Abstract. In this paper we prove that any asymptotically sectional-hyperbolic (ASH) attractor associated to a three-dimensional $C^1$ vector field $X$ is entropy-expansive and has periodic orbits. The ASH property means that any point outside the stable manifolds of the singularities has arbitrarily large hyperbolic times. The results obtained in this paper extend, to this context, those obtained for singular-hyperbolic attractors.

1. Introduction

The hyperbolic theory of dynamical systems in the last decades has proven to be a fruitful research field. As one of its start points, we can mention the seminal work of S. Smale [22] where the concept of hyperbolic set was introduced. Since then, a very rich theory was developed and several important properties of hyperbolic systems were obtained both for the discrete-time and continuous-time systems, such as stability, robustness, expansiveness and existence of physical measures, just to cite a few.

Some years later, motivated by the works of E. Lorenz in weather forecast, it was introduced independently in [1] and [9] a geometrical construction that nowadays is called the Geometric Lorenz Attractor (GLA for short). This geometrical model has proven to display very rich dynamics. In particular, it is robustly transitive, sensitive to initial conditions and has dense periodic orbits. Nevertheless, since it contains a singularity accumulated by regular orbits, this set fails to be hyperbolic. This motivated the notion of singular-hyperbolic set introduced by M.J. Pacífico, C.A. Morales and E. Pujals in [14]. Among the main results they proved that, indeed, this class of dynamics contains the GLA, and it extends properly the class of hyperbolic sets. In [2] was proved that three-dimensional attractors with this kind of hyperbolicity are $k^*$-expansive and support physical measures.

In 2008, Morales and R.J. Metzger introduced in [12] the concept of sectional-hyperbolic set. The notions of sectional-hyperbolicity and singular-hyperbolicity agree in the three-dimensional setting, but in general sectional-hyperbolicity is stronger than singular-hyperbolicity. Furthermore, the class of sectional-hyperbolic sets contains the multidimensional GLA as a prototype example of these dynamics in higher-dimensional manifolds (see [12] for details). In addition, as in the singular-hyperbolic case, any hyperbolic set is sectional-hyperbolic.

In 1993, A. Rovella performed a slight modification of the GLA, introducing in [18] a new example of system that nowadays is called the Rovella attractor.

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new set fails to be sectional-hyperbolic, but it still shares many interesting properties with the GLA. Indeed, both sets are homoclinic classes, support physical measures and are $k^*$-expansive. This motivated the authors in [15] to introduce a new class of systems that extends properly the class of sectional-hyperbolic systems, the so-called Asymptotically Sectional-Hyperbolic systems (ASH for short). Recently, in [19] it was proved that the Rovella attractor is a representative example of this kind of dynamics.

In this paper we deal with ASH attractors in three-dimensional manifolds and we present results about the entropy theory these sets, as well as the existence of periodic orbits for such systems. Versions of these results were previously obtained in [6] and [16], respectively, in the singular-hyperbolic setting. Besides, it is worth to mention that here we will prove our results by different approaches. As we will see in the next sections, this is due to the lack of uniform area-expansion in the central bundle. In fact, this represents the main challenge when one tries to extend results from the sectional-hyperbolic to the ASH setting. This motivates us to develop new tools to deal with these issues. In section 2 we will precisely introduce our main results, whereas in Sections 3 and 4 we will devote to their proofs.

2. Statements of the results

In what follows we denote by $M$ to a three-dimensional compact Riemannian manifold endowed with metric $d$, which is induced by the Riemannian metric $\| \cdot \|$. We denote by $X$ to a $C^1$ vector field on $M$, and we will refer by flow on $M$ to the family of maps $\{X_t\}_{t \in \mathbb{R}}$, induced by $X$. If $t = 1$, the map $X_1$ is called the time-one map of $X$. For $x \in M$, the orbit of $x$ is denoted by $\mathcal{O}(x)$. For $a, b \in \mathbb{R}$, the orbit segment from $a$ to $b$ of a point $x$ is defined by $X_{[a,b]}(x) = \{X_t(x) : t \in [a,b]\}$. We will denote the singularities (or zeros) of a vector field $X$ by $\text{Sing}(X)$, and the set of periodic orbits of $X$ by $\text{Per}(X)$. The orbits that are not associated to either periodic orbits or singularities we will called regular orbits. As always, we say that a subset $\Lambda$ of $M$ is invariant if $X_t(\Lambda) = \Lambda$ for any $t \in \mathbb{R}$. We say that a compact invariant set $\Lambda$ is attracting if there exists a neighborhood $U_0$ (called trapping region) of $\Lambda$ such that $X_t(U_0) \subset U_0$, for any $t > 0$, and

$$\Lambda = \bigcap_{t \geq 0} X_t(U_0),$$

and we call it attractor if it is also transitive, i.e., there is $z \in \Lambda$ such that $\omega(z) = \Lambda$.

Now, we say that a compact invariant set $\Lambda$ has a dominated splitting if there are a continuous invariant splitting $T\Lambda = E \oplus E^c$ (respect to $DX_t$) and constants $K, \lambda > 0$ satisfying

$$\frac{\|DX_t(x)|_{E^c}\|}{m(DX_t(x)|_{E^c})} \leq Ke^{-\lambda t}, \quad \forall x \in \Lambda, \forall t > 0,$$

where $m(A)$ denotes the minimum norm of $A$. In this case we say that $E^c$ is dominated by $E$. We say that $\Lambda$ is partially hyperbolic if $E$ is a subbundle of contracting type, i.e., $\|DX_t(x)|_{E^c}\| \leq Ke^{-\lambda t}$ for every $t > 0$ and $x \in \Lambda$.

Motivated by the construction given by Turaev and Shilnikov in [23], in [14] was defined the notion of singular-hyperbolic set as a compact invariant partially hyperbolic set with hyperbolic singularities (if it has) and volume expanding central
subbundle $E_c$, i.e., there are $K, \lambda > 0$ such that

$$\det DX_t(x)_{|E_c} \geq Ke^{\lambda t}, \quad \forall x \in \Lambda, \forall t > 0.\quad (2.1)$$

It is easy to see that any hyperbolic set is singular-hyperbolic. Besides, it is shown that the geometric Lorenz attractor is an archetypal example of these systems, which is not hyperbolic. A more restrictive notion of hyperbolicity was introduced in [12] in order to get a better understanding of the dynamic behavior of certain higher-dimensional flows. More precisely, a compact invariant partially hyperbolic set $\Lambda$ for a vector field $X$ is sectiona

sectional-hyperbolic if the singularities are hyperbolic and the central subbundle $E_c$ is sectional expanding, i.e, there are $K, \lambda > 0$ such that for every two-dimensional subspace $L_x$ of $E_c$ one has

$$\det DX_t(x)_{|L_x} \geq Ke^{\lambda t}, \quad \forall x \in \Lambda, \forall t > 0.\quad (2.1)$$

For three-dimensional manifolds, both notions agree (sectional and singular hyperbolicity), but in higher dimensions is showed that the system exhibited in [23] is a singular-hyperbolic set which is not sectional-hyperbolic (see [12] for more details).

It is well known from hyperbolic theory that every hyperbolic singularity $\sigma$ of $X$ has associated its stable manifold, denoted by $W^s(\sigma)$, which is tangent to $E^s$. Denote by $W^s(Sing(X))$ the union of the stable manifolds of the singularities of $X$. Let us recall the main notion of this paper, which was introduced in [15]:

**Definition 2.1.** Let $\Lambda$ be an invariant partially hyperbolic set of a vector field $X$ whose singularities are hyperbolic. We say that $\Lambda$ is asymptotically sectional-hyperbolic (ASH for short) if its central subbundle is eventually asymptotically expanding outside the stable manifolds of the singularities, i.e, there exists $C > 0$ such that

$$\limsup_{t \to +\infty} \frac{\log \det DX_t(x)_{|E_c}}{t} \geq C,$$

for every $x \in \Lambda' = \Lambda \setminus W^s(Sing(X))$ and every two-dimensional subspace $L_x$ of $E_c$. We say that an ASH set is non-trivial if it is not reduced to a singularity.

From ASH property we see that any point outside $W^s(Sing(X))$ has arbitrarily large hyperbolic times, i.e., if $x \in \Lambda'$ there is an unbounded increasing sequence of positive numbers $t_k = t_k(x) > 0$ such that

$$\det DX_{t_k}(x)_{|E_c} \geq e^{Ct_k}, \quad k \geq 1.\quad (2.3)$$

Any unbounded increasing sequence satisfying the relation $(2.3)$ will be called $C$-hyperbolic times for $x$.

**Remark 2.1.** We have the following remarks:

- Every sectional-hyperbolic set is asymptotically sectional-hyperbolic. On the other hand, there are ASH sets which are not sectional-hyperbolic. This shows that asymptotic sectional-hyperbolicity extends properly the sectional-hyperbolicity.
- Any asymptotically sectional-hyperbolic set satisfies the Hyperbolic lemma. The proof of this result can be found in [20].

