SPLITTING SUBMANIFOLDS OF FAMILIES OF FAKE ELLIPTIC CURVES

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INTRODUCTION

Let $M_m$ be a compact complex manifold and $N_n$ a compact complex submanifold, $0 < n < m$. We say $N$ splits in $M$ if the holomorphic tangent bundle sequence

$$(0.1) \quad 0 \to T_N \to T_M|_N \to N_{N/M} \to 0$$

splits holomorphically. Recall that this is the case if and only if $id_{N_{N/M}}$ is mapped to zero under $H^0(N, N_{N/M}^* \otimes N_{N/M}) \to H^1(N, N_{N/M}^* \otimes T_N)$.

The splitting of (0.1) is not particularly geometric in certain cases ([J05]). The situation is different if one imposes strong conditions on $M$. The following result is due to Mok:

0.2. Proposition. ([Mok05]) Let $M$ be compact Kähler–Einstein with constant holomorphic sectional curvature. Then a submanifold $N$ splits in $M$ if and only if $N$ is totally geodesic.

The manifolds $M$ in question here are $M = \mathbb{P}_m(\mathbb{C})$, finite étale quotients of complex tori and ball quotients, i.e., manifolds whose universal covering space $\tilde{M}$ is $\mathbb{B}_m(\mathbb{C}) = \{z \in \mathbb{C}^m | |z| < 1\}$, the non–compact dual of $\mathbb{P}_m(\mathbb{C})$ in the sense of hermitian symmetric spaces. The compact submanifolds $N$ that split in $M$ can be described quite explicitly: in the case of $M = \mathbb{P}_m(\mathbb{C})$, $N$ splits if and only if $N$ is a linear subspace ([VdV58]); if $M$ is a torus, then $N$ splits if and only if $N$ is a subtorus ([J05]). For the case $M$ a ball quotient see [Yeung04].

The three types of $M$ share the common property that the universal covering space $\tilde{M}$ can be embedded into $\mathbb{P}_m(\mathbb{C})$ such that $\pi_1(M)$ acts as a subgroup of $PGL_m(\mathbb{C})$. Manifolds with this property carry a holomorphic projective structure in the sense of Gunning ([Gu78]). This property is important in Mok’s proof.

In the case $M$ a compact Riemann surface or $M$ a projective complex surface ([KO80]), the existence of such a structure already implies $M$
Kähler Einstein and of constant holomorphic sectional curvature. In three dimensions this is no longer true ([JR04]):

0.3. Theorem. A projective threefold $M$ carries a holomorphic projective connection if and only if up to finite étale coverings

1.) $M \cong \mathbb{P}_m(\mathbb{C})$ or
2.) $M$ is an abelian threefold or
3.) $M$ is a modular family of fake elliptic curves or
4.) $M$ is ball quotient.

In any case the connection is flat.

The above case 3.) is not covered by Mok’s result. A modular family of fake elliptic curves for our purposes is a projective threefold $M$ that admits a holomorphic submersion

$$\pi : M \rightarrow C$$

onto a compact Shimura curve $C$ such that every fiber is a smooth abelian surface $A$ and such that for the general fiber $\text{End}_\mathbb{Q}(A)$ is an indefinite division quaternion algebra over $\mathbb{Q}$. We recall the construction in detail in §3. What we prove here is the following result which completes the classification in the projective case up to dimension three:

0.4. Proposition. Let $\pi : M \rightarrow C$ be a modular family of fake elliptic curves. A compact submanifold $N$ of $M$ of dimension $0 < \dim N < 3$ splits in $M$ if and only if

1.) $N$ is an étale multisection of $\pi$ or
2.) $N$ is an elliptic curve in a fiber of $\pi$.

Notations. Manifolds are complex manifolds, $\Omega^1_X$ denotes the bundle of holomorphic 1–forms, $T_X$ the holomorphic tangent bundle. We do not distinguish between Cartier divisors and line bundles, e.g., $K_X = \omega_X = \det \Omega_X^1$.

For a vector bundle $E$, $\mathbb{P}(E)$ denotes the “hyperplane” bundle (defined by projectivizations of the transition functions of $E^*$).

1. Holomorphic normal projective connections

There are definitions of projective structures and connections in the language of principal bundles. We essentially follow Kobayashi and Ochiai ([KO80]).

We first recall the notion of the Atiyah class ([A57]) associated to a holomorphic vector bundle $E$ on the complex manifold $M$: It is the
splitting class \( b(E) \in H^1(M, \text{Hom}(E, E) \otimes \Omega^1_M) \) of the first jet sequence
\[
(1.1) \quad 0 \rightarrow \Omega^1_M \otimes E \rightarrow J_1(E) \rightarrow E \rightarrow 0,
\]
i.e., the image of \( id_E \) under the first connecting homomorphism of (1.1) tensorized with \( E^* \).

If \( \Theta^{1,1} \) denotes the \((1, 1)\)-part of the curvature tensor of some differentiable connection on \( E \), then, under the Dolbeault isomorphism, \( b(E) \) corresponds to \([\Theta^{1,1}] \in H^{1,1}(M, \text{Hom}(E, E)) \). In particular, for \( M \) Kähler
\[
(1.2) \quad \text{trace}(b(E)) = -2i\pi c_1(E) \in H^1(M, \Omega^1_M).
\]
Because of (1.2) we consider \( a(E) := -\frac{1}{2\pi} b(E) \), the normalized Atiyah class. For properties and functorial behaviour of these classes see [A57].

