Complex sliding flows of yield-stress fluids

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A theoretical and numerical study of complex sliding flows of yield-stress fluids is presented. Yield-stress fluids are known to slide over solid surfaces if the tangential stress exceeds the sliding yield stress. The sliding may occur due to various microscopic phenomena such as the formation of an infinitesimal lubrication layer of the solvent and/or elastic deformation of the suspended soft particles in the vicinity of the solid surfaces. This leads to a ‘stick-slip’ law which complicates the modelling and analysis of the hydrodynamic characteristics of the yield-stress fluid flow. In the present study, we formulate the problem of sliding flow beyond one-dimensional rheometric flows. Then, a numerical scheme based on the augmented Lagrangian method is presented to attack these kind of problems. Theoretical tools are developed for analysing the flow/no-flow limit. The whole framework is benchmarked in planar Poiseuille flow and validated against analytical solutions. Then two more complex physical problems are investigated: slippery particle sedimentation and pressure-driven sliding flow in porous media. The yield limit is addressed in detail for both flow cases. In the particle sedimentation problem, method of characteristics—slipline method—in the presence of slip is revisited from the perfectly-plastic mechanics and used as a helpful tool in addressing the yield limit. Finally, flows through model and randomized porous media are studied. The randomized configuration is chose to capture more sophisticated aspects of the yield-stress fluid flows in porous media at the yield limit—channelization.

Key words: non-Newtonian flows, particle/fluid flow, plastic materials

1. Introduction

Slip behaviour in complex fluids has been attributed to various microscopic phenomena such as chain detachment/desorption at the polymer-wall interface in polymer melts (Hatzikiriakos 2012), migration/depletion of disperse phase away from the vicinity of the solid boundaries in suspensions (Vand 1948), and elastic deformation of the suspended soft particles lying on the smooth solid boundaries (Meeker et al. 2004b) in pastes—yield-stress fluids. In the present study, however, we rather investigate the macroscopic consequences of slip on hydrodynamic features of yield-stress fluid flows, in order to form a bridge over these two distinct scales.

In experiments, roughened (e.g. using sandpapers) or chemically treated surfaces are usually used to suppress slip in measurements of yield-stress fluid flows. For instance, Christel et al. (2012) proposed a Polymethyl methacrylate (PMMA) treatment to reduce slip by exciting positive surface charges providing electrostatic interactions and squashing the lubricating layer. This is important since in a great number of laboratory experiments,

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PMMA is used due to its transparency properties which enable flow visualizations. In industrial scales/processes, however, chemical treatments are not always feasible, and therefore it is crucial to analyse the hydrodynamic consequences of sliding flows. Moreover, slip alters rheometric measurements, causing rheological properties of the samples to be inaccurately evaluated, sometimes even by one order of magnitude (Poumaere et al. 2014). This issue has been discussed extensively in the literature, for instance recently by Medina-Bañuelos et al. (2017, 2019). Hence, it would be constructive to have a generic analysis framework of sliding flows in yield-stress fluids, and this is the aim of the present paper.

In the present study, we formulate the well-known ‘stick-slip’ law for yield-stress fluids beyond 1D rheometric flows and then generalise the previously proposed numerical algorithm (Roquet & Saramito 2008; Muravleva 2018) based on the augmented Lagrangian method for attacking this kind of problems. Theoretical tools are developed in the presence of slip. The whole framework is first benchmarked by a simple channel Poiseuille flow. Then we proceed to investigate flows in complex geometries: the creeping flows about a circular cylinder and the pressure-driven flows in model and randomized porous media.

1.1. General slip law for yield-stress fluids

Characterising the slip behaviour of yield-stress fluids has been the main objective of a large number of studies from theoretical attempts to experimental measurements. Meeker et al. (2004b) and Piau (2007) tried to theoretically describe the slip origin in soft particle pastes/microgels from an elastohydrodynamic perspective. Rheometric tests with slippery plates in the cone-plate/plate-plate geometries have been conducted to quantify the slip behaviours of yield-stress fluids ranging from Carbopol gels to emulsions (Meeker et al. 2004a,b; Poumaere et al. 2014; Zhang et al. 2018). Moreover, very recently, in a series of experiments using Optical coherence tomography (OCT) and particle tracking velocimetry (PTV), Daneshi et al. (2019) characterised the slip behaviour of Carbopol gel (a model ‘simple’ yield-stress fluid) inside a capillary tube. The sliding characteristics of yield-stress fluids observed in all the mentioned 1D experimental studies can be summarized in a general slip law as,

\[
\hat{u}_s = \begin{cases} 
\hat{\beta}_s (\hat{\tau}_w - \hat{\tau}_s)^k, & \text{iff } |\hat{\tau}_w| > \hat{\tau}_s, \\
0, & \text{iff } |\hat{\tau}_w| \leq \hat{\tau}_s,
\end{cases}
\]

(1.1)

usually termed as ‘stick-slip’ law, where \(\hat{u}_s\) is the slip velocity on the solid surface and \(\hat{\tau}_w\) is the shear stress at the solid boundaries. The sliding threshold is called the sliding yield stress—\(\hat{\tau}_s\). The slip coefficient and the power index are designated by \(\hat{\beta}_s\) and \(k\), respectively. Hence, \(\hat{\beta}_s\) has the dimension of \(m^{2k+1}/N^k.s\) or \(m/Pa^k.s\). Its physical interpretation (for the case \(k = 1\)) is the slip length over the local effective viscosity of the fluid, \(\hat{\ell}_s/\hat{\mu}_{eff}\). In the entire paper, quantities with a ‘hat’ symbol (\(\hat{\cdot}\)) are dimensional and others are dimensionless.

For a deeper understanding of the physical meaning of the slip law (1.1), we consider the simple unidirectional channel Poiseuille flow in the presence of slip of a Bingham fluid,

\[
\begin{align*}
\hat{\tau}_{xy} &= \mu \frac{d\hat{u}}{d\hat{y}} + \hat{\tau}_y \text{sgn} \left( \frac{d\hat{u}}{d\hat{y}} \right) & \text{iff } |\hat{\tau}_{xy}| > \hat{\tau}_y, \\
\frac{d\hat{u}}{d\hat{y}} &= 0 & \text{iff } |\hat{\tau}_{xy}| \leq \hat{\tau}_y,
\end{align*}
\]

(1.2)

where \(\hat{\mu}\) is the plastic viscosity of the fluid, \(\hat{u}\) the streamwise velocity, \(\hat{\tau}_y\) the material’s
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Figure 1. Schematic of the sliding channel flow of a yield-stress fluid; the x-axis is aligned and put at the axial centreline of the channel which is formed by the two infinite parallel plates separated by the gap $H$ and y-axis is in the wall-normal direction: (a) when $\tau_w \leq \tau_s$ there is no flow, (b) when $\tau_s < \tau_w \leq \tau_y$ then the fluid in the whole gap slides as an unyielded plug, (c) when $\tau_y < \tau_w$ then the fluid in the vicinity of the walls ($y_p < y$ where $\tau_y < \tau_{xy}$) yields and at the same time slides over the walls. There is a core unyielded region as well ($y \leq y_p$).

yield stress, $\tau_{xy}$ the shear stress, and $\text{sgn}(\cdot)$ is the sign function. Figure 1 represents the slip law schematically. When the applied pressure gradient is small enough, the wall shear stress ($\tau_w$) lies below the sliding yield stress (panel (a)) and the slip velocity is zero. Since $\tau_s$ is always less than the material’s yield stress, then there is no flow in the channel. When the pressure gradient is increased beyond $(\Delta\hat{p}/\hat{L})_c$ where subscript ‘c’ indicates the critical value, the wall shear stress grows beyond $\tau_s$ and the fluid slides over the walls. If the wall shear stress is still less than $\tau_y$, then the material moves as a sliding unyielded plug with a constant velocity in the entire gap (see panel (b)). When the pressure gradient is increased, at some point, the wall shear stress exceeds the fluid yield stress and the sheared/yielded regions appear, which at the same time slide over the walls; see panel (c). Yet, in the vicinity of the centreline, the shear stress drops below the yield stress and a core unyielded region is formed. The flows corresponding to panels (b) and (c) are usually termed the fully plugged regime and the deformation regime, respectively.

In the present study, the objective is to systematically address the effect of slip on yield-stress fluid flows and its consequences on the yield limit; not only by conducting numerical simulations, but also by forming a theoretical framework for analysing this type of problems. Although some attempts have been made to solve simple rheometric flows in the previous years (Philippou et al. 2016; Panaseti & Georgiou 2017; Damianou et al. 2019), still the lack of generic numerical implementation frameworks (especially non-regularised methods) and the absence of adequate theoretical frameworks to attack sliding yield-stress fluid flows are discernible. Hence, in what follows, we aim to partially fill this gap in the literature. Beyond that, we study complex sliding flows such as particle sedimentation and pressure-driven flows in porous media within these frameworks.

The outline of the paper is as follows. In §2, we set out the slip law beyond 1D for the yield-stress fluid flows and generalise the numerical algorithm previously proposed for simple 1D problems. We then benchmark the numerical algorithm for simple channel Poiseuille flow in §3 and derive some theoretical tools for investigating the sliding flows. This will be followed by addressing the sliding flow about a circular particle and the aspects of the particle yield limit in §4. The next two sections are devoted to the sliding flows in porous media: in §5 some general hints are made by studying the flow through model porous media and in §6 some deeper investigations are carried out by performing numerical simulations in randomized porous media. Finally, some conclusions are drawn in the closing section §7.
2. Slip law and the numerical algorithm beyond 1D

In this section, we set out and formulate a sample problem of sliding flow of a yield-stress fluid and generalise the slip law and numerical algorithm for solving such problems beyond 1D.

