DUAL BRAIDED QUANTUM E(2) GROUPS

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Abstract. An explicit construction of the braided dual of quantum E(2) groups is described over the circle group $\mathbb{T}$, with respect to a specific $R$-matrix $R$. Additionally, the corresponding bosonization is also described.

1. Introduction

Along with a clear formula for the bicharacter characterising the duality of quantum groups in the operator algebraic context, a beautiful duality for the quantum E(2) group for non-zero real deformation parameters was discovered in [15]. It is shown in [11] that the bidual of quantum E(2) is again isomorphic to quantum E(2) group. This article is the culmination of an effort to expand this duality to complex deformation parameters.

The $q$-deformations of the (double cover of) E(2) group, which Woronowicz first described in [16] for real deformation parameters, are classic examples of locally compact quantum groups. These deformations were then extended in [10] for complex values of the deformation parameter $q$ to produce braided locally compact quantum E(2) groups over the circle group $\mathbb{T}$, viewed as a quasitriangular quantum group with respect to a certain $R$-matrix $R$.

The category $\mathcal{C}^{\ast}\text{alg}(\mathbb{T})$, whose objects are $C^{\ast}$-algebras with an action of $\mathbb{T}$, called $\mathbb{T}$-$C^{\ast}$-algebras, and the morphisms are the $\mathbb{T}$-equivariant morphisms between two $\mathbb{T}$-$C^{\ast}$-algebras, plays a pivotal role in the construction of both braided $E_{q}(2)$ and its dual, henceforth denoted by $\hat{E}_{q}(2)$. However, unlike the case of braided $E_{q}(2)$, the monoidal product $\boxtimes_{R}$, in this case, is governed by the dual braiding induced by the dual $R$-matrix $\hat{R}$ in $\mathcal{R}\mathcal{e}\mathcal{p}(\mathbb{T})$, the representation category of $\mathbb{T}$. Therefore, we provide a complete description of the dual of the braided quantum E(2) group building on the general construction of braided locally compact quantum groups over a quasitriangular quantum group following [5], taking in particular the quasitriangular quantum group to be $\mathbb{T}$, inside the general framework of braided multiplicative unitaries. It turns out that the bidual of braided quantum E(2) group coincides with braided quantum E(2) group $E_{q}(2)$ is made to be a braided $C^{\ast}$-quantum group in an appropriate sense by the change from real to complex values of the deformation parameter $q$, as shown in [10], and the associated multiplicative unitary then becomes a braided multiplicative unitary [10, Theorem 4.6]. The underlying $C^{\ast}$-algebra of $E_{q}(2)$ groups is generated by a unitary $v$ and an unbounded normal element $n$ with spectrum contained in the set defined and denoted by $\mathcal{D}^{(q)} := \{ \lambda \in \mathbb{C} : |\lambda| \in |q|^{\mathbb{Z}} \} \cup \{ 0 \}$, satisfying the formal commutation relation $vnv^{*} = qn$ [10].

We face similar kinds of technical limitations when trying to extend this duality for purely complex values of $q$ as we did while building braided $E_{q}(2)$ groups and

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we refer to introductory section in [10] for details. In what follows we use a similar
approach to overcome these limitations.

The article is organised as follows: the required preliminaries are gathered in the
next section. The main content is contained in section 3. In the last section we
construct the bosonization, an analogous construction of the semidirect products
of groups, of the dual quantum group which yields an ordinary noncompact locally
compact C∗-quantum group.

2. Preliminaries

The generators of the C∗-algebras we shall encounter here are unbounded and
hence we shall use the notions of C∗-algebras generated by unbounded elements
affiliated with the C∗-algebra [10] in the sense of Woronowicz [13].

The notations used in [10] are significantly relied upon in this text. As a result,
we try to keep the preliminaries brief. Therefore, for any undefined notation we refer to the aforementioned articles. Nevertheless, we reiterate some of the ideas
here for the readers’ convenience.

For a C∗-algebra A, we denote by $\mathcal{M}(A)$, the multiplier algebra of A and by $\mathcal{U}(A)$ the group of unitary multipliers of A. For a Hilbert space $\mathcal{H}$, the identity operator on $\mathcal{H}$ is denoted by $1_\mathcal{H}$ whereas the unit element of $\mathcal{B}(\mathcal{H}) = \mathcal{M}(\mathcal{K}(\mathcal{H}))$ is
denoted by $1_\mathcal{H}$. Also for two norm closed subsets $X, Y \subset \mathcal{B}(\mathcal{H})$

$$X \cdot Y = \left\{ xy \mid x \in X \text{ and } y \in Y \right\}^{\text{CLS}},$$

where “CLS” stands for the closed linear span. A morphism from A to B is a
nondegenerate *-homomorphism $\varphi: A \to \mathcal{M}(B)$ such that $\varphi(A)B = B$. Any
quantum group G is a pair $(C_0(G), \Delta_G)$ consisting of a C∗-algebra $C_0(G)$ and a
morphism $\Delta_G$ satisfying the coassociativity condition $(\text{id} \otimes \Delta_G) \circ \Delta_G = (\Delta_G \otimes \text{id}) \circ \Delta_G$ and the cancellation conditions $\Delta_G(C_0(G))(1 \otimes C_0(G)) = C_0(G) \otimes C_0(G) = \Delta_G(C_0(G))(1)$. For a complete definition of a C∗-quantum

We write $G = (C_0(G), \Delta_G)$ to associate the underlying C∗-algebra and the comultiplication map.

Quantum group representations. A (right) representation of G on a C∗-algebra
$D$ is an element $U \in \mathcal{U}(D \otimes C_0(G))$ with

$$(\text{id}_D \otimes \Delta_G)U = U_{12}U_{13} \quad \text{in} \quad \mathcal{U}(D \otimes C_0(G) \otimes C_0(G)).$$

In particular, if $D = \mathbb{K}(\mathcal{L})$ for some Hilbert space $\mathcal{L}$, then U is said to be a (right)
representation of G on $\mathcal{L}$.

The tensor product of two representations $U^i \in \mathcal{U}(\mathbb{K}(\mathcal{L}_i) \otimes C_0(G))$ of G on $\mathcal{L}_i$
for $i = 1, 2$ is a representation on $\mathcal{L}_1 \otimes \mathcal{L}_2$ defined by

$$U^1 \otimes U^2 := U^1_{13}U^2_{23} \quad \text{in} \quad \mathcal{U}(\mathbb{K}(\mathcal{L}_1 \otimes \mathcal{L}_2) \otimes C_0(G)).$$

According to the notations above, if $(t \otimes 1_G)U^1 = U^2(t \otimes 1_G)$ for some $t \in \mathbb{B}(\mathcal{L}_1, \mathcal{L}_2)$, then we say that $t$ intertwines $U^1$ and $U^2$ and such an element is referred to as an intertwiner. The set of all intertwiners between $U^1$ and $U^2$ is denoted as $\text{Hom}^G(U^1, U^2)$. A simple calculation demonstrates that the $\otimes$ is associative, and the trivial 1-dimensional representation is a tensor unit. This establishes a $W^*$-category structure on the collection of representations, which we denote by $\mathfrak{Rep}(G)$; for additional information, see [8] Section 3.1–2. Pairs $(\mathcal{L}, U)$ consisting of a Hilbert space $\mathcal{L}$ and a representation of G on $\mathcal{L}$ are objects of $\mathfrak{Rep}(G)$.
Quantum group actions. A (right) action of $G$ on a C$^*$-algebra $C$ is an injective morphism $\Gamma \in \text{Mor}(C, C \otimes C_0(G))$ with the following properties:

1. $\Gamma$ is a comodule structure, that is, 
\[
(id_C \otimes \Delta_G) \circ \Gamma = (\Gamma \otimes id_{C_0(G)}) \circ \Gamma; \quad (2.2)
\]

2. $\Gamma$ satisfies the Podleś condition: 
\[
\Gamma(C)(1_C \otimes C_0(G)) = C \otimes C_0(G). \quad (2.3)
\]

The pair $(C, \Gamma)$ is referred to as a $G$-C$^*$-algebra. When it is obvious from the context, we shall drop $\Gamma$ from our notation and merely write $C$ is a $G$-C$^*$-algebra.

