A generalized robust optimization approach for right-hand-side uncertainty with application to power dispatch

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Abstract

Robust optimization (RO) tackles data uncertainty by finding an optimal solution that is protected against any realization of the uncertain parameter(s). In the literature of RO, a budget of uncertainty can be used to adjust the level of conservatism (robustness) such that higher budgets of uncertainty correspond to more conservative solutions. In this paper, we show that this approach may produce non-intuitive results in problems with right-hand side (RHS) uncertainty since increasing the budget of uncertainty by more than a certain threshold may not further impact the level of conservatism. We refer to this phenomenon as “partially-ineffective budgets” and propose a new tractable two-stage robust optimization model that effectively incorporates the budget of uncertainty in problems with RHS uncertainty. The proposed approach accurately controls the level of conservatism and provides insights on the trade-off between robustness and economy in such problems. We examine the applicability of the proposed model on a power dispatch problem with wind power uncertainty. The numerical results demonstrate the merits of the proposed approach from various aspects such as the effectiveness of the budget of uncertainty, robustness, and cost efficiency and reliability against randomly-simulated scenarios.

Keywords: robust optimization, budget of uncertainty, right-hand side uncertainty, ineffective budget, wind power uncertainty

1. Introduction

Traditionally, deterministic decision-making models assume that the parameters of optimization problems are accurate. However, due to measurement errors, round-off computational errors, and even forecasting inaccuracies, such perfect knowledge is rarely available, and data uncertainty is inevitable in most optimization problems. Uncertainties associated with parameters can significantly degrade the solution performance and lead to potentially sub-optimal or infeasible solutions (Ben-Tal et al., 2009).

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Robust optimization (RO) (Soyster, 1973; Ben-Tal and Nemirovski, 1998) has recently emerged as a technique to manage data uncertainty. It considers an uncertain parameter within a given uncertainty set and finds the optimal solution under the worst-case scenario of the uncertain parameter. The applications of RO have been studied in various problems including portfolio selection (e.g., Hassanzadeh et al. (2014)), network flows (e.g., Atamtürk and Zhang (2007)), inventory management (e.g., Ang et al. (2012)), radiation treatment planning (e.g., Chan et al. (2014)), and power dispatch problems (e.g., Li et al. (2015, 2016)). See Bertsimas et al. (2011) and Gabrel et al. (2014) for comprehensive reviews on the applications of robust optimization methods in different areas.

Depending on the position of the uncertain parameters in the model, robust optimization models are often classified as having “row-wise” or “column-wise” uncertainty. Ben-Tal and Nemirovski (1998, 2002), El Ghaoui and Lebret (1997), and Bertsimas and Sim (2003) studied “row-wise” uncertainty, i.e., where the rows of the constraint matrix belong to a given set, for different shapes of the uncertainty set (e.g., ellipsoidal or polyhedral). On the other hand, “column-wise” uncertainty, i.e., where the columns of the constraint matrix belong to a given set, was first studied by Soyster (1973) and was further developed by Falk (1976) and Singh (1982). A special case of column-wise uncertainty is when the uncertain parameter appears in the right-hand side (RHS) of a problem. Different shapes of RHS uncertainty sets, such as polyhedral and ellipsoidal, were considered by Ouorou (2016) and Minoux (2008, 2012). Traditional RO models may produce over-conservative solutions since they seek a solution that is feasible under any realization of the uncertain parameter. Bertsimas and Sim (2004) investigated the issue of conservatism for row-wise uncertainty and proposed a tractable model that controls the level of conservatism using a parameter called “the budget of uncertainty”. The budget of uncertainty is defined for each row to control the level of uncertainty that one would like to consider for that row. Particularly, a zero budget corresponds to the deterministic problem with no uncertainty, and a larger budget corresponds to a higher level of uncertainty. Ultimately, a full budget refers to complete protection against uncertainty, which is equivalent to the traditional worst-case approach.

Accounting for the budget of uncertainty in problems with column-wise RHS uncertainty is more challenging. In this paper, we demonstrate that the conventional budget-of-uncertainty approach proposed by Bertsimas and Sim (2004) may not produce intuitively-expected results for problems with uncertainty in the RHS. The intuition behind this issue is as follows: We often expect to see a change in the level of conservatism of the solution by changing the budget of uncertainty; thus, the higher the budget, the more conservative the solution is expected to become. However, this behavior may not be explicitly observed in robust problems with RHS uncertainty. That is, deviating the value of the RHS parameters from their nominal value by more than a certain threshold may not have any further effect on the robust solution, simply because the corresponding robust constraint would become redundant. Therefore, any consideration of the budget of uncertainty beyond such threshold would be “ineffective” and the robust solution would remain unchanged regardless of the
higher level of uncertainty (budget) considered. In this paper, we refer to this phenomenon as “partially-ineffective budgets” and propose a new approach to generalize the conventional budget-of-uncertainty approach for such settings. The specific contributions of this paper are as follows:

- We propose a column-wise budget-of-uncertainty approach for a class of problems with RHS uncertainty that have “ineffective budgets”. We show that our proposed approach generalizes the conventional budget-of-uncertainty approach and performs the same way as the conventional budget approach defined for the row-wise uncertainty.

- We propose a two-stage optimization framework for accurately controlling the level of robustness of the optimal solution without changing the set of feasible solutions.

- We examine the applicability of the proposed approach on power dispatch optimization problems with RHS uncertainty and demonstrate the merits of the proposed approach in terms of effectiveness, robustness, cost efficiency, and reliability against randomly-simulated scenarios within an uncertainty set.

The rest of this paper is organized as follows: Section 2 motivates the problem using power dispatch optimization under wind uncertainty as an example application. In Section 3, the proposed two-stage robust optimization model is presented and discussed in details. Section 4 provides numerical results and analyses. Finally, concluding remarks are presented in Section 5.

2. Motivation: Power Dispatch under Wind Uncertainty

Modern power systems rely on the integration of low-carbon renewable resources to meet the demand for electricity. Among these renewable resources, wind energy is of special importance due to increasing penetration into power systems in the recent years so that the U.S. energy’s plan for 2030, wind power is expected to have a significant contribution (20%) in providing electricity (Lopez et al., 2012). However, due to the inherent uncertainty in wind, it is not possible to forecast the exact amount of available wind power in the existing day-ahead electricity market. If the power system is planned based on the day-ahead wind power prediction, even small prediction errors and uncertainties associated with the amount of available wind power can make the planned power dispatch infeasible by violating the operational limits of the system, and therefore, can potentially lead to security and reliability issues in the power system in real-time (Lorca and Sun, 2015).

Reliability and security concerns have been the focus of power systems since the 1960s (Billinton and Bollinger, 1968). Particularly, the security-constrained economic dispatch (SCED) problem is concerned with operational security and reliability of the system to mitigate the risk of a system failure under unforeseen contingencies (Frank and Rebenнак, 2016). In SCED, the goal is to find the most economical power dispatch plan while considering the operational constraints of the system (e.g., power balance, generation and ramp limits, reserve requirements, and power flow transmission constraints). When there is an excessive amount of wind power that cannot be absorbed by the system,
the wind power can be “curtailed” by shutting down some or all of the wind turbines in order to maintain the system’s operational security. However, wind curtailment is not desirable from an economical point of view since wind power is a free renewable resource while the alternatives rely mainly on costly fossil fuels. Therefore, the objective function of SCED models typically contains a penalty for wind curtailment to encourage the power system to utilize as much wind power as possible without violating the system’s security requirements (Li et al., 2016).

Given wind power uncertainty, the SCED problem is categorized as an optimization problem with RHS uncertainty where the optimal wind power generation has an uncertain upper bound. Intuitively, a higher value selected for the budget of uncertainty should correspond to more wind power uncertainty in the system. However, the potential challenge in this problem is that the available wind power may never be entirely utilized due to the operational limits of the system; Thus, when the system reaches its maximum wind power admissibility, increasing the budget of uncertainty for the available wind power does not change the optimal solution – hence, the budget would be “ineffective”. This behavior is the motivation behind our proposed approach and will be elaborated in more details in the numerical results in Section 4.

