ON THE UNIRATIONALITY OF SUPERSINGULAR K3 SURFACES

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ABSTRACT. We prove that supersingular K3 surfaces over algebraically closed fields of characteristic at least 5 are unirational, following a simplified form of Liedtke's strategy.

CONTENTS

1. Introduction 1
2. The existing literature 2
3. Notation and assumptions 2
4. Artin-Tate families 3
5. Families of elliptic torsors of order $p$ 3
6. Unirationality of supersingular K3s 4
References 14

1. INTRODUCTION

The main purpose of this paper is to prove the following conjecture of Artin, which is modestly stated as an almost-hidden question in the last sentence of Section 2 of the seminal paper [2].

Conjecture 1.1 (Artin). Any supersingular K3 surface over an algebraically closed field is unirational.

Our strategy is a modification of the strategy taken in a recent preprint of Liedtke (see Section 2). Fix an algebraic closure $k_\infty$ of $k((t))$.

(1) By algebraizing formal Brauer elements and using a relative form of the Artin-Tate isomorphism, one can produce families of supersingular K3 surfaces that move between Artin-invariant strata. In particular, fixing a Jacobian elliptic fibration $X \to \mathbb{P}^1$ on a single supersingular K3 surface, the family deforms as an elliptic pencil over $k[[t]]$ in such a way that the fiber $X_\infty \to \mathbb{P}^1_{k_\infty}$ over $k_\infty$ is a non-trivial torsor under the base change of the Jacobian of $X \to \mathbb{P}^1$ to $k_\infty$. (It is clear from the footnote on page 552 of [2] that Artin was well aware of this construction.)
(2) The geometric generic fibration $X_\infty \to P_{k_\infty}^1$ has a multisection that is purely inseparable over $P_{k_\infty}^1$. This relates the special fiber and the geometric generic fiber, up to inseparable extensions. Thus, if the special fiber is unirational, so is the generic fiber.

(3) One can apply this construction at enough generic points of the moduli space of supersingular K3 surfaces to account for everything.

An outline of the paper: in Section 3 we fix a few conventions; in Section 4 we recall some results on constructing families of elliptic K3 surfaces parametrized by Brauer classes; in Section 5 we analyze the generic fibers of those families and give a correct proof that the torsors admit inseparable splittings; in Section 6 we use this analysis to prove that supersingular K3 surfaces are unirational.

2. The existing literature

There is a long history of proving various cases of this conjecture under various conditions [5-7, 17, 18, 21-23]. As far as I can tell, Liedtke’s preprint [11] is the first to claim a proof in full generality (for $p \geq 5$). Liedtke uses a more elaborate form of the strategy taken here that appears to yield stronger results; his approach to families of torsors and curves in the moduli space is quite similar to that taken in [9]. I have an extremely difficult time following many of the details in [11], especially those related to Step 2 of the strategy.

This paper represents my attempt to write a complete, self-contained, and efficient proof of Artin’s conjecture following this strategy. The fundamental construction used here in Step 1 is described in [9] and will not be repeated in detail in this paper; the manuscript [9] will be made public shortly. Most of the work in the present manuscript is in the proof of Step 2 (which we prove here using purity of the branch locus) and the careful use of 1-parameter deformations to achieve Step 3, given the very subtle nature of the Ogus space, relative crystals, and the crystalline period map.

3. Notation and assumptions

Throughout this paper, we fix a choice of algebraic closure $k((t)) \hookrightarrow k_\infty$.

We will assume that $p \geq 5$ in order to use Ogus’s papers [14, 15]. Note that the Tate conjecture has been proven for K3 surfaces in these characteristics [3, 12, 16], so that every supersingular K3 surface we consider over an algebraically closed field has Picard number 22 (allowing us to use the results in [15], for example).

In particular, while one can deduce that any unirational K3 must have Picard number 22, we cannot leverage this fact here, as we are using that fact to begin with!

1Also available in video form from a 2012 Banff lecture at http://videos.birs.ca/2012/12w5027/201203271601-Lieblich.mp4
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4. ARTIN-TATE FAMILIES

In this section, we recall a few results about families of K3 surfaces parametrized by elements of the formal Brauer group of a supersingular K3 surface. This can be viewed as a continuous form of the Artin-Tate isomorphism. A careful and detailed write-up of these results (and significantly more general results) will be found in [9].

The main results relevant to this paper are the following. Fix a supersingular K3 surface (in the sense of Artin) $X$ over $k$.

**Proposition 4.1.** There is a class $\alpha \in Br(X \otimes_k \mathbb{k}[t])$ such that

1. there is an Azumaya algebra of degree $p$ with class $\alpha$;
2. $\alpha|_{t=0} = 0 \in Br(X)$;
3. the restriction of $\alpha$ to the formal scheme $X \times_k \text{Spf} \ k[\![t]\!]$ gives an isomorphism $\text{Spf} \ k[\![t]\!] \sim \widehat{Br}(X)$.