A morphism $f : C \to D$ between two $G$-C$^*$-algebras $(C, \Gamma_C)$ and $(D, \Gamma_D)$ is said to be $G$-equivariant if 
\[
\Gamma_D \circ f = (f \otimes id_G) \circ \Gamma_C.
\]

The set of $G$-equivariant morphisms from $C$ to $D$ is denoted by $\text{Mor}^G(C, D)$, while the category containing $G$-C$^*$-algebras as objects and $G$-equivariant morphisms as arrows is denoted by $\text{CAlg}^G(G)$.

For all $f \in C_0(G)$, $g \in G$, every continuous action $\phi$ of a locally compact group $G$ on a C$^*$-algebra $D$ produces an action $\Gamma_D$ of $G=(C_0(G), \Delta_{C_0(G)})$ on $D$ given by $(\Gamma_D(d)f)(g) := f(\phi(g)(d))$ and vice versa. Additionally, when $G$-C$^*$-algebras are used as objects and $G$-equivariant morphisms are used as arrows, the category $\text{CAlg}^G(G)$ is equivalent to the category $\text{Rep}(G)$. Analogously, $\text{Rep}(G)$ is equivalent to the representation category $\text{Rep}(G)$.

Quasitriangular quantum groups. Let $\hat{G} = (C_0(\hat{G}), \Delta_{\hat{G}})$ be the dual of $G$. An element $R \in \mathcal{U}(C_0(\hat{G}) \otimes C_0(\hat{G}))$ is said to be an $R$-matrix on $\hat{G}$ if it is a bicharacter, that is, it satisfies the following:

\[
(id_{C_0(\hat{G})} \otimes \Delta_{\hat{G}})R = R_{12}R_{13}, \quad (\Delta_{\hat{G}} \otimes id_{C_0(\hat{G})})R = R_{23}R_{13}.
\]

in $\mathcal{U}(C_0(\hat{G}) \otimes C_0(\hat{G}) \otimes C_0(\hat{G}))$ and satisfies the $R$-matrix condition:

\[
R(\sigma \Delta_{\hat{G}}(\hat{a})) R^* = \Delta_{\hat{G}}(\hat{a}) \quad \text{for all } \hat{a} \in C_0(\hat{G}). \quad (2.4)
\]

A quasitriangular quantum group is a quantum group $G$ with an $R$-matrix $R \in \mathcal{U}(C_0(\hat{G}) \otimes C_0(\hat{G}) \otimes C_0(\hat{G}))$. [3] Section 3).

The categories $\text{Rep}(G)$ and $\text{CAlg}^G(G)$ are particularly interesting when $G$ is quasitriangular. More specifically, the representation category $\text{Rep}(G)$ is a braided monoidal category and $\text{CAlg}^G(G)$ is a monoidal category by virtue of [3] Proposition 3.2 & Theorem 4.3. The braiding isomorphism in $\text{Rep}(G)$ is a unitary operator. Such a category is referred to as a unitarily braided monoidal category. We take into consideration the monoidal product $\boxtimes^R$ on $\text{CAlg}^G(G)$ and the dual braiding corresponding to the unitary braiding $\times$ on $\text{Rep}(G)$ whose explicit construction is described in [3][4][10]. The latter construction was motivated by [9]. We briefly recall the construction for $G=\mathbb{T}$ in the following subsection.

Duality of braided quantum groups. The circle group $\mathbb{T}$ can be viewed as a quasitriangular quantum group. Every bicharacter $\chi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{T}$ satisfies the $R$-matrix condition \[because C_0(\hat{\mathbb{T}}) = C_0(\mathbb{Z}) is a commutative C$^*$-algebra. Thus $\mathbb{T}$ is quasitriangular with respect to any bicharacter on $\mathbb{Z}$. In particular, for $q \in \mathbb{C} \setminus \{0\}$ the bicharacter $R : \mathbb{Z} \times \mathbb{Z} \to \mathbb{T}$ defined by $R(m, n) = \zeta^{mn}$ where $\zeta = \frac{q}{q}$, is an $R$-matrix on $\mathbb{Z}$.
Let $U \in \mathcal{U}(\mathbb{K}(\mathcal{L}) \otimes \mathcal{C}(\mathcal{T}))$ be a representation of $\mathcal{T}$ on $\mathcal{L}$. Then there is a unique solution $\tilde{Z} \in \mathcal{U}(\mathcal{L} \otimes \mathcal{L})$ such that

$$U_{13}U_{29}\tilde{Z}_{12} = U_{29}U_{13} \text{ in } \mathcal{U}(\mathcal{L} \otimes \mathcal{L} \otimes \mathcal{K})$$

(2.5)

for any $\mathcal{R}$-Heisenberg pair $(\beta, \alpha)$ on any Hilbert space $\mathcal{K}$, see [3] Theorem 4.1]. Hence, the unitary $\tilde{Z}$ determines the dual braiding $\times := \tilde{Z} \circ \Sigma$. For more information on Heisenberg pairs see [3], and for the braiding operator see [4].

Recall that the unitary braiding operator $\times$ on $\text{Rep}(\mathcal{T})$ induced by the $R$-matrix $\mathcal{R}$ is given by $\times := Z \circ \Sigma$, where $Z$ satisfies an appropriate variant of (2.5) for any $\mathcal{R}$-Heisenberg pair $(\alpha, \beta)$ on $\mathcal{L}$ [10].

Now suppose $(\mathcal{C}_i, \Gamma_i)$ are two objects in $\mathcal{C}^\ast\mathfrak{R}f(\mathbb{T})$ for $i = 1, 2$ and let $\times$ be the unitary dual braiding. Let $\mathcal{C}_i \hookrightarrow \mathcal{B}(\mathcal{L}_i)$ be the $\mathcal{T}$-equivariant representations of $\mathcal{C}_i$ on $\mathcal{L}_i$ for $i = 1, 2$ respectively. Then the monoidal product $\boxtimes_\mathcal{R}$ on $\mathcal{C}^\ast\mathfrak{R}f(\mathbb{T})$ is given by

$$C_1 \boxtimes_\mathcal{R} C_2 := \left\{ \hat{j}_1(x)\hat{j}_2(y) : x \in C_1, y \in C_2 \right\} \subset \mathcal{B}(\mathcal{L}_1 \otimes \mathcal{L}_2),$$

where $\hat{j}_1(x) = x \otimes \text{id}_{\mathcal{L}_2}$ and $\hat{j}_2(y) = \times(y \otimes \text{id}_{\mathcal{L}_1}) \times$ for all $x \in C_1$ and $y \in C_2$.

Then $\mathcal{C}_1 \boxtimes_\mathcal{R} C_2 := j_1(C_1)j_2(C_2)$ is a $C^\ast$-algebra [3] Theorem 4.6] and

$$\Gamma_1 \otimes \Gamma_2(\hat{j}_1(c_1)\hat{j}_2(c_2)) := (\hat{j}_1 \otimes \text{id}_A)\Gamma_1(c_1)(\hat{j}_2 \otimes \text{id}_A)\Gamma_2(c_2)$$

defines the diagonal action of $\mathbb{T}$ on $C_1 \boxtimes_\mathcal{R} C_2$ [3] Proposition 4.1]. Hence, $C_1 \boxtimes_\mathcal{R} C_2$ is an object in $\mathcal{C}^\ast\mathfrak{R}f(\mathbb{T})$. For two objects $(\mathcal{C}_i, \Gamma_i)$ in $\mathcal{C}^\ast\mathfrak{R}f(\mathbb{T})$ and $f_i \in \text{Mor}(\mathcal{C}_i, \mathcal{C}_j)$ for $i = 1, 2$, the monoidal product $f_1 \boxtimes_\mathcal{R} f_2$ is defined in a canonical way.

