Primary and reciprocal space-time experiments, relativistic reciprocity relations and Einstein’s train-embankment thought experiment

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Abstract

The concepts of primary and reciprocal experiments and base and travelling frames in special relativity are concisely described and applied to several different space-time experiments. These include Einstein’s train/embankment thought experiment and a related thought experiment, due to Sartori, involving two trains in parallel motion with different speeds. Spatially separated clocks which are synchronised in their common proper frame are shown to be so in all inertial frames and their spatial separation to be Lorentz invariant. The interpretations given by Einstein and Sartori of their experiments, as well as those given by the present author in previous papers, are shown to be erroneous.

PACS 03.30.+p
1 Introduction

The aim of the present paper is to present a concise summary and some applications of a nomenclature and notation for the general description of space-time experiments introduced and explained in detail in Ref. [1]. The latter is the third in a series of recently written papers [2, 3, 1], devoted to space-time physics in the absence of gravitation, that correct several misconceptions about the subject originating in Einstein’s seminal Special Relativity paper [4]. The most important of these, the spurious nature of the ‘relativity of simultaneity’ (RS) and ‘length contraction’ (LC) effects was explained in Ref. [5] and further discussed from different points-of-view in Refs. [6, 7, 8]. At the time of writing, there is ample and precise experimental confirmation of the time dilatation (TD) effect, predicted as a consequence of the space-time Lorentz Transformation (LT) in Ref. [4], but none for RS or LC [5]. Earth-satellite based experiments to test for the existence of RS have been proposed by the present author [9].

The present paper contains, in the following section, definitions of the concepts of base and travelling frames and a space-time experiment and its reciprocal, introduced in Ref. [1]. The following sections contain applications of these concepts: time dilatation and the simultaneity of spatially separated events in different inertial frames, the Lorentz invariance of spatial intervals, velocity transformation formulae and reciprocity relations, Einstein’s train-embankment experiment [10] and a thought experiment involving two trains moving on parallel tracks at different speeds due to Sartori [11]. As explained below, these two thought experiments were incorrectly analysed in previous papers by the present author.

2 Base and Travelling frames; Primary and Reciprocal space-time experiments

An experiment is considered where a ponderable physical object at some fixed position in an inertial frame, S’, is in uniform motion relative to another inertial frame S. The frame S is denoted as the base frame of the experiment, S’ as a travelling frame. As is conventional the origin of S’ moves along the positive $x$-axis in S with speed $v_B$, the $x$- and $x'$-axes being parallel, and the object lying on the $x'$ axis. The above configuration describes a primary experiment; the value of $v_B$ is a fixed initial condition specified in the frame S. An experiment with a reciprocal configuration is one in which the origin of S moves along the negative $x'$-axis with speed $v'_B$. S’ is now the base frame and S the travelling frame. The value of $v'_B$ is a fixed initial condition (in general not equal to $v_B$) specified in the frame S’. In the special case that $v_B = v'_B \equiv v$, the experiment with the reciprocal configuration is termed reciprocal to the primary experiment, and vice versa.

3 Time dilatation and invariance of simultaneity

The nomenclature introduced above is now applied to a primary experiment and its
reciprocal, in which similar clocks C, C’ are situated at the origins of S and S’ respectively. In the primary experiment C’ moves with speed \( v_B = v \) along the positive \( x \)-axis in S, and in the reciprocal experiment C moves with speed \( v_B' = v \) along the negative \( x' \)-axis in S’. The Lorentz transformations (and their inverses) describing the experiments are as follows:

**Primary Experiment**

Transformation:

\[
x'(C')_T = \gamma [x'(C)_B - vt(C)_B] = 0, \quad \rightarrow x'(C)_B = vt(C)_B
\]

\[
t'(C')_T = \gamma [t(C)_B - \frac{vx'(C)_B}{c^2}], \quad \rightarrow t'(C')_T = \frac{t(C)_B}{\gamma}
\]

Inverse Transformation:

\[
x(C)_B = \gamma [x'(C')_T + vt'(C')_T], \quad \rightarrow x(C)_B = \gamma vt'(C')_T = vt(C)_B
\]

\[
t(C)_T = \gamma [t'(C')_B + \frac{vx'(C)'_B}{c^2}], \quad \rightarrow t(C)_T = \gamma t'(C')_T
\]

**Reciprocal Experiment**

Transformation:

\[
x(C)_T = \gamma [x'(C)_B + vt'(C)_B] = 0, \quad \rightarrow x'(C)_B = -vt'(C)_B
\]

\[
t(C)_T = \gamma [t'(C)_B + \frac{vx'(C)_B}{c^2}], \quad \rightarrow t(C)_T = \frac{t'(C)_B}{\gamma}
\]

