Spectral asymptotics for a class of integro-differential equations arising in the theory of fractional Gaussian processes

Alexander I. Nazarov

Abstract

We study the spectral problems for integro-differential equations arising in the theory of Gaussian processes similar to the fractional Brownian motion. We generalize the method of Chigansky–Kleptsyna and obtain the two-term eigenvalues asymptotics for such equations. Application to the small ball probabilities in $L_2$-norm is given.

1 Introduction

The spectral analysis of Gaussian processes is intensively studied in last two decades, in particular, in the context of the problem of small deviation asymptotics in Hilbert norm for such processes.

It is known, see [14], that to obtain the logarithmic $L_2$-small ball asymptotics of a Gaussian process $X$, we need just one-term asymptotics of the counting function for the eigenvalues of its covariance operator. However, to manage the exact asymptotics (up to a constant), we need at least two-term asymptotics of the eigenvalues with the remainder estimate ([11], see also [9]).

The last problem is quite delicate and was solved only for several special processes. Most of them are so-called Green Gaussian processes. This means that the covariance function $G_X$ is the Green function for the ordinary differential operator (ODO) subject to proper homogenous boundary conditions.

*St.Petersburg Department of Steklov Institute, Fontanka 27, St.Petersburg, 191023, Russia, and St.Petersburg State University, Universitetskii pr. 28, St.Petersburg, 198504, Russia. E-mail: al.il.nazarov@gmail.com.
This class contains many classical processes, e.g., the integrated Brownian motion, the Slepian process and the Ornstein-Uhlenbeck process. The special nature of the Green Gaussian processes allows to use the well-developed techniques of spectral theory for ODOs, see, e.g., [13]. This approach was elaborated in [18], [15] and was used in a number of papers, see, e.g., [20] and references therein. We mention also the papers [16], [21], [24] where the two-term spectral asymptotics was obtained for finite-dimensional perturbations of the Green Gaussian processes.

The case of fractional Gaussian processes is much more complicated. Until recently, only the main term of spectral asymptotics was known, and thus, only logarithmic small ball asymptotics was obtained for such processes. Namely, in the pioneer paper [4] the one-term spectral asymptotics was calculated for the fractional Brownian motion (FBM) $W^H$, that is the zero mean-value Gaussian process with covariance function

$$G(x, y) := G_{W^H}(x, y) = \frac{1}{2} \left( x^{2H} + y^{2H} - |x-y|^{2H} \right)$$

(here $H \in (0, 1)$ is the so-called Hurst index, the case $H = \frac{1}{2}$ corresponds to the standard Wiener process).

A general approach was developed in [19]. This approach is based on the powerful theorems on spectral asymptotics of integral operators [1], see also [2, Appendix 7], and covers many fractional processes. However, it also gives only the one-term eigenvalues asymptotics.

A fundamental step was managed in the breakthrough paper [6]. The eigenproblem for the covariance operator of $W^H$ on $[0, 1]$ was reduced to the generalized eigenproblem

$$\langle K_\alpha \psi \rangle(x) = -\lambda \psi''(x), \quad x \in (0, 1)$$

with boundary conditions $\psi'(0) = \psi(1) = 0$. Here we use the notation $\alpha = 2 - 2H \in (0, 2) \setminus \{1\}$, and

$$\langle K_\alpha \psi \rangle(x) = (1 - \alpha/2) \frac{d}{dx} \int_0^1 \text{sign}(x - y) |x - y|^{1-\alpha} \psi(y) \, dy.$$

By the Laplace transform

$$\hat{\psi}(z) = \int_0^1 \psi(y) \exp(-zy) \, dy$$
the problem \( (1) \) was reduced to the Riemann–Hilbert problem which, in turn, was solved asymptotically using the ideas of \([25], [23]\) (see some additional references in \([6]\)). In this way the two-term asymptotics of the eigenvalues with the remainder estimate was obtained for the FBM in the full range of the Hurst index. On this base, the exact \( L_2 \)-small ball asymptotics for \( W^H \) was established for the first time, along with some other applications. Notice also that the eigenfunctions asymptotics for FBM was obtained in \([6]\) as well.

In later papers \([7], [8]\) similar results were obtained for some other fractional Gaussian processes.

In this paper we suggest a slightly more general point of view and consider the eigenproblem \( (1) \) with general self-adjoint boundary conditions. This gives a unified approach to the previous results and covers several new fractional Gaussian processes.

The paper is organized as follows. In Section 2 we calculate the two-term spectral asymptotics of the problem \( (1) \) with arbitrary self-adjoint boundary conditions that do not contain the spectral parameter. Here we mainly follow the line of \([6]\). It turns out that there are three possible “shifts” of the second term in the asymptotics depending on the sum of orders of boundary conditions. It is well known, see \([18, \text{Theorem 7.1}]\) and \([15, \text{Theorem 1.1}]\), that this parameter drives the second term of spectral asymptotics for ODOs of arbitrary order. We conjecture that this is the case also for general eigenproblems of the type \( (1) \) with ODO of arbitrary order in the right-hand side.

In Section 3 we consider a more general eigenproblem

\[
(K_\alpha \psi)(x) = \lambda \left( -\psi''(x) + p(x)\psi(x) \right), \quad x \in (0, 1), \tag{2}
\]

with self-adjoint boundary conditions. We prove that the additional term in \( (2) \) can be considered as a weak perturbation of the problem \( (1) \) which does not influence upon the two-term eigenvalues asymptotics.

In Section 4 we give several examples of fractional Gaussian processes covered by the results of Sections 2 and 3.

Finally, in Section 5 we collect the results on \( L_2 \)-small ball probabilities for the fractional processes considered in Section 4.
2 Analysis of the problem (1) with general boundary conditions

First we consider in detail the case $\alpha < 1$. In this case the problem (1) reads as follows:

$$(1 - \alpha/2)(1 - \alpha) \int_{0}^{1} |x - y|^{-\alpha} \psi(y) \, dy = \lambda \psi''(x)$$

with the same boundary conditions.

2.1 Transformation of the problem

Following [6, Sec. 5.1] we define 

$$u(x, t) := \int_{0}^{1} \exp(-t|x - y|) \psi(y) \, dy; \quad u_0(x) = \int_{0}^{\infty} t^{\alpha-1} u(x, t) \, dt.$$ 

Then (3) becomes

$$c_\alpha u_0(x) = -\lambda \psi''(x), \quad c_\alpha = \frac{(1 - \alpha/2)(1 - \alpha)}{\Gamma(\alpha)}.$$

The Laplace transform gives 

$$\hat{u}_0(z) = -\frac{\lambda}{c_\alpha} \left( z^2 \hat{\psi}(z) + \exp(-z)(\psi'(1) + z\psi(1)) - (\psi'(0) + z\psi(0)) \right).$$

On the other hand, 

$$(z^2 - t^2)\hat{u}(z, t) = u(0, t)(z + t) - \exp(-z)u(1, t)(z - t) - 2t\hat{\psi}(z),$$

i.e. for $z \notin \mathbb{R}$

$$\hat{u}_0(z) = \int_{0}^{\infty} \frac{t^{\alpha-1}}{z - t} u(0, t) \, dt - \exp(-z) \int_{0}^{\infty} \frac{t^{\alpha-1}}{z + t} u(1, t) \, dt - \hat{\psi}(z) \int_{0}^{\infty} \frac{2t^\alpha}{z^2 - t^2} \, dt.$$
So, we obtain
\[
\left( \frac{\lambda}{c_\alpha} z^2 - \int_0^\infty \frac{2t^\alpha}{z^2 - t^2} \, dt \right) \hat{\psi}(z) = \frac{\lambda}{c_\alpha} (\psi'(0) + z\psi(0)) + \int_0^\infty \frac{t^{\alpha-1}}{t - z} u(0, t) \, dt
\]
\[- \exp(-z) \left( \frac{\lambda}{c_\alpha} (\psi'(1) + z\psi(1)) - \int_0^\infty \frac{t^{\alpha-1}}{z + t} u(1, t) \, dt \right),
\]
and thus
\[
z\hat{\psi}(z) = \frac{1}{\Lambda(z)} (\exp(-z)\Psi(-z) + \Phi(z)),
\]
where
\[
\Lambda(z) = \frac{\lambda}{c_\alpha} z + 1^\infty \int_0^\infty \frac{2t^\alpha}{t^2 - z^2} \, dt
\]
\[
= \frac{\lambda}{c_\alpha} z + z^{\alpha-2} \frac{\pi \exp(\pm i\pi (1 - \alpha)/2)}{\cos(\pi \alpha/2)} , \quad \Im(z) \geq 0;
\]
\[
\Phi(z) = \frac{\lambda}{c_\alpha} (\psi'(0) + z\psi(0)) + \int_0^\infty \frac{t^{\alpha-1}}{t - z} u(0, t) \, dt;
\]
\[
\Psi(z) = - \frac{\lambda}{c_\alpha} (\psi'(1) - z\psi(1)) + \int_0^\infty \frac{t^{\alpha-1}}{z + t} u(1, t) \, dt.
\]

