Abstract

We consider the recent works of [3, 31, 2] that provide tools for analyzing focused stochastic local search algorithms that arise from algorithmizations of the Lovász Local Lemma [17] (LLL) in general probability spaces. These are algorithms which search a state space probabilistically by repeatedly selecting a “flaw” that is currently present and moving to a random nearby state in an effort to address it and, eventually, reach a flawless state.

While the original Moser-Tardos [41] (MT) algorithm is amenable to the analysis of these abstract frameworks, many follow-up results [23, 29, 35, 9, 12, 26] that further enhance, or exploit, our understanding of the MT process are not transferable to these general settings. Mainly, this is because a key ingredient of the original analysis of Moser and Tardos, the witness tree lemma, does not longer hold.

In this work, we show that we can recover the witness tree lemma in the “commutative setting”. The latter was recently introduced by Kolmogorov [17] and captures the vast majority of the LLL applications. Armed with it, we focus on studying properties of commutative algorithms and give several applications. Among other things, we are able to generalize and extend to the commutative setting the main result of [23] which states that the output of the MT algorithm well-approximates the conditional LLL-distribution, i.e., the distribution obtained by conditioning on all bad events being avoided.

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1 Introduction

Many problems in combinatorics and computer science can be phrased as finding an object that lacks certain bad properties, or “flaws”. In this paper we study algorithms that take as input a flawed object and try to rid of all the flaws by repeated probabilistic action.

Concretely, let \( \Omega \) be a set of objects and let \( F = \{ f_1, f_2, \ldots, f_m \} \) be a collection of subsets of \( \Omega \). We will refer to each \( f_i \in F \) as a flaw to express that its elements share some negative feature. For example, if a CNF formula \( F \) on \( n \) variables has clauses \( c_1, c_2, \ldots, c_m \), we can define for each clause \( c_i \) the flaw (subcube) \( f_i \subseteq \{0,1\}^n \) whose elements violate \( c_i \). Following linguistic rather than mathematical convention we say that \( f \) is present in \( \sigma \) if \( f \supseteq \sigma \) and that \( \sigma \in \Omega \) is flawless (perfect) if no flaw is present in \( \sigma \).

To prove the existence of flawless objects we can often use the Probabilistic Method. As a matter of fact, in many interesting cases, this is the only way we know how to do so. To employ the Probabilistic Method, we introduce a probability measure \( \mu \) over \( \Omega \) and consider the collection of “bad” events corresponding to flaws. If we are able to show that the probability to avoid all bad events is strictly positive, then this implies the existence of a flawless object. A trivial example is the case where all the bad events are independent of one another and none of them has probability one. One of the most powerful tools of the Probabilistic Method is the Lovász Local Lemma \([17]\) which weakens the latter restrictive condition of independence to a condition of limited dependence.

**General LLL.** Let \((\Omega , \mu)\) be a probability space and \( A = \{ A_1, A_2, \ldots, A_m \} \) be a set of \( m \) (bad) events. For each \( i \in [m] \), let \( D(i) \subseteq [m] \setminus \{i\} \) be such that \( \mu(A_i \cap \bigcap_{j \in S}A_j) = \mu(A_i) \) for every \( S \subseteq [m] \setminus (D(i) \cup \{i\}) \).

If there exist positive real numbers \( \{ \psi_i \}_{i=1}^m \) such that for all \( i \in [m] \),

\[
\frac{\mu(A_i)}{\psi_i} \sum_{S \subseteq D(i) \cup \{i\}} \prod_{j \in S} \psi_j \leq 1 ,
\]

then the probability that none of the events in \( A \) occurs is at least \( \prod_{i=1}^m 1/(1 + \psi_i) > 0 \).

**Remark 1.** Condition (1) above is equivalent to the more well-known form \( \mu(A_i) \leq x_i \prod_{j \in D(i)}(1 - x_j) \), where \( x_i = \psi_i/(1 + \psi_i) \). As we will see, formulation (1) facilitates refinements.

Let \( G \) be the digraph over the vertex set \([m]\) with an edge from each \( i \in [m] \) to each element of \( D(i) \cup \{i\} \). We call such a graph a dependency graph. Therefore, at a high level, the LLL states that if there exists a sparse dependency graph and each bad event is not too likely, then perfect objects exist.

After proving that \( \Omega \) contains flawless objects via the LLL it is natural to ask if some flawless object can be found efficiently. Making the LLL constructive has been the subject of intensive research, starting with the result of Beck \([7]\) and followed by works of others \([4, 39, 15, 45, 40]\). In a landmark paper \([41]\), Moser and Tardos made the general LLL constructive for product measures over explicitly presented variables. Specifically, in the so-called variable setting of \([41]\), each event \( A_i \) is determined by a set of variables \( \text{vbl}(A_i) \) so that \( j \in D(i) \) iff \( \text{vbl}(A_i) \cap \text{vbl}(A_j) \neq \emptyset \). Moser and Tardos proved that if the general LLL condition holds, then the algorithm that repeatedly selects any occurring event \( A_i \) (flaw present) and resamples every variable in \( \text{vbl}(A_i) \) independently of all others converges quickly to a flawless object.

The first result that made the LLL constructive in a non-product probability space was due to Harris and Srinivasan in \([28]\), who considered the space of permutations endowed with the uniform measure. More recent works of Achlioptas and Iliopoulos \([3, 2]\), and Harvey and Vondrák \([51]\), develop frameworks for making the LLL constructive in abstract probability spaces. At this point, there exist efficient algorithms for all the applications of the LLL that we are aware of.

The results in \([3, 31, 2]\) can be seen as giving tools for analyzing focused stochastic local search algorithms in arbitrary spaces. These are algorithms that search a state space probabilistically by repeatedly
selecting a flaw that is currently present and moving to a random nearby state in an effort to address it and, eventually, reach a perfect object. To be more precise, these frameworks study a class of algorithms that are defined by (i) a family of “resampling” probability distributions $\rho_i(\sigma, \cdot)$ for each $i \in [m]$ and for each $\sigma \in f_i$; (ii) an initial distribution $\theta$; (iii) a strategy according to which they choose which flaw to address at each step. An algorithm in this class starts at a state (object) distributed as $\theta$ and, at each subsequent step, uses its strategy to choose a flaw $f_i$ that is currently present and addresses it by moving to a state $\sigma'$ with probability $\rho_i(\sigma, \sigma')$. Given a probability measure $\mu$ over the space $\Omega$, if one can design a family of efficiently samplable resampling probability distributions for addressing each flaw in a way that is, in a certain sense, “perfectly compatible” with $\mu$, then one has an algorithm that converges quickly to a perfect object whenever the LLL conditions hold. More generally, one can design the algorithm independently of $\mu$, and then get a generalized Local Lemma criterion for fast convergence that depends on the compatibility between $\mu$ and the algorithm.

While the original Moser-Tardos (MT) [41] algorithm is amenable to the analysis of these abstract frameworks, many follow-up results that further enhance, or exploit, our understanding of the MT process are not transferable to these general settings. As a prominent example, the result of Haeupler, Saha and Srinivasan [23], as well as follow-up works of Harris and Srinivasan [29, 25], allow one to argue about the dynamics of the MT process, resulting in several new applications such as estimating the entropy of the output distribution, partially avoiding bad events, dealing with super-polynomially many bad events, and even new frameworks [27, 10].

The key ingredient that makes the above results attainable is the witness tree lemma; a technical result that is at the heart of the original MT analysis. Roughly, it states that any tree of bad events growing backwards in time from a certain root bad event $A_i$, with the children of each node $A_j$ being bad events that are adjacent to $A_j$ in the dependency graph, has probability of being consistent with the trajectory of the algorithm that is bounded by the product of the probabilities of all events in this tree. The witness tree lemma (and its variations [35, 22]) has been used for several other purposes besides those already mentioned, such as designing deterministic, parallel and distributed algorithms for the LLL [41, 9, 12, 22, 26]. Thus, it would be desirable to establish that it holds for algorithms in the most general frameworks of [3, 31, 2].

Unfortunately, Harvey and Vondrák show [31] that this is impossible under the most general assumptions. On the other hand, Harris and Srinivasan in [28] do manage to prove the witness tree lemma for their algorithm for the LLL on the space of permutations, via an analysis that is tailored specifically to this setting. Although their proof does not seem to be easily generalizable to general spaces, their success makes it natural to ask if we can impose mild assumptions in the general settings of [3, 31, 2] under which the witness tree lemma (and most of its byproducts) can be established.

The main contribution of this paper is to answer this question positively by showing that it is possible to prove the witness tree lemma in the commutative setting. The latter was recently introduced by Kolmogorov [37], who showed that under its assumptions one can obtain parallel algorithms, as well as the flexibility of having arbitrary flaw choice strategy in the frameworks of [3, 31, 2]. We note that the commutative setting captures the vast majority of LLL applications, including but not limited to both the variable and the permutation settings.

Armed with the witness tree lemma, we are able to study properties of algorithms in the commutative setting and give several applications.

**Distributional Properties** As already mentioned, one of the most important applications of the witness tree lemma is given in the paper of Haeupler, Saha and Srinivasan [23], where they study properties of the MT-distribution; the output distribution of the MT algorithm. Their main result is that the MT-distribution well-approximates the LLL-distribution, i.e., the distribution obtained by conditioning on all bad events being avoided. As an example, an immediate consequence of this fact is that one can argue about the expected
weight of the output of the MT algorithm, given a weighting function over the space $\Omega$. Furthermore, as shown in the same paper [23] and follow-up papers by Harris and Srinivasan [29, 25], one can lower bound the entropy of the MT distribution, go beyond the LLL conditions (if one’s willing to partially avoid bad events), and deal with applications with super-polynomially many number of bad events.

Here we extend the result (and its applications) of [23] to the commutative setting: Given a commutative algorithm that is perfectly compatible with the underlying probability measure, its output well-approximates the LLL distribution in the same sense the MT-distribution does in the variable setting. For arbitrary commutative algorithms, the quality of the approximation additionally depends on the compatibility of the algorithm with the measure in the event(s) of interest.

Moreover, we quantitatively improve the bounds of [23] under the weaker assumptions of Shearer’s condition [44], the most general LLL criterion (under certain assumptions in form of the input). This allows us to study distributional properties of commutative algorithms using criteria that lie between the General LLL and Shearer’s condition such as the Clique LLL [36].

Algorithmic LLL without a slack and Arbitrary Flaw Choice Strategy The works of Achlioptas, Iliopoulos and Kolmogorov [3, 2, 37] require a multiplicative slack in the [generalized] LLL conditions in order to establish fast convergence to a perfect object. On the other hand, Harvey and Vondrák [31] dispense with this requirement in the important case of algorithms that are perfectly compatible with the underlying measure under the mild assumption that the [generalized] dependency graph is undirected.

Using the witness lemma, we are able to dispense with the multiplicative slack requirement for arbitrary algorithms in the commutative setting and also have the flexibility of arbitrary flaw choice strategy (as in the result of Kolmogorov [37]).

Improved Running Time Bounds We are able to improve the running time bounds of Harvey and Vondrák [31] for commutative algorithms, matching the ones of Kolipaka and Szegedy [35] for the MT algorithm. Whether this could be done was left as an open question in [31]. We note that while the results of Achlioptas, Iliopoulos and Kolmogorov [3, 2, 37] do manage to give improved running bounds as well, they require a multiplicative-slash in the LLL conditions.

Concrete Applications We give concrete applications showing new results for the problems of rainbow matchings, proper vertex coloring and acyclic edge coloring.

The first problem is in the space of matchings of a complete graph. We use this problem as an example that allows us to show how several byproducts of the witness tree lemma can be applied to a setting that is not captured neither by the variable nor the permutation setting, and for which we know [3, 31, 37] how to design commutative algorithms that are perfectly compatible with the uniform measure over the state space.

The second problem is a case where in order to get the best convergence guarantees one has to use algorithms that are not perfectly compatible with the underlying measure over the state space. We use the witness tree lemma to show how one can get guarantees about the expected weight of the output of such an algorithm (for a family of weight function over solutions), as well as a lower bound on the number of solutions of a given instance of the problem.

Finally, in the third problem we show how one can use bounds on the output distribution of commutative algorithms that are induced by the Shearer’s condition in order to analyze applications of the Clique version of the Local Lemma.

1.1 Overview of the paper

The rest of the paper is organized as follows. In Section 2 we give the necessary background and describe our setting in detail. We also rigorously state the witness tree lemma. In Section 3 we formally describe our
main results: In Subsection 3.1 we outline the applications of the witness tree lemma that have to do with approximating the LLL distribution, improved running bounds, and dispensing with the slack requirement. In Subsections 3.2 and 3.3 we state and discuss two important byproducts of approximating the LLL distribution which we will be using in our concrete applications: bounding the entropy of the output distribution of commutative algorithms and partially avoiding flaws. In Section 4 we give the proofs of the main results. In Section 5 we discuss how we can deal with settings in which the number of flaws is super-polynomial in the natural size of the problem in the commutative setting. Finally, in Section 6 we employ our results to give new applications for the problems of rainbow matchings, proper vertex coloring and acyclic edge coloring.

2 Background and Preliminaries

In this section we present the necessary background and definitions to describe our setting. In Subsections 2.1 and 2.2 we formally outline the algorithmic assumptions of [3, 31, 2, 37], while in Subsection 2.3 we state the witness tree lemma.

2.1 Algorithmic Framework

Here we describe the class of algorithms we will be considering as well as the algorithmic LLL criteria for fast convergence to a perfect object. Since we will be interested in algorithms that search for perfect objects, we sometimes refer to $\Omega$ as a state space and to its elements as states.

For a state $\sigma$, we denote by $U(\sigma) = \{ j \in [m] \; s.t. \; f_j \ni \sigma \}$ the set of indices of flaws that are present at $\sigma$. We consider algorithms which at each flawed state $\sigma$ choose an element of $U(\sigma)$ and randomly move to a nearby state in an effort to address the corresponding flaw. Concretely, we will assume that for every flaw $f_i$ and every state $\sigma \in f_i$ there is a probability distribution $\rho_i(\sigma, \cdot)$ with a non-empty support $A(i, \sigma) \subseteq \Omega$ such that addressing flaw $f_i$ at state $\sigma$ amounts to selecting the next state $\sigma'$ from $A(i, \sigma)$ with probability $\rho_i(\sigma, \sigma')$. We call $A(i, \sigma)$ the set of actions for addressing flaw $f_i$ at $\sigma$ and note that potentially $A(i, \sigma) \cap f_i \neq \emptyset$, i.e., addressing a flaw does not necessarily imply removing it. The actions for flaw $f_i$ form a digraph $D_i$ on $\Omega$ having an arc $\sigma \xrightarrow{i} \sigma'$ for each pair $(\sigma, \sigma') \in f_i \times A(i, \sigma)$. Let $D$ be the multi-digraph on $\Omega$ that is the union of all $D_i$. Thus, the main goal is to analyze a walk on $D$ that starts at a state $\sigma_j \in \Omega$, selected according to some probability distribution $\theta$, which at each non-sink vertex $\sigma$ first selects a flaw $f_i \ni \sigma$ to address and then selects an arc leaving $\sigma$ in $D_i$, each such arc selected with probability $\rho_i(\sigma, \sigma')$. Finally, when we refer to running “time”, we will refer to the number of steps on $D$, without concern for exactly how long it takes to perform a single step, i.e., to identify a flaw present and select from its actions, e.g., to find a violated constraint and resample its variables.

