Comment on Gouriéroux, Monfort, Renne (2019): Identification and Estimation in Non-Fundamental Structural VARMA Models

Bernd Funovits
University of Helsinki and TU Wien

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Abstract

This comment points out a serious flaw in the article ‘Gouriéroux, Monfort, Renne (2019): Identification and Estimation in Non-Fundamental Structural VARMA Models’, henceforth abbreviated as GMR, with regard to mirroring complex-valued roots with Blaschke polynomial matrices. Moreover, the (non-)feasibility of the proposed method (if the handling of Blaschke transformation were not prohibitive) for cross-sectional dimensions greater than two and vector moving average (VMA) polynomial matrices of degree $q$ greater than one is discussed.

1 Introduction

Blaschke matrices are at the center of GMR’s estimation strategy. Whenever the VMA polynomial matrix has complex valued determinantal roots, GMR leaves the specified (real) parameter space and result in complex-valued estimates. While the VMA matrix polynomial in GMR’s empirical application indeed has complex-valued determinantal roots, the complex valued nature of the intermediate estimates does not propagate to the final (published) estimates because they are “covered up” at multiple locations. GMR discard the imaginary part whenever they result in complex-valued parameter matrices or, if some quantity (similar to the condition number of some “Blaschke-transformed” matrix) is sufficiently large, replace a complex-valued matrix with the identity matrix.

It thus seems likely that the authors were aware of the fact that their Blaschke matrices are incorrectly complex-valued (which implies that the parameter space is left) and that they have not dealt with the problem appropriately. Of course, the obtained (mirrored) points in the parameter space do not correspond to observationally equivalent models in terms of second moments. Moreover, even the number of determinantal roots inside and outside the unit circle (GMR call this “fundamentalness regime”) may be incorrect.

To fix ideas, consider the univariate MA(2) model

$$y_t = (1 - \frac{3}{4}L) (1 - \frac{1}{4}L) \varepsilon_t = \Theta(L) \varepsilon_t$$

where $(\varepsilon_t)$ is non-Gaussian i.i.d. with unit variance and $L$ is the lag-operator such that $L(y_t) = (y_{t-1})$. Blaschke factors of the form $g_1(z) = \frac{1+i}{4+i z}$, $g_2(z) = \frac{1-i}{4-i z}$, $g_3(z) = \frac{1-i}{4+i z}$, $g_4(z) = \frac{1+i}{4-i z}$, for which $g_i(z) g_i(\frac{1}{z}) = 1$ holds and which mirror roots at the unit circle, can be used to obtain different representations of $(y_t)$ with identical second order properties but different higher order properties, i.e. the processes $(\Theta(L)g_i(L)) (g_i(L)^{-1} \varepsilon_t) = \Theta^{(i)}(L) \varepsilon_t^{(i)}$ are indistinguishable from $(y_t)$ in terms of second moments. GMR’s approach comprises obtaining all

1 Another example is that the imaginary part of the log-likelihood function is discarded.
representations which are observationally equivalent in terms of second moments and subsequent optimisation in terms of their objective function (which uses more than second order information) starting from all different combinations of determinantal zeros inside or outside of the unit circle. While this is easy in the univariate case with real roots, there are some non-trivial problems in the multivariate case with complex determinantal roots of the VMA matrix polynomial \( \Theta(z) \). Mirroring complex-conjugated roots separately results necessarily in complex-valued parameter matrices. Even when mirroring both complex-conjugated roots together, the resulting VMA matrix polynomial is in general not real-valued and it is non-trivial to ensure that the VMA matrix polynomial is real-valued after mirroring both complex-conjugated roots (meaning that an additional step is required).

Another issue, which does not invalidate GMR’s approach as does the first issue but still warrants mentioning, is that the method is presented as if it were feasible for VARMA(p,q) models of arbitrary polynomial degree \( q \) and cross-sectional dimension \( n \). This is not the case. The article should at best be understood as a proposal of a method for estimating VARMA(p,1) models of small dimensions (given a correct handling of mirroring determinantal roots such that the transformed system generates the same second moments as the original one). Likely, it is not a coincidence that applications and examples consider at most a cross-sectional dimension of two. In the code it is even explicitly stated that it only works for \( q = 1 \). The computational complexity of calculating all Blaschke matrices would be prohibitively expensive when estimating models with a reasonable number of VMA lags or cross-sectional dimension because for each choice of determinantal zeros of the VMA polynomial matrix \( \Theta(z) \) inside or outside the unit circle an optimisation has to be performed. If there are \( n \cdot q \) real determinantal zeros, there are up to \( 2^{nq} \) options, which is prohibitive even for modest dimensions and lags. For 4 variables and VMA order 8 (or similar), we would result in a maximal number of \( 2^{48} \) representations (which are observationally equivalent in terms of second moments but have different combinations of zeros inside and outside the unit circle).\(^2\) If one optimisation based on a starting value of this kind were to take one second (GMR’s procedure is more costly), this would result in a computing time of \( 2^{32}/(3600 \cdot 24 \cdot 365) \approx 136 \) years. The fact that GMR’s method is essentially only applicable for \( q = 1 \), is alluded to shortly in Section 4.1.4 Extension to the SSVARMA(p,q) case (which comprises 8 lines)\(^3\), where the authors do mention that their representation (4.18) is not the same as a VARMA(p,q).

The remainder of this comment is structured as follows: In Section 2, we summarise the use of Blaschke transformations of VMA(1) polynomial matrices (their method does not allow for VMA orders \( q > 1 \)) in GMR and point out shortcomings. In Section 3, we provide real-valued constructions of Blaschke polynomial matrices (for arbitrary VMA order \( q \)). In Section 4, we illustrate and parametrise the extent to which the use of Blaschke matrices in GMR is incorrect with various examples. Lastly, we discuss GMR’s estimation strategy and how the fact that the real-valued model class is left is covered up when complex-valued Blaschke polynomial matrices are post-multiplied on the VMA(1) polynomial matrices.

\(^2\)Of course, one may only mirror pairs of complex-conjugated roots jointly in order not to leave the (real-valued) model class, which reduces this number. However, not even this fact is recognized in GMR. They mirror complex-conjugated roots separately.

\(^3\)In these 8 lines is another trivial error. GMR write that the “eigenvalues of \( \tilde{\Theta} \) are the roots of \( \det (\Theta(z)) \)”. However, the eigenvalues of \( \tilde{\Theta} \) are the solutions of \( \det (\Theta(e^{\pi i/2})) \) and (except for zeros at zero or infinity) correspond to the inverses of the determinantal roots of \( \Theta(z) \), see e.g. Hannan, Deistler (2012, Theorem 1.2.2, page 19)
2 GMR’s Incorrect Blaschke Procedure

GMR’s parametrization of the VMA polynomial matrix is of the form $\Theta(L) = (I_n - \Theta L) C$. Their Blaschke procedure for obtaining observationally equivalent VMA polynomial matrices involves the following three functions:

- **build.orthonormal.basis(...)**: It takes an eigenvector (pertaining to the determinantal root to be mirrored) of the VMA(1) coefficient matrix $\Theta$ as input, “normalises” it (in an incorrect way), and attempts to build an orthonormal basis in an iterative process by projecting a basis vector on the orthogonal complement of orthonormal vectors from the previous iterative step. The following problems occur at this stage:
  - “Normalising” does not take the possibly complex nature of eigenvectors into account. According to this logic, the vector $\left(\frac{1}{i}\right)$ would have length zero.
  - Multiplicity of eigenvalues greater than one are ignored (which is a minor problem because generically the multiplicity of eigenvalues is equal to one).

