ABSTRACT

Online game playing algorithms produce high-quality strategies with a fraction of memory and computation required by their offline alternatives. Continual Resolving (CR) is a recent theoretically sound approach to online game playing that has been used to outperform human professionals in poker. However, parts of the algorithm were specific to poker, which enjoys many properties not shared by other imperfect information games. We present a domain-independent formulation of CR applicable to any two-player zero-sum extensive-form games that works with an abstract resolving algorithm. We further describe and implement its Monte Carlo variant (MCCR) which uses Monte Carlo Counterfactual Regret Minimization (MCCFR) as a resolver. We prove the correctness of CR and show an $O(T^{-1/2})$-dependence of MCCFR’s exploitability on the computation time. Furthermore, we present an empirical comparison of MCCFR with incremental tree building to Online Outcome Sampling and Information-set MCTS on several domains.

KEYWORDS

Monte Carlo; counterfactual regret; resolving; imperfect information; online play; extensive form games; approximate Nash equilibrium

1 INTRODUCTION

Strategies for playing games can be pre-computed offline for all possible situations, or they can be computed online only for the situations that occur in a particular match. The advantage of the offline computation are stronger bounds on the quality of the computed strategy. Therefore, it is preferable if we want to solve a game optimally. On the other hand, online algorithms can produce strong strategies with a fraction of memory and computation time required by the offline approaches. Online game playing algorithms have outperformed humans in Chess [14], Go [27], and no-limit Poker [3, 22], long before solving these games is feasible.

While online approaches have always been the method of choice for strong play in perfect information games, it is less clear how to apply them in imperfect information games (IIGs). To find the optimal strategy for a specific situation in an IIG, a player has to reason about the unknown parts of the game state. They depend on the (possibly observable) actions of the opponent prior to the situation, which in turn depends on what the optimal decisions are for both players in many other parts of the game. This makes the optimal strategies in distinct parts of the game closely interdependent and makes correct reasoning about the current situation difficult without solving the game as a whole.

Existing online game playing algorithms for imperfect information games either do not provide any guarantees on the quality of the strategy they produce [8, 9, 21], or require the existence of a compact heuristic evaluation function and a significant amount of computation to construct it [4, 22]. Moreover, the algorithms that are theoretically sound were developed primarily for Texas hold’em poker, which has a very particular information structure. After the initial cards are dealt, all of the actions and chance outcomes that follow are perfectly observable. Furthermore, since the players’ moves alternate, the number of actions taken by each player is always known. None of this holds in general for games that can be represented as two-player zero-sum extensive-form games (EFGs).

In a blind chess [8], we may learn we have lost a piece, but not necessarily which of the opponent’s pieces took it. In visibility-based pursuit-evasion [25], we may know the opponent remained hidden, but not in which direction she moved. In phantom games [28], we may learn it is our turn to play, but not how many illegal moves has the opponent attempted. Because of these complications, the previous theoretically sound algorithms for imperfect-information games are no longer directly applicable.

The sole exception is Online Outcome Sampling (OOS) [20]. It is theoretically sound, completely domain independent, and it does not use any pre-computed evaluation function. However, it starts all its samples from the beginning of the game, and it has to keep sampling actions that cannot occur in the match anymore. As a result, its memory requirements grow as more and more actions are taken in the match, and the high variance in its importance sampling corrections slows down the convergence.

We revisit the Continual Resolving algorithm (CR) introduced in [22] for poker and show how it can be generalized in a way that can handle the complications of general two-player zero-sum EFGs. Based on this generic algorithm, we introduce Monte Carlo Continual Resolving (MCCR), which combines MCCFR [18] with incremental construction of the game tree, similarly to OOS, but replaces its targeted sampling scheme by Continual Resolving. This leads to faster sampling since MCCFR starts its samples not from the root, but from the current point in the game. It also decreases the memory requirements by not having to maintain statistics about parts of the game no longer relevant to the current match. Furthermore, it allows evaluating continual resolving in various domains, without the need to construct expensive evaluation functions.
We prove that MCCR’s exploitability approaches 0 with increasing computational resources and verify this property empirically in multiple domains. We present an extensive experimental comparison of MCCR with OOS, Information Set Monte Carlo Tree Search (Information Set Monte Carlo Tree Search (IS-MCTS)) [9] and MCCFR. We show that MCCR’s performance heavily depends on its ability to quickly estimate key statistics close to the root, which is good in some domains, but insufficient in others.

2 BACKGROUND

We now describe the standard notation for IIGs and MCCFR.

2.1 Imperfect Information Games

We focus on two-player zero-sum extensive-form games with imperfect information. Based on [24], game $G$ can be described by

- $H$ - the set of histories, representing sequences of actions.
- $Z$ - the set of terminal histories (those $z \in H$ which are not a prefix of any other history). We use $g \subseteq h$ to denote the fact that $g$ is equal to or a prefix of $h$.
- $A(h) := \{a \mid ha \in H\}$ denotes the set of actions available at a non-terminal history $h \in H \setminus Z$.
- $P : H \times Z \rightarrow \{1, 2, c\}$ is the function partitioning non-terminal histories into $H_{1}$, $H_{2}$ and $H_{c}$ depending on which player chooses an action at $h$.
- The strategy of chance is a fixed probability distribution $\pi_{c}$ over actions in chance player’s histories.
- The utility function $u = (u_{1}, u_{2})$ assigns to each terminal history $z$ the rewards $u_{1}(z), u_{2}(z) \in \mathbb{R}$ received by players 1 and 2 upon reaching $z$. We assume that $u_{2} = -u_{1}$.
- The information-partition $I = (I_{1}, I_{2})$ captures the imperfect information of $G$. For each player $i \in \{1, 2\}$, $I_{i}$ is a partition of $H_{i}$. If $g, h \in H_{i}$ belong to the same $i \in I_{i}$ then $i$ cannot distinguish between them. We only consider games with perfect recall, where the players always remember their past actions and the information sets visited so far.

A behavioral strategy $\sigma_{i} \in \Sigma_{i}$ of player $i$ assigns to each $l \in I_{i}$ a probability distribution $\sigma(l)$ over available actions $a \in A(l)$. A strategy profile $\sigma = (\sigma_{1}, \sigma_{2}) \in \Sigma$ consists of strategies of players 1 and 2. For a player $i \in \{1, 2\}$, $-i$ will be used to denote the other two actors $\{1, 2\} \setminus \{i\}$ in $G$ (for example $H_{-1} := H_{2} \cup H_{c}$ and opp$_{i}$ denotes $i$’s opponent (opp$_{1}$ := 2).

2.2 Nash Equilibria and Counterfactual Values

The reach probability of a history $h \in H$ under $\sigma$ is defined as $\pi(\sigma) := \pi_{1}^{\sigma}(h)\pi_{2}^{\sigma}(h)\pi_{c}^{\sigma}(h)$, where each $\pi_{i}^{\sigma}(h)$ is a product of probabilities of the actions taken by player $i$ between the root and $h$. The reach probabilities $\pi_{1}^{\sigma}(h)$ and $\pi_{2}^{\sigma}(h)$ conditional on being in some $g \subseteq h$ are defined analogously, except that the products are only taken over the actions on the path between $g$ and $h$. Finally, $\pi_{c}^{\sigma}(h)$ is defined like $\pi^{\sigma}(\cdot)$, except that in the product $\pi_{1}^{\sigma}(\cdot)\pi_{2}^{\sigma}(\cdot)\pi_{c}^{\sigma}(\cdot)$ is replaced by 1.

The expected utility for player $i$ of a strategy profile $\sigma$ is $u_{i}(\sigma) = \sum_{z \in Z} \pi(\sigma)u_{i}(z)$. The profile $\sigma$ is an $\epsilon$-Nash equilibrium ($\epsilon$-NE) if

$$(\forall i \in \{1, 2\}) : u_{i}(\sigma) \geq \max_{\sigma'_{i} \in \Sigma_{i}} u_{i}(\sigma'_{i}, \sigma_{opp_{i}}) - \epsilon.$$

A Nash equilibrium (NE) is an $\epsilon$-NE with $\epsilon = 0$. It is a standard result that in two-player zero-sum games, all $\sigma^{*} \in \Sigma$ have the same $u_{i}(\sigma^{*})$ [24]. The exploitability $\exp_{i}(\sigma)$ of $\sigma \in \Sigma$ is the average of exploitabilities $\exp_{i}(\sigma) = \min_{\sigma_{opp_{i}}} u_{i}(\sigma_{i}, \sigma_{opp_{i}})$.

The expected utility conditional on reaching $h \in H$ is

$u_{i}(h) = \sum_{z \in Z} \pi(\sigma)u_{i}(z).$

An ingenious variant of this concept is the counterfactual value (CFV) of a history, defined as $\pi^{\sigma}(\cdot) := \pi_{1}^{\sigma}(h)\pi_{2}^{\sigma}(h)$, and the counterfactual value $\sigma(h, a)$ of taking some action, defined as $\sigma(h, a)$ where $\sigma(h-a)$ is the same strategy as $\sigma$ except that in $a$, $h$ is taken with probability 1. We set $\sigma(l) := \sum_{h \in I_{l}} \sigma(l, h)$ for $l \in I_{2}$ and define $\sigma^{*}(l, a)$ analogously. A strategy $\sigma^{*} \in \Sigma_{2}$ is a counterfactual best response CBR($\sigma_{1}$) to $\sigma_{1} \in \Sigma_{1}$ if $u_{2}(\sigma^{*}(l, a)) \max_{\sigma_{a} \in \mathcal{A}(l)} u_{2}(\sigma^{*}(l, a))$ holds for each $l \in I_{2}$ [7].

