Truncated moments of perpetuities and a new central limit theorem for GARCH processes without Kesten’s regularity

Adam Jakubowski* and Zbigniew S. Szewczak†
Nicolaus Copernicus University, Toruń, Poland

Abstract

We consider a class of perpetuities which admit direct characterization of asymptotics of the key truncated moment. The class contains perpetuities without polynomial decay of tail probabilities and thus not satisfying Kesten’s theorem. We show how to apply this result in deriving a new weak law of large numbers for solutions to stochastic recurrence equations and a new central limit theorem for GARCH(1,1) processes in the critical case.

Keywords: stochastic recurrence equation; central limit theorem; weak law of large numbers; GARCH processes; perpetuities.

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1 Truncated moments of perpetuities

Let \((A_j, B_j), j = 1, 2, \ldots\) be a sequence of independent, identically distributed random vectors with non-negative coordinates: \(A_j \geq 0, B_j \geq 0\). Suppose that the series

\[ U_\infty = \sum_{k=1}^{\infty} B_k \prod_{j=1}^{k-1} A_j \tag{1} \]

is almost surely finite (by definition \(\prod_{\emptyset} \equiv 1\)). Then \(U_\infty\) is called perpetuity for it admits a natural interpretation in insurance and finance. We refer to [14] for an excellent primer on perpetuities and to the articles [20] and [1] for an in-depth discussion of existence, uniqueness and related properties. If \(U_\infty\) exists, then its law is a distributional solution to the equation

\[ U = \mathcal{D} AU + B, \tag{2} \]

where \(U\) and \((A, B)\) are copies of \(U_\infty\) and \((A_1, B_1)\), respectively, and \(U\) and \((A, B)\) are independent. See [10] for an extensive treatment of this stochastic recurrence equation.

We will assume the following non-degeneracy conditions:

\[ \mathbb{P}(A = 1) < 1, \tag{3} \]
\[ \mathbb{P}(B = 0) < 1. \tag{4} \]

*E-mail: adjakubo@mat.umk.pl
†E-mail: zssz@mat.umk.pl
We will also assume that there exists a constant $\kappa > 0$ such that

$$\mathbb{E}A^\kappa = 1, \quad (5)$$

$$\mathbb{E}B^\kappa < +\infty. \quad (6)$$

Relations (5) and (6) imply that

$$\mathbb{E}A^p < 1, \quad (7)$$

whenever $p > 0$, $p < \kappa$. The function $\psi(p) = \mathbb{E}A^p I(A > 0)$ is strictly convex in $(0, \kappa)$, $\psi(\kappa) = 1$ and $\psi(p) < 1$ in $(0, \kappa)$. Hence

$$\mathbb{E}A^\kappa \log A \in (0, +\infty]. \quad (8)$$

Moreover, for $p \in (0, \kappa)$ (5) and (7) imply convergence in $L^p$ of the series defining the perpetuity and therefore $U_\infty$ is a distributional solution to (2). This solution is unique by [38], since $\mathbb{P}(A > 1) > 0$, $\mathbb{P}(A < 1) > 0$ and $B \geq 0$, $\mathbb{P}(B = 0) < 1$, imply that there is no $c \in \mathbb{R}$ such that $B = c(1 - A)$. In particular, the solution satisfies

$$\mathbb{E}U^p < +\infty, \quad 0 < p < \kappa. \quad (9)$$

It is well-known that condition (5) is crucial for power-like behavior of tail probabilities of perpetuities. If $\log A$ conditioned on $\{A > 0\}$ has a non-arithmetic distribution and

$$\mathbb{E}A^\kappa \log^+ A < +\infty,$$

then

$$\mathbb{P}(U > t) \asymp Ct^{-\kappa}, \text{ as } t \to \infty. \quad (10)$$

(here and in the sequel $f(t) \asymp g(t)$ means $f(t)/g(t) \to 1$ as $t \to \infty$ and $f(\varepsilon) \asymp g(\varepsilon)$ means $f(\varepsilon)/g(\varepsilon) \to 1$ as $\varepsilon \searrow 0$). This result essentially belongs to Kesten [30]. We refer to [19] for a completely elaborated proof of this fact, benefitting from a method developed by Grincevičius [21]. Notice that easy examples (arithmetic) show that in general Kesten’s result is not valid, i.e. (10) fails to hold.

But (10) can fail for non-arithmetic $\log A$, as well. Kevei [31] explored the case

$$\mathbb{E}A^\kappa = 1 \text{ and } \mathbb{E}A^\kappa \log^+ A = +\infty,$$

and his main assumption was

$$\overline{\Pi}_A(x) := \mathbb{E}A^\kappa I\{\log A > x\} = \ell_0(x)x^{-\alpha}, \quad (11)$$

where $\alpha \in (0, 1]$ and $\ell_0(x)$ is a slowly varying function. Under the extra assumption that $\mathbb{E}|B|^\nu < +\infty$, for some $\nu > \kappa$, and a highly technical condition related to the strong renewal theorem, Kevei [31, Theorem 1.1] proved that it is possible to obtain regularly varying tails of the form

$$\mathbb{P}(U > t) \asymp D t^{\kappa-\nu}m(\log t),$$

where $m(x) = \int_0^x \overline{\Pi}_A(s) \, ds$.

In [32] Kevei extended the results of [21] and [28] and gave the corresponding theory for the case of arithmetic distribution of $\log A$ conditioned on $\{A > 0\}$. As expected, within such a framework the tails of the generated perpetuity are not regularly varying. For refinements in other directions we refer to [11].

In view of the above discussion it is interesting that the truncated $\kappa$-th moment of $U$ exhibits remarkable regularity under minimal conditions.
1.1 Theorem Let \((A, B)\) be a random vector with nonnegative components satisfying conditions \(\mathcal{C} - \mathcal{G}\).

Let \(U\) represent the unique distributional solution to equation \((2)\) (with \(U\) and \(A, B\) independent).

Suppose that
\[
\mathbb{E} A^\kappa \log^+ (A \wedge t) = h_A(\log t),
\]
where
\[
h_A(x) = x^\rho \ell(x),
\]
\(0 \leq \rho < 1\) and \(\ell(x)\) is a slowly varying function. Then, as \(t \to \infty\),
\[
\mathbb{E} U^\kappa I\{U \leq t\} \asymp \mathbb{E} ((AU + B)^\kappa - (AU)^\kappa) g_A(t),
\]
where
\[
g_A(t) = \begin{cases} 
\frac{\log t}{\ell(\log t)}, & \text{if } \rho = 0; \\
\frac{\sin(\pi \rho)}{\pi \rho} \frac{(\log t)^{1-\rho}}{\ell(\log t)}, & \text{if } \rho \in (0,1). 
\end{cases}
\]

In particular, if
\[
\mathbb{E} A^\kappa \log^+ A < +\infty,
\]
then, as \(t \to \infty\),
\[
\mathbb{E} U^\kappa I\{U \leq t\} \asymp \frac{\mathbb{E} ((AU + B)^\kappa - (AU)^\kappa) \kappa A^\kappa \ln A}{\kappa B^\kappa \ln B}.
\]

The proof is based on multiple application of the Karamata Tauberian Theorem. We postpone it till the end of this section.

