Spherically symmetric solutions and gravitational collapse in brane-worlds

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Abstract

We consider spherically symmetric solutions within the context of brane-world theory without mirror symmetry or any form of junction conditions. For a constant curvature bulk, we obtain the modified Tolman-Oppenheimer-Volkoff (TOV) interior solutions in two cases where one is matched to a Schwarzschild-de Sitter exterior while the other is consistent with an exterior solution whose structure can be used to explain the galaxy rotation curves without postulating dark matter. We also find the upper bound to the mass of a static brane-world star and show that the influence of the bulk effects on the interior solutions is small. Finally, we investigate the gravitational collapse on the brane and show that the exterior of a collapsing star can be static in this scenario.

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1 Introduction

The idea that our familiar 4-dimensional (4D) space-time is a hypersurface (brane) in a 5-dimensional space-time (bulk) [1, 2, 3] has been under detailed elaboration during the last decade. According to this brane-world scenario, all matter and gauge interactions reside on the brane, while gravity can propagate in the whole 5-dimensional space-time. Several brane-world cosmologies have been proposed in the context of Randall-Sundrum (RS) formulations [2], defined in a 5-dimensional anti-de Sitter space-time (AdS5). The dynamics of these models feature boundary terms in the action and sometimes mirror symmetry, such that bulk gravitational waves interfere with the brane-world motion. This usually comes together with junction conditions producing an algebraic relationship between the extrinsic curvature and the confined matter [4, 5]. The consequence is that the Friedman equation acquires an additional term which is proportional to the square of energy density of the confined matter field [6, 7]. This term was initially considered as a possible solution to the accelerated expansion of the universe. However, soon it was realized to be incompatible with the big bang nucleosynthesis, requiring additional fixes [7].

Brane-world scenarios under more general conditions and still compatible with the brane-world program have also been rather extensively studied over the past decade where it has been shown that it is possible to find a richer set of cosmological solutions in accordance with the current observations [8]. Under these conditions, without using Z2 symmetry or without postulating any junction condition, Friedman equation is modified by a geometrical term which is defined in terms of the extrinsic curvature, leading to a geometrical interpretation for dark energy [9]. There have also been arguments concerning the uniqueness of the junction conditions. Indeed, other forms of junction conditions exist, so that different conditions may lead to different physical results [5]. Furthermore, these conditions cannot be used when more than one non-compact extra dimension is involved. Against
this background, an interesting higher-dimensional model was introduced in [11] where particles are trapped on a 4-dimensional hypersurface by the action of a confining potential. The dynamics of test particles confined to a brane by the action of such potential at the classical and quantum levels were studied in [12]. In [13], a brane-world model was studied in which matter is confined to the brane through the action of such a potential without using any junction conditions, offering a geometrical explanation for the accelerated expansion of the universe. A geometrical explanation for the generalized Chaplygin gas was considered in [14] along the same line. We have also studied exact solutions of the vacuum field equations on the brane for two interesting cases. The first solution can be used to explain the galaxy rotation curves without assuming the existence of dark matter and without having to resort to the Modified Newtonian Dynamics (MOND), and the second solution represents a black hole in an asymptotically de Sitter space-time [15].

One physically important problem in brane-world scenarios is the development of a full understanding of stellar structures and black holes. Static, spherically symmetric exterior vacuum solutions of the brane-world models were first proposed by Dadhich and co-workers [16] and Germani and Maartens [17]. In [16], the authors obtained an exact black hole solution of the effective Einstein equation on the brane under the condition that the bulk has non-zero Weyl curvature and the brane spacetime satisfies the null energy condition. The solution is given by the usual Reissner-Nordstrom (RN) metric where the charge parameter is thought of as a tidal charge arising from the projection of the Weyl curvature of the bulk onto the brane. The tidal charge, like the RN electric charge, would generate a $1/r^2$ term in the potential while the high energy modification to the Newtonian potential cannot be any stronger than $1/r^3$ [2, 18]. The cause for this disagreement is the presence of tidal charges which is a measure of the bulk Weyl curvature. The main drawback of the solution is that we do not know the corresponding bulk solution. It is however agreed that the RN metric is a good approximation to a black hole on the brane near the horizon [19]. It has also been shown that the vacuum field equations on the brane reduce to a system of two ordinary differential equations which describe all the geometrical properties of the vacuum as functions of dark pressure and dark radiation terms [20]. Stellar structure in brane-world models is very different from that in ordinary general relativity. An exact interior uniform density stellar solution on the brane has been found in [17]. In this model the general relativistic upper bound for the mass-radius ratio, $M < 4R/9$, is reduced by 5-dimensional high-energy effects [17]. Spherically symmetric brane-world solutions when there is a contribution from the brane intrinsic curvature invariant in the dynamics have been studied in [21, 22].

