ON FORMAL SCHUBERT POLYNOMIALS

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Abstract. Present notes can be viewed as an attempt to extend the notion of Schubert/Grothendieck polynomial of Lascoux-Schützenberger to the context of an arbitrary formal group law and of an arbitrary oriented cohomology theory.

Let \( F \in R[[x, y]] \) be a commutative one-dimensional formal group law over a commutative unital ring \( R \) and let \( h \) be an algebraic oriented cohomology theory with the coefficient ring \( h(pt) = R \). According to Levine-Morel [LM07] there is a 1-1 correspondence between such \( F \)'s and universal \( h \)'s. Indeed, given an oriented theory \( h \) the respective formal group law is determined by the Quillen formula for the first characteristic classes in the theory \( h \) of the tensor product of line bundles

\[
c_1^h(L_1 \otimes L_2) = F(c_1^h(L_1), c_1^h(L_2))
\]

and given a formal group law \( F \) over \( R \) one obtains the respective universal theory \( h \) by tensoring with the algebraic cobordism \( \Omega \), i.e.

\[
h(-) := \Omega(-) \otimes_{\Omega(pt)} R,
\]

where \( \Omega(pt) \to R \) is obtained by specializing coefficients of the universal formal group law.

For example, additive formal group law \( F_a(x, y) = x + y \) corresponds to the Chow theory \( h = CH \), multiplicative \( F_m(x, y) = x + y - xy \) to the Grothendieck \( h = K_0 \) and the universal one \( F_u \) to the algebraic cobordism \( h = \Omega \). Observe that in the first two cases \( h(pt) = \mathbb{Z} \) and in the last case the coefficient ring is the Lazard ring which is infinitely generated over \( \mathbb{Z} \).

Let \( G \) be a split semisimple linear group over a field \( k \) containing a split maximal torus \( T \). Following [CPZ, §2] consider the formal group algebra

\[
R[[M]]_F := R[[x_\omega]]_{\omega \in M}/(x_0, x_{\omega+\omega'} - F(x_\omega, x_{\omega'})),
\]

where \( M \) is the weight lattice, together with the augmentation map \( \epsilon: R[[M]]_F \to R, x_\omega \to 0 \). From the geometric point of view \( R[[M]]_F \) models the completion of the equivariant cohomology \( h_T(pt) \) and the map \( \epsilon \) is the forgetful map. Algebraically, \( R[[M]]_F \) is non-canonically isomorphic to the ring of formal power series in \( rk(M) \) variables.

Consider the algebra of formal divided difference operators \( D(M)_F \) on \( R[[M]]_F \) and let \( \epsilon D(M)_F := Hom_R(\epsilon \circ D(M)_F, R) \) denote the dual of the algebra of augmented operators. The main result of [CPZ, §13] says that if \( h \) is a (weakly birationally invariant) oriented cohomology theory corresponding to \( F \) (e.g. \( h = CH \),...
$K_0$ or $\Omega$), then there is an $R$-algebra isomorphism
\begin{equation}
\epsilon \mathcal{D}(M)^*_F \simeq \mathfrak{h}(X),
\end{equation}
where $X$ is the variety of Borel subgroups of $G$. Moreover, it was shown that the $R$-basis of $\mathfrak{h}(X)$ consisting of classes of Bott-Samelson resolutions of Schubert varieties corresponds to the basis of $\epsilon \mathcal{D}(M)^*_F$ constructed as follows:

First, for each element of the Weyl group $w \in W$ one chooses a reduced decomposition $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ into a product of simple reflections and denotes by $I_w = (i_1, \ldots, i_r)$ the respective reduced word. Then one shows that the $R$-linear operators $\epsilon C^F_{I_w}$ defined by composing $\epsilon$ with the composite of the formal divided difference operators $C^F_{I_w} = C^F_{i_1} \circ \cdots \circ C^F_{i_r}$ form a basis of $\epsilon \mathcal{D}(M)^*_F$ \cite[Prop. 5.4]{CPZ}.

Finally, the elements $A^*_{I_w}(z_0)$ give the desired basis, where $z_0$ is the element of $\epsilon \mathcal{D}(M)^*_F$ dual to $\epsilon C^F_{I_w}(u_0)$ for some specially chosen $u_0 \in (\ker \epsilon)^{\dim X}$, $w_0$ is the element of maximal length and $A^*$ is the operator on $\epsilon \mathcal{D}(M)^*_F$ given by composition on the right with $C^F_i$ \cite[Thm. 13.13]{CPZ}.