1.1. Definitions. Let \( M \) be some \( m \)-dimensional compact complex Kähler manifold. Then \( M \) carries a holomorphic normal projective connection if the normalised Atiyah class of the holomorphic cotangent bundle has the form
\[
(1.3) \quad a(\Omega^1_M) = \frac{c_1(K_M)}{m+1} \otimes id_{\Omega^1_M} + id_{\Omega^1_M} \otimes \frac{c_1(K_M)}{m+1} \in H^1(M, \Omega^1_M \otimes T_M \otimes \Omega^1_M)
\]
where we use the identities \( \Omega^1_M \otimes T_M \otimes \Omega^1_M \simeq \text{End}(\Omega^1_M) \otimes \Omega^1_M \simeq \Omega^1_M \otimes \text{End}(\Omega^1_M) \).

A Čech-solution to (1.3) can be interpreted as a connection on \( T_M \) satisfying certain conditions similar to the Schwarzian derivative ([MM96]). We will not make use of this fact.

\( M \) is said to carry a projective structure if there exists a holomorphic projective atlas, i.e., an atlas whose charts can be embedded into \( \mathbb{P}_m(\mathbb{C}) \) such that the coordinate change is given by restrictions of projective automorphisms. A manifold with a projective structure carries a (flat) projective connection, meaning that zero is a cocycle solution to (1.3).

1.4. Example. Compact complex manifolds that admit a flat holomorphic projective connection:
1. \( M = \mathbb{P}_m(\mathbb{C}) \).
2. Any manifold \( M \) whose universal covering space \( \tilde{M} \) can be embedded into \( \mathbb{P}_m(\mathbb{C}) \) such that \( \pi_1(M) \) acts by restrictions of projective transformations admits a projective structure. In particular
   2.1. Ball quotients \( \mathbb{B}_m(\mathbb{C})/\Gamma \), where \( \Gamma \subset SU(1, m) \) is discrete and torsion free and
2.2.) tori $\mathbb{C}^m/\Lambda$ where $\Lambda \simeq \mathbb{Z}^{2m}$, carry a projective structure ([KO80]). Note that 1) and 2) covers all compact Kähler Einstein manifolds with constant holomorphic sectional curvature.

3.) If $M$ carries a holomorphic projective connection and $M' \to M$ is finite étale, then $M'$ carries a holomorphic projective connection.

2. Holomorphic projective connections and splitting submanifolds

A general remark on coverings: Let $N$ (resp. $N'$) be a compact submanifold of some compact manifold $M$ (resp. $M'$). Let $\nu : M' \to M$ be finite étale such that $\nu(N') = N$. Then $N$ splits in $M$ if and only if $N'$ splits in $M'$.

Indeed, the tangent bundle sequence (0.1) of $N$ in $M$ is the pull back of the tangent bundle sequence of $N$ in $M$. By the trace map, $\nu^* \Omega_M$ is a direct summand of $\nu^* \Omega_{M'}$. We have an inclusion in cohomology:

$$H^1(N, N_{N/M} \otimes T^*_N) \hookrightarrow H^1(N', N'_{N'/M'} \otimes T^*_{N'})$$

coming from Leray spectral sequence. The splitting class of (0.1) of $N$ in $M$ is mapped to the splitting class of (0.1) of $N'$ in $M'$. In this sense, we can always replace a pair $(M, N)$ by $(M', N')$.

We give a necessary but not sufficient condition for a submanifold to be split (see example 2.9):

2.5. Proposition. Let $M_m$ be compact Kähler carrying a holomorphic projective connection. Let $\iota : N_n \hookrightarrow M_m$ be a compact submanifold that splits in $M_m$. Then:

1.) $N$ carries a holomorphic projective connection.
2.) $c_1(\iota^* K_M)_{m+1} = c_1(K_N)_{n+1}$ in $H^1(N, \Omega^1_N)$.
3.) $a(N^*_{N/M}) = \text{id}_{N_{N/M}} \otimes c_1(K_N)_{n+1} \in H^1(M, N^*_{N/M} \otimes N_{N/M} \otimes \Omega^1_N)$.

Proof. Sequence (1.1) for a direct sum $E_1 \oplus E_2$ of vector bundles is the direct sum of the sequences associated to $E_1$ and $E_2$, respectively ([A57], proposition 8). In particular $a(E_1 \oplus E_2)$ is the direct sum of $a(E_1)$ and $a(E_2)$ in a natural way.

By assumption, $\iota^* \Omega^1_M \simeq \Omega^1_N \oplus N^*_{N/M}$. Then we can compute $a(\Omega^1_N)$ and $a(N_{N/M})$ from $a(\iota^* \Omega^1_M)$ by projecting the class to the corresponding summands.

The class $a(\Omega^1_M)$ is given by (1.3) and hence

$$a(\iota^* \Omega^1_M) = \frac{i^* c_1(K_M)_{m+1}}{m+1} \otimes dt + i^* \iota^* \Omega^1_N \otimes \frac{c_1(\iota^* (K_M))_{m+1}}{m+1}$$
in $H^1(N, \iota^*\Omega^1_M \otimes \iota^*T_M \otimes \Omega^1_N)$, where $dt : \iota^*\Omega^1_M \to \Omega^1_N$. Note that we distinguish between pull back of cohomology classes and pull back of forms. Now denote the splitting map by $s : \iota^*T_M \to T_N$.

