GENERAL POLYTOPIAL $H(\text{div})$ – CONFORMAL FINITE ELEMENTS AND THEIR DISCRETISATION SPACES

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Abstract. We present a class of discretisation spaces and $H(\text{div})$ – conformal elements that can be built on any polytope. Bridging the flexibility of the Virtual Element spaces towards the element’s shape with the divergence properties of the Raviart – Thomas elements on the boundaries, the designed frameworks offer a wide range of $H(\text{div})$ – conformal discretisations. As those elements are set up through degrees of freedom, their definitions are easily amenable to the properties the approximated quantities are wished to fulfill. Furthermore, we show that one straightforward restriction of this general setting share its properties with the classical Raviart – Thomas elements at each interface, for any order and any polytopial shape. Then, to close the introduction of those new elements by an example, we investigate the shape of the basis functions corresponding to particular elements in the two dimensional case.

1 Introduction Several classes of schemes adapted to hyperbolic problems, like Discontinuous Galerkin [5], Flux Reconstruction [9, 16, 11] or Residual Distribution use the Finite Elements framework. The approximated solution is considered piecewise and is split onto a tessellation of the computational domain, using usually simplicial or quadrangular tiles.

In order to provide an adequate spatial approximation of complex shape domains, extensive efforts have been made on designing strategies allowing various element shapes [14]. Also, some variational formulations, spaces included, have been redesigned to handle non simplicial elements, as by example [15]. Our contribution is along similar lines, going beyond simplicial elements by providing a framework for Flux Reconstruction methods on arbitrary polytopes.

The classical Flux Reconstruction starts from a point-wise approximation of the flux and modify it in order to take care of local conservation. This amounts to introducing a modification of the centred approximation. When dealing with triangles, quad, hex or tets, this correction can be written as adding an element of the Raviart-Thomas [12] space associated to this element. Going further and using the Residual Distribution framework, we have shown in [1] how to construct schemes similar to those occurring in the Flux Reconstruction framework for arbitrary polytopes, convex or non-convex. This method enjoys a non-linear entropy stability property, but this is only seen algebraically. It is possible to recover a clean variational setting if one is able to construct space sharing the properties of the Raviart-Thomas setting on the boundaries of arbitrary polygons. This is the purpose of this paper.

The theory of $H(\text{div})$-conformal element has already been studied by Raviart, Thomas [12] and later generalized by Nédélec [10] or Brezzi, Douglas and Marini in the context of mixed finite element method [3]. More recently, a mixed Petrov-Galerkin scheme using Raviart-Thomas elements has also been investigated in [7]. However, up to the authors knowledge, those elements are limited to simplicial and quadrangular shapes.

Several attempts to use general polygons have been made [8, 13], but they usually make use of generalized barycentric coordinates and are delicate to handle in distorted non-convex elements. A first polygonal $H(\text{div})$-conformal element has been proposed in [6] using gradient reconstruction and pyramidal sub-meshes tessellation. However, their construction requires some shape regularity within the mesh, and the parallel with Raviart-Thomas spaces is limited to the lowest order space. Some other approaches as Virtual Elements Method [15] introduced approximation spaces based on Poisson’s
solutions. Although more flexible towards the element’s shape, those are scalar and not \( \text{H(div)} \)-conformal.

In this paper, we propose a construction that inherits the interface properties of the Raviart-Thomas elements and benefits from the shape flexibility of the Virtual Element discretisation. Moreover, more than defining a basis on which the correction functions can be decomposed, this new setting offers a new element class that can be used as such in the construction of further numerical schemes.

After briefly recalling the key ideas of the Raviart-Thomas elements, we introduce our new class of discretisation spaces. In a second time, we detail a possible definition of \( \text{H(div)} \)-conformal element to finally test them through the behaviour of their corresponding basis functions in the numerical results. For the sake of readability, proofs and further element examples will be given in the appendix. For interested readers, more details on the construction can be found in the extended technical report [2] and an application of those elements can be found in [1].

**Notations** Throughout the paper, our notation will be the following:

**Geometrical notations**
- \( x \): spatial variable \( x = (x_1, \ldots, x_d)^T \in \mathbb{R}^d \)
- \( K \): polytopial shape with interior \( K \) and boundary \( \partial K \)
- \( n \): number of faces of the polytope \( K \)
- \( f \): generic face of the boundary \( \partial K \)
- \( \partial_j K \): hyper-face of \( f \in \partial K \) for which the variable \( x_j \) is fixed

**Monomials and Polynomial spaces**
- \( \alpha \): multi-index \( \alpha = (\alpha_1, \ldots, \alpha_d) \) defining the monomial \( x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d} \)
- \( \mathbb{Q}_k \): space of polynomials of degree \( \max_i (\alpha_i) \) is at most \( k \), of dimension \( \binom{k+d}{k} \)
- \( \mathbb{P}_k \): space of polynomials of degree \( \sum_{i=1}^d \alpha_i \leq k \), of dimension \( \binom{k+d}{k} \)
- \( \mathbb{Q}[k] \): space of polynomials of degree \( \max_i (\alpha_i) = k \)
- \( \mathbb{P}[k] \): space of polynomials of degree \( \sum_{i=1}^d \alpha_i = k \)
- \( \mathbb{P}_{k_1,k_2,\ldots,k_d} \): space of polynomials with degree in \( x_i \) is \( k_i \)
- \( \mathbb{Q}_{-1} = \mathbb{P}_{-1} = \{0\} \)

**Functional spaces**
- \( \mathbb{R}_k(\partial K) = \{ p \in L^2(K), p|_{f_i} \in \mathbb{P}_k(f_i) \text{ for every face } f_i \in \partial K \} \)
- \( \text{H(div, K)} = \{ u \in (L^2(K))^d, \nabla u \in L^2(K) \} \)

**Operators**
- \( \mathbb{X}_1^d \): Cartesian product: \( \mathbb{X}_1^d(x_i) = (x_1, \ldots, x_d) \)
- \( \{ \zeta_i \}_{d-1} \): set containing the \( d \) cyclic permutations of \( \{ k + 1, k, \ldots, k \} \)
- \( \int_{\partial K} \cdot dx \): group of moments \( \int_f \cdot dx \) for all faces \( f \in \partial K \).
- \( \mathbb{1} \): indicator function

### 2 Classical Raviart-Thomas elements.

The spirit of the Raviart-Thomas elements is to work in a vectorial polynomial discretisation subspace of \( \text{H(div, K)} \) in which the functions are characterised separately on the boundary and within the elements. Doing so, the enforcement of the \( \text{H(div)} \)-conformity can be done at the interfaces by a specific choice of moment-based degrees of freedom acting only on the boundaries. As formalised by Nédelec [10], its definition on any simplicial reference
shape $K$ contained in $\mathbb{R}^d$ reads
\begin{equation}
RT_k(K) = (\mathbb{P}_k(K))^d \oplus x \mathbb{P}_{[k]}(K).
\end{equation}
There, any element $p \in RT_k(K) \subset (\mathbb{P}_{k+1}(K))^d$ writes under the form
\begin{equation}
p = \begin{pmatrix} p_1 + x_1 q \\ p_2 + x_2 q \\ \vdots \\ p_d + x_d q \end{pmatrix} := \prod_{i=1}^d (p_i + x_i q) = \prod_{i=1}^d p_i + \prod_{i=1}^d x_i q
\end{equation}
for some $p_i \in \mathbb{P}_k(K), i \in [1, d]$ and $q \in \mathbb{P}_{[k]}(K)$. Up to some straightforward computations, the dimension of $RT_k(K)$ can be formulated as
\begin{equation}
dim RT_k(K) = \dim(\mathbb{P}_{k+1}(K))^d + (d+1) \dim \mathbb{P}_k(f).
\end{equation}
Therefore, the definition of the element is done by setting internal and normal moments projecting respectively on the spaces $\mathbb{P}_{k-1}(K)^d$ and $\mathcal{R}_k(\partial K)$.

**Definition 2.1 (Degrees of freedom).** Any $q \in RT_k(K)$ is determined by
\begin{align}
\text{(2.4a) Normal moments} & \quad q \mapsto \int_{\partial K} q \cdot n_{p_k} d\gamma(x), \quad \forall p_k \in \mathcal{R}_k(\partial K), \\
\text{(2.4b) Internal moments} & \quad q \mapsto \int_K q \cdot p_{k-1} \, dx, \quad \forall p_{k-1} \in (\mathbb{P}_{k-1}(K))^d,
\end{align}
where $d\gamma$ represents the Lebesgue measure on the faces.
The basis functions of $RT_k(K)$ that are dual to those degrees of freedom verify
\begin{equation}
q \cdot n|_{\partial K} \in \mathbb{P}_k(\partial K) \quad \text{or} \quad q \cdot n|_{\partial K} = 0
\end{equation}
and are classified respectively as normal and internal basis functions. One can observe the $H(\text{div})$-conformity property: it reduces here to the continuity of the normal component across the boundary. Further divergence properties also hold directly by the nature of the approximation space [12], being a subspace of $H(\text{div}, K)$.

**Property 1 (Divergence properties).** For any $q \in RT_k(K)$, it holds:
\begin{equation}
\begin{cases}
\text{div } q \in \mathbb{P}_k(K) \\
q \cdot n|_{\partial K} \in \mathcal{R}_k(\partial K).
\end{cases}
\end{equation}
However, as the relation $(d+1) \dim \mathbb{P}_k(f) = \dim \mathcal{R}_k(\partial K)$ is only valid when the number of edges is $d+1$, this definition is very specific to the simplicial case. Therefore, when going to quads, the definition is changed by modifying the meaning of the polynomial degree, i.e. using $\mathbb{Q}_k$ spaces instead of $\mathbb{P}_k$ spaces. The $RT_k(K)$ space then reads
\begin{equation}
RT_k(K) = (\mathbb{Q}_k(K))^d \oplus x \mathbb{P}_{[k]}(K) = \prod_{i=1}^d \mathbb{P}_{\zeta_i(k+1, k, \ldots, k)}(K)
\end{equation}
and benefits from a dimensional split similar to (2.3);
\begin{equation}
dim RT_k(K) = 2d \dim \mathbb{Q}_k(f) + \dim \prod_{i=1}^d \mathbb{P}_{\zeta_i((k-1, k, \ldots, k))}(K).
\end{equation}
A definition of degrees of freedom analogous to (2.4) can then be set up. However, this extension is very specific to quads and cannot be adapted to offer a discretisation framework for arbitrary polytopes (see [2] for details).

**3 A framework for arbitrary polytopes** In order to build a unifying discretisation framework, we have to define spaces $\mathbb{H}_k(K)$ that fulfils the following
property:

**Requirement 1** (Requirements on the discretisation space). The space $H_k(K)$ is a finite dimensional vectorial subspace of $H(\text{div}, K)$ which dimension adapts to both the number of the polygon’s faces and the discretisation order.

In addition, to be able to endow $H(\text{div}, K)$-conformal elements $E_k(K)$ through definitions of degrees of freedom, we further ask the requirement 2.

**Requirement 2** (Requirements on the elements).
1. For any space $\mathbb{H}_k(K)$, there exists a unisolvent set of degrees of freedom $\{\sigma\}$ that can be split into internal and normal subsets so that both the number of internal degrees of freedom and of the normal degrees of freedom per face do not depend on the shape of $K$.
2. The number of internal and normal degrees of freedom both increase strictly monotonously with the discretisation order.

Last, to ensure the existence of a split into internal and normal subsets of degrees of freedom that matches the classification (2.5) at the level of the dual basis functions, the feasibility of the requirement 3 in $H_k(K)$ is also needed.

**Requirement 3** (Requirement on the basis functions). For any polytope $K$, the internal basis functions vanish on every face of the element.