2.3 Monte Carlo CFR

For a strategy $\sigma \in \Sigma$, $l \in I_{1}$ and $a \in \mathcal{A}(l)$, we set the counterfactual regret for not playing $a$ in $l$ under strategy $\sigma$ to

$$r^{\sigma}_{1}(l, a) := u_{1}(\sigma) - u_{1}(l, a).$$

The Counterfactual Regret minimization (CFR) algorithm [29] generates a sequence of strategies $\sigma^{0}, \sigma^{1}, \ldots$ in such a way that the immediate counterfactual regret

$$R^{T}_{i, imm}(l) := \max_{a \in \mathcal{A}(l)} \frac{1}{T} \sum_{t=1}^{T} r^{\sigma^{t}}_{i, l}(a)$$

is minimized for each $l \in I_{1}$, $i = 1, 2$. It does this by using the Regret Matching update rule [1, 12]: let $x^{*} := \max(x, 0)$, then

$$\sigma^{t+1}(a) := \frac{\frac{R^{T}_{i, imm}(l) + \sigma^{t, l}(a)}{\sum_{a' \in \mathcal{A}(l)} R^{T}_{i, imm}(l, a')} \sigma^{t, l}(a)}{\sum_{i \in I_{1}} \sum_{l \in I_{2}} \pi^{t}(l)}$$

where $l \in I_{1}$.

Since the overall regret is bounded by the sum of immediate counterfactual regrets [29, Theorem 3], this causes the average strategy $\bar{\sigma}^{T}$ (defined by (2)) to converge to a NE [18, Theorem 1]:

$$\bar{\sigma}^{T}(l)(a) := \frac{\sum_{t=1}^{T} \pi^{t}(l)\sigma^{t}(l, a)}{\sum_{t=1}^{T} \pi^{t}(l)}$$

The disadvantage of CFR is the (costly) need to traverse the whole game tree during each iteration. Monte Carlo CFR [18] works similarly, but only samples a small portion of the game tree each iteration. It calculates sample variants of CFR’s variables, each of which is an unbiased estimate of the original [18, Lemma 1]. We use a particular variant of MCCFR called Outcome Sampling (OS) [18]. OS only samples a single terminal history $z$ each iteration, using the sampling strategy $\sigma_{l, c} := \epsilon - \epsilon \sigma_{l, c} + \epsilon \cdot \text{rnd}$ (where $\epsilon \in (0, 1)$ controls the exploration and $\text{rnd}(l)(a) := \frac{1}{|A(l)|})$.

This $z$ is then traversed forward (to compute each player’s probability $\pi^{\sigma}_{i}(z)$ of playing to reach each prefix of $z$) and backward (to compute each player’s probability $\pi^{\sigma}_{i}(z)$ of playing the remaining actions of the history). During the backward traversal, the
sampled counterfactual regrets at each visited \( l \in I \) are computed according to (3) and added to \( \hat{R}_{i, \text{mun}}^I (l) \):

\[
\hat{r}_i^O (l, a) := \begin{cases} 
  w_l \cdot \left( \pi_i^O (z | ha) - \pi_i^O (z | h) \right) & \text{if } ha \subseteq z \\
  w_l \cdot (1 - \pi_i^O (z | h)) & \text{otherwise}
\end{cases}
\]

(3)

where \( h \) denotes the prefix of \( z \) which is in \( I \) and \( w_l \) stands for

\[
\frac{1}{\pi_i^O (z | h) \cdot u_h(z)}
\]

where \( u_h(z) \) is the utility of \( z \) in \( h \).

3 Domain-Independent Formulation of Continual Resolving

The only domain for which continual resolving has been previously defined and implemented is poker. All information sets in poker include a fixed number of histories of the same length. Public states have the same size and only a single player always chooses an action in each public state. None of these properties has to hold in general in extensive-form games. In this section, we show that the high-level structure of continual resolving can still be applied in general, but notions like the public tree and resolving gadget have to be defined more carefully. Since most of these concepts have already appeared in various forms in an earlier literature, Section 3.4 explains the relation of the presented definitions to the preexisting ones.

3.1 Subgames and the Public Tree

To speak about the information available to player \( i \) in histories where he doesn’t act, we will use augmented information sets. For player \( i \) \( 1, 2 \) and history \( h \in H \setminus Z \), the \( i \)’s observation history \( \tilde{O}_i (h) \) in \( h \) is the sequence \( (l_1, a_1, l_2, a_2, \ldots) \) of the information sets visited and actions taken by \( i \) on the path to \( h \) (incl. \( l \supset h \) if \( h \in H_i \)). Two histories \( g, h \in H \setminus Z \) belong to the same augmented information set \( I \in I^{\text{aug}} \) if \( \tilde{O}_i (g) = \tilde{O}_i (h) \) [7]. Since we assume perfect recall, \( I^{\text{aug}} \) coincides with \( I \) on \( H_i \). We write \( g \sim h \) when there is a player who cannot distinguish the two:

\[
g \sim h \iff \tilde{O}_i (g) = \tilde{O}_i (h) \lor \tilde{O}_2 (g) = \tilde{O}_2 (h).
\]

Remark 3.1 (Alternatives to \( I^{\text{aug}} \)). \( I^{\text{aug}} \) isn’t the only viable way of generalizing information sets. As an alternative, one could also consider some further-unrefineable perfect-recall partition \( I^{\text{aug}}_r \) of \( H \) which coincides with \( I \) on \( H_i \), and many other variants between the two extremes. We focus only on \( I^{\text{aug}} \), since an in-depth discussion of the general topic would be outside of the scope of this paper.

We denote \( \equiv \) the transitive closure of \( \sim \). Formally, \( g \equiv h \) iff

\[
(\exists n) (h_1, \ldots, h_n) : g \sim h_1 \sim h_2 \sim \ldots \sim h_{n-1} \sim h_n \sim h.
\]

If two states do not satisfy \( g \equiv h \), then both players know that everybody can tell them apart and everybody knows it.

Definition 3.2 (Public state). Public partition is any partition \( S \) of \( H \setminus Z \) whose elements are closed under \( \sim \) (cf. [15]). An element \( S \) of any such \( S \) is called a public state. The common knowledge partition \( \mathcal{S}_h \) is the one consisting of the equivalence classes of \( \sim \).

We endow any \( S \) with the tree structure inherited from \( H \). Clearly, \( \mathcal{S}_h \) is the finest public partition. Using the concept of a public state, a subgame in imp. inf. games can be defined as follows (cf. [15]).

3.2 Aggregation and the Upper Frontier

Often, it is useful to aggregate reach probabilities and counterfactual values over (augmented) information sets or public states. In general EFGs, an augmented information set \( I \in I^{\text{aug}} \) can be “thick”, i.e. it can contain both some \( ha \in H \) and it’s parent \( h \). Indeed, this happens e.g. if we are unsure about the number of successive moves made by the opponent. For such \( I \), we only aggregate over the “upper frontier” \( I := \{ h \in I \mid \exists g \in I : g \subset h \land g \neq \emptyset \} \) of \( I \) [10, 11]: We overload \( \pi^O (\cdot) \) as \( \pi_i^O (\cdot) := \sum_h \pi^O (h) \) and \( \psi^O (\cdot) \) as \( \psi_i^O (\cdot) := \sum_h \psi_i^O (h) \). We define \( \tilde{S} \) for \( S \in S \), \( \pi_i^O (\cdot) \) and \( \psi_i^O (\cdot) \) analogously. The presence of “thick” information sets causes various complications further discussed in Section 3.4. By \( \tilde{S} (i) := \{ I \in I^{\text{aug}} \mid I \subseteq S \} \) we denote the topmost (augmented) information sets of player \( i \) in \( S \).

3.3 Resolving Gadget Game

We describe a generalization of the resolving gadget game from [7] (cf. [3, 23]) for resolving Player 1’s strategy (see Figure 1).

