Scale-dependent planar Anti-de Sitter black hole

Ángel Rincón\textsuperscript{a,1}, Ernesto Contreras\textsuperscript{c,3}, Pedro Bargueño\textsuperscript{b,2}, Benjamin Koch\textsuperscript{d,1}

\textsuperscript{1}Instituto de Física, Pontificia Universidad Católica de Chile, Av. Vicuña Mackenna 4860, Santiago, Chile.
\textsuperscript{2}Departamento de Física, Universidad de los Andes, Cra.1E No.18A-10, Bogotá, Colombia.
\textsuperscript{3}Yachay Tech University, School of Physical Sciences & Nanotechnology, 100119-Urcuquí, Ecuador.

Abstract In this work, we investigate four-dimensional planar black hole solutions in anti-de Sitter spacetimes in light of the so-called scale-dependent scenario. To obtain this new family of solutions, the classical couplings of the theory, i.e., the gravitational coupling $G_0$ and the cosmological constant $\Lambda_0$, are not taken to be fixed values anymore. Thus, those classical parameters evolve to functions which change along the “height” coordinate, $z$. The effective Einstein field equations are solved, and the results are analyzed and compared with the classical counterpart. Finally, some thermodynamic properties of the presented scale–dependent black hole are investigated.

1 Introduction

Although a consistent formulation of quantum gravity remains an open task, there are several promising approaches in this direction. Even though those candidate theories differ in their approach, their variables, and techniques, they have a useful common feature. Their low energy effective action for the gravitational field acquires a scale dependence. This is observed through the coupling constants which evolve from constant values to scale–dependent functions with respect to certain energy scale. Similar approaches have been considered before, but the motivation and implementation in those approaches is quite different. This is the case of the Brans–Dicke (BD) theory \cite{Brans1961}, which treats the Newton coupling constant as an auxiliary scalar field. Thus, adopting this formalism, the link between $G$ and $\phi$ is just $\phi \to G^{-1}$ which means that the Einstein coupling constant takes the equivalent form $\kappa \equiv 8\pi \phi^{-1}$. This deviation from the classical Einstein Gravity take into account that Newton coupling could be a field and not a fixed value. Despite of it, BD theory is still a classical theory and it does not include the possibility for the other parameters included in the action to evolve to scale–dependent functions. What is more, it is very–well known that an effective description takes the effective action as a functional whose coefficients show a scale dependence, which is a generic result of quantum field theory.

In this sense, the aforementioned effective action $\Gamma[g_{\mu\nu}, k]$ contains a set of couplings inherited from the classical theory but incorporating the scale dependence, where $k$ stands for an undetermined scale–dependence. Specifically, $\{G_k, (\cdots)_k\}$ comes from $\{G_0, (\cdots)_0\}$ (where $(\cdots)$ denotes any other coupling present in the theory). The probably most successful implementation of those ideas was achieved within the so called Asymptotic Safety (AS) program, where a non-trivial ultra violet fixed point for the leading dimensionless gravitational couplings was conjectured \cite{Friedemann2008} and found \cite{Reuter2009–2016}.

Recently, scale–dependent gravity has been used to construct black hole backgrounds both by improving classical solutions with the scale dependent couplings from AS \cite{Hennig2017–2018} and by solving the gap equations of a generic scale dependent action \cite{Alberte2015–2018}. Even more, regular black holes \cite{Ramos2019} and traversable (vacuum) wormholes \cite{Koch2018} have been shown to exist within this approach. In this sense, scale–dependent gravity might shed light on how to cure, in an effective way, some of the classical problems which appear in classical general relativity. From the cosmological side, the impact of scale dependence has been explored in various ways \cite{Mottola1988–1990, Kehagias1997, Chimento2000}.

