The second order dense ferromagnetic-ferromagnetic phase transition

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The fcc spin-1 Ising (BEG) model has a dense ferromagnetic (df) ground state instead of the ferromagnetic ground state at low temperature region and exhibits the dense ferromagnetic (df) - ferromagnetic (F) phase transition for
\[ d = D/J = 2.9, \ k = K/J = -0.5, \ \ell = L/J = 0 \ \text{and} \ h = H/J = 0. \]
The critical behavior of the dense ferromagnetic (df) - ferromagnetic (F) phase transition has been investigated using the cellular automaton cooling and heating algorithms. The universality class and the type of the dense ferromagnetic (df) - ferromagnetic (F) phase transition have been researched within the framework of the finite - size scaling, the power law relations and the probability distribution. The results show that the dense ferromagnetic- ferromagnetic phase transition is of the second order and the model shows universal second order Ising critical behavior at \( d = 2.9 \) parameter value through \( k = -0.5 \) line.

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1. INTRODUCTION

In recent years, some of the studies indicated that the spin-1 Ising model has a ground state ordered structure which is named the dense ferromagnetic (df) [1 – 4]. In absence of the df ordered structure, the phase diagrams were formed for some phase transitions which has been assumed the weak first order transition instead of the second order transition. The presence of the df ordered structure can be caused to some changes on the global phase diagrams [4]. This case clarifies the differences among the results of the previous studies for the $(kT_C/J, d)$ phase diagram through the $k$ line. [5 – 11]. For example, the phase diagram has exhibited a tricritical point (TCP) instead of a critical end point (CEP) for $k = -0.5$. While the MFA [7] and RG [8] studies exhibited a critical end point (CEP) for $k=-0.5$ on the BEG model global phase diagram, CA [5,6], MCRG [9], TPCA [10] and CVM [11] studies show that there is the tricritical point (TCP). Through the $k$ line, the $df$ - $F$ phase transition is very important for determining the type of the phase boundary. The purpose of this study is to define the $df$ ordered phase and is to investigate the nature of $df$ - $F$ phase transition at $d = 2.9$ value through $k = -0.5$ line. This point creates the type of the special point of the $(kT_C/J, d)$ phase space for $k = -0.5$. Therefore, we have found the $df$ - $F$ phase transition worthy of investigation in depth.
Furthermore, the universality class of the $df - F$ phase transition has not been investigated so far. The critical temperature and the statical critical exponents are estimated by analyzing the data within the framework of the finite-size scaling theory and the power law relations.

The spin-1 Ising model, which is known as the generalized Blume-Emery-Griffiths (BEG) model, can be used to simulate many physical systems. The model firstly has been presented for describing phase separation and superfluid ordering in He mixtures [12]. The versions of the model have been applied to the physical systems such as the solid-liquid-gas systems [13], the multicomponent fluids [14], the microemulsions [15], the semiconductor alloys [16–18], He$^3$-He$^4$ mixtures [12,19] and the binary alloys [20].

The BEG model Hamiltonian is defined as

$$H_I = -\sum_{\langle ij \rangle} J S_i S_j - K \sum_{\langle ij \rangle} S_i^2 S_j^2 + L \sum_{\langle ij \rangle} (S_i^2 S_j + S_i S_j^2) + D \sum_i S_i^2 + h \sum_i S_i$$

which is equivalent to the lattice gas Hamiltonian under some transformations [21–23]. $\langle ij \rangle$ denotes summation over all nearest-neighbor (nn) pairs of sites and $S_i = -1, 0, 1$. The parameters $J$, $K$, $L$, $D$ and $h$ are bilinear, biquadratic, dipole-quadrupole interaction terms, the single-ion anisotropy constant and the
field term. The BEG model for \( k \geq 0 \) has been studied by mean field approximation (MFA) [12 − 14], the transfer matrix method [24], series expansion method [25], the constant coupling approximation [26], the position-space renormalization method [27], cluster variation method (CVM) [1], linear-chain approximation [2], Monte Carlo method (MC) [3] and Cellular Automaton (CA) [4 − 6, 28, 29].