One of the major difficulties in extending the Schubert calculus to such generalized theories $\mathfrak{h}$ (and, hence, the Schubert polynomials) is the fact that all mentioned bases are non-canonical, i.e. depend on choices of reduced decompositions. Moreover, according to \cite{BE90} they are canonical if and only if $F$ has the form $F(u,v) = u + v - \beta uv$ for some $\beta \in R$. In other words, the Bott-Samelson resolutions of Schubert varieties provide a canonical basis of $\mathfrak{h}(X)$ only for Chow groups ($\beta = 0$), Grothendieck $K_0$ ($\beta$ is invertible) and connective $K$-theory ($\beta \neq 0$ is non-invertible).

In the present notes we try to overcome this difficulty and, hence, provide a canonical basis of $\mathfrak{h}(X)$ by either

1. averaging over all reduced decompositions, i.e. over all classes of Bott-Samelson resolutions; or
2. exploiting the Kazhdan-Lusztig theory in the case of a special elliptic formal group law.

Observe that approach (1) works only after inverting the Hurwitz numbers, e.g. over $\mathbb{Q}$, but over $\mathbb{Q}$ all formal group laws become isomorphic. Therefore, one may suspect that we simply reduce to the known cases of additive or multiplicative formal group laws. But this isomorphism does not preserve the formal difference operators as well as many other structures, so this is not the case.

Approach (2) seems to be even more interesting as it gives a canonical basis integrally. However, we don’t know how to extend it to other examples of formal group laws.

\section{Averaging over reduced expressions}

Consider the evaluation map $\epsilon^F : R[[M]]_F \to \epsilon \mathcal{D}(M)^*_F$ of \cite[§6]{CPZ}. Observe that on the level of cohomology (after identifying with $\mathfrak{h}(X)$) it coincides with the characteristic map induced by $\omega \mapsto \epsilon^F_\omega(L(\omega))$. In the case of the additive or multiplicative formal group law it gives the characteristic map described by Demazure in \cite{De74}, \cite{De73}. Moreover, according to \cite[Thm. 6.9]{CPZ} if the Grothendieck torsion index $\tau$ of $G$ is invertible (this always holds for Dynkin types A and C), then the kernel of $\epsilon^F$ is the ideal $I^W_\tau$ generated by augmented $W$-invariant elements, and we
obtain an $R$-algebra isomorphism

\[(2)\quad R[[M]]_F/\mathcal{I}_F^W \simeq \epsilon \mathcal{D}(M)_F^*,\]

which in view of the results of [HMSZ] and [CZZ] relate the invariant theory of $W$ with an $F$-version of the Hecke ring of Kostant-Kumar [KK86], [KK90], [Ku02] and Bressler-Evens [BE87]. In general, though the kernel of $\epsilon^F$ always contains the invariant ideal $\mathcal{I}_F^W$, the induced map $\epsilon^F : R[[M]]_F/\mathcal{I}_F^W \to \epsilon \mathcal{D}(M)_F^*$ is neither injective nor surjective.

Observe also that if $\tau$ is invertible, then we can identify the basis $A_{I_{w}}(z_0)$ of $\epsilon \mathcal{D}(M)_F^*$ with the basis $C_{I_{w}^{\text{rev}}}(u_0)$ of $R[[M]]_F/\mathcal{I}_F^W$, where $u_0 = [pt]$ corresponds to the class of a point (see [CPZ, Thm. 6.7]). The latter suggest to define for each $w \in W$ the following element in $R_Q[[M]]_F/\mathcal{I}_F^W$:

\[(3)\quad P^F_w := \frac{1}{\text{red}(w)} \sum_{I_w \in \text{red}(w)} C_{I_w^{\text{rev}}}([pt]),\]

where the sum is taken over the set $\text{red}(w)$ of all reduced words of $w$.

The results of [HMSZ] and [CZZ] then imply that $\{P^F_w\}_{w \in W}$ is the desired canonical basis over $Q$:

**Theorem 1.** The elements $\{P^F_w\}_{w \in W}$ form a $R_Q = R \otimes_Z Q$-basis of $R[[M]]_F/\mathcal{I}_F^W$ and, hence, of $h(X)_Q$.

