Topological phases, topological flat bands, and topological excitations in a one-dimensional dimerized lattice with spin-orbit coupling

ZHONGBO YAN and SHAOLONG WAN (a)

Institute for Theoretical Physics and Department of Modern Physics, University of Science and Technology of China - Hefei, 230026, PRC

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Abstract – The Su-Schrieffer-Heeger (SSH) model describes a one-dimensional $Z_2$ topological insulator, which has two topological distinct phases corresponding to two different dimerizations. When a modulated spin-orbit coupling is introduced into the SSH model, we find that the structure of the Bloch bands can be greatly changed, and most interestingly, a new topological phase with single zero-energy bound state which exhibits non-Abelian statistics at each end emerges, which suggests that a new topological invariant is needed to fully classify all phases. In a comparatively large range of parameters, we find that the modulated spin-orbit coupling induces a completely flat band with non-trivial topology. For the case with non-uniform dimerization, we find that the modulated spin-orbit coupling changes the symmetric structure of topological excitations known as solitons and antisolitons and when the modulated spin-orbit coupling is strong enough to induce a topological phase transition, the whole system is topologically non-trivial with the disappearance of solitons and antisolitons, consequently, the system is a real topological insulator with well-protected end states.

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Introduction. – Since the discovery of integer quantum Hall effect [1], the concept of topology has become increasingly popular and important in condensed-matter physics [2]. Topological phases, usually characterized by topological invariants which are connected to the energy spectrum and the nature of the wave function [3,4], can appear in different dimensional systems with higher or lower symmetries [5,6], and due to the robustness against disorder, they have the appealing potential application in topological quantum computation [7].

In the past few years, the theoretical predictions [8,9] and the experimental observations [10,11] of topological insulators have stimulated strong and continuous interest in predicting new classes of materials with non-trivial topological properties. From the lessons of topological insulators, we have learned that spin-orbit coupling is usually a natural ingredient to generate topological phases, e.g., topological superconductors which host desirable Majorana fermions [12,13]. Inspired by the recent experimental realization of direct measurement of the Zak phase in one-dimensional topological Bloch bands [14], and the realization of spin-orbit coupling in a one-dimensional cold atomic system [15–17], in this work we investigate the influence of a modulated spin-orbit coupling on the topology of Bloch bands in the one-dimensional double-well optical lattice used in the experiment [14] where the well-known Su-Schrieffer-Heeger (SSH) model [18] is realized.

Recently, due to the probability of hosting various kinds of topological phase, the SSH model and its various extensions have been investigated extensively both from the theoretical perspective and the experimental perspective [19–32]. The original SSH model describes a $Z_2$ topological insulator, which has two topological distinct phases corresponding to two different dimerizations. In one dimension, the Zak phase which determines the topology of Bloch bands [33], can only take two values for a Bloch band, either $-\pi/2$ or $\pi/2$ (here we follow the choice of unit

(a)E-mail: slwan@ustc.edu.cn
Within the unit cell, $\hat{\pi}$ is lifted by this spin-orbit coupling, and the structures of Bloch bands are also greatly changed, consequently, the physics of the system are greatly enriched. The main results induced by the spin-orbit coupling we consider include: i) a series of topological phase transitions; ii) the reappearance of solitons and antisolitons with the introduction of a modulated spin-orbit coupling; iii) the formation of a completely flat band with non-trivial topology; iv) the change of the symmetric form of solitons and antisolitons; v) the disappearance and the reappearance of solitons and antisolitons with an increase of the strength and the asymmetry the modulated spin-orbit coupling.

**Theoretical model.** We consider the one-dimensional double-well lattice realized in the experiment [14], but with the introduction of a modulated spin-orbit coupling. The Hamiltonian describing the system is given by

$$\hat{H} = -\sum_{i,\sigma} \left( J\hat{a}_{i,\sigma}^\dagger \hat{b}_{i+1,\sigma} + J'\hat{a}_{i,\sigma}^\dagger \hat{b}_{i-1,\sigma} + h.c. \right) + \sum_{i,\sigma} \left( \lambda\hat{a}_{i,\sigma}^\dagger \hat{b}_{i-\sigma} - \lambda'\hat{a}_{i,\sigma}^\dagger \hat{b}_{i-1,-\sigma} + h.c. \right),$$  

(1)

where $J$, $J'$ denote modulated tunneling amplitudes within the unit cell, $\hat{a}_{i,\sigma}^\dagger$ ($\hat{b}_{i,\sigma}^\dagger$) are the particle creation operators for an atom with spin $\sigma$ ($\uparrow$ or $\downarrow$) in the $i$-th lattice cell. The terms in the second line denote the modulated spin-orbit coupling, $\lambda$, $\lambda'$ denote the strength.

