CHEBYSHEV UPPER ESTIMATES FOR BEURLING’S GENERALIZED PRIME NUMBERS

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Abstract. Let \( N \) be the counting function of a Beurling generalized number system and let \( \pi \) be the counting function of its primes. We show that the \( L^1 \)-condition

\[
\int_1^\infty \left| \frac{N(x) - ax}{x} \right| \, dx < \infty
\]

and the asymptotic behavior

\[
N(x) = ax + O \left( \frac{x}{\log x} \right),
\]

for some \( a > 0 \), suffice for a Chebyshev upper estimate

\[
\frac{\pi(x) \log x}{x} \leq B < \infty.
\]

1. INTRODUCTION

Let \( P = \{p_k\}_{k=1}^\infty \) be a set of Beurling generalized primes, namely, a non-decreasing sequence of real numbers \( 1 < p_1 \leq p_2 \leq \cdots \leq p_k \to \infty \). The sequence \( \{n_k\}_{k=1}^\infty \) denotes its associated set of generalized integers \([2, 3]\). Consider the counting functions of generalized integers and primes

\[
N(x) = N_P(x) = \sum_{n_k < x} 1 \quad \text{and} \quad \pi(x) = \pi_P(x) = \sum_{p_k < x} 1.
\]

Beurling’s problem consists in finding mild conditions over \( N \) that ensure a certain asymptotic behavior for \( \pi \). This problem has been extensively investigated in connection with the prime number theorem (PNT), i.e.,

\[
\pi(x) \sim \frac{x}{\log x}, \quad x \to \infty,
\]

and Chebyshev two-sided estimates, that is,

\[
0 < \liminf_{x \to \infty} \frac{\pi(x) \log x}{x} \quad \text{and} \quad \limsup_{x \to \infty} \frac{\pi(x) \log x}{x} < \infty.
\]

2000 Mathematics Subject Classification. Primary 11N80. Secondary 11N05, 11M41.

Key words and phrases. Chebyshev upper estimates; Beurling generalized primes.

The author gratefully acknowledges support by a Postdoctoral Fellowship of the Research Foundation–Flanders (FWO, Belgium).
On the other hand, there are no mild hypotheses in the literature for Chebyshev upper estimates,

\[(3) \limsup_{x \to \infty} \frac{\pi(x) \log x}{x} < \infty.\]

The purpose of this article is to study asymptotic requirements over \(N\) that imply the Chebyshev upper estimate \(3\).

Beurling \[3\] proved that

\[(4) N(x) = ax + O \left( \frac{x}{\log^2 x} \right), \quad x \to \infty \quad (a > 0),\]

where \(\gamma > 3/2\), suffices for the PNT \(1\) to hold. See \[3, 10, 13\] for more general PNT. Beurling’s condition is sharp, because when \(\gamma = 3/2\) there are generalized number systems for which the PNT fails \[3, 5\]. For \(\gamma < 1\), not even Chebyshev estimates need to hold, as follows from an example of Hall \[9\] (see also \[1\]). Diamond has shown \[6\] that \(4\) with \(\gamma > 1\) is enough to obtain Chebyshev two-sided estimates \(2\). Furthermore, he conjectured \[7\] that the weaker hypothesis

\[(5) \int_1^\infty \left| \frac{N(x) - ax}{x} \right| \frac{dx}{x} < \infty, \quad \text{with } a > 0,\]

would be enough for \(2\). His conjecture was shown to be false by Kahane \[11\]. Nevertheless, the author has recently shown \[15\] that if one adds to \(5\) the condition

\[(6) N(x) = ax + o \left( \frac{x}{\log x} \right), \quad x \to \infty,\]

then \(2\) is fulfilled, extending thus earlier results from \[6, 18\].

It is natural to replace the little \(o\) symbol in \(6\) by an \(O\) growth estimate and investigate the effect of this new condition on the asymptotic distribution of the generalized primes. It turns out that one gets a Chebyshev upper estimate in this case. Our main goal is to give a proof of the following theorem.