The idea of the proof is to start with a certain Azumaya algebra of degree $p$ and deform it over the formal Brauer group of $X$ as the universal Brauer class deforms. This becomes a calculation in the deformation theory of twisted vector bundles with trivial determinant on a K3 surface, which is formally smooth. Algebraizing the Azumaya algebra gives the first two parts. In fact, more is true: one can make a class over $\mathbb{A}^1$ whose restriction to $k[\![t]\!]$ is $\alpha$ above, but this is a more involved argument and is unnecessary for us here.

The second main result is the relative Artin-Tate isomorphism. Fix an elliptic fibration $X \to P^1$.

**Proposition 4.2.** Given $\alpha$ as in Proposition 4.1, there is a morphism $X \to P^1 \times \text{Spec} \ k[\![t]\!]$ such that the special fiber is isomorphic to $X \to P^1$ and the geometric generic fiber $X_{k_{\infty}} \to P^1_{k_{\infty}}$ is an étale form of $X_{k_{\infty}} \to P^1_{k_{\infty}}$ over $P^1_{k_{\infty}}$ that corresponds to $\alpha_{k_{\infty}}$ via the Artin-Tate isomorphism (Section 3 of [25]).

As one can imagine, the proof is essentially that of Artin and Tate: over the locus of $P^1$ where $X$ has smooth fibers, one can use the same Leray spectral sequence argument. The difficult lies in filling in the singular fibers. An argument that proceeds using the theory of stable sheaves is carefully written in [9].

We will call the families arising as in Proposition 4.2 Artin-Tate families in this paper.
5. Families of elliptic torsors of order $p$

In this section we study what happens to families of genus 1 curves that arise as the fibers of Artin-Tate families. As we will see, deformations of the trivial torsor always possess purely inseparable sections of degree $p$, and this will be useful when we study the unirationality of supersingular K3 surfaces in Section 6.

Fix an algebraically closed field $k$ of characteristic $p$ and a finitely generated regular extension field $L/k$ (e.g., the function field of a geometrically integral scheme over $k$).

Given a field extension $k \subset M$, let $\text{Dvr}^M_k$ denote the category of dvrs over $k$ with residue field $M$. A morphism in $\text{Dvr}^M_k$ is a commutative diagram

\[
\begin{array}{c}
R \\
\downarrow \\
M \\
\uparrow \\
R'
\end{array}
\]

in which the horizontal arrow is necessarily a local homomorphism respecting the identifications of the residue fields with $M$.

Define a functor $S : \text{Dvr}^k_k \to \text{Dvr}^L_k$ by

\[ S(R) := (L \otimes_k R)_t. \]

That $S(R)$ is in fact a dvr with residue field $L$ follows from the fact that $L \otimes_k R$ is a domain ($L$ is regular over $k$), and the fact that $R$ has residue field $k$ (so that $S(R)/tS(R) \cong L$, making $t$ generate a maximal ideal).

Fix an elliptic curve $E$ over $L$ with identity section $0$.

**Definition 5.1.** A family of $E$-torsors over $L$ parametrized by $R$ is an $E \otimes_k R$-torsor over $L \otimes_k R$.

Write $C_0 = C \otimes_R R/tR$. This section is primarily concerned with invertible sheaves on families of $E$-torsors parametrized by $R$, with implications for specialization of divisors and the existence of inseparable splittings.

**Lemma 5.2.** Suppose $\mathcal{L}$ is an invertible sheaf on a family $C$ of $E$-torsors parametrized by $R$ such that $H^1(C_0, \mathcal{L}|_{C_0}) = 0$. Let $f : C_{S(R)} \to \text{Spec} S(R)$ be the restriction of the projection. Then

1. the sheaf $f_* \mathcal{L}_{S(R)}$ is locally free and of formation compatible with arbitrary base change on $S$;
2. any section of $\mathcal{L}|_{C_0}$ lifts to a section of $\mathcal{L}_{S(R)}$.

**Proof.** This is a standard application of cohomology and base change; it is a special case of Corollary 2, Section II.5 of [13]. □
Notation 5.3. Write $K$ for the fraction field of $R$ and choose an algebraic closure $\overline{K}$. Let $\eta$ be the generic point of $\text{Spec} \, S(R)$ and $\eta_\infty$ the generic point of $\text{Spec} \, S(R) \otimes_R \overline{K} = \text{Spec} \, L \otimes_k \overline{K}$.

Lemma 5.4. Suppose $L$ has transcendence degree 1 over $k$. Let $C \to \text{Spec} \, S(R)$ be a smooth proper relative curve whose Jacobian fibration $\text{Jac}(C) \to \text{Spec} \, S(R)$ has the form $J \otimes_L S(R)$ for a single abelian variety $J$ over $L$. Given an invertible sheaf $\lambda$ on $C_0$ of degree 0, there is a morphism $R \to R'$ in $\text{Dvr}_k^1$ and a lift $\Lambda$ of $\lambda$ to an invertible sheaf on $\mathcal{E}_{S(R')}$. 