1.2 Remark It is easy to check that \((10)\) implies
\[
\mathbb{E} U^\kappa I\{U \leq t\} \asymp \kappa C \ln t.
\]

It follows that using \((16)\) we can identify the well-known constant in \((10)\):
\[
C = \frac{\mathbb{E} ((A + BU)^\kappa - (BU)^\kappa)}{\kappa \mathbb{E} B^\kappa \log B}.
\]

1.3 Remark Now suppose that
\[
\mathbb{P}(U > t) = t^{-\kappa} \ell_1(t),
\]
for some slowly varying \(\ell_1\) (e.g. as in \([31]\)). Then by \([5\] Proposition 1.5.9a\]
\[
\ell_2(t) = \int_0^t \frac{\ell_1(s)}{s} ds
\]
is slowly varying and \(\ell_1(t)/\ell_2(t) \to 0\). Therefore
\[
\mathbb{E} U^\kappa I\{U \leq t\} \asymp \kappa \int_0^t s^{\kappa-1} \mathbb{P}(U > s) ds = \int_0^t \frac{\ell_1(s)}{s} ds.
\]

It follows that we are able to identify the asymptotics of \(\ell_1(x)\) (up to equivalence).
1.4 Remark Let us consider Kevei’s assumption (11):

$$\mathbb{E} A^\kappa I\{\log A > x\} = \ell_0(x)x^{-\alpha},$$

where $\alpha \in (0, 1]$. By the direct part of the Karamata Theorem

$$\mathbb{E} A^\kappa \log^+ (A \wedge e^x) = \int_0^x \mathbb{E} A^\kappa I\{\log A > v\} dv \asymp \ell_0(x)x^{1-\alpha}/(1 - \alpha),$$

if $\alpha \in (0, 1)$ and

$$\mathbb{E} A^\kappa \log^+ (A \wedge e^x) \asymp \int_1^x \ell_0(v) dv/v$$

is slowly varying, if $\alpha = 1$.

Proof of Theorem 1.1 First we shall establish the relation

$$\lim_{\varepsilon \searrow 0} \mathbb{E} \left[ (AU + B)^{\kappa - \varepsilon} - (AU)^{\kappa - \varepsilon} \right] =$$

$$= \mathbb{E} \left[ (AU + B)^\kappa - (AU)^\kappa \right] < +\infty. \tag{18}$$

If $\kappa \leq 1$, then $0 \leq (AU + B)^{\kappa - \varepsilon} - (AU)^{\kappa - \varepsilon} \leq B^{\kappa - \varepsilon} \leq 1 + B^\kappa$ and therefore (18) holds. Now assume that $\kappa > 1$ and set

$$k_0 = \begin{cases} \lfloor \kappa \rfloor & \text{if } \kappa \notin \mathbb{N} \\ \kappa - 1 & \text{if } \kappa \in \mathbb{N}, \end{cases}$$

so that $\kappa - k_0 > 0$ and $\kappa - \varepsilon - k_0 < 1$ for $\varepsilon > 0$. Then for small $\varepsilon$

$$0 \leq (AU + B)^{\kappa - \varepsilon} - (AU)^{\kappa - \varepsilon}$$

$$= (AU + B)^{\kappa - \varepsilon - k_0}(AU + B)^{k_0} - (AU)^{\kappa - \varepsilon - k_0}(AU)^{k_0}$$

$$\leq (AU)^{\kappa - \varepsilon - k_0}(AU + B)^{k_0} + B^{\kappa - \varepsilon - k_0}(AU + B)^{k_0} - (AU)^{\kappa - \varepsilon - k_0}(AU)^{k_0}$$

$$= (AU)^{\kappa - \varepsilon - k_0} \sum_{j=0}^{k_0-1} \binom{k_0}{j} (AU)^j B^{k_0-j} + B^{\kappa - \varepsilon - k_0} \sum_{j=0}^{k_0-1} \binom{k_0}{j} (AU)^j B^{k_0-j}$$

$$\leq (1 + (AU)^{\kappa - k_0}) \sum_{j=0}^{k_0-1} \binom{k_0}{j} (AU)^j B^{k_0-j} + (1 + B^{\kappa - k_0}) \sum_{j=0}^{k_0-1} \binom{k_0}{j} (AU)^j B^{k_0-j}.$$

The reader may verify that the finite sum in the last line above is integrable by (5), (6) and (9). It follows that (18) is valid also for $\kappa > 1$.

Let us now denote $G(\varepsilon) = 1 - \mathbb{E} A^{\kappa - \varepsilon} = \mathbb{E} (A^\kappa - A^{\kappa - \varepsilon})$ and $H(\varepsilon) = \mathbb{E} U^{\kappa - \varepsilon}$. Then we have for small $\varepsilon > 0$

$$G(\varepsilon)H(\varepsilon) = (1 - \mathbb{E} A^{\kappa - \varepsilon})\mathbb{E} U^{\kappa - \varepsilon} = \mathbb{E} (AU + B)^{\kappa - \varepsilon} - \mathbb{E} (AU)^{\kappa - \varepsilon}$$

$$= \mathbb{E} \left[ (AU + B)^{\kappa - \varepsilon} - (AU)^{\kappa - \varepsilon} \right],$$

and relation (18) states that

$$\lim_{\varepsilon \searrow 0} G(\varepsilon)H(\varepsilon) = \mathbb{E} \left[ (AU + B)^\kappa - (AU)^\kappa \right] =: D > 0. \tag{19}$$
In order to examine the asymptotics of $G(\varepsilon)$ at 0 set

$$G_1(\varepsilon) = \mathbb{E} \left( A^\kappa - A^{\kappa - \varepsilon} \right) I\{A > 1\}$$

and $G_2(\varepsilon) = G(\varepsilon) - G_1(\varepsilon)$. Direct calculation shows that

$$G_2(\varepsilon) = \varepsilon \mathbb{E} \left( A^{\kappa - \varepsilon} I\{A \leq 1\} \right) = -\varepsilon \mathbb{E} A^\kappa \ln A I\{A \leq 1\} = -\varepsilon \mathbb{E} A^\kappa \ln^{-} A,$$

where $0 \leq \mathbb{E} A^\kappa \ln^{-} A < +\infty$. If (15) holds, then also

$$G_1(\varepsilon) \approx \varepsilon \mathbb{E} A^\kappa \ln A I\{A > 1\} = \varepsilon \mathbb{E} A^\kappa \ln^{+} A,$$

and we obtain that

$$H(\varepsilon) \approx \varepsilon^{-1} \mathbb{E} ((AU + B)^\kappa - (AU)^\kappa) = \varepsilon^{-1} C'.$$

But $H(\varepsilon)$ is asymptotically the Laplace transform $L_R(\varepsilon)$ of a measure $R$ on $[0, +\infty)$ given by the formula

$$R([0, x]) = \int_{[0, x]} e^{\kappa u} P_{\ln U}(du).$$

In fact,

$$H(\varepsilon) = \mathbb{E} U^{\kappa - \varepsilon} I\{U \geq 1\} + \mathbb{E} U^{\kappa - \varepsilon} I\{U < 1\} = H_1(\varepsilon) + H_2(\varepsilon),$$

where

$$\lim_{\varepsilon \downarrow 0} H_2(\varepsilon) = \mathbb{E} U^\kappa I\{U < 1\}$$

and

$$H_1(\varepsilon) = \mathbb{E} e^{-\varepsilon \ln U} e^{\kappa \ln U} I\{\ln U \geq 0\} = \int_{\ln 1}^{\infty} e^{-\varepsilon u} R(du).$$

By the Karamata Tauberian Theorem (see e.g. [5, Theorem 1.7.1]) relation (20) is equivalent to

$$R([0, x]) = \mathbb{E} U^\kappa I\{1 \leq U \leq e^x\} \asymp C' \cdot x, \quad x \to \infty.$$ 

In other words

$$\mathbb{E} U^\kappa I\{U \leq t\} \asymp \mathbb{E} U^\kappa I\{1 \leq U \leq t\} \asymp C' \ln t,$$

what gives (16).