The function \( \Lambda \) is defined in \( \mathbb{C} \setminus \mathbb{R} \), has two purely imaginary zeros
\[
\pm z_0 = i\nu, \quad \nu^{\alpha-3} = \frac{\lambda \cos(\pi \alpha/2)}{c_\alpha \pi}
\]
and has limits on the real axis
\[
\Lambda^\pm(t) := \lim_{z \to t \mp i0} \Lambda(z) = \frac{\lambda}{c_\alpha} t \pm |t|^{\alpha-2} \begin{cases} 
\frac{\pi \exp(i\pi (1 \mp \alpha)/2)}{\cos(\pi \alpha/2)}, & t > 0; \\
\frac{\pi \exp(i\pi (1 \pm \alpha)/2)}{\cos(\pi \alpha/2)}, & t < 0.
\end{cases}
\]
The following relations hold true:

\[ \Lambda^-(t) = \Lambda^+(t) = -\Lambda^+(-t). \quad (7) \]

We introduce the function \( \theta(t) = \text{arg}(\Lambda^+(t)) = \pi - \theta(-t) \) and notice that (6) implies

\[ \theta_0(t) := \theta(\nu t) = \arctan \frac{\sin\left(\frac{\pi(1-\alpha)}{2}\right)}{\cos\left(\frac{\pi(1-\alpha)}{2}\right) + t^{3-\alpha}}, \quad t > 0. \quad (8) \]

Evidently, \( \theta_0 \) is independent on \( \nu \), positive and decreasing, \( \theta_0(0^+) = \frac{\pi(1-\alpha)}{2} \) and \( \theta_0(+\infty) = 0 \). Moreover, integration by parts and [10, 3.252.12] give

\[ \int_0^\infty \theta_0(t) \, dt = \int_0^\infty \frac{\left(\sin\left(\frac{\pi(1-\alpha)}{2}\right)s\right)^{1-\alpha}}{s^2 + 2s \cot\left(\frac{\pi(1-\alpha)}{2}\right) + \csc^2\left(\frac{\pi(1-\alpha)}{2}\right)} \, dt = \pi \frac{\sin\left(\frac{\pi(1-\alpha)}{2(3-\alpha)}\right)}{\sin\left(\frac{\pi}{3-\alpha}\right)} = \pi \cot\left(\frac{\pi}{3-\alpha}\right) =: \pi b_\alpha. \]

Now we look to the relation on the real line. The equation (4) shows that the right-hand side is continuous on \( \mathbb{R} \), and we obtain for \( t > 0 \) and \( t < 0 \) respectively

\[ \frac{1}{\Lambda^+(t)} \left( \exp(-t)\Psi(-t) + \Phi^+(t) \right) = \frac{1}{\Lambda^-(t)} \left( \exp(-t)\Psi(-t) + \Phi^-(t) \right); \]
\[ \frac{1}{\Lambda^+(t)} \left( \exp(-t)\Psi^-(t) + \Phi(t) \right) = \frac{1}{\Lambda^-(t)} \left( \exp(-t)\Psi^+(t) + \Phi(t) \right), \]

or, equivalently, with regard of (7),

\[
\begin{align*}
\Phi^+(t) - \frac{\Lambda^+(t)}{\Lambda^-(t)} \Phi^-(t) &= \exp(-t)\Psi(-t)\left(\frac{\Lambda^+(t)}{\Lambda^-(t)} - 1\right); \\
\Psi^+(t) - \frac{\Lambda^+(t)}{\Lambda^-(t)} \Psi^-(t) &= \exp(-t)\Phi(-t)\left(\frac{\Lambda^+(t)}{\Lambda^-(t)} - 1\right),
\end{align*}
\]

Since

\[ \frac{\Lambda^+(t)}{\Lambda^-(t)} = \frac{\Lambda^+(t)}{\Lambda^+(t)} = \exp(2i\theta(t)), \]

6
we can rewrite (9) as follows:

\[
\begin{align*}
\Phi^+(t) - \exp(2i\theta(t))\Phi^-(t) &= 2i \exp(-t) \exp(i\theta(t)) \sin(\theta(t))\Psi(-t); \\
\Psi^+(t) - \exp(2i\theta(t))\Psi^-(t) &= 2i \exp(-t) \exp(i\theta(t)) \sin(\theta(t))\Phi(-t).
\end{align*}
\]

(10)

We also know from definition that \( \Phi(z) \) and \( \Psi(z) \) behave as polynomials of order not greater than one at infinity while they are \( O(z^{\alpha-1}) \) at the origin.

We introduce the function \( X_0(z) \) with the cut at positive semiaxis such that

\[
X_0^+(t) = \exp(2i\theta_0(t)), \quad t > 0; \quad X_0(z) \asymp \begin{cases} 1, & z \to \infty; \\ z^{\alpha-1}, & z \to 0. \end{cases}
\]

(11)

The first relation in (11) is satisfied by the Sokhotski–Plemelj formula

\[ X_0(z) := \exp\left(\frac{1}{\pi} \int_0^\infty \frac{\theta_0(s)}{s-z} ds\right). \]

(12)

It is easy to see that

\[
X_0(z) = \exp\left( -\frac{b_0}{z} + O\left(\frac{1}{z^2}\right) \right) = 1 - \frac{b_0}{z} + O\left(\frac{1}{z^2}\right), \quad z \to \infty; \\
X_0(z) \asymp \exp\left( -\frac{\theta_0(0+) \log(z)}{\pi} \right) = z^{\alpha-1}, \quad z \to 0.
\]

(13)

Using (11) we rewrite (10) as follows:

\[
\begin{align*}
\frac{\Phi_0^+(t)}{X_0^+(t)} - \frac{\Phi_0^-(t)}{X_0^-(t)} &= 2i \exp(-\nu t) \exp(i\theta_0(t)) \sin(\theta_0(t)) \frac{X_0(-t)\Psi_0(-t)}{X_0^+(t)X_0^-(t)}; \\
\frac{\Psi_0^+(t)}{X_0^+(t)} - \frac{\Psi_0^-(t)}{X_0^-(t)} &= 2i \exp(-\nu t) \exp(i\theta_0(t)) \sin(\theta_0(t)) \frac{X_0(-t)\Phi_0(-t)}{X_0^+(t)X_0^-(t)};
\end{align*}
\]

(14)

where \( \Phi_0(t) = \Phi(\nu t) \) and \( \Psi_0(t) = \Psi(\nu t) \). Therefore, functions

\[
S(z) = \frac{\Phi_0(z) + \Psi_0(z)}{2X_0(z)}, \quad D(z) = \frac{\Phi_0(z) - \Psi_0(z)}{2X_0(z)}
\]
satisfy equations for $t > 0$

\[
S^+(t) - S^-(t) = 2i \exp(-\nu t) h_0(t) S(-t);
\]

\[
D^+(t) - D^-(t) = -2i \exp(-\nu t) h_0(t) D(-t),
\]

where

\[
h_0(t) = \exp(i\theta_0(t)) \frac{X_0^+(t)}{X_0^-(t)}
\]

\[
= \sin(\theta_0(t)) \exp \left( -\frac{1}{\pi} \int_0^\infty \theta''_0(s) \log \left| \frac{s+t}{s-t} \right| ds \right)
\]

(here we used (12) and integration by parts).

Since $S(z)$ and $D(z)$ behave as polynomials of order not greater then one at infinity, the Sokhotski–Plemelj formula yields

\[
S(z) = \frac{1}{\pi} \int_0^\infty \frac{\exp(-\nu s) h_0(s)}{s-z} S(-s) ds + C_1 + C_2 z;
\]

\[
D(z) = -\frac{1}{\pi} \int_0^\infty \frac{\exp(-\nu s) h_0(s)}{s-z} D(-s) ds + C_3 + C_4 z.
\]

Substituting $z = -t$, $t > 0$, we obtain the integral equations

\[
\hat{S}(t) - (\mathcal{A}\hat{S})(t) = C_1 - C_2 t;
\]

\[
\hat{D}(t) + (\mathcal{A}\hat{D})(t) = C_3 - C_4 t,
\]

where $\hat{S}(t) = S(-t)$, $\hat{D}(t) = D(-t)$, and $\mathcal{A}$ is the integral operator with the kernel $A(t,s) = \frac{\exp(-\nu s) h_0(s)}{\pi(s+t)}$, $s, t \in \mathbb{R}_+$.

By [6, Lemma 5.6] the operator $\mathcal{A}$ is contracting in $L_2(\mathbb{R}_+)$ for $\nu$ large enough, and maps arbitrary polynomial into $L_2(\mathbb{R}_+)$. Therefore, these equations are uniquely solvable. Moreover, the relation $h_0(0) = \sin(\theta_0(0+)) = \sin(\frac{\pi(1-\alpha)}{2})$ shows that (see [10, 3.241.2])

\[
\hat{S}(t), \hat{D}(t) = O(t^{\alpha-1}) \text{ as } t \to 0 \implies S(z), D(z) = O(z^{\alpha-1}) \text{ as } z \to 0,
\]

and therefore, [13] implies $\Phi_0(z), \Psi_0(z) = O(z^{\alpha-1})$ as $z \to 0$, as required.

