Next-to-Leading Order QCD Corrections to Jet Cross Sections and Jet Rates in Deeply Inelastic Electron Proton Scattering

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Abstract

Jet cross sections in deeply inelastic scattering in the case of transverse photon exchange for the production of (1+1) and (2+1) jets are calculated in next-to-leading order QCD (here the ‘+1’ stands for the target remnant jet, which is included in the jet definition for reasons that will become clear in the main text). The jet definition scheme is based on a modified JADE cluster algorithm. The calculation of the (2+1) jet cross section is described in detail. Results for the virtual corrections as well as for the real initial- and final state corrections are given explicitly. Numerical results are stated for jet cross sections as well as for the ratio $\frac{\sigma_{(2+1)\text{ jet}}}{\sigma_{\text{tot}}}$ that can be expected at E665 and HERA. Furthermore the scale ambiguity of the calculated jet cross sections is studied and different parton density parametrizations are compared.

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1 Introduction

Recent results from E665 and HERA show that events with a clear jet-like structure are present in deeply inelastic electron proton scattering \[1, 2\]. With sufficient luminosity it should therefore be possible to study jet cross sections to use them for a test of QCD and an independent determination of its fundamental parameter \(\Lambda_{\text{QCD}}\). Jet cross sections may even be useful to extract some information on the gluon density at small \(x\), because the gluon density is important in \((2+1)\) jet production.

Because of the strong scale dependence of fixed order cross sections calculated in perturbative QCD, the possibility of a large size of the corrections and because a determination of \(\alpha_s\) must be based on a next-to-leading (NLO) order cross section, the calculation of higher order corrections is well motivated. Since now experimental results are available, it is worthwhile to give a detailed account of the technical problems of the calculation of the \((2+1)\) jet cross section. A second goal of the present work is to study the jet cross sections in detail numerically. In deeply inelastic scattering, the \(\mathcal{O}(\alpha_s)\) corrections to the \(\mathcal{O}(\alpha_s^0)\) Born term are well known (see \[3, 4, 5, 6, 7, 8, 9\]). In addition, the \(\mathcal{O}(\alpha_s^2)\) Born terms for the production of \((3+1)\) jets have been calculated \[10, 11\].

In this paper the calculation of the cross section for the production of \((2+1)\) jets to \(\mathcal{O}(\alpha_s^2)\) in the case of transverse photon exchange is described. For more details of the calculation see \[12, 13\]. The contributions with a transverse exchanged photon dominate the cross section, as is shown in Section 8. More recently, the other parity-conserving helicity cross sections have been calculated based on \[12\], see \[14\].

One of the main features of hadronic events in high-energy collisions is the pronounced jet-like structure. These jets are attributed to the production of partons in the fundamental QCD process \[15\]. Due to the presence of infrared divergences in QCD, a suitable prescription has to be given in order to define the finite parts that arise from divergent terms after the singularities between real and virtual corrections have been cancelled. Such a prescription is related to the jet definition that is used on the parton level to calculate jet cross sections. In Section 2 jet definitions in deeply inelastic scattering are discussed and the jet definition scheme based on a modification of the JADE cluster algorithm that is used in this calculation is defined.

In Section 3 the calculation of the \((1+1)\) jet cross section in next-to-leading order is reviewed and the results for the \((2+1)\) jet Born terms are stated. In Section 4 the results for the virtual corrections to \((2+1)\) jet production are given. In Sections 5 and 6 the calculation of the real corrections to \((2+1)\) jet production is described (separated according to singularities in the final and initial state). The sum of virtual and real corrections gives the finite jet cross section after renormalisation of the parton densities. The flavour factors are listed in Section 7. In Section 8 the numerical results for jet cross sections and for the ratio \(\sigma_{(2+1)\text{ jet}}/\sigma_{\text{tot}}\) are presented. The dependence of the jet cross sections on the renormalisation and factorization scales, the dependence on the jet definition scheme and the dependence on the parametrization of the parton densities is also studied. The appendix contains explicit results for massless 1-loop tensor structure integrals, phase space integrals and the virtual and real corrections.

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1 The target remnant is counted as a jet, so “2+1” stands for the production of 2 partons in the hard QCD process, possibly accompanied by additional soft or collinear partons.

2 Here, the cross section for transverse photon exchange is defined by the helicity cross section obtained by a contraction of the hadron tensor with the metric tensor \((-g_{\mu\nu})\).
2 Jet Cross Sections

Jets must be defined in terms of experimentally observable quantities. An experimental event is characterised by the energies and momenta of the outgoing particles. Since at high energies the multiplicity of the events is large and since presently there is no practical way (based on QCD) to describe hadron dynamics on the level of observable particles, one has to try to extract information from experimental data in a form that can be compared with theoretical results from perturbative QCD. At $e^+e^-$-colliders it has been observed that outgoing hadrons very often appear as clusters of particles concentrated in a small cone in momentum space. These clusters were called jets. Given an experimental event and a resolution parameter $c$ (the “jet cut”), a suitable algorithm is applied to the event giving the number of jets of the event and the particles associated with each of the jets. Therefore, the algorithm that is used defines what is meant by a “jet”. To compare experiment and theory, one must use the same algorithm in theoretical calculations. One should expect that experiment and theory are comparable as long as the same algorithm is used in the experimental analysis and in the theoretical calculation. Of course, the problem is that in realistic events the final state consists of hadrons, whereas in theoretical calculations (based on QCD) the outgoing “particles” are partons. Therefore the crucial hypothesis is that jets on the hadron level and jets on the parton level can be identified.

The first jet definition that has been given is that of Sterman and Weinberg [15]. It is based on cones in momentum space defined by an opening angle $\delta$ and an energy fraction $\epsilon$. This definition, which is well suited for $e^+e^-$-annihilation for a small number of jets becomes complicated if a larger number of jets is produced. In addition, it is not Lorentz invariant (this is not a problem in the case of $e^+e^-$-annihilation, since here the CM system is a unique frame of reference). Later, another type of algorithm was proposed, the cluster algorithm first used by the JADE collaboration [16]. This algorithm combines successively two particles into a jet, if their invariant mass squared $s_{ij}$ is smaller than a fraction $c$ of a typical mass scale:

$$s_{ij} \leq cM^2.$$  \hspace{1cm} (1)

In $e^+e^-$-annihilation, $c$ is of the order of $10^{-2}$, and $M^2$ is set to $Q^2$, the total invariant mass of the hadronic final state.

For hadron colliders, jet definitions in terms of “cones” in rapidity and azimuthal angle are favoured (UA1-type algorithms). Such a jet definition singles out a particular axis (namely, the direction of the two colliding beams). In the case of pp-events the incoming partons are assumed to have only small transverse momentum, and therefore the situation is symmetric with respect to this particular axis.

The situation is quite different for electron-proton colliders. Here the interaction is mediated by the exchange of a virtual photon with momentum $q$ (and $Q^2 := -q^2 > 0$). This photon hits a proton with momentum $P$. Therefore, the interaction should be described in the CM system with $\vec{P} + \vec{q} = \vec{0}$. This system varies from event to event, and this is the reason why a Lorentz invariant jet definition should be used [11].

Before a suggestion for a suitable jet algorithm in eP-scattering is given, one should have a closer look at the target remnant. An interesting question is: “Is it reasonable to include the target jet in a jet analysis?” One should consider the process in fig. [1]. This Feynman graph
describes initial state radiation of a gluon with momentum $p_2$ from the initial quark line with momentum $p_0$. It is assumed that the gluon is emitted collinearly with a large energy in the direction of the incoming quark causing a strong enhancement of the cross section for this process because of the pole of the quark propagator. Since all partons are assumed to be massless the gluon will go in the same direction as the target remnant. Under the assumption that “parton jets” roughly correspond to “hadron jets”, there will be no possibility to disentangle the hadrons from the debris of the proton and those coming from the fragmentation of the gluon. To be consistent, one therefore should define “parton jets” in the following way: If an outgoing gluon can be separated from the remnant by a suitable condition (e.g. invariant mass), the gluon and the remnant are considered to be two jets with momenta $p_2$ and $p_r$. If the gluon and the remnant cannot be separated, they count as one jet whose momentum $p_\ast = p_2 + p_r$ is the sum of the gluon momentum and the momentum of the remnant. In an experiment, however, one cannot measure the momentum of the remnant directly since most of the hadrons from this jet are lost in the beam pipe. The momentum of the target remnant jet must therefore be determined in an indirect way. The jet algorithm should also have the property that all collinear singularities are treated in such a way that they factorize the corresponding Born term. This allows for a process independent definition of the renormalised scale dependent parton densities.

For the experimental analysis it is proposed to use a modified JADE cluster algorithm (mJADE algorithm) consisting of two steps (see also [11]):

1. Define a precluster of longitudinal momentum (in the direction of the beam pipe) $p_r$ that is given by the missing longitudinal momentum of the event.
2. Apply the JADE cluster algorithm to the set of momenta

$$\{p_1, p_2, \ldots, p_n, p_r\},$$

where $p_1, p_2, \ldots, p_n$ are the momenta of the visible hadrons in the detector and $p_r$ is the momentum of the precluster.

It remains to define the order of magnitude of the jet cut and the mass scale $M^2$ to be used in the mJADE algorithm. The jet cut should be such that it is small enough to ensure properly separated jets in the detector (otherwise the jets coming from the real corrections could become too broad), but large enough to avoid large logarithms that could spoil a fixed order result in perturbation theory. One can think of $cM^2$ as a new mass scale that is needed to specify the cross section completely. In a deeply inelastic process there are several mass scales given by an event, namely the virtuality $Q^2$ of the exchanged vector boson, the invariant mass $W^2$ of the hadronic final state and, in general, several invariant masses $p_{i\perp}^2$ from transverse momenta of outgoing particles. Therefore large logarithms are expected if the quotient of any two of these scales becomes small (or large). So one should avoid kinematical regions where this could happen. Because of the jet cut an additional scale $cM^2$ enters the calculation, where one could, for example, use $Q^2$, $W^2$ or some $p_{i\perp}^2$ as $M^2$. Since it is suggested to include the target remnant in the jet analysis, it is natural to use the scale $M^2 = W^2$ in the jet algorithm (analogous to the situation in $e^+e^-$-annihilation). If the scales $p_{i\perp}^2$ are omitted from the discussion, the scales $Q^2 = S_{\mu\nu}x_{\mu\nu}$, $W^2 = S_{\mu\nu}(1 - x_{\mu\nu})$ and $cW^2 = cS_{\mu\nu}(1 - x_{\mu\nu})$ are relevant. Here $\sqrt{S_{\mu\nu}}$ is the total CM energy of the collider, $x_{\mu\nu} = Q^2/2Pq$ is the Bjorken variable and $y$ is the usual lepton variable $y = Pq/Pk$, with $P$ the proton momentum and $k$ the momentum of the incoming electron.

This is not the only reasonable choice for the mass scale used in the jet definition. In fact, in Section 3 it is pointed out that a scale like $Q^2$ or $\left(W^\alpha Q^2 \sqrt{S_{\mu\nu}y}^{1-\alpha-\beta}\right)^2$ with parameters
\(\alpha, \beta\) may be reasonable if the parton densities have to be probed at small \(x\). In contrast, the considerations here are based on the simple observation that the observable invariants (including those with the remnant jet) sum up to \(W^2\), and it is somehow natural to use this particular scale. For another possible jet definition in eP scattering see [17].

In the cross section large logarithms of \(Q^2/W^2 \approx x_B, cW^2/W^2 = c\) and \(cW^2/Q^2 \approx c/x_B\) (the approximation of small \(x_B\) is made since most events are expected in this region) are expected. For cuts of the order of 0.01 the logarithms in \(c\) should be comparable to those encountered in \(e^+e^-\)-annihilation because of a similar structure of the matrix elements.

Berger and Nisius have studied the effect of the inclusion of the target remnant in the jet analysis [18] by using the Monte Carlo generator LEPTO5.2. They come to the conclusion that the correlation of the number of jets on the parton level and the number of jets after fragmentation is much stronger if the remnant jet is included. For more details, see [11].

3 (1+1) and (2+1) Jets: Cross Sections to \(\mathcal{O}(\alpha_s)\)

In this section the calculation of the (1+1) jet cross section in NLO is reviewed. This is done for two reasons: these results are needed to calculate jet rates, and their calculation serves as an illustration of the more complicated case of the NLO corrections to (2+1) jet production. As a byproduct, one gets the results for the (2+1) jet Born terms. The total cross section to \(\mathcal{O}(\alpha_s)\) for photon exchange has been calculated in [4], and the (1+1) jet cross section to this order for all neutral and charged current processes can be found in [3]. Here the special case of the (1+1) jet cross section for the exchange of a transverse photon is discussed in detail and results for a longitudinally polarized photon are stated. It is safe to focus on the QCD corrections for the transverse polarization of the virtual photon, since the cross section for longitudinally polarized photons contributes only about 20\% to the Born term cross section (see Section 8), and this is expected to be true for the relative contribution of the longitudinal terms to the NLO corrections as well.

The cross section for eP-scattering differential in \(x_B\) and \(y\) is given by

\[
\frac{d\sigma_{\text{H}}}{d x_B d y} = \sum_i \int_{x_B}^1 \frac{d \xi}{\xi} f_i(\xi) \frac{(4\pi)^\epsilon (S_H x_B)^{-\epsilon} (y(1-y))^{-\epsilon} \mu^{4\epsilon}}{\Gamma(1-\epsilon)} \alpha^2 \frac{1}{2} S_H x_B \cdot d\text{PS}_{\text{parton}} \frac{1}{e^2 \mu^{2\epsilon} (2\pi)^{2d} Q^2 N'} \int d\Omega' l'^{\mu\nu} H_{\mu\nu},
\]

where \(d = (4 - 2\epsilon)\) is the space-time dimension \((\epsilon \neq 0\) regularises the ultraviolet and infrared divergences [19, 20, 21]), \(\mu\) is a mass scale for making the coupling constants dimensionless in \(d\) dimensions, \(d\text{PS}_{\text{parton}}\) is the n-parton phase space

\[
d\text{PS}_{\text{parton}}^{(n)} = (2\pi)^d \prod_{i=1}^n \frac{d^d p_i \delta(p_i^2)}{(2\pi)^{d-1}} \delta(p_0 + q - \sum_{i=1}^n p_i),
\]

\(\Omega'\) is the volume element of \(d-3\) angles specifying the direction of the outgoing lepton relative to the outgoing partons, and

\[
N' = \int d\Omega'
\]
is a normalisation constant. The incoming parton carries a fraction \( \xi \) of the proton momentum, the \( f_i(\xi) \) are the bare parton densities for partons of flavour \( i \),

\[
l^{\mu\nu} = k^{\mu}k^{\nu} + k^{\nu}k^{\mu} - kk'g^{\mu\nu}
\]

is the lepton tensor and \( H_{\mu\nu} \) is the hadron tensor (including coupling constants, colour factors, etc.).