- **compute.other.basic.form(...)**: Given the original VMA(1) coefficient matrix $\Theta$, the original static shock transmission matrix $C$, and a vector $w$ of length $n$ indicating whether a certain determinantal zero of $\Theta(z)$ should be mirrored at the unit circle, this function computes an VMA matrix polynomial $\Theta^{(1)}(L) = (I_n - \Theta^{(1)} L) C^{(1)}$ which generates the same spectral density (given a vector mutually and temporally vector of inputs with variance one) as the original $\Theta(z)$ but whose determinantal zeros are mirrored at the unit circle according to the vector $w$. The vector $w$ is the eigenvector pertaining to the determinantal zero of the VMA polynomial matrix to be mirrored. It serves as input to build.orthonormal.basis(...)

- **compute.all.forms(...)**: This function is a wrapper which calls compute.other.basic.form(...) for all possible combinations of zeros and ones in $w$. It generates a matrix of dimension $(2^n \times n)$ where each row corresponds to a particular selection of determinantal zeros of $\Theta(z)$ to be mirrored. In particular, complex-conjugated roots are mirrored separately, which results necessarily in complex-valued parameter matrices. Obviously, the dimension of this matrix makes GMR’s approach prohibitively costly for VMA(q) processes with output dimension $n > 2$ and even moderately high $q > 1$ (which is excluded in GMR’s code).

For more detail regarding versions of these functions, where variable names (but no functionality) have been changed in order to increase readability and comments have been added for clarification, we refer to the Appendix.

The incorrect normalisation in build.orthonormal.basis(...), which does not take the complex-valued nature of eigenvectors into account, could be considered an implementation error (albeit with serious implications). While the facts that normalisation is done incorrectly and that complex-conjugated roots are mirrored...
separately frankly add insult to injury, they are not the focus of this comment. Much rather, this comment criticizes that GMR ignored (possibly intentionally) a difficult theoretical problem: The VMA matrix polynomial is in general not real-valued even when complex-conjugated roots are mirrored jointly. Without further adjustments, the parameter space is thus left and the obtained intermediate estimators are non-sensical.

In the following, it is shown that it is indeed possible to obtain real-valued VMA polynomial matrices with mirrored complex-conjugated roots but otherwise the same second order properties. This can be achieved either with an additional unitary transformation after flipping both complex-conjugated roots or with a state space construction.

### 3 Real-Valued Approaches

Here, we present correct real-valued approaches for applying all-pass transformations (to be defined below) to a polynomial matrix $\Theta(z)$ of dimension $(n \times n)$ and degree $q$ such that the transformed polynomial matrix is again part of the model class (and thus has real-valued coefficients). Like GMR, we assume that $\Theta(z)$ is non-singular at zero and non-singular on the unit circle.

First, we provide some minimal definitions for all-pass transformations. Subsequently, we discuss an approach which involves the singular value decomposition (SVD) instead of the eigendecomposition in GMR. In contrast to GMR, there is an additional step after mirroring a pair of complex-conjugated roots of $\det(\Theta(z))$ which ensures that the transformed matrix polynomial has real coefficients and is thus still part of the model class under investigation. While this approach is quite straight-forward, it is difficult to prove formally that the coefficients of the transformed matrix polynomial are indeed real.

Lastly, we describe an approach based on state space methods which does not take a detour into the complex plane and thus results by construction in an all-pass filter with real coefficients. For details, see Scherrer, Funovits (2020). Intermediate results (in connection to the QR decomposition of the complex-conjugated right-singular vectors) can be used to show that the coefficients in the polynomial approach are real.

#### 3.1 Definitions

A **multivariate rational all-pass filter** is an $(n \times n)$-dimensional matrix $V(z)$ whose entries are rational functions and which satisfies $V(z)V^*(\frac{1}{z}) = V^*(\frac{1}{z})V(z) = I_n$. The superscript asterisk takes an (arbitrary) matrix function $m(z) = \sum_{j=-\infty}^{\infty} m_j z^j$ to its version with complex conjugated and transposed coefficient matrices, i.e. $m^*(z) = \sum_{j=-\infty}^{\infty} m^*_j z^j$.

Obviously, the spectral densities of $y_t = \Theta(z)\eta_t$ and $y_t = (\Theta(z)V(z))\left(V^*(\frac{1}{z})\eta_t\right) = \tilde{\Theta}(z)\tilde{\eta}_t$, where $\eta_t$ is serially and mutually i.i.d. as in GMR, are identical.

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7This point is not addressed in the proofs of the spectral factorization theorem in Rozanov (1967) and Hannan (1970), which use Blaschke matrices to mirror zeros of the VMA polynomial matrix outside the unit circle, either. Of course, this is not a problem in these books because the obtained spectral factors (which are in general not real-valued) do not serve as estimators but are rather an intermediate step in their proofs.

8For notational simplicity, we incorporate the static shock transmission matrix $C$ into $\Theta(z) = \Theta_0 + \Theta_1 z + \cdots + \Theta_q z^q$.

9Presented at the 12th International Conference on Computational and Financial Econometrics in Pisa in December 2018.

10Using the SVD has the advantage that it is numerically more stable and that it directly generates an orthogonal (or unitary) transformation (which is obviously all-pass). The orthogonalization step in GMR (starting from a normalised eigenvector and obtaining the remaining orthonormal basis vectors iteratively as orthogonal complements of the projection on the orthonormal vectors of the previous step) seems computationally costly and likely worse conditioned than the orthogonal or unitary matrices occurring in the SVD. Assuming of course a correct implementation, which is not the case for the code provided on the website of the Review of Economic Studies.
An *elementary Blaschke factor* at $\alpha$ (which is obviously all-pass) is of the form\(^{11}\)

$$B(z, \alpha) = \frac{1 - \bar{\alpha}z}{-\alpha + z}.$$  

A *squared Blaschke factor* for the pair of complex-conjugated roots $\alpha_{\pm} = \alpha_r \pm i\alpha_i$ (in obvious notation) is defined as

$$B_{sq}(z, \alpha_{\pm}) = \frac{1 - \bar{\alpha}z}{-\alpha + z} = \frac{1 - 2\alpha_r z + |\alpha|^2}{|\alpha|^2 - 2\alpha_r z + z^2}.$$  

Lastly, a *bivariate Blaschke factor* pertaining to the pair of complex-conjugated roots $\alpha_{\pm} = \alpha_r \pm i\alpha_i$, where $\alpha_i > 0$, and the non-zero vector $w \in \mathbb{C}^{2 \times 1}$ is given as

$$B_2(z, \alpha_{\pm}, w) = a^{-1}(z, \alpha_{\pm})b(z, \alpha_{\pm}, w),$$

where $a(z, \alpha_{\pm})$ is a diagonal matrix with entries $B_{sq}(z, \alpha_{\pm})$, and $b(z, \alpha_{\pm}, w)$ is a $(2 \times 2)$ polynomial matrix with highest degree 2 and which is of reduced rank at $z = \alpha^{-1}$, $z = \bar{\alpha}^{-1}$, $z = \alpha$ and $z = \bar{\alpha}$. We construct $b(z, \alpha_{\pm}, w)$ such that the column space of $b(\alpha_{+}, \alpha_{\pm}, w)$ is spanned by a given (non-trivial) vector $w \in \mathbb{C}^{2 \times 1}$.