Inverse Transformation:

\[
x'(C)_B = \gamma [x(C)_T - vt(C)_T], \quad \rightarrow x'(C)_B = -\gamma vt(C)_T = -vt'(C)_B
\]

\[
t'(C)_B = \gamma [t(C)_T - \frac{vx(C)_T}{c^2}], \quad \rightarrow t'(C)_B = \gamma t(C)_T
\]

where \( \gamma \equiv \frac{1}{\sqrt{1 - (v/c)^2}} \). \( t(C) \) and \( t'(C') \) are the times recorded by C and C’ respectively and the subscripts B and T specify whether the space or time coordinate is defined in a base frame or a travelling frame, respectively. Thus \( t(C)_B \) and \( t'(C')_B \) are times recorded by clocks at rest in primary and reciprocal experiments, respectively while \( t(C)_T \) and \( t'(C')_T \) are the respective times recorded by clocks in motion in the two experiments. The time dilatation (TD) relations given by the second equations in (3.2), (3.4), (3.6) and (3.8) are obtained by using the equations of motion in (3.1), (3.3), (3.5) and (3.7) respectively to eliminate the spatial coordinates on the right sides of the first equations in (3.2), (3.4), (3.6) and (3.8).

The following remarks may be made concerning Eqns.(3.1)-(3.8)

(i) The primary experiment and its reciprocal are physically independent. The LT equations for the primary experiment contain only the spatial coordinates of the
travelling clock C’, the position of the stationary base-frame clock C being arbitrary. The LT equations for the reciprocal experiment contain only the spatial coordinates of the travelling clock C, the position of the stationary base frame clock C’ being arbitrary.

(ii) In both experiments, clocks in the travelling (base) frame appear to be running slower (faster) to observers in the base (travelling) frames.

(iii) Identical predictions are given, in both the primary and reciprocal experiments, by the transformation and the inverse transformation.

(iv) The TD relations:

\[ t(C)_B = \gamma t'(C')_T; \quad t'(C')_B = \gamma t(C)_T \]

are translationally invariant (do not depend on the spatial positions of the clocks).

(v) The equations of motion of the clocks:

\[ x'(C')_T = 0, \quad x(C')_B = vt(C)_B; \quad x(C)_T = 0, \quad x'(C)_B = -vt'(C')_B \]

are the same as in Galilean relativity.

Because of (iv) the TD relations hold for pairs of clocks, at arbitrary positions in S and S’; that is:

\[ t(C_1)_B = \gamma t'(C'_1)_T \]
\[ t(C_2)_B = \gamma t'(C'_2)_T \]

(3.9) (3.10)

where C₁ and C₂ are at arbitrary positions in S and C’₁ and C’₂ are at arbitrary positions in S. If now C’₁ and C’₂ are synchronised so that, at any instant in the frame S’:

\[ t'(C'_1)_T = t'(C'_2)_T = t'_T \]

(3.11)

it follows from (3.9) and (3.10) that:

\[ t(C_1)_B = t(C_2)_B = \gamma t'_T = t_B \]

(3.12)

There is therefore no ‘relativity of simultaneity’ effect for a pair of synchronised clocks at different positions in S — they are also observed to be synchronised in the frame S. How this spurious effect arises from misuse of the space-time Lorentz transformation is explained elsewhere [1, 2, 5, 6, 7, 8].

4 Lorentz invariance of spatial separations

To discuss spatial intervals using the Lorentz transformation, an abbreviated notation is used where the clock at the origin of S’ with \( x'_T = 0 \) is given the label 1 and a second clock, on the \( x' \) axis, with \( x'_T = L' \) the label 2. Assuming the same initial conditions for
the primary experiment as in Eqns(3.1) and (3.2) and dropping, for simplicity, the clock, base frame and travelling frame labels, Eqns(3.1) and (3.2) are written:

\[ x_1' = \gamma(x_1 - vt_1) = 0, \quad \rightarrow \quad x_1 = vt_1 \]  

(4.1)

\[ t_1' = \gamma \left( t_1 - \frac{vx_1}{c^2} \right) \quad \rightarrow \quad t_1' = \frac{t_1}{\gamma} \]  

(4.2)

The equation of motion in S of the clock at \( x_T' = L' \) is

\[ x_2 = vt_2 + L \]  

(4.3)

where \( L \equiv x_2(t_2 = 0) \) is a constant, independent of the value of \( v \), depending on the choice of spatial coordinates in S. The space transformation equation for the clock 2, consistent with (4.1) in the limit \( L = L' = 0 \), and therefore using the same spatial coordinate system in S as clock 1, is:

\[ x_2' - L' = \gamma(x_2 - L - vt_2) = 0, \quad \rightarrow \quad x_2 = vt_2 + L \]  

(4.4)

The corresponding time transformation equation, given by the replacement \( x \rightarrow x - L \), in (4.2) is

\[ t_2' = \gamma \left[ t_2 - \frac{v(x_2 - L)}{c^2} \right] \quad \rightarrow \quad t_2' = \frac{t_2}{\gamma} \]  

(4.5)

Considering now simultaneous events in the frame S'; \( t_1' = t_2' = t' \), (4.1)-(4.5) yield:

\[ x_1(\beta) = \beta \gamma ct_1 = \gamma \beta ct' \]  

(4.6)

\[ t_1(\beta) = \gamma t' \]  

(4.7)

\[ x_2(\beta) - L = \beta \gamma ct_2 = \gamma \beta ct' \]  

(4.8)

\[ t_2(\beta) = \gamma t' \]  

(4.9)

where \( \beta \equiv v/c \), and the \( \beta \) dependences of \( x \) and \( t \), for a fixed value of \( t' \), are explicitly indicated.