The integration over \( \Omega' \) can be performed. It can be shown by a direct evaluation of the integrals in \( d \)-dimensional space that

\[
\frac{1}{Q^2 N'} \int dQ' l^{\mu\nu} = \frac{1 + (1 - y)^2}{2(1 - \epsilon)y} - \epsilon y^2 (g^{\mu\nu} - \epsilon^{\mu\nu})
+ \frac{4(1 - \epsilon)(1 - y) + 1 + (1 - y)^2 - \epsilon y^2}{2(1 - \epsilon)y^2} (\epsilon^{\mu\nu}_R + \epsilon^{\mu\nu}_0),
\]

where

\[
\begin{align*}
\epsilon^{\mu\nu}_q &= \frac{1}{Q^2} q^{\mu}q^{\nu}, \\
\epsilon^{\mu\nu}_R &= \frac{1}{Q^2} (q^{\mu}q^{\nu} + 2x_p(p_0^{\mu}q^{\nu} + q^{\mu}p_0^{\nu})), \\
\epsilon^{\mu\nu}_0 &= \frac{4x_p^2}{Q^2} p_0^{\mu}p_0^{\nu},
\end{align*}
\]

and \( x_p := x_B/\xi \). Because of current conservation \( q^{\mu}H_{\mu\nu} = 0 \) one has

\[
\epsilon^{\mu\nu}_q H_{\mu\nu} = \epsilon^{\mu\nu}_R H_{\mu\nu} = 0.
\]

With the definition \( \text{tr}H := h_g := (-g^{\mu\nu}) H_{\mu\nu}, h_0 := \epsilon^{\mu\nu}_0 H_{\mu\nu} \) one obtains

\[
\frac{1}{Q^2 N'} \int dQ' l^{\mu\nu} H_{\mu\nu} = \frac{1 + (1 - y)^2}{2(1 - \epsilon)y} h_g + \frac{4(1 - \epsilon)(1 - y) + 1 + (1 - y)^2 - \epsilon y^2}{2(1 - \epsilon)y^2} h_0.
\]

By defining

\[
\sigma_\lambda = \sum_i \int_{x_B}^1 \frac{d\xi}{\xi} f_i(\xi) (4\pi)^e (S_\lambda x_B)^{-\epsilon} (y(1 - y))^{-\epsilon} \mu^{4\epsilon} \frac{1}{\Gamma(1 - \epsilon)} \frac{1}{2 S_\mu x_B} \cdot d\text{PS}^{(n)}_{\text{parton}} \frac{1}{\epsilon^2 \mu^2 (2\pi)^{2d}} h_\lambda
\]

with \( \lambda \in \{ g, 0 \} \) one arrives at

\[
\frac{d\sigma_\mu}{dx_B dy} = \frac{1 + (1 - y)^2 - \epsilon y^2}{2(1 - \epsilon)y^2} \sigma_g
+ \frac{4(1 - \epsilon)(1 - y) + 1 + (1 - y)^2 - \epsilon y^2}{2(1 - \epsilon)y^2} \sigma_0.
\]
In the literature cross sections \( \sigma_U, \sigma_L \) for unpolarized and longitudinally polarized photons have been defined. They are related to the definitions used in this paper by \( \sigma_g = 2(1-\epsilon)\sigma_U - \sigma_L \), \( \sigma_0 = \sigma_L \).

Now the special case of \((1+1)\) jet production is considered. Fig. 3 depicts the Feynman diagram to \( O(\alpha_s^0) \). The diagram for the 1-loop virtual correction is shown in fig. 3. The result for the sum of these diagrams is well known [4]:

\[
\frac{d\sigma_{\text{Born&virt.}}}{dx_Bdy} = \sum_i \int_{x_B}^{1} \frac{dx_p}{x_p} \frac{f_i^{\text{ren}}}{x_p} \left( \frac{x_B}{x_p} M_f^2 \right) \frac{(4\pi)^\epsilon (S_i x_B)^{-\epsilon} (y(1-y))^{-\epsilon} x^{\mu 4\epsilon} \alpha_s^2}{2 S_{\text{ren}} x_B} \cdot \frac{1 + (1-y)^2 - \epsilon y^2}{2(1-\epsilon)y^2} 2\pi 4(1-\epsilon) Q_i^2 
\cdot \left\{ \delta_{qi} \left[ 1 + \left( \frac{4\pi\mu^2}{Q_i^2} \right)^\epsilon \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \frac{\alpha_s}{2\pi} C_F \left( -\frac{2}{\epsilon^2} - \frac{3}{\epsilon} \right) + \frac{\alpha_s}{2\pi} C_F \left( -8 - 2\zeta(2) \right) \right] \delta(1-x_p) 
+ \left( \frac{4\pi\mu^2}{M_f^2} \right)^\epsilon \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \frac{\alpha_s}{2\pi} \epsilon P_{q\rightarrow i}(x_p) \right\} + O(\epsilon). \tag{13}
\]

The integration over \( \xi \) is rewritten in terms of the variable \( x_p = x_B/\xi \), and the symbol \( \delta_{qi} \) restricts the summation to quark initiated terms. \( \zeta(2) = \pi^2/6 \), and \( Q_i \) is the charge of the quark with flavour \( i \) normalised to \( e \). \( \mu \) is the renormalisation scale and \( M_f \) is the factorization scale. The pole terms in \( \epsilon \) proportional to \( \delta(1-x_p) \) are infrared singularities that will cancel against infrared singularities in the real corrections, and the term proportional to the Altarelli-Parisi splitting function \( P_{q\rightarrow i}(x_p) \) will cancel against collinear singularities in the real corrections. Note that the bare parton densities \( f_i(\xi) \) are already expressed in terms of the renormalised parton densities [4, 22] in the \( \overline{\text{MS}} \)-scheme

\[
f_i^{\text{ren}}(\xi, M_f^2) = \int_{\xi}^{1} \frac{du}{u} \left[ \delta_{ij} \delta(1-u) + \frac{\alpha_s}{2\pi} \left( -\frac{1}{\epsilon} \right) P_{i\rightarrow j}(u) \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left( \frac{4\pi\mu^2}{M_f^2} \right)^\epsilon \right] f_j(\frac{\xi}{u}). \tag{14}\]

The Altarelli-Parisi kernels are

\[
\begin{align*}
P_{q\rightarrow q}(u) &= C_F \left[ 1 + \frac{u^2}{(1-u)_+} + \frac{3}{2} \delta(1-u) \right], \\
P_{g\rightarrow q}(u) &= C_F \left[ 1 + \frac{u}{(1-u)_+} \right], \\
P_{g\rightarrow g}(u) &= 2NC_F \left[ \frac{1}{(1-u)_+} + \frac{1}{u} + u(1-u) - 2 \right] + \left( \frac{11}{6} NC - \frac{1}{3} N_f \right) \delta(1-u), \\
P_{q\rightarrow g}(u) &= \frac{1}{2} \left[ u^2 + (1-u)^2 \right]. \tag{15}\end{align*}
\]

\( N_f \) is the number of quark flavours.

Now the real NLO corrections to \((1+1)\) jet production will be calculated. The Born terms of \( O(\alpha_s) \) have to be integrated over the phase space region that, by the jet definition, is considered to be a \((1+1)\) jet region. In a first step, suitable variables are defined, then the \((1+1)\) jet region
is determined and finally the integration is performed. The phase space for the production of 2 partons is constructed in the following way. As usual, the variable

\[ z = \frac{Pp_1}{Pq}, \]  

is defined, where \( p_1 \) is one of the outgoing partons. By defining \( t = s_{12}/W^2 \) (\( p_2 \) is the momentum of the second outgoing parton) and \( a = x_B + (1 - x_B)t \) one obtains

\[
\int d\text{PS}^{(2)} = \int d\text{PS}^*(2) \xi \delta(\xi - a),
\]

\[
\int d\text{PS}^*(2) = \int \frac{(4\pi)^{\epsilon}}{\Gamma(1 - \epsilon)} \left( W^2 \right)^{-\epsilon} t^{-\epsilon} (z(1 - z))^{-\epsilon} \frac{1 - x_B}{a} dz dt,
\]

(17)

where the fact that \( \xi = a \) if \( p_0 = \xi P \) is being used. By means of the factor \( \xi \delta(\xi - a) \) the integral \( \int d\xi/\xi f(\xi) \) in the cross section formula (3) can be performed trivially. The ranges of integration are \( z \in [0, 1] \) and \( t \in [0, 1] \). The invariants \( s_{ij} = 2p_ip_j \) expressed in terms of \( z \) and \( t \) are

\[
s_{01} = S_Hy(x_B + (1 - x_B)t)z, \\
s_{02} = S_Hy(x_B + (1 - x_B)t)(1 - z), \\
s_{12} = S_Hy(1 - x_B)t.
\]

(18)

The momentum of the target remnant is \( p_r = (1 - \xi)P \). The observable momenta are \( p_1, p_2 \) and \( p_r \). Invariants for these momenta are defined by \( u_{ij} = 2p_ip_j/W^2 \). In terms of the variables \( z \) and \( t \) they read

\[
u_{r1} = (1 - t)z, \\
u_{r2} = (1 - t)(1 - z), \\
u_{12} = t.
\]

(19)

Let us define the (2+1) jet region by the condition \( s_{ij} > cW^2 \), with \( i,j \in \{1, 2, r\} \). It is easy to see that in the \((t, z)\)-plane where the allowed phase space is the square \([0, 1] \times [0, 1]\) the (2+1) jet region is a triangle given by the following conditions:

(a) \( c < t < 1 - 2c \),

(b) \( z_c(t) < z < 1 - z_c(t) \), where \( z_c(t) = \frac{c}{1 - t} \).

The region within the square surrounding this triangle is therefore the (1+1) jet region. All regions where the cross section becomes singular are within the (1+1) jet region thus allowing the factorization of the collinear singularities and their absorption into the renormalised parton densities. It should be noted that for (2+1) jet production the minimal momentum fraction \( \xi \) of the parton densities that can be probed is \( \xi_{\text{min}}(x_B) = x_B + (1 - x_B)c \) because of the cut condition on \( s_{12} \). The minimum is \( \xi_{\text{min}} = c \) for \( x_B \to 0 \) and is therefore of order 0.01. If one wants to use the gluon initiated (2+1) jet events to determine gluon densities at small \( x \) a different scale for the jet definition could be chosen. If the scale is \( Q^2 \) or \( (W^2 \alpha^3 \sqrt{S_Hy}^{1-\alpha-\beta})^2 \), the minimal
Depending on $x_B$ turns out to be $\xi_{\text{min}}(x_B) = x_B(1 + c)$ and $\xi_{\text{min}}(x_B) = x_B + (1 - x_B)^2 x_B^2 c$, respectively, and for sufficiently small $x_B$ and $c$ both give small $\xi_{\text{min}}$. If $x_B$ is fixed, these definitions are of course equivalent to the choice of a much smaller cut $c$ in the jet definition scheme based on $W^2$, and it is therefore important to check whether the $(2+1)$ jets are still calculated in a regime where perturbation theory is still valid, i.e., whether the $(2+1)$ jet rate is small enough compared to the total cross section.

After this short digression the results for the Born cross sections for the production of 2 partons are stated. The relevant diagrams are depicted in fig. 4 with the graph $G$ replaced by the diagrams of fig. 5. Of course, in fig. 4 one would have an additional diagram from an incoming antiquark, but since the resulting expressions are (except for the charges) the same as those for incoming quarks, these diagrams are omitted in the sequel.

The traces of the hadron tensor for quark and gluon initiated processes are

$$
\text{tr}H_{\text{Born, } q \text{ inc.}} = L_1 Q_j^2 s(1 - \epsilon) C_F T_q,
$$

$$\text{tr}H_{\text{Born, } g \text{ inc.}} = L_1 Q_j^2 s(1 - \epsilon) \frac{1}{2} T_g
$$

with

$$T_q = (1 - \epsilon) \left( \frac{s_{02}}{s_{12}} + \frac{s_{12}}{s_{02}} \right) + \frac{2Q^2 s_{01}}{s_{02} s_{12}} + 2\epsilon,
$$

$$T_g = \frac{1}{1 - \epsilon} \left\{ (1 - \epsilon) \left( \frac{s_{01}}{s_{02}} + \frac{s_{02}}{s_{01}} \right) - \frac{2Q^2 s_{12}}{s_{01} s_{02}} - 2\epsilon \right\}
$$

and

$$L_1 = (2\pi)^2 g^2 e^2 \mu^4 \epsilon.
$$

$p_0$ is always the momentum of the incoming parton, and $p_1$ is the momentum of the outgoing quark.

These formulae are already averaged over the colour degree of freedom of the incoming partons. The factor of $1/(1 - \epsilon)$ in the expression for $T_g$ comes from the fact that a gluon has $2(1 - \epsilon)$ helicity states in $d = (4 - 2\epsilon)$ dimensions compared to only 2 helicity states in 4 dimensions.

Now the integration of the cross sections over the (1+1) jet region of the 2-parton phase space can be done by subtracting the (2+1) jet cross section from the total cross section to $\mathcal{O}(\alpha_s)$. The technical reason is that this subtraction makes the explicit universal factorization of the singular parts in the form of a product of an Altarelli-Parisi splitting function and the Born term more transparent than the direct integration over the (1+1) jet region.

The total cross section from the real corrections for transverse photons and quark initiated processes is

$$
\frac{d\sigma_{\text{tot. t.q}}}{dx_B dy} = \sum_i \int_{x_B}^{1} \frac{dx_B}{x_B} \frac{\text{ren} \left( \frac{x_B}{x_p}, \frac{M_i^2}{x_p} \right)}{\frac{4\pi^4}{\Gamma(1 - \epsilon)} (S_{n x_B})^{-\epsilon} (y(1 - y))^{-\epsilon} \mu^{4\epsilon} \alpha_s^{2} 2 1 \frac{1}{2 S_{n x_B}} 1 + (1 - y)^2 - \frac{c y^2}{2(1 - \epsilon)y^2} 2\pi 4(1 - \epsilon) Q_i^2 \left( \frac{4\pi\mu^2}{Q^2} \right)^{\epsilon} \frac{\Gamma(1 - \epsilon)}{\Gamma(1 - 2\epsilon)} \frac{\alpha_s}{2\pi} C_F \delta_{qi}}
$$

8
\[
\left\{ \frac{2}{\epsilon^2} \delta(1 - x_p) + \frac{1}{\epsilon} \left[ - \frac{1}{C_F} P_{qg} + 3 \delta(1 - x_p) \right] \right. \\
+ 2 \left( \ln(1 - x_p) \right) - (1 + x_p) \ln(1 - x_p) - \frac{3}{2} \frac{1}{1 - x_p} + \frac{1 + x_p^2}{1 - x_p} \ln x_p \\
+ 3 - x_p + \frac{7}{2} \delta(1 - x_p) \right\} + \mathcal{O}(\epsilon).
\]

(23)

The “+”–prescriptions are defined in [23] (they are implicitly used for functions defined on the interval [0,1]) and are the result of the subtraction in the collinear regime. The bare parton densities are expressed in terms of the renormalised ones and terms of \(\mathcal{O}(\alpha_s^2)\) are dropped. The corresponding expression for the gluon initiated process is

\[
\frac{d\sigma_{\text{tot,t,g}}}{dx_B dy} = \sum_{i=1}^{2N_f} \int_{x_B}^{1} \frac{dx_p}{x_p} \frac{d\sigma_{t_i}^\text{ren}}{dx_p} \left( \frac{x_B}{x_p}, M_f^2 \right) \left( 4\pi \right)^\epsilon \left( \frac{S_{hi} x_B}{(1 - y)^{\epsilon}} \right)^\epsilon \frac{\Gamma(1 - \epsilon)}{\Gamma(1 - 2\epsilon)} \frac{\alpha_s}{2} \frac{1}{\pi} \frac{1}{S_{hi} x_B} \cdot \frac{1 + (1 - y)^2 - \epsilon y^2}{2(1 - e)y^2} 2\pi 4(1 - \epsilon) \tilde{Q}_i^2 \left( \frac{4\pi \mu^2}{Q^2} \right)^\epsilon \frac{\Gamma(1 - \epsilon)}{\Gamma(1 - 2\epsilon)} \frac{\alpha_s}{2} \frac{1}{\pi} \frac{1}{1 - \epsilon} \cdot \left\{ - 2 \frac{1}{\epsilon} P_{qg}(x_p) + \left( (1 - x_p)^2 + x_p^2 \right) \ln \frac{1 - x_p}{x_p} \right\} + \mathcal{O}(\epsilon).
\]

(24)

Note that the sum over quark flavours reflects the different flavours that are produced in this process.\[\]

