Morphology of small snarks

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Abstract

The aim of this paper is to classify all snarks up to order 36 and explain the reasons of their uncolourability. The crucial part of our approach is a computer-assisted structural analysis of cyclically 5-connected critical snarks, which is justified by the fact that every other snark can be constructed from them by a series of simple operations while preserving uncolourability. Our results reveal that most of the analysed snarks are built up from pieces of the Petersen graph and can be naturally distributed into a small number of classes having the same reason for uncolourability. This sheds new light on the structure of all small snarks. Based on our analysis, we generalise certain individual snarks to infinite families and identify a rich family of cyclically 5-connected critical snarks.

1 Introduction

In this paper we attempt to provide insight into the structure of snarks of small order. The ultimate aim of our endeavour is, of course, to contribute to understanding the nature of snarks in general. Snarks are — in essence — connected cubic graphs whose edges cannot be properly coloured with three colours. In recent years, snarks have been attracting considerable attention, mainly because this family might contain counterexamples to several profound and long-standing conjectures such as the cycle double cover conjecture, the 5-flow conjecture, or the Berge-Fulkerson conjecture [19, 25, 26]. Understanding the structure of snarks is therefore crucial for proving or disproving any of these conjectures.

Snarks are very difficult to find since almost all cubic graphs are hamiltonian and hence 3-edge-colourable [40]. This asymptotic behaviour manifests itself already at very small orders: on 26 vertices, there exist fewer than three cyclically 4-edge-connected snarks per million of cubic graphs, and this ratio seems to exponentially decrease with increasing order. On the other hand, deciding whether a cubic graph is 3-edge-colourable or not is an NP-complete problem [22], which means that the class of snarks is still sufficiently rich: for instance, there are more than 400 millions of cyclically 4-edge-connected snarks on at most 36 vertices [6, 20].
Snarks are also difficult to understand because the reasons that force the absence of 3-edge-colourings in cubic graphs are in general unknown. Numerous constructions of snarks have been presented by various authors (see [1, 23, 45, 31, 32, 29, 42] for some examples), often aiming at proving the existence of snarks possessing certain special properties. Despite this effort, very little is known about the intrinsic structure of snarks. For example, it is not known whether small edge-cuts in snarks are unavoidable: a conjecture of Jaeger and Swart [24] states that every snark contains a cycle-separating edge-cut comprising at most six edges, but this conjecture has been open for almost 40 years without any visible progress. Thus no general approach to classification of the entire family of snarks seems to be within the reach.

The evidence drawn from the published lists of small snarks (see, for example, [6, 7, 8]) reveals that most of them are composed of several common construction components, typically obtained from the Petersen graph. This phenomenon has neither been formalised nor thoroughly studied yet, and it is unclear whether anything similar holds for snarks of large order. Although we believe that these questions are well worth of investigation, it does not seem that the currently available methods are powerful enough to attack them in full generality. Neither probabilistic nor constructive methods provide us with good insight into the structure of snarks. In particular, no uniform random model for snarks is currently known, which leaves us without strong theoretical tools for studying the typical behaviour of snarks. The NP-completeness of the problem of edge-colourability [22] also indicates that no quick progress is likely. This is why we focus on what we have at hand, which is the complete list of all cyclically 4-edge-connected snarks on up to 36 vertices, recently produced by Brinkmann et al. [6] and completed in [20], by employing exhaustive computer search.

Suppose, for a moment, that we would like to move a step further and produce a list of all snarks on 38 vertices. An obvious way to do it would be to generate all cubic graphs on 38 vertices and discard those that are colourable. Unfortunately, there are just too many of them in comparison with the computing power presently available for research, even if we restrict ourselves to those with cyclic connectivity at least 4 and girth at least 5. Such an approach is therefore very unlikely to work. In this situation it may be useful to realise that a vast majority of known snarks contain an edge whose removal followed by the suppression of the resulting 2-valent vertices again leaves a snark. Conversely, most known snarks arise from a smaller snark by choosing a suitable pair of edges, subdividing each of them with one additional vertex, and connecting the resulting 2-valent vertices with a new edge; this operation is called an I-extension. The meaning of “suitable” in order for the operation of I-extension to be feasible is easy to explain (see Proposition 4), which suggests that this approach might be promising. The hard part of the problem, however, are the snarks that cannot be obtained by a series of I-extensions from a smaller snark in such a way that each member of the extension series is a snark. Such snarks indeed exist and have been already studied [11, 12, 37, 42], in fact, they have been rediscovered several times [14, 16, 41]: they are known as critical snarks and are characterised by the property that for each edge the inverse of I-extension produces a colourable graph.

Critical snarks are known to be cyclically 4-edge-connected with girth at least 5 [37, Proposition 4.8], and thus can be regarded as “proper snarks” by the usual standards. A decomposition theory developed by Chladný and Škoviera in [12] suggests that critical snarks that possess a cycle-separating 4-edge-cut can be explained via the reversal of the well-known operation of dot product. This is especially true for bicritical snarks, an important subclass of critical snarks: every bicritical snark containing a cycle-separating 4-edge-cut admits a decomposition into a unique collection of cyclically 5-edge-connected bicritical snarks, and conversely, it can be reconstructed from them by a repeated appli-
cation of dot product [12, Theorem C].

By contrast, a decomposition process along 5-edge-cuts is much more complicated [10, 37, 38]. Moreover, it only works in one way and cannot be easily used for constructing snarks. Indeed, as argued in [37, p. 273], the original snark cannot be reconstructed from decomposition factors by using any collection of well-defined simple operations. Thus, from this point of view, the most fundamental and at the same time most enigmatic snarks are those that are critical and cyclically 5-edge-connected, also known under the term 5-simple [37]. Since all cyclically 4-edge-connected snarks can be obtained from them by applying I-extensions and dot products, it is natural to start the structural analysis of snarks by investigating cyclically 5-edge-connected critical snarks.

Our aim in this paper is, therefore, to analyse and classify all 5-simple snarks of order not exceeding 36. The list of such snarks is known and contains exactly 2110 graphs. We have extracted it from the complete list of all nontrivial snarks of order up to 36 which was produced by Brinkmann et al. [6] in 2013. The list of all critical snarks of order up to 36 was previously compiled by Carneiro et al. [9], however, those with cyclic connectivity at least 5 have not been singled out.

The method which we apply to the analysis of snarks is similar to what biologists have been doing for centuries in morphology. By discovering and investigating more and more species they have been constantly improving and refining the hierarchy of organisms, making it more complete with every new species examined. Our aim is to describe the structure of each 5-simple snark of order not exceeding 36 in a manner comprehensible to a human, with uncolourability readily verifiable by hand, as opposed to having a proof that relies on an exhaustive enumeration carried out by a computer. It transpires that uncolourability of small snarks can be conveniently explained in terms of multipoles (subgraphs with dangling edges) and their interconnections. Snarks with similar structure of multipoles, similar interconnections, and similar reasons for uncolourability are collected into families. As we analyse larger and larger snarks, the set of multipoles with known colouring properties grows. Whenever we encounter a snark whose uncolourability cannot be fully explained by previously discovered multipoles, we analyse it, extend the list of known multipoles, and employ it in the further analysis.

Results of our analysis are summarised in Section 6. Amongst the snarks up to 30 vertices we have not discovered any example that could not be easily explained in terms of multipoles arising from the Petersen graph. A new phenomenon arises on 32 vertices with class denoted by 32-A and schematically depicted in Fig. 17. The 7-pole $M_{11}$ contained in these snarks is perhaps the most interesting specimen of all — it is the only multipole that we have not been able to generalise.

On 34 and 36 vertices, there are several families of interest not described before; all of them contain several disjoint copies of 5-poles each consisting of a pair of 5-cycles sharing two edges. These 5-poles can be obtained from the Petersen graph by removing a path of length 2 and are known as negators.

Perhaps the most aesthetically pleasing family is illustrated in Fig. 24. The smallest snark from this family can also be regarded as a cleverly arranged tangle of 6-cycles complemented by a 6-pole arising from the Petersen graph by removing a 6-cycle; it is displayed in Fig. 1.

Despite our success in explaining the uncolourability of the investigated graphs, we have not accomplished that much in terms of truly understanding their criticality. In
particular, we can easily generalise any of the described families of snarks into an infinite class of new snarks, but we know very little about which of these new snarks are critical or even bicritical. All our achievements in this direction are collected in Section 8. On the one hand, we observe that construction components for critical snarks need not come from critical snarks. On the other hand, we give examples of constructions for which obvious necessary conditions, such as criticality of the constituting multipoles, are not sufficient to ensure criticality of the resulting snarks. In spite of that, we are able to describe a new rich infinite family of bicritical snarks. This family demonstrates that many (perhaps even all) of the families described in this paper can be turned into infinite families of bicritical snarks by imposing additional restrictions on the construction blocks.

Finally, we employ the results of our analysis to give a negative answer to a question posed by Chladný and Škoviera in [12, Problem 5.7] about pairs of edges essential for a dot product of bicritical snarks to be bicritical. According to the theory developed therein, an essential pair of edges must be non-removable (that is, its removal leaves a colourable graph). However, it was left open whether there exists a non-removable pair of edges that is not essential. An example of such a pair is provided in Section 9.

We conclude this section with a short list of definitions. We assume that the reader has the basic knowledge related to graph colourings and flows. For more information on this matter we recommend consulting [26].

Our graphs are finite and may contain parallel edges and loops. A connected 2-regular graph is called a cycle. A cubic graph $G$ is said to be cyclically $k$-connected (or cyclically $k$-edge-connected, to be more precise) if no set of fewer than $k$ edges separates two cycles of $G$ from each other. The cyclic connectivity of $G$ is the smallest integer $k$ such that $G$ is cyclically $k$-connected. A connected uncolourable cubic graph is a snark. The graph which consists of two vertices joined by an edge and has a loop attached at each vertex is called the dumbbell graph; it is denoted by $Db$. Note that $Db$ is a snark according to our definition. It is well known that the smallest 2-connected snark is the Petersen graph, denoted here by $Pg$. Cyclically 4-connected snarks with girth at least 5 will be called nontrivial and the remaining ones will be trivial.

2 Multipoles and their Tait colourings

Snarks are often described as combinations of graph-like structures called multipoles. In contrast to graphs, multipoles are permitted to contain dangling edges or even isolated edges, see e. g. [17]. Formally, a multipole is a pair $M = (V(M), E(M))$, where $V(M)$ is a set of vertices and $E(M)$ is a set of edges. Every edge $e \in E(M)$ has two ends which may, or may not, be incident with a vertex. An edge whose ends are incident with two distinct vertices is called a link. If only one end of an edge is incident with a
vertex, then the edge is a dangling edge, and if neither end of an edge is incident with a vertex, it is called an isolated edge. A semiedge is an end of an edge that is incident with no vertex. The set of all semiedges of a multipole \( M \) is denoted by \( S(M) \). A multipole with \( k \) semiedges is called a \( k \)-pole. The order \( |M| \) of a multipole \( M \) is the number of its vertices. In this paper, we will only consider cubic multipoles, that is, multipoles where each vertex is incident with three edge ends.

It is often convenient to partition the set \( S(M) \) into pairwise disjoint sets \( S_1, \ldots, S_n \) called connectors. Although semiedges in a connector are unordered, sometimes it is useful to endow a connector with a linear order. Such a connector \( S = (e_1, \ldots, e_k) \) is called an ordered connector. A multipole \( M \) with \( n \) connectors \( S_1, S_2, \ldots, S_n \) such that \( |S_i| = c_i \) for \( i \in \{1, 2, \ldots, n\} \) is denoted by \( M(S_1, S_2, \ldots, S_n) \) and called a \((c_1, c_2, \ldots, c_n)\)-pole. If a connector \( S \) contains only one semiedge \( s \), we will usually only write \( s \) in place of \( \{s\} \).

The junction of semiedges \( e \) and \( f \) is an operation under which the end-vertices of the corresponding dangling edges are joined to produce a new edge. The junction of two connectors \( S = \{e_1, e_2, \ldots, e_k\} \) and \( T = \{f_1, f_2, \ldots, f_k\} \) of the same size \( k \) consists of performing \( k \) individual junctions of \( e_i \) and \( f_i \) for \( i \in \{1, 2, \ldots, k\} \). If \( S \) and \( T \) are not ordered, prior to performing the junction we can enumerate their semiedges in an arbitrary order. Although different orderings may lead to several different multipoles, in a vast majority of cases our results do not depend on the order in which the junctions of semiedges of two connectors have been performed. A junction of two \((c_1, \ldots, c_n)\)-poles \( M(S_1, \ldots, S_n) \) and \( N(T_1, \ldots, T_n) \) consists of \( n \) individual junctions of \( S_i \) and \( T_i \) for each \( i \in \{1, 2, \ldots, n\} \). Hence, performing a junction of \( M \) and \( N \) requires to join the corresponding connectors \( S_i \) and \( T_i \), while the order of semiedges in \( S_i \) and \( T_i \) may be arbitrary, unless the connectors are ordered.

A natural approach to explaining the uncolourability of a snark is to split it into a set of multipoles and to study interactions between their colourings. The aim is to show that any combination of colourings of the constituting multipoles gives rise to a conflict within the snark. By an edge colouring of a multipole \( M \) we mean a mapping \( \varphi : E(M) \to X \) from the edge set of \( M \) to a certain set \( X \) of colours. An edge colouring naturally induces a colouring of edge ends. If the ends of all edges incident with any vertex \( v \) of \( M \) receive distinct colours, the colouring is said to be proper. If \( |X| = k \), the colouring is a \( k \)-edge-colouring.

Since multipoles in this paper are all cubic, colourings considered in this paper will mostly be proper \( 3 \)-edge-colourings. This permits us to abbreviate the term “proper 3-edge-colouring” to just “colouring”. We therefore say that a multipole \( M \) is colourable whenever it has a colouring; otherwise \( M \) is uncolourable.

A convenient set of colours for the study of snarks is provided by the set \( K \) of nonzero elements of the Klein four-group \( \mathbb{Z}_2 \times \mathbb{Z}_2 \). An edge colouring using \( K \) as the colour set is often termed a Tait colouring because the usage of such colourings can be traced back to Tait’s paper [14] on the Four-Colour Problem. One of the advantages of using \( K \) is that we can use addition in the group \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) to express the properties of a colouring. Indeed, a colouring of a multipole \( M \) is proper if and only if for every vertex \( v \) of \( M \) the three colours meeting at \( v \) sum to 0 in \( \mathbb{Z}_2 \times \mathbb{Z}_2 \). In other words, a proper \( 3 \)-edge-colouring of a cubic multipole is a nowhere-zero \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-flow and vice versa. Now, if we regard a colouring \( \varphi \) of a multipole \( M \) as a flow, we can use the Kirchhoff law to conclude that \( \sum_{e \in E(M)} \varphi(e) = 0 \). This fact has a useful consequence commonly known as the parity lemma, first proved by Tutte [15] under the pseudonym of Blanche Descartes.