The multiplicative unitary, which simultaneously encodes much of the information of a quantum group and it’s dual, is one of the key elements in the theory of quantum groups in the operator algebraic framework [11][12][13]. Along with a full investigation of a more generic entity known as the braided multiplicative unitary, the theory of braided $C^\ast$-quantum groups is explored in [5]. More explicitly, for a Hilbert space $\mathcal{L}$, a unitary operator $\mathcal{F}$ on $\mathcal{L} \otimes \mathcal{L}$ is said to be a multiplicative unitary if it satisfies the pentagonal equation

$$\mathcal{F}_{12}\mathcal{F}_{13} = \mathcal{F}_{13}\mathcal{F}_{12} \text{ in } \mathcal{U}(\mathcal{L} \otimes \mathcal{L} \otimes \mathcal{L}).$$

More assumptions such as modularity and manageability are required on $\mathcal{F}$ to construct $C^\ast$-quantum groups. Braided analogues of a multiplicative unitary and its manageability are defined in [5] Definition 3.2] and [5] Definition 3.5] respectively. We will, however, use an alternative but equivalent approach to the one established in [5]. As a result, our definition of the notion of braided multiplicative unitary over a regular quasitriangular quantum group becomes more concise one, see [10] Definition3.1 and Definition 3.4.

**Theorem 2.6.** Let $\mathcal{F} \in \mathcal{U}(\mathcal{L} \otimes \mathcal{L})$ be a manageable braided multiplicative unitary over $\mathcal{T}$ with respect to $\mathcal{(U, \mathcal{R})}$. Then the dual $\mathcal{\hat{F}} := \times\mathcal{F}^\ast\times^\ast$ is a manageable braided multiplicative unitary over $\mathcal{T}$ with respect to $\mathcal{(U, \mathcal{R})}$. 

*Proof.* The proof essentially follows the arguments used in the proof of [5] Proposition 3.6]. Let $\mathcal{F}$ and $\mathcal{Q}$ be the operators that witness the manageability of the braided multiplicative unitary $\mathcal{F}$. And let $\mathcal{Z}$ and $\tilde{\mathcal{Z}}$ be the operators appearing in [10] Equation 3.5].

The $R$-matrix $\mathcal{R}$ being a bicharacter, it induces a quantum group homomorphism from $\mathcal{T}$ to $\mathcal{Z}$ [2] and hence induces a representation $\mathcal{V} \in \mathcal{U}(\mathcal{K}(\mathcal{L}) \otimes \mathcal{C}_0(\mathcal{Z}))$ satisfying [2] Equation (32)]. Recall that $\mathcal{W}$ is a multiplicative unitary generating $\mathcal{T}$. Let $\pi \in \text{Mor}(\mathcal{C}(\mathbb{T}), \mathcal{K}(\mathcal{H}))$ and $\tilde{\pi} \in \text{Mor}(\mathcal{C}_0(\mathcal{Z}), \mathcal{K}(\mathcal{H}))$ be the embeddings as
The triple \( C \) generated by the braided multiplicative unitary

\[
\begin{align*}
\text{Theorem 2.8.} & \quad \text{Let } \mathbb{F} \text{ be a manageable braided multiplicative unitary over } T \text{ relative to } (U, R) \text{ and } \mathbb{F} = \times^{*}\mathbb{F}^{*}. \text{ Let }
\end{align*}
\]
\[
\begin{align*}
C_{0}(\widehat{H}) & := \{(\omega \otimes \text{id}_{K(L)})\mathbb{F} | \omega \in \mathbb{B}(L), \}^{\text{CLS}} \subset \mathbb{B}(L).
\end{align*}
\]

Then

\( (1) C_{0}(\widehat{H}) \) is a nondegenerate, separable C*-subalgebra of \( \mathbb{B}(L) \);

\( (2) \) Define \( \widehat{\Gamma}(b) := U(b \otimes 1_{A})U^{*} \) for all \( b \in C_{0}(\widehat{H}) \). Then \( \widehat{\Gamma} \in \text{Mor}(C_{0}(\widehat{H}), C_{0}(\widehat{H}) \otimes C_{0}(G)) \) and \( (C_{0}(\widehat{H}), \widehat{\Gamma}) \) is an object of \( \mathfrak{C}^{*}\mathfrak{d}(G) \);

\( (3) \) \( \mathbb{F} \in \mathcal{U}(\mathfrak{K}(L) \otimes C_{0}(\widehat{H})) \);

Consider the twisted tensor product \( C_{0}(\widehat{H}) \otimes^{R} C_{0}(\widehat{H}) \). Suppose \( \widehat{j}_{1}, \widehat{j}_{2} \) are the canonical embeddings : \( \widehat{j}_{1}, \widehat{j}_{2} \in \text{Mor}^{G}(C_{0}(\widehat{H}), C_{0}(\widehat{H}) \otimes^{R} C_{0}(\widehat{H})) \).

\( (4) \) There exists a unique \( \Delta_{\widehat{H}} \in \text{Mor}^{G}(C_{0}(\widehat{H}), C_{0}(\widehat{H}) \otimes^{R} C_{0}(\widehat{H})) \) characterised by

\[
\text{id}_{K(L)} \otimes \Delta_{\widehat{H}} \mathbb{F} = ((\text{id}_{K(L)} \otimes \widehat{j}_{1}) \mathbb{F}) ((\text{id}_{K(L)} \otimes \widehat{j}_{2}) \mathbb{F}).
\]

Moreover, \( \Delta_{\widehat{H}} \) is coassociative: \( (\Delta_{\widehat{H}} \otimes \text{id}_{C_{0}(\widehat{H})}) \circ \Delta_{\widehat{H}} = (\text{id}_{C_{0}(\widehat{H})} \otimes \Delta_{\widehat{H}}) \circ \Delta_{\widehat{H}} \)

and satisfy the cancellation conditions: \( \Delta_{\widehat{H}}(C_{0}(\widehat{H}))j_{1}(C_{0}(\widehat{H})) = C_{0}(\widehat{H}) \otimes^{R} C_{0}(\widehat{H}) \).

The triple \( (C_{0}(\widehat{H}), \Delta_{\widehat{H}}, \widehat{\Gamma}) \) is said to be the braided C*-quantum group (over \( G \)) generated by the braided multiplicative unitary \( \mathbb{F} \) relative to \( (U, \widehat{R}) \).

Proof. See [6] Theorem 5.1. \( \square \)

Remark 2.11. Theorem [20] and Theorem [48] remain valid when \( T \) is replaced by a regular quasitriangular quantum group \( G \), see for example [6][10].
3. DUAL OF $E_q(2)$

The construction of the braided dual of $E_q(2)$ groups in the sense of Theorem 2.8 will be the focus of this section. We begin by constructing the dual braided multiplicative unitary.

Let $q \in \mathbb{C} \setminus \{0\}$ such that $|q| < 1$. The rest of the article makes implicit use of this assumption without explicitly mentioning it.

3.1. The multiplicative unitary $\hat{F}$. Let $\mathcal{H} = \ell^2(\mathbb{Z})$ with an orthonormal basis \( \{ e_i \}_{i \in \mathbb{Z}} \) and identify $\mathcal{L} = \mathcal{H} \otimes \mathcal{H}$. Then, \( \{ e_{i,j} = e_i \otimes e_j \}_{i,j \in \mathbb{Z}} \) is an orthonormal basis for $\mathcal{L}$. Recall that the unitary operator $v$, the invertible normal operator $n$ can be realised concretely as operators on $\mathcal{L}$ by $ve_{ij} = e_{i-1,j}$ and $ne_{ij} = q^{e_{i,j+1}}$, see [10] Equation 1.6 and $P \in \mathcal{U}(\mathcal{L})$ defined by $Pe_{ij} = \zeta^{1-i}e_{ij}$. Also the action of $n^{-1}$ on the basis elements is given by $n^{-1}e_{ij} = q^{-e_{i,j-1}}$. Define the operators $N$ and $b$ on $\mathcal{L}$ by

\[
N e_{i,j} = (i+j)e_{i,j}, \quad b e_{i,j} = q^{\frac{i+j}{2}}e_{i-1,j-1}. \tag{3.1}
\]

Then $N$ is self-adjoint with spectrum $\mathbb{Z}$, $b$ is normal and injective, implying that it is invertible. Moreover, one can easily check that $n^{-1}P = q^{-\frac{1}{2}}b$.