In the absence of spin-orbit coupling, i.e. $\lambda = \lambda' = 0$, the above Hamiltonian corresponds to the well-known SSH model of polyacetylene [18]. In the SSH model, spin degrees are decoupled and the Bloch bands for spin-up and spin-down are degenerate. For each spin degree, the topology of the Bloch bands is classified by the Zak phase. As the bands are degenerate, then if the lower band (the occupied band) is of non-trivial topology for one spin degree, the lower band for the other spin degree is also topologically non-trivial, this implies that the number of bound states for each end is even. Due to the degeneracy, the upper Bloch band and the lower Bloch band for spin-up and spin-down are simultaneously touched at the only one topological phase transition point $J = J'$, which means there is only one topological phase transition for either the spinful or spinless SSH model. Therefore, the SSH model (spinful or spinless) hosts only two topologically distinct phases, which means it can be fully classified by a $Z_2$ invariant.

With the introduction of the modulated spin-orbit coupling, the spin is no longer a quantum number, and most importantly we find the degeneracy of the Bloch bands will be lifted. This indicates that by tuning the strength and asymmetry of the modulated spin-orbit coupling, only one of the lower band corresponding to one of the two helicities will undergo the topologically trivial to non-trivial or non-trivial to trivial transition, and the other lower band will keep its topology. This means that if the system is of non-trivial topology in the absence of the modulated spin-orbit coupling, then even the modulated spin-orbit coupling drives a topological phase transition, the system is still topologically non-trivial, but with a change of the number of end bound states. This suggests that using a $Z_2$ invariant can no longer fully characterize all phases of the extended SSH model, and we need to introduce a new topological invariant. In the following, we will give a detailed investigation of the influence of the modulated spin-orbit coupling.

The Hamiltonian (1) in momentum space takes the form

$$\hat{H} = \sum_{k} \hat{h}(k) = \sum_{k} \left\{ \left[ (\text{Re} \rho_k) \sigma_x - (\text{Im} \rho_k) \sigma_y \right] - \left[ (\text{Re} \delta_k) \sigma_x - (\text{Im} \delta_k) \sigma_y \right] \tau_x \right\},$$  

(2)

where $\sigma_x, \sigma_y$ are the Pauli matrices for sublattice and $\tau_x$ is the Pauli matrix for spin, and

$$\rho_k = \lambda e^{ikd/2} + J'e^{-ikd/2} = |e^{i\theta_1(k)}|, \quad \delta_k = \lambda' e^{ikd/2} - J'e^{-ikd/2} = |e^{i\theta_2(k)}|.$$  

(3)

The Hamiltonian obviously has time-reversal symmetry, particle-hole symmetry and chiral symmetry,

$$\hat{h}(k) = I\hat{h}^T(-k)I, \quad -\hat{h}(k) = \sigma_z\hat{h}^T(-k)\sigma_z, \quad -\hat{h}(k) = I\sigma_\lambda \hat{h}(k) I.$$  

(4)

The time-reversal operator $T = I$ ($I$ is a $4 \times 4$ identity matrix) satisfies $T^2 = 1$, besides, $\sigma_x^T = 1$ and $\sigma_z^T = \sigma_x$, then according to ref. [5], we know that the Hamiltonian belongs to the symmetry class $BDI$ (chiral orthogonal), and the phases are classified by $Z$, an integer. In fact, the SSH model also belongs to this symmetry class, however, as we have discussed before, due to the degeneracy of the Bloch bands, a $Z_2$ (a subset of $Z$) invariant can fully classify the phases.