Proof. The invertible sheaf $\lambda$ gives rise to a point $[\lambda] \in \text{Jac}(C)(L)$. Since the Jacobian is constant, this lifts to a point $P \in \text{Jac}(C)(S(R))$, taking some generic value $P^0 \in \text{Jac}(C)(\eta)$. By assumption, $\eta_\infty$ has transcendence degree 1 over an algebraically closed field, hence $\text{Br}(\eta_\infty) = 0$ by Tsen’s theorem. On the other hand, the obstruction to lifting $P^0|_{\eta_\infty}$ to an invertible sheaf lies in that Brauer group. It follows that there is a finite extension $R \to R'$ in $\text{Dvr}_k^1$ such that $P^0|_{\kappa(S(R'))}$ comes from an invertible sheaf $\mathcal{L} \in \text{Jac}(C)(\kappa(S(R')))$. 

Taking a reflexive hull of $\mathcal{L}$ over $C_0 \otimes \mathbb{R}$ and using the fact that the Jacobian is separated, we see that there is an invertible sheaf $\Lambda \in \text{Pic}(C_0 \otimes \mathbb{R})$ whose restriction $\Lambda|_{\kappa_0}$ gives the same point in $\text{Jac}(C)(\kappa(S(R')))$. Since the Jacobian stack is a $\mathbb{G}_m$-gerbe over the Jacobian scheme, we see that $\lambda$ and $\Lambda_0$ must differ by tensoring with an invertible sheaf pulled back from $\text{Spec} \, L$. By Hilbert’s Theorem 90, we have that $\Lambda_0 \cong \lambda$, as desired. □

Corollary 5.5. Suppose $C$ is a family of $E$-torsors parametrized by $R$. Assume that $L$ has transcendence degree 1 over $k$. If $C_0$ has a section $s$ and the fiber $C_{\eta_\infty}$ has index $p$ over $\kappa(\eta_\infty)$ then there is a morphism $R \to R'$ in $\text{Dvr}_k^1$ such that the invertible sheaf $\mathcal{E}_{C_0}(ps)$ lifts to an invertible sheaf on $C_{S(R')}$. 

Proof. Given an invertible sheaf $\mathcal{M}_\infty$ of degree $p$ on $C_{\eta_\infty}$, its closure gives an invertible sheaf on $C_L \otimes \overline{K}$ with degree $p$ on each fiber. Writing $\overline{K}$ as a union of finite extensions of $K$, standard finite-presentation methods show that there is a finite extension $K'$ of $K$ such that $\mathcal{M}_\infty$ is the base change of an invertible sheaf $\mathcal{M}$ on $C_L \otimes K'$. 

Letting $R'$ be a localization of the normalization of $R$ in $K'$ that dominates $R$ (and recalling that the residue field of $R$ is the algebraically closed field $k$), we get a morphism $R \to R'$ in $\text{Dvr}_k^1$ such that $C_{S(R)}$ admits an invertible sheaf $\mathcal{M}$ of degree $p$ on the fibers. The sheaf $\mathcal{M}_{C_0}(ps)$ has degree 0, hence lifts to some $\mathcal{L}$ on $C_{S(R')}$ for some finite $R \to R'$ in $\text{Dvr}_k^1$ by Lemma[5.4]. 

Replacing $\mathcal{M}$ by $\mathcal{M} \otimes \mathcal{L}'$ yields an invertible sheaf lifting $\mathcal{E}_{C_0}(ps)$, as desired. □

If $R$ is Henselian, we can define a specialization map 

$$\text{Pic}(C_{\eta_\infty}) \to \text{Pic}(C_0)$$

as follows: for any finite extension $K'$ of $K$, let $A$ be the normalization of $R$ in $K'$. Since $R$ is a Henselian dvr with residue field $k$, $R'$ is a Henselian dvr with residue field $k$. 
Taking closure of divisors defines an isomorphism
\[
\text{Pic}(C_{k(S(R'))}) \to \text{Pic}(C_{S(R')}),
\]
and the restriction map defines a morphism
\[
\text{Pic}(C_{S(R')}) \to \text{Pic}(C_0).
\]
These maps are compatible with finite extensions $K' \subset K''$. Taking the colimit over all finite subextensions of $K \subset \overline{K}$ thus gives a well-defined map.

We retain Notation 5.3 in the following.

**Proposition 5.6.** Fix $R \in \text{Dvr}_k$. Suppose $C \to \text{Spec}(R \otimes_k L)$ is a family of $E$-torsors over $L$ parametrized by $R$ such that

(a) the curve $C_{\eta_\infty}$ has index $p$ over $\kappa(\eta_\infty)$ (and thus has order $p$ in $H^1(\eta_\infty, E_{\eta_\infty})$);

(b) there is an isomorphism of $E$-torsors
\[
\tau : E \sim \to C_0;
\]

Then

(1) there is a point $P$ on $C_{\eta_\infty}$ that is purely inseparable of degree $p$ over $\eta_\infty$;

(2) if $R$ is Henselian and $L$ has transcendence degree 1 over $k$ then the specialization map
\[
\text{Pic}(C_{\eta_\infty}) \to \text{Pic}(C_0)
\]
has image of index $p$.

