Another Realization of Kerr/CFT Correspondence

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Abstract

We study another realization of the Kerr/CFT correspondence. By imposing new asymptotic conditions for the near horizon geometry of Kerr black hole, an asymptotic symmetry which contains all of the exact isometries can be obtained. In particular, the Virasoro algebra can be realized as an enhancement of $SL(2,\mathbb{R})$ symmetry of the AdS geometry. By using this asymptotic symmetry, we discuss finite temperature effects and show the correspondence concretely.

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1 Introduction

Recently, the correspondence between the Kerr black hole and conformal field theory was studied \[1\]. They investigated the near horizon geometry which had $SL(2, \mathbb{R}) \times U(1)$ isometries \[2\], and considered the asymptotic symmetry following the work by Brown and Henneaux \[3\]. The Virasoro algebra was realized from an enhancement of the rotational $U(1)$ isometry, not from the $SL(2, \mathbb{R})$. It is natural to ask if there exists another symmetry promoted from the $SL(2, \mathbb{R})$ symmetry of the near horizon geometry. This symmetry is likely relevant to the deviation from the extremality.

In this paper, we investigate another realization of an asymptotic symmetry. Imposing new asymptotic condition which is stronger than that in \[1\], we obtain an asymptotic symmetry which contains all of the exact isometries. The $SL(2, \mathbb{R})$ isometry could be enhanced to the Virasoro algebra. We define the energy-momentum tensor following the studies on the quasi-local energy \[4\]. Then we calculate the finite temperature effect for the mass and the angular momentum by using this symmetry and show agreements with those of the near extremal Kerr black hole. The entropy can be also discussed by using the Cardy formula.

2 Kerr Black Hole

We start by introducing the Kerr metric:

$$
\begin{align*}
\text{d}s^2 &= -\text{d}t^2 + \frac{2mr}{r^2 + a^2 \cos^2 \theta} (\text{d}t - a \sin^2 \theta \text{d}\phi)^2 + (r^2 + a^2 \sin^2 \theta \text{d}\phi^2 \\
&\quad + \frac{r^2 + a^2 \cos^2 \theta}{r^2 - 2mr + a^2} \text{d}r^2 + (r^2 + a^2 \cos^2 \theta) \text{d}\theta^2,
\end{align*}
$$

(2.1)

where we have used Boyer-Lindquist coordinate. The parameters $m$ and $a$ are related to the ADM mass and the angular momentum as

$$
M = \frac{m}{G_N}, \quad J = \frac{am}{G_N},
$$

(2.2)

The position of the horizon and the Hawking temperature are given by

$$
\begin{align*}
r_\pm &= m \pm \sqrt{m^2 - a^2}, \\
T_H &= \frac{r_+ - m}{4\pi mr_+}.
\end{align*}
$$

(2.3)

We consider the near horizon geometry of the Kerr geometry. We define new coordinates

$$
\begin{align*}
t &= 2\epsilon^{-1} a \hat{t}, \\
r &= a (1 + \epsilon \hat{r}), \\
\phi &= \hat{\phi} + \frac{t}{2a},
\end{align*}
$$

(2.4)
and take the limit of $\epsilon \to 0$ to obtain the near horizon geometry. For extremal case $a = m$, the near horizon geometry becomes

$$ds^2 = -f_0(\theta)\hat{r}^2d\hat{t}^2 + f_0(\theta)\frac{d\hat{r}^2}{\hat{r}^2} + f_\phi(\theta)\left(d\hat{\phi} + k \hat{r} d\hat{t}\right)^2 + f_\theta(\theta)d\theta^2,$$

(2.5)

with

$$f_0(\theta) = f_\theta(\theta) = a^2 \left(1 + \cos^2 \theta\right), \quad f_\phi(\theta) = \frac{4a^2 \sin^2 \theta}{1 + \cos^2 \theta}, \quad k = 1.$$  

(2.6)

Hereafter, we consider this near horizon geometry and omit “^” of the coordinates. The near horizon geometry has $SL(2, \mathbb{R}) \times U(1)$ isometries generated by the following four Killing vectors:

$$\xi_1 = \partial_t, \quad \xi_0 = t\partial_t - r\partial_r, \quad \xi_1 = \left(t^2 + \frac{1}{r^2}\right)\partial_t - 2t\partial_r - \frac{2k}{r}\partial_\phi,$$

(2.7a)

$$\xi_\phi = \partial_\phi.$$  

(2.7b)

where $\xi_1, \xi_0$ and $\xi_1$ form the $SL(2, \mathbb{R})$, and $\xi_\phi$ is the $U(1)$ rotational symmetry.

