A NOTE ON CUSP FORMS AND REPRESENTATIONS OF $\text{SL}_2(\mathbb{F}_p)$

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Abstract. Cusp forms are certain holomorphic functions defined on the upper half-plane, and the space of cusp forms for the principal congruence subgroup $\Gamma(p)$, $p$ a prime, is acted by $\text{SL}_2(\mathbb{F}_p)$. Meanwhile, there is a finite field incarnation of the upper half-plane, the Deligne–Lusztig (or Drinfeld) curve, whose cohomology space is also acted by $\text{SL}_2(\mathbb{F}_p)$. In this note we study the relation between these two spaces in the weight 2 case.

Contents

1. Introduction
2. Comparing the spaces
3. A further remark
References

1. Introduction

Given a prime $p$ — for convenience we assume $p \geq 7$ — the cusp forms of weight $k$ for the principal congruence subgroup $\Gamma(p) := \text{Ker}(\text{SL}_2(\mathbb{Z}) \to \text{SL}_2(\mathbb{F}_p))$ form a finite dimensional linear space over $\mathbb{C}$, denoted by $S_k(\Gamma(p))$; these holomorphic functions defined on the upper half-plane are objects of considerable interests in number theory. Here we focus on the case $k = 2$. The space $S_2(\Gamma(p))$ is acted by $\text{SL}_2(\mathbb{F}_p)$ in a natural way. We want to understand this space by viewing $\text{SL}_2(\mathbb{F}_p)$ as a finite reductive group.

On the other hand, there is a finite field analogue of the upper half-plane, $\mathbb{P}^1 \backslash \mathbb{P}^1(\mathbb{F}_p)$, which is an algebraic curve over $\mathbb{F}_p$. The group $\text{SL}_2(\mathbb{F}_p)$ also acts on this curve and its $\ell$-adic cohomology in a natural way. This is one of the starting points of Deligne–Lusztig theory, a geometric approach to the representations of reductive groups over finite fields. Indeed, $\mathbb{P}^1 \backslash \mathbb{P}^1(\mathbb{F}_p)$ is a very special example of Deligne–Lusztig varieties, and also referred to as Drinfeld curve. The original reference for this beautiful subject is [DL76].

Consider the algebraic group $G = \text{SL}_2$ over $\overline{\mathbb{F}}_p$. Let $F$ be the standard geometric Frobenius endomorphism on $G$ over $\overline{\mathbb{F}}_p$, so we have $G^F := G(\overline{\mathbb{F}}_p)^F = \text{SL}_2(\mathbb{F}_p)$. In the below we give a brief review on our basic objects.

Cusp form representations. Let $Z = \{\pm 1\}$ be the centre of $G$, then $\text{PSL}_2(\mathbb{F}_p) = G^F/Z$ is the Galois group of the finite cover $X(p) \to X(1)$, where $X(\_)$ denotes the corresponding modular curve of the principal congruence subgroup $\Gamma(\_)$, and $\Gamma(p)$ via the identification $S_2(\Gamma(p)) \cong H^0(X(p),\Omega^1)$, where $\Omega^1$ is the sheaf of relative differentials of degree 1. Explicitly, the action of a matrix $g$ on a 1-form $f(z)dz$ on $X(p)$ is given by

$$g : f(z)dz \to f(g^{-1}(z))dg^{-1}(z),$$
where $g^{-1}(z)$ is the corresponding Möbius transformation. (This action is well-defined by basic properties of factors of automorphy.) We denote by $S_2(\Gamma(p))$ the dual space of $S_2(\Gamma(p))$. More details can be found e.g. in [DS05].