The gravitational collapse on the brane has been widely studied by many authors [26]-[48]. Based on the tidal charge scenario, Oppenheimer-Snyder type [25] gravitational collapse of spherically symmetric objects was analyzed in [26]. This was formulated by a no-go theorem that indicates a non-static exterior for the collapsing sphere on the brane. The non-static exterior of the collapsing brane star could be the Vaidya radiating solution on the brane [27]. The non-static nature of a collapsing brane star for induced gravity with or without the Gauss-Bonnet term have also been studied in [28]. However, it was demonstrated in [29] that a static exterior can be obtained by relaxing the idea of dust inside the star, thereby introducing a non-vanishing surface pressure, and by ignoring the tidal effect. It has also been shown that a generalized non-empty bulk may lead to a manifestly static exterior for a collapsing spherical star on the brane [30].

In this paper, following the model introduced in [8, 9], we consider a 4-dimensional brane embedded in a 5-dimensional bulk, without using the $Z_2$ symmetry or without postulating any junction condition. Taking a constant curvature bulk, the effective field equations on the brane are modified by an extra term, $Q_{\mu\nu}$, which is a geometrical quantity. We study the interior space-time of stars in this scenario and derive the modified Tolman-Oppenheimer-Volkoff (TOV) equations on the brane in two cases. We investigate gravitational collapse of spherical objects on the brane and show the possibility of having a static exterior for a collapsing sphere in this scenario.
2 Field equations and matching conditions

The embedding of the brane-world in the bulk plays an essential role on the covariant formulation of the brane-world gravity, since it tells us how the Einstein-Hilbert dynamics of the bulk is transferred to the brane-world. However, there are many different ways to embed a manifold into another, classified as local, global, isometric, conformal, rigid, deformable, analytic or differentiable. The choice of one or other depends on what the embedded manifold is supposed to do.

Generally, there are three basic postulates in the geometrical approach considered in brane-world scenarios, that is, the confinement of the standard gauge interactions to the brane, the existence of quantum gravity in the bulk and finally, the embedding of the brane-world. All other model dependent properties such as warped metric, mirror symmetries, radion or extra scalar fields, fine tuning parameters like the tension of the brane and the choice of a junction condition are left out as much as possible in our calculations [8].

In the following we present a brief review of the model proposed in [9, 10]. Consider a 4-dimensional brane (Σ, g_{μν}) embedded in a m-dimensional bulk (M, G_{AB}). The components of the Riemann tensor of the bulk written in the embedding vielbein Z_A^A, lead to the Gauss-Codazzi equations, respectively 1 [49]

\[ R_{μβγδ} = 2g_{ab}K_μ[αa]K_δβ + R_{ABCD}Z_A^A Z_B^B Z_C^C Z_D^D, \]  

\[ D_δK_μγc - D_γK_μδc = 2g_{ab}A_μ[αbc] K_δ[αc] + R_{ABCD}Z_A^C Z_B^D Z_C^D, \]

where \( D_μ \) is the covariant differentiation with respect to \( g_{μν} \). Also, \( N_a^A \) (\( a, b = 4, ..., m \)) are the components of the \((m - 4)\)-independent normal vectors to \( Σ \) and the induced metric on \( Σ \) is \( g_{μν} = G_{AB} - g_{ab}N_a^A N_b^B \). Contracting the Gauss equation (1) on \( α \) and \( γ \) we find

\[ R_{μν} = (K_{μαc} K_ν^α - K_μ K_ν^c) + R_{AB} Z_μ^A Z_ν^B - g_{αβ} R_{ABCD} N_a^A Z_μ^B Z_ν^C Z_a^D. \]

A further contraction gives the Ricci scalar

\[ R = (K_α K^α - K_α K_α) + R - 2g_{ab} R_{AB} N_a^A N_b^B + g_{αβ} g_{bc} R_{ABCD} N_a^A N_b^B N_c^C N_δ^D. \]

Therefore the Einstein-Hilbert action for the bulk geometry in \( m \)-dimensions can be written as [9]

\[ \int (R - 2Λ^{(b)}) \sqrt{G} d^m x \equiv \int \left[ R - 2Λ^{(b)} - (K_μ K_μ - K_μ K_μ) \right] \sqrt{G} d^m x \]

\[ + \int \left[ 2g_{ab} R_{AB} N_a^A N_b^B - g_{αβ} g_{bc} R_{ABCD} N_a^A N_b^B N_c^C N_δ^D \right] \sqrt{G} d^m x. \]

The covariant equations of motion for a brane-world in a \( m \)-dimensional bulk can be derived by taking the variation of (5) with respect to \( g_{μν} \) and \( g_{ab} \), noting that the Lagrangian depends on these variables through \( Z_μ^A \) [9]. Thus, the field equation for \( g_{μν} \), with the confined matter represented by \( τ_μ^μ \) is

\[ R_{μν} - \frac{1}{2} R g_{μν} = α* τ_μ^μ - Λ^{(b)} g_{μν} + Q_{μν} + S_{μν}, \]

where we have denoted

\[ Q_{μν} = g_{ab} (K_μ^a K_ν^b - K_a K_μ^b) - \frac{1}{2} (K_μ^α K_ν^β - K_μ K_β) g_{μν}, \]

and

\[ S_{μν} = g_{ab} R_{AB} N_a^A N_b^B g_{μν} - g_{αβ} R_{ABCD} N_a^A Z_μ^B Z_ν^C N_δ^D. \]

