Multi-observable Uncertainty Equality based on the sum of standard deviations in the qubit system

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Abstract: We construct a multi-observable uncertainty equality as well as an inequality based on the sum of standard deviations in the qubit system. The obtained equality indicates that the uncertainty relation can be expressed more accurately, and also can be used to detect the mixedness of the system. Meanwhile, the new uncertainty inequality can provide a tighter lower bound, and the tightness can be maintained at a high level even in an open system. Furthermore, the deficiency of the uncertainty relation, that the lower bound of the product form uncertainty relations can be null even for two incompatible observables, can be completely fixed by the new uncertainty relation.

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I. Introduction

Quantum uncertainty relations are one of the most fundamental differences between quantum and classical mechanics [1-3]. Any pair of incompatible observables admits a certain form of uncertainty relation [4-8], which expresses the impossibility of the jointly sharp preparation of these incompatible observables [9-12]. The uncertainty relation has been widely used in the quantum information science, such as entanglement detection [1,13-15], quantum squeezing [16-19], and quantum metrology [20-22]. The initial investigation of the uncertainty relation was mainly focused on the product form, such as the Schrödinger uncertainty relation (SUR) [3]:

\[ \Delta A^2 \Delta B^2 \geq \frac{1}{4\hbar} \left| \left\langle [A, B] \right\rangle \right|^2 + \frac{1}{2} \left| \left\langle \{A, B\} \right\rangle \right|^2 \]  \hspace{1cm} (1)

where the variance \( \Delta \mathcal{O}^2 \) and expected value \( \langle \mathcal{O} \rangle \) are calculated on the state \( \rho \), and \( \tilde{\mathcal{O}} = \mathcal{O} - \langle \mathcal{O} \rangle \). However, the product form uncertainty relations cannot fully capture the concept of incompatible observables because the lower bounds of them may be null even for two incompatible observables. This deficiency is referred to as the triviality problem of the uncertainty relation. In order to fix the

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The uncertainty relation based on the sum of variances have been investigated [23]:

$$\Delta A^2 + \Delta B^2 \geq |\langle \psi | A \pm iB | \psi^\perp \rangle|^2 \pm i\langle [A, B] \rangle,$$

(2)

where $| \psi^\perp \rangle$ is the state orthogonal to the state of the system $| \psi \rangle$. The lower bound (2) can be guaranteed to be non-zero for incompatible observables. Thus the triviality problem can be fixed by the sum uncertainty relation (2). However, due to the existence of $| \psi^\perp \rangle$, the lower bound (2) is difficult to be extended to the mixed state [23].

In addition to the uncertainty relations for two incompatible observables, there exist the ones for three or more incompatible observables. The multi-observable uncertainty relations have a wider application than the one for two incompatible observables. The recently derived uncertainty relations for three or N observables are [24, 25]:

$$\Delta A_1^2 + \Delta A_2^2 + \Delta A_3^2 \geq \frac{1}{3} \Delta \{\{A_1, A_2, A_3\}\}$$

(3)

$$\Delta A_1^2 + \Delta A_2^2 + \Delta A_3^2 \geq \frac{\sqrt{\pi}}{3} \{\{A_1, A_2, A_3\}\} + \{\{A_2, A_3\}\} + \{\{A_3, A_1\}\}$$

(4)

$$\sum_{m=1}^{N} \Delta A_m^2 \geq \frac{1}{N-2} \sum_{1 \leq m < n \leq N} \Delta(A_m + A_n)^2 - \frac{1}{(N-1)^2(N-2)} \left[\sum_{1 \leq m < n \leq N} \Delta(A_m + A_n)^2\right]^2$$

(5)

where $[A_1, A_2, A_3] = [A_1, A_2] + [A_2, A_3] + [A_3, A_1]$.