Eventually, one may also ask for one further requirement ensuring a parallel with the Raviart-Thomas setting from the lowest order on.

**Requirement 4** (Optional requirement on the basis functions). The lowest order element has no internal degrees of freedom.

3.1 A class of admissible approximation spaces.

**Construction of spaces of discretisation.** In order to design a subspace of $H(\text{div}, K)$ that satisfies Condition 1 we are led to define a space $H_k(K)$ with the same architecture as the classical Raviart-Thomas space. Thus, we look for spaces in the form $H_k(K) = (A_k)^d + xB_k$ for two given functional sets $A_k$ and $B_k$. In order to design those two sets, we start by observing that the use of polynomial spaces is excluded by the condition 3, being required for any number of edges. Therefore, we consider the spaces $A_k$ and $B_k$ based on solutions to Poisson’s problems as in the context of the VEM method [15]. There, a way to allow the existence in $H_k(K)$ of smooth internal basis functions is to use the set of solutions to the boundary problems $\{u|_{\partial K} = 0, \Delta u = p_k\}$ for any $p_k$ belonging to $Q_m(K), m \in \mathbb{N} \cup \{-1\}$.

In addition, as the $H(\text{div}, K)$-conformity will be enforced by normal quantities that are tested only on the boundaries, we also consider the set of Poisson’s problems $\{u|_{\partial K} = p_k I_f, \Delta u = 0\}$ defined from polynomial boundary functions $p_k \in Q_l(f)$, $l \in \mathbb{N} \cup \{-1\}$ for each face $f$ of $\partial K$. Thus, seeing the boundary $\partial K$ face-wise, we define the set

$$H_k(\partial K) = \{u|_{\partial K} \in L^2(\partial K), u|_f \in Q_k(f), \forall f \in \partial K\}$$

and build the space $H_k(K)$, for integers $l_1$, $l_2$, $m_1$ and $m_2$, by the following direct sum.

**Definition 3.1** ($H_k(K)$ space).

$$H_k(K) = \{u \in H^1(K), u|_{\partial K} \in H_{l_1}(\partial K), \Delta u \in Q_{m_1}(K)\}^d$$

$$\oplus x \{u \in H^1(K), u|_{\partial K} \in H_{l_2}(\partial K), \Delta u \in Q_{m_2}(K)\}.$$

The choice of $l_1$, $l_2$, $m_1$ and $m_2$ is related to $k$ and will be discussed below.
Properties of $\mathbb{H}_h(K)$ spaces. The space $\mathbb{H}_h(K)$ is constructed from four independent blocks which definitions are driven by the independent coefficients $l_1, l_2, m_1$ and $m_2$. The couple $(m_1, m_2)$ drives the discretisation quality exclusively within the cell while $(l_1, l_2)$ takes care only of the boundary. Thus, the separation between internal and normal basis functions is natural. Furthermore, it holds the Property 3.2 emphasising that the $H(\text{div}, K)$-conformity is ensured by the definition of $\mathcal{H}_h(\partial K)$, while the inner smoothness is provided through the Laplacian.

**Proposition 3.2.** For any function $q$ belonging to any space $\mathbb{H}_h(K)$, it holds:
(3.3) $\quad q \cdot n|_{\partial K} \in \mathcal{H}_{\max \{l_1, l_2\}}(\partial K)$ and $\text{div} \ q \in L^2(K)$. 
It comes the following inclusion allowing $H(\text{div}, K)$-conformity.

**Corollary 3.3.** For any couples $(l_1, l_2)$ and $(m_1, m_2)$, 
$\mathbb{H}_h(K) \subset H(\text{div}, K)$.

Those spaces being furthermore of dimension
(3.4) $\quad \dim \mathbb{H}_h(K) = n(d(l_1+1)^{d-1} + (l_2+1)^{d-1}) + (d(m_1+1)^d + (m_2+1)^d - m_2^d)$,
their structure make them a-priori suitable to be used as discretisation spaces endowing $H(\text{div}, K)$-conformal elements.

**Example 1.** When $K$ is a two-dimensional simplex and when $l_1, l_2$ are chosen as $(l_1, l_2) = (-1, k)$, the discretisation quality of the normal component matches the one of the Raviart-Thomas setting.

Admissibility of the spaces for building $H(\text{div})$-conformal elements. In order to define elements in the spirit of Raviart-Thomas, we need to set $(d(l_1+1)^{d-1} + (l_2+1)^{d-1})$ normal degrees of freedom per face and $(d(m_1+1)^d + (m_2+1)^d - m_2^d)$ internal degrees of freedom. While this split does not impact the set of admissible coefficients $(m_1, m_2)$, it reduces the range of coefficients $(l_1, l_2)$ that can be used. Indeed, the space $\mathbb{H}_h(K)$ is constructed from four independent blocks providing two distinct discretisations: on the boundary and within the element. Thus, when testing a function of $\mathbb{H}_h(K)$ through normal degrees of freedom, one can only retrieve the polynomial obtained from the two boundary conditions defining the sets $A_k$ and $B_k$. On each face, this polynomial is of the form $p_k,A \in (A_k)^d|_f$, where the function $p_{k,A} \in (A_k)^d|_f$ reads

(3.5) $\quad p_{k,A} = \sum_{j=1}^{d} \left( \sum_{|\alpha_j| \leq l_1} a_{ij}x^{\alpha_j} \right)$
for a given set of multi-index $\{\alpha_j\}_i$ and coefficients $\{a_{ij}\}_{i,j}$ depending on the coordinates $x_j$. The function $p_{k,B} \in x.B_k|_f$ reads however

(3.6) $\quad p_{k,B} = \sum_{j=1}^{d} \left( \sum_{|\beta_j| \leq l_2} b_{ij}x^{\beta_j} \right)$
for a given set of multi-indices $\{\beta_j\}_i$ and coefficients $\{b_{ij}\}_i$ independent of the coordinates $x_j$. Therefore, denoting by $\{\xi_j\}_{j \in [1, d]}$ the coordinates permutation that allows to shift the lowest orders terms of $x.B_k|_f$ to $(A_k)^d|_f$, $p \in \mathbb{H}_h(K)|_f$ can be written as follows.

If $l_2 \geq l_1$, $\quad p = \sum_{j=1}^{d} \left( \sum_{|\alpha_j| \leq l_1} a_{ij}x^{\alpha_j} \right) + \sum_{j=1}^{d} \left( \sum_{|\beta_j| \leq l_2} b_{ij}x^{\beta_j} \right)$
3.2 Definition of admissible elements. Under the admissibility conditions 1, the spaces $\mathbb{H}_k(K)$ allow the construction of $H(\text{div})$-conformal elements through the definition of normal degrees of freedom enforcing the conformity and internal ones preserving it. We propose here a possible construction of such sets.

Definition of admissible normal degrees of freedom. The role of the normal degrees of freedom is to determine vectorial polynomials on the boundaries and to enforce the $H(\text{div})$-conformity of the element. We define them as the normal component of the tested quantities projected against polynomials of $\mathbb{H}_k(K)|_f$. We focus on the

\begin{equation}
\dim \mathbb{H}_k(K)|_f = \sum_{j=1}^{d} \left( \sum_{|\alpha_i|\leq l_1, |\alpha_i|\neq 0} (a_{ij} + b_{\xi_j(i)}) x^{\alpha_i} + x_j \sum_{l_1 \leq |\beta_i| \leq l_2} b_i x^{\beta_i} \right) + \sum_{j=1}^{d} a_{0j} x_j^0.
\end{equation}

\begin{equation}
\text{If } l_1 \geq l_2 + 1, \quad p = \sum_{j=1}^{d} \left( \sum_{|\alpha_i|\leq l_1} (a_{ij} + b_{\xi_j(i)}) x^{\alpha_i} + a_{0j} x_j^0 \right).
\end{equation}

The structure of those relations implies that the terms $a_{ij}$ and $b_{\xi_j(i)}$ are combined into a single coefficient and cannot be specified individually from further normal degrees of freedom. Indeed, the remaining freedom can only be seen inside the polytope, as a consequence of the boundary conditions on the Poisson’s solutions in either $A_k$ or $B_k$.

To prevent any over-determination by the normal degrees of freedom in $\mathbb{H}_k(K)|_f$, we therefore have to make sure that the dimension of the boundary part (3.7)-(3.8) of any function living in $\mathbb{H}_k(K)$ is larger than the number of wished normal degrees of freedom per face. By reading out the structure of (3.7)-(3.8) it comes

\begin{equation}
\dim \mathbb{H}_k(K)|_f = \begin{cases} 
   d(l_1 + 1)^{d-1} + (l_2 + 1)^{d-1} - l_1^{d-1} & \text{if } l_2 \geq l_1, \\
   d(l_1 + 1)^{d-1} & \text{otherwise}.
\end{cases}
\end{equation}

We thus restrict the admissible couples $(l_1, l_2)$ to those verifying the admissibility condition 1, preventing any over-determination.

Admissibility Conditions 1 (Necessary condition for using conformal elements). It should hold:

\[
\frac{\dim N}{\dim \mathbb{H}_k(K)|_f} \leq \frac{\dim \mathbb{H}_k(K)|_f}{\text{Available tuning coefficients on one face}}.
\]

In the case $l_2 \geq l_1$, it reduces to:

\[
d(l_1 + 1)^{d-1} + (l_2 + 1)^{d-1} \leq d(l_1 + 1)^{d-1} + (l_2 + 1)^{d-1} - l_1^{d-1} \quad (\Leftrightarrow l_1^{d-1} \leq 0)
\]

while otherwise it comes

\[
d(l_1 + 1)^{d-1} + (l_2 + 1)^{d-1} \leq d(l_1 + 1)^{d-1} \quad (\Leftrightarrow l_2 = -1).
\]

Regarding the internal characterisation, any couple of coefficients $(m_1, m_2)$ is allowed.

Definition of series of spaces. While fulfilling the above conditions, one can set a specific discretisation framework within which the spaces share a predefined structure. By example, defining the four coefficients $l_1, l_2, m_1$ and $m_2$ through affine relations of the type $l = ak + b$ for some index $k \in \mathbb{N}$, the range of discretisation qualities achievable within the framework is predetermined by a refinement sequence in each block, and the order of each space can be simply defined as the index $k$ generating each of the four coefficients. A typical working example is obtained by defining $m_1 = m_2 = k - 1$, $l_1 = 0$ and $l_2 = k$, leading to a series of discretisation spaces of order $k$. This case is specifically detailed in the section 3.4.
following possibilities:

**Available types of degrees of freedom.** For any \( q \in \mathbb{H}_k(K) \), we define:

1. The face integral of coordinate-wise components tested against polynomials:

   \[(3.10a) \quad q \mapsto \int_f q_i n_{ix} p \, d\gamma(x), \quad \forall p \in \mathbb{Q}_{\max(l_1, l_2)}(f),\]

2. The face integral of a function in \( \mathbb{H}_k(K) \) projected onto the face normal, and tested against polynomials:

   \[(3.10b) \quad q \mapsto \int_f q \cdot n_p d\gamma(x), \quad \forall p \in \mathbb{Q}_{\max(l_1, l_2)}(f), \]

3. The pointwise values of the discretised quantity tested against the face’s normal:

   \[(3.10c) \quad q \mapsto q(x_{im}) \cdot n_i, \quad \text{for sampling points }\{x_{im}\}_m \text{ on the face } f.\]

Defining the normal degrees of freedom then reduces to choosing \(d(1+d-1) + (l_2+1)d-1\) among the possibilities (3.10) so that their set is unisolvent for \( \mathbb{H}_k(K)|_f \). To ensure this, preventing any under-determination is sufficient. Therefore, we need to avoid the selection of projectors that are linearly dependent, and pay attention to determining both global and coordinate-wise behaviours of any vector polynomial \( q \in \mathbb{H}_k(K)|_f \).