**Problem P:** we consider the Stokes flow of a Bingham fluid in $\Omega \setminus X$ (see figure 2):

\[
- \nabla \hat{p} + \nabla \cdot \hat{\tau} + \hat{\rho} \hat{f} = 0,
\]

\[
\begin{cases}
\hat{\tau} = \left( \hat{\mu} + \frac{\hat{\tau}_y}{\|\hat{\gamma}\|} \right) \hat{\gamma} & \text{iff } \|\hat{\tau}\| > \hat{\tau}_y, \\
\hat{\gamma} = 0 & \text{iff } \|\hat{\tau}\| \leq \hat{\tau}_y,
\end{cases}
\]

where $\hat{p}$ is the pressure, $\hat{\tau}$ the deviatoric stress tensor, $\hat{\rho} \hat{f}$ the body force, $\hat{\rho}$ the fluid density, and $\hat{\gamma}$ is the rate of deformation tensor. Without loss of generality, we consider no-slip & no-penetration boundary condition on $\partial \Omega$ as $\hat{u} = \hat{u}_0$ and the slip boundary condition,

\[
\hat{u}_s = \hat{u}_{ns} \cdot t - \delta \hat{u} \cdot t = \begin{cases}
\hat{\beta}_s \hat{A} \left( 1 - \frac{\hat{\tau}_s}{|\hat{A}|} \right), & \text{iff } |\hat{A}| > \hat{\tau}_s, \\
0, & \text{iff } |\hat{A}| \leq \hat{\tau}_s,
\end{cases}
\]

on $\partial X$, where $\hat{u}_s$ is the slip velocity, $\hat{u}_{ns}$ the velocity of the solid boundary $\partial X$, and $\delta \hat{u}$ is the restriction of $\hat{u}$ on $\partial X$: $\hat{u} \to \delta \hat{u}$ as $\hat{x} \to \partial X$. The value of the tangential traction vector (i.e. tangential force per unit area on the solid surface) is shown by $\hat{A} = [(\hat{\rho} \hat{1} + \hat{\tau}) \cdot \hat{n}] \cdot \hat{t}$, where the normal and tangential unit vectors to the solid surface $\partial X$ are represented by $\hat{n}$ and $\hat{t}$, respectively. The no-penetration condition on $\partial X$ reads $\delta \hat{u} \cdot \hat{n} = \hat{u}_{ns} \cdot \hat{n}$.

Without loss of generality, we fixed $k$ at unity in the slip law (2.3) and will do so in the rest of the present study. This is supported by experimental observations as well: Seth et al. (2012) validated the slip power index of unity for emulsions contacting non-adhering surfaces. Moreover, in separate experimental studies, Poumaere et al. (2014) and Daneshi et al. (2019) reported $k \approx 1$ for the pressure-driven flows of Carbopol gels with different concentrations and stirring rates during the sample preparation.
2.1. Numerical algorithm

The minimisation principle for the sliding problem $\mathcal{P}$ can be derived as:

$$
\mathcal{J}(\hat{\mathbf{u}}) = \frac{\hat{\mu}}{2} \int_{\Omega \setminus \bar{X}} \hat{\gamma}(\hat{\mathbf{u}}) \cdot \hat{\mathbf{v}} \, dV + \hat{\tau}_y \int_{\Omega \setminus \bar{X}} \| \hat{\gamma}(\hat{\mathbf{u}}) \| \, dV - \int_{\Omega \setminus \bar{X}} \hat{\sigma} \cdot \hat{\mathbf{u}} \, dV 
+ \frac{1}{2\beta_s} \int_{\partial X} \hat{u}_s^2 \, dA + \hat{\tau}_s \int_{\partial X} |u_s| \, dA.
$$

(2.4)

For the augmented Lagrangian approach we require to relax the rate of strain tensor to the auxiliary tensor $\hat{\mathbf{q}}$ and the restriction of the velocity vector on $\partial X$ to the auxiliary vector $\hat{\mathbf{q}}$. Then the saddle-point problem associated with $\mathcal{P}$ is:

$$
\mathcal{J}(\hat{\mathbf{u}}, \hat{\mathbf{q}}, \delta \hat{\mathbf{\xi}}) = \frac{\hat{\mu}}{2} \int_{\Omega \setminus \bar{X}} \hat{\gamma}(\hat{\mathbf{u}}) \cdot \hat{\mathbf{q}} \, dV + \hat{\tau}_y \int_{\Omega \setminus \bar{X}} \| \hat{\gamma}(\hat{\mathbf{u}}) \| \, dV + \frac{1}{2} \int_{\Omega \setminus \bar{X}} \left[ \hat{\gamma}(\hat{\mathbf{u}}) - \hat{\mathbf{q}} \right] : \hat{\tau} \, dV 
- \int_{\Omega \setminus \bar{X}} \hat{\sigma} \cdot \hat{\mathbf{u}} \, dV + \frac{1}{2\beta_s} \int_{\partial X} \hat{\xi}_s^2 \, dA + \hat{\tau}_s \int_{\partial X} |\hat{\mathbf{\xi}}| \, dA + \int_{\partial X} \hat{\lambda} \cdot (\hat{\mathbf{\xi}} - \delta \hat{\mathbf{\xi}}) \, dA
+ \frac{a \mu}{2} \int_{\Omega \setminus \bar{X}} (\hat{\gamma}(\hat{\mathbf{u}}) - \hat{\mathbf{q}}) : (\hat{\gamma}(\hat{\mathbf{u}}) - \hat{\mathbf{q}}) \, dV + \frac{b}{2\beta_s} \int_{\partial X} (\hat{\mathbf{\xi}} - \delta \hat{\mathbf{\xi}})^2 \, dA,
$$

(2.5)

where $a$ and $b$ are arbitrary constants (augmentation parameters); $\hat{T}$ and $\hat{\lambda}$ are the Lagrange multipliers.

Hence, the Uzawa algorithm in the presence of slip takes the form of algorithm [1]. Upon convergence of the algorithm [1] with the free augmentation parameters $a$ and $b$, the Lagrange multiplier $\hat{T}$ converges to the true stress field, $\hat{q}$ to the true rate of strain tensor, Lagrange multiplier $\hat{\lambda}$ to the traction vector on $\partial X$, and auxiliary variable $\hat{\mathbf{\xi}}$ to the velocity on $\partial X$.

A mesh adaptation procedure, the same as the one proposed by [Roquet & Saramito 2003], could be coupled with the above algorithm to obtain a fine resolution of the yield surfaces. Based on this procedure, the adapted mesh is stretched anisotropically in the direction of the eigenvectors of the Hessian of $\sqrt{\mu \hat{\gamma} : \hat{\gamma} + \tau_y \| \hat{\gamma} \|}$, which is the square root of the local energy dissipation. In this study, we implement the entire numerical algorithm in an open-source C++ finite element environment—FreeFEM++ ([Hecht 2012]). We have previously validated our numerical implementation and mesh adaptation widely within various studies ([Chaparian & Frigaard 2017b] [Chaparian & Tammissola 2019] [Chaparian et al. 2020]). In what follows, we quickly validate the algorithm [1] with the extra steps due to the sliding flow.

3. Benchmark problem: sliding channel Poiseuille flow

In this section we consider the sliding flow of a Bingham fluid in a channel (Poiseuille flow); the same as the one shown in figure [1]. The walls are denoted by $\Gamma$ and the full domain by $\Omega$. We consider two classic formulations: (i) [M]obility problem where the pressure gradient is applied and the flow rate can be computed and (ii) [R]esistance problem in which flow rate is set and as a result pressure gradient can be computed. We also validate the presented numerical algorithm and derive some variational tools useful for the rest of this study.

3.1. [M] problem

In this subsection, we use $\hat{G}\hat{H}^2/\hat{\mu}$ as the velocity scale, $\hat{G}\hat{H}$ as the characteristic viscous stress to scale the deviatoric stress tensor and $\hat{H}/\hat{\mu}$ to scale the slip coefficient where $\hat{G}$
Algorithm 1

1: procedure (solving yield-stress fluid flow with slip yield stress boundary condition)
2: \( n \leftarrow 0 \)
3: \( \hat{q}^0, \hat{T}^0, \lambda^0, \xi^0 \leftarrow 0 \) (or any other initial guess)
4: loop (Uzawa algorithm):
5: \[\text{if residual} < \text{convergence then close.}\]
6: find \( \hat{u}^{n+1} \) and \( \hat{p}^{n+1} \) which satisfy,
\[
\begin{cases}
- a \hat{\mu} \Delta \hat{u}^{n+1} = -\nabla \hat{p}^{n+1} + \nabla \cdot \left( \hat{T}^n - a \hat{\mu} \hat{q}^n \right) + \hat{\rho} \hat{f}, & \text{in } \Omega \setminus \bar{X},
\end{cases}
\]
\[
a \hat{\mu} \hat{\gamma}(\hat{u}^{n+1} \cdot n + a \delta \hat{u}^{n+1} = a \hat{\mu} \hat{q}^n \cdot n + a \hat{\xi}^n + \left[ \lambda^n - \left( -\hat{p}^{n+1} + \hat{T}^n \right) \cdot n \right] \text{ on } \partial X,
\]
\[
\nabla \cdot \hat{u}^{n+1} = 0, \text{ in } \Omega \setminus \bar{X},
\]
with given B.C.: \( \hat{u}^{n+1} = \hat{u}_0 \) on \( \partial \Omega \).
7: \( \hat{q}^{n+1} \leftarrow \begin{cases}
0, & \text{iff } \| \hat{\Sigma} \| \leq \hat{\tau}_y,
\left( 1 - \frac{\hat{\tau}_y}{\| \hat{\Sigma} \|} \right) \frac{\hat{\Sigma}}{(1 + a)\hat{\mu}}, & \text{iff } \| \hat{\Sigma} \| > \hat{\tau}_y.
\end{cases}\)
where \( \hat{\Sigma} = \hat{T}^n + a \hat{\mu} \hat{\gamma}(\hat{u}^{n+1}) \).
8: \( \hat{\xi}^{n+1} \leftarrow \begin{cases}
(\hat{u}_{ns} \cdot n) n + (\hat{u}_{ns} \cdot t) t, & \text{iff } |\hat{\Phi}| \leq \hat{\tau}_s,
(\hat{u}_{ns} \cdot n) n + (\hat{u}_{ns} \cdot t) t + \frac{\hat{\beta}_s}{1 + b} \hat{\Phi} \left( -\hat{\Phi} \hat{\tau}_s \right) t, & \text{iff } |\hat{\Phi}| > \hat{\tau}_s.
\end{cases}\)
where \( \hat{\Phi} = -\hat{\lambda}^n \cdot t + \frac{b}{\hat{\beta}_s} \left( \delta \hat{u}^{n+1} \cdot t - \hat{u}_{ns} \cdot t \right) \).
9: \( \hat{T}^{n+1} \leftarrow \hat{T}^n + a \hat{\mu} \left[ \hat{\gamma}(\hat{u}^{n+1}) - \hat{q}^{n+1} \right] \)
10: \( \hat{\lambda}^{n+1} \leftarrow \hat{\lambda}^n - \frac{b}{\hat{\beta}_s} \left[ \delta \hat{u}^{n+1} - \hat{\xi}^{n+1} \right] \)
11: residual \( \leftarrow \)
\[
\max \left( \int_{\Omega \setminus \bar{X}} |\hat{u}^{n+1} - \hat{u}^n| \, d\hat{A}, \int_{\Omega \setminus \bar{X}} |\hat{\gamma}(\hat{u}^{n+1}) - \hat{q}^{n+1}| \, d\hat{A}, \int_{\Omega \setminus \bar{X}} |\hat{\xi}^{n+1} - \hat{\xi}^n| \, d\hat{S} \right)
\]
12: \( n \leftarrow n + 1 \)
13: goto loop.

