ON THE $H^1 – L^1$ BOUNDEDNESS OF OPERATORS

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(Communicated by Andreas Seeger)

Abstract. We prove that if $q$ is in $(1, \infty)$, $Y$ is a Banach space, and $T$ is a linear operator defined on the space of finite linear combinations of $(1, q)$-atoms in $\mathbb{R}^n$ with the property that

$$\sup \{ \| Ta \|_Y : a \text{ is a (1, q)-atom} \} < \infty,$$

then $T$ admits a (unique) continuous extension to a bounded linear operator from $H^1(\mathbb{R}^n)$ to $Y$. We show that the same is true if we replace (1, q)-atoms by continuous $(1, \infty)$-atoms. This is known to be false for (1, $\infty$)-atoms.

1. Introduction

In a recent paper, M. Bownik [3] showed that there exists a linear functional $F$ defined on finite linear combinations of $(1, \infty)$-atoms in $\mathbb{R}^n$ with the property that

$$\sup \{ | F(a) | : a \text{ is a (1, \infty)-atom} \} < \infty,$$

but which does not admit a continuous extension to $H^1(\mathbb{R}^n)$. If $v$ is a fixed function in $L^1(\mathbb{R}^n) \setminus \{0\}$, then the operator $B$, defined on finite linear combinations of $(1, \infty)$-atoms by $Bf = F(f) v$, satisfies

$$\sup \{ \| Ba \|_{L^1(\mathbb{R}^n)} : a \text{ is a (1, \infty)-atom} \} < \infty$$

but does not admit an extension to a bounded operator from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. This shows that the argument “the operator $T$ maps $(1, \infty)$-atoms uniformly into $L^1(\mathbb{R}^n)$, and hence it extends to a bounded operator from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$” is fallacious.

Fortunately, if $T$ is a Calderón–Zygmund operator, then the uniform boundedness of $T$ on $(1, \infty)$-atoms implies the boundedness from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ (see, for instance, [11] Ch. 7.3, Lemma 1, [2] Ch. 1.9, [7] Ch. III.7 and [8] Thm 6.7.1).

The purpose of this paper is to show that the operator $B$ constructed above is, to a certain extent, pathological. Indeed, we prove that if $q$ is in $(1, \infty)$, $Y$ is a Banach space, and $T$ is a linear operator defined on finite linear combinations of $(1, q)$-atoms in $\mathbb{R}^n$ with the property that

$$\sup \{ \| Ta \|_Y : a \text{ is a (1, q)-atom} \} < \infty,$$

then $T$ admits a unique continuous extension to a bounded linear operator from $H^1(\mathbb{R}^n)$ to $Y$. The same conclusion holds if we assume that $T$ is a linear operator...
on finite linear combinations of continuous \((1, \infty)\)-atoms in \(\mathbb{R}^n\) with the property that

\[
\sup \{\|Ta\|_Y : a \text{ is a continuous } (1, \infty)\text{-atom}\} < \infty.
\]

Note that this does not contradict Bownik’s example. Indeed, the restriction of the operator \(B\) to continuous \((1, \infty)\)-atoms extends to a bounded operator \(\tilde{B}\) from \(H^1(\mathbb{R}^n)\) to \(L^1(\mathbb{R}^n)\). However, \(B\) and \(\tilde{B}\) will agree on continuous \((1, \infty)\)-atoms but not on all \((1, \infty)\)-atoms.

To explain the idea of the proofs of these results, we need more notation. Suppose that \(q\) is in \((1, \infty]\), and denote by \(H^1_{\text{fin}}(\mathbb{R}^n)\) the vector space of all finite linear combinations of \((1,q)\)-atoms. Notice that \(H^1_{\text{fin}}(\mathbb{R}^n)\) consists of all \(L^q(\mathbb{R}^n)\) functions with compact support and integral 0. Clearly, \(H^1_{\text{fin}}(\mathbb{R}^n)\) is a dense subspace of \(H^1(\mathbb{R}^n)\). We may define a norm on \(H^1_{\text{fin}}(\mathbb{R}^n)\) as follows:

\[
\|f\|_{H^1_{\text{fin}}(\mathbb{R}^n)} = \inf \left\{ \sum_{j=1}^N |\lambda_j| : f = \sum_{j=1}^N \lambda_j a_j, \ a_j \text{ is a } (1,q)\text{-atom, } N \in \mathbb{N} \right\}.
\]

Obviously \(\|f\|_{H^1(\mathbb{R}^n)} \leq \|f\|_{H^1_{\text{fin}}(\mathbb{R}^n)}\) for every \(f\) in \(H^1_{\text{fin}}(\mathbb{R}^n)\). An example due to Y. Meyer (see [12] p. 513, Bownik’s paper [3] or [7] p. 370) shows that \(\cdot\|_{H^1(\mathbb{R}^n)}\) and \(\cdot\|_{H^1_{\text{fin}}(\mathbb{R}^n)}\) are inequivalent norms on \(H^1_{\text{fin}}(\mathbb{R}^n)\). This is the starting point of Bownik’s construction.