Let \( S \in S \) be a public state to resolve from, \( \sigma \in \Sigma \), and let \( \sigma (I) \in \mathbb{R} \) for \( I \in \tilde{S} (i) \) be the required counterfactual values. First, the upper frontier of \( S \) is duplicated as \( \{ h | h \in \tilde{S} \} = : \hat{S} \). Player 2 is the acting player in \( \hat{S} \), and from his point of view, nodes \( h \) are partitioned according to \( \{ I := \{ h | h \in I \} | I \in \tilde{S} (2) \} \). In \( \hat{h} \in \tilde{S} \) corresponding to \( h \in I \), he can choose between “following” (F) into \( h \) and “terminating” (T), which ends the game with utility \( \check{u}_2 (hT) := \check{u}_2 (I) \pi^O_2 (S) / \pi^O_2 (I) \). From any \( h \in \hat{S} \) onward, the game is identical to \( G (S) \), except that the utilities are multiplied by a constant: \( \check{u}_2 (z) := u(z) \pi_2 (S) / \pi_2 (I) \). To turn this into a well-defined game, a root chance node is added and connected to each \( h \in \hat{S} \), with transition probabilities \( \pi^O_2 (h) / \pi^O_2 (S) \).
This game is called the resolving gadget game $\tilde{G}(S, \sigma, \tilde{v})$, or simply $\tilde{G}(S)$ when there is no risk of confusion, and the variables related to it are denoted by tilde. If $\tilde{\rho} \in \tilde{S}$ is a “resolved” strategy in $\tilde{G}(S)$, we denote the new combined strategy in $G$ as $\sigma^{\text{new}} := \sigma|_{(G(S))} \circ \tilde{\rho}$.

The added value of our generalization of [7] is the ability to handle thick information sets and public states. While tedious, it is straightforward to check that $\tilde{G}(S, \sigma, \tilde{v})$ has all the properties proved in [6, 7, 22].

The following properties are helpful to get an intuitive understanding of gadget games. Their more general versions and proofs (resp. references for proofs) are listed in the appendix.

Lemma (Gadget game preserves opponent’s values). For each $I \in \mathcal{I}_2^{\text{aug}}$ with $I \subset G(S)$, we have $v^0_2(\tilde{I}) = v^0_2(I)$.

Note that the conclusion does not hold for counterfactual values of the (resolving) player 1! (This is can be easily verified on a simple specific example such as Matching Pennies.)

Lemma (Optimal resolving). If $\sigma$ and $\tilde{\rho}$ are both Nash equilibria and $\tilde{v}(I) = v_2^0(I)$ for each $I \in \bar{S}(2)$, then $\sigma^{\text{new}}$ is not exploitable.

### 3.4 Obstacles in General EFGs

In [7], subgames are defined as “a forest of trees, closed under both the descendant relation and membership within $I_2^{\text{aug}}$ for any player”. For any $h \in S \in \mathcal{S}_k$, the subgame rooted at $S$ is the smallest $[7]$-subgame containing $h$. As a result, [7]-subgames are “forests of subgames rooted at common-knowledge public states”.

We can see that finer public partitions lead to smaller subgames, which are easier to solve. In this sense, the common-knowledge partition is clearly the “best one”. However, finding $S_{\text{thick}}$ is sometimes non-trivial, which is why the definition of general public states from [15] is important. The problem with this definition of public states, as well as with our Definition 3.2, is ambiguity — indeed, both allow for extremes such as grouping the whole $\mathcal{H}$ into a single public state, without giving a practical recipe for arriving at the “intuitively correct” public partition.

To the best of our knowledge, the issue of “thick” information sets has only been discussed in the context of non-augmented games with imperfect recall [11]. One scenario where thick augmented information sets cause problems is the resolving gadget game. If we copied the whole $S$ rather than just $S$ in its construction, some nodes would have two parents and the game would be ill-defined. Moreover, if we defined $\pi_2^0(S)$ as the sum over the whole $S$, the chance probabilities in the root wouldn’t sum to 1. Similarly, aggregating $v_2^0$ over whole thick $I$’s would destroy the theoretical properties of the original construction from [7].

If Continual Resolving is to be generalized beyond Poker, we need to watch out for several more complications. Firstly, we might be asked to take several turns within a single public state (e.g. in phantom games). When we are not the acting player, we might be unsure whether it is the opponent’s or chance’s turn. Finally, both players might be acting within the same public state (e.g. because a secret chance roll determines whether we get to act or not).

#### 3.5 Continual Resolving

Domain-independent continual resolving closely follows the structure of continual resolving for poker [22], but uses the generalized resolving gadget and handles situations which do not arise in poker, such as multiple moves in one public state. We explain the algorithm from the perspective of Player 1. The abstract CR keeps track of strategy $\sigma_1$ it has computed in previous moves. Whenever it gets to a public state $S$, where $\sigma_1$ has not been computed, it resolves the subgame $G(S)$. As a by-product of this resolving, it estimates opponent’s counterfactual values $v_2^{\sigma_1, \text{CR}}(s)$ for all public states that might come next, allowing it to keep resolving as the game progresses.

CR repetitively calls a Play function which takes the current information set $I \in \mathcal{I}_1$ as the input and returns an action $a \in \mathcal{A}(I)$ for Player 1 to play. It maintains the following variables:

- $S \in S$ – the current public state,
- $\mathcal{KPS} \subset S$ – the public states where strategy is known,
- $\sigma_1$ – a strategy defined for every $I \in \mathcal{I}_1$ in KPS,
- $\mathcal{NPS} \subset S$ – the public states where CR may resolve next,
- $D(S')$ for $S' \in \mathcal{NPS}$ – data allowing resolving at $S'$, such as the estimates of opponent’s counterfactual values.

The pseudo-code for CR is described in Algorithm 1. If the current public state belongs to KPS, then the strategy $\sigma_1(I)$ is defined, and we sample action $a$ from it. Otherwise, we should have the data necessary to build some resolving game $\tilde{G}(S)$ (line 3). We then determine the public states $\mathcal{NPS}$ where we might need to resolve next (line 5). We solve $\tilde{G}(S)$ via some resolving method which also computes the data necessary to resolve from any $S'$ in $\mathcal{NPS}$ (line 6). Finally, we save the resolved strategy in $S$ and update the data needed for resolving (line 7-9). To initialize the variables before the first resolving, we set KPS and $\sigma_1$ to $\emptyset$, find appropriate NPS, and start solving the game from the root using the same solver as Play, i.e.: $D \leftarrow \text{Resolve}(G, \mathcal{NPS})$.

We now consider CR variants that use the gadget game from Section 3.3 and data of the form $D = (\rho, \tilde{v})$, where $\rho(S') = (\rho_1(h))_h$ is CR’s range and $\tilde{v}(S') = (\tilde{v}(j))_j$ estimates opponent’s counterfactual value at each $j \in S'(2)$. We shall use the following notation: $S_n$ is the $n$-th public state from which CR resolves; $\tilde{\rho}_n$ is the corresponding strategy in $\tilde{G}(S_n)$; $\sigma_1^n$ is CR’s strategy after $n$-th resolving,
defined on KPS_n; the optimal extension of $\sigma^n_I$ is

$$
\sigma^n_{I} := \arg\min_{v_1 \in S_1} \exp_1 \left( \sigma^n_{I,KPS_n} \cup v_1 \mid S \setminus KPS_n \right).
$$

Lemmata 24 and 25 of [22] (summarized into Lemma A.5 in our Appendix A) give the following generalization of [22, Theorem S1]:

**Theorem 3.4 (Continual Resolving bound).** Suppose that CR uses $D = (r, \tilde{v})$ and $\tilde{G}(S, \sigma_1, \tilde{v})$. Then the exploitability of its strategy is bounded by

$$
\exp_1(\sigma) \leq \hat{\epsilon}_1 + \hat{\epsilon}_R + \frac{\hat{\epsilon}_S}{N},
$$

where $N$ is the number of resolving steps and $\hat{\epsilon}_S := \exp_1(\hat{\rho})$. $\hat{\epsilon}_n := \sum_{j \in \tilde{S}_n \setminus D} \hat{\epsilon}(j) - \bar{\sigma}_n^{CRR}(J)$ are the exploitability (in $\tilde{G}(S_n)$) and value estimation error made by the $n$-th resolver (resp. initialization for $n = 0$).

## 4 MONTE CARLO CONTINUALLY RESOLVING

Monte Carlo Continual Resolving is a specific instance of CR which uses Outcome Sampling MCCFR for game (re)resolving. Its data are of the form $D = (r, \tilde{v})$ described above and it resolves using the gadget game from Section 3.3. We first present an abstract version of the algorithms that we formally analyze and then add improvements that make it practical. To simplify the theoretical analysis, we assume MCCFR computes the exact counterfactual value of resulting average strategy $\hat{\sigma}^T$ for further resolving. It is costly and we discuss the alternatives later. The following theorem shows that MCCR’s exploitability converges to 0 at the rate of $O(T^{-1/2})$.

**Theorem 4.1 (MCCR bound).** With probability at least $(1-p)^{N+1}$, the exploitability of strategy $\sigma$ computed by MCCR satisfies

$$
\exp_1(\sigma) \leq \sqrt{2 \sqrt{p} + 1} \cdot \left| I_t \right| \Lambda_{u,i} \sqrt{A_t} \frac{2N-1}{\sqrt{L_0} + \sqrt{R}}.
$$

where $L_0$ and $R$ are the numbers of MCCR’s iterations in pre-play and each resolving, $N$ is the number of resolving steps, $\delta = \min_{z', z} q_i(z)$ where $q_i(z)$ is the probability of sampling $z \in Z$ at iteration $t$, $A_{u,i} = \max_{z', z} |u_i(z) - u_i(z')|$ and $A_t = \max_{i \in I_t} |A(I)|$.

