Windowing and random weighting based cubature RTS smoothing for target tracking

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Abstract—This paper presents windowing and random weighting (WRW) based adaptive cubature Rauch–Tung–Striebel (CRTS) smoother (WRWACRTS). The Unscented KF (WRWUKF) has already existed as an alternative to nonlinear smoothing solutions. In the proposed method, both windowing and random weighted estimation methods are combined together, and used to estimate the noise statistics. Subsequently, the weights of each window are adjusting randomly, and update the process and measurement noise covariances matrices at each epoch. The developed WRWACRTS algorithm overcomes the limitation of developing adaptive cubature Rauch–Tung–Striebel smoother (WRWACRTS). The Unscented KF (WRWUKF) and cubature Rauch–Tung–Striebel smoother (CRTSS) [13] have been developed and proved that URTSS makes the better estimation accuracy than that of the conventional UKF and CKF, respectively. In practice, priori knowledge of the system and noise models are unknown and the uncertainties in measurement model may leads to large errors, even the filter becomes diverge [15].

Several nonlinear adaptive CKF (ACKF) have been developed based on innovation or residual based adaptive estimation (RAE). The innovation or residual vector is a additional information to the filter, used to estimate the noise statistics and, followed by sliding window average method. Random weighting estimation (RWE) is an adaptive method has been developed based on statistics and probability theory [16], [17]. The RWE method has a simple computation method, unbiased estimation and easy to solve large sample problems. Moreover, noise parameters are updated without considering an exact probability distribution of the state characteristic [18] [16]. In the RWE analysis, random variables, \( X_1, X_2, ..., X_n \) are independent and identically distributed (i.i.d) observations. \( F(x) \) and \( F_n(x) \) are the common distribution function and corresponding empirical distribution function of random variables. Let \( x_1, x_2, ..., x_n \) are the sample realization. From which, an empirical distribution function is calculated [18] as

\[
F_n(x) = \frac{1}{n} \sum_{i=1}^{n} I(x_i < x)
\]

and also \( F(x) \) is the random weighting estimation, it can defined as

\[
H_n(x) = \sum_{i=1}^{n} \lambda_i I(x_i < x)
\]

where \( I(x_i < x) \) is the indicator function, it is defined as

\[
I_{X_i < x} = \begin{cases} 1, & I_{X_i < x} \\ 0, & I_{X_i > x} \end{cases}
\]

and \( [\lambda_1, \lambda_2, ......., \lambda_n] \) is the random vector. It follows the Dirichlet distribution \( D(1, 1, ..., 1) \), and that is \( \sum_{i=1}^{n} \lambda_i = 1 \). The joint density function of \( [\lambda_1, \lambda_2, ......., \lambda_n] \) is \( f(\lambda_1, \lambda_2, ......., \lambda_n) = \tau_n \) where \( (\lambda_1, \lambda_2, ......., \lambda_n) \in D_n \)

I. INTRODUCTION

In several practical problems of target tracking has been an active research area for many wide range application in radar, wireless sensor network, navigation and sonar are in [1]. Estimating the kinematic states (e.g., position, velocity and acceleration) including in target tracking and is the main objective for a moving object under the the noisy measurements. Target tracking typically involves arrival time, arrival time difference, bearing angle (angle of arrival) and received signal strength, which are all provided by the sensors. Bearings-only target tracking is the main challenging problem that indicates the nonlinear relationship between target dynamics and bearing only angle measurements in the 2D plane. Several state estimation methods have been developed for bearings-only tracking as in the literature, Kalman filter [2], Extended Kalman filter (EKF) [3], Unscented Kalman filter (UKF) [4] and cubature Kalman filter (CKF) [5] have been developed. The basic idea of the nonlinear estimators are to approximate mean and covariances of the state. The performance of nonlinear KF depends on the priori knowledge of the system and measurement models. As compared to other estimators, the CKF has better than EKF and the same with UKF for solving the higher order nonlinear systems. Moreover, it can avoid linearization of the nonlinear system. The accuracy of the CKF and UKF are the same, but better than the EKF algorithm [5].

Because of these merits, the CKF is widely used in control and tracking applications [6], [2], and used in robotics and Inertial [8], control and guidance navigation [9], [10], [11], [12].

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As mentioned before, the EKF, UKF and CKF are developed based on the idea of forward filtering, whereas in the smoothing estimation, the state is estimated by use of full observation information of each moment. The accuracy of the optimal smoothing algorithm is higher than that of the CKF [13]. In [14], unscented Rauch-Tung-Striebel smoother (URTSS) and cubature Rauch-Tung-Striebel smoother (CRTSS) [13] have been developed and proved that URTSS makes the better estimation accuracy than that of the conventional UKF and CKF, respectively. In practice, priori knowledge of the system and noise models are unknown and the uncertainties in measurement model may leads to large errors, even the filter becomes diverge [15].
and $D_{n-1} = \{[\lambda_1, \lambda_2, \ldots, \lambda_n : \lambda_k > 0 \text{ (k = 1, 2, \ldots, n - 1)}, \sum_{i=1}^{n-1} \lambda_k < 1\}$.

Recently, the windowing and random weighted estimation (WRWE) theory was explored in the nonlinear UKF [16]. To the best of authors knowledge, there has been very limited research regarding the use of WRWE in ACKF for adjusting the filter parameters and in case of application too. The RWE based cubature KF as been developed with bias estimator in [19] for improving the accuracy of nonlinear dynamical system. We extend the same work into adaptive cubature RTS smoothing with noise statistics estimators.

This paper presents windowing and random weighting-estimation (WRWE) based adaptive Cubature RTS smoothing (WRWACRTS). In the proposed method, WRWE method is combined together, and used to adjust the weights randomly at each window for updating the noise statistics. Moreover, presenting the convergence proof of the WRWACRTS algorithm. To demonstrate the performance improvement of the proposed WRWACRTS algorithm with numerical example.

The rest of the paper was organized as follows; The related work is studied in Section II. The description of the traditional cubature smoothing filter is presented in Section III. Section IV provides the proposed algorithm. Numerical examples along with performance analysis of the developed algorithm is given in Section IV. Numerical simulation results are demonstrated in Section V. Section VI conclusions of the paper.

