Close-packed dimers on nonorientable surfaces

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Abstract

The problem of enumerating dimers on an \( M \times N \) net embedded on non-orientable surfaces is considered. We solve both the Möbius strip and Klein bottle problems for all \( M \) and \( N \) with the aid of imaginary dimer weights. The use of imaginary weights simplifies the analysis, and as a result we obtain new compact solutions in the form of double products. The compact expressions also permit us to establish a general reciprocity theorem.

keywords: close-packed dimers, non-orientable surfaces, reciprocity theorem

1 Introduction

A seminal development in modern lattice statistics is the solution of enumerating close-packed dimers, or perfect matchings, on a finite \( M \times N \) simple-quartic net obtained by Kasteleyn [1] and by Temperley and Fisher [2, 3] more than 40 years ago. In their solutions the simple-quartic net is assumed to possess free or periodic boundary conditions [1]. In view of the connection with the conformal field theory [4], where the boundary conditions play a crucial role, there has been considerable renewed interest to consider lattice models on non-orientable surfaces [5, 6, 7, 8, 9].

For close-packed dimers the present authors [6] have obtained the generating function for the Möbius strip and the Klein bottle for even \( M \) and \( N \). Independently, Tesler [10] has solved the problem of perfect matchings on Möbius strips and deduced solutions in terms of a \( q \)-analogue of the Fibonacci numbers for all \( \{M, N\} \). It turns out that the explicit expression of the generating function depends crucially on whether \( M \) and \( N \) being even or odd, and the analysis differs considerably when either \( M \) or \( N \) is odd. In this paper we consider the general \( \{M, N\} \) problems for both the Möbius strip and the Klein bottle by introducing imaginary dimer weights. The use of imaginary weights simplifies the analysis and as a result we obtain compact expressions of the solutions without the recourse of Fibonacci numbers. The compact expressions of the solutions also permit us to establish a reciprocity theorem on the enumeration of dimers.

2 Summary of results

For the convenience of references, we first summarize our main results. Details of derivation will be presented in subsequent sections.

Consider an \( \mathcal{M} \times \mathcal{N} \) simple-quartic net consisting \( \mathcal{M} \mathcal{N} \) sites arranged in an array of \( \mathcal{M} \) rows and \( \mathcal{N} \) columns. The net forms a Möbius strip if there is a twisted boundary condition in the horizontal direction as shown in Fig. 1, and a Klein bottle if, in addition to the twisted boundary condition, there is also a periodic boundary condition in the vertical direction. Let the dimer weights be \( z_h \) and \( z_v \), respectively, in the horizontal and vertical directions. We are interested in the close-form evaluation of the dimer generating function

\[
Z_{\mathcal{M}, \mathcal{N}}(z_h, z_v) = \sum z_h^{a_h} z_v^{a_v}
\]  

(1)
where \(n_h, n_v\) are, respectively, the number of horizontal and vertical dimers, and the summation is taken over all close-packed dimer coverings of the net.

Our results are as follows: For both \(M\) and \(N\) even, we have

\[
Z_{M,N}^{\text{Mob}}(z_h, z_v) = \prod_{m=1}^{M/2} \prod_{n=1}^{N/2} \left[ 4z_h^2 \sin^2 \left( \frac{(4n-1)}{2N} \pi \right) + 4z_v^2 \cos^2 \frac{m\pi}{M+1} \right],
\]

(2)

\[
Z_{M,N}^{\text{Kln}}(z_h, z_v) = \prod_{m=1}^{M/2} \prod_{n=1}^{N/2} \left[ 4z_h^2 \sin^2 \left( \frac{(4n-1)}{2N} \pi \right) + 4z_v^2 \sin^2 \frac{(2m-1)}{M} \pi \right],
\]

(3)

where the superscripts refer to the type of the nonorientable surface under consideration.

For \(M\) even and \(N\) odd, we have

\[
Z_{M,N}^{\text{Mob}}(z_h, z_v) = \text{Re} \left[ (1 - i) \prod_{m=1}^{M/2} \prod_{n=1}^{N} \left( 2i(-1)^{m+1} z_h \sin \left( \frac{(4n-1)}{2N} \pi \right) + 2z_v \cos \frac{m\pi}{M+1} \right) \right],
\]

(4)

\[
Z_{M,N}^{\text{Kln}}(z_h, z_v) = \text{Re} \left[ (1 - i) \prod_{m=1}^{M/2} \prod_{n=1}^{N} \left( 2i(-1)^{m+1} z_h \sin \left( \frac{(4n-1)}{2N} \pi \right) + 2z_v \sin \frac{(2m-1)}{M} \pi \right) \right],
\]

