Integral $p$-adic cohomology theories

Tomoyuki Abe and Richard Crew

1 Introduction

Suppose $\Lambda$ is a complete local ring with residue field $k$. Can we expect that there is a “reasonable” cohomology theory $H(X, \Lambda)$ with its values in $\Lambda$-modules for separated schemes of finite type over a field $k$? Here “reasonable” means, in a first approximation that it has all the usual properties of integral Betti cohomology when $k = \mathbb{C}$. For example if $\Lambda = \mathbb{Z}_\ell$ with $\ell$ different from $p = \text{char}(k)$, $\ell$-adic cohomology $H^\text{et}_\Lambda(X, \mathbb{Z}_\ell)$ is such a theory. If $\Lambda = W$ is a Cohen ring of $k$, crystalline cohomology gives such a theory when $X/k$ is proper and smooth, but not in more general situations: Berthelot found that the torsion of $H^1_{\text{crys}}(X, W)$ could be infinite if $X$ was even mildly singular. Of course if we replace $W$ by its fraction field $K$, rigid cohomology has the desired properties.

In the article in which he first discussed such theories Grothendieck emphasized the importance of $p$-torsion phenomena, which is only visible in an “integral” $p$-adic theory such as crystalline cohomology:

Such a theory should associate to each scheme $X$ of finite type over a perfect field $k$ of characteristic $p > 0$, cohomology groups which are modules over an integral domain, whose quotient field is of characteristic 0, and which satisfy all the desirable formal properties (functoriality, finite-dimensionality...). This cohomology should also, most importantly, explain torsion phenomena, and in particular $p$-torsion. ([9, §1.7], emphasis in the original).

The work of Illusie and others on the de Rham-Witt complex shows the immense richness of $p$-torsion phenomena, but of course only in the proper smooth case.

When $k$ is perfect and $X/k$ is smooth Davis, Langer and Zink [6] have defined an overconvergent version of the de Rham-Witt complex $W^+\Omega^1_{X/W(k)}$ and constructed an isomorphism

$$H^r(X, W^+\Omega^1_{X/W(k)}) \otimes \mathbb{Q} \simeq H^r_{\text{rig}}(X)$$
when $X$ is quasiprojective. It’s evidently too much to hope that the $W(k)$-modules $H^i(X, W^\dagger \Omega^\cdot_{X/W(k)})$ are finitely generated, and we will see that this is never true if $X$ is a smooth affine curve. Davis, Langer and Zink made the more reasonable conjecture that the image of $H^i(X, W^\dagger \Omega^\cdot_{X/W(k)})$ in $H^i_{\text{rig}}(X)$ is finitely generated as a $W(k)$-module, but in a recent preprint Ertl and Shiho [7] have produced counterexamples to this assertion as well. In any case one loses all torsion information by replacing $H^i(X, W^\dagger \Omega^\cdot_{X/W(k)})$ by its image in $H^i_{\text{rig}}(X)$.

The main result of this paper is a nonexistence theorem for certain theories of this sort. We assume that a theory has a comparison theorem with rigid cohomology (or rigid cohomology with compact support), that the cohomology groups are finitely generated $\Lambda$-modules, and – a natural but important additional restriction – the theory is compatible with finite étale descent in a manner to be explained later. The result is that there is no such theory even for affine curves. We do not rule out the possibility of theories satisfying weaker descent conditions, such as cdh-descent. In fact in another recent preprint [8] Ertl, Shiho and Sprang construct a “good” integral $p$-adic theory under certain assumptions concerning resolution of singularities in positive characteristic. This construction uses cdh-descent and so does not contradict our result if their hypotheses on resolution hold. They also consider a theory based on simplicial generically étale hypercovers, and show that it is not independent of the choice of hypercover, and thus does not provide a theory compatible with finite étale descent. We should remark finally that Bhargav Bhatt has also given a proof of the nonexistence of this sort of theory (private communication) by considering Artin-Schreier covers of the the affine line by itself; he shows that the existence of such a theory would imply that the affine line has Euler characteristic 0. The method of the present paper on the other hand shows that the misbehavior of $p$-adic cohomology theories is ubiquitous, at least as far as curves are concerned; see the example at the end of section [4].