Passing to the general case let us assume that

$$\mathbb{E} A^\kappa \log^{+} (A \wedge t) = h_A(\log t) \to +\infty.$$ 

Let us notice that

$$G_1(\varepsilon) = \varepsilon \mathbb{E} \left( A^{\kappa - \varepsilon} I\{A > 1\} \right)$$

and

$$G_2(\varepsilon) = \varepsilon \mathbb{E} \ln A \left( \int_{\ln A}^{\infty} e^{-\varepsilon v} dv \right) I\{\ln A > 0\}.$$

Therefore

$$G_2(\varepsilon) = \varepsilon \int_{\ln 1}^{\infty} e^{-\varepsilon v} (1 - F(v)) dv = \varepsilon L_Q(\varepsilon),$$

where $L_Q(\varepsilon)$ is a Laplace transform of a measure $Q$ on $[0, +\infty)$. If (15) holds, then also

$$G_1(\varepsilon) \approx \varepsilon \mathbb{E} A^\kappa \ln A I\{A > 1\} = \varepsilon \mathbb{E} A^\kappa \ln^{+} A,$$

and we obtain that

$$H(\varepsilon) \approx \varepsilon^{-1} \mathbb{E} ((AU + B)^\kappa - (AU)^\kappa) = \varepsilon^{-1} C'.$$ 

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Therefore

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and we obtain that

$$H(\varepsilon) \approx \varepsilon^{-1} \mathbb{E} ((AU + B)^\kappa - (AU)^\kappa) = \varepsilon^{-1} C'.$$ 

By the Karamata Tauberian Theorem (see e.g. [5, Theorem 1.7.1]) relation (20) is equivalent to

$$R([0, x]) = \mathbb{E} U^\kappa I\{1 \leq U \leq e^x\} \asymp C' \cdot x, \quad x \to \infty.$$ 

In other words

$$\mathbb{E} U^\kappa I\{U \leq t\} \asymp \mathbb{E} U^\kappa I\{1 \leq U \leq t\} \asymp C' \ln t,$$

what gives (16).
where $F(v) = \mathbb{E}A^\kappa I\{A \leq e^v\}$, and

$$Q([0,x]) = \int_0^x (1 - F(v)) \, dv = \int_0^x \mathbb{E}A^\kappa I\{A > e^v\} \, dv = \int_1^{e^x} \mathbb{E}A^\kappa I\{A > u\} \, du = \mathbb{E}A^\kappa \log^+ (A \wedge e^x) = h_A(\log(e^x)) = x^\rho \ell(x).$$

Again, by the Karamata Tauberian Theorem,

$$\mathcal{L}_Q(\varepsilon) \asymp \Gamma(1 + \rho)\varepsilon^{-\rho} \ell(1/\varepsilon),$$

hence $G_1(\varepsilon) \asymp \Gamma(1 + \rho)\varepsilon^{-\rho} \ell(1/\varepsilon)$. We have

$$\frac{G_2(\varepsilon)}{G_1(\varepsilon)} \asymp \frac{-\varepsilon \mathbb{E}A^\kappa \log^- A}{\Gamma(1 + \rho)\varepsilon^{-\rho} \ell(1/\varepsilon)} = -\frac{\mathbb{E}A^\kappa \log^- A}{\Gamma(1 + \rho)} \frac{1}{\ell(1/\varepsilon)} \to 0, \quad \text{as } \varepsilon \searrow 0,$$

because (23) implies either $\rho \in (0,1)$ or $\rho = 0$ and $\ell(x) \to \infty$, $x \to \infty$. It follows that $G(\varepsilon) \asymp G_1(\varepsilon)$ and finally

$$H_1(\varepsilon) \asymp \frac{D}{\Gamma(1 + \rho)\ell(1/\varepsilon)} \varepsilon^{\rho - 1}.$$

Similarly as in the previous case, by the Karamata Tauberian Theorem we obtain

$$R([0,x]) = EU^\kappa I\{1 \leq U \leq e^x\} \asymp \frac{D}{\Gamma(1 + \rho)\Gamma(2 - \rho)} x^{1 - \rho} \ell(x), \quad x \to \infty,$$

or

$$\mathbb{E}U^\kappa I\{U \leq t\} \asymp \mathbb{E}U^\kappa I\{1 \leq U \leq t\} \asymp \frac{D}{\Gamma(1 + \rho)\Gamma(2 - \rho)} \left(\frac{\log t}{\ell(\log t)}\right)^{1 - \rho}.$$

This proves (13) and (14), for $\Gamma(1 + \rho)\Gamma(2 - \rho) = 1$ if $\rho = 0$ and

$$\Gamma(1 + \rho)\Gamma(2 - \rho) = \rho \Gamma(\rho)(1 - \rho)\Gamma(1 - \rho) = \rho(1 - \rho) \frac{\pi}{\sin(\pi \rho)},$$

if $\rho \in (0,1)$. □

2 A consequence: a weak law of large numbers for stochastic recursions

Let $\{(A_j, B_j)\}$ be an i.i.d. sequence of random vectors distributed like $(A, B)$ that satisfies (13)-(16) and let $\{U_j\}$ be a sequence given by the stochastic recursion equation

$$U_j = A_j U_{j-1} + B_j, \quad j = 1, 2, \ldots,$$

where $U_0$ is independent of $\{(A_j, B_j)\}$ and distributed according to the stationary distribution (11). Theorem 1.1 leads us to the following weak law of large numbers.
2.1 Theorem In assumptions and notation of Theorem 1.1 we have

\[ \frac{U_1 + U_2 + \ldots + U_n}{n g_A(\log n)} \xrightarrow{p} \mathbb{E}((AU + B)^\kappa - (AU)^\kappa). \]  

We shall obtain this theorem from a more general result that might be of independent interest.

2.2 Theorem Let \( \{Y_j\} \) be a sequence of non-negative random variables with identical distribution \( Y_j \sim Y_j, j = 1, 2, \ldots \). We assume that \( \ell(x) = \mathbb{E}I(Y \leq x) \) is slowly varying and satisfies both

\[ \lim_{x \to \infty} \frac{\ell(x)}{\ell(x)} = 1, \]  

and

\[ \lim_{x \to \infty} \frac{\ell(x/\log x)}{\ell(x)} = 1. \]  

Moreover, we assume that there are numbers \( 0 < \eta < 1, h_0 > 0 \) and \( C_\infty > 0 \) such that for all \( i, j \in \mathbb{N} \) and \( h \geq h_0 \)

\[ \mathbb{E}(\chi_h(Y_i)\chi_h(Y_j)) - \mathbb{E}(\chi_h(Y_i))\mathbb{E}(\chi_h(Y_j)) \leq C_\infty h^2 \eta |j-i|, \]

where for \( h > 0 \)

\[ \chi_h(x) = \begin{cases} x, & \text{if } |x| < h; \\ h, & \text{if } x \geq h; \\ -h, & \text{if } x \leq -h. \end{cases} \]

Then

\[ \frac{Y_1 + Y_2 + \ldots + Y_n}{n\ell(n)} \xrightarrow{p} 1. \]

Before proving both theorems let us make some comments.

2.3 Remark Conditions (26) and (27) are independent. For example, function \( \exp(\log x/\log \log x) \) satisfies (26) and does not satisfy (27), while the function given by formula (55) does not satisfy (27) and it does (26).