(5)

and for \(M\) odd and \(N\) even, we have

\[
Z_{M,N}^{\text{Mob}}(z_h, z_v) = z_h^{-N/2} \prod_{n=1}^{N/2} \prod_{m=1}^{(M+1)/2} \left[ 4z_h^2 \sin^2 \left( \frac{(4n-1)}{2N} \pi \right) + 4z_v^2 \cos^2 \frac{m\pi}{M+1} \right],
\]

(6)

\[
Z_{M,N}^{\text{Kln}}(z_h, z_v) = z_h^{-N/2} \prod_{n=1}^{N/2} \prod_{m=1}^{(M+1)/2} \left[ 4z_h^2 \sin^2 \left( \frac{(4n-1)}{2N} \pi \right) + 4z_v^2 \sin^2 \frac{(2m-1)}{M} \pi \right].
\]

(7)

For \(M\) and \(N\) both odd, the generating function is zero.

### 3 The Möbius strip

It is well-known that there is a one-one correspondence between dimer coverings and terms in a Pfaffian defined by the dimer weights. However, since terms in the Pfaffian generally do not possess the same sign, the evaluation of the Pfaffian does not necessarily produce the desired dimer generating function. The crux of matter is to attach signs, or more generally factors, to the dimer weights so that all terms in the Pfaffian have the same sign, and the task is reduced to that of evaluating a Pfaffian.

These tasks were first achieved by Kasteleyn [1] for the simple-quartic dimer lattice with free and periodic boundary conditions, who showed how to attach signs to dimer weights and how to evaluate the Pfaffian. Soon after the publication of [1], T. T. Wu [11] pointed out that the structure of the Pfaffian, and hence its evaluation, can be simplified if a factor \(i\) is associated to dimer weights in one spatial direction. Indeed, the Wu prescription requires only uniformly directed lattice edges with the association of a factor \(i\) to dimer weights in the direction in which the number of lattice sites is odd. (If the number of lattice sites is even in both directions, then the factor \(i\) cannot be associated to dimers in either direction.) For \(\{M, N\} = \{\text{even, odd}\}\) for example, one replaces \(z_h\) by \(iz_h\).
To see that the Wu procedure is correct, one considers a standard dimer covering $C_0$ in which the lattice is covered only by parallel dimers with real weights. Then, the two terms in the Pfaffian corresponding to $C_0$ and any other dimer covering $C_1$ have the same sign, since the superposition polygon produced by $C_0$ and $C_1$ contains an even number of arrows pointing in one direction as well as a factor $i^{4n+2} = -1$, where $n$ is a nonnegative integer. Namely, the superposition polygon is “clockwise-odd” [11]. This use of imaginary dimer weights is the starting point of our analysis.

### 3.1 $M = \text{even M"obius strip}$

For $M$ even and $N = \text{even or odd}$, we write for definiteness $M = 2M$, $N = N$ where $M, N$ are positive integers. We orient lattice edges as shown in Fig. 1(a). For the time being consider more generally that the horizontal dimers connecting the first and $N$-th columns have weights $z$ and the associated generating function

$$Z_{M,N}^{\text{Mob}}(z_h, z_v; z) = \sum_{m=0}^{2M} z^m T_m$$  \hspace{1cm} (8)

where $T_m \equiv T_m(z_h, z_v)$ is a multinomial in $z_h$ and $z_v$ with strictly positive coefficients. The desired result is then obtained by setting $z = z_h$. Note that $T_0$ is precisely the generating function with free boundary conditions.

![Edge orientations of Möbius strips with twisted boundary conditions in the horizontal direction. A, B, C, D, E denote repeated sites. (a). $M = 4$, $N = 5$. (b). $M = 5$, $N = 4$.](image)

Attach a factor $i$ to all (horizontal) dimer weights $z_h$. This leads us to consider the antisymmetric matrix

$$A(z) = iz_h(F_N - F_N^T) \otimes I_{2M} + z_v I_N \otimes (F_{2M} - F_{2M}^T) + z(K_N + K_N^T) \otimes J_{2M}$$ \hspace{1cm} (9)

where $I_N$ is the $N \times N$ identity matrix, $F^T$ is the transpose of $F$, and $F_{2M}$, $K_N$ and $J_{2M}$ are matrices of the order given by the subscripts,

$$F_N = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad K_N = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix}, \quad J_{2M} = \begin{pmatrix} & & & 1 \\ & \cdots & -1 \\ -1 & \cdots & \end{pmatrix}.$$  \hspace{1cm} (10)

Now, the Pfaffian of $A(z)$ gives the correct generating function $T_0$ in the case of $z = 0$ [11]. For general $z$ we have the following result:
Theorem: The dimer generating function for the simple-quartic net with a twisted boundary condition in the horizontal direction is

$$Z_{M,N}^{Mob}(z_h, z_v; z_h) = \frac{1}{2} \left[ (1 - i)\text{Pf}A(i z_h) + (1 + i)\text{Pf}A(-i z_h) \right]$$

$$= \text{Re} \left[ (1 - i)\text{Pf}A(i z_h) \right]. \quad (11)$$

Proof: It is clear that the term in (11) corresponding to the configuration $C_0$ ($m = 0$) has the correct sign. For any other dimer configuration $C_1$, the superposition of $C_0$ and $C_1$ forms superposition polygons containing $z$ edges. We have the following facts which can be readily verified:

(i) The sign of a superposition polygon remains unchanged under deformation of its border which leaves $n_z$, the number of $z$ edges it contains, invariant.