**Proof.** By Corollary 5.5, after possibly replacing $R$ by a finite extension $R'$ we may assume that there is an invertible sheaf $\mathcal{L}$ on $C$ lifting $\mathcal{O}_{C_0}(p\tau(0))$. Since $C$ has relative genus 1 and positive degree divisors are unobstructed on genus 1 curves, there is a universal divisor
\[
\mathcal{D} \subset C \times_{\text{Spec} S(R)} \text{P}^{p-1}_{\text{Spec} S(R)},
\]
where $\text{P}^{p-1}$ is given by a choice of basis for the free $S(R)$-module $\Gamma(C_{S(R)}, \mathcal{L})$. (In other words, we are looking at the relative linear system.) The divisor $\mathcal{D}$ is finite and flat over $\text{P}^{p-1}_{S}$.

Since positive degree divisors move in basepoint free linear systems on genus 1 curves, we see that $\mathcal{D} \to \text{P}^{p-1}_{S(R)}$ is generically separable over the special fiber $\text{P}^{p-1}_{L}$, and that $\mathcal{D}$ is regular. By purity of the branch locus, there is a divisor $\mathcal{R} \subset \text{P}^{p-1}_{S(R)}$ over which $\mathcal{D}$ ramifies. Via the natural projection
\[
\text{P}^{p-1}_{S(R)} \sim \to \text{P}^{p-1}_{R} \times_{\text{Spec} k} \text{Spec} L \to \text{Spec} R
\]
the divisor $p0$ gives a point $r$ of $R$ in the fiber over $k$. Moreover, since $\mathcal{D}$ is generically reduced in the special fiber, the divisor $\mathcal{R}$ does not contain the entire special fiber. Since $R$ is a dvr, it follows that there is a point $\tilde{r}$ of $\mathcal{R}$ supported over $\text{Spec} K \subset \text{Spec} R$.

The point $\tilde{r}$ gives rise by base change (and appropriate choices) to a point of $\text{P}^{p-1}$ over $\overline{K} \otimes_k L$ that specializes to a point over $\eta_\infty$. This is a divisor $D \subset C_{\eta_\infty}$ of degree $p$ that is
ramified over $\eta_\infty$. On the other hand, $C_{\eta_\infty}$ has index $p$ over $\kappa(\eta_\infty)$, so $C_{\eta_\infty}$ cannot contain any divisor of order prime to $p$. It follows that $D$ consists of a single closed point with residue field of degree $p$ over $\kappa(\eta_\infty)$. Since the divisor is ramified, we conclude that $\kappa(D)$ is purely inseparable of degree $p$ over $\kappa(\eta_\infty)$, establishing the first statement.

To prove the second one, note that the hypotheses imply that the torsor $C_{\eta_\infty}$ has order $p$ in $H^1(\eta_\infty, E_{\eta_\infty})$ and possesses a multisection $\Xi \subset C_{\eta_\infty}$ of degree $p$ over $\eta_\infty$. Consider the diagram

$$
\begin{array}{c}
\text{Pic}^0(C_{\eta_\infty}) \rightarrow \text{Pic}^0(C_0) \\
\downarrow \quad \downarrow \\
\text{Pic}(C_{\eta_\infty}) \rightarrow \text{Pic}(C_0) \\
\downarrow \quad \downarrow \\
\mathbb{Z} \rightarrow \mathbb{Z} \\
\downarrow \quad \downarrow \\
0 \rightarrow 0
\end{array}
$$

Since $C$ is a family of $E$-torsors (for the constant family with fiber $E$), the top horizontal arrow is surjective by Lemma 5.4. The middle arrow is of interest to us. The bottom arrow has image $p\mathbb{Z}$, since its image is contains $p\mathbb{Z}$ and cannot contain 1 (as $C_{\eta_\infty}$ has index $p$). The desired result follows from the Snake Lemma. □

There are some immediate corollaries of Proposition 5.6 for Artin-Tate families. In the following, suppose $R$ is Henselian and fix an Artin-Tate family

$$X \rightarrow \mathbb{P}^1_R \rightarrow \text{Spec } R$$

and an algebraic closure $\kappa(R) \subset \bar{\kappa}$. Call the special fiber $\pi_0 : X_0 \rightarrow \mathbb{P}^1$ and the geometric generic fiber $\pi_\infty : X_\infty \rightarrow \mathbb{P}^1_{\bar{\kappa}}$.

**Lemma 5.7.** Suppose $X \rightarrow \mathbb{P}^1$ is a non-Jacobian elliptic fibration on a supersingular K3 surface such that there is a purely inseparable multisection $\Sigma \subset X$. Then

1. the Jacobian fibration $J(X)$ is a supersingular K3 surface over strictly smaller Artin invariant;
2. if $J(X)$ is unirational then so is $X$.

**Proof.** First, since $\Sigma$ is purely inseparable over $\mathbb{P}^1$, it has degree $p^d$ for some $d$. In particular, since $X \rightarrow \mathbb{P}^1$ is non-Jacobian (hence cannot have index 1), the index $i$ of the generic fiber $X_\eta$ must be a power of $p$, say $p^b$. We claim that the Artin invariant of $J(X)$ is $\sigma_0(X) - b$; in other words, we claim that the discriminant of $\text{Pic}(X)$ is $i^2 \text{disc } \text{Pic}(J(X))$. The analogous statement is well-known over $\mathbb{C}$ (due to Keum) and the generalization
to arbitrary base fields is hinted at in Remark 4.7 of [4]. A proof will be written out in [9].

