A NEW MONOTONICITY FORMULA FOR THE SPATIALLY HOMOGENEOUS LANDAU EQUATION WITH COULOMB POTENTIAL AND ITS APPLICATIONS

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Abstract. We describe a time-dependent functional involving the relative entropy and the $H^1$ seminorm, which decreases along solutions to the spatially homogeneous Landau equation with Coulomb potential. The study of this monotone functional sheds light on the competition between the dissipation and the nonlinearity for this equation. It enables to obtain new results concerning regularity/blowup issues for the Landau equation with Coulomb potential.

1. Introduction

We consider the spatially homogeneous Landau equation with Coulomb potential

$$\partial_t f = Q(f, f)(v),$$

complemented with initial data $f_0 = f_0(v) \geq 0$. Here $f := f(t, v) \geq 0$ stands for the distribution of particles that at time $t \in \mathbb{R}_+$ possess the velocity $v \in \mathbb{R}^3$. The Landau operator (with Coulomb potential) $Q$ is a bilinear operator acting only on the velocity variable $v$. It writes

$$Q(g, h) = \nabla \cdot \left( [a * g] \nabla h - [a * \nabla g] h \right),$$

with

$$a(z) = |z|^{-1} \left( Id - \frac{z \otimes z}{|z|^2} \right).$$

This equation, first obtained by Landau in 1936, is used to describe the evolution in time of a (spatially homogeneous) plasma due to collisions between charged particles under the Coulomb potential.

Introducing the quantity

$$b_i(z) := \sum_{j=1}^3 \partial_j a_{ij}(z) = -2 z_i |z|^{-3},$$

the Landau operator with Coulomb potential can also be written as

$$Q(f, f) = \sum_{i=1}^3 \partial_i \left( \sum_{j=1}^3 (a_{ij} * f) \partial_j f - (b_i * f) f \right)$$

$$= \sum_{i=1}^3 \sum_{j=1}^3 (a_{ij} * f) \partial_{ij} f + 8\pi f^2,$$

where we used the identity $\sum_{i=1}^3 \partial_i b_i(z) = -8\pi \delta_0(z).$
1.1. Basic properties of the equation and notations. The weak formulation of the Landau operator $Q$, for a suitable test function $\varphi$, is written in the following way:

\begin{equation}
\int_{\mathbb{R}^3} Q(f, f)(v) \varphi(v) \, dv = -\frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} \int_{\mathbb{R}^3 \times \mathbb{R}^3} a_{ij}(v - v_*) \left\{ \frac{\partial_i f}{f} (v) - \frac{\partial_i f}{f}(v_*) \right\} \left\{ \frac{\partial_j f}{f} (v) - \frac{\partial_j f}{f}(v_*) \right\} f(v_*) \, dv_*. \end{equation}

From formula (1.6), we can obtain the fundamental properties of the Landau operator $Q$. The operator indeed conserves (at the formal level) mass, momentum and energy, more precisely

\begin{equation}
\int_{\mathbb{R}^3} Q(f, f)(v) \varphi(v) \, dv = 0 \quad \text{for} \quad \varphi(v) = 1, \, v_i, \, \frac{|v|^2}{2}, \, i = 1, 2, 3.
\end{equation}

We also deduce from formula (1.6) the entropy structure of the operator (still at the formal level) by taking the test function $\varphi(v) = \log f(v)$, that is

\begin{equation}
D(f) := -\int_{\mathbb{R}^3} Q(f, f)(v) \log f(v) \, dv
= -\frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} \int_{\mathbb{R}^3 \times \mathbb{R}^3} a_{ij}(v - v_*) \left\{ \frac{\partial_i f}{f} (v) - \frac{\partial_i f}{f}(v_*) \right\} \left\{ \frac{\partial_j f}{f} (v) - \frac{\partial_j f}{f}(v_*) \right\} f(v_*) \, dv_*.
\end{equation}

Note that $D(f) \geq 0$ since the matrix $a$ is (semi-definite) positive. Note also that for any $f$ such that $D(f) = 0$, it can be shown (cf. [8] and [9] for a rigorous statement and proof) that $f$ is a Maxwellian distribution, that is $f = \mu_{\rho, u, T}$, with

\begin{equation}
\mu_{\rho, u, T}(v) = \frac{\rho}{(2\pi T)^{3/2}} e^{-\frac{|v-u|^2}{2T}}.
\end{equation}

where $\rho \geq 0$ is the density, $u \in \mathbb{R}^3$ is the mean velocity and $T > 0$ is the temperature of the plasma. They are defined by

\begin{equation}
\rho = \int_{\mathbb{R}^3} f(v) \, dv, \quad u = \frac{1}{\rho} \int_{\mathbb{R}^3} v f(v) \, dv, \quad T = \frac{1}{3\rho} \int_{\mathbb{R}^3} |v - u|^2 f(v) \, dv.
\end{equation}

Thanks to the conservation of mass, momentum and energy, we have (when $f := f(t, v)$ is a solution of eq. (1.1) - (1.3) and $\rho, u, T$ are defined above, at the formal level),

\begin{equation}
\forall t \geq 0, \quad \rho(t) = \rho(0), \quad u(t) = u(0), \quad T(t) = T(0),
\end{equation}

which implies that the parameters $\rho, u, T$ are constant (along solutions of eq. (1.1) - (1.3)).

Denoting (when $f := f(t, v)$ is a solution of eq. (1.1) - (1.3)) by

\begin{equation}
H(t) := H(f|\mu_{\rho, u, T}(t)) := \int_{\mathbb{R}^3} \left( f(t, v) \log \left( \frac{f(t, v)}{\mu_{\rho, u, T}} \right) - f(t, v) + \mu_{\rho, u, T} \right) \, dv,
\end{equation}

the relative entropy with respect to $\mu_{\rho, u, T}$ (defined by (1.9), (1.10)), we see that (still at the formal level),

\begin{equation}
\frac{d}{dt} H(t) = -D(f(t, \cdot)) \leq 0.
\end{equation}

Note that in the above definition, $H(t)$ differs from the usual (non relative) entropy $\int f(t, v) \log f(t, v) \, dv$ only by a constant, thanks to identities (1.11).

Throughout this paper, we shall assume that $f_0 \geq 0$ and $f_0 \in L^1_{\infty} \cap L \log L(\mathbb{R}^3)$. Furthermore, without loss of generality, we shall also assume that $f_0$ satisfies the normalization identities

\begin{equation}
\int_{\mathbb{R}^3} f_0(v) \, dv = 1, \quad \int_{\mathbb{R}^3} f_0(v) \, v \, dv = 0, \quad \int_{\mathbb{R}^3} f_0(v) \, |v|^2 \, dv = 3,
\end{equation}
which can be rewritten as
\[ \mu(v) := (2\pi)^{-3/2}e^{-|v|^2/2}, \]
the Maxwellian distribution (centred reduced Gaussian) with same mass, momentum and energy as \( f_0 \) satisfying (1.14).

Next we introduce some function spaces which will be used throughout the paper:

- Let \( \langle v \rangle := (1 + |v|^2)^{1/2} \) denote the Japanese bracket. For any \( p \in [1, +\infty], l \in \mathbb{R} \), the \( L^p_l \) norm is defined by
  \[ \|f\|^p_{L^p_l} := \int_{\mathbb{R}^3} |f(v)|^p \langle v \rangle^l dv. \]

- The following quantity, for functions of \( L \log L \), is written as if it were a norm, and defined by
  \[ \|f\|_{L \log L} := \int_{\mathbb{R}^3} |f| \log(1 + |f|) dv. \]

- For any \( p \in (1, \infty), q \in [1, +\infty] \), the standard Lorentz space \( L^{p,q} \) is defined by the norm
  \[ \|f\|_{L^{p,q}} := \left( \sup_{t \geq 0} \int_0^t |f^{**}(t)|^q dt \right)^{1/q} \cdot \int_0^\infty t^{1/p} f^{**}(t) dt, \quad q = \infty, \]
  where \( f^{**}(t) := t^{-1} \int_0^t f^{*}(s) ds \), and \( f^{*} \) is the decreasing rearrangement of \( f \). We also denote when \( l \in \mathbb{R} \) the weighted Lorentz norm by
  \[ \|f\|_{L^{p,q}_l} := \|f(\cdot)\langle \cdot \rangle^l\|_{L^{p,q}}. \]

More details on Lorentz spaces including the case when \( p = 1 \), \( p = \infty \) can be found in the Appendix.

- The homogeneous Sobolev norm \( \dot{H}^m \) with \( m \in \mathbb{R} \) is defined by
  \[ \|f\|_{\dot{H}^m}^2 := \int_{\mathbb{R}^3} |\xi|^{2m} |\hat{f}(\xi)|^2 d\xi, \]
  while the weighted inhomogeneous Sobolev norm \( H^m_l \) with \( m \in \mathbb{N}, l \in \mathbb{R} \) is defined by
  \[ \|f\|^2_{H^m_l} := \sum_{|\alpha| \leq m} \int_{\mathbb{R}^3} |\partial^\alpha (f(v)^l)|^2 dv. \]

1.2. Short review on the Landau equation with Coulomb potential. We briefly review the works on the Landau equation with Coulomb potential (1.1) – (1.3).

- Existence and uniqueness of solutions: In [37], Villani proved the global existence of the so-called \( H \)-solutions for equation (1.1) – (1.3) when the initial data have finite mass, energy and entropy. The key part of the proof lies in the use of the entropy dissipation \( D(f) \), rewritten as
  \[ D(f(t)) = 2 \int \int \frac{1}{|v - v_*|} \Pi(v - v_*) \nabla_{v - v_*} \sqrt{f(t,v)f(t,v_*)} dv dv_. \]
  Here \( \Pi(\cdot) = (I_d - \frac{\cdot}{|\cdot|^2} \otimes \frac{\cdot}{|\cdot|^2}) \nabla \), is called the weak projection gradient (see [22] and [37]).

In generality (when an estimate for \( \nabla_v f \) is not available), \( \Pi(v - v_*) \nabla_{v - v_*} \) is not equal to \( \Pi(v - v_*) \nabla_v - \Pi(v - v_*) \nabla_{v_*} \). This means that the construction of the approximated solutions to an \( H \)-solution plays a significant role. When the solutions are well-constructed (that is, using a suitable approximation process), we have
  \[ \Pi(v - v_*) \nabla_{v - v_*} = \Pi(v - v_*) \nabla_v - \Pi(v - v_*) \nabla_{v_*}. \]

We refer readers to [22] for more details. When (1.18) holds, we can use the estimate for the entropy dissipation \( D(f) \) in [8] to show that an \( H \)-solution is a weak solution of the equation.
More precisely, there is an explicitly computable constant $C_0 = C_0(\bar{H}) > 0$ such that, for all (normalized) $f \geq 0$ satisfying $H(f) \leq \bar{H}$, the following inequality holds:

$$\|f\|_{L^3_{\alpha,3}} \leq C_0 (1 + D(f)).$$

Therefore, we know that such an $H$-solution of equation (1.11) lies in $L^1_{loc}((0, \infty); L^3_{\alpha,3}(\mathbb{R}^3))$, and this estimate is sufficient to show that it is indeed a weak solution in the usual sense.

Fournier [13] showed that uniqueness holds for the solutions of (1.11) – (1.3) lying in the class $L^\infty_{loc}([0, \infty); L^2(\mathbb{R}^3)) \cap L^\infty_{loc}([0, \infty); L^\infty(\mathbb{R}^3))$, and this result implies a local well-posedness result assuming further that the initial data lie in $L^\infty(\mathbb{R}^3)$, thanks to the local existence result of Arsenev-Peskov [2] for such initial data. We also refer to [6] for uniqueness of higher integrable solutions, and to [26] for the study of an equation sharing significant features with eq. (1.1) – (1.3).

In the spatially inhomogeneous context, we quote [35] for the existence of renormalized solutions and [17] and [23] for the global well-posedness near Maxwellian and the local well-posedness in weighted Sobolev spaces. We finally refer to [3] for a general perturbation result, and to [24], [26] for conditional regularity results.

- **Long time behavior**: In a perturbative and spatially inhomogeneous framework, Guo and Strain [18] proved for solutions of (1.11) – (1.3) the stretched exponential decay to equilibrium in a high-order Sobolev space with fast decay in the velocity variable. For (uniformly w.r.t time) a priori smooth solutions with large initial data, LD and Villani [12] proved the algebraic convergence to equilibrium.

In the homogeneous setting, Carrapatoso, LD and LH proved the following result which plays an essential role in the present paper:

**Theorem 1.1.** (Cf. Theorem 2 and Lemma 8 of [4]) Let $f_0 \in L^\log L(\mathbb{R}^3) \cap L^1_\ell(\mathbb{R}^3)$ with $\ell > \frac{10}{9}$ satisfy the normalization (1.14), and consider a (well-constructed) weak (or $H$-) solution $f$ to eq. (1.1) – (1.3) with initial datum $f_0$. Then for any strictly positive $\beta < \frac{24^2 - 25 \ell + 57}{9(\ell - 2)}$, there exists some computable constant $C_\beta > 0$ (depending only on $\beta$, and $K > 0$ such that $\|f_0\|_{L^1_\ell} + \|f_0\|_{L^\log L} \leq K$), such that the relative entropy satisfies

$$\forall t \geq 0, \quad H(t) \leq C_\beta (1 + t)^{-\beta}.$$  

Moreover, for all $\ell > 2$, there exists $C_\ell > 0$ (which only depends on $\ell$ and $K$ such that $\|f_0\|_{L^1_\ell(\mathbb{R}^3)} + \|f_0\|_{L^\log L} \leq K$ ), such that

$$\forall t > 0, \quad \|f(t, \cdot)\|_{L^1_\ell(\mathbb{R}^3)} \leq C_\ell (1 + t).$$

- **Functional estimates**: In [8], it is shown that (for normalized $f \geq 0$) the following estimate holds,

$$D(f) + 1 \geq C_{D,1} \|\sqrt{f}\|_{H^{-\frac{1}{2}}_\ell}^2,$$

where $C_{D,1} > 0$ depends only on an upper bound of $H(f|\mu)$.

Using the precisied Sobolev embedding inequality $\|f\|_{L^{6,2}} \leq C \|\nabla f\|_{L^2}$ (see [1]) and the O’Neil inequality in Lorentz spaces (see Proposition 6.2 in the Appendix), we end up with the following inequality (holding for normalized $f \geq 0$):

$$D(f) + 1 \geq C_{D,2} \|f\|_{L^{3,1,1}}.$$

where $C_{D,2} > 0$ depends only on an upper bound of $H(f|\mu)$.

We refer to [8], [9] and [4] for variants of inequality (1.23).
• **Partial regularity issue:** Very recently Golse, Gualdani, Imbert and Vasseur [14] proved that the set of singular times for (suitable) weak solutions of the spatially homogeneous Landau equation with Coulomb potential has Hausdorff dimension at most 1/2 if the initial data possesses all polynomial moments. The key ingredient of the proof lies in the application of De Giorgi’s method to a scaled suitable solution. They also observed that the solution to Landau equation with Coulomb interaction enjoys a scaling property which is similar to that of the 3D incompressible Navier-Stokes equation. This explains the link between the bound on the Hausdorff dimension of the set of singular times in both equations. We also cite the papers [15] and [10] where Gualdani and Guillen provide estimates which are useful to understand the issues of regularity/appearance of blowup and the role played by the various terms in the Landau equation with Coulomb potential.

1.3. **Main result.** A very challenging problem for the (spatially homogeneous) Landau equation with Coulomb potential (1.1)–(1.3) is to answer whether the smoothness is propagated for all positive times, or if some blowup may occur after a finite time. If such a blowup appears, a further challenging issue is to understand what really happens at the blowup time (Cf. §1.3 (2) in Chapter 5 of Villani’s monograph [39]). The main result of this paper provides new partial answers to the first question, while another result of this paper deals with the second question. In particular, our results shed some new light on the competition between the dissipation and the nonlinearity (see more details at the end of this section) for Landau equation with Coulomb potential.

Our main result is concerned with the new monotonicity formula for equation (1.1) – (1.3) announced in the title, and its byproducts:

**Theorem 1.2.** Let \( f_0 \in L \log L(\mathbb{R}^3) \cap L^1_{SG}(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3) \) be a nonnegative initial datum satisfying the normalization (1.14).

Then there exist (explicitly computable) constants \( B^*, C_6 > 0, k_2 > 7/2, k > 0 \) (depending only on \( K \) satisfying \( \| f_0 \|_{L^1_{SG}(\mathbb{R}^3)} + \| f_0 \|_{L \log L(\mathbb{R}^3)} \leq K \)) such that the three following statements hold:

(i) (Monotonicity of a functional). We consider \( T > 0 \) and denote by \( f := f(t, v) \) a smooth and quickly decaying (\( C^2(S) \)) nonnegative solution on the interval \( [0, T] \) to eq. (1.1) – (1.3) with initial datum \( f_0 \). We define \( h := f - \mu \), where \( \mu \) is given by eq. (1.12) (recall also that \( H(t) \) is the relative entropy given by (1.12)).

Then the following a priori estimate (that we call monotonicity property) holds for \( t \in [0, T] \):

\[
\frac{d}{dt} \left[ H(t) - \frac{5}{2} \left( \| h(t) \|_{\dot{H}^1}^2 + B^* (1 + t)^{-k_2 + 1} \right)^{-\frac{5}{2}} \right] + C_6 (1 + t)^k \leq 0. \tag{1.24}
\]

(ii) (Global regularity for initial data below threshold). If moreover \( H(0) \left( \| h(0) \|_{\dot{H}^1}^2 + B^* \right)^{\frac{5}{2}} \leq \frac{5}{2} \), then eq. (1.1) – (1.3) admits a (unique) global and strong (that is, lying in \( L^\infty(\mathbb{R}^+; \dot{H}^1(\mathbb{R}^3)) \)) nonnegative solution satisfying that

\[
\forall t > 0, \quad \| h(t) \|_{\dot{H}^1} \left( H(t) + \frac{C_6}{k + 1} \left[ (1 + t)^{1+k} - 1 \right] \right)^{\frac{5}{2}} \leq \left( \frac{2}{3} \right)^{\frac{5}{2}} \leq \frac{5}{2}, \tag{1.25}
\]

where we used the same notations for \( h, \mu, H \) as in statement (i).

(iii) (No blowup after a finite time). If finally \( H(0) \left( \| h(0) \|_{\dot{H}^1}^2 + B^* \right)^{\frac{5}{2}} > \frac{5}{2} \), we denote

\[
T^* := \left( \frac{1 + k}{C_6} \left[ H(0) - \frac{5}{2} \left[ \| h(0) \|_{\dot{H}^1}^2 + B^* \right]^{-2/5} \right] + 1 \right)^{\frac{1}{1 + k}} - 1. \tag{1.26}
\]
Then one can construct a global weak (or \(H\)-) nonnegative solution of eq. (1.1) – (1.3), such that for \(t > T^*\), it becomes global and strong (that is, it lies in \(L^\infty([T^*, \infty); H^1(\mathbb{R}^3))\), and satisfies the estimates

\begin{align}
  & (1.27) \quad H(t) \left[ \|h(t)\|_{H^1}^2 + B^* (1 + t)^{1-k_2} \right]^{-2/5} \leq \frac{\beta}{2}, \\
  & (1.28) \quad \|h(t)\|_{H^1} \left( \frac{C_6}{k+1} \left( (1 + t)^{1+k} - (1 + T^*)^{1+k} \right) \right)^{\frac{7}{2}} \leq \left( \frac{2}{\beta} \right)^{\frac{7}{2}},
\end{align}

where we used the same notations for \(h, \mu\) and \(H\) as in statement (i).

Using variants of the estimates above, it is possible to get more standard results of local (in time) well-posedness for large data (in \(H^1\) norm), and global (in time) well-posedness for small initial data (in \(H^1\) norm). It is also possible to give estimates concerning a possible blowup (of the \(H^1\) norm). These results are stated in the three following propositions, where we recall that \(\mu\) is the Maxwellian given by eq. (1.15), and we denote \(h := f - \mu\) and \(h_0 := f_0 - \mu\).

