The mass-extended 't Hooft-Nobbenhuis complex transformations and their consequences

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received 25 August 2010; accepted in final form 4 October 2010
published online 15 November 2010

PACS 31.30.J- – Relativistic and quantum electrodynamic (QED) effects in atoms, molecules, and ions
PACS 04.20.-q – Classical general relativity
PACS 03.50.De – Classical electromagnetism, Maxwell equations

Abstract – We have extended the 't Hooft-Nobbenhuis complex transformations to include mass. Under these new transformations, Schrödinger, Dirac, Klein-Gordon and Einstein general-relativity equations are invariant. The non-invariance of the cosmological constant in Einstein field equations dictates it to vanish thus solving the long-standing cosmological-constant problem.

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Introduction. – Recently 't Hooft and Nobbenhuis have introduced complex space-time transformations under which the invariance of the ground state would associate the vacuum state to a zero cosmological constant [1]. These transformations are such that the space-time coordinates \(x^\mu \rightarrow ix^\mu\). Under these transformations, the Hamiltonian of a non-relativistic particle \(H \rightarrow -H\) and the boundary conditions of the physical states do not become invariant. This is because while the real part of the states goes to zero as \(x \rightarrow \infty\), the imaginary part does not. Consequently, all states except the ground state (\(\psi(x) = \text{const}\)) will break this symmetry.

In the context of quantum theory, Hermiticity, normalization and boundary conditions will not transform as in the usual symmetry transformations for a non-relativistic particle.

In as much as in quantum mechanics energy and momentum are described by \(p = -i\hbar \nabla\) and \(E = i\hbar \frac{\partial}{\partial t}\), any coordinates transformations will inevitably transform \(p\) and \(E\). Moreover, in relativity theory mass and energy are related. Hence, the coordinate transformations will necessarily require the mass transformations too. This latter transformations have been overlooked by 't Hooft and Nobbenhuis in their complex transformations [1].

We argue that the inclusion of mass transformation will resolve the shortcomings of the theory pertaining to the Hermiticity, normalization and boundary conditions. The resulting complex transformations, referred to the 't Hooft-Nobbenhuis mass-extended transformations, lead to interesting physics when applied to Schrödinger, Dirac, Klein-Gordon and Einstein general-relativity and Maxwell equations.

We assert that the transformations we introduce in this letter are applicable to massive fields, unlike those of 't Hoof-Nobbenhuis which are only applicable to massless fields/particles. Under these transformations, the Hermitian operators are transformed into anti-Hermitian operators. Moreover, the wave functions that are periodic in real spaces are also periodic in imaginary space. These pretty transformations urge us to formulate the laws of nature in a complex space instead of the present real space.

The mass-extended 't Hooft-Nobbenhuis transformations. – 't Hooft and Nobbenhuis have recently introduced complex space-time transformations, and identified it as a symmetry of the laws of nature. In quantum mechanics, the space-time transformations will transform momentum \((p)\) and energy \((E)\) since the latter are expressed by

\[
\vec{p'} = -i\hbar \nabla', \quad E' = i\hbar \frac{\partial}{\partial t}'. \tag{1}
\]

From the theory of relativity, one knows that mass \((m)\) and energy are related by

\[
E = mc^2, \quad p = mv. \tag{2}
\]

Hence, under \(x^\mu \rightarrow ix^\mu\), eqs. (1) and (2) will yield

\[
E' = -iE, \quad \vec{p'} = -i\vec{p}, \quad m' = -im. \tag{3}
\]

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Therefore, the full complex transformations will become
\[
\vec{r}' = i \vec{r}, \quad t' = it, \quad m' = -im. \tag{4}
\]
We refer to these transformations as 't Hooft-Nobbenhuis mass-extended transformations. These transformations do not alter the Einstein mass-energy equation, \( E = \sqrt{c^2p^2 + m_0^2c^4} \) and the Lorentz transformations. According to eq. (4), one notices that
\[
E't' = Et, \quad \vec{p}' \cdot \vec{r}' = \vec{p} \cdot \vec{r}, \quad \hbar' = \hbar, \quad c' = c, \tag{5}
\]
so that the wave phase does not change. This guarantees the normalization and boundary conditions of all physical states. Hence, eq. (4) is a symmetry of laws of nature.