In this paper, we construct a multi-observable uncertainty equality as well as an inequality based on the sum of standard deviations in the qubit system. The outline of the paper is as follows. In section II, the uncertainty equality as well as an inequality based on the sum of standard deviations is formulated in the qubit system. We demonstrate that the new uncertainty inequality is tighter than other recent uncertainty relations, and the tightness can be maintained at a high level even in the open system. Then, a proof, that the triviality problem of the product form uncertainty relation can be completely fixed by the obtained uncertainty relation, is presented in section III. In section IV, we show that the uncertainty equality can be used as a measure of the mixedness of the system which usually is expensive in terms of resources involved. Finally, the last section is devoted to the discussion and conclusion.

II. Uncertainty Equality Based on the Sum of Standard Deviations

Consider N arbitrary Hermitian operators $A_1, A_2, ..., A_N$ in the qubit system, and the uncertainties of the corresponding outcomes when they are measured admit the following uncertainty equality:

$$\sum_{m=1}^{N} \Delta A_m = \frac{1}{2(N-1)} \sum_{j=1, j \neq 1}^{N} \sqrt{M_F([A_i, A_j])} + |G(A_i, A_j)|^2 + \Delta^2_{ms} A_i + \Delta^2_{ms} A_j -$$
\[ \zeta(A_i,A_j)^{1/2} \] (6)

where \( F([A_i,A_j]) = \text{tr} \left( [A_i,A_j][A_i,A_j]^\dagger \right) /4 \), \( G(A_i,A_j) = \langle A_i,A_j \rangle - \langle A_i \rangle \langle A_j \rangle \), and \( \zeta(A_i,A_j) = \langle (A_i - \langle A_i \rangle_{\text{cms}})^2 + (A_j - \langle A_j \rangle_{\text{cms}})^2 \rangle. \) \( M = 1 - \text{tr}(\rho^2) \) is the mixedness of the state \( \rho \) [26], and the mixedness is equal to zero for pure state and greater than zero for mixed state. The variance \( \Delta^2_{\text{cms}}A_i \) and the expected value \( \langle A_i \rangle_{\text{cms}} \) are calculated on the completely mixed state \( \rho_{\text{cms}} = I/2 \), namely the state which possesses the maximum mixedness [26], with \( I \) being the identity matrix. Based on the definition of \( F([A_i,A_j]) \), \( F([A_i,A_j]) \) can be rewritten as \( ||[A_i,A_j]||/4 \), namely the modulus of commutator \([A_i,A_j]\), with \( ||O|| \) being defined as \( \text{tr}(OO^\dagger) \). Thus, \( F([A_i,A_j]) \) is used to quantify the noncommutativity between \( A_i \) and \( A_j \). \( G(A_i,A_j) = \langle A_i,A_j \rangle - \langle A_i \rangle \langle A_j \rangle \) is the covariance between \( A_i \) and \( A_j \), and can thus be used to measure the correlation between \( A_i \) and \( A_j \). \( \langle (A_i - \langle A_i \rangle_{\text{cms}})^2 \rangle \) is the difference between the expected value of \( A_i \) on the state of the system and the expected value of \( A_i \) on the completely mixed state, and \( \zeta(A_i,A_j) \) thus represents the sum of the differences related to \( A_i \) and \( A_j \).

**Proof:** In Bloch sphere representation, the density matrix of the qubit system can be expressed as [27, 28]:

\[ \rho = \frac{1}{2} \left( I + p_1 \sigma_x + p_2 \sigma_y + p_3 \sigma_z \right), \] (7)

where \( \sigma_x, \sigma_y, \sigma_z \) are standard Pauli matrices and \( p_1, p_2, p_3 \) denote real parameters with \( p_1^2 + p_2^2 + p_3^2 \leq 1 \). Additionally, an arbitrary two-dimension Hermitian operator can be written as a linear combination of \( \{\sigma_x, \sigma_y, \sigma_z, I\} \):

\[ A_m = a_{m1} \sigma_x + a_{m2} \sigma_y + a_{m3} \sigma_z + a_{m4} I, \] (8)

where \( a_{mi} \) is real parameters \( (i = 1,2,3,4) \). Based on the assumption above, we have:

\[ \Delta A_m = \langle \sum_{i=1}^{3}(1 - p_i^2)a_{mi}^2 - 2[p_2p_3a_{m2}a_{m3} + p_1a_{m1}(p_2a_{m2} + p_3a_{m3})]\rangle^{1/2} \] (9)

