THE LIOUVILLE PARAMETRIZATION OF A TRIAXIAL ELLIPSOID

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Abstract. In this article we will construct the Liouville parametrization of the triaxial ellipsoid. In the literature quadrics are given as examples of Liouville surfaces, yet no one gives such a parametrization.

1. Introduction

In the literature that you will find at the end of this article (see [2], [3], [5]), the authors describe how to map a triaxial ellipsoid conformally to a plane. The best paper (to my knowledge) on this matter is [3] because it actually computes (making use of elliptic integrals) the integrals already given by Jacobi in his “Lectures on Dynamics”. In this article we want to go in the opposite direction and map a plane rectangle conformally to a triaxial ellipsoid in such a way that the map has a Liouville line element. The result can be seen in the right image of figure 1.

Figure 1. Standard curvature line (left image) and Liouville (right image) parametrization of a triaxial ellipsoid

2. Standard curvature line parametrization of the triaxial ellipsoid

We will start here with the standard curvature line parametrization of the triaxial ellipsoid with semi-axes $0 < c < b < a$:

$\text{Ellipsoid}(u, v) = \left( \frac{a^2(u - a^2)(v - b^2)(a^2 - c^2)}{(a^2 - b^2)(a^2 - c^2)} \right)^{1/2} \cdot \left( \frac{b^2(b^2 - u)(b^2 - v)}{(b^2 - c^2)(b^2 - a^2)} \right)^{1/2} \cdot \left( \frac{c^2(c^2 - u)(c^2 - v)}{(c^2 - a^2)(c^2 - b^2)} \right)^{1/2}$

where $0 < c^2 < v < b^2 < u < a^2$ (see left image of figure 1).

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The coefficients of the first fundamental form are computed as follows:

\[ g_{11}(u, v) = \langle \partial_u \text{Ellipsoid}(u, v), \partial_u \text{Ellipsoid}(u, v) \rangle = \frac{1}{4} (u - v) f(u) \]
\[ g_{12}(u, v) = \langle \partial_u \text{Ellipsoid}(u, v), \partial_v \text{Ellipsoid}(u, v) \rangle = g_{21}(u, v) = 0 \]
\[ g_{22}(u, v) = \langle \partial_v \text{Ellipsoid}(u, v), \partial_v \text{Ellipsoid}(u, v) \rangle = \frac{1}{4} (u - v)(-f(v)) \]

with the function \( f \) defined as:

\[ f(t) = \frac{t}{(a^2 - t)(b^2 - t)(c^2 - t)} \]

The line element of the ellipsoid is:

\[ ds^2 = g_{11}(u, v) du^2 + g_{22}(u, v) dv^2 = \frac{1}{4} (u - v)(f(u)du^2 - f(v)dv^2) \] (2.1)

3. Conformal map from ellipsoid to plane

What we want to achieve is the following Liouville form of this line element 2.1:

\[ ds^2 = \frac{1}{4} (U(x) - V(y))(dx^2 + dy^2) \] (3.1)

If formulas 2.1 and 3.1 are to be the same we must have:

\[ dx = \sqrt{+f(u)} du \quad \text{and} \quad dy = \sqrt{-f(v)} dv \]

By integrating we get formulas corresponding to (7) and (8) from [3]:

\[ X(u) = \int_{\varphi_1}^{u} \sqrt{+f(t)} dt = F_1(u) - F_1(b^2) = F_1(u) \]
\[ Y(v) = \int_{\varphi_2}^{v} \sqrt{-f(t)} dt = F_2(v) - F_2(c^2) = F_2(v) \]

with

\[ F_1(t) = \frac{2b^2 i}{c \sqrt{a^2 - b^2}} \Pi(n_1; \varphi_1(t)|m_1) \]
\[ F_2(t) = \frac{2c^2}{b \sqrt{a^2 - c^2}} \Pi(n_2; \varphi_2(t)|m_2) \]

where \( i = \sqrt{-1} \) and

\[ n_1 = 1 - \frac{b^2}{c^2} \quad \varphi_1(t) = \arcsin \left( -ic \sqrt{\frac{t - b^2}{(b^2 - c^2)t}} \right) \quad m_1 = \frac{a^2}{c^2} (c^2 - b^2) \]
\[ n_2 = 1 - \frac{c^2}{b^2} \quad \varphi_2(t) = \arcsin \left( i \sqrt{\frac{t - c^2}{(b^2 - c^2)t}} \right) \quad m_2 = \frac{a^2}{b^2} (b^2 - c^2) \]

and the incomplete elliptic integral of the third kind is defined as follows:

\[ \Pi(n; \varphi|m) = \int_{0}^{\varphi} \frac{d\theta}{(1 - n \sin^2 \theta) \sqrt{1 - m \sin^2 \theta}} \]
4. Liouville map from plane to ellipsoid