### 3 Asymptotic symmetry

Let us move on an asymptotic symmetry of the near horizon geometry of the Kerr metric. The asymptotic symmetry is defined by using the asymptotic boundary condition. The theory has an asymptotic symmetry if it has symmetry in the asymptotic region up to the small perturbations which satisfy the asymptotic boundary condition. For geometries, asymptotic symmetries are specified by asymptotic Killing vectors which satisfy

$$\mathcal{L}_\xi g_{\mu\nu} = \mathcal{O}(\chi_{\mu\nu}),$$

(3.1)

where $\mathcal{L}_\xi$ is the Lie derivative along $\xi$. Here, the asymptotic boundary condition is given by

$$h_{\mu\nu} = \mathcal{O}(\chi_{\mu\nu}),$$

(3.2)

where $h_{\mu\nu}$ is a perturbation of the metric. We now impose the following boundary condition:

$$h_{\mu\nu} = \begin{pmatrix}
  t & r & \phi & \theta \\
  t & \mathcal{O}(r^0) & \mathcal{O}(r^{-3}) & \mathcal{O}(r^{-3}) \\
  r & \mathcal{O}(r^{-4}) & \mathcal{O}(r^{-3}) & \mathcal{O}(r^{-3}) \\
  \phi & \mathcal{O}(r^{-2}) & \mathcal{O}(r^{-3}) & \mathcal{O}(r^{-3}) \\
  \theta & \mathcal{O}(r^{-2}) & \mathcal{O}(r^{-3}) & \mathcal{O}(r^{-3}) 
\end{pmatrix}.$$  

(3.3)
The most general form of the asymptotic Killing vector which satisfies (3.1) is
\[ \xi = \left( \epsilon_\xi(t) + \frac{\epsilon_\xi'(t)}{2r^2} + O(r^{-3}) \right) \partial_t + \left( -r \epsilon_\xi'(t) + \frac{\epsilon_\xi''(t)}{2r} + O(r^{-2}) \right) \partial_r \\
+ \left( C - \frac{k \epsilon_\xi'(t)}{r} + O(r^{-3}) \right) \partial_\phi + O(r^{-3}) \partial_\theta, \]
where \( \epsilon_\xi(t) \) is an arbitrary function of \( t \), and \( C \) is an arbitrary constant. We define \( \xi_n \) as the asymptotic Killing with \( \epsilon_\xi(t) = t^{1+n} \) and \( C = 0 \) in (3.4), since the Hawking temperature is zero for the extremal Kerr black hole (or equivalently time is non-compact). Then these vectors form the Virasoro algebra
\[ [\xi_n, \xi_m]_{LB} = \mathcal{L}_{\xi_n} \xi_m = (m - n) \xi_{m+n}. \]

We should mention key differences from the original Kerr/CFT correspondence [1]. Our asymptotic condition (3.3) is stronger than the original one. The constraint (3.1) can be divided into that from the background \( g_{\mu\nu} \) and the perturbation, \( \mathcal{L}_\xi g_{\mu\nu} = \mathcal{L}_\xi \bar{g}_{\mu\nu} + \mathcal{L}_\xi h_{\mu\nu} \). The constraint from the perturbation is automatically satisfied when we impose the constraint from the background, differently from [1] and similarly to [3]. Our asymptotic symmetry (3.4) contains all of the original symmetries (2.7a) and (2.7b). This is because our perturbation is small enough not to break the original symmetry asymptotically. Contrary, in [1], the perturbation is so large such that it breaks original symmetry even asymptotically. The rotational Killing vector (2.7b) is just realized as \( \xi \) with \( \epsilon_\xi(t) = 0 \) and \( C = 1 \). It is crucial that the \( SL(2, \mathbb{R}) \) Killing vectors (2.7a) are identical to \( \xi_n \) with \( n = -1, 0, 1 \). In our asymptotic symmetry, this \( SL(2, \mathbb{R}) \) is enhanced to the Virasoro algebra.