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2 Globally Perfect Models

In what follows $\Lambda$ is a complete noetherian local ring with residue field $k$ and fraction field $K$. As usual $D_{\text{perf}}(\Lambda)$ is the triangulated category of perfect complexes of $\Lambda$-modules, which since $\Lambda$ is local means that an object of $D_{\text{perf}}(\Lambda)$ is quasi-isomorphic to a bounded complex of free $\Lambda$-modules.

We have in mind the following requirements on a cohomology theory $H(X)$ on a subcategory of the category of $k$-schemes of finite type:

- $H(X)$ can be used to compute rigid cohomology (or rigid cohomology with compact supports) via a suitable comparison theorem.
- $H(X)$ may be computed as the cohomology of an object of $D_{\text{perf}}(\Lambda)$;
- The $H(X)$ are compatible with finite étale descent, in a sense to be explained presently.

When $\Lambda$ is regular the second requirement reduces to the condition that the $H^n(X)$ be finitely generated $\Lambda$-modules, and vanish for $|n| \gg 0$.

To explain the last condition we use the finite étale site, referring to the book of Abbes, Gros and Tsuji [1, Ch. VI §9] for proofs of the assertions used below. If $X$ is a scheme the site $\text{Fet}(X)$ is the category of finite étale morphisms $Y \to X$, and the coverings are surjective morphisms. The associated topos will be written $X_{\text{fet}}$. A morphism $\pi : Y \to X$ induces a morphism $\pi_{\text{fet}} : Y_{\text{fet}} \to X_{\text{fet}}$ of topoi; in what follows we will abbreviate the associated functors $\pi_{\text{fet}}^* \text{ and } \pi_{\text{fet}}_* \text{ by } \pi^* \text{ and } \pi_* \text{ (in other words the latter refer to the morphism of finite étale topoi, not étale topoi). There is also a projection } \rho_X : X_{\text{et}} \to X_{\text{fet}} \text{ such that } \pi_{\text{fet}} \rho_Y \simeq \rho_X \pi_{\text{et}} \text{ in } \text{Fet}(X) \text{ for any } \pi : Y \to X. \text{ If } X \text{ is a coherent scheme with finitely many components (e.g. if } X \text{ is of finite type over a field) the inverse image } \rho_X^*: X_{\text{fet}} \to X_{\text{et}} \text{ is fully faithful (} [1 \text{, Prop. VI.9.18}]).$

2.1 Definition An object $M$ of $D^+(X_{\text{fet}}, \Lambda)$ is globally perfect if there are integers $a \leq b$ such that for all $\pi : Y \to X$ in $\text{Fet}(X)$, $R\Gamma(Y, \pi^*M)$ is an object of $D_{\text{perf}}^{[a,b]}(\Lambda)$.

We denote by $\text{Mod}_K$ the category of $\mathbb{Z}$-graded $K$-vector spaces. The next definition formulates our notion of what it means for a cohomology theory with values in $K$-vector spaces to have a “good” integral model compatible with finite étale descent.
2.2 Definition Suppose $X$ is a $k$-scheme of finite type and $H^i : \text{Fet}(X) \to \text{Mod}_{K^e}$ is a functor. A globally perfect model of $H^i$ is a globally perfect $M$ in $D^+(X_{\text{fet}}, \Lambda)$ and a functorial isomorphism

$$H^i(Y) \cong H^i(R\Gamma(Y, \pi^*M)) \otimes_{\Lambda} K$$

for all $Y \to X$ in $\text{Fet}(X)$.

Note that if $\Lambda$ is regular the condition on $R\Gamma(Y, \pi^*M)$ is equivalent to saying that the $H^p(Y, \pi^*M)$ are finitely generated and vanish outside of a finite range depending only on $X$.

2.3 Theorem Suppose $\Lambda$ is noetherian, $X$ is affine, $k_{\text{et}}$ has finite cohomological dimension, $M$ is a globally perfect object of $D(X_{\text{fet}}, \Lambda)$ and $\pi : Y \to X$ is finite étale Galois with group $G$. There is an object $M_Y$ of $D_{\text{perf}}(\Lambda[G])$ whose image under the forgetful functor $D_{\text{perf}}(\Lambda[G]) \to D_{\text{perf}}(\Lambda)$ is isomorphic to $R\Gamma(Y, \pi^*M)$, and $H^i(M_Y) \cong H^i(Y, \pi^*M)$ as $\Lambda[G]$-modules.

