Non–Hyperbolic Dynamics: a Family of Special Functions

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Abstract. We study the iterative dynamics of a family of special functions from $\mathbb{R}^2$ into $\mathbb{R}^2$ with a non–hyperbolic fixed point in the origin. The characterization by the eigenvalues is analyzed and discussed.

1 Introduction

The last twenty years have seen an explosion of interest in the study of nonlinear systems. Discrete dynamical systems, in particular one–dimensional, have been the subject of much active research, but only few results are known to be valid in any dimension. Moreover, very little is known so far on non–hyperbolic dynamics [1–4].

In this paper we introduce some theorems for a class of non–hyperbolic fixed points on $\mathbb{R}^N$ and then analyze a family of functions $f_\theta$ on the plane which have a non–hyperbolic fixed point in the origin. The dynamical properties of the family near the fixed point, like the basin of attraction, are studied. Finally the limits of applicability of the characterization by the eigenvalues are discussed.
2 Hyperbolic and non–hyperbolic points

The basic goal of the theory of dynamical systems is to understand the eventual or asymptotic behaviour of an iterative process. If the process is the iteration of a function $f$, then the theory hopes to understand the eventual behaviour of the points $x, f(x), f^2(x), ..., f^n(x)$ as $n$ becomes large. Functions which determine dynamical systems are also called maps. This terminology connotes the geometric process of taking one point to another [1,2].

We start with some elementary definitions for the discrete dynamics of maps on the plane.

**Definition 2.1** Let $f : \mathbb{R}^N \rightarrow \mathbb{R}^N$. $f$ is a homeomorphism if is one–to–one, onto, and continuous, and $f^{-1}$ is also continuous. $f$ is a $C^r$–diffeomorphism if $f$ and $f^{-1}$ are homeomorphism of class $C^r$.

**Definition 2.2** The forward orbit of $p \in \mathbb{R}^N$ for $f$ is the set of points $O^+(p) = \{p, f(p), f^2(p), \ldots\}$. If $f$ is an homeomorphism, we define the full orbit of $p$ as the set of points $O(p) = \{f^n(p), \forall n \in \mathbb{Z}\}$, and the backward orbit of $p$ as the set of points $O^-(p) = \{p, f^{-1}(p), f^{-2}(p), \ldots\}$.

**Definition 2.3** A point $p$ is a fixed point for $f$ if $f(p) = p$. We denote the set of fixed points by $\text{Fix}(f)$.

**Definition 2.4** A point $p$ is a periodic point for $f$ with period $n$ if $f^n(p) = p$. The last positive $n$ for which $f^n(p) = p$ is called the prime period of $p$. We denote the set of periodic points of period $n$ by $\text{Per}_n(f)$.

**Remark 2.5** A fixed point is a periodic point of prime period 1.

**Definition 2.6** Let $p$ a periodic point of prime period $n$ for $f$, and $Df : \mathbb{R}^N \rightarrow \mathbb{R}^N \times \mathbb{R}^N$ the Jacobian of the first partial derivatives of $f$. The point $p$ is hyperbolic if $Df^n(p)$ does not have eigenvalues of unitary modulus. The point $p$ is non–hyperbolic if $Df^n(p)$ has at least one eigenvalue of unitary modulus.
**Definition 2.7** Let $p$ a fixed point for $f$. It is called

i) attractor, if exists a neighbour $U$ of $p$ so that

$$\forall q \in U, \lim_{k \to \infty} f^k(q) = p;$$

ii) source, if exists a neighbour $U$ of $p$ so that

$$\forall q \in U, \lim_{k \to \infty} f^{-k}(q) = p;$$

iii) saddle point, in all the other cases.

**Definition 2.8** Let $p$ an attractor for $f$. The basin of attraction of $p$ is the set of points

$$w^+(p) = \{ q \in \mathbb{R}^N : \lim_{n \to \infty} f^n(q) = p \}.$$  

It is easy to characterize the properties of hyperbolic periodic points.

**Theorem 2.9** Let $p$ a hyperbolic periodic point of prime period $n$ for $f$. Then the point $p$ is

(i) attractor, if all the eigenvalues of $Df^n(p)$ have modulus less than 1;

(ii) source, if all the eigenvalues of $Df^n(p)$ have modulus greater than 1;

(iii) saddle point, in all the other cases.