1Capita Latin indices refer to the bulk dimensions. Small case Latin indices refer to the extra dimensions and all Greek indices refer to the brane.
The last term $S_{\mu\nu}$ in equation (6) depends on the definition of the geometry of the bulk [10]. Now, we restrict our analysis to a 5-dimensional bulk with a constant curvature characterized by the Riemann tensor

$$R_{ABCD} = k_s(g_{AC}g_{BD} - g_{AD}g_{BC}),$$

(9)

where $k_s$ denotes the bulk constant curvature. In the flat case $k_s = 0$ and in the de Sitter and anti-de Sitter cases we may write $k_s = \pm \frac{\Lambda^{(b)}}{6}$ respectively. In the normal Gaussian frame defined by the embedded space-time the bulk metric may be decomposed as

$$\mathcal{G}_{AB} = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & g_{55} \end{pmatrix}, \quad g_{55} = +1,$$

(10)

with this assumptions the Gauss-Codazzi equations reduce to

$$R_{\alpha\beta\gamma\delta} = (K_{\alpha\gamma}K_{\beta\delta} - K_{\alpha\delta}K_{\beta\gamma}) + k_s(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}),$$

(11)

$$K_{\alpha[\beta\gamma]} = 0,$$

(12)

and $S_{\mu\nu} = 3k_s g_{\mu\nu}$. Thus the dynamical equation (6) given by

$$G_{\mu\nu} = \alpha^* \tau_{\mu\nu} - \lambda g_{\mu\nu} + Q_{\mu\nu},$$

(13)

where

$$Q_{\mu\nu} = \left( K^\rho_{\mu\nu}K_{\rho\mu} - KK_{\mu\nu} \right) - \frac{1}{2} \left( K_{\alpha\beta}K^{\alpha\beta} - K^2 \right) g_{\mu\nu}. $$

(14)

Here, $\tau_{\mu\nu}$ is the confined matter energy-momentum tensor on the brane and $\lambda = -3k_s + \Lambda^{(b)}$. Using equation (12) and considering the definition of $Q_{\mu\nu}$, we find

$$Q^{\mu\nu}_{\mu} = 0.$$  

(15)

Thus, $Q_{\mu\nu}$ is independently a conserved quantity so that there is no exchange of energy between this geometrical correction and the confined matter. Such an aspect has one important consequence, that is, if $Q_{\mu\nu}$ is to be related to dark energy, it does not exchange energy with ordinary matter, much the same as in coupled quintessence models [50]. The confined matter source on the brane is considered to be an isotropic perfect fluid

$$\tau_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}, \quad p = (\gamma - 1)\rho,$$

(16)

where $\rho = p = 0$ for the exterior solutions ($r > R$).

In order to solve the Codazzi equation (12), we choose the static spherically symmetric metric on the brane in the form

$$ds^2 = -e^{\mu(r)}dt^2 + e^{\nu(r)}dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right).$$

(17)

The York relation

$$K_{\mu\nu} = -\frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial \xi},$$

(18)

then shows that in a diagonal metric, $K_{\mu\nu}$ are diagonal. Now, separating the spatial components, the Codazzi equation (12) reduces to

$$K_{\mu\nu,\sigma} - K_{\nu\rho} \Gamma^{\rho}_{\mu\sigma} = K_{\mu\sigma,\nu} - K_{\sigma\rho} \Gamma^{\rho}_{\mu\nu},$$

(19)
\[ K_{00,1} - \left( \frac{\mu'}{2} \right) K_{00} = - \left( \frac{\mu' e^\mu}{2e^\nu} \right) K_{11}, \]  

\[ K_{22,1} - \left( \frac{1}{r} \right) K_{22} = (re^{-\nu}) K_{11}, \]  

where a prime represents differentiation with respect to \( r \). The first equation gives \( K_{00,\sigma} = K_{11,\sigma} = K_{22,\sigma} = K_{33,\sigma} = 0 \) for \( \sigma = 0, 3 \). Repeating the same procedure for \( \sigma = 2 \), we obtain \( K_{00,\sigma} = K_{11,\sigma} = K_{22,\sigma} = 0 \). This shows that \( K_{11} \) depends only on the variable \( r \) and the choice \( K_{11} = \alpha e^{\nu(r)} \) would simplify our analysis. Using equations (20),(21) and \( K_{11} \), one finds

\[ K_{00}(r) = -\alpha e^{\mu(r)} + ce^{\mu(r)/2}, \]  

\[ K_{22}(r) = \alpha r^2 + \beta r. \]  