**Schrödinger equation.** – Applying the transformations in eq. (4) to Schrödinger equation
\[
i \hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V \psi \tag{6}
\]
shows that Schrödinger equation is invariant provided \( V(i \vec{x}) = -iV(\vec{x}) \). The probability and current densities in Schrödinger formalism are defined by
\[
\rho = \psi^* \psi, \quad \vec{J} = \frac{\hbar}{2mi} \left( \psi^* \nabla \psi - \psi \nabla \psi^* \right),
\tag{7}
\]
so that the continuity equation reads
\[
\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0. \tag{8}
\]
While the derivation of the current density transformation in eq. (9) is straightforward, explaining that of the probability, namely \( \rho' = i \rho \), is noteworthy. Since the probability, \( P_r = \int \rho dV \), is a pure number (dimensionless), then the transformations in eq. (4) should dictate that \( P'_r = P_r \). This implies that \( \rho dV = \rho' dV' \), and since \( V' = -iV \), then \( \rho' = i \rho \).

Applying the transformations (4) in eq. (7) yields
\[
\vec{J}' = i \vec{J}, \quad \rho' = i \rho. \tag{9}
\]
This implies that the continuity equation, eq. (8), is invariant under the transformation in eq. (4) too.

**Non-relativistic particle motion.** – Following 't Hooft and Nobbenhuis, we discuss here a one-dimensional motion of a non-relativistic particle. Consider the Hamiltonian
\[
H = \frac{p^2}{2m} + V(x). \tag{10}
\]
Under the transformations in eq. (4), the above equation yields
\[
H' = -\frac{p^2}{2mi} + V(ix) = -i \left( \frac{p^2}{2m} + V(x) \right) = -iH, \tag{11}
\]
where, \( V(ix) = -iV(x) \). This can be realized for certain potentials.

If we now consider a harmonic oscillator where the Hamiltonian is given by
\[
H = \hbar \omega \left( a^+ a + \frac{1}{2} \right), \tag{12}
\]
where
\[
a = \sqrt{\frac{m \omega}{2\hbar}} \left( x + \frac{ip}{m \omega} \right), \quad a^+ = \sqrt{\frac{m \omega}{2\hbar}} \left( x - \frac{ip}{m \omega} \right), \tag{13}
\]
then under the transformations in eq. (4), one has
\[
a' = -a, \quad a'^+ = -a^+, \quad H' = -iH, \quad m' \omega' = -m \omega. \tag{14}
\]
The state wave function, \( \psi_n(x) \), which is given by
\[
\psi_n(x) = \frac{1}{\sqrt{2^n n!}} \left( \frac{m \omega}{\pi \hbar} \right)^{1/4} \exp \left( -\frac{m \omega}{2\hbar} x^2 \right) H_n \left( \sqrt{\frac{m \omega}{\hbar}} x \right), \tag{15}
\]
where
\[
H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} \left( e^{-x^2} \right), \quad n = 0, 1, 2, \ldots, \tag{16}
\]
transforms as, \( \psi'_n(x) = (-1)^n \sqrt{i} \psi_n(x) \). This is also consistent with eqs. (7) and (9). Hence, the boundary and normalization conditions are satisfied for all states. This is unlike the original 't Hooft-Nobbenhuis transformations, where these two properties of the wave function are lost. Moreover, the transformed energy and momentum operators are anti-Hermitian.

**Classical scalar field.** – Consider a real scalar field \( \Phi(x) \) defined by the Lagrangian density \[2\]
\[
\mathcal{L} = -\frac{1}{2} (\partial \mu \Phi)^2 - V(\Phi), \quad V(\Phi) = \frac{1}{2} m^2 \Phi^2 + \lambda \Phi^4 \tag{17}
\]
and the Hamiltonian density
\[
\mathcal{H} = \frac{1}{2} \Pi^2 + \frac{1}{2} (\nabla \Phi)^2 + V(\Phi), \quad \Pi(x) = \partial_0 \Phi, \tag{18}
\]
where \( \lambda \) is a constant. Now if \( \Phi(x) \) transforms as
\[
\Phi'(x) \equiv \Phi(ix) = -i \Phi(x) \quad \Rightarrow \quad \Pi'(ix) = -i \Pi(x), \tag{19}
\]
the above Lagrangian and Hamiltonian will be invariant, i.e., \( \mathcal{L}' = \mathcal{L} \) and \( \mathcal{H}' = \mathcal{H} \). Moreover, the action \( S = \int \mathcal{L} d^4x = S' \).