**Example 2.** In two dimensions and for \( l_1 = l_2 = 0 \), any \( p \in \mathbb{H}_k(K)|_f \) reads

\[ q = \begin{pmatrix} A \\ B \end{pmatrix} + C \begin{pmatrix} x \\ y \end{pmatrix} \]

for some constants \( A, B \) and \( C \). The characterisation of \( q \) can be done by selecting two component-wise moments involving \( A n_{ix} \) or \( B n_{iy} \) tested against the constant polynomial \( p = 1 \) and one global moment that tests \( q \cdot n = C (n_{ix} + n_{iy}) + A n_{ix} + B n_{iy} \) against the polynomial \( p = x \). One could also choose two global moments and one coordinate-wise.

In practice, the selection of degrees of freedom reduces to choosing the polynomials \( p \) on which the function \( q \) will be tested coordinate-wise. The other polynomials \( p \) play the role of test functions for the global normal component \( q \cdot n \). The unisolvency of the set is then ensured by the following admissibility conditions.

**Admissibility conditions 2.**

1. The projection polynomials \( p \), and all the polynomials \( \sigma: q \mapsto \sigma(q(x_{im})) \) that define the point values must be linearly independent.

2. When using a coordinate-wise degree of freedom of the type (3.10a), polygonal shapes \( K \) containing a face parallel to any axis are not allowed. The term \( n_{ix} \) or \( n_{iy} \) would indeed always vanish for some \( i \in [1, n] \), thus not describing any function of \( \mathbb{H}_k(K)|_f \).

To help the construction of an element on \( K \) through the selection of degrees of freedom among those fulfilling the admissibility conditions 2, we recall that the chosen set of degrees of freedom imposes the shape of the dual basis functions. We can therefore select the degrees of freedom depending on the wished properties of the basis functions.

More crucially, the selection of global and/or coordinate-wise normal degrees of freedom leads to the reclassification of some basis functions as internal ones. Indeed,
as the face-wise normal component of any function \( q \) in \( \mathbb{H}_k(K)|_f \) is only of degree 
\[ \text{max}\{l_1, l_2\}, \] 
the term \( q \cdot n|_f \) requires only \( (\text{max}\{l_1, l_2\} + 1)^{d-1} \) basis functions to be 
decomposed on. Therefore, up to \( d(l_1 + 1)^{d-1} \) basis functions may see their global 
normal component vanishing on every face. Their coordinate-wise components will 
however not vanish, as they take care of the coordinate-wise behaviours that cannot 
be determined solely through the expression of \( q \cdot n|_f \).

\textbf{Remark 1.} Typically, the more global degrees of freedom are designed, the more 
the representation of \( p \cdot n \) is completed globally. As a consequence, more basis functions 
have a vanishing normal component as they are forced to take care only of coordinate-
wise behaviours, forcing them to be reclassified into internal basis function. The reverse 
scenario may also be considered.

To avoid this reclassification and allow a parallel with the Raviart-Thomas elements 
from \( \mathbb{H}_k(K)|_f \) for integrands the second terms of the right hand side of (3.7) when 
\[ (l_1 + 1)^{d-1} \] 
for any \( \tilde{x} \in Q_{l_2}(\mathbb{R}^{d-2}) \) is not involving the variable \( x_j \) so that the moment (3.11d) has 
for integrands the second terms of the right hand side of (3.7) when |\( \beta_i \)| = \( l_2 \). Note 
that the set (3.11) is of dimension \( d(l_1 + 1)^{d-1} + (l_2 + 1)^{d-1} - (l_1 + 1)^{d-1} - (d-1)(l_2 + 1)^{d-2} \) 
though we require \( d(l_1 + 1)^{d-1} + (l_2 + 1)^{d-1} \) moments. Thus, this configuration can only 
be used when \( l_1 \) and \( l_2 \) verify the feasibility condition:
\[ (l_1 + 1)^{d-1} \leq (d-1)(l_2 + 1)^{d-2} \] 
which is a reduction of the admissibility conditions 1. In two dimensions the above 
relation reduces to an equality, and all the degrees of freedom presented in (3.11) are 
considered. In higher dimensions, a further selection from the set (3.11) is required. 
There, we consider the sets (3.11a)-(3.11c) fully and select any \( (l_1 + 1)^{d-1} \) moments 
from (3.11d).

\textbf{Definition 3.4.} Any choice of \( (l_1 + 1)^{d-1} \) moments among (3.11d) is denoted as 
the “configuration Ia”. Associated with any admissible internal degrees of freedom, its 
unisolvency is given by the lemma B.1.

Up to the additional coordinate-wise moments, the configuration Ia is close to 
the Raviart-Thomas setting. However, the scaling of the dual basis functions does not 
match the one of the Raviart-Thomas basis. In order to obtain a similar scaling, one
should rather scale the above degrees of freedom with respect to each edge’s length
and orientation, or consider in place of the moments (3.11d) the point-wise values
\[ q \mapsto q(x_{im}) \cdot n, \]
where \( i \in [1, d], m \in [1, (l_1 + l_2)^{-1}] \) and \( x_{im} \) is any sampling point on the face \( f_i \).

**Definition 3.5.** Any selection of \((l_1 + 1)^{d-1}\) degrees of freedom among the sets
(3.11a),(3.11b),(3.11c) and (3.13) is labelled as the “configuration Ib”. Associated with
any admissible internal degrees of freedom, its unisolvence is given by the lemma B.1.

**Definition of admissible internal degrees of freedom.** In order to define
admissible internal degrees of freedom, we have to make sure that the corresponding
internal basis functions vanish on every face. We therefore stick to the idea of Raviart-
Thomas and define moment based degrees of freedom that reads for any \( q \in H^k(K) \)
\[ \sigma(q) \mapsto \int_K q \cdot p_k \, dx, \quad \text{for all } p_k \in \mathcal{P}(K) \]
for some function space \( \mathcal{P}(K) \) of dimension \((m_1 + 1)^d + (m_2 + 1)^d\). Considered as a
test space, \( \mathcal{P}(K) \) may simply gather polynomial functions used in the definition of
the Poisson’s problems generating \( H^k(K) \). The discretised quantities would then be
determined through their polynomial projections. An other choice is to test against
the set of Poisson’s solutions to the problems \( \{ \Delta p_k \in \mathcal{P}, p_k|_{\partial K} = 0 \} \).

Using one or the other possibility for \( \mathcal{P}(K) \), the unisolvence of the set of internal
degrees of freedom in \( H^k(K) \) is ensured by the following admissibility conditions
(see the proof B.1, Part 3):

**Admissibility conditions 3.**
1. The polynomials \( \{ p_l \} \) generating \( \mathcal{P}(K) \) are linearly independent.
2. No polynomial \( p_l \) is of degree larger than \( \max\{m_1, m_2 + 1\} \).

**Definition of the elements.** Combining the two previous paragraphs with the
definition of the space \( H^k(K) \), \( H(\text{div}, K) \)-conformal elements can be set up.

**Proposition 3.6.** Let \( K \) be any polytope satisfying the second item of the admissi-
ability conditions 2 and \( H^k(K) \) be any admissible space built on it. Let also \( \{ \sigma_N \} \) be any
selection of \( d(l_1 + 1)^{d-1} + (l_2 + 1)^{d-1} \) degrees of freedom from the set (3.10) fulfilling
first item of the admissibility conditions 2, and \( \{ \sigma_I \} \) the set of internal moments
built through the expression (3.14) for any of the projection sets \( \mathcal{P}(K) \) fulfilling the
admissibility conditions 3. Then, the set \( \{ \sigma_N \} \cup \{ \sigma_I \} \) is unisolvent for \( H^k(K) \) and
defines a \( H(\text{div}, K) \)-conformal element.

This well-possessedness property is an immediate corollary of the following proposition,
proven in the appendix B.

**Proposition 3.7.** Let \( q \in H^k(K) \), and denote \( \sigma_N(q) \) the \( n \)-tuple of normal degrees
of freedom extracted from the set (3.10). If
\[ \sigma_N(q) = 0 \quad \text{and} \quad \int_K q \cdot p_k \, dx = 0 \quad \text{for all } p_k \in \mathcal{P} \]
then \( q = 0 \).

At this point, any admissible definition leads to \( H(\text{div}, K) \)-conformal elements.

**3.3 Summary of the construction.** Let us summarize the spaces construction
and the example of normal degrees of freedom that has been detailed above. To begin
Thus, conformal elements can be defined through normal degrees of freedom enforcing
with, the class of discretisation spaces reads

$$
\mathbb{H}_k(K) = \{ u \in H^1(K), u|_{\partial K} \in \mathcal{H}_l(\partial K), \Delta u \in Q_{m_1}(K) \}^d
\oplus x \{ u \in H^1(K), u|_{\partial K} \in \mathcal{H}_l(\partial K), \Delta u \in Q_{m_2}(K) \},
$$

with the convention that $Q_{-1} = \{ 0 \}$ and where the integers $l_1, l_2, m_1$ and $m_2$ verify

$\quad m_1, m_2, l_2 \geq -1 \quad \text{and} \quad -1 \leq l_1 \leq 0.$

So defined, it holds:
- $\dim \mathbb{H}_k(K) = n(d(l_1 + 1)^{d-1} + (l_2 + 1)^{d-1}) + ((m_1 + 1)^d + (m_2 + 1)^d - m_2^2)$
- For all $q \in \mathbb{H}_k(K)$, $q \cdot n|_{\partial K} \in \mathcal{H}_{\max(l_1, l_2)}(\partial K)$ and $\mathbb{H}_k(K) \subset H(\text{div}, K)$.

Thus, conformal elements can be defined through normal degrees of freedom enforcing the $H(\text{div})$-conformity and internal ones preserving it, provided that the polytope $K$ satisfies the two conditions
- The polytope $K$ is decently squeezed, ensuring existence of Poisson’s solutions
- (Using component-wise degrees of freedom) No face is parallel to any axis.

### Internal degrees of freedom.
It is set

$$
\sigma : q \mapsto \int_K q \cdot p_k \, dx, \quad \forall p_k \in \mathcal{P}
$$

for any space $\mathcal{P}$ defined either as a polynomial space or as any subspace of Poisson’s solutions, having for dimension $(m_1 + 1)^d + (m_2 + 1)^d - m_2^2$ and fulfilling the assumptions 3.

### Normal degrees of freedom.
The selection of degrees of freedom among the bold

| Representation | of low order | of $(A_l \cap B_k)|_{\partial K}$ | of higher orders |
|----------------|--------------|-----------------------------------|-----------------|
| available when $(d - 1)(l_1 + 1)^{d-1} \geq (l_1 + 1)^{d-1}$ Select $(l_1 + 1)^{d-1}$ moments per face from the bold ones. | Inherited from the highest order representation (see the remark ??) | $\int q_i \, n_{x_i} p_k \partial K$ $\int q_i \, x_{j} t_{i}^{l_1+1} f_j \forall i \in [1, d]$, $\forall j \in [1, n]$, $p_k \in \mathcal{H}_{l_1}(\partial K) \setminus \mathcal{H}_0(\partial K)$ | $\int q \cdot n p_k \partial K$ $\int q \cdot n_{x_j} x_{j}^{l_2 - i} \forall p_k \in \mathcal{H}_{l_2}(\partial K) \setminus \mathcal{H}_{l_1}(\partial K)$ $\forall e \in Q_{l_2}(\partial K)$ |

**Table 3.1**

Summary of the used degrees of freedom for the configurations Ia.

ones is a matter of taste, possibly directed by properties of the discretised quantities that are known *a priori*. Note also that one could project on any other polynomial basis rather than using projections over monomials.