### 3.2 SVD Approach for Constructing All-Pass Filters

Irrespective of whether or not the root $\alpha$ to be mirrored is real or complex, we perform an SVD on $\Theta(\alpha) = UDV^*$, where $U$ and $V$ are orthogonal or unitary matrices, where the asterisk denotes transposition and complex conjugation, and where $D$ is a diagonal matrix of non-negative and non-increasing elements containing the singular values. For simplicity of presentation, we assume that the rank deficiency of $\Theta(\alpha)$ is equal to one and thus that $d_{nn}$ is equal to zero (and $d_{n-1,n-1}$ is positive).

Post-multiplying $V$ on $\Theta(\alpha)$ results in a matrix whose Lastly column is zero. This implies that each element in the Lastly column of $\Theta(z)V$ has as polynomial factor $(z - \alpha)$. Therefore, post-multiplying $\text{diag}(I_{n-1}, B(z, \alpha))$ mirrors the zero of $\Theta(z)$ at $\alpha$ to a zero at $\frac{1}{\alpha}$. Obviously, orthogonal (or unitary) transformations and transformations involving Blaschke factors are all-pass.

In the case that $\alpha_+$ is complex with positive imaginary part, we perform the same procedure additionally on $\Theta(z)V_+\text{diag}(I_{n-1}, B(z, \alpha_+))$ such that also the root with negative imaginary part, say $\alpha_-$, is mirrored at the unit circle. We obtain $\hat{\Theta}(z) = \Theta(z)V_+\text{diag}(I_{n-1}, B(z, \alpha_+))V_-\text{diag}(I_{n-1}, B(z, \alpha_-))$ where the unitary matrix $V_-$ is obtained from the SVD of $\Theta(\alpha_-)V_+\text{diag}(I_{n-1}, B(\alpha_-, \alpha_+))$. Otherwise (if only one determinantal root from the pair of complex conjugated roots is mirrored into the unit circle), it is impossible to obtain a real-valued polynomial matrix. Be that as it may, in GMR, complex-conjugated roots are reflected separately and therefore the model class is necessarily left.

So far, we have used two static, in general complex-valued, unitary transformations $V_+$ and $V_-$, and two dynamic (rational) transformations involving $B(z, \alpha_+)$ and $B(z, \alpha_-)$ whose product has real-valued coefficients. This suggests that another static transformation may take us back into the model class with real-valued parameter matrices. We construct a unitary transformation by using the fact that $\hat{\Theta}(1) (\hat{\Theta}(1))^* = \Theta(1) (\Theta(1))^*$ is real-valued together with the polar decomposition of $\hat{\Theta}(1)$. Indeed, the real-valued orthogonal eigenbasis of $\Theta(1) (\Theta(1))^*$ corresponds to the left singular vectors of $\Theta(1)$. Given the SVD $\hat{\Theta}(1) = WSX^*$, its polar

\(^{11}\)Sometimes, the Blaschke factor is defined with an additional factor $\alpha/|\alpha|$. However, this factor is not well defined if $\alpha = 0.$
decomposition is \(\tilde{\Theta}(1) = (W D W') (W X^*)\). It follows that right-multiplying the unitary matrix \(X W'\) results in \(\tilde{\Theta}(1) (X W')\) being real. 

While this argument seems compelling, and while the transformations are obviously all-pass and the implementation based on these derivations has turned out to result in real-valued parameter matrices, it is difficult to prove that \(\Theta(z)\) is real-valued whenever \(z \in \mathbb{R}\) and \(z \neq 1\) by analysing the specific SVDs and the polar decomposition of the matrices involved.

Therefore, we outline a different construction (for more detail see Scherrer, Funovits (2020)) which involves the bivariate Blaschke factor \(B_2(z, \alpha_{\pm}, w)\) and which does not take a detour into complex-valued matrices (implying that the model class is left temporarily). A key step in this construction is the QR decomposition of the real and imaginary part of the (complex-valued) vector in the right-kernel of \(\Theta(\alpha_{+})\). It is possible to use the upper triangular matrix \(R\) from this QR decomposition to parametrise the unitary matrices involved in the SVD construction above and prove that \(\Theta(z)\) is indeed real-valued whenever \(z \in \mathbb{R}\).

### 3.3 Proving Realness

#### 3.3.1 QR Decomposition of Complex-Valued Right-Kernel

We start from a (normalised) vector \(v = v_r + iv_i \in \mathbb{C}^{n \times 1}\) in the right-kernel of \(\Theta(\alpha_{+})\), where \(v_r, v_i \in \mathbb{R}^{n \times 1}\) are linearly independent. Eventually, we will result in the transformed polynomial matrix (with identical second order properties as the original one)

\[
\Theta(z) \tilde{Q} \begin{bmatrix}
\text{diag}\left\{B_2 \left(z, \alpha_{\pm}, R \left(\frac{1}{i}\right)\right), I_{n-2}\right\}
\end{bmatrix}
\]

where the orthogonal real matrix \(\tilde{Q}\) and the upper-triangular matrix \(R\) with positive diagonal elements are obtained from the QR decomposition

\[
\begin{pmatrix}
v_r \\ v_i
\end{pmatrix} = \tilde{Q} \tilde{R} = \begin{pmatrix}
Q \\ \tilde{Q}_2
\end{pmatrix} \begin{pmatrix}
R \\ 0_{(n-2)\times 2}
\end{pmatrix}.
\]

Note that \(R \left(\frac{1}{i}\right)\) is in the right-kernel of \(\Theta(\alpha_{+}) Q\) and that, if the column space of \(b(\alpha_{+}, \alpha_{\pm}, w)\) is spanned by \(w = R \left(\frac{1}{i}\right)\), we have that \(\Theta(\alpha_{+}) Q b(\alpha_{+}, \alpha_{\pm}, R \left(\frac{1}{i}\right)) = \Theta(\alpha_{-}) Q b(\alpha_{-}, \alpha_{\pm}, R \left(\frac{1}{i}\right)) = 0\). Hence, all entries of \(\Theta(z) Q b(\alpha_{-}, \alpha_{\pm}, R \left(\frac{1}{i}\right))\) and \(\Theta(z) Q b(\alpha_{+}, \alpha_{\pm}, R \left(\frac{1}{i}\right))\) are divisible by the diagonal element of \(a(z, \alpha_{\pm})\).

