Fundamental Limits of Multiple-Access Integrated Sensing and Communication Systems

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Abstract

A memoryless state-dependent multiple access channel is considered to model an integrated sensing and communication system, where two transmitters wish to convey messages to a receiver while simultaneously estimating the sensing state sequences through echo signals. In particular, the sensing states are assumed to be correlated with the channel state, and the receiver has imperfect channel state information. In this setup, improved inner and outer bounds for capacity-distortion region are derived. The inner bound is based on an achievable scheme that combines message cooperation and joint compression via distributed Wyner-Ziv coding at each transmitter, resulting in unified cooperative communication and sensing. The outer bound is based on the ideas of dependence balance for communication rate and rate-limited constraints on sensing distortion. The proposed inner and outer bounds are proved to improve the state-of-the-art bounds. Finally, numerical examples are provided to demonstrate that our new inner and outer bounds strictly improve the existing results.

Index Terms

Integrated sensing and communication, multiple access channels, correlated sensing and channel states, imperfect channel state information at the receiver, capacity-distortion region.

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I. INTRODUCTION

Future 6G mobile networks aim to integrate the functions of communication and sensing to provide advanced intelligent sensing services, such as smart vehicular networks and smart homes. As mobile networks progress towards millimeter wave (mmWave) bands and embrace massive multi-input multi-output (MIMO) techniques, communication signals tend to have higher resolution in both time and angular domains, and this opens doors for highly accurate sensing via mobile networks. Integrated sensing and communication (ISAC), in which sensing and communication share the same frequency band and hardware, has thus emerged as a pivotal technology for 6G networks [2], [3].

A number of previous studies have investigated ISAC in various scenarios and system architectures [4]–[6], demonstrating the advantages of integration. Nonetheless, the optimality of these schemes and the fundamental tradeoff between sensing and communication performance in ISAC systems are worth further studied. To elucidate the tradeoff, Bliss in [7] has introduced the notion of “estimation information rate” to quantify the sensing estimation performance and examined the tradeoff between estimation information rate and communication rate for the system considered. Kumari et al. [8] have instead proposed to convert the communication rate into a mean-square-error (MSE) equivalent quantity, providing a framework for representing the sensing-communication tradeoff under the MSE metric. More recently, Xiong et al. [9], [10] have utilized the Cramer-Rao bound (CRB), a lower bound of MSE, as the performance metric for sensing, and investigated the sensing-communication tradeoff through CRB-communication rate region. However, the aforementioned studies [7]–[10] while insightful, often assume Gaussian parameters for sensing or Gaussian channels, limiting their applicability to more general scenarios.

To address the broader scope of ISAC systems, Kobayashi et al. [11] drew from rate-distortion theory [12] and introduced a pivotal performance metric known as the capacity-distortion tradeoff. In this framework, the precision of parameter sensing is quantified using general distortion functions, while the effectiveness of communication is evaluated through the classical Shannon communication rate. The authors modeled the sensing echo signals as strictly causal feedback and made the assumption that the sensing state coincides with the channel state, with perfect channel state information available at the receiver (CSIR). This allowed them to establish the optimal
capacity-distortion tradeoff for ISAC systems employing monostatic sensing over memoryless point-to-point channels. In contrast, our recent work \[13\] has characterized the optimal capacity-distortion tradeoff for point-to-point channels where the sensing and channel states are correlated, and CSIR is imperfect. While channel state information (CSI) governs how transmitted signals propagate, combine, and are received at their destinations, the sensing operation primarily seeks to detect physical phenomena (represented as sensing states) within the channel. These sensing states are often correlated with but not necessarily identical to the CSI. Furthermore, achieving perfect CSIR in practice is challenging due to channel estimation errors.

References \[14\]–\[18\] instead delve deeper into the capacity-distortion tradeoff for multi-terminal ISAC systems. In particular, Kobayashi \textit{et al.} in \[14\] have considered a multiple-access ISAC model where two transmitters wishes to convey messages to a receiver while simultaneously sensing the respective channel states through echo signals. By leveraging the Willem’s coding scheme \[19\], they have demonstrated that two transmitters can cooperate by decoding and transmitting partial messages (called common message) of the other transmitter through echo signals, which can then be leveraged for both state sensing and communication. This approach enlarged the achievable rate-distortion region compared to the conventional time-sharing approach. They have also established an outer bound for ISAC over multiple access channels (MAC) by combining the principles of dependence balance constraints \[20\]–\[22\] on the allowable input distributions and genie-aided side information regarding sensing. More recently, Ahmadipour \textit{et al.} \[17\], \[18\] have proposed a collaborative integrated sensing and communication scheme, where each transmitter conveys information pertaining to the echo signals to the other transmitter. In addition to the message cooperation \[14\], they have demonstrated that sending quantized information related to the echo signals as part of the common message, decodable by the other transmitter, can further enhance sensing performance.

Other related studies have also appeared to explore the fundamental limits of ISAC systems, each addressing distinct considerations \[23\]–\[33\]. For instance, references \[23\]–\[26\] have investigated the capacity-distortion tradeoff in bistatic sensing scenarios where state estimation occurs at the receiver. Additionally, references \[27\]–\[29\] have considered scenarios where the sensing state remains a fixed parameter correlated with channel states, shedding light on the tradeoff between the classical communication rate and the state detection-error exponent. Furthermore, references \[30\]–\[32\] have addressed security concerns in ISAC systems, establishing the capacity-
distortion tradeoff while adhering to secure constraints, with sensing operations carried out at the transmitter and receiver, respectively. Inspired by the application of ISAC in mmWave communication, the authors in [33] have analyzed a binary beam-pointing channel with in-block memory and feedback, deriving its capacity subject to peak transmission cost constraints in closed-form.

A. Contributions

In this paper, we develop improved inner and outer bounds on the capacity-distortion region for ISAC over MACs. The summary of contributions is as follows.

- For inner bound, we propose a new achievable scheme that combines the concepts of message cooperation [14], [19] and joint compression of past transmitted codeword and echo signals through distributed Wyner-Ziv coding [34]. Our scheme differs from the existing one [18] in two main ways. First, both transmitters send compressed information related to the echo signals as part of private message that can only be decoded by the receiver, in addition to the common message. This conveys additional information about the channel state to the receiver, similar to relevant works’ schemes for state-dependent MACs with strictly causal state information at the transmitters [34]–[36]. Second, we carefully design the cooperative signals to maximize the benefits of cooperation between two transmitters to transmit compressed information in the common message part. Compared with the existing scheme [18], our approach allows for more compressed information to be sent in the common message part, thereby facilitating sensing. Overall, our achievable scheme leads to a unified cooperative communication and sensing approach to leverage the cooperation between the transmitters. The corresponding achievable rate-distortion region is derived and proved to include the region established by [18].

- For outer bound, in addition to dependence balance constraints [20]–[22], we leverage the idea of rate-limited constraints to bound the performance of sensing instead of genie-aided side information [14]. The proposed rate-limited constraints on sensing performance use the idea of joint source-channel coding. In ISAC over MAC, apart from the transmitted codeword and received feedback, each transmitter can also obtain some useful information from the other transmitter. One can treat the amount of useful information successfully conveyed from the other transmitter as a joint source-channel coding problem and further
bound the amount of such useful information by the source rate and channel capacity. The obtained outer bound is tighter than the existing one [14] on sensing performance, and is thus a tighter bound on capacity-distortion tradeoff.

• Several numerical examples are constructed to show the advantages of our improved inner and outer bounds compared to the existing results. We first provide three examples to intuitively demonstrate the advantages of introducing unified cooperative communication and sensing scheme in inner bound and rate-limited constraints on sensing in outer bound. Then, a general example is presented to show that our proposed scheme can achieve better performance in both sensing and communication compared to the existing one [18], and the proposed outer bound is strictly tighter than that of [14].

B. Organization and Notations

The rest of this paper is organized as follows. Section II describes the general model for ISAC over MAC considered in this work. Section III summarizes the latest results on inner and outer bounds for ISAC over MAC in [18] and [14], respectively. Section IV presents the main results of improved inner and outer bounds, as well as the theoretically comparison with related works. Section V constructs several numerical examples to demonstrate that our inner and outer bounds strictly improve the existing results. Section VI concludes the paper.

Notation: Throughout the paper, we use calligraphic letters, uppercase letters, and lowercase letters to denote sets, random variables, and the realizations, respectively, e.g., $\mathcal{X}, X, x$. The probability distributions are denoted by $P$ with the subscript indicating the corresponding random variables, e.g., $P_X(x)$ and $P_{Y|X}(y|x)$ are the probability of $X = x$ and conditional probability of $Y = y$ given $X = x$. We use $x^i$ to denote the vector $[x_1, x_2, \cdots, x_i]$, $[1 : L]$ to denote the set $\{1, 2, \cdots, L\}$ for integer $L$, and $\mathbb{E}(X)$ to denote the expectation of random variable $X$. For $k \in \{1, 2\}$, we define $\bar{k} = 3 - k$. For a event $\mathcal{A}$, we use $\mathcal{A}^c$ to denote its complement. Logarithms are taken with respect to base 2.

II. System Model

Consider a general ISAC over discrete memoryless (DM) MAC as shown in Fig. 1. Over $n$ uses of such a channel, transmitter $k \in \{1, 2\}$ wishes to convey a message $W_k \in [1 : 2^{nR_k}] \triangleq \{1, 2, \cdots, 2^{nR_k}\}$ to the receiver over a state-dependent DM MAC while simultaneously
estimating the sensing state sequence $S^n_{T_k}$ via output feedback $Z^n_k$. Here output feedback models the communication echo signal reflected back to the transmitter.

The state-dependent DM-MAC considered in Fig. 1 is denoted by

$$(X_1 \times X_2, S, P_{Y|Z_1, Z_2|X_1, x_2, s}, Y, Z_1, Z_2)$$

with input alphabets $X_1 \times X_2$, state alphabet $S$, output alphabet $Y$, feedback alphabets $Z_1 \times Z_2$, and the conditional probability mass function $P_{Y|Z_1, Z_2|X_1, x_2, s}$. The channel state sequence $S^n$ is assumed to be i.i.d. with $P_{S^n}(s^n) = \prod_{i=1}^n P_S(s_i)$. The channel is memoryless in the sense that at each time instance $i \in [1 : n],$

$$P(y_i, z_{1,i}, z_{2,i}|x_1^i, x_2^i, s^n, y^{i-1}, z_{1,i}^{i-1}, z_{2,i}^{i-1}) = P(y_i, z_{1,i}, z_{2,i}|x_1^i, x_2^i, s_i).$$

In the ISAC model considered, the joint distribution of channel state $S$, CSIR $S_R$, and sensing states $S_{T_k}, k \in \{1, 2\}$ is given by $P_{S^n S^n_{T_k} S^n_R}$, which is i.i.d. according to

$$P_{S^n S^n_{T_k} S^n_R}(s^n_{S^n_{T_k} S^n_R}) = \prod_{i=1}^n P_{S^n_{T_k} S^n_R}(s_i s_{S^n_{T_k} S^n_R}).$$

More specifically, the CSIR sequence $S^n_R$ is i.i.d. according to $P_{S^n_R}(s^n_R) = \prod_{i=1}^n P_{S^n_R}(s_i)$ and the associated $S_R$ is assumed to be correlated with channel state $S$ but not necessarily the same. This modeling thus includes imperfect CSIR ($S_R \neq S$) in general and perfect CSIR ($S_R = S$) as a special case. Moreover, the sensing state sequence $S^n_{T_k}, k \in \{1, 2\}$ is also i.i.d. according to $P_{S^n_{T_k}}(s^n_{T_k}) = \prod_{i=1}^n P_{S^n_{T_k}}(s_i)$, and the associated $S_{T_k}$ is correlated with the channel state $S$.

**Definition 1.** For the model considered in Fig. 1 a $(2^nR_1, 2^nR_2, n)$ code consists of

1) two message sets $W_k = [1 : 2^nR_k], k \in \{1, 2\}$ where the messages $W_1, W_2$ are uniformly distributed;

2) two encoders where encoder $k \in \{1, 2\}$ assigns a symbol $x_{k,i} = f_{k,i}(w_k, z_{i-1}^k)$ to each message $w_k \in W_k$ and delayed feedback $z_{i-1}^k \in Z_{i-1}^k$;

3) a decoder that produces a message pair $(\hat{w}_1, \hat{w}_2) = h(y^n, s^n_R) \in W_1 \times W_2$ upon observing $y^n$ and $s^n_R$;

4) two state estimators where estimator $k \in \{1, 2\}$ assigns an estimated sensing sequence $\hat{s}_{T_k}^n = g_k(x^n_k, z_{-k}^n) \in \hat{S}^n_{T_k}$ based on the codeword $x^n_k \in X^n_k$ and receiving feedback $z_{-k}^n \in Z_{-k}^n$. 


The sensing performance is measured by the expected distortion of the state estimated, i.e.,

$$\mathbb{E}[d_k(S^n_{T_k}, \hat{S}^n_{T_k})] = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[d_k(S_{T_k,i}, \hat{S}_{T_k,i})],$$  \hspace{1cm} (4)

where $d_k : S_{T_k} \times \hat{S}_{T_k} \rightarrow [0, \infty)$ is a bounded distortion function.

**Definition 2.** A rate-distortion tuple $(R_1, R_2, D_1, D_2)$ is said to be achievable if there exist a sequence of $(2^{nR_1}, 2^{nR_2}, n)$ codes with arbitrarily small error probability for decoding, i.e.,

$$\lim_{n \to \infty} P_r(\hat{W}_1, \hat{W}_2) \neq (W_1, W_2) = 0,$$  \hspace{1cm} (5)

and the sensing distortion constraints

$$\limsup_{n \to \infty} \mathbb{E}[d_k(S^n_{T_k}, \hat{S}^n_{T_k})] \leq D_k, k \in \{1, 2\}$$  \hspace{1cm} (6)

are satisfied. For any $(D_1, D_2)$, the capacity-distortion region $C(D_1, D_2)$ is defined as the closure of achievable rate tuple $(R_1, R_2)$ such that $(R_1, R_2, D_1, D_2)$ is achievable.

**Remark 1.** Different from the existing studies [14], [18], the considered model in this work introduce three random variables $S_{T_1}, S_{T_2}$ and $S_R$ to denote the two sensing states and CSIR, respectively. The correlation among sensing states, CSIR, and channel states is thus explicitly modeled through $P_{SS_{T_1}S_{T_2}S_R}$, while the sensing states in [14], [18] are denoted as $S_1, S_2$, and
the CSIR in [18] is absorbed into channel output. As claimed in [18], both our model and that in [18] can capture the general sensing states and CSIR. The results in the following sections are thus summarized based on the model in Fig. 7.

III. Preliminaries

A. Inner Bound Established by [18]

The proposed scheme in [18] is a modified version of Willem’s coding scheme [19] by combining the ideas of message cooperation and transmitting the compressed information related to the echo signals. To achieve the message cooperation, the message $W_k \in [1 : 2^{nR_k}], k \in \{1, 2\}$ is divided into two parts $(W_{k,c}, W_{k,p})$. $W_{k,c} \in [1 : 2^{nR_{k,c}}]$ is the part of common message sent by transmitter $k$ which can be decoded by both transmitter $\bar{k}$ through the delayed echo signals and the receiver through channel outputs. $W_{k,p} \in [1 : 2^{nR_{k,p}}]$ is the part of private message sent by transmitter $k$ that can only be decoded by the receiver. Then, two transmitters send both common messages to the receiver to realize the cooperative gain. In addition to message cooperation, transmitter $k \in \{1, 2\}$ also sends a compressed information $V_{k}$ related to the echo signals that can be decoded by both the other transmitter $\bar{k}$ and the receiver. Since the compressed information $V_{k}$ is obtained from the echo signals which is correlated with the channel states, the transmitter $\bar{k}$ and the receiver can leverage $V_{k}$ for sensing and decoding the messages, respectively.

The above operations can be achieved by block Markov encoding at the transmitters and backward decoding at the receiver. In particular, a sequence of $B$ messages $\{W_{1}^{(b)}, W_{2}^{(b)}\}_{b=1}^{B}$ will be transmitted over $B + 1$ blocks, each of $N$ channel uses. In block $b \in [1 : B]$, each transmitter $k$ divides the message $W_{k}^{(b)}$ into a common part $W_{k,c}^{(b)}$ and a private part $W_{k,p}^{(b)}$, and the following four types of message components are transmitted:

- The common part (to be decoded by the other transmitter) of its own message $W_{k,c}^{(b)}$;
- The private part of its own message $W_{k,p}^{(b)}$;
- Compressed information $V_{k}^{(b-1)}$ (to be decoded by the other transmitter) related to the delayed echo signal from the previous block $b - 1$;
- Cooperative signal about common messages $W_{1,c}^{(b-1)}, W_{2,c}^{(b-1)}$ of both two transmitters from the previous block $b - 1$. 
At the beginning of block $b+1$, thanks to the delayed echo signal, each transmitter $k$ can decode the common message $W_{k,c}^{(b)}$ and compressed information $V_{k}^{b-1}$ sent by the other transmitter $\bar{k}$, where the obtained common message of transmitter $\bar{k}$ is used to construct the cooperative signal to achieve message cooperation, and the obtained compressed information is used to enhance sensing. In particular, the state estimation at the transmitter $k \in \{1, 2\}$ is performed in a block-by-block manner. To obtain an estimate of sensing state sequence corresponding to block $b$, transmitter $k$ can leverage not only its own codeword and echo signal, but also the common message part $W_{k,c}^{(b)}$ and compressed information $V_{k}^{(b)}$ sent by transmitter $\bar{k}$ in block $b$ and $b+1$, respectively.

The decoding of the receiver is performed in a backward manner, which starts by decoding the last block $B+1$, then block $B$, and so on. For the decoding in each block $b$, the receiver uses the channel output and CSIR in block $b$ and the common messages of two transmitters $W_{1,c}^{(b)}$, $W_{2,c}^{(b)}$ that are decoded by the receiver in block $b+1$ to simultaneously decode the common messages $W_{1,c}^{(b-1)}$, $W_{2,c}^{(b-1)}$, private messages $W_{1,p}^{(b)}$, $W_{2,p}^{(b)}$ and the compressed information $V_{1}^{(b-1)}$, $V_{2}^{(b-1)}$. Such operations are conducted until $b = 1$, and the receiver can construct all messages sent by two transmitters.

An illustration of key ideas in the achievable scheme proposed by [18] is shown in Fig. 2, and the corresponding inner bound of capacity-distortion region for ISAC over MAC considered is presented in Theorem 1. Note that the advantages of such a scheme are two-fold. First, the message cooperation between two transmitters facilitates the communication. Second, by leveraging the message cooperation, each transmitter can successfully decode the common message and compressed information sent by the other transmitter, which consequently improves the state estimation.

**Theorem 1** [18]. Considering the estimator for $k \in \{1, 2\}$:

$$\hat{s}_{T_k}(x_k, u_k, z_k, v_k) = \arg \min_{s_{T_k} \in S_{T_k}} \sum_{s_{T_k} \in S_{T_k}} P_{S_{T_k}}|X_k u_k z_k V_{k}(s_{T_k}|x_k z_k v_k)d_k(s_{T_k}, s_{T_k}'),$$

an achievable rate-distortion region $I_{R:D}^{awk}$ includes any rate-distortion tuple $(R_1, R_2, D_1, D_2)$ that for some choice of variables

$$UU_1U_2 X_1X_2S S_{T_1}S_{T_2}Z_1Z_2 Y V_1 V_2 \hat{S}_{T_1} \hat{S}_{T_2}$$

(8)
Fig. 2. An illustration of the key ideas and random variables in the achievable scheme proposed by [18].

with joint distribution

\[
P_{U|P_{U_1}|U}P_{U_2|U}P_{X_1|U}P_{X_2|U}P_{S_{R_1}S_{R_2}X_{12}}P_{Y|Z_{12}|X_{12}}S
\]

\[
P_{V_1|U}P_{U_1|U}P_{V_2|U}P_{V_1|U_2|X_{12}}P_{S_{R_1}X_{12}V_1}P_{S_{R_2}X_{12}V_2}P_{V_2|X_{12}V_1}
\]

satisfying

\[
R_k \leq I(U_k; X_kZ_k|UU_k) + I(V_k; X_kZ_k|UU_1U_2) - I(V_k; X_kZ_k|UU_1U_2)
\]

\[
+ \min \left\{ I(X_k; YS_{R}|UX_k) + I(V_k; X_1X_2YS_{R}|UU_1U_2) + I(V_k; X_1X_2YS_{R}V_k|UU_1U_2)
\]

\[
- I(V_k; X_kZ_k|UU_1U_2),
\]

\[
I(X_1X_2; YS_{R}|UU_k) + I(V_k; X_1X_2YS_{R}|UU_1U_2) + I(V_k; X_1X_2YS_{R}V_k|UU_1U_2)
\]

\[
- I(V_k; X_kZ_k|UU_1U_2),
\]

\[
I(X_1X_2; YS_{R}|U) + I(V_k; X_1X_2YS_{R}|UU_1U_2) + I(V_k; X_1X_2YS_{R}V_k|UU_1U_2)
\]

\[
- I(V_k; X_kZ_k|UU_1U_2) - I(V_k; X_kZ_k|UU_1U_2),
\]
\[
I(X_k; YS_RV_1V_2|UU_1U_2X_k), \quad k \in \{1, 2\}, \quad (10a)
\]

\[
R_1 + R_2 \leq I(U_2; X_1Z_1|UU_1) + I(V_2; X_1Z_1|UU_1U_2) - I(V_2; X_2Z_2|UU_1U_2) + I(U_1; X_2Z_2|UU_2) + I(V_1; X_2Z_2|UU_1U_2) - I(V_1; X_1Z_1|UU_1U_2) + \min \left\{ I(X_1X_2; YS_R|UU_2) + I(V_1; X_1X_2YS_R|UU_1U_2) + I(V_2; X_1X_2YS_RV_1|UU_1U_2) - I(V_1; X_1Z_1|UU_1U_2), 
I(X_1X_2; YS_R|UU_1) + I(V_1; X_1X_2YS_R|UU_1U_2) + I(V_2; X_1X_2YS_RV_1|UU_1U_2) - I(V_2; X_2Z_2|UU_1U_2), 
I(X_1X_2; YS_R|U) + I(V_1; X_1X_2YS_R|UU_1U_2) + I(V_2; X_1X_2YS_RV_1|UU_1U_2) - I(V_1; X_1Z_1|UU_1U_2) - I(V_2; X_2Z_2|UU_1U_2), 
I(X_1X_2; YS_RV_1V_2|UU_1U_2) \right\}, \quad (10b)
\]

\[
R_1 + R_2 \leq I(X_1X_2; YS_R) + I(V_1; X_1X_2YS_R|UU_1U_2) + I(V_2; X_1X_2YS_RV_1|UU_1U_2) - I(V_1; X_1Z_1|UU_1U_2) - I(V_2; X_2Z_2|UU_1U_2) \quad (10c)
\]

and

\[
I(U_k; X_kZ_k|UU_k) + I(V_k; X_kZ_k|UU_1U_2) \geq I(V_k; X_kZ_k|UU_1U_2), \quad k \in \{1, 2\}, \quad (11a)
\]

\[
I(X_k; YS_R|UX_k) + I(V_1; X_1X_2YS_R|UU_1U_2) + I(V_2; X_1X_2YS_RV_1|UU_1U_2) \geq I(V_k; X_kZ_k|UU_1U_2), \quad k \in \{1, 2\}, \quad (11b)
\]

\[
I(X_1X_2; YS_R|U) + I(V_1; X_1X_2YS_R|UU_1U_2) + I(V_2; X_1X_2YS_RV_1|UU_1U_2) \geq I(V_1; X_1Z_1|UU_1U_2) + I(V_2; X_2Z_2|UU_1U_2), \quad (11c)
\]

as well as sensing distortion constraints

\[
\mathbb{E}[d_k(S_{T_k}^n, \hat{S}_{T_k}^n)] \leq D_k, \quad k \in \{1, 2\}. \quad (12)
\]
B. Outer Bound Established by [14]

The key ideas of [14] for establishing the outer bound on capacity-distortion region for ISAC over MAC are based on dependence balance [20]–[22] and genie-aided side information. By treating the echo signals as the delayed generalized feedback, [14] introduced dependence balance constraints on the set of allowable input distributions and further bounded the communication rates. As for the sensing of each transmitter \( k \), in addition to its own side information \( X_k, Z_k \), [14] introduced genie-aided side information by assuming that transmitter \( k \) also has perfect knowledge of the side information \( \bar{X}_k, \bar{Z}_k \) of the other transmitter \( \bar{k} \). The idea of dependence balance and the idealized assumptions of sensing estimators provide an outer bound for communications and sensing performance, respectively, and thus establish an outer bound on capacity-distortion region. The results of the corresponding outer bound on capacity-distortion region for ISAC over MAC are summarized in Theorem 2.

**Theorem 2** [14]. Considering the genie-aided state estimator for transmitter \( k \in \{1, 2\} \):

\[
\hat{s}^*_T(x_1, x_2, z_1, z_2) = \arg \min_{s'_{T_k} \in S_{T_k}} \sum_{s_{T_k} \in S_{T_k}} P_{S_{T_k}|X_1X_2Z_1Z_2}(s_{T_k}|x_1x_2z_1z_2)d_k(s_{T_k}, s'_{T_k}),
\]

an outer bound \( O^{\text{blk}}_{R_D} \) for capacity-distortion region \( C(D_1, D_2) \) is the set of all tuples \( (R_1, R_2, D_1, D_2) \) satisfying

\[
R_1 \leq I(X_1; YZ_1Z_2|S_RS_2T), \quad R_2 \leq I(X_2; YZ_1Z_2|S_RX_1T), \quad (14a)
\]

\[
R_1 + R_2 \leq I(X_1X_2; YZ_1Z_2|S_RT), \quad (14b)
\]

\[
R_1 + R_2 \leq I(X_1X_2; Y|S_R), \quad (14c)
\]

where \( T - X_1X_2S_R - YZ_1Z_2 \) forms a Markov chain, and there is the dependence balance constraint

\[
I(X_1; X_2|T) \leq I(X_1; X_2|Z_1Z_2T) \quad (15)
\]

as well as the average distortion constraints

\[
\mathbb{E}[d_k(S_{T_k}, \hat{s}^*_T(X_1, X_2, Z_1, Z_2))] \leq D_k, \ k \in \{1, 2\}, \quad (16)
\]

where it suffices to consider \( T \) whose alphabet \( \mathcal{T} \) has cardinality \( |\mathcal{T}| \leq |X_1||X_2| + 3 \).
IV. Main Results

In this section, we present the main results of our improved inner and outer bounds for ISAC over MAC. For each bound, we first elaborate the key ideas of our proposed scheme. Then, the results and theoretical comparison with the existing ones are provided.

A. Improved Inner Bound

We propose a new achievable scheme for the model considered. The scheme is based on the idea of combining the message cooperation [14], [19] and joint compression of past transmitted codeword and echo signals through distributed Wyner-Ziv coding [34] to fully utilize the delay echo signals for both communication and sensing.

The proposed scheme is also based on block-Markov encoding at the transmitters and backward decoding at the receiver, where both two transmitters send messages and the compressed version related to the received echo signals (called as compressed information). The message and the compressed information of each transmitter are both divided into common and private parts. Then, in each block, transmitter $k$ sends the following six types of message components:

(a) The common part (to be decoded by the other transmitter) of its own message $W^{(b)}_{k,c}$;
(b) The private part of its own message $W^{(b)}_{k,p}$;
(c) The common part (to be decoded by the other transmitter) of its compressed information $V^{(b)}_{k,c}$ related to the delayed echo signal from the previous block $b-1$;
(d) The private part of its compressed information $V^{(b)}_{k,p}$ related to the delayed echo signal from the previous block $b-1$;
(e) Cooperative signal about common messages $W^{(b-1)}_{1,c}, W^{(b-1)}_{2,c}$ of both two transmitters from the previous block $b-1$;
(f) Cooperative signal about common compressed information $V^{(b-1)}_{1,c}, V^{(b-1)}_{2,c}$ of both two transmitters which are related to the delayed echo signal from the previous block $b-2$.

A brief illustration of codeword construction is presented in Fig. 3.

Thanks to the delayed echo signal, each transmitter $k$ can decode the common message and compressed information sent by the other transmitter $\bar{k}$, which can be used to construct the cooperative signal between two transmitters and enhance sensing. The decoding procedure of the receiver is performed backwards, similar to that in [18]. The key difference is that in each
block $b$, in addition to the channel output and CSIR in block $b$ and the common messages of two transmitters $W_{1,c}^{(b)}, W_{2,c}^{(b)}$, the receiver also utilizes the common compressed information of two transmitters $V_{1,c}^{(b)}, V_{2,c}^{(b)}$ as well as the compressed information $V_{1,c}^{(b+1)}, V_{2,c}^{(b+1)}, V_{1,p}^{(b)}, V_{2,p}^{(b)}$ of two transmitters sent in block $b+1$ that are related to the echo signals in block $b$ to perform the decoding of common messages $W_{1,c}^{(b-1)}, W_{2,c}^{(b-1)}$, common compressed information $V_{1,c}^{(b-1)}, V_{2,c}^{(b-1)}$, private messages $W_{1,p}^{(b)}, W_{2,p}^{(b)}$, and the private compressed information $V_{1,p}^{(b)}, V_{2,p}^{(b)}$. An illustration of our proposed achievable scheme is shown in Fig. 4. We note that in Fig. 4, only the details of the first $B$ blocks are provided. There are also $\tilde{B}$ blocks called “termination blocks” which are necessary to guarantee the success of backward decoding. The discussion of these termination blocks can be found in Appendix A.