It is important to note that almost all the exact black hole solutions found in the context of scale–dependent gravity (but the cosmological and a rotating scale–dependent BTZ black hole which has been recently reported \cite{Ramos2019} belong to
the spherically symmetric case. Therefore, the role of different geometries for scale–dependent black hole solutions (if any), remains to be investigated. This is the purpose of the present work, with emphasis in planar black hole geometries. Although this work could be easily extended to the the toroidal case, we do not expect substantial differences with respect to the spherically symmetric case. On the contrary, the planar nature of the scale–dependent black hole we will present in the present work makes it an ideal candidate to see the effects of scale dependence when a non–compact event horizon is present.

The manuscript is organized as follows. In Sect. 2 we review the main aspects of the classical planar AdS black hole solution. Section 3 is devoted to introduce the scale–dependent model. In sections 4 and 5 we obtain the scale–dependent solution and study their geometrical and thermodynamical aspects. Some final comments are given in the last section.

2 Classical planar Anti-de Sitter theory and black hole solution

The Einstein-Hilbert action is, in four dimensions, given by

$$I_0[g_{\mu\nu}] = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa_0} \left( R - 2\Lambda_0 \right) \right],$$

(1)

where $\kappa_0 \equiv 8\pi G_0$ is the gravitational coupling, $G_0$ is Newton’s constant, $\Lambda_0$ is the cosmological coupling, $g$ is the determinant of the metric and $R$ the Ricci scalar. In what follows we assume that the space-time is plane-symmetric and time-independent. Besides, we assume the coordinate set $x^\mu = \{t, x, y, z\}$, we use the metric signature $\{+, +, +, -\}$, and natural units ($c = \hbar = k_B = 1$) such that the action is dimensionless. The line element is then written according to

$$d\tau^2 = -f_0(z)dt^2 + f_0(z)^{-1}dz^2 + (L_z)^2(dx^2 + dy^2).$$

(2)

Please, note that the term $L_z$ is dimensionless. In addition, it is remarkable that the cosmological coupling is usually related to $L$ by $3L_z^2 \equiv -\Lambda_0 > 0$ (where $\Lambda_0$ denotes the negative cosmological constant). This constraint is, however, relaxed in order to obtain a more general set of solutions. To be consistent with the classical scale setting, we take $G_0 = 1$. Varying the classical action 1 yields the equations of motion, i.e.,

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = -\Lambda_0 g_{\mu\nu}.$$ 

(3)

For a vacuum solution we only have the cosmological constant contribution, and the lapse function becomes

$$f_0(z) = -\frac{1}{3}\Lambda_0 z^2 - \frac{4M_0}{L_z}.$$ 

(4)

or, in terms of the event horizon, $z_0$, we have

$$f_0(z) = -\frac{1}{3}\Lambda_0 z^2 \left[ 1 - \left( \frac{z_0}{z} \right)^3 \right] ,$$

(5)

where the aforementioned horizon is then given by

$$z_0^3 = \frac{12M_0}{\Lambda_0 L_z}.$$ 

(6)

Due to the cubic nature of the line element there are three possible horizons, however, only one of them is real and it is defined as the classical event horizon. The two extra imaginary roots of (4) have no physical meaning. Notice that $M_0$ is the classical black hole mass. What is more, given the non-compactibility of the coordinates $x$ and $y$, we only consider the mass per unit area in the $x - y$ plane [74].

At this point we move to thermodynamics of the black plane solutions. The starting point is the Euclidean action method [75, 76]. First, note that the metric can be written in terms of the Euclidean time $\tau$ after the change $t \rightarrow -i\tau$

$$ds^2 = f_0(z)dt^2 + f_0(z)^{-1}dz^2 + (L_z)^2(dx^2 + dy^2).$$

(7)

Thus, in order to obtain the Hawking temperature, the requirement of the absence of the conical singularity in the Euclidean space-time (7) causes the Euclidean time $\tau$ to have a period $\bar{\sigma}_0$, which verifies that the temperature is given by

$$T_0(z_0) = \frac{1}{4\pi} \lim_{\tau \rightarrow 0} \left| \frac{\partial g_{tt}}{\sqrt{-g}} \right| = \frac{1}{2\pi} \left| \left( \frac{3\Lambda_0}{L_z} \right)^{1/3} \left( \frac{1}{2}M_0 \right)^{1/3} \right| \propto M_0^{1/3}.$$ 