3. The Proposed Robust Optimization Approach

In this section, we first describe a basic robust optimization problem with budget of uncertainty in mathematical terms and explain the challenges with having RHS uncertainty. We then present the proposed approach by defining “admissible” and “effective” uncertainty sets and developing a two-stage robust model for dealing with RHS uncertainty.

3.1. Mathematical Problem Description

Consider the conventional row-wise budget-of-uncertainty approach (Bertsimas and Sim, 2004) and assume that an uncertain parameter \( \hat{a}_{ij} \) can take any value within an uncertainty set \([\underline{a}_{ij}, \overline{a}_{ij}]\) where \( \underline{a}_{ij} = \hat{a}_{ij} - \hat{a}_{ij} \) and \( \overline{a}_{ij} = \hat{a}_{ij} + \hat{a}_{ij} \). Parameters \( \hat{a}_{ij} \) and \( \hat{a}_{ij} \) are the nominal value and the maximum error in estimating the nominal value, respectively. Denote parameter \( \Gamma_i \) as the budget of uncertainty for each row \( i \) and variables \( z_{ij}^+ \) and \( z_{ij}^- \) as the positive and negative scaled deviations from the nominal values. The budget \( \Gamma_i \) limits the total scaled deviations of all uncertain parameters within the same row \( i \). The mathematical formulation of the budget of uncertainty for row \( i \) can be written as follows:

\[
\hat{a}_{ij} = \hat{a}_{ij} + z_{ij}^+ (\overline{a}_{ij} - \hat{a}_{ij}) + z_{ij}^- (\underline{a}_{ij} - \hat{a}_{ij}), \quad \forall i, j,
\]
\[
\sum_{j=1}^n (z_{ij}^+ + z_{ij}^-) \leq \Gamma_i, \quad \forall i, \quad 0 \leq z_{ij}^+, z_{ij}^- \leq 1, \quad \forall i, j. \quad (1)
\]

Parameter \( \Gamma_i \) can take any value within \([0, |J_i|]\) where \( J_i \) is the set of coefficients that are subject to uncertainty in row \( i \). In particular, a zero budget, i.e., \( \Gamma_i = 0 \), corresponds to the deterministic
problem where the uncertain parameters do not deviate from the nominal values. If \( \Gamma_i \) increases, the solution becomes more conservative against uncertainty and results in a worse value of the objective function. On the other hand, a full budget, i.e., \( \Gamma_i = |J_i| \), refers to the satisfaction of constraint \( i \) under all scenarios of the uncertainty set, which is equivalent with traditional RO approaches with over-conservative solutions.

However, RHS uncertainty cannot be directly handled using this approach for two main reasons. First, the duality approach for reformulating the worst-case scenario of the uncertain RHS does not directly apply to such problems (Minoux, 2008). Second, the regular budget-of-uncertainty approach cannot always adjust the level of conservatism in these problems. To elaborate on the second point, consider formulation \([M]\) with an uncertain RHS, i.e., \( \tilde{y} \in \mathcal{U} \), where \( \mathcal{U} = [\underline{y}, \bar{y}] \):

\[
[M] : \min_{x,y} c_1(x) + c_2(y), \quad (2a)
\]
\[
\text{s.t.} \quad Ax + By \leq g, \quad (2b)
\]
\[
y \leq \tilde{y}, \quad \forall \tilde{y} \in \mathcal{U} \quad (2c)
\]
\[
y, x \geq 0. \quad (2d)
\]

Problem \([M]\) corresponds to an optimization problem with two types of resources. The objective function consists of linear cost functions \( c_1(x) \), i.e., the cost of using resource 1, and \( c_2(y) \), i.e., the penalty for not utilizing resource 2, where decision variables \( x \) and \( y \) are vectors of size \( m \times 1 \). Let \( c_2(y) = \max_{\tilde{y}} d^T(\tilde{y} - y) \) be the penalty cost for non-utilized resource 2 under the worst-case scenario. Thus, the objective function maximizes the amount of resource 2 used under the worst-case scenario while considering the uncertainty in its available amount in the RHS of constraint \((2c)\). Constraint \((2b)\) captures the limitations of the system on both \( x \) and \( y \), where matrices \( A \) and \( B \) are of size \( n \times m \), and, for simplicity, consider \( B \geq 0 \). Particularly, \( B = 0 \) corresponds to the upper limit of vector \( x \). Formulation \([M]\) represents a general model for RHS uncertainty that can be used for the SCED problem where vectors \( x \) and \( y \) correspond to the power generation of conventional generators and wind power plants, respectively. Thus, \( c_2(y) \) corresponds to the penalty for the worst-case wind power curtailment, and the uncertainty of available wind power is considered in the RHS of constraint \((2c)\).

Following the concept of the budget of uncertainty proposed by Bertsimas and Sim (2004), consider the uncertainty set \( \mathcal{U}^B \) with budget \( \Gamma \) as follows:

\[
\mathcal{U}^B = \left\{ \tilde{y} \in \mathbb{R}^m : \tilde{y} = \hat{y} + z \odot (\underline{y} - \hat{y}), \sum_{i=1}^m z_i \leq \Gamma, \ 0 \leq z \leq 1 \right\}, \quad (3)
\]

where \( \hat{y} \) is the nominal value of the uncertain parameter. Note that, without loss of generality, we only allow positive deviations from the nominal – this point is later proved in Proposition 4. In \((3)\), operator \( \odot \) is the Hadamard Product (Horn, 1990) denoting element-wise multiplication of
vectors with equal dimension. Incorporating $U^B$ in formulation $[M]$, we have:

$$\min_{x,y,z} \quad c_1(x) + c_2(y),$$  \hspace{1cm} (4a)

$$\text{s.t} \quad Ax + By \leq g,$$  \hspace{1cm} (4b)

$$y \leq \hat{y} + z \odot (\overline{y} - \hat{y}),$$  \hspace{1cm} (4c)

$$\sum_{i=1}^{m} z_i \leq \Gamma,$$  \hspace{1cm} (4d)

$$0 \leq z \leq 1,$$  \hspace{1cm} (4e)

$$x, y \geq 0.$$  \hspace{1cm} (4f)

Based on constraints (4b) and (4c), two upper bounds can be derived for $Ax + By$ as follows:

$$Ax + By \leq g,$$  \hspace{1cm} (5a)

$$Ax + By \leq Ax + B(\hat{y} + z \odot (\overline{y} - \hat{y})).$$  \hspace{1cm} (5b)

Depending on the value of $z$, one of the constraints (5a) and (5b) would become redundant and would not impact the solution. Consider a feasible $z^0$ such that (5a) is binding and $g = Ax + B(\hat{y} + z^0 \odot (\overline{y} - \hat{y}))$. Denote the budget and the robust solution corresponding to $z^0$ as $\Gamma^0$ and $y^0$, respectively. When $B > 0$, for all $z^1 \in (z^0, 1]$, we can show that $g < Ax + B(\hat{y} + z^1 \odot (\overline{y} - \hat{y}))$, and hence constraint (5b) becomes redundant. Let $\Gamma^1$ and $y^1$ be the budget and the robust solution corresponding to $z^1$, respectively; For $\Gamma^1 > \Gamma^0$ (as $z^1 > z^0$), we have $y^1 = y^0$ since the additional budget does not have any further impact on the solution. Thus, even though there is a penalty in the objective function for under-utilization of resource 2, constraint (5a) may not allow $U = [\underline{y}, \overline{y}]$ to be entirely utilized. We denote the value of $\Gamma^1 - \Gamma^0$ as an “ineffective” budget which can be calculated as $\sum_{i=1}^{m} (z_i^1 - z_i^0)$.