4 Asymptotic charge

We first consider a charge which is associated with the asymptotic symmetry [5, 6]. The asymptotic charge is defined as the deviation of the charge from the background \( \bar{g}_{\mu\nu} \). For infinitesimal perturbation \( h_{\mu\nu} \), that is given by
\[ Q^\mathcal{L}_{\xi} [h] = \frac{1}{8\pi G_N} \int_{\partial \Sigma_\infty} k_\xi[h, \bar{g}], \]
where the integration is taken over the boundary of a time slice. The two-form \( k_\xi \) is defined by
\[ k_\xi[h, \bar{g}] = \sqrt{-\bar{g}} \epsilon_{\mu\nu\rho\sigma} \tilde{k}^{\mu\nu}_{\xi}[h, \bar{g}] dx^\rho \wedge dx^\sigma, \]

4
with
\[
\tilde{k}^{\mu\nu}[h, \bar{g}] = \frac{1}{2} \left[ \xi^\mu D^\nu h - \xi^\mu D_\lambda h^{\lambda\nu} + (D^\mu h^{\nu\lambda}) \xi_\lambda + \frac{1}{2} hD^\mu \xi^\nu 
- h^{\mu\lambda} D_\lambda \xi^\nu + \frac{1}{2} h^{\mu\lambda} (D^\nu \xi_\lambda + D^\lambda \xi^\nu) - (\mu \leftrightarrow \nu) \right],
\]
(4.3)
where \( D_\mu \) is a covariant derivative on the background geometry, and we denote \( \bar{g} = \det \bar{g}_{\mu\nu} \) and \( h = \bar{g}^{\mu\nu} h_{\mu\nu} \). The Dirac bracket of the asymptotic charge can be calculated by varying the charge as
\[
\{ Q^A_\xi, Q^A_\zeta \} = \delta_\xi Q^A_\zeta
= \frac{1}{8\pi G_N} \int_{\partial \Sigma} k_\xi [\mathcal{L}_\zeta \bar{g}, \bar{g}] + \frac{1}{8\pi G_N} \int_{\partial \Sigma} k_\xi [\mathcal{L}_\zeta h, \bar{g}].
\]
(4.4)
The second term is the charge \( Q^A_{[\xi, \bar{g}]} \), which is proportional to the perturbation \( h \). The first term is an additional constant term, which provides the central charge of the algebra.

For the near horizon geometry (2.5), we obtain
\[
\tilde{k}^{\mu r}[\mathcal{L}_\zeta \bar{g}, \bar{g}] = - \frac{k}{f_0(\theta)} \epsilon_\alpha(t) \epsilon^{\alpha}(t) + \mathcal{O}(r^{-2}), \quad (\mu = \phi)
\]
(4.5a)
\[
\tilde{k}^{\mu r}[\mathcal{L}_\zeta \bar{g}, \bar{g}] = \mathcal{O}(r^{-2}). \quad \text{(other components)}
\]
(4.5b)
Since we get the charge which depends on \( t \), we consider the analytic continuation of \( t \) and define the generators as\[
L_n = \frac{1}{2\pi i} \oint dt \ Q^A_{\zeta_n}.
\]
(4.6)
The central charge \( c \) is then calculated as
\[
\frac{c}{12} = \frac{1}{2\pi i} \oint dt \ \frac{1}{8\pi G_N} \int_{\partial \Sigma} k_{\zeta_m} [\mathcal{L}_{\zeta_n} \bar{g}, \bar{g}] = 0,
\]
(4.7)
namely, this algebra does not have the central extension within the framework to use the “asymptotic charge”, and satisfies the commutation relation,
\[
[L_n, L_m] = (n - m) L_{n+m}.
\]
(4.8)

5 Energy-momentum tensor in CFT

As shown in the previous section, the algebra of the asymptotic charge does not have a central extension. This fact makes it difficult to discuss the correspondence of physical

\footnote{This definition is an analogue of that for the Virasoro algebra in two-dimensional CFT i.e. \( L_n = \frac{1}{2\pi i} \oint dz \ z^{1+n}T(z) \). We identify \( Q_\epsilon_n \sim \epsilon_n(z)T(z) \) with \( \epsilon_n(z) = z^{1+n} \).}
quantities between CFT and Kerr black holes. Hence, we use the “quasi-local charge”
defined by using the surface energy momentum tensor \[4\], and obtain anomalous trans-
formations with a plausible cut-off.