The class $a(\Omega^1_N)$ is the image of $a(\iota^*\Omega^1_M)$ under the map induced by $dt \otimes id$:

$$H^1(N, \iota^*\Omega^1_M \otimes \iota^*T_M \otimes \Omega^1_N) \to H^1(\Omega^1_M \otimes T_N \otimes \Omega^1_N).$$

Applying $dt$ to the first factor maps $a(\iota^*\Omega^1_M)$ to the following class in $H^1(M, \Omega^1_N \otimes \iota^*T_M \otimes \Omega^1_N)$:

$$\frac{c_1(\iota^*K_M)}{m+1} \otimes dt + dt \otimes \frac{c_1(\iota^*K_M)}{m+1}.$$

The map $s$ induces a map $H^0(N, \iota^*T_M \otimes \Omega^1_N) \to H^0(N, T_N \otimes \Omega^1_N)$, mapping $dt$ to $id_{\Omega^1_N}$ by definition. We therefore obtain

$$a(\Omega^1_N) = \frac{c_1(\iota^*K_M)}{m+1} \otimes id_{\Omega^1_N} + id_{\Omega^1_N} \otimes \frac{c_1(\iota^*K_M)}{m+1}.$$

The trace obtained by contracting the first two factors $\Omega^1_N \otimes T_N$ yields $c_1(K_N) \in H^1(N, \Omega^1_N)$. Then [2.7] shows

$$trace(a(\Omega^1_N)) = \frac{c_1(\iota^*K_M)}{m+1} + n \frac{c_1(\iota^*K_M)}{m+1} = \frac{n+1}{m+1} c_1(\iota^*K_M)$$

and we obtain 2.) By (2.7) $N$ carries a holomorphic projective connection. Formula 3.) is obtained in the same way using

$$H^1(N, \iota^*\Omega^1_M \otimes \iota^*T_M \otimes \Omega^1_N) \to H^1(\Omega^1_M \otimes T_N \otimes \Omega^1_N).$$

We only remark that $\frac{\iota^*c_1(K_M)}{m+1} \otimes dt$ is mapped to zero. \qed

2.8. Example. The Kähler–Einstein case:

1.) By a result of Van de Ven ([VdV58]), a compact complex submanifold $N_n$ of $\mathbb{P}_m(\mathbb{C})$ splits if and only if $N$ is a linear subspace. Here we have $O_{\mathbb{P}_m}(1) = O_{\mathbb{P}_m}(1)|_N$ and $N_{N/M} \cong O_{\mathbb{P}_m}(1)^{\oplus m-n}$.

In the case of tori $\mathbb{C}^m/\Gamma$ and ball quotients $\mathbb{B}_m(\mathbb{C})/\Gamma$ a splitting submanifold may or may not exist depending on the choice of $\Gamma$:

2.1.) By a result of one of the authors ([J05]), a compact complex submanifold $N_n$ of a torus $M = \mathbb{C}^m/\Lambda$ splits if and only if $N$ is a subtorus. Here $O_N = O_T|_N$ and the normal bundle is trivial.

2.2.) See [Yeung04] for an example in the case $\mathbb{B}_2(\mathbb{C})/\Gamma$.

2.9. Example. The three conditions in proposition 2.5 are not sufficient for a submanifold to split: Let $\Gamma$ be some torsion free congruence...
subgroup of $Sl_2(\mathbb{Z})$. Then $\Gamma$ acts without fixed points on $\mathcal{H}_1$ as a group of Moebius transformations. The set of matrices

$$\gamma_{m,n} = \begin{pmatrix} 1 & m & n \\
0 & a & b \\
0 & c & d \end{pmatrix}, \quad \gamma = \begin{pmatrix} a & b \\
& & & c & d \end{pmatrix} \in \Gamma,$$

is a subgroup $\Gamma_\Lambda$ of $Sl_3(\mathbb{C})$ that acts projectively on $\mathbb{C} \times \mathcal{H}_1$, i.e., $\gamma_{m,n}(z,\tau) = \left(\frac{z+m\tau+n}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right)$. The action is free, $U = \mathbb{C} \times \mathcal{H}_1/\Gamma_\Lambda$ is a smooth (non–compact) manifold. The canonical map

$$\pi : U \longrightarrow \mathcal{H}_1/\Gamma$$

is proper holomorphic with elliptic fibers $\pi^{-1}(\tau) \simeq \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z}) =: E_\tau$. The map has sections. For the standard groups $\Gamma = \Gamma_0(n), \Gamma_1(n)$, we obtain the usual `universal’ families of elliptic curves.

By construction, $U$ has a projective structure. Any fiber $E_\tau$ satisfies the three conditions of proposition 2.5. Nevertheless $E_\tau$ does not split in $U$ (This is well known or can be proved along the arguments of lemma 5.2).

### 3. Modular families of fake elliptic curves

In this section we recall the construction and basic properties of families of fake elliptic curves following Shimura ([Sh59]). The construction depends on the choice of a PEL datum.