### 3.4 Two examples in two dimensions.
We first detail an example of a discretisation framework contained in the previously presented setting for which a parallel with the Raviart-Thomas elements can be drawn from the order $k = 1$ on. In a second time, we present an example of a reduced framework where a parallel with the Raviart-Thomas is achieved at any order.

#### 3.4.1 An example of a general setting.
We consider a series of discretisation spaces by indexing the coefficients $l_1 = 0$, $l_2 = k$, $m_1 = k - 1$ and $m_2 = k - 1$ for any $k \in \mathbb{N}$, seen here as the space order. The space $\mathbb{H}_k(K)$ is then defined as

$$
\mathbb{H}_k(K) = \{ u \in H^1(K), u|_{\partial K} \in \mathcal{H}_0(\partial K), \Delta u \in Q_{m_1}(K) \}^2
\oplus x \{ u \in H^1(K), u|_{\partial K} \in \mathcal{H}_k(\partial K), \Delta u \in Q_{m_2}(K) \},
$$

(3.16)
By a straightforward application of the previous section, it comes
(3.17) \( \dim \mathbb{H}_k(K) = n(k + 3) + 2k(k + 1) - \mathbbm{1}_{k>0} \).

Example of a two dimensional element. The example of selected normal
degrees of freedom defining the elements \( E_k = (K, \mathbb{H}_k(K), \{\sigma\}) \) presented in the
previous section then reduces to the expressions given in the table 3.2. The internal

| Representation | of low order | of \((A_k \cap B_k)_{\partial K}\) | of higher orders |
|----------------|-------------|----------------|-----------------|
| Moments        | \( \int_{\partial K} q \cdot n \) | \( \int q \cdot n_{x_i} \) | \( \int q \cdot n_p_k \) |
|                | \( l_j \in [1, 2], \forall j \in [1, n] \) |               | \( \forall p_k \in \mathcal{H}_k(\partial K) \backslash \mathcal{H}_0(\partial K) \) |

Table 3.2

Definition of the degrees of freedom in the 2D case for the configuration 1a.

degrees of freedom are set as
(3.18) \( \sigma(q) \mapsto \int_K q \cdot p_k \, dx, \) for all \( p_k \in \mathcal{P} \),

where \( \mathcal{P} \) is chosen as the symmetric space
\[ \mathcal{P} = \mathbb{P}_{k, k-1} \times \mathbb{P}_{k-1, k} \backslash \left( \mathbb{P}_{[k], [k-1]} \times \mathbb{P}_{[k-1], [k]} \right) \cup \left\{ (x, y)^T \mapsto \begin{pmatrix} x^k y^{k-1} \\ x^{k-1} y^k \end{pmatrix} \right\}. \]

Though the internal projection space is less refined than the one set on the edges,
this is not bothersome as the impact of the divergence within the cell is less dramatic.
Note also that in practice, for defining the projections (3.18) one can work with any
basis of \( \mathcal{P} \).

3.4.2 An example of a reduced setting. As quickly addressed in the section
3.2 and as it will be shown in the numerical results, a classical construction of the
space \( \mathbb{H}_k(K) \) implies the degeneration of some normal functions into internal ones.
This is a consequence of the coordinate-wise freedom provided on the boundary from
the definition of the set \( A_k \). Therefore, to allow a parallel with the Raviart-Thomas
elements from the lowest order on and to fulfill the optional condition 4, one can
consider replacing the boundary conditions \( u|_{\partial K} \in \mathcal{H}_{l_1}(\partial K) \) in \( A_k \) to obtain the
reduced space
(3.19) \( \mathbb{H}_k(K) = \left\{ u \in H^1(K), u|_{\partial K} \equiv 1, \Delta u \in \mathbb{Q}_{m_1}(K) \right\}^d \)

\( \oplus x \left\{ u \in H^1(K), u|_{\partial K} \in \mathcal{H}_{l_2}(\partial K), \Delta u \in \mathbb{Q}_{[m_2]}(K) \right\} \).

There, the coordinate-wise freedom on the boundary is reduced and the normal
degrees of freedom can be set as in the classical Raviart-Thomas setting. Furthermore,
contrarily to the general case, any definition of \( l_2, m_1 \) and \( m_2 \) leads to an \( H(\text{div}) \)-
conformity ready space.

Example of a reduced two dimensional element. To emphasise the parallel
with the Raviart-Thomas setting on the boundary, we reduce the previous example
and derive the corresponding reduced discretisation framework.

Definition 3.8 (Reduced space).
\( \mathbb{H}_k(K) = \left\{ u \in H^1(K), u|_{\partial K} \equiv 1, \Delta u \in \mathbb{Q}_{k-1}(K) \right\}^2 \)
\( \oplus \begin{pmatrix} x \\ y \end{pmatrix} \left\{ u \in H^1(K), u|_{\partial K} \in \mathcal{H}_k(\partial K), \Delta u \in \mathbb{Q}_{[k-1]}(K) \right\} \).
Its dimension then naturally reads:

$$\dim \mathbb{H}_k(K) = n(k + 1) + 2k(k - 1) - \mathbb{1}_{k > 0}. $$

Therefore, exactly $k + 1$ normal functions per edge can be designed, fitting the framework of Raviart-Thomas. As this matches the dimension of $Q_k(f)$, all the freedom is required to entirely determine the global normal component. Thus, as a straightforward reduction of the general case, the $H(\text{div}, K)$-conformal element presented in the section 3.4 simplify to the following degrees of freedom. Note that

| Core normal Moments | Misc moment | Core internal moment | Misc internal moment |
|---------------------|-------------|----------------------|----------------------|
| $\int_{\partial K} x^i q \cdot n d\gamma, \quad i \in [1, k]$ | $\int_{\partial K} q \cdot n d\gamma$ | $\int_{K} \left( \begin{array}{l} x^iy^j \\ 0 \end{array} \right) q dxdy$ and $\int_{K} \left( \begin{array}{l} 0 \\ x^hy^j \end{array} \right) q dxdy, \quad i \in [0, k], m \in [0, k - 1]$ | $\int_{K} \left( \begin{array}{l} x^iy^j \\ x^jy^i \end{array} \right) q dxdy$ |
| $\int_{K} \left( \begin{array}{l} x^iy^j \\ x^jy^i \end{array} \right) q dxdy, \quad (l, m) \neq (k, k - 1)$ | | | |

Table 3.3

*Degrees of freedom of the element Ia defined within the reduced setting*

here too, for defining the projections (3.18) one can work with any basis of $P$ instead of the presented canonical basis.

4 Numerical results. We explore the properties of the main element $Ia$ and its variant $Ib$ presented in the previous section by investigating their basis functions, for both the general framework and the reduced one. As an example, we consider the non-convex nine-edges polygon presented in the figure 4.1 on which the elements are built. In all the results, the polynomial projectors used in the definition of the degrees of freedom were chosen as Hermite polynomials, as experimentally we observed that this improves the conditioning of the linear system.

4.1 General setting. We start by considering the spaces and elements described in the section 3.4.1. First of all, we have investigated the behaviour of the internal basis functions. The normal component of them is shown in the right of figure 4.1. As wished, the basis functions corresponding to internal degrees of freedom vanish on the boundary.
In order to study the behaviour of the normal basis functions on the boundaries, we have considered the case $k = 2$ where we expect five basis function to have a quadratic normal component. We have plotted in the most left side of the figure 4.2 the normal component of one of the normal basis functions associated to the element $Ib$. One can observe that its support is contained on one single edge.

We then have plotted all the basis functions associated to the sixth edge on the figure 4.2, for both the configurations $Ib$ (middle) and $Ia$ (right). One can first notice that their normal components are all polynomial of degree $k \leq 2$. Observing further, it appears that two basis functions have a vanishing normal component, i.e. $q \cdot n = q_1 \cdot n_{6,x} + q_2 \cdot n_{6,y} = 0$. Indeed, those two basis functions characterise the coordinate-wise liberties $q_1 \cdot n_{6,x} \neq 0$ and $q_2 \cdot n_{6,y} \neq 0$ which are not reflected through the global term $q \cdot n$ only (as addressed in the example 2 and in the remark 1). As those functions are nevertheless regular within the polygon $K$ and not identically vanishing on $K$ (see figure 4.3), they can be reclassified as internal basis functions. Note that this can be suppressed when using the reduced setting, as one can observe below in the section 4.2.

Figure 4.2. Left: normal component of one representative of the normal basis functions for the element $Ib$ and $k = 2$ along the edges. Middle: normal component of all the functions generated from the sixth edge, plotted on the first edge. Right: as middle, for the element $Ia$.

Figure 4.3. Degeneration of a degenerating normal basis functions’ representative in the case $k = 2$, for the element $Ib$. Left: normal component on all the boundaries. Right: internal behaviour of the basis functions.

Lastly, one can consider the scaling of the basis functions by plotting the normal basis functions corresponding to the configurations $Ia$ and $Ib$ in the lowest order case, i.e. for $k = 0$ (see figure 4.4). There, only the configuration $Ib$-using a point-wise valuescales to one. The fully moment-based configuration $Ia$ scales to another constant.
that depends on the edge’s length and orientation with respect to the axes. This example emphasises that the configuration $Ib$ leads to basis functions which share similar properties like the Raviart-Thomas elements.

**Figure 4.4.** Scaling of the non-vanishing basis functions generated from the first edge when $k = 0$. From left to right: $Ia$ and $Ib$.

### 4.2 Reduced setting

As a last example, we derive some results obtained for the reduced element $Ib$, offering a complete parallel with the Raviart-Thomas elements on the boundary by suppressing the further coordinate-wise liberty provided by the general setting. The internal basis functions being unchanged from the general setting, they are not represented.

Indeed, one can observe on the bottom of the figure 4.5 that there is no more degenerating normal basis functions. Therefore, all normal basis functions are acting globally to characterise the polynomial behaviour of functions of the reduced $H^k(K)$ space on the boundary. Furthermore, one can observe that the scaling of the basis functions corresponding to lowest order element, as well as the amplitude of the basis functions describing the higher order ones, make the discretisation framework reliable.

**Remark 2.** The shape of the normal component of the basis functions is driven by the definition of the projectors $p$ in the normal degrees of freedom. Changing the basis of the projectors then allows to enforce wished shape of the basis functions of $H^k(K)$ while keeping the regularity and order of the discretisation. Shifting them by modulating the offset directly from the definition of the degrees of freedom to enforce their positivity is equally possible.

### 5 Conclusion

Motivated by defining a flux reconstruction scheme on general polytopes [1], we have developed a new $H(\text{div})$-conformal discretisation framework that can be set up on any polytope, not necessarily convex. It merges the flexibility of the Virtual Element setting with the properties of the Raviart-Thomas elements on the boundaries.

The introduced finite dimensional spaces are vectorial an allow a lot of flexibility in the definition of the degrees of freedom. In particular, the choices of discretisation quality and degrees of freedom on the boundary are independent from the ones made within the element.

The discretised quantities benefit from an extensive coordinate-wise liberty. Therefore, upon the choice made while selecting the degrees of freedom, some dual normal basis functions may be reclassified into internal ones. Thus, to allow a complete parallel with the Raviart-Thomas setting on the boundary from the lowest order on, one may construct straightforwardly a reduced space, along with reduced elements.
Last, we detailed a particular example of a discretisation framework through a series of spaces and the definition of a particular element. It could be observed that in both general and reduced frameworks, the type of degrees of freedom (point-wise values or moments) impacts the scaling of the dual basis functions. This can typically be observed in the lowest order case of the given examples, where only the dual basis functions of the element $I_b$ scale to one.