To determine the topology of the system, we need to determine the topology of the occupied Bloch bands, which can be realized by calculating the Zak phase of the corresponding band,

$$\varphi_{\text{Zak}} = i \sum_{\sigma} \int_{-G/2}^{G/2} (\alpha_{k,\sigma}^* \partial_k \alpha_{k,\sigma} + \beta_{k,\sigma}^* \partial_k \beta_{k,\sigma}) dk, \quad (5)$$

where $\alpha_{k,\sigma}$ and $\beta_{k,\sigma}$ are the four components of a spinor $\mathbf{u}_k = (\alpha_{k,\uparrow}, \beta_{k,\uparrow}, \alpha_{k,\downarrow}, \beta_{k,\downarrow})^T$, and $\mathbf{u}_k$ is the cell-periodical eigenstate of the corresponding band, satisfying $\hat{h}(k)\mathbf{u}_k = E_k \mathbf{u}_k$. Following ref. [34], we require that $\alpha_{k,\sigma} = \alpha_{k+G,\sigma}$
and \( \beta_{k,\sigma} = -\beta_{k+G,\sigma} \), where \( G = 2\pi/d \) is the reciprocal lattice vector.

From eq. (2), we obtain the energy spectra

\[
E_k = \pm \sqrt{|\rho_k|^2 + |\delta_k|^2 \pm \sqrt{2( |\rho_k|^2 |\delta_k|^2 + \text{Re}(\rho_k^2 \delta_k^2))}}. \tag{6}
\]

We know that the necessary condition for topological phase transition is the closure of the energy gap. When \( \lambda = \lambda' = 0 \), the energy spectra reduce to \( E_k = \pm |\rho_k| \), it is direct to find that the energy gap is closed only when \( J = J' \) at \( k = \pi/d \). In ref. [14], it is shown that when \( J > J' \), each of the two occupied bands has \( \varphi_{Zak} = \pi/2 \), and the system is topologically trivial; when \( J < J' \), each band has \( \varphi_{Zak} = -\pi/2 \), and the system is topologically non-trivial. To characterize the topology of the Bloch bands conveniently, we define a new topological invariant \( \nu \),

\[
(-1)^\nu = \text{sgn}(\varphi_{Zak}). \tag{7}
\]

\( \nu = 0 \) corresponds to a trivial band and \( \nu = 1 \) corresponds to a topologically non-trivial band. Based on this definition, the phases of the system are classified by an integer defined as \( Z = \nu_1 + \nu_2 \), where \( \nu_1 \) is the topological invariant of the lower occupied band, and \( \nu_2 \) is the topological invariant of the upper occupied band. However, in the following we use \( Z(\nu_1, \nu_2) \), instead of \( Z \), to characterize the phases, because we can directly read the topology of every occupied band from \( Z(\nu_1, \nu_2) \).

### Topological phase transitions induced by the modulated spin-orbit coupling.

When \( \lambda = \lambda' \neq 0 \), the degeneracy of the bands is lifted, and only the energy gap between the upper occupied band and the lower unoccupied band can be closed (see fig. 1(a)–(d)). The condition for the closure of energy gap is now modified. The new criterion is given by

\[
|\epsilon_1(k_c)| = |\epsilon_2(k_c)|, \quad \theta_1(k_c) - \theta_2(k_c) = 0 \quad (\text{mod } \pi). \tag{8}
\]

A little investigation shows that the two conditions can only be simultaneously satisfied when the parameters satisfy the relations \((\lambda^2 + \lambda'^2) - (J^2 + J'^2) = \pm 2(\lambda \lambda' + J J') \).

For weak modulated spin-orbit coupling \((\lambda^2 + \lambda'^2 \ll J^2 + J'^2) \), the Bloch bands do not change much and the upper occupied band and the lower unoccupied band still touch at \( k_c = \pi/d \) (see fig. 1(a), (b)). At the gap-closure point, the parameters should satisfy the relation \((\lambda^2 + \lambda'^2) - (J^2 + J'^2) = -2(\lambda \lambda' + J J') \). If we take \( \lambda = \lambda' = 0 \), this relation just gives \( J = J' \), agreeing with the result discussed above. From the criterion above, the parameter relation for the closure of the energy gap is given by

\[
|J - J'| = \lambda + \lambda'. \tag{9}
\]

For the sake of discussion, here we first assume \( J > J' \). This assumption gives \( J = J' + \lambda + \lambda' \). This simple relation indicates that the modulated spin-orbit coupling changes the transition point in a linear way.