We begin with the local well-posedness for small initial data:

**Proposition 1.1.** Let \(f_0 \in L \log L(\mathbb{R}^3) \cap L_5^1(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)\) be a nonnegative initial datum satisfying the normalization (1.14). Then there exists a time \(T := \frac{5}{2}(\|h_0\|_{H^1}^2 + C_7^{-1})^{\frac{1}{2}}\) (where \(C_7 > 0\) only depends on \(K\) such that \(\|f_0\|_{L_5^1} + \|f_0\|_{\log L} \leq K\)), such that the Landau equation (1.1) – (1.3) admits a unique strong solution on the interval \([0, T]\). By strong solution, we mean here that \(f \in C([0, T]; H^1) \cap L^2([0, T]; H^2_{\log 2/3});\)

We turn then to the global well-posedness for small initial data:

**Proposition 1.2.** Let \(f_0 \in L \log L(\mathbb{R}^3) \cap L_5^1(\mathbb{R}^3) \cap H_3^1(\mathbb{R}^3)\) be a nonnegative initial datum satisfying the normalization (1.14), and \(h_0 := f_0 - \mu\).

Then there exists a (small) constant \(\epsilon_0 > 0\) (depending only on \(K > 0\) such that \(\|f_0\|_{L_5^1} + \|f_0\|_{\log L} \leq K\)), such that if \(\|h_0\|^2_{H^1} \leq \epsilon_0\), the Landau equation with Coulomb potential (1.1) – (1.3) admits a (unique) global smooth (that is, lying in \(L^\infty([0, +\infty); H_3^1(\mathbb{R}^3))\)) and nonnegative solution, denoted by \(f := f(t, v)\). Moreover, (under the same assumption on the initial datum) there exists a constant \(C > 0\) only depending on \(K\) such that (with the notation \(h := f - \mu\))

\[ \|h(t, \cdot)\|_{H^1} \leq C (1 + t)^{\frac{k}{2}}. \]

Finally, we give some clues about the behavior of solutions close to a potential blowup:

**Proposition 1.3.** Let \(f := f(t, v)\) be a nonnegative solution of the Landau equation with Coulomb potential (1.1) – (1.3), corresponding to initial data satisfying the assumptions of Theorem 1.2 We suppose that \(f \in L^\infty([0, t]; H^1(\mathbb{R}^3))\) for all \(t \in [0, \bar{T}]\), and that \(\|\nabla f(t)\|_{L^2(\mathbb{R}^3)}\) blows up at time \(\bar{T}\). Then for \(\bar{T} - t \ll 1\) and some explicitly computable constants \(c, C > 0, C_1, C_2 > 0\) (depending only on \(K\) satisfying \(\|f_0\|_{L \log L(\mathbb{R}^3)} + \|f_0\|_{L_5^1(\mathbb{R}^3)} \leq K\),

\[ \|h(t)\|_{H^1} \geq C(H(t) - \bar{H})^{-\frac{k}{2}} \quad \text{with} \quad H(t) - \bar{H} \geq C(\bar{T} - t)(1 + \bar{T})^{k+1}; \]

\[ \inf_{s \in [t, \bar{T}]} \|h(s)\|_{H^1} \leq \left( B(c(\bar{T} - t)) \frac{2}{C_1}(1 + \bar{T})^{-(k_1 + k_2)} \right)^{\frac{2}{k}} \]

where \(\bar{H} := \lim_{t \to \bar{T}} H(t)\) and \(B(x) := C_2 x^{-13} \exp\{7 x^{-\frac{44}{13}}\}\).

1.4. Comments.
1.4.1. Comment on the monotonicity formula (1.24). To the best of our knowledge, inequality (1.24) of statement (i) of Theorem 1.2 is a new monotonicity formula for the (smooth solutions of the) Landau equation with Coulomb potential. The explicit increasing rate $C_6 (1 + t)^k$ comes from the dissipation effect of the equation. We denote the monotone functional by

$$M(t) := H(t) - \frac{5}{2} \left( \|h(t)\|_{\dot{H}^1}^2 + B^* (1 + t)^{-k+1}\right)^{\frac{2}{5}},$$

and notice that the differential inequality (1.24) formally allows $\|h(t)\|_{\dot{H}^1}$ to blow up. The global dynamics of $M(t)$ described by inequality (1.24) gives clues about the global dynamics for the original solution:

- When $M(t_0)$ is below its critical value, (that is, $M(t_0) \leq 0$, cf. comment below), the solution to equation (1.1) – (1.3) after time $t_0$ will remain bounded in $\dot{H}^1$ and converge to the equilibrium; this is indicated in Statement (ii) of Theorem 1.2. We call this situation the “stable regime”.
- When $M(t)$ is above its critical value (that is, $M(t_0) \geq 0$), some blowup may occur, but there exists a computable time $T^*$ (strictly bigger than the blowup time if it occurs) such that $M(t)$ gets inside the stable regime for any $t > T^*$; this is indicated in Statement (iii) of Theorem 1.2.

Finally, note that in Statement (i) of Theorem 1.2 the differential inequality (1.24) is shown to rigorously hold for all smooth and quickly decaying (when $|v| \to \infty$) solutions of eq. (1.1) – (1.3). It also rigorously holds for (smooth and quickly decaying when $|v| \to \infty$) solutions to an approximated problem (that is, problem (2.42) described in subsection 2.6), of equation (1.1) – (1.3). Finally, when it is integrated with respect to time (see (5.8)), it is shown in the proof of Proposition 1.3 that it also rigorously holds for strong (that is, lying in $L^\infty_t(H^m_1)$ for $m$ large enough) solutions to the original equation (1.1) – (1.3), such as those appearing (on suitable time intervals) in Propositions 1.1 and 1.2.

1.4.2. Comment on the non optimality of the presented results. We notice that the lifespan of local wellposedness is not optimal in Proposition 1.1. For example, it can be extended by the effect of the dissipation term $C_6 (1 + t)^k$, which is not used in the proof of this Proposition.

It is also possible to use Proposition 1.1 in order to relax the condition $M(0) \leq 0$ in statement (ii) of Theorem 1.2. Using the fact that the Landau equation with Coulomb interaction admits a local solution $f \in C([0, T]; \dot{H}^1)$ where $T$ depends only on the initial data $f_0$ (see Proposition 1.1 for more details), this condition can be transformed in

$$M(0) \leq \frac{C_6}{k+1} (1 + T)^{k+1} - 1).$$

Indeed, thanks to estimates (1.24) and (1.29), one gets

$$M(T) + C_6 \int_0^T (1 + t)^k dt \leq M(0),$$

so that $M(T) \leq 0$ and we can use statement (ii) of Theorem 1.2 starting at time $T$ (the equation being invariant by translation in time).

1.4.3. Comment on the impossibility of blowup after a finite time. This is a direct consequence of inequality (1.24) since after the time $T^*$, the monotone functional $M(t)$ will enter the stable regime (defined in Comment 1.4.1).

- If the solution has not blown up in $\dot{H}^1$ before the time $T^*$, then the solution will remain strong (that is, will lie in $\dot{H}^1$) for all time thanks to inequality (1.28). Then thanks to the uniqueness result established in [30] and the regularity obtained in Proposition 1.1.
the constructed solution is the unique strong solution with the initial data \( f_0 \) satisfying the conditions stated in Theorem 1.2.

- Looking at definition (1.20), we see that \( \mathcal{M}(t) \) is still well-defined if \( \|h(t)\|_{L^1} = \infty \). When such a blowup (in \( H^1 \) norm) happens, the constructed solution is the unique strong solution before the first blowup time, and becomes strong again after time \( T^* \). Note that in order to give a rigorous proof of these facts, we apply the estimates obtained in this paper to solutions of an approximated problem and then pass to the limit.

Finally, combining our result with the previous result in [14], we see that the set of singular times for weak solutions is included in a subset of the interval \([T, T^*]\) whose Hausdorff dimension is at most 1/2.

1.4.4. Comment on the description of the potential blowup. Proposition 1.3 describes a potential blowup phenomenon for solutions to the Landau equation with Coulomb potential. We recall that restrictions are given in [15] and [14] on the possible appearance of such a blowup. Our lower bound for the blowup rate is given in terms of relative entropy. Our upper bound enables to exclude a double exponential (that is, exponential of an exponential) growth of the \( H^1 \) norm of the solution close to the first blowup time. This bound heavily depends on the number of initial moments which are assumed.

1.4.5. Comment on the dependence of the coefficients appearing in the main Theorem with respect to the \( L^1 \) moments. We can provide estimates for the explicit dependence of all coefficients in the Theorem 1.2. Moreover we can extend the validity of this Theorem somewhat, when the initial data have less than 55 moments. Indeed, when \( f_0 \in L^1 \), let us define

\[
q_{\ell,\eta} := -2 \frac{\ell^2 - 5 \ell + 57}{18 (l - 2)} \left( 1 - \frac{\theta}{l} \right) + \frac{\theta}{l},
\]

and choose \( \ell > 31 \) and \( \tau \in [31, \ell) \) such that \( q_{\ell,99/4} > 7/4, q_{\ell,\tau} > 0 \). Then one can check that it is possible to take \( k := \min\{k_1, \frac{\eta}{5} k_2 - \frac{\tau}{5}\} \) with \( k_1 = 2 q_{\ell,14/5}, k_2 = q_{\ell,99/4} \), in such a way that estimate (1.24) holds. In our main Theorem, we selected \( \ell = 55 \) and \( \tau = 45 \), for the sake of readability.

1.4.6. Comment on Landau equation with very soft potentials. We can generalize the result of Theorem 1.2 to the Landau equation with very soft potential in the range \( \gamma \in [-3, -2] \), that is, when

\[
(1.31) \quad a(z) := |z|^\gamma +\left( Id - \frac{z \otimes z}{|z|^2} \right).
\]

Indeed, the main difference in the proof with the Coulomb case lies in the estimate of the term \( \int \int_{|v - v_*| \leq 1} f(v_*) |v - v_*|^\gamma + |\nabla h(v)||\nabla^2 h(v)| \, dv_* \, dv \), which appears in Proposition 2.3. The Coulomb potential case \( \gamma = -3 \) is critical in the sense that it requires (in order to close the differential inequality (1.24)) the use of the Lorentz space \( L^{3,1} \), since

\[
\int \int_{|v - v_*| \leq 1} f(v_*) |v - v_*|^\gamma + |\nabla h(v)||\nabla^2 h(v)| \, dv_* \, dv \leq C \|f\|_{L^3} \|\nabla h\|_{L^2} \|\nabla^2 h\|_{L^2}
\]

holds when \( \gamma > -3 \) but not when \( \gamma = -3 \).

We think therefore that when \( \gamma \in [-3, -2] \), it is possible to avoid the use of the Lorentz spaces and still get a closed inequality in the same spirit as inequality (1.24).

We also believe that if \( \gamma = -2 - \eta \) with \( \eta > 0 \) sufficiently small, then the equation will generate a global and bounded solution if initially \( \|\nabla h_0\|^2 \leq (C_1 \eta - C_2 \eta^{-1} + C_3 \eta_0 - C_4 \eta_0^{-1} + C_5 \eta) \) for some \( C_1, C_2, C_3 > 0 \) (depending on \( H(0) \)). This is coherent with the existing theory of existence of global strong solutions when \( \gamma \in [-2, 0] \), cf. [30] for example.
1.4.7. **Comment on the comparison with Leray’s work for 3D Incompressible Navier-Stokes.** We recall that the 3D incompressible Navier-Stokes equations reads

\[
\begin{aligned}
&\partial_t u + u \cdot \nabla u - \Delta u + \nabla p = 0; \\
&\text{div } u = 0; \\
&u|_{t=0} = u_0.
\end{aligned}
\]

(1.32)

In the classical work [28] (see also [30] and reference therein), Leray proved the following results:

(i) If \(|u_0|_{L^2}||\nabla u_0||_{L^2} \ll 1\), the 3D incompressible Navier-Stokes equations admits a global smooth solution, which nowadays are called Leray solutions.

(ii) He also considered the potential blowup phenomenon. Using the lower bound of the blowup rate for the potential singularity, one can show that the set of singular times for suitable weak solutions has Hausdorff dimension at most 1/2.

For a result about longtime regularity, we refer to [31, 36].

We are in a position to compare our results with Leray’s.

If we consider that the relative entropy \(H\) plays for the Landau equation with Coulomb interaction the same role as the energy \(||u||_{L^2}\) for the Navier-Stokes equations, it is natural to compare the Leray condition \(||u_0||_{L^2}||\nabla u_0||_{L^2} \ll 1\) to the condition \(M(0) \leq 0\) written under the form \(H(0)(|h(0)|^{2}_{H^{1}+} + B^*)^{2/5} \leq 5/2\). We see then that as in the Navier Stokes equation, the \(L^2\) norm of the gradient of the solution plays a decisive role. Note however that no equivalent of the term \(B^*\) exists in Leray’s condition for Navier Stokes equation, which constitutes a significant difference.

The condition \(M(0) \leq 0\) includes the case in which the initial relative entropy \(H(0)\) is small, while \(||h(0)||_{\dot{H}^1}\) may be large. Note that such (normalized) initial data exist. Indeed one can take initial data \(f(0)\) close (in weighted \(L^1\)) to the Maxwellian \(\mu\), but having quick oscillations, so that \(||h(0)||_{\dot{H}^1}\) is large (see Proposition 6.6 for a concrete example).

Note that in Proposition 1.3, we get not only a lower bound, but also an upper bound for the rate at which a potential blowup occurs. However, the lower bound is given in terms of relative entropy and thus probably cannot be used to estimate the size of singular times. We recall that the size of that set for the Landau equation with Coulomb potential is anyway estimated in [14].

1.5. **Sketch of the proof.** We present here the main ideas which are used in the proofs of Theorem 1.2 and the related results. In particular, we point out that the mechanisms which enable to build the global strong solutions to eq. (1.31) – (1.3) when \(M(0) \neq 0\) (in Theorem 1.2 statement (ii)) and when \(||h(0)(\cdot)||_{\dot{H}^1}\) is small (in Proposition 1.3), are quite different.

Let us first recall that thanks to a previous study of the large time behavior of the Landau equation with Coulomb potential (cf. [4]), the \(L^1\) moments of \(h = f - \mu\) decrease with a power law (cf. Theorem 1.1).

As a consequence, by interpolation, we see that the dissipation of \(\dot{H}^1\) energy typically increases as time goes. Roughly speaking, for some \(C, k_1 > 0\), this dissipation is lower bounded in the following way:

\[
||\nabla^2 h||_{L^2}^2 \geq C ||h||_{L^1_{13/4}}^{-\frac{4}{5}} ||\nabla h||_{L^2}^{15} \geq C (1 + t)^{k_1} ||\nabla h||_{L^2}^{15}.
\]

- **When the initial data is far from equilibrium regime (measured in terms of \(\dot{H}^1\) norm):** In this situation, the main challenge is to show that the nonlinear terms can be controlled. Indeed, by interpolation, the behavior of the nonlinear term with respect to the \(\dot{H}^1\) energy is of the same order as the dissipation term in the following sense:

\[
\text{Nonlinearity} \lesssim D(f)||\nabla h||_{L^2}^{15}.
\]
These observations suggest that the competition between the dissipation and nonlinearity in $\dot{H}^1$ energy can be characterized as $(1 + t)^{k_1} \|\nabla h\|_{L^2}^{\frac{4}{3}}$ versus $D(f)\|\nabla h\|_{L^2}^{\frac{4}{3}}$. Since it is expected that the dissipation will dominate the nonlinear term after some time (remembering that $(1 + t)^{k_1} \to +\infty$ and $\int_0^t D(f)(s)\, ds < +\infty$), one can understand the emergence of the new monotonicity formula that we propose. The detailed arguments are included in Section 2.

- **When the initial data is close to equilibrium regime (measured in terms of $\dot{H}^1$ norm):** In this situation, we have

  Tail of linear term plus nonlinearity $\lesssim \|\nabla h\|_{L^2}^{\frac{4}{3}} + \|\nabla h\|_{L^2}^2$,

as we can observe from equation (2.17) by neglecting the weights. Since we assumed that the $\dot{H}^1$ norm of the initial data is sufficiently small, we see that the competition occurs between $(1 + t)^{k_1} \|\nabla h\|_{L^2}^{\frac{4}{3}}$ and $\|\nabla h\|_{L^2}^2$.

  Suppose now that $\|\nabla h_0\|_{L^2}^2 \sim \epsilon$. Then the smallness of $\|\nabla h\|_{L^2}^2$ can be kept at least for an interval of time of length $|\ln \epsilon|$ (cf. the proof of Proposition 1.2). It implies that at some point, the dissipation will be lower bounded in the following way:

  $$(1 + t)^{k_1} \|\nabla h\|_{L^2}^{\frac{4}{3}} \geq C (1 + |\log \epsilon|)^{k_1} \|\nabla h\|_{L^2}^2.$$ 

  Then, when $\|\nabla h\|_{L^2}^{\frac{4}{3}}$ is not small, the dissipation still prevails and prevents a blowup of the $\dot{H}^1$ norm. We refer readers to the content of Section 4 for detailed and rigorous arguments. Note however that in the description above, weights are not taken into account, whereas they play a significant role in the proof of Proposition 1.2. Finally, we refer to [16] for extra considerations on the competition between dissipation and nonlinearities.

2. $\dot{H}^1$ estimate and the proof of Theorem 1.2

This section is devoted to the $\dot{H}^1$ estimate for the Landau equation with Coulomb potential, which leads to the monotonicity formula (1.24). We first provide a set of *a priori* estimates for the terms appearing in the equation (this is done in Subsections 2.4 to 2.5). Then we show that all estimates rigorously hold by passing to the limit in an approximated problem (this is done in Subsection 2.6), which enables us to complete the proof of Theorem 1.2.