The proof is presented in the appendix. Essentially, it inductively combines the OS bound (Lemma A.1) with the guarantees available for resolving games in order to compute the overall exploitability bound.\(^\text{1}\) For specific domains, a much tighter bound can be obtained by going through our proof in more detail and noting that the size of subgames decreases exponentially as the game progresses (whereas the proof assumes that it remains constant). In effect, this would replace the $N$ in the bound above by a small constant.

### 4.1 Practical Modifications

Above, we describe an abstract version of MCCR optimized for clarity and ease of subsequent theoretical analysis. We now describe the version of MCCR that we implemented in practice. The code used for the experiments is available online at https://github.com/aicenter/gtlibrary-java/tree/mccr.

\(^\text{1}\)Note that Theorem 4.1 isn’t a straightforward corollary of Theorem 3.4, since calculating the numbers $\epsilon^n_S$ does require non-trivial work. In particular, $\hat{\sigma}^T$ from the $n$-th resolving isn’t the same as $\sigma^n_{I,CRR}(\pi^n)$ and the simplifying assumption about $\hat{\epsilon}$ is not equivalent to assuming that $\epsilon^n_h = 0$.

#### 4.1.1 Incremental Tree-Building

A massive reduction in the memory requirements can be achieved by building the game tree incrementally, similarly to Monte Carlo Tree Search (MCTS) [5]. We start with a tree that only contains the root. When an information set is reached that is not in memory, it is added to it and a playout policy (e.g., uniformly random) is used for the remainder of the sample. In playout, information sets are not added to memory. Only the regrets in information sets stored in the memory are updated.

#### 4.1.2 Counterfactual Value Estimation

Since the computation of exact counterfactual values of the average strategy requires the traversal of the whole game tree, we have to work with their estimates instead. To this end, our MCCFR additionally computes the opponent’s sampled counterfactual values

$$
\hat{v}^O(i) := \frac{1}{z_{2-h}(h)} \pi_{2-h}^O(z)\nu_2(z).
$$

It is not possible to compute the exact counterfactual value of the average strategy just from the values of the current strategies. Once the $T$ iterations are complete, the standard way of estimating the counterfactual values of $\hat{\sigma}^T$ is using arithmetic averages

$$
\hat{v}(i) := \frac{1}{T} \sum_{t} \hat{v}^O(i).
$$

However, we have observed better results with weighted averages

$$
\hat{v}(h) := \frac{1}{\sum_i \hat{v}^O(i)} \sum_i \hat{v}^O(i) \hat{v}(i).
$$

The stability and accuracy of these estimates is experimentally evaluated in Section 5 and further analyzed in Appendix B. We also propose an unbiased estimate of the exact values computed from the already executed samples, but its variance makes it impractical.

#### 4.1.3 Root Distribution of Gadgets

As in [22], we use the information about opponent’s strategy in previous resolving in constructing the gadget game. Instead of being proportional to $\pi_{2-h}(h)$, it is proportional to $\pi_{2-h}(h) + \pi_{2-h}(h) + \epsilon$. This modification is sound, as long as $\epsilon > 0$.

#### 4.1.4 Custom Sampling Scheme

To improve the efficiency of resolving by MCCFR, we use a custom sampling scheme which differs from OS in two aspects. First, we modify the above sampling scheme such that with probability 90% we sample a history that belongs to the current information set $I$ that CR obtained as an input. This allows us to focus on the most relevant part of the game. Second, whenever $h \in S$ is visited by MCCFR, we sample both of the available actions (terminate and follow). This increases the transparency and stability of the algorithm, since each iteration now does a similar amount of work, as opposed to some iterations terminating right at the start. These modifications are theoretically sound, since the resulting sampling scheme still satisfies the assumptions of the general MCCFR bound from [17].

#### 4.1.5 Keeping the Data between Successive Resolvings

Both in pre-play and subsequent resolvings, MCCFR operates on successively smaller and smaller subsets of the game tree. In particular, we don’t need to start each resolving from scratch, but we can re-use the previous computations. To do this, we initialize each resolving MCCFR with the MCCFR variables (regrets, average strategy and the corresponding value estimates) from the previous resolving.
When initiated from a non-empty match history, it starts samples (resp. pre-play). In practice this is accomplished by simply not resetting the data from the previous MCCFR. While not being backed up by theory, this approach worked better in most practical scenarios, and we believe it can be made correct with the use of warm-starting [2] of the resolving gadget.

5 EXPERIMENTAL EVALUATION

After brief introduction of competing methods and explaining the used methodology, we focus on evaluating the alternative methods to estimate the counterfactual values required for resolving during MCCFR. Next, we evaluate how quickly and reliably can these values be estimated in different domains, since these values are crucial for good performance of MCCFR. Finally, we compare exploitability and head-to-head performance to competing methods.

5.1 Competing Methods

**Information-Set Monte Carlo Tree Search.** IS-MCTS [19] runs MCTS samples as in a perfect information game, but computes statistics for the whole information set and not individual states. When initiated from a non-empty match history, it starts samples uniformly from the states in the current information set. We use two selection functions: Upper Confidence bound applied to Trees (UCT) [16] and Regret Matching (RM) [13]. We use the same settings as in [20]: UCT constant 2x the maximal game outcome, and RM with exploration 0.2. In the following, we refer to IS-MCTS with the corresponding selection function by only UCT or RM.

**Online Outcome Sampling.** OOS [20] is an online search variant of MCCFR. MCCFR samples from the root of the tree and needs to pre-build the whole game tree. OOS has two primary distinctions from MCCFR: it builds its search tree incrementally and it can bias samples with some probability to any specific part of the game tree. This is used to target the information sets (OOS-IST) or the public states (OOS-PST) where the players act during a match.

We do not run OOS-PST on domain of IIGS, due to non-trivial biasing of sampling towards current public state.

We further compare to MCCFR with incremental tree building and the random player denoted RND.

5.2 Computing Exploitability

Since the online game playing algorithms do not compute the strategy for the whole game, evaluating exploitability of the strategies they play is more complicated. One approach, called brute-force in [20], suggest ensuring that the online algorithm is executed in each information set in the game and combining the computed strategies. If the thinking time of the algorithm per move is \( t \), it requires \( O(t|I|) \) time to compute one combined strategy and multiple runs are required to achieve statistical significance for randomized algorithms. While this is prohibitively expensive even for the smaller games used in our evaluation, computing the strategy for each public state, instead of each information set is already feasible. We use this approach, however, it means we have to disable the additional targeting of the current information set in the resolving gadget proposed in Section 4.1.4.

There are two options how to deal with the variance in the combined strategies in different runs of the algorithm in order to compute the exploitability of the real strategy realized by the algorithm. The pessimistic option is to compute the exploitability of each combined strategy and average the results. This assumes the opponent knows the random numbers used by the algorithm for sampling in each resolving. A more realistic option is to average the combined strategies from different runs into an expected strategy \( \bar{\sigma} \) and compute its exploitability. We use the latter.

5.3 Domains

We choose the same domains as in [20] for direct comparison with prior work, Phantom Tic-Tac-Toe with thick public states. Detailed rules are in Appendix C and their sizes are summarized in Table 1.

**Biased Rock Paper Scissors** B-RPS is a version of standard game of rock-paper-scissors with the reward for the first player winning with rock over second player’s scissors 100 times larger than all the other rewards.

**Phantom Tic-Tac-Toe** PTTT is partially observable tic-tac-toe. Each player can see only his marks and learns the position of opponent’s marks if he attempts to play in an occupied cell. If the player is successful in marking a cell, the opponent takes an action in the next round. Otherwise, the player has to choose a cell again, until he makes a successful move. Hence, players might not know how many marks has the opponent discovered.

**Imperfect Information Goofspiel** IIGS(N) is a card game where each player is given a private hand of bid cards with values 0 to \( N − 1 \). Players simultaneously bid in tricks worth different number of points. After each trick players learn whether they won it, lost it, or it was a tie, but not which bid cards was used by their opponent. **Liar’s Dice** LD(D1, D2, F), also known as Dudo, Perudo, and Bluff is a dice-bidding game. Each die has \( F \) faces. Player \( i \) rolls \( D_i \) dice without showing them to their opponent. Each round, players alternate by bidding on the outcome of all dice in play until one player “calls liar”, i.e. claims that their opponent’s latest bid does not hold. **Generic Poker** GP(T, C, R, B) is a simplified poker game inspired by Leduc Hold’em. There are \( C \) cards of \( T \) different types, two betting rounds with at most of \( R \) raises limited to \( B \) different sizes.

There might be domains where MCCFR performs much better than competing algorithms, such as small variants of computer games (like Heroes of Might & Magic, Civilization) and we intend to evaluate these in future as well.