The most elegant approach to constructing a polynomial matrix \(b(z)\) with real coefficients is a state space construction. However, it is also possible to parametrise unitary \((2 \times 2)-\)dimensional matrices \(V_{\beta}, V_{\gamma},\) and \(V_\delta\) with the parameters in \(R\) and \(\alpha_{+} = r \cdot e^{i\phi}\) such that

\[
\tilde{Q} \cdot \text{diag}(V_{\beta}, I_{n-2}) \cdot \text{diag}(B(z, \alpha_{+}), I_{n-1}) \cdot \text{diag}(V_{\gamma}, I_{n-2}) \cdot \text{diag}(B(z, \alpha_{-}), I_{n-1}) \cdot \text{diag}(V_\delta, I_{n-2})
\]

is real. We will now give a succinct overview of both approaches and refer to Scherrer, Funovits (2020) for details.
3.3.2 Summary of State Space Construction

The \((2 \times 2)\)-dimensional, real-valued, rational, all-pass filter \(B_2(z, \alpha_{\pm}, w) = C(z^{-1}I_2 - A)^{-1}B + D\) is constructed in the following steps.

First, the poles are fixed by setting \(A = \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_0 \end{pmatrix}\), where \(\lambda_\pm = \lambda_0 + i\lambda_i = \alpha_\pm^{-1}\). Second, \(C\) is determined by requiring that the column-space of \(b(\alpha_+, \alpha_\pm, w)\) be equal to a vector \(w \in \mathbb{C}^{2 \times 1}\) (and thus that the column-space of \(b(\alpha_-, \alpha_\pm, w) = \bar{b}(\alpha_+, \alpha_\pm, w)\) be equal to \(\bar{w} \in \mathbb{C}^{2 \times 1}\). Lastly, \(B\) and \(D\) are determined such that \(B_2(z, \alpha_{\pm}, w)B_2'(\frac{1}{z}, \alpha_{\pm}, w)\) is constant and that \(B_2(z, \alpha_{\pm}, w)B_2'(\frac{1}{z}, \alpha_{\pm}, w) = I_n\). For this Lastly step, it is necessary to apply a state transformation and solve a Lyapunov equation.

3.3.3 Summary of Parametrization of Unitary Matrices

Every unitary matrix can be parametrized through four parameters \((\phi_0, \phi_1, \phi_2, \phi_3)\) as \(e^{i\phi_i} \begin{pmatrix} e^{i\phi_0} \cos(\phi_3) & e^{i\phi_2} \sin(\phi_3) \\ -e^{-i\phi_2} \sin(\phi_3) & e^{-i\phi_0} \cos(\phi_3) \end{pmatrix}\).

Thus, we obtain \(V_{\beta}\) with \((\beta_0, \beta_1) = (0, 0)\) by choosing\(^{12}\) \(\beta_2, \beta_3\) such that \(R\left(\begin{array}{c} 1 \\ 1 \end{array}\right)\) is in the span of \(\begin{pmatrix} \cos(\beta_3)e^{i\phi_2} \\ \sin(\beta_3) \end{pmatrix}\).

Similarly, \(V_{\gamma}\) with \((\gamma_0, \gamma_1) = (0, 0)\) is determined by setting \(\gamma_2, \gamma_3\) such that \(R\left(\begin{array}{c} 1 \\ -1 \end{array}\right)\) is in the span of \(V_{\beta} \cdot (R(\alpha_{-} \alpha_{+})_{01}) \cdot V_{\gamma} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}\). Note that the parameters \(\gamma_2, \gamma_3\) are functions of \(\beta_2, \beta_3\), and \(\alpha_{-}\). Lastly, \(V_{\delta}\) is chosen such that \(V_{\beta} \cdot (R(\alpha_{+} \alpha_{-})_{01}) \cdot V_{\gamma} \cdot (R(1 \alpha_{-})_{01}) \cdot V_{\delta}\) is equal to the identity matrix. Using this parametrisation, it can be verified by straightforward calculations that the obtained transformation has real coefficients.

4 Examples

In this section, we parametrise matrices in order to analyze how the construction in GMR fails. We start with the purely complex case, go on to skew-symmetric VMA(1) coefficients, and finally consider some general coefficient matrices which result in complex-valued determinantal zeros of \(\Theta(z)\). We will consider the 2-dimensional case, because this is the one used in GMR. Code for these examples can be found in the Appendix.

4.1 Incorrect “Normalisation”

4.1.1 Purely Complex Case

Generating a VMA(1) coefficient matrix of the kind \(\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}\) (randomly) results in an error due to the fact that in GMR, the squared length of a vector (which is subsequently used to “normalise” the vector) is incorrectly calculated by summing the squared components. Since the eigenvectors of these matrices are of the form \(\frac{1}{\pm i}\), the implementation in GMR is non-functional because calculating the sum of the squared components of this vector is zero.

4.1.2 Skew-Symmetric Complex Case

Next, we consider VMA(1) coefficient matrices of the kind \(\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\). In exact arithmetic, the eigenvectors are again of the form \(\frac{1}{\pm i}\). Since GMR check whether the sum of squared components is exactly\(^{13}\) equal to zero, this sometimes does not result in an error but rather in very large (and non-sensical) entries of the VMA(1) coefficient matrix \(\Theta\) and the static shock transmission matrix \(C\). For example, setting \(a = b = 1\) results in an

\(^{12}\)The parameters \(\beta_2, \beta_3\) can be determined as a function of the parameters in \(R\).

\(^{13}\)More commonly, one checks whether a quantity is numerically zero by comparing it to a certain small threshold. A common way to check whether a quantity is numerically zero is to compare its absolute value to a certain numerical tolerance level.
error (or rather a message printed to the console and an error triggered at later stage) but setting $a = 2$ and $b = 5$ results in very large entries and incorrect results (in the sense that the obtained polynomial matrix is not all-pass and does not have its roots mirrored).

### 4.1.3 General Complex Case

Whenever the VMA(1) coefficient matrix is not exactly skew-symmetric, the procedure does not throw an error. However, depending on the extent to which the matrix is not skew-symmetric, the results remain incorrect. In order to understand this, one may consider matrices of the kind \( \begin{pmatrix} a & b+c \\ -b & a \end{pmatrix} \), where \( c \cdot b > 0 \) such that we obtain complex-valued eigenvalues. Depending on the absolute value of \( c \), it is possible to parametrise the distance from transformations which are actually all-pass.

### 4.2 Fundamental Issues

Even without the problems mentioned above, the strategy in GMR would still fail for two reasons. First, complex-conjugated roots $\alpha_+$ and $\alpha_-$ are mirrored separately in GMR. This results in complex-conjugated parameter matrices which cannot be made real and are thus not part of the parameter space specified in GMR. Second, even when both complex-conjugated roots are mirrored (and a real-valued construction would be possible), the method used in GMR still results in complex-valued parameter matrices $\Theta$ and $C$. Given that GMR go to great lengths to deal with complex-valued quantities in their code, it is hard to imagine that GMR were not aware of this shortcoming. Discarding the imaginary part of these matrices results in parameters which do not correspond to observationally equivalent models in terms of second moments. In addition, the number of determinantal roots inside the unit circle may change when discarding imaginary parts.

### 5 Issues in GMR’s Estimation Strategy

In this section, we discuss in detail where the questionable steps in GMR’s code (Supplementary data, downloaded from the website of The Review of Economic Studies) occur. In particular, we describe where complex-valued quantities are replaced with (ad-hoc choices of) real ones. We emphasize that in every evaluation of the likelihood function, an adjustment for the possibility of complex-valued matrices is made. All line references refer to GMR’s script `set.of.procedures.R`.