In the sum of ([13], [23] and [24]) the infrared and collinear singularities cancel, and one is left with a finite total cross section to \(\mathcal{O}(\alpha_s)\). To obtain the (1+1) jet cross section one has to subtract the (2+1) jet cross section from the total cross section. Since there are no singularities in the (2+1) jet region, \(\epsilon\) can be set to 0. The integration of (24) over the (2+1) jet region is not difficult. If all contributions are summed up one arrives at the finite (1+1) jet cross section. The quark initiated part is given by

\[
\frac{d\sigma_{\text{tot,t,q}}}{dx_B dy} = \sum_{i=1}^{2N_f} \int_{x_B}^{1} \frac{dx_p}{x_p} Q_i^2 \frac{d\sigma_{t_i}^\text{ren}}{dx_p} \left( \frac{x_B}{x_p}, M_f^2 \right) \alpha_s \frac{1}{2} \frac{1}{S_{hi} x_B} \cdot 2\pi 4(1 - \epsilon) \tilde{Q}_i^2 \left( \frac{4\pi \mu^2}{Q^2} \right)^\epsilon \frac{\Gamma(1 - \epsilon)}{\Gamma(1 - 2\epsilon)} \frac{\alpha_s}{2} \frac{1}{\pi} \frac{1}{1 - \epsilon} \cdot \left\{ 1 + C_F \frac{\alpha_s}{2\pi} \left[ (-8 - 2\zeta(2)) \delta(1 - x_p) \right. \\
+ 2 \left( \ln(1 - x_p) \right) - (1 + x_p) \ln(1 - x_p) - \frac{3}{2} \frac{1}{1 - x_p} + \frac{1 + x_p^2}{1 - x_p} \ln x_p \\
+ 3 - x_p + \frac{7}{2} \delta(1 - x_p) + \frac{1}{C_F} \ln \frac{Q^2}{M_f^2} P_{qg}(x_p) \\
- \left( \frac{1}{2} \frac{x_p}{1 - x_p} - \frac{2}{1 - x_p} \frac{x_p}{1 - x_p} \right) (1 - 2z_c(t(x_p))) \\
+ \left[ \frac{1}{2} \frac{x_p}{1 - x_p} - \frac{2}{1 - x_p} \frac{x_p}{1 - x_p} \right] \ln \frac{1 - z_c(t(x_p))}{z_c(t(x_p))} \Xi_{c \leq t(x_p) \leq 1 - 2c} \right\},
\]

(25)

\[\]

\[\]

\[\]Compared to [\] there is a difference in the finite parts. It results from the average over the helicities of an initial gluon. Here it is assumed that the gluon has \(d - 2 = 2(1 - \epsilon)\) polarization states instead of 2.

\[\]To make the cancellation of the collinear divergence transparent, a “double counting” is included which is cancelled by a factor of 1/2.
and the gluon initiated processes contribute

\[
\frac{d\sigma_i^{(1+1),t,g}}{dx_{\beta}dy} = \sum_{i=1}^{2N_f} \int_{x_{\beta}}^{1} \frac{dx_{p}}{x_{p}} Q_i^2 f_{i}^{\text{ren}} \left( \frac{x_{p}}{x_{\beta}}, M_f^2 \right) \alpha_s^2 \frac{1}{2 S_{i \beta}} \frac{1 + (1 - y)^2}{2 y^2} \cdot 2\pi \cdot 4
\]

\[
\frac{1}{2} \frac{\alpha_s}{2\pi} \left[ \frac{1}{2} \left( 1 - x_{p} \right)^2 + x_{p}^2 \left( \ln \frac{1 - x_{p}}{x_{\beta}} - 1 \right) + 2 \ln \frac{Q_i^2}{M_f^2} P_{q_t-g}(x_p) \right] - \left( 1 - z_c(t(x_p)) + (1 - 2x_p(1 - x_p)) \right)
\]

\[
\cdot \ln \frac{1 - z_c(t(x_p))}{z_c(t(x_p))} \Xi_{c \leq t(x_p) \leq 1 - 2c} \]  \quad (26)

where \( \Xi_A \) is the characteristic function of the set specified in \( A \) restricting the integration to that set and \( t(x_p) \) is the variable \( t \) introduced above given by

\[
t(x_p) = \frac{x_{\beta} - 1 - x_{p}}{x_{p}}. \quad (27)
\]

The logarithms depending on the factorization scale that cancel part of the scale dependence of the parton densities are indicated explicitly. There are no such terms depending on the renormalisation scale because the Born term is of \( \mathcal{O}(\alpha_s^0) \).

The corresponding terms for a virtual photon with longitudinal polarization arise at \( \mathcal{O}(\alpha_s) \). They are given by

\[
\frac{d\sigma_i^{(1+1),1,q}}{dx_{\beta}dy} = \sum_{i=1}^{2N_f} \int_{x_{\beta}}^{1} \frac{dx_{p}}{x_{p}} Q_i^2 f_{i}^{\text{ren}} \left( \frac{x_{p}}{x_{\beta}}, M_f^2 \right) \alpha_s^2 \frac{1}{2 S_{i \beta}} \frac{4(1 - y) + 1 + (1 - y)^2}{2 y^2} \cdot 2\pi \cdot 4
\]

\[
\cdot C_F \frac{\alpha_s}{2\pi} \left[ x_{p} - \left( x_{p} \left( 1 - 2z_c(t(x_p)) \right) \right) \Xi_{c \leq t(x_p) \leq 1 - 2c} \right] \]  \quad (28)

and

\[
\frac{d\sigma_i^{(1+1),1,g}}{dx_{\beta}dy} = \sum_{i=1}^{2N_f} \int_{x_{\beta}}^{1} \frac{dx_{p}}{x_{p}} Q_i^2 f_{i}^{\text{ren}} \left( \frac{x_{p}}{x_{\beta}}, M_f^2 \right) \alpha_s^2 \frac{1}{2 S_{i \beta}} \frac{4(1 - y) + 1 + (1 - y)^2}{2 y^2} \cdot 2\pi \cdot 4
\]

\[
\cdot \frac{1}{2} \frac{\alpha_s}{2\pi} \left[ 2x_{p}(1 - x_{p}) - \left( 2x_{p}(1 - x_{p}) \left( 1 - 2z_c(t(x_p)) \right) \right) \Xi_{c \leq t(x_p) \leq 1 - 2c} \right] \]  \quad (29)

for incoming quarks and gluons, respectively.

Now the jet recombination scheme ambiguity which arises from the problem of the mapping of a “jet phase space” of massive jets in NLO onto a phase space of massless partons in leading order is discussed. The real corrections for the (1+1) jet cross section have been obtained by an integration of the Born terms for the production of 2 partons over some region of phase space specified by the jet cut definition. To cancel the infrared and collinear singularities, one has to define effective (1+1) jet variables that allow the identification of the corresponding singularities in the virtual corrections and in the contribution from the redefinition of the parton densities. This procedure is a map of the (1+1) jet region of the 2-parton phase space onto the (1+1) jet phase space. In principle this map is ambiguous, but fortunately the the difference between
different maps is 0 for vanishing jet cut. Here the following scheme is tacitly assumed: If a cluster algorithm is applied to a 2-parton event that looks like a (1+1) jet event, then the final result of this clustering will be two clusters with momenta \( p_A \) and \( p_B \), one of them being massless and one of them being massive. The (1+1) jet Born term and the virtual correction always result in two massless clusters \( p_C \) and \( p_D \). The identification is done by mapping \( (p_A + p_B)^2 \) onto \( (p_C + p_D)^2 \). In this special case both expressions are equal to \( W^2 \). This quantity can be determined from electron variables alone, so what is essentially done is the identification of processes in which the electrons have the same momenta. This, however, may correspond to very different situations on the parton level, since e.g. a radiated final state gluon can be collinear to the remnant jet resulting in a remnant with a large momentum or can be collinear to the outgoing quark. The situation is similar in the case of the corrections to the (2+1) jet cross section. In that case, however, the identification involves two variables and not only one, and so the identification using \( W^2 \) is not sufficient.

4 (2+1) Jets: Virtual Corrections

In this section the calculation of the virtual corrections to the production of (2+1) jets is described. The one-loop corrections to the graphs from fig. 5 are given in fig. 6. The one-loop diagrams from fig. 7 contribute to the wave function renormalisation and are taken into account in the counter term. To obtain the \( \mathcal{O}(\alpha_s^2) \) corrections the diagrams in fig. 6 must be multiplied by the Born diagrams. The resulting topologies can be divided into three classes:

(I) QED-like graphs with colour factor \( N_C C_F^2 \),

(II) QED-like graphs with colour factor \( N_C C_F \left( C_F - \frac{1}{2} N_C \right) \),

(III) non-abelian graphs with colour factor \(-\frac{1}{2} N_C^2 C_F\),

resulting in terms proportional to the colour factors \( N_C C_F^2 \) and \( N_C^2 C_F \). The sum of the virtual \( \mathcal{O}(\alpha_s^2) \) corrections averaged over the colour degree of freedom of the incoming parton is

\[
\text{tr} H_v, \text{q inc.} = L_1 L_2 Q_f^2 S(1 - \epsilon) \left\{ C_F^2 E_{1,q} - \frac{1}{2} N_C C_F E_{2,q} \right\} \\
+ \left( \frac{1}{\epsilon} + \log \frac{Q^2}{\mu^2} \right) \left\{ \frac{1}{3} N_f - \frac{11}{6} N_C \right\} C_F T_q \right\} + \mathcal{O}(\epsilon), \tag{30}
\]

\[
\text{tr} H_v, \text{g inc.} = L_1 L_2 Q_f^2 S(1 - \epsilon) \left\{ \frac{1}{2} C_F E_{1,g} - \frac{1}{4} N_C E_{2,g} \right\} \\
+ \left( \frac{1}{\epsilon} + \log \frac{Q^2}{\mu^2} \right) \left\{ \frac{1}{3} N_f - \frac{11}{6} N_C \right\} \frac{1}{2} T_g \right\} + \mathcal{O}(\epsilon), \tag{31}
\]
where
\[
L_2 = \frac{\alpha_s}{2\pi} \left( \frac{4\pi \mu^2}{q^2 - i\eta} \right)^\epsilon \frac{\Gamma(1 - \epsilon)}{\Gamma(1 - 2\epsilon)}.
\] (32)

Here \( N_f \) is the number of active quark flavours in the fermion loops and \( \mu^2 \) is the renormalisation scale (it is understood that the running \( \alpha_s \) is always evaluated at the scale \( \mu^2 \)). The contributions \( E_{1,q}, E_{2,q}, E_{1,g} \) and \( E_{2,g} \) come from the calculation of the matrix elements of the graphs in fig. 3. The explicit expressions for the \( E_{i,q} \) and \( E_{i,g} \) are collected in Appendix B. The trace calculations of the matrix elements were done with the help of REDUCE [24]. Then the loop integrals were performed by an insertion of one-loop tensor structure integrals (see appendix A). Here one has to be careful with respect to the imaginary parts of Spence functions and logarithms which are important because \( q^2 < 0 \). The results obtained here have been checked against those of the \( e^+ e^- \)-case [25]. This was possible because all infinitesimal imaginary parts from the propagators were kept in the formulae.

In (30), (31) the counter terms (in the \( \overline{\text{MS}} \)-scheme) to cancel UV singularities (see [25]) is already added. Some \( 1/\epsilon^2 \) and \( 1/\epsilon \)-poles remain. These divergences are due to the IR singularities of the loop corrections.

For the processes with an incoming quark the following variables are defined:
\[
z_q := \frac{p_1 p_0}{p_0 q}, \quad z_g := \frac{p_2 p_0}{p_0 q}, \quad x_p := \frac{x_B}{a}.
\] (33)

The divergent parts are
\[
\text{tr}H_{v, q \text{ inc.}} = L_1 L_2 Q_f^2 8 (1 - \epsilon) C_F T_q
\cdot \left( -\frac{1}{\epsilon^2} + \frac{1}{\epsilon} \left( \ln \frac{1 - z_q}{x_p} - \frac{3}{2} \right) \right)
+ \left( -\frac{1}{\epsilon^2} + \frac{1}{\epsilon} \left( \ln \frac{z_g a}{x_g} - \frac{3}{2} \right) \right)
- \frac{1}{2} N_C \left( \frac{2}{\epsilon^2} + \frac{1}{\epsilon} \left( \ln \frac{x_p^2}{(1 - x_p)(1 - z_q)} + 2 \right) \right) + \frac{1}{\epsilon} \cdot \frac{5}{3}
+ \frac{1}{\epsilon} \ln \frac{1 - z_g}{1 - x_p} + \frac{1}{\epsilon} \ln \frac{z_q}{1 - z_q}
+ N_f \cdot \frac{1}{3} \cdot \frac{1}{\epsilon} + \mathcal{O}(\epsilon^0).
\] (34)

For the processes with an incoming gluon the variables are similar:
\[
z_q := \frac{p_1 p_0}{p_0 q}, \quad z_{\pi} := \frac{p_2 p_0}{p_0 q}, \quad x_p := \frac{x_B}{a}.
\] (35)

The divergent parts are
\[
\text{tr}H_{v, g \text{ inc.}} = L_1 L_2 Q_f^2 8 (1 - \epsilon) \frac{1}{2} T_g
\]
\begin{align*}
&\left\{ C_F \left[ 2 \left( \frac{-1}{\epsilon^2} + \frac{1}{\epsilon} \left( \ln \frac{1-x_p}{x_p} - \frac{3}{2} \right) \right) \right] \\
-\frac{1}{2} N_C \left[ \frac{1}{\epsilon} \ln \frac{1-x_p}{1-z_q} + \frac{1}{\epsilon} \ln \frac{1-x_p}{1-z_q} \\
+ \left( \frac{2}{\epsilon^2} + \frac{1}{\epsilon} \left( -\ln \frac{a^2 z_q (1-z_q)}{x_p^2} + \frac{11}{3} \right) \right) \right] \\
+N_f \cdot \frac{1}{3} \cdot \frac{1}{\epsilon} \right \} + \mathcal{O}(\epsilon^0).
\end{align*}

They will cancel against divergent terms from the real corrections.

\section{(2+1) Jets: Final State Real Corrections}

To $\mathcal{O}(\alpha_s^2)$ one has to consider the contributions from the Born terms in fig. 8 integrated over the (2+1) jet phase space region in the 3-parton phase space.

Again there are graphs with an incoming quark and an incoming gluon. The generic diagrams are shown in fig. 9. There are, of course, additional contributions with incoming antiquarks; their structure is identical to the quark-initiated processes.

The integrations become singular if the integrand contains a propagator whose denominator vanishes in the integration region. The method of partial fractions is used to separate initial and final state singularities. This allows the identification of the terms proportional to $\mathcal{O}(\epsilon^0)$, $\mathcal{O}(\ln c)$ and $\mathcal{O}(\ln^2 c)$ ($c$ is the jet cut). In this section the final state singularities are considered, the initial state singularities are treated in the next section.

For the process

\[ e^-(k) + \text{proton}(P) \rightarrow e^-(k') + \text{target remnant}(p_r) + \text{parton}(p_1) + \text{parton}(p_2) + \text{parton}(p_3) \quad (37) \]

a parametrization of the phase space of the outgoing partons with momenta $p_i$ is needed. The target remnant with momentum $p_r = (1 - \xi)P$ is described by the variable $\xi$. The parametrization is chosen such that the integration over the region $s_{12} < c$ ($p_1$ and $p_2$ being collinear or $p_1$ being soft or $p_2$ being soft) is simple. In close analogy to calculations in $e^+e^-$-annihilation it is reasonable to describe the two particle phase space of $p_1$ and $p_2$ in the CM frame of these momenta (see fig. 10).