5.4 Results

**Averaging of Sampled CFVs.** As mentioned in Section 4.1.2, computing the exact counterfactual values of the average strategy \( \bar{\sigma}^T \) is often prohibitive, and we replace it by using the arithmetic or weighted averages instead. To compare the two approaches, we run MCCFR on the B-RPS domain (which only has a single NE \( \sigma^* \) to which \( \bar{\sigma}^T \) converges) and measure the distance \( \Delta \sigma(t) \) between the estimates and the correct action-values \( \nu^*_t \)\( (\text{root}, a) \).

| Game                  | \(|S|\) | \(|I|\) | \(|H|\) | \(|Z|\) | \(|\Omega|\) |
|-----------------------|--------|--------|--------|--------|--------|
| IIGS(5)               | 363    | 9948   | 41331  | 14400  | 13     |
| LD(1,1,6)             | 4098   | 24576  | 147456 | 147420 | 396    |
| GP(3,3,2,2)           | 2671   | 7920   | 23760  | 44883  | 45     |

Table 1: Sizes of the small variants of the evaluated games.
We compare exploitability (top) with expected “instability” of CFVs differences (middle) and its variance (bottom).

However, the MCCFR resolver will typically not have enough time to find such \( \sigma^T \). If MCCFR is to work, the CFVs computed by MCCFR need to be close to those of an approximate equilibrium CFVs even though they correspond to an exploitable strategy.

We run MCCFR in the root of the games and focus on CFVs in the public states where MCCFR might need to construct the resolving gadget for the first time, i.e. where player 1 acts for the 2nd time (the gadget is not necessary for the first action):

\[ \Omega := \{ J \in I_{\text{aug}}^T \cup S', h \in S : J \subset S' \& \text{pl. 1 played once in } h \}. \]

Since there are multiple equilibria, MCCFR might converge to, we cannot measure the convergence exactly. Instead, we measure the “instability” of CFV estimates by saving the weighted averages \( \tilde{v}^T_t(J) \) for \( t = 1, \ldots, T \), tracking how \( \Delta_t(J) := |\tilde{v}^T_t(J) - \tilde{v}^T_0(J)| \) changes over time, and aggregating it into \( \frac{1}{T} \sum_t \Delta_t(J) \). Finally, we run the experiment with 100 different MCCFR seeds and measure the expectation of the aggregates and, for the small domains, the expected exploitability of \( \sigma^T \). Large expectations indicate that the CFVs change a lot as we are approaching equilibrium. Ideally, we’d like to see these values to be very small at the point of \( 10^5 \) samples. This is the typical number of samples we can do with our time budgets. Small values long before the total number of samples is reached indicate that the CFVs already converged.

Figure 3 confirms that in small domains of LD and GP, CFVs are stabilized long before the exploitability of the computed strategy is low. This is not the case for IIGS and PTTT, where the error decreases, but not fast enough.

5.4.3 Comparison of Exploitability with Other Algorithms. We compare the exploitability of MCCR to OOS-PST and MCCFR, and include random player for reference. We do not include OOS-IST, whose exploitability is comparable to that of OOS-PST [20]. For an evaluation of IS-MCTS’s exploitability (which is very high, with the exception of poker) we refer the reader to [19, 20].

Figure 2 (right) confirms that for all algorithms, the exploitability decreases with the increased time per move. MCCCR is better than MCCFR on LD and worse on IIGS. The keep variant of MCCCR has smaller exploitability initially, but it decreases more slowly than the reset variant, which indicates the keep variant could be improved.

5.4.4 Influence of the Exploration Rate. One of MCCCR’s parameters is the exploration rate \( \epsilon \) of its MCCFR resolver. When measuring the exploitability of MCCR we observed no noteworthy influence of \( \epsilon \) (for \( \epsilon = 0.2, 0.4, 0.6, 0.8 \), across all of the evaluated domains).

5.4.5 Head-to-head Performance. For each pair of algorithms, thousands of matches have been played on each domain, alternating positions of player 1 and 2. We played more matches in domains where it was needed to ensure statistical significance of the winning player. In the smaller domains, players have 0.3s of pre-play computation and 0.1s per move. In the larger domains, it is 10 times as much. We report the results as percentages of the maximum achievable payoff in Table 2.

Note that the results of the matches are not necessarily transitive because they are not representative of what is algorithm’s strength.

The fact that “A beats B”, “B beats C” does not necessarily imply “A beats C”.
exploitability. For this reason previous experiment is a much better comparison of algorithm’s performance for domains where exact best response can be calculated.

The proposed algorithm significantly outperforms the random opponent in all games. MCCR outperforms MCCFR significantly in all domains besides small GS (where we know it is slightly more exploitable, according to previous experiment). MCCR is better than OOS-PST only in large LD. MCCR has similar performance, but winner is not statistically significant. MCCR does not outperform OOS-IST and the heuristic IS-MCTS variants.

6 CONCLUSION
We propose generalization of Continual Resolving from poker [22] to other extensive-form games. We show that the structure of the public tree may be more complex in general and we propose an extended version of the resolving gadget necessary to handle this complexity. Furthermore, both players may play in the same public state (possibly multiple times), and we extend the definition of Continual Resolving to allow this case as well. As a result, we present a completely domain-independent version of the algorithm that can be applied to any EFG, is sufficiently robust to use variable resolving schemes, and can be coupled with different resolving games and algorithms (including classical CFR, depth-limited search utilizing neural networks, or other domain-specific heuristics). We show that the existing theory naturally translates to this generalized case.

We further introduce Monte Carlo CR as a specific instance of this abstract algorithm that uses MCCFR as a resolver. It allows deploying continual resolving on any domain, without the need for expensive construction of evaluation functions. MCCR is theoretically sound as demonstrated by Theorem 4.1, constitutes an improvement over MCCFR in the online setting in terms head-to-head performance, and doesn’t have the restrictive memory requirements of OOS. The experimental evaluation shows that MCCCR is very sensitive to the quality of its counterfactual value estimates. With good estimates, its worst-case performance (i.e. exploitability) improves faster than that of OOS. In head-to-head matches MCCCR plays similarly to OOS, but it only outperforms IS-MCTS in one of the smaller tested domains. Note however that the lack of theoretical guarantees of IS-MCTS often translates into severe exploitability in practice [20], and this cannot be alleviated by increasing IS-MCTS’s computational resources [19]. In domains where MCCR’s counterfactual value estimates are less precise, its exploitability still converges to 0, but at a slower rate than OOS, and its head-to-head performance is noticeably weaker than that of both OOS and IS-MCTS. In the future work, the quality of MCCR’s estimates might be improved by techniques such as variance reduction [26], exploring ways of improving these estimates over the

Table 2: Head-to-head performance. Positive numbers mean that the row algorithm is winning against the column algorithm by the given percentage of the maximum payoff in the domain. Gray numbers indicate the winner isn’t statistically significant.
course of the game, or by finding an alternative source from which they can be obtained.

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APPENDIX

ABBREVIATIONS

CFR CounterFactual Regret minimization
CFV CounterFactual Value v
EFG Extensive Form Game
IIG Imperfect-Information Game
IS-MCTS Information Set Monte Carlo Tree Search
MCCFR Monte Carlo CounterFactual Regret minimization
MCCR Monte Carlo Continual Resolving
MCTS Monte Carlo Tree Search
NE Nash Equilibrium
OOS Online Outcome Sampling
OOS-IST OOS with Information Set Targeting
OOS-PST OOS with Public State Targeting
OS Outcome Sampling
RM Regret Matching
UCT Upper Confidence bound applied to Trees

A THE PROOF OF THEOREM 4.1

We now formalize the guarantees available for different ingredients of our continual resolving algorithm, and put them together to prove Theorem 4.1.

A.1 Monte Carlo CFR

The basic tool in our algorithm is the outcome sampling (OS) variant of MCCFR. In the following text, p ∈ (0, 1) and δ > 0 will be fixed, and we shall assume that the OS’s sampling scheme is such that for each z ∈ Z, the probability of sampling z is either 0 or higher than δ.

In [17, Theorem 4], it is proven that the OS’s average regret decreases with increasing number of iterations. This translates to the following exploitability bound, where ∆_u,i := max_z1,z2 [u_i(z1) - u_i(z2)] is the maximum difference between utilities and A_i := max_h |A(h)| is the player i branching factor.