is the absolute value of the applied pressure-gradient to drive the flow from left to right (in the positive direction of the \( x \)-axis). Hence, the non-dimensional governing and constitutive equations take the forms:

\[ e_x + \nabla \cdot \tau = 0 \]  \hspace{1cm} (3.1)
and,

\[
\tau = \begin{cases} 
1 + \frac{\text{Od}}{\|\gamma\|} \hat{\gamma} & \text{iff } \|\tau\| > \text{Od}, \\
\hat{\gamma} = 0 & \text{iff } \|\tau\| \leq \text{Od},
\end{cases}
\tag{3.2}
\]

respectively, and the slip law,

\[
u_s = \begin{cases} 
\beta_s A \left(1 - \frac{\text{Od}_s}{|A|}\right) & \text{iff } |A| > \text{Od}_s, \\
0 & \text{iff } |A| \leq \text{Od}_s,
\end{cases}
\tag{3.3}
\]

where \(\text{Od} = \frac{\hat{\tau}_v}{H\dot{G}}\) and \(\text{Od}_s = \frac{\hat{s}_v}{H\dot{G}}\). In what follows, the ratio \(\frac{\hat{\tau}_v}{\hat{s}_v}\) is denoted by \(\alpha_s\). The analytical solutions to the velocity and stress fields are derived in Appendix A.1.

The energy balance equation in the presence of slip is:

\[
a(\mathbf{u}, \mathbf{u}) + \text{Od} j(\mathbf{u}) + \int_{\Gamma} |\sigma_{nt} u_s| \, dS = \]

\[
a(\mathbf{u}, \mathbf{u}) + \text{Od} j(\mathbf{u}) + \frac{1}{\beta_s} \int_{\Gamma} u_s^2 \, dS + \text{Od}_s \int_{\Gamma} |u_s| \, dS = \]

\[
a(\mathbf{u}, \mathbf{u}) + \text{Od} j(\mathbf{u}) + a_s(u_s, u_s) + \text{Od}_s j_s(u_s) = \int_{\Omega} \mathbf{u} \cdot \mathbf{e}_x \, dA, \tag{3.4}
\]

where \(a(\mathbf{u}, \mathbf{u}) = \int_{\Omega} \hat{\gamma}(\mathbf{u}) \, dA\) and \(\text{Od} j(\mathbf{u}) = \text{Od} \int_{\Omega} \|\hat{\gamma}(\mathbf{u})\| \, dA\) are the viscous and plastic dissipations, respectively. Please note that the slip dissipation \(\int_{\partial\Omega} |\sigma_{nt} u_s| \, dS\) can be split into two terms: the ‘viscous’ slip dissipation \((1/\beta_s) \int_{\Gamma} u_s^2 \, dS\) which is designated by \(a_s(u_s, u_s)\) in this study and the ‘plastic’ slip dissipation \(\text{Od}_s \int_{\Gamma} |u_s| \, dS\) with \(\text{Od}_s j_s(u_s)\). Moreover, \(j(\mathbf{u})\) is the normalized plastic dissipation and \(j_s(u_s)\) the normalized ‘plastic’ slip dissipation.

We can rearrange the energy balance equation and form a set of inequalities:

\[
0 \leq a(\mathbf{u}, \mathbf{u}) = -j(\mathbf{u}) \left( \text{Od} - \frac{\int_{\Omega} \mathbf{u} \cdot \mathbf{e}_x \, dA - a_s(u_s, u_s) - \text{Od}_s j_s(u_s)}{j(\mathbf{u})} \right) \leq \\
-j(\mathbf{u}) \left( \text{Od} - \frac{\int_{\Omega} \mathbf{u} \cdot \mathbf{e}_x \, dA - \text{Od}_s j_s(u_s)}{j(\mathbf{u})} \right) \leq \\
-j(\mathbf{u}) \left( \text{Od} - \sup_{\mathbf{v} \in \mathbf{V}, \mathbf{v} \neq 0} \frac{\int_{\Omega} \mathbf{v} \cdot \mathbf{e}_x \, dA - \text{Od}_s j_s(\mathbf{v})}{j(\mathbf{v})} \right), \tag{3.5}
\]

to find the definition of the critical Oldroyd number in the presence of slip as,

\[
\text{Od}_c = \sup_{\mathbf{v} \in \mathbf{V}, \mathbf{v} \neq 0} \left( \frac{\int_{\Omega} \mathbf{v} \cdot \mathbf{e}_x \, dA - \text{Od}_s j_s(\mathbf{v})}{j(\mathbf{v})} \right), \tag{3.6}
\]

where \(\mathbf{v}\) is a velocity test function from the set of all admissible velocity fields \(\mathbf{V}\). Then it enforces that for \(\text{Od}_c \leq \text{Od}\), \(a(\mathbf{u}, \mathbf{u}) = 0\) or \(\mathbf{u} = 0\) (i.e. no flow condition).

We may use \(3.6\) for calculating \(\text{Od}_c\) in simple flows such as Poiseuille flow here: the
Figure 3. [M] problem features: (a) velocity profile of the case $Od = 1, \alpha_s = 0.2, \beta_s = 0.1$ (fully sliding plug), (b) velocity profile of the case $Od = 0.3, \alpha_s = 0.2, \beta_s = 0.1$ (deforming regime), (c) the mesh after 5 cycles of adaptation associated with panel (b), (d) Velocity contour associated with panel (b); the green lines show the yield surfaces, (e,f) zoom in of the cyan windows in the panels (b,c) respectively. Please note that in panels (a,b,e) the blue line shows the numerical data corresponding to the initial mesh, the red line corresponds to the adapted mesh (the mesh shown in the panel (c)), and the red circles are the analytical solution; see Appendix A.1.

Flow can be postulated by two boundary layers with thickness $\delta$ attached to the walls, through which the slip velocity $U_1$ is connected to the plug velocity $U_2$, then,

$$j(v) \approx 2\frac{U_2 - U_1}{\delta} \delta \ell = 2(U_2 - U_1)\ell \quad \text{and} \quad j_s(v_s) = 2U_1\ell,$$

where $\ell$ is the length of the chosen control volume. This suggests,

$$Od_c = \sup_{0 < U_1 \leq U_2, (U_1, U_2) \in \mathbb{R}} \frac{U_2 - 2Od_s U_1}{2(U_2 - U_1)} = \sup_{0 < U_1 \leq U_2, (U_1, U_2) \in \mathbb{R}} \frac{U_2 - 2\alpha_s Od_c U_1}{2(U_2 - U_1)},$$

or,

$$Od_c = \sup_{0 < U_1 \leq U_2, (U_1, U_2) \in \mathbb{R}} \frac{1}{2[1 + (\alpha_s - 1)U_1/U_2]}.$$ (3.7)

The maximal value of the argument is $1/2\alpha_s$ which occurs at $U_1/U_2 = 1$. Its physical interpretation is that the supremum occurs in the fully sliding regime which is intuitive. Hence, $Od_c = 1/2\alpha_s$ or in the other words, there is no flow if $1/2 \leq Od_s$, which is in agreement with the solution of the full equations derived in §A.1.

