GLOBAL AND EXPONENTIAL ATTRACTORS FOR A NONLINEAR POROUS ELASTIC SYSTEM WITH DELAY TERM

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Abstract. This paper is concerned with the study on the existence of attractors for a nonlinear porous elastic system subjected to a delay-type damping in the volume fraction equation. The study will be performed, from the point of view of quasi-stability for infinite dimensional dynamical systems and from then on we will have the result of the existence of global and exponential attractors.

1. Introduction. In recent years the study of continuous models of deformable bodies has intensified, in particular we have the elastic solids with voids. Due to its great applicability as for example in soil mechanics, petroleum industry, materials sciences and biomechanics, porous solids now play a prominent role in scientific research (cf. [27]).

Among the various theories dealing with porous material, we can find a linear theory proposed by Cowin and Nunziato [14, 42] which is a generalization of the elastic theory for materials with voids, considering besides the material elasticity property, the volume fraction of the voids in the material. In this theory, the bulk density \( \rho = \rho(x,t) \) is given by the product of matrix density of the material \( \gamma = \gamma(x,t) \) and the volume fraction \( \nu = \nu(x,t) \)

\[
\rho(x,t) = \gamma(x,t)\nu(x,t).
\]

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They also consider a reference configuration (generally considered as initial configuration)
\[ \rho_0(x) = \gamma_0(x) \nu_0(x). \]

Let \( u_i = u_i(x, t) \) denote the components of the displacement vector field and so the components of the infinitesimal strain field are given by
\[ e_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}), \]
where the comma after the letter indicates the partial derivative with respect to the indicated coordinate. In addition, \( \phi = \phi(x, t) \) represents the change in volume fraction with respect to reference configuration. In a framework of evolution equations and in a setting of three-dimensional theory, the porous-elastic theory is described as
\[ \rho \ddot{u}_i = T_{ij,j} + \rho f_i, \]
\[ \rho k \ddot{\phi} = h_{i,i} + g + \rho \ell, \]
which are called balance of linear momento and balance of equilibrated force equations, respectively. In the above system, \( T_{ij} \) are the components of the stress tensor, \( f_i \) is the body force vector, \( h_i \) are the components of the equilibrated stress vector, \( k \) is the equilibrated inertia, \( g \) is the intrinsic equilibrated body force and \( \ell \) is the extrinsic equilibrated body force.

The constitutive equations for homogeneous and isotropic elastic bodies are (cf. [14])
\[ T_{ij} = \lambda \delta_{ij} e_{rr} + 2\mu e_{ij} + \beta \phi \delta_{ij}, \]
\[ h_i = \alpha \phi_x, \]
\[ g = -\omega \dot{\phi} - \xi \phi - \beta e_{rr}, \]
where \( \lambda, \mu, \beta, \alpha, \omega, \xi \) and \( \omega \) are constitutive constants that depend on the reference state \( \nu_0 \) and \( \delta_{ij} \) is Kronecker’s delta. The necessary and sufficient conditions for the internal energy density to be positive definite quadratic form are (cf. [14])
\[ \mu > 0, \alpha > 0, \xi > 0, \kappa > 0, \omega > 0, 3\lambda + 2\mu > 0 \text{ and } 3b^2 \leq (3\lambda + 2\mu)\xi. \]

In recent years, a large number of studies have been performed related to the asymptotic behavior of solutions of elastic pore models subject to the most diverse damping mechanisms (cf. [5, 10, 21, 29, 36, 37, 44, 48, 49, 50]).

For long-time dynamics of porous elastic system, there are few results till now, among them we can consider the work of M. Freitas et al. in [22]. In this paper, the authors considered a porous elastic system with nonlinear damping and sources terms
\[
\begin{cases}
\rho u_{tt} - \mu u_{xx} - b \phi_x + g_1(u_t) = f_1(u, \phi), \\
J \phi_{tt} - \delta \phi_{xx} + bu_x + \xi \phi + g_2(\phi_t) = f_2(u, \phi).
\end{cases}
\]
They proved global well-posedness, blow up of solutions and obtained the existence of global attractors and exponential attractors.