In this paper, the fcc BEG model for \( d = D/J = 2.9, k = K/J = -0.5, \ell = L/J = 0 \) and \( h = H/J = 0 \) is simulated using cooling and heating algorithm improved from Creutz Cellular Automaton. In the previous papers, the Creutz cellular automaton (CCA) algorithm and its improved versions have been used successfully to study the properties of the critical behaviors of the Ising model Hamiltonians [4 − 6, 28 − 50]. The CCA algorithm, which was first introduced by Creutz [51], is a microcanonical algorithm interpolating between the conventional Monte Carlo and the molecular dynamics techniques. The Creutz cellular automaton (CCA) is faster than the conventional Monte Carlo method (MC). The CCA does not need high quality random numbers and it is a new and an alternative simulation method for physical systems. It has another advantage allowing the specific heat to be computed from internal energy fluctuations. Our previous studies showed that the heating and the cooling algorithms im-
proved from the Creutz Cellular Automaton algorithm are effective to study the phase space and the critical behavior of the Blume Emery Griffiths model [4 – 6, 28 – 30, 35, 36].

II . RESULTS AND DISCUSSION

The CA algorithm of spin-1 Ising model is a microcanonical algorithm. The total energy $H$, which is conserved, is given by

$$H = H_I + H_K$$

where $H_I$ is Ising energy which is given by equation 1 and $H_K$ is kinetic energy.

The kinetic energy $H_K$ is an integer, equal to the change in the Ising spin energy for any spin flip and its value lie in the interval $(0, m)$. $m$ is equal to $24J$ for $d = D/J = 2.9$, $k = K/J = -0.5$, $\ell = L/J = 0$ and $h = H/J = 0$ on fcc lattice. For a site to be updated, its spin is changed to one of the other two states with $1/2$ probability. If this energy is transferable to or from the kinetic energy variable of the site, such that the total energy $H$ is conserved, then this change is done and kinetic energy is appropriately changed. Otherwise the spin is not change [30, 34 – 36].

At the heating and the cooling algorithms, the simulation consist of two parts, the initialization procedure and the computation of the thermodynamic
quantities. The initial configuration for heating and cooling algorithms can be set in different shapes. In this study, the initial configurations are obtained at three different shapes for heating algorithm during 20,000 CA steps. Firstly, all the spins are up ($S = +1$) at the absolute zero temperature for both algorithms. The initial configuration of the heating algorithm has been obtained at low temperature ordered phase ($df$) adding kinetic energy which is equal to the maximum change in the Ising spin energy for the any spin flip to the spin system for set I and II. The another initial configuration (set III) has been obtained flipping 8% of the spins to $S = 0$ state. The heating rate is realized by increasing of 8% in the kinetic energy ($H_k$) of 15% of the fcc lattice for two sets at the computation of the thermodynamic quantities. At set III, the heating rate is realized by increasing of 8% in the kinetic energy ($H_k$) of all site of the fcc lattice.

On the other hand, the initial configuration for the cooling algorithm is obtained adding energy to the 70% of the spin system for getting the disordered phase ($P$) at high temperature. During the cooling cycle, the cooling rate is realized by decreasing of 8% in the kinetic energy ($H_k$) from 25% of the spin system. The initial configurations are run during the 20,000 Cellular Automaton time steps. Instead of resetting the starting configuration at each energy, it
was used the final configuration at a given energy as the starting point for the next at both heating and cooling algorithms. The computed values of the thermodynamic quantities (the order parameters \((M, Q)\), the susceptibility \((\chi)\), the Ising energy \((H_I)\) and the specific heat \((C)\)) are averages over the lattice and over the number of time steps \((2,000,000)\) with discard of the first \(100,000\) time steps during the cellular automaton develops \([4−6, 28, 29]\).

They have been computed on the fcc lattice with \(L = 8, 9, 10, 11\) and 12 (The total number of sites is \(N = 4L^3\)) for periodic boundary conditions. The fcc lattice was formed in a simple cubic (sc) lattice. (The total number of sites is \(N = 4L^3 = 6912\) for \(L = 12\) fcc lattice, this total site number equals to \(L = 19\) for the simple cubic lattice). The presented figures are set III (heating algorithm) results.