It is well known that any non trivial hyperbolic isolated set contains periodic orbits. Morales in [13] showed that this is not the case for singular-hyperbolic sets by exhibiting an isolated transitive set whose dynamics is similar to that given by the Cherry flow, but in [6] was proved that singular-hyperbolic attracting sets contains...
periodic orbits. Nevertheless, there exists ASH attracting sets without periodic orbits. Indeed, consider a vector field $X$ in $S^2 \times [-1, 1]$ whose attracting set consists of one source $S$, one sink $N$, and regular orbits contained in $W^s(S) \cap W^u(N)$. On the other hand, it is not hard to see that the Rovella attractor [18] contains periodic orbits. Besides, in [20] was showed that this attractor is a representative example of ASH systems. So, a natural question arises: Any ASH attractor has a periodic orbit? In this paper we give an affirmative answer for three-dimensional $C^1$ vector fields:

**Theorem 2.2.** Every asymptotically sectional-hyperbolic attractor $\Lambda$ associated to a $C^1$ vector field $X$ on a three-dimensional manifold $M$ has a periodic orbit.

In this case, we denote $\text{Per}_\Lambda(X) = \text{Per}(X) \cap \Lambda$.

Now, recall that for a hyperbolic periodic point $p$ associated to a $C^1$ vector field $X$, the strong stable and strong unstable manifolds of $p$ are defined, respectively, by

$$W^{ss}(p) = \{y \in M; \lim_{t \to \infty} d(X_t(p), X_t(y)) = 0, \}$$

and

$$W^{uu}(p) = \{y \in M; \lim_{t \to \infty} d(X_t(p), X_t(y)) = 0, \}.$$  

We then define the stable and unstable manifolds of $p$, respectively, by

$$W^s(p) = \bigcup_{t \in \mathbb{R}} W^{ss}(X_t(p))$$

and

$$W^u(p) = \bigcup_{t \in \mathbb{R}} W^{uu}(X_t(p)).$$

Thus, the homoclinic class of $p$, denoted by $H(p)$, is defined as

$$H(p) := W^s(p) \cap W^u(p).$$

It is clear that $O(p) \subset H(p)$. We say that a homoclinic class $H(p)$ is non trivial if $H(p) \neq O(p)$. In [21] was showed that every singular-hyperbolic attractor associated to a $C^1$ vector field is a homoclinic class. In this paper we have the following result:

**Theorem 2.3.** Let $\Lambda$ be an asymptotically sectional-hyperbolic associated to a $C^1$ vector field $X$ on a three-dimensional manifold $M$. If $\text{Per}_\Lambda(X) = \Lambda$, then there is $p \in \text{Per}_\Lambda(X)$ such that $\Lambda = H(p)$.

**Remark 2.2.** For singular-hyperbolic sets, the above theorems are obtained by using the uniform volume-expansion along the central subbundle. Indeed, in [20] was obtained a triangular map, whose domain is formed by a finite collection of cross-sections, with markovian properties. Nevertheless, this technique cannot be applied for ASH attractors by the lack of uniform expansion of $|\det DX_t(\cdot)|_{E^c}$. Therefore, the proof of the results stated here are made by a different approach.

Next we discuss the entropy theory of ASH attractors. Before to state our results, we precise the concept of entropy-expansiveness for flows. Let $(M, d)$ be a compact metric space and let $f : M \to M$ be a homeomorphism. Recall that the $\delta$-dynamical ball of a point $x \in M$ is the set

$$B^\delta_{\infty}(x, f) = \{y \in M : d(f^n(x), f^n(y)) \leq \delta, \forall n \in \mathbb{Z}\}.$$  

Let $A \subset M$ and fix $\varepsilon, n > 0$. we say that $K \subset A$ is an $n$-$\varepsilon$-separated set of $A$ if for any pair of distinct points $x, y \subset K$ there is some $0 \leq n_0 \leq n$ such that
Let $S(n, \varepsilon, A)$ denotes the maximal cardinality of an $n$-\(\varepsilon\)-separated subset of $A$.

We say that $K \subset A$ is a $n$-\(\varepsilon\)-generator for $A$ if for every $x \in A$, there exists $y \in K$ such that $d(f^i(x), f^i(y)) \leq \varepsilon$, for every $0 \leq i \leq n$. Denote by $R(n, \varepsilon, A)$ the minimum cardinality of the $n$-\(\varepsilon\)-generators for $A$.

Note that both $S(n, \varepsilon, A)$ and $R(n, \varepsilon, A)$ are always finite due the compactness of $M$. Then we define the topological entropy of $A$ as the number

$$h_{top}(f, A) = \lim_{n \to \infty} \limsup_{\varepsilon \to 0} \frac{1}{n} \log(S(n, \varepsilon, A)) = \lim_{n \to \infty} \limsup_{\varepsilon \to 0} \frac{1}{n} \log(R(n, \varepsilon, A)).$$

As we will see in Section 3 during the proof of Theorem 2.2 we obtain that three-dimensional ASH attractors have positive topological entropy. This motivate us to ask what other entropy properties one can obtain through the ASH property. In this direction, we studied the entropy-expansiveness property of ASH attractors.

**Definition 2.4.** A continuous flow $X_t$ on a compact metric space $M$ is said to be **entropy-expansive** if its time-one map is entropy-expansive, i.e., there exists $\delta > 0$ such that $h_{top}(X_t, B^S_T(x, X_t)) = 0$, for every $x \in M$.

We then obtain the following result for ASH attractors.

**Theorem 2.5.** Every ASH attractor $\Lambda$ associated to a $C^1$ vector field $X$ on a three-dimensional manifold $M$ is entropy-expansive.

It is worth mentioning that a similar result was obtained in [16] in the context of sectional hyperbolicity. Unfortunately, the techniques developed there do not suit to our case. This is due to the absence of uniform area expansion in the center-unstable bundle in the ASH setting. This motivated us to obtain Theorem 2.5 through techniques similar to that developed in [17]. By following [16] we obtain the following consequence:

**Corollary 2.1.** Let $\Lambda$ be an ASH attractor for a $C^1$ vector field on a three-dimensional manifold $M$. Then the metric entropy function of $X$ is upper semi-continuous. In particular, there exists a measure of maximal entropy for $X$ in $\Lambda$.

### 3. Existence of Periodic Orbits

Before to present the proof of Theorem 2.2 it is necessary to introduce some tools. Let $E^s \oplus E^c$ be the partially hyperbolic splitting associated to the asymptotically sectional-hyperbolic attractor $\Lambda$, and consider a continuous extension $\tilde{E}^s \oplus \tilde{E}^c$ to the trapping region $U_0$. It is known that $\tilde{E}^s$ can be chosen $DX_t$-invariant for positive $t$. Nevertheless, the subbundle $\tilde{E}^c$ is not invariant in general, but we can consider a cone field $C^c_a$ of size $a > 0$ around $\tilde{E}^c$ on $U_0$ defined by

$$C^c_a(x) := \{v = v_s + v_c : v_s \in \tilde{E}^s, v_c \in \tilde{E}^c \text{ and } \|v_s\| \leq a\|v_c\|\}, \quad \forall x \in U_0,$$

which is invariant for $t > 0$ large enough, i.e., there is $T_0 > 0$ such that

$$DX_t(x) C^c_a(x) \subset C^c_a(X_t(x)), \quad \forall t \geq T_0, \forall x \in U_0.$$

**Remark 3.1.** By possibly shrinking the neighborhood $U_0$, the number $a > 0$ can be taken arbitrarily small.

In order to simplify the notation, we write $E^s$ and $E^c$ for $\tilde{E}^s$ and $\tilde{E}^c$ respectively in what follows.

Let $\sigma$ be a singularity of $\Lambda$. We say that $\sigma$ is
Lemma 3.2. Let $\lambda$ be an asymptotically sectional-hyperbolic attractor associated to a $C^1$ vector field $X$. Then, we have

$$\limsup_{t \to \infty} \frac{1}{t} \log |\det DX_t(x)|_{E^s} > 0, \quad \forall x \in U' = U_0 \setminus W^s(Sing(x)).$$