**Deligne–Lusztig representations.** Fix a prime $\ell \neq p$. In our case, there are two types of $F$-stable maximal tori of $G$ involved, the anisotropic type and the split type; we denote a fixed anisotropic torus by $T_a$ and a fixed split one by $T_s$. Note that $T_a \cap T_s = \mathbb{Z}$. For an irreducible $\overline{\mathbb{Q}_\ell}$-character $\theta_s \in \hat{T}_s^F$, we put $R^\theta_{Ts} := \text{Ind}_{B^F}^{G^F}\hat{\theta}_s$, where $B$ is an $F$-stable Borel subgroup containing $T_s$, and $\hat{\theta}_s$ is the trivial extension of $\theta_s$; they provide the principal series representations of $G^F$. The non-principal series representations are called cuspidal representations, which are far more interesting and can be constructed via $\ell$-adic characters of $T_a^F$ on the curve $\mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{F}_p)$: For each $\theta_a \in \hat{T}_a^F$ there is an $\ell$-adic local system $\mathcal{F}_{\theta_a}$ on $\mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{F}_p)$, such that

$$H^i_c(\mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{F}_p), \mathcal{F}_{\theta_a}) \cong H^i_c(xy^p - x^py - 1 = 0, \overline{\mathbb{Q}_\ell}) \otimes_{\mathbb{Q}[T_a^F]} \theta_a$$

as representations of $G^F$; we denote the alternating sum $\sum (-1)^i H^i_c(\mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{F}_p), \mathcal{F}_{\theta_a})$, a virtual representation of $G^F$, by $R^\theta_{Ta}$. These $R^\theta_{Ta}$ and $R^\theta_{Ts}$ are called Deligne–Lusztig representations of $G^F$. More details can be found in [Bon11].

We show that (see Theorem 2.7), as a representation of $\text{SL}_2(\mathbb{F}_p)$, the structure of $S_2(\Gamma(p)) + S_2(\Gamma(p))$ depends on the residue of $p$ modulo 12, and this space is a linear combination of Deligne–Lusztig representations, whose coefficients can be chosen to be linear polynomials in $p$ and can be determined explicitly. Moreover, the involved coefficients imply that the single space $S_2(\Gamma(p))$ is usually not uniform (see Corollary 2.8), and every non-trivial irreducible representation of $\text{PSL}_2(\mathbb{F}_p)$ appears in $S_2(\Gamma(p)) + S_2(\Gamma(p))$ when $p$ is big enough (see Corollary 2.10). Our argument is computational, and based on a formula due to Jared Weinstein and a property of the Steinberg representation.

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### 2. Comparing the spaces

Let $T$ be an $F$-stable maximal torus of $G$, and let $\theta \in \hat{T}^F$ be such that $\theta|_Z = 1$; we always assume $T$ is $T_a$ or $T_s$. We denote by $\epsilon(T)$ the $\mathbb{F}_p$-rank of $T$, that is, $\epsilon(T) = 0$ if $T = T_a$, and $\epsilon(T) = 1$ if $T = T_s$. We first give the decomposition rule of $\text{St} \otimes R^\theta_T$, where $\text{St}$ is the Steinberg representation.

**Lemma 2.1.** Let $T_i, i = 1, 2$, be two $F$-stable maximal tori of $G$, and pick $\theta_i \in \hat{T}_i^F$ with $\theta_i|_Z = 1$. We have

$$(-1)^{(T_1)+\epsilon(T_2)} \cdot \langle \text{St} \otimes R^\theta_{T_1}, R^\theta_{T_2} \rangle_{G^F} = \begin{cases} 2, & T_1 \neq T_2 \\ 2 + (\theta_1, \theta_2)_{T_s^F} + (\theta_1, \theta_2^{-1})_{T_s^F}, & T_1 = T_2 = T_s \\ 2 - (\theta_1, \theta_2)_{T_s^F} - (\theta_1, \theta_2^{-1})_{T_s^F}, & T_1 = T_2 = T_a \end{cases}$$