Taking \( \mu, \nu = 3 \) in the first equation we obtain

\[ K_{33,1} - \left( \frac{1}{r} \right) K_{33} = \left( e^{-\nu} r \sin^2 \theta \right) K_{11} = \alpha r \sin^2 \theta, \]  

\[ K_{33,2} - (\cot \theta) K_{33} = (\sin \theta \cos \theta) K_{22}. \]  

Using equations (22), (23) and (24), we find

\[ K_{33}(r, \theta) = \alpha r^2 \sin^2 \theta + r\beta \sin^2 \theta + rc_1 \sin \theta. \]  

Now, use of equation (14) leads to the components of \( Q_{\mu\nu} \)

\[ Q_{00} = \frac{-g_{00}}{r^2} \left[ 3\alpha^2 r^2 + 4\alpha \beta r + \beta^2 + \frac{c_1}{\sin \theta} (2\alpha r + \beta) \right], \]

\[ Q_{11} = \frac{-g_{11}}{r^2} \left[ 3\alpha^2 r^2 + 4\alpha \beta r + \beta^2 + \frac{c_1}{\sin \theta} (2\alpha r + \beta - cr e^{-\mu/2}) - 2ce^{-\mu/2} (\alpha r + \beta) \right], \]

\[ Q_{22} = \frac{g_{22}}{r} \left[ -3\alpha^2 r - 2\alpha \beta + ce^{-\mu/2} (2\alpha r + \beta) + \frac{c_1}{\sin \theta} (-2\alpha + ce^{-\mu/2}) \right], \]

\[ Q_{33} = \frac{g_{33}}{r} \left[ -3\alpha^2 r - 2\alpha \beta + ce^{-\mu/2} (2\alpha r + \beta) \right]. \]  

Since \( G_2^2 = G_3^3 \) and thus \( Q_2^2 = Q_3^3 \), one obtains \( c_1 = 0 \). Now, using these relations and equation (13), the gravitational field equations become

\[ e^{-\nu} \left( -\frac{1}{r^2} + \frac{\mu'}{r} \right) + \frac{1}{r^2} = \alpha^\rho + \lambda + 3\alpha^2 + \frac{4\alpha \beta}{r} + \frac{\beta^2}{r^2}, \]  

\[ e^{-\nu} \left( \frac{1}{r^2} + \frac{\mu'}{r} \right) - \frac{1}{r^2} = \alpha^p - \lambda - 3\alpha^2 - \frac{4\alpha \beta}{r} - \frac{\beta^2}{r^2} + 2ce^{-\mu/2} \left( \alpha + \frac{\beta}{r} \right), \]  

\[ e^{-\nu} \left( -\frac{\mu' - \nu'}{2r} - \frac{\mu' \nu'}{4} + \frac{\mu''}{2} + \frac{\mu''}{4} \right) = \alpha^p - \lambda - 3\alpha^2 - \frac{2\alpha \beta}{r} + 2ce^{-\mu/2} \left( \alpha + \frac{\beta}{2r} \right). \]  

We again note that the exterior solutions are characterized by \( \rho = p = 0 \). Using the contracted Bianchi identities and equation (15) the conservation equations is given by

\[ \tau_{\mu\nu};\nu = 0. \]
For the static, spherical symmetry metric (17), these equations give
\[ p' + \frac{\mu'}{2} (\rho + p) = 0. \]  
(32)

The Israel-Darmois matching conditions at the stellar surface \( \Sigma \) are given by [23]
\[ [G_{\mu\nu} r^\nu]_\Sigma = 0, \]  
(33)

where \([f]_\Sigma \equiv f(R^+) - f(R^-)\). Using equations (13) and (33) and taking \( \lambda = 0 \), one finds
\[ [\alpha^* \tau_{\mu\nu} r^\nu + Q_{\mu\nu} r^\nu]_\Sigma = 0, \]  
(34)

which, upon using the second equation in (27), leads to
\[ p(R) = 0. \]  
(35)

In our model this result coincides with that of general relativity whereas in brane-world models where a delta-function in the energy-momentum is used, we take this as an assumption [17]. In the next section, we study the influence of \( Q_{\mu\nu} \) term on the interior brane-world solutions \((r \leq R)\).

3 Exact solutions with uniform-density

Equations (28)-(30) and (32) determine the system of field equations on the brane. In what follows we will consider the cases where there are three independent field equations which imply energy-momentum conservation, namely, \( c = 0 \) and \( c = \beta = 0 \). One may then either use the three independent field equations or two of the field equations together with the energy-momentum conservation equation. For the purpose of this paper it is more convenient to follow the latter approach. Now, taking \( c = 0 \) and \( \lambda = 0 \), the field equations become
\[ e^{-\nu} \left( -\frac{1}{r^2} + \frac{\nu'}{r} \right) + \frac{1}{r^2} = \alpha^* \rho + 3\alpha^2 + \frac{4\alpha\beta}{r} + \frac{\beta^2}{r^2}, \]  
(36)

\[ e^{-\nu} \left( \frac{1}{r^2} + \frac{\mu'}{r} \right) - \frac{1}{r^2} = \alpha^* p - 3\alpha^2 - \frac{4\alpha\beta}{r} - \frac{\beta^2}{r^2}, \]  
(37)

\[ p' + \frac{\mu'}{2} (\rho + p) = 0. \]  
(38)

