Fast decomposition of $p$-groups in the Roquette category, for $p > 2$

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Abstract: Let $p$ be a prime number. In [9], I introduced the Roquette category $\mathcal{R}_p$ of finite $p$-groups, which is an additive tensor category containing all finite $p$-groups among its objects. In $\mathcal{R}_p$, every finite $p$-group $P$ admits a canonical direct summand $\partial P$, called the edge of $P$. Moreover $P$ splits uniquely as a direct sum of edges of Roquette $p$-groups.

In this note, I would like to describe a fast algorithm to obtain such a decomposition, when $p$ is odd.

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1. Introduction

Let $p$ be a prime number. The Roquette category $\mathcal{R}_p$ of finite $p$-groups, introduced in [9], is an additive tensor category with the following properties:

- Every finite $p$-group can be viewed as an object of $\mathcal{R}_p$. The tensor product of two finite $p$-groups $P$ and $Q$ in $\mathcal{R}_p$ is the direct product $P \times Q$.

- In $\mathcal{R}_p$, any finite $p$-group has a direct summand $\partial P$, called the edge of $P$, such that

$$P \cong \bigoplus_{N \unlhd P} \partial(P/N) \ .$$

Moreover, if the center of $P$ is not cyclic, then $\partial P = 0$.

- In $\mathcal{R}_p$, every finite $p$-group $P$ decomposes as a direct sum

$$P \cong \bigoplus_{R \in S} \partial R \ ,$$

where $S$ is a finite sequence of Roquette groups, i.e. of $p$-groups of normal $p$-rank 1, and such a decomposition is essentially unique. Given the group $P$, such a decomposition can be obtained explicitly from the knowledge of a genetic basis of $P$.

- The tensor product $\partial P \times \partial Q$ of the edges of two Roquette $p$-groups $P$ and $Q$ is isomorphic to a direct sum of a certain number $\nu_{P,Q}$ of copies of the edge $\partial(P \circ Q)$ of another Roquette group (where both $\nu_{P,Q}$ and $P \circ Q$ are known explicitly).
The additive functors from $\mathcal{R}_p$ to the category of abelian groups are exactly the rational $p$-biset functors introduced in [4].

The latter is the main motivation for considering this category: any structural result on $\mathcal{R}_p$ will provide for free some information on such rational functors for $p$-groups, e.g. the representation functors $R_K$, where $K$ is a field of characteristic 0 (see [2], [3], and L. Barker's article [1]), the functor of units of Burnside rings ([6]), or the torsion part of the Dade group ([5]).

The decomposition of a finite $p$-group $P$ as a direct sum of edges of Roquette $p$-groups can be read from the knowledge of a genetic basis of $P$. The problem is that the computation of such a basis is rather slow, in general. For most purposes however, the full details encoded in a genetic basis are useless, and it would be enough to know the direct sum decomposition.

Hence it would be nice to have a fast algorithm taking any finite $p$-group $P$ as input, and giving its decomposition as direct sum of edges of Roquette groups in the category $\mathcal{R}_p$. This note is devoted to the description of such an algorithm, when $p > 2$.

2. Rational $p$-biset functors

2.1. Recall that the characteristic property of the edge $\partial P$ of a finite $p$-group in the Roquette category $\mathcal{R}_p$ is that for any rational $p$-biset functor $F$

$$\partial F(P) = \hat{F}(\partial P),$$

where $\partial F(P)$ is the faithful part of $F(P)$, and $\hat{F}$ denotes the extension of $F$ to $\mathcal{R}_p$. Also recall the following criterion ([7], Theorem 3.1):

2.2. **Theorem**: Let $p$ be a prime number, and $F$ be a $p$-biset functor. Then the following conditions are equivalent:

1. The functor $F$ is a rational $p$-biset functor.
2. For any finite $p$-group $P$, the following conditions hold:
   
   - if the center of $P$ is non-cyclic, then $\partial F(P) = \{0\}$.
   - if $E \trianglelefteq P$ is a normal elementary abelian subgroup of rank 2, and $Z \leq E$ is a central subgroup of order $p$ of $P$, then the map

   $$\text{Res}^P_{C_P(E)} \oplus \text{Def}^P_{P/Z} : F(P) \rightarrow F(C_P(E)) \oplus F(P/Z)$$

   is injective.
2.3. Let $K$ be a commutative ring in which $p$ is invertible. When $P$ is a finite group, denote by $\text{CF}_K(P)$ the $K$-module of central functions from $P$ to $K$. The correspondence sending a finite $p$-group $P$ to $\text{CF}_K(P)$ is a rational $p$-biset functor:

$$\ovalbox{$\text{REJECT}_{p\text{-biset functor, denoted by } \text{CF}_K}$With this definition, the correspondence $P \mapsto \text{CF}_K(P)$ becomes a rational $p$-biset functor, denoted by $\text{CF}_K$}$$