We perform a few simulations for the [M] problem to benchmark the presented algorithm. Figure 3 illustrates different regimes: panel (a) compares the computed velocity profile (blue line) in the gap with the analytical solution (red circles) in the fully sliding plug regime ($Od = 1, \alpha_s = 0.2 \& \beta_s = 0.1$) and panel (b) within the deforming regime. Utilizing the mesh adaptation seems more indispensable in the deforming regime compared to the fully sliding plug regime where velocity gradient is zero. It is clear that the velocity distribution after adaptation (red line) is much closer to the analytical solution (red circles) than the solution of the initial mesh (blue line), and also yields a fine resolution of the yield surfaces; see panel (d). We shall mention that the focus of the present study is not on quantifying these improvements by the mesh adaptation as it has been previously studied in details [Roquet & Saramito 2003 2008 Chaparian & Tammisola 2019].
3.2. [R] problem

Here, we reformulate the problem in the resistance scalings and the quantities are designated with asterisk sign (*) to prevent any potential confusion with the [M] problem. In the resistance formulation, the velocity is scaled with the mean bulk velocity, \( \hat{U} = \hat{Q}/\hat{H} \), and as a result, there is always a non-zero flow with the flow rate equals to unity (\( Q^* = 1 \)), whether the flow is in the deforming regime or the fully sliding plug regime. The deviatoric stress and pressure are scaled with the characteristic viscous stress, \( \hat{\mu} \). The absolute value of the pressure gradient versus the Bingham number under the no-slip condition and the sliding flow are shown with the continuous black and red lines, respectively. While for both type of flows an increase in \( G^* \) by increasing Bingham number can be observed, they display different trends. This can be explained by knowing that beyond a critical Bingham
number \((\bar{B} = 1/\beta_s (1 - \alpha_s) \leq B)\), the sliding flow state switches from the deforming regime to the fully-sliding plug regime. Only at large enough Bingham numbers \((B \to \infty)\), we can use the numerical data to extract the yield limit. This is illustrated in figure 4 by the dotted lines, where at \(B \to \infty\), \(G^*\) asymptotes to \(2B\) and \(2\alpha_s B = 0.4B\) for no-slip and the sample sliding flow considered, respectively. In this limit, \(j(\mathbf{u}^*)\) (dashed lines in figure 4) asymptotes to 2 and 0 for the no-slip and sliding flows: this is equivalent to \(Od_c = 1/2\) and \(1/2\alpha_s\), respectively, knowing that \(j_s(u_s)\) asymptotes to 2 for the sliding flows at sufficiently large Bingham numbers (follow the red dashed-dotted line in figure 3). These results are the same as the ones derived in Appendix A.1 considering the \([M]\) formulation. Please note that due to the log-log scaling used in figure 4, \(j(\mathbf{u}^*)\) corresponding to the sliding flow, which is exactly equal to zero for \(\bar{B} \leq B\) is not visible; however, its fast decay is clear: \(j(\mathbf{u}^*)\) is almost equal to \(10^{-2}\) for \(B \approx 11.85\) where \(\bar{B}(\alpha_s = 0.2 \& \beta_s = 0.1) = 12.5\). From a different perspective, figure 4 inset shows how mesh adaptation helps us to achieve more accurate results.

4. Slippery particle motion

Particle motion in yield-stress fluids has been considered in numerous studies ranging from numerical simulations \((\text{Beres et al. 1985} \quad \text{Tokpavi et al. 2008} \quad \text{Putz & Frigaard 2010} \quad \text{Nirmalkar et al. 2012} \quad \text{Chaparian & Frigaard 2017} \quad \text{Fraggedakis et al. 2016} \quad \text{Izbassarov et al. 2018})\) to experimental validations/extensions \((\text{Jossic & Magnin 2001} \quad \text{Putz et al. 2008} \quad \text{Tokpavi et al. 2009})\). Unfortunately, less studies are devoted to investigate the effect of slip on the flow field around the particle surface. Although Fraggedakis et al. (2016) used Navier-slip law in their simulations, the emphasis of their study was on the effect of elasticity of the yield-stress fluid in the particle sedimentation problem. On the experimental side, Jossic & Magnin (2001) reported the drag coefficient of the particle...
moving in a Carbopol gel and showed that for particles with smooth surfaces, the drag coefficient is smaller compared to the rough particles that prevent slip which is intuitive. However, the mentioned authors did not quantify the slip velocity/law on the particle surface and rather qualitatively addressed the effect of slip.

In this section, we will shed light on the effect of slippery motion of the yield-stress fluid about a circular 2D particle and in details will address the yield limit under this condition. The horizontal symmetry axis is aligned with the $x$–axis (positive direction to the right) and the vertical one by $y$–axis (positive direction upward) where the origin of the coordinate system is fixed at the particle centre. The fluid flow about the particle ($X$) with its radius $R$ as the length scale and its velocity $V_p^*$ as the velocity scale—[$R$] formulation—is governed by:

$$- \nabla p^* + \nabla \cdot \tau^* = 0 \text{ in } \Omega \setminus X \quad \& \quad u_{ns}^* = -e_y \text{ on } \partial X$$

and the slip law is the same as the expression (3.10). Hence, the drag force on the particle can be computed from,

$$a(u^*, u^*) + Bj(u^*) + a_s(u_s^*, u_s^*) + B_s j_s(u_s^*) = \int_{\partial X} (\sigma^* \cdot n) \cdot e_y \, dS = F_D^*$$

which can be converted to ‘plastic drag coefficient’ using,

$$C_{D,c}^p = \frac{F_D^*}{\ell_\perp B}$$

where $\ell_\perp$ is the width of the particle perpendicular to the flow direction; $\ell_\perp = 2$. Then the mapping between [R] and [M] problems reduces to $Y = \pi B/F_D^*$ or $B = Y/V_p^*$, where $V_p$ is the particle sedimentation velocity in the [M] problem which is scaled with $\hat{V}_p = \Delta \hat{\rho} \hat{g} \hat{R}^2/\hat{\mu}$; the velocity scale obtained via balancing the buoyancy stress with the characteristic viscous stress. Please note that here $\hat{g}$ is the gravitational acceleration and $\Delta \hat{\rho}$ is the density difference between the particle and the suspending fluid in the [M] problem. Then, the critical plastic drag coefficient can be converted easily to the critical yield number,

$$C_{D,c}^p = \left[ C_{D,c}^p \right]_{[R]} \xrightarrow{B \to \infty} \left[ C_{D,c}^p \right]_{[M]} \xrightarrow{Y \to Y_c^-} = \frac{\pi}{\ell_\perp Y_c}.$$ 

For more details please see Chaparian & Frigaard (2017b). For instance, for a 2D circle, as has been extensively validated (Tokpavi et al. 2008; Putz & Frigaard 2010; Chaparian & Frigaard 2017a, b), the critical plastic drag coefficient under the no-slip condition is 11.94 which yields to $Y_c = 0.1316$.

Regarding the computational details, we conduct the same strategy as in our previous studies (Chaparian & Frigaard 2017a, b; Iglesias et al. 2020). The computational box ($\Omega$) is chosen large enough to ensure independency from far-field conditions (Chaparian et al. 2018). The boundary condition $u^* = 0$ is enforced on $\partial \Omega$.

The yield limit of particle motion in a quiescent yield-stress fluid is directly relevant to the lateral resistance of an object (e.g. a pile) in a perfectly-plastic medium (e.g. soil) which can be investigated by method of characteristics (i.e. slipline solution) due to the hyperbolic nature of the equations. We briefly revisit this problem in the next subsection.

4.1. Revisit slipline solution and lower & upper bound calculations

In a deformed perfectly-plastic medium, the second invariant of the stress is equal to $B$ everywhere. The equilibrium equations for the unrestricted 2D plastic flow then form a closed set of hyperbolic equations for which there are two family of orthogonal
characteristic lines: $\alpha -$ and $\beta -$family (Hill 1950, Chakrabarty 2012). Physically, these lines (termed as sliplines) show the direction of the maximum shear stress which is equal to $B$. Properties of these lines facilitate studying the ‘yield limit’ in various viscoplastic problems (Chaparian & Frigaard 2017a, b; Dubash et al. 2009, Chaparian & Nasouri 2018, Hewitt & Balmforth 2018). However, finding the sliplines itself is not trivial in some complex problems. Indeed, the sliplines assist us in postulating admissible stress and velocity fields which yield to finding the lower and upper bounds, respectively, of the load limit.

Initially investigated by Randolph & Houlsby (1984), a slipline solution was devised for calculating the lateral resistance of a circular pile in soil (considered as a perfectly-plastic material), taking into account the effect of soil adhesion at the pile-soil interface, $\tilde{\alpha}_s$. In other words, the tangential shear stress at the pile-soil interface is assumed to be equal to $\tilde{\alpha}_s B$. Although widely believed that the Randolph & Houlsby (1984) solution is exact (lower and upper bounds of the load/drag are the same for the whole range of $\tilde{\alpha}_s$), an issue was firstly detected by Murff et al. (1989): for $\tilde{\alpha}_s < 1$, there is a region in the vicinity of the pile surfaces in which the rate of strain is negative and its absolute value should be taken into account in calculating the upper bound to avoid negative plastic dissipation. If one do so, then there is a discrepancy between the lower and upper bounds. Later, Martin & Randolph (2006) proposed two new postulated velocity fields which improved the upper bound predictions to some extent; one for small ($\tilde{\alpha}_s \rightarrow 0$) and the other one for large ($\tilde{\alpha}_s \rightarrow 1$) soil adhesion factors. By combining these two mechanisms, Martin & Randolph (2006) proposed an alternative mechanism which markedly shrinks the uncertainty between the lower and upper bounds predictions for the entire range of $\tilde{\alpha}_s$. We will quickly review the lower (Randolph & Houlsby 1984) and upper (Martin & Randolph 2006) bound solutions here; however, for more details readers are refereed to the original references.

The first family of sliplines in this problem are straight lines that make $\pi/4 - \psi_2$ angle with the pile surface (say $\alpha -$lines) where $\psi = \sin^{-1} \tilde{\alpha}_s$. Figure 5 represents these lines in cyan colour. The initial $\alpha -$line is $AB$ which makes $\pi/4$ angle with the vertical symmetry line since $\tilde{\tau}_{xy}$ is zero on $OB$. Hence, the region enclosed between $AB$ and the pile surface is the plug region shown in light gray. The $\alpha -$lines from $AB$ to $CD$ can be found easily as discussed above; the corresponding $\beta -$lines are curved lines which are the involutes unwrapped from an imaginary co-centre circle—the evolute—with radius $\eta = \cos \left(\frac{1}{2} \cos^{-1} \tilde{\alpha}_s\right)$ shown with a dashed white line. Please note that indeed $\alpha -$lines can be introduced as tangents of this imaginary circle as well. From $CD$ to $CE$, $\alpha -$lines are spokes of a fan centred at $C$; hence, the $\beta -$lines in this part are arcs of co-centred circles at point $C$.