To state the algorithmic LLL criteria for fast convergence which we will be using we need to introduce two key ingredients. The first one is a notion of causality among flaws that will be used to induce a graph over $[m]$, which will play a role similar to the one of the dependency graph in the existential Local Lemma formulation. We note that there is a formal connection between causality graphs and (generalizations of) dependency graphs (for more details see [31]).

Causality. For an arc $\sigma \xrightarrow{i} \sigma'$ in $D$ and a flaw $f_j$ present in $\sigma'$ we say that $f_i$ causes $f_j$ if $f_i = f_j$ or $f_j \not\ni \sigma$. If $D$ contains any arc in which $f_i$ causes $f_j$ we say that $f_i$ potential causes $f_j$.

Causality Digraph. Any digraph $C = C(\Omega, F, D)$ on $[m]$ where $i \rightarrow j$ exists whenever $f_i$ potentially causes $f_j$ is called a potential causality digraph. The neighborhood of a flaw $f_i$ in $C$ is $\Gamma(i) = \{ j : i \rightarrow j \text{ exists in } C \}$. 

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In the interest of brevity we will be using the term causality graph, instead of a potentially causality graph.

The second ingredient is a measure of compatibility between the actions of the algorithm for addressing each flaw $f_i$ (that is, digraph $D_i$) and the probability measure $\mu$ over $\Omega$ which we will be using for the analysis. As was shown in [31, 2, 37] one can capture compatibility by letting

$$d_i = \max_{\sigma \in \Omega} \frac{\nu_i(\sigma)}{\mu(\sigma)} \geq 1,$$

where $\nu_i(\sigma)$ is the probability of ending up at state $\sigma$ at the end of the following experiment: sample $\omega \in f_i$ according to $\mu$ and address flaw $f_i$ at $\omega$. An algorithm achieving perfect compatibility for flaw $f_i$, i.e., $d_i = 1$, is a resampling oracle for flaw $f_i$ (observe that the Moser and Tardos algorithm is trivially a resampling oracle for every flaw). More generally, ascribing to each flaw $f_i$ the charge

$$\gamma(f_i) = d_i \cdot \mu(f_i) = \max_{\sigma' \in \Omega} \frac{1}{\mu(\sigma')} \sum_{\sigma \in f_i} \mu(\sigma) \rho_i(\sigma, \sigma') ,$$

yields the following algorithmization condition. If for every flaw $f_i \in F$,

$$\frac{\gamma(f_i)}{\psi_i} \sum_{S \subseteq \Gamma(i)} \prod_{j \in S} \psi_j < 1$$

then there exists a flaw choice strategy under which the algorithm will reach a perfect object quickly (that is, in $O(\log |\Omega| + \text{poly}(m))$ steps) with high probability.

Throughout the paper we will assume that we are given an undirected causality graph $C$ (and thus the relation $\Gamma(\cdot)$ is symmetric) and we will sometimes write $i \sim j$ if $j \in \Gamma(i) \leftrightarrow j \in \Gamma(j)$. Furthermore, for a set $S \subseteq [m]$ we define $\Gamma(S) = \bigcup_{i \in S} \Gamma(i)$. Finally, we denote by $\text{Ind}(S) = \text{Ind}_C(S)$ the set of independent subsets of $S$ with respect to $C$.

### 2.2 Commutativity

We will say that $\sigma \xrightarrow{i} \sigma'$ is a valid trajectory if it is possible to get from state $\sigma$ to state $\sigma'$ by addressing flaw $f_i$ as described in the algorithm, i.e., if two conditions hold: $i \in U(\sigma)$ and $\sigma' \in A(i, \sigma)$. Kolmogorov [37] described the following “commutativity” condition. We call the setting in which Definition 1 holds the commutative setting.

**Definition 1** (Commutativity [37]). A tuple $(D, F, \sim, \rho)$ is called commutative if there exists a mapping Swap that sends any trajectory $\Sigma = \sigma_1 \xrightarrow{i} \sigma_2 \xrightarrow{j} \sigma_3$ with $i \sim j$ to another valid trajectory $\text{Swap}(\Sigma) = \sigma_1 \xrightarrow{j} \sigma'_2 \xrightarrow{i} \sigma_3$, and:

1. Swap is injective,
2. $\rho_i(\sigma_1, \sigma_2) \rho_j(\sigma_2, \sigma_3) = \rho_j(\sigma_1, \sigma'_2) \rho_i(\sigma'_2, \sigma_3)$ .

It is straightforward to check that the Moser Tardos algorithm satisfies the commutativity condition. As a matter of fact, any algorithm (i) for which we use the same causality graph as the one in the MT analysis; (ii) which addresses constraints (flaws) using any (state independent) resampling probability distribution
that assigns strictly positive probability mass to each possible assignment for the variables of the constraints, satisfies the commutativity condition. Furthermore, Kolmogorov showed that the same is true for resampling oracles in the permutation [28] and perfect matchings [31] settings.

Finally, as already mentioned, Kolmogorov showed that in the commutativity setting one may choose an arbitrary flaw choice strategy which is a function of the entire past execution history. The same will be true for our results, so we make the convention that given a tuple \((D, F, \sim, \rho)\) we always fix some arbitrary flaw choice strategy to get a well-defined, “commutative” algorithm \(\mathcal{A} = (D, F, \sim, \rho)\).

2.3 The Witness Tree Lemma

Given a trajectory \(\Sigma = \sigma_1 \xleftarrow{w_1} \cdots \xleftarrow{w_{i+1}} \sigma_i \xrightarrow{w} \sigma_{i+1}\) we will be denoting by \(W(\Sigma) = (w_1, \ldots, w_t)\) the witness sequence of \(\Sigma\) (recall that according to our notation, \(w_i\) denotes the index of the flaw that was addressed at the \(i\)-th step).

To state the lemma, we will first need to recall the definition of witness trees from [41] (slightly reformulated to fit our setting). A witness tree \(\tau = (T, \ell_T)\) is a finite rooted, unordered, tree \(T\) along with a labelling \(\ell_T: V(T) \to [m]\) of its vertices with indices of flaws such that the children of a vertex \(v \in V(T)\) receives labels from \(\Gamma(\ell(v))\). To lighten the notation, we will sometime be writing \([v]\) to denote \(\ell(v)\) and \(V(\tau)\) instead of \(V(T)\). Given a witness sequence \(W = (w_1, w_2, \ldots, w_t)\) we associate with each \(i \in [t]\) a witness tree \(\tau_W(i)\) that is constructed as follows: Let \(\tau_W^{(1)}(i)\) be an isolated vertex labelled by \(w_i\). Then, going backwards for each \(j = i - 1, i - 2, \ldots, 1\): if there is a vertex \(v \in \tau_W^{(j+1)}(i)\) such that \([v] \sim w_j\) then we choose among those vertices the one having the maximum distance (breaking ties arbitrarily) from the root and attach a new child vertex \(u\) to \(v\) that we label \(w_j\) to get \(\tau_W^{(j)}(i)\). If there is no such vertex \(v\) then \(\tau_W^{(j+1)}(i) = \tau_W^{(j)}(i)\). Finally, let \(\tau_W(i) = \tau_W^{(1)}(i)\).

We will say that a witness tree \(\tau\) occurs in a trajectory \(\Sigma\) if \(W(\Sigma) = (w_1, w_2, \ldots, w_t)\) and there is \(k \in [t]\) such that \(\tau_W(k) = \tau\).

Finally, we will be using the notation \(\Pr[\cdot] = \Pr_{\mathcal{A}}[\cdot]\) to refer to the probability of events that can happen in an execution of \(\mathcal{A}\). For example, the probability that \(\mathcal{A}\) follows a certain trajectory \(\Sigma\) will be denoted by \(\Pr[\Sigma]\), while the probability that a specific witness tree \(\tau\) occurs in the trajectory followed by \(\mathcal{A}\) will be denoted by \(\Pr[\tau]\). We are now ready to state the witness tree lemma.

**Definition 2** (The witness tree lemma). We will say that the witness tree lemma holds if for every witness tree \(\tau\) we have that:

\[
\Pr[\tau] \leq \lambda_{\text{init}} \prod_{v \in V(\tau)} \gamma(f[v]),
\]

where \(\lambda_{\text{init}} = \max_{\sigma \in \Omega} \frac{\theta(\sigma)}{\mu(\sigma)}\).

We note that the witness lemma is usually stated with the assumption that the initial distribution \(\theta = \mu\) (and, thus, \(\lambda_{\text{init}} = 1\)), and this is typically crucial for its proof. As we will see, we will be able to show the more general state of Definition 2 for the commutative setting. While this can be important in principle since sampling from \(\mu\) can be hard, in all of our applications we will have that \(\lambda_{\text{init}} = 1\).

3 Statement of Results

Our main result is that the witness tree lemma holds in the commutative setting.

**Theorem 1** (Main Result). Assume that \(\mathcal{A} = (D, F, \sim, \rho)\) is commutative. Then, the witness tree lemma holds.
3.1 Distributional Properties and Improved Running Time Bounds

Assuming that the LLL conditions (1) hold, the LLL-distribution \((\mu_{\text{flawless}})\), which we will be denoting by \(\mu\), is defined as the distribution induced by measure \(\mu\) conditional on no bad event (flaw) occurring. The following proposition connects the LLL distribution with measure \(\mu\) making it a powerful tool that can be used to argue about properties of flawless objects. The idea is that if an (not necessarily bad) event \(E\) is independent from most bad events, then its probability under the LLL distribution is not much larger than its probability under the probability measure \(\mu\).

Proposition 1 ([23]). If the LLL conditions (1) hold then for any event \(E\):

\[
\mu_{\text{flawless}}(E) \leq \mu(E) \sum_{S \subseteq D(i) \cup \{i\}} \prod_{j \in S} \psi_j .
\]

The key result of [23] is that the Moser-Tardos algorithm approximates well the LLL distribution in the sense that one can bound the probability of any event \(E \subseteq \Omega\) occurring during its execution (and, thus, also in its output) by the left-hand side of (1). Building on this fact, [23] and followup works [29, 25] manage to show several new applications.

Here we extend the latter result to arbitrary commutative algorithms: Given an arbitrary set \(E \subseteq \Omega\) and a commutative algorithm \(\mathcal{A}\), consider an extension \(\mathcal{A}'\) of \(\mathcal{A}\) by defining an extra flaw \(f_{m+1} := E\) with its own set of resampling probability distributions. Define also an extended causality graph so that \(m + 1\) is also included. If \(\mathcal{A}'\) is commutative with respect to this new causality graph and \(\Gamma(E) := \Gamma(f_{m+1})\) is small enough, then the probability that the trajectory of the algorithm will reach a state \(\sigma \in E\) is approximately bounded by \(\gamma(E)\), i.e., the charge of \(f_{m+1}\) induced by \(\mathcal{A}'\).

We note that, although we will not always explicitly mention it, whenever we talk about the charge or the neighbourhood of a set \(E \notin F\), we will be implicitly assuming that they are induced by a “commutative extension” of the original algorithm per the above description.

Below, we state our results assuming improved LLL criteria: the cluster expansion and Shearer’s conditions.

Cluster Expansion Condition The cluster expansion criterion strictly improves upon the General LLL criterion (1) by taking advantage of the local density of the dependency graph. It was first showed as an existential result in [8] and was later made constructive under the most general algorithmic assumptions [35, 43, 1, 3, 31, 37]. Below we state it in the context of our setting.

Definition 3 (Cluster Expansion Condition). Given a sequence of positive real numbers \(\{\psi_i\}_{i=1}^m\), we say that the cluster expansion condition is satisfied if for each \(i \in [m]\):

\[
\frac{\gamma(f_i)}{\psi_i} \sum_{S \in \text{Ind}(\Gamma(i))} \prod_{j \in S} \psi_j \leq 1 .
\]

Remark 2. Observe that

\[
\sum_{S \in \text{Ind}(\Gamma(i))} \prod_{j \in S} \psi_j \leq \sum_{S \subseteq \Gamma(i)} \prod_{j \in S} \psi_j = \prod_{j \in \Gamma(i)} (1 + \psi_j) ,
\]

and thus, the cluster expansion condition is strictly stronger than the General LLL.

In Theorem 2 below we assume that we are given a commutative algorithm \(\mathcal{A}\) and we give two results. We bound the expected number of times each flaw is addressed during an execution of \(\mathcal{A}\) and also the probability that any arbitrary event will ever occur during an execution of \(\mathcal{A}\).
\textbf{Theorem 2.} If \( \mathcal{A} = (D, F, \sim, \rho) \) is commutative and the cluster expansion condition is satisfied then:

1. for each \( i \in [m] \): \( \mathbb{E}[N_i] \leq \lambda_{\text{init}} \psi_i \);
2. for each \( E \subseteq \Omega \): \( \Pr[E] \leq \lambda_{\text{init}} \gamma(E) \sum_{S \subseteq \text{Ind}(\Gamma(E))} \prod_{j \in S} \psi_j \);

where \( N_i \) is the number of times flaw \( f_i \) is addressed during the execution of \( \mathcal{A} \).

We note that the first part of Theorem 2 allows us to guarantee fast convergence of \( \mathcal{A} \) to a perfect object without having to assume a “slack” in the cluster expansion condition (unlike the works of \([2, 37]\)) and also to match the running time guarantee given by the original work of Moser Tardos \([41]\), per the following straightforward corollary.

\textbf{Corollary 1.} Algorithm \( \mathcal{A} \) terminates after \( O\left( \lambda_{\text{init}} \sum_{i \in [m]} \psi_i \right) \) steps in expectation.

\textbf{Shearer’s Condition} In \([44]\), Shearer gave a sufficient and necessary LLL criterion for perfect objects to exist given the probabilities of each flaw and an undirected dependency graph with respect to a certain measure \( \mu \). This means that when the Shearer’s condition is violated, there exists a probability space \((\Omega, \mu)\) and a set of events \( \{f_1, \ldots, f_n\} \) for which \( \mu \left( \bigcap_{i=1}^n f_i \right) = 0 \).

Kolipaka and Szegedy were the first to show in \([35]\) that this condition suffices for the Moser-Tardos algorithm to converge to a solution, while Harvey and Vondrák \([31]\) and Kolmogorov \([37]\) extend the result to arbitrary spaces and charges of flaws. Finally, as far as the task of verifying Shearer’s condition is concerned, we note that very recent works \([30, 20]\) show approximation algorithms and hardness results (depending on the region of the parameters of Shearer’s theorem).

To state the conditions, given \((D, F, \sim, \rho)\) let \( \gamma \in \mathbb{R}^m \) be the real vector such that \( \gamma_i = \gamma(f_i) \). Furthermore, for \( S \subseteq [m] \) define \( \gamma_S = \prod_{j \in S} \gamma_j \), as well as, the polynomial \( q_S \):

\[
q_S(q) = \sum_{I \in \text{Ind}([m])} (-1)^{|I|-|S|} \gamma_I .
\]

\textbf{Definition 4.} We say that the Shearer’s condition is satisfied if \( q_S(\gamma) \geq 0 \) for all \( S \subseteq [m] \), and \( q_0(\gamma) > 0 \).

Theorem 3 is the analogue of Theorem 2 in the Shearer’s condition regime.