The order parameters, the Ising energy, susceptibility and specific heat are calculated from

\[
M = \frac{1}{N} \sum_i S_i, \quad Q = \frac{1}{N} \sum_i S_i^2 \quad (3)
\]

\[
U_I = (-J \sum_{<ij>} S_i S_j - K \sum_{<ij>} S_i^2 S_j^2 + D \sum_i S_i^2)/U_0 \quad (4)
\]

\[
\chi = N \frac{\langle M^2 \rangle - \langle M \rangle^2}{kT} \quad (5)
\]
\[ C_1/k = N \frac{\langle U^2 \rangle - \langle U \rangle^2}{(kT)^2} \]  

(6)

where \( U_0 \) is the ground state energy at \( kT/J = 0 \).

The ferromagnetic (F) and the paramagnetic (P) phases can be determined with the average occupation of the states \( \langle P_{\pm 1,0} \rangle \). As the projectors for the states \( S = +1, -1 \) and 0 are \( P_{+1} = \frac{1}{2}S(S+1), P_{-1} = \frac{1}{2}S(S-1) \) and \( P_0 = 1-S^2 \), the average occupation of the states are \( \langle P_{+1} \rangle = \frac{1}{2}(Q+M), \langle P_{-1} \rangle = \frac{1}{2}(Q-M) \) and \( \langle P_0 \rangle = 1-Q \), respectively. With considering the average occupation of the states, another ferromagnetic phase can be determined as the dense ferromagnetic phase \( (df) \).

- Ferromagnetic (F): \( \langle P_{+1} \rangle \neq \langle P_{-1} \rangle \neq \langle P_0 \rangle \neq 0, (M \neq Q \neq 0) \)
- Dense ferromagnetic (df): \( \langle P_{-1} \rangle \rightarrow 0; \langle P_{+1} \rangle \neq \langle P_0 \rangle \neq 0, (M \approx Q \neq 0) \)
- Paramagnetic (P): \( \langle P_{-1} \rangle = \langle P_{+1} \rangle \neq \langle P_0 \rangle \neq 0, (M = 0, Q \neq 0) \).

II.1 Temperature Variations of Thermodynamic Quantities for the \( dF - F - P \) Phase Transitions

The temperature variation of the order parameters \( (M, Q) \), the susceptibility \( (\chi) \), the Ising energy \( (H_I) \) and the specific heat \( (C_I) \) are illustrated in figure 1 for exhibiting the general aspect of the successive \( df - F - P \) phase transitions at \( d = 2.9 \) parameter value through \( k = -0.5 \) line. As it is seen in figure 1(a) and
figure 1(c), the order parameters and the Ising energy appear continuously for
df − F and F − P phase transitions. Therefore both phase transitions are of the
second order as functional behavior. The susceptibility (\(\chi\)) and the specific heat
\((C_1, C)\) exhibit two peaks at \(T_{C1}\) and \(T_{C2}\) temperatures corresponding to \(df − F\)
and \(F − P\) phase transitions (Figure 1(b), 1(d) and figure 2). It can be seen from
figure 1(c), the functional change of the Ising energy from order to order (\(df
− F\)) phase transition is different from the view of the order to disorder (\(F − P\))
phase transition. For \(df - F\) phase transition, the Ising energy difference (\(\Delta U\)) is
greater than for the \(F - P\) phase transition. The estimated critical temperature
from susceptibility and specific heat maxima is compatible with each other for
\(F - P\) phase transition. But the critical temperature values for \(df - F\) phase
transition are not compatible (Figure 1(b) and 2(a)). Therefore specific heat
\((C)\) has been recalculated for only spin-spin interaction energy (Figure 2(b)).

The spin-spin interaction energy \(U\) is determined as

\[
U = (-J \sum_{<ij>} S_i S_j)/U_0
\]  
(7)

The specific heat calculated from \(U\) can show a sharp peak for \(df - F\) phase
transition. Because, the first sum (\(U\)) in Ising energy (\(U_I\)) distinguish the \(S =
+1\) and \(-1\) states. Indeed, the obtained infinite critical temperature (\(T_{C1}(\infty)=

9
1.52±0.04) from the susceptibility ($\chi$) and specific heat ($C$) peak temperatures are compatible with each other. $T_{C2}(\infty)$ is obtained from the susceptibility ($\chi$) and specific heat ($C_I$ and $C$) peak temperatures as 3.20±0.02.