2.4 Remark Condition (28) resembles the well-known \( \alpha \)-mixing at exponential rate. Notice, however, that it is automatically satisfied by pairwise independent sequences and, more generally, by negative quadrant dependent sequences. See [27] for an example of a stationary sequence of bounded random variables that is pairwise independent and is not exponentially \( \alpha \)-mixing.

Proof of Theorem 2.1

Let \( C_{(A,B)} = \mathbb{E}((AU + B)^\kappa - (AU)^\kappa) \). By Theorem 1.1 \( \ell(x) = \mathbb{E}I(U \leq x) = C_{(A,B)} g_A(\log x) \) is slowly varying. The reader may directly verify that both (26) and (27) are satisfied. It remains to prove that (28) holds.

By stationarity it is enough to estimate from above the quantity

\[ \sigma_j^h = \mathbb{E}(\chi_h(Y_j)\chi_h(U_0)) - \mathbb{E}(\chi_h(Y_j))\mathbb{E}(\chi_h(U_0)). \]
Iterating (24) and using the independence of \( U_0 \) and \( \{ (A_j, B_j) \} \) we get

\[
\mathbb{E}(\chi_h(U_j)\chi_h(U_0)) = \mathbb{E}(\chi_h(U_0 \prod_{i=1}^{j} A_i + \sum_{k=1}^{j} B_k \prod_{i=k+1}^{j} A_i)\chi_h(U_0)) \\
\leq \mathbb{E}(\chi_h(U_0 \prod_{i=1}^{j} A_i)\chi_h(U_0)) + \mathbb{E}(\chi_h(\sum_{k=1}^{j} B_k \prod_{i=k+1}^{j} A_i)\chi_h(U_0)) \\
= \mathbb{E}(\chi_h(U_0 \prod_{i=1}^{j} A_i)\chi_h(U_0)) + \mathbb{E}(\chi_h(U_0)\mathbb{E}(\chi_h(U_0))) \\
\leq \mathbb{E}(\chi_h(U_0 \prod_{i=1}^{j} A_i)\chi_h(U_0)) + \mathbb{E}(\chi_h(U_0))^2.
\]

Therefore

\[
\sigma_t^h = \mathbb{E}(\chi_h(U_j)\chi_h(U_0)) - \mathbb{E}(\chi_h(U_j))\mathbb{E}(\chi_h(U_0)) \leq \mathbb{E}(\chi_h(U_0 \prod_{i=1}^{j} A_i)\chi_h(U_0)).
\]

Let us notice that for \( x \geq 0 \) and \( \epsilon \in (0, 1) \)

\[
\chi_h(x) = h \cdot \chi_1(x/h) \leq h(x/h)^\epsilon = h^{1-\epsilon}x^\epsilon.
\]

It follows that for \( \epsilon \in (0, \kappa \wedge 1) \) and \( h \geq 1 =: h_0 \) we have

\[
\mathbb{E}(\chi_h(U_0 \prod_{i=1}^{j} A_i)\chi_h(U_0)) \leq h^{2-\epsilon}\mathbb{E}U_0^\epsilon(\mathbb{E}A_1^\epsilon)^j \leq C_\infty h^2\eta^j,
\]

where \( C_\infty = \mathbb{E}U_0^\epsilon < +\infty \) by \((9)\) and \( \eta = \mathbb{E}A_1^\epsilon < 1 \) by \((7)\). \( \square \)

**Proof of Theorem 2.2**

By Theorem \([A.1]\) we have

\[
\ell(x) = \mathbb{E}YI(Y \leq x) \asymp \ell_1(x) = \mathbb{E}\chi_x(Y),
\]

so it is enough to prove 2.7 with \( \ell_1(x) \) in place of \( \ell(x) \). Moreover, it is easy to see that \( \ell_1(x) \) satisfies both \((20)\) and \((27)\). Set \( b_n = n\ell_1(n) \).

By \((20)\), \((27)\) and \((54)\) we have:

\[
\ell_1(b_n / \log n) = \ell_1(n\ell_1(n)/\log n) = \ell_1\left(\frac{n}{\log n} - \ell_1(n/\log n) \frac{\ell_1(n)}{\ell_1(n/\log n)}\right) \\
\asymp \ell_1\left(\frac{n\ell_1(n/\log n)}{\log n}\right) \asymp \ell_1(n/\log n) \\
\asymp \ell_1(n) \asymp \ell_1(n\ell_1(n)) = \ell_1(b_n).
\]

8
This implies that for every \( t > 0 \)
\[
\frac{\ell_1(tb_n / \log n)}{\ell_1(b_n)} \asymp \frac{\ell_1(b_n / \log n)}{\ell_1(b_n)} \asymp 1.
\]

Therefore there exists a sequence \( a_n \downarrow 0 \) such that
\[
\frac{\ell_1(a_nb_n / \log n)}{\ell_1(b_n)} \to 1, \text{ as } n \to \infty. \tag{30}
\]

For the sake of clarity, let us denote
\[
Y^{(h)} = \chi_h(Y).
\]

**2.5 Lemma**
\[
\frac{Y_1 + Y_2 + \ldots + Y_n}{b_n} \overset{p}{\to} 1
\]
if, and only if,
\[
\frac{Y_1^{((a_n / \log n)b_n)} + Y_2^{((a_n / \log n)b_n)} + \ldots + Y_n^{((a_n / \log n)b_n)}}{b_n} \overset{p}{\to} 1.
\]

**Proof.** By Corollary A.2, \( n\mathbb{P}(Y > b_n) \to 0 \), hence
\[
\mathbb{P}\left( Y_1 + Y_2 + \ldots + Y_n \neq Y_1^{(b_n)} + Y_2^{(b_n)} + \ldots + Y_n^{(b_n)} \right)
= \mathbb{P}\left( \bigcup_{j=1}^n \{Y_j > b_n\} \right) \leq n\mathbb{P}(Y > b_n) \to 0.
\]

Next let us consider
\[
D_n = \frac{Y_1^{(b_n)} + \ldots + Y_n^{(b_n)} - Y_1^{((a_n / \log n)b_n)} - \ldots - Y_n^{((a_n / \log n)b_n)}}{b_n} \geq 0.
\]

We have by (30)
\[
\mathbb{E}D_n = \frac{n}{b_n} \left( \ell(b_n) - \ell((a_n / \log n)b_n) \right) = \frac{n\ell(b_n)}{b_n} \left( 1 - \frac{\ell((a_n / \log n)b_n)}{\ell(b_n)} \right) \to 0. \square
\]

Let us denote
\[
T_n = \frac{Y_1^{((a_n / \log n)b_n)} + Y_2^{((a_n / \log n)b_n)} + \ldots + Y_n^{((a_n / \log n)b_n)}}{b_n}.
\]

We have
\[
\mathbb{E}T_n = \frac{n\ell_1((a_n / \log n)b_n)}{b_n} = \frac{\ell_1((a_n / \log n)b_n)}{\ell_1(b_n)} \frac{n\ell_1(b_n)}{b_n} \to 1.
\]

It follows that we shall complete the proof of Theorem 2.2 by showing that \( T_n - \mathbb{E}T_n \to L^2 0 \).