The key differences between our coding scheme and that in [18] are as follows. In our scheme, the common part of compressed information $V_{k,c}$ is sent not only through component (c), but also through component (f), leveraging the cooperation between the transmitters and enabling them to send a larger amount of compressed information, facilitating both sensing and communication. Moreover, the private part of compressed information $V_{k,p}$ is sent through component (d), which can be decoded by the receiver and facilitates communication. By comparison, in [18], the compressed information of transmitter $k$ denoted as $V_k$ is sent only through component (c), which cannot achieve the cooperation for the transmission of compressed information between
two transmitters.

The inner bound on the capacity-distortion region achieved by our proposed scheme is given as follows.

**Theorem 3.** Considering the state estimator for transmitter $k \in \{1, 2\}$

$$\hat{S}_k(x_k, u_k, z_k, v_{k,c}) = \arg \min_{s_{T_k}^1 \in S_{T_k}} \sum_{s_{T_k}^2 \in S_{T_k}} P_{S_{T_k}}|x_k u_k z_k v_{k,c} (s_{T_k}|x_k u_k z_k v_{k,c}) d_k (s_{T_k}, s_{T_k}^2),$$

(17)

an achievable rate-distortion region $\mathcal{T}_{\text{our}}^{R_D}$ includes any rate-distortion tuple $(R_1, R_2, D_1, D_2)$ that for some choice of variables

$$UU_1U_2X_1X_2SSRS_{T_1}S_{T_2}Z_1Z_2YV_{1,c}V_{1,p}V_{2,c}V_{2,p}\hat{S}_{T_1}\hat{S}_{T_2}$$

(18)

with joint distribution

$$(PU|PV|PU)_1 |PV|PV|UU_1|PU_2|UU_2|UU_2|PSSRS_{T_1}S_{T_2}PY_z|z_2|X_1|X_2$$

$$PV_{1,c}V_{1,p} |UU_1U_2X_1Z_1P_{V_{2,c}}V_{2,p}|UU_1U_2X_2Z_2P_{S_{T_1}}|X_1Z_1U_2V_{2,c}P_{S_{T_2}}|Z_2U_1V_{1,c}$$

(19)
satisfying
\[ R_k \leq I(U_k; Z_k|UU_kX_k) - I(V_{k,c}; X_k Z_k|UU_1U_2X_k Z_k) \]
\[ + \min \left\{ I(X_k; X_k Y S_R Y S_R V_{1,c} V_{2,c} V_{k,p}|UU_1U_2) - I(V_{k,p}; Z_k|UU_1U_2X_k Y S_R V_{1,c} V_{2,c} V_{k,p}) , \right. \]
\[ \left. I(X_k; X_k Y S_R Y S_R V_{1,c} V_{2,c} V_{k,p}|UU_1U_2) - I(V_{k,p}; Z_k|UU_1U_2X_k Y S_R V_{1,c} V_{2,c} V_{k,p}) \right\} , \quad k \in \{1, 2\}, \quad (20a) \]
\[ R_1 + R_2 \leq I(U_1; Z_2|UU_2X_2) - I(V_{1,c}; X_2 Z_2|UU_1U_2X_2 Z_2) \]
\[ + I(U_2; Z_1|UU_1X_1) - I(V_{2,c}; X_2 Z_2|UU_1U_2X_2 Z_2) \]
\[ + I(X_1X_2; Y S_R Y S_R V_{1,c} V_{2,c}|UU_1U_2) - I(V_{1,p}; Z_1|UU_1U_2X_2 Y S_R V_{1,c} V_{2,c} V_{1,p}) - I(V_{2,p}; Z_2|UU_1U_2X_2 Y S_R V_{1,c} V_{2,c} V_{1,p}) , \quad (20b) \]
\[ R_1 + R_2 \leq I(X_1X_2; Y S_R Y S_R V_{1,c} V_{2,c}|UU_1U_2) - I(V_{1,c}; Z_1|UU_1U_2X_2 Y S_R V_{1,c} V_{2,c} V_{1,p}) - I(V_{2,c}; Z_2|UU_1U_2X_2 Y S_R V_{1,c} V_{2,c} V_{1,p}) , \quad (20c) \]
and
\[ I(U_k; Z_k|UU_kX_k) - I(V_{k,c}; X_k Z_k|UU_1U_2X_k Z_k) \geq 0 , \quad k \in \{1, 2\} , \quad (21a) \]
\[ I(X_k; X_k Y S_R Y S_R V_{1,c} V_{2,c} V_{k,p}|UU_1U_2) - I(V_{k,p}; Z_k|UU_1U_2X_2 Y S_R V_{1,c} V_{2,c} V_{k,p}) \geq 0 , \quad k \in \{1, 2\} , \quad (21b) \]
\[ I(X_1X_2; Y S_R Y S_R V_{1,c} V_{2,c}|UU_1U_2) - I(V_{1,p}; Z_1|UU_1U_2X_2 Y S_R V_{1,c} V_{2,c} V_{1,p}) \]
\[ - I(V_{2,p}; Z_2|UU_1U_2X_2 Y S_R V_{1,c} V_{2,c} V_{1,p}) \geq 0 , \quad (21c) \]
\[ I(X_1X_2; Y S_R Y S_R V_{1,c} V_{2,c}|UU_1U_2) - I(V_{1,c}; Z_1|UU_1U_2X_2 Y S_R V_{1,c} V_{2,c} V_{1,p}) - I(V_{2,c}; Z_2|UU_1U_2X_2 Y S_R V_{1,c} V_{2,c} V_{1,p}) \geq 0 \quad (21d) \]
as well as the average distortion constraints
\[ \mathbb{E}[d_k(S_{T_k}^n, \hat{S}_{T_k}^n)] \leq D_k , \quad k \in \{1, 2\} . \quad (22) \]

**Proof:** See Appendix A.
Remark 2. Both Theorem 1 and Theorem 3 include the results of [14] as a special case, where no compressed information is transmitted, that is, by setting $V_k = \phi$ in Theorem 1 and $V_{k,c} = V_{k,p} = \phi$ in Theorem 3 respectively.

Remark 3. When the feedback coincides with the channel state, such that $Z_1 = S_1$, $Z_2 = S_2$, and no sensing tasks are required at the transmitters, Theorem 3 specializes to the results for conventional state-dependent MAC with strictly causal state information [36].

Remark 4. The constraints in Theorem 3 can be interpreted as follows. For single-user rate constraints in (20a), the term $I(U_k; Z_k | U_k X_k)$ represents the achievable rate constraint for common message part of transmitter $k$, and $I(V_{k,c}; X_k Z_k | UU_1 U_2 X_k Z_k)$ is the reduction in message rate due to the transmission of common part of compressed information. The terms in min(·, ·) function represent the achievable rate constraints of private message sent by transmitter $k$, where the first term $I(X_k; X_k Y S V_1 V_2 V_{k,p} | UU_1 U_2) - I(V_{k,p}; Z_k | UU_1 U_2 X_k Y S V_1 V_2 V_{k,p})$ is the rate constraint for private message part of transmitter $k$ when the receiver has already decoded the codeword $X_k$, and the second term represents the achievable rate constraint for private message part when the receiver jointly decodes the private message parts of both two transmitters. The sum rate constraints in (20b) and (20c) can be explained in a similar manner. The inequality constraints in (21a), (21b), (21c), and (21d) are imposed to guarantee the successful transmission of compressed information.

While the joint distribution factorization (19) in our scheme is more general than (9) in Theorem 1, the different mutual information expressions and inequality constraints make the immediate comparison between our result $I_{our}^{R-D}$ and the existing one $I_{awk}^{R-D}$ difficult. The following theorem indicates that in fact the achievable rate-distortion region $I_{our}^{R-D}$ of our proposed scheme always includes that $I_{awk}^{R-D}$ in [18].

Theorem 4. Let $I_{our}^{R-D, com}$ denote the achievable rate-distortion of Theorem 3 with $V_{1,p} = V_{2,p} = \phi$, i.e., transmitting no private compressed information. Such a region always includes the achievable region $I_{awk}^{R-D}$, and is a subset of $I_{our}^{R-D}$, i.e.,

$$I_{awk}^{R-D} \subseteq I_{our}^{R-D, com} \subseteq I_{our}^{R-D}. \tag{23}$$

Proof: See Appendix B.
B. Improved Outer Bound

Before presenting the new results of our outer bound, we first define the following two rate-distortion functions $f_{1,R-D}(D_1)$ and $f_{2,R-D}(D_2)$ as

$$f_{1,R-D}(D_1) = \min_{P\hat{S}_{T_1}|X_1Z_1S_{T_1}} \sum_{s_{T_1}} P_{X_1Z_1S_{T_1}} \min_{\hat{s}_{T_1}} I(S_{T_1};\hat{S}_{T_1}),$$

$$f_{2,R-D}(D_2) = \min_{P\hat{S}_{T_2}|X_2Z_2S_{T_2}} \sum_{s_{T_2}} P_{X_2Z_2S_{T_2}} \min_{\hat{s}_{T_2}} I(S_{T_2};\hat{S}_{T_2}).$$

Such rate-distortion functions $f_{1,R-D}(D_1)$ and $f_{2,R-D}(D_2)$ characterize the minimum rates for estimators to guarantee the sensing distortion constraints, in a similar manner to the rate-distortion theory [12]. With the help of $f_{1,R-D}(D_1)$ and $f_{2,R-D}(D_2)$, we proceed to demonstrate the key idea of our outer bound.

Instead of introducing genie-aided side information to bound the sensing performance [14], in our improved outer bound, we consider the following rate-limited constraints. Take transmitter 2 for instance, where its own channel input $X_2$ and feedback $Z_2$ provide certainly some information (called side sensing information) about sensing state $S_{T_2}$. Thanks to the generalized feedback $Z_2$, transmitter 2 can also obtain some useful information (called as cooperative sensing information) about transmitter 1’s input $X_1$ and feedback $Z_1$, which can be leveraged for sensing given $X_2, Z_2$. For example, if $Z_1 = X_1 + S_1$, $Z_2 = X_1 + X_2 + S_2$, and $S_{T_2} = S_1$, transmitter 1 has perfect knowledge of $S_1$ by performing $S_1 = Z_1 - X_1$ and can convey a compressed version of $S_1$ to transmitter 2, which can be decoded by feedback $Z_2$. The actual amount of cooperative sensing information that transmitter 2 can obtain relies on the coding schemes. Assuming that transmitter 2 has perfect knowledge of $X_1, Z_1$, reference [14] provides an outer bound. To get a tighter bound, we treat the transmission of cooperative sensing information as a joint source-channel coding problem, where transmitter 1 has a virtual source containing the cooperative sensing information. Transmitter 1 encodes such information and communicates it to the transmitter 2 by the channel from transmitter 1 to transmitter 2 with input $X_1$ and output $Z_2$, i.e., $P_{Z_2|X_1X_2S}$. An illustration of such a treatment is presented in Fig. 5.

Now we can bound the amount of cooperative sensing information that transmitter 2 can obtain. By joint source-channel coding, we know that:

1) The amount of cooperative sensing information is bounded by the source rate;
2) The amount of cooperative sensing information is bounded by the capacity of channel
\( P_{Z_2|X_1X_2S} \). For example, if \( Z_2 = X_2 + S_2 \), transmitter 2 can obtain no information from
transmitter 1. Consequently, this situation leads to a cooperative sensing information level
of zero.

Therefore, the whole useful information for the sensing of transmitter 2 is bounded by the
sum of cooperative sensing information and side sensing information. To guarantee the sensing
distortion constraints, such amount of information should be no less that the rate-distortion
functions \( f_{1,R-D}(D_1) \) and \( f_{2,R-D}(D_2) \). Based on the above observations and combining with the
idea of dependence balance, one can obtain an improved outer bound.

**Theorem 5.** Let \((D_1, D_2)\) be the given distortion constraints for sensing states. An outer bound
\( C_{\text{out}}^{R,D} \) for capacity-distortion region \( C(D_1, D_2) \) is the set of all tuples \((R_1, R_2, D_1, D_2)\) satisfying

\[
\begin{align*}
R_1 &\leq I(X_1; YZ_1Z_2|S_RX_2T), \\
R_2 &\leq I(X_2; YZ_1Z_2|S_RX_1T), \\
R_1 + R_2 &\leq I(X_1X_2; YZ_1Z_2|S_RT), \\
R_1 + R_2 &\leq I(X_1X_2; Y|S_R),
\end{align*}
\]

with dependence balance constraint

\[
I(X_1; X_2|T) \leq I(X_1; X_2|Z_1Z_2T)
\]
and sensing constraints

\[ f_{k,R-D}(D_k) \leq I(S_{T_k}X_k; Z_k|X_kQ), k \in \{1, 2\}, \quad (27a) \]

\[ f_{k,R-D}(D_k) \leq I(S_{T_k}; Z_1Z_2|X_1X_2Q), k \in \{1, 2\}, \quad (27b) \]

where \( T = (Q, Q_Z) \), \( Q - T - S_RX_1X_2 - YZ_1Z_2 \) forms a Markov chain, and it suffices to consider \( Q, T \) whose alphabet \( Q, T \) has cardinality \(|Q| \leq |T| \leq |X_1||X_2| + 3\).

\[\text{Proof: See Appendix C.}\]

The following theorem shows that in fact our new outer bound is tighter than the existing one in Theorem 2.

**Theorem 6.** The proposed outer bound \( O_{our}^{R-D} \) is a subset of that \( O_{R-D}^{khk} \) in Theorem 2, i.e.,

\[ O_{our}^{R-D} \subseteq O_{R-D}^{khk}. \quad (28)\]

\[\text{Proof: See Appendix D.}\]

V. NUMERICAL EXAMPLES

In this section, we first provide three examples to intuitively demonstrate the improvement of our results. The first two examples are constructed to show the benefits of introducing a unified cooperative communication and sensing scheme in the inner bound. The third example is to demonstrate the improvement of introducing the rate-limited constraints on sensing in the outer bound. Then, an example is constructed to show that our proposed scheme strictly includes the existing one [18], which achieves better performance in both sensing and communication, and the proposed outer bound is strictly tighter than that in [14].

A. The Advantages of Unified Cooperative Communication and Sensing Scheme

**Example 1.** Consider a MAC where the inputs \( X_1, X_2 \) are binary. The channel output \( Y = (Y_1, Y_2) \) is a binary tuple with orthogonal components \( Y_1 = X_1 \oplus S_1, Y_2 = X_2 \oplus S_2 \), where the channel states \( S_1, S_2 \) are independent binary random variables with \( H(S_1) = H(S_2) = 0.5 \). Transmitter 1’s feedback is \( Z_1 = X_1 \oplus S_2 \), and transmitter 2’s feedback \( Z_2 = X_2 \oplus S_1 \). The

\[ P_{S_1}(1) = P_{S_2}(1) = 0.11. \]
receiver is assumed to have no CSI, i.e., $S_R = \phi$. The sensing states of two transmitters are assumed as $S_{T_1} = S_2, S_{T_2} = S_1$, and the Hamming distortion $d(s_T, \hat{s}_T) = s_T \oplus \hat{s}_T$ is considered as the distortion metric. In this example, both two transmitters can always achieve zero distortion for sensing states by conducting $\hat{S}_{T_k} = Z_k \oplus X_k, \ k \in \{1, 2\}$.

In our proposed scheme corresponding to Theorem 3 transmitter $k$ can send the compressed information related to echo signal as part of private message by choosing $V_{k,p} = S_k$ to improve the decoding of the receiver. For example, let transmitter 1 send the compressed information about $S_2$, i.e.,

$$X_k = U_k \oplus \Theta_k = U \oplus \Sigma_k \oplus \Theta_k, \ k \in \{1, 2\}, \quad (29a)$$

$$V_{1,p} = S_2, V_{1,c} = V_{2,c} = V_{2,p} = \phi, \quad (29b)$$

where $U, \Sigma_1, \Sigma_2, \Theta_1, \Theta_2$ are independent binary random variables. One can find that the rate-distortion tuple $(R_1, R_2, D_1, D_2) = (0, 1, 0, 0)$ lies in $\mathcal{I}^{ou}_{R-D}$. However, $(R_1, R_2, D_1, D_2) = (0, 1, 0, 0)$ is not in the inner bound $\mathcal{I}^{aw}_R$ of Theorem 1. In fact, in $\mathcal{I}^{aw}_R$, the achievable rate of transmitter 2 is no more than 0.5. Together with Theorem 4 we establish $\mathcal{I}^{aw}_R \subset \mathcal{I}^{ou}_{R-D}$ for this channel.

**Proof:** See Appendix E.

**Remark 5.** In Example 1 the echo signals of two transmitters $Z_1 = X_1 \oplus S_2, Z_2 = X_2 \oplus S_1$ provide only channel state information, where two transmitters cannot decode any common messages of the other, and thus no direct cooperation through message transmission. However, as shown in Example 1 our scheme allows transmitter 2 to send state information acquired from the echo signals as part of private messages, which can improve the decoding of receiver to achieve a larger transmission rate for transmitter 1. This operation can also be performed by transmitter 1. In this way, two transmitters achieve the cooperation in an implicit manner.

**Example 2.** Consider a MAC where the inputs $X_1, X_2$ are binary. The channel output $Y = (X_1 \oplus S_1 \oplus N, X_2 \oplus S_2)$. Transmitter 1’s feedback is $Z_1 = (X_1 \oplus S_1, X_1 \oplus S_2)$, and transmitter 2’s feedback $Z_2 = X_1 \oplus B$. The channel states $S_1, S_2, B, N$ are independent binary random variables with entropies $H(S_1) = H(S_2) = H(B) = 0.5$ and $H(N) = 1$. The receiver is assumed to have no CSI, i.e., $S_R = \phi$. The sensing states of two transmitter are assumed

$P_{S_1}(1) = P_{S_2}(1) = P_B(1) = 0.11, P_N(1) = 0.5$. 

$^2$
as $S_{T1} = S_2, S_{T2} = S_1$, and the Hamming distortion $d(s_T, \hat{s}_T) = s_T \oplus \hat{s}_T$ is considered as the distortion metric. In this example, transmitter 1 can always achieve zero distortion by conducting $\hat{S}_{T1} = (X_1 \oplus S_2) \oplus X_1$, while transmitter 2 cannot directly get any information about $S_{T2} = S_1$ as $Z_2 = X_1 \oplus B$.

In our proposed scheme corresponding to Theorem 3, transmitter 1 can send compressed information related to echo signals in common message part to improve the state estimation of transmitter 2. One can choose

$$X_k = U_k \oplus \Theta_k = U \oplus \Sigma_k \oplus \Theta_k, \quad k \in \{1, 2\},$$

$$V_{1,c} = S_1, V_{1,p} = V_{2,c} = V_{2,p} = \phi,$$

where $U, \Sigma_1, \Sigma_2, \Theta_1, \Theta_2$ are independent binary random variables. It can be checked that constraints (21a), (21b), (21c), (21d) can be satisfied for certain distributions of $U, \Sigma_1, \Sigma_2, \Theta_1, \Theta_2$. As a result, transmitter 2 obtains $V_{1,c} = S_1$ and thus achieves zero distortion. Therefore, the rate-distortion tuple

$$(R_1, R_2, D_1, D_2) = (0, 0, 0, 0)$$

lies in $I_{RD}$ (also in $I_{RD,com}$) of Theorem 3. However, $(R_1, R_2, D_1, D_2) = (0, 0, 0, 0)$ is not in the inner bound $I_{RD,ack}$ of Theorem 1. Together with Theorem 4, we have $I_{RD,ack} \subsetneq I_{RD,com} \subseteq I_{RD}$ for this channel.

**Proof:** See Appendix F.

**Remark 6.** In Example 2, transmitter 1 can obtain the perfect sensing states of both two transmitters, while transmitter 2 can only obtain a noisy version of codeword sent by transmitter 1. The channel from transmitter 1 to transmitter 2 with input $X_1$ and output $Z_2$ is with capacity 0.5 as $Z_2 = X_1 \oplus B$ and $H(B) = 0.5$. The communication channel from the transmitters to the receiver are with capacity 0 and 0.5, respectively, as $H(N) = 1$ and $H(S_2) = 0.5$.

Both our proposed scheme and that in [18] consider letting the transmitter 1 send compressed information to improve the sensing of transmitter 2 as well as the decoding of receiver. In our proposed scheme, the cooperation between transmissions among messages and compressed information is achieved. As a result, the compressed information $V_{1,c} = S_1$ without noise can be transmitted to transmitter 2 through the feedback, then transmitter 2 sends this compressed...
information to the receiver through the communication channel, guaranteeing that the receiver can also successfully decode $V_{1,c} = S_1$. However, the scheme proposed in [18] only consider the cooperation among messages, which results in a single-user rate-constraint on the transmission of compressed information $V_1$ to the receiver, i.e., the inequality constraint (11b). As a result, transmitter 1 cannot send a noiseless version of $S_1$ to both transmitter 2 and the receiver in the scheme of [18].

B. The Advantage of Introducing Rate-Limited Constraints in Our Outer Bound

**Example 3.** Consider a MAC where the inputs $X_1, X_2$ are binary. The channel output $Y = S_1 X_1 + S_2 X_2$ is ternary, where the channel states $S_1, S_2$ are independent binary random variables with $H(S_1) = H(S_2) = 0.5$. Transmitter 1’s feedback is $Z_1 = X_1 \oplus S_1$, and transmitter 2’s feedback $Z_2 = X_2 \oplus S_2$. The receiver is assumed to have no CSI, i.e., $S_R = \phi$. The sensing states of two transmitters are assumed as $S_{T_1} = S_2, S_{T_2} = S_1$, and the Hamming distortion $d(s_{T_k}, \hat{s}_{T_k}) = s_{T_k} \oplus \hat{s}_{T_k}$ is considered as the distortion metric.

In this example, the rate-distortion tuple

$$(R_1, R_2, D_1, D_2) = (0, 0, 0, 0)$$

(32)

lies in the $O_{khkc}^{R=D}$ of Theorem 2 but not in our outer bound $O_{our}^{R=D}$ of Theorem 5. Together with Theorem 6, we establish $O_{our}^{R=D} \subset O_{khkc}^{R=D}$ for this channel. More specifically, when choosing the genie-aided estimators (13) in Theorem 2, both two transmitters can obtain zero distortion for sensing states. While in our outer bound, based on sensing constraints (27a), we have

$$f_{k,R-D}(D_k) \leq I(S_{T_k}X_{3-k}; Z_k|X_kQ)$$

$$= H(Z_k|X_kQ) - H(Z_k|X_1X_2S_{T_k}Q)$$

$$\leq H(Z_k|X_k) - H(Z_k|X_1X_2S_{T_k})$$

$$= 0, \ k \in \{1, 2\},$$

(33)

where (a) follows from that conditioning reduces entropy and Markov chain $Q - X_1X_2S_{T_k} - Z_k$. Combing with the numerical calculations, one can obtain that in our outer bound,

$$D_{1, \text{min}} = \min\{P_{S_2}, 1 - P_{S_2}\}, \ D_{2, \text{min}} = \min\{P_{S_1}, 1 - P_{S_1}\}. \quad (34)$$
C. The Advantages Can Be Obtained Simultaneously

Example 4. Consider a MAC where the inputs $X_1, X_2$ are binary. The channel output $Y = (Y_1, Y_2)$ is a binary tuple with channel law $Y_1 = X_1 \oplus S_1, Y_2 = X_2 \oplus S_2$, where the channel states $S_1, S_2$ are independent binary random variables with $P_{S_1}(1) = 0.24, P_{S_2}(1) = 0.05$. Transmitter 1’s feedback is $Z_1 = X_1 \oplus N$, and transmitter 2’s feedback $Z_2 = (B X_1, X_2 \oplus S_1)$ is a binary tuple, where $N, B$ are independent binary random variables with $P_N(1) = 0.3, P_B(1) = 0.5$. The receiver is assumed to have no CSI, i.e., $S_R = \phi$. The sensing states of two transmitters are assumed as $S_{T_1} = S_{T_2} = N$, and the Hamming distortion $d(s_T, \hat{s}_T) = s_T \oplus \hat{s}_T$ is considered.

1) Characterization of Inner Bounds: In this example, transmitter 1 can always achieve zero distortion, and we thus focus on the achievable region $(R_1, R_2, D_2)$. Note that even for the considered binary example, the explicit characterizations of both our results $I^{\text{our}}_{R-D}, I^{\text{com}}_{R-D}$ and the existing one $I^{\text{awk}}_{R-D}$ of [18] are intractable due to the large number of choices of random variables in these schemes. To cope with this challenge, we first evaluate $I^{\text{awk}}_{R-D}, I^{\text{our}}_{R-D}, I^{\text{com}}_{R-D}$ by considering some particular choice of random variables involved in the rate-distortion characterization. Then, we theoretically prove that $I^{\text{awk}}_{R-D} \subsetneq I^{\text{our}}_{R-D} \subsetneq I^{\text{our}}_{R-D}$, i.e., our results strictly include the existing one, regardless of the choice of the random variables.

We consider the following choice of random variables to evaluate region $I^{\text{our}}_{R-D}$:

$$X_k = U_k \oplus \Theta_k = U \oplus \Sigma_k \oplus \Theta_k, \ k \in \{1, 2\},$$  \hspace{1cm} (35a)

$$V_{1, c} = \tilde{N}, \ V_{1, p} = V_{2, c} = \phi, \ V_{2, p} = S_1,$$  \hspace{1cm} (35b)

where $U, \Sigma_1, \Sigma_2, \Theta_1, \Theta_2$ are independent binary random variables, $N = \tilde{N} \oplus E$, and $E \sim \text{Bern}(P_E)$ is a binary quantization variable with $P_E(1) \in [0, P_N(1)]$. For $I^{\text{com}}_{R-D}$, we still consider (35a) and (35b) except that $V_{2, p} = \phi$. To characterize region $I^{\text{awk}}_{R-D}$, we consider (35a) and set variables $V_1 = \tilde{N}, V_2 = \phi$. For the above choices of random variables, the achievable regions of all three schemes are presented in Fig. 6.

It can be shown that tuples $(R_1, R_2, D_2) = (0, 0, 0.13072)$ and $(0.11697, 0, 0.1783)$ are in $I^{\text{com}}_{R-D}$ (as shown in Fig. 6) but not in $I^{\text{awk}}_{R-D}$, and tuple $(0.918563, 0, 0.3)$ is in $I^{\text{our}}_{R-D}$ (as shown in Fig. 6) but not in $I^{\text{our}}_{R-D}$. Together with Theorem 4, we have

$$I^{\text{awk}}_{R-D} \subsetneq I^{\text{our}}_{R-D} \subsetneq I^{\text{our}}_{R-D}$$  \hspace{1cm} (36)
for this channel.

2) Characterization of Outer Bounds: Transmitter 1 can always achieve zero distortion, and we notice that both $O_{R,D}^{\text{our}}$ and $O_{R,D}^{\text{khkc}}$ are based on the idea of dependence balance and have the same results on communication when ignoring the sensing task. Therefore, we focus on the tradeoff between symmetric rate $R_1 = R_2 = R$ and distortion $D_2$.

It should be noted that the evaluations of outer bounds $O_{R,D}^{\text{our}}$ and $O_{R,D}^{\text{khkc}}$ are intractable due to the fact that the large cardinality of involved auxiliary random variables $T$ prohibits an exhaustive search [21]. To address these challenges, we apply the technique of [21] by introducing an adaptive parallel channel extension for the dependence balance bound and using a composite function $\phi(t) = \frac{1 - \sqrt{1 - 2t}}{2}$, $t \in [0, 1]$ and its properties to explicitly evaluate both our outer bound $O_{R,D}^{\text{our}}$ and the existing one $O_{R,D}^{\text{khkc}}$.