For strong modulated spin-orbit coupling \((\lambda^2 + \lambda'^2 > J^2 + J'^2) \), the form of the Bloch bands has changed a lot and the upper occupied band and the lower unoccupied band now touch at \( k_c = 0 \) (see fig. 1(c), (d)). The criterion for the gap closure is \((\lambda^2 + \lambda'^2) - (J^2 + J'^2) = 2(\lambda \lambda' + J J') \). The parameter relation deduced from the criterion above is given by

\[
|\lambda - \lambda'| = J + J'. \tag{10}
\]

To confirm whether the gap closure discussed above corresponds to a topological phase transition, we need to calculate the Zak phase of the upper occupied band before and after the closure of the gap. We first consider the case with weak modulated spin-orbit coupling. Before the discussion, we remind the fact that only when the energy gap gets closed, the topology of the bands can change, and therefore, we can choose a special value for the parameters in every gapped region to calculate the Zak phase. Before the closure of the gap, based on eq. (9), we choose \( J = 2, J' = 0, \lambda = 1, \lambda' = 0 \), a few steps of calculation show that the four-component spinor \( \mathbf{u}_{\epsilon} \) of the upper occupied Bloch band takes the form

\[
\mathbf{u}_{\epsilon} = \frac{1}{2}(1, e^{-i k d/2}, 1, e^{-i k d/2})^T. \tag{11}
\]

then based on eq. (5), it is direct to obtain \( \varphi_{Zak}(J > J_c) = \pi/2 \) (where \( J_c = J + \lambda + \lambda' \)), or equivalently, \( \nu_2 = 0 \). This indicates that the system is a trivial insulator, agreeing with the fact that the parameters we choose above can be continuously transited to parameters \( \lambda = \lambda' = 0 \) and \( J > J' \), without the closure of the gap. After the closure of the gap and the re-opening of the gap, based on eq. (9), we choose the parameters of fig. 1(c): \( J = 2, J' = 1, \lambda = 2, \lambda' = 1 \), then \( \mathbf{u}_k \) of the upper occupied Bloch band.

Fig. 1: (Color online) Energy dispersion relation with parameters: (a) \( J = 2, J' = 1, \lambda = 0.2, \lambda' = 0.1 \). (b) \( J = 2, J' = 1, \lambda = 0.6, \lambda' = 0.4 \). (c) \( J = 2, J' = 1, \lambda = 2, \lambda' = 1 \). (d) \( J = 2, J' = 1, \lambda = 4, \lambda' = 1 \).
is given by
\[ u_k = \frac{1}{2} \left( 1, e^{ikd/2}, 1, e^{-ikd/2} \right)^T, \]  
(12)
then we obtain \( \varphi_{Zak}(J < J_c) = -\pi/2 \), or equivalently, \( \nu_2 = 1 \). This indicates that the upper occupied band which is completely flat (see fig. 1(c)) is of non-trivial topology. If we keep \( J = \lambda \), we find that varying \( \lambda' \) with the constraint \( \lambda' < \lambda \) does not change the completely flat form of the upper occupied band. This suggests that the topological flat band can exist in a broad region. Flat bands are of great interest because they play an important role in the study of strong correlated phenomena. One-dimensional flat bands with non-trivial topology have already been considered in models similar to the SSH one (only similar in form, the underlying topology is different) and the authors have found the existence of fractional topological phase [35–37].

\( \nu_2 = 1 \) indicates when \( \lambda + \lambda' > J - J' \), the system is driven into the topological phase \( Z(0, 1) \). As \( J \) keeps larger than \( J' \), the dimerization of the topological phase \( Z(0, 1) \) is the same as the trivial phase \( Z(0, 0) \). This suggests an important fact that with the appearance of modulated spin-orbit coupling, the re-opened gap will get closed after \( \lambda ' < J \) (mod 2π).

Based on the analysis above, when \( J = J_c \), accompanying the closure of the energy gap, a topological phase transition takes place, with a change of the Zak phase \( \delta \varphi_{Zak} = \pi \) (mod 2π).

With continuously increasing the strength of modulated spin-orbit coupling, the energy gap will get closed again. In the region \( |\lambda - \lambda'| > J + J' \), we choose \( J = 1 \), \( J' = 0 \), \( \lambda = 1 \), \( \lambda' = 0 \) to calculate the Zak phase. The \( u_k \) of the upper occupied Bloch band is given by

\[ u_k = \frac{1}{2} \left( -1, e^{-ikd/2}, -1, e^{-ikd/2} \right)^T, \]  
(13)
then the same procedure produces \( \varphi_{Zak}(\lambda > \lambda') = \pi/2 \), or equivalently, \( \nu_2 = 0 \). The system returns to the trivial phase \( Z(0, 0) \). Therefore, the two closures of the gap both correspond to topological phase transition with a change of the Zak phase \( \delta \varphi_{Zak} = \pi \) (mod 2π).