**Proof.** By [HMSZ, Prop. 5.8] and [CZZ, Lem. 7.1] the difference $\left(C_i C_j\right)^{m_{ij}} - \left(C_j C_i\right)^{m_{ij}}$ is a linear combination of terms of length strictly smaller than $2m_{ij}$ (here $m_{ij}$ is the exponent in the respective Coxeter relation). Therefore, each $P^F_w$ can be written as $P^F_w = (C_{I_w} + (\text{products of smaller length}))([pt])$. So the matrix expressing $P^F_w$ in terms of the usual basis $\{C_{I_w}^{\text{rev}}\}_{w \in W}$ corresponding to a fixed choice of reduced decompositions $\{I_w\}_{w \in W}$ is upper-triangular with 1’s on the main diagonal. \hfill $\square$

Consider a root system of Dynkin type $A_n$. Let $\{e_1, \ldots, e_{n+1}\}$ be the standard basis with $\alpha_i = e_i - e_{i+1}$ the set of simple roots. Consider a ring homomorphism

$$R[t_1, t_2, \ldots, t_{n+1}] \to R[[\Lambda]]_F, \text{ given by } t_i \mapsto x_{-e_i}.$$ 

It is $S_{n+1}$-equivariant, therefore, it induces a map on quotients

\[(4)\quad R[t_1, t_2, \ldots, t_{n+1}]/I \to R[[\Lambda]]_F/\mathcal{I}_F^W,\]

where $I$ is the ideal generated by symmetric functions. By Hornbostel-Kiritchenko [HK, Thm. 2.6] this is an $R$-algebra isomorphism.

**Definition 2.** We define an $F$-Schubert polynomial $\pi^F_w$ to be the image of $P^F_w$ in the quotient $R[t_1, \ldots, t_{n+1}]/I$ via the isomorphism (4).

If $F$ has the form $F(u, v) = u + v - \beta uv$ for some $\beta \in R$, i.e. exactly the formal group law for which the respective composites $C_{I_w}$ do not depend on choices of reduced words of $w$, then for $\beta = 0$ (resp. $\beta = 1$) $\pi^F_w$ coincide with the respective Schubert (resp. Grothendieck) polynomials of Lascoux-Schützenberger (e.g. see [Fo94], [FK]) and for arbitrary $\beta$ we obtain polynomials studied in [FK] and [Hu12].

Observe that under this isomorphism the class of the point $[pt]$ corresponds to the class of the polynomial $t_1^{n+1}t_2^{n-1} \ldots t_n$ and the formal divided divided difference operator
(5)

where \( \sigma_i \) swaps \( t_i \) and \( t_{i+1} \) and \( \rho_i \) is the formal power series given by \( t_i - F t_{i+1} = F(t_i, t(t_{i+1})) \), where \( t(x) \) is the formal inverse series of \( x \).

By definition, operators \( C_i \) are \( \mathcal{R}[|M|]\) linear (here \( W_i = (s_i) \) ) and satisfy [CPZ, Prop. 3.13]:

\[
C_i(uf) = C_i(u)v + s_i(u)C_i(v) - \kappa_i s_i(u)v \quad \text{and} \quad C_i(1) = \kappa_i,
\]

where \( \kappa_i = \frac{1}{\rho_i} + \frac{1}{\tau(\rho_i)} \in \mathcal{R}[|M|]. \) We also have \( C_i(t_j f) = t_j C_i(f) \) for \( j \neq i, i+1 \) and

\[
C_i(t_i) = \frac{\tau_i}{\tau_i - \rho_i \tau_{i+1}} + \frac{\tau_{i+1}}{\tau_{i+1} - \rho_i \tau_i} = \frac{t_i - \rho_i t_{i+1} + \rho_i t_{i+1}}{t_i - \rho_i t_i} + \frac{t_{i+1} - \rho_i t_{i+1} + \rho_i t_i}{t_{i+1} - \rho_i t_{i+1}} = F(\rho_i, t_{i+1}) - t_{i+1} + t_i + 1 \kappa_i.
\]

Using these formulas one can compute the polynomials \( \pi_w^F \).

Our goal now is (using these formulas) to express each polynomial \( \pi_w^F \) as a linear combination of sub-monomials of \( t_1^{n-1} \ldots t_n \) with coefficients from \( \mathcal{R} \).

