Boundary layer flow of nanofluid over an exponentially stretching surface

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Abstract
The steady boundary layer flow of nanofluid over an exponential stretching surface is investigated analytically. The transport equations include the effects of Brownian motion parameter and thermophoresis parameter. The highly nonlinear coupled partial differential equations are simplified with the help of suitable similarity transformations. The reduced equations are then solved analytically with the help of homotopy analysis method (HAM). The convergence of HAM solutions are obtained by plotting $h$-curve. The expressions for velocity, temperature and nanoparticle volume fraction are computed for some values of the parameters namely, suction injection parameter $a$, Lewis number $Le$, the Brownian motion parameter $Nb$ and thermophoresis parameter $Nt$.

Keywords: nanofluid, porous stretching surface, boundary layer flow, series solutions, exponential stretching

1 Introduction
During the last many years, the study of boundary layer flow and heat transfer over a stretching surface has achieved a lot of success because of its large number of applications in industry and technology. Few of these applications are materials manufactured by polymer extrusion, drawing of copper wires, continuous stretching of plastic films, artificial fibers, hot rolling, wire drawing, glass fiber, metal extrusion and metal spinning etc. After the pioneering work by Sakiadis [1], a large amount of literature is available on boundary layer flow of Newtonian and non-Newtonian fluids over linear and nonlinear stretching surfaces [2-10]. However, only a limited attention has been paid to the study of exponential stretching surface. Mention may be made to the works of Magyari and Keller [11], Sanjayanand and Khan [12], Khan and Sanjayanand [13], Bidin and Nazar [14] and Nadeem et al. [15,16].

More recently, the study of convective heat transfer in nanofluids has achieved great success in various industrial processes. A large number of experimental and theoretical studies have been carried out by numerous researchers on thermal conductivity of nanofluids [17-22]. The theory of nanofluids has presented several fundamental properties with the large enhancement in thermal conductivity as compared to the base fluid [23].

In this study, we have discussed the boundary layer flow of nanofluid over an exponentially stretching surface with suction and injection. To the best of our knowledge, the nanofluid over an exponentially stretching surface has not been discussed so far. However, the present paper is only a theoretical idea, which is not checked experimentally. The governing highly nonlinear partial differential equation of motion, energy and nanoparticle volume fraction has been simplified by using suitable similarity transformations and then solved analytically with the help of HAM [24-39]. The convergence of HAM solution has been discussed by plotting $h$-curve. The effects of pertinent parameters of nanofluid have been discussed through graphs.

2 Formulation of the problem
Consider the steady two-dimensional flow of an incompressible nanofluid over an exponentially stretching surface. We are considering Cartesian coordinate system in such a way that $x$-axis is taken along the stretching surface in the direction of the motion and $y$-axis is normal to it. The plate is stretched in the $x$-direction with a velocity $U_w = U_0 \exp (x/l)$, defined at $y = 0$. The flow and heat transfer characteristics under the boundary layer approximations are governed by the following equations.
\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \]  
(1)

\[ \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}, \]  
(2)

\[ \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{\partial^2 T}{\partial y^2} + \left( \frac{\rho c_p}{\rho c_p} \right) \frac{\partial C}{\partial y}, \]  
(3)

\[ \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = \frac{\partial^2 T}{\partial y^2}, \]  
(4)

where \((u, v)\) are the velocity components in \((x, y)\) directions, \(\rho\) is the fluid density of base fluid, \(v\) is the kinematic viscosity, \(T\) is the temperature, \(C\) is the nanofluid temperature, \((\rho c_p)\) is the effective heat capacity of nanofluids, \((\rho c_p)\) is the heat capacity of the fluid, \(\alpha = k/(\rho c_p)\) is the thermal diffusivity of the fluid, \(D_B\) is the Brownian diffusion coefficient and \(D_T\) is the thermophoretic diffusion coefficient.

The corresponding boundary conditions for the flow problem are

\[ u = U_0, \quad v = 0, \quad T = T_w, \quad C = C_w \text{ at } y = 0, \]
\[ u = 0, \quad T = T_\infty, \quad C = C_\infty \text{ at } y = \infty, \]  
(5)

in which \(U_0\) is the reference velocity, \(\beta(x)\) is the suction and injection velocity when \(\beta(x) > 0\) and \(\beta(x) < 0\), respectively, \(T_w\) and \(T_\infty\) are the temperatures of the sheet and the ambient fluid, \(C_w, C_\infty\) are the nanoparticles volume fraction of the plate and the fluid, respectively.