Denote by $p_0^\pm(t)$ and $p_1^\pm(t)$ the (unique) solutions of the equations on $\mathbb{R}_+$

\[
p_0^\pm(t) = (\mathcal{A}p_0^\pm)(t) = 1; \quad p_1^\pm(t) = (\mathcal{A}p_1^\pm)(t) = t,
\]

8
and extend them analytically to $\mathbb{C} \setminus \mathbb{R}$. Then evidently

$$S(z) = C_1 p_+^0(-z) - C_2 p_+^1(-z), \quad D(z) = C_3 p_-^0(-z) - C_4 p_-^1(-z),$$

whence

$$\Phi_0(z) = X_0(z) (C_1 p_+^0(-z) - C_2 p_+^1(-z) + C_3 p_-^0(-z) - C_4 p_-^1(-z));$$
$$\Psi_0(z) = X_0(z) (C_1 p_+^0(-z) - C_2 p_+^1(-z) - C_3 p_-^0(-z) + C_4 p_-^1(-z)). \quad (15)$$

Since $\hat{\varphi}$ is an entire function, the following relation should be fulfilled:

$$\exp(-z_0) \Psi(-z_0) + \Phi(z_0) \equiv \exp(-i \nu) \Psi_0(-i) + \Phi_0(i) = 0, \quad (16)$$

where $z_0$ is introduced in (6). So, every eigenvalue of the original problem generates a root of (16) by relation (6). It is easy to show that, vice versa, every root of (16) (except $\nu = 0$, if arises) generates an eigenvalue by relation (6).

We multiply (16) by $\exp(i \nu/2)$ and obtain

$$C_1 (\exp(i \nu/2) X_0(i) p_+^0(-i) + \exp(-i \nu/2) X_0(-i) p_+^0(i))$$
$$- C_2 (\exp(i \nu/2) X_0(i) p_+^1(-i) + \exp(-i \nu/2) X_0(-i) p_+^1(i))$$
$$+ C_3 (\exp(i \nu/2) X_0(i) p_-^0(-i) - \exp(-i \nu/2) X_0(-i) p_-^0(i))$$
$$- C_4 (\exp(i \nu/2) X_0(i) p_-^1(-i) - \exp(-i \nu/2) X_0(-i) p_-^1(i)) = 0. \quad (17)$$

By [6, Lemma 5.5] we have

$$X_0(\pm i) = \sqrt{\frac{3 - \alpha}{2}} \exp(\pm i \pi(1 - \alpha)/8),$$

and [6, Lemma 5.7] claims

$$p_+^0(i) = 1 + O(\nu^{-1}), \quad p_+^0(-i) = 1 + O(\nu^{-1}),$$
$$p_+^1(i) = i + O(\nu^{-2}), \quad p_+^1(-i) = -i + O(\nu^{-2}), \quad \text{as } \nu \to \infty.$$ 

Thus, (17) is equivalent to

$$C_1 \left[ \cos \left( \frac{\nu + \rho}{2} \right) \right] - C_2 \left[ \sin \left( \frac{\nu + \rho}{2} \right) \right]$$
$$+ i \left( C_3 \left[ \sin \left( \frac{\nu + \rho}{2} \right) \right] + C_4 \left[ \cos \left( \frac{\nu + \rho}{2} \right) \right] \right) = 0$$

9
(here and elsewhere \( \rho = \pi (1 - \alpha)/4 \) and we use the notation \([a] = a + O(\nu^{-1})\), see [13, §4]).

By (5) and (13), all coefficients \( C_j \) are real, therefore, (17) is equivalent to the real system

\[
\begin{align*}
C_1 \left[ \cos \left( \frac{\nu + \rho}{2} \right) \right] - C_2 \left[ \sin \left( \frac{\nu + \rho}{2} \right) \right] &= 0; \\
C_3 \left[ \sin \left( \frac{\nu + \rho}{2} \right) \right] + C_4 \left[ \cos \left( \frac{\nu + \rho}{2} \right) \right] &= 0.
\end{align*}
\] (18)

Now we compare the behavior of \( \Phi(\nu z) \) and \( \Psi(\nu z) \) at infinity provided by (15), with (5). By [6, Lemma 5.7] we have

\[
p_{\pm}^0(z) = 1 + O(z^{-1}), \quad p_{\pm}^1(z) = z + O(z^{-1}), \quad \text{as } z \to \infty.
\]

Using (5) and (13) we obtain

\[
\begin{align*}
C_1 - C_2 b_\alpha + C_3 - C_4 b_\alpha &= \frac{\lambda}{c_\alpha} \psi'(0); \\
C_2 + C_4 &= \frac{\lambda}{c_\alpha} \psi(0) \nu;
\end{align*}
\]

\[
\begin{align*}
C_1 - C_2 b_\alpha - C_3 + C_4 b_\alpha &= -\frac{\lambda}{c_\alpha} \psi'(1); \\
C_2 - C_4 &= \frac{\lambda}{c_\alpha} \psi(1) \nu.
\end{align*}
\]

We solve these equations and substitute to (18). This gives

\[
\begin{align*}
(\psi'(0) - \psi'(1)) [A] + (\psi(0) + \psi(1)) \nu (b_\alpha [A] - [B]) &= 0; \\
(\psi'(0) + \psi'(1)) [B] + (\psi(0) - \psi(1)) \nu ([A] + b_\alpha [B]) &= 0
\end{align*}
\] (19)

(here \( A = \cos \left( \frac{\nu + \rho}{2} \right) \) and \( B = \sin \left( \frac{\nu + \rho}{2} \right) \)).

The equations (19) complemented by the boundary conditions of the original problem generate a \((4 \times 4)\) homogeneous system. Standard argument based on the Rouché theorem shows that the roots of its determinant are approximations of the solutions of (16) for large \( |\nu| \).

### 2.2 Separated boundary conditions

Separated boundary conditions (or Sturm type conditions) for the second order operator can be written as follows\(^1\)

\[
\begin{align*}
\beta_0 \psi'(0) - \gamma_0 \psi(0) &= 0; \\
\beta_1 \psi'(1) + \gamma_1 \psi(1) &= 0
\end{align*}
\] (20)

\(^1\)Recall that here we consider the boundary conditions that do not contain the spectral parameter \( \lambda \).
(one of two coefficients in every condition may vanish).

We denote by $\kappa$ the sum of orders of the derivatives in boundary conditions (20). It was mentioned in the Introduction that this quantity plays an important role in the spectral asymptotics of ordinary differential operators, see, e.g., [18]. In our case, evidently, $\kappa \in \{0, 1, 2\}$.

1. Let $\kappa = 0$. Then (20) reads $\psi(0) = \psi(1) = 0$, and (19) is reduced to

$$\begin{align*}
[A] \psi'(0) - [A] \psi'(1) &= 0; \\
[B] \psi'(0) + [B] \psi'(1) &= 0.
\end{align*}$$

This system has nontrivial solutions iff

$$2A \mathcal{B} \equiv \sin(\nu + \rho) = O(\nu^{-1}), \quad \text{as } \nu \to \infty.$$ 

Thus, if we enumerate the roots of (16) in increasing order of absolute values then

$$\nu_{n+k} = \pi n - \frac{\pi(1-\alpha)}{4} + O(n^{-1}), \quad \text{as } n \to \infty$$

for some $k$.

By verbatim repetition of [6, Section 5.1.7] we show that $k$ is independent on $\alpha$. Therefore, it can be calculated by comparing with the case $\alpha = 1$ where the original problem becomes standard Sturm–Liouville problem. Thus, we obtain $k = 0$.

2. Let $\kappa = 1$. By symmetry we can suppose without loss of generality that (20) reads $\psi(0) = \psi'(1) + \gamma \psi(1) = 0$, and (19) is reduced to

$$\begin{align*}
[A] \psi'(0) + \nu \left((b_\alpha + \gamma \nu^{-1}) [A] - [B]\right) \psi(1) &= 0; \\
[B] \psi'(0) - \nu \left([A] + (b_\alpha + \gamma \nu^{-1}) [B]\right) \psi(1) &= 0.
\end{align*}$$

This system has nontrivial solutions iff

$$\mathcal{A}^2 - \mathcal{B}^2 + 2b_\alpha \mathcal{A} \mathcal{B} \equiv \cos(\nu + \rho) + b_\alpha \sin(\nu + \rho) = O(\nu^{-1}).$$

Recalling that $b_\alpha = \cot \left(\frac{\pi}{3-\alpha}\right)$ we conclude that in this subcase

$$\nu_{n+k} = \pi n - \frac{\pi(1-\alpha)}{4} - \frac{\pi}{3-\alpha} + O(n^{-1}), \quad \text{as } n \to \infty$$

for some $k$. Comparing with the case $\alpha = 1$ we obtain $k = 0$. 