(ii) Deformations of the border of a superposition polygon can change $n_z$ only by multiples of 2, and the sign of the superposition polygon reverses whenever $n_z$ changes by 2.

(iii) Superposition polygons having 0 or 1 $z$ edges have the sign +.

As a result, we obtain

$$\text{Pf}A(z) \equiv \sqrt{|A(z)|}$$

$$= X_0 + zX_1 - z^2X_2 - z^3X_3 \quad (12)$$

where $| \cdot |$ denotes the determinant of $\cdot$ and

$$X_\alpha = T_\alpha + z^4T_{\alpha+4} + z^8T_{\alpha+8} + \cdots, \quad \alpha = 0, 1, 2, 3. \quad (13)$$

The theorem is now a consequence of the fact $Z_{M,N}^{Mob}(z_h, z_v; z_h) = X_0 + z_hX_1 + z_h^2X_2 + z_h^3X_3$.

Remarks: The theorem holds also for the Klein bottle which, in addition to a twisted boundary condition in the horizontal direction, has a periodic boundary condition in the vertical direction (see below). For the Möbius strip we have $X_1 = X_3 = 0$ when $N = \text{even}$.

It now remains to evaluate $\text{Pf}A(\pm iz_h)$. To evaluate $\text{Pf}A(\pm iz_h) = \sqrt{|A(\pm iz_h)|}$, we make use of the fact that, since $F_{2M} - F_{2M}^T$ commutes with $J_{2M}$, the $2MN \times 2MN$ matrix $A(z)$ can be diagonalized in the $2M$-subspace [8]. Introducing the $2M \times 2M$ matrix $U$ whose elements are

$$U_{m,m'} = i^m \sqrt{\frac{2}{2M+1}} \sin \left( \frac{mm'\pi}{2M+1} \right)$$

$$\quad (U^{-1})_{m,m'} = (-i)^{m'} \sqrt{\frac{2}{2M+1}} \sin \left( \frac{mm'\pi}{2M+1} \right), \quad m, m' = 1, 2, \ldots, 2M, \quad (14)$$

we find

$$\left( U^{-1}F_{2M} - F_{2M}^T U \right)_{m,m'} = (2i \cos \phi_m) \delta_{m,m'}$$

$$\left( U^{-1}J_{2M} U \right)_{m,m'} = i (-1)^{M+m} \delta_{m,m'}, \quad m, m' = 1, 2, \ldots, 2M, \quad (15)$$

where $\phi_m = m\pi/(2M+1)$. Thus, we can replace the $2M \times 2M$ matrices in (11) by their respective eigenvalues, and express $|A(z)|$ as a product of the replaced determinants, namely,

$$|A(z)| = i^{2MN} \prod_{m=1}^{2M} |A_N^{(m)}(z)| \quad (16)$$
where we have taken out a common factor $i$ from each element of the $N \times N$ matrix

$$A_N^{(m)}(z) = 2zv \cos \phi_m I_N + z_h(F_N - F_N^T) + (-1)^{M+m} z(K_N + K_N^T). \quad (17)$$

The matrix $A_N^{(m)}(z)$ can be evaluated for general $z$ in terms of a $q$-analogue of Fibonacci numbers, but for our purposes when $z = \pm i z_h$, the matrix can be diagonalized directly.

Define the $N \times N$ matrix

$$T_N = F_N + i(-1)^{M+m}K_N = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ i(-1)^{M+m} & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad (18)$$

we can rewrite $(17)$ when $z = iz_h$ as

$$A_N^{(m)}(iz_h) = 2zv \cos \phi_m I_N + z_h(T_N - T_N^T). \quad (19)$$

Now $T_N$ and $T_N^\dagger$ commute so they can be diagonalized simultaneously with respective eigenvalues $e^{i\theta_n}$ and $e^{-i\theta_n}$, where

$$\theta_n = (-1)^{M+m} + 1/2N \pi. \quad (20)$$

Thus, we obtain

$$|A_N^{(m)}(iz_h)| = \prod_{n=1}^N \left[ 2zv \cos \frac{m\pi}{2M+1} + 2i(-1)^{M+m+1}z_h \sin \frac{(4n-1)\pi}{2N} \right], \quad (21)$$