II. RELATED WORK

When the system and noise models are not known exactly in practice, even-though the system models are nonlinear behaviour. Over years, extended KF (EKF), Unscented Kalman filter (UKF) and cubature Kalman filter (CKF) have been developed for estimating the state of a nonlinear dynamical system [20]. In the EKF, Jacobin/Taylor series can be used to approximate the mean and covariance of state vector. The performance of the EKF is limited by the Jacobian matrices calculation that leads large errors. To do this, UKF has been developed based on Unscented Transform (UT) [4], for addressing the EKF limitations. In practice, system and noise models are vary with time. Moreover, the UKF accuracy is limited for higher order system analysis. Alternatively, CKF has been developed for solving the higher order nonlinear system analysis. However, the computation load and accuracy of CKF and UKF are similar, but the filter converges slowly.

Aforementioned literature, optimal smoothing algorithms can be divided into (i) two-filter smoother and (ii) the Rauch–Tung–Striebel (RTS) smoother [13]. In the two-filter smoother, combined two filters and run them in both forward and backward direction. The performance of two filters smoother can provide poor results in practice [13]. In RTS smoother [14], [21], forward filter and backward smoother were combined both algorithms and widely uses in practical analysis. Recently, RTS type of CKS has been developed based on high-degree of the spherical and the radial rules. In this analysis, Genz’s and Mysovskikh’s methods have been used to generate the number of spherical points and update the weights too [15]. In recently, several nonlinear smoothing algorithms have been developed on extended RTS (ERTS) and unscented Rauch-Tung-Striebel smoother (URTSS) [22], [14] for smoothing characteristics of a nonlinear systems. The estimation accuracy of URTSS is much better than that of the EKF and UKF. In addition, there are other smoothing filter developed with central difference RTS (CDRTS) smoothing [23], cubature RTS (CRTS) smoothing algorithm [24], [9] also have been developed for nonlinear systems. As Compared with the URTS and CDRTS, CRTS achieves the same accuracy level, but higher numerical stability. In the later development, moment matching method has been utilized to generate the radial rule points and weights with arbitrary degree of accuracy. Although the third-degree CKS works well but when dynamical system has high non-linearity and large uncertainty [19], [6], [22], it may not provide accurate results. The high-degree cubature rule based CKS has developed for maintaining the high accuracy.

Fading factor based adaptive unscented two-filter smoother (AUTFS) [12] has been proposed for updating the covariance matrices. In addition, the WSLR linearized model was used to formulate the adaptive backward filter and it can run in the forward direction. However, adaptive forward and backward filters were combined together for estimating the measurement noise matrix to obtain the smoothed solution. Sage windowing method [17] has been developed for estimating noise statistics based on windowing approximation. This method has many advantages in the filter estimation and nonlinear systems application. In practice, noise statistics were obtained within time window that may not accurate, eventough the filter can biased or even divergent. Random weighted estimation (RWE) method has been explored into adaptive UKF by for estimation of system noise statistics. Subsequently, the random weighting concept is adapted by adjusting random weights of each window and estimate the noise statistics online [16]. The developed RWE based adaptive UKF overcomes the limitation of the conventional UKF statistics [17]. Adaptive random weighting based cubature Kalman filter (ARWCKF) has been developed by adopting the concept of random weighting. This method is used to construct the random weighting estimations for estimating the noise statistics, and it adaptively adjusts the weights of cubature points to inhibit the disturbances of system noises on state estimation, leading to improve the robustness of the ARWCKS algorithm [19].

III. CUBATURE KALMAN SMOOTHING

In the CKS, 2L cubature points are required to approximate the state vector mean and error covariance and followed by the spherical radial cubature criterion [5], [13].

Let us consider a discrete time stochastic nonlinear dynamic system and measurement equations:

\[ x_k = f(x_{k-1}, u_{k-1}) + w_{k-1} \]  
\[ z_k = h(x_k) + v_k \]

where, $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^r$ is the input vector, $z_k \in \mathbb{R}^m$ is the measurement vector at time $k$.
$k$, $f(x_{k-1})$ and $h(x_k)$ are the nonlinear system dynamic and measurement functions. The process and measurement noise are assumed to be white Gaussian noise with zero mean and finite variance, represented as $w_k = N(0, Q_k)$ and $v_k = N(0, R_k)$, respectively. The step-wise implementation of CKS algorithm is follows \[13\]

A. Forward estimation with cubature Kalman filter

The detailed algorithm of the CKS is given as follows. Step 1: Initialize the state estimation $\hat{x}_0$ and error covariance matrix $\hat{P}_0$ as

\[
\begin{align*}
\hat{x}_0 &= E[x_0] \\
\hat{P}_0 &= E[(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)^T]
\end{align*}
\] (5)

A set of $2L$ cubature points are given by set $[(\chi_k)_i, \bar{W}_i]$, where $(\chi_k)_i$ is the $i$th cubature point and corresponding weights are represented as

\[
\chi_i = \sqrt{L}[1]_i, \\
\bar{W}_i = \frac{1}{2L}, \quad i = 1, 2..2L
\] (6)

where, $[1]_i$ denotes its $i$-th column vector of the identity matrix. The steps involved in the predicted (time-update) and the measurement-update of the CKS are summarised in below.

Prediction:
1. Factorise and evaluate the cubature points:

\[
\hat{P}_{k-1} = S_{k-1}S_{k-1}^T, \\
(\chi_k)_i = S_{k-1}\chi_i + \hat{x}_{k-1}, \quad i = 1, 2..2L
\] (7)

2. Propagate each sigma points through the nonlinear system as

\[
(X_{k-1})_i = f((\chi_k)_i, u_k), \quad i = 1, 2.....2L
\] (8)

3. Evaluate the predicted state $\hat{x}_{k-1}$ and state error covariance $\hat{P}_{k-1}$ based on the transformed sigma points as

\[
\hat{x}_{k-1} = \frac{1}{2L} \sum_{i=1}^{2L} (X_{k-1})_i, \\
\hat{P}_{k-1} = \frac{1}{2L} \sum_{i=1}^{2L} (X_{k-1})_i(X_{k-1})_i^T - \hat{x}_{k-1}\hat{x}_{k-1}^T + Q_{k-1}
\] (9)

Measurement update:
1. Factorise and evaluate the new cubature points:

\[
\hat{P}_{k-1} = S_{k-1}S_{k-1}^T, \\
(Z_{k-1})_i = S_{k-1}\xi_i + \hat{x}_{k-1}, \quad i = 1, 2..2L
\] (10)