**Lemma A.1 (MCCFR exploitability bound).** Let σ_T be the average strategy produced by applying T iterations of OS to some game G. Then with probability at least 1 − p, we have

\[
\hat{R}_T^i \leq \left( \frac{2}{p} + 1 \right) |I_i| \frac{\Delta_{u,i} \sqrt{A_i}}{\delta} \cdot \frac{1}{\sqrt{T}} \quad (6)
\]

**Proof.** The exploitability expl_i(σ_T) of the average strategy σ_T is equal to the average regret \( \hat{R}_T^i \). By Theorem 4 from [17], after T iterations of OS we have

\[
\hat{R}_T^i \leq \left( \frac{\sqrt{2} |I_i| |B_i|}{\sqrt{p}} + M_i \right) \frac{\Delta_{u,i} \sqrt{A_i}}{\delta} \cdot \frac{1}{\sqrt{T}}
\]

with probability at least 1 − p, where M_i, |B_i| ≤ |I_i| are some domain specific constants. The regret can then be bounded as

\[
\hat{R}_T^i \leq \left( \frac{2}{p} + 1 \right) |I_i| \frac{\Delta_{u,i} \sqrt{A_i}}{\delta} \cdot \frac{1}{\sqrt{T}},
\]

which concludes the proof. □

**Lemma A.2 (MCCFR value approximation bound).** Let \( S \in \mathcal{S} \). Under the assumptions of Lemma A.1, we further have

\[
\sum_{I \in \mathcal{S}(2)} |v_2^T(I) - v_2^T(CBR(\sigma_T))(I)| \leq \left( \sqrt{\frac{2}{p} + 1} \right) |I| \frac{\Delta_{u,i} \sqrt{A_i}}{\delta} \cdot \frac{1}{\sqrt{T}}
\]

**Proof.** Consider the full counterfactual regret of player 2’s average strategy, defined in [29, Appendix A.1]:

\[
\hat{R}_{2,full}^i(I) := \frac{1}{T} \max_{\sigma' \in \mathcal{S}} \sum_{t=1}^T \left( v_2(I) - v_2(\sigma_T)(I) \right) .
\]

where D(I) ⊂ I_2 contains I and all its descendants. Since CBR_2(σ_T) is one of the strategies σ'_T which maximize the sum in (7), we have

\[
\hat{R}_{2,full}^i(I) \leq v_2^T(CBR(\sigma_T))(I) - v_2^T(I),
\]

Consider now any \( S \in \mathcal{S} \). By [29, Lemma 6], we have

\[
\sum_{I \in \mathcal{S}(2)} |v_2^T(I) - v_2^T(CBR(\sigma_T))(I)| \leq \sum_{I \in \mathcal{S}(2)} \hat{R}_{2,full}^i(I)
\]

\[
\leq \sum_{I \in \mathcal{S}(2)} \sum_{J \in D(I)} \hat{R}_{2,full}^i(J) \leq \sum_{J \in I_2} \hat{R}_{2,full}^{i+}(J).
\]

The proof is now complete, because the proof of [17, Theorem 4], which we used to prove Lemma A.1, actually shows that the sum \( \sum_{J \in I_2} \hat{R}_{2,full}^{i+}(J) \) is bounded by the right-hand side of (6). □

A.2 Gadget Game Properties

The following result is a part of why resolving gadget games are useful, and the reason behind multiplying all utilities in \( G(S, \sigma, v_2) \) by \( \pi_2^o(S) \).

**Lemma A.3 (Gadget game preserves opponent’s values).** Let \( S \in \mathcal{S}, \sigma, \bar{\sigma} \in \Sigma \) and let \( \tilde{\sigma} \) be any strategy in the resolving game \( G(S, \sigma, \bar{\sigma}) \) (where \( \bar{\sigma} \) is arbitrary). Denote \( \sigma_{new} := \sigma_{G(S,\bar{\sigma})}^\tilde{\sigma} \). Then for each \( I \in T_{2}^{aug}(G(S)) \), we have \( v_2^\sigma_{new}(I) = v_2^\sigma(I) \).

Note that the conclusion does not hold for counterfactual values of the (resolving) player 1! (This can be easily verified by hand in any simple game such as matching pennies.)

**Proof.** In the setting of the lemma, it suffices to show that \( v_2^\sigma(h) \) is equal to \( v_2^\sigma_{new}(h) \) for every \( h \in G(S) \). Let \( h \in G(S) \) and denote by \( g \) the prefix of \( h \) that belongs to \( S \).

Recall that \( \tilde{g} \) is the parent of \( g \) in the resolving game, and that the reach probability \( \pi_{\tilde{g}}^\sigma(h) \) of \( g \) in the resolving game (for any strategy \( \tilde{\sigma} \)) is equal to \( \pi_{\tilde{g}}^\sigma(h) / \pi_{\tilde{g}}^\sigma(S) \). Since \( \hat{u}_2(z) = u_2(z) \pi_{\tilde{g}}^\sigma(S) \) for any \( h \in z \in Z \), the definition of \( \sigma_{new} \), gives \( \hat{u}_2^\sigma(h) = u_2^\sigma_{new}(h) \pi_{\tilde{g}}^\sigma(S) \). The following series of identities then completes the proof:
Under the assumptions of Lemma A.3, we have the following.

\[
\sigma \in \pi_1 G(S),
\]

Suppose that MCCR is run with parameters \(T \) and \(R \) is the required number of resolvings, \( \delta = \min_{z \in \mathbb{Z}} q_i(z) \) where \( q_i(z) \) is the probability of sampling \( z \in \mathbb{Z} \) at iteration \( t \). Let \( \Delta_{u,1} = \max_{z \in \mathbb{Z}} |u(z) - u(z')| \) and \( A_i = \max_{z \in \mathbb{Z}} |A(z)| \).

Proof. Without loss of generality, we assume that MCCR acts as player 1. We prove the theorem by induction. The initial step of the proof is different from the induction steps, and goes as follows. Let \( S_0 \) be the root of the game and denote by \( \sigma^0 \) the strategy obtained by applying \( T_0 \) iterations of MCCFR to \( G \). Denote by \( \epsilon^0 \) the upper bound on \( \exp_1(\sigma^0) \) obtained from Lemma A.1. If \( S_1 \) is the first encountered public state where \( \rho(\sigma^0) = \epsilon^0 \), then by Lemma A.2 \( \sum_{I \in S_1} \left| \sigma^0(I) - \sigma^0(I) \right| \) is bounded by some \( \epsilon^{A-1} \). This concludes the initial step.

For the induction step, suppose that \( n \geq 1 \) and player 1 has already acted \( (n-1) \)-times according to some strategy \( \sigma^{n-1} \) with \( \exp_1(\sigma^{n-1}) \leq \epsilon^{A-1} \), and is now in a public state \( S_n \) where he needs to act again. Moreover, suppose that there is some \( \epsilon^{A-1} \geq 0 \) s.t.

\[
\sum_{S_n} \left| \sigma^{n-1}(I) - \sigma^{n-1}(I) \right| \leq \epsilon^{A-1}.
\]

We then obtain some strategy \( \rho_n \) by resolving the game \( \tilde{G}_n := G \{ S_n, \sigma^*, \tilde{v}^2 \} \) by \( T_R \) iterations of MCCFR. By Lemma A.1, the exploitability \( \exp_1(\rho_n) \) in \( \tilde{G}_n \) is bounded by some \( \epsilon^{A-1} \). We choose our next action according to the strategy

\[
\sigma^n := \sigma^{n-1} \big|_{G(S_n), \sigma^{n-1}} - \rho_n.
\]

By Lemma A.5, the exploitability \( \exp_1(\sigma^n) \) is bounded by \( \epsilon^{A-1} + \epsilon^{A-1} + \epsilon^{A-1} \). The game then progresses until it either ends without player 1 acting again, or reaches a new public state \( S_{n+1} \) where player 1 acts.

If such \( S_{n+1} \) is reached, then by Lemma A.2, the value approximation error \( \sum_{S_{n+1}} \left| \tilde{v}^2(I) - \tilde{v}^2(I) \right| \) in the resolving gadget game is bounded by some \( \epsilon^{A-1} \). By Lemma A.3, this sum is equal to the value approximation error

\[
\sum_{S_{n+1}} \left| \tilde{v}^2(I) - \tilde{v}^2(I) \right| \geq \epsilon^{A-1}.
\]

in the original game. This concludes the inductive step.

Eventually, the game reaches a terminal state after visiting some sequence \( S_1, \ldots, S_n \) of public states where player 1 acted by using the strategy \( \sigma := \sigma^N \). We now calculate the exploitability of \( \sigma \). It follows from the induction that

\[
\exp_1(\sigma) \leq \epsilon^0 + \epsilon^0 + \epsilon^1 + \epsilon^1 + \epsilon^2 + \epsilon^2 + \cdots + \epsilon^{A-1} + \epsilon^{A-1}.
\]

To emphasize which variables come from using \( T_0 \) iterations of MCCFR in the original game and which come from applying \( T_R \),

\[
\exp_1(\sigma) \leq \epsilon^0 + \epsilon^0 + \epsilon^1 + \epsilon^1 + \epsilon^2 + \epsilon^2 + \cdots + \epsilon^{A-1} + \epsilon^{A-1}.
\]
iterations of MCCFR to the resolving game, we set $\epsilon^{R}_n := \epsilon_n^{R}$ and $\bar{\epsilon}_n := \bar{\epsilon}_n$ for $n \geq 1$. We can then write

$$\exp_1(\sigma) \leq \epsilon_N^{E} + \epsilon_1^{A} + \sum_{n=1}^{N-1} (\epsilon_n^{R} + \bar{\epsilon}_n^{A}) + \epsilon_N^{R}. \quad (9)$$

Since the bound from Lemma A.1 is strictly higher than the one from Lemma A.2, we have $\epsilon_0^{E} \leq \epsilon_0^{R}$ and $\epsilon_1^{A} \leq \epsilon_1^{R}$. Moreover, we have $G(S_1) \supset G(S_2) \supset \ldots \supset G(S_N)$, which means that $\epsilon_1^{R} \geq \epsilon_2^{R} \geq \ldots \geq \epsilon_N^{R}$. It follows that $\exp_1(\sigma) \leq 2\epsilon_0^{E} + (2N-1)\epsilon_1^{R}$. Finally, we clearly have $N \leq D_1$, where $D_1$ is the “player 1 depth of the public tree of $G$”. This implies that

$$\exp_1(\sigma) \leq 2\epsilon_0^{E} + (2D_1-1)\epsilon_1^{R}. \quad (10)$$

Plugging in the specific numbers from Lemma A.1 for $\epsilon_0^{E}$ and $\epsilon_1^{R}$ gives the exact bound, and noticing that we have used the lemma $(D_1 + 1)$-times implies that the result holds with probability $(1 - p)^{D_1} \geq 1$. □

Note that a tighter bound could be obtained if we were more careful and plugged in the specific bounds from Lemma A.1 and A.2 into (9), as opposed to using (10). Depending on the specific domain, this would yield something smaller than the current bound, but higher than $\epsilon_0^{E} + \epsilon_1^{R}$.

