 ± 0.63    | 3.07            | -
| 45-50        | 10.69         | 1.18       | 0.98 ± 0.57    | 2.78            | -

**TABLE II:** Listed are some estimates of QGP viscosity to entropy ratio from experimental data in Au+Au collisions at RHIC. The observables analyzed are also listed.

| Sl. no. | 4πη/s | Experimental observable |
|---------|-------|-------------------------|
| 1       | 0.88 ± 0.38 ± 1.76 | φ meson’s ⟨N⟩, ⟨pT⟩ and v2 [14] |
| 2       | 1.0-3.77 | pT fluctuations [26] |
| 3       | 1.4-2.4 | v2 scaling violation [27] |
| 5       | 1.3 ± 0.19 ± 1.26 | v2 scaling violation [28] |
| 6       | 1.51 ± 0.38 | v2 scaling violation [29] |
| 7       | 1.3-2.0 | heavy quark energy loss [31] |
| 8       | 1.45 ± 0.06 | v2 [32] |
| 9       | ≤ 1.51 | pT spectra of π, K and p [33] |
| 10      | 2.14 ± 1.26 ± 2.56 | v2 & dNch/dη |

*present work*
tion of the parameter \( \left( \frac{1}{\tau_i}, 2 \right) \) is shown. The solid line in Fig. 2 is a straight line fit, \( \eta/s = 0.2 \left( \frac{1}{\tau_i} \right)^2 \). The relation can be used to convert extracted \( \left( \frac{1}{\tau_i}, 2 \right) \) to \( \eta/s \). We obtain, \( \eta/s = 0.17 \pm 0.10 \). From the PHO- 
BOS data, viscosity to entropy ratio can be determined only within \( \sim 60\% \) accuracy. Evidently, much better quality data are required for more precise determination of viscosity to entropy ratio. Thus even if viscous effects are neglected in the 
experimental data include the effect of entropy generation. As mentioned earlier, the estimate was obtained with 
Knudsen ansatz, the fitted parameters 
are statistical, the second is systematic. Systematic un-
certainty in \( \eta/s \) is \( \sim 120\% \) due to a factor of 2 un-
certainty in \( K_0 \sigma_c \). We then estimate QGP viscosity to 
temperature ratio as \[ \eta/s = \frac{K_0}{\sigma_c} \approx 0.121. \] 
Estimate of viscosity depend on the value of \( K_0 \sigma_c \). Systematic un-
certainty in \( \eta/s \) is \( \sim 120\% \) due to a factor of 2 un-
certainty in \( K_0 \sigma_c \). We then estimate QGP viscosity to 
temperature ratio as \[ \eta/s = 0.17 \pm 0.10 \pm 0.20, \] the ﬁrst error is 
statistical, the second is systematic. Systematic un-
certainty will increase if uncertainty in initial time and 
temperature scale is included. The estimated value is 
well with the two bounds, \[ 1 \leq 4\eta/s \leq 5 \]. In table II present estimate for QGP viscosity is compared 
with some recent estimates. One may note that presently 
estimated \( \eta/s \) is similar to the values obtained in previous 
extractions \[ \left[ 27, 28 \right] \], which disregarded entropy generation 
in the Knudsen ansatz. The reason is understood. Experimental data include the effect of entropy generation. 
Thus even if viscous effects are neglected in the 
Knudsen ansatz, the ﬁtted paratemers \( \left( \frac{1}{\tau_i} \right)^{th} \) and \( K_0 \sigma_c \) 
(see Eq. 8) will include the effect.

We note that the present estimate should be consid-
ered as an upper limit for QGP viscosity. We have neg-
lected bulk viscosity. Experimental data include the 
effect of bulk viscosity. Neglect of bulk viscosity will 
be compensated by increasing \( \eta/s \). Also, we have neg-
lected transverse expansion. Experimental data also in-
clude the effect of transverse expansion. One observes 
from Eq. 7 that \( K^{-1} \) will decrease if transverse expansion 
is included (transverse area of system at freeze-out will 
be larger than the initial area). Neglect of the transverse expansion will be compensated again by increasing \( \eta/s \).

To conclude, taking into account that entropy is gen-
erated during evolution of a viscous ﬂuid, we have gener-
alized a relation between inverse Knudsen number \( K^{-1} \) 
and particle density \( \frac{1}{N} \frac{dN}{dy} \). PHOBOS data on the 
centrality dependence of elliptic ﬂow indicate that \( K^{-1} \) 
is overestimated by a factor \( \sim 2-7 \) if entropy generation is 
neglected. We have also estimated ideal hydrodynamic 
limit for elliptic ﬂow \( \left( \frac{\rho}{\eta} \right)^{th} \) and QGP viscosity to 
temperature ratio \( \eta/s \). Estimated \( \frac{\rho}{\eta}^{th} \) is \( \sim 40\% \) larger than 
the scaled ﬂow in a central collision. Estimated viscosity 
to entropy ratio, \( 4\eta/s = 2.14 \pm 1.26 \) is well within the 

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