An inseparable multisection gives a diagram

\[
\begin{array}{ccc}
X \\
\downarrow \\
P^1 & \rightarrow & \rightarrow \\
\end{array}
\]

in which the horizontal arrow is a power of the Frobenius. In particular, the function fields \( K \) and \( K_0 \) of \( X \) and \( J(X) \), respectively, become isomorphic after adjoining the \( p^b \)th root of a coordinate on \( P^1 \) to each, so that there are inclusions

\[
K \subset K_0(t^{1/p^b}) \subset K_0^{1/p^b}.
\]

If \( J(X) \) is unirational, there is an inclusion \( K_0 \subset \kappa(x,y) \), and the final statement follows from the fact that \( P^2 \) is defined over \( F_p \), so it is isomorphic to its Frobenius twist.

\[\square\]

**Corollary 5.8.** Given an Artin-Tate family \( X \to P^1_R \) such that \( \pi_0 \) has a section there is a diagram

\[
\begin{array}{ccc}
X_\infty \\
\downarrow \\
P^1_\pi & \rightarrow & \rightarrow \\
\end{array}
\]

in which the horizontal arrow is the relative Frobenius of \( P^1_\pi \). In particular,

1. \( X_\infty \) and \( (X_0)_\pi \) are isomorphic after pullback by the relative Frobenius of \( P^1_\pi \) and thus there is an inclusion of function fields

\[
\kappa(X_\infty) \subset \kappa((X_0)_\pi)^{1/p}
\]

over \( \kappa \);

2. if \( X_0 \) is unirational than \( X_\infty \) is unirational.

**Proof.** By Proposition 5.6 applied with \( L = k(t) \), the non-Jacobian pencil \( X_\infty \to P^1_\pi \) admits a purely inseparable multisection of degree \( p \). (In this case, we know explicitly that the Jacobian is \( X_0 \to P^1 \), which has a smaller Artin invariant, without needing to invoke a general result on Jacobians.) The rest follows from Lemma 5.7. \[\square\]

### 6. Unirationality of Supersingular K3s

In this section, we prove that K3 surfaces are unirational using the following inductive procedure.

Let \( \text{YES}(s) \) denote the statement “Every supersingular K3 surface of Artin invariant at most \( s \) is unirational.” Given a supersingular K3 surface \( X \), let \( \text{HELS}(X) \) denote the statement “\( X \) admits a non-Jacobian elliptic fibration \( X \to P^1 \) that has a purely
inseparable multisection” and let HELS(s) denote the statement “HELS(X) holds for every supersingular K3 surface X of Artin invariant s.”

**Proposition 6.1.** For any s between 1 and 9, we have that HELS(s + 1) and YES(s) implies YES(s + 1).

**Lemma 6.2.** HELS(s) holds for all s between 2 and 10.

We defer the proofs of these two results to the end of this section.

**Corollary 6.3.** Every supersingular K3 surface is unirational.

**Proof.** Any supersingular K3 surface of Artin invariant 1 is isomorphic to the Kummer surface of $E \times E$, where $E$ is a supersingular elliptic curve, and we know that this is unirational by Theorem 1.1 of [23]. Proposition 6.1 provides the necessary induction. □

Before proving Proposition 6.1, we prove a few results on Artin-Tate families.

**Proposition 6.4.** Let $\eta$ denote the generic point of $\mathbb{P}^1_{k_{\infty}}$. Given an Artin-Tate family $X \to \mathbb{P}^1 \times \text{Spec} k[t]$ whose special fiber $X \to \mathbb{P}^1_k$ has a section $\Sigma \subset X$ (i.e., is a Jacobian fibration), the generic fiber $X_\eta$ has index p over $\eta$.

**Proof.** Let $E/k(t)$ denote the Jacobian of $X_{k(t)}$; a choice of section identifies $E$ and $X_{k(t)}$. The Artin-Tate isomorphism makes $X_\eta$ a torsor under $E_\eta$ with order $p$ in $H^1(\eta, E_\eta)$. (By standard period-index results for genus 1 curves Theorem 8 of [8], we know that the index of $X_\eta$ divides $p^2$, even without knowing it is the fiber of a K3 surface).

Artin proved (Corollary 1.3 of [2]) that the specialization map

$$\text{Pic}(X_{k_{\infty}}) \to \text{Pic}(X)$$

has $p$-elementary cokernel. In particular, the invertible sheaf $\mathcal{O}_X(p\Sigma)$ lifts to $X_{k_{\infty}}$. Restricting to $X_\eta$, we see that the index divides $p$.

Since $\alpha_{k_{\infty}} \neq 0$, the $E$-torsor $X_\eta$ cannot be trivial. It follows that the index is exactly $p$, as desired. □

In order to avoid getting mired too deeply in subtleties of the theory of moduli of supersingular K3 surfaces, we abstract a few statements about generic surfaces. Recall that Ogus defined a period space $M_T$ for each isomorphism class $T$ of supersingular K3 lattices. The space has the following properties:

1. $M_T$ is smooth and projective over $F_p$ of dimension $\sigma_0(T) - 1$ and the algebraic closure of $F_p$ in $\Gamma(M_T, \mathcal{O})$ is isomorphic to $F_{p^2}$, so that $M_T$ admits an $F_{p^2}$-structure of which it is geometrically irreducible.