According to the GKPW relation \[7, 8\] in the AdS/CFT correspondence \[9\], an
expectation value of energy-momentum tensor in the CFT is given by

\[
\langle T_{\mu\nu} \rangle = \frac{2}{\sqrt{-\gamma}} \frac{\delta S_{cl}}{\delta \gamma_{\mu\nu}},
\]

where \(\gamma_{\mu\nu}\) is the metric on the boundary. This can be identified with the quasi-local
energy-momentum tensor defined by Brown and York \[4\] (see also \[10\]),

\[
T_{\mu\nu} = \frac{2}{\sqrt{-\gamma}} \frac{\delta S_{grav}}{\delta \gamma_{\mu\nu}}.
\]

The gravitational action \(S_{grav}\) is the Einstein-Hilbert action with the Gibbons-Hawking
term,

\[
S_{grav} = \frac{1}{16\pi G_N} \int_M d^4x \sqrt{-g} R + \frac{1}{8\pi G_N} \int_{\partial M} d^3x \sqrt{-\gamma} K,
\]

where \(K = K^\mu_{\mu}\) and \(K_{\mu\nu}\) is the extrinsic curvature. Let us consider following decomposition of the metric:

\[
ds^2 = N^2 dr^2 + \gamma_{\mu\nu}(dx^\mu + N^\mu dr)(dx^\nu + N^\nu dr).
\]

We focus on the variation of the action with respect to the induced metric \(\gamma_{\mu\nu}\). Since
we only consider an on-shell action, the variation of the action is reduced to boundary
terms,

\[
\delta S = \int_{\partial M} d^3x \pi^{\mu\nu} \delta \gamma_{\mu\nu},
\]

where \(\pi^{\mu\nu}\) is the conjugate momentum of \(\gamma_{\mu\nu}\). Then the energy-momentum tensor on
the boundary is expressed as

\[
T_{\mu\nu} = \frac{2}{\sqrt{-\gamma}} \pi^{\mu\nu} = \frac{1}{8\pi G_N} (K^{\mu\nu} - \gamma^{\mu\nu} K).
\]

We next consider the ADM decomposition of the boundary metric \(\gamma_{\mu\nu}\),

\[
ds^2_{\partial M} = -N_{\partial}^2 dt^2 + \sigma_{ab}(dx^a + N_a^\alpha dt)(dx^b + N_b^\alpha dt),
\]

where \(\Sigma\) is a time slice and \(\partial \Sigma\) is that on the boundary. Then the quasi-local charge on
the boundary can be defined by

\[
Q_{\xi}^{QL} = \int_{\partial \Sigma} d^2x \sqrt{\sigma} u^\mu T_{\mu\nu} \xi^\nu,
\]
where $u^\mu$ is a future pointing timelike unit normal to $\partial \Sigma$. This definition allows for the boundary $\partial \Sigma$ at a finite $r$. Here, we shall redefine the energy-momentum tensor on the boundary by its deviation from the background,

$$\tau^{\mu\nu}[h] = T^{\mu\nu}|_{g=\bar{g}+h} - T^{\mu\nu}|_{g=\bar{g}}. \quad (5.9)$$

This redefinition is equivalent to introducing the boundary term to the action such that $\delta S = 0$ also at the boundary for $g_{\mu\nu} = \bar{g}_{\mu\nu}$. Then charges in the boundary theory become

$$Q^\xi_{\text{QL}} = \int_{\partial \Sigma} d^2x \sqrt{\sigma} u^\mu \tau_{\mu\nu}[h] \xi^{\nu}(t). \quad (5.10)$$

This definition of charges is slightly different from that for the asymptotic charge (4.1).\footnote{The relation between the quasi-local charge and the asymptotic charge is discussed in Appendix A.}

Let us consider the anomalous transformation of the energy-momentum tensor which is given by

$$\delta \xi T^{\mu\nu} = \tau^{\mu\nu}[L\xi \bar{g}]. \quad (5.11)$$

The mass and the angular momentum are defined as $M = Q^\xi_{\partial t}$ and $J = Q^\xi_{\partial \phi}$, respectively.\footnote{In [4], the mass is defined by using $\xi^\mu = N_{\partial \Sigma} u^\mu$, instead of the Killing vector. However, this definition cannot give the ADM mass if the Killing vector is not orthogonal to the time slice. Hence we define the mass by using the Killing vector.}