2. Propagate new cubature points through the nonlinear system as

\[
(z_{k-1})_i = h((Z_{k-1})_i), \quad i = 1, 2.....2L
\] (11)

3. Evaluate the predicted measurements $\hat{z}_{k-1}$ based on the new cubature points as

\[
\hat{z}_{k-1} = \frac{1}{2L} \sum_{i=1}^{2L} (Z_{k-1})_i
\] (12)

(4) calculate the the cross and auto covariance of state and measurement values of $P_{xx,k}$ and $P_{zz,k}$ are

\[
P_{xx,k-1} = \frac{1}{2L} \sum_{i=1}^{2L} (X_{k})_i(X_{k})_i^T - \hat{x}_{k-1}\hat{x}_{k-1}^T
\]

\[
P_{zz,k-1} = \frac{1}{2L} \sum_{i=1}^{2L} (Z_{k})_i(Z_{k})_i^T - \hat{z}_{k-1}\hat{z}_{k-1}^T + R_{k-1}
\] (13)

(5) evaluate the Kalman gain and the updated stat and error covariance are

\[
K_k = P_{xz,k-1}P_{zz,k-1}^{-1}, \\
\hat{x}_k = \hat{x}_{k-1} + K_k(z_k - \hat{z}_{k-1})
\] (14)

\[
\hat{P}_k = \hat{P}_{k-1} - K_kP_{zz,k-1}K_k^T.
\]

B. Backward estimation with cubature Kalman filter

Once forward estimation is processed, the RTS smoother is applied after the measurements. In this step, the smoother is initialized from the last time step, i.e., $\hat{x}_N = \hat{x}_k$ and $\hat{P}_N = \hat{P}_k$. The recursive process runs backwards for $k = N - 1, ..., 0$, and computes the smoother gain $K_k^s$, the smoothed mean and the covariance are represented as follows \[13, 23\]:

Evaluate the Kalman gain and the updated stat and error covariance are

\[
K_k^s = D_{k+1}P_{zz,k+1}^{-1}, \\
\hat{x}_k^s = \hat{x}_{k+1}^s + K_{k+1}(z_{k+1} - \hat{z}_{k+1})
\] (15)

\[
\hat{P}_k^s = \hat{P}_{k+1}^s - K_{k+1}P_{zz,k+1}K_{k+1}^T.
\]

where, $(z_{k} = z_k - \hat{z}_{k-1})$ is the innovation sequence. $\hat{P}_k$, is the posterior state estimate of state. More detailed explanation of CKS can be found in \[13\].

IV. WINDOW AND RANDOM WEIGHTED ADAPTIVE CUBATURE RTS SMOOTHING

In the CKS, the system and noise models are not known exactly, hence, the filter becomes sub-optimal. In practice, system models are nonlinear and statistical noise characteristics of the these models may vary with time. Thus, the performance of CKS can be degraded. Consequently, the filter becomes divergence. To address divergence issue, adaptive CKS (ACKS) has been developed based on innovation or residual for estimating the noise statistics \[19\]. In the ACKS, innovation vector is used for determining the noise statistics \[17\]. To improve the practicability and adaptability, windowing and random weighting estimation method has been developed. It is a promising method which can used for estimating the covariance of process and measurement noise matrices \[17\], \[19\]. We utilized the same method and develop adaptive CKS algorithm.


A. Windowing-based noise statistic estimation

From the nonlinear system described by (3), it is obvious that
\[
\begin{align*}
E[w_k] &= E[(x_k - f(x_k))] = 0 \\
E[v_k] &= E[z_k - h(x_k)) = 0 \\
E[w_k w_k^\top] &= Q_k = E[(x_k - f(x_k))(x_k - f(x_k))^\top] \\
E[v_k v_k^\top] &= R_{k-1} = E[(z_k - h(x_k))(z_k - h(x_k))^\top]
\end{align*}
\]

(16)

Where, \(x_k\) is actual state, it cannot measured directly used in the filtering process because of the state is unobservable. Moreover, the variance of noise statistics are very small in the considering window width \(N_w\) and uses expectation to the predicted state error \(\hat{x}_{k-1}\) and its estimated \(\hat{x}_k\). Then, the sub-optimal estimator is defined as
\[
\begin{align*}
\hat{R}_{k-j} &= \frac{1}{N_w} \sum_{j=1}^{N_w} E[(x_k - h(\hat{x}_{k-j}))(x_k - h(\hat{x}_{k-j}))^\top] \\
\hat{Q}_{k-j} &= \frac{1}{N_w} \sum_{j=1}^{N_w} E[(x_k - f(\hat{x}_{k-j}))(x_k - f(\hat{x}_{k-j}))^\top]
\end{align*}
\]

(17)

where \(f(\hat{x}_{k-j})\) is the posteriori mean of the estimated state through nonlinear function \(f(.)\). For nonlinear CKS, \(f(\hat{x}_{k-j})\) is approximated by each cubature points through nonlinear functions, that is
\[
\begin{align*}
f(\hat{x}_{k-j}) &= \frac{1}{2L} \sum_{i=1}^{2L} f((\hat{x}_{k-j})_i), \\
h(\hat{x}_{k-1-j}) &= \frac{1}{2L} \sum_{i=1}^{2L} h((\hat{x}_{k-1-j})_i)
\end{align*}
\]

(18)

**Theorem 1.** Suppose the system and measurement noise statistics and its variances are constants or varied with time within the window width \(N\). Then, moving window method based sub-optimal noise statistic estimator (19) is unbiased for \(\hat{R}_k\) and \(\hat{Q}_k\).

\[
\begin{align*}
\hat{R}_k &= \frac{1}{N_w} \sum_{j=0}^{N_w} [v_{k-j}v_{k-j}^\top - H_{k-j}p_{k-j}H_{k-j}^\top] \\
\hat{Q}_{k-1} &= \frac{1}{N_w} \sum_{j=1}^{N_w} \hat{P}_{k-j} + K_{k-j}v_{k-j}v_{k-j}^\top K_{k-j}^\top \\
&\quad - \sum_{i=0}^{2L} w_{j}^\top [(X_{k-j})_i - \hat{x}_{k-j}][(X_{k-j})_i - \hat{x}_{k-j}]^\top
\end{align*}
\]

