Symbolic computation of moments of sampling distributions

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Abstract

By means of the notion of umbrae indexed by multisets, a general method to express estimators and their products in terms of power sums is derived. A connection between the notion of multiset and integer partition leads immediately to a way to speed up the procedures. Comparisons of computational times with known procedures show how this approach turns out to be more efficient in eliminating much unnecessary computation.

keywords: Umbral calculus; symmetric functions; moments of moments; sampling distributions; U-statistics

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1 Introduction

It is acknowledged that an appropriate choice of language and notation can simplify and clarify many statistical calculations. In recent years, most of the work has been done by symbolic computation, main references are [14], [2]. These books offer a variety of applications of symbolic methods, from asymptotic expansions to the Edgeworth series, from likelihood functions to the saddlepoint approximations. In [25], Zeilberger describes a methodology for using computer algebra systems to automatically derive moments, up to order 4, of interesting combinatorial random variables. Such a methodology is applied to pattern statistics of permutations. For different applications of computer algebra in statistics, see also [18].

The aim of this paper is to show the computational efficiency of umbral calculus in manipulating expressions involving random variables.

The umbral calculus to which we refer is the version featured by Rota and Taylor in [20]. The basic device is the representation of a unital sequence of numbers by a symbol α, called an umbra, that is, the sequence 1, a_1, a_2, ... is represented by the sequence 1, α, α^2, ... of powers of α via an operator E resembling the expectation operator of random variables. This approach has led to a finely adapted language for random variables by Di Nardo and Senato, see [7]. In [8], attention is focused on cumulants since a random variable is often better described by its cumulants than by its moments, as it happens for the family of Poisson random variables. Moreover, due to the properties of additivity and invariance under translation, cumulants are not necessarily connected with moments of any probability distribution. As a matter of fact, an umbra seems to have the structure of a random variable but with no reference to a probability space, bringing us closer to statistical methods. In [9], it is shown that classical umbral...
calculus provides a unifying framework for unbiased estimators of cumulants, called $k$-statistics, and their multivariate generalizations. Moreover, within the umbral framework, a statistical result does not require a check of background details by hand, but becomes a corollary of a more general theorem.

Here, we focus attention on the more general problem of calculations of algebraic expressions such as the variance of a sample mean or, more generally, moments of sampling distributions, which have a variety of applications within statistical inference [23]. The field has, in the past, been marred by the difficulty of manual computations. Symbolic computations have removed many of such difficulties, leaving some issues unresolved. One of the most intriguing questions is to explain why symbolic procedures, which are straightforward in the multivariate case, turn out to be obscure in the simpler univariate one, see [2].

We show that the notion of multiset is the key for dealing with symbolic computation in multivariate statistics. Actually, at the root of the question, there are some aspects of the combinatorics of symmetric functions that would benefit if the attention is shifted from sets to the more general notion of multiset. On the other hand, due to its generality, umbral calculus reduces the combinatorics of symmetric functions, commonly used by statisticians, to few relations which cover a great variety of calculations. In particular, umbral equivalences [21] smooth the way to handle any kind of product of sums. As example, by using equivalences [21], we evaluate the mean of product of augmented polynomials in separately independent and identically distributed random variables. The resulting umbral strategy is completely different from those recently proposed in the literature, see for instance [24], and computationally more efficient, as we show in the last section.

Moreover, by means of the notion of umbrae indexed by multiset, we remove the necessity to specify if the random variables of a vector are identically distributed or not. The basic procedure consists in finding multiset subdivisions, which suitably extends the notion of set partitions. The strategy here proposed is a sort of iterated inclusion-exclusion rule [1, 3], whose efficiency is improved taking into account the structure of multiset and its relation with integer partitions. The result is the algorithm makeTab, given in the appendix.

The paper is structured as follows. Section 2 is provided for readers unaware of classical umbral calculus. We resume terminology, notation and some basic definitions. In Section 3, we give a general procedure for writing down $U$-statistics. Recall that many statistics of interest may be exactly represented or approximated by $U$-statistics [11]. Such a procedure is based on the umbral relation between moments and augmented symmetric functions. The connection with power sums is analyzed in Section 4. The effectiveness of umbral methods is shown in several examples, proposed with the intention of helping the reader unaware of umbral calculus to understand the basic algebraic rules necessary to work with this syntax. Section 5 is devoted to umbral formulae giving power sums in terms of augmented symmetric functions. These formulae provide the most natural way to form the product of augmented symmetric functions by using a suitable umbral substitution. The consequent reduction of the computational time is made clear through some examples, which point out the role played by the singleton umbra in selecting the suitable variables. Section 6 is devoted to computational comparisons with the procedures known in the literature in dealing with moments of sampling distributions. The speed up of umbral methods is evident. Some concluding remarks end the paper.

Although, by a numerical point of view, MAPLE seems to work less efficiently respect to MATHEMATICA, we have implemented our algorithms in MAPLE because the syntax is more comfortable for symbolic computation. All tasks have been performed on a PC Pentium(R)4 Intel(R), CPU 3.00 Gzh, 480MB Ram with MAPLE version 10.0 and MATHEMATICA version 4.2.
2 The classical umbral calculus

Classical umbral calculus is a syntax consisting of the following data:

i) a set $A = \{\alpha, \beta, \ldots\}$, called the alphabet, whose elements are named umbrae;

ii) a commutative integral domain $R$ whose quotient field is of characteristic zero;

iii) a linear functional $E$, called an evaluation, defined on the polynomial ring $R[A]$ and taking values in $R$ such that
   
   \begin{enumerate}
   \item $E[1] = 1$;
   \item $E[\alpha^i \beta^j \cdots \gamma^k] = E[\alpha^i]E[\beta^j] \cdots E[\gamma^k]$ for any set of distinct umbrae in $A$ and for $i, j, \ldots, k$ non-negative integers (uncorrelation property);
   \end{enumerate}

iv) an element $\varepsilon \in A$, called an augmentation, such that $E[\varepsilon^n] = 0$ for all $n \geq 1$;

v) an element $u \in A$, called a unity umbra, such that $E[u^n] = 1$ for all $n \geq 1$.

Note that, for statistical applications, $R$ is the field of real numbers.

An umbral polynomial is a polynomial $p \in R[A]$. The support of $p$ is the set of all umbrae occurring in $p$.

If $p$ and $q$ are two umbral polynomials, then

i) $p$ and $q$ are uncorrelated if and only if their supports are disjoint;

ii) $p$ and $q$ are umbrally equivalent iff $E[p] = E[q]$, in symbols $p \simeq q$.

The basic idea of the classical umbral calculus is to associate a sequence of numbers $1, a_2, a_3, \ldots$ to an indeterminate $\alpha$, which is said to represent the sequence. This device is familiar in statistics, when $a_i$ represents the $i$-th moment of a random variable $X$. In this case, the sequence $1, a_1, a_2, \ldots$ results from applying the expectation operator $E$ to the sequence $1, X, X^2, \ldots$ consisting of powers of $X$. This is why the elements $a_n \in R$ such that

$$E[\alpha^n] = a_n, \ n \geq 0$$

are named moments of the umbra $\alpha$ and we say that the umbra $\alpha$ represents the sequence of moments $1, a_1, a_2, \ldots$. The umbra $\varepsilon$ plays the same role of a random variable which takes the value 0 with probability 1 and the umbra $u$ plays the same role of a random variable which takes the value 1 with probability 1. The uncorrelation property among umbrae parallels the analogue one for random variables. In this setting no attention must be paid to the well-known “moment problem”.

In parallel with random variable theory, the factorial moments of an umbra $\alpha$ are the elements $a_{(n)} \in R$ corresponding to the umbral polynomials $(\alpha)_n = \alpha(\alpha - 1) \cdots (\alpha - n + 1), n \geq 1$, via the evaluation $E$, that is $E[(\alpha)_n] = a_{(n)}$.

There are umbrae playing a special role in the umbral calculus. Their properties have been investigated with full particulars in \[7, 8\].