\[ F([A_m,A_n]) = 2\langle (a_{m3}a_{n2} - a_{m2}a_{n3})^2 + (a_{m2}a_{n1} - a_{m1}a_{n2})^2 + (a_{m3}a_{n1} - a_{m1}a_{n3})^2 \rangle \] (10)

\[ M = \frac{1}{2} \langle 1 - p_1^2 - p_2^2 - p_3^2 \rangle \] (11)

\[ \Delta^2_{\text{cms}}A_m = a_{m1}^2 + a_{m2}^2 + a_{m3}^2 \] (12)

\[ \zeta(A_m,A_n) = \langle \sum_{i=1}^{3}p_i(a_{mi})^2 + (\sum_{i=1}^{3}p_i(a_{ni})^2 \rangle \] (13)

Here, the analytic expression of \( G(A_m,A_n) \), due to its complex form, will not be presented. Combining above formulas, one can obtain uncertainty equality (6), and therefore the proof is completed. The uncertainty equality indicates that the uncertainty relation can be expressed exactly in the qubit system.
Based on the definition of $\mathcal{F}([A_i, A_j]) = \|[A_i, A_j]\|/4$, we can see that $\mathcal{F}([A_i, A_j])$ is in fact a state-independent measure to quantify the noncommutativity between $A_i$ and $A_j$. To investigate the effect of such a state-independent noncommutativity measure on the uncertainty relation, we eliminate the non-negative and state-independent term $|G(A_m, A_n)|^2$ in the uncertainty equality (6). Based on the uncertainty equality (6) and using $|G(A_m, A_n)|^2 \geq 0$, one can obtain a multi-observable uncertainty inequality based on the sum form of standard deviations:

$$\sum_{m=1}^{N} \Delta A_m \geq \frac{1}{2(N-1)} \sum_{i,j=1,i \neq j}^{N} [2\sqrt{M\mathcal{F}([A_i, A_j]) + \Delta^2 c_m, A_i + \Delta^2 c_m, A_j - \zeta(A_i, A_j)}]^{1/2}$$

(14)

The uncertainty equality (6) is saturated for any qubit state, and thus we mainly focus on the performance of the new uncertainty inequality (14).

The comparison between the new uncertainty relation (14) and other recent ones will be presented in the following. It turns out to be a relatively reasonable way to compare different types of uncertainty relations by their tightness, which is defined as dividing both sides of the inequalities by their own lower bound [28]. Thus the tightness of (14), (3), (4), and (5) are obtained as:

$$T1 = \frac{\sum_{m=1}^{N} \Delta A_m}{\sum_{i,j=1,i \neq j}^{N} [2\sqrt{M\mathcal{F}([A_i, A_j]) + \Delta^2 c_m, A_i + \Delta^2 c_m, A_j - \zeta(A_i, A_j)}]^{1/2}}$$

(15)

$$T2 = \frac{\Delta A_i^2 + \Delta A_j^2 + \Delta A_k^2}{\frac{3}{2}(|(A_i, A_j, A_k)|)}$$

(16)

$$T3 = \frac{\Delta A_i^2 + \Delta A_j^2 + \Delta A_k^2}{\frac{3}{2}(|(A_i, A_j, A_k)|)}$$

(17)

$$T4 = \frac{\sum_{m=1}^{N} \Delta A_m^2}{N^2 - \sum_{i=1}^{N} \Delta A_m^2}$$

(18)

As shown in Fig.1, we can see that uncertainty relation (14) is tighter than the other three ones.