#### 3.1. Quaternions

Let $B$ be an indefinite division quaternion algebra over $\mathbb{Q}$. Then $B \otimes_\mathbb{Q} \mathbb{R} \simeq M_{2 \times 2}(\mathbb{R})$. Fix once and for all an embedding

$$B \hookrightarrow M_{2 \times 2}(\mathbb{R})$$

such that the reduced norm and trace are given by usual determinant and trace, respectively. From now on think of $B$ as a matrix group generated over $\mathbb{Q}$ by

$$x = \begin{pmatrix} \sqrt{a} & 0 \\
0 & -\sqrt{a} \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 0 & b \\
1 & 0 \end{pmatrix}$$

where $a, b \in \mathbb{Q}$, $a > 0, b < 0$. We have $x^2 = aid = a, y^2 = b, xy = -yx$ and $B = \mathbb{Q} + \mathbb{Q}x + \mathbb{Q}y + \mathbb{Q}xy$. The usual quaternion (anti-) involution $'$ on $B$ given by $(k + lx + my + nx)y) = k - lx - my - nx$. It extends to an involution on $M_{2 \times 2}(\mathbb{R})$ which we also denote by $'$. 
3.2. The abelian surface $A_\tau$. Any $\tau \in \mathcal{H}_1$, the upper half plane of $\mathbb{C}$, endows $\mathbb{R}^4 \cong M_{2 \times 2}(\mathbb{R})$ with a complex structure

\begin{equation}
M_{2 \times 2}(\mathbb{R}) \to \mathbb{C}^2, \quad m \mapsto m_\tau := m \begin{pmatrix} \tau \\ 1 \end{pmatrix}.
\end{equation}

For the construction we may take a maximal order $\mathcal{O}_B \subset B$. Define

\begin{equation}
\mathcal{O}_{B,\tau} := \mathcal{O}_B \begin{pmatrix} \tau \\ 1 \end{pmatrix}, \quad A_\tau := \mathbb{C}^2/\mathcal{O}_{B,\tau}.
\end{equation}

Then $A_\tau$ is an abelian surface and there is a natural inclusion map $\mathcal{O}_B \hookrightarrow \text{End}(A_\tau)$

given by multiplication from the left. The induced embedding $B \hookrightarrow \text{End}_\mathbb{Q}(A_\tau)$ is an isomorphism iff $A_\tau$ is simple (see for example the classification of possible $\text{End}_\mathbb{Q}$’s in [BiLa04]).

Projectivity can be seen as follows: Choose $\rho \in B$ such that $\rho = -\rho'$ and $\rho^2 < 0$. Then

$$E(m_1, m_2) := \text{trace}(\rho m_1 m_2')$$

is a symplectic, non degenerate form on $M_{2 \times 2}(\mathbb{R})$. Some rational multiple of $E$ takes integral values on $\mathcal{O}_B$ and satisfies the Riemann conditions for any $\tau$, defining a polarization $H_\tau$ on $A_\tau$.

3.3. Isomorphic $A_\tau$'s. Let $\mathcal{O}_{B, +}^* \subset \mathcal{O}_B$ be the group of units of positive norm. Then $\mathcal{O}_{B, +}^* \subset SL_2(\mathbb{R})$. The group therefore acts on $\mathcal{H}_1$ as a group of Moebius transformations.

If $\tau' = \gamma(\tau)$ for some $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{O}_{B, +}^*$, then

$$\mathcal{O}_{B, \tau'} = \mathcal{O}_B \begin{pmatrix} \gamma(\tau) \\ 1 \end{pmatrix} = \frac{1}{cT + d} \mathcal{O}_B \gamma \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \frac{1}{cT + d} \mathcal{O}_B \begin{pmatrix} \tau \\ 1 \end{pmatrix}$$

Multiplication by $\frac{1}{cT + d}$ therefore induces an isomorphism $A_\tau \cong A_{\tau'}$. On the underlying real vector space $M_{2 \times 2}(\mathbb{R})$, this map is given by multiplication by $\gamma$ from the right. The form $E$ is invariant under this action, i.e. $E(m_1 \gamma, m_2 \gamma) = E(m_1, m_2)$ for any $m_1, m_2 \in M_{2 \times 2}(\mathbb{R})$. The isomorphism therefore is an isomorphism of polarized abelian varieties $(A_\tau, H_\tau) \cong (A_{\tau'}, H_{\tau'})$.

3.4. The modular family $\pi : M \to C$. Let $\Gamma \subset \mathcal{O}_{B, +}^*$ be of finite index and torsion free. The set of matrices

\begin{equation}
\gamma_\lambda := \begin{pmatrix} \text{id}_{2 \times 2} & \lambda \\ 0_{2 \times 2} & \gamma \end{pmatrix} \in \text{SL}_4(\mathbb{R}) \quad \lambda \in \mathcal{O}_B, \gamma \in \Gamma
\end{equation}

defines a group $\Gamma_{O_B}$. Think of $\mathbb{C}^2 \times \mathcal{H}_1$ as embedded into the standard chart $\{X_3 = 1\}$ of $\mathbb{P}_3$. $\Gamma_{O_B}$ acts projectively on $\mathbb{C}^2 \times \mathcal{H}_1$:

$$\gamma\lambda(z, \tau) = \left( \frac{z + \lambda \tau}{c \tau + d}, \frac{a \tau + b}{c \tau + d} \right), \quad \lambda \in O_B, \gamma = \left( \begin{array}{cc} a & b \\
 c & d \end{array} \right) \in \Gamma$$

The action is free and properly discontinuously; the quotient $M$ is a smooth threefold carrying a holomorphic projective structure.

The homomorphism $\Gamma_{O_B} \rightarrow \Gamma, \gamma \rightarrow \gamma$ gives a holomorphic map $\pi : M \rightarrow C = \Gamma/\mathcal{H}_1$. The fiber over $[\tau]$ is isomorphic to $A_\tau$. Both $M$ and $C$ are compact and the $H_\tau$ glue to a $\pi$–ample $H_\pi$ on $M$. The map $\mathcal{H}_1 \rightarrow \mathbb{C}^2 \times \mathcal{H}_1, \tau \mapsto (\tau, 0)$ induces a section $C \rightarrow M$ of $\pi$. In other words, $M$ is an abelian scheme.