We then obtain their anomalous transformations for the geometry (2.5),

$$\delta \xi M = -\frac{k^2}{8\pi G_N} \int d\phi d\theta \frac{(f_0(\theta))^{3/2}}{2f_0(\theta)} \frac{\sqrt{f_0(\theta)}}{2\Lambda f_0(\theta)} \epsilon^\mu(t), \quad (5.12a)$$

$$\delta \xi J = -\frac{k}{8\pi G_N} \int d\phi d\theta \frac{(f_0(\theta))^{3/2}}{2\Lambda f_0(\theta)} \frac{\sqrt{f_0(\theta)}}{2\Lambda^2 f_0(\theta)} \epsilon^\mu(t), \quad (5.12b)$$

where we introduced a regularization by putting the boundary at $r = \Lambda$ with large but finite $\Lambda$. These terms vanish for $\Lambda \to \infty$. The regularization of $\Lambda$ can be interpreted as a cut-off in the field theory side. In the case of the Kerr black hole (2.6), we obtain

$$\delta \xi M = -\frac{a^2}{G_N \Lambda} \epsilon^\mu(t), \quad (5.13a)$$

$$\delta \xi J = -\frac{a^2}{G_N \Lambda^2} \epsilon^\mu(t). \quad (5.13b)$$

We assume this energy-momentum tensor corresponds to that of CFT, which obeys the Virasoro algebra coming from the transformation of $t$. We could define the generators as

$$L_n[h] = \frac{1}{2\pi i} \int dt \int_{\partial \Sigma} d^2x \sqrt{\sigma} u^\mu \tau_{\mu\nu}[h] \xi^\mu(t), \quad (5.14)$$
where we have considered the analytic continuation of $t$. We have replaced the Killing vector by $\epsilon_\xi(t)\partial_t$, or equivalently picked up the leading term. The central charge can be read off from $L_n[\mathcal{L}_\xi \tilde{g}]$ as

$$c = \frac{12a^2}{G_N \Lambda}.$$ (5.15)

6 Correspondence

We now show the correspondence between the Kerr black hole and the boundary theory which is a (chiral) CFT. Charges in the boundary theory are identified with the asymptotic charges of the Kerr black hole. Since we take the Kerr geometry itself as the background, expectation values of charges should be zero at zero temperature. Hence, we consider finite temperature effects by using the conformal transformation from the zero temperature system.

We first consider the analytic continuation of $t$ into a complex $z$ whose imaginary part is $t$. We map the complex plane of $z$ to cylinder by using the transformation,

$$z \to w = \frac{\beta}{2\pi} \log z,$$ (6.1)

where $\beta$ is the period of the imaginary part of $w$, so that we can obtain the finite temperature system with temperature $T = 1/\beta$. The charges being zero at zero temperature, only anomalous effects have nontrivial contributions at finite temperature. Through the anomaly of mass (5.13a) and angular momentum (5.13b), we obtain the finite temperature effect,

$$M = \frac{a^2}{2G_N \Lambda} (2\pi T)^2,$$ (6.2a)

$$J = \frac{a^2}{2G_N \Lambda^2} (2\pi T)^2.$$ (6.2b)

An entropy in CFT can be calculated by using the Cardy formula [11] which relates the central charge $c$ to the density of high-energy states:

$$S = 2\pi \sqrt{\frac{c L_0}{6}}.$$ (6.3)

At finite temperature, $\epsilon_\xi$ should be expanded in Fourier modes $e^{-int/\beta}$, and $L_0$ corresponds to the energy. From (5.15) and (6.2a), we obtain the entropy

$$S = \frac{(2\pi)^2 a^2 T}{G_N \Lambda}.$$ (6.4)
In the gravity side, the finite temperature effects could be observed from the non-extremal case. In order to obtain the near horizon geometry with the AdS-like structure, we have to take the near extremal limit in which the difference between energy and angular momentum is infinitesimally small. We here consider the following relation:

$$m = a \left(1 + \epsilon^2 \frac{\Delta}{2}\right),$$  \hspace{1cm} (6.5)

where the parameter $\Delta$ implies the deviation from the extremality. Then the temperature is now given by

$$T_H = \epsilon \frac{\sqrt{\Delta}}{4\pi a} + O(\epsilon^2).$$  \hspace{1cm} (6.6)