**Theorem 1.** Diamond’s \(L^1\)-condition \(5\) and the asymptotic behavior

\[(7) N(x) = ax + O \left( \frac{x}{\log x} \right), \quad x \to \infty,\]

suffice for the Chebyshev upper estimate \(3\).

2. Notation

We will give an analytic proof of Theorem 1. Our technique follows distributional ideas already used in \[13, 15, 16\]. It employs the Wiener division theorem \[12\] Chap. 2 and the operational calculus for the Laplace transform of Schwartz distributions \[3, 17\]. The Schwartz spaces of test functions and distributions are denoted as \(D(\mathbb{R})\), \(S(\mathbb{R})\), \(D'(\mathbb{R})\) and \(S'(\mathbb{R})\), see \[8, 14, 17\] for
their properties. If \( f \in S'(\mathbb{R}) \) has support in \([0, \infty)\), its Laplace transform is well defined as
\[
\mathcal{L} \{ f; s \} = \langle f(u), e^{-su} \rangle, \quad \Re s > 0,
\]
and the Fourier transform \( \hat{f} \) is the distributional boundary value \([4]\) of \( \mathcal{L} \{ f; s \} \) on \( \Re s = 0 \). We use the notation \( H \) for the Heaviside function, it is simply the characteristic function of \((0, \infty)\).

Observe that (3) is equivalent to
\[
(8) \limsup_{x \to \infty} \frac{\psi(x)}{x} < \infty,
\]
where \( \psi \) is the Chebyshev function
\[
\psi(x) = \psi_p(x) = \sum_{n_k < x} \Lambda(n_k),
\]
as follows from \([2, \text{Lem. 2E}]\).

3. Proof of Theorem 1

Assume (5) and (7). Set \( T(u) = e^{-u} \psi(e^u) \). We must show (8), that is,
\[
(9) \limsup_{u \to \infty} T(u) < \infty.
\]
The crude inequality \( T(u) \leq ue^{-u}N(e^u) = O(u) \) implies that \( T \in S'(\mathbb{R}) \).

The proof of (9) depends upon estimates on convolution averages of \( T \):

**Lemma 1.** There exists \( c > 0 \) such that
\[
(10) \int_{-\infty}^{\infty} T(u)\hat{\phi}(u - h)du = O(1),
\]
whenever \( \phi \in \mathcal{D}(-c, c) \).

Indeed, suppose that Lemma 1 has been already established. Choose then in (10) a test function \( \phi \in \mathcal{D}(-c, c) \) such that \( \phi \) is non-negative. Since \( \psi(e^u) \) is non-decreasing, we have \( e^{-u}T(h) \leq T(u + h) \) whenever \( u \) and \( h \) are positive. Setting \( C = \int_0^\infty e^{-u}\hat{\phi}(u)du > 0 \), we obtain that
\[
T(h) \leq \frac{1}{C} \int_0^\infty T(u + h)\hat{\phi}(u)du = O(1),
\]
and Theorem 1 follows at once. It remains to prove the lemma.

**Proof of Lemma 1.** Set \( E_1(u) := e^{-u}N(e^u) - aH(u) \) and \( E_2(u) = uE_1(u) \). The assumptions (5) and (7) take the form \( E_1 \in L^1(\mathbb{R}) \) and \( E_2 \in L^\infty(\mathbb{R}) \). Consider
\[
G(s) = \zeta(s) - \frac{a}{s - 1} = s\mathcal{L} \{ E_1; s - 1 \} + a.
\]
Taking \( \Re s \to 1^+ \), in the distributional sense, we obtain \( G(1 + it) = (1 + it)\tilde{E}_1(t) + a \). Since \( E_1 \in L^1(\mathbb{R}) \), \( \tilde{E}_1 \) is continuous; therefore \( G(s) \) extends to a continuous function on \( \Re s = 1 \). Consequently, \( (s - 1)\zeta(s) \) is continuous on
Next, we study the boundary values, on the line segment $1 + i(-c, c)$, of

$$
\mathcal{L}\{T(u); s - 1\} = \mathcal{L}\{\psi(e^u); s\} = -\frac{\zeta'(s)}{s\zeta(s)}.
$$

