EXPLICIT FORMULA FOR WITTEN-KONTSEVICH TAU-FUNCTION

JIAN ZHOU

Abstract. We present an explicit formula for Witten-Kontsevich tau-function.

1. Introduction

By the famous Witten Conjecture/Kontsevich Theorem \cite{12,9}, the generating series of intersection numbers of $\psi$-classes on moduli spaces of algebraic curves is a tau-function of the KdV hierarchy. Furthermore, as Witten \cite{12} pointed out, this fact together with the string equation completely determines this tau-function. Finding the explicit formula of this Witten-Kontsevich tau-function has remained an open problem for a long time. We will solve this problem in this paper.

In Kyoto school’s approach to the integrable hierarchy \cite{11}, there are three pictures to describe the tau-function of the KdV hierarchy: (a) as an element of the Sato infinite-dimensional Grassmannian; (b) as a vector in the bosonic Fock space, i.e. the space of symmetric functions; (c) as a vector in the fermionic Fock space, i.e., the semi-infinite wedge space. Hence one can describe the Witten-Kontsevich tau-function from each of these pictures. For the first picture, it was given by Kac-Schwarz \cite{8} using the asymptotic expansion of the Airy function. For the second picture, Alexandrov \cite{2} recently gave a recursive solution in terms of a cut-and-join type differential operator:

\begin{equation}
Z_{WK} = e^W 1 = 1 + \sum_{n \geq 1} \frac{1}{n!} \hat{W}^n 1,
\end{equation}

where the operator $\hat{W}$ is defined by

\[
\hat{W} = \frac{2}{3} \sum_{k,l \geq 0} (k + \frac{1}{2})u_k \cdot (l + \frac{1}{2})u_l \cdot \frac{\partial}{\partial u_{k+l-1}} + \frac{\lambda^2}{12} \sum_{k,l \geq 0} (k + l + \frac{5}{2})u_{k+l+2} \frac{\partial^2}{\partial u_k \partial u_l} \left( \frac{1}{\lambda^2} \frac{u_0^3}{3!} + \frac{u_1}{16} \right).
\]
Inspired by these results, we study the Witten-Kontsevich tau-function in the third picture and obtain the following result: The Witten-Kontsevich tau-function is a Bogoliubov transformation:

\[ Z_{WK} = e^A|0\rangle, \quad A = \sum_{m,n \geq 0} A_{m,n} \psi_{-m-\frac{1}{2}} \psi^*_{-n-\frac{1}{2}}, \]

where \( A_{m,n} = 0 \) if \( m + n \not\equiv -1 \pmod{3} \) and

\[
A_{3m-1,3n} = A_{3m-3,3n+2} = (-1)^n \left( -\frac{\sqrt{-2}}{144} \right)^{m+n} \frac{(6m+1)!!}{(2(m+n))!} \\
\cdot \prod_{j=0}^{n-1} (m+j) \cdot \prod_{j=1}^{n} (2m+2j-1) \cdot (B_n(m) + \frac{b_n}{6m+1}),
\]

\[
A_{3m-2,3n+1} = (-1)^{n+1} \left( -\frac{\sqrt{-2}}{144} \right)^{m+n} \frac{(6m+1)!!}{(2(m+n))!} \\
\cdot \prod_{j=0}^{n-1} (m+j) \cdot \prod_{j=1}^{n} (2m+2j-1) \cdot (B_n(m) + \frac{b_n}{6m-1}),
\]

where \( B_n(m) \) is a polynomial in \( m \) of degree \( n-1 \) defined by:

\[ B_n(x) = \frac{1}{6} \sum_{j=1}^{n} 108^j b_{n-j} \cdot (x+n)_{[j-1]}, \]

where

\[ (a)_{[j]} = \begin{cases} 1, & j = 0, \\ a(a-1) \cdots (a-j+1), & j > 0, \end{cases} \]

and \( b_n \) is a constant depending on \( n \) defined by:

\[ b_n = \frac{2^n \cdot (6n+1)!!}{(2n)!}. \]

After using the boson-fermion correspondence, our result can be translated into the bosonic picture to give an explicit formula for the Witten-Kontsevich tau-function, and it takes a very simple form. The Witten-Kontsevich tau-function is given in suitable coordinates as follows:

\[ Z_{WK} = \sum_{|\mu| \equiv 0 \pmod{3}} A_{\mu} \cdot s_{\mu}, \]

where \( s_{\mu} \) is the Schur function indexed by a partition \( \mu \), and \( A_{\mu} \) is a specialization of \( s_{\mu} \) given as follows. Let \( \mu = (m_1, \ldots, m_k | n_1, \ldots, n_k) \) in Frobenius notation, then

\[ A_{\mu} = (-1)^{n_1+\cdots+n_k} \cdot \det(A_{m_i,n_j})_{1 \leq i,j \leq k}. \]
Originally we have attempted to derive the fermionic operator from Alexandrov’s result by boson-fermion correspondence. This is a natural approach but unfortunate we did not succeed.

Our result solves a special case of a remarkable conjecture of Aganagic-Dijkgraaf-Klemm-Mariño-Vafa [1]. In joint work with Fusheng Deng [3, 4], we have established some other cases of this conjecture. In [14] we have generalized Alexandrov’s result to Witten’s $r$-spin intersection numbers (this corresponds to simple singularities of type A) and further generalization to the simple singularities of types D and E can also be made [15]. In a work in progress [16], we will generalize our result to these cases. Working on the general $r$-spin case has provided with us some important insight on the original Witten-Kontsevich tau-function, which corresponds to the $r = 2$ case. In particular, a choice of coordinates that make the idea of Bogoliubov transformation work comes from the consideration of the $r$-spin case in [14].

The rest of this paper is arranged as follows. In Section 2 we review some preliminary backgrounds on partitions, symmetric functions, and boson-fermion correspondence. In Section 3 we recall Witten Conjecture/Kontsevich Theorem, DVV Virasoro constraints, and Alexandrov’s bosonic operator formula for the Witten-Kontsevich tau-function. In Section 4 we present our main result, whose proof is presented in the last two Sections.

2. Preliminaries on Boson-Fermion Correspondence

2.1. Partitions. A partition $\mu$ of a nonnegative integral number $n$ is a decreasing finite sequence of nonnegative integers $\mu_1 \geq \cdots \geq \mu_l > 0$, such that $|\mu| := \mu_1 + \cdots + \mu_l = n$. It is very useful to graphically represent a partition by its Young diagram, e.g., $\mu = (3, 2)$ is represented by:

This leads to many natural definitions. First of all, by transposing the Young diagram one can define the conjugate $\mu^t$ of $\mu$, e.g., for $\mu = (3, 2)$, $|\mu^t| = (2, 2, 1)$:

Secondly, assume the Young diagram of $\mu$ has $k$ boxes in the diagonal. Define $m_i = \mu_i - i$ and $n_i = \mu_i^t - i$ for $i = 1, \cdots, k$, then it is clear that $m_1 > \cdots > m_k \geq 0$ and $n_1 > \cdots > n_k \geq 0$. The partition $\mu$ is completely determined by the numbers $m_i, n_i$. One can denote
the partition $\mu$ by $(m_1, \ldots, m_k|n_1, \ldots, n_k)$, this is called the Frobenius notation. A partition of the form $(m|n)$ in Frobenius notation is called a hook partition, for example:

```
+---+---+---+
|   |   |   |
+---+   +---+
|   |   |
+---+---+---+
```

2.2. **Schur functions.** Let $\Lambda$ be the space of symmetric functions in $x = (x_1, x_2, \ldots)$. For a partition $\mu$, let $s_\mu := s_\mu(x)$ be the Schur function in $\Lambda$. If we write $\mu = (m_1, \ldots, m_k|n_1, \ldots, n_k)$ in Frobenius notation, then there is a determinantal formula that expresses $s_\mu$ in terms of $s_{(m|n)}$ ([10], p. 47, Example 9):

$$s_\mu = \det(s_{(m_i|n_j)})_{1 \leq i,j \leq k}.$$ 

where $\mu = (m_1, \ldots, m_k|n_1, \ldots, n_k)$ be a partition in Frobenius notation.

2.3. **Fermionic Fock space.** We say a sequence of half-integers $a_1 > a_2 > \cdots$ is admissible if both the set $\mathbb{Z}_- + \frac{1}{2}\setminus\{a_1, a_2, \ldots\} \subset \mathbb{Z} + \frac{1}{2}$ and the set $\mathbb{Z}_- + \frac{1}{2}\setminus\{a_1, a_2, \ldots\}$ are finite, where $\mathbb{Z}_-$ is the set of negative integers.