Hencky’s equations (Hill 1950),

$$\dot{p} + 2B\theta = \text{const.} \text{ along an } \alpha-\text{line} \quad \& \quad \dot{p} - 2B\theta = \text{const.} \text{ along a } \beta-\text{line} \quad (4.5)$$

provide the tool for finding the admissible stress field associated with these sliplines which can be used for calculating the lower bound of the plastic drag coefficient. Here, $\theta$ is the counterclockwise orientation of the $\alpha -$lines made with the $x$-axis (aligned with $OE$). The individual stress contributions can be written as,

$$\tilde{\sigma}_x = -\dot{p} - B \sin(2\theta), \quad \tilde{\sigma}_y = -\dot{p} + B \sin(2\theta), \quad \tilde{\tau}_{xy} = B \cos(2\theta) \quad (4.6)$$

in the new curvilinear coordinates $\alpha$ and $\beta$. Then the lower bound can be calculated as,

$$\left[ C_{D,c}^{\rho} \right]_L = \frac{\tilde{F}}{\ell_B B} = \frac{\tilde{F}}{2B} = \int_{\partial X} \tilde{\sigma} \cdot n \, dS \quad (4.7)$$
which basically consists of four individual contributions in the shown quadrant: shear force acting on $AB$, $B \sin(\psi/2)$; shear force acting on $AC$, $B \cos(\psi/2)$; normal force acting on $AB$, $B \left[ c + \frac{3\pi}{2} \right] \sin\left(\frac{\psi}{2}\right)$; and normal force acting on $AC$,

$$B \left[ c \left( 1 - \sin\left(\frac{\psi}{2}\right) \right) + \left[ 2 \cos\left(\frac{\psi}{2}\right) + \frac{\pi}{2} + \psi + \cos \psi - \left( \frac{3\pi}{2} + \cos \psi \right) \sin\left(\frac{\psi}{2}\right) \right] \right],$$

where $cB$ is the reference mean stress, $(\tilde{\sigma}_x + \tilde{\sigma}_y)/2$, on $CE$ (position of the centre of the Mohr’s circle). Please note that $c$ will be eliminated if all four quadrants are taken into account.

Hence, the total lower bound of the plastic drag coefficient will be,

$$\left[ C_{D,c}^p \right]_L = \pi + 2\psi + 2 \cos \psi + 4 \left( \cos \frac{\psi}{2} + \sin \frac{\psi}{2} \right). \quad (4.8)$$

The upper bound calculation can be performed by postulating admissible velocity fields. Geiringer equations (Hill 1950) make it feasible via slipline solution,

$$d\tilde{u}_\alpha - \tilde{u}_\beta d\theta = 0 \text{ along an } \alpha-\text{line} \& \ d\tilde{u}_\beta + \tilde{u}_\alpha d\theta = 0 \text{ along a } \beta-\text{line}, \quad (4.9)$$

where $\tilde{u}_\alpha$ and $\tilde{u}_\beta$ are the velocities in the directions $\alpha$ and $\beta$ in the new curvilinear coordinate system. Based on these quations, Randolph & Houlsby (1984) proposed a mechanism (see figure 6(a)) in which velocity along $\alpha-$lines is zero and along each $\beta-$line is a constant which can be found by considering no-penetration condition on the pile or fore-aft plugs surface. As mentioned above, this mechanism leads to a relatively large uncertainty with the lower bound solution for $\tilde{\alpha}_s \neq 1$. Martin & Randolph (2006) improved this mechanism by substituting a rotating rigid block (blue region in figure 6(b)) in the middle of the domain with angular velocity $\omega$ compatible with the no-penetration boundary condition on the pile surface; the centre of that is determined by the angle $\gamma$ and the imaginary evolute circle of radius $\eta$. By optimizing for $\gamma$, one can markedly reduce the uncertainty between the lower and upper bounds of $C_{D,c}^p$.

It is worth mentioning that although $\tilde{\alpha}_s$ and $\alpha_s$ have not exactly the same physical origins, in what follows, we show that both display the same effect in the current problem and can be used interchangeably. Nevertheless, we should note that, $\alpha_s B$ in the viscoplastic fluid mechanics context not only marks the boundary between the presence of sliding or no-slip flow on the solid surface, but actually also controls the yield limit (see figure 4): for instance in the case of $\alpha_s = 1$, the limit of flow/no-flow reduces to having stress as large as $B$ on the solid surfaces. For sure, when there is a flow, it slides over solid boundaries depending on $\beta_s$ in this case. Yet, the case of $\alpha_s = 1$ shares the same
Figure 6. Schematic of the upper-bound mechanisms: (a) associated with the sliplines (Randolph & Houlsby 1984), (b) combined mechanism (Martin & Randolph 2006) with the blue region rotating as an unyielded block. Black arrows show velocity vectors. For a better representation, in this schematic the pile velocity is assumed to be $e_y$. Nevertheless, it should be noted that the flow has fore-aft symmetry.

yield limit characteristics with the no-slip condition. In the perfectly-plastic mechanics context, $\hat{\alpha}_s B$ roughly has the same interpretation since if the tangential shear stress on the solid boundary is less than that, then there is no deformed regions. Indeed, this is the strict condition at the yield limit in a deformed perfectly-plastic solid.

4.2. Results

Figure 7(a) represents the plastic drag coefficient versus the Bingham number under different conditions. As shown by Putz & Frigaard (2010) and Chaparian & Frigaard (2017b), with the no-slip condition on a circular particle surface, $\lim_{B \to \infty} C^p_D = C^p_{D,c} \approx 1.94$ which is the value that can be obtained from the expression (4.8) as well when $\hat{\alpha}_s = 1$. Three main points can be drawn from figure 7(a) as follows:

(i) Plastic drag coefficients associated with the sliding flows over the particle surface are smaller compared to the same Bingham number flows with the no-slip condition. This seems intuitive and as mentioned has been validated experimentally by Jossic & Magnin (2001).

(ii) The plastic drag coefficient decreases by increasing Bingham number and reaches a plateau when $B \to \infty$. More interestingly, the critical plastic drag coefficient associated with the flows of the same $\alpha_s$ asymptote to a same limiting value (see asterisks and circles). Indeed, $C^p_{D,c}$ is only a function of $\alpha_s$. However, in the Newtonian limit, $B \to 0$, this is $\beta_s$ which controls $C^p_D$ (see stars and circles).

(iii) The lower-bound estimations from the slipline solutions accurately predicts $C^p_{D,c}$.

Figure 7(b) compares the lower- and upper-bounds of the critical plastic drag coefficient with the data obtained by the numerical computations. It clearly shows that even within the sliding flow context, perfectly-plastic theories could be useful in investigating the yield limit. It is worth mentioning that as investigated by Chaparian & Frigaard (2017b) in detail, although the computations and slipline analysis display same behaviours to some extent for the no-slip case, the envelope of the characteristics does not agree fully with the yield surfaces in the viscoplastic computations. Both characteristics solution and the viscoplastic computation predict rigid caps fore and aft of the particle (although they are not exactly the same size), whereas the viscoplastic solution also has kidney plug regions along the side of the cylinder. The velocity fields are not exactly equal either; see figure 4 of the mentioned reference. The perfectly-plastic velocity has discontinuities which are diffused in the viscoplastic field by the narrow viscous boundary layers. Given
Figure 7. (a) Plastic drag coefficient versus the Bingham number for different $\alpha_s$ and $\beta_s$. The continuous blue lines are lower-bound estimations for $\tilde{\alpha}_s = 0.2$, $0.5$, and $1$. (b) Critical plastic drag coefficient versus $\alpha_s$. The blue line is the lower-bound estimation and the red line shows the upper-bound estimation by the combined mechanism \cite{Martin2006} with the same symbol interpretation as panel (a).

Figure 8. Slip lines for the case (a) $\tilde{\alpha}_s = 0.2$ and (b) no-slip ($\tilde{\alpha}_s = 1$); $\alpha$–lines in blue and $\beta$–lines in red while the rigid caps stock to the particle surface represented in light gray. The other panels show the contour of $|\mathbf{u}^*|$ at $B = 10^5$: (c) $\alpha_s = 0.2$ & $\beta_s = 0.1$ and (d) no-slip flow. Green lines in panels (c,d) represent the yield surfaces.

The discrepancies in both stress and velocity fields for the no-slip case, it is perhaps interesting that slip line solutions still predicts the yield limit very well. Figure 8 reveals that these differences exist in the sliding flows as well. Panel (a) sketches the slip lines for the case $\tilde{\alpha}_s = 0.2$ and panel (c) shows the computed contours of velocity for the case $\alpha_s = 0.2$ & $\beta_s = 0.1$ at $B = 10^5$. Panels (b) and (d) make the comparison possible with the no-slip condition. The frontal caps are significantly smaller for $\alpha_s = 0.2$ while the side kidney are bigger and are in touch with the particle surface.

Figure 9 closely monitors these differences for the case $\alpha_s = 0.5$. Panel (a) shows the
Figure 9. Features of the flow about a particle $\alpha_s = 0.5$ & $\beta_s = 0.05$: (a) velocity contour ($|u^*|$) at $B = 10^5$; the green lines show the yield surfaces and blue discontinuous lines are yield surfaces predicted in combined upper-bound mechanism [Martin & Randolph 2006] with $\eta = 0.866$ & $\gamma = 20.8^\circ$ and cyan discontinuous lines mark the start and end spokes of the rigid zones. (b) velocity profile comparison in simulations with increasing Bingham number ($B = 10, 100, and 10^5$ from cyan to dark blue color, respectively) and upper-bound mechanisms [Martin & Randolph 2006] combined mechanism with continuous red line and Randolph & Houlsby (1984) mechanism with discontinuous dashed red line) over the horizontal symmetry line. Please note that particle surface velocity is $v^* = -1$.

contour of velocity in the viscoplastic problem with yield surfaces in continuous green, whereas ‘yield surfaces’ predicted by the combined upper-bound mechanism [Martin & Randolph 2006] are shown with discontinuous blue lines. Moreover, panel (b) compares velocity distributions along the horizontal symmetry line: the two red lines represent the ones associated with the upper-bound mechanisms by Martin & Randolph (2006) (continuous) and Randolph & Houlsby (1984) (discontinuous), while the computed velocities for different Bingham numbers are shown in blue colours; the darker the Bingham number higher. As $B$ increases, the velocity distributions get closer to the ones predicted by upper-bound mechanisms, especially the combined mechanism where even the slip velocity on the particle surface at the equator is predicted correctly. Nevertheless, again the discontinuities in the perfectly-plastic solution are replaced by the viscous boundary layers in the viscoplastic solutions.