**Lemma 1** (Parity lemma). Let \( M \) be a \( k \)-pole and let \( k_1, k_2, \) and \( k_3 \) be the numbers of
semiedges of colour \((0, 1), (1, 0), \) and \((1, 1)\), respectively. Then
\[ k_1 \equiv k_2 \equiv k_3 \equiv k \pmod{2}. \]

3 Reducibility and criticality of snarks

We have already indicated why critical snarks are a natural place to start investigation of the intrinsic structure of snarks. In this section we discuss this matter in a greater detail and explain important relations between criticality of snarks, their nontriviality, and reducibility.

A natural way of approaching the idea of nontriviality of snarks is by asking whether the snark in question contains vertices that do not contribute to its uncolourability. According to the parity lemma, removing just one vertex from a snark leaves an uncolourable graph, so one has to remove at least two vertices to make the graph colourable. A pair of distinct vertices \(\{u, v\}\) of a snark \(G\) will be called removable if \(G - \{u, v\}\) is not 3-edge-colourable; otherwise it will be called non-removable. A snark \(G\) is critical if every pair of adjacent vertices in \(S\) is non-removable; it is called bicritical if every pair of distinct vertices in \(G\) is non-removable.

The concept of criticality of snarks can also be developed by interpreting 3-edge-colourings of a cubic graph as nowhere-zero flows. This direction has been explored by several authors, see for example [9, 13, 14, 16, 18]. Following da Silva et al. [13] we define a graph to be \(k\)-flow-edge-critical if it does not admit a nowhere-zero \(k\)-flow but the graph obtained by the contraction of any edge does. We further define a graph to be \(k\)-flow-vertex-critical if it does not admit a nowhere-zero \(k\)-flow but the graph obtained by the identification of any two distinct vertices does. These definitions apply to snarks with \(k = 4\) since no snark admits a nowhere-zero 4-flow. If we take into account the fact that contracting an edge has the same effect on the existence of a nowhere-zero flow as identifying its end-vertices, 4-flow-edge-critical snarks and 4-flow-vertex-critical snarks are natural counterparts of critical and bicritical snarks, respectively. Nevertheless, it has been only recently shown [33] that in spite of different formal definitions flow-critical snarks are exactly the same as critical snarks.

**Theorem 2.** A snark is 4-flow-edge-critical if and only if it is critical. A snark is 4-flow-vertex-critical if and only if it is bicritical.

Consider a pair of adjacent vertices \(u\) and \(v\) forming an edge \(e\) of a cubic graph \(G\), and let \(G \sim e\) denote the cubic graph homeomorphic to \(G - e\). The operation that transforms \(G\) into \(G \sim e\) is called an edge reduction, and its reverse is called an edge extension or an I-extension. To perform an edge extension of a cubic graph \(G'\) one picks in \(G'\) two edges \(e_1\) and \(e_2\) (not necessarily distinct), subdivides each of them with a new vertex, and adds an edge \(e\) joining the two new vertices (if \(e_1 = e_2\), this results in a digon, that is, a pair of parallel edges). The resulting graph is denoted by \(G'(e_1, e_2)\).

It turns out that removing a pair of adjacent vertices \(u\) and \(v\) from a snark has the same effect on colourability as reducing the edge \(e\) joining them. This fact was first observed in [37].

**Proposition 3.** Let \(G\) be a snark and \(e = uv\) an edge of \(G\). Then \(G \sim e\) is 3-edge-colourable if and only if \(G - \{u, v\}\) is 3-edge-colourable.

Proposition [3] implies that a snark \(G\) is critical if and only if \(G \sim e\) is colourable for each edge \(e\). Every noncritical snark thus contains an edge whose reduction leaves a
smaller snark. If the resulting snark is still not critical, we can repeat the process and continue until we eventually produce a critical snark. Reversing this process shows that every snark can be constructed from a critical snark by a series of edge extensions, with all intermediate graphs being snarks that contain a subdivision of the initial critical snark. All this tells us that critical snarks can be regarded as basic building blocks of all snarks.

In order to make the extension process work, it is important to know under what conditions an edge extension \( G(e_1, e_2) \) of a snark \( G \) is again a snark. The answer requires one more definition. A pair \( \{e_1, e_2\} \) of edges of a snark \( G \) is said to be **removable** if \( G - \{e_1, e_2\} \) is 3-edge-colourable.

**Proposition 4.** Let \( G \) be a snark, and let \( e_1 \) and \( e_2 \) be distinct edges of \( G \). Then the edge extension \( G(e_1, e_2) \) of \( G \) is a snark if and only if the pair \( \{e_1, e_2\} \) is removable.

**Proof.** If \( \{e_1, e_2\} \) is a removable pair of edges of \( G \), then \( G - \{e_1, e_2\} \) is uncolourable. Since \( G - \{e_1, e_2\} \subseteq G(e_1, e_2) \), we conclude that so is \( G(e_1, e_2) \). Thus \( G(e_1, e_2) \) is a snark.

Conversely, let \( G(e_1, e_2) \) be a snark and suppose to the contrary that \( \{e_1, e_2\} \) is non-removable, that is, \( G - \{e_1, e_2\} \) is colourable. Let \( e'_1, e''_1 \) and \( e'_2, e''_2 \) be the edges of \( G(e_1, e_2) \) obtained by subdividing \( e_1 \) and \( e_2 \), respectively. Every 3-edge-colouring \( \varphi \) of \( G - \{e_1, e_2\} \) forces at least one of the pairs \( \{e'_1, e''_1\} \) and \( \{e'_2, e''_2\} \) to receive distinct colours, say \( \varphi(e'_1) = a \) and \( \varphi(e''_1) = b \), otherwise \( \varphi \) would induce a colouring of \( G \). By the parity lemma, the other pair also receives colours \( a \) and \( b \). Thus the edge of \( G(e_1, e_2) \) added across \( e_1 \) and \( e_2 \) can be coloured \( a + b \) to produce a 3-edge colouring of \( G(e_1, e_2) \), which is a contradiction. \( \square \)

Another possibility to capture the notion of nontriviality of snarks is to identify edge-cuts whose removal from a snark produces an uncolourable component. The aim is to generalise the well-known fact that snarks with short cycles and small edge-cuts are just trivial modifications of smaller snarks. For this purpose Nedela and Škoviera \[37\] proposed the following definitions. Let \( G \) be a snark which can be expressed as a junction \( M \ast N \) of two \( k \)-poles \( M \) and \( N \) for some \( k \geq 0 \). If one of \( M \) and \( N \), say \( M \), is uncolourable, we can extend \( M \) to a snark \( \tilde{M} \) of order not greater than \( |G| \) by adding to \( M \) a small number of vertices and edges; possibly \( \tilde{M} = G \). By creating the graph \( \tilde{M} \) we have reduced a snark \( G \) to a new snark which is called a \( k \)-**reduction** of \( G \). (Note that an edge reduction \( G \sim e \) is a special case of a 4-reduction.) A \( k \)-reduction \( \tilde{M} \) of \( G \) is **proper** if \( |\tilde{M}| < |G| \). If \( G \) admits a proper \( k \)-reduction for some \( k \geq 0 \), the essence of uncolourability of the smaller snark is the same as the one that can be found in \( G \). A snark is **\( k \)-irreducible**, for \( k \geq 1 \), if it has no proper \( m \)-reduction for any \( m < k \). A snark is **irreducible** if it is \( k \)-irreducible for every \( k > 0 \), that is, if it admits no proper reductions at all. Observe that a \( k \)-irreducible snark is also \( r \)-irreducible for every \( r \leq k \).

The following theorem, proved in \[37\], puts the concept of criticality of snarks into the perspective of various ranks of irreducibility. Among others, it tells us that there are, surprisingly, only finitely many different degrees of irreducibility, with bicritical snarks holding the highest position.

**Theorem 5.** Let \( G \) be a snark. Then the following statements hold true.

(i) If \( 1 \leq k < 4 \), then \( G \) is \( k \)-irreducible if and only if it is either cyclically \( k \)-connected or the dumbbell graph.

(ii) If \( k \in \{5, 6\} \), then \( G \) is \( k \)-irreducible if and only if it is critical.

(iii) If \( k \geq 7 \), then \( G \) is \( k \)-irreducible if and only if it is bicritical.
Theorem 5 implies that irreducible snarks coincide with bicritical ones, and that critical
snarks are just one step away from being irreducible. Critical snarks that are not bicritical,
called strictly critical, appear to be very rare. This can be observed already among snarks
of small order: there are exactly 55172 critical snarks of order not exceeding 36, but only
846 of them are strictly critical, just slightly over 1.5 percent [5, 33].

Another important consequence of Theorem 5 is that every critical snark is cyclically
4-edge-connected and has girth at least 5, see [37, Proposition 4.1]. Thus critical snarks
are nontrivial by all generally accepted standards.

| Order | $\lambda_c = 4$ | $\lambda_c = 5$ | $\lambda_c = 6$ | Total |
|-------|-----------------|-----------------|-----------------|-------|
| 10    | 0               | 1               | 0               | 1     |
| 18    | 2               | 0               | 0               | 2     |
| 20    | 0               | 1               | 0               | 1     |
| 22    | 0               | 2               | 0               | 2     |
| 24    | 0               | 0               | 0               | 0     |
| 26    | 103             | 8               | 0               | 111   |
| 28    | 31              | 1               | 1               | 33    |
| 30    | 104             | 11              | 0               | 115   |
| 32    | 16              | 13              | 0               | 29    |
| 34    | 38827           | 1503            | 0               | 40330 |
| 36    | 14063           | 568             | 1               | 14548 |

Table 1: Numbers of critical snarks by connectivity

Table 1 indicates that among critical snarks those with cyclic connectivity 4 signifi-
cantly prevail. It transpires, however, that critical snarks whose cyclic connectivity equals
4 can be reasonably well understood through the concept of a dot product. Given two
snarks $G$ and $H$, their dot product $G \cdot H$ is defined as follows (see [2, 23]): Choose two
independent edges $e = ab$ and $f = cd$ in $G$ and two adjacent vertices $u$ and $v$ in $H$. Let
$a'$, $b'$ and $v$ be the neighbours of $u$, and let $c'$, $d'$ and $u$ be the neighbours of $v$. Remove
the edges $e$ and $f$ from $G$ and the vertices $u$ and $v$ from $H$. Finally, connect $a$ to $a'$, $b$
to $b'$, $c$ to $c'$, and $d$ to $d'$. It is well known that $G \cdot H$ is indeed a snark, and that it is
cyclically 4-edge-connected provided that both $G$ and $H$ are (see [2, Theorem 2]).

Note that the added edges $aa'$, $bb'$, $cc'$, and $dd'$ of $G \cdot H$ form a cycle-separating
4-edge-cut, called the principal 4-edge-cut of $G \cdot H$. Thus the cyclic connectivity of a dot
product snark cannot exceed 4. Various authors observed that the converse holds as well:
every snark that contains a cycle-separating 4-edge-cut can be expressed as a dot product
of two smaller snarks (see for example [10, 21]).

**Theorem 6.** Every cycle-separating 4-edge-cut $S$ in a snark $G$ gives rise to a decompo-
sition of $G$ into a dot product $G = G_1 \cdot G_2$ in such a way that the principal cut of $G_1 \cdot G_2$
coincides with $S$. Moreover, if $G - S$ is 3-edge-colourable, then $G_1$ and $G_2$ are uniquely
determined by $S$.

The previous theorem raises the following natural question: What can be said about
the decomposition $G = G_1 \cdot G_2$ when $G$ is critical or bicritical? The following three
theorems, proved in [12], provide answers.

**Theorem 7.** Let $G$ and $H$ be snarks different from the dumbbell graph. Then $G \cdot H$ is
critical if and only if $H$ is critical, $G$ is nearly critical, and the pair of edges of $G$ involved
in this dot product is essential in $G$.  

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A nearly critical snark is one where every pair of adjacent vertices is non-removable except possibly the pairs of endvertices of the edges $e = ab$ and $f = cd$ of $G$ involved in the dot product. Being essential is a rather technical local property which will be defined and discussed in Section 9.

For dot products of bicritical snarks we only have a partial result, nevertheless, one of fundamental importance. Its essence is the fact that the class of bicritical snarks is closed under decompositions into a dot product.

**Theorem 8.** Let $G$ and $H$ be snarks different from the dumbbell graph. If $G \cdot H$ is bicritical, then both $G$ and $H$ are bicritical. Moreover, the pair of edges of $G$ involved in this dot product is essential in $G$.

Let $G$ be a bicritical snark that contains a cycle-separating 4-edge-cut. By Theorems 6 and 8, we can decompose $G$ into a dot product $G = G_1 \cdot G_2$ of two smaller bicritical snarks different from the dumbbell graph. If one of these graphs again contains a cycle-separating 4-edge-cut, we can continue the process. After a finite number of steps we eventually obtain a collection $H_1, H_2, \ldots, H_r$ of cyclically 5-edge-connected bicritical snarks which cannot be further decomposed. Note that the decomposition process is not uniquely determined. The edge-cuts used on the way to a final collection of cyclically 5-edge-connected bicritical snarks may intersect in a very complicated fashion, and choosing one particular 4-edge-cut at a certain stage of the decomposition process may exclude certain cuts from a later use in the decomposition. This concerns especially the cuts which do not exist in the original snark but might be, and often are, created during the process. It is therefore rather unexpected that the following theorem is true.

**Theorem 9.** Every bicritical snark $G$ different from the dumbbell graph can be decomposed into a collection $\{H_1, \ldots, H_n\}$ of cyclically 5-connected bicritical snarks such that $G$ can be reconstructed from them by repeated dot products. Moreover, such a collection is unique up to isomorphism and ordering of the factors.

The assumption of Theorem 9 requiring a snark $G$ to be bicritical cannot be relaxed. Indeed, in [12, Section 12] it is shown that there exist critical snarks with substantially different decompositions, and even with decompositions having different numbers of factors.

The results discussed in the preceding paragraphs can be summarised as follows. For every snark $G$, there exists a sequence of edge reductions

$$G_0 = G, G_1 = G_0 \sim e_1, \ldots, G_r = G_{r-1} \sim e_r$$

such that each $G_i$ is a snark and the terminal member $G_r$ of the sequence is a critical snark.

If $G_r$ is bicritical, then it has a unique decomposition into a collection of cyclically 5-edge-connected critical snarks (all of which are even bicritical). Therefore $G$ can be reconstructed from a collection of cyclically 5-edge-connected critical snarks by using repeated dot products and by a series of edge extensions of snarks.

If $G_r$ is strictly critical, the situation is more complicated, because $G_r$ can possibly be expressed as a dot product $H_1 \cdot H_2$ where $H_2$ is critical but $H_1$ is only nearly critical. Nevertheless, as shown in [12, Section 6], strictly critical snarks whose cyclic connectivity equals 4 can still be fairly well understood. This brings us back to cyclically 5-edge-connected critical snarks even in the latter case.
4 Methods of analysis

Exhaustive computer search performed by Brinkmann et al. [6] reveals that there are very few 5-simple snarks with girth greater than 5 on up to 36 vertices. These can be put aside and discussed separately. The remaining ones have girth 5, and hence contain a 5-cycle. If \( C \) is a 5-cycle in a snark \( G \) and \( e = uv \) and \( e' = u'v' \) are two edges of \( C \), then according to Kászonyi [28] (see also Bradley [4]) the number of 3-edge-colourings of \( G \sim e \) is the same as that of \( G \sim e' \). In particular, the pairs \( \{u, v\} \) and \( \{u', v'\} \) are either both non-removable or both removable. It follows that connected components of the subgraph \( K \) formed by the union of all 5-cycles of \( G \) are subgraphs that play a fundamental structural role in \( G \). Any such component will be called a 5-cycle cluster of \( G \).