(8)

Following the same line, the Bekenstein–Hawking entropy is given by the usual relation

$$S_0(z_0) = \frac{1}{4} \sigma_0 = \frac{3}{2} \pi \left( \frac{L_z}{3\Lambda_0^2} \right)^{1/3} \left( 4M_0 \right)^{2/3} \propto M_0^{2/3},$$

(9)

where the area of the horizon $\sigma_0$, per unit length, is in this case given by

$$\sigma_0 = 2\pi L_z z_0^2.$$ 

(10)

Finally, The heat capacity is

$$C_0(z_0) = T \left. \frac{\partial S}{\partial T} \right|_{z_0} = -S_0.$$ 

(11)

It is remarkable that the temperature goes as $M_0^{1/3}$, which strongly differs from the Schwarzschild black hole [77]. In this sense, both the negative cosmological constant and the planar topology of the horizon introduce a strong deviation from the Schwarzschild black hole case. In addition, it should be noted that, when $3L_z^2 = -\Lambda_0$, the aforementioned results are precisely those given in Ref. [78].
3 Scale–dependent gravity

As was previously commented in the introduction, one possible way of obtaining a self–consistent theory beyond General Relativity is, roughly speaking, by promoting the classical coupling constants to scale–dependent quantities. In this sense, effective quantum corrections to well–known black hole solutions are typically incorporated in two different ways: i) starting from the effective action, we vary $\Gamma[g_{\mu\nu}, k]$ to obtain the effective Einstein equations, and ii) starting from the solution, we replace the classical couplings with scale–dependent couplings. In particular, we focus on the first situation. The purpose of this section is to summarize the equations of motion for the scale–dependent Anti–de Sitter theory. Along this paper, we will follow the idea and notation adopted in Ref. [52–54, 58–60, 71, 79, 80]. After recognizing both the scale–dependent couplings of the theory, which are the Newton’s coupling $G_k$ (which can be related with the gravitational coupling by $k_0 \equiv 8\pi G_k$), and the cosmological coupling $\Lambda_k$ and the two independent fields, i.e., the metric field $g_{\mu\nu}(x)$ and the energy scale $k$, the scale–dependent Einstein–Hilbert effective action reads

$$\Gamma[g_{\mu\nu}, k] = \int d^4 x \sqrt{\gamma} \left[ \frac{1}{2k_0} \left( R - 2\Lambda_k \right) \right],$$

where $k$ is a scale-dependent field related to a renormalization scale, and $G_k$ and $\Lambda_k$ stand for the scale–dependent gravitational and cosmological couplings, respectively. First, taking variations with respect to the metric field $g_{\mu\nu}$ leads to modified Einstein’s equations

$$G_{\mu\nu} + g_{\mu\nu}A_k = -\Delta t_{\mu\nu},$$

where we use the so–called non–matter energy–momentum tensor, $\Delta t_{\mu\nu}$, defined according to [66, 82]

$$\Delta t_{\mu\nu} = G_k \left( g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu \right) G_k^{-1}. $$

We note that the strength of the gravitational and cosmological couplings, $G_k$ and $\Lambda_k$, determine the deviation of the theory with respect to the corresponding classical case, as expected. Second, taking the variation of the effective action with respect to the field $k(x)$, one imposes [80]

$$\frac{d}{dk} \Gamma[g_{\mu\nu}, k] = 0.$$  

This condition can be seen as an a posteriori condition towards background independence [81–87]. We must emphasize that the aforementioned equation gives a restriction between $G_k$ and $\Lambda_k$ which reveals that the cosmological parameter is indeed required to produce self–consistent scale–dependent solutions, at least when the matter sector is absent.

However, in order to solve these equations, we need the knowledge of the precise beta functions of the problem. Given that, in general, an unique solution for the beta functions is still an open question, we can avoid to assume any particular form for those. This means that we do not have enough information in order to find both $g_{\mu\nu}(x)$ and $k(x)$. One possibility to circumvent this issue is by considering that the couplings $\{G_k, \Lambda_k\}$ inherit the dependence on space–time coordinates from the space–time dependence of $k(x)$, thus the couplings are written as $\{G(z), \Lambda(z)\}$ [51, 52, 80], in combination with a simplifying ansatz for the line element. Although this procedure allows to solve the problem, a high degree of symmetry is usually necessary in order to obtain exact solutions.