Similar to the case of elastic pore systems, in recent years, PDE containing time delay effects has become a very active area of research thanks to its applications in, for example, control theory, biology, economics, physiology, epidemiology and neural networks (see eg [25, 51]). It is well known (see [17, 38]) that terms of delay acting on mechanical systems, for example in the boundary of the wave equation, can cause instability to the system, that is, can lose its robustness. It is worth mentioning that
to stabilize hyperbolic systems containing delay terms it is necessary to introduce additional control terms (cf. [16]).

There are several results of stabilization and instabilization of the wave equation containing internal or boundary delay terms, among them we can cite [1, 9, 16, 17, 38, 39, 41, 54]. Nicaise and Pignotti in [38] consider the following equation of the wave with a term of delayed velocity and mixed boundary conditions of Dirichlet-Neumann

\[ u_{tt} + \Delta u + a(x)[\mu_1 u_t + \mu_2 u_t(x, t - \tau)] = 0 \quad \text{in} \quad \Omega \times (0, +\infty) \]

\[ u = 0 \quad \text{in} \quad \Gamma_D \times (0, +\infty) \]

\[ \frac{\partial u}{\partial n} = 0 \quad \text{in} \quad \Gamma_N \times (0, +\infty) \]

\[ u(x, t) = u_0(x), \quad u_t(x, 0) = u_1(x) \quad \text{in} \quad \Omega \]

\[ u(x, t - \tau) = f_0(x, t - \tau) \quad \text{in} \quad \Omega \times (0, \tau) \]

where \( a \in L^\infty(\Omega) \) is a function such that \( a(x) \geq 0 \) a.e. in \( \Omega \) and \( a(x) > a_0 > 0 \) in \( \omega \subset \Omega \), where \( \omega \) is an open neighborhood of \( \Gamma_N \). When \( \mu_2 = 0 \), it is a well known fact that the solutions of the system (1)-(5) decay exponentially (cf. [26, 31, 56]). The authors showed that if

\[ 0 < \mu_2 < \mu_1, \]  

then the system is exponentially stable. If (6) does not occur, it is possible to construct some sequence of delays for which the energy of the solution does not tend to zero. The same authors in [39], studied the system (1)-(5) with the term

\[ \mu_1 u_t + \int_{\tau_0}^{\tau_1} a(x)\mu_2(s)u_t(t - s)ds, \quad \text{in place of} \quad a(x)[\mu_1 u_t + \mu_2 u_t(x, t - \tau)], \]

where \( \mu_2(s) \geq 0 \), they showed that if

\[ \mu_1 > \|a\|_\infty \int_{\tau_0}^{\tau_1} \mu_2(s)ds, \]

then the energy of the system decays exponentially. The relations (6) and (8) were also used to obtain the exponential decay of the respective transmission problems associated with these systems (cf. [9, 30]). As early as [40] they considered the delay \( \tau \) as a function of the time, the so-called time-varying delay.

Kirane and Said-Houari in [28] used the inequality (6) to obtain exponential decay of the system

\[ u_{tt} - \Delta u + \int_0^t g(t - s)\Delta u ds + \mu_1 u_t + \mu_2 u_t(x, t - \tau) = 0. \]

W. Liu in [33] improved the result of Kirane and Said-Houari by considering the equation (9) with time-varying delay term and \( \mu_2 \) not necessarily positive. Later, Dai and Yang in [15] established the existence of the global solution to the problem (9) with \( \mu_1 \) and \( \mu_2 \) arbitrary, they obtained exponential decay of the energy for \( \mu_2 = 0 \). Wang et al. in [53] obtains exponential energy decay to the transmission system associated with Dai and Yang work.

There is a lot of work considering the presence of delay term in the Timoshenko system, among them [3, 4, 6, 19, 20, 47, 55]. Feng and Yang in [20], using (6) and the equality of wave velocities

\[ \frac{\rho_1}{\kappa} = \frac{\rho_2}{b}, \]
obtained the existence of a global and exponential attractor for the following non-linear Timoshenko system with delay term

$$\rho_1 \varphi_{tt} - \kappa (\varphi_x + \psi)_x = h(x)$$  \hspace{1cm} (11)

$$\rho_2 \psi_{tt} - b \psi_{xx} + \kappa (\varphi_x + \psi) + \mu_1 \phi_t + \mu_2 \psi_t(x, t-\tau) + f(\psi) = g(x).$$ \hspace{1cm} (12)

For $h = g = f = 0$, Said-Houari and Laskri in [47] obtained exponential decay assuming (6) and (10). In [55], J. Zhang et al., by using (6) and (10) they obtained the existence of global solution and exponential attractor for the system (11)-(12) considering time-varying delay, that is, $\tau = \tau(t)$.