Let $p_0$ be the momentum of the incoming parton. One can define a variable $z$ by

\[ z := \frac{p_0 p_3}{p_0 q} \quad (38) \]

that describes the phase space of $p_3$ (the azimuthal dependence is contained in the lepton phase space, and the remaining third integration is trivial because of energy conservation). One can define a polar angle $\theta$ given by $\theta := \angle(p_1, p_0)$ in the CM frame of $p_1$ and $p_2$ and an azimuthal
angle $\varphi$ by the angle between the planes spanned by $\tilde{p}_0, \tilde{p}_1$ and $\tilde{p}_0, \tilde{p}_3$, respectively. Let $\chi$ be defined by $\chi := \angle (\tilde{p}_0, \tilde{p}_3)$ and

\begin{align}
 b & := \frac{1}{2}(1 - \cos \theta) \\
 d & := \frac{1}{2}(1 - \cos \chi) \\
 e & := b + d - 2bd - 2\sqrt{b(1-b)d(1-d)} \cos \varphi.
\end{align}

(39)  
(40)  
(41)

With the normalisation factors

\begin{align}
 N_\varphi & := \frac{\pi^{4\epsilon}}{\Gamma(1-2\epsilon)} = \int_0^\pi \sin^{-2\epsilon} \varphi d\varphi \\
 N_b & := \frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)} = \int_0^1 (b(1-b))^{-\epsilon} db
\end{align}

(42)  
(43)

the 3-parton phase space in $d = 4 - 2\epsilon$ dimensions is

\begin{align}
 \int dPS^{(3)} = \int \frac{(16\pi^2)^\epsilon}{128\pi^3 \Gamma(2-2\epsilon)} s_{12}^{-\epsilon} ds_{12} z^{-\epsilon} (s_{12} y (\xi - x_B)(1-z) - s_{12})^{-\epsilon} \\
 \cdot d\varphi \frac{1}{N_\varphi} \sin^{-2\epsilon} \varphi d\varphi \frac{1}{N_b} (b(1-b))^{-\epsilon} db \\
 = \int dPS^{(2)} \int dPS^{*(r)}.
\end{align}

(44)

$dPS^{(2)}$ is defined in eq. (17), and

\begin{align}
 dPS^{*(r)} & = a_\delta (\xi - a)L_2 \frac{2\pi}{\alpha_s} \mu^{-2\epsilon} \frac{1}{2} \frac{1}{8\pi^2} \frac{1}{x_p} Q^2 \frac{1}{1-2\epsilon} d\mu_F, \\
 d\mu_F & = \left(1 - \frac{s_{12}}{s_{12} y (\xi - x_B)(1-z)}\right)^{-\epsilon} r_{12}^{-\epsilon} dr_{12} \frac{1}{N_\varphi} \sin^{-2\epsilon} \varphi d\varphi \\
 r_{ij} & := \frac{s_{ij}}{s_{12} y \xi}.
\end{align}

(45)  
(46)  
(47)

$dPS^{(2)}$ contains the variables $z$ and $t = s_{12}/W^2$ that are identified with the corresponding $(2+1)$ jet variables. The phase space for 3 particles factorizes as a product of a phase space for 2 particles $dPS^{*(r)}$ and an effective phase space for a particle and a cluster $dPS^{(2)}$. The invariants $r_{ij}$ can be expressed in the variables $t, z, b, x_p = x_B/\xi, \varphi$ and $r_{12}$:

\begin{align}
 r_{01} & = (1 - z)b, \\
 r_{02} & = (1 - z)(1 - b), \\
 r_{03} & = z, \\
 r_{13} & = (1 - x_p - r_{12})e, \\
 r_{23} & = (1 - x_p - r_{12})(1 - e).
\end{align}

(48)
The variable $d$ in eq. (40) is given by

$$d = \frac{z}{1 - z} \frac{r_{12}}{1 - x_p - r_{12}}. \quad (49)$$

The phase space boundaries are

$$x_B \in [0, 1], \quad \xi \in [x_B, 1], \quad t \in [0, 1],$$

$$\varphi \in [0, \pi], \quad b \in [0, 1], \quad r_{12} \in [0, (1 - x_p)(1 - z)]. \quad (50)$$

The products of the diagrams in fig. 8 with the complex conjugated diagrams can be classified in classes A, B, ..., H with different colour factors (see tab. I). An explicit calculation shows that the colour classes G and H are regular when integrated over the 2-particle phase space $dPS^{*r}$ and therefore vanish for $c \to 0$. Therefore these classes are not considered here.

| I  | II | III | IV | V  | VI  | VII |
|----|----|-----|----|----|-----|-----|
| I  | A  | *   | *  |    |     |     |
| II | B  | A   | *  |    |     |     |
| III| C  | E   | D  |    |     |     |
| IV | F  | *   | *  |    |     |     |
| V  | G  | F   | *  |    |     |     |
| VI | G  | H   | F  | *  |     |     |
| VII| H  | G   | G  | F  |     |     |

Table 1: Colour classes and colour factors of the real corrections.

The calculation of the spin sum for external gluons has been performed with the formula

$$\sum_{\lambda=0}^{d-1} \epsilon_{\mu}^{\lambda} \epsilon_{\nu}^{\lambda} = -g_{\mu\nu}. \quad (51)$$

To cancel the contributions from scalar and longitudinal gluons one has to subtract diagrams with external ghost lines. The longitudinal and scalar contributions then drop out because of the Slavnov-Taylor identities (see [26]).

The matrix elements have been calculated with REDUCE in $d = 4 - 2\epsilon$ dimensions. In principle they could be obtained from the results in $e^+e^-$-scattering [27, 28, 29]. However, the results of the procedure to obtain partial fractions are different here since some of the invariants pick up a sign because of the crossing prescriptions. Here only the results of the factorization of the IR divergent terms

$$M_{\text{singular}} = K \cdot T_{q/g}. \quad (52)$$

are stated. $T_{q/g}$ is the Born term [21] with incoming quark and gluon, respectively, and $K$ is a singular kernel whose integration is divergent in $d = 4$ dimensions. In $d = 4 - 2\epsilon$ dimensions the result of the phase space integral is of the form $a/\epsilon^2 + B/\epsilon + C$. $A$ and $B$ do not depend on the invariant mass cut $c$. $C$ contains terms of the form $\ln c$ and $\ln^2 c$ which diverge for $c \to 0$. The
Table 2: Colour factors.

| Class | incoming parton | product of diagrams | colour factor |
|-------|-----------------|---------------------|---------------|
| $F_1$ | quark           | I-I, II-II, II-I    | $N_C C_F^2/N_C$ |
| $F_2$ | quark           | II-I, III-I, III-II | $(-1/2)N_C^2 C_F/N_C$ |
| $F_3$ | quark           | III-I, III-II      | $(-1/2)N_C^2 C_F/N_C$ |
| $F_4$ | quark           | III-III            | $N_C^2 C_F/N_C$ |
| $F_5$ | quark           | IV-IV, V-V, VI-VI, VII-VII | $(1/2)N_C C_F/N_C$ |
| $F_6$ | gluon           | I-I, II-II, II-I   | $N_C C_F^2/(2N_C C_F)$ |
| $F_7$ | gluon           | II-I, III-I, III-II | $(-1/2)N_C^2 C_F/(2N_C C_F)$ |

Contributions from the final state singularities are divided into seven classes (see tab. 3). The results for the traces of the hadronic tensor $\text{tr}H_{F_i}$ for the seven colour classes are given explicitly in appendix C.

The singular kernels are integrated over the (2+1) jet like region in phase space. The condition for partons $j, k$ to form a jet is $s_{jk} \leq cW^2$. This condition can be rephrased in terms of the variables

$$r_{jk} := \frac{s_{jk}}{S_h y^\xi}$$

as

$$r_{jk} \leq c \frac{1-x_B}{x_B} x_p.$$ (54)

The phase space boundary is given by $r_{jk} \leq (1-x_p)(1-z)$. So the (2+1) jet region is specified by

$$r_{jk} \leq \alpha := \min \left\{ (1-x_p)(1-z), c \frac{1-x_B}{x_B} x_p \right\}, \quad b \in [0,1], \quad \varphi \in [0,\pi].$$ (55)

The phase space integrals are listed in appendix D. One obtains

$$\int_{(2+1) \text{ jet}} dPS^{i(r)} \text{tr}H_{F_i} = a \delta(\xi - a) L_1 L_2 S(1-\epsilon) F_i.$$ (56)

for the corrections from the final state singularities. The explicit expressions are collected in appendix E.

6 (2+1) Jets: Initial State Real Corrections

In this section the contributions from the initial state singularities are described. In the case of the final state singularities the momentum fraction $\xi$ of the incoming parton is a fixed parameter in the matrix elements. In the case of the initial state singularities, however, there is the additional problem that $\xi$ is an integration variable and that $\xi$ is an argument of the parton densities $f_i(\xi)$. Although there is no expression for $f_i(\xi)$ in a closed form, the problem can be solved \cite{23} by a Taylor expansion around the singular point and the well known “+”-prescriptions \cite{23}

$$D_+(g) := \int_0^1 du D(u) (g(u) - g(1)).$$ (57)
A similar parametrization of the phase space as in the case of the final state singularities is used, but the effective (2+1) jet variables are defined in a different way. Let \( p_0 \) be the momentum of the incoming parton and \( p_3 \) the momentum of an outgoing parton. For the calculation of the singularity resulting from \( s_{03} \rightarrow 0 \) \( p_1 \) and \( p_2 \) are the momenta of the partons to be identified with the effective (2+1) jet momenta. Variables \( z' \), \( z \) and \( t \) are defined by

\[
z' := \frac{p_0 p_3}{p_0 q}, \quad z := \frac{p_0 p_1}{p_0 q}, \quad t := \frac{s_{12}}{W^2}.
\]

With the conventions from Section 3 one obtains

\[
b = \frac{z}{1 - z'}.
\]

A factor of unity

\[
1 = \int d\xi' \delta(\xi - \xi')
\]

is inserted in eq. (54) to display a change of variables

\[
\sigma := \frac{1 - t - \frac{1 - z'}{1 - z'}}{1 - \frac{1 - z'}{1 - z'}}
\]

explicitly. As a result one obtains

\[
\int dPS^{(3)} = \int dPS^{(2)} \int dPS^{(r)},
\]

\[
\int dPS^{(r)} = \delta(\xi - \xi') L_2 \frac{2\pi}{\alpha_s} \mu^{-2} \frac{1}{28\pi^2} W^2 H(z') d\mu_I,
\]

\[
H(z') = (1 - z')^{-2 + 2\epsilon} \left(1 - \frac{z'}{1 - t}\right)^{1 - \epsilon} \left(1 - \frac{z'}{1 - z}\right)^{-\epsilon} = 1 + O(z'),
\]

\[
d\mu_I = \frac{\Gamma(1 - 2\epsilon)}{\Gamma^2(1 - \epsilon)} a \left(\frac{1 - x_B}{x_B}\right)^{-\epsilon} (1 - t)^{1 - \epsilon} \sigma^{-\epsilon} d\sigma z'^{-\epsilon} dz' \frac{1}{N_\varphi} \sin^{-2\epsilon} \varphi d\varphi.
\]

The \( \delta \)-function \( \delta(\xi - \xi') \) can be used to perform the integration over the parton densities. \( d\mu_I \) is the measure for the singular integrations. The invariants \( t_{ij} := s_{ij}/W^2 \) can be expressed in terms of the phase space variables:

\[
t_{01} = (\nu - \zeta)z, \\
t_{02} = (\nu - \zeta)(1 - z - z'), \\
t_{03} = (\nu - \zeta)z', \\
t_{12} = t, \\
t_{13} = (1 - \zeta - t)e, \\
t_{23} = (1 - \zeta - t)(1 - e),
\]

where

\[
\nu = \frac{1}{1 - x_B}, \quad \zeta = (1 - \sigma) \left(1 - \frac{t}{1 - z'}\right),
\]

\[
e = b + d + 2bd - 2\sqrt{b(1 - b)d(1 - d)} \cos \varphi,
\]

\[
d = \frac{z'}{1 - z' \frac{1 - b}{b} - \varphi}.
\]
It should be noted that
\[ \xi = a + (1 - a)\sigma + \mathcal{O}(z'). \] (70)

The phase space boundaries are given by
\[ x_B \in [0, 1], \quad \xi, \xi' \in [x_B, 1], \quad t \in [0, 1], \quad z \in [0, 1], \]
\[ \varphi \in [0, \pi], \quad \sigma \in [0, 1], \quad 0 \leq z' \leq \min \{1 - z, 1 - t\}. \] (71)

The factorization of the divergent parts is performed in the form
\[ \mathcal{M}_{\text{singular}} = K \cdot T_{q/g}. \] (72)

up to terms that vanish for \( c \to 0 \) after the integration. Therefore one can, for example, set \( H(z') \) identically to 1.

The contributions from the initial state singularities are divided into seven classes (see tab. 3). In the case of the initial state singularities the incoming parton of the (3+1) jet graph is not necessarily the incoming parton of the (factorized) (2+1) jet process. This is the familiar fact that the quark parton densities modify the evolution of the gluon density and vice versa. In tab. 3 a list of the incoming partons of both processes is added. The explicit expressions of the singular kernels \( trH_i \) are collected in appendix F.

| Class | (3+1) | (2+1) | product of diagrams | colour factor |
|-------|-------|-------|--------------------|--------------|
| \( I_1 \) | quark | quark | I-I, II-II, II-I | \( N_C C_F^2 / N_C \) |
| \( I_2 \) | quark | quark | II-I, III-I, III-II | \( (-1/2) N_C C_F / N_C \) |
| \( I_3 \) | quark | gluon | IV-IV, V-V, VI-VI, VII-VII | \( (1/2) N_C C_F / N_C \) |
| \( I_4 \) | gluon | quark | I-I, II-II, II-I | \( N_C C_F^2 / (2 N_C C_F) \) |
| \( I_5 \) | gluon | quark | II-I, III-I, III-II | \( (-1/2) N_C C_F / (2 N_C C_F) \) |
| \( I_6 \) | gluon | gluon | II-I, III-I, III-II | \( (-1/2) N_C C_F / (2 N_C C_F) \) |
| \( I_7 \) | gluon | gluon | III-III | \( N_C C_F / (2 N_C C_F) \) |

Table 3: Colour factors.

The boundaries of the (2+1) jet phase space region are given by the invariant mass cut condition. For the initial state singularities, the variable \( z' \) is the crucial variable that determines the singularity structure. The matrix elements become singular for \( z' = 0 \). Let "rt" be the label for the target remnant jet. The invariant \( t_{rj} \) is given by \( t_{rj} = \left( (1 - \xi)/\xi \right) t_{0j} \). If \( p_r \) and \( p_3 \) are the momenta combined into a jet, then the invariants of the effective (2+1) jet event are \( t_{12}, t_{1r3}, t_{2r3} \). The cut conditions read
\[ t_{r3} = \frac{1 - \xi}{1 - x_B} z' \leq c, \]
(73)
\[ t_{12} = t \geq c, \]
(74)
\[ t_{1r3} = t_{13} + t_{1r} + t_{3r} = (1 - \zeta - t)e + \frac{1 - \xi}{1 - x_B} (z + z') \geq c, \]
(75)
\[ t_{2r3} = t_{23} + t_{2r} + t_{3r} = (1 - \zeta - t)(1 - e) + \frac{1 - \xi}{1 - x_B} (1 - z) \geq c. \]
(76)
Especially the last two of these conditions are too complicated to be used in an analytical calculation since they involve a restriction of the azimuthal angle integration. Therefore the contributions from the initial state singularities are integrated up to the phase space boundary (so $\varphi \in [0, \pi], z' \in [0, \min\{1-z, 1-t\}]$) by keeping the effective (2+1) jet variables $z$ and $t$ fixed. The (3+1) jet contribution is then subtracted after a numerical integration. Since the integral including the parton densities cannot be performed analytically, this is not a serious restriction. The shape of the phase space regions in the $(z', \sigma)$-plane is given in fig. 11.