Proof. First, we need to get an estimate of $|\det DX_t(\cdot)|_{\bar{E}^s}$ in a neighborhood of the singularities of $\Lambda$. Let $\sigma$ be either a Rovella-like or resonant singularity and let $\bar{V}_\sigma$ be a neighborhood of $\sigma$. By shrinking $\bar{V}_\sigma$, if it is necessary and by continuity of $DX_t(x)$ and the choice of the cone field there are $\theta_\sigma' \leq 0$ and $T_0 > 0$ such that for every $x \in \bar{V}_\sigma$,

$$|\det DX_t(x)|_{L_x} \geq \left( \frac{1}{2} e^{\theta_\sigma' t} \right) \left( \frac{1}{2} e^{\theta_\sigma' T_0} \right)^m \geq K_-(\sigma) e^{\theta_\sigma t},$$

for every plane $L_x \subset C^\alpha_\sigma(x)$ and every $t \in [0, T_0]$ such that $X_t(x) \in \bar{V}_\sigma$. So, for $x \in \bar{V}_\sigma$ and every $t > 0$ such that $X_t(x) \in \bar{V}_\sigma$, we have that $t = mT_0 + r$, $m \in \mathbb{N}$ and $0 \leq r < T_0$, so that by the above estimation we have for every plane $L_x \subset C^\alpha_\sigma(x)$ that

$$|\det DX_t(x)|_{L_x} \geq K_+(\sigma) e^{\theta_\sigma t},$$

where $\theta_\sigma = \frac{1}{T_0} \log \left( \frac{1}{2} e^{\theta_\sigma' T_0} \right) < 0, K_-(\sigma) > 0$. In a similar way, by taking $T_0$ large enough, there are $K_+ > 0$ and $\theta_\sigma > 0$ such that

$$|\det DX_t(x)|_{L_x} \geq K_+(\sigma) e^{\theta_\sigma t},$$

for every Lorenz-like singularity $\sigma$, every $x \in \bar{V}_\sigma$ such that $X_t(x) \in \bar{V}_\sigma$, and for every plane $L_x \subset C^\alpha_\sigma(x)$. In this case, we set $V_{Sing(X)} = \bar{V}_R \cup \bar{V}_L$, where $\bar{R}$ denotes...
By definition of these numbers, for every \( t > r \) singularities. In this case, let
\[ \theta_+ = \min_{\sigma \in L} \theta_+ > 0, \quad \theta_- = \min_{\sigma \in R} \{ \theta_- \} \leq 0, \quad \text{and} \quad K = \min_{\sigma \in V_{Sing}(X)} \{ K_-(\sigma), K_+(\sigma) \} > 0. \]

On the other hand, by ASH property there is a positive constant \( c < C \), where \( C \) is given by \( (2.2) \), and \( T_1 > T_0 \) large enough with the following property: for every \( x \in \Lambda' = \Lambda \setminus W^s(Sing(X)) \) there is a neighborhood \( V_x \) of \( x \) such that
\[ |\det DX_t(y)|_{L^p} \geq e^{c \sigma}, \quad \forall y \in V_x, t_1 = t_1(x) \geq T_1, \]
for every plane \( L_y \subseteq C^0_\alpha(y) \), where \( t_1 \) is the first hyperbolic time for \( x \). In this case, denote
\[ W_x = \bigcup_{0 \leq t \leq t_1(x)} X_t(V_x). \]

Then, the set \( W = \bigcup_{x \in \Lambda'} W_x \) defines an open cover of \( \Lambda' \).

By compactness of \( \overline{U_0 \setminus (V_{Sing}(X) \cup \overline{W})} \), there are \( T_2 > 0 \) and \( b \in \mathbb{R} \) such that
\[ |\det DX_t(z)|_{L^p} \geq e^{b \sigma}, \quad \forall (z, r) \in \left( \overline{U_0 \setminus (V_{Sing}(X) \cup \overline{W})} \right) \times [0, T_2], \quad L_z \subseteq C^0_\alpha(z). \]

Now, for every \( x \in U_0 \), let consider the following numbers:
\[
\begin{align*}
\beta_0(x) &= \limsup_{t \to +\infty} \frac{1}{t} \int_0^t \chi_{U_0 \setminus (V_{Sing}(X) \cup \overline{W})}(X_s(x))ds, \\
\beta_1(x) &= \liminf_{t \to +\infty} \frac{1}{t} \int_0^t \chi_{V_L \cup W}(X_s(x))ds, \quad \text{and} \\
\beta_2(x) &= \limsup_{t \to +\infty} \frac{1}{t} \int_0^t \chi_{V_R \setminus W}(X_s(x))ds.
\end{align*}
\]

By definition of these numbers, for every \( \varepsilon > 0 \) there is \( r > 0 \) such that for any \( t > r \), one has
\[
\frac{1}{t} \int_0^t \chi_{U_0 \setminus (V_{Sing}(X) \cup \overline{W})}(X_s(x))ds \leq \beta_0(x) + \varepsilon, \quad \frac{1}{t} \int_0^t \chi_{V_L \cup W}(X_s(x))ds \geq \beta_1(x) - \varepsilon \quad \text{and} \quad \frac{1}{t} \int_0^t \chi_{V_R \setminus W}(X_s(x))ds \leq \beta_2(x) + \varepsilon.
\]

So, for any \( x \in U_0 \), by splitting the orbit in orbit segments and by joining the above estimations, we have that
\[
|\det DX_t(x)|_{E^1_x} \geq e^{\psi(x, t)t}, \quad \forall t > r,
\]
with
\[ \psi(x, t) = \frac{(k_1(t) + k_2(t)) \ln K}{t} + b \beta_0(x) + c' \beta_1(x) + \theta_- \beta_2(x) + (\theta_- - c' + b) \varepsilon. \]

Here \( c' = \min\{c, \theta_+\} > 0 \) and the numbers \( k_1(t) \) and \( k_2(t) \) denote how many times the orbit of \( x \) hits \( V_R \) and \( V_L \) in the interval \([0, t]\), respectively, and these hitting points does not belong to \( W \).

Now, let
\[ Z = \{ x \in U_0 : d(x) = \rho(x) + b \beta_0(x) + c' \beta_1(x) + \theta_- \beta_2(x) \leq 0 \} \subset U_0, \]
where \( \rho(x) = \limsup_{t \to +\infty} ((k_1(t) + k_2(t)) \ln K/t) < \infty. \) Since \( \rho(\cdot), \beta_0(\cdot), \beta_1(\cdot) \) and \( \beta_2(\cdot) \) are invariant by the flow we have that \( Z \) is an invariant subset of \( U_0 \). So, since \( \Lambda \) is attracting it follows that \( Z \subset \Lambda \). But, in \( \Lambda' \cup L \) one has \( k_1(t) = 0, \)

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\(k_2(t) = 0\) or 1, \(\beta_0(x) = 0, \beta_2(x) = 0\) and \(\beta_1(x) = 1\), so that \(d(x) = c' > 0\). In particular, \(Z \subset W^s(R)\), so that \(d(x) > 0\) in \(U'\). Moreover, note that \(U' = \bigcup_{n \in \mathbb{N}} U'_n\), where \(U'_n = \{ x \in U' : d(x) > 1/n \}\). Therefore, if \(x \in U'\) there is \(n \in \mathbb{N}\) such that \(d(x) > 1/n\). Thus, if we choose \(\varepsilon_n > 0\) satisfying
\[
1/n + (\theta_- + c')\varepsilon_n > d_n > 0,
\]
we obtain by (3.2) that
\[
\limsup_{t \to \infty} \frac{1}{t} \log |\det Df_t(x)|_{E^c} | \geq d_n > 0.
\]
This concludes the proof. \(\square\)

Now, recall that for a continuous map \(f : M \to M\), the empirical probabilities of orbit of a point \(x \in M\) are defined as
\[
m_{n,x} = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)}, \quad n \in \mathbb{N},
\]
where \(\delta_y\) is the Dirac measure supported at \(y \in M\). Let \(p_\omega(x)\) be the set of accumulation points of the sequence \(\{m_{n,x}\}\) in the weak* topology. Recall that a probability measure \(\mu\) is atomic if its support consists of atoms. Besides, we say that a probability measure \(\mu\) has no atoms if \(\mu(\{x\}) = 0\) for every \(x \in \text{supp}(\mu)\).

**Lemma 3.3.** For every point \(x \in \Lambda \setminus W^*(\text{Sing}(X))\), any measure \(\nu \in p_\omega(x)\), where \(f = X_1\), has no atoms in \(R\).

**Proof.** Let \(E\) be the set of atoms of \(\nu\), which is formed by countable many periodic points because of the invariance and finiteness of \(\nu\). Assume that \(R_E = R \cap E \neq \emptyset\) and let \(k = \#R_E \geq 1\), where \(\#A\) denotes the cardinality of \(A\). For every \(\sigma_i \in R_E\), let \(V_i\) a neighborhood of \(\sigma_i\) such that \(V_i \cap V_j \neq \emptyset\), if \(i \neq j\). Let us consider \(V = \bigcup_{i=1}^k V_i\). Since the singularities in \(V\) are either Rovella-like or resonant, by shrinking \(V\) if necessary, there exists \(C_0 > 0\) and \(N > 0\) such that
\[
|\det Df(x)|_{E^c} | \leq C_0 e^{\alpha t} \leq e^{\overline{\alpha} t},
\]
where \(0 \leq \overline{\alpha} \leq \alpha\), for every \(n \geq N\) and every \(x \in V\) satisfying \(f^i(x) \in V\), for \(i = 0, \ldots, n\).