2. If $U$ is the set of points of $M_T$ parametrizing rigidified crystals of Artin invariant exactly $\sigma_0$ then $M_T \setminus U \cong M_{T'}$, where $T'$ is the lattice of Artin invariant $\sigma_0 - 1$. 

Here $\sigma_0(T)$ is the Artin invariant of the lattice $T$, characterized as follows: the discriminant of $T$ is $-p^{2\sigma_0}$.

Given a supersingular K3 surface and an isometric embedding $\varphi : T \hookrightarrow \text{Pic}(X)$, there is an associated point $[(X, \varphi)] \in M_T$. More generally, a family of $T$-marked supersingular K3 surfaces over a smooth base $B$ gives rise to a morphism $B \to M_T$. (The smoothness is required due to subtleties in the theory of relative crystals.)

**Definition 6.5.** A supersingular K3 surface $X$ over $k$ will be called *generic* if it admits a marking $\varphi : T \to \text{Pic}(X)$ such that the associated point $[(X, \varphi)] \in M_T$ is a generic point of $M_T$.

The basic lemma is that being generic is a geometric property of $X$, depending only upon the isomorphism class of its Néron-Severi group.

**Lemma 6.6.** If $X$ is generic with respect to one marking $\varphi$, then it is generic with respect to any marking by the same lattice.

*Proof.* In order for $X$ to be generic, $\varphi$ must be an isomorphism. Given another marking $\psi : T \to \text{Pic}(X)$ there is thus an automorphism $\gamma : T \to T$ such that $\psi = \varphi \gamma$. The automorphism $\gamma$ acts on $M_T$ as an algebraic automorphism, so it preserves the generic points. $\square$

**Lemma 6.7.** If $X_1$ and $X_2$ are generic of the same Artin invariant over the algebraically closed field $k$ then there is a commutative diagram

$$
\begin{array}{ccc}
X_1 & \longrightarrow & X_2 \\
\downarrow & & \downarrow \\
\text{Spec } k & \longrightarrow & \text{Spec } k
\end{array}
$$

in which the horizontal arrows are isomorphisms of schemes and the vertical arrows are the structure maps. In particular, if $X_1$ is unirational then so is $X_2$.

*Proof.* Let $T$ be the supersingular K3 lattice of Artin invariant $\sigma_0(X_1) = \sigma_0(X_2)$. Choose markings $\varphi : T \xrightarrow{\sim} \text{Pic}(X_i)$. Let $K$ be the function field of the integral $\mathbb{F}_p$-scheme $M_T$. By the genericity assumption, the marked surfaces $(X_1, \varphi_1)$ and $(X_2, \varphi_2)$ give two embeddings $\varepsilon_i : K \hookrightarrow k$, $i = 1, 2$. By standard field theory, there is a field automorphism $\alpha : k \to k$ such that $\varepsilon_2 = \alpha \varepsilon_1$. Since $M_T(k)$ parametrizes marked K3 crystals $k$, it follows from Theorem I of [15] that changing the $k$-structure on $X_1$ via $\alpha$ yields $X_2$ (as it yields the same marked crystal, hence the same crystal), up to isomorphism. This gives the desired diagram.

The last statement follows from the fact that $\mathbb{P}^2$ is defined over $\mathbb{F}_p$, so that changing the $k$-structure leaves the variety invariant up to isomorphism. $\square$

Next, we remark on deformations of generic surfaces.
Lemma 6.8. Given a supersingular K3 surface $X$ with a polarization $L$ of degree prime to $p$, any deformation $(X, L) \rightarrow B$ of the polarized surface $(X, L)$ over a smooth scheme $B$ that is versal in a neighborhood of $X$ contains a generic supersingular K3 surface of each Artin invariant between $\sigma_0(X)$ and 10.

Note that such deformations exist by Theorem 1.6 of [1] (i.e., the moduli space of polarized K3 surfaces is algebraic) and Theorem 3.4 of [14] (the intersection form on the Picard group is not divisible by $p$).

Proof. By assumption, completing $B$ at the point corresponding to $(X, L)$ gives a versal deformation of $(X, L)$. Let $S \subset B$ be the locus over which the $X$ has supersingular fibers. As explained in Section 7 of [2], $S$ is closed of dimension 9. By Theorem 5.6 of [14], each irreducible component of the closed subscheme

$$\text{Spec } \hat{\mathcal{O}}_{S, b} \subset \text{Spec } \hat{\mathcal{O}}_{B, b}$$

is formally smooth. Passing to an étale neighborhood of $b \in B$, we may assume that $S \subset B$ is a union of smooth irreducible closed subschemes $\bigcup \Sigma_i$. Since $\Sigma_i$ is smooth, there is a map $\chi_i : \Sigma_i \rightarrow M_T$, and Ogus checks in Theorem 5.6 of [14] that $\chi_i$ is formally étale at $b$. It thus follows that $\Sigma_i$ contains a generic K3 surface of Artin invariant 10. Since the Artin invariant stratification is a stratification by divisors (Remark 4.8 of [14] or Section 7 of [2], being careful about the duality assumptions), the smaller Artin invariants must all generically occur, as desired.