The poles and double poles in $\epsilon$ characterising IR ($z' = \sigma = 0$) and collinear ($z' = 0$) singularities can be calculated in the integration over the full $(z', \sigma)$-plane. The phase space integrals are given in appendix G. In the formulae given there the upper limit of the $z'$-integration is $\beta = \min\{1-z, 1-t\}$. In the integrals $\sigma$ is used as an integration variable. The integral involving the parton densities is of the form

$$\int_0^1 d\sigma \frac{f(\xi(\sigma, z', \ldots), Q^2)}{\xi(\sigma, z', \ldots)} D(\sigma).$$

(77)

Here $D$ is a generalized function depending on $\sigma$ and the other jet variables. $f(\xi(\sigma, z', \ldots), Q^2)$ is expanded in a Taylor series in $z'$ and all terms of order $O(z')$ that do not contribute in the approximation used here are neglected. One obtains

$$f(\xi(\sigma, z', \ldots)) = f(a + (1-a)\sigma, Q^2) + O(z').$$

(78)

$a$ is the momentum fraction of the incoming parton of the factorized Born term. With the definition

$$u := \frac{a}{\xi} = \frac{a}{a + (1-a)\sigma} + O(z')$$

(79)

eq (77) can be rewritten as

$$\int_0^1 \frac{du}{u} f(a, Q^2) D(\sigma(u)) \frac{1}{1-a}.$$  

(80)

Since $D$ is a generalized function, one has to take care for the boundary terms of the variable transformation $\sigma \rightarrow u$.

Finally one obtains for the real corrections from the initial state singularities

$$\int_{(2+1) \text{ jet}}^{(3+1) \text{ jet}} dPS^{(r)} \text{tr} \mathcal{H}_i = \int_a^1 \frac{du}{u} \xi \delta \left( \xi - \frac{a}{u} \right) L_1 L_2 8(1-\epsilon) I_i.$$  

(81)

The explicit expressions for the $I_i$ are given in appendix H.

7 (2+1) Jets: Finite Cross Sections

In the preceeding sections the calculation of the Born terms $O(\alpha_s)$, the virtual corrections $O(\alpha_s^2)$ and the real corrections $O(\alpha_s^3)$ has been described. In the sum of the virtual and real corrections the IR singularities cancel, and the remaining collinear singularities are absorbed into the parton densities by the redefinition eq. (14). The final cross sections is then free of divergencies. The
partonic cross sections must be multiplied with the different flavour factors of the 14 classes of diagrams and integrated over the momentum fraction of the incoming parton. Let the charge of the quark of flavour \( i \) be \( q_i = Q_i e \), where \( i = 1 \) stands for d-quarks, \( i = 2 \) for \( \bar{d} \), \( i = 3 \) for u-quarks, and so on. Let \( f_i(\xi, M_f^2) \) be the parton density of flavour \( i \), \( f_g(\xi, M_f^2) \) the gluon density and \( N_f \) the number of flavours. Then the flavour factors are

\[
H_{F_1}, H_{F_2}, H_{F_3}, H_{F_4}, H_{I_1}, H_{I_2} : \sum_{i=1}^{2N_f} Q_i^2 f_i(\xi, M_f^2),
\]

\[
H_{F_5} : N_f \sum_{i=1}^{2N_f} Q_i^2 f_i(\xi, M_f^2),
\]

\[
H_{F_6}, H_{F_7}, H_{I_4}, H_{I_5}, H_{I_6}, H_{I_7} : \sum_{i=1}^{N_f} Q_{2i-1}^2 f_g(\xi, M_f^2),
\]

\[
H_{I_3} : \sum_{i=1}^{N_f} Q_{2i-1}^2 \sum_{j=1}^{2N_f} f_j(\xi, M_f^2). \tag{82}
\]

Here the factors \( Q_j^2 \) are included which are already present in the terms \( H_{F_i} \) and \( H_{I_i} \).

The Born terms and virtual corrections with incoming quarks are multiplied by

\[
\sum_{i=1}^{2N_f} Q_i^2 f_i(\xi, M_f^2), \tag{83}
\]

those with an incoming gluon by

\[
\sum_{i=1}^{N_f} Q_{2i-1}^2 f_g(\xi, M_f^2). \tag{84}
\]

8 Numerical Results

In this section the results of the numerical evaluation of the jet cross sections are presented. The finite (1+1) jet cross section has been calculated in Section 3, the (2+1) jet cross section in Section 7.

The matrix elements for (1+1) and (2+1) jet production are implemented in the program PROJET 3.3 [30] which uses the multidimensional adaptive integration routine VEGAS [31, 32] for the numerical integrations. The parton density parametrizations are from the package PAKPDF [33]. PROJET 3.3 allows the integration over bins in \( x_B \), \( y \), \( W^2 \) and \( Q^2 \). Furthermore, acceptance cuts on the angles of the outgoing lepton and the outgoing jets in the laboratory frame can be applied.

To be definite, the following parameters are used. In the case of HERA the CM energy is \( E_{\text{CM}} = 295 \text{ GeV} \), whereas the CM energy of the E665 experiment is \( E_{\text{CM}} = 31 \text{ GeV} \). In the latter case the lepton phase space is restricted by \( 2 \text{ GeV} < Q < 5 \text{ GeV} \), \( 20 \text{ GeV} < W < 40 \text{ GeV} \) and \( 0.05 < y < 0.95 \). Unless otherwise stated, the renormalisation and the factorization scales
are set to \( \mu = M_f = Q \), and the parton densities are from the MRS, set D– parametrization \[34\], which is presently favoured by structure function measurements at HERA. The number of flavours in the final state is set to \( 5 \).

The dependence of the jet cross sections on the jet cut \( c \) is shown in fig. [12]. The jet definition scheme is \( s_{ij} \geq cW^2 \) (\( s_{ij} \geq M^2 \) means that two clusters have to be combined if their invariant mass is smaller than \( M^2 \) and have to be considered as two separate clusters if their invariant mass is larger than \( M^2 \)). The phase space of the outgoing lepton for HERA is assumed to be \( 0.001 < x_g < 1, 10 \text{ GeV} < W < 295 \text{ GeV}, \) and \( 3.16 \text{ GeV} < Q < 10 \text{ GeV} \) (a), \( 10 \text{ GeV} < Q < 31.6 \text{ GeV} \) (b), \( 31.6 \text{ GeV} < Q < 100 \text{ GeV} \) (c). The graph in fig. [12] (d) is for E665. The \((1+1)\) jet Born cross section does not depend on the jet cut, because the condition \( s_{r1} > cW^2 \) is always trivially satisfied for \( c < 1 \). The \((1+1)\) jet cross section in NLO decreases with decreasing cut \( c \), because the total cross section to \( \mathcal{O}(\alpha_s) \) is independent of \( c \) while the \((2+1)\) jet cross section to \( \mathcal{O}(\alpha_s) \) strongly increases with decreasing cut. The \((2+1)\) jet cross section in NLO is comparable to the \((2+1)\) jet cross section on the Born level as long as the jet cut is not too small. At \( c \approx 0.007 \) the NLO \((2+1)\) jet cross section starts to decrease, and will go to \(-\infty \) for \( c \rightarrow 0 \) because of dominant terms \( -\ln^2 c \) in the NLO corrections. If one thinks of \( cW^2 \) as a new scale in the cross section, this behaviour (an extremum of the cross section at some value of this scale) can be interpreted as a stabilisation with respect to a change in this scale. If the difference between the cross section on the Born level and the cross section in NLO is too large this is a sign of the breakdown of fixed order perturbation theory. However, if \( c \geq 0.01 \) this does not seem to be the case, and therefore the region of values for \( c \) for phenomenological studies should start here. The contribution from the longitudinal polarisation of the virtual photon is always small (of the order of 20%) compared to the transverse cross section. Therefore using the transverse contributions in NLO and the longitudinal contributions on the Born level should be accurate (the relative magnitude of the corrections to the longitudinal cross sections are expected to be of the same order as in the transverse case, see also \[34\]). The cut dependence of the jet cross sections in all the three regions in \( Q \) for HERA studied here is similar, up to the absolute normalisation.

The jet rates \( R_{(2+1)} \) in fig. [13] are defined by \( R_{(2+1)} = \sigma_{(2+1)}/\sigma_{\text{tot}} \). In the case of Born terms, \( \sigma_{\text{tot}} \) is given by the total cross section to \( \mathcal{O}(\alpha_s^0) \), which is equal to the \((1+1)\) jet cross section on the Born level. In the case of the NLO corrections, \( \sigma_{\text{tot}} \) is given by the total cross section to \( \mathcal{O}(\alpha_s) \), which is equal to the sum of the \((1+1)\) and \((2+1)\) jet cross sections to \( \mathcal{O}(\alpha_s) \). This definition has the advantage that the denominator is always independent of the jet cut. \( R_{(2+1)} \), Born is strongly increasing with decreasing jet cut \( c \). \( R_{(2+1)} \), NLO is smaller than \( R_{(2+1)} \), Born in the bin of smaller values of \( Q \) (a). In the bin of larger values of \( Q \) (c), \( R_{(2+1)} \), NLO is larger than \( R_{(2+1)} \), Born for \( c > 0.015 \) and smaller for \( c < 0.015 \).

Now two different jet definition schemes are compared, fig. [14]. Here we use the HERA CM energy. The lepton phase space is given by \( 3.16 \text{ GeV} < Q < 20 \text{ GeV}, \) \( 10 \text{ GeV} < W < 295 \text{ GeV} \). In (a) the scheme is \( s_{ij} \geq cW^2 \), and in (b) the cut condition is \( s_{ij} \geq c \left( W^\alpha Q^\beta \sqrt{S_{ij}}^{1-\alpha-\beta} \right)^2 \) with \( \alpha = 0.8, \beta = 0.2 \). The \((2+1)\) jet cross sections on the Born level in the scheme (b) are always larger (for the same value of \( c \)) because the absolute scale of the cut in (b) is always smaller. Furthermore, the NLO starts to deviate considerably from the Born level at \( c \approx 0.02 \) in the jet definition scheme (b). Therefore, if this scheme is used, larger values of \( c \) are advised. If
the parameter $\beta$ is too small, then the NLO cross sections are frequently negative (for small absolute values of the cut scale, fixed order perturbation theory breaks down, this is similar to the case of small $c$ in the $cW^2$ scheme, see above).

The results for the scale dependence of the cross sections at HERA energies are shown in fig. 15 (a)–(c) ($0.001 < x_B < 1$, $5 \text{ GeV} < Q < 100 \text{ GeV}$, $10 \text{ GeV} < W < 295 \text{ GeV}$, $c = 0.02$). In principle, the renormalisation scale $\mu$ and the factorization scale $M_f$ are arbitrary. These scales give rise to logarithms of the form $\ln(\mu^2/M^2)$ and $\ln(M_f^2/M^2)$ in the cross section, where $M$ is some mass scale in the process. The logarithms are potentially large and spoil perturbation theory if the renormalisation and factorization scales are not of the same order of magnitude as $M$. In order to study the behaviour of the cross sections for a change of the scale, the renormalisation scale (fig. 15 (a), (c)) and the factorization scale (fig. 15 (b), (c)) are varied in the form of $\rho Q$, where $\rho$ is a parameter in the range 0.2 to 5. If only the renormalisation scale is varied (a), the $(1+1)$ jet cross section on the Born level is constant, because it is of $\mathcal{O}(\alpha_s^0)$. The $(2+1)$ jet cross section on the Born level possesses a large scale dependence of $\pm 40\%$ in the range of $\rho$ given above. The NLO correction to the $(2+1)$ jet cross section reduces this scale dependence considerably, because there is a term logarithmic in $\mu$ that cancels a part of the scale dependence of the running coupling in the Born term such that the overall dependence on $\mu$ is (formally) of $\mathcal{O}(\alpha_s^3)$. The $(1+1)$ jet cross section in NLO is scale dependent because of the running coupling constant, and there is no mechanism (i.e., no explicit logarithmic term in $\mu$) that would cancel this dependence (the reason for this is discussed in Section 3). Fortunately, the total cross section to $\mathcal{O}(\alpha_s)$ is less scale dependent than the $(1+1)$ jet cross section. The dependence on the factorization scale is shown in fig. 15 (b). The $(1+1)$ jet Born term is strongly scale dependent. Because the parton densities are redefined when the NLO contributions are calculated, there is a term that makes the dependence on $M_f$ formally of $\mathcal{O}(\alpha_s^2)$ in NLO. A similar cancellation takes place for the $(2+1)$ jet cross section. If both the renormalisation and the factorization scale are varied (c), the overall picture is that, compared to the Born level, the NLO cross sections are less scale dependent. In fig. 15 (a) the scale dependence of the jet rates is shown. It is evident that the NLO results have a much smaller scale dependence than the results on the Born level. The same graphs for HERA for the range of $Q$ given by $100 \text{ GeV} < Q < 200 \text{ GeV}$ are shown in fig. 15 (d)–(f), and the corresponding graph for the jet rates is fig. 16 (b). The scale dependent cross sections and jet rates for the kinematical region of the E665 experiment (for a jet cut $c = 0.04$) are shown in fig. 15 (g)–(i) and fig. 16 (c).

Finally the dependence on the parton densities at HERA for two different jet definition schemes (fig. 17) is discussed, with parameters $5 \text{ GeV} < Q < 295 \text{ GeV}$, $10 \text{ GeV} < W < 295 \text{ GeV}$, $c = 0.02$. The parametrizations HMRS set B [35], MT set B1 [36] and the more recent ones MRS sets D0 and D– are chosen for comparison. The two sets of curves in fig. 17 are for the cross sections $x_B d\sigma/dx_B$ for $(1+1)$ and $(2+1)$ jets in NLO. The jet definition scheme in (a) is $s_{ij} \geq cW^2$. For values of $x_B$ smaller than 0.01 the different parametrizations clearly predict different $(1+1)$ jet cross sections. The $(2+1)$ jet cross section differential in $x_B$ is insensitive to a variation of the parametrization. The reason is that in the $cW^2$ scheme all contributions to the $(2+1)$ jet cross section come from $\xi > c$, $\xi$ being the momentum fraction of the incoming parton, as discussed in Section 3. Since $c = 0.02$, there is very small variation in the $(2+1)$ jet

5 The three curves for the NLO corrected terms should intersect at $\rho = 1$. The small difference at $\rho = 1$ is due to small fluctuations from the spline fit to the Monte Carlo results.
cross section because there is not much difference in the parametrizations for $\xi > 0.02$. The situation is different in the scheme (b) $s_{ij} > c \left( W^\alpha Q^\beta \sqrt{S_H y^{1-\alpha-\beta}} \right)^2$ with $\alpha = 0.7$, $\beta = 0.3$. Here the $(2+1)$ jet cross section receives contributions from the parton densities at $\xi < c$ as well, and therefore the $(2+1)$ jet cross section depends on the parametrization. Using such a scheme might therefore be a possibility to measure the gluon density $f_g(\xi, M^2_f)$ for small $\xi$ via $(2+1)$ jet cross sections by a subtraction of the quark initiated contribution from the total (quark and gluon initiated) $(2+1)$ jet cross section.

9 Summary and Conclusions

In this paper the calculation of $(1+1)$ and $(2+1)$ jet cross sections in deeply inelastic electron proton scattering has been described. The jet definition includes the target remnant jet and is based on a modified JADE cluster algorithm. The inclusion of the proton remnant in the jet definition scheme is a consistent way to define 'exclusive' jet cross section for the production of $(n+1)$ jets because of the possibility of collinear emission of partons in the direction of the target remnant jet.

The cross sections are studied for HERA and E665 energies in detail. The jet cut dependence suggests that, if $c W^2$ is used as the mass scale in the invariant jet definition, the jet cut $c$ should be larger than 0.01 to avoid large NLO corrections that could invalidate a fixed order perturbative expansion. In the $c W^2$ scheme, the $(2+1)$ jet cross section depends on the parton densities $f_i(\xi, M^2_f)$ for $\xi > c$ only, even for very small $x_B$. If one wishes to probe the parton densities at smaller values of $\xi$, a different jet definition scheme has to be used. In the proposed region for $c$, the NLO corrections are at most of the order of 30%, and even smaller for very large $Q^2$.

A set of jet definition schemes that could be useful is given by the cut condition $s_{ij} > c \left( W^\alpha Q^\beta \sqrt{S_H y^{1-\alpha-\beta}} \right)^2$, where $\alpha$ and $\beta$ are some parameters in the range of $[0,1]$. By comparing the results for different parametrizations of parton densities it is explicitly shown that such a jet definition scheme gives a strong dependence of the $(2+1)$ jet cross section on the chosen parametrization. However, it must be studied whether such a jet definition is experimentally feasible and useful for the determination of the gluon density.