\textbf{Theorem 3.} If \( \mathcal{A} = (D, F, \sim, \rho) \) is commutative and the Shearer’s condition is satisfied then:

1. for each \( i \in [m] \): \( \mathbb{E}[N_i] \leq \lambda_{\text{init}} \frac{q_i(\gamma)}{q_0(\gamma)} \);
2. for each \( E \subseteq \Omega \): \( \Pr[E] \leq \lambda_{\text{init}} \gamma(E) \sum_{S \subseteq \text{Ind}(\Gamma(E))} \prod_{j \in S} \frac{q_j(\gamma)}{q_0(\gamma)} \);

where \( N_i \) is the number of times flaw \( f_i \) is addressed during the execution of \( \mathcal{A} \).

We note that the first part of Theorem 3 allows us to guarantee that \( \mathcal{A} \) converges quickly to a perfect object without assuming a slack in the Shearer’s condition as in \([37]\) and, moreover, improves upon the (roughly quadratically worse) running bound of \([31]\), matching the one of \([35]\). Whether the latter could be done was left as an open question in \([31]\). The reason we manage to answer positively in the commutative setting is exactly because the witness tree lemma holds (a variation of the witness tree lemma was also used in \([35]\)).

\textbf{Corollary 2.} Algorithm \( \mathcal{A} \) terminates after \( O\left( \lambda_{\text{init}} \sum_{i \in [m]} \frac{q_i(\gamma)}{q_0(\gamma)} \right) \) steps in expectation.
3.2 Entropy of the Output Distribution

An elegant application of the known bounds for the Moser Tardos distribution is estimating its randomness. In particular, Harris and Srinivasan [29] show that one can give lower bounds on the Rényi entropy of the output of the Moser Tardos algorithm.

**Definition 5.** [11] Let \( \nu \) be a probability measure over a finite set \( S \). The Rényi entropy with parameter \( \rho \) of \( \nu \) is defined to be

\[
H_\rho[\nu] = \frac{1}{1 - \rho} \ln \sum_{s \in S} \nu(s)^\rho.
\]

The min-entropy \( H_\infty \) is a special case defined as \( H_\infty[\nu] = \lim_{\rho \to \infty} H_\rho[\nu] = -\ln \max_{v \in S} \nu(v) \).

Using the results of Section 3.1 we can show the analogous result in our setting.

**Theorem 4.** Assume that \( A = (D, F, \sim, \rho) \) is commutative, the cluster expansion condition is satisfied and \( \lambda_{\text{init}} = 1 \). Let \( \nu \) be the output distribution of \( A \). Then, for \( \rho > 1 \),

\[
H_\rho[\nu] \geq H_\rho[\mu] - \frac{n}{\rho - 1} \ln \left( \sum_{S \in \text{Ind}([m])} \prod_{j \in S} \psi_j \right).
\]

Given Theorem 2, the proof is akin to the analogous result in [29] and can be found in the appendix.

**Remark 3.** Using the Shearer’s condition we can replace \( \psi_i \) with \( \frac{q_i(\gamma)}{q_i(\gamma)} \), \( i \in [m] \).

A straightforward application of having a lower bound on \( H_\rho[\nu] \) (for any \( \rho \)), where \( \nu \) is the output distribution of the algorithm, is that there exist at least \( \exp(H_\rho[\nu]) \) flawless objects. Before [29], the authors in [38] also used the (existential) LLL for enumeration of combinatorial structures by exploiting the fact that it guarantees a small probability \( p \) of avoiding all flaws when sampling from the uniform measure (and, thus, their number is at least \( p|\Omega| \)).

3.3 Partially Avoiding Flaws

On of the main results of [23, 29] are “Local Lemma criteria” for the existence of objects that avoid a large portion of the bad events. For example, given a sparse \( k \)-SAT formula that violates the original LLL conditions, one can still find an assignment that satisfies many clauses. Using the results of Section 3.1 we are able to extend the (most general) result of Harris and Srinivasan [29] to the commutative setting.

Given a sequence of positive numbers \( \{\psi_i\}_{i=1}^m \), for each \( i \in [m] \) define:

\[
\zeta_i := \sum_{S \in \text{Ind}(\Gamma(i))} \prod_{j \in S} \psi_j,
\]

and notice that the cluster expansion condition can be expressed as requiring that for each \( i \in [m] \) we have that \( \gamma(f_i) \zeta_i \leq \psi_i \).
Theorem 5. Assume that $A = (D, F, \sim, \rho)$ is commutative and $\lambda_{\text{init}} = 1$. Let $\{\psi_i\}_{i=1}^m$ be a sequence of positive numbers. Then there is an algorithm $\mathcal{A}'$ (which is a modification of $\mathcal{A}$) and whose output distribution $\nu$ has the property that for each $i \in [m]$

$$\nu(f_i) \leq \max\{0, \gamma(f_i)\zeta_i - \psi_i\}.$$ 

Furthermore, the expected number of times a flaw $f_i$ is addressed is $\psi_i$.

In words, given a commutative algorithm $\mathcal{A}$, we can come up with an efficient algorithm $\mathcal{A}'$ with the property that the probability of a flaw being present in its output is a function of “how much” it violates the cluster expansion condition. Given Theorem 2, the proof of Theorem 5 is akin to the one of [29] and can be found in the appendix.

Remark 4. Using the Shearer’s condition we can replace $\psi_i$ with $\frac{q_i(\gamma)}{q_k(\gamma)}$, $i \in [m]$.

4 Proof of Main Results

4.1 Witness Trees and Stable Set Sequences

In this section we prove some properties of witness trees (which are induced by witness sequences of the algorithm) that will be useful to us later. We also draw a connection between witness trees and stable set sequences (a notion which have been used to make the Shearer’s criterion constructive), which we will need in the proof of Theorem 1.

4.1.1 Properties of Witness Trees

The following propositions capture the main properties of witness trees we will need.

Proposition 2. For a witness tree $\tau = (T, \ell, \sim)$ let $L_i = L_i(\tau)$ denote the set of labels of the nodes at distance $i$ from the root. For each $i \geq 0$, $L_i \in \text{Ind}([m])$.

Proof. Let $W = (w_1, w_2, \ldots, w_t)$ be a witness sequence. Define $L_{i,k}^{(j)} := L_i \left( \tau_{W}^{(j)}(k) \right)$ to be the set of labels of nodes at distance $i$ from the root of $\tau_{W}^{(j)}(k)$ for some fixed $k \in [t]$. We will show that for every possible values of $i,j$ we have that $L_{i,k}^{(j)} \in \text{Ind}([m])$.

We will fix a specific $i$ and use induction. For $j = k$ it is straightforward to see the claim because, for every $i$, $L_{i,k}^{(j)}$ is either empty or contains a single element of $[m]$. Assume now that the hypothesis is true for all $j \in [p,k]$. We will show that it holds for $p - 1$. In particular, at step $p - 1$ consider node $v_{p-1}$ labelled by $w_{p-1}$. It suffices to show that if there exists a node $u$ of $\tau_{W}^{(j-1)}(k)$ at distance $i$ from thee root such that $[u] \sim w_{p-1}$, then $v_{p-1}$ cannot be at distance $i$ from the root in $\tau_{W}^{(j-1)}(k)$. To see this, notice that $v_{p-1}$ is eligible for being a child of $u$.

Proposition 3. For a witness sequence $W$ of length $t$ and any two distinct $i, j \in [t]$ we have that $\tau_W(i) \neq \tau_W(j)$.

Proof. Assume w.lo.g. that $i < j$. If $w_i \neq w_j$ then the claim is straightforward because the root of $\tau_W(i)$ is $w_i$ while the root of $\tau_W(j)$ is $w_j$. If $w_i = w_j = w$, then there are two cases. In the first case, $w \in \Gamma(w)$, and so tree $\tau_W(j)$ has at least one more vertex than $\tau_W(i)$. In the second case $w \notin \Gamma(w)$. This implies that at the $i$-th step of any trajectory $\Sigma$ such that $W(\Sigma) = W$, flaw $f_w$ was addressed and removed. However, the fact
that \( w_j = w \) implies that there has to be \( k \in (i, j) \) such that addressing \( w_k \) introduced \( w \) and thus, \( w_k \sim w \). Again, this means that \( \tau_W(j) \) has at least one more vertex than \( \tau_W(i) \).

\( \square \)

### 4.1.2 Stable Set Sequences

We will now recall the definition of stable sequences, that was first introduced in [35] to make the Shearer’s criterion constructive in the variable setting. Variations of this notion have also been used by [31] [37] to make the Shearer’s criterion constructive in the more general settings of the algorithmic Local Lemma. Here we will use the following definition:

**Definition 6.** A sequence of subsets \( (I_1, \ldots, I_k) \) of \([m]\) with \( k \geq 1 \) is called stable if

1. \( I_r \in \text{Ind}([m]) \setminus \{\emptyset\} \) for each \( r \in [k] \)
2. \( I_{r+1} \subseteq \Gamma(I_r) \) for each \( r \in [k-1] \).

**Definition 7.** A witness sequence \( W = (w_1, \ldots, w_t) \) is called stable if it can be partitioned into non-empty sequences as \( W = (W_1, \ldots, W_k) \) such that the elements of each sequence \( W_r \) are distinct, and the sequence \( \phi_W := (I_1, \ldots, I_k) \) is stable, where \( I_r \) is the set of indices of flaws in \( W_r \) (for \( r \in [k] \)).

For any arbitrary ordering \( \pi \) among indices of flaws, if in addition each sequence \( W_r = (w_i, \ldots, w_j) \), satisfies \( w_i \prec \pi \ldots \prec \pi w_j \) then \( W \) is called \( \pi \)-stable.

**Proposition 4.** ([37]) For a stable witness sequence the partitioning in Definition 7 is unique.

**Proof.** Let \( W = (w_1, \ldots, w_t) \) be a stable sequence and consider the following algorithm. We start with a single segment containing \( w_1 \). For \( i = 2 \) to \( t \), if there exists index \( w_k \) in the currently last segment such that \( w_k \sim w_i \) then we start a new segment containing \( w_i \). Otherwise, we add \( w_i \) to the currently last segment. \( \square \)

For a witness sequence \( W = (w_1, \ldots, w_t) \) let \( \text{Rev}[W] = (w_t, \ldots, w_1) \) denote the reverse sequence. Let also \( \chi_W \) denote the first set (the “root”) of the stable sequence \( \phi_W := (I_1, \ldots, I_k) \), i.e., \( \chi_W = I_1 \). Finally, let \( \mathcal{R}^\pi_W \) be the set of witness sequences \( W \) such that \( \text{Rev}[W] \) is \( \pi \)-stable and \( R_{\text{Rev}[W]} = \{i\} \).

There is a connection between stable sequences and witness trees that we will need for the proofs of Theorem 1 and which we will describe below.

Let \( \mathcal{W}_i \) denote the set of witness trees with root labelled by \( i \). For each \( \tau \in \mathcal{W}_i \), let \( \chi_\pi(\tau) \) be the ordered witness tree that is induced by ordering the children of each node in \( \tau \) from left to right, increasingly according to \( \pi \). Define \( \mathcal{W}_\pi^i := \chi_\pi(\mathcal{W}_i) \) and observe that \( \chi_\pi \) is a bijection. Finally, recall that for a witness tree \( \tau \) we denote by \( L_j(\tau) \) the set of labels of the nodes at distance \( j \) from the root.

**Lemma 1.** There is a bijection \( \chi^\pi_i \) mapping \( \mathcal{R}^\pi_i \) to \( \mathcal{W}_\pi^i \) with the following property: Let \( W \in \mathcal{R}^\pi_i \) and let \( (I_1 = \{i\}, I_2, \ldots, I_k) \) be the unique partitioning of \( \text{Rev}[W] \) guaranteed by Definition 7. Then, \( I_j = L_{j-1}(\chi^\pi_i(W)) \) for each \( j \in [k] \).

**Proof.** Consider a witness sequence \( W \in \mathcal{R}^\pi_i \) of length \( t \). We define \( \chi^\pi_i(W) := \chi_\pi(\tau_W(t)) \). That is, we map \( W \) to the \( \pi \)-ordered witness tree that is induced by applying the algorithm that constructs witness trees to the final element of the sequence. Recall now the algorithm in the proof of Proposition 4. The key observation is that when this algorithm is applied to \( \text{Rev}[W] \), it is essentially equivalent to the algorithm that constructs \( \tau_W(t) \) in the sense that the decisions taken for partitioning \( \text{Rev}[W] \) to segments by the algorithm of Proposition 4 are identical to the decisions taken by the algorithm that constructs \( \tau_W(t) \) for forming
$L_j(\tau_W(t)), j \geq 0$. In particular, if $\phi_{Rev[W]} = (I_1 = \{i\}, I_2, \ldots, I_k)$ is the unique partitioning of $Rev[W]$, then $I_j = L_{j-1}(\tau_W(t)) = L_{j-1}(\chi^\pi_W(t))$ for each $j \in [k]$.

It remains to show that $\chi^\pi_W$ is bijection. To see this, at first observe that from $\chi^\pi_W(t)$ one can uniquely reconstruct $\phi_{Rev[W]}$. Given $\phi_{Rev[W]}$ one can reconstruct $Rev[W]$ (and, thus, $W$) by ordering each segment of $\phi_{Rev[W]}$ according to $\pi$.

4.1.3 Counting Witness Trees

In our proofs we will need to bound the sum over all trees $\tau \in W_i$ of the product of charges of the (labels of) the nodes of each tree $\tau$. Fortunately, the method for doing that is well trodden by now (see for example [41, 43]). Here we show the following lemma whose proof can be found in the appendix. Recall that $W_i$ denotes the set of all possible witness trees with root that is labelled by $i$.

Lemma 2. If the Cluster Expansion condition is satisfied then:

$$\sum_{\tau \in W_i} \prod_{v \in V(\tau)} \gamma(f_{[v]}) \leq \psi_i.$$ 

We also show the following lemma that can be used whenever the Shearer’s condition applies.

Lemma 3. If the Shearer’s condition is satisfied then:

$$\sum_{\tau \in W_i} \prod_{v \in V(\tau)} \gamma(f_{[v]}) \leq \frac{q_{\{i\}}(\gamma)}{q_0(\gamma)}.$$ 

Proof. We first observe that due to Lemma 1 we have that

$$\sum_{\tau \in W_i} \prod_{v \in V(\tau)} \gamma(f_{[v]}) = \sum_{\phi \in \text{Stab}_i} \prod_{i \in I} \gamma(f_{[v]}) = \sum_{W \in \mathcal{R}_i^\pi} \prod_{w \in W} \gamma(f_{[w]}).$$

Now let $\text{Stab}_i$ denote the set of stable set sequences whose first segment is $\{i\}$ and also every segment is non-empty. For $\phi = (I_1, I_2, \ldots, I_k) \in \text{Stab}_i$ define $\gamma_\phi = \prod_{i=1}^k \prod_{i \in I} \gamma(f_i)$. Observe that there is a natural injection from $\mathcal{R}_i^\pi$ to $\text{Stab}_i$ which maps each sequence $W \in \mathcal{R}_i^\pi$ to $\phi_{Rev[W]}$. This is because given $\phi_{Rev[W]}$ one can reconstruct $Rev[W]$ (and thus, $W$) by ordering each segment of $\phi_{Rev[W]}$ according to $\pi$. The latter observation implies that:

$$\sum_{\tau \in W_i} \prod_{v \in V(\tau)} \gamma(f_{[v]}) \leq \sum_{\phi \in \text{Stab}_i} \gamma_\phi = \frac{q_{\{i\}}(\gamma)}{q_0(\gamma)},$$

where the proof of the last inequality can be found in Theorem 14 of [35] and in Lemmata 5.26, 5.27 of [31].