In the next section we shall apply this method in order to obtain planar black hole solutions.

4 Scale–dependent planar AdS black hole

In order to obtain the complete solution with planar symmetry, we need to find the set $\{G(z), \Lambda(z)\}$. The running of the gravitational coupling introduces the tensor $\Delta t_{\mu\nu}$ and the effective Einstein field equations are

$$\Delta t_{\mu\nu}(x) \equiv G_{\mu\nu} + g_{\mu\nu}\Lambda(z) + \Delta t_{\mu\nu} = 0.$$  

The so–called non–matter energy momentum tensor, which encodes the running of the Newton coupling, is demanded to be zero in the classical limit. Therefore, a well–defined classical limit for the gravitational coupling should be imposed. This is achieved through the integration constants which play a crucial role here. Now we will move to the line element used to properly describe the geometry of this problem. Specifically, we will consider the line element parametrized as

$$d\tilde{s}^2 = -f(z)dr^2 + f(z)^{-1}dz^2 + (Lz)^2(dx^2 + dy^2),$$

where both $\Lambda$ and $G$ depend only on the $z$–coordinate due to the planar symmetry. First, the scale–dependent gravitational coupling solving one of the gravitational field equations has to obey

$$G(z) \frac{d^2 G(z)}{dz^2} - 2 \left( \frac{dG(z)}{dz} \right)^2 = 0,$$

which allows us to obtain the now well–know scale–dependent solution

$$G(z) = \frac{G_0}{1 + \varepsilon z},$$

where $\varepsilon$ controls the intensity of the running of the gravitational coupling. The rest of the field equations allow us to find the solution for the lapse function, which we write as

$$f(z) = f_0(z) + \frac{6M_0}{L} \varepsilon Y(z).$$
where the auxiliary function $Y(z)$ is defined to be

$$Y(z) \equiv 1 - 2\varepsilon z + 2(\varepsilon z)^2 \ln \left(1 + \frac{1}{\varepsilon z}\right).$$  \hspace{1cm} (21)

Finally, the cosmological scale–dependent coupling is obtained when the corresponding algebraic equation is used, which gives

$$\Lambda(z) = \Lambda_0 + \varepsilon \left(\frac{1}{Lz(1 + \varepsilon z)^2}\right)\lambda(z),$$  \hspace{1cm} (22)

where we have defined another supplementary function, $\lambda(z)$, written as

$$\lambda(z) = \Lambda_0 L^2(z + 2\varepsilon z(1 + 12\varepsilon(1 + \varepsilon z)) - 36M_0\varepsilon^2(\varepsilon z + 1)(2\varepsilon z + 1) \ln \left(\frac{1}{\varepsilon z} + 1\right).$$  \hspace{1cm} (23)

Note that the integration constants have been chosen such that we recover the classical solution after turning off the running parameter in the functions involved, as can be revealed in Fig. (1). Specifically, taking $\varepsilon \to 0$ in the scale–dependent solution we recover

$$\lim_{\varepsilon \to 0} G(z) = G_0 = 1,$$  \hspace{1cm} (24)

$$\lim_{\varepsilon \to 0} f(z) = f_0(z) = \left(Lz\right)^2 \left[1 - \left(\frac{z_0}{z}\right)^3\right],$$  \hspace{1cm} (25)

$$\lim_{\varepsilon \to 0} \Lambda(z) = \Lambda_0.$$  \hspace{1cm} (26)

Even more, as in general the scale–dependent effects are assumed to be weak, the running parameter is assumed to be small with respect to the other scales entering the problem such as $M_0$ and $G_0$ [80], we can write

$$G(z) \approx G_0(1 - \varepsilon z) + O(\varepsilon^2),$$

$$f(z) \approx f_0(z) + \frac{6M_0}{L}\varepsilon + O(\varepsilon^2),$$

$$\Lambda(z) \approx \Lambda_0(1 + \varepsilon z) + O(\varepsilon^2).$$

Interestingly, as pointed out in [80] regarding other scale–dependent geometries, the solution employed reveals novel long-range effects due to the scale–dependence because, for $\varepsilon \to 0$, the coordinate $z$ would have to be very large in order to note a deviation from the classical solution.