For an admissible sequence $a_1 > a_2 > \cdots$, let $A = \{a_1, a_2, \ldots\}$, associate an element $\underline{A}$ in the half-infinite wedge space as follows:

$$\underline{A} = a_1 \wedge a_2 \wedge \cdots.$$ 

The free fermionic Fock space $\mathcal{F}$ is defined as

$$\mathcal{F} = \text{span}\{\underline{A} : A \subset \mathbb{Z} + \frac{1}{2} \text{ is admissible}\}.$$ 

One can define an inner product on $\mathcal{F}$ by taking $\{\underline{A} : A \subset \mathbb{Z} + \frac{1}{2} \text{ is admissible}\}$ as an orthonormal basis.

2.4. **Charge decomposition.** There is a natural decomposition

$$\mathcal{F} = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}^{(n)},$$

where $\mathcal{F}^{(n)} \subset \mathcal{F}$ the subspace spanned by $\underline{A}$ such that

$$|A|\mathbb{Z}_- + \frac{1}{2} - |\mathbb{Z}_- + \frac{1}{2}\setminus A| = n.$$ 

An operator on $\mathcal{F}$ is called charge 0 if it preserves the above decomposition.
The charge 0 subspace \( \mathcal{F}^{(0)} \) has a basis indexed by partitions:

\[
|\mu\rangle := \mu_1 \left(\frac{1}{2}\right) \wedge \cdots \wedge \mu_l \left(\frac{1}{2}\right) \wedge -l \left(\frac{1}{2}\right) \wedge \cdots,
\]

where \( \mu = (\mu_1, \ldots, \mu_l) \). If \( \mu = (m_1, \ldots, m_k | n_1, \ldots, n_k) \) in Frobenius notation, then

\[
|\mu\rangle = m_1 \left(\frac{1}{2}\right) \wedge \cdots \wedge m_k \left(\frac{1}{2}\right) \wedge -\frac{3}{2} \wedge \cdots \wedge -n_k \left(\frac{1}{2}\right) \wedge \cdots.
\]

In particular, when \( \mu \) is the empty partition, we get:

\[
|0\rangle := -\frac{1}{2} \wedge -\frac{3}{2} \wedge \cdots \in \mathcal{F}.
\]

It is called the fermionic vacuum vector.

### 2.5. Creators and annihilators on \( \mathcal{F} \)

For \( r \in \mathbb{Z} + \frac{1}{2} \), define operators \( \psi_r \) and \( \psi^*_r \) by

\[

\psi_r(A) = \begin{cases} 
(-1)^k a_1 \wedge \cdots \wedge a_k \wedge r \wedge a_{k+1} \wedge \cdots, & \text{if } a_k > r > a_{k+1} \text{ for some } k, \\
0, & \text{otherwise},
\end{cases}
\]

\[

\psi^*_r(A) = \begin{cases} 
(-1)^{k+1} a_1 \wedge \cdots \wedge a_k \wedge \cdots, & \text{if } a_k = r \text{ for some } k, \\
0, & \text{otherwise}.
\end{cases}
\]

The anti-commutation relations for these operators are

\[
[\psi_r, \psi^*_s]_+ := \psi_r \psi^*_s + \psi^*_s \psi_r = \delta_{r,s} i \text{id}
\]

and other anti-commutation relations are zero. It is clear that for \( r > 0 \),

\[
\psi_r |0\rangle = 0,
\]

\[
\psi^*_r |0\rangle = 0,
\]

so the operators \( \{\psi_r, \psi^*_r\}_{r>0} \) are called the fermionic annihilators. For a partition \( \mu = (m_1, m_2, \ldots, m_k | n_1, n_2, \ldots, n_k) \), it is clear from the definition of \( |\mu\rangle \) that

\[
|\mu\rangle = (-1)^{n_1+n_2+\cdots+n_k} \prod_{i=1}^{k} \psi_{m_i+\frac{1}{2}} \psi^*_{-n_i-\frac{1}{2}} |0\rangle.
\]

So the operators \( \{\psi_{-r}, \psi^*_r\}_{r>0} \) are called the fermionic creators.
2.6. **The boson-fermion correspondence.** For any integer $n$, define an operator $\alpha_n$ on the fermionic Fock space $\mathcal{F}$ as follows:

$$\alpha_n = \sum_{r \in \mathbb{Z} + \frac{1}{2}} : \psi_{-r} \psi^*_{r+n} :$$

Let $\mathcal{B} = \Lambda[z, z^{-1}]$ be the bosonic Fock space, where $z$ is a formal variable. Then the boson-fermion correspondence is a linear isomorphism $\Phi: \mathcal{F} \rightarrow \mathcal{B}$ given by

$$u \mapsto z^m \langle 0^m | e^{\sum_{n=1}^{\infty} \frac{p_n}{n} \alpha_n} u \rangle, \quad u \in \mathcal{F}^{(m)}$$

where $|0^m\rangle = -\frac{1}{2} + m \wedge -\frac{3}{2} + m \wedge \cdots$. It is clear that $\Phi$ induces an isomorphism between $\mathcal{F}^{(0)}$ and $\Lambda$. Explicitly, this isomorphism is given by

$$|\mu\rangle \longleftrightarrow s_\mu.$$

3. **Bosonic Representation of Witten-Kontsevich Tau-Function**

3.1. **Free energy of 2D topological gravity.** The correlators of the 2D topological gravity are defined as the following intersection numbers on the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable algebraic curves:

$$\langle \tau_{a_1} \cdots \tau_{a_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_n}.$$ 

Because the dimension of $\overline{\mathcal{M}}_{g,n}$ is given by the following formula:

$$\dim \overline{\mathcal{M}}_{g,n} = 3g - 3 + n,$$

the following selection rule is satisfied by the correlators: The correlator $\langle \tau_{a_1} \cdots \tau_{a_n} \rangle_g \neq 0$ only if

$$a_1 + \cdots + a_n = 3g - 3 + n.$$

This suggests the following grading:

$$\deg t_a = 2a + 1.$$ 

In the literature, the following gradings have also been used: $\deg t_a = a + \frac{1}{2}$ or $\deg t_a = \frac{1}{2}(2a + 1)$. The reason for our choice of the grading will be made clear later in this paper.

The free energy is defined by

$$F(t) = F_0(t) + F_1(t) + \cdots,$$
where the genus $g$ part of the free energy is defined by:

\[(20) \quad F_g(t) = \langle \exp \sum_{a \geq 0} t_a \tau_a \rangle_g = \sum_{n \geq 0} \frac{1}{n!} \sum_{a_1, \ldots, a_n \geq 0} t_{a_1} \cdots t_{a_n} \langle \tau_{a_1} \cdots \tau_{a_n} \rangle_g.\]

It is a common practice to introduce a genus tracking parameter $\lambda$ and set

\[(21) \quad F(t; \lambda) = \sum_{g \geq 0} \lambda^{2g-2} F_g(t).\]

For our purpose in this paper, it is crucial to suppress the explicit dependence on $\lambda$. However, this will not cause any loss of information. Indeed, by the selection rule (17), one can recover $F(t; \lambda)$ from $F(t)$ as follows:

\[(22) \quad F(t; \lambda) = F(\lambda^{-2/3} t_0, t_1, \ldots, \lambda^{(2n-2)/3} t_n, \ldots).\]

For the reader’s convenience, we recall the first few terms of $F_g(t)$ here:

\[
F_0 = \frac{1}{6} t_0^3 + \frac{1}{6} t_0^3 t_1 + \frac{1}{6} t_0^3 t_2 + \cdots,
F_1 = \frac{1}{24} t_1 + \frac{1}{24} t_0 t_2 + \frac{1}{48} t_1^2 + \frac{1}{12} t_0^2 t_1 + \frac{1}{48} t_0^2 t_2 + \cdots,
F_2 = \frac{1}{1152} t_4 + \frac{29}{5760} t_2 t_3 + \frac{1}{384} t_1 t_4 + \frac{1}{1152} t_0 t_5 + \cdots.
\]

3.2. Partition function of 2D topological gravity. It is defined by

\[(23) \quad Z_{WK}(t) = \exp F(t).\]

This is also called the Witten-Kontsevich tau-function. The following are the first few terms of $Z_{WK}(t)$:

\[
Z_{WK}(t) = 1 + \left( \frac{1}{6} t_0^3 + \frac{1}{24} t_1 \right) + \left( \frac{25}{144} t_0^3 t_1 + \frac{1}{24} t_0 t_2 + \frac{25}{1152} t_1^2 + \frac{1}{72} t_0^2 t_1 \right) + \cdots
\]