Let \( K > 0 \) be such that
\[
K \log \eta < -1,
\]
and let 

\[ m_n = \lceil K \log n \rceil. \]

We shall split the components in \( \mathbb{E}(T_n - \mathbb{E}T_n)^2 \) into two groups.

\[
\sum_{1 \leq i,j \leq n} \mathbb{E}\left( \frac{Y_i (\frac{a_n}{n \log n} b_n) - \mathbb{E}Y_i (\frac{a_n}{n \log n} b_n)}{b_n} \right) \left( \frac{Y_j (\frac{a_n}{n \log n} b_n) - \mathbb{E}Y_j (\frac{a_n}{n \log n} b_n)}{b_n} \right) \leq C \frac{a_n^2}{n \log n} b_n^2 \sum_{1 \leq i,j \leq n} \eta |i-j| \leq \frac{2C \infty}{1 - \eta} (n - m_n) \eta^m_n
\]

\[
\leq \frac{2C \infty}{1 - \eta} \exp (\log n + (\log \eta) K \log n) \to 0.
\]

So we have to consider the remaining covariances only.

\[
\left| \sum_{1 \leq i,j \leq n} \mathbb{E}\left( \frac{Y_i (\frac{a_n}{n \log n} b_n) - \mathbb{E}Y_i (\frac{a_n}{n \log n} b_n)}{b_n} \right) \left( \frac{Y_j (\frac{a_n}{n \log n} b_n) - \mathbb{E}Y_j (\frac{a_n}{n \log n} b_n)}{b_n} \right) \right| \leq \sum_{i=1}^{n-1} \sum_{j=i+1}^{n-1} \mathbb{E}\left( \frac{Y_i (\frac{a_n}{n \log n} b_n) - \mathbb{E}Y_i (\frac{a_n}{n \log n} b_n)}{b_n} \right) \left( \frac{Y_j (\frac{a_n}{n \log n} b_n) - \mathbb{E}Y_j (\frac{a_n}{n \log n} b_n)}{b_n} \right)
\]

\[
\leq n \mathbb{E}\left( \frac{Y_i (\frac{a_n}{n \log n} b_n)}{b_n} \right)^2 + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \mathbb{E}\left( \frac{Y_i (\frac{a_n}{n \log n} b_n)}{b_n} \right)^2 + \mathbb{E}\left( \frac{Y_j (\frac{a_n}{n \log n} b_n)}{b_n} \right)^2 \leq (n + 2(n - 1)m_n) \mathbb{E}\left( \frac{Y_i (\frac{a_n}{n \log n} b_n)}{b_n} \right)^2 \leq 3 \frac{a_n}{n \log n} m_n n \mathbb{E}\left( \frac{Y_i (\frac{a_n}{n \log n} b_n)}{b_n} \right)
\]

\[
= 3 \left( \frac{a_n}{n \log n} \lceil K \log n \rceil \right) \frac{\ell (\frac{a_n}{n \log n} b_n)}{\ell (b_n)} \frac{\ell (b_n)}{b_n} \times 3Kn \to 0.
\]

This finishes the proof of Theorem 2.2. \( \square \)

2.6 Remark It should be pointed out that in the above proof properties (26) and (27) are used in order to cope with the convergence \( T_n - \mathbb{E}T_n \to L^2 \) only. If we know more on \( \{Y_j\} \) (e.g. pairwise independence) we obtain a complete analogue of the independent case, as the next theorem shows. This is not surprising, for many of results on the a.s. convergence or the convergence in probability rely on two-dimensional joint distributions only (see e.g. [34, 22 Remark 3.2, p. 276]).

Recall that two random variables \( X \) and \( Y \) are negatively quadrant dependent (NQD) (see [34, also 29]), if

\[
\mathbb{P}(X > x, Y > y) \leq \mathbb{P}(X > x) \mathbb{P}(Y > x), \quad x, y \in \mathbb{R}^1.
\]
If \( X \) and \( Y \) are NQD, then by the well-known Hoeffding’s identity
\[
E\chi_h(X)\chi_h(Y) - E\chi_h(X)E\chi_h(Y) = \int_{-h}^{h} \int_{-h}^{h} \left( P(X > x, Y > y) - P(X > x)P(Y > y) \right) dx\,dy \leq 0,
\]
and therefore, if \( Y_1, \ldots, Y_n \) are NQD
\[
\text{Var}(\chi_h(Y_1) + \ldots + \chi_h(Y_n)) \leq \text{Var}(\chi_h(Y_1)) + \ldots + \text{Var}(\chi(Y_n)).
\]

2.7 Theorem Let \( \{Y_j\} \) be a sequence of non-negative and NQD random variables with identical distribution \( Y_j \sim Y \), \( j = 1, 2, \ldots \). Suppose that \( \ell(x) = EYI(Y \leq x) \) is slowly varying and \( \{b_n\} \) satisfies
\[
\frac{n\ell(b_n)}{b_n} \to 1, \text{ as } n \to \infty.
\]
Then
\[
\frac{Y_1 + Y_2 + \ldots + Y_n}{b_n} \xrightarrow{p} 1. \tag{31}
\]

Proof. Let \( a_n \to 0 \) be such that
\[
\frac{\ell(a_nb_n)}{\ell(b_n)} \to 1.
\]
Then by arguments identical as in the proof of Lemma 2.5 (with \( \{a_n\} \) alone replacing \( \{a_n/\log n\} \)) convergence (31) holds if, and only if,
\[
\frac{Y_1^{(a_nb_n)} + Y_2^{(a_nb_n)} + \ldots + Y_n^{(a_nb_n)}}{b_n} \xrightarrow{p} 1.
\]
Notice that also
\[
E\left(\frac{Y_1^{(a_nb_n)} + Y_2^{(a_nb_n)} + \ldots + Y_n^{(a_nb_n)}}{b_n}\right) = \frac{n\ell(a_nb_n)}{b_n} = \frac{n\ell(b_n)}{b_n}\frac{\ell(a_nb_n)}{\ell(b_n)} \to 1.
\]
Therefore the following natural estimate completes the proof.
\[
\text{Var}\left(\frac{Y_1^{(a_nb_n)} + Y_2^{(a_nb_n)} + \ldots + Y_n^{(a_nb_n)}}{b_n}\right) \leq \frac{n\text{Var}(Y_1^{(a_nb_n)})}{b_n^2} \leq a_n\frac{n\ell(a_nb_n)}{b_n} \to 0.
\]

\[
\square
\]

3 Another consequence: a central limit theorem for GARCH(1,1) processes

A sequence \( \{X_j\} \) of random variables is said to be a GARCH(1,1) process if
\[
X_j = \sigma_j Z_j, \tag{32}
\]
\[
\sigma_j^2 = \beta + \lambda X_{j-1}^2 + \delta \sigma_{j-1}^2, \tag{33}
\]
where the constants \( \beta, \lambda, \delta \) are nonnegative, \( \{Z_j\} \) is an i.i.d. multiplicative noise, \( \sigma_j \geq 0 \) and \( X_0 \) and \( \sigma_0^2 \) are given and independent of \( \{Z_j\}_{j \geq 1} \). If \( \delta = 0 \) in (33) then the corresponding process is called ARCH(1) process.

The terminology (ARCH stands for “Autoregressive Conditionally Heteroskedastic” while GARCH is the “Generalized ARCH”) was introduced by Engle [15] and Bollerslev [7] in the context of modeling volatility phenomena in econometric time series. Engle considered only normally distributed noise variables, but this is too restrictive and it is reasonable to assume only that

\[
EZ_j = 0, \quad EZ_j^2 = 1. \tag{34}
\]

There exists a huge literature on both theoretical and practical aspects of GARCH processes. As an excellent mathematical introduction to ARCH(1) processes may serve [14]. Mathematics of GARCH(1,1) processes is studied in detail in [8], [33], [4], see also [10, Chapters 2 and 3]. For financial aspects of modeling with GARCH(1,1) processes we refer to the extensive sources [2] and [18].