We do not proof this well–know theorem (see [1] or [4]) but we observe that they are valid also for periodic fixed point of prime period $n$, if one substitutes $f$ with $f^n$.

A more difficult question is: How can we characterize the non–hyperbolic fixed point? *Id est*, what happen if some eigenvalues have unitary modulus?

### 3 Some theorems for non–hyperbolic fixed points

A partial answer to the question of the characterization of the non–hyperbolic fixed point is given by the following theorem.
Theorem 3.1 Let \( f : \mathbb{R}^N \rightarrow \mathbb{R}^N \) and \( p \in \mathbb{R}^N \) a non–hyperbolic fixed point for \( f \). Let \( df(p) : \mathbb{R}^N \rightarrow \mathbb{R}^N \) the differential of \( f \) in \( p \) so that \( ||df(p)|| = 1 \) and \( \forall q \in B_p/p \ ||df(q)|| < 1 \), where \( B_p \) is an open ball centered in \( p \). Then \( p \) is an attractor and \( B_p \subset w^+(p) \), where \( w^+(p) \) is the basin of attraction of \( p \).

Proof. Let \( q \in B_p \). From the definition of fixed point and the mean–value theorem we have
\[
||f(q) - p|| = ||f(q) - f(p)|| \leq ||q - p|| sup\{||df(x)|| : x \in B^q_p\},
\]
where \( B^q_p \) is the open ball centered in \( p \) with radius \( q \). We have that \( sup\{||df(x)|| : x \in B^q_p\} = ||df(p)|| = 1 \) and so
\[
\forall q \in B_p/p \ ||f(q) - p|| < ||q - p||,
\]
i.e the iterations of \( q \) by \( f \) remain in \( B_p \) and their distance from \( q \) decreases: \( ||f^n(q) - p|| < ||f^{n-1}(q) - p|| \). As a consequence \( \forall \epsilon > 0 \) exists a \( m > 0 \) so that \( ||f^m(q) - p|| < \epsilon \), thus the limit of \( f^n(q) \) for \( n \rightarrow \infty \) is \( p \). \( \square \)

Now by using the theorem 3.1 we introduce another theorem that will be useful in the next section to study our family of special functions on the plane. First we proof the following lemma.

Lemma 3.2 Let \( A \) a symmetric \( 2 \times 2 \) matrix. If its eigenvalues are less than one in modulus than \( A \) has the property that
\[
\forall v \neq (0,0) \ ||Av|| < ||v||,
\]
where \( || \cdot || \) is the Euclidean norm.

Proof. We know from the Cauchy-Schwarz inequality that for any compatible norm of the matrix \( A \)
\[
||Av|| \leq ||A|| ||v||.
\]
Let \( \lambda_1 \) and \( \lambda_2 \) the eigenvalues of \( A \) and \( \rho(A) = max\{|\lambda_1|, |\lambda_2|\} \) the spectral radius of \( A \). We have that \( ||A|| = \sqrt{\rho(A^T A)} \) for the Euclidean norm by definition, but \( A^T A = A^2 \) because the matrix \( A \) is symmetric. This means that \( ||A|| = \sqrt{\rho(A^2)} = \rho(A) \), and because both the eigenvalues of \( A \) are less than one in modulus \( ||A|| = \rho(A) < 1 \). In conclusion we obtain
\[
||Av|| \leq ||A|| ||v|| = \rho(A) ||v|| < ||v||.
\]
This completes the proof. \( \square \)
Theorem 3.3 Let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) and \((0, 0) \in \mathbb{R}^2\) a non–hyperbolic fixed point for \( f \), such as the eigenvalues of the Jacobian \( Df(0, 0) \) are \( \lambda_1, \lambda_2 \in \mathbb{C} \); \( |\lambda_1| = |\lambda_2| = 1 \). If exists an open ball \( B_{(0,0)} \) centered in \((0, 0)\) such as \( \forall (x, y) \in B_{(0,0)}/(0, 0) \) the eigenvalues of \( Df(x, y) \) are less than 1 in modulus, then \((0, 0)\) is an attractor and \( B \subset w^+(0,0) \), where \( w^+(0,0) \) is the basin of attraction of \((0,0)\).