The smallest 5-cycle clusters can be found in the Petersen graph. We call them basic or Petersen clusters. It is convenient to view them as cubic multipoles with natural partition of their semiedges into connectors, which is determined by the way in which they were constructed from the Petersen graph. Clearly, every 5-cycle cluster contained in a cyclically 5-conneced snark must have at least five semiedges and girth 5. To start our analysis we have therefore identified all 5-cycle clusters of order up to 10 with at least five semiedges. There are seven such 5-cycle clusters, six of which are Petersen clusters.

- The \textit{pentagon} \( P \) is the smallest 5-cycle cluster. It consists of a single cycle of length 5 together with 5 dangling edges forming its unique connector (see Figure 2a). It can be constructed from the Petersen graph by removing any 5-cycle.

- The \textit{dyad} \( D \) (or Petersen negator) is a 5-cycle cluster consisting of two 5-cycles sharing a path of length 2. It can be constructed by removing a path \( uuvv \) of length 2 from the Petersen graph. The natural distribution of semiedges into connectors makes it a \((2, 2, 1)\)-pole \( D(I, O, R) \), with 2-connectors \( I \) and \( O \) containing the dangling edges formerly incident with \( u \) and \( v \) respectively, and the 1-connector \( R \) containing the only dangling edge formerly incident with \( w \) (see Figure 2b). The dyad has 7 vertices, 8 edges, and 5 semiedges.

- The \textit{triad} \( T \) is a 5-cycle cluster formed by three 5-cycles \( C_1 \), \( C_2 \), and \( C_3 \) such that \( C_1 \) and \( C_2 \) have exactly one edge in common while \( C_3 \) contains the common edge of \( C_1 \) and \( C_2 \) and one additional edge of each \( C_1 \) and \( C_2 \). It can be constructed from the Petersen graph by removing one vertex and severing an edge not incident with it. The natural distribution of semiedges into connectors turns it into a \((2, 3)\)-pole \( T(B, C) \), shown in Figure 2c, where the connector \( B \) corresponds to the severed edge and the connector \( C \) corresponds to the removed vertex. The triad has 9 vertices, 11 edges and 5 semiedges.

- The \textit{quasitriad} \( qT \) is a 5-cycle cluster consisting of three 5-cycles \( C_1 \), \( C_2 \), and \( C_3 \) such that \( C_1 \) and \( C_2 \) share two edges and \( C_3 \) shares one edge with each \( C_1 \) and \( C_2 \). One can simply check that performing the junction of a pair of semiedges and connecting the remaining three semiedges to a new vertex always yields a cycle of length smaller than 5. Hence, the quasitriad is not a Petersen cluster. Like triad, quasitriad has 9 vertices, 11 edges, and 5 dangling edges. The quasitriad contains exactly one pair of dangling edges at distance 1, which distinguishes it from the triad containing two such pairs.

- The \textit{double pentagon} \( dP \) is a 5-cycle cluster containing two 5-cycles sharing an edge. It can be obtained from the Petersen graph by removing two adjacent vertices \( u \) and
v and severing an edge $e$ at distance 2 from $uv$. The natural distribution of semiedges turns it into a $(2, 2, 2)$-pole $dP(A, B, C)$, shown in Figure 2e, where the connectors $A$ and $B$ correspond to the vertices $u$ and $v$ respectively, and $C$ corresponds to the edge $e$. The double pentagon has 8 vertices, 9 edges and 6 dangling edges.

- The triple pentagon $tP$ is a 5-cycle cluster consisting of three 5-cycles, each pair having two edges in common. It can be constructed from the Petersen graph by severing three pairwise non-adjacent edges lying on a 6-cycle in an alternating order, making the triple pentagon a $(2, 2, 2)$-pole $tP(A, B, C)$ as depicted in Figure 2g. Note that the three severed edges cannot be extended to a perfect matching of the Petersen graph. The triple pentagon has 10 vertices, 12 edges and 6 dangling edges.

- The tricell $tC$ is a 5-cycle cluster containing three 5-cycles $C_1$, $C_2$ and $C_3$, where $C_1$ and $C_2$ share one edge, $C_2$ and $C_3$ share two edges, and $C_1$ and $C_3$ are disjoint. Like the triple pentagon, it arises from the Petersen graph by severing three pairwise non-adjacent edges, however, in this case these edges do not lie on a 6-cycle and can be extended to a perfect matching of the Petersen graph. The 3-cell has a natural representation as a $(2, 2, 2)$-pole $tC(A, B, C)$ as shown in Figure 2g. It has 10 vertices, 12 edges and 6 dangling edges. In contrast to the triple pentagon, the tricell has one dangling edge whose distance to each other dangling edge is at least 2.

The key step of our analysis of 5-simple snarks is the identification of 5-cycle clusters. They are easy to find by a limited-depth breadth-first search in any given graph; we do it by employing a computer program. Obviously, identifying the 5-cycle clusters is not sufficient since many snarks contain a significant number of vertices belonging to no clusters. This is why for every given snark we determine the structure of its 5-cycle clusters, the vertices not contained in clusters, and the connections between them. This allows us to distribute almost all 5-simple snarks into a small number of classes depending on which basic clusters they contain and how they are interconnected. Finally, for each class we theoretically explain the reasons why their members are not colourable.

It is worth mentioning that quasitriads, triple pentagons, and tricells do not occur in the analysed snarks, and thus they are listed mainly for completeness. While the latter two clusters have not been observed only because the analysed critical snarks are too
small, quasitriads cannot occur in critical snarks at all. Indeed, one can easily check that the quasitriad is a colour-closed 5-pole in the sense of \[37\], which means that in every snark of the form \(G = qT \ast M\) the complementary 5-pole \(M\) must be uncolourable. It follows that \(G\) has a proper 5-reduction, and by Theorem 3 such a snark cannot be critical.

5 Commonly used multipoles

In this section we develop tools for determining the reasons why the analysed graphs fail to admit a 3-edge-colouring. Since most our graphs are built up from 5-cycle clusters and a number of additional vertices, the crucial point is to analyse the colouring properties of the basic 5-cycle clusters and their combinations. Since our arguments only use the fact that the Petersen graph is a snark, one can replace the Petersen graph with a larger snark to construct a multipole in a similar manner and with similar colouring properties as the given basic 5-cycle cluster. Regarding the basic 5-cycle clusters as special cases of these multipoles enables us to generalise small snarks to infinite classes that cover almost all 5-simple snarks up to order 36.

We start with several technical definitions that are necessary for this purpose.

Consider a multipole \(M(S_1, \ldots, S_n)\) with connectors \(S_1, \ldots, S_n\) and a 3-edge-colouring \(\varphi\). Define the flow through a connector \(S_i\) of \(M\) to be the value \(\varphi_*(S_i) = \sum_{e \in S_i} \varphi(e)\). Note that \(\varphi_*(S_i)\) may happen to be 0. A connector \(S_i\) is called proper if \(\varphi_*(S_i) \neq 0\) for every colouring \(\varphi\) of \(M\); it is called improper if \(\varphi_*(S_i) = 0\) for every colouring \(\varphi\) of \(M\). A multipole is called proper if all of its connectors are proper; similarly, it is improper if all its connectors are improper.

Now let \(M\) be an arbitrary \(k\)-pole and let \(S(M) = \{e_1, e_2, \ldots, e_k\}\) be the set of its semiedges listed in a specific order given by increasing indices. Although semiedges in connectors are generally not ordered, we fix this order only to avoid ambiguity. For any 3-edge-colouring \(\varphi\) of \(M\) and any subset \(T = \{f_1, f_2, \ldots, f_l\}\) of \(S(M)\) we set \(\varphi(T) = (\varphi(f_1), \varphi(f_2), \ldots, \varphi(f_l))\), where the order of semiedges in \(T\) agrees with the chosen order of \(S(M)\). We define the colouring set of \(M\) to be the set

\[
\text{Col}(M) = \{\varphi(S(M)) \mid \varphi \text{ is a Tait colouring of } M\}.
\]

Two \(k\)-poles \(M\) and \(N\) are called colour-disjoint if \(\text{Col}(M) \cap \text{Col}(N) = \emptyset\). Note that if \(M\) and \(N\) are colour-disjoint, then \(M \ast N\) is a snark, and vice versa.

Many constructions of snarks can be conveniently described in terms of multipole substitution. This is a very natural concept and as such it has previously occurred in the literature under various disguises and different names, see for example \[17\].

Consider two \(k\)-poles \(M_1\) and \(M_2\). We say that \(M_1\) is colour-contained in \(M_2\) if \(\text{Col}(M_1) \subseteq \text{Col}(M_2)\). If \(\text{Col}(M_1) = \text{Col}(M_2)\), we say that \(M_1\) and \(M_2\) are called equivalent. Now, let \(G = M \ast N\) be a snark expressed as a junction of two \(k\)-poles \(M\) and \(N\), and let \(M'\) be a \(k\)-pole colour-contained in \(M\). We say that the graph \(G' = M' \ast N\) is obtained from the snark \(G\) by a substitution of \(M'\) for \(M\). Observe that \(G'\) is again a snark: if \(G'\) was colourable, then any colouring of \(G'\) could be modified to a colouring of \(G\) in a straightforward manner.

Substitution is a generic method of constructing new snarks from old ones. It can be used in two ways: either to produce smaller snarks with a more transparent structure, or to create larger snarks from snarks already known. Applying substitution requires having suitable pairs of multipoles. In the rest of this section we describe several examples of such pairs which occur in 5-simple snarks. As we shall see, most of them arise from generalisations of the basic 5-cycle clusters.
5.1 Negators

Let $G$ be a snark and let $uvw$ be a path of length two in $G$. Removing $uvw$ from $G$ leaves a multipole $M$ whose semiedges can be naturally distributed to two 2-connectors and one 1-connector, each of them consisting of the semiedges formerly incident with the same vertex of the path. Let $I = \{e_1, e_2\}$ and $O = \{e_3, e_4\}$ be the connectors consisting of the semiedges formerly incident with $u$ and $v$, respectively, and let $R = \{e_5\}$ be the 1-connector containing the remaining semiedge of $M$. If $G$ is the Petersen graph, then $M(I, O, R)$ is the dyad.

Observe that for each colouring of $M$ the total flow through exactly one of its 2-connectors is zero — otherwise the colouring could be extended to a colouring of $G$. The flow through the other connector is the same as that through the residual semiedge due to the parity lemma. In other words, the colouring set of $M$ is a subset of

$$C = \{(x, x, a, b, a + b), (a, b, x, x, a + b) \in \mathbb{Z}^5 | a \neq b\}.$$

It means that $M$ behaves like an inverting gadget which inverts matching colours in the input connector $I$ to mismatching colours in the output connector $O$, and vice versa. For this reason $M$ is referred to as a negator and is denoted by $\text{Neg}(G; u, v)$. The edge $e_5$ does not contribute to the inverting property of $M$ and therefore it is called residual.

The notation $\text{Neg}(G; u, v)$ is ambiguous if $u$ and $v$ have more than one common neighbour $w$. This can only happen if the girth of $G$ does not exceed 4, in which case $G$ is a trivial snark. As we are primarily interested in nontrivial snarks, in our work this ambiguity will be marginal.

A negator whose colouring set is identical with $C$ is called perfect, otherwise it is called imperfect. For an imperfect negator $N$, it is possible that one of its 2-connectors is improper, which means that the other 2-connector is proper. If such a negator $N$ additionally admits all such colourings, it is called semiperfect. The following theorem proved by Mácajová and Škoviera [32] implies that each negator is either perfect, semiperfect or uncolourable, and provides a characterisation of perfect and semiperfect negators.

**Theorem 10.** Let $N = \text{Neg}(G; u, v)$ be a negator and let $w$ a common neighbour of $u$ and $v$. If $N$ is colourable, then it is either perfect or semiperfect. Moreover, the following hold.

(i) $N$ is perfect if and only if each of the pairs $\{u, w\}$ and $\{v, w\}$ of adjacent vertices is nonremovable.

(ii) $N$ is semiperfect if and only if at least one of the pairs $\{u, w\}$ and $\{v, w\}$ is removable.

The smallest (connected) example of a negator can be constructed from the Petersen graph. Since the Petersen graph graph is 2-arc transitive, there is, up to isomorphism, only one way of removing a path of length 2 and thus there is, up to isomorphism, only one Petersen negator — the dyad. Obviously, the dyad is a perfect negator.
The parity lemma implies that every $(2,2,1)$-pole which is colour-disjoint from a perfect negator must be a proper $(2,2,1)$-pole. The smallest such $(2,2,1)$-pole is the path of length 2 with dangling edges retained and distributed into connectors in the usual way. It consists of two endvertices $u$ and $v$ and their common neighbour $w$. We will denote it by $P_2(I,O,r)$, where the connector $I$ corresponds to the dangling edges incident with $u$, $O$ corresponds to the dangling edges incident with $v$, and the remaining semiedge $r$ arises from the dangling edge incident with $w$. The colouring set of $P_2$ is

$$\text{Col}(P_2) = \{ (a,b,c,d,e) \in \mathbb{K}^5 \mid a \neq b, c \neq d, a + b + c + d + e = 0 \}.$$  

5.2 Proper $(2,3)$-poles

Take an arbitrary snark $G$ and choose a vertex $v$ and an edge $e$ in $G$. Form a $(2,3)$-pole $T(B,C)$ by removing $v$ and severing $e$. Let $B$ be the set of semiedges arising from severing $e$ and let $C$ be the set of semiedges formerly incident with $v$. Observe that if $G$ is the Petersen graph and $e$ is not incident with $v$, then the result is the triad. For connectivity reasons we only consider proper $(2,3)$-poles that arise from a snark $G$ where the removed vertex $v$ and edge $e$ are not incident, although the colouring properties of $T(B,C)$ hold in the general case as well.

We claim that $T$ is a proper $(2,3)$-pole. Suppose not. Then $T$ admits a colouring $\varphi$ such that the flow through one of the connectors is zero. The parity lemma then implies that $\varphi_*(B) = \varphi_*(C) = 0$ and allows extending the colouring of $T$ to a colouring of the original snark $G$, which is impossible. Therefore $T$ is proper. The colouring set of each proper $(2,3)$-pole $T$ is clearly a subset of

$$\mathcal{C} = \{ (a_1,a_2,b_1,b_2,b_3) \in \mathbb{K}^5 \mid a_1 + a_2 = b_1 + b_2 + b_3 \neq 0 \}.$$  

If a proper $(2,3)$-pole $T$ admits all colourings such that the flow through each of the connectors is nonzero — that is to say, if its colouring set coincides with $\mathcal{C}$ — then $T$ is called perfect; otherwise it is imperfect. As one can expect, the triad is a perfect proper $(2,3)$-pole, which can easily be verified by hand or with the help of a computer. An example of an imperfect proper $(2,3)$-pole will be discussed in Section 5.8.