Proof: A straightforward argument shows that $\text{CF}_K(U)(f)$ is indeed a central function on $Q$, hence the map $\text{CF}_K(U)$ is well defined. It is also clear that this map only depends on the isomorphism class of the biset $U$, and that for any two finite $(H, G)$-bisets $U$ and $U'$, we have

$$\text{CF}_K(U \sqcup U') = \text{CF}_K(U) + \text{CF}_K(U') .$$

Moreover if $U$ is the identity biset at $P$, i.e. if $U = P$ with biset structure given by left and right multiplication, then for $f \in \text{CF}_K(P)$ and $s \in P$

$$\text{CF}_K(U)(f)(s) = \frac{1}{|P|} \sum_{u \in U, x \in P \atop su = ux} f(x) .$$

hence $\text{CF}_K(U)$ is the identity map.

Now if $R$ is a third finite $p$-group, and $V$ is a finite $(R, Q)$-biset, then for any $t \in R$, setting $\lambda = \text{CF}_K(V) \circ \text{CF}_K(U)(f)(t)$, we have that

$$\lambda = \frac{1}{|Q|} \sum_{v \in V, s \in Q \atop tv = vs} \frac{1}{|P|} \sum_{u \in U, x \in P \atop su = ux} f(x)$$

$$= \frac{1}{|Q||P|} \sum_{(v, u) \in V \times U \atop s \in Q, x \in P \atop tv = vs, su = ux} f(x)$$

$$= \frac{1}{|Q||P|} \sum_{(v, u) \in V \times U, x \in P \atop t(v, u) = (v, u)x} |\{s \in Q \mid tv = vs, su = ux\}| f(x)$$
\[
\lambda = \frac{1}{|Q||P|} \sum_{(v_Q u) \in V \times Q U, x \in P, t(v_Q u) = (v_Q u)x} |Q : Q_v \cap uP||Q_v \cap uP| f(x)
\]

\[
= \frac{1}{|P|} \sum_{(v_Q u) \in V \times Q U, x \in P, t(v_Q u) = (v_Q u)x} f(x) = CF_K(V \times Q U)(f)(t) .
\]

Hence $CF_K(V) \circ CF_K(U) = CF_K(V \times Q U)$, and $CF_K$ is a $p$-biset functor.

To prove that this functor is rational, we use the criterion given by Theorem 2.2. Suppose first that the center $Z(P)$ of $P$ is non-cyclic. Let $E$ denote the subgroup of $Z(P)$ consisting of elements of order at most $p$. Then saying that $\partial CF_K(P) = \{0\}$ amounts to saying that for any $f \in CF_K(P)$, the sum

\[
S = \sum_{Z \leq E} \mu(1, Z) \text{Inf}^P_{P/Z} \text{Def}^P_{P/Z} f
\]

is equal to 0, where $\mu$ denotes the Möbius function of the poset of subgroups of $P$ (or of $E$). Equivalently, for any $s \in P$

\[
S(s) = \sum_{Z \leq E} \mu(1, Z) \frac{1}{|P|} \sum_{aZ \in P, x \in P} f(x) = 0 .
\]

This also can be written as

\[
S(s) = \sum_{Z \leq E} \mu(1, Z) \frac{1}{|Z|} \sum_{a \in P, x \in P} f(sa.z)
\]

To prove this, we need to show that for any $z \in E$, the sum

\[
\sum_{z \leq E} \frac{\mu(1, Z)}{|Z|} = 0
\]

for any $z \in E$. This follows from the fact that $E$ is an elementary abelian $p$-group of rank at least 2.
Proof: For $z \in E$, set $\sigma(z) = \sum_{z \in Z \leq E} \frac{\mu(1, Z)}{|Z|}$. Assume first that $z \neq 1$, i.e. $|z| = p$. If $Z \ni z$ is elementary abelian of rank $r$, then $\mu(1, Z) = (-1)^{r}p^{\binom{r}{2}}$, hence $\frac{\mu(1, Z)}{|Z|} = (-1)p = \frac{-1}{p} \mu(1, Z/<z>)$. Hence setting $\overline{Z} = Z/<z>$ and $\overline{E} = E/<z>$, 

$$\sigma(z) = \frac{-1}{p} \sum_{1 \leq \overline{Z} \leq \overline{E}} \mu(1, \overline{Z}) = 0 \ ,$$

since $|\overline{E}| > 1$. Now

$$\sum_{z \in E} \sigma(z) = \sigma(1) + \sum_{e \in E - \{1\}} \sigma(z) = \sum_{z \in Z} \sum_{z \in Z \leq E} \frac{\mu(1, Z)}{|Z|} = \sum_{1 \leq Z \leq E} \mu(1, Z) = 0$$

hence $\sigma(1) = 0$, completing the proof of the lemma. \hfill $\square$

It follows that $S(s) = 0$, hence $S = 0$, as was to be shown.