In what follows, we first introduce the admissible and effective uncertainty sets (regardless of the budget $\Gamma$) in Section 3.2, and then we identify an effective budget of uncertainty in Section 3.3. Finally, in Section 3.4, we propose a two-stage robust optimization approach to tackle the challenges of RHS uncertainty with the ineffective budget.

### 3.2. Admissible and Effective Uncertainty Sets

In this section, we define “admissible” and “effective” uncertainty sets which will be used in the development of our proposed approach. Note that similar to the conventional budget-of-uncertainty approach, we use an interval (box) uncertainty and therefore may use the terms “interval” and “uncertainty set” interchangeably throughout this paper.

The admissible interval $[\underline{s}, \overline{s}]$ is a subset of $[0, \overline{y}]$ such that for any solution $y \in [\underline{s}, \overline{s}]$ problem $[M]$ is feasible. The effective interval $[\hat{s}, \overline{s}]$ is a subset of the admissible interval within which the worst-
case scenario of the admissible interval always occurs. Figure 1 shows the admissible and effective intervals. Note that the budget of uncertainty does not play a role in defining these intervals. In what follows, we first propose an optimization problem to find the admissible interval. Next, we identify the effective interval, accordingly.

3.2.1. Admissible Uncertainty Set

The admissible interval \([s, \bar{s}]\) is a subset of \([0, y]\) (the potential domain for variable \(y\)) and may partially or entirely lay outside of the uncertainty set \([y, \bar{y}]\). Thus, for the upper and lower limits of the admissible interval we have:

\[
\begin{align*}
\bar{s} &\leq y, \\
\underline{s} &\leq y,
\end{align*}
\]

where \(\bar{s} = \bar{y}\) means that the RHS can be fully utilized and the entire uncertainty set is admissible. Proposition 1 demonstrates how to identify whether a given interval \([s, \bar{s}]\) is entirely admissible. Remark 1 further explains how to find the largest possible such interval that is as close as possible to the uncertainty set \(U = [y, \bar{y}]\).

**Proposition 1.** A solution \(y \in [s, \bar{s}]\) satisfies constraint \((2b)\) of formulation \([M]\) under any scenario of the uncertainty set if \(\underline{s}\) and \(\bar{s}\) meet the following conditions:

\[
\begin{align*}
Ax + Bs + \alpha &\leq g, \\
\alpha &\geq B(\bar{s} - s), \\
\alpha &\geq 0.
\end{align*}
\]

**Proof.** Variable \(y \in [s, \bar{s}]\) can be written as \(y = s + r \odot (\bar{s} - s)\), where \(0 \leq r \leq 1\). Thus, constraint \((2b)\) can be reformulated as \(Ax + By \leq Ax + B(s + r \odot (\bar{s} - s)) \leq g\). To meet the mentioned
constraint under its worst-case scenario, the following constraint should be satisfied:

\[ Ax + By \leq Ax + \beta(r) \leq g, \]  

where

\[ \beta(r) = \max_r B \left( s + r \odot (s - g) \right), \]  

s.t. \( 0 \leq r \leq 1. \)

Considering \( \alpha \) as the dual vector for constraint (9b) and replacing the dual of problem (9) into (8), we can recover formulation (7).

**Remark 1.** The following optimization problem finds the largest admissible interval \( [s, \bar{s}] \) that has the smallest distance from the uncertainty set \( \mathcal{U} = [\underline{y}, \overline{y}] \).

\[
\begin{align*}
\min_{\underline{s}, \bar{s}, \alpha, x} & \quad (\overline{y} - \bar{s}) + (\underline{y} - \underline{s}), \\
\text{s.t.} & \quad Ax + B\bar{s} + \alpha \leq g, \\
& \quad \alpha \geq B(s - \underline{s}), \\
& \quad \underline{s} \leq \underline{y}, \\
& \quad \bar{s} \leq \overline{y}, \\
& \quad x, \underline{s}, \bar{s}, \alpha \geq 0.
\end{align*}
\]

**Proof.** As shown in Proposition 1, constraints (10b), (10c), and \( \alpha \geq 0 \) in (10f) identify whether a given interval \( [s, \bar{s}] \) is admissible. Constraints (10d) and (10e) demonstrate that the admissible interval may lay outside of the uncertainty set since the RHS parameters might not be fully utilized. Given that the objective function (10a) minimizes the gap between \( [\underline{y}, \overline{y}] \) and \( [s, \bar{s}] \), formulation (10) always obtains the largest possible admissible interval that has the smallest distance from the initial uncertainty set.

**Proposition 2.** The admissible interval \( [s, \bar{s}] \) can always be categorized as one of the following four cases:

a) \( \underline{s} = \underline{y} \) and \( \bar{s} = \overline{y} \)

b) \( \underline{s} = \underline{y} \) and \( \hat{y} \leq \underline{s} < \overline{y} \)

c) \( \underline{s} = \underline{y} \) and \( \underline{y} < \underline{s} < \hat{y} \)

d) \( \underline{s} = \underline{s} \leq \underline{y} \)

**Proof.** For any feasible solution of problem (10), \( \exists \alpha \geq 0 \), such that \( \alpha \geq B(s - \underline{s}) \). By assumption, \( B \geq 0 \). Let us separate the case of \( B = 0 \) and \( B > 0 \):
(i) If $B = 0$, constraint (10c) would become redundant since $B(s - \bar{s}) = 0$. Constraint (10b) is also independent from $s$ since $Bs = 0$. Thus, due to the minimization objective function, constraints (10d) and (10e) are binding at optimality, and the admissible interval always corresponds to case (a), where $s = \bar{y}$ and $\bar{s} = \bar{y}$.

(ii) If, on the other hand, $B > 0$, we again separate the cases in which $\alpha > 0$ or $\alpha = 0$.

First, for $\alpha > 0$, it is obvious that $s > \bar{s}$. Thus, replacing (10c) in (10b), we can conclude that $Ax + Bs < g$ always holds for any value of $s \in [0, \bar{y}]$. Due to the minimization objective function in (10a), $s = \bar{y}$ at optimality. Depending on the value of $\alpha$, by substitution, we observe from (10b) and (10c) that:

- If $B(\bar{y} - y) \leq \alpha$, then $Ax + Bs \leq Ax + By \leq g$ is satisfied for any value of $s \leq \bar{y}$. Thus, due to the minimization objective function (10a), $s = \bar{y}$ at optimality, which again corresponds to case (a).
- Similarly, if $B(\bar{y} - y) \leq \alpha < B(\hat{s} - y)$, then $\hat{y} \leq s < \bar{y}$ at optimality, which corresponds to case (b).
- If $0 < \alpha < B(\hat{s} - y)$, then $\bar{y} < s < \hat{y}$ at optimality, which corresponds to case (c).

Second, for $\alpha = 0$, we have $s = \bar{s}$. We show that $s \leq \bar{y}$ by contradiction: Define sets $I = [0, \bar{y}]$ and $J = (\bar{y}, \bar{y}]$ where $I \cap J = \emptyset$ and $I \cup J = [0, \bar{y}]$. Assume $\exists s, \bar{s} \in J$, such that $s = \bar{s}$. This contradicts constraint $s \leq \bar{y}$ and thus $s, \bar{s} \notin J$. Therefore, $s, \bar{s} \in I$ and $s = \bar{s}$, which corresponds to case (d).

Now consider formulation $[M']$ where the uncertainty set $U = [\bar{y}, \bar{y}]$ of problem $[M]$ is substituted with the admissible uncertainty set $[s, \bar{s}]$ denoted by $U^A$:

$$[M'] : \begin{array}{ll}
\min_{x,y} & c_1(x) + c_2(y), \\
\text{s.t.} & Ax + By \leq g, \\
& y \leq \bar{s}, \quad \forall s \in U^A = [s, \bar{s}] \\
x, y \geq 0.
\end{array}$$

(11a) (11b) (11c) (11d)

The following proposition shows the equivalency of problems $[M]$ and $[M']$.