We are interested in the functions $U(x)$ and $V(y)$. But we have $X(u)$ and $Y(v)$, which cannot be inverted easily. We have two alternatives:

(1) The first alternative is to define a generalized Jacobi amplitude $\text{am}(n; z|m)$ as inverse function of the elliptic integral of the third kind. That means

$$z = \Pi(n; \varphi|m)$$

$$\text{am}(n; z|m) = \varphi$$

The Jacobi amplitude as special case can be expressed in terms of this generalized Jacobi amplitude as $\text{am}(z|m) = \text{am}(0; z|m)$. With the generalized Jacobi amplitude we can invert the elliptic integrals of the third kind and get:

$$U(x) = \frac{b^2}{1 - n_1 \sin^2 \left( \text{am} \left( n_1; \frac{b \sqrt{a^2 - b^2}}{2b} \right) | m_1 \right)}$$

$$V(y) = \frac{c^2}{1 - n_2 \sin^2 \left( \text{am} \left( n_2; \frac{a \sqrt{b^2 - c^2}}{2c} \right) | m_2 \right)}$$

We can introduce the generalized Jacobi elliptic function $\text{sn}(n; z|m) = \sin(\text{am}(n; z|m))$ and then we have:

$$U(x) = \frac{b^2}{1 - n_1 \sin^2 \left( \text{am} \left( n_1; \frac{b \sqrt{a^2 - b^2}}{2b} \right) | m_1 \right)}$$

$$V(y) = \frac{c^2}{1 - n_2 \sin^2 \left( \text{am} \left( n_2; \frac{a \sqrt{b^2 - c^2}}{2c} \right) | m_2 \right)}$$

(2) The other alternative is to use a series representation for $X(u)$ and $Y(v)$ and then compute the reverse/inverse series, giving a series representation for $U(x)$ and $V(y)$. We first expand $X(u)$ in a series about the point $u_0 = b^2$ and $Y(v)$ in a series about the point $v_0 = c^2$:

$$X(u) = \sum_{k=0}^{\infty} A_{2k+1} \left( \frac{\sqrt{u - b^2}}{\sqrt{(a^2 - b^2) (b^2 - c^2)}} \right)^{2k+1}$$

$$Y(v) = \sum_{k=0}^{\infty} B_{2k+1} \left( \frac{\sqrt{v - c^2}}{\sqrt{(a^2 - c^2) (b^2 - c^2)}} \right)^{2k+1}$$

The first three coefficients are:

$$A_1 = 2b,$$

$$A_3 = \frac{b^4 - a^2 c^2}{3b},$$

$$A_5 = \frac{-a^4 c^4 + 4a^4 b^2 c^2 - 10a^2 b^4 c^2 + 4a^2 b^2 c^4 + 3b^8}{20b^3},$$

$$B_1 = 2c,$$

$$B_3 = \frac{-c^4 - a^2 b^2}{3c},$$

$$B_5 = \frac{-a^4 b^4 + 4a^4 b^2 c^2 + 4a^2 b^4 c^2 - 10a^2 b^2 c^4 + 3c^8}{20c^3}.$$
By computing the reverse/inverse series we get:

\[ U(x) = b^2 + (a^2 - b^2) (b^2 - c^2) \sum_{k=1}^{\infty} C_{2k} x^{2k}, \]

\[ V(y) = c^2 + (c^2 - a^2) (c^2 - b^2) \sum_{k=1}^{\infty} D_{2k} y^{2k} \]

where the first three coefficients are:

\[ C_2 = \frac{1}{4b^2}, \quad D_2 = \frac{1}{4c^2} \]

\[ C_4 = \frac{b^4 - a^2c^2}{48b^6}, \quad D_4 = \frac{c^4 - a^2b^2}{48c^6} \]

\[ C_6 = \frac{11a^4c^4 - 9a^4b^2c^2 - 9a^2b^4c^2 + 5a^2b^4c^2 + 2b^6}{2880b^{10}}, \]

\[ D_6 = \frac{11a^4b^4 - 9a^4b^2c^2 - 9a^2b^4c^2 + 5a^2b^4c^2 + 2c^6}{2880c^{10}} \]

Then the Liouville parametrization of the ellipsoid is given by:

\[ \text{Ellipsoid}(U(x), V(y)) \]

where \( 0 = X(b^2) < x < X(a^2) \) and \( 0 = Y(c^2) < y < Y(b^2) \).