3.3. Witten Conjecture/Kontsevich Theorem and Virasoro constraints. Witten [12] conjectured that $Z_{WK}$ is a tau-function of the KdV hierarchy. He also pointed out that together with the string equation satisfied by $Z_{WK}$, this hierarchy of nonlinear differential equations uniquely determines $Z_{WK}$. For a proof of this conjecture, see Kontsevich [9]. Dijkgraaf, E. Verlinde, H. Verlinde [5] and independently Fukuma, Kawai, Nakayama [6] showed that that condition of being a
solution of the KdV hierarchy satisfying the string equation is equivalent to satisfying a sequence of linear differential equations called the Virasoro constraints. In terms of correlators they are given by:

\[
\langle \tilde{\tau}_{a_0} \prod_{i=1}^{n} \tilde{\tau}_{a_i} \rangle_g = \sum_{i=1}^{n} (2a_i + 1) \langle \tilde{\tau}_{a_0+a_i-1} \prod_{j \in [n]_i} \tilde{\tau}_{a_j} \rangle_g \\
+ \frac{1}{2} \sum_{b_1+b_2=a_0-2} \left( \langle \tilde{\tau}_{b_1} \tilde{\tau}_{b_2} \prod_{i=1}^{n} \tilde{\tau}_{a_i} \rangle_{g-1} \right) \\
+ \sum_{A_1 \cup A_2 = [n]}^{s} \langle \tilde{\tau}_{b_1} \prod_{i \in A_1} \tilde{\tau}_{a_i} \rangle_{g_1} \cdot \langle \tilde{\tau}_{b_2} \prod_{i \in A_2} \tilde{\tau}_{a_i} \rangle_{g_2},
\]

where \( \tilde{\tau}_a = (2a+1)!! \cdot \tau_a \) and \([n] = \{1, \ldots, n\}\), \([n]_i = [n] - \{i\}\). Here the summation \( \sum_{A_1 \cup A_2 = [n]}^{s} \) is taken over pairs \((g_1, A_1)\) and \((g_2, A_2)\) that satisfy the stability conditions:

\[ 2g_1 - 1 + |A_1| > 0, \quad 2g_2 - 1 + |A_2| > 0. \]

With the following change of coordinates:

\[ t_k = \frac{1}{2^k} (2k + 1)!! u_k, \]

the Virasoro constraints can be written as the following sequence of differential equations:

\[
\frac{\partial}{\partial u_{n+1}} Z_{WK} = \hat{L}_n Z_{WK}, \quad n \geq -1,
\]

where the operators \( \hat{L}_n \) are defined by:

\[
\hat{L}_n = \sum_{k=1}^{\infty} (k+1/2) u_k \frac{\partial}{\partial u_{k+n}} + \frac{\lambda^2}{8} \sum_{k=0}^{n-1} \frac{\partial^2}{\partial u_k \partial u_{n-k-1}} + \frac{u_{n+1}^2}{2\lambda^2} \delta_{n,-1} + \frac{\delta_{n,0}}{16}.
\]

3.4. Kontsevich model and Alexandrov’s solution. The Witten-Kontsevich tau-function can be represented by Kontsevich matrix integral \( \mathcal{Z} \):

\[
Z_{WK} = \frac{\int [dA] \exp \left( -\lambda^{-1} \text{Tr} \left( \frac{A^3}{3!} + \frac{\Lambda A}{2} \right) \right) }{\int [dA] \exp \left( -\lambda^{-1} \text{Tr} \left( \frac{A^3}{3!} \right) \right) },
\]

with times given by the Miwa variables:

\[ u_k = \frac{\lambda}{2k+1} \text{Tr} \frac{1}{\Lambda^{2k+1}}. \]
One can write
\[ Z_{WK} = \sum_{n=0}^{\infty} Z_{WK}^{(n)}, \]
where
\[ Z_{WK}^{(n)} = \frac{(-1)^n}{n!(3!\lambda)^n} \int [dA](\text{Tr} A^3)^n \exp \left(-\lambda^{-1} \text{Tr} \left(\frac{\lambda A}{2}\right)\right) \int [dA] \exp \left(-\lambda^{-1} \text{Tr} \left(\frac{A^3}{3!}\right)\right). \]

Alexandrov [2] noticed that if one defines \( \deg u_k = \frac{2k+1}{3} \), then one has
\[ \deg Z_{WK}^{(n)} = n. \]
Therefore, by (26),
\[ \hat{W} Z_{WK} = \hat{D} Z_{WK}, \]
where
\[ \hat{D} = \frac{2}{3} \sum_{k=0}^{\infty} (k + \frac{1}{2}) u_k \frac{\partial}{\partial u_k} \]
is the Euler vector field for the grading, and the operator
\[ \hat{W} = \frac{2}{3} \sum_{k=1}^{\infty} (k + \frac{1}{2}) u_k \hat{L}_{k-1} \]
\[ = \frac{2}{3} \sum_{k,l \geq 0} (k + \frac{1}{2}) u_k \cdot (l + \frac{1}{2}) u_l \cdot \frac{\partial}{\partial u_{k+l-1}} \]
\[ + \frac{\lambda^2}{12} \sum_{k,l \geq 0} (k + l + \frac{5}{2}) u_{k+l+2} \frac{\partial^2}{\partial u_k \partial u_l} + \frac{1}{\lambda^2} \frac{u_0^3}{3!} + \frac{u_1}{16} \]
has
\[ \deg \hat{W} = 1. \]
Using the grading decomposition (30), one has
\[ \hat{W} Z_{WK}^{(n)} = (n + 1) Z_{WK}^{(n+1)}, \quad n \geq 0. \]
Alexandrov then gets the following solution:
\[ Z_{WK} = e^{\hat{W}} 1. \]
We will not use this result in this paper, but as mentioned in the Introduction, the search for the fermionic version of this result serves as a motivation for our work.
4. Main Theorem

4.1. A natural coordinate. If we use the following change of coordinates introduced in [14]:

\begin{equation}
\label{natural-coordinate}
t_n = (-1)^n \sqrt{-2} T_{2n+1} \cdot \prod_{j=0}^{n} (j + \frac{1}{2}),
\end{equation}

then we get

\begin{align*}
Z_{WK} &= 1 - \sqrt{-2} \left( \frac{1}{24} T_1^3 + \frac{1}{32} T_3 \right) \\
&\quad - \frac{25}{384} T_1^3 T_3 + \frac{5}{64} T_1 T_5 + \frac{25}{1024} T_3^2 + \frac{1}{576} T_6^1 \\
&\quad + \sqrt{-2} \left( \frac{105}{4096} T_9 + \frac{1225}{98304} T_3^3 + \frac{245}{2048} T_3 T_1 T_5 \right) \\
&\quad + \frac{35}{512} T_1^2 T_7 + \frac{1225}{24576} T_3 T_1^3 + \frac{35}{1536} T_4^1 T_5 \\
&\quad + \frac{49}{18432} T_1^6 T_3 + \frac{1}{41472} T_1^9 + \cdots
\end{align*}

Note

\begin{equation}
\label{un-coordinate}
u_n = \frac{(-1)^n \sqrt{-2}}{2} T_{2n+1},
\end{equation}

one can rewrite the operators \( L_n = -\frac{\partial}{\partial u_{n+1}} + \hat{L}_n \) and \( \hat{W} \) in the \( T \)-coordinates as follows:

\begin{align*}
L_n &= (-1)^{n+1} \sqrt{-2} \frac{\partial}{\partial T_{2n+3}} + \frac{(-1)^n}{2} \sum_{k=0}^{\infty} (2k + 1) T_{2k+1} \frac{\partial}{\partial T_{2k+2n+1}} \\
&\quad + \frac{(-1)^n \lambda^2}{4} \sum_{k=0}^{n-1} \frac{\partial^2}{\partial T_{2k+1} \partial T_{2n-2k-1}} - \frac{T_1^2}{4 \lambda^2} \delta_{n,-1} + \frac{\delta_{n,0}}{16},
\end{align*}

\begin{align*}
\hat{W} &= -\frac{\sqrt{-2}}{12} \sum_{k,l \geq 0} (2k + 1) T_{2k+1} \cdot (2l + 1) T_{2l+1} \cdot \frac{\partial}{\partial T_{2k+2l-1}} \\
&\quad - \frac{\sqrt{-2}}{24} \sum_{k,l \geq 0} (2k + 2l + 5) T_{2k+2l+5} \frac{\partial^2}{\partial T_{2k+1} \partial T_{2l+1}} \\
&\quad - \frac{\sqrt{-2}}{24} T_1^3 - \frac{\sqrt{-2}}{32} T_3.
\end{align*}
In particular,

\[ L_{-1} = \sqrt{-2} \frac{\partial}{\partial T_1} - \frac{1}{2} \sum_{k=1}^{\infty} (2k + 1) T_{2k+1} \frac{\partial}{\partial T_{2k-1}} - \frac{T_1^2}{4 \lambda^2}, \]

and

\[ L_0 = -\sqrt{-2} \frac{\partial}{\partial T_3} + \frac{1}{2} \sum_{k=0}^{\infty} (2k + 1) T_{2k+1} \frac{\partial}{\partial T_{2k+1}} + \frac{1}{16}. \]