We could express the conclusion of the theorem by saying that “$R\Gamma(Y, \pi^*M)$ with its $G$-action is a perfect complex of $\Lambda[G]$-modules.”

For the proof of theorem 2.3 we will need some facts about cohomological dimension. Recall that if $R$ is a ring and $X$ is a topos, the $R$-cohomological dimension of $X$ is the smallest integer $d$ such that $H^n(X, F) = 0$ for all $n > d$ and every $R$-module $F$ in $X$. The $\mathbb{Z}$-cohomological dimension of $X$ will also be called simply the cohomological dimension and will be denoted by coh.dim($X$).

2.4 Lemma Suppose $k$ is a field of characteristic $p > 0$ such that $k_{\text{et}}$ has finite cohomological dimension. Let $X$ be a $k$-scheme of finite type.

- The cohomological dimension of $X_{\text{et}}$ is finite.
- If $X$ is affine the cohomological dimension of $X_{\text{fet}}$ is finite.

Proof. If $k$ is separably closed, the first assertion is a theorem of Gabber whose proof can be found in [1, §1.1]. The general case follows by the Hochschild-Serre spectral sequence.

For the second we can assume that $X$ is connected; in this case [1, VI.9.8] shows that $X_{\text{fet}}$ is equivalent to the classifying topos $B_{\pi_1(X)}$ and that

$$H^n(X_{\text{fet}}, F) \cong H^n(\pi_1(X), F)$$

for all abelian sheaves $F$ on $X_{\text{fet}}$. It thus suffices to show that $H^n(\pi_1(X), F)$ vanishes for $|n| \gg 0$ and all continuous $\pi_1(X)$-modules $F$. 

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Serre [10, Ch. 1 §2.2 Cor. 3] showed that $H^n(\pi_1(X), F)$ is torsion for $n > 0$, so $H^n(\pi_1(X), F) = 0$ if $F$ is a $\mathbb{Q}$-module and $n > 0$. Again by Abbes-Gros-Tsuji [1, Prop. VI.9.12] $X_{\text{fet}}$ is a coherent topos and thus $H^n(\pi_1(X), \_)$ commutes with filtered inductive limits. A devissage using the exact sequence

$$0 \to F_{\text{tor}} \to F \to F \otimes \mathbb{Q} \to F \otimes (\mathbb{Q}/\mathbb{Z}) \to 0$$

reduces to the case when $F$ is torsion. Since $\pi_1(X)$ acts continuously on $F$, the latter is a filtered inductive limit of $\pi_1(X)$-modules corresponding to locally constant constructible torsion sheaves. So we can assume that $F$ is constructible, and as $X$ is affine we can use a result of Achinger [2, Thm. 1.1.1] showing that there are isomorphisms

$$H^i(X_{\text{et}}, F) \simeq H^i(\pi_1(X), F).$$

Thus the assertion follows from the étale case.

In what follows $\Lambda_X$ denotes the sheafification in $\text{Fet}(X)$ of the constant presheaf with value $\Lambda$ (here $X$ is any scheme). If $\pi : Y \to X$ in $\text{Fet}(X)$ then $\Lambda_Y = \pi^*\Lambda_X$.

2.5 Proposition If $M$ is a globally perfect object of $\mathcal{D}^+(X_{\text{fet}}, \Lambda)$ then $M$ has finite Tor-dimension. Suppose conversely that $k_{\text{et}}$ has finite cohomological dimension, $X$ is affine and $M$ satisfies

- $M$ has finite Tor-dimension, and
- for every $\pi : Y \to X$ in $\text{Fet}(X)$, $R\Gamma(Y, \pi^*M)$ is in $\mathcal{D}_{\text{perf}}(\Lambda)$.

Then $M$ is globally perfect.