Remark 5. We note that if we have assumed a stronger “cluster expansion condition” (namely, in [4] we have $\Gamma(i) \cup \{i\}$ instead of $\Gamma(i)$ then Corollary 2 could have also been shown as an immediate application of Lemma 3 since it is known ([8, 37, 37]) that, in this case, for every $i \in [m]$ we have that $\frac{q_{\{i\}}(\gamma)}{q_0(\gamma)} \leq \psi_i$. 

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4.2 Proof of Theorems 2 and 3

We first prove Theorem 2. The first part follows by Theorem 1, Lemma 2 and Proposition 3 that suggest:

\[ \mathbb{E}[N_i] \leq \lambda_{\text{init}} \sum_{\tau \in \mathcal{W}_i} \prod_{v \in V(\tau)} \gamma(f_v) \leq \lambda_{\text{init}} \psi_i. \]

To see the second part of Theorem 2, consider the new set of flaws \( F' = F \cup \{ f_{m+1} \} \), where \( f_{m+1} = E \), as well as a “truncated” commutative extension \( \mathcal{A}' \) of \( \mathcal{A} \) with the following properties:

(i) For each state \( \sigma \notin f_{m+1} \) algorithm \( \mathcal{A}' \) invokes \( \mathcal{A} \) to choose the next state.

(ii) \( \gamma(E) := \gamma_{\mathcal{A}'}(f_{m+1}) \).

(iii) \( f_{m+1} \) is always of the highest priority: when at a state \( \sigma \in f_{m+1} \), \( \mathcal{A}' \) chooses to address \( f_{m+1} \).

(iv) \( \mathcal{A}' \) stops after it addresses \( f_{m+1} \) for the first time.

By coupling \( \mathcal{A} \) and \( \mathcal{A}' \) we see that \( \Pr[\mathcal{A}[E]] = \Pr[\mathcal{A}'[f_{m+1}]] \). Let \( \mathcal{W}_E \) be the set of witness trees that might occur in an execution of \( \mathcal{A}' \) and whose root is labelled by \( m+1 \). Notice that due to property (iv) of \( \mathcal{A}' \) every tree \( \tau \in \mathcal{W}_E \) contains exactly one node (the root) labelled by \( m+1 \), while every other node is labelled by elements in \( [m] \). Furthermore, the set of labels of the children of the root of \( \tau \) is an element of \( \text{Ind}(\Gamma(E)) \).

Finally, if \( v \) is a node that corresponds to a child of the root in \( \tau \), then the subtree \( \tau_v \) that is rooted at \( v \) is an element of \( \mathcal{W}[v] \). Using Theorem 1 and the fact that \( \mathcal{A}' \) is commutative we get:

\[ \Pr[\mathcal{A}[E]] \leq \sum_{\tau \in \mathcal{W}_E} \Pr[\mathcal{A}'[\tau]] \leq \lambda_{\text{init}} \gamma(E) \sum_{S \in \text{Ind}(\Gamma(E))} \left( \prod_{j \in S} \sum_{\tau \in \mathcal{W}_j} \prod_{v \in \tau} \gamma([v]) \right) \leq \lambda_{\text{init}} \gamma(E) \sum_{S \in \text{Ind}(\Gamma(E))} \prod_{j \in S} \psi_j, \]

where the last equality follows from Corollary 2. The proof of Theorem 3 is the same as the one of Theorem 2 where instead of Lemma 2 we use Lemma 3.

4.3 Proof of Theorem 1

Throughout the proof, we will be using ideas and definitions from [37]. We also note that we will assume w.l.o.g. that algorithm \( \mathcal{A} \) follows a deterministic flaw choice strategy. This is because randomized flaw choice strategies can equivalently be interpreted as convex combination of deterministic ones (and therefore, randomized strategies can be seen as taking expectation over deterministic ones).

For a trajectory \( \Sigma \) of length \( t \) we define

\[ p(\Sigma) = \lambda_{\text{init}} \prod_{i=1}^{t} \rho_{w_i}(\sigma_i, \sigma_{i+1}) \]

and notice that \( \Pr[\Sigma] \leq p(\Sigma) \). Furthermore, we say that a trajectory \( \Sigma' \) is a proper prefix of \( \Sigma \) if \( \Sigma' \) is a prefix of \( \Sigma \) and \( \Sigma' \neq \Sigma \).

**Definition 8 ([37]).** A set \( \mathcal{X} \) of trajectories of the algorithm will be called valid if (i) all trajectories in \( \mathcal{X} \) follow the same deterministic flaw choice strategy (not necessarily the same used by \( \mathcal{A} \)), and (ii) for any \( \Sigma, \Sigma' \in \mathcal{X} \) trajectory \( \Sigma \) is not a proper prefix of \( \Sigma' \).
Lemma 4 ([37]). Consider a witness sequence $W = (w_1, \ldots, w_t)$ and a valid set of trajectories $\mathcal{X}$ such that $W$ is a prefix of $W(\Sigma)$ for every $\Sigma \in \mathcal{X}$. Then:

$$\sum_{\Sigma \in \mathcal{X}} p(\Sigma) \leq \lambda_{\text{init}} \prod_{i=1}^{t} \gamma(f_{w_i}) ,$$

The main idea now will be to construct a “swapping mapping” whose goal will be to transform trajectories of the algorithm to a form that satisfies certain properties by applying “swaps”.

For a trajectory $\Sigma$ in which a tree $\tau \in \mathcal{W}_i$ occurs, we denote by $W_{\tau}^{\Sigma}$ the prefix of $W(\Sigma)$ up to the step that corresponds to the root of $\tau$ (observe that Proposition 5 mandates that there exists a unique such step). Notice that, since $\tau \in \mathcal{W}_i$, the algorithm addresses flaw $f_i$ at this step, and thus the final element of $W_{\tau}^{\Sigma}$ is $\{i\}$. Finally, recall the definitions of $\mathcal{R}_i^\Sigma, \chi_\pi$ and $\chi_\pi^f$.

Lemma 5. Fix a witness tree $\tau \in \mathcal{W}_i$ and let $\mathcal{X}_\tau^\Sigma$ be a valid set of trajectories in which $\tau$ occurs. If $\mathcal{A} = (D, F, \sim, \rho)$ is commutative then there exists a set of trajectories $\mathcal{X}_\tau^\Sigma$ and a swapping mapping $\Phi^\tau : \mathcal{X}^\tau \rightarrow \mathcal{X}_\tau^\Sigma$ which is a bijection such that

(a) for any $\Sigma \in \mathcal{X}_\tau^\Sigma$ we have that $W_{\tau}^{\Sigma}$ is the unique witness sequence in $\mathcal{R}_i^\Sigma$ such that $\chi_\pi^f(W_{\tau}^{\Sigma}) = \chi_\pi(\tau)$;

(b) for any witness sequence $W$ the set $\{\Sigma \in \mathcal{X}_\tau^\Sigma \mid \text{Rev}[W_{\tau}^{\Sigma}] = W\}$ is valid.

We prove Lemma 5 in Section 4.4. To see Theorem 1, consider a witness tree $\Sigma$ which there exists a trajectory $\Sigma$ such that $\Sigma$ is a proper prefix of $\Sigma$ to get $\mathcal{X}_\tau^\Sigma$. Clearly, this is a valid set and so recalling that $\chi_\pi$ is a bijection and applying Lemma 5 we have that:

$$\Pr[\tau] = \sum_{\Sigma \in \mathcal{X}_\tau^\Sigma} \Pr[\Sigma] \leq \sum_{\Sigma \in \mathcal{X}_\tau^\Sigma} p(\Sigma) = \sum_{\Sigma \in \mathcal{X}_\tau^\Sigma} p(\Sigma) ,$$

where to get the second equality we use the second requirement of Definition 1. Lemma 5 further implies that for every trajectory $\Sigma \in \mathcal{X}_\tau^\Sigma$ we have that $W_{\tau}^{\Sigma}$ is the (unique) witness sequence in $\mathcal{R}_i^\Sigma$ such that $\chi_\pi^f(W_{\tau}^{\Sigma}) = \chi_\pi(\tau)$, i.e., $W_{\tau}^{\Sigma} = (\chi_\pi^f)^{-1}(\chi_\pi(\tau))$. This means that the witnesses of the trajectories in $\mathcal{X}_\tau^\Sigma$ have $W := (\chi_\pi^f)^{-1}(\chi_\pi(\tau))$ as a common prefix. Since part (b) of Lemma 5 implies that $\mathcal{X}_\tau^\Sigma$ is valid, applying Lemma 4 we get:

$$\sum_{\Sigma \in \mathcal{X}_\tau^\Sigma} p(\Sigma) \leq \lambda_{\text{init}} \prod_{w \in W} \gamma(f_w) = \lambda_{\text{init}} \prod_{v \in V(\tau)} \gamma(f_{[v]}) ,$$

where the second inequality follows from the fact that $\chi_\pi^f(W) = \chi_\pi(\tau)$ and $V(\tau) = V(\chi_\pi(\tau))$, concluding the proof.

4.4 Proof of Lemma 5

Our proof builds on the proof of Theorem 19 in [37]. We will be denoting witness sequences $W = \{w_1, w_2, \ldots, w_t\}$ as a sequence of named indices of flaws $W = \{w_1, \ldots, w_t\}$ where $w_j = (w_j, n_j)$ and $n_j = \{k \in [j] \mid w_k = w_j\}$ is the number of occurrences of $w_j$ in the length-$j$ prefix of $W$. Note that a named index $w$ cannot appear twice in a sequence $W$. Finally, if $w$ is a named index of flaw we will be denoting by $w$ (that is, without bold font) the flaw index that is associated with it.

For a trajectory $\Sigma$ such that $W(\Sigma) = \{w_1, \ldots, w_t\}$ we define a directed acyclic graph $G(\Sigma) = (V(\Sigma), E(\Sigma))$ where $V(\Sigma) = \{w_1, \ldots, w_t\}$ and $E(\Sigma) = \{(w_j, w_k) \mid w_j \sim w_k \text{ and } j < k\}$. This
means that we have an edge from a named flaw \( w_i \) to another flaw \( w_j \) whenever their corresponding flaw indices are related according to \( \sim \) and \( w_j \) occurs in \( \Sigma \) before \( w_k \).

By Proposition \([3] \) for any trajectory \( \Sigma \) in which \( \tau \) occurs there is a unique step \( t^* = t^*(\Sigma) \) such that \( \tau_W(\Sigma)(t^*) = \tau \). For such a trajectory \( \Sigma \), let \( Q(\Sigma) \subseteq V(\Sigma) \) be the set of flaws from which the node \( w_{\tau^*} \) can be reached in \( G(\Sigma) \), where \( w_{\tau^*} \) is the named flaw index that corresponds to the step \( t^* \). Notice that \( w_{\tau^*} = i \) (since \( \tau \in \mathcal{W} \)). For \( w \in Q(\Sigma) \) let \( d(w) \) be the length of the longest path from \( w \) to \( w_{t^*} \) in \( G(\Sigma) \) plus one.

For example, \( d(w_{t^*}) = 1 \).

Let \( Q(\Sigma) \) denote the sequence consisting of the named flaws in \( Q(\Sigma) \) listed in the order they appear in \( \Sigma \). The idea is to repeatedly apply the operation Swap to \( \Sigma \) so that we reach a trajectory \( \Sigma' \) that has a permutation \( Q_\pi(\Sigma) \) of \( Q(\Sigma) \) as a prefix. In particular, we will show that \( Q_\pi(\Sigma) \in R_\pi^{\tau} \) and \( \chi_\pi^\tau(Q_\pi(\Sigma)) = \tau \).

To that end, for an integer \( r \geq 1 \) define \( I_r = \{ w \in Q(\Sigma) \mid d(w) = r \} \), and let \( Q_r \) be the sequence consisting of the named flaw indices in \( I_r \) sorted in decreasing order with respect to \( \pi \). Then, we define \( Q_\pi(\Sigma) = (Q_s, \ldots, Q_1) \) where \( s = \max\{d(w) \mid w \in Q(\Sigma)\} \).

**Lemma 6.** \( Q_\pi(\Sigma) \in R_\pi^{\tau} \) and \( \chi_\pi^\tau(Q_\pi(\Sigma)) = \chi_\pi(\tau) \).

**Proof.** Let \( Y = \tau(\Sigma) = \text{Rev}([Q_\pi(\Sigma)] = (Q_1, Q_2, \ldots, Q_s) \) be the reverse sequence of \( Q_\pi(\Sigma) \). By definition, \( R_Y = R_1 \) it suffices to show that \( Q_I \subseteq \Gamma(Q_i) \) for each \( i \in [s - 1] \).

To see this, recall the definitions of \( I_{r+1} \) and \( Q_{r+1} \) and observe that, for each \( i_r, i_{r+1} \in Q_{r+1} \), there must be a path of \( r \) indices of flaws \( i_r, i_{r-1}, \ldots, i_1 \) such that for every \( j \in [r - 1] \) we have that \( i_j \in Q_j \) and \( i_j \sim i_{j+1} \).

Let \( k \) be the number of elements in witness sequence \( Q(\Sigma) \). Recall that \( \chi_\pi(Q_\pi(\Sigma)) := \chi_\pi(\tau_{Q_\pi(\Sigma)}(k)) \) (proof of Lemma \([1] \)). The proof is concluded by also recalling the algorithm for constructing witness trees and observing that \( \tau_{Q_\pi(\Sigma)}(k) = \tau_W(\Sigma)(t^*) = \tau \).

Note that applying Swap to \( \Sigma \) does not affect graph \( G(\Sigma) \) and set \( Q(\Sigma) \) and, thus, neither the sequence \( Q_\pi(\Sigma) \). With that in mind, we show next how we could apply Swap repeatedly to \( \Sigma \in \mathcal{X}^\tau \) to reach a \( \Sigma' \) such \( Q_\pi(\Sigma) \) is a prefix of its witness sequence (that is, \( W(\Sigma') = (Q_\pi(\Sigma), U) \)). We will do this by applying swaps to swappable pairs in \( \Sigma \).

**Definition 9.** Consider a trajectory \( \Sigma \in \mathcal{X}^\tau \). A pair \((w, y)\) of named indices of flaws is called a swappable pair in \( \Sigma \) if it can be swapped in \( \Sigma \) (i.e., \( W(\Sigma) = (\ldots, w, y \ldots) \) and \( w \sim y \)) and either

1. \((w, y) \in (V(\Sigma) \setminus Q(\Sigma)) \times Q(\Sigma), \) or
2. \((w, y) \in Q(\Sigma) \times Q(\Sigma) \) and their order in \( Q_\pi(\Sigma) \) is different: \( Q_\pi(\Sigma) = (\ldots, y, w, \ldots) \)

The position of the rightmost swappable pair in \( \Sigma \) will be denoted as \( k(\Sigma) \), where the position of \((w, y)\) in \( \Sigma \) is the number of named indices of flaws that precede \( y \) in \( W(\Sigma) \). If \( \Sigma \) does not contain a swappable pair then \( k(\Sigma) = 0 \). Thus, \( k(\Sigma) \in [0, |\Sigma| - 1] \).