2.1. Decomposition of the derivative in time of the $\dot{H}^1$ norm of the solutions to the Landau equation. To make full use of the results on the long-time behavior of the solution (cf. Theorem 1.1), we write the Landau equation with Coulomb potential as follows (at the formal level), setting $h := f - \mu$, with $\mu$ defined by (1.15):

$$\partial_t h = Q(f, h) + Q(h, \mu).$$

(2.1)

Then we focus (at the formal level) on the $\dot{H}^1$ norm of $h$. We write the equation (for $k = 1, 2, 3$) satisfied by $\partial_k h$:

$$\partial_k \partial_t h = Q(f, \partial_k h) + Q(\partial_k f, h) + Q(\partial_k h, \mu) + Q(h, \partial_k \mu).$$

(2.2)

Then we multiply it by $\partial_k h$, integrate with respect to $v$, and sum over all $k$. It gives

$$\frac{1}{2} \frac{d}{dt} \|\nabla h\|_{L^2}^2 = I_1 + I_2 + I_3 + I_4,$$

(2.3)

where $I_1, I_2, I_3$ and $I_4$ are defined (and subdivided) as follows:

1. $I_1 := \sum_{k=1}^3 \int_{\mathbb{R}^3} Q(f, \partial_k h) \partial_k h \, dv$. We also write

2. $I_1 := -I_{1,1} + I_{1,2},$
where

\[ I_{1,1} := \sum_{k=1}^{3} \int_{\mathbb{R}^3} (a \ast f) : \nabla \partial_k h \otimes \nabla \partial_k h \, dv, \quad I_{1,2} := \sum_{k=1}^{3} \int_{\mathbb{R}^3} (b \ast f) \cdot \nabla \partial_k h \, \partial_k h \, dv. \]

(2) \( I_2 = \sum_{k=1}^{3} \int_{\mathbb{R}^3} Q(\partial_k f, h) \partial_k h \, dv. \) We also write

\[ I_2 := -I_{2,1} + I_{2,2}, \]

where

\[ I_{2,1} := \sum_{k=1}^{3} \int_{\mathbb{R}^3} (a \ast \partial_k f) : \nabla \partial_k h \otimes \nabla h \, dv = \sum_{k=1}^{3} \int_{\mathbb{R}^3} (\partial_k a \ast f) : \nabla \partial_k h \otimes \nabla h \, dv, \]

\[ I_{2,2} := \sum_{k=1}^{3} \int_{\mathbb{R}^3} (b \ast \partial_k f) \cdot \nabla \partial_k h \, h \, dv = \sum_{k=1}^{3} \int_{\mathbb{R}^3} (\partial_k b \ast f) \cdot \nabla \partial_k h \, h \, dv \]

\[ = \sum_{k=1}^{3} \sum_{i=1}^{3} \left( - \int_{\mathbb{R}^3} (\partial_k \partial_i b \ast f) \partial_k h \, h \, dv - \int_{\mathbb{R}^3} (\partial_k b \ast f) \partial_k h \, \partial_i h \, dv \right) \]

\[ = \sum_{k=1}^{3} \left( -8 \pi \int_{\mathbb{R}^3} f (h \partial_k^2 h + (\partial_k h)^2) \, dv + \int_{\mathbb{R}^3} (b \ast f) \cdot (\nabla h \partial_k^2 h + \nabla \partial_k h \partial_k h) \, dv \right) \]

\[ \leq \sum_{k=1}^{3} \left( -8 \pi \int_{\mathbb{R}^3} f h \partial_k^2 h \, dv + \int_{\mathbb{R}^3} (b \ast f) \cdot (\nabla h \partial_k^2 h + \nabla \partial_k h \partial_k h) \, dv \right). \]

(3) \( I_3 := \sum_{k=1}^{3} \int_{\mathbb{R}^3} Q(\partial_k h, \mu) \partial_k h \, dv. \) We also write

\[ I_3 := -I_{3,1} + I_{3,2}, \]

where \( I_{3,1} := \sum_{k=1}^{3} \int_{\mathbb{R}^3} (a \ast \partial_k h) : \nabla \partial_k h \otimes \nabla \mu \, dv, \)

\[ I_{3,2} := \sum_{k=1}^{3} \int_{\mathbb{R}^3} (b \ast \partial_k h) \cdot \nabla \partial_k h \, \mu \, dv \]

\[ = \sum_{k=1}^{3} \sum_{i=1}^{3} \left( - \int_{\mathbb{R}^3} (\partial_k \partial_i b \ast f) \partial_k h \, \mu \, dv - \int_{\mathbb{R}^3} (b \ast f) \cdot (\partial_k \partial_i h) \partial_k h \, \partial_i \mu \, dv \right) \]

\[ = \sum_{k=1}^{3} \left( 8 \pi \int_{\mathbb{R}^3} \mu |\partial_k h|^2 \, dv + \int_{\mathbb{R}^3} (b \ast h) \cdot (\partial_k h \partial_k \nabla \mu + \partial_k^2 h \nabla \mu) \, dv \right). \]

(4) \( I_4 := \sum_{k=1}^{3} \int_{\mathbb{R}^3} Q(h, \partial_k \mu) \partial_k h \, dv. \) We also write

\[ I_4 := -I_{4,1} + I_{4,2}, \]

where

\[ I_{4,1} := \sum_{k=1}^{3} \int_{\mathbb{R}^3} (a \ast h) : \nabla \partial_k h \otimes \nabla \partial_k \mu \, dv, \quad I_{4,2} := \sum_{k=1}^{3} \int_{\mathbb{R}^3} (b \ast h) \cdot \nabla \partial_k h \, \partial_k \mu \, dv. \]

In subsections 2.1 to 2.5 the a priori estimates are proven as if the considered functions are smooth and quickly decaying (when \(|v| \to \infty\)). They are used later for solutions of an approximated problem which satisfy those properties.
2.2. Coercivity estimate for $I_{1,1}$. In order to treat the term $I_{1,1}$, we prove the following (rather classical) coercivity estimate:

**Proposition 2.1.** For all $j \in \{1, 2, 3\}$, $m \in \mathbb{R}$, $f \geq 0$, $p \in W^{1,1}_{\text{loc}}(\mathbb{R}^3)$, we have

\begin{equation}
\int_{\mathbb{R}^3} |\nabla p(v)|^2 \langle v \rangle^{m-3} \, dv \leq 4 \|f\|_{L^1_\mu(\mathbb{R}^3)} A_j(f)^{-2}
\end{equation}

\begin{equation}
\times \int_{\mathbb{R}^3} |v-v_*|^{-3} \left\{ |v-v_*|^2 - (v-v_*) \otimes (v-v_*) \right\} : \nabla p(v) \otimes \nabla p(v) f(v_*) \langle v \rangle^m \, dv \, dv_*. \tag{2.8}
\end{equation}

where $A_j(f) := \int_{\mathbb{R}^3} f v_j^2 \, dv$.

**Proof.** We denote

$q_{ij}(v, v_*) = (v_i - v_{*i}) \partial_j p(v) - (v_j - v_{*j}) \partial_i p(v)$.

As observed in [35], by choosing a suitable system of coordinates, we can assume that $\int_{\mathbb{R}^3} f v_j v_3 \, dv = \delta_{ij} A_i(f)$. Then

\begin{equation}
\int_{\mathbb{R}^3} q_{ij}(v, v_*) f(v_*) v_{*j} \, dv_* = A_j(f) \partial_i p(v),
\end{equation}

and for all $n \in \mathbb{R}$, $i, j \in \{1, 2, 3\}$, $i \neq j$, thanks to Cauchy-Schwarz inequality,

\begin{equation}
A_j(f)^2 \int_{\mathbb{R}^3} |\partial_i p(v)|^2 \langle v \rangle^n \, dv \leq \int_{\mathbb{R}^3} \langle v \rangle^n \left\{ \int_{\mathbb{R}^3} q_{ij}(v, v_*)^2 f(v_*) \frac{\langle v \rangle^{m-n}}{|v-v_*|^3} \, dv_* \right\} \times \int_{\mathbb{R}^3} f(v_*) |v-v_*|^3 |v_{*j}|^2 \langle v \rangle^{n-m} \, dv_* \right\} \, dv.
\end{equation}

We end up with estimate [2.8] by using $n = m - 3$, the bound

\begin{equation}
\sup_{v \in \mathbb{R}^3} \int_{\mathbb{R}^3} f(v_*) |v-v_*|^3 |v_{*j}|^2 \langle v \rangle^{-3} \, dv_* \leq 4 \|f\|_{L^1_\mu(\mathbb{R}^3)},
\end{equation}

and the identity

\begin{equation}
\left\{ |v-v_*|^2 - (v-v_*) \otimes (v-v_*) \right\} : \nabla p(v) \otimes \nabla p(v) = |(v-v_*) \times \nabla p(v)|^2.
\end{equation}

We now state two corollaries which can easily be obtained from Proposition 2.1.

**Corollary 2.1.** Let $f \geq 0$ be such that $\int_{\mathbb{R}^3} f(v) \, dv = 1$, $\int_{\mathbb{R}^3} f(v) \langle v \rangle^2 \, dv = 3$, and such that $\|f\|_{L^1_\mu(\mathbb{R}^3)} + \|f\|_{L^\infty(\mathbb{R}^3)} \leq K$ for some $K > 0$. We denote $h = f - \mu$. Then there exists a constant $C(K) > 0$ (depending only on $K$) such that for all $h \in L^1_{\text{loc}}(\mathbb{R}^3)$, $m \in \mathbb{R}$,

\begin{equation}
\sum_{k=1}^3 \int_{\mathbb{R}^3} (a * f) : \nabla \partial_k h \otimes \nabla h \langle v \rangle^m \, dv \geq C(K) \|\nabla^2 h\|_{L^{2/3-2/3}_{m/2}}^2.
\end{equation}

**Proof.** We just observe that under the assumption $\int_{\mathbb{R}^3} f(v) \, dv = 1$, $\int_{\mathbb{R}^3} f(v) \langle v \rangle^2 \, dv = 3$, and $\|f\|_{L^\infty(\mathbb{R}^3)} \leq K$, the quantity $A_j(f)$ is bigger than some strictly positive quantity (which depends only on $K$).

**Corollary 2.2.** Let $f \geq 0$ be such that $\int_{\mathbb{R}^3} f(v) \, dv = 1$, $\int_{\mathbb{R}^3} f(v) \langle v \rangle^2 \, dv = 3$, and such that $\|f\|_{L^1_\mu(\mathbb{R}^3)} + \|f\|_{L^\infty(\mathbb{R}^3)} \leq K$ for some $K > 0$. We denote $h = f - \mu$. Then there exist constants $C(K), C^*(K) > 0$ depending only on $K$, such that for all $h \in L^1_{\text{loc}}(\mathbb{R}^3)$, $m \in \mathbb{R}$,

\begin{equation}
I_{1,1} \geq C(K) \|\nabla^2 h\|_{L^{2/3-2/3}_{m/2}}^2 + C(K) \|h\|_{L^1_{\mu/2}}^{-\frac{2}{3}} \|\nabla h\|_{L^2}^{\frac{4}{3}} - C^*(K) \|h\|_{L^1_{\mu/2}}^2.
\end{equation}
Proof. Note first that by taking $m = 0$ in Corollary 2.1 we get that
\[ I_{1,1} \geq C(K) \| \nabla^2 h \|_{L^2_{3/2}}^2. \]
Then using Proposition 6.4 with $m = 0$, we see that for some constant $C > 0$,
\[ \| \nabla h \|_{L^2} \leq C \| h \|_{L^{3/4}_{1/2}} \left( \| h \|_{L^1} + \| \nabla^2 h \|_{L^2_{3/2}} \right)^{5/7}. \]
This inequality implies that (for some constant $C^* > 0$)
\[ \| \nabla^2 h \|_{L^2}^2 \geq C \| h \|_{L^{3/4}_{1/2}} \| \nabla h \|_{L^2}^2 - C^* \| h \|_{L^2}^4, \]
whence estimate (2.10). \hfill \Box

2.3. Estimates for the remainder terms.

2.3.1. Estimates for $I_3$ and $I_4$.

**Proposition 2.2.** Let $f \geq 0$ and $h = f - \mu$. Then for all $\eta \in ]0,1]$, (2.11)
\[ I_3 + I_4 \leq C \eta^{-1} \| h \|_{L^2}^2 + C \| \nabla h \|_{L^2}^2 + \frac{\eta}{4} \| \nabla^2 h \|_{L^2_{3/2}}^2, \]
for some absolute constant $C > 0$.

**Proof.** Indeed, for some constant $C > 0$,
\[ I_3 + I_4 \leq C \int_{\mathbb{R}^3} |\nabla h|^2 \mu \, dv + C \int_{\mathbb{R}^3} \int_{\mathbb{R}^6} |v - v_*|^{-2} |h_*| (|\nabla^2 h| + |\nabla h|) \mu^2 \, dv_* \, dv \\
\leq C \int_{\mathbb{R}^3} |\nabla h|^2 \, dv + \frac{\eta}{4} \| \nabla^2 h \|_{L^2_{3/2}}^2 + C \eta^{-1} \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^6} |v - v_*|^{-2} |h_*| \, dv_* \right)^2 \langle v \rangle^3 \mu \, dv \\
+ C \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} |v - v_*|^{-2} |h_*| \, dv_* \right)^2 \mu \, dv.
\]
Then, we see that
\[ \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} |v - v_*|^{-2} |h_*| \, dv_* \right)^2 \langle v \rangle^3 \mu \, dv \leq C \int_{\mathbb{R}^3} \left( \int_{|v - v_*| \leq 1} |v - v_*|^{-2} |h_*| \, dv_* \right)^2 \langle v \rangle^3 \mu \, dv \\
+ 2 \int_{\mathbb{R}^3} \left( \int_{|v - v_*| \geq 1} |v - v_*|^{-2} |h_*| \, dv_* \right)^2 \langle v \rangle^3 \mu \, dv \\
\leq C \int_{v_* \in \mathbb{R}^3} |h_*|^2 \int_{|v - v_*| \leq 1} |v - v_*|^{-2} \langle v \rangle^3 \mu \, dv \, dv_* + 2 \left( \int_{\mathbb{R}^3} |h_*| \, dv_* \right)^2 \int_{\mathbb{R}^3} \langle v \rangle^3 \mu \, dv.
\]
We conclude thanks to Cauchy-Schwarz inequality. \hfill \Box

Then we observe that for some constant $C > 0$,
\[ I_{1,2} + I_2 \leq C \int_{\mathbb{R}^6} |v - v_*|^{-2} f_* |\nabla h| |\nabla^2 h| \, dv_* \, dv + C \int f |\nabla^2 h| |h| \, dv, \]
and we define (with the constant $C$ being the same as in the inequality above)
\[ \mathcal{I} := C \int_{\mathbb{R}^6} |v - v_*|^{-2} f_* |\nabla h| |\nabla^2 h| \, dv_* \, dv, \]
and
\[ \mathcal{II} := C \int_{\mathbb{R}^3} f |\nabla^2 h| |h| \, dv. \]
2.3.2. Estimate for $\mathcal{I}$. We state the:

**Proposition 2.3.** Let $f \geq 0$ (and $h = f - \mu$) such that $\|f\|_{L^1(\mathbb{R}^3)} = 4$ and $\|f\|_{L^1(\mathbb{R}^3)} + \|f\|_{L^\infty L^2 \log L} \leq K$ for some $K > 0$, $\tau > 31$. Then for all $\eta \in [0,1]$ and $A > 1$, the following estimate holds:

$$
\mathcal{I} \leq \frac{\eta}{2} \|\nabla^2 h\|_{L^2_{-\frac{3}{2}}}^2 + (C\eta + C(K)\eta^{-1}(\log A)^{-\frac{\tau - 5}{2\tau - 5}})(D(f) + 1)\|\nabla h\|_{L^2}^2 + C(\eta + \eta^{-1} + \eta^{-3}A^2)\|\nabla h\|_{L^4_{\infty}}^2 + C(K)\eta^{-1}(\log A)^{-\frac{\tau - 5}{2\tau - 5}}\|\nabla h\|_{L^2}^2,
$$

where $C(K) > 0$ is a constant depending only on $K$, and $C > 0$ is an absolute constant.

**Proof.** We write $\mathcal{I} = \mathcal{I}_1 + \mathcal{I}_2$, with

$$
\mathcal{I}_1 := \int \int (|v - v_*|^{-2}f_* |\nabla h| |\nabla^2 h| \, dv_* \, dv,
$$

$$
\mathcal{I}_2 := \int \int (|v - v_*|^{-2}f_* |\nabla h| |\nabla^2 h| \, dv_* \, dv.
$$

We see that

$$
\mathcal{I}_2 \leq C \int \int <v>^{-2} (f_* |\nabla h| |\nabla^2 h| \, dv_* \, dv
$$

$$
\leq C \|f\|_{L^2} \|\nabla h\|_{L^2} \|\nabla^2 h\|_{L^2_{-\frac{3}{2}}}
$$

$$
\leq \frac{\eta}{4} \|\nabla^2 h\|_{L^2_{-\frac{3}{2}}}^2 + C \eta^{-1} \|\nabla h\|_{L^2}^2.
$$

We now turn to $\mathcal{I}_1$. We note first that in the region $\{|v - v_*| \leq 1\}$ holds the estimate

$$
\frac{1}{\sqrt{3}} \langle v \rangle \leq \langle v_* \rangle \leq \sqrt{3} \langle v \rangle.
$$

Thus by Cauchy-Schwartz inequality, we have

$$
\mathcal{I}_1 \leq C \left( \int \int \frac{|v - v_*|^{-2} (f_* |\nabla h| |\nabla^2 h| |\nabla^2 h| |\nabla^2 h| \, dv_* \, dv) }{\nabla h| |\nabla^2 h|} \right)^{\frac{1}{2}} \quad \text{def}_{B_1}
$$

$$
\times \left( \int \int \frac{|v - v_*|^{-2} (f_* |\nabla h| |\nabla^2 h| |\nabla^2 h| |\nabla^2 h| \, dv_* \, dv) }{\nabla h| |\nabla^2 h|} \right)^{\frac{1}{2}} \quad \text{def}_{B_2}
$$

We observe that for $k \in [0,3/\tau], 1(|v| \leq 1) \cdot |v| \in L^{3/k, \infty}$. Therefore, we get

$$
B_1 \leq C \|f\|_{L^\infty_{-\frac{3}{2}}} \|\nabla h\|_{L^2} \|\nabla^2 h\|_{L^2_{-\frac{3}{2}}},
$$

thanks to O’Neill inequality (cf. Proposition 6.2 in the Appendix).

Concerning $B_2$ (and for any $A > 1$), we split $f$ into two parts: $f_A = f \chi(f/A)$ and $f^A := f - f_A$, where $\chi$ is a nonnegative $C^1$ function satisfying that $\chi = 1$ in $B_1$ and $\chi = 0$ outside of $B_2$. Thanks to this decomposition, we get

$$
B_2 \leq C \|f^A\|_{L^\infty_{-\frac{3}{2}}} \|\nabla h\|_{L^2} \|\nabla^2 h\|_{L^2_{-\frac{3}{2}}} + C A \|\nabla h\|_{L^\infty_{-\frac{3}{2}}} \|\nabla^2 h\|_{L^2_{-\frac{3}{2}}}.
$$

Here we use again O’Neill inequality (cf. once again Proposition 6.2 in the Appendix) for $f^A$, and the bound $f_A \leq C A$. 


Putting together the estimates for $B_1$ and $B_2$ yields
\[
I_1 \leq C \left( \| f \|_{L^3_1}^{\frac{5}{3}} \| f^A \|_{L^3_1}^{\frac{1}{3}} \| \nabla h \|_{L^2} \| \nabla^2 h \|_{L^2_{-\frac{1}{2}}} + A^{\frac{2}{3}} \| f \|_{L^3_1}^{\frac{5}{3}} \| \nabla h \|_{L^2} \| \nabla^2 h \|_{L^2_{-\frac{1}{2}}} \right)
\]
(2.17) \leq \frac{\eta}{4} \| \nabla^2 h \|_{L^2_{-\frac{1}{2}}}^2 + \eta \| f \|_{L^3_2}^2 \| \nabla h \|_{L^2}^2 + C \left( \eta^{-1} \| f \|_{L^3_2} \| f^A \|_{L^3_{-1}} \| \nabla h \|_{L^2}^2 + \eta^{-3} A^2 \| \nabla h \|_{L^2_{-\frac{1}{2}}}^2 \right).

In what follows, we estimate the quantities $\| f \|_{L^3_2}^2 \| \nabla h \|_{L^2}^2$ and $\| f \|_{L^3_{-1}} \| f^A \|_{L^3_{-1}} \| \nabla h \|_{L^2}^2$.

*Estimate of $\| f \|_{L^3_2}^2 \| \nabla h \|_{L^2}^2$.* Remembering that $\| f \|_{L^2} = 4$ and using Proposition 6.3 we get the estimate
\[
\| f \|_{L^3_2}^2 \| \nabla h \|_{L^2}^2 \leq C \| f \|_{L^3_2}^{-\frac{3}{2}} \| \nabla h \|_{L^2}^2 \leq C \| \nabla h \|_{L^2}^2 + 1,
\]
(2.18) where we used the interpolation estimate
\[
\| h \|_{L^2} \leq C \left( \| h \|_{L^1} + \| \nabla h \|_{L^2} \right) \leq C \left( 1 + \| \nabla h \|_{L^2} \right).
\]
(2.19) It yields
\[
\| f \|_{L^3_2}^2 \| \nabla h \|_{L^2}^2 \leq C \left( \| \nabla h \|_{L^2}^2 + 1 \right) \| f \|_{L^3_2} \| \nabla h \|_{L^2}^2.
\]
(2.20) We end up with the bound
\[
\| f \|_{L^3_2}^2 \| \nabla h \|_{L^2}^2 \leq C \left( \| f \|_{L^3_2} \| \nabla h \|_{L^2}^2 + \| f \|_{L^3_2} \| \nabla h \|_{L^2}^2 \right)
\]
\leq C \left( D(\eta) \| \nabla h \|_{L^2}^2 + \| \nabla h \|_{L^2}^2 + \| \nabla h \|_{L^2}^2 \right),
\]
where we used the estimates (2.19) for the first term, and (2.18) for the second term.