Singleton umbra. The singleton umbra $\chi$ is the umbra whose moments are all zero, except the first $E[\chi] = 1$. Its factorial moments are $x_{(n)} = (-1)^{n-1}(n-1)!$ As we will see later on, this umbra is the keystone for managing symmetric umbral polynomials.
Bell umbra. The Bell umbra $\beta$ is the umbra whose factorial moments are all equal to 1, that is $E[(\beta)_n] = 1$ for all $n \geq 1$. Its moments are the Bell numbers, that is the number of partitions of a finite nonempty set with $n$ elements, or the $n$-th coefficient in the Taylor series expansion of the function $\exp(e^t - 1)$. So $\beta$ is the umbral counterpart of a Poisson random variable with parameter 1.

It is possible that two distinct umbrae represent the same sequence of moments, in such case these are called similar umbrae. More formally two umbrae $\alpha$ and $\gamma$ are said to be similar when

$$E[\alpha^n] = E[\gamma^n] \quad \forall n \geq 0, \text{ in symbols } \alpha \equiv \gamma.$$ 

Furthermore, given a sequence 1, $a_1, a_2, \ldots \in R$, there are infinitely many distinct, and thus similar umbrae representing the sequence. So, the umbral counterpart of a univariate random sample is a n-vector $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, where $\alpha_i$, $i = 1, 2, \ldots, n$ are uncorrelated umbrae, similar to the same umbra $\alpha$. Thanks to the notion of similar umbra, it is possible to extend the alphabet $A$ with the so-called auxiliary umbrae resulting from operations among similar umbrae. This leads to construct a saturated umbral calculus in which auxiliary umbrae are handled as elements of the alphabet [20]. In the following, we focus the attention on auxiliary umbrae which play a special role. Let $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a set of $n$ uncorrelated umbrae similar to an umbra $\alpha$. The symbol $n.\alpha$ denotes an auxiliary umbra similar to the sum $\alpha_1 + \alpha_2 + \cdots + \alpha_n$. So $n.\alpha$ is the umbral counterpart of a sum of independent and identically distributed random variables. The symbol $\alpha^n$ is an auxiliary umbra denoting the product $\alpha_1 \alpha_2 \cdots \alpha_n$.

Moments of $\alpha^n$ can be easily recovered from its definition. Indeed, if the umbra $\alpha$ represents the sequence 1, $a_1, a_2, \ldots$, then $E[(\alpha^n)^k] = a_k^n$ for nonnegative integers $k$ and $n$.

Moments of $n.\alpha$ can be expressed through integer partitions. Recall that a partition of an integer $i$ is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$, where $\lambda_j$ are weakly decreasing integers and $\sum_{j=1}^t \lambda_j = i$. The integers $\lambda_j$ are named parts of $\lambda$. The length of $\lambda$ is the number of its parts and will be indicated by $\nu_\lambda$. A different notation is $\lambda = (1^{r_1}, 2^{r_2}, \ldots)$, where $r_j$ is the number of parts of $\lambda$ equal to $j$ and $r_1 + r_2 + \cdots = \nu_\lambda$. For example, $(1^3, 2^1, 3^2)$ is a partition of the integer 11. We use the classical notation $\lambda \vdash i$ to denote that “$\lambda$ is a partition of $i$”. By using an umbral version of the well-known multinomial expansion theorem [8], we have

$$ (n.\alpha)^i \simeq \sum_{\lambda \vdash i} (n)_{\nu_\lambda} d_\lambda \alpha_\lambda, \quad (1) $$

where the sum is over all partitions $\lambda = (1^{r_1}, 2^{r_2}, \ldots)$ of the integer $i$, $(n)_{\nu_\lambda} = 0$ for $\nu_\lambda > n$,

$$ d_\lambda = \frac{i!}{r_1! r_2! \cdots (1!)^{r_1} (2!)^{r_2} \cdots} \quad \text{and} \quad \alpha_\lambda \equiv (\alpha_{j_1})^{r_1} (\alpha_{j_2})^{r_2} \cdots, \quad (2) $$

with $\{j_i\}$ distinct integers chosen in $\{1, 2, \ldots, n\} = [n]$.

The reader interested in proofs of identities involving auxiliary umbrae is referred to [7].

A feature of the classical umbral calculus is the construction of new auxiliary umbrae by suitable symbolic substitutions. For example, in $n.\alpha$ replace the integer $n$ by an umbra $\gamma$. From [11], the new auxiliary umbra $\gamma.\alpha$ has moments

$$ (\gamma.\alpha)^i \simeq \sum_{\lambda \vdash i} (\gamma)_{\nu_\lambda} d_\lambda \alpha_\lambda \quad (3) $$

and it is called dot-product of $\gamma$ and $\alpha$. The auxiliary umbra $\gamma.\alpha$ is the umbral counterpart of a random sum. In the following, we recall some useful dot-products of umbrae, whose properties have been investigated with full particulars in [5].
**α-factorial umbra.** The umbra $α.χ$ is called the $α$-factorial umbra. Its moments are the factorial moments of $α$, that is $(α.χ)^i \simeq (α)_i$. If $α \equiv χ$, then $E[(χ.χ)^i] = E[(χ)_i] = x(i) = (-1)^{i-1}(i-1)!$.

**α-cumulant umbra.** The umbra $χ.α$, with $χ$ the singleton umbra, is called the $α$-cumulant umbra. By virtue of (3), its moments are

$$
(χ.α)^i \simeq \sum_{\lambda \vdash i} x(\nu_\lambda) d_\lambda α_\lambda \simeq \sum_{\lambda \vdash i} (-1)^{r_\lambda - 1}(ν_\lambda - 1)! d_\lambda α_\lambda.
$$

(4)

Since the second equivalence in (4) recalls the well-known expression of cumulants in terms of moments of a random variable, it is straightforward to refer the moments of the α-cumulant umbra $χ.α$ as cumulants of the umbra $α$.

3 **$U$-statistics**

In the following, we focus our attention on two kinds of auxiliary umbrae: $n.α$ and $n.(χα)$. Such umbrae, and their products, are similar to some well-known symmetric polynomials. Indeed, by definition we have

$$
n.α^r \equiv α_1^r + \cdots + α_n^r,
$$

where $α_1, α_2, \ldots, α_n$ are uncorrelated umbrae, similar to the umbra $α$. Since the umbrae $α_i$ for $i = 1, 2, \ldots, n$ can be rearranged without effecting the evaluation $E$, the auxiliary umbra $n.α^r$ is similar to the $r$-th power sum symmetric polynomial in the indeterminates $α_1, α_2, \ldots, α_n$.

Moreover, since

$$
n.(χα) \equiv χ_1 α_1 + \cdots + χ_n α_n,
$$

powers of $n.(χα)$ are umbrally equivalent to the umbral elementary symmetric polynomials $[n.(χα)]^k \simeq k! e_k(α_1, α_2, \ldots, α_n)$, where

$$
e_k(α_1, α_2, \ldots, α_n) = \sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq n} α_{j_1} α_{j_2} \cdots α_{j_k}, \quad k = 1, 2, \ldots, n.
$$

The proof is given in [9] and it relies on the role played by the umbra $χ$ in picking out the indeterminates.

**Example 3.1** If $k = 2$, then

$$
n.(χα)^2 \simeq (χ_1 α_1 + \cdots + χ_n α_n)^2 \simeq \sum_{i=1}^{n}(χ_i α_i)^2 + 2 \sum_{1 \leq j_1 < j_2 \leq n} χ_{j_1} α_{j_1} χ_{j_2} α_{j_2}
$$

$$
\simeq 2 \sum_{1 \leq j_1 < j_2 \leq n} α_{j_1} α_{j_2}.
$$

The last equivalence follows by observing that $(χ_i α_i)^2 \simeq 0$ for $i = 1, 2, \ldots, n$ since $E[(χ_i α_i)^2] = E[χ_i^2] E[α_i^2]$ for the uncorrelation property between $α_i$ and $χ_i$, and $E[χ_i^2] E[α_i^2] = 0$ because $E[χ_i^2] = 0$. On the other hand, we have $χ_{j_1} α_{j_1} χ_{j_2} α_{j_2} \simeq α_{j_1} α_{j_2}$ for $1 \leq j_1 < j_2 \leq n$ since the uncorrelation property among $χ_{j_1}, α_{j_1}, χ_{j_2}, α_{j_2}$ implies $E[χ_{j_1} α_{j_1} χ_{j_2} α_{j_2}] = E[χ_{j_1}] E[α_{j_1}] E[χ_{j_2}] E[α_{j_2}]$, but $E[χ_{j_1}] E[α_{j_1}] E[χ_{j_2}] E[α_{j_2}] = E[α_{j_1}] E[α_{j_2}]$ because $E[χ_{j_1}] = E[χ_{j_2}] = 1$.