With the aid of the identity: \( \gamma^2 - \gamma^2 \beta^2 \equiv 1 \), (4.6),(4.7) and (4.8),(4.9) yield identical hyperbolic curves on the \( ct \) versus \( x \) plot for a given value of \( t' \):

\[ c^2 t_1(\beta)^2 - x_1(\beta)^2 = c^2 (t')^2 = c^2 t_2(\beta)^2 - (x_2(\beta) - L)^2 \]  

(4.10)

Since (4.7) and (4.9) give

\[ t_1(\beta) = \gamma t' = t_2(\beta) \]  

(4.11)

(4.10) simplifies to

\[ x_2(\beta) - x_1(\beta) = L \]  

(4.12)

The spatial separation of the clocks in S is therefore independent of the value of \( \beta \). Since, for \( \beta \rightarrow 0 \), \( x \rightarrow x' \) it follows from (4.12) that:

\[ x_2(0) - x_1(0) = x'_2 - x'_1 \equiv L' = L \]  

(4.13)

The spatial separation of the clocks in S and S' is therefore the same for all values of \( \beta \) —there is no ‘relativistic length contraction’. How the latter spurious effect —correlated with ‘relativity of simultaneity’ — arises is also discussed in Refs. [1, 2, 5, 6, 7, 8].
5 Velocity transformation formulae and relativistic reciprocity relations

Two, physically distinct, kinds of velocity addition formulae are considered in this section. The first, corresponding to the well-known relativistic velocity addition formulae as derived by Einstein in Ref. [4], gives relations between the base frame velocities of a single object in different inertial frames. The second gives the transformation of the relative velocity of two objects in a given inertial frame into the similarly defined relative velocity between them in another inertial frame. For the first type of transformation, since only base frame velocities are involved, the ‘travelling frame’ concept plays no role, whereas it is essential for the second (relative velocity) transformation in order to correctly understand the physical basis of the TD effect.

Suppose that the frame $S'$ moves with speed $v_B = v$ in the positive $x$-direction in $S$ and that an object moves with velocity components $u_B^{(x)}$ and $u_B^{(y)}$ in the directions of the $x$- and $y$-axes in $S$. The first type of calculation predicts the corresponding base frame velocities \( \bar{w}_B^{(x)} \) and \( \bar{w}_B^{(y)} \) in the frame $S'$. The bar on a symbol denotes that it is a derived quantity rather than an assumed initial value of a parameter of the problem. The appropriate differential LT formulae are:

\[
\begin{align*}
\frac{d}{dt'} B^{(x)} & = \gamma \left[ \frac{d}{dt} B^{(x)} - v \frac{d}{dt} t_B \right] \\
\frac{d}{dt'} B^{(y)} & = \frac{d}{dt} B^{(y)} \\
\frac{d}{dt'} B & = \gamma \left[ \frac{d}{dt} B - \frac{v}{c^2} \frac{d}{dt} x_B \right]
\end{align*}
\]  

where

\[
\frac{d}{dt} B^{(x)} = u_B^{(x)}, \quad \frac{d}{dt} B^{(y)} = u_B^{(y)} \\
\frac{d}{dt} B^{(x)} = \bar{w}_B^{(x)}, \quad \frac{d}{dt} B^{(y)} = \bar{w}_B^{(y)}
\]

Dividing (5.1) or (5.2) by (5.3) and substituting, in the equations so obtained, the base frame velocities defined in (5.4) and (5.5) gives the longitudinal and transverse base frame velocity addition formulae:

\[
\begin{align*}
\bar{w}_B^{(x)} & = \frac{u_B^{(x)} - v_B}{1 - \frac{v_B^2}{c^2}} \\
\bar{w}_B^{(y)} & = \frac{u_B^{(y)}}{\gamma(1 - \frac{v_B^2}{c^2})}
\end{align*}
\]