Equation (36) can easily be integrated to give
\[ e^{\nu(r)} = \left[ 1 - \frac{m(r)}{r} \right]^{-1}, \]  
(39)

where the mass function is
\[ m(r) = \alpha^* \int_s^r \left[ \rho(r') + \frac{3\alpha^2}{\alpha^*} + \frac{4\alpha\beta}{\alpha^* r'} + \frac{\beta^2}{\alpha^* r'^2} \right] r'^2 dr', \]  
(40)

and \( s = 0, s = R \) represent the interior and exterior solutions respectively. If an equation of state of the form \( \rho = \rho(p) \) is assumed for the interior, the conservation equation (38) can be integrated to give
\[ e^{\mu(r)} = \exp \left[ -2 \int_{p_c}^{p(r)} \frac{dp}{p + \rho(p)} \right], \]  
(41)

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where $p_c$ denotes the central pressure. The function $\mu'$ can be eliminated from equations (29) and (32), yielding the modified TOV equation

$$\frac{p'}{p + \rho} = \frac{1}{2} \left[ \frac{\alpha^* pr + \frac{m(r)}{r} - 3\alpha^2 r}{1 - \frac{m(r)}{r}} - 4\alpha\beta - \frac{\beta^2}{r} + 2 c r e^{-\mu/2} (\alpha r + \beta) \right].$$

(42)

In the general relativity limit of $\alpha, \beta, c \to 0$, we regain the usual TOV equation. Taking the simplest case of a static spherical star with uniform density given by $\rho(r) = \rho_0 = \frac{3M}{4\pi R^3}$, we can find the exact interior solutions, generalizing the interior solutions predicted in general relativity. Stars with such a uniform distribution are of interest, not only because they actually exist, but also because they are simple enough to allow an exact solution of the Einstein field equations and set an upper limit to the gravitational red shift of spectral lines from the surface of any star [24].

Integration of equation (40) immediately gives

$$e^{\nu(r)} = \frac{1}{1 - \frac{\alpha^* M}{4\pi r} (\frac{r}{R})^3 - (\alpha r + \beta)^2}, \quad r \leq R.$$  

(43)

Furthermore, integration of equation (38) results in

$$e^{\mu(r)} = \frac{B}{(\rho_0 + p(r))^2}, \quad r \leq R,$$

(44)

where $B$ is a constant of integration. Also the pressure is given by

$$p(r) = -\rho_0 b e^{\nu(r)} \left[ \alpha \beta b_1 r^3 + (6\alpha^2 \beta^2 b_3 - \frac{b_3}{b_2}) r^2 - 5\alpha \beta \left( \frac{18\alpha^2 b_4}{\alpha^* \rho_0} + b_2 b_3 \right) r + b_2 b_3 (1 - \beta^2) \right] + I(r)$$

$$I(r) \left[ 1 + b e^{\nu(r)} (\alpha \beta r + \beta^2 - 1) \right],$$

(45)

where

$$I(r) = b_1 r^2 + 6b_3 \alpha \beta r + 3b_3 (\beta^2 - 1),$$

(46)

and

$$b_1 = \alpha^* \rho_0 + 6\alpha^2 + \frac{9\alpha^4}{\alpha^* \rho_0},$$

$$b_2 = 1 - \beta^2 - \frac{6\alpha^2}{\alpha^* \rho_0},$$

$$b_3 = 1 + \frac{3\alpha^2}{\alpha^* \rho_0}.$$  

(47)

In the above equation, $b$ is the constant of integration and is evaluated by taking $p(R) = 0$. The vanishing of the pressure at the surface, which is certainly physically reasonable, is a consequence of the application of the Israel matching conditions at the stellar surface. In the limit $\alpha, \beta \to 0$, equation (45) reduces to the pressure for a uniform-density star in general relativity.

The exterior solutions of equations (28)-(30) are found to be [15]

$$e^{-\nu(r)} = 1 - \frac{A_1}{r} - \alpha^2 r^2 - 2\alpha \beta r - \beta^2,$$

(48)

and

$$e^{\mu(r)} = \frac{f(r)}{4r} \left( -A_2 + 2\alpha c \int \frac{r^{5/2} dr}{f(r)^{3/2}} + 2\beta c \int \frac{r^{3/2} dr}{f(r)^{3/2}} \right)^2,$$

(49)
where $A_1$ and $A_2$ are integration constants and

$$f(r) = -r + A_1 + \alpha^2 r^3 + 2\alpha \beta r^2 + \beta^2 r.$$  \hfill (50)

There are a number of arbitrary constants which do not let us find the unique vacuum solution of the gravitational field equations on the brane because the Birkhoff theorem does not apply here [17, 20].