There is an inclusion $O_B \hookrightarrow End(M)$, i.e., any $\lambda \in O_B$ induces a $C$–endomorphism of $M$.

We call $\pi : M \rightarrow C$ a modular family of fake elliptic curves. For our purposes, $M$ depends on a choice of $B, B \hookrightarrow M_{2 \times 2}(\mathbb{R}), O_B$ and $\Gamma$ as above.

3.5. Factors of automorphy. The group $\Gamma$ is torsion free. Then $a(\tau, \gamma) = (c \tau + d) \in H^1(\Gamma, O_{\mathcal{H}_1}^*)$ is a well defined factor of automorphy defining a theta characteristic on $C$ which we denote by $\frac{K_C}{2}$ for simplicity.

The holomorphic tangent bundle $T_M$ is defined by the following factor in $H^1(\Gamma_{O_B}, Gl_3(O_{\mathbb{C}^2 \times \mathcal{H}_1}))$:

$$a(\gamma, (z, \tau)) := \frac{1}{c \tau + d} \begin{pmatrix} id_{2 \times 2} & \frac{i \lambda - c(z + \lambda \tau)}{c \tau + d} \\ 0_{1 \times 2} & \frac{i \lambda - c(z + \lambda \tau)}{c \tau + d} \end{pmatrix},$$

where we introduce the following notation:

$1m$ denotes the first column of $m \in M_{2 \times 2}(\mathbb{R})$.

The block form of $a$ corresponds to the exact sequence

$$0 \rightarrow T_{M/C} \simeq \pi^* \left( \frac{K_C}{2} \oplus \frac{-K_C}{2} \right) \rightarrow T_M \rightarrow \pi^* T_C \rightarrow 0.$$

Taking determinants we find in particular

$$K_M = 2\pi^* K_C.$$
4. Splitting submanifolds

From Proposition 2.5 we get the following possibilities:

4.1. Lemma. Let $\pi : M \rightarrow C$ be a modular family of fake elliptic curves as in §3. Let $N$ be a compact submanifold that splits in $M$. Then either

1.) $N$ is a fiber of $\pi$ or
2.) $N$ is a smooth elliptic curve in a fiber of $\pi$ or
3.) $N$ is an étale multisection of $\pi$.

In Lemma 5.2 below we show that, as in the case of modular families of elliptic curves, a fiber in fact never splits.

Proof. For the proof we may assume $g_C > 1$. We first show that $N$ cannot be a surface:

1. Case. Let $N$ be a complex surface that splits in $M$. By proposition 2.5 $N$ carries a holomorphic projective connection. By [KOS80], $N \cong \mathbb{P}_2(\mathbb{C})$ or $N$ is a finite étale quotient of an abelian surface or $N$ is a ball quotient. As $M$ does not contain a rational curve, $N \neq \mathbb{P}_2(\mathbb{C})$.

If $N$ is not a fiber of $\pi$, then $\pi : N \rightarrow C$ is surjective. By proposition 2.5, $K_N \equiv \frac{3}{4} K_M|_N$. Since $K_M$ is trivial on every fiber of $\pi$ by (3.5), $K_N$ is (numerically) trivial on every fiber of $\pi|_N$. The adjunction formula shows that $\pi_N$ defines an elliptic fibration on $N$.

A ball quotient is hyperbolic and cannot contain an elliptic curve (any holomorphic map $\mathbb{C} \rightarrow \mathbb{B}_2(\mathbb{C})$ is constant). A torus does not have a surjective map to a curve of genus $> 1$ for the same reason (any holomorphic map $\mathbb{C}^2 \rightarrow \mathbb{B}_1(\mathbb{C})$ is constant). Therefore, $N$ must be a fiber of $\pi$.

2. Case. Let $N$ be a compact Riemann surface that splits in $M$. Then $g_N > 0$, as $M$ does not contain rational curves. By Proposition 2.5, $K_N \equiv \frac{1}{2} K_M|_N$.

If $N$ is contained in a fiber of $\pi$, then $\deg K_N = \frac{1}{2} K_M.N = 0$ and $N$ is an elliptic curve. If $\pi : N \rightarrow C$ is surjective of degree $d$, then

$$K_N \sim \pi_N^* K_C + R$$

by Hurwitz formula, where $R$ is effective and $R = 0$ iff $\pi_N$ is étale. By (3.5), $K_M \equiv 2 \pi^* K_C$. Then Hurwitz' formula reads

$$K_N \equiv \frac{1}{2} K_M|_N + R.$$ 

Combining this with $K_N \equiv \frac{1}{2} K_M|_N$ from above we find $R = 0$ and $\pi_N$ étale.
5. Fibers are non–split

Fix some \( \tau \in \mathfrak{H}_1 \). Then \( A_\tau \) from \((3.2)\) can be viewed as the fiber of \( \pi \) over \( \{ \tau \} \in \mathfrak{H}_1/\Gamma \). Denote by \( \iota : A_\tau \rightarrow M \) the inclusion map. Then we have

\[
0 \rightarrow N^*_{A_\tau/M} \simeq \mathcal{O}_{A_\tau} \rightarrow \iota^* \Omega^1_M \rightarrow \Omega^1_{A_\tau} \simeq \mathcal{O}^{\oplus 2}_{A_\tau} \rightarrow 0.
\]

The following lemma seems to be well-known within the theory of Kuga fiber spaces. Since we could not find a reference, we give a proof.