*Estimate of $\| f \|_{L^3_2} \| f^A \|_{L^3_{-1}} \| \nabla h \|_{L^2}^2$.* Thanks to the definition of $f^A$ and the interpolation estimate (2.19), we first see that
\[
\| f^A \|_{H^1} \leq C \sqrt{1 + A^{-2}} \left( \| \nabla h \|_{L^2} + 1 \right).
\]
(2.21) For $R > 0$ and $A > 1$, we know that
\[
\| f^A \|_{L^3_1} \leq R^{31} (\log A)^{-1} \int_{f \geq 1} f \log f \, dv + R^{-(r-31)} \| f \|_{L^1},
\]
so that
\[
\| f^A \|_{L^3_1} \leq C(K)(\log A)^{-\frac{r-31}{3}}.
\]
(2.22) Using Proposition 6.3 one gets
\[
\| f \|_{L^3_2} \| f^A \|_{L^3_1} \| \nabla h \|_{L^2}^2 \leq C \| f \|_{L^3_2} \| f^A \|_{H^1} \| f^A \|_{L^3_1} \| \nabla h \|_{L^2}^2.
\]
Then, using estimates (2.21) and (2.22),
\[
\| f \|_{L^3_2} \| f^A \|_{L^3_1} \| \nabla h \|_{L^2}^2 \leq C(K)(\log A)^{-\frac{r-31}{3}} \int \left( D(\eta) + 1 \right) \| \nabla h \|_{L^2}^2 + \| \nabla h \|_{L^2}^2.
\]
(2.23) Finally, using both estimates (1.23) and (2.18), we end up with
\[
\| f \|_{L^3_2} \| f^A \|_{L^3_1} \| \nabla h \|_{L^2}^2 \leq C(K)(\log A)^{-\frac{r-31}{3}} \left( D(\eta) + 1 \right) \| \nabla h \|_{L^2}^2 + \| \nabla h \|_{L^2}^2.
\]
(2.24) Finally, we see that
\[
\| f \|_{L^3_2} \leq \frac{\eta}{4} \| \nabla^2 h \|_{L^2_{-\frac{1}{2}}}^2 + C \eta^{-3} A^2 \| \nabla h \|_{L^2}^2 + C \eta (D(\eta) \| \nabla h \|_{L^2}^2 + \| \nabla h \|_{L^2}^2).
\]
(2.25)
\[ + C(K) \eta^{-1} (\log A)^{-\frac{4}{7\tau^2}} \left[(D(f) + 1)\|\nabla h\|_{L^2}^{4\over 3} + \|\nabla h\|_{L^2}^2\right] \]

\begin{align*}
\leq \frac{\eta}{4} & \|\nabla^2 h\|_{L^2}^2 + (C \eta + C(K) \eta^{-1} (\log A)^{-\frac{4}{7\tau^2}}) (D(f) + 1)\|\nabla h\|_{L^2}^{4\over 3} \\
& \quad + C (\eta + \eta^{-3} A^2) \|\nabla h\|_{L^2}^2 + C(K) \eta^{-1} (\log A)^{-\frac{4}{7\tau^2}} \|\nabla h\|_{L^2}^2.
\end{align*}

We deduce the desired result by combining this estimate with the estimate for \(I_2\).

\[ \square \]

2.3.3. Estimate for \(I_2\). We now prove the following bound:

**Proposition 2.4.** Consider \(f \geq 0\) (and \(h = f - \mu\)) such that \(\|f\|_{L^1_\tau(R^3)} = 4\) and \(\|f\|_{L^1_\tau(R^3)} + \|f\|_{L^1_\tau} \leq K\) for some \(K > 0, \tau > 31\). Then for all \(\eta \in \{0,1\} [A > 1\), the following estimate holds:

\[ I_2 \leq \frac{\eta}{4} \|\nabla^2 h\|_{L^2}^2 + C(K) \eta^{-1} (\log A)^{-\frac{4}{7\tau^2}} (D(f) + 1)\|\nabla h\|_{L^2}^{4\over 3} + C \eta^{-1} A^2 \|\nabla h\|_{L^2}^2 \]

\begin{align*}
& \quad + C(K) \eta^{-1} (\log A)^{-\frac{4}{7\tau^2}} \|\nabla h\|_{L^2}^2,
\end{align*}

where \(C(K) > 0\) only depend on \(K\) (and \(\tau\)), and \(C > 0\) only depends on \(\tau\).

**Proof.** We recall that

\[ I_2 \leq \int f \partial^2 h h \, dv = \int f_A \partial^2 h h \, dv + \int f^A \partial^2 h h \, dv. \]

Then

\[ I_2 \leq C A \|\nabla^2 h\|_{L^2} \|h\|_{L^3}^2 + C \|f^A\|_{L^3} \|\nabla^2 h\|_{L^2} \|h\|_{L^6} \]

\begin{align*}
\leq \frac{\eta}{4} \|\nabla^2 h\|_{L^2}^2 + C \eta^{-1} (A^2 \|h\|_{L^2}^2 + \|f\|_{L^1_\tau} \|f^A\|_{L^3_\tau} \|\nabla h\|_{L^2}^2).
\end{align*}

The desired result is obtained by using an estimate almost identical to that of (2.24). \( \square \)

2.3.4. Summary of the estimate for the remainder terms. We regroup the results of Propositions 2.2, 2.3 and 2.4 in the

**Proposition 2.5.** Let \(f \geq 0\) (and \(h = f - \mu\)) such that \(\|f\|_{L^1_\tau(R^3)} = 4\) and \(\|f\|_{L^1_\tau(R^3)} + \|f\|_{L^1_\tau} \leq K\) with \(K > 0, \tau > 31\). Then for all \(\eta \in \{0,1\} [A > \epsilon\), the following estimate holds:

\[ I_{1,2} + I_2 + I_3 + I_4 \leq \frac{\eta}{4} \|\nabla^2 h\|_{L^2}^2 + (C \eta + C(K) \eta^{-1} (\log A)^{-\frac{4}{7\tau^2}}) (D(f) + 1)\|\nabla h\|_{L^2}^{4\over 3} \]

\begin{align*}
& \quad + C \eta^{-1} A^2 \|h\|_{L^2}^2 + C(K) (\eta^{-1} + \eta^{-3} A^2) \|\nabla h\|_{L^2}^2,
\end{align*}

where \(C(K) > 0\) only depend on \(K\) and \(\tau\), and \(C > 0\) only depends on \(\tau\).

**Proof.** This estimate is directly obtained from Propositions 2.2, 2.3 and 2.4 remembering that \(\eta < 1, \log A > 1\) and \(-\frac{4}{7\tau^2} < 0\). \( \square \)

2.3.5. Summary of the estimate for all terms. We now regroup the results of Proposition 2.4 and Corollary 2.2. From now on, we typically denote by \(C^*\) constants which can be replaced by a larger constant, and by \(C\) constants which can be replaced by a smaller (strictly positive) constant. We get the
Proposition 2.6. Let \( f \geq 0 \) (and \( h = f - \mu \)) such that \( \int_{\mathbb{R}^3} f(v) \, dv = 1 \), \( \int_{\mathbb{R}^3} f(v) \, |v|^2 \, dv = 3 \), and \( \|f\|_{L^1(\mathbb{R}^3)} + \|f\|_{L^\infty} \leq K \) with \( \tau > 31 \). Then for all \( \eta \in [0, 1] \), the following estimate holds:

\[
(2.27) \quad I_1 + I_2 + I_3 + I_4 \leq - \frac{C(K)}{2} \|\nabla^2 h\|_{L^2}^2 - \frac{C(K)}{2} \|h\|_{L^{15/4}} \|\nabla h\|_{L^2}^{14/3} + C^*(K) \eta^{13} \exp\left(7 \eta^{-10/31} \right) \|h\|_{L^{60/47}}^2 + C^*(K) \eta (1 + D(f)) \|\nabla h\|_{L^2}^{14/3},
\]

where \( C(K), C^*(K) > 0 \) only depend on \( K \) and \( \tau \).

Proof. Using Proposition 2.5 and Corollary 2.2, we see that

\[
(2.28) \quad I_1 + I_2 + I_3 + I_4 \leq - C(K) \|\nabla^2 h\|_{L^2}^2 - C(K) \|h\|_{L^{15/4}}^{5/7} \|\nabla h\|_{L^2}^{14/7} + C^*(K) \|h\|_{L^{60/47}}^2 + C^*(K) \eta^{-1} A^2 \|h\|_{L^2}^2,
\]

Taking \( \eta := C^* \eta^{10/7} A^{-10/7} \), we see that

\[
(2.29) \quad C^* \eta^{-1} A^2 \|h\|_{L^2}^2 \leq C^* \eta^{-1} A^2 \left(1 + (1 - 5 \eta^5 A^5) \right) \|h\|_{L^{60/47}}^2 + \frac{\eta}{2} \|\nabla^2 h\|_{L^2}^2,
\]

while taking \( \zeta := C^*(K) \eta^{20/7} A^{-10/7} \) (and observing that \( \eta^{-1} \leq \eta^{-3} A^2 \)), we see that

\[
(2.30) \quad C^*(K) \eta^{-1} A^2 \|\nabla h\|_{L^2}^2 \leq C^*(K) \eta^{-3} A^2 \left(1 + \eta^{-10} A^5 \right) \|h\|_{L^{60/47}}^2 + \frac{\eta}{2} \|\nabla^2 h\|_{L^2}^2.
\]

Using this bound in estimate (2.28), we see that

\[
(2.31) \quad I_1 + I_2 + I_3 + I_4 \leq (2\eta - C(K)) \|\nabla^2 h\|_{L^2}^2 - C(K) \|h\|_{L^{15/4}}^{5/7} \|\nabla h\|_{L^2}^{14/7} + C^*(K) \|h\|_{L^{60/47}}^2 + \frac{\eta}{2} \|\nabla^2 h\|_{L^2}^2
\]

\[
+ \left[C^*(K) + C^* \eta^{-1} A^2 \left(1 + \eta^{-5} A^5 \right) + C^*(K) \eta^{-3} A^2 \left(1 + \eta^{-10} A^5 \right) \right] \|h\|_{L^{60/47}}^2
\]

\[
+ (C^* \eta + C^*(K) \eta^{-1} (\log A)^{-\frac{11}{31} A^5}) \left( D(f) + 1 \right) \|\nabla h\|_{L^2}^{14/3},
\]

so that when \( \eta < C(K)/4 \),

\[
(2.32) \quad I_1 + I_2 + I_3 + I_4 \leq - \frac{C(K)}{2} \|\nabla^2 h\|_{L^2}^2 - C(K) \|h\|_{L^{15/4}}^{5/7} \|\nabla h\|_{L^2}^{14/7} + C^*(K) \eta^{-13} A^7 \|h\|_{L^{60/47}}^2
\]

\[
+ (C^* \eta + C^*(K) \eta^{-1} (\log A)^{-\frac{11}{31} A^5}) \left( D(f) + 1 \right) \|\nabla h\|_{L^2}^{14/3}.
\]

We now select \( A > e \) such that \( (\log A)^{-\frac{11}{31} A^5} = \eta \), and get (changing the names of the constants) estimate (2.27). \( \square \)
2.4. Application of the estimates to the solutions of Landau equation.

**Lemma 2.1.** Let $\ell > 19/2$, $f_0 \in L^1_0 \cap L \log L$ be a nonnegative function such that $\int_{\mathbb{R}^3} f_0(v) \, dv = 1$, $\int_{\mathbb{R}^3} f_0(v) \, |v|^2 \, dv = 3$. We consider $f := f(t, v)$ a weak (well constructed) nonnegative solution to the Landau equation with Coulomb potential (1.1) – (1.3), and $h = f - \mu$.

Then for all $\theta \in [0, \ell]$, $q < q_\ell, \theta$ with

$$q_{\ell, \theta} := -\frac{2 \ell^2 - 25 \ell + 57}{18 (\ell - 2)} \left(1 - \frac{\theta}{\ell}\right) + \frac{\theta}{\ell},$$

there exists $C > 0$ (depending on $\theta$, $\ell$ and $K$ such that $\|f_0\|_{L^1_0} + \|f_0\|_{L \log L} \leq K$) such that

$$\forall t \geq 0, \quad \|h(t, \cdot)\|_{L^1_\theta} \leq C (1 + t)^{\theta}.$$  

More specifically, if $\ell \geq 55$, then for some $r_1 > 7/4$, $r_2 > 0$,

$$\forall t \geq 0, \quad \|h(t, \cdot)\|_{L^{r_1/2}} \leq C (1 + t)^{-r_1}, \quad \|h(t, \cdot)\|_{L^{r_2}} \leq C (1 + t)^{-r_2}.$$  

**Proof.** We first recall that thanks to Theorem 1.1 for $\beta < \frac{2 \ell^2 - 25 \ell + 57}{3 \ell (\ell - 2)}$, the relative entropy decays according to the inequality

$$\forall t > 0, \quad H(t) \leq C_\beta (1 + t)^{-\beta},$$

where $C_\beta > 0$ only depends on $\ell$ and $K$ such that $\|f_0\|_{L^1_0} + \|f_0\|_{L \log L} \leq K$.

Using Csiszar-Kullback-Pinsker inequality (cf. [7, 27]), we see that

$$\forall t > 0, \quad \|h(t, \cdot)\|_{L^1_\theta} \leq C_\beta (1 + t)^{-\beta/2}.$$

Then, (using Theorem 1.1 again) for all $\ell > 2$, there exists $C_\ell > 0$ (which only depends on $\ell$ and $K$ such that $\|f_0\|_{L^2_0} + \|f_0\|_{L \log L} \leq K$), such that

$$\forall t > 0, \quad \|h(t, \cdot)\|_{L^1_\ell} \leq C_\ell (1 + t).$$

Finally, we interpolate between the two previous inequalities, for $\theta \in [0, \ell]$:

$$\|h\|_{L^1_\theta} \leq \|h\|_{L^1_\ell}^{1 - \theta/\ell} \|h\|_{L^1_\ell}^{\theta/\ell} \leq C (1 + t)^{\theta},$$

for $q < q_\ell, \ell$, and $C > 0$ as described in the Lemma.

The special case (when $\theta = 99/4$, or $\theta = 45$) is directly obtained thanks to this estimate. □

We now write the $H^1$ estimate that will yield the differential inequality (1.24).

**Proposition 2.7.** Let $f_0 \in L^1_{55}(\mathbb{R}^3) \cap L \log L(\mathbb{R}^3)$ be a nonnegative function such that $\int_{\mathbb{R}^3} f_0(v) \, dv = 1$ and $\int_{\mathbb{R}^3} f_0(v) \, |v|^2 \, dv = 3$. We consider $f := f(t, v)$ a nonnegative smooth and quickly decaying when $|v| \to \infty$ ($C^2(S)$) solution (on an interval of time $[0, T]$) to the Landau equation (1.1) – (1.3), and $h = f - \mu$.

Then for some $k_1 > 0$, $k_2 > 7/2$, $C_1, C_2, C_3 > 0$ (depending only on $K$ such that $\|f_0\|_{L^1_{55}} + \|f_0\|_{L \log L} \leq K$), the following differential inequality holds (on $[0, T]$) for all $\eta \in [0, 1]$ sufficiently small (depending on $K$):

$$\frac{d}{dt} \|\nabla h\|_{L^2}^2 + C_1 (1 + t)^{k_1} \|\nabla h\|_{L^2}^{14} \leq \eta C_3 D(f) \|\nabla h\|_{L^2}^{14} + C_2 \eta^{-13} \exp\left(7\eta^{-\frac{450}{11}}\right) (1 + t)^{-k_2}.$$  

**Proof.** We consider a smooth and quickly decaying when $|v| \to \infty$ solution $f := f(t, v) \geq 0$ to (1.1) – (1.3) (on a given interval of time $[0, T]$). According to Lemma 2.1 (more precisely to the special case described in this Lemma), this solution is bounded in $L^1_{45}(\mathbb{R}^3)$ (with a bound controlled by $K$ such that $\|f_0\|_{L^1_{55}(\mathbb{R}^3)} + \|f_0\|_{L \log L} \leq K$).
Recalling the computation (2.3), which rigorously holds, we can use Proposition 2.6 with \( \tau = 45 \) (for a smooth solution to eq. (1.1) – (1.3)), and we see that (for some \( C, C_4, C_5 > 0 \) depending only on \( K \) such that \( \| f_0 \|_{L^1} + \| f_0 \|_{L \log L} \leq K \)),

\[
(2.38) \quad \frac{d}{dt} \| \nabla h \|^2_{L^2} + \| h \|_{L_{15/4}^4} \| \nabla h \|^2_{L^2} \leq C_4 \eta^{-13} \exp(7 \eta^{-450}) \| h \|^2_{L_{10}^{39/4}} + C_3 \eta (1 + D(f)) \| \nabla h \|^2_{L^2}.
\]

Using again the special case described at the end of Lemma 2.1 (and observing that \( \| h \|_{L_{15/4}^4} \leq \| h \|_{L_{10}^{39/4}} \)), we complete the proof of the differential inequality (2.37).

Thanks to Proposition 2.7, we now can reduce the main results in Theorem 1.2 to the analysis of some ordinary differential inequality.

2.5. Analysis of a differential inequality. We start with the following Lemma, which corresponds to the special case \( \eta = C_3^{-1}, B^* := B(\eta), B(x) := C_2 x^{-13} \exp \{ 7 x^{-450} \} \) in Proposition 2.7.

**Lemma 2.2.** Let \( X, H \) be \( C^1 \) functions from \( [0, T] \) to \( \mathbb{R}_+ \) (for \( T \in [0, +\infty) \)), \( C_1, B^*, k_1 > 0, k_2 > 7/2, \) and \( D := -H' \) such that

\[
(2.39) \quad \forall t \in [0, T], \quad \frac{d}{dt} X(t)^2 + C_1 (1 + t)^{k_1} X(t)^{4/3} \leq D(t) X(t)^{4/3} + B^*(1 + t)^{-k_2}.
\]

Then for \( k := \min(2k_2^{-1}, k_1) \) and some constant \( C_6 > 0 \) depending only on \( C_1, B^*, k_2 \), the following differential inequality holds:

\[
(2.40) \quad \forall t \in [0, T], \quad \frac{d}{dt} \left( H(t) \frac{5}{2} [X(t)^2 + B^*(1 + t)^{1-k_2}]^{-2/5} \right) + C_6 (1 + t)^k \leq 0.
\]

**Proof.** We first observe that denoting \( Y(t) := B^*(1 + t)^{-k_2+1} \) and \( c_1 := (B^*)^{-\frac{5}{2}} \), the following differential inequality holds:

\[
\forall t \in [0, T], \quad \frac{d}{dt} Y(t) + c_1 (1 + t)^{2k_2^{-1}} Y^\frac{5}{2}(t) \leq -B^*(1 + t)^{-k_2}.
\]

Therefore for some \( C_6 > 0 \) depending only on \( C_1, B^*, k_2 \), the following differential inequality also holds:

\[
\forall t \in [0, T], \quad \frac{d}{dt} \left( X(t)^2 + B^*(1 + t)^{1-k_2} \right) + C_6 (1 + t)^k \left[ X(t)^2 + B^*(1 + t)^{1-k_2} \right]^{7/5} \leq D(t) \left[ X(t)^2 + B^*(1 + t)^{1-k_2} \right]^{7/5}.
\]

The differential inequality stated in the Lemma is then obtained by dividing this differential inequality by \( X(t)^2 + B^*(1 + t)^{1-k_2} \) \( 7/5 \).

Next we turn to the following consequence of Lemma 2.2.

**Lemma 2.3.** Let \( X, H \) be \( C^1 \) functions from \( [0, T] \) to \( \mathbb{R}_+ \) (for \( T \in [0, +\infty) \)), \( C_1, B^*, k_1 > 0, k_2 > 7/2, \) and \( D := -H' \), such that the differential inequality (2.39) holds.

