ON MOTOHASHI'S FORMULA

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Abstract. We complement and offer a new perspective of the proof of a Motohashi-type formula relating the fourth moment of $L$-functions for $GL_1$ with the third moment of $L$-functions for $GL_2$ over number fields, studied earlier by Michel-Venkatesh and Nelson. Our main tool is a new type of pre-trace formula with test functions on $M_2(\mathbb{A})$ instead of $GL_2(\mathbb{A})$, on whose spectral side the matrix coefficients are replaced by the standard Godement-Jacquet zeta integrals. This is also a generalization of Bruggeman-Motohashi’s other proof of Motohashi’s formula. We give a variation of our method in the case of division quaternion algebras instead of $M_2$, yielding a new spectral reciprocity, for which we are not sure if it is within the period formalism given by Michel-Venkatesh. We also indicate a further possible generalization, which seems to be beyond what the period method can offer.

Contents

1. Introduction
   1. History and New Perspective
   1.2. Notations and Conventions
   1.3. Main Result
   1.4. Compact Variation
   1.5. List of Tempered Distributions
   1.6. List of Tempered Distributions: Compact Variation
2. Third Moment (Spectral) Side
   2.1. Some Spectral Theory
   2.2. Godement-Jacquet Pre-trace Formula
3. Fourth Moment (Geometric) Side
4. Analytic Continuation
   4.1. Third Moment Side
   4.2. Geometric Aspect of Residues
   4.3. Fourth Moment Side
5. Analysis of Degenerate Terms
6. Proof of Compact Variation
   6.1. Mixed Moment Side
   6.2. Second Moment Side
7. Appendix: Comparison with Period Approach
   7.1. Recall of Period Approach
   7.2. Comparison of Geometric Sides
   7.3. Comparison of Spectral Sides
Acknowledgement
References

1. Introduction

1.1. History and New Perspective. In a series work (one collaborated with Ivic) culminating in [17], Motohashi established an explicit formula relating the fourth moment of Riemann zeta function with the cubic moment of $L$-functions related to modular forms (holomorphic and Maass forms and Eisenstein
series) for the full modular group $\text{SL}_2(\mathbb{Z})$. In particular, the transform formula from the weight function on the fourth moment side to the weight function on the cubic moment side is described with explicit formula. A further extension to $\mathbb{Q}(i)$ can be found in [4]. In terms of automorphic representation theoretic language, this is a relation between the fourth moment of $L$-functions for $\text{GL}_1$ and the third moment of $L$-functions for $\text{GL}_2$ over $\mathbb{Q}$, hence has natural generalization in the level aspect.

Conrey-Iwaniec [6] noticed that the continuous part of the cubic moment side becomes a sixth moment of $\text{GL}_1$ $L$-functions. They studied Motohashi’s formula in the level aspect for the inverse direction of the weight transform formula. They partially succeeded in doing this, and obtained a Weyl-type subconvexity in the level aspect for quadratic Dirichlet characters. This approach was recently extended by Petrow-Young [21] to cube-free level Dirichlet characters in the hybrid aspect, then to all Dirichlet characters in [20]. Partial inversion in the archimedean aspect for Motohashi’s original formula is also known for Maass forms by Ivić [11], and more recently for holomorphic forms by Frolenkov [7], which also imply the relevant Weyl-type subconvexity. However, in these great achievements, an explicit transform formula of the weight functions in the inverse direction remains mysterious.

In [12, 4.3.3] and [16, §4.5.3] Michel-Venkatesh proposed a sketch of period approach to the Motohashi formula relating the fourth moment of the Riemann zeta function and some cubic moment of $L$-functions of automorphic forms for $\text{GL}_2$, which exploits the Gan-Gross-Prasad conjecture (GGP) via two different paths of automorphic restriction of (a product of) Eisenstein series as follows

$$\begin{align*}
\text{GL}_2 \times \text{GL}_2 & \to \\
\text{GL}_1 \times \text{GL}_1 & \leftarrow \\
\text{GL}_1 & \to \text{GL}_2.
\end{align*}$$

This **Michel-Venkatesh sketch** is not a proof of a Motohashi-type formula because of serious convergence issue. Recently, Nelson [19] announced a solution to the convergence issue, a version of the transform formula of weight functions in the inverse direction which generalizes Conrey-Iwaniec’s result over general number fields. In the current version, his inversion formula seems to work only for a special type of test functions, which is insufficient for our purpose in the subsequent work [3].

Inserting Godement’s construction of Eisenstein series, we discover a new perspective of the Michel-Venkatesh and Nelson’s (incomplete) treatment. Precisely, we realize Motohashi’s formula as the equality of two different decompositions of a tempered distribution $\Theta(\lambda, \cdot)$ on $\mathcal{S}(\mathbb{M}_2(\mathbb{A}))$ (see Theorem 1.5). This distribution satisfies a certain co-variance property under the action of $H(\mathbb{A}) = \text{GL}_1(\mathbb{A})^3$, offered by the following companion graph of (1.1)

$$\begin{align*}
\text{GL}_2 & > \text{GL}_1 \\
\text{GL}_1 & < \text{GL}_2
\end{align*}$$

(1.2)

where the two actions of $\text{GL}_2$ on $\mathbb{M}_2$ are the natural actions by multiplication and the bottom action of $\text{GL}_1$ is the multiplication by the center of $\text{GL}_2$. In such a way, we make appear the Godement-Jacquet zeta integrals on the cubic moment side to obtain the analytic continuation of all degenerate terms. The class of test functions is the whole $\mathcal{S}(\mathbb{M}_2(\mathbb{A}))$, and the degenerate terms are explicitly related to the main terms via concise formulas (see Proposition 1.6 below). We do not think that the theory of regularized integrals (as developed in [16, 27, 28]) alone can offer such results with the same depth as conveniently as we do here. The full power (Schwartz functions, invariance of distributions, formulas for degenerate terms) of the results obtained in this paper is important for the subsequent joint work of the author with Olga Balkanova and Dmitry Frolenkov [3], where we generalize Petrow-Young’s cube-free Weyl-type subconvex bound [21] over totally real number fields, via a proper inverse transform formula of weight functions at the real places. Incidentally, we point out that our method can be viewed as a generalization of Bruggeman-Motohashi’s other proof [8] of Motohashi’s formula.
Remark 1.1. Although we feel that the graph (1.3) might be the companion graph of
\[ \begin{align*}
& \mathbf{D}^\times \times \mathbf{D}^\times \\
& \mathbf{E}^\times \times \mathbf{E}^\times \\
& \mathbf{E}^\times \\
& \mathbf{D}^\times
\end{align*} \]
as in the previous case, the precise period approach to our spectral reciprocity relation in Theorem 1.10 remains unclear (to us). The difficulty is to associate an automorphic form \( \vartheta_\Phi(\Omega) \) on \( \mathbf{D}^\times \) with a Hecke character \( \Omega \) of \( \mathbf{E} \) and a Schwartz function \( \Phi \in \mathcal{S}(\mathbb{A}_E) \) in a canonical way. Some ideas in two special cases are as follows:

(1) In the case \( \Omega |_{\mathbb{A}^\times} = 1 \), \( \Omega \) can be viewed as an automorphic representation of \( \mathbf{F}^\times \backslash \mathbf{E}^\times \simeq \mathbf{E}^1 \), where \( \mathbf{E}^1 \) is the \( \mathbf{F} \)-subgroup of elements in \( \mathbf{E}^\times \) with norm 1. Then we have the theta lift \( \Theta_1(\Omega) \) to the metaplectic group \( \mathbf{Mp} \) via the Weil representation \( r_1 \) of \( \mathbf{E}^1 \times \mathbf{Mp} \). We also have the theta lift \( \Theta_2(\sigma) \) for any cuspidal automorphic representation \( \sigma \) of \( \mathbf{Mp} \) to any quaternionic group \( \mathbf{D}^\times \) via the Weil representation \( r_2 \) of \( \mathbf{D}^\times \times \mathbf{Mp} \). One may expect that \( \Theta_2(\Theta_1(\Omega)) \) is well-defined, independent of the additive characters chosen in the above two Weil representations, and is the cuspidal representation containing \( \vartheta_\Phi(\Omega) \).

(2) In the case \( \mathbf{D} = \mathbb{M}_2 \), the automorphic representation containing \( \vartheta_\Phi(\Omega) \) should be the automorphic induction of \( \Omega \) to \( \mathbf{GL}_2 \).

However we do not know what role should \( \Phi \) play in either case.

Since our proof relies on a new type of pre-trace formulas (see Theorem 2.10 and Theorem 5.31), whose spectral side has the Godement-Jacquet zeta functions instead of the usual matrix coefficients, we point out that a further possible application of Theorem 2.10 is to combine it with Jacquet-Zagier’s approach of the trace formula for \( \mathbf{GL}_2 \) (see [12, 27]), to study the mixed moment of \( L(1/2, \pi)L(1/2, \pi, \text{Ad}) \). This possibility does not seem to be within the scope of Michel-Venkatesh’s period formalism. Such moment possibility was recently studied in [1, 2]. It should be possible to generalize these results over general number fields in the light of this possible application of Theorem 2.10. We also reserve it for another paper.

1.2. Notations and Conventions. Throughout the paper, \( \mathbf{F} \) is a (fixed) number field with ring of integers \( o \). \( V_F \) denotes the set of places of \( \mathbf{F} \). For any \( v \in V_F \), \( p \in V_F, p < \infty, F_v \) resp. \( o_p \) is the completion of \( \mathbf{F} \) resp. \( o \) with respect to the absolute value \( |.|_v \), corresponding to \( v \) resp. \( v = p \). \( \mathbb{A} = \mathbb{A}_F \) is the ring of adeles of \( \mathbf{F} \), while \( \mathbb{A}^\times \) denotes the group of ideles. We fix a section \( s_F \) of the adelic norm map \( |.|_\mathbb{A} : \mathbb{A}^\times \rightarrow \mathbb{R}_+ \), identifying \( \mathbb{R}_+ \) as a subgroup of \( \mathbb{A}^\times \). Hence any character of \( \mathbb{R}_+ \mathbf{F}^\times \backslash \mathbb{A}^\times \) is identified with a character of \( \mathbf{F}^\times \backslash \mathbb{A}^\times \). We put the standard Tamagawa measure \( dx = \prod_v d^v x_v \) on \( \mathbb{A}^\times \). We recall their constructions. Let \( \text{Tr} = \text{Tr}^F \) be the trace map, extended to \( \mathbb{A} \rightarrow \mathbb{A}_Q \). Let \( \psi_Q \) be the additive character of \( \mathbb{A}_Q \) trivial on \( Q \), restricting to the infinite place as \( Q_\infty = \mathbb{R} \rightarrow \mathbb{C}^{(1)}, x \mapsto e^{2\pi ix} \).
We put \( \psi = \psi_0 \circ \text{Tr} \), which decomposes as \( \psi(x) = \prod_v \psi_v(x_v) \) for \( x = (x_v)_v \in A \). \( dx_v \) is the additive Haar measure on \( F_v \), self-dual with respect to \( \psi_v \). Precisely, if \( F_v = \mathbb{R} \), then \( dx_v \) is the usual Lebesgue measure on \( \mathbb{R} \); if \( F_v = \mathbb{C} \), then \( dx_v \) is twice the usual Lebesgue measure on \( \mathbb{C} \cong \mathbb{R}^2 \); if \( v = p < \infty \) such that \( \mathfrak{o}_p \) is the valuation ring of \( F_p \), then \( dx_p \) gives the mass \( D_p^{-1/2} \), where \( D_p = D(\mathbb{F}_p) \) is the local component at \( p \) of the discriminant \( D(F) \) of \( F/\mathbb{Q} \) such that \( D(F) = \prod_{p < \infty} D_p \). Consequently, the quotient space \( F/\mathbb{A} \) with the above measure quotient by the discrete measure on \( F \) admits the total mass \( 1 \) \([13, \text{Ch.XIV Prop.7}]\). Recall the local zeta-functions: if \( F_v = \mathbb{R} \), then \( \zeta_v(s) = \Gamma_R(s) = \pi^{−s/2} \Gamma(s/2) \); if \( F_v = \mathbb{C} \), then \( \zeta_v(s) = \Gamma_C(s) = (2\pi)^{−1/2} \Gamma(s) \); if \( v = p < \infty \) then \( \zeta_p(s) = (1 - q_p^{−s})^{−1} \), where \( q_p := \text{Nr}(p) \) is the cardinality of \( \mathfrak{o}/\mathfrak{p} \). We then define

\[
d^x x_v := \zeta_v(1) \frac{dx_v}{|x_v|}.
\]

In particular, \( \text{Vol}(\mathfrak{o}_p^\times, d^x x_p) = \text{Vol}(\mathfrak{o}_p, dx_p) \) for \( p < \infty \). Their product gives a measure of \( \mathbb{A}^\times \). Equip \( \mathbb{R}_p(\mathbb{R}_+) \) with the measure \( dt/t \) on \( \mathbb{R}_+ \), where \( dt \) is the (restriction of the) usual Lebesgue measure on \( \mathbb{R} \), and \( \mathbb{F}^\times \) with the counting measure. Then we have (see \([13, \text{Ch.XIV Prop.13}]\))

\[
\text{Vol}(\mathbb{R}_p, \mathbb{F}^\times \setminus \mathbb{A}^\times) = \zeta_p^\ast,
\]

where \( \zeta_p^\ast \) is the residue at \( 1 \) of the Dedekind zeta function \( \zeta_p(s) \).

For \( p < \infty \), let \( \mathcal{S}(\mathbb{F}_p) = C_0(\mathbb{F}_p) \). We call \( \mathcal{S}(\mathbb{A}) = \otimes_v \mathcal{S}(\mathbb{F}_v) \) the space of Schwartz functions over \( \mathbb{A} \). Let \( \chi \in \mathbb{R}_+^{\mathbb{F}^\times \setminus \mathbb{A}^\times} \) viewed as a unitary Hecke character of \( \mathbb{F}^\times \setminus \mathbb{A}^\times \), and let \( f \in \mathcal{S}(\mathbb{A}) \). Tate’s global zeta function is defined by

\[
Z(s, \chi, f) := \int_{\mathbb{A}} f(x) \chi(x) |x|^s \cdot d^x x.
\]

These are integral representations of the complete \( L \)-function \( \Lambda(s, \chi) \). They are absolutely convergent for \( \Re s > 1 \) with meromorphic continuation to \( s \in \mathbb{C} \) and rapid decay for \( C(\chi|_\mathbb{A}^\times) \to \infty \), satisfy the global functional equation

\[
Z(s, \chi, f) = Z(1 - s, \chi^{-1}, \overline{\mathfrak{f}}) \quad \text{with} \quad \overline{\mathfrak{f}} f(x) := \int_{\mathbb{A}} f(y) \psi(-xy) dy,
\]

and admit possible simple poles at \( s \in \{0, 1\} \) for \( \chi = 1 \) with residues

\[
\text{Res}_{s=1} Z(s, 1, f) = \zeta_p^\ast \int_{\mathbb{A}} f(x) dx, \quad \text{Res}_{s=0} Z(s, 1, f) = -\zeta_p^\ast f(0) .
\]

For \( R \in \{ \mathbb{F}_v \mid v \in V_F \} \cup \{ \mathbb{A} \} \), we define the following subgroups of \( \text{GL}_2(R) \)

\[
\mathbb{Z}(R) = \left\{ z(u) := \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \mid u \in R^\times \right\}, \quad \mathbb{N}(R) = \left\{ n(x) := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in R \right\},
\]

\[
\mathbb{A}(R) = \left\{ a(y) := \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \mid y \in R^\times \right\}, \quad \mathbb{A}(R) \mathbb{Z}(R) = \left\{ d(t_1, t_2) := \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \mid t_1, t_2 \in R^\times \right\},
\]

and equip them with the Haar measures on \( R^\times, R, R^\times \times R^\times \) respectively. The long Weyl element \( w \) in \( \text{GL}_2(R) \) is specified as

\[
w := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

The product \( B := \mathbb{Z} \mathbb{N} \mathbb{A} \) is a Borel subgroup of \( \text{GL}_2 \). We pick the standard maximal compact subgroup \( K = \prod_v K_v \) of \( \text{GL}_2(\mathbb{A}) \) by defining

\[
K_v = \begin{cases} 
\text{SO}_2(\mathbb{R}) & \text{if } F_v = \mathbb{R} \\
\text{SU}_2(\mathbb{C}) & \text{if } F_v = \mathbb{C} \\
\text{GL}_2(\mathfrak{o}_p) & \text{if } v = p < \infty
\end{cases}.
\]
and equip it with the Haar probability measure $dk_v$. At every place $v$, we define a height function

$$H_t_v : Z(F_v) \setminus GL_2(F_v)/K_v \to \mathbb{R}_{>0}, \quad \begin{pmatrix} t_1 & x \\ 0 & t_2 \end{pmatrix} \mapsto \frac{|t_1|}{|t_2|}. $$

Their tensor product $H_t(g) := \prod_v H_t_v(g_v)$ for $g = (g_v)_v \in GL_2(\mathcal{A})$ is the height function on $GL_2(\mathcal{A})$.

$M_2(\mathcal{A})$ admits an action of $GL_2(\mathcal{A}) \times GL_2(\mathcal{A})$ which induces an action on $S(M_2(\mathcal{A}))$ by

$$L_{g_1} R_{g_2} \Psi(x) := \Psi(g_1^{-1} x g_2), \quad \forall g_1, g_2 \in GL_2(\mathcal{A}), \Psi \in S(M_2(\mathcal{A})).$$

This is a smooth Fréchet-representation.

1.3. **Main Result.** Let $R \in \{ F_v \mid v \in V_F \} \cup \{ \mathcal{A} \}$. Consider the (right) action of $H(R) = R^x \times R^x \times R^x$ on $M_2(R)$ given for $x \in M_2(R)$ and $t_j, z \in R^x$ by

$$x^{(t_1, t_2, z)} := a(t_1)^{-1} x d(t_2 z, z).$$

**Definition 1.2.** A Motohashi distribution is a tempered distribution $\Theta$ on $M_2(R)$ satisfying the co-invariance property under the above action of $H(R) = (R^x)^3$

$$\Theta(h \cdot \Psi) = \lambda(h) \Theta(\Psi), \quad \forall \Psi \in S(M_2(R)) \& h.\Psi(x) := \Psi(x^h),$$

where $\lambda$ is a quasi-character of $H(R)$, called the parameter of the distribution $\Theta$. According as $R = \mathcal{A}$ or $F_v$, we say $\Theta$ is global or local. For a global Motohashi distribution, we require its parameter $\lambda$ to be automorphic, i.e., to be a quasi-character of $H(F) \setminus H(\mathcal{A}) \simeq (F^x \setminus \mathcal{A}^x)^3$.

The set of parameters $\lambda$ for Motohashi distributions (global or local) has the structure of a (trivial) complex vector bundle over the discrete set of a compact abelian group, with bundles isomorphic to $\mathbb{C}^3$.

In the global case, we shall fix three unitary characters $\chi_1, \chi_2, \omega$ of $R_+ F^x \setminus \mathcal{A}^x$, the relevant bundle at which is identified with $\mathbb{C}^3$ via a shift

$$\lambda = (s_1, s_2, s_0) \in \mathbb{C}^3 \leftrightarrow (t_1, t_2, z) \in (\mathcal{A}^x)^3 \mapsto \chi_1(t_1)\chi_2(t_2)\omega(z) \cdot |t_1|_{\mathcal{A}}^{s_1-1} |t_2|_{\mathcal{A}}^{s_2+1} |z|_{\mathcal{A}}^{s_0+2}.$$

To any $\Psi \in S(M_2(\mathcal{A}))$, we associate a kernel function (regarded as a tempered distribution)

$$KK(x; y) := \sum_{0 \neq \xi \in M_2(F)} \Psi(x^{-1} \xi y), \quad x, y \in GL_2(\mathcal{A}).$$

It is a smooth function on $GL_2(F) \setminus GL_2(\mathcal{A})$. Hence we can define

$$NK(x; y) := \int_{F \setminus \mathcal{A}} KK \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} x; y \, du,$$

$$KN(x; y) := \int_{F \setminus \mathcal{A}} KK \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} y; x \, du,$$

$$NN(x; y) := \int_{F \setminus \mathcal{A}} KN \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} x; y \, du.$$  

From these functions, we form a smooth function on $B(F) \setminus GL_2(\mathcal{A}) \times B(F) \setminus GL_2(\mathcal{A})$

$$\Delta \Delta (x; y) := KK(x; y) - NK(x; y) - KN(x; y) + NN(x; y).$$

Note that we have chosen our notation so that “$K$” resp. “$N$” resp. “$\Delta$” stands for “kernel” resp. “constant term” resp. “difference (non constant term)”, and have omitted the dependence on $\Psi$ for simplicity for this set of notations.

The global Motohashi distribution we are interested in is

$$\Theta(\lambda, \Psi) := \int_{(F^x \setminus \mathcal{A}^x)^3} \Delta \Delta(a(t_1); d(t_2 z, z)) \omega(z) |z|_\mathcal{A}^{s_0+2} \chi_1(t_1)|t_1|_\mathcal{A}^{s_1-1} \chi_2(t_2)|t_2|_\mathcal{A}^{s_2+1} d^x z d^x t_1 d^x t_2,$$

whose fundamental properties are summarized in the following proposition.
Theorem 1.5. We get a Motohashi-type formula as the following equality of tempered distributions by the application in [3], we will give the answer in the case \( \omega \)

\[
\mathbf{1.11} \quad \lambda \in D := \left\{ (s_1, s_2, s_0) \in \mathbb{C}^3 \mid \Re s_0 > 2, \Re s_1 - 1 - \frac{\Re s_0 + 1}{2} \& \Re s_2 + 1 - \frac{\Re s_0 + 1}{2} > 1 \right\}.
\]

(2) It has a meromorphic continuation to \( \mathbb{C}^3 \).

Remark 1.4. Throughout the paper, we hide the dependence on \( \chi_1, \chi_2, \omega \) in the notation for simplicity.

