FINITELY GENERATED ABELIAN GROUPS OF UNITS

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Abstract. In 1960 Fuchs posed the problem of characterizing the groups which are the groups of units of commutative rings. In the following years, some partial answers have been given to this question in particular cases.

In this paper we address Fuchs’ question for finitely generated abelian groups and we consider the problem of characterizing those groups which arise in some fixed classes of rings $C$, namely the integral domains, the torsion free rings and the reduced rings.

To determine the realizable groups we have to establish what finite abelian groups $T$ (up to isomorphism) occur as torsion subgroup of $A^*$ when $A$ varies in $C$, and on the other hand, we have to determine what are the possible values of the rank of $A^*$ when $(A^*)_{\text{tors}} \cong T$.

Most of the paper is devoted to the study of the class of torsion-free rings, which needs a substantially deeper study.

1. Introduction

1.1. General introduction to the problem. The study of the group of units of a ring is an old problem. The first general result is the classical Dirichlet’s Unit Theorem (1846), which describes the group of units of the ring of integers $\mathcal{O}_K$ of a number field $K$: the group of units $\mathcal{O}_K^*$ is a finitely generated abelian group of the form $C_{2n} \times \mathbb{Z}^g$ where $n \geq 1$ and $g$ is explicit in terms of the structure of the field $K$.

In 1940 G. Higman discovered a perfect analogue of Dirichlet’s Unit Theorem for a group ring $\mathbb{Z} T$ where $T$ is a finite abelian group: $(\mathbb{Z}T)^* \cong \pm T \times \mathbb{Z}^g$ for a suitable explicit constant $g$.

In 1960 Fuchs in [Fuc60, Problem 72] raised explicitly the question posing the following problem.

Characterize the groups which are the groups of all units in a commutative and associative ring with identity.

In the subsequent years, this question has been considered by many authors. A first result is due to Gilmer [Gil63], who considered the case of finite commutative rings, classifying the possible cyclic groups that arise in this case. An important contribution to the problem can be derived from the results by Hallett and Hirsch [HH65], and subsequently by Hirsch and Zassenhaus [HZ66], combined with [Cor63].

Key words and phrases. Commutative algebra; Group theory; Fuchs’ Problem; Units groups; Torsion-free rings; Cyclotomic extensions; Cyclotomic polynomials.
From their study it is possible to deduce that if a finite group is the group of units of a reduced and torsion free ring, then it must satisfy some necessary conditions, namely, it must be a subgroup of a direct product of groups of a given family.

Later on, Pearson and Schneider [PS70] combined the result of Gilmer and the result of Hallett and Hirsch to describe explicitly all possible finite cyclic groups that can occur as $A^*$ for a commutative ring $A$.

Recently, Chebolu and Lockridge [CL15] were able to classify the indecomposable abelian groups which occur as group of units of a ring.

In the papers [DCD18a] [DCD18b], R. Dvornicich and the author studied Fuchs’ question for finite abelian groups and for a general ring of any characteristic, obtaining necessary conditions for a group to be realizable, and producing infinite families of both realizable and non-realizable groups. Moreover, they got a complete classification of the group of units realizable in some particular classes of rings (integral domains, torsion-free rings and reduced rings).

The study of groups of units has been investigated also for non abelian groups. Much has been said about the units of group rings. Recently, the finite dihedral groups and the simple groups that are realizable as the group of units of a ring have been classified (see [CL17] and [DO14]).

1.2. The questions studied in the paper. In this paper we consider Fuchs’ question for finitely generated abelian groups and we consider the problem of characterizing those groups which arise in some fixed classes of rings $C$, namely the integral domains, the torsion free rings and the reduced rings.

This question is twofold: on the one hand, we have to establish what finite abelian groups $T$ (up to isomorphism) occur as the torsion subgroup of $A^*$ when $A$ varies in $C$. On the other hand, we have to determine what are the possible values of the rank of $A^*$ when $(A^*)_{\text{tors}} \cong T$. Therefore, the situation becomes substantially different from the case when the group of units is finite and abelian, which has been studied already in [DCD18a] and [DCD18b].

1.3. Integral domains: result and idea of proof. In Section 3 we focus on the study of groups of units of integral domains. Our main tools are Dirichlet’s Unit Theorem and the properties of cyclotomic extensions. The principal result is the following theorem in which we collect the results of Theorems 3.1 and 3.4.

**Theorem A:** The finitely generated abelian groups that occur as group of units of an integral domain $A$ are:

i) the groups of the form $C_{2n} \times \mathbb{Z}^g$, with $n \in \mathbb{N}$, $g \geq \frac{\phi(2n)}{2} - 1$, if $\text{char}(A) = 0$;

ii) the groups of the form $\mathbb{F}_p^* \times \mathbb{Z}^g$ with $n \geq 1$ and $g \geq 0$, if $\text{char}(A) = p$. 
As a particular case we get the characterization of the finite abelian groups which are realizable as group of units of an integral domain (see Corollary 3.2).

Finally, in Proposition 3.3 we describe the finitely generated abelian groups that occur as group of units of an integral domain \( A \) which is integral over \( \mathbb{Z} \).

1.4. **Torsion-free rings: result and idea of proof.** The most relevant part of the paper is the classification of the finitely generated abelian groups of units realizable with torsion-free rings (Section 4).

We remark that the study of the group of units of torsion free rings has become classical in the literature (see the aforementioned papers by Hallett, Hirsch and Zassenhaus) and that the finitely generated abelian group rings belong to this class.

In Theorem 4.12 we prove the following

**Theorem B:** Let \( T \) be a finite abelian group of even order. Then there exists an explicit constant \( g(T) \) depending on \( T \) (see \( \delta \) for the explicit value of \( g(T) \)) such that the following holds: the group \( T \times \mathbb{Z}^r \) is the group of units of a torsion free ring if and only if \( r \geq g(T) \).

The proof is rather long and requires many steps. The first step is the reduction to the study of the subring of \( A \) generated over \( \mathbb{Z} \) by the torsion units. This ring has the same torsion units as \( A \) and is finitely generated and integral over \( \mathbb{Z} \). Restricting to study these rings, in Proposition 4.3 we show that the \( \mathbb{Q} \)-algebra \( A \otimes \mathbb{Z} \mathbb{Q} \) is semisimple and is a finite product of cyclotomic fields (for short, a cyclotomic \( \mathbb{Q} \)-algebra). The next important step is the study of the units of the subrings of \( A \) of type \( \mathbb{Z}[\alpha] \) with \( \alpha \) a torsion unit of \( A \) (see Proposition 4.10). Once this preliminary results are established, we pass to the proof of the theorem, which requires two parts.

On the one hand, we have to show that if \( A \) is a torsion-free ring with \( (A^*)_{\text{tors}} \cong T \), then \( \text{rank}(A^*) \geq g(T) \). This is done through the analysis of the possible maximal order of \( T \)-admissible cyclotomic \( \mathbb{Q} \)-algebras (namely, cyclotomic \( \mathbb{Q} \)-algebras which could admit a subring with \( (A^*)_{\text{tors}} \cong T \)). This gives a first lower bound on the rank of the group of units (Proposition 4.16). This “natural” bound works only if the 2-Sylow subgroup of \( T \) has “enough” cyclic factors of minimal order in its decomposition. If not, the actual bound is bigger than the natural one: this is described in Propositions 4.17.

On the other hand, for each \( T \) we have to construct a torsion-free ring \( A \) with \( A^* \cong T \times \mathbb{Z}^{g(T)} \); the construction of orders with a bigger rank can then be obtained via localization. In the previous part for a given \( T \) we have identified a maximal order \( \mathcal{M}_T \) of a cyclotomic \( \mathbb{Q} \)-algebra with \( \text{rank}(\mathcal{M}_T^*) = g(T) \). We construct \( A \) as an order of \( \mathcal{M}_T \), hence \( \text{rank}(A) = \text{rank}(\mathcal{M}_T^*) = g(T) \) (see Lemma 4.2). The group \( (\mathcal{M}_T^*)_{\text{tors}} \) contains a subgroup isomorphic to \( T \) and it differs from \( T \) only in the
2-Sylow subgroup: our task is to construct an order with a 2-Sylow as small as possible.

We note that also in this case the result of [DCD18b] on finite abelian groups of units are recovered as a corollary of this more general result.

1.5. **Reduced rings: result and idea of proof.** In Section 5 we deal with the units of reduced rings.

For a non reduced ring \( R \) with nilradical \( N \), it is known that the \( R^* \) is an extension of \( (R/N)^* \) by \( 1+N \) (see Proposition 5.1). So the study of units of reduced rings is also a step towards the understanding of the units of general rings.

In Theorem 5.4 we prove the following.

**Theorem C:** The finitely generated abelian groups that occur as group of units of a reduced ring are those of the form

\[
\prod_{i=1}^{k} \mathbb{F}_{p_i}^{*n_i} \times T \times \mathbb{Z}^g
\]

where \( k, n_1, \ldots, n_k \) are positive integers, \( \{p_1, \ldots, p_k\} \) are not necessarily distinct primes, \( T \) is any finite abelian group of even order and \( g \geq g(T) \).

The proof is achieved by using a result by Pearson and Schneider [PS70, Prop. 1] which allows to split a generic reduced ring \( A \) as a direct sum \( A_1 \oplus A_2 \) where \( A_1 \) is finite and \( A_2 \) is torsion-free. Putting together our previous results on torsion-free rings with some properties of the finite rings we get the classification of the groups of units in this case.

**Acknowledgement:** I wish to thank Cornelius Greither for his careful reading of the paper and for suggesting to me stylistic improvements and a refinement of the proof of Proposition 4.3.

2. **Notation and preliminary results**

Let \( A \) be a ring with 1: throughout the paper we will assume that its group of units \( A^* \) is finitely generated and abelian. Let \( (A^*)_\text{tors} \) denote its torsion subgroup and let \( g_A \) be its rank so that

\[
A^* \cong (A^*)_\text{tors} \times \mathbb{Z}^{g_A}.
\]

Let \( A_0 \) be the fundamental subring of \( A \), namely \( A_0 = \mathbb{Z} \) or \( \mathbb{Z}/n\mathbb{Z} \) depending on whether the characteristic of \( A \) is 0 or \( n \). It is immediate to check that the ring \( A_0[A^*] \) has the same group of units as \( A \). Since we are interested in the classification of the possible groups of units, we can assume without loss of generality that \( A \) is a ring of type \( A_0[A^*] \). In particular, we will always assume that \( A \) is commutative and that it is finitely generated over \( A_0 \).

Let \( B \) the subring of \( A \) generated over \( A_0 \) by the torsion units of \( A \), namely \( B \cong A_0[(A^*)_\text{tors}] \). It is important to note that all the elements of
(A^*_{\text{tors}}) are integral over $A_0$, since they have finite order. This ensures that $B$ is commutative, finitely generated and integral over $A_0$.

**Lemma 2.1.** $B^* \cong (A^*_{\text{tors}}) \times \mathbb{Z}^{g_B}$ and $g_B \leq g_A$. Moreover, if the characteristic of $A$ is positive, then $B^* = (A^*_{\text{tors}})$.

**Proof.** $B$ is a subring of $A$, hence $B^* < A^*$: in particular $B^*$ is finitely generated and $g_B \leq g_A$. On the other hand, $(A^*_{\text{tors}}) < (B^*_{\text{tors}}) < (A^*_{\text{tors}})$ and equality holds.

Moreover, when the characteristic of $A_0$ is positive, then $B^*$, being integral and finitely generated over $A_0$, is itself finite, so $B^* = (A^*_{\text{tors}})$. □

**Remark 2.2.** The previous lemma shows that all possible torsion parts occur already when restricting to consider rings which are generated over $A_0$ by a finite number of integral elements verifying an equation of type $x^n - 1$ for some $n$.

The lemma also shows that there is a completely different behavior between the characteristic zero and positive characteristic rings. In fact, a finite abelian group $T$ can be isomorphic to the torsion subgroup of the group of units of a ring $A$ of positive characteristic only if it is also the group of units of a finite ring and all the results of [DCD18a] apply in this case. In particular, not all finite abelian group can occur.

Instead, when $A_0 = \mathbb{Z}$ it will turn out that the torsion subgroup of $A^*$ can be any finite abelian groups of even order, whereas this is not true if we also require that $A^*$ is finite (see Theorem [DCD18b]). Nevertheless, to determine the minimum rank $g(T)$ such that $T \times \mathbb{Z}^{g(T)}$ is the group of units of some ring $A$, it is sufficient to restrict to consider the finitely generated integral extensions of $\mathbb{Z}$.