The auxiliary umbra $n.(χα)$ enables us to rewrite umbral augmented symmetric polynomials in a very compact expression. Let $λ = (1^r, 2^r, \ldots)$ be a partition of the integer $i \leq n$. Augmented monomial
symmetric polynomials in the indeterminates \( \alpha_1, \alpha_2, \ldots, \alpha_n \) are defined as
\[
\tilde{m}_\lambda(\alpha_1, \alpha_2, \ldots, \alpha_n) = \sum_{j_1 \neq \cdots \neq j_r \in \mathbb{N}} \alpha_{j_1} \cdots \alpha_{j_r} \alpha_{j_{r_1+1}}^2 \cdots \alpha_{j_{r_1+r_2}}^2 \cdots.
\]

In statistical literature, a more common notation is \([1^{r_1}2^{r_2} \ldots] \chi\). For instance, \([1^23]\) denotes
\[
\sum_{1 \leq j_1 \neq j_2 \neq j_3 \leq n} \alpha_{j_1} \alpha_{j_2} \alpha_{j_3}^3.
\]

If \( \lambda = (1^{r_1}, 2^{r_2}, \ldots) \vdash i \), then
\[
[n.(\chi\alpha)]^{r_1} [n.(\chi\alpha^2)]^{r_2} \cdots \simeq [1^{r_1}2^{r_2} \ldots],
\]

(5)
taking into account the role played by the umbra \( \chi \) in selecting variables. We point out that the umbral notation is very similar to the notation \([1^{r_1}2^{r_2} \ldots] \). As before, we give an example in order to clarify equivalence \( \Box \).

Example 3.2 If \( \lambda = (1^2, 3) \), then
\[
[n.(\chi\alpha)]^2 [n.(\chi\alpha^3)] \simeq (\chi_1 \alpha_1 + \cdots + \chi_n \alpha_n)^2 (\chi_1 \alpha_1^3 + \cdots + \chi_n \alpha_n^3)
\]
\[
\simeq \left( \sum_{1 \leq j_1 \neq j_2 \leq n} \chi_{j_1} \alpha_{j_1} \chi_{j_2} \alpha_{j_2} \right) \left( \sum_{1 \leq j_3 \leq n} \chi_{j_3} \alpha_{j_3}^3 \right) \simeq \sum_{1 \leq j_1 \neq j_2 \neq j_3 \leq n} \alpha_{j_1} \alpha_{j_2} \alpha_{j_3}^3.
\]
The last equivalence follows by observing that \( E[\chi_{j_1} \chi_{j_2} \chi_{j_3}] \) vanishes where there is at least one pair of equal indexes \( \Box \).

In the following theorem, we give the umbral formulation of a fundamental expectation result in statistics, see \( \Box \). This is a deep result because it lies at the core of unbiased estimation and moments of moments literature.

Theorem 3.1 If \( \lambda = (1^{r_1}, 2^{r_2}, \ldots) \) is a partition of the integer \( i \leq n \), then
\[
\alpha_\lambda \simeq \frac{1}{(n)^\nu_\lambda} [n.(\chi\alpha)]^{r_1} [n.(\chi\alpha^2)]^{r_2} \cdots.
\]
(6)
See \( \Box \) for the proof.

Equivalence \( \Box \) states how to estimate products of moments \( \alpha_\lambda \) by means of only \( n \) bits of information drawn from the population. In umbral terms, the population is represented by \( \alpha \) and the \( n \) bits of information are the uncorrelated umbrae \( \alpha_1, \alpha_2, \ldots, \alpha_n \), coming into \( [n.(\chi\alpha^i)]^r \). Moreover, having shown that the umbral polynomials \( [n.(\chi\alpha^i)]^r \) are similar to elementary polynomials, Theorem \( \Box \) discloses a more general result: products of moments are unbrally equivalent to products of umbral elementary polynomials. The symmetric polynomial on the right side of equivalence \( \Box \) is named \( U \)-statistic of uncorrelated and similar umbrae \( \alpha_1, \alpha_2, \ldots, \alpha_n \). We take a moment to motivate this denomination. Usually an \( U \)-statistic has the form
\[
U = \frac{1}{(n)^k} \sum \Phi(X_{j_1}, X_{j_2}, \ldots, X_{j_k}),
\]
where \( X_1, X_2, \ldots, X_n \) are \( n \) independent random variables, and the sum ranges in the set of all permutations \( (j_1, j_2, \ldots, j_k) \) of \( k \) integers with \( 1 \leq j_i \leq n \). If \( X_1, X_2, \ldots, X_n \) have the same cumulative distribution function \( F(x) \), \( U \) is an unbiased estimator of the population parameter
\[
\theta(F) = \int \cdots \int \Phi(x_1, \ldots, x_k) dF(x_1) \cdots dF(x_k).
\]
In this case, the function $\Phi$ may be assumed to be a symmetric function of its arguments. Often, in the applications, $\Phi$ is a polynomial in $X_i$’s so that the $U$-statistic is a symmetric polynomial. Hence, by virtue of the fundamental theorem on symmetric polynomials, such an $U$-statistic can be expressed as a polynomial in elementary symmetric polynomials.

**Example 3.3** Moment powers. Let us consider the partition $\lambda = (1^2)$ of the integer 2. The symmetric polynomial

$$U = \frac{1}{(n)_2} [n.(\chi \alpha)]^2, \quad n \geq 2,$$

is the $U$-statistic related to $\alpha^2 \simeq a^2$. Indeed, setting $r_1 = 2$ and $\nu_\lambda = 2$ in (4), we have

$$\alpha^2 \simeq \frac{1}{(n)_2} \sum_{i \neq j} \alpha_i \alpha_j \simeq U,$$

where the last equivalence follows by expanding the square of $n.(\chi \alpha)$.

**Example 3.4** $k$-statistics. The $i$-th $k$-statistic $k_i$ is the unique symmetric unbiased estimator of the cumulant $\kappa_i$ of a given statistical distribution, that is $E[k_i] = \kappa_i$. In umbral terms, we have

$$(\chi \alpha)^i \simeq \sum_{\lambda \vdash i} \frac{x_{\nu_\lambda}}{(n)_{\nu_\lambda}} d_\lambda [n.(\chi \alpha)]^{r_1} [n.(\chi \alpha^2)]^{r_2} \cdots, \quad (7)$$

by using equivalence (4) and Theorem 3.1. Equivalence (7) is the umbral version of the $i$-th $k$-statistic $k_i$.

Usually $k$-statistics are expressed in terms of power sums in the data points, $S_r = \sum_{i=1}^n X_i^r$. Umbrally, this is equivalent to expressing $k$-statistics in terms of $n.\alpha^r$, that is to expressing products of auxiliary umbrae such as $[n.(\chi \alpha^i)]^{r_1}$ in terms of $n.\alpha^j$, for some $j$. Next section is devoted to exploring such relations, which also allow us to express moments of sampling distributions in terms of population moments.

### 4 Augmented and power sums symmetric functions

In this section we turn our attention to symmetric functions useful in computing moments of sampling distributions, i.e augmented monomial symmetric functions and power sums, with special care in formula converting the former in terms of the latter and vice versa. Such polynomials are classical bases of the algebra of symmetric polynomials. The well-known changes of bases involve the lattice of partitions, see [21]. Several packages are available aiming to implement changes of bases (see http://garsia.math.yorku.ca/MPWP/). For instance, the SF package [22] is an integrated MAPLE package devoted to symmetric functions. The use of such packages requires a good knowledge of symmetric function theory and is not so obvious. Moreover, due to their generality, such packages are slow when applied to large variable sets.

The connection between augmented symmetric functions and power sums has been given in umbral terms, this because umbral notation simplifies the changes of bases, taking advantage of multiset notion. In the following we summarize the steps necessary to construct such formulae in the most general case, which have applications in multivariate statistics. The reader interested in proofs is referred to [9].