We will establish Proposition 1.3 by two different methods. One method (Proposition 1.4) uses the spectral theory for \( GL_2 \) while the other (Proposition 4.1) uses (partial) Poisson summation formula over \( M_2(\mathbb{A}) \). They allow us to write \( \Theta(\lambda, \Psi) \) for \( \lambda \) near \( \vec{0} \) as

\[
M_3(\lambda, \Psi) + \sum_{j=0}^{3} DS_j(\lambda, \Psi), \quad \text{resp.} \quad M_4(\lambda, \Psi) + \sum_{j=1}^{8} DG_j(\lambda, \Psi) - \sum_{j=4}^{5} DS_j(\lambda, \Psi),
\]

where \( M_3(\vec{0}, \cdot) \) resp. \( M_4(\vec{0}, \cdot) \) is a Motohashi distribution of parameter \( \vec{0} \) representing some cubic moment of \( GL_2 \)-functions resp. fourth moment of \( GL_1 \)-functions (in particular \( M_3(\lambda, \Psi) \) resp. \( M_4(\lambda, \Psi) \) is regular at \( \vec{0} \)), while \( DS_j(\lambda, \cdot) \) resp. \( DG_j(\lambda, \cdot) \) are degenerate distributions in the sense that

1. Each of them is supported, up to partial Fourier transforms, in a single \( H(\mathbb{A}) \)-orbit, a singularity (in some sense) in the variety of \( H(\mathbb{A}) \)-orbits on \( M_2(\mathbb{A}) \);
2. Each is meromorphic at \( \vec{0} \) but not necessarily regular there, and \( DS := \sum_j DS_j \) resp. \( DG := \sum_j DG_j \) is regular at \( \vec{0} \);
3. \( DS_j \) resp. \( DG_j \) is expressible in terms of \( M_3(\lambda, \Psi \mid \chi, s) \), the projection onto the principal series representation \( \pi(\chi \mid \chi^t \omega \chi^{-1} \mid \chi^t \omega \chi^{-1} \mid \chi^t \omega \chi^{-1}) \) of \( M_3(\lambda, \Psi) \) resp. \( M_4(\lambda, \Psi \mid \chi, s) \), the projection onto the Hecke character \( \chi \) of \( M_4(\lambda, \Psi) \).

Our main result is thus the equality at \( \vec{0} \) of the two expressions of meromorphic continuation.

Theorem 1.5. We get a Motohashi-type formula as the following equality of tempered distributions

\[
M_3(\vec{0}, \Psi) + DS(\vec{0}, \Psi) = M_4(\vec{0}, \Psi) + DG(\vec{0}, \Psi), \quad \forall \Psi \in S(M_2(\mathbb{A})).
\]

We will not give full detail for the analysis of the degenerate terms in all cases. However, motivated by the application in [3], we will give the answer in the case \( \omega = \chi_1 = \chi_2 = \mathbb{1} \) and establish

Proposition 1.6. In the case \( \omega = \chi_1 = \chi_2 = \mathbb{1} \), we have

\[
DG(\vec{0}, \Psi) = \frac{1}{jF} \left\{ \text{Res}_{s=1} M_4(\vec{0}, \Psi \mid \mathbb{1}, s) - \text{Res}_{s=0} M_4(\vec{0}, \Psi \mid \mathbb{1}, s) \right\},
\]

\[
DS(\vec{0}, \Psi) = \frac{1}{jF} \text{Res}_{s=\frac{1}{2}} M_3(\vec{0}, \Psi \mid \mathbb{1}, s).
\]

Remark 1.7. Our method works for all other cases, and is simpler than the original work of Motohashi.

Remark 1.8. The formula of \( DS(\vec{0}, \Psi) \) is new. In fact, in Motohashi’s original work [17, Theorem 4.2], a precise explicit formula for this term (the first component of \( L_2(r) \)) is incomplete, and has not so far been completed in the literature.

1.4. Compact Variation. We re-interpret our graph of invariance [12] by viewing the split torus \( GL_1 \times GL_1 \) as \( E^\times \) for the split algebra \( E \cong F \oplus F \). We regard \( E \) resp. \( E^\times \) as \( F \)-algebra resp. \( F \)-group embedded in the split quaternion algebra \( M_2 \). Then the group \( H \) is canonically isomorphic to \( (E^\times \times E^\times) / F^\times \), where \( F^\times \) is the group of centers of \( GL_2 \) diagonally embedded in \( E^\times \times E^\times \).

We consider the case where \( E \) is non-split, and \( M_2 \) is replaced by a division quaternion algebra \( D \) over \( F \) containing \( E \). The group \( H = (E^\times \times E^\times) / F^\times \) acts on \( D \) (from right) by

\[
H \times D \to D, \quad (t_1, t_2) \times x \mapsto t_1^{-1} x t_2.
\]
Let $\Omega$ be a (quasi-)character of $\mathbf{E}^\times \backslash \mathbb{A}_E^\times$, whose restriction to $\mathbf{F}^\times \backslash \mathbb{A}_E^\times$ is $\omega$. It defines a (quasi-)character of $H(\mathbf{F}) \backslash H(\mathbb{A})$ by

\[(1.12) \quad H(\mathbb{A}) \to \mathbb{C}^\times, (t_1, t_2) \mapsto \Omega(t_1)^{-1}\Omega(t_2).\]

From now on, we fix $\Omega$ a unitary character of $\Theta (\mathbf{E} \times \mathbb{A}_E^\times)^2$ with central character $\omega$ traverses cuspidal representations of $\mathbf{D}^\times$ and write

\[\Theta_D(\Omega, \psi) := \Theta_D(\Omega|_{\mathbf{E} \backslash \mathbb{A}_E^\times}, \psi).\]

For simplicity, we assume $\Omega \neq 1$ is not trivial.

**Proposition 1.9.** The integral defining $\Theta_D(s_0, \psi)$ is absolutely convergent for $\Re s_0 > 1$, and admits a meromorphic continuation to $s_0 \in \mathbb{C}$.

Similarly to Proposition 1.3, we will establish Proposition 1.9 with two different methods. They allow us to write $\Theta_D(s_0, \psi)$ for $|s_0| < 1/4$ as

\[M(s_0, \psi) + DS(s_0, \psi) \quad \text{resp.} \quad M_2(s_0, \psi) + \sum_{j=1}^{4} DG_j(s_0, \psi).\]

Every term is regular at $s_0 = 0$. $M(0, \psi)$ represents some mixed moment of $L(1/2, \pi)L(1/2, \pi_\mathbf{E} \otimes \Omega^{-1})$ as $\pi$ traverses cuspidal representations of $\mathbf{D}^\times$ with central character $\omega$. $M_2(0, \psi)$ represents the mean value of $L(1/2 - it, \Omega_{\Xi}^{-1})L(1/2 + it, \Xi)$ as $\Xi$ traverses the unitary characters of $[\mathbf{E}_1 \setminus \mathbf{E}^\times]$ and $\tau \in \mathbb{R}$.

**Theorem 1.10.** We get an equality of tempered distributions

\[M(0, \psi) + DS(0, \psi) = M_2(0, \psi) + \sum_{j=1}^{4} DG_j(0, \psi), \quad \forall \psi \in \mathcal{S}(\mathbf{D}(\mathbb{A})).\]

### 1.5. List of Tempered Distributions.

#### 1.5.1. Geometric Main Term.

We can identify $M_2(\mathbb{A})$ with $\mathbb{A}^4$ via

\[M_2(\mathbb{A}) \simeq \mathbb{A}^4, \quad \left(\begin{array}{llll}
x_1 & x_2 & x_3 & x_4 \\
x_3 & x_4 & x_1 & x_2
d\end{array}\right) \mapsto (x_1, x_2, x_3, x_4).\]

In particular, we transport $\mathcal{F}_j$, the partial Fourier transform on $\mathbb{A}^4$ with respect to the $j$-th variable, to $\mathcal{S}(M_2(\mathbb{A}))$. Tate’s integral (see (1.13)) for $\mathcal{S}(\mathbb{A})$ admits obvious multi-dimensional generalization. In particular, it generalizes to $\mathcal{S}(M_2(\mathbb{A}))$. Define (recall we have identified $\lambda$ with $(s_1, s_2, s_0) \in \mathbb{C}^3$)

\[\eta_1 = \omega^{-1}\lambda_1 \chi_2, \quad \eta_2 = \omega^{-1}\lambda_2, \quad \eta_3 = \omega^{-1}\lambda_3^{-1}, \quad \eta_4 = 1,
\]

\[s'_1 = s_2 - s_1, \quad s'_2 = s_2 - s_0, \quad s'_3 = -s_0 - s_1, \quad s'_4 = 0.\]

We form a tempered distribution for $\lambda$ near $\lambda$ using four dimensional Tate’s integrals (see (1.13))

\[(1.14) \quad M_4(\lambda, \psi) = \frac{1}{\sqrt{\pi}} \sum_{\xi \in \mathbb{F}} \int_{\mathbb{R}^n \setminus 1/2} \int_{(\mathbf{A}^\times)^n} \mathcal{F}_2 \mathcal{F}_3 \psi \left(\begin{array}{llll}x_1 & x_2 & x_3 & x_4 \\
x_3 & x_4 & x_1 & x_2
d\end{array}\right) \left(\prod_{i=1}^{4} \eta_i \chi(x_i)|x_i|^{s + s'_i}d^nx_i\right) \frac{ds}{2\pi i} .\]

which is absolutely convergent by the rapid decay recalled after (1.13). Note that the inner integral

\[(1.15) \quad M_4(\lambda, \psi | \chi, s) = \int_{(\mathbf{A}^\times)^n} \mathcal{F}_2 \mathcal{F}_3 \psi \left(\begin{array}{llll}x_1 & x_2 & x_3 & x_4 \\
x_3 & x_4 & x_1 & x_2
d\end{array}\right) \left(\prod_{i=1}^{4} \eta_i \chi(x_i)|x_i|^{s + s'_i}d^nx_i\right),\]
naturally defines a meromorphic function in $(\lambda, s) \in \mathbb{C}^d$.

1.5.2. **Spectral Main Term.** Let $\pi$ be a cuspidal representation in the (right) regular representation $R_\omega$ of $GL_2(\mathbb{A})$ on $V_\omega = L^2(GL_2, \omega)$ given by

$$L^2(GL_2, \omega) := \left\{ \varphi : GL_2(F) \backslash GL_2(\mathbb{A}) \to \mathbb{C} \bigg| \varphi(gz(u)) = \omega(u)\varphi(g), \quad \forall g \in GL_2(\mathbb{A}), u \in \mathbb{A}^\times \right\},$$

where we have written

$$[PGL_2] := Z(\mathbb{A})GL_2(F) \backslash GL_2(\mathbb{A}).$$

If $V_\omega$ denotes the underlying Hilbert space of $\pi$ with subspace $V_\omega^\infty$ of smooth vectors realized as smooth functions on $GL_2(\mathbb{A})$, then we have the Hecke-Jacquet-Langlands zeta integrals (for all $s \in \mathbb{C}$)

$$Z(s, \varphi, \chi) := \int_{F^\times \backslash \mathbb{A}^\times} \varphi(a(t))\chi(t)|t|_\mathbb{A}^{s-\frac{1}{2}}dt, \quad \forall \varphi \in V_\omega^\infty, \chi \in \mathbb{R}_+F^\times \backslash \mathbb{A}^\times.$$

Similarly, let $\chi$ be a unitary character of $\mathbb{R}_+F^\times \backslash \mathbb{A}^\times$, we can associate the principal series representation $\pi_s := \pi(\chi|\cdot|_\mathbb{A}^\times, \omega^{-1}|\cdot|_\mathbb{A}^{-s})$, whose underlying Hilbert space is

$$V_{\chi, \omega^{-1}} := \left\{ f : K \to \mathbb{C} \bigg| f \left( \begin{array}{cc} t_1 \\ 0 \\ t_2 \\ 0 \end{array} \right) g = \chi(t_1)\omega^{-1}(t_2)f(g) \right\}.$$

Note that this space is the common one for all $s \in \mathbb{C}$, and the subspaces of smooth vectors are identical for any $s$, although the actions $\pi_s$ differ as $s$ varies. In particular, $\pi_s$ is unitary for $s \in i\mathbb{R}$. To any smooth vector $e \in V_{\chi, \omega^{-1}}$, is associated a flat section $e(s)$ in $\pi_s$, from which we construct an Eisenstein series

$$E(s, e)(g) = E(e(s))(g) := \sum_{e \in \mathbb{B}(F)\backslash GL_2(F)} e(s)(\gamma g),$$

convergent for $\Re s \gg 1$ and admitting meromorphic continuation to $s \in \mathbb{C}$. Defining the constant term

$$E_N(s, e)(g) := \int_{F \backslash \mathbb{A}} E(s, e)(n(x)g)dx,$$

we have the extended Hecke-Jacquet-Langlands zeta integrals

$$Z(s', E(s, e), \eta) := \int_{F^\times \backslash \mathbb{A}^\times} (E - E_N)(s, e)(a(t))\eta(t)|t|_\mathbb{A}^{s'-\frac{1}{2}}dt, \quad \forall e \in V_{\chi, \omega^{-1}}, \eta \in \mathbb{R}_+F^\times \backslash \mathbb{A}^\times,$$

which is convergent for $\Re s' \gg 1$ and admits meromorphic continuation to $s' \in \mathbb{C}$.

For $V = V_\pi$ or $V_{\chi, \omega^{-1}}$, the underlying inner product $\langle \cdot, \cdot \rangle$ identifies $V$ with its dual space $V^\vee$. Define

$$\varphi^\vee := \overline{\varphi}/\|\varphi\|^2 \in V^\vee, \quad \forall 0 \neq \varphi \in V.$$

The **matrix coefficient** $\beta(e_2, e_1^\vee)$ or $\beta_s(e_2, e_1^\vee)$ associated with a pair of nonzero vectors $e_1, e_2 \in V$ is defined to be

$$\beta(e_2, e_1^\vee)(g) := \frac{\langle \pi(g)e_2, e_1 \rangle}{\|e_1\|^2} \quad \text{or} \quad \beta_s(e_2, e_1^\vee)(g) := \frac{\langle \pi_s(g)e_2, e_1 \rangle}{\|e_1\|^2}, \quad g \in GL_2(\mathbb{A}).$$

We have the Godement-Jacquet zeta integral for $\beta \in \{\beta(e_2, e_1^\vee), \beta_s(e_2, e_1^\vee)\}$

$$Z(s', \Psi, \beta) := \int_{GL_2(\mathbb{A})} \Psi(g)\beta(g)|\det g|_\mathbb{A}^{s'+\frac{1}{2}}dg,$$

which is convergent for $\Re s' \gg 1$ and admits a meromorphic continuation to $s' \in \mathbb{C}$. 

Let $\mathcal{B} = B(\pi)$ resp. $B(\chi, \omega \chi^{-1})$ be an orthogonal basis of $V_\pi$ resp. $V_{\chi, \omega \chi^{-1}}$. Then $B^\vee := \{ \varphi^\vee | \varphi \in \mathcal{B} \}$ is the dual basis of $\mathcal{B}$ in $V^\vee$. Define the \textit{(cuspidal resp. continuous) spectral Motohashi distributions}

$$M_3(\lambda, \Psi | \pi) := \sum_{\varphi_1, \varphi_2 \in B(\pi)} Z \left( \frac{s_0 + 1}{2}, \Psi, \beta(\varphi_2, \varphi_1^\vee) \right),$$

$$Z \left( s_1 + \frac{s_0 + 1}{2}, \varphi_1, \chi_1 \right) \cdot Z \left( s_2 - \frac{s_0 - 1}{2}, \varphi_2^\vee, \chi_2 \right),$$

$$M_3(\lambda, \Psi | \chi, s) := \sum_{e_1, e_2 \in \mathcal{B}(\chi, \omega \chi^{-1})} Z \left( \frac{s_0 + 1}{2}, \Psi, \beta_s(e_2, e_1^\vee) \right),$$

$$Z \left( s_1 + \frac{s_0 + 1}{2}, E(s, e_1), \chi_1 \right) \cdot Z \left( s_2 - \frac{s_0 - 1}{2}, E(-s, e_2^\vee), \chi_2 \right),$$

which are Motohashi distributions of parameter $\lambda = (s_1, s_2, s_0)$ and are meromorphic in $(\lambda, s) \in \mathbb{C}^4$. $M_3(\bar{\theta}, \Psi)$ is defined for $\lambda$ near $\bar{\theta}$ via

$$M_3(\lambda, \Psi) := \sum_{\pi \text{ cuspidal } \omega = \omega_0} M_3(\lambda, \Psi | \pi) + \sum_{\chi \in \mathbb{R}_+ \times \mathbb{A}^\times} \int_{-\infty}^{\infty} M_3(\lambda, \Psi | \chi, i \tau) \, d\tau \div 4\pi.$$

$1.5.3. \text{Geometric Degenerate Terms.}$ Using Tate’s integral, we define eight Motohashi distributions of parameter $\lambda = (s_1, s_2, s_0)$ near $\bar{\theta}$ as

$$DG_1(\lambda, \Psi) := \frac{1}{\zeta_F} \operatorname{Res}_{s = 1 - s_1} M_4(\lambda, \Psi | \chi, s) = \int_{(\mathbb{A}^\times)^3} \mathcal{F}_0 \mathcal{F}_3 \mathcal{F}_4 \left( \frac{0}{x_3}, \frac{x_2}{x_4} \right)$$

$$\omega_1 \chi_1^2 \chi_2^2(x_2) | x_2 |^{s_1 + 1} \chi_2^{-1}(x_3) | x_3 |^{1 - s_2} \omega \chi_1 \chi_2^{-1}(x_4) | x_4 |^{1 + s_0 + s_1 - s_2} \prod_{i \neq 1} d^x x_i,$$

$$DG_2(\lambda, \Psi) := \frac{1}{\zeta_F} \operatorname{Res}_{s = 1 - s_2} M_4(\lambda, \Psi | \chi, s) = \int_{(\mathbb{A}^\times)^3} \mathcal{F}_3 \mathcal{F}_4 \left( \frac{x_1}{x_3}, \frac{0}{x_4} \right)$$

$$\omega_1 \chi_1^{-2} \chi_2^{-2}(x_1) | x_1 |^{1 - s_1} \omega^{-1} \chi_1 \chi_2(x_3) | x_3 |^{1 - s_1 - s_2} \chi_1^{-1} \chi_2^{-1}(x_4) | x_4 |^{1 + s_0 - s_2 + 1} \prod_{i \neq 2} d^x x_i,$$

$$DG_3(\lambda, \Psi) := \frac{1}{\zeta_F} \operatorname{Res}_{s = 1 - s_3} M_4(\lambda, \Psi | \chi, s) = \int_{(\mathbb{A}^\times)^3} \mathcal{F}_3 \mathcal{F}_4 \left( \frac{x_1}{x_3}, \frac{x_2}{x_4} \right)$$

$$\chi_2^{x_1} | x_1 |^{1 + s_1 + 1} \omega \chi_1 \chi_2^{-1}(x_2) | x_2 |^{1 + s_1 + s_2} \omega \chi_1^{-1}(x_4) | x_4 |^{s_0 + s_1 + 1} \prod_{i \neq 3} d^x x_i,$$

$$DG_4(\lambda, \Psi) := \frac{1}{\zeta_F} \operatorname{Res}_{s = 1 - s_4} M_4(\lambda, \Psi | \chi, s) = \int_{(\mathbb{A}^\times)^3} \mathcal{F}_2 \mathcal{F}_3 \mathcal{F}_4 \left( \frac{0}{x_3}, \frac{x_2}{x_4} \right)$$

$$\omega_1 \chi_1^{-1} \chi_2(x_1) | x_1 |^{1 + s_2 - s_0 - s_1} \chi_1 \chi_2^{-1}(x_2) | x_2 |^{1 + s_2 - s_0 - s_1} \omega^{-1} \chi_1^{-1}(x_3) | x_3 |^{1 - s_0 - s_1} \prod_{i \neq 4} d^x x_i,$$

$$DG_5(\lambda, \Psi) := \frac{1}{\zeta_F} \operatorname{Res}_{s = 1 - s_4} M_4(\lambda, \Psi | \chi, s) = \int_{(\mathbb{A}^\times)^3} \mathcal{F}_2 \mathcal{F}_3 \mathcal{F}_4 \left( \frac{x_1}{x_3}, \frac{0}{x_4} \right)$$

$$\chi_1(x_2) | x_2 |^{s_4} \chi_2(x_3) | x_3 |^{s_3 + 1} \omega_1 \chi_1 \chi_2^{-1}(x_4) | x_4 |^{2 - s_0 - s_1} \prod_{i \neq 1} d^x x_i,$$

$$DG_6(\lambda, \Psi) := \frac{1}{\zeta_F} \operatorname{Res}_{s = 1 - s_4} M_4(\lambda, \Psi | \chi, s) = \int_{(\mathbb{A}^\times)^3} \mathcal{F}_2 \mathcal{F}_3 \mathcal{F}_4 \left( \frac{x_1}{x_3}, \frac{x_2}{x_4} \right)$$

$$\chi_1(x_1) | x_1 |^{s_3 + 1} \omega \chi_2(x_3) | x_3 |^{s_3 + s_1 + 1} \omega^{-1} \chi_2(x_4) | x_4 |^{s_0 - s_0 + 1} \prod_{i \neq 2} d^x x_i,$$
Let \( \pi \) be any cuspidal representation of \( \mathbb{A} \), \( \nu_\mathbb{D} \) be the reduced norm map of \( \mathbb{D} \), whose restriction to \( \mathbb{E} \) is the usual norm map \( \nu_\mathbb{E} \) from \( \mathbb{E} \) to \( \mathbb{F} \). We introduce the (compact) automorphic quotient spaces as

\[
\mathbb{D}^\times := \mathbb{D}^\times(\mathbb{F})\mathbb{A}^\times(\mathbb{A}), \quad \mathbb{E}^\times := \mathbb{E}^\times(\mathbb{F})\mathbb{A}^\times(\mathbb{A}).
\]

Let \( \pi \) be any cuspidal representation of \( \mathbb{D}^\times \) with central character \( \omega \). Let \( \eta \) be any Hecke character of \( \mathbb{F}^\times \mathbb{R}_+ \backslash \mathbb{A}^\times \). We introduce

\[
M(s_0, \Psi | \eta) := \sum_{\varphi_1, \varphi_2 \in \hat{\kappa}(\pi)} Z \left( s_0 + \frac{1}{2}, \Psi, \beta(\varphi_2, \varphi_1^* \sigma) \right) \int_{[\mathbb{E}^\times]} \varphi_1(t_1) \Omega(t_1)^{-1} d^\times t_1 \int_{[\mathbb{E}^\times]} \varphi_2(t_2) \Omega(t_2) d^\times t_2,
\]

\[
DS(s_0, \Psi | \eta) := \left( \int_{\mathbb{D}^\times(\mathbb{A})} \Psi(g) \eta(\nu_\mathbb{D}(g)) \nu_\mathbb{D}(g)^{s_0+1} dg \right) \cdot 1_{\Omega = \eta \nu_\mathbb{E}}.
\]

The integrals in \( DS(s_0, \Psi | \eta) \) resp. \( M(s_0, \Psi | \pi) \) with parameter \( s_0 \) have meromorphic continuation to \( s_0 \in \mathbb{C} \), and are integral representations for \( L(s_0 + 1/2, \eta) \) resp. \( L(s_0 + 1/2, \pi) \) (see [3, Proposition 6.9 & Theorem 13.8]). By Waldspurger’s famous formula [22], the product of integrals over \( \mathbb{E}^\times \) is an integral representation for \( L(1/2, \pi \mathbb{E} \otimes \Omega^{-1}) \), where \( \pi_\mathbb{E} \) is the base change lift of \( \pi \) to \( (\mathbb{D} \otimes \mathbb{F})^\times \mathbb{A} \approx \text{GL}_2(\mathbb{A}_\mathbb{E}) \). We define the tempered distributions

\[
M(s_0, \Psi | \pi) := \sum_{\omega \in \pi \text{ cuspidal}} M(s_0, \Psi | \pi), \quad DS(s_0, \Psi) := \sum_{\eta} DS(s_0, \Psi | \eta).
\]

Their value at \( s_0 = 0 \) represent some mixed moment of \( L(1/2, \pi)L(1/2, \pi_\mathbb{E} \otimes \Omega^{-1}) \) resp. first moment of \( L(1/2, \eta) \) with \( \eta \circ \nu_\mathbb{D} \) distinguished by \( (\mathbb{E}^\times, \Omega^{-1}) \).
1.6.2. Geometric Terms. By the structure theory (see Definition 1.1) there exists \( j \in D(F) \) so that \( D = E \oplus E_j, j^2 \in F^\times \) and \( jxj^{-1} = \bar{x} \) for any \( x \in E \), where \( x \mapsto \bar{x} \) is the action of the non trivial element in the Galois group of \( E/F \). Consequently, we have \( D(A) \simeq E(A)^2 \), hence \( S(D(A)) \simeq S(E(A)^2) \). Under this identification, we can view \( \Psi \in S(E^2) \) by writing \( \Psi(x_1 + x_2j) \) for \( x_j \in A_E \). We denote by \( \hat{\delta}_j \) the partial Fourier transform with respect to the variable \( x_j \) for the additive character \( \psi \circ \text{Tr}_{E/F} \). Let the \( F \)-group \( E^1 \) be the subgroup of elements \( t \in E^\times \) such that \( tf \equiv 1 \). Define a compact group

\[
[E^1 \backslash E^\times] := E^\times \mathbb{R}_+ E^1(\hat{A}) \backslash A_E^\times,
\]

where \( \mathbb{R}_+ \) is viewed as the image of a fixed section map \( s_E \) of the adelic norm map \( E^\times \backslash A_E^\times \to \mathbb{R}_+ \).