**Example 3.** We can write an arbitrary formal group law \( F \) as

\[
F(x, y) = x + y - xyg(x, y).
\]

This implies that

\[
\frac{1}{x} + \frac{1}{y} = g(z, u(z)).
\]

If \( F(x, y) = x + y + a_{11}xy + a_{12}xy(x + y) + O(4) \), then

\[
g(x, y) = -a_{11} - a_{12} + a_{13}(x^2 + y^2) - a_{22}xy + O(3)
\]

and

\[
u(x) = -x + a_{11}x^2 - a_{11}x^3 + O(4).
\]

So, after substituting, we obtain

\[
x - F y = x + (-y + a_{11}y^2 - a_{11}^2 y^4) + a_{11}x(-y + a_{11}y^2) + a_{12}x(-y)(x - y) + O(4) =
\]

\[
= (x - y) - a_{11}y(x - y) + a_{11}^2 y^2(x - y) - a_{12}xy(x - y) + O(4),
\]

and, hence,

\[
(x - F y) + \frac{(y - F y)}{x} = a_{11}(x - y)^2 - a_{11}^2 (x + y)(x - y)^2 + O(4),
\]

\[
(x - F y)^2 + \frac{(y - F y)^2}{x^2} = 2(x - y)^2 - 2a_{11}(x + y)(x - y)^2 + O(4),
\]

\[
(x - F y)(y - F x) = -(x - y)^2 + a_{11}(x + y)(y - x)^2 + O(4).
\]

Combining these together we get

\[
g(x - F y, y - F x) = -a_{11} - a_{12}a_{11}(x - y)^2 - 2a_{13}(x - y)^2 + a_{22}(x - y)^2 + O(3) =
\]

\[
- a_{11} - (a_{11}a_{12} + 2a_{13} - a_{22})(x - y)^2 + O(3).
\]

We then obtain

\[
r(x, y) = \frac{F(x,y) - y}{x} = 1 + a_{11}y + a_{12}y(x + y) + a_{13}y(x^2 + y^2) + a_{22}xy^2 + O(4).
\]

Hence,

\[
r(x-y, y) = 1 + a_{11}y + a_{12}y(x + a_{11}y(x - y)) + a_{13}y((x - y)^2 + y^2) + a_{22}(x - y)^2 + O(4) =
\]

\[
= 1 + a_{11}y + a_{12}xy + (a_{22} - a_{12}a_{11})(x - y)y^2 + a_{13}y((x - y)^2 + y^2) + O(4),
\]
and, therefore,
\[ r(x - Fy, y) + yg(x - Fy, y - Fx) = 1 + a_{12}xy + (a_{22} - a_{12}a_{11})xy(x - y) + a_{13}xy(2y - x) + O(4). \]
Observe that there is a relation \(2(a_{22} - a_{11}a_{12}) = 3a_{13}\) in the Lazard ring (the only relation in degree 4) which gives
\[ 2[(a_{22} - a_{12}a_{11})xy(x - y) + a_{13}xy(2y - x)] = 2a_{13}xy(x + y). \]
In particular, if 2 is invertible and \(x = t_i, y = t_{i+1}\) for the type \(A_2\), then \(t_i t_{i+1}(t_i + t_{i+1})\) is in the ideal \(I\) generated by symmetric functions in \(t_1, t_2, t_3\), meaning that for \(i = 1, 2\)
\[ C_i(t_i) = \frac{t_i}{t_i - F t_{i+1}} + \frac{t_{i+1}}{t_{i+1} - F t_i} = r(t_i - F t_{i+1}, t_{i+1}) + t_{i+1} \kappa_i = 1 + a_{12} t_i t_{i+1}. \]