An important point is the scale dependence of the calculated cross section. It is a general phenomenon that leading order cross sections that depend on the strong coupling constant and scale dependent parton densities are strongly scale dependent. This leads to a theoretical uncertainty because, in principle, the renormalisation and factorization scales are arbitrary (although they should be chosen to be of the order of some physical scale in the process) and the variation of the cross section with respect to changes in the scales can be interpreted as being due to (unknown) higher order corrections because the cross section to all orders must be independent of the scales. The NLO corrections usually improve the situation because terms arise that cancel part of the scale dependence of the leading order. This desirable feature is present in the calculation described here, as has been shown explicitly by a variation of the scales as multiples of $Q^2$. The scale dependence is significantly reduced. It can be concluded that the NLO corrections reduce the theoretical uncertainties of the leading order and should provide well defined jet cross sections that could be useful in experimental analyses.
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A Massless 1-Loop Tensor Structure Integrals

This appendix contains the results for the tensor structure integrals

\[ I_{2(0,1\mu)}(r) := \int d^d k \frac{\{1, k_\mu\}}{(k^2 + i\eta)((k - r)^2 + i\eta)}, \]

\[ I_{3(0,1\mu,2\nu)}(r, p) := \int d^d k \frac{\{1, k_\mu, k_\nu\}}{(k^2 + i\eta)((k - r)^2 + i\eta)((k - r - p)^2 + i\eta)}, \]

\[ I_{40}(p_1, p_2, p_3) := \int d^d k \frac{1}{(k^2 + i\eta)((k - p_2)^2 + i\eta)((k - p_1)^2 + i\eta)((k - p_1 - p_3)^2 + i\eta)} \]

needed for the evaluation of the virtual corrections for graphs with massless particles. It is assumed that \( p^2 = p_1^2 = p_2^2 = p_3^2 = 0 \), but not necessarily \( r^2 = 0 \). The integrals are regularized by dimensional regularisation \( (d = 4 - 2\epsilon) \) to take care of UV and IR divergences. \( i\eta \) is an infinitesimal imaginary part, \( \epsilon > 0 \). It is convenient to define the functions

\[ F(r, p) := \left( \frac{r^2 + 2pr + i\eta}{q^2 + i\eta} \right)^{-\epsilon} - \left( \frac{r^2 + i\eta}{q^2 + i\eta} \right)^{-\epsilon}, \]

\[ G(r, p) := \frac{-q^2}{2pr} \left[ \left( \frac{r^2 + 2pr + i\eta}{q^2 + i\eta} \right)^{1-\epsilon} - \left( \frac{r^2 + i\eta}{q^2 + i\eta} \right)^{1-\epsilon} \right], \]

\[ H(r, p) := \left( \frac{-q^2}{2pr} \right)^2 \left[ \left( \frac{r^2 + 2pr + i\eta}{q^2 + i\eta} \right)^{2-\epsilon} - \left( \frac{r^2 + i\eta}{q^2 + i\eta} \right)^{2-\epsilon} \right]. \]

The calculation with Feynman parametrizations gives (compare [25])

\[ I_{20}(r) = \frac{i\pi^{-\epsilon} \Gamma(1 + \epsilon) \Gamma^2(1 - \epsilon)}{\epsilon^2 \Gamma(1 - 2\epsilon)} (-q^2 - i\eta)^{-\epsilon} (\epsilon + 2\epsilon^2) \left( \frac{r^2 + i\eta}{q^2 + i\eta} \right)^{-\epsilon} + O(\epsilon), \]

\[ I_{21\mu}(r) = \frac{i\pi^{-\epsilon} \Gamma(1 + \epsilon) \Gamma^2(1 - \epsilon)}{\epsilon^2 \Gamma(1 - 2\epsilon)} (-q^2 - i\eta)^{-\epsilon} (\frac{\epsilon}{2} + \epsilon^2) \left( \frac{r^2 + i\eta}{q^2 + i\eta} \right)^{-\epsilon} r_\mu + O(\epsilon), \]

\[ I_{30}(r, p) = \frac{i\pi^{-\epsilon} \Gamma(1 + \epsilon) \Gamma^2(1 - \epsilon)}{\epsilon^2 \Gamma(1 - 2\epsilon)} (-q^2 - i\eta)^{-\epsilon} \frac{1}{2pr} F(r, p) + O(\epsilon), \]

\[ I_{31\mu}(r, p) = \frac{i\pi^{-\epsilon} \Gamma(1 + \epsilon) \Gamma^2(1 - \epsilon)}{\epsilon^2 \Gamma(1 - 2\epsilon)} (-q^2 - i\eta)^{-\epsilon} \cdot \left\{ (1 + \epsilon + 2\epsilon^2) \frac{1}{2pr} F(r, p) r_\mu \right\} + O(\epsilon), \]

\[ I_{32\mu\nu}(r, p) = \frac{i\pi^{-\epsilon} \Gamma(1 + \epsilon) \Gamma^2(1 - \epsilon)}{\epsilon^2 \Gamma(1 - 2\epsilon)} (-q^2 - i\eta)^{-\epsilon} \left( 1 + \frac{3}{2} \epsilon + 3\epsilon^2 \right) \frac{1}{2pr} \]

\[ \cdot \left\{ -\frac{1}{4} (\epsilon + \frac{3}{2} \epsilon^2) 2pr G(r, p) g_{\mu\nu} + F(r, p) r_\mu r_\nu \right\} \]
\[ I_{40}(p_1, p_2, p_3) = \frac{r^2}{2pr} F(r, p) + (\epsilon + \epsilon^2) G(r, p) \left( p_\mu r_\nu + p_\nu r_\mu \right) + \mathcal{O}(\epsilon), \]  
(93)

\[ \left\{ 1 - \epsilon \left( l(y_{12}) + l(y_{13}) \right) \right\} + \epsilon^2 \left( \frac{1}{2} l^2(y_{12}) + \frac{1}{2} l^2(y_{13}) + R(y_{12}, y_{13}) \right) \} + \mathcal{O}(\epsilon). \]  
(94)

In \( I_{40} \) the momentum \( q \) is defined by \( q = p_1 + p_2 + p_3 \), and the invariants \( y_{ij} \) are given by \( y_{ij} := 2p_ip_j/q^2 \). The function \( R \) is given by

\[ R(x, y) = l(x) l(y) - l(1 - x) l(1 - y) - l(1 - y) - S(x) - S(y) + \zeta(2). \]  
(95)

\( l(x) \) is the natural logarithm with an additional prescription for arguments on the cut \([-\infty, 0] \]

\[ l(x) := \lim_{\eta \to 0} \ln(x + \text{sgn}(q^2)\text{sgn}(1 - x)i\eta), \]  
(96)

and \( S \) is defined by

\[ S := \lim_{\eta \to 0} \mathcal{L}_2(x + \text{sgn}(q^2)\text{sgn}(1 - x)i\eta), \]  
(97)

where \( \mathcal{L}_2 \) is the complex dilogarithm

\[ \mathcal{L}_2(z) = -\int_0^z du \frac{\ln(1 - u)}{u}. \]  
(98)

It can easily be seen where the \( i\eta \)-prescription is important. For \( q^2 > 0 \) it fixes the imaginary part of the factor \((-q^2 - i\eta)^{-\epsilon}\). Expanded up to \( \mathcal{O}(\epsilon^2) \) this gives the well known \( \pi^2 \)-terms (in combination with \( 1/\epsilon^2 \)-poles) in \( e^+e^- \)-annihilation and in the Drell-Yan process. In deeply inelastic scattering \((-q^2 - i\eta)^{-\epsilon} \) has no imaginary part. However, the functions \( F, G \) and \( H \) give rise to \( \pi^2 \)-terms in combination with poles \( 1/\epsilon, 1/\epsilon^2 \).

### B Virtual Corrections

In this appendix the explicit expressions for the virtual corrections are given. For the processes with an incoming quark the following invariants are defined:

\[ y_{qi} = \frac{z_q}{x_p} = \frac{1 - z_g}{x_p}, \]

\[ y_{ig} = \frac{1 - z_q}{x_p} = \frac{z_g}{x_p}, \]

\[ y_{qg} = \frac{1 - x_p}{x_p}, \]  
(99)
where the variables \( z_q \) and \( z_g \) are defined in eq. \((33)\) and \( x_p = x_\beta/\xi \). One then obtains

\[
E_{1,q} = \left[ -\frac{2}{e^2} + \frac{1}{e} (2 \ln y_{qi} - 3) \right] \cdot T_q
+ \left( -2\zeta(2) - \ln^2 y_{qi} - 8 \right) \cdot T_q
+ 4 \ln(y_{qi}) \left( \frac{2y_{qi}}{-y_{iq} + y_{ig}} + \frac{y_{qi}^2}{(-y_{iq} + y_{ig})^2} \right)
\]

\[
+ \ln(y_{gg}) \left( \frac{4y_{qi} - 2y_{gg}}{y_{qi} + y_{ig}} + \frac{y_{gg}y_{ig}}{(y_{qi} + y_{ig})^2} \right)
\]

\[
+ \ln(y_{ig}) \left( \frac{4y_{qi} + 2y_{ig}}{y_{qi} - y_{gg}} + \frac{y_{gg}y_{ig}}{(y_{qi} - y_{gg})^2} \right)
\]

\[
+ 2 \frac{y_{qi}^2 + (y_{qi} + y_{ig})^2}{y_{gg}y_{ig}} R'(y_{qi}, -y_{gg}) + 2 \frac{y_{qi}^2 + (y_{qi} - y_{gg})^2}{y_{gg}y_{ig}} R'(y_{qi}, y_{ig})
\]

\[
+ y_{qi} \left( \frac{4}{-y_{iq} + y_{ig}} + \frac{1}{y_{qi} - y_{gg}} + \frac{1}{y_{qi} + y_{ig}} \right)
\]

\[
+ \frac{y_{qi}}{y_{gg}} - \frac{y_{qi}}{y_{ig}} + \frac{y_{gg}}{y_{ig}} + \frac{y_{ig}}{y_{gg}}, \tag{100}
\]

\[
E_{2,q} = \left[ \frac{2}{e^2} + \frac{1}{e} (2 \ln y_{qi} - 2 \ln y_{gg} - 2 \ln y_{ig}) \right] \cdot T_q
+ \left( 2\zeta(2) - \ln^2 y_{qi} + \left( \ln^2 y_{gg} - \pi^2 \right) + \ln^2 y_{ig} + 2R'(-y_{gg}, y_{ig}) \right) \cdot T_q
- \left[ \ln(y_{gg}) \frac{2y_{gg}}{y_{qi} + y_{ig}} + \ln(y_{ig}) \frac{-2y_{ig}}{y_{qi} - y_{gg}} \right]
\]

\[
+ 4 \ln(y_{qi}) \left( \frac{y_{qi}^2}{(-y_{iq} + y_{ig})^2} + \frac{2y_{qi}}{-y_{iq} + y_{ig}} \right)
\]

\[
- 2 \left( \frac{y_{gg}}{y_{ig}} - \frac{y_{ig}}{y_{gg}} - \frac{y_{qi}}{y_{ig}} - \frac{y_{qi}}{y_{gg}} - \frac{2y_{qi}}{-y_{iq} + y_{ig}} \right)
\]

\[
+ 2R'(y_{qi}, -y_{gg}) \frac{y_{qi}^2 + (y_{qi} + y_{ig})^2}{y_{gg}y_{ig}}
\]

\[
+ 2R'(y_{qi}, y_{ig}) \frac{y_{qi}^2 + (y_{qi} - y_{gg})^2}{y_{gg}y_{ig}} \right]. \tag{101}
\]

For the processes with an incoming gluon the following variables are defined:

\[
y_{qi} = \frac{z_q}{x_p} = \frac{1 - z_{\overline{q}}}{x_p},
\]

\[
y_{\overline{q}i} = \frac{z_{\overline{q}}}{x_p} = \frac{1 - z_q}{x_p},
\]

\[
y_{qg} = \frac{1 - x_p}{x_p}. \tag{102}
\]
The variables \( z_q \) and \( z_{q'} \) are defined in eq. (33). For these processes one obtains

\[
E_{1,g} = \left[\frac{2}{e^2} + \frac{1}{e} (2 \ln y_{q\overline{q}} - 3)\right] \cdot T_g \\
+ \left(-2\zeta(2) - \left(\ln^2 y_{q\overline{q}} - \pi^2\right) - 8\right) \cdot T_g \\
+ 4 \ln(y_{q\overline{q}}) \left(-\frac{y_{q\overline{q}}}{y_{q_i} + y_{q_i}} + \frac{y_{q\overline{q}}^2}{(y_{q_i} + y_{q_i})^2}\right) \\
+ \ln(y_{q_i}) \left(-\frac{4y_{q\overline{q}} + 2y_{q_i}}{-y_{q\overline{q}} + y_{q_i}} - \frac{y_{q_i}y_{q_i}}{(-y_{q\overline{q}} + y_{q_i})^2}\right) \\
+ \ln(y_{q_i}) \left(-\frac{4y_{q\overline{q}} + 2y_{q_i}}{-y_{q\overline{q}} + y_{q_i}} - \frac{y_{q_i}y_{q_i}}{(-y_{q\overline{q}} + y_{q_i})^2}\right) \\
\frac{-2y_{q\overline{q}}^2 + (-y_{q\overline{q}} + y_{q_i})^2}{y_{q_i}y_{q_i}} R'(-y_{q\overline{q}}, y_{q_i}) \\
\frac{-2y_{q\overline{q}}^2 + (-y_{q\overline{q}} + y_{q_i})^2}{y_{q_i}y_{q_i}} R'(-y_{q\overline{q}}, y_{q_i}) \\
\frac{-y_{q\overline{q}}^2}{y_{q_i}^2} + \frac{y_{q\overline{q}}^2}{y_{q_i}} - \frac{y_{q_i}y_{q_i}}{y_{q_i}}. 
\] (103)

\[
E_{2,g} = \left[\frac{2}{e^2} + \frac{1}{e} (2 \ln y_{q\overline{q}} - 2 \ln y_{q_i} - 2 \ln y_{q_i})\right] \cdot T_g \\
+ \left(2\zeta(2) - \left(\ln^2 y_{q\overline{q}} - \pi^2\right) + \ln^2 y_{q_i} + \ln^2 y_{q_i} + 2 R'(y_{q_i}, y_{q_i})\right) \cdot T_g \\
+ 4 \ln(y_{q\overline{q}}) \left(-\frac{2y_{q_i}}{-y_{q\overline{q}} + y_{q_i}} + \frac{2y_{q_i}}{-y_{q\overline{q}} + y_{q_i}}\right) \\
+ \ln(y_{q_i}) \left(-\frac{2y_{q_i}}{-y_{q\overline{q}} + y_{q_i}} + \frac{2y_{q_i}}{-y_{q\overline{q}} + y_{q_i}}\right) \\
\frac{-2y_{q\overline{q}}^2 + (-y_{q\overline{q}} + y_{q_i})^2}{y_{q_i}y_{q_i}} R'(-y_{q\overline{q}}, y_{q_i}) \\
\frac{-2y_{q\overline{q}}^2 + (-y_{q\overline{q}} + y_{q_i})^2}{y_{q_i}y_{q_i}} R'(-y_{q\overline{q}}, y_{q_i}) \\
\frac{-y_{q\overline{q}}^2}{y_{q_i}^2} + \frac{y_{q\overline{q}}^2}{y_{q_i}} - \frac{y_{q_i}y_{q_i}}{y_{q_i}}. 
\] (104)

The function \( R' \) is given by

\[
R'(x, y) = \ln |x| \ln |y| - \ln |x| \ln |1 - x| - \ln |y| \ln |1 - y| \\
- \lim_{\eta \to 0} \text{Re } (\mathcal{L}_2(x + i\eta) + \mathcal{L}_2(y + i\eta)) + \zeta(2). 
\] (105)
C Factorized Integration Kernels for Final State Singularities