The script `run.VARMA.BQ.dataset.R` calls `run.estim.QZ.GMM.MLE.R` (in the global environment) with different values for the AR order (1 to 6). The VMA order is always equal to one. For each AR order, the latter script calls first the GMM method `estim.VARMAp1.2SLS.GMM(…)` (serving as initial estimates of their maximum likelihood (ML) approach) and the the ML optimisation routine `estim.MA.inversion(…).` These two main estimation procedures (as well as the parameter choices in the two scripts `run.estim.QZ.GMM.MLE.R` and `run.VARMA.BQ.dataset.R`) are described in detail in the Appendix.

### 5.1 Issues in GMM Procedure

In the main function `estim.VARMAp1.2SLS.GMM(…)` (after computing “all observationally equivalent” VMA polynomial matrices in line 1051), GMR apply $Re(\cdot)$, which returns the real part of a complex number,
to all.MA$all.Theta and all.MA$all.C (which are the arrays which describing all putative observationally
equivalent VMA polynomial matrices). There is of course no foundation for doing so.\textsuperscript{14}

5.2 Issues in ML Procedure

Similar to the GMM procedure, in the main function \texttt{estim.MA.inversion(\ldots)} (after computing “all observationally equivalent” VMA polynomials in line 274), GMR apply \texttt{Re(\cdot)} to all.MA$all.Theta and all.MA$all.C. Again, naturally, there is no foundation for doing so.

An even more serious shortcoming appears in the function \texttt{estim.struct.shocks(\ldots)} (which can be considered to be at the core of the implementation in GMR since it uses the Schur decomposition to “invert” the VMA polynomial), where the static shock transmission matrix \( C \) is replaced with an identity matrix whenever a quantity similar to the condition number\textsuperscript{15} is above a certain large threshold or any element of the vector containing the absolute value of the ratios of the eigenvalues of the imaginary part of \( C \) to the real part of \( C \) has a large element. This is, of course, an ad-hoc fix of a non-functional procedure and readers are misled to believe that Blaschke matrices are used even though the obtained values can, at best, be interpreted as ad-hoc selected new starting values for the optimisation of an objective function. Whenever none of the two conditions above is satisfied, GMR continue with a complex-valued static shock transmission matrix \( C \) and VMA(1) coefficient matrix \( \Theta \). The reason why no complex-valued parameters appear at the end of the procedure is found in various calls to \texttt{Re(\cdot)}, in order to discard imaginary parts (e.g. in the likelihood evaluation function \texttt{ML.inversion.loglik(\ldots)} on line 171).

Furthermore, we want to emphasize that \texttt{estim.struct.shocks(\ldots)} is at the center of GMR’s implementation (which can also be seen from the dependency graphs in the Appendix). Every time the likelihood is evaluated (e.g. in the optimisation routine \texttt{stats::optim(\ldots)}, \texttt{estim.struct.shocks(\ldots)} is called by \texttt{ML.inversion.loglik.aux(\ldots)}. The latter function calls \texttt{g(\ldots)} (line 2401), which in turn calls the respective class of log-densities (usually a mixture of Gaussians). There is an additional ad-hoc fix\textsuperscript{16} in \texttt{g(\ldots)} (next to setting \( C \) equal to the identity matrix if one of the two conditions mentioned above is not satisfied): Whenever the sum of the absolute values of the eigenvalues of \( C \) is small (i.e. the matrix \( C \) is close to singular), the value of the log density is set to \(-100000\). Similar to \texttt{estim.struct.shocks(\ldots)}, the function \texttt{g(\ldots)} is called whenever the likelihood is evaluated and is therefore also at the center of the procedure in GMR.

6 Acknowledgments

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\textsuperscript{14}An issue unrelated to Blaschke polynomial matrices appears in line 1470 where the absolute value function \texttt{abs(\cdot)} is only applied to the diagonal elements the sign change should be applied to the whole column. This seems problematic because it happens directly before the variables are returned, i.e. there is no optimisation which could alleviate wrong parameter values.

\textsuperscript{15}The condition number of a matrix is usually defined as the ratio of the largest to the smallest singular value. GMR calculate the ratio of the largest to the smallest absolute value of the eigenvalues.

\textsuperscript{16}This might be reasonable to avoid singular cases. Note however, that singular \( C \) matrices only occur because of an incorrect implementation of mirroring roots with Blaschke matrices.
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8 Appendix

In this Appendix, we provide more details regarding GMR’s implementation of the Blaschke procedure (i.e. we discuss GMR’s code), provide code for and run the examples mentioned in the main text, and describe the estimation procedure in detail.

9 GMR’s Implementation of Blaschke Matrices

The functions below are taken from the website of the Review of Economic Studies. Variable names have been changed in order to increase readability and comments have been added for clarification. For testing purposes, we have also included the original functions where no names have been changed and no comments have been added.

```r
if(params$use_GMRcode_commented){

# ' Build Orthonormal Basis
# '
# ' This function creates an orthonormal matrix \code{orth} whose first column
# ' is the "normalised" value of \code{vec}.
# '
# ' @param vec Vector to be "normalised" of dimension \code{dim_out}
# '
# ' @return Orthonormal matrix \code{K} (no mentioning about the complex case,
# ' seems like it either fails or result is incorrect in the complex case)
# ' @export
build.orthonormal.basis <- function(vec){

# Integer-valued parameters
dim_out <- length(vec)

# Allocate other columns to be orthogonalized w.r.t. input *vec*
Id <- diag(dim_out)
if (vec[1]!=0) {
  basis_non_orth <- matrix(Id[,2:dim_out], dim_out, dim_out-1)
} else {
  basis_non_orth <- matrix(Id[,1:(dim_out-1)], dim_out, dim_out-1)
}

# In the following if-condition, no thought is spent on the complex case.
# (sum(c(1, 1i)^2) == 0 is TRUE)
if(sum(vec^2) == 0){

  # This should throw an error, not print a message.
  # Of course, an error is thrown later in the estimation procedure

```
```r
# because of a dimension mismatch.
print("z should not be 0")
orth <- NaN

} else {

  # Normalise the column and initialize successive orthonormalisation
  orth <- matrix(vec/sqrt(sum(vec^2)), ncol = 1)

  # Successively orthonormalise columns of *basis_non_orth*
  for(ix_col in 2:dim_out){

    # Columns that are already orthonormal
    X <- orth[, 1:(ix_col-1)]

    # Next column to be orthonormalised
    # Note that *basis_non_orth* is a matrix with *dim_out-1* columns!
    y <- matrix(basis_non_orth[, ix_col-1], ncol = 1)

    # New orthogonal column
    orth_col_new <- (Id - X %*% solve(t(X) %*% X) %*% t(X)) %*% y

    # Normalise
    orth_col_new <- orth_col_new/sqrt(sum(orth_col_new^2))

    # Bind old and now orthonormal columns together
    orth <- cbind(orth, orth_col_new)
}

return(orth)
}
```

```
#' GMR Function: Helper for computing Blaschke Transformations
#'
#' For given vector \code{w} of dimension \code{dim_out}, calculate
#' a new MA polynomial matrix and associated static shock transmission matrix.
#' #'
#' @section GMR "documentation":
#' Entries are 1 or -1 (see Lippi and Reichlin 1994).
#' If \code{w[i]=-1} then the output MA representation corresponds
#' to an MA representation
#' with the same spectral density as the input one,
```

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but the \(i\)-th pole of \(\det(I - \Theta \cdot L)\) would have been replaced by its inverse conjugate.