(19)

**Proof.** The innovation sequence is described as
\[
v_k = z_k - \hat{z}_k
\]

(20)

and substituting the measurement equation in (2) into (21), we have
\[
v_k = h(.)((x_k - \hat{x}_k) + v_k
\]

(21)

By taking the expectation of the innovation sequence is
\[
\begin{align*}
E[v_k] &= E[z_k - \hat{z}_k] = 0 \\
E[v_k v_k^\top] &= E[(z_k - \hat{z}_k)(z_k - \hat{z}_k)^\top] = P_{z_kz_k}
\end{align*}
\]

(22)

The window width is \(N_w\) and there are \(N_w\) measurements within \(t_{k-N}\) to \(t_k\). In this stage, the noise statistics is very small variation within a window width. Define the predicted and estimated state error are
\[
\Delta \hat{x}_{k-j} = \hat{x}_{k-j} - \hat{x}_{k-1-j},
\]

(23)

From equations (21) and (22), we can apply the expectation
\[
E[w_{k-1}] = \frac{1}{N} \sum_{i=1}^{N} E[z_{k-i} - \hat{z}_{k-i}]
\]

(24)

\[
E[w_{k-1}] = \frac{1}{N} \sum_{i=1}^{N} E[z_{k-i} - \hat{z}_{k-i}]
\]

(25)

and, in the above equation represents the mean and covariance of the predicted state after transferred by nonlinear function. The sub-optimal unbiased estimator under the measurement noise vector can be obtained as follows:
\[
E[v_k] = \frac{1}{N} \sum_{i=1}^{N} E[z_{k-i} - \hat{z}_{k-i}]
\]

(25)
Thus, the unbiased estimation for $Q_{k-1}$ can be represented as

$$
\hat{Q}_{k-1} = \frac{1}{N} \sum_{i=1}^{N} \left[ \hat{P}_{k-j} + K_{k-j}v_{k-j}v_{k-j}^T K_{k-j}^T \right] - \frac{1}{2L} \sum_{i=1}^{2L} (X_{k-1})_i (X_{k-1})_i^T - \hat{x}_{k-1} \hat{x}_{k-1}^T
$$

From equation (22), we can apply the expectation

$$
E[\hat{R}_k] = \frac{1}{N} \sum_{j=1}^{N} E[Z_{k-j} - \sum_{i=1}^{2L} h((\xi_{k-j})_i)]
$$

$$
= \frac{1}{N} \sum_{j=1}^{N} E[v_{k-j}v_{k-j}^T]
$$

$$
= \frac{1}{N} \sum_{j=1}^{N} P_{zz,k-j}
$$

$$
= \frac{1}{N} \sum_{j=1}^{N} 2L \sum_{i=1}^{2L} (Z_{k-1})_i (Z_{k-1})_i^T - \hat{z}_{k-1} \hat{z}_{k-1}^T + R_{k-1}
$$

$$
= \frac{1}{N} \sum_{j=1}^{N} 2L \sum_{i=1}^{2L} (Z_{k-1})_i (Z_{k-1})_i^T - \hat{z}_{k-1} \hat{z}_{k-1}^T + R_{k-1}
$$

Thus, the unbiased estimation for $R_{k-1}$ can be represented as

$$
\tilde{R}_{k-1} = \frac{1}{N} \sum_{i=1}^{N} [v_{k-j}v_{k-j}^T - \sum_{i=1}^{2L} (Z_{k-1})_i (Z_{k-1})_i^T] - \hat{z}_{k-1} \hat{z}_{k-1}^T
$$

$$
\hat{R}_{k-1} = \frac{1}{N} \sum_{i=1}^{N} [v_{k-j}v_{k-j}^T - \sum_{i=1}^{2L} (Z_{k-1})_i (Z_{k-1})_i^T] - \hat{z}_{k-1} \hat{z}_{k-1}^T
$$

where $v_{k-j}$ is the innovation vector. The proof the unbiasedness of the estimator (25) can be easily proved. The remark of a nonlinear Gaussian system, the posterior mean and covariance approximation of the state through cubature propagation, which means the residual sequence, i.e., $E[v_{k-j}] = 0$, of filter is an approximate Gaussian white noise sequence. [19].

B. Noise Statistic Estimator Based on Moving Windowing and Random Weighting Methods:

In theorem 1, we presented a simple moving window based noise statistic estimator [17]. However, system and measurement noise statistics are changing rapidly in practice, thus, the moving window based estimator is used to compute the noise statistics based on true condition of the system’s noises at the current epoch. Moreover, the residual weights are the same and also too much historical information are required in this case. To address this issue, the combined windowing and random weighting method is used to adjust the weights on the residuals, which will shows the performance improvement of the filter.

Theorem 2. To utilize the theorem 1, noise statistic estimator of the moving window and random weighting methods can be described as

$$
\hat{Q}_{k-1} = \sum_{j=1}^{N} \lambda_j \hat{P}_{k-j} + K_{k-j} \hat{x}_{k-j} \hat{x}_{k-j}^T K_{k-j}^T
$$

$$
- \sum_{i=0}^{2L} W_i^T [(X_{k-j})_i - \hat{x}_{k-j}][(X_{k-j})_i - \hat{x}_{k-j}]^T
$$

$$
\hat{R}_{k-1} = \sum_{j=0}^{N} \lambda_j [v_{k-j}v_{k-j}^T - H_{k-j} \hat{P}_{k-j} \hat{H}_{k-j}^T]
$$

where, $\lambda_1, \lambda_2, \ldots, \lambda_n$ is are the random vector and it follows Dirichlet distribution, D(1, 1, ..., 1), i.e., $\sum_{i=1}^{n} \lambda_i = 1$. Then, evaluate the joint density function of $[\lambda_1, \lambda_2, \ldots, \lambda_n]$ is $f(\lambda_1, \lambda_2, \ldots, \lambda_n) = \tau_n$. where $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathcal{D}_m$ and $\mathcal{D}_n = \{ (\lambda_1, \lambda_2, \ldots, \lambda_n) : \lambda_k > 0 \ (k = 1, 2, \ldots, n-1), \sum_{i=1}^{n-1} \lambda_k = \frac{1}{2} \sum_{i=1}^{n-1} \lambda_k < 1 \}$. If the noise statistics are constants or the variations of them are very small in the window width $N$, satisfying (18), then the moving window and random weighting methods of the noise statistic estimator is unbiased.