**5.2. Lemma.** The induced map \( H^0(A_\tau, N^*_{A_\tau/M}) \hookrightarrow H^0(A_\tau, \iota^* \Omega^1_M) \) is an isomorphism. In particular, \( A_\tau \) is non–split in \( M \).

Before the proof we first illustrate the general method. We have (see section 3 for notations)

\[
\iota_* : \pi_1(A_\tau) \simeq \mathcal{O}_B \rightarrow \pi_1(M) \simeq \Gamma_{\mathcal{O}_B}, \quad \iota_*(\lambda) = \text{id}_\lambda.
\]

The bundle \( \iota^* \Omega^1_M \) is given by the pull back to \( A_\tau \) of the factor of automorphy dual to \((3.4)\) (i.e., the pull back of \( (a(\gamma_\lambda, (z, \tau))^{-1})^t \)). We get the following representation of the fundamental group

\[
\rho_A : \pi_1(A_\tau) \simeq \mathcal{O}_B \rightarrow \text{GL}_3(\mathbb{C}), \quad \lambda \mapsto \begin{pmatrix}
  id_{2\times 2} & 0_{2\times 1} \\
  -1 & 1
\end{pmatrix}.
\]

The bundle \( \iota^* \Omega^1_M \) is a flat bundle and \((5.1)\) is a sequence of flat bundles. In terms of representations, \( \mathbb{C}^3 \) as a \( \pi_1(A_\tau) \) module is the extension of the trivial one dimensional and the trivial two dimensional module.

**5.4. Remark.** Let \( 0 \rightarrow K \rightarrow V \rightarrow Q \rightarrow 0 \) be a short exact sequence of complex vector spaces which are \( G \) modules, where \( G = \pi_1(G) \) for some complex manifold \( M \). We get a short exact sequence of flat vector bundles \( 0 \rightarrow K \rightarrow V \rightarrow Q \rightarrow 0 \).

If \( V \simeq K \oplus Q \) as \( G \)–modules, then the sequence of flat vector bundles splits holomorphically. The convers, however, need not be true, the map

\[
H^1(G, Q^* \otimes K) \rightarrow H^1(M, Q^* \otimes K)
\]

of obstruction spaces might have a non trivial kernel.

**Proof.** (of Lemma 5.2) The space \( H^0(A_\tau, \iota^* \Omega^1_M) \) is isomorphic to the space of holomorphic \( f = (f_1, f_2, f_3) \in \mathcal{O}^3_{A_\tau} \) satisfying \((z = (z_1, z_2)^t)\)

\[
f(z + \lambda_\tau) = \rho_A(\lambda)f(z) \quad \text{for any } \lambda \in \mathcal{O}_B.
\]

The sections corresponding to \( H^0(A_\tau, N^*_{A_\tau/M}) \hookrightarrow H^0(A_\tau, \iota^* \Omega^1_M) \) are \( f = (0, 0, b)^t \), \( b \in \mathbb{C} \) constant. We have to show that these are all.

Let \( f \) satisfy \((5.3)\). Then \( f_i(z + \lambda_\tau) = f_i(z) \) for \( i = 1, 2 \) implies \( f_1, f_2 \) constant. The third equation reads \( f_3(z + \lambda_\tau) = -\lambda_{11} f_1 - \lambda_{21} f_2 + f_3(z) \),
\[ \lambda \in \mathcal{O}_B \text{ arbitrary. Then } \frac{\partial f_3}{\partial z_i} \text{ are constant and hence } f_3(z_1, z_2) = a_1 z_1 + a_2 z_2 + b, a_1, a_2, b \in \mathbb{C}. \text{ After subtracting } (0, 0, b)^t \text{ we may assume } b = 0. \]

The third equation now reduces to
\[ 0 = \lambda_{11} f_1 + \lambda_{21} f_2 + (a_1, a_2) \lambda_{\tau} \text{ for any } \lambda \in \mathcal{O}_B. \]

The four generators \( \lambda_1, \lambda_2, \lambda_2, \lambda_4 \) of \( \mathcal{O}_B \) yield four linear equations in \( (f_1, f_2, a_1, a_2) \). The defining matrix in \( M_{4 \times 4}(\mathbb{C}) \) is
\[
\begin{pmatrix}
\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\
\lambda_{1, \tau} & \lambda_{2, \tau} & \lambda_{3, \tau} & \lambda_{4, \tau}
\end{pmatrix} = 
\begin{pmatrix}
id_{2 \times 2} & 0 \\
\tau \cdot \id_{2 \times 2} & \id_{2 \times 2}
\end{pmatrix} \cdot
\begin{pmatrix}
\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\
2 \lambda_1 & 2 \lambda_2 & 2 \lambda_3 & 2 \lambda_4
\end{pmatrix}.
\]

The determinant is nonzero since \( \lambda_1, \lambda_2, \lambda_2, \lambda_4 \) are \( \mathbb{R} \)-independent. The only solution therefore is \( (f_1, f_2, a_1, a_2) = (0, 0, 0, 0) \).