For \( |s_0| < 1/4 \), we define for any \( \Xi \in [E^1 \backslash E^\times] \)

\[
M_2(s_0, \Psi \mid \Xi) := \frac{1}{\text{Vol}([E^1 \backslash E^\times])} \int_{[A_E^1 \backslash A_E^\times]} \Psi(t_1 + t_2j) \Omega_{\Xi}^{-1}(t_1)|t_1|_{A_E}^{s_0 + 1} \Xi(t_2)|t_2|_{A_E}^{s_1} d^x t_1 d^x t_2 \frac{ds \Omega_{\Xi}}{2\pi i},
\]

\[
DG_1(s_0, \Psi) = \text{Vol}([E^1 \backslash E^\times]) \int_{[A_E^1 \backslash A_E^\times]} \Psi(t_1 + t_2j) \Omega(t_1)|t_1|_{A_E}^{s_0 + 1} d^x t_1,
\]

\[
DG_2(s_0, \Psi) = \text{Vol}([E^1 \backslash E^\times]) \int_{[A_E^1 \backslash A_E^\times]} \Psi(t_1 + t_2j) \Omega(t_1)|t_1|_{A_E}^{s_0} d^x t_1,
\]

\[
DG_3(s_0, \Psi) = \frac{\zeta_E}{\text{Vol}([E^1 \backslash E^\times])} \int_{[A_E^1 \backslash A_E^\times]} \left( \int_{[A_E^1 \backslash A_E^\times]} \Psi(t_1 + t_2j) d^x t_1 \right) \Omega(t_1)|t_1|_{A_E}^{s_0} d^x t_2,
\]

\[
DG_4(s_0, \Psi) = \frac{\zeta_E}{\text{Vol}([E^1 \backslash E^\times])} \int_{[A_E^1 \backslash A_E^\times]} \left( \int_{[A_E^1 \backslash A_E^\times]} \Psi(t_1 + t_2j) d^x t_1 \right) \Omega(t_2)|t_2|_{A_E}^{s_1} d^x t_2.
\]

The geometric main term is defined to be

\[
M_2(s_0, \Psi) := \sum_{\Xi \in [E^1 \backslash E^\times]} M_2(s_0, \Psi \mid \Xi),
\]

whose value at \( s_0 = 0 \) represents the mean value of \( L(1/2 + it, \Omega^{-1}) L(1/2 + it, \Xi) \). The degenerate terms \( DG_j(s_0, \Psi) \) have obvious meromorphic continuation to \( s_0 \in \mathbb{C} \) and are regular at \( s_0 \) under the assumption \( \Omega \neq 1 \).

2. Third Moment (Spectral) Side

2.1. Some Spectral Theory. Recall the representation \( (\pi, V) \) of \( GL_2(\mathbb{A}) \) given by (1.10).

**Definition 2.1.** Let \( \varphi \in V^\infty \) be a smooth vector represented by a smooth function on \( GL_2(\mathbb{A}) \), so that for any \( X \) in the universal enveloping algebra of the Lie algebra of \( GL_2(\mathbb{A}_\infty) \), we have

\[
|\text{Re}(X) \varphi(g)| \ll H(t)^{1/2 - \epsilon}
\]

for any \( \epsilon > 0 \) sufficiently small, uniformly in \( g \) lying in any Siegel domain, then we call \( \varphi \) “nice”.

**Remark 2.2.** \( V^\infty \) is the Sobolev space of order \( \infty \) for the \( L^2 \)-norm.

**Theorem 2.3.** (automorphic Fourier inversion formula) For “nice” \( \varphi \in V^\infty \), we have a decomposition

\[
\varphi(g) = \sum_{\psi \in \text{cusp}_\omega} \sum_{\psi \in \text{B}(\pi)} \langle \varphi, \psi \rangle \psi(g)
\]

\[
+ \sum_{\chi \in \mathbb{F}_\chi \backslash \mathbb{A}_\chi} \sum_{f \in \mathbb{B}(\chi, \omega \chi^{-1})} \int_{\mathbb{R}^\times} \langle \varphi, E(i\tau, f) \rangle E(i\tau, f)(g) d\tau
\]

\[
+ \frac{1}{\text{Vol}([PGL_2])} \sum_{\chi \in \mathbb{F}_\chi \backslash \mathbb{A}_\chi} \int_{[PGL_2]} \varphi(x)(\det x) dx \cdot \chi(\det g)
\]

with normal convergence in \([PGL_2]\). Here, \( \text{B}(\pi) \) resp. \( \mathbb{F}(\chi, \omega \chi^{-1}) \) is an orthonormal basis of \( K_{\infty}-\text{isotypic} \) and \( K_{\text{fin}} \)-finite vectors of \( V \) resp. \( V_{\chi, \omega \chi^{-1}} \) and the automorphic Fourier coefficients, namely the above terms written via inner product such as \( \langle \varphi, E(i\tau, f) \rangle \), are given by the usual convergent integrals.
Proof. For general \( \varphi \in V_c^\infty \), the above formula is established in [24, Theorem 2.16] with \( \langle \varphi, E(\tau, f) \rangle \) replaced by the existence of some \( a(\tau, f) \in \mathbb{C} \) (see the clarification in [25]). The justification of \[ a(\tau, f) = \langle \varphi, E(\tau, e) \rangle \quad \text{a.e. } \tau \in \mathbb{R} \]
literally follows the proof of [25, Lemma 3.24], replacing \( h \in C_c^\infty(\text{GL}_2, \omega) \) there by “nice” \( \varphi \) here. \qed

Remark 2.4. For a clarification of the terminology “automorphic Fourier inversion” as well as its relation with “spectral decomposition”, please see [25, §1.2 & 1.3] and [14, §2.1].

Theorem 2.5. Let notations be as in the previous theorem. Write
\[ \varphi_n(g) := \int_{F \setminus \mathbb{A}} \varphi(n(u)g) du, \]
\[ W_e(g) := \int_{F \setminus \mathbb{A}} e(n(u)g) \psi(-u) du \quad \text{resp.} \quad W(\tau, f)(g) := \int_{F \setminus \mathbb{A}} E(\tau, f)(n(u)g) \psi(-u) du \]
for the Whittaker function of \( e \) resp. \( E(\tau, f) \). Then we have an equality as functions on \( B(F) \setminus \text{GL}_2(\mathbb{A}) \)
\[ \varphi(g) - \varphi_n(g) = \sum_{\tau \in \text{cuspidal}} \sum_{e \in B(\tau)} \langle \varphi, e \rangle \sum_{\alpha \in F^\times} W_e(e(a(\alpha)g)) \]
\[ + \sum_{\chi \in \mathbb{R}, F^\times \setminus \mathbb{A}^\times} \sum_{f \in B(\chi, \omega^{-1})} \int_{-\infty}^\infty \langle \varphi, E(\tau, f) \rangle \sum_{\alpha \in F^\times} W(\tau, f)(a(\alpha)g) \frac{4\pi du}{d\tau} \]
with absolute and uniform convergence in any Siegel domain. Moreover, the dominating sum by replacing each coefficient \( \ell \) summand \( \ell \) integrand with its absolute value is of rapid decay with respect to \( \text{Ht}(g) \).

Proof. This is [24, Theorem 2.18]. Note that the “moreover” part is implicit in the proof. \qed

Definition 2.6. Let \( \omega \) be a unitary character of \( F^\times \setminus \mathbb{A}^\times \). Let \( \varphi \) be a continuous function on \( \text{GL}_2(F) \setminus \text{GL}_2(\mathbb{A}) \) with central character \( \omega \). We call \( \varphi \) finitely regularizable if there exist characters \( \chi_i : F^\times \setminus \mathbb{A}^\times \rightarrow \mathbb{C}^\times \), \( \alpha_i \in \mathbb{C}, n_i \in \mathbb{N} \) and continuous functions \( f_i \in \text{Ind}_{B(\mathbb{A})}^{K} \chi_i, \omega^{-1} \), such that for any \( M \gg 1 \)
\[ \varphi(n(x)a(y)k) = \varphi_n(n(x)a(y)k) + O(|y|_A^{-M}) , \text{ as } |y|_A \rightarrow \infty, \]
where we have written the essential constant term
\[ \varphi_n(n(x)a(y)k) = \varphi_n(a(y)k) = \sum_{i=1}^l \chi_i(y)|y|_A^{-\alpha_i} \log^{n_i}|y|_A f_i(k). \]
We call \( \mathcal{E}(\varphi) = \{ |\chi_i|_A^{1+\alpha_i} : 1 \leq i \leq l \} \) the exponent set of \( \varphi \), and define
\[ \mathcal{E}(\varphi) = \{ |\chi_i|_A^{1+\alpha_i} \in \mathcal{E}(\varphi) : \Re \alpha_i \geq 0 \} \quad \text{and} \quad \mathcal{E}(\varphi) = \{ |\chi_i|_A^{1+\alpha_i} \in \mathcal{E}(\varphi) : \Re \alpha_i < 0 \} \]
The space of finitely regularizable functions with central character \( \omega \) is denoted by \( A^R(\text{GL}_2, \omega) \).

Proposition 2.7. Let \( \xi_1, \xi_2, \omega \) be Hecke characters with \( \xi_1 \xi_2 \omega = 1 \). Let \( f \in \pi(\xi_1, \xi_2) \). Suppose \( \varphi \in A^R(\text{GL}_2, \omega) \) as in Definition 2.4. For \( \Re s \gg 1 \) sufficiently large,
\[ R(s, \varphi; f) := \int_{F^\times \setminus \mathbb{A}^\times} \left( \varphi_N - \varphi_n \right)(a(y)k) f(\kappa) \xi_1(y)|y|_A^{\frac{1}{2} - \frac{\Theta}{2}} dk dy \]
is absolutely convergent. It has a meromorphic continuation to \( s \in \mathbb{C} \). If in addition
\[ \Theta := \max_j \{|\Re \alpha_j| \} < 0, \]
then we have, with the right hand side absolutely converging
\[ R(s, \varphi; f) = \int_{[\pi, \text{GL}_2]} \varphi \cdot E(s, f), \quad \Theta < \Re s < -\Theta. \]

Proof. This is (part of) [26, Proposition 2.5]. \qed
2.2. Godement-Jacquet Pre-trace Formula. Above all, we mention the following elementary estimation without proof (see \[27\], Lemma 5.37 or \[9\], Lemma 11.4).

**Proposition 2.8.** Let \( \Phi \in \mathcal{S}(\mathbb{A}) \). Then we have for any \( N > 1 \)
\[
\sum_{\alpha \in F^\times} |\Phi(\alpha t)| \ll_N \min(\|t\|_{\mathbb{A}}^{-1}, |t|_{\mathbb{A}}^{-N}),
\]
where the implied constant depends only on \( F \) and some Schwartz norm of \( \Phi \) with order depending on \( N \).

For fixed \( x \), we propose to study the function in \( y \)
\[
KK \left( \frac{x^2 y}{\omega, s_0} \right) := \int_{F^\times \setminus \mathbb{A}^\times} KK \left( x; yz(u) \right) \frac{u \|s_0+2d_z \|_{\mathbb{A}}}{\omega(u)} du,
\]
or equivalently \( \widetilde{KK} \left( \frac{x^2 y}{\omega, s_0} \right) := KK \left( \frac{x^2 y}{\omega, s_0} \right) |\det x^{-1} y|_{\mathbb{A}}^{\frac{2}{n}+1} \).

**Remark 2.9.** The partition of \( M_2(F) - \{0\} \) into \( GL_2(F) \times GL_2(F) \)-orbits
\[
M_2(F) - \{0\} = \bigsqcup_{i=1}^{2} M_2(i)(F) \quad \text{with} \quad M_2(i)(F) := \{ \xi \in M_2(F) \mid \text{rank}(\xi) = i \}.
\]
implies a decomposition
\[
KK(x; y) = \sum_{i=1}^{2} KK^i(x; y), \quad KK^i(x; y) := \sum_{\xi \in M_2(i)(F)} \Psi(x^{-1} \xi y).
\]
They give the corresponding functions \( KK^i \left( \frac{x^2 y}{\omega, s_0} \right), \widetilde{KK}^i \left( \frac{x^2 y}{\omega, s_0} \right) \) and \( \widetilde{\Delta}^i \left( \frac{x^2 y}{\omega, s_0} \right), \) etc.

We would like to show that \( y \mapsto KK^i \left( \frac{x^2 y}{\omega, s_0} \right) \) lie in \( \mathcal{A}^{fr}(GL_2, \omega^{-1}) \). This is obvious for \( i = 1 \). In fact, if we define
\[
RR^1(x; y) := \sum_{\alpha \in F^\times} \int_{\mathbb{A}^\times} R_{\gamma} \Psi \left( \begin{array}{cc} 0 & \alpha \\ 0 & 0 \end{array} \right) y \omega(z) \|s_0+2d_z \|_{\mathbb{A}} \|z\|_{\mathbb{A}} \|y\|_{\mathbb{A}}^{\frac{2}{n}+1},
\]
then it is easy to verify that
\[
x \mapsto \widetilde{RR}^1 \left( \frac{x^2 y}{\omega, s_0} \right) \quad \text{resp.} \quad y \mapsto \widetilde{RR}^1 \left( \frac{x^2 y}{\omega, s_0} \right)
\]
lies in the induced model of the principal series representation \( \pi(\omega; |s_0+1/2|, \omega^{-1}; |s_0+1/2|) \) resp.
\( \pi(\omega; |s_0+1/2|, \omega^{-1}; |s_0+1/2|) \). Note that \( M_2^1(F) \) is a single orbit under the action of \( GL_2(F) \times GL_2(F) \), and the stabilizer group of the line consisting of the elements
\[
\left( \begin{array}{cc} 0 & \alpha \\ 0 & 0 \end{array} \right), \quad \alpha \in F^\times
\]
is \( B(F) \times B(F) \). Therefore we get (with absolute convergence for \( \Re s_0 > 1 \))
\[
\widetilde{KK}^1 \left( \frac{x^2 y}{\omega, s_0} \right) = \sum_{[\gamma_1], [\gamma_2] \in C(F) \setminus GL_2(F)} \widetilde{RR}^1 \left( \frac{x^2 y}{\omega, s_0} \right)
\]
is an Eisenstein series in either variable, hence lies in \( \mathcal{A}^{fr}(GL_2, \omega^{-1}) \).

It remains to show that \( y \mapsto KK \left( \frac{x^2 y}{\omega, s_0} \right) \) lie in \( \mathcal{A}^{fr}(GL_2, \omega^{-1}) \). We assume
\[
y = n(u) a(t) \kappa, \quad u \in F \setminus A, t \in F^\times \setminus A^\times, \kappa \in K
\]
For any $\Psi \in \mathcal{S}$, hence we can rewrite

\[ KK(x; y) = KN(x; y) + \sum_{\delta \in F^\times} KW(x; a(\delta)y), \]

where $KN(x; y)$ is given in (1.8) and

\[ KW(x; y) := \int_{F^\times} KK(x; n(u)y) \psi(-u) du. \]

(2.5)

We first deal with the sum over $\delta \in F^\times$. The orbital decomposition of $M_2^*(F) := M_2^{(1)}(F) \cup M_2^{(2)}(F)$ by right multiplication by $N(F)$ (or $B(F)$)

\[ M_2^*(F) = \bigsqcup_{\alpha, \beta \in F^\times} \left( \begin{array}{cc} \alpha & \gamma \\ \beta & 0 \end{array} \right) N(F) \bigsqcup \bigsqcup_{\gamma \in F} \left( \begin{array}{cc} 0 & \gamma \\ \alpha & 0 \end{array} \right) N(F) \bigsqcup \bigsqcup_{\gamma \in F} \left( \begin{array}{cc} 0 & \alpha \\ \gamma & 0 \end{array} \right) N(F) \]

implies the following decomposition

\[ KW(x; y) = \sum_{i=1}^{5} KW_i(x; y), \quad KW_i(x; y) := \int_{F^\times} \sum_{\xi \in O_i} \Psi(x^{-1} \xi n(u)y) \psi(-u) du. \]

For the first term, we have

\[ KW_1(x; n(u)a(t)\kappa) = \psi(u) \int_{\mathbb{A}} \sum_{\alpha, \beta \in F^\times} L_d(\alpha, \beta)^{-1} L_x R_n \Psi(t \left( \begin{array}{cc} \gamma + v \\ v \end{array} \right) ) \psi(-v) dv. \]

For any $\Psi \in \mathcal{S}(M_2(\mathbb{A}))$, Poisson summation formula implies

\[ \int_{\mathbb{A}} \sum_{\gamma \in F} \Psi(t \left( \begin{array}{cc} \gamma + v \\ v \end{array} \right) ) \psi(-v) dv = \sum_{\gamma \in F} \mathcal{F}_4 \mathcal{F}_2^\times \Psi(t \left( \begin{array}{cc} \gamma \\ 1 - \gamma \end{array} \right) ). \]

Hence we can rewrite

\[ KW_1(x; n(u)a(t)\kappa d(z, z)) = \frac{\psi(u)}{|z|^2} \sum_{\alpha, \beta \in F^\times} \mathcal{F}_4 \mathcal{F}_2^\times \Psi \left( \begin{array}{cc} \alpha t z \\ \beta t z \end{array} \right) \frac{\gamma \alpha^{-1} z^{-1}}{(1 - \gamma) \beta^{-1} z^{-1}}, \]

from which we deduce

\[ \sum_{\delta \in F^\times} |KW_1(x; a(\delta)n(u)a(t)\kappa d(z, z))| \leq |z|^{-2} \sum_{\alpha, \beta, \delta \in F^\times} \left| \mathcal{F}_4 \mathcal{F}_2^\times \Psi \left( \begin{array}{cc} \alpha t z \\ \beta t z \end{array} \right) \gamma \delta^{-1} \alpha^{-1} z^{-1} \right| \leq |z|^{-2} \sum_{\alpha, \beta \in F^\times} \sum_{\gamma, \delta \in F \setminus \{0\}} \left| \mathcal{F}_4 \mathcal{F}_2^\times \Psi \left( \begin{array}{cc} \alpha t z \\ \beta t z \end{array} \right) \gamma \delta^{-1} \alpha^{-1} z^{-1} \right|. \]

We split the sum over $\gamma, \delta$ as

\[ \sum_{\gamma, \delta \in F \setminus \{0\}} = \sum_{\gamma + \delta \neq 0} + \sum_{\gamma + \delta = 0}. \]

By Proposition 2.8, the first part is bounded as

\[ \ll N_1 N_2 |z|^{-2} \min(|t z|^{-1}, |t z| N_1) \min(|z|, |z|^N_2); \]

similarly the second part is dominated and bounded as

\[ \ll N_1 N_2 |z|^{-2} \min(|t z|^{-1}, |t z|^N_1) \min(|z|, |z| N_2). \]
We deduce that for any $N_1, N_2 > 1$

$$\sum_{\delta \in \mathbb{F}^x} |KW_1(x; a(\delta)n(u)a(t)\kappa d(z, z))| \ll_{N_1, N_2} |z|^{-2} \min(|tz|_\lambda^{-1}, |tz|_\lambda^{-N_1}) \min(|z|_\lambda, |z|_\lambda^{N_2}),$$

where the implied constant depends only on $\mathbb{F}$ and some Schwartz norm of $\Psi$ with order depending only on $x, N_1$ and $N_2$ (not on $u$ nor $\kappa$).

The treatment of the second and third terms being similar, we only deal with the second one. We have

$$KW_2(x; n(u)a(t)\kappa) = \psi(u) \int_\Lambda \sum_{\beta \in \mathbb{F}^x} \Psi \left( x^{-1} \begin{pmatrix} 0 & 0 & 0 & t \nu \\ 0 & 1 & 0 & 1 \kappa \end{pmatrix} \right) \psi(-v) dv$$

$$= \psi(u) \sum_{\beta \in \mathbb{F}^x} \tilde{\Phi}_4 L_x R_\kappa \Psi \left( \begin{pmatrix} 0 & 0 \\ \beta t & \beta^{-1} \end{pmatrix} \right).$$

It follows that for any $N_1, N_2 > 1$

$$\sum_{\delta \in \mathbb{F}^x} |KW_2(x; a(\delta)n(u)a(t)\kappa d(z, z))| \ll_{N_1, N_2} |z|^{-1} \min(|tz|_\lambda^{-1}, |tz|_\lambda^{-N_1}) \min(|z|_\lambda, |z|_\lambda^{N_2}) \{1 + |z|^{-1}\}.$$  \hspace{1cm} (2.6)

We leave it as a simple exercise to check $KW_4(x; y) = 0$.