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**Appendix A. Further configurations.**

**A.1 An alternative approach** As further examples of the case $l_2 \geq l_1$, we propose a different approach furnishing another type of configuration, $II$, leading to two further elements $IIa$ and $IIb$. This configuration focuses on a coordinate-wise characterisation of the normal component and arises naturally from the equation (3.7) that provides the following layout of degrees of freedom.

\[
\sigma : q \mapsto \int_{\partial K} q_i n_{x_i} \, d\gamma(x), \quad i \in [1, d] 
\]

\[
\sigma : q \mapsto \int_{\partial K} q_i n_{x_i} x^{\alpha_j} \, d\gamma(x), \quad i \in [1, d], \quad \forall 1 \leq |\alpha_j| \leq l_1
\]

\[
\sigma : q \mapsto \int_{\partial K} q \cdot v x^{\alpha_j} \, d\gamma(x), \quad |\alpha_j| = l_1 + 1, \quad \text{for any vector } v \neq n
\]
There, the moment (A.1c) has been slightly modified from (A.1d) in order to ensure the assumption 2.1, without having any impact on the $H(\text{div})$-conformity. So defined, the total number of degrees of freedom available in the layout (A.1) is $\dim(A.1) = n[d(l_1 + 1)^{d-1} + (d - 1)(l_2 + 1)^{d-1} - (d - 1)l_1^{d-1}]$, larger than the required $d(l_1 + 1)^{d-1} + (l_2 + 1)^{d-1} - (d - 1)(l_2 + 1)^{d-1} - (d - 1)l_1^{d-1}$, i.e.

$$(A.2) \quad (d - 1)l_1^{d-1} \leq (d - 2)(l_2 + 1)^{(d-1)}.$$  

As $l_1 \leq 0$ by the admissibility conditions 1, this inequality always hold. In particular, in two dimensions the bound (A.2) reduces to an equality and the set of degrees of freedom (A.1) has to be considered fully. In higher dimensions, not achieving the bound provides liberty in the choice of the projection space, as any extraction of $(l_2 + 1)^{d-1}$ moments is admissible. A natural choice is to observe the relation (A.2) and select fully the sets (A.1a) and (A.1b) while extracting any $(l_2 + 1)^{d-1}$ elements from (A.1d). The focus is therefore made on the coordinate-wise determination of the normal component, limiting the number of degenerating polynomials in the dual basis.

**Definition A.1.** Any extraction of $(l_2 + 1)^{d-1}$ moments among (A.1) is labelled as the “configuration IIa”, whose unisolvence is given by the lemma B.1.

Note that one can set an alternative definition using some point values, where instead of the moments (A.1a) the following expression is used for any $j \in [1, d]$, $i \in [1, n]$ and $x_{im}$ middle or barycentric point in $f_i$

$$(A.3) \quad \sigma: q \mapsto q_i(x_{im}) n_{ix_j}.$$  

There, the point-wise values allows to drive the offset of the dual polynomials in every direction, which may be used to enforce positivity of the basis functions.

**Definition A.2.** Any extraction of $(l_2 + 1)^{d-1}$ degrees of freedom among the sets (A.3),(A.1b),(A.1c) and (A.1d) is labelled as the “configuration IIb”, whose unisolvence is given by the lemma B.1.

**A.2 Additional numerical results.** For the sake of completeness, let us represent the shape of the normal component of the normal basis functions obtained with the configurations IIa and IIb in the figure A.1. There, contrarily to the configuration I, only one function is degenerating as an internal one (in IIb, two quadratic functions have an identical normal component, making them not distinct in the second picture). Indeed, the focus is made on the coordinate-wise determination of the normal component, limiting the number of degenerating polynomials in the dual basis.

![Figure A.1](image-url)
of the normal component. As we are in two dimensions, two functions are required to
determine the coordinate-wise constants. The similarity with the configuration 1 lies
in the scaling. Indeed, only the type b using pointwise value forces the scaling of the
dual basis functions to one in the lowest order case.

A.3 Summary of the four investigated configurations
For the sake convenience, we wrap up the definition of the normal degrees of freedom of the four
elements investigated in the classical setting in the table below. Their expression in
two dimensions as well as their definition in the case of the reduced setting can be
found in the technical report [2].

| Configuration | IIa | IIb | Ia | Ib |
|---------------|-----|-----|----|----|
| Available when | \( \frac{(d-2)}{(d-1)} (l_2 + 1) \) \(d-1 \geq l_1^{d-1} \) | \( (d - 1)(l_2 + 1)^{d-2} \geq (l_1 + 1)^{d-1} \) |
| Low order representation | \( \int q_j n_{x_j} \) \( \frac{\partial}{\partial K} \) | \( q_i(x_{im}) n_{x_j} \) \( \frac{\partial}{\partial K} \) | Inherited from the highest order representation | Inherited from the highest order representation |
| \( \forall j \in [1, d] \) | \( \forall v \in [1, n] \) | \( \forall j \in [1, d] \) | \( x_m \) midpoint |
| Representation of elements in \( A_k \cap B_k, | \( \frac{\partial}{\partial K} q \cdot v \cdot z_{a j} \) | \( \frac{\partial}{\partial K} q(x) \cdot v \cdot z_{a j} \) | \( \frac{\partial}{\partial K} q \cdot n_{x_j} \cdot p_k \) | \( \frac{\partial}{\partial K} q \cdot n_{x_j} \cdot p_k \) |
| \( \forall l \in [1, d] \) | \( \forall j \in [1, n] \) | \( \forall j \in [1, n] \) | \( \forall v \in [1, n] \) | \( p_k \in H_1(\partial K) \setminus H_0(\partial K) \) |
| \( \forall l \leq |x_j| \leq l_1 \) | \( \forall j \in [1, n] \) | \( \forall j \in [1, n] \) | \( \forall v \in [1, n] \) | \( p_k \in H_1(\partial K) \setminus H_0(\partial K) \) |
| \( \forall l \leq |x_j| \leq l_1 \) | \( \forall j \in [1, n] \) | \( \forall j \in [1, n] \) | \( \forall v \in [1, n] \) | \( p_k \in H_1(\partial K) \setminus H_0(\partial K) \) |
| Higher orders representation: | \( \int q \cdot n_{x_j} \cdot z_{a j} \) | \( \int q \cdot n_{x_j} \cdot z_{a j} \) | \( \int q \cdot n \cdot p_k \) | \( \frac{\partial}{\partial K} q(x_{im}) \cdot n \) |
| \( \forall l \leq |x_j| \leq l_1 \) | \( \forall j \in [1, d - 1] \) | \( \forall j \in [1, d - 1] \) | \( \forall p_k \in H_2(\partial K) \setminus H_1(\partial K) \) | \( \forall \alpha j \in [1, n] \) |
| Pick \( N_f \) elements per face from the bold elements | \( N_f = (l_2 + 1)^{d-1} \) | \( N_f = (l_2 + 1)^{d-1} \) | \( N_f = (l_1 + 1)^{d-1} \) | \( N_f = (l_1 + 1)^{d-1} \) |

### Summary of the used degrees of freedom for the four considered configurations.

**Appendix B. Proofs.**

*Proof of Proposition 3.2.* We start by deriving the first statement. By construction,
any \( q \in H_k(K) \) can be decomposed into \( q = q_0 + x \cdot q_1 \) for some \( q_0 \in (A_k)^d \) and
\( q_1 \in B_k \). Therefore, on the boundary of \( K \) one has \( q \cdot n|_{\partial K} = q_0 \cdot n|_{\partial K} + \langle xq_1 \cdot n \rangle|_{\partial K} \). As
the functions \( q_1 \) is scalar, this quantity can also read \( q \cdot n|_{\partial K} = q_0 \cdot n|_{\partial K} + \langle x \cdot q_1 \cdot n \rangle|_{\partial K} \)
by linearity and commutativity of the dot product.

Since for every face \( f \) of \( K \) the term \( x \cdot n|_{f} \) is constant, it reduces to \( q \cdot n|_{f} = q_0 \cdot n|_{f} + cf \cdot q_1|_{f} \) on each face \( f \) for a constant \( cf \in \mathbb{R} \) depending only on the face layout
and position with respect to the axes and origin. Therefore, since \( q_0|_{f} \in (\mathbb{Q}_{l_2}(f))^d \) and \( q_1|_{f} \in \mathbb{Q}_{l_2}(f) \), \( q \cdot n|_{f} \in \mathbb{Q}_{\max(l_1, l_2)}(f) \). And since it is valid for any face \( f \in \partial K \),
we finally get that \( q \cdot n|_{\partial K} \in H_{\max(l_1, l_2)}(\partial K) \).

Let us now derive the divergence property within the cell. Any \( u \in H_k(K) \)
can be written under the form \( u = \tilde{q} + xq \) for some functions \( q \in H^1(K) \) and 
\( \tilde{q} = (\tilde{q}_1, \cdots, \tilde{q}_d)^T \in (H^1(K))^d \) such that
\[
\begin{aligned}
\Delta q &\in \mathcal{Q}_{[m_2]}(K) \\
q|_{\partial K} &\in \mathcal{Q}_{l_2}^{\partial}(\partial K)
\end{aligned}
\] and
\[
\begin{aligned}
\Delta \tilde{q}_i &\in \mathcal{Q}_{m_1}(K) \\
\tilde{q}_i|_{\partial K} &\in \mathcal{Q}_{l_1}(\partial K), \quad \forall i \in [1, d].
\end{aligned}
\]
We have
\[
\text{div}(u) = \sum_{i=1}^{d} \partial_{x_i}(x_i, q) + \sum_{i=1}^{d} \partial_{x_i} \tilde{q}_i = \sum_{i=1}^{d} (q + x_i \partial_{x_i} q) + \sum_{i=1}^{d} \partial_{x_i} \tilde{q}_i.
\]
Since by (B.1) we have \( \nabla \cdot q \in L^2(K) \), it comes that for any \( i \in [1, d] \); \( x_i \partial_{x_i} q \in L^2_{\text{loc}}(K) \).
As \( K \) is compact and bounded, we have \( L^2_{\text{loc}}(K) = L^2(K) \) and \( \text{div} q \in L^2(K) \). As a by-product, note that we can derive \( \nabla \cdot (x \nabla q) = \nabla \cdot q + x \Delta q \), where \( \Delta q \in \mathcal{Q}_{\max \{m_1, m_2+1\}} \) and \( x\Delta q \in C^\infty(K) \).

**Lemma B.1.** The configurations \( I_a, I_b, I_{IIa} \) and \( I_{IIb} \) are sets of degrees of freedom leading to unisolvent elements when endowed in \( \mathbb{H}_k(K) \).

**Proof of the lemma B.1.** We refer to the functions \( p_k \) by the term “kernel”, while using the term “integrand” to represent the term \( q \cdot p_k \). Immediate transfer of this designation apply to the normal moment based degrees of freedom.

We first sketch the proof. To begin with, let us point out that the key lies in the assumptions 1 ensuring the linear independence of the set of point-wise values and moment’s integrands. The linearity of the integral operators transfers then this independence to the moments themselves, characterising any function of \( \mathbb{H}_k(K) \) independently on the boundary and within the cell. We proceed in three steps.

1. First, we show that the internal characterisation of the function does not impact the normal one, allowing the determination to be done distinctively within the element and on the boundary.
2. Then, we show that selecting the appropriate number of degrees of freedom in any of the sets \( I_a, I_b, I_{IIa} \) or \( I_{IIb} \) ensures a unique characterisation on the boundary. We use the fact that the kernels are scalar polynomials while the functions of \( \mathbb{H}_k(K) \) are vector polynomials.
3. Lastly, we consider the interior of the element where the characterisation is done through projections over linearly independent sets. Those projections of functions in \( \mathbb{H}_k(K) \) are indeed neither identically null nor identically identical (i.e. they differ at least on a subset of non-zero measure).