The adaptive parallel channel extension of our outer bound is given as follows. Let $\Delta(\mathcal{U})$ denote the set of all distributions of $U$ and $\Delta(\mathcal{U}|\mathcal{V})$ denote the set of all conditional distributions...
of $U$ given $V$. Then for any mapping $F : \Delta(\mathcal{X}_1 \times \mathcal{X}_2) \to \Delta(\mathcal{Z}_{pc} | \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{Y} \times \mathcal{Z}_1 \times \mathcal{Z}_2)$, the optimal capacity-distortion region $C(D_1, D_2)$ of ISAC over MAC is contained in $\mathcal{O}^{|T|}_{R-D-PC}$ where

$$R_1 \leq I(X_1; Y Z_1 Z_2 Z_{pc} | S_R X_2 T),$$  \hfill (37a)

$$R_2 \leq I(X_2; Y Z_1 Z_2 Z_{pc} | S_R X_1 T),$$  \hfill (37b)

$$R_1 + R_2 \leq I(X_1 X_2; Y Z_1 Z_2 Z_{pc} | S_R T),$$  \hfill (37c)

$$R_1 + R_2 \leq I(X_1 X_2; Y | S_R),$$  \hfill (37d)

with the dependence balance constraint

$$I(X_1; X_2 | T) \leq I(X_1; X_2 | Z_1 Z_2 Z_{pc} T)$$  \hfill (38)

and sensing constraints

$$f_{k,R-D-PC}(D_k) \leq I(S_{T_k}; X_{3-k} Z_k Z_{pc} | X_k Q), k \in \{1, 2\},$$  \hfill (39a)

$$f_{k,R-D-PC}(D_k) \leq I(S_{T_k}; Z_1 Z_2 Z_{pc} | X_1 X_2 Q), k \in \{1, 2\},$$  \hfill (39b)

where $T = (Q, Q_Z)$ with cardinality $|T| \leq |\mathcal{X}_1| |\mathcal{X}_2| + 3$, the new rate-distortion function is given by

$$f_{1,R-D-PC}(D_1) = \min_{P_{S_{T_1} | X_1 Z_1 Z_{pc} S_{T_1}} \sum_{s_1 z_1 z_{pc} s_{T_1}} P_{X_1 Z_1 Z_{pc} s_{T_1}} P_{S_{T_1} | X_1 Z_1 Z_{pc} s_{T_1}} d_1(s_1, s_{T_1}) \leq D_1} I(S_{T_1}; \hat{S}_{T_1}),$$  \hfill (40a)

$$f_{2,R-D-PC}(D_2) = \min_{P_{S_{T_2} | X_2 Z_2 Z_{pc} S_{T_2}} \sum_{s_2 z_2 z_{pc} s_{T_2}} P_{X_2 Z_2 Z_{pc} s_{T_2}} P_{S_{T_2} | X_2 Z_2 Z_{pc} s_{T_2}} d_2(s_2, s_{T_2}) \leq D_2} I(S_{T_2}; \hat{S}_{T_2}),$$  \hfill (40b)

and the random variables $QT X_1 X_2 Y Z_1 Z_2 Z_{pc} S_R S_{T_1} S_{T_2}$ have the joint distribution

$$P_{QT X_1 X_2 Y Z_1 Z_2 Z_{pc} S_R S_{T_1} S_{T_2}} = P_Q T P_{X_1 X_2 | T} P_{S_R S_{T_1} S_{T_2}} P_{Y Z_1 Z_2 | X_1 X_2 S_{pc} T} P_{Z_{pc} | T X_1 X_2 Y Z_1 Z_2},$$  \hfill (41)

such that for all $t$

$$P_{Z_{pc} | T X_1 X_2 Y Z_1 Z_2}(z_{pc} | t x_1 x_2 y z_1 z_2) = F(P_{X_1 X_2 | T}(x_1 x_2 | t)).$$  \hfill (42)

The idea of choosing the parallel channel $P_{Z_{pc} | T X_1 X_2 Y Z_1 Z_2}^+(z_{pc} | t x_1 x_2 y z_1 z_2)$ is to reduce the amount of dependence, which makes the characterization more tractable. In other words, one can choose the parallel channel $P_{Z_{pc} | T X_1 X_2 Y Z_1 Z_2}^+(z_{pc} | t x_1 x_2 y z_1 z_2)$ to make the dependence balance...
constraint more stringent, consequently reducing the set of allowable input distributions. In this example, we consider the same technique used in [21] with the choice of \( Z_{pc} = X_2 \). Given such a choice, we have

\[
I(X_1; X_2|T) \leq I(X_1; Z_1Z_2Z_{pc}|T) = 0,
\]

which implies only distributions of the type

\[
P_{X_1X_2|T} = P_{X_1|TP_{X_2|T}}
\]

are allowed. The corresponding result is given by

\[
\mathcal{C}_{R-D-PC}^{our,X_2} = \left\{ (R_1, R_2, D_1, D_2) : R_1 \leq I(X_1; YZ_1Z_2|S_RX_2T), \quad R_2 \leq H(X_2|S_RT), \quad R_1 + R_2 \leq I(X_1X_2; Y|S_R), \quad f_{1,R-D-PC}(D_1) \leq I(S_{T_1}X_2; Z_1X_1|X_1Q), \quad f_{2,R-D-PC}(D_2) \leq I(S_{T_2}X_1; Z_2X_2Q), \quad f_{k,R-D-PC}(D_k) \leq I(S_{T_k}X_2; Z_1Z_2|X_1X_2Q), k \in \{1, 2\} \right\},
\]

where the sum rate (37c) is redundant [21] and the region is evaluated over the set of input distributions of the form \( P_{QTX_1X_2} = P_{QT}P_{X_1|T}P_{X_2|T} \) and the auxiliary random variables \( Q, T \) are subject to cardinality constraint of \( |Q| \leq |T| \leq |X_1||X_2| + 3 \).

We consider the composite function \( \omega(t) = \frac{1-\sqrt{1-2t}}{2} \) for \( 0 \leq t \leq 1 \). We refer to the entropy function as which is defined as

\[
h^{(k)}(t_1, \ldots, t_k) = -\sum_{i=1}^{k} t_i \log(t_i),
\]

for \( t_i \geq 0, i \in [1 : k], \) and \( \sum_{i=1}^{k} t_i = 1 \). Specifically, we denote \( h^{(2)}(t, 1-t) \) simply as \( h(t) \). Note that for composite function \( \omega(t) = \frac{1-\sqrt{1-2t}}{2} \) for \( 0 \leq t \leq 1 \), the following property holds:

\[
\omega(2t(1-t)) = \min(t, 1-t).
\]

As a consequence, the following holds:

\[
h(\omega(2t(1-t))) = h(t).
\]
Now we characterize the outer bound result. Let the cardinality of the auxiliary random variables $Q$ and $T$ be fixed and arbitrary, say $Q, T$. Then, the joint distribution $P_{QT} P_{X_1|T} P_{X_2|T}$ can be described by the following variables

$$\pi_{1t} = P_{X_1|T}(0|t), \quad t = 1, \ldots, |T|,$$

(49a)

$$\pi_{2t} = P_{X_2|T}(0|t), \quad t = 1, \ldots, |T|,$$

(49b)

$$\kappa_t = P_T(t) = \sum_q P_{QT}(qt), \quad t = 1, \ldots, |T|,$$

(49c)

$$\kappa_{qt} = P_{QT}(qt), \quad q = 1, \ldots, |Q|, \quad t = 1, \ldots, |T|,$$

(49d)

Our outer bound can be characterized in terms of three variables $\alpha_1, \alpha_2, \alpha$, which are functions of $P_{QT} P_{X_1|T} P_{X_2|T}$, and are given as

$$\alpha_1 = \sum_t \kappa_t \pi_{1t} (1 - \pi_{1t}) = \sum_t \kappa_t \alpha_{1t},$$

(50a)

$$\alpha_2 = \sum_t \kappa_t \pi_{2t} (1 - \pi_{2t}) = \sum_t \kappa_t \alpha_{2t},$$

(50b)

$$\alpha = \sum_t \kappa_t \pi_{1t} = \sum_t \kappa_t \alpha_t.$$  

(50c)

It should be noted that $\alpha_1, \alpha_2$ both lie in the range $[0, \frac{1}{4}]$ as $0 \leq \pi_{jt} \leq 1$ for $j = 1, 2, t = 1, \ldots, |T|$, and $\alpha$ lies in the range $[\alpha_1, 1]$. The upper bounds for terms in $O^{our,X_2}_{R,D-PC}$ are given as follows.

$$R_1 \leq I(X_1; Y Z_1 Z_2 | X_2 S R T)$$

$$\overset{(a)}{=} H(X_1|T)$$

$$= \sum_t \kappa_t h(\pi_{1t})$$

$$\overset{(b)}{=} \sum_t \kappa_t h(\omega(2\pi_{1t}(1 - \pi_{1t})))$$

$$\overset{(c)}{=} \sum_t \kappa_t h(\omega(2\alpha_{1t}))$$

$$\overset{(d)}{\leq} h(\omega(2\alpha_1)),$$

(51)

where (a) follows from that $S_R = \phi$, $Y = (X_1 \oplus S_1, X_2 \oplus S_2)$, $Z_1 = X_1 \oplus N$, $Z_2 = (BX_1, X_2 \oplus S_1)$, and $X_1$ is independent of $X_2$ given $T$ for $O^{our,X_2}_{R,D-PC}$, (b) follows from (48), (c) follows from (50).
(d) follows from the application of Jensen’s inequality. Similarly, we have
\[
R_2 \leq H(X_2|S_RT) = H(X_2|T) \leq h^{(2)}(\omega(2\alpha_2)),
\]
and
\[
R_1 + R_2 \leq I(X_1X_2;Y|S_R) = H(Y) - H(Y|X_1X_2) \\
= h^{(4)}(P_{Y_1Y_2}(0,0), P_{Y_1Y_2}(0,1), P_{Y_1Y_2}(1,0), P_{Y_1Y_2}(1,1)) - H(S_1) - H(S_2) \\
\leq h(P_{Y_1Y_2}(0,0) + P_{Y_1Y_2}(0,1)) + 1 - H(S_1) - H(S_2) \\
= h(P_{Y_1}(0)) + 1 - H(S_1) - H(S_2) \\
= h\left(P_{S_1}(1) + (1 - 2P_{S_1}(1))\alpha\right) + 1 - H(S_1) - H(S_2),
\]
where (a) follows from the fact that
\[
h^{(4)}(a, b, c, d) = \frac{1}{2} h^{(4)}(a, b, c, d) + \frac{1}{2} h^{(4)}(b, a, d, c) \\
\leq h^{(4)}\left(\frac{a + b}{2}, \frac{a + b}{2}, \frac{c + d}{2}, \frac{c + d}{2}\right) \\
= h(a + b) + 1
\]
due to the concavity of the entropy function and the application of Jensen’s inequality. For for sensing constraints, we have
\[
I(S_{T_2}X_1;Z_2X_2Q) \overset{(a)}{=} H(Z_2|X_2Q) - H(Z_2|X_1X_2Q) \\
= H(B \cdot X_1|X_2Q) - H(B \cdot X_1|X_1Q) \\
\overset{(b)}{\leq} H(B \cdot X_1) - H(B \cdot X_1|X_1) \\
= h\left(P_B(1)(1 - \alpha)\right) - (1 - \alpha)H(B),
\]
and similarly,
\[
I(S_{T_2};Z_1X_1X_2Q) = h(N).
\]
where (a) follows from that \(S_{T_2} = N\), \(Z_2 = (BX_1, X_2 \oplus S_1)\), (b) follows from that conditioning reduces entropy and \(Q\) is independent of channel variables \(B, N, S_1, S_2\).

Based on the above results, one can conduct numerical characterization of symmetric-rate-distortion region \((R_1 = R_2 = R, D_2)\) for \(O^\text{our}_{R-D-PC}\), which is an outer bound of \(O^\text{our}_{R-D}\) in Theorem 5.
The same technique can be applied to get a valid outer bound $\mathcal{O}_{R-D-PC}^{khkc,X_2}$ for the results in Theorem 2 and to characterize the symmetric–rate-distortion region $(R_1 = R_2 = R, D_2)$ in $\mathcal{O}_{R-D-PC}^{total,X_2}$. We note that outer bounds $\mathcal{O}_{R-D}^{our}$ and $\mathcal{O}_{R-D}^{khkc}$ both use the idea of dependence balance. Thus, such a parallel channel extension results in the same performance on communication. Moreover, since the genie-aided state estimators are considered in $\mathcal{O}_{R-D}^{khkc}$ of Theorem 2, the parallel channel extension $Z_{PC} = X_2$ does not effect sensing performance for $\mathcal{O}_{R-D}^{khkc}$. Therefore, such a characterization can demonstrate the advantages of our improved outer bound on sensing performance.

The results are shown in Fig. 7 which shows that our outer bound is tighter than state-of-the-art [14]. Specifically, one can easily check that tuple $(R = 0, D_2 = 0)$ is in $\mathcal{O}_{R-D}^{khkc}$ but not in $\mathcal{O}_{R-D}^{our}$ which reveals that

$$\mathcal{O}_{R-D}^{our} \subsetneq \mathcal{O}_{R-D}^{khkc}. \quad (57)$$

Moreover, it can also be seen that our proposed scheme can achieve the sensing-optimal point $(D_{2,min}, 0)$ and communication-optimal point $(D_{2,max}, R_{max})$ in this example, while for certain distortion constraints such as $D_2 = 0.2$, how to achieve the optimal performance for ISAC still remains open.
Proof: The proof of $T_{R-D}^{\text{awk}} \subset T_{R-D}^{\text{our,com}} \subset T_{R-D}^{\text{our}}$ can be found in Appendix G.

Remark 7. The results and analysis presented in Example 4 demonstrate the following facts:

1) For inner bound, the performance gain of our proposed scheme is two-fold. First, by allowing transmitters to send the compressed information as part of common messages, our achievable scheme can get a better sensing performance (point A in Fig. 6), and enlarge the rate-region (point B) for the same distortion (point D $(0, 0, 0.1783)$ is in $T_{R-D}^{\text{awk}}$) that the existing one can achieve. The reason behind this improvement is that our scheme allows for the transmission of compressed information related to echo signals leveraging cooperation between the transmitters. More specifically, the proposed scheme in [17] requires the receiver to successfully decode compressed information $\tilde{N}$ sent by transmitter 1 and results in that amount of compressed information $\tilde{N}$ is bounded by single-user rate constraint of transmitter 1, $I(X_1; Y_{SR}|UX_2)$. While in our scheme, the cooperation enables transmitter 2 to send compressed information $\tilde{N}$ of transmitter 1 instead of its own messages. Such an operation results in that compressed information $\tilde{N}$ is bounded by the sum rate of two transmitters, i.e., $I(X_1X_2; Y_{SR}|U)$. Therefore, with the help of transmitter 2, transmitter 1 can send a greater amount of compressed information thus improving sensing performance or send more messages while achieving the same sensing performance. Second, allowing the transmitter 2 to send compressed information as part of private messages $V_{2,p} = S_1$ conveys information about channel states, facilitating the decoding of $W_1$ at the receiver and enlarging the rate region (point C).

2) For outer bound, our results adopts the rate-limited constraints on sensing performance, which provides a tighter bound for sensing performance, and thus a better rate-distortion outer bound for the ISAC model considered.

VI. Conclusion

In this paper, we have investigated the fundamental limits of ISAC over DM MACs with correlated sensing and channel states as well as imperfect CSIR. A new achievable scheme that combines message cooperation and joint compression of past transmitted codewords and echo signals using Wyner-Ziv coding has been proposed, and the corresponding inner bound of the capacity-distortion region has been proved to include that of [18]. We have also established an
improved outer bound of the capacity-distortion region by introducing the rate-limited constraints on sensing. It has been demonstrated, through several numerical examples, that the improved inner bound can achieve better communication and sensing performance, and the proposed outer bound strictly improves the existing one. We finally remark that while the inner and outer bounds presented in this paper improve upon the existing results [14], [18], the optimal capacity-distortion region for the model considered remains open. Future works include further tightening the bounds for ISAC over MACs and investigation of the fundamental limits of other multi-terminal ISAC building blocks, such as in broadcast channels and relay channels.

APPENDIX A

PROOF OF THEOREM 3

The achievable scheme proposed combined with distributed Wyner-Ziv coding, block Markov encoding, and backward decoding that consists of $B + \tilde{B}$ blocks. The last $\tilde{B}$ blocks called “termination blocks” are necessary to guarantee that the receiver can successfully obtain the compressed information corresponding to blocks $B$ and $B - 1$. We first focus on the first $B$ blocks by assuming that the receiver can successfully obtain compressed information corresponding to blocks $B$ and $B - 1$. Then, the detailed discussion of termination blocks is presented.

A. The Proposed Coding Scheme

Each of the first $B$ blocks is of length $N$ channel uses. Transmitter $k$ transmits $B - 1$ i.i.d. messages $\{w_k(b)\}_{b=1}^{B-1}$ over $B + \tilde{B}$ blocks. Each message $w_k(b)$ is partitioned into two messages $\{w_k(b)_{p}, w_k(b)_{c}\}$. The subscripts “p” and “c” here stand for “private” and “common”, respectively. The messages $w_{k,p}^{(b)} \in [1 : 2^{NR_{k,p}}]$ and $w_{k,c}^{(b)} \in [1 : 2^{NR_{k,c}}]$ are uniformly distributed and mutually independent for all $k \in \{1, 2\}$ and $b \in [1 : B - 1]$.

1) Codebook Generation: Pick non-negative real numbers $R_{s_{1,c}}, R_{s_{1,p}}, R_{s_{2,c}}, R_{s_{2,p}}, R_{v_{1,c}}, R_{v_{1,p}}, R_{v_{2,c}}$, and $R_{v_{2,p}}$ satisfying $R_{v_{k,c}} \leq R_{s_{k,c}}, R_{v_{k,p}} \leq R_{s_{k,p}}$, $k \in \{1, 2\}$, where $R_{s_{k,c}}$ and $R_{s_{k,p}}$ denote the quantization rates of common and private compressed sequences, and $R_{v_{k,c}}$ and $R_{v_{k,p}}$ denote the binning rates of common and private compressed sequences, respectively. For block $b \in [1 : B]$, an illustration of constructed codebook is presented in Fig. 8, and the detailed codebook is as follows.
Fig. 8. An illustration of codebook generation in block $b$

- Generate $2^{N(R_1,c+R_{e_1,c}+R_2,c+R_{e_2,c})}$ sequences $u^N(j_{b-1},l_{b-1},k_{b-1},m_{b-1})$, $j_{b-1} \in [1 : 2^{NR_1,c}]$, $l_{b-1} \in [1 : 2^{NR_{e_1,c}}]$, $k_{b-1} \in [1 : 2^{NR_2,c}]$, $m_{b-1} \in [1 : 2^{NR_{e_2,c}}]$, i.i.d. according to $P_U^f(\cdot)$.

- For each tuple $(j_{b-1},l_{b-1},k_{b-1},m_{b-1})$, generate $2^{N(R_1,c+R_{e_1,c})}$ sequences $u_1^N(j_{b-1},l_{b-1},k_{b-1},m_{b-1})$, $j_{b-1} \in [1 : 2^{NR_1,c}]$, $l_{b-1} \in [1 : 2^{NR_{e_1,c}}]$, i.i.d. according to $P_{U_1}^{f\cup}(\cdot|u)$. Similarly, generate $2^{N(R_2,c+R_{e_2,c})}$ sequences $u_2^N(j_{b-1},l_{b-1},k_{b-1},m_{b-1})$, $j_{b-1} \in [1 : 2^{NR_2,c}]$, $l_{b-1} \in [1 : 2^{NR_{e_2,c}}]$ i.i.d. according to $P_{U_2}^{f\cup}(\cdot|u)$.

- For each tuple $(j_{b-1},l_{b-1},k_{b-1},m_{b-1},j_b,l_b)$, generate $2^{N(R_{1,p}+R_{e_1,p})}$ sequences $x_1^N(j_{b-1},l_{b-1},k_{b-1},m_{b-1},j_b,l_b,r_b,\beta_{1,b})$, $r_b \in [1 : 2^{NR_{1,p}}]$, $\beta_{1,b} \in [1 : 2^{NR_{e_1,p}}]$ i.i.d. according to $P_{X_1|UU_1}(\cdot|uu_1)$. Similarly, generate $2^{N(R_{2,p}+R_{e_2,p})}$ sequences $x_2^N(j_{b-1},l_{b-1},k_{b-1},m_{b-1},j_b,l_b,m_b,s_b,\beta_{2,b})$ i.i.d. according to $P_{X_2|UU_2}(\cdot|uu_2)$.

- For each tuple $(j_{b-1},l_{b-1},k_{b-1},m_{b-1},j_b,l_b,\beta_{1,b})$, generate $2^{NR_{1,c}}$ sequences $v_{1,c}^N(j_{b-1},l_{b-1},k_{b-1},m_{b-1},j_b,l_b,k_b,m_b,\Delta_{1,c,b})$, $\Delta_{1,c,b} \in [1 : 2^{NR_{1,c}}]$ i.i.d. according to $P_{V_{1,c}|UU_1U_2}(\cdot|uu_1u_2)$. Randomly partition the indices $\{\Delta_{1,c,b} : \Delta_{1,c,b} \in [1 : 2^{NR_{1,c}}]\}$ into $2^{NR_{1,c}}$ bins. Denote by $\eta_{1,c,b}(\Delta_{1,c,b})$ the index of bin to which $\Delta_{1,c,b}$ belongs, and by $\alpha_{1,c,b}(\eta_{1,c,b})$ the set of indices $\Delta_{1,c,b}$ corresponding to the bin number $\eta_{1,c,b}$. Similarly, generate $2^{NR_{2,c}}$ sequences $v_{2,c}^N(j_{b-1},l_{b-1},k_{b-1},m_{b-1},j_b,l_b,k_b,m_b,\Delta_{2,c,b})$, $\Delta_{2,c,b} \in [1 : 2^{NR_{2,c}}]$ i.i.d. according to $P_{V_{2,c}|UU_1U_2}(\cdot|uu_1u_2)$. Randomly partition the indices $\{\Delta_{2,c,b} : \Delta_{2,c,b} \in [1 : 2^{NR_{2,c}}]\}$
For each tuple $(j_{b-1}, l_{b-1}, k_{b-1}, m_{b-1}, j_b, l_b, k_b, m_b, r_b, \beta_{1,b}, \Delta_{1,c,b})$, generate $2^{NR_{1,p}}$ sequences $u_{1,p}(j_{b-1}, l_{b-1}, k_{b-1}, m_{b-1}, j_b, l_b, k_b, m_b, r_b, \beta_{1,b}, \Delta_{1,c,b}, \Delta_{1,p,b}, \Delta_{2,p,b})$, $\Delta_{1,p,b} \in [1 : 2^{NR_{1,p}}]$ i.i.d. according to $P_{V_{1,p} \mid UU_1 U_2 X_{1,c}}(\cdot \mid uu_1 u_2 x_1 v_1,c)$. Randomly partition the indices $\{\Delta_{1,p,b} : \Delta_{1,p,b} \in [1 : 2^{NR_{1,p}}]\}$ into $2^{NR_{1,p}}$ bins. Denote by $\eta_{1,p,b}(\Delta_{1,p,b})$ the index of bin to which $\Delta_{1,p,b}$ belongs, and by $\alpha_{1,p,b}(\eta_{1,p,b})$ the set of indices $\Delta_{1,p,b}$ corresponding to the bin number $\eta_{1,p,b}$.

Similarly, generate $2^{NR_{2,p}}$ sequences $u_{2,p}(j_{b-1}, l_{b-1}, k_{b-1}, m_{b-1}, j_b, l_b, k_b, m_b, s_b, \beta_{2,b}, \Delta_{2,c,b}, \Delta_{2,p,b})$, $\Delta_{2,p,b} \in [1 : 2^{NR_{2,p}}]$ i.i.d. according to $P_{V_{2,p} \mid UU_1 U_2 X_{2,c}}(\cdot \mid uu_1 u_2 x_2 v_2,c)$. Randomly partition the indices $\{\Delta_{2,p,b} : \Delta_{2,p,b} \in [1 : 2^{NR_{2,p}}]\}$ into $2^{NR_{2,p}}$ bins. Denote by $\eta_{2,p,b}(\Delta_{2,p,b})$ the index of bin to which $\Delta_{2,p,b}$ belongs, and by $\alpha_{2,p,b}(\eta_{2,p,b})$ the set of indices $\Delta_{2,p,b}$ corresponding to the bin number $\eta_{2,p,b}$.

2) Encoding: We set $j_0 = k_0 = 1$, $l_0 = m_0 = 1$, $l_1 = m_1 = 1$, $\beta_{1,1} = \beta_{2,1} = 1$, $j_B = k_B = r_B = s_B = 1$ and $\Delta_{1,c,0} = \Delta_{2,c,0} = 1$.

- **Block $b = 1$**. Two transmitters send $x_1^N(1, 1, 1, 1, j_1, 1, r_1, 1)$ and $x_2^N(1, 1, 1, 1, k_1, 1, s_1, 1)$, respectively.

- **Block $b \in [2 : B+1]$**. At the beginning of block $b$, transmitter 1 obtains generalized feedback $z_1^N(b-1)$ and the correct indices tuple $(j_{b-2}^*, l_{b-2}^*, k_{b-2}^*, m_{b-2}^*, j_{b-1}^*, l_{b-1}^*, r_{b-1}^*, \beta_{1,b-1}^*)$. If $b \in [3 : B]$, it also knows $(j_{b-3}^*, l_{b-3}^*, k_{b-3}^*, m_{b-3}^*, r_{b-2}^*, \beta_{1,b-2}^*)$. For $B \in [3 : B+1]$, transmitter 1 finds a unique triple $((\hat{k}_{b-1}, \hat{m}_{b-1}, \hat{\Delta}_{2,c,b-2}) \in [1 : 2^{NR_{2,c}}] \times [1 : 2^{NR_{2,c}}] \times \alpha_{2,c,b-1}(\hat{m}_{b-1})$ satisfying

\[
\begin{align*}
&\left(u_1^N(j_{b-2}^*, l_{b-2}^*, k_{b-2}^*, m_{b-2}^*), u_2^N(j_{b-2}^*, l_{b-2}^*, k_{b-2}^*, m_{b-2}^*, k_{b-1}^*, m_{b-1}^*), x_1^N(j_{b-2}^*, l_{b-2}^*, k_{b-2}^*, m_{b-2}^*, r_{b-1}^*, \beta_{1,b-1}^*),
\right. \\
&\left.z_1^N(b-1) \right) \in \mathcal{T}_{\epsilon}^N(UU_1 U_2 X_1 Z_1),
\end{align*}
\]

and

\[
\left(u_1^N(j_{b-3}^*, l_{b-3}^*, k_{b-3}^*, m_{b-3}^*),
\right. 
\]

into $2^{NR_{2,c}}$ bins. Denote by $\eta_{2,c,b}(\Delta_{2,c,b})$ the index of bin to which $\Delta_{2,c,b}$ belongs, and by $\alpha_{2,c,b}(\eta_{2,c,b})$ the set of indices $\Delta_{2,c,b}$ corresponding to the bin number $\eta_{2,c,b}$. 

\[(58)\]
Block \( b = 1 \) \( b = 2 \) \( b = 3 \) \( \cdots \) \( b = B - 1 \) \( b = B \)

| Enc 1 | | | | | |
|-------|---|---|---|---|---|
| \( 1 \) | \( 1 \) | \( j_1 \) | \( 1 \) | \( k_1 \) | \( 1 \) |
| \( 1 \) | \( 1 \) | \( j_2 \) | \( 1 \) | \( k_2 \) | \( m_2 \) |

| Enc 2 | | | | | |
|-------|---|---|---|---|---|
| \( 1 \) | \( 1 \) | \( j_3 \) | \( 1 \) | \( k_3 \) | \( m_3 \) |
| \( 1 \) | \( 1 \) | \( j_4 \) | \( 1 \) | \( k_4 \) | \( m_4 \) |

Common part obtained from feedback

| Message indices | Compressed information obtained from previous blocks |
|----------------|---------------------------------------------------|
| \( j_1 \) | \( 1 \) |
| \( j_2 \) | \( 1 \) |
| \( j_3 \) | \( 1 \) |
| \( j_4 \) | \( 1 \) |

Fig. 9. Encoding operations in the first \( B \) blocks.

\[
\begin{align*}
\hat{u}_1^N &= (j_{b-3}^*, l_{b-3}^*, k_{b-3}^*, m_{b-3}^*, j_{b-2}^*, l_{b-2}^*), \\
\hat{u}_2^N &= (j_{b-3}^*, l_{b-3}^*, k_{b-3}^*, m_{b-3}^*, k_{b-2}^*, m_{b-2}^*), \\
x_1^N &= (j_{b-3}^*, l_{b-3}^*, k_{b-3}^*, m_{b-3}^*, j_{b-2}^*, l_{b-2}^*, r_{b-2}^*, \beta_{1,b-2}), \\
v_{2,c}^N &= (j_{b-3}^*, l_{b-3}^*, k_{b-3}^*, m_{b-3}^*, j_{b-2}^*, l_{b-2}^*, k_{b-2}^*, m_{b-2}^*, \\
\Delta_{2,c,b-2}, z_{1}^{N}(b-2) \in T_{e}^{N}(UU_1U_2X_1V_2cZ_1) \quad (59)
\end{align*}
\]

simultaneously. For \( b = 2 \), only (58) needs to be satisfied.

If there is exactly one triple \((\hat{k}_{b-1}, \hat{m}_{b-1}, \hat{\Delta}_{2,c,b-2})\) satisfying the conditions, transmitter 1 sets \( k_{b-1}^* = \hat{k}_{b-1}, m_{b-1}^* = \hat{m}_{b-1}, \) and \( \Delta_{2,c,b-2}^* = \hat{\Delta}_{2,c,b-2} \). Otherwise, an error is declared. Once obtaining the correct indices \( k_{b-1}^*, m_{b-1}^* \), transmitter 1 finds a pair \((\hat{\Delta}_{1,c,b-1}, \hat{\Delta}_{1,p,b-1}) \in [1 : 2^{NR_{s1,c}}] \times [1 : 2^{NR_{s1,p}}] \) satisfying

\[
\begin{align*}
(u^N(j_{b-2}^*, l_{b-2}^*, k_{b-2}^*, m_{b-2}^*), \\
u_1^N(j_{b-2}^*, l_{b-2}^*, k_{b-2}^*, m_{b-2}^*, j_{b-1}^*, l_{b-1}^*),
\end{align*}
\]
and sets $\Delta_{1,c,b-1}^* = \hat{\Delta}_{1,c,b-1}$ and $\Delta_{1,p,b-1}^* = \hat{\Delta}_{1,p,b-1}$. If there is no such pair, set $\Delta_{1,c,b-1}^* = 1$ and $\Delta_{1,p,b-1}^* = 1$. Once obtaining the indices $\Delta_{1,c,b-1}^*$, $\Delta_{1,p,b-1}^*$, denote by $l_b^*$ and $b_{1,b}$ the bin numbers to which the indices $\Delta_{1,c,b-1}^*$ and $\Delta_{1,p,b-1}^*$ belong, i.e., $l_b^* = m_{1,c,b-1}(\Delta_{1,c,b-1}^*)$, $b_{1,b} = m_{1,p,b-1}(\Delta_{1,p,b-1}^*)$. Then, transmitter 1 sends $x_1^N(j_{b-1}^*, l_{b-1}^*, k_{b-1}^*, m_{b-1}^*, j_b, l_b^*, r_b, \beta_{1,b})$ in block $b \in [2 : B]$. In particular, the successful transmission of indices $l_B^*, l_{B+1}^*$, $\beta_{1,B+1}^*$ is guaranteed by termination blocks. The operations at transmitter 2 are analogous and omitted here.