Let us consider the multipole which is removed from a snark $G$ in order to construct the proper $(2,3)$-pole $T(B,C)$. It is a disconnected $(2,3)$-pole $M_{ev}(B,C)$ with connectors $B = (b_1,b_2)$ and $C = (c_1,c_2,c_3)$ where $b_1$ and $b_2$ are the ends of an isolated edge and $c_1$, $c_2$, and $c_3$ are ends of three dangling edges incident with one common vertex (see also Figure 12). Its colouring set is

$$\text{Col}(M_{ev}) = \{ (x,x,a,b,c) \in \mathbb{K}^5 \mid a + b + c = 0 \}.$$  

5.3 Proper $(3,3)$-poles

Let $u$ and $v$ be two vertices of a snark $G$. Construct a $(3,3)$-pole $M(I,O)$ by removing $u$ and $v$ from $G$ and putting the three semiedges formerly incident with $u$ and $v$ into the connectors $I$ and $O$, respectively. The $(3,3)$-pole $M$ is proper: Kirchhoff’s law implies that $\varphi_*(I) = \varphi_*(O)$ for every 3-edge-colouring $\varphi$ of $M$, and if any of these values equals
zero, then $\varphi$ can be extended to a colouring of $G$, which is a contradiction. Again, for connectivity reasons we only consider proper $(3,3)$-poles where the vertices $u$ and $v$ are not adjacent.

In the Petersen graph there is only one way of removing a pair of non-adjacent vertices because the graph is 2-arc-transitive and has diameter 2. The result is a unique proper $(3,3)$-pole $M_8$ of order 8. It is easy to see that $M_8$ can be constructed from the dyad by attaching the residual semiedge to a new vertex incident with two additional dangling edges, one contributing to the input and the other one to the output.

### 5.4 Even $(2,2,2)$-poles

An even $(2,2,2)$-pole is one where the number of connectors having nonzero total flow is even. The term for this multipole was coined by Goldberg [21] who was the first to study this kind of multipoles. An even $(2,2,2)$-pole can be constructed as follows. Take a snark $G$ and a vertex $v$ with neighbours $u_1$, $u_2$ and $u_3$. Remove $v$, $u_1$, $u_2$, and $u_3$, and for each $i \in \{1,2,3\}$ let $S_i = \{e_i, f_i\}$ be the set containing the semiedges formerly incident with $u_i$ that do not arise from $u_i v$. We claim that the resulting $(2,2,2)$-pole $H(S_1, S_2, S_3)$ is even. Kirchhoff’s law tells us that it is impossible for a flow to have a nonzero total flow because the graph is not adjacent. Hence, no flow can be extended to a colouring of $G$, which is a contradiction. Therefore $H(S_1, S_2, S_3)$ is indeed an even $(2,2,2)$-pole.

A simple example of an even $(2,2,2)$-pole arises from a cycle of length six with a dangling edge at every vertex by forming each connector from a pair of opposite dangling edges of the 6-cycle. It is derived from the Petersen graph. Note that every even $(2,2,2)$-pole can be constructed from a snar by deleting the residual semiedge along with its end-vertex.

Denote by $V_4(S_1, S_2, S_3)$ the complementary $(2,2,2)$-pole we removed from the snark $G$ to create $H$. It consists of a vertex $v$ and its three neighbours $u_1$, $u_2$ and $u_3$, each of them with two dangling edges attached. The connector $S_i$ contains the semiedges belonging to the dangling edges incident with $u_i$ (for $i \in \{1,2,3\}$). The colouring set of $V_4$ is the set

$$\text{Col}(V_4) = \{(a_1, b_1, a_2, b_2, a_3, b_3) \in \mathbb{K}^6 \mid a_1 + b_1 + a_2 + b_2 + a_3 + b_3 = 0, (\forall i)(a_i \neq b_i)\}.$$

Large even $(2,2,2)$-poles can be constructed from smaller ones as follows. Let $M$ be an arbitrary 3-pole, possibly containing loops and parallel edges, with three pairwise distinct vertices $v_1$, $v_2$, and $v_3$, each being incident with one dangling edge. Replace each vertex $v$ of $M$ with an even $(2,2,2)$-pole $H_v$ and each edge (including the dangling ones) with the $(2,2)$-pole consisting of two isolated edges which have their ends in different connectors. The result is a $(2,2,2)$-pole $H_M(S_1, S_2, S_3)$, where each connector $S_i$ consists of two dangling edges that replace the dangling edge incident with $v_i$. We claim that $H_M$ is an even $(2,2,2)$-pole. Assume that $\varphi_\ast(S_i) = a \neq 0$ for some connector $S_i$ of $H_M$. Observe that each even $(2,2,2)$-pole $H$ must have an even number of connectors with flow $a$ through it. By applying this argument inductively one can readily conclude that there exists another connector $S_j$ of $H_M$ with $\varphi_\ast(S_j) = a$. Consequently, $\varphi_\ast(S_k) = 0$ for $k \in \{1,2,3\} - \{i,j\}$, and therefore the number of connectors of $H_M$ with nonzero flow is either zero or two. In other words, $H_M$ is an even $(2,2,2)$-pole.

For illustration, consider the cubic graph $B_2$ consisting of two vertices and three parallel edges joining them, subdivide each edge with a new vertex, and attach a dangling
edge to all three 2-valent vertices, thereby producing a 3-pole $M$ on five vertices. If we apply the construction described above to this 3-pole, we obtain an even $(2, 2, 2)$-pole $H_M(S_1, S_2, S_3)$ composed from five smaller ones. The result is represented in Figure 5, with the $(2, 2, 2)$-poles denoted by $H_i$ for $i \in \{1, \ldots, 5\}$. Now, if for each of the five even $(2, 2, 2)$-poles we take the hexagonal $(2, 2, 2)$-pole derived from the Petersen graph and perform the junction $H_M(S_1, S_2, S_3) \ast V_4(S_1, S_2, S_3)$ we obtain the snark displayed in Figure 1. Clearly, an analogous construction can be performed starting from any cubic graph in place of $B_2$.

![Figure 5: An even $(2, 2, 2)$-pole constructed from five smaller ones](image)

5.5 Isaacs $(3, 3)$-poles

Flower snarks are a well-known and in a certain sense exceptional family of snarks introduced by Isaacs in [23]. Their basic building block is the $(3, 3)$-pole $Y(I, O)$ with two ordered connectors $I = (i_1, i_2, i_3)$ and $O = (o_1, o_2, o_3)$ shown in Figure 6. It can be constructed from the complete bipartite graph $K_{3,3}$ by deleting two vertices from the same partite set, forming the connectors in the usual way, and ordering their semiedges in such a way that $\{i_1, o_2\}$, $\{i_2, o_1\}$, and $\{i_3, o_3\}$ are adjacent pairs of dangling edges.

The Isaacs flower snark $J_n$, where $n \geq 3$ is odd, can be produced by taking the disjoint union on $n$ copies $Y(I_i, O_i)$ of $Y(I, O)$, where $i \in \{1, 2, \ldots, k\}$ and by joining $O_i$ to $I_{i+1}$ following the order of semiedges in the connectors; of course, the subscripts in the definition are taken modulo $n$. The snark $J_3$ is the only trivial Isaacs snark.

Let $Y_k(I_1, O_k)$ denote the $(3, 3)$-pole arising similarly from the union of $k$ disjoint copies $Y(I_i, O_i)$ of $Y(I, O)$ and by performing the junction of $O_i$ and $I_{i+1}$ only for $i \in \{1, 2, \ldots, k-1\}$. It is known that $\text{Col}(Y_{2m}) = \text{Col}(Y_2)$ for every integer $m \geq 1$, see [37].

Note that all the connectors considered in this subsection are ordered, since the arguments of the uncolourability of the Isaacs snarks require this fixed order of the semiedges in the connectors of $Y$, see for example [23].
5.6 Proper \((2, 2, 1)\)-poles of type NN

Take two negators \(N_1(I_1, O_1, r_1)\) and \(N_2(I_2, O_2, r_2)\) and perform the junction of the connectors \(O_1\) and \(I_2\). Add one vertex \(v\) incident with the semiedges \(r_1, r_2\) and one new dangling edge producing a semiedge \(r_3\) (see Figure 7). Denote the resulting \((2, 2, 1)\)-pole \(M(I_1, O_2, r_3)\) by \(NN(N_1, N_2)\).

We prove that \(\text{Col}(M) \subseteq \text{Col}(P_2)\), and if \(N_1\) and \(N_2\) are perfect negators, then \(\text{Col}(M) = \text{Col}(P_2)\). Let \(\varphi\) be a colouring of \(M\). The flow through the connectors \(O_1\) and \(I_2\) has to be zero, otherwise \(\varphi(r_1) = \varphi(0_1) = \varphi(I_2) = \varphi(r_2)\), a contradiction. Consequently, the flows through \(I_1\) and \(O_2\) have to be nonzero. This implies that \(\varphi(S(M)) \in \text{Col}(P_2)\). Now assume that \(N_1\) and \(N_2\) are perfect negators. Let \(\varphi\) be an assignment of colours to the dangling edges of \(M\) such that \(\varphi(S(M)) \in \text{Col}(P_2)\), that is, \(\varphi(I_1) = a \neq 0\), \(\varphi(I_2) = b \neq 0\) and \(\varphi(r_3) = a + b \neq 0\). We extend \(\varphi\) to a colouring of the multipole \(M\).

5.7 Proper \((2, 2, 1)\)-poles of type TT

Take two proper \((2, 3)\)-poles \(T_1(B_1, C_1)\) and \(T_2(B_2, C_2)\) and perform the junction of \(C_1\) to \(C_2\). Pick one of the newly created edges, subdivide it with a vertex \(v\) and attach a dangling edge to \(v\), producing a semiedge \(r\) (see Figure 9). Denote the \((2, 2, 1)\)-pole \(M(B_1, B_2, r)\) constructed in this way by \(TT(T_1, T_2)\).

We prove that \(\text{Col}(TT(T_1, T_2)) \subseteq \text{Col}(P_2)\) with equality attained if both \(T_1\) and \(T_2\) are perfect (see Section 5.2). Since \(T_1\) and \(T_2\) are proper, any colouring \(\varphi\) of \(M\) satisfies \(\varphi_s(B_1) = a \neq 0\) and \(\varphi_s(B_2) = b \neq 0\), so \(\varphi(S(M)) \in \text{Col}(P_2)\). Assume that \(T_1\) and \(T_2\) are perfect. Let \(\varphi\) be an assignment of colours to the dangling edges of \(M\) such that \(\varphi(S(M)) \in \text{Col}(P_2)\). Then there exist distinct colours \(a\) and \(b\) such that \(\varphi_s(B_1) = a\), \(\varphi_s(B_2) = b\), and \(\varphi(r) = a + b \neq 0\). Assign the edges joining \(v\) to \(T_1\) and \(T_2\) colours \(b\) and \(a\), respectively. Now, if we colour the remaining two edges of \(M\) joining \(T_1\) to \(T_2\) with colours \(a\) and \(b\) arbitrarily, we obtain admissible assignments of colours for the semiedges.
of both $T_1$ and $T_2$. Since both $T_1$ and $T_2$ are perfect, the assignments extend to colourings of $T_1$ and $T_2$ and hence to a colouring of the entire $M$.

Since $P_2$ is a proper $(2, 2, 1)$-pole, so is $TT(T_1, T_2)$. If both proper $T_1$ and $T_2$ are obtained from the Petersen graph — that is, if they are triads — we get a perfect proper $(2, 2, 1)$-pole $P_{TT} = TT(T, T)$ of order 19 with Col($P_{TT}$) = Col($P_2$).

Observe that the substitution of $TT(T_1, T_2)$ for $P_2$ produces a snark consisting of two proper $(2, 3)$-poles $T_1$ and $T_2$ and one negator $N$ as depicted in Figure 10.

![Figure 10: Structure of a snark consisting of two proper (2,3)-poles $T_1$ and $T_2$ and one negator $N$](image)

5.8 Improper $(2, 3)$-poles of type NT

Let $N(I, O, r)$ be a negator and $T(B, C)$ be a proper $(2, 3)$-pole. Perform the junction of $O$ and $B$, subdivide one of the dangling edges belonging to the 3-connector $C$ of $T$, say $e$, with a new vertex $v$, and attach the residual semiedge $r$ of $N$ to $v$ (see Figure 11). The resulting $(2, 3)$-pole $M(I, C')$ has its 2-connector $I$ inherited from $N$ while its 3-connector $C'$ has two semiedges $e_1$ and $e_2$ inherited from the output connector $C$ of $T$, and the third semiedge $e_3$ arises from the subdivision of $e$ with $v$. Denote this $(2, 3)$-pole by $NT(N, T)$.

We show that Col(NT($N, T$)) $\subseteq$ Col($M_{ev}$), where $M_{ev}$ is a $(2, 3)$-pole shown in Figure 12 (see also Section 5.2). We also show that if both $N$ and $T$ are perfect, then Col(NT($N, T$)) = Col($M_{ev}$).

Let $\varphi$ be a colouring of the multipole $M$. Since $T$ is a proper $(2, 3)$-pole, $\varphi_{\ast}(B) = \varphi_{\ast}(O) \neq 0$ and thus $\varphi_{\ast}(I) = 0$. The parity lemma implies that $\varphi_{\ast}(C') = 0$ and therefore $\varphi(S(M)) \in$ Col($M_{ev}$).

Assume that $N$ is a perfect negator and $T$ is a perfect proper $(2, 3)$-pole. Let $\varphi$ be an assignment of colours to the dangling edges of $M$ such that $\varphi_{\ast}(I) = 0$ and $\varphi(C') = (\varphi(e_1), \varphi(e_2), \varphi(e_3)) = (a, b, c)$ where $a + b + c = 0$. To produce a colouring of $M(I, C')$, set $\varphi(r) = a$, and assign the colours $b$ and $c$ to the dangling edges of $B$ in any order. Then $\varphi_{\ast}(O) = \varphi(r) = a = \varphi_{\ast}(C)$, and since $N$ and $T$ are both perfect, this assignment extends to a colouring of $M$, as required. Therefore $\varphi(S(M)) \in$ Col($M_{ev}$) in this case.
As we have seen, the flow through both connectors of \( NT(N, T) \) is always zero, which means that \( NT(N, T) \) is an improper \((2, 3)\)-pole. The improper \((2, 3)\)-pole \( P_{NT} = NT(D, T) \) is obtained by taking the negator and the proper \((2, 3)\)-pole from the Petersen graph. It has 17 vertices and \( \text{Col}(P_{NT}) = \text{Col}(M_{ev}) \).