Here we shall focus on seldom investigated properties of GARCH processes related to the threshold condition \( \lambda + \delta = 1 \).

It is well known that if \( \lambda + \delta < 1 \), then there exists a strictly stationary sequence \( \{(X_j, \sigma_j^2)\} \) built on the i.i.d. noise \( \{Z_j\}_{j \in \mathbb{Z}} \), satisfying (32) and (33) and such that

\[
E\sigma_j^2 = EX_j^2 = \frac{\beta}{1 - \lambda - \delta}. \tag{35}
\]

If \( \lambda + \delta > 1 \) and the stationary solution exists, then it has heavy tails (see [3], [10] for the corresponding limit theory with stable limits).

When \( \lambda + \delta = 1 \) and \( \beta > 0 \), a simple choice \( Z_j = \pm 1 \) with probability \( 1/2 \) provides an example with no stationary solution. It is not difficult to show (see e.g. [10]) that a necessary and sufficient condition for the existence of a (unique in law) stationary distribution for (32) and (33) is that

\[
\beta > 0, \quad \text{and} \quad E \log (\lambda Z_1^2 + \delta) < 0. \tag{36}
\]

In any case, if the stationary solution exists, it is of infinite variance. This makes the modeling with GARCH a delicate problem, for estimates performed on real data often give the value of \( \lambda + \delta \) very close to 1 (e.g. 0.995 - see [36], also [16]). It follows that the critical case \( \lambda + \delta = 1 \) is interesting from the point of view of both mathematics and econometrics.

Here we are going to prove a central limit theorem for GARCH(1,1) processes in the case when \( \lambda + \delta = 1 \) and under minimal assumptions on the marginal distribution of the noise sequence \( \{Z_j\} \).

To give a flavor of necessary reasoning we begin with discussion of two central limit theorems for ARCH(1) processes (\( \delta = 0 \)). For the time being we shall assume that

the noise i.i.d. variables \( \{Z_j\} \) are standard normal.

Then \( \lambda < 1 \) implies

\[
\frac{X_1 + X_2 + \ldots + X_n}{\sqrt{n}} \xrightarrow{D} \mathcal{N}(0, \frac{\beta}{1 - \lambda}), \tag{37}
\]

while \( \lambda = 1 \) implies

\[
\frac{X_1 + X_2 + \ldots + X_n}{\sqrt{n \ln n}} \xrightarrow{D} \mathcal{N}(0, C_{\beta,1}), \tag{38}
\]
where

\[ C_{\beta,1} = \frac{\beta}{E[(Z_1^2 \ln(Z_1^2))] \approx 1.3705 \cdot \beta. \]

(37) can be proved in various ways. One possible direction is based on mixing properties of GARCH processes. Mikosch and Stáricá [33] proved that GARCH(1,1) processes with Gaussian noise are strongly (or \(\alpha\)-)mixing with exponential rate. This means that \(\alpha(n) \leq Kn^\eta\) for some constants \(K > 0\) and \(\eta \in [0,1]\), where for a stochastic process \(\{Y_k\}_{k \in \mathbb{N}}\) the well-known coefficient \(\alpha(n) = \alpha(n,\{Y_k\})\) is defined as

\[ \alpha(n) = \sup \{|P(A \cap B) - P(A)P(B)| : A \in \mathcal{F}_1^n, B \in \mathcal{F}_{m+n}^\infty, m \in \mathbb{N} \}, \]

with \(\mathcal{F}_1^n = \sigma\{Y_k : k \leq m\}\) and \(\mathcal{F}_m^n = \{Y_k : k \geq m + n\}\) (see e.g. [9] or [12] for properties and examples). Since we have exponential \(\alpha\)-mixing and there exist moments higher than 2 (due to \(\lambda < 1\)), (37) is a direct consequence of Ibragimov’s CLT for strongly mixing sequences (see e.g. [23] Theorem 18.5.3, p. 346).

On the other hand \(\{X_{n,k} = \frac{X_n}{\sqrt{n}} : k = 1,2,\ldots,n, n \in \mathbb{N}\}\) is a square integrable martingale difference array, so one might also use a suitable version of the Martingale CLT, as it is done later in this section.

To avoid technicalities we prefer another proof, based on the fact that the regular conditional distribution of \(X_n\) with respect to the “past” is \(\mathcal{N}(0,1+n\lambda X_{n-1}^2)\). Let us recall a device related to the Principle of Conditioning (see [25], [26] and [13] Appendix) for an extended version.

3.1 Lemma Let \(\{X_{n,k} : k = 1,2,\ldots,k_n, n \in \mathbb{N}\}\) be an array of random variables which are row-wise adapted to a sequence of filtrations \(\{\mathcal{F}_{n,k}\}_{n \in \mathbb{N}}\). Define

\[ \phi_{n,k}(\theta) = E(e^{i\theta X_{n,k}} | \mathcal{F}_{n,k-1}), \quad \phi_n(\theta) = \phi_{n,1}(\theta) \cdot \phi_{n,2}(\theta) \cdot \ldots \cdot \phi_{n,n}(\theta). \]

If \(\phi_n(\theta) \xrightarrow{p} C(\theta) \neq 0\), then also

\[ Ee^{i\theta (X_{n,1} + X_{n,2} + \ldots + X_{n,n})} \rightarrow C(\theta). \]

Given the above lemma, the proof of (37) is in one line: setting \(X_{n,k} = X_k/\sqrt{n}\) and applying the individual ergodic theorem one obtains:

\[ -\ln \phi_n(\theta) = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{2} \theta^2 (\beta + \lambda X_{k-1}^2) \rightarrow \frac{1}{2} \theta^2 (\beta + \lambda E X_0^2) = \frac{1}{2} \theta^2 \frac{\beta}{1 - \lambda} \quad \text{a.s.} \]

When we try to prove (38) the same way, we obtain (\(\lambda = 1\)):

\[ -\ln \phi_n(\theta) = \frac{1}{n \ln n} \sum_{k=1}^{n} \frac{1}{2} \theta^2 (\beta + X_{k-1}^2) \rightarrow \frac{1}{2} \theta^2 \frac{1}{n \ln n} \sum_{k=1}^{n} X_{k-1}^2 \]

and the convergence in probability of \(\phi_n(\theta)\) is not obvious, unless we have at disposal a weak law of large numbers for \(\{X_k^2\}\)! A suitable law of large numbers and the corresponding central limit theorem (38) were proved in [37] Example 1.

It should be pointed out that the results of (37) rely heavily on the assumption of exponential \(\alpha\)-mixing as well as on (10) held for \(\{X_k^2\}\) with \(\kappa = 1\) (Kesten’s regularity). We know
from Section 1 that the power tail decay does not hold in many cases. Similarly, there seems to be no general result on exponential $\alpha$-mixing valid for all GARCH(1,1) processes (see \[4\] or \[10\] Proposition 2.2.4, p. 23).

The main advantage of our approach is that we can use the weak law of large numbers given in Theorem 2.1, where we need only natural non-degeneracy assumptions (3)–(6) and a weak one-sided covariance bound given by (28), ideally suited for stationary solutions to stochastic recurrence equations, hence also for GARCH processes.