Eqs(5.1)-(5.3) can also be used to derive transformation equations for the 4-vector velocity, $U$, of the object. If $d\tau$ denotes an interval of the proper time of the object, the TD relations $dt_B = \gamma_{\bar{u}B} d\tau$ and $dt'_{B} = \gamma_{\bar{u}B} d\tau$ ($\gamma_U \equiv 1/\sqrt{1 - (u/c)^2}$) give, on dividing (5.1)-(5.3) throughout by $d\tau$ and using (5.4) and (5.5), the relations:

\[
\gamma_{\bar{u}B} \bar{w}_B^{(x)} = \gamma \left[ \gamma_{uB} u_B^{(x)} - v \gamma_{uB} \right]
\]
The Lorentz invariance, (4.10), of the spatial separation of the object from the origin of the TD relation:

\[ \gamma_{\bar{w}B} \bar{w}^{(y')} = \gamma_{uB} u^{(y)} \]
\[ \gamma_{\bar{w}B} = \gamma \left[ \gamma_{uB} - \frac{v \gamma_{uB} u^{(x)}}{c^2} \right] \]  

or

\[ U'(x') = \gamma [U^{(x)} - \beta U^{(0)}] \]  
\[ U'(y') = U^{(y)} \]  
\[ U^{(0)} = \gamma [U^{(0)} - \beta U^{(x)}] \]

where the 4-vector velocities \( U \) and \( U' \) of the object in \( S \) and \( S' \) are defined as:

\[ U = (U^{(0)}; U^{(x)}, U^{(y)}, U^{(z)}) \equiv (c \gamma_{wB} \gamma_{uB} u^{(x)}, \gamma_{uB} u^{(y)}, 0) \]  
\[ U' = (U'^{(0)}; U'^{(x')}, U'^{(y')}, U'^{(z')}) \equiv (c \gamma_{\bar{w}B} \gamma_{\bar{w}B} \bar{w}^{(x')}, \gamma_{\bar{w}B} \bar{w}^{(y')}, 0) \]

The velocity addition relations (5.6) and (5.7) are recovered by dividing (5.8) and (5.9) respectively by (5.10) or dividing (5.11) and (5.12) respectively by (5.13) and using the definitions of the components of the 4-vector velocities in (5.14) and (5.15).

It is interesting to note that, although space-time events in an experiment and its reciprocal are physically independent, the initial kinematical configurations in the two experiments are related by the kinematical LT (5.11)-(5.13) that yields the parallel velocity addition formula when \( u^{(x)}_B = u_B \) and \( u^{(y)}_B = 0 \):

\[ \bar{w}_B = \frac{u_B - v_B}{1 - \frac{v_B u_B}{c^2}} \]  

If \( u_B = 0 \) (for example an object at rest at the origin of \( S \)) than (5.16) gives \(-\bar{w}_B = v'_B = v_B\), which describes the kinematical configuration of the reciprocal experiment — the object moves with speed \( v_B \) along the negative \( x' \)-axis in \( S' \).

Consider now an experiment in which an object moving with the specified speed \( u_B \) along the positive \( x \)-axis in \( S \) is observed in the travelling frame \( S' \). Since the relative velocity of the object and the frame \( S' \), in \( S \), is \( u_B - v_B \), the speed of the object, as observed in \( S' \), is the transformed value of this relative velocity. If the origins of \( S \) and \( S' \) and the moving object all have the same \( x \)-coordinate at time \( t_B = 0 \), and \( u_B > v_B \), the separation, \( \Delta x_B \), in the frame \( S \), of the object from the origin of \( S' \) at time \( t_B \) is

\[ \Delta x_B = (u_B - v_B)t_B \]  

If \( \bar{w}'_T \) is the velocity of the object in \( S' \), in the positive \( x' \) direction, in the primary experiment, the separation of the object from the origin of \( S' \) at time \( t'_B \) is

\[ \Delta x'_T = \bar{w}'_T t'_B \]

The Lorentz invariance, (4.10), of the spatial separation of the object from the origin of \( S' \), at the corresponding times \( t_B \) and \( t'_B \), which implies, for \( u_B > v_B \), \( \Delta x_B = -\Delta x'_T \), and the TD relation: \( t_B = \gamma_B t'_B \) (c.f. Eqn (3.9)) then gives the the transformation law for the relative velocity of the object and \( S' \) as:

\[ -\bar{w}'_T = \gamma_B (u_B - v_B) \]  

where \( \gamma_B \equiv \frac{1}{\sqrt{1 - (v_B/c)^2}} \). As above, the bar in the symbol \( \bar{w}'_T \) indicates that it is a calculated quantity as contrasted with the assumed initial values, in this case, of \( v_B \) and \( u_B \). In the special case \( u_B = 0 \), (5.19)\(^1\) is the transformation law of the relative velocity

\(^1\) Note that for \( u_B < v_B \), \( \Delta x_B = +\Delta x'_T \) so that there is no minus sign on the left side of (5.19).
of S and S’ between the base frame S and the travelling frame S’:

\[-\bar{u}'_T \equiv \bar{v}'_T = \gamma_B v_B \]  