A quick calculation shows that

$$
(11) \quad \frac{\zeta'(s)}{s\zeta(s)} = \frac{\mathcal{L}\{E'_2; s - 1\}}{(s-1)\zeta(s)} - \frac{(2s-1)\mathcal{L}\{E_1; s - 1\} + a}{s(s-1)\zeta(s)} - \frac{1}{s} + \frac{1}{s-1},
$$

Consider the boundary distributions

$$
g_1(t) = \lim_{\sigma \to 1^+} \frac{\mathcal{L}\{E'_2; \sigma - 1 + it\}}{(\sigma - 1 + it)\zeta(\sigma + it)} \quad \text{in} \quad S'(\mathbb{R}),
$$

and

$$
g_2(t) = -\lim_{\sigma \to 1^+} \left( \frac{(2\sigma - 1 + 2it)\mathcal{L}\{E_1; \sigma - 1 + it\} + a}{(\sigma + it)(\sigma - 1 + it)\zeta(\sigma + it)} + \frac{1}{\sigma + it} \right) \quad \text{in} \quad S'(\mathbb{R}).
$$

Taking boundary values in (11), we have $\hat{T}(t) = g_1(t) + g_2(t) + \hat{H}(t)$, where $\hat{H}$ is the Heaviside function. Fix $\phi \in \mathcal{D}(-c, c)$. Notice that $g_2$ is actually a continuous function on $(-3c, 3c)$, thus,

$$
\int_{-\infty}^{\infty} T(u)\hat{\phi}(u-h)du = \left\langle g_1(t), e^{iht}\phi(t) \right\rangle + \int_{-c}^{c} e^{iht}g_2(t)\phi(t)dt + \int_{-h}^{\infty} \hat{\phi}(u)du = \left\langle g_1(t), e^{iht}\phi(t) \right\rangle + o(1) + O(1).
$$

Our task is then to demonstrate that $\left\langle g_1(t), e^{iht}\phi(t) \right\rangle = O(1)$. Let $M \in S'(\mathbb{R})$ be the distribution supported in the interval $[0, \infty)$ that satisfies $\mathcal{L}\{M; s - 1\} = ((s-1)\zeta(s))^{-1}$. Notice that $g_1 = (E'_2 * M)$. Fix an even function $\eta \in \mathcal{D}(-3c, 3c)$ such that $\eta(t) = 1$ for all $t \in (-2c, 2c)$. Then, $\eta(t)it\zeta(1 + it) \neq 0$ for all $t \in (-2c, 2c)$; moreover, it is the Fourier transform of the $L^1$-function $\chi_1 * E_1 + \chi_2$, where $\hat{\chi}_1(t) = it(1 + it)\eta(t)$ and $\hat{\chi}_2(t) = a(1 + it)\eta(t)$. We can therefore apply the Wiener division theorem [12 p. 88] to $\eta(t)it\zeta(1 + it)$ and $\phi(t)$. So we find $f \in L^1(\mathbb{R})$ such that

$$
\hat{f}(t) = \frac{\phi(t)}{\eta(t)it\zeta(1 + it)}.
$$

Hence,

$$
\left\langle g_1(t), e^{iht}\phi(t) \right\rangle = \left\langle (E'_2 * M)(u), \hat{\phi}(u-h) \right\rangle = (E_2 * (\eta)' \ast f)(h) = O(1),
$$

because $E_2 \in L^\infty(\mathbb{R})$ and $(\eta)' \ast f \in L^1(\mathbb{R})$, whence (10) follows.

□
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