11
3. Let $\nu = 2$. In this case (20) reads

$$\psi'(0) - \gamma_0 \psi(0) = \psi'(1) + \gamma_1 \psi(1) = 0,$$

and (19) is reduced to

$$\left( (b_\alpha + \gamma_0 \nu^{-1}) [\mathcal{A}] - [\mathcal{B}] \right) \psi(0) + \left( (b_\alpha + \gamma_1 \nu^{-1}) [\mathcal{A}] - [\mathcal{B}] \right) \psi(1) = 0;$$

$$\left( [\mathcal{A}] + (b_\alpha + \gamma_0 \nu^{-1}) [\mathcal{B}] \right) \psi(0) - \left( [\mathcal{A}] + (b_\alpha + \gamma_1 \nu^{-1}) [\mathcal{B}] \right) \psi(1) = 0.$$

This system has nontrivial solutions iff

$$b_\alpha (2\nu^2 - \nu^2) + (b_\alpha^2 - 1) \mathcal{A} \mathcal{B} \equiv b_\alpha \left( \cos(\nu + \rho) + \cot \left( \frac{2\pi}{3 - \alpha} \right) \sin(\nu + \rho) \right) = O(\nu^{-1}),$$

and we conclude that in this subcase

$$\nu_{n+k} = \pi n - \frac{\pi (1 - \alpha)}{4} - \frac{2\pi}{3 - \alpha} + O(n^{-1}), \quad \text{as} \quad n \to \infty \quad (23)$$

for some $k$. Comparing with the case $\alpha = 1$ we obtain $k = 0$.

Now we can formulate the final result.

**Theorem 2.1** The eigenvalues of the problem (3) supplemented by separate boundary conditions have the following asymptotics as $n \to \infty$:

$$\lambda_n = \sin(\pi \alpha / 2) \Gamma(3 - \alpha) \left( \pi n - \frac{\pi (1 - \alpha)}{4} - \frac{\pi \pi}{3 - \alpha} + O(n^{-1}) \right)^{\alpha - 3}, \quad (24)$$

where $\nu$ stands for the sum of orders of the derivatives in conditions (20).

This statement easily follows from relation (6) and the obtained asymptotics of $\nu_n$.

**Remark 2.2** Notice that in general zero root of (16) can arise (say, for $\alpha = 2$, $\gamma_0 = \gamma_1 = 0$). Since $\nu = 0$ does not generate any eigenvalue by formula (6), this forces us to renumerate eigenvalues.

**2.3 Almost separated boundary conditions**

For the second order operator, almost separated (or separated in the principal order) boundary conditions can be written as follows:

$$\psi'(0) - \gamma_0 \psi(0) = \hat{\gamma} \psi(1) = 0; \quad \psi'(1) + \gamma_1 \psi(1) + \hat{\gamma} \psi(0) = 0.$$

Analysis of this case repeats mostly the subcase $\nu = 2$, and the eigenvalues asymptotics coincides with (24) for $\nu = 2$.

**Remark 2.3** In this case zero root of (16) can arise (say, for $\gamma_0 = \gamma_1 = -\hat{\gamma}$), which forces to renumerate eigenvalues.
2.4 Non-separated boundary conditions

For the second order operator, non-separated boundary conditions can be written as follows:

\[ \beta \psi'(0) + \gamma \psi'(1) + \delta \psi(0) = 0; \quad \gamma \psi(0) + \beta \psi(1) = 0. \quad (25) \]

In this case (19) is reduced to

\[
\frac{(\beta + \gamma)^2}{\beta^2 + \gamma^2} \mathcal{A}^2 - \frac{(\beta - \gamma)^2}{\beta^2 + \gamma^2} \mathcal{B}^2 + 2b_a \mathcal{A} \mathcal{B} \\
\equiv \cos(\nu + \rho) + b_a \sin(\nu + \rho) + \frac{2\beta \gamma}{\beta^2 + \gamma^2} = O(\nu^{-1}).
\]

This system has nontrivial solutions iff

\[
\frac{(\beta + \gamma)^2}{\beta^2 + \gamma^2} \mathcal{A}^2 - \frac{(\beta - \gamma)^2}{\beta^2 + \gamma^2} \mathcal{B}^2 + 2b_a \mathcal{A} \mathcal{B} \\
\equiv \cos(\nu + \rho) + b_a \sin(\nu + \rho) + \frac{2\beta \gamma}{\beta^2 + \gamma^2} = O(\nu^{-1}).
\]

Therefore, in this case the sequence \( \nu_n \) can be split into two subsequences \( \nu_n', \nu_n'' \) such that, as \( n \to \infty \),

\[
\nu_{n+k'} = (2\pi - 1)n - \frac{\pi(1 - \alpha)}{4} - \frac{\pi}{3 - \alpha} \\
+ \arcsin \left( \frac{2\beta \gamma}{\beta^2 + \gamma^2} \sin \left( \frac{\pi}{3 - \alpha} \right) \right) + O(n^{-1});
\]

\[
\nu_{n+k''} = 2\pi n - \frac{\pi(1 - \alpha)}{4} - \frac{\pi}{3 - \alpha} \\
- \arcsin \left( \frac{2\beta \gamma}{\beta^2 + \gamma^2} \sin \left( \frac{\pi}{3 - \alpha} \right) \right) + O(n^{-1}),
\]

for some \( k', k'' \). Comparing with the case \( \alpha = 1 \) we obtain \( k' - k'' = 0 \), so without loss of generality we can put \( k' = k'' = 0 \).

Now we can formulate the final result.

**Theorem 2.4** The eigenvalues of the problem (3) supplemented by non-separated boundary conditions have the following asymptotics as \( n \to \infty \):

\[
\lambda_n = \sin(\pi \alpha/2) \Gamma(3 - \alpha) \left( \pi n - \frac{\pi(1 - \alpha)}{4} - \frac{\pi}{3 - \alpha} \right) \left( \frac{2\beta \gamma}{\beta^2 + \gamma^2} \sin \left( \frac{\pi}{3 - \alpha} \right) \right) + O(n^{-1}) \] - \left( -1 \right)^n \arcsin \left( \frac{2\beta \gamma}{\beta^2 + \gamma^2} \sin \left( \frac{\pi}{3 - \alpha} \right) \right) + O(n^{-1}) \right)^{\alpha - 3}. \quad (27)
\]
This statement easily follows from relations (6) and (26).

**Remark 2.5** In the case \( \beta \gamma = 0 \) the boundary conditions (25) are in fact separated, with \( \varkappa = 1 \). So, two subsequences (26) can be merged, and the result coincides with (22). In general case two subsequences of eigenvalues have the opposite shifts with respect to (22), cf. [13, Theorem 1.1].

In the cases \( \beta = \pm \gamma \) one of the subsequences (26) has the second term as in (22) while another one has the second term as in (23). Notice that in contrast to the case \( \alpha = 1 \) two subsequences in (26) cannot be asymptotically close or coincide.

Also a zero root of (16) can arise (say, for \( \beta = -\gamma, \delta = 0 \)), which forces to renumerate eigenvalues.

### 2.5 The case \( \alpha > 1 \)

Repeating the argument of [6, Sec. 5.2], we arrive at the relation (4) with

\[
\Lambda(z) = \frac{\lambda}{|c_\alpha|} z - z \int_0^\infty \frac{2t^{\alpha-2}}{t^2 - z^2} dt
\]

\[
= \frac{\lambda}{|c_\alpha|} z + z^{\alpha-2} \frac{\pi \exp(\pm i\pi(1 - \alpha)/2)}{|\cos(\pi \alpha/2)|}, \quad \Im(z) \geq 0;
\]

\[
\Phi(z) = \frac{\lambda}{|c_\alpha|} (\psi'(0) + z\psi(0)) + \int_0^\infty \frac{t^{\alpha-1}}{t - z} u(0, t) dt;
\]

\[
\Psi(z) = -\frac{\lambda}{|c_\alpha|} (\psi'(1) - z\psi(1)) - \int_0^\infty \frac{t^{\alpha-1}}{t - z} u(1, t) dt.
\]

Following the same line as in previous subsections we again obtain formulae (24) and (27).

### 3 A more general problem

As it was explained in the Introduction, we wish to consider the problem (2) as a perturbation of the problem (1). For simplicity only, we assume that the operator \(-\psi''\) with given boundary conditions is positive definite, otherwise the argument should be changed in a standard way.
We begin with the estimate for the eigenfunctions of the problem (1) with arbitrary self-adjoint boundary conditions that do not contain the spectral parameter.