We can only apply finite many swaps to swappable pairs in \( \Sigma \). This is because swapping pairs of the first form moves a named index in \( Q(\Sigma) \) to the left, while swapping pairs of the second one decrease the number of pairs whose relative order in \( Q(\Sigma) \) is not consistent with the one in \( Q_\pi(\Sigma) \). Clearly, both of these actions can be performed only a finite number of times.

The following lemma shows how we can obtain a mapping \( \Phi^\tau \) such that \( \mathcal{X}_\pi^\tau := \Phi^\tau(\mathcal{X}^\tau) \) satisfies the first condition of Lemma \([5] \). The proof is identical (up to minor changes) to the one of Lemma 27 of \([37] \).

We include it in Section \([A] \) of the appendix for completeness.
Lemma 7. Consider a trajectory $\Sigma \in \mathcal{X}^\tau$ such that $W(\Sigma) = (A,U)$ where $A$ and $U$ are some sequences of indices of flaws, and there are no swappable pairs inside $U$. Then $U = (B,C)$ where sequence $B$ is a subsequence of $Q_{\pi}(\Sigma)$ and $C$ does not contain named indices of flaws from $Q(\Sigma)$.

In particular, if $|A| = 0$ and $W(\Sigma) = U$ does not contain a swappable pair then $W(\Sigma) = (Q_{\pi}(\Sigma), C)$.

It remains to show that $\Phi^\tau$ can be constructed so that it is also a bijection and that it satisfies the second condition of Lemma 5. To do so, consider the following algorithm.

- Let $X_0^\tau = X$.
- While $k = \max_{\Sigma \in X_p} k(\Sigma) \neq 0$
  - For each $\Sigma \in X_p$: if $k(\Sigma) = k$ then swap the pair $(w, y)$ at position $k$ in $\Sigma$, otherwise leave $\Sigma$ unchanged.
  - Let $X_{p+1}$ the new set of trajectories.

For a witnesses sequence $W$ define $X_p^W = \{ \Sigma \in X_p \mid Q_{\pi}(\Sigma) = W \}$ for an index $p \geq 0$. Now the following lemma concludes the proof since $X_0^W \subseteq \mathcal{X}^\tau$ is valid. Its proof is identical (up to minor changes) to the proof of Lemma 28 in [37]. We also include it in Section A of the appendix for completeness.

Lemma 8. If set $X_p^W$ is valid then so is $X_{p+1}^W$, and the mapping from $X_p^W$ to $X_{p+1}^W$ defined by the algorithm above is injective.

5 Dealing with Super-Polynomially Many Flaws

In this section we discuss how one can deal with problems where the number of flaws is super-polynomial in the natural size of the problem using commutative algorithms. For example, such a problem is the one of acyclic edge coloring which we will see in section 6.3.

In such a setting, there are two issues to be resolved. The first issue is that one should be able to show that the expected number of steps until convergence is polynomial, and thus, much less than $\Theta(|F|)$. The second issue is that one should have an efficient procedure for finding a flaw that is present in the current state, or decide that no such flaw exists.

Polynomial-Time Convergence As far as the issue of polynomial-time convergence is concerned, there are at least three approaches one can follow. A first one is to start the algorithm at a state $\sigma_1$ in which the set of flaws present is of polynomial size, and then employ the main results from [3, 2, 37] which guarantee that the algorithm will terminate after $O \left( |U(\sigma_1)| + \max_{\sigma \in \Omega} \log_2 \frac{1}{p(\sigma)} \right)$ steps with high probability. This approach does not require the algorithm to be commutative, but it does require a multiplicative slack in the condition used to establish quick termination. A second approach, which was first applied in the context of the Moser Tardos algorithm by Haeupler, Saha and Srinivasan [23], is to find a core set of flaws of polynomial size and apply a modified version of the algorithm that effectively ignores any non-core flaw. The hope is that non-core flaws will never occur during the execution of this modified algorithm. Extended to our setting, one uses the following theorem which is a straightforward corollary of Theorem 2.

Theorem 6. Assume that $A = (D, F, \sim, \rho)$ is commutative. Let $I \subseteq [m]$ be a set of indices that corresponds to a core subset of $F$ and assume there exist a sequence of positive real numbers $\{\psi_i\}_{i=1}^m$ such that for each $i \in [m]$: \[\sigma_i = \max_{\sigma : \psi_i(\sigma) < \psi_i(\sigma')} \psi_i(\sigma') \]
Theorem 8. Let \( A = (D, F, \sim, \rho) \) be a commutative algorithm such that the causality graph induced by \( \sim \) can be decomposed into \( n \) cliques covering all its vertices. If there exists a sequence of positive reals \( \{x_i\}_{i=1}^m \) in \( (0, 1) \) such that for each \( i \in [m] \) we have that:

\[
\gamma(f_i) \leq x_i \prod_{j \in \Gamma(i)} (1 - x_j)
\]

then algorithm \( A \) terminates in at most \( n \log(1/\delta) \max_{i \in [m]} \frac{1}{1 - x_i} \) expected number of steps, where \( \delta = \min_{i \in [m]} x_i \prod_{j \in \Gamma(i)} (1 - x_j) \).

Remark 6. In [23] it is argued that in the vast majority of applications \( \delta = O(n \log n) \) and many times even linear in \( n \).

Fast Search for Flaws  Searching for occurring flaws efficiently can be a major obstacle in getting polynomial time algorithms, even in the case where convergence is guaranteed after a polynomial number of steps. Again, there is more than one approach one can follow to deal with this issue.

A first approach was introduced in [23] where it is shown that the “variable setting version” of Theorems 6 and 7 can be combined into a single theorem that guarantees the existence of a Monte Carlo algorithm which runs in polynomial time, even in the presence of super-polynomially many flaws. The theorem assumes the existence of a polynomial size decomposition of the causality graph into cliques and, moreover, an exponential slack in the General LLL conditions. Using Theorem 2 we can extend the latter result in a straightforward way to our setting to get:

**Theorem 7.** Let \( A = (D, F, \sim, \rho) \) be a commutative algorithm such that the causality graph induced by \( \sim \) can be decomposed into \( n \) cliques covering all its vertices. If there exists a sequence of positive reals \( \{x_i\}_{i=1}^m \) in \( (0, 1) \) such that for each \( i \in [m] \) we have that:

\[
\gamma(f_i) \leq x_i \prod_{j \in \Gamma(i)} (1 - x_j)
\]

Then there exists a modification of \( A \) that terminates in an expected number of \( O \left( \sum_{i \in I} x_i \right) \) steps and outputs a flawless element with probability at least \( 1 - \sum_{i \in [m] \setminus I} \mu(f_i) \sum_{S \in \Ind(\Gamma(f_i))} \prod_{j \in S} \psi_j \).

Finally, a third approach is to show that the causality graph can be decomposed into a set of cliques of polynomial size and then apply a result (Theorem 3.1) of [23] that says that, in this case, the running time of the algorithm is polynomial (roughly quadratic) in the size of the decomposition. To be more precise, we note that in [23] the latter result is shown for the Moser-Tardos algorithm under the General LLL conditions, where the clique decomposition considered is induced by the random variables that form the probability space (one clique per variable). However, the proof for the general case is identical. Using Theorem 2 and Remark 6 we can extend the latter result to our setting to get:

\[
\gamma(f_i) \sum_{S \subseteq \Ind(\Gamma(i) \setminus I)} \prod_{j \in S} \psi_j \leq \psi_i.
\]
In a follow-up work [29], Harris and Srinivasan describe a general technique that yields efficient procedures for searching for flaws. The main building blocks of their technique is a “witness tree lemma for internal states” and problem-specific, possibly randomized, data-structures that contain the flaws that are present in each state. We refer the reader to [29] for more details, but we note that combining the proof of [29] with the proof of Theorems 1 and 2 one can show that the “witness tree lemma for internal states” holds for commutative algorithms.

6 Applications

6.1 Rainbow Matchings

In an edge-colored graph $G = (V, E)$, say that $S \subseteq E$ is rainbow if its elements have distinct colors. In this section we consider the problem of finding rainbow matchings in complete graphs of size $2n$, where each color appears a limited amount of times.

Applying the cluster expansion condition, it can be shown [3, 31] that any edge-coloring of a complete graph of size $2n$ in which each color appears on at most $\frac{27}{128} n \approx 0.211n$ edges admits a rainbow perfect matching that can be found efficiently. Furthermore, in [37] it is shown that the resampling oracles defined by [31] for the space of matchings in a clique of even size, and which are used in this particular application, induce commutative algorithms. The latter fact that will allow us to use our results to further study this problem.

6.1.1 Finding Rainbow Perfect Matchings

We first formulate the problem to fit our setting and use Theorem 2 to show that the algorithm of [3, 31] finds a perfect rainbow matching efficiently. Assuming a multiplicative slack in the cluster expansion conditions, a running time (number of steps) of $O(n)$ can be given using the results of [3, 31, 37]. However, the best known upper bound without this assumption was given in [31] to be $O(n^2)$. Here we improve the latter to $O(n)$.

Let $\phi$ be any edge-coloring of $K_{2n}$ in which each color appears on at most $\lambda n$ edges. Let $P = P(\phi)$ be the set of all pairs of vertex-disjoint edges with the same color in $\phi$, i.e., $P = \{\{e_1, e_2\} : \phi(e_1) = \phi(e_2)\}$. Let $\Omega$ be the set of all perfect matchings of $K_{2n}$. For each $\{e_i, e_j\} \in P$ let

$$f_{i,j} = \{M \in \Omega : \{e_i, e_j\} \subset M\}.$$

Thus, an element of $\Omega$ is flawless if it is a rainbow perfect matching. The algorithm that finds a rainbow perfect matching starts at a state of $\Omega$ chosen uniformly at random and, at every subsequent step, it chooses (arbitrarily) a flaw to address. Algorithm 1 describes the probability distributions $\rho_{i,j}(M, \cdot)$, where $M \in f_{i,j}$. This is a special case of the implementation of a general resampling oracle with respect to the uniform measure over $\Omega$ for perfect matchings described in [31]. For the problem of rainbow matchings, the latter implies that $\gamma(f_{i,j}) = \mu(f_{i,j}) = \frac{1}{(2n-1)(2n-3)}$. 

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Furthermore, there exists an algorithm that outputs each one of them with positive probability.

For a vertex \( v \) let \( \Gamma(v) \) denote the set of indices of flaws that correspond to edges adjacent to \( v \). By observing the algorithm it’s not hard to verify (and is also proved in \([2, 31, 37]\)) that the graph \( C \) over indices of flaws such that for each \((e_i = (v_1, v_2), e_j = (v_3, v_4)) \in P \) we have that

\[
\Gamma(i, j) = \bigcup_{i=k}^{4} \Gamma(v_k)
\]

is a causality graph. Furthermore, if \( S \in \text{Ind}(\Gamma(i, j)) \) then for each \( k \in [4] \) we have that \( |S \cap \Gamma(v_k)| \leq 1 \). This means that \( |S| \leq 4 \) and, moreover, for each \( j \in [0, 4] \) there are at most \( \binom{4}{j}(2n-1)^j(\lambda n - 1)^j \) subsets \( S \in \text{Ind}(\Gamma(i, j)) \) of size \( j \). Choosing parameters \( \psi_{i, j} = \psi = \frac{3}{4n^2} \) we have that:

\[
\gamma(f_{i, j})\xi_{i, j} := \gamma(f_{i, j}) \sum_{S \in \text{Ind}(\Gamma(i, j))} \psi^{|S|} \leq \frac{1}{(2n-3)(2n-1)}(1 + (2n-1)(\lambda n - 1))^4,
\]

from which it can be seen that whenever \( \lambda \leq \frac{27}{128} \) we have that \( \gamma(f_{i, j})\xi_{i, j} \leq 1 \) and so the cluster expansion condition is satisfied.

Since \( |P| \leq (2n)^2 \cdot (\lambda n - 1) < 4\lambda n^3 \) Theorem\([2]\) implies that the algorithm terminates after an expected number of \( 3\lambda n \) steps. Overall, we have showed the following theorem.

**Theorem 9.** For any \( \lambda \leq \frac{27}{128} \), given any edge-coloring of the complete graph on \( 2n \) vertices in which each color appears on at most \( \lambda n \) edges, there exists an algorithm that terminates in an expected number of at most \( 3\lambda n \) steps and outputs a rainbow perfect matching.

### 6.1.2 Number of Rainbow Perfect Matchings

In this subsection we use Theorem\([4]\) to give an exponential lower bound on the number of perfect matchings when each color appears at most \( \lambda n \) times, where \( \lambda \leq \frac{27}{128} \), by bounding the entropy of the output distribution of the algorithm described in the previous subsection.

**Theorem 10.** For any \( \lambda \leq \frac{27}{128} \), given any edge-coloring of the complete graph on \( 2n \) vertices in which each color appears on at most \( \lambda n \) edges, there exist at least \( \frac{1}{2} e^{-3\lambda n} \cdot (2n - 1)!! \) rainbow perfect matchings. Furthermore, there exists an algorithm that outputs each one of them with positive probability.

**Proof.** To apply Theorem\([4]\) we will need to estimate (give an upper bound for) \( \sum_{S \in \text{Ind}(|m|)} \psi^{|S|} \). Similarly to applications in \([29]\), we will find useful the following crude, but general upper bound:

\[
\text{Recall that } (2n - 1)!! = 1 \cdot 3 \cdot \ldots \cdot (2n - 1) = \frac{(2n)!}{2^n n!}.
\]
\[
\sum_{S \in \text{Ind}([m])} \prod_{j \in S} \psi_j \leq \sum_{S \subseteq [m]} \prod_{j \in S} \psi_j \leq \prod_{i \in [m]} (1 + \psi_i) \leq \exp \left( \sum_{i \in [m]} \psi_i \right).
\]

(7)

Since the number of perfect matching in \(K_{2n}\) is \((2n - 1)!!\) and also \(|P| < 4\lambda n^3\), Theorem 4 and (7) imply that the number of rainbow perfect matchings is at least

\[
\exp \left( \ln |\Omega| - \sum_{j \in F} \psi \right) \geq \exp \left( \ln ((2n - 1)!!) - 3\lambda n \right) = \frac{(2n - 1)!!}{e^{3\lambda n}}.
\]

\[ \square \]

### 6.1.3 Low Weight Rainbow Perfect Matchings

Consider an arbitrary weighting function \(W : E \rightarrow \mathbb{R}\) over the edges of \(K_{2n}\). Here we consider the problem of finding rainbow perfect matchings of low weight, where the weight of a matching is defined as the sum of weights of its edges. Clearly, there is a selection of \(n\) edges of \(K_{2n}\) whose total weight is at most \(\frac{1}{2n-1} \sum_{e \in K_{2n}} W(e)\). We use Theorem 2 to show that, whenever \(\lambda \leq \frac{27}{128}\), the algorithm of subsection 6.1.1 outputs a rainbow perfect matching of similar expected weight.