**Proof.** We use extensively the character table of $\text{SL}_2(\mathbb{F}_p)$ (which can be found e.g. in [DM91, Chapter 15] and [Bon11, Chapter 5]). First, as the character values of $\text{St}$ are zero at the non-semisimple elements, we have:
(-1)^{(T_1)+e(T_2)} \cdot \langle \text{St} \otimes R_{T_1}^{\theta_1}, R_{T_2}^{\theta_2} \rangle_{G^F} \\
= \frac{1}{|G^F|} \cdot (-1)^{(T_1)+e(T_2)} \cdot \left( \sum_{g \in G^F} \text{Tr}(g, \text{St} \otimes R_{T_1}^{\theta_1}) \cdot \text{Tr}(g^{-1}, R_{T_2}^{\theta_2}) \right) \\
= \frac{1}{|G^F|} \cdot (-1)^{(T_1)+e(T_2)} \cdot \left( \sum_{g \in (G_{ss})^F} \text{Tr}(g, \text{St}) \cdot \text{Tr}(g, R_{T_1}^{\theta_1}) \cdot \text{Tr}(g^{-1}, R_{T_2}^{\theta_2}) \right),

where $G_{ss} \subseteq G$ denotes the subset of semisimple elements.

Note that, when $f$ is a class function on $G^F$, we have

\[
\sum_{g \in (G_{ss})^F} f(g) = \sum_{g \in Z} f(g) + \frac{p(p+1)}{2} \sum_{g \in T^F \setminus Z} f(g) + \frac{p(p-1)}{2} \sum_{g \in T^F \setminus Z} f(g).
\]

Using this decomposition we get: (Let $e'$ be short for $(-1)^{(T_1)+e(T_2)}$)

\[
\sum_{g \in (G_{ss})^F} \text{Tr}(g, \text{St}) \cdot \text{Tr}(g, R_{T_1}^{\theta_1}) \cdot \text{Tr}(g^{-1}, R_{T_2}^{\theta_2}) \\
= 2p \frac{(p^2 - 1)^2}{|T_1^F||T_2^F|} \cdot e' + \frac{p(p+1)}{2} \sum_{g \in T^F \setminus Z} \text{Tr}(g, R_{T_1}^{\theta_1}) \cdot \text{Tr}(g^{-1}, R_{T_2}^{\theta_2}) \\
- \frac{p(p-1)}{2} \sum_{g \in T^F \setminus Z} \text{Tr}(g, R_{T_1}^{\theta_1}) \cdot \text{Tr}(g^{-1}, R_{T_2}^{\theta_2}) \\
= 2p \frac{(p^2 - 1)^2}{|T_1^F||T_2^F|} \cdot e' + \frac{p(p+1)}{2} \sum_{g \in T^F \setminus Z} e(T_1)e(T_2) \cdot (\theta_1(g) + \theta_1(g^{-1})) \cdot (\theta_2(g) + \theta_2(g^{-1})) \cdot (1 - e(T_1))(1 - e(T_2)) \\
- \frac{p(p-1)}{2} \sum_{g \in T^F \setminus Z} (1 - e(T_1))(1 - e(T_2)) \cdot (\theta_1(g) + \theta_1(g^{-1})) \cdot (\theta_2(g) + \theta_2(g^{-1})) \\
= 2p \frac{(p^2 - 1)^2}{|T_1^F||T_2^F|} \cdot e' + p|T^F_{a^1}| \sum_{g \in T^F \setminus Z} e(T_1)e(T_2) \cdot (\theta_1(g)\theta_2(g^{-1}) + \theta_1(g)\theta_2(g)) \\
- p|T^F_{a^1}| \sum_{g \in T^F \setminus Z} (1 - e(T_1))(1 - e(T_2)) \cdot (\theta_1(g)\theta_2(g^{-1}) + \theta_1(g)\theta_2(g)).
\]
By putting the above formula into (1) we see that (recall that $|G^F| = p|T_a^F||T_s^F|$)
\[ (1) \cdot \epsilon' = \frac{2(p^2 - 1)}{|T_s^F||T_a^F|} \cdot \epsilon + \frac{1}{|T_a^F|} \sum_{g \in T_a^F \setminus Z} \epsilon(T_1)\epsilon(T_2) \cdot (\theta_1(g)\theta_2(g^{-1}) + \theta_1(g)\theta_2(g)) \\
- \frac{1}{|T_a^F|} \sum_{g \in T_a^F \setminus Z} (1 - \epsilon(T_1))(1 - \epsilon(T_2)) \cdot (\theta_1(g)\theta_2(g^{-1}) + \theta_1(g)\theta_2(g)) \\
= \frac{2(p^2 - 1)}{|T_s^F||T_a^F|} \cdot \epsilon' + \epsilon(T_1)\epsilon(T_2) \cdot \left( \langle \theta_1, \theta_2 \rangle_{T_s^F} + \langle \theta_1, \theta_2^{-1} \rangle_{T_a^F} - \frac{4}{|T_a^F|} \right) \\
- (1 - \epsilon(T_1))(1 - \epsilon(T_2)) \cdot \left( \langle \theta_1, \theta_2 \rangle_{T_s^F} + \langle \theta_1, \theta_2^{-1} \rangle_{T_a^F} - \frac{4}{|T_a^F|} \right),
\]
from which the assertion follows by specialising $T_1$ to $T_s$ and $T_a$ respectively. \(\square\)