Equation (49) cannot be solved in closed form. The choice $c = 0$ leads to the exact exterior solution

$$e^{\mu(r)} = e^{-\nu(r)} = 1 - \frac{A_1}{r} - \alpha^2 r^2 - 2\alpha \beta r - \beta^2, \quad r \geq R.$$ \hfill (51)

This solution can be used to explain the galactic rotation curves without relying on the existence of dark matter and without assuming any new modified theory, e.g. MOND. The matching of the interior solution to that of the exterior then determines the integration constant as $A_1 = 2GM$.

A second class of solutions of the system of equations (28)-(30) can be obtained by the choice $c = \beta = 0$ and $\alpha \neq 0$. Substituting $c = \beta = 0$ into equation (27), one obtains

$$Q_{00} = -3\alpha^2 g_{00}, \quad Q_{\mu\nu} = -3\alpha^2 g_{\mu\nu}, \quad \mu, \nu = 1, 2, 3.$$ \hfill (52)

As we noted before, $Q_{\mu\nu}$ is an independently conserved quantity, suggesting an analogy with the energy-momentum tensor of an uncoupled non-conventional energy source. Let us define $Q_{\mu\nu}$ as an isotropic perfect fluid and write

$$Q_{\mu\nu} \equiv \frac{1}{\alpha^*} \left[ (\rho_{\text{extr}} + p_{\text{extr}})u_{\mu} u_{\nu} + p_{\text{extr}} g_{\mu\nu} \right],$$ \hfill (53)

where we have denoted the “geometric pressure” associated with the extrinsic curvature by $p_{\text{extr}}$ and the “geometric energy density” by $\rho_{\text{extr}}$. The geometric fluid can be implemented by the equation of state

$$p_{\text{extr}} = (\gamma_{\text{extr}} - 1) \rho_{\text{extr}},$$ \hfill (54)

where $\gamma_{\text{extr}}$ may be a function of the radius. Comparing $Q_{\mu\nu}$ and $Q_{00}$ from equation (53) with the components of $Q_{\mu\nu}$ given by equation (52), we obtain

$$p_{\text{extr}} = \frac{-3\alpha^2}{\alpha^*}, \quad \rho_{\text{extr}} = \frac{3\alpha^2}{\alpha^*}.$$ \hfill (55)

Equation (54) then gives $\gamma_{\text{extr}} = 0$, showing that the geometrical matter may play the role of a positive cosmological constant ($3\alpha^2 = \Lambda \approx 3 \times 10^{-56} \text{cm}^2$) on the interior space-time [51]. In this case, substituting $\beta = 0$ into equations (43) and (44), the interior line element takes the form

$$ds^2 = -\frac{B}{(\rho_0 + p(r))^2} dt^2 + \frac{dr^2}{1 - (\frac{\alpha^2 \rho_0}{3} + \alpha^2)r^2} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2).$$ \hfill (56)

Since $\frac{\alpha^2 \rho_0}{3} + \alpha^2 > 0$ the spatial geometry of this metric describes part of a 3-sphere of radius $\frac{1}{\sqrt{\alpha^2 \rho_0 / 3 + \alpha^2}}$ and has a coordinate singularity at $r = \hat{r} = \frac{1}{\sqrt{\alpha^2 \rho_0 / 3 + \alpha^2}}$. The metric is well defined for radii less than $\hat{r}$. The pressure is given by

$$p(r) = \rho_0 \left( 1 - \frac{6\alpha^2}{\alpha^2 \rho_0} \right) - \rho_0 \sqrt{1 - \left( \frac{\alpha^2 \rho_0}{3} + \alpha^2 \right)r^2}.$$

$$-3 + \rho_0 \sqrt{1 - \left( \frac{\alpha^2 \rho_0}{3} + \alpha^2 \right)r^2},$$ \hfill (57)

where the integration constant $\rho_0$, evaluated by defining $p(r = 0) = p_c$ to be the central pressure, is given by

$$p_0 = \frac{3p_c + \rho_0 - 6\alpha^2}{p_c + \rho_0}.$$ \hfill (58)
Figure 1: Pressure as a function of radius in general relativity (bottom curve) and in the brane-world model with \((1 - \frac{6\alpha^2}{\alpha^*\rho_0}) > 0\) (top curve).

For coordinate singularity \(\hat{r}\) the pressure is

\[
p(\hat{r}) = p_s \equiv -\frac{\rho_0}{3} \left( 1 - \frac{6\alpha^2}{\alpha^*\rho_0} \right),
\]

which is negative if \((1 - \frac{6\alpha^2}{\alpha^*\rho_0}) > 0\). Since \(p(\hat{r}) < 0\) there exists an \(R\) such that \(p(R) = 0\), consistent with equation (35). Taking \(p(R) = 0\), \(R\) can be derived from equation (57) as

\[
\sqrt{1 - \left( \frac{\alpha^*\rho_0}{3} + \alpha^2 \right) R^2} = \frac{1}{p_0} \left( 1 - \frac{6\alpha^2}{\alpha^*\rho_0} \right).
\]