We define the following operator on the space \( \mathbb{C}[T_1, T_3, \ldots] \):

\[ \gamma_n = \begin{cases} (-n) \cdot T_{-n}, & n < 0, \\ \frac{\partial}{\partial T_n}, & n > 0. \end{cases} \] (42)

Then we get

\[ L_n = (-1)^{n+1} \sqrt{-2}\gamma_{2n+3} + \frac{(-1)^n}{4} \sum_{a+b=-n-1} \gamma_{2a+1}\gamma_{2b+1} : + \frac{1}{16} \delta_{n,0}, \] (43)

in particular,

\[ L_{-1} = \sqrt{-2}\gamma_1 - \frac{1}{4} \sum_{a+b=-2} \gamma_{2a+1}\gamma_{2b+1} : \] (44)

and

\[ L_0 = -\sqrt{-2}\gamma_3 + \frac{1}{4} \sum_{a+b=-1} \gamma_{2a+1}\gamma_{2b+1} : + \frac{1}{16}. \] (45)

We also have

\[ \hat{W} = -\frac{\sqrt{-2}}{12} \sum_{k,l \geq 0 \atop k+l \geq 0} \gamma_{-(2k+1)}\gamma_{-(2l+1)}\gamma_{2k+2l-1} \]

\[ - \frac{\sqrt{-2}}{24} \sum_{k,l \geq 0 \atop k+l \geq 0} \gamma_{-(2k+2l+5)}\gamma_{2k+1}\gamma_{2l+1} - \frac{\sqrt{-2}}{24} \gamma_{-1}^3 - \frac{\sqrt{-2}}{96} \gamma_{-3}. \]

It can be written in a more compact form:

\[ \hat{W} = -\frac{\sqrt{-2}}{24} \sum_{k \geq 0} \gamma_{-(2k+1)} \left( \sum_{a+b=k-2} \gamma_{2a+1}\gamma_{2b+1} : + \frac{1}{4} \delta_{k,1} \right). \] (46)
4.2. Reformulation of Witten-Kontsevich tau-function in terms of Schur functions. We make a further change of variables:

\[
T_n = \frac{1}{n} p_n.
\]

Now we have

\[
Z_{WK} = 1 - \sqrt{-2} \left( \frac{1}{96} p_3 + \frac{1}{24} p_1^3 \right) - \left( \frac{25}{1152} p_1^3 p_3 + \frac{1}{64} p_1 p_5 + \frac{25}{9216} p_3^2 + \frac{1}{576} p_6^1 \right) \\
+ \sqrt{-2} \left( \frac{35}{12288} p_9 + \frac{1}{1225} p_1 p_5 + \frac{6144}{2654208} p_3^3 + \frac{7}{1536} p_1^3 p_5 \\
+ \frac{5}{512} p_5^2 p_7 + \frac{1225}{221184} p_1^3 p_5^2 \right) + \cdots.
\]

We understand \( p_n \) as the Newton power function:

\[
p_n = x_1^n + x_2^n + \cdots,
\]

and so each \( Z_{n, WK} \) lies in \( \Lambda^{3n} \), the degree 3n subspace of the space \( \Lambda \) of symmetric functions in \( x_1, x_2, \ldots \).

Recall that for a partition \( \nu = (\nu_1, \ldots, \nu_l) \) of a positive integer,

\[
p_\nu = p_{\nu_1} \cdots p_{\nu_l},
\]

and for the empty partition \( \emptyset \), \( p_\emptyset = 1 \). It is well-known that \( \{p_\nu\}_\nu \) form an additive basis of \( \Lambda \). Another additive basis is provided by the Schur functions \( \{s_\mu\}_\mu \), and they are related to each other by the Frobenius formula \[10\]:

\[
p_\nu = \sum_\mu \lambda_\nu^\mu s_\mu.
\]
We use this formula to rewrite $Z_{W_K}$ in terms of Schur functions:

\[
Z = 1 - \frac{\sqrt{-2}}{96} (5s_{(3)} + 7s_{(2,1)} + 5s_{(1^3)}) \\
- \frac{1}{9216} (385s_{(6)} + 455s_{(5,1)} + 0 \cdot s_{(4,2)} + 385s_{(4,1^2)} \\
- 70s_{(3,3)} - 50s_{(3,2,1)} - 70s_{(2,2,2)} \\
+ 385s_{(3,1^3)} + 455s_{(2,1^4)} + 385s_{(1^9)}) \\
+ \frac{\sqrt{-2}}{2654208} (85085s_{(9)} + 95095s_{(8,1)} + 0 \cdot s_{(7,2)} \\
+ 85085s_{(7,1^2)} - 8085s_{(6,3)} - 5775s_{(6,2,1)} \\
+ 87010s_{(6,1^3)} + 6825s_{(5,4)} + 0 \cdot s_{(5,3,1)} \\
- 6825s_{(5,2^2)} + 0 \cdot s_{(5,2,1^2)} + 91910s_{(5,1^4)} \\
+ 5775s_{(4^2,1)} + 8085s_{(4,3,2)} + 0 \cdot s_{(4,3,1^2)} \\
+ 0 \cdot s_{(4,2^2,1)} + 0 \cdot s_{(4,2,1^3)} + 87010s_{(4,1^5)} \\
- 1050s_{(3^3)} + 8085s_{(3^2,2,1)} - 6825s_{(3^2,1^3)} \\
+ 5775s_{(3,2^3)} + 0 \cdot s_{(3,2^2,1^2)} - 5775s_{(3,1^2,1^4)} \\
+ 85085s_{(3,1^6)} + 6825s_{(2^4,1)} - 8085s_{(2^3,1^3)} \\
+ 0 \cdot s_{(2^4,1^5)} + 95095s_{(2,1^7)} + 85085s_{(1^9)}) + \cdots.
\]
4.3. Fermionic formula for $W_{WK}$. Now we use the formula for boson-fermion correspondence to rewrite $W_{WK}$ in the fermionic picture:

$$Z_{WK} = 1 - \frac{\sqrt{-2}}{96}(5\psi_{-\frac{1}{2}}\psi^{*}_{\frac{1}{2}} - 7\psi_{-\frac{1}{2}}\psi^{*}_{\frac{1}{2}} + 5\psi_{-\frac{1}{2}}\psi^{*}_{\frac{1}{2}})$$

$$- \frac{1}{9216}(385\psi_{-\frac{11}{2}}\psi^{*}_{\frac{1}{2}} - 455\psi_{-\frac{7}{2}}\psi^{*}_{\frac{5}{2}} + 385\psi_{-\frac{3}{2}}\psi^{*}_{\frac{7}{2}})$$

$$+ 70\psi_{-\frac{5}{2}}\psi^{*}_{\frac{5}{2}} + 35\psi_{-\frac{9}{2}}\psi^{*}_{\frac{9}{2}} - 50\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}\psi^{*}_{\frac{1}{2}} + 70\psi_{-\frac{3}{2}}\psi^{*}_{\frac{5}{2}} - 35\psi_{-\frac{1}{2}}\psi^{*}_{\frac{1}{2}} + 385\psi_{-\frac{3}{2}}\psi^{*}_{\frac{1}{2}}$$

$$- 385\psi_{-\frac{1}{2}}\psi^{*}_{\frac{3}{2}} + 455\psi_{-\frac{3}{2}}\psi^{*}_{\frac{7}{2}} - 385\psi_{-\frac{1}{2}}\psi^{*}_{\frac{1}{2}})$$

$$+ \frac{\sqrt{-2}}{2654208}(85085\psi_{-\frac{11}{2}}\psi^{*}_{\frac{1}{2}} - 95095\psi_{-\frac{13}{2}}\psi^{*}_{\frac{3}{2}} + 85085\psi_{-\frac{13}{2}}\psi^{*}_{\frac{5}{2}})$$

$$+ 8085\psi_{-\frac{11}{2}}\psi^{*}_{\frac{3}{2}} - 8085\psi_{-\frac{11}{2}}\psi^{*}_{\frac{3}{2}} - 5775 s_{(6,2,1)}$$

$$- 87010\psi_{-\frac{11}{2}}\psi^{*}_{\frac{3}{2}} + 6825 s_{(5,4)}$$

$$- 6825 s_{(5,2,1)} + 91910\psi_{-\frac{9}{2}}\psi^{*}_{\frac{9}{2}}$$

$$+ 5775 s_{(4,2,1)} + 8085 s_{(4,3,2)}$$

$$- 87010\psi_{-\frac{7}{2}}\psi^{*}_{\frac{7}{2}}$$

$$- 1050 s_{(3,1)} + 8085 s_{(3,2,1)} - 8085 s_{(3,2,1)}$$

$$+ 5775 s_{(3,2,1)} - 5775 s_{(3,2,1)}$$

$$+ 8085\psi_{-\frac{3}{2}}\psi^{*}_{\frac{13}{2}} + 6825 s_{(21,1)} - 8085 s_{(23,13)}$$

$$- 95095\psi_{-\frac{3}{2}}\psi^{*}_{\frac{13}{2}} + 85085\psi_{-\frac{1}{2}}\psi^{*}_{\frac{13}{2}} + \cdots$$