We summarize the above estimation as: for any $N_1, N_2 > 1$

$$\sum_{\delta \in \mathbb{F}^x} |KW_4(x; a(\delta)n(u)a(t)\kappa d(z, z))| \ll_{N_1, N_2} \min(|tz|_\lambda^{-1}, |tz|_\lambda^{-N_1}) \min(|z|_\lambda, |z|_\lambda^{N_2}) = 0.$$

For $KN(x; y)$, the same classification of orbits implies the decomposition

$$KN(x; y) = \sum_{i=1}^{4} KN_i(x; y), \hspace{1cm} KN_i(x; y) := \int_{\mathbb{F}^x \setminus \xi \in \mathcal{O}_i} \Psi(x^{-1} \xi n(u)y) du.$$  \hspace{1cm} (2.7)

For the first term, we have

$$KN_1(x; y) = \sum_{\alpha, \beta \in \mathbb{F}^x} \int_\Lambda L_x R_y \Psi \left( \begin{pmatrix} \alpha & \gamma \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & u \\ 0 & 1 \end{pmatrix} \right) du$$

$$= \sum_{\alpha, \beta \in \mathbb{F}^x} \int_\Lambda L_d(\alpha, \beta)^{-1} L_x R_y \Psi \left( \begin{pmatrix} 1 & u + \gamma \\ 0 & u \end{pmatrix} \right) du$$

$$= \sum_{\alpha, \beta \in \mathbb{F}^x} \int_\Lambda \sum_{\gamma \in \mathbb{F}} \tilde{\Phi}_2 L_d(\alpha, \beta)^{-1} L_x R_y \Psi \left( \begin{pmatrix} 1 & \gamma \\ 0 & u \end{pmatrix} \right) \psi(\gamma u) du$$

$$= \sum_{\alpha, \beta \in \mathbb{F}^x} \tilde{\Phi}_4 \tilde{\Phi}_2 L_d(\alpha, \beta)^{-1} L_x R_y \Psi \left( \begin{pmatrix} 1 & \gamma \\ 0 & -\gamma \end{pmatrix} \right).$$

We distinguish the terms for which $\gamma \neq 0$ from $\gamma = 0$, splitting the above sum as

$$\sum_{\alpha, \beta \in \mathbb{F}^x} \sum_{\gamma \in \mathbb{F}} + \sum_{\alpha, \beta \in \mathbb{F}^x} \sum_{\gamma = 0 \cdot \mathbb{F}}.$$

Denote the first resp. second part by $KN_{1,1}(x; y)$ resp. $KN_{1,2}(x; y)$. For the first part, we have the bound for any $N_1, N_2 > 1$

$$|KN_{1,1}(x; n(u)a(t)\kappa d(z, z))| = |z|^{-2} \sum_{\alpha, \beta \in \mathbb{F}^x} \left| \tilde{\Phi}_4 \tilde{\Phi}_2 L_x R_\kappa \Psi \left( \begin{pmatrix} \alpha t z & \gamma (\alpha z)^{-1} \\ \beta t z & -\gamma (\beta z)^{-1} \end{pmatrix} \right) \right|$$

$$\ll_{N_1, N_2} |z|^{-2} \sum_{\alpha, \beta \in \mathbb{F}^x} \left| \tilde{\Phi}_4 \tilde{\Phi}_2 L_x R_\kappa \Psi \left( \begin{pmatrix} \alpha t z & \gamma (\alpha z)^{-1} \\ \beta t z & \delta (\beta z)^{-1} \end{pmatrix} \right) \right| \ll_{N_1, N_2} \min(|tz|_\lambda^{-1}, |tz|_\lambda^{-N_1}) \min(|z|_\lambda, |z|_\lambda^{N_2}).$$
For the second term, we have for all $\Re s_0 > 2$

$$K_{N1,2} \left( \frac{x; n(u)a(t)\kappa}{\omega, s_0} \right) := \int_{\mathbb{F}_2} K_{N1,2} (x; n(u)a(t)\kappa(d(z, z)) \omega(z) |z|^{s_0+2} d^x z$$

(2.8)

$$= \omega(t)^{-1} |t|^{-s_0}_{\mathbb{A}} \int_{\mathbb{F}_2} \left( \sum_{\alpha, \beta \in \mathbb{F}_2} \mathfrak{f}_2 L_2 R_\alpha \Psi \left( \frac{\alpha z}{\beta z} \right) \right) \omega(z) |z|^{s_0} d^x z.$$  

For the second term, we have by Poisson summation

$$K_{N2} (x; y) = \sum_{\beta \in \mathbb{F}_2} \mathfrak{f}_2 L_2 R_\beta \Psi \left( \frac{0}{\beta} \right) = K_{N2,1} (x; y) + K_{N2,2} (x; y),$$

where

$$K_{N2,1} (x; y) := \sum_{\beta, \gamma \in \mathbb{F}_2} \mathfrak{f}_2 L_2 R_\gamma \Psi \left( \frac{0}{\beta} \right),$$

$$K_{N2,2} (x; y) := \sum_{\beta \in \mathbb{F}_2} \mathfrak{f}_2 L_2 R_\gamma \Psi \left( \frac{0}{\beta} \right).$$

It follows easily that

$$K_{N2,1} (x; n(u)a(t)\kappa(d(z, z)) = |z|^{-2}_{\mathbb{A}} \sum_{\beta, \gamma \in \mathbb{F}_2} \mathfrak{f}_2 L_2 R_\gamma \Psi \left( \frac{0}{\beta} \right)$$

(2.9)

$$\ll_{N1, N2} |z|^{-2}_{\mathbb{A}} \min(|tz|^{-1}_{\mathbb{A}}, |tz|^{-1}_{\mathbb{A}^N}) \min(|z|, |z|^{N_2}),$$

$$K_{N2,2} \left( \frac{x; n(u)a(t)\kappa}{\omega, s_0} \right) := \int_{\mathbb{F}_2} K_{N2,2} \left( x; n(u)a(t)\kappa(d(z, z)) \omega(z) |z|^{s_0+2} d^x z$$

(2.10)

$$= \omega(t)^{-1} |t|^{-s_0}_{\mathbb{A}} \int_{\mathbb{F}_2} \mathfrak{f}_2 L_2 R_\gamma \Psi \left( \frac{0}{\beta} \right) \omega(z) |z|^{s_0} d^x z, \quad \Re s_0 > 1.$$  

For the third term, we have similarly

$$K_{N3} (x; y) = \sum_{\alpha \in \mathbb{F}_2} \mathfrak{f}_2 L_2 R_\alpha \Psi \left( \frac{0}{\alpha} \right) = K_{N3,1} (x; y) + K_{N3,2} (x; y),$$

where

$$K_{N3,1} (x; y) := \sum_{\alpha, \gamma \in \mathbb{F}_2} \mathfrak{f}_2 L_2 R_\gamma \Psi \left( \frac{0}{\alpha} \right),$$

$$K_{N3,2} (x; y) := \sum_{\alpha \in \mathbb{F}_2} \mathfrak{f}_2 L_2 R_\gamma \Psi \left( \frac{0}{\alpha} \right).$$

It follows easily that

$$K_{N3,1} (x; n(u)a(t)\kappa(d(z, z)) = |z|^{-2}_{\mathbb{A}} \sum_{\alpha, \gamma \in \mathbb{F}_2} \mathfrak{f}_2 L_2 R_\gamma \Psi \left( \frac{0}{\alpha} \right)$$

(2.11)

$$\ll_{N1, N2} |z|^{-2}_{\mathbb{A}} \min(|tz|^{-1}_{\mathbb{A}}, |tz|^{-1}_{\mathbb{A}^N}) \min(|z|, |z|^{N_2}),$$

$$K_{N3,2} \left( \frac{x; n(u)a(t)\kappa}{\omega, s_0} \right) := \int_{\mathbb{F}_2} K_{N3,2} \left( x; n(u)a(t)\kappa(d(z, z)) \omega(z) |z|^{s_0+2} d^x z$$

(2.12)

$$= \omega(t)^{-1} |t|^{-s_0}_{\mathbb{A}} \int_{\mathbb{F}_2} \mathfrak{f}_2 L_2 R_\gamma \Psi \left( \frac{0}{\beta} \right) \omega(z) |z|^{s_0} d^x z, \quad \Re s_0 > 1.$$  

For the fourth term, we leave the reader to check the simple formula

(2.13) \quad K_{N4} \left( \frac{x; n(u)a(t)\kappa}{\omega, s_0} \right) = \int_{\mathbb{F}_2} \left( \sum_{(\alpha, \beta) \in \mathbb{F}_2^2 \setminus \{0\}} L_2 R_\alpha \Psi \left( \frac{0}{\beta} \right) \right) \omega(z) |z|^{s_0+2} d^x z, \quad \Re s_0 > 0.$$
Theorem 2.10. (Godement-Jacquet pre-trace formula) Assume $\Re s_0 > 2$.

(1) For fixed $x$, the function (see (2.2)) $y \mapsto \widetilde{KK}^{(2)}_{\gamma s}(x; y/\omega, s_0)$ lies in $A^{fr}(GL_2, \omega^{-1})$ and is bounded by $\ll \text{Ht}(y)^{1 - \frac{\Re s_0}{2}}$ in any Siegel domain.

(2) Its Fourier inversion with respect to $y$ in $L^2(GL_2(F) \backslash GL_2(\mathbb{A}), \omega^{-1})$ converges normally for $(x, y) \in (GL_2(F) \backslash GL_2(\mathbb{A}))^2$, and takes the form

$$
\widetilde{KK}^{(2)}_{\gamma s}(x; y/\omega, s_0) = \sum_{\chi \in \text{cuspida}(\mathbb{A})} \chi(x) \left( \frac{y}{\omega, s_0} \right) \int_{\mathbb{R}_+} \widetilde{KK}^{(2)}_{\gamma s}(x; y/\omega, s_0) \frac{\chi(\tau)}{\tau} d\tau + \frac{1}{\text{Vol}(PGL_2(\mathbb{A}^\mathbb{A}))} \sum_{\eta \in F^\omega(\mathbb{A})} \left( \int_{GL_2(\mathbb{A})} \Psi(y) \eta(\text{det } y) |\text{det } gj|^{s_0+1} dE \right) \eta(\text{det } x) \eta(\text{det } y),
$$

where $\widetilde{KK}^{(2)}_{\gamma s}(x; y/\omega, s_0)$ is the function (see (2.2)). The Fourier inversion with respect to $y$ in $L^2(GL_2(F) \backslash GL_2(\mathbb{A}), \omega^{-1})$ converges normally for $(x, y) \in (GL_2(F) \backslash GL_2(\mathbb{A}))^2$, and takes the form

$$
\widetilde{KK}^{(2)}_{\gamma s}(x; y/\omega, s_0) = \sum_{\chi \in \text{cuspida}(\mathbb{A})} \chi(x) \left( \frac{y}{\omega, s_0} \right) \left( \int_{\mathbb{R}_+} \widetilde{KK}^{(2)}_{\gamma s}(x; y/\omega, s_0) \frac{\chi(\tau)}{\tau} d\tau \right) + \frac{1}{\text{Vol}(PGL_2(\mathbb{A}^\mathbb{A}))} \sum_{\eta \in F^\omega(\mathbb{A})} \left( \int_{GL_2(\mathbb{A})} \Psi(y) \eta(\text{det } y) |\text{det } gj|^{s_0+1} dE \right) \eta(\text{det } x) \eta(\text{det } y),
$$

where $\widetilde{KK}^{(2)}_{\gamma s}(x; y/\omega, s_0)$ is the function (see (2.2)).
(2) For any $X$ in the universal enveloping algebra of $\mathfrak{sl}_2(\mathbb{A}_\infty)$, $R_X \Psi \in \mathcal{S}(M_2(\mathbb{A}))$. Hence the assertions in (1) remain valid if we replace $\Psi$ with $R_X \Psi$. In other words, the function $y \mapsto \widetilde{K}_2^{(2)} \left( \frac{x: y}{\omega, s_0} \right)$ is “nice” in the sense of Definition 2.1 to which Theorem 2.3 applies. We need to compute the automorphic Fourier coefficients.

If $\varphi_2 \in \mathcal{B}(\pi)$ for a cuspidal $\pi$ of central character $\omega$, then

$$
\int_{[\mathrm{PGL}_2]} \widetilde{K}_2^{(2)} \left( \frac{x: y}{\omega, s_0} \right) \varphi_2(y)dy = \int_{\mathrm{GL}_2(\mathbb{F})} \widetilde{K}_2^{(2)} (x: y) |\det x|_{\mathbb{A}}^{-\frac{s_0 + 1}{2}} \varphi_2(y)dy
$$

$$
= \int_{\mathrm{GL}_2(\mathbb{A})} \Psi(g)|\det g|_{\mathbb{A}}^{\frac{s_0 + 1}{2}} \varphi_2(xg)dy
$$

$$
= \sum_{\varphi_1 \in \mathcal{B}(\pi)} \mathcal{Z} \left( \frac{s_0 + 1}{2}, \Psi, \beta(\varphi_2, \varphi_1^*) \right) \varphi_1(x),
$$

justifying the automorphic Fourier coefficients for the cuspidal spectrum.

Take $e_2 \in \mathcal{B}(\chi, \omega \chi^{-1})$. Proposition 2.7 identifies the automorphic Fourier coefficient

$$
\int_{[\mathrm{PGL}_2]} \widetilde{K}_2^{(2)} \left( \frac{x: y}{\omega, s_0} \right) E(i\tau, e_2)(y)dy
$$

with the analytically continued value at $s = i\tau$ of

$$
R \left( s, \widetilde{K}_2^{(2)} \left( \frac{x: y}{\omega, s_0} \right), e_2 \right)
$$

$$
:= \int_{\mathbb{F}^\times \mathbb{A}^\times \mathbb{K}} \left( \widetilde{K}N^{(2)} \left( \frac{x: a(t)\kappa}{\omega, s_0} \right) \right) e_2(\kappa) \chi(t)|t|_{\mathbb{A}}^{s_0 + \frac{1}{2}} d\kappa dt
$$

$$
= \sum_{i=1}^3 \int_{\mathbb{F}^\times \mathbb{A}^\times \mathbb{K}} \left( \widetilde{K}N_{3, i} \left( \frac{x: a(t)\kappa}{\omega, s_0} \right) \right) e_2(\kappa) \chi(t)|t|_{\mathbb{A}}^{s_0 + \frac{1}{2}} d\kappa dt,
$$

$\Re s \gg 1$.

Inserting (2.11), we get

$$
f_s(x) := \int_{\mathbb{F}^\times \mathbb{A}^\times \mathbb{K}} \left( \widetilde{K}N_{3, 1} \left( \frac{x: a(t)\kappa}{\omega, s_0} \right) \right) e_2(\kappa) \chi(t)|t|_{\mathbb{A}}^{s_0 + \frac{1}{2}} d\kappa dt
$$

$$
= \int_{(\mathbb{A}^\times)^2 \mathbb{K}} \widetilde{\mathfrak{S}}_2 \mathfrak{S}_4 L_{x R_\kappa} \Psi \left( \frac{tz}{z} \right) e_2(\kappa) \omega(z)|z|_{\mathbb{A}}^{s_0 + \frac{1}{2}} \chi(t)|t|_{\mathbb{A}}^{s_0 + \frac{1}{2}} d\kappa d|z|_{\mathbb{A}}^{1/2}.
$$

The easy to check property

$$
\widetilde{\mathfrak{S}}_2 \mathfrak{S}_4 L_{n(u)\nu(t, z, s_0)} L_{x R_\kappa} \Psi \left( \frac{tz}{z} \right) = \widetilde{\mathfrak{S}}_2 \mathfrak{S}_4 L_{x R_\kappa} \Psi \left( \frac{t_1^{-1} t_0 z_0^{-1} z}{z_0^{-1}} \right)
$$

readily implies

$$
f_s \left( \left( \begin{array}{c} t \cr z \end{array} \right) \right) = \omega(z) \chi(t)|t|_{\mathbb{A}}^{s_0 + \frac{1}{2}} f_s(x).
$$

Hence $f_s \in \pi(\chi, |z|_{\mathbb{A}}^{s_0} \omega \chi^{-1}, |z|_{\mathbb{A}}^{-s})$ (not necessarily a flat section). Note that for any $\alpha \in \mathbb{F}$

$$
\widetilde{\mathfrak{S}}_2 \mathfrak{S}_4 L_{wn(\alpha)} L_{x R_\kappa} \Psi \left( \frac{tz}{z} \right) = \int_{\mathbb{A}^2} L_{x R_\kappa} \Psi \left( \frac{-\alpha t z}{t} \right) x_4 \psi(-x_4 z^{-1}) dx_2 dx_4
$$

$$
= \widetilde{\mathfrak{S}}_2 \mathfrak{S}_4 L_{x R_\kappa} \Psi \left( \frac{-\alpha t z}{t} \right) \psi(-x_4 z^{-1}) dx_2 dx_4,
$$

From which we easily verify

$$
f_s(wx) = \int_{\mathbb{F}^\times \mathbb{A}^\times \mathbb{K}} \left( \widetilde{K}N_{3, 1} \left( \frac{x: a(t)\kappa}{\omega, s_0} \right) \right) e_2(\kappa) \chi(t)|t|_{\mathbb{A}}^{s_0 + \frac{1}{2}} d\kappa dt \cdot |\det x|_{\mathbb{A}}^{\frac{s_0 + 1}{2}},
\[
\sum_{\alpha \in \mathcal{F}^x} f_s(w(\alpha)x) = \int_{\mathcal{F}^x \setminus \mathbb{A}^x \times \mathbb{K}} \hat{K} \hat{N}_{1,1} (\frac{x_1 a(t) \kappa}{\omega, s_0}) e_2(\kappa) \gamma(t) \frac{|\chi(t)|^{s_0-\frac{1}{2}}}{|\mathbb{A}_\kappa^s|} d\kappa \cdot |\det x|_\mathbb{A}^{-\frac{d-2}{2}}.
\]

We obtain, with absolute convergence for \( \Re s \gg 1 \)
\[
R \left( s, \hat{K} \hat{K}^{(2)} \left( \frac{x_1}{\omega, s_0} \right), e_2 \right) = \sum_{\kappa \in \mathcal{B}(\mathbb{F}) \setminus \mathcal{GL}_2(\mathbb{F})} f_s(\gamma(x)) = \sum_{e_1 \in \mathcal{B}(\chi, \omega \chi^{-1})} \int_{\mathbb{K}} f_s(\kappa) e_1(\kappa) d\kappa \cdot E(s, e_1(x)).
\]

Note that (the global version of) \([10, (11.9.4)]\) gives
\[(2.16) \quad Z(s'_1, \Psi, \beta, e_2, e_1') = \int_{(\mathbb{A}^x)^2} \hat{\mathcal{F}}_2 (e_1\Psi e_2) \left( \begin{array}{cc} t_1 & 0 \\ 0 & t_2 \end{array} \right) \chi(t_1) |t_1|^{s'_1} \omega \chi^{-1}(t_2) |t_2|^{s'_2} d^x t_1 d^x t_2,
\]

where \( e_1\Psi e_2(x) := \int_{\mathbb{K} \times \mathbb{K}} e_1'(\kappa_1) \Psi(\kappa_1^{-1} x \kappa_2) e_2(\kappa_2) d\kappa_1 d\kappa_2. \)

By a change of variables and the global functional equation of Tate's integral, \( f_s(x) \) is equal to

\[
\int_{(\mathbb{A}^x)^2 \times \mathbb{K}} \hat{\mathcal{F}}_2 (e_1\Psi e_2) \left( \begin{array}{c} x_1 \\ x_4 \end{array} \right) e_2(\kappa) \omega \chi^{-1}(x) |x_4|^{\frac{s_0-1}{2}} \omega \chi(x) |x_1|^{\frac{1}{2}} d^x x_1 d^x x_4 \cdot |\det x|^{-\frac{2-d}{2}}.
\]

We deduce that
\[
R \left( s, \hat{K} \hat{K}^{(2)} \left( \frac{x_1}{\omega, s_0} \right), e_2 \right) = \sum_{e_1 \in \mathcal{B}(\chi, \omega \chi^{-1})} Z \left( \frac{s_0+1}{2}, \Psi, \beta, e_2, e_1' \right) E(s, e_1(x)),
\]
justifying the automorphic Fourier coefficients for the continuous spectrum.

We leave the justification for the residue spectrum as an exercise (unimportant for this paper). \( \square \)

We introduce (temporarily) the absolutely convergent integral for \( \lambda \in D \)
\[
(2.17) \quad DS_0(\lambda; \Psi) := \int_{(\mathbb{F}^x \setminus \mathbb{A}^x)^2} \Delta_\Delta(\lambda; \Psi) \left( \frac{a(t_1); a(t_2)}{\omega, s_0} \right) \chi(t_1) |t_1|^{s_0-\frac{1}{2}} \omega \chi(t_2) |t_2|^{s_0+1} d^x t_1 d^x t_2.
\]

An obvious variant of Theorem \(2.10\) (just as Theorem \(2.3\) v.s. Theorem \(2.3\)) implies Proposition \(1.3\) \((1)\) via (recall \(1.17\) & \(1.18\)) absolutely convergent sum & integral for \( \lambda \in D \) of
\[
(2.18) \quad \Theta(\lambda, \Psi) - DS_0(\lambda; \Psi) = \sum_{\pi \text{ cuspidal} \atop \omega_\pi = 1} M_3(\lambda, \Psi \mid \pi) + \sum_{\chi \in \mathcal{R} \setminus \mathcal{F}^x \setminus \mathbb{A}^x} \int_{-\infty}^{\infty} M_3(\lambda, \Psi \mid \chi, i\tau) \frac{d\tau}{4\pi}.
\]

3. Fourth Moment (Geometric) Side

We would like to write in another way the following distribution
\[
\Delta_\Delta(1; 1) = KK(1; 1) - KN(1; 1) - N \Delta(1; 1).
\]

Remark 3.1. For simplicity of notation,
- the summation symbol “\( \sum \)” below means, by default, summing over \( \xi_i \in \mathbb{F} \) for those variables \( \xi_i \) appearing in the summands. Only extra conditions, such as “\( \xi_1 \in \mathbb{F}^x \)” will be explicitly written;
- the summation symbol “\( \sum \)” below means, by default, summing over \( \xi_i \in \mathbb{F}^x \) for those variables \( \xi_i \) appearing in the summands. Only extra conditions, such as “\( \xi_1 \in \mathbb{F}^x \)” will be explicitly written.

Lemma 3.2. We have
\[
KN(1; 1) = \sum_{\xi_1 \xi_2 + \xi_3 \xi_4 = 0} \hat{\mathcal{F}}_2 \hat{\mathcal{F}}_1 \Psi(\xi_1 \xi_2) - \Psi(0), \quad NK(1; 1) = \sum_{\xi_1 \xi_3 + \xi_2 \xi_4 = 0} \hat{\mathcal{F}}_1 \hat{\mathcal{F}}_2 \Psi(\xi_1 \xi_2) - \Psi(0).
\]
Proof. Applying the Poisson summation formula with respect to the variables $\xi_2, \xi_4$, we get

$$KK\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) + \Psi(0) = \sum \Psi\left(\begin{array}{c} \xi_1 \\ \xi_3 \end{array} \right) \xi_2 u \xi_3 + \Psi(0) = \sum \tilde{\Psi} \tilde{\Phi} \left(\begin{array}{c} \xi_1 \\ \xi_3 \end{array} \right) \psi(u(\xi_1 \xi_2 + \xi_3 \xi_4)).$$

Integrating against $u \in F \setminus A$ gives the first equation. The second one is proved similarly.