Now, let us refer to the operator $N + 2I$, where $I$ is the identity operator on $\mathcal{L}$, as $\hat{N}$ throughout this article. Define the operators $\hat{X}$ and $\hat{Y}$ on $\mathcal{L} \otimes \mathcal{L}$ as well by

\[
\hat{X} := Pb^{-1}q^{\frac{q^2}{2}} \otimes q^{\frac{q^2}{2}} \, b, \\
\hat{Y} := (v \otimes 1)^{1 \otimes N}.
\]

Then $\hat{Y}$ is a unitary and $\hat{X}$ is a closed operator with spectrum $\text{Sp}(\hat{X})$ contained in $\mathbb{T}^{|q|}$. Additionally, on the basis elements the actions are given by $\hat{X} e_{ij} \otimes e_{k,l} = \zeta^{1-i} q^{1+j+1-i} e_{i+1,j+1} \otimes e_{k-1,l-1}$ and $\hat{Y} e_{ij} \otimes e_{k,l} = e_{i-k,j} \otimes e_{k,l}$. Recall that the manageable braided multiplicative unitary $\mathcal{F}^*$ generating the braided quantum $E(2)$ groups from [10] Theorem 4.6. Then with these operators at our disposal we construct the corresponding dual braided multiplicative unitary in the next theorem.

Also recall the quantum exponential function $F_{|q|}: \mathbb{T}^{|q|} \to \mathbb{T}$ defined in [14] Equation 1.2 by

\[
F_{|q|}(\lambda) = \prod_{k=0}^{\infty} 1 + |q|^{2k+1} \frac{\lambda}{1 + |q|^{2k+1}} \text{ if } k \in \mathbb{T}^{|q|} \setminus \{-|q|^{-2k} \mid k = 0, 1, 2, \ldots\}
\]

\[
\text{otherwise.}
\]

Since $F_{|q|}$ is a unitary multiplier of $C_0(\mathbb{T}^{|q|})$ we observe that $F_{|q|}(\hat{X}) \in \mathcal{B}(\mathcal{L} \otimes \mathcal{L})$.

**Theorem 3.2.** The operator $\mathcal{F}^* := F_{|q|}(\hat{X})^* \hat{Y}$ is a manageable braided multiplicative unitary on $\mathcal{L} \otimes \mathcal{L}$ over $\mathbb{T}$ with respect to $(U, \hat{R})$.

*Proof.* The dual braiding induced by the dual $R$-matrix $\hat{R}$ is given by $\times = \hat{Z} \circ \Sigma$. Equation (2.5) and a computation analogous to [10] Equation 4.2 indicate that $\hat{Z}$ operates on $\mathcal{L} \otimes \mathcal{L}$ by

\[
\hat{Z} e_{ij} \otimes e_{kl} = \zeta^{ij} e_{ij} \otimes e_{kl}.
\]
Again, simple computation yields $\hat{Z}(\nu n^{-1}vP)\hat{Z}^* = Pvn^{-1}v$. Indeed,

\[
\hat{Z}(\nu n^{-1}vP)\hat{Z}^* e_{i,j} \otimes e_{k,l} = \zeta^{-j}\hat{Z}(\nu n^{-1}vP)e_{i,j} \otimes e_{k,l} = \zeta^{-j}\hat{Z}q^i e_{i-1,j+1} \otimes q^{-k+1}e_{k-1,l-1} = \zeta^{-j-i}q^i e_{i-1,j+1} \otimes q^{-k+1}e_{k-1,l-1} = (Pvn \otimes n^{-1}v)e_{i,j} \otimes e_{k,l}.
\]

Since by definition $\hat{Y}$ shifts only the first factor of the basis vectors, $\hat{Z}$ commutes with $\hat{Y}$. Now,

\[
\hat{Y}(Pvn \otimes n^{-1}v)\hat{Y}^* e_{i,j} \otimes e_{k,l} = \hat{Y}(Pvn \otimes n^{-1}v)e_{i+k+l,j} \otimes e_{k,l} = \hat{Y}\zeta^{-j-i}q^{i+k+l}e_{i+k+l-1,j+1} \otimes q^{-k+1}e_{k-1,l-1} = \zeta^{-j-i}q^i e_{i+1,j+1} \otimes q^{-1}e_{k-1,l-1} = (Pv^*n \otimes q^2q^\frac{2}{2})b e_{i,j} \otimes e_{k,l}.
\]

By [10 Theorem 4.6], the braided multiplicative unitary for the braided $E_q(2)$ groups is explicitly given by $F = F_q(vn \otimes n^{-1}vP)\hat{Y}$. Consequently, by combining the previous computations with the definition of $\hat{F}$, we have

\[
\hat{F} = \times_{(P^*F^*F)^*} = \hat{Z}\Sigma^*F^*\Sigma^*\hat{Z}^* = \hat{Y}F_q(vn \otimes n^{-1}vP)^*\hat{Z}^* = \hat{Y}F_q(Pvn \otimes q^{-\frac{2}{2}}b)^* = F_q(Pv^*n \otimes q^2q^\frac{2}{2}b)^*\hat{Y}.
\]

It is now sufficient to show that $v^*n = b^{-1}q^\frac{2}{2}$. Since $v$ is unitary, it is easy to see that $v^*n = b^{-1}q^\frac{2}{2}$. As a result, the equality follows.

By employing a technique similar to that described in [10 Proposition 4.8] for $\lambda = 1$, we see that $\hat{F}$ satisfies the braided pentagon equation given below:

\[
\hat{F}_{23} \hat{F}_{12} = \hat{F}_{12} \hat{F}_{12} \hat{F}_{23} ^* = \hat{F}_{12} \times_{(P^*F^*F)^*} \hat{F}_{12} \times_{(P^*F^*F)^*} \hat{F}_{23} ^*.
\]

Since $F$ is manageable by [10 Theorem 4.6], $\hat{F}$ is manageable by Theorem 2.6. □

3.2. The $C^*$-algebra $C_0(E_q(2))$. Let $C$ be the $C^*$-algebra generated by an element $\xi$ subject to the commutation relation $\xi^*\xi = |q|^{-2}\xi^*\xi$ and $\text{Sp}(\xi) = \mathbb{T}$, Then by definition, $C$ is the closed linear span of the set of all finite linear combinations of the form $\sum_k \Phi^b_k g_k(\xi)$ where $\xi = \Phi^b(\xi)$ is the polar decomposition of $\xi$, $k$ runs over a finite subset of $\mathbb{Z}$ and $g_k \in C_0(\text{Sp}(\xi))$ for all $k \neq 0$ and $g_0 = 0$. Define $\alpha: \mathbb{T} \to \text{Aut}(C)$ by $\alpha_z(f)(t) = f(zt)$. Then the $C^*$-algebra $C_0(E_q(2))$ of continuous functions vanishing at infinity on $E_q(2)$ is isomorphic to the crossed product $C^*$-algebra $C \rtimes \mathbb{T}$.

Let $b$ denote the operator $q^\frac{2}{2}b$ and $\bar{b} = \Phi^b_0 |\bar{b}|$ be its polar decomposition. Then the actions of $\Phi^b_k$ and $|\bar{b}|$ on $L$ is given by $\Phi^b_0 e_{ij} = q^{-j-1} e_{i-1,j-1}$ and $|\bar{b}| e_{ij} = |q|^{-j} e_{ij}$ respectively. Hence, it follows that $\Phi^b_0 |\bar{b}| \Phi^b_0 = |q| |\bar{b}|$. Consequently, we have $\bar{b}^*b = |q|^{-2}\bar{b}^*b$. By the universal property of $C^*$-algebra generated by a finite set of elements, there exists a $^*-$homomorphism

\[
\sum_k \Phi^b_k g_k(\xi) \to \sum_k \Phi^b_k g_k(|\bar{b}|).
\]
Therefore, $\hat{b}$ is affiliated with $C$. Since, $\mathbb{T}$ is commutative, $C \rtimes \mathbb{T}$ contains $C_0(\mathbb{Z})$ as a subalgebra and by definition $N_\eta C_0(\mathbb{Z})$.