If we distribute the semiedges of the connector \( C' \) of the improper \((2, 3)\)-pole \( M(I, C') \) into a 2-connector and a 1-connector, we obtain a semiperfect negator \( M'(I, C' - \{s\}, s) \) where \( I \) is its improper connector and \( s \in C' \) takes the role of a residual semiedge. Furthermore, if the residual semiedge \( s \) of \( M' \) is adjoined to \( I \) to make a 3-connector, a proper \((3, 2)\)-pole \( M''(I \cup \{s\}, C' - \{s\}) \) is obtained. Indeed, for every colouring \( \varphi \) of \( M'' \) we have \( \varphi_*(I \cup \{s\}) = \varphi_*(I) + \varphi(s) = 0 + \varphi(s) \neq 0 \). However, \( M'' \) is an imperfect \((3, 2)\)-pole, because the two semiedges of \( I \) always receive the same colour.

Note that if \( G = M_{ev} * T_2 \) is a snark, then \( T_2 \) is a proper \((2, 3)\)-pole. Hence, the substitution of an improper \((2, 3)\)-pole \( NT(N, T_1) \) for \( M_{ev} \) yields a snark consisting of two proper \((2, 3)\)-poles \( T_1 \) and \( T_2 \), and the negator \( N \) as shown in Figure 10.

5.9 Proper \((2, 2, 2)\)-poles of type TTT

Take three proper \((2, 3)\)-poles \( T_i(B_i, C_i) \), where \( i \in \{1, 2, 3\} \), and a \((3, 3, 3)\)-pole \( W = W(D_1, D_2, D_3) \) formed from a single vertex \( w \) with three dangling edges by adding three isolated edges in such a way that each isolated edge contributes to different 3-connectors of \( W \). Perform the junctions \( C_i \) to \( D_i \) for each \( i \in \{1, 2, 3\} \) to obtain a \((2, 2, 2)\)-pole \( M(B_1, B_2, B_3) \) shown in Figure 13. Denote the result by \( \text{TTT}(T_1, T_2, T_3) \).

We prove that \( \text{Col}(\text{TTT}(T_1, T_2, T_3)) \subseteq \text{Col}(V_4) \), with equality attained if all of \( T_1 \), \( T_2 \), and \( T_3 \) are perfect (see Section 5.4 for the definition of \( V_4 \)). Let \( \varphi \) be a colouring of \( M(B_1, B_2, B_3) \). Since \( B_i \) is a connector of a proper \((2, 3)\)-pole, we see that \( \varphi_*(B_i) \neq 0 \) for all \( i \in \{1, 2, 3\} \). This fact implies that \( \varphi(S(M)) \in \text{Col}(V_4) \). Next, assume that \( T_1 \), \( T_2 \), and \( T_3 \) are perfect. Consider an assignment \( \varphi \) of colours to the semiedges of \( M \) such that \( c_i = \varphi_*(B_i) \neq 0 \) for \( i \in \{1, 2, 3\} \), and \( c_1 + c_2 + c_3 = 0 \). Let \( e_1 \), \( e_2 \), and \( e_3 \) be those semiedges from \( C_1 \), \( C_2 \), and \( C_3 \), respectively, that are incident with the vertex \( w \). If we assign the same colour, chosen arbitrarily, to all the remaining semiedges of the connectors \( C_i \) and set \( \varphi(e_i) = c_i \) for \( i \in \{1, 2, 3\} \), we get an admissible colouring of the semiedges for each perfect \((2, 3)\)-pole \( T_i \). Therefore \( \varphi \) can be extended to the entire multipole \( M \).

We have thus showed that \( \text{TTT}(T_1, T_2, T_3) \) is a proper \((2, 2, 2)\)-pole. By choosing the triads for all three proper \((2, 3)\)-poles we obtain a proper \((2, 2, 2)\)-pole \( P_{TTT} = \text{TTT}(T, T, T) \) of order 28 with \( \text{Col}(P_{TTT}) = \text{Col}(V_4) \).

Note that after removing the \((2, 2, 2)\)-pole \( V_4 \) from any snark \( G \) we obtain an even \((2, 2, 2)\)-pole. It follows that the snark resulting from a substitution of \( \text{TTT}(T_1, T_2, T_3) \) for \( V_4 \) consists of an even \((2, 2, 2)\)-pole connected to a proper \((2, 2, 2)\)-pole of type TTT.

Figure 13: A \((2, 2, 2)\)-pole of type TTT

Figure 14: A \((2, 2, 2, 1)\)-pole of type 3NT
5.10 Panchromatic (2,2,2,1)-poles of type 3NT

Take three negators \( N_i(I_i, O_i, r_i) \), for \( i \in \{1, 2, 3\} \), and one proper (2,3)-pole \( T(B, C) \). Arrange them as depicted in Figure 14 and denote the resulting (2,2,2,1)-pole \( M(O_1, O_2, O_3, r) \) by 3NT\((N_1, N_2, N_3, T)\).

Let \( M_2(\{e_1, e_2\}, I, O, r) \) denote a (2,2,2,1)-pole consisting of two components, an isolated edge, whose semiedges \( e_1 \) and \( e_2 \) constitute the first 2-connector, and path of length 2 with the standard distribution of semiedges into two 2-connectors and a 1-connector. We show that \( \text{Col}(M) \subseteq \text{Col}(M_2) \). Moreover, if the negators \( N_1, N_2, N_3 \), and the proper (2,3)-pole \( T \) are all perfect, then \( \text{Col}(M) = \text{Col}(M_2) \).

Let \( \varphi \) be a colouring of \( M \), let \( e \) be the edge joining \( I_3 \) and the 3-connector \( C \) of \( T \), and let \( \varphi(e) = a \). Since the connector \( I_1 \) is joined to the proper (2,3)-pole \( T \), there exist an element \( b \in \mathbb{K} \) such that \( \varphi_s(I_1) = b \), whence \( \varphi_s(O_1) = 0 \) and \( \varphi(r_1) = b \).

Suppose that \( a + b = \varphi_s(I_3) \neq 0 \). Since \( N_3 \) is a negator, \( \varphi(r_3) = a + b \). On the other hand, the parity lemma applied to \( T \) implies that \( \varphi_s(I_2) = \varphi_s(B) + \varphi(e) = a + b \neq 0 \). However, \( N_3 \) is a negator, so \( \varphi(r_2) = a + b = \varphi(r_3) \). Thus both \( r_2 \) and \( r_3 \) receive the same colour, which is impossible, because they are adjacent. Hence, \( \varphi_s(I_3) = 0 \) and \( a = b \).

Now, \( \varphi_s(O_3) \neq 0 \) whence \( \varphi(e) = b = \varphi(r_1) = \varphi(I_1) \). If we apply the parity lemma to \( T \) again, we conclude that \( b = \varphi_s(C) = \varphi_s(I_2) + \varphi(e) = \varphi_s(I_2) + b \). Therefore \( \varphi_s(I_2) = 0 \) and so \( \varphi_s(O_2) \neq 0 \). Summing up, \( \varphi_s(O_1) = 0 \) while \( \varphi_s(O_2), \varphi_s(O_3) \), and \( \varphi(r) \) are all nonzero. By the parity lemma, the connectors of \( M \) receive from \( \varphi \) all four values of \( \mathbb{Z}_2 \times \mathbb{Z}_2 \), which justifies calling \( M \) panchromatic. Furthermore, we can immediately see that \( \varphi(S(M)) \subseteq \text{Col}(M_2) \).

Now assume that all the multipoles \( N_1, N_2, N_3, \) and \( T \) are perfect. If \( \varphi \) is an assignment of colours to the dangling edges of \( M \) such that \( \varphi(S(M)) \in \text{Col}(M_2) \), we can extend it to a colouring of dangling edges of each of the 5-poles \( N_1, N_2, N_3 \), and \( T \) in such a way that the flows through their connectors are the same as the flows in the proof that \( \text{Col}(M) \subseteq \text{Col}(M_2) \). Such a colouring is admissible for all of the connectors of multipoles \( N_1, N_2, N_3, \) and \( T \), and since they are perfect, it can be extended to the entire \( M \).

By plugging in dyads and triads we obtain a panchromatic (2,2,2,1)-pole \( P_{3NT} = 3NT(D, D, D, T) \) of order 31 with \( \text{Col}(P_{3NT}) = \text{Col}(M_7) \).

5.11 Superpentagons

Let \( C_5 = C_5(e_0, \ldots, e_4) \) denote the 5-pole consisting of a 5-cycle having vertices \( v_0, \ldots, v_4 \), arranged cyclically, with five semiedges \( e_0, \ldots, e_4 \) attached to them correspondingly. We define a superpentagon to be any 5-pole \( M \) with \( \text{Col}(M) \subseteq \text{Col}(C_5) \). The importance of superpentagons consists in the fact that substituting a superpentagon \( M \) for a 5-cycle \( K \) in a snark \( G \) produces another snark \( G' \). It may be worth mentioning that substituting a superpentagon for a 5-cycle is equivalent to a 5-product of \( G \) with a snark \( M \) obtained from \( M \) by joining \( M \) to a 5-cycle in a Petersen-like manner (that is, as a pentagram). For the definition of a 5-product of snarks see [10] last paragraph of Section 3] or [12] pp. 51-52).

It is a well known fact, proved in [37] Lemmas 6.2-6.5], that for an arbitrary 5-pole \( M \) with \( \text{Col}(M) \subseteq \text{Col}(C_5) \) only two possibilities can occur: either \( \text{Col}(M) = \emptyset \) or \( \text{Col}(M) = \text{Col}(C_5) \). In the latter case we call \( M \) a perfect superpentagon. A familiar example of a perfect superpentagon has 15 vertices and can be obtained from the Isaacs flower snark \( J_5 \) by removing the unique 5-cycle \( C \) of \( J_5 \) and changing the cyclic order \( e_0, e_1, e_2, e_3, e_4 \) of the resulting dangling edges to \( e_0, e_2, e_3, e_1, e_4 \).

We now describe a superpentagon \( Q(f_0, \ldots, f_4) \) that can be observed in the analysed...
Figure 15: A superpentagon

graphs. Let $T = T(B, C)$ be a proper $(2,3)$-pole with $B = \{d_1, d_2\}$. By distributing the semiedges of $B$ into two 1-connectors we obtain a $(3,1,1)$-pole $T(C, d_1, d_2)$, which is proper as well. We further need a proper $(3,3)$-pole $R = R(I, O)$ and a $(3,3,1)$-pole $U = U(S_1, S_2, r)$ with $S_i = \{e_i, f_i, g_i\}$ and $i \in \{1, 2\}$. It consists of one vertex $v$ incident with three dangling edges whose semiedges are $e_1$, $e_2$ and $r$, and two isolated edges with semiedges $f_1$, $f_2$ and $g_1$, $g_2$, respectively (see $U_2$ and $U_3$ in Figure 15).

To construct $Q(f_0, \ldots, f_4)$ proceed as follows (see Figure 15):

- Set $f_0 = e_0$.
- For $i \in \{1, 4\}$ substitute a copy $T_i$ of $T(C, d_1, d_2)$ for the vertex $v_i$ of $C_5$. Define $f_i$ to be the copy of the semiedge $d_1$ in $T_i$.
- For $i \in \{2, 3\}$ substitute a copy $U_i$ of $U(S_1, S_2, r)$ for the vertex $v_i$ of $C_5$. Define $f_i$ to be the copy of the semiedge $r$ in $U_i$.
- Substitute the proper $(3,3)$-pole $R = R(I, O)$ for the edge $v_2v_3$ of $C_5$. Join $I$ to the copy of the connector $S_2$ in $U_2$, and further join $O$ to the copy of $S_1$ in $U_3$.
- Join the copy of $S_1$ in $U_2$ to the 3-connector of $T_1$. Join the copy of $S_2$ in $U_3$ to the 3-connector of $T_4$.
- Attach the copy of $d_2$ in both $T_1$ and $T_4$ to $v_0$.

It is easy to check that $Q(f_0, \ldots, f_4)$ is a superpentagon. Moreover, if the involved $(3,3)$-pole $R$ is colourable and both proper $(2,3)$-poles corresponding to $T_1$ and $T_4$ are perfect, then $Q$ is a perfect superpentagon.

The smallest example that can be produced by this method has 29 vertices. It is constructed by choosing $T$ to be the triad and $R$ to be the proper $(3,3)$-pole on eight vertices. Clearly, this superpentagon is proper.

6 Results of analysis

Having specified all necessary tools, we present results of our analysis. In the present section we describe only bicritical snarks since critical snarks that are not bicritical deserve special attention, which is why they will be treated in a separate section.

For a given snark $G$ we start by identifying all 5-cycle clusters as explained in Section 4. Subsequently we check whether $G$ contains any of the 5-poles $P_{NN} = NN(D, D)$, $P_{NT} =$
NT(D, T), \( P_{TT} = TT(T, T) \), \( P_{TTT} = TTT(T, T, T) \), and \( P_{3NT} = 3NT(D, D, D, T) \) which we have described in Section 5. As we have proved in Section 5, the multipoles \( P_{NN} \), \( P_{NT} \), \( P_{TT} \), \( P_{TTT} \), and \( P_{3NT} \) are colour-equivalent to \( P_2 \), \( M_{ee} \), \( P_2 \), \( V_4 \), and \( M_7 \), respectively. Here we make use of the fact that both \( D \) and \( T \) are perfect. Whenever a snark \( G \) contains any of those multipoles, it can be constructed from a smaller snark \( G' \) by a suitable substitution. The order of \( G' \) is easy to compute: if \( |G| \leq 36 \), then \( |G'| \leq 24 \). There are not so many snarks of order at most 24, thus it is easy to check if \( G' \) is isomorphic to one of them. It may be useful to note that in all cases where this procedure could be applied, the snark \( G' \) was nontrivial, although sometimes its cyclic connectivity was smaller (necessarily 4) or \( G' \) was reducible.

If we discard the snarks arising from smaller snarks by one of the substitutions mentioned above, some snarks will remain. We distribute them, for each particular order, into several classes. Each class can be characterised by specific junctions of suitable multipoles. Most of these multipoles are just 5-cycle clusters, but several more interesting ones have emerged, too (for example, \( M_{11} \) in Figure 17). In contrast to 5-cycle clusters, those have been analysed by hand. We present each class as an infinite family of snarks containing the desired small snarks.

Up to isomorphism, there is only one Petersen negator \( N_P \), the dyad \( D \), and only one Petersen proper \((2,3)\)-pole \( T_P \), the triad \( T \). However, when performing a junction of two connectors containing more than one semiedge, the result may depend on the particular order of semiedges in the connectors. Typically, the order does not affect our uncolourability arguments because they involve the connectors in their entirety, the only exception being the Isaacs snarks. Nevertheless, choosing different order of semiedges in connectors may lead to several non-isomorphic variations of the multipoles \( P_{NN}, P_{NT}, P_{TT}, \) and \( P_{TTT} \). It would be possible to take this into account in our classification, but such level of detail would just obscure the analysis without tangible benefits. Perhaps the only case where one might be interested in distinguishing those non-isomorphic variants occurs when using them as construction blocks in order to obtain larger snarks with specific properties.