As in Section 2, we consider the case $\alpha < 1$; for $\alpha > 1$ the argument is similar, and the result is the same. Following the proof of [6, Sec. 5.1.5] we write

$$\hat{\psi}'(z) = z\hat{\psi}(z) - \psi(0) + \exp(-z)\psi(1).$$

By construction, $\hat{\psi}'(z)$ is an entire function, and we can restore it by integrating over the imaginary axis. Using the relation (4) we obtain

$$\psi'(x) = -\frac{1}{2\pi i} \lim_{R \to \infty} \int_{-iR}^{iR} (\mathcal{F}_1(z) \exp(z(x-1)) + \mathcal{F}_2(z) \exp(zx)) \, dz,$$

where

$$\mathcal{F}_1(z) = \frac{\Psi(-z)}{\Lambda(z)} + \psi(1), \quad \mathcal{F}_2(z) = \frac{\Phi(z)}{\Lambda(z)} - \psi(0).$$

The integral in (28) does not depend on the constant terms in $\mathcal{F}_1$ and $\mathcal{F}_2$. It was calculated in the proof of [6, Lemma 5.8]. Up to a multiplicative constant, we obtain

$$\psi'(x) = -\frac{2}{3-\alpha} \Re\left(\exp(i\nu x)\Phi_0(i)\right) + \frac{1}{\pi} \int_0^\infty \sin(\theta_0(t)) \left( \exp(-\nu t(1-x))\Psi_0(-t) - \exp(-\nu tx)\Phi_0(-t) \right) dt,$$

where $\nu$ is related to the eigenvalue $\lambda$ by (5), $\theta_0$ is introduced in (8), $\Phi_0$ and $\Psi_0$ are defined after (14), and

$$\tau_0(t) = \frac{\cos(\pi \alpha/2)}{\pi \nu^{\alpha-2}} |\Lambda_+(\nu t)| = |t + t^{\alpha-2} \exp(i\pi(1-\alpha)/2)|.$$

Taking into account the behavior of ingredients in (29) at zero and infinity we derive

$$\psi'(x) = A \left( \cos(\nu x + \phi(\nu, \alpha)) + F(\nu, \alpha, x) \right),$$

where $A$ is a multiplicative constant.

We limit ourselves to the eigenfunction estimate though its asymptotics can be also obtained from (29) as it is done in [6]–[8]. In particular, the phase shift $\phi$ can be written explicitly.

---

\[\text{We limit ourselves to the eigenfunction estimate though its asymptotics can be also obtained from (29) as it is done in [6]–[8]. In particular, the phase shift $\phi$ can be written explicitly.} \]
where

$$|F(\nu, \alpha, x)| \leq \int_0^\infty \frac{c(\alpha)t}{1 + t^{4-\alpha}} \left( \exp(-\nu t(1-x)) + \exp(-\nu tx) \right) \, dt.$$  

It is easy to see that

$$\int_0^1 |F(\nu, \alpha, x)| \, dx \leq c_1(\alpha)^{-1}. \quad (31)$$

Now we introduce the energy space $\mathcal{H}$ of the problem (1). For smooth functions $h_1$ and $h_2$ satisfying boundary conditions, we set

$$(h_1, h_2)_{\mathcal{H}} := -\int_0^1 h_1''(x)h_2(x) \, dx$$

and define the Hilbert space $\mathcal{H}$ as the completion of the set of such functions with respect to the norm generated by the scalar product $(\cdot, \cdot)_{\mathcal{H}}$. It is well known that, depending on the boundary conditions, $\mathcal{H}$ coincides either with standard Sobolev space $W^1_2(0,1)$ or with its subspace of codimension 1 or 2. Corresponding norm is given by

$$\|h\|^2_{\mathcal{H}} = \int_0^1 (h'(x))^2 \, dx + q(h, h), \quad (32)$$

where $q(h, h)$ is a quadratic form of the variables $h(0)$ and $h(1)$.

In a standard way, we rewrite the problems (1) and (2) as the equations in $\mathcal{H}$

$$\mathcal{K}\psi = \lambda\psi; \quad \mathcal{K}\psi = \lambda(\psi + \mathcal{B}\psi), \quad (33)$$

where $\mathcal{K}$ and $\mathcal{B}$ are compact self-adjoint operators in $\mathcal{H}$ defined by relations

$$(\mathcal{K}\psi, \eta)_{\mathcal{H}} := \int_0^1 (\mathcal{K}_\alpha \psi)(x)\eta(x) \, dx; \quad (\mathcal{B}\psi, \eta)_{\mathcal{H}} := \int_0^1 p(x)\psi(x)\eta(x) \, dx.$$

We are in the position to formulate an important abstract statement.
Proposition 3.1 (Theorem 1 in [17]). Let \( K \) and \( B \) be self-adjoint compact operators in the Hilbert space \( H \). Suppose that \( K \) and \( I + B \) are positive. Denote by \( \lambda_n \) the eigenvalues of \( K \) enumerated in the decreasing order taking into account the multiplicities, and by \( \psi_n \) corresponding normalized eigenfunctions. Finally, suppose that
\[
\lambda_n = (an + b + O(n^{-\varepsilon}))^{-r}, \quad \text{as} \quad n \to \infty, \quad (34)
\]
\[
|\langle B\psi_n, \psi_m \rangle_H| \leq c(mn)^{\frac{1-\varepsilon}{2}}, \quad (35)
\]
where \( a, c, \varepsilon, r > 0, b \in \mathbb{R} \). Then the eigenvalues \( \lambda_n \) of generalized eigenproblem
\[
K\psi_n = \lambda_n(\psi_n + B\psi_n)
\]
have the same two-term asymptotics as \( n \to \infty \):
\[
\lambda_n = (an + b + O(n^{-\varepsilon}))^{-r}.
\]

First, let the boundary conditions be separated or almost separated. As we have proved in Section 2, see (24), the eigenvalues of the first equation in (33) satisfy the relation (34) with \( \varepsilon = 1 \) and \( r = 3 - \alpha \).

To obtain the estimate (35) we need to normalize \( \psi_n \). Since all eigenfunctions except the first one change the sign, the relation (30) and the estimate (31) imply
\[
|\psi_n(x)| = O(A\nu_n^{-1}) = O(An^{-1}), \quad n \to \infty, \quad (36)
\]
uniformly in \( x \in [0, 1] \).

This implies in view of (32)
\[
||\psi_n(x)||_H^2 = A^2\left( \int_0^1 \cos^2(\nu_n x) + O(n^{-1}) \right), \quad n \to \infty,
\]
so for the normalized eigenfunctions we have \( A = \pm \sqrt{2} + O(n^{-1}) \). Finally, taking into account (36) we obtain
\[
|\langle B\psi_n, \psi_m \rangle_H| \leq \max_{x \in [0,1]} |\psi_n(x)\psi_m(x)|\int_0^1 |p(x)| dx \leq c(mn)^{-1}
\]
for any $p \in L_1(0, 1)$. Thus, the estimate (35) fulfills with $\varepsilon = 1$, and Proposition 3.1 ensures the two-term eigenvalues estimate (24) for the problem (2).

For the non-separated boundary conditions the eigenvalues of the operator $\mathcal{K}$ are organized in two sequences, see (27). However, the estimate (35) also holds, and Remark 2 in [17] ensures that the asymptotics (27) persists for the problem (2).

Now we can formulate the final result of this section.

**Theorem 3.2** Let $p \in L_1(0, 1)$. Then the two-term eigenvalues asymptotics of the problem (2) with self-adjoint boundary conditions does not depend on $p$ and is given in (24) or (27), depending on boundary conditions.

4 Gaussian processes related to the problems (1) and (2) with various boundary conditions

We recall that $\mathcal{G}$ stands for the covariance function $G_{W^H}$.

1. **Fractional Brownian bridge.** This process is defined as

$$B^H(x) = W^H(x) - a(x)W^H(1), \quad a(x) = \frac{\mathcal{G}(x, 1)}{\mathcal{G}(1, 1)} = \mathcal{G}(x, 1).$$

Its covariance function reads

$$G_{B^H}(x, y) = \mathcal{G}(x, y) - \mathcal{G}(x, 1)\mathcal{G}(1, y),$$

and corresponding operator can be considered as a critical one-dimensional perturbation of the covariance operator of $W^H$, see [16].

In [7] this approach was applied to obtain the two-term spectral asymptotics for $B^H$. Moreover, it was mentioned that the direct method developed in [6] for $W^H$ does not produce results quite as explicit as those in [6]. However, we show that it is not the case, and the direct method works as well.

Notice that $G_{B^H}(0, y) = G_{B^H}(1, y) \equiv 0$, and therefore any eigenfunction of

$$\int_0^1 G_{B^H}(x, y)\varphi(y) \, dy = \lambda \varphi(x)$$

For some boundary conditions the assumption on $p$ can be weakened.
satisfies \( \varphi(0) = \varphi(1) = 0 \).