Proof. By taking the random weighting estimation principle, it can be easily obtained noise statistics, Here, we prove that an estimator satisfies the unbiasedness.

Taking the expectation of equation (33) yields
The equations (33) and (34) imply that an estimator is unbiased.

\[
E[\hat{Q}_{k-1}^1] = \frac{1}{N} \sum_{i=1}^{N} \lambda_j E[\hat{\xi}_{k-j} - f(\hat{\xi}_{k-1-j})] \\
= \frac{1}{N} \sum_{i=1}^{N} \lambda_j E[\hat{\xi}_{k-j} - \hat{\xi}_{k-1-j}] \\
= \frac{1}{N} \sum_{i=1}^{N} \lambda_j [K_{k-j} v_{k-j}^T] \\
= \frac{1}{N} \sum_{i=1}^{N} \lambda_j [K_{k-j} \hat{P}_{zz,k-j} K_{k-j}^T] \\
= \frac{1}{N} \sum_{i=1}^{N} \lambda_j [\hat{P}_{k-1-j} - \hat{P}_{k-j}] \\
= \frac{1}{N} \sum_{i=1}^{N} \lambda_j \left[ \frac{1}{2L} \sum_{i=1}^{2L} (X_{k-1})_i (X_{k-1})_i^T \right] \\
- \hat{\xi}_{k-1} \hat{\xi}_{k-1}^T + Q_{k-1} - \hat{P}_{k-j} \\
= \frac{1}{N} \sum_{i=1}^{N} \lambda_j Q_{k-1} \\
= Q_{k-1}
\]

and

\[
E[\hat{R}_{k-1}] = \frac{1}{N} \sum_{j=1}^{N} \lambda_j E[Z_{k-j} - \sum_{i=1}^{z} h((\xi_{k-j})_i)] \\
= \frac{1}{N} \sum_{j=1}^{N} \lambda_j E[v_{k-j} v_{k-j}^T] \\
= \frac{1}{N} \sum_{j=0}^{N-1} P_{zz,k-j} \\
= \frac{1}{N} \sum_{j=1}^{N} \lambda_j \left[ \frac{2L}{2L} \sum_{i=1}^{2L} (Z_{k-1})_i (Z_{k-1})_i^T \right] \\
- \hat{z}_{k-1} \hat{z}_{k-1}^T + \hat{R}_{k-1} \\
- \hat{R}_{k-1} \\
= \frac{1}{N} \sum_{j=1}^{N} \lambda_j \hat{R}_{k-1} \\
= \hat{R}_{k-1}
\]

B. Determination of Random Weighting Factors

Suppose the predicted ($\hat{x}_{k-1}$) and estimated of the state ($\hat{x}_{k-j/k-1-j}$) at epoch $k-j$ ($j=1,2,..,n$), respectively. The residual vector ($\Delta x_{k-1}$) of the prediction of the state is assumed as

\[
\Delta x_{k-1} = \hat{x}_{k-1} - \hat{x}_{k-j/k-1-j}
\]

. The residual vector of the measurement is expressed as

\[
\Delta z_{k-1} = \hat{z}_{k-1} - \hat{z}_{k-j/k-1-j}
\]

If noise statistics of the system is changed, then the predicted state, $\hat{x}_{k-j/k-1-j}$, will decrease, which means leading the predicted state to be biased. As a result, the magnitude of the residual vector of predicted state $\Delta x_{k-1}$ will increase. Similarly, when the measurement noise statistics are changed, the residual vector of measurement $\Delta z_{k-1}$ will be biased and also magnitude of residual vector of measurements will increase. Hence, random weighting factors are required to capture the changes of noise of the system and also which can satisfy as

\[
\lambda_j \propto \|\Delta x_{k-j}\| \|\Delta z_{k-j}\|
\]

where $\|\Delta x_{k-1}\| = \sqrt{\Delta x_{k-1} \Delta x_{k-1}^T}$, $\|\Delta z_{k-1}\| = \sqrt{\Delta z_{k-1} \Delta z_{k-1}^T}$ and the symbol $\propto$ indicates the proportional operation.

In the above Equation, weighting factor is proportional to the $\|\Delta x_{k-1}\|$ and $\|\Delta z_{k-1}\|$. If weighting factor is increased whereas an residual error values can be increased because of proportional. In general, covariance matching method is an effective way to detect and eliminate the disturbance and abnormality in measurements by adjusting their weights to the filter, getting the solution through the following inequality.

\[
[v_{k-j}^T v_{k-j}] \leq S \text{ tr}[E[v_{k-j} v_{k-j}^T]], (j = 1, 2, 3, ..., N) \\
= S \text{ tr}[\sum_{i=1}^{2L} (Z_{k-1})_i (Z_{k-1})_i^T - \hat{z}_{k-1} \hat{z}_{k-1}^T + \hat{R}_{k-1}] \\
= S \text{ tr}[\sum_{j=1}^{N} \lambda_j \left[ \frac{2L}{2L} \sum_{i=1}^{2L} (Z_{k-1})_i (Z_{k-1})_i^T \right] - \hat{R}_{k-1}] \\
\]

where $v_{k-j}$ is the innovation vector and S is an adjustable factor satisfying $S \geq 1$

In this paper, because $\hat{R}_{k-1}$ is unknown, by replacing $\hat{R}_{k-1}$ with its estimate $\hat{R}_{k-1}$, (38) can be rewritten as

\[
[v_{k-j}^T v_{k-j}] \leq S \text{ tr}[\sum_{i=1}^{2L} (Z_{k-1})_i (Z_{k-1})_i^T - \hat{z}_{k-1} \hat{z}_{k-1}^T + \hat{R}_{k-1}] \\
\]

If the equation (39) is not satisfied, the weight on the $k-j$ th residual in measurements, that shows abnormal measurement and ought to be small. Thus, the random weighting factors are required to satisfy
\[ S \propto \text{tr}\left(\sum_{i=1}^{2L} (Z_{k-1}i)(Z_{k-1}i)_{\mathbf{T}} - \hat{z}_{k-1\mathbf{T}} - \hat{\mathbf{R}}_{k-1}\right) \]

Therefore, weights on \( k - j \)th residual is evaluated and corresponding the random weighting factor (\( \lambda_j \)) can be determined as follows.

\[ \omega_j = \|\Delta x_{k-j}\|\|\Delta z_{k-j}\|\Delta S(j) \quad (j = 1, 2, ..N) \]

Normalizing the \( \omega_j(j=1,2,..n) \) of the random weighting factors are obtained as

\[ \lambda_j = \frac{\omega_j}{\sum_{j=1}^{N} \omega_j} \]

where \( \lambda_1, \lambda_2, ... \lambda_N \) obeys Dirichlet distribution, D(1,1,1,1,...). The innovation vector is used to estimated the noise statistics and followed by the windowing and random weighted estimation. It enables adaptively adjust the weights on each residual or innovation vector to improve the filter accuracy. However, the process noise and measurement noise on state estimation can also improve the reliability of the filter.