By applying the approach mentioned above we have analysed all cyclically 5-connected bicritical snarks with at most 36 vertices. The list of such snarks was obtained from [5]. The results are summarised in Table 2. Uncolourability of certain snarks can be explained in several different ways. Consequently, they are included in more than one of our classes, which explains why the numbers in Table 2 do not add up.

| Order | 10 | 20 | 22 | 24 | 26 | 28 | 30 | 32 | 34 | 36 | 38 |
|-------|----|----|----|----|----|----|----|----|----|----|----|
| NN substitution | 0 | 0 | 2 | 0 | 0 | 0 | 10 | 11 | 26 | 10 | ≥ 39 |
| TT substitution | 0 | 0 | 0 | 0 | 8 | 0 | 0 | 0 | 84 | 69 | ≥ 3 |
| NT substitution | 0 | 0 | 0 | 0 | 8 | 0 | 0 | 0 | 1084 | 396 | ≥ 17 |
| TTT substitution | 0 | 0 | 0 | 0 | 8 | 0 | 0 | 0 | 22 | 0 | ≥ 0 |
| Superpentagon subst. | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 72 | 0 | ≥ 0 |
| Other | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 2 | 215 | 9 | ≥ 4 |
| TOTAL | 1 | 1 | 2 | 0 | 8 | 1 | 11 | 13 | 1503 | 484 | ≥ 56 |

Table 2: Classification of all cyclically 5-connected bicritical snarks of order up to 36
Isaacs’ flower snarks

We have already defined the Isaacs flower snarks \( J_n \), where \( n \geq 3 \) is odd, in Section 5.5. An alternative approach to describing them uses substitutions starting from the Petersen graph. The flower snark \( J_3 \) arises from the Petersen graph by substituting a triangle for a vertex. For each \( k \geq 2 \), the snark \( J_{2k+1} \) can be constructed from \( J_{2k-1} \) by substituting the \((3,3)\)-pole \( Y_4 \) for a copy of the \((3,3)\)-pole \( Y_2 \) contained in it, both \((3,3)\)-poles having the same colouring set (for the definition of \( Y_i \) see Section 5.5). It is well known that the flower snarks \( J_n \) are cyclically 6-edge-connected for all \( n \geq 7 \) and bicritical for \( n \geq 5 \) (see [37, Proposition 4.7]).

There are three Isaacs flower snarks up to order 36, namely \( J_5, J_7, \) and \( J_9 \). The snark \( J_5 \) with 20 vertices contains single cycle-separating 5-cut; the next one, \( J_7 \) with 28 vertices, is the smallest cyclically 6-connected snark, and \( J_9 \) is also cyclically 6-connected. The Isaacs snarks will not be mentioned anymore in the rest of this classification.

Order 22

The only cyclically 5-connected bicritical snarks of order 22 are the two Loupekine snarks (shown in Figure 8). Both of them contain the 5-pole \( P_{NN} \), so they can be constructed from the Petersen graph by a substitution of \( P_{NN} \) for \( P_2 \).

Order 26

There are eight cyclically 5-connected bicritical snarks of order 26. All of them contain a proper \((2,2,1)\)-pole \( P_{NT} \) and also a proper \((2,2,1)\)-pole \( P_{TT} \). Hence they can be obtained by a substitution from the Petersen graph. Also, each of them is spanned by the uncolourable 7-pole from Figure 16 consisting of one dyad \( N \) connected to two triads \( T_1 \) and \( T_2 \). This multipole was used by Steffen and others [31, 42] to construct cyclically 5-connected snarks with small order and large resistance.

Order 30

On 30 vertices, there is one cyclically 5-connected snark of girth 6, the double-star snark discovered by Isaacs [23]. It can be described as a 5-product of \( J_5 \) with itself. All the remaining snarks of order 30 arise from the Blanuša snarks by substituting \( P_{NN} \) for \( P_2 \); six snarks from the Type 1 Blanuša snark and four snarks from the Type 2. By the Type 1 Blanuša snark we mean a nontrivial snark of order 18 discovered in 1946 by Blanuša [3], and by the Type 2 Blanuša snark we mean the other nontrivial snark on 18 vertices, the Blanuša double as it is called in [32], where the history and properties of these two snarks are discussed in detail. Note that this substitution increases cyclic connectivity.
Order 32

There are 13 cyclically 5-connected bicritical snarks of order 32. From among them, 11 contain the 5-pole $P_{NN}$. All of them can be constructed from the flower snark $J_5$ by a substitution. The remaining two constitute Class 32-A which is in detail described below.

![Figure 17: The structure of Class 32-A](image1)

![Figure 18: The 7-pole $M_{11}$ constructed from $J_3$](image2)

Class 32-A

The two snarks of this class consist of three Petersen negators $N_i(I_i, O_i; r_i)$ for $i \in \{1, 2, 3\}$ and one 7-pole $M_{11}$, which are combined as shown in Figure 17. In general, the negators $N_1$, $N_2$, and $N_3$ can be taken from an arbitrary snark. The 7-pole $M_{11}$ can be derived from the flower snark $J_3$ by removing one vertex and severing two edges as indicated in Figure 18.

We now explain why any graph $G$ represented by Figure 17 is a snark. By contradiction, suppose that $G$ has a 3-edge-colouring $\varphi$. To avoid ambiguity assume that, in $G$, the connector $O_2$ is joined to $I_1$ and the connector $O_1$ is joined to $I_3$. Let $a = \varphi(r_1)$. One of the connectors of $N_1$, say, $I_1$, must have zero flow. Then $\varphi^*(O_1) = a = \varphi^*(I_3)$. From the negator $N_3$, we get that $\varphi^*(O_3) = 0$ and $\varphi(r_3) = a$. Now, consider the 7-pole $M_{11}$. The semiedges $e_1$ and $e_2$ connected to the connector $O_3$ have the same colour and so have the semiedges $e_3$ and $e_4$ connected to $r_3$ and $r_1$, respectively. The parity lemma implies that the sum of the flows through the remaining three semiedges $e_5$, $e_6$, and $e_7$ is zero. Therefore, we can perform junctions of $e_1$ with $e_2$, $e_3$ with $e_4$ and add one new vertex incident with $e_5$, $e_6$ and $e_7$, giving rise to a graph $H$ with edges coloured by $\varphi$. However, the graph $H$ is isomorphic to the flower snark $J_3$, which is a contradiction.

Note that we require that $M_{11}$ can be extended to a snark in two symmetric ways, so we cannot replace it with an arbitrary 7-pole constructed from a snark by removing a vertex and cutting two edges.

Order 34

Among the 1503 bicritical cyclically 5-connected snarks of order 34, we have found 26 that contain the 5-pole $P_{NN}$. They can be constructed from the Loupekine snarks (both types) by a substitution for $P_3$. The 5-pole $P_{NT}$ is contained in 1084 snarks which arise from the Blanuša snark (both types) by a substitution for one vertex and one edge. The next 84 snarks are spanned by the 5-pole $P_{TT}$ and can be constructed from the Blanuša snark (both types) by a substitution of $P_2$. Further 72 snarks arise from the Petersen graph by substituting a superpentagon described in Section 5.11 for a pentagon.
After analysing the structure of the remaining snarks we have decided to categorise them into six classes. The classification for order 34 is summarised in Table 3.

| Type of a snark                  | Number of snarks |
|----------------------------------|------------------|
| Containing $P_{NN}$              | 26               |
| Containing $P_{NT}$              | 1084             |
| Containing $P_{TT}$              | 84               |
| Containing $P_{TTT}$             | 22               |
| Containing superpentagon         | 72               |
| Class 34-A                       | 21               |
| Class 34-B                       | $18 + 18 (P_{TT}) + 54 (P_{NT}) = 90$ |
| Class 34-C                       | $162 + 18 (P_{NT}) = 180$ |
| Class 34-D                       | 5                |
| Class 34-E                       | 7                |
| Class 34-F                       | 2                |
| TOTAL                            | 1503             |

Table 3: Structure of cyclically 5-connected bicritical snarks of order 34.

Class 34-A

Take two negators $N_1$ and $N_2$ and two proper (2, 3)-poles $T_1$ and $T_2$, and construct a 9-pole $M_1$ as shown in Figure 19. We prove that $M_1$ is uncolourable. Suppose to the contrary that $M_1$ admits a colouring $\varphi$. Let $e$ denote the dangling edge which is incident with none of the multipoles $N_1$, $N_2$, $T_1$, and $T_2$. If a connector of a negator is joined to a proper connector, then its other connector must have zero flow through it. In our case, this is true for both $N_1$ and $N_2$. After applying the parity lemma to the (2, 2, 1)-pole connecting $N_1$ to $N_2$, which contains one vertex with three dangling edges and one isolated edge, we obtain $\varphi(e) = 0$, which is a contradiction.

Among the studied snarks of order 34, there are 21 snarks containing $M_1$. In all of them, $M_1$ is built from dyads and triads and has 33 vertices.

Class 34-B

Assume that we have two negators $N_1$ and $N_2$ and two proper (2, 3)-poles $T_1$ and $T_2$ which are assembled to a 9-pole $M_2$ as depicted in Figure 20. We prove that $M_2$ is uncolourable. Let $v$ denote the vertex of $M_2$ not belonging to any of $N_1$, $N_2$, $T_1$, and $T_2$, and let $e$ be the edge between $v$ and $T_1$. Suppose to the contrary that there is a colouring $\varphi$ of $M_2$. Since a
connector of $N_2$ is joined with a proper connector, the flow through its other connector is zero. If we apply the parity lemma first to the $(2, 2, 1)$-pole containing $v$ (which is joined to $N_1$, $N_2$, and $T_1$) and then to $N_1$, we can conclude that the flow through $e$ is the same as the flow through the residual semiedge of $N_1$. These values force a zero flow through a connector of $T_1$, which is impossible.

The multipole $M_2$ contained in the studied snarks of order 34 consists of dyads and triads and has 33 vertices. We have identified 90 snarks containing this 9-pole. Of them, 54 snarks contain also the 5-pole $P_{NT}$, and 18 snarks contain the 5-pole $P_{TT}$.

**Class 34-C**

Let $G$ be a snark. Delete a path $uv$ from $G$, sever an edge $e \neq uv$ of $G$, and denote the resulting $(2, 2, 2)$-pole by $R(A, B, C)$, where the connectors $A$ and $B$ contain the two semiedges formerly incident with $u$ and $v$, and $C$ contains two semiedges that arose from severing $e$. Should $R$ be contained in a cyclically 5-connected graph, $e$ cannot be incident with any of $u$ and $v$. The crucial property of $R$ is that it admits no colouring $\varphi$ such that $\varphi_*(A) \neq 0$ and $\varphi_*(C) = 0$. Clearly, any such colouring could be extended to the entire snark $G$. Observe that if we choose the Petersen graph for $G$, we obtain the double pentagon as the $(2, 2, 2)$-pole $R$.

Take the $(2, 2, 2)$-pole $R(A, B, C)$, a negator $N(I, O)$, and two proper $(2, 3)$-poles $T_1(B_1, C_1)$ and $T_2(B_2, C_2)$. Join $B_1$ to $A$, $C$ to $I$, $O$ to $B_2$, and denote the resulting 9-pole $M_3$ (see Figure [21]). Assume that $M_3$ has a colouring $\varphi$. The negator $N$ is connected to the proper $(2, 3)$-pole $T_2$, hence $\varphi_*(I) = \varphi_*(C) = 0$. Since $T_1$ is also proper, we have $\varphi_*(B_1) = \varphi_*(A) \neq 0$. This colouring is impossible for $R$—a contradiction.

![Figure 21: Uncolourable 9-pole $M_3$ (Class 34-C)](image)

There are 72 bicritical 5-connected snarks of order 34 belonging to this class (18 of them also contain $P_{NT}$). In all of them, the negators and proper $(2, 3)$-poles are derived from the Petersen graph.

**Class 34-D**

Take four negators $N_i(I_i, O_i, r_i)$ for $i \in \{1, 2, 3, 4\}$, connect them as shown in Figure [22] and denote the resulting graph by $G_4$. We prove that $G_4$ is a snark. If it is not, then it has a colouring $\varphi$. Without loss of generality we may assume that $\varphi_*(O_1) = 0$. Let $\varphi(uw_1) = a \in \mathbb{K}$; then $\varphi_*(I_2) = \varphi_*(O_1) + \varphi(uw_1) = a \neq 0$, so $\varphi_*(O_2) = 0$. Analogously, we get $\varphi_*(O_3) = \varphi_*(O_4) = 0$. By repeatedly applying the parity lemma we get $\varphi(w_3u) = \varphi(I_1) = \varphi(r_1) = \varphi(r_2) = \varphi(I_2) = a$, thus there is a colour conflict at $u$.

We have identified six snarks having this structure, with all four negators taken from the Petersen graph. All of them are permutation snarks, which means that they admit a 2-factor consisting of two induced cycles. For more information about permutation snarks one can consult [6, 34, 35].
Class 34-E

The four negators $N_1$, $N_2$, $N_3$ and $N_4$ can also be arranged in a different way shown in Figure 23. The proof of uncolourability is similar to the one for the previous class. There are six snarks of order 34 having this structure. The 12 snarks constituting the classes 34-D and 34-E form a complete set of all cyclically 5-connected permutation snarks of order 34, see Brinkmann et al. [6].

Class 34-F

We take five even $(2,2,2)$-poles $H_1$, $H_2$, $H_3$, $H_4$ and $H_5$, and from them we construct a larger $(2,2,2)$-pole $H_M$ as explained in Section 5.4. Since $H_M$ is an even $(2,2,2)$-pole, $H_M \ast V_4$ is a snark. Class 34-F consists of snarks of the form $H_M \ast V_4$ whose scheme can be seen in Figure 24. Since each even $(2,2,2)$-pole $H_i$, for $i \in \{2,3,4\}$, has its semiedges from one of its connectors joined to a vertex, which produces a negator $N_i$, the structure of Class 34-F snarks can be described using two even $(2,2,2)$-poles $H_1$ and $H_5$ and three negators $N_2$, $N_3$ and $N_4$ as depicted in Figure 25.

Among the studied snarks of order 34, there are two snarks of this class, both using even $(2,2,2)$-poles derived from the Petersen graph (i.e. hexagons); or alternatively, two hexagons and three dyads.

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Figure 22: The structure of Class 34-D snarks

Figure 23: The structure of Class 34-E snarks

Figure 24: The structure of Class 34-F snarks

Figure 25: Negators in Class 34-F snarks
Order 36

Out of the 484 bicritical cyclically 5-connected snarks of this order, 396 snarks contain the 5-pole $P_{NT}$ and all of them arise from the flower snark $J_5$ by a substitution for $P_2$. In 69 snarks we have identified the 5-pole $P_{TT}$ and all of them arise from the $J_5$ by a substitution for $P_2$.

The 5-pole $P_{NN}$ is contained in ten snarks. They are constructed by substitution from two smaller snarks of order 24 whose structure can be seen in Figure 26. Their structure is similar to Loupekine snarks; they only contain two additional vertices $u$ and $v$ joined by an edge. The pair of $\{u, v\}$ is removable, so these snarks are reducible, although after a substitution of $P_{NN}$ for a path $P_2$ the resulting snarks become irreducible. In all ten cases, the $(2,2)$-pole $P_2$ used for the substitution contains one of the vertices $u$ or $v$.