A brief illustration of encoding operations is provided in Fig. 9.

3) Decoding: Decoding begins at the block $B$ and proceeds backward. In block $b \in [B : 1]$, the receiver has channel output $y_b^N(b)$, CSIR $s_{b}^N(b)$. Assuming that the receiver has found the correct indices tuple $(l_{b+1}^*, m_{b+1}^*, b_{1,b+1}, b_{2,b+1})$ and $(j_{b}^*, k_{b}^*, m_{b}^*)$, it then finds a unique tuple $(\hat{j}_{b-1}, \hat{l}_{b-1}, \hat{k}_{b-1}, \hat{m}_{b-1}, \hat{r}_{b}, \hat{\beta}_{1,b}, \hat{\beta}_{2,b}) \in [1 : 2^{NR1,c}] \times [1 : 2^{NR1,c}] \times [1 : 2^{NR2,c}] \times [1 : 2^{NR2,c}]$ such that

\begin{align*}
&u_2^N(j_{b-2}^*, l_{b-2}^*, k_{b-2}^*, m_{b-2}^*, k_{b-1}^*, m_{b-1}^*), \\
u_1^N(j_{b-2}^*, l_{b-2}^*, k_{b-2}^*, m_{b-2}^*, j_{b-1}^*, l_{b-1}^*, r_{b-1}^*, \beta_{1,b-1}^*), \\
v_1^N(j_{b-2}^*, l_{b-2}^*, k_{b-2}^*, m_{b-2}^*, j_{b-1}^*, l_{b-1}^*, k_{b-1}^*, m_{b-1}^*), \\
v_1^N(j_{b-2}^*, l_{b-2}^*, k_{b-2}^*, m_{b-2}^*, j_{b-1}^*, l_{b-1}^*, k_{b-1}^*, m_{b-1}^*, \hat{\Delta}_{1,c,b}), \\
v_2^N(j_{b-2}^*, l_{b-2}^*, k_{b-2}^*, m_{b-2}^*, j_{b-1}^*, l_{b-1}^*, k_{b-1}^*, m_{b-1}^*, \hat{\Delta}_{2,c,b}), \\
v_1^N(j_{b-2}^*, l_{b-2}^*, k_{b-2}^*, m_{b-2}^*, j_{b-1}^*, l_{b-1}^*, k_{b-1}^*, m_{b-1}^*, \hat{r}_{b}, \hat{\beta}_{1,b})
\end{align*}
It can also obtain the indices \( j^{*} \) for some \( \tilde{\Delta} \) and \( \hat{\alpha} \). Otherwise, an error is declared. The output of receiver is

\[
\hat{b}^{*}, \beta^{2, b}, \tilde{\Delta}, \hat{\alpha}, y^{N}(b), s_{R}^{N}(b)
\]

\( \in T_{e}^{N}(UU_{1}U_{2}X_{1}X_{2}V_{1,c}V_{2,c}V_{1,p}V_{2,p}Y_{S_{R}}) \) (61)

for some \( \tilde{\Delta}_{1,c,b} \in \alpha_{1,c,b+1}(l_{b+1}), \tilde{\Delta}_{2,c,b} \in \alpha_{2,c,b+1}(m_{b+1}), \hat{\alpha}_{1,p,b} \in \alpha_{1,p,b+1}(\hat{\alpha}_{b+1}), \) and \( \tilde{\Delta}_{2,p,b} \in \alpha_{2,p,b+1}(\hat{\alpha}_{b+1}) \). If there is exactly one tuple \( (j^{*}_{b-1}, l^{*}_{b-1}, \hat{k}_{b-1}, \hat{m}_{b-1}, \beta^{*}_{1,b}, \beta^{*}_{2,b}) \) satisfying (61), receiver sets \( j^{*}_{b-1} = j_{b-1}, l^{*}_{b-1} = l_{b-1}, k^{*}_{b-1} = \hat{k}_{b-1}, m^{*}_{b-1} = \hat{m}_{b-1}, r^{*}_{b} = \hat{r}_{b}, \beta^{*}_{1,b} = \hat{\beta}_{1,b}, \beta^{*}_{2,b} = \hat{\beta}_{2,b} \). Otherwise, an error is declared. The output of receiver is \( (j^{*}_{b}, r^{*}_{b}, k^{*}_{b}, m^{*}_{b}) \), \( b \in [1 : B - 1] \). A brief illustration of decoding operations is provided in Fig. 10.

4) **State Estimation:** Transmitter 1 can successfully obtain the \( k^{*}_{b}, m^{*}_{b}, \tilde{\Delta}_{2,c,b} \) for \( b \in [1 : B - 1] \). It can also obtain the indices \( j^{*}_{b-1}, l^{*}_{b-1}, k^{*}_{b-1}, m^{*}_{b-1}, \beta^{*}_{1,b}, \beta^{*}_{2,b} \) and generalized feedback \( z_{1}^{N}(b) \) for \( b \in [1 : B - 1] \). Therefore, transmitter 1 outputs the estimated state sequence

\[
\hat{S}_{T_{1}}(b) = \hat{S}_{T_{1}}(x_{1}^{N}(j^{*}_{b-1}, l^{*}_{b-1}, k^{*}_{b-1}, m^{*}_{b-1}, \hat{j}_{b}, \hat{l}_{b}, r^{*}_{b}, \beta^{*}_{1,b}), u_{2}^{N}(j^{*}_{b-1}, l^{*}_{b-1}, k^{*}_{b-1}, m^{*}_{b-1}, k_{b}, m_{b}), \beta^{2,b})
\]
\[ z_1^N(b), v_2^N(j_{b-1}^*, l_{b-1}^*, k_{b-1}^*, m_{b-1}^*, j_b^*, l_b^*, k_b^*, m_b^*, \Delta_{2,c,b}^*) \]  

(62)

for block \( b \in [1 : B - 1] \). The operations at transmitter 2 are analogous.

### B. Discussion of Termination Blocks

The termination blocks are used to guarantee that the receiver can conduct the backward decoding \((61)\). In the following part, we show that for most cases, the termination blocks can be carefully constructed to achieve this goal, while for the other cases, one can modify the achievable scheme to guarantee the achievability of Theorem 3.

Based on the \((20c)\) in Theorem 3, we have

\[ R_1 + R_2 \leq I(X_1X_2; Y|S_R). \]  

(63)

1) \( I(X_1X_2; Y|S_R) = 0 \): Based on \((63)\), we have

\[ R_1 = R_2 = 0. \]  

(64)

In this case, the achievability of Theorem 3 can be guaranteed by the proposed scheme in Appendix A-A without the decoding operations, and there is no need to consider the termination blocks.

2) \( I(X_1X_2; Y|S_R) > 0 \): Since

\[ I(X_1X_2; Y|S_R) = I(X_1; Y|S_R) + I(X_2; Y|X_1S_R), \]  

(65)

we know that \( I(X_1; Y|S_R) \) and \( I(X_2; Y|X_1S_R) \) cannot be both zero when \( I(X_1X_2; Y|S_R) > 0 \). This reveals that at least one transmitter can achieve a positive transmission rate to the receiver.

Without loss of generality, we consider that

\[ I(X_1; Y|S_R) > 0 \]  

(66)

as the case that \( I(X_2; Y|X_1S_R) > 0 \) can be addressed in a similar manner. We consider the following three cases: (a) \( I(U_2; Z_1|UU_1X_1) > 0 \); (b) \( I(U_2; Z_1|UU_1X_1) = 0 \) and \( I(X_2; UU_1U_2X_1Z_1YS_R) > 0 \); (c) \( I(U_2; Z_1|UU_1X_1) = 0 \) and \( I(X_2; UU_1U_2X_1Z_1YS_R) = 0 \). The discussions for the above three cases are as follows.

3If \( I(X_1; Y|S_R) > 0 \), one can treat the MAC as a point-to-point channel from transmitter 1 to the receiver. If \( I(X_2; Y|X_1S_R) > 0 \), transmitter 2 can achieve a positive rate by letting transmitter 1 send a deterministic sequence.
(a) $I(U_2; Z_1|UU_1X_1) > 0$: Thanks to the feedback $Z_1$, transmitter 1 can obtain part information sent by transmitter 2 potentially through message cooperation, i.e., $U_2$, and term $I(U_2; Z_1|UU_1X_1)$ represents the rate that transmitter 1 can successfully decode. If $I(X_1; Y|S_R) > 0$ and $I(U_2; Z_1|UU_1X_1) > 0$, transmitter 1 can achieve a positive rate $I(X_1; Y|S_R)$ to the receiver, and transmitter 2 can achieve a positive rate $I(U_2; Z_1|UU_1X_1)$ to transmitter 1. The termination blocks can be constructed as follows.

There are $\tilde{B} = 3$ blocks in termination blocks. In block $B + 1$, transmitter 1 conveys its own compressed information corresponding to blocks $B$ and $B - 1$, i.e., $l^*_B, l^*_{B+1}, \beta^*_{1,B+1}$, as messages to the receiver by treating the MAC as a point-to-point channel. The number of channel uses in block $B + 1$ is roughly

$$n_1 = \frac{N(2R_{v_1,c} + R_{v_1,p})}{I(X_1; Y|S_R)},$$

where $\frac{n_1}{N} = \frac{(2R_{v_1,c} + R_{v_1,p})}{I(X_1; Y|S_R)}$ is finite as $I(X_1; Y|S_R) > 0$. In block $B + 2$, transmitter 2 conveys its own compressed information corresponding to blocks $B$ and $B - 1$, i.e., $m^*_B, m^*_{B+1}, \beta^*_{2,B+1}$, as messages to transmitter 1 by letting transmitter 1 sends a deterministic sequence. The number of channel uses in block $B + 2$ is roughly

$$n_2 = \frac{N(2R_{v_2,c} + R_{v_2,p})}{I(U_2; Z_1|UU_1X_1)},$$

where $\frac{n_2}{N} = \frac{(2R_{v_2,c} + R_{v_2,p})}{I(U_2; Z_1|UU_1X_1)}$ is finite as $I(U_2; Z_1|UU_1X_1) > 0$. In block $B + 3$, transmitter 1 conveys transmitter 2’s compressed information $m^*_B, m^*_{B+1}, \beta^*_{2,B+1}$ as messages to the receiver. The number of channel uses in block $B + 3$ is roughly

$$n_3 = \frac{N(2R_{v_2,c} + R_{v_2,p})}{I(X_1; Y|S_R)},$$

where $\frac{n_3}{N} = \frac{(2R_{v_2,c} + R_{v_2,p})}{I(X_1; Y|S_R)}$ is also finite as $I(X_1; Y|S_R) > 0$. Based on the above operations, the receiver can successfully obtain the compressed information $l^*_B, m^*_B, l^*_{B+1}, m^*_{B+1}, \beta^*_{1,B+1}$, $\beta^*_{2,B+1}$, and the scheme proposed in Appendix A-A can be conducted. In particular, the total number of channel uses in the proposed scheme is

$$NB + n_1 + n_2 + n_3.$$

As a result, the rate tuple of our scheme is

$$\left(\frac{N(B - 1)}{NB + n_1 + n_2 + n_3}\right) = \left(\frac{(B - 1)}{B + n_1/N + n_2/N + n_3/N}\right).$$
which approaches \((R_1, R_2)\) for \(B \to \infty\). The distortion tuple of our scheme is bounded by

\[
\frac{(D_1, D_2) \cdot (B - 1) + (d_{1,\text{max}}, d_{2,\text{max}})(N + n_1 + n_2 + n_3)}{NB + n_1 + n_2 + n_3} = \frac{(D_1, D_2) \cdot (B - 1) + (d_{1,\text{max}}, d_{2,\text{max}})(1 + n_1/N + n_2/N + n_3/N)}{B + n_1/N + n_2/N + n_3/N},
\]

which approaches \((D_1, D_2)\) for \(B \to \infty\).

(b) \(I(U_2; Z_1|UU_1X_1) = 0\) and \(I(X_2; UU_1U_2X_1Z_1YS_R) > 0\): In this case, transmitter 1 can decode no message sent by transmitter 2, and the compressed information of transmitter 2 cannot be transmitted to transmitter 1. Thanks to \(I(X_2; UU_1U_2X_1Z_1YS_R) > 0\) and \(I(X_1; Y|S_R) > 0\), transmitter 1 can send a lossless description of codewords \(U, U_1, U_2, X_1\) and echo signal \(Z_1\) to the receiver, then transmitter 2 can achieve a positive rate to receiver as \(I(X_2; UU_1U_2X_1Z_1YS_R) > 0\) by letting the receiver use \(UU_1U_2X_1Z_1\) to improve the decoding. The detailed termination blocks are constructed as follows.

There are \(\tilde{B} = 3\) blocks in termination blocks. In block \(B + 1\), transmitter 1 conveys its own compressed information corresponding to blocks \(B\) and \(B - 1\), i.e., \(l^*_B, l^*_{B+1}, \beta^*_{1,B+1}\), as messages to the receiver by treating the MAC as a point-to-point channel. The number of channel uses in block \(B + 1\) is roughly

\[
n_1 = \frac{N(2R_{v_1,\epsilon} + R_{v_1,p})}{I(X_1; Y|S_R)},
\]

where \(\frac{n_1}{N} = \frac{(2R_{v_1,\epsilon} + R_{v_1,p})}{I(X_1; Y|S_R)}\) is finite as \(I(X_1; Y|S_R) > 0\).

In block \(B + 2\), transmitter 2 conveys its own compressed information corresponding to blocks \(B\) and \(B - 1\), i.e., \(m^*_B, m^*_{B+1}, \beta^*_{2,B+1}\), as messages to receiver. The number of channel uses in block \(B + 2\) is roughly

\[
n_2 = \frac{N(2R_{v_2,\epsilon} + R_{v_2,p})}{I(X_2; UU_1U_2X_1Z_1YS_R)},
\]

where \(\frac{n_2}{N} = \frac{(2R_{v_2,\epsilon} + R_{v_2,p})}{I(X_2; UU_1U_2X_1Z_1YS_R)}\) is finite as \(I(X_2; UU_1U_2X_1Z_1YS_R) > 0\). It should be noted that at the end of block \(B + 2\), the receiver cannot decode such compressed information sent by transmitter 2 because the rate \(I(X_2; UU_1U_2X_1Z_1YS_R)\) is not achievable unless the receiver is informed of \(UU_1U_2X_1Z_1\).

This is done in block \(B + 3\), which is roughly of length

\[
n_3 = \frac{n_2(H(UU_1U_2X_1Z_1) + \delta)}{I(X_1; Y|S_R)} = \frac{N(2R_{v_2,\epsilon} + R_{v_2,p})(H(UU_1U_2X_1Z_1) + \delta)}{I(X_2; UU_1U_2X_1Z_1YS_R)I(X_1; Y|S_R)},
\]
where the transmitter 1 sends a lossless description of sequences $UU_1U_2X_1Z_1$ corresponding to block $B + 2$. We also notice that $\frac{r_1}{N}$ is finite.

The receiver first decodes block $B + 3$ and thus losslessly learns the sequences $UU_1U_2X_1Z_1$ corresponding to block $B + 2$, which can be leveraged for the decoding of block $B + 2$ to obtain the compressed information sent by transmitter 2. As a result, receiver can successfully obtain the compressed information $l_B^*, m_B^*, l_{B+1}^*, m_{B+1}^*, \beta_{1,B+1}^*, \beta_{2,B+1}^*$, and the scheme proposed in Appendix A-A can be conducted. In particular, the total number of channel uses in the proposed scheme is

$$NB + n_1 + n_2 + n_3,$$

(76)

where the rate and distortion tuples also approach $(R_1, R_2)$ and $(D_1, D_2)$ for $B \to \infty$.

(c) $I(U_2; Z_1|UU_1X_1) = 0$ and $I(X_2; UU_1U_2X_1YS_R) = 0$: In this case, we show that letting transmitter 1 send the common compressed information of both two transmitters, i.e., $l_B^*, m_B^*, l_{B+1}^*, m_{B+1}^*$, is enough to guarantee the achievability of Theorem 3. We first prove that $R_2$ in Theorem 3 must be 0 in this case. The details are as follows.

Based on the (21a) for $k = 2$, we have

$$0 = I(U_2; Z_1|UU_1X_1) \geq I(V_{2,c}; X_2Z_2|UU_1U_2X_1Z_1).$$

(77)

Next, based on the (20a) for $k = 2$, we have

$$R_2 \leq I(U_2; Z_1|UU_1X_1) - I(V_{2,c}; X_2Z_2|UU_1U_2X_1Z_1)$$

$$+ I(X_2; YS_RV_{1,c}V_2,cV_1,p|UU_1U_2X_1) - I(V_{2,p}; YS_RV_{1,c}V_2,cV_1,p)$$

$$\leq I(X_2; YS_RV_{1,c}V_2,cV_1,p|UU_1U_2X_1),$$

$$\leq I(X_2; YS_RV_{1,c}V_2,cV_1,p|Z_1|UU_1U_2X_1),$$

$$= H(X_2|UU_1U_2X_1) - H(X_2|UU_1U_2X_1Z_1YS_RV_{1,c}V_2,cV_1,p)$$

$$\leq H(X_2|UU_1U_2X_1) - H(X_2|UU_1U_2X_1Z_1YS_RV_{1,c}V_2,c)$$

$$= I(X_2; Z_1YS_RV_{2,c}|UU_1U_2X_1)$$

$$= I(X_2; Z_1YS_R|UU_1U_2X_1) + I(X_2; V_{2,c}|UU_1U_2X_1Z_1YS_R)$$

$$\leq I(X_2; UU_1U_2X_1Z_1YS_R) + I(X_2; V_{2,c}|UU_1U_2X_1Z_1YS_R)$$

\(\leq I(X_2; V_{2,c}|UU_1U_2X_1Z_1YS_R)\)
\[
\begin{align*}
&= H(V_{2,c}|UU_1U_2X_1Z_1YS_R) - H(V_{2,c}|UU_1U_2X_1YS_RX_2) \\
&\leq H(V_{2,c}|UU_1U_2X_1Z_1) - H(V_{2,c}|UU_1U_2X_1YS_RX_2Z_2) \\
&= I(V_{2,c}; X_2Z_2|UU_1U_2X_1Z_1) \\
&\leq 0, \\
\end{align*}
\]

where (a) follows from \(I(U_2; Z_1|UU_1X_1) = I(V_{2,c}; X_2Z_2|UU_1U_2X_1Z_1) = 0\) and mutual information is nonnegative, (b) follows from the Markov chain \(X_2 - UU_1U_2X_1Z_1 - V_{1,c}V_{1,p}\), (c) follows from that \(I(X_2; UU_1U_2X_1Y'S_R) = 0\), (d) follows from that conditioning reduces entropy, (e) and (f) follow from the Markov chain \(X_1Z_1YS_R - UU_1U_2X_2Z_2 - V_{2,c}\), and (g) follows from \(I(V_{2,c}; X_2Z_2|UU_1U_2X_1Z_1) = 0\).

Such a result means that when \(I(U_2; Z_1|UU_1X_1) = 0\) and \(I(X_2; UU_1U_2X_1YS_R) = 0\), the achievable distortion-rate region is reduced to the convex hull of tuple \((R_1, R_2 = 0, D_1, D_2)\).

For \(R_1\), based on (20a), (20c), \(I(U_2; Z_1|UU_1X_1) = 0\) and \(I(X_2; UU_1U_2X_1Z_1YS_R) = 0\), we have

\[
\begin{align*}
R_1 &\leq I(X_1X_2; YS_R) - I(V_{1,c}; Z_1|UU_1U_2X_1X_2YS_R) - I(V_{2,c}; Z_2|UU_1U_2X_1X_2YS_RV_{1,c}) \\
&\quad - I(V_{1,p}; Z_1|UU_1U_2X_1X_2YS_RV_{1,c}V_{2,c}) - I(V_{2,p}; Z_2|UU_1U_2X_1X_2YS_RV_{1,c}V_{2,c}V_{1,p}) \\
&\quad \leq I(X_1X_2; YS_R) - I(V_{1,c}; Z_1|UU_1U_2X_1X_2YS_R) - I(V_{2,c}; Z_2|UU_1U_2X_1X_2YS_RV_{1,c}) \\
&\quad - I(V_{1,c}; Z_1|UU_1U_2X_1X_2YS_R) - I(V_{2,c}; Z_2|UU_1U_2X_1YS_RV_{1,c}) \\
&\quad \leq I(X_1; YS_R) + I(X_2; UU_1U_2X_1Z_1YS_R) \\
&\quad - I(V_{1,c}; Z_1|UU_1U_2X_1X_2YS_R) - I(V_{2,c}; Z_2|UU_1U_2X_1X_2YS_RV_{1,c}) \\
&\quad \leq I(X_1; YS_R) - I(V_{1,c}; Z_1|UU_1U_2X_1X_2YS_R) - I(V_{2,c}; Z_2|UU_1U_2X_1X_2YS_RV_{1,c}).
\end{align*}
\]

where (a) follows from the nonnegativity of mutual information, and (b) follows from that
\[ I(X_2; UU_1 U_2 X_1 Z_1 Y S_R) = 0, \text{ and} \]
\[ R_1 \leq I(U_1; Z_2 | UU_2 X_2) - I(V_{1,c}; X_1 Z_1 | UU_1 U_2 X_2 Z_2) + I(X_1 X_2; Y S_R V_{1,c} V_{2,c} | UU_1 U_2) \]
\[ - I(V_{1,p}; Z_1 | UU_1 U_2 X_1 Y S_R V_{1,c} V_{2,c}) - I(V_{2,p}; Z_2 | UU_1 U_2 X_2 Y S_R V_{1,c} V_{2,c} V_{1,p}) \]
\[ \leq I(U_1; Z_2 | UU_2 X_2) - I(V_{1,c}; X_1 Z_1 | UU_1 U_2 X_2 Z_2) + I(X_1 X_2; Y S_R V_{1,c} V_{2,c} | UU_1 U_2) \]
\[ \leq I(U_1; Z_2 | UU_2 X_2) - I(V_{1,c}; X_1 Z_1 | UU_1 U_2 X_2 Z_2) + I(X_1; Y S_R V_{1,c} V_{2,c} | UU_1 U_2 X_2) \]
(80)

where (a) follows from the nonnegativity of mutual information, and (b) follows from

\[ I(X_2; Y S_R V_{1,c} V_{2,c} | UU_1 U_2) \leq I(X_2; UU_1 U_2 X_1 Z_1 Y S_R V_{1,c} V_{2,c}) \]
\[ \leq I(X_2; V_{1,c} V_{2,c} | UU_1 U_2 X_1 Z_1 Y S_R) \]
\[ = H(V_{2,c} | UU_1 U_2 X_1 Z_1 Y S_R) - H(V_{2,c} | UU_1 U_2 X_1 Z_1 Y S_R X_2) \]
\[ \leq H(V_{2,c} | UU_1 U_2 X_1 Z_1) - H(V_{2,c} | UU_1 U_2 X_1 Z_1 Y S_R X_2) \]
\[ \leq H(V_{2,c} | UU_1 U_2 X_1 Z_1) - H(V_{2,c} | UU_1 U_2 X_1 Z_2) \]
\[ = I(V_{2,c}; X_2 Z_2 | UU_1 U_2 X_1 Z_1) \]
\[ \leq 0, \quad (81) \]

where (a) follows from that \( I(X_2; UU_1 U_2 X_1 Z_1 Y S_R) = 0 \), (b) follows from the Markov chain \( X_2 - UU_1 U_2 X_1 Z_1 Y S_R V_{2,c} - V_{1,c} \), (c) follows from that conditioning reduces entropy, (d) follows from the Markov chain \( Y S_R - UU_1 U_2 X_1 Z_1 X_2 Z_2 - V_{2,c} \), and (e) follows from that \( I(X_2; UU_1 U_2 X_1 Z_1 Y S_R) = 0 \).

Let \( \mathcal{I}_{R-D}(R_1, R_2 = 0, D_1, D_2) \) denote the union of (79), (80), (21a), (21d) and the sensing distortion constraints. The results in Theorem 3 is achievable if one can prove that \( \mathcal{I}_{R-D}(R_1, R_2 = 0, D_1, D_2) \) is achievable. In fact, \( \mathcal{I}_{R-D}(R_1, R_2 = 0, D_1, D_2) \) can be obtained by the similar achievability scheme as that in Appendix A-A. The key difference is that we need to take \( R_{2,c} = R_{2,p} = 0, V_{i,p} = V_{2,p} = \text{const} \), i.e., transmitter 2 sends no message and both two transmitters do not send compressed information as part of private messages.

In such a scheme, the termination blocks are constructed as follows.
There are two blocks $\tilde{B} = 2$. In block $B + 1$ of length $n_1 = N$ channel uses, two transmitters send codewords $x_1^N(1, l_B, 1, m_B, 1, l_{B+1}, 1, 1)$ and $x_2^N(1, l_B, 1, m_B, 1, m_{B+1}, 1, 1)$, respectively. Then, at the beginning of block $B + 2$, transmitter 1 obtains $Z_1^N(B + 1)$ and can decode $m_{B+1}$ according to (58). Then, in block $B + 2$, it conveys the compressed information $l_B^*, l_{B+1}^*, \beta_{1,B+1}^*$ of both two transmitters as messages to the receiver. The number of channel uses in block $B + 2$ is roughly

$$n_2 = \frac{N(2R_{v_1,c} + 2R_{v_2,c})}{I(X_1; Y | S_R)},$$

(82)

where $\frac{n_2}{N}$ is also finite. In this way, the total number of channel uses in the proposed scheme is

$$NB + N + n_2,$$

(83)

where the rate and distortion tuples also approach $(R_1, R_2)$ and $(D_1, D_2)$ for $B \rightarrow \infty$.

C. Error Analysis

We focus on the error analysis of the first $B$ blocks. Denote by $\mathcal{E}_{\text{Tx}, k}$ and $\mathcal{E}_{\text{Rx}}$ the “encoding error” events of transmitter $k \in \{1, 2\}$ and “decoding error” of the receiver, respectively. By the union bound, we have

$$P_{\mathcal{E}} \leq P(\mathcal{E}_{\text{Tx},1}) + P(\mathcal{E}_{\text{Tx},2}) + P(\mathcal{E}_{\text{Rx}}).$$

(84)

We proceed to derive an upper bound for each term in the right hand side of (84).

**Error Analysis of Transmitter 1:** The error probability of transmitter 1 can be bounded as

$$P(\mathcal{E}_{\text{Tx},1}) \leq P(\mathcal{E}_{\text{Tx},1,(2)}) + \sum_{b=3}^{B} P(\mathcal{E}_{\text{Tx},1,(b)}) + P(\mathcal{E}_{\text{Tx},1,(B+1)}),$$

(85)

where $\mathcal{E}_{\text{Tx},1,(b)}$ represents the error event of transmitter 1 in block $b$.

We first bound $P(\mathcal{E}_{\text{Tx},1,(b)})$ for $b \in [3 : B]$. Denote by $\mathcal{A}_{1,(b)}$ the error events corresponding to (58) and (59), and by $\mathcal{B}_{1,(b)}$ the error event corresponding to (60). By the union bound, we have

$$P(\mathcal{E}_{\text{Tx},1,(b)}) \leq P(\mathcal{A}_{1,(b)}) + P(\mathcal{B}_{1,(b)}).$$

(86)

For term $P(\mathcal{A}_{1,(b)})$, define

$$\mathcal{A}_{1,(b)}(\hat{k}_{b-1}, \hat{m}_{b-1}, \hat{\Delta}_{2,c,b-2}) = \left\{ \right.$$
\[
\left( u^N(j^*_b - 1, t^*_b - 2, k^*_b - 2, m^*_b - 2), u^N(j^*_b - 1, t^*_b - 2, k^*_b - 2, \hat{b}_{b - 1}, \hat{m}_{b - 1}), u^N(j^*_b - 2, l^*_b - 2, k^*_b - 2, \hat{b}_{b - 1}, \hat{m}_{b - 1}), u^N(j^*_b - 1, l^*_b - 1, k^*_b - 2, \hat{b}_{b - 1}, \hat{m}_{b - 1}) \right) \in T_e^N(UU_1 U_2 X_1 Z_1),
\]

and

\[
\left( u^N(j^*_b - 3, l^*_b - 3, k^*_b - 3, m^*_b - 3), u^N(j^*_b - 3, l^*_b - 3, k^*_b - 3, \hat{b}_{b - 2}, \hat{m}_{b - 2}), u^N(j^*_b - 3, l^*_b - 3, k^*_b - 3, \hat{b}_{b - 2}, \hat{m}_{b - 2}), u^N(j^*_b - 3, l^*_b - 3, k^*_b - 3, \hat{b}_{b - 2}, \hat{m}_{b - 2}) \right) \in T_e^N(UU_1 U_2 X_1 V_2 c Z_1),
\]

(87)

and we have

\[
A_{1,(b)} = A_{1,(b)}^c \bigcup \left( \bigcup \left( \hat{b}_{b - 1}, \hat{m}_{b - 1}, \hat{A}_{2,c,b - 1} \right) \left( \hat{b}_{b - 1}, \hat{m}_{b - 1}, \hat{A}_{2,c,b - 1} \right) \right).
\]

(88)

By the union bound, we have

\[
P(A_{1,(b)}) \leq P\{A_{1,(b)}(k^*_b - 1, m^*_b - 1, \Delta^*_b,c,b - 2)\} + \sum_{(k^*_{b - 1}, m^*_{b - 1}, \Delta^*_{2,c,b - 2}) \neq (k^*_b - 1, m^*_b - 1, \Delta^*_{2,c,b - 2})} P\{A_{1,(b)}(k^*_b - 1, m^*_b - 1, \Delta^*_{2,c,b - 2})\}.
\]

(89)

By the law of large numbers, as \( N \to \infty \), there is \( P\{A_{1,(b)}(k^*_b - 1, m^*_b - 1, \Delta^*_{2,c,b - 2})\} \to 0 \).