We sum up the above observations in the following

**Proposition 3.4.** The $C^*$-algebra $C_0(\mathbb{E}_q(2))$ is generated by $N$ and $\hat{b}$.

Let $\hat{B} := \{ (\omega \otimes \text{id})\hat{F} \mid \omega \in \mathbb{B}(L)_* \}^{\text{CLS}}$. The general theory of manageable multiplicative unitaries ensures that $\hat{B}$ is a $C^*$-algebra. Therefore $\hat{B} \subseteq \mathcal{M}(C_0(\mathbb{E}_q(2)))$.

It is already observed that $v^* n = b^{-1}q^N$. Now, since $F_{|q|} \in \mathcal{M}(C_0(\mathbb{E}_q(2)))$ and the $z$-transform $z_{P \times n} = P v^* z_n$, we obtain $P v^* \eta C_0(\mathbb{E}_q(2))$ and hence $P b^{-1}q^N \otimes q^2 b \eta K(L) \otimes C_0(\mathbb{E}_q(2))$. Consequently $F_{|q|}(P b^{-1}q^N \otimes q^2 b) \in \mathcal{U}(K(L) \otimes C_0(\mathbb{E}_q(2)))$.

Also observe that $\hat{Y}$ is a manageable multiplicative unitary that generates $\mathbb{T}$ as a quantum group. Hence, its dual $\hat{\mathcal{Y}} \in \mathcal{U}(\mathbb{L} \otimes L)$ is also a manageable multiplicative unitary and generates the dual $\mathcal{Z}$ of $\mathbb{T}$ as a quantum group. Therefore, $\hat{\mathcal{Y}} \in \mathcal{U}(K(L) \otimes C_0(\mathbb{Z})) \subset \mathcal{U}(K(L) \otimes C_0(\mathbb{E}_q(2)))$. As a consequence, we have $\hat{F} \in \mathcal{U}(K(L) \otimes C_0(\mathbb{E}_q(2)))$ and therefore,

$$\hat{B} C_0(\mathbb{E}_q(2)) = \{(\omega \otimes \text{id})\hat{F}((1x(L) \otimes \hat{b})) \mid \omega \in \mathbb{B}(L)_*, \hat{b} \in C_0(\mathbb{E}_q(2))\}^{\text{CLS}}$$

$$= \{(\omega \otimes \text{id})\hat{F}(m \otimes \hat{b}) \mid m \in \mathbb{K}(L), \omega \in \mathbb{B}(L)_*, \hat{b} \in C_0(\mathbb{E}_q(2))\}^{\text{CLS}}$$

$$= \{(\omega \otimes \text{id})(m \otimes \hat{b}) \mid m \in \mathbb{K}(L), \omega \in \mathbb{B}(L)_*, \hat{b} \in C_0(\mathbb{E}_q(2))\}^{\text{CLS}}$$

$$= C_0(\mathbb{E}_q(2))$$

**Proposition 3.5.** The $C^*$-algebra $\hat{B}$ coincides with $C_0(\mathbb{E}_q(2))$.

*Proof.* It is enough to show that $N$ and $\hat{b}$ are affiliated with $\hat{B}$. The braided pentagon equation \[X_{23}\hat{F}_{12}X_{12}^* \] of $\hat{F}$ implies that

$$\hat{F}_{12} \hat{F}_{23} \hat{F}_{12}^* \hat{F}_{23} = \hat{X}_{23} \hat{F}_{12}^* \hat{X}_{23}^*.$$ (3.6)

The unitary $\hat{\mathcal{Y}}$ commutes with $\hat{Z}$, and we can infer from the proof of Theorem 3.22 that

$$\hat{Z}((1 \otimes q^2 b)Z^* = P \otimes q^N b.$$ (3.7)

Using this observation we compute

$$X_{23}\hat{F}_{12}X_{12}^* = \Sigma_{23} Z_{23}^* F_{|q|}(P b^{-1} q^N \otimes q^2 b)_{12} \hat{Y}_{12} Z_{23} \Sigma_{23}$$

$$= \hat{Z}_{23} F_{|q|}(P b^{-1} q^N \otimes q^2 b)_{13} \hat{Y}_{13} \hat{Z}_{23}^*$$

$$= F_{|q|}(P b^{-1} q^N \otimes P^* \otimes \hat{q}^2 b)_{13} \hat{Y}_{13}.$$ (3.8)

Let $\hat{F}^\lambda = F_{|q|}(\lambda P b^{-1} q^N \otimes \hat{q}^2 b) \hat{\mathcal{Y}}$ and $S'(|\lambda|) = F_{|q|}(\lambda P b^{-1} q^N \otimes P^* \otimes \hat{q}^2 b) \hat{\mathcal{Y}}$. Then using the above analysis we have,

$$\hat{F}^\lambda \hat{F}_{12}^\lambda \hat{F}_{23} \hat{F}_{12}^{\lambda^*} \hat{F}_{23} = X_{23} \hat{F}_{12} \hat{X}_{12} X_{23} = F_{|q|}(\lambda P b^{-1} q^N \otimes P^* \otimes \hat{q}^2 b) \hat{\mathcal{Y}}_{13} = S'(|\lambda|).$$ (3.8)

It follows that since the expression on the left hand side belongs to $\mathcal{U}(K(L) \otimes K(L) \otimes \hat{B})$, $F_{|q|}(\lambda P b^{-1} q^N \otimes P^* \otimes \hat{q}^2 b) \hat{\mathcal{Y}}_{13}$ must also belong to $\mathcal{U}(K(L) \otimes K(L) \otimes \hat{B})$ for all $\lambda \in \mathbb{Z}$. Now $\hat{\mathcal{Y}} \in \mathcal{U}(L \otimes L)$ and as already observed $F_{|q|}(P b^{-1} q^N \otimes \hat{q}^2 b) \in \mathcal{U}(K(L) \otimes C_0(\mathbb{E}_q(2)))$, $P$ is unitary implies that $P \eta K(L)$. Hence, the expression on the right hand side belongs to $\mathcal{U}(K(L) \otimes K(L) \otimes \hat{B})$ for all $\lambda \in \mathbb{Z}$. The terms on the right hand side of equation (3.8) constitute a strictly continuous family of elements of $\mathcal{U}(K(L) \otimes K(L) \otimes \hat{B})$ due to strict continuity of the map
Thus, we draw the conclusion that
\[ \lambda \to F_{|q|}(\lambda P b^{-1} \eta \overline{\eta} \otimes q \overline{\eta} b) \in \mathcal{M}(\mathcal{L} \otimes \hat{B}). \]

By Proposition 5.2, we see \( P b^{-1} \eta \overline{\eta} \otimes q \overline{\eta} b \in \mathcal{K}(\mathcal{L} \otimes \mathcal{K}(\mathcal{L}) \otimes \hat{B}). \) Hence, \( q \overline{\eta} b \eta \hat{B}. \)

Finally, note that \( S'(0) = \hat{\gamma}. \) Hence, \( \hat{\gamma} \in \mathcal{U}(\mathcal{K}(\mathcal{L}) \otimes \hat{B}). \) Since \( \hat{\gamma} \) generates \( \mathcal{Z} \) as a quantum group, the left slices \( \{ (\omega \otimes \text{id}_{\mathcal{K}(\mathcal{L})}) \hat{\gamma} \} \) for \( \omega \in \mathcal{B}(\mathcal{L}), \) are dense in \( \mathcal{C}_0(\mathcal{Z}). \) Moreover, by definition, \( \text{Sp}(\mathcal{N}) = \mathcal{Z} \) and \( \mathcal{C}_0(\mathcal{Z}) \) is generated by \( N \eta \mathcal{C}_0(\mathcal{Z}). \)

Thus, we draw the conclusion that \( N \eta \hat{B} \) and this completes the proof.