One can check that:

$$Z = \exp\left(-\frac{\sqrt{-2}}{96}(5\psi_{-\frac{1}{2}}\psi^{*}_{\frac{1}{2}} - 7\psi_{-\frac{1}{2}}\psi^{*}_{\frac{1}{2}} + 5\psi_{-\frac{1}{2}}\psi^{*}_{\frac{1}{2}})\right)$$

$$- \frac{1}{9216}(385\psi_{-\frac{11}{2}}\psi^{*}_{\frac{1}{2}} - 455\psi_{-\frac{7}{2}}\psi^{*}_{\frac{5}{2}} + 385\psi_{-\frac{3}{2}}\psi^{*}_{\frac{7}{2}})$$

$$- 385\psi_{-\frac{3}{2}}\psi^{*}_{\frac{1}{2}} + 455\psi_{-\frac{3}{2}}\psi^{*}_{\frac{7}{2}} - 385\psi_{-\frac{1}{2}}\psi^{*}_{\frac{1}{2}})$$

$$+ \frac{\sqrt{-2}}{2654208}(85085\psi_{-\frac{11}{2}}\psi^{*}_{\frac{1}{2}} - 95095\psi_{-\frac{13}{2}}\psi^{*}_{\frac{3}{2}} + 85085\psi_{-\frac{13}{2}}\psi^{*}_{\frac{5}{2}})$$

$$- 87010\psi_{-\frac{11}{2}}\psi^{*}_{\frac{3}{2}} + 6825 s_{(5,4)}$$

$$+ 5775 s_{(3,2,1)} - 8085 s_{(23,13)}$$

$$- 95095\psi_{-\frac{3}{2}}\psi^{*}_{\frac{13}{2}} + 85085\psi_{-\frac{1}{2}}\psi^{*}_{\frac{13}{2}} + \cdots \right) 1.$$
where the operator

\[ A = \sum_{m,n \geq 0} A_{m,n} \psi_{-m-\frac{1}{2}} \psi^*_{-n-\frac{1}{2}} \]

with the coefficients \( A_{m,n} \) as given in the Introduction.

By applying (12) and (14), one easily derives the formula (6) in the Introduction.

The proof of this Theorem will occupy the next two Sections.

5. Derivation of the Explicit Formula for \( A_{m,n} \)

5.1. Fermionic formulation of the Virasoro operators. To make the computations easier, we make use of fields of operators and operator product expansions [7]. Consider the fields of fermionic operators:

\[ \psi(z) = \sum_{r \in \mathbb{Z}+\frac{1}{2}} \psi_r z^{-r-\frac{1}{2}}, \]

\[ \psi^*(z) = \sum_{s \in \mathbb{Z}+\frac{1}{2}} \psi^*_s z^{-s-\frac{1}{2}}, \]

The commutation relations

\[ [\psi_r, \psi_s]_+ = 0, \quad [\psi^*_r, \psi^*_s] = 0, \quad [\psi_r, \psi^*_s] = \delta_{r+s,0} \]

are equivalent to the following OPE’s:

\[ \psi(z)\psi^*(w) \sim \psi(z)\psi^*(w) : + \frac{1}{z-w}, \]

\[ \psi^*(z)\psi(w) \sim \psi^*(z)\psi(w) : + \frac{1}{z-w}, \]

Write

\[ \gamma(z) =: \psi(z)\psi^*(z) := \sum_{n \in \mathbb{Z}} \gamma_n z^{-n-1}, \]

where

\[ \gamma_n = \sum_{r,s \in \mathbb{Z}+1/2 \atop r+s=n} : \psi_r \psi^*_s :. \]

Consider

\[ \eta(z) = \frac{1}{2}(\gamma(z) + \gamma(-z)) = \sum_{n \in \mathbb{Z}} \gamma_{2n+1} z^{-2n-2}. \]
By Wick’s theorem,
\[ \gamma(z)\gamma(w) = :\psi(z)\psi^*(z) : :\psi(w)\psi^*(w) : \]
\[ = :\psi(z)\psi^*(z)\psi(w)\psi^*(w) : + \frac{\psi(z)\psi^*(w)}{z-w} \]
\[ + \frac{\psi^*(z)\psi(w)}{z-w} + \frac{1}{(z-w)^2} \]
\[ = \frac{1}{(z-w)^2} :\partial_w\psi(w)\psi^*(w) : + :\partial_w\psi^*(w)\psi(w) : + \cdots , \]

and so
\[ \eta(z)\eta(w) \]
\[ = \frac{1}{4}(\gamma(z)\gamma(w) + \gamma(z)\gamma(-w) + \gamma(-z)\gamma(w) + \gamma(-z)\gamma(-w)) \]
\[ = \frac{1}{4}\left( \frac{1}{(z-w)^2} + :\partial_w\psi(w)\psi^*(w) : + :\partial_w\psi^*(w)\psi(w) : + \cdots \right) \]
\[ + \frac{1}{(z+w)^2} + :\partial_w\psi(-w)\psi^*(-w) : + :\partial_w\psi^*(-w)\psi(-w) : + \cdots \]
\[ + \frac{1}{(-z-w)^2} + :\partial_w\psi(w)\psi^*(w) : + :\partial_w\psi^*(w)\psi(w) : + \cdots \]
\[ + \frac{1}{(z-w)^2} + :\partial_w\psi(-w)\psi^*(-w) : + :\partial_w\psi^*(-w)\psi(-w) : + \cdots \]
\[ = \left( \frac{z^2 + w^2}{(z^2 - w^2)^2} \right) + \frac{1}{2} :\partial_w\psi(w)\psi^*(w) : + :\partial_w\psi^*(w)\psi(w) : \]
\[ + :\partial_w\psi(-w)\psi^*(-w) : + :\partial_w\psi^*(-w)\psi(-w) : + \cdots . \]

Hence
\[ :\eta(w)\eta(w) : = \frac{1}{2}( :\partial_w\psi(w)\psi^*(w) : + :\partial_w\psi^*(w)\psi(w) : \]
\[ + :\partial_w\psi(-w)\psi^*(-w) : + :\partial_w\psi^*(-w)\psi(-w) : ) \]
\[ = \sum_{n \in \mathbb{Z}} \sum_{r+s=2n} (s-r) :\psi_r\psi_s^* : w^{-2n-2} . \]

It follows that
\[ \sum_{a+b=n} :\gamma_{2a+1}\gamma_{2b+1} := \sum_{r+s=2n+2} (s-r) :\psi_r\psi_s^* : . \]

Recall the formula for \( L_n \) is
\[ L_n = (-1)^{n+1}\sqrt{-2}\gamma_{2n+3} + \frac{(-1)^n}{4} \sum_{a+b=n-1} :\gamma_{2a+1}\gamma_{2b+1} : + \frac{1}{16}\delta_{n,0} , \]

\[ (56) \]

\[ (57) \]
because we also have
\[ \gamma_{2n+1} = \sum_{r, s \in \mathbb{Z}+1/2} \psi_r \psi_s^* : , \]
we have:
\[ L_n = (-1)^{n+1} \sqrt{-2} \sum_{r+s=2n+1} : \psi_r \psi_s^* : \]
\[ + \frac{(-1)^n}{4} \sum_{r+s=2n} (s-r) : \psi_r \psi_s^* : + \frac{1}{16} \delta_{n,0}. \]
This was also obtained by Kac and Schwarz \[8\] in a different normalization.