There are not many studies considering the elastic pore system with delay mechanisms, among which we can consider [2, 32, 45]. In the references [2, 32] the authors studying the elastic porous system subjected to thermal mechanisms and time-varying delay.

C. Raposo et al. in [45] studied the following dimensional transmission problem for a elastic pore system with delay in the equation of transverse displacement

$$u_{tt} - u_{xx} - b \varphi_x + \mu_1 u_t + \mu_2 u_t(x, t-\tau) = 0 \hspace{1cm} (x, t) \in \Omega \times (0, +\infty),$$ \hspace{1cm} (13)

$$\varphi_{tt} - \varphi_{xx} + bu_x + a \varphi + \xi \varphi_t = 0 \hspace{1cm} (x, t) \in \Omega \times (0, +\infty),$$ \hspace{1cm} (14)

$$v_{tt} - v_{xx} - \beta \phi_x = 0 \hspace{1cm} (x, t) \in (L_1, L_2) \times (0, +\infty),$$ \hspace{1cm} (15)

$$\phi_{tt} - \phi_{xx} + \beta v_x + \alpha \phi + \xi \phi_t = 0 \hspace{1cm} (x, t) \in (L_1, L_2) \times (0, +\infty),$$ \hspace{1cm} (16)

where $0 < L_1 < L_2 < L$ and $\Omega = (0, L_1) \cup (L_2, L)$. By using a method developed by Z. Liu and S. Zheng (cf.[34]) and the inequality (6), they showed that the semigroup associated with the system is analytic and therefore exponentially stable.

In this paper we study a nonlinear porous-elastic system with delay given by

$$\rho u_{tt} - \mu u_{xx} - b \phi_x + u_t + f_1(u, \phi) = h_1(x),$$ \hspace{1cm} (17)

$$J\phi_{tt} - \delta \phi_{xx} + bu_x + \xi \phi + \mu_1 \phi_t + \mu_2 \phi_t(x, t-\tau) + f_2(u, \phi) = h_2(x),$$ \hspace{1cm} (18)

$(x, t) \in (0, 1) \times (0, \infty), \text{subject to the boundary conditions}$

$$u(0, t) = u(1, t) = \phi(0, t) = \phi(1, t) = 0, \hspace{1cm} t > 0,$$ \hspace{1cm} (19)

and initial boundary conditions

$$u(x, 0) = u_0(x), \hspace{0.2cm} u_t(x, 0) = u_1(x), \hspace{0.2cm} \phi(x, 0) = \phi_0(x),$$ \hspace{1cm} (20)

$$\phi_t(x, 0) = \phi_1(x), \hspace{1cm} x \in (0, 1),$$

$$\phi_t(x, t-\tau) = f_0(x, t-\tau), \hspace{0.2cm} (x, t) \in (0, 1) \times (0, \tau).$$ \hspace{1cm} (21)

It is noteworthy that the Timoshenko system when subjected to a single damping of the form

$$\mu_1 \psi_t(x, t) + \mu_2 \psi_t(x, t-\tau)$$ \hspace{1cm} (22)

in the equation of filament rotation, is exponentially stable (or in the case of non-linear systems, there are attractors), if equality of wave velocities is valid, but the equality of speeds for the Timoshenko system is not physically possible. Therefore, in order not to adopt equal velocities for the porous elastic system and of this form to restrict the system, the term $u_t$ was added to equation (17).
2. Well-posedness. In this section, we will establish the existence and uniqueness of global solution for the porous elastic system (17)-(21). In order to obtain the well posedness of the system (17)-(21), we will use a procedure found in [23, 47] to perform a convenient variable change

\[ z(x, y, t) = \phi_1(x, t - \tau y), \quad (x, y, t) \in (0, 1) \times (0, 1) \times (0, \infty), \]  