The temperature variations of the $\langle P_{+1} \rangle$, $\langle P_{-1} \rangle$ and $\langle P_0 \rangle$ are given in figure 3. The initial configuration for the heating algorithm is created as all spins are up ($S = +1$) at absolute zero temperature ($T = 0$). If the enough energy is added to the spin system, the $S = 0$ begins to arise. The excitation energy of the single spin flipping from $S = +1$ to $S = 0$ is $3.1J$ while it is $24J$ for the single spin flipping from $S = +1$ to $S = -1$. Therefore, at low temperature region, the rising probability of the $S = -1$ state has to be lower than $S = 0$ state. It can be seen in figure 3 that the spin system includes $S = +1$ and $S = 0$ states predominantly. The value of $\langle P_{+1} \rangle$ is about 1 and $\langle P_0 \rangle$ is different from zero, while $\langle P_{-1} \rangle$ appears almost zero indicating the $df$ phase for $T < T_{C1}(L)$. At the same time, $M$ is almost equal to the $Q$ ($M \cong Q \neq 0$). As $\langle P_{-1} \rangle$ increases above $T_{C1}(L)$, the $df$ ordered phase changes to the $F$ ordered phase. At high temperature region, the ferromagnetic - the paramagnetic phase transition ($F$ - $P$) occurs. $\langle P_{+1} \rangle$ is equal to $\langle P_{-1} \rangle$ and the system is in the paramagnetic ($P$) disordered phase above $T_{C2}(L)$ temperature ($M \neq Q \neq 0$). Therefore the phase space is divided into three regions ($df$, $F$ and $P$). It is obvious that to
follow the temperature variation of $\langle P_{-1} \rangle$ is a useful way to prove the existence of the $df$ ordered phase.

II .2 Probability Distribution of Order Parameter

The another useful procedure to distinguish the phase transition type is to calculate the probability distributions of the order parameter ($P(M)$). In our study the probability distribution is calculated by

$$P_L(M) = \frac{N_M}{N_{CCAS}}$$  \hspace{1cm} (8)

where $N_M$ is the number of times that magnetization $M$ appears, and $N_{CCAS}$ is the total number of the cellular automaton steps. The histogram with 200 bins are used for plotting the probability distribution of the magnetization [24, 31]. The probability distribution of the order parameter ($P(M)$) near the phase transition temperature shows two peaks in the second phase transitions.

The probability distributions of the order parameter ($P(M)$) are shown for different temperature values in figure 4. The peaks of the order parameter probability distribution exhibits minimum with increasing temperature at the low temperature region. This minimum corresponds to the second order $df - F$ phase transition at the $T^x_{C1}(L = 12) = 1.479$. Although the phase transition is of the second order, the probability distribution shows the single peak near the
phase transition temperature $T_{C1}(L)$. Because the system has $S = +1$ and 0 spins below $T_{C1}(L)$ and the transition is from order ($df$) to order ($F$). However, the probability distributions in the $F - P$ phase transition region exhibit the two peaks with the contribution of the $S = -1$ state near the $T_{C2}(L = 12) = 3.187$. For $T > T_{C2}$, there is a single peak focused to $M = 0$ indicating the disordered ($P$) phase.

II.3 Finite - Size Scaling Analyses and the Statical Critical Exponents

The values of the statical critical exponents ($\nu, \beta, \gamma, \alpha$) are estimated within the framework of the finite - size scaling theory and the power laws. The infinite lattice critical temperature $T_C(\infty)$ has been obtained from the susceptibility and the specific heat peak temperatures for the successive second order $df - F - P$ phase transitions and from the intersection point of Binder cumulant curves ($U_L$) for the second order $F - P$ phase transition [52]. The finite - size scaling relations of the Binder cumulant ($U_L$), the order parameter ($M$), the susceptibility ($\chi$) and the specific heat ($C$) are given by

$$U_L = G^\alpha (\varepsilon L^{1/\nu})$$

(9)
\[ M = L^{-\beta/\nu} X^o(\varepsilon L^{1/\nu}) \] (10)

\[ kT_X = L^{\gamma/\nu} Y^o(\varepsilon L^{1/\nu}) \] (11)

\[ C = L^{\alpha/\nu} Z^o(\varepsilon L^{1/\nu}) \] (12)