**Example 4.** Using the formulas above we obtain the following expressions for \(\pi_{w_0}^F\) in the \(A_2\)-case for an arbitrary \(F\) (this agrees with computations at the end of [HR] and [CPZ])
\[ \pi_1^F = C_1([pt]) = t_1 t_2, \quad \pi_2^F = C_2([pt]) = t_1^2, \]
Indeed,
\[ C_1(t_1^2 t_2) = \frac{t_1^2 t_2}{t_1 - F t_2} + \frac{t_1 t_2^2}{t_2 - F t_1} = t_1 t_2 C_1(t_1) = t_1 t_2 \]
and
\[ C_2(t_1^2 t_2) = \frac{t_1^2 t_2}{t_2 - F t_1} + \frac{t_1 t_2^2}{t_1 - F t_2} = t_1^2 \frac{t_1}{t_2 - F t_1} + \frac{t_2}{t_1 - F t_2} = t_1^2 \left(1 + a_{12} t_2 t_3\right) = t_1^2. \]
\[ \pi_{21}^F = C_{12}([pt]) = t_1 + t_2 + a_{11} \pi_1^F, \quad \pi_{12}^F = C_{21}([pt]) = C_2(t_1 t_2) = t_1. \]
Indeed,
\[ C_1(t_1^2) = C_1(t_1) t_1 + t_2 C_1(t_1) - \kappa_1 t_1 t_2 = (1 + a_{12} t_1 t_2)(t_1 + t_2) + a_{11} t_1 t_2 = t_1 + t_2 + a_{11} t_1 t_2 \]
and
\[ C_2(t_1 t_2) = t_1 C_2(t_2) = t_1 \left(1 + a_{12} t_2 t_3\right) = t_1 \]
And for the element of maximal length \(w_0 = (121) = (212)\) we obtain
\[ C_{212}([pt]) = 1 + a_{12} \pi_{12}^F, \quad C_{121}([pt]) = C_1(t_1) = 1 + a_{12} t_1 t_2 = 1 + a_{12} \pi_1^F. \]
Indeed,
\[ C_2(t_1 + t_2 + a_{11} t_1 t_2) = t_1 C_2(1) + C_2(t_2) + a_{11} t_1 C_2(t_2) = t_1 \left(-a_{11} + (a_{11} a_{12} + 2a_{12} - a_{22})(t_2 - t_3)^2 + 1 + a_{12} t_2 t_3 + a_{11} t_1 (1 + a_{12} t_2 t_3) = 1 + a_{12} t_2 t_3 - \frac{1}{2} a_{13} t_1 (t_2 - t_3)^2 + 1 + a_{12} t_2 t_3 = 1 + a_{12} t_1^2 \right) \]
as \(t_1^2 \equiv t_2 t_3\) and \(t_1 (t_2 - t_3)^2\) is in the ideal \(\left(t_1 (t_2 - t_3)^2 \equiv t_1 (t_2^2 + t_3^2) \equiv t_1^3 \equiv 0\right)\).
Therefore,
\[ \pi_{w_0}^F = 1 + \frac{1}{a_{12}} (t_1^2 + t_1 t_2). \]

Observe that the twisted braid relation (which leads to the dependence on choices) of [HMSZ, Prop. 5.8] then coincides with
\[ C_{121} - a_{12} C_1 = C_{212} - a_{12} C_2. \]
2. A special elliptic formal group law and the Kazhdan-Lusztig basis

Consider an elliptic curve given in Tate coordinates by

\[(1 - \mu_1 x - \mu_2 x^2)y = x^3.\]

The corresponding formal group law over the coefficient ring \(R = \mathbb{Z}[\mu_1, \mu_2]\) is given by (e.g. [BB10, Example 63]),

\[F(x, y) := \frac{x + y - \mu_1 xy}{1 + \mu_2 xy}\]

and will be called a special elliptic formal group law. Observe that by definition, we have

\[F(x, y) = x + y - xy(\mu_1 + \mu_2 F(x, y)), \text{ so } a_{11} = -\mu_1 \text{ and } a_{12} = -\mu_2.\]

By [HMSZ, Theorem 5.14] for the type \(A_n\) the algebra \(D(M)_F\) is generated by operators \(C_i, i \in \ldots, n\), and multiplications by elements \(u \in R[[M]]_F\) subject to the following relations:

(a) \(C_i^2 = \mu_1 C_i\)
(b) \(C_{ij} = C_{ji}\) for \(|i - j| > 1\),
(c) \(C_{ij} C_{ji} = \mu_2 (C_j - C_i)\) for \(|i - j| = 1\) and
(d) \(C_i u = s_i(u) C_i + \mu_1 u - C_i(u)\).