(23)

it is easy to verify that

\[ \tau z_t(x, y, t) + z_y(x, y, t) = 0 \quad \text{em} \quad (0, 1) \times (0, 1) \times (0, \infty). \]  

(24)

Replacing \( z \) in (17)-(21) we get

\[ \rho u_{tt} - \mu u_{xx} - b \phi_x + u_t + f_1(u, \phi) = h_1(x), \]  

(25)

\[ J\phi_{tt} - \delta \phi_{xx} + bu_x + \xi \phi + \mu_1 \phi_t + \mu_2 z(1, \cdot) + f_2(u, \phi) = h_2(x), \]  

(26)

\[ \tau z_t + z_y = 0, \]  

(27)

with initial conditions

\[ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad \phi(x, 0) = \phi_0(x), \quad \phi_t(x, 0) = \phi_1(x), \; x \in (0, 1), \]  

(28)

and boundary conditions

\[ u(0, t) = u(1, t) = \phi(0, t) = \phi(1, t) = 0, \quad t > 0, \]  

\[ z(x, 0, t) = \phi_t(x, t), \quad x \in (0, 1), \quad t > 0. \]  

(30)

For reasons of economy of notation, we will write \( z_1 = z_1(x, t) \) to represent \( z = z(x, y, t) \) when \( y = 1 \) is constant. In this way, we can write the system (3)-(8) in the form of a semilinear abstract initial value problem, in the unknown \( \Psi \) as follows

\[
\begin{aligned}
\left\{
\begin{array}{l}
\Psi_t(t) = A\Psi(t) + \mathcal{F}(\Psi(t)) \quad t > 0, \\
\Psi(0) = \Psi_0,
\end{array}
\right.
\end{aligned}
\]  

(31)

where \( \Psi_0 = (u_0, u_1, \phi_0, \phi_1, f_0) \), \( A : D(A) \subset \mathcal{H} \to \mathcal{H} \) and \( \mathcal{F} : \mathcal{H} \to \mathcal{H} \) are operators given by

\[
AW = \begin{pmatrix}
\frac{\nu}{\rho} v_{xx} + \frac{b}{\rho} \varphi_x - \frac{1}{\rho} w \\
\frac{\delta}{\tau} \varphi_{xx} - \frac{b}{\rho} v_x - \frac{\xi}{\rho} \varphi - \frac{\mu_1}{\rho} \psi - \frac{\mu_2}{\rho} p(x, 1)
\end{pmatrix}, \quad \mathcal{F}(W) = \begin{pmatrix}
\frac{1}{\rho} [h_1 - f_1(\varphi, \psi)] \\
\frac{1}{\tau} [h_2 - f_2(\varphi, \psi)]
\end{pmatrix},
\]

with

\[ W \in D(A) = \{(v, w, \varphi, \psi, p) \in \mathcal{H} : \quad p(x, 0) = \psi(x) \in (0, 1)\} \]  

(32)

where \( p = p(x, y) \) and

\[ \mathcal{H} = (H^2(0, 1) \cap H^1_0(0, 1)) \times H^1_0(0, 1) \times (H^2(0, 1) \cap H^1_0(0, 1)) \times H^1_0(0, 1) \times L^2(0, 1; H^1_0(0, 1)). \]  

(33)

The phase space \( \mathcal{H} \) is given by

\[ \mathcal{H} = H^1_0(0, 1) \times L^2(0, 1) \times H^1_0(0, 1) \times L^2(0, 1) \times L^2((0, 1) \times (0, 1)). \]  

(34)

Consider \( \vartheta \) a positive constant satisfying

\[ \tau \mu_2 < \vartheta < \tau (2\mu_1 - \mu_2). \]  

(35)
The choice of \( \vartheta \) is possible, since \( \mu_2 < \mu_1 \). In addition, for economy of notation, let us consider
\[
\chi := \mu - \frac{b^2}{\xi} \geq 0. \tag{36}
\]