□

Lemma 6.9. If $\mathcal{X} \rightarrow \text{Spec } k[[t]]$ is a family of supersingular K3 surfaces such that the specialization maps

$$\text{Pic}(\mathcal{X}) \rightarrow \text{Pic}(\mathcal{X}_{k_{\infty}})$$

and

$$\text{Pic}(\mathcal{X}) \rightarrow \text{Pic}(\mathcal{X}_0)$$

are isomorphisms then HELS($\mathcal{X}_{k_{\infty}}$) implies HELS($\mathcal{X}_0$).

Proof. Let $\mathcal{X}_{k_{\infty}} \rightarrow \mathbb{P}^1_{k_{\infty}}$ be a non-Jacobian elliptic fibration with an inseparable multisection $\Sigma \subset \mathcal{X}_{k_{\infty}}$. We assume that $\Sigma$ is an integral curve (that is dominated by some relative Frobenius of $\mathbb{P}^1$, but this is unimportant). By the specialization assumption, after possibly making a finite extension of $k[[t]] \hookrightarrow R$, the fibration extends to an elliptic fibration

$$\mathcal{X}_R \rightarrow \mathbb{P}^1_R,$$

and closure of $\Sigma$ gives an inseparable multisection $\Sigma_0 \subset \mathcal{X}_0$. If $\Sigma_0$ were non-integral (e.g., a $p$th power in the Picard group) then $\Sigma$ would be non-integral by the assumption on Picard groups. Thus, $\Sigma_0$ is integral, giving a purely inseparable multisection of $\mathcal{X}_0 \rightarrow \mathbb{P}^1$. Similarly, by Lemma 2.3 of [10], any section of $\mathcal{X}_0 \rightarrow \mathbb{P}^1$ would deform to a section of $\mathcal{X}_R \rightarrow \mathbb{P}^1_R$, contradicting the non-Jacobian assumption on $\mathcal{X}_{k_{\infty}}$. This proves that HELS($\mathcal{X}_0$) holds, as desired. □
Lemma 6.10. If a generic K3 surface of Artin invariant $\sigma_0$ satisfies HELS then any supersingular K3 surface over Artin invariant $\sigma_0$ satisfies HELS.

Proof. First, note that HELS($X$) holds if and only if HELS($X_K$) holds for some (in fact, any) algebraically closed extension field $k \subset K$. Fix a supersingular K3 surface $X$ of Artin invariant $\sigma_0$ and a prime-to-$p$ polarization $L$. By Lemma 6.8, the versal deformation of $(X, L)$ contains a generic supersingular K3 surface $X_{gen}$ of Artin invariant $\sigma_0$. At the expense of possibly enlarging $k$, we can dominate the specialization $X_{gen} \to X$ by a dvr and get a family $X \to \mathbb{P}^1_{k[t]}$ with special fiber $X_K$ and generic fiber a generic surface with Artin invariant $\sigma_0$. Making another finite extension if necessary, we can assume that the specialization maps $\text{Pic}(X) \to \text{Pic}(X_\infty)$ and $\text{Pic}(X) \to \text{Pic}(X_K)$ are isomorphisms. By Lemma 6.9, we conclude that HELS($X_K$) holds, whence HELS($X$) holds, as desired.

Lemma 6.11. If $X$ is a family of supersingular K3 surfaces over $k[[t]]$ such that $X_0$ is generic of Artin invariant $\sigma_0$ and $X_\infty$ has Artin invariant at least $\sigma_0$, then $X_\infty$ is generic.

Proof. By Popescu’s theorem (Theorem 1.1 of [24]), there is an essentially smooth local $k$-algebra $A$ with a local homomorphism $A \to k[[t]]$ and a family $X$ of supersingular K3 surfaces $X_A$ such that $X_A \otimes_A k[[t]] \cong X$. In particular, the geometric generic fiber of $X$ has Artin invariant at least $\sigma_0$. Moreover, after replacing $A$ with a finite extension, we may assume that there is a global marking

$$T \hookrightarrow \text{Pic}(X_A),$$

where $T$ is the K3 lattice of Artin invariant $\sigma_0(X_\infty) \geq \sigma_0$ (by assumption). This gives rise to a supersingular K3 crystal with $T$-structure on $(\text{Spec} A/W)$, yielding a map $\text{Spec} A \to M_T$ by Theorem 5.3 of [14]. By assumption, the closed point of $A$ maps to the generic point of a subscheme of $M_T$ of codimension at most 1. It thus follows that the generic point of $A$ maps to the generic point of $M_T$, as desired.

Corollary 6.12. Given an Artin-Tate family $X \to \mathbb{P}^1 \times \text{Spec} k[[t]]$ with Jacobian special fiber, we have that

$$\sigma_0(X_\infty) = \sigma_0(X_0) + 1.$$ 

Moreover, if $X_0$ is generic with Artin invariant $\alpha$ over $k$, then $X_\infty$ is generic with Artin invariant $\alpha + 1$ over $k_\infty$ and HELS($X_\infty$) holds.