We prove that Meyer’s example itself is somewhat exceptional. Indeed, by using the maximal characterisation of \(H^1(\mathbb{R}^n)\), we show that if \(q < \infty\), then \(\cdot\|_{H^1(\mathbb{R}^n)}\) and \(\cdot\|_{H^1_{\text{fin}}(\mathbb{R}^n)}\) are equivalent norms on \(H^1_{\text{fin}}(\mathbb{R}^n)\) (see Section 3). Similarly, we prove that \(\cdot\|_{H^1_{\text{fin}}(\mathbb{R}^n)}\) and \(\cdot\|_{H^1_{\text{fin}}(\mathbb{R}^n) \cap C(\mathbb{R}^n)}\) are equivalent norms on \(H^1_{\text{fin}}(\mathbb{R}^n) \cap C(\mathbb{R}^n)\).

This immediately implies that operators defined on \(H^1_{\text{fin}}(\mathbb{R}^n)\) which have either property (1.1) or property (1.2) automatically extend to bounded operators from \(H^1(\mathbb{R}^n)\) to \(L^1(\mathbb{R}^n)\).

As discussed briefly in Section 3, this equivalence of norms remains true for \(H^p(\mathbb{R}^n)\) with \(0 < p < 1\) and \((p,q)\)-atoms.

The extension property for operators was also proved, by different methods, for \(0 < p \leq 1\) and \((p,2)\)-atoms and operators taking values in quasi-Banach spaces, by D. Yang and Y. Zhou [17].

A theory of Hardy spaces has been developed in spaces of homogeneous type; see R.R. Coifman and G. Weiss [4]. It is, however, not evident whether our results extend to this case in general. Nevertheless, let \(M\) be such a space. By a simple functional analysis argument, we show that if \(q\) is in \((1, \infty)\) and \(T\) is an operator defined on \(H^1_{\text{fin}}(M)\) satisfying the analogue of (1.1), then \(T\) automatically extends to a bounded operator from \(H^1(M)\) to \(L^1(M)\) (see Section 4). It may be worth noticing that the proof of this result also applies to certain metric measured spaces \((M, \rho, \mu)\) where \(\mu\) is only “locally doubling” [10], [4], and [6].

For so-called RD-spaces, which are spaces of homogeneous type having “dimension \(n\)” in a certain sense, our complete results were recently extended in the paper [9] by L. Grafakos, L. Liu and Yang. These authors consider \(n/(n+1) < p \leq 1\) and quasi-Banach-valued operators.

The authors wish to thank N. Th. Varopoulos for useful conversations on the subject of this paper.
2. Notation and terminology

Suppose that \((M, \rho, \mu)\) is a space of homogeneous type in the sense of Coifman and Weiss [5] and that \(\mu\) is a \(\sigma\)-finite measure. For the sake of simplicity, we shall assume that \(\mu(M)\) is infinite.

Suppose that \(q\) is in \((1, \infty]\). For each closed ball \(B\) in \(M\), we denote by \(L^q_0(B)\) the space of all functions in \(L^q(M)\) which are supported in \(B\) and have integral 0. Clearly \(L^q_0(B)\) is a closed subspace of \(L^q(M)\). The union of all spaces \(L^q_0(B)\) as \(B\) varies over all balls coincides with the space \(L^q_{c,0}(M)\) of all functions in \(L^q(M)\) with compact support and integral 0. Fix a reference point \(o\) in \(M\) and for each positive integer \(k\) denote by \(B_k\) the ball centred at \(o\) with radius \(k\). A convenient way of topologising \(L^q_{c,0}(M)\) is to interpret \(L^q_{c,0}(B_k)\) as the strict inductive limit of the spaces \(L^q_{c,0}(B_k)\) (see [II, p. 33] for the definition of the strict inductive limit topology). We denote by \(X^q\) the space \(L^q_{c,0}(M)\) with this topology, and write \(X^q_k\) for \(L^q_{c,0}(B_k)\).