Now, using equations (58) and (60) the central pressure is

\[
p_c = \frac{\rho_0}{3} \frac{\left( 1 - \frac{6\alpha^2}{\alpha^*\rho_0} \right) \left[ 1 - \sqrt{1 - \left( \frac{\alpha^*\rho_0}{3} + \alpha^2 \right) R^2} \right]}{\left( 1 - \frac{6\alpha^2}{\alpha^*\rho_0} \right)},
\]

and

\[
p(r) = \rho_0 \left( 1 - \frac{6\alpha^2}{\alpha^*\rho_0} \right) - \left( 1 - \frac{6\alpha^2}{\alpha^*\rho_0} \right) \left( \frac{\epsilon^{\nu}(R)}{\epsilon^{\nu}(r)} \right)^{\frac{1}{2}} - 3 + \left( 1 - \frac{6\alpha^2}{\alpha^*\rho_0} \right) \left( \frac{\epsilon^{\nu}(R)}{\epsilon^{\nu}(r)} \right)^{\frac{1}{2}}.
\]

We can also obtain an upper limit on compactness from the requirement that \(p(r)\) must be finite. Figure 1 shows the behavior of pressure as a function of \(r\). As can be seen the pressure is a decreasing function of \(r\), similar to what one obtains in general relativity. This is equivalent to the conditions that \(p_c\) is finite and positive which gives the following condition for the mass

\[
MG < \frac{4}{9} R - \frac{2}{9} R \left( 1 - \sqrt{1 - \frac{9}{4\alpha^2 R^2}} \right).
\]

For a given \(R\) there is an upper bound to the mass of a static star where the central pressure becomes infinite as \(M \to M_{max}\). For the choice \(3\alpha^2 = \Lambda\), the correction to the general relativity limit of \(\frac{4}{9}\) is small in this scenario. The square root term is real if

\[
R \leq \frac{2}{3\alpha}.
\]
Also, definition of $p_s$ above implies $\alpha^2 < \frac{\alpha^2 \rho_0}{6}$. Definition of mass can then be used to re-write this to give

$$3\alpha^2 R^2 < \frac{\alpha^2 \rho_0 R^2}{2} < \frac{2}{3} + \sqrt{\frac{4}{9} - \alpha^2 R^2},$$

which reduces to

$$R < \frac{1}{\sqrt{3\alpha}},$$

indicating that the boundary of the stellar object is located before the event horizon is reached. Also, if we take $\alpha = 0$ we recover the standard general relativity result, namely

$$\frac{MG}{R} < \frac{4}{9}.$$ (67)

For the case $c = \beta = 0$ and $\alpha \neq 0$ the exterior solution is given by

$$e^{\mu(r)} = e^{-\nu(r)} = 1 - \frac{A_1}{r} - \alpha^2 r^2, \quad r \geq R,$$ (68)

where $A_1$ is an integration constant. The matching condition implies $A_1 = 2GM$ and therefore

$$B = \rho_0^2 \left( 1 - \frac{2GM}{R} - \alpha^2 R^2 \right).$$ (69)

The corresponding line element now takes the form

$$ds^2 = - \left( 1 - \frac{2GM}{r} - \alpha^2 r^2 \right) dt^2 + \frac{dr^2}{\left( 1 - \frac{2GM}{r} - \alpha^2 r^2 \right)} + r^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right).$$ (70)

Comparing the above result with the line element for the black hole solution in an asymptotically de Sitter space, the cosmological constant is found to be $3\alpha^2 = \Lambda$. This positive value is in agreement with present observations. Note that for $\sqrt{27GM} < \frac{1}{\alpha}$ there are two horizons $2GM < r_1 < 6GM$ and $\frac{1}{\sqrt{3\alpha}} < r_2 < \frac{1}{\alpha}$, while for $\sqrt{27GM} = \frac{1}{\alpha}$, $r_1$ and $r_2$ coincide and there is only one horizon $r = \frac{1}{\sqrt{3\alpha}}$. Note also that there is no horizon for $\sqrt{27GM} > \frac{1}{\alpha}$.

In the special case where $1 - \frac{6\alpha^2}{\alpha^2 \rho_0} = 0$, the line element takes the form

$$ds^2 = - \frac{B}{(\rho_0 + p(r))^2} dt^2 + \frac{dr^2}{(1 - 3\alpha^2 r^2)} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2),$$ (71)

and the pressure is given by

$$p(r) = \rho_0 \frac{p_0 \sqrt{1 - 3\alpha^2 r^2}}{3 - p_0 \sqrt{1 - 3\alpha^2 r^2}}.$$ (72)

We note that the pressure vanishes at the coordinate singularity which means that the stellar surface coincides with the coordinate singularity. Thus the interior solution cannot be joined to the schwarzschild-de Sitter exterior solution.