5. Sliding flow in model porous media

Non-Newtonian fluid flows through porous media, especially yield-stress fluids, are of great practical importance for numerous industries such as filtration. Due to the complexities of these type of geometries and their randomness, it is expensive to conduct numerical simulations/experiments in the full scale. Hence, a common way is to model the medium as arrays of cylinders or other model geometries. For instance, Chevalier et al. (2013) studied the flow of a Herschel-Bulkley fluid through confined packings of glass beads experimentally and proposed a semi-empirical model for the pressure drop versus the flow rate. Chaparian et al. (2020) and De Vita et al. (2018), recently, studied the effect of elasticity of the yield-stress fluid in flows through model porous media. Another related topic is the flow along uneven channels: in a series of studies Roustaei et al. (Roustaei & Frigaard 2013; Roustaei et al. 2015; Roustaei & Frigaard 2015; Roustaei et al. 2016) investigated in details the yield-stress fluid flow inside fractures and washouts for a wide range of flow configurations. They showed that large-amplitude variations in
the duct walls lead to the formation of static unyielded zones (termed as fouling layers) adjacent to the walls. Lubrication approximation (which is based on a constant pressure gradient along the channel length), hence, is an inadequate approach to study yield-stress fluid flows along wavy channels.

It should be mentioned that all the above studies are limited to the no-slip condition. Here, in the continuation of the previous sections, we investigate the effect of slip on the pressure-driven yield-stress fluid flows in porous media.

In this section, we consider flows through two model geometries mimicking porous media; the same ones as studied by Chaparian et al. (2020). The schematic is represented in figure 10: both geometries are designed to have the same porosity $\phi = 0.38$ (i.e. $\hat{\ell}/\hat{R} = 0.25$ and 1.25 in panels (a) and (b), respectively). The obstacles radius, $\hat{R}$, is used as the length scale. We use the $[R]$ formulation in this section; hence, for all the cases of the same geometry, the flow rate is constant and the pressure drop is a function of the Bingham number, $\alpha_s$ and $\beta_s$. We consider an inertialess fully developed flow with the sliding boundary condition (3.10) on the solid topologies ($\partial X$). Fixed flow rate condition is enforced using a Lagrange multiplier and periodic boundary conditions are applied at the inlet and outlet. On the top and bottom sides of the computational cell (horizontal faces), a symmetry boundary condition is imposed. For more details about the computational method and its validations please see Chaparian et al. (2020).

Figure 11 compares different flows within a wide range of Bingham numbers. Dead zones (shaded in light grey) are smaller in the sliding flows and unyielded regions in the middle of the passage (yield surfaces in green) are mostly larger compared to the no-slip condition, which is intuitive. Interestingly, because of the slip, unyielded zones are prone to attach to the solid surfaces (e.g. see figure 11(p)).

Figure 12 reveals various features of the flow in the model porous media quantitatively. Panel (a) sketches $G^*$ with respect to the Bingham number; zoomed-in insets are provided in panels (b-e). In the range of small Bingham numbers—Newtonian limit—the pressure gradient is a function of $B$ and $\beta_s$ alone; see panels (b) and (d) in which the cyan and yellow curves ($\beta_s = 0.05$) converge together and also red and purple curves ($\beta_s = 0.1$) together. On the other hand, in the limit of large Bingham numbers (i.e. no flow or yield limit; panels (c) and (e)), the critical pressure gradient is a function of $B$ and $B_s$ or indeed $B$ and $\alpha_s$: yellow and red curves with $\alpha_s = 0.2$ are almost indistinguishable; the same for the purple and cyan curves with $\alpha_s = 0.8$. Moreover, as shown by Chaparian et al. (2020), in the yield limit ($B \to \infty$), the pressure gradient scales with $\sim B$; this is true for both the no-slip and sliding flows, yet with different pre-factors. Indeed, as
Figure 11. Velocity magnitude contour $|\mathbf{u^*}|$: Panels (a-d) and (i-l) show the no-slip flow in the regular and staggered geometries, respectively. Panels (e-h) and (m-p) illustrate the sliding flow ($\alpha_s = 0.2$, $\beta_s = 0.1$) in regular and staggered geometries, respectively. Columns from left to right are associated with $B = 1, 10, 100, \text{and } 1000$, respectively. The dead zones are shown in light grey and green lines represent the yield surfaces in the middle of the flow passage.

shown in §3.2 for the simple channel flow,

$$\lim_{B \to \infty} \frac{G^*(\text{no-slip})}{G^*(\alpha_s)} = \frac{1}{\alpha_s},$$

which is the case in the complex flows through model porous media as well.

Panel (f) confirms that the viscous dissipation, even in sliding flows, is at least one order of magnitude less than the plastic and slip dissipations. Interestingly, for sliding
flows, $a(u^*, u^*)$ converges to its limiting value when $B \to \infty$ and the limiting value is a function of $\alpha_s$ only. Whereas, in the no-slip flow, the viscous dissipation increases by increasing Bingham number but with a much smaller rate compared to the plastic dissipation. Panel (f) also shows that the leading order dissipation terms (i.e. $B_j(u^*)$ and $\int |\sigma_{nt} u^*_s| \, dS$) scales with $\sim B$ at the yield limit for sliding flows which is the main reason of yielding to the same scaling for $G^*$ at this limit; see expression (3.11). Panel (g) also establishes that the leading order term in the slip dissipation is the ‘plastic’ slip dissipation $B_s j_s(u^*_s)$.

Please note that although $j(u^*)$ of the no-slip flow is not shown in panel (h), the trend is the same as the sliding flows, but of course with larger values as it is clear from panel (f) where $B_j(u^*)$ is shown.

Under the no-slip condition, the asymptotic values of $j(u^*)$ at $B \to \infty$ are equal to 3.42 and 12.57 for regular and staggered geometries which yield to $O(d_c) \approx 0.1645$ and
Chaparian and Tammisola

\[
\begin{array}{|c|c|c|c|}
\hline
\text{no-slip} & 0.1645 & -- & 0.1645 \\
\text{\(\alpha_s = 0.8\)} & 0.1781 & 0.1553 & 0.1553 \\
\text{\(\alpha_s = 0.2\)} & 0.2581 & 0.2578 & 0.2578 \\
\hline
\end{array}
\]

Table 1. Values of \(O_d\) for regular and staggered geometries

\[\approx 0.2237, \text{ respectively, using expression } (3.12) \text{ in the absence of any slip dissipation. The critical Oldroyd number in the presence of slip is calculated using the same expression in Table 1. Please note that as an approximation, one may alternatively use,}
\]

\[
O_d = \lim_{B \to \infty} \frac{\int_{\Omega \setminus X} \mathbf{u}^* \cdot \mathbf{e}_x \, dA}{j_s(u^*) + \frac{1}{B} \int_{\partial X} |\sigma^*_{nt} u^*_s| \, dS} \approx \lim_{B \to \infty} \frac{\ell L}{(1 + \alpha_s) j(u^*)},
\]

in which \(j_s(u^*)\) is approximated by \(j(u^*)\) as is suggestive by panel (h), especially for small values of \(\alpha_s\) (i.e. more slippery flows). These values are reported in Table 1 as well. Negligible difference between \(O_d\) calculated using expressions (3.12) and (5.1) for \(\alpha_s = 0.2\) shows the validity of this approximation when the ‘sliding yield stress’ is much smaller than the material yield stress.

We shall mention that as it is clear from figure 12(c,e,h), \(O_d\) only depends on \(\alpha_s\) as reported in Table 1 (at least for \(0 \leq \beta_s \leq 0.1\)).

6. Towards more complex geometries

Although previous studies on the yield-stress fluid flows in the model porous media uncovered some generic interesting aspects of the problem, yet they were not capable of capturing the whole physics behind these type of complex flows. For instance, Talon & Bauer (2013) utilized the Lattice Boltzmann Method (LBM) to study the Bingham fluid flow in an stochastically reconstructed porous media. These authors showed that due to the yield stress, channelization can happen: the material will flow only in self-selected paths depending on the imposed driving pressure gradient; at the yield limit it is only one path and by increasing the pressure gradient gradually, other paths will open up. This has been supported by the work of Liu \textit{et al.} (2019) in model pore networks connected by straight tubes.

To capture this kind of fascinating and more realistic behaviours, in this section, we will complement the findings in §5 by studying the flow in random-designed porous media with a wide range of porosities. The computational domain is a box of dimensions 50 × 50 where the pores are 2D circles of radius unity. Again we use the [R] formulation in this section and the same numerical method as in the previous section. To have a better comparison, in addition to the random porous media, we simulate the flow again in the two introduced model porous media (figure 10); yet with the same porosity as the random cases: \(L = \sqrt{\frac{\pi}{1 - \phi}}\).

6.1. Baseline no-slip flow

Figure 13 illustrates the no-slip flow in the case \(\phi = 0.7\) for a wide range of Bingham numbers from unity to \(10^4\). Top panels show the velocity magnitude contours and the bottom row the log-scale of the second invariant of the rate of strain tensor. When the
Bingham number increases, the flow becomes more and more localized, and at the yield limit \((B \to \infty)\), the flow only passes through a single channel. Please note that the black level contours represent the quiescent parts. It is worth mentioning that due to the \([R]\) scaling, the flow rate in all the simulations is the same (for a fixed geometry regardless of the Bingham number, \(\alpha_s\) and \(\beta_s\)), hence, in panel (c) the whole fluid passes through the single open channel which results in extremely high velocities (compare the level of contours in the top panels). By looking at the panels from left to right sequentially, the channelization characteristic is well observable: by increasing the Bingham number (i.e. moving to the yield limit), the flow occurs in lesser pathes, until only one path remains at the yield limit.