We recall the basic definitions and results concerning the atomic Hardy space \(H^1(M)\). The reader is referred to [5] and the references therein for this and more on Hardy spaces defined on spaces of homogeneous type. Suppose that \(q\) is in \((1, \infty]\). A \((1, q)\)-atom is a function \(a\) in \(L^q(M)\) supported in a ball \(B\), with mean value 0 and such that

\[
\left( \frac{1}{\mu(B)} \int_B |a|^q \, d\mu \right)^{1/q} \leq \mu(B)^{-1}
\]

if \(q\) is finite, and \(\|a\|_\infty \leq \mu(B)^{-1}\) if \(q = \infty\). We denote by \(H^{1,q}(M)\) the space of all functions \(g\) in \(L^1(M)\) which admit a decomposition of the form \(g = \sum_j \lambda_j a_j\), where the \(a_j\) are \((1, q)\)-atoms and the \(\lambda_j\) are complex numbers such that \(\sum_j |\lambda_j| < \infty\). The norm \(\|g\|_{H^{1,q}}\) of \(g\) in \(H^{1,q}(M)\) is the infimum of \(\sum_j |\lambda_j|\) over all such decompositions. It is well known that all the spaces \(H^{1,q}(M)\) with \(q \in (1, \infty)\) coincide with \(H^{1,\infty}(M)\), and we denote them all by \(H^1(M)\).

Clearly, the vector space \(H^1_{\text{fin}}(M)\) of all finite linear combinations of \((1, q)\)-atoms is dense in \(H^1(M)\) with respect to the norm of \(H^1(M)\), for \(q\) in \((1, \infty]\). Observe also that \(H^1_{\text{fin}}(M)\) and \(L^q_{c,0}(M)\) agree as vector spaces, and so do the space of finite linear combinations of continuous \((1, \infty)\)-atoms and \(H^1_{\text{fin}}(M) \cap C(\mathbb{R}^n)\).

For each ball \(B\) and each locally integrable function \(f\), we denote by \(f_B\) the average of \(f\) on \(B\). Recall that \(BMO\) is the Banach space of all locally integrable functions \(f\), defined modulo constants, such that

\[
\|f\|_{BMO} = \sup_B \frac{1}{\mu(B)} \int_B |f - f_B| \, d\mu < \infty.
\]

The dual of \(H^1(M)\) may be identified with \(BMO\).

There are several characterisations of the space \(H^1(\mathbb{R}^n)\). We shall make use of the so-called maximal characterisation, which we briefly recall. Suppose that \(m\) is an integer with \(m > n\), and denote by \(\mathcal{A}_m\) the set of all functions \(\varphi\) in the Schwartz space \(S(\mathbb{R}^n)\) such that

\[
\sup_{|\beta| \leq m} \sup_{x \in \mathbb{R}^n} (1 + |x|)^m |D^\beta \varphi(x)| \leq 1,
\]

where \(|\beta|\) denotes the length of the multi-index \(\beta\). For \(\varphi\) in \(S(\mathbb{R}^n)\) denote by \(\varphi_t\) the function \(t^{-n} \varphi(\cdot/t)\). Given \(f\) in \(L^1(\mathbb{R}^n)\), define the “grand maximal function”
Suppose that $f$ is in $L^1(\mathbb{R}^n)$. The following are equivalent:

(i) $f$ is in $H^1(\mathbb{R}^n)$;

(ii) the grand maximal function $M_m f$ is in $L^1(\mathbb{R}^n)$.

Furthermore, $f \mapsto \|M_m f\|_{L^1(\mathbb{R}^n)}$ is an equivalent norm on $H^1(\mathbb{R}^n)$.

The letter $C$ will denote a positive constant, which need not be the same at different occurrences. Given two positive quantities $A$ and $B$, we shall mean by $A \sim B$ that there exists a constant $C$ such that $1/C \leq A/B \leq C$.

3. The Euclidean case

In this section we work in the classical setting of $\mathbb{R}^n$.

Theorem 3.1. The following hold:

(i) if $q < \infty$, then $\|f\|_{H^{1,q}_\text{fin}(\mathbb{R}^n)}$ and $\|f\|_{H^1(\mathbb{R}^n)}$ are equivalent norms on $H^{1,q}_\text{fin}(\mathbb{R}^n)$;

(ii) the two norms $\|f\|_{H^{1,\infty}_\text{fin}(\mathbb{R}^n)}$ and $\|f\|_{H^1(\mathbb{R}^n)}$ are equivalent on $H^{1,\infty}_\text{fin}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$.

Proof. Clearly, $\|f\|_{H^1(\mathbb{R}^n)} \leq \|f\|_{H^{1,q}_\text{fin}(\mathbb{R}^n)}$ for $f$ in $H^{1,q}_\text{fin}(\mathbb{R}^n)$ and for $q$ in $(1,\infty]$. Thus, we have to show that for every $q$ in $(1,\infty)$ there exists a constant $C$ such that

$$\|f\|_{H^{1,q}_\text{fin}(\mathbb{R}^n)} \leq C \|f\|_{H^1(\mathbb{R}^n)} \quad \forall f \in H^{1,q}_\text{fin}(\mathbb{R}^n),$$

and that a similar estimate holds for $q = \infty$ and all $f$ in $H^{1,\infty}_\text{fin}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$.