In the open system, the ubiquitous interaction with the environment inevitably deduces the increasing of the mixedness of the system [29], and thus it becomes important to investigate the performance of the uncertainty relations in the mixed states. However, as shown in Fig.2, the tightness of the uncertainty relations (3), (4), and (5) becomes worse and worse with the mixedness increasing. In other words, the tightness of traditional uncertainty relations performed poorly in the open system. As shown in Fig.2, as the mixedness increases, the tightness of uncertainty relation (14) becomes better and better with. That is to say, the tightness of the new uncertainty inequality can be maintained at a high level even in an open system.
We take \( A_1 = \sigma_x, A_2 = \sigma_y, A_3 = \sigma_z \), and the state of the system is parameterized by \((\gamma, \theta, \varphi)\) as \( \rho = 1/2 \left[ \gamma \cos \varphi \sin \theta \sigma_x + \gamma \varphi \cos(\theta) \sigma_y + \gamma \sin(\varphi) \sigma_z + I \right] \), where \( \gamma \in [0,1] \), and \( \varphi, \theta \in [0, \pi] \). Here \( \gamma = \sqrt{\text{Tr} \rho} \), and then \( M = (1 - \gamma^2)/2 = 0.1 \).

### III. Fixing the triviality problem

Fig.1: Evolution of the tightness T1, T2, T3, and T4 with respect to the state are shown in the (a), (b), (c) and (d), respectively. We take \( A_1 = \sigma_x, A_2 = \sigma_y, A_3 = \sigma_z \), and the state of the system is parameterized by \((\gamma, \theta, \varphi)\) as \( \rho = 1/2 \left[ \gamma \cos \varphi \sin \theta \sigma_x + \gamma \varphi \cos(\theta) \sigma_y + \gamma \sin(\varphi) \sigma_z + I \right] \), where \( \gamma \in [0,1] \), and \( \varphi, \theta \in [0, \pi] \). Here \( \gamma = \sqrt{\text{Tr} \rho} \), and then \( M = (1 - \gamma^2)/2 = 0.1 \).

Fig.2: Evolution of the tightness T1, T2, T3, and T4 with respect to the mixedness M of the system are shown in the (a), (b), (c) and (d), respectively. We take \( A_1 = \sigma_x, A_2 = \sigma_y, A_3 = \sigma_z \), and the state of the system is parameterized by \((\gamma, \theta, \varphi)\) as \( \rho = 1/2 \left[ \gamma \cos \varphi \sin \theta \sigma_x + \gamma \varphi \cos(\theta) \sigma_y + \gamma \sin(\varphi) \sigma_z + I \right] \), where \( \gamma \in [0,1] \), and \( \varphi, \theta \in [0, \pi] \). The mixedness of the system can be obtained as \( M = (1 - \gamma^2)/2 \), and we take \( \theta = 3\pi/4 \) and \( \varphi = \pi/4 \).
The product form uncertainty relation cannot fully capture the concept of the incompatible observables, because its lower bound can be null even for incompatible observables [23]. Then we will demonstrate that this triviality problem can be completely fixed by the new uncertainty relation. Take N=2, and the new uncertainty relation (14) then turns into:

$$\Delta A + \Delta B \geq \left[ 2\sqrt{MF([A,B])} + \Delta_{\text{cms}}^2 A + \Delta_{\text{cms}}^2 B - \zeta(A,B) \right]^{1/2} \quad (19)$$

where $A = a_1 \sigma_x + a_2 \sigma_y + a_3 \sigma_z + a_4 I$ and $B = b_1 \sigma_x + b_2 \sigma_y + b_3 \sigma_z + b_4 I$ stand two arbitrary observables with $a_i, b_i \in \mathbb{R}$. Then, we can deduce:

$$\Delta_{\text{cms}}^2 A + \Delta_{\text{cms}}^2 B - \zeta(A,B) = \sum_{i=1}^{3} (a_i^2 + b_i^2) \geq \left[ 1 - (p_x^2 + p_y^2 + p_z^2) \right] \geq 0 \quad (20)$$

where $p_x^2 + p_y^2 + p_z^2 \leq 1$ is used. It worth mentioning that the necessary condition for $\Delta_{\text{cms}}^2 A + \Delta_{\text{cms}}^2 B - \zeta(A,B) = \sum_{i=1}^{3} (a_i^2 + b_i^2) - (p_x^2 + p_y^2 + p_z^2) [\sum_{i} (a_i^2 + b_i^2)]$ is $a_i/b_i = a_2/b_2 = a_3/b_3$, which indicates $[A,B] = 0$. Thus, we have $\Delta_{\text{cms}}^2 A + \Delta_{\text{cms}}^2 B - \zeta(A,B) > 0$ when $[A,B] \neq 0$.