Recall that the Iwahori-Hecke algebra \(H\) of the symmetric group \(S_{n+1}\) is (after the respective normalization) an \(\mathbb{Z}[t, t^{-1}]\)-algebra with generators \(T_i, i \in \ldots, n\), subject to the following relations:

(A) \((T_i - t^{-1})(T_i + t) = 0\) or, equivalently, \(T_i^2 = (t^{-1} - t)T_i + 1\),
(B) \(T_{ij} = T_{ji}\) for \(|i - j| > 1\) and
(C) \(T_{ij} = T_{ji}\) for \(|i - j| = 1\).

Observe that \(T_i\)'s appearing in the classical definition of the Iwahori-Hecke algebra in [CG10, Def. 7.1.1] correspond to \(tT_i\) in our notation, where \(t = q^{-1/2}\).

Following [HMSZ, Def. 5.3] let \(D_F\) denote the \(R\)-subalgebra of \(D(M)_F\) generated by the elements \(C_i, i = \ldots, n\), only. In [HMSZ, Prop. 6.1] it was shown that for \(F = F_n\) (resp. \(F = F_m\)) \(D_F\) is isomorphic to the nil-Hecke algebra (resp. the 0-Hecke algebra) of Kostant-Kumar.

Comparing the relations for \(D_F\) and \(H\) we see that for \(R = \mathbb{Z}[t, t^{-1}][\frac{1}{t + t^{-1}}]\), \(\mu_1 = 1\) and \(\mu_2 = -\frac{1}{(t + t^{-1})}\) there is an isomorphism of \(R\)-algebras (see [CZZ1] and [LNZ] for the case of an arbitrary root system)

\[H[\frac{1}{t + t^{-1}}] \simeq D_F \quad \text{given on generators by } T_i \mapsto (t + t^{-1})C_i - t, i = \ldots, n.\]

By definition of (6) the involution on \(H\) (sending \(t \mapsto t^{-1}\) and \(T_i \mapsto T_i^{-1}\)) corresponds to the involution on \(D_F\) obtained by extending the involution \(t \mapsto t^{-1}\) on the coefficient ring. Observe that each push-pull element \(C_i = \frac{1}{t + t^{-1}}(T_i + t)\) is invariant under this involution.

Consider the Kazhdan-Lusztig basis \(\{C'_w\}_{w \in W}\) on \(H\) (e.g. see [CG10]). Recall that it is unique and does not depend on choices of reduced decompositions. After the respective normalization we have

\[C'_w = T_w + \sum_{v < w} t\pi_{v, w}(t)T_v,\]
where deg $\pi_{v,w} \leq l(w) - l(v) - 1$ and $\pi_{v,w}$ are the Kazhdan-Lusztig polynomials. For instance, $C_l^0 = T_i + t$, $C_l^{ij} = T_{ij} + l(T_i + T_j) + t^2$ and $C_l^{iji} = T_{iji} + l(T_{ij} + T_{ji}) + t^2(T_i + T_j) + t^3$.

Let $C'_w$ denote the element in $D_F$ that corresponds to $C'_w$ via (6). Choose a reduced word $I_w$ for each $w \in W$. Then

$$C_w = (t + t^{-1})^{l(w)}(C'_w + \text{lower degree terms}) + \sum_{v < w} t\pi_{v,w}(t)(t + t^{-1})^{l(v)}(C'_v + \text{lower degree terms})$$

where the right hand side does not depend on choices of reduced decompositions. This suggests the following

**Definition 5.** We define the special elliptic polynomial $\pi^{se}_w$ to be the image in $\mathbb{Z}[t, t^{-1}][t_1, \ldots, t_n]/I$ of the element $\frac{1}{(t + t^{-1})^{l(w)}}C_w([pt])$ via (4).

We expect polynomials $\pi^{se}_w$ to play the same role (in the special elliptic case) as the Schubert (resp. Grothendieck) polynomials for Chow groups (resp. $K_0$).

**Example 6.** For the type $A_2$ we obtain

$$\pi^{se}_i = C_i([pt]), \quad \pi^{se}_{ij} = C_jC_i([pt])$$

and for the element of maximal length we obtain exactly the twisted braid relation

$$\pi^{se}_{w_0} = (C_{121} + \mu_2 C_1)([pt]) = (C_{212} + \mu_2 C_2)([pt]) = 1.$$

**Remark 7.** It would be interesting to see

1. that $\pi^{se}_{w_0} = 1$, for the element of maximal length $w_0$,
2. whether $\pi^{se}_{w_0}$ corresponds to the class of an actual resolution of the respective Schubert variety $X_w$.