Proof. Suppose first that $M$ is globally perfect, and let $a \leq b$ be as in definition 2.1. If $\bar{x}$ runs through the set of geometric points of $X$, the family of fiber functors associated to the points $\rho_X(\bar{x})$ of $X_{\text{fet}}$ is conservative by [1, Lemma VI.9.6]. Since $(M \otimes_{\Lambda} N)_x \simeq M_x \otimes_{\Lambda} N$ for any such point it suffices to show that $M_x$ has finite Tor-dimension as a $\Lambda$-module for all $x$, but this is true since $M_x$ is a direct limit of complexes in $\mathcal{D}^{[a,b]}_{\text{fdr}}(\Lambda)$.

Suppose conversely that $X$ is affine, $k_{\text{et}}$ has finite cohomological dimension and $M$ satisfies the two conditions. We can then apply lemma 2.4 to conclude that $X_{\text{fet}}$ has finite cohomological dimension. As in the proof of lemma 2.3 we can identify $X_{\text{fet}}$ with $B_{\pi_1(X)}$ and $M$ with an object of
$D(\Lambda[\pi_1(X)])$ whose image in $D(\Lambda)$ belongs to $D_{ftd}^{[a,b]}(\Lambda)$. If $Y \to X$ is in $\text{Fet}(X)$, $\pi_1(Y)$ is a subgroup of $\pi_1(X)$ and thus

$$\text{coh.dim}(Y_{\text{fet}}) \leq \text{coh.dim}(X_{\text{fet}})$$

by [10, 3.3 Prop. 14]. Since $X_{\text{fet}}$ has finite cohomological dimension we can apply [SGA4, Exp. XVII Thm. 5.2.11] to conclude that $R\Gamma(Y, M)$ is in $D_{ftd}^{[a,b+\text{coh.dim}(X)]}(\Lambda)$, and since $R\Gamma(Y, \pi^*M)$ is in $D_{\text{perf}}(\Lambda)$ by hypothesis, $M$ is globally perfect.

**2.6 Lemma** For any $\pi : Y \to X$ in $\text{Fet}(X)$ and $M$ in $D^+(X_{\text{fet}}, \Lambda)$ there is a functorial isomorphism

$$M \otimes^L_{\Lambda_X} R\pi_*\Lambda_Y \simto R\pi_*\pi^*(M).$$

**Proof.** The adjunction $\pi^* R\pi_*(\Lambda_Y) \to \Lambda_Y$ yields a morphism

$$\pi^*(M \otimes^L_{\Lambda_X} R\pi_*(\Lambda_Y)) \simeq \pi^*(M) \otimes^L_{\Lambda_Y} \pi^* R\pi_*(\Lambda_Y)$$

$$\to \pi^*(M) \otimes^L_{\Lambda_Y} \Lambda_Y \simeq \pi^*(M)$$

and applying the adjunction to this yields the morphism in the lemma. It will be an isomorphism if it is after pulling it back by a covering morphism $g : W \to X$ in $\text{Fet}(X)$. By [11 VI.9.4] $g^*$ has an exact left adjoint, so $g^*$ sends injectives to injectives and it follows that $g^*$ and $R\pi_*$ commute. We may therefore replace $\pi : Y \to X$ by its base change $W \times_X Y \to W$. Now there is a finite étale $g : W \to X$ such that $W \times_X Y$ is a disjoint sum of copies of $W$, and in this case the assertion is clear.

The same argument as in the proof of the last lemma shows:

**2.7 Lemma** Suppose $\pi : Y \to X$ is finite étale and Galois with group $G$. The natural morphism

$$\pi_*\Lambda_Y \to R\pi_*\Lambda_Y$$

is an isomorphism, and $\pi_*\Lambda_Y$ with its natural $G$-action is a flat $\Lambda_X[G]$-module.