The starting point is the expression of moments of $n.(\chi \alpha)$ in terms of $n.\alpha$ and vice versa:

$$[n.(\chi \alpha)]^i \simeq \sum_{\lambda \vdash i} d_\lambda (\chi \cdot \chi)_\lambda (n.\alpha)^{r_1} (n.\alpha^2)^{r_2} \cdots, \quad (8)$$

$$(n.\alpha)^i \simeq \sum_{\lambda \vdash i} d_\lambda [n.(\chi \alpha)]^{r_1} [n.(\chi \alpha^2)]^{r_2} \cdots. \quad (9)$$
Such equivalences involve integer partitions and are very easy to implement since there is at least one procedure devoted to integer partitions in any symbolic package. Note that equivalences (8) and (9) may be rewritten replacing $\alpha$ with any power $\alpha^k$. For instance, in (9) we have

$$ (n.\alpha^k)^i \simeq \sum_{\lambda \vdash i} d_{\lambda}[n.(\chi \alpha^k)]^{r_1}[n.(\chi^2 \alpha^2)]^{r_2} \cdots. $$

The next step is to express more general products $[n.(\chi \alpha)]^{r_1}[n.(\chi^2 \alpha^2)]^{r_2} \cdots$ (that is augmented symmetric polynomials) in terms of power sums. With this aim, equivalences (8) and (9) must be rewritten by using set partitions instead of integer partitions. We say in advance that the final step will consist in replacing the set with the more general structure of multiset.

Let $C$ be a subset of $R[A]$ with $n$ elements. Recall that a partition $\pi$ of $C$ is a collection $\pi = \{B_1, B_2, \ldots, B_k\}$ with $k \leq n$ disjoint and not-empty subsets of $C$ whose union is $C$. We denote by $\Pi_n$ the set of all partitions of $C$. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a set of $n$ uncorrelated umbrae similar to an umbra $\alpha$. The symbol $\alpha^\pi$ denotes the umbra

$$ \alpha^\pi \equiv \alpha_{i_1}^{\lfloor B_1 \rfloor} \alpha_{i_2}^{\lfloor B_2 \rfloor} \cdots \alpha_{i_k}^{\lfloor B_k \rfloor}, \quad (10) $$

where $\pi = \{B_1, B_2, \ldots, B_k\}$ is a partition of $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $i_1, i_2, \ldots, i_k$ are distinct integers chosen in $\{1, 2, \ldots, n\}$. Note that $\alpha^\pi \equiv \alpha_\lambda$, when $\lambda$ is the partition of the integer $n$ determined by $\pi$. Indeed, a set partition is said to be of type $\lambda = (1^{r_1}, 2^{r_2}, \ldots)$ if there are $r_1$ blocks of cardinality 1, $r_2$ blocks of cardinality 2 and so on. The number of set partitions of type $\lambda$ is $d_\lambda$, as given in (2). By using set partitions, equivalences (8) and (9) may be rewritten as

$$ (n.\alpha)^i \simeq \sum_{\pi \in \Pi_n} [n.(\chi \alpha)]^{r_1}[n.(\chi^2 \alpha^2)]^{r_2} \cdots, \quad (11) $$

$$ [n.(\chi \alpha)]^i \simeq \sum_{\pi \in \Pi_n} (\chi^\pi \cdot n.\alpha)^{r_1}(n.\alpha^2)^{r_2} \cdots, \quad (12) $$

where $E[(\chi^\pi \cdot n.\alpha)] = E [(\chi \cdot n.\alpha)^{B_1} (\chi^2 \cdot n.\alpha)^{B_2} \cdots (\chi^n \cdot n.\alpha)^{B_k}] = \prod_{i=1}^k x^{B_i}$ from (10). From a computational point of view, equivalences (11) and (12) are less efficient than equivalences (8) and (9). The computational cost is $O(B_n)$, where $B_n$ is the $n$-th Bell number whose growth is greater than $e^n$. Anyway, equivalences (11) and (12) smooth the way to generalize such computations to the multivariate case, by using the notion of multiset.

A multiset $M$ is a pair $(\bar{M}, f)$, where $\bar{M} \subset R[A]$ is a set, called the support of the multiset, and $f$ is a function from $\bar{M}$ to the non-negative integers. For each $\mu \in \bar{M}$, $f(\mu)$ is called the multiplicity of $\mu$. If the support of $M$ is a finite set, say $\bar{M} = \{\mu_1, \mu_2, \ldots, \mu_k\}$, we write

$$ M = \{\mu_1^{(f(\mu_1))}, \mu_2^{(f(\mu_2))}, \ldots, \mu_k^{(f(\mu_k))}\} \text{ or } M = \{\mu_1^{f(\mu_1)}, \mu_2^{f(\mu_2)}, \ldots, \mu_k^{f(\mu_k)}\}. $$

The length of the multiset $M$ is the sum of multiplicities of all elements of $\bar{M}$, that is

$$ |M| = \sum_{\mu \in \bar{M}} f(\mu). $$

From now on, we denote a multiset $(\bar{M}, f)$ simply by $M$. For instance the multiset

$$ M = \{\alpha, \ldots, \alpha\} = \{\alpha^{(i)}\} $$
has length \( i \), support \( \hat{M} = \{ \alpha \} \) and \( f(\alpha) = i \). In the following, we set
\[
\mu_M = \prod_{\mu \in M} \mu^{f(\mu)} \quad \text{and} \quad (n.\mu)_M = \prod_{\mu \in M} (n.\mu)^{g(\mu)}.
\] (13)

For instance, if \( M = \{ \alpha(i) \} \) then \( (n.\alpha)_{\hat{M}} \simeq (n.\alpha)^{i} \) and \( [n.(\chi\alpha)]_{\hat{M}} \simeq [n.(\chi\alpha)]^{i} \). Note that this notation can be easily extended to umbral polynomials.

If \( \lambda = (1^{r_1}, 2^{r_2}, \ldots) \) is an integer partition, set
\[
P_\lambda = \{ \alpha_1, \alpha_2, \ldots, \alpha_2, \ldots \}.
\] (14)

By using the notation (13), we have \( (n.\alpha)_{P_\lambda} \simeq (n.\alpha)^{r_1}(n.\alpha^2)^{r_2} \) and \( [n.(\chi\alpha)]_{P_\lambda} \simeq [n.(\chi\alpha)]^{r_1}[n.(\chi\alpha^2)]^{r_2} \), so equivalences (11) and (12) may be more compressed
\[
(n.\alpha)_{\hat{M}} \simeq \sum_{\pi \in \Pi_i} [n.(\chi\alpha)]_{\pi} \quad \text{and} \quad [n.(\chi\alpha)]_{\hat{M}} \simeq \sum_{\pi \in \Pi_i} (\chi.\chi)^{\pi}(n.\alpha)_{P_\lambda},
\] (15)

where \( \lambda \) is the type of the set partition \( \pi \).

In equivalences (15), we have \( M = \{ \alpha(i) \} \). The last step consists in generalizing such equivalences to any multiset \( M \). To this aim, we recall the notion of multiset subdivision. Such a notion is quite natural and it is equivalent to splitting the multiset into disjoint blocks (submultisets) whose union gives the whole multiset.

A subdivision of a multiset \( M \) is a multiset \( S = (\hat{S}, g) \) of \( k \leq |M| \) non-empty submultisets \( M_i = (\hat{M}_i, f_i) \) of \( M \) such that
i) \( \cup_{i=1}^{k} \hat{M}_i = \hat{M} \);  
ii) \( \sum_{i=1}^{k} f_i(\mu) = f(\mu) \) for any \( \mu \in \hat{M} \).

Recall that a multiset \( M_i = (\hat{M}_i, f_i) \) is a submultiset of \( M = (\hat{M}, f) \) if \( \hat{M}_i \subseteq \hat{M} \) and \( f_i(\mu) \leq f(\mu) \), \( \forall \mu \in \hat{M}_i \).

If \( M = \{ \alpha(i) \} \), then subdivisions are of type
\[
S = \{ \{ \alpha \}, \ldots , \{ \alpha \}, \{ \alpha^{(2)} \}, \ldots , \{ \alpha^{(2)} \}, \ldots \}
\]
with \( r_1 + 2r_2 + \cdots = i \), and we will say that the subdivision \( S \) is of type \( \lambda = (1^{r_1}, 2^{r_2}, \ldots) \vdash i \). The support of \( S \) is \( \hat{S} = \{ \{ \alpha \}, \{ \alpha^{(2)} \}, \ldots \} \).

If \( S = \{ M_1, \ldots , M_k \} \)
\[
\mu_S = \prod_{M_i \in \hat{S}} \mu_{M_i}^{g(M_i)} \quad \text{and} \quad (n.\mu)_S = \prod_{M_i \in \hat{S}} (n.\mu_{M_i})^{g(M_i)}.
\] (17)

we set

extending the notation (13).

When integer partitions are replaced by multiset subdivisions, the fundamental expectation result (6) becomes
\[
[n.(\chi\mu)]_{S} \simeq (n)_{S}\mu_{S},
\] (18)

with \( S \) given in (16) and
\[
\mu_{S} \equiv (\mu_{M_1})^{g(M_1)} \cdots (\mu_{M_k})^{g(M_k)},
\] (19)
where \( \mu_{M_{1}} \) are uncorrelated umbral monomials.

By using the notation \((\ref{17})\) and recalling \((\ref{14})\), we have \((n.\alpha)p_{\lambda} \equiv (n.\alpha)S\) and \(n.(\chi\alpha)p_{\lambda} \equiv n.(\chi\alpha)S\), with \(\lambda = (1^{r_{1}}, 2^{r_{2}}, \ldots) \vdash i\) and \(S\) the subdivision of type \(\lambda\). Then equivalences \((\ref{15})\) may be written as follows

\[
(n.\alpha)_{M} \simeq \sum_{\pi \in \Pi_{i}} [n.(\chi\alpha)]_{S_{\pi}} \quad \text{and} \quad [n.(\chi\alpha)]_{M} \simeq \sum_{\pi \in \Pi_{i}} (\chi\cdot\chi)^{\pi}(n.\alpha)_{S_{\pi}}.
\]

(20)

One more remark allows us to remove integer partitions from \((\ref{20})\) which is necessary when the multiset \(M = \{\alpha^{(i)}\}\) is replaced by an arbitrary multiset. Let us observe that a subdivision of the multiset \(M\) may be constructed in the following way: suppose the elements of \(M\) are distinct, build a set partition \(\pi\) and then replace each element in any block by the original one. In this way, any subdivision corresponds to a set partition \(\pi\) and we will write \(S_{\pi}\). Note that it is \(|S_{\pi}| = |\pi|\) and it could be \(S_{\pi_{1}} = S_{\pi_{2}}\) for \(\pi_{1} \neq \pi_{2}\), as the following example shows.

**Example 4.1** If \(M = \{\alpha, \alpha, \gamma, \delta, \delta\}\), label each element of \(M\) in order to have the set \(C = \{\alpha_{1}, \alpha_{2}, \gamma_{1}, \delta_{1}, \delta_{2}\}\). The subdivision \(S_{1} = \{\{\alpha, \gamma\}, \{\alpha\}, \{\delta, \delta\}\}\) corresponds to the partition \(\pi_{1} = \{\{\alpha_{1}, \gamma_{1}\}, \{\alpha_{2}\}, \{\delta_{1}, \delta_{2}\}\}\) of \(C\). It is \(|S_{1}| = |\pi_{1}|\). Note that the subdivision \(S_{1}\) also corresponds to the partition \(\pi_{2} = \{\{\alpha_{2}, \gamma_{1}\}, \{\alpha_{1}\}, \{\delta_{1}, \delta_{2}\}\}\). 

Finally, equivalences \((\ref{15})\) may be rewritten as follows

\[
(n.\mu)_{M} \simeq \sum_{\pi \in \Pi_{i}} [n.(\chi\mu)]_{S_{\pi}} \quad \text{and} \quad [n.(\chi\mu)]_{M} \simeq \sum_{\pi \in \Pi_{i}} (\chi\cdot\chi)^{\pi}(n.\mu)_{S_{\pi}},
\]

(21)

where \(S_{\pi}\) is the subdivision of \(M\) corresponding to the partition \(\pi\) of a set \(C\) such that \(|C| = |M| = i\). The symbolic expression of such equivalences does not change if the multiset \(M = \{\alpha^{(i)}\}\) is replaced by an arbitrary multiset.

The following example shows the effectiveness of umbral notation in managing moments of sampling distributions.

**Example 4.2** If \(M = \{\mu_{1}, \mu_{1}, \mu_{2}\}\) then \((n.\mu)_{M} = (n.\mu_{1})^{2}(n.\mu_{2})\). In statistical terminology, moments of \((n.\mu)_{M}\) correspond to moments of the product of sums \((\sum_{i=1}^{n}X_{i})^{2}(\sum_{i=1}^{n}Y_{i})\), where \((X_{1}, Y_{1}), \ldots, (X_{n}, Y_{n})\) are separately independent and identically distributed random variables. In order to apply the first part of \((\ref{21})\), we need to compute all subdivisions of \(M\). These are given in Table 1.

| Subdivision | \# Subdivision | \([n.(\chi\mu)]_{S_{\pi}}\) |
|-------------|---------------|---------------------|
| \{\{\mu_{1}, \mu_{1}, \mu_{2}\}\} | 1 | \((n.\mu_{1})^{2}(n.\mu_{2})\) |
| \{\{\mu_{1}\}, \{\mu_{1}, \mu_{2}\}\} | 2 | \([n.(\chi\mu_{1})][n.(\chi\mu_{1}\mu_{2})]\) |
| \{\{\mu_{2}\}, \{\mu_{1}, \mu_{2}\}\} | 1 | \([n.(\chi\mu_{2})][n.(\chi\mu_{1}^{2})]\) |
| \{\{\mu_{1}\}, \{\mu_{1}, \mu_{2}\}\} | 1 | \([n.(\chi\mu_{1})^{2}][n.(\chi\mu_{2})]\) |

Table 1: Subdivisions of \(M = \{\mu_{1}, \mu_{1}, \mu_{2}\}\).

The last column in Table 1 has been constructed by the following considerations. Suppose to consider the second row: \(S_{1} = \{\{\mu_{1}\}, \{\mu_{1}, \mu_{2}\}\}\) is a subdivision of \(M = \{\mu_{1}, \mu_{1}, \mu_{2}\}\). The support of \(S_{1}\) consists of two multisets, \(M_{1} = \{\mu_{1}\}\) and \(M_{2} = \{\mu_{1}, \mu_{2}\}\), each of one with multiplicity 1 so that \([n.(\chi\mu)]_{S_{1}} = n.(\chi\mu_{M_{1}})n.(\chi\mu_{M_{2}})\). Since \(n.(\chi\mu_{M_{1}}) = n.(\chi\mu_{1})\) and \(n.(\chi\mu_{M_{2}}) = n.(\chi\mu_{1}\mu_{2})\), we have \([n.(\chi\mu)]_{S_{1}} = n.(\chi\mu_{1})\ n.(\chi\mu_{1}\mu_{2})\). Repeating the same arguments for all subdivisions of \(M\), we get the results of Table 1. By using Table 1
and the first part of (21), we have
\[
(n, \mu_1)^2(n, \mu_2) \simeq n.(\chi_2 \mu_2) + 2[n.(\chi_1)[n.(\chi_1 \mu_2)] + [n.(\chi_2)]^2[n.(\chi_2)],
\]
that is
\[
\left(\sum_{i=1}^{n} X_i\right)^2 \left(\sum_{i=1}^{n} Y_i\right) = \sum_{i=1}^{n} X_i^2 Y_i + 2 \sum_{1 \leq i \neq j \leq n} X_i X_j Y_j + \sum_{1 \leq i \neq j \neq k \leq n} X_i X_j Y_k.
\tag{22}
\]
In order to evaluate the mean of the sums, on the right hand side of (22), in terms of population moments, we have to use (18) and finally we have
\[
E \left[ \left(\sum_{i=1}^{n} X_i\right)^2 \left(\sum_{i=1}^{n} Y_i\right) \right] = nE[X^2Y] + 2(n_2)E[X]E[XY] + (n_2)E[X]^2E[Y].
\]

□

In the previous example, we have shown the usefulness of the first part of (21) in evaluating the mean of product of power sums. In the next example, we show the usefulness of the second part of (21) in order to express any $U$-statistic in terms of power sums. Indeed, first we make use of equivalence (18), in order to translate products of moments - also multivariate - in terms of augmented symmetric polynomials. Then we apply the change of bases given by the second part of (21).

The following example shows how to construct multivariate $k$-statistics.