Define

\[
\psi(x) = \int_x^1 \varphi(y) \, dy - c, \quad c = \int_0^1 G(1,y) \varphi(y) \, dy.
\] (37)

Then evidently \( \psi'(0) = \psi'(1) = 0 \), and

\[
\lambda \psi'(x) = \int_0^1 G_B(x,y) \psi'(y) \, dy
\]

\[
= -\int_0^1 (G_y(x,y) - G(x,1)G_y(1,y)) \psi(y) \, dy.
\]

The last term vanishes by the choice of \( c \):

\[
\int_0^1 G_y(1,y) \psi(y) \, dy = G(1,1)\psi(1) + \int_0^1 G(1,y) \varphi(y) \, dy = 0,
\] (38)

and we obtain

\[
\lambda \psi'(x) = -\int_0^1 G_y(x,y) \psi(y) \, dy
\]

\[
= H \int_0^1 \left( y^{2H-1} + \text{sign}(x-y)|x-y|^{2H-1} \right) \psi(y) \, dy.
\]

Differentiation gives (1) with \( \alpha = 2 - 2H \). Since boundary conditions are separated, we obtain the spectral asymptotics (24) with \( \kappa = 2 \). By Remark 2.2, we should exclude zero root of (16). This changes \( n \to n + 1 \) in the right-hand side of (24) and yields

\[
\lambda_n = \sin(\pi H) \Gamma(1 + 2H) \left( \pi n + \frac{\pi(2H - \frac{1}{2})(\frac{3}{2} - H)}{2(H + \frac{1}{2})} + O(n^{-1}) \right)^{-1-2H},
\]

that coincides with the result of [7, Theorem 2.2] and even gives a slightly better estimate of the remainder term.
2. Centered FBM. This process is defined as

$$\overline{W^H}(x) = W^H(x) - \int_0^1 W^H(t) \, dt,$$

and its covariance function reads

$$G_{\overline{W^H}}(x, y) = G(x, y) - \int_0^1 G(x, y) \, dy - \int_0^1 G(x, y) \, dx + \int_0^1 \int_0^1 G(x, y) \, dxdy. $$

Notice that $\int_0^1 G_{\overline{W^H}}(x, y) \, dy = 0$. Therefore the equation

$$\int_0^1 G_{\overline{W^H}}(x, y) \varphi(y) \, dy = \lambda \varphi(x)$$

has a zero eigenvalue corresponding to the constant eigenfunction, and all other eigenfunctions satisfy $\int_0^1 \varphi(y) \, dy = 0$.

Define $\psi(x) = \int_0^1 \varphi(y) \, dy$. Then evidently $\psi(0) = \psi(1) = 0$, and

$$\lambda \psi'(x) = \int_0^1 G_{\overline{W^H}}(x, y) \psi'(y) \, dy$$

$$= - \int_0^1 \left( G_y(x, y) - \int_0^1 G_y(x, y) \, dx \right) \psi(y) \, dy.$$

Differentiation gives (11) with $\alpha = 2 - 2H$. Since boundary conditions are separated, we obtain the spectral asymptotics (24) with $\kappa = 0$. This yields

$$\lambda_n = \sin(\pi H) \Gamma(1 + 2H) \left( \pi n - \frac{\pi (H - \frac{1}{2})}{2} + O(n^{-1}) \right)^{-1 - 2H}.$$

3. Centered Brownian bridge. This process is defined similarly:

$$\overline{B^H}(x) = B^H(x) - \int_0^1 B^H(t) \, dt.$$
and its covariance function reads
\[
G_{B|H}(x, y) = G_{BH}(x, y) - \int_0^1 G_{BH}(x, y) \, dy - \int_0^1 G_{BH}(x, y) \, dx + \int_0^1 \int_0^1 G_{BH}(x, y) \, dx \, dy.
\]

As in the previous example, \( \int_0^1 G_{B|H}(x, y) \, dy = 0 \), and thus the equation
\[
\int_0^1 G_{B|H}(x, y) \varphi(y) \, dy = \lambda \varphi(x)
\] (39)

has a zero eigenvalue corresponding to the constant eigenfunction, while all other eigenfunctions satisfy \( \int_0^1 \varphi(y) \, dy = 0 \). Therefore, (39) can be rewritten as
\[
\int_0^1 \left( G_{BH}(x, y) - \int_0^1 G_{BH}(x, y) \, dx \right) \varphi(y) \, dy = \lambda \varphi(x),
\]
and from \( G_{BH}(0, y) = G_{BH}(1, y) \equiv 0 \) we conclude that \( \varphi(0) = \varphi(1) \).

We define \( \psi \) by formula (37). Then evidently \( \psi(0) = \psi(1), \psi'(0) = \psi'(1), \) and
\[
\lambda \psi'(x) = \int_0^1 \left( \mathcal{G}_{y}(x, y) - \mathcal{G}(x, 1) \mathcal{G}_{y}(1, y) - \int_0^1 \mathcal{G}_{B|H}(x, y) \, dx \right) \psi(y) \, dy.
\]
The second term vanishes by (38), and differentiation gives (11) with \( \alpha = 2 - 2H \). Since boundary conditions are periodic, we obtain the spectral asymptotics (27) with \( \beta = -\gamma \). By Remark 2.5 we should exclude zero root
of \((16)\). This changes \(n \to n + 1\) in the right-hand side of \((27)\) and yields

\[
\lambda_n = \sin(\pi H) \Gamma(1 + 2H) \left( \pi n - \frac{\pi (H - \frac{1}{2})}{2} + \frac{\pi}{2} (1 - (-1)^n) - \frac{\pi (H - \frac{1}{2})}{2(H + \frac{1}{2})} (1 - (-1)^n) + O(n^{-1}) \right)^{-1 - 2H}.
\]

4. **Fractional Slepian process.** The conventional Slepian process \(S\) on \([0, 1]\) can be defined in several ways:

1) \(S\) is a zero mean-value stationary Gaussian process with correlation function \(1 - |x - y|\);
2) \(S(x) = W(x + 1) - W(x)\) where \(W\) is the Wiener process;
3) \(S(x) = W_1(x) + W_2(1 - x)\) where \(W_1\) and \(W_2\) are independent Wiener processes.

Fractional Slepian processes defined by analogy in these three ways are different, see, e.g. [12] for the first one. We define the process \(S^H\) as the mixture of two independent FBMs:

\[
S^H(x) = W_1^H(x) + W_2^H(1 - x), \quad x \in [0, 1].
\]

Its covariance function reads

\[
G_{S^H}(x, y) = G(x, y) + G(1 - x, 1 - y).
\]

We emphasize that, as in the case \(H = 1/2\), the following relation holds:

\[
S^H(x) - S^H(0) \overset{d}{=} 2W^H(x).
\]

Notice also that \(G_{S^H}(0, y) + G_{S^H}(1, y) \equiv 1\).

By symmetry of the kernel, any eigenfunction of

\[
\int_0^1 G_{S^H}(x, y) \varphi(y) \, dy = \lambda \varphi(x)
\]

satisfies either \(\varphi(x) = \varphi(1 - x)\) or \(\varphi(x) = -\varphi(1 - x)\).

In the first case we define

\[
\psi(x) = \int_{\frac{1}{2}}^x \varphi(y) \, dy.
\]

(40)
Evidently $\psi(0) + \psi(1) = 0$, and

$$\lambda(\varphi(0) + \varphi(1)) = \int_0^1 (G_{Sh}(0, y) + G_{Sh}(1, y)) \varphi(y) \, dy = \int_0^1 \varphi(y) \, dy,$$

that is equivalent to

$$\psi(0) + \psi(1) = 0; \quad \psi'(0) + \psi'(1) + \frac{2}{\lambda} \psi(0) = 0. \tag{41}$$

In the second case we define

$$\psi(x) = \int_0^x \varphi(y) \, dy. \tag{42}$$

Then evidently $\psi(0) = \psi(1) = 0$ and $\psi'(0) + \psi'(1) = 0$, so the boundary conditions (41) are satisfied as well.

Further,

$$\lambda \psi'(x) = \int_0^1 G_{Sh}(x, y) \psi'(y) \, dy = G_{Sh}(x, 1) \psi(1) - G_{Sh}(x, 0) \psi(0)$$

$$- \int_0^1 (G_y(x, y) - G_y(1 - x, 1 - y)) \psi(y) \, dy.$$

The double substitution is equal to $\psi(1)$ and vanishes after differentiation. So, we obtain

$$(\mathbb{K}_\alpha \psi)(x) = -\frac{\lambda}{2} \psi''(x) \tag{43}$$

with $\alpha = 2 - 2H$ and boundary conditions (41).