Within this limit we have

$$f(z) = -\frac{1}{3}\Lambda_0 z^2 - \frac{3M_0}{Lz^2} + O(z^{-3}),$$  \hspace{1cm} (30)

which indicates that the AdS radius is not modified, in contrast with [80], but an effective electric charge appears when $\varepsilon \to 0$, as can be shown due to the planar charged black hole $z^{-2}$ dependence for the lapse function. Finally, we note that the singular behaviour at $z \to 0$ persists, as a straightforward computation of the curvature invariants reveals.

5 Invariants and Thermodynamics

A useful way of exploring possible problems in a black hole solution is to investigate the corresponding invariants of the geometry. In principle, they can reveal if any problem arises in certain sector of the solution. For instance, the Ricci scalar for the classical black hole solution is given in terms of the lapse function as:

$$R_0 = -f_0''(z) - \frac{4f_0'(z)}{z} - \frac{2f_0(z)}{z^2},$$  \hspace{1cm} (31)

and it turns out that $R_0 = 4\Lambda_0$ is a constant for any value of $z$. In contrast, in the scale–dependent scenario, the Ricci scalar becomes extremely complicated and indeed we observe that the $z = 0$ singularity, which was already present in the Kretschmann scalar for the classical solution, now appears also in the Ricci scalar. This characteristic is intrinsically related to our formalism and, as far as we known, cannot be avoided.

Before analyzing the thermodynamics, we must focus our attention on the horizon radius. In this case, the event horizon is obtained using the condition $f(z_H) = 0$. In general, the task of obtaining an exact horizon is not always possible. This is our case because there is a logarithmic contribution to the line element. Still, we can obtain a numerical solution for the event horizon and, using that, we can analyse the effect of scale–dependent couplings on the AdS planar black hole solution. Moreover, we still can make some progress if we take advantage of the small parameter $\varepsilon$. As has been mentioned, any deviation with respect to the classical solution should be very small, reason why we can assume that $\varepsilon$ small provide us an acceptable solution. Thus, using the approximation given by Eq. (28) we obtain

$$z_H = z_0 \left(\frac{1}{2} \varepsilon z_0 - \frac{1}{2}\varepsilon^2 + O(\varepsilon^2),$$  \hspace{1cm} (32)

where we again observe that the horizon is smaller that the one corresponding to the classical AdS planar solution. Note that this can also be shown in Fig. 2 (left).

Although it is important to note the appearance of a shifted horizon with respect to its classical counterpart, we do not expect substantial deviations from classical black hole thermodynamics since, as commented previously, only long–range effects might show scale–dependent modifications in an appreciable way. In this sense, black hole thermodynamics remains robust [46–59, 71].

Regarding black hole thermodynamics, some comments are in order. First, the Hawking temperature is given by

$$T_H(z_H) = \frac{1}{4\pi} \left|\frac{12M_0}{Lz_H^2(1 + \varepsilon z_H^2)}\right|,$$  \hspace{1cm} (33)

showing that it has a correction via the scale–dependent gravitational coupling. When we demand $\varepsilon \to 0$, the standard solution is, of course, recovered. In order to get some insight
about how the scale–dependent scenario affects the temperature with respect to the classical solution, we expand for small values of \( \varepsilon \) to obtain

\[
T_H(z_H) = T_0(z_0) \left( 1 - \frac{3}{4} (\varepsilon z_0)^2 \right) + \mathcal{O}(\varepsilon^3). \tag{34}
\]