In this appendix the results for the singular kernels from the final state singularities are summarised. It is convenient to add additional indices to the integration variables \( r, z \) and \( b \). Let

\[
z_j := \frac{p_0 p_j}{p_0 q},
\]

(106)

where \( p_0 \) is the momentum of the incoming parton, and

\[
r_{jk} := \frac{s_{jk}}{S_{ij} y \xi}.
\]

(107)

If the integration variable \( b \) is fixed in the \((p_j, p_k)\) CM system, let

\[
b_j := \frac{1}{2}(1 - \cos \theta_j), \quad \theta_j := \angle(\vec{p_j}, \vec{p_0}).
\]

(108)

Let \( T_{q/g}(x_p, z_j) \) be the Born term with incoming quark/gluon, expressed in the variables \( x_p = x_0/\xi \) and \( z_j \). For the singular matrix elements one obtains (including the average over colour degrees of freedom for incoming partons and the symmetry factor for identical particles in the final state)

\[
\text{tr} H_{F_1} = L_1 8\pi^2 \frac{\alpha_s}{2\pi} \mu^{2\varepsilon} 16(1 - \epsilon) C_F^2 Q^2 \frac{x_p}{Q^2}
\]

\[
\frac{1}{r_{qq}} \left[ (1 - b_q)(1 - \epsilon) \right] \quad \cdot \quad \frac{1}{1 - x_p - r_{gg}} \left[ \frac{1}{r_{gg} + (1 - z_q)(1 - b_q)} \right] T_q(x_p, z_q),
\]

(109)

\[
\text{tr} H_{F_2} = L_1 8\pi^2 \frac{\alpha_s}{2\pi} \mu^{2\varepsilon} (-16)(1 - \epsilon) N_C C_F Q^2 \frac{x_p}{Q^2}
\]

\[
\frac{1}{r_{qq}} \left[ \frac{1 - z_q}{r_{gg} + (1 - z_q)(1 - b_q)} \right] \quad \cdot \quad \frac{1}{1 - x_p - r_{gg}} \left[ \frac{1}{r_{gg} + (1 - z_q)(1 - b_q)} \right] T_q(x_p, z_q),
\]

(110)

\[
\text{tr} H_{F_3} = L_1 8\pi^2 \frac{\alpha_s}{2\pi} \mu^{2\varepsilon} 8(1 - \epsilon) N_C C_F Q^2 \frac{x_p}{Q^2}
\]

\[
\frac{1}{r_{qq}} \left[ \frac{1 - x_p - r_{gg}}{r_{gg} + (1 - x_p - r_{gg})} \right] \quad + \quad \frac{1}{1 - x_p - r_{gg}} \left[ \frac{1 - x_p - r_{gg}}{r_{gg} + (1 - z_q)(1 - b_q)} \right] \quad + \quad \frac{1}{r_{gg} + (1 - x_p - r_{gg})(1 - b_q)} \quad - \quad 2 \right] T_q(x_p, z_q),
\]

(111)

\[
\text{tr} H_{F_4} = L_1 8\pi^2 \frac{\alpha_s}{2\pi} \mu^{2\varepsilon} (-16)(1 - \epsilon) N_C C_F Q^2 \frac{x_p}{Q^2}
\]

\[
\frac{1}{r_{qq}} \left[ \frac{1 - 2b_q(1 - b_q)(1 + \epsilon)}{r_{gg} + (1 - b_q)} \right] T_q(x_p, z_q)
\]

+ terms \( \sim \left( 1 - 2(1 - \epsilon) \cos^2 \varphi \right) \),

(112)

\[
\text{tr} H_{F_5} = L_1 8\pi^2 \frac{\alpha_s}{2\pi} \mu^{2\varepsilon} 8(1 - \epsilon) C_F Q^2 \frac{x_p}{Q^2}
\]

\[
\frac{1}{r_{qq}} \left[ \frac{1 - 2b_q(1 - b_q)(1 + \epsilon)}{r_{gg} + (1 - b_q)} \right] T_q(x_p, z_q)
\]

+ terms \( \sim \left( 1 - 2(1 - \epsilon) \cos^2 \varphi \right) \),

(113)

\[
\text{tr} H_{F_6} = L_1 8\pi^2 \frac{\alpha_s}{2\pi} \mu^{2\varepsilon} 8(1 - \epsilon) C_F Q^2 \frac{x_p}{Q^2}
\]

29
\[
\frac{1}{rrg} \left[ (1 - b_q)(1 - \epsilon) - 2 + 2 \frac{1 - x_p - r_qg}{r_qg + (1 - x_p - r_qg)(1 - b_q)} \right] T_g(x_p, z_q)
\]
\[+(q \leftrightarrow \bar{q}), \quad (114)\]
\[\text{tr} H_F = L_1 8\pi^2 \frac{\alpha_s}{2\pi} 2 \epsilon (-8)(1 - \epsilon) N_C Q^2 \frac{x_p}{Q^2}
\]
\[
\frac{1}{rrg} \left[ 1 - x_p - r_qg \right] \left[ r_qg + (1 - x_p - r_qg)(1 - b_q) \right] - \frac{1 - z_q}{r_qg + (1 - z_q)(1 - b_q)} \right] T_g(x_p, z_q)
\]
\[+(q \leftrightarrow \bar{q}). \quad (115)\]

In this factorization terms that vanish (after the integration) for \(c \to 0\) are neglected. Therefore the cut should not be too large in the numerical evaluation. For small values of the cut the cross section is dominated by terms \(\sim (-\ln^2 c)\).

## D Phase Space Integrals for Final State Singularities

In this appendix the results for the real corrections of the terms involving final state singularities are collected. A measure \(d\mu_F\) is defined by

\[
\int d\mu_F := \int_0^\alpha dr^{-\epsilon} \left[ 1 - \frac{r}{h} \right]^{-\epsilon} \int_0^1 db \frac{b^{-\epsilon}(1 - b)}{N_b} \int_0^\pi d\varphi \frac{\sin^{-2\epsilon} \varphi}{N_\varphi}.
\]  

Integrals of terms with singularities for \(r \to 0\), \(b \to 0\) are needed in the integrations over the singular region of phase space. \(N_b\) and \(N_\varphi\) are normalisation constants from the phase space, and \(h\) is an arbitrary parameter which does not show up in the results up to order \(O(\epsilon^0)\). The upper limit \(\alpha\) of the \(r\)-integration has the meaning of a jet cut. The integrals that were needed were solved by use of \([37, 38, 39, 40]\) and are of the following type:

\[
f_1(y) := \int d\mu_F \frac{y}{r} \frac{r + y}{yb} = \frac{1}{2\epsilon^2} + \frac{1}{\epsilon} \left( -1 - \frac{1}{2} \ln(y) \right) + \ln y - \frac{1}{2} \ln^2 \frac{\alpha}{y} + \frac{1}{4} \ln^2 y
\]
\[-S \left( -\frac{\alpha}{y} \right) - \zeta(2) + O(\epsilon), \quad (117)\]
\[
f_2(y) := \int d\mu_F \frac{y - r}{r} \frac{r + (y - r)b}{b} = \frac{1}{2\epsilon^2} + \frac{1}{\epsilon} \left( -1 - \frac{1}{2} \ln(y) \right) + \ln y - \frac{1}{2} \ln^2 \frac{\alpha}{y} + \frac{1}{4} \ln^2 y - \zeta(2) + O(\epsilon), \quad (118)\]
\[
f_3 := \int d\mu_F \frac{1}{r(1 - b)(1 - \epsilon)} = -\frac{1}{2\epsilon} + \frac{1}{2} + \frac{1}{2} \ln \alpha + O(\epsilon), \quad (119)\]
\[
f_4 := \int d\mu_F \frac{1}{r} = -\frac{1}{\epsilon} + \ln \alpha + O(\epsilon), \quad (120)\]
\[ f_5 := \int d\mu_F \frac{1}{r} (1 - b + b^2) \]
\[ = -\frac{51}{6\epsilon} + \frac{5}{6} \ln \alpha - \frac{1}{18} + \mathcal{O}(\epsilon), \quad (121) \]
\[ f_6 := \int d\mu_F \frac{1}{r} b(1 - b)(1 + \epsilon) \]
\[ = -\frac{11}{6\epsilon} + \frac{1}{9} \ln \alpha + \mathcal{O}(\epsilon). \quad (122) \]

The function \( f_1(y) \) can be checked against the result in [27].

\section*{E Real Corrections, Final State Singularities}

By means of the basic integrals from appendix \[ \text{[3]} \] one obtains the explicit expressions for the final state real corrections:

\[ F_1 = C_F^2 Q_T^2 T_q(x_p, z_g) \]
\[ \cdot \left\{ \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \left( \frac{3}{2} - \ln \frac{1 - z_g}{x_p} \right) + \frac{7}{2} - \frac{3}{2} \ln \frac{\alpha}{x_p} - \ln x_p \ln (1 - z_g) \right. \]
\[ + \frac{1}{2} \ln^2 x_p - \ln^2 \frac{\alpha}{1 - z_g} + \left. \frac{1}{2} \ln^2 (1 - z_g) - 2S \left( -\frac{\alpha}{1 - z_g} \right) - 2\zeta(2) \right\} + \mathcal{O}(\epsilon), \quad (123) \]
\[ F_2 = -\frac{1}{2} N_C C_F Q_T^2 T_q(x_p, z_g) \]
\[ \cdot \left\{ \frac{1}{\epsilon} \ln \frac{1 - x_p}{1 - z_g} + \ln x_p \ln \frac{1 - x_p}{1 - z_g} + \ln^2 \frac{\alpha}{1 - x_p} - \ln^2 \frac{\alpha}{1 - z_g} \right. \]
\[ + \left. \frac{1}{2} \left( \ln^2 (1 - z_g) - \ln^2 (1 - x_p) \right) - 2S \left( -\frac{\alpha}{1 - z_g} \right) \right\} + \mathcal{O}(\epsilon), \quad (124) \]
\[ F_3 = -\frac{1}{2} N_C C_F Q_T^2 T_q(x_p, z_q) \]
\[ \cdot \left\{ -\frac{2}{\epsilon^2} + \frac{1}{\epsilon} \left( -2 - \ln \frac{x_p^2}{(1 - z_g)(1 - x_p)} \right) - 4 + 2 \ln \frac{\alpha}{x_p} \right. \]
\[ + \ln x_p \ln ((1 - z_g)(1 - x_p)) - \ln^2 x_p + \ln^2 \frac{\alpha}{1 - x_p} + \ln^2 \frac{\alpha}{1 - z_q} \]
\[ - \frac{1}{2} \left( \ln^2 (1 - x_p) + \ln^2 (1 - z_q) \right) + 4\zeta(2) + 2S \left( -\frac{\alpha}{1 - z_q} \right) \right\} + \mathcal{O}(\epsilon), \quad (125) \]
\[ F_4 = -\frac{1}{2} N_C C_F Q_T^2 T_q(x_p, z_q) \]
\[ \cdot \left\{ -\frac{51}{3\epsilon} + \frac{5}{3} \ln \frac{\alpha}{x_p} - \frac{31}{9} \right\} + \mathcal{O}(\epsilon), \quad (126) \]
\[ F_5 = C_F Q_T^2 T_q(x_p, z_q) \]
\[ \cdot \left\{ -\frac{11}{3\epsilon} + \frac{5}{9} + \frac{1}{3} \ln \frac{\alpha}{x_p} \right\} + \mathcal{O}(\epsilon), \quad (127) \]
\[ F_6 = \frac{1}{2} C_F Q_j^2 T_g(x_p, z_q) \]
\[ \cdot \left\{ \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \left( \frac{3}{2} - \ln \frac{1-x_p}{x_p} \right) + \frac{7}{2} - \frac{3}{2} \ln \frac{\alpha}{x_p} - \ln x_p \ln(1-x_p) \right. \]
\[ + \frac{1}{2} \ln^2 x_p + \frac{1}{2} \ln^2(1-x_p) - \ln^2 \frac{\alpha}{1-x_p} - 2\zeta(2) \left. \right\} + \mathcal{O}(\epsilon) \]
\[ + (q \leftarrow q), \quad (128) \]
\[ F_7 = -\frac{1}{4} N_C Q_j^2 T_g(x_p, z_q) \]
\[ \cdot \left\{ \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \left( \frac{3}{2} - \ln \frac{1-z_q}{x_p} \right) + \frac{7}{2} - \frac{3}{2} \ln \frac{\alpha}{1-z_q} - \ln x_p \ln(1-x_p) \right. \]
\[ + \frac{1}{2} \ln^2 x_p + \frac{1}{2} \ln^2(1-x_p) - \ln^2 \frac{\alpha}{1-x_p} - 2\zeta(2) \left. \right\} + \mathcal{O}(\epsilon) \]
\[ + (q \leftarrow q). \quad (129) \]

**F  Factorized Integration Kernels for Initial State Singularities**

This section contains the singular kernels from the factorization of the matrix elements with initial state singularities. Let \( T_{qg}(t_{jk}, z_l) \) be the Born term with incoming quark/gluon, expressed in the variables \( t_{jk} \) and \( z_l \). The singular matrix elements then read

\[ \text{tr} H_1 = L_1 8\pi^2 \frac{\alpha_s}{2\pi} \mu^2 16(1-\epsilon) C_F \frac{1}{W^2} Q_j^2 \]
\[ \cdot \frac{1}{z^l} \left[ \frac{1}{-(1-\nu-t_{qg})} - \frac{1}{-(1-\nu-t_{qg}) + (1-t_{qg})\sigma} \right] (1-\epsilon) \]
\[ + 2\frac{z_q}{-(1-\nu-t_{qg})z^l + (1-t_{qg})z_q\sigma} T_{qg}(t_{qg}, z_q), \quad (130) \]

\[ \text{tr} H_2 = L_1 8\pi^2 \frac{\alpha_s}{2\pi} \mu^2 (-16)(1-\epsilon) N_C C_F \frac{1}{W^2} Q_j^2 \]
\[ \cdot \frac{1}{z^l} \left[ \frac{z_q}{-(1-\nu-t_{qg})z^l + (1-t_{qg})z_q\sigma} \right. \]
\[ \left. - \frac{1}{1-z_q} -(1-\nu-t_{qg})z^l + (1-t_{qg})(1-z_q)\sigma \right] T_{qg}(t_{qg}, z_q), \quad (131) \]

\[ \text{tr} H_3 = L_1 8\pi^2 \frac{\alpha_s}{2\pi} \mu^2 Q_j^2 \]
\[ \cdot \left\{ 8(1-\epsilon) C_F \frac{1}{W^2} T_{qg}(t_{qg}, z_q)(1-\epsilon) \right. \]
\[ \left. + \frac{1}{z^l} \left[ \frac{1}{-(1-\nu-t_{qg})} + \frac{2(1-t_{qg})\sigma}{(1-\nu-t_{qg})^2(1+\epsilon)} \right] - 16 C_F \frac{1}{W^2} z_q^l(1-\nu-t_{qg})^2 z_q(1-z_q) \left( 1 - 2(1-\epsilon) \cos^2 \varphi \right) \right\}, \quad (132) \]

\[ \text{tr} H_4 = L_1 8\pi^2 \frac{\alpha_s}{2\pi} \mu^2 8(1-\epsilon) C_F \frac{1}{W^2} Q_j^2 \frac{1}{1-\epsilon} \]
\[
\frac{1}{z'} \left[ -\frac{1}{(1 - \nu - t_{qg})} (1 - \epsilon) - 2 \frac{\sigma(1 - t_{qg})}{(-(1 - \nu - t_{qg}) + (1 - t_{qg}) \sigma)^2} \right] T_g(t_{qg}, z_q) \\
+ (q \leftrightarrow \bar{q}),
\]

\[
\text{tr} H_{I_5} = 0,
\] (133)

\[
\text{tr} H_{I_6} = L_1 8 \pi^2 \frac{\alpha_s}{2 \pi} \mu^{2\epsilon} 8(1 - \epsilon) N_C \frac{1}{W^2} Q_j^2 \\
\left\{ \frac{1}{z'} \left[ \frac{1 - z_q}{(1 - \nu - t_{qg}) z' + (1 - t_{qg})(1 - z_q) \sigma} + \frac{z_q}{(1 - \nu - t_{qg}) z' + (1 - t_{qg}) z_q \sigma} \\
- 2 \frac{(-1 - \nu - t_{qg}) + (1 - t_{qg}) \sigma}{(-1 - \nu - t_{qg}) + (1 - t_{qg}) \sigma} \right] T_g(t_{qg}, z_q), \right\}
\] (134)

\[
\text{tr} H_{I_7} = L_1 8 \pi^2 \frac{\alpha_s}{2 \pi} \mu^{2\epsilon} Q_j^2 \\
\left\{ 16(1 - \epsilon) N_C \frac{1}{W^2} T_g(t_{qg}, z_q) \frac{1}{z'} \left[ \frac{1}{-(1 - \nu - t_{qg}) + (1 - t_{qg}) \sigma} + \frac{(1 - t_{qg}) \sigma}{(1 - \nu - t_{qg})^2} \right] \\
- 16 N_C \frac{1}{W^2} z' \left\{ (1 - \nu - t_{qg})^2 z_q \right. \left[ 1 - 2(1 - \epsilon) \cos^2 \varphi \right] \right\}.
\] (135)

In these formulae the terms proportional to \(1 - 2(1 - \epsilon) \cos^2 \varphi\) are stated explicitly, since for invariant mass cuts the integration over the azimuthal angle is (in general) not over the range \([0, \pi]\), and as a consequence

\[
\int d\varphi \sin^{-2\epsilon} \varphi (1 - 2(1 - \epsilon) \cos^2 \varphi)
\] (136)

does not vanish. Additional factors of \((1 - \epsilon)\) and \(1/\epsilon\) in \(\text{tr} H_{I_5}\) and \(\text{tr} H_{I_4}\), respectively, are due to the fact that in \(d\) dimensions a gluon has \((d - 2)\) (physical) helicity states, whereas a quark has only 2, and the incoming particle in the \((3+1)\) jet Born term is not the same as that in the factorized \((2+1)\) jet Born term.