**Proof of theorem 2.3:** By lemma 2.7, the sheaf

$$G := \pi_*\Lambda_Y$$

is a flat sheaf of $\Lambda_X[G]$-modules. Since $M$ is globally perfect it has finite Tor-dimension as an object of $D(X_{\text{fet}}, \Lambda_X)$ by proposition 2.5. Therefore
\(M \otimes_{\Lambda_X} G\) has finite Tor-dimension as an object of \(D(X_{fet}, \Lambda_X[G])\). Since \(X_{fet}\) has finite cohomological dimension by lemma 2.4 we can appeal once again to [SGA4, Exp. XVII Thm. 5.2.11] to conclude that

\[M_Y := R\Gamma(X_{fet}, M \otimes_{\Lambda_X} G)\]

is in \(D_{prd}(\Lambda[G])\). By the same reasoning the image of \(M_Y\) in \(D(\Lambda)\) lies in \(D^b(\Lambda[G])\). Since \(\Lambda\) is noetherian it follows that \(M_Y\) is an object of \(D_{perf}(\Lambda[G])\).

On the other hand lemmas 2.7 and 2.6 show that

\[M \otimes_{\Lambda_X} G \simeq M \otimes_{\Lambda_X} R\pi_*\Lambda_Y \simeq R\pi_*\pi^* M\]

in \(D(X_{fet}, \Lambda_X)\), whence isomorphisms

\[M_Y = R\Gamma(X_{fet}, M \otimes_{\Lambda_X} G) \simeq R\Gamma(X_{fet}, R\pi_*\pi^* M) \simeq R\Gamma(Y_{fet}, \pi^* M).\]

Since these isomorphisms are functorial and the \(G\)-module structure of \(G = \pi_*\Lambda_Y\) was induced by the action of \(G\) on \(Y\) the induced isomorphisms

\[H^\cdot(M_Y) \simeq H^\cdot(Y_{fet}, \pi^* M)\]

are isomorphisms of \(\Lambda[G]\)-modules.

### 3 The main theorem

Denote by \(K_0\) the fraction field of a Cohen ring of \(k\), and recall that \(K\) is the fraction field of \(\Lambda\), which we can make into an extension field of \(K_0\). In what follows rigid cohomology has coefficients in \(K_0\).

**3.1 Lemma** Suppose \(f : X \to Y\) is a morphism of schemes. If \(M\) is a globally perfect object of \(D(X_{fet}, \Lambda)\), \(Rf_* M\) is a globally perfect object of \(D(Y_{fet}, \Lambda)\).

**Proof.** If \(V \to Y\) is in \(\text{Fet}(Y)\), \(X \times_Y V \to X\) is in \(\text{Fet}(X)\) and

\[R\Gamma(V, Rf_* M) \simeq R\Gamma(X \times_Y V, M)\]

and the assertion follows. \(\blacksquare\)
3.2 Theorem Suppose $k$ is a field of characteristic $p > 0$ and $\Lambda$ is a complete noetherian local ring with residue field $k$ and fraction field $K$. Suppose $X$ is a smooth affine curve over $k$. Neither of the functors

$$H_{\text{rig}}(\_ \otimes_{K_0} K), \ H_{\text{rig},c}(\_ \otimes_{K_0} K) : \text{Fet}(X) \to \text{Mod}_K$$

have a globally perfect model.

Proof: By duality it suffices to show this for cohomology with compact supports, so we treat the case of $H_{\text{rig},c}(\_ \otimes_{K_0} K)$. We first reduce to the case when $k_{\text{et}}$ has finite cohomological dimension. By standard arguments there is a subfield $k_0 \subseteq k$ finitely generated over $\mathbb{F}_p$ and a smooth affine curve $X_0$ over $k_0$ such that $X \cong X_0 \otimes_{k_0} k$. Suppose $M$ is a globally perfect model of $H_{\text{rig},c}(\_ : \text{Fet}(X) \to \text{Mod}_K$; if $f : X \to X_0$ is the projection then

$$R\Gamma(X_0, Rf_* M) \simeq R\Gamma(X, M)$$

and $Rf_* M$ is a globally perfect model of

$$H_{\text{rig},c}(\_ \otimes W K) : \text{Fet}(X_0) \to \text{Mod}_K$$

where $W$ is a Cohen ring for $k_0$. Thus it suffices to show that this functor does not have a globally perfect model; i.e. we can replace $X/k$ by $X_0/k_0$. Equivalently we can assume that $k_{\text{et}}$ has finite cohomological dimension, and as $X$ is affine we can apply theorem [2,3].