C. Forward estimation with WRWACRTS algorithm

In the proposed method, predicted and measurement update phase are updated adaptively by random weighted factors. The estimated state and its error covariance equations are involved in the measurement updated equations. The updated predicted equations are

\[ \hat{x}_{k-1} = \frac{1}{2L} \sum_{i=1}^{2L} (X_{k-1}i) \]
\[ \hat{P}_{k-1} = \frac{1}{2L} \sum_{i=1}^{2L} (X_{k-1}i)(X_{k-1}i)_{\mathbf{T}} - \hat{z}_{k-1\mathbf{T}} + \hat{\mathbf{R}}_{k-1} \]

In this step, adaptive Kalman gain \( \mathbf{K}_k \) is updated by

\[ \mathbf{K}_k = \mathbf{P}_{xz,k}\mathbf{P}_{zz,k}^{-1} \]

Where \( \mathbf{P}_{xz,k} \) is the cross covariance of state and measurement values.

\[ \mathbf{P}_{xz,k} = \sum_{i=0}^{2L} \mathbf{W}_{i}[(X_{k-1}i) - \hat{X}_k][Z_{k}i]_{\mathbf{T}} \]

and \( \mathbf{P}_{zz,k} \) is the auto covariance of innovation sequence evaluated as

\[ \mathbf{P}_{zz,k} = \sum_{i=0}^{2L} \mathbf{W}_{i}[(Z_{k}i) - \hat{Z}_k][Z_{k}i]_{\mathbf{T}} + \mathbf{R}_k \]

D. Backward estimation with WRWACRTS algorithm

Once forward estimation is processed, the RTS smoother is applied after the measurements. In this step, recursive process runs backwards for \( k = N - 1, ..., 0 \) and computes the smoother gain \( \mathbf{K}_k^s \), the smoothed mean and the covariance are represented as follows. The Kalman gain, the updated state and error covariance are evaluated as

\[ \mathbf{K}_k^s = \mathbf{D}_{k+1}\mathbf{P}_{zz,k+1}^{-1} \]
\[ \hat{x}_k^s = \hat{x}_{k+1} + \mathbf{K}_k^s(z_{k+1} - \hat{z}_{k+1}) \]
\[ \hat{P}_k^s = \hat{P}_{k+1}^s - \mathbf{K}_k^s\mathbf{P}_{zz,k+1}\mathbf{K}_k^s_{\mathbf{T}} \]

where, \( (v_k = z_k - \hat{z}_{k-1}) \) is the innovation sequence. \( \hat{P}_k \), is the posterior state estimate of state. More detailed explanation of CKF can be found in [5].

V. NUMERICAL SIMULATION

In this section, to show the performance of the proposed adaptive algorithm is demonstrated by a nonlinear target tracking example has been considered. It is a benchmark problem that has been used to test the effectiveness of different nonlinear adaptive filters[13]. The nonlinear state and measurement model of tracking example can be expressed as follows [4]:

\[ \left\{ \begin{array}{l}
  x_1(k+1) = x_1(k) + T_s x_3(k) + w_{1,k} \\
  x_2(k+1) = x_2(k) + T_s (-k_2 x_3^2)(k) + w_{2,k} \\
  x_3(k+1) = x_3(k) + T_s x_4(k) + w_{3,k} \\
  x_4(k+1) = x_4(k) + T_s (k_3 x_3^2 - g)(k) + w_{4,k}
\end{array} \right. \]

The model of the target motion state is as follows:

\[ \mathbf{x}_k = \mathbf{F}\mathbf{x}_{k-1} + \mathbf{w}_k \]

The target motion state vector is a linear cases, \( \mathbf{F} \) is linear state transition matrix is designed based on equation (48), which describes the motion of state. Where, the state, \( x_k = [x_{1,k} \, x_{2,k} \, x_{3,k} \, x_{4,k}]^T \) are the vehicle position and velocity in \( x - y \) plane and its constant coefficients. \( w_{1,k} = [w_{1,k} \, w_{2,k} \, w_{3,k} \, w_{4,k}] \) are the process noises. \( T_s \) is the step size is set to 0.1 s. The \( g = 9.8 m/s^2 \). The \( w_k \) is the assumed to be white Gaussian process noise with zero mean and covariance.

The nonlinear measurement model is given by

\[ \left\{ \begin{array}{l}
  z_1(k) = \sqrt{(x_1(k))^2 + (x_3(k))^2 + v_{1,k}} \\
  z_2(k) = \tan^{-1}\frac{x_1(k)}{x_3(k)} + v_{2,k}
\end{array} \right. \]

where, \( z_k = [z_{1,k} \, z_{2,k}] \) are the range and angle measurements. \( \tan^{-1} \) is the four-quadrant inverse tangent function; \( v_{1,k} = [v_{1,k} \, v_{2,k}] \) are the measurement noises and is assumed to be the white Gaussian measurement noise with zero mean and its covariance. Equations (49) and (50) are the state and measurement equations, state is a linear, whereas the measurement equation is a non-linear. Since the measurement equation is a non-linear, then, the problem of target tracking is a non-linear system in this study [12]. The process noise covariance matrix is \( Q_k \) is represents as
The initial measurement noise covariance matrix, \( R_k = diag([100, 3 \times 10^{-3}]) \) is selected. The initial estimate \( \hat{x}_k \) is generated randomly from the normal distribution \( N(\hat{x}_0; x_0, P_0) \). Where, \( x_0 \) is the actual initial state \( x_0 = [0 \text{m}, 0 \text{m/s} 0 \text{m}]^\top \) and initial error covariance, \( P_0 = diag([100 \text{m}, 100 \text{m/s}, 100 \text{m/s}]) \). The proposed algorithm is applied to a target tracking example in comparison with the standard CKS and ACKS. The true trajectory and estimated target trajectory is shown on Figure 1. In this analysis, We selected an optimal window width is \( N=15 \) samples.