(5.20)

where \(\bar{v}'_T\) is defined as the velocity of S relative to S’, in S’, in the direction of the negative \(x'\)-axis in the primary experiment. Thus the Reciprocity Principle (RP) [12], that “the velocity of an inertial frame of reference S’, with respect to another inertial frame of reference S, is equal and opposite to velocity of S relative to S’”, although true in Galilean relativity, no longer holds in special relativity, being replaced by the reciprocity relation (5.20), when the space-time LT is used to transform events, in a particular space-time experiment, from one frame into another. As derivation of the latter equation shows, the breakdown of the RP is a necessary consequence of the definition of a relative velocity, the invariance of spatial intervals, and TD. The reciprocity relation for an experiment with a reciprocal configuration where S’ is the base frame and S the travelling frame is

\[\bar{v}_T = \gamma'_B v'_B \]  

(5.21)

where \(\gamma'_B \equiv 1/\sqrt{1 - (v'_B/c)^2}\). Eqn(5.21) is obtained from (5.20) by exchange of primed and unprimed quantities. Reciprocal experiments correspond to the special case where \(v'_B = v_B \equiv v\).

Thus, in special relativity, the RP should be replaced a ‘Kinematical Reciprocity Principle’ (KRP) [1]: ‘the velocity of an inertial frame of reference S’ relative to another such frame S in a space-time experiment is equal and opposite to the velocity of S relative to S’ in the reciprocal experiment’. This statement, which is actually the definition of a reciprocal experiment, rather than a relation between velocities in different frames in the same space-time experiment, is applicable in both special and Galilean relativity.

### 6 Einstein’s train-embankment thought experiment

A straightforward application of the relative velocity transformation law (5.19) is to the analysis of the much-discussed train-embankment thought experiment[10]. This was introduced by Einstein in the popular book ‘Relativity, the Special and General Theory’ with the intention to illustrate, in a simple way, ‘relativity of simultaneity’. Light signals are produced by lightning strikes which simultaneously hit an embankment at positions coincident with the front and back ends of a moving train. The signals are seen by an observer, \(O_T\), at the middle of the train and an observer, \(O_B\), on the embankment, aligned with \(O_T\) at the instant of the lightning strikes. The light signals are observed simultaneously by \(O_B\) who concludes that the lightning strikes are simultaneous. Because of the relative motion of \(O_T\) and the light signals, the latter are not observed by \(O_T\) at the same time. Invoking the constancy of the speed of light in the train frame, Einstein concludes that \(O_T\) would not judge the strikes to be simultaneous, giving rise to a ‘relativity of simultaneity’ effect between the train and embankment frames.

This train-embankment thought experiment (TETE) is now analysed in terms of the concepts and nomenclature introduced above\(^2\). The observer \(O_T\) is replaced by a two-sided light detector, D, at the middle of the train. The latter moves to the right with

\(^2\)A similar analysis is presented in Ref. [1].
Figure 1: Analysis of Einstein’s train-embankment thought experiment. Configurations a), b) and c) in the embankment frame (S); c), d) and e) in the train frame (S’). \(v = c/2, \gamma = 2/\sqrt{3}\). See text for discussion.

speed \(v\). The embankment frame, S, is the base frame of the experiment, the train frame, S’, is the travelling frame. At time \(t_0\) in S (Fig. 1a) light signals moving at speed \(c\) in the embankment frame are emitted, and move towards D. The light signals are also ‘travelling objects’ in the source frame S. The essential input parameters of the problem, \(v\) and \(c\) are therefore fixed in the frame S. In accordance with Eqn(4.13) the length of the train, \(L\), is invariant. At time in S \(t_1 = L/[2(c + v)] + t_0\) (Fig. 4b) the left-moving light signal strikes D, and at time in S \(t_2 = L/[2(c - v)] + t_0\) (Fig. 1c) the right-moving light signal strikes D. The configurations in S’ corresponding to those in S in Figs. 1a,b,c are shown in Figs. 1d,e,f respectively. The velocity transformation formula (5.19) implies that the speed in S’ of the right-moving light signal relative to D is \(v_T(-) = \gamma(c - v)\) while that of the left-moving light signal is \(v_T(+) = \gamma(c + v)\). The pattern of detection events in S and S’ is then the same, the only difference being that that the velocities of the light signals relative to D are greater in S’ by the factor \(\gamma\) — a necessary consequence of time dilatation and the invariance of length intervals. The left-moving light signal is then observed in S’ at the time:

\[
t_1' = \frac{L}{2\gamma(c + v)} + t_0' = \frac{t_1 - t_0}{\gamma} + t_0' \quad (6.1)
\]

and the right-moving one at the time:

\[
t_2' = \frac{L}{2\gamma(c - v)} + t_0' = \frac{t_2 - t_0}{\gamma} + t_0' \quad (6.2)
\]

The time dilatation effect for the travelling frame S’ is manifest in these equations. It
is seen to be a consequence of the relative velocity transformation formula (5.19), not of LC.