**Example 4.3 Multivariate $k$-statistics.** In umbral terms, a multivariate cumulant is the element of $R$ corresponding to $E[(\chi.\mu)_M] = \kappa_{i_1,...,i_r}$, where $M = \{\mu_1^{(i_1)}, \mu_2^{(i_2)}, \ldots, \mu_r^{(i_r)}\}$ is a multiset and $(\chi.\mu)_M$ is the symbol denoting the product $(\chi.\mu_1)^{i_1}(\chi.\mu_2)^{i_2} \cdots (\chi.\mu_r)^{i_r}$. Suppose $|M| = i$. The extension of (1) to the multivariate case is
\[
(\chi.\mu)_M \simeq \sum_{\pi \in \Pi_i} (\chi.\mu)^{|\pi|} \mu^S_\pi,
\]
where $S_\pi$ is the subdivision of the multiset $M$ corresponding to the partition $\pi \in \Pi_i$. By equivalence (18), we write
\[
(\chi.\mu)_M \simeq \sum_{\pi \in \Pi_i} (\chi.\mu)^{|\pi|} \frac{1}{|n|}[n.(\chi.\mu)]_{S_\pi},
\]
and by the second part of (21) we have the umbral version of multivariate $k$-statistics in terms of power sums, that is
\[
(\chi.\mu)_M \simeq \sum_{\pi \in \Pi_i} \frac{(\chi.\mu)^{|\pi|}}{|n|} \sum_{\tau \in \Pi_{|\tau|}} (\chi.\mu)^\tau (n.\mu)_{S_\tau},
\tag{23}
\]
where $S_\pi$ is the subdivision of $M$ corresponding to the partition $\tau$ of a set having the same cardinality of $\pi$. □


5 Products of augmented polynomials

This section is devoted to a different application of equivalences (21), necessary to evaluate the mean of product of augmented polynomials in separately independent and identically distributed random variables.

We borrow the following example from the paper of Vrbik [24].

Suppose, for instance, to need the mean of

\[
\left( \sum_{i \neq j}^{n} X_i^2 X_j \right) \left( \sum_{i=1}^{n} X_i^2 Y_i \right)^2 = S_{\{2,0\},\{1,0\}} S_{\{2,1\}} S_{\{2,1\}},
\]

where \((X_1, Y_1), \ldots, (X_n, Y_n)\) are separately independent and identically distributed random variables and

\[
S_{\{k_1, l_1\}, \{k_2, l_2\}, \ldots} = \sum_{i_1 \neq i_2 \neq \cdots} X_{i_1}^{k_1} Y_{i_1}^{l_1} X_{i_2}^{k_2} Y_{i_2}^{l_2} \cdots,
\]

by means of the notation introduced by Vrbik in [24].

If we expand the product (24) as a linear combination of augmented symmetric polynomials (25), then we are able to apply the fundamental expectation result (18) and to evaluate the mean of (24). The umbral tools, we have introduced up to now, are sufficient to do such a work. Therefore, since

\[
\left( \sum_{i \neq j}^{n} X_i^2 X_j \right) \text{ corresponds to } [n.(\chi_{\mu_1}^2) n.(\chi_{\mu_2})], \quad \text{and}
\]

\[
\left( \sum_{i=1}^{n} X_i^2 Y_i \right) \text{ corresponds to } [n.(\chi_{\mu_1}^2 \mu_2)],
\]

the product (24) is umbrally represented by

\[
n.(\chi_{\mu_1}^2) n.(\chi_{\mu_2}) n.(\chi_{\mu_1} \mu_2) n.(\chi_{\mu_1} \mu_2),
\]

where \(\{\chi_i\}, i = 1, 2, 3\) are uncorrelated singleton umbrae. Indeed, a product of uncorrelated singleton umbrae does not “delete” the same indexed umbrae. Moreover, the sum (25) is umbrally represented by

\[
n.(\chi_{\mu_1}^{k_1} \mu_2^{l_1}) n.(\chi_{\mu_1}^{k_2} \mu_2^{l_2}) \cdots.
\]

monomials involving correlated singleton umbrae, the auxiliary umbra \( S \) multiset subdivisions similar to those in the first column of Table 1 and Table 2.

via the first part of equivalence (21), we have

\[
\chi \simeq 0
\]

In conclusion, when in one or more blocks of the subdivision \( S \), there are at least two umbral monomials involving correlated singleton umbrae, the auxiliary umbra \( n(\chi)_{S_{\pi}} \) has the evaluation equal to zero. If within every block of the subdivision \( S \) there are only uncorrelated singleton umbrae, then \( n(\chi)_{S_{\pi}} \) gives rise expressions like \( (27) \), umbrally representing \( (25) \).

We do all computation in Table 2. We give the corresponding sum of independent random variables \( (25) \), instead of \( n(\chi)_{S_{\pi}} \). Subsection 6.1 is devoted to the algorithm \( \text{makeTab} \), which allows us to construct multiset subdivisions similar to those in the first column of Table 1 and Table 2.
6 Computational results

6.1 Find subdivisions of a multiset: the procedure makeTab

Equivalences \( (21) \) have been implemented in MAPLE. These equivalences share the procedure \( \text{makeTab} \) necessary to construct multiset subdivisions.

When the multiset is of type \( \{ \alpha^{(k)} \} \), an efficient way is to resort the partitions of the integer \( k \), as equivalences \( (15) \) show. In general, as already stressed in Section 4, we may construct multiset subdivisions by using suitable set partitions, but this approach has a computational cost proportional to the \( n \)-th Bell number \( B_n \), so it is not efficient. Indeed, examples have shown how subdivisions may occur more than one time in the same formula (see Table 1), so that it is necessary to build a procedure generating only different subdivisions together with their multiplicity, that is the number of corresponding set partitions. To accomplish this task, the algorithm \( \text{makeTab} \) takes into account the connection between multisets and integer partitions, reducing the overall computational complexity.

In the following, we illustrate the main steps of \( \text{makeTab} \) by an example. Suppose to need subdivisions of the multiset

\[
M = \{ \alpha, \alpha, \alpha, \gamma, \gamma \}.
\]

We compute all different subdivisions of \( \{ \alpha^{(3)} \} \) by using all partitions \( \lambda \) of the integer 3, that is

\[
\{\{\alpha\}\}, \{\{\alpha\}\}, \{\{\alpha\}\}, \{\{\alpha, \alpha\}\}, \{\{\alpha^{(3)}\}\}.
\]

(30)

The same we do for \( \{\gamma^{(2)}\} \), that is

\[
\{\{\gamma\}\}, \{\{\gamma\}\}, \{\{\gamma^{(2)}\}\}.
\]

(31)

Now, we insert every element of \( (31) \) in every element of \( (30) \) one at a time and recursively, as the following example shows. Suppose to do the insertion of \( \{\gamma\} \) in every block of \( \{\{\alpha\}\}, \{\{\alpha\}\}, \{\{\alpha\}\} \), that is

\[
\{\{\alpha\}\}, \{\{\alpha\}\}, \{\{\alpha\}\} \leftarrow \gamma.
\]

Then, we insert a second time \( \gamma \) in the output subdivision, that is

\[
\left(\{\{\alpha\}\}, \{\{\alpha\}\}, \{\{\alpha\}\} \leftarrow \gamma\right) \leftarrow \gamma.
\]

The insertion \( \leftarrow \) is a kind of iterated inclusion-exclusion rule \( [1] \), but with some more constraints:

i) the insertion of a submultiset of \( \{\gamma^{(2)}\} \) in a submultiset of \( \{\alpha^{(3)}\} \) must be done only if it does not generate a new submultiset equal to a previous one or it has not yet inserted;

ii) at the end, every submultiset of \( \{\gamma^{(2)}\} \) is simply appended to every subdivision of \( \{\alpha^{(3)}\} \).

Table 3 gives the results of the double insertion of \( \{\gamma\}, \{\gamma\} \) in every submultiset of \( \{\alpha^{(3)}\} \), according to rules \( i) \) and \( ii) \).
F or example, if we have an insertion procedure, we can delete the subdivision from the list and reduce the overall computational time. This strategy is speedier than the iterated full partition of Andrews and Stafford [1], given that it takes into account the multiplicity of all elements of $M$. The higher this multiplicity is, the more the insertion procedure gives efficient results, considering that it involves more than one element of $M$.