*Strong* The condition is rather:

If \(w[i] \neq 1\), the zero gets flipped.

@inheritParams compute.all.forms

@param \(w\) Vector of length \(dim_out\).

Indicates whether a certain zero is mirrored inside or not.

@return List with slots

\begin{itemize}
  \item \(\Theta\): \(MA(1)\) coefficient matrix
  \item \(C\): Static shock transmission matrix
\end{itemize}

from which the new \(MA\) polynomial with flipped roots can be obtained.

@export

\begin{verbatim}
compute.other.basic.form <- function(Theta, C, w){

  n_zeros <- dim(C)[1]

  all_zeros_initial <- 1/eigen(Theta)$values

  Theta_step <- Theta
  C_step <- C

  for(ix_zero in 1:n_zeros){

    if(w[ix_zero] != 1){

      zero_this <- all_zeros_initial[ix_zero]
      all_zeros_step <- 1/eigen(Theta_step)$values
      zero_this_position <- which(abs(all_zeros_step-zero_this) == min(abs(all_zeros_step-zero_this)))[1]

      # Choose zero to be mirrored and get its position
    }

  }
}
\end{verbatim}
# Get the corresponding eigenvector and transform it with inverse of C

```r
ev_theta_step <- eigen(Theta_step)$vectors[, zero_this_position]
ev_theta_step_transformed <- solve(C_step) %*% ev_theta_step
```

```r
trafo_orth <- build.orthonormal.basis(ev_theta_step_transformed)
```

# Transformation of the static shock transmission matrix C corresponds to:
# 1) Post-multiply orthogonal transformation
# 2) Post-multiply identity matrix with (1,1) element replaced
#    by 1/Conj(zero_this)

```r
CK.mod <- C_step %*% trafo_orth
a <- CK.mod[, 1]
CK.mod[, 1] <- a * (1/Conj(zero_this))
```

# Transformation of MA(1) coefficient corresponds to:
# 1) Post-multiply orthogonal transformation
# 2) Post-multiply identity matrix with (1,1) element replaced
#    by -zero_this
# 3) Post-multiply newly transformed inverse of C_step

```r
Theta.mod <- Theta_step %*% C_step %*% trafo_orth
Theta.mod[,1] <- -a
```

```r
C_step <- CK.mod
Theta_step <- Theta.mod %*% solve(CK.mod)
```

```r
return(list(Theta = Theta_step,
             C = C_step))
```

# GMR Function: Obtain All Observationally Equivalent MA Polynomials
#
# Generates all combinations of zeros inside and outside the unit circle.
#
# This function is called by GMR's `estim.MA.inversion`.
#
# @param Theta MA(1) Parameter matrix
# @param C Static shock transmission matrix
#
# @return List with two slots, `all.Theta` and `all.C`.
# Each slot contains an array of
# dimension `((dim_out, dim_out, 2^dim_out),`
where the third dimension corresponds to the choice of zeros inside and outside the unit circle.

@export
compute.all.forms <- function(Theta, C) {
  
dim_out <- dim(C)[1]

  if(dim_out == 1) {
    all.Theta <- array(c(1/Theta, Theta), c(1,1,2))
    all.C <- array(c(C*Theta, C), c(1,1,2))
  } else {

    # Create a matrix with $2^{(number of roots)}$ rows...
    combi <- matrix(0, 2^dim_out, dim_out)

    basic <- matrix(c(0,1), ncol = 1)

    # Fill columns of possibly ridiculously large matrix
    for(ix_col in 1:dim_out) {
      combi[,ix_col] <- rep(basic, 2^(ix_col-1)) %x% rep(1,2^(dim_out-ix_col))
    }

    # Allocate all possible choices of zeros inside and outside the unit circle
    all.Theta <- array(NA, c(dim_out, dim_out, 2^dim_out))
    all.C <- array(NA, c(dim_out, dim_out, 2^dim_out))

    # Compute all possible MA polynomials
    for(ix_flip_choice in 1:(2^dim_out)) {
      Theta_C_tmp <- compute.other.basic.form(Theta, C, combi[ix_flip_choice, ])
      all.Theta[, , ix_flip_choice] <- Theta_C_tmp$Theta
      all.C[, , ix_flip_choice] <- Theta_C_tmp$C
    }

    return(list(all.Theta = all.Theta,
                 all.C = all.C))
  }
}

} else {

  # This is the GMR version with original variable names and no added comments by BF

  build.othonormal.basis <- function(z) {
    # z is a vector of dimension n

# This function creates an orthonormal matrix K whose first column
# is the normalised value of z.

n <- length(z)
I <- diag(n)
if(z[1]! = 0){
  Z <- matrix(I[,2:n],n,n-1)
}else{
  Z <- matrix(I[,1:(n-1)],n,n-1)
}
if(sum(z^2) == 0){
  K <- NaN
  print("z should not be 0")
}else{
  K <- matrix(z/sqrt(sum(z^2)),ncol=1)
  for(i in 2:n){
    y <- matrix(Z[,i-1],ncol=1)
    X <- K[,1:(i-1)]
    k <- (I - X %*% solve(t(X) %*% X)) %*% y
    k <- k/sqrt(sum(k^2))
    K <- cbind(K,k)
  }
  return(K)
}

compute.other.basic.form <- function(Theta,C,w){
  # w is a vector whose entries are 1 or -1 (see Lippi and Reichlin 1994)
  # w is of dimension n, where Theta is n x n.
  # If w_i=-1 then the output MA representation corresponds to an MA representation
  # with the same spectral density as the input one,
  # but the i-th pole of det(I - Theta.L) would have been replaced
  # by its inverse conjugate.

  n <- dim(C)[1]

  poles.ini <- 1/eigen(Theta)$values

  Theta.k <- Theta
  C.k <- C

  for(nb.pole in 1:n){
    if(w[nb.pole] != 1){
      pole <- poles.ini[nb.pole]
    }
  }
}
poles.k <- \text{1/eigen(Theta.k)}$values
indic.pole.k <- \text{which(abs(poles.k-pole)==min(abs(poles.k-pole)))}[1]

k <- \text{eigen(Theta.k)}$vectors[,indic.pole.k]
k.star <- \text{solve(C.k) \times k}

K <- \text{build.othonormal.basis(k.star)}

CK.mod <- C.k \times K
a <- CK.mod[,1]
CK.mod[,1] <- - a \times (1/\text{Conj}(pole))

Theta.mod <- \text{Theta.k} \times K
Theta.mod[,1] <- - a

C.k <- CK.mod
Theta.k <- \text{Theta.mod} \times \text{solve(CK.mod)}
}
}
\text{return(list(Theta=Theta.k,C=C.k))}

compute.all.forms <- \text{function(Theta,C){}

n <- \text{dim(C)[1]}

if(n==1){
  all.Theta <- \text{array(c(1/Theta,Theta),c(1,1,2))}
  all.C <- \text{array(c(C*Theta,C),c(1,1,2))}
}else{
  combi <- \text{matrix(0,2^n,n)}
  basic <- \text{matrix(c(0,1),ncol=1)}
  \text{for}(i \text{ in } 1:n){
    combi[,i] <- \text{rep(basic,2^(i-1)) \times rep(1,2^(n-i))}
  }
  all.Theta <- \text{array(\text{NaN},c(n,n,2^n))}
  all.C <- \text{array(\text{NaN},c(n,n,2^n))}

  \text{for}(i \text{ in } 1:(2^n)){
    aux <- \text{compute.other.basic.form(Theta,C,combi[i,])}
    all.Theta[,i] <- aux$Theta
    all.C[,i] <- aux$C
  }
}
}
10 Code for Examples

Using GMR’s implementation of Blaschke matrices, we provide examples for the cases described in the main text.