Lemma 3.3. We have

$$N\Delta (\mathbb{I}; \mathbb{I}) = \sum \tilde{\Psi} \tilde{\Phi} + \sum \tilde{\Psi} \tilde{\Phi} + \sum \int_{A} \tilde{\Psi} \tilde{\Phi} (-\xi_2/\xi_3, \xi_2/\xi_3) \psi(-v\xi) dv.$$

Proof. We combine the following elementary relation

$$\tilde{\Psi} \tilde{\Phi}(n(u)) \left(\begin{array}{c} x_1 \\ x_3 \end{array} \right) = \int_{A} \Psi\left(\begin{array}{c} y_1 \\ y_3 \end{array} \right) \psi(-y_1 x_1 - y_2 x_2) dy_1 dy_2$$

$$= \int_{A} \Psi\left(\begin{array}{c} y_1 \\ y_3 \end{array} \right) \psi(-y_1 (x_1 - x_2) - y_2 x_2) dy_1 dy_2$$

$$= \tilde{\Psi} \tilde{\Phi}(x_1 - u x_2, x_2)$$

with the second equation in Lemma 3.2 to get

$$NK(\mathbb{I}; n(u)) + \Psi(0) = \sum \tilde{\Psi} \tilde{\Phi} \left(\begin{array}{c} \xi_1 - u \xi_2 \\ \xi_3 \end{array} \right) \xi_2 \xi_4 + \Psi(0) \psi(u \xi_3 \xi_4) + \sum \tilde{\Psi} \tilde{\Phi} \left(\begin{array}{c} \xi_1 \\ \xi_3 \end{array} \right) \xi_2 \xi_4 + \sum \tilde{\Psi} \tilde{\Phi} \left(\begin{array}{c} \xi_1 \xi_2 \\ \xi_3 \end{array} \right) \xi_2 \xi_4,$$

where we distinguish the cases $\xi_2 = \xi_3 = 0, \xi_2 = 0 \& \xi_3 \neq 0, \xi_2 \neq 0 \& \xi_3 = 0$ and $\xi_2 \xi_3 \neq 0$ and apply partial Poisson summation in passing from the first line to the second. The desired equation then follows at once by definition

$$N\Delta (\mathbb{I}; \mathbb{I}) = NK(\mathbb{I}; \mathbb{I}) - \int_{F \setminus A} NK(\mathbb{I}; n(u)) du.$$

Lemma 3.4. We have the decomposition of tempered distributions

$$\Delta \Delta (\mathbb{I}; \mathbb{I}) = \sum \Delta \Delta_1 (\mathbb{I}; \mathbb{I}),$$

where each term is defined as

$$\Delta \Delta_1 (\mathbb{I}; \mathbb{I}) := \sum \tilde{\Psi} \tilde{\Phi} \left(\begin{array}{c} \xi_1 \\ \xi_3 \end{array} \right) \xi_2 \xi_4,$$

$$\Delta \Delta_2 (\mathbb{I}; \mathbb{I}) := \sum \tilde{\Psi} \tilde{\Phi} \left(\begin{array}{c} \xi_1 \\ \xi_3 \end{array} \right) \xi_2 \xi_4 + \sum \tilde{\Psi} \tilde{\Phi} \left(\begin{array}{c} \xi_1 \\ \xi_3 \end{array} \right) \xi_2 \xi_4,$$

$$\Delta \Delta_3 (\mathbb{I}; \mathbb{I}) := - \sum \tilde{\Psi} \tilde{\Phi} \left(\begin{array}{c} \xi_1 \\ \xi_3 \end{array} \right) \xi_2 \xi_4.$$
Each integral as \((3.1)\) is absolutely convergent in the following domain

\[ \int_{\mathbb{R}} \mathcal{F} \sum_{\xi} \mathcal{F} = 0 \]

Lemma 3.5. Recall the new Hecke characters

\[ \mathcal{F} = \sum_{\xi} \mathcal{F} \]

and applying partial Poisson summation formula to \(KK(\mathbb{R})\), we rewrite \(\Delta \Delta (\mathbb{R}) \) as

\[ \sum_{\xi} \mathcal{F} \sum_{\xi} \mathcal{F} \]

The two terms in the last column are \(\Delta \Delta (\mathbb{R}) \) resp. \(\Delta \Delta (\mathbb{R}) \). We identify terms in the first two columns with \(\Delta \Delta (\mathbb{R}) + \Delta \Delta (\mathbb{R}) \) by summing the following partial Poisson summation

\[ \sum_{\xi} \mathcal{F} \sum_{\xi} \mathcal{F} \]

According to the above decomposition, we define for \(1 \leq i \leq 4\)

\[(3.1) \quad I_i(\lambda, \Psi) := \int_{(\mathbb{R}^\times \setminus A^\times)^3} \Delta \Delta (a(1_t) \cdot d(t_2, z)) \omega(z) |z|^{\rho_0 + 2} \chi_1(t_1) |t_1| \chi_2(t_2) |t_2| d^\times d^\times d^\times.

Lemma 3.5. Each integral as \((3.1)\) is absolutely convergent in the following domain

\[ D' := \{(s_0, s_1, s_2) \in \mathbb{C}^3 \mid R s_0 > 2, R s_1 > 1, R s_2 > R s_0 + R s_1 + 1\} .\]

Remark 3.6. Recall the new Hecke characters \(\eta_i\) and complex numbers \(s_i\) are given by

\[ \eta_1 = \omega^{-1} \chi_1^{-1} \chi_2, \quad \eta_2 = \omega^{-1} \chi_2, \quad \eta_3 = \omega^{-1} \chi_1^{-1}, \quad \eta_4 = 1, \]

\[ s_1 = s_2 - s_0 - s_1, \quad s_2 = s_2 - s_0, \quad s_3 = -s_0 - s_1, \quad s_4 = 0. \]

It is convenient to record

\[ D' = \{(s_1', s_2', s_3') \in \mathbb{C}^3 \mid -R s_3' - 2 > R s_2' - R s_1' > 1, R s_1' > 1\} .\]

Proof. We give the treatment for \(I_1\) and \(I_2\) here. The remaining two can be directly treated in a similar way. But we prefer to give another proof in \(\text{4.3}\).

(1) \(I_1\). Assume \((s_0, s_1, s_2) \in D'\). Some basic properties of Fourier transform yield

\[ \Delta \Delta (a(1_t) \cdot d(t_2, z)) \]

Consider the following function on \((\mathbb{F}^\times \setminus A^\times)^4\)

\[ f(z, t_1, t_2, t) := \frac{|t_1|}{|t_2|} \sum_{\xi} \mathcal{F} \sum_{\xi} \mathcal{F} \]

It is smooth in each variable. With the change of variables

\[ \left( \begin{array}{c} t_1^{-1} t_2 z \\ t_2 z^{-1} \\ t z^{-1} \\ z^{-1} \end{array} \right) = \left( \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \right) .\]
we have for any real numbers $M_j$ the relation
\[
|t_1|_{A}^{M_1}|t_2|^2_{A}z_{A}^{M_2}|t_{3}|_{A}^{M_3}|t_{4}|_{A}^{M_4} = \left|x_1\right|_{A}^{(M_4+M_2-M_3-M_1)}\left|x_2\right|_{A}^{-M_1}\left|x_3\right|_{A}^{M_3}\left|x_4\right|_{A}^{(M_4-M_3-M_1)}.
\]
Let $\epsilon > 0$ be small and choose $M_j$ to be
\[
M_1 \in \{\Re s_1 \pm \epsilon\}, \quad M_2 \in \{\Re s_2 \pm \epsilon\}, \quad M_3 \in \{\Re s_0 \pm \epsilon\}, \quad M_4 + \Re s'_i \geq 1 + 3\epsilon, \forall 1 \leq i \leq 4.
\]
Proposition \textbf{2.3} implies the following bound
\[
|f(t_1, t_2, z, t)| \ll \frac{|t_1|_{A}}{|t_2|_{A}z^{2}}|t_1|_{A}^{-M_1}|t_2|_{A}^{-M_2}|z|_{A}^{-M_3}|t_{3}|_{A}^{-M_4}.
\]
Consequently, we can break the integral defining the following function on $\mathbf{F}^{\times} \setminus \mathbb{A}^{\times}$
\[
g(t):= \int_{(\mathbf{F}^{\times} \setminus \mathbb{A}^{\times})^3} f(z, t_1, t_2, t)\omega(z)|z|_{A}^{s_0+2}\chi_1(t_1)|t_1|_{A}^{s_1-1}\chi_2(t_2)|t_2|_{A}^{s_2+1}d^{\times}z d^{\times} t_1 d^{\times} t_2
\]
according as $|t_1|_{A} \geq 1 \text{ or } |t_1|_{A} < 1 \ (8 \text{ cases in total})$, apply to each sub-integral suitable $M_j$ chosen above, and see $g(t)$ is well-defined with absolutely convergent integral, smooth, and satisfies the bound
\[
|g(t)| \ll |t|_{A}^{-M_4}, \quad M_4 + \Re s'_i > 1, \forall 1 \leq i \leq 4.
\]
The same bound holds if we replace $g(t)$ by $D^n g(t)$ for the invariant differential $D$ on $s_{\mathbf{F}}(\mathbb{R}_+)\), where $s_{\mathbf{F}}$ is a arbitrary section map of the adelic norm map on $\mathbf{F}^{\times} \setminus \mathbb{A}^{\times}$. Taking into account the decomposition as a direct product of
\[
\mathbf{F}^{\times} \setminus \mathbb{A}^{\times} \simeq \mathbf{F}^{\times} \setminus \mathbb{A}^{\times(1)} \times \mathbb{R}_{>0},
\]
where $\mathbb{A}^{\times(1)}$ is the subgroup of elements in $\mathbb{A}^{\times}$ with adelic norm 1 (hence $\mathbf{F}^{\times} \setminus \mathbb{A}^{\times(1)}$ is compact), we see that Mellin inversion on $\mathbf{F}^{\times} \setminus \mathbb{A}^{\times}$ holds for $g(t)$. In particular, we have
\[
g(1) = \sum_{\chi} \int_{\Re s=c} \int_{(\mathbf{F}^{\times} \setminus \mathbb{A}^{\times})^4} g(t)\chi(t)|t|_{A}^{s}d^{\times} \frac{ds}{2\pi i},
\]
where $c + \Re s'_i > 1$ and $\chi$ traverses the unitary characters of $\mathbf{F}^{\times} \setminus \mathbb{A}^{\times(1)} \simeq \mathbb{R}_{>0}\mathbf{F}^{\times} \setminus \mathbb{A}^{\times}$. The innermost integral is identified as
\[
\int_{(\mathbb{A}^{\times})^4} f(z, t_1, t_2, t)\omega(z)|z|_{A}^{s_0+2}\chi_1(t_1)|t_1|_{A}^{s_1-1}\chi_2(t_2)|t_2|_{A}^{s_2+1}\chi(t)|t|_{A}^{s}d^{\times}z d^{\times} t_1 d^{\times} t_2 d^{\times} t = \int_{(\mathbb{A}^{\times})^4} \overline{\mathfrak{F}}\mathfrak{F} \Psi \left(\begin{array}{cccc} x_1 & x_2 & \cdots & x_4 \\ x_3 & x_4 & \cdots & x_1 \end{array} \right) \prod_{i=1}^{4} \eta_i(\chi(x_i)|x_i|_{A}^{s+s'_i}d^{\times} x_i.
\]
Hence we get
\[
(3.2) \quad g(1) = II_1(\lambda, \Psi) = \sum_{\chi} \int_{\Re s=c} \int_{(\mathbb{A}^{\times})^4} \mathfrak{F}\mathfrak{F} \Psi \left(\begin{array}{cccc} x_1 & x_2 & \cdots & x_4 \\ x_3 & x_4 & \cdots & x_1 \end{array} \right) \prod_{i=1}^{4} \eta_i(\chi(x_i)|x_i|_{A}^{s+s'_i}d^{\times} x_i) \frac{ds}{2\pi i}.
\]
(2) $\Delta \Delta_2$ has an obvious decomposition into the sum of four terms $\Delta \Delta_{2,i}$, yielding
\[
II_2(\lambda, \Psi) = \sum_{i=5}^{8} DG_i(\lambda, \Psi),
\]
where $DG_i$’s are given by the same formulas as \textbf{1.24} to \textbf{1.27}, absolutely convergent in $D'$. \hfill \Box

We summarized the above discussion in the following equality for $\lambda \in D'$ with absolute convergence
\[
(3.3) \quad \Theta(\lambda, \Psi) = \sum_{i=1}^{4} II_i(\lambda, \Psi) + \sum_{j=5}^{8} DG_j(\lambda, \Psi),
\]
where the terms on the right hand side are given by \textbf{1.51}, \textbf{3.2} and \textbf{1.24} - \textbf{1.27}.

4. Analytic Continuation

4.1. Third Moment Side. Based on \([2,18]\), we shall give a meromorphic continuation of \(\Theta(\lambda, \Psi)\) from \(\lambda \in D\) to a small neighborhood of \(0\), precisely to
\[
D_0 := \{(s_1, s_2, s_0) \mid |s_0|, |s_1|, |s_2| < 1/6\}.
\]

Lemma 4.1. Recall the continuous spectral Motohashi distributions \([LJ3]\). There is a functional equation
\[
\mathcal{K} \mathcal{K} \left( \frac{x, y}{\omega, s_0} \right) \chi(s) = \mathcal{K} \mathcal{K} \left( \frac{x, y}{\omega, s_0} \omega^{s-1}, -s \right),
\]
\[
\mathcal{M}_3(\lambda, \Psi \mid \chi, s) = \mathcal{M}_3(\lambda, \Psi \mid \omega^{-1}, -s).
\]

Proof. The second equality obviously follows from the first one. The intertwining operator
\[
\mathcal{M} : \pi(\chi, |\omega^{-1}|, \lambda, \psi) \rightarrow \pi(\omega^{-1}, |\omega^{-1}|, \psi)
\]
has the following two properties:

1. (See \([8, (4.4) \& (4.17)]\)) For any \(e \in \pi(\chi, \omega^{-1})\), \(e' \in \pi(\chi^{-1}, \omega^{-1})\),
\[
\langle \mathcal{M} e(s), \mathcal{M} e'(-s) \rangle = \langle e(s), e'(-s) \rangle.
\]
2. (See \([8, (5.15)]\)) For any \(e \in \pi(\chi, \omega^{-1})\),
\[
E(s, e) = E(e(s)) = E(\mathcal{M} e(s)).
\]
The first property, together with the \(GL_2(\mathbb{A})\)-intertwining property, implies
\[
\beta_s(e_2, e_1^\prime) = \beta(e_2(s), e_1^\prime(-s)) = \beta(\mathcal{M} e_2(s), \mathcal{M} e_1^\prime(-s)).
\]
Moreover, if \(e^\prime\) is the dual element of \(e\), then \(\mathcal{M} e^\prime(-s)\) is the dual element of \(\mathcal{M} e(s)\). Hence we get
\[
\sum_{e_1, e_2 \in B(\chi, \omega^{-1})} Z \left( \frac{s_0 + 1}{2}, \psi, \beta(e_2(s), e_1^\prime(-s)) \right) \cdot \mathcal{M} e_1(s) \cdot \mathcal{M} e_2^\prime(-s)(y)
\]
\[
= \sum_{e_1, e_2 \in B(\chi, \omega^{-1})} Z \left( \frac{s_0 + 1}{2}, \psi, \beta(\mathcal{M} e_2(s), \mathcal{M} e_1^\prime(-s)) \right) \cdot \mathcal{M} e_1(s) \cdot \mathcal{M} e_2^\prime(-s)(y)
\]
\[
= \sum_{e_1, e_2 \in B(\chi, \omega^{-1})} Z \left( \frac{s_0 + 1}{2}, \psi, \beta(\tilde{e}_2(-s), \tilde{e}_1^\prime(s)) \right) \cdot \tilde{e}_1(-s)(x) \cdot \tilde{e}_2^\prime(s)(y).
\]
Forming Eisenstein series in both variables yields the desired equality. \( \Box \)

Lemma 4.2. Let \((K\text{-finite}) f \in \pi(\chi_1, \chi_2)\) with \(\chi_j \in \mathbb{R}_+ \mathbb{F}^x \backslash \mathbb{A}^x\) and let \(\chi \in \mathbb{R}_+ \mathbb{F}^x \backslash \mathbb{A}^x\). Fix \(s_0 \in \mathbb{C}\) and let \(s \mapsto Z(s_0, E(s, f), \chi)\) has possible poles at

1. \((\rho - 1)/2\) as \(\rho\) runs over the zeros of \(GL_1\) \(L\)-function \(L^{(S)}(s, \chi \chi^{-1}_1)\) (partial \(L\)-function defined in \([4, 2]\)), where \(S\) is a finite set of finite places such that \(p \leq \infty\) and \(p \notin S\) implies \(f\) is \(K_p\)-invariant;
2. simple poles at \(s = -s_0\) and \(s = 1 - s_0\) if \(\chi = \chi^{-1}_1\) with
\[
\text{Res}_{s = -s_0} Z(s_0, E(s, f), \chi^{-1}_1) = -\psi^{s_0}_f(1), \quad \text{Res}_{s = 1 - s_0} Z(s_0, E(s, f), \chi^{-1}_1) = -\zeta^{s_0}_f \mathcal{M} f_{1-s_0}(w),
\]
where \(\zeta^{s_0}_f\) is the residue at \(s = 1\) of the Dedekind zeta function \(\zeta_f(s)\);
3. simple poles at \(s = s_0\) and \(s = s_0 - 1\) if \(\chi = \chi^{-1}_2\) with
\[
\text{Res}_{s = s_0} Z(s_0, E(s, f), \chi^{-1}_2) = \psi^{s_0}_f \mathcal{M} f_{s_0}(1), \quad \text{Res}_{s = s_0 - 1} Z(s_0, E(s, f), \chi^{-1}_2) = -\zeta^{s_0}_f f_{s_0 - 1}(w).
\]
Proof. First consider the Godement sections

\[ f_\Phi(s, g) = f_\hat{\Phi}(s, \chi_1, \chi_2; g) := \chi_1(\det g)\det g_\chi^{1/2+s} \int_{\mathbb{A}_\chi} \Phi((0, t)g)\chi_1\chi_2^{-1}(t)|t|_\chi^{1+2s}dt, \quad \Phi \in S(A_\chi^2) \]

with the related Eisenstein series \( E(s, \Phi) \). It is easy to identify the relevant zeta integral

\[ Z(s_0, E(s, \Phi), \chi) = \int_{(\mathbb{A}_\chi^2)^2} \Phi(t_1, t_2)\chi_1(t_1)|t_1|_\chi^{1+2s_0}\chi_2(t_2)|t_2|_\chi^{1-2s_0}dt_1dt_2 \]

as a two dimensional Tate's zeta integral. Thus we detect the type (2) and (3) poles. We have by (1.6)

\[ \text{Res}_{s=-s_0} Z(s_0, E(s, \Phi), \chi_1^{-1}) = -\zeta_s \int_{\mathbb{A}_\chi} \Phi(0, t_2)\chi_1^{-1}(t_2)|t_2|_\chi^{1-2s_0}dt_2 = -\zeta_s f_\Phi(-s_0, 1) \]

which gives the first formula of residues. We leave the formulas for other residues as an exercise, providing the following formula for the intertwining operator

\[ \mathcal{M}f_\Phi(s, \chi_1, \chi_2; g) = f_\hat{\Phi}(-s, \chi_2, \chi_1; g), \quad \Phi(x_1, x_2) := \Phi(-x_2, x_1). \]

To see the type (1) poles, we claim that for any \( K \)-isotypic \( f \), we can find \( \Phi \in S(A_\chi^2) \) such that

\[ f_\Phi(s, g) = \prod_{v|\infty} P_v(s)L_v(1+2s, \chi_1, \chi_2^{-1}) \cdot L^{(S)}(1+2s, \chi_1\chi_2^{-1}) \cdot f_s(g), \]

where

(a) \( P_v(s) \) is a polynomial in \( s \) with simple zeros, whose set of zeros is included in the set of poles of \( L_v(1+2s, \chi_1, \chi_2^{-1}) \);

(b) \( S \) is a large finite set of finite places \( p < \infty \) at which \( f_\Phi \) is not \( K_p \)-invariant and

\[ L^{(S)}(s, \chi_1\chi_2^{-1}) = \prod_{p < \infty, p \notin S} L_p(s, \chi_1, \chi_2^{-1}). \]

In fact, such \( \Phi = \otimes_v \Phi_v \) can be chosen decomposable. The existence of \( P_v(s) \) and \( \Phi_v \) for \( v | \infty \) such that

\[ f_\Phi(s, g) = P_v(s)L_v(1+2s, \chi_1, \chi_2^{-1})f_v(s) \]

is proved in \[25\] Lemma 3.5 (1) & Lemma 3.8 (1)]. At \( p < \infty \) and \( p \notin S \), we choose \( \Phi_p = \mathbb{I}_{\delta_x^1 \times \delta_y^2} \) so that

\[ f_\Phi(s, g) = L_p(1+2s, \chi_1, \chi_2^{-1})f_p(s). \]

At \( p \in S \), by the isomorphism map \( \iota \) defined in \[25\] §3.4.2 we deduce the existence of \( \Phi_p \) with support in

\[ F_0 := \{(x, y) \in \mathbb{F}_p^2 \mid \max(|x_p|, |y_p|) = 1\}, \]

such that \( f_{\Phi_p}(0, \kappa) = f_{p, 0}(\kappa) \) for all \( \kappa \in K_p \). Note that \( F_0 \) is stable by \( K_p \), therefore the integral

\[ f_{\Phi_p}(s, \kappa) = \chi_1\chi_p(\det \kappa) \int_{F_p^c} \Phi((0, t)\kappa)\chi_1\chi_2^{-1}(t)|t_p|^{1+2s}dt \]

is non-vanishing only for \( t \in \delta_x^1 \) thus independent of \( s \), hence \( f_{\Phi_p}(s, g) \) is already a flat section, implying

\[ f_{\Phi_p}(s, g) = f_{p, s}(g), \]

and conclude the proof of the claim \[4.1\]. The type (1) poles are precisely the zeros of \( L^{(S)}(1+2s, \chi_1\chi_2^{-1}) \) for all possible \( S \).

\[ \square \]

Corollary 4.3. The possible poles of \( s \mapsto M_3(\lambda, \Psi \mid \chi, s) \) are classified as

(0) \( (p-1)/2 \) resp. \( (1-p)/2 \) as \( p \) runs over the zeros of \( L^{(S)}(s, \omega^{-1}\chi^2) \) resp. \( L^{(S)}(s, \omega\chi^{-2}) \) (see \[4.2\]), where \( S \) is a finite set of finite places such that \( p < \infty \) and \( p \notin S \) implies \( \Psi \) is bi-\( K_p \)-invariant;

(1) \( (1-s_0)/2, -(1+s_0)/2 \) if \( \chi = 1 \); \( (s_0 - 1)/2, (s_0 + 1)/2 \) if \( \chi = \omega \).

(2) \( -(s_1 + (s_0 + 1)/2), -(s_1 + (s_0 - 1)/2) \) if \( \chi = \chi_1^{-1}; s_1 + (s_0 - 1)/2, s_1 + (s_0 + 1)/2 \) if \( \chi = \omega\chi_1 \).
where $H$ which has measure $0$ avoids any zero of $L$.