3.3. The Comultiplication map \( \Delta_{\mathcal{E}_n(2)}. \) In this section we provide the quantum group structure to the \( C^* \)-algebra constructed above inside the category \( \mathcal{C}^*\text{alg}(\mathbb{T}). \)

Let us recall that the quantum group \( \mathbb{T} = (C(\mathbb{T}), \Delta_{\mathbb{T}}), \) constructed from the manageable multiplicative unitary \( \mathbb{W} \) on \( \mathcal{H} \otimes \mathcal{H} \) defined by \( \mathbb{W}e_i \otimes e_j = e_i + e_{i+j}, \) see \( \mathbb{T} \) Section 4, can be viewed as a quasitriangular quantum group with respect to the \( R \)-matrix \( R \) that appears in section \( \mathbb{T}. \) Also, we can readily see that \( U = \mathbb{W}_{23} \) acts on \( \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \) as \( L \otimes \mathcal{H} \) by \( U e_i \otimes e_j = e_{i+k} \).

A simple computation reveals that \( \mathbb{U}(N \otimes 1)U^* = N \otimes 1 \) and \( \mathbb{U}(\tilde{b} \otimes 1)U^* = \tilde{b} \otimes z^*. \)

Hence, the map \( \hat{\Gamma}: \mathbb{C}_0(\mathcal{E}_n(2)) \to \mathbb{C}_0(\mathcal{E}_n(2)) \otimes C(\mathbb{T}) \) given by
\[ \hat{\Gamma}(N) = N \otimes 1, \quad \hat{\Gamma}(\tilde{b}) = \tilde{b} \otimes z^* \]
yields a well defined action of \( \mathbb{T} \) on \( \mathbb{C}_0(\mathcal{E}_n(2)). \) Therefore, \( \mathbb{C}_0(\mathcal{E}_n(2)) \) is an object in \( \mathcal{C}^*\text{alg}(\mathbb{T}). \)

Consider the monoidal product \( \boxtimes_R \) on \( \mathcal{C}^*\text{alg}(\mathbb{T}) \) induced by the dual braiding in \( \mathcal{R} \mathcal{P}(\mathbb{T}) \) with respect to the dual \( R \)-matrix \( \hat{R} \) and define the two-fold twisted tensor product
\[ \mathbb{C}_0(\mathcal{E}_n(2)) \boxtimes_R \mathbb{C}_0(\mathcal{E}_n(2)) := (\mathbb{C}_0(\mathcal{E}_n(2)) \otimes 1) \tilde{Z}(1 \otimes \mathbb{C}_0(\mathcal{E}_n(2))) \tilde{Z}^*. \]

The canonical embeddings \( \hat{j}_1, \hat{j}_2 \in \text{Mor}^*(\mathbb{C}_0(\mathcal{E}_n(2)), \mathbb{C}_0(\mathcal{E}_n(2)) \boxtimes_R \mathbb{C}_0(\mathcal{E}_n(2))) \) into the first and second factor of the twisted tensor product are given by:
\[ \hat{j}_1(\tilde{b}) = \tilde{b} \otimes 1, \quad \hat{j}_2(\tilde{b}) = \tilde{Z}(1 \otimes \tilde{b}) \tilde{Z}^* \quad \text{for all } \tilde{b} \in \mathbb{C}_0(\mathcal{E}_n(2)). \]

Then, explicitly, on the distinguished generators \( N \) and \( \tilde{b}, \) the embeddings \( \hat{j}_1 \) and \( \hat{j}_2 \) are expressed as
\[ \hat{j}_1(N) = N \otimes 1, \quad \hat{j}_1(\tilde{b}) = \tilde{b} \otimes 1, \quad \hat{j}_2(N) = 1 \otimes N, \quad \hat{j}_2(\tilde{b}) = P \otimes \tilde{b}. \]

Define \( \Delta_{\mathcal{E}_n(2)}(\tilde{b}) = \hat{\Gamma}(\tilde{b} \otimes 1)\hat{\Gamma}^* \) for all \( \tilde{b} \in \mathbb{C}_0(\mathcal{E}_n(2)). \) Then it is sufficient to compute the action of \( \Delta_{\mathcal{E}_n(2)} \) on the distinguished generators of \( \mathbb{C}_0(\mathcal{E}_n(2)) \): \( N \) and \( \tilde{b} = q \overline{\eta} b. \) We have already observed that \( \text{Sp}(\hat{X}) = \mathbb{C}^{|q|} \) and have \( F_{|q|}(\hat{X}) \in \mathcal{U}(\mathcal{L} \otimes \mathcal{L}). \)

Firstly, we observe that \( \Delta_{\mathcal{E}_n(2)}(N) = \hat{\Gamma}(N \otimes 1)\hat{\Gamma}^* \). Now, \( \tilde{Y}(N \otimes 1)\tilde{Y}^* = N \otimes 1 + 1 \otimes N \), where \( \tilde{Y} \) denotes the closure of the sum of two normal operators. Therefore,
\[ \Delta_{\mathcal{E}_n(2)}(N) = F_{|q|}(\hat{X})^* (N \otimes 1 + 1 \otimes N) F_{|q|}(\hat{X}) \quad \text{(3.9)} \]

The commutation relations \( Nb = b(N - 2I) \) and \( N^{-1}b = b^{-1}(N + 2I) \) together imply that \( N \otimes 1 + 1 \otimes N \) commutes with \( \hat{X}. \) Consequently, the equation \( \text{(3.9)} \) reduces to
\[ \Delta_{\mathcal{E}_n(2)}(N) = N \otimes 1 + 1 \otimes N = \hat{j}_1(N) + \hat{j}_2(N). \]
Secondly, again using the definitions, we have $\hat{\mathcal{Y}}(q^{2} b \otimes 1)\hat{\mathcal{Y}}^{*} = q^{2} b \otimes 1$. Therefore, substituting $\hat{b} = b$ in the formula for $\Delta_{E_{q}(2)}$, we have

$$\Delta_{E_{q}(2)}(\hat{b}) = \hat{\mathcal{Y}}(q^{2} b \otimes 1)\hat{\mathcal{Y}}^{*} = F_{|\varphi|}(\hat{X})^{*}(q^{2} b \otimes 1)F_{|\varphi|}(\hat{X}). \quad (3.10)$$

Let $\mathcal{Y}$ be a normal operator such that the spectrum $\text{Sp}(\mathcal{Y})$ is contained in $\mathbb{T}^{[q]}$. Let $R = \mathcal{Y} \otimes b^{*}(\mathcal{Y})\hat{\mathcal{Y}} \otimes 1$ and $S = \mathcal{Y} \otimes (\mathcal{Y})^{N} P^{*} \otimes b^{*}(\mathcal{Y})\hat{\mathcal{Y}}$. Then,

$$R^{-1}S = (T^{-1} \otimes (\mathcal{Y})^{-}\hat{\mathcal{Y}})\mathcal{Y}^{-1}(\mathcal{Y})^{N} P^{*} \otimes b^{*}(\mathcal{Y})\hat{\mathcal{Y})} \quad = 1 \otimes (\mathcal{Y})^{-}\hat{\mathcal{Y}}(b^{*})^{-1}(\mathcal{Y})^{N} P^{*} \otimes b^{*}(\mathcal{Y})\hat{\mathcal{Y})} \quad = 1 \otimes (\mathcal{Y})^{-}\hat{\mathcal{Y}}(q^{2} b^{-1} q^{N}) P^{*} \otimes b^{*}(\mathcal{Y})\hat{\mathcal{Y})} \quad = 1 \otimes (\mathcal{Y})^{-}\hat{\mathcal{Y}}(b^{-1})^{*} P^{*} \otimes b^{*}(\mathcal{Y})\hat{\mathcal{Y})} \quad = \hat{X}_{23}^{*}.$$ 

