Nontrivial solution for Klein-Gordon equation coupled with Born-Infeld theory with critical growth

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Abstract

In this paper, we study the following system

\[
\begin{aligned}
-\Delta u + V(x)u - (2\omega + \phi)\phi u &= \lambda f(u) + |u|^4 u, \quad \text{in } \mathbb{R}^3, \\
\Delta \phi + \beta \Delta_4 \phi &= 4\pi(\omega + \phi)u^2, \quad \text{in } \mathbb{R}^3,
\end{aligned}
\]

where \( f(u) \) without any growth and Ambrosetti-Rabinowitz conditions. We use cut-off function and Moser iteration to obtain the existence of nontrivial solution. Finally, as a by-product of our approaches, we get the same result for Klein-Gordon-Maxwell system.

Key words: Klein-Gordon equation · Born-Infeld theory · Moser iteration · Mountain pass theorem

1 Introduction

This paper studies the Klein-Gordon equation coupled with Born-Infeld theory with critical growth

\[
\begin{aligned}
-\Delta u + V(x)u - (2\omega + \phi)\phi u &= \lambda f(u) + |u|^4 u, \quad \text{in } \mathbb{R}^3, \\
\Delta \phi + \beta \Delta_4 \phi &= 4\pi(\omega + \phi)u^2, \quad \text{in } \mathbb{R}^3,
\end{aligned}
\]

where \( \omega > 0 \) is a constant, \( \lambda > 0 \) is a positive parameter. Klein-Gordon equation can be used to develop the theory of electrically charged fields (see \[13\]) and study the interaction with an assigned electromagnetic field (see \[11\]). The Born-Infeld (BI) electromagnetic theory \[2, 3\] was originally proposed as a nonlinear correction of the Maxwell theory in order to overcome the problem of infiniteness in the classical electrodynamics of point particles(see \[14\]). Klein-Gordon equation

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coupled with Born-Infeld theory system has attracted many theoretic physicists. For more physical
applications, please refer to reference [16, 25] and the references therein.

In the past decades, many people have studied this system through using variational methods,
and have also obtained existence of nontrivial solutions under different assumptions. Let us recall
some previous results which give an inspiration to the present research.

The first result is due to d’Avenia and Pisani, in which the existence of infinitely many radially
symmetric solution for the following form

\[
\begin{cases}
-\Delta u + [m^2 - (\omega + \phi)^2]u = |u|^{p-2}u, & \text{in } \mathbb{R}^3, \\
\Delta \phi + \beta \Delta_4 \phi = 4\pi(\omega + \phi)u^2, & \text{in } \mathbb{R}^3,
\end{cases}
\]  

(1.2)

was proved when \(4 < p < 6\) and \(|\omega| < |m_0|\) in [12]. Mugnai [16] get the same result when
\(2 < p \leq 4\) and \(0 < \omega < \sqrt{\frac{1}{2}p - 1}|m|\). Afterwards, Wang [21] use Pohožaev identity to improve literature [12],
[16] and obtains the solitary wave solution when the one of the following conditions is satisfied

(i) \(3 < p < 6\) and \(m > \omega > 0\),

(ii) \(2 < p \leq 3\) and \((p-2)(4-p)m^2 > \omega^2 > 0\).

Yu [25] get the existence of the least-action solitary wave in both bounded smooth case and \(\mathbb{R}^3\) case. Moreover, replacing \(|u|^{p-2}u\) by \(|u|^{p-2}u + h(x)\), Chen and Li in [8] get the existence of multiple
solution if one of the following condition holds

(i) \(4 < p < 6\) and \(|m| > \omega\),

(ii) \(2 < p \leq 4\) and \(\sqrt{\frac{1}{2}p - 1}|m| > \omega\).

Later, Chen and Song [9] studied the following Klein-Gordon equation with concave and convex
nonlinearities coupled with Born-Infeld theory

\[
\begin{cases}
-\Delta u + V(x)u - (2\omega + \phi)\phi u = \lambda k(x)|u|^{q-2}u + g(x)|u|^{p-2}u, & \text{in } \mathbb{R}^3, \\
\Delta \phi + \beta \Delta_4 \phi = 4\pi(\omega + \phi)u^2, & \text{in } \mathbb{R}^3.
\end{cases}
\]  

(1.3)

Under some appropriate assumptions on \(V(x)\), \(\lambda\), \(k(x)\) and \(g(x)\), the obtain the existence of multiple nontrivial solutions when \(1 < q < 2 < p < 6\). Recently, for general potential \(V(x)\) and \(|u|^{p-2}u\) by
a continuous nonlinearity \(f(x, u)\) with polynomial growth, Wen and Tang [22] obtained infinitely
many solutions and least energy solutions, Che and Chen [7] use genus theory to obtain nontrivial
solutions.

We know that a large number of predecessors have studied the problem of subcritical growth like
the above papers. Furthermore, when the nonlinearity term is accompanied by critical growth, it is
one of the most dramatic cases of loss of compactness. To my best knowledge, there is only one work
about the Klein-Gordon-Born-Infeld system with critical growth. Teng and Zhang [19] investigated the
following system

\[
\begin{cases}
-\Delta u + [m^2 - (\omega + \phi)^2]u = |u|^{p-2}u + |u|^{2^* - 2}u, & \text{in } \mathbb{R}^3, \\
\Delta \phi + \beta \Delta_4 \phi = 4\pi(\omega + \phi)u^2, & \text{in } \mathbb{R}^3.
\end{cases}
\]  

(1.4)
They obtained it has at least a nontrivial solution when \(4 < p < 6\) and \(m < \omega\).

Motivated by the aforementioned works, in this paper, we will use some new tricks to generalize the above results to problem (1.1) under the following conditions:

**\((V_1)\)** \(V \in C^1(\mathbb{R}^3, \mathbb{R})\) and there is a \(V_0 > 0\) such that \(V(x) \geq V_0\) for all \(x \in \mathbb{R}^3\).

**\((V_2)\)** \(V(x) \to \infty\) as \(|x| \to \infty\).

**\((f_1)\)** \(f \in C(\mathbb{R})\), \(\lim_{u \to 0} \frac{f(u)}{u} = 0\).

**\((f_2)\)** \(\lim_{|u| \to \infty} \frac{f(u)}{u} = +\infty\).

Our first result is as follows.

**Theorem 1.1.** Assume that \((V_1) - (V_2)\) and \((f_1) - (f_2)\) hold. then there exists a constant \(\lambda^*_1 \geq 0\) such that, for any \(\lambda \in (0, \lambda^*_1)\), system (1.1) has a nontrivial solution.

**Remark 1.2.** We all know that if the nonlinear term is \(|u|^4u\), we can use the Pohožaev identity and the classical variational method to know whether the system has no nontrivial solutions. Therefore, when studying the nonlinear term is a critical growth case, it is usually to add a high energy lower-order perturbation term like \([4, 19, 20]\). By comparison with the above papers, the result of this paper is that \(\lambda\) is small enough, that is, the lower-order perturbation is a lower energy perturbation.

**Remark 1.3.**

(i) The condition \((V_2)\) was first introduced by P.H.Rabinowitz in [17] to overcome the lack of compactness.

(ii) It is worth noting that in this paper we did not require any growth conditions and the Ambrosetti-Rabinowitz condition, and the function \(f\) can also be sign-changing.

(iii) There are many functions that can satisfy the condition \((f_1) - (f_2)\), the most typical example is \(f(t) = |t|^{p-2}t, p > 6\). Moreover, our result is valid for general supercritical nonlinearity.

We emphasize that our result requires no growth conditions. To prove the existence of nontrivial solution, we adapt a similar argument as in [15, 24]. Here we briefly explain the process. Firstly, we make a suitable cut-off function to replace \(f(u)\) in problem (1.1), so we can get a new system. Secondly, we prove the new system have nontrivial solution. Finally we use the Moser iteration to obtain the existence of nontrivial solution to original Klein-Gordon equation coupled with Born-Infeld theory.

In the second part of this paper, It is worthy of our special attention that when \(\beta = 0\), a small modification to problem (1.1) will become a Klein-Gordon-Maxwell system with critical growth, namely:

\[
\begin{cases}
-\Delta u + V(x)u - (2\omega + \phi)\phi u = \lambda f(u) + |u|^4u, & \text{in } \mathbb{R}^3, \\
\Delta \phi = (\omega + \phi)u^2, & \text{in } \mathbb{R}^3,
\end{cases}
\]  
(1.5)
which has been extensively studied by many authors. A pioneer work is due to Cassani \cite{6} who considered the following critical Klein-Gordon-Maxwell system:

\[
\begin{align*}
-\Delta u + [m^2 - (\omega + \phi^2)]u &= \lambda |u|^{p-2}u + |u|^4u, \quad \text{in } \mathbb{R}^N, \\
\Delta \phi &= (\omega + \phi)u^2, \quad \text{in } \mathbb{R}^N.
\end{align*}
\]

where \( \lambda > 0, \ 2 < p < 6 \) and \( 0 < \omega < m \). When \( N = 3 \) he obtained the existence of a radially symmetric solution for any \( \lambda > 0 \) if \( p \in (4, 6) \) and for \( \lambda \) is sufficiently large if \( p = 4 \). Afterwards C.Carriao, L.Cunha and H.Miyagaki \cite{4} complement the result of \cite{6} and also extend it in higher dimensions. They obtained the same result provided one of those conditions satisfies

(i) \( N = 4 \) and \( N \geq 6 \) for \( 2 < p < 2^* \) and \( |m| > \omega \) if \( \lambda > 0 \);

(ii) \( N = 5 \) and either \( 2 < p < \frac{8}{3} \) if \( \lambda > 0 \) or \( \frac{8}{3} \leq p < 2^* \) if \( \lambda \) is sufficiently large;

(iii) \( N = 3 \) and either \( 4 < p < 2^* \) if \( \lambda > 0 \) or \( 2 < p \leq 4 \) if \( \lambda \) is sufficiently large.