We define in \( \mathcal{H} \) the following inner product \( (\cdot, \cdot)_{\mathcal{H}} \) and norm \( \| \cdot \|_{\mathcal{H}} \),
\[
(W, W)_{\mathcal{H}} = \rho(w, \bar{w})_2 + J(\psi, \dot{\psi})_2 + \delta(\varphi_x, \dot{\varphi}_x)_2 + \chi(v_x, \dot{v}_x)_2 + \vartheta(p, \dot{p})_2,
\]
\[
\|W\|_{\mathcal{H}}^2 = \rho\|w\|_2^2 + J\|\psi\|_2^2 + \delta\|\varphi_x\|_2^2 + \chi\|v_x\|_2^2 + \vartheta\|p\|_2^2,
\]
respectively, for \( W = (v, w, \varphi, \psi, p) \), \( \bar{W} = (\bar{v}, \bar{w}, \bar{\varphi}, \bar{\psi}, \bar{p}) \) in \( \mathcal{H} \), where \( (\cdot, \cdot)_2 \) and \( \| \cdot \|_2 \) are inner product and norm in \( L^2(0,1) \) (or in \( L^2((0,1) \times (0,1)) \) in the case of the function \( p \)), respectively.

2.1. **Assumptions.** In order to obtain the results of existence and uniqueness of solution as well as the existence of attractors, consider the following assumptions about the functions present in (17)-(18).

(A1): \( h_i \in L^2(0,1) \) for \( i = 1,2; \)

(A2): The functions \( f_i : \mathbb{R}^2 \to \mathbb{R} \), \( i = 1,2 \) are locally Lipschitz continuous on each of its arguments, namely, there exist a constant \( \gamma_i \geq 1 \) and a continuous function \( \sigma_i : \mathbb{R} \to \mathbb{R}_+ \) such that
\[
|f_i(s_1, r) - f_i(s_2, r)| \leq \sigma_i(|r|)(1 + |s_1|^{\gamma_i} + |s_2|^{\gamma_i})|s_1 - s_2|, \tag{38}
\]
\[
|f_i(s, r_1) - f_i(s, r_2)| \leq \sigma_i(|s|)(1 + |r_1|^{\gamma_i} + |r_2|^{\gamma_i})|r_1 - r_2|, \tag{39}
\]
for every \( (s, r), (s_j, r_j) \in \mathbb{R}^2 \), \( j = 1,2. \)

(A3): There is a \( C^2 \) function \( F : \mathbb{R}^2 \to \mathbb{R} \) such that
\[
\nabla F = (f_1, f_2) \tag{40}
\]
and
\[
F(s, r) \geq -\theta_2 - \alpha_1 |r|^2 - \theta_1 |s|^2, \quad \forall (s, r) \in \mathbb{R}^2, \tag{41}
\]
\[
F(s, r) \leq f_1(s, r)s + f_2(s, r)r + \theta_1 |s|^2 + \alpha_1 |r|^2 + \theta_2, \quad \forall (s, r) \in \mathbb{R}^2, \tag{42}
\]
for some constants
\[
0 \leq \theta_1 \leq \frac{\chi}{8}, \quad 0 \leq \alpha_1 \leq \frac{\delta}{8}, \quad \text{and} \quad \theta_2 \geq 0. \tag{43}
\]

2.2. **Strong and mild solutions.**

**Lemma 2.1.** Assume that \( \mu_2 < \mu_1 \), then the operator \( \mathcal{A} \) defined in (10) is an infinitesimal generator of a \( C_0 \)-semigroup of contractions \( e^{\mathcal{A}t} \) in \( \mathcal{H} \).

**Proof.** Let us first observe that \( D(A) \) is dense in \( \mathcal{H} \) and for all \( W = (v, w, \varphi, \psi, p) \in D(A) \), we have
\[
(\mathcal{A}W, W)_{\mathcal{H}} = -\|w\|^2 - \left( \mu_1 - \frac{\vartheta}{2\tau} \right) \|\psi\|^2 - \frac{\vartheta}{2\tau} \|p_1\|^2 - \mu_2 \int_0^1 p_1 \psi dx,
\]
where \( p_1(x) = p(x, 1) \) for all \( x \in (0,1) \). From Young’s inequality and (13) we obtain
\[
(\mathcal{A}W, W)_{\mathcal{H}} \leq -\|w\|^2 - \left( \mu_1 - \frac{\vartheta}{2\tau} - \frac{\mu_2}{2} \right) \|\psi\|^2 - \left( \frac{\vartheta}{2\tau} - \frac{\mu_2}{2} \right) \|p_1\|^2 \leq 0,
\]
that is, the operator $A$ is dissipative. Let us now prove that $0 \in g(A)$ where $g(A)$ is the resolvent set of $A$. In fact, given $F = (g^1, g^2, g^3, g^4, g^5) \in \mathcal{H}$, we must obtain $W = (v, w, \varphi, \psi, p) \in D(A)$ such that