The boundary conditions in this problem are non-separated but contain the spectral parameter $\lambda$. So, formula (27) is not applicable. However, the basic scheme runs without essential changes. We change $\lambda \to \lambda/2$ in (4) and arrive at (19). Using (41) we rewrite (19) as follows:

$$2 [\mathfrak{A}] \psi'(0) + \nu^{3-\alpha} \frac{\cos(\pi \alpha/2)}{c_\alpha \pi} [\mathfrak{A}] \psi(0) = 0;$$

$$\left( \nu^{3-\alpha} \frac{\cos(\pi \alpha/2)}{c_\alpha \pi} [\mathfrak{B}] - 2\nu ([\mathfrak{A}] + b_\alpha [\mathfrak{B}]) \right) \psi(0) = 0.$$
This system has nontrivial solutions iff
\[ 2\mathcal{A}\mathcal{B} \equiv \sin(\nu + \rho) = O(\nu^{-\min\{1,2-\alpha\}}), \quad \text{as } \nu \to \infty, \]
and we conclude
\[ \nu_{n+k} = \pi n - \frac{\pi(1 - \alpha)}{4} + O(n^{-\min\{1,2-\alpha\}}), \quad \text{as } n \to \infty \]
for some \( k \). Comparing this result with the eigenvalues of the conventional Slepian process, see [22], we obtain \( k = 1 \), and
\[ \lambda_n = 2 \sin(\pi H) \Gamma(1 + 2H) \left( \pi(n - 1) - \frac{\pi(H - \frac{1}{2})}{2} + O(n^{-\min\{1,2H\}}) \right)^{-1-2H}. \]

5. The spectral analysis of some other mixtures of fractional processes can be reduced to the problems of the same type, cf. [20, Section 2] for the mixtures of the Green Gaussian processes. We consider here only one modification of the previous example.

Let \( S^H_\gamma(x) = S^H(x) - \gamma(S^H(0) + S^H(1)) \). This process is a one-dimensional perturbation of the fractional Slepian process, and its covariance function reads
\[ G_{S^H_\gamma}(x, y) = G(x, y) + G(1 - x, 1 - y) + 2(\gamma^2 - \gamma). \]
For \( \gamma \neq \frac{1}{2} \) this perturbation is non-critical, see [16], and the two-term spectral asymptotics does not change. The case \( \gamma = \frac{1}{2} \) is critical and should be studied separately, cf. [15, Theorem 2.1] and [16, Example 8] for conventional Slepian process.

To manage the equation for eigenfunctions in the case \( \gamma = \frac{1}{2} \)
\[ \int_0^1 \left( G_{S^H_\gamma}(x, y) - \frac{1}{2} \right) \varphi(y) dy = \lambda \varphi(x) \]
we notice that \( \varphi(0) + \varphi(1) = 0 \) and make the change of function (40), (42). This gives (43) with \( \alpha = 2 - 2H \) and anti-periodic boundary conditions
\( \psi(0) + \psi(1) = 0; \quad \psi'(0) + \psi'(1) = 0. \)
So we obtain the spectral asymptotics (47) with \( \beta = \gamma \). This yields
\[ \lambda_n = 2 \sin(\pi H) \Gamma(1 + 2H) \left( \pi n - \frac{\pi(H - \frac{1}{2})}{2} - \frac{\pi}{2} (1 + (-1)^n) \right. \]
\[ + \left. \frac{\pi(H - \frac{1}{2})}{2(H + \frac{1}{2})} (1 + (-1)^n) + O(n^{-1}) \right)^{-1-2H}. \]
6. The fractional Ornstein–Uhlenbeck process beginning at zero. In the fractional setting, the Ornstein–Uhlenbeck process can be defined in a number of nonequivalent ways, see, e.g., [5]. Following [7], we consider the solution of the Langevin equation driven by the FBM:

\[ U_\beta^H(x) = \xi - \beta \int_0^x U_\beta^H(t) \, dt + W^H(x), \]  

(44)

where \( \beta \in \mathbb{R} \) is the drift parameter and \( \xi \sim \mathcal{N}(0, \sigma^2) \) is the initial condition independent of \( W^H \). The covariance function is given by the formula

\[ G_\beta(x, y) = \exp(-\beta(x + y)) \times \left[ \sigma^2 + \int_0^x \exp(\beta s) \, ds \int_0^y H|s - t|^{2H-1}\text{sign}(s - t) \exp(\beta t) \, dt \, ds \right]. \]

We begin with \( \sigma = 0 \), i.e., \( \xi = 0 \). This case was considered in [8, Sec. 6]. By separate fine analysis the following expression for eigenvalues was derived:

\[ \lambda_n = \sin(\pi H)\Gamma(1 + 2H) \frac{\nu_n^{1-2H}}{\nu_n^2 + \beta^2}, \quad n \in \mathbb{N}, \]  

(45)

where the sequence \( \nu_n \) satisfies \([22]\) with \( k = 0 \) and \( \alpha = 2 - 2H \).

We claim that this result is covered by our Theorem 3.2. Indeed, the change of function

\[ \psi(x) = \exp(\beta x) \int_x^1 \exp(-\beta y) \varphi(y) \, dy \]  

(46)

(cf. the proof of Lemma 6.1 in [8]) reduces the equation

\[ \int_0^1 G_\beta(x, y) \varphi(y) \, dy = \lambda \varphi(x) \]

to the problem

\[ (\mathcal{K}_\alpha \psi)(x) = \lambda \left( -\psi''(x) + \beta^2 \psi(x) \right), \quad x \in (0, 1) \]  

(47)
with boundary conditions \((\psi' - \beta \psi)(0) = \psi(1) = 0\).

Theorem 3.2 shows that 
\[
\lambda_n = \sin(\pi H) \Gamma(1 + 2H) \left( \pi n - \frac{\pi(H - \frac{1}{2})}{2} - \frac{\pi}{2(H + \frac{1}{2})} + O(n^{-1}) \right)^{-1-2H},
\]
which coincides with (45) taking into account that \(\nu_n^2 + \beta^2 = \nu_n^2(1 + O(n^{-2}))\). Thus, the claim follows.

7. Now we consider the Ornstein–Uhlenbeck process (44) with \(\sigma \neq 0\).

The change of function (46) gives the same equation (47) and the boundary condition \(\psi(1) = 0\). To obtain the boundary condition at zero we write 
\[
\lambda \phi(0) = \int_0^1 G_\beta(0, y) \phi(y) \, dy = \sigma^2 \int_0^1 \exp(-\beta y) \phi(y) \, dy = \sigma^2 \psi(0),
\]
and the evident relation \(\beta \psi(x) - \psi'(x) = \varphi(x)\) implies 
\[
\psi'(0) - (\beta - \frac{\sigma^2}{\lambda}) \psi(0) = 0; \quad \psi(1) = 0. \tag{48}
\]

First, we consider the problem (1) with the same boundary condition (48). These boundary conditions are separated but contain the spectral parameter \(\lambda\). So, as in the example 4, formula (24) is not applicable. However, the basic scheme again runs without changes. Using (48) we rewrite (19) as follows:
\[
\begin{align*}
\left(\sigma^2 \nu^{3-\alpha} \frac{\cos(\pi \alpha/2)}{c_{\alpha} \pi} [\mathcal{A}] - \nu (b_{\alpha} [\mathcal{A}] - [\mathcal{B}]) - \beta [\mathcal{A}]\right) \psi(0) + [\mathcal{A}] \psi'(1) &= 0; \\
\left(\sigma^2 \nu^{3-\alpha} \frac{\cos(\pi \alpha/2)}{c_{\alpha} \pi} [\mathcal{B}] - \nu ([\mathcal{A}] + b_{\alpha} [\mathcal{B}]) - \beta [\mathcal{B}]\right) \psi(0) - [\mathcal{B}] \psi'(1) &= 0.
\end{align*}
\]
This system has nontrivial solutions iff
\[
2\mathcal{A}\mathcal{B} \equiv \sin(\nu + \rho) = O(\nu^{-\min\{1,2-\alpha\}}), \quad \text{as } \nu \to \infty,
\]
and we conclude 
\[
\nu_{n+k} = \pi n - \frac{\pi(1-\alpha)}{4} + O(n^{-\min\{1,2-\alpha\}}), \quad \text{as } n \to \infty
\]
for some \(k\).
We postpone the specification of $k$ and turn to the problem \(47\). Theorem 3.2 is not applicable directly because of the spectral parameter in the boundary conditions \(48\). However, basic relations \(30\) and \(31\), and therefore the estimate \(36\) hold regardless of the boundary conditions. To apply Proposition 3.1 we need only to redefine the operator $K$ in \(33\) setting

$$(K\psi, \eta)_H := \int_0^1 (K_\alpha \psi)(x)\eta(x) \, dx + \sigma^2 \psi(0)\eta(0).$$

The estimate \(35\) with $\varepsilon = 1$ persists, and Proposition 3.1 shows that the term $\beta^2 u$ in \(17\) does not influence upon the two-term eigenvalues asymptotics.

Finally, comparing this result with the eigenvalues of the conventional Ornstein–Uhlenbeck process corresponding to $\alpha = 1$ and $\sigma^2 = 1/2\beta$, see, e.g., \[18, Proposition 5.5\], we obtain $k = 1$, and

$$\lambda_n = \sin(\pi H)\Gamma(1 + 2H) \left( \pi(n - 1) - \frac{\pi(H - \frac{1}{2})}{2} + O(n^{-\min\{1, 2H\}}) \right)^{-1 - 2H}.$$

5 Application to the small ball probabilities

As it was mentioned in the Introduction, the results of the previous section give rise to the exact (up to a constant) $L_2$-small ball asymptotics of all considered Gaussian process.