From [DL76] we know that:

(i) If $\theta^2 \neq 1$, then $(-1)^{c(T)+1} R_T^\theta \cong (-1)^{c(T)+1} R_T^{\theta^{-1}}$ is an irreducible representation;
(ii) if $\theta = 1$, then $R_T^{\theta} = 1 + (-1)^{c(T)+1} \text{St}$;
(iii) if $\theta^2 = 1$ but $\theta \neq 1$, then $(-1)^{c(T)+1} R_T^\theta$ is the sum of two non-isomorphic irreducible representations.

The character in (iii) is the unique character of order 2; we denote it by $\alpha$ (and, when specialising $T$ to $T_s$ or $T_a$, we also use the notation $\alpha_s$ or $\alpha_a$). We shall need some complementary rules for the representations in (ii) and (iii).

**Lemma 2.2.** Let $(T_1, \theta_1)$ be as in Lemma 2.1. We have $(-1)^{c(T_1)+1} \langle \text{St} \otimes R_{T_1}^\theta, 1 \rangle = \langle \theta_1, 1 \rangle$ and $(-1)^{c(T_1)+1} \langle \text{St} \otimes R_{T_1}^\theta, \text{St} \rangle = 2 + (-1)^{c(T_1)+1} \langle \theta_1, 1 \rangle$. And, if $\alpha|_Z = 1$, then the two irreducible constituents of $(-1)^{c(T_1)+1} R_{T_1}^\alpha$ have same multiplicities in $(-1)^{c(T_1)+1} \text{St} \otimes R_{T_1}^\theta$.

**Proof.** The first two assertions follow from the same method of Lemma 2.1. For the last assertion, note that the character values of the two constituents of $R_{T_1}^\alpha$ are only different on non-semisimple elements, on which the Steinberg character vanishes, so we see from the argument of Lemma 2.1 that the multiplicities are the same. \(\square\)

**Remark 2.3.** Note that $\alpha$ is actually the “quadratic residue symbol”, i.e. $\alpha(t) = 1$ if and only if $t$ is a square in $T^F$. In particular, we see that: If $T = T_s$, then $\alpha|_Z = 1$ if and only if $p = 1 \mod 4$; if $T = T_a$, then $\alpha|_Z = 1$ if and only if $p = 3 \mod 4$.

Summarising the above results we obtain:

**Lemma 2.4.** Let $(T_1, \theta_1)$ be as in Lemma 2.1. The virtual representation $\text{St} \otimes R_{T_1}^\theta$ is a $\mathbb{Z}$-linear combination of Deligne–Lusztig representations, and the coefficient of $R_T^\theta$ for each $\theta$ can be arranged to be: (Note that we do not identify $R_T^\theta$ with $R_T^{\theta^{-1}}$ unless $\theta = \alpha$.)