- If \( H(0) [X(0)^2 + B^*]^{2/5} \leq \frac{2}{5} \), then for some constant \( C_6 > 0 \) depending only on \( C_1, B^*, k_2 \),

\[
\forall t \in [0, T], \quad X(t) \leq \left( \frac{2}{5} \right)^{-\frac{5}{2}} \left( H(t) + \frac{C_6}{k+1} (1 + t)^{1+k} - 1 \right)^{-\frac{5}{2}}.
\]
If $H(0) \left[ X(0)^2 + B^* \right]^{2/5} > \frac{2}{5}$, then for
\[
T^* := \left( \frac{1+k}{C_6} \left[ H(0) - \frac{5}{2} \left[ X(0)^2 + B^* \right]^{-2/5} \right] + 1 \right)^{-\frac{1}{1+k}} - 1,
\]
one has (for $T > T^*$) $H(T^*) \leq \frac{2}{5} \left[ X(T^*)^2 + B^* (1+T^*)^{1-k_2} \right]^{-2/5}$ and for $t \in [0, T-T^*]$,\[
X(T^* + t) \leq \left( \frac{2}{5} \right)^{-\frac{2}{5}} \left( H(T^*) + \frac{C_6}{k+1} \left[ (1+T^* + t)^{1+k} - (1+T^*)^{1+k} \right] \right)^{-\frac{2}{5}}.
\]

**Proof.** By integrating both sides of inequality (2.40) on the interval $[t_1, t_2]$, $T > t_2 > t_1 \geq 0$, we see that
\[
H(t_2) - \frac{5}{2} \left[ X(t_2)^2 + B^* (1+t_2)^{1-k_2} \right]^{-2/5} + \frac{C_6}{1+k} \left[ (1+t_2)^{1+k} - (1+t_1)^{1+k} \right] \leq H(t_1) - \frac{5}{2} \left[ X(t_1)^2 + B^* (1+t_1)^{1-k_2} \right]^{-2/5}.
\]
(2.41)

Taking $t_1 = 0$, $t_2 = t$, and using the condition $\frac{5}{2} \left[ X(0)^2 + B^* \right]^{-2/5} \geq H(0)$, we rewrite the above inequality as
\[
\frac{5}{2} \left[ X(t)^2 + B^* (1+t)^{1-k_2} \right]^{-2/5} \geq H(t) + \frac{C_6}{1+k} \left[ (1+t)^{1+k} - 1 \right].
\]
From this, we get
\[
X(t) \leq \left[ \left( \frac{2}{5} \right)^{-\frac{2}{5}} \left( H(t) + \frac{C_6}{k+1} \left[ (1+t)^{1+k} - 1 \right] \right)^{-\frac{2}{5}} - B^* (1+t)^{1-k_2} \right]^\frac{1}{2},
\]
which proves the first result.

The second result follows from estimate (2.41) by taking $t_1 = 0$ and $t_2 = T^*$, and solving
\[
\frac{C_6}{1+k} \left[ (1+T^*)^{1+k} - 1 \right] = H(0) - \frac{5}{2} \left[ X(0)^2 + B^* \right]^{-2/5}.
\]
Now let $t_1 = T^*$ and $t_2 = T^* + t$, $t > 0$, then $H(T^*) - \frac{5}{2} \left[ X(T^*)^2 + B^* (1+T^*)^{1-k_2} \right]^{-2/5} \leq 0$ implies that
\[
X(T^* + t) \leq \left( \frac{2}{5} \right)^{-\frac{2}{5}} \left( H(T^*) + \frac{C_6}{k+1} \left[ (1+T^* + t)^{1+k} - (1+T^*)^{1+k} \right] \right)^{-\frac{2}{5}},
\]
which gives the estimate for $X$ after the time $T^*$ described in the Lemma. \qed

### 2.6. End of the proof of Theorem 1.2
We now are in a position to prove Theorem 1.2. We show that the *a priori* estimates obtained in Subsections 2.1 to 2.5 can be used to build a solution to eq. (1.1) – (1.3) thanks to their application to the smooth solutions of an approximated equation.

We introduce therefore the unique solution $f^* := f^*(t, v) \geq 0$ to the approximated equation
\[
\partial_t f^* = Q^*(f^*, f^*),
\]
where $Q^*$ is defined by
\[
Q^*(g, h) = \nabla_v \cdot \left( [a^* \ast g] \nabla_v h - [a^* \ast \nabla g] h \right),
\]
with
\[
a^*(z) = (|z|^2 + \epsilon^2)^{-\frac{1}{2}} \left( Id - \frac{z \otimes z}{|z|^2} \right).
\]
We are therefore still considering a Landau equation, but with a regularized cross section. We also introduce smooth and quickly decaying (when $|v| \to \infty$) initial data, converging when $\varepsilon \to 0$ towards the original initial data $f_0$. The problem (2.42) – (2.43) satisfies the same conservation properties (propagation of nonnegativity, conservation of mass, momentum and kinetic energy, decay of the entropy) as the original equation (1.1) – (1.3).

Next we briefly explain how to prove the

**Proposition 2.8.** For $\varepsilon > 0$, estimates (1.22), (1.23) and (1.27) – (1.28) hold for the unique smooth ($C^2_t(S)$) solution of equation (2.42) – (2.43) (with smoothed initial data), with constants which do not depend on $\varepsilon$.

**Proof.** Step 1: Since (for $\varepsilon > 0$), there is no singularity in $a^\varepsilon$, equation (2.42) – (2.43) behaves (from the point of view of regularity) like the Landau equation with Maxwell molecules (that is, when $\gamma = 0$ in (1.31)). Hence, smoothness and moments can be proved to be propagated globally for this equation. This is easily checked by following the strategy used in [19, 20]. Thus equation (2.42) – (2.43) admits a unique (global) smooth solution (the initial data being themselves smooth).

Step 2: Using Theorem 3 in [8], we see that estimates (1.22) and (1.23) hold when $a$ is replaced by $a^\varepsilon$, with a constant that does not depend on $\varepsilon$. It is then possible to show, using the same method as in [1], that the long-time behavior estimates are the same for the solution to equation (2.42) – (2.43) as those for the solution to Landau equation with Coulomb potential (1.1) – (1.3), with constants which do not depend on $\varepsilon$.

Step 3: We show that Proposition 2.7 holds for the unique smooth solutions of equation (2.42) – (2.43), with constants in the estimate which do not depend on $\varepsilon$.

This amounts to showing that the estimates in the proof still hold when $a$ is replaced by $a^\varepsilon$. Noticing that $b^\varepsilon_i(z) := \sum_{j=1}^3 \partial_j a^\varepsilon_{ij}(z) = -2 z_i |z|^2 (|z|^2 + \varepsilon^2)^{-1} \sum_{i=1}^3 \partial_i b^\varepsilon_i(z) = -2 \varepsilon^{-3} |z|^{-2} \times (\varepsilon^{-1}|z|^2 + 1)^{-1/2}$, we see that $|a'| \leq |a|$, $|b_i'| \leq |b_i|$, and those inequalities can be used to show that the estimates from above in Subsections 2.3, 2.4 can be reproduced with the same constants for the approximated problem as for the original problem.

We then can directly check by inspecting the proofs that the coercivity estimate appearing in Proposition 2.1 and Corollary 2.1 can be reproduced when $a$ is replaced by $a^\varepsilon$, with constants that do not depend on $\varepsilon$.

Since for $\varepsilon > 0$, the solution $f^\varepsilon$ is smooth and quickly decaying when $|v| \to \infty$, the assumptions of Proposition 2.7 are fulfilled, so that estimate (2.37) holds (for this solution), with constants which do not depend on $\varepsilon$.

**Step 4:** We now can apply Lemmas 2.2 and 2.3 to $X = \|\nabla h^\varepsilon\|_{L^2}$, and obtain the estimates of Theorem 1.2 for the unique smooth solution of equation (2.42) – (2.43), with constants in the estimate which do not depend on $\varepsilon$.

Finally we give the end of the proof of Theorem 1.2.

**End of the proof of Theorem 1.2:** Note first that part (i) of Theorem 1.2 is immediately obtained (without using the approximation problem) by the use of Proposition 2.7 and Lemma 2.2.

We now turn to parts (ii) and (iii). As in [8], we consider $f^\varepsilon$ the unique smooth solution of eq. (2.42) – (2.43) with initial data strongly converging to $f_0$. It is then possible to pass to the limit (in a weighted weak $L^1$ space, and up to extracting a subsequence) when $\varepsilon \to 0$ in $f^\varepsilon$, and get in this way a (well constructed) weak solution $f$ to the original equation eq. (1.1) – (1.3) with initial data $f_0$. 

□
Due to the convexity of $x \mapsto x \log x$ and the lower semi-continuity of the weak convergence in $H^1$, we obtain that
\[ H(t) \leq \liminf_{\epsilon \to 0} H'(t), \quad \|\nabla h\|_{L^2} \leq \liminf_{\epsilon \to 0} \|\nabla h'\|_{L^2}, \]
where $H'(t)$ is the relative entropy of $f'$ at time $t$.

Thanks to these properties, we can pass to the limit in the following estimates:

- For the initial data under the threshold,
  \[ \forall t \geq 0, \quad \|h'(t)\|_{H^1} \leq \left( \frac{2}{5}\right)^{-\frac{5}{4}} \left( H'(t) + \frac{C_6}{k+1} \left[ (1+t)^{1+k} - 1 \right] \right)^{-\frac{5}{4}}; \]
- For general suitable initial data and $t > \tau$,
  \[ H'(t) \leq \frac{5}{2} \left[ \|h'(t)\|_{H^1}^2 + B^* (1+t)^{-k_2} \right]^{-2/5}. \]

\[ \|h'(t)\|_{H^1} \leq \left( \frac{2}{5}\right)^{-\frac{5}{4}} \left( \frac{C_6}{k+1} \left[ (1+t)^{1+k} - (1+T^*)^{1+k} \right] \right)^{-\frac{5}{4}}. \]

We conclude thus the proof of statements (ii) and (iii) of Theorem 1.2.

\[
3. \ \text{LOCAL SOLUTIONS: PROOF OF PROPOSITION 1.1}
\]

We present in this section the Proof of Proposition 1.1. We start with the following Proposition, which is a variant of Proposition 2.6

\textbf{Proposition 3.1.} Let $f \geq 0$ (and $h = f - \mu$) such that $\|f\|_{L^1_{\log L}(\mathbb{R}^3)} = 4$ and $\|f\|_{L^1_{\log L}(\mathbb{R}^3)} + \|f\|_{L^1_{\log L}} \leq K$ with $K > 0$ and $\tau > 31$. Then the following estimate holds:

\begin{equation}
I_1 + I_2 + I_3 + I_4 \leq - \frac{C(K)}{4} \|\nabla^2 h\|_{L^2}^2 - \frac{C(K)}{2} \|h\|_{L^2}^2 \|\nabla h\|_{L^2}^{18/5} - \frac{1}{2} I_{1,1}
+ C^*(K) \|h\|_{L^2}^{18/5} + \|\nabla h\|_{L^2}^{18/5},
\end{equation}

where $C^*(K), C(K) > 0$ only depend on $K$ and $\tau$.

\textit{Proof.} Using estimates (2.18) and (2.20), we see that

\[ \|f\|_{L^2_{\log L}}^2 \|\nabla h\|_{L^2}^2 \leq C \left( \|\nabla h\|_{L^2}^2 + \|\nabla h\|_{L^2}^{18/5} \right). \]

Then, recalling estimates (2.18) and (2.28), we also see that (for $A > \epsilon$)

\[ \|f\|_{L^2_{\log L}}^2 \|f^A\|_{L^1_{\log L}} \|\nabla h\|_{L^2}^2 \leq C(K) (\log A)^{-\frac{16}{57}} \left( \|\nabla h\|_{L^2}^2 + \|\nabla h\|_{L^2}^{18/5} \right). \]

Using the notation (2.38) and bounds (2.10) and (2.17), this leads to the bound (for all $\eta \in [0,1]$ and $A > \epsilon$)

\[ I \leq \frac{\eta}{2} \|\nabla^2 h\|_{L^2}^2 + \left( C_\eta + C(K)\eta^{-1} (\log A)^{-\frac{16}{57}} \right) \|\nabla h\|_{L^2}^{18/5} + C (\eta + \eta^{-1} + \eta^{-3} A^2) \|\nabla h\|_{L^2}^2 + C(K) \eta^{-1} (\log A)^{-\frac{16}{57}} \|\nabla h\|_{L^2}^2. \]
Using the notation (2.14) and estimate (2.20), we also get the estimate (for all \( \eta \in [0,1] \) and \( A > e \))

\[
\begin{align*}
\mathcal{II} & \leq \frac{\eta}{4} \| \nabla^2 h \|_{L^2}^2 + C(K) \eta^{-1}(\log A)^{-\frac{2}{31}} \| \nabla h \|_{L^2}^{\frac{18}{5}} + C \eta^{-1} A^2 \| h \|_{L^2}^2 \\
& \quad + C(K) \eta^{-1}(\log A)^{-\frac{2}{5}} \| \nabla h \|_{L^2}^2,
\end{align*}
\]

where \( C(K) > 0 \) only depend on \( K \) and \( \tau \), and \( C > 0 \) only depends on \( \tau \).

Recalling now estimates (2.11) and inequality (2.12) (together with notations (2.13) and (2.14)), and remembering that \( \eta < 1 \), \( \log A > 1 \) and \( -\frac{2}{5} < 0 \), we end up with the estimate

\[
\begin{align*}
I_1 + I_2 + I_3 + I_4 & \leq \eta \| \nabla^2 h \|_{L^2}^2 + (C + C(K) \eta^{-1}(\log A)^{-\frac{2}{31}}) \| \nabla h \|_{L^2}^{\frac{18}{5}} \\
& \quad + C \eta^{-1} A^2 \| h \|_{L^2}^2 + (C + C(K) \eta^{-1}(\log A)^{-\frac{2}{5}}) \| \nabla h \|_{L^2}^2,
\end{align*}
\]

where \( C(K) > 0 \) only depend on \( K \) (and \( \tau \)) and \( C > 0 \) only depends on \( \tau \).

Using estimates (2.10), (2.28) and (3.3), we see that (using \( C^* \) for constants which can be replaced by larger constants, and \( C \) for constants which can be replaced by smaller constants)

\[
\begin{align*}
I_1 + I_2 + I_3 + I_4 & \leq (2\eta - \frac{1}{2} C(K)) \| \nabla^2 h \|_{L^2}^2 - \frac{1}{2} C(K) \| h \|_{L^{\infty}/4}^{\frac{8}{5}} \| \nabla h \|_{L^2}^2 - \frac{1}{2} I_{1,1} \\
& \quad + \left[ \frac{1}{2} C^*(K) + C^* \eta^{-1} A^2 (1 + \eta^{-5} A^5) \right] \left[ \| h \|_{L^{\infty}/4}^2 \right] \\
& \quad + (C^* \eta + C^*(K) \eta^{-1}(\log A)^{-\frac{2}{31}}) \| \nabla h \|_{L^2}^2,
\end{align*}
\]

so that when \( \eta < C(K)/8 \),

\[
\begin{align*}
I_1 + I_2 + I_3 + I_4 & \leq -\frac{C(K)}{4} \| \nabla^2 h \|_{L^2}^2 - \frac{C(K)}{2} \| h \|_{L^\infty/4}^{\frac{8}{5}} \| \nabla h \|_{L^2}^2 - \frac{1}{2} I_{1,1} \\
& \quad + C^*(K) \eta^{-13} A^7 \| h \|_{L^\infty/4}^2 + (C^* \eta + C^*(K) \eta^{-1}(\log A)^{-\frac{2}{31}}) \| \nabla h \|_{L^2}^2.
\end{align*}
\]

Selecting \( \eta > 0 \) sufficiently small, and \( A > e \) such that \( (\log A)^{-\frac{2}{31}} = \eta \), we see that estimate (3.1) holds.

**End of the proof of Proposition 2.7.** We observe that inequality (3.1) still holds when the kernel of the Landau equation is replaced by the kernel of the approximated equation (2.42) – (2.43), with all constants not depending on \( \epsilon \). Then, when \( f^* \) (and \( h^* = f^* - \mu \)) is the unique smooth solution of eq. (2.42) – (2.43) (with regularized initial data), and proceeding as in the proof of Proposition 2.7 we get the estimate

\[
\begin{align*}
\frac{d}{dt} \| \nabla h^* \|_{L^2}^2 + C_{12} \| \nabla^2 h^* \|_{L^2}^2 + \frac{1}{4} I_{1,1} + \frac{1}{4} C_{10} (1 + t)^{k_1} \| \nabla h^* \|_{L^2}^{\frac{18}{5}} \leq \| \nabla h^* \|_{L^2}^{\frac{18}{5}} + C_{11} (1 + t)^{-k_2},
\end{align*}
\]

where \( k_1 > 0 \) and \( k_2 > 7/2 \) are defined as in Prop. 2.7 and \( C_{10}, C_{11}, C_{12} > 0 \) only depend on \( K \) such that \( \| f_0 \|_{L^\infty} + \| f_0 \|_{L \log L} \leq K \).

This differential inequality implies that

\[
\frac{d}{dt} \| \nabla h^* \|_{L^2}^2 \leq \| \nabla h^* \|_{L^2}^{\frac{18}{5}} + C_{11},
\]

so that

\[
\frac{d}{dt} \left( \| \nabla h^* \|_{L^2}^2 + C_{11} \right) \leq \left( \| \nabla h^* \|_{L^2}^2 + C_{11} \right)^{9/5}.
\]
Therefore, for $t \leq T := \frac{2}{5} \left( \| \nabla h'(0) \|^2_{L^2} + 2C_{11}^{5/9} \right)^{-4/5}$,

\begin{equation}
\| \nabla h'(t) \|^2_{L^2} \leq \left[ \left( \| \nabla h'(0) \|^2_{L^2} + C_{11}^{5/9} \right)^{-4/5} - \frac{4}{5} t \right]^{-5/4} - C_{11}^{5/9}.
\end{equation}

Passing to the limit when $\epsilon \to 0$ as in the end of the Proof of Theorem 1.2, we get the existence of a weak solution of Landau equation (1.1) – (1.3) on the interval $[0, T]$ which is in fact strong in the sense that it lies in $L^\infty([0, T]; H^1(\mathbb{R}^3))$. Note indeed that the first time of blowup (in $H^1$ norm) is strictly bigger than $T$ since part of the dissipative terms were not used in the differential inequality in order to get the bound (3.6).

We now focus on the regularity of the obtained solution, and the consequences concerning the issue of uniqueness. Using Theorem 1.1 we see that on the time interval $[0, T]$, one has $h \in L^\infty_t(L^1_{x})$. Then the estimates (3.6) and (3.5) imply that $\nabla h \in H^1(L^2_t)$, and $\nabla^2 h \in L^2_t(L^2_x)$.

Thanks to a Sobolev embedding, we see that $h \in L^\infty_t(L^6)$. Interpolating with the estimate stating that $h \in L^\infty_t(L^{25}_x)$, we see that $h \in L^\infty_t(L^7_{x})$. Interpolating again this estimate with the statement $\nabla^2 h \in L^2_t(L^2_x)$, we see that $h \in L^6_t(L^4_x)$.

Thanks to yet another Sobolev embedding, we obtain that $h \in L^7_t(L^\infty)$, which is sufficient to apply the stability result in [13], and get the uniqueness of the strong solution built above, on the concerned interval of time.

We finally prove that $f \in C([0, T]; \dot{H}^1)$. Using estimate 3.20, we see that $\nabla^2 h \in L^2([0, T]; L^2_x)$, and $I_{1,1} \in L^1([0, T])$. Recalling identities (2.3), (2.4) and estimate 3.20, we observe that $\frac{d}{dt} \| \nabla h \|^2_{L^2} \in L^1([0, T])$, so that $s \mapsto \| \nabla h(t) \|^2_{L^2}$ is continuous on $[0, T]$.

Remembering the weak formulation (1.6) and the fact $\nabla h \in H^1_t(L^2)$, it is not difficult to check that $t \mapsto \int_{\mathbb{R}^3} \partial_t h(t, \nu) \phi(\nu) d\nu$ is continuous on $[0, T]$, for any smooth and compactly supported function $\phi$. We can conclude that $h \in C([0, T]; \dot{H}^1)$ by patching together the above facts. Indeed, thanks to the continuity of $t \mapsto \| \nabla h(t) \|^2_{L^2}$, we know that

$$\lim_{s \to t} \| \nabla (h(t) - h(s)) \|^2_{L^2} = 2 \| \nabla h(t) \|^2_{L^2} - 2 \lim_{s \to t} \| \nabla h(s) \|^2_{L^2}.$$ 

We conclude by approximating $\nabla h(t)$ in $L^2$ by a sequence $\phi_n \in C_c^\infty$. Note finally that the formula appearing in the definition of $T$ in Proposition 1.1 is obtained by defining $C_T := \frac{1}{2} C_{11}^{5/9}$.

\section{4. Weighted $H^1$ estimates and proof of Proposition 1.2}

The main goal of this section is to get estimates for weighted $H^1$ norms of solutions to the Landau equation with Coulomb potential (1.1) – (1.3), and then to use them in order to prove Proposition 1.2.