Proof. We want to show that \( \forall q \in B_{(0,0)}, q \neq (0,0), ||f(q)|| < ||q|| \). We consider the curve \( \gamma(t) : [0, 1] \to B_{(0,0)}, \gamma(t) = t \cdot q \). The curve is monotonic, \( f(\gamma(0)) = 0, f(\gamma(1)) = f(q) \), and \( \gamma(t) \in B_{(0,0)} \) \( \forall t \in [0,1] \). Then we have
\[
||f(q)|| = || \int_{[0,1]} f'(\gamma(t))dt || \leq \int_{[0,1]} ||Df(\gamma(t))\gamma'(t)||dt < \int_{[0,1]} ||\gamma'(t)||dt = \int_{[0,1]} ||q||dt = ||q||,
\]
because the Jacobian \( Df \) is a symmetric matrix. \( \square \)

Remark 3.4 This theorem is true also if we eliminate a set of zero–measure from the domain of integration.

The theorem can be used to address the problem of the characterization of fixed points to the classical study of maxima and minima of the determinant of the Jacobian \( Df(x, y) \) [5].

Lemma 3.5 Let \( A \) a 2 \( \times \) 2 matrix. If the eigenvalues \( \lambda_1, \lambda_2 \in \mathbb{C}/\mathbb{R} \) then \( |\lambda_1| = |\lambda_2| = \sqrt{\det(A)} \).

Proof. The determinant is given by \( \det(A) = \lambda_1 \lambda_2 \). The non–real roots of any equation of degree 2 are complex conjugate: \( \bar{\lambda}_1 = \lambda_2 \). We have
\[
|\lambda_1|^2 = \lambda_1 \bar{\lambda}_1 = \lambda_1 \lambda_2 = \det(A).
\]
In conclusion \( |\lambda_1| = \sqrt{\det(A)} \). The same proof can be applied to \( \lambda_2 \). \( \square \)

Theorem 3.6 Let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) and \((0, 0) \in \mathbb{R}^2\) a non–hyperbolic fixed point for \( f \) such as the eigenvalues of the Jacobian \( Df(0,0) \) are \( \lambda_1, \lambda_2 \in \mathbb{C}/\mathbb{R}, |\lambda_1| = |\lambda_2| = 1 \). If the function \( \det(Df(x,y)) \) has a local maximum in \((0,0)\), then \((0,0)\) is an attractor.
Figure 1: The dynamics of the 4–polypous (left) and the 12–polypous (right). Initial condition: \((1/2, 1)\). Number of iterations: \(5 \times 10^3\).

**Proof.** Because the eigenvalues of \(Df(0, 0)\) are complex but non–real, there is a neighbour of \((0,0)\) in which such property is also true for continuity. If \(\det(Df(x, y))\) is a local maximum in \((0,0)\) from the Lemma 3.5 we have that \(\det(Df(0, 0)) = 1\) and that also the modulus of the eigenvalues is maximum in \((0,0)\). Now by using the theorem 3.3 we obtain the thesis. □

4 A family of special functions: the \(n\)–polypous

Now we analyze a family of functions on the plane which has a non–hyperbolic fixed point in the origin. By using the theorems of the previous section we study analytically and numerically the dynamical properties of the family near the fixed point.

**Definition 4.1** Let \(f_\theta : \mathbb{R}^2 \to \mathbb{R}^2\) such as \(f_\theta(x, y) = (f^{(1)}_\theta(x, y), f^{(2)}_\theta(x, y))^T\),
with $\theta \in [-2\pi, 2\pi]$ and:

$$f^{(1)}_\theta(x, y) = x \cos \theta + y \sin \theta - \frac{x^3}{3} - \frac{y^3}{3},$$

$$f^{(2)}_\theta(x, y) = -x \sin \theta + y \cos \theta + \frac{x^3}{3} - \frac{y^3}{3}.$$

We call this family of functions $n$–polypous.

To characterize the $n$–polypous we can calculate the Jacobian

$$Df_\theta(x, y) = \begin{pmatrix} \cos \theta - x^2 & \sin \theta - y^2 \\ -\sin \theta + x^2 & \cos \theta - y^2 \end{pmatrix}. \quad (1)$$

The trace of $Df$ is given by

$$Tr(Df_\theta(x, y)) = 2 \cos \theta - (x^2 + y^2), \quad (2)$$

and the determinant

$$det(Df_\theta(x, y)) = 2x^2y^2 - (\cos \theta + \sin \theta)(x^2 + y^2) + 1. \quad (3)$$

**Theorem 4.2** The $n$–polypous has a non–hyperbolic fixed point in $(0, 0) \in \mathbb{R}^2$. This fixed point is an attractor if $\theta \in ]-\pi/4, 3\pi/4[$, but $\theta \neq 0$.