The previous expression reveals that the temperature decreases with respect to the classical case, \( T_0(z_0) \), which is in agreement with the behaviour shown in Fig. 2 (right). Second, the Bekenstein-Hawking entropy have the well-known relation inherited from Brans-Dicke theory \cite{88} which, in 3+1 dimensions, reads

\[
S(z_H) = S_0(z_H)(1 + \varepsilon z_H) \tag{35}
\]

and this quantity, as opposed to the temperature, increases when \( \varepsilon > 0 \) and decreases when \( \varepsilon < 0 \). It is thus remarkable that, although the expression for entropy admits both positive and negative values for the parameter \( \varepsilon \), we must be careful since \( S \) must be positive. Therefore, this could be considered a point against considering negative values for \( \varepsilon \). Finally, the heat capacity is easily computed with help of the relation

\[
C_H(z_H) = T \frac{dS}{dT} \bigg|_{z_H} = -S_H(z_H), \tag{36}
\]

where we notice that always \( C_H < 0 \), which means that the black hole is indeed unstable. We shown the entropy in Fig. 3 (left) and the heat capacity in Fig. 3 (right) for different values of the running parameter \( \varepsilon \). In these figures we can see that the scale–dependent effect is only appreciated when \( M_0 \) is large.

---

**Fig. 1** Left panel: Lapse function \( f(z) \) versus \( z \) for different values of the running parameter \( \varepsilon \). Right panel: Cosmological function \( \Lambda(z) \) versus \( z \) for different values of the running parameter \( \varepsilon \). The color code correspond: i) \( \varepsilon = 0.0 \) (solid black line) ii) \( \varepsilon = 0.1 \) (short dashed blue line) iii) \( \varepsilon = 0.3 \) (dotted red line) iv) \( \varepsilon = 0.5 \) (dotted-dashed green line) v) \( \varepsilon = 0.8 \) (long dashed cyan line) The classical mass \( M_0 = 1 \), the parameters \( L = 1/\sqrt{3} \) and \( \Lambda_0 = -1 \) were used in the aforementioned figures.

**Fig. 2** Left panel: Horizon \( z_H \) versus \( M_0 \) for different values of the running parameter \( \varepsilon \). Right panel: Hawking temperature \( T_H \) versus \( M_0 \) for different values of the running parameter \( \varepsilon \). The color code corresponds to: i) \( \varepsilon = 0.0 \) (solid black line) ii) \( \varepsilon = 0.1 \) (short dashed blue line) iii) \( \varepsilon = 0.3 \) (dotted red line) iv) \( \varepsilon = 0.5 \) (long dashed green line) v) \( \varepsilon = 0.8 \) (dotted-dashed cyan line) The classical mass has been taken as \( M_0 = 1 \), while the parameters \( L = 1/\sqrt{3} \) and \( \Lambda_0 = -1 \) were used in the aforementioned figures.
Fig. 3 Left panel: Bekenstein Hawking entropy $S_H$ versus $M_0$ for different values of the running parameter $\varepsilon$. Right panel: Heat capacity $C_H$ versus $M_0$ for different values of the running parameter $\varepsilon$. The color code correspond: i) $\varepsilon = 0.0$ (solid black line) ii) $\varepsilon = 0.1$ (short dashed blue line) iii) $\varepsilon = 0.3$ (dashed red line) iv) $\varepsilon = 0.5$ (long dashed green line) v) $\varepsilon = 0.8$ (dotted dashed cyan line). The classical mass $M_0 = 1$, the parameters $L = 1/\sqrt{3}$ and $\Lambda_0 = -1$ were used in the aforementioned figures.

6 Concluding remarks

In this article we have studied the scale dependence of four dimensional Anti de–Sitter Planar black holes. After presenting the model and the classical black hole solution, we have allowed for a scale dependence of the cosmological as well as the gravitational coupling, and we have solved the corresponding generalized field equations in four-dimensional spacetimes with planar symmetry. We have analysed in detail some black hole properties such as horizon structure, Hawking temperature, Bekenstein-Hawking entropy as well as the heat capacity. In the previous thermodynamics quantities we observe that the running correction appears when $M_0$ is large, opposite to what is usually found in solutions based on the asymptotic safety program.