**Figure 1: Trajectory of the maneuvering target.** CKS: Cubature Kalman smoothing; ACKS: Adaptive cubature Kalman smoothing; WRW ACRTS: Windowing and random weighted estimation based adaptive cubature RT smoothing

\[
Q_k = \begin{bmatrix}
2 \times 10^{-1} & 0 & \ldots & 0 \\
0 & 2 \times 10^{-1} & \ldots & 0 \\
0 & 0 & 2 \times 10^{-1} & 0 \\
0 & 0 & \ldots & 2 \times 10^{-1}
\end{bmatrix}
\]

The root mean square error (RMSE) is used to compare the position and velocity estimation of proposed algorithms. The RMSE formulation is given by

\[
\text{RMSE} = \sqrt{\frac{1}{N_{sim}} \sum_{i=1}^{N_{sim}} (x_i - \hat{x}_i)^2}
\]

where \( N_{sim} = 100 \) is the number of runs. \( x_k \) and \( \hat{x}_k \) are the actual and estimated values of the position, respectively. The similar formula is used for RMSE velocity calculation.

Fig. 1 illustrates the actual and estimated target trajectory performance with the CRTS and the proposed algorithm. It can be seen that the estimation accuracy of the conventional CKS is not exactly follow the true state of \( x_{1,k} \) and filter becomes diverge because of inaccurate of process and measurement noise covariance matrices. Moreover, the adaptive CRTS and proposed algorithms, the noise statistics are updated adaptively. It shows that the ACKS and WRWACKS can track the true state \( x_{1,k} \) exactly and filter accuracy is improved. Fig. 2 and Fig.3 illustrates position and speed error of the filters for target tracking examples. In CKS, the process noise covariance is constant value. From Fig.2, conventional CKS leads to large estimation error in position.. However, WRWACKS algorithm is the good performance than the CKS and ACKS and its outstanding merits in target tracking. The RMSE values of proposed algorithms are tabulated in Table I.

**Table I: Average RMSE values of the proposed algorithm**

|                | Position RMSE [m] | Speed RMSE[m/s] |
|----------------|-------------------|-----------------|
| CKS            | 1.758             | 0.604           |
| ACKS           | 1.391             | 0.507           |
| WRWACRTS       | 0.550             | 0.389           |

This demonstrates that the proposed algorithm can track the true trajectory and also updated the noise statistic online. The performance of the proposed algorithm is improved in terms minimum RMSE as compared to conventional CKS and adaptive CKS.

**VI. CONCLUSION**

In this paper, we combine windowing and random weighting concepts and applied into cubature Kalman smoothing.
to developed a novel adaptive cubature Kalman smoothing (WRWACRTS) algorithm. Windowing theory is firstly addressed and then to extend the random windowing concept to adaptive cubature Kalman smoothing. secondly, combine windowing and random weighted theories are utilized and to adjust random weights dynamically based on historical residuals for estimation of system and measurement noise statistics. However, the proposed WRWACRTS method overcomes the problem of the conventional CKS and adaptive CKS in requiring precise knowledge on statistical characteristics of system noise. Moreover, it can provide higher filtering accuracy over the CKS and ACKS algorithm. The algorithm can also requires small computation time, sufficient to achieve the good performance in simulation example. Based on the RMSE values are observed in simulation result reveal that the proposed algorithms outperforms the other algorithms.

APPENDIX

Convergence analysis of proposed algorithm: In this section, we used Lyapunov stability for analyzing the convergence of the proposed algorithm and following proposition [25, 12]. Consider the WRWACRTS described in previous section. The convergence of the WRWACRTS is ensured if the following holds

\[
(1 - \frac{2}{\alpha_k})\tilde{R}_k^{-1} + \frac{1}{\alpha_k^2}\tilde{R}_k^{-1}H\tilde{P}_kH^\top\tilde{R}_k^{-1} \leq 0 \]

\[
\alpha_k^{2}\tilde{F}_k^{\top}\tilde{C}_k(\tilde{P}_k^{-1} - \tilde{P}_k^{-1}) \leq 0,
\]

where,

\[
\tilde{C}_k = \text{diag}\{\tilde{\zeta}_1, \ldots, \tilde{\zeta}_m\}.
\]

\[
\tilde{P}_k^{-1} = \frac{1}{2L} \sum_{i=1}^{2L} (X_k)_i(X_k)_i^\top - \tilde{x}_k\tilde{x}_k^\top + Q_k.
\]

**Proof.** The state estimation and prediction error vectors are defined as

\[
\tilde{x}_k = x_k - \hat{x}_k
\]

\[
\tilde{x}_k^- = x_k - \hat{x}_k^-
\]

Consider the candidate Lyapunov function

\[
V_k = \tilde{x}_k^\top P_k^{-1} \tilde{x}_k.
\]

The objective is to derive conditions under which the sequence \(\{V_k\}_{k=1}^{\infty}\) is decreasing \(V_{k+1} - V_k \leq 0\) along the trajectories of the cubature Kalman filter

\[
x_k = f(x_{k-1}) + w_{k-1}
\]

\[
z_k = h(x_k) + v_k
\]

For convenience, we use the presented approach in [24, 25], to simplify the error expression

\[
\tilde{F}_k = \left[\frac{\partial f(x_{k-1})}{\partial x_k}\right]_{x_k=\hat{x}_k}
\]

\[
\tilde{H}_k = \left[\frac{\partial h(x_{k-1})}{\partial x_k}\right]_{x_k=\hat{x}_k}
\]

where, \(w_{k-1}\) and \(v_k\) are the white noise sequence with zero mean. The innovation error, defined as the difference between measured output and the predicted measurement is

\[
e_k = z_k - \hat{z}_k
\]

\[
= H_k(x_k + v_k) - H_k\hat{x}_k^-
\]

\[
= H_k\tilde{x}_k^- + v_k.
\]