On the assumption that an experimenter analyzing the signals received by D knows the essential parameters of the problem, \( L, v, \) and \( c \), the measured times \( t_{1}' \) and \( t_{2}' \) in the train frame can be used to decide whether the left and right moving light signals were emitted simultaneously in this frame or not. If the right-moving and left-moving signals are emitted at times \( t_{0}'(-) \) and \( t_{0}'(+) \) respectively then (6.1) and (6.2) are modified to:

\[
t_{1}' = \frac{L}{2\gamma(c + v)} + t_{0}'(+) = \frac{t_{1} - t_{0}}{\gamma} + t_{0}'(+) \tag{6.3}
\]

and

\[
t_{2}' = \frac{L}{2\gamma(c - v)} + t_{0}'(-) = \frac{t_{2} - t_{0}}{\gamma} + t_{0}'(-) \tag{6.4}
\]

Subtracting (6.3) from (6.4) and rearranging:

\[
t_{2}' - t_{1}' = t_{0}'(-) - t_{0}'(+) + \frac{\gamma\beta L}{c} \tag{6.5}
\]

The observed time difference \( t_{2}' - t_{1}' \) and knowledge of the value of \( \gamma\beta L/c \) then enables determination of \( t_{0}'(-) - t_{0}'(+) \) so that the simultaneity of emission of the light signals can be tested. For the event configurations shown in Fig. 1 it would be indeed concluded that \( t_{0}'(-) = t_{0}'(+) \), so the emission of the signals is found to be simultaneous in the train frame, contrary to Einstein’s assertion in Ref. [10]. The essential flaw in Einstein’s argument was the failure to distinguish between the speed of light, relative to some fixed object in an inertial frame, and the speed of light relative to some moving object in the same frame, which is what is relevant for the analysis of the TETE. Einstein’s interpretation corresponds to replacing \( t_{2}' \) and \( t_{1}' \) by \( t_{2} \) and \( t_{1} \), so that only events in the embankment frame are considered, and making the replacements, (confusing the speed of light in an inertial frame, with the relative speed of light and a moving object in the same frame): \( \gamma(c \pm v) \to c \) in (6.1) and (6.2) giving:

\[
t_{0}'(-) - t_{0}'(+) = t_{2} - t_{1} = \frac{\gamma^{2}\beta L}{c} \tag{6.6}
\]

This leads to Einstein’s false conclusion that the light signal emission events would be found to be non-simultaneous in the train frame.

An analysis of the TETE in a previous paper [13] by the present author also concluded that the train observer would judge the lightning strikes to be simultaneous, but the reasoning leading to this conclusion was fallacious. At the time of writing Ref. [13] I had not understood correctly the distinction between an experiment and its reciprocal and the difference in physical interpretation of a space-time and a kinematical LT explained in Sections 3 and 5 above. I incorrectly assumed that the kinematical LT relating two base frame velocities was valid in a single space-time experiment, that is, for transformation between a base frame and a travelling frame. Thus the velocities of both photons in \( S' \) in Fig. 1 of the present paper were assumed to be \( c \). Since the lightning strikes are (see Section 3 above) simultaneous in both \( S \) and \( S' \) the train observer would then see the light signals they emit at the same time and conclude that the strikes are simultaneous. This is a correct description, in the train frame, of the physically independent experiment that is
Table 1: Base frame (B) and travelling frame (T) velocities in various frames. The base frame velocities are related by the parallel velocity addition formula (5.16) while the travelling frame velocities are derived from base frame velocities using the relative velocity transformation formula (5.19). Each row of velocities specifies a physically-independent space-time experiment. The experiment shown in Figs. 2-4 is that shown in the first row.

Sartori’s two-train thought experiment

Sartori [11] proposed the thought experiment shown in Fig. 2. In the base frame S, the rest frame of the platform P, two trains T1 and T2, with proper frames S' and S” respectively, move to the right with speeds v and u respectively, where \( u > v \). These base frame velocities are the fixed input parameters of the problem. As in the previous Section, to lighten the notation, the base frame labels on these quantities are omitted. Initially (Fig. 2a) when \( t = t_1 = 0 \), T1 is aligned with P and distant \( L_1 \) from T2. At time \( t = t_2 = L_1/u \), T2 is aligned with P and T1 is distant \( L_2 \) from P (Fig. 1b). Since \( u > v \), T1 is aligned with T2 at time \( t = t_3 = L_1/(u - v) \) when T1 and T2 are distant \( L_3 \) from P (Fig. 2c). The corresponding configurations as observed in the travelling frames S’ and S” are shown in Figs. 3 and 4. Because of the invariance of spatial separations the corresponding spatial configurations are identical to those of Fig. 2, whereas the times of the corresponding events are scaled by the TD factors \( 1/\gamma_v \), \( 1/\gamma_u \) respectively. The travelling frame velocities as given by the relative velocity transformation formula (5.19) are shown in Figs. 3 and 4.