In the following subsections we collect some classical results we will need in the paper.

2.1. **Units of Laurent polynomials.** Let $R$ be a reduced ring, namely a ring without non-zero nilpotents. Then the polynomial ring $R[x]$ is reduced and has the same units as $R$ and the ring of Laurent polynomials $R[x, x^{-1}]$ has group of units $(R^*, x)$. Inductively we get that the group of units of the ring of Laurent polynomials in $k$ indeterminates $R[x_1, \ldots, x_k, x_1^{-1}, \ldots, x_k^{-1}]$ is isomorphic to $R^* \times \mathbb{Z}^k$.

2.2. **Dirichlet’s Unit Theorem.** The fundamental results to study of the groups of units of an integral domain is the classical Dirichlet’s Theorem. Let $K$ be a number field, and let $\mathcal{O}_K$ be its ring of integers. An order of $K$ is a subring of $\mathcal{O}_K$ which spans $K$ over $\mathbb{Q}$.

**Proposition 2.3** (Dirichlet’s Unit Theorem). Let $K$ be a number field such that $[K : \mathbb{Q}] = n$ and assume that among the $n$ embeddings of $K$
in $\bar{\mathbb{Q}}$, $r$ are real (namely map $K$ into $\mathbb{R}$) and $2s$ are non-real ($n = r + 2s$). Let $R$ be an order of $K$. Then

$$R^* \cong T \times \mathbb{Z}^{r+s-1}$$

where $T$ is the group of the roots of unity contained in $R$.

For a proof see [Neu99, Ch.1, §12].

The Dirichlet’s units Theorem admit the following genaralization (see for example [Jan96] p.207).

**Proposition 2.4** (S-units Theorem). Let $K$ be a number field such that $[K : \mathbb{Q}] = n = r + 2s$ and let $S$ be a finite set of prime ideals of $O_K$ and let $O_{K,S} = \{ \alpha \in K \mid v_P(\alpha) \geq 0 \forall P \notin S \}$. Then

$$O_{K,S}^* \cong T \times \mathbb{Z}^{r+s-1+|S|}$$

where $T$ is the group of the roots of unity contained in $K$.

2.3. **Cyclotomic polynomials.** For $n \geq 1$ let $\zeta_n = e^{2\pi i/n}$, then $\zeta_n$ is a primitive $n$-th root of unity. Denote by $\Phi_n(x)$ its minimal polynomial over $\mathbb{Q}$: as it is well known, $\Phi_n(x) \in \mathbb{Z}[x]$ and

$$\Phi_n(x) = \prod_{j=1,\ldots,n}^{(j,n)=1} (x - \zeta_n^j).$$

Moreover, $\mathbb{Q}(\zeta_n)$ is a Galois extension of $\mathbb{Q}$ of degree $\phi(n)$, where $\phi$ is the Euler totient function, and its ring of integers is $\mathbb{Z}[\zeta_n]$.

The root of units contained in $\mathbb{Z}[\zeta_n]$ are the $n$-th roots of unity if $n$ is even and the $2n$-roots of unity if $n$ is odd and by Dirichlet’s Unit Theorem $\mathbb{Z}[\zeta_n]^* \cong (-\zeta_n) \times \mathbb{Z}^{\frac{\phi(n)}{2}-1}$ for each $n \geq 3$. In the following we will use the notation $\left(\frac{\phi(n)}{2} - 1\right)^*$ for the rank of $\mathbb{Z}[\zeta_n]^*$, namely, $\left(\frac{\phi(n)}{2} - 1\right)^* = \frac{\phi(n)}{2} - 1$ for $n \geq 3$ and $\left(\frac{\phi(n)}{2} - 1\right)^* = 0$ for $n = 1, 2$. We will omit the * when $n > 2$.

In the paper we will need the following classical property of the cyclotomic polynomials. Most of the results could be generalized, but we give only those necessary for our purposes.

**Lemma 2.5.**

(1) Suppose that $n$ has at least two distinct prime factors. Then $1 - \zeta_n$ is a unit of $\mathbb{Z}[\zeta_n]$ and $\Phi_n(1) = \prod_{(j,n)=1}^{1 \leq j \leq n} (1 - \zeta_n^j) = 1$.

(2) For $p$ prime and $e > 0$, then $1 - \zeta_{pe}$ is a generator of the prime of $\mathbb{Z}[\zeta_{pe}]$ over $p$ and $\Phi_{pe}(1) = \prod_{(j,pe)=1}^{1 \leq j \leq pe} (1 - \zeta_{pe}^j) = p$.

Proof. For part (1) see [Was87, Lemma 2.8]. For part (2) see [Lan94, IV, 1, Thm 1].

**Lemma 2.6.** Let $l > 1$ and let $\Psi_{n,l}(x)$ denote the minimal polynomial of $\zeta_n$ over $K = \mathbb{Q}(\zeta_l)$.
(1) Suppose that \( n \) has at least two distinct prime factors. Then the algebraic integer \( \Psi_{n,l}(1) \) is a unit.

(2) If \( n = p^a \), where \( p \) is a prime and \( a > 0 \), and \( l = l_1p^b \), with \((l_1, p) = 1\) and \( 0 \leq b \leq a \), then \( \Psi_{p^a,l}(1) \) is a generator of the prime over \( p \) of \( \mathbb{Z}[\zeta_p] \).

Proof. \( \Psi_{n,l}(x) \) divides \( \Phi_{n}(x) \), hence \( \Psi_{n,l}(1) \) is a unit since it divides the unit \( \Phi_{n}(1) \) (this actually holds for any number field \( K \)).

For part (2) note that \( \Psi_{p^a,l}(x) = \Phi_{p^a} \) if \( b = 0 \) and \( \Psi_{p^a,l}(x) = x^{p^a-b} - \zeta_p^b \) if \( b > 0 \), hence \( \Psi_{p^a,l}(1) \) is equal to \( p \) or \( 1 - \zeta_p^b \) according to \( b = 0 \) or \( b > 0 \), namely it is a generator for the prime over \( p \) of \( \mathbb{Z}[\zeta_p] \).

\( \square \)

Lemma 2.7. Let \( n > m \geq 1 \). The algebraic integer \( \Phi_{n}(\zeta_m) \) is a unit in \( \mathbb{Z}[\zeta_m] \) if \( n/m \) is not a prime power.

In the case when \( n/m = p^a \) for a prime \( p \) and an integer \( a > 0 \), then \( \Phi_{n}(\zeta_m) \) is associated to \( p \).

Proof. The first part of is an immediate consequence of [Apo70] Thms 1 and 4.

For the second part (which could also be deduced from [Apo70]), we note that

\[
\Phi_{n}(\zeta_m) = \prod_{j=1, \ldots, n} (\zeta_m - \zeta_m^j) = \prod_{j=1, \ldots, n} \zeta_m(1 - \zeta_m^{j-p^a}).
\]

From Lemma 2.5 we have that \( (1 - \zeta_m^{j-p^a}) \) is invertible if \( \frac{n}{(n,j-p^a)} \) is not a prime power. On the other hand, \( \frac{n}{(n,j-p^a)} \) is a prime power only if it is a power of \( p \) and \( j \equiv p^a \pmod{m_1} \), where \( m = p^bm_1 \) and \( (m_1, p) = 1 \). Taking into account that \( (j, n) = 1 \), an easy computation shows that there are \( \phi(p^{a+b}) \) values of \( j \) with this property. For these values \( 1 - \zeta_m^{j-p^a} \) is a generator of the ideal \( (1 - \zeta_m^{p^{a+b}}) \), namely

\[
(\Phi_{n}(\zeta_m)) = (1 - \zeta_m^{p^{a+b}})^{\phi(p^{a+b})} = p\mathbb{Z}[\zeta_m^{p^{a+b}}].
\]

\( \square \)

3. Integral domains

In this section we characterize the finitely generated groups which occur as group of units of an integral domain of any characteristic and in Proposition 3.3 those which are the group of units of integral extensions of \( \mathbb{Z} \).

Theorem 3.1. The finitely generated abelian groups that occur as group of units of an integral domain of characteristic zero are the groups of the form \( C_{2n} \times \mathbb{Z}^g \), with \( n \in \mathbb{N} \), \( g \geq \frac{\phi(2n)}{2} - 1 \).

Proof. Let \( A \) be an integral domain and let \( K \) be its quotient field. Let \( A^* \cong T \times \mathbb{Z}^{d_A} \) where \( T \) denotes the (finite) torsion subgroup. Clearly \( T \) is a finite multiplicative subgroup of \( K \), hence it is s a cyclic group.
As noted in Section 2, the ring $B = A_0[T]$ has group of units isomorphic to $T \times \mathbb{Z}^g$ with $g_B \leq g_A$ and $B^*$ differs from $A^*$ only by a, possibly trivial, power of $\mathbb{Z}$. It turns out that to prove that $A^*$ has the required form it is enough to restrict to the case when $A = B$, namely it is finitely generated and integral over $\mathbb{Z}$. In this case, its quotient field $K$ is a number field and $A$ is an order of $K$. By Dirichlet’s Unit Theorem $A^* \cong T \times \mathbb{Z}^{r+s-1}$ where $T$ is the (cyclic) group of roots of unity contained in $A$ and $r$ and $2s$ are the number of real and non-real embeddings of $K$, respectively. Clearly, $|T|$ is even since $-1 \in A^*$. Let $T = \langle \zeta_{2n} \rangle$, then $\mathbb{Z}[\zeta_{2n}] \subseteq A$, so $\mathbb{Q}(\zeta_{2n}) \subseteq K$. For $n = 1$ we have nothing to prove. If $n > 1$, then all embeddings of $K$ in $\mathbb{Q}$ must be non-real, so $r = 0$ and $2s = [K : \mathbb{Q}]$. Since $\mathbb{Q}(\zeta_{2n}) \subseteq K$ then $\phi(2n) | s$ so the rank of $A^*$ is $g = s - 1 \geq \frac{\phi(2n)}{2} - 1$.

As to the converse, let $n \geq 1$ and let $K = \mathbb{Q}(\zeta_{2n})$. Then $\mathcal{O}_K \cong C_{2n} \times \mathbb{Z}[\zeta_{2n}]^{-1}$ and for any $k \geq 1$ the ring of Laurent polynomials in $k$ indeterminates $\mathcal{O}_K[x_1, \ldots, x_k, x_1^{-1}, \ldots, x_k^{-1}]$ has group of units isomorphic to $C_{2n} \times \mathbb{Z}[\zeta_{2n}]^{-1+k}$. □

As a corollary we recover the characterization of the finite abelian groups which are group of units of an integral domain.

**Corollary 3.2.** The finite abelian groups that occur as group of units of an integral domain $A$ of characteristic 0 are the cyclic groups of order 2, 4, or 6.

**Proof.** From Theorem 3.1 we know that the finitely generated groups of units of a domain are of the form $C_{2n} \times \mathbb{Z}^g$ with $g \geq \left(\frac{\phi(2n)}{2} - 1\right)^*$, hence we can have $g = 0$ only for $n = 1, 2, 3$. □

In Theorem 3.1 we have seen that among the rings with finitely generated group of units and torsion subgroup isomorphic to $C_{2n}$, the ring $A = \mathbb{Z}[\zeta_{2n}]$ has the minimum possible rank. The example of rings whose group of units has the same torsion subgroup, but a greater rank are constructed in the theorem by localizing polynomial rings. In particular, the rings of our example are no longer integral over $\mathbb{Z}$. Actually, only some of these groups can also be obtained with units that are integral over $\mathbb{Z}$. The following proposition characterizes these cases.

**Proposition 3.3.** The finite generated abelian groups that can be realized as group of units of an integral domain $A$, with $A$ integral over $\mathbb{Z}$ are the groups of the type $C_{2n} \times \mathbb{Z}^g$, with $n \geq 1$, $g \geq 0$ and $\phi(2n) | 2(g + 1)$.

**Proof.** Up to replacing $A$ with $\mathbb{Z}[(A^*)^\text{tor}, 1]$, we can assume that its quotient field $K$ is a number field and $A$ is an order of $K$. By Dirichlet’s Unit Theorem, $A^* \cong T \times \mathbb{Z}^{r+s-1}$ where $T = \langle \zeta_{2n} \rangle$ for some $n \geq 1$ and $[K : \mathbb{Q}] = r + 2s$. It follows that $\mathbb{Z}[\zeta_{2n}] \subseteq A$, so $\mathbb{Q}(\zeta_{2n}) \subseteq K$. This
shows that if $n > 1$ all the embeddings of $K$ in $\overline{Q}$ must be non-real ($r = 0$) and $[Q(\zeta_{2n} : Q) = \phi(2n) | 2s = [K : Q] = 2(g + 1)$ where $g = s - 1$ is the rank of $A^\ast$. For $n = 1$ the divisibility condition is trivial.