For large \( x = \varepsilon L^{1/\nu} \), the finite lattice critical behaviors must be asymptotically reproduced, that is,

\[ X^o(x) \propto A x^\beta \] (13)

\[ Y^o(x) \propto B x^{-\gamma} \] (14)

\[ Z^o(x) \propto C x^{-\alpha} \] (15)

According to the finite size scaling theory, the data for the finite - size lattices of the thermodynamic quantities should lie on a single curve for the temperatures both below and above \( T_C(\infty) \) with universal critical exponents. The critical exponents \( \beta, \gamma \) and \( \alpha \) have been obtained from the Log-Log plots of the asymptomatic functions.
The temperature variation of the Binder cumulant is shown for the different lattice sizes in figure 5. The Binder cumulant curves intersect at the $T_{C2}^{UL}(\infty) = 3.20 \pm 0.02$ corresponding to the $F-P$ phase transition (Figure 5(a)). This value is compatible with $T_{C2}(\infty)$ which is extrapolated according to the finite size scaling theory from the susceptibility and the specific heat peak temperatures ($T_C(L)$) of the finite lattices, respectively.

$$T_C(L) = T_C(\infty) + aL^{-1/\nu}$$

(16)

It can be seen in the inset of the figure 5(a) that there is no intersection at the Binder cumulant for the data of the $df-F$ phase transition region. However, the Binder cumulant curves exhibit a plateau near the infinite lattice critical temperature ($T_{C1}(\infty)$) which is obtained from the susceptibility ($\chi$) and the specific heat ($C$) peak temperatures as $T_{C1}^{\chi}(\infty) = 1.52 \pm 0.04$. In figure 5(b), the scaling data of the Binder cumulants are shown for the second order phase transition from $F$ to $P$. Near the $T_{UL}^{C2}(\infty)$, Binder cumulant curves have been scaled well for $T_C=T_{C2}(\infty)$ with $\nu = 0.64$. It can be seen in the inset of the figure 5(b) that the data corresponding to $df$ ordered phase could not be scaled with the $T_{C2}^{UL}(\infty)$ critical temperature value. However, the scaling data of the binder cumulant corresponding to the $df$ ordered phase lie on a single curve for
\[ \varepsilon = (T - T_{C_1}(\infty))/T_{C_1}(\infty) \] at \( T < T_{C_1}(\infty) \) region using \( \nu = 0.64 \) (Figure 5(c)) and the finite size scaling relations validate for the \( df \) ordered region.

In figure 6(a), the scaling data of the order parameter is illustrated at the successive \( df - F - P \) phase transitions for \( L = 8, 9, 10, 11 \) and 12 at \( T_C = T_{C_2}(\infty) \). The order parameter data lie on the two different curves with slope=\( \beta/\nu = 0.31 \) and \( \beta'/\nu = -0.55 \) for the temperatures both below and above \( T_{C_2}(\infty) \) respectively except for \( df \) ordered region with \( \beta = 0.31 \) and \( \nu = 0.64 \). As it is seen in the inset of the figure 6(a), the data of \( df - F \) phase transition region have not been scaled with \( T_C = T_{C_2}(\infty) \). However, the data corresponding to the \( df - F \) phase transition region have been scaled well with \( T_C = T_{C_1}(\infty) \) for \( T < T_{C_1}(\infty) \) using \( \beta = 0.31 \) and \( \nu = 0.64 \) in figure 6(b).

The scaling data of the susceptibility have been shown in figure 7 with the straight lines describing the theoretically predicted behavior for large \( x \) (Equation 8). The susceptibility data for the temperatures both below and above \( T_{C_2}(\infty) \) agrees with the asymptotic form except for the \( df - F \) phase transition region and so with the \( T_C = T_{C_2}(\infty) \), \( \gamma = \gamma' = 1.25 \) and \( \nu = 0.64 \) in figure 7(a) and 7(c). However, the data of \( df \) ordered phase \( (T < T_{C_1}) \) have been scaled with the \( T_C = T_{C_1}(\infty) \) using \( \gamma = 1.25 \) and \( \nu = 0.64 \) in figure 7(b).