Furthermore, the second term in the right hand side of (89) can be expressed as

\[
\sum_{(k^*_{b - 1}, m^*_{b - 1}, \Delta^*_{2,c,b - 2})} P\{A_{1,(b)}(k^*_b - 1, m^*_b - 1, \Delta^*_{2,c,b - 2})\} = \sum_{\hat{b}_{b - 1} \neq k^*_{b - 1}} P\{A_{1,(b)}(k^*_b - 1, m^*_b - 1, \Delta^*_{2,c,b - 2})\} + \sum_{\hat{m}_{b - 1} \neq m^*_{b - 1}} P\{A_{1,(b)}(k^*_b - 1, m^*_b - 1, \Delta^*_{2,c,b - 2})\} + \sum_{\hat{\Delta}_{2,c,b - 2} \neq \Delta^*_{2,c,b - 2}} P\{A_{1,(b)}(k^*_b - 1, m^*_b - 1, \Delta^*_{2,c,b - 2})\} + \sum_{\hat{\Delta}_{2,c,b - 2} \neq \Delta^*_{2,c,b - 2}} P\{A_{1,(b)}(k^*_b - 1, m^*_b - 1, \Delta^*_{2,c,b - 2})\},
\]

(90)
According to the codebook generation and standard information-theoretical arguments [12], the right hand side of (90) tends to 0 as \( N \to \infty \) if

\[
R_{2,c} < I(U_2; Z_1|U_1X_1),
\]

\[
R_{2,c} + R_{s_{2,c}} - R_{v_{2,c}} < I(U_2; Z_1|U_1X_1) + I(V_{2,c}; X_1Z_1|U_1U_2),
\]

\[
R_{2,c} + R_{s_{2,c}} < I(U_2; Z_1|U_1X_1) + I(V_{2,c}; X_1Z_1|U_1U_2),
\]

\[
R_{s_{2,c}} - R_{v_{2,c}} < I(V_{2,c}; X_1Z_1|U_1U_2),
\]

\[
R_{s_{2,c}} < I(U_2; Z_1|U_1X_1) + I(V_{2,c}; X_1Z_1|U_1U_2).
\]

For term \( P(\mathcal{B}_{1,(b)}) \), define the following events \( \mathcal{B}_{1,(b),0}, \mathcal{B}_{1,(b),c}, \) and \( \mathcal{B}_{1,(b),p} \):

\[
\mathcal{B}_{1,(b),0} = \left\{ \left( w_N(j_{b-2}, l_{b-2}, k_{b-2}, m_{b-2}), u_1^N(j_{b-2}, l_{b-2}, k_{b-2}, m_{b-2}, j_{b-1}, l_{b-1}),ight. \right.
\]

\[
\left. u_2^N(j_{b-2}, l_{b-2}, k_{b-2}, m_{b-2}, j_{b-1}, l_{b-1}, r_{b-1}^*, \beta_{1,b-1}^*), \right. \right.
\]

\[
\left. z_1^N(b - 1) \right) \notin \mathcal{T}_e^N(UU_1U_2X_1Z_1) \right\},
\]

\[
\mathcal{B}_{1,(b),c} = \left\{ \left( w_N(j_{b-2}, l_{b-2}, k_{b-2}, m_{b-2}), u_1^N(j_{b-2}, l_{b-2}, k_{b-2}, m_{b-2}, j_{b-1}, l_{b-1}),ight. \right.
\]

\[
\left. u_2^N(j_{b-2}, l_{b-2}, k_{b-2}, m_{b-2}, j_{b-1}, l_{b-1}, r_{b-1}^*, \beta_{1,b-1}^*), \right. \right.
\]

\[
\left. v_1^N(j_{b-2}^*, l_{b-2}^*, k_{b-2}^*, m_{b-2}, \hat{\Delta}_{1,c,b-1}, \hat{\Delta}_{1,c,b-1}^*, z_1^N(b - 1) \right) \notin \mathcal{T}_e^N(UU_1U_2X_1V_1,cZ_1), \right. \right.
\]

\[
\text{for all } \hat{\Delta}_{1,c,b-1} \in [1 : 2^{NR_{s_{1,c}}}]
\]

\[
\mathcal{B}_{1,(b),p} = \left\{ \left( w_N(j_{b-2}, l_{b-2}, k_{b-2}, m_{b-2}), u_1^N(j_{b-2}, l_{b-2}, k_{b-2}, m_{b-2}, j_{b-1}, l_{b-1}),ight. \right.
\]

\[
\left. u_2^N(j_{b-2}^*, l_{b-2}^*, k_{b-2}^*, m_{b-2}, \hat{\Delta}_{1,c,b-1}, \hat{\Delta}_{1,c,b-1}^*, z_1^N(b - 1) \right) \notin \mathcal{T}_e^N(UU_1U_2X_1V_1,cV_1,pZ_1), \right. \right.
\]

\[
\text{for all } \hat{\Delta}_{1,p,b-1} \in [1 : 2^{NR_{s_{1,p}}}]
\].

By the union bound, there is

\[
P(\mathcal{B}_{1,(b)}) \leq P(\mathcal{B}_{1,(b),0}) + P(\mathcal{B}_{1,(b),0} \cap \mathcal{B}_{1,(b),c}) + P(\mathcal{B}_{1,(b),c} \cap \mathcal{B}_{1,(b),p}).
\]
According to the codebook generation and standard information-theoretic arguments \cite{12}, we can obtain that $P(B_{1,(b)})$ tends to 0 as $N \to \infty$ if

\begin{align}
R_{s1,c} &> I(V_{1,c}; X_1 Z_1 | UU_1 U_2), \quad (94a) \\
R_{s1,p} &> I(V_{1,p}; Z_1 | UU_1 U_2 X_1 V_{1,c}). \quad (94b)
\end{align}

Therefore, there is

$$\lim_{N \to \infty} P(\mathcal{E}_{Tx,1,(b)}) = 0,$$

whenever

\begin{align}
R_{2,c} &< I(U_2; Z_1 | UU_1 X_1), \quad (96a) \\
R_{2,c} + R_{s2,c} - R_{v2,c} &< I(U_2; Z_1 | UU_1 X_1) + I(V_{2,c}; X_1 Z_1 | UU_1 U_2), \quad (96b) \\
R_{2,c} + R_{s2,c} &< I(U_2; Z_1 | UU_1 X_1) + I(V_{2,c}; X_1 Z_1 | UU_1 U_2), \quad (96c) \\
R_{s2,c} - R_{v2,c} &< I(V_{2,c}; X_1 Z_1 | UU_1 U_2), \quad (96d) \\
R_{s2,c} &< I(U_2; Z_1 | UU_1 X_1) + I(V_{2,c}; X_1 Z_1 | UU_1 U_2), \quad (96e) \\
R_{s1,c} &> I(V_{1,c}; X_1 Z_1 | UU_1 U_2), \quad (96f) \\
R_{s1,p} &> I(V_{1,p}; Z_1 | UU_1 U_2 X_1 V_{1,c}). \quad (96g)
\end{align}

Following the similar analysis, one can also prove that when \cite{96} holds, $P(\mathcal{E}_{Tx,1,(2)})$ and $P(\mathcal{E}_{Tx,1,(B+1)})$ tends to 0 as $N \to \infty$.

Combining all above results, we know that

$$\lim_{N \to \infty} P(\mathcal{E}_{Tx,1}) = 0,$$

whenever

\begin{align}
R_{2,c} &< I(U_2; Z_1 | UU_1 X_1), \quad (98a) \\
R_{2,c} + R_{s2,c} - R_{v2,c} &< I(U_2; Z_1 | UU_1 X_1) + I(V_{2,c}; X_1 Z_1 | UU_1 U_2), \quad (98b) \\
R_{2,c} + R_{s2,c} &< I(U_2; Z_1 | UU_1 X_1) + I(V_{2,c}; X_1 Z_1 | UU_1 U_2), \quad (98c) \\
R_{s2,c} - R_{v2,c} &< I(V_{2,c}; X_1 Z_1 | UU_1 U_2), \quad (98d) \\
R_{s2,c} &< I(U_2; Z_1 | UU_1 X_1) + I(V_{2,c}; X_1 Z_1 | UU_1 U_2), \quad (98e)
\end{align}
Furthermore, we note that equations (98b) and (98e) are obsolete in view of (98c). Therefore, we conclude that

\[
\lim_{N \to \infty} P(\mathcal{E}_{\text{Tx},1}) = 0, \quad (99)
\]

whenever

\[
R_{2,c} < I(U_2; Z_1|UU_1X_1), \quad (100a)
\]
\[
R_{2,c} + R_{s2,c} < I(U_2; Z_1|UU_1X_1) + I(V_2,c; X_1Z_1|UU_1U_2), \quad (100b)
\]
\[
R_{s2,c} - R_{v2,c} < I(V_2,c; X_1Z_1|UU_1U_2), \quad (100c)
\]
\[
R_{s1,c} > I(V_1,c; X_1Z_1|UU_1U_2), \quad (100d)
\]
\[
R_{s1,p} > I(V_1,p; Z_1|UU_1U_2X_1V_1,c). \quad (100e)
\]

**Error Analysis of Transmitter 2:** The analysis of error probability of transmitter 2 is analogous to that of transmitter 1. One can prove that

\[
\lim_{N \to \infty} P(\mathcal{E}_{\text{Tx},2}) = 0, \quad (101)
\]

whenever

\[
R_{1,c} < I(U_1; Z_2|UU_2X_2), \quad (102a)
\]
\[
R_{1,c} + R_{s1,c} < I(U_1; Z_2|UU_2X_2) + I(V_1,c; X_2Z_2|UU_1U_2), \quad (102b)
\]
\[
R_{s1,c} - R_{v1,c} < I(V_1,c; X_2Z_2|UU_1U_2), \quad (102c)
\]
\[
R_{s2,c} > I(V_2,c; X_2Z_2|UU_1U_2), \quad (102d)
\]
\[
R_{s2,p} > I(V_2,p; Z_2|UU_1U_2X_2V_2,c). \quad (102e)
\]

**Error Analysis of the Receiver:** The error probability of receiver can be bounded as

\[
P(\mathcal{E}_{\text{Rx}}) \leq \sum_{b=3}^{B} P(\mathcal{E}_{\text{Rx},(b)}) + P(\mathcal{E}_{\text{Rx},(2)}) + P(\mathcal{E}_{\text{Rx},(1)}), \quad (103)
\]

where $\mathcal{E}_{\text{Rx},(b)}$ represents the error event of receiver in block $b$. 
We first bound $P(\mathcal{E}_{\text{Rx},(b)})$ for $b \in [3 : B]$. Define two events $\mathcal{H}_{(b),1}$ and $\mathcal{H}_{(b),2}(\hat{j}_{b-1}, \hat{l}_{b-1}, \hat{k}_{b-1}, \hat{m}_{b-1}, \hat{r}_b, \hat{\beta}_{1,b}, \hat{s}_b, \hat{\beta}_{2,b}, \hat{\Delta}_{1,c,b}, \hat{\Delta}_{1,p,b}, \hat{\Delta}_{2,c,b}, \hat{\Delta}_{2,p,b})$ as

$$\mathcal{H}_{(b),1} = \left\{ \left( u^N(\hat{j}_{b-1}, \hat{l}_{b-1}, k_{b-1}^*, m_{b-1}^*), u^N_1(\hat{j}_{b-1}, \hat{l}_{b-1}, k_{b-1}^*, m_{b-1}^*, j_b^*, l_b^*), u^N_2(\hat{j}_{b-1}, \hat{l}_{b-1}, k_{b-1}^*, m_{b-1}^*, k_b^*, m_b^*), \right. \right.$$

$$x_1^N(\hat{j}_{b-1}, \hat{l}_{b-1}, k_{b-1}^*, m_{b-1}^*, j_b^*, l_b^*, r_b^*, \beta_{1,b}^*), x_2^N(\hat{j}_{b-1}, \hat{l}_{b-1}, k_{b-1}^*, m_{b-1}^*, k_b^*, m_b^*, s_b^*, \beta_{2,b}^*),$$

$$v_1^N(\hat{j}_{b-1}, \hat{l}_{b-1}, k_{b-1}^*, m_{b-1}^*, j_b^*, l_b^*, k_b^*, m_b^*, \hat{\Delta}_{1,c,b}), v_2^N(\hat{j}_{b-1}, \hat{l}_{b-1}, k_{b-1}^*, m_{b-1}^*, j_b^*, l_b^*, k_b^*, m_b^*, \hat{\Delta}_{2,c,b}),$$

$$v_1^N_1(\hat{j}_{b-1}, \hat{l}_{b-1}, k_{b-1}^*, m_{b-1}^*, j_b^*, l_b^*, k_b^*, m_b^*, \hat{\Delta}_{1,c,b}), v_2^N_1(\hat{j}_{b-1}, \hat{l}_{b-1}, k_{b-1}^*, m_{b-1}^*, j_b^*, l_b^*, k_b^*, m_b^*, \hat{\Delta}_{1,p,b}),$$

$$v_2^N_2(\hat{j}_{b-1}, \hat{l}_{b-1}, k_{b-1}^*, m_{b-1}^*, j_b^*, l_b^*, k_b^*, m_b^*, \hat{\Delta}_{2,c,b}),$$

$$y^N(b), s_R^N(b) \notin T_e^N(UU_1U_2X_1X_2V_{1,c}V_{2,c}V_1pV_2pY S_R), \quad \text{for all } \hat{\Delta}_{1,c,b} \in \alpha_{1,c,b+1}(l_{b+1}^*),$$

$$\hat{\Delta}_{2,c,b} \in \alpha_{2,c,b+1}(m_{b+1}), \hat{\Delta}_{1,p,b} \in \alpha_{1,p,b+1}(\beta_{1,b+1}^*), \text{ and } \hat{\Delta}_{2,p,b} \in \alpha_{2,p,b+1}(\beta_{2,b+1}^*) \} \right\},$$

and

$$\mathcal{H}_{(b),2}(\hat{j}_{b-1}, \hat{l}_{b-1}, \hat{k}_{b-1}, \hat{m}_{b-1}, \hat{r}_b, \hat{\beta}_{1,b}, \hat{s}_b, \hat{\beta}_{2,b}, \hat{\Delta}_{1,c,b}, \hat{\Delta}_{1,p,b}, \hat{\Delta}_{2,c,b}, \hat{\Delta}_{2,p,b})$$

$$= \left\{ \left( u^N(\hat{j}_{b-1}, \hat{l}_{b-1}, \hat{k}_{b-1}, \hat{m}_{b-1}), u^N_1(\hat{j}_{b-1}, \hat{l}_{b-1}, \hat{k}_{b-1}, \hat{m}_{b-1}, \hat{j}_b^*, \hat{l}_b^*), u^N_2(\hat{j}_{b-1}, \hat{l}_{b-1}, \hat{k}_{b-1}, \hat{m}_{b-1}, \hat{k}_b^*, \hat{m}_b^*), \right. \right.$$

$$x_1^N(\hat{j}_{b-1}, \hat{l}_{b-1}, \hat{k}_{b-1}, \hat{m}_{b-1}, \hat{j}_b^*, \hat{l}_b^*, \hat{r}_b, \hat{\beta}_{1,b}^*), x_2^N(\hat{j}_{b-1}, \hat{l}_{b-1}, \hat{k}_{b-1}, \hat{m}_{b-1}, \hat{k}_b^*, \hat{m}_b^*, \hat{s}_b, \hat{\beta}_{2,b}^*),$$

$$v_1^N(\hat{j}_{b-1}, \hat{l}_{b-1}, \hat{k}_{b-1}, \hat{m}_{b-1}, \hat{j}_b^*, \hat{l}_b^*, \hat{k}_b^*, \hat{m}_b^*, \hat{\Delta}_{1,c,b}), v_2^N(\hat{j}_{b-1}, \hat{l}_{b-1}, \hat{k}_{b-1}, \hat{m}_{b-1}, \hat{j}_b^*, \hat{l}_b^*, \hat{k}_b^*, \hat{m}_b^*, \hat{\Delta}_{2,c,b}),$$

$$v_1^N_1(\hat{j}_{b-1}, \hat{l}_{b-1}, \hat{k}_{b-1}, \hat{m}_{b-1}, \hat{j}_b^*, \hat{l}_b^*, \hat{k}_b^*, \hat{m}_b^*, \hat{\Delta}_{1,c,b}), v_2^N_1(\hat{j}_{b-1}, \hat{l}_{b-1}, \hat{k}_{b-1}, \hat{m}_{b-1}, \hat{j}_b^*, \hat{l}_b^*, \hat{k}_b^*, \hat{m}_b^*, \hat{\Delta}_{1,p,b}),$$

$$v_2^N_2(\hat{j}_{b-1}, \hat{l}_{b-1}, \hat{k}_{b-1}, \hat{m}_{b-1}, \hat{j}_b^*, \hat{l}_b^*, \hat{k}_b^*, \hat{m}_b^*, \hat{s}_b, \hat{\beta}_{2,b}^*, \hat{\Delta}_{2,c,b}, \hat{\Delta}_{2,p,b}),$$

$$y^N(b), s_R^N(b) \in T_e^N(UU_1U_2X_1X_2V_{1,c}V_{2,c}V_1pV_2pY S_R) \right\}.\quad (105)$$

Then, we have

$$\mathcal{E}_{\text{Rx}} = \mathcal{H}_{(b),1} \cup \left( \bigcup_{\hat{\Delta}_{1,c,b}, \hat{\Delta}_{1,p,b}, \hat{\Delta}_{2,c,b}, \hat{\Delta}_{2,p,b}} \mathcal{H}_{(b),2}(\hat{j}_{b-1}, \hat{l}_{b-1}, \hat{k}_{b-1}, \hat{m}_{b-1}, \hat{r}_b, \hat{\beta}_{1,b}, \hat{s}_b, \hat{\beta}_{2,b}, \hat{\Delta}_{1,c,b}, \hat{\Delta}_{1,p,b}, \hat{\Delta}_{2,c,b}, \hat{\Delta}_{2,p,b}) \right).\quad (106)$$

By the union bound, we have

$$P(\mathcal{E}_{\text{Rx},(b)}) \leq P\{\mathcal{H}_{(b),1}\} + \sum \left( \bigcup_{\hat{\Delta}_{1,c,b}, \hat{\Delta}_{1,p,b}, \hat{\Delta}_{2,c,b}, \hat{\Delta}_{2,p,b}} \mathcal{H}_{(b),2}(\hat{j}_{b-1}, \hat{l}_{b-1}, \hat{k}_{b-1}, \hat{m}_{b-1}, \hat{r}_b, \hat{\beta}_{1,b}, \hat{s}_b, \hat{\beta}_{2,b}, \hat{\Delta}_{1,c,b}, \hat{\Delta}_{1,p,b}, \hat{\Delta}_{2,c,b}, \hat{\Delta}_{2,p,b}) \right).$$
According to the codebook generation and standard information-theoretic arguments [12], the
By the law of large numbers, as $N \to \infty$, we know $P\{\mathcal{H}_{(b),1}\} \to 0$. Moreover, the second term
in the right hand side of (107) can be expressed as shown in (108)

$$
P\{\mathcal{H}_{(b),2}(\hat{j}_{b-1}, \hat{l}_{b-1}, \hat{k}_{b-1}, m_{b-1}, \hat{r}_b, \hat{b}_1, \hat{s}_b, \hat{b}_2, \bar{\Delta}_{1,c,b}, \bar{\Delta}_{1,p,b}, \bar{\Delta}_{2,c,b}, \bar{\Delta}_{2,p,b})\}.
$$

(107)

Moreover, the second term in the right hand side of (107) can be expressed as shown in (108)

$$
\sum_{(\hat{j}_{b-1}, \hat{l}_{b-1}, \hat{k}_{b-1}, m_{b-1}, \hat{r}_b, \hat{b}_1, \hat{s}_b, \hat{b}_2, \bar{\Delta}_{1,c,b}, \bar{\Delta}_{1,p,b}, \bar{\Delta}_{2,c,b}, \bar{\Delta}_{2,p,b})}
$$

(108)

According to the codebook generation and standard information-theoretic arguments [12], the
first term in (108) tends to zero as $N \to \infty$ whenever

$$
R_{1,c} + R_{s_{1,c}} + R_{2,c} + R_{s_{2,c}} + R_{1,p} + R_{s_{1,p}} + R_{2,p} + R_{s_{2,p}}
$$

$$
< I(X_1 X_2; Y S_R) + I(V_{1,c}; X_1 X_2 Y S_R | U U_1 U_2) + I(V_{2,c}; X_1 X_2 Y S_R V_{1,c} | U U_1 U_2)
$$

$$
+ I(V_{1,p}; X_2 Y S_R V_{2,c} | U U_1 U_2 X_1 V_{1,c}) + I(V_{2,p}; X_1 Y S_R V_{1,c} V_{1,p} | U U_1 U_2 X_2 V_{2,c}).
$$

(109)

The second term in (108) tends to zero as $N \to \infty$ whenever

$$
R_{1,p} + R_{s_{1,p}} + R_{2,p} - R_{v_{1,c}} + R_{s_{2,c}} - R_{v_{2,c}} + R_{s_{2,p}} - R_{v_{2,p}}
$$

$$
< I(X_1; X_2 Y S_R | U U_1 U_2) + I(V_{1,c}; X_1 X_2 Y S_R | U U_1 U_2) + I(V_{2,c}; X_1 X_2 Y S_R V_{1,c} | U U_1 U_2)
$$

$$
+ I(V_{1,p}; X_2 Y S_R V_{2,c} | U U_1 U_2 X_1 V_{1,c}) + I(V_{2,p}; X_1 Y S_R V_{1,c} V_{1,p} | U U_1 U_2 X_2 V_{2,c}),
$$

(110a)
Similarly, the third term in (108) tends to zero as $N \to \infty$ whenever

\[
R_{1,p} + R_{s_{1},p} + R_{s_{1},c} - R_{v_{1},c} < I(X_{1}; X_{2} Y S_{R} V_{2,c} V_{2,p} | UU_{1} U_{2}) \\
+ I(V_{1,c}; X_{1} X_{2} Y S_{R} V_{2,c} V_{2,p} | UU_{1} U_{2}) + I(V_{1,p}; X_{2} Y S_{R} V_{2,c} V_{2,p} | UU_{1} U_{2} X_{1} V_{1,c}),
\]

\[
(110b)
\]

\[
R_{1,p} + R_{s_{1},p} + R_{s_{1},c} - R_{v_{1},c} + R_{s_{2},p} - R_{v_{2},p} < I(X_{1}; X_{2} Y S_{R} V_{1,c} V_{2,c} | UU_{1} U_{2}) \\
+ I(V_{1,c}; X_{1} X_{2} Y S_{R} V_{1,c} V_{2,c} | UU_{1} U_{2}) + I(V_{1,p}; X_{2} Y S_{R} V_{2,c} V_{2,p} | UU_{1} U_{2} X_{1} V_{1,c}) \\
+ I(V_{2,p}; X_{1} Y S_{R} V_{1,c} V_{1,p} | UU_{1} U_{2} X_{2} V_{2,c}),
\]

\[
(110c)
\]

\[
R_{1,p} + R_{s_{1},p} + R_{s_{2},c} - R_{v_{2},c} + R_{s_{2},p} - R_{v_{2},p} < I(X_{1}; X_{2} Y S_{R} V_{1,c} V_{2,c} | UU_{1} U_{2}) \\
+ I(V_{1,c}; X_{1} X_{2} Y S_{R} V_{1,c} V_{2,c} | UU_{1} U_{2}) + I(V_{1,p}; X_{2} Y S_{R} V_{1,c} V_{1,p} | UU_{1} U_{2} X_{2} V_{2,c}),
\]

\[
(110d)
\]

\[
R_{1,p} + R_{s_{1},p} < I(X_{1}; X_{2} Y S_{R} V_{1,c} V_{2,c} V_{2,p} | UU_{1} U_{2}) + I(V_{1,p}; X_{2} Y S_{R} V_{2,c} V_{2,p} | UU_{1} U_{2} X_{1} V_{1,c}),
\]

\[
(110e)
\]

\[
R_{1,p} + R_{s_{1},p} + R_{s_{2},p} - R_{v_{2},p} < I(X_{1}; X_{2} Y S_{R} V_{1,c} V_{2,c} | UU_{1} U_{2}) \\
+ I(V_{1,p}; X_{2} Y S_{R} V_{2,c} | UU_{1} U_{2} X_{1} V_{1,c}) + I(V_{2,p}; X_{1} Y S_{R} V_{1,c} V_{1,p} | UU_{1} U_{2} X_{2} V_{2,c}).
\]

\[
(110f)
\]
\[ R_{2,p} + R_{s2,p} < I(X_2; X_1 Y S_R V_{1,c} V_{2,c} V_{1,p} | U_1 U_2) + I(V_{2,p}; X_1 Y S_R V_{1,c} V_{1,p} | U_1 U_2 X_2 V_{2,c}), \]
\[ (111e) \]

\[ R_{2,p} + R_{s2,p} + R_{s1,p} - R_{e1,c} < I(X_2; X_1 Y S_R V_{1,c} V_{2,c} | U_1 U_2) \]
\[ + I(V_{1,p}; X_2 Y S_R V_{2,c} | U_1 U_2 X_1 V_{1,c}) + I(V_{2,p}; X_1 Y S_R V_{1,c} V_{1,p} | U_1 U_2 X_2 V_{2,c}), \]
\[ (111f) \]

and we can prove that last term in (108) tends to zero as \( N \to \infty \) whenever

\[ R_{1,p} + R_{s1,p} + R_{2,p} + R_{s2,p} < I(X_1 X_2; Y S_R V_{1,c} V_{2,c} | U_1 U_2) \]
\[ + I(V_{1,p}; X_2 Y S_R V_{2,c} | U_1 U_2 X_1 V_{1,c}) + I(V_{2,p}; X_1 Y S_R V_{1,c} V_{1,p} | U_1 U_2 X_2 V_{2,c}), \]
\[ (112a) \]

\[ R_{1,p} + R_{s1,p} + R_{2,p} + R_{s2,p} + R_{s1,c} - R_{e1,c} < I(X_1 X_2; Y S_R V_{1,c} | U_1 U_2) \]
\[ + I(V_{1,c}; X_1 X_2 Y S_R V_{2,c} | U_1 U_2) + I(V_{1,p}; X_2 Y S_R V_{2,c} | U_1 U_2 X_1 V_{1,c}) \]
\[ + I(V_{2,p}; X_1 Y S_R V_{1,c} V_{1,p} | U_1 U_2 X_2 V_{2,c}), \]
\[ (112b) \]

\[ R_{1,p} + R_{s1,p} + R_{2,p} + R_{s2,p} + R_{s2,c} - R_{e2,c} < I(X_1 X_2; Y S_R | U_1 U_2) \]
\[ + I(V_{2,c}; X_1 X_2 Y S_R V_{1,c} | U_1 U_2) + I(V_{1,c}; X_2 Y S_R V_{2,c} | U_1 U_2 X_1 V_{1,c}) \]
\[ + I(V_{2,p}; X_1 Y S_R V_{1,c} V_{1,p} | U_1 U_2 X_2 V_{2,c}), \]
\[ (112c) \]

\[ R_{1,p} + R_{s1,p} + R_{2,p} + R_{s2,p} + R_{s1,c} - R_{e1,c} + R_{s2,c} - R_{e2,c} \]
\[ < I(X_1 X_2; Y S_R | U_1 U_2) + I(V_{1,c}; X_1 X_2 Y S_R | U_1 U_2) + I(V_{2,c}; X_1 X_2 Y S_R V_{1,c} | U_1 U_2) \]
\[ + I(V_{1,p}; X_2 Y S_R V_{2,c} | U_1 U_2 X_1 V_{1,c}) + I(V_{2,p}; X_1 Y S_R V_{1,c} V_{1,p} | U_1 U_2 X_2 V_{2,c}). \]
\[ (112d) \]

Combining these results, we conclude that \( P(\mathcal{E}_{Rx,(b)}), b \in [3 : B] \) tends to zero as \( N \to \infty \) if conditions (109), (110), (111), and (112) are satisfied. For term \( P(\mathcal{E}_{Rx,(2)}) \) and \( P(\mathcal{E}_{Rx,(1)}) \), we note that the decoding operations in blocks \( b = 1 \) and \( b = 2 \) are the special cases that are contained in block \( b \in [3 : B] \). Therefore, if conditions (109), (110), (111), and (112) are satisfied, \( P(\mathcal{E}_{Rx,(2)}) \) and \( P(\mathcal{E}_{Rx,(1)}) \) also tend to 0 as \( N \to \infty \).