Excluding the flower snark $J_9$, there remain eight snarks of order 36, which fall into two classes.

Class 36-A

Take three negators $N_i(I_i, O_i, r_i)$, where $i \in \{1, 2, 3\}$, with unordered connectors (as usual), arrange them cyclically and join each $O_i$ to $I_{i+1}$, where the indices are reduced modulo 3. Subdivide one of the two edges connecting $N_i$ to $N_{i+1}$ with a new vertex $v_i$ and attach a dangling edge $e_i$ to $v_i$, thereby producing a cubic 6-pole. Turn it into a $(3,3)$-pole $M_{24}(I, O)$ by distributing the semiedges into two ordered connectors $I = (r_1, r_2, r_3)$ and $O = (e_3, e_1, e_2)$; see Figure 27.

Consider the $(3,3)$-pole $Y_3(J_Y, O_Y)$ consisting of three copies of the Isaacs $(3,3)$-pole $Y$, defined in Section 5.5. We prove that $M_{24}$ and $Y_3$ are colour-disjoint, implying that $M_{24} \ast Y_3$ is a snark.

Let $\varphi$ be a colouring of $M_{24}$ and assume that $\varphi(I) = (a, b, c) \in \mathbb{K}^3$. It is not difficult to see that for the connectors of the three negators we have either $\varphi_*(I_1) = \varphi(I_2) =$...
\( \varphi_*(I_3) = 0 \) or \( \varphi_*(O_1) = \varphi_*(O_2) = \varphi_*(O_3) = 0 \); the proof is similar to that for Class 34-D. If the former case occurs, then \( \varphi_*(O_1) = \varphi(r_1) = a \) and the parity lemma implies that \( \varphi(e_3) = \varphi_*(O_1) + \varphi_*(I_2) = a \). In a similar manner we get \( \varphi(e_1) = b \) and \( \varphi(e_2) = c \), whence \( \varphi(O) = (b, c, a) \). If the latter case occurs, we similarly get that \( \varphi(O) = (c, a, b) \). If follows that each element of Col(M24) is a 6-tuple \( (a, b, c, a', b', c') \) of elements of \( K \) where \( (a', b', c') \) arises from \( (a, b, c) \) by cyclically permuting its entries. (Obiously, certain 6-tuples of this form may be absent in Col(M24) when some of the negators are imperfect.) All of such 6-tuples are also contained in Col(Y2), except \( (a, a, a, a, a, a) \). However, \( (a, a, a, a, a, a) \not\in \text{Col}(Y3) \). Since \( Y_2 \) and \( Y_3 \) are colour-disjoint, so are \( M_{24} \) and \( Y_3 \). Therefore \( M_{24} * Y_3 \) is a snark. With the exception of the Isaacs flower snarks this is the first family of snarks known to us where ordered connectors emerge in explaining uncolourability.

There are six snarks of order 36 belonging to Class 36-A.

**Class 36-B**

Take five negators \( N_i(I_i, O_i, r_i) \), where \( i \in \{1, 2, 3, 4, 5\} \), arrange them in a cyclic manner, and for \( 1 \leq i \leq 4 \) join \( O_i \) to \( I_{i+1} \). Let \( I_1 = \{i_1, i_2\} \) and \( O_5 = \{o_1, o_2\} \). Join \( i_1 \) to \( o_1 \), subdivide the resulting edge with a new vertex \( v \), and attach to \( v \) a dangling edge with a semiedge \( r_0 \). Finally, join \( r_0 \) to \( r_3 \), \( r_1 \) to \( r_5 \), \( i_2 \) to \( r_4 \), and \( o_2 \) to \( r_2 \) to obtain a cubic graph denoted by \( G \), see Figure 28.

![Figure 28: The structure of Class 36-B snarks](image)

We show that \( G \) is a snark. Suppose to the contrary that \( G \) has a 3-edge-colouring \( \varphi \). Since \( N_3 \) is a negator, the flow through one its connectors is nonzero. If \( \varphi_*(O_3) \neq 0 \), then there exists an element \( a \in K \) such that \( \varphi_*(O_3) = a \). It follows that \( \varphi_*(O_4) = 0 \) and \( \varphi(r_4) = a \). Moreover, \( \varphi_*(I_2) \neq 0 \), so \( \varphi_*(I_1) = 0 \). Now, the edge leading from \( N_4 \) to \( N_1 \) through \( I_1 \) has colour \( \varphi(r_4) = a \), which implies that the edge connecting \( N_1 \) to \( v \) through \( I_1 \) has the same colour. However, the edge leading from \( N_3 \) to \( v \) has colour \( \varphi(r_3) = a \), too, and this is a contradiction. If \( \varphi_*(I_3) \neq 0 \), a contradiction is derived similarly.

This family of snarks, built from five negators, can easily be generalised to a similar family built from a larger number negators, namely \( 4k + 1 \), where \( k \geq 1 \). In this case we connect the vertex \( v \) to \( N_1 \) through \( I_1 \), to \( N_{4k+1} \) through \( O_{4k+1} \), and to \( N_{2k+1} \) by using \( r_{2k+1} \). We further join \( r_2 \in I_1 \) with \( r_{2k+2} \) and \( o_2 \in O_{2k+1} \) with \( r_{2k} \). The proof that the resulting graph \( G \) is a snark is similar to the one above. Clearly, \( G \) can be made cyclically 5-edge-connected by identifying pairs of residual semiedges appropriately.

Among the snarks of order 36 we have identified two snarks of this type.
7 Strictly critical snarks

We conclude our investigation of small cyclically 5-connected critical snarks by turning our attention to those that are strictly critical. In Section 3, we have explained that strictly critical snarks are of special interest, partially due to the fact that some of them can be derived from noncritical snarks.

By Theorem 6.1 in [12], a strictly critical snark of order \(n\) exists if and only if \(n \geq 32\). Among the snarks of order at most 36, there are only 84 cyclically 5-connected strictly critical snarks, all having 36 vertices. Of those, 77 arise by an NT-expansion from a non-critical snark of order 20. The structure of the remaining seven snarks is very similar: they all arise from the Petersen graph by a substitution of a suitable proper \((2, 2, 2)\)-pole for the \((2, 2, 2)\)-pole \(V_4\). The multipole \(M\) is constructed as follows. Let \(T_1, T_2,\) and \(T_3\) be three perfect proper \((2, 3)\)-poles. Add three new vertices and produce \(M\) by connecting them to \(T_1, T_2,\) and \(T_3\) in the manner indicated in Figure 29. For the reasons similar to those valid for the \(TTTT (2, 2, 2)\)-pole (Section 5.9), the multipole \(M\) is colour-equivalent to the \((2, 2, 2)\)-pole \(V_4\) consisting of one vertex and its three neighbours. However, if we remove from \(M\) any two of the three new vertices, the colouring set will not change. Consequently, no snark containing \(M\) is bicritical. The \((2, 2, 2)\)-pole \(M\) contained in all of the seven remaining snarks consist of three triads.

8 An infinite family of bicritical snarks

In the previous sections we have presented a number of new constructions of snarks which mimic the structure of small snarks. We now deal with the problem of their bicriticality.

As mentioned in the discussion of the results for order 36, the base components used in constructions of bicritical snarks need not come from bicritical snarks. There are ten bicritical snarks of order 36 that arise from noncritical snarks of order 24 by a substitution of \(P_{NN}\) for \(P_2\). Obviously, it is necessary to eliminate all pairs of removable vertices from the used construction blocks, but in general we do not understand under what conditions it would be sufficient.

On the other hand, taking all construction blocks from bicritical snarks — a slightly stronger requirement than just the absence of removable pairs of vertices — does not ensure the resulting snark to be bicritical. For instance, a proper \((2, 3)\)-pole constructed even from a bicritical snark can be uncolourable (see results for order 26). Such a proper \((2, 3)\)-pole cannot be used in any substitution which should yield a bicritical snark. In general, we do not know much about the circumstances under which proper \((2, 3)\)-poles are colourable or perfect. This is a significant difference from negators whose colouring properties are characterised by Theorem 10.
The purpose of this section is to illustrate that imposing certain additional requirements on the construction blocks can assure bicriticality of the resulting snark in a fairly general setting. The described requirements are not overly restrictive and it is even possible that most construction blocks taken from bicritical snarks (of any given order) satisfy them.

For our demonstration, we have chosen snarks constructed by an NN-substitution (see Section 5.6). This is perhaps the simplest class of infinite classes which we have described, nevertheless, a similar approach works for the rest of them as well. We view the snarks arising by an NN-substitution as consisting of three negators \( N_i(I_i, O_i, r_i) = \text{Neg}(G_i; u_i, v_i) \) for \( i \in \{1, 2, 3\} \) arranged along a circle with an additional vertex attached to the residual semiedges (see Figure 30). We denote the resulting graph by \( \text{NNN}(N_1, N_2, N_3) \).

![Figure 30: A schematic drawing of a snark \( \text{NNN}(N_1, N_2, N_3) \).](image)

As discussed in the beginning of this section, our restrictions have to be imposed on the construction blocks, not on the snarks they originate from. We will call a negator \( N = \text{Neg}(G; u, v) \) bicritical if the multipole \( G - \{x, y\} \) is colourable for every two distinct vertices \( x, y \in V(N) \). The following proposition shows that this property is necessary.

**Proposition 11.** If \( N_1, N_2, \) and \( N_3 \) are negators such that \( G = \text{NNN}(N_1, N_2, N_3) \) is a bicritical snark, then all of them bicritical.

**Proof.** Consider a negator \( N_i = N(G_i; u_i, v_i) \) and choose an arbitrary pair \( \{x, y\} \) of its vertices. Since \( G \) is a bicritical snark, the multipole \( G - \{x, y\} \) is colourable. By replacing the \( (2, 2, 1) \)-pole \( \text{NN}(N_2, N_3) \) with the colour-equivalent \( (2, 2, 1) \)-pole \( P_2 \), we get the multipole \( G_i - \{x, y\} \) which is colourable as well. It follows that each negator \( N_i \) is bicritical, as claimed.

We do not know whether the property stated in Proposition 11 — or a stronger property that all three negators are taken from bicritical snarks — is sufficient. We have constructed all the negators from bicritical cyclically 5-connected snarks on up to 30 vertices. Using them we have created all possible snarks of class \( \text{NNN} \) that contain at most two different negators; there are approximately 600,000 such snarks. With the help of a computer we have verified that all of them are bicritical.

In order to specify a sufficient condition, we introduce the following rather technical property of negators. We will henceforth assume that the negators in question are constructed from snarks of girth at least 5, in order to avoid ambiguity of notation and certain marginal cases.

**Definition 12.** Let \( G \) be a snark of girth at least 5. A negator \( N(\{i_1, i_2\}, \{o_1, o_2\}, r) = \text{Neg}(G; u, v) \) is called feasible if it is bicritical and possesses the following properties:

(i) For every pair of vertices \( \{x, y\} \) where \( x \in \{u, v\} \) and \( y \in V(N) \) and for any two dangling edges \( e \) and \( f \) of the 6-pole \( G - \{x, y\} \) formerly incident with \( x \) there exists a colouring \( \varphi \) of the 6-pole \( G - \{x, y\} \) such that \( \varphi(e) = \varphi(f) \).
(ii) For any vertex \( y \in V(N) \) there exist colourings \( \varphi \) and \( \psi \) of the 8-pole \( N - y \) such that \( \varphi(i_1, i_2, o_1, o_2, r) = (a, a, b, b, a) \) and \( \psi(i_1, i_2, o_1, o_2, r) = (a, a, b, b, b) \) for some \( a, b \in K \) with \( a \neq b \).

Here we regard each multipole \( M = G - \{x, y\} \) always as a 6-pole — if the vertices \( x \) and \( y \) are adjacent, we keep the edge \( xy \) as an isolated edge in \( M \). If \( G \) is bicritical, Property (i) is easily satisfied. Indeed, the 6-pole \( M \) is colourable because \( G \) is bicritical; by the parity lemma, the two dangling edges formerly incident with \( x \) have the same colour which can be assigned to the isolated edge in \( M \) so that all three semiedges formerly incident with \( x \) have the same colour for a suitable colouring of \( M \).

If \( G \) is bicritical, \( M \) is also colourable for each pair of non-adjacent vertices \( x \) and \( y \) of \( G \). From the parity lemma we deduce that in every 3-edge-colouring of \( M \) two of the semiedges formerly incident with \( x \) must have the same colour. Property (i) requires more: the two semiedges having the same colour can be arbitrarily prescribed. We have tested all cyclically 5-connected bicritical snarks of order up to 36 and only six of them support a negator violating Property (i). We describe them in Section 9.

From the definition, it is not clear whether a feasible negator has to be perfect or not. Amongst the negators constructed from nontrivial snarks up to order 28, there is no example of an imperfect feasible negator.

Property (i) of feasible negators is related to a similar technical property introduced by Chladný and Škoviera [12] in their study of criticality and bicriticality of dot products of snarks.

**Definition 13 ([12])**. A pair \( \{e, f\} \) of edges of a snark \( G \) is called essential if it is non-removable and, moreover, if for every 2-valent vertex \( v \) of \( G - \{e, f\} \), the graph obtained from \( G - \{e, f\} \) by suppressing \( v \) is colourable.

**Lemma 14.** Let \( N = \text{Neg}(G; u, v) \) be a snark, \( x \in \{u, v\} \), and \( y \in V(N) \). Assume that for every edge \( e \) incident with \( x \) in \( G \) there is an edge \( f \) incident with \( y \) in \( G \) such that \( \{e, f\} \) is an essential pair of edges in \( G \). Then the negator \( N \) satisfies Property (i) of Definition [12].

**Proof.** Let \( e_1, e_2, \) and \( e_3 \) denote the edges incident with \( x \) listed in an arbitrary order. From our assumption it follows that there is an edge \( f \) incident with \( y \) such that \( \{e_1, f\} \) is an essential pair of edges in \( G \). Therefore, the multipole \( G - \{e_1, f\} \) with \( x \) suppressed has a colouring \( \varphi \). If we cut the edge resulting from the suppression of \( x \) into two dangling edges corresponding to \( e_2 \) and \( e_3 \) and remove \( y \), we get the \((3, 3)\)-pole \( G - \{x, y\} \) with a colouring in which the dangling edges corresponding to \( e_2 \) and \( e_3 \) have the same colour, where \( e_2 \) and \( e_3 \) can be chosen arbitrarily. Thus the negator \( N \) satisfies Property (i) of Definition [12].

We have tested negators for Property (ii). Although there are many negators violating (ii), more than 90% of all negators created from bicritical cyclically 5-connected snarks with at most 34 vertices are feasible. For instance, for every such snark \( G \) of order 34, there are, if we ignore possible isomorphisms, 102 possible negators which can be constructed from \( G \); the number of the feasible ones among them ranges from 74 to 102.