We define two important quantities:

$$D(H) := \frac{H}{(2H + 1) \sin\left(\frac{\pi}{2H + 1}\right) \left( \frac{\sin(\pi H)\Gamma(2H + 1)}{(2H + 1) \sin\left(\frac{\pi}{2H + 1}\right)} \right)^{1/2H}};$$

$$B(H) := \frac{(H - \frac{1}{2})^2}{2H}.$$

We substitute the two-term spectral asymptotics into the general result of \[18, Theorem 6.2\]. For the examples 3 and 5 we use in addition the Lifshits lemma, see, e.g., \[15, Lemma 0.1\]. This gives us the following statement.

**Theorem 5.1** The exact small ball asymptotics for the fractional processes considered in Section 4 read as follows:

$$\mathbb{P}\{ \int_0^1 X^2(x) \, dx \leq \varepsilon^2 \} \sim C(X) \cdot \varepsilon^{B_X} \exp\left(-D_X \varepsilon^{-\frac{1}{H}}\right).$$
where the values of $B_X$ and $D_X$ are collected in the table.

| $X$       | $B_X$                        | $D_X$                        |
|-----------|------------------------------|------------------------------|
| $W^H$     | $B(H) + \frac{1}{2H}$       | $D(H)$                      |
| $B^H$     | $B(H) + \frac{1}{2H} - 1$   | $D(H)$                      |
| $W^H$     | $B(H)$                       | $D(H)$                      |
| $B^H$     | $B(H) - 1$                   | $D(H)$                      |
| $S^H_{\gamma}, \gamma \neq \frac{1}{2}$ | $B(H) + \frac{1}{2H} + 1$   | $2\pi D(H)$                |
| $S^H_{\frac{1}{2}}$ | $B(H) + \frac{1}{2H}$       | $2\pi D(H)$                |
| $U^H_{\beta}, \sigma = 0$  | $B(H) + \frac{1}{2H}$       | $D(H)$                      |
| $U^H_{\beta}, \sigma \neq 0$ | $B(H) + \frac{1}{2H} + 1$   | $D(H)$                      |

**Remark 5.2**

1. It is well known that the centered Wiener process coincides in distribution with the Brownian bridge. The table shows that for $H \neq \frac{1}{2}$ this is not the case, and even $L_2$-small ball asymptotics for $B^H$ and $W^H$ differ at the power level.

2. In contrast, $L_2$-small ball asymptotics for $W^H$ and $U^H_{\beta}$ in the case $\sigma = 0$ coincide up to a constant for all $H \in (0, 1)$.

**Acknowledgements.** I am grateful to P. Chigansky, M. Kleptsyna and Ya. Nikitin for useful discussions.

**References**

[1] Birman, M.S., and Solomyak, M.Z. Asymptotic behavior of the spectrum of weakly polar integral operators. Izv. AN SSSR. Ser. matem., 34 (1970), N6, 1143–1158 (Russian); English transl.: Math. of the USSR-Izvestiya, 4 (1970), N5, 1151-1168.

[2] Birman, M.S., and Solomyak, M.Z. Quantitative analysis in Sobolev imbedding theorems and applications to spectral theory. In: Proceed. of X Summer Mathematical School. Yu.A. Mitropol’skiy and A.F. Shestopal (Eds), 1974, 5–189 (Russian); English transl. in: AMS Translations, Series 2, 114. AMS, Providence, R.I. 1980.

---

4The result for $W^H$ is given for the comparison.
[3] Birman, M.S., and Solomyak, M.Z. Spectral theory of self-adjoint operators in Hilbert space, 2nd ed., revised and extended. Lan’, St.Petersburg, 2010 [in Russian]; English transl. of the 1st ed.: Mathematics and Its Applications. Soviet Series. 5, Kluwer, Dordrecht etc. 1987.

[4] Bronski, J.C. Small ball constants and tight eigenvalue asymptotics for fractional Brownian motions. J. Theor. Probab., 16 (2003), 87–100.

[5] Cheridito, P., Kawaguchi, H., and Maejima, M. Fractional Ornstein–Uhlenbeck processes. Electron. J. Probab., 8 (2003), N3, 1–14.

[6] Chigansky, P., and Kleptsyna, M. Exact Asymptotics in Eigenproblems for Fractional Brownian Covariance Operators. Stoch. Proc. Appl., 128 (2018), N6, 2007–2059.

[7] Chigansky, P., Kleptsyna, M., and Marushkevych, D. On the eigenproblem for Gaussian bridges. Preprint available at https://arxiv.org/pdf/1706.09298.pdf. 18 pp.

[8] Chigansky, P., Kleptsyna, M., and Marushkevych, D. Exact spectral asymptotics of fractional processes. Preprint available at https://arxiv.org/pdf/1802.09045.pdf. 52 pp.

[9] Gao, F., Hannig, J., and Torcaso, T. Comparison theorems for small deviations of random series. Electron. J. Probab. 8 (2003), paper N21, 1–17.

[10] Gradshteyn, I.S., and Ryzhik, I.M. Tables of integrals, sums, series and products, 5th ed. Moscow, Nauka, 1971 (Russian); English transl.: Table of integrals, series, and products. Corr. and enl. ed. by Alan Jeffrey. New York – London – Toronto: Academic Press, 1980.

[11] Li, W.V. Comparison results for the lower tail of Gaussian seminorms. J. Theor. Probab. 5 (1992), N1, 1–31.

[12] Molchan, G. Survival Exponents for Some Gaussian Processes. Int. J. Stoch. Analysis, 2012 (2012), ID 137271, 1–20.

[13] Naimark, M.A. Linear Differential Operators. 2nd ed. Moscow, Nauka, 1969 (Russian); English transl. of the 1st ed.: Naimark M.A. Linear Differential Operators. Part I (1967): Elementary Theory of Linear Differential Operators. With add. material by the author. N.Y.: F. Ungar Publishing Co. XIII.

[14] Nazarov, A.I. Log-level comparison principle for small ball probabilities. Stat. & Prob. Letters. 79 (2009), N4, 481–486.
[15] Nazarov, A.I. Exact $L_2$-Small Ball Asymptotics of Gaussian Processes and the Spectrum of Boundary-Value Problems. J. Theor. Probab. 22 (2009), N3, 640–665.

[16] Nazarov, A.I. On a set of transformations of Gaussian random functions. Teor. Ver. Primen., 54 (2009), N2, 209–225 (Russian); English transl.: Theor. Probab. Appl., 54 (2010), N2, 203–216.

[17] Nazarov, A.I. Some lemmata on the perturbation of the spectrum. Preprint available at https://arxiv.org/abs/1908.09365. 6pp.

[18] Nazarov, A.I., and Nikitin, Ya.Yu. Exact small ball behavior of integrated Gaussian processes under $L_2$-norm and spectral asymptotics of boundary value problems. Probab. Theory and Rel. Fields, 129 (2004), N4, 469–494.

[19] Nazarov, A.I., and Nikitin, Ya.Yu. Logarithmic $L_2$-small ball asymptotics for some fractional Gaussian processes. Teor. Ver. Primen., 49 (2004), N4, 695–711 (Russian); English transl.: Theor. Probab. Appl., 49 (2005), N4, 645–658.

[20] Nazarov, A.I., and Nikitin, Ya.Yu. On Small Deviation Asymptotics in $L_2$ of Some Mixed Gaussian Processes. Mathematics, 6 (2018), N4, paper N55, 1–9.

[21] Nazarov, A.I., and Petrova, Yu.P. The small ball asymptotics in Hilbertian norm for the Kac–Kiefer–Wolfowitz processes. Teor. Ver. Primen., 60 (2015), N3, 482–505 (Russian); English transl.: Theor. Probab. Appl., 60 (2016), N3, 460–480.

[22] Nikitin, Ya.Yu., and Orsingher, E. Sharp small ball asymptotics for Slepian and Watson processes in Hilbert norm. ZNS POMI, 320 (2004), 120–128 (Russian); English transl.: J. Math. Sci. (N.Y.), 137 (2006), N1, 4555–4560.

[23] Pal’tsev, B.V. Asymptotic behavior of the spectrum and eigenfunctions of convolution operators on a finite interval with the kernel having a homogeneous Fourier transform. Dokl. Akad. Nauk SSSR, 218 (1974), N1, 28–31.

[24] Petrova, Yu.P. $L_2$-small ball asymptotics for a family of finite-dimensional perturbations of Gaussian functions. Preprint available at https://arxiv.org/pdf/1905.07804.pdf. 17 pp.

[25] Ukai, S. Asymptotic distribution of eigenvalues of the kernel in the Kirkwood–Risman integral equation. J. Math. Phys., 12 (1971), N1, 83–92.