**Proof.** Write $\Omega := \{(1, \omega, \chi_1^{-1}, \omega \chi_1, \chi_2, \omega \chi_2^{-1}) \}$ and define for $\lambda \in D_0$, and the sums/integrals are absolutely convergent. But the resulting function, denoted by

$$I_0(\lambda, \Psi) := \sum_{\pi \text{ cuspidal}} \sum_{\begin{subarray}{c} \omega > \omega \pi = \omega \\
\chi \in \mathbb{R}^2 \setminus \lambda \times \chi \notin \Omega \end{subarray}} M_3(\lambda, \chi, \tau) \int_{-\infty}^{\infty} M_3(\lambda, \chi, \tau) \frac{d\tau}{4\pi}, \quad \lambda \in D_0$$

is not the meromorphic continuation of $\Theta(\lambda, \Psi) - DS_0(\lambda, \Psi)$ from $\lambda \in D$, because as $\lambda$ goes continuously from $D$ to $D_0$, it hits several “walls”/hyperplanes of singularities, such as $\Re s_0 = 1$ (see Corollary 4.3).

**Proposition 4.4.** The meromorphic continuation of $\Theta(\lambda, \Psi) - DS_0(\lambda, \Psi)$ from $\lambda \in D$ to $D_0 - H$ is given by

$$\Theta(\lambda, \Psi) - DS_0(\lambda, \Psi) = I_0(\lambda, \Psi) + \sum_{\begin{subarray}{c} \omega > \omega \pi = \omega \\
\chi \in \mathbb{R}^2 \setminus \lambda \times \chi \notin \Omega \end{subarray}} M_3(\lambda, \chi, \tau) \int_{-\infty}^{\infty} M_3(\lambda, \chi, \tau) \frac{d\tau}{4\pi}.$$ 

Then every summand/integrand on the right hand side is entire in $\lambda = (s_0, s_1, s_2) \in \mathbb{C}^3$. Hence it defines an analytic function in $D_0$, which is its meromorphic continuation from $D$. Since

$$\Theta(\lambda, \Psi) - DS_0(\lambda, \Psi) = I_0(\lambda, \Psi) + \sum_{\chi \in \Omega} \int_{\Re s_0 = 0} M_3(\lambda, \chi, \tau) \frac{d\tau}{4\pi},$$

for $(s_1, s_2, s_0) \in D$, we only need to give the analytic continuation for each $\chi \in \Omega$ of

$$\int_{\Re s = 0} M_3(\lambda, \chi, s) \frac{d\tau}{4\pi}.$$

Let $B \gg 1$ and define a box region

$$X_B := \{(s_1, s_2, s_0) \in \mathbb{C}^3 \mid |\Re s_0|, |3s_1| < B \}.$$ 

Let $Q > 3B + 1$ and take $L_Q$ to be the polygon line joining $-i\infty$, $-iQ, Q - iQ, Q + iQ, iQ$ and $i\infty$. Let $S$ be the set of finite places $p < \infty$ at which $\Psi$ is not bi-$K_p$-invariant. We choose $Q$ carefully so that $L_Q$ avoids any zero of $L^{(s)}(1 - 2s, \omega \chi_2^{-1})$ for $\chi \in \Omega$. Then for $(s_1, s_2, s_0) \in D \cap X_B$

$$\int_{\Re s = 0} M_3(\lambda, \chi, s) \frac{d\tau}{4\pi} = \int_{L_Q} M_3(\lambda, \chi, s) \frac{d\tau}{4\pi}$$

$$+ \frac{1}{2} \sum_{\rho \in L^{(s)}(1 - 2s, \omega \chi_2^{-1})} \Res_{s = \rho} M_3(\lambda, \chi, s),$$

where

$$\Res_{s = \rho} M_3(\lambda, \chi, s) = -\frac{1}{2} \sum_{\rho \in \mathbb{Z}_\chi} \Res_{s = \rho} M_3(\lambda, \chi, s).$$
where $Z_\chi$ is the set of non type (0) poles (see Corollary 3.3) encountered in the contour shift. Precisely, we have (if multiple conditions are satisfied, take the union of the corresponding sets on the right hand side)

$$Z_\chi = \left\{ \begin{array}{ll}
\{ \rho_\chi, (s_0 + 1)/2 \} & \text{if } \chi = \omega \\
\{ \rho_\chi, s_1 + (s_0 + 1)/2 \} & \text{if } \chi = \omega \chi_1 \\
\{ \rho_\chi, s_2 - (s_0 - 1)/2 \} & \text{if } \chi = \omega \chi_2 \\
\emptyset & \text{otherwise}
\end{array} \right.\text{ with } \rho_\chi := \left\{ \begin{array}{ll}
(s_0 - 1)/2 & \text{if } \chi = \omega \\
(s_1 + (s_0 - 1)/2 & \text{if } \chi = \omega \chi_1 \\
(s_2 - (s_0 + 1)/2 & \text{if } \chi = \omega \chi_2 \\
\end{array} \right.$$

The meromorphic continuation from $D \cap X_B$ to $D_0$ of the right hand side of (4.4) has the same expression. In particular, the residues appearing on the right hand side of (4.4) are naturally meromorphic functions in $(s_1, s_2, s_0) \in \mathbb{C}^3$. For example, we have for $L(\lambda; 1 - 2\rho, \omega \chi^{-2})$ and by (2.10)

$$\text{Res}_{s=\rho} M_3(\lambda, \Psi \mid \chi, s) = \sum_{e_1, e_2 \in B(\chi, \omega \chi^{-1})} \int_{(\mathbb{R}^+)^2} \tilde{\mathcal{F}}_2 (e_1, \Psi e_2) \left( \begin{array}{c}
t_1 \\
t_2
\end{array} \right) \chi(t_1)[t_1 - s_0 - 1] \chi^{-1}(t_2) \frac{\omega \chi^{-1}(t_2)}{t_2} \frac{\rho_\chi - \rho}{\lambda} d^2 t_1 d^2 t_2.$$

Fix $(s_0, s_1, s_2) \in D_0$, we shift the contour back. The right hand side of (4.4) becomes

$$\int_{\mathbb{R}^3} M_3(\lambda, \Psi \mid \chi, s) \frac{ds}{4\pi i} - \frac{1}{2} \text{Res}_{s=\rho_\chi} M_3(\lambda, \Psi \mid \chi, s) + \frac{1}{2} \text{Res}_{s=\rho'_\chi} M_3(\lambda, \Psi \mid \chi, s),$$

where $\rho'_\chi$ is the pole we meet during the backwards contour shift whose residue does not cancel out with those residues already appeared. Precisely

$$\rho'_\chi = -\rho \omega \chi^{-1} = \left\{ \begin{array}{ll}
(1 - s_0)/2 & \text{if } \chi = \lambda \\
-(s_1 + (s_0 - 1)/2) & \text{if } \chi = \omega \chi_1^{-1} \\
-(s_2 - (s_0 + 1)/2) & \text{if } \chi = \omega \chi_2^{-1} \\
\text{does not exist} & \text{otherwise}
\end{array} \right.$$

But the functional equation in Lemma 4.1 implies

$$\text{Res}_{s=\rho_\chi} M_3(\lambda, \Psi \mid \chi, s) = -\text{Res}_{s=\rho'_\chi} M_3(\lambda, \Psi \mid \omega \chi^{-1}, s).$$

Hence we only need to sum over $\rho'_\chi$, giving the stated expression. We conclude since $B$ is arbitrary. \qed

4.2. Geometric Aspect of Residues. The residues of $M_3(\lambda, \Psi \mid \chi, s) = M_3(\lambda, \Psi \mid \chi, s)$ appearing in Proposition 4.3 are at $s = 1/2$ for $\lambda = \tilde{0}$. There are three more such residues which do not show up in Proposition 4.3

(4.5) \quad \frac{1}{\gamma} \text{Res}_{s=\tilde{0} + 1} M_3(\lambda, \Psi \mid \omega, s),

(4.6) \quad \frac{1}{\gamma} \text{Res}_{s=s_1 + \frac{1}{2} + \tilde{0}} M_3(\lambda, \Psi \mid \omega \chi_1, s),

(4.7) \quad \frac{1}{\gamma} \text{Res}_{s=s_2 - \tilde{0}} M_3(\lambda, \Psi \mid \chi_2, s).

We are going to give alternative expressions of them in some subregion of $D'$ of parameters. It will be convenient to use the new parameters in computation

$$\tilde{s}_0 := \frac{s_0}{2}, \quad \tilde{s}_1 := s_1 + \frac{s_0}{2}, \quad \tilde{s}_2 := s_2 - \frac{s_0}{2}.$$
Hence the domains \(D\) and \(D'\) become
\[
D = \left\{ (s_0, s_1, s_2) \in \mathbb{C}^3 \mid \Re s_0 > 1, \Re s_1 - 2\Re s_0 - 2 > \frac{1}{2}, \Re s_2 > \frac{1}{2} \right\},
\]
\[
D' = \left\{ (s_0, s_1, s_2) \in \mathbb{C}^3 \mid \Re s_0 > 1, \Re s_1 - \Re s_0 > 1, \Re s_2 - \Re s_1 > 1 \right\}.
\]

We first record some elementary results concerning the zeta integrals.

**Lemma 4.5.** Let \(\chi_1, \chi_2\) and \(\chi\) be quasi-characters of \(F^\times \backslash \mathbb{A}^\times\). Let \(f \in \pi(\chi_1, \chi_2)\) be a smooth vector.

1. If \(\Re s_0 > (1 + \Re(\chi_1^{-1}\chi_2))/2\), then for \(1 - \Re s_0 - \Re(\chi_1) < \Re s < \Re s_0 - \Re(\chi_2)\) we have
   \[
   Z(s, E(s_0, f), \chi) = \int_{\mathcal{A}^\times} f_{s_0}(wn(u))\chi^{-1}_2(1-u)|u|^{s_0-\delta}d^\times u,
   \]
   where the integral is absolutely convergent. Here, \(\Re(\chi) \in \mathbb{R}\) is defined by \(|\chi(t)| = |t|_{\mathcal{A}}^{\Re(\chi)}\).

2. Let \(W_{s_0}\) be the Whittaker function of \(f_{s_0} \in \pi(\chi_1|_{\mathcal{A}}^{s_0}, \chi_2|_{\mathcal{A}}^{-s_0})\). If \(\Re s + \Re(\chi) - 1 > \max(\Re(\chi_1) + \Re s_0, \Re(\chi_2) - \Re s_0)\), then we have
   \[
   Z(s, E(s_0, f), \chi) = \int_{\mathcal{A}^\times} W_{s_0}(a(y))\chi(y)|y|^{s_0-\delta}d^\times y,
   \]
   where the integral is absolutely convergent.

**Proof.** We only prove (1). Both sides being linear in \(f_{s_0} \in \pi(\chi_1|_{\mathcal{A}}^{s_0}, \chi_2|_{\mathcal{A}}^{-s_0})\), we can replace the flat section \(f_{s_0}\) by a Godement section constructed from some \(\Phi \in \mathcal{S}(\mathcal{A}^2)\)
\[
f_{\Phi}(s_0, g) := \chi_1(\det g)|\det g|^{1/2 + s_0} \int_{\mathcal{A}^\times} \Phi((0, t)g)\chi_1^{-1}(t)|t|^{1-2s_0}d^\times t.
\]
Routine computation gives the Whittaker function of \(f_{\Phi}(s_0, g)\)
\[
W_{\Phi}(s_0, a(y)) = \chi_1(y)|y|^{1/2 + s_0} \int_{\mathcal{A}^\times} \mathcal{F}_2\Phi(ty, 1/t)\chi_1^{-1}(t)|t|^{2s_0}d^\times t.
\]
Hence for \(\Re s \gg 1\), we get
\[
Z(s, E(f_{\Phi}(s_0, \cdot), \chi) = \int_{\mathcal{A}^\times} W_{\Phi}(s_0, a(y))\chi(y)|y|^{s_0-1/2}d^\times y
= \int_{\mathcal{A}^\times} \mathcal{F}_2\Phi(t_1, t_2)\chi_1(t_1)^{|t_1|^{s_0}}\chi_2(t_2)^{|t_2|^{s_0}}d^\times t_1d^\times t_2.
\]
Applying the global functional equation to \(t_2\), we get for \(1 - \Re s_0 - \Re(\chi_1) < \Re s < \Re s_0 - \Re(\chi_2)\)
\[
Z(s, E(f_{\Phi}(s_0, \cdot), \chi) = \int_{\mathcal{A}^\times} \mathcal{F}_2\Phi(t_1, t_2)\chi_1(t_1)^{|t_1|^{s_0}}\chi_2^{-1}(t_2)^{|t_2|^{s_0}}d^\times t_1d^\times t_2
= \int_{\mathcal{A}^\times} f_{\Phi}(s_0, wn(u))\chi_2^{-1}(1-u)|u|^{s_0-\delta}d^\times u,
\]
since \(f_{\Phi}(s_0, wn(u)) = \int_{\mathcal{A}^\times} \Phi(t, tu)\chi_2^{-1}(t)|t|^{1+2s_0}d^\times t\).

**Lemma 4.6.** Let \(f \in \pi(\chi_1, \chi_2)\). Then we have \(W_f = W_{\mathcal{M}f}\), where \(\mathcal{M} : \pi(\chi_1, \chi_2) \to \pi(\chi_2, \chi_1)\) is the intertwining operator.

**Proof.** We have the functional equation of Eisenstein series \(E(f) = E(\mathcal{M}f)\). The desired equality follows since by definition we have
\[
W_f(g) = \int_{F \backslash \mathbb{A}} E(f)(u)g\psi(-u)du.
\]
We then notice that for quasi-characters $\chi, \chi_1, \chi_2$ of $F^\times \backslash A^\times$ the following function on $GL_2(A) \times GL_2(A)$

$$f_\Psi(g_1, g_2; \chi_1, \chi_2) := \int_{(A^\times)^2} \mathfrak{F}_2 L(g_1) R(g_2) \Psi \left( \begin{array}{cc} t_1 & \chi_1(t_1) \chi_2(t_2) | t_1 t_2 | F \end{array} \right) \frac{1}{|A|^{\frac{5}{2}}} d^x t_1 d^x t_2,$$

resp.

$$f_\Psi(g_1, g_2; \chi, \ast) := \int_{A^\times} \mathfrak{F}_2 L(g_1) R(g_2) \Psi \left( \begin{array}{cc} t & 0 \\ 0 & 0 \end{array} \right) \chi(t) |t| \frac{1}{|A|^{\frac{5}{2}}} d^x t$$
defines a vector in the tensor product $\pi(\chi_1, \chi_2) \otimes \pi(\chi_1^{-1}, \chi_2^{-1})$ resp. $\pi(\chi, \ast) \otimes \pi(\chi^{-1}, \ast)$ realized in the induced model, where the integrals in $t_1, t_2$ are Tate’s integral.

**Definition 4.7.** We also write $(f \times f)\Psi(\ast)$ for $f_\Psi(\ast)$. We denote by $M$ resp. $W$ the image under the intertwining operator resp. Whittaker functional of $f$ with respect to one variable. For example, $(f \times W)(g_1, g_2; \chi_1, \chi_2) \in \pi(\chi_1, \chi_2) \otimes \mathcal{W}(\chi_1^{-1}, \chi_2^{-1}; \psi)$ is so defined that for any fixed $g_1$, the function $g_2 \mapsto (f \times W)(g_1, g_2; \chi_1, \chi_2)$ is the Whittaker function with respect to the additive character $\psi$ of $F \backslash A$ of the vector in the induced model given by $g_2 \mapsto (f \times f)\Psi(g_1, g_2; \chi_1, \chi_2)$.

**Lemma 4.8.** We have the functional equation

$$(M \times f)\Psi(g_1, g_2; \chi_1, \chi_2) = (f \times M)\Psi(g_1, g_2; \chi_2, \chi_1).$$

**Proof.** It suffices to prove the equation for $g_1 = g_2 = 1$, since the general case follows from this special case by taking $L(g_1) R(g_2) \Psi$ instead of $\Psi$. By definition, we have in the absolutely convergent region

$$(M \times f)\Psi(1, 1; \chi_1, \chi_2) = \int_{A^\times} \int_{A^\times} \int_{A^\times} \Psi \left( \frac{-1}{0} \right) \chi_1(t_1) \chi_2(t_2) | t_1 t_2 | \frac{1}{|A|^{\frac{5}{2}}} d^x t_1 d^x t_2 d u = \int_{A^\times} \int_{A^\times} \int_{A^\times} \Psi \left( \frac{-1}{0} \right) \chi_1(t_1) \chi_2(t_2) | t_1 t_2 | \frac{1}{|A|^{\frac{5}{2}}} d^x t_1 d^x t_2 d u$$

The equation is thus proved by the uniqueness of analytic continuation. \qed

**Lemma 4.9.**

1. If $|\chi(t)| = |t\chi(z)|^\sigma$ for some $\sigma > 5/2$, then the Whittaker function $(W \times W)\Psi(g_1, g_2; \chi, \ast)$ is given by the absolutely convergent integral

$$\int_{A^\times} \int_{A^\times} L(g_1) R(g_2) \Psi \left( \frac{-1}{0} \right) \psi(-u_1 - u_2) du_1 du_2 \chi(z) |z\chi(z)|^{\frac{1}{2}} d^x z.$$

2. If $|\chi_j(t)| = |t\chi(z)|^\sigma$ for some $\sigma_1 > 5/2, \sigma_2 < 1/2$, then $(M \times W)\Psi(g_1, g_2; \chi_1, \chi_2)$ is given by the absolutely convergent integral

$$\int_{(A^\times)^2} \mathfrak{F}_1 \mathfrak{F}_2 L(g_1) R(g_2) \Psi \left( \begin{array}{cc} u_1 & -t_1 \\ -t_2 & u_2 \end{array} \right) \psi(-u) \chi_1^{-1}(t_1) |t_1| \frac{1}{|A|^{\frac{5}{2}}} d^x t_1 d^x t_2.$$

**Proof.** (1) Fix $g_1$. Then the function $g_2 \mapsto f\Psi(g_1, g_2; \chi, \ast)$ is a meromorphic section in $\pi(\chi^{-1}, \ast)$.

Its Whittaker function in $W(\chi^{-1}, \ast)$, i.e. $(f \times W)\Psi(g_1, g_2; \chi, \ast)$ can be computed via the defining integral first assuming $\sigma < -1/2$, then by meromorphic continuation to $\sigma > 1/2$. For $\sigma < -1/2$, we have

$$(f \times W)\Psi(g_1, g_2; \chi, \ast) = \int_{A^\times} f\Psi(g_1, u \chi_1^{-1} g_2; \chi, \ast) \psi(-u) du = \int_{A^\times} \int_{A^\times} \mathfrak{F}_1 \mathfrak{F}_2 L(g_1) R(u \chi_1^{-1} g_2) \Psi \left( \begin{array}{cc} t & 0 \\ 0 & 0 \end{array} \right) \chi^{-1}(t) |t| \frac{1}{|A|^{\frac{5}{2}}} \psi(-u) d^x t d u$$
by Tate’s global functional equation. For any $\Psi \in S(M_2(\mathbb{A}))$, we have

$$
\mathfrak{F}_1\mathfrak{F}_2 R(wn(u))\Psi \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \int_{K^2} \Psi \begin{pmatrix} -y_2 & y_1 - uy_2 \\ -x_4 & x_3 - ux_4 \end{pmatrix} \psi(-y_1x_1 - y_2x_2)dy_1dy_2
$$

\[= \mathfrak{F}_1\mathfrak{F}_2 \Psi \begin{pmatrix} -x_2 - ux_1 & x_1 \\ -x_4 & x_3 - ux_4 \end{pmatrix}.\]

(4.8)

Hence we obtain

$$
(f \times W)\varphi(g_1, g_2; \chi, *) = \int_{\mathbb{A}} \int_{\mathbb{A}^*} \mathfrak{F}_1\mathfrak{F}_2 L(g_1)R(g_2)\Psi \begin{pmatrix} -ut & t \\ 0 & 0 \end{pmatrix} \chi^{-1}(t)|t|^\frac{2}{\sigma_\mathbb{A}} \psi(-u)d^\times tdu
$$

\[= \int_{\mathbb{A}^*} \mathfrak{F}_2 L(g_1)R(g_2)\Psi \begin{pmatrix} t^{-1} & 0 \\ 0 & 0 \end{pmatrix} \chi^{-1}(t)|t|^\frac{2}{\sigma_\mathbb{A}} d^\times t
\]

\[= \int_{\mathbb{A}^*} \mathfrak{F}_2 L(g_1)R(g_2)\Psi \begin{pmatrix} t & t^{-1} \\ 0 & 0 \end{pmatrix} \chi(t)|t|^{-\frac{2}{\sigma_\mathbb{A}}} d^\times t.
$$

Since the right hand side is absolutely convergent for all $\chi$, it gives the desired meromorphic continuation to $\sigma > 1/2$, where we have

$$
(W \times W)\varphi(g_1, g_2; \chi, *) = \int_{\mathbb{A}} (f \times W)\varphi(wn(u_1)g_1, g_2; \chi, *)\psi(-u_1)du_1
$$

\[= \int_{\mathbb{A}^*} \mathbb{I} L(g_1)R(g_2)\Psi \begin{pmatrix} (n(-u_1))w^{-1} & t \\ 0 & u_2 \end{pmatrix} \psi(-u_1 - u_2t^{-1})\chi(t)|t|^{-\frac{2}{\sigma_\mathbb{A}}} du_1du_2d^\times t,
$$

which is precisely the desired formula by the change of variables $u_2 = u_2t$, $t = z$. The order change of integrations is justified by

$$
\int_{\mathbb{A}^2} \max_{x \in \mathbb{A}} \left| L(g_1)R(g_2)\Psi \begin{pmatrix} -u_1t & x \\ u_2t \end{pmatrix} \right| du_1du_2 \ll \prod_v \min(1, |t_v|^{-N}) \cdot |t|^{-\frac{2}{\sigma_\mathbb{A}}},
$$

whose integral against $|t|^{\sigma_\mathbb{A}+1/2}$ over $\mathbb{A}^\times$ is absolutely convergent if $\sigma + 1/2 - 2 > 1$, i.e. $\sigma > 5/2$.

