Extending the GKZ limit without breaking Lorentz Invariance

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A scenario is presented on how to shift the predicted cutoff in the cosmic ray spectrum at $10^{20}$ eV, called the Greisen-Zatsepin-Kuzmin limit (GKZ), to larger energies without breaking the Lorentz invariance. The formulation is based on a pseudo-complex extension of standard field theory. The dispersion relation of particles can be changed, leading to a modification of the GKZ limit. Maximal shifts are determined.

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The GZK limit (Greisen-Zatsepin-Kuzmin) corresponds to a cutoff in high energy cosmic rays at $10^{20}$ eV. Experiments were performed to measure the spectrum of these energy cosmic rays. In AGASA, several events breaking this limit were observed. The new experiment of Auger in Argentina intends to verify these data. First results are reported in Ref. [8].

Thus, a change in the GZK limit is associated with a change of this dispersion relation. The main stream of recent events breaking this limit were observed. The new experiment of Auger in Argentina intends to verify these data. First results are reported in Ref. [8]. If confirmed, it will be generally considered as a clear evidence for the breaking of Lorentz symmetry.

The GZK limit can be calculated considering a head on collision of a proton with a photon from the Cosmic Microwave back ground (CMB), producing a pion, for the breaking of Lorentz symmetry.

The pseudo-complex number are defined via $X = x_1 + i x_2 = x_+ \sigma_+ + x_- \sigma_-$, with $I^2 = 1$ and $\sigma_\pm = \frac{1}{2} (1 \pm i)$. This is similar to the complex notation except for the different behavior of $I$. The $\sigma_\pm$ obey the relations $\sigma_\pm^2 = \sigma_\mp$ and $\sigma_\pm \sigma_- = 0$. The last expression of $X$ above is in terms of the zero divisor basis (a linear combination in terms of $\sigma_\pm$), which turns out to be extremely useful.

The pseudo-complex conjugate of a pseudo-complex number is $X^* = x_1 - i x_2 = x_+ \sigma_- + x_- \sigma_+$. The norm of a pseudo-complex number is given by the square root of $|X|^2 = XX^* = x_1^2 - x_2^2$. There are three different possibilities: $|X|^2 > 0$, $|X|^2 < 0$ and $|X|^2 = 0$. The structure of this space is isomorphic to $O(1,1)$.

Calculations, like differentiation and integration, can be done in complete analogy to the case of complex numbers. The pseudo-complex derivative is denoted by $D \frac{\partial}{\partial X}$ and the rules of application are the same. The derivative can be directly extended to variables with an additional index $\alpha\beta \frac{\partial}{\partial X^\alpha}$ and to functional derivatives.

However, no residual theorem exists. Thus, the structure of pseudo-complex numbers is very similar to the usual complex numbers but not completely, due to the appearance of the zero divisor branch $\mathcal{P}^0 = \mathcal{P}^0_+ \cup \mathcal{P}^0_-$, with $\mathcal{P}^0_\pm = \{ \sigma \pm | a e R \}$. It implies that the set of pseudo-complex numbers is just a ring. This reflects the less stringent algebraic structure.

In the pseudo-complex Lorentz group $SOP(1,3)$, finite transformations are given by $exp(\imath \omega_{\mu\nu} M^{\mu\nu})$, with $\omega_{\mu\nu}$ a pseudo-complex number and $M^{\mu\nu}$ the generators of the Lorentz group. The usual Lorentz group is obtained by restricting to real numbers.
The particular combination $M_{\pm}^{\mu\nu} = \frac{1}{2} (M^{\mu\nu} \pm i M^{\mu\nu})$, leads to commuting generators $M^{\mu\nu}_{+}$ with $M^{\mu\nu}_{-}$. This implies that the pseudo-complex Lorentz group is the direct product of two algebras, each with the commutation relations of a $SO(1,3)$ group. In group notation we write

$$SO_{P}(1,3) \simeq SO_{+}(1,3) \otimes SO_{-}(1,3) \subset SO(1,3).$$

This is the larger group we were looking for.

The pseudo-complex Poincaré group is generated by the pseudo-complex four momentum

$$P^{\mu} = iD^{\mu} = i \frac{D}{DX_{\mu}} = P^{\mu}_{+}\sigma_{+} + P^{\mu}_{-}\sigma_{-},$$

and generators of the pseudo-complex Lorentz algebra.

A Casimir operator of the pseudo-complex Poincaré group is

$$P^{2} = \sigma_{+}P^{2}_{+} + \sigma_{-}P^{2}_{-}.$$  

Its eigenvalue is $M^{2} = \sigma_{+}M_{+}^{2} + \sigma_{-}M_{-}^{2}$, i.e., a pseudo-complex mass associated to each particle. Similarly, the Pauli-Ljubanski vector $[\Sigma]$ can be defined.