Later, when \( N = 3 \) Wang \cite{20} improved the result of \cite{4, 6} to the case when one of the following holds:

(i) \( 4 < p < 6, \ 0 < \omega < m \) and \( \lambda > 0 \);

(ii) \( 3 < p \leq 4, \ 0 < \omega < m \) and \( \lambda \) is sufficiently large;

(iii) \( 2 < p \leq 3, \ 0 < \omega < \sqrt{(p-2)(4-p)m} \) and \( \lambda \) is sufficiently large.

In recent paper \cite{10}, Chen uses some analytical skills and variational method that is different from \cite{20} to get the same result of it. The authors of \cite{4} have also studied that for problem (1.5), \( N = 3, \ V(x) \) is a periodic function and \( f(u) = |u|^{p-2}u \) in \cite{5}. They use the minimization of the corresponding Euler-Lagrange functional on the Nehari manifold and the Brézis and Nirenberg technique to get a positive ground state solution for each \( \lambda > 0 \) if \( p \in (4, 6) \) and for \( \lambda \) sufficiently large if \( p \in (2, 4) \).

Moreover when the potential well is steep, namely

\[
\begin{align*}
-\Delta u + \mu V(x)u - (2\omega + \phi)\phi u &= \lambda f(x, u) + |u|^4u, \quad \text{in } \mathbb{R}^3, \\
\Delta \phi &= (\omega + \phi)u^2, \quad \text{in } \mathbb{R}^3.
\end{align*}
\]

where \( \mu, \lambda \) are positive parameters and \( \omega > 0 \), there exist \( \hat{\mu}_0, \hat{\lambda}_0 > 0 \) such that for \( \mu > \hat{\mu}_0 \) and \( \lambda > \hat{\lambda}_0 \) problem (1.6) admits a nontrivial solution has been proved by Zhang in \cite{26}. At the same time, he also obtained a nontrivial solution when the potential well may be not steep. Instead of the expression "\( \lambda \) sufficiently large" in the above existing works, Tang, Wen and Chen \cite{18} give a certain range \( \lambda \geq \lambda_0 \) which admits a ground state solution when \( V \) is positive and periodic.

Similarly to the method of Theorem 1.1, we can also get a nontrivial solution. Compared with the hypothesis of subcritical perturbation in the above article, in this paper the perturbation term \( f(u) \) can be not only a subcritical perturbation but also a supercritical perturbation. What’s more the restriction on \( \lambda \) is no longer sufficiently large or greater than a certain number, we can only require \( \lambda \in (0, \lambda_2^* \) where \( \lambda_2^* \geq 0 \). Our second result is as follows.
Theorem 1.4. Assume that \( (V_1) \) – \( (V_2) \) and \( (f_1) \) – \( (f_2) \) hold. then there exists a constant \( \lambda_2^* \geq 0 \) such that, for any \( \lambda \in (0, \lambda_2^*) \), system \( (1.5) \) has a nontrivial solution.

Remark 1.5. We underline that the existence of nontrivial solution for problem \( (1.5) \) it has been proved by above papers with a different approach in this paper. However, it is interesting that we do not need \( \lambda \) is sufficiently large or greater than a certain number.

This paper is organized as follows. In Section 2, we give some preliminary lemmas. In section 3, we prove Theorems [1.1] In section 4, we prove Theorems [1.4]

2 Preliminaries

In this section we explain the notations and some auxiliary lemmas which are useful later.

\( H^1(\mathbb{R}^3) \) denotes the usual Sobolev space equipped with the standard norm.

\( L^q(\mathbb{R}^3), \ell \in [1, +\infty) \) denotes the Lebesgue space with the norm \( |u|_\ell = (\int_{\mathbb{R}^3}|u|^{\ell}\,dx)^{\frac{1}{\ell}} \).

Under \( (V_1) \) and \( (V_2) \), we define the Hilbert space

\[
E = \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2\,dx < \infty \right\},
\]

with respect the norm

\[
||u|| = \left( \int_{\mathbb{R}^3} \left( |\nabla u|^2 + V(x)u^2 \right)\,dx \right)^{\frac{1}{2}}.
\]

Then, the embedding \( E \hookrightarrow H^1(\mathbb{R}^3) \) is continuous. The embedding from \( E \) into \( L^q(\mathbb{R}^3) \) is compact for \( q \in [2, 6) \) and its detailed proof process can be seen in Lemma 3.4 in [28].

Denote by \( D(\mathbb{R}^3) \) the completion of \( C_0^\infty(\mathbb{R}^3) \) with respect to the norm

\[
\|\phi\|_{D(\mathbb{R}^3)} = \left( \int_{\mathbb{R}^3} |\nabla \phi|^2\,dx \right)^{\frac{1}{2}} + \left( \int_{\mathbb{R}^3} |\phi|^4\,dx \right)^{\frac{1}{4}}.
\]

It is easy to know that \( D(\mathbb{R}^3) \) is continuously embedded in \( D^{1,2}(\mathbb{R}^3) \), where \( D^{1,2}(\mathbb{R}^3) \) is the completion of \( C_0^\infty(\mathbb{R}^3) \) with respect to the norm \( \|\phi\|_{D^{1,2}(\mathbb{R}^3)} = \left( \int_{\mathbb{R}^3} |\nabla \phi|^2\,dx \right)^{\frac{1}{2}} \). Moreover, \( D^{1,2}(\mathbb{R}^3) \) is continuously embedded in \( L^6(\mathbb{R}^3) \) by Sobolev inequality and \( D(\mathbb{R}^3) \) is continuously embedded in \( L^\infty(\mathbb{R}^3) \).

\( C_1, C_2, \cdots \) denote positive constant possibly different in different places.

Indeed, solutions of \( (1.1) \) are critical point of functional \( G_\lambda : E(\mathbb{R}^3) \times D(\mathbb{R}^3) \rightarrow \mathbb{R}, \) defined by

\[
G_\lambda(u, \phi) = \frac{1}{2} \int_{\mathbb{R}^3} \left( |\nabla u|^2 + V(x)u^2 - (2\omega + \phi)u^2 \right)\,dx - \frac{1}{8\pi} \int_{\mathbb{R}^3} |\nabla \phi|^2\,dx
- \frac{\beta}{16\pi} \int_{\mathbb{R}^3} |\nabla \phi|^4\,dx - \int_{\mathbb{R}^3} \left( \lambda F(u) + \frac{1}{6}|u|^6 \right)\,dx.
\]

Due to the strong indefiniteness of functional \( (2.1) \), we use the reduction method which can reduces the study of \( G_\lambda(u, \phi) \) to study a new functional \( I_\lambda(u) \) as in \([1]\).

We state some properties of the second equation of problem \( (1.1) \).
Lemma 2.1. For any $u \in H^1(\mathbb{R}^3)$, we have:

(i) there exists a unique $\phi_u \in D(\mathbb{R}^3)$ which solves the second equation of problem (1.1).

(ii) in the set $\{ X : u(x) \neq 0 \}$, we have $-\omega \leq \phi_u \leq 0$.

(iii) $\|\phi_u\|_D \leq C \|u\|^2$ and $\int_{\mathbb{R}^3} |\phi_u|^2 dx \leq C \|u\|_{12}^4$.