**Proof.** The point $(0, 0) \in \mathbb{R}$ is a fixed point because $\forall \theta \in [-2\pi, 2\pi] \ f_\theta(0, 0) = (0, 0)$. From the equation $(2)$ and $(3)$ we obtain $\lambda_{1,2}(0, 0) = (\cos \theta \pm i \sin \theta)$. If $\theta \neq 0, \pi$, the eigenvalues are complex numbers but not real and $|\lambda_{1,2}| = 1$.

We call $d(x, y) = det(Df_\theta(x, y))$ and obtain

$$\nabla d(x, y) = (4xy^2 - 2(\cos \theta + \sin \theta)x, 4x^2y - 2(\cos \theta + \sin \theta)y)^T,$$

and $\nabla d(0, 0) = (0, 0)$. Then

$$\frac{\partial^2 d}{\partial x^2} = 4y^2 - 2(\cos \theta + \sin \theta), \quad \frac{\partial^2 d}{\partial y^2} = 4x^2 - 2(\cos \theta + \sin \theta), \quad \frac{\partial^2 d}{\partial x \partial y} = 8xy$$

and the eigenvalues of the Hessian of $d$ in $(0,0)$ are equal to $-2(\cos \theta + \sin \theta)$. The Hessian $Hd(0,0)$ is negative definite if $(\cos \theta + \sin \theta) > 0$, thus if $\theta \in ]-\pi/4, 3\pi/4[$. For these values of $\theta$ the origin is a local maximum and by using the theorem 3.6 we have the thesis. \qed
Figure 2: Basin of attraction $w^+(0,0)$. 4-polyous (left) and 12-polyous (right). Lattice of $300 \times 300$ initial conditions on the region $[-3,3] \times [-3,3]$. Number of iterations: $10^4$. 
Remark 4.3 Let $\theta = 2\pi/n$, $n \geq 3$. The dynamics of $f_\theta$ is attractive and, because $Df_\theta(0, 0)$ is a rotation of $\theta$, it looks like a polypous with $n$ branches.

This remark justifies the name used for the family of these special functions. In Figure 1 we show the dynamics of 4-polypous and the 12-polypous for the same initial condition.

The basin of attraction $w^+(0, 0)$ of the origin for the 4-polypous and the 12-polypous is shown in Figure 2. This Figure is obtained by choosing a lattice of initial conditions on the plane and then by applying the definition 2.8. Our numerical calculations suggest that $w^+(0, 0)$ is simply connected but it seems hard to find an analytical description for $w^+(0, 0)$.

In Figure 3 we plot for the 12-polypous the region $S$ of the plane $\mathbb{R}^2$ where the eigenvalues of the Jacobian $Df_\theta(x, y)$ are in modulus less than one. This region is very different from the basin of attraction. In fact the theorems 3.1 and 3.3 are valid only for open balls $B_{(0,0)}$ centered in $(0,0)$; thus if $B_{(0,0)} \subset S$ than $B_{(0,0)} \subset w^+(0,0)$. 

Figure 3: Region $S$ of the plane where the eigenvalues of the 12-polypous’s Jacobian are less than one in modulus. Lattice of $300 \times 300$ testing points on the region $[-3,3] \times [-3,3]$. 

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5 Conclusions

We have studied some properties of a class of non–hyperbolic fixed points on $\mathbb{R}^N$ by introducing some theorems to characterize the attractors by the eigenvalues of the Jacobian matrix. We have analyzed the family of maps $f_\theta$, called $n$–polypous, showing that such maps have a non–hyperbolic fixed point in the origin. This fixed point is an attractor for many values of the parameter $\theta$ and numerical calculations suggest that its basin of attraction $w^+(0,0)$ is simply connected. The basin $w^+(0,0)$ changes by changing the values of the parameter $\theta$ and it is very different from the region $S$ of the plane where the eigenvalues of the Jacobian are in modulus less than one.

Our theorems, based on the study the eigenvalues of the Jacobian, are valid only for open balls centered in the origin of the fixed point. We thus conclude that the study of the eigenvalue of the Jacobian of the maps can be useful only for a local analysis of the fixed points.

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