In this new approach, fields are direct extensions from standard field theory: If a field transforms under a given representation of the Lorentz group, the pseudo-complex extension of the field transforms in the same way in both components ($\sigma_{+}$ and $\sigma_{-}$). For example, a Weyl-spinor transforms as $(\frac{1}{2}, 0) + (0, \frac{1}{2})$ under the Lorentz group. The same transformation property holds in the pseudo-complex field with respect to the $SO_{+}(1,3)$ and $SO_{-}(1,3)$ groups. One consequence is that in a field $\Psi = \Psi_{+}\sigma_{+} + \Psi_{-}\sigma_{-}$ both components ($\Psi_{+}$ and $\Psi_{-}$) have the same spin.

A new variational principle is introduced, in order to connect both zero divisor components, i.e., $\delta S \in \mathcal{P}^{0}$, a number in the zero divisor branch. If this condition would not be imposed, the result is two equations of motion of two independent systems, i.e., no new field theory would be obtained.

When physical observables are calculated, a projection to the pseudo-real part is applied. E.g., $P_{\mu}^{0}$ the pseudo-real part of the linear momentum, related to acceleration, is set to zero. It corresponds to go into an inertial system, where the vacuum is well defined. This is interpreted as going back to the inertial system, which is connected to an accelerated system via a pseudo-imaginary transformation.

For the case of a Dirac field, the Lagrange density is given by $\mathcal{L} = \Psi(\gamma_{\mu}P^{\mu} - M)\Psi$ and the resulting field equation, after variation, is

$$\left(\gamma_{\mu}P^{\mu} - M\right) \epsilon \mathcal{P}^{0},$$

with $\mathcal{P}^{0}$ being the set of zero divisors, as defined in the introduction. Multiplying with the pseudo-conjugate $(\gamma_{\nu}P^{\nu} - M)^{\star}$ leads to

$$\left(\gamma^{\nu}P^{\nu}_{+} - M_{+}\right)\left(\gamma_{\mu}P^{\mu}_{-} - M_{-}\right)\Psi = 0,$$

which is the final field equation for free fields.

Now, inspecting Eq. (5), a solution is given either by setting the first or the second factor equal to zero, having substituted before $P^{\mu}_{\pm}$ by $p^{\pm}$. Without loss of generality, let us take the first choice. Then, the solution describes a propagating particle with mass $M_{+} = m$, which is identified as the physical mass. The other mass parameter, $M_{-}$ is related to the regulating mass of the theory, of the order of $\frac{1}{4}$. Due to the appearance of $M_{-}$ the theory is regularized a la Pauli-Villars \cite{14}, which is an important detail.

In this theory, the photon is described with $M_{+} = 0$ but $M_{-} = \frac{1}{4}$, a large regularizing mass. This generates a term in the Lagrange density of the form $\sim M_{2}A_{\mu}A^{\mu}\sigma$, which assures gauge invariance because of the appearance of $\sigma_{-}$. This is due to the property that the gauge angle $\alpha(x)$ has the same value in the $\sigma_{+}$ and $\sigma_{-}$ component, thus, $\alpha(x) \sim \sigma_{+}$, but $\sigma_{+}$ and $\sigma_{-}$ commute.

In conclusion, the advantages of the pseudo-complex field theory are: i) It is regularized, ii) stays gauge invariant and, the most important point, iii) it maintains known symmetries and thus permits to proceed in a very similar fashion to the standard field theory and the determination of cross sections for different processes.

Having resumed the most important characteristics of the extended field theory, we proceed to the main part. Our proposal is to add to the Lagrange density an interaction of the type $\Psi^{\star}\gamma^{\mu}f_{\mu}\Psi$, which is a scalar under the Lorentz group $SO(1,3)$, diagonally embedded in $SO_{+}(1,3) \otimes SO_{-}(1,3)$.

A possible reason for this term is as follows: It is led by the assumption that during a collision new effective interactions result from contributions of accelerated systems, connected to the inertial in- and outgoing systems. Because there is a minimal length scale, the interaction is distributed over a finite size of space-time, which is a consequence of a finite, maximal acceleration, not permitting an instant interaction. A transformation to an accelerated system is given by $exp(lI\omega \cdot L) = exp(l\omega \cdot L)\sigma_{+} + exp(-l\omega \cdot L)\sigma_{-}$, with $L_{i}$ being a generator of the Poincaré group. The smallness of $l$ indicates the order of contribution. Applying it to $\Psi(x)$, expanding up to first order, yields (note that $I = \sigma_{+} - \sigma_{-}$ and define $\mu = exp(l\omega \cdot L)$)

$$\left(\psi(\mu_{x}x) + \psi(\mu_{-}^{-1}x)\right) \approx \psi(x) + lI\overline{\psi}(x),$$

where the first part is an average over field values from neighboring systems, and the second term gives the contribution to the field, due to the transformation to accelerated/rotated systems. It describes the differences of fields to neighboring accelerated systems. The action of

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this transformation is symbolically expressed by the operator $\Pi$ on the state vector. The arrow indicates the direction of action. In principle, one should model the interaction by weighting over the contributions of different accelerated systems. The exact form depends on the transformation parameters and their range, i.e., on the model used.