**Theorem 11.** For any \(\lambda \leq \frac{27}{128}\), given any edge-coloring of the complete graph on \(2n\) vertices in which each color appears on at most \(\lambda n\) edges there exists an algorithm that outputs a perfect rainbow matching \(M\) such that

\[
\mathbb{E}[W(M)] \leq \frac{(1 + \frac{3}{2}\lambda)^2}{2n - 1} \sum_{e \in K_{2n}} W(e).
\]

**Proof.** Let \(A_e\) be the subset of \(\Omega\) that consists of the matchings that contain \(e\). It is proven in [31], and it’s also not hard to verify, that Algorithm \([1]\) with \(A = \{e\}\) is a resampling oracle for this type of flaw. Moreover, using an identical counting argument to the one of subsection 6.1.1 we get that:

\[
\sum_{S \in \text{Ind}(\Gamma(A_e))} \psi^{|S|} \leq (1 + (2n - 1)(\lambda n - 1)\psi)^2.
\]

Applying Theorem 2 we get that:

\[
\mathbb{E}[W(M)] \leq \sum_{e \in K_{2n}} W(e) \Pr[A_e]
\]

\[
\leq \sum_{e \in K_{2n}} W(e) \mu(A_e) (1 + (2n - 1)(\lambda n - 1)\psi)^2
\]

\[
< \frac{(1 + \frac{3}{2}\lambda)^2}{2n - 1} \sum_{e \in K_{2n}} W(e),
\]

concluding the proof. \[ \square \]
6.1.4 Finding Rainbow Matchings with many edges

In this subsection we use Theorem 5 to show that whenever $\lambda < 0.5$ we can find rainbow matchings with a linear number of edges.

**Theorem 12.** Given any edge-coloring of the complete graph on $2n$ vertices in which each color appears on at most $\lambda n$ edges, where $\lambda < 0.5$ and $n$ is sufficiently large, there exists an algorithm that terminates within $O(n)$ steps in expectation and finds a rainbow matching with an expected number of edges that is at least $n \min \left(1, 0.94 \sqrt[3]{\frac{2}{\lambda}} - 1\right)$.

**Proof.** Let $\phi$ be any edge-coloring of $K_{2n}$ in which each color appears on at most $\lambda n$ edges and recall the definitions of $P = P(\phi)$, $\Omega$, and $f_{i,j} = \{M \in \Omega : \{e_i, e_j\} \subseteq M\}$ from the proof of Theorem 10.

The idea is to apply Theorem 5 that guarantees that we can come up with a “truncated version” $A'$ of our algorithm for finding perfect rainbow matchings. In particular, if $\nu$ is the output probability distribution of $A'$ and for each flaw $f_{i,j}$ we set $\psi_{i,j} = \alpha$ then:

$$
\nu(f_{i,j}) \leq \max (0, \gamma(f_{i,j})\zeta_{i,j} - \alpha) \\
\leq \max \left(0, \frac{(1 + (2n - 1)(\lambda n - 1)\alpha)^4}{(2n - 3)(2n - 1)} - \alpha\right).
$$

(8)

Consider now the following strategy: We first execute algorithm $A'$ to get a perfect, possibly non-rainbow, matching $M$ of $K_{2n}$. Then, for each flaw $f_{i,j}$ that appears in $M$, we choose arbitrarily one of its corresponding edges and remove it from $M$, to get a non-perfect, but rainbow, matching $M'$. If $S = S(M')$ is the random variable that equals the size (number of edges) of $M'$ then by setting $\alpha = \frac{1}{(2n - 1)(\lambda n - 1)} \left(3\sqrt[3]{\frac{2n - 3}{4(\lambda n - 1)}} - 1\right)$

we get:

$$
E[S] = n - \sum_{(e_i, e_j) \in P} \nu(f_{i,j}) \\
\geq n - \max \left(0, |P| \left(\frac{(1 + (2n - 1)(\lambda n - 1)\alpha)^4}{(2n - 3)(2n - 1)} - \alpha\right)\right) \\
= n \min \left(1, 1 - \frac{4n}{2n - 1} \left(1 - \frac{3}{4 \cdot 2^{2/3} \sqrt[3]{\lambda n - 1}}\right)\right).
$$

For large enough $n$, the latter is $\min \left(1, 0.94 \sqrt[3]{\frac{2}{\lambda}} - 1\right)$. Finally, notice that (for large $n$) $\alpha$ is positive whenever $\lambda < 0.5$. \hfill \Box

6.2 Proper Vertex Colorings

A vertex $q$-coloring of a graph $G = G(V, E)$ is a labeling $\phi : V \rightarrow [q]$ of its vertices. A proper vertex $q$-coloring $\phi$ has the extra property that no edge is monochromatic, i.e., $\phi(u) \neq \phi(v)$ for each edge $(u, v) \in E$. 

While the problem of finding a proper coloring is trivial whenever \( q \geq \Delta + 1 \), the problem of sampling proper vertex \( q \)-colorings is way more challenging and has been a subject of intense study \([34, 16, 33, 32, 46]\). The currently best result is due to Vigoda \([46]\) who shows that there exists an MCMC algorithm that (approximately) samples from the uniform distribution over proper vertex \( q \)-colorings in \( O(nq \log n) \) steps, provided that \( q > \frac{11}{6} \Delta \), where \( n \) is the number of vertices of the graph.

In this section we use Theorem 1 to compare the performance of an algorithm that samples proper colorings versus a (randomized) greedy algorithm in the context of finding a low weight proper coloring, given some weighting function \( W : \Omega^* \to \mathbb{R} \), where \( \Omega^* \) is the set of proper \( q \)-colorings of \( G \). In particular, we consider a class of weighting functions that can be written as \( W = \sum_{v \in A} W_v \), where \( A \subseteq V \) is a set of vertices that are “far apart” from each other, and where each function \( W_v \) depends on the colors of the vertices in \( N_d(v) \); the set of vertices at distance \( d \) from \( v \). By “far apart” here we mean that for any \( u, v \in A \) sets \( N_u + 1 \) and \( N_v + 1 \) are disjoint.

Let \( G_v = G_v(V_v, E_v) \) denote the subgraph of \( G \) that is induced by \( N_d(v) \) and let \( L_v \) be its corresponding line graph. We call an independent set \( S \) of \( L_v \) proper if each vertex of \( G_v \) appears at most once in an edge of \( S \). Let \( \text{PrInd}(L_v) \) denote the set of proper independent sets of \( L_v \). Finally, let \( r_v = \frac{|\Omega_v|}{|\Omega_v^*|} \) and \( s_v = |G_v| \), where \( \Omega_v, \Omega_v^* \) are the set of vertex \( q \)-colorings and proper vertex \( q \)-colorings of \( G_v \), respectively.

We show the following theorem whose proof can be found in the appendix.

**Theorem 13.** Given a graph \( G(V, E) \) on \( n \) vertices with maximum degree \( \Delta \) and \( q \geq (1 + \epsilon)\Delta \) colors, \( \epsilon > 0 \), there exists a greedy algorithm that terminates in at most \( 2n \) coloring steps and outputs a proper vertex \( q \)-coloring \( \phi_{\text{out}} \) of expected weight at most:

\[
\mathbb{E}[W(\phi_{\text{out}})] \leq \sum_{v \in A} r_v a_v \mathbb{E}_{\phi \sim \nu}[W_v(\phi)],
\]

where

\[
a_v = \sum_{S \in \text{PrInd}(L_v)} \left( \frac{1 + \epsilon}{\epsilon^2 \Delta} \right)^{|S|} \left( 1 + \frac{1}{\epsilon} + \frac{1}{\epsilon^2} \right)^{s_v} < \left( 1 + \frac{1}{\epsilon} + \frac{1}{\epsilon^2} \right)^{s_v},
\]

and \( \nu \) is the uniform measure over \( \Omega^* \).

Notice that our bound improves both as the the number of colors and the density of graphs \( \{G_v\}_{v \in A} \) increases, as it is intuitively expected. In particular, observe that the density of \( G_v \) improves factor \( a_v \) in two ways: by reducing the number of independent sets in \( L_v \) (which is typically denser than \( G_v \)), and by reducing \( r_v \). Finally, we get the following straightforward corollary.

**Corollary 3.** Given a graph \( G \) on \( n \) vertices with maximum degree \( \Delta \) and \( q = (1 + \epsilon)\Delta \) colors, there exist at least

\[
|\Omega^*| \geq q^n \cdot \left( \sum_{S \in \text{PrInd}(L(G))} \left( \frac{1 + \epsilon}{\epsilon^2 \Delta} \right)^{|S|} \right)^{-1},
\]

proper colorings of \( G \), where \( L(G) \) is the line graph of \( G \).
Proof. Let $v$ be an arbitrary vertex, $A = \{v\}$, $W_v(\phi) = 1$ for every $\phi \in \Omega^*$ and $G_v = G$. Applying Theorem 13 and noticing that $\mathbb{E}[W(\phi_{\text{out}})] = \mathbb{E}_{\phi \sim \nu}[W(\phi)]$ we get that

$$|\Omega^*| \geq q^n \cdot \left( \sum_{S \in \text{PrInd}(L(G))} \left( \frac{1 + \epsilon}{\epsilon^2 \Delta} \right)^{|S|} \right)^{-1}.$$

\[\square\]

6.3 Acyclic Edge Coloring via the Clique Lovász Local Lemma

An edge-coloring of a graph is proper if all edges incident to each vertex have distinct colors. A proper edge coloring is acyclic if it has no bichromatic cycles, i.e., no cycle receives exactly two (alternating) colors. The smallest number of colors for which a graph $G$ has an acyclic edge-coloring is denoted by $\chi'_a(G)$.

Acyclic Edge Coloring (AEC), was originally motivated by the work of Coleman et al. [14, 13] on the efficient computation of Hessians and, since then, there has been a series of works [5, 39, 42, 23, 36, 19] that upper bound $\chi'_a(G)$ for graphs with bounded degree. The current best result was given recently by Giotis et al. in [21] who showed that $\chi'_a(G) \leq 3.74\Delta$ in graphs with maximum degree $\Delta$.

The analysis of [21], while inspired by the algorithmic LLL, uses a custom argument that does not correspond to any of its known versions. Furthermore, their algorithm does not correspond to an instantiation of the Moser Tardos algorithm and does not seem to be commutative (assuming the natural formulation in our setting) and, thus, it’s not amenable to our analysis.

On the other hand, Kolipaka, Szegedy and Yixin Xu show in [36] that $8.6(\Delta - 1)$ colors suffice for the Moser Tardos algorithm to converge in this setting. They do this by introducing the Clique the Lovász Local Lemma, a criterion that is typically stronger than (although, technically, incomparable to) the cluster expansion condition, but weaker than the Shearer’s condition. In fact, the Clique Lovász Local Lemma is a member of a “hierarchy” of Local Lemma criteria that are increasingly complex and use an increasing amount of information about the structure of the dependency graph. On the limit, they give the Shearer’s condition.

While the use of the Clique Lovász Local Lemma (or any other criterion in the hierarchy of [36]) makes the results of [23] inapplicable, it does allow us to use Theorems 4 and 3 (along with Remark 3) which capture the cases where the Shearer’s criterion is satisfied. We note though that very recently Harris [25] also showed how one can use Shearer’s condition to improve the known bounds on the output of the Moser Tardos algorithm, so the same results could be obtained employing this work.

Overall, we exchange a strengthening of the convergence criterion (number of colors) by a constant factor for access to the properties of the output distribution of our algorithm. We show two results. Our first theorem says that the Moser Tardos algorithm applied on the acyclic edge coloring problem converges in polynomial time and has high output entropy whenever $q \geq 8.6(\Delta - 1)$.

**Theorem 14.** Given a graph $G = (V, E)$ with maximum degree $\Delta$ and $q \geq 8.6(\Delta - 1)$ colors, there exist at least $(\frac{q}{\Delta})^{|E|}$ acyclic edge colorings of $G$. Furthermore, there exists an algorithm with expected polynomial running time that outputs each one of them with positive probability.

Our second theorem considers a problem of weighted acyclic edge colorings. In particular, given a graph $G(V, E)$ let $W = \sum_{v \in V} W_v$ be a weighting function over edge $q$-colorings of $G$ such that each $W_v, v \in V$, is a function of the colors of the edges adjacent to $v$. By sampling uniformly at random, one can find an edge coloring $\phi$ of weight $\mathbb{E}_{\phi \sim \mu}[W(\phi)]$, where $\mu$ is the uniform distribution over the edge $q$-colorings of $G$. Using Theorem 3, we show that whenever $q \geq 8.6(\Delta - 1)$ we can use the Moser Tardos algorithm to find an acyclic edge coloring of similar weight (assuming that $\Delta$ is constant).
Theorem 15. Given a graph $G(V, E)$ with maximum degree $\Delta$ and $q \geq 8.6(\Delta - 1)$ colors, there exist an algorithm with expected polynomial running time that outputs an acyclic edge coloring $\phi_{\text{out}}$ of expected weight at most

$$E[\phi_{\text{out}}] < 1.3^\Delta \cdot E_{\phi \sim \mu}[W(\phi)].$$

The proofs of Theorems 14, 15 as well as more details on the Clique Lovász Local Lemma, can be found in the appendix.

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A Omitted Proofs

A.1 Proof of Theorem 4

To lighten the notation, let \( u := \sum_{S \in \text{Ind}([m])} \prod_{j \in S} \psi_j \). For each \( \sigma \in \Omega \), define a flaw \( f_\sigma = \{ \sigma \} \) and consider the extended algorithm that addresses it by sampling from \( \mu \), as well as the extended causality graph that connects \( f_\sigma \) with every flaw in \( F \). Clearly, we have that \( \gamma(f_\sigma) = \mu(\sigma) \). Moreover, if the original algorithm is commutative, so is the extended one since the commutativity condition is trivially true for flaws of the form \( f_\sigma \), where \( \sigma \in \Omega \). Observe now that for every \( \sigma \in \Omega \), Theorem 2 yields \( \nu(\sigma) \leq \Pr[\sigma] \leq u \cdot \mu(\sigma) \).

Thus:

\[
H_\rho[\nu] = \frac{1}{1 - \rho} \ln \sum_{\sigma \in \Omega} \nu(\sigma)^\rho \geq \frac{1}{1 - \rho} \ln \sum_{\sigma \in \Omega} (u \mu(\sigma))^\rho = \frac{1}{1 - \rho} \ln \sum_{\sigma \in \Omega} \mu(\sigma)^\rho + \frac{\rho}{1 - \rho} \ln u ,
\]

concluding the proof.

A.2 Proof of Theorem 5

For each flaw \( f_i \) we define a Bernoulli variable \( Y_i \) with probability of success \( p_i = \min \left\{ 1, \frac{\psi_i}{\gamma(\Omega)} \right\} \). The sequence \( \{Y_i\}_{i=1}^m \) and \( \Omega \) induce a new space \( \Omega' = \Omega \times \{0, 1\}^m \) which can be thought as a “labelled” version of \( \Omega \), where each state \( \sigma \) is labelled with a binary vector of length \( m \), whose \( i \)-th bit describes the state of \( Y_i \). Similarly, measure \( \mu \) and \( \{Y_i\}_{i=1}^m \) induce a measure \( \mu' \) over \( \Omega' \).