$$-AW = F,$$  

in terms of coordinates of the equation (22) we have

$$-w = g^1,$$  

$$-\mu v_{xx} - b\varphi_x + w = \rho g^2,$$  

$$-\psi = g^3,$$  

$$-\delta \varphi_{xx} + bv_x + \xi \varphi + \mu_1 \psi + \mu_2 p(\cdot, 1) = Jg^4,$$  

$$p_y = \tau g^5,$$

we get this way

$$w = -g^1, \in H^1_0(0, 1)$$  

$$\psi = -g^3, \in H^1_0(0, 1)$$

$$p(x, y) = \tau \int_0^y g^5(x, s)ds + g^3(x) \in L^2(0, 1; H^1_0(0, 1)),$$

replacing (28) in (24) and (29), (30) in (26) we obtain

$$-\mu v_{xx} - b\varphi_x = g^6,$$  

$$-\delta \varphi_{xx} + bv_x + \xi \varphi = g^7,$$

where

$$g^6 = \rho g^2 + g^1 \in L^2(0, 1),$$  

$$g^7 = Jg^4 + \mu_1 g^3 - \mu_2 \tau \int_0^1 g^5(\cdot, s)ds - \mu_2 g^3 \in L^2(0, 1).$$

Multiplying (31) by $\hat{v} \in H^1_0(0, 1)$, (31) by $\hat{\varphi} \in H^1_0(0, 1)$, integrating over $[0, 1]$ with respect to $x$ and adding the results we obtain

$$a((v, \varphi), (\hat{v}, \hat{\varphi})) = h(\hat{v}, \hat{\varphi}),$$

where $a : H^1_0(0, 1) \times H^1_0(0, 1) \rightarrow \mathbb{R}$ is a bilinear functional and $h : H^1_0(0, 1) \rightarrow \mathbb{R}$ is a linear functional given by

$$a((v, \varphi), (\hat{v}, \hat{\varphi})) = \delta(\varphi_x, \hat{\varphi}_x) + \mu(v_x, \hat{\varphi}_x) + b(v_x, \varphi) + \mu_2 \hat{\varphi}_x + \xi(\varphi, \hat{\varphi})_2$$

$$h(\hat{v}, \hat{\varphi}) = (g^6, \hat{v})_2 + (g^7, \hat{\varphi})_2.$$

It is not difficult to show that $a$ is continuous and coercive and $h$ is continuous. It follows from the Lax-Milgram Theorem that there exists $(v, \varphi) \in H^1_0(0, 1) \times H^1_0(0, 1)$ such that (33) is verified for all $(\hat{v}, \hat{\varphi}) \in H^1_0(0, 1) \times H^1_0(0, 1)$ and from (31)-(32), we obtain

$$\mu v_{xx} = -g^6 - b\varphi_x \in L^2(0, 1),$$

$$\delta \varphi_{xx} = -g^7 + bv_x + \xi \varphi \in L^2(0, 1),$$

this means that $v, \varphi \in H^2(0, 1)$ and thus $(v, w, \varphi, \psi, p) \in D(A)$ is solution of (22), which implies $0 \in g(A)$. It follows from Lumer-Phillips’s Theorem [34, Theorem 1.2.4] that $A$ is the infinitesimal generator of a $C_0$ semigroup of contractions thus proving the Lemma 2.1.

The proof of the following Lemma can be found in [18].
Lemma 2.2. Assume that (A1)-(A2) are valid, then the operator $\mathcal{F}$ defined in (10) is locally Lipschitz, that is, for all $M > 0$, there exists a constant $L_M > 0$ (depending on $M$) such that, if $W, \dot{W} \in \mathcal{H}$, with $\|W\|_{\mathcal{H}}, \|\dot{W}\|_{\mathcal{H}} \leq M$ we have
\[
\|\mathcal{F}(W) - \mathcal{F}(\dot{W})\|_{\mathcal{H}} \leq L_M\|W - \dot{W}\|_{\mathcal{H}}.
\] (56)

Consider below the concepts of strong solution and mild solution that will be used in the remainder of this work.