As for the converse, we have to construct examples of orders in number fields realizing all the listed groups. One possible construction is the following.

Let $d, n \geq 1$ and let $p$ be a prime such that

$$p \equiv 1 \pmod{2d};$$

since there are infinitely many such primes (see for example [Was87, Corollary 2.11] or use Dirichlet’s Prime Number Theorem) we can assume $p \nmid n$.

The congruence condition guarantees that inside the cyclotomic extension $Q(\zeta_p)$ there is a unique subextension, $K_{d,p}$, of degree $d$ over $Q$. Now, $d \mid \frac{2n}{p}$ hence $K_{d,p} \subseteq Q(\zeta_p + \zeta_p^{-1})$ and $K_{d,p}$ is totally real; hence $r = d, s = 0$ and the only roots of units in $K_{p,d}$ are $\pm 1$. This shows that the group of units of the integers of $K_{d,p}$ is isomorphic to $C_2 \times Z^{d-1}$. This family gives the examples for $n = 1$.

Consider now the case $n > 1$. Put $L = L_{d,p,n} = K_{d,p}Q(\zeta_{2n})$ and denote by $O_L = O_{L_{d,p,n}}$ its ring of integers. We claim that

$$O_L^\ast \cong C_{2n} \times Z^{\frac{d\phi(2n)}{2} - 1}.$$ 

First, we note that the torsion subgroup of $O_L^\ast$ is $\langle \zeta_{2n} \rangle$. In fact, $Q(\zeta_{2n}) \subset L \subset Q(\zeta_p)Q(\zeta_{2n}) = Q(\zeta_{2np})$ ($p \nmid n$) and a degree argument shows that $\zeta_p$ can not belong to $L$, proving that the only roots of unity in $O_L^\ast$ are the powers of $\zeta_{2n}$.

To compute the rank of $O_L^\ast$, we note that $Q(\zeta_p)$ is arithmetically disjoint from $Q(\zeta_{2n})$ since $(p, 2n) = 1$, hence also $K_{d,p}$ is arithmetically disjoint from $Q(\zeta_{2n})$ and $[L : Q] = [K_{d,p} : Q][Q(\zeta_{2n}) : Q] = d\phi(2n)$. Moreover, $L$ is Galois over $Q$ and all its embeddings are non-real, so the rank of its group of units is $s - 1 = \frac{d\phi(2n)}{2} - 1$.

□

To complete the description of the finitely generated group of units of integral domains, in the following theorem we present the simple result for finite characteristic rings.

**Theorem 3.4.** The finite generated abelian groups that occur as group of units of an integral domain of characteristic $p$ are the groups of the form $\mathbb{F}_p^\ast \times Z^g$ with $n \geq 1$ and $g \geq 0$.

**Proof.** Let $A$ be a domain and let $A^\ast \cong (A^\ast)_{\text{tors}} \times F$ with $(A^\ast)_{\text{tors}}$ finite $F \cong Z^g$ for some $g \geq 0$. Let $B = \mathbb{F}_p[(A^\ast)_{\text{tors}}]$, then, as noted in Lemma 2.1, $B^\ast = (A^\ast)_{\text{tors}}$. Since $B$ is a finitely generated algebraic extension of $\mathbb{F}_p$, it is a finite ring, whence a finite field, say $B \cong \mathbb{F}_{p^n}$ for some
\[ n \geq 1. \] It follows that \((A^\ast)_{\text{tors}} = B^\ast \cong \mathbb{F}_p^\ast,\) hence \(A^\ast \cong \mathbb{F}_p^\ast \times \mathbb{Z}^g\) as required.

Conversely, for \(n \geq 1\) and \(g \geq 0,\) the group \(\mathbb{F}_p^\ast \times \mathbb{Z}^g\) is isomorphic to the group of units of the ring of Laurent polynomials with coefficients in \(\mathbb{F}_p\) and \(g\) indeterminates. \(\square\)

4. Torsion-free rings

A commutative ring \(A\) is called torsion-free if its only element of finite additive order is 0. Clearly, a torsion-free ring has characteristic zero.

For a torsion-free ring \(A\) we put \(Q_A = A \otimes_\mathbb{Z} \mathbb{Q}.\) We note that in this case the map \(\iota: A \rightarrow Q_A\) defined by \(a \mapsto a \otimes 1\) is an embedding, so we will say that \(A \subseteq Q_A.\)

As noted in Section 2 (Lemma 2.1 and Remark 2.2), to characterize the possible finitely generated abelian groups \(T \times \mathbb{Z}^g\) that arise as group of units of torsion-free rings, a substantial step is the study of the subrings that are generated over \(\mathbb{Z}\) by units of finite order. In fact, the groups of units of rings in this subclass realize all possible torsion subgroups \(T\) and, for each of them, the minimum rank \(g(T)\) with which it can appear in the group of units of a torsion-free ring. This case is much easier to study since if \(A\) is integral over \(\mathbb{Z}\) then \(Q_A\) is a finite dimensional \(\mathbb{Q}\)-algebra and \(A\) is an order of \(Q_A.\) In the following we will restrict to this case and we will generalize the result in the end.

We begin with two lemmas which allow to describe the ring \(A\) when it is generated by one torsion unit. The next one is a slight variation of \([\text{DCD18b, Lemma 4.2}].\)

**Lemma 4.1.** Let \(K\) be a number field and let \(O_K\) be its ring of integers. Assume that \(O_K \subseteq A.\) Let \(\alpha \in (A^\ast)_{\text{tors}},\) and let \(\varphi_\alpha: O_K[x] \rightarrow A\) be the homomorphism defined by \(p(x) \mapsto p(\alpha).\) Then \(\ker(\varphi_\alpha)\) is a principal ideal generated by a non-empty product of distinct polynomials, cyclotomic over \(K.\)

**Proof.** Let \(n = \text{ord}(\alpha),\) then \(x^n - 1 \in \ker(\varphi_\alpha).\) Denote by \(\hat{\varphi}_\alpha: K[x] \rightarrow Q_A\) the extension of \(\varphi_\alpha.\) Then, there exists a monic polynomial \(\mu(x) \in K[x]\) such that \(\ker(\hat{\varphi}_\alpha) = (\mu(x)).\) Clearly, \(\mu(\alpha) = 0\) and \(\mu(x)\) divides the separable polynomial \(x^n - 1\) in \(K[x].\) Now, the monic irreducible factors of \(x^n - 1\) in \(K[x]\) are the minimal polynomials over \(K\) of some \(n\)-th root of units, so they belong to \(O_K[x]\). It follows that \(\mu(x) \in O_K[x],\) hence \(\mu \in \ker(\varphi_\alpha).\) On the other hand, for each \(f(x) \in \ker(\varphi_\alpha)\) we have \(\mu(x)|f(x)\) in \(K[x]\) and since \(\mu(x) \in O_K[x]\) is a monic polynomial, then it divides \(f(x)\) in \(O_K[x].\) This proves that \(\ker(\varphi_\alpha) = (\mu(x))\).

Finally, \(\mu(x)\) is separable since \(x^n - 1\) is such. \(\square\)
Lemma 4.2. Let $\alpha \in A^*$ be an element of order $n$, let $\varphi_\alpha : O_K[x] \to A$ be the evaluation homomorphism and put $\ker(\varphi_\alpha) = (\mu_\alpha(x))$. Then
\[ \mu_\alpha(x) = \Psi_{m_1}(x) \cdots \Psi_{m_r}(x) \]
where, for each $i$, $\Psi_{m_i}(x) \in O_K[x]$ denotes the minimal polynomial over $K$ of a primitive $m_i$-th root of unity. Moreover the $\Psi_{m_i}(x)$’s are pairwise distinct and $[m_1, \ldots, m_r] = \text{lcm}\{m_1, \ldots, m_r\} = n$.

Proof. The element $\alpha$ has order $n$, so $\ker(\varphi_\alpha) = (\mu_\alpha(x)) \supseteq (x^n - 1)$ and
\[ \mu_\alpha(x) | (x^n - 1) = \prod_{m | n} \Phi_m(x). \]

Now, each $\Phi_m$ factors as a product of distinct cyclotomic polynomials over $K$, hence $\mu_\alpha(x)$ factors in $K[x]$ as
\[ \mu_\alpha(x) = \Psi_{m_1}(x) \cdots \Psi_{m_r}(x), \]
where $\Psi_{m_i}(x) \in O_K[x]$ denotes the minimal polynomial over $K$ of a primitive $m_i$-th root of unity. The $\Psi_{m_i}(x)$’s are pairwise distinct since $x^n - 1$ is separable.

Let $[m_1, \ldots, m_r] = m$. Since $m_i | n$ for all $i$, then $m \leq n$. In fact $m = n$, since otherwise $\mu_\alpha(x) | x^m - 1$ and therefore $\alpha^m = 1$, contrary to our assumption. \hfill \Box

Let $A = \mathbb{Z}[\alpha_1, \ldots, \alpha_s]$, where, for all $i$, $\alpha_i$ is a unit of finite order; then $Q_A = A \otimes_{\mathbb{Z}} Q \cong Q[\alpha_1, \ldots, \alpha_s]$. The following proposition shows that $Q_A$ is a semisimple $Q$-algebra.

Proposition 4.3. The $Q$ algebra $Q_A$ is a finite direct product of cyclotomic fields.

Proof. Let $T \cong \langle \alpha_1, \ldots, \alpha_s \rangle$. For $\alpha = \alpha_i$, we have $\mathbb{Z}[\alpha] \cong \mathbb{Z}[x]/(\mu_\alpha(x))$ where $\mu_\alpha(x) = \Phi_{m_1}(x) \cdots \Phi_{m_r}(x)$ for some distinct $m_1, \ldots, m_r$, with $[m_1, \ldots, m_r] = \text{ord}(\alpha)$. Then the Chinese Remainder Theorem gives the following isomorphism
\[ Q[\alpha] = \mathbb{Z}[\alpha] \otimes_{\mathbb{Z}} Q \cong Q[x]/(\mu_\alpha(x)) \cong \prod_{i=1}^r Q[x]/(\Phi_{m_i}(x)) \cong \prod_{i=1}^r Q(\zeta_{m_i}). \]

It follows that also $Q = Q[\alpha_1] \otimes_Q \cdots \otimes_Q Q[\alpha_s]$ is a product of cyclotomic fields, and, in turn, the same is true for $Q_A = Q[\alpha_1, \ldots, \alpha_s]$ since it is the epimorphic image of $Q$ via the $Q$-algebra homomorphism defined by $\alpha_1 \otimes \cdots \otimes \alpha_s \mapsto \alpha_1 \cdots \alpha_s$. \hfill \Box

Remark 4.4. The last proposition shows that the $Q$ algebra $Q_A$ is isomorphic to $\prod_{i=1}^t Q(\zeta_{n_i})$ for some $n_1, \ldots, n_t$. Now, the $Q$-algebra $Q_A$ is commutative and separable, so it has a unique maximal order $M_A$, which is the integral closure of $\mathbb{Z}$ in $Q_A$, namely
\[ M_A \cong \prod_{i=1}^t \mathbb{Z}[\zeta_{n_i}]. \]
Since $A$ is an order of $Q_A$, then $A$ is a subring of $M_A$, therefore the rings we are taking into account are subrings of finite products of cyclotomic rings.

The next lemma shows that the groups of units of all orders of $Q_A$ have the same rank.

**Lemma 4.5.** Let $R$ be an order of a commutative and finitely generated $\mathbb{Q}$-algebra $Q$ and let $M$ denote its maximal order. Then $R^*$ has the same rank of $M^*$.

**Proof.** Each order $R$ of $Q$ is a subring of finite index of $M$, since both are $\mathbb{Z}$-modules of the same finite rank. Let $[M : R] = m$, then the ideal $mM$ is contained in $R$ and $M/mM$ is a finite ring.

Consider the projection $\pi : M \to M/mM$. Since $\pi$ is a ring homomorphism, the image of a unit of $M$ is a unit of the quotient, so the restriction of $\pi$ is a group homomorphism $M^* \to (M/mM)^*$.

Let $|(M/mM)^*| = c$. For each $\varepsilon \in M^*$ we have that $\varepsilon^c \equiv 1 \pmod {mM}$ so $\varepsilon^c - 1 \in mM \subseteq R$. Now, $R$ and $M$ have the same identity, hence $\varepsilon^c \in R$ and $(M^*)^c \subseteq R^* \subseteq M^*$. Finally, since $(M^*)^c$ and $M^*$ have the same rank, this is also the rank of $R^*$.