**References**

[BE92] Bressler, P.; Evens, S. Schubert calculus in complex cobordisms. Trans. Amer. Math. Soc. 331 (1992), no.2, 799–813.

[BE90] P. Bressler, S. Evens, *The Schubert calculus, braid relations and generalized cohomology*. Trans. Amer. Math. Soc. 317 (1990), no.2, 799–811.

[BE87] Bressler, P.; Evens, S. On certain Hecke rings. Proc. Nat. Acad. Sci. USA 84 (1987), 624–625.

[BB10] V. Buchstaber, E Bunkova, *Elliptic formal group laws, integral Hirzebruch genera and Krichever genera*, Preprint arXiv.org 1010.0944v1, 2010.

[CZZ] B. Calmès, K. Zainoulline and C. Zhong, *A coproduct structure on the formal affine Demazure algebra*, Preprint arXiv:1209.1676, 2013.

[CZZZ1] B. Calmès, K. Zainoulline and C. Zhong, *Push-pull operators on the formal affine Demazure algebra and its dual*, Preprint arXiv.org 1312.0019.

[CPZ] B. Calmès, V. Petrov and K. Zainoulline, *Invariants, torsion indices and oriented cohomology of complete flags*, Ann. Sci. École Norm. Sup. (4) 46, no.3, 2013, 405–448.

[CG10] N. Chriss, V. Ginzburg, Representation theory and complex geometry. Modern Birkhauser Classics. Birkhauser Boston Inc., Boston, MA, 2010. Reprint of the 1997 edition.

[De74] M. Demazure. Désingularisation des variétés de Schubert généralisées. Ann. Sci. École Norm. Sup. (4) 7 (1974), 53–88.

[De73] M. Demazure, *Invariants symétriques entiers des groupes de Weyl et torsion*, Invent. Math. 21:287–301, 1973.

[Ga09] N. Gantzer, *Hecke operators in equivariant elliptic cohomology and generalized moonshine*, 1-39, American Mathematical Society, 2009.
[GR12] N. Ganter, A. Ram, Generalized Schubert Calculus. J. Ramanujan Math. Soc. 28A (Special Issue-2013), 149-190.

[Ha78] M. Hazewinkel, Formal groups and applications, Pure and Applied Mathematics, 78. Acad. Press. New-York-London, 1978. xxii+573pp.

[HMSZ] A. Hoffmann, J. Malagón-López, A. Savage, and K. Zainoulline, Formal Hecke algebras and algebraic oriented cohomology theories, to appear in Selecta Math. (see also Preprint arXiv:1208.4114), 2013.

[HK] J. Hornbostel, V. Kiritchenko, Schubert calculus for algebraic cobordism, J. Reine Angew. Math. (Crelle) 656 (2011), 59-86.

[Hu12] T. Hudson, Thom-Pontrjagin formula in Algebraic cobordism, Preprint arxiv.org 1206.2514, 2012.

[LM07] M.-A. Leclerc, E. Neher, K. Zainoulline, A special elliptic Demazure algebra for a Kac-Moody root system, Work in progress.

[KK86] B. Kostant and S. Kumar, The nil Hecke ring and cohomology of G/P for a Kac-Moody group G*, Advances in Math. 62:187–237, 1986.

[KK90] B. Kostant and S. Kumar, T-equivariant K-theory of generalized flag varieties, J. Differential geometry 32 (1990), 549–603.

[Ku02] S. Kumar, Kac-Moody groups, their flag varieties and representation theory. Progress in Mathematics, vol. 204, Birkhäuser, Boston, MA, 2002.

[SGA] Schémas en groupes III: Structure des schémas en groupes réductifs. (SGA 3, Vol. 3), Lecture Notes in Math. 153, Springer-Verlag, Berlin-New York, 1970, viii+529 pp.

[Fo94] S. Fomin, Schubert Polynomials and the nilCoxeter Algebra, Advances in Math, 103: 196-207, 1994.

[FK] S. Fomin and A. N. Kirillov, Grothendieck polynomials and the Yang-Baxter equation, in Formal power series and algebraic combinatorics/Séries formelles et combinatoire algébrique, DIMACS, Piscataway, NJ, sd, pp. 183189.