(S1): A function $\Psi : [0, T) \rightarrow \mathcal{H}$, with $T > 0$, is a strong solution of (9), if $\Psi$ is continuous on $[0, T)$, continuously differentiable on $(0, T)$, with $\Psi(t) \in D(\mathcal{A})$ for all $t \in (0, T)$ and satisfying (9) on $[0, T)$ almost everywhere.

(S1i): A function $\Psi \in C([0, T), \mathcal{H})$, $T > 0$, satisfying the integral equation
\[
\Psi(t) = e^{At}\Psi_0 + \int_0^t e^{A(t-s)}\mathcal{F}(\Psi(s))ds, \quad t \in [0, T),
\] (57)

is called mild solution of (9).

Lemma 2.3 (Local solution). Suppose $\mu_2 < \mu_1$, (A1)-(A2) are valid. If $U_0 \in \mathcal{H}$, then there exists $T_{\text{max}} > 0$ such that (9) has a unique mild solution $U : [0, T_{\text{max}}) \rightarrow \mathcal{H}$. In addition, if $U_0 \in D(\mathcal{A})$, then the mild solution is strong solution.

Proof. The result follows directly from Lemmas 2.1, 2.2 and of [43, Chap. 6, Theorems 1.4 and 1.5].

Remark 1. It is worth noting that by density arguments, any mild solution of (3)-(8) can be approximated by strong solutions. The calculations that will be performed in this work will not need regularity better than that of the strong solutions.

2.3. Global solution. In this subsection, we will define the energy associated with the system (3)-(8) as well as prove the existence of an inequality involving this energy that will enable us to obtain a global solution for the system (3)-(8).

Let $U(t) = (u(t), u_t(t), \phi(t), \dot{\phi}(t), z(t))$ be a strong or mild solution of (3)-(8) defined on $[0, T_{\text{max}})$. The energy $E(t)$, associated with $U(t)$ is the functional defined by
\[
E(t) = \frac{\rho}{2}\|u_t(t)\|_2^2 + \frac{J}{2}\|\phi_t(t)\|_2^2 + \frac{\delta}{2}\|\phi_x(t)\|_2^2 + \frac{\chi}{2}\|u_x(t)\|_2^2 + \frac{1}{2}\left(\frac{b}{\sqrt{\xi}}\|u_x(t) + \sqrt{\xi}\phi(t)\|_2^2 + \frac{\vartheta}{2}\|z(t)\|_2^2\right) + \int_0^1 F(u(t), \phi(t))dx + \int_0^1 h_1u_t(t)dx + \int_0^1 h_2\phi(t)dx,
\] (58)

for all $t \in [0, T_{\text{max}})$. 

Lemma 2.4. Assume that $\mu_2 < \mu_1$ and (A1)-(A3) are valid. Then the functional energy (36) is non-decreasing, more precisely, for any strong solution $U(t) = (u(t), u_t(t), \phi(t), \dot{\phi}(t), z(t))$ of (3)-(8), defined in $[0, T_{\text{max}})$ we have
\[
\frac{dE(t)}{dt} \leq -\|u_t(t)\|_2^2 - \left(\mu_1 - \frac{\vartheta}{2\tau} - \frac{\mu_2}{2}\right)\|\phi_t(t)\|_2^2 - \left(\frac{\vartheta}{2\tau} - \frac{\mu_2}{2}\right)\|z_1(t)\|_2^2 \leq 0,
\] (59)

t \in [0, T_{\text{max}}). In addition, there exists a constant $K = K(||h_1||_2, ||h_2||_2) > 0$ such that
\[
\|U(t)\|_\mathcal{H}^2 \leq 4E(t) + K, \quad \forall t \in [0, T_{\text{max}}).
\] (60)
Proof. Multiplying (3) by $u_t$ and (4) by $\phi_1$, integrating over $[0, 1]$ with respect to $x$ and adding the results and applying the Young’s inequality, we obtain