We are interested in similarity solution of the above boundary value problem; therefore, we introduce the following similarity transformations

\[ u = U_0 \exp \left( \frac{x}{l} \right)^{\frac{1}{2}}, \quad v = -\beta \left( \frac{x}{l} \right)^{\frac{1}{2}}, \quad T = T_w, \quad C = C_w \text{ at } y = 0, \]
\[ u = 0, \quad T = T_\infty, \quad C = C_\infty \text{ at } y = \infty. \]  
(6)

Making use of transformations (6), Eq. (1) is identically satisfied and Equations (2)-(4) take the form

\[ f_{\eta\eta} + f f_{\eta} - 2f_\eta^2 = 0, \]  
(7)

\[ \theta_{\eta\eta} + Pr \left( \frac{\partial \theta}{\partial \eta} - f \frac{\partial \theta}{\partial \eta} + Nb \beta \theta \right) = 0 \]  
(8)

\[ g_{\eta\eta} + Le \left( \frac{\partial g}{\partial \eta} - f \frac{\partial g}{\partial \eta} \right) + \frac{Nt}{Nb} g_{\eta\eta} = 0, \]  
(9)

\[ f = -v_w, \quad f_\eta = 1, \quad \theta = 1, \quad g = 1 \text{ at } \eta = 0, \]
\[ f_\eta \to 0, \quad \theta \to 0, \quad g \to 0 \text{ as } \eta \to \infty, \]  
(10)

where

\[ Nt = \frac{D_B (\rho c_p)}{\rho c_p} (C_w - C_\infty), \quad Nb = \frac{D_T (\rho c_p)}{T_w (\rho c_p)} \frac{T_w - T_\infty}{v}, \quad Le = \frac{\nu}{D_B} \frac{Pr}{\alpha} \]  

The physical quantities of interest in this problem are the local skin-friction coefficient \(C_f\), Nusselt number \(Nu_r\), and the local Sherwood number \(Sh_{w}\), which are defined as

\[ C_f = \left. \frac{\partial u}{\partial y} \right|_{\eta=0}, \quad Nu_r = \left. \frac{\partial \theta}{\partial y} \right|_{\eta=0}, \quad Sh_{w} = \left. \frac{\partial \phi}{\partial y} \right|_{\eta=0}. \]  
(11)

where \(Re_s = U_0 x/\nu\) is the local Reynolds number.

### 3 Solution by homotopy analysis method

For HAM solutions, the initial guesses and the linear operators \(L_i\) \((i = 1 - 3)\) are

\[ f_0 (\eta) = 1 - v_w e^{-\eta}, \quad \theta_0 (\eta) = e^{-\eta}, \quad g_0 (\eta) = e^{-\eta}, \]  
(12)

\[ L_1 (f) = f'' - f', \quad L_2 (\theta) = \theta'' - \theta, \quad L_3 (g) = g'' + g \]  

The operators satisfy the following properties

\[ L_1 \left[ c_1 e^{-\eta} + c_2 e^\eta + c_3 \right] = 0, \]  
(14)

\[ L_2 \left[ c_4 e^{-\eta} + c_5 e^\eta \right] = 0, \]  
(15)

\[ L_3 \left[ c_6 e^{-\eta} + c_7 e^\eta \right] = 0, \]  
(16)

in which \(c_1\) to \(c_7\) are constants. From Equations (7) to (9), we can define the following zeroth-order deformation problems

\[ (1 - p) L_1 \left[ \hat{f} (\eta, p) - f_0 (\eta) \right] = ph_1 H_1 \hat{N}_1 \left[ \hat{f} (\eta, p) \right], \]  
(17)

\[ (1 - p) L_2 \left[ \hat{\theta} (\eta, p) - \theta_0 (\eta) \right] = ph_2 H_2 \hat{N}_2 \left[ \hat{\theta} (\eta, p) \right], \]  
(18)

\[ (1 - p) L_3 \left[ \hat{g} (\eta, p) - g_0 (\eta) \right] = ph_3 H_3 \hat{N}_3 \left[ \hat{g} (\eta, p) \right], \]  
(19)

\[ \hat{f} (0, p) = -v_w, \quad \hat{f}' (0, p) = 1, \quad \hat{f}' (\infty, p) = 0, \]  
(20)

\[ \hat{\theta} (0, p) = 1, \quad \hat{\theta} (\infty, p) = 0, \]  
(21)

\[ \hat{g} (0, p) = 1, \quad \hat{g} (\infty, p) = 0. \]  
(22)