\subsection{4.1. Weighted $\dot{H}^1$ estimate}

Multiplying the equation for the derivatives of the Landau equation with Coulomb potential (1.1) – (1.3), that is (remembering that $h = f - \mu$ and that $\mu$ is the normalized Maxwellian given by (1.5)),

\begin{equation}
\partial_t (\partial_t h) = Q(f, \partial_t h) + Q(\partial_t f, h) + Q(\partial_t h, \mu) + Q(h, \partial_t \mu),
\end{equation}

by $\langle v \rangle^m \partial_t h$, integrating with respect to $v$ and summing for $k = 1, 2, 3$, we obtain (at the formal level)

\begin{equation}
\frac{1}{2} \frac{d}{dt} \| \nabla h \|^2_{L^2_{x/v}} = W_1 + W_2 + W_3 + W_4,
\end{equation}

where $W_1, W_2, W_3$ and $W_4$ correspond to the terms of the right-hand side of (1.1).

We start our study by estimating the most significant terms, that is $W_1$ and $W_2$. 

4.1.1. Estimate for $W_1$ and $W_2$. The following proposition enables to treat a large part of the terms coming out of $W_1$ and $W_2$:

**Proposition 4.1.** Let $f$ be a nonnegative function satisfying the normalization (1.14), and $h = f - \mu$.

Then, the following estimates hold (for all $m \geq 0$ and some (absolute) constant $C > 0$):

\[
\int\int [v - v_* |^{-1} + |v - v_* |^{-2}] f(v_*) |\nabla h(v)|^2 \langle v \rangle^m dv_* dv \leq C (1 + \|\nabla h\|_{L^2}) \|\nabla h\|^2_{L^2_{m/2}},
\]

and

\[
\int\int |v - v_* |^{-2} |\nabla f(v_*)| |h(v)| |\nabla h(v)| \langle v \rangle^m dv_* dv \leq C (1 + \|\nabla h\|_{L^2}) \|\nabla h\|_{L^2_{m/2}}.
\]

**Proof.** For estimate (4.3), we bound the integral over $|v - v_* | \leq 1$ in the following way:

\[
\int\int_{|v - v_* | \leq 1} \left[ |v - v_* |^{-1} + |v - v_* |^{-2} \right] f(v_*) |\nabla h(v)|^2 \langle v \rangle^m dv_* dv \leq \left\| \| (|\cdot|^{-1} + |\cdot|^{-2}) \* f \|_{L^\infty} \right\| \nabla h\|_{L^2_{m/2}}^2
\]  
\[
\leq \left\| \| \cdot \|_{L^1}^{-1} + \| \cdot \|_{L^1}^{-2} \right\|_{L^\infty} \left\| \| h \|_{L^6} \right\| \|\nabla h\|_{L^2_{m/2}}^2
\]  
\[
\leq \left\| \| \nabla f \|_{L^2} \right\| \|\nabla h\|_{L^2_{m/2}}^2
\]  
\[
\leq C (1 + \|\nabla h\|_{L^2}) \|\nabla h\|_{L^2_{m/2}}^2.
\]

The integral over $|v - v_* | \geq 1$ satisfies

\[
\int\int_{|v - v_* | \geq 1} \left[ |v - v_* |^{-1} + |v - v_* |^{-2} \right] f(v_*) |\nabla h(v)|^2 \langle v \rangle^m dv_* dv \leq C \| f \|_{L^1} \|\nabla h\|^2_{L^2_{m/2}}.
\]

Then, estimate (4.3) is a consequence of the bounds (4.5) and (4.6).

For estimate (4.4), using $\langle v \rangle \leq \sqrt{3} \langle v \rangle$ when $|v - v_* | \leq 1$, we see that the integral over $|v - v_* | \leq 1$ is bounded by

\[
\int\int_{|v - v_* | \leq 1} |v - v_* |^{-2} |\nabla f(v_*)| |h(v)| |\nabla h(v)| \langle v \rangle^m dv_* dv
\]  
\[
\leq \left\| \| (|\cdot|^{-1} + |\cdot|^{-2}) \* f \|_{L^\infty} \right\| \| h \|_{L^6} \|\nabla h\|_{L^2_{m/2}}^2
\]  
\[
\leq \left\| \| \nabla f \|_{L^2} \right\| \|\nabla h\|_{L^2_{m/2}}^2 \|\nabla h\|_{L^2_{m/2}}^2
\]  
\[
\leq C (1 + \|\nabla h\|_{L^2}) \|\nabla h\|_{L^2_{m/2}}^2 \|\nabla h\|_{L^2_{m/2}}^2.
\]

Since $|\cdot|_{L^2_{m/2}}^{-2}$ lies in $L^2$, the integral over $|v - v_* | \geq 1$ is bounded in the following way:

\[
\int\int_{|v - v_* | \geq 1} |v - v_* |^{-2} |\nabla f(v_*)| |h(v)| |\nabla h(v)| \langle v \rangle^m dv_* dv
\]  
\[
\leq C \| \nabla f \|_{L^2} \|\nabla h\|_{L^2_{m/2}} \|\nabla h\|_{L^2_{m/2}}^2
\]  
\[
\leq C \|\nabla h\|_{L^2_{m/2}} \|\nabla h\|_{L^2_{m/2}}^2 + C \|\nabla h\|_{L^2} \|\nabla h\|_{L^2_{m/2}} \|\nabla h\|_{L^2_{m/2}}^2.
\]

We get estimate (4.4) by collecting the bounds (4.7) and (4.8).

Next we estimate the terms $W_1$ and $W_2$. We start with the...
Proposition 4.2. Let \( f \geq 0 \) be such that \( \int_{\mathbb{R}^3} f(v) \, dv = 1, \int_{\mathbb{R}^3} f(v) \, |v|^2 \, dv = 3 \), and such that \( \|f\|_{L^{1/2}} + \|f\|_{L^{\log L}} \leq K \), for some \( K > 0 \). We denote \( h = f - \mu \). Then for all \( m \geq 0 \) and some constants \( C'(K), C(K) > 0 \) depending only on \( K \):

\[
(4.9) \quad W_1 := \langle Q(f, \partial_h h), \langle \cdot \rangle^m \partial_h h \rangle \leq -\frac{7}{8} C(K) \|\nabla^2 h\|^2_{L^2_{m/2 - 3/2}} + C^*(K) \left( 1 + \|\nabla h\|^2_{L^2_{m/2}} \right) \|\nabla h\|^2_{L^2_{m/2}}.
\]

Proof. Using an integration by parts, we see that

\[
W_1 = \langle Q(f, \partial_h h), \langle \cdot \rangle^m \partial_h h \rangle = -\sum_{k,i,j} \int (a_{ij} \ast f)(\partial_j \partial_h h)[\partial_i(\langle \cdot \rangle^m \partial_h h)] \, dv + \left( \sum_k \int (b_k \ast f)(\partial_h h)[\partial_i(\langle \cdot \rangle^m \partial_h h)] \, dv \right)
\]

\[
= -\sum_k \int (a \ast f) : (\nabla \partial_h h) \otimes (\nabla \partial_h h) \langle \cdot \rangle^m \, dv - \frac{1}{2} \sum_k \int (b_k \ast f) [\partial_h h]^2 \partial_i \langle \cdot \rangle^m \, dv
\]

\[
- \frac{1}{2} \sum_{i,j,k} (a_{ij} \ast f) [\partial_h h]^2 \partial_j \partial_i \langle \cdot \rangle^m \, dv + \left( \frac{1}{2} \sum_k \int (b_k \ast f) [\partial_h h]^2 \partial_i \langle \cdot \rangle^m \, dv \right)
\]

\[
+ 4\pi \sum_k \int f [\partial_h h]^2 \langle \cdot \rangle^m \, dv.
\]

Thanks to Corollary 2.1, the first term of the expression above satisfies

\[
\sum_k \int (a \ast f) : (\nabla \partial_h h) \otimes (\nabla \partial_h h) \langle \cdot \rangle^m \, dv \geq C(K) \|\nabla^2 h\|^2_{L^2_{m/2 - 3/2}}.
\]

Then, thanks to Hölder’s inequality and Sobolev embedding \((\dot{H}^1 \subset L^6)\),

\[
\left| \sum_k \int f [\partial_h h]^2 \langle \cdot \rangle^m \, dv \right| \leq C^* \|f\|^{1/5}_{L^{15/2}} \|f\|^{4/5}_{L^4} \|\nabla h\|_{L^2} \|\nabla^2 h\|_{L^2_{m/2}} \|\langle \nabla h\rangle \langle \cdot \rangle^{m/2 - 3/2} \|_{L^2}
\]

\[
\leq C^* \|f\|^{1/5}_{L^{15/2}} \|f\|^{4/5}_{L^4} \|\nabla h\|_{L^2} \|\nabla^2 h\|_{L^2_{m/2}} \|\nabla h\|_{L^{2/5}_{m/2}} + \|\nabla h\|_{L^2_{m/2}}
\]

\[
\leq C^* \left( \frac{K}{8} \|\nabla^2 h\|_{L^2_{m/2}}^2 + C^*(K) \left( 1 + \|\nabla h\|_{L^2} \right)^{8/5} \|\nabla h\|_{L^2_{m/2}}^2 + C^*(K) \left( 1 + \|\nabla h\|_{L^2} \right)^{4/5} \|\nabla h\|_{L^2_{m/2}}^2 \right).
\]

Using the estimates above and Proposition 4.1, eq. (4.3), we end up with estimate (4.9). \(\square\)

We now turn to the

Proposition 4.3. Let \( f \geq 0 \) be such that \( \int_{\mathbb{R}^3} f(v) \, dv = 1, \int_{\mathbb{R}^3} f(v) \, |v|^2 \, dv = 3 \), and such that \( \|f\|_{L^{1/2}} + \|f\|_{L^{\log L}} \leq K \) for some \( K > 0 \). We denote \( h = f - \mu \). Let \( C(K) \) be the same constant as in Proposition 4.2 and \( 0 \leq m \leq 76 \). Then there exists some constant \( C^*(K) \) depending only on \( K \) such that:

\[
W_2 := \langle Q(\partial_h h), \langle \cdot \rangle^m \partial_h h \rangle \leq \frac{3C(K)}{8} \|\nabla^2 h\|_{L^2_{m/2 - 3/2}}^2
\]

\[
+ C^*(K) \left( 1 + \|\nabla h\|_{L^2} \right)^{2/5} \|\nabla h\|_{L^2_{m/2}}.
\]

(4.10)
Proof. Using an integration by parts, we see that

\[ W_2 = \langle Q(\partial_k f, h), (v)^m \partial_k h \rangle 
\]

\[ = \sum_{k,i,j} \int - (\partial_k a_{i,j} \ast f)(\partial_j h) \left[ (\partial_i \partial_k h) (\cdot)^m + (\partial_k h) \partial_i (\cdot)^m \right] dv 
\]

\[ + \sum_{k,i} \int (b_i \ast \partial_k f)(h) \left[ (\partial_i (\cdot)^m) \partial_k h + (\partial_i \partial_k h) (\cdot)^m \right] dv. \]

Using first Proposition 4.1, eq. (4.3), we obtain the estimate

\[ \left| \sum_{k,i,j} \int (\partial_k a_{i,j} \ast f)(\partial_j h)(\partial_k h) (\cdot)^m dv \right| \leq C(1 + \| \nabla h \|_{L^2}) \| \nabla h \|_{L^m/2}^2. \]

Also, still treating separately \( |v - v_*| \leq 1 \) and \( |v - v_*| > 1 \), and observing that \( | \cdot |_{1 < 1} \in L^{4/3} \), we compute

\[ \left| \sum_{k,i,j} \int (\partial_k a_{i,j} \ast f)(\partial_j h)(\partial_k h) (\cdot)^m dv \right| 
\]

\[ \leq C^* \left( \| f \|_{L^4_{m/2}} \| \nabla h \|_{L^2_{m/2}} \| \nabla^2 h \|_{L^2_{m/2-3/2}} + \| f \|_{L^{9/10}_{m/2}} \| \nabla h \|_{L^2_{m/2-1/2}} \| \nabla^2 h \|_{L^2_{m/2-3/2}} \right) 
\]

\[ \leq C^* \left( \| f \|_{L^1_{m/2}} \| f \|_{L^{9/10}_{m/2}} \| \nabla h \|_{L^2_{m/2}} + \| \nabla h \|_{L^2_{m/2-1/2}} \right) \| \nabla^2 h \|_{L^2_{m/2-3/2}} 
\]

\[ \leq C(K) \frac{\| \nabla^2 h \|_{L^2_{m/2-3/2}}^2}{8} + C^* (K) (1 + \| \nabla h \|_{L^2})^{9/5} \| \nabla h \|_{L^2_{m/2}}^{3/2}. \]

Then, thanks to Proposition 4.1 again,

\[ \left| \sum_{k,i} \int (b_i \ast \partial_k f)(h)(\partial_i (\cdot)^m) \partial_k h dv \right| \leq C^* \left( 1 + \| \nabla h \|_{L^2} \right) \| h \|_{H^{1/2}_{m/2}} \| \nabla h \|_{L^2_{m/2}}. \]

Finally, we estimate \( \left| \sum_{k,i} \int (b_i \ast \partial_k f)(h)(\partial_i (\cdot)^m) \partial_k h dv \right| \). Recall that \( f = h + \mu \). Thanks to an integration by parts, we have

\[ \sum_{k,i} \int (b_i \ast \partial_k \mu)(h)(\partial_i (\cdot)^m) \partial_k h dv 
\]

\[ = - \sum_{k,i} \int (b_i \ast \partial_k \mu)(\partial_k h) (\partial_i (h(\cdot)^m)) dv + 8\pi \sum_{k} \int (\partial_k \mu)(\partial_k h) (h(\cdot)^m) dv. \]

Therefore, we deduce that

\[ \left| \sum_{k,i} \int (b_i \ast \partial_k \mu)(h)(\partial_i (\cdot)^m) \partial_k h dv \right| \leq C^* \| h \|_{H^{1/2}_{m/2}} \| \nabla h \|_{L^2_{m/2}}. \]
We now turn to the term $| \sum_{k,i} \int (b_i * \partial_k h)(\partial_t \partial_k h) \langle \cdot \rangle^m dv |$. The integral over $|v - v_*| \leq 1$ is bounded by

$$C^* \int_{|v - v_*| \leq 1} |v - v_*|^{-2} |\nabla h(v_*)| |h(v)| |\nabla^2 h(v)| \langle \cdot \rangle^m dv \, dv \, dv$$

$$\leq C^* ||| \cdot |||_1^2 + |\langle \cdot \rangle^m \nabla h |||_{L^4} ||| h |||_{L^4} |\nabla^2 h| \langle \cdot \rangle^m |||_{L^4}$$

$$\leq C^* ||\nabla h||_{L^2_{m/2}} ||| \cdot |||_1^2 + |\langle \cdot \rangle^m \nabla h |||_{L^4} ||| h |||_{L^4} |\nabla^2 h| \langle \cdot \rangle^m |||_{L^4}$$

$$\leq C^* ||\nabla h||_{L^2_{m/2}} ||| \cdot |||_1^2 + |\langle \cdot \rangle^m \nabla h |||_{L^4} ||| h |||_{L^4} |\nabla^2 h| \langle \cdot \rangle^m |||_{L^4}$$

$$\leq \frac{C(K)}{8} \| \nabla^2 h \|_{L^2_{m/2}}^2 + C^*(K)(1 + \| h^2 \|_{L^2_{m/2}^{2/3}}) \| \nabla h \|_{L^2_{m/2}}^2,$$

Notice now that $|\cdot|_{1}^2 \in L^{14/9}$. Then, the integral over $|v - v_*| \geq 1$ is bounded by

$$C^* \int_{|v - v_*| \geq 1} |v - v_*|^{-2} |\nabla h(v_*)| |h(v)| |\nabla^2 h(v)| \langle \cdot \rangle^m dv \, dv \, dv$$

$$\leq \frac{C^*}{8} \| \nabla^2 h \|_{L^2_{m/2}}^2 + C^*(K)(1 + \| h^2 \|_{L^2_{m/2}^{2/3}}) \| \nabla h \|_{L^2_{m/2}}^2,$$

since $m/2 + 105/16 < 45$ when $m \leq 76$.

Finally, we get estimate (4.10) by regrouping all the estimates above.

4.1.2. Estimates for $W_3$ and $W_4$. We now estimate jointly the terms $W_3$ and $W_4$.

**Proposition 4.4.** Let $f \geq 0$ be such that $\int_{\mathbb{R}^3} f(v) \, dv = 1$, $\int_{\mathbb{R}^3} f(v) \, |v|^2 \, dv = 3$. Then for all $m \geq 2$, and some (absolute) constant $C > 0$:

$$W_3 + W_4 := \langle Q(\partial_t h, \mu), \langle \cdot \rangle^m \partial_k h \rangle + \langle Q(h, \partial_k \mu), \langle \cdot \rangle^m \partial_k h \rangle \leq C \| \nabla h \|_{L^2} \| h \|_{H^1_{m/2}}.$$

**Proof.** Using integrations by parts, we compute

$$W_3 = \langle Q(\partial_k h, \mu), \langle \cdot \rangle^m \partial_k h \rangle$$

$$= \sum_{k,j} \int (b_j * \partial_k h)(\partial_t \partial_k h) \langle \cdot \rangle^m \partial_k h \, dv + \sum_{k,i,j} \int (\partial_k a_{ij} * h)(\partial_t \partial_k h) \langle \cdot \rangle^m \partial_k h \, dv$$

$$+ \sum_k 8\pi \int |\partial_k h|^2 \langle \cdot \rangle^m \, dv - \sum_{k,i} \int (b_i * \partial_k h)(\partial_t \partial_k h) \langle \cdot \rangle^m \, dv,$$

and

$$W_4 = \langle Q(h, \partial_k \mu), \langle \cdot \rangle^m \partial_k h \rangle$$

$$= \sum_{k,j} \int (b_j * h)(\partial_j \partial_k \mu)(\partial_k h) \langle \cdot \rangle^m \, dv + \sum_{k,i,j} \int (a_{ij} * h)(\partial_i \partial_j \partial_k \mu)(\partial_k h) \langle \cdot \rangle^m \, dv$$

$$- \sum_{k,i} \left( \int (b_i * \partial_k h)(\partial_i \partial_k \mu)(\partial_k h) \langle \cdot \rangle^m \, dv + \int (b_i * h)(\partial_i \partial_k \mu)(\partial_k h) \langle \cdot \rangle^m \, dv \right).$$
As a consequence, using the elementary inequality \(<v_\ast>^2 \ 1_{|v_\ast|\leq 1} \leq |v_\ast|^2 e^{v_\ast^2/4} 1_{|v_\ast|\leq 1},\)
\[
W_3 + W_4 \leq C \left[ \int |b| \cdot |\nabla h| \cdot |\nabla h| \mu^{1/2} + \int |b| \cdot |h| \cdot |\nabla h| \mu^{1/2} + \int |\nabla h|^2 \mu^{1/2} \right] \\
\leq C \left[ \|b\|_{L^1} \cdot \|h\|_{L^2} \cdot \|\nabla h\|_{L^2} \right] \\
+ C \int_{v_\ast \geq 1} (v_\ast)^{-1} \left[ |h(v_\ast)| + |\nabla h(v_\ast)| \right] |\nabla h(v)| \mu^{1/2} dv \\
\leq C \|\nabla h\|^2_{L^2} + C \|h\|_{L^2} \cdot \|\nabla h\|_{L^2} + C \|\nabla h\|^2_{L^2} (\|h\|_{L^1} + \|\nabla h\|_{L^2}) \\
\leq C \|\nabla h\|^2_{L^2} (\|h\|_{L^2}^2 + \|\nabla h\|_{L^2}^2) \leq C \|\nabla h\|_{L^2} \cdot \|h\|_{H^1_{L^2}}^2,
\]
remembering that \(m \geq 2.\)

4.2. \(L^2\) estimate.