Suppose that $q$ is in $(1,\infty]$ and that $f$ is in $H^{1,q}_\text{fin}(\mathbb{R}^n)$ with $\|f\|_{H^1(\mathbb{R}^n)} = 1$. By the translation invariance of Lebesgue measure, we may assume that the support of $f$ is contained in the closed ball $B = B(0,R)$ centred at 0 with radius $R$. For each $k$ in $\mathbb{Z}$, denote by $\Omega_k$ the level set $\{x \in \mathbb{R}^n : M_m f(x) > 2^k\}$ of the grand maximal function $M_m f$ of $f$. We choose Whitney cubes $Q^i_k$, $i \in \mathbb{N}$, with disjoint interiors satisfying $\Omega_k = \bigcup_i Q^i_k$ and

$$\text{diam}(Q^i_k) \leq \eta \text{dist}(Q^i_k, \Omega_k) \leq 4 \text{diam}(Q^i_k),$$

where $\eta$ is a suitable constant in $(0,1)$. Except for the factor $\eta$, this is Theorem VI.1 of [14] p. 167. The only modification needed in the proof of [14] concerns the choice of the constant denoted by $c$.

By following closely the proof of [15] Theorem III.2, p. 107] or [13] Theorem 3.5, pp. 12-18, we produce an atomic decomposition of $f$ of the form

$$f = \sum_{i,k} \lambda^i_k a^k_i,$$

such that the following hold:

(a) $|\lambda^i_k a^k_i| \leq C 2^k$ for every $k$ in $\mathbb{Z}$;

(b) for each $k$ in $\mathbb{Z}$, the atoms $a^k_i$ are supported in balls $B^k_i$ concentric with the $Q^i_k$ and contained in $\Omega_k$. By choosing the constant $\eta$ in (3.1) small enough, depending on the dimension, we can also ensure that the family $\{B^k_i\}_i$ has the bounded overlap property, uniformly with respect to $k$. 

(c) there exists a constant $C$ independent of $f$ such that
\[
\sum_{i,k} |\lambda_i^k| \leq C \|f\|_{H^1(\mathbb{R}^n)} = C.
\]

We write $2B$ for the closed ball concentric with $B$ whose radius is twice as large. For $\varphi$ in $A_m$ and $x$ in $\mathbb{R}^n \setminus (2B)$ one then has
\[
|\varphi_t \ast f(x)| \leq t^{-n} \sup_{y \in B} \varphi(y/t) \|f\|_{L^1(\mathbb{R}^n)}
\leq t^{-n} (1 + R/t)^{-m} \|f\|_{L^1(\mathbb{R}^n)} \quad \forall t \in \mathbb{R}^+,
\]
so that
\[
\mathcal{M}_m f(x) = \sup_{\varphi \in A_m} \sup_{t > R} |\varphi_t \ast f(x)| \leq R^{-n},
\]
since $m > n$. Now, if $x$ is in $\Omega_k \setminus (2B)$, the above inequality and the definition of $\Omega_k$ force $2k < R^{-n}$; denote by $k'$ the largest integer $k$ such that $2k < R^{-n}$. Then $\Omega_k$ is contained in $2B$ for $k > k'$.

Next we define the functions $h$ and $\ell$ by
\[
(3.3) \quad h = \sum_{k \leq k'} \sum_i \lambda_i^k a_i^k \quad \text{and} \quad \ell = \sum_{k > k'} \sum_i \lambda_i^k a_i^k.
\]

Observe that both these series converge in $L^1(\mathbb{R}^n)$, simply because $\sum_{i,k} |\lambda_i^k| < \infty$, so that $h$ and $\ell$ have integral 0. Clearly, $f = h + \ell$. Furthermore, the support of $\ell$ is contained in $2B$, because it is contained in $\Omega_k$ by (b) above, and $\Omega_k$ is contained in $2B$ for all $k > k'$. Therefore $h = f = 0$ in $(2B)^c$.

To estimate the size of $h$ in $2B$, we use (a) above and the bounded overlap property of (b), getting
\[
|h| \leq C \sum_{k \leq k'} 2^k \leq C 2^{k'} \leq C |2B|^{-1}.
\]
This proves that $h/C$ is a $(1, \infty)$-atom, where $C$ is independent of $f$.

Now we assume that $q < \infty$ and conclude the proof of (i). Observe that $\ell$ is in $L^q(\mathbb{R}^n)$, because $\ell = f - h$, and both $f$ and $h$ are in $L^q(\mathbb{R}^n)$.

We claim that the series $\sum_{k > k'} \sum_i \lambda_i^k a_i^k$ converges to $\ell$ in $L^q(\mathbb{R}^n)$.