4 Gravitational collapse on the brane

In this section, we discuss the gravitational collapse for spherically symmetric objects in our model. For a sphere undergoing Oppenheimer-Snyder collapse, the collapsing region can be conveniently expressed by a Robertson-Walker metric

$$ds^2 = -d\tau^2 + a(\tau)^2 \left( 1 + \frac{k\chi^2}{4} \right)^{-2} \left( d\chi^2 + \chi^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) \right).$$ (73)
The modified Friedman equation from equation (13) is \[ \dot{a}^2 = \frac{\alpha^2 \rho_0}{3} a^{-3} + b_0^2 a^{-3 \gamma_{extr}} + \frac{\lambda}{3} - \frac{k}{a^2}. \] (74)

Also, the Raychaudhuri equation can be written as \[ \frac{\ddot{a}}{a} = -\frac{\alpha^2 \rho_0}{6} a^{-3} + b_0^2 a^{-3 \gamma_{extr}} \left( 1 - \frac{3}{2} \gamma_{extr} \right) + \frac{\lambda}{6}, \] (75)

where $\gamma_{extr}$ is defined by equation (54) and $b_0$ is an integration constant. The modified Friedman equation can also be written in terms of the proper radius, $r(\tau) = \chi a(\tau)/(1 + k \chi^2)$, of the collapsing boundary surface at $\chi = \chi_0$, that is

\[ \dot{r}^2 = \frac{\alpha^2 \rho_0 \chi_0^3}{3} \left( 1 + \frac{k \chi_0^2}{4} \right)^{-3} \frac{1}{r} + b_0^2 \chi_0^{3 \gamma_{extr}} \left( 1 + \frac{k \chi_0^2}{4} \right)^{-3 \gamma_{extr}} r^{-3 \gamma_{extr} + 2} + \frac{\lambda}{3} r^2 + E, \] (76)

where

\[ E = -\frac{k \chi_0^2}{(1 + \frac{k \chi_0^2}{4})^2}, \] (77)

is the energy per unit physical mass. Let us express the static spherically symmetric metric for the vacuum exterior by

\[ ds^2 = -F(r)^2 e^{\mu(r)} dt^2 + e^{-\mu(r)} dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right). \] (78)

In order for a metric of the form (78) to become the exterior to the region described by metric (73), the metric and extrinsic curvature have to be continuous across the collapsing boundary surface. The two conditions are simultaneously satisfied by expressing the metrics (73) and (78) in terms of null coordinates by adopting the method of [26], which results in $F(r) = 1$ and

\[ e^{\mu(r)} = 1 - \frac{\alpha^2 \rho_0 \chi_0^3}{3} \left( 1 + \frac{k \chi_0^2}{4} \right)^{-3} \frac{1}{r} - b_0^2 \chi_0^{3 \gamma_{extr}} \left( 1 + \frac{k \chi_0^2}{4} \right)^{-3 \gamma_{extr}} r^{-3 \gamma_{extr} + 2}. \] (79)

The choice $\gamma_{extr} = 0$, $b_0^2 = \alpha^2$ and $\frac{\alpha^2 \rho_0 \chi_0^3}{3} \left( 1 + \frac{k \chi_0^2}{4} \right)^{-3} = A_1$, then leads to

\[ e^{\mu(r)} = 1 - \frac{A_1}{r} - \alpha^2 r^2. \] (80)

Equations (78) and (80) imply the brane Ricci scalar is

\[ R^\mu_\mu = 12 \alpha^2. \] (81)

Equation (13) for a vacuum exterior implies

\[ R^\mu_\mu = -Q^\mu_\mu = 12 \alpha^2 + \frac{12 \alpha \beta}{r} + \frac{2 \beta^2}{r^2} - 2 ce^{-\mu(r)}/ \left( 3 \alpha + \frac{2 \beta}{r} \right). \] (82)

Comparing equations (81) and (82) we see that a static exterior is possible only if $c = \beta = 0$. It is now obvious that the static exterior solution (79), surrounding the collapsing region, can take the form of equation (68). Considering the universe as a 3-dimensional brane embedded in a 5-dimensional bulk of constant curvature, without $Z_2$ symmetry or any form of junction conditions, the vacuum exterior of a spherical cloud can be static [27,28], which is similar to the standard general relativity and is different from the brane-world models where a delta-function is used [26] to confine matter on the brane.
5 Conclusions

In this paper, we have studied spherically symmetric solutions in a brane-world model without mirror symmetry or any form of junction conditions. We have shown that within the context of the model presented here, the matching conditions lead to a vanishing pressure at the surface of the star. This result is different from those obtained in Randall-Sundrum type brane-world models where the vanishing of pressure at the surface of the star is simply assumed. We have obtained exact uniform-density stellar solutions localized on a 3-brane in two cases by considering a constant curvature bulk. The first solution is consistent with an exterior solution whose structure can be used to explain the galaxy rotation curves and the second solution represents a stellar model with the exterior Schwarzschild-de Sitter space-time. We have also obtained the upper bound to the mass of a static brane-world star, for the case \( c = \beta = 0 \), and shown that the influence of the bulk on the interior solutions is small. Finally, we have studied the fate of a collapsing star on the brane and shown that the exterior of a collapsing star can be static in our model.

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