The expectation of innovation error \(e_k\) is zero and its covariance is

\[
E[e_k e_k^\top] = E[(H_k\tilde{x}_k^- + v_k)(H_k\tilde{x}_k^- + v_k)^\top]
\]

\[
= E[H_k\tilde{x}_k^-\tilde{x}_k^-^\top H_k^\top + H_k\tilde{x}_k^- v_k^\top + v_k\tilde{x}_k^-^\top H_k^\top + v_k v_k^\top]
\]

\[
= H_k E[\tilde{x}_k^-\tilde{x}_k^-^\top] H_k^\top + E[v_k v_k^\top]
\]

\[
= H_k P_k H_k^\top + R_k
\]

\[
= \sum_{i=1}^{2L} (Z_k - Z_k^-) (Z_k - Z_k^-)^\top + \hat{z}_k^- \hat{z}_k^-^\top + R_k
\]

\[
= P_{zz,k}.
\]

The predicted error is

\[
\tilde{x}_k^- = x_k - \hat{x}_k^- = f(x_{k-1}) + w_{k-1} - f(\hat{x}_k^-)
\]

\[
= F_k(x_{k-1} - \hat{x}_{k-1}) + w_{k-1}
\]

\[
= F_k(\hat{x}_{k-1}) + w_{k-1}.
\]

Due to the classical approximation of equation (57), the noise sequence \(w_{k-1}\) can be changed to exact equality as

\[
\tilde{x}_k^- \equiv \beta_k F_k \tilde{x}_k^-.
\]

Now, we take residual account at each iteration, in order to obtain an exact equality, we introduced unknown time-varying matrix \(\beta_k\) as

\[
\tilde{x}_k^- \equiv \tilde{\zeta}_k F_k \tilde{x}_k^- - 1
\]

Next, from (24), we have \(\hat{x}_k - x_k = \hat{x}_k^- + K_k e_k - x_k\) and further

\[
\hat{x}_k = (\tilde{x}_k^- - \frac{1}{\alpha_k} \hat{F}_k H^\top \hat{R}_k^{-1}) e_k.
\]

Now, we can use the auto-covariance of state and measurements are

\[
\frac{1}{\alpha_k} \hat{F}_k H^\top \hat{R}_k^{-1} = \frac{1}{\alpha_k} P_{xz,k} [\frac{1}{\alpha_k} P_{zz,k} + \hat{R}_k^{-1}]^{-1}
\]

and equation (60) becomes

\[
\hat{x}_k = \tilde{x}_k^- - \frac{1}{\alpha_k} \hat{P}_k H^\top \hat{R}_k^{-1} e_k
\]

while

\[
\hat{P}_k^{-1} = \frac{1}{\alpha_k} (\tilde{P}_k^{-1} + H^\top \hat{R}_k^{-1} H)
\]
Substituting (62) into (53), it follows

\[
V_k = (\tilde{x}_k - \frac{1}{\alpha_k} P_{xz,k} R_k^{-1} e_k)^\top \tilde{P}_k^{-1} \tilde{x}_k + \frac{1}{\alpha_k} P_{xz,k-1} R_k^{-1} e_k
\]

\[
= \frac{1}{\alpha_k} \frac{V_k}{\alpha_k} = \frac{1}{\alpha_k} V_k \tilde{P}_k^{-1} \tilde{x}_k + \frac{1}{\alpha_k} \frac{1}{\alpha_k} P_{xz,k-1} R_k^{-1} e_k
\]

\[
= \frac{1}{\alpha_k} \frac{V_k}{\alpha_k} \tilde{P}_k^{-1} \tilde{x}_k + \frac{1}{\alpha_k} \frac{1}{\alpha_k} P_{xz,k-1} R_k^{-1} e_k
\]

\[
= \frac{1}{\alpha_k} \frac{V_k}{\alpha_k} \tilde{P}_k^{-1} \tilde{x}_k + \alpha_k \tilde{z}_k \tilde{P}_k^{-1} H \tilde{P}_k^{-1} e_k
\]

\[
= \frac{1}{\alpha_k} \frac{V_k}{\alpha_k} \tilde{P}_k^{-1} \tilde{x}_k + \alpha_k \tilde{z}_k \tilde{P}_k^{-1} H \tilde{P}_k^{-1} e_k
\]

\[
= \frac{1}{\alpha_k} \frac{V_k}{\alpha_k} \tilde{P}_k^{-1} \tilde{x}_k + \alpha_k \tilde{z}_k \tilde{P}_k^{-1} H \tilde{P}_k^{-1} e_k
\]

\[\cdots\]

\[\cdots\]

where \( V_k = \frac{1}{\alpha_k} \frac{V_k}{\alpha_k} \tilde{P}_k^{-1} \tilde{x}_k \). Using (58) and the identity

\[
\tilde{P}_k^{-1} = \frac{1}{2L} \left( \sum_{i=1}^{2L} (X_k)_i (X_k)_i^\top - \tilde{x}_k \tilde{x}_k^\top + Q_k \right)
\]

we have

\[
V_k = \alpha_k \frac{1}{2L} \left( \sum_{i=1}^{2L} (X_k)_i (X_k)_i^\top - \tilde{x}_k \tilde{x}_k^\top + Q_k \right)
\]

\[\cdots\]

and (64) becomes

\[
V_k = \frac{1}{\alpha_k} \frac{V_k}{\alpha_k} \tilde{P}_k^{-1} \tilde{x}_k + \frac{1}{\alpha_k} \frac{1}{\alpha_k} P_{xz,k-1} R_k^{-1} e_k
\]

A decreasing sequence \( \{V_k\}_{k=1}^{\infty} \) implies

\[
V_k - V_{k-1} = V_k - V_k + V_k - V_{k-1} \leq 0.
\]

Thus

\[
V_k - V_{k-1} = e_k^\top \left( \frac{1}{\alpha_k} \frac{1}{\alpha_k} P_{xz,k-1} R_k^{-1} e_k + \tilde{x}_k \tilde{x}_k^\top - \tilde{x}_k \tilde{x}_k^\top - Q_k \right)^{-1} \tilde{z}_k \tilde{P}_{k-1}^{-1} \tilde{x}_k - 1 \leq 0.
\]

in view of the inequality (52).