Fixing $s$ in $\mathbb{Z}$, we shall estimate $\sum_{k > k'} \sum_i |\lambda_i^k a_i^k|$ in $\Omega_s \setminus \Omega_{s+1}$. First observe that all terms with $k > s$ vanish outside $\Omega_{s+1}$. Then apply (a) and (b) to get the pointwise bound
\[
\sum_{k > k'} \sum_i |\lambda_i^k a_i^k| \leq C \sum_{k \leq s} 2^k \leq C 2^s \leq C \mathcal{M}_m f.
\]
The constants $C$ above are independent of $f$ and $s$, so that
\[
\sum_{k > k'} \sum_i |\lambda_i^k a_i^k| \leq C \mathcal{M}_m f
\]
in all of $\mathbb{R}^n$, with $C$ independent of $f$. Note that $\mathcal{M}_m f$ is in $L^q(\mathbb{R}^n)$, since $f$ is. This implies that the series defining $\ell$ converges almost everywhere and the limit must coincide with the $L^1$ limit $\ell$. The Lebesgue dominated convergence theorem now implies that $\sum_{k > k'} \sum_i \lambda_i^k a_i^k$ converges to $\ell$ in $L^q(\mathbb{R}^n)$, and the claim is proved.

Finally, for each positive integer $N$ we denote by $F_N$ the finite set of all pairs of integers $(i,k)$ such that $k > k'$ and $|i| + |k| \leq N$, and by $\ell_N$ the function $\sum_{(i,k) \in F_N} \lambda_i^k a_i^k$. The function $\ell_N$ is in $H^1_{\text{fin}}(\mathbb{R}^n)$, and $f = h + \ell_N + (\ell - \ell_N)$.
Observe that \( \ell - \ell_N \) will be a small multiple of a \((1,q)\)-atom for large \(N\). Indeed, by taking \(N\) large enough, we can make the corresponding coefficient less than any given \(\varepsilon\) in \(\mathbb{R}^+\). Then

\[
\|f\|_{H_{\text{fin}}^{1,q}(\mathbb{R}^n)} \leq C + \sum_{(i,k) \in F_N} |\lambda_i^k| + \varepsilon,
\]

so that

\[
\|f\|_{H_{\text{fin}}^{1,q}(\mathbb{R}^n)} \leq C + \sum_{(i,k) \in F_N} |\lambda_i^k| \leq C,
\]

by property (c) above, as required to conclude the proof of (i).

Now we finish the proof of (ii). Assume that \(f\) is a continuous function in \(H_{\text{fin}}^{1,\infty}(\mathbb{R}^n)\). A careful examination of the proof of [15] Theorem III.2, pp. 107-8 or [13] Theorem 3.5, pp. 12-18 shows that the atoms \(a_i^k\) that appear in the decomposition (3.3) are then continuous. Furthermore, we see that for each \(k\) and \(i\) the function \(\lambda_i^k a_i^k\) depends only on the restriction of \(f\) to a ball \(\hat{B}_i^k\) which is a concentric enlargement of the ball \(B_i^k\) from (b) above, by a fixed scaling factor. It is straightforward to check that if \(f\) is constant in \(\hat{B}_i^k\), then \(\lambda_i^k a_i^k = 0\) and that there exists an absolute constant \(C\) such that if \(|f| < \varepsilon\) in \(\hat{B}_i^k\), then \(|\lambda_i^k a_i^k| < C \varepsilon\).

Since trivially \(M_{\text{fin}} f \leq C_n \|f\|_{\infty}\), where the constant \(C_n\) depends only on \(n\), the level set \(\Omega_k\) is empty for all \(k\) such that \(2^k \geq C_n \|f\|_{\infty}\). We denote by \(k''\) the largest integer for which the last inequality does not hold. Then the index \(k\) in the sum defining \(\ell\) in (3.3) will run only over \(k' < k \leq k''\).

Let \(\varepsilon\) be positive. Since \(f\) is uniformly continuous, there exists a positive \(\delta\) such that \(|x - y| < \delta\) implies

\[
|f(x) - f(y)| < \varepsilon.
\]

Write \(\ell = \ell_1^* + \ell_2^*\) with

\[
\ell_1^* = \sum_{(i,k) \in F_1} \lambda_i^k a_i^k \quad \text{and} \quad \ell_2^* = \sum_{(i,k) \in F_2} \lambda_i^k a_i^k,
\]

where \(F_1 = \{(i,k) : \text{diam}(\hat{B}_i^k) \geq \delta, k' < k \leq k''\}\) and \(F_2 = \{(i,k) : \text{diam}(\hat{B}_i^k) < \delta, \ k < k \leq k''\}\). Since \(F_1\) is a finite set, \(\ell_1^*\) is continuous.