(a) The coefficient for $R_T^\theta$ with $\theta|_Z \neq 1$ is zero;
(b) let $\theta_s \neq 1 \in \hat{T}_s^F$ be such that $\theta_s|_Z = 1$. If $T_1 = T_s$, then the coefficient for $R_{T_1}^\theta$ is $1 + \langle \theta_1, \theta_s \rangle_{T_s^F}$;
(c) let $\theta_a \neq 1 \in \hat{T}_a^F$ be such that $\theta_a|_Z = 1$. If $T_1 = T_a$, then the coefficient for $R_{T_1}^\theta$ is $-1$;
(d) let $\theta_a \neq 1 \in \hat{T}_a^F$ be such that $\theta_a|_Z = 1$. If $T_1 = T_a$, then the coefficient for $R_{T_1}^\theta$ is $1 - \langle \theta_1, \theta_a \rangle_{T_a^F}$;

Remark 2.3...
(e) let \( \theta_a \neq 1 \in \hat{T}_a^F \) be such that \( \theta_a|_Z = 1 \). If \( T_1 = T_s \), then the coefficient for \( R_{T_s}^\theta \) is \(-1\);
(f) if \( T_1 = T_s \), then the coefficient of \( R_{T_s}^1 \) is \( 1 + \langle \theta_1, 1 \rangle_T \) and the coefficient of \( R_{T_s}^1 \) is \(-1\);
(g) if \( T_1 = T_a \), then the coefficient of \( R_{T_a}^1 \) is \(-1\) and the coefficient of \( R_{T_a}^1 \) is \( 1 - \langle \theta_1, 1 \rangle_T \).

**Proof.** This is a simple combination of Lemma 2.1, Lemma 2.2, the above (i)-(iii), and the character table of \( SL_2(\mathbb{F}_p) \) (note that \( 1 = \langle R_{T_s}^1 + R_{T_s}^1 \rangle / 2 \) and \( St = \langle R_{T_s}^1 - R_{T_s}^1 \rangle / 2 \)).

Using the equivariant Riemann–Roch formula, Weinstein find a nice expression of the space \( S_2(\Gamma(p)) + S_2(\Gamma(p)) \) in terms of representations induced from certain small subgroups of \( G^F/Z = \text{PSL}_2(\mathbb{F}_p) \). More precisely, let \( G_{1728} \subseteq G^F/Z \) be the subgroup (of order 2) generated by \( \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \), \( G_0 \subseteq G^F/Z \) the subgroup (of order 3) generated by \( \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \), and \( G_\infty \subseteq G^F/Z \) the subgroup (of order \( p \)) generated by \( \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \), then according to the argument in [Wei07, Page 31] we have:

\[
(2) \quad S_{2,p} \cong \mathbb{Q}_\ell[G^F/Z] - \text{Ind}_{G_{1728}}^{G^F/Z} 1_{G_{1728}} - \text{Ind}_{G_0}^{G^F/Z} 1_{G_0} - \text{Ind}_{G_\infty}^{G^F/Z} 1_{G_\infty} + 2 \cdot 1_{G^F/Z},
\]

where \( S_{2,p} := S_2(\Gamma(p)) + S_2(\Gamma(p)) \). (Note that this is the corrected version of the formula for a single \( S_2(\Gamma(p)) \) appeared in [Wei07, 3.4.1]; see also [Wei09, 4.3], in which the formula is established in the framework of parabolic cohomology.)