and as a result

$$|A(iz_h)| = \prod_{m=1}^M \prod_{n=1}^N \left[ 2zv \cos \frac{m\pi}{2M+1} + 2i(-1)^{M+m+1}z_h \sin \frac{(4n-1)\pi}{2N} \right]^2, \quad (22)$$

where we have made use of the fact that $\cos \phi_{2M+1-m} = -\cos \phi_m$, $(-1)^{2M+1-m} = (-1)^m$, and $i^{2MN} = (-1)^{MN}$. We thus obtain after taking the square root of $(22)$

$$\text{Pf}(iz_h) = \prod_{m=1}^M \prod_{n=1}^N \left[ 2zv \cos \frac{m\pi}{2M+1} + 2i(-1)^{M+m+1}z_h \sin \frac{(4n-1)\pi}{2N} \right]. \quad (23)$$

The substitution of $(23)$ into $(11)$ now yields $(4)$. For $N = \text{even}$ the Pfaffian $(23)$ is real and $(4)$ reduces to $(2)$. There is no such simplification for $N = \text{odd}$.

### 3.2 $\mathcal{M} = \text{odd}$ Möbius strip

For $\mathcal{M} = \text{odd}$ and $\mathcal{N} = \text{even}$ or odd, we write, for definiteness, $\mathcal{M} = 2M+1$, $\mathcal{N} = N$. Since the number of rows $\mathcal{M}$ is odd, we now attach a factor $i$ to dimers in the vertical direction. Again, we assign weights $z$ to horizontal dimers connecting the first and $N$-th columns and consider the generating function

$$Z_{\mathcal{M},\mathcal{N}}^{\text{Mob}}(zh, zv; z) = \sum_{m=0}^{2M+1} z^m T_m \quad (24)$$

$^1$A similar expression of $\theta_n$ given in $(17)$ of Ref. $[6]$ contains a typo where $M + m + 1$ in the exponent should read $M + m$. This does not alter the results of Ref. $[6]$. 

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defined similar to (8). It is readily verified that, with lattice edge orientations shown in Fig. 1(b), all terms in $T_0$ have the same sign. It follows that we can use the theorem of the preceding subsection where $\text{Pf}(A)$ is the Pfaffian of the antisymmetric matrix

$$A(z) = z_h(F_N - F_N^T) \otimes I_{2M+1} - iz_v I_N \otimes (F_{2M+1} - F_{2M+1}^T) + zG_N \otimes H_{2M+1},$$

with $G_N = K_N - K_N^T$ and

$$H_{2M+1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$  

(26)

Apply to (25) the unitary transformation (14) (with $2M$ replaced by $2M + 1$). The transformation diagonalizes $(F_{2M+1} - F_{2M+1}^T)$ as in (15) and, in addition, produces

$$U^{-1}H_{2M+1}U = (-1)^M J_{2M+1}.$$  

(27)

Thus, we obtain

$$\bar{A}(z) \equiv (I_N \otimes U^{-1})A(z)(I_N \otimes U)$$

$$= \left[z_h(F_N - F_N^T) + 2z_v \cos \bar{\phi}_m I_N \right] \otimes I_{2M+1} - zG_N \otimes J_{2M+1}$$

(28)

where $\bar{\phi}_m = m\pi/(2M + 2)$.

Writing

$$B_N^{(m)} \equiv z_h(F_N - F_N^T) + 2z_v \cos \bar{\phi}_m I_N,$$

(29)

then the matrix $\bar{A}(z)$ assumes a form shown below in the case of $2M + 1 = 5$,

$$\bar{A}(z) = \begin{pmatrix} B_N^{(1)} & B_N^{(2)} & -zG_N \\ B_N^{(2)} & B_N^{(3)} + zG_N & -zG_N \\ -zG_N & B_N^{(4)} & B_N^{(5)} \end{pmatrix}.$$  

(30)

Interchanging rows and columns, this matrix can be changed into a block-diagonal form having the same determinant,

$$\begin{pmatrix} B_N^{(1)} & zG_N \\ zG_N & B_N^{(5)} \end{pmatrix}$$

$$\begin{pmatrix} B_N^{(2)} & -zG_N \\ -zG_N & B_N^{(4)} \end{pmatrix}$$

$$\begin{pmatrix} B_N^{(3)} + zG_N \end{pmatrix}.$$  

(31)

For general $M$ we define the $2N \times 2N$ matrix

$$A_{2N}^{(m)}(z) = \begin{pmatrix} B_N^{(m)} \\ (-1)^{M+m+1}zG_N \\ (-1)^{M+m+1}zG_N \end{pmatrix},$$

(32)

and use the result

$$|B_N^{(M+1)} + zG_N| = \frac{1}{2}[1 + (-1)^N]z_h^{N-2}(z_h + z)^2$$

(33)
to obtain

$$|A(z)| = |\bar{A}(z)| = \frac{1}{2}[1 + (-1)^N]z_h^{N-2}(z_h + z)^2 \cdot \prod_{m=1}^M |A_{2N}^{(m)}(z)|.$$ (34)

It therefore remains to evaluate $|A_{2N}^{(m)}(z)|$.