Let \(x \in \Lambda \setminus W^*(\text{Sing}(X))\) and let \(t_k\) be a sequence of \(C\)-hyperbolic times for \(x\). It is easy to check that one can take \(t_k = n_k \in \mathbb{N}\), so that
\[
|\det Df^{n_k}(x)|_{E^c} | \geq e^{Cn_k}, \quad k \geq 1.
\]
Since \(V^c\) is a compact set, there exists \(a > 1\) such that
\[
|\det Df(z)|_{E^c} | \leq a \forall z \in V^c.
\]
So, by (3.4), (3.5) and (3.6), we have that
\[
a^{\#B_n} e^{\#B_n a} \geq \prod_{i \in B_n} |\det DX_i(f^i(x))|E^{c_i}_{f^i(x)} \prod_{i \in B_n} |\det Df(X_i(x))|E^{c_i}_{f^i(x)}
\]
(3.7)
\[
= \prod_{i=0}^{n_k-1} |\det Df^i(x)|E^{c_i}_x |
\]
\[
\geq e^{Cn_k},
\]
where \(B_n = \{1 \leq m \leq n : f^m(x) \in V^c\}\). So,
\[
\frac{\#B_n}{n_k} \geq \frac{\#B_n}{n_k} + \frac{\#B_n^c}{n_k} \frac{\theta}{\log a} \geq \frac{C}{\log a} > 0,
\]
and hence
\[
\limsup_{n \to \infty} \frac{\#B_n}{n} = \alpha(x) > 0.
\]
Now, by above relation,
\[
0 \leq \alpha'(x) = \limsup_{n \to \infty} \frac{\#B_n^c}{n} < 1.
\]
In particular, if \(\varepsilon > 0\) satisfies \(\alpha'(x) + \varepsilon < \alpha < 1\), there is \(N \in \mathbb{N}\) such that

\[
\frac{\#B_n^c}{n} < \alpha, \quad \forall n \geq N.
\]

Let \(V'\) be a compact neighborhood of \(R_E\) contained in \(V\). By Urysohn’s lemma there is a continuous function \(\varphi : M \to \mathbb{R}\) such that \(\varphi(V') = 1\) and \(\varphi(M \setminus V) = 0\). So, it follows that \(\int \varphi dv \geq k\). Therefore,
\[
\left| \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) - \int \varphi dv \right| \geq 1 - \frac{\#B_n^c}{n} > 1 - \alpha > 0, \quad n \geq N.
\]
This shows that \(\nu \notin p\omega(x)\), which lead us to a contradiction. This proves the result.

Proof of Theorem 2.2. By Lemma 3.3 we have that \(\text{supp}(\nu) \subset \Lambda' \cup L\) for every \(\nu \in p\omega(x), x \in U'\), so that, by Theorem F in [8] and by following the argument given in Theorem 2.4 in [19], we deduce that there exists an SRB-like measure \(\nu\) such that \(h_\nu(X_\Lambda) > 0\). So, by Variational Principle we have that \(h_{top}(X) > 0\). In particular, there is an ergodic measure \(\mu\) with \(h_\mu(X) > 0\). Since \(\text{supp}(\mu) \subset \Lambda\) it follows that \(\text{supp}(\mu) \subset \Lambda'\), so that \(\mu\) is a hyperbolic measure with splitting induced by the splitting on \(\Lambda'\). Moreover, this hyperbolic splitting induces a dominated splitting \(E \oplus F\) on \(\text{supp}(\mu)\) whose index is \(\dim E\). So, since \(\Lambda\) is attracting, it follows from Theorem 4.1 in [16] that \(\Lambda\) contains one periodic orbit.

Next, we are ready to prove Theorem 2.3.

Proof of Theorem 2.3. Let \(z \in \Lambda\) such that \(\omega(z) = \Lambda\). By hypothesis, there is a sequence of periodic orbits \(p_n\), with period \(\tau(p_n)\), such that \(p_n \to z\). In particular, we have \(\tau(p_n) \to \infty\). So, by the hyperbolic lemma we can to find \(N \in \mathbb{N}\) large enough such that \(p_n \sim p_m\) for \(n, m \geq N\). This shows that \(z \in H(p_N)\) since \(H(p_N)\) is closed. Therefore, the result is obtained by denseness of the orbit of \(z\).
4. Entropy Expansiveness

Next we proceed with the proof of Theorem 2.5. Our proof is essentially based in three results. The first two are elementary facts from flow theory and we present the proofs by the sake of completeness.

**Lemma 4.1.** Let \( \{X_t\} \) be a continuous flow on a compact metric space \( M \). For every \( \alpha > 0 \), there exists \( \beta > 0 \) such that if \( y \in B^\beta_\alpha(x, X_1) \), then

\[
d(X_t(x), X_t(y)) \leq \alpha
\]

**Proof.** Fix \( \alpha > 0 \). Since \( M \) is compact and \( X_t \) is a continuous flow, we can find \( \beta > 0 \) such that if \( d(x, y) \leq \beta \), then \( d(X_t(x), X_t(y)) \leq \alpha \), for \( t \in [0, 1] \). Take \( y \in B^\beta_\alpha(x, X_1) \) and fix \( t \in \mathbb{R} \). One can write \( t = n_t + r_t \) with \( n_t \in \mathbb{Z} \) and \( 0 \leq r_t \leq 1 \). By hypothesis we have \( d(X_{n_t}(x), X_{n_t}(y)) \leq \beta \) and therefore

\[
d(X_t(x), X_t(y)) = d(X_{n_t+r_t}(x), X_{n_t+r_t}(y)) = d(X_{r_t}(X_{n_t}(x)), X_{r_t}(X_{n_t}(y))) \leq \alpha,
\]

and this concludes the proof. \( \square \)

If one denotes

\[
B^\alpha_\beta(x, X) = \{ y \in M; d(X_t(x), X_t(y)) \leq \alpha, \forall t \in \mathbb{R} \},
\]

then the previous lemma says that \( B^\beta_\alpha(x, X_1) \subset B^\alpha_\beta(x, X) \), if \( \beta > 0 \) is small enough.

**Lemma 4.2.** Let \( \{X_t\}_{t \in \mathbb{R}} \) be a continuous flows on a compact metric space \( M \). Then, \( h_{\text{top}}(X_{[-\varepsilon,\varepsilon]}(x)) = 0 \) for every \( \varepsilon > 0 \) and \( x \in M \).

**Proof.** First, we claim that for every \( \eta > 0 \) there is \( \varepsilon_0 > 0 \) such that if \( y \in X_{[-\varepsilon_0,\varepsilon_0]}(x) \), then \( d(X_1(x), X_1(y)) \leq \eta \), for any \( t \in \mathbb{R} \). Indeed, otherwise there is \( \eta > 0 \) and sequences \( x_n \in M \), \( t_n \in \mathbb{R} \) and \( s_n \in \mathbb{R} \) with \( s_n \to 0 \) such that

\[
d(X_{t_n}(x_n), X_{t_n+s_n}(x_n)) > \eta, \quad \forall n \geq 1.
\]

By compactness of \( M \) and the continuity of the flow there is \( s > 0 \) such that

\[
d(X_t(x), x) < \eta \quad \text{for any} \quad x \in M \quad \text{and} \quad |t| < s.
\]

So, by (4.1)

\[
\eta > d(X_{t_n+s_n}(x_n), X_{t_n(x_n)}) = d(X_{s_n+t_n}(x_n), X_{t_n}(x_n)) \geq \eta,
\]

which is a contradiction for \( n \) enough large.

Now, we fix \( \eta > 0 \). By the above claim, there is \( \varepsilon_0 > 0 \) such that

\[
d(X_t(x), X_t(y)) \leq \eta, \quad \forall t \in \mathbb{R},
\]

for every \( x \in M \) and \( y \in X_{[-\varepsilon_0,\varepsilon_0]}(x) \). So, if \( \varepsilon_0 > \varepsilon \), we have that \( S(t, \eta) = 1 \), where \( S(t, \eta) \) denotes the minimum cardinality of a \( (t, \eta) \)-spanning set, whereas if \( \varepsilon_0 < \varepsilon \), there is \( N \in \mathbb{N} \) such that \( N\varepsilon_0 > \varepsilon \), so that \( S(t, \eta) \leq N \). Then, we have the desired result by definition of topological entropy. \( \square \)

In light of above lemmas, in order to prove Theorem 2.5 we need the following result:

**Theorem 4.3.** Every asymptotically sectional-hyperbolic attractor \( \Lambda \) associated to \( C^1 \) vector field \( X \) on a three-dimensional manifold \( M \) is kinematic expansive, i.e., for every \( \varepsilon > 0 \), there is \( \delta > 0 \) such that if \( x, y \in \Lambda \) satisfy

\[
d(X_t(x), X_t(y)) \leq \delta \quad \forall t \in \mathbb{R},
\]

then \( y \in X_{[-\varepsilon,\varepsilon]}(x) \).
Remark 4.1. Note that kinematic expansiveness is a weaker form of expansiveness, since it does not care about reparametrizations. For a more detailed discussion about properties of kinematic expansive flows we refer the reader to the work [4].

We explain why these results implies Theorem 2.5.