In order to put the space of cusp forms into the picture of representation theory of a finite reductive group, we need to decompose the above large representations; there is the following nice property of the Steinberg representation:

**Lemma 2.5.** We have \((-1)^{1+\psi(T)} St \otimes R_T^\theta = \text{Ind}_{T_s}^{G^F} \theta\).

**Proof.** See [DL76, 7.3].

Now let \( \tilde{G}_s \) be the preimage of \( G_s \) along the surjection \( G^F \to G^F/Z \) for each \( * \in \{1728, 0, \infty\} \), then (2) becomes

\[
(3) \quad S_{2,p} \cong \text{Ind}^{G^F}_Z 1_Z - \text{Ind}^{G_{1728}}_{G_{1728}} 1_{G_{1728}} - \text{Ind}^{G_0}_{G_0} 1_{G_0} - \text{Ind}^{G_\infty}_{G_\infty} 1_{G_\infty} + 2 \cdot 1_{G^F}.
\]

Here a basic observation is that the generators of \( G_{1728} \) and \( G_0 \) are semisimple (as elements in the algebraic group \( G \)), so we can conjugate \( \tilde{G}_{1728} \) and \( \tilde{G}_0 \) into \( T_s^F \cong \mathbb{F}_p^\times \) or \( T_a^F \cong \mu_{p+1} \), which depends on \( p \mod 12 \):

**Lemma 2.6.** We have (up to conjugations in \( G^F \)):

- If \( p = 1 \mod 12 \), then both \( \tilde{G}_{1728} \) and \( \tilde{G}_0 \) are in \( T_s^F \);
- if \( p = 5 \mod 12 \), then \( \tilde{G}_{1728} \) is in \( T_s^F \) and \( \tilde{G}_0 \) is in \( T_a^F \);
- if \( p = 7 \mod 12 \), then \( \tilde{G}_{1728} \) is in \( T_a^F \) and \( \tilde{G}_0 \) is in \( T_a^F \);
- if \( p = 11 \mod 12 \), then both \( \tilde{G}_{1728} \) and \( \tilde{G}_0 \) are in \( T_a^F \).

**Proof.** This follows from direct computations.

\[\square\]
For \(* \in \{1728, 0\}\), let \(T_s\) be one of \(T_s\) and \(T_a\), and suppose \(\tilde{G}_s\) lies in \(T_s\). Then (3) becomes (note that \(B^F/\tilde{G}_\infty = T_s^F/\mathbb{Z}\))

\[
S_{2,F} \cong \text{Ind}_{\mathbb{Z}}^F 1_{\mathbb{Z}} - \text{Ind}_{\tilde{G}_\infty}^F 1_{\tilde{G}_\infty} + 2 \cdot 1_{G^F} - \sum_{\theta \in \tilde{T}_{1728}} \text{Ind}_{\tilde{T}_{1728}}^F \theta - \sum_{\theta \in \tilde{T}_0} \text{Ind}_{\tilde{T}_0}^F \theta
\]

\[(4) \cong \sum_{\theta \in \tilde{T}_0^F; \theta \mid z = 1} \text{St} \otimes R^\theta_{T_s} - \sum_{\theta \in \tilde{T}_0^F; \theta \mid z = 1} R^\theta_{T_a} + 2 \cdot 1_{G^F} - (-1)^{\epsilon(T_{1728})+1} \sum_{\hat{\theta} \in \tilde{T}_0^F; \hat{\theta} \mid \tilde{G}_0 = 1} \text{St} \otimes R^\hat{\theta}_{T_{1728}} - (-1)^{\epsilon(T_0)+1} \sum_{\hat{\theta} \in \tilde{T}_0^F; \hat{\theta} \mid \tilde{G}_0 = 1} \text{St} \otimes R^\hat{\theta}_{T_0},
\]

where the second equality follows from Lemma 2.5.