Above, we have assumed \( J > J' \); if we instead assume \( J' > J \); we find the gap closure also always corresponds to topological phase transition with a change of the Zak phase \( \delta \varphi_{Zak} = \pi \) (mod 2π). A big difference from the case with \( J > J' \) is that, for \( J' > J \), the topological phase transition is between two topological phases with different topological invariants, and therefore, the system always hosts end bound states. The only change is the number of the end bound states.

Based on the analysis above, the phase diagrams can be directly obtained. The phase diagrams for \( J > J' \) and \( J < J' \) have similar form. From fig. 2(a), (b), we can see by continuously increasing the strength and asymmetry of the modulated spin-orbit coupling, the system first undergoes a topological phase transition from the trivial phase \( Z(0,0) \) (topological phase \( Z(1,1) \)) to the topological phase \( Z(0,1) \) (topological phase \( Z(1,0) \)) and then undergoes another topological phase transition from the topological phase \( Z(0,1) \) (topological phase \( Z(1,0) \)) back to the trivial phase \( Z(0,0) \) (topological phase \( Z(1,1) \)). Figure 2(c) shows that in the trivial phase \( Z(0,0) \), there is no zero-energy bound state at the end of the one-dimensional system. Figures 2(d), (f) show that the topological phase \( Z(0,1) \) or \( Z(1,0) \) hosts a single zero-energy bound state at each end. Figure 2(e) shows that the topological phase \( Z(1,1) \) hosts two zero-energy bound states at each end. The difference of the number of edge states suggests that the topological phase \( Z(0,1) \) is a new topological phase. As the Hamiltonian belongs to the symmetry class \( BDI \), the single zero-energy end state should be the same as the one found in ref. [38] and therefore exhibits non-Abelian statistics. Consequently, the single zero-energy end state is similar to an unpaired Majorana fermion [39] and has a potential application in topological quantum computation.

**The effect of the modulated spin-orbit coupling to topological excitations.** — It is well known that the most famous excitations in the SSH model are solitons and antisolitons which are movable [18]. They are the domain walls of the two topologically distinct phases with different dimerizations. The physics of such domain walls is
captured by the famous Jackiw-Rebbi model [40] and the TLM model [41]. Based on eq. (1), for weak modulated spin-orbit coupling, by an expansion of small momentum at $k = \pi/d$ and a series rotation in sublattice space (rotating $\sigma_x$ to $\sigma_z$, $\sigma_y$ to $\sigma_z$), we obtain a low-energy continuum Hamiltonian in real space, 

$$\hat{h}(x) = -iv\partial_x \sigma_z + \Delta(x) \sigma_z + [iv'\partial_x \sigma_z - \Delta'(x) \sigma_z] \tau_z,$$ (14)

where $v = (J + J')d/2$, $v' = (\lambda - \lambda')d/2$, and $\Delta(x) = J - J'$, $\Delta'(x) = \lambda + \lambda'$ in the case of uniform dimerization. By making a rotation of spin, $\hat{h}(x) = e^{i\frac{\pi}{4} \tau_z} \hat{h}(x) e^{-i\frac{\pi}{4} \tau_z}$, we obtain

$$\hat{h}(x) = -iv\partial_x \sigma_z + \Delta(x) \sigma_z + [iv'\partial_x \sigma_z - \Delta'(x) \sigma_z] \tau_z,$$ (15)

the Hamiltonian now can be decoupled into two parts corresponding to two different helicities,

$$\hat{h}_+ = -i(v + v')\partial_x \sigma_z + (\Delta(x) + \Delta'(x)) \sigma_z,$$
$$\hat{h}_- = -i(v - v')\partial_x \sigma_z + (\Delta(x) - \Delta'(x)) \sigma_z.$$ (16)

Based on our knowledge of the Bloch bands, we know that for $J > J'$, $\hat{h}_+(x)$ always describes a trivial phase, and $\hat{h}_-(x)$ may describe a trivial phase or a topological phase depending on the sign of $(\Delta(x) - \Delta'(x))$. For $J < J'$, on the contrary, $\hat{h}_-(x)$ always describes a topological phase, and $\hat{h}_+(x)$ may describe a trivial phase or a topological phase depending on the sign of $(\Delta(x) + \Delta'(x))$. The system undergoes a topological phase transition when $\Delta(x) - \Delta'(x) = 0$ for $J > J'$ and $\Delta(x) + \Delta'(x) = 0$ for $J < J'$. From the two equations, the parameter relation for uniform dimerization in eq. (9) is reobtained.