**B COMPUTING COUNTERFACTUAL VALUES ONLINE**

For the purposes of MCCR, we require that our solver (MCCFR) also returns the counterfactual values of the average strategy. The straightforward way of ensuring this is to simply calculate the counterfactual values once the algorithm has finished running. However, this might be computationally intractable in larger games, since it potentially requires traversing the whole game tree. One straightforward way of fixing this issue is to replace this exact computation by a sampling-based evaluation of $\bar{\sigma}^T$. With a sufficient number of samples, the estimate will be reasonably close to the actual value.

In practice, this is most often solved as follows. During the normal run of MCCFR, we additionally compute the opponent’s sampled counterfactual values

$$\tilde{\sigma}_2^T(i) := \frac{1}{\pi^\sigma(z)} \sum_{z} \pi_2^T(h) \pi^\sigma(z|h) u_2(z).$$

Once the $T$ iterations are complete, the counterfactual values of $\bar{\sigma}^T$ are estimated by $\tilde{\bar{\sigma}}(i) := \frac{1}{T} \sum_{t=1}^{T} \tilde{\sigma}_2^T(i)$. While this arithmetical average is the standard way of estimating $\bar{\sigma}_2^T(i)$, it is also natural to consider alternatives where the uniform weights are replaced by either $\pi_i^T(i)$ or $\pi\sigma^T(i)$. In principle, it is possible for all of these weighting schemes to fail (see the counterexample in Section B.1). We experimentally show that all of these possibilities produce good estimates of $\bar{\sigma}_2^T$ in many practical scenarios, see Figure 4. Even when this isn’t the case, one straightforward way to fix this issue is to designate a part of the computation budget to a sampling-based evaluation of $\bar{\sigma}^T$. Alternatively, in Lemma B.2 we present a method inspired by lazy-weighted averaging from [17] that allows for computing unbiased estimates of $\bar{\sigma}_2^T$ on the fly during MCCFR.

In the main text, we have assumed that the exact counterfactual values are calculated, and thus that $\tilde{\bar{\sigma}}(i) \approx \bar{\sigma}_2^T(i)$. Note that this assumption is made to simplify the theoretical analysis – in practice, the difference between the two terms can be incorporated into Theorem 4.1 (by adding the corresponding term into Lemma A.5).

**B.1 The Counterexample**

In this section we show that no weighting scheme can be used as a universal method for computing the counterfactual values of the average strategy on the fly. We then derive an alternative formula for $\bar{\sigma}_2^T$ and show that it can be used to calculate unbiased estimates of $\bar{\sigma}_2^T$ in MCCFR. Note that this problem is not specific to MCCFR, but also occurs in CFR (although it is not so pressing there, since CFR’s iterations are already so costly that the computation of exact counterfactual values of $\bar{\sigma}^T$ is not a major expense). But since these issues already arise in CFR, we will work in this simpler setting.

Suppose we have a history $h \in \mathcal{H}$, strategies $\sigma_1, \ldots, \sigma_T$ and the average strategy $\bar{\sigma}^T$ defined as

$$\bar{\sigma}^T(i) := \sum_{T} \pi_i^\sigma(h) \sigma^T(i) / \sigma_i^\sigma(h). \quad (11)$$

Figure 4: An extension of experiment 5.4.1 and Figure 3 for small domains. We calculate the exact values $\bar{u}(h)$, and compute the absolute differences of each weighing sampling scheme ($\bar{u}^1(h)$ and $\bar{u}^2(h)$) to exact values. Those differences are then averaged across information sets and seeds. In each domain, the weighted averages (solid) have smaller error than ordinary averages (dashed) and are thus better approximations of exact values.
for $l \in I_i$. First, note that we can easily calculate $\pi^T_{\bar{o}(h)}(h)$. Since $u^T_i(h) = \pi^T_{\bar{o}}(h)u^T_i(h)$, an equivalent problem is that of calculating the expected utility of the average strategy at $h$ on the fly, i.e. by using $u^T_i(h)$ and possibly some extra variables, but without having to traverse the whole tree below $h$.

Looking at the definition of the average strategy, the most natural candidates for an estimate of $u^T_i(h)$ are the following weighted averages of $u^T_i(h)$:

$$\hat{u}^T_i(h) := \frac{\sum_t u^T_i(h)}{2}$$

$$\hat{u}^T_i(h) := \frac{\sum_t \pi^T_{\bar{o}}(h) u^T_i(h)}{2}$$

$$\hat{u}^T_i(h) := \frac{\sum_t \pi^T(h) u^T_i(h)}{2}$$

Figure 5: A domain where weighting schemes for $u^T_i$ fail.

Example B.1 (No weighting scheme works). Each of the estimates $\hat{u}(i, j)$, $j = 1, 2, 3$, can fail to be equal to $u^T_i$. Yet worse, no similar works reliably for every sequence $(\sigma^T_i)_t$.

Consider the game from Figure 5 with $T = 2$, where under $\sigma^T_i$, each player always goes right (R) and under $\bar{\sigma}^T_i$, the probability of going right is $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$ and $\frac{1}{3}$ at $h_0, h_1, h_2$ and $h_3$ respectively. A straightforward application of (11) shows that the probabilities of $R$ under $\bar{\sigma}^T_i$ are $\frac{3}{4}, \frac{3}{4}, \frac{3}{4}$ and $\frac{3}{4}$ and hence $u^T_i(h) = \frac{15}{12} \frac{11}{12} = \frac{121}{144}$. On the other hand, we have $\hat{u}^T_i(h_2) = \frac{108}{121} \frac{11}{12} = \frac{140}{121}$ and $\hat{u}^T_i(h_3) = \frac{181}{121} \frac{11}{12}$.

To prove the “yet worse” part, consider also the sequence of strategies $v^T_i$, $v^T_i$, where $v^T_i = \sigma^T_i$ and under $v^T_i$, the probabilities of going right are $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$ and $\frac{1}{3}$ at $h_0, h_1, h_2$ and $h_3$ respectively. The probabilities of $R$ under $v^T_i$ are $\frac{3}{4}, \frac{3}{4}, \frac{3}{4}$ and $\frac{3}{4}$ and hence $u^T_i(h_2) = \frac{121}{121} \frac{11}{12} = \frac{121}{144} \frac{11}{12} = \frac{140}{121}$ and $\hat{u}^T_i(h_3) = \frac{181}{121} \frac{11}{12}$.

Using the definition of the expected utility, we can rewrite $\pi_{\bar{o}}^T_i(h)$ as

$$\pi_{\bar{o}}^T_i(h) = \frac{\sum_t \pi^T_i(h) u^T_i(h)}{\sum_t \pi^T_i(h)}$$

for $z \in \mathbb{Z}$. Then we remark that during MCCFR it suffices to keep track of $\bar{\pi}_{\bar{o}}^T_i(ha)$ where $a \in \mathcal{A}(h)$ and $h$ is in the tree built by MCCFR, and show how these values can be calculated similarly to the lazy-weighted averaging of [17]. Lastly, we note that a sampled variant can be used in order to get an unbiased estimate of $u^T_i$.

Recall the standard fact that the average strategy satisfies

$$u^T_i(h) = \frac{\sum_t \pi^T_i(h) u^T_i(h)}{\sum_t \pi^T_i(h)}$$

for every $h \in \mathcal{H}$, and has the analogous property for $\bar{\sigma}^T_i$. Indeed, this follows from the formula

$$\pi^T_i(h) = \frac{1}{T} \sum_t \pi^T_i(h)$$

which can be proven by induction over the length of $h$ using (11).

Lemma B.2. For any $h \in \mathcal{H}$, $i$ and $\sigma^T_i, \ldots, \sigma^T_T$, we have $\bar{\pi}_{\bar{o}}^T_i(h) = \frac{1}{T} \sum_t \pi^T_i(z) c_{p}(s) c_{p}(s) c_{p}(s) - \pi^T_i(z) \pi_{\pi}(x) u^T_i(z)$.