In Equations (17)-(22), \(H_1, H_2, H_3\) denote the non-zero auxiliary parameters, \(H_1 = H_2 = H_3 = 1\) and
\[
\tilde{N}_1 \left[ \tilde{f}(\eta, p) \right] = \frac{\partial^3 f}{\partial \eta^3} - 2 \left( \frac{\partial f}{\partial \eta} \right)^2 + f \frac{\partial^2 f}{\partial \eta^2}, \tag{23}
\]
\[
\tilde{N}_2 \left[ \tilde{\theta}(\eta, p) \right] = \frac{\partial^2 g}{\partial \eta^2} + \Pr \left( \frac{\partial^2 \phi}{\partial \eta^2} + \frac{\partial f}{\partial \eta} \frac{\partial \phi}{\partial \eta} + \frac{\partial N_t}{\partial \eta} \right) + \frac{\partial^2 g}{\partial \eta^2} \tilde{N}_1 \left[ \tilde{g}(\eta, p) \right], \tag{24}
\]
\[
\tilde{N}_1 \left[ \tilde{g}(\eta, p) \right] = \frac{\partial^2 \tilde{g}}{\partial \eta^2} + \nu \left( \frac{\partial \phi}{\partial \eta} + \frac{\partial^2 \phi}{\partial \eta^2} \right) + \frac{\partial \tilde{g}}{\partial \eta} \frac{\partial^2 \phi}{\partial \eta^2} + \frac{\partial \tilde{g}}{\partial \eta} \frac{\partial^2 \phi}{\partial \eta^2}. \tag{25}
\]

Obviously
\[
\tilde{f}(\eta, 0) = f_0(\eta), \quad \tilde{f}(\eta, 1) = f(\eta), \tag{26}
\]
\[
\tilde{\theta}(\eta, 0) = \theta_0(\eta), \quad \tilde{\theta}(\eta, 1) = \theta(\eta), \tag{27}
\]
\[
\tilde{g}(\eta, 0) = g_0(\eta), \quad \tilde{g}(\eta, 1) = g(\eta). \tag{28}
\]

When \( p \) varies from 0 to 1, then \( \tilde{f}(\eta, p), \tilde{\theta}(\eta, p), \tilde{g}(\eta, p) \) vary from initial guesses \( f_0(\eta), \theta_0(\eta) \) and \( g_0(\eta) \) to the final solutions \( f(\eta), \theta(\eta) \) and \( g(\eta) \), respectively. Considering that the auxiliary parameters \( h_1, h_2 \) and \( h_3 \) are so properly chosen that the Taylor series of \( \tilde{f}(\eta, p), \tilde{\theta}(\eta, p) \) and \( \tilde{g}(\eta, p) \) expanded with respect to an embedding parameter converge at \( p = 1 \), hence Equations (17)-(19) become
\[
\tilde{f}(\eta, p) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta) p^m, \tag{29}
\]
\[
\tilde{\theta}(\eta, p) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta) p^m, \tag{30}
\]
\[
\tilde{g}(\eta, p) = g_0(\eta) + \sum_{m=1}^{\infty} g_m(\eta) p^m, \tag{31}
\]
\[
f_m(\eta) = \frac{1}{m!} \left. \frac{\partial^m \tilde{f}(\eta, p)}{\partial p^m} \right|_{p=0}, \tag{32}
\]
\[
\theta_m(\eta) = \frac{1}{m!} \left. \frac{\partial^m \tilde{\theta}(\eta, p)}{\partial p^m} \right|_{p=0}, \tag{33}
\]
\[
g_m(\eta) = \frac{1}{m!} \left. \frac{\partial^m \tilde{g}(\eta, p)}{\partial p^m} \right|_{p=0}. \tag{34}
\]