**Corollary 4.6.** Let $Q_A = \prod_{i=1}^t \mathbb{Q}(\zeta_{n_i})$. Then, $A^* \cong T \times \mathbb{Z}^g$ where

$$g = \sum_{i=1}^t \left( \frac{\phi(n_i)}{2} - 1 \right)^*$$

and $T$ is a subgroup of even order of $U = \prod_{i=1}^t (-\zeta_{n_i})$.

**Proof.** The order $A$ is contained in the maximal order $M_A \cong \prod_{i=1}^t \mathbb{Z}[\zeta_{n_i}]$, hence $\{ \pm 1 \} < A^* < M_A^*$ and by Lemma 4.5 the two groups have the same rank $g$. The result follows since

$$M_A^* \cong \prod_{i=1}^t Z[\zeta_{n_i}]^* \cong \prod_{i=1}^t \left( (-\zeta_{n_i}) \times Z^{\left( \frac{\phi(n_i)}{2} - 1 \right)^*} \right) \cong U \times \mathbb{Z}^g.$$

**4.1. Study of $\mathbb{Z}[\alpha]$.** In this subsection we introduce some instruments that will be useful in the following and will describe, in some particular cases, the group of units of a torsion ring $A$ in the case when $s = 1$, namely $A = \mathbb{Z}[\alpha]$. Although not all the results are actually needed in the rest of the paper, we include them since we believe that they have an intrinsic interest and can suggest some motivations for the construction we will make in the proof of Theorem 4.12.

With the notation of Lemma 4.2 let $\mu_n(x) = \Phi_{m_1} \cdots \Phi_{m_k}(x)$. Then $Q_{\mathbb{Z}[\alpha]} = \prod_{i=1}^r \mathbb{Q}(\zeta_{n_i})$ and Corollary 4.6 shows that

$$\text{rank}(\mathbb{Z}[\alpha]^*) = \sum_{i=1}^r \left( \frac{\phi(m_i)}{2} - 1 \right)^*$$

(2)
and \((\mathbb{Z}[\alpha])_{\text{tors}}^*\) is a subgroup of \(\prod_{i=1}^{r} \langle -\zeta_m \rangle\).

To completely determine \((\mathbb{Z}[\alpha])_{\text{tors}}^*\) we need a more detailed analysis. We introduce the general argument by giving two examples.

**Example 1.** Let \(\mathcal{M} = \mathbb{Z}[\zeta_3] \times \mathbb{Z}[i]\) and let \(\alpha = (\zeta_3, i) \in \mathcal{M}\). The element \(\alpha\) is a unit of order \(12\). Let \(\varphi_\alpha : \mathbb{Z}[x] \to \mathcal{M}\) the substitution homomorphism sending \(\varphi_\alpha(1) = (1, 1)\) and \(\varphi_\alpha(x) = \alpha = (\zeta_3, i)\). We have \(\ker(\varphi_\alpha) = (\Phi_3(x)\Phi_4(x))\) and \(\mathbb{Z}[\alpha] \cong \mathbb{Z}[x]/(\Phi_3(x)\Phi_4(x))\).

Moreover, \(\mathbb{Z}[\alpha] \cong \mathbb{Z}[x]/(\Phi_3(x)\Phi_4(x)) \cong \mathbb{Z}[x]/(\Phi_3(x)) \times \mathbb{Z}[x]/(\Phi_4(x)) \cong \mathcal{M}\) where the second isomorphism is given by the Chinese Remainder Theorem since \((\Phi_3(x), \Phi_4(x)) = \mathbb{Z}[x]\) (see Proposition 4.7).

**Example 2.** Let \(\mathcal{M} = \mathbb{Z}[\zeta_3] \times \mathbb{Z}[\zeta_9]\) and let \(\alpha = (\zeta_3, \zeta_9) \in \mathcal{M}\). Clearly, \(\alpha\) is a unit of order \(9\). Denoting again by \(\varphi_\alpha : \mathbb{Z}[x] \to \mathcal{M}\) the substitution homomorphism, we have \(\ker(\varphi_\alpha) = (\Phi_3(x)\Phi_9(x))\).

However, in this case the map \(\varphi_\alpha\) is not onto \(\mathcal{M}\), since for example \((\zeta_3, 1)\) is not in the image. This can also be seen via Chinese Remainder Theorem since \((\Phi_3(x), \Phi_9(x)) \not\subseteq \mathbb{Z}[x]\).

The following proposition generalizes the previous examples.

**Proposition 4.7.** Let \(K\) be a number field and let \(\mathcal{O}_K\) be its ring of integers.

Let \(n > m \geq 1\) and denote by \(\Psi_m(x)\) and \(\Psi_n(x)\) the minimal polynomials over \(K\) of \(\zeta_m\) and \(\zeta_n\) respectively. Then the following are equivalent:

1. \(\mathcal{O}_K[x]/(\Psi_m(x)\Psi_n(x)) \cong \mathcal{O}_K[\zeta_m] \times \mathcal{O}_K[\zeta_n]\);
2. \((\Psi_m(x), \Psi_n(x)) = \mathcal{O}_K[x]^{*}\);
3. \(\Psi_n(\zeta_m)\) is invertible.

**Proof.** Assume that there exists an injective homomorphism \(\psi : \mathcal{O}_K[x]/(\Psi_m(x)\Psi_n(x)) \to \mathcal{O}_K[\zeta_m] \times \mathcal{O}_K[\zeta_n]\), and let \(\tilde{\psi} : \mathcal{O}_K[x] \to \mathcal{O}_K[\zeta_m] \times \mathcal{O}_K[\zeta_n]\) be the lifting of \(\psi\) to \(\mathcal{O}_K[x]\).

If \(\tilde{\psi}(x) = (\alpha, \beta)\), then \(\tilde{\psi}(\Psi_m(x)\Psi_n(x)) = (\Psi_m(\alpha)\Psi_n(\alpha), \Psi_m(\beta)\Psi_n(\beta)) = (0, 0)\), hence one between \(\Psi_m(\alpha)\) and \(\Psi_n(\alpha)\) and one between \(\Psi_m(\beta)\) and \(\Psi_n(\beta)\) is 0. This means that \(\alpha\) and \(\beta\) are primitive \(m\)-th or \(n\)-th roots of unity. Now, \(\alpha\) and \(\beta\) can not have the same order, since otherwise \(\Psi_m(x)\) or \(\Psi_n(x)\) would belong to \(\ker(\psi)\). Moreover, since \(\zeta_n \not\in \mathcal{O}_K[\zeta_m]\), or \(\zeta_m \not\in \mathcal{O}_K[\zeta_n]\), then \(\alpha\) must be a primitive \(m\)-th roots of unity and \(\beta\) must be a primitive \(n\)-th roots of unit. Therefore, up to an isomorphism fixing \(\mathcal{O}_K\), we can assume \(\alpha = \zeta_m\) and \(\beta = \zeta_n\).

This shows that there exists an isomorphism as in (i) if and only if the Chinese Remainder map is an isomorphism and this holds if and only if (ii) is true.
Remark 4.9. For convenient but non necessary for the conditions (i) and only if \( n/m \) is not a prime. This means that, if \( n/m \) is not a prime power, than each \( \Lambda \)-module extension of \( \mathbb{Z}[\zeta_m] \) by \( \mathbb{Z}[\zeta_n] \)

\[
1 \to \mathbb{Z}[\zeta_n] \to E \to \mathbb{Z}[\zeta_m] \to 1
\]
splits. Now, the ring \( \mathbb{Z}[x]/(\Phi_m(x)\Phi_n(x)) \) is actually a \( \Lambda \)-module extension of \( \mathbb{Z}[\zeta_n] \) by \( \mathbb{Z}[\zeta_m] \), so it should be isomorphic to their direct product. The basic idea of the two proofs is similar, however, we decided to detail the proof since our argument is completely elementary.

In the general case the condition \( n/m \) is not a prime power is sufficient but non necessary for the conditions (i), (ii), (iii) to hold.

Remark 4.9. For \( K = \mathbb{Q} \) the last proposition could also be deduced from [CR81, Theorem 25.26], where the following is proved. Let \( \Lambda = \mathbb{Z}[\zeta_l] \) and \( n, m | l \), then \( \text{Ext}_1^\Lambda(\mathbb{Z}[\zeta_m], \mathbb{Z}[\zeta_n]) = 0 \) unless \( \frac{n}{m} \) is the power of a prime. This means that, if \( \frac{n}{m} \) is not the power of a prime, than each \( \Lambda \)-module extension of \( \mathbb{Z}[\zeta_n] \) by \( \mathbb{Z}[\zeta_m] \)

\[
\frac{n}{m} \text{ is not a power of a prime;}
\]

\[
T \cong \begin{cases} 
C_{[l,m]} \times C_{[l,n]} & \text{if } \frac{n}{m} \text{ is not a power of a prime;} \\
C_{[l,m]} & \text{if } \frac{n}{m} = p^a \text{ with } p \neq 2; \\
C_2 \times C_{[l,n]} & \text{if } \frac{n}{m} = 2^a.
\end{cases}
\]

Proof. The conditions \( (l, m) \leq 2 \) and \( (l, n) \leq 2 \) ensure that \( \Phi_m(x) \) and \( \Phi_n(x) \) are irreducible over \( K \). In this case the maximal order of \( \mathbb{O}_K[x]/(\Phi_m(x)\Phi_n(x)) \) is isomorphic to \( \mathcal{M} = O_K[\zeta_m] \times O_K[\zeta_n] \) and the value of the rank follows from \( \textbf{[2]} \).

Consider the injection given by the Chinese Remainder Theorem:

\[
\phi : \mathbb{O}_K[x]/(\Phi_m(x)\Phi_n(x)) \rightarrow \mathbb{O}_K[x]/(\Phi_m(x)) \times \mathbb{O}_K[x]/(\Phi_n(x)) \cong \mathcal{M}.
\]
The map $\varphi$ can be restricted to the group of units, hence

$$T = \varphi((\mathcal{O}_K[x]/(\Phi_n(x)\Phi_n(x)))^*_{tor})$$

is a subgroup of $U = (\zeta_{[l,m]}) \times (\zeta_{[l,n]}) \cong C_{[l,m]} \times C_{[l,n]}$.

If $\frac{n}{m}$ is not a power of a prime, then, by Corollary 4.8, the map $\varphi$ is an isomorphism so $T \cong C_{[l,m]} \times C_{[l,n]}$.

Assume now $\frac{n}{m} = p^a$ for a prime $p$ and $a > 0$. To describe $T$ we will characterize the units of $U$ which belongs to $\text{Im}(\varphi) = \{(a(\zeta_m),a(\zeta_n)) \mid a(x) \in \mathcal{O}_K[x]\}$.

Consider first the subgroup $T_0$ of $T$ made by the trivial units, namely the subgroup generated by $\varphi(\zeta_l) = (\zeta_l, \zeta_l)$ and $\varphi(x) = (\zeta_m, \zeta_n)$.

If $(l,n) = 1$ then $T_0 \cong C_l \times C_n \cong C_{ln}$.

If $(l,n) = 2$ the subgroup $\langle(\zeta_l,\zeta_l)\rangle \cap \langle(\zeta_m,\zeta_n)\rangle$ can be trivial or of order 2, according to $(-1,-1) = (\zeta_m^{n/2},\zeta_n^{n/2})$ or not. The equality holds if and only if $\zeta_m^{n/2} = -1$. Now, $n = mp^a$, so if $p = 2$ then $\zeta_m^{n/2} = 1$ hence $T_0 \cong C_l \times C_n \cong C_2 \times C_{[l,n]}$. If $p \neq 2$, then $m$ is even and $(\zeta_m^{n/2})^{p^a} = -1$. In this case $T_0 \cong C_{ln/2} = C_{[l,n]}$.