**Proposition 3.** Problems $[M]$ and $[M']$ have the exact same feasible region.

**Proof.** Let $X$ and $X'$ be the feasible sets of $[M]$ and $[M']$, respectively. To conclude that the two feasible sets are equal, it is sufficient to show that any feasible solution in $X$ is also feasible for $X'$ and vice-versa.

First, let $x', y' \in X'$, where $y' \leq s \in [s, \bar{s}]$. Since $[s, \bar{s}]$ falls into one of the cases of Proposition 2, it satisfies the conditions of Proposition 1. Thus, $x', y' \in X$ as well.
Similarly, let \( x, y \in X \), where \( y \leq \tilde{y} \in [\underline{y}, \overline{y}] \). From the constraints of (5), we observe that for values of \( z \) corresponding to \( \tilde{y} \in (\underline{y}, \overline{y}] \) constraint (5b) becomes redundant. Thus, \( y \leq \underline{s} \) and it demonstrates that \( x, y \in X' \) as well.

From Proposition 3, we can conclude that instead of incorporating a budget of uncertainty in problem \([M]\) with partially-ineffective uncertainty set, we can use problem \([M']\) without changing the feasible region. The admissible uncertainty set of problem \([M']\) can be formulated as follows.

\[
U^A = \left\{ s \in \mathbb{R}^m : \ s = \tilde{s} + z^+ \odot (\underline{s} - \tilde{s}) + z^- \odot (\overline{s} - \tilde{s}), \ 0 \leq z^+, z^- \leq 1 \right\},
\]

where \( \tilde{s} \) is the middle point of the interval \([\underline{s}, \overline{s}]\), and \( z^+ \) and \( z^- \) denote positive and negative scaled deviations from \( \tilde{s} \), respectively. We next show that the worst-case scenario of the admissible uncertainty set always occurs within \( \tilde{s}, \overline{s} \), called the “effective” uncertainty set.

### 3.2.2. Effective Uncertainty Set

We first identify a subset of the admissible uncertainty set, as shown in Figure 2, within which the worst-case scenario of the uncertain parameter \( \tilde{y} \) would always occur.

**Proposition 4.** The worst-case realization of the admissible uncertainty set \( U^A \) always occurs within interval \([\tilde{s}, \overline{s}]\), where \( \tilde{s} \) is the middle point of the interval \([\underline{s}, \overline{s}]\).

**Proof.** Using the definition of \( U^A \), we note that \([\underline{s}, \overline{s}] = [\tilde{s}, \overline{s}] \cup [\tilde{s}, \overline{s}]\). Thus, we can re-write the worst-case of constraint (8) by considering \( \beta(z^+, z^-) \) instead of \( \beta(\mathbf{r}) \), where:

\[
\beta(z^+, z^-) = \max_{z^+, z^-} B \left( \tilde{s} + z^+ \odot (\underline{s} - \tilde{s}) + z^- \odot (\overline{s} - \tilde{s}) \right),
\]

\[
\text{s.t.} \quad 0 \leq z^+ \leq 1, \quad 0 \leq z^- \leq 1.
\]

Separating the constant \( B\tilde{s} \) from the objective function and considering dual vectors \( \eta \) and \( \zeta \) for constraints (13b) and (13c), respectively, the dual formulation of problem (13) is:

\[
B\tilde{s} + \min_{\eta, \zeta} \eta + \zeta,
\]

\[
\text{s.t.} \quad \eta \geq B(\underline{s} - \tilde{s}),
\]

\[
\zeta \geq B(\overline{s} - \tilde{s}),
\]

\[
\eta, \zeta \geq 0.
\]

Since \( \underline{s} - \tilde{s} \leq 0 \) and \( B \geq 0 \), constraint (14c) is redundant and thus \( \zeta = 0 \). By removing the redundant constraint (14c) and taking the dual of model (14) with vector \( \mathbf{r} \) as the dual vector...
Figure 2: Comparison of the initial uncertainty set $\mathcal{U} = [\mathbf{y}, \bar{\mathbf{y}}]$, the admissible uncertainty set $\mathcal{U}^A = [\mathbf{s}, \bar{\mathbf{s}}]$, and the effective uncertainty set $\mathcal{U}^E = [\hat{s}, \bar{s}]$ for the four possible cases of Proposition 2.

corresponding to constraint (14b), the following formulation is obtained:

$$
\begin{align*}
\mathbf{B}\hat{s} + \max_r \mathbf{B}\left(r \odot (\bar{s} - \hat{s})\right), \\
\text{s.t.} \quad 0 \leq r \leq 1,
\end{align*}
$$

which shows $\hat{s} \in [\bar{s}, \bar{s}]$.

Proposition 4 showed the equivalency of the uncertainty sets $[\mathbf{s}, \bar{\mathbf{s}}]$ and $[\hat{s}, \bar{s}]$ in terms of their worst-case scenarios. We denote $\mathcal{U}^E$ as the effective uncertainty set $[\hat{s}, \bar{s}]$ where:

$$
\mathcal{U}^E := \left\{ \hat{s} \in \mathbb{R}^m : \hat{s} = \hat{s} + r \odot (\bar{s} - \hat{s}), \ 0 \leq r \leq 1 \right\}.
$$

Therefore, we can use the effective uncertainty set $\mathcal{U}^E = [\hat{s}, \bar{s}]$ in formulation $[\mathbf{M}']$.

So far, we introduced the admissible and effective uncertainty sets without considering any budget of uncertainty and showed that $\mathcal{U}^E$ can be used instead of $\mathcal{U}$ without affecting the solution. In the following section, we identify an effective budget of uncertainty using the definitions made so far.

3.3. Effective Budget of Uncertainty

In the conventional budget approach, the budget $\Gamma$ controls the sum of scaled deviations from the nominal value $\hat{\mathbf{y}}$, which is the middle point of the uncertainty set $[\mathbf{y}, \bar{\mathbf{y}}]$. In Proposition 4 we showed that the worst-case scenario always occurs within the effective uncertainty set $\mathcal{U}^E = [\hat{s}, \bar{s}]$ and that we can replace the original uncertainty set with $\mathcal{U}^E$ without changing the feasible region. However, for $\mathcal{U}^E$, $\hat{s}$ is the nominal value which is often not equal to $\hat{\mathbf{y}}$ as shown in Fig 2, so the conventional definition of $\Gamma$ cannot be directly used. In this section, we propose a new definition of effective budget of uncertainty on the set $\mathcal{U}^E$ which has the following properties:

- For $\Gamma = 0$ and $\Gamma = m$ (zero budget and full budget, respectively), it generates the same solutions as the conventional budget approach.
For other values of $0 < \Gamma < m$, depending on the corresponding case from Proposition 2, it accounts for the ineffective budgets meaning that it explicitly controls the level of conservatism with an increase in the budget.

Before presenting the new definition of budget, let us first present an intuition behind what is expected. Recall that the conventional approach with zero uncertainty ($\Gamma = 0$) corresponds to solution $y = \bar{s}$ in cases (c) and (d) since $\bar{s} < \bar{y} \in [\bar{y}, \bar{y}]$. To generate the same solution in $U^E$, a one-directional positive deviation $r$ from the new nominal value $\hat{s}$ is required since $y \leq \hat{s} + r \circ (\bar{s} - \hat{s})$. Otherwise, zero uncertainty in $U^E$ would result in $y = \hat{s}$ (as $r = 0$), which is different than the solution of the conventional approach with no uncertainty. Thus, to keep the properties of the conventional approach, we should allow such deviations in the proposed approach but should not take them into account in the budget of uncertainty constraint for cases (c) and (d).