**Theorem 2.7.** As a representation of \(G^F = \text{SL}_2(F_p)\), the structure of the space \(S_2(\Gamma(p)) + S_2(\Gamma(p))\) depends on \(p\) mod 12, and it can be written as a linear combination of \(R^\theta_{T}\) for various \((T, \theta)\) with \(\theta \mid z = 1\) (hence uniform in the sense of [Lus78, 2.15]), whose coefficients can be chosen to be rational linear polynomials in \(p\):

\[
S_2(\Gamma(p)) + S_2(\Gamma(p)) = \sum_{\theta \in \tilde{T}_0^F; \theta \mid z = 1} c_0 R^\theta_{T_s} + \sum_{\theta \in \tilde{T}_0^F; \theta \mid z = 1} c_0 R^\theta_{T_a},
\]

where \(c_0 \in \frac{1}{12}\mathbb{Z}[p]/p^2\) are linear polynomials in \(p\) depending on \(p\) mod 12.

**Proof.** We can write out these \(c_0\). Consider the following (possibly empty) subsets of \(\tilde{T}_s^F\) for each \(* \in \{s, a\}\): First, let \(A_*\) be consisting of those \(\theta\) such that \(\theta\) is defined and non-trivial on both \(\tilde{G}_{1728}\) and \(\tilde{G}_0\), then let \(B_*\) be consisting of those \(\theta \neq 1\) such that \(\theta\) is defined and trivial on both \(\tilde{G}_{1728}\) and \(\tilde{G}_0\); let \(C_* \subseteq \tilde{T}_s^F \setminus (A_* \cup B_*)\) be consisting of those \(\theta \neq 1\) such that \(\theta\) is defined and trivial on \(\tilde{G}_{1728}\); let \(D_* \subseteq \tilde{T}_s^F \setminus (A_* \cup B_* \cup C_*)\) be consisting of those \(\theta \neq 1\) such that \(\theta\) is defined and trivial on \(\tilde{G}_0\); let \(E_* = \{1\}\).

Then, by applying Lemma 2.4, Lemma 2.6, and Remark 2.3 to (4) we see

\[
S_2(\Gamma(p)) + S_2(\Gamma(p)) = \sum_{\theta \in \tilde{T}_0^F; \theta \mid z = 1} c_0 R^\theta_{T_s} + \sum_{\theta \in \tilde{T}_0^F; \theta \mid z = 1} c_0 R^\theta_{T_a},
\]

where the non-zero \(c_0\) can be chosen as:

- The case \(p = 1\) mod 12:
  - If \(\theta \in A_*,\) then \(c_0 = \frac{\nu_1}{12} + 1;\)
  - If \(\theta \in B_*,\) then \(c_0 = \frac{\nu_1 - 1}{12} - 2;\)
  - If \(\theta \in C_* \cup D_*,\) then \(c_0 = \frac{\nu_1 - 1}{12} - 1;\)
  - If \(\theta \in E_*,\) then \(c_0 = \frac{\nu_1 - 1}{12} - 1;\)
  - If \(\theta \neq 1 \in \tilde{T}_a^F,\) then \(c_0 = -\frac{\nu_1 - 1}{12};\)
  - If \(\theta \in E_a,\) then \(c_0 = 1 - \frac{\nu_1 - 1}{12};\)

- The case \(p = 5\) mod 12:
  - If \(\theta \in C_*,\) then \(c_0 = -\frac{5}{12} - 1;\)
if \( \theta \in \widehat{T}^F_s \setminus (C_s \cup E_s) \), then \( c_\theta = \frac{p-5}{12} \);
if \( \theta \in E_s \), then \( c_\theta = \frac{p-5}{12} \);
if \( \theta \in D_s \), then \( c_\theta = -\frac{p-5}{12} \);
if \( \theta \in \widehat{T}^F_a \setminus (D_a \cup E_a) \), then \( c_\theta = -\frac{p-5}{12} + 1 \);
if \( \theta \in E_a \), then \( c_\theta = -\frac{p-5}{12} \).