In this new state space we introduce a new family of flaws \( F' = \{f'_1, f'_2, \ldots, f'_m\} \), where \( f'_i \) is defined as the subset of \( \Omega' \) where \( f_i \) is present and \( Y_i = 1 \). Consider now the algorithm \( \mathcal{A}' \) that is induced by \( \mathcal{A} \) as follows: Each time we want to address flaw \( f'_i \) we move in \( \Omega \) by invoking \( \mathcal{A} \) to address \( f_i \) and also take a sample from \( Y_i \) to update the value of the \( i \)-th entry of label-vector.

It’s not hard to verify that (i) the charge of each flaw \( f'_i \) is \( \gamma(f'_i) = \gamma(f_i)p_i \); (ii) any causality graph for \( (\Omega, F, \mathcal{A}) \) is also a causality graph for \( (\Omega', F', \mathcal{A}') \) (and, in particular, so is the one induced by \( \sim \)); (iii) if \( \mathcal{A} \) is commutative then so is \( \mathcal{A}' \); and that (iv) the cluster expansion condition is satisfied (with respect to the causality graph induced by \( \sim \)).
To conclude the proof, consider a flaw $f'_i$ and notice that in order for $f_i$ to be present in the output of $A'$ it has to be the case that $Y_i = 0$. Notice now that Theorem 2 implies that for each flaw $f'_i$:

$$\nu(f_i \cap Y_i = 0) \leq \mu'(f_i \cap Y_i = 0) \sum_{S \in \text{Ind}(\Gamma(i))} \prod_{j \in S} \psi_j = (1 - p_i) \mu(f_i) \zeta_i \leq (1 - p_i) \gamma(f_i) \zeta_i = \max \{0, \gamma(f_i) \zeta_i - \psi_i\}. $$

### A.3 Proof of Lemma 2

We show the following more general lemma. The claim follows by applying this general lemma with $\text{List}(j) = \text{Ind}(\Gamma(i))$ for every $j \in [m]$.

**Lemma 9.** Assume that for every $i \in [m]$ were are given a set $\text{List}(i) \subseteq 2^{[m]}$ and there exist positive numbers $\{\psi_i\}_{i=1}^m$ such that for each $i$:

$$\frac{\gamma(f_i)}{\psi_i} \sum_{S \in \text{List}(i)} \prod_{j \in S} \psi_j \leq 1. $$

Let $L_i$ be the set of witness trees whose root is labelled by $i$ and such that the set of labels of every node $v$ with $[v] = j$ is in $\text{List}(j)$, for every $j \in [m]$. Then:

$$\sum_{\tau \in L_i} \prod_{v \in V(\tau)} \gamma(f_{[v]}) \leq \psi_i. $$

**Proof of Lemma 9.** To proceed, we use ideas from [41, 43]. Specifically, we introduce a branching process that produces only trees in $L_i$ and bound $\sum_{\tau \in L_i} \prod_{v \in V(\tau)} \gamma(f_{[v]})$ by analyzing it.

In particular, we start with a single node labelled by $i$. In each subsequent round each leaf $u$ "gives birth" to a set of nodes whose set of (distinct) labels is a set $S \in \text{List}([u])$ with probability proportional to $\prod_{j \in S} \psi_j$. It is not hard to see that this process creates every tree in $L_i$ with positive probability. To express the exact probability received by each $S \subseteq [m]$ we define

$$Q(S) := \prod_{j \in S} \psi_j $$

and let $Z = \sum_{S \in \text{List}([u])} Q(S)$. Clearly, each $S \in \text{List}([v])$ receives probability equal to $\frac{Q(S)}{Z}$. We now show the following lemma.

**Proposition 5.** The branching process described above produces every tree $\tau \in L_i$ with probability

$$p_\tau = \frac{1}{\psi_i} \prod_{v \in V(\tau)} \frac{\psi_v}{\sum_{S \in \text{List}([v])} Q(S) \prod_{j \in S} \psi_j}. $$

**Proof.** For each tree $\tau \in L_i$ and each node $v$ of $\tau$, let $N(v)$ denote the set of labels of its children. Then:

$$p_\tau = \prod_{v \in \phi} \frac{Q(N(v))}{\sum_{S \in \text{List}([v])} Q(S)} \psi_v \prod_{v \in \phi} \frac{Q(S)}{Q(S)}. $$

\[\square\]
Notice now that:

\[
\sum_{\tau \in \mathcal{L}} \prod_{v \in V(\tau)} \gamma(f_{[v]}) \leq \sum_{\tau \in \mathcal{L}, v \in V(\tau)} \prod_{v \in V(\tau)} \frac{\psi_{[v]}}{\sum_{S \in \text{List}([v])} \prod_{j \in S} \psi_j}
\]

(10)

\[
= \psi_i \sum_{\tau \in \mathcal{L}_i} p_\tau
\]

(11)

\[
= \psi_i ,
\]

where (10) follows by the hypothesis of Lemma 9 while (11) by Proposition 5.

\[\Box\]

### A.4 Proof of Lemma 7

Let \( u_1, \ldots, u_m \) be the named indices of flaws of \( Q(\Sigma) \) that occur in \( U \) listed in the order of their appearance in \( U \). We claim that \( u_1, \ldots, u_m \) is a prefix of \( U \). To see this, assume for the sake of contradiction that \( U = \ldots wu_1 \ldots \) where \( w \notin Q(\Sigma) \) and \( u_1 \in Q(\Sigma) \). Thus, \( (w, u_1) \notin E(\Sigma) \) and so \( w \sim u_1 \). Therefore, \( (w, u_1) \) is a swappable pair in \( U \), which is a contradiction.

Note that the latter observation implies that \( W(\Sigma) = (A, B, C) \) where \( B = (u_1, u_2, \ldots, u_m) \). It remains to show that \( B \) is a subsequence of \( Q_\pi(\Sigma) \). In particular, it suffices to show that for any \( i \in [m-1] \) the relative order of \( u_i \) and \( u_{i+1} \) in \( B \) is the same as in \( Q_\pi(\Sigma) \). Assume the opposite, i.e., \( Q_\pi(\Sigma) = \ldots u_{i+1} \ldots u_i \ldots \). It has to be that \( u_i \sim u_{i+1} \). For otherwise, \( (u_i, u_{i+1}) \) would be a swappable pair, contradicting the assumption. This means that \( (u_i, u_{i+1}) \in E(\Sigma) \), implying that \( d(u_i) > d(u_{i+1}) \). Recalling the definition of \( Q_\pi(\Sigma) \) we see that \( u_i \) should be to the left of \( u_{i+1} \) in \( Q_\pi(\Sigma) \), a contradiction.

### A.5 Proof of Lemma 8

First we prove that the mapping is injective. In particular, assume that two distinct trajectories \( \Sigma_1, \Sigma_2 \in \mathcal{X}_p[W] \) are transformed to the same trajectory \( \Sigma \in \mathcal{X}_{p+1}[W] \). At least one of \( \Sigma_1, \Sigma_2 \) must have changed. Without loss of generality, assume \( \Sigma \neq \Sigma_1 \). The latter implies that \( W(\Sigma_1) = (A, w, y, B) \) with \( w \sim y \) and \( \Sigma_1 \) is transformed to a trajectory \( \Sigma \) with \( W(\Sigma) = (A, w, y, B) \). Notice that it cannot be the case that \( \Sigma_2 = \Sigma \) since then \( \Sigma_1 \) and \( \Sigma \) would both be in \( \mathcal{X}_p[W] \) without following the same deterministic flaw choice strategy, a contradiction. Thus, \( W(\Sigma_2) = (A, w, y, B) \), and \( \Sigma \) was obtained from \( \Sigma_2 \) by swapping \( w \) and \( y \). Recalling that the Swap operation is an injection, we get that \( \Sigma_1 = \Sigma_2 \).

We will assume that \( \mathcal{X}_{p+1}[W] \) is not valid and reach a contradiction. The latter assumption implies that there should be trajectories \( \Sigma, \Sigma' \in \mathcal{X}_{p+1}[W] \) such that

\[
W(\Sigma) = (w_1, \ldots, w_\ell, w, y, \ldots)
\]

\[
W(\Sigma') = (w_1, \ldots, w_\ell, w, \bar{y}, \ldots)
\]

with \( w \neq \bar{w} \) and the states in \( \Sigma \) to the left of \( w \) match the corresponding states in \( \Sigma' \) to the left of \( \bar{w} \). Here it is assumed that some of \( w, y, \bar{w}, \bar{y} \) may equal \( \emptyset \), which means they don’t exist. It is also assumed that \( w = \emptyset \) also implies that \( y = \emptyset \) and similarly for \( \bar{w} \) and \( \bar{y} \).

Let \( \Sigma_1 \) and \( \Sigma_2 \) be respectively the trajectories in \( \mathcal{X}_p[W] \) that were transformed to \( \Sigma \) and \( \Sigma' \). Since \( \mathcal{X}_p[W] \) is valid, at least one of them must have changed. Assume, without loss of generality, \( \Sigma \neq \Sigma_1 \). We know that (i) \( \Sigma_1 \) and \( \Sigma_2 \) follow the same deterministic flaw choice strategy, and they are not proper prefixes of each other, as well as, that (ii) named flaws indices \( w_1, \ldots, w_\ell \) are distinct.

Some other useful facts to have in mind (and which we will implicitly use) are that the first \( \ell + 1 \) states of \( \Sigma \) match those of \( \Sigma' \), and also that swapping adjacent indices of flaws only affects the state between them in a deterministic way. Now there are four cases:
(a) The swapped pair in $\Sigma$ was $(w_i, w_{i+1})$ for $i \in [\ell - 1]$. Thus, $W(\Sigma_1) = (w_1, \ldots, w_{i+1}, w_i, \ldots, w_\ell, w, y, \ldots)$. Using (i) and (ii), we conclude that $\Sigma' \neq \Sigma_2$ and, thus, $W(\Sigma_2) = (w_1, \ldots, w_{i+1}, w_i, \ldots, w_\ell, w, y, \ldots)$. To see this, recall that swaps are applied at the same positions for trajectories in $X_p$. Condition (i) now implies that $w = \bar{w}$.

(b) The swapped pair in $\Sigma$ was $(w_\ell, w)$, and so $W(\Sigma) = (w_1, \ldots, w_\ell, y, w, \ldots)$. Thus, $\Sigma' \neq \Sigma_2$ by (i) and (ii). Now condition (i) implies that $w = \bar{w}$.

(c) The swapped pair in $\Sigma$ was $(w, y)$, and so $W(\Sigma_1) = (w_1, \ldots, w_\ell, y, w, \ldots)$ and $w \in Q(\Sigma_1)$. We now apply Lemma 7 to $\Sigma_1$ and we notice that since there are no swappable pairs in $\Sigma_1$ to the right of $(y, w)$ it should be that $W(\Sigma_1) = (w_1, \ldots, w_\ell, B, C)$ where $B$ is a subsequence of $Q_\pi(\Sigma_1)$ and $C$ does not contain named indices of flaws from $Q_\pi(\Sigma)$. Observe that $B$ should start with $w$. Using (i) we get that $W(\Sigma_2) = (w_1, \ldots, w_\ell, y, \ldots)$ and again by Lemma 7 $W(\Sigma_2) = (w_1, \ldots, w_\ell, y, \bar{B}, \bar{C})$ where $\bar{B}$ is a subsequence of $Q_\pi(\Sigma_2)$ and $\bar{C}$ does not contain named indices of flaws from $Q(\Sigma_2)$.

By the definition of $X_p[W]$ we know that $Q_\pi(\Sigma_1) = Q_\pi(\Sigma_2) = W$. The latter fact, along with the forms of $\Sigma_1, \Sigma_2$ imply that $B$ should be a permutation of $\bar{B}$. Notice though that $B$ and $\bar{B}$ are subsequences of $W$ and, furthermore, all elements of $W$ are distinct. Therefore, it has to be that $B = \bar{B}$.

The latter observation implies that $W(\Sigma_2) = (w_1, \ldots, w_\ell, y, w, \ldots)$. Now since $(y, w)$ is a swappable pair in $\Sigma_1$ it should also be a swappable pair in $\Sigma_2$. Thus, $\Sigma' = (w_1, \ldots, w_\ell, w, y, \ldots)$. This means that $w = \bar{w}$.

(d) The swapped pair in $\Sigma$ was to the right of $(w, y)$. In this case condition (i) implies that $w = \bar{w}$.

B Proofs of Theorems 14 and 15

B.1 The Clique Lovász Local Lemma

We first state the Clique Lovász Local Lemma reformulated to the language of our setting, assuming a commutative algorithm $(D, F, C, \rho)$, where $F = \{f_1, \ldots, f_m\}$ is the set of flaws and $C$ is a causality graph.

**Theorem 16** (The Clique Lovász Local Lemma). Let $\{K_1, K_2, \ldots, K_n\}$ be a set of cliques in $C$ covering all the edges (not necessarily disjointly). If there exists a set of vectors $\{x_1, \ldots, x_n\}$ from $(0, 1)^m$ such that the following condition are satisfied

- For each $v \in [n]$: $\sum_{i \in K_v} x_{i,v} < 1$.
- For each $i \in [m]$, $\forall v$ such that $i \in K_v$:

\[
\gamma(f_i) \leq x_{i,v} \prod_{u \neq v : K_u \ni i} (1 - \sum_{j \in K_u \setminus \{i\}} x_{j,u})
\]

then:

1. $\mu\left(\bigcap_{v \in [n]} f_i\right) \geq \prod_{v \in [m]} (1 - \sum_{i \in K_v} x_{i,v}) > 0$

2. The algorithm terminates after an expected number of at most

\[
\sum_{i \in [m]} \min_{v : K_v \ni i} \frac{x_{i,v}}{1 - \sum_{j \in K_v \setminus \{i\}} x_{j,v}}
\]

steps.
We note that in [36] the authors first prove the first part of their theorem, which implies the existence of perfect objects, and then they invoke the results of [35] which imply that the Moser Tardos algorithm converges under the Shearer’s criterion. In particular, they use the fact (which we will also find useful) that
\[ \frac{q(i)}{q_0} \leq \min_{v:K_v \ni i} 1 - \sum_{j \in K_v \setminus \{i\}} x_{j,v}. \]  

(12)

To prove Theorem 16 in our setting, we can follow the same strategy, invoking Theorem 3 instead of [35]. In fact, the proof of the first part of Theorem 16 is identical to the one of [36] assuming that our algorithm is a resampling oracle with respect to \( \mu \) for each flaw \( f_i \). In the general case, some extra work is required (although nothing sophisticated). Since in our application we will be using the Moser Tardos algorithm (which is a resampling oracle for every flaw), and in the interest of brevity, we leave this as an exercise.

Finally, we note that the authors provide a canonical way of decomposing the causality graph and applying the Clique Lovász Local Lemma in the variable setting of Moser and Tardos. Specifically, recall that in the variable setting the flaws/events are determined by a set of independent discrete random variables \( \{v_1, \ldots, v_m\} \) and two events are connected in the dependency graph whenever they share a variable. Thus, each variable \( v \) forms a clique \( K_v \) in the causality graph consisting of the flaws that contain this variable.

B.2 Finding Acyclic Edge Colorings

We now recall the proof of [36] for Acyclic Edge Coloring. The proof is the same as the one in [6, 39] and the improvement comes from the use of Clique LLL instead of the general Local Lemma.