Similarly, in case (b), when $\bar{s} < \bar{y}$, for any value of $\Gamma$, the conventional approach corresponds to $y = \bar{y}$. Consider Figure 3 which elaborates on case (b) and compares the uncertain parameters in $U^E$ and $U^B$ based on their scaled deviations. The shaded regions in Figure 3 show the areas within which the budget of uncertainty of the conventional approach does not impact the solutions. Hence, only the scaled deviations in $U^B$ corresponding to line segment $A'C$ are effective; therefore, a complete scaled deviation (1) should correspond to point $C$, since any further budget would not change the solution. In order to keep the properties of the conventional budget approach, both the scaled deviations and the total budget should be adjusted to correspond to the new length of the uncertainty set as well as the one-sided nature of it.

Based on the properties discussed above, in what follows, we formally present an effective uncertainty set $U^{EB}$ with an effective budget $\Gamma^E$ in Definition 1. Proposition 2 then describes the linear mapping between $\Gamma$ and $\Gamma^E$ and elaborates on the properties of the proposed budget.

**Definition 1.** $U^{EB}$ is the effective uncertainty set with an effective budget of uncertainty $\Gamma^E$ and
is defined as

\[
U^{EB} = \left\{ \tilde{s} \in \mathcal{R}^m : \tilde{s} = \hat{s} + r \odot (\bar{s} - \tilde{s}), \sum_{i=1}^{m} e_i r_i \leq \Gamma^E, v \leq r \leq 1 \right\}
\] (17a)

where vectors \( v \), \( e \) and the new budget \( \Gamma^E \) are parameters calculated based on existing information as follows

\[
h = \frac{1}{2} \left( 1 - \text{sgn}(\bar{y} - \bar{s}) \right),
\] (18a)

\[
e = h \odot \left( \frac{\bar{s} - \hat{s}}{\bar{y} - \bar{y}} \right),
\] (18b)

\[
v = h \odot \left( \frac{\hat{y} - \hat{s}}{\bar{s} - \hat{s}} \right),
\] (18c)

\[
\Gamma^E = \Gamma + \sum_{i=1}^{m} v_i \left( \frac{\bar{s}_i - \hat{s}_i}{\bar{y}_i - \bar{y}_i} \right).
\] (18d)

Proposition 5. The effective uncertainty set \( U^{EB} \) only considers deviations within \( U^E \) into the effective budget \( \Gamma^E \).

Proof. Consider \( h = \frac{1}{2} (1 - \text{sgn}(\bar{y} - \bar{s})) \). Based on the definition of \( h \), we observe that \( h = 1 \) for cases (a) and (b), and \( h = 0 \) for cases (c) and (d). Since \( e \) is a function of \( h \), constraint (17b) ensures that \( \Gamma^E \) only considers effective deviations by letting the required deviations of cases (c) and (d) happen but not allocating a budget of uncertainty for such deviations.

Similarly, consider \( v = h \odot \left( \frac{\bar{s} - \hat{s}}{\bar{y} - \bar{y}} \right) \), which takes a value of 0 for all cases except case (b), due to \( h = 0 \) in cases (c) and (d), and \( \bar{y} = \hat{s} \) in case (a). The nonzero value of \( v \) is the scaled deviation required to map \( \hat{s} \) to \( \hat{y} \). Thus, \( v \leq r \) ensures \( \hat{s} \) is mapped to \( \hat{y} \). To ensure the proposed approach allows this mapping without using the budget of uncertainty, \( \Gamma \) is linearly mapped to \( \Gamma + v \), where

\[
\sum_{i=1}^{m} h_i r_i \leq \Gamma + v
\] (19)

Note that \( v \) and \( r \) are scaled deviations based on the length of \( (\bar{s} - \hat{s}) \), while \( \Gamma \) is a scaled parameter based on the magnitude of \( (\bar{y} - \bar{y}) \). To normalize the scaled deviations, the factor \( \frac{\bar{y} - \bar{s}}{\bar{y} - \bar{y}} \) is multiplied by \( v \) and \( r \). Doing so, formulations (17) and (18) can be recovered.

Remark 2. The partially-ineffective budget \( \Gamma \) in the uncertainty set \( U^B \) is linearly mapped into an
Proof. It is sufficient to note that all functions mapping \( \Gamma \) to \( \Gamma^E \) in Definition 1 are linear.

Finally, we note that the range of values that \( \Gamma \) and \( \Gamma^E \) can take are different from each other. Recall that in the conventional budget approach, \( \Gamma \) takes a value within \([0, m]\). (Bertsimas and Sim, 2004). In the proposed approach, however, the effective budget is defined as \( \Gamma^E = \Gamma + \sum_{i=1}^{m} v_i \), and therefore, \( \Gamma^E \in \left[ \sum_{i=1}^{m} v_i, \sum_{i=1}^{m} v_i + m \right] \). The term \( \sum_{i=1}^{m} v_i \) is constant and is only positive when the uncertain parameter corresponds to the effective interval of case (b). Particularly, \( \Gamma = 0 \) in the conventional approach would generate the same solution as \( \Gamma^E = \sum_{i=1}^{m} v_i \) in the proposed approach (point \( A' \) of Figure 3). On the other hand, when the entire uncertainty set is admissible, (i.e., case (a)), for each \( i \), we conclude that: \( e_i = 1, v_i = 0, \hat{s}_i = \hat{y}_i, \bar{s}_i = \bar{y}_i \), and hence \( \Gamma^E = \Gamma \) and the proposed approach becomes the same as the conventional budget approach.

3.4. The Proposed Two-Stage Approach

Based on the definition of \( U^E \) and \( U^{EB} \), we now propose a two-stage robust approach to solve robust problems with RHS uncertainty while considering an effective budget of uncertainty.

Stage (I): In this stage, we solve the auxiliary optimization problem (10) to find the admissible uncertainty set \( U^A = [\bar{s}, \bar{s}] \) that has the smallest distance from the initial uncertainty set \( U = [\bar{y}, \bar{y}] \). We then form the effective uncertainty set \( U^E = [\hat{s}, \hat{s}] \) and calculate parameter \( \Gamma^E \) and vectors \( \mathbf{v} \) and \( \mathbf{e} \) using the set of equations in (18).

Stage (II): In this stage, we use the output of Stage (I) and solve the following optimization problem (20) which incorporates the effective budget of uncertainty into the robust model.

\[
\begin{align*}
\min_{x,y,r} & \quad c_1(x) + c_2(y), \\
\text{s.t} & \quad Ax + By \leq g, \\
& \quad y \leq \bar{s} \\
& \quad \bar{s} = \hat{s} + r \odot (\bar{s} - \hat{s}), \\
& \quad \sum_{i=1}^{m} e_i r_i \leq \Gamma^E, \\
& \quad \mathbf{v} \leq r \leq 1, \\
& \quad x, y \geq 0.
\end{align*}
\]

The proposed two-stage approach provides insights on the price of robustness in problems with uncertain parameters in the RHS of constraints. From a managerial point of view, the trade-off between the robustness and objective function value can explicitly be observed since changing the uncertainty budget would change the objective function. This would allow for a more intuitive way
for decision makers to determine the level of conservatism of the robust solutions and hence, the value of the budget of uncertainty in the robust model.

4. Numerical Results

In this section, we examine the performance of the proposed robust model on an example of the previously-described SCED problem with wind uncertainty. The SCED problem can be presented in the form of model \([M]\) where vectors \(x\) and \(y\) correspond to the power generation of conventional generators and wind turbines, respectively. The objective function aims to minimize the total operational cost, i.e., the generation cost of conventional generators, plus a penalty cost associated with non-utilized wind power, under the worst-case scenario of wind availability over a given time horizon. Constraint (2b) captures the supply-demand balance equations and operational limits (e.g., limits of transmission lines, generators, and reserve requirements) of the power system. Finally, the uncertainty associated with available wind power can be presented in the RHS of constraint (2c), where the utilized wind power is less than or equal to the available amount of wind power.