Proof. (i) is proved in Lemma 3 of [12] and (ii) can be found in Lemma 2.1 of [16].

\[
\int_{\mathbb{R}^3} |\nabla \phi_u|^2 dx + \beta \int_{\mathbb{R}^3} |\nabla \phi_u|^4 dx = - \int_{\mathbb{R}^3} 4\pi \omega \phi_u u^2 dx - \int_{\mathbb{R}^3} 4\pi \phi_u^2 u^2 dx \\
\leq 4\pi \omega \int_{\mathbb{R}^3} |\phi_u|^2 dx \\
\leq 4\pi \omega \|\phi_u\|_D \|u\|_{12}^2 .
\]

we can get $\|\phi_u\|_D \leq C \|u\|^2$ and $\int_{\mathbb{R}^3} |\phi_u|^2 dx \leq C \|u\|_{12}^4$. \qed

From the second equation in (1.1) and Lemma 2.1, we get

\[
\frac{1}{4\pi} \int_{\mathbb{R}^3} |\nabla \phi_u|^2 dx + \frac{\beta}{4\pi} \int_{\mathbb{R}^3} |\nabla \phi_u|^4 dx = - \int_{\mathbb{R}^3} (\omega \phi_u + \phi_u^3) u^2 dx .
\]

Consider the functional $I_\lambda(u) : E \to \mathbb{R}$ defined by $I_\lambda(u) = G_\lambda(u, \phi_u)$ and combine (2.3), we obtain

\[
I_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x) u^2 - (2\omega + \phi_u) \phi_u u^2) dx - \frac{1}{8\pi} \int_{\mathbb{R}^3} |\nabla \phi_u|^2 dx - \frac{\beta}{16\pi} \int_{\mathbb{R}^3} |\nabla \phi_u|^4 dx \\
- \int_{\mathbb{R}^3} \left( \lambda F(u) + \frac{1}{6} |u|^6 \right) dx \\
= \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x) u^2) dx - \frac{3}{4} \int_{\mathbb{R}^3} \omega \phi_u u^2 dx - \frac{1}{4} \int_{\mathbb{R}^3} \phi_u^2 u^2 dx - \frac{1}{16\pi} \int_{\mathbb{R}^3} |\nabla \phi_u|^2 dx \\
- \int_{\mathbb{R}^3} \left( \lambda F(u) + \frac{1}{6} |u|^6 \right) dx \\
= \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x) u^2) dx - \frac{1}{2} \int_{\mathbb{R}^3} \omega \phi_u u^2 dx + \frac{\beta}{16\pi} \int_{\mathbb{R}^3} |\nabla \phi_u|^4 dx \\
- \int_{\mathbb{R}^3} \left( \lambda F(u) + \frac{1}{6} |u|^6 \right) dx .
\]

(2.4)

Of course $I_\lambda(u) \in C^1(E, \mathbb{R})$ and for any $u, v \in E$, we have

\[
\langle I'_\lambda(u), v \rangle = \int_{\mathbb{R}^3} \{ \nabla u \cdot \nabla v + V(x) uv - (2\omega + \phi_u) \phi_u uv - \lambda f(u)v - |u|^4 uv \} dx .
\]

(2.5)

Lemma 2.2. (16) The following statements are equivalent:

(i) $(u, \phi) \in E(\mathbb{R}^3) \times D(\mathbb{R}^3)$ is a critical point of $G_\lambda$, i.e. $(u, \phi)$ is a solution of problem (1.1);

(ii) $u$ is a critical point of $I_\lambda$ and $\phi = \phi_u$. 

6
Lemma 2.3. The functional \( J_{\lambda,T}(u) \) satisfies the following conditions:
(i) there exists \( \alpha, \rho > 0 \) such that \( J_{\lambda,T}(u) \geq \alpha \) when \( \|u\| = \rho \);

(ii) there exists \( e \in E \) such that \( \|e\| > \rho \) and \( J_{\lambda,T}(e) < 0 \).

Proof. From \((h_1)\) and \((h_3)\), there exists a \( \varepsilon > 0 \) small such that

\[
|h_T(t)| \leq \varepsilon |t| + C_{\varepsilon} |t|^5,
\]

and

\[
|H_T(t)| \leq \frac{\varepsilon}{2} |t|^2 + \frac{C_{\varepsilon}}{6} |t|^6.
\]  

(2.10)

Then from Lemma \((2.7)\) \((2.8)\), \((2.10)\) and Sobolev embedding theorem, for every \( u \in E \setminus \{0\} \) we can deduce

\[
J_{\lambda,T}(u) = \frac{1}{2} \int_{\mathbb{R}^3} \left( |\nabla u|^2 + V(x)u^2 \right) dx - \frac{1}{2} \int_{\mathbb{R}^3} \omega \phi_u u^2 dx + \frac{\beta}{16\pi} \int_{\mathbb{R}^3} |\nabla \phi_u|^4 dx 
- \int_{\mathbb{R}^3} \left( \lambda H_T(u) + \frac{1}{6} |u|^6 \right) dx 
\geq \frac{1}{2} \int_{\mathbb{R}^3} \left( |\nabla u|^2 + V(x)u^2 \right) dx - \int_{\mathbb{R}^3} \left( \lambda H_T(u) + \frac{1}{6} |u|^6 \right) dx 
\geq \frac{1}{2} \|u\|^2 - \int_{\mathbb{R}^3} \left( \frac{\lambda \varepsilon}{2} |u|^2 + \frac{\lambda C_{\varepsilon}}{6} |u|^6 + \frac{1}{6} |u|^6 \right) dx 
\geq \frac{1}{2} \|u\|^2 - C_{\varepsilon} \|u\|^2 - C \|u\|^6.
\]  

(2.11)

Since \( \varepsilon \) is arbitrarily small, there exists \( \rho > 0 \) and \( \alpha > 0 \) such that \( J_{\lambda,T}(u) \geq \alpha > 0 \) for \( \|u\| = \rho \). Hence, \( J_{\lambda,T}(u) \) satisfied \((i)\).

From \((h_1)\) \((h_2)\) and \((h_3)\) we get for any \( M > 0 \) there exists a positive constant \( C_M > 0 \) such that

\[
H_T(u) \geq Mt^4 - C_M t^2, \quad \forall t > 0.
\]  

(2.12)

So, fix \( u \in E \setminus \{0\} \) and \( t > 0 \), From \((2.8)\) and \((2.12)\) we obtain

\[
J_{\lambda,T}(tu) = \frac{t^2}{2} \int_{\mathbb{R}^3} \left( |\nabla u|^2 + V(x)u^2 \right) dx - t^2 \int_{\mathbb{R}^3} \omega \phi_{tu} u^2 dx - \frac{t^2}{2} \int_{\mathbb{R}^3} \phi_{tu} u^2 dx - \frac{1}{8\pi} \int_{\mathbb{R}^3} |\nabla \phi_{tu}|^2 dx 
- \frac{\beta}{16\pi} \int_{\mathbb{R}^3} |\nabla \phi_{tu}|^4 dx - \int_{\mathbb{R}^3} \left( \lambda H_T(tu) + \frac{t^6}{6} |u|^6 \right) dx 
\leq \frac{t^2}{2} \int_{\mathbb{R}^3} \left( |\nabla u|^2 + V(x)u^2 \right) dx - t^2 \int_{\mathbb{R}^3} \omega \phi_{tu} u^2 dx + \lambda C_M t^2 \int_{\mathbb{R}^3} |u|^2 dx 
- \lambda M t^4 \int_{\mathbb{R}^3} |u|^4 dx - \frac{t^6}{6} \int_{\mathbb{R}^3} |u|^6 dx.
\]  

(2.13)

we can easily to know that \( J_{\lambda,T}(tu) \to -\infty \) for \( t \to +\infty \). The step \((ii)\) is proved by taking \( e = t_0 u \) with \( t_0 > 0 \) large enough. \(\square\)
From Lemma 2.3 we can easily get a PS sequence, namely there exists a sequence \(\{u_n\} \subset E\) satisfying
\[
J_{\lambda,T}(u_n) \to c_{\lambda,T}, \quad J'_{\lambda,T}(u_n) \to 0,
\]
where
\[
c_{\lambda,T} := \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} J_{\lambda,T}(\gamma(t)),
\]
\[
\Gamma := \{ \gamma \in C([0,1], H^1(\mathbb{R}^3)) : \gamma(0) = 0, \quad \gamma(1) = e \}.
\]

**Lemma 2.4.** The sequence \(\{u_n\}\) defined by (2.14) is bounded in \(E\).

**Proof.** From (2.8), (2.9), (2.14) and \((h_4)\), we obtain that
\[
c_{\lambda,T} + o_n(1) \|u_n\|
\geq J_{\lambda,T}(u_n) - \frac{1}{4} J'_{\lambda,T}(u_n)
\geq \frac{1}{4} \int_{\mathbb{R}^3} \left( |\nabla u_n|^2 + V(x)u_n^2 \right) dx + \frac{\beta}{16\pi} \int_{\mathbb{R}^3} |\nabla \phi_{u_n}|^4 dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_{u_n}^2 u_n^2 dx
\geq \frac{1}{4} \int_{\mathbb{R}^3} \left( |\nabla u_n|^2 + V(x)u_n^2 \right) dx + \frac{1}{12} \int_{\mathbb{R}^3} |u_n|^6 dx
\geq \frac{1}{4} \|u_n\|^2 - \frac{\lambda \mu}{4} \int_{\mathbb{R}^3} |v_n^+|^2 dx,
\]
where \(u_n(x) = u_n^+(x) + u_n^-(x)\), \(u_n^+(x) = \max\{u_n(x), 0\}\), \(u_n^-(x) = \min\{u_n(x), 0\}\). We use the contradiction method that assume \(\|u_n\| \to +\infty\) as \(n \to \infty\). Let \(v_n = \frac{u_n}{\|u_n\|}, n \geq 1\).