For strong modulated spin-orbit coupling, as the energy bands now touch at $k_c = 0$, therefore, the roles played by the kinetic term $-iv\partial_x \sigma_z$ and the order parameter term $\Delta(x) \sigma_z$ are exchanged. The discussion of the case with strong modulated spin-orbit coupling is straightforward, and we neglect it here.

For non-uniform dimerization, $\hat{h}_+(x)$ describes topological excitations with one of the helicities labeled as $"+$" and $\hat{h}_-(x)$ describes topological excitations with the other one labeled as $"-"$. Based on eq. (16), the decay properties of the wave functions take the form

$$\varphi_\pm(x) \propto \exp \left(-\frac{|\Delta(x) \pm \Delta'(x)|}{\nu}|x|\right),$$ (17)

where we have used the assumption $\Delta(x) = -\Delta(-x) = J - J' > 0$ for $x < 0$, $\Delta'(x) = \lambda + \lambda' < J - J'$ and Dirichlet boundary condition for simplicity, and $\nu = v \pm v'$ for $\hat{h}_\pm(x)$, respectively. From eq. (17), it is found that when $\Delta'(x) = 0$, the topological excitations have a symmetric structure, however, once $\Delta'(x) \neq 0$, topological excitations with helicity $"+"$ ("-"") become more localized at the left (right) side and more extended at the right (left) side of the domain.

Another important result induced by the modulated spin-orbit coupling is that by increasing spin-orbit coupling to make $\lambda + \lambda' > |J - J'|$ (but $|\lambda - \lambda'| < J + J'$), although the dimerization configuration keeps the same, the topological properties of both sides are changed (as shown in fig. 2(a), (b)). A direct result of the simultaneous change is the disappearance of the moving topological excitations. It is direct to confirm this by noting that in this parameter region, the sign of $(\Delta(x) + \Delta'(x))$ is always positive and the sign of $(\Delta(x) - \Delta'(x))$ is always negative. Unlike the SSH model, where the existence of moving topological excitations makes the system actually conducting and the end states in fact unprotected (as both the end state and the soliton are zero-energy states, from the degenerate perturbation theory we know that a small coupling between them will induce a great split of their energy), the disappearance of the moving topological excitations indicates that the system turns out to be a real insulator in bulk, and consequently, the system is a real topological insulator with well-protected end states. The disappearance of the moving topological excitations also suggest the two topological phases denoted by $Z(0, 1)$ and $Z(1, 0)$ are the same, agreeing with the fact that the Hamiltonian is classified by $Z = \nu_1 + \nu_2$. Base on fig. 2(a), (b), topological excitations will reappear by further increasing the strength and the asymmetry of the modulated spin-orbit coupling.

**Conclusion.** — The introduction of a modulated spin-orbit coupling greatly enriches the physics of SSH model. First, with the lift of the degeneracy, we find the usual $Z_2$ invariant classifying the phases of SSH model can no longer fully classify all phases of the extended SSH model. The new topological phase corresponding to $Z = 1$ hosts an interesting single zero-energy bound state which exhibits non-Abelian statistics at each end and is stable against the variation of dimerization, therefore, it can have great potential application in topological quantum computation. Second, in the region of topological phase $Z = 1$, a completely flat band with non-trivial topology can be formed by tuning the modulated spin-orbit coupling. Third, for the case with non-uniform dimerization, the modulated spin-orbit coupling in the weak region changes the symmetric form of topological excitations, and with increasing the strength and the asymmetry of the modulated spin-orbit coupling, the topological excitations will disappear when a topological phase transition takes place and then re-appear when another topological phase transition takes place. These results suggest that spin-orbit coupling can be used to control the phases and the topological excitations of the system. The progress on experiments makes the observation of the new topological phase and the topological phase transitions induced by the modulated spin-orbit coupling in experiments within current ability.

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