3.2 Theorem Suppose that $\lambda > 0$, $\lambda + \delta = 1$, $\mathbb{P}(Z_1^2 \neq 1) > 0$ and that

$$E(1 + \lambda(Z_1^2 - 1)) \log^+ ((1 + \lambda(Z_1^2 - 1)) \wedge t) = h_A(\log t),$$

where

$$h_A(x) = x^\rho \ell(x),$$

$0 \leq \rho < 1$ and $\ell(x)$ is a slowly varying function. Then we have

$$\frac{X_1 + X_2 + \ldots + X_n}{\sqrt{n} g_A(\log n)} \to_D \mathcal{N}(0, \beta),$$

$$\frac{X_1^2 + X_2^2 + \ldots + X_n^2}{n g_A(\log n)} \to_\mathcal{P} \beta,$$

$$\frac{\sigma_1^2 + \sigma_2^2 + \ldots + \sigma_n^2}{n g_A(\log n)} \to_\mathcal{P} \beta,$$

where $g_A(x)$ is given by (14).

In particular, if

$$E(1 + \lambda(Z_1^2 - 1)) \log^+ (1 + \lambda(Z_1^2 - 1)) < +\infty,$$

then

$$\frac{X_1 + X_2 + \ldots + X_n}{\sqrt{n} \log n} \to_D \mathcal{N}(0, \beta C_{\lambda,Z}),$$

$$\frac{X_1^2 + X_2^2 + \ldots + X_n^2}{n \log n} \to_\mathcal{P} \beta C_{\lambda,Z},$$

$$\frac{\sigma_1^2 + \sigma_2^2 + \ldots + \sigma_n^2}{n \log n} \to_\mathcal{P} \beta C_{\lambda,Z},$$

where

$$C_{\lambda,Z} = \frac{1}{E(1 + \lambda(Z_1^2 - 1)) \log(1 + \lambda(Z_1^2 - 1))}.$$
3.3 Theorem In assumptions and notation of Theorem 3.2, relation (39) implies that
\[ \frac{S_n(t)}{\sqrt{n} g_A(\log n)} \rightarrow_D \sqrt{\beta} W(t), \] (47)
and (43) implies that
\[ \frac{S_n(t)}{\sqrt{n \log n}} \rightarrow_D \sqrt{\beta C_{\lambda,Z}} W(t). \] (48)
where in both cases the convergence in law holds on the space \( D([0, +\infty)), J_1 \).

3.4 Remark Clearly, from the point of view of possible applications relations (44)–(46) and (48) are the most important, for they refer to common noise variables \( \{Z_j\} \) (like \( N(0, 1) \), normalized \( t \)-Student’s distributions, etc.). On the other hand, more general relations (40)–(42) and (47) illustrate the remarkable flexibility of the model.

3.5 Remark With the law of \( Z \) fixed, function \( y(\lambda) = C_{\lambda,Z} \) is strictly decreasing from \(+\infty\) (for \( \lambda = 0^+ \)) to \( 1/\mathbb{E}Z^2 \log Z^2 \) (for \( \lambda = 1 \)). As an example may serve
\[ C_{\lambda,Z} = \frac{2}{(1 + \lambda) \log (1 + \lambda) + (1 - \lambda) \log (1 - \lambda)}, \]
obtained for
\[ Z = \begin{cases} 
\sqrt{2}, & \text{with probability } 1/4, \\
0, & \text{with probability } 1/2, \\
-\sqrt{2}, & \text{with probability } 1/4.
\end{cases} \]
Notice that in this simple example Kesten’s regularity does not hold.

Proofs of Theorems 3.2 and 3.3
Let us consider the stochastic recurrence equation implied by (32) and (33) and specified for the case \( \lambda > 0, \lambda + \delta = 1 \).
\[ \sigma_j^2 = (\lambda Z_j^2 + \delta)\sigma_{j-1}^2 + \beta = (1 + \lambda (Z_j^2 - 1))\sigma_{j-1}^2 + \beta = A_j\sigma_{j-1}^2 + B_j. \]
Condition (3) is satisfied if \( \mathbb{P}(X_j^2 \neq 1) > 0 \) and (4) holds if \( \beta > 0 \). Notice that these non-degeneracy assumptions exclude the trivial case when \( X_j^2 \equiv 1 \) and when there is no stationary solution to (32)–(33). Condition (5) holds for \( \kappa = 1 \) and therefore
\[ \mathbb{E}((AU + B)^\kappa - (AU)^\kappa) = \beta. \]
Finally (6) is trivial.

It follows that that we may apply Theorem 2.1 to the sequence \( \{\sigma_j^2\} \):
\[ \frac{\sigma_1^2 + \sigma_2^2 + \ldots + \sigma_n^2}{n g_A(\log n)} \rightarrow_P \beta. \] (49)
Let
\[ s_{n,j} = \sqrt{\chi_1 \left( \frac{\sigma_j^2}{n g_A(\log n)} \right)}, \quad n, j \in \mathbb{N}. \]
We have by (49) and Corollary A.2
\[ \sum_{j=1}^{n} \varsigma_{n,j}^2 \xrightarrow{p} \beta. \]

In fact, by the row-wise stationarity of \( \{\varsigma_{n,j}^2\} \) and the 1-regular variation of \( b_n = ng_{A}(\log n) \) we have more:
\[ Q_n(t) = \sum_{j=1}^{[nt]} \varsigma_{n,j}^2 \xrightarrow{p} t\beta, \quad t \geq 0. \]
(50)

Set
\[ Y_{n,j} = \varsigma_{n,j}Z_j, \quad j, n \in \mathbb{N}, \quad \Sigma_n(t) = \sum_{j=1}^{[nt]} Y_{n,j}, \quad t \geq 0, n \in \mathbb{N}. \]

By Corollary A.2 we have for each \( T > 0 \)
\[ \mathbb{P}(\exists t \in [0,T] \frac{S_n(t)}{\sqrt{ng_{A}(\log n)}} \neq \Sigma_n(t)) \leq (nT)\mathbb{P}(\sigma_1^2 > b_n) \rightarrow 0, \]
(51)
and so it is enough to prove a functional limit theorem for processes \( \Sigma_n(t) \). Notice that \( \{Y_{n,j}\} \) is a martingale difference array for which (50) gives the convergence of conditional variances:
\[ \sum_{j=1}^{[nt]} \mathbb{E}(Y_{n,j}^2 | \mathcal{F}_{j-1}) = Q_n(t) \xrightarrow{p} t\beta, \quad t \geq 0, \]

where \( \mathcal{F}_j = \sigma(\sigma_1^2, Z_1, Z_2, \ldots, Z_j), \quad j \in \mathbb{N} \). By [24, Theorem 3.33, p. 478] we have to check the Lindeberg condition in the conditional form.

Let \( a_n \xrightarrow{\gamma} 0 \) be given by (59). Let us notice that
\[ I(\varsigma_{n,j}^2 Z_j^2 > \epsilon) = I(\varsigma_{n,j}^2 Z_j^2 > \epsilon, \varsigma_{n,j}^2 \leq a_n) + I(\varsigma_{n,j}^2 Z_j^2 > \epsilon, \varsigma_{n,j}^2 > a_n) \leq I(Z_j^2 > \epsilon/a_n) + I(\varsigma_{n,j}^2 > a_n). \]

Therefore
\[ \mathbb{E}(Y_{n,j}^2 I(Y_{n,j}^2 > \epsilon) | \mathcal{F}_{j-1}) = \mathbb{E}(\varsigma_{n,j}^2 Z_j^2 I(\varsigma_{n,j}^2 Z_j^2 > \epsilon) | \mathcal{F}_{j-1}) \leq \varsigma_{n,j}^2 \mathbb{E}(Z_j^2 > \epsilon/a_n) + \varsigma_{n,j}^2 I(\varsigma_{n,j}^2 > a_n), \]
and
\[ \sum_{j=1}^{[nt]} \mathbb{E}(Y_{n,j}^2 I(Y_{n,j}^2 > \epsilon) | \mathcal{F}_{j-1}) \leq Q_n(t)\mathbb{E}(Z_j^2 > \epsilon/a_n) + \sum_{j=1}^{[nt]} \varsigma_{n,j}^2 I(\varsigma_{n,j}^2 > a_n). \]