Due to \(E \hookrightarrow L^q(\mathbb{R}^3), q \in [2,6)\) is compact, it is easy to assume \(v_n \rightharpoonup v\) in \(E\), then
\[
v_n \to v, \quad \text{in } L^q(\mathbb{R}^3), \quad 2 \leq q < 6,
\]
\[
v_n \to v, \quad \text{a.e. in } \mathbb{R}^3.
\]

What’s more, we deduce
\[
v_n^+ \rightharpoonup v^+, \quad \text{in } E,
\]
\[
v_n^+ \to v^+, \quad \text{in } L^q(\mathbb{R}^3), \quad 2 \leq q < 6,
\]
\[
v_n^+ \to v^+, \quad \text{a.e. in } \mathbb{R}^3.
\]

Divide both sided of (2.15) by \(\|u_n\|^2\), we obtain
\[
o_n(1) \geq \frac{1}{4} - \frac{\lambda \mu}{4} \int_{\mathbb{R}^3} |v_n^+|^2 dx
\geq \frac{1}{4} - \frac{\lambda \mu}{4} \int_{\mathbb{R}^3} |v^+|^2 dx + o(1).
\]
we can deduce \( v^+ \neq 0 \). Due to \( u_n^+ = u_n^+ \|u_n\| \to +\infty \), \((2.20), (2.14)\) and Lemma 2.1, we get
\[
\frac{J'_{\lambda,T}(u_n)}{\|u_n\|^2} u_n = \frac{\|u_n\|^2}{\|u_n\|^4} - \frac{\int_{\mathbb{R}^3} 2\omega \phi u_n u_n^+ dx}{\|u_n\|^4} - \frac{\int_{\mathbb{R}^3} \phi^2 \phi u_n^+ dx}{\|u_n\|^4} - \frac{\int_{\mathbb{R}^3} \lambda h_T(u_n) u_n dx}{\|u_n\|^4} - \frac{\int_{\mathbb{R}^3} |u_n|^6 dx}{\|u_n\|^4} \quad (2.19)
\]

\[
\leq o_n(1) + \frac{\int_{\mathbb{R}^3} 2\omega |\phi u_n| u_n^2 dx}{\|u_n\|^4} - \int_{\mathbb{R}^3} \frac{\lambda h_T(u_n^4) u_n^4}{(u_n^4)^4} dx.
\]

From Lemma 2.1(iii), \((2.17)\) and \((h_2)\), we know that \( \int_{\mathbb{R}^3} 2\omega |\phi u_n| u_n^2 dx \to 2\omega \) and \( \int_{\mathbb{R}^3} \lambda h_T(u_n^4) u_n^4 (u_n^4)^4 dx \to +\infty \). Taking the limit of \((2.19)\) we get \( 0 \leq -\infty \) which have a contradiction. Therefore, \( u_n \) is bounded in \( E \).

**Lemma 2.5.** If \( u_n \) is bounded in \( E \) then, up to subsequence, \( \phi u_n \to \phi u \) in \( D \).

**Proof.** Due to \( u_n \) is bounded in \( E \), we know
\[
u_n \to u \quad \text{weakly in } E
\]
\[
u_n \to u \quad \text{in } L^q(\mathbb{R}^3), \quad 2 \leq q < 6.
\]

From \((2.2)\), we can easily know \( \{\phi u_n\} \) is bounded in \( D(\mathbb{R}^3) \). So, there exists \( \phi_0 \in D \) such that \( \phi_{u_n} \to \phi_0 \) in \( D \), as a consequence,
\[
\phi_{u_n} \to \phi_0 \quad \text{weakly in } L^6(\mathbb{R}^3),
\]
\[
\phi_{u_n} \to \phi_0 \quad \text{in } L^q(\mathbb{R}^3), \quad 1 \leq q < 6.
\]

Next we will show \( \phi_u = \phi_0 \). By Lemma 2.1, it suffices to show that
\[
\Delta \phi_0 + \beta \Delta_4 \phi_0 = 4\pi(\omega + \phi_0)u^2.
\]

Let \( \varphi \in C^\infty(\mathbb{R}^3) \) be a test function. Since \( \Delta \phi_{u_n} + \beta \Delta_4 \phi_{u_n} = 4\pi(\omega + \phi_{u_n})u^2 \), we get
\[
- \int_{\mathbb{R}^3} \langle \nabla \phi_{u_n}, \nabla \varphi \rangle dx - \beta \int_{\mathbb{R}^3} \langle |\nabla \phi_{u_n}|^2 \nabla \phi_{u_n}, \nabla \varphi \rangle dx = \int_{\mathbb{R}^3} 4\pi \omega u_n^2 \varphi dx + \int_{\mathbb{R}^3} 4\pi \omega u_n^2 \varphi dx.
\]

From \((2.20)\) and \((2.21)\) and the boundedness of \( \{\phi u_n\} \) in \( D \), the following formulas are all true, namely
\[
\int_{\mathbb{R}^3} \langle \nabla \phi_{u_n}, \nabla \varphi \rangle dx \to_{n \to \infty} \int_{\mathbb{R}^3} \langle \nabla \phi_0, \nabla \varphi \rangle dx,
\]
\[
\int_{\mathbb{R}^3} \langle |\nabla \phi_{u_n}|^2 \nabla \phi_{u_n}, \nabla \varphi \rangle dx \to_{n \to \infty} \int_{\mathbb{R}^3} \langle |\nabla \phi_0|^2 \nabla \phi_0, \nabla \varphi \rangle dx,
\]
\[
\int_{\mathbb{R}^3} u_n^2 \varphi dx \to_{n \to \infty} \int_{\mathbb{R}^3} u^2 \varphi dx,
\]
\[
\int_{\mathbb{R}^3} 4\pi \phi_{u_n} u_n^2 \varphi dx \to_{n \to \infty} \int_{\mathbb{R}^3} 4\pi \phi_0 u^2 \varphi dx,
\]
proving that \( \phi_u = \phi_0 \).
Since $\phi_{u_n}$ and $\phi_u$ satisfies the second equation in problem (1.1), let us take the difference between them, we get

$$
\int_{\mathbb{R}^3} \{ \nabla (\phi_{u_n} - \phi_u) \nabla v + \beta (|\nabla \phi_{u_n}|^2 \nabla \phi_{u_n} - |\nabla \phi_u|^2 \nabla \phi_u) v \} \, dx
$$

$$
= -4\pi \int_{\mathbb{R}^3} \{ \omega (u_n^2 - u^2) v + (\phi_{u_n} u_n^2 - \phi_u u^2) v \} \, dx
$$

(2.22)

for any $v \in D$. Let $v = \phi_{u_n} - \phi_u$ and using the inequality

$$
(|x|^{p-2} x - |y|^{p-2} y)(x - y) \geq c_p |x - y|^p, \quad \text{for any } x, y \in \mathbb{R}^N, \ p \geq 2
$$

the following hold

$$
C(||\nabla \phi_{u_n} - \nabla \phi_u||^2_2 + ||\nabla \phi_{u_n} - \nabla \phi_u||^2_4)
\leq 4\pi \int_{\mathbb{R}^3} (\omega |u_n^2 - u^2| |\phi_{u_n} - \phi_u| + |\phi_{u_n} - \phi_u| u_n^2 + |\phi_u||\phi_{u_n} - \phi_u| u^2) \, dx.
$$

(2.23)

By the Hölder inequality, Sobolev’s inequality and (2.20), we can complete the statement. \hfill \square

**Lemma 2.6.** $J_{\lambda,T}$ satisfies the (PS)$_c$ condition at any level $c \in (0, \frac{1}{3}S^3)$, where $S$ is the best constant of the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, i.e.,

$$
S = \inf_{u \in D^{1,2}(\mathbb{R}^3)} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 \, dx}{(\int_{\mathbb{R}^3} |u|^6 \, dx)^{\frac{1}{3}}},
$$

Proof. Let $u_n$ be a PS sequence satisfying (2.11). Form Lemma 2.4 we know that $\{u_n\}$ is bounded in $E$, then up to a subsequence, we get

$$
u_n \rightarrow u, \quad \text{in } E,
$$

$$
u_n \rightarrow u, \quad \text{in } L^q(\mathbb{R}^3), \ 2 \leq q < 6,
$$

$$
u_n \rightarrow u, \quad \text{a.e. in } \mathbb{R}^3.
$$

(2.24)

Assume $\nu_n = u_n - u$, From the Brezis-Lieb lemma in [23], we obtain

$$
\int_{\mathbb{R}^3} |\nabla u_n|^2 \, dx = \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + \int_{\mathbb{R}^3} |\nabla \nu_n|^2 \, dx + o(1),
$$

$$
\int_{\mathbb{R}^3} |u_n|^2 \, dx = \int_{\mathbb{R}^3} |u|^2 \, dx + \int_{\mathbb{R}^3} |\nu_n|^2 \, dx + o(1),
$$

$$
\int_{\mathbb{R}^3} |u_n|^6 \, dx = \int_{\mathbb{R}^3} |u|^6 \, dx + \int_{\mathbb{R}^3} |\nu_n|^6 \, dx + o(1).
$$

(2.25)

Due to [23] Theorem A.1, for any $\varphi \in C^\infty_0 \subset E$. we deduce

$$
\int_{\mathbb{R}^3} h_T(u_n) \varphi \, dx \rightarrow \int_{\mathbb{R}^3} h_T(u) \varphi \, dx.
$$

It is easy to know

$$
\int_{\mathbb{R}^3} (h_T(u_n)u_n - h_T(u)u) \, dx = \int_{\mathbb{R}^3} (h_T(u_n) - h_T(u))u_n \, dx + \int_{\mathbb{R}^3} h_T(u)(u_n - u) \, dx
$$

$$
\leq \int_{\mathbb{R}^3} (h_T(u_n) - h_T(u))u_n \, dx + \left( \int_{\mathbb{R}^3} (h_T(u))^2 \, dx \right)^{\frac{1}{2}} |u_n - u|_{L^2}.
$$