For the second condition of Theorem 2.2, suppose that $E$ is a normal elementary abelian subgroup of $P$ of rank 2, and that $Z$ is a central subgroup of $P$ of order $p$ contained in $E$. Let $f \in CF_K(P)$ which restricts to 0 to $C_P(E)$, and such that

$$\forall sZ \in P/Z, \ (Def_{P/Z}^{P}f)(sZ) = \frac{1}{|P|} \sum_{z \in Z} f(sz) = 0 \ .$$

Thus $f(s) = 0$ if $s \in C_P(E)$. Assume that $s \notin C_P(E)$. Then for $e \in E$, the commutator $[s, e]$ lies in $Z$. Moreover the map $e \in E \mapsto [s, e] \in Z$ is surjective. it follows that for any $z \in Z$, there exists $e \in E$ such that $s^e = sz$. Thus $f(sz) = f(s^e) = f(s)$. Hence $Def_{P/Z}^{P}f(s) = f(s) = 0$. Hence $f = 0$, as was to be shown. \hfill $\square$

3. Action of p-adic units

Let $\mathbb{Z}_p$ denote the ring of p-adic integers, i.e. the inverse limit of the rings $\mathbb{Z}/p^n\mathbb{Z}$, for $n \in \mathbb{N} - \{0\}$. The group of units $\mathbb{Z}_p^\times$ is the inverse limits of the unit groups $(\mathbb{Z}/p^n\mathbb{Z})^\times$, and it acts on the functor $CF_K$ in the following way: if $\zeta \in \mathbb{Z}_p^\times$ and $P$ is a finite p-group, choose an integer $r$ such that $p^r$ is a multiple of the exponent of $P$, and let $\zeta_{p^r}$ denote the component of $\zeta$ in $(\mathbb{Z}/p^r\mathbb{Z})^\times$. For $f \in CF_K(P)$, define $\hat{\zeta}_p(f) \in CF_K(P)$ by

$$\forall s \in P, \ \hat{\zeta}_p(f)(s) = f(s^{\zeta_{p^r}}) \ .$$
Then clearly $\hat{\zeta}_P(f)$ only depends on $\zeta$, and this gives a well defined map

$$\hat{\zeta}_P : CF_K(P) \to CF_K(P).$$

One can check easily (see [8] Proposition 7.2.4 for details) that if $Q$ is a finite $p$-group, and $U$ is a finite $(Q, P)$-biset, then the square

$$\begin{array}{ccc}
CF_K(P) & \xrightarrow{\hat{\zeta}_P} & CF_K(P) \\
\downarrow^{CF_K(U)} & & \downarrow^{CF_K(U)} \\
CF_K(Q) & \xrightarrow{\hat{\zeta}_Q} & CF_K(Q)
\end{array}$$

is commutative. In other words, we have an endomorphism $\hat{\zeta}$ of the functor $CF_K$. It is straightforward to check that for $\zeta, \zeta' \in \mathbb{Z}_p^\times$, we have $\zeta \zeta' = \hat{\zeta} \circ \hat{\zeta}'$, and that $\hat{1}$ is the identity endomorphism of $CF_K$. So this yields an action of the group $\mathbb{Z}_p^\times$ on $CF_K$.

It follows in particular that when $n \in \mathbb{N} - \{0\}$, and $P$ is a finite $p$-group, if we set

$$F_n(P) = \{ f \in CF_K(P) \mid \forall s \in P, f(s^{1+p^n}) = f(s) \},$$

then the correspondence $P \mapsto F_n(P)$ is a subfunctor of $CF_K$: indeed $F_n$ is the subfunctor of invariants by the element $1 + p^n$ of $\mathbb{Z}_p^\times$.

It follows that $F_n$ is a rational $p$-biset functor, for any $n \in \mathbb{N} - \{0\}$, hence it factors through the Roquette category $\mathcal{R}_p$. In particular, for any finite $p$-group $P$, if $P$ splits as a direct sum

$$P \cong \bigoplus_{R \in S} \partial R$$

of edges of Roquette groups in $\mathcal{R}_p$, then there is an isomorphism

$$F_n(P) \cong \bigoplus_{R \in S} \partial F_n(R).$$

3.1. **Notation:** For a finite $p$-group $P$, and an integer $n \in \mathbb{N} - \{0\}$, let $l_n(P)$ denote the number of conjugacy classes of elements $s$ of $P$ such that $s^{1+p^n}$ is conjugate to $s$ in $P$. Also set $l_0(P) = 1$.