The finite size scaling data of the singular portion of the specific heat (\( C \))
have been exhibited in figure 8. The data of $F - P$ phase transition region of $C_I$ have been scaled well both below and above $T_{C_2}(\infty)$ using $\alpha = 0.12$, $\nu = 0.64$ and the correction terms, $b^- = -70$ and $b^+ = -8$ (Figure 8(a) and 8(b)). Although the data of the $df$ phase region could not been scaled with $T_C = T_{C_2}(\infty)$ in figure 8(a), the data lie on single curves with slope $=-\alpha/\nu = -0.12$ at both side of the $T_{C_1}(\infty)$, using $\alpha = 0.12$, $\nu = 0.64$ and the correction terms, $b^- = -1$ and $b^+ = -0.3$ in figure 8(c) and (d). On the other hand, the specific heat ($C_I$) data calculated from $U_I$ scales well at $df - F - P$ phase transitions using $\alpha = 0.12$, $\nu = 0.64$ for $T < T_{C_1}(\infty)$ and $T < T_{f+}(\infty)$ and $T > T_{C_2}(\infty)$.

The $M$, $\chi$ and $C$ data have been analyzed within the framework of the finite size scaling theory for the successive $df - F - P$ phase transitions. The estimated values of the statical critical exponents are in good agreement with the universal values ($\alpha = 0.12$, $\beta = 0.31$, $\gamma = 1.25$, $\nu = 0.64$) for the $df - F$ and the $F - P$ phase transitions.

II.4 Power Law Relations and the Infinite Lattice Statical Critical Exponents

On the other hand, the critical exponent values for $df - F$ phase transition can be obtained using the following power law relations [53].
where \( \varepsilon = (T - T_C(L))/T_C(L) \). The finite lattice critical exponents \( \beta(L) \), \( \beta'(L) \), \( \gamma(L) \), \( \gamma'(L) \), \( \alpha(L) \) and \( \alpha'(L) \) of the order parameter \( (M) \), susceptibility \( (\chi) \) and the specific heat \( (C) \) quantities are obtained from the slope of the log-log plot of the power laws relations for each finite lattices in the interval \( 0.05 \leq \varepsilon \leq 0.2 \).

The infinite lattice critical exponents are obtained using linear extrapolation and their values are given in Table I. The estimated values for cooling algorithm and three simulation sets of heating algorithm are in good agreement with the finite size scaling critical exponent estimations and the universal values for 3d Ising model \( (\beta = 0.31, \gamma = 1.25, \alpha = \alpha' = 0.12 \text{ and } \nu = 0.64) \).

Table 1. The estimated values of the infinite lattice critical exponents and the critical temperatures \( (\alpha, \beta, \gamma \text{ and } T_{C1}^{x,C}(\infty)) \) using linear extrapolation.
### III. SUMMARY

The \((kT_c/J, d)\) phase diagrams is illustrated in figure 9 for the presence of \(df\) order. The type of special point is determined by the \(d = 2.9\) parameter. The calculations show that model exhibits the phase transition from order to order for \(d = 2.9\) and the first order phase transition from order to disorder in the \(3 \leq d < 4\) parameter region. If the model doesn’t exhibit the \(df\) ordered phase instead of the \(F\) ordered phase in the low temperature region, the continuous phase transition from \(F\) order to \(F\) order \((F - F)\) is considered as the weak first order. This constitutes the part of the first order phase transition line which creates the critical end point \((CEP)\) [7, 8]. However, CA results show that, the model has a \(df\) ordered phase for the parameters in the \(2.9 \leq d < 4.0\) region. The spin system contains \(S = +1\) and \(S = 0\) states. As a result of this, the order parameters \(M\) and \(Q\) are almost equal each other \((M \cong Q \neq 0)\) at low temperatures. With increasing temperature, the dense ferromagnetic \((df)\) ordered phase changes continuously to ferromagnetic \((F)\) ordered phase at the