When there are monomials involving correlated singleton umbrae in the multiset $M$, see equivalence \([29]\), we have further speeded up the procedure. When in the same block of the subdivision, there are monomials involving more than one correlated singleton umbra, the evaluation of umbrae indexed by this subdivision does not give contribution. Then, if we check the indexes of singleton umbrae before the insertion procedure, we can delete the subdivision from the list and reduce the overall computational time. For example, if we have \(\{\chi_1\alpha_1, \chi_2\alpha_2\}, \{\chi_2\alpha_2\}\), the insertion of $\chi_1\alpha_1$ may be done only in the second set, because the first one gives a zero contribution in the overall evaluation.

| Subdivision          | Output first ← | Output second ← |
|----------------------|----------------|-----------------|
| \(\{\alpha, \{\alpha, \{\alpha\}\}\}\) | \(\{\alpha, \gamma, \{\alpha, \{\alpha\}\}\}\) | \(\{\alpha, \gamma, \{\alpha, \gamma, \{\alpha\}\}\}\) |
| \(\{\alpha, \{\alpha, \{\alpha, \gamma\}\}\}\) | \(\{\alpha, \gamma, \{\alpha, \{\alpha, \gamma\}\}\}\) | \(\{\alpha, \gamma, \{\alpha, \gamma, \{\alpha, \gamma\}\}\}\) |
| \(\{\alpha, \{\alpha^{(2)}\}\}\) | \(\{\alpha, \gamma^{(2)}, \{\alpha\}\}\) | \(\{\alpha, \gamma, \{\alpha^{(2)}, \gamma\}\}\) |
| \(\{\alpha, \{\alpha^{(2)}\}\}\) | \(\{\alpha, \gamma^{(2)}, \{\alpha\}\}\) | \(\{\alpha, \gamma, \{\alpha^{(2)}, \gamma\}\}\) |
| \(\{\alpha^{(3)}\}\) | \(\{\alpha^{(3)}, \gamma\}\) | \(\{\alpha^{(3)}, \gamma, \gamma, \{\alpha\}\}\) |

Table 3: Insertion of \(\{\gamma, \{\gamma\}\}\) in every submultiset of \(\alpha^{(3)}\).

Table 4 gives the results of the insertion of \(\{\gamma^{(2)}\}\) in every submultiset of \(\alpha^{(3)}\), according to rules \(i)\) and \(ii)\).

| Subdivision          | Output first ← |
|----------------------|----------------|
| \(\{\alpha, \{\alpha, \{\alpha\}\}\}\) ← \(\gamma^{(2)}\) | \(\{\alpha, \gamma^{(2)}, \{\alpha\}\}\) |
| \(\{\alpha, \{\alpha^{(2)}\}\}\) ← \(\gamma^{(2)}\) | \(\{\alpha, \gamma^{(2)}, \{\alpha^{(2)}\}\}\) |
| \(\{\alpha^{(3)}\}\) ← \(\gamma^{(2)}\) | \(\{\alpha^{(3)}, \gamma^{(2)}\}\) |

Table 4: Insertion of \(\{\gamma^{(2)}\}\) in every submultiset of \(\alpha^{(3)}\).
### 6.2 U-statistics

Table 5 shows computational times of three procedures implementing the change of bases from augmented symmetric polynomials versus power sums, which is at the bottom of the construction of U-statistics. The three procedures are the function AugToPowerSum given in MathStatica (release 1.0) [19], the function TOP given in SF (version 2.4) [22], and our MAPLE function augToPs, with which we have implemented the second part of [21]. Comparisons have shown how augToPs performs its task using less computational time than all the others.

|      | TOP  | AugToPowerSum | augToPs |
|------|------|---------------|---------|
| $[1^2 2^1 3^1]$ | 0.78 | 0.18          | 0.13    |
| $[1^3 2^1 3^2]$ | 0.08 | 0.01          | 0.01    |
| $[2^1 3^1]$      | 2.57 | 0.03          | 0.01    |
| $[1^5 2^1 3^1]$ | 6.15 | 1.20          | 0.65    |
| $[1^2 2^1 3^2 4^2]$ | 2.75 | 0.11          | 0.09    |

Table 5: Comparison of computational times.

Unlike our MAPLE algorithm, note that AugToPowerSum and TOP do not work on multiple sets of variables so no comparisons can be done.

In order to compare the results achieved by means of umbral methods with those of [2], we have performed symbolic computations involved in unbiased estimators of product of univariate and multivariate cumulants.

The symmetric statistic $k_{r_1 \ldots, t}$ such that $E[k_{r_1 \ldots, t}] = \kappa_r \cdots \kappa_t$, where $\kappa_r, \ldots, \kappa_t$ are univariate cumulants, is known as polykay. Being a product of cumulants, the umbral expression of a polykay is simply

$$k_{r_1 \ldots, t} \simeq (\chi, \alpha)^r \cdots (\chi', \alpha')^t,$$

where $\chi, \ldots, \chi'$ being uncorrelated singleton umbrae and $\alpha, \ldots, \alpha'$ satisfying $\alpha \equiv \ldots \equiv \alpha'$. If $r + \cdots + t \leq n$, the right-hand product of (32) has the following umbral expression in terms of power sums:

$$k_{r_1 \ldots, t} = \sum_{(\lambda \vdash r_1, \ldots, n+t)} \frac{(\chi, \chi)^n \cdots (\chi, \chi')^n}{(n)_{\nu_1 + \cdots + \nu_n}} d_\lambda \cdots d_n \sum_{\pi \in \Pi_{\nu_1 + \cdots + \nu_n}} (\chi, \chi)^\pi (n.\alpha)_{S_\pi},$$

where $S_\pi$ is the subdivision of the multiset $P_{\nu_1 + \cdots + \nu_n} = \{\alpha^{(r_1 + \cdots + t_1)}, \alpha^{2(r_2 + \cdots + t_2)}, \ldots\}$, corresponding to the partition $\pi \in \Pi_{\nu_1 + \cdots + \nu_n}$. In analogy with (32), a multivariate polykay is a product of multivariate cumulants, that is $E[k_{t_1 \ldots, t_r, l_1 \ldots, l_m}] = \kappa_{t_1 \ldots, t_r} \cdots \kappa_{l_1 \ldots, l_m}$, where $\kappa_{t_1 \ldots, t_r}, \ldots, \kappa_{l_1 \ldots, l_m}$ are multivariate cumulants. Products of multivariate cumulants are represented by products of uncorrelated multivariate $\alpha$-cumulant umbrae, that is

$$k_{t_1 \ldots, t_r, l_1 \ldots, l_m} \simeq (\chi, \mu)^T \cdots (\chi', \mu')_L,$$

where $\chi$ and $\chi'$ are uncorrelated and the umbral monomials $\mu \in T$ and $\mu' \in L$ are such that

$$T = \{\mu_1^{(t_1)}, \ldots, \mu_r^{(t_r)}\}, \ldots, L = \{\mu_1^{(l_1)}, \ldots, \mu_m^{(l_m)}\}.$$

If $|T| + \cdots + |L| \leq n$, the right-hand product of (34) has the following umbral expression in terms of multivariate power sums:

$$k_{t_1 \ldots, t_r, l_1 \ldots, l_m} = \sum_{(\pi \in \Pi_{|T|}, \ldots, \pi \in \Pi_{|L|})} \frac{(\chi, \chi)^{\pi_1} \cdots (\chi', \chi')^{\pi_2}}{(n)_{|\pi| + \cdots + |\pi|}} \sum_{\pi \in \Pi_{|\pi| + \cdots + |\pi|}} (\chi, \chi)^\pi (n.\mu)_S \cdots (n.\mu')_S,$$
where $S_\tau$ is the subdivision of the multiset obtained by the disjoint union of $T, \ldots, L$ with no uncorrelation labels and corresponding to the partition $\tau$ of the set built with the blocks of $\{\pi, \ldots, \tilde{\pi}\}$. The umbral formulae here recalled are stated in [9].