10.1 Purely Complex Case

We define a particular matrix of the kind \( \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \).

```r
if (params$generate_random_examples) {
  # Randomly generate imaginary part:
  a = 0
  b = floor(runif(1, min = -1, max = 1) * 10 + 1)
} else {
  a = 0
  b = 1
}

# MA(1) coefficient matrix:
(Theta = matrix(c(a, b, -b, a), 2, 2))
# [1,] [2,]
# [1,] 0 -1
# [2,] 1 0

# Static shock transmission matrix:
(C = diag(2))
# [1,] [2,]
# [1,] 1 0
# [2,] 0 1
```

Next, we calculate its eigenvalues.

```r
# Its eigenvalues:
(ev_Theta = eigen(Theta)$values)
# [1] 0+1i 0-1i
```

# Determinantal roots of the MA polynomial:
\[(ev_{\Theta_{inv}} = ev_{\Theta}_{(-1)})\]

Calling GMR’s procedure results in an error.

\[(ex\_trivial\_rand\_purelycplx\_conj\_10 = \text{try}(\text{compute.\_other.\_basic.\_form}(\Theta = \text{Theta}, C = C, \text{w} = \text{c}(1,0))))\]

10.2 Skew-Symmetric Complex Case

Next, we generate two matrices of the kind \((a \ b \\
-b \ a)\). For \(a = b = 1\) an error is thrown, whereas for \(a = 2\) and \(b = -6\) obviously incorrect results are obtained.

\# Choose parameter values such that an error is thrown
a = 1
b = 1

\# MA(1) coefficient matrix:
(\text{Theta} = \text{matrix}(\text{c}(a,b,-b,a), 2, 2))

\# Static shock transmission matrix:
(C = \text{diag}(2))

Calling GMR’s procedure results in an error.

\[(ex1\_skewsymm\_10 = \text{try}(\text{compute.\_other.\_basic.\_form}(\Theta = \text{Theta}, C = C, \text{w} = \text{c}(1,0))))\]
## <simpleError in C_step %*% trafo_orth: non-conformable arguments>

While the next example does not throw an error, the obtained matrices are not all-pass and the eigenvalues are not mirrored at the unit circle.

# Choose parameter values such that no error is thrown but that results are incorrect
a = 2
b = -6

# MA(1) coefficient matrix:
(Theta = matrix(c(a,b,-b,a), 2, 2))
## [,1] [,2]
## [1,] 2 6
## [2,] -6 2

# Static shock transmission matrix:
(C = diag(2))
## [,1] [,2]
## [1,] 1 0
## [2,] 0 1

Calling GMR’s procedure results in obviously incorrect values.

(ex2_skewsymm_10 = try(compute.other.basic.form(Theta = Theta, C = C, w = c(1,0)))
## $Theta
## ## [,1] [,2]
## ## [1,] -2.639553e+15+1.273073e+16i 1.273073e+16+2.639553e+15i
## ## [2,] 1.273073e+16+2.639553e+15i 2.639553e+15-1.273073e+16i

## $C
## ## [,1] [,2]
## ## [1,] -94906266-284718797i 0+47453133i
## ## [2,] -284718797+ 94906266i 47453133+ 0i

There is no apparent connection of the eigenvalues of the original MA(1) coefficient to the ones of the newly transformed system.

# Eigenvalues of original MA(1) coefficient matrix
eigen(Theta)$values
## [1] 2+6i 2-6i

# Eigenvalues of original MA(1) coefficient matrix
eigen(ex2_skewsymm_10$Theta)$values
## [1] -295256766-93315703i 295256766+93315703i
10.3 General Complex Case

Here, we consider matrices of the kind \( \begin{pmatrix} a & b+c \\ -b & a \end{pmatrix} \), where \( c \cdot b > 0 \) such that we obtain complex-valued eigenvalues.

```r
if(params$generate_random_examples){
  # Randomly generate imaginary part:
  a = floor(runif(1, min = -1, max = 1)*10+1)
  b = floor(runif(1, min = -1, max = 1)*10+1)
  while(b == 0){
    b = floor(runif(1, min = -1, max = 1)*10+1)
  }
  c = floor(runif(1, min = -1, max = 1)*10+1)
  while(c == 0){
    c = floor(runif(1, min = -1, max = 1)*10+1)
  }
  if(b*c < 0){
    c = -c
  }
} else {
  a = 4
  b = 3
  c = 2
}

# MA(1) coefficient matrix:
(Theta = matrix(c(a, b+c, -b, a), 2, 2))
## [,1] [,2]
## [1,] 4  -3
## [2,] 5   4

# Eigenvalues of Theta:
eigen(Theta)$values
## [1] 4+3.872983i 4-3.872983i

# Static shock transmission matrix:
(C = diag(2))
## [,1] [,2]
## [1,] 1 0
## [2,] 0 1
```

While no error is thrown, we will show that when both complex-conjugated roots are mirrored at the unit circle, the result does not constitute an observationally equivalent matrix polynomial (even though the eigenvalues of the original MA(1) coefficient matrix are mirrored correctly). First, we mirror both eigenvalues.
The eigenvalues of the transformed MA(1) coefficient matrix are the inverses of the eigenvalues of the original MA(1) coefficient matrix.

```
# Eigenvalues of original MA(1) coefficient matrix
eigen(Theta)$values
## [1] 4+3.872983i 4-3.872983i
# Inverses of eigenvalues of original MA(1) coefficient matrix
eigen(Theta)$values^(-1)
## [1] 0.1290323-0.1249349i 0.1290323+0.1249349i
# Eigenvalues of transformed MA(1) coefficient matrix
eigen(ex_nonskew_00$Theta)$values
## [1] 0.1290323-0.1249349i 0.1290323+0.1249349i
```

However, the original and transformed MA(1) polynomial matrices are not observationally equivalent. This is a consequence of the fact that the complex nature of the eigenvectors is ignored in GMR’s function `build.orthonormal.basis(...)`.

```
# Generate (equally-spaced) points on the unit circle:
z = exp(1i*2*pi*seq(0, 1, length.out = 11))[-11]
# Functions to evaluate original and transformed MA(1) polynomial matrices:
polymat_original = function(z){
  (diag(2) - Theta * z) %*% C
}
polymat_transformed = function(z){
  (diag(2) - ex_nonskew_00$Theta * z) %*% ex_nonskew_00$C
}
# Function for comparing polynomial matrices evaluated on the unit circle
# in terms of many different distance measures
```
compare_original_vs_transformed = function(original, transformed){