In the above computation we use the commutation relation $q^{N} b^{-1} = q^{2} b^{-1} q^{N}$, because $q^{N} b^{-1} e_{i,j} = q^{2} b^{-1} q^{N} e_{i,j}$. Replacing $\hat{b}$ by $\hat{b}^{*}$ in equation (4.10) and applying Theorem 2.7 and Proposition 3.11 we have

$$F_{|\varphi|}(\hat{X})_{23}(T \otimes b^{*}(\mathcal{Y})\hat{\mathcal{Y}} \otimes 1)F_{|\varphi|}(\hat{X})_{23} = \mathcal{Y} \otimes b^{*}(\mathcal{Y})\hat{\mathcal{Y}} \otimes 1 + \mathcal{Y} \otimes (\mathcal{Y})^{N} P^{*} \otimes b^{*}(\mathcal{Y})\hat{\mathcal{Y}).$$

Since $\mathcal{Y}$ is chosen arbitrarily, we get that

$$\Delta_{E_{q}(2)}(\hat{b}^{*}) = b^{*}(\mathcal{Y})\hat{\mathcal{Y}} \otimes 1 + (\mathcal{Y})^{N} P^{*} \otimes b^{*}(\mathcal{Y})\hat{\mathcal{Y})} = \hat{j}_{1}(\hat{b})^{*} + \hat{j}_{1}(q^{N})^{*} \hat{j}_{2}(\hat{b})^{*}. \quad (3.11)$$

Coassociativity and cancellation conditions for $\Delta_{E_{q}(2)}$ follows from Theorem 2.8.

In summary we have,

**Theorem 3.11.** There exists $\Delta_{E_{q}(2)} \in \text{Mor}^{\mathcal{T}}(C_{0}(E_{q}(2)), C_{0}(E_{q}(2)) \boxtimes_{R} C_{0}(E_{q}(2)))$ such that

$$\Delta_{E_{q}(2)}(N) = \hat{j}_{1}(N) + \hat{j}_{2}(N) \quad \text{and} \quad \Delta_{E_{q}(2)}(\hat{b}) = \hat{j}_{1}(\hat{b}) + \hat{j}_{1}(q^{N})^{*} \hat{j}_{2}(\hat{b}).$$

Moreover, $\Delta_{E_{q}(2)}$ satisfies the coassociativity condition:

$$(\text{id}_{C_{0}(E_{q}(2)))} \boxtimes_{R} \Delta_{E_{q}(2)}) \circ \Delta_{E_{q}(2)} = (\Delta_{E_{q}(2))} \boxtimes_{R} \text{id}_{C_{0}(E_{q}(2)))} \circ \Delta_{E_{q}(2)};$$

and the cancellation condition:

$$\hat{j}_{1}(C_{0}(E_{q}(2))) (1_{C_{0}(E_{q}(2)))} \boxtimes_{R} C_{0}(E_{q}(2))) = C_{0}(E_{q}(2)) \boxtimes_{R} C_{0}(E_{q}(2)) = (C_{0}(E_{q}(2)) \boxtimes_{R} 1_{C_{0}(E_{q}(2)))} \hat{j}_{2}(C_{0}(E_{q}(2))) .$$

Finally combining Proposition 3.4 and Theorem 3.11 we obtain the dual of braided quantum E(2) group.

**Theorem 3.12.** The triple $(C_{0}(E_{q}(2)), \Delta_{E_{q}(2)}^{\ast}, \hat{\Gamma})$ is a braided $C^{*}$-quantum group over $\mathcal{T}$ with respect to $(U, \hat{R})$ and the dual of $E_{q}(2)$ is isomorphic to the braided $E_{q}(2)$ over $\mathcal{T}$ with respect to $(U, R)$.
The comultiplication ∆

The \( C \) is a subalgebra of \( T \) the unitary generator of \( C(\mathbb{T}) \) using the definitions of \( \tilde{E} \). Let \( \tilde{E} \) is a manageable multiplicative unitary . Let \( \hat{Y} \) is a projection in the sense of \([5]\), hence explains our notation. Observe that \( \hat{d} = \hat{Y} \), clearly , for \( W \) Moreover,

The \( C^* \)-algebra \( C_0(\mathbb{Q}(2) \times \mathbb{T}) \) is a subalgebra of \( \mathcal{B}(\mathbb{H} \otimes \mathcal{L}) \), where \( j_\mathbb{T}(z) = z \otimes 1 \) and \( j_{\mathbb{Q}(2)}(\hat{b}) = \hat{V}^* (1 \otimes \hat{b}) \hat{V}^* \), \( z \) is the unitary generator of \( C(\mathbb{T}) \) and \( \hat{b} \in C_0(\mathbb{Q}(2)) \subset \mathcal{B}(\mathbb{L}) \). Then a short calculation using the definitions of \( \hat{b}, N, z \) yields

The comultiplication \( \Delta_{\mathbb{Q}(2) \times \mathbb{T}}: C_0(\mathbb{Q}(2) \times \mathbb{T}) \to \mathcal{M}(C_0(\mathbb{Q}(2) \times \mathbb{T}) \otimes C_0(\mathbb{Q}(2) \times \mathbb{T})) \) is given by

Notice that \( \hat{W}_{25} \hat{W}_{26} \) acts trivially on the fifth and sixth tensor factor. Hence, it commutes with \( \hat{V}_{456} \). Also, the commutation relation \( \hat{V}^* (1 \otimes q^{\frac{2}{3}} b) \hat{V} = P^* \otimes q^{\frac{2}{3}} b \) allows us to rewrite equation (4.2) as

Moreover, \( \hat{W}_{14}(z \otimes 1_{H^{\otimes 2}} \otimes 1_{H^{\otimes 2}}) \hat{W}_{14}^* = z \otimes 1_{H^{\otimes 2}} \otimes z \otimes 1_{H^{\otimes 2}} = j_\mathbb{T}(z) \otimes j_\mathbb{T}(z) \). Then clearly, for \( d = j_\mathbb{T}(z) = z \otimes 1 \), we obtain

Now for \( d = j_{\mathbb{Q}(2)}(N) \) we first see that

\[
\hat{W}_{25} \hat{W}_{26} (1 \otimes N \otimes 1_{H^{\otimes 2}}) \hat{W}_{25}^* e_{ij} \otimes e_q \otimes e_m = (1 \otimes N \otimes 1_{H^{\otimes 2}}) e_{ij} \otimes e_q \otimes e_m + e_p \otimes e_{i,j} \otimes (m+n)e_m.
\]
We express $\Delta_{E^7(2)\times T}$ as $\Delta_{E^7(2)\times T}(j_{E^7(2)}(N)) = \Delta^1_{E^7(2)\times T} + \Delta^2_{E^7(2)\times T}$ where

$$\Delta^1_{E^7(2)\times T} = \hat{W}_{14}\hat{W}_{34}F_{[q]}(Pb^{-1}q^{\hat{\Delta}} \otimes P' \otimes q^{\hat{\Delta}} b)_{23456}(1 \otimes N \otimes 1_{\hat{H}^{\otimes 3}})$$

$$F_{[q]}(Pb^{-1}q^{\hat{\Delta}} \otimes P' \otimes q^{\hat{\Delta}} b)_{23456}\hat{W}^{*}_{34}\hat{W}^{*}_{14}. \quad (4.3)$$

$$\Delta^2_{E^7(2)\times T} = \hat{W}_{14}\hat{W}_{34}F_{[q]}(Pb^{-1}q^{\hat{\Delta}} \otimes P' \otimes q^{\hat{\Delta}} b)_{23456}(1_{1_{\hat{H}^{\otimes 3}}} \otimes 1 \otimes N)$$

$$F_{[q]}(Pb^{-1}q^{\hat{\Delta}} \otimes P' \otimes q^{\hat{\Delta}} b)_{23456}\hat{W}^{*}_{34}\hat{W}^{*}_{14}. \quad (4.4)$$