Considering now the term $\bar{\Psi} \gamma_\mu p^\mu \Psi$, with the usual linear momentum $p_\mu$. Changing $\Psi$ by the expression in (3), leads in zero order in $l$ to the old term and a correction $\sim \Pi$ of first order in $l$:

$$\bar{\Psi} \gamma_\mu p^\mu \Psi \rightarrow \bar{\Psi} \left(p^\mu + IP_2^\mu\right) \Psi + l \bar{\Psi} \gamma_\mu \left(\bar{\Pi} p^\mu + p^\mu \bar{\Pi}\right) \Psi$$  (7)

where $IP_2^\mu$ comes from the pseudo-complexification of the linear momentum. As an alternative, the last term can be modeled as $\bar{\Psi} \gamma_\mu l f^\mu \Psi$. Because of dimensional reasons the $f^\mu$ has the dimensions of a force which leads to a particular model. A more strict derivation is needed, but for the moment this phenomenologic argument suffices to present our point.

Instead of this heuristic explanation, we can skip it and start from the ad hoc addition of the term just discussed.

The advantage lies in the possibility to include this within a generalization of the minimal coupling scheme. The extended minimal coupling, concerning the linear momentum, reads

$$P_\mu \rightarrow P_\mu + l f_\mu$$  (8)

where the $l$ is due to dimensional reasons and indicates that the correction should be of the order of the minimal length scale. After having projected to an inertial system ($P_2^\mu = 0$), the $f_\mu$ is still present and describes the role of a non-vanishing, pseudo-imaginary component of the linear momentum, reflecting the effect of the acceleration.

Let us suppose that $f_\mu$ is given by

$$f_\mu \sim F_{\mu\nu} p^\nu$$  (9)

with $F_{\mu\nu}$ being an antisymmetric tensor not defined yet. This choice of $f_\mu$ contains the possibility to interpret $f_\mu$ as a force ($f_\mu = dU/d\tau$, $\tau$ being the eigen-time, thus, $p_\mu f^\mu = 0$). Then, under the modification in (3) the gauge transformation for the electro-magnetic vector field $A_\mu$ is also changed adding to the transformation a term $l/2 f_\mu \alpha(x)$, with $\alpha(x)$ as the gauge angle and $g$ the coupling strength to the electro-magnetic field.

This is not the only possible choice. The $f_\mu$ may in general depend on a certain power $\beta$ in the linear momentum, i.e. symbolically

$$|f_\mu| \sim p^\beta$$  (10)

The effects, as a function in $\beta$ will be discussed further below for two cases.

This leaves a rich choice of possible interactions which can be invented, studying their effects on the dispersion relation.

The resulting equation of motion for $\Psi$, setting $A_\mu$ to zero, is modified to

$$(\gamma^\mu (P_\mu + l f_\mu) - M) \Psi \in \mathcal{P}^0$$  (11)

with $\mathcal{P}^0 = \mathcal{P}_+^0 \cup \mathcal{P}_-^0$ (see the definition given above), is the set of zero divisors. Multiplying by the pseudo-complex conjugate of the operator, multiplying by $\gamma_\mu \left(P_\mu - l f_\mu + M_\nu \right)$ and using the properties of the $\gamma^\mu$ matrices, we arrive at the equation

$$(P_{+\mu} P_-^\mu - M_+^2) (P_{-\mu} P_-^\mu - M_-^2) = 0$$  (12)

We project to an inertial system ($P_2^\mu = 0$), i.e. $\mathcal{P}^0_\mu = p^\mu + l f^\mu$. Selecting the first factor, using $P_{+\mu} P_-^\mu = E^2 - p^2 + l^2 f_\mu f^\mu + l (p_\mu f^\mu + f_\mu p^\mu)$, we arrive at the dispersion relation

$$E^2 = p^2 + (lf)^2 + (lp f + fp) + M^2$$  (13)

with $f^2 = - f_\mu f^\mu > 0$. and $p f = - p_\mu f^\mu$, $f p = - f_\mu p^\mu$. Interpreting $f_\mu$ as a force, i.e. a derivative of $p_\mu$, with respect to the eigen-time, we have $p_\mu f^\mu = 0$, which eliminates the term proportional to the first order in the length scale.