With this notation, for any finite $p$-group $P$, and any $n \in \mathbb{N} - \{0\}$, the $K$-module $F_n(P)$ is a free $K$-module of rank $l_n(P)$. In particular, if $P = C_{p^m}$ is cyclic of order $p^m$, then $F_n(P)$ has rank $l_n(P) = p^{\min(m,n)}$. Thus if $m > 0$,
then $\partial F_n(C_{p^m})$ has rank $p^{\min(m,n)} - p^{\min(m-1,n)}$, since $C_{p^m} \cong \partial C_{p^m} \oplus C_{p^{m-1}}$ in $\mathcal{R}_p$.

**3.2. Theorem:** Assume that a $p$-group $P$ splits as a direct sum

$$P \cong 1 \oplus \bigoplus_{m=1}^{\infty} a_m \partial C_{p^m}$$

of edges of cyclic groups in the Roquette category $\mathcal{R}_p$, where $a_m \in \mathbb{N}$. Then

$$\forall m \geq 1, \ a_m = \frac{l_m(P) - l_{m-1}(P)}{p^{m-1}(p-1)}.$$

**Proof:** For any $n \in \mathbb{N} - \{0\}$, we have

$$l_n(P) = 1 + \sum_{m=1}^{\infty} a_m (p^{\min(m,n)} - p^{\min(m-1,n)}) = 1 + \sum_{m=1}^{n} a_m (p^m - p^{m-1}) .$$

For $n \in \mathbb{N} - \{0\}$, this gives $l_n(P) - l_{n-1}(P) = a_n (p^n - p^{n-1})$. $\square$

**3.3. Corollary:** Suppose $p > 2$. If $P$ is a finite $p$-group, then

$$P \cong 1 \oplus \bigoplus_{m=1}^{\infty} \frac{l_m(P) - l_{m-1}(P)}{p^{m-1}(p-1)} \partial C_{p^m}$$

in the Roquette category $\mathcal{R}_p$.

**Proof:** Indeed for $p$ odd, all the Roquette $p$-groups are cyclic, hence the assumption of Theorem 3.2 holds for any $P$. $\square$

**Appendix**

**3.1. A GAP function:** The following function for the GAP software ([10]) computes the decomposition of $p$-groups for $p > 2$, using Corollary 3.3:

```
#
# Roquette decomposition of an odd order p-group g
# output is a list of pairs of the form [p^n, a_n]
# where a_n is the number of summands of g
# isomorphic to the edge of the cyclic group of order p^n
#
```
roquette_decomposition:=function(g)
local prem, cg, s, i, x, y, z, pn, u;
if IsTrivial(g) then return [[1,1]]; fi;
prem:=PrimeDivisors(Size(g));
if Length(prem)>1 then
  Print("Error : the group must be a p-group\n");
  return fail;
fi;
prem:=prem[1];
if prem=2 then
  Print("Error : the order must be odd\n");
  return fail;
fi;
cg:=ConjugacyClasses(g);
s:=[ ];
for i in [2..Length(cg)] do
  x:=cg[i];
  y:=Representative(x);
  pn:=1;
  u:=y;
  repeat
    pn:=pn*prem;
    u:=u^prem;
    z:=y*u;
    until z in x;
  Add(s,pn);
  od;
s:=Collected(s);
s:=List(s,x->[x[1],x[2]*prem/(prem-1)/x[1]]);
s:=Concatenation([[1,1],s]);
return s;
end;

3.2. Example :

gap> l:=AllGroups(81);;
gap> for g in l do
> Print(roquette_decomposition(g),"\n");
> od;
[[1, 1], [3, 1], [9, 1], [27, 1], [81, 1]]
[[1, 1], [3, 4], [9, 12]]
[[1, 1], [3, 7], [9, 3]]
[[1, 1], [3, 7], [9, 3]]
[[1, 1], [3, 4], [9, 3], [27, 3]]
[[1, 1], [3, 4], [9, 4]]
[[1, 1], [3, 8]]
[[1, 1], [3, 5], [9, 1]]
[[1, 1], [3, 5], [9, 1]]
[[1, 1], [3, 5], [9, 1]]
For example, the group on line 6 of the previous list, isomorphic to the semidirect product $C_{27} \rtimes C_3$, is isomorphic to $1 \oplus 4\partial C_3 \oplus 4\partial C_9$ in $\mathcal{R}_3$.

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