The \( m \)-th order problems are defined as follow
\[
\mathcal{L}_1 \left[ f_m(\eta) - \chi_m f_{m-1}(\eta) \right] = h_1 \tilde{R}_m^1(\eta), \tag{35}
\]
\[
\mathcal{L}_2 \left[ \theta_m(\eta) - \chi_m \theta_{m-1}(\eta) \right] = h_2 \tilde{R}_m^2(\eta), \tag{36}
\]
\[
\mathcal{L}_3 \left[ g_m(\eta) - \chi_m g_{m-1}(\eta) \right] = h_3 \tilde{R}_m^3(\eta), \tag{37}
\]
\[
f_m(0) = f^r_m(0) = f_m^r(\infty) = 0, \tag{38}
\]
\[
\theta_m(0) = \theta_m(\infty) = 0, \tag{39}
\]
\[
g'_m(0) = g_m(\infty) = 0, \tag{40}
\]

where
\[
\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \tag{41}
\]

Employing MATHEMATICA, Equations (35)-(40) have the following solutions
\[
f(\eta) = \sum_{n=0}^{\infty} f_n(\eta) = \lim_{M \to \infty} \left[ \sum_{n=0}^{M} a_{0,n} \sum_{k=0}^{\infty} \sum_{m=1}^{M} \epsilon_m \left( \sum_{n=0}^{M} \sum_{m=1}^{M} \sum_{k=0}^{\infty} \sum_{m=1}^{M} A_{m,n} \eta^k \right) \right]. \tag{42}
\]
\[
\theta(\eta) = \lim_{M \to \infty} \left[ \sum_{n=0}^{M} \sum_{m=1}^{M} \sum_{n=0}^{M} \sum_{m=1}^{M} \sum_{n=0}^{M} \sum_{m=1}^{M} \sum_{k=0}^{\infty} \sum_{m=1}^{M} \sum_{k=0}^{\infty} \sum_{m=1}^{M} A_{m,n} \eta^k \right]. \tag{43}
\]
\[
g(\eta) = \lim_{M \to \infty} \left[ \sum_{n=0}^{M} \sum_{m=1}^{M} \sum_{n=0}^{M} \sum_{m=1}^{M} \sum_{k=0}^{\infty} \sum_{m=1}^{M} \sum_{k=0}^{\infty} \sum_{m=1}^{M} A_{m,n} \eta^k \right]. \tag{44}
\]

in which \( a_{0,n}^k, a_{m,n}^k, A_{m,n}^k, F_{m,n}^k \) are the constants and the numerical data of above solutions are shown through graphs in the following section.

**4 Results and discussion**

The numerical data of the solutions (45)-(47), which is obtained with the help of Mathematica, have been discussed through graphs. The convergence of the series solutions strongly depends on the values of non-zero auxiliary parameters \( h_i (i = 1, 2, 3, h_1 = h_2 = h_3) \), which can adjust and control the convergence of the solutions. Therefore, for the convergence of the solution, the h-curves is plotted for velocity field in Figure 1. We have found the convergence region of velocity for different values of suction injection parameter \( v_w \). It is seen that
with the increase in suction parameter $v_{w}$, the convergence region become smaller and smaller. Almost similar kind of convergence regions appear for temperature and nanoparticle volume fraction, which are not shown here. The non-dimensional velocity $f'$ against $\eta$ for various values of suction injection parameter is shown in Figure 2. It is observed that velocity field increases with the increase in $v_{w}$. Moreover, the suction causes the reduction of the boundary layer. The temperature field $\theta$ for different values of Prandtl number $Pr$, Brownian parameter $Nb$, Lewis number $Le$ and thermophoresis parameter $Nt$ is shown in Figures 3, 4, 5 and 6. In Figure 3, the temperature is plotted for different values of $Pr$. It is observed that with the increase in $Pr$, there is a very slight change in temperature; however, for very large $Pr$, the solutions seem to be unstable, which are not shown here. The variation of $Nb$ on $\theta$ is shown in Figure 4. It is depicted that with the increase in $Nb$, the temperature profile increases. There is a minimal change in $\theta$ with the increase in $Le$ (see Figure 5). The results remain unchanged for very large values of $Le$. The effects of $Nt$ on $\theta$ are seen in Figure 6. It is seen that temperature profile increases with the increase in $Nt$; however, the thermal boundary layer thickness reduces. The nanoparticle volume fraction $g$ for different values of $Pr$, $Nb$, $Nt$ and $Le$ is plotted in Figures 7, 8, 9 and 10. It is observed from Figure 7 that with the increase in
Nb, g decreases and boundary layer for g also decreases. The effects of Pr on g are minimal. (See Figure 8). The effects of Le on g are shown in Figure 9. It is observed that g decreases as well as layer thickness reduces with the increase in Le. However, with the increase in Nt, g increases and layer thickness reduces (See Figure 10).