Combining all above error analysis, we conclude that the error probability of the proposed scheme \( P_{E} \) tends to zeros as \( N \to \infty \) is conditions (100), (102), (109), (110), (111), and (112) are satisfied. As there are too many parameters and mutual information terms in rate region (100), (102), (109), (110), (111), and (112), directly applying Fourier-Motzkin Elimination is very complex, we refer the reader to next subsection for the transformation of obtained region.
D. Fourier-Motzkin Elimination

Consider the following transformation. Let \( \mathcal{I}_1(R_{11}, R_{12}, R_{21}, R_{22}, R_{31}, R_{32}, R_{41}, R_{42}) \) denote the region corresponding to (100), (102), (109), (110), (111), (112). Let \( \mathcal{I}_2(R_{11}, R_{12}, R_{21}, R_{22}, R_{31}, R_{32}, R_{41}, R_{42}, R_{v1}, R_{v2}) \) denote the region corresponding to (100), (102), (109), (110), (111), (112), and the following constraints:

\[
R_{sk,c} - R_{vk,c} < \min \left \{ I(V_{k,c}; X_k Y S R_{V_{k,c}} V_{k,p}|UU_1 U_2), I(V_{k,c}; Y S R_{V_{k,c}}|UU_1 U_2) \right \},
\]

(113a)

\[
R_{sk,p} - R_{vk,p} < I(V_{k,p}; Y S R_{V_{k,c}}|UU_1 U_2 X_k V_{k,c}),
\]

(113b)

\[
R_{sk,c} - R_{vk,c} + R_{sk,p} - R_{vk,p} < I(V_{k,c}; X_k Y S R_{V_{k,c}}|UU_1 U_2) + I(V_{k,p}; Y S R_{V_{k,c}}|UU_1 U_2 X_k V_{k,c}),
\]

(113c)

\[
R_{sk,c} - R_{vk,c} + R_{sk,p} - R_{vk,p} < I(V_{k,c}; X_k Y S R_{V_{k,c}}|UU_1 U_2) + I(V_{k,p}; Y S R_{V_{k,c}}|UU_1 U_2 X_k V_{k,c}),
\]

(113d)

\[
R_{s1,c} - R_{v1,c} + R_{s2,c} - R_{v2,c} < I(V_{1,c}; Y S R|UU_1 U_2) + I(V_{2,c}; Y S R_{V_{1,c}}|UU_1 U_2),
\]

(113e)

\[
R_{s1,c} - R_{v1,c} + R_{s2,c} - R_{v2,c} + R_{sk,p} - R_{vk,p} < I(V_{k,c}; X_k Y S R|UU_1 U_2) + I(V_{k,c}; X_k Y S R_{V_{k,c}}|UU_1 U_2 X_k V_{k,c}).
\]

(113f)

Since we introduce additional constraints, there must be

\[
\mathcal{I}_2 \subseteq \mathcal{I}_1.
\]

(114)

We also note that by introducing (113), in region \( \mathcal{I}_2 \), several constraints are redundant, and region \( \mathcal{I}_2 \) can be written by the union of

\[
R_{sk,c} > I(V_{k,c}; X_k Z_k|UU_1 U_2),
\]

(115a)

\[
R_{sk,p} > I(V_{k,p}; Z_k|UU_1 U_2 X_k V_{k,c}),
\]

(115b)

\[
R_{k,c} < I(U_k; Z_k|UU_k X_k),
\]

(115c)

\[
R_{k,c} + R_{sk,c} < I(U_k; Z_k|UU_k X_k) + I(V_{k,c}; X_k Z_k|UU_1 U_2),
\]

(115d)

\[
R_{k,p} + R_{sk,p} < I(X_k; X_k Y S R_{V_{1,c}} V_{2,c} V_{k,p}|UU_1 U_2) + I(V_{k,p}; X_k Y S R_{V_{k,c}} V_{k,p}|UU_1 U_2 X_k V_{k,c}),
\]

(115e)
\[ R_{1,p} + R_{s1,p} + R_{2,p} + R_{s2,p} < I(X_1X_2; YS_RV_{1,c}V_{2,c}|UU_1U_2) \]
\[ + I(V_{1,p}; X_2YS_RV_{2,c}|UU_1U_2X_1V_{1,c}) + I(V_{2,p}; X_1YS_RV_{1,c}V_{1,p}|UU_1U_2X_2V_{2,c}), \]  
\[ (115f) \]

\[ R_{1,c} + R_{s1,c} + R_{2,c} + R_{s2,c} + R_{1,p} + R_{s1,p} + R_{2,p} + R_{s2,p} \]
\[ < I(X_1X_2; YS_R) + I(V_{1,c}; X_1X_2YS_R|UU_1U_2) + I(V_{2,c}; X_1X_2YS_RV_{1,c}|UU_1U_2) \]
\[ + I(V_{1,p}; X_2YS_RV_{2,c}|UU_1U_2X_1V_{1,c}) + I(V_{2,p}; X_1YS_RV_{1,c}V_{1,p}|UU_1U_2X_2V_{2,c}), \]  
\[ (115g) \]

and

\[ R_{sk,c} - R_{vk,c} < \min \left\{ I(V_{k,c}; X_kYS_RV_{k,c}V_{k,p}|UU_1U_2), I(V_{k,c}; YS_RV_{k,c}|UU_1U_2), I(V_{k,c}; X_kZ_k|UU_1U_2) \right\}, \]  
\[ (116a) \]

\[ R_{sk,p} - R_{vk,p} < I(V_{k,p}; YS_RV_{k,c}|UU_1U_2X_kV_{k,c}), \]  
\[ (116b) \]

\[ R_{sk,c} - R_{vk,c} + R_{sk,p} - R_{vk,p} \]
\[ < I(V_{k,c}; X_kYS_RV_{k,c}|UU_1U_2) + I(V_{k,p}; YS_RV_{k,c}|UU_1U_2X_kV_{k,c}), \]  
\[ (116c) \]

\[ R_{sk,c} - R_{vk,c} + R_{sk,p} - R_{vk,p} \]
\[ < I(V_{k,c}; X_kYS_RV_{k,c}|UU_1U_2) + I(V_{k,p}; YS_RV_{k,c}|UU_1U_2X_kV_{k,c}), \]  
\[ (116d) \]

\[ R_{s1,c} - R_{v1,c} + R_{s2,c} - R_{v2,c} < I(V_{1,c}; YS_R|UU_1U_2) + I(V_{2,c}; YS_RV_{1,c}|UU_1U_2), \]  
\[ (116e) \]

\[ R_{s1,c} - R_{v1,c} + R_{s2,c} - R_{v2,c} + R_{sk,p} - R_{vk,p} < I(V_{k,c}; X_kYS_R|UU_1U_2) \]
\[ + I(V_{k,c}; X_kYS_RV_{k,c}|UU_1U_2) + I(V_{k,p}; YS_RV_{k,c}|UU_1U_2X_kV_{k,c}). \]  
\[ (116f) \]

We now apply Fourier-Motzkin Elimination to project out \( R_{vk,c}, R_{vk,p} \) for \( k \in \{1, 2\} \) from \( \mathcal{I}_2 \).

As

\[ R_{vk,c} \leq R_{sk,c}, \quad R_{vk,p} \leq R_{sk,p}, \quad k \in \{1, 2\} \]  
\[ (117) \]

by projecting \( R_{vk,c}, R_{vk,p}, k \in \{1, 2\} \), we obtain

\[ R_{sk,c} > I(V_{k,c}; X_kZ_k|UU_1U_2), \]  
\[ (118a) \]

\[ R_{sk,p} > I(V_{k,p}; Z_k|UU_1U_2X_kV_{k,c}), \]  
\[ (118b) \]

\[ R_{k,c} < I(U_k; Z_k|UU_1X_k), \]  
\[ (118c) \]
\[ R_{k,c} + R_{s,k,c} < I(U_k; Z_k | UU_k X_k) + I(V_{k,c}; X_k Z_k | UU_1 U_2), \]  
(118d)

\[ R_{k,p} + R_{s,k,p} < I(X_k; X_k Y S_R V_{1,c} V_{2,c} V_{k,p}) | UU_1 U_2) + I(V_{k,p}; X_k Y S_R V_{k,c} V_{k,p} | UU_1 U_2 X_k V_{k,c}), \]  
(118e)

\[ R_{1,p} + R_{s_1,p} + R_{2,p} + R_{s_2,p} < I(X_1 X_2; Y S_R V_{1,c} V_{2,c} | UU_1 U_2) \]
\[ + I(V_{1,p}; X_2 Y S_R V_{2,c} | UU_1 U_2 X_1 V_{1,c}) + I(V_{2,p}; X_1 Y S_R V_{1,c} V_{1,p} | UU_1 U_2 X_2 V_{2,c}), \]  
(118f)

\[ R_{1,c} + R_{s_1,c} + R_{2,c} + R_{s_2,c} + R_{1,p} + R_{s_1,p} + R_{2,p} + R_{s_2,p} \]
\[ < I(X_1 X_2; Y S_R) + I(V_{1,c}; X_1 X_2 Y S_R | UU_1 U_2) + I(V_{2,c}; X_1 X_2 Y S_R V_{1,c} | UU_1 U_2) \]
\[ + I(V_{1,p}; X_2 Y S_R V_{2,c} | UU_1 U_2 X_1 V_{1,c}) + I(V_{2,p}; X_1 Y S_R V_{1,c} V_{1,p} | UU_1 U_2 X_2 V_{2,c}). \]  
(118g)

Let \( \mathcal{I}_p(R_{1,c}, R_{1,p}, R_{2,c}, R_{2,p}, R_{s_1,c}, R_{s_1,p}, R_{s_2,c}, R_{s_2,p}) \) denote the region by projecting \( R_{v_{1,c}}, R_{v_{1,p}}, R_{v_{2,c}}, R_{v_{2,p}} \) out from \( \mathcal{I}_2 \), i.e., \( \Box \). Moreover, considering the following region \( \mathcal{I}_3(R_{v_{1,c}}, R_{v_{1,p}}, R_{2,c}, R_{2,p}, R_{s_1,c}, R_{s_1,p}, R_{s_2,c}, R_{s_2,p}, R_{v_{1,c}}, R_{v_{1,p}}, R_{v_{2,c}}, R_{v_{2,p}}) \) where constraints (118) are satisfied and variables \( (R_{v_{1,c}}, R_{v_{1,p}}, R_{v_{2,c}}, R_{v_{2,p}}) \) can take arbitrary values. It is no doubt that \( \mathcal{I}_p \) is also the result of projecting \( R_{v_{1,c}}, R_{v_{1,p}}, R_{v_{2,c}}, R_{v_{2,p}} \) out from \( \mathcal{I}_3 \). On the other hand, we notice that

\[ \mathcal{I}_1 \subseteq \mathcal{I}_3 \]  
(119)

since any tuple \( (R_{1,c}, R_{1,p}, R_{2,c}, R_{2,p}, R_{s_1,c}, R_{s_1,p}, R_{s_2,c}, R_{s_2,p}, R_{v_{1,c}}, R_{v_{1,p}}, R_{v_{2,c}}, R_{v_{2,p}}) \) in \( \mathcal{I}_1 \) must satisfy constraints (118). Thus, we conclude that \( \mathcal{I}_p(R_{v_{1,c}}, R_{v_{1,p}}, R_{v_{2,c}}, R_{v_{2,p}}) \) (i.e., \( \Box \)) is the result of projecting \( R_{v_{1,c}}, R_{v_{1,p}}, R_{v_{2,c}}, R_{v_{2,p}} \) out from \( \mathcal{I}_1 \). Such a result reveals that in our achievable scheme, the binning rates \( R_{v_{1,c}}, R_{v_{1,p}}, R_{v_{2,c}}, R_{v_{2,p}} \) are not unique. For each achievable rate tuple \( (R_{1,c}, R_{1,p}, R_{2,c}, R_{2,p}, R_{s_1,c}, R_{s_1,p}, R_{s_2,c}, R_{s_2,p}, R_{v_{1,c}}, R_{v_{1,p}}, R_{v_{2,c}}, R_{v_{2,p}}) \), one can arbitrarily choose \( R_{v_{1,c}}, R_{v_{1,p}}, R_{v_{2,c}}, R_{v_{2,p}} \) as long as (116) is satisfied.

We now can continue to apply Fourier-Motzkin elimination on (118). Defining \( R_1 = R_{1,c} + R_{1,p}, R_2 = R_{2,c} + R_{2,p} \), and one can eliminate auxiliary random variables \( R_{s_1,c}, R_{s_1,p}, R_{s_2,c}, R_{s_2,p} \) and obtain the achievable rate region as stated in Theorem 3 by automatic tools like [37].

E. Analysis of Expected Distortion

Define \( w_k = \{w_k^{(b)}\}_{b=1}^{B-1}, k \in \{1, 2\} \) with \( |\mathcal{W}_k| = 2^{N(B-1)(R_{k,p}+R_{k,c})} \) where \( w_k^{(b)} = \{w_k^{(b)}\}, w_k^{(b)} \). For any given message pair \( (w_1, w_2) \), define \( n = N(B-1) \), the expected distortion of transmitter 1
is

\[
\limsup_{n \to \infty} d_1^{(n)}(w_1, w_2) = \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} d_1(S_{T_1,i}, \hat{S}_{T_1,i}) \right] \\
\overset{(a)}{\leq} \limsup_{n \to \infty} \left( P_{E} d_{\text{max}} + (1 - P_{E})(1 + \epsilon) \cdot \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} d_1(S_{T_1,i}, \hat{S}_{T_1,i}) \right] \right) \\
\overset{(b)}{\leq} \limsup_{n \to \infty} \left( P_{E} d_{\text{max}} + (1 - P_{E})(1 + \epsilon) D_1 \right) \\
\overset{(c)}{=} D_1,
\]

(120)

where (a) follows by applying the upper bound of the distortion function to the decoding error event and the typical average lemma \cite{12} to the successful decoding event; (b) follows from the random codebook generation and state estimating function that achieves \(D_1\); (c) follows because \(P_{E}\) tends to zeros as \(n \to \infty\) if the rate constraints (20) and (21) in Theorem 3 holds. Since the uniformly distributed messages are considered, it is easy to show that

\[
\limsup_{n \to \infty} d_1^{(n)} \leq D_1.
\]

(121)

The similar analysis can be conducted for transmitter 2 to verify that

\[
\limsup_{n \to \infty} d_2^{(n)} \leq D_2.
\]

(122)

**APPENDIX B**

**PROOF OF THEOREM 4**

By the definition of \(\mathcal{I}^{\text{our,com}}_{R-D}\), it is straightforward to have

\[
\mathcal{I}^{\text{our,com}}_{R-D} \subseteq \mathcal{I}^{\text{our}}_{R-D}.
\]

(123)

We proceed to prove that \(\mathcal{I}^{\text{awk}}_{R-D} \subseteq \mathcal{I}^{\text{our,com}}_{R-D}\). Consider

\[
P_{V_1,c|U_1U_2X_1Z_1} = P_{V_1|UU_1U_2X_1Z_1},
\]

(124a)

\[
P_{V_2,c|U_1U_2X_2Z_2} = P_{V_2|UU_1U_2X_2Z_2},
\]

(124b)

\[
V_1,p = V_2,p = \phi.
\]

(124c)

For single-user rate bound on \(R_1\), we have

\[
R_1^{\text{awk}} \leq I(U_1; X_2Z_2|UU_2) + I(V_1; X_2Z_2|UU_1U_2) - I(V_1; X_1Z_1|UU_1U_2)
\]
+ \min \left \{ I(X_1; YS_R|UX_2) + I(V_1; X_1X_2YS_R|UU_1U_2) + I(U_2; X_1X_2YS_RV_1|UU_1U_2) - I(V_1; X_1Z_1|UU_1U_2), \\
I(X_1X_2; YS_R|UU_1) + I(V_1; X_1X_2YS_R|UU_1U_2) + I(V_2; X_1X_2YS_RV_1|UU_1U_2) - I(V_2; X_2Z_2|UU_1U_2), \\
I(X_1X_2; YS_R|U) + I(V_1; X_1X_2YS_R|UU_1U_2) + I(V_2; X_1X_2YS_RV_1|UU_1U_2) - I(V_1; X_1Z_1|UU_1U_2) - I(V_2; X_2Z_2|UU_1U_2), \\
I(X_1; YS_RV_1V_2|UU_1U_2X_2) \right \} \\
\leq I(U_1; Z_2|UU_2X_2) - I(V_1; X_1Z_1|UU_1U_2X_2Z_2) + I(X_1; YS_RV_1V_2|UU_1U_2X_2) \tag{a} \\
= I(U_1; Z_2|UU_2X_2) - I(V_1; X_1Z_1|UU_1U_2X_2Z_2) + \min \left \{ I(X_1; YS_RV_1V_2|UU_1U_2), I(X_1X_2; YS_RV_1V_2|UU_1U_2) \right \} \tag{b} \\
\leq I(U_1; Z_2|UU_2X_2) - I(V_1; X_1Z_1|UU_1U_2X_2Z_2) + \min \left \{ I(X_1; X_2YS_RV_1V_2|UU_1U_2), I(X_1X_2; YS_RV_1V_2|UU_1U_2) \right \} \tag{c} \\
= I(U_1; Z_2|UU_2X_2) - I(V_1; X_1Z_1|UU_1U_2X_2Z_2) + \min \left \{ I(X_1; X_2YS_RV_1,cV_2,cV_2,p|UU_1U_2) - I(V_1,p; Z_1|UU_1U_2X_2YS_RV_1,cV_2,cV_2,p), \\
I(X_1X_2; YS_RV_1,cV_2,c|UU_1U_2) - I(V_1;p; Z_1|UU_1U_2X_2YS_RV_1,cV_2,c) \\
- I(V_2,p; Z_2|UU_1U_2X_2YS_RV_1,cV_2,cV_1),c \right \} \tag{d} \end{align*}
+ \min \left\{ I(X_1X_2; YS_R|UU_2) + I(V_1; X_1X_2YS_R|UU_1U_2) + I(V_2; X_1X_2YS_RV_1|UU_1U_2) \\
- I(V_1; X_1Z_1|UU_1U_2), \\
I(X_1X_2; YS_R|UU_1) + I(V_1; X_1X_2YS_R|UU_1U_2) + I(V_2; X_1X_2YS_RV_1|UU_1U_2) \\
- I(V_2; X_2Z_2|UU_1U_2), \\
I(X_1X_2; YS_R|U) + I(V_1; X_1X_2YS_R|UU_1U_2) + I(V_2; X_1X_2YS_RV_1|UU_1U_2) \\
- I(V_1; X_1Z_1|UU_1U_2) - I(V_2; X_2Z_2|UU_1U_2), \\
I(X_1X_2; YS_RV_1V_2|UU_1U_2) \right\}

\leq (a) I(U_1; Z_2|UU_2X_2) - I(V_1; X_1Z_1|UU_1U_2X_2Z_2) \\
+ I(U_2; Z_1|UU_1X_1) - I(V_2; X_2Z_2|UU_1U_2X_1Z_1) + I(X_1X_2; YS_RV_2|UU_1U_2) \\
\overset{(b)}{=} I(U_1; Z_2|UU_2X_2) - I(V_{1,c}; X_1Z_1|UU_1U_2X_2Z_2) \\
+ I(U_2; Z_1|UU_1X_1) - I(V_{2,c}; X_2Z_2|UU_1U_2X_1Z_1) \\
+ I(X_1X_2; YS_RV_1cV_2|UU_1U_2) - I(V_{1,p}; Z_1|UU_1U_2X_1YS_RV_{1,c}V_{2,c}) \\
- I(V_{2,p}; Z_2|UU_1U_2X_1YS_RV_{1,c}V_{2,c}V_{1,p}), \quad (126)

where (a) follows from the Markov chains \( X_1U_1 - U - U_2X_2, X_2Z_2 - UU_1U_2X_1Z_1 - V_1, X_1Z_1 - UU_1U_2X_2Z_2 - V_2 \), and (b) follows from (124c). For another sum-rate bound (10c), we also have

\[ R_{1}\text{awk} + R_{2}\text{awk} \leq I(X_1X_2; YS_R) + I(V_1; X_1X_2YS_R|UU_1U_2) + I(V_2; X_1X_2YS_RV_1|UU_1U_2) \\
- I(V_1; X_1Z_1|UU_1U_2) - I(V_2; X_2Z_2|UU_1U_2), \]

\overset{(a)}{=} I(X_1X_2; YS_R) - I(V_1; Z_1|UU_1U_2X_1X_2YS_R) - I(V_2; Z_2|UU_1U_2X_1X_2YS_RV_1) \\
\overset{(b)}{=} I(X_1X_2; YS_R) - I(V_{1,c}; Z_1|UU_1U_2X_1X_2YS_R) - I(V_{2,c}; Z_2|UU_1U_2X_1X_2YS_RV_{1,c}) \\
- I(V_{1,p}; Z_1|UU_1U_2X_1X_2YS_RV_{1,c}V_{2,c}) - I(V_{2,p}; Z_2|UU_1U_2X_1X_2YS_RV_{1,c}V_{2,c}V_{1,p}), \quad (127)

where (a) follows from the Markov chains \( X_2YS_R - UU_1U_2X_1Z_1 - V_1, X_1YS_RV_1 - UU_1U_2X_2Z_2 - V_2 \), and (b) follows from (124c).

For the remaining constraints, we can also obtain that when (124a), (124b), and (124c) hold,
the constraints (21a), (21b), (21c), and (21d) in Theorem 3 become

$$I(U_k; Z_k | UU_k X_k) - I(V_{k;c}; X_k Z_k | UU_1 U_2 X_k Z_k) \geq 0, \; k \in \{1, 2\},$$

(128a)

$$I(X_1 X_2; Y S_R) - I(V_{1;c}; Z_1 | UU_1 U_2 X_1 X_2 Y S_R) - I(V_{2;c}; Z_2 | UU_1 U_2 X_1 X_2 Y S_R V_{1,c}) \geq 0,$$

(128b)

where (128a) and (128b) correspond to the constraints (11a) and (11c) in Theorem 1 due to the Markov chains $X_1 U_1 - U_2 X_2$, $X_2 Z_2 - UU_1 U_2 X_1 Z_1 - V_1$, $X_1 Z_1 - UU_1 U_2 X_2 Z_2 - V_2$, $X_2 Y S_R - UU_1 U_2 X_1 Z_1 - V_1$, and $X_1 Y S_R V_1 - UU_1 U_2 X_2 Z_2 - V_2$.

Moreover, we note that both the schemes in Theorem 1 and Theorem 3 use the same estimating functions. Combining the above analysis, one can obtain that

$$\mathcal{I}_{\text{awk}}^{\text{cur,com}} \subseteq \mathcal{I}_{R-D}^{\text{our,com}}.$$

(129)

APPENDIX C

CONVERSE PROOF OF THEOREM 5

We first derive an upper bound for $R_1$ as

$$n R_1 = H(W_1) = H(W_1 | W_2)$$

$$= I(W_1; Y^m S^n_{R_1} Z_1^n Z_2^n | W_2) + H(W_1 | W_2 Y^n S^n_{R_1} Z_1^n Z_2^n)$$

$$\leq I(W_1; Y^n S^n_{R_1} Z_1^n Z_2^n | W_2) + n \epsilon_1^{(n)}$$

$$= \sum_{i=1}^{n} I(W_1; Y_i S_{R,i} Z_{1,i} Z_{2,i} | W_2 Y^{i-1} S_{R}^{i-1} Z_1^{i-1} Z_2^{i-1}) + n \epsilon_1^{(n)}$$

$$= \sum_{i=1}^{n} H(Y_i S_{R,i} Z_{1,i} Z_{2,i} | W_2 Y^{i-1} S_{R}^{i-1} Z_1^{i-1} Z_2^{i-1})$$

$$- H(Y_i S_{R,i} Z_{1,i} Z_{2,i} | W_1 W_2 Y^{i-1} S_{R}^{i-1} Z_1^{i-1} Z_2^{i-1}) + n \epsilon_1^{(n)}$$

$$\leq \sum_{i=1}^{n} H(Y_i S_{R,i} Z_{1,i} Z_{2,i} | W_2 X_{2,i} Y^{i-1} S_{R}^{i-1} Z_1^{i-1} Z_2^{i-1})$$

$$- H(Y_i S_{R,i} Z_{1,i} Z_{2,i} | W_1 W_2 X_{1,i} X_{2,i} Y^{i-1} S_{R}^{i-1} Z_1^{i-1} Z_2^{i-1}) + n \epsilon_1^{(n)}$$

$$\leq \sum_{i=1}^{n} H(Y_i S_{R,i} Z_{1,i} Z_{2,i} | X_{2,i} Z_1^{i-1} Z_2^{i-1})$$

$$- H(Y_i S_{R,i} Z_{1,i} Z_{2,i} | W_1 W_2 X_{1,i} X_{2,i} Y^{i-1} S_{R}^{i-1} Z_1^{i-1} Z_2^{i-1}) + n \epsilon_1^{(n)}$$
where \((a)\) follows from the fact that conditioning reduces entropy and Fano’s inequality \([12]\), \((b)\) follows from the fact that \(X_{1,i}\) and \(X_{2,i}\) are functions of \((W_1, Z_1^{i-1})\) and \((W_2, Z_2^{i-1})\), respectively, \((c)\) follows from the fact that conditioning reduces entropy and we drop \((W_2, Y^{i-1}, S_R^{i-1})\) from the conditioning in the first term, \((d)\) follows form the Markov chain \((W_1 W_2 Y^{i-1} S_R^{i-1}) - (X_{1,i} X_{2,i} Z_1^{i-1} Z_2^{i-1}) - (Y_i S_{R,i} Z_1, Z_2, i)\). Finally, we define \(X_1 = X_{1,Q}, X_2 = X_{2,Q}, Y = Y_Q, Z_1 = Z_{1,Q}, Z_2 = Z_{2,Q}, S_R = S_{R,Q}, Q_Z = (Z_1^{Q-1} Z_2^{Q-1}), T = (Q, Q_Z)\), where \(Q\) is a random variable which is uniformly distributed over \([1 : n]\). Similarly, we have

\[
R_2 \leq I(X_2; Y Z_1 Z_2 | S_R X_1 T),
\]

\[
R_1 + R_2 \leq I(X_1 X_2; Y Z_1 Z_2 | S_R T).
\]

In addition to \((132)\), we also have the following sum-rate constraint which appears in the cut-set outer bound,

\[
R_1 + R_2 \leq I(X_1 X_2; Y S_R) = I(X_1 X_2; Y | S_R).
\]

The proof of dependence balance constraint \((26)\) follows the same step as in \([22]\). The detailed proof is provided as follows,

\[
(a) \sum_{i=1}^{n} H(Y_i S_{R,i} Z_{1,i} Z_{2,i} | X_{2,i} Z_{1,i}^{i-1} Z_{2,i}^{i-1}) = H(Y_i S_{R,i} Z_{1,i} Z_{2,i} | X_{1,i} X_{2,i} Z_{1,i}^{i-1} Z_{2,i}^{i-1}) + n \epsilon_1 (n)
\]

\[
= \sum_{i=1}^{n} I(X_{1,i}; Y_i S_{R,i} Z_{1,i} Z_{2,i} | X_{2,i} Z_{1,i}^{i-1} Z_{2,i}^{i-1}) + n \epsilon_1 (n)
\]

\[
= \sum_{i=1}^{n} I(X_{1,i}; Y_i Z_{1,i} Z_{2,i} | S_{R,i} X_{2,i} Z_{1,i}^{i-1} Z_{2,i}^{i-1}) + n \epsilon_1 (n)
\]

\[
= n I(X_1; Y Z_1 Z_2 | S_R X_2 T) + n \epsilon_1 (n).
\]

(130)
\[ -H(Z_{1,i}Z_{2,i}|Z_1^{i-1}Z_2^{i-1}) + H(Z_{1,i}Z_{2,i}|W_1 Z_1^{i-1}Z_2^{i-1}) \]

\[ \leq \sum_{i=1}^{n} H(Z_{1,i}Z_{2,i}|W_2 X_{i,2},Z_1^{i-1}Z_2^{i-1}) - H(Z_{1,i}Z_{2,i}|W_1 W_2 X_{i,1},X_{2,i}Z_1^{i-1}Z_2^{i-1}) \]

\[ - H(Z_{1,i}Z_{2,i}|Z_1^{i-1}Z_2^{i-1}) + H(Z_{1,i}Z_{2,i}|W_1 X_{1,i}Z_1^{i-1}Z_2^{i-1}) \]

where (a) follows because \( W_1 \) and \( W_2 \) are independent, (b) follows from the fact that \( X_{1,i} \) and \( X_{2,i} \) are functions of \( (W_1, Z_1^{i-1}) \) and \( (W_2, Z_2^{i-1}) \), respectively, (c) follows from that conditioning reduces entropy and the Markov chains \( W_k \rightarrow (X_{k,i}Z_1^{i-1}Z_2^{i-1}) \rightarrow (Z_{1,i}Z_{2,i}), k \in \{1,2\} \).