To estimate \(\ell_2^*\), we denote by \(x_i^k\) the centre of the ball \(B_i^k\) and write for \((i,k)\) in \(F_2\)

\[
f(x) = f(x_i^k) + f(x) - f(x_i^k).
\]

Then \(|\lambda_i^k a_i^k| < C \varepsilon\), because \(|f(x) - f(x_i^k)| < \varepsilon\) for \(x\) in \(\hat{B}_i^k\). For fixed \(k\) the balls \(\{B_i^k\}\) have uniformly bounded overlap, so there exists an absolute constant \(C\) such that

\[
|\ell_2^*| \leq C \sum_{k' < k \leq k''} \varepsilon \leq C (k'' - k') \varepsilon.
\]

Since \(\varepsilon\) is arbitrary, we can thus split \(\ell\) into a continuous part and a part that is uniformly arbitrarily small. It follows that \(\ell\) is continuous. But then \(h = f - \ell\) is also continuous, so that \(h\) is a continuous \((1,\infty)\)-atom, multiplied by a factor \(C\).

To find a finite atomic decomposition of \(\ell\), we again use the splitting \(\ell = \ell_1^* + \ell_2^*\). Clearly \(\ell_1^*\) is for each \(\varepsilon\) a finite linear combination of continuous \((1,\infty)\)-atoms, and the \(\ell^1\) norm of the coefficients is controlled by \(\|f\|_{H^1}\), in view of (c). Observe that \(\ell_2^* = \ell - \ell_1^*\) is continuous. Further, \(\ell_2^*\) is supported in \(2B\), has integral 0 and satisfies \(|\ell_2^*| \leq C (k'' - k') \varepsilon\). Choosing \(\varepsilon\), we can thus make \(\ell_2^*\) into an arbitrarily small multiple of a continuous \((1,\infty)\)-atom.
To sum up, \( f = h + \ell_1^j + \ell_2^j \) gives the desired finite atomic decomposition of \( f \), with coefficients controlled by \( \|f\|_{H^1} \).

We have completed the proof of (ii) and that of the theorem. □

**Remark 3.2.** Theorem 3.1 (ii) implies that any function \( f \) in \( H_{\text{fin}}^1(\mathbb{R}^n) \cap C(\mathbb{R}^n) \) admits a finite decomposition in \((1, \infty)\)-atoms such that the sum of the corresponding coefficients is \( \leq C \|f\|_{H^1(\mathbb{R}^n)} \). Actually, the proof of Theorem 3.1 (ii) shows that we can construct this finite decomposition in such a way that it involves only continuous \((1, \infty)\)-atoms.

**Remark 3.3.** Theorem 3.1 extends to \( H^p(\mathbb{R}^n) \) with \( 0 < p < 1 \) and \((p, q)\)-atoms, where one can now have \( 1 \leq q \leq \infty \). The proof is rather similar to the one given above, so we only briefly describe the modifications needed for part (i). Thus let \( f \in H^p_{\text{fin}}(\mathbb{R}^n) \) supported in a ball \( B_R \), the first step is the inequality \( M_{m}f \leq CR^{-n/p}\|f\|_{H^p(\mathbb{R}^n)} \), valid outside a larger ball \( B_{CR} \). One proves this by comparing the values of \( M_m f \) at different points and using the fact that \( \|M_m f\|_{L^p(\mathbb{R}^n)} \sim \|f\|_{H^p(\mathbb{R}^n)} \). Then the \( \Omega_k \) and the decompositions \( f = \sum \lambda_i a_i = h + \ell \) are introduced as above. The sum \( \ell \) now converges in \( S' \) and is dominated by \( M_m f \). If \( q > 1 \), we have \( \mathcal{M}_m f \in L^q(\mathbb{R}^n) \) and conclude as before that \( \ell \) converges in \( L^q(\mathbb{R}^n) \). For \( q = 1 \), the tail sum \( S_\kappa = \sum_{k \geq \kappa} \sum_j \lambda_i a_i \) tends to 0 in \( L^1(\mathbb{R}^n) \) as \( \kappa \to +\infty \), because \( S_\kappa \) is nonzero only in \( \Omega_\kappa \) and not larger than \( |f| + C2^n \) there, and \( |\Omega_\kappa| = o(2^{-\kappa}) \) as \( \kappa \to +\infty \). The rest of the proof proceeds as before. See also [9] Theorem 5.6.

**Corollary 3.4.** Suppose that \( Y \) is a Banach space and that one of the following holds:

(i) \( q \) is in \((1, \infty)\) and \( T : H_{\text{fin}}^{1,q}(\mathbb{R}^n) \to Y \) is a linear operator such that
\[
A := \sup \{ \| Ta \|_Y : a \text{ is a } (1, q)\text{-atom} \} < \infty;
\]

(ii) \( T \) is a \( Y \)-valued linear operator defined on continuous \((1, \infty)\)-atoms such that
\[
A := \sup \{ \| Ta \|_Y : a \text{ is a continuous } (1, \infty)\text{-atom} \} < \infty.
\]

Then there exists a unique bounded linear operator \( \tilde{T} \) from \( H^1(\mathbb{R}^n) \) to \( Y \) which extends \( T \).