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SN done the major part of the article, however, the funding and computational suggestions and proof reading has been done by CL. All authors read and approved the final manuscript.
Competing interests
This is just the theoretical study, every experimentalist can check it experimentally with our consent.

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References
1. Sakiadis BC. Boundary layer behavior on continuous solid surfaces: I Boundary layer equations for two dimensional and axisymmetric flow. AIChE J 1961, 7:26-28.
2. Liu IC. Flow and heat transfer of an electrically conducting fluid of second grade over a stretching sheet subject to a transverse magnetic field. Int J Heat Mass Transf 2004, 47:4427-4437.
3. Vajravelu K, Rollins D. Convective heat transfer at a stretching sheet. Acta Mech 1993, 96(1-4):47-54.
4. Nadeem S, Hussain A. Heat and mass transfer in the boundary layer on an exponentially stretching sheet. Int J Heat Mass Transf 2003, 46:47-57.
5. Cornell R. Effects of viscous dissipation and work done by deformation on the MHD flow and heat transfer of a viscoelastic fluid over a stretching surface. Phys Lett A 2006, 357:298-305.
6. Dandapat BS, Santra B, Vajravelu K. The effects of variable fluid properties and thermocapillarity on the flow of a thin film on an unsteady stretching surface. Int J Heat Mass Transf 2007, 50:991-996.
7. Nadeem S, Hussain A, Malik MY, Hayat T. Series solutions for the stagnation flow of a second-grade fluid over a shrinking sheet. Appl Math Mech Eng Ed 2009, 30:1255-1262.
8. Nadeem S, Hussain A, Khan M. HAM solutions for boundary layer flow in the region of the stagnation point towards a stretching sheet. Commun Nonlinear Sci Numer Simul 2010, 15:475-481.
9. Afzal N. Heat transfer from a stretching surface. Int J Heat Mass Transf 1993, 36:1128-1131.
10. Magarini E, Keller B. Heat and mass transfer in the boundary layer on an exponentially stretching continuous surface. J Phys D Appl Phys 1999, 32:577-585.
11. Sanjayanand E, Khan SK. On heat and mass transfer in a viscoelastic boundary layer flow over an exponentially stretching sheet. Int J Therm Sci 2006, 45:819-828.
12. Khan SK, Sanjayanand E. Viscoelastic boundary layer flow and heat transfer over an exponential stretching sheet. Int J Heat Mass Transf 2005, 48:1541-1542.
13. Binod B, Nazar R. Numerical solution of the boundary layer flow over an exponentially stretching sheet with thermal radiation. Eur J Sci Res 2009, 33:710-717.
14. Nadeem S, Hayat T, Malik MY, Rajput SA. Thermal radiations effects on the flow by an exponentially stretching surface: a series solution. Zeitschrift fur Naturforschung 2010, 65a:1-9.
15. Nadeem S, Zaeher S, Fang T. Effects of thermal radiations on the boundary layer flow of a Jeffrey fluid over an exponentially stretching surface. Numer Algir 2011, 57:187-205.
16. Bachok N, Ishak A, Pop I. Boundary Layer flow of nanofluid over a moving surface in a flowing fluid. Int J Therm Sci 2010, 49:1663-1668.
17. Choi SUS. Enhancing thermal conductivity of fluids with nanoparticle. In Developments and Applications of Non-Newtonian Flows Edited by: Siginer DA, Wang HP 1995, 66:99-105, ASME FED 231/MD.
18. Khabba K, VafiA K, Lightstone M. Buoyancy driven heat transfer enhancement in a two dimensional enclosure utilizing nanofluids. Int J Heat Mass Transf 2005, 46:3639-3651.
19. Makinde OD, Azi A. Boundary layer flow of a nano fluid past a stretching sheet with a convective boundary condition. Int J Therm Sci 2011, 50:1326-1332.
20. Bayat J, Niskiriheit AA. Investigation of the different base fluid effects on the nanofluids heat transfer and pressure drop. Heat Mass Transf, doi:10.1007/s00231-011-0773-0.
21. Hlaht M, Etemad SG, Baghri R. Laminar heat transfer of nanofluid in a circular tube. Korean J Chem Eng 2010, 27(5):1391-1396.
22. Fan J, Wang L. Heat conduction in nanofluids: structure-property correlation. Int J Heat Mass Transf 2011, 54:4349-4359.