(2) It suffices to prove the equation for $g_1 = g_2 = \mathbb{I}$. By Lemma 4.6 and 4.8, we have

$$
(M \times W)\varphi(\mathbb{I}, \mathbb{I}; \chi_1, \chi_2) = (f \times W)\varphi(\mathbb{I}, \mathbb{I}; \chi_2, \chi_1).
$$

The right hand side can be computed by definition for $\sigma_1 - \sigma_2 > 1$ and 4.8 as

$$
(f \times W)\varphi(\mathbb{I}, \mathbb{I}; \chi_2, \chi_1) = \int_{\mathbb{A}} (f \times f)\varphi(\mathbb{I}, wn(u); \chi_2, \chi_1)\psi(-u)du
$$

\[= \int_{\mathbb{A}^2} \mathfrak{F}_1\mathfrak{F}_2 R(wn(u))\Psi \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \chi_2^{-1}(t_1)|t_1|^{\frac{2}{\sigma_\mathbb{A}}} \chi_1(t_2)|t_2|^{\frac{2}{\sigma_\mathbb{A}}} \psi(-u)d^\times t_1d^\times t_2du
\]

\[= \int_{\mathbb{A}^2} \mathfrak{F}_1\mathfrak{F}_2 \Psi \begin{pmatrix} -ut_1 & t_1 \\ -t_2 & -ut_2 \end{pmatrix} \chi_2^{-1}(t_1)|t_1|^{\frac{2}{\sigma_\mathbb{A}}} \chi_1(t_2)|t_2|^{\frac{2}{\sigma_\mathbb{A}}} \psi(-u)d^\times t_1d^\times t_2du.
$$

The above integral is absolutely convergent by

$$
\int_{\mathbb{A}^2} \max_{x \in \mathbb{A}} \mathfrak{F}_1\mathfrak{F}_2 \Psi \begin{pmatrix} x \\ -t_2 \end{pmatrix} \psi(-u)du \ll \prod_v \min(1, |t_1|^{-N}) \min(1, |t_2|^{-N}) |t_2|_{\mathbb{A}^2}^{-1},
$$

whose integral against $|t_1|^{3/2-\sigma_1}|t_2|_{\mathbb{A}^2}^{\sigma_1-1/2}$ over $\mathbb{A}^\times \times \mathbb{A}^\times$ is absolutely convergent if $\sigma_1 > 5/2, \sigma_2 < 1/2$. The desired equation follows by order change of integrations and the change of variables $t_j \rightarrow -t_j$.

**Corollary 4.10.** Assume $\lambda \in D'$. Recall the summation convention in Remark 2.7.

(1) Let $\ell_0$ be the tempered distribution given by

$$
\ell_0(\Psi) := \sum_{\mathbb{A}^2} L \left( \frac{\xi_1}{1} \right) R \left( \frac{\xi_2}{\xi} \right) \Psi \begin{pmatrix} n(-u_1) & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \psi(-u_1 - u_2)du_1du_2.
$$
Then \( \text{(4.9)} \) is equal to the absolutely convergent integral
\[
\int_{(\mathbb{F} \setminus \mathbb{A})^3} \ell_0 \left( L \begin{pmatrix} t_1 & 1 \\ t_2 & t \end{pmatrix} \right) \Psi(t) |t|^{|s_0+2,1-\frac{a}{2}|} \chi_1(t_1)|t_1|^{|s_1-1,\frac{a}{2}+\frac{a}{2}+\frac{a}{2}|} dt_1 dt_2 dt_3.
\]

(2) Let \( \ell_1 \) be the tempered distribution given by
\[
\ell_1(\Psi) := \sum_{\alpha} \int_{\mathbb{A}} \tilde{s}_1 \tilde{s}_2 L_p \begin{pmatrix} \xi_1 & 1 \\ 1 & 0 \end{pmatrix} R \begin{pmatrix} \xi_2 \xi & \xi \\ 0 & 1 \end{pmatrix} \Psi \begin{pmatrix} u & 1 \\ 1 & u \end{pmatrix} \psi(-u) du.
\]
Then \( \text{(4.10)} \) is equal to the absolutely convergent integral
\[
\int_{(\mathbb{F} \setminus \mathbb{A})^3} \ell_1 \left( L \begin{pmatrix} t_1 & 1 \\ t_2 & t \end{pmatrix} \right) \Psi(t) |t|^{|s_0+2,1-\frac{a}{2}|} \chi_1(t_1)|t_1|^{|s_1-1,\frac{a}{2}+\frac{a}{2}+\frac{a}{2}|} dt_1 dt_2 dt_3.
\]

(3) Let \( \ell_2 \) be the tempered distribution given by
\[
\ell_2(\Psi) := \sum_{\alpha} \tilde{s}_1 \tilde{\omega}_2 L_p \begin{pmatrix} \xi_1 & 1 \\ 1 & 0 \end{pmatrix} R \begin{pmatrix} \xi_2 \xi & \xi \\ 0 & 1 \end{pmatrix} \Psi \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.
\]
Then \( \text{(4.7)} \) is equal to the absolutely convergent integral
\[
\int_{(\mathbb{F} \setminus \mathbb{A})^3} \ell_2 \left( L \begin{pmatrix} t_1 & 1 \\ t_2 & t \end{pmatrix} \right) \Psi(t) |t|^{|s_0+2,1-\frac{a}{2}|} \chi_1(t_1)|t_1|^{|s_1-1,\frac{a}{2}+\frac{a}{2}+\frac{a}{2}|} dt_1 dt_2 dt_3.
\]

Proof. (1) Applying \( \text{(1.6)}, \text{(2.16)} \) and Lemma \( \text{4.5} \) we identify \( \text{(4.5)} \) with
\[
\sum_{e_1,e_2 \in \mathcal{B}(\omega_1)} \int_{\mathbb{A}^2} \tilde{s}_2 \left( e_2 \Psi e_1 \right) \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix} |t|^{|s_0,1-\frac{a}{2}|} dt.
\]

\[
\int_{\mathbb{A}^2} W_{e_1} \begin{pmatrix} \frac{1}{2} + \tilde{s}_0, a(y) \end{pmatrix} \chi_1(y) |y|^{|s_1,1-\frac{a}{2}|} dy \cdot \int_{\mathbb{A}^2} W_{e_2} \begin{pmatrix} -\frac{1}{2} - \tilde{s}_0, a(y) \end{pmatrix} \chi_2(y) |y|^{|s_2,1-\frac{a}{2}|} dy
\]
\[
= \sum_{e_1,e_2 \in \mathcal{B}(\omega_1)} \int_{\mathbb{A}^2} (f \times f) \phi \begin{pmatrix} \kappa_1, \kappa_2; \omega_1 \end{pmatrix} |y|^{|s_0,2+\frac{a}{2}|} \kappa_1 \kappa_2.
\]

In the above we need the tricky twists by \( |y|^{|s_1,1-\frac{a}{2}|} \) because \( (f \times f) \phi \begin{pmatrix} \kappa_1, \kappa_2; \omega_1 \end{pmatrix} |y|^{|s_0,2+\frac{a}{2}|} \) has central characters \( \omega_1 |\mathbb{A}^2| \) resp. \( \omega_1 |\mathbb{A}^2| \) in the first resp. second variable. While the flat section \( e_1(\frac{1}{2} + \tilde{s}_0) \in \pi(\omega_1 |\mathbb{A}^2|, |\mathbb{A}^2|) \) resp. \( e_2(\frac{1}{2} + \tilde{s}_0) \in \pi(\omega_1 |\mathbb{A}^2|, |\mathbb{A}^2|) \), we must apply the indicated twists to make the relevant representations dual to each other. This subtlety re-occurs in the proof of (2) and (3) below, and we will not mention it again. Inserting Lemma \( \text{4.9} \) (1), this is equal to
\[
\int_{\mathbb{A}^2} \int_{\mathbb{A}^2} L(a(t_1)) R(a(t_2)) \Psi \begin{pmatrix} n(-u_1) & 0 \\ 0 & 0 \end{pmatrix} n(u_2) \psi(-u_1 - u_2) du_1 du_2 \omega(t) |t|^{|s_0,2+\frac{a}{2}|} dt.
\]

\[
\chi_1(t_1)|t_1|^{|s_1,1-\frac{a}{2}|} \chi_2(t_2)|t_2|^{|s_2,1-\frac{a}{2}|} dt_1 dt_2 dt_3.
\]

It is equal to the stated formula by the absolute convergence.
(2) Similarly we identify (4.6) with
\[
\sum_{e_1, e_2 \in B(\omega \chi_1, \chi_1^{-1})} \int_{(\mathbb{A}^\times)^2} \tilde{\mathcal{F}}_2 (e_1 \Psi_{e_2}) \left( \begin{array}{c} t_1 \\ 0 \\ t_2 \end{array} \right) \omega \chi_1(t_1) |t_1|^{\frac{1}{2} + \tilde{s}_0 + \frac{1}{2} - \tilde{s}_1} \chi_2(t_2) |t_2|^{\frac{1}{2} - \tilde{s}_1} d^x t_1 d^x t_2.
\]
We get the desired formula by inserting Lemma 4.9 (2).

(3) Similarly we identify (4.7) with
\[
\sum_{e_1, e_2 \in B(\omega \chi_2, \chi_2^{-1})} \int_{(\mathbb{A}^\times)^2} \tilde{\mathcal{F}}_2 (e_1 \Psi_{e_2}) \left( \begin{array}{c} t_1 \\ 0 \\ t_2 \end{array} \right) \chi_2(t_1) |t_1|^{1 + \frac{1}{2} + \tilde{s}_0 + \frac{1}{2} - \tilde{s}_1} \omega \chi_2(t_2) |t_2|^{\frac{1}{2} - \tilde{s}_1} d^x t_1 d^x t_2.
\]
where we have applied the following elementary relation, whose proof is similar to (4.8).

\[
\chi_2(t_1) |t_1|^{1 + \frac{1}{2} + \tilde{s}_0 + \frac{1}{2} - \tilde{s}_1} \omega \chi_2(t_2) |t_2|^{\frac{1}{2} - \tilde{s}_1} d^x t_1 d^x t_2.
\]

We get the desired formula by inserting Lemma 4.9 (2).

The last expression is easily identified with the desired formula. \(\Box\)

To end this subsection, we give the meromorphic continuation of \(DS_0(\lambda, \Psi)\) defined in (2.17). By definition, we have for \(\lambda \in D\)

\[
DS_0(\lambda, \Psi) = \int_{(\mathbb{A}^\times)^2} \tilde{\mathcal{F}}_2 \tilde{\mathcal{F}}_4 (\omega \chi_2) \left( \begin{array}{c} t_1 \\ 0 \\ t_2 \end{array} \right) \chi_2(t_1) |t_1|^{1 + \frac{1}{2} + \tilde{s}_0 + \frac{1}{2} - \tilde{s}_1} \omega \chi_2(t_2) |t_2|^{\frac{1}{2} - \tilde{s}_1} d^x t_1 d^x t_2.
\]

via the Whittaker-Fourier expansions in each variable, where

\[
\tilde{\mathcal{F}}_2 \tilde{\mathcal{F}}_4 (\omega) = \int_{(\mathbb{A}^\times)^2} \tilde{\mathcal{F}}_2 \tilde{\mathcal{F}}_4 (\omega) \left( \begin{array}{c} u_1 \\ 0 \\ u_2 \end{array} \right) \psi(-u_1 - u_2) du_1 du_2.
\]

The equality of matrices

\[
\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\]
identifies \(DS_0\) via (4.3) and Corollary 4.10 (1) with

\[
DS_0(\lambda, \Psi) = \frac{1}{\zeta}\ Res_{s = \frac{1}{2} + a} M_3(\lambda, \Psi | \mathbb{I}, s), \quad \lambda \in D'.
\]

The right hand side gives the meromorphic continuation of \(DS_0(\lambda, \Psi)\).
4.3. Fourth Moment Side. Based on (3.3), we shall give another expression of the meromorphic continuation of $\Theta(\lambda, \Psi)$ from $\lambda \in D$ to $D_0$. Along the way, we complete the proof of Lemma 3.3.

$DG_j$, $5 \leq j \leq 8$ are given as higher dimensional global Tate integrals in (1.24) to (1.27), hence have obvious meromorphic continuation to $\mathbb{C}^3$. We are left with $II_i$ for $i \neq 2$.

(3.2) is still valid in $D_0$, giving the analytic continuation of $II_1(\lambda, \Psi)$ therein. Fix $\lambda = (s_0, s_1, s_2) \in D_0$.

We shift the contour back to $\Re s = 1/2$, crossing four poles of the Tate integrals (because $|\Re s_i'| < 1/2$ for $(s_0, s_1, s_2) \in D_0$) and get

\begin{equation}
II_1(\lambda, \Psi) = M_4(\lambda, \Psi) + \sum_{j=1}^{4} DG_j(\lambda, \Psi),
\end{equation}

where $M_4$ and $DG_j$ are given by (1.14) and (1.20) to (1.23).

We can easily identify the distributions $\ell_1(\Psi)$ resp. $\ell_2(\Psi)$ in Corollary 4.10 as

\begin{align*}
\ell_1(\Psi) &= \sum_{\lambda} \int_{\Lambda} \tilde{\mathcal{F}}_1 \tilde{\mathcal{F}}_2 \Psi \left( \frac{-\xi_2 u}{\xi_3} \right) \psi(-u\xi) du = -\Delta \Delta (\lambda; \lambda), \\
\ell_2(\Psi) &= \sum_{\xi_1 \xi_2 + \xi_3 \xi_4 = 0} \tilde{\mathcal{F}}_2 \tilde{\mathcal{F}}_4 \Psi \left( \frac{\xi_1}{\xi_3} \frac{\xi_2}{\xi_4} \right) = -\Delta \Delta (\lambda; \lambda).
\end{align*}

Hence $II_3$ and $II_4$ are absolutely convergent in $D'$ by Corollary 4.10, and we have

\begin{equation}
II_3(\lambda, \Psi) = -DS_4(\lambda, \Psi) = -\frac{1}{\zeta_F} \text{Res}_{s = s_2 + \frac{i}{\zeta_F}} M_3(\lambda, \Psi | \chi_2, s),
\end{equation}

\begin{equation}
II_4(\lambda, \Psi) = -DS_5(\lambda, \Psi) = -\frac{1}{\zeta_F} \text{Res}_{s = s_2 + \frac{i}{\zeta_F}} M_3(\lambda, \Psi | \omega \chi_1, s),
\end{equation}

justifying their meromorphic continuation.

We summarize the above discussion in the following proposition.

**Proposition 4.11.** The meromorphic continuation of $\Theta(\lambda, \Psi)$ from $\lambda \in D'$ to $D_0 - H'$ is given by

$$\Theta(\lambda, \Psi) = M_4(\lambda, \Psi) + \sum_{j=1}^{8} DG_j(\lambda, \Psi) - DS_4(\lambda, \Psi) - DS_5(\lambda, \Psi),$$

where $M_4$ and $DG_j$ are given by (1.14) and (1.20) to (1.23), and $DS_4 \cup DS_5$ are given by (1.14) and (4.13); and $H' = H'(\chi_1, \chi_2)$ is the union of hyperplanes where two of the above residues in the degenerate terms may merge, which has measure 0.

5. Analysis of Degenerate Terms

We prove Proposition 4.11 in this section. Henceforth, we assume $\omega = \chi_1 = \chi_2 = 1$. In this case, we can work out

$$H \cup H' = \{ \lambda = (s_0, s_1, s_2) \in \mathbb{C}^3 \mid s_1(s_1 + s_0 - s_2)(s_2 - s_0) \cdot s_2(s_2 + s_1)(s_0 + s_1) = 0 \}.$$

Suppose $\delta > 0$ is small enough so that $\zeta_F(s)\zeta_F(1 + s) \neq 0$ for $|s| < 2\delta$. Assume $|\lambda| < \delta/4$ with $\lambda \notin H \cup H'$. The terms $DG_j(\lambda)$, $1 \leq j \leq 4$ are the residues of all possible poles of $s \mapsto M_4(\lambda, \Psi | \lambda, s)$ in the region $|s - 1| < \delta$. On the circle $|s - 1| = \delta$, $M_4(\lambda, \Psi | \lambda, s)$ is regular and continuous in $\lambda$.

Thus

$$\sum_{j=1}^{4} DG_j(\lambda, \Psi) = \frac{1}{\zeta_F} \int_{|s - 1| = \delta} M_4(\lambda, \Psi | \lambda, s) \frac{ds}{2\pi i},$$

which holds as $\lambda \to 0$, proving the regularity with

$$\sum_{j=1}^{4} DG_j(0, \Psi) = \frac{1}{\zeta_F} \int_{|s - 1| = \delta} M_4(0, \Psi | \lambda, s) \frac{ds}{2\pi i} = \frac{1}{\zeta_F} \text{Res}_{s=1} M_4(0, \Psi | \lambda, s).$$
Similarly, we have
\[
\sum_{j=5}^{8} DG_j(\lambda, \Psi) = -\frac{1}{\zeta_F} \int_{|s|=\delta} M_4(\lambda, \Psi \mid \mathbb{1}, s) \frac{ds}{2\pi i},
\]
which holds as \( \lambda \to \overline{0} \), proving the regularity with
\[
\sum_{j=5}^{8} DG_j(\overline{0}, \Psi) = -\frac{1}{\zeta_F} \int_{|s|=\delta} M_4(\overline{0}, \Psi \mid \mathbb{1}, s) \frac{ds}{2\pi i} = -\frac{1}{\zeta_F} \text{Res}_{\alpha=0} M_4(\overline{0}, \Psi \mid \mathbb{1}, s).
\]
The first formula in Proposition 1.6 readily follows. The second formula in Proposition 1.6 can be proved in the same way. We leave the details to the reader.

**Remark 5.1.** The regularity at \( \lambda = (0,0,0) \) for other \( \omega, \chi_1, \chi_2 \) can be proven in a similar way. We leave the case-by-case details to the reader.

## 6. Proof of Compact Variation

### 6.1. Mixed Moment Side

**Theorem 6.1** (Godement-Jacquet pre-trace formula: compact variation). The integral
\[
\overline{KK_D} \left( \frac{x,y}{\omega}, s_0 \right) := \int_{\mathbf{F} \times \mathbf{A}^\times} \overline{KK_D}(x,yz) \omega(z) z_{\mathbf{A}}^{s_0+2d^\times} \cdot |\nu_D(x^{-1}y)|_{\mathbf{A}}^{s_0}
\]
is absolutely convergent for \( \Re s_0 > 1 \), and defines a smooth function in \( \mathbf{D}^\times(\mathbf{F}) \setminus \mathbf{D}^\times(\mathbf{A}) \) with central character \( \omega \) for \( x \) resp. \( \omega^{-1} \) for \( y \). Its Fourier inversion with respect to \( y \) in \( L^2(\mathbf{D}^\times(\mathbf{F}) \setminus \mathbf{D}^\times(\mathbf{A}), \omega^{-1}) \) converges normally for \( (x,y) \in (\mathbf{D}^\times(\mathbf{F}) \setminus \mathbf{D}^\times(\mathbf{A}))^2 \), and takes the form
\[
\overline{KK_D} \left( \frac{x,y}{\omega}, s_0 \right) = \sum_{\pi \ cusp \ \mathbb{R} \times \mathbb{R}} \sum_{\varphi_1, \varphi_2 \in \mathbb{R}(\pi)} Z \left( s_0 + \frac{1}{2}, \varphi_1 \right) \varphi_1(x) \varphi_2(y) + \frac{1}{\text{Vol}([\mathbf{D}^\times])} \sum_{\eta \in \mathbf{F} \times \mathbf{A}^\times} \left( \int_{\mathbf{D}^\times(\mathbf{A})} \Psi(g) |\nu_D(g)|^{s_0+1} \, dg \right) \eta(\nu_D(x)) \eta(\nu_D(y)).
\]

**Proof.** There exist \( i,j,k = ij \in \mathbf{D}^\times(\mathbf{F}) \) so that \( \mathbf{D} = \mathbf{F} \oplus \mathbf{F}^i \oplus \mathbf{F}^j \oplus \mathbf{F}^k \) as \( \mathbf{F} \)-vector spaces. Hence \( S(\mathbf{D}(\mathbf{A})) \simeq S(\mathbf{A}^4) \). We have by definition
\[
\overline{KK_D} \left( \frac{x,y}{\omega}, s_0 \right) \cdot |\nu_D(x^{-1}y)|_{\mathbf{A}}^{-s_0} = \int_{\mathbf{F} \times \mathbf{A}^\times} \left( \sum_{\xi \in \mathbf{D}^\times(\mathbf{F})} L_x R_y \Psi(\xi z) \right) \left| z_{\mathbf{A}}^{s_0+2d^\times} z \right| \left| z_{\mathbf{A}}^{-s_0} \right| \left| z_{\mathbf{A}}^{-s_0} \right| \left| z_{\mathbf{A}}^{-s_0} \right| 
\]
where we have written out the most problematic term (for convergence) and omitted the others. Proposition 2.8 implies for any \( N > 1 \)
\[
\sum_{\alpha_0, \alpha_1, \alpha_2, \alpha_3 \in \mathbf{F}^\times} L_x R_y \Psi(z(\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k)) \ll_N \left( \min(|z_{\mathbf{A}}^{-1}|, |z_{\mathbf{A}}^{-N}|) \right)^4.
\]
Hence the above integral converges absolutely for \( \Re s_0 > 1 \). The proofs of the other assertions are straightforward analogues of those of Theorem 2.10. We leave the details to the reader. \( \square \)
Theorem 6.1 implies readily the decomposition in the region $\Re s_0 > 1$ of
$$\Theta_D(s_0, \Psi)$$

$$= \sum_{\omega_\mathcal{D}} \sum_{\varphi_1, \varphi_2 \in \mathcal{E}^1(\mathcal{D})} \mathcal{Z}(s_0 + \frac{1}{2}, \varphi_1, \varphi_2, \varphi_1^\gamma) \int_{\mathcal{E}^1(\mathcal{D})} \varphi_1(t_1) \Omega(t_1)^{-1} t_1^x \int_{\mathcal{E}^1(\mathcal{D})} \varphi_2^\gamma(t_2) \Omega(t_2) t_2^x t_2^y$$
$$+ \frac{\text{Vol}([\mathcal{E}^1])^2}{\text{Vol}([\mathcal{D}])} \sum_{\eta \in \mathcal{E}^1(\mathcal{D})} \left( \int_{\mathcal{E}^1(\mathcal{D})} \Psi(g) \eta(tD(g)) |tD(g)|^{s_0 + 1} \eta dg \right) \cdot \Omega_{\eta \oplus \eta}$$
$$= M(s_0, \Psi) + DS(s_0, \Psi).$$

The terms on the right hand side have meromorphic continuation to $s_0 \in \mathbb{C}$ as we explained in §6.3.