The ADM mass and the angular momentum get the following corrections, respectively,

$$M = \frac{a}{G_N} \left(1 + \epsilon^2 \frac{\Delta}{2}\right),$$  \hspace{1cm} (6.7a)

$$J = \frac{a^2}{G_N} \left(1 + \epsilon^2 \frac{\Delta}{2}\right).$$  \hspace{1cm} (6.7b)

The near horizon geometry becomes

$$ds^2 = -f_0(\theta) \left(r^2 - \Delta\right) dt^2 + f_0(\theta) \frac{dr^2}{r^2 - \Delta} + f_\phi(\theta) \left(d\phi + k r dt\right)^2 + f_\theta(\theta) d\theta^2,$$  \hspace{1cm} (6.8)

where $f_0(\theta), f_\phi(\theta), f_\theta(\theta)$ and $k$ are same as in (2.6). Due to the rescaling of time (2.4), the Hawking temperature of the near horizon geometry is

$$T_H = \frac{\sqrt{\Delta}}{2\pi}.$$  \hspace{1cm} (6.9)

The mass of the near horizon geometry is defined by the charge associated to the timelike Killing $\partial_t$ in the geometry, and is related to that of the original Kerr metric as

$$M_{\text{(near horizon)}} = 2ae^{-1}M_{\text{(original)}} - e^{-1}J_{\text{(original)}}.$$  \hspace{1cm} (6.10)

We consider the energy and the angular momentum given by their deviation from the extremality and obtain

$$M = \epsilon \frac{a^2 \Delta}{2G_N} = \epsilon \frac{a^2}{2G_N} \left(2\pi T_H\right)^2,$$  \hspace{1cm} (6.11a)

$$J = \epsilon^2 \frac{a^2 \Delta}{2G_N} = \epsilon^2 \frac{a^2}{2G_N} \left(2\pi T_H\right)^2,$$  \hspace{1cm} (6.11b)

where we wrote only the leading term in $\epsilon$. Therefore, we conclude that the mass and the angular momentum obtained in the CFT agree with those for the Kerr black hole,
if we identify $\Lambda = \epsilon^{-1}$. The entropy of CFT also corresponds to its deviation from the extremality in the gravity side. The Bekenstein-Hawking entropy of the Kerr black hole is given by

$$S = \frac{2\pi mr^+}{G_N} = \frac{2\pi a^2}{G_N} \left(1 + 2\pi\epsilon T_H + \mathcal{O}(\epsilon^2)\right). \quad (6.12)$$

Its deviation from the extremality agrees with (6.4).

The relation between $\epsilon$ and $\Lambda$ can be understood as follows: The near horizon geometry comes from an infinitesimal region around the horizon. If the parameter $\epsilon$ in (2.4) is kept finite, the geometry cannot be exactly equivalent to (2.5) but can be approximated by (2.5) in the near horizon region $r - r_+ \ll \epsilon^{-1} a$. Since we cannot go out of this near horizon region as long as we use (2.5), the boundary of this geometry should be taken around $\hat{r} \lesssim \epsilon^{-1}$. Therefore we identify the position of boundary as $\Lambda = \epsilon^{-1}$.

7 Conclusions and discussions

We have considered another realization of the Kerr/CFT correspondence. We have constructed new asymptotic Killing vectors by imposing a stronger constraint in the asymptotic region of the near horizon geometry of the Kerr black hole. Perturbations which satisfy this constraint do not break the symmetry of the original geometry. Then our Killing vectors give an asymptotic symmetry which contains all of the exact isometries. These forms the Virasoro algebra which is an extension of the $SL(2,\mathbb{R})$ symmetry of the geometry. We have calculated the central extension of this algebra by using the "asymptotic charge" and it turns out to be zero. However this is not the end of the story. Indeed we found anomalies when we consider the "quasi-local charge", instead.

In order to see the anomalies in our case, we need some regularizations. Even if we apply the same regularization for the quasi-local charge into the asymptotic charge, we cannot obtain anomalous effects. Since the asymptotic charge is defined up to some asymptotic condition, it might not be appropriate to estimate quantities which asymptotically vanish. In other words, the asymptotic charge should be defined at the boundary $r \to \infty$. On the other hand, the quasi-local charge can be defined at arbitrary $r$, and does not require any asymptotic condition. We could then introduce a regularization and obtain anomalies.