To conclude the proof we have to show that $T$ contains only trivial units. Let $u = (\zeta_l^d \zeta_m^k, \zeta_n^j)$ then $u$ is equivalent to $v = (\zeta_l^{e-f} \zeta_m^{i-j},1)$ modulo $T_0$. Let $h$ be the representative of the class $e-f$ modulo $l$ with $0 \leq h < l$ and let $k$ be the representative of the class $i-j$ modulo $m$ with $0 \leq k < m$. Then

$$v \in T \iff v - (1,1) = (\zeta_l^h \zeta_m^k - 1,0) \in \text{Im}(\varphi).$$

This means that there exists $a(x) \in \mathcal{O}_K[x]$ such that

$$\zeta_l^h \zeta_m^k - 1 = a(\zeta_m) \Phi_n(\zeta_m).$$

By Lemma 2.7, $\Phi_n(\zeta_m) \in p\mathcal{O}_K[\zeta_m]$, hence the last equation implies

$$\zeta_l^h \zeta_m^k - 1 \in p\mathcal{O}_K[\zeta_m].$$

Consider the element $\zeta_l^h \zeta_m^k = \zeta_{lm+lk}^\nu$: it is a primitive $\nu$-th root of units where $\nu = lm/(lm,mh+lk)$ and, up to a conjugation, equation (3) can be rewritten as

$$\zeta_\nu - 1 \in p\mathbb{Z}[\zeta_{[l,m]}].$$

By Lemma 2.7, (4) can hold only if $\nu$ is 1 or is a prime power. If $\nu = 1$ then $v = (1,1)$, hence $u \in T_0$.

Let now $\nu = q^b$. In this case $\zeta_\nu - 1$ is a generator of the prime of the ring of integers $\mathbb{Z}[\zeta_\nu]$ over $q$, so (4) can have solution only in the case $q = p$. For $\nu = q^b$ ($b \geq 1$), equation (4) can be rewritten as $\zeta_{\nu} - 1 \in (p) = (\zeta_p^b - 1)^{\nu(p^b)}$ and this is true only for $\phi(p^b) = 1$, i.e., if $\nu = p^b = 2$. This is the case only if $v = (-1,1)$ which for $p = 2$ belongs to $T_0$.

This proves that $T = T_0$ and has the required decomposition. □

**Proposition 4.11.** Let $l$ be a positive even integer, let $K = \mathbb{Q}(\zeta_l)$ and let $\mathcal{O}_K$ be its ring of integers.
Let $p$ be a prime, let $a > 0$ and assume that $l = l_1 p^b$ with $(l_1, p) = 1$ and $b \leq a$. Denote by $\Psi_p(x)$ the minimal polynomial of $\zeta_p$ over $K$. Then

$$(\mathcal{O}_K[x]/((x - 1)\Psi_p(x)))^* \cong C_l \times C_p \times \mathbb{Z}^b$$

where $g = \left(\frac{\varphi(p)}{2} - 1\right) + \left(\frac{\varphi(l_1 p^b)}{2} - 1\right)$.  

**Proof.** The $\mathcal{O}_K[x]/((x - 1)\Psi_p(x))$ can be embedded in the maximal order $\mathcal{M} = \mathcal{O}_K \times \mathcal{O}_K[\zeta_p] \cong \mathbb{Z}[\zeta_l] \times \mathbb{Z}[\zeta_{l_1 p^b}]$ via the Chinese Remainder Theorem:

$$\varphi: \mathcal{O}_K[x]/((x - 1)\Psi_p(x)) \to \mathcal{O}_K[x] \times \mathcal{O}_K[x]/(\Psi_p(x)) \cong \mathcal{M}.$$  

As for the groups of units we get that

$$T = \varphi((\mathcal{O}_K[x]/((x - 1)\Psi_p(x)))^*_{tors}$$

is a subgroup of $U = \langle \zeta_l \rangle \times \langle \zeta_{l_1 p^b} \rangle \cong C_l \times C_{l_1 p^b}$.

As in the previous case, we will show that all the units of $T$ are trivial units, namely they belong to the subgroup $T_0 = \langle (\zeta_l, \zeta_{l_1 p^b}) \rangle$. We note that $T_0 \cong C_l \times C_p$, since $\langle (\zeta_l, \zeta_l) \rangle \cap \langle (1, \zeta_p) \rangle = (1, 1)$.

Let $u = (\zeta_l^i, \zeta_{l_1 p^b}^j) \in T$, then $u$ is equivalent to $v = (\zeta_l^{i-j}, 1)$ modulo $T_0$, hence $u \in T$ if and only if $(\zeta_l^{i-j} - 1, 0) \in \text{Im}(\varphi)$.

This means that there exists $a(x) \in \mathcal{O}_K[x]$ such that

$$\zeta_l^{i-j} - 1 = a(1)\Psi_p^0(1).$$

By Lemma 2.6 $(\Psi_p(1)) = P_b$ where $P_b = (1 - \zeta_p)$ if $b \geq 1$ and $P_0 = (p)$, hence last equation implies

$$\zeta_l^{i-j} - 1 \in P_b. \tag{5}$$

The element $\zeta_l^{i-j}$: it is a primitive $\nu$-th root of units where $\nu = l/(l_1, i - j)$ and $\zeta_l - 1 \in P_b$ if and only if $\nu|p^b$.

If $b \geq 1$, this holds exactly when $i \equiv j \pmod{l_1}$. With this condition we have $u = (\zeta_l^i, \zeta_{l_1 p^b}^j)$ and this element is clearly in $T_0$.

If $b = 0$ equation (5) can hold only for $p = 2$, $\nu = 2$ and $i \equiv j \pmod{l_1 2^{b-1}}$. This corresponds to the unit $u = (\zeta_l^i, -\zeta_l^j \zeta_{2^n}^k) = (\zeta_l^i, \zeta_l^{j+k+2n})$ which belongs to $T_0$. This proves that $T = T_0$ and hence it has the required decomposition.

4.2. **The classification theorem.** Our aim is to classify the abelian and finitely generated groups which arise as group of units of a torsion-free ring. This question is twofold: on the one hand, we have to establish what finite groups $T$ (up to isomorphism) can be the torsion subgroup of $A^*$ when $A$ is a torsion-free ring. On the other hand, we have to determine what are the possible values of the rank, $g(A)$, of $A^*$ when $(A^*)_{tors} \cong T$. In Theorem 4.12 give a complete answer to both questions.
For every finitely generated abelian group $T$ define $\varepsilon = \varepsilon(T)$ as the minimum exponent of 2 in the decomposition of the 2-Sylow of $T$ as direct sum of cyclic groups.

If $T$ is a finite abelian group of even order, then $T$ can be uniquely written as

$$T \cong \prod_{i=1}^{s} C_{p_i^{a_i}} \times \prod_{i=1}^{\rho} C_{2^{\varepsilon_i}} \times C_{2^{\varepsilon}}$$

(6)

where $s, \rho \geq 0$, $\sigma \geq 1$ and
- for all $i = 1, \ldots, s$ the $p_i$’s are odd prime numbers not necessarily distinct and $a_i \geq 1$;
- $\varepsilon = \varepsilon(T) \geq 1$ and $\varepsilon_i > \varepsilon$ for all $i = 1, \ldots, \rho$.

Assume that $p_1, \ldots, p_{s_0}$ are the distinct primes in the set $\{p_1, \ldots, p_s\}$. Denoting by $T_{p_i}$ the $p_i$-Sylow of $T$, for $i = 1, \ldots, s_0$, and by $T_2$ its 2-Sylow, we can also write $T$ as

$$T \cong \prod_{i=1}^{s_0} T_{p_i} \times T_2.$$  

(7)

**Theorem 4.12.** Let $T$ be a finite abelian group of even order as above and put

$$g(T) = \sum_{i=1}^{s} \left( \frac{\phi(2^{\varepsilon p_i^{a_i}})}{2} - 1 \right) + \sum_{i=1}^{\rho} \left( \frac{\phi(2^{\varepsilon_i})}{2} - 1 \right) + c(T)$$

(8)

where

$$c(T) = \begin{cases} (\sigma - s)(\frac{\phi(2^{s})}{2} - 1)^* & \text{for } s < \sigma \\ 0 & \text{for } s_0 \leq \sigma \leq s \\ (\frac{\phi(2^{\varepsilon})}{2} - 1)^* & \text{for } \sigma < s_0. \end{cases}$$

Then there exists a torsion free ring $A$ with

$$A^* \cong T \times \mathbb{Z}^r$$

if and only if $r \geq g(T)$.

As a particular case of this theorem we re-obtain the classification of finite groups which occur as group of units of a torsion-free ring, already found in [DCD18b, Thm 4.1].

**Corollary 4.13.** The finite abelian groups which are the group of units of a torsion-free ring are all those of the form

$$C_{2^a} \times C_{4^b} \times C_{3^c}$$

where $a, b, c \in \mathbb{N}$, $a + b \geq 1$ and $a \geq 1$ if $c \geq 1$.

**Proof.** A finite abelian groups $T$ of even order is the group of units of a torsion-free ring if and only if $g(T) = 0$. In the notation of (6), this means that $\frac{\phi(2^{\varepsilon p_i^{a_i}})}{2} - 1 = 0$ for all $i = 1, \ldots, s$ and $\frac{\phi(2^{\varepsilon_i})}{2} - 1 = 0$ for each $i = 1, \ldots, \rho$. This implies that $\varepsilon_i \leq 2$ for all $i$ and that, if $s > 0$, then $p_i = 3$ for all $i$ and $\varepsilon = 1$. $\square$
Before to proceed with the proof we want to point out that all the complication in relation to the realization of a group \( T \) are related to its 2-torsion part. The following examples show a phenomenon which at first sight may seem paradoxical: sometimes for a groups \( T \) with subgroup \( T' \) one can have that \( g(T) < g(T') \).

**Example 3.** Let \( T = C_2 \times C_8 \times C_5 \). In this case \( \varepsilon = 1 \) and \( g(T) = 2 \); in fact, choosing \( A \) equal to the maximal order \( \mathcal{M} = \mathbb{Z}[\zeta_8] \times \mathbb{Z}[\zeta_5] \) we have \( A^* \cong T \times \mathbb{Z}^2 \).

**Example 4.** Let \( T \cong C_8 \times C_5 \) and let \( A \) be a torsion-free ring such that \((A^*)_{\text{tors}} \cong T\). Then, \( A \) contains a unit \( \alpha \) of order 8 and a unit \( \beta \) of order 5. If we denote by \( \mathcal{M} \) the maximal order of \( A \), then \( \mathcal{M} \) must contain a direct factor with a subring isomorphic to \( \mathbb{Z}[\zeta_8] \) and one which contains \( \mathbb{Z}[\zeta_5] \). There are two minimal possibilities: \( \mathcal{M} = \mathbb{Z}[\zeta_8] \times \mathbb{Z}[\zeta_5] \) or \( \mathcal{M} = \mathbb{Z}[\zeta_{40}] \). The first possibility has to be excluded since each order of a maximal order containing \( \mathbb{Z}[\zeta_8] \times \mathbb{Z}[\zeta_5] \) has at least 3 units of order 2 (this will be clear after Lemma 4.14). In this case Theorem 4.12 shows that \( g(T) = \phi(40)/2 - 1 = 7 \).

The proof of Theorem 4.12 is quite long. For the convenience of the reader, we separate the "only if" part and the "if" part. Both parts require a number of auxiliary results that we will prove separately, in order to make it easier to follow the main argument.

### 4.3. Proof of Theorem 4.12 the "only if" part

Let \( A \) be a torsion free ring with finitely generated group of units, such that \((A^*)_{\text{tors}} \cong T\). We have to prove the rank\((A^*) \geq g(T)\).

To this aim, by Lemma 2.1 we can assume that \( A = \mathbb{Z}[(A^*)_{\text{tors}}] \) and Proposition 4.3 says that there exist \( n_1, \ldots, n_t \) such that \( Q_A = A \otimes \mathbb{Q} \cong \prod_{j=1}^t \mathbb{Q}(\zeta_{n_j}) \). Now, by Lemma 4.5 the rank of \( A^* \) is equal to the rank of the maximal order \( \mathcal{M}_A = \prod_{j=1}^t \mathbb{Z}[\zeta_{n_j}] \) which is known by Dirichlet’s Unit Theorem.

In order that \( \mathcal{M} = \prod_{j=1}^t \mathbb{Z}[\zeta_{n_j}] \) contains an order \( A \) such that \((A^*)_{\text{tors}} \cong T\), the \( n_j \)'s must fulfill the following necessary conditions (see Lemma 4.14 below):

- \( t \geq \rho + \sigma \);
- \( 2^e | n_j \) for all \( j = 1, \ldots, t \);
- \( \text{for each } i = 1, \ldots, s \) there exists an index \( j_i \in \{1, \ldots, t\} \) such that \( p_i^{n_i} | n_{j_i} \); moreover, \( j_i \neq j_h \) if \( p_i = p_h \);
- \( \text{for each } i = 1, \ldots, \rho \) there exists an index \( l_i \in \{1, \ldots, t\} \) such that \( 2^e | n_{l_i} \) and \( l_i \neq l_h \) if \( i \neq h \).

We will say that the maximal order \( \mathcal{M} = \prod_{j=1}^t \mathbb{Z}[\zeta_{n_j}] \) is \( T \)-admissible if \( \{n_1, \ldots, n_t\} \) fulfills the conditions (i)-(iv).
Define
\[ \mathcal{M}_{0,T} = \prod_{i=1}^{s} \mathbb{Z}[\zeta_{2^{e_{i}}}] \times \prod_{i=1}^{\rho} \mathbb{Z}[\zeta_{2^{e_{i}}}] \times \mathbb{Z}[\zeta_{2^{e}}]^d, \]
where \(d = \max\{\sigma - s, 0\}\). \(\mathcal{M}_{0,T}\) is \(T\)-admissible and in Proposition 4.16 we prove that \(\mathcal{M}_{0,T}\) has minimum rank among the groups of units of all \(T\)-admissible maximal orders. This ensures that
\[ \text{rank}(A^*) = \text{rank}(\mathcal{M}_{0,T}^*) \geq \text{rank}(\mathcal{M}_{0,T}). \]

Now,
\[ \text{rank}(\mathcal{M}_{0,T}) = \sum_{i=1}^{s} \left( \frac{\phi(2^{\epsilon_{i}})}{2} - 1 \right) + \sum_{i=1}^{\rho} \left( \frac{\phi(2^{\epsilon_{i}})}{2} - 1 \right) + d \left( \frac{\phi(2^{\epsilon})}{2} - 1 \right)^*, \]

hence
\[ \text{rank}(\mathcal{M}_{0,T}^*) = \begin{cases} g(T) & \text{for } \sigma \geq s_0 \\ g(T) - \left( \frac{\phi(2^{\epsilon})}{2} - 1 \right)^* & \text{for } \sigma < s_0. \end{cases} \]

If \(\sigma \geq s_0\) or if \(\epsilon = 1\) we got the required bound on \(\text{rank}(A^*)\).

On the other hand, Proposition 4.17 shows that, if \(\sigma < s_0\), then \(\mathcal{M}_{0,T}\) does not contain any order \(A\) with \((A^*)_{\text{tors}} \cong T\). Now, by Proposition 4.16, for \(\epsilon > 1\), \(\mathcal{M}_{0,T}\) is the only \(T\)-admissible maximal order of minimum rank, hence, if \(\sigma < s_0\) and \(\epsilon > 1\), then \(\text{rank}(A^*) > \text{rank}(\mathcal{M}_{0,T}^*)\) and \(\text{rank}(A^*) \geq \text{rank}(\mathcal{M}_{0,T}) + \left( \frac{\phi(2^{\epsilon})}{2} - 1 \right)^* = g(T)\) (see again Proposition 4.16).

We now state and prove the results quoted above.

**Lemma 4.14.** Let \(\mathcal{M} = \prod_{j=1}^{t} \mathbb{Z}[\zeta_{n_j}]\). If \(\mathcal{M}\) contains a subring \(A\) with \((A^*)_{\text{tors}} \cong T\), then the following hold:

i) \(t \geq \rho + \sigma\);

ii) \(2^{\epsilon} \mid n_j\) for all \(j = 1, \ldots, t\);

iii) for each \(i = 1, \ldots, s\) there exists an index \(j_i \in \{1, \ldots, t\}\) such that \(p_i^{\epsilon_i} \mid n_{j_i}\); moreover, \(j_i \neq j_h\) if \(p_i = p_h\);

iv) for each \(i = 1, \ldots, \rho\) there exists an index \(l_i \in \{1, \ldots, t\}\) such that \(2^{\epsilon_i} \mid n_{l_i}\) and \(l_i \neq l_h\) if \(i \neq h\).

**Proof.** For each prime \(q\), the \(q\)-Sylow subgroup of \(\mathcal{M}^*\) is the direct product of the (cyclic) \(q\)-Sylow of its cyclic factors \(\langle \zeta_{n_j} \rangle\), hence every of its \(q\)-Sylow has at most \(t\) cyclic components. Looking at the 2-Sylow of \(T\) we get \(t \geq \sigma + \rho\). Moreover, if \(T\) has an element of order \(q^k\), for some \(k \geq 1\), then the \(q\)-Sylow of \(\mathcal{M}^*\) has a cyclic component of order at least \(q^k\), namely, \(q^k \mid n_j\) for some \(j \in \{1, \ldots, t\}\); this proves the first part of (iii) and (iv). The last part of these statements follows by noticing that the \(q\)-Sylow of \(\langle \zeta_{n_j} \rangle\) is cyclic.

We are now left with proving (ii). By identifying \(A\) with its image in \(\prod_{j=1}^{t} \mathbb{Z}[\zeta_{n_j}]\), we have that the opposite of the identity \((-1, \ldots, -1)\) is an element of order 2 in \((A^*)_{\text{tors}} \cong T\) which is in turn a subgroup of
\[ \prod_{j=1}^{t} \langle \zeta_{n_j} \rangle. \] Now the 2-Sylow of \((A^*)_{\text{tors}}\) is isomorphic to \(C_2^t \times \prod_{i=1}^{t} C_{2^{e_i}}\), and all elements of order 2 of such a group belong to the subgroup \((C_2^t)^{\sigma} \times \prod_{i=1}^{t} C_{2^{e_i}}^{\rho_i}\), hence they are \(2^{e_i-1}\)-powers since \(\varepsilon_i > \varepsilon\) for all \(i\). In particular,

\[ (-1, \ldots, -1) = \gamma^{2^{e_i-1}} = (\gamma_{i_1}^{2^{e_i-1}}, \ldots, \gamma_{t}^{2^{e_i-1}}) \]

with \(\gamma_j \in \langle \zeta_{n_j} \rangle, \forall j\). It follows that \(\text{ord}(\gamma_j) = 2^\varepsilon\) (in fact, \(\text{ord}(\gamma_j) \mid 2^\varepsilon\) and \(\text{ord}(\gamma_j) \nmid 2^\varepsilon\)) so \(2^\varepsilon \mid n_j\) for all \(j\). \(\square\)

**Remark 4.15.** Point (ii) of the previous lemma shows that each \(T\)-admissible maximal order is a \(\mathbb{Z}[\zeta_{2^t}]\)-algebra.

**Proposition 4.16.** Let \(M = \prod_{j=1}^{t} \mathbb{Z}[\zeta_{n_j}]\) be \(T\)-admissible. Then,

\[ \text{rank}(M^*) \geq \sum_{i=1}^{k} \left( \frac{\phi(2^\varepsilon q_i^{\varepsilon_i})}{2} - 1 \right) + \sum_{i=1}^{k} \left( \frac{\phi(q_i^{\varepsilon_i})}{2} - 1 \right) \]

and equality holds only for \(M = M_{0,T}\) or, in the case when \(\varepsilon = 1\), for \(M = M_{0,T} \times \mathbb{Z}^k\) and \(k \geq 0\).

Moreover, if \(M \neq M_{0,T}\), then \(\text{rank}(M^*) \geq \text{rank}(M_{0,T}^*) + (\frac{\phi(2^\varepsilon)}{2} - 1)^{\ast}\).

**Proof.** Assume the \(M = \prod_{j=1}^{t} \mathbb{Z}[\zeta_{n_j}]\) is \(T\)-admissible. Then, by Lemma 4.14 all the \(n_j\)'s are divisible at least by \(2^\varepsilon\) and, up to reordering, we can assume that \(n_1, \ldots, n_t\) are divisible by \(2^{\varepsilon_1}, \ldots, 2^{\varepsilon_t}\), respectively.

Now, if \(m = 2^{d_1} q_1^{e_1} \cdots q_k^{e_k}\), with the \(q_i\)'s pairwise distinct odd primes, we can prove that, for \(\delta \geq \varepsilon\) we have

\[ \frac{\phi(2^\delta q_1^{e_1} \cdots q_k^{e_k})}{2} - 1 \geq \sum_{i=1}^{k} \left( \frac{\phi(2^\varepsilon q_i^{e_i})}{2} - 1 \right) \tag{10} \]

and for \(\delta > \varepsilon\) we have

\[ \frac{\phi(2^\delta q_1^{e_1} \cdots q_k^{e_k})}{2} - 1 \geq \sum_{i=1}^{k} \left( \frac{\phi(2^\varepsilon q_i^{e_i})}{2} - 1 \right) + \frac{\phi(2^\delta)}{2} - 1. \tag{11} \]

Let us prove (11); the easy check of (10) is left to the reader. For \(k = 0\) the formula is an equality. Let \(k > 0\); noting that for all \(a, b > 1\) it holds \(\phi(ab) \geq \phi(a) + \phi(b)\) and using the condition \(\delta \geq \varepsilon + 1\), we have

\[ 2^{d-1} \phi(q^e) = 2^{\delta-2} \phi(q^e) + 2^{\delta-2} \phi(q^e) \geq \phi(2^\varepsilon q^e) + \phi(2^\delta), \]

whence:

\[ \phi(2^\delta q_1^{e_1} \cdots q_k^{e_k}) \geq 2^{\delta-1} \sum_{i=1}^{k} \phi(q_i^{e_i}) \geq \sum_{i=1}^{k} \phi(2^\varepsilon q_i^{e_i}) + k \phi(2^\delta) \]

and (11) follows. Now, \(\text{rank}(M^*) = \sum_{j=1}^{t} \text{rank}(\mathbb{Z}[\zeta_{n_j}]^*) = \sum_{j=1}^{t} \left( \frac{\phi(n_j)}{2} - 1 \right)^{\ast}\).
By our initial assumption, for $j = 1, \ldots, \rho$ we can bound from below the term $\frac{\phi(n_j)}{2} - 1$ by using (11), while for $j = \rho + 1, \ldots, t$ we can use (10). Hence,

$$\sum_{j=1}^{\rho} \text{rank}(\mathbb{Z}[\xi_{n_j}]) \geq \sum_{j=1}^{\rho} \sum_{q \text{ odd prime } q || n_j} \left(\frac{\phi(2^\varepsilon q^e)}{2} - 1\right) + \sum_{i=1}^{\rho} \left(\frac{\phi(2^\varepsilon)}{2} - 1\right)$$

and

$$\sum_{j=\rho+1}^{t} \text{rank}(\mathbb{Z}[\xi_{n_j}]) \geq \sum_{j=\rho+1}^{t} \sum_{q \text{ odd prime } q || n_j} \left(\frac{\phi(2^\varepsilon q^e)}{2} - 1\right).$$

By Lemma 4.14, each $p_i^{a_i}$ divides some $n_j$, hence all terms $\frac{\phi(2^\varepsilon p_i^{a_i})}{2} - 1$ occur at least once in the previous formulas, hence a fortiori we get

$$\text{rank}(\mathcal{M}^*) \geq \sum_{i=1}^{s} \left(\frac{\phi(2^\varepsilon p_i^{a_i})}{2} - 1\right) + \sum_{i=1}^{\rho} \left(\frac{\phi(2^\varepsilon)}{2} - 1\right) + d\left(\frac{\phi(2^\varepsilon)}{2} - 1\right)^*, \quad (12)$$

where the summand $d\left(\frac{\phi(2^\varepsilon)}{2} - 1\right)^*$ can be explained as follows. If $s + \rho < t$ then at least $t - \rho - s$ of the $n_j$ contribute to the right hand side of (12) neither with a term $\frac{\phi(2^\varepsilon p_i^{a_i})}{2} - 1$ nor with a term $\frac{\phi(2^\varepsilon p_i^{a_i})}{2} - 1$: for these $n_j$ (they are at least $t - \rho - s \geq \sigma - s$ of them and $d = \max\{\sigma - s, 0\}$) we use the inequality

$$\text{rank}(\mathbb{Z}[\xi_{n_j}]) \geq \left(\frac{\phi(2^\varepsilon)}{2} - 1\right)^*.$$  

Equation (12) shows that $\mathcal{M}_{0,T}$ has the minimum possible rank among the $T$-admissible maximal orders and, when $\varepsilon = 1$, the same is clearly true for $\mathcal{M} = \mathcal{M}_{0,T} \times \mathbb{Z}^k$.

Finally, if $\mathcal{M} = \prod_{j=1}^{t} \mathbb{Z}[\xi_{n_j}]$ is $T$-admissible, then $\mathcal{M}$ properly contains $\mathcal{M}_{0,T}$ or at least one of the following holds:
- $2^\varepsilon p_i^{a_i} | n_j$ for some $j$ and two coprime factors $p_i^{a_{i1}}, p_i^{a_{i2}}$;
- $2^\varepsilon p_i^{b_i} | n_j$ for some $i$.

In all these cases it easy to prove that

$$\text{rank}(\mathcal{M}^*) \geq \text{rank}(\mathcal{M}_{0,T}^*) + \left(\frac{\phi(2^\varepsilon)}{2} - 1\right)^*.$$  

□

The last proposition shows that the group of units of $\mathcal{M}_{0,T}$ has minimum rank among the $T$-admissible maximal orders. However, in the following proposition we show that, for some $T$, the maximal order $\mathcal{M}_{0,T}$ does not contain an order $A$ with $(A^*)_{tors} \cong T$.

**Proposition 4.17.** If $\sigma < s_0$, then $\mathcal{M}_{0,T}$ does not contain an order $A$ with $(A^*)_{ tors} \cong T$. 
Proof. For brevity, we will write $\mathcal{M}$ for $\mathcal{M}_{0,T}$. Recall that $(\mathcal{M}^*)_{\text{tors}} \cong T \times C_{2^\sigma - \sigma}$. Assume, by contradiction, that $\mathcal{M}$ contains an order $A$ with $(A^*)_{\text{tors}} \cong T$. In the notation of (7), we have $T \cong \prod_{i=1}^{s_0} T_{p_i} \times T_2$, where

$$T_{p_i} = \prod_{j=1}^{v_i} C_{p_i} \quad \text{and} \quad T_2 = \prod_{i=1}^{p} C_{2^\sigma} \times C_{2^\sigma}.$$

(13)

For $i = 1, \ldots, s_0$, put $\mathcal{M}_{p_i} = \prod_{j=1}^{v_i} \mathbb{Z}[\zeta_{p_i}]$ and let $\mathcal{M}_2 = \prod_{i=1}^{p} \mathbb{Z}[\zeta_{2^\sigma}]$. In this case $d = 0$, hence

$$\mathcal{M} \cong \left( \prod_{i=1}^{s_0} \mathcal{M}_{p_i} \right) \times \mathcal{M}_2.$$

We first consider the case when $\rho = 0$, so $\mathcal{M}_2$ is trivial.

For each $i = 1, \ldots, s_0$ let $\alpha_{p_i} = (\zeta_{p_i}, \ldots, \zeta_{p_i}) \in \mathcal{M}_{p_i}$ and put $\alpha = (\alpha_{p_1}, \ldots, \alpha_{p_{s_0}}) \in \mathcal{M}$. Clearly, $\alpha$ is a unit of $\mathcal{M}$ of order $p_1 \cdots p_{s_0}$, so it belongs also to $A^*$.

We claim that

$$\mathbb{Z}[\alpha] \cong \mathbb{Z}[x]/(\Phi_{p_1}(x) \ldots \Phi_{p_{s_0}}(x)) \cong \prod_{i=1}^{s_0} \mathbb{Z}[x]/(\Phi_{p_i}(x)) \cong \prod_{i=1}^{s_0} \mathbb{Z}[\zeta_{p_i}].$$

In fact, denoting by $\varphi_{\alpha} : \mathbb{Z}[x] \rightarrow A$ the substitution homomorphism $x \mapsto \alpha$ we have $\mathbb{Z}[\alpha] \cong \mathbb{Z}[x]/(\ker \varphi_{\alpha})$ and, using Lemmas 4.1 and 4.2, it is easy to check that $\ker \varphi_{\alpha} = (\Phi_{p_1}(x) \ldots \Phi_{p_{s_0}}(x))$. Since last isomorphism is trivial, we are left to consider the isomorphism in the middle. Nothing has to be proven for $s_0 = 1$; for $s_0 \geq 2$ we proceed by induction. The base step $s_0 = 2$ follows from the Chinese Remainder Theorem since $(\Phi_{p_1}(x), \Phi_{p_2}(x)) = \mathbb{Z}[x]$ (see Corollary 1.7). For $s_0 > 2$ we have that $(\Phi_{p_1}(x), \Phi_{p_{s_0}}(x)) = \mathbb{Z}[x]$ for all $i = 1, \ldots, s_0 - 1$, so there exist $a_i(x), b_i(x) \in \mathbb{Z}[x]$ such that

$$a_i(x)\Phi_{p_1}(x) + b_i(x)\Phi_{p_{s_0}}(x) = 1.$$

Multiplying the $s_0 - 1$ equations we get

$$a(x)\Phi_{p_1}(x) \cdots \Phi_{p_{s_0-1}}(x) + b(x)\Phi_{p_{s_0}}(x) = 1,$$

for some $a(x), b(x) \in \mathbb{Z}[x]$, hence the Chinese Remainder Theorem gives

$$\mathbb{Z}[x]/(\Phi_{p_1}(x) \ldots \Phi_{p_{s_0}}(x)) \cong \mathbb{Z}[x]/(\Phi_{p_1}(x) \ldots \Phi_{p_{s_0-1}}(x)) \times \mathbb{Z}[x]/(\Phi_{p_{s_0}}(x))$$

and we can conclude by induction.

It follows that

$$(\mathbb{Z}[\alpha]^*)_{\text{tors}} \cong C_{2^{s_0}} \times C_{p_1} \times \cdots \times C_{p_{s_0}}$$

and this gives a contradiction since $(\mathbb{Z}[\alpha]^*)_{\text{tors}} < (A^*)_{\text{tors}} \cong T$ and $\sigma < s_0$.

In the case when $\rho > 0$, we have to slightly modify the previous argument to obtain the contradiction.
As in the previous case, for each $i = 1, \ldots, s_0$ let $\alpha_{p_i} = (\zeta_{p_i}, \ldots, \zeta_{p_i}) \in \mathcal{M}_{p_i}$ and consider in addition the unit element $v_0$ of $\mathcal{M}_2$ and its opposite $\alpha_2 = -v_0 = (-1, \ldots, -1) \in \mathcal{M}_2$. Put $\alpha' = (\alpha_{p_1}, \ldots, \alpha_{p_{s_0}}, v_0) \in \mathcal{M}$: $\alpha' \in A$ since it is a unit of $\mathcal{M}$ of odd order. Moreover, $\alpha = (\alpha_{p_1}, \ldots, \alpha_{p_{s_0}}, \alpha_2) = (-\alpha')^e$, where $e$ is any odd integer such that $e \equiv 1 \pmod{p_i}$ for all $i = 1, \ldots, s_0$, so also $\alpha \in A$. As before, we can show that $\mathbb{Z}[\alpha] \cong \mathbb{Z}[x]/(\Phi_{p_i}(x) \ldots \Phi_{p_{s_0}}(x) \Phi_2(x))$: the same argument gives

$$\mathbb{Z}[\alpha] \cong \mathbb{Z} \times \prod_{i=1}^{s_0} \mathbb{Z}[\zeta_{p_i}]$$

(in fact, the previous argument does not require the $p_i$’s to be odd, but only distinct primes), and

$$(\mathbb{Z}[\alpha]^*)_{\text{tors}} \cong C_2^{s_0+1} \times C_{p_1} \times \cdots \times C_{p_{s_0}}$$

is a subgroup of $(A^*)_{\text{tors}}$.

On the other hand, for each $i = 1, \ldots, \rho - 1$, let $\beta_i$ be the element of $\mathcal{M}$ with all coordinates 1, but whose coordinate in $\mathbb{Z}[\zeta_{2^i}]$ is equal to $-1$. These elements generate a subgroup of $(\mathcal{M}^*)$ isomorphic to $C_2^{\rho-1}$. Moreover, all the $\beta_i$’s belong to $A^*$: in fact, $\beta_i$ is a $2^i$ power of an element of $(\mathcal{M}^*)_{\text{tors}}$ so it belongs to $(\mathcal{M}^*)_{\text{tors}}^e = (A^*)_{\text{tors}}^e$ (here an inclusion is trivial and equality follows by a cardinality argument). Now, we have that $(\mathbb{Z}[\alpha]^*)_{\text{tors}} \cap \langle \beta_1, \ldots, \beta_{\rho-1} \rangle = \{(1, \ldots, 1)\}$: in fact, the torsion units of $\mathbb{Z}[\alpha]$ are of type $((-\alpha_{p_1})^{e_1}, \ldots, (-\alpha_{p_{s_0}})^{e_0}, \alpha_2^{e_2})$ with $e_0, e_1, \ldots, e_{s_0} \in \mathbb{Z}$, so their coordinates in $\mathcal{M}_2$ are all 1 or all -1. It follows that $(A^*)_{\text{tors}}$ contains a subgroup isomorphic to

$$(\mathbb{Z}[\alpha]^*)_{\text{tors}} \times \langle \beta_1, \ldots, \beta_{\rho-1} \rangle \cong C_2^{s_0+\rho} \times C_{p_1} \times \cdots \times C_{p_{s_0}}$$

and this is not possible since $A^*$ has $\sigma + \rho$ cyclic factors of order a power of 2, and $s_0 + \rho > \sigma + \rho$ since $s_0 > \sigma$.

### 4.4. Proof of Theorem 4.12: the ”if” part

Let $T$ be any finite abelian group of even order as in (6). For each $g \geq g(T)$ we will construct an example of a torsion free ring $A$ with $A^* \cong T \times \mathbb{Z}^2$.

The first and most substantial step is the construction for $g = g(T)$. The following propositions deal with two particular cases.

**Proposition 4.18.** Let $p$ be an odd prime and let $\varepsilon, b_1, \ldots, b_v$ be integers, with $0 < b_1 \leq b_2 \leq \cdots \leq b_v$. The maximal order $\mathcal{M} = \prod_{j=1}^v \mathbb{Z}[\zeta_{2^{b_j}}]$ contains a subring $A$ with $(A^*)_{\text{tors}} \cong C_2^{\varepsilon} \times \prod_{j=1}^v C_{p_j^{b_j}}$.

**Proof.** For $j = 2, \ldots, v$ let $\beta^{(j)} = (\beta_1^{(j)}, \ldots, \beta_v^{(j)}) \in \mathcal{M}$, where $\beta_1^{(j)} = 1$ for $i \neq j$ and $\beta_j^{(j)} = \zeta_{p_j^{b_j}}$, and put

$$A = \mathbb{Z}[\zeta_{2^\varepsilon}]^v[\beta^{(2)}, \ldots, \beta^{(v)}]$$
where we are identifying $A$ with a subring of $\mathcal{M}$ via the diagonal embedding of $\mathbb{Z}[\zeta_{2^r}]$. This means that we identify $\zeta_{2^r}$ with $\alpha = (\zeta_{2^r}, \ldots, \zeta_{2^r})$.

We claim that $(A^*)_{\text{tors}} \cong V = C_{2^r} \times \prod_{j=1}^{v} C_{p_j}$. It is clear that the elements $\alpha, \beta^{(2)}, \ldots, \beta^{(v)} \in A$ are multiplicatively independent units and that they generate a subgroup of $(A^*)_{\text{tors}}$ isomorphic to $V$. On the other hand, $(A^*)_{\text{tors}} \cong \prod_{j=1}^{v} C_{2^r p_j}$, hence

$$(A^*)_{\text{tors}} < V \times C_{2^r - 1}.$$ 

To prove that $(A^*)_{\text{tors}} \cong V$ it is enough to show that the 2-Sylow of $(A^*)_{\text{tors}}$ is cyclic, or equivalently, that $(-1, \ldots, -1)$ is the only element of order 2 of $(A^*)_{\text{tors}}$.

For each $i = 2, \ldots, s$ define $\mathcal{M}_i = \mathbb{Z}[\zeta_{2^r p_1}] \times \mathbb{Z}[\zeta_{2^r p_i}]$ and denote by $\pi_i : \mathcal{M} \to \mathcal{M}_i$ the canonical projection. Put $A_i = \pi_i(A)$ and $\beta_{0,i} = (1, \zeta_{p_i})$, then

$$A_i = \mathbb{Z}[\zeta_{2^r p_i}]$$

Now, the kernel of the evaluation homomorphism $\varphi_{\beta_{0,i}}$, defined on $\mathbb{Z}[\zeta_{2^r}]$ is generated by $(x - 1)\Psi_{p_1, p_i}(x)$ (see Lemma 4.11), hence

$$A_i = \mathbb{Z}[\zeta_{2^r p_i}]$$

and, by Proposition 4.11, $(A^*)_i_{\text{tors}} \cong C_{2^r p_i} \times \mathbb{Z}[\zeta_{2^r p_i}]$. This ensures that, for all indices $i$, the 2-Sylow of $\pi_i((A^*)_{\text{tors}})$, which is a subgroup of $(A^*)_{\text{tors}}$, is cyclic and this allows to conclude the proof. In fact, let $u = (u_1, \ldots, u_v) \in \mathcal{M}$ be such that $u^2 = (1, \ldots, 1)$; if $u \in A$, then $\pi_i(u) = (u_1, u_v)$ is an element of exponent 2 of $(A^*)_{\text{tors}}$, so $(u_1, u_v)$ must be equal to $(1, 1)$ or $(-1, -1)$, in particular, $u_i = u_1$ for all $i = 1, \ldots, v$. This ensures that $u = (1, \ldots, 1)$ or $u = (-1, \ldots, -1)$, and $A^*$ has only one element of order 2 and we get

$$(A^*)_{\text{tors}} \cong C_{2^r} \times C_{p_1} \times \cdots \times C_{p_v} = V.$$ 

When the group $T$ has too few 2-cyclic factors of minimal order, then Proposition 4.17 shows that $\mathcal{M}_{0,T}$ does not contain any order with $(A^*)_{\text{tors}} \cong T$. In this case we have to add to $\mathcal{M}_{0,T}$ an extra direct factor which works as a “control” factor on the 2-torsion. The following proposition deals with the case $\sigma = 1$.

**Proposition 4.19.** The maximal order $\mathcal{M} = \mathbb{Z}[\zeta_{2^r}] \times \prod_{i=1}^{s} \mathbb{Z}[\zeta_{2^r p_i}]$ contains a subring $A$ with $(A^*)_{\text{tors}} \cong C_{2^r} \times \prod_{i=1}^{s} C_{p_i}^{a_i}$. 

**Proof.** For each $i = 1, \ldots, s$, let $\beta^{(i)} = (1, \beta^{(i)}_1, \ldots, \beta^{(i)}_s) \in \mathcal{M}$, where $\beta^{(i)}_j = 1$ for all $j \neq i$ and $\beta^{(i)}_i = \zeta_{p_i}^{a_i}$. Put

$$A = \mathbb{Z}[\zeta_{2^r}][\beta^{(1)}, \ldots, \beta^{(s)}]$$
viewed as a subring of $\mathcal{M}$. We claim that $(A^*)_{\text{tors}} \cong C_{2^s} \times \prod_{i=1}^{s} C_{p_i^a_i}$.

Clearly, the elements $\alpha = (\zeta_2, \ldots, \zeta_2), \beta^{(1)}, \ldots, \beta^{(s)} \in A$ are multiplicatively independent units which generate a subgroup of $(A^*)_{\text{tors}}$ isomorphic to $C_{2^s} \times \prod_{i=1}^{s} C_{p_i^a_i}$.

On the other hand, $(\mathcal{M}^*)_{\text{tors}} \cong C_{2^s+1} \times \prod_{i=1}^{s} C_{p_i^a_i}$, then to prove our claim it is enough to show that the 2-Sylow of $(A^*)_{\text{tors}}$ is cyclic, or equivalently that $(-1, \ldots, -1)$ is the only element of order 2 of $(A^*)_{\text{tors}}$. This can be proved arguing as in the previous proposition. In fact, for each $i = 1, \ldots, s$ define $\mathcal{M}_i = \mathbb{Z}[\zeta_2] \times \mathbb{Z}[\zeta_2^{p_i^a_i}]$ and denote by $\pi_i : \mathcal{M} \to \mathcal{M}_i$ the canonical projection. Put $A_i = \pi_i(A)$ and $\beta_{0,i} = (1, \zeta_2^{p_i^a_i})$, then

$$A_i = \mathbb{Z}[\zeta_2][\pi_i(\beta^{(1)}), \ldots, \pi_i(\beta^{(s)})] = \mathbb{Z}[\zeta_2][\beta_{0,i}].$$

The kernel of the evaluation homomorphism $\varphi_{\beta_{0,i}} : \mathbb{Z}[\zeta_2][x] \to A$ is generated by $\Phi_1(x)\Phi_{p_i^{a_i}}(x)$: in fact, since $p_i^{a_i}$ is odd, the polynomial $\Phi_{p_i^{a_i}}(x)$ is irreducible in $\mathbb{Z}[\zeta_2]$. Thus

$$A_i = \mathbb{Z}[\zeta_2][\beta_{0,i}] \cong \mathbb{Z}[\zeta_2][x]/(\Phi_1(x)\Phi_{p_i^{a_i}}(x))$$

and, by Proposition 4.10 $(A^*)_{\text{tors}} \cong C_{2^s}^{p_i^{a_i}}$. This implies that also its subgroup $\pi_i((A^*)_{\text{tors}})$ is cyclic and this allows to conclude the proof. In fact, let $u = (u_0, \ldots, u_s) \in \mathcal{M}^*$ be such that $u^2 = (1, \ldots, 1)$; if $u \in A$, then, for all $i$, $\pi_i(u) = (u_0, u_i)$ is an element of exponent 2 of the cyclic group $\pi_i((A^*)_{\text{tors}})$, so $(u_0, u_i)$ must be equal to $(1, 1)$ or $(-1, -1)$. In particular, $u_i = u_0$ for all $i = 1, \ldots, s$. This ensures that $u = (1, \ldots, 1)$ or $u = (-1, \ldots, -1)$, and $A^*$ has only one element of order 2, as required.

We are now ready for the general construction for $g = g(T)$. Let

$$\mathcal{M}_T = \begin{cases} 
\mathcal{M}_{0,T} & \text{for } \sigma \geq s_0 \\
\mathcal{M}_{0,T} \times \mathbb{Z}[\zeta_2] & \text{for } \sigma < s_0,
\end{cases} \tag{14}$$

then $\text{rank}(\mathcal{M}_T) = g(T)$ for all $T$. We will construct $A$ as an order in $\mathcal{M}_T$.

The case when $s \leq \sigma$ is very easy: we can simply take $A = \mathcal{M}_T$ since $\mathcal{M}_T^* \cong T \times \mathbb{Z}^{g(T)}$.

Consider now the more general case when $\sigma \geq s_0$. We can write the group $T$ as

$$T = V_2 \times \prod_{i=1}^{s_0} V_{p_i},$$

where $V_{p_i} = C_{2^s} \times T_{p_i} = C_{2^s} \times \prod_{j=1}^{v_i} C_{e_{p_i^{a_j}}}$ and $V_2 = C_{2^s}^{\sigma-s_0} \times \prod_{i=1}^{\rho} C_{2^{s_i}}$.
For \(i = 1, \ldots, s_0\), let \(M_{p_i} = \prod_{j=1}^{s_0} \mathbb{Z}[\zeta_{2^{p_i} p_j}]\) and \(M_2 = \mathbb{Z}[\zeta_2]^{s-s_0} \times \prod_{i=1}^p \mathbb{Z}[\zeta_2]\). Then
\[
M_T \cong M_2 \times \prod_{i=1}^{s_0} M_{p_i}.
\]

By Proposition 4.18, for all \(p = p_1, \ldots, p_{s_0}\), the maximal order \(M_p\) contains a subring \(A_p\) such that \((A_p^*)_{\text{tors}} \cong V_p\). It follows that \(A = M_2 \times \prod_{i=1}^{s_0} A_{p_i}\) is an order of \(M_T\) with \((A^*)_{\text{tors}} \cong T\).

Consider now the case \(\sigma < s_0\). We write the group \(T\) as \(T_0 \times T_1\) where
\[
T_0 = \prod_{i=1}^{\sigma-1} C_{2^{p_i} p_i} \times \prod_{\sigma} C_{2^\sigma}, \quad \text{and} \quad T_1 = C_{2^\sigma} \times C_{p_1^{\sigma}} \times \cdots \times C_{p_{s_0}^{\sigma}}.
\]

By Proposition 4.19 the order \(M_1 = \mathbb{Z}[\zeta_2] \times \prod_{i=\sigma}^p \mathbb{Z}[\zeta_{2^{p_i} p_i}]\) contains a subring \(A_1\) with \((A_1^*)_{\text{tors}} \cong T_1\).

On the other hand,
\[
M_T = M_1 \times \prod_{i=1}^{\sigma-1} \mathbb{Z}[\zeta_{2^{p_i} p_i}] \times \prod_{i=1}^p \mathbb{Z}[\zeta_2]\n\]
and its subring
\[
A = A_1 \times \prod_{i=1}^{\sigma-1} \mathbb{Z}[\zeta_{2^{p_i} p_i}] \times \prod_{i=1}^p \mathbb{Z}[\zeta_2]
\]
is such that \((A^*)_{\text{tors}} \cong T\). Moreover, \(\text{rank}(A^*) \leq \text{rank}(M_T)\), hence they must be equal. This also proves that \(A\) is an order of \(M_T\).

The final step is the construction of torsion-free rings with group of units isomorphic to \(T \times \mathbb{Z}^g\) for all \(g > g(T)\). Also in this case if \(A\) is a torsion-free ring with \((A^*)_{\text{tors}} = T\) and minimal rank \(g(T)\), then \(A = A[x_1, \ldots, x_k, x_1^{-1}, \ldots, x_k^{-1}]\) is torsion-free and has group of units isomorphic to \(T \times \mathbb{Z}^{g(T)+k}\).

5. Reduced rings

In this section we classify the finitely generated abelian groups which arise as group of units of a reduced ring. The next proposition describe the relation between the units of a ring and those of its reduced quotient, showing that the study of reduced rings is a substantial step to the study of units of a general ring.

**Proposition 5.1.** Let \(A\) be a commutative ring and let \(\mathfrak{N}\) be its nilradical. Then the sequence
\[
1 \to 1 + \mathfrak{N} \to A^* \xrightarrow{\phi} (A/\mathfrak{N})^* \to 1,
\]
where \(\phi(x) = x + \mathfrak{N}\), is exact.
We note that for finite characteristic rings the exact sequence (15) always splits (see [DCD18a, Thm 3.1]). This is no longer true in general, as shown in [DCD18b, Ex 2]).

The units of a reduced ring of finite characteristic rings are characterized as follows.

**Proposition 5.2.** The finitely generated abelian groups which are the group of units of a reduced ring $A$ of positive characteristic are exactly those of the form

$$\prod_{i=1}^{k} \mathbb{F}_{p_{i}^{n_{i}}} \times \mathbb{Z}^{g}$$

where $k, n_1, \ldots, n_k$ are positive integers, $\{p_1, \ldots, p_k\}$ are not necessarily distinct prime numbers and $g \geq 0$.

**Proof.** Let $A$ be a reduced ring of characteristic $n$, such that $A^* \cong (A^*)_{\text{tors}} \times F$, with $(A^*)_{\text{tors}}$ finite and $F \cong \mathbb{Z}^g$ for some $g \geq 0$. The ring $B = \mathbb{Z}/n\mathbb{Z}[(A^*)_{\text{tors}}]$ is a finite ring and by Lemma 2.1 $B^* = (A^*)_{\text{tors}}$. Since $B$ is finite, $B$ is artinian and so it is a product of local artinian rings. Moreover, a reduced local artinian ring is a field, hence $B$ is a product of finite fields (see also [DCD18a Corollary 3.2]) and we get that $(A^*)_{\text{tors}} = B^*$ has the required form.

On the other hand, let the $p_i$’s, $n_i$’s and $g$ be as in the statement and put $R = \prod_{i=1}^{k} \mathbb{F}_{p_{i}^{n_{i}}}$. Then the ring $R[x_1, \ldots, x_g, x_{g+1}^{-1}, \ldots, x_{g+k}^{-1}]$ has group of units isomorphic to $\prod_{i=1}^{k} \mathbb{F}_{p_{i}^{n_{i}}} \times \mathbb{Z}^{g}$. □

The following proposition together with the results of the previous section allows to classify the finitely generated abelian groups which arise as group of units of a reduced ring.

**Proposition 5.3.** ([PS70 Prop. 1]) Let $A$ be a commutative ring which is finitely generated and integral over its fundamental subring. Then $A = A_1 \oplus A_2$, where $A_1$ is a finite ring and the torsion ideal of $A_2$ is contained in its nilradical.

Now, if $A$ is reduced then the finite ring $A_1$ is reduced and $A_2$ is torsion-free. Then, Theorems 4.12 and 5.2 immediately gives the following.

**Theorem 5.4.** The finitely generated abelian groups that occur as group of units of a reduced ring are those of the form

$$\prod_{i=1}^{k} \mathbb{F}_{p_{i}^{n_{i}}} \times T \times \mathbb{Z}^{g}$$

where $k, n_1, \ldots, n_k$ are positive integers, $\{p_1, \ldots, p_k\}$ are the not necessarily distinct prime numbers, $T$ is any finite abelian group of even order and $g \geq g(T)$. 
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