- The case \( p = 7 \mod 12 \):
  If \( \theta \in D_s \), then \( c_\theta = \frac{p-7}{12} - 1 \);
if \( \theta \in \widehat{T}^F_s \setminus (D_s \cup E_s) \), then \( c_\theta = \frac{p-7}{12} \);
if \( \theta \in E_s \), then \( c_\theta = \frac{p-7}{12} \);
if \( \theta \in C_s \), then \( c_\theta = -\frac{p-7}{12} - 1 \);
if \( \theta \in \widehat{T}^F_s \setminus (C_s \cup E_a) \), then \( c_\theta = -\frac{p-7}{12} \);
if \( \theta \in E_a \), then \( c_\theta = -\frac{p-7}{12} \).

- The case \( p = 11 \mod 12 \):
  If \( \theta \neq 1 \in \widehat{T}^F_s \), then \( c_\theta = \frac{p-11}{12} \);
if \( \theta \in E_s \), then \( c_\theta = 1 + \frac{p-11}{12} \);
if \( \theta \in A_s \), then \( c_\theta = -\frac{p-11}{12} \);
if \( \theta \in B_s \), then \( c_\theta = -\frac{p-11}{12} - 2 \);
if \( \theta \in C_s \cup D_s \), then \( c_\theta = -\frac{p-11}{12} - 1 \);
if \( \theta \in E_s \), then \( c_\theta = -1 - \frac{p-11}{12} \).

So the theorem follows. \( \square \)

The coefficients in the above argument also imply that, unlike the sum \( S_2(\Gamma(p)) + \overline{S}_2(\Gamma(p)) \), the single space \( S_2(\Gamma(p)) \) is usually not uniform. For instance, we have:

**Corollary 2.8.** The representation \( S_2(\Gamma(p)) \) of \( G^F = \text{SL}_2(\mathbb{F}_p) \) is not a linear combination of Deligne–Lusztig representations of \( \text{SL}_2(\mathbb{F}_p) \) if \( p = 23 \mod 24 \).

**Proof.** From the argument of Theorem 2.7, we see that the multiplicity of each irreducible constituent of \( R^\alpha_{T_s} \) in \( S_2(\Gamma(p)) + \overline{S}_2(\Gamma(p)) \) is an odd integer. As these constituents are not linear combinations of the \( R^\alpha_{T_s} \)'s, the corollary follows. \( \square \)

**Example 2.9.** There is an accidental case: Let \( p = 7 \), then \( S_2(\Gamma(7)) \) is an irreducible constituent of \( R^\alpha_{T_s} \), hence not uniform. However, note that \( \text{PSL}_2(\mathbb{F}_7) \cong \text{GL}_3(\mathbb{F}_2) \), so we can also view \( S_2(\Gamma(7)) \) as a representation of \( \text{GL}_3(\mathbb{F}_2) \), of which it is a cuspidal Deligne–Lusztig representation of dimension 3.

**Corollary 2.10.** Suppose \( p \geq 23 \). An irreducible representation \( \rho \) of \( G^F/Z = \text{PSL}_2(\mathbb{F}_p) \) appears in \( S_2(\Gamma(p)) + \overline{S}_2(\Gamma(p)) \) if and only if \( \rho \neq 1_{G^F/Z} \).

**Proof.** The representations of \( \text{PSL}_2(\mathbb{F}_p) \) can be viewed as the representations of \( G^F \) factored through \( Z \), so the corollary follows from the coefficients in the argument of Theorem 2.7. \( \square \)

3. A further remark

It would be interesting to know whether there is a similar result for the principal congruence subgroup \( \Gamma(p^r), r \in \mathbb{Z}_{>0} \), in which case the representations of \( \text{SL}_2(\mathbb{Z}/p^r) \cong \text{SL}_2(\mathbb{Z}/p) \) are involved. Note that there are generalisations of Deligne–Lusztig theory to this setting;
see e.g. [Lus04], [Sta11], and [Che18]. Moreover, Weinstein’s formula (2) still holds, and there are also possible candidates of the Steinberg representation, like the ones given in [Lee78] and [Cam07]. However, we are yet lacking of a good knowledge of values of the generalised Deligne–Lusztig characters.

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