Proof of Theorem 2.5. Fix $\eta > 0$. By Theorem 4.3 there is $\varepsilon > 0$ such that $B_\varepsilon^\infty(x) \subset X_{[-\eta,\eta]}(x)$ for any $x \in M$. Let $\beta > 0$ be as in Lemma 4.1 with respect to $\varepsilon$. Therefore, for every $x \in M$ one has

$$B_\beta^\infty(x_1) \subset B_\beta^\infty(x, X) \subset X_{[-\eta,\eta]}(x)$$

and the proof follows from Lemma 4.2.□

From now on we will devote to obtain a proof of Theorem 4.3. Many parts of the arguments presented here resembles that of the proof of Theorem 2.5 in [17]. First, we state some known results that will be used in the proof. We begin by considering a special kind of neighborhoods of the singularities contained in an ASH attractor. Let $\Lambda$ be an ASH attractor on a three-dimensional manifold $M$. Let $x \in \Lambda$ and let $\Sigma'$ be a cross-section to $X$ containing $x$ in its interior. Define $W_s(x, \Sigma')$ as the connected component of $W_s(x) \cap \Sigma'$.

This gives us a foliation $F_{\Sigma'}$ of $\Sigma'$. We can construct a smaller cross section $\Sigma$, which is the image of a diffeomorphism $h: [-1, 1] \times [-1, 1] \to \Sigma'$, that sends vertical lines inside $F_{\Sigma'}$ in a such way that $x$ belongs to the interior of $h([-1, 1] \times [-1, 1])$. In this case, the $s$-boundary $\partial^s \Sigma$ and $cu$-boundary $\partial^{cu} \Sigma$ of $\Sigma$ are defined by

$$\partial^s \Sigma = h([-1, 1] \times [-1, 1]) \quad \text{and} \quad \partial^{cu} \Sigma = h([-1, 1] \times \{-1, 1\})$$

respectively. We say that $\Sigma$ is $\eta$-adapted if

$$d(\Lambda \cap \Sigma, \partial^{cu} \Sigma) > \eta.$$

A consequence of the hyperbolic lemma (see [20]) is the following:

Proposition 4.1. Let $\Lambda$ be an asymptotically sectional-hyperbolic attractor and let $x \in \Lambda$ be a regular point. Then, there exists a $\eta_0$-adapted cross-section $\Sigma$ at $x$ for some $\eta_0 > 0$.

Remark 4.2. From any $x \in M$ and any cross-section containing $x$ in its interior, one can obtain an $\eta_0$-adapted cross-section containing $x$ in its interior.

Next we recall the construction performed in [21] of partitions for singular cross sections. These partitions give us a very detailed picture of the flow dynamics inside small neighborhoods of the singularities of $\Lambda$. According to that reference we can find $\beta_1 > 0$ such that:

(a) $B_{\beta_1}(\sigma) \cap B_{\beta_1}(\sigma') = \emptyset$, where $B_r(a)$ denotes the open ball centered in $a$ and radius $r > 0$ and $\sigma, \sigma' \in Sing_\Lambda(X) = Sing(X) \cap \Lambda$.

(b) The map $exp_{\sigma}$ is well defined on $\{v \in TM_{\sigma} : \|v\| \leq \beta_1\}$ for every $\sigma \in Sing_\Lambda(X)$.

(c) There are $L_0, L_1 > 0$ such that

$$L_0 \leq \frac{\|X(x)\|}{d(x, \sigma)} \leq L_1, \quad \forall x \in B_{\beta_1}(\sigma), \quad \forall \sigma \in Sing_\Lambda(X).$$

(d) The flow in $B_{\beta_1}(\sigma)$ is a small $C^1$ perturbation of the linear flow.
For every $\sigma \in \text{Sing}_\Lambda(X)$ define the singular cross-section

$$D_\sigma = \exp_\sigma \{ v = (v^s, v^u) \in TM_\sigma : ||v|| \leq \beta_1, ||v^s|| = ||v^u|| \} \subset M,$$

and the following partition of $D_\sigma$

$$D_n = D_\sigma \cap (B_{e^{-n}}(\sigma) \setminus B_{e^{-(n+1)}}(\sigma)), \quad \forall n \geq n_0,$$

where $n_0$ is large enough. As noticed by the authors in [21], the partition $\{D_n\}_{n \geq n_0}$ induces a partition of the cross sections $\Sigma_{i,o,\pm}^{\sigma_i,\pm}$ given in [2]. More precisely, assume

$$D^+_n = \bigcup_{x \in D_n} X_{t^+_x}(x), \quad D^-_n = \bigcup_{x \in D_n} X_{t^-_x}(x), \quad \forall n \geq n_0,$$

where

$$t^+_x = \inf\{ \tau > 0 : X_\tau(x) \in \Sigma^{o,\pm}_\sigma \},$$

and

$$t^-_x = \inf\{ \tau > 0 : X_{-\tau}(x) \in \Sigma^{i,\pm}_\sigma \}.$$
• We can assume without loss of generality that every cross section in \( \Sigma_{\hat{\beta}}^{\pm} \) is \( \eta_0 \) adapted for some \( \eta_0 > 0 \).
• The above construction can be made by taking \( \hat{\beta} < \beta_1 \). In this case, we denote
\[
\Sigma_{\hat{\beta}}^{\pm} = \bigcup_{\sigma \in \text{Sing}_{\Lambda}(X)} \Sigma_{\sigma,\hat{\beta}}^{\pm} \quad \text{and} \quad V_{\hat{\beta}} = \bigcup_{\sigma \in \text{Sing}_{\Lambda}(X)} V_{\sigma,\hat{\beta}}.
\]

Before to keep going with the proof, we will make a little break to give a brief outline of the next lemmas in order to make our argument more clear. Our main goal is to obtain some sort of hyperbolicity for Poincaré maps from the ASH property.

In [2] this property is obtained from the uniform area expansion property of the sectional hyperbolicity. Nevertheless, for ASH attractors we only can see area expansion on the center bundle during hyperbolic times.

In what follows, we proceed to prove that if two points satisfy the shadowing condition given in the definition of kinematic expansiveness, then they share the same hyperbolic times. To do this, we need some previous lemmas that will help us to control the hyperbolic times of a pair of points that are close, depending on where they are located in the phase space. Precisely, Lemma 4.4 controls the hyperbolic times of points away from singularities, Lemma 4.5 controls the hyperbolic times of points crossing a neighborhood of a singularity and Lemma 4.8 helps us to control the hyperbolic times of points crossing a neighborhood of a singularity.

First, denote \( \Lambda_+ = \bigcap_{t \geq 0} \chi^\ast_t(\Lambda \setminus V) \). Then, we have the following lemma.

**Lemma 4.4** (Lemma 3.2 in [17]). Given \( \varepsilon_0 > 0 \), there are positive numbers \( \delta_1(\varepsilon_0), T_0 \) and \( K_2(\varepsilon_0) \approx 1 \), and a neighborhood \( W_0 \) of \( \Lambda_+ \) such that for any \( x, y \in W_0 \) with \( d(x,y) < \delta_1(\varepsilon_0) \), then

\[
\frac{|\det DX_{t_1}(y)|_{E_x^c}|}{|\det DX_{t_1}(x)|_{E_x^c}|} \geq K_2(\varepsilon_0), \quad 0 < t_1 \leq T_0,
\]

where \( t_1 \leq T_0 \) is a first hyperbolic time for \( x \) and \( y \).

Let us consider the following compact set \( \Lambda'' = \Lambda \setminus (V \cup W_0) \), where \( W_0 \) is as the above lemma.

**Lemma 4.5** (Lemma 3.3 in [17]). Given \( \varepsilon_1 > 0 \), there are positive numbers \( \delta_2(\varepsilon_1), T_1 \) and \( K_3(\varepsilon_1) \approx 1 \), and a neighborhood \( W_1 \) of \( \Lambda'' \) such that for every \( x, y \in W_1 \), with \( d(x,y) \leq \delta_2(\varepsilon_1) \), there is \( 0 < s \leq T_1 \) such that \( X_s(x), X_s(y) \in V \) and

\[
\frac{|\det DX_s(y)|_{E_x^c}|}{|\det DX_s(x)|_{E_x^c}|} \geq K_3(\varepsilon_1).
\]

We need, in addition the following result

**Lemma 4.6** (Lemma 3.4 in [17]). Let \( \tilde{U} = \bigcup_{\sigma \in \text{Sing}_{\Lambda}(X)} V \), where \( U \) is the basin of attraction of \( \Lambda \). There are \( \beta' > 0 \) and \( \varepsilon_2 > 0 \) such that if \( x \in \tilde{U} \) and \( y, z \in B_{\varepsilon_2}(x) \), \( z \in O(y) \), then \( z = X_u(y) \), \( |u| < 2\beta' \).

**Lemma 4.7** (Lemma 3.5 in [17]). There exists \( \delta_3 > 0 \) with the following property: For \( x \in \Sigma_{\beta}^{\pm}, \sigma \in \text{Sing}_{\Lambda}(X) \) and \( z \in M \) with \( d(x,z) < \delta_3 \) exist \( l \in \mathbb{R} \), with \( |l| < L \), such that \( X_l(z) \in \Sigma_{\beta}^{\pm} \), where \( L = \frac{\Lambda_+ + 1}{\Lambda_+} \).
Next lemma is contained in the proof of Theorem 2.5 in [17], for the reader convenience we state it here separately and provide a proof.