5. Differential equations

If we plug \( u = U(x) \) and \( v = V(y) \) in the equation [2.1] of the line element of the ellipsoid we get:

\[ ds^2 = \frac{1}{4} (U(x) - V(y)) \left( f(U(x)) \left( \frac{dU(x)}{dx} \right)^2 + f(V(y)) \left( \frac{dV(y)}{dy} \right)^2 \right) \]

Comparing this formula with [3.1] we see that the functions \( U(x) \) and \( V(y) \) satisfy the following differential equations:

\[ f(U(x)) \left( \frac{dU(x)}{dx} \right)^2 = +1 \quad \text{and} \quad f(V(y)) \left( \frac{dV(y)}{dy} \right)^2 = -1 \]

6. Remark about the figure

Because we don’t have the complete series \( U(x) \) and \( V(y) \) (but only an approximation, with a few terms) we need another method for drawing a quite good figure of the Liouville ellipsoid for \( a = 3, b = 2, c = 1 \):

1. Collect the points \( (X(u_k), y_k) \) at the values \( b^2 = u_0 < u_1 < \ldots < u_k = (n - k)u_n + ku_n < \ldots < u_n = a^2 \).
2. Interpolate these points with some smooth function (possibly piecewise defined), giving a good approximation \( \tilde{U}(x) \approx U(x) \).
3. Do the same with the points \( (Y(u_k), v_k) \) and get \( \tilde{V}(y) \approx V(y) \).
4. Draw the figure with the parametrization \( \text{Ellipsoid}(\tilde{U}(x), \tilde{V}(y)) \) for \( 0 = X(b^2) < x < X(a^2) \) and \( 0 = Y(c^2) < y < Y(b^2) \).

One could also try to numerically invert the functions \( X(u) \) and \( Y(v) \) as described in [1] to get approximations to \( U(x) \) and \( V(y) \).
Look at the series representations of the (haversed sine)/haversine and inverse haversine functions:

\[
\text{hav}(z) = \sin^2\left(\frac{z}{2}\right) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2(2k)!} z^{2k}
\]

\[
= \frac{z^2}{4} - \frac{z^4}{48} + \frac{z^6}{1440} - \frac{z^8}{80640} + \frac{z^{10}}{7257600} + \ldots
\]

\[
\text{hav}^{-1}(z) = 2 \arcsin\left(\sqrt{\frac{z}{2}}\right) = \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k-1}(2k+1)} (\sqrt{z})^{2k+1}
\]

\[
= 2\sqrt{z} + \frac{z^{3/2}}{3} + \frac{z^{5/2}}{20} + \frac{5z^{7/2}}{56} + \frac{35z^{9/2}}{576} + \ldots
\]

These coefficients/numbers also appear in the series of \(U(x), V(y)\) and \(X(u), Y(v)\). The series for \(U(x)\) and \(V(y)\) can be written now as:

\[
U(x) = b^2 + (a^2 - b^2) (b^2 - c^2) \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \gamma_{2k}}{2(2k)! b^{2(2k-1)} x^{2k}}
\]

\[
V(y) = c^2 + (c^2 - a^2) (c^2 - b^2) \sum_{k=1}^{\infty} \frac{\delta_{2k}}{2(2k)! c^{2(2k-1)} y^{2k}}
\]

where the first three coefficients are:

\[
\gamma_2 = 1, \quad \delta_2 = 1
\]

\[
\gamma_4 = b^4 - a^2 c^2, \quad \delta_4 = c^4 - a^2 b^2
\]

\[
\gamma_6 = \frac{11a^4 c_4 - 9a^4 b^2 c^2 - 9a^2 b^2 c^4 + 5a^2 b^4 c^2 + 2b^6}{2}, \quad \delta_6 = \frac{11a^4 b^4 c^2 - 9a^2 b^4 c^2 + 5a^2 b^4 c^4 + 2b^6}{2}
\]

The series representations for \(X(u)\) and \(Y(v)\) are as follows:

\[
X(u) = \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k-1}(2k+1)b^{2k-1}} \left(\frac{\sqrt{u - b^2}}{\sqrt{(a^2 - b^2)(b^2 - c^2)}}\right)^{2k+1}
\]

\[
Y(v) = \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k-1}(2k+1)c^{2k-1}} \left(\frac{\sqrt{v - c^2}}{\sqrt{(a^2 - c^2)(b^2 - c^2)}}\right)^{2k+1}
\]

The first three coefficients are:

\[
\alpha_1 = 1, \quad \beta_1 = 1
\]

\[
\alpha_3 = b^4 - a^2 c^2, \quad \beta_3 = c^4 - a^2 b^2
\]

\[
\alpha_5 = \frac{-a^4 c^4 + 4a^4 b^2 c^2 - 10a^2 b^4 c^2 + 4a^2 b^2 c^4 + 3b^6}{3}, \quad \beta_5 = \frac{-a^4 b^4 c^2 + 4a^4 b^2 c^2 + 4a^2 b^4 c^2 - 10a^2 b^2 c^4 + 3c^8}{3}
\]

It is possible to calculate the first coefficients \(\alpha_{2k+1}, \beta_{2k+1}, \gamma_{2k}\) and \(\delta_{2k}\) of the series expansions of \(X(u), Y(v), U(x)\) and \(V(y)\). But I have not been able to get the closed general form of these coefficients. This is an open problem and I would like to hear from you, if you make progress on it.
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