Figure 14 shows the pressure drop against the Bingham number. The lines are computed data in the present study and the symbols are computational and experimental data by Bauer et al. (2019) for 3D flows. The blue lines are acquired from the randomized porous media and the red lines from the two model porous media for a wide range of porosities. To have a better comparison, we combined \(G^*\) from the regular and staggered models in the red lines by averaging. This figure demonstrates that the model porous media closely predicts the pressure drop in the randomized porous media, however, there are small discrepancies, more pronounced in the two limits of the problem—Newtonian and the yield limit. Yet, in the intermediate regime of the Bingham numbers, the predictions are more satisfactory. Table 2 compares the critical Oldroyd number under the no-slip condition computed in different configurations. As can be seen in figure 14, the model porous media slightly over predicts the pressure drop which results in smaller \(Od_c\) compared to the randomized geometries.

Another interesting point in figure 14 is that the 2D simulations in the present study

\[\text{Figure 13. } \phi = 0.7: \text{ panels (a-c) illustrate } |\mathbf{u}^*| \text{ and (d-f) contour of } \log_{10}(\|\dot{\gamma}^*\|) \text{ for flows with no-slip condition. Please note that the range of colourbars are different in velocity contours whereas it is the same in the bottom row. Columns from left to right are associated with } B = 1, 100 \text{ and } 10^4, \text{ respectively.}\]
Figure 14. Pressure drop with respect to the Bingham number under the no-slip condition. The red colour stands for simulations of the model porous media, blue colour the simulations of the random porous media, and the cyan colour experimental data. The lines represent our computations: the continuous line corresponds to \( \phi = 0.38 \), dashed line to \( \phi = 0.5 \), dashed-dotted line to \( \phi = 0.7 \), and the dotted line to \( \phi = 0.9 \). The symbols are data borrowed from Bauer et al. (2019): simulations for FCC packing of spheres (\( \Delta; \phi = 0.218 \)), simulations for random array of overlapping spheres (\( \triangledown; \phi = 0.429 \)), simulations (\( \langle \rangle \)) and experiments (\( \triangleright \)) of porous structure of a sandstone (\( \phi = 0.855 \)).

\[
\begin{array}{cccc}
\phi = 0.38 & \phi = 0.5 & \phi = 0.7 & \phi = 0.9 \\
\text{Randomized} & 0.3511 & 0.7678 & 2.2096 \\
\text{Regular geo.} & 0.1645 & 0.2804 & 0.6163 & 1.7996 \\
\text{Staggered geo.} & 0.2237 & 0.3308 & 0.6365 & 1.8001 \\
\end{array}
\]

Table 2. Comparison of \( Od_c \) (under no-slip condition) for different geometries

and 3D simulations by Bauer et al. (2019) are partially comparable for different porosities; specifically in the large Bingham number limit where a large portion of domain is filled with static/fouling regions. However, a closer comparison in the Newtonian limit is suggestive of different trends, which could be the consequence of different flow structures in 2D and 3D flows in this limit.

More sophisticated analyses/comparisons of the model and random porous media under the no-slip condition are presented in Appendix B as the main focus of the present study is on the sliding flows.

6.2. Effect of slip

In this subsection, we take a close look at the effect of slip in the randomized porous media. Figures 15 and 16 compare the velocity and the rate of strain fields, respectively, between the flows under the no-slip condition and the sliding flows (\( \alpha_s = 0.2 \) & \( \beta_s = 0.1 \)) at \( B = 10^4 \). These two figures show that when the flow can slide over the pores surfaces, the channelization is not strong compared to the no-slip condition. Indeed, the flow slides over the solid surfaces which cover a large portion of the domain in the small porosities.
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Figure 15. $|u^*|$ at $B = 10^4$: (a-c) no-slip flow, (d-f) sliding flow ($\alpha_s = 0.2$ and $\beta_s = 0.1$). Columns from left to right correspond to $\phi = 0.5, 0.7,$ and $0.9$, respectively.

Limit and consequently there is a flow in most of the passages. However, in the limit of high porosities (i.e. less solid surfaces), the effect of slip is negligible: compare panels (c) and (f) in figures 15 and 16. The static regions under the no-slip condition and in the sliding flows are almost the same when porosity is as high as $\phi = 0.9$. There are slightly sheared parts out of the main open path (see panel (f) of figure 16), but the flow mainly passes through the open channel whose shape and location are identical to the no-slip case.

Figure 17 looks at this phenomenon from the pressure drop perspective. The lines (no-slip simulations) are borrowed from figure 14 for reference. The symbols refer to sliding flows ($\alpha_s = 0.2$ & $\beta_s = 0.1$); the blue ones collected from the simulations in the randomized porous media and the red ones from the model porous media: please note that again we plot the average value of $G^*$ corresponding to the two model porous media (normal and staggered cases). This figure demonstrates that as the porosity increases, the effect of slip on the pressure drop decreases, in agreement with figures 15 and 16 for the model porous media. Indeed, the symbols are much closer to their corresponding lines with the same porosity as $\phi \to 1$. Moreover, in this high porosity limit, the model porous media predictions are appropriate approximations of the pressure drop in the randomized porous media: the asterisks and pluses with different colours are not distinguishable. However, in the other limit (low porosities), the effect of slip on the pressure drop is more pronounced: the pressure drop in the sliding flows are smaller compared to no-slip flows which is intuitive. Moreover, the intensity of the slip effect on the pressure drop depends on the geometric structures since there is a significant discrepancy between the predictions of the model porous media and the randomized porous media: the blue and red full circles are relatively far from each other specifically in the low Bingham number regime; please note the logarithmic scale.
Figure 16. $\log_{10} (||\dot{\gamma}^*||)$ at $B = 10^4$; (a-c) no-slip flow, (d-f) sliding flow ($\alpha_s = 0.2$ and $\beta_s = 0.1$). Columns from left to right correspond to $\phi = 0.5, 0.7,$ and $0.9$, respectively. Please note that for each row, we have used the same range of colourbar.

Figure 17. Pressure drop with respect to the Bingham number. The red colour stands for the simulations of the model porous media and blue colour the simulations of the random porous media. The lines are borrowed from figure 14 (no-slip condition): the continuous line corresponds to $\phi = 0.38$, dashed line to $\phi = 0.5$, dashed-dotted line to $\phi = 0.7$, and the dotted line to $\phi = 0.9$. The symbols are computations of the sliding flow ($\alpha_s = 0.2$ & $\beta_s = 0.1$): stars correspond to $\phi = 0.38$, full circles to $\phi = 0.5$, asterisks to $\phi = 0.7$, and the pluses to $\phi = 0.9$. 
7. Summary & discussion

The hydrodynamic features of sliding flows of yield-stress fluids are investigated in the present study. It has been well-documented in the literature that yield-stress fluids slide over solid surfaces due to microscopic effects: formation of a lubrication layer of solvent and elastic deformation of soft particles in the vicinity of the solid surface (Meeker et al. 2004b). Firstly, we formulated a general vectorial form of the well-known ‘stick-slip’ law and presented a numerical algorithm based on the augmented Lagrangian scheme combined with anisotropic mesh adaptation to attack these kind of problems. We derived some theoretical tools as well to formulate the yield limit in the presence of slip. The whole framework was benchmarked in a simple channel Poiseuille flow. Then we have moved forward to address more complicated problems including moving solid surfaces and the sliding flow over complex topologies.

Firstly, we simulated the slippery particle sedimentation problem and addressed the yield limit in detail. Slipline solutions from the perfectly-plastic mechanics were revisited and utilized to find the yield limit in the presence of slip. The slipline method has proven capable of finding the yield limit, even though the velocity/stress fields obtained from slipline solutions can differ from the viscoplastic solutions at large, yet finite Bingham numbers; see Chaparian & Frigaard (2017b). The conclusions from the present study are similar, showing again that the lower and upper bounds of the plastic drag coefficient are excellent estimations of the yield limit of slippery particle motion.

Secondly, we addressed the complex sliding flows in the model and randomized porous media. Different hydrodynamic features of the flow were investigated and the critical pressure gradient to ensure continuous non-zero flow was reported. Moreover, some general conclusions were drawn about modelling the flow in porous media by comparing the results computed from the model configurations and more realistic random-designed porous media; both under the no-slip condition and sliding flows. Furthermore, it was shown that the effect of slip is more severe in the low porosity limit where a larger portion of the domain is occupied by solids (i.e. a large portion of the flow is in contact with the solid surfaces). In this limit, the pressure drop of a slippery flow is decreased compared to the flow under the no-slip condition. For instance, the pressure drop under the no-slip condition at $\phi = 0.5$ closely follows the pressure drop of the sliding flow for the case $\phi = 0.38$ when $\alpha_s = 0.2$ & $\beta_s = 0.1$ (see figure 17). Indeed, an effective porosity can be defined for the sliding flows in porous media to predict the pressure drop: $G^*(\phi_{eff};\text{no-slip}) = G^*(\phi;\alpha_s,\beta_s)$ where $\phi_{eff} > \phi$. In the high porosity limit, however, $\phi_{eff} \approx \phi$ since the solid surfaces are occupied a small portion of the whole domain.

By investigating different physical problems in the presence of slip from particle motion to pressure-driven flows in complex geometries, the following general conclusions can be drawn about sliding flows of viscoplastic fluids:

(i) There are three source of dissipation in sliding flows: viscous, plastic and slip dissipations. Slip dissipation itself can be split into two main contributions: ‘viscous’ slip dissipation which comes from the slip coefficient $\beta_s$ and ‘plastic’ slip dissipation which is the contribution of the sliding yield stress.