**Proposition 4.5.** Let \(f \geq 0\) be such that \(\int_{\mathbb{R}^3} f(v) \, dv = 1, \int_{\mathbb{R}^3} f(v) |v|^2 \, dv = 3,\) and such that \(\|f\|_{L^1_{\mathbb{R}^3}} + \|f\|_{L^2_{\mathbb{R}^3}} \leq K\) for some \(K > 0.\) We denote \(h = f - \mu.\) Let \(C(K)\) be the same constant as in Proposition 4.3. Then for all \(m \geq 4,\) and some constant \(C^*(K)\) depending only on \(K:\)
\[
(4.12) \quad \langle Q(f, h), (\cdot)^m h \rangle + \langle Q(h, \mu), (\cdot)^m h \rangle \\
\leq -C(K) \|\nabla h\|_{L^2}^2 (\|h\|_{L^2}^2 + \|\nabla h\|_{L^2}^2).
\]

**Proof.** Using integration by parts, we obtain the decomposition:
\[
\langle Q(f, h), (\cdot)^m h \rangle = -\int (a \ast f) : (\nabla h) \otimes (\nabla h) (\cdot)^m \, dv \\
- \sum_{i,j} \int (a_{ij} \ast f)(\partial_j h) h (\partial_i (\cdot)^m) \, dv + \sum_i \int (b_i \ast f)(h)(\partial_i (\cdot)^m h) \, dv \\
=: -E_1 - E_2 - E_3.
\]
Using Proposition 2.1 (and keeping in mind the arguments used in the proof of Corollary 2.1), we see that
\[
(4.13) \quad E_1 \geq C(K) \|\nabla h\|_{L^2}^2.
\]
Using the same computations as in the proof of Proposition 4.3, we see that
\[
(4.14) \quad |E_2| \leq C \|\nabla f\|_{L^2} \|h\|_{L^2} \|\nabla h\|_{L^2} + C \|f\|_{L^1} \|h\|_{L^2} \|\nabla h\|_{L^2} \\
\leq C (1 + \|\nabla h\|_{L^2}) \|h\|_{L^2} \|\nabla h\|_{L^2}.
\]
Similarly
\[
(4.15) \quad |E_3| \leq C (1 + \|\nabla h\|_{L^2}) (\|h\|_{L^2} + \|\nabla h\|_{L^2}^2).
\]
Using again an integration by parts, we see also that
\[
(4.16) \quad |\langle Q(h, \mu), (\cdot)^m h \rangle| = \left| \sum_i \int \left( \sum_j -(a_{ij} \ast h)(\partial_j \mu) [\partial_i (\cdot)^m h] + (b_i \ast h)(\mu) [\partial_i (\cdot)^m h] \right) \right| dv \\
\leq C \int (|a| + |b|) \cdot |h| (|h| + |\nabla h|) \mu^{1/2} dv \\
\leq C \|[(|a| + |b|) 1_{|\cdot| \leq 1}] \|h\|_{L^2} (\|h\|_{L^2} + \|\nabla h\|_{L^2}) \\
+ C \int_{v_\ast \geq 1} (v_\ast)^{-1} |h(v_\ast)| \left[ |h(v)| + |\nabla h(v)| \right] \mu^{1/4} dv dv_\ast.
Using Proposition 4.5 and computation (4.18), we get the differential inequality
\[ l \leq C \left( \|h\|_{L^2} + \|\nabla h\|_{L^2} \right) \|h\|_{L^2}, \]
where \(|v - v_*|^{-1}|v - v_*|_t \leq \langle v_*^{-1}(v) \rangle \) is used.

Collecting all terms and remembering that \( m \geq 4 \), we conclude the proof of Proposition 4.5\( \Box \)

4.3. End of the proof of Proposition 1.2

For the end of the proof, we perform the computations for a smooth \( C^2(S) \) solution \( f \geq 0 \) of Landau equation with Coulomb potential (1.1) – (1.3). We should in fact repeat here the process of approximation presented in the proofs of Theorem 1.2 and Proposition 1.1. We do not write it for the sake of readability, since no new argument is used to deal with the approximation process.

We first observe that thanks to the assumptions of Theorem 1.2 and Lemma 2.1, there exists a constant \( K > 0 \) such that
\[ \sup_{t > 0} (\|f(t)\|_{L^4_{55}} + \|f(t)\|_{L^2_{\log L}}) \leq K. \]

Then we compute (for \( 4 \leq m \leq 76 \)) the quantity \( \frac{1}{2} \frac{d}{dt} \|\nabla h\|_{L^2_{m/2}}^2 \). By using the computations (4.1), (4.2) and Proposition 4.3 Proposition 4.3 Proposition 4.4 Proposition 4.4, we end up with the estimate
\[ \frac{d}{dt} \|\nabla h\|_{L^2_{m/2}}^2 + \frac{C(K)}{2} \|\nabla^2 h\|_{L^2_{m/2 - 3/2}}^2 \leq C^*(K) (1 + \|\nabla h\|_{L^2}) (\|\nabla h\|_{L^2_{m/2}}^2 + \|h\|_{L^2_{m/2}}^2). \]

Then, multiplying eq. (4.17) by \( \langle \cdot \rangle^m \), and integrating with respect to \( v \), we compute
\[ \frac{d}{dt} \|h\|_{H^1_{m/2}}^2 \leq \langle Q(h, \mu), \langle \cdot \rangle^m h \rangle. \]

Using Proposition 4.5 and computation (4.18), we get the differential inequality
\[ \frac{d}{dt} \|h\|_{H^1_{m/2}}^2 + C(K) \|\nabla h\|_{L^2_{m/2 - 3/2}}^2 \leq C^*(K) (1 + \|\nabla h\|_{L^2}) (\|\nabla h\|_{L^2_{m/2}}^2 + \|h\|_{L^2_{m/2}}^2). \]

Patch together inequalities (4.19) and (4.17), we finally obtain the differential inequality
\[ \frac{d}{dt} \|h\|_{H^1_{m/2}}^2 + \frac{C(K)}{2} \|\nabla h\|_{H^1_{m/2 - 3/2}}^2 \leq C^*(K) (1 + \|\nabla h\|_{L^2}) \|h\|_{H^1_{m/2}}^2. \]

We emphasize that from the proof of Proposition 1.2 Proposition 3.3 Proposition 4.4 and Proposition 4.5, the constants \( C^*(K) \), \( C(K) > 0 \) in the above inequality only depend on \( K \) such that \( \|f_0\|_{L^5_{55}} + \|f_0\|_{L^2_{\log L}} \leq K. \)

Thanks to Proposition 6.4, we know that, for some \( C, C_3, C_4 > 0 \), and some \( k_3 > 2/5 \) (we take \( l = 55 \), \( \theta = 15/4 + 7 \) and \( q_{1, \theta} \sim -3.79 \) with the notations of Lemma 2.1, then \( k_3 > 3 \),
\[ \frac{C(K)}{2} \|\nabla h\|_{H^1_{m/2}}^2 \geq C_3 \|h\|_{H^1_{m/2}}^2 |h|_{L^{15/4 + 7}}^2 - C \|\nabla h\|_{L^2_{m/2}}^2 \geq C_4 (1 + t)^{k_3} \|h\|_{H^1_{m/2}}^2 - C \|h\|_{L^2_{m/2}}^2. \]

In the inequality above and in the rest of the proof, we do not make explicit the (existing) dependence of \( C, C_3, C_4 > 0 \), and \( k_3 > 2/5 \) with respect to \( K. \)

Denoting \( Y^2(t) := \|h(t)\|_{H^1_{m/2}}^2 \), we therefore get the differential inequality (for some \( C_5 > 0 \) only depending on \( K)\):
\[ \frac{d}{dt} Y^2(t) + C_4 (1 + t)^{k_3} Y(t) \leq C_5 (Y^4(t) + Y^2(t)). \]

Remembering that \( \|h_0\|_{L^5_{55}} \) is bounded and that the initial condition is supposed to satisfy \( \|h_0(\cdot)\|_{H^1} \leq \epsilon_0 < 1 \), we see that by interpolation, the differential inequality (4.21) is complemented with the initial datum \( Y^2(0) = \tilde{\epsilon} \ll 1 \) (note that here and in the sequel, the way in which \( \tilde{\epsilon} \) is small depends in fact (only) on \( K \)).
We now consider \( T^* := \sup\{ t > 0 \mid Y^4(t) \leq Y^2(t) \} = \sup\{ t > 0 \mid Y(t) \leq 1 \} \). For \( t \in [0, T^*] \), the differential inequality
\[
\frac{d}{dt} Y^2(t) \leq 2 C_5 Y^2(t)
\]
holds. It implies that \( \forall t \in [0, T_1] := (2 C_5)^{-1} |\log(\frac{1}{2} |\log \bar{\epsilon}|^{-1})| \), the inequality \( Y^2(t) \leq \frac{1}{2} |\log \bar{\epsilon}|^{-1} \) also holds. Thus, \( T^* \geq T_1 \).

We now use a contradiction argument in order to show that solutions of inequality (4.21) globally exist. If the set \( \{ t > 0 \mid Y^2(t) = |\log \bar{\epsilon}|^{-1} \} \) is empty, then this is automatically true. If it is not the case, we define \( T^{**} := \inf\{ t > 0 \mid Y^2(t) = |\log \bar{\epsilon}|^{-1} \} \). Then there exists a time \( T_2 := \sup\{ t \leq T^{**} \mid Y^2(t) = \frac{1}{2} |\log \bar{\epsilon}|^{-1} \} \). Because of the definition of \( T_1 \) and \( T_2 \), we see that \( T^* > T^{**} > T_2 \geq T_1 \), and \( Y^2(t) |\log \bar{\epsilon}| \in [1/2, 1] \) when \( t \in [T_2, T^{**}] \). In particular, in the interval \([T_2, T^{**}]\), we have
\[
\frac{d}{dt} Y^2(t) + C_4(1 + T_1)^{k_3} Y(t)^{\frac{4}{3}} Y^2(t) \leq 2 C_5 Y^2(t),
\]
where
\[
(1 + |\log \bar{\epsilon}|)^{k_3} |\log \bar{\epsilon}|^{-\frac{4}{3}} = O_{\bar{\epsilon} \to 0} \left[ C_4(1 + T_1)^{k_3} Y(t)^{\frac{4}{3}} \right].
\]
It implies that if \( k_3 > \frac{2}{5} \) and \( \bar{\epsilon} > 0 \) is sufficiently small (depending on \( K \) again), \( C_4(1 + T_1)^{k_3} Y(t)^{\frac{4}{3}} \geq 2 C_5 \). Then \( Y^2 \) is decreasing on the interval \([T_2, T^{**}]\), so that \( Y^2(T^{**}) \leq Y^2(T_2) = \frac{1}{2} |\log \bar{\epsilon}|^{-1} \). This is not compatible with the definition of \( T^{**} \), which entails that the set \( \{ t > 0 \mid Y^2(t) = |\log \bar{\epsilon}|^{-1} \} \) is empty. As a consequence, we get the global existence for solutions of (4.21), and those solutions moreover satisfy the bound \( \sup_{t \geq 0} Y^2(t) \leq |\log \bar{\epsilon}|^{-1} \).

They satisfy therefore the following modified differential inequality
\[
\frac{d}{dt} Y^2(t) + C_4(1 + t)^{k_3} Y(t)^{\frac{4}{3}} \leq 2 C_5 Y^2(t).
\]
Splitting the interval \([0, \infty)\) into the two sets \( \{ t > 0 \mid C_4(1 + t)^{k_3} Y(t)^{\frac{4}{3}} \leq 4 C_5 Y^2(t) \} \) and \( \{ t > 0 \mid C_4(1 + t)^{k_3} Y(t)^{\frac{4}{3}} > 4 C_5 Y^2(t) \} \), we conclude that for some constant \( C > 0 \) (remembering that \( k_3 > 3 \))
\[
Y(t) \leq C (1 + t)^{-\frac{4}{3} k_3} \leq C (1 + t)^{-\frac{15}{4}}.
\]

We recall that the estimates obtained in this subsection hold for a smooth solution of the Landau equation (1.1) - (1.3), and that, as in Proposition 2.8, they also hold uniformly w.r.t. \( \epsilon \in [0, 1] \) for smooth solutions of the approximated equation (2.42) - (2.43), with suitably mollified initial datum (we recall that such solutions are known to exist and be unique). It is then possible to pass to the (weak weighted \( L^1 \)) limit in the final estimate
\[
\|h'\|_{H^1_2} \leq C (1 + t)^{-\frac{4}{3} k_3} \leq C (1 + t)^{-\frac{15}{4}},
\]
and get the existence of the strong global nonnegative solution to Landau equation (1.1) - (1.3), announced in the Proposition 1.1. The uniqueness is obtained thanks to a variant of the arguments used in the proof of Theorem 1.2 and Proposition 1.1.
5. Investigation of a potential blowup

Here, we prove Proposition 1.3 which provides estimates describing the potential blowup (in $H^1$) of solutions to eq. (1.1) – (1.3).

We present first the following (abstract) Lemma:

**Lemma 5.1.** Let $\bar{T} > 0$, $X, H$ be $C^1$ functions from $[0, \bar{T}]$ to $\mathbb{R}_+$, $C_1, C_3, k_1 > 0$, $k_2 > 7/2$, and $D := -H'$. We suppose that $X$ is solution to the following ordinary differential inequality for all $\eta > 0$ small enough:

\[
(5.1) \quad \frac{d}{dt} X^2(t) + C_1(1 + t)^{k_1} X(t)^{\frac{5}{2}} \leq \eta C_3 D(t) X(t)^{\frac{5}{2}} + B(\eta)(1 + t)^{-k_2},
\]

and that $\lim_{t \to \bar{T}} X(t) = +\infty$. In the estimate above, $B$ is a continuous decreasing nonnegative function.

Then the following quantitative estimates hold for some $c, C > 0$ (depending on $C_1, C_3, k_1, k_2$ and $B$) and $k := \min\left(\frac{2k_2-7}{2}, k_1\right)$, when $\bar{T} - t > 0$ is small enough:

\[
(5.2) \quad X(t) \geq C \left( H(t) - \bar{H} \right)^{-\frac{7}{2}} \quad \text{while} \quad H(t) - \bar{H} \geq C \left( \bar{T} - t \right)(1 + \bar{\bar{T}})^k,
\]

\[
(5.3) \quad \inf_{s \in [t, \bar{T}]} X(s) \leq \left( B(c [\bar{T} - t]) \frac{2}{C_1} (1 + \bar{T})^{- (k_1 + k_2)} \right)^{\frac{5}{7}}.
\]

In the estimates above, we used the notation $\bar{H} := \lim_{t \to \bar{T}} H(t)$.

**Proof.** We can first use Lemma 2.2 with $\eta = C_3^{-1}$ (up to choosing $C_3 > 0$ large enough), $B^* := B(\eta)$. Estimate (2.40) implies that for $\delta > 0$ small enough, and some $C_6 > 0$ given by Lemma 2.2

\[
(5.4) \quad H(\bar{T} - \delta) + C_6 \int_{t}^{\bar{T} - \delta} (1 + s)^k ds \leq H(t) - \frac{5}{2} [X(t)^2 + B^* (1 + t)^{1-k_2}]^{-\frac{7}{2}},
\]

which is enough to get the first part of estimate (5.2), by letting $\delta \to 0$.

Using again estimate (5.4) and letting $\delta \to 0$, we see that

\[
X(t)^2 + B^* (1 + t)^{1-k_2} \geq \left[ \frac{5}{2} \left( H(t) - \bar{H} \right) \right]^{-5/2}.
\]

By definition, $\lim_{t \to \bar{T}} H(t) = \bar{H}$ so that $(1 + t)^{1-k_2} = o_{t \to \bar{T}} (H(t) - \bar{H})^{-5/2}$, and we get therefore the second part of estimate (5.2).

In order to prove estimate (5.3), we go back to assumption (5.1). Dividing it by $X^{-\frac{5}{2}}$, we get

\[
-\frac{5}{2} \frac{d}{dt} X(t)^{-\frac{7}{2}} + C_1(1 + t)^{k_1} \leq \eta C_3 D(t) X(t)^{-\frac{5}{2}} + B(\eta)(1 + t)^{-k_2} X(t)^{-\frac{5}{2}},
\]

which gives

\[
\frac{5}{2} X(t)^{-\frac{7}{2}} - \frac{5}{2} X(\bar{T} - \delta)^{-\frac{7}{2}} + C_1 \int_{t}^{\bar{T} - \delta} (1 + s)^k ds
\]

\[
\leq \eta C_3 H(0) + B(\eta) \int_{t}^{\bar{T} - \delta} (1 + s)^{-k_2} X(s)^{-\frac{5}{2}} ds.
\]

From this, we deduce (remember that $\lim_{t \to \bar{T}} X(t) = +\infty$) that

\[
\sup_{s \in [t, \bar{T}]} \left( X^{-\frac{5}{2}}(s) \right)(1 + t)^{-k_2}(\bar{T} - t) B(\eta) \geq C_1(1 + t)^{k_1}(\bar{T} - t) - \eta C_3 H(0).
\]
Let \( \eta := c (\bar{T} - t) \), for \( c > 0 \) chosen small enough. Then for \( \bar{T} - t \) small enough, we get
\[
\sup_{s \in [t, \bar{T}]} X^{-1} (s) \geq \left( (B(c |\bar{T}-t|)) \frac{-C_1}{2} (1 + \bar{T})^{k_1 + k_2} \right)^{\frac{1}{k_2}},
\]
which yields estimate (5.3).

\[ \square \]

**Remark 5.1.** The entropy \( H \) plays an important role in the estimates giving hints about the way that a possible blowup could occur for eq. (1.1). We observe that this quantity is continuous (with respect to time, on \([0, \bar{T}]\)) under our assumptions (namely when \( f := f(t, v) \) is a nonnegative solution to eq. (1.1) lying in \( C([0, \bar{T}); H^1)) \cap L_{loc}^\infty([0, \bar{T}); L_2^2) \)). From the inequality
\[
|a \log a - b \log b| \leq C_p |a - b|^{1/p} + |a - b| \log^+ |a - b| + 2 \sqrt{a} \land b \sqrt{|a - b|},
\]
which is proved in Proposition 6.5 (for \( p > 1 \), \( C_p = p/(e(p - 1)) \) and \( a \land b = \min \{a, b\} \), used when \( p := 4/3 \), we see that (for \( 0 \leq t_1, t_2 < \bar{T} \))
\[
|H(t_1) - H(t_2)| \leq C \left( \|f(t_1) - f(t_2)\|_{L_2^2}^{\frac{3}{2}} + \|f(t_1) - f(t_2)\|_{L_2^1}^{\frac{1}{2}} + \|f(t_1) - f(t_2)\|_{L_2^1}^{\frac{3}{2}} + \|f(t_1) + f(t_2)\|_{L_2^2}^{\frac{1}{2}} \right).
\]

Thanks to the interpolation inequalities (based on Hölder’s inequality and Sobolev embeddings),
\[
\|f\|_{L_2^{\frac{3}{2}}} \leq \|f\|_{L_2^{\frac{5}{4}}} \|f\|_{L_2^{\frac{8}{5}}} \leq C \|f\|_{L_2^{\frac{5}{4}}} \|\nabla f\|_{L_2^{\frac{3}{2}}}, \quad \|f\|_{L_2^1} \leq \|f\|_{L_2^{\frac{3}{2}}} \|\nabla f\|_{L_2^{\frac{3}{2}}},
\]
we finally get the estimate (for some \( C \) depending on \( \|f\|_{L_{loc}^\infty(L_2^1)} \) and \( \|f\|_{L_{loc}^\infty(H^1)} \), those norms being taken on \([0, \sup(t_1, t_2)]\))
\[
(H(t_1) - H(t_2)) \leq C \left( \|f(t_1) - f(t_2)\|_{H_1^1}^{\frac{3}{2}} + \|f(t_1) - f(t_2)\|_{H_1^1}^{\frac{1}{2}} + \|f(t_1) - f(t_2)\|_{H_1^1}^{\frac{3}{2}} \right),
\]
which is sufficient to conclude.

We are in a position to prove the result.

**Proof of Proposition 1.3.** We begin with this proof in the case when \( f \) is a smooth and quickly decaying (when \( |v| \to \infty \)) solution to eq. (1.1) on a time interval \([0, \bar{T}]\). Thanks to estimate (2.37) in Proposition 2.7, we see that assumption (5.1) holds with \( B(x) := C_2 x^{-13} \exp \{7 x - \frac{40}{7} \} \). We can then apply Lemma 5.1 to \( X(t) := \|\nabla f(t)\|_{L_2^1} \).