The first term on the right hand side trivially converges in probability to 0, while for the second term we obtain by (59)
\[ \mathbb{P}\left( \sum_{j=1}^{[nt]} \varsigma_{n,j}^2 I(\varsigma_{n,j}^2 > a_n) > 0 \right) \leq nt \mathbb{P}(\sigma_1^2 > a_n b_n) \rightarrow 0, \quad t > 0. \]
It follows that on the space \( \left( D([0, +\infty)), J_1 \right) \)
\[
\Sigma_n(t) \rightarrow_D \sqrt{\beta}W(t),
\]
hence by \( \text{(51)} \) we have also \( \text{(47)} \). Applying again \[24, \text{Theorem 3.33, p. 478}\] we obtain \( \text{(41)} \).
\[\square.\]

A On slowly varying functions

In this section we gather some properties of slowly varying functions which are crucial for our reasoning.

A measurable positive function \( \ell : [x_0, +\infty) \rightarrow \mathbb{R}^+, x_0 > 0 \), is slowly varying, if for every \( t > 0 \)
\[
\lim_{x \rightarrow \infty} \frac{\ell(tx)}{\ell(x)} = 1.
\]
By \[5, \text{Theorem 1.5.13}\] there exists the Bruin conjugate \( \ell^\#(x) \) of \( \ell(x) \) that is determined uniquely up to the asymptotic equivalence by the relations
\[
\lim_{x \rightarrow \infty} \ell(x)\ell^\#(x\ell(x)) = 1, \quad \lim_{x \rightarrow \infty} \ell^\#(x)\ell(x\ell^\#(x)) = 1.
\]
If we set \( b_n = n\ell_0^\#(n) \), where \( \ell_0(x) = 1/\ell(x) \), then by the second relation above \( \{b_n\} \) satisfies
\[
\frac{n\ell(b_n)}{b_n} \rightarrow 1, \quad \text{as } n \rightarrow \infty. \quad (52)
\]
In particular, there exists a sequence \( a_n \downarrow 0 \) such that
\[
\frac{\ell(a_nb_n)}{\ell(b_n)} \rightarrow 1, \quad \text{as } n \rightarrow \infty. \quad (53)
\]
Indeed, \( b_n \rightarrow \infty \), hence we have \( \ell(tb_n)/\ell(b_n) \rightarrow 1 \) for every \( t > 0 \). Therefore \( \text{(53)} \) holds if \( a_n \downarrow 0 \) slowly enough.

The construction of \( b_n \) is considerably easier, if \( \ell(x) \) satisfies \( \text{(26)} \). In such a case
\[
\frac{n\ell(n\ell(n))}{n\ell(n)} = \frac{\ell(n\ell(n))}{\ell(n)} \rightarrow 1, \quad \text{as } n \rightarrow \infty,
\]
and it is enough to set
\[
b_n = n\ell(n). \quad (54)
\]
It should be pointed out that not all slowly varying functions satisfy \( \text{(26)} \). A suitable example can be taken from \[6, \text{p. 302}\]:
\[
\ell(x) = \exp \left( (\log x)^\beta \right), \quad \frac{1}{2} < \beta < 1. \quad (55)
\]

The next fact can be deduced from \[17, \text{p. 283, Theorem 2}\]. Since it is of crucial importance for our reasoning and since the proofs in \[17\] are a bit informal we provide here a direct proof based on core properties of slowly varying functions.
A.1 Theorem  Let \( Y \) be a non-negative random variable such that
\[
\ell(x) = \mathbb{E}Y I(Y \leq x)
\]
is a slowly varying function. Then
\[
\lim_{x \to \infty} \frac{xP(Y > x)}{\mathbb{E}Y I(Y \leq x)} = 0. \tag{56}
\]
In particular
\[
\ell_1(x) = EY \vee x = \mathbb{E}Y I(Y \leq x) + xP(Y > x) \asymp \ell(x). \tag{57}
\]

PROOF.  By the Fubini theorem we have for \( x > 0 \)
\[
\int_x^\infty \left( \mathbb{E}Y I(Y \leq y) - \mathbb{E}Y I(Y \leq x) \right) \frac{dy}{y^2} = EY \int_x^\infty \frac{dy}{y^2} - EY I(Y \leq x) \frac{1}{x} = EY \frac{x}{x \vee Y} - EY I(Y \leq x) \frac{1}{x} = EY I(Y \leq x) \frac{1}{x} + P(Y > x) - EY I(Y \leq x) \frac{1}{x} = P(Y > x).
\]
Therefore
\[
\frac{xP(Y > x)}{\mathbb{E}Y I(Y \leq x)} = \frac{x}{\ell(x)} \int_x^\infty \left( \ell(x) - \ell(x) \right) \frac{dy}{y^2} = \frac{x}{\ell(x)} \int_x^\infty \left( \ell(tx) - \ell(x) \frac{1}{t} \right) \frac{dt}{t^2} = \int_1^\infty \left( \frac{\ell(tx)}{\ell(x)} - \frac{1}{t} \right) \frac{dt}{t^2}.
\]
Take \( \delta \in (0, 1) \). By the Potter Theorem [5 Theorem 1.5.6] there exist constants \( C_1, C_2 \geq 1 \) such that for \( u \geq v \geq C_2 \) we have
\[
\frac{\ell(u)}{\ell(v)} \leq C_1 \frac{u^\delta}{v^\delta}.
\]
Take \( \varepsilon > 0 \) and let \( T \geq C_2 \) be such that
\[
C_1 \int_T^\infty \frac{dt}{t^{2-\delta}} < \varepsilon/2.
\]
Further, by the Uniform Convergence Theorem [5 Theorem 1.2.1], let \( x_0 \geq C_2 \) be such that
\[
\sup_{t \in [1,T]} \left( \frac{\ell(tx)}{\ell(x)} - 1 \right) < \varepsilon/2, \quad x \geq x_0.
\]
Then for \( x \geq x_0 \) we have
\[
\int_1^\infty \left( \frac{\ell(tx)}{\ell(x)} - 1 \right) \frac{dt}{t^2} = \int_1^T \left( \frac{\ell(tx)}{\ell(x)} - 1 \right) \frac{dt}{t^2} + \int_T^\infty \left( \frac{\ell(tx)}{\ell(x)} - 1 \right) \frac{dt}{t^2} \leq \int_1^T \frac{\varepsilon}{2} \frac{dt}{t^2} + \int_T^\infty C_1 \ell^\delta \frac{dt}{t^2} = \varepsilon.
\]
A.2 Corollary In assumptions of Theorem A.1, if \( b_n \) is given by (52), then
\[
nP(Y > b_n) \to 0, \text{ as } n \to \infty, \tag{58}
\]
and there exists a sequence \( a_n \downarrow 0 \) such that still
\[
nP(Y > a_nb_n) \to 0, \text{ as } n \to \infty. \tag{59}
\]

Proof. We have for each \( t > 0 \)
\[
nP(Y > tb_n) = \frac{n\ell(b_n)}{b_n} \cdot \frac{\ell(tb_n)}{\ell(b_n)} \cdot \frac{tb_nP(Y > tb_n)}{\ell(tb_n)} \to 0, \text{ as } n \to \infty.
\]
Therefore it is enough to take \( a_n \downarrow 0 \) slowly enough. □

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