Let us detail this determination process more in details.

**Step 1.** Let us first recall that the space \( \mathbb{H}_k(K) \) is built from blocks of independent functions. In particular, the boundary behaviour of functions living in \( \mathbb{H}_k(K) \) is independent of their behaviour within the inner cell. Therefore, by the structure of \( \mathbb{H}_k(K) \) and making use of the superposition theorem, any function \( q \in \mathbb{H}_k(K) \) reads
\[
(B.2) \quad q = f 1_{\partial K} + g 1_{K}
\]
for two functions \( f \) and \( g \) belonging to \( \mathbb{H}_k(K) \). As a consequence, characterising a function \( q \in \mathbb{H}_k(K) \) comes down to characterising the independent functions \( f \) and \( g \) on the distinct supports \( \partial K \) and \( K \), respectively. Note also that necessarily, \( f |_{f_j} \in \mathcal{Q}_{\max \{t_1, t_2+1\}}(\mathbb{R}^{d-1}) \) for any face \( f_j \in \partial K \). We show that any admissible
extraction (in the sense of the admissibility conditions 1) from either of the four sets of degrees of freedom (IIa, internal), IIb, Ia, internal), (Ib, internal) fully characterises the functions $f$ and $g$, independently. In all the following, the notation IIa, IIb, Ia or Ib refers to the corresponding set of normal degrees of freedom while "internal" refers to the set (3.14) and is identical to any of the four configurations under consideration.

We first show that any above defined set of degrees of freedom preserve the independence of the boundary and inner characterisations. To this aim, we combine the relation (B.2) with the all possible definitions of the degrees of freedom. It comes that all global normal moments lead to an expression of the form

$$\sigma(q) = \int_{\partial K} q \cdot n p_k = \int_{\partial K} (f \mathbb{1}_{\partial K} + g \mathbb{1}_K) \cdot n p_k = \int_{\partial K} f \cdot n p_k$$

for some polynomial function $p_k$ living on $\partial K$. On the other side, as $x_{im} \in \partial K$, the global degrees of freedom that are built from point-wise values read

$$\sigma(q) = q(x_{im}) \cdot n = f(x_{im}) \cdot n \mathbb{1}_{\partial K}(x_{im}) + g(x_{im}) \cdot n \mathbb{1}_K(x_{im}) = f(x_{im}) \cdot n.$$

Similar relations for coordinate-wise degrees of freedom can be derived, that is;

\begin{align*}
\sigma(q) &= \int_{\partial K} q_n \cdot n = \int_{\partial K} (f_n \mathbb{1}_{\partial K} + g_n \mathbb{1}_K) \cdot n = \int_{\partial K} f_n \cdot n = \int_{\partial K} f_n \cdot n \mathbb{1}_K(x_{im}), \\
\text{and } \sigma(q) &= q_n(x_{im}) \cdot n = f_n(x_{im}) \cdot n \mathbb{1}_{\partial K}(x_{im}) + g_n(x_{im}) \cdot n \mathbb{1}_K(x_{im}) \\
&= f_n(x_{im}) \cdot n.
\end{align*}

Therefore, in any of the configurations IIa, IIb, Ia and Ib no contribution of the function $g$ representing the inner part of the cell is involved in the normal degrees of freedom. The mirror case is obtained with the internal moments, leading via (B.2) to

$$\sigma(q) = \int_K q \cdot p_k = \int_K (f \mathbb{1}_K + g \mathbb{1}_K) \cdot p_k = \int_K g \cdot p_k,$$

where $p_k$ stands for any Poisson’s function living in $\mathbb{H}_k(K)$ or any polynomial function defining the second member of a Poisson’s problem involved in the definition of $\mathbb{H}_k(K)$. There, the function $f$ representing the boundary part of the function $q$ is not involved, that for any definition of the space $P_k$ generating the internal moments. Thus, by linearity we can decompose the degrees of freedom \{q $\mapsto \sigma_i(q)\}_i$ in the following matrix.

\[
\begin{pmatrix}
\sigma_1 \\
\vdots \\
\sigma_{N_N} \\
\sigma_{N_N+1} \\
\vdots \\
\sigma_{N_I}
\end{pmatrix}
\begin{pmatrix}
\text{Normal moments applied to } f \\
0
\end{pmatrix}
\begin{pmatrix}
f \\
g
\end{pmatrix}
\begin{pmatrix}
\text{Internal moments applied to } g \\
0
\end{pmatrix}
\begin{pmatrix}
\end{pmatrix}
\]

Clearly, there is no interconnection between the function’s characterisation on the boundaries and the one performed within the element. Thus, showing the proposition 3.7 reduces to show independently that IIa $= 0$ or IIb $= 0$ or Ia $= 0$ or Ib $= 0$ implies $f|_{\partial K} = 0$ and $\int_{K} g \cdot p_k x = 0$, for all $p_k \in P_k$ implies $g|_{K} = 0$.

**Step 2.** Let us now consider the boundary characterisation. There, by definition of the
The structure of the retrieved form makes clearly emerge the coefficients that should be used depending on the coordinate-wise behaviour of the polygon. Thus, we can write

\[ f \big|_{\partial K} = \sum_{j=1}^{n} r_j \mathbb{I}_f \]

with \( r_j \in \mathbb{R}^{\max\{l_1, l_2+1\}}(f_j) \) and \( f_j \) any face belonging to \( \partial K \). With a similar argument than in the previous point, the characterisation of \( f \big|_{\partial K} \) can therefore be done edge-wise, and the determination matrix becomes block-diagonal. We discuss here the characterisation on one particular edge \( f_j \) by showing the invertibility of the corresponding matrix block. The arguments naturally transpose to the other ones.

In this perspective, let us show that for any \( r_j \in \mathbb{R}^{\max\{l_1, l_2+1\}}(f_j) \), it holds \( \{ (IIa) \mid f_j = 0 \} \) or \( \{ (IIb) \mid f_j = 0 \} \) if \( r_j = 0 \), where \( \{ \cdot \} \mid f_j = 0 \) represents the subset of the degrees of freedom \( \{ \cdot \} \) whose support (or evaluation point for point-values) matches (or lies on) \( f_j \).

First of all, we recall that on the face \( f_j \) the function \( r_j \) is a multi-valued polynomial of the form

\[ r_j \big|_{f_j} = \left( \begin{array}{c}
  a_{0,1} \\
  \vdots \\
  a_{0,d}
\end{array} \right) + \sum_{i=\dim(\mathcal{H}_0)}^{\dim(\mathcal{H}_{l_1} \cap \mathcal{H}_{l_2})} \left( \begin{array}{c}
  b_{i_1}(i) + a_{i,1} \\
  \vdots \\
  b_{i_d}(i) + a_{i,d}
\end{array} \right) \sum_{j=\dim(\mathcal{H}_0)}^{\dim(\mathcal{H}_{l_1} \cap \mathcal{H}_{l_2})} x_j b_j,
\]

where \( m_{\alpha_i} \) represents a monomial of \( \mathbb{Q}_{\max\{l_1, l_2\}} \) of multi-index degree \( \alpha_i \) such that the set \( \{ m_{\alpha_i} \}_{i=\dim(\mathcal{H}_0)}^{\dim(\mathcal{H}_{l_1} \cap \mathcal{H}_{l_2})} \) forms a base of \( \mathcal{H}_m = \mathcal{H}_{l_1} \cap \mathcal{H}_{l_2} \). Note that the coefficients \( \{ a_{ij} \} \) are defined coordinate-wise while the coefficients \( \{ b_{ij} \} \) are identical for all the components. The function \( r_j \) is therefore determined by

\[ \dim(\{ \{ a_{i,m} \}_{i=\dim(\mathcal{H}_0)}^{\dim(\mathcal{H}_{l_1} \cap \mathcal{H}_{l_2})}, \{ b_{i,j} \}_{i=\dim(\mathcal{H}_0)},\{ h_{ij} \}_{i=\dim(\mathcal{H}_0)}^{\dim(\mathcal{H}_{l_1} \cap \mathcal{H}_{l_2})} \} \]

coefficients.

As in all configurations the function \( r_j \) is determined only through its normal components, let us use the above expression to derive them more specifically. With the normal \( n_j = (n_{jx_1}, \ldots, n_{jx_d}) \) to the face \( f_j \), it comes

\[ r_j \cdot n_j \big|_{f_j} = \sum_{m=1}^{d} a_{0,m} n_{jx_m} + \sum_{i=\dim(\mathcal{H}_0)}^{\dim(\mathcal{H}_{l_1} \cap \mathcal{H}_{l_2})} \left( b_{i_m}(i) + a_{i,m} \right) n_{jx_m} m_{\alpha_i}(x) + \sum_{i=\dim(\mathcal{H}_0)}^{\dim(\mathcal{H}_{l_1} \cap \mathcal{H}_{l_2})} c_j b_{i_m} m_{\alpha_i}(x), \]

where \( c_j = x \cdot n_{j} \) is a constant term on the face \( f_j \). Reordering the terms, we end up with the formulation

\[ r_j \cdot n_j \big|_{f_j} = \sum_{m=1}^{d} \left( a_{0,m} + \sum_{i=\dim(\mathcal{H}_0)}^{\dim(\mathcal{H}_{l_1} \cap \mathcal{H}_{l_2})} a_{i,m} m_{\alpha_i}(x) \right) n_{jx_m} + \left( \sum_{i=\dim(\mathcal{H}_0)}^{\dim(\mathcal{H}_{l_1} \cap \mathcal{H}_{l_2})} b_{i_m}(i) m_{\alpha_i}(x) + \sum_{i=\dim(\mathcal{H}_0)}^{\dim(\mathcal{H}_{l_1} \cap \mathcal{H}_{l_2})} c_j b_{i_m} m_{\alpha_i}(x) \right). \]

The structure of the retrieved form makes clearly emerge the coefficients that should be used depending on the coordinate-wise behaviour of the polygon.