**Lemma 4.8.** Let $\varepsilon_0 > 0$. There are a positive number $\delta_0$, independent of $\varepsilon_0$, and a positive number $\bar{\beta}$ such that if $x, y \in \Sigma^{i, \pm}_\beta$ satisfy
\[
d(X_t(x), X_t(y)) \leq \delta_0
\]
for every $t \in \mathbb{R}$, then
\[
\frac{|\det DX_t(y)|_{E_\varepsilon}|}{|\det DX_t(x)|_{E_\varepsilon}|} \geq K_1C^\sigma_0(x),
\]
where $C_0 = C_0(\varepsilon_0)$ and $K_1$ is a fixed positive constant.

**Proof.** Let $\sigma$ be an attached singularity of $\Lambda$. First recall that since $\sigma$ is hyperbolic, we can use the Grobman-Hartman Theorem to conjugate $X_t$ with its linear part in a small neighborhood of $\sigma$. More precisely, there are a neighborhood $U_\sigma$ of $\sigma$, a neighborhood $U_0$ of $0 \in T_\sigma M$ and a homeomorphism $h : U_\sigma \to U_0$ that conjugates $X_t$ with its linear flow $L_t$. Furthermore, this homeomorphism can be chosen to be Hölder continuous (see [5] for more details). Thus there are $C = C(\sigma) > 0$ and $\alpha(\sigma) > 0$ such that
\[
|h(x) - h(y)| \leq C d(x, y)^\alpha, \quad \forall x, y \in U_\sigma.
\]
Since we have only finitely many singularities in $\Lambda$, we can shrink $U_0$, if it is necessary, to obtain uniform Hölder constants for every singularity, i.e., denoting
\[
C = \max_{\sigma \in \text{Sing}(X) \cap \Lambda}\{C(\sigma)\} \quad \text{and} \quad \alpha = \min_{\sigma \in \text{Sing}(X) \cap \Lambda}\{\alpha(\sigma)\}.
\]

Now let us consider
\[
W^{ss}(0) = \{(a, b, c) : b = c = 0\}, \quad W^s(0) = \{(a, b, c) : b = 0\}
\]
and
\[
W^u(0) = \{(a, b, c) : a = c = 0\}.
\]
inside of the neighborhood $U_0$. We can assume that $V_\sigma \subset U_\sigma$. In particular, we have that $\Sigma^{i, \pm}_{\sigma} \subset U_\sigma$ for every $\sigma \in \Lambda$.

Since $h$ is a homeomorphism, it induces a partition of $h(\Sigma^{i, \pm}_{\sigma}) \subset U_0$. Indeed, we just need to consider the family
\[
\mathcal{F} = \{h(D_n^i \cap \Sigma^{i, \pm}_{\sigma})\}_{n \geq n_0}.
\]

By the remarks in page 380 of [21], there is $K' > 0$ such that for every $x \in D_n^i \cap \Sigma^{i, \pm}_{\sigma}$, we have
\[
d(x, W^s(\sigma)) \leq K' e^{-(\lambda_u + 1)n}.
\]
On the other hand, since $h(W^s(\sigma)) = W^s(0)$, the holder estimates of $h$ gives us
\[
d(h(x), W^s(0)) \leq C d(x, W^s(\sigma))^\alpha
\]
\[
\leq CK' e^{-\alpha(1 + \lambda_u)n}
\]
\[
= e^{\left(\ln(CK') - \alpha(1 + \lambda_u)\right)n}.
\]

Let $0 < \varepsilon_0 < \alpha(1 + \lambda_u)$ and $n_0 \geq 1$ such that
\[
|\ln(CK')/n| < \varepsilon_0, \forall n \geq n_0.
\]
Then
\[ d(h(x), W^s(0)) \leq e^{\rho n}, \quad \forall x \in D^i_n \cap \Sigma^i_{\sigma^t}, \]
where \( \rho = \varepsilon_0 - \alpha(1 + \lambda_u) < 0 \). In fact, since \( F \) is a partition of \( h(\Sigma^i_{\sigma^t}) \) we deduce that
\[ e^{\rho(n+1)} \leq d(h(x), W^s(0)) \leq e^{\rho n}, \quad \forall x \in D^i_n \cap \Sigma^i_{\sigma^t}, \forall n \geq n_0. \]
Therefore, if
\[ h(\Sigma^i_{\sigma^t}) = \Sigma^i_0 = \{ p = (\pm 1, b, c) : |b|, |c| < 1 \} \subset U_0, \]
we have by (4.8) that the flight time \( \tau(p) \), \( p \in h(\Sigma^i_{\sigma^t}) \), to go from \( h(\Sigma^i_{\sigma^t}) \) to \( \Sigma^i_0 \) satisfies
\[ -\frac{\rho}{\lambda_u} n \leq \tau(p) \leq -\frac{\rho}{\lambda_u} (n + 1). \]

Since \( h \) conjugates \( X_t \) and \( L_\sigma \), by applying Lemma 4.6 we obtain that for every \( \beta > 0 \) there is \( \varepsilon > 0 \) such that if \( v \in V_0 \) and \( w \in h(\Sigma^i_{\sigma^t}) \) satisfy \( \| v - w \| < \varepsilon \), then \( L_u(v) \in h(\Sigma^i_{\sigma^t}) \), with \( u \in (-\beta, \beta) \).

Take a compact neighborhood \( C' \) of \( V_0 \) inside \( U_0 \). By uniform continuity of \( h \) on \( h^{-1}(C') \), there is \( \delta > 0 \) such that \( \| h(z) - h(w) \| < \varepsilon \) if \( z, w \in h^{-1}(C') \) satisfies \( d(z, w) < \delta \).

Now observe that by uniform continuity of \( DX_t(\cdot) \) and \( E^c \) on \( \mathcal{U} \) there exists \( 0 < \delta_0 < \delta \) such that
\[ d(x, y) < \delta_0, \text{ then } \frac{|\det DX_t(y)|_{E^c}}{|\det DX_t(x)|_{E^c}} \geq C_0 \approx 1, \]
for some \( \sigma \in \text{Sing}(X) \cap \Lambda \) and \( n \geq n_0 \). Then there is \( s \in \mathbb{R} \) such that \( y' = X_s(y) \in \Sigma^i_{\sigma^t} \). Moreover, both points \( y' \) and \( X_t(x) \) belongs to the same connected component of \( \Sigma^i_{\sigma^t} \) \( \setminus \ell_{\pm} \).

**Claim:** \( y' \in \left(D^i_{n-1} \cap \Sigma^i_{\sigma^t}\right) \cup \left(D^i_{n} \cap \Sigma^i_{\sigma^t}\right) \cup \left(D^i_{n+1} \cap \Sigma^i_{\sigma^t}\right) \).

Indeed, assume that \( y' \in D^i_{n+k} \cap \Sigma^i_{\sigma^t} \) for some \( k > 1 \). By the choice of \( \delta \) there is \( u \in (-\beta, \beta) \) such that
\[ h(y') = (a_0, b_0, c_0) = (e^{\lambda_u t} a, e^{\lambda_u b}, e^{\lambda_u c}) \in h(D^i_{n+k} \cap \Sigma^i_{\sigma^t}). \]

Moreover, assume that \( a, c > 0 \). Since \( h(x) \in D^i_{n} \cap \Sigma^i_{\sigma^t} \) we have by (4.8) and (4.9) that
\[ \varepsilon > \| L_{e(h(x))}(h(x)) - L_{e(h(x))}(h(y)) \| \geq 1 - e^{\lambda_u \tau(h(x))} a \]
\[ = 1 - e^{-\lambda_u \varepsilon} e^{\lambda_u \tau(h(x))} a_0 \]
\[ \geq 1 - e^{\rho + \lambda_u \beta}, \]
which contradicts the choice of \( \varepsilon \) and the claim is proved. So, by the above claim we have that
Let $n_1 \geq 0$ be the largest integer such that $X_{n_1}(y) \in V_\alpha$. Thus we have $\tau(y) = n_1 + r_y$ with $0 \leq r_y \leq 1$ and $\tau(x) = n_1 + r_x$ with $|r_x| \leq L + 1$. Therefore, by the chain rule, we have

$$\frac{|\tau(x) - \tau(y)|}{\lambda u + 1} = L.$$

Once we have the previous lemmas, we can easily adapt the proof of lemma 3.6 in [7] to obtain the desired result.