(ii) In the yield limit, the leading order contribution to the total slip dissipation is the ‘plastic’ slip dissipation. Exploiting that viscous dissipation is at least one order of magnitude less than the plastic dissipation, we can conclude that the yield limit of sliding flows is indeed controlled by $\alpha_s$ or sliding yield stress. This was evidenced both for particle sedimentation and pressure-driven flows in porous media.

(iii) In the other limit (Newtonian limit), however, the flow characteristics are determined by the slip coefficient $\beta_s$. For instance, the plastic drag coefficient in the particle
sedimentation problem and the pressure drop in the porous media are the same for the flows with the same $\beta_s$ as $B \to 0$.

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**Appendix A. Analytical solution of the sliding Poiseuille flow**

We derive analytical solutions for the sliding Poiseuille flow in a channel with gap width unity considering both [M] and [R] formulations.

A.1. [M] problem

For the [M] problem, the only non-zero stress component is the shear stress which can be derived from,

$$\frac{d\tau_{xy}}{dy} = 1,$$

and the slip velocity can be calculated from,

$$u_s = \begin{cases} 
\beta_s (|\tau_w| - Od_s), & \text{iff } |\tau_w| > Od_s, \\
0, & \text{iff } |\tau_w| \leq Od_s,
\end{cases}$$

where $\tau_w$ is the wall shear stress. From the constitutive equation, we know that in the yielded regions (if there are any),

$$Od \ sgn(y) - y = \dot{\gamma}_{xy} = \frac{du}{dy}.$$  

Hence, if $|y| \geq Od$,

$$u = Od \ |y| - \frac{y^2}{2} + A,$$

with the boundary condition,

$$u \left( y = \pm \frac{1}{2} \right) = u_s = \beta_s \left( \frac{1}{2} - Od_s \right)$$

Therefore, three different scenarios can happen based on values of $Od$ and $Od_s$:

$$u = \begin{cases} 
0, & \text{iff } 1/2 \leq Od_s \text{ (i.e. no flow)}, \\
\beta_s \left( \frac{1}{2} - Od_s \right), & \text{iff } Od_s < 1/2 \leq Od \text{ (i.e. fully sliding plug)}, \\
u^y, & \text{iff } Od < 1/2 \text{ (i.e. deforming regime)},
\end{cases}$$

where,

$$u^y = \begin{cases} 
\frac{1}{2} \left( Od - \frac{1}{2} \right)^2 + \beta_s \left( \frac{1}{2} - Od_s \right), & \text{iff } |y| \leq Od \text{ (i.e. core plug region)}, \\
Od \ (|y| - \frac{1}{2}) + \frac{y^2}{2} + \beta_s (\frac{1}{2} - Od_s), & \text{iff } Od < |y| \text{ (i.e. sliding sheared region)}.
\end{cases}$$

Hence, $Od_c = 1/2\alpha_s$. 

A.2. [R] problem

In the [R] formulation, we always have a non-zero flow rate due to the velocity scaling. Hence, we shall identify two different regimes: deforming regime at small and moderate Bingham numbers ($B < \bar{B}$) and a fully sliding plug at $B \to \infty$ (or strictly when $B > \bar{B}$). The shear stress satisfies,

$$\frac{d\tau^*_{xy}}{dy^*} = G^*, \quad (A 8)$$

with the slip velocity,

$$u^*_s = \begin{cases} \beta_s (|\tau^*_w| - B_s), & \text{iff } |\tau^*_w| > B_s, \\ 0, & \text{iff } |\tau^*_w| \leq B_s. \end{cases} \quad (A 9)$$

In deforming regime, in the yielded regions,

$$B \text{ sgn}(y^*) - G^* y^* = \dot{\tau}^*_{xy} = \frac{du^*_s}{dy^*}. \quad (A 10)$$

Hence,

$$u^* = \begin{cases} B |y^*| - \frac{G^*}{2} y^* + A, & \text{iff } |y^*| > y^*_p, \\ U_p^*, & \text{iff } |y^*| \leq y^*_p. \end{cases} \quad (A 11)$$

where $y^*_p = \frac{B}{G^*}$ and $u(y^*_p) = U_p^*$. The boundary condition for velocity is:

$$u^*(\pm \frac{1}{2}) = u^*_s = \frac{\beta_s G^*}{2} \left(1 - \frac{2B_s}{G^*}\right) \quad (A 12)$$

therefore,

$$u^* = \begin{cases} B \left(|y^*| - \frac{1}{2}\right) + \frac{G^*}{2} \left[\frac{1}{4} - y^* + \beta_s \left(1 - \frac{2B_s}{G^*}\right)\right], & \text{iff } |y^*| > y^*_p, \\ U_p^*, & \text{iff } |y^*| \leq y^*_p. \end{cases} \quad (A 13)$$

Please note that still $G^*$ is unknown and should be calculated from continuity:

$$1 = 2 \int_{0}^{1/2} u^* \, dy^* = G^* \left(\frac{1}{12} + \beta_s \frac{2}{G^*}\right) + B \left(\frac{B^2}{3 G^*} - \frac{1}{4}\right) - \beta_s B_s. \quad (A 14)$$

The individual dissipation terms can be calculated as:

$$a(u^*, u^*) = \int_{-1/2}^{1/2} \left(\frac{du^*_s}{dy^*}\right)^2 \, dy^* = \frac{2}{3 G^*} \left(\frac{G^*}{2} - B\right)^3, \quad (A 15)$$

$$B j(u^*) = \int_{-1/2}^{1/2} |\frac{du^*_s}{dy^*}| \, dy^* = B \left(\frac{G^*}{4} - B + \frac{B^2}{G^*}\right), \quad (A 16)$$

and,

$$\int |\sigma^*_{nt} u^*_s| \, dS = \frac{\beta_s G^*}{2} \left(1 - \frac{2B_s}{G^*}\right). \quad (A 17)$$

However, in the case of fully sliding plug, the slip velocity is,

$$u^*_s = 1 = \frac{\beta_s G^*}{2} \left(1 - \frac{2B_s}{G^*}\right). \quad (A 18)$$

Hence,

$$G^* = \frac{2(1 + \beta_s B_s)}{\beta_s} = \frac{2(1 + \alpha_s \beta_s B)}{\beta_s}. \quad (A 19)$$
Therefore, the critical Bingham number beyond which the fully sliding plug occurs is,

\[ \bar{B} = \frac{1}{\beta_s(1 - \alpha_s)}. \quad (A 20) \]

**Appendix B. Flow rate curve in porous media**

Bauer et al. (2019) considered 3D flow in model and random porous media using regularized Lattice-Boltzmann approach. The model geometry used was a FCC maximum-packing arrangement of spheres whereas, a random array of overlapping spheres was designed for more realistic porous media. Interestingly, Bauer et al. (2019) reported that the model and random porous media display different flow rate dependencies: i.e. flow rate \( \sim B^{-1} \) as a function of the excessive pressure gradient \( \sim Od^{-1} - Od_c^{-1} \). They observed that the model porous media predicts a linear increase while the random porous media exhibits two distinctive scalings in the range of moderate and low/high excessive pressure gradients. In other words, in random porous media, viscous limit or yield (i.e. plastic) limit shares similar characteristics/slopes in predicting the flow rate as a function of the excessive pressure gradient; however, in viscoplastic regime, the slope is higher. This has been approved by experimental data collected by considering Carbopol flow through sandstone bed.

Using the computational data of the present study under the no-slip condition, we consider how the 2D model porous media and the 2D randomized porous media predicts the flow rate curves and how it differs with Bauer et al. (2019) simulations/experiments. Moreover, we investigate if the model and randomized porous media shares the same characteristic in converging to the yield limit at large \( B \).

Figure 18(a) illustrates the convergence to the yield limit \( 1 - \frac{Od}{Od_c} \) versus the Bingham number. All the simulations display more or less same scalings:

\[ 1 - \frac{Od}{Od_c} \sim B^{-\nu} \quad (B 1) \]

where \( \nu \approx -1 \). However, two more conclusions can be drawn from figure 18(a):

(i) At small porosities, we need to go to the higher Bingham numbers to get close enough to the yield limit. This can be attributed to the highly localized and heterogeneous/channelized flow at small porosities. Data borrowed from Bauer et al. (2019) also show same characteristics, yet a slower convergence rate \( |\nu| < 0.9 \) with the Bingham number compared to our results, which could be the consequence of 3D flows.

(ii) At high porosities, the model and randomized porous media display closely matched convergence to the yield limit by increasing the Bingham number. For instance, see the dotted blue line (randomized) and the two dotted red lines decorated with symbols (regular and staggered models with circles and squares). However, at the low porosities, the model porous media displays faster convergence compared to the randomized porous media (compare the red continuous lines+symbols (\( \phi = 0.38 \)) with the dashed line (\( \phi = 0.5 \)).

Another interesting feature to compare, as discussed above, is the flow rate curve; see panel (b) which shows the flow rate versus the excessive pressure gradient. Contradictory to what has been reported by Bauer et al. (2019), this figure shows that the model porous media (at least in 2D), predicts same behaviour: at small and large excessive pressure gradients (yield and Newtonian limit, respectively), the flow rate scales linearly with the excessive pressure gradient; however, at the intermediate regime, the flow rate growth as a function of \( \frac{1}{Od} - \frac{1}{Od_c} \) is faster. This behaviour is manifested by the randomized porous media as well.

Figure 18. Flow features in model and random-designed porous media. The lines and symbols interpretation is the same as figures 14 and 17. Please note that in both panels, the lines with circles correspond to the regular model geometry (figure 10(a)) and those with squares to the staggered model geometry (figure 10(b)). The dotted black line in panel (a) marks $B^{-0.95}$ scaling and the two dotted black lines in panel (b) 1:1 scaling for reference.

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