Any smooth affine curve $X$ has an Artin-Schreier cover $\pi : Y \to X$ that ramifies at infinity. So it suffices to invoke the following lemma:

3.3 Lemma With the hypotheses of the theorem, suppose

- $\pi : Y \to X$ is a finite étale Galois cover whose group $G$ is a $p$-group;
- $M$ is a globally perfect model of the functor $H_{\text{rig}}(\_ \otimes_{K_0} K : \text{Fet}(X) \to \text{Mod}_K$ or of $H_{\text{rig},c}(\_ \otimes_{K_0} K : \text{Fet}(X) \to \text{Mod}_K$.

Then $\pi : Y \to X$ extends to a finite étale morphism $\bar{\pi} : \bar{Y} \to \bar{X}$ of smooth projective curves.

Proof. We first consider the case of the functor $H_{\text{rig},c}(\_ \otimes_{K_0} K$. If it has a globally perfect module then by the previous theorem there is a perfect complex $M_Y$ of $\Lambda[G]$-modules such that

$$H^n(M_Y \otimes_{\Lambda} K) \simeq H_{\text{rig},c}^n(Y) \otimes_{K_0} K$$
and then
\[ H^n(M_Y)^G \simeq H^n_{\text{rig},c}(X) \otimes_{K_0} K \]
since \( \pi : Y \to X \) is finite étale.

Since \( G \) is a \( p \)-group, the group ring \( \Lambda[G] \) is local, so a finitely generated projective \( \Lambda[G] \)-module is free. Therefore these last isomorphisms imply that
\[ \chi_c(X) = |G| \chi_c(Y) \quad (3.3.1) \]
where
\[ \chi_c(Z) = \sum_i (-1)^i \dim_{K_0} H^i_{\text{rig},c}(Z) \]
for any separated \( k \)-scheme \( Z \) of finite type.

Comparing (3.3.1) with the Grothendieck-Ogg-Shafarevich formula shows that the ramification of \( \pi : Y \to X \) is tame at infinity. But since \( G \) is a \( p \)-group, \( \pi \) must be totally wild at infinity if it is ramified at all. It is therefore unramified at infinity, which is the conclusion of the lemma.

Finally if \( H^i_{\text{rig}}(\, \cdot \, ) \otimes_{K_0} K \) has a global perfect model the same argument shows that \( \chi(Y) = |G| \chi(X) \) where \( \chi \) is now the Euler characteristic for rigid cohomology without supports. But for any smooth variety \( Z \), \( \chi(Z) = \chi_c(Z) \) and we conclude as before.

4 Examples

The “standard” cohomology theories can all be computed by complexes of étale sheaves, and therefore satisfy étale cohomological descent. Thus when finite they have globally perfect models. For the sake of completeness let’s review the basic examples. The reader should have no problem thinking of global perfect models for various sorts of cohomology theories: \( \ell \)-adic cohomology with or without supports, crystalline cohomology of proper smooth varieties (use the crystalline-étale topos \((X/V)_{\text{crys-et}} \) of Berthelot-Breen-Messing [3], or else the de Rham-Witt complex). The case of \( p \)-adic étale cohomology with compact supports (where \( p \) is the residual characteristic) was used in [5].

Suppose on the other hand that \( k \) is perfect and \( W^\dagger \Omega_X/W \) is the complex of Davis, Langer and Zink [6] and let \( \Lambda = W(k) := W \). If the \( W \)-modules \( H(Y, W^\dagger \Omega_Y/W) \) were finitely generated for all \( Y \to X \) in \( \text{Fet}(X) \) then \( W^\dagger \Omega_Y/W \) would be a global perfect model of \( H^i_{\text{rig}}(\, \cdot \, ) : \text{Fet}(X) \to \text{Mod}_K \) where \( K \) is the fraction field of \( W = W(k) \). We conclude that for every smooth affine curve \( X \) there is a finite étale \( Y \to X \) such that \( H(Y, W^\dagger \Omega_Y/W) \) is not a finitely generated \( W \)-module.
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Tomoyuki Abe:
Kavli Institute for the Physics and Mathematics of the Universe (WPI)
The University of Tokyo, 5-1-5 Kashiwanoha, Kashiwa, Chiba, 277-8583, Japan
e-mail: tomoyuki.abe@ipmu.jp