The matrix $A_{2N}^{(m)}(z)$ can be diagonalized for $z = \pm iz_h$. To proceed, it is convenient to multiply from the right by a $2N \times 2N$ matrix (whose determinant is $(-1)^N$) to obtain

$$A_{2N}^{(m)}(iz_h) = A_{2N}^{(m)}(iz_h)\left[I_N \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] = 2z_v \cos \tilde{\phi}_m I_{2N} + z_h (F_N - F_N^T) \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + i(-1)^{M+m+1}z_h G_N \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where

$$Q_{2N} = \left(\begin{array}{cc} F_N & i(-1)^{M+m+1}K_N \\ -i(-1)^{M+m+1}K_N & -F_N \end{array}\right).$$ (36)

Now $Q_{2N}$ commutes with $Q_{2N}^\dagger$, so they can be diagonalized simultaneously with the respective eigenvalues $e^{i\bar{\theta}_n}$ and $e^{-i\bar{\theta}_n}$, where $\bar{\theta}_n = (2n-1)/2N, n = 1, 2, \cdots, 2N$. Thus we obtain

$$|A_{2N}^{(m)}(iz_h)| = (-1)^N|A_{2N}^{(m)}(iz_h)| = (-1)^N \prod_{n=1}^N (4z_h^2 \sin^2 \bar{\theta}_n + 4z_v^2 \cos^2 \bar{\phi}_m),$$ (37)

and, taking the square root of (34) with $z$ replaced by $iz_h$,

$$\text{Pf}A(iz_h) = \frac{1}{2}[1 + (-1)^N](1 + i)\frac{N/2}{2} \cdot \prod_{n=1}^{N/2} \prod_{m=1}^M \left[4z_h^2 \sin^2 \frac{(2n-1)\pi}{2N} + 4z_v^2 \cos^2 \frac{m\pi}{2M+2}\right].$$ (38)

Therefore the generating function vanishes identically for $N = odd$. For $N = even$ we substitute (38) into (11) and make use of the fact that the two sets $\sin^2 (2n-1)\pi/2N$ and $\sin^2 (4n-1)\pi/2N$, $n = 1, 2, \cdots, N/2$ are identical. This leads to the result (8).

4 The Klein bottle

A Klein bottle is constructed by inserting $N$ extra vertical edges with weight $z_v$ to the boundary of the Möbius strips of Figs. 1(a) and 1(b), so that there is a periodic boundary condition in the vertical direction. The extra edges are oriented upward as the other vertical edges. The consideration of the Klein bottle parallels that of the Möbius strip. Again, we need to consider the cases of even and odd $M$ separately.

4.1 $M = even$ Klein bottle

For a $2M \times N$ Klein bottle with horizontal edges connecting the first and $N$-th row having weights $z$, we have as in (8) the generating function

$$Z_{M,N}^{\text{Kln}}(z_h, z_v; z) = \sum_{m=0}^{2M} z^m T_m.$$ (39)
The dimer weights now generate the antisymmetric matrix

\[ A^{\text{Kln}}(z) = A(z) - z_v I_N \otimes (K_{2M} - K_{2M}^T) \quad (40) \]

where \( A(z) \) is given by (39). Following the same analysis as in the case of the Möbius strip, since (i) - (iii) still hold we find the desired dimer generating function again given by the theorem (31) or, explicitly,

\[ Z_{M,N}^{\text{Kln}}(z_h, z_v; z_h) = \frac{1}{2} \left[(1 - i) \text{Pf} A^{\text{Kln}}(i z_h) + (1 + i) \text{Pf} A^{\text{Kln}}(-i z_h)\right]. \quad (41) \]

To evaluate \( \text{Pf} A^{\text{Kln}}(z) \), we note that the \( 2M \times 2M \) matrices \( (F_{2M} - K_{2M} - F_{2M}^T + K_{2M}^T) \) and \( J_{2M} \) commute, and can be diagonalized simultaneously by the \( 2M \times 2M \) matrix \( W \) whose elements are

\[ W_{mn'} = \frac{1}{\sqrt{4M}} \left[e^{i(2m-1)(2m'-1)\pi/4M} - i(-1)^{m+m'+M} e^{-i(2m-1)(2m'-1)\pi/4M}\right], \]

\[ (W^{-1})_{mn'} = \frac{1}{\sqrt{4M}} \left[e^{-i(2m-1)(2m'-1)\pi/4M} + i(-1)^{m+m'+M} e^{i(2m-1)(2m'-1)\pi/4M}\right]. \quad (42) \]