(2.26)
From the Hölder inequality, one have
\[
\left| \int_{\mathbb{R}^3} (\phi_n u_n^2 - \phi u^2) \, dx \right| \leq \int_{\mathbb{R}^3} |\phi_n| u_n + u - u \, dx + \int_{\mathbb{R}^3} |\phi_n - \phi u^2| \, dx
\leq |\phi_n| |L^6(\mathbb{R}^3)| u_n + u |L^2(\mathbb{R}^3)| u_n - u |L^3(\mathbb{R}^3)|
+ |\phi_n - \phi u| |L^6(\mathbb{R}^3)| u_n |L^2(\mathbb{R}^3)|, \tag{2.27}
\]
and
\[
\left| \int_{\mathbb{R}^3} (\phi_n^2 u_n^2 - \phi_n^2 u^2) \, dx \right| \leq \int_{\mathbb{R}^3} |\phi_n|^2 |u_n| u_n - u \, dx + \int_{\mathbb{R}^3} |\phi_n + \phi_n| |\phi_n - \phi u| |u_n| \, dx
\leq |\phi_n| |L^6(\mathbb{R}^3)| u_n + u |L^1(\mathbb{R}^3)| u_n - u |L^3(\mathbb{R}^3)|
+ |\phi_n - \phi u| |L^6(\mathbb{R}^3)| u_n + \phi u |L^6(\mathbb{R}^3)| u_n |L^2(\mathbb{R}^3)|. \tag{2.28}
\]
Combining (2.24) and (2.28) with Lemma (2.5), up to subsequence, we get
\[
\langle J'_{\lambda,T}(u_n), u_n \rangle - \langle J'_{\lambda,T}(u), u \rangle = \int_{\mathbb{R}^3} (|\nabla u_n|^2 - |\nabla u|^2) \, dx + \int_{\mathbb{R}^3} V(x)(u_n^2 - u^2) \, dx - \int_{\mathbb{R}^3} 2\omega(\phi_n u_n^2 - \phi u^2) \, dx
- \int_{\mathbb{R}^3} (\phi_n u_n^2 - \phi_n^2 u^2) \, dx - \int_{\mathbb{R}^3} \lambda(h_T(u_n) u_n - h_T(u) u) \, dx
- \int_{\mathbb{R}^3} (|u_n|^6 - |u|^6) \, dx \tag{2.29}
= \int_{\mathbb{R}^3} \nu_n^2 \, dx + V(x) \int_{\mathbb{R}^3} \nu_n^2 \, dx \to b, \quad \int_{\mathbb{R}^3} |\nu_n|^6 \, dx \to b, \quad \text{as } n \to \infty.
\]
It is easy to know that \( \langle J'_{\lambda,T}(u_n), u_n \rangle \to \langle J'_{\lambda,T}(u), u \rangle = 0 \), we assume that
\[
\int_{\mathbb{R}^3} |\nabla\nu_n|^2 \, dx + V(x) \int_{\mathbb{R}^3} \nu_n^2 \, dx \to b, \quad \int_{\mathbb{R}^3} |\nu_n|^6 \, dx \to b,
\]
where \( b \) is nonnegative constant.

We assert that \( b = 0 \). If \( b \neq 0 \), under the definition of \( S \) we get
\[
\int_{\mathbb{R}^3} |\nabla\nu_n|^2 \, dx \geq S \left( \int_{\mathbb{R}^3} |\nu_n|^2 \, dx \right)^{\frac{1}{2}}.
\]
Then
\[
\int_{\mathbb{R}^3} |\nabla\nu_n|^2 \, dx + V(x) \int_{\mathbb{R}^3} \nu_n^2 \, dx \geq S \left( \int_{\mathbb{R}^3} |\nu_n|^2 \, dx \right)^{\frac{1}{2}}.
\]
which means \( b \geq S^\frac{1}{2} \). Thus \( b \geq S^\frac{1}{2} \).

As discussed above and follows from \( \phi_u \leq 0 \) and \( b \geq S^\frac{1}{2} \), we can know
\[
c = \lim_{n \to \infty} J_{\lambda,T}(u_n)
\geq \lim_{n \to \infty} \left\{ \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla\nu_n|^2 + V(x)\nu_n^2) \, dx - \frac{1}{6} \int_{\mathbb{R}^3} |\nu_n|^6 \, dx \right\}
= \frac{1}{3} b \geq \frac{1}{3} S^\frac{3}{2}.
\]
There exists a sequence $\nu_n$ such that $0 \leq \|\nu_n\| = \left[ \int_{\mathbb{R}^3} \left( |\nabla \nu_n|^2 + V(x)\nu_n^2 \right) dx \right]^{\frac{1}{2}} \to 0$.

**Lemma 2.7.** $c_{\lambda,T} < \frac{3}{2}S^2$, where $c_{\lambda,T}$ and $S$ are respectively defined in (2.14) and Lemma 2.6.

**Proof.** Let $\varphi \in C_0^\infty$ is a cut-off function satisfying that there exists $R > 0$ such that $\varphi|_{B_R} = 1$, $0 \leq \varphi \leq 1$ in $B_{2R}$ and $\text{supp}\varphi \subset B_{2R}$. Let $\epsilon > 0$ and define $u_\epsilon := w_\epsilon \varphi$ where $w_\epsilon \in D^{1,2}(\mathbb{R}^3)$ is the Talenti function $w_\epsilon(x) = \left( \frac{3\epsilon^2}{4} \right)^{\frac{1}{2}} \frac{1}{(\epsilon^2 + |x|^2)^\frac{3}{2}}$. From estimates obtained in [23] we get if $\epsilon$ is small enough,

\begin{equation}
\int_{\mathbb{R}^3} |\nabla u_\epsilon|^2 dx = S^2 + O(\epsilon), \tag{2.30}
\end{equation}

\begin{equation}
\int_{\mathbb{R}^3} |u_\epsilon|^6 dx = S^2 + O(\epsilon^3), \tag{2.31}
\end{equation}

\begin{equation}
\int_{\mathbb{R}^3} |u_\epsilon|^q dx = \begin{cases} O(\epsilon^{\frac{q}{2}}), & q \in [2, 3), \\ O(\epsilon^{\frac{3}{2}} \ln \epsilon), & q = 3, \\ O(\epsilon^{\frac{q-3}{2}}), & q \in (3, 6). \end{cases} \tag{2.32}
\end{equation}

Since for any $\epsilon > 0$, $\lim_{t \to \infty} J_{\lambda,T}(t_\epsilon u_\epsilon) = -\infty$. We can assume there exists $t_\epsilon \geq 0$ such that $\sup_{t \geq 0} J_{\lambda,T}(t u_\epsilon) = J_{\lambda,T}(t_\epsilon u_\epsilon)$ and without loss of generality we let $t_\epsilon \geq C_0 > 0$. In fact, suppose there exists a sequence $\epsilon_n \subset \mathbb{R}^+$ such that $\lim_{n \to \infty} t_\epsilon = 0$ and $J_{\lambda,T}(t_\epsilon_n u_\epsilon_n) = \sup_{t \geq 0} J_{\lambda,T}(t u_\epsilon)$. We can deduce $0 < \alpha < c_{\lambda,T} \leq \lim_{n \to \infty} J_{\lambda,T}(t_\epsilon_n u_\epsilon_n) = 0$ which have a contradiction.

Moreover, we claim that $\{t_\epsilon\}_{\epsilon > 0}$ is bounded from above. Otherwise, there exists a subsequence $t_{\epsilon_n}$ such that $\lim_{n \to \infty} t_{\epsilon_n} = +\infty$. From (2.8), (2.12), (2.30), (2.31) and Lemma 2.1 we get

\begin{align*}
0 &< c_{\lambda,T} \\
&\leq J_{\lambda,T}(t_{\epsilon_n} u_{\epsilon_n}) \\
&\leq \frac{t_{\epsilon_n}^2}{2} \int_{\mathbb{R}^3} (|\nabla u_{\epsilon_n}|^2 + V(x)u_{\epsilon_n}^2) dx - t_{\epsilon_n}^2 \int_{\mathbb{R}^3} \omega \phi_{\epsilon_n} u_{\epsilon_n}^2 dx \\
&\quad - \int_{\mathbb{R}^3} \lambda H_T(t_{\epsilon_n} u_{\epsilon_n}) dx - \frac{t_{\epsilon_n}^6}{6} \int_{\mathbb{R}^3} |u_{\epsilon_n}|^6 dx \\
&\leq \frac{t_{\epsilon_n}^2}{2} \int_{\mathbb{R}^3} (|\nabla u_{\epsilon_n}|^2 + V(x)u_{\epsilon_n}^2) dx - t_{\epsilon_n}^2 \int_{\mathbb{R}^3} \omega \phi_{\epsilon_n} u_{\epsilon_n}^2 dx \\
&\quad + t_{\epsilon_n}^2 \int_{\mathbb{R}^3} C_M u_{\epsilon_n}^2 dx - t_{\epsilon_n}^4 \int_{\mathbb{R}^3} \lambda Mu_{\epsilon_n}^4 dx - \frac{t_{\epsilon_n}^6}{6} \int_{\mathbb{R}^3} |u_{\epsilon_n}|^6 dx \\
&\leq C_1 t_{\epsilon_n}^2 - C_2 t_{\epsilon_n}^4 - C_3 t_{\epsilon_n}^6 \to -\infty, \quad \text{as } n \to \infty.
\end{align*}

Therefore $0 < -\infty$ is a contradiction.
Let
\[ g(t) = \frac{t^2}{2} \int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 dx - \frac{t^6}{6} \int_{\mathbb{R}^3} |u_\varepsilon|^6 dx. \]

It is easy to know
\[ \sup_{t \geq 0} g(t) = \frac{1}{3} S^\frac{2}{3} + O(\varepsilon). \]  

(2.33)

According to assumption (V2), for $|x| < r$ there exists $\xi > 0$ such that
\[ |V(x)| \leq \xi. \]  

(2.34)

Form (2.12) and (2.30)-(2.34) we obtain
\[
J_{\lambda,T}(t_\varepsilon u_\varepsilon) = \frac{t^2}{2} \int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 dx + \frac{t^2}{2} \int_{\mathbb{R}^3} V(x) u_\varepsilon^2 dx - \frac{3t^2}{4} \int_{\mathbb{R}^3} \omega \phi_{t_\varepsilon u_\varepsilon} u_\varepsilon^2 dx - \frac{t^6}{4} \int_{\mathbb{R}^3} \phi_{t_\varepsilon u_\varepsilon}^2 u_\varepsilon^2 dx \\
- \frac{1}{16\pi} \int_{\mathbb{R}^3} |\nabla \phi_{t_\varepsilon u_\varepsilon}|^2 dx - \int_{\mathbb{R}^3} \lambda H_T(t_\varepsilon u_\varepsilon) dx - \frac{t^6}{6} \int_{\mathbb{R}^3} |u_\varepsilon|^6 dx \\
\leq \frac{t^2}{2} \int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 dx + \frac{t^2}{2} \int_{\mathbb{R}^3} V(x) u_\varepsilon^2 dx - \frac{3t^2}{4} \int_{\mathbb{R}^3} \omega \phi_{t_\varepsilon u_\varepsilon} u_\varepsilon^2 dx - \frac{t^6}{4} \int_{\mathbb{R}^3} \phi_{t_\varepsilon u_\varepsilon}^2 u_\varepsilon^2 dx - \lambda Mt_\varepsilon^4 \int_{\mathbb{R}^3} u_\varepsilon^4 dx \\
+ \lambda C_M t_\varepsilon^2 \int_{\mathbb{R}^3} u_\varepsilon^2 dx - \frac{t^6}{6} \int_{\mathbb{R}^3} |u_\varepsilon|^6 dx \\
\leq \sup_{t \geq 0} g(t) + \frac{t^2}{2} \xi \int_{\mathbb{R}^3} u_\varepsilon^2 dx - \frac{3t^2}{4} \int_{\mathbb{R}^3} \omega \phi_{t_\varepsilon u_\varepsilon} u_\varepsilon^2 dx - \lambda Mt_\varepsilon^4 \int_{\mathbb{R}^3} u_\varepsilon^4 dx + \lambda C_M t_\varepsilon^2 \int_{\mathbb{R}^3} u_\varepsilon^2 dx \\
\leq \sup_{t \geq 0} g(t) + C|u_\varepsilon|^2 + C|u_\varepsilon|^2 - \lambda MC|u_\varepsilon|^4 \\
\leq \frac{1}{3} S^\frac{2}{3} + CO(\varepsilon) - \lambda MC(\varepsilon).
\]

When $M$ large enough, we know $CO(\varepsilon) - \lambda MC(\varepsilon) \to -\infty$ for small enough $\varepsilon > 0$. 

\[ \square \]

**Theorem 2.8.** Assume $\lambda > 0$ $T > 0$, problem (2.7) has a nontrivial solution $u_{\lambda,T}$ with $J_{\lambda,T}(u_{\lambda,T}) = c_{\lambda,T}$.

**Proof.** First of all, we know that the function $J_{\lambda,T}$ satisfies Lemma [2.3] that is, the geometric structure of the mountain pass. Then the PS sequence can be obtained. Secondly, because of Lemma [2.6] it can be know that function $J_{\lambda,T}$ satisfies the PS condition. According to the mountain pass theorem, there exists a critical point $u_{\lambda,T} \in E$. Moreover, $J_{\lambda,T} = c_{\lambda,T} \geq \alpha > 0 = J(0)$, so that $u_{\lambda,T}$ is a nontrivial solution. 

\[ \square \]

### 3 Proof of the Theorem 1.1

In this section, we will prove Theorem [1.1]. First, prove the solution of problem (2.7) satisfied $|u_{\lambda,T}|_{\infty} \leq T$, which means that the solution at this time is the solution of problem (1.1). The proof method is similar to the document [15][24] using the Nash-Moser method.
Lemma 3.1. If \( u \) is a critical point of \( J_{\lambda,T} \), then \( u \in L^\infty(\mathbb{R}^3) \) and

\[
|u|_\infty \leq C_0^{\frac{1}{2(\zeta-1)}} \frac{\zeta}{(\zeta-1)^2} \left[ \frac{1}{(\lambda C_T^* + \alpha(\varepsilon,u))(1 + |u|^2)^2 + \lambda C_T|u|_6^{p-2}} \right]^{\frac{1}{2(\zeta-1)}} |u|^\kappa,
\]

where \( C_0 > 0 \) and \( \kappa \leq 1 \) are constants independent of \( \lambda \) and \( T \), \( \zeta = \frac{8-p}{2} \).

Proof. Assume \( A_k = \{ x \in \mathbb{R}^3 : |u|^{s-1} \leq k \} \), \( B_k = \mathbb{R}^3 \setminus A_k \), where \( s > 1, k > 0 \). Let

\[
u_k = \begin{cases} u|u|^{2(s-1)}, & x \in A_k, \\ k^2 u, & x \in B_k, \end{cases}
\]

and

\[
\chi_k = \begin{cases} u|u|^{s-1}, & x \in A_k, \\ ku, & x \in B_k. \end{cases}
\]

It is easy to know \( |u_k| \leq |u|^{2s-1} \) if \( u_k, \chi_k \in E \), and \( \chi_k^2 = uu_k \leq |u|^{2s} \). Through direct calculation, the following formula can be obtained:

\[
\nabla u_k = \begin{cases} (2s-1)|u|^{2s-2}\nabla u, & x \in A_k, \\ k^2 \nabla u, & x \in B_k, \end{cases}
\]

and

\[
\nabla \chi_k = \begin{cases} s|u|^{s-1}\nabla u, & x \in A_k, \\ k \nabla u, & x \in B_k, \end{cases}
\]

and

\[
\int_{\mathbb{R}^3} (|\nabla \chi_k|^2 - \nabla u \nabla u_k) \, dx = (s-1)^2 \int_{A_k} |u|^{2(s-1)} |\nabla u|^2 \, dx. \tag{3.5}
\]

Due to its definition, we get

\[
\int_{\mathbb{R}^3} \nabla u \nabla u_k \, dx
= (2s-1) \int_{A_k} |u|^{2(s-1)} |\nabla u|^2 \, dx + k^2 \int_{B_k} |\nabla u|^2 \, dx
\geq (2s-1) \int_{A_k} |u|^{2(s-1)} |\nabla u|^2 \, dx. \tag{3.6}
\]

From (3.5) and (3.6), we get \( \int_{\mathbb{R}^3} \nabla u \nabla u_k \, dx \geq 0 \) and

\[
\int_{\mathbb{R}^3} |\nabla \chi_k|^2 \, dx \leq s^2 \int_{\mathbb{R}^3} \nabla u \nabla u_k \, dx. \tag{3.7}
\]

Since \( u \) is the critical point, Let \( u_k \) is a test function in (2.9), we can deduce

\[
\int_{\mathbb{R}^3} (\nabla u \nabla u_k + V(x)uu_k - 2\omega \phi u uu_k - \phi^2 u uu_k) \, dx = \int_{\mathbb{R}^3} \lambda h_T(u)u_k \, dx + \int_{\mathbb{R}^3} |u|^4 uu_k \, dx. \tag{3.8}
\]
When through the interpolation inequality, we know from the Sobolev embedding theorem, Hölder inequality, and (3.9)-(3.10), we get
\[ |\nabla \chi_k|^2 \leq \varepsilon \int_{\mathbb{R}^N} |\nabla \chi_k|^2 + \alpha(\varepsilon, u) \int_{\mathbb{R}^N} |\chi_k|^2 dx. \]

By a version of the Brézis-Kato lemma as done in [27, Lemma 2.5], for any \( \varepsilon > 0 \), we can find \( \alpha(\varepsilon, u) \) such that
\[ \int_{\mathbb{R}^N} |u|^4 \chi_k^2 dx \leq \varepsilon \int_{\mathbb{R}^N} |\nabla \chi_k|^2 + \alpha(\varepsilon, u) \int_{\mathbb{R}^N} |\chi_k|^2 dx. \]

Let \( \varepsilon = \frac{1}{2s^2} \), from \( \chi_k^2 = uu_k \) and \((h_3)\) we deduce
\[ \int_{\mathbb{R}^N} |\nabla \chi_k|^2 dx \leq 2s^2 \left( \int_{\mathbb{R}^N} \lambda h_T(u) u_k dx + \alpha(\varepsilon, u) \int_{\mathbb{R}^N} |\chi_k|^2 dx \right), \tag{3.9} \]
and
\[ |h_T(u) u_k| \leq C_T^* \chi_k^2 + C_T|u|^{p-2} \chi_k^2. \tag{3.10} \]

From the Sobolev embedding theorem, Hölder inequality, and (3.9)-(3.10), we get
\[ \left( \int_{A_k} |\chi_k|^6 dx \right)^{\frac{1}{6}} \leq S^{-1} \int_{\mathbb{R}^N} |\nabla \chi_k|^2 dx \]
\[ \leq S^{-1} 2s^2 \left[ \int_{\mathbb{R}^N} \lambda C_T^* \chi_k^2 + C_T|u|^{p-2} \chi_k^2 \right] dx + \alpha(\varepsilon, u) \int_{\mathbb{R}^N} |\chi_k|^2 dx \]
\[ \leq S^{-1} 2s^2 \left[ (\lambda C_T^* + \alpha(\varepsilon, u)) |\chi_k|^2 + \lambda C_T|u|^{p-2} |\chi_k|^2 \right], \tag{3.11} \]
where \( q = \frac{6}{3-p} \in (\frac{3}{2}, 3) \) and \( S \) defined in Lemma 2.6 Due to \( |\chi_k| \leq |u|^s \) and \( |\chi_k| = |u|^s \) for \( x \in A_k \), together with (3.11), we easily get
\[ \left( \int_{A_k} |u|^{6s} dx \right)^{\frac{1}{6}} \leq S^{-1} 2s^2 \left[ (\lambda C_T^* + \alpha(\varepsilon, u)) |u|^{2s} + \lambda C_T|u|^{p-2} |u|^{2s} \right]. \tag{3.12} \]

Through the interpolation inequality, we know \( |u|_{2s} \leq |u|^{1-\sigma}_{2s} |u|^{\sigma}_{2s} \), where \( \sigma \in (0, 1) \) and \( \frac{1}{2s} = \frac{1-\sigma}{2s} + \frac{\sigma}{2s} \), so \( \sigma = \frac{q(s-1)}{6q-1} \). Moreover since \( 2s(1-\sigma) = 2 + \frac{2(1-\sigma)}{4q-1} < 2 \), we know
\[ |u|_{2s} \leq |u|^{2s(1-\sigma)}_{2s} |u|^{2s\sigma}_{2s} \leq (1 + |u|_2^2) |u|^{2s\sigma}_{2s}. \tag{3.13} \]

When \( k \to \infty \), together with (3.12) and (3.13), we deduce
\[ |u|_{6s} \leq (S^{-1} 2s^2) \frac{1}{2s} \left( \lambda C_T^* + \alpha(\varepsilon, u) \right) (1 + |u|_2^2) |u|^{2s\sigma}_{2s} + \lambda C_T |u|^{p-2} |u|^{2s^2}_{2s}. \tag{3.14} \]

where \( \kappa \in \{ \sigma, 1 \} \), \( C_0 = \max \{ 2S^{-1}, 1 \} \). Let \( \zeta = \frac{6}{2q} \), then \( \zeta \in (1, 2) \). Now we use \( j \) iterations by letting \( s_j = \zeta^j \) in (3.14), then we get
\[ |u|_{6s^j} \leq C_0 \frac{1}{\zeta \sum_{j=1}^{\infty} \frac{1}{\zeta^j}} \left( \lambda C_T^* + \alpha(\varepsilon, u) \right) (1 + |u|_2^2) + \lambda C_T |u|^{p-2} \frac{1}{\zeta^j} \sum_{j=1}^{\infty} \frac{1}{\zeta^j} |u|^{k_1 \cdots k_j}_{6s^j}. \tag{3.15} \]
where $\sigma_j = \frac{q(\zeta_j - 1)}{q\zeta_j - 1} < 1$, $\kappa_j \in \{\sigma_j, 1\} \leq 1$. From a easy calculation, we get

$$\sum_{j=1}^{\infty} \frac{1}{\zeta_j^2} = \frac{1}{\zeta - 1}, \quad \sum_{j=1}^{\infty} \frac{j}{\zeta_j} = \frac{\zeta}{(\zeta - 1)^2}.$$ 

The estimates of $|u|_{\infty}$ will be divided into two cases.

(i) When $|u|_6 \geq 1$, $|u|_{6}^{\kappa_1 \cdots \kappa_j} \leq |u|_6$. If $j \to \infty$ in equation (3.15), we can get

$$|u|_{\infty} \leq C_0^{\frac{1}{2(\zeta - 1)}} \zeta \left[ (\lambda C_T^2 + \alpha(\varepsilon, u))(1 + |u|_2^2 + \lambda C_T|u|_6^{p-2}) \right]^{\frac{1}{2(\zeta - 1)}} |u|_6. \quad (3.16)$$

(ii) When $|u|_6 < 1$, by $\sigma_j = \frac{q(\zeta_j - 1)}{q\zeta_j - 1} \geq 1 - \frac{1}{\zeta_j}$ and $\kappa_j \in \{\sigma_j, 1\}$, for any $j \in \mathbb{N}$, we deduce

$$0 < \sigma_1 \sigma_2 \cdots \sigma_j \leq \kappa_1 \kappa_2 \cdots \kappa_j.$$ 

For $s \in (0, 1)$, we claim that $\ln(1 - s) \geq -s - \frac{s^2}{2(1-s)^2}$. Then we can easy to get

$$\sum_{i=1}^{j} \ln \kappa_i \geq \sum_{i=1}^{j} \ln \sigma_i \geq -j \frac{1}{\zeta} - \frac{1}{2} \sum_{i=1}^{j} \frac{1}{(\zeta_i - 1)^2}.$$ 

From direct calculation, we get

$$\sum_{i=1}^{j} \frac{1}{(\zeta_i - 1)^2} \leq \frac{\zeta^2}{(\zeta^2 - 1)(\zeta - 1)^2}.$$ 

So

$$\sum_{i=1}^{\infty} \ln \kappa_i \geq \frac{1}{\zeta - 1} - \frac{\zeta^2}{2(\zeta^2 - 1)(\zeta - 1)^2} := \theta.$$ 

Therefore, $\kappa_1 \kappa_2 \cdots \kappa_j \geq e^{\theta}$ for any $j \in \mathbb{N}$. Due to $|u|_6 < 1$ then $|u|_{6}^{\kappa_1 \kappa_2 \cdots \kappa_j} \leq |u|_6^\theta$. If $j \to \infty$ in equation (3.15), we can get

$$|u|_{\infty} \leq C_0^{\frac{1}{2(\zeta - 1)}} \zeta \left[ (\lambda C_T^2 + \alpha(\varepsilon, u))(1 + |u|_2^2 + \lambda C_T|u|_6^{p-2}) \right]^{\frac{1}{2(\zeta - 1)}} |u|_6^\theta. \quad (3.17)$$

Combining the two cases, we know that the proof is complete if $\kappa = 1$ or $\kappa = e^{\theta} \leq 1$.

Proof of Theorem 1.1. Let $u \in C_0^\infty(\mathbb{R}^3)$ and $u(x) \leq 0$, then from the definition of equation $h_T(t)$ in (2.6), we know $H_T(tu) = 0$ for any $t > 0$. Hence,

$$J_{\lambda, T}(tu) = \frac{t^2}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) - \frac{1}{2} \int_{\mathbb{R}^3} \omega \phi_{tu} u^2 dx - \frac{\beta}{2} \int_{\mathbb{R}^3} \phi_{tu} u^2 dx - \frac{1}{8\pi} \int_{\mathbb{R}^3} |\nabla \phi_{tu}|^2 dx$$

$$- \frac{\beta}{16\pi} \int_{\mathbb{R}^3} |\nabla \phi_{tu}|^4 dx - \frac{t^6}{6} \int_{\mathbb{R}^3} |u|^6 dx.$$ 