Observe that $Nb^{-1} = b^{-1}(N + 2I)$ and $Nb = b(N - 2I)$. Therefore the above equations reduce to

$$\Delta^1_{E^7(2)\times T} = \hat{W}_{14}\hat{W}_{34}(1 \otimes (N + 2) \otimes 1_{1_{\hat{H}^{\otimes 3}}})\hat{W}^{*}_{34}\hat{W}^{*}_{14} = 1 \otimes N \otimes 1_{1_{\hat{H}^{\otimes 3}}} + 2 \cdot 1_{1_{\hat{H}^{\otimes 3}}},$$

$$\Delta^2_{E^7(2)\times T} = \hat{W}_{14}\hat{W}_{34}(1_{1_{\hat{H}^{\otimes 3}}} \otimes 1 \otimes (N - 2))\hat{W}^{*}_{34}\hat{W}^{*}_{14} = 1_{1_{\hat{H}^{\otimes 3}}} \otimes 1 \otimes N - 2 \cdot 1_{1_{\hat{H}^{\otimes 3}}}.$$ 

Summing up we have,

$$\Delta_{E^7(2)\times T}(j_{E^7(2)}(N)) = j_{E^7(2)}(N) \otimes 1 + 1 \otimes j_{E^7(2)}(N).$$

Now we compute $\Delta_{E^7(2)\times T}(d)$ for $d = j_{E^7(2)}(\vec{b}^{\star})$. We choose to calculate the action of the counit map on $\vec{b}^{\star}$ instead of $\vec{b}$ to avoid some computational hurdles.

By definition $\Delta_{E^7(2)\times T}(j_{E^7(2)}(\vec{b}^{\star})) = \hat{W}(P' \otimes q^{\hat{\Delta}} b \otimes 1_{1_{\hat{H}^{\otimes 3}}})^{*}\hat{W}^{*}$. By analogous computation as above we see that $\hat{W}_{20}\hat{W}_{25}$ commutes with $(P' \otimes q^{\hat{\Delta}} b \otimes 1_{1_{\hat{H}^{\otimes 3}}})^{*}$. Therefore we have

$$\Delta_{E^7(2)\times T}(j_{E^7(2)}(\vec{b}^{\star})) = \hat{W}_{14}\hat{W}_{34}F_{[q]}(Pb^{-1}q^{\hat{\Delta}} \otimes P' \otimes q^{\hat{\Delta}} b)_{23456}(P' \otimes q^{\hat{\Delta}} b \otimes 1_{1_{\hat{H}^{\otimes 3}}})^{*}$$

$$F_{[q]}(Pb^{-1}q^{\hat{\Delta}} \otimes P' \otimes q^{\hat{\Delta}} b)_{23456}\hat{W}^{*}_{34}\hat{W}^{*}_{14}. \quad (4.5)$$

Now consider $R = P^{\star} \otimes b^{\star}(\vec{q})^{\hat{\Delta}} \otimes 1_{1_{\hat{H}^{\otimes 3}}}$ and $S = P^{\star} \otimes (\vec{q})^{N}P^{\star} \otimes P^{\star} \otimes b^{\star}(\vec{q})^{\hat{\Delta}}$. Then

$$R^{-1}S = 1 \otimes (\vec{q})^{\hat{\Delta}}(b^{\star})^{-1}P^{\star} \otimes P^{\star} \otimes b^{\star}(\vec{q})^{\hat{\Delta}}.$$ 

Then again invoking [13] Theorem 2.2 and Theorem 3.1 equation (4.5) becomes

$$\Delta_{E^7(2)\times T}(j_{E^7(2)}(\vec{b}^{\star}))$$

$$= \hat{W}_{14}\hat{W}_{34}(P^{\star} \otimes b^{\star}(\vec{q})^{\hat{\Delta}} \otimes 1_{1_{\hat{H}^{\otimes 3}}} + P^{\star} \otimes (\vec{q})^{N}P^{\star} \otimes P^{\star} \otimes b^{\star}(\vec{q})^{\hat{\Delta}})\hat{W}^{*}_{34}\hat{W}^{*}_{14}. \quad (4.6)$$

We compute each term on the right hand side of the above equation separately. Note that $\hat{W}_{14}\hat{W}_{34}$ effects a change only in the fourth leg, hence it commutes with $P^{\star} \otimes b^{\star}(\vec{q})^{\hat{\Delta}} \otimes 1_{1_{\hat{H}^{\otimes 3}}}$. Therefore the first term remains $P^{\star} \otimes b^{\star}(\vec{q})^{\hat{\Delta}} \otimes 1_{1_{\hat{H}^{\otimes 3}}}$. As for the second term, we have

$$\hat{W}_{14}\hat{W}_{34}(P^{\star} \otimes (\vec{q})^{N}P^{\star} \otimes P^{\star})\hat{W}^{*}_{34}\hat{W}^{*}_{14}e_{p} \otimes e_{ij} \otimes e_{k}$$

$$= \hat{W}_{14}\hat{W}_{34}(P^{\star} \otimes (\vec{q})^{N}P^{\star} \otimes P^{\star})e_{p} \otimes e_{ij} \otimes e_{k-p-j}$$

$$= \hat{W}_{14}\hat{W}_{34}(\zeta^{P}e_{p} \otimes \zeta^{ij}e_{ij} \otimes \zeta^{k-p-j}e_{k-p-j})$$

$$= e_{p} \otimes (\vec{q})^{N}e_{ij} \otimes (\vec{q})^{k}e_{k}$$

$$= (1 \otimes (\vec{q})^{N} \otimes P^{\star})e_{p} \otimes e_{ij} \otimes e_{k}.$$
There exists a unique self-adjoint operator $N$ with an idempotent Hopf $^\ast$-algebra $C$ and $N$ generated by as a closed quantum subgroup of $\hat{\Gamma}$ subject to the commutation relations

We sum up the above calculations in the following theorem yielding the bosonization of the dual braided quantum $E(2)$ group.

**Theorem 4.7.** $C_0(\hat{E}_q(2) \rtimes \hat{\Gamma})$ is the universal $C^\ast$-algebra generated by a unitary $u$, a self-adjoint operator $N'$ with integer spectrum and an operator $b'$ with $Sp(b') = [q]^Z$ subject to the commutation relations

$$uN' = N'u, \quad ub' = \zeta b'u, \quad N'b' = b'(N' - 2I).$$

There exists a unique $\Delta_{E_q(2) \rtimes \hat{\Gamma}^Z} : C_0(\hat{E}_q(2) \rtimes \hat{\Gamma}) \to M(C_0(\hat{E}_q(2) \rtimes \hat{\Gamma})) \otimes C_0(\hat{E}_q(2) \rtimes \hat{\Gamma})$ such that

$$\Delta_{E_q(2) \rtimes \hat{\Gamma}^Z}(u) = u \otimes u,$$

$$\Delta_{E_q(2) \rtimes \hat{\Gamma}^Z}(N') = N' \otimes 1 + 1 \otimes N',$$

$$\Delta_{E_q(2) \rtimes \hat{\Gamma}^Z}(b') = b' \otimes 1 + q^{N'} \otimes b',$$

and $(C_0(\hat{E}_q(2) \rtimes \hat{\Gamma}), \Delta_{E_q(2) \rtimes \hat{\Gamma}^Z})$ is a $C^\ast$-quantum group. Moreover, there exists an idempotent Hopf $^\ast$-homomorphism $g : C_0(\hat{E}_q(2) \rtimes \hat{\Gamma}) \to M(C_0(\hat{E}_q(2) \rtimes \hat{\Gamma}))$ with $g(u) = u$, $g(N') = 0$ and $g(b') = 0$. Its image is the copy of $C(\hat{T})$ generated by $u$ as a closed quantum subgroup of $\hat{E}_q(2) \rtimes \hat{\Gamma}$ and its kernel is the copy of $C_0(\hat{E}_q(2))$ generated by $N', b'$ as the braided $E(2)$ group over $\mathbb{T}$.

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