6. Elliptic curves in fake elliptic curves

Let \( \pi : M \to C \) be a modular family of fake elliptic curves. We call matrices in the kernel of \( B^\times \to PGL_2(\mathbb{C}) \) projectively trivial. Under this map, any \( b \in B^\times \) acts on \( \mathbb{P}_1(\mathbb{C}) \). Identify
\[ \mathcal{H}_1 = \{ [\tau : 1] \mid \tau \in \mathcal{H}_1 \}. \]

If some \( b \in B^\times \) fixes some \( \tau \in \mathcal{H}_1 \), then \( \det b > 0 \) and \( b \) fixes \( \mathcal{H}_1 \).

The next proposition shows the existence of elliptic curves in special fibers of \( \pi \):

**6.1. Proposition.** The abelian surface \( A_\tau \) from (3.2) is isogeneous to a product of elliptic curves if and only if \( \tau \) is a fixed point of some projectively non-trivial \( b \in B^\times \).

For a given elliptic curve \( E \) the following conditions are equivalent:

1. There exists a non constant holomorphic map \( f : E \to A_\tau \).
2. \( A_\tau \) is isogeneous to \( E \times E \).
3. \( \text{End}_\mathbb{Q}(A_\tau) \simeq M_{2 \times 2}(\text{End}_\mathbb{Q}(E)) \).

In any case \( E \) and \( A_\tau \) are CM.

For an elliptic curve \( E \), \( \text{End}_\mathbb{Q}(E) \simeq \mathbb{Q} \) or \( \text{End}_\mathbb{Q}(E) \) is an imaginary quadratic extension of \( \mathbb{Q} \). A given elliptic curve \( E \) appears in a fiber of the modular family if and only if \( \text{End}_\mathbb{Q}(E) \) is a splitting field of \( B \).

**Proof.** \( A_\tau \) is isogeneous to a product of elliptic curves if and only if there exists a non constant homomorphism \( \varphi : E_{\tau'} \to A_\tau \) for some
\[ \tau' \in \mathcal{H}_1, \text{ where } E_{\tau'} = \mathbb{C}/\Lambda_{\tau'} \text{ for } \Lambda_{\tau'} = \mathbb{Z}\tau' + \mathbb{Z}. \] Such a homomorphism exists if and only if we find \(0 \neq \lambda, \mu \in \mathcal{O}_B\) such that

\[
(6.2) \quad \tau'\lambda \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \mu \begin{pmatrix} \tau \\ 1 \end{pmatrix}.
\]

Indeed, a given \(\varphi\) may be interpreted as a \(\mathbb{C}\)-linear map \(\varphi : \mathbb{C} \rightarrow \mathbb{C}^2\) satisfying \(\varphi(\mathbb{Z}\tau' + \mathbb{Z}) \subset \mathcal{O}_{B,\tau}\) and then \(\lambda\) and \(\mu\) are induced by \(\varphi(1)\) and \(\varphi(\tau')\), respectively. Conversely, given \(\mu, \lambda\) as above, \(\varphi\) is defined by \(\mathbb{C} \rightarrow \mathbb{C}^2, z \mapsto z\lambda\). The map \(\varphi\) is non-constant if and only if \(\lambda \neq 0\).

The existence of a non constant \(\varphi\) then implies \(\tau\) is a fixed point of \(b := \lambda^{-1}\mu \in B^\times\). If conversely \(\tau\) is a fixed point of some projectively non trivial \(b \in B^\times\), we immediately get (6.2).

The equivalence of the four points is not difficult to prove. We skip details since it is not important for our purposes. We only recall that \(End_Q(A_{\tau})\) always contains \(B\)

6.3. Remark. Let \(\varphi : E_{\tau'} \rightarrow A_{\tau}\) be as above induced by \(0 \neq \lambda, \mu \in \mathcal{O}_B\) such that (6.2) holds. Modulo isogenies we may assume \(\lambda = id_{2 \times 2}\).

Indeed, choose \(n \in \mathbb{N}\) such that \(\mu' := n\lambda^{-1}\mu \in \mathcal{O}_B\), consider \(\tilde{\varphi} : E_{n\tau'} \rightarrow A_{\tau}\) induced by \(z \mapsto (\tau, 1)^t z\). Then \(\lambda \circ \tilde{\varphi} = \varphi \circ p\) where \(p : E_{n\tau'} \rightarrow E_{\tau'}\) is the canonical map and \(\lambda \in End(A_{\tau})\).

Note that \(\lambda\) also induces an isogeny of \(M\).

7. Elliptic curves in fibers split

Let \(\iota : E \hookrightarrow A_{\tau}\) be an elliptic curve in the fiber \(A_{\tau}\) of \(\pi\). Then we have

\[
(7.1) \quad 0 \rightarrow N^*_{E/M} \rightarrow \iota^*\Omega^1_M \rightarrow \Omega^1_E = \mathcal{O}_E \rightarrow 0
\]

Our aim is to prove that an elliptic curve in a fiber of \(\pi : M \rightarrow C\) splits in \(M\). Choose \(\tau' \in \mathcal{H}_1\) and

\[ \varphi : E_{\tau'} \rightarrow A_{\tau} \]

such that \(E = \varphi(E_{\tau'})\). Then (7.1) splits holomorphically if and only if \(\varphi^*\) of (7.1) splits. By Remark 6.3 we may assume that \(\varphi\) as a map \(\mathbb{C} \rightarrow \mathbb{C}^2\) is given by \(z \mapsto (\tau, 1)^t z\), sending \(\tau'\) to

\[
(7.2) \quad \tau' \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \mu \tau
\]