    # Since the spectral densities of the original and transformed polynomial matrix
    # should be identical,
    # we evaluated them on various points on the unit circle,
    # calculate the value of the spectral density for the original and transformed one,
    # and compare their values

    original_herm = original %*% Conj(t(original))
    transformed_herm = transformed %*% Conj(t(transformed))
    diff_herm = original_herm - transformed_herm

    return(list(original_herm = original_herm,
                 transformed_herm = transformed_herm,
                 diff_herm= diff_herm,
                 dist_herm = sum(diff_herm * Conj(diff_herm))))
}

for (ix_z in seq_along(z)){
    original_evaluated = polymat_original(z[ix_z])
    transformed_evaluated = polymat_transformed(z[ix_z])

    comparison_out_list = compare_original_vs_transformed(original_evaluated, transformed_evaluated)

    cat(paste0("Distance measure when evaluated at exp(2i pi x) for x equal to ", (ix_z - 1)/10, " is : ",
    comparison_out_list$dist_herm, 
    ))
}
## Distance measure when evaluated at exp(2i pi x) for x equal to 0 is : 29924.897951836+0i
## Distance measure when evaluated at exp(2i pi x) for x equal to 0.1 is : 19163.9222490856+0i
## Distance measure when evaluated at exp(2i pi x) for x equal to 0.2 is : 19947.6366425084+0i
## Distance measure when evaluated at exp(2i pi x) for x equal to 0.3 is : 32593.8082149725+0i
## Distance measure when evaluated at exp(2i pi x) for x equal to 0.4 is : 58200.1756711532+0i
## Distance measure when evaluated at exp(2i pi x) for x equal to 0.5 is : 90032.6530612241+0i
## Distance measure when evaluated at exp(2i pi x) for x equal to 0.6 is : 111887.119096838+0i
## Distance measure when evaluated at exp(2i pi x) for x equal to 0.7 is : 109869.165310613+0i
## Distance measure when evaluated at exp(2i pi x) for x equal to 0.8 is : 85366.7012425608+0i
## Distance measure when evaluated at exp(2i pi x) for x equal to 0.9 is : 53666.9816063492+0i

Even if these issues were solved, the matrices would still not be real-valued and therefore outside of GMR's model class.
10.4 Effects of Discarding Imaginary Part

Here, we show that the number of roots inside the unit circle changes when the imaginary part of the MA parameter matrix is discarded.

```r
if (params$generate_random_examples) {
  # Randomly generate imaginary part:
  a = floor(runif(1, min = -1, max = 1) * 10 + 1)
  b = floor(runif(1, min = -1, max = 1) * 10 + 1)
  while (b == 0) {
    b = floor(runif(1, min = -1, max = 1) * 10 + 1)
  }
  c = floor(runif(1, min = -1, max = 1) * 10 + 1)
  while (c == 0) {
    c = floor(runif(1, min = -1, max = 1) * 10 + 1)
  }
  if (b * c < 0) {
    c = -c
  } else {
    a = 2
    b = 2
    c = 4
  }

  # MA(1) coefficient matrix:
  (Theta = matrix(c(a, b + c, -b, a), 2, 2))
  ## [,1] [,2]
  ## [1,] 2 -2
  ## [2,] 6 2

  # Eigenvalues of Theta:
  eigen(Theta)$values
  ## [1] 2 +3.464102i 2 -3.464102i

  # Static shock transmission matrix:
  (C = diag(2))
  ## [,1] [,2]
  ## [1,] 1 0
  ## [2,] 0 1
}
```

Mirroring one determinantal root inside the unit circle results in a complex-valued parameter matrix. The complex-valued parameter matrix has one determinantal root inside the unit circle. However, when the imaginary parts are discarded, all roots are outside the unit circle.
11 Description of GMR’s Main Procedures

11.1 GMM Method

Let us start by considering the dependency graphs of GMR’s GMM method `estim.VARMAp.2SLSGMM(...)`: 

![Calling tree for GMM procedure](image)

The procedure has the following stages.

1. Two stage least squares, implemented in `TSLS(...)`, is used to obtain initial estimates for the AR parameters.

2. The function `gmm.estim.sigma.theta(...)` is used to obtain initial estimates for the MA parameter matrix (and the error covariance matrix).

3. At this stage the weighting matrix is calculated for the first time (with `compute.asymptotic.distri.2SLSGMM(...)`).
Figure 2: Calling tree for two Stage Least Squares

Figure 3: Calling tree for computing the asymptotic distribution
4. The main worker function `compute.distance.moments(…)` is called iteratively `nb.loops.GMM = 5` times (a parameter specified in the script `run.VARMA.BQ.dataset.R`) by the main GMM function `estim.VARMAp1.2SLS.GMM(…)` (which itself is called in the script `run.estim.BQ.GMM.MLE.R`) to optimise the parameters. In each of the `nb.loops.GMM = 5` iterations, `optimx::optimx(…)` is called first with method `BFGS` and then with method `Nelder-Mead`. For both methods, the maximal number of iterations is set to `maxitNM = 2000` (default parameter value, but also in the script `run.estim.BQ.GMM.MLE.R`). The name as well as the value of `maxitNM` suggests that this is done by accident for the `BFGS` method where a different (lower) number of maximal iterations should have been chosen (default in R for derivative based methods is 100 iterations).

![Call tree](call_tree.png)

Figure 4: Calling tree for computing the distance between theoretical and empirical moments

5. Now, all possible Blaschke matrices are calculated. In GMR’s empirical application there are $2^2 - 1 = 3$ flips. For each of these flips, the same procedure is applied (5 iterations of `BFGS` and `Nelder-Mead` optimisations). Only if the obtained likelihood value is improved, is the “optimal model” updated.

6. The weighting matrix is calculated a second time (with `compute.asymptotic.distri.2SLS.GMM(…)`).

7. The 5 iterations of `compute.distance.moments(…)` are performed one Lastly time with the currently best model.

8. The weighting matrix is calculated a third and Lastly time (again with `compute.asymptotic.distri.2SLS.GMM(…)`).

11.2 ML Method

The optimal parameter from the GMM method serves as initial value for the ML method. The main work is done by the function `estim.MA.inversion.aux(…)` (which is called by `estim.MA.inversion(…)`). The function called/optimised in `optimx::optimx(…)` below is `ML.inversion.loglik(…)` (which uses the Schur-factorization for calculating the residuals).

1. In the first call to `estim.MA.inversion.aux(…),` the optimiser is called `nb.loops.Lastly = 4` times (this time with `nlminb` and `Nelder-Mead` with respectively `MAXIT.nlminb.Lastly = 300` and `MAXIT.NMLastly = 1000` iterations). These parameters are defined in the script `run.VARMA.BQ.dataset.R`. 
Figure 5: Calling tree for ML procedure

Figure 6: Calling tree of main worker for ML procedure

Figure 7: Calling tree for likelihood evaluation
2. Then, all Blaschke flips are obtained and the same optimisation procedure is performed but only with $nb.loops.BM = 2$ iterations for each new starting value obtained by flipping. The fact that the Blaschke matrices are complex-valued is covered up at this step. Tracking the zero location and the likelihood values is therefore particularly interesting at this step. The output of the optimiser is saved but the optimum is only updated if there is an improvement.

3. Lastly, the covariance is calculated through an optim(.)-call to $ML.inversion.loglik(.)$ with argument $hessian = TRUE$. 