We find

\[
[W^{-1}(F_{2M} - K_{2M} - F_{2M}^T + K_{2M}^T)W]_{m,m'} = (2i \sin \alpha_m) \delta_{m,m'}
\]

\[
(W^{-1}J_{2M}W)_{m,m'} = i(-1)^{M+m} \delta_{m,m'},
\]

where \( \alpha_m = (2m - 1)\pi/2M \). Diagonalizing \( A(z) \) in the \( 2M \)-subspace, we obtain

\[ |A^{\text{Kln}}(z)| = i^{2MN} \prod_{m=1}^{2M} |A^{(m)}_N(z)| \quad (44) \]

where

\[ A^{(m)}_N(z) = 2z_v \sin \alpha_m I_N + z_h (F_N - F_N^T) + (-1)^{M+m} z (K_N + K_N^T). \quad (45) \]

This expression is the same as (17) for the Möbius strip, except with the substitution of \( \cos \phi_m \) by \( \sin \alpha_m \). Thus, following the same analysis, we obtain

\[ \text{Pf} A(i z_h) = \prod_{m=1}^{M} \prod_{n=1}^{N} \left[2z_v \sin \frac{(2m-1)\pi}{2M} + 2i(-1)^{M+m+1} z_h \sin \frac{4n-1)\pi}{2N}\right]. \quad (46) \]

The substitution of (46) into (31) now gives the result (3). For \( N \) even, the Pfaffian (46) is real and (3) reduces to (2).

**4.2 M = odd Klein bottle**

For a \((2M + 1) \times N \) Klein bottle, the inserted vertical edges have dimer weights \( iz_v \). The consideration then parallels that of the preceding sections. Particularly, the desired dimer generating function is also given by (31), but now with

\[ A^{\text{Kln}}(z) = A(z) + iz_v I_N \otimes (K_{2M+1} - K_{2M+1}^T). \quad (47) \]
To evaluate $\text{Pf} A^{\text{Kin}}(z)$, one again applies in the $2M + 1$ subspace the unitary transformation which diagonalizes $F_{2M+1} - K_{2M+1} - F_{2M+1}^T + K_{2M+1}^T$. Define

\begin{align*}
V_{mm'} &= \frac{1}{\sqrt{2M+1}} e^{im(2m'-1)\pi/(2M+1)}, \\
(V^{-1})_{mm'} &= \frac{1}{\sqrt{2M+1}} e^{-im'(2m-1)\pi/(2M+1)}, \quad m, m' = 1, 2, ..., 2M + 1. \tag{48}
\end{align*}

Using the result

\begin{align*}
[V^{-1}(F_{2M+1} - K_{2M+1} - F_{2M+1}^T + K_{2M+1}^T)\ V]_{m,m'} &= (2i \sin \bar{\alpha}_m) \delta_{m,m'} \\
(V^{-1}H_{2M+1}V)_{mm'} &= -e^{-i\bar{\alpha}_m} \delta_{m,2M+2-m'}, \tag{49}
\end{align*}

where $\bar{\alpha}_m = (2m-1)\pi/(2M+1)$, $m = 1, 2, \cdots, 2M + 1$, then the matrix $\bar{A}(z) = (I_N \otimes V^{-1})A(z)(I_N \otimes V)$ assumes the form in the case of $2M + 1 = 5$, \begin{align*}
\bar{A}(z) &= \begin{pmatrix}
B_{N}^{(1)} & & & & -ze^{-i\bar{\alpha}_1}G_N \\
& B_{N}^{(2)} & & & \\
& & B_{N}^{(3)} + zG_N & & \\
& & & -ze^{-i\bar{\alpha}_2}G_N & \\
-ze^{i\bar{\alpha}_1}G_N & & & & B_{N}^{(5)}
\end{pmatrix}.
\tag{50}
\end{align*}

Here $B_{N}^{(m)} = z_h Q_N + 2z_v \sin \bar{\alpha}_m$. Again, interchanging rows and columns, one changes $\bar{A}(z)$ into the block-diagonal form

\begin{align*}
\begin{pmatrix}
B_{N}^{(1)} & & & & -ze^{-i\bar{\alpha}_1}G_N \\
& B_{N}^{(2)} & & & \\
& & B_{N}^{(3)} + zG_N & & \\
& & & -ze^{-i\bar{\alpha}_2}G_N & \\
-ze^{i\bar{\alpha}_1}G_N & & & & B_{N}^{(5)}
\end{pmatrix}.
\tag{51}
\end{align*}

Explicitly, for general $M$, the $m$-th block is

\begin{align*}
A_{2N}^{(m)}(z) &= \begin{pmatrix}
B_{N}^{(m)} & -ze^{-i\bar{\alpha}_m}G_N \\
-ze^{i\bar{\alpha}_m}G_N & B_{N}^{(2M+1-m)}
\end{pmatrix} \\
&= z_h (F_N - F_N^T) \otimes I_2 + 2z_v \sin \bar{\alpha}_m I_N \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - zG_N \otimes \begin{pmatrix} 0 & e^{-i\bar{\alpha}_m} \\ e^{i\bar{\alpha}_m} & 0 \end{pmatrix}.
\tag{52}
\end{align*}