If one compares our result for the running gravitational coupling with the corresponding results provided by the AS program [3–25] one finds that a matching is straightforward for the scale setting choice $k(z) \sim z$. This choice seems peculiar, since one usually expects $k \sim 1/z$ for dimensional reasons. Similar results have been found in [46–59] but the deeper reason behind this result is still unknown. An important hint for solving this riddle could come from considering the dimensionless product $G(k) \cdot \Lambda(k)$ instead of the individual dimensionful quantities as discussed in [73].

Another interesting feature of our solution is that the event horizon is attenuated in the scale–dependent scenario, which means that the black hole is smaller than the classical solution. Regarding the temperature, we notice that it is lower than in the classical case, whereas the entropy is larger than that of the non-running case. Finally, we have noted that the heat capacity is negative, which implies that the black hole is unstable. All these features give a better comprehension of the effect of scale–dependent couplings in well known black hole solutions.

Acknowledgments

The author A. R. was supported by the CONICYT-PCHA/Doctorado Nacional/2015-21151658. The author B. K. was supported by Fondecyt 1161150 and Fondecyt 1181694. The author P. B. was supported by the Faculty of Science and Vicerrectoría de Investigaciones of Universidad de los Andes, Bogotá, Colombia.

References

1. C. Brans and R. H. Dicke, Phys. Rev. 124, 925 (1961). doi:10.1103/PhysRev.124.925
2. S. Weinberg, doi:10.1007/978-1-4684-0931-4 1
3. C. Wetterich, Phys. Lett. B 301, 90 (1993) doi:10.1016/0370-2693(93)90726-X [arXiv:1710.05815 [hep-th]].
4. T. R. Morris, Int. J. Mod. Phys. A 9, 2411 (1994) doi:10.1142/S0217751X94000972 [hep-ph/9308265].
5. M. Reuter, Phys. Rev. D 57, 971 (1998) doi:10.1103/PhysRevD.57.971 [hep-th/9605030].
6. M. Reuter and F. Saueressig, Phys. Rev. D 65, 065016 (2002) doi:10.1103/PhysRevD.65.065016 [hep-th/0110054].
7. D. F. Litim and J. M. Pawlowski, Phys. Rev. D 66, 025030 (2002) doi:10.1103/PhysRevD.66.025030 [hep-th/0202188].
8. D. F. Litim, Phys. Rev. Lett. 92, 201301 (2004) doi:10.1103/PhysRevLett.92.201301 [hep-th/0312114].
9. M. Niedermaier and M. Reuter, Living Rev. Rel. 9, 5 (2006), doi:10.12942/lrr-2006-5
10. M. Niedermaier, Class. Quant. Grav. 24, R171 (2007) doi:10.1088/0264-9381/24/18/R01 [gr-qc/0610018].
82. M. Reuter and H. Weyer, Phys. Rev. D 69, 104022 (2004) doi:10.1103/PhysRevD.69.104022 [hep-th/0311196].
83. D. Becker and M. Reuter, Annals Phys. 350, 225 (2014) doi:10.1016/j.aop.2014.07.023 [arXiv:1404.4537 [hep-th]].
84. J. A. Dietz and T. R. Morris, JHEP 1504, 118 (2015) doi:10.1007/JHEP04(2015)118 [arXiv:1502.07396 [hep-th]].
85. P. Labus, T. R. Morris and Z. H. Slade, Phys. Rev. D 94, no. 2, 024007 (2016) doi:10.1103/PhysRevD.94.024007 [arXiv:1603.04772 [hep-th]].
86. T. R. Morris, JHEP 1611, 160 (2016) doi:10.1007/JHEP11(2016)160 [arXiv:1610.03081 [hep-th]].
87. N. Ohta, PTEP 2017, no. 3, 033E02 (2017) doi:10.1093/ptep/ptx020 [arXiv:1701.01506 [hep-th]].
88. G. Kang, Phys. Rev. D 54, 7483 (1996) doi:10.1103/PhysRevD.54.7483 [gr-qc/9606020].