In addition, as all the coefficients determining \( r_j \) appear in this expression, using degrees of freedom defined only from the normal components of tested functions is admissible. Thus, the four configurations fitting this framework, we only have to make
sure that the set of extracted degrees of freedom are uniquely characterising each of the involved coefficients. To this aim, we explicit all the possible degrees of freedom when applied to the function $r_j$. For the sake of clarity, we denote by $\{\sigma_{M_i}\}$ the moments designed coordinate-wise, being of the form

$$\sigma_{M_i} : q \mapsto q(x_j) n_j x_i,$$

for some scalar polynomial $p_i$, and by $\{\sigma_{T_i}\}$ the ones acting globally, reading

$$\sigma_{T_i} : q \mapsto q(x_i) \cdot n$$

for some scalar polynomial $p_i$. Further, for convenience we denote by $\{\sigma_{V_i}\}$ the global degrees of freedom that comes into play to determining the coordinate-wise moments designed coordinate-wise, being of the form

$$\sigma_{V_i} : q \mapsto \int f_j q(x_i) \cdot p_i$$

for some scalar polynomial $p_i$. Further, for convenience we denote by $\{\sigma_{V_i}\}$ the global degrees of freedom that comes into play to determining the coordinate-wise moments designed coordinate-wise, being of the form

$$\sigma_{V_i} : q \mapsto \int f_j q(x_i) \cdot p_i$$

and setting the permutation operator directly on the multi-indices $\alpha_i$ instead of the coefficients $a_i$, we can rewrite the moments as follows.

$$\sigma_{M_{m,i}} : \{\{a_{i,m}\}, \{b_i\}\} \mapsto b_{0,m} \int_{f_j} n_{jx_m} p_i + \sum_{i=H_0}^{\dim(H_{I_i} \cap H_{I_2})} a_{i,m} \int_{f_j} m_{\alpha_i}(x) n_{jx_m} p_i$$

$$+ \sum_{i=H_0}^{\dim(H_{I_2})} b_i \int_{f_j} n_{jx_m} m_{\xi_m(\alpha_i)}(x) p_i + \sum_{i=\dim(H_{I_1} \cap H_{I_2})}^{\dim(H_{I_2})} b_i \int_{f_j} (x_m n_{jx_m} m_{\alpha_i}(x)) p_i$$

$$\sigma_{T_{i}} : \{\{a_{i,m}\}, \{b_i\}\} \mapsto \sum_{m=1}^{d} a_{0,m} \int_{f_j} n_{jx_m} p_i + \sum_{m=1}^{d} \sum_{i=H_0}^{\dim(H_{I_2})} a_{i,m} \int_{f_j} m_{\alpha_i}(x) n_{jx_m} p_i$$

$$+ \sum_{i=H_0}^{\dim(H_{I_2})} b_i \sum_{m=1}^{d} \left( \int_{f_j} n_{jx_m} m_{\xi_m(\alpha_i)}(x) p_i \right) + \sum_{i=\dim(H_{I_1} \cap H_{I_2})}^{\dim(H_{I_2})} b_i \int_{f_j} (c_m m_{\alpha_i}(x)) p_i$$

Thus, defining the component-wise parts of the global moments $\sigma_{T_i}$ by $\sigma_{T_{m,i}}(q) = \int_{f_j} n_{jx_m} q p_i$ such that $\sigma_{T_i} = \sum_{m=1}^{d} \sigma_{T_{m,i}}$, one can express any degrees of freedom of the four considered sets as

$$\sigma_{M_{m,i}} : q \mapsto a_{0,m} \sigma_{M_{m,i}}(1) + \sum_{i=H_0}^{\dim(H_{I_2} \cap H_{I_2})} a_{i,m} \sigma_{M_{m,i}}(m_{\alpha_i})$$

$$+ \sum_{i=H_0}^{\dim(H_{I_2})} b_i \sigma_{M_{m,i}}(m_{\xi_m(\alpha_i)}) + \sum_{i=\dim(H_{I_1} \cap H_{I_2})}^{\dim(H_{I_2})} b_i \sigma_{M_{m,i}}(x_m m_{\alpha_i})$$

and

$$\sigma_{T_i} : q \mapsto \sum_{m=1}^{d} a_{0,m} \sigma_{T_{m,i}}(1) + \sum_{m=1}^{d} \sum_{i=H_0}^{\dim(H_{I_2})} a_{i,m} \sigma_{T_{m,i}}(m_{\alpha_i})$$

$$+ \sum_{i=H_0}^{\dim(H_{I_2})} b_i \sum_{m=1}^{d} \sigma_{T_{m,i}}(m_{\xi_m(\alpha_i)}) + \sum_{i=\dim(H_{I_1} \cap H_{I_2})}^{\dim(H_{I_2})} a_{i} c_j \sigma_{T_{i}}(m_{\alpha_i}).$$

Note that in view of deriving the determination matrix, the last term can also be
decomposed as follows.

\[ \sum_{i=\dim(H_{l_1} \cap H_{l_2})}^{\dim(H_{l_2})} b_i c_j \sigma_{T_i}(m_{\alpha_i}) = \sum_{i=\dim(H_{l_1} \cap H_{l_2})}^{\dim(H_{l_1})} b_i \sum_{m=1}^{d} \sigma_{T_{i,m}}(x_m m_{\alpha_i}). \]

Similar relations for \( \sigma_V \) can be derived from the expression of \( \sigma_T \). Thus, we can rewrite the degrees of freedom as a dot product and derive the characterisation matrix \( \Sigma \)

\[ (\sigma_{M_{1,1}}, \ldots, \sigma_{M_{d,1}}, \sigma_{V_1}, \ldots, \sigma_{V_1}, \sigma_{T_1}, \ldots, \sigma_{T_1})^T = \Sigma(\{a_{i,m}\}_{i,m}, \{b_i\}_i)^T \]

whose shape is given in the following page. We now investigate its structure.

First of all, as the number of extracted degrees of freedom from the four sets \( IIa, IIb, Ia \) and \( Ib \) matches the number of coefficients determining \( r_j \), the matrix \( \Sigma \) is a square matrix.

Let us focus on the top two-by-two left blocks, surrounded in blue. They correspond to the coefficients that should be determined coordinate-wise. Thus, by construction, there are \( \dim(\{a_{i,m}\}_{i,m}) = d(l_1 + 1)^{d-1} \) columns. And by the definition of the configurations \( II \) and \( I \), we have \( \dim(\{\sigma_{M_{i,j}}\}_{i,j}) = d + d((l_1 + 1)^{d-1} - 1) = d(l_1 + 1)^{d-1}. \)

Therefore, this submatrix is a square matrix. Furthermore, each subblock corresponds to one member of the decomposition of \( q \) tested through coordinate-wise degrees of freedom whose kernels are built on the same monomial. Therefore, as the degrees of freedom \( \{\sigma_{M_{i,j}}\} \) consider one normal component only, the coefficients \( \{a_{i,m}\}_{i,m} \) for \( m \neq j \) are not involved, and the subblocks are diagonal. Thus, those submatrices are invertible and in particular their columns and rows are linearly independent.

On the other side of the matrix, the last bottom block surrounded in deep red matches the Raviart-Thomas moments tuning members of \( H_k(K)|_{\partial K} \) living exclusively in \( x_{B_{k,l}}f_j \). It is then a submatrix of the classical Raviart-Thomas’ one, and is thus invertible. In particular, its rows and columns are linearly independent.

The extended bottom right submatrix highlighted in dashed red corresponds to the previously described high-order submatrix of the Raviart-Thomas’s setting, enriched by the moments \( \{\sigma_V\} \) tuning the behaviour of members of \( H_k(K)|_{f_j} \) falling in the intersection \( H_{l_1} \cap H_{l_2}(f_j) \).

This matrix is equivalent to the full Raviart-Thomas setting. Indeed, even if the moments \( \{\sigma_V\} \) have to be slightly modified from the Raviart-Thomas setting in the configuration \( I \), this modification leaves the projection order unchanged and the integrand still belongs to \( H_k(K)|_{f_j} \). Therefore, the dashed line block is invertible and its columns and rows are linearly independent.

There is only left to show that there is no linear dependence between rows of different row blocks. As the degrees of freedom are linear forms, it is enough to show that the integrand of moments (or polynomials constructing the point-wise values) that involve the same monomial are linearly independent.

Indeed, being linear forms whose kernels are polynomials, the degrees of freedom can combine each other only if their integrand (\( q \) tested against the kernel) involve – up to constants – the same monomials. We then have to show that in both configurations, the rows involving terms whose projection onto the kernel can be expressed from a same monomial are linearly independent.
Corresponds to \( \{b_1, \ldots, b_d, \alpha\} \)

Single block of coordinate-wise constants

| \(\sigma_{M_1,1}(1)\) | 0 | \(\sigma_{M_1,1}(m_{\alpha_m})\) | 0 | \(\sigma_{M_1,1}(x_1m_{\alpha_m})\) |
| \(\sigma_{M_1,2}(1)\) | 0 | \(\sigma_{M_1,2}(m_{\alpha_m})\) | 0 | \(\sigma_{M_1,2}(x_1m_{\alpha_m})\) |
| \(\sigma_{M_2,1}(1)\) | 0 | \(\sigma_{M_2,1}(m_{\alpha_m})\) | 0 | \(\sigma_{M_2,1}(x_1m_{\alpha_m})\) |
| \(\sigma_{M_2,2}(1)\) | 0 | \(\sigma_{M_2,2}(m_{\alpha_m})\) | 0 | \(\sigma_{M_2,2}(x_1m_{\alpha_m})\) |
| \(\sigma_{V_1,1}(1)\) | \(\sigma_{V_1,1}(m_{\alpha_m})\) | \(\sigma_{V_1,1}(m_{\alpha_m})\) | \(\sum_{i=1}^{d} \sigma_{V_1,1}(x_im_{\alpha_m})\) |
| \(\sigma_{V_2,1}(1)\) | \(\sigma_{V_2,1}(m_{\alpha_m})\) | \(\sigma_{V_2,1}(m_{\alpha_m})\) | \(\sum_{i=1}^{d} \sigma_{V_2,1}(x_im_{\alpha_m})\) |
| \(\sigma_{T_{1,1}}(1)\) | \(\sigma_{T_{1,1}}(m_{\alpha_m})\) | \(\sigma_{T_{1,1}}(m_{\alpha_m})\) | \(\sum_{i=1}^{d} \sigma_{T_{1,1}}(x_im_{\alpha_m})\) |
| \(\sigma_{T_{1,M}}(1)\) | \(\sigma_{T_{1,M}}(m_{\alpha_m})\) | \(\sigma_{T_{1,M}}(m_{\alpha_m})\) | \(\sum_{i=1}^{d} \sigma_{T_{1,M}}(x_im_{\alpha_m})\) |

| \(\sigma_{M_1,1}(1)\) | 0 | \(\sigma_{M_1,1}(m_{\alpha_m})\) | 0 | \(\sigma_{M_1,1}(x_1m_{\alpha_m})\) |
| \(\sigma_{M_1,2}(1)\) | 0 | \(\sigma_{M_1,2}(m_{\alpha_m})\) | 0 | \(\sigma_{M_1,2}(x_1m_{\alpha_m})\) |
| \(\sigma_{M_2,1}(1)\) | 0 | \(\sigma_{M_2,1}(m_{\alpha_m})\) | 0 | \(\sigma_{M_2,1}(x_1m_{\alpha_m})\) |
| \(\sigma_{M_2,2}(1)\) | 0 | \(\sigma_{M_2,2}(m_{\alpha_m})\) | 0 | \(\sigma_{M_2,2}(x_1m_{\alpha_m})\) |
| \(\sigma_{V_1,1}(1)\) | \(\sigma_{V_1,1}(m_{\alpha_m})\) | \(\sigma_{V_1,1}(m_{\alpha_m})\) | \(\sum_{i=1}^{d} \sigma_{V_1,1}(x_im_{\alpha_m})\) |
| \(\sigma_{V_2,1}(1)\) | \(\sigma_{V_2,1}(m_{\alpha_m})\) | \(\sigma_{V_2,1}(m_{\alpha_m})\) | \(\sum_{i=1}^{d} \sigma_{V_2,1}(x_im_{\alpha_m})\) |
| \(\sigma_{T_{1,1}}(1)\) | \(\sigma_{T_{1,1}}(m_{\alpha_m})\) | \(\sigma_{T_{1,1}}(m_{\alpha_m})\) | \(\sum_{i=1}^{d} \sigma_{T_{1,1}}(x_im_{\alpha_m})\) |
| \(\sigma_{T_{1,M}}(1)\) | \(\sigma_{T_{1,M}}(m_{\alpha_m})\) | \(\sigma_{T_{1,M}}(m_{\alpha_m})\) | \(\sum_{i=1}^{d} \sigma_{T_{1,M}}(x_im_{\alpha_m})\) |

- C: Group of coordinate-wise moments associated to a same quantification process (same testing monomial in moments or evaluation at a same point-value).
- (*) Tuning globally coefficients that have repercussion on the coordinate-wise coefficients \(a_i\) (misc moments).
- As many rows as moments \(\sigma_{V_{\alpha}}\), with \(\{\dim(\mathcal{H}_i \setminus \mathcal{H}_0) + 1, \dim(\mathcal{H}_i)\}\)
In the configuration \( I \), this property comes automatically. Indeed, the only interaction between degrees of freedom having integrands sharing the same monomial order (and then possibly being based on the same monomial) is possible between (3.11b) and (3.11c) when \(|p_k| = l_1 + 1\). Indeed, by definition of \( \mathbb{H}_k(K) \), the polynomial \( p_k \cdot n \) in (3.11b) is only of order \( l_1 \). However, no combination of (3.11c) can form the moments (3.11b). Indeed, for any real coefficients \( c_i \) it holds

\[
\sum_{i=1}^{d} c_i q_{x_i} n_{x_i} x_i^{l_1+1} \neq q \cdot n p_k
\]

for any monomial \( p_k \) such that \(|p_k| = l_1 + 1\). Note that in the left hand side, all the \( c_i \) should be non-null to reconstruct the term \( p \cdot n \). However, doing so no factorisation by a single monomial such that

\[
\sum_{i=1}^{d} c_i q_{x_i} n_{x_i} x_i^{l_1+1} = \left( \sum_{i=1}^{d} c_i q_{x_i} n_{x_i} \right) p_k
\]

is possible. Thus, the designed moments are linearly independent, and no row combination can occur for any tested polynomial belonging to \( \mathbb{H}_k(K) |_{\partial K} \).