That is to say, the triviality problem can be completely fixed by the new uncertainty relation (14). For instance, we take $A = \sigma_x, B = \sigma_y$ and a set of states parameterized by $\theta$ as $|\psi\rangle = \cos(\theta) |1\rangle + \sin(\theta) |0\rangle$ with $|1\rangle$ and $|0\rangle$ being the eigenstates of $\sigma_z$. Obviously, $[A,B] \neq 0$ and there exists no common eigenstate between $A$ and $B$. Then, one can obtain that:

$$L_{\text{new}} = \sqrt{1 + \cos(2\theta)^2} \quad (21)$$

$$L_{\text{SUR}} = \cos(2\theta)^2 \quad (22)$$

where $L_{\text{new}} = [2\sqrt{MF([A,B])} + \Delta_{\text{cms}}^2 A + \Delta_{\text{cms}}^2 B - \zeta(A,B)]^{1/2}$ is the lower bound of the new uncertainty inequality (19), and $L_{\text{SUR}} = |\langle [A,B] \rangle|^2/4 + |\langle [\vec{A}, \vec{B}] \rangle|^2/4$ is the lower bound of the SUR. The evolutions of the two lower bounds are shown in Fig.3. It can be seen that SUR will be trivial for $\theta = \pi/4$ and $3\pi/4$, and the triviality problem can be easily fixed by the new uncertainty inequality.
IV. Detection of the mixedness

As mentioned above, the ubiquitous interaction with the environment inevitably affects the purity of a quantum system at the practical level, and it becomes important for an experimenter to test the mixedness of the system so as to use it effectively as a resource for the quantum information processing [29]. But in the practical applications, the mixedness of the system usually is impossible to be obtained by direct calculations and is expensive in term of the physical resources and measurements involved [27, 29]. Therefore, it is become meaningful to find an effective method to detect the mixedness of the system [29]. The obtained uncertainty equality can be used to measure the mixedness of the system. Take $N=2$, and then the expression of the mixedness can be obtained by a deformation of Eq. (6):

$$M = \frac{(\Delta A + \Delta B)^2 - \Delta_{\text{CMS}}^2 A - \Delta_{\text{CMS}}^2 B + \zeta(A, B))^2 - 4|\mathcal{G}(A, B)|^2}{4\mathcal{F}([A, B])}$$  \quad (23)

The terms in the right hand of the Eq. (23) can be divided into four categories: (i) the operator function $\Delta_{\text{CMS}}^2 A$, $\Delta_{\text{CMS}}^2 B$ and $\mathcal{F}([A, B])$; (ii) the function of expectation $\zeta(A, B)$; (iii) the standard deviation $\Delta A$ and $\Delta B$; (iv) the covariance $\mathcal{G}(A, B)$. The value of the operator function, which has no relationship with the system, can be obtained by direct calculations. The expectation, standard deviation and covariance can be obtained by measuring the related observables and observing the expectation, standard deviation as well as covariance of the measurement result, such as $\Delta A$ can be obtained by making measurement $A$ on the system and observing the standard deviation of the measurement result. Thus the mixedness of the system can be easily obtained by detecting the expectation, standard deviation and covariance involved.

V. Conclusion

In conclusion, we have constructed a multi-observable uncertainty equality as well as an inequality based on the sum of standard deviations in the qubit system. The uncertainty equality
indicates that the uncertainty relation can be expressed exactly. Meanwhile, the obtained uncertainty equality can also be used as a measure of the mixedness which usually is expensive in terms of resources involved. As for the uncertainty inequality, we demonstrate that the new uncertainty inequality is tighter than other uncertainty relations, and the tightness can be maintained as a high level in the mixed state, which means the new uncertainty relation has a good performance even in the open system. Furthermore, we prove that the deficiency in the product form uncertainty relation can be completely fixed by the new uncertainty inequality.

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