Table 6 shows computational times obtained by the procedure PolyK of MathStatica by our MAPLE function polyk implementing (34), and by the procedures proposed by Andrews and Stafford [1] (in particular, see [4] for multiple sums). Note that (35) gives as special case both (32) and univariate and multivariate $k$-statistics (23). We remark that MathStatica has no procedure to handle multivariate polykays. The computational times of Andrews and Stafford’s procedures have been obtained by the code available at http://fisher.utstat.toronto.edu/david/SCSI/chap.3.nb.

| $k_1, \ldots, l$ | Andrews-Stafford | MathStatica | MAPLE |
|------------------|------------------|-------------|-------|
| $k_8$            | 0.10             | 0.09        | 0.05  |
| $k_{10}$         | 0.35             | 0.25        | 0.15  |
| $k_{12}$         | 1.19             | 0.84        | 0.42  |
| $k_{14}$         | 3.93             | 2.67        | 1.29  |
| $k_{16}$         | 12.74            | 8.54        | 3.86  |
| $k_{18}$         | 40.44            | 30.32       | 12.3  |
| $k_{6,6}$        | 39.84            | 0.81        | 0.51  |
| $k_{9,3}$        | 18.32            | 0.84        | 0.46  |
| $k_{9,6}$        | 676.14           | 5.02        | 3.13  |
| $k_{9,9}$        | > 1.5 hh         | 29.10       | 23.19 |
| $k_{3,3}k_{2,2}$ | 4.67             | n.c.        | 1.71  |
| $k_{3,3}k_{3,3}$ | 32.16            | n.c.        | 15.87 |
| $k_{2,1,1}k_{2,1,1}$ | 1.031        | n.c.        | 0.52  |

Table 6: Comparison of computational times. The acronym “n.c.” stands for not calculable.

Remark that the output expressions of Andrews and Stafford’s code are unpractical. These are very different from the output expressions of PolyK of MathStatica and polyk in MAPLE. For example, the output of Andrews and Stafford’s code for $k_3$ is

$$\frac{x^3}{(1-\frac{1}{3})(1-\frac{1}{5})} + \frac{1 - \frac{3}{5} - \frac{6}{5}}{(1-\frac{1}{5})}n x^2 + \frac{1 + \frac{4}{5} - \frac{6}{5}}{(1-\frac{1}{3})(1-\frac{1}{5})}n^2 x^3,$$

whereas the output of PolyK of MathStatica and polyk in MAPLE is

$$\frac{n^2 S_3 - 3nS_1S_2 + 2S_1^3}{n(n-1)(n-2)}.$$

In $k_{12}$, the expression of Andrews and Stafford’s code consists of 602 terms compared with 77 terms of the expressions obtained by PolyK of MathStatica and polyk in MAPLE. In order to recover the same output of PolyK of MathStatica and polyk in MAPLE in (36), we must group the terms in parenthesis over a common denominator, deleting equal factors in the results. This operation increases the overall computational time. For example the computational time of $k_{10}$ grows from 0.35 to 2.693, the one of $k_{12}$

1 Of course, if single $k$-statistics are enough to be computed, the function k-stat of MathStatica is suitable and faster.

2 In the forthcoming MathStatica, release 2, the procedure to handle multivariate polykays is now available (C. Rose, private communication).
grows from 1.191 to 14.56. For $k$-statistics of order greater than 14, an error occurs since the recursion exceeds a depth of 256.

6.3 Product of augmented symmetric functions

The MAPLE routine Pam implements equivalences such as (29). Pam calls the routine makeTab. Considering that there are monomials involving singleton umbrae in the multiset $M$, the efficiency of makeTab improves, as we have mentioned at the end of Subsection 6.1. In Table 7, we compare some computational times of Pam with those of the routine SIP, written in MATHEMATICA language by Vrbik [24], and exclusively devoted to products of augmented symmetric functions.

| $[1^{i}2^{j}3^{k}\cdots]$ | SIP | MAPLE |
|--------------------------|-----|-------|
| $[5^{3}9^{10}[1\ 2\ 3\ 4\ 5]]$ | 0.7 | 0.1   |
| $[5^{3}8^{9}10[12\ 3\ 4\ 5]]$ | 5.6 | 0.4   |
| $[6^{7}8^{9}10][1\ 2\ 3\ 4\ 5]]$ | 2.2 | 0.1   |
| $[6^{7}8^{9}10][1\ 2][3\ 4\ 5]]$ | 3.1 | 0.4   |
| $[6^{7}][8^{9}10][1\ 2][3\ 4\ 5]]$ | 4.7 | 1.3   |
| $[5^{6}7^{8}9^{10}][1\ 2\ 3\ 4\ 5]]$ | 16.7 | 0.3 |
| $[5^{6}7^{8}9^{10}][1\ 2\ 3\ 4\ 5\ 6]]$ | 348.7 | 1.5 |
| $[6^{7}8^{9}10][6^{7}][3\ 4\ 5][1\ 2]]$ | 125.6 | 16.4 |

Table 7: Comparison of computational times.

Note that the computational time of SIP depends heavily on the number of variables involved in the brackets, because it resorts set permutations, whose computation cost is factorial in its cardinality.

7 Concluding remarks

This paper focuses attention on a symbolic calculation of products of statistics related to cumulants or moments. Undoubtedly, an enjoyable challenge is to find efficient procedures to deal with the necessarily huge amount of algebraic and symbolic computations involved in such a kind of calculations. The methods we propose result more efficient compared with those available in the literature. Note that high order statistics have a variety of applications. Recently, Rao [17] have shown applications of high order cumulants in statistical inference and time series. Indeed, there are different areas, such as astronomy (see [16] and references therein), astrophysics [10] and biophysics [15], where one computes high order $k$-statistics in order to recognize a gaussian population or characterizes asymptotic behavior of high order $k$-statistics if the population is gaussian. Indeed, $k$-statistics are independent from the sample mean if and only if the population is gaussian [13] and in such a case $k$-statistics of order greater than 2 should be nearly to zero. For such applications, increasing speed and efficiency is a significant investment.

As we have shown, the codes of Andrews and Stafford are quite inefficient for the problems posed here. This paper has pointed out the role played by the notion of subdivision in speeding up the calculations resulting by multiplying sums of random variables and the role played by the umbra $\chi$ in selecting the involved variables. The symbolic algorithm we propose, in order to evaluating the mean of product of augmented polynomials in random variables, relies on this innovative strategy.

In closing, we would like to emphasize that classical umbral calculus not only decreases the computational time, but offers a theory to prove more general results. Recently, L-moments and trimmed L-moments have been noticed as appealing alternatives to conventional moments, see [12] and [6]. We believe that the handling of these number sequences would benefit by an umbral approach.
8 Appendix

In the following, we present the MAPLE code of the procedure giving subdivisions of a multiset. In order to have the results of Table 3 and 4, the calling syntax is `makeTab(3,2)`.

```maple
nRep := proc(u)
    mul(x[2]^!, x=convert(u,multiset)); end:
URv := proc(u,v)
    local U, u, i, ptr_i, vI;
    ou:=NULL; U:=[ ]; vI:=indets(v);
    for ptr_i from nops(u) by -1 to 2 do
        if has(u[ptr_i], v) then break; fi; od;
    for i from ptr_i to nops(u) do
        if not (u[i]=ou or has(u[i], vI)) then ou:=u[i];
            U:=[op(U), [op(u[1..(i-1)]), u[i]*v, op(u[(i+1)..-1])]];
        fi;
    od;
    op(U), [op(u), v]; end:
URV := proc()
    local U, V, i;
    U:= [args[1,1]]; V:= args[2,1];
    for i from 1 to nops(V) do
        U := [ seq( URv(u,V[i]), u=U ) ];
    od;
    seq( [x, args[1,2]*args[2,2]/nRep(x)] , x=U ) ; end:
URmV := proc()
    local U, i, nbin;
    if nargs=1 then
        U:= args;
    else U:= URV( args[1], args[2]);
        for i from 3 to nargs do
            U:= seq( URV( u, args[i] ), u=[U] );
        od;
    fi;
    U; end:
comb := proc(V, ptr, Y)
    if ptr=nops(V)+1 then return(Y); fi;
    seq( comb(V, ptr+1, [ op(Y), L ] ), L=V[ptr] ); end:
makeTab := proc()
    local U;
    U:= [ seq( [ seq( [ seq(P||i^z, z=y) ],
                   combinat['multinomial'](args[i], seq(r,r=y)) ),
                   y=combinat['partition'](args[i]))],
        i=1..nargs)];
    if nops(U)=1 then
        seq([x[1], (x[2]/nRep(x[1])), x=op(U)]]
    else seq(URmV(op(x), x=[comb(U,1,[])])
    fi; end:
```

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