5.2. Constraints from \( L_{-1} \). Now we have
\[ L_{-1} = \sqrt{-2} \sum_{r_1+s_1=1} : \psi_{r_1} \psi_{s_1}^* : - \frac{1}{4} \sum_{r_2+s_2=-2} (s_2-r_2) : \psi_{r_2} \psi_{s_2}^* : \]
\[ = \sqrt{-2} \left( \psi_{\frac{1}{2}} \psi_{\frac{1}{2}}^* + \sum_{k=0}^{\infty} \left( \psi_{-\frac{k-1}{2}} \psi_{\frac{k+3}{2}} - \psi_{-\frac{k-1}{2}}^* \psi_{\frac{k+3}{2}}^* \right) \right) \]
\[ - \frac{1}{4} \left( \psi_{-\frac{1}{2}} \psi_{-\frac{1}{2}}^* - \psi_{-\frac{3}{2}} \psi_{-\frac{3}{2}}^* + \sum_{l=0}^{\infty} (2l+3) \left( \psi_{-l-\frac{3}{2}} \psi_{\frac{l+1}{2}}^* + \psi_{-l-\frac{3}{2}}^* \psi_{l+\frac{1}{2}} \right) \right). \]
Under the assumption that:
\[ Z = \exp \left( \sum_{m,n \geq 0} A_{m,n} \psi_{-m-\frac{1}{2}} \psi_{-n-\frac{1}{2}} \right) \langle 0 \rangle, \]
where \( A_{m,n} \neq 0 \) only if \( m, n \geq 0 \) and \( m + n \equiv -1 \pmod{3} \), we have
\[ e^{-A} L_{-1} Z = \left[ \sqrt{-2} \left( - \sum_{m,n \geq 0} A_{m,0} A_{0,n} \psi_{-m-\frac{1}{2}} \psi_{-n-\frac{1}{2}} \right) \right. \]
\[ + \sum_{k,l \geq 0} \left( \psi_{-k-\frac{1}{2}} A_{k+1,l} \psi_{-l-\frac{1}{2}} + \psi_{-k-\frac{1}{2}}^* A_{l,k+1} \psi_{-l-\frac{1}{2}}^* \right) \]
\[ - \frac{1}{4} \left( \psi_{-\frac{3}{2}} \psi_{\frac{3}{2}}^* - \psi_{-\frac{1}{2}}^* \psi_{-\frac{1}{2}} \right) \]
\[ + \sum_{k,l \geq 0} (2l+3) \left( \psi_{-l-\frac{3}{2}} A_{l,k} \psi_{-k-\frac{1}{2}}^* + \psi_{-l-\frac{3}{2}}^* A_{l,k} \psi_{-k-\frac{1}{2}} \right) \left. \right] \langle 0 \rangle. \]
By comparing the coefficients of $\psi_{-m - \frac{1}{2}} \psi^*_{-n - \frac{1}{2}}$, we get the following equations:

$$\sqrt{-2}( - A_{m,0}A_{0,n} + A_{m+1,n} - A_{m,n+1})$$

$$- \frac{1}{4} \left( \delta_{m,1} \delta_{n,0} - \delta_{m,0} \delta_{n,1} \right)$$

$$+ (2m - 1)A_{m-2,n} + (2n - 1)A_{m,n-2}) = 0.$$  \hspace{1cm} (60)

This can be converted into the following recursion relations:

$$A_{m,n+1} = (A_{m+1,n} - A_{m,0}A_{0,n}) - \frac{1}{4\sqrt{-2}} \left( \delta_{m,1} \delta_{n,0} - \delta_{m,0} \delta_{n,1} \right)$$

$$+ (2m - 1)A_{m-2,n} + (2n - 1)A_{m,n-2}) .$$ \hspace{1cm} (61)

We will make an extra assumption on the initial values:

$$A_{0,3m-1} = (-1)^{3m-1} \left( - \frac{\sqrt{-2}}{144} \right)^m (6m - 1)!! \frac{(2m)!}{2^m} .$$ \hspace{1cm} (62)

This will uniquely determine all $A_{m,n}$’s. One can also convert it into the following recursion relations:

$$A_{m+1,n} = (A_{m,n+1} + A_{m,0}A_{0,n}) + \frac{1}{4\sqrt{-2}} \left( \delta_{m,1} \delta_{n,0} - \delta_{m,0} \delta_{n,1} \right)$$

$$+ (2m - 1)A_{m-2,n} + (2n - 1)A_{m,n-2}) .$$ \hspace{1cm} (63)

This will also determine all $A_{m,n}$’s, together with the initial value:

$$A_{3m-1,0} = \left( - \frac{\sqrt{-2}}{144} \right)^m (6m - 1)!! \frac{(2m)!}{2^m} .$$ \hspace{1cm} (64)

One then notice that $F_n(m) = A_{m,n}$ and $G_n(m) = (-1)^n A_{n,m}$ satisfying the same recursion relations in $n$ and the same initial values for $n = 0$, hence we have

$$A_{m,n} = (-1)^n A_{n,m} .$$ \hspace{1cm} (65)

5.3. **Ansatz for** $A_{m,n}$. We separate the recursion relations (61) into the following three cases:

$$A_{3m-1,3n} = A_{3m,3n-1} - A_{3m-1,0}A_{0,3n-1}$$

$$- \frac{6m - 3}{4\sqrt{-2}} A_{3m-3,3n-1} - \frac{6n - 3}{4\sqrt{-2}} A_{3m-1,3n-3} .$$ \hspace{1cm} (66)
\[ A_{3m-2,3n+1} = A_{3m-1,3n} \]
\[ - \frac{6m - 5}{4\sqrt{-2}} A_{3m-4,3n} - \frac{6n - 1}{4\sqrt{-2}} A_{3m-2,3n-2}, \]
\[ A_{3m-3,3n+2} = A_{3m-2,3n+1} \]
\[ - \frac{6m - 7}{4\sqrt{-2}} A_{3m-5,3n+1} - \frac{6n + 1}{4\sqrt{-2}} A_{3m-3,3n-1}. \]

The following are the first few examples that we obtain:

\[ A_{3m-1,0} = \left( -\sqrt{-2} \right)^m \left( \frac{6m + 1}{(2m)!} \right) \cdot \frac{1}{6m + 1}, \]
\[ A_{3m-2,1} = \left( -\sqrt{-2} \right)^m \left( \frac{6m + 1}{(2m)!} \right) \cdot \frac{1}{6m - 1}, \]
\[ A_{3m-3,2} = \left( -\sqrt{-2} \right)^m \left( \frac{6m + 1}{(2m)!} \right) \cdot \frac{1}{6m + 1}, \]
\[ A_{3m-1,3} = \left( -\sqrt{-2} \right)^{m+1} \left( \frac{6m + 1}{(2m+1)!} \right) m(2m+1)(18 + \frac{105}{6m+1}), \]
\[ A_{3m-2,4} = \left( -\sqrt{-2} \right)^{m+1} \left( \frac{6m + 1}{(2m+1)!} \right) m(2m+1)(18 + \frac{105}{6m-1}), \]
\[ A_{3m-3,5} = \left( -\sqrt{-2} \right)^{m+1} \left( \frac{6m + 1}{(2m+1)!} \right) m(2m+1)(18 + \frac{105}{6m+1}), \]
\[ A_{3m-1,6} = \left( -\sqrt{-2} \right)^{m+2} \left( \frac{6m + 1}{(2m+2)!} \right) m(m+1) \cdot (2m+3)(2m+1) \]
\[ \cdot \left( 1944m + 5778 + \frac{45045}{2(6m+1)} \right), \]
\[ A_{3m-2,7} = \left( -\sqrt{-2} \right)^{m+2} \left( \frac{6m + 1}{(2m+2)!} \right) m(m+1) \cdot (2m+1)(2m+3) \]
\[ \cdot \left( 1944m + 5778 + \frac{45045}{2(6m-1)} \right), \]
\[ A_{3m-3,8} = \left( -\sqrt{-2} \right)^{m+2} \left( \frac{6m + 1}{(2m+2)!} \right) m(m+1) \cdot (2m+1)(2m+3) \]
\[ \cdot \left( 1944m + 5778 + \frac{45045}{2(6m+1)} \right)). \]

Based on these concrete results, we make the following assumption on the form of \( A_{m,n} \): For each \( n \geq 0 \), there are a polynomial

\[ B_n(x) = B_0^{(n)} x^{n-1} + B_1^{(n)} x^{n-2} + \cdots + B_{n-1}^{(n)} \]
and a constant $b_n$ such that

$$A_{3m-1,3n} = A_{3m-3,3n+2} = (-1)^n \left( -\frac{\sqrt{-2}}{144} \right)^{m+n} \frac{(6m+1)!}{(2(m+n))!} \cdot \prod_{j=0}^{n-1} (m+j) \cdot \prod_{j=1}^{n} (2m+2j-1) \cdot (B_n(m) + \frac{b_n}{6m+1}),$$

$$A_{3m-2,3n+1} = (-1)^{n+1} \left( -\frac{\sqrt{-2}}{144} \right)^{m+n} \frac{(6m+1)!}{(2(m+n))!} \cdot \prod_{j=0}^{n-1} (m+j) \cdot \prod_{j=1}^{n} (2m+2j-1) \cdot (B_n(m) + \frac{b_n}{6m-1}).$$