**Theorem 15.** If \( N_1, N_2, \) and \( N_3 \) are any three feasible perfect negators, then \( G = \text{NNN}(N_1, N_2, N_3) \) is a bicritical snark.

**Proof.** For \( j \in \{1, 2, 3\} \) let \( N_j = \text{Neg}(G_j; u_j, v_j) \). Let \( I_j = \{i_j, i'_j\} \) and \( O_i = \{o_j, o'_j\} \) be the connectors of \( N_j \), let \( r_j \) be its residual edge, and let \( w_j \) be the the common neighbour
of \( u_j \) and \( v_j \) in \( G_j \). Choose any two distinct vertices \( x \) and \( y \) of \( G \). We show that the multipole \( G - \{x, y\} \) is colourable, implying that \( G \) is bicritical.

**Case 1.** If both vertices \( x \) and \( y \) belong to the same negator \( N_j \), we can replace the other two negators with the colour-equivalent path \( P_2 \) (path of length two), completing the negator \( N_j \) into a snark \( G_j \). Since \( N_j \) is bicritical, \( G_j - \{x, y\} \) is colourable, hence so is \( G - \{x, y\} \).

**Case 2.** Assume that \( x \) and \( y \) belong to different negators, say, \( x \in V(N_1) \) and \( y \in V(N_2) \). Remove the vertices \( v_1 \) and \( x \) from the snark \( G_1 \) and denote the semiedges formerly incident with \( v_1 \) by \( e_1, e_2, \) and \( e_3 \) in such a way that \( e_3 \) is incident with \( w_1 \). According to Property (i) of the feasible negator \( N_1 \), there exists a colouring \( \varphi_1 \) of \( G_1 - \{v_1, x\} \) such that \( \varphi_1(e_1) = \varphi_1(e_2) \). Let \( a = \varphi_3(e_3) \) and \( b = \varphi_1(u_1 w_1) \); obviously \( a \neq b \) (see Figure 31).

We can simply restrict the colouring \( \varphi_1 \) to a colouring of the multipole \( N_1 - x \) for which \( \varphi_1(I_1) = \varphi_1(u_1 w_1) = b, \varphi_1(r_1) = a + b \) and \( \varphi_1(O_1) = 0 \).

![Figure 31: The colouring \( \varphi_1 \) of \( N_1 \).](image)

![Figure 32: The colouring \( \varphi_2 \) of \( N_2 \).](image)

Let \( p = \varphi_1(o_1) \). The negator \( N_2 \) is feasible, so according to Property (ii) there exists a colouring \( \varphi_2 \) of \( N_2 - y \) such that \( \varphi_2(r_2) = a, \varphi_2(i_2) = \varphi_2(i'_2) = p, \) and \( \varphi(o_2) = \varphi(o'_2) = q \). If \( p = a \), then \( q = b \); if \( p \neq a \) then \( q = a \). The colourings \( \varphi_1 \) and \( \varphi_2 \) are compatible and can be glued together to form a colouring \( \varphi \) of the multipole \( M = NN(N_1, N_2) - \{x, y\} \) depicted in Figure 33. Since \( N_3 \) is perfect, the colouring \( \varphi \) of \( M \) can be extended to the entire 6-pole \( G - \{x, y\} \).

**Case 3.** If one of the vertices \( x \) and \( y \), say \( y \), does not belong to any of the negators, it must be the vertex attached to the residual semiedges of \( N_1, N_2, \) and \( N_3 \). Let \( x \in V(N_1) \). Property (ii) applied to the feasible negator \( N_1 \) guarantees there exists a colouring \( \varphi_1 \) of \( N_1 - x \) such that \( \varphi_1(i_1) = \varphi_1(i'_1) = a \) and \( \varphi_1(r_1) = \varphi(o_1) = \varphi(o'_1) = b \neq a \). Since \( N_2 \) is perfect, it admits a colouring \( \varphi_2 \) such that \( \varphi_2(I_2) = 0 \) and \( \varphi_2(O_2) \neq 0 \). Finally, \( N_3 \)

![Figure 33: Colouring of the 11-pole \( NN(N_1, N_2) - \{x, y\} \).](image)
is also perfect, and hence it admits a colouring \( \varphi_3 \) that is compatible with both \( \varphi_1 \) and \( \varphi_2 \). The partial colourings \( \varphi_1, \varphi_2, \) and \( \varphi_3 \) can be combined to colour the entire 6-pole \( G - \{x, y\} \). This completes the proof of the theorem.

In order to create an infinite class of bicritical snarks, we need an infinite family of feasible negators. As one could expect, all the negators constructed from Isaacs snarks are feasible. Our proof is based on the fact, established in [12], that every pair of non-adjacent edges in any Isaacs snark except \( J_3 \) is essential.

**Proposition 16.** Every negator constructed from the snark \( J_k \) is feasible.

*Proof.* Since every pair of edges of \( J_k \), with \( k \geq 5 \), is essential, it follows from Lemma 14 that each negator of \( J_k \) has Property (i) of a feasible negator. The fact that these negators possess also Property (ii) will be established by induction on \( k \).

For the induction basis we have checked that all negators derived from the snarks \( J_5, J_7, \) and \( J_9 \) satisfy Property (ii); we did it with the help of a computer.

Now, consider the Isaacs snark \( J_k \) for an odd \( k \geq 11 \). Remove an arbitrary path \( uvw \) from \( J_k \) to produce a negator \( N = \text{Neg}(J_k; u, v) \) whose dangling edges are denoted by \( i_1, i_2, o_1, o_2 \) and \( r \) in the usual way. Remove from \( N \) an arbitrary vertex \( x \) and denote the resulting 8-pole by \( M \). The path \( uvw \) intersects at most three consecutive copies of the Isaacs \( (3,3) \)-pole \( Y \) and the removal of the vertex \( x \) corrupts at most one other copy of \( Y \). Consequently, there are at least four consecutive copies of \( Y \) in \( M \) that remain intact; let \( Y_4 \) denote the \( (3,3) \)-pole which they induce. We replace them with a \( (3,3) \)-pole \( Y_2 \) consisting of two copies of \( Y \) and denote the resulting multipole \( M' \). Clearly, \( M' \) is isomorphic to the multipole obtained from \( J_{k-2} \) by removal of a certain path of length two and a certain additional vertex. By the induction hypothesis, there exists a colouring of \( M' \) in which the dangling edges corresponding to \( i_1, i_2, o_1, o_2, r \) have colours exactly as desired for either \( \varphi \) or \( \psi \) from Property (ii). Since the multipoles \( Y_4 \) and \( Y_2 \) are colour-equivalent [37], the desired colours can also be assigned to the semiedges \( i_1, i_2, o_1, o_2, r \) of \( M \). Hence, any negator constructed from the Isaacs snark \( J_k \) satisfies Property (ii). Consequently, every negator constructed from the snark \( J_k \) is feasible, as claimed.

Using feasible negators from the Isaacs snarks \( J_k \), with \( k \geq 5 \), we can construct an infinite class of bicritical snarks. All such snarks are cyclically 5-connected. Since 5-cycles can only occur in negators constructed from \( J_5 \), avoiding such negators will lead to bicritical snarks of girth 6.

9 Non-removable edges that are not essential

As anticipated in the previous section, we wish to investigate snarks containing negators that violate Property (i) of Definition 12. Property (i) of a feasible negator is related to the concept of an essential pair of edges, which in turn plays a crucial role in factorisation of a critical snark into a dot product of two smaller snarks (see Theorem 7).

Among the 5-simple snarks of order up 36 there are exactly six snarks supporting a negator violating Property (i), all of them belonging to the class NNN. They consist of two Petersen negators and one negator constructed from a reducible snark of order 24 (see Figure 26). One of these snarks, denoted by \( G_{36} \), is illustrated in Figure 34. If we remove the pair of vertices 12 and 3 (or 12 and 8), we get a 6-pole \( M \) such that for each colouring of \( M \), the dangling edges incident with the vertices 20 and 4 (or 20 and 28) have different colours; this property has been verified by exhaustive computer search.
we construct a negator from the snark $G_{36}$ by removing a path of length 2 starting from the vertex 12, the result violates Property (i).

The snark $G_{36}$ has another interesting property. If we take the 6-pole $G_{36} - \{3, 12\}$ and perform the junction of the semiedges (4) and (20), we get a 4-pole which is uncolourable (because the two joined semiedges have different colours in any possible 3-edge-colouring). Furthermore, we can add one vertex incident with semiedges (8) and (23); the resulting uncolourable multipole is isomorphic to $G_{36} - \{12-28, 3-7\}$ with the vertex 12 suppresed. This implies that the pair of edges $\{12-28, 3-7\}$ is not essential in $G_{36}$. On the other hand, with the help of a computer we have found a colouring which proves that this pair of edges is non-removable. This solves Problem 5.7 proposed by Chladný and Škoviera in [12] by showing that there exists a pair of non-removable edges in an irreducible snark which is not essential. The same holds for the pairs of edges $\{12-28, 3-23\}$, $\{12-4, 8-0\}$ and $\{12-4, 8-2\}$.

10 Beyond order 36

We conclude our paper by analysing the currently known 5-simple snarks of order 38. We also explain how our results can be used to generate some of such snarks of higher orders. At present, there are 19,775,768 known nontrivial snarks of order 38 (see [3], section Snarks). Of them, 56 are 5-simple snarks, all being bicritical. In the latter set we have identified the 5-pole $P_{NN}$ in 39 snarks, the 5-pole $P_{NT}$ in 22 snarks, and the 5-pole $P_{TT}$ in 7 snarks, while there are 10 snarks containing both $P_{NN}$ and $P_{NT}$ and 6 snarks containing both $P_{NN}$ and $P_{TT}$. In three snarks we have found the panchromatic $(2,2,2,1)$-pole $M_{3NT} = 3NT(D,D,D,T)$ (see Section 5.10). All three snarks arise from the Petersen graph by substitution of $M_{3NT}$ for a path of length two and one edge. The only remaining snark gives rise to an infinite family, which we are just about to describe.

Class 38-A

Take four negators $N_i(I_i, O_i; r_i)$, for $i \in \{1, 2, 3, 4\}$, and one proper (2,3)-pole $T(B, C)$, connect them as shown in Figure 35, and denote the resulting graph by $G$.

We show that $G$ is a snark. The connector $B$ is proper, so the edges $r_1$ and $r_2$ contained in $B$ have different colours, say $\varphi(r_1) = a$ and $\varphi(r_2) = b$, where $a \neq b$. It follows that $\varphi_*(O_1) = \varphi_*(I_2) = 0$, for otherwise we would have $a = b$. Moreover, $\varphi_*(I_1) = a$ and $\varphi_*(O_2) = \varphi_*(I_3) = b$. Since $N_3$ is a negator, we infer that $\varphi(r_3) = b$ and $\varphi_*(O_3) = 0$. Knowing the colour of three semiedges of the proper (2,3)-pole $T$, we can use the Kirchhoff law to determine the flow through the remaining two semiedges, which coincides with the flow through $O_4$. We have $\varphi_*(O_4) = \varphi(r_1) + \varphi(r_2) + \varphi(r_3) = a + b + b = a \neq 0$, so
Figure 35: The structure of Class 38-A snarks

\[ \varphi_*(I_4) = 0. \] Therefore \( \varphi(r_4) = a = \varphi_*(I_1) \), which forces the flow value on the edge of \( I_1 \) different from \( r_4 \) to be zero. This contradiction proves that \( G \) is a snark.

The one remaining 5-simple snark of order 38 consists of four dyads and one triad. One can clearly see that permuting the semiedges in the connector \( C \) of the triad leads to other 5-simple snarks of order 38. Moreover, permutations of the semiedges in the connectors of dyads may also give rise to additional nonisomorphic 5-simple snarks of order 38.

Class 42-A

This family illustrates the fact mentioned in Section 4 that, unlike quasitriads, both triple pentagons and tricells may occur in 5-simple snarks. From among the six 5-cycle clusters on up to 10 vertices shown in Figure 36 quasitriad is thus the only one that cannot occur in 5-simple snarks.

Let \( R \) be a \((2, 2, 2)\)-pole obtained from a snark \( G \) by severing three pairwise nonadjacent edges; if \( G \) is the Petersen graph, then \( R \) is either the triple pentagon or the tricell, depending on the choice of the three edges. Let \( N_1 \) and \( N_2 \) be two negators, and \( T_1 \) and \( T_2 \) two proper \((2, 3)\)-poles. Construct a 10-pole \( M \) as depicted in Figure 36. The 10-pole \( M \) is uncolourable: for each \( i \in \{1, 2\} \) the flow between \( N_i \) and \( R \) has to be zero, which is impossible for \( R \). Thus, to obtain a snark it is sufficient to perform junctions of the 10 semiedges of \( M \). If we take all the multipoles from the Petersen graph, we obtain several 5-simple snarks of order 42. Somewhat surprisingly, it is also possible to perform the junctions in such a way that the outcome will be cyclically 5-edge connected, but not critical.

Figure 36: A new infinite class of snarks containing snarks with triplepentagons and tricells.

Constructions and analysis of larger critical snarks

Infinite families described in our paper can be used to generate new cyclically 5-connected critical snarks of orders greater than 36. We briefly sketch the ideas that can be used to
construct reasonable amounts of such snarks.

To construct a member of an infinite family we choose snarks for the construction of the desired multipoles. Cyclically 5-connected critical snarks are a sensible choice, nevertheless, we have seen examples where the multipoles were constructed from non-critical snarks or from snarks with smaller cyclic connectivity. These snarks might produce new members of the families, however, it requires more computational time. One way or another, we still need to check if the resultant snarks are cyclically 5-connected and critical, which is not guaranteed just by the membership in any of our families. Another possibility to construct new families of snarks that can contain cyclically 5-connected critical ones is to use ideas which occurred in our proofs of uncolourability.

Currently we are not able to construct any sort of a complete list of snarks of order 38 (or more). If a complete list of all the cyclically 5-connected critical snarks of a given order \( n \geq 38 \) eventually becomes available, it is possible to employ our methods to analyse them. We expect that, at least for reasonably small \( n \), a large number of these snarks will fall into one of already described families, and the remaining snarks give a rise to new infinite families of snarks. With increasing \( n \), however, the approach based on the analysis of 5-cycle clusters will become less efficient. For instance, we would not be able to identify important multipoles, such as negators or proper \((2, 3)\)-poles, arising from Isaacs flower snarks. It might be therefore useful to extend the analysis of the computer generated snarks by including the search for certain subgraphs of flower snarks, for instance the iterated Isaacs \((3, 3)\)-poles \( Y_k \) (which can be regarded as clusters of 6-cycles). In general, however, there is no known method for the analysis of cyclically 6-connected snarks. In spite of the efforts of Karabáš et al. [27], a decomposition theorem for cyclically 6-connected snarks similar to decomposition theorems for lower connectivities, such as those proved in [10, 37], is not known. Furthermore, small cyclically 6-connected snarks different from the flower snarks seem to be very difficult to find: the smallest known example was constructed in 1996 by Kochol in [30] and has 118 vertices.

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