We proceed as in (35) by multiplying a $2N \times 2N$ matrix whose determinant is $(-1)^N$ from the right, and obtain

\begin{align*}
A_{2N}^{(m)}(iz_h) &= A_{2N}^{(m)}(iz_h) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= 2z_v \sin \bar{\alpha}_m J_{2N} + z_h (F_N - F_N^T) \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + izh G_N \otimes \begin{pmatrix} 0 & e^{-i\bar{\alpha}_m} \\ e^{i\bar{\alpha}_m} & 0 \end{pmatrix} \\
&= 2z_v \sin \bar{\alpha}_m J_{2N} + z_h (Q_{2N} - Q_{2N}^T), \tag{53}
\end{align*}

9
where
\[ Q_{2N} = \begin{pmatrix} F_N & ie^{-i\alpha_m}K_N \\ -ie^{i\alpha_m}K_N & -F_N \end{pmatrix}. \] (54)

Now \( Q_{2N} \) commutes with \( Q_{2N}^\dagger \), and they can be diagonalized simultaneously with the respective eigenvalues \( e^{i\beta_n} \) and \( e^{-i\beta_n} \) where \( \beta_n = (2n - 1)/2N, n = 1, 2, \ldots, 2N. \) Thus we obtain
\[
\left|A_{2N}^{(m)}(iz_h)\right| = \prod_{n=1}^{2N} \left[ 4z_h^2 \sin^2 \left( \frac{(2n - 1)\pi}{2N} \right) + 4z_v^2 \sin^2 \left( \frac{(2m - 1)\pi}{2M} \right) \right].
\] (55)

Using (53) and (53), we get
\[
PfA(iz_h) = \frac{1}{2} \left( 1 + (-1)^N \right) (1 + i) z_h^{N/2} \prod_{n=1}^{M} \prod_{m=1}^{N/2} \left[ 4z_h^2 \sin^2 \left( \frac{(2n - 1)\pi}{2N} \right) + 4z_v^2 \sin^2 \left( \frac{(2m - 1)\pi}{2M} \right) \right].
\] (56)

Thus, the generating function (51) vanishes identically for \( N = \text{odd} \). For \( N = \text{even} \), we replace as before \( \sin^2(2n - 1)\pi/N \) by \( \sin^2(4n - 1)\pi/N \). The substitution of (56) into (51) now leads to the result (7).

5 A reciprocity theorem

Using the explicit expression of dimer enumerations on a simple-quartic lattice with free boundaries, Stanley [12] has shown that the enumeration expression satisfies a certain reciprocity relation, a relation rederived recently by Propp [13] from a combinatorial approach. Here, we show that our solutions of dimer enumerations lead to an extension of the reciprocity relation to enumerations on cylindrical, toroidal, and nonorientable surfaces.

We first consider solutions (2), (3), (6) and (7) for \( N = \text{even} \). Writing
\[
T_{\text{Mob}}(\mathcal{M}, \mathcal{N}) = Z_{\mathcal{M}, \mathcal{N}}^{\text{Mob}}(1, 1), \quad T_{\text{Kln}}(\mathcal{M}, \mathcal{N}) = Z_{\mathcal{M}, \mathcal{N}}^{\text{Kln}}(1, 1)
\] (57)
and using the identity [14]
\[
\prod_{k=0}^{n-1} \left[ x^2 - 2x \cos \left( \frac{2k\pi}{n} \right) + 1 \right] = x^{2n} - 2x^n \cos(n\alpha) + 1
\] (58)
repeatedly, we can rewrite our solutions (for general \( z_h \) and \( z_v \) ) in the form of

\[
Z_{\mathcal{M}, \mathcal{N}}^{\text{Mob}}(z_h, z_v) = z_h^{\mathcal{M}N/2} \prod_{m=1}^{(\mathcal{M}+1)/2} (x_m^{\mathcal{N}} + x_m^{-\mathcal{N}})
\]
\[
= \sqrt{\frac{3 - (-1)^\mathcal{M}}{2}} z_v^{\mathcal{M}N/2} \prod_{n=1}^{\mathcal{N}/2} \left[ \frac{y_n^{\mathcal{M}+1} + (-1)^\mathcal{M} y_n^{-\mathcal{M}-1}}{y_n + y_n^{-1}} \right]
\] (59)