**Proof.** We consider the case (i). Suppose that \( f \) is in \( H_{\text{fin}}^{1,q}(\mathbb{R}^n) \), \( f = \sum_{j=1}^{N} \lambda_j a_j \) say, where \( a_j \) are \((1, q)\)-atoms. Then the assumption and the triangle inequality give
\[
\|Tf\|_Y \leq A \sum_{j=1}^{N} |\lambda_j|.
\]

By taking the infimum of the right-hand side with respect to all decompositions of \( f \) as a finite sum of \((1, q)\)-atoms, we obtain
\[
\|Tf\|_Y \leq A \|f\|_{H_{\text{fin}}^{1,q}(\mathbb{R}^n)}.
\]

Now, Theorem 3.1 (i) implies that the right-hand side is dominated by \( CA \|f\|_{H^1(\mathbb{R}^n)} \), where \( C \) does not depend on \( f \), and a density argument completes the proof of the corollary.

The case (ii) is similar. □
we assume that $\mu$ also that Corollary 3.4 applies to linear functionals.

**Theorem 4.1.** Suppose that $T$ is a linear operator defined on $H^1_{\nu}(M)$ with the property that

$$A := \sup\{\|Ta\|_{L^1(M)} : a \text{ is a } (1,q)\text{-atom}\} < \infty.$$ 

Then there exists a unique bounded linear operator $\tilde{T}$ from $H^1(M)$ to $L^1(M)$ which extends $T$.

**Proof.** We prove the result in the case where $q = 2$. The proof in the other cases is similar.

Suppose that $B$ is a ball. For each $f$ in $L^2_0(B)$ such that $\|f\|_{L^2(M)} = 1$, the function $\mu(B)^{-1/2} f$ is a $(1,2)$-atom, so that

$$\|Tf\|_{L^1(M)} \leq A \mu(B)^{1/2} \quad \forall f \in L^2_0(B)$$

by the assumption. In particular, the restriction of $T$ to $X^2_k$ is bounded from $X^2_k$ to $L^1(M)$ for each $k$. Thus, $T$ is bounded from $X^2$ to $L^1(M)$. It follows that $T^*$ is bounded from $L^\infty(M)$ to the dual of $X^2$. But the dual of $X^2$ is the quotient space $L^2_{\text{loc}}(M)/\mathbb{C}$, since that of $L^2_0(B_k)$ is $L^2(B_k)/\mathbb{C}$. Now, for every $f$ in $L^\infty(M)$ and for every $(1,2)$-atom $a$,

$$\langle Ta, f \rangle = \langle a, T^* f \rangle = \int_M a T^* f \, d\mu,$$

so that

$$\left| \int_M a T^* f \, d\mu \right| = |\langle Ta, f \rangle| \leq A \|f\|_\infty.$$

A standard argument then shows that $T^* f$ belongs to $BMO(M)$ and that

$$\|T^* f\|_{BMO(M)} \leq 2A \|f\|_\infty \quad \forall f \in L^\infty(M).$$

We give the details for the reader’s convenience. Suppose that $B$ is a ball and observe that

$$\left[ \int_B |T^* f - (T^* f)_B|^2 \, d\mu \right]^{1/2} = \sup_{\|\varphi\|_{L^2(B)}=1} \left| \int_B \varphi \, (T^* f - (T^* f)_B) \, d\mu \right|.$$

But

$$\int_B \varphi \, (T^* f - (T^* f)_B) \, d\mu = \int_B (\varphi - \varphi_B) \, (T^* f - (T^* f)_B) \, d\mu = \int_B (\varphi - \varphi_B) \, T^* f \, d\mu,$$

and since $\|\varphi\|_{L^2(B)} = 1$,

$$|\varphi_B| \leq \left[ \frac{1}{\mu(B)} \int_B |\varphi|^2 \, d\mu \right]^{1/2} \leq \mu(B)^{-1/2}.$$
Write $\psi$ instead of $\varphi - \varphi_B$. Then
\[ \|\psi\|_{L^2(B)} \leq \|\varphi\|_{L^2(B)} + |\varphi_B| \mu(B)^{1/2} \leq 2, \]
so that $\psi/(2 \mu(B)^{1/2})$ is a $(1,2)$-atom. Therefore
\[ \left| \int_B \psi \, T^* f \, d\mu \right| \leq 2A \mu(B)^{1/2} \|f\|_\infty. \]
Combining the above, we conclude that for every ball $B$
\[ \left[ \frac{1}{\mu(B)} \int_B |T^* f - (T^* f)_B|^2 \, d\mu \right]^{1/2} \leq 2A \|f\|_\infty, \]
and (4.1) follows.