The configurations shown in Figs. 2-4 correspond to \( u = 0.8c \), \( v = 0.4c \). By performing a kinematical transformation according to Eqns(5.8)-(5.10) or (5.11)-(5.13), between S and S’ or S and S” configurations may be obtained in which T1 is at rest (configuration in S’) or T2 at rest (configuration in S”). In these cases the parallel base frame velocity

\[
\gamma_v v \quad \gamma_u u
\]

is reciprocal to the one proposed by Einstein (i.e. the one where S’ is the the base, not the travelling frame) not the correct description, in the train frame, of Einstein’s experiment. The mistake in Ref. [13] was the hitherto universal, but erroneous, assumption that events defined in the base frames of an experiment and its reciprocal are related by the space-time LT.
Figure 2: Spatial coincidence events as observed in the base frame $S$ (the rest frame of $P$) in which the velocities $v = 0.4c$ and $u = 0.8c$ of $T_1$ and $T_2$, respectively, are specified. a) Event1, $T_1$ opposite $P$, b) Event2, $T_2$ opposite $P$, c) Event3, $T_1$ opposite $T_2$.

Figure 3: Spatial coincidence events as observed in the travelling frame $S'$ (the rest frame of $T_1$). a) Event1, $T_1$ opposite $P$, b) Event2, $T_2$ opposite $P$, c) Event3, $T_1$ opposite $T_2$. 
Figure 4: Spatial coincidence events as observed in the travelling frame rest $S''$ (the rest frame of T2). a) Event1, T1 opposite P, b) Event2, T2 opposite P, c) Event3, T1 opposite T2.

Table 2: Times and spatial separations of the coincidence events 2 and 3 in different frames for the three space-time experiments specified in the rows of Table 1.
transformation (5.16) is applicable. In S’ P has velocity \( -v \) and T2 velocity \( w = (u - v)/(1 - uv/c^2) \), while in S” P has velocity \( -u \) and T1 velocity \( -w \). These configurations constitute base frame parameters for two independent space-time experiments each of which are reciprocal to the primary experiment shown in Figs.2-4. In the first reciprocal experiment S’ is the base frame and S and S” are travelling frames while in the second S” is the base frame and S and S’ are the travelling frames. The travelling and base frame velocities for all three experiments are presented in Table 1; the base frame velocities transform according to Eqn (5.16), while the travelling frame velocities are calculated using the relative velocity transformation formula (5.19). In Table 2 are presented, for the primary experiment and the two reciprocal experiments, the values of \( t_2 \), \( t'_2 \) and \( t''_2 \) (when P and T2 are aligned) and \( t_3 \), \( t'_3 \) and \( t''_3 \) (when T1 and T2 are aligned) in the frames S, S’ and S”, respectively, as well as the (invariant) separations: \( L_2 \) of P and T1 at the instant of P2-T2 alignment and \( L_3 \) of P and T1 at the instant of T1-T2 alignment. The different values of the times and separations in the primary and two reciprocal experiments make manifest the physical independence of these experiments even though their kinematical configurations are related by the kinematical LT of Eqns(5.8)-(5.10). The initial spatial separation of P and T1, \( L_1 \), is the same in all three experiments. Several different TD relations can be read off from the entries of Table 2:

\[
\begin{align*}
S(B), S'(T), S''(T) & : \quad t(B) = \gamma_v t'(T), \quad t(B) = \gamma_u t''(T) \\
S'(B), S(T), S''(T) & : \quad t'(B) = \gamma_v t(T), \quad t'(B) = \gamma_w t''(T) \\
S''(B), S(T), S'(T) & : \quad t''(B) = \gamma_u t(T), \quad t''(B) = \gamma_w t''(T)
\end{align*}
\]

It is interesting to compare these predictions with assumption made by Sartori in the paper where the thought experiment shown in Figs. 1-3 was first proposed [11]. The aim of this paper was to present a simple derivation of the parallel velocity addition formula (5.16) without direct use of the LT. In the paper it was claimed to derive (5.16) taking as initial postulate the TD effect. The first incorrect assumption of Ref. [11] was that the base frame configurations of the primary experiment \(^3\) \([S(B), S'(T)]\) and the reciprocal experiment \([S'(B), S(T)]\), which are indeed related by a kinematical LT, are also related by a space-time LT. Thus it was assumed that in the relations \(^4\) that may be derived from the entries of Table 2:

\[
\begin{align*}
t_3(B) &= \frac{ut_2(B)}{u - v} \\
t'_3(B) &= \frac{(w + v)t'_2(B)}{w}
\end{align*}
\]

that \( t_2(B) \) and \( t'_2(B) \) and \( t_3(B) \) and \( t'_3(B) \) are related by TD relations:

\[
\begin{align*}
t'_2(B) &= \gamma_v t_2(B) \\
t_3(B) &= \gamma_v t'_3(B)
\end{align*}
\]

Taking the ratio of (7.4) to (7.5) and using (7.6) and (7.7) to eliminate the ratios \( t_2(B)/t'_2(B) \) and \( t_3(B)/t'_3(B) \) from the resulting equation gives:

\[
\gamma_v^2(u - v)(w + v) = uu
\]

\(^3\)The symbols \([S(B), S'(T)]\) and \([S'(B), S(T)]\) specify, in an evident notation, an experiment and its (physically independent) reciprocal.