The proof of sensing constraint (27a) is as follows. For \( k = 1 \), there is

\[ I(S^n_{T_1}; X^n_1 Z^n_1) \geq I(S^n_{T_1}; \hat{S}^n_{T_1}). \] (135)

The inequality holds due to the Markov chain \( S^n_{T_1} \rightarrow (X^n_1, Z^n_1) \rightarrow \hat{S}^n_{T_1} \). For term \( I(S^n_{T_1}; \hat{S}^n_{T_1}) \), we have

\[ I(S^n_{T_1}; \hat{S}^n_{T_1}) = H(S^n_{T_1}) - H(S^n_{T_1}|\hat{S}^n_{T_1}) \]

\[ \overset{(a)}{=} \sum_{i=1}^{n} H(S_{T_1,i}) - H(S_{T_1,i}|\hat{S}^n_{T_1}, S_{T_1}^{i-1}) \]

\[ \geq \sum_{i=1}^{n} H(S_{T_1,i}) - H(S_{T_1,i}|\hat{S}_{T_1,i}) \]

\[ = \sum_{i=1}^{n} I(S_{T_1,i}; \hat{S}_{T_1,i}) \]

\[ \overset{(c)}{=} \sum_{i=1}^{n} f_{1,R,D}(E[d_1(S_{T_1,i}; \hat{S}_{T_1,i})]) \]
where (a) follows from the fact that the sensing state sequence \( S^n_{T_1} \) is i.i.d.; (b) follows because conditioning reduces entropy, (c) follows from the definition of rate-distortion function, (d) follows from the convexity of rate-distortion function and Jensen’s inequality, (e) follows from the definition (4) of distortion for blocks of length \( n \), (f) follows from the fact that rate-distortion function \( f_{1,R-D}(\cdot) \) is a nonincreasing function of \( D_1 \) and \( E[d_1(S^n_{T_1}; \hat{S}_T^n)] \leq D_1 \).

For term \( I(S^n_{T_1}; X^n_1 Z^n_1) \), we have

\[
I(S^n_{T_1}; X^n_1 Z^n_1) \leq \sum_{i=1}^{n} I(S^n_{T_1}; X^n_1 Z^n_1 | W_1)
\]

\[
= H(X^n_1 Z^n_1 | W_1) - H(X^n_1 Z^n_1 | W_1 S^n_{T_1})
\]

\[
= \sum_{i=1}^{n} H(Z_{1,i}|W_1 X_{1,i} X_{1,i}^{-1} Z_1^{-1}) - H(Z_{1,i}|W_1 S^n_{T_1} X_{1,i} X_{1,i}^{-1} Z_1^{-1}) \tag{137}
\]

where (a) follows from the facts that \( W_1 \) and \( S^n_{T_1} \) are independent and conditioning reduces entropy, (b) follows from the fact that \( X_{1,i} \) is a function of \( (W_1, Z_1^{-1}) \), (c) follows from that
conditioning reduces entropy, (d) follows from the fact that conditioning reduces entropy and we introduce \(X_{2,i}\) in the conditioning of the second term, (e) follows from the Markov chain 
\((W_1\{S_{T_1,i}\}_{i \neq i}X_1^{i-1}Z_1^{i-1}) - (S_{T_1,i}X_1X_2)\) - \(Z_{1,i}\). Combining the above results, we obtain that 
\[
I(X_2S_{T_1}; Z_1|X_1Q) \geq f_{1,R-D}(D_1). \tag{138}
\]
Similarly, there is 
\[
I(S_{T_1}^n; X_1^nZ_1^n) = I(S_{T_1}^n; X_1^nZ_1^n|W_1W_2)
\leq I(S_{T_1}^n; X_1^nZ_1^nW_1W_2)
\leq H(X_1^nZ_1^nZ_2^n|W_1W_2) - H(X_1^nZ_1^nZ_2^n|W_1W_2S_{T_1}^n)
= \sum_{i=1}^{n} H(Z_{1,i}Z_{2,i}|W_1W_2X_{1,i}X_{2,i}X_1^{i-1}Z_1^{i-1}Z_2^{i-1}) - H(Z_{1,i}Z_{2,i}|W_1W_2S_{T_1}^nX_{1,i}X_{2,i}X_1^{i-1}Z_1^{i-1}Z_2^{i-1})
\leq I(S_{T_1,i}; Z_{1,i}Z_{2,i}|X_{1,i}X_{2,i}) - H(Z_{1,i}Z_{2,i}|S_{T_1,i}X_{1,i}X_{2,i})
= nI(S_{T_1,Q}; Z_{1,Q}Z_{2,Q}|X_{1,Q}X_{2,Q}Q),
= nI(S_{T_1}; Z_1Z_2|X_1X_2Q), \tag{139}
\]
where (a) follows from the fact that \(X_{1,i}\) is a function of \((W_1, Z_1^{i-1})\) and \(X_{2,i}\) is a function of \((W_2, Z_2^{i-1})\), (b) follows from the Markov chain 
\((W_1W_2X_1^{i-1}Z_1^{i-1}Z_2^{i-1}) - X_{1,i}X_{2,i} - Z_{1,i}Z_{2,i}\),
(c) follows from the Markov chain 
\((W_1W_2\{S_{T_1,i}\}^{i \neq i}X_1^{i-1}Z_1^{i-1}Z_2^{i-1}) - (S_{T_1,i}X_1X_{2,i}) - Z_{1,i}Z_{2,i}\).
Combining with (136), we obtain that 
\[
I(S_{T_1}; Z_1Z_2|X_1X_2Q) \geq f_{1,R-D}(D_1). \tag{140}
\]
For \(k = 2\), we can easily obtain the similar result.
To prove Theorem \[\text{6}\] we define two rate-distortion functions as

\[
\begin{align*}
  f_{1,R-D}^{\text{aux}}(D_1) &= \min_{P_{\hat{S}_T_1|X_1X_2Z_1Z_2S_{T_1}}} \sum_{s_1z_1z_2s_{T_1}} P_{X_1X_2Z_1Z_2S_{T_1}} P_{\hat{S}_T_1|X_1X_2Z_1Z_2S_{T_1}} d_1(s_1, \hat{s}_{T_1}) \leq D_1, \\
  f_{2,R-D}^{\text{aux}}(D_2) &= \min_{P_{\hat{S}_T_2|X_1X_2Z_1Z_2S_{T_2}}} \sum_{s_2z_1z_2s_{T_2}} P_{X_1X_2Z_1Z_2S_{T_2}} P_{\hat{S}_T_2|X_1X_2Z_1Z_2S_{T_2}} d_2(s_2, \hat{s}_{T_2}) \leq D_2.
\end{align*}
\]

and introduce an auxiliary rate-distortion region \(\mathcal{O}_{R-D}^{\text{aux}}\) defined as the set of all tuples \((R_1, R_2, D_1, D_2)\) satisfying

\[
\begin{align*}
  R_1 &\leq I(X_1; YZ_1Z_2|S_RX_2T), \\
  R_2 &\leq I(X_2; YZ_1Z_2|S_RX_1T), \\
  R_1 + R_2 &\leq I(X_1X_2; YZ_1Z_2|S_RT), \\
  R_1 + R_2 &\leq I(X_1X_2; Y|S_R),
\end{align*}
\]

with dependence balance constraint

\[
I(X_1; X_2|T) \leq I(X_1; X_2|Z_1Z_2T)
\]

and sensing constraints

\[
f_{k,R-D}^{\text{aux}}(D_k) \leq I(S_{T_k}; Z_1Z_2|X_1X_2Q), k \in \{1, 2\},
\]

where \(T = (Q, Q_Z)\), \(Q - T - S_RX_1X_2 - YZ_1Z_2\) forms a Markov chain, and it suffices to consider \(Q, T\) whose alphabet \(Q, T\) has cardinality \(|Q| \leq |T| \leq |X_1||X_2| + 3\).

The proof consists of two steps. In the first step, we show that \(\mathcal{O}_{R-D}^{\text{our}} \subseteq \mathcal{O}_{R-D}^{\text{aux}}\), and in the second step, we show that \(\mathcal{O}_{R-D}^{\text{aux}} \subseteq \mathcal{O}_{R-D}^{\text{khh}}\).

**Step 1:** By the definition of \(f_{k,R-D}^{\text{aux}}(D_k)\), \(f_{k,R-D}^{\text{our}}(D_k)\) and the fact that

\[
\mathbb{E}[d(S_{T_k}, \hat{S}_{T_k})] = \mathbb{E}[d(S_{T_k}, \hat{S}_{T_k})|X_1X_2Z_1Z_2]
\]

\[
= \sum_{x_1x_2z_1z_2s_{T_k}^k} P_{X_1X_2Z_1Z_2S_{T_k}} P_{\hat{S}_{T_k}|X_1X_2Z_1Z_2S_{T_k}} d_k(s_{T_k}, \hat{s}_{T_k})
\]
\[ f_{k,R-D}^{\text{aux}}(D_k) \leq f_{k,R-D}^{\text{our}}(D_k). \]

The reason is as follows. Given the distortion \( D_k \), the rate distortion functions \( f_{k,R-D}^{\text{aux}}(D_k) \) and \( f_{k,R-D}^{\text{our}}(D_k) \) are both the optimal values of optimization problems with the same objective functions \( I(S_{T_k}; \hat{S}_{T_k}) \) and constraints \( \mathbb{E}[d(S_{T_k}, \hat{S}_{T_k})] \leq D_k \). The difference is that optimization variables in the optimization problem corresponding to \( f_{k,R-D}^{\text{aux}}(D_k) \) are \( P_{S_{T_k} \mid X_1 X_2 Z_1 Z_2 S_{T_k}} \), \( \forall X_1 X_2 Z_1 Z_2 S_{T_k} \), while the optimization variables in the optimization problem corresponding to \( f_{k,R-D}^{\text{our}}(D_k) \) are \( P_{S_{T_k} \mid X_1 X_2 Z_1 Z_2 S_{T_k}} \), \( \forall X_1 X_2 Z_1 Z_2 S_{T_k} \). Since

\[ P_{\hat{S}_{T_k} \mid X_k Z_k S_{T_k}} = \sum_{x_k z_k s_{T_k}} P_{\hat{S}_{T_k} \mid X_1 X_2 Z_1 Z_2 S_{T_k}} \), \( \forall X_1 X_2 Z_1 Z_2 S_{T_k} \),

we know that the optimal value of optimization problem corresponding to \( f_{k,R-D}^{\text{aux}}(D_k) \) is no more than that corresponding to \( f_{k,R-D}^{\text{our}}(D_k) \).

Now we proceed to consider two rate-distortion regions \( O_{R-D}^{\text{our}} \) and \( O_{R-D}^{\text{aux}} \). We find that two regions have the same rate constraints and dependence balance constraint. Moreover, for the sensing constraints \( (27) \) in \( O_{R-D}^{\text{our}} \) and \( (144) \) in \( O_{R-D}^{\text{aux}} \), we have

\[ f_{k,R-D}^{\text{aux}}(D_k) \leq f_{k,R-D}^{\text{our}}(D_k) \leq \min\{ I(S_{T_k} X_{3-k}; Z_k \mid X_k Q), I(S_{T_k}; Z_1 Z_2 \mid X_1 X_2 Q) \}
\]

\[ \leq I(S_{T_k}; Z_1 Z_2 \mid X_1 X_2 Q), \quad k \in \{1, 2\}, \]

which means that all rate-distortion tuples \((R_1, R_2, D_1, D_2)\) satisfying the constraints in \( O_{R-D}^{\text{our}} \)
also satisfy the constraints in \( O_{R-D}^{\text{aux}} \), i.e.,

\[ O_{R-D}^{\text{our}} \subseteq O_{R-D}^{\text{aux}}. \]

**Step 2:** We notice that two regions \( O_{R-D}^{\text{aux}} \) and \( O_{R-D}^{\text{khc}} \) have the same rate constraints and dependence balance constraint. Therefore, the allowable input distributions \( P_{TX_1 X_2} \) of two regions \( O_{R-D}^{\text{aux}} \) and \( O_{R-D}^{\text{khc}} \) are the same, and two regions can achieve the same rate region \((R_1, R_2)\) when ignoring the sensing performance. Thus, to prove \( O_{R-D}^{\text{aux}} \subseteq O_{R-D}^{\text{khc}} \), it is enough to show that for arbitrary rate pair \((R_1, R_2)\), the corresponding distortion in \( O_{R-D}^{\text{aux}} \) is no less than that in \( O_{R-D}^{\text{khc}} \), i.e., \( D_k^{\text{aux}} \geq D_k^{\text{khc}}, \quad k \in \{1, 2\}. \)
To prove this, we can arbitrarily choose the allowable distribution \( P_{TX_1X_2} \). Let \( \tilde{R}_1 \) and \( \tilde{R}_2 \) denote the corresponding rates of transmitters 1 and 2, and let \( \hat{S}^{\text{aux}}_{Tk} \) and \( \hat{S}^{\text{khkc}}_{Tk} \) denote the estimated sensing state of transmitter \( k \in \{1, 2\} \), respectively. Based on the definition of \( O^{\text{aux}}_{R-D} \), when fixed \( P_{TX_1X_2} \), one can obtain the values of \( I(S_{Tk}; Z_1Z_2|X_1X_2Q) \) and thus determine the minimum distortion of sensing states that two transmitters can achieve based on (144). Let \( D_k^{\text{aux}} \) denote the obtained distortion for region \( O^{\text{aux}}_{R-D} \).

In capacity-distortion region \( O^{\text{khkc}}_{R-D} \), it is assumed that each transmitter \( k \in \{1, 2\} \) has the full information of \( X_1, X_2, Z_1, Z_2 \) with the aid of a genie, and the estimated sensing state \( \hat{S}^{\text{khkc}}_{Tk} \) is a deterministic function of \( X_1, X_2, Z_1, Z_2 \). Thus, we have

\[
\mathbb{E}[d_k(S_{Tk}, \hat{S}^{\text{khkc}}_{Tk})] = \mathbb{E}[d_k(S_{Tk}, \hat{S}^*_{Tk}(X_1, X_2, Z_1, Z_2))] = \sum_{x_1, x_2, z_1, z_2} P_{X_1X_2Z_1Z_2} \sum_{S_{Tk} \in \hat{S}^{*}_{Tk}} P_{S_{Tk}|X_1X_2Z_1Z_2} d_k(S_{Tk}, \hat{S}^*_k(X_1, X_2, Z_1, Z_2)) \leq \sum_{x_1, x_2, z_1, z_2} P_{X_1X_2Z_1Z_2} \min_{\hat{S}_{Tk} \in S_{Tk}} \sum_{S_{Tk} \in \hat{S}^{*}_{Tk}} P_{S_{Tk}|X_1X_2Z_1Z_2} d_k(S_{Tk}, \hat{S}^*_k)
\]

\[
= \mathbb{E}[d(S_{Tk}, \hat{S}_{Tk})|X_1X_2Z_1Z_2] = \mathbb{E}[d(S_{Tk}, \hat{S}_{Tk})],
\]

(150)

where (a) follows by the definition of genie-aided state estimator (13) and \( \hat{S}'_{Tk} \) is the possible output for estimated sensing states in \( O^{\text{aux}}_{R-D} \). (b) follows from the Markov chain \( \hat{S}'_{Tk} \rightarrow (X_1X_2Z_1Z_2) \rightarrow S_{Tk} \). This result reveals that for certain rate region \( (\tilde{R}_1, \tilde{R}_2) \), the genie-aided state estimators in Theorem 2 can always achieve a distortion \( D_k^{\text{khkc}} \) which is no more than \( D_k^{\text{aux}} \), i.e.,

\[
D_k^{\text{aux}} \geq D_k^{\text{khkc}}, \quad k \in \{1, 2\}.
\]

(151)

Since two regions can achieve the same rate region, we conclude that

\[
O^{\text{aux}}_{R-D} \subseteq O^{\text{khkc}}_{R-D}.
\]

(152)
APPENDIX E
PROOF OF EXAMPLE

Given \( Z_1 = X_1 \oplus S_2 \) and \( Z_2 = X_2 \oplus S_1 \), we have

\[
I(U_1; X_2 Z_2 | UU_2) = I(U_1; X_2 S_1 | UU_2) = 0, \tag{153a}
\]

\[
I(U_2; X_1 Z_1 | UU_1) = I(U_2; X_1 S_2 | UU_1) = 0, \tag{153b}
\]

since \( X_1 U_1 - U - U_2 X_2 \) forms a Markov chain and \( S_1, S_2 \) are independent of \( UU_1 U_2 X_1 X_2 \).

Then, consider the inequality constraint (11a) for \( k = 1 \),

\[
I(U_2; X_1 Z_1 | UU_1) + I(V_2; X_1 Z_1 | UU_1 U_2) \geq I(V_2; X_2 Z_2 | UU_1 U_2). \tag{154}
\]

Given \( I(U_2; X_1 Z_1 | UU_1) = 0 \), we have

\[
0 \geq I(V_2; X_2 Z_2 | UU_1 U_2) - I(V_2; X_1 Z_1 | UU_1 U_2)
= -H(V_2|UU_1 U_2 X_2 Z_2) + H(V_2|UU_1 U_2 X_1 Z_1)
\leq H(V_2|UU_1 U_2 X_1 Z_1) - H(V_2|UU_1 U_2 X_1 Z_1 X_2 Z_2)
= I(V_2; X_2 Z_2 | UU_1 U_2 X_1 Z_1) \geq 0, \tag{155}
\]

where (a) follows from that \( V_2 - UU_1 U_2 X_2 Z_2 - X_1 Z_1 \) forms a Markov chain. We can obtain a same result for \( k = 2 \) as

\[
I(V_1; X_1 Z_1 | UU_1 U_2) - I(V_1; X_2 Z_2 | UU_1 U_2) = I(V_1; X_1 Z_1 | UU_1 U_2 X_2 Z_2) = 0. \tag{156}
\]

Then, considering the first term in function min(\cdot) of (10a) for \( k = 2 \), we have

\[
Raw_2 \leq I(U_2; X_1 Z_1 | UU_1) + I(V_2; X_1 Z_1 | UU_1 U_2) - I(V_2; X_2 Z_2 | UU_1 U_2) + I(X_2; YS_R | UX_1) \\
+ I(V_2; X_1 X_2 YS_R | UU_1 U_2) + I(V_1; X_1 X_2 YS_R V_2 | UU_1 U_2) - I(V_2; X_2 Z_2 | UU_1 U_2)
= I(X_2; YS_R | UX_1) + I(V_2; X_1 X_2 YS_R | UU_1 U_2) + I(V_1; X_1 X_2 YS_R V_2 | UU_1 U_2)
- I(V_2; X_2 Z_2 | UU_1 U_2)
= I(X_2; YS_R | UX_1) + I(V_2; X_1 X_2 YS_R V_1 | UU_1 U_2) + I(V_1; X_1 X_2 YS_R | UU_1 U_2)
- I(V_2; X_2 Z_2 | UU_1 U_2)
\leq I(X_2; YS_R | UX_1) + I(V_1; X_1 X_2 YS_R | UU_1 U_2) - I(V_2; Z_2 | UU_1 U_2 X_1 X_2 YS_R V_1) \tag{a}
\]
\[ I(U_2; Y_{12}S)(X_{12}) + I(V_1; X_{12}Y_{12}S)(UU_1U_2) \]

\[ (\text{b}) \quad H(Y_1, Y_2|UX_1) - H(Y_1, Y_2|UX_2) + I(V_1; X_1X_2Y_{12}|UU_1U_2) \]

\[ (\text{c}) \quad H(S_1) + H(Y_2|UX_1Y_1) - H(S_1) - H(S_2) + I(V_1; X_1X_2S_1S_2|UU_1U_2) \]

\[ (\text{d}) \quad 1 - 0.5 + I(V_1; X_1X_2S_1S_2|UU_1U_2) \]

\[ (\text{e}) \quad 0.5 + I(V_1; X_1X_2Z_1Z_2|UU_1U_2) \]

\[ (\text{f}) \quad 0.5 + I(V_1; X_2Z_2|UU_1U_2) \]

\[ = 0.5 + H(X_2Z_2|UU_1U_2) - H(X_2Z_2|UU_1U_2V_1) \]

\[ (\text{g}) \quad 0.5 + H(X_2Z_2|UU_1U_2) - H(X_2Z_2|UU_1U_2X_1Z_1V_1) \]

\[ (\text{h}) \quad 0.5 + H(X_2Z_2|UU_1U_2) - H(X_2Z_2|UU_1U_2X_1Z_1) \]

\[ = 0.5 + I(X_1Z_1; X_2Z_2|UU_1U_2) \]

\[ = 0.5 + I(X_1S_2; X_2S_1|UU_1U_2) \]

\[ (\text{i}) \quad 0.5 \quad (157) \]

where (a) follows from the Markov chain \( V_2 - UU_1U_2X_2Z_2 - X_1Y_{12}S_1V_1 \), (b) follows from that \( S_R = \phi \), (c) follows from that \( Y_1 = X_1 \oplus S_1, Y_2 = X_2 \oplus S_2 \), (d) follows from that the entropy of binary variable is no more than 1 and \( H(S_2) = 0.5 \), (e) follows from that \( Z_1 = X_1 \oplus S_2, Z_2 = X_2 \oplus S_1 \), (f) follows from that \( I(V_1; X_1Z_1|UU_1U_2X_2Z_2) = 0 \), (g) follows from that conditioning reduces entropy, (h) follows from the Markov chain \( V_1 - UU_1U_2X_1Z_1 - X_2Z_2 \), and (i) follows from that given \( UU_1U_2, X_1S_2 \) and \( X_2S_1 \) are independent.

**APPENDIX F**

**PROOF OF EXAMPLE 2**

Consider the inequality constraints (11a) and (11b) for \( k = 1 \),

\[ I(U_1; X_2Z_2|UU_2) + I(V_1; X_2Z_2|UU_1U_2) \geq I(V_1; X_1Z_1|UU_1U_2), \quad (158a) \]

\[ I(X_1; Y_{12}S)(X_{12}) + I(V_1; X_1X_2Y_{12}S)(UU_1U_2) + I(V_2; X_1X_2Y_{12}S)(V_1|UU_1U_2) \]

\[ \geq I(V_1; X_1Z_1|UU_1U_2), \quad (158b) \]
we know that the sum of the right hand side of (158a) and (158b) is no more than the sum of
the left hand side, i.e.,
\[
I(U_1; X_2 Z_2 | UU_1 U_2) + I(V_1; X_2 Z_2 | UU_1 U_2) + I(X_1; Y S_R | U X_2) + \\
I(V_1; X_1 X_2 Y S_R | UU_1 U_2) + I(V_2; X_1 X_2 Y S_R | UU_1 U_2) \\
\geq I(V_1; X_1 Z_1 | UU_1 U_2) + I(V_1; X_1 Z_1 | UU_1 U_2). \tag{159}
\]
Given \( Y = X_1 \oplus S_1 \oplus N, X_2 \oplus S_2, Z_1 = (X_1 \oplus S_1, X_1 \oplus S_2), Z_2 = X_1 \oplus B, \) and \( S_R = \phi, \) one can prove that
\[
I(U_1; X_2 Z_2 | UU_2 X_2) \overset{(a)}{=} I(U_1; Z_2 | UU_2 X_2), \tag{160}
\]
where \( (a) \) follows from the Markov chain \( X_1 U_1 - U - U_2 X_2, \) and
\[
I(X_1; Y S_R | UX_2) = I(X_1; Y | UX_2) \overset{(a)}{=} 0, \tag{161}
\]
where \( (a) \) follows from that \( H(N) = 1. \) Thus, (159) becomes
\[
I(U_1; Z_2 | UU_2 X_2) + I(V_1; X_2 Z_2 | UU_1 U_2) + \\
I(V_1; X_1 X_2 Y S_R | V_2 | UU_1 U_2) + I(V_2; X_1 X_2 Y S_R | UU_1 U_2) \\
\geq I(V_1; X_1 Z_1 | UU_1 U_2) + I(V_1; X_1 Z_1 | UU_1 U_2). \tag{162}
\]
Based on the Markov chains \( X_2 Z_2 - UU_1 U_2 X_1 Z_1 - V_1 \) and \( X_2 Y S_R V_2 - UU_1 U_2 X_1 Z_1 - V_1, \) inequality (162) can be further written as
\[
I(U_1; Z_2 | UU_2 X_2) + I(V_2; X_1 X_2 Y S_R | UU_1 U_2) \\
\geq I(V_1; X_1 Z_1 | UU_1 U_2 X_2 Z_2) + I(V_1; Z_1 | UU_1 U_2 X_1 X_2 Y S_R V_2). \tag{163}
\]
For the right hand side of (163), we have
\[
I(V_1; X_1 Z_1 | UU_1 U_2 X_2 Z_2) \geq I(V_1; Z_1 | UU_1 U_2 X_1 X_2 Z_2) \\
\overset{(a)}{=} I(V_1; S_1 S_2 | UU_1 U_2 X_1 X_2, X_1 \oplus B) \\
\geq I(V_1; S_1 | UU_1 U_2 X_1 X_2, X_1 \oplus B) \\
\overset{(b)}{=} H(S_1) - H(S_1 | UU_1 U_2 X_1 X_2, X_1 \oplus B, V_1) \\
\overset{(c)}{=} H(S_1) - H(S_1 | U_1 X_2, X_1 \oplus B, V_1) 
\]
\begin{align*}
&= I(S_1; U_1 X_2 V_1 Z_2) \\
&\geq I(S_1; \hat{S}_{T_2})
\end{align*}

where \((a)\) follows from that \(Z_1 = (X_1 \oplus S_1, X_1 \oplus S_2)\) and \(Z_2 = X_1 \oplus B\), \((b)\) follows from that \(S_1\) is independent of \(UU_1 U_2 X_1 X_2, X_1 \oplus B\), \((c)\) follows from that conditioning reduces entropy, \((d)\) follows from the Markov chain \(S_1 - U_1 X_2 V_1 Z_2 - \hat{S}_{T_2}\) and data processing inequality \([12]\), and

\[I(V_1; Z_1|UU_1 U_2 X_1 X_2 Y S_R V_2) \overset{(a)}{=} I(V_1; S_1 S_2|UU_1 U_2 X_1 X_2, S_1 \oplus N, S_2 S_R V_2)\]

\[= H(S_1) - H(S_1|UU_1 U_2 X_1 X_2, S_1 \oplus N, S_2 S_R V_1 V_2)\]

\[\overset{(c)}{=} H(S_1) - H(S_1|UU_1 U_2 X_1 X_2, S_1 \oplus N, S_2 S_R V_1 V_2 B)\]

\[= H(S_1) - H(S_1|UU_1 U_2 X_1 X_2, S_1 \oplus N, S_2 S_R V_1 V_2 B, X_1 \oplus B)\]

\[\overset{(d)}{=} H(S_1) - H(S_1|U_1 X_2 V_1, X_1 \oplus B)\]

\[\overset{(e)}{=} I(S_1; U_1 X_2 V_1 Z_2)\]

where \((a)\) follows from that \(Y = X_1 \oplus S_1 \oplus N, X_2 \oplus S_2\) and \(Z_1 = (X_1 \oplus S_1, X_1 \oplus S_2)\), \((b)\) follows from that \(S_1\) is independent of \(UU_1 U_2 X_1 X_2, S_1 \oplus N, S_2, S_R = \phi, B\) as \(H(N) = 1\) and Markov chain \(S_1 - UU_1 U_2 X_2 Z_2 - V_2\) with \(Z_2 = X_1 \oplus B\), \((c)\) follows from that \(S_1\) is independent of \(B\) given \(UU_1 U_2 X_1 X_2, S_1 \oplus N, S_2 S_R V_1 V_2\), \((d)\) follows from that conditioning reduces entropy, and \((e)\) follows from the Markov chain \(S_1 - U_1 X_2 V_1 Z_2 - \hat{S}_{T_2}\) and data processing inequality \([12]\).