Since the definition of the surface energy-momentum tensor has the same form to the GKPW relation, we have used it as the energy-momentum tensor in the CFT side. We have calculated finite temperature effects on the energy and angular momentum through their anomalies and shown the correspondence concretely. The entropy has been also
derived by using the Cardy formula and has agreed with that from the extremal case. This fact is consistent in the following sense: Since charges are also defined as their deviation from the extremality, we can expect that the CFT which obeys our Virasoro algebra describes the deviation from the extremality.

We have defined the energy-momentum tensor in the field theory side by its deviation from the background. This is related to the boundary condition, $\delta S$ on the boundary. Instead of this, one can introduce boundary counter terms related to the boundary condition of the background. If geometry approaches asymptotically to the background $\bar{g}$, the corresponding boundary counter terms are defined such that $\delta S = 0$ also on the boundary for $g = \bar{g}$. By using this counter term, we can obtain the same energy-momentum tensor without taking the deviation from the background. However, the explicit form of the counter terms are not derived in this paper. This is left for future studies.

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Appendix A  Asymptotic and quasi-local charges

We consider the relation between the asymptotic charge and the quasi-local energy. The asymptotic charge can be rewritten in the following form,

$$Q^\lambda_\xi = \int_{\partial \Sigma_\infty} d^2 x \sqrt{-g} \tilde{k}^{\mu\nu}_\xi = \lim_{r \to \infty} \int_{\partial \Sigma} d^2 x \sqrt{\sigma} u_\mu n_\nu \tilde{k}^{\mu\nu}_\xi, \quad (A.1)$$

where $n^\mu$ is an outward pointing unit normal to the boundary $\partial \Sigma$. Comparing (5.8) and (A.1), we can read off the relation between the "flux" $\tilde{k}^{\mu\nu}_\xi$ and the surface energy-momentum tensor $T^{\mu\nu}$ as

$$\tilde{k}^{\mu\nu}_\xi n_\nu \bigg|_{r \to \infty} \sim T^{\mu\nu} \xi_\nu. \quad (A.2)$$

This relation is analogous to the GKPW relation, in which the flux is related to the current in boundary theory. In the case of one-form gauge field, it has an expansion in the form of

$$A_\mu = A_\mu^{(0)} + r^{-a} \langle J_\mu \rangle + \cdots, \quad (A.3)$$

where $A_\mu^{(0)}$ is related to the source in the boundary theory. Then the flux which goes through the boundary is related to the current in the boundary theory,

$$F_{\mu\nu} \sim -\partial_\nu A_\mu \sim a \langle J_\mu \rangle r^{-a-1}. \quad (A.4)$$
Let us check this relation for the anomalous transformation of the “flux” \( k_\xi \). We consider the variation of the metric under the asymptotic symmetry (3.4), and take \( \mathcal{L}_\xi \bar{g} \) as a perturbation. We define

\[
\tilde{\kappa}^{\mu \nu} [\mathcal{L}_\xi \bar{g}] = N^{-1} T^{\mu \nu} \zeta^\nu \bigg|_{\bar{g} = \bar{g} + \mathcal{L}_\xi \bar{g}} - N^{-1} T^{\mu \nu} \zeta^\nu \bigg|_{\bar{g}}.
\]  

(A.5)

Again, we consider its deviation from the background. Then, for the near horizon geometry (2.5), we obtain

\[
\tilde{\kappa}^{\mu \nu} [\mathcal{L}_\xi \bar{g}] = \frac{k}{f_0(\theta)} \epsilon_\xi(t) \epsilon_\nu(t), \quad (\mu = \phi)
\]  

(A.6a)

\[
\tilde{\kappa}^{\mu \nu} [\mathcal{L}_\xi \bar{g}] = 0, \quad (\text{other components})
\]  

(A.6b)

By using the identification of (A.2), this expression agrees with (4.5). Then, we can define a charge in analogy with the asymptotic charge as

\[
\bar{Q}_\xi^A = \int_{\partial \Sigma} d^2 x \sqrt{\sigma} u_\mu n_\nu \tilde{\kappa}^{\mu \nu} [h].
\]  

(A.7)

However, this is not identical to \( Q_\xi^{QL} \). When we define the charge, we consider the deviation from the background. For the asymptotic charge we take the difference of the charge itself. On the other hand, we take the difference of the energy-momentum tensor for the quasi-local charge. Then, these two definitions give different charge, even though they are same if we do not take the background into account.

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