Proof. For $h \in \mathcal{H}$, we can rewrite $\bar{\pi}_{\bar{o}}^T_i(h)$ as

$$\bar{\pi}_{\bar{o}}^T_i(h) = \frac{\pi^T_i(h) u^T_i(h)}{\pi^T_i(h)} = \frac{\pi^T_i(h) u^T_i(h)}{\pi^T_i(h)}$$

Using the definition of the expected utility, we can rewrite the numerator $N$ as

$$N = \sum_{s,t} \pi^T_i(h) \pi^T_i(h) \sum_{z \in \mathcal{H}} \pi^T_i(z) c_{p}(s) c_{p}(s) c_{p}(s) \pi_{\pi}(x) u^T_i(z)$$

$$= \sum_{s,t} \pi^T_i(z) c_{p}(s) c_{p}(s) c_{p}(s) \pi_{\pi}(x) u^T_i(z)$$

The double sum over $s$ and $t$ can be rewritten using the formula

$$\sum_t \sum_{x \in \mathcal{X}} x_t y_t = \sum_t \sum_{x \in \mathcal{X}} [x_t (y_1 + \cdots + y_t) + (x_1 + \cdots + x_t) y_t - x_t y_t]$$

which yields $\sum_{s,t} \pi^T_i(z) c_{p}(s) c_{p}(s) c_{p}(s)$ as

$$= \sum_t \pi^T_i(z) c_{p}(s) c_{p}(s) c_{p}(s) + \cdots + \pi^T_i(z) c_{p}(s) c_{p}(s) c_{p}(s)$$

Substituting this into the formula for $N$ and $\pi^T_i$ concludes the proof.
B.3 Computing Cumulative Reach Probabilities

While it is intractable to store \( \text{crp}_t(z) \) in memory for every \( z \in \mathcal{Z} \), we can store the cumulative reach probabilities for nodes in the tree \( \mathcal{T}_t \) built by MCCFR at time \( t \). We can translate these into \( \text{crp}_t(z) \) with the help of the uniformly random strategy \( \text{rnd} \):

Lemma B.3. Let \( z \in \mathcal{Z} \) be s.t. \( z \ni h \), where \( h \) is a leaf of \( \mathcal{T}_t \) and \( a \in \mathcal{A}(h) \). Then we have \( \text{crp}_t(z) = \text{crp}_t(h) \pi_t^{\text{rnd}}(z|a) \).

Proof. This immediately follows from the fact that for any \( g \not\in \mathcal{T}_t \), \( \pi_t(g) = \text{rnd}(g) \) for every \( s = 1, 2, \ldots, t \). \( \square \)

To keep track of \( \text{crp}_t(h) \) for \( h \in \mathcal{T}_t \), we add to it a variable \( \text{crp}_t(h) \) and auxiliary variables \( \text{w}_i(h), a \in \mathcal{A}(h) \), which measure the increase in cumulative reach probability since the previous visit of \( ha \). All these variables are initially set to 0 except for \( \text{w}_i(\emptyset) \) which is always assumed to be equal to 1. Whenever MCCFR visits some \( h \in \mathcal{T}_t \), is visited, \( \text{crp}_t(h) \) is increased by \( \text{w}_i(h) \) (stored in \( h \)'s parent), each \( \text{w}_i(h) \) is increased by \( \text{w}_i(h) \pi_t^q(h|a) \) and \( \text{w}_i(h) \) (in the parent) is set to 0. This ensures that whenever a value \( \text{w}_i(h) \) gets updated without being reset, it contains the value \( \text{crp}_t(h) = \text{crp}_t(ha) \), where \( t_{ha} \) is the previous time when \( ha \) got visited. As a consequence, the variables \( \text{crp}_t(h) \) that do get updated are equal to \( \text{crp}_t(h) \). Note that this method is very similar to the lazy-weighted averaging of [17].

Finally, we observe that the formula from Lemma B.2 can be used for on-the-fly calculation of an unbiased estimate of \( u^*_t(h) \). Indeed, it suffices to replace the sum over \( z \) by its sampled variant \( \tilde{s}_t^i(h) := \frac{1}{q^t(z)} \left( \pi^q_t(z) \text{crp}_t^i(z) + \text{crp}_t^i(z) \pi^g_t(z) - \pi^g_t(z) \right) \pi_c(z|h) u_i(z) \), (14) where \( z \) is the terminal state sampled at time \( t \) and \( q^t(z) \) is the probability that it got sampled with \( z \). We keep track of the cumulative sum \( \sum_t \tilde{s}_t^i(h) \) and, once we reach iteration \( T \), we do one last update of \( h \) and set \( \bar{u}_i(h) := \frac{\sum_t \tilde{s}_t^i(h)}{\text{crp}_T^i(h)} \bar{c}_t^i(h) \) and \( \bar{v}_i(h) := \pi_c^T(h) \bar{u}_i(h) \).

By (14) and Lemma B.2, we have \( \text{E}\bar{u}_i(h) = u^*_i(h) \) and thus \( \text{E}\bar{v}_i(h) = v^*_i(h) \). Note that \( \bar{v}_i(h) \) might suffer from a very high variance and devising its low-variance modification (or alternative) would be desirable.

C. GAME RULES

Biased Rock Paper Scissors B-RPS is a version of standard game of Rock-Paper-Scissors with modified payoff matrix:

|     | R   | P   | S   |
|-----|-----|-----|-----|
| R   | 0   | -1  | 100 |
| P   | 1   | 0   | -1  |
| S   | -1  | 1   | 0   |

This variant gives the first player advantage and breaks the game action symmetry.

Phantom Tic-Tac-Toe PTTT Phantom Tic-Tac-Toe is a partially observable variant of Tic-Tac-Toe. It is played by two players on 3x3 board and in every turn one player tries to mark one cell. The goal is the same as in perfect-information Tic-Tac-Toe, which is to place three consecutive marks in a horizontal, vertical, or diagonal row.

Player can see only his own marked cells, or the marked cells of the opponent if they have been revealed to him by his attempts to place the mark in an occupied cell.

If the player is successful in marking the selected cell, the opponent takes an action in the next round. Otherwise, the player has to choose a cell again, until he makes a successful move.

The opponent receives no information about the player’s attempts at moves.

Imperfect Information Goofspiel In 11-GS(N), each player is given a private hand of bid cards with values 0 to \( N-1 \). A different deck of \( N \) point cards is placed face up in a stack. On their turn, each player bids for the top point card by secretly choosing a single card in their hand. The highest bidder gets the point card and adds the point total to their score, discarding the points in the case of a tie. This is repeated \( N \) times and the player with the highest score wins.

In Il-Goofspiel, the players only discover who won or lost a bid, but not the bid cards played. Also, we assume the point-stack is strictly increasing: 0, 1, \ldots, \( N-1 \). This way the game does not have chance nodes, all actions are private and information sets have various sizes.

Liar’s Dice LD(D1,D2,F), also known as Dudo, Perudo, and Bluff is a dice-bidding game. Each die has faces 1 to \( F-1 \) and a star \( \star \). Each player rolls \( D_i \) of these dice without showing them to their opponent. Each round, players alternate by bidding on the outcome of all dice in play until one player “calls liar”, i.e. claims that their opponent’s latest bid does not hold. If the bid holds, the calling player loses; otherwise, she wins. A bid consists of a quantity of dice and a face value. A face of \( \star \) is considered wild and counts as matching any other face. To bid, the player must increase either the quantity or face value of the current bid (or both).

All actions in this game are public. The only hidden information is caused by chance at the beginning of the game. Therefore, the size of all information sets is identical.

Generic Poker GP(T, C, R, B) is a simplified poker game inspired by Leduc Hold’em. First, both players are required to put one chip in the pot. Next, chance deals a single private card to each player, and the betting round begins. A player can either fold (the opponent wins the pot), check (let the opponent make the next move), bet (add some amount of chips, as first in the round), call (add the amount of chips equal to the last bet of the opponent into the pot), or raise (match and increase the bet of the opponent).

If no further raise is made by any of the players, the betting round ends, chance deals one public card on the table, and a second betting round with the same rules begins. After the second betting round ends, the outcome of the game is determined - a player wins if: (1) her private card matches the table card and the opponent’s card does not match, or (2) none of the players’ cards matches the table card and her private card is higher than the private card of the opponent. If no player wins, the game is a draw and the pot is split.
The parameters of the game are the number of types of the cards $T$, the number of cards of each type $C$, the maximum length of sequence of raises in a betting round $R$, and the number of different sizes of bets $B$ (i.e., amount of chips added to the pot) for bet/raise actions.

This game is similar to Liar’s Dice in having only public actions. However, it includes additional chance nodes later in the game, which reveal part of the information not available before. Moreover, it has integer results and not just win/draw/loss.

No Limit Leduc Hold’em poker with maximum pot size of $N$ and integer bets is $GP(3, 2, N, N)$.

D EXTENDED RESULTS
Table 3: Comparison of counterfactual values for different domains by tracking how absolute differences $\Delta_r(J) = |\hat{v}_2(J) - \hat{v}_1(J)|$ change over time.

Table 4: Comparison of counterfactual values for different domains by tracking how relative differences $\Delta_r(J) = \hat{v}_2(J) - \hat{v}_1(J)$ change over time.
Table 5: Sensitivity to exploration parameter. Top row is "reset" variant, bottom row is "keep" variant.