**Lemma 4.9.** There exist positive numbers $\delta_4$, $T$, $c_*$ such that if $x \in \Lambda'$, and $y \in U$ satisfy

$$d(X_t(x), X_t(y)) \leq \delta_4, \quad \forall t \in \mathbb{R},$$

and given a $C$-hyperbolic time $t_x \geq T$, we have

$$|\det DX_{t_x}(y)| \geq c_2 \epsilon^t_x.$$  

**Proof.** Let $\tilde{\epsilon}_0, \epsilon_0, \epsilon_1$ be given by the previous lemmas and let $c^*, \alpha > 0$ satisfying

$$C - (\alpha + |\ln(C_0)| + |\ln K_2(\epsilon_0)| + |\ln K_3(\epsilon_1)|) > c^* > 0.$$

Since $C_0 \approx 1$, $K_2(\epsilon_0) \approx 1$, $K_3(\epsilon_1) \approx 1$, it is possible to choose such $c^*$. The choice of $\alpha$ will be specified later. Let $\delta_0, \delta_1(\epsilon_0), \delta_2(\epsilon_1)$ and $\delta_3$ be given by Lemmas 4.8, 4.4 and 4.7 respectively. Let consider

$$0 < \delta_4 < \min \{\delta_0, \delta_1(\epsilon_0), \delta_2(\epsilon_1), \delta_3, \delta',\},$$

where $\delta'$ is the Lebesgue’s number of the open cover $V; W_0, W_1$ of $\Lambda$ and fix $T = \max\{T_0, T_1\}$.

Let $x \in \Lambda'$ and $y \in U$ be two points satisfying the condition given the statement of lemma. Next, we will take a sequence of points $x_n$ in the orbit of $x$ which is contained in the union of

$$((W_0 \cup W_1) \setminus V) \cup \Sigma_{\tilde{\beta}}^\pm.$$  

Let $s_0 \geq 0$ (if it exists) be the first time satisfying

$$x_0 = X_{s_0}(x) \in \Sigma_{\tilde{\beta}}^\pm,$$

where $\tilde{\beta}$ is given by lemma 4.8. Then, define

$$x_1 = X_{s_0 + \tau(x_0)}(x)$$

and note that $x_1 \in (W_0 \cup W_1) \setminus V$. Now we split the definition of $x_2$ in two cases:

1. If $x_1 \in W_1 \setminus V$, then Lemma 4.5 implies the existence of $0 < s < T$ such that $X_s(x) \in V$, and therefore we can take $0 < s_1 < s$ such that $X_{s_1}(x_1) \in \Sigma_{\tilde{\beta}}^\pm$. In this case, define $x_2 = X_{s_1}(x_1)$.
(2) If \( x_1 \in W_0 \setminus V \), then we consider \( 0 < t_1 \leq T \) given by Lemma 4.4 and define \( x_2 = X_t(x_1) \), if \( X_t(x_1) \notin V \) or \( x_2 = X_{t_1}(x_1) \), if \( X_{t_1}(x_1) \in V \) and \( 0 < r_1 < t_1 \) is such that

\[ X_{r_1}(x_1) \in \Sigma_{i}^{i} \].

Proceeding inductively for \( n \geq 3 \) we define

\[
X_n = \begin{cases} 
X_{t_{n-1}}(x_{n-1}) & \text{if } x_{n-1} \in \Sigma^{i}_{i}, \\
X_{s_{n-1}}(x_{n-1}) & \text{if } x_{n-1} \in W_1 \setminus V, \\
X_{t_{n-1}}(x_{n-1}) & \text{if } x_{n-1} \in W_0 \setminus V \text{ and } X_{t_{n-1}}(x_{n-1}) \notin V, \\
X_{r_{n-1}}(x_{n-1}) & \text{if } x_{n-1} \in W_0 \setminus V \text{ and } X_{r_{n-1}}(x_{n-1}) \in V.
\end{cases}
\]

For a fixed \( n \geq 1 \), let us consider the following sets

- \( O_n = \{0 \leq i \leq n : x_i \in \Sigma^{i}_{i}\} \), \( n_O = \#O_n \),
- \( A_n = \{0 \leq i \leq n : x_i \in W_1\} \), \( n_A = \#A_n \),
- \( B_n = \{0 \leq i \leq n : x_i \in W_0 \text{ and } X_{t_i}(x_i) \notin V\} \), \( n_B = \#B_n \),
- \( C_n = \{0 \leq i \leq n : x_i \in W_0 \text{ and } X_{r_i}(x_i) \in V\} \) and \( n_C = \#C_n \).

Note that for every \( x_n \), there exists \( t'(n) > 0 \) such that \( x_n = X_{t'(n)}(x) \) where

\[
t'(n) = s_0 + \sum_{i \in A_n} s_i + \sum_{i \in B_n} t_i + \sum_{i \in C_n} r_i + \sum_{i \in O_n} \tau(x_i).
\]

If \( y \) is such that

\[
d(X_t(x), X_t(y)) \leq \delta_4, \forall t \in \mathbb{R},
\]

define \( y_n = X_{t'(n)}(y) \).

Consider \( x_n \in \Sigma_{i}^{i} \). By hypothesis we have that

\[
d(x_n, y_n) \leq \delta_4 \text{ and } d(X_t(x_n), X_t(y_n)) \leq \delta_4.
\]

By using twice the lemma 3.5, there exists \( l_0, l_1 \in (-L, L) \) such that

\[
y' = X_{l_0}(y_n) \in \Sigma^{i}_{i} \text{ and } X_{t(x_n) + l_1}(y_n) \in \Sigma_{i}^{i}.
\]

Thus,

\[
\tau(x_n) = \tau(y') + l_0 - l_1,
\]

and by (4.16) and lemma (4.8) we have that

\[
\det DX_{t(x_n)}(y_n)|_{E_{y_n}} \geq \det DX_{-l_0}(X_{t(y')} + l_0(y_n))|_{E_{x_{t(y')} + l_0(y_n)}} \geq K' \det DX_{t(y')}|_{E_{y'}} \det DX_{l_0}(y_n)|_{E_{y_n}} \geq K' C_0 \det DX_{t(x_n)}(x_n)|_{E_{x_n}},
\]

where

\[
K' = K_0^2 K_1 \text{ and } K_0 = \min_{(z,a) \in U \times [-L, L]} |\det DX_{t}(z)|_{E_{z}}|.
\]

Let \( (t_k)_{k \geq 1} \) be an unbounded and increasing sequence of \( C \)-hyperbolic times for \( x \). Up to a slight change of \( C \), we can assume that \( X_{t_k}(x) \notin V \). Thus, for every \( k \geq 1 \)
one can write $t_k = t'_k + u_k$, with $u_k \in [0, T)$ and $t'_k$ of the form \((4.14)\). Let us denote $\varphi_t(z) = |\det DX_t(z)|_{E_\gamma}^\varphi$. By the relations \((4.4)\), \((4.5)\) and \((4.17)\),

$$|\det DX_{t_x}(y)|_{E_\gamma}^\varphi = |\det DX_{t_x}(X_{t'_x}(y))|_{E_\gamma}^\varphi |\det DX_{t'_x}(y)|_{E_\gamma}^\varphi \geq K_0 \prod_{i \in O_0} \varphi_{t_x}(y_i) \prod_{i \in A_0} \varphi_{s_i}(y_i) \prod_{i \in B_0} \varphi_{t'_x}(y_i) \prod_{i \in C_0} \varphi_{t_x}(y_i),$$

where

$$K = \frac{|\det DX_{s_0}(y)|_{E_\gamma}^\varphi}{|\det DX_{s_0}(x)|_{E_\gamma}^\varphi} \text{ and } K_5 = \max_{(z, s) \in \partial \times [0, T]} |\det DX_s(z)|_{E_\gamma}^\varphi.$$

Since $t_x$ is a $C$-hyperbolic time for $x$, by the above estimate we have that

$$|\det DX_{t_x}(y)|_{E_\gamma}^\varphi \geq K e^{(C+N(C_0)+N(K'+K)+N(K_2)+N(K_3)+N(K_0,K_5))t_x},$$

where

- $N(C_0) = \sum_{i \in O_0} \tau(x_i) \ln(C_0)$,
- $N(K') = \frac{n_0}{t_x} \ln(K')$,
- $N(K_2) = \frac{n_0}{t_x} \ln(K_2(\varepsilon_0))$,
- $N(K_3) = \frac{n_0}{t_x} \ln(K_3(\varepsilon_0))$ and
- $N(K_0, K_5) = \frac{n_0}{t_x} \ln\left(\frac{K_0}{K_5}\right)$.