6.2. Second Moment Side. The group action of $H = (\mathbb{E}^x \times \mathbb{E}^y)/\mathbb{F}^x$ on $\mathbb{E}^2 \simeq \mathcal{D}, (x, y) \mapsto x + yj$ is identified as
$$H \times \mathbb{E}^2 \to \mathbb{E}^2, \quad (t_1, t_2) \times (x_1, x_2) \mapsto (t_1^{-1} t_2 x_1, t_1^{-1} t_2 x_2).$$

The above action of $H$ is identified as the action of
$$\tilde{H} := \left\{ (t_1, t_2) \in \mathbb{E}^x \times \mathbb{E}^y \mid t_1 t_2^{-1} \in \mathbb{E}^1 \right\} \subset \mathbb{E}^x \times \mathbb{E}^y$$
by Hilbert’s Theorem 90, where $\mathbb{E}^x \times \mathbb{E}^y$ acts component by component on $\mathbb{E}^2$ as
$$\mathbb{E}^x \times \mathbb{E}^y \to \mathbb{E}^2, \quad (t_1, t_2) \times (x_1, x_2) \mapsto (t_1 x_1, t_2 x_2).$$

Our chosen character of $H$ defined in (1.12) is identified with the character on $\tilde{H}$
(6.1)
$$\Omega_{\tilde{H}} : \tilde{H} \to \mathbb{C}^\times, \quad (t_1, t_2) \mapsto \Omega(t_1).$$

Since $\mathcal{D}^x(\mathbb{F}) = (\mathbb{E} + \mathbb{E}j) - \{0\}$, we have the orbital decomposition for the action of $\mathbb{E}^x \times \mathbb{E}^y$
$$\mathcal{D}^x(\mathbb{F}) = (\mathbb{E}^x + \mathbb{E}^x j) \bigoplus \mathbb{E}^x \bigoplus \mathbb{E}^x j.$$

Consequently, we have another decomposition in the region $\Re s_0 > 1$ of
$$\Theta_D(s_0, \Psi) = \int_{\mathbb{E}^x \times \mathbb{E}^y \backslash \mathbb{A}^\times} \left( \sum_{\alpha, \beta \in \mathbb{E}^x} \Psi(t_1^{-1} t_2 \alpha + t_1^{-1} t_2 \beta j) \right) \Omega(t_1^{-1} t_2) |t_1^{-1} t_2|^{s_0 + 1} t_1^x t_2^x t_2^y$$
$$\int_{\mathbb{E}^x \times \mathbb{E}^y \backslash \mathbb{A}^\times} \left( \sum_{\alpha \in \mathbb{E}^x} \Psi(t_1^{-1} t_2 \alpha) \right) \Omega(t_1^{-1} t_2) |t_1^{-1} t_2|^{s_0 + 1} t_1^x t_2^x t_2^y$$
$$\int_{\mathbb{E}^x \times \mathbb{E}^y \backslash \mathbb{A}^\times} \left( \sum_{\beta \in \mathbb{E}^y} \Psi(t_1^{-1} t_2 \beta) \right) \Omega(t_1^{-1} t_2) |t_1^{-1} t_2|^{s_0 + 1} t_1^x t_2^x t_2^y$$
$$\int_{\mathbb{E}^x \times \mathbb{E}^y \backslash \mathbb{A}^\times} \left( \sum_{\alpha, \beta \in \mathbb{E}^x} \Psi(t_1 \alpha + t_1 \beta j) \right) \Omega(t_1) |t_1|^{s_0 + 1} t_1^x t_2^x t_2^y$$
$$\int_{\mathbb{E}^x \times \mathbb{E}^y \backslash \mathbb{A}^\times} \left( \sum_{\alpha, \beta \in \mathbb{E}^x} \Psi(t_1 \alpha + t_1 \beta j) \right) \Omega(t_1) |t_1|^{s_0 + 1} t_1^x t_2^x t_2^y \cdot \text{Vol}([\mathcal{E}^1]) \cdot \int_{\mathbb{A}^\times} \Psi(t) \Omega(t) |t|^{s_0 + 1} t_1 + \text{Vol}([\mathcal{E}^1]) \cdot \int_{\mathbb{A}^\times} \Psi(t) \Omega(t) |t|^{s_0 + 1} t_1 + \text{Vol}([\mathcal{E}^1]) \cdot \int_{\mathbb{A}^\times} \Psi(t) \Omega(t) |t|^{s_0 + 1} t_1 + \text{Vol}([\mathcal{E}^1]) \cdot \int_{\mathbb{A}^\times} \Psi(t) \Omega(t) |t|^{s_0 + 1} t_1 \cdot \Omega_{\eta \oplus \eta} = 1.$$

The second resp. third term is simply $DG_1(s_0, \Psi)$ resp. $DG_2(s_0, \Psi)$, which has obvious meromorphic continuation to $s_0 \in \mathbb{C}$ regular at $s_0 = 0$. For the first term, we have by Proposition 2.8
$$\sum_{\alpha, \beta \in \mathbb{E}^x} |\Psi(t_1 \alpha + t_1 \beta j)| \ll_{p, N} \min(|t_1|_{\mathcal{A}_E^1}^{-1} |t_1|_{\mathcal{A}_E^{-N}}) |t_1 t_2|_{\mathcal{A}_E}^{-p} \cdot \forall p, N \geq 1.$$

Thus for any $p > 1$ and $\Re s_0 > 1 + p$, the function
$$g : \mathbb{E}^x \backslash \mathbb{A}^\times \to \mathbb{C}, \quad g(t) = \int_{\mathbb{E}^x \backslash \mathbb{A}^\times} \left( \sum_{\alpha, \beta \in \mathbb{E}^x} \Psi(t_1 \alpha + t_1 \beta j) \right) \Omega(t_1) |t_1|^{s_0 + 1} t_1^x t_2^x t_2^y$$
is smooth, invariant by $E^1(F) \backslash E^1(\mathbb{A}) < E^\times \backslash E^\times_F$ and satisfies the bound

$$|g(t)| \ll_p \min(|t|_F^{-1}, |t|_{\kappa}^{-p}).$$

The same bound holds if we replace $g(t)$ by $D^n g(t)$ for the invariant differential $D$ on $s_E(\mathbb{R}_+)$, where $s_E$ is a/any section map of the adelic norm map on $E^\times \backslash E^\times_F$. Thus $g(t)$ is Mellin invertible with with

$$g(1) = \frac{1}{\text{Vol}(E^\times \backslash E^1(\mathbb{A}) \backslash E^\times_F)} \sum_{E^\times \backslash E^1(\mathbb{A}) \backslash E^\times_F} \int_0^1 \int_{E^\times \backslash E^1(\mathbb{A}) \backslash E^\times_F} g(t) \Xi(s) t^s d^x t ds \frac{ds}{2\pi i},$$

where $1 < c < \Re s_0 - 1$ is arbitrary. Write $\Omega = \Omega_0 |_{\kappa}^{s_0} / \Psi$ for a unique $s_0 \in \mathbb{R}$ such that $\Omega_0 |_{\kappa} = 1$. Moreover, we can rewrite the above double integral with a contour shift to $c' > \Re s_0 + 1$, obtaining

$$\int_{c'} \int_{A_{E} \times A_{E}} g(t) \Xi(s) t^s d^x t ds \frac{ds}{2\pi i}$$

$$= \int_{c'} \int_{A_{E} \times A_{E}} \Psi(t_1 + t_2 j) \Omega^{-1}(t_1 |_{A_{E}}^{s_0 + 1 - s} \Xi(t_2) |_{A_{E}}^s t_1 d^x t_2 ds \frac{ds}{2\pi i}$$

$$= \int_{c'} \int_{A_{E} \times A_{E}} \Psi(t_1 + t_2 j) \Omega^{-1}(t_1 |_{A_{E}}^{s_0 + 1 - s} \Xi(t_2) |_{A_{E}}^s t_1 d^x t_2 ds + \zeta_E^{s} \Omega_0 \cdot \int_{A_{E}} \int_{A_{E}} \Psi(x_1 + t_2 j) dx_1 \Omega(t_2) |_{A_{E}}^{s_0} t_2 - \zeta_E^{s} \Omega_0 \cdot \int_{A_{E}} \Psi(t_2 j) \Omega(t_2) |_{A_{E}}^{s_0 + 1} d^x t_2.$$

All three terms have meromorphic continuation to $s_0 \in \mathbb{C}$. We have proved the meromorphic continuation of $\Theta_D(s_0, \Psi)$ to $s_0 \in \mathbb{C}$.

Assuming $|s_0| < 1/4$, we shift the contour to get

$$\int_{c'} \int_{A_{E} \times A_{E}} \Psi(t_1 + t_2 j) \Omega^{-1}(t_1 |_{A_{E}}^{s_0 + 1 - s} \Xi(t_2) |_{A_{E}}^s t_1 d^x t_2 ds \frac{ds}{2\pi i}$$

$$= \int_{c} \int_{A_{E} \times A_{E}} \Psi(t_1 + t_2 j) \Omega^{-1}(t_1 |_{A_{E}}^{s_0 + 1 - s} \Xi(t_2) |_{A_{E}}^s t_1 d^x t_2 ds + \zeta_E^{s} \Omega_0 \cdot \int_{A_{E}} \int_{A_{E}} \Psi(t_1 + x j) dx \Omega(t_1) |_{A_{E}}^{s_0} t_1.$$

We thus get

$$\Theta_D(s_0, \Psi) = \frac{1}{\text{Vol}([E^1(\mathbb{E}) \backslash E^1 \backslash E^\times_F])} \sum_{E^\times \backslash E^1(\mathbb{E})} \int_{(c)} \int_{A_{E} \times A_{E}} \Psi(t_1 + t_2 j) \Omega^{-1}(t_1 |_{A_{E}}^{s_0 + 1 - s} \Xi(t_2) |_{A_{E}}^s t_1 d^x t_2 ds \frac{ds}{2\pi i}$$

$$+ \frac{\zeta_E^{s}}{\text{Vol}([E^1(\mathbb{E}) \backslash E^1 \backslash E^\times_F])} \int_{A_{E}} \int_{A_{E}} \Psi(t_1 + x j) dx \Omega(t_1) |_{A_{E}}^{s_0} d^x t_1$$

$$+ \frac{\zeta_E^{s}}{\text{Vol}([E^1(\mathbb{E}) \backslash E^1 \backslash E^\times_F])} \int_{A_{E}} \int_{A_{E}} \Psi(t_1 + x j) dx \Omega(t_2) |_{A_{E}}^{s_0} d^x t_2$$

$$+ \text{Vol}([E^\times_F]) \cdot \int_{A_{E}} \Psi(t) \Omega(t) |_{A_{E}}^{s_0 + 1} d^x t_1 + \text{Vol}([E^\times_F]) \cdot \int_{A_{E}} \Psi(t) \Omega(t) |_{A_{E}}^{s_0 + 1} d^x t_1 \cdot \Omega_1 |_{\kappa}^{s_0} \cdot \Omega_0 |_{\kappa}^{s_0}$$

$$= M_2(s_0, \Psi) + \sum_{j=1}^{4} DG_j(s_0, \Psi).$$

Note that the terms containing $\Omega_1 |_{\kappa}^{s_0} = \Omega_0 |_{\kappa}^{s_0}$, in which case the restriction of $\Omega$ to $E^1(\mathbb{A})$ necessarily has to be trivial, since $\Xi$ is so.
7. Appendix: Comparison with Period Approach

7.1. Recall of Period Approach. For simplicity, we consider the case where $\omega = \chi_1 = \chi_2 = 1$ are trivial character. The period approach, proposed by Michel-Venkatesh, considers the regularized integral

$$\int_{\mathbb{F} \setminus \mathbb{A}} \text{E}(s_1, f_1) \cdot \text{E}(s_2, f_2)(a(t))d^ \times t$$

along the diagonal torus of the product of two Eisenstein series constructed from smooth vectors $f_1, f_2 \in \mathcal{B}(1, 1)$. One expects a suitable Fourier inversion formula for this product, so that the projection to the space of a cuspidal representation $\pi$ gives the contribution

$$(7.1) \quad \sum_{\varphi \in \mathcal{B}(\pi)} \langle \text{E}(s_1, f_1) \cdot \text{E}(s_2, f_2), \varphi \rangle \int_{\mathbb{F} \setminus \mathbb{A}} \varphi(a(t))d^ \times t.$$ 

By Hecke-Jacquet-Langlands' theory, the above integral represents $L(1/2, \pi)$. By the Rankin-Selberg theory, the automorphic Fourier coefficient represents $L(1/2 + s_1, \pi^\vee)L(1/2 + s_2, \pi^\vee)$. Hence $(7.1)$ represents a certain third moment. This approach has been made rigorous by Nelson $[19]$. Precisely, Nelson

- defined the above regularized integral as the special value at $s = 0$ of the following integral (equivalent to $\ell_s$ defined by $[19$, (10.3)]

$$I(s, s_1, s_2) := \int_{\mathbb{F} \setminus \mathbb{A}} \langle \text{E}(s_1, f_1) \cdot \text{E}(s_2, f_2) - E_N(s_1, f_1) \cdot E_N(s_2, f_2)(a(t)) |t|^s d^\times t,$$

which converges for $\Re s \gg 1$ (depending on $s_1, s_2$) and has meromorphic continuation$^4$ to $s \in \mathbb{C}$;
- restricted to $|s_1|, |s_2|$ small;
- expanded $I(s, s_1, s_2)$ in two different ways and analytically continue the obtained equation to $s = s_1 = s_2 = 0$ to deduce the relevant Motohashi-type formula.

We compare the period approach with our approach in this section. The comparison seems to be more convenient in the following region

$$\{(s_1, s_2) \in \mathbb{C}^2 \mid \Re s_1, \Re s_2 - \Re s_1 > 1/2\}$$

instead of the original region$^4$ considered by Nelson $[19$, Theorem 10.2], where $|s_1|$ and $|s_2|$ are small.

The proofs are easy computation hence left to the reader.

7.2. Comparison of Geometric Sides. Recall the Godement section constructed from $\Phi \in \mathcal{S}(\mathbb{A}^2)$, absolutely convergent for $\Re s > 0$

$$f_{\Phi}(s, g) := |\det g|_A^{1+s} \int_{\mathbb{A}} \Phi((0, t)g)|t|^1 |^1_{\mathbb{A}} d^\times t.$$ 

Lemma 7.1. Let $E(s, \Phi)$ be the Eisenstein series formed from $f_{\Phi}(s, \cdot)$. Then its constant term is

$$E_N(s, \Phi)(g) = f_{\Phi}(s, g) + f_{\Phi}(-s, g),$$

with $f_{\Phi}(-s, g) = |\det g|_A^{1+s} \int_{\mathbb{A}} \tilde{\Phi}_2 L_{t^{-1}} R_{g} \Phi(1, 0)|t|^1 |^1_{\mathbb{A}} d^\times t,$

where by abuse of notations we have denoted by $L$ (and $R$) the action

$$L_t R_{g} \Phi(x, y) := \Phi(t^{-1}x, t^{-1}y)g).$$

We also have the difference, absolutely convergent for all $s \in \mathbb{C}$

$$(E(s, \Phi) - E_N(s, \Phi))(g) = |\det g|_A^{1+s} \int_{\mathbb{F} \setminus \mathbb{A}} \left( \sum_{\alpha_1, \alpha_2 \in \mathbb{F}_x} \tilde{\Phi}_2 L_{t^{-1}} R_{g} \Phi(\alpha_1, \alpha_2) \right) |t|^1 |^1_{\mathbb{A}} d^\times t.$$ 

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$^1$In fact, this follows also from the theory of zeta function for finitely regularizable functions in $[26, \S 2.3]$. 
$^2$We do not know how to work directly with the original region. Nor can we unfortunately match the degenerate terms with Nelson’s version.
Corollary 7.2. Let $\Phi_1, \Phi_2 \in S(\mathcal{A}^2)$ and define $\Psi \in S(M_2(\mathcal{A}))$ by

$$\Psi \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} := \Phi_1(x_1, x_2)\Phi_2(x_3, x_4).$$

Then for $\Re s \gg 1$, we have a decomposition of

$$\tilde{I}(s, s_1, s_2) := \int_{F \times \mathcal{A}^\times} (E(s_1, \Phi_1) \cdot E(s_2, \Phi_2) - E_N(s_1, \Phi_1) \cdot E_N(s_2, \Phi_2)) \sigma(t)|t|^\Re s dt$$

as $\tilde{I}(s, s_1, s_2) = \sum_{j=0}^{4} \tilde{I}_j(s, s_1, s_2)$, where (recall the convention $[7, 7]$),

$$\tilde{I}_0(\cdot) = \int_{(F \times \mathcal{A}^\times)^3} \sum^* \mathfrak{S}_2 \mathfrak{S}_4 L_d(t_1, t_2) \cdot 1 \cdot R_{\mathfrak{a}}(t) \Psi \begin{pmatrix} \xi_1 & \xi_2 \\ \xi_3 & \xi_4 \end{pmatrix} |t_1|^{1+2s_1} |t_2|^{1+2s_2} |t_3|^{1+s_1+s_2+s} d^\times t_1 d^\times t_2 d^\times t,$n

$$\tilde{I}_1(\cdot) = \int_{(F \times \mathcal{A}^\times)^3} \sum^* \mathfrak{S}_2 L_d(t_1, t_2) \cdot 1 \cdot R_{\mathfrak{a}}(t) \Psi \begin{pmatrix} \xi_1 & \xi_2 \\ \xi_3 & \xi_4 \end{pmatrix} |t_1|^{1+2s_1} |t_2|^{1+2s_2} |t_3|^{1+s_1+s_2+s} d^\times t_1 d^\times t_2 d^\times t,$n

$$\tilde{I}_2(\cdot) = \int_{(F \times \mathcal{A}^\times)^3} \sum^* \mathfrak{S}_2 L_d(t_1, t_2) \cdot 1 \cdot R_{\mathfrak{a}}(t) \Psi \begin{pmatrix} \xi_1 & \xi_2 \\ \xi_3 & \xi_4 \end{pmatrix} |t_1|^{1+2s_1} |t_2|^{1+2s_2} |t_3|^{1+s_1+s_2+s} d^\times t_1 d^\times t_2 d^\times t,$n

$$\tilde{I}_3(\cdot) = \int_{(F \times \mathcal{A}^\times)^3} \sum^* \mathfrak{S}_4 L_d(t_1, t_2) \cdot 1 \cdot R_{\mathfrak{a}}(t) \Psi \begin{pmatrix} \xi_1 & \xi_2 \\ \xi_3 & \xi_4 \end{pmatrix} |t_1|^{1+2s_1} |t_2|^{1+2s_2} |t_3|^{1+s_1+s_2+s} d^\times t_1 d^\times t_2 d^\times t,$n

$$\tilde{I}_4(\cdot) = \int_{(F \times \mathcal{A}^\times)^3} \sum^* \mathfrak{S}_2 L_d(t_1, t_2) \cdot 1 \cdot R_{\mathfrak{a}}(t) \Psi \begin{pmatrix} \xi_1 & \xi_2 \\ \xi_3 & \xi_4 \end{pmatrix} |t_1|^{1+2s_1} |t_2|^{1+2s_2} |t_3|^{1+s_1+s_2+s} d^\times t_1 d^\times t_2 d^\times t.$n

We apply to $\tilde{I}_0(s, s_1, s_2)$ the method leading to the decomposition (41.10), and will get four other degenerate terms. Together with $\tilde{I}_j(s, s_1, s_2)$ for $1 \leq j \leq 4$, these eight degenerate terms correspond precisely to our $D_{\mathfrak{a}}$ for $1 \leq j \leq 8$ given in (1.20) to (1.27). The geometric side of the period approach does not contain terms corresponding to our $H_3$ & $H_4$, which are moved to the spectral side.

7.3. Comparison of Spectral Sides. The relation of $I(s, s_1, s_2)$ or $\tilde{I}(s, s_1, s_2)$ to the third moment of $L$-functions for $\text{GL}_2$ is given by the automorphic Fourier inversion of

$$E(s_1, \Phi_1) \cdot E(s_2, \Phi_2) = \tilde{E},$$n

where $\tilde{E} = \tilde{E}_1 + \tilde{E}_2$ and $\tilde{E}_j$ are Eisenstein series constructed from

$$f_{\Phi_1}(s_1, \cdot) f_{\Phi_2}(s_2, \cdot), \quad \text{resp.} \quad f_{\Phi_1}(-s_1, \cdot) f_{\Phi_2}(s_2, \cdot).$$n

First, we notice that for $\Re s > 1/2$

$$E(s, \Phi)(g) = \sum_{\gamma \in B(F) \setminus \text{GL}_2(F)} f_{\Phi}(s, \gamma g) = |\text{det } \gamma|^{\Re s} \int_{F \times \mathcal{A}^\times} \sum_{\mathfrak{a} \in F^2 \setminus \{0, 0\}} \Phi(t\mathfrak{a}g)|t|^{1+2s} d^\times t.$$n

Hence we get

$$(E(s_1, \Phi_1)E(s_2, \Phi_2))(g) = |\text{det } \gamma|^{1+s_1+s_2} \int_{(F \times \mathcal{A}^\times)^2} \sum_{\xi \in M_2^*(F)} L_d(t_1, t_2) \cdot 1 \cdot R_g(\xi)|t_1|^{1+2s_1}|t_2|^{1+2s_2} d^\times t_1 d^\times t_2,$n

where $M_2^*(F) \subset M_2(F)$ is defined by

$$M_2^*(F) = \left\{ \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in M_2(F) \mid \xi_j \neq (0, 0), j = 1, 2 \right\}.$$n

By definition and the formulas in Lemma (7.1), it is easy to deduce that $\tilde{E}_1$ and $\tilde{E}_2$ are given by

$$|\text{det } \gamma|^{1+s_1+s_2} \int_{(F \times \mathcal{A}^\times)^2} \sum_{\gamma \in B(F) \setminus \text{GL}_2(F)} \sum^* L_d(t_1, t_2) \cdot 1 \cdot R_g(\xi)|t_1|^{1+2s_1}|t_2|^{1+2s_2} d^\times t_1 d^\times t_2,$n

$$|\text{det } \gamma|^{1+s_1+s_2} \sum_{\gamma \in B(F) \setminus \text{GL}_2(F)} \int_{(F \times \mathcal{A}^\times)^2} \sum^* \mathfrak{S}_2 L_d(t_1, t_2) \cdot 1 \cdot R_g(\xi)|t_1|^{1+2s_1}|t_2|^{1+2s_2} d^\times t_1 d^\times t_2.$$n

ON Motohashi’s Formula 37
It is easy to verify
\[ \bigcup_{\xi_2, \xi_4 \in F^x} \begin{bmatrix} 0 & \xi_2 \\ 0 & \xi_4 \end{bmatrix} \operatorname{GL}_2(F) = M_2^+(F) - \operatorname{GL}_2(F). \]

Hence we identify the following difference as an absolutely convergent integral
\[
\left( E(s_1, \Phi_1) \cdot E(s_2, \Phi_2) - \tilde{\Delta} \right)(g) = \int_{F^x \setminus \Delta^x} \Delta \tilde{R}^{(2)} \left( \frac{a(t_1); g}{1, 2(s_1 + s_2)} \right) |t_1|^{s_2 - s_1} d^x t_1,
\]
whose contribution to the regularized integral \[19\, (10.6)\]
\[
\int_{F^x \setminus \Delta^x} \left( \left( E(s_1, \Phi_1) \cdot E(s_2, \Phi_2) - \tilde{\Delta} \right) - \left( E(s_1, \Phi_1) \cdot E(s_2, \Phi_2) - \tilde{\Delta} \right)_{\mathbf{L}} \right)(a(y)) |y|^a d^x y
\]
\[
= \int_{(F^x \setminus \Delta)^2} \Delta \tilde{R}^{(2)} \left( \frac{a(t_1); a(t_2)}{1, 2(s_1 + s_2)} \right) |t_1|^{s_2 - s_1} |t_2|^a d^x t_1 d^x t_2
\]
is the essential part of the main Mochishka distribution we considered in Proposition 13.

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