We use an IEEE reliability test system (RTS) with multiple wind farms to perform numerical testing of our methodology. Detailed data of the test system can be found in Grigg et al. (1999). The hourly load (demand) is shown in Figure 4. We consider a total of four wind farms in the system, two with wind profile #1 and another two with wind profile #2. These wind profiles and their corresponding uncertainty sets on the available wind power are shown in Figure 5.

We solve a day-ahead optimization model for the SCED problem with a 24-hour time horizon. The budget of uncertainty is defined for each one-hour time period across all wind farms. A number of power dispatch studies in the literature neglect the impacts of the unutilized wind power – which is called wind power curtailment – in the optimal solution and assume that the nominal wind power can be fully utilized regardless of how volatile and large the wind power is (Wu et al., 2014). We call this approach the “naive approach” where instead of optimizing over \(y\), we assume \(y = \hat{y}\) is given. For comparison, we will use three variants of modeling the SCED optimization.
problem with wind power curtailment in this section: (i) the deterministic model, which does not account for wind uncertainty, (ii) the conventional robust model, which considers wind uncertainty but does not account for effective budgets, and (iii) the proposed two-stage robust model with effective budgets. Detailed formulations of these three models for the SCED problem are provided in Appendix A. For further details on the robust formulation of the SCED problem, the reader is referred to Dehghani Filabadi (2019).

In the rest of this section, we first perform day-ahead analyses where the uncertainty set on the available wind power for the next 24 hours is used to solve the SCED problem before knowing the actual realization of the uncertain parameter. We explicitly show all potential cases of the admissible interval, demonstrate the performance of the proposed approach in terms of the effectiveness of the budget of uncertainty, and study the trade-off between operational cost and budget of uncertainty. Next, to verify the reliability and cost efficiency of the day-ahead solutions of the proposed approach, we compare the solutions of each approach with a “prescient” solution in which we assume we have the perfect information of real-time wind power availability when planning.

4.1. Admissible Wind Power Interval

Figure 6 shows the admissible wind power intervals for all time periods across all wind farms. First, consider the shaded region which shows the admissible interval; we can observe examples of all four possible cases of Proposition 2 during this 24-hour time horizon. For instance, case (a) is observed for wind farm A during periods 9 to 12, where the available wind power can be entirely absorbed. Cases (b) and (c) are observed in wind farm B during periods 16 and 20, respectively, in which a part of the available wind power cannot be utilized. Case (d) is observed in wind farm A during period 2, where the admissible wind power interval is outside of the uncertainty set.

Figure 6 also shows the day-ahead solution of the proposed approach with \( \Gamma_t = 4 \) (full budget) and \( \Gamma_t = 0 \) (no budget) in comparison with that of the naive approach. Note that the proposed
4.2. Performance of The Proposed Approach versus The Conventional Budget Approach

In this section, we compare the effectiveness of the proposed robust approach with that of the conventional approach when we modify the budget of uncertainty. Recall that the budget of uncertainty is defined per time period. Here, we focus on comparing results for a sample time period

---

### Figure 6: Wind power output comparison between the proposed robust, deterministic, and naive approaches. Shaded regions correspond to the wind power admissible intervals.

- **Wind farm A**
- **Wind farm B**
- **Wind Farm C**
- **Wind farm D**

The robust solution with \( \Gamma_t = 0 \) corresponds to the deterministic solution with no wind power uncertainty. From Figure 6, we observe that the proposed robust solutions lie within the admissible wind power interval (shaded region) and thus guarantee feasibility (system security) under any scenario of the actual wind power in real-time. During some periods (e.g., periods 1 and 2), even though the naive solution corresponds to more wind power utilization, it is outside of the admissible interval and may violate the operational limits of the systems in real-time.
(time period \( t = 17 \)), but we note that similar observations can be made for other time periods as well. Figure 7 shows the wind power utilization versus the budget of uncertainty for the sample time period. Overall, Figure 7 shows that the proposed approach results in larger wind power utilization compared to the conventional approach and controls the trade-off between the budget of uncertainty and wind power utilization, as intuitively expected. In what follows, we explain the detailed reasoning behind this behaviour for \( t = 17 \).

For \( \Gamma_{17} \in [0,1] \), we observe that both the conventional and proposed approaches result in the same wind power utilization, and an increase in the budget of uncertainty leads to higher wind power utilization. Here, the conventional approach uses the first unit of the budget for deviations of the uncertain parameter from the nominal value in wind farm A, whose initial uncertainty set corresponds to case (a) during time 17 and can be entirely utilized. Thus, the uncertain parameters are within the admissible interval for both approaches, and larger budgets of uncertainty correspond to more wind power utilization without causing infeasibility.

For \( \Gamma_{17} \in (1,1.35] \), the uncertain parameter of the conventional approach corresponds to the admissible interval of wind farm B and results in the same solution as the proposed approach. On the other hand, for \( \Gamma_{17} \in (1.35,2] \), the conventional approach leads to an ineffective budget that cannot be further utilized in wind farm B, and changing the budget does not impact the robust solution. In this case, the uncertain parameters corresponding to the available wind power lie outside of the admissible interval and cannot be utilized. In contrast, the proposed robust approach leads to a higher wind power utilization since it allocates the budget of uncertainty only where it can be used. This means that the proposed approach would allocate the budget to a different wind farm for which the uncertain parameter is admissible.

For \( \Gamma_{17} \in (2,2.47] \) an increase in the budget of uncertainty results in higher wind power utilization in both approaches since the uncertain parameter corresponding to the power output of both wind farms are within the admissible interval of wind farm C. However, the total utilized
wind power in the conventional approach is less than that of the proposed approach due to previous units of budget being ineffectively used in wind farms A and B. Similar arguments can be made for $\Gamma_{17} \in (2.47, 3]$, $\Gamma_{17} \in (3, 3.59]$, and $\Gamma_{17} \in (3.59, 4]$. Finally, for $\Gamma_{17} = 4$ (full budget), both approaches lead to the same overly-conservative solution, where the uncertain parameters have maximum deviations from their nominal values in all wind farms.

4.3. Total Cost versus Budget of Uncertainty

Figure 8 shows the trade-off between the total operational cost over the 24-hour time horizon of the day-ahead robust and deterministic solutions versus the budget of uncertainty. In this case, we adjust the budget of uncertainty for all time periods simultaneously to see how it would affect the total operational cost. This trade-off is often referred to as “the price of robustness” (Bertsimas and Sim, 2004), and it is expected that requiring a higher level of uncertainty would lead to a more conservative solution and thus a higher total cost.

For the deterministic approach, the optimal operational cost is constant and does not depend on the budget of uncertainty. We note that this is the expected day-ahead cost based on the deterministic model and is not necessarily equal to the actual cost when the actual wind power is different than the nominal one. We will elaborate on the comparison of how close of an estimate of the actual cost each of these methods would provide, when we later compare each solution with a prescient solution in which we assume to have the perfect prediction of the wind power in advance (Section 4.4).

For the robust approaches, on the other hand, as the budget of uncertainty increases, the operational cost consistently increases. The reason is that a higher budget of uncertainty corresponds to a larger interval for the actual wind power (more uncertainty on the RHS). Therefore, the larger wind power interval results in a more conservative solution that potentially results in a higher day-ahead wind power curtailment during critical periods—where the excessive wind power can not be absorbed by the system—to guarantee the security of the system under all wind power scenarios in real-time. This, in turn, increases the objective function value due to high curtailment costs (Li et al., 2016).