The following are some examples:

$$B_0 = 0,$$
$$B_1 = 18,$$
$$B_2 = 1944x + 5778,$$
$$B_3 = 209952x^2 + 1253880x + 2277477,$$
$$B_4 = 22674816x^3 + 2261183604x^2 + 787643676x + 1114815879,$$
$$B_5 = 2448880128x^4 + 36665177472x^3 + 207169401168x^2 + 54572769972x + 2633883829515/4,$$
$$B_6 = 264479053824x^5 + 5546713489920x^4 + 4613330328000x^3 + 193184363553840x^2 + 424746412978761x + 1828597219279695/4,$$

and

$$b_0 = 1,$$
$$b_1 = 105,$$
$$b_2 = 45045/2,$$
$$b_3 = 14549535/2,$$
$$b_4 = 25097947875/8,$$
$$b_5 = 13537833083775/8,$$
$$b_6 = 17531493843488625/16,$$

5.4. **Recursion relations for $B_n$ and $b_n$ and their solutions.** The three types of recursion relations for $A$ lead to three types of recursion
relations for $B$ as follows:

\[
B_n(x) + \frac{b_n}{6x+1} = -3(B_{n-1}(x+1) + \frac{b_{n-1}}{6x+7}) \cdot \frac{(6x+5)(6x+7)}{x} + \frac{2^n(x+n)}{6x+1} \cdot \frac{(6n-1)!}{(2n)! \cdot x} + 216(x+n) \cdot (B_{n-1}(x) + \frac{b_{n-1}}{6x+1}),
\]

(69)

\[
B_n(x) + \frac{b_n}{6x-1} = -(B_n(x) + \frac{b_n}{6x+1}) + \frac{12(x+n)(x-1)(6x-5)}{(x+n-1)(6x+1)(6x-1)} \cdot (B_n(x-1) + \frac{b_n}{6x-5}) + 18(6n-1) \cdot \frac{2(x+n)}{x+n-1} \cdot (B_n(x) + \frac{b_n-1}{6x-1}),
\]

(70)

\[
B_n(x) + \frac{b_n}{6x+1} = -(B_n(x) + \frac{b_n}{6x-1}) + \frac{12(x+n)(x-1)(6x-7)}{(x+n-1)(6x+1)(6x-1)} \cdot (B_n(x-1) + \frac{b_n}{6x-7}) + 18(6n+1) \cdot \frac{2(x+n)}{x+n-1} \cdot (B_n(x) + \frac{b_n-1}{6x+1}),
\]

(71)

We now find their solutions. We take $\text{res}_{x=-1/6}$ on both sides of (69) to get:

\[
b_n = 36(6n-1)b_{n-1} - \frac{2^n(6n-1) \cdot (6n-1)!}{(2n)!}.
\]

(72)

One easily checks that

\[
b_n = \frac{2^n \cdot (6n+1)!}{(2n)!}
\]

(73)

is a solution. Now we can rewrite (69) as follows:

\[
B_n(x) = -108(x+2) \cdot B_{n-1}(x+1)
- \frac{105}{x} \cdot B_{n-1}(x+1) + \frac{18(n-1)b_{n-1}}{x}
+ 216(x+n) \cdot B_{n-1}(x) + 18b_{n-1},
\]

Subtracting (71) from (70) and changing $x$ to $x+1$, one gets:

\[
B_n(x) = 108(x+2) \cdot B_{n-1}(x+1)
+ \frac{105 B_{n-1}(x+1)}{x} - \frac{18(n-1)b_{n-1}}{x} + 18b_{n-1}.
\]
Adding up the above two equations we get:

\[ B_n(x) = 108(x + n) \cdot B_{n-1}(x) + 18b_{n-1}. \]

From this one easily gets the following solution:

\[ B_n(x) = \frac{1}{6} \sum_{j=1}^{n} 108^j b_{n-j} \cdot (x + n)_{[j-1]} . \]

where

\[ (a)_j = \begin{cases} 1, & j = 0, \\ a(a-1) \cdots (a-j+1), & j > 0. \end{cases} \]

6. Verifications of the Virasoro Constraints

In this section we verify that the recursion relations from all \( L_n \) are indeed satisfied for \( e^A|0\rangle \), with the operator \( A \) as derived in last section, hence completing the proof of Theorem 4.1.

6.1. Checking the equation (69). We plug the formula (75) into (69). The light-hand side becomes:

\[ \text{LHS} = \frac{1}{6} \sum_{j=1}^{n} 108^j b_{n-j}(x + n)_{[j-1]} + \frac{b_n}{6x + 1}. \]

the right-hand side becomes

\[ \text{RHS} = - \frac{1}{2} \sum_{j=1}^{n-1} 108^j b_{n-1-j}(x + n)_{[j-1]} \cdot \left( 36(x + 2) + \frac{35}{x} \right) \]

\[ = 3b_{n-1} \left( 6 + \frac{5}{x} \right) + \frac{2^n(6n-1)!!}{(2n)!} \cdot \left( \frac{n}{x} - \frac{6n-1}{6x + 1} \right) \]

\[ + 216(x + n) \cdot \frac{1}{6} \sum_{j=1}^{n-1} 108^j b_{n-1-j} \cdot (x + n - 1)_{[j-1]} \]

\[ + 36b_{n-1} + \frac{36(6n-1)b_{n-1}}{6x + 1}. \]

Note

\[ \frac{15b_{n-1}}{x} + \frac{2^n(6n-1)!!}{(2n)!} \cdot \frac{n}{x} \]

\[ = \frac{15b_{n-1}}{x} + 3(6n-1)b_{n-1} \]

\[ = 18(n-1)b_{n-1}. \]
Hence we need to show that

\[
\frac{1}{2} \sum_{j=1}^{n-1} 108^j b_{n-1-j} (x + n)_{[j-1]} \cdot \left(36(x + 2) + \frac{35}{x}\right) - \frac{18(n - 1)b_{n-1}}{x} = \frac{1}{6} \sum_{j=1}^{n-1} 108^{j+1} b_{n-1-j} \cdot (x + n)_{[j]}.
\]

After multiplying by 6 on both sides it becomes:

\[
(x + 2) \sum_{j=1}^{n-1} 108^{j+1} b_{n-1-j} \cdot (x + n)_{[j-1]} + \frac{105}{x} \sum_{j=1}^{n-1} 108^j b_{n-1-j} \cdot (x + n)_{[j-1]} - \frac{108(n - 1)b_{n-1}}{x} = \sum_{j=1}^{n-1} 108^{j+1} b_{n-1-j} \cdot (x + n)_{[j]}.
\]

We multiply both sides of this equation by \(x\), and note that

\[
(x + n)_{[j]} - (x + n)_{[j-1]} = (n - 1 - j) \cdot (x + n)_{[j-1]},
\]

so one needs to check that:

\[
105 \sum_{j=1}^{n-1} 108^j b_{n-1-j} \cdot (x + n)_{[j-1]} = 108(n - 1)b_{n-1} + \sum_{j=1}^{n-2} 108^{j+1} b_{n-1-j} \cdot (n - 1 - j) \cdot x(x + n)_{[j-1]}.
\]

This can be done by rewriting the right-hand side in the basis \(\{(x + n)_{[j]}\}_{j \geq 0}\) for \(\mathbb{C}[x]\). More precisely, the right-hand side of this equation
can be rewritten as:

\[
RHS = 108(n - 1)b_{n-1} + \sum_{j=1}^{n-2} 108^j b_{n-1-j} \cdot (n - 1 - j) \cdot (x + n)_{[j-1]} \\
\quad \cdot [(x + n - j + 1) - (n - j + 1)] \\
= 108(n - 1)b_{n-1} + \sum_{j=1}^{n-2} 108^j b_{n-1-j} \cdot (n - 1 - j) \cdot (x + n)_{[j]} \\
\quad - \sum_{j=1}^{n-1} 108^j b_{n-j} \cdot (n - j) \cdot (x + n)_{[j-1]} \\
= \sum_{j=1}^{n-1} 108^j b_{n-j} \cdot (n - j) \cdot (x + n)_{[j-1]} \\
\quad - \sum_{j=1}^{n-2} 108^j b_{n-1-j} \cdot (n - 1 - j)(n - j + 1) \cdot (x + n)_{[j-1]}.
\]

By comparing the coefficients of \((x + n)_{[j-1]}\) on both sides, one needs to check that

\[
105 \cdot b_{n-1-j} = b_{n-j} \cdot (n - j) - 108b_{n-1-j} \cdot (n - 1 - j)(n - j + 1).
\]

After changing \(n - j\) to \(m\), one gets an equivalent equation:

\[
105 \cdot b_{m-1} = mb_m - 108b_{m-1} \cdot (m - 1)(m + 1).
\]

This last equation is equivalent to

\[
(77) \quad b_m = \frac{3(6m + 1)(6m - 1)}{m} b_{m-1}.
\]

This is readily checked by (73).