Given \( q \) colors and a graph \( G \) with maximum degree \( \Delta \) let \( \Omega \) be the set of all edge \( q \)-colorings of \( G \). We identify the two following types of flaws:

1. For a path \( P \) of length 2 let \( f_P \) be the set of states in \( \Omega \) in which \( P \) is monochromatic.
2. For a cycle \( C \) of even length let \( f_C \) comprise the set of states in \( \Omega \) in which \( C \) is bicolored.

Clearly a flawless element of \( \Omega \) is an acyclic edge coloring of \( G \). Our algorithm is the MT algorithm (the variables that correspond to each event are the edges of the path/cycle), \( \mu \) is the uniform measure and \( \theta = \mu \). Therefore:

\[ \gamma(f_P) = \mu(f_P) = \frac{1}{q}, \]
\[ \gamma(f_C) = \mu(f_C) \leq \frac{1}{q|C|-2}. \]

Furthermore, two flaws are connected in the causality graph iff they share an edge. We now follow the canonical way of decomposing the causality graph into cliques by having one clique \( K_e \) for each edge \( e \) of \( G \). Moreover:

- For each path \( P \ni e \) of length 2 we set \( x_{f_P,e} = x_{P,e} = \frac{e}{1+e}\frac{1}{\Delta - 2} \)
- For a cycle \( C \) of even length we set \( x_{f_C,e} = x_{C,e} = \frac{e}{(1+e)|C|-2}\frac{1}{\Delta - 1)|C|-2} \)

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for some positive $c, \epsilon$ to be determined later.

Observe that the number of cycles of length $2\ell$, where $\ell \geq 2$, that contain any given edge $e$ is at most $(\Delta - 1)^{2\ell - 2}$, while the number of paths of length 2 that contain $e$ is at most $2\Delta - 2$. Thus, it suffices to show that for each edge $e$ and each path of length $P$ and cycle $C$ of length $2\ell$ that contain $e$ we have that:

$$\gamma(f_P) \leq x_{P,e} \prod_{e' \in P \setminus \{e\}} \left(1 - \sum_{j \in K_e} x_{j,e'}\right),$$

$$\gamma(f_C) \leq x_{C,e} \prod_{e' \in C \setminus \{e\}} \left(1 - \sum_{j \in K_e} x_{j,e'}\right),$$

which is

$$\frac{1}{q} \leq \frac{c}{1 + \epsilon} \frac{1}{2\Delta - 2} \left(1 - c \sum_{j=1}^{\infty} (1 + \epsilon)^{-j}\right),$$

$$\frac{1}{q^{2\ell - 2}} \leq \frac{c}{(1 + \epsilon)^{\ell}} \frac{1}{(\Delta - 1)^{2\ell - 2}} \left(1 - c \sum_{j=1}^{\infty} (1 + \epsilon)^{-j}\right)^{2\ell - 1}.$$  

The latter imply that for the Moser Tardos algorithm to converge it suffices to have $c < \epsilon$ and also

$$\frac{q}{\Delta - 1} \geq \max \left\{ \frac{2}{c} \frac{(1 + \epsilon)}{\epsilon - c}, \max_{\ell \geq 2} \left\{ \frac{1}{c^{2\ell - 2}} \frac{1}{(1 + \epsilon)^{2\ell - 2}} \left(\frac{\epsilon}{\epsilon - c}\right)^{\frac{2\ell - 1}{2\ell - 2}} \right\} \right\}$$

$$\geq \max \left\{ \frac{2}{c} \frac{(1 + \epsilon)}{\epsilon - c}, \frac{(1 + \epsilon)}{\sqrt{c}} \left(\frac{\epsilon}{\epsilon - c}\right)^{\frac{3}{2}} \right\} \geq 8.59.$$  

for $\epsilon = 2.05869$ and $c = 0.8282$.

We now show that the expected running time of our algorithm is polynomial. To do so, we will need to be careful in our flaw choice strategy and also deal with the fact that the number of flaws is super-polynomial.

As far as the flaw choice strategy is concerned, we choose to give priority to flaws of the form $f_P$ whose presence in the current state can be verified in polynomial time in the number of edges of $G$. This means we only address bichromatic cycles when the underlying edge-coloring is proper. Note, though, that in order to find bichromatic cycles in a properly edge-colored graph we can just consider each of the $\binom{q}{2}$ pair of distinct colors and seek cycles in the subgraph of the correspondingly colored edges.

One way to address the fact that the number of flaws is super-polynomial is to invoke Theorem 7 of [35] that states that whenever the LLL criterion is satisfied with a multiplicative slack (notice that we have showed that the Clique LLL is satisfied with $q \geq 8.59$) then the expected number of resampling of the Moser Tardos algorithm is polynomial in the number of the independent random variables (in our case, the edges of $G$) with constant probability. The probability of polynomial convergence can be boosted by repetition. Overall, we have shown the following theorem.

**Theorem 17.** Given a graph $G$ with maximum degree $\Delta$ and $q \geq 8.6$ colors there exists an algorithm that outputs an acyclic edge coloring in expected polynomial time.
B.3 Proof of Theorem 14

To prove Theorem 14 we need to estimate \( \sum_{S \in \text{Ind}(F)} \prod_{f \in S} q_{\{f\}} \) per Theorem 4 and Remark 3 (we slightly abuse the notation and indicate \( \psi \)'s using flaws instead of indices of flaws). To do so, we will use a technical result proved in [29], (trivially) reformulated to fit our needs.

**Theorem 18 ([29]).** Consider a constraint satisfaction problem on a set of variables \( V \) and set of constraints \( C \). Assume we have a flaw \( f_c \) for each constraint \( c \), which comprises the set of states that violate \( c \). We are also given an undirected causality graph such that two constraints are connected with an edge iff they share variables. For each constraint \( c \) define

\[
y_c = \left( 1 + \frac{q_{\{c\}}}{q_0} \right)^{\frac{1}{|\text{var}(c)|}} - 1,
\]

where \( \text{var}(c) \) denotes the set of variables that correspond to constraint \( c \). Then:

\[
\sum_{S \in \text{Ind}(C)} \prod_{c \in S} q_{\{c\}} q_0 \leq \prod_{v \in V} \left( 1 + \sum_{c \in C, v \in \text{var}(c)} y_c \right).
\]

Using the result of the previous subsection along with (12) we have that for each path \( P \) of length two and each cycle \( C_\ell \) of length \( \ell \):

\[
y_P \leq \left( 1 + \frac{c}{1 + \epsilon} \frac{1}{2(\Delta - 1)} \frac{\epsilon}{\epsilon - c} \right)^{\frac{1}{2\ell}} - 1 < \left( 1 + \frac{1}{\Delta - 1} \right)^{1/2} - 1 < \frac{1}{2(\Delta - 1)}
\]

\[
y_{C_\ell} \leq \left( 1 + \frac{c}{(1 + \epsilon)^{2\ell}} \frac{1}{(\Delta - 1)^{2\ell - 2}} \right)^{\frac{1}{2\ell}} - 1 < \left( 1 + \frac{3^{-\ell}}{(\Delta - 1)^{2\ell - 2}} \right)^{\frac{1}{2\ell}} - 1 < \frac{1}{(2(\Delta - 1))^{2\ell - 2}}
\]

and, thus, we get:

\[
\sum_{S \in \text{Ind}(F)} \prod_{f \in S} q_{\{f\}} q_0 \leq \prod_{e \in E} \left( 1 + 2(\Delta - 1) \frac{1}{2(\Delta - 1)} + \sum_{i=2}^{\infty} (\Delta - 1)^{2\ell - 2} \frac{1}{(2(\Delta - 1))^{2\ell - 2}} \right) < 4^{|E|}.
\]

Now Theorem 4 and the fact that \( |\Omega| = q^{|E|} \) conclude the proof.

B.4 Proof of Theorem 15

For a vertex \( v \) and a function \( W_v \) let \( N(v) \) denote the set of edges adjacent to \( v \) and \( A_v \) denote the set of possible edge \( q \)-colorings of the edges in \( N(v) \). For \( \alpha \in A_v \) let \( E_v(\alpha) \) be the subset of \( \Omega \) whose elements assign \( \alpha \) to the edges in \( N(v) \). Moreover, consider the resampling probability distributions induced my Moser Tardos for \( E_v(\alpha) \). Observe that using (12) we get (again, slightly abusing the notation):
\[
\sum_{S \in \text{Ind}(\Gamma(E_{\nu}(\alpha))))} \prod_{f \in S} \frac{q(f)}{q_0} \leq \prod_{e \in N(v)} \left( 1 + \sum_{P \ni e} x_{P,e} + \sum_{C \ni e} x_{C,e} \right)
\]

\[
\leq \prod_{e \in N(v)} \left( 1 + \left( \frac{c}{1 + \epsilon} + \sum_{\ell=2}^{\infty} \frac{c}{(1 + \epsilon)^{\ell}} \right) \frac{\epsilon}{\epsilon - c} \right)
\]

\[
< 1.3^\Delta .
\]

Now applying Theorem\[3\] we get:

\[
\mathbb{E}[W_v] = \sum_{\alpha \in A_v} \Pr [E_{\nu}(\alpha)] W_v(\alpha)
\]

\[
< 1.3^\Delta \sum_{\alpha \in A_v} \mu(E_{\nu}(\alpha))W_v(\alpha)
\]

\[
= 1.3^\Delta \cdot \mathbb{E}_{\phi \sim \mu}[W_v(\phi)] .
\]

Linearity of expectation concludes the proof.

C Proof of Theorem[13]

Let \( \Omega \) be the set of \( q \)-colorings of \( V \). For each edge \( e \in E \) and each color \( c \in [q] \) define the flaw \( f_{e,c} \) to be the set of colorings in which both vertices of \( e \) are colored with \( c \). Clearly, a flawless object is a proper coloring of \( G \).

To address a flaw \( f_{e,c} \) at state \( \sigma \), we fix choose an arbitrary ordering among the vertices of \( e \). For each vertex of \( e \), in that order, we choose uniformly at random among the \( q - \Delta \) lowest indexed colors that do not appear in its neighbourhood. Choosing \( \mu \) to be the uniform measure over \( \Omega \), we have that

\[
\gamma(f_{e,c}) = \frac{1}{(q - \Delta)^2} = \frac{1}{(\epsilon \Delta)^2} ,
\]

since for every transition \( \sigma \xrightarrow{(e,c)} \sigma' \) we have that \( \rho(e,c)(\sigma, \sigma') = \frac{1}{(q - \Delta)^2} \) and for each state \( \sigma' \) there exists at most one incoming arc labelled by \( (e, c) \).

Furthermore, it’s straightforward to see that an algorithm that addresses the flaws in this greedy fashion terminates in at most \( n \) steps, independently of the initial distribution \( \theta \) and the strategy with which we choose which flaw to address at each step. However, to get the best result our analysis can give we will fix \( \theta = \mu \) and choose an arbitrary ordering among flaws, so that flaws that correspond to edges in \( \bigcup_{v \in A} E_v \) have the highest priority.

To analyze the properties of the output of the algorithm we will use the witness tree lemma, and thus, we will need to show that our algorithm is commutative with respect to a certain causality graph. While it’s not hard to see that the empty graph is a valid causality graph for our algorithm, to guarantee commutativity we will need to be more conservative. In particular, given \( f_{e,c} \) and \( f_{e',c'} \) we have that \( (e, c) \sim (e', c') \) iff \( e \) and \( e' \) are adjacent in \( L^2(G) \).

To see that the algorithm is commutative with respect to this causality graph, fix two flaws \( f_{e,c}, f_{e',c'} \) such that \( (e, c) \sim (e', c') \). Since \( e \) and \( e' \) are not adjacent in \( L^2 \), it is straightforward to verify that for
every trajectory $\sigma_1 f_{e,c} \rightarrow \sigma_2 \rightarrow \sigma_3 \rightarrow \sigma_3$ there exists a unique trajectory $\sigma_1 f_{e,c} \rightarrow \sigma_2 \rightarrow \sigma_3$ and also $\rho_{(e,c)}(\sigma_1, \sigma_2) \rho_{(e,c)}(\sigma_2, \sigma_3) = \rho_{(e,c)}(\sigma_1, \sigma_2') \rho_{(e,c)}(\sigma_2', \sigma_3) = \frac{1}{(q-\Delta)^2}$. This is because the recoloring of vertices of $e$ does not affect in any way the choices for the vertices for $e'$ and vice versa. This establishes the existence of an injective Swap operator.

Recall now that $\phi_{\text{out}}$ denotes the random variable that equals the output of our algorithm. For each $v \in A$, let $C_v$ be the set of possible proper colorings for the the vertices in $V_v$ that are induced by the set of proper colorings of $G$. For $\alpha \in C_v$, let $B_v(\alpha)$ be the subset of $\Omega$ whose elements assign $\alpha$ to vertices in $V_v$. Moreover, consider the extension of the original algorithm that addresses each flaw $B_v(\alpha)$ by choosing a color for the vertices of $V_v$ uniformly at random, as well as, the extended causality graph, where two flaws determined by a set of edges $A, B$ are adjacent iff $A, B$ are adjacent in $L^2(G)$. The extended algorithm is also commutative for the same reason as the original one.

Now an identical argument to the one of proof of Theorem 2 shows that:

$$
\Pr [B_v(\alpha)] \leq \mu(B_v) \sum_{\tau \in W_{B_v}} \prod_{u \in V(\tau)} \gamma(f_{[u]}) = r_v \nu(B_v) \sum_{\tau \in W_{B_v}} \frac{1}{(\epsilon \Delta)^2 |\tau|},
$$

where $W_{B_v}$ is the set of witness trees with root labelled by $B_v$ that might occur during an execution of the algorithm. The key observation now is $W_{B_v}$ can be injected into the set of proper independent sets of $L_v$. To see this, at first notice that our algorithm has the property that each each edge is considered at most once, and once it does, flaws that corresponding to neighbouring edges will never be addressed. Furthermore, observe that the fact that sets $\{N^d(\nu)\}_v \in A$ are disjoint, the definition of our causality graph, and the fact that we give priority to flaws in $\bigcup_{v \in A} E_v$, imply that the labels of each witness tree form a proper independent set in $L_v$. Finally, since the ordering according to which we address the flaws is fixed, given a set of flaws that corresponds to a proper independent set in $L_v$, we can uniquely reconstruct a witness tree $\tau \in W_{B_v}$ (assuming that the way we break ties is also fixed). Overall, we have that:

$$
\sum_{\tau \in W_{B_v}} \prod_{u \in V(\tau)} \gamma(f_{[u]}) \leq \sum_{S \in \text{PrInd}(L_v)} \left( \frac{q}{(\epsilon \Delta)^2} \right)^{|S|} = a_v,
$$

where the $q$-factor comes from the fact that we have $q$ flaws for each edge. Finally:

$$
\mathbb{E} [W_v(\phi_{\text{out}})] = \sum_{\alpha \in C_v} \Pr [B_v(\alpha)] W_v(\alpha)
\leq a_v r_v \sum_{\alpha \in C_v} \nu(B_v(\alpha)) \mathbb{E} [W_v(\phi)]
\leq a_v r_v \mathbb{E} \phi \cdot \mathbb{E} [W_v(\phi)] .
$$

Linearity of expectation concludes the proof of the claim for the weight of the resulting coloring. The claim for the number of steps follows from the fact that we color edge vertex at most twice: once for the initialization step, and at most once during the execution of the algorithm.