for some \(0 \neq \mu \in \mathcal{O}_B\). The matrix \(\mu\) remains fixed for the rest of this section.
The idea is the same as in §4: we will view (7.1) as a sequence of flat bundles coming from representations of \( \pi_1(E_{\tau'}) \) and compute dimensions of spaces of global sections. We have \( \varphi_* : \pi_1(E_{\tau'}) = \mathbb{Z}\tau' + \mathbb{Z} \longrightarrow \pi_1(M) \simeq \Gamma_{\mathcal{O}_B}, \quad \varphi_*(m\tau' + n) = id_{M + n\text{id}_{2 \times 2}} \) in the notation (3.3). The flat bundle \( \varphi^*\Omega^1_M \) is given by the pull back of the factor of automorphy dual to (3.4). We get the representation

\[
\rho_E : \pi_1(E_{\tau'}) \longrightarrow \text{Gl}_3(\mathbb{C}), \quad \rho(m\tau' + n) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -m\mu_{11} - n & -m\mu_{21} & 1 \end{pmatrix}
\]

as in (5.3). It turns \( \mathbb{C}^3 \) into a \( \pi_1(E_{\tau'}) \)-module. The bundle \( \Omega^1_{E_{\tau'}} \) is given by the trivial one dimensional module, and \( d\varphi \) is induced by the \( \pi_1(E_{\tau'}) \)-map

\[
\mathbb{C}^3 \longrightarrow \mathbb{C}, \quad (z_1, z_2, z_3) \longmapsto (z_1\tau' + z_2).
\]

The kernel is the \( \pi_1(E_{\tau'}) \)-module corresponding to \( \varphi^*N^*_{E/M} \). Note that there is an inclusion map

\[
(7.3) \quad \varphi^*N^*_{A_{\tau}/M} \simeq \mathcal{O}_{E_{\tau'}} \hookrightarrow \varphi^*N^*_{E/M}
\]

into the subbundle \( \varphi^*N^*_{E/M} \) of \( \varphi^*\Omega^1_{M} \).

**7.4. Proposition.** Let \( \varphi : E_{\tau'} \rightarrow A_{\tau} \) be an elliptic curve in the fiber \( A_{\tau} \) of \( \pi \). Then \( \text{dim}_\mathbb{C} H^0(E_{\tau'}, \varphi^*\Omega^1_M) = 2 \) and the global differential map \( D\varphi : H^0(E_{\tau'}, \varphi^*\Omega^1_M) \longrightarrow H^0(E_{\tau'}, \Omega^1_{E_{\tau'}}) \) is surjective. In particular, elliptic curves in fibers of \( \pi \) split in \( M \).

**Proof.** The space \( H^0(E_{\tau'}, \varphi^*\Omega^1_M) \) is isomorphic to the space of holomorphic \( f = (f_1, f_2, f_3) \in \mathcal{O}^3_{E_{\tau'}} \) satisfying

\[
(7.5) \quad f(z + m\tau' + n) = \rho_E(m\tau' + n)f(z) \quad \text{for any } m, n \in \mathbb{Z}.
\]

The sections coming from \( \varphi^*N^*_{A_{\tau}/M} \simeq \mathcal{O}_{E_{\tau'}} \hookrightarrow \varphi^*N^*_{E/M} \) correspond to \( f = (0, 0, b)^t, b \in \mathbb{C} \) constant. We have to show that there is an additional dimension.

Let \( f = (f_1, f_2, f_3) \) satisfy (7.5). As in the proof of lemma 5.2 \( f_1, f_2 \) are constant while \( f_3(z) = az + b \) for some \( a, b \in \mathbb{C} \). We may assume \( b = 0 \). Then (7.5) reduces to

\[
amr' + an = -(m\mu_{11} + n)f_1 - m\mu_{21}f_2 \quad \text{for any } m, n \in \mathbb{Z}.
\]

After eliminating \( a \) from the equations obtained for \( (m, n) = (1, 0) \) and \( (0, 1) \) we get

\[
(7.6) \quad \tau'f_1 = \mu_{11}f_1 + \mu_{21}f_2.
\]
Conversely, if $f_1, f_2$ satisfies (7.6), then $f = (f_1, f_2, -f_1z)$ is a solution to (7.5). By (7.2), $\tau'$ is an eigenvalue of $\mu$ and therefore also of $\mu^t$. A nonzero vector $(f_1, f_2)$ satisfies (7.6) if and only if $(f_1, f_2)^t$ is an eigenvector of $\mu$ to the eigenvalue $\tau'$. Therefore $\text{h}^0(E_{\tau'}, \varphi^*\Omega^1_M) = 2$.

The eigenspace of $\mu^t$ to $\tau'$ is
\[
\left\langle \left( \frac{1}{\bar{\tau}} \right) \right\rangle_C.
\]
Since $d\varphi(1, -\bar{\tau}, -z) = \tau - \bar{\tau} = 23m\tau \neq 0$, $D\varphi$ is surjective. □

**Proof of Proposition 0.4.** Lemma 4.1 and 5.2 show that a compact submanifold $N$ that splits in $M$ is an étale multisection of $\pi$ or an elliptic curve in a fiber of $\pi$.

Étale multisections $\tilde{C}$ split because of
\[
0 \to \pi^*K_C \to \Omega^1_M \to \Omega^1_{M/C} \to 0.
\]
The restriction to $\tilde{C}$ shows $d\pi|_{\tilde{C}}$ splits $\Omega^1_M|_{\tilde{C}} \to K_{\tilde{C}}$. Elliptic curves in fibers split in $M$ by Proposition 7.4. □

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