In the configuration \( II \), the integrands involving the same monomials are found in the definitions (A.3) and (A.1d) when \( l_1 = 0 \) (resp. (A.1a) and (A.1d)). As the problem arises on constants, linear combinations would be possible in a classical Raviart – Thomas setting as the terms \( p_{x_i} n_{x_i} \) and \( p_{y_i} n_{y_i} \) can be combined to form \( p \cdot n \). However, the use of the vector \( v \) not collinear to \( n \) in place of \( n \) for the lowest order moment in the equation (A.1d) makes the setting fulfil the assumptions 1. Therefore, those degrees of freedom are not linearly dependent.

All in all, for both configurations all the rows are linearly independent. As by construction we have as many relations as unknowns, the matrix is invertible. Thus, we get a null kernel, meaningly

\( IIa = 0 \) or \( IIb = 0 \) or \( Ia = 0 \) or \( Ib = 0 \) implies \( f |_{\partial K} = 0 \).

Step 3. Let us now consider the internal characterisation of functions living in \( \mathbb{H}_k(K) \). From the first point of the proof, it is enough to study the characterisation of \( g \) within the inner cell. By definition of \( \mathbb{H}_k(K) \), any function \( g \in \mathbb{H}_k(K) |_{K} \) can be decomposed over a set of Poisson’s solutions as follows.

\[
g = \sum_{i=1}^{\dim A_k} \sum_{j=1}^{\dim A_k} a_{i,j} e_j u_i + \sum_{i=1}^{\dim B_k} b_j x \tilde{u}_i
\]

Here, the vector \( e_j \) stands for \( e_j = (0, \cdots, 1, 0, \cdots, 0)^T \) where the 1 is in the \( j \)th position. The functions \( u_i \) and \( \tilde{u}_i \) represents the Poisson’s solutions of the problems

\[
\begin{cases}
\Delta u_i = p_i, & p_i \in \mathbb{Q}_{m_1}(K) \\
u_i |_{\partial K} = 0
\end{cases}
\quad
\begin{cases}
\Delta \tilde{u}_i = \tilde{p}_i, & \tilde{p}_i \in \mathbb{Q}_{[m_2]}(K) \\
\tilde{u}_i |_{\partial K} = 0
\end{cases}
\]

where \( \{p_i\}_i \) and \( \{\tilde{p}_i\}_i \) form respectively a basis of \( \mathbb{Q}_{m_1}(K) \) and \( \mathbb{Q}_{[m_2]}(K) \). In any presented definition of the degrees of freedom, the internal characterisation is done through moment - based degrees of freedom of the form

\[
\sigma_{f_k} : q \mapsto \int_K q \cdot p_k \ dx
\]

where the kernels \( p_k \in \mathcal{P}_k \) consist of linearly independent polynomials belonging to \( \mathbb{Q}_{\max\{m_1, m_2+1\}}(K) \), or of the solution of their corresponding problems of the form (B.4). Therefore, we can derive a characterisation matrix in the same spirit than in
the case of the normal characterisation.

(B.5)

\[
\begin{pmatrix}
  a_1 & \cdots & \cdots & a_m \\
  b_1 & \cdots & \cdots & b_p
\end{pmatrix}
\begin{pmatrix}
  c_1, u_{i,1} & \cdots & c_1, u_{i,m} \\
  \vdots & \ddots & \vdots \\
  c_m, u_{i,1} & \cdots & c_m, u_{i,m}
\end{pmatrix}
= \begin{pmatrix}
  f(x, u_{i,1}) & \cdots & f(x, u_{i,m}) \\
  \vdots & \ddots & \vdots \\
  f(x, u_{i,1}) & \cdots & f(x, u_{i,m})
\end{pmatrix}
\begin{pmatrix}
  x u_{i,1} & \cdots & x u_{i,m} \\
  \vdots & \ddots & \vdots \\
  x u_{i,1} & \cdots & x u_{i,m}
\end{pmatrix}
\begin{pmatrix}
  p_1 \\
  \vdots \\
  p_p
\end{pmatrix}
\]

Let us consider the case where \( \mathcal{P}_k \) forms a polynomial projection space. There, none of the \( p_k \in \mathcal{P}_k \) is the zero function. In the same time, the functions \( \{u_i\}_i, \{\tilde{u}_i\}_i \) are linearly independent, and being solutions to some Poisson’s problem with non-zero second member, they are by construction not identically vanishing on \( K \). Indeed, even when \( m_2 < m_1 \) where second members of the problems (B.4) lives both in \( Q_{m_1} \) and \( Q_{m_2} \), it holds \( \Delta(\tilde{u}_i) = 2\nabla \cdot \tilde{u}_i + \Delta(\tilde{u}_i) \). Thus, it is impossible to combine linearly the function \( \tilde{u}_i \) with functions of the set \( \{u_i\}_i \).

Furthermore, the degrees of the polynomials belonging to the space \( \mathcal{P}_k \) are lower or equal than the highest degree of the second members of the Poisson’s problem defining the space \( \mathbb{H}_k(K) \). Thus, every projection of function of \( \mathbb{H}_k(K) \) onto the space \( \mathcal{P}_k \) is not null. And as the internal moments are linear forms, any linear combination of those moments at fixed \( p_k \) could have its integrand factorised by the kernel \( p_k \) for any \( p_k \in \mathcal{P}_k \), transferring the linear independency of the set \( \{u_i\}_i, \{\tilde{u}_i\}_i \) to the terms \( \{f(u_i, p_k)\}_i \) for any fixed \( p_k \in \mathcal{P}_k \).

Lastly, as the space \( \mathcal{P}_k \) contains only linearly independent functions the previous argument can be repeated for each row of the matrix defined in (B.5). And as by construction the number of internal degrees of freedom matches the dimension of the space \( \mathbb{H}_k(K) \), the linear independence of functions of \( \mathcal{P}_k \) combined with the linear independence of the tested functions transfer automatically to the moments tested against a basis of \( \mathbb{H}_k(K) \). Thus, the internal submatrix is invertible. The same reasoning can be applied when \( \mathcal{P}_k \) is built from the Poisson’s solutions themselves, as the projections of functions would decompose the functions directly.

Merging the above points together, we get that \( \text{IIa} = 0 \) or \( \text{Iib} = 0 \) or \( \text{Ia} = 0 \) or \( \text{Ib} = 0 \) implies \( f|_K = 0 \) and \( \int_K g \cdot p_k x = 0 \) for all \( p_k \in \mathcal{P}_k \) implies \( g|_K = 0 \). From this, we get that for \( g \in \mathbb{H}_k(K) \) \( \{\text{IIa} = 0 \) or \( \text{Iib} = 0 \) or \( \text{Ia} = 0 \) or \( \text{Ib} = 0 \) and \( \int_K q \cdot p_k x = 0 \) for all \( p_k \in \mathcal{P}_k \) implies \( q = 0 \).

**Proof of the Propositions 3.6 and 3.7.** The proof of the proposition 3.6 is a straightforward generalisation of the one presented for the four examples \( \text{IIa}, \text{Iib}, \text{Ia} \) and \( \text{Ib} \) in the lemma B.1. The only change lies in the extraction of the degrees of freedom, which impacts the matrix only on the top left two by two blocks describing the coordinate-wise behaviours. As the extraction fulfils the assumptions 1, the same arguments can be applied and the conclusion follows.

In particular, by this admissibility criterion there cannot be more than \( d + 1 \) polynomials reducing to the same moments’ kernel or to an equivalent point-value quantifier. Thus, as we have \( d + 1 \) coordinate-wise moments to tune per decomposed monomial, there is no over-determination at a fixed polynomial degree. The constraint on the extraction of degrees of freedom ensures the non over-determination overall. Further, the linear independence of the sub-matrix’s columns is ensured as those polynomials cannot be linearly dependent. Thus, by linearity of the degrees of freedom, the independence of the kernels transfers to the moments and there is no row dependency. The submatrix block corresponding to any specific order is therefore
invertible, and the same conclusion as in the proof B.1 follows.

The proposition 3.6 thus holds by the number of degrees of freedom, matching the dimension of the space $\mathbb{H}_k(K)$. Indeed, as by the proposition 3.7 the kernel of the linear operator defined by the set of degrees of freedom has a null kernel providing their unisolvence when enclosed within the space $\mathbb{H}_k(K)$.

Proof of the relation (3.4). As the two natural subspaces are in direct sum, recalling the block construction of $\mathbb{H}_k(K)$ allows the dimension of the space $\mathbb{H}_k(K)$ to be easily derived. We can simply add the dimension of the two main subspaces $(A_k)^d$ and $x B_k$ to retrieve the dimension of $\mathbb{H}_k(K)$. Let us derive their respective dimensions.

First, we compute the dimension of $A_k$. In the way presented in [15], one can get it by using the superposition theorem. Indeed, for any second member belonging to $Q_{m_1}$ and any boundary function $p_k \mathbb{1}_f \in L^2(K)$, there exists a unique solution to the Poisson’s problems defining $A_k$ (see e.g. [4]). Thus, reading out the structure of the set $A_k$ implies the following relation.

$$
\dim A_k = \dim \mathcal{H}_1(\partial K) + \dim Q_{m_1}(K) = n(l_1 + 1)^{d-1} + (m_1 + 1)^d.
$$

Therefore, as $(A_k)^d$ is a simple Cartesian product of $d$ copies of $A_k$, we have immediately

$$
\dim A_k = d(\dim A_k) = d(n(l_1 + 1)^{d-1} + (m_1 + 1)^d).
$$

In the exact same way, we retrieve the dimension of $B_k$ by

$$
\dim B_k = \dim \mathcal{H}_2(\partial K) + \dim Q_{m_2}(K) = n(l_2 + 1)^{d-1} + (m_2 + 1)^d - m_2^d.
$$

Last, we recall that the space $x B_k$ simply corresponds to an identical $d$-duplication of the space $B_k$ where each coordinate has been multiplied by the corresponding spatial variable. Thus, there is no liberty adjunction during its construction, and the dimension of $x B_k$ equals the one of $B_k$. By combining this, we finally get

$$
\dim \mathbb{H}_k(K) = d \dim A_k + \dim B_k = d(n(l_1 + 1)^{d-1} + (m_1 + 1)^d) + n(l_2 + 1)^{d-1} + (m_2 + 1)^d - m_2^d.
$$

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