We now briefly explain how to prove Proposition 1.3 without assuming that \( f \) is smooth and quickly decaying (when \( |v| \to \infty \)). We consider an interval of time on which \( f \in L_1^\infty (H^1 \cap L_1^{55}) \). We first observe that thanks to Proposition 1.1, we have \( f \in L_1^2 (H_{-3/2}^2) \). Since \( f \in L_1^\infty (L_{55}) \), we see that thanks to Proposition 6.3, \( f \in L_1^2 (H_{1/2}) \). Using now estimate (4.20) and the uniqueness result, we see then that \( f \in L_1^\infty (H_{1/2}^{1/2}) \cap L_1^2 (H_{1/2}^2) \) for all compact intervals of \([t_0, \bar{T}]\) where \( t_0 > 0 \).

Using the equation satisfied by second order derivatives of \( f \) and computing the time derivative of the square of the \( H^2 \) norm of \( f \), we can use Corollary 2.1 and estimates like in Propositions 4.1 to 4.3 and end up with the bound
\[
\frac{1}{2} \frac{d}{dt} \|f\|_{H_2^2}^2 + C(K) \|\nabla f\|_{H_2^1}^2 \leq C(\|f\|_{H_2^2}^2 + \|f\|_{L_2^1}^4) \|f\|_{H_2^2}^2.
\]

Thanks to the fact \( f \in L_1^2 (H_{1/2}^2) \), we see that \( f \in L_1^\infty (H^2) \cap L_1^2 (H_{-3/2}^2) \) on all compact intervals of \([t_0, \bar{T}]\) where \( t_0 > 0 \).
Using the estimates above for solutions $f^\epsilon$ of the approximated problem (2.42) – (2.43), we get that $f^\epsilon$ is bounded in $L^\infty(H^2)$ on any interval $[t_1, t_2] \subset [0, \bar{T}]$. Using also (5.6), this is sufficient to pass to the limit in the inequality (5.8)

$$M(t_2) + C_6 \int_{t_1}^{t_2} (1 + t)^k dt \leq M(t_1),$$

where $M(t) = H^\epsilon(t) - \frac{\epsilon}{2} \left( \|h^\epsilon(t)\|_{H^1}^2 + B^* (1 + t)^{-k_2 + 1} \right)^{-\frac{\epsilon}{2}}$. We end up with the inequality (5.9) in an "integrated in time" form:

$$M(t_2) + C_6 \int_{t_1}^{t_2} (1 + t)^k dt \leq M(t_1).$$

The same construction can be used to obtain estimates (5.4) and (5.5) and conclude the proof of Proposition 1.3 when $f \in L^\infty(\bar{H}^1 \cap L^1_{55})$ on all compact intervals of $[0, \bar{T}]$.

\[ \square \]

6. Appendix

In this appendix, we present some results which are used in the paper. We start with interpolation results and properties of Lorentz spaces.

6.1. Dyadic decompositions. We start by recalling some aspects of the Littlewood-Paley decomposition. Let $B_{4/3} := \{ x \in \mathbb{R}^3 \mid |x| < 4/3 \}$ and $R_{3/4,8/3} := \{ x \in \mathbb{R}^3 \mid 3/4 < |x| < 8/3 \}$. Then one introduces two radially symmetric functions $\psi \in C_0^\infty(B_{4/3})$ and $\varphi \in C_0^\infty(R_{3/4,8/3})$ which satisfy

$$\psi, \varphi \geq 0, \quad \text{and} \quad \psi(x) + \sum_{j \geq 0} \varphi(2^{-j} x) = 1, \quad x \in \mathbb{R}^3.$$  

The dyadic operator $P_j$ is defined for $j \geq -1$ by

$$P_{-j} f(x) := \psi(x) f(x), \quad P_j f(x) := \varphi(2^{-j} x) f(x), \quad (j \geq 0).$$

We recall that $P_j P_k = 0$ if $|j - k| > N_0$, for some $N_0 \in \mathbb{N}$.

We present a norm based on the dyadic decomposition which is equivalent to the usual norm of the weighted Sobolev spaces $H^s_t$.

**Proposition 6.1.** (21) Let $s, l \in \mathbb{R}$. Then for $f \in H^s_t$, $\sum_{k=\substack{-\infty \leq k \leq \infty \\cdot}} 2^{2k l} \| P_k f \|_{H^s_t}^2 \sim \| f \|_{H^s_t}^2$.

6.2. Definition, norms and quasi-norms of Lorentz spaces. For the convenience of the readers and the sake of self content, we collect some facts about Lorentz spaces from [1, 33] which are useful for us. Considering $\mathbb{R}^n$ with Lebesgue measure $|\cdot|$. In Section 1, we define the norm in Lorentz space $L^{p,q}$, $p \in [1, \infty]$, $q \in [1, \infty]$ (or $p = q = \infty$, using the convention $t^{1/\infty} = 1$, $t \geq 0$)

$$\| f \|_{L^{p,q}} := \left\{ \left( \int_0^\infty \left( t^{1/p} f^{**}(t) \right)^q \frac{dt}{t} \right)^{1/q} \right\}, 1 \leq q < \infty$$

$$\sup_{t > 0} t^{1/p} f^{**}(t), \quad q = \infty,$$

which is different from the following (commonly used) definition

$$\| f \|_{L^{p,q}}^* := \left\{ \left( \int_0^\infty \left( t^{1/p} f^{**}(t) \right)^q \frac{dt}{t} \right)^{1/q} \right\}, 1 \leq q < \infty$$

$$\sup_{t > 0} t^{1/p} f^{**}(t), \quad q = \infty.$$
Here

\[ f^{**}(t) = \frac{1}{t} \int_0^t f^*(s)ds, \quad f^*(s) = \inf\{t \geq 0 : a_f(t) \leq s\}, \]

where \( a_f \) is the distribution function of \( f \) given by

\[ a_f(t) = \{x \in \mathbb{R}^n : |f(x)| > t\}. \]

For \( p \in (1, \infty) \) and \( q \in [1, \infty] \), we note that the functional \( \| \cdot \|_{L^{p,q}}^* \) is a norm only when \( q \leq p \) and a quasi-norm otherwise, on the other hand \( \| \cdot \|_{L^{p,q}} \) is always a norm. For \( p \in (1, \infty) \) and \( q \in [1, \infty] \), the following comparison inequality holds:

\[ \|f\|_{L^{p,q}} \leq \|f\|_{L^{p,q}}^* \leq \frac{p}{p-1} \|f\|_{L^{p,q}}^*. \]

Clearly for \( 1 < p < \infty \), we have \( \|f\|_{L^{p,p}}^* = \|f\|_{L^p} \) and thus \( L^{p,p} = L^p \). For \( p = 1 \) the situation is different (See also [1] p. 224), one can indeed check that

\[ \|f\|_{L^{1,\infty}} = \sup_{t>0} tf^{**}(t) = \sup_{t>0} \int_0^t f^*(s)ds = \int_0^\infty f^*(s)ds = \|f\|_{L^1}. \]

Finally, for \( p = \infty \) (See also [1] p. 224), one can also check that

\[ \|f\|_{L^{\infty,\infty}} = \sup_{t>0} f^{**}(t) = \sup_{t>0} \frac{1}{t} \int_0^t f^*(s)ds = f^*(0) = \|f\|_{L^\infty}. \]

6.3. Inequalities and Interpolation. We begin with Sobolev embedding theorem and O’Neil inequality in Lorentz spaces.

**Proposition 6.2** (see [1] and [34]). (i). If \( f \in H^1(\mathbb{R}^3) \), then \( f \in L^{6,2}(\mathbb{R}^3) \) and

\[ \|f\|_{L^{6,2}(\mathbb{R}^3)} \leq C \|f\|_{H^1(\mathbb{R}^3)}. \]

(ii). For \( p_1, p_2, q_1, q_2 \in [1, \infty] \) with \( 1/p = 1/p_1 + 1/p_2 \) and \( 1/q = 1/q_1 + 1/q_2 \), there exists a computable constant \( C \) depending only on \( p_1, q_1, p_2, q_2 \) such that

\[ \|fg\|_{L^{p,q}} \leq C \|f\|_{L^{p_1,q_1}} \|g\|_{L^{p_2,q_2}}. \]

(iii). If \( f \in L^{p,q}, g \in L^{p',q_2} \) where \( p, p', q_1, q_2 \in [1, \infty] \) such that \( 1/p + 1/p' = 1 \) and \( 1/q_1 + 1/q_2 \geq 1 \), then \( f \ast g \in L^\infty \) and

\[ \|f \ast g\|_{L^\infty} \leq \|f\|_{L^{p,q}} \|g\|_{L^{p',q_2}}. \]

Next we will prove some useful interpolation inequalities which are widely used throughout the paper.

**Proposition 6.3.** For \( m \in \mathbb{R} \), and some constant \( C > 0 \) depending only on \( m \),

\[ \|f\|_{L^{5,1}_m} \leq C \|f\|_{L^{5,1}_{5m+4}} \|f\|_{H^1}^{5}. \]

**Proof.** We split the proof into two parts. The first step is devoted to showing that

\[ \|f\|_{L^{5,1}} \leq \|f\|_{L^{5,1}_0} \|f\|_{L^{5,2}} \leq C \|f\|_{L^1} \|f\|_{H^1}^{5}. \]

By the definition of Lorentz spaces, one gets

\[ \|f\|_{L^{5,1}} = \int_0^\infty t^{\frac{1}{2}} f^{**}(t) \frac{dt}{t} \]

\[ \leq (\int_0^R (t^{\frac{1}{2}} f^{**}(t))^2 \frac{dt}{t})^{\frac{1}{2}} (\int_0^R t^{\frac{1}{2}} \frac{dt}{t})^{\frac{1}{2}} + (\sup_{t>0} tf^{**}(t)) \int_R^\infty t^{-\frac{1}{2}} \frac{dt}{t} \]

\[ \leq \|f\|_{L^{0,2}} R^{\frac{1}{2}} + \|f\|_{L^{1,\infty}} R^{-\frac{1}{2}}. \]
We conclude by optimizing $R$ and by using the identity $\|f\|_{L^1} = \|f\|_{L^1}$ (see subsection 6.2 and p. 224).

In next step, we extend the above result to the general case (the one with weights appearing in the norms) using a dyadic decomposition. We observe that

$$\|f\|_{L^{3,1}_m} \leq \sum_{k=1}^{\infty} \|P_k f\|_{L^{3,1}_m} \leq 2 \sum_{k=1}^{\infty} \|P_k f\|_{L^{3,1}_m} \|P_k f\|_{L^{3,1}_m} \leq C \sum_{k=1}^{\infty} (2^{5km}\|P_k f\|_{L^1})^{\frac{5}{5}} \|P_k f\|_{L^1}^{\frac{3}{5}}$$

where we use O'Neil inequality (6.2) and the fact $\|f\|_{L^\infty} = \|f\|_{L^\infty}$ (see subsection 6.2 and p. 224). From this together with the computation

$$\sum_{k=1}^{\infty} 2^{5km}\|P_k f\|_{L^1}^{\frac{3}{5}} \leq C \sum_{k=1}^{\infty} 2^{5km} 2^{-\frac{1}{2}(5m+1)k} \|f\|_{L^1_{5m+1}} \leq C \|f\|_{L^1_{5m+1}}^\frac{5}{3},$$

we finally get the inequality

$$\|f\|_{L^{3,1}_m} \leq C \|f\|_{L^1_{5m+1}} \|f\|_{L^1_{5m+1}}.$$

\[\square\]

**Proposition 6.4.** For $m \in \mathbb{R}$,

$$\|f\|_{H^1_m} \leq C \|f\|_{L^1_{5/4+7m/2}} \left(\|f\|_{L^1_{-3/2}} + \|\nabla^2 f\|_{L^2_{-3/2}}\right)^{\frac{3}{5}},$$

$$\|f\|_{H^1_m} \leq C \|f\|_{L^1_{5/4+7m/2}} \left(\|f\|_{L^1_{-1/2}} + \|\nabla^2 f\|_{L^2_{-1/2}}\right)^{\frac{3}{5}},$$

where $C > 0$ is a constant depending only on $m$.

**Proof.** We first claim that

$$\|f\|_{H^1} \leq C \|f\|_{L^1} \left(\|f\|_{L^1} + \|\nabla^2 f\|_{L^2}\right)^{\frac{3}{5}}.$$}

Indeed, since $\|f\|_{H^1} \sim \int_{\mathbb{R}} (1 + |\xi|)^2 \hat{f}(|\xi|^2) d\xi$, then (for $R \geq 1$)

$$\|f\|_{H^1} \leq C (R^5 \|f\|_{L^1} + R^{-2} \|f\|_{H^2}).$$

We conclude by taking $R^7 = \|f\|_{H^2} / \|f\|_{L^1} + 1$, recalling that $\|f\|_{H^1} \sim \|f\|_{L^1} + \|\nabla^2 f\|_{L^2}$.

Thanks to Proposition 6.1, we see that

$$\|f\|_{H^1_m}^2 \sim \sum_{k=1}^{\infty} 2^{5km} \|P_k f\|_{L^1}^2 \leq \sum_{k=1}^{\infty} 2^{5km} \|P_k f\|_{L^1}^2 (\|P_k f\|_{H^1} + \|P_k f\|_{L^1})^{\frac{5}{3}}$$

$$\leq C \left(\sum_{k=1}^{\infty} 2^{7m+15/2k} \|P_k f\|_{L^1}^2\right)^{\frac{5}{7}} \left(\sum_{k=1}^{\infty} 2^{-3k} \|P_k f\|_{H^1}^2 + \|P_k f\|_{L^1}^2\right)^{\frac{5}{3}}$$

$$\leq C \|f\|_{L^1_{5/4+7m/2}} \left(\|f\|_{L^1_{-3/2}} + \|\nabla^2 f\|_{L^2_{-3/2}}\right)^{\frac{5}{3}}.$$

The proof of the second inequality is similar. \[\square\]
Proposition 6.5 ([22]). For $a, b \geq 0$ and $1 < p < \infty$, the following inequality holds:
\begin{equation}
|a \log a - b \log b| \leq C_p |a - b|^{1/p} + |a - b| \log^+(|a - b|) + 2\sqrt{a \wedge b} \sqrt{|a - b|},
\end{equation}
where $a \wedge b = \min\{a, b\}$, $C_p := \frac{p}{e(p-1)}$ and
\[
\log^+ |x| = \begin{cases} \log x & \text{if } x \geq 1 \\ 0 & \text{if } x < 1. \end{cases}
\]

Proof. We first observe that the following inequalities hold:
\begin{equation}
\log(1 + x) \leq \sqrt{x}, \quad x \geq 0, \quad \log x \leq \frac{1}{e\alpha} x^{-\alpha}, \quad 0 < x \leq 1, \alpha > 0.
\end{equation}
Then, let $q > 1$ satisfy $1/p + 1/q = 1$. In what follows, we assume that $a > b > 0$.

We first observe that
\[|a \log a - b \log b| \leq (a - b) |\log a| + b \log \left(\frac{a}{b}\right).\]
Using estimate (6.4), we see that
\[b \log \left(\frac{a}{b}\right) = b \log \left(1 + \frac{a - b}{b}\right) \leq b \sqrt{\frac{a - b}{b}} = \sqrt{b} \sqrt{a - b}.\]
Next we compute
\[|\log a| = \left| \log \left( (a - b) \left(1 + \frac{b}{a - b}\right) \right) \right| \leq \frac{q}{e} (a - b)^{-1/q} + \log^+ (a - b) + \sqrt{\frac{b}{a - b}},\]
where in the case when $a - b \leq 1$, we use estimate (6.4) with $\alpha = 1/q$. This gives
\[(a - b) |\log a| \leq \frac{q}{e} (a - b)^{1/p} + (a - b) \log^+ (a - b) + \sqrt{b} \sqrt{a - b},\]
which enables to conclude. \hfill \Box

6.4. A remark on initial data. Finally we show that there exist initial data for Theorem 1.2 whose initial relative entropy $H(0)$ is not big, while their $H^1$ norm is large. See also the last comment of Theorem 1.2 in the introduction.

Proposition 6.6. Let $\epsilon, \eta \ll 1$ and $\eta := \epsilon^{1/9}$. Assume the Maxwellian $\mu$ and a smooth $\phi_0 \geq 0$ both satisfy the normalization (1.14). Then
\begin{equation}
f_0(v) := (1 - \eta + \eta \epsilon^2)^{\frac{3}{2}} \left(1 - \eta \right) \mu \left( (1 - \eta + \eta \epsilon^2)^{\frac{3}{2}} v \right) + \eta \epsilon^{-1} \phi_0 \left( \epsilon^{-1} (1 - \eta + \eta \epsilon^2)^{\frac{3}{2}} v \right)
\end{equation}
also satisfies the normalization (1.14), and
\[\mathcal{M}(0) := H(0) - \frac{5}{2} (\|h(0)\|_{H^1}^2 + B)^{-\frac{3}{2}} \leq 0,\]
while $\|h_0\|_{H^\frac{3}{2}} \sim c^{-\frac{3}{2}}$ (where $h_0 = f_0 - \mu$, and $H(0)$ is the relative entropy of $f_0$).

Proof. We check that $f_0$ satisfies the third condition of normalization (1.14) since the other two are easier to check. Thanks to a change of variables,
\[
(1 - \eta + \eta \epsilon^2)^{\frac{3}{2}} \left( \int (1 - \eta) \mu((1 - \eta + \eta \epsilon^2)^{\frac{3}{2}} v) |v|^2 dv + \eta \epsilon^{-3} \int \phi_0(\epsilon^{-1} (1 - \eta + \eta \epsilon^2)^{\frac{3}{2}} v) |v|^2 dv \right)
\]
\[= \frac{1 - \eta}{1 - \eta + \eta \epsilon^2} \int \mu(v) |v|^2 dv + \frac{\eta \epsilon^2}{1 - \eta + \eta \epsilon^2} \int \phi_0(v) |v|^2 dv = 3.\]
Next let us estimate $\mathcal{M}(0)$. We first observe that $\eta \epsilon^{-\frac{2}{9}} \geq 1$. Then for any $s > 0$, $\|h_0\|_{H^s} \sim \eta \epsilon^{-\frac{s}{2} - \frac{4}{9}}$. The relative entropy $H(0)$ is bounded from above by

$$H(0) = \int_{\mathbb{R}^3} \left( \frac{f_0}{\mu} \log \left( \frac{f_0}{\mu} \right) - \frac{f_0}{\mu} + 1 \right) \mu \, dv$$

$$\leq \int_{\mathbb{R}^3} \int_0^1 \left| \log \left( \frac{f_0}{\mu} \right) \right| \left| \frac{f_0}{\mu} - 1 \right| \mu \, d\theta dv,$$

for some $\theta \in [0,1]$, with the notation $f_0 = (1 - \theta) f_0 + \theta \mu$. We from now on denote by $C$ any strictly positive constant.

At points where $f_0 \geq \mu$, we see that

$$| \log \left( \frac{f_0}{\mu} \right) | = \log \left( \frac{f_0}{\mu} \right) \leq C \left( \|v\|^2 + \log(\eta \epsilon^{-3}) \right),$$

while at points where $f_0 \leq \mu$,

$$| \log \left( \frac{f_0}{\mu} \right) | = \log \left( \frac{\mu}{f_0} \right) \leq \log(\mu f_0) \leq C \left( \|v\|^2 + 1 \right).$$

From these estimates, we deduce that

$$H(0) \leq C \left( 1 + \log(\eta \epsilon^{-3}) \right) ||f_0 - \mu||_{L^\infty} \leq C \eta \left( 1 + \log(\eta \epsilon^{-3}) \right).$$

Thus,

$$\mathcal{M}(0) \leq C \eta \left( 1 + \log(\eta \epsilon^{-3}) \right) - C \eta^{-\frac{4}{5}} \epsilon^2.$$  

Remembering that $\eta \sim \epsilon^{11/9}$, and $\epsilon \ll 1$, we see that

$$\mathcal{M}(0) \leq C \epsilon^{11/9} \log(\epsilon^{-1}) - C \epsilon^{46/45} \leq 0.$$

Finally, $\|h_0\|_{H^{\frac{1}{2}}} \sim \epsilon^{-\frac{7}{9}}$, so that $h_0$ is a large initial datum for Landau equation in $H^{\frac{1}{2}}$ (the critical space for incompressible Navier-Stokes equations). \qed

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