|                  | Heating                | Cooling               |
|------------------|------------------------|-----------------------|
|                  | Set I                  | Set II                | Set III               | Average of sets |                  | Cooling               |
| \(T_{C1}^\lambda(\infty)\) | 1.50 ± 0.02            | 1.50 ± 0.03           | 1.52 ± 0.03           | 1.51 ± 0.03     | 1.52 ± 0.04     |
| \(T_{C1}^\beta(\infty)\) | -                      | -                     | 1.52 ± 0.01           | -                | 1.52 ± 0.01     |
| \(\beta(T < T_{C1})\)  | 0.31 ± 0.01            | 0.31 ± 0.01           | 0.30 ± 0.01           | 0.31 ± 0.01     | 0.31 ± 0.01     |
| \(\gamma(T < T_{C1})\)  | 1.25 ± 0.01            | 1.23 ± 0.03           | 1.23 ± 0.02           | 1.24 ± 0.03     | 1.23 ± 0.02     |
| \(\alpha(T < T_{C1})\)  | 0.12 ± 0.01            | 0.12 ± 0.01           | 0.11 ± 0.01           | 0.12 ± 0.01     | 0.12 ± 0.01     |
| \(\alpha'(T > T_{C1})\) | 0.12 ± 0.01            | 0.12 ± 0.01           | 0.12 ± 0.01           | 0.12 ± 0.01     | 0.12 ± 0.01     |
$d = 2.9, k = -0.5$ with the enough contribution of $S = -1$ state. Therefore, near the $d = 2.9$, the first order phase transition line have been changed to second order phase transition line, and there occurs the tricritical point $TCP$ (Figure 9). In order to determine the universality class of the successive $df - F - P$ second order phase transitions, the static critical exponents ($\alpha, \beta, \gamma$ and $\nu$) are estimated within the framework of the finite-size scaling theory. The estimated values of the critical exponents ($\alpha = 0.12, \beta = 0.31, \gamma = 1.25$ and $\nu = 0.64$) near the $T_{C1}$ and $T_{C2}$ temperatures are in good agreement with the theoretical values for three sets. The $df - F$ phase transition is analyzed with the power laws for comparing with the critical exponent values estimated from the finite-size scaling theory. The obtained values are in compatible with the finite-size scaling analyze results and the universal values for the 3d Ising model. The obtained results have shown that the $df - F$ phase transition is of the second order and it is compatible with the universal Ising critical behavior for $d = 2.9$ parameter value through $k = -0.5$ line. As a result of this, the definition of the $df$ phase changes the phase transition type and the special point type in the phase space for the BEG model. This result will lead to reexamine the structure of phase spaces.

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Figure Captions

Figure 1. For $(d = 2.9, k = -0.5)$, the temperature dependence of (a) the order parameters $(M, Q)$, (b) the susceptibility $(\chi)$, (c) the Ising energy $(H_I)$ and (d) the specific heat $(C_I/k)$.

Figure 2. For $(d = 2.9, k = -0.5)$, the specific heat $(C/k)$ calculated from $U$.

Figure 3. For $(d = 2.9, k = -0.5)$, the temperature dependences of the $\langle P \rangle$, $\langle P_+ \rangle$, $\langle P_- \rangle$ and $\langle P_0 \rangle$ correspond to the $S = +1$, $-1$ and 0 spin states, respectively.

Figure 4. The probability distribution of the $M$ for $(d = -0.5, k = 0.9)$ on $L = 12$.

Figure 5. For $(d = 2.9, k = -0.5)$, (a) the temperature dependence of the Binder cumulant $(U_L)$, (b) the finite-size scaling of the Binder cumulant near
the $df - F - P$ phase transition with $T_{C2}^f(\infty)$, (c) the finite-size scaling of
the Binder cumulant near the $df - F$ phase transition with $T_{C1}(\infty)$.

Figure 6. For $(d = 2.9, k = -0.5)$, the finite-size scaling plots of (a) the
order parameter with $T_{C2}(\infty)$, (b) the order parameter with $T_{C1}(\infty)$ near the
$df - F$ phase transition for $T < T_{C1}(\infty)$.

Figure 7. For $(d = 2.9, k = -0.5)$, the finite-size scaling plots of the
susceptibility (a) with $\varepsilon = (T - T_{C2}(\infty))/T_{C2}(\infty)$ for $T < T_{C2}(\infty)$, (b) near the
$df - F$ phase transition with $\varepsilon = (T - T_{C1}(\infty))/T_{C1}(\infty)$ for $T < T_{C1}(\infty)$, (c)
with $\varepsilon = (T - T_{C2}(\infty))/T$ for $T > T_{C2}(\infty)$.