### G Phase Space Integrals for Initial State Singularities

Here the results for the integrals needed for the real corrections with initial state singularities are stated. A measure \(d\mu_I\) can be defined by

\[
\int d\mu_I := \int_0^\beta d' z'^{-\epsilon} \int_0^1 d\sigma^{-\epsilon} \int_0^\pi d\varphi \sin^{-2\epsilon} \varphi \Gamma(1 - 2\epsilon) a \left( \frac{1 - x_\mu}{x_\mu} \right)^{-\epsilon} (1 - t)^{1-\epsilon}.
\] (137)

For an arbitrary \(C^\infty\)-function \(g : [0, 1] \to C\) let

\[
F(\sigma)[g] := \int_0^1 \! d\sigma F(\sigma) g(\sigma).
\] (138)
Then the following integrals are given by
\[ i_1(y) := \int d\mu g(\sigma) \frac{y}{y' + y\sigma} \]
\[ = a(1-t) \left\{ \left( \frac{1}{2\epsilon^2} + \frac{1}{\epsilon} \left( -\frac{1}{2} \ln \frac{(1-a)y}{x_B} + \frac{1}{4} \ln^2 \frac{(1-a)y}{x_B} \right. \right. \right. \]
\[ \left. \left. \left. - \frac{1}{2} \ln^2 \frac{\beta}{y} - S \left( \frac{-\beta}{y} \right) \right) \delta_0[g] \right. \right. \right. \]
\[ + \left\{ - \frac{1}{\epsilon} + \ln \frac{(1-a)y}{x_B} \right\} \left( \frac{1}{\sigma} \right) [g] + 2 \left( \frac{\ln \sigma}{\sigma} \right) + [g] \]
\[ - \left( \frac{\ln \left( 1 + \frac{\alpha^2}{\beta^2} \right)}{\sigma} \right)[g] + \mathcal{O}(\epsilon), \]  
(140)

\[ i_2 := \int d\mu g(\sigma) \frac{y}{y' + y\sigma} \]
\[ = a(1-t) \left\{ - \frac{1}{\epsilon} 1[g] + \left( \frac{\ln \frac{(1-a)\beta\sigma}{x_B}}{x_B} \right) [g] \right\} + \mathcal{O}(\epsilon), \]  
(141)

\[ i_3(y) := \int d\mu g(\sigma) \frac{y}{y' + y\sigma} \]
\[ = a(1-t) \left\{ - \frac{1}{\epsilon} \left( \frac{y}{1 + y\sigma} \right) [g] + \left( \frac{y}{1 + y\sigma} \ln \frac{(1-a)\beta\sigma}{x_B} \right) [g] \right\} + \mathcal{O}(\epsilon), \]  
(142)

\[ i_4 := \int d\mu g(\sigma) \frac{y^2}{y' + y^2} \]
\[ = a(1-t) \left\{ - \frac{1}{\epsilon} \sigma [g] + \left( \sigma \ln \frac{(1-a)\beta\sigma}{x_B} \right) [g] \right\} + \mathcal{O}(\epsilon), \]  
(143)

\[ i_5(y) := \int d\mu g(\sigma) \frac{y^2}{y' + y^2} \]
\[ = a(1-t) \left\{ - \frac{1}{\epsilon} \left( \frac{y^2}{1 + y^2} \right) [g] + \left( \frac{y^2}{1 + y^2} \ln \frac{(1-a)\beta\sigma}{x_B} \right) [g] \right\} + \mathcal{O}(\epsilon), \]  
(144)

\[ i_6(y) := \int d\mu g(\sigma) \frac{y^2}{y' + y^2} \]
\[ = a(1-t) \left\{ - \frac{1}{\epsilon} \left( \frac{y^2}{1 + y^2} \right) [g] + \left( \frac{y^2}{1 + y^2} \ln \frac{(1-a)\beta\sigma}{x_B} \right) [g] \right\} + \mathcal{O}(\epsilon). \]  
(145)

Here the distributions
\[ \delta_0[g] := g(0), \quad 1[g] := \int_0^1 d\sigma g(\sigma) \]  
(146)

have been used.

The “+”-prescription for the \(\sigma\)-integration is defined by
\[ \int_0^1 d\sigma D_+(\sigma) g(\sigma) := \int_0^1 d\sigma D(\sigma) (g(\sigma) - g(0)). \]  
(147)
The explicit expressions of the sum of the (2+1) and (3+1) jet contributions from the initial state singularities read

\[ I_1 = Q_j^2 T_q(t_{qq}, z_q) \]
\[ \cdot \left\{ \left( \frac{Q^2}{M_f^2} \right)^\epsilon \left( -\frac{1}{\epsilon} + \log \frac{Q^2}{M_f^2} \right) P_{q\rightarrow q}(u) C_F \right. \]
\[ + C_F \left[ \frac{1}{\epsilon^2} \delta(1-u) + \frac{1}{\epsilon} \left( -\ln \frac{a z_q}{x_q} \delta(1-u) + \frac{3}{2} \delta(1-u) \right) \right. \]
\[ + S_1 \left( \frac{a \beta}{x_q}, \frac{a z_q}{x_q}, \frac{z_q}{\beta} \right) \left\} + O(\epsilon), \right. \]
\[ (148) \]

\[ I_2 = Q_j^2 T_q(t_{qg}, z_q) \cdot \left( -\frac{1}{2} N_C C_F \right) \]
\[ \cdot \left\{ \frac{1}{\epsilon} \ln \frac{1 - z_q}{z_q} \delta(1-u) + S_2 \left( \frac{a z_q}{x_q}, \frac{a(1 - z_q)}{x_q}, \frac{z_q}{\beta}, \frac{1 - z_q}{\beta} \right) \right\} + O(\epsilon), \]
\[ (149) \]

\[ I_3 = Q_j^2 T_q(t_{qg}, z_q) \cdot \left( -\frac{1}{2} N_C C_F \right) \]
\[ \cdot \left\{ \left( \frac{Q^2}{M_f^2} \right)^\epsilon \left( -\frac{1}{\epsilon} + \log \frac{Q^2}{M_f^2} + 1 \right) P_{g\rightarrow q}(u) \cdot \frac{1}{2} \right. \]
\[ + \frac{1}{2} C_F S_3 \left( \frac{a \beta}{x_q} \right) \left\} + O(\epsilon), \right. \]
\[ (150) \]

\[ I_4 = Q_j^2 T_q(t_{qq}, z_q) \]
\[ \cdot \left\{ \left( \frac{Q^2}{M_f^2} \right)^\epsilon \left( -\frac{1}{\epsilon} + \log \frac{Q^2}{M_f^2} - 1 \right) P_{q\rightarrow g}(u) C_F \right. \]
\[ + C_F S_4 \left( \frac{a \beta}{x_q} \right) \left\} + (q \leftrightarrow g) + O(\epsilon), \right. \]
\[ (151) \]

\[ I_5 = O(\epsilon), \]
\[ (152) \]

\[ I_6 + I_7 = Q_j^2 T_q(t_{q\bar{q}}, z_q) \]
\[ \cdot \left\{ \left( \frac{Q^2}{M_f^2} \right)^\epsilon \left( -\frac{1}{\epsilon} + \log \frac{Q^2}{M_f^2} \right) P_{g\rightarrow q}(u) \cdot \frac{1}{2} \right. \]
\[ + \left( -\frac{1}{4} N_C \right) \left[ \frac{1}{\epsilon^2} \delta(1-u) \right. \]
\[ + \frac{1}{\epsilon} \left( \ln \frac{a z_q(1 - z_q)}{x_q^2} \delta(1-u) - \frac{2}{N_C} \left( 11 \cdot 6 N_C - \frac{1}{3} N_F \right) \delta(1-u) \right) \]
\[ + S_6 \left( \frac{a z_q}{x_q}, \frac{a(1 - z_q)}{x_q}, \frac{z_q}{\beta}, \frac{1 - z_q}{\beta}, \frac{a \beta}{x_q} \right) \left\} + O(\epsilon), \right. \]
\[ (153) \]

The functions \( P_{B\rightarrow A} \) are the Altarelli-Parisi kernels and the functions \( S_i \) are defined by

\[ S_1(A, B, C) = R_2 + R_5(A) - R_6(A) + \left( \frac{1}{2} \ln^2 B + 6 \zeta(2) \right) R_1 \]
\[ + 2 R_3(B) - 2 R_4(C), \]
\[ (154) \]
\[ S_2(A, B, C, D) = R_1 \left( \frac{1}{2} \ln^2 A - \frac{1}{2} \ln^2 B \right) + 2 \left( R_3(A) - R_3(B) \right) \]
\[ -2 \left( R_4(C) - R_4(D) \right), \]  
(155)

\[ S_3(A) = R_6(A) + 2R_7(A) - 2R_{10}, \]  
(156)

\[ S_4(A) = \frac{1}{2} R_0 + \frac{1}{2} R_5(A) - R_8(A), \]  
(157)

\[ S_6(A, B, C, D, E) = -R_1 \left( \frac{1}{2} \ln^2 A + \frac{1}{2} \ln^2 B + 12\zeta(2) \right) - 2 \left( R_3(A) + R_3(B) \right) \]
\[ + 2 \left( R_4(C) + R_4(D) \right) + 4R_9(E) - 4R_6(E) - 4R_7(E), \]  
(158)

and the functions \( R_i \) are distributions in the variable \( u \) given by

\[ R_0 = 1, \]  
(159)

\[ R_1 = \delta(1 - u), \]  
(160)

\[ R_2 = 1 - u, \]  
(161)

\[ R_3(\lambda) = \frac{\left( \ln \left[ \lambda \left( \frac{1}{u} \right) \right]^2 \right)}{1 - u} - \ln \left( \lambda \left( \frac{1}{u} \right) \right), \]  
(162)

\[ R_4(\lambda) = \frac{u}{1 - u} \ln \left( 1 + \lambda \frac{1 - u}{u} \right), \]  
(163)

\[ R_5(\lambda) = \ln \left( \lambda \frac{1 - u}{u} \right), \]  
(164)

\[ R_6(\lambda) = u \ln \left( \lambda \frac{1 - u}{u} \right), \]  
(165)

\[ R_7(\lambda) = \frac{1 - u}{u} \ln \left( \lambda \frac{1 - u}{u} \right), \]  
(166)

\[ R_8(\lambda) = u(1 - u) \ln \left( \lambda \frac{1 - u}{u} \right), \]  
(167)

\[ R_9(\lambda) = u^2 \ln \left( \lambda \frac{1 - u}{u} \right), \]  
(168)

\[ R_{10}(\lambda) = \frac{1 - u}{u}. \]  
(169)
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Figure Captions:

Fig. 1 Diagram for initial state radiation.

Fig. 2 Diagram for 1-parton production.

Fig. 3 Diagram for the virtual correction to 1-parton production.

Fig. 4 Generic diagrams for 2-parton production.

Fig. 5 Born terms for 2-parton production (hadron tensor).

Fig. 6 Virtual corrections to 2-parton production (hadron tensor).

Fig. 7 Diagrams contributing to the wave function renormalisation.

Fig. 8 Born terms of $\mathcal{O}(\alpha_s^2)$ (the roman numbering labels the 7 different colour classes).

Fig. 9 Generic diagrams for 3-parton production.

Fig. 10 CM frame of $p_1$ and $p_2$.

Fig. 11 (2+1) and (3+1) jet regions in phase space.

Fig. 12 Cut dependence of jet cross sections. “tr.” stands for transverse, “long.” for longitudinal polarisation of the exchanged virtual photon. Contributions marked as “NLO” are given by the sum of the Born term and the next-to-leading order contribution. (a)–(c) HERA kinematics with (a) $3.16 \text{ GeV} < Q < 10 \text{ GeV}$, (b) $10 \text{ GeV} < Q < 31.6 \text{ GeV}$, (c) $31.6 \text{ GeV} < Q < 100 \text{ GeV}$. (d) is the graph for the E665 kinematics.

Fig. 13 The ratio $\sigma_{2+1}/\sigma_{tot}$. (a)–(c) HERA kinematics with (a) $3.16 \text{ GeV} < Q < 10 \text{ GeV}$, (b) $10 \text{ GeV} < Q < 31.6 \text{ GeV}$, (c) $31.6 \text{ GeV} < Q < 100 \text{ GeV}$, (d) E665 kinematics.

Fig. 14 Dependence of the cross section on the jet definition scheme. The jet definition scheme is (a) $s_{ij} \gtrsim cW^2$, (b) $s_{ij} \gtrsim c \left( W^\alpha Q^\beta \sqrt{\mu_1 y_1 - \alpha - \beta} \right)^2$ with $\alpha = 0.8$, $\beta = 0.2$.

Fig. 15 Dependence on the renormalisation and factorization scales. (a) renormalisation scale $\mu = \rho Q$, factorization scale $M_f = Q$; (b) $\mu = Q$, $M_f = \rho Q$; (c) $\mu = \rho Q$, $M_f = \rho Q$. (d)–(f) are the same, but for much larger Q. (g)–(i) are the graphs for E665 energies. To avoid scales that are too low for perturbation theory to be valid, the scales are clipped at a lower bound of 2 GeV (HERA) and 1 GeV (E665).

Fig. 16 Dependence of the jet rate $\sigma_{2+1}/\sigma_{tot}$ on the renormalisation and factorization scales. The indices (a)–(i) refer to the indices of Fig. 12 (a)–(i), respectively.

Fig. 17 Dependence on the parton densities. The plotted cross section is $x_\beta d\sigma/dx_\beta$, the jet definition scheme is (a) $s_{ij} \gtrsim cW^2$, (b) $s_{ij} \gtrsim c \left( W^\alpha Q^\beta \sqrt{\mu_1 y_1 - \alpha - \beta} \right)^2$ with $\alpha = 0.7$, $\beta = 0.3$. 

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Figure 1:

Figure 2:

Figure 3:
Figure 4:

Figure 5:

Figure 6:
Figure 7:

Figure 8:
