ON VARIETIES IN AN ORBITAL VARIETY CLOSURE IN
SEMISIMPLE LIE ALGEBRAS

ANNA MELNIKOV

ABSTRACT. In this note we discuss the closure of an orbital variety as a union of varieties. We show that if semisimple Lie algebra \( g \) contains factors not of type \( A_n \) then there are orbital varieties whose closure contains components which are not Lagrangian. We show that the argument does not work if all the factors are of type \( A_n \) and provide the facts supporting the conjecture claiming that if all the factors of \( g \) are of type \( A_n \) then the closure of an orbital variety is a union of orbital varieties.

1. Introduction

1.1. Let \( G \) be a connected simply-connected complex algebraic group. Set \( g = \text{Lie}(G) \). Consider the co-adjoint action of \( G \) on \( g^* \). Identify \( g^* \) with \( g \) through the Killing form. A \( G \) orbit \( O \) in \( g \) is called nilpotent if it consists of ad-nilpotent elements.

Fix a triangular decomposition \( g = n^- \oplus h \oplus n \).

Let \( O \) be some nilpotent orbit. Consider an intersection \( O \cap n \). As it was shown by N. Spaltenstein \([7]\) and R. Steinberg \([8]\), this is an equidimensional variety of the dimension \( 0.5 \dim O \). Moreover, it was shown by A. Joseph \([2]\) that this is a Lagrangian subvariety of \( O \).

An irreducible component of \( O \cap n \) is called an orbital variety associated to \( O \).

According to orbit method philosophy, one would like to attach an irreducible representation of the enveloping algebra \( U(g) \) to an orbital variety. This should be a simple highest weight module. The results of A. Joseph and T. A. Springer provide a one to one correspondence between the set of primitive ideals of \( U(g) \) containing the augmentation ideal of its centre (thus, corresponding to integral weights) and the set of orbital varieties in \( g \) corresponding to Lusztig’s special orbits. Thus, orbital varieties play a key role in the study of primitive ideals in \( U(g) \). The details can be found in \([1, 2]\) and \([3]\).

1.2. The closure of a nilpotent orbit is a union of nilpotent orbits. The combinatorial description of this union was given by M. Gerstenhaber for \( g = sl_n \). Further H. Karft and C. Procesi described this union for other simple Lie algebras.
Generalizing the results of N. Spaltenstein, D. Mertens [6] showed that \( \mathcal{O} \cap n = \mathcal{O} \cap n \). Thus, \( \mathcal{O} \cap n \) is a union of intersections of \( n \) with corresponding orbits defined by the results of Gerstenhaber and Kraft-Procesi.

The question is to give a description of an orbital variety closure in the spirit of Gerstenhaber theory. This question is much more involved than the question on a nilpotent orbit closure. It has two components. The first one, a purely geometrical component, is to describe the type of varieties which constitute the closure of an orbital variety. This question can be formulated as following. Let \( \mathcal{V} \) be an orbital variety. Then its \( G \)-saturation \( \mathcal{O}_\mathcal{V} \) is a nilpotent orbit, to which \( \mathcal{V} \) is associated. Let us take some nilpotent orbit \( \mathcal{O} : \mathcal{O} \subset \mathcal{O}_\mathcal{V} \) and consider \( \mathcal{V} \cap \mathcal{O} \). As we show in 3.1 this intersection is always not empty. So, a natural task is to describe the irreducible components of this intersection. Is this intersection equidimensional? Is this intersection Lagrangian?

Here we show that if \( g \) contains factors not of type \( A_n \), there exist orbital varieties in \( g \) such that the intersection mentioned above is not Lagrangian.

We explain why the same argument does not work if all factors are of type \( A_n \). Moreover, the study of special cases in [4] and [5] shows that at least for some special types of orbital varieties in \( sl_n \) (that is \( A_{n-1} \)) the intersection of an orbital variety closure \( \mathcal{V} \) with any nilpotent orbit in the closure of \( \mathcal{O}_\mathcal{V} \) is Lagrangian. Together with computations in low rank cases (for \( n \leq 6 \)) this supports

**Conjecture.** In \( sl_n \) the closure of an orbital variety is a union of orbital varieties.

Note that in any case the straight generalization of Gerstenhaber theory cannot work for orbital varieties. As it is shown in [4, 4.1], even if \( V \) is of the most simple type (that is, when \( V \) is a nilradical) in \( sl_n \) and \( \mathcal{O} \subset \mathcal{O}_\mathcal{V} \) one has that in general \( \overline{\mathcal{V} \cap \mathcal{O}} \neq \mathcal{V} \cap \mathcal{O} \).

1.3. The second component of the description of an orbital variety closure is to give a combinatorial algorithm describing all the orbital varieties included in the closure of a given one. This is a very complex combinatorial question. The only general description of an orbital variety is provided by Steinberg’s construction (cf. 2.1). It is given via surjection from the Weyl group onto the set of orbital varieties. But even the description of the fibers of this map is highly non trivial outside of type \( A_n \). For the type \( A_n \) the picture is much nicer and simpler. Here the fibers are described by Robinson-Schensted procedure and the question can be formulated in terms of partial ordering of Young tableaux. Partial results in this case are provided in [3], [4], [5].

1.4. The body of the paper consists of two sections. In section 2 we give all the essential background to make this note self-contained. In section 3 we provide the results on the orbital variety closures.

2. Preliminaries
2.1. The definition of an orbital variety does not provide any way to construct it. The only general construction of an orbital variety belongs to R. Steinberg [8]. We explain it here in short.

Let $g$ be any semisimple Lie algebra. Fix its triangular decomposition $g = n^- \oplus h \oplus n$. For any $X \in n$ put $O_X := \{gXg^{-1} : g \in G\}$ to be its nilpotent orbit.

Let $B$ be the Borel subgroup of $G$ with $\text{Lie}(B) = h \oplus n$ and let $B$ act adjointly on $n$.

Let $R \subset h^*$ denote the set of non-zero roots, $R^+$ the set of positive roots corresponding to $n$ in the triangular decomposition of $g$, and $\Pi \subset R^+$ the resulting set of simple roots. Let $X_\alpha$ denote the root subspace for $\alpha \in R$. One has $n = \bigoplus_{\alpha \in R^+} X_\alpha$.

Let $W$ be the Weyl group of $\langle n, h \rangle$. The action of $w \in W$ on root subspace $X_\alpha$ is defined (in a standard way) by $w(X_\alpha) = X_{w(\alpha)}$. Consider the following subspace of $n$:

$$n \cap^w n = \bigoplus_{\{\alpha \in R^+ \mid w^{-1}(\alpha) \in R^+\}} X_\alpha.$$ 

Consider $\overline{G(n \cap^w n)}$. Since the number of orbits is finite, this is a closure of the unique orbit which we denote by $O_w$. By R. Steinberg [8], one has

**Theorem.** For each $w \in W$ there exists an orbital variety $\mathcal{V}$ and for each orbital variety $\mathcal{V}$ there exists $w \in W$ such that

$$\mathcal{V} = \overline{B(n \cap^w n)} \cap O_w.$$ 

In what follows we will denote $\mathcal{V}_w := \mathcal{V}$ in that case. Obviously, $\mathcal{V}_w$ is associated to $O_w$.

2.2. For any $\alpha \in \Pi$ let $P_\alpha$ be the standard parabolic subgroup of $G$ such that $\text{Lie}(P) = b \oplus X_{-\alpha}$.

Given $I \subset \Pi$, let $P_I$ denote the unique standard parabolic subgroup of $G$ such that $P_{-\alpha} \subset P_I$ iff $\alpha \in I$. Let $M_I$ be the unipotent radical of $P_I$ and $L_I$ a Levi factor. Let $p_I$, $m_I$, $l_I$ denote the corresponding Lie algebras. Set $B_I := B \cap L_I$ and $n_I := n \cap l_I$. We have decompositions $B = M_I \ltimes B_I$ and $n = n_I \oplus m_I$. They define projections $B \to B_I$ and $n \to n_I$ which we denote by $\pi_I$.

Set $W_I := \langle s_\alpha \mid \alpha \in I \rangle$ to be a parabolic subgroup of $W$. Set $D_I := \{w \in W : w(\alpha) \in R^+ \\forall \alpha \in I\}$. Set $R^+_I := R^+ \cap \text{span}(I)$. A well known result provides that each $w \in W$ has a unique expression of the form $w = w_d \ell$ where $d \in D_I$, $w \in W_I$ and $\ell(w) = \ell(w_d) + \ell(d)$. Moreover, for any $w \in W$ one has

$$R^+_I \cap^w R^+ = R^+_I \cap^w R^+_I.$$ 

Thus, a decomposition $W = W_I \times D_I$ defines a projection $\pi_I : W \to W_I$. For $w \in W$ set $w_I := \pi_I(w)$. This element can be regarded as an element of $W_I$ and as an element of $W$. 


Let $V_{wz}$ be the corresponding orbital variety in $\mathfrak{g}$ and $V^\Sigma_{wz}$ be the corresponding orbital variety in $\mathfrak{I}$. As it is shown in [3, 4.1.1] all the projections are in correspondence on orbital varieties, namely

**Theorem.** Let $\mathfrak{g}$ be a reductive Lie algebra. Let $\mathfrak{I} \subset \Pi$. For every orbital variety $V_w \subset \mathfrak{g}$ one has $\pi_\mathfrak{I}(V_w) = V^\mathfrak{I}_{wz}$.

2.3. In what follows we need also the notion of $\tau$-invariant. Let $w$ be any element of $W$. Set $S(w) := R^+ \cap w^{-1}R^- = \{ \alpha \in R^+ : w^{-1}(\alpha) \in R^- \}$. Set $\tau(w) = \Pi \cap S(w)$.

As it can be seen immediately from Steinberg’s construction for orbital variety closures, one has

**Proposition.** Let $w, y \in W$. If $V_w \subset V_y$ then $\tau(w) \supset \tau(y)$.

Note that as a trivial corollary we get that if $V_w = V_y$ then $\tau(w) = \tau(y)$. In other words, $\tau$ invariant is constant on an orbital variety.

### 3. An orbital variety closure

3.1. We begin with a simple

**Lemma.** Let $\mathcal{O}, \mathcal{O}' \subset \mathfrak{g}$ be two nilpotent orbits such that $\overline{\mathcal{O}'} \subset \overline{\mathcal{O}}$. Let $V$ be an orbital variety associated to $\mathcal{O}$. Then $\overline{V} \cap \mathcal{O}' \neq \emptyset$ and in particular there exist an orbital variety $V'$ associated to $\mathcal{O}'$ such that $\overline{V} \cap V' \neq \emptyset$.

**Proof.**

Indeed since there exist $w \in W$ such that $\overline{V} = B(n \cap w n)$ and since $G/B$ is projective we get

$$\mathcal{O}' \subset \overline{\mathcal{O}} = G(n \cap w n) = G(B(n \cap w n) = G(\overline{V})$$

which proves the first part.

Since $\mathcal{O}' \cap \overline{V} = \mathcal{O}' \cap n \cap \overline{V}$ we get the existence of $V'$.

3.2. Our strategy is to show that if in $\mathfrak{g}$ not all the factors are of type $A_n$, there exist nilpotent orbits $\mathcal{O}_1, \mathcal{O}_2$ such that $\overline{\mathcal{O}_2} \subset \overline{\mathcal{O}_1}$ and there exists $V_w$ associated to $\mathcal{O}_1$ such that for every $V_z$ associated to $\mathcal{O}_2$ one has $\tau(w) \not\supset \tau(z)$. Then on one hand by lemma 3.1 there exist at least one $V_z$ associated to $\mathcal{O}_2$ such that $\overline{V}_w \cap V_z \neq \emptyset$. On the other hand by proposition 2.3 if $\tau(z) \not\supset \tau(w)$ then $V_z \not\subset \overline{V}_w$ for every $V_z$ associated to $\mathcal{O}_2$. We get that $\overline{V}_w \cap \mathcal{O}_2$ is a non empty variety of dimension less than $0.5 \dim \mathcal{O}_2$. 

3.3. Consider the algebras of type $B_2$ and $G_2$. They are fully described in [9] and we just follow these computations.

$B_2$ : Let $s$ be a reflection corresponding to the long root $\beta$ and $t$ be the reflection corresponding to the short root $\alpha$. Consider $O_s$ and $O_{ts}$. One has $O_s \supseteq O_{ts}$. Moreover, $O_{ts} \cap \mathfrak{n}$ is irreducible so $\mathcal{V}_{ts}$ is the unique orbital variety associated to $O_{ts}$. Consider $\mathcal{V}_s$. If its closure is a union of orbital varieties then by lemma 3.1 $\mathcal{V}_{ts}$ must be included in it. But $\tau(s) = \{\beta\}$ and $\tau(tst) = \{\alpha\}$ so this inclusion contradicts proposition 2.3.

In what follows we will need also the following fact about these orbits: there is no intermediate nilpotent orbit between $O_s$ and $O_{ts}$, that is if $O'$ is such that $O_{ts} \subset O' \subset O_s$ then $O' = O_{ts}$ or $O' = O_s$.

$G_2$ : In that case the picture is very similar to $B_2$. Let $s$ be a reflection corresponding to the long root $\beta$ and $t$ be the reflection corresponding to the short root $\alpha$. Once more $O_s \supseteq O_{ts}$ and $O_{ts} \cap \mathfrak{n}$ is irreducible so that $\mathcal{V}_{ts}$ is the unique orbital variety associated to $O_{ts}$. Hence if $\mathcal{V}_s$ is a union of orbital varieties then $\mathcal{V}_s \supseteq \mathcal{V}_{ts}$ which is again impossible by proposition 2.3 since $\tau(s) = \{\beta\}$ and $\tau(tst) = \{\alpha\}$.

3.4. For searching the situation in $D_4$ we use the calculations in [7]. Let $s_3$ be the reflection giving $s_3(\alpha_i) = \alpha_i + \alpha_3$ for $i = 1, 2, 4$ and $s_1, s_2, s_4$, the rest fundamental reflections. Let us parameterize nilpotent orbits in $D_4$ by the partitions corresponding to their Jordan form. Then there are $O_1 \leftrightarrow (3,3,1,1)$ and $O_2 \leftrightarrow (3,2,2,1)$ such that $O_2 \subset O_1$. There are only 2 orbital varieties associated to $O_2$. The elementary calculations show that these are $\mathcal{V}_{s_1 s_2 s_4 s_3 s_1 s_2 s_4}$ and $\mathcal{V}_{s_3 s_1 s_2 s_4 s_3 s_1 s_2 s_4}$. Note that $\tau(s_1 s_2 s_4 s_3 s_1 s_2 s_4, s_3 s_1 s_2 s_4 s_3 s_1 s_2 s_4) = \{\alpha_1, \alpha_2, \alpha_4\}$ and $\tau(s_1 s_2 s_4 s_3 s_1 s_2 s_4, s_3 s_1 s_2 s_4 s_3 s_1 s_2 s_4) = \{\alpha_3\}$. We also have that $\mathcal{V}_{s_1 s_3 s_1}$ is associated to $O_1$. Note that $\tau(s_1 s_3 s_1) = \{\alpha_1, \alpha_3\}$.

Again, by 3.1 if $\mathcal{V}_{s_1 s_3 s_1}$ is a union of orbital varieties it must include at least one of $\mathcal{V}_{s_1 s_2 s_4 s_3 s_1 s_2 s_4}$, $\mathcal{V}_{s_3 s_1 s_2 s_4 s_3 s_1 s_2 s_4}$, which is impossible by proposition 2.3.

Again, as in the case of $B_2$, there is no intermediate nilpotent orbit $O'$ between $O_1$ and $O_2$.

3.5. Now we are ready to show

**Proposition.** In a semi-simple Lie algebra $\mathfrak{g}$ having a factor not of type $A_n$ there exists an orbital variety such that its closure is not a union of orbital varieties.

**Proof.**

Our proof is based on the previous computations and proposition 2.2.

Indeed, since orbital variety as well as its closure in a semisimple Lie algebra is just a direct product of corresponding simple factors, it is enough to prove the proposition for a simple Lie algebra not of type $A_n$. So, let $\mathfrak{g}$ be a simple Lie algebra not of type $A_n$.

For $\mathfrak{g}$ of type $G_2$ we have shown the existence of such orbital variety in 3.3.
If $\mathfrak{g}$ is not of type $G_2$, then there exist $\mathcal{I} \subset \Pi$ of type $B_2$ or of type $D_4$. Let us denote simple reflections in $\mathcal{I}$ as in case of $B_2$ in 3.3 if $\mathfrak{g}$ is of type $B_n$, $C_n$ or $F_4$, and as in case of $D_4$ in 3.4 otherwise. Let us denote $\mathcal{O}_2 := \mathcal{O}_{\text{tst}}$ in the case of $B_2$ and keep the notation $\mathcal{O}_2$ in the case of $D_4$ as in 3.4. Set
\[ w_I := \begin{cases} 
 s & \text{if } \mathfrak{g} \text{ is of type } B_n, C_n, \text{ or } F_4, \\
 s_1s_3s_1 & \text{otherwise.} 
\end{cases} \]
Recall the notion of $D_I$ from 2.2 and set $d_m$ to be the maximal element of $D_I$. Such element is unique by the uniqueness of the longest element in $W$.

We will show that $\overline{V}_{w_I d_m}$ is not a union of orbital varieties.

By the construction
\[ \overline{V}_{w_I d_m} = \overline{B}(n_I \cap w_I n_I). \] (\*)
Hence for every $\mathcal{V}$ such that $\pi_I(\mathcal{V}) = \overline{V}_{w_I}$ one has $\mathcal{V} \supset \mathcal{V}_{w_I d_m}$.

Assume that $\overline{V}_{w_I d_m} \setminus \mathcal{V}_{w_I d_m} = \bigcup \mathcal{V}_i$ for some orbital varieties $\mathcal{V}_i$. By the previous note
\[ \pi_I(\mathcal{V}_i) \subset \overline{V}_{w_I}. \] (**) 

Now take some point $X \in \overline{V}_{w_I} \cap \mathcal{O}_2$. Consider it as a point of $n_I \subset n$. We denote it by $\hat{X}$ when we consider it as a point of $\mathfrak{g}$. Then $\hat{X} \in \overline{V}_{w_I d_m}$ by (\*). Moreover, $\mathcal{O}_{\hat{X}} \neq \mathcal{O}_{w_I d_m}$, hence, $\hat{X} \in \overline{V}_{w_I d_m} \setminus \mathcal{V}_{w_I d_m}$. By our assumption there exist $\mathcal{V}_i$ such that $\hat{X} \in \mathcal{V}_i$. By theorem 2.2 $\pi_I(\mathcal{V}_i)$ is a closure of an orbital variety, and by (**) $\pi_I(\mathcal{V}_i) \subset \overline{V}_{w_I}$. Since $X \in \pi_I(\mathcal{V}_i)$ and there is no intermediate nilpotent orbits between $\mathcal{O}_{\overline{V}_{w_I}}$ and $\mathcal{O}_2$ (by the notes in the end of case $B_2$ in 3.3 and of 3.4) we get that $\pi_I(\mathcal{V}_i)$ is the closure of some orbital variety associated to $\mathcal{O}_2$. This contradicts our computations in 3.3, 3.4. □

3.6. Now let us consider the situation $\mathfrak{g} = \mathfrak{sl}_n$. Here, as it is shown in [3, 4.1.8], one has

**Proposition.** Let $\mathfrak{g} = \mathfrak{sl}_n$. Then for every two nilpotent orbits $\mathcal{O}_1, \mathcal{O}_2$ such that $\mathcal{O}_2 \subsetneq \mathcal{O}_1$ and for every $\mathcal{V}_1$ associated to $\mathcal{O}_1$ there exist $\mathcal{V}_2$ associated to $\mathcal{O}_2$ such that $\mathcal{V}_2 \subset \mathcal{V}_1$.

Therefore, the argument we use in other cases cannot work for $\mathfrak{sl}_n$. Moreover, modulo this proposition conjecture 1.2 is equivalent to the equidimensionality of $\overline{\mathcal{V}} \cap \mathcal{O}$ for any $\mathcal{O}$ in the closure of the nilpotent orbit, $\mathcal{V}$ is associated to.

As it is shown in [4, 2.3], if $\mathcal{V}$ is a Richardson component (that is its closure is a nilradical of a standard parabolic subgroup) then its closure is a union of orbital varieties. Note that in our counterexamples 3.3, 3.4 all the orbital varieties in question are Richardson. This demonstrates again, that the situation in $\mathfrak{sl}_n$ is different from other cases. As well it is shown in [5, 3.15] that if $\mathcal{V}$ is associated to a nilpotent orbit of nilpotent order 2 then
its closure is a union of orbital varieties. As we mentioned in \[1.2\] these results together with computations for low ranks support conjecture \[1.2\].

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Department of Mathematics, University of Haifa, Haifa 31905, Israel

E-mail address: melnikov@math.haifa.ac.il