Now, since $t_x > \sum_{i \in O_0} \tau(x_i)$ it follows that

$$|N(C_0)| \leq |\ln C_0|.$$ 

Furthermore, since $n_C + 1 \leq n_0$ by the construction of the sequence $x_n$ we have

$$\frac{n_0}{t_x} \ln K' + \frac{n_C + 1}{t_x} \ln K_0 \leq \frac{n_0}{t_x} \ln K' + \ln K_0 \leq \frac{|\ln K'| + |\ln K_0|}{n_0}.$$ 

Then, by shrinking $V_\delta$ if it is necessary, we have

$$\frac{n_0}{t_x} \ln K' + \frac{n_C + 1}{t_x} \ln K_0 \leq \alpha.$$ 

On the other hand, since $\frac{(n_A+n_B+n_C+n_0)}{t_x} \leq 1$, we obtain

$$|N(K_2)| \leq |\ln K_2(\varepsilon_0)| \text{ and } |N(K_3)| \leq |\ln K_3(\varepsilon_1)|,$$

and by \((4.18)\) and \((4.13)\) we have

$$|\det DX_{t_x}(y)|_{E_\gamma}^\varphi \geq K e^{(C-|\ln C_0|+|\ln K_2(\varepsilon_0)|+|\ln K_3(\varepsilon_1)|)t_x} \geq K e^{c_x t_x}.$$

Finally, there exist $0 < c_s < e^* \text{ and } T > 0$ such that $Ke^{c_s t} \geq e^{c_s t}$ for all $t \geq T$. Therefore, for every $y$ satisfying \((4.15)\) and every $C$-hyperbolic time $t_x \geq T$ we have

$$|\det DX_{t_x}(y)|_{E_\gamma}^\varphi \geq Ke^{c_x t_x} \geq e^{c_x t_x},$$

and this concludes the proof. \(\square\)
Next, we can finally give the proof of the kinematic expansiveness of $\Lambda$. The proof that we will present here is very similar to the proof of Theorem A of \cite{2}, in a sense that most of the steps given in that work can be directly followed here. The only exception is the Theorem 3.1 of \cite{2} which cannot be replicated directly to our context. Hence, we will only briefly explain the steps of the proof which are analogous to the proof of Theorem A in \cite{2} and we will be mainly focused in to give a detailed proof of a version of the Theorem 3.1 in \cite{2} to our setting.

**Proof of Theorem 4.3.** We begin supposing that $X$ is not kinematic expansive on an ASH attractor $\Lambda$. Then, there are $\varepsilon > 0$, $x_n, y_n \in \Lambda$ and $\delta_n \to 0$ such that $y_n \notin X_{[-\varepsilon, \varepsilon]}(x_n)$ and

$$d(X_t(x_n), X_t(y_n)) \leq \delta_n \text{ for every } t \in \mathbb{R}. \quad (4.19)$$

Following the arguments in \cite{2} we can find a regular point $z \in \Lambda$ and $z_n \in \omega(x_n)$ such that $z_n \to z$. Let us consider $\Sigma_\eta$ an $\eta$-adapted cross-section trough $z$. By using flow boxes in a small neighborhood of $\Sigma_\eta \cup \Sigma_i,o,$ we can find positive numbers $\delta', t_0$ such that for any $\Sigma' \subset \Sigma_\eta \cup \Sigma_i,o,$ $z \in \Sigma'$ and $w \in M$ with $d(z, w) < \delta'$, there is $t_w \leq t_0$ such that $w' = X_{t_w}(w) \in \Sigma'$ and $d_{\Sigma'}(z, w') < K\delta'$, where $d_{\Sigma'}$ is the intrinsic distance in $\Sigma'$, for some constant positive constant $K'$ which depends on $\Sigma_\eta \cup \Sigma_i,o.$

Let $N > 0$ be large enough such that

$$0 < \delta_N < \min \{\delta_4, \eta, \eta_0, \delta'\},$$

where $\eta_0$ and $\delta_4$ are given by Remark 4.2, Lemma 4.9 respectively. Let us denote $x = x_N$ and $y = y_N$. The next claim is our version of the Theorem 3.1 in \cite{2}.

**Claim:** There is $s \in \mathbb{R}$ such that $X_s(y) \in W^s_\varepsilon(X_{[s-\varepsilon, s+\varepsilon]}(x))$.

Once the claim holds, by following step by step the proof of Theorem A in \cite{2} one can conclude the proof of Theorem 4.2. Because of this, we now devote to prove the previous claim. By construction, the orbit of $x$ must intersect $\Sigma_\eta$ in infinitely many positive times $t_j$. Let us denote $x_j = X_{t_j}(x)$. Thus we can find a sequence of times $s_j$ close to $t_j$ such that $y_j = X_{s_j}(y)$ also intersect $\Sigma_\eta$. Now we briefly recall the construction of the tube-like domain presented in \cite{2}. For any $j \geq 0$ we can find a smooth immersion

$$\rho^j : [0, 1] \times [0, 1] \to M$$

such that the following holds:

1. $\rho^j([0, 1] \times \{0\})$ is the orbit arc from $x_j$ to $x_{j+1}$ and $\rho^j([0, 1] \times \{1\})$ is the orbit arc from $y_j$ to $y_{j+1}$ and
2. $\rho^j(\{0\} \times [0, 1])$ is a curve contained in $\Sigma_\eta$, everywhere tranverse to $W^s(\Sigma_\eta)$ and joining $x_j$ and $y_j$.
3. $\rho^j(\{1\} \times [0, 1])$ is a curve contained in $\Sigma_\eta$, everywhere tranverse to $W^s(\Sigma_\eta)$ and joining $x_{j+1}$ and $y_{j+1}$.
4. Denote $S_j = \rho^j([0, 1] \times [0, 1])$. Then the intersection of $S_j$ with any $\Sigma_i,o,$ is transverse to stable foliation of $\Sigma_i,o.$

Denote

$$T_j = \bigcup_{p \in S_j} W^s_{loc}(p).$$
Then, by the results in [2] one can see that \( T_j \) does not contain singularities. Moreover, if \( T_j \) intersects some \( \Sigma_{i,o,\pm} \), this intersection is totally contained in the same connected component of \( \Sigma_{i,o,\pm} \setminus W^s(\sigma) \). Finally, in [2] it is also showed the existence of a Poincaré map \( R_j \) between the whole strip between the stable manifolds of \( x_j \) and \( y_j \) inside \( \Sigma_\eta \). Unfortunately, we cannot guarantee that these Poincaré maps are hyperbolic in the same way as in [2] since we are in the ASH setting. Nevertheless, we are going to obtain some expansion for these maps using the previous lemmas. To prove the claim we first assume that

\[
X_s(y) \notin W^s_\varepsilon(X_{[s-\varepsilon,s+\varepsilon]}(x)),
\]

for every \( s \in \mathbb{R} \). In this case, we have \( y \notin \mathcal{O}(x) \), otherwise by Lemma 4.6 and Remark 4.3 we obtain that \( y \) belongs to \( X_{[s-\varepsilon,s+\varepsilon]}(x) \) and this is a contradiction.

In addition, unless to take a subsequence of \( x_j \) we can find a sequence of arbitrarily large \( \mathcal{C} \)-hyperbolic times \( \{t_j\}_{j \geq 1} \) of \( x \) such that \( x_j = X_{t_j+r_j}(x) \in \Sigma_\eta \), where \( 0 \leq r_j < T \), with

\[
T = \max\{T_0, T_1\}.
\]

Besides, we have that \( y_j = X_{t_j+r_j+v_j}(y) \in \Sigma_\eta \), where \( |v_j| \leq t_0 \). Moreover, by shrinking \( U \) if it is necessary, by Lemma 2.7 in [2] there is \( \kappa_0 > 0 \) such that

\[
(4.20)
\ell(\gamma_n) \leq \kappa_0 d_{\Sigma_\eta}(x_j,y_j) \leq \kappa_0 K'\delta',
\]

where \( \gamma_j \) is any curve joining \( x_j \) and \( y_j \), for every \( n \geq 1 \). In particular this holds for the curves

\[
\gamma_j = \rho^j([1] \times [0,1]) \subset \Sigma_\eta.
\]

On the other hand, take

\[
\kappa = \min_{(z,s) \in B_0(\Sigma_\eta) \times [-t_0,t_0]} |\det DX_s(z)|_{E^s_\varepsilon} \cdot \min_{(z,s) \in U \times [0,T]} |\det DX_s(z)|_{E^s_\varepsilon} > 0.
\]

Let \( \lambda > \kappa_0 K'\delta' \), and let \( j_1 \) large enough such that \( \kappa e^{\varepsilon t_1} > \lambda \). Let \( R \) be a Poincaré map with return time

\[
s(x) \approx t_{j_2} + T \quad s(z') \approx t_0 + t_{j_2} + T, \quad \forall z' \in \gamma_0,
\]

where \( j_2 > j_1 \) is large enough. Figure 2 helps to visualize the definition of \( R \).
Following the proof of Lemma 4.9 we deduce, by shrinking $\varepsilon'$ if it is necessary, that the relation (4.12) is satisfied for any $z \in \gamma_{j_1}$. So, by definition of $R$ we have that $R(\gamma_{j_1})$ is a curve in $\Sigma_\eta$ that connects $x_{j_1+j_2}$ with $y_{j_1+j_2}$ and satisfies
\[
\ell(R(\gamma_{j_1})) \geq ke^{t_1} \geq \lambda > \kappa_0K'd',
\]
which contradicts (4.20). So, we have that $y_{j_1+j_2} \in W^s(x_{j_1+j_2}, \Sigma_\eta)$. Therefore, we deduce the claim by following step by step the argument given in [2], p. 2456. □

5. Declarations

**Ethical Approval:** This declaration is not applicable.

**Competing Interests:** We hereby declare that the authors have no competing interests as defined by Springer, or other interests that might be perceived to influence the results and/or discussion reported in this paper.

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