\(^4\)Eqns.(7.4) and (7.5) correspond to Eqns.(2) and (8), respectively, of Ref. [11].
which when solved for \( w \) in terms of \( v \) and \( u \) yields the parallel velocity addition formula (5.16).

The following comments may be made on this calculation:

(i) The base frame configurations of the independent experiments [\( S(B), S'(T) \)] and [\( S'(B), S(T) \)] although related by the kinematical LT of Eqns(5.8)-(5.9) are not connected by the space-time LT —for [\( S(B), S'(T) \)] the speed of \( P \) in the frame \( S' \) is (see Fig. 3a) \( \gamma_v v \) not \( v \).

(ii) Comparison of (7.6) and (7.7) with (7.1) and (7.2) above shows that the former formulae are inconsistent. For Event 2 the formula (7.6) corresponds to the TD effect for the experiment [\( S'(B), S(T) \)], as in (7.2), whereas for Event 3 the formula (7.7) corresponds to the independent experiment [\( S(B), S'(T) \)] as in (7.1). It is clear that, for example, the times \( t'_2 \) and \( t'_3 \) must both be times of clocks at rest in \( S' \) but observed in motion from \( S \) in the primary experiment shown in Figs 1-3, [\( S(B), S'(T) \)], and times of clocks at rest in \( S' \) and observed in the same frame for the reciprocal experiment [\( S'(B), S(T) \)].

The argument given by Sartori for (7.6) and (7.7) is that, for (7.6) ‘the Events 1 and 2 occur at the same position in \( S' \) so that \( t_2 \) is a proper time interval’, and for (7.7) that ‘the Events 1 and 3 occur at the same position in \( S' \) so that \( t'_3 \) is a proper time interval’. Actually all the times in the TD relations shown in (7.1)-(7.3) are ‘proper time intervals’ recorded by some clock. Given the existence of an array of synchronised clocks in each frame, it is of no importance, for the timing of two events whether, or not, their times are both measured locally by the same clock. Suppose that in Fig. 3 there is a clock \( C' \) at rest in \( S' \), synchronised with a clock at \( T_1 \), distant \( L_2 \) from it, such that Event 2 in Fig. 3b is local at \( C' \). Because \( C' \) and a synchronised clock at \( T_1 \) both record \( t' = 0 \) at the epoch of Event 1, the time interval in \( S' \), between events 1 and 2 in \( S' \), is correctly given by the epoch \( t'_2 \), as measured by \( C' \), of the Event 2 which is local at \( C' \). Since \( t'_2 \) is a ‘proper time interval’ measured by the clock \( C' \), then if \( t_3 = \gamma_v t'_3 \) as in (7.6), as measured by a local clock at \( T_1 \), then also \( t_2 = \gamma_v t'_2 \) as measured by the local clock \( C' \), in contradiction with Sartori’s assumption (7.6).

Summarising, Sartori’s analysis assumes incorrectly that the base frames of an experiment and its reciprocal are related by the space-time LT and uses TD relations in an inconsistent manner, (7.7) being applicable to the primary experiment shown in Figs 2-4: [\( S(B), S'(T) \)], and (7.6) to the reciprocal experiment: [\( S'(B), S(T) \)]. That the algebraic manipulation of (7.4)-(7.7) yields the correct velocity addition formula (5.16) must then be considered as purely fortuitous.

In my previous analysis [14] of Sartori’s thought experiment, the same mistake of principle was made as in that of the train/embankment experiment in Ref. [13]. In common with Sartori, it was assumed that the kinematical configurations of a primary experiment and its reciprocal correspond to the base and travelling frame configurations of the primary experiment. That is (see Figs. 1 and 2 of Ref. [11]) that the velocities in the frame \( S' \) in the experiment shown in Figs. 2-4 and the first row of Table 1 of the present paper, were given instead by the base frame velocities in \( S' \) of the reciprocal experiment shown in the second row of Table 1. Thus, it is falsely assumed that the
base frame configurations of an experiment and its reciprocal are actually the base and travelling frame configurations of the primary experiment, and that events in these two frames are connected by the space-time LT. This leads to false predictions [14] of the breakdown of the Lorentz invariance of spatial intervals in different inertial frames and ratios between time intervals observed in different inertial frames differing from the TD effect.
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