\[
Z_{\mathcal{M}, \mathcal{N}}^{\text{Kln}}(z_h, z_v) = z_h^{\mathcal{M}N/2} \prod_{m=1}^{(\mathcal{M}+1)/2} (t_m^{\mathcal{N}} + t_m^{-\mathcal{N}})
\]
\[
= \sqrt{\frac{3 - (-1)^\mathcal{M}}{2}} z_v^{\mathcal{M}N/2} \prod_{n=1}^{\mathcal{N}/2} \left[ \frac{y_n^{\mathcal{M}} + (-y_n)^{-\mathcal{M}} + 1 + (-1)^\mathcal{M}}{y_n + (-y_n)^{-\mathcal{M}}} \right],
\] (60)
where
\[ x_m = G \left( \frac{z_v}{z_h} \cos \frac{m\pi}{M+1} \right), \quad y_n = G \left( \frac{z_h}{z_v} \sin \frac{(2n-1)\pi}{2N} \right), \quad t_m = G \left( \frac{z_v}{z_h} \sin \frac{(2m-1)\pi}{M} \right) \]

and \( G(y) = y + \sqrt{y^2 + 1} \). Thus, the following reciprocity relations are obtained by inspection:

\[
\begin{align*}
T_{\text{Mob}}(M, N) &= T_{\text{Mob}}(M, -N) = \epsilon_{N,M} T_{\text{Mob}}(-M-2, N) \\
T_{\text{Kln}}(M, N) &= T_{\text{Kln}}(M, -N) = \epsilon_{N,M} T_{\text{Kln}}(-M, N),
\end{align*}
\]

where
\[
\epsilon_{N,M} = \begin{cases} (-1)^M, & N \equiv 2 \pmod{4} \\ +1, & \text{otherwise.} \end{cases}
\]

There are no reciprocity relations for \( N = \text{odd} \).

We have carried out similar analyses for dimer enumerations on a simple-quartic net embedded on a cylinder and a torus, using the solutions given in [13, 1], and have discovered universal rules of associating reciprocity relations to specific boundary conditions. Generally, there are 3 different boundary conditions, or “matchings”, that can be imposed between 2 opposite boundaries of an \( M \times N \) net. The conditions can be twisted such as those shown in the horizontal direction in Fig. 1, periodic such as on a torus, or free which means free standing. To establish the convention we shall refer to the boundary condition between the first and the \( N \)-th columns as (the boundary condition) in the \( N \) direction, and similar that between the first and the \( M \)-th rows as in the \( M \) direction. Then, our findings together with those of Ref. [13] lead to the following theorem applicable to all cases:

Reciprocity theorem: Let \( T(M, N) \) be the number of close-packed dimer configurations (perfect matchings) on an \( M \times N \) simple-quartic lattice with free, periodic, or twisted boundary conditions in either direction. (The case of twisted boundary conditions in both directions is excluded). If the twisted boundary condition, if occurring, is in the \( M \) (\( N \)) direction, we restrict to \( M \) (\( N \)) = even. Then, we have

1. \( T(M, N) = \epsilon_{N,M} T(-2-M, N) \) if the boundary condition in the \( M \) direction is free.
2. \( T(M, N) = \epsilon_{N,M} T(-M, N) \) if the boundary condition in the \( M \) direction is periodic or twisted.

6 Summary and Discussions

We have evaluated the dimer generating function \( [1] \) for an \( M \times N \) simple-quartic net embedded on a Möbius strip and a Klein bottle for all \( M, N \). The results are given by \( [2] - [7] \). Our results can also be written in terms of the \( q \)-analogue of the Fibonacci numbers \( F_n(q) \) defined by

\[
\frac{1}{1 - qs - s^2} = \sum_{n=0}^{\infty} F_n(q)s^n.
\]

Using the first line of \( [29] \), for example, and the identity
\[
F_n(q) + F_{n-2}(q) = x^n + (-x)^{-n} \quad q \equiv x - x^{-1},
\]
one can verify that our results (2) and (3) for the Möbius strip are the same as those given by Tesler [10]. Details of the proof which also lead to some new product identities involving the Fibonacci numbers will be given elsewhere [16]. We have also deduced a reciprocity theorem for the enumeration $T(M,N)$ of dimers on an $M \times N$ lattice under arbitrary including free, periodic, and twisted boundary conditions.

Finally, we point out that the results (2) - (7) can be put in a compact expression valid for all cases as

$$Z_{M,N}(z_h,z_v) = z_h^{MN/2} \text{Re} \left[ (1 - i) \prod_{m=1}^{[(M+1)/2]} \prod_{n=1}^N \left( 2i(-1)^{M+m+1} \sin \frac{(4n - 1)\pi}{2N} + 2 X_m \right) \right], \quad (66)$$

where $[x]$ is the integral part of $x$, and

$$X_m = \begin{cases} \frac{z_v}{z_h} \cos \frac{m\pi}{M+1} & \text{for the Möbius strip} \\ \frac{z_v}{z_h} \sin \frac{(2m - 1)\pi}{M} & \text{for the Klein bottle.} \end{cases} \quad (67)$$

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