6.2. Checking the equation (70) and (71). We multiply both sides of (70) by \((x + n - 1)(36x^2 - 1)\) to get:

\[
2(x + n - 1)(36x^2 - 1)B_n(x) + 12b_n x(x - n + 1)
\]
= \[
12(x + n)(x - 1)(6x - 5)B_n(x - 1) + 12b_n (x + n)(x - 1)
\]
+ \[
36(6n - 1) \cdot (x + n)(36x^2 - 1) \cdot B_{n-1}(x)
\]
+ \[
36(6n - 1) \cdot (x + n)(6x + 1)b_{n-1}.
\]

This can be checked using similar expansions as in last subsection. Similarly for (71).
6.3. Checking constraints from $L_n \ (n \geq 0)$. Recall

$$L_0 = -\sqrt{-2} \sum_{r_1+s_1=3} \psi_{r_1} \psi_{s_1}^* + \frac{1}{4} \sum_{r_2+s_2=0} (s_2 - r_2) \psi_{r_2} \psi_{s_2}^* + \frac{1}{16}$$

$$= -\sqrt{-2} \left( \psi_{\frac{3}{2}} \psi_{\frac{1}{2}}^* + \psi_{\frac{1}{2}} \psi_{\frac{3}{2}}^* + \psi_{\frac{1}{2}} \psi_{\frac{3}{2}}^* + \sum_{k=0}^{\infty} (\psi_{-\frac{k-1}{2}} \psi_{\frac{k+1}{2}} - \psi_{-\frac{k-1}{2}} \psi_{\frac{k+1}{2}}) \right)$$

$$+ \frac{1}{4} \sum_{l=0}^{\infty} (2l + 1) (\psi_{-\frac{l-1}{2}} \psi_{\frac{l+1}{2}} + \psi_{-\frac{l-1}{2}} \psi_{\frac{l+1}{2}}) + \frac{1}{16},$$

and so

$$L_0 Z = -e^A \left[ \sqrt{-2} \left( A_{2,0} + A_{1,1} + A_{0,2} \right) - \sum_{m,n \geq 0} (A_{m,0} A_{2,n} + A_{m,1} A_{1,n} + A_{m,2} A_{0,n}) \psi_{m-\frac{1}{2}} \psi_{n-\frac{1}{2}} \right.$$}

$$\left. + \sum_{k,l \geq 0} (\psi_{-\frac{k-1}{2}} A_{k+3,l} \psi_{-\frac{l-1}{2}} + \psi_{-\frac{k-1}{2}} A_{l+3,k} \psi_{-\frac{l-1}{2}}) \right)$$

$$- \frac{1}{4} \sum_{k,l \geq 0} (2l + 1) (\psi_{-\frac{l-1}{2}} A_{k,l} \psi_{-\frac{k-1}{2}} - \psi_{-\frac{l-1}{2}} A_{k,l} \psi_{-\frac{k-1}{2}}) - \frac{1}{16} \right] \langle 0 \rangle.$$

hence the constraints from $L_0$ are

$$\sqrt{-2} (A_{2,0} + A_{1,1} + A_{0,2}) - \frac{1}{16} = 0,$$

which is readily checked, and

$$\sqrt{-2} \left( - \sum_{i=0}^{2} A_{m,i} A_{2-i,n} + A_{m+3,n} - A_{m,n+3} \right)$$

$$- \frac{1}{4} \left( 2m + 1 \right) A_{m,n} + (2n + 1) A_{m,n} = 0.$$

The latter can be divided into three cases:

$$A_{3m-1,3n+3} = A_{3m+2,3n} - A_{3m-1,0} A_{2,3n} - \frac{6(m+n)}{4\sqrt{-2}} A_{3m-1,3n},$$

$$A_{3m-2,3n+4} = A_{3m+1,3n+1} - A_{3m-2,1} A_{1,3n+1} - \frac{6(m+n)}{4\sqrt{-2}} A_{3m-2,3n+1},$$

$$A_{3m-3,3n+5} = A_{3m,3n+2} - A_{3m-3,2} A_{0,3n+2} - \frac{6(m+n)}{4\sqrt{-2}} A_{3m-3,3n+2}. $$
They are equivalent to the following two relations for $B_n(x)$:

$$B_{n+1}(x) + \frac{b_{n+1}}{6x+1}$$

$$= -\frac{3(6x+5)(6x+7)}{x} \left( B_n(x+1) + \frac{b_n}{6x+7} \right)$$

$$+ 216(x+n+1) \left( B_n(x) + \frac{b_n}{6x+1} \right)$$

$$+ \frac{2^{n+1}(6n+5)!!}{(2n+2)!} \cdot \frac{x+n+1}{x(6x+1)},$$

and

$$B_{n+1}(x) + \frac{b_{n+1}}{6x-1}$$

$$= -\frac{3(6x+5)(6x+7)}{x} \left( B_n(x+1) + \frac{b_n}{6x+5} \right)$$

$$+ 216(x+n+1) \left( B_n(x) + \frac{b_n}{6x-1} \right)$$

$$- \frac{2^{n+1}(6n+7)!!}{(2n+2)!(6n+5)} \cdot \frac{x+n+1}{x(6x-1)}.$$

They can be checked by the same method. Similarly, one can also check the constraints for $L_1$ and $L_2$. Because the Virasoro commutation relation

$$(79) \quad [L_m, L_n] = (m-n)L_{m+n},$$

all $L_n$-constraints are satisfied. This completes the proof.

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References

[1] M. Aganagic, R. Dijkgraaf, A. Klemm, M. Mariño, C. Vafa, Topological strings and integrable hierarchies, Commun. Math. Phys. 261(2006), no. 2, 451-516.

[2] A. Alexandrov, Cut-and-join operator representation for Kontsevich-Witten tau-function. Modern Phys. Lett. A 26 (2011), no. 29, 2193-2199.

[3] F. Deng, J. Zhou, On fermionic representation of the framed topological vertex, arXiv:1111.0415

[4] F. Deng, J. Zhou, Fermionic gluing principle of the topological vertex. J. High Energy Phys. 2012, no. 6, 166, front matter+26 pp. arXiv:1204.5067
[5] R. Dijkgraaf, H. Verlinde, E. Verlinde, *Loop equations and Virasoro constraints in nonperturbative two-dimensional quantum gravity*. Nuclear Phys. B 348 (1991), no. 3, 435-456.

[6] M. Fukuma, H. Kawai, R. Nakayama, *Continuum Schwinger-Dyson equations and universal structures in two-dimensional quantum gravity*. Internat. J. Modern Phys. A 6 (1991), no. 8, 1385-

[7] V. Kac, Vertex algebras for beginners. Second edition. University Lecture Series, 10. American Mathematical Society, Providence, RI, 1998.

[8] V. Kac, A. Schwarz, *Geometric interpretation of the partition function of 2D gravity*. Phys. Lett. B 257 (1991), no. 3-4, 329-334.

[9] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*. Comm. Math. Phys. 147 (1992), no. 1, 1–23.

[10] I.G. MacDonald, Symmetric functions and Hall polynomials, 2nd edition. Claredon Press, 1995.

[11] T. Miwa, M. Jimbo, E. Date, Solitons. Differential equations, symmetries and infinite-dimensional algebras. Translated from the 1993 Japanese original by Miles Reid. Cambridge Tracts in Mathematics, 135. Cambridge University Press, Cambridge, 2000.

[12] E. Witten, *Two-dimensional gravity and intersection theory on moduli space*, Surveys in Differential Geometry, vol.1, (1991) 243–310.

[13] E. Witten, *Algebraic geometry associated with matrix models of two-dimensional gravity*. Topological methods in modern mathematics (Stony Brook, NY 1991), 235-269, Publish or Perish, Houston, TX, 1993.

[14] J. Zhou, *Solution of W-constraints for r-spin intersection numbers*, arXiv:1305.6991

[15] J. Zhou, *Solution of W-constraints for simple singularities*, in preparation.

[16] J. Zhou, *Explicit formula for the partition function of simple singularities*, in preparation.

**Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, China**

*E-mail address: jzhou@math.tsinghua.edu.cn*