Most importantly, the shaded region in Figure 8 demonstrates the difference between the proposed and conventional robust approaches in terms of total cost. For $0 < \Gamma_t < 4$, the proposed approach leads to a lower operational cost since it effectively allocates the budget to wind farms that can utilize it (where the uncertain parameter is within the admissible intervals), and hence results in lower wind curtailment. However, for $\Gamma_t = 0$ (no budget) and $\Gamma_t = 4$ (full budget), both robust approaches are equivalent since they both correspond to the deterministic and worst-case solutions, respectively.
4.4. Simulation Results and Cost Verification

One advantage of using a Robust Optimization methodology is that the resulting optimal solutions are often less sensitive to changes in the uncertain parameter compared to that of a deterministic approach and provides a better estimate of the realized cost, regardless of the realized scenario of the uncertain parameter. To demonstrate this advantage in our specific application, we compare the day-ahead solution ($y$) of each approach with a prescient solution which assumes that perfect information of $y^{Actual}$ is known in advance and is obtained by solving the deterministic model with $\tilde{y} = y^{Actual}$. To obtain a set of simulated prescient solutions, we generate randomized values of the actual wind power within the uncertainty set $[\underline{y}, \bar{y}]$ for each time period and calculate the scaled deviation $z = \frac{|y^{Actual} - \tilde{y}|}{\bar{y} - \underline{y}}$ from the nominal value for each period. This process is repeated until there are 100 scenarios (for each time interval) where we have $\sum_{i=1}^{m} z_i \leq \Gamma_t$, for values of $\Gamma_t = 1, 2, 3,$ and $4$. Note that for each time period, a scenario is a collection of randomized values for $y^{Actual}$ of different wind farms such that the total deviations from the nominal values fits the budget constraint. For each scenario, we solve the deterministic SCED problem with $\tilde{y} = y^{Actual}$ and calculate the absolute difference between the total cost of the day-ahead and prescient solutions, denoted as $\Delta C$, for each of the three approaches.

Figure 9 compares the performance of the three approaches in terms of $\Delta C$ which shows the sensitivity of day-ahead solutions to simulated realizations of wind power and provides a basis to compare which model gives a better estimate of the prescient solution. We can observe that the deterministic approach corresponds to the largest $\Delta C$ values since this approach does not take wind power uncertainty into account, and the day-ahead scheduled (expected) cost is much lower than the prescient (actual) cost. On the other hand, the conventional and proposed robust approaches corresponds to smaller $\Delta C$ values as they account for uncertainty, and hence their day-ahead solutions are closer to the prescient solutions.
The proposed robust approach has the smallest $\Delta C$ since it allocates the uncertainty budget to wind farms that can utilize the power and hence, generally results in lower $\Delta C$ which indicates the proposed approach correspond to a robust solution closer to the prescient solution. Particularly, for $\Gamma_t = 1$, the average $\Delta C$ for the proposed robust approach is approximately 25% and 40% less than those of the conventional robust and deterministic approaches, respectively. This is a desirable outcome since it helps planners to have a much better estimate of the actual costs when performing day-ahead planning.

5. Conclusion

In this paper, we proposed a robust optimization approach for considering budget of uncertainty in a class of problems with right-hand side (RHS) uncertainty. We discussed limitations of the conventional robust approach with budget of uncertainty in such problems where adjusting the budget may not have an explicit effect on the level of conservatism. We showed that this class of problems can be thought of having partially-ineffective budgets and proposed a two-stage robust approach that has the same properties of the conventional robust approach while accounting for partially-ineffective budgets.

Our proposed approach provides managerial insights on the trade-off between the robustness and economy and allows the managers to make more informed decisions on the level of conservatism they would like to consider for the system. We demonstrated the practical merits of the proposed approach in a security-constrained economic dispatch (SCED) problem with wind power uncertainty and numerically compared our results with those of the deterministic approach and the conventional
robust approach.

There are many real-world applications with uncertain RHS such as project management, scheduling, dynamic inventory management, and telecommunication problems. An area of further research is to extend the proposed approach for such problem settings in order to more clearly understand the trade-off between the level of conservatism and the total cost.

Appendix A. The SCED Problem

In this section, we present three variants of modeling the security-constrained economic dispatch (SCED) problem based on settings from Li et al. (2016). In the SCED models presented here, parameters $W_{k,t}$, $\hat{W}_{k,t}$, $\tilde{W}_{k,t}$, and $\overline{W}_{k,t}$ correspond to the uncertain vectors $\mathbf{y}$, $\hat{\mathbf{y}}$, $\tilde{\mathbf{y}}$ and $\overline{\mathbf{y}}$ used in this paper, respectively. In what follows, we first present all notations used in the SCED models. Next, we present the three SCED optimization models used in the numerical results in the paper, namely, the deterministic model, the conventional robust model, and the proposed two-stage robust model.

The Deterministic Model

The deterministic model considers the predicted wind power and assumes the prediction is perfect. It then finds the power dispatches based on the predicted wind power output, since the deterministic model does not account for wind uncertainty. The mathematical formulation of the deterministic SCED problem is as follows:

$$\min \sum_{t \in T} \sum_{i \in N} \left( \sum_{g \in G_i} C_g p_{g,t} + \sum_{k \in K_i} \sigma_k \left( \hat{W}_{k,t} - p_{k,t}^W \right) \right),$$ (A.1)

subject to:

$$\sum_{i \in N} \left( \sum_{g \in G_i} p_{g,t} + \sum_{k \in K_i} p_{k,t}^W \right) = \sum_{i \in N} D_{i,t}, \quad \forall t \in T$$ (A.2)

$$f_{f} \leq \sum_{i \in N} G_{f,i} \left( \sum_{g \in G_i} p_{g,t} + \sum_{k \in K_i} p_{k,t}^W - D_{i,t} \right) \leq P_{f}, \quad \forall f \in F, \forall t \in T$$ (A.3)

$$\sum_{i \in N} \sum_{g \in G_i} r_{g,t}^+ \geq P_{t}^u, \quad \forall t \in T$$ (A.4)

$$\sum_{i \in N} \sum_{g \in G_i} r_{g,t}^- \geq P_{t}^d, \quad \forall t \in T$$ (A.5)

$$0 \leq r_{g,t}^+ \leq \min \left\{ P_{g} - p_{g,t}, U_{g} \Delta t \right\}, \quad \forall g \in G_i, i \in N, t \in T$$ (A.6)

$$0 \leq r_{g,t}^- \leq \min \left\{ p_{g,t} - P_{g}, D_{g} \Delta t \right\}, \quad \forall g \in G_i, i \in N, t \in T$$ (A.7)

$$- U_{g} \Delta t \leq p_{g,t} - p_{g,t-1} \leq U_{g} \Delta t, \quad \forall g \in G_i, i \in N, t \in T$$ (A.8)

$$p_{g} \leq p_{g,t} \leq P_{g}, \quad \forall g \in G_i, i \in N, t \in T$$ (A.9)

$$0 \leq p_{k,t} \leq \hat{W}_{k,t}, \quad \forall k \in K_i, i \in N, t \in T$$ (A.10)

The objective function (A.1) minimizes the total operational cost consisting of generation cost of
conventional generators and wind power curtailment cost. Constraint (A.2) ensures the power balance between generation and load. In constraint (A.3), the power flows limits are considered. The requirements for upward and downward spinning reserve are considered in constraints (A.4) and (A.5), respectively. Constraints (A.6) and (A.7) address the capacity of upward and downward reserves of each conventional generator. Constraint (A.8) ensures that the dynamic changes in power outputs are limited with respect to the ramping rate of conventional generators. The generation limits of conventional generators and wind farms are shown in (A.9) and (A.10), respectively.