**Maxwell equations.** – Defining the electromagnetic tensor \( F_{\mu\nu} \) as \[3\]
\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{20}
\]
Maxwell equations read
\[
\partial_\mu F^{\mu\nu} = \mu_0 j^\nu, \quad \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0. \tag{21}
\]
\(1/|\omega| = T^{-1}\).
Under the transformations in eq. (4), the charge \( q \), current \( J \) and vector potential \( A_\mu \) transform as
\[
q' = q, \quad J' = -iJ, \quad A'_\mu = -iA_\mu.
\]
(22)

Therefore, the electromagnetic tensor, the charge and current densities transform as
\[
F'_{\mu\nu} = -F_{\mu\nu}, \quad j' = iJ, \quad \rho' = i\rho.
\]
(23)

Hence, Maxwell equations are invariant under eq. (4). If we extend our analysis to Yang-Mills field, the Lagrangian will involve quadratic, cubic and quartic couplings of the filed \( A_\mu \). In this case, one has \( F_{\mu\nu} \rightarrow F'_{\mu\nu} \), where
\[
F''_{\mu\nu} = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + if^{abc} A_\mu^b A_\nu^c,
\]
(24)

where \( f^{abc} \) are the structure constants, we will get an invariant Lagrangian, since
\[
F''_{\mu\nu} = -F''_{\mu\nu}.
\]
(25)

**Quantum electrodynamics (QED).** – The QED Lagrangian density for a free particle with rest mass \( m_0 \) is given by [2]
\[
\mathcal{L} = i\bar{\psi}\gamma^\mu D_\mu\psi - m_0\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu},
\]
(26)

where \( D_\mu = \partial_\mu - i\frac{e}{\hbar} A_\mu \). This Lagrangian is invariant under eqs. (4) and (17), viz., \( \mathcal{L}' = \mathcal{L} \), provided that \( \psi' = i\psi \), \( \gamma' = i\gamma \). This can be satisfied if \( \psi' = \sqrt{i}\psi \) and \( \gamma' = \sqrt{i}\gamma \). This is in agreement with the transformations in eqs. (9) and (17).

**General theory of relativity.** – Einstein equations for general relativity are [4]
\[
R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu}.
\]
(27)

The particle equation of motion is given by the geodesic equation
\[
\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}u^\nu u^\lambda = 0,
\]
(28)

where \( \Gamma^\mu_{\nu\lambda} \) are the Christoffel symbols, \( \tau \) and \( u^\mu \) are the proper time and velocity, respectively. Under the transformations (4), eq. (28) yields
\[
\Gamma^\mu_{\nu\lambda} = -i\Gamma^\mu_{\nu\lambda}.
\]
(29)

Using the transformations in eq. (4), one finds that
\[
\rho'_m = \rho_m, \quad G' = -G, \quad R''_{\mu\nu} = -R_{\mu\nu}, \quad T''_{\mu\nu} = T_{\mu\nu}.
\]
(30)

where \( \rho_m \) is the matter density and \( G \) is Newton constant. Hence, Einstein general-relativity equations are invariant under the transformations in eq. (4). However, the existence of the cosmological constant in the Einstein field equations will violate the invariance. The way out of this is that the cosmological constant must be zero. Thus, the vanishing of the cosmological constant is necessary because its existence violates the symmetry defined in eq. (4). But if the cosmological constant has to be present in the Einstein field equations, it must change sign under the transformations in eq. (4). In this case, the Einstein field equations are invariant. This global invariance is a quite interesting feature that the original 't Hooft-Nobbenhuis transformations had wished for. Hence, the reason for the vanishing of the cosmological constant is now understood.

**Concluding remarks.** – We have extended in this work the complex space-time transformation postulated by 't Hooft and Nobbenhuis to include mass. This extended transformation remedied the problems of the original transformations. These transformations can be considered as a special case of scale transformation. We have found in this work that all physical laws are invariant under complex space-time and mass transformations. Hence, the extended complex transformations signify the symmetry of the laws of nature.

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We are grateful to the anonymous referees for their useful and critical comments.

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\[2(G) = M^{-1}L^3T^{-2} \text{ and } |\rho_m| = ML^{-3}.\]