Proof. Combining Proposition 6.4 and Proposition 5.6(2), we see that the specialization map

$$\text{Pic}(X_\infty) \to \text{Pic}(X_0)$$
has cyclic $p$-torsion cokernel. Indeed, restricting to generic fibers gives a diagram

\[
\begin{align*}
0 & \quad 0 \\
& \quad \downarrow \\
L_\infty & \quad L_0 \\
& \quad \downarrow \\
\text{Pic}(X_\infty) & \quad \text{Pic}(X_0) \\
& \quad \downarrow \\
\text{Pic}(C_\eta) & \quad \text{Pic}(C_0) \\
& \quad \downarrow \\
0 & \quad 0
\end{align*}
\]

with exact columns. Proposition 5.6(2) tells us that the bottom horizontal arrow has cokernel isomorphic to $\mathbb{Z}/p\mathbb{Z}$. On the other hand, cohomology and base change tells us that the kernel of the restriction map is the sublattice generated by the components of the singular fibers. Since $X_\infty$ and $X_0$ have singular fibers isomorphic over $k_\infty$, it follows that the top horizontal arrow is an isomorphism. The conclusion follows from the Snake Lemma (which also tells us that the bottom horizontal map is injective!).

The Artin invariant statement follows immediately from the claim about the index of the specialization map, by the definition of the Artin invariant.

It remains to show that HELS($X_{k_\infty}$) holds. But this is precisely Corollary 5.8. □

**Lemma 6.13.** Any supersingular K3 surface of Artin invariant strictly less than 10 admits a Jacobian elliptic fibration.

*Proof.* As in Section 12 of [20], it is enough to know that the hyperbolic lattice embeds in the Néron-Severi group of any supersingular K3 surface with Artin invariant less than 10. Thus, it is enough to show the same thing for Artin invariant 9, as every lower Artin invariant gives an overlattice of the Artin invariant 9 lattice, and then it suffices to show it for a single surface of Artin invariant 9. On page 1480 of [19], one finds a classification of $p$-elementary even lattices (for odd $p$). In particular, the K3 lattice of Artin invariant 9 can be written as an orthogonal direct sum

\[L = U \oplus H_p(-1) \oplus (I(-p)^{16})_*,\]

with the following notation.

1. $U$ is the hyperbolic plane.
2. $H_p$ is even, positive definite, has rank 4 and discriminant $p^2$, and $H_p(-1)$ has bilinear form $x \cdot y = -(x \cdot H_p y)$.
3. $I^{16}$ is the lattice with diagonal form $x_1^2 + \cdots + x_{16}^2$. 
(4) For a lattice $Q$, the notation $Q(a)$ means that the bilinear form on $Q$ is multiplied by $a$.

(5) For a lattice $Q$ with diagonal form $a_1x_1^2 + \cdots + a_nx_n^2$, the lattice $Q_*$ is given by the sublattice of $Q \otimes Q$ generated by

(a) vectors $\langle x_1, \ldots, x_n \rangle \in Q$ with $\sum x_i$ even;
(b) the vector $\langle 1/2, \ldots, 1/2 \rangle$.

In particular, we have $Q(a)_* = Q_*(a)$, and $(I^{16})_*$ is even and unimodular of rank 16. It follows that $(I(-p)^{16})_*$ is even and has discriminant $p^{16}$.

The orthogonal direct sum multiplies discriminants, and the discriminant of $U$ is $-1$, so we see that this lattice indeed is even, $p$-elementary, of rank 22, and has discriminant $-p^{18}$. Since the lattice is unique given its rank and discriminant (the main theorem of Section 1 of [19]), this must describe the K3 lattice of Artin invariant 9 up to isomorphism.

A final note: as pointed out to me by Schütt, the paper [19] contains several typographical errors on page 1480. In particular, the congruence conditions are flawed – the first displayed congruence should have 2 and not 3 on the right, and the third displayed congruence has a 1 in the denominator instead of a 2.

**Proof of Lemma 6.2** By Lemma 6.10, it is enough to check this for generic surfaces. By Corollary 6.12, Artin-Tate families produce generic surfaces satisfying HECS, and by Lemma 6.13 for each $\sigma_0 > 1$ there is an Artin-Tate family with generic fiber of Artin invariant $\sigma_0$. The result follows.

**Proof of Proposition 6.1** Assume that YES($s$) holds and $1 < s < 10$. Let $X$ be a supersingular K3 surface of Artin invariant $s + 1$. By Lemma 6.2, there is a non-Jacobian elliptic fibration $X \to \mathbb{P}^1$ with an inseparable multisection of degree $p$. In particular, by Lemma 5.7, if the Jacobian fibration $J(X) \to \mathbb{P}^1$ is unirational then so is $X$. But $J(X)$ is a supersingular K3 surface of strictly smaller Artin invariant, whence it is unirational by YES($s$). This proves YES($s + 1$).

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ON THE UNIRATIONALITY OF SUPERSINGULAR K3 SURFACES

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