Let us now investigate the consequences for the threshold momentum for the production of pions, without specifying $f_\mu$. Using energy and momentum conservation in a head-on collision, it was shown in (10) that the threshold linear momentum of a reaction $B_1 + \gamma \rightarrow B_2 + M_3 \left(B_k\text{ being baryons, e.g. protons, and } M_3\text{ being a meson, e.g. pion, and } \gamma \text{ is a soft photon from the CMB}\right)$ is given by

$$p_{1,\text{thr.}} \approx \frac{(m_2 + m_3)^2 - m_k^2}{4\omega} = 10^{11} \text{ GeV}$$  (14)

with $\omega$ being the energy of a soft photon from the CMB and $m_k$ the masses of the participating particles.

With the modified dispersion relation, the energy changes to ($p f = f p = 0$)

$$E_k \approx p_k + \frac{m_k^2}{2p_k} + \frac{l^2 f_k^2}{2p_k} = p_k + \frac{\tilde{m}_k^2}{2p_k}$$  (15)

with $\tilde{m}_k^2 = m_k^2 + l^2 f_k^2$ and $f_k^2 = - f_{k\mu} f_k^\mu$ for particle number $k = (1,2,3)$. The result for the threshold momentum is now

$$p_{1,\text{thr.}} \approx \frac{(\tilde{m}_2 + m_3)^2 - \tilde{m}_1^2}{4\omega} \approx \frac{(m_2 + m_3)^2 - m_k^2}{4\omega} + \frac{l^2}{4\omega} \left[(m_2 + m_3)\left(f_1^2 + f_k^2\right) - f_1^2\right]$$  (16)
The new threshold momentum is shifted to larger values if the expression in the square bracket is positive. Similar terms are added, when we skip the above constriction of \( pf = f_p = 0 \).

### We can estimate how far the threshold momentum can be shifted

As one example, assume that the last term in Eq. (16) is proportional to \( p^2 (\beta = 2) \), which corresponds to the first choice of \( f_\mu \) above. We arrive at a quadratic equation \( p - Xp^2 = Y \), where \( Y \approx 10^{11} \text{ GeV} \). The dependence on \( l \) is hidden in \( X \). The solution is \( p_{1/2} = \frac{1}{2X} (1 \pm \sqrt{1 - 4XY}) \).

For each given \( X \) we have, thus, two solutions. Taking the minus sign, we reproduce in the limit of \( X \to 0 \) the solution \( p = Y \). The largest value we obtain for \( X \approx 15 \) and the threshold momentum is \( 2 \times 10^{11} \text{ GeV} \), i.e., just the double value. We stress, that this result is only valid when \( f^2 \) is proportional to \( p^2 \). It changes when the dependence in the power of \( p \) is different. The maximal shift by a factor of 2 is the consequence of the second order equation in \( p \). The exact value of the shift depends on \( X \sim \frac{1}{25} \), which may serve to determine \( l \).

The \( f_\mu \) may be approximated by the average force acting on the particle (\( \beta = 0 \)). The expression in Eq. (16) stays the same, with the difference that it adds some constant to the right hand side of the equation, implying a shift of the threshold momentum to higher values. Assuming \( f_k^2 = (m_k \alpha)^2 \), \( f_\mu^2 \ll f_k^2 = f_\alpha^2 \) and the acceleration "\( \alpha \" as maximal, i.e. \( \frac{1}{\alpha^2} \), leads only to a shift to \( 1.5 \times 10^{11} \text{ GeV} \). When the terms proportional to \( pf \) and \( fp \) are included, the change involves also a rescaling of \( p_{1,\text{thr}} \), which corresponds to the case of \( \beta = 1 \). However, it implies a different interpretation than \( f_\mu \) being a force.

Which form of \( f_\mu \) finally describes the physical situation depends on several assumptions, which may change the present results. The explanation given here is of phenomenological nature and not much can be said about the exact structure of \( f_\mu \). It would be more attractive to get \( f_\mu \) from basic principles. For that, we have to complete the formulation of the pseudo-complex field theory. Nevertheless, phenomenologic considerations shed some light on possible processes. Work is still in progress.

The main point given here is that the GZK limit can be shifted to larger values without assuming a violation of Lorentz invariance, if one accepts to change standard field theory. The pseudo-complex field theory seems to be one candidate. It has several advantages, which we list here again: The theory is regularized, maintains gauge invariance and it keeps the known symmetries. This facilitates the calculation of cross sections, following standard procedures.

Note, that most models explaining a breaking of Lorentz invariance, like given in (10), are phenomenologic in nature. Considering the lack of a fundamental theory, this is the only viable way up to now. The pseudo-complex extension of field theory seems to be the correct direction to go.

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