For the left hand side of \((163)\), we have

\[I(U_1; Z_2|UU_2 X_2) + I(V_2; X_1 X_2 Y S_R|UU_1 U_2)\]

\[\overset{(a)}{=} I(U_1; Z_2|UU_2 X_2) + I(V_2; X_1 X_2|UU_1 U_2) + I(V_2; S_1 \oplus N, S_2|UU_1 U_2 X_1 X_2)\]

\[\overset{(b)}{\leq} I(U_1; Z_2|UU_2 X_2) + I(V_2; X_1 X_2|UU_1 U_2) + H(S_1 \oplus N, S_2|UU_1 U_2 X_1 X_2 B)\]

\[\quad - H(S_1 \oplus N, S_2|UU_1 U_2 X_1 X_2 V_2 B)\]

\[= I(U_1; Z_2|UU_2 X_2) + I(V_2; X_1 X_2|UU_1 U_2) + I(V_2; S_1 \oplus N, S_2|UU_1 U_2 X_1 X_2 B)\]

\[\overset{(c)}{=} I(U_1; Z_2|UU_2 X_2) + I(V_2; X_1 X_2|UU_1 U_2)\]
\[ = I(U_1; Z_2|UU_2X_2) + I(V_2; X_1|UU_1U_2) + I(V_2; X_2|UU_1U_2X_1) \]

\[ = I(U_1; Z_2|UU_2X_2) + H(X_1|UU_1U_2) - H(X_1|UU_1U_2V_2) + I(V_2; X_2|UU_1U_2X_1) \]

\[ \leq I(U_1; Z_2|UU_2X_2) + H(X_1|UU_1U_2) - H(X_1|UU_1U_2X_2Z_2V_2) + I(V_2; X_2|UU_1U_2X_1) \]

\[ \equiv I(U_1; Z_2|UU_2X_2) + I(X_1; X_2Z_2|UU_1U_2) + I(V_2; X_2|UU_1U_2X_1) \]

\[ \equiv I(U_1; Z_2|UU_2X_2) + I(X_1; Z_2|UU_1U_2X_2) + I(V_2; X_2|UU_1U_2X_1) \]

\[ = H(Z_2|UU_2X_2) - H(Z_2|UU_1U_2X_1X_2) + I(V_2; X_2|UU_1U_2X_1) \]

\[ \leq H(X_1 \oplus B) - H(X_1 \oplus B|X_1) + I(V_2; X_2|UU_1U_2X_1) \]

\[ \leq 0.5 + I(V_2; X_2|UU_1U_2X_1), \quad (166) \]

where \((a)\) follows from that \(S_2 = \phi, Y = X_1 \oplus S_1 \oplus N, X_2 \oplus S_2, \) \((b)\) follows from the that \(S_1 \oplus N, S_2\) are independent of \(B\) given \(UU_1U_2X_1X_2\) and conditioning reduces entropy, \((c)\) follows from the Markov chain \(S_1 \oplus N, S_2 - UU_1U_2X_2Z_2 - V_2\) with \(Z_2 = X_1 \oplus B, \) \((d)\) follows from that conditioning reduces entropy, \((e)\) follows from the Markov chain \(X_1 - UU_1U_2X_2Z_2 - V_2, \) \((f)\) follows from the Markov chain \(X_1U_1 - U - U_2X_2, \) \((f)\) follows from \(Z_2 = X_1 \oplus B\) and conditioning reduces entropy, \((g)\) follows from that the entropy of binary variable is no more than 1 and \(H(B) = 0.5.\) Moreover, based on the inequality constraint \((11a)\) for \(k = 2,\) one can know that

\[ 0 \equiv I(U_2; X_1Z_1|UU_2) \]

\[ \geq I(V_2; X_2Z_2|UU_1U_2) - I(V_2; X_1Z_1|UU_1U_2) \]

\[ \equiv I(V_2; X_2Z_2|UU_1U_2) - I(V_2; X_1|UU_1U_2) - I(V_2; S_1S_2|UU_1U_2X_1) \]

\[ \geq I(V_2; X_2Z_2|UU_1U_2) - I(V_2; X_1|UU_1U_2) - H(S_1S_2|UU_1U_2X_1X_2B) \]

\[ + H(S_1S_2|UU_1U_2X_1X_2V_2B) \]

\[ = I(V_2; X_2Z_2|UU_1U_2) - I(V_2; X_1|UU_1U_2) - I(V_2; S_1S_2|UU_1U_2X_1X_2B) \]

\[ \equiv I(V_2; X_2Z_2|UU_1U_2) - I(V_2; X_1|UU_1U_2) \]

\[ = H(V_2|UU_1U_2X_1) - H(V_2|UU_1U_2X_2Z_2) \]

\[ \equiv H(V_2|UU_1U_2X_1) - H(V_2|UU_1U_2X_2Z_2X_1) \]
holds for

\[ I \left( V_2; X_2 \mid UU_1U_2X_1 \right) \]
\[ \geq I \left( V_2; X_2 \mid UU_1U_2X_1 \right), \quad (167) \]

where \((a)\) follows from the Markov chain \(X_1 U_1 - U - U_2 X_2\) and \(Z_1 = (X_1 \oplus S_1, X_1 \oplus S_2)\), \((b)\) follows from \((11a)\) for \(k = 2\), \((c)\) follows from that \(Z_1 = (X_1 \oplus S_1, X_1 \oplus S_2)\), \((d)\) follows from that \(S_1, S_2\) is independent of \(UU_1U_2X_1X_2B\) and conditioning reduces entropy, \((e)\) follows from the Markov chain \(S_1 S_2 - UU_1U_2X_2Z_2 - V_2\) and \(Z_2 = X_1 \oplus B\), \((f)\) follows from the Markov chain \(X_1 - UU_1U_2X_2Z_2 - V_2\).

Therefore, the inequality \((163)\) means that

\[ I(S_1; \hat{S}_{T_2}) \leq 0.25 < 0.5 = H(S_1), \quad (168) \]

i.e., the mutual information \(I(S_1; \hat{S}_{T_2})\) is strictly smaller than the entropy \(H(S_1)\), which shows that transmitter 2 cannot achieve zero distortion based on the rate-distortion theory \([12]\).

**APPENDIX G**

**PROOF OF EXAMPLE 4**

We first prove that tuple \((0.918563, 0.3)\) is not in \(\mathcal{T}_{R-D}^{our, com}\) by proving that \(R_1^{our, com} < 0.918563\) holds for \(\mathcal{T}_{R-D}^{our, com}\) and tuple \((0, 0, 0.13072)\) is not in \(\mathcal{T}_{R-D}^{awk}\) by proving that \(D_2^{awk} > 0.13072\) holds for \(\mathcal{T}_{R-D}^{awk}\). Then, we show that tuple \((0.11697, 0, 0.1783)\) is not in \(\mathcal{T}_{R-D}^{awk}\) by proving that \(R_1^{awk} < 0.11697\) when \(D_2^{awk} = 0.1783\).

**A. Proof of \(R_1^{our, com} < 0.918563\)**

Due to the fact that \(\min\{a, b\} \leq a\) and \(V_{1,p} = V_{2,p} = \phi\) in \(\mathcal{T}_{R-D}^{our, com}\), we obtain a relaxed single-user bound for \(R_1\) in \(\mathcal{T}_{R-D}^{our, com}\) based on \((20a)\) as

\[
R_1^{our, com} \leq I(U_1; Z_2|UU_2X_2) - I(V_{1,c}; X_1Z_1|UU_1U_2X_2Z_2) + I(X_1; X_2YS_RV_{1,c}V_{2,c}|UU_1U_2)
\]
\[
\overset{(a)}{=} I(U_1; Z_2|UU_2X_2) + I(V_{1,c}; X_2Z_2|UU_1U_2) - I(V_{1,c}; X_1Z_1|UU_1U_2)
\]
\[
+ I(X_1; YV_{1,c}V_{2,c}|UU_1U_2X_2)
\]
\[
= I(U_1; Z_2|UU_2X_2) + I(V_{1,c}; X_2Z_2|UU_1U_2) - I(V_{1,c}; X_1Z_1|UU_1U_2)
\]
\[
+ I(X_1; Y|UU_1U_2X_2) + I(X_1; V_{2,c}|UU_1U_2X_2Y) + I(X_1; V_{1,c}|UU_1U_2X_2YV_{2,c}), \quad (169)
\]
where \((a)\) follows from the Markov chains \(X_2Z_2 - UU_1U_2X_1Z_1 - V_{1,c}\) and \(X_1U_1 - U - U_2X_2\) as well as \(S_R = \phi\) in the example.

Now we proceed to obtain upper bounds of terms in (169). For term \(I(U_1; Z_2|UU_2X_2)\), we have

\[
I(U_1; Z_2|UU_2X_2) \overset{(a)}{=} I(U_1; B \cdot X_1, X_2 \oplus S_1|UU_2X_2) = I(U_1; B \cdot X_1|UU_2X_2) + I(U_1; X_2 \oplus S_1|UU_2X_2, B \cdot X_1) \overset{(b)}{=} I(U_1; B \cdot X_1|UU_2X_2),
\]

where \((a)\) follows from the fact that \(Z_2 = (BX_1, X_2 \oplus S_1)\) in the example, \((b)\) follows from the fact that \(S_1\) is independent of \(U, U_1, U_2, X_1, X_2, B, B \cdot X_1\).

For term \(I(V_{1,c}; X_2Z_2|UU_1U_2)\), we have

\[
I(V_{1,c}; X_2Z_2|UU_1U_2) = H(X_2Z_2|UU_1U_2) - H(X_2Z_2|UU_1U_2V_{1,c}) \overset{(a)}{=} H(X_2Z_2|UU_1U_2) - H(X_2Z_2|UU_1U_2X_1V_{1,c}) \overset{(b)}{=} H(X_2Z_2|UU_1U_2) - H(X_2Z_2|UU_1U_2X_1Z_1) = I(X_1Z_1; X_2Z_2|UU_1U_2) = I(X_1; X_2Z_2|UU_1U_2) + I(Z_1; X_2Z_2|UU_1U_2X_1) \overset{(c)}{=} I(X_1; X_2Z_2|UU_1U_2) + I(N; X_2Z_2|UU_1U_2X_1) \overset{(d)}{=} I(X_1; X_2Z_2|UU_1U_2) \overset{(e)}{=} I(X_1; B \cdot X_1, X_2 \oplus S_1|UU_1U_2X_2) \overset{(f)}{=} I(X_1; B \cdot X_1|UU_1U_2X_2),
\]

where \((a)\) follows from that conditioning reduces entropy, \((b)\) follows from the Markov chain \(X_2Z_2 - UU_1U_2X_1Z_1 - V_{1,c}\), \((c)\) follows from that \(Z_1 = X_1 \oplus N\) in the example, \((d)\) follows from that \(N\) is independent of \(UU_1U_2X_1Z_2\) as \(Z_2 = (BX_1, X_2 \oplus S_1)\), \((e)\) follows from the Markov chain \(X_1U_1 - U - U_2X_2\) and \(Z_2 = (BX_1, X_2 \oplus S_1)\), \((f)\) follows from that \(S_1\) is independent of \(X_1\) given \(UU_1U_2X_2, B \cdot X_1\).

For term \(I(X_1; Y|UU_1U_2X_2)\), we have

\[
I(X_1; Y|UU_1U_2X_2) \overset{(a)}{=} I(X_1; X_1 \oplus S_1, X_2 \oplus S_2|UU_1U_2X_2)
\]
\[ I(X_1; X_1 \oplus S_1 | UU_1 U_2 X_2) + I(X_1; X_2 \oplus S_2 | UU_1 U_2 X_2, X_1 \oplus S_1) \]
\[ = I(X_1; X_1 \oplus S_1 | UU_1 U_2 X_2) + I(X_1; S_2 | UU_1 U_2 X_2, X_1 \oplus S_1) \]
\[ = I(X_1; X_1 \oplus S_1 | UU_1 U_2 X_2) \]
\[ \leq H(X_1 \oplus S_1 | UU_1 U_2 X_2) - H(X_1 \oplus S_1 | UU_1 U_2 X_1 X_2) \]
\[ \leq H(X_1 \oplus S_1) - H(S_1) \]
\[ \leq 1 - H(S_1) \]
\[ = 0.204959720615478 < 0.205, \] (172)

where (a) follows from the fact that \( Y = (Y_1, Y_2) = (X_1 \oplus S_1, X_2 \oplus S_2) \) in the example, (b) follows from the fact that \( S_2 \) is independent of \( UU_1 U_2 X_1 X_2 S_1 \), (c) follows from that conditioning reduces entropy and \( S_1 \) is independent of \( UU_1 U_2 X_1 X_2 \), (d) follows from that the entropy of binary random variable is no more than 1, (e) follows from that \( P_{S_1}(1) = 0.24 \) in the example.

For term \( I(X_1; V_{2,c} | UU_1 U_2 X_2 Y) \), we have

\[ I(X_1; V_{2,c} | UU_1 U_2 X_2 Y) \leq I(V_{2,c}; X_1 X_2 Y | UU_1 U_2) \]
\[ \stackrel{(a)}{=} I(V_{2,c}; X_1 X_2, X_1 \oplus S_1, X_2 \oplus S_2 | UU_1 U_2) \]
\[ = I(V_{2,c}; X_1 X_2 S_1 S_2 | UU_1 U_2) \]
\[ = I(V_{2,c}; X_1 X_2 S_1 | UU_1 U_2) + I(V_{2,c}; S_2 | UU_1 U_2 X_1 X_2 S_1) \]
\[ \leq I(V_{2,c}; X_1 X_2 S_1 | UU_1 U_2) + H(S_2 | UU_1 U_2 X_1 X_2 S_1 B) - H(S_2 | UU_1 U_2 X_1 X_2 S_1 V_{2,c} B) \]
\[ \stackrel{(c)}{=} I(V_{2,c}; X_1 X_2 S_1 | UU_1 U_2) + I(S_2; V_{2,c} | UU_1 U_2 X_1 X_2 S_1 B Z_2) \]
\[ \leq H(X_1 | UU_1 U_2) - H(X_1 | UU_1 U_2 V_{2,c}) + I(V_{2,c}; X_2 S_1 | UU_1 U_2 X_1) \]
\[ \leq H(X_1 | UU_1 U_2) - H(X_1 | UU_1 U_2 X_2 Z_2 V_{2,c}) + I(V_{2,c}; X_2 S_1 | UU_1 U_2 X_1) \]
\[ \leq H(X_1 | UU_1 U_2) - H(X_1 | UU_1 U_2 X_2 Z_2) + I(V_{2,c}; X_2 S_1 | UU_1 U_2 X_1) \]
\[ \geq H(X_1 | UU_1 U_2) - H(X_1 | UU_1 U_2, B \cdot X_1, X_2 \oplus S_1) + I(V_{2,c}; X_2 S_1 | UU_1 U_2 X_1) \]
\[ \geq H(X_1 | UU_1 U_2) - H(X_1 | UU_1 U_2, B \cdot X_1) + I(V_{2,c}; X_2 S_1 | UU_1 U_2 X_1) \]
\[ I(B \cdot X_1; X_1|UU_1U_2) + I(V_{2,c}; X_2S_1|UU_1U_2X_1), \]  

(173)

where \((a)\) follows from that \(Y = (Y_1, Y_2) = (X_1 \oplus S_1, X_2 \oplus S_2)\) in the example, \((b)\) follows from that \(S_2\) is independent of \(UU_1U_2X_1S_1B\) and conditioning reduces entropy, \((c)\) follows from that \(Z_2 = (B \cdot X_1, X_2 \oplus S_1)\) in the example, \((d)\) follows from the Markov chain \(S_2 - UU_1U_2X_1Z_2 - V_{2,c}\), \((e)\) follows from that conditioning reduces entropy, \((f)\) follows from the Markov chain \(X_1 - UU_1U_2X_2Z_2 - V_{2,c}\), \((g)\) follows from that \(Z_2 = (B \cdot X_1, X_2 \oplus S_1)\), \((h)\) follows from the Markov chain \(X_2, S_1 - UU_1U_2 - X_1\).

For term \(I(X_1; V_{1,c}|UU_1U_2X_2YV_{2,c}) - I(V_{1,c}; X_1Z_1|UU_1U_2)\), we have

\[
I(X_1; V_{1,c}|UU_1U_2X_2YV_{2,c}) - I(V_{1,c}; X_1Z_1|UU_1U_2) = H(V_{1,c}|UU_1U_2X_2YV_{2,c}) - H(V_{1,c}|UU_1U_2X_1Z_1) \\
\leq H(V_{1,c}|UU_1U_2) - H(V_{1,c}|UU_1U_2X_1Z_1) + H(V_{1,c}|UU_1U_2X_1Z_1) \\
= -H(V_{1,c}|UU_1U_2X_1Z_1) + H(V_{1,c}|UU_1U_2X_1Z_1) \\
= -I(V_{1,c}; Z_1|UU_1U_2X_1Z_1) \\
\leq 0,
\]

(174)

where \((a)\) follows from that conditioning reduces entropy, \((b)\) follows from the Markov chain \(X_2YV_{2,c} - UU_1U_2X_1Z_1 - V_{1,c}\).

Combining (169), (170), (171), (172), (173), and (174), we have

\[
R_{\text{sur,com}}^0 \leq I(U_1; B \cdot X_1|UU_2X_2) + I(X_1; B \cdot X_1|UU_1U_2X_2) + I(B \cdot X_1; X_1|UU_1U_2) \\
+ I(V_{2,c}; X_2S_1|UU_1U_2X_1) + 0.205.
\]

(175)

Next, we show that \(I(V_{2,c}; X_2S_1|UU_1U_2X_1) = 0\). The detailed proof is as follows. In the example, \(Z_1 = X_1 \oplus N\). Therefore,

\[
I(U_2; Z_1|UU_1X_1) = I(U_2; N|UU_1X_1) \overset{(a)}{=} 0,
\]

(176)

where \((a)\) follows from that \(N\) is independent of \(UU_1U_2X_1\). Consider (21a) for \(k = 2\), we have

\[
0 \geq I(V_{2,c}; X_2Z_2|UU_1U_2X_1Z_1) \\
= H(V_{2,c}|UU_1U_2X_1Z_1) - H(V_{2,c}|UU_1U_2X_1Z_1X_2Z_2)
\]
\( (a) \quad H(V_{2,c}|UU_1U_2X_1Z_1) - H(V_{2,c}|UU_1U_2X_2Z_2) \\
= H(V_{2,c}|UU_1U_2X_1Z_1) - H(V_{2,c}|UU_1U_2) + H(V_{2,c}|UU_1U_2) - H(V_{2,c}|UU_1U_2X_2Z_2) \\
= I(V_{2,c}; X_2Z_2|UU_1U_2) - I(V_{2,c}; X_1Z_1|UU_1U_2), \\
(b) \quad I(V_{2,c}; X_2Z_2|UU_1U_2) - I(V_{2,c}; X_1, X_1 \oplus N|UU_1U_2), \\
= I(V_{2,c}; X_2Z_2|UU_1U_2) - I(V_{2,c}; X_1|UU_1U_2) - I(V_{2,c}; N|UU_1U_2X_1) \\
(c) \quad I(V_{2,c}; X_2Z_2|UU_1U_2) - I(V_{2,c}; X_1|UU_1U_2) \\
(d) \quad I(V_{2,c}; X_2Z_2|UU_1U_2X_1) \\
(e) \quad I(V_{2,c}; X_2, B \cdot X_1, X_2 \oplus S_1|UU_1U_2X_1) \\
= I(V_{2,c}; X_2, B \cdot X_1, S_1|UU_1U_2X_1) \\
\geq I(V_{2,c}; X_2S_1|UU_1U_2X_1), \quad (177)

\]

where \((a)\) follows from the Markov chain \(X_1Z_1 - UU_1U_2X_2Z_2 - V_{2,c}\), \((b)\) follows from that \(Z_1 = X_1 \oplus N\) in the example, \((c)\) follows from that \(N\) is independent of \(V_{2,c}\) given \(UU_1U_2X_1\), \((d)\) follows from the Markov chain \(X_1 - UU_1U_2X_2Z_2 - V_{2,c}\), \((e)\) follows from that \(Z_2 = (BX_1, X_2 \oplus S_1)\) in the example.

Given \((177)\), we have

\[
R_1^{\text{our.com}} \leq I(U_1; B \cdot X_1|UU_2X_2) + I(X_1; B \cdot X_1|UU_1U_2X_2) + I(B \cdot X_1; X_1|UU_1U_2) + 0.205 \\
= H(B \cdot X_1|UU_2X_2) - H(B \cdot X_1|UU_1U_2X_2) \\
\quad + H(B \cdot X_1|UU_1U_2X_2) - H(B \cdot X_1|UU_1U_2X_1) \\
\quad + H(B \cdot X_1|UU_1U_2) - H(B \cdot X_1|UU_1U_2X_1) + 0.205 \\
= H(B \cdot X_1|UU_2X_2) - H(B \cdot X_1|UU_1U_2X_1) \\
\quad + H(B \cdot X_1|UU_1U_2) - H(B \cdot X_1|UU_1U_2X_1) + 0.205 \\
\overset{(a)}{\leq} H(B \cdot X_1) - H(B \cdot X_1|UU_1U_2X_1X_2) + H(B \cdot X_1) - H(B \cdot X_1|UU_1U_2X_1) + 0.205 \\
\overset{(b)}{=} H(B \cdot X_1) - H(B \cdot X_1|X_1) + H(B \cdot X_1) - H(B \cdot X_1|X_1) + 0.205 \\
= 2I(X_1; B \cdot X_1) + 0.205 \\
\overset{(c)}{=} 2 \cdot 0.322 + 0.205
\]
where (a) follows from that conditioning reduces entropy, (b) follows from the Markov chain $UU_1U_2X_2 - X_1 - B \cdot X_1$, (c) follows from the numerical computation with $P_B = 0.5$ and $X$ is a binary random variable.

B. Proof of $D_{2^{\text{awk}}} > 0.13072$

Considering the inequality constraints (11a) and (11b) for $k = 1$, i.e.,

\[ I(U_1; X_2Z_2|UU_2) + I(V_1; X_2Z_2|UU_1U_2) \geq I(V_1; X_1Z_1|UU_1U_2), \quad (179a) \]

\[ I(X_1; YS_R|UX_2) + I(V_1; X_1X_2YS_R|UU_1U_2) + I(V_2; X_1X_2YS_RV_1|UU_1U_2) \]

\[ \geq I(V_1; X_1Z_1|UU_1U_2), \quad (179b) \]

we know that the sum of the right hand side of (179a) and (179b) is no more than the sum of the left hand side, i.e.,

\[ I(U_1; X_2Z_2|UU_2) + I(V_1; X_2Z_2|UU_1U_2) + I(X_1; YS_R|UX_2) + I(V_1; X_1X_2YS_R|UU_1U_2) + I(V_2; X_1X_2YS_RV_1|UU_1U_2) \]

\[ \geq I(V_1; X_1Z_1|UU_1U_2) + I(V_1; X_1Z_1|UU_1U_2), \quad (180) \]

which can be equivalently transformed to

\[ I(U_1; Z_2|UU_2X_2) + I(X_1; YS_R|UX_2) + I(V_2; X_1X_2YS_R|UU_1U_2) \]

\[ \geq I(V_1; X_1Z_1|UU_1U_2X_2Z_2) + I(V_1; Z_1|UU_1U_2X_1X_2YS_RV_2) \quad (181) \]

based on the Markov chains $X_2Z_2YS_RV_2 - UU_1U_2X_1Z_1 - V_1$ and $X_1U_1 - U - U_2X_2$. By the similar procedures as those in (170), (172), (173), (177), and (178), we have

\[ I(U_1; Z_2|UU_2X_2) + I(X_1; YS_R|UX_2) + I(V_2; X_1X_2YS_R|UU_1U_2) \]

\[ < I(U_1; B \cdot X_1|UU_2X_2) + 0.205 + I(B \cdot X_1; X_1|UU_1U_2) \]

\[ \leq H(B \cdot X_1) - H(B \cdot X_1|X_1) + 0.205 \]

\[ = I(X_1; B \cdot X_1) + 0.205 \]
< 0.322 + 0.205 \\
= 0.527 < 0.6. \quad (182)

For the left hand side of (181), by the similar procedures as those in (164) and (165), we have

\[
I(V_1; X_1 Z_1 | UU_1 U_2 X_2 Z_2) \overset{(a)}{=} I(V_1; X_1, X_1 \oplus N | UU_1 U_2 X_2 Z_2) \\
\geq I(V_1; N | UU_1 U_2 X_1 X_2 Z_2) \\
= H(N | UU_1 U_2 X_1 X_2 Z_2) - H(N | UU_1 U_2 X_1 X_2 Z_2 V_1) \\
\overset{(b)}{=} H(N) - H(N | UU_1 U_2 X_1 X_2 Z_2 V_1) \\
\overset{(c)}{=} H(N) - H(N | U_1 Z_2 V_1) \\
= I(N; U_1 Z_2 V_1) \\
\overset{(d)}{=} I(N; \hat{S}_{T_2}) \quad (183)
\]

where (a) follows from that \( Z_1 = X_1 \oplus N \) in the example, (b) follows from that \( N \) is independent of \( UU_1 U_2 X_1 X_2 Z_2 \) as \( Z_2 = (B \cdot X_1, X_2 \oplus S_1) \) in the example, (c) follows from that conditioning reduces entropy, (d) follows from that \( N - U_1 Z_2 V_1 - \hat{S}_{T_2} \) forms a Markov chain, and

\[
I(V_1; Z_1 | UU_1 U_2 X_1 X_2 Y S_R V_2) \overset{(a)}{=} I(V_1; N | UU_1 U_2 X_1 X_2 Y S_R V_2) \\
= H(N | UU_1 U_2 X_1 X_2 Y S_R V_2) - H(N | UU_1 U_2 X_1 X_2 Y S_R V_1 V_2) \\
\overset{(b)}{=} H(N) - H(N | UU_1 U_2 X_1 X_2 Y S_R V_1 V_2) \\
\overset{(c)}{=} H(N) - H(N | UU_1 U_2 X_1 X_2 Y S_R V_1 V_2 B) \\
\overset{(d)}{=} H(N) - H(N | UU_1 U_2 X_1 X_2 Y S_R V_1 V_2 B, B \cdot X_1, X_2 \oplus S_1) \\
= H(N) - H(N | UU_1 U_2 X_1 X_2 Y S_R V_1 V_2 B Z_2) \\
\overset{(e)}{=} H(N) - H(N | U_1 X_2 Z_2 V_1) \\
= I(N; U_1 X_2 Z_2 V_1) \\
\overset{(f)}{=} I(N; \hat{S}_{T_2}), \quad (184)
\]

where (a) follows from that \( Z_1 = X_1 \oplus N \) in the example, (b) follows from that \( N \) is independent of \( UU_1 U_2 X_1 X_2 Y S_R B \) as \( Y = (X_1 \oplus S_1, X_2 \oplus S_2), S_R = \phi \) in the example and
$N - UU_1U_2X_2Z_2 - V_2$ forms a Markov chain as $Z_2 = (BX_1, X_2 \oplus S_1)$, (c) follows from that $N$ is independent of $B$ given $UU_1U_2X_1YS_RV_1V_2$ for the numerical example, (d) follows from that given $B, X_1, X_2, Y$, one can exactly know $B \cdot X_1$ and $X_2 \oplus S_1$, (e) follows from that conditioning reduces entropy, (f) follows from that $N - U_1X_2Z_2V_1 - \hat{S}_{T_2}$ forms a Markov chain.

Combining the above results, one can obtain that

$$I(N; \hat{S}_{T_2}) < 0.3$$

holds for $T_{R-D}^{\text{awk}}$ in Theorem 1. According to the rate-distortion theory \cite{12}, one can know that the distortion between $N$ and $\hat{S}_{T_2}$ must be strictly larger than 0.138\footnote{When the distortion is equal to 0.138, the mutual information is $I(N; \hat{S}_{T_2}) = 0.3023 > 0.3.$}, i.e.,

$$D_{2}^{\text{awk}} > 0.138 > 0.13072.$$

C. Proof of $R_{1}^{\text{awk}} < 0.11697$ when $D_{2}^{\text{awk}} = 0.1783$

Since $\min\{a, b\} \leq a$, we can obtain a relaxed single-user bound for $R_{1}$ in $T_{R-D}^{\text{awk}}$ based on (10a) for $k = 1$ as

$$R_{1}^{\text{awk}} \leq I(U_1; X_2Z_2|UU_2) + I(V_1; X_2Z_2|UU_1U_2) - I(V_1; X_1Z_1|UU_1U_2)$$

$$+ I(X_1; YS_R|UX_2) + I(V_1; X_1X_2YS_R|UU_1U_2) + I(V_2; X_1X_2YS_RV_1|UU_1U_2)$$

$$- I(V_1; X_1Z_1|UU_1U_2)$$

$$\overset{(a)}{=} I(U_1; X_2Z_2|UU_2) + I(X_1; YS_R|UX_2) + I(V_2; X_1X_2YS_R|UU_1U_2)$$

$$- I(V_1; X_1Z_1|UU_1U_2X_2Z_2) - I(V_1; Z_1|UU_1U_2X_2YS_RV_2)$$

$$\overset{(b)}{\leq} I(X_1; B \cdot X_1) + H(X_1 \oplus S_1) - H(S_1)$$

$$- I(V_1; X_1Z_1|UU_1U_2X_2Z_2) - I(V_1; Z_1|UU_1U_2X_1X_2YS_RV_2)$$

$$\overset{(c)}{\leq} I(X_1; B \cdot X_1) + H(X_1 \oplus S_1) - H(S_1) - I(N; \hat{S}_{T_2}) - I(N; \hat{S}_{T_2}),$$

where (a) follows from the Markov chain $X_2Z_2YS_RV_2 - UU_1U_2X_1Z_1 - V_1$, (b) follows the similar procedures as these in (170), (172), (173), (177), (178), (c) follows based on (183) and (184).
Given $P_{S_1}(1) = 0.24$, $P_{S_2}(1) = 0.05$, $P_N(1) = 0.3$, $P_B(1) = 0.5$ in the example, we have

$$I(X_1; B \cdot X_1) \leq 0.321928094887362$$

(188)

by numerical computation, where the inequality holds with equality if and only if $P_X(1) = 0.4$. We can also know that

$$H(X_1 \oplus S_1) - H(S_1) \leq 0.204959720615478,$$

(189)

where the inequality holds with equality if and only if $P_X(1) = 0.5$. Moreover, when $D_2^{awk} \approx 0.1783$, there is $I(N; \hat{S}_T) = \max(H(X_1 \oplus S_1) - H(S_1))$. Combing with these results, we have

$R_1^{awk} < 0.321928094887362 - 0.204959720615478$

$$= 0.116968374271884$$

(190)

when $D_2^{awk} \approx 0.1783$. In particular, we note that the numerical result of $R_1 = 0.11697$ in point B shown in Fig. 6 comes from the computational accuracy. When choosing

$$X_k = U_k \oplus \Theta_k = U \oplus \Sigma_k \oplus \Theta_k, \ k \in \{1, 2\},$$

(191a)

$$V_{1,c} = \tilde{N}, \ V_{1,p} = V_{2,c} = \phi, \ V_{2,p} = \phi,$$

(191b)

for $T_{R-D, our}^{com}$, one can find that by taking

$$P_U = 0, P_{\Sigma_1} = 0.4, P_{\Theta_1} = 0, P_{\Sigma_2} = 0, P_{\Theta_2} = 0.5,$$

(192)

and choosing $V_{1,c} = \tilde{N}$ to make that $I(N; \tilde{N}) = \max(H(X_1 \oplus S_1) - H(S_1)) = 1 - H(S_1)$, $D_2^{our, com} \approx 0.1783$ and $R_1^{our, com} = 0.116968374271884$ can be achieved simultaneously.

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