Now we show that $T$ extends to a bounded operator from $H^1(M)$ to $L^1(M)$ with norm at most $2A$. Observe that $X^2$ and $H^1_{\text{fin}}(M)$ coincide as vector spaces. For every $g$ in $H^1_{\text{fin}}(M)$ and for every $f$ in $L^\infty(M)$
\[ |\langle Tg, f \rangle| = |\langle g, T^* f \rangle| \leq \|g\|_{H^1(M)} \|T^* f\|_{\text{BMO}(M)} \leq 2A \|g\|_{H^1(M)} \|f\|_{L^\infty(M)}. \]
By taking the supremum of both sides over all functions $f$ in $L^\infty(M)$ with $\|f\|_{L^\infty(M)} = 1$, we obtain that
\[ \|Tg\|_{L^1(M)} \leq 2A \|g\|_{H^1(M)} \quad \forall g \in H^1_{\text{fin}}(M). \]
Finally we observe that $H^1_{\text{fin}}(M)$ is dense in $H^1(M)$ (with respect to the norm of $H^1(M)$), and the required conclusion follows by a density argument. 

Quite often one encounters the following situation. Suppose that $T$ is a bounded linear operator on $L^2(M)$. Then $T$ is automatically defined on $H^1_{\text{fin}}(M)$. Assume that
\[ A := \sup\{\|Ta\|_{L^1(M)} : a \text{ is a } (1,2)\text{-atom}\} < \infty. \]
By the previous result, the restriction of $T$ to $H^1_{\text{fin}}(M)$ has a unique extension to a bounded linear operator $\widetilde{T}$ from $H^1(M)$ to $L^1(M)$. The question is whether the operators $T$ and $\widetilde{T}$ are consistent, i.e., whether they coincide on the intersection $H^1(M) \cap L^2(M)$ of their domains. The answer to this question is in the affirmative, as the following proposition shows.

**Proposition 4.2.** Suppose that $T$ is bounded on $L^2(M)$ and that
\[ A := \sup\{\|Ta\|_{L^1(M)} : a \text{ is a } (1,2)\text{-atom}\} < \infty. \]
Denote by $\widetilde{T}$ the unique continuous linear extension of the restriction of $T$ to $H^1_{\text{fin}}(M)$ to an operator from $H^1(M)$ to $L^1(M)$. Then the operators $T$ and $\widetilde{T}$ agree on $H^1(M) \cap L^2(M)$.

**Proof.** Suppose that $f$ is in $L^2(M) \cap L^\infty(M)$ and that $g$ is in $L^2_{c,0}(M)$. Denote by $T^*$ the transpose operator of $T$ (as an operator on $L^2(M)$). Then
\[ (4.2) \int_M g \, T^* f \, d\mu = \int_M T g \, f \, d\mu. \]
Since \( g \) is in \( H_{lin}^{1,2}(M) \) and the operators \( T \) and \( \widetilde{T} \) agree on \( H_{lin}^{1,2}(M) \), we see that

\[
\int_M T g f \, d\mu = \int_M \widetilde{T} g f \, d\mu = \langle g, (\widetilde{T})^* f \rangle,
\]

where \( (\tilde{T})^* \) denotes the transpose of the operator \( \tilde{T} \) from \( H^1(M) \) to \( L^1(M) \). Note that \( (\tilde{T})^* f \) is in \( BMO(M) \) and \( g \) is a multiple of an atom. Thus the above scalar product \( \langle g, (\tilde{T})^* f \rangle \) (with respect to the duality between \( H^1(M) \) and \( BMO(M) \)) may be written as \( \int_M g (\tilde{T})^* f \, d\mu \). Therefore, (4.2) and (4.3) imply that

\[
\int_M g \left[ T^* f - (\tilde{T})^* f \right] \, d\mu = 0 \quad \forall g \in L^2_c(M),
\]

i.e., for all \( g \) in \( X^2 \). Therefore \( T^* f - (\tilde{T})^* f = 0 \) in the dual space of \( X^2 \), i.e., in \( L^2_{loc}(M)/\mathbb{C} \). This implies that \( T^* f - (\tilde{T})^* f \) is constant.

Now, suppose that \( g \) is in \( H^1(M) \cap L^2(M) \) and that \( f \) is in \( L^2(M) \cap L^\infty(M) \). Then

\[
\int_M T g f \, d\mu = \int_M g T^* f \, d\mu = \int_M g (\tilde{T})^* f \, d\mu = \int_M \tilde{T} g f \, d\mu.
\]

Since \( f \) is an arbitrary function in \( L^2(M) \cap L^\infty(M) \), \( Tg - \tilde{T}g = 0 \) almost everywhere, as required. \( \Box \)

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