(3.18)
it shows \( J_{\lambda,T}(tu) \to -\infty \) for \( t \to +\infty \). Then it can be find a \( t_0 > 0 \) such that \( J_{\lambda,T}(t_0 u) < 0 \). We can assume \( \gamma(\cdot) = t_0 u, t \in [0,1], \) so \( \gamma(t) \in \Gamma \). Because \( H_T(u) = 0, t \in [0,1], \) we get

\[
c_{\lambda,T} \leq \max_{t \in [0,1]} J_{\lambda,T}(\gamma(t)) \leq \max_{t \geq 0} \left\{ \frac{t^2}{2} \int_{\mathbb{R}^3} \left( |\nabla u|^2 + V(x)u^2 \right) dx - t^2 \int_{\mathbb{R}^3} \omega \phi u u^2 dx - \frac{t^6}{6} \int_{\mathbb{R}^3} |u|^6 dx \right\} := D > 0,
\]

where \( D \) is constant independent of \( \lambda \) and \( T \). From Theorem 1.5, (h.4) and \( (V_1) - (V_2) \), we know

\[
4D \geq 4c_{\lambda,T} \geq 4J_{\lambda,T} - \langle J'_{\lambda,T}(u_{\lambda,T}), u_{\lambda,T} \rangle
= \int_{\mathbb{R}^3} |\nabla u_{\lambda,T}|^2 + V(x)u_{\lambda,T} dx + \frac{\beta}{4\pi} \int_{\mathbb{R}^3} |\nabla \phi u_{\lambda,T}|^4 dx + \int_{\mathbb{R}^3} \phi^2_{u_{\lambda,T}} u^2_{\lambda,T} dx
+ \int_{\mathbb{R}^3} \lambda (h_T(u_{\lambda,T})u_{\lambda,T} - 4H_T(u_{\lambda,T})) dx + \frac{1}{3} \int_{\mathbb{R}^3} |u_{\lambda,T}|^6 dx
\geq \int_{\mathbb{R}^3} |\nabla u_{\lambda,T}|^2 + V(x)u_{\lambda,T} dx - \lambda \mu \int_{\mathbb{R}^3} u^2_{\lambda,T} dx
\geq \frac{1}{2} \|u_{\lambda,T}\|^2 + (\frac{V_0}{2} - \lambda \mu)\|u_{\lambda,T}\|^2.
\]

We can find a \( \lambda_0 \) such that \( \frac{V_0}{2} - \lambda_0 \mu > 0 \). Then from (3.20), \( \|u_{\lambda,T}\| \leq 8D \). Hence, we deduce

\[
|u_{\lambda,T}|_2 \leq C_5, \quad |u_{\lambda,T}|_6^2 \leq C_6,
\]

where \( C_5, C_6 > 0 \) independent of \( \lambda, T \).

From Lemma (3.1), we get

\[
|u_{\lambda,T}|_\infty \leq C_0^{\frac{1}{2(\xi-1)}} \zeta^{\frac{\xi}{(\xi-1)^2}} \left[ (\lambda C_5^p + \alpha(\varepsilon, u))(1 + C_5)^2 + \lambda C_5 C_6^{p-2} \right]^{\frac{1}{2(\xi-1)}} C_6^p.
\]

So, we can choose \( T > 0 \) large enough such that

\[
C_0^{\frac{1}{2(\xi-1)}} \zeta^{\frac{\xi}{(\xi-1)^2}} \left[ (\lambda(\varepsilon, u))(1 + C_5)^2 \right]^{\frac{1}{2(\xi-1)}} C_6^p \leq \frac{T}{2}.
\]

Since \( C_5^p, C_T \) are fixed constants for above \( T \), we can choose \( \lambda_1 \) such that

\[
|u_{\lambda,T}|_\infty \leq C_0^{\frac{1}{2(\xi-1)}} \zeta^{\frac{\xi}{(\xi-1)^2}} \left[ (\lambda_1^p C_5^p + \alpha(\varepsilon, u))(1 + C_5)^2 + \lambda_1^p C_5 C_6^{p-2} \right]^{\frac{1}{2(\xi-1)}} C_6^p \leq T.
\]

Then, for \( \lambda \in (0, \lambda_1^+) \), we can get \( |u_{\lambda,T}|_\infty \leq T \), \( u_{\lambda,T} \) is also a solution for the problem (1.1). \( \square \)

### 4 Proof of the theorem 1.4

**Proof.** When \( \beta = 0 \) check the proof of theorem 1.1 we can easy to get theorem 1.4, hence we omit the details. \( \square \)

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References

[1] V. Benci, D. Fortunato, A. Masiello, and L. Pisani. Solitons and the electromagnetic field. *Math. Z.*, 232(1):73–102, 1999.

[2] Max Born. On the quantum theory of the electromagnetic field. *Proc. R. Soc. Lond. Ser.*, 143(849):410–437, 1934.

[3] Max Born and Leopold Infeld. Foundations of the new field theory. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 144(852):425–451, 1934.

[4] Paulo C. Carrião, Patrícia L. Cunha, and Olímpio H. Miyagaki. Existence results for the Klein-Gordon-Maxwell equations in higher dimensions with critical exponents. *Commun. Pure Appl. Anal.*, 10(2):709–718, 2011.

[5] Paulo C. Carrião, Patrícia L. Cunha, and Olímpio H. Miyagaki. Positive ground state solutions for the critical Klein-Gordon-Maxwell system with potentials. *Nonlinear Anal.*, 75(10):4068–4078, 2012.

[6] Daniele Cassani. Existence and non-existence of solitary waves for the critical Klein-Gordon equation coupled with Maxwell’s equations. *Nonlinear Anal.*, 58(7-8):733–747, 2004.

[7] Guofeng Che and Haibo Chen. Infinitely many solutions for the Klein-Gordon equation with sublinear nonlinearity coupled with Born-Infeld theory. *Bull. Iranian Math. Soc.*, 46(4):1083–1100, 2020.

[8] Shang-Jie Chen and Lin Li. Multiple solutions for the nonhomogeneous Klein-Gordon equation coupled with Born-Infeld theory on $\mathbb{R}^3$. *J. Math. Anal. Appl.*, 400(2):517–524, 2013.

[9] Shang-Jie Chen and Shu-Zhi Song. The existence of multiple solutions for the Klein-Gordon equation with concave and convex nonlinearities coupled with Born-Infeld theory on $\mathbb{R}^3$. *Nonlinear Anal. Real World Appl.*, 38:78–95, 2017.

[10] Zhi Chen, Xianhua Tang, Lei Qin, and Dongdong Qin. Improved results for Klein-Gordon-Maxwell systems with critical growth. *Appl. Math. Lett.*, 91:158–164, 2019.

[11] J. M. Combes, R. Schrader, and R. Seiler. Classical bounds and limits for energy distributions of Hamilton operators in electromagnetic fields. *Ann Physics*, 111(1):1–18, 1978.

[12] Pietro d’Avenia and Lorenzo Pisani. Nonlinear Klein-Gordon equations coupled with Born-Infeld type equations. *Electron. J. Differential Equations*, pages No. 26, 13, 2002.

[13] B. Felsager and B. R. Holstein. Geometry, particles and fields. *American Journal of Physics*, 52(6):573–573, 1997.

[14] D. Fortunato, L. Orsina, and L. Pisani. Born-Infeld type equations for electrostatic fields. *J. Math. Phys.*, 43(11):5698–5706, 2002.
[15] Yuhua Li and Qian Geng. The existence of nontrivial solution to a class of nonlinear Kirchhoff equations without any growth and Ambrosetti-Rabinowitz conditions. Appl. Math. Lett., 96:153–158, 2019.

[16] Dimitri Mugnai. Coupled Klein-Gordon and Born-Infeld-type equations: looking for solitary waves. Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 460(2045):1519–1527, 2004.

[17] Paul H. Rabinowitz. On a class of nonlinear Schrödinger equations. Z. Angew. Math. Phys., 43(2):270–291, 1992.

[18] Xianhua Tang, Lixi Wen, and Sitong Chen. On critical Klein-Gordon-Maxwell systems with super-linear nonlinearities. Nonlinear Anal., 196:111771, 21, 2020.

[19] Kaimin Teng and Kejing Zhang. Existence of solitary wave solutions for the nonlinear Klein-Gordon equation coupled with Born-Infeld theory with critical Sobolev exponent. Nonlinear Anal., 74(12):4241–4251, 2011.

[20] Feizhi Wang. Solitary waves for the Klein-Gordon-Maxwell system with critical exponent. Nonlinear Anal., 74(3):827–835, 2011.

[21] Feizhi Wang. Solitary waves for the coupled nonlinear Klein-Gordon and Born-Infeld type equations. Electron. J. Differential Equations, pages No. 82, 12, 2012.

[22] Lixi Wen, Xianhua Tang, and Sitong Chen. Infinitely many solutions and least energy solutions for Klein-Gordon equation coupled with Born-Infeld theory. Complex Var. Elliptic Equ., 64(12):2077–2090, 2019.

[23] Michel Willem. Minimax theorems, volume 24 of Progress in Nonlinear Differential Equations and their Applications. Birkhäuser Boston, Inc., Boston, MA, 1996.

[24] Jie Yang, Haibo Chen, and Senli Liu. The existence of nontrivial solution of a class of Schrödinger-Bopp-Podolsky system with critical growth. Bound. Value Probl., pages Paper No. 144, 16, 2020.

[25] Yong Yu. Solitary waves for nonlinear Klein-Gordon equations coupled with Born-Infeld theory. Ann. Inst. H. Poincaré Anal. Non Linéaire, 27(1):351–376, 2010.

[26] Jian Zhang. Solutions to the critical Klein-Gordon-Maxwell system with external potential. J. Math. Anal. Appl., 455(2):1152–1177, 2017.

[27] Jianjun Zhang, David G. Costa, and João Marcos do Ó. Semiclassical states of p-Laplacian equations with a general nonlinearity in critical case. J. Math. Phys., 57(7):071504, 12, 2016.

[28] Wenming Zou and Martin Schechter. Critical point theory and its applications. Springer, New York, 2006.