Figure 8. For $(d = 2.9, k = -0.5)$, the finite-size scaling plots of the specific
heat (a) for $T < T_{C2}(\infty)$ with $T_{C2}(\infty)$, (b) for $T > T_{C2}(\infty)$ with $T_{C2}(\infty)$, (c)
for $T < T_{C1}^C(\infty)$ with $T_{C1}^C(\infty)$, (d) for $T > T_{C1}^C(\infty)$ with $T_{C1}^C(\infty)$.

Figure 9. The phase diagram for $k = -0.5$. The phase space with $df$ phase
contains a $TCP$ at $d = 2.9$. 

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Figure 1(a)

\( d = 2.9 \)

\( k = -0.5 \)
Figure 1(b)

The graph shows the magnetic susceptibility $\chi$ as a function of $kT/J$, with peaks at $T_{C1}$ and $T_{C2}$. The peaks indicate critical temperatures $T_{C1}$ and $T_{C2}$.
Figure 1(c)
Figure 1(d)
Figure 2
Figure 3

L=12
Heating Algorithm

- $S=+1$
- $S=0$
- $S=-1$

$kT/J$

$\langle P \rangle$

$T_{C1}$

$T_{C2}$

Figure 3
$T_{c1}(L=12)=1.479$

$T_{c2}(L=12)=3.187$
Figure 5(a)
Figure 5(a)
$\nu=0.64$

$T_C = T_{C2(\infty)} = 3.2$

Figure 5(b)
Figure 5(c)

\[ \nu = 0.64 \]

\[ T_C = T_{C1}(\infty) = 1.52 \]

\[ T < T_{C1} \]
slope $= \beta/\nu = 0.31$

slope $= -\beta/\nu = -0.55$

$\nu = 0.64$

$\beta = 0.31$

$T_C = T_{C2}(\infty)$

Figure 6(a)
\[ \nu = 0.64 \]

\[ \beta = 0.31 \]

\[ T_C = T_{C1}(\infty) \]

slope = $\beta/\nu = 0.31$

\[ T < T_{C1} \]
slope $= -\gamma/\nu = -1.25$

$\nu = 0.64$

$\gamma = 1.25$

$T_C = T_{C2}(\infty)$

$T < T_{C2}$

$\text{df region}$
\[ \text{slope} = -\frac{\gamma}{\nu} = -1.25 \]

\[ T < T_{C1} \]

\[ \nu = 0.64 \]
\[ \gamma = 1.25 \]
\[ T_C = T_{C1}(\infty) \]
Figure 7(c)

\[ v = 0.64 \]
\[ \gamma = 1.25 \]
\[ T_C = T_{C2}(\infty) \]

\[ T > T_{C2} \]

slope = \(-\gamma/v = -1.25\)
slope = \(-\alpha/\nu\) = -0.12

Log\((C-b)^L\) vs. Log\((\epsilon L^{1/\nu})\)

T < T_{C2}

slope = -\alpha/\nu = -0.12

T_C = T_{C2}(\infty)

\(\alpha = 0.12\)

\(\nu = 0.64\)

b = -70

Figure 8(a)
\[ \text{Log}((C-b)L L^{\alpha/\nu}) \]

\[ T_C = T_{C2}(\infty) \]

\[ \alpha = 0.12 \]

\[ \nu = 0.64 \]

\[ b^+ = -8 \]

\[ \text{slope} = -\alpha/\nu = -0.12 \]

Figure 8(b)
slope $= -\frac{\alpha}{\nu} = -0.12$

$\alpha = 0.12$
$\nu = 0.64$
$b^- = -1$

$T < T_{C1}$

$slope = -\frac{\alpha}{\nu} = -0.12$

Figure 8(c)
\( \frac{\alpha}{\nu} = -0.12 \)

\[ \log((C-b^+)L^{1/\nu}) \]

\( T > T_{C1} \)

slope = \(-\frac{\alpha}{\nu} = -0.12 \)

\( \alpha = 0.12 \)
\( \nu = 0.64 \)
\( b^+ = -0.3 \)

Figure 8(d)
Figure 9