The Conventional Robust Model

The conventional robust model accounts for uncertainty but does not account for effective budgets. The objective function of the conventional robust approach is

$$
\min \sum_{t \in T} \sum_{i \in N} \left( \sum_{g \in G_i} C_g p_{g,t} + \max_{W_{k,t}} \sum_{k \in K_i} \sigma_k \left( \hat{W}_{k,t} - p^W_{k,t} \right) \right),
$$

where the inner maximization finds the wind power curtailment under the worst-case scenario, and then the optimal solution is obtained under the worst-case scenario. In the conventional robust model, constraint (A.12) corresponds to the wind power dispatch limit, and constraints (A.13)-(A.15) correspond to a polyhedral uncertainty set incorporating the budget of uncertainty $\Gamma_t$ in the model.

$$
0 \leq p^W_{k,t} \leq \hat{W}_{k,t}, \quad \forall k \in K_i, i \in N, t \in T
$$

$$
\hat{W}_{k,t} = \bar{W}_{k,t} + z^+_{k,t}(\bar{W}_{k,t} - \hat{W}_{k,t}) + z^-_{k,t}(- \hat{W}_{k,t} + \bar{W}_{k,t}), \quad \forall k \in K_i, i \in N^f, t \in T
$$

$$
\sum_{t \in T} \sum_{k \in K_i} (z^+_{k,t} + z^-_{k,t}) \leq \Gamma_t, \quad \forall t \in T
$$

$$
0 \leq z^+_{k,t} \leq 1, \quad \forall k \in K_i, i \in N^f, t \in T
$$

$$
0 \leq z^-_{k,t} \leq 1, \quad \forall k \in K_i, i \in N^f, t \in T
$$

Using Duality Theorems, a linear reformulation of the conventional robust model is obtained as follows, where $\xi$ and $\mu$ are dual vectors corresponding to constraints (A.14) and (A.15). Note that the constraint corresponding to (A.16) becomes redundant, as shown in Proposition 4, and is removed:

$$
\min \sum_{t \in T} \sum_{i \in N} \left( \sum_{g \in G_i} C_g p_{g,t} + \sum_{k \in K_i} \sigma_k (\hat{W}_{k,t} + \mu_{k,t} + \Gamma t \xi_t - p^W_{k,t}) \right),
$$

s.t.: 

$$
\mu_{k,t} + \xi_t \geq (\bar{W}_{k,t} - \hat{W}_{k,t}), \quad \forall k \in K_i, i \in N^f, t \in T
$$

$$
\mu, \xi \geq 0
$$

(A.2) – (A.9), (A.12) – (A.16).
The Proposed Robust Model

The proposed robust model considers effective budgets of uncertainty and uses the proposed two-stage approach. **Stage (I):** The auxiliary optimization problem (A.20) is used to find the largest admissible wind power interval $[\underline{s}_{k,t}, \bar{s}_{k,t}]$:

$$\min \sum_{t \in T} \sum_{i \in N^t} \sum_{k \in K_i} (W_{k,t} - \bar{s}_{k,t}) + (W_{k,t} - \underline{s}_{k,t})$$

s.t.  
$$\sum_{i \in N^t} G^f_{i,t} \left( \sum_{g \in G_i} p_{g,t} - D_{i,t} + \sum_{k \in K_i} \underline{s}_{k,t} \right) + \sum_{i \in N^t} \sum_{k \in K_i} \alpha_{k,f,t} \leq F_f,$$  
$$\forall k \in K_i, i \in N^t, f \in F, t \in T$$ (A.20b)

$$\alpha_{k,f,t} \geq G^f_{f,t}(\underline{s}_{k,t} - \bar{s}_{k,t}),$$  
$$\forall k \in K_i, i \in N^t, f \in F, t \in T$$ (A.20c)

$$\sum_{i \in N^t} G^f_{i,t} \left( \sum_{g \in G_i} p_{g,t} - D_{i,t} + \sum_{k \in K_i} \bar{s}_{k,t} \right) - \sum_{i \in N^t} \sum_{k \in K_i} \alpha_{k,f,t} \geq E_f,$$  
$$\forall k \in K_i, i \in N^t, f \in F, t \in T$$ (A.20d)

$$\zeta_{k,f,t} \geq -G^f_{f,t}(\bar{s}_{k,t} - \underline{s}_{k,t}),$$  
$$\forall k \in K_i, i \in N^t, f \in F, t \in T$$ (A.20e)

$$\sum_{i \in N^t} \left( \sum_{g \in G_i} p_{g,t} + \sum_{g \in G_i} r_{g,t}^+ - D_{i,t} + \sum_{k \in K_i} \bar{s}_{k,t} - \sum_{k \in K_i} \eta_{k,t} \right) \geq R^u_t,$$  
$$\forall t \in T$$ (A.20f)

$$\eta_{k,t} \leq -\bar{G}_{f,t}(\bar{s}_{k,t} - \underline{s}_{k,t}),$$  
$$\forall k \in K_i, i \in N^t, t \in T$$ (A.20g)

$$\sum_{i \in N^t} \left( \sum_{g \in G_i} p_{g,t} - \sum_{g \in G_i} r_{g,t}^- - D_{i,t} + \sum_{k \in K_i} \underline{s}_{k,t} + \sum_{k \in K_i} \beta_{k,t} \right) \leq R^d_t,$$  
$$\forall t \in T$$ (A.20h)

$$\beta_{k,t} \geq \bar{s}_{k,t} - \underline{s}_{k,t},$$  
$$\forall k \in K_i, i \in N^t, t \in T$$ (A.20i)

$$\bar{s}_{k,t} \leq W_{k,t},$$  
$$\forall k \in K_i, i \in N^t, t \in T$$ (A.20j)

$$\underline{s}_{k,t} \leq W_{k,t},$$  
$$\forall k \in K_i, i \in N^t, t \in T$$ (A.20k)

$$\alpha, \zeta, \eta, \beta, r^+, r^-, \bar{s}, \underline{s}, \rho \geq 0,$$  

where vectors $\alpha, \zeta, \eta, \beta$ are auxiliary variables use in the robust counterpart reformulation. Given the admissible interval $[\underline{s}_{k,t}, \bar{s}_{k,t}]$, the effective uncertainty set $[\hat{s}_{k,t}, \bar{s}_{k,t}]$ is obtained.

**Stage (II):** Given the optimal solution of Stage I, the effective budget of uncertainty $\Gamma^E$ is obtained and incorporated in the Stage II formulation as follows:

$$\min \left\{ \sum_{t \in T} \sum_{i \in N^t} \left( \sum_{g \in G_i} C_{g} p_{g,t} + \sum_{k \in K_i} \sigma_k (\hat{s}_{k,t} + \mu_{k,t} - v_{k,t} \lambda_{k,t} + \Gamma^E_t \xi_t - p_{k,t}) \right) \right\}$$

s.t.  
$$\mu_{k,t} - \lambda_{k,t} + \epsilon_{k,t} \xi_t \geq (\bar{s}_{k,t} - \hat{s}_{k,t}),$$  
$$\forall k \in K_i, i \in N^t, t \in T$$ (A.22)

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\[ p_{k,t}^W \leq \hat{s}_{k,t}, \quad (A.23) \]
\[ \tilde{s}_{k,t} = \hat{s}_{k,t} + r_{k,t}(\bar{s}_{k,t} - \hat{s}_{k,t}), \quad \forall k \in K_i, i \in N^I, t \in T \quad (A.24) \]
\[ \sum_{i \in N^I} \sum_{k \in K_i} e_{k,t} r_{k,t} \leq \Gamma^E_t, \quad \forall t \in T \quad (A.25) \]
\[ v_{k,t} \leq r_{k,t} \leq 1, \quad \forall t \in T \quad (A.26) \]
\[ \mu, \lambda, \xi \geq 0, \quad (A.27) \]
\[ \mu, \lambda, \xi \geq 0, \quad (A.28) \]

where parameters \( e_{k,t}, v_{k,t}, \) and \( \Gamma^E_t \) are calculated based on Section 3.3.
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