$$
\frac{d}{dt} \left\{ \frac{\rho}{2} \| u_t(t) \|^2 + \frac{J}{2} \| \phi_1(t) \|^2 + \frac{\delta}{2} \| \phi_x(t) \|^2 + \frac{\chi}{2} \| u_x(t) \|^2 \right\} + \frac{1}{2} \left\| \frac{b}{\sqrt{\xi}} u_x(t) + \sqrt{\xi} \phi(t) \right\|^2 + \int_0^1 F(\varphi(t), \psi(t))dx - \int_0^1 h_1 \varphi(t)dx - \int_0^1 h_2 \psi(t)dx \right\}
$$

$$
\leq -\| u_t(t) \|^2 - \left( \mu_1 - \frac{\mu_2}{2} \right) \| \phi(t) \|^2 + \frac{\mu_2}{2} \| z_1(t) \|^2.
$$

Multiplying (5) by $\frac{\partial}{\partial t} \varphi$ and integrating over $[0, 1] \times [0, 1]$ with respect to $y$ and $x$ we arrive at

$$
\frac{\partial}{\partial t} \int_0^1 \frac{1}{2} \| \varphi(t) \|^2 dydx = \frac{\partial}{\partial t} \| \phi(t) \|^2 - \frac{\partial}{\partial t} \| \phi(t) \|^2.
$$

Combining (39) and (40), we obtain (37). On the other hand, it follows from (15) and (36) that

$$
E(t) = \frac{1}{2} \| U(t) \|_{\mathcal{H}}^2 + \int_0^1 F(\varphi(t), \psi(t))dx - \int_0^1 h_1 \varphi(t)dx - \int_0^1 h_2 \psi(t)dx.
$$

From (19) and Poincaré’s inequality we have

$$
\int_0^1 F(u(t), \phi(t))dx \geq -\theta_2 - \alpha_1 \| \phi(t) \|^2 - \theta_1 \| u(t) \|^2
$$

$$
\geq -\theta_2 - \alpha_1 \| \phi_x(t) \|^2 - \theta_1 \| u_x(t) \|^2.
$$

Using the Young’s and Poincaré’s inequalities, we obtain

$$
\int_0^1 h_1 \varphi(t)dx \leq \frac{1}{2 \varepsilon_1} \| h_1(t) \|^2 + \frac{\varepsilon_1}{2} \| u_x(t) \|^2
$$

$$
\int_0^1 h_2 \psi(t)dx \leq \frac{1}{2 \varepsilon_2} \| h_2(t) \|^2 + \frac{\varepsilon_2}{2} \| \phi_x(t) \|^2.
$$

Combining (41), (42), (43) and (44), we arrive at

$$
\frac{1}{2} \| U(t) \|^2 - \left[ \theta_2 + \frac{1}{2 \varepsilon_1} \| h_1(t) \|^2 + \frac{1}{2 \varepsilon_2} \| h_2(t) \|^2 \right] \leq E(t) + \left( \alpha_1 + \frac{\varepsilon_2}{2} \right) \| \phi_x(t) \|^2 + \left( \theta_1 + \frac{\varepsilon_1}{2} \right) \| u_x(t) \|^2.
$$

Taking appropriate $\varepsilon_1, \varepsilon_2 > 0$, we obtain

$$
\frac{1}{2} \| U(t) \|^2 - 2K \leq 2E(t),
$$

which proves (38) thus completing the proof of Lemma 2.4. \qed

**Theorem 2.5 (Global Solutions)**. Suppose $\mu_2 < \mu_1$ and (A1)-(A3) are valid.

(i): The local solutions obtained in Lemma 2.3 are global solutions, that is, $T_{\text{max}} = \infty$;

(ii): If $U_1$ and $U_2$ are two mild solutions of the problem (9) and $T > 0$, then there exists a positive constant $C_0 = C_0(\| U_1(0) \|_{\mathcal{H}}, \| U_2(0) \|_{\mathcal{H}})$ such that

$$
\| U_1(t) - U_2(t) \|_{\mathcal{H}} \leq e^{C_0 t} \| U_1(0) - U_2(0) \|_{\mathcal{H}}, \quad \forall t \in [0, T].
$$
Proof. (i) From (37), we have
\[ E(t) \leq E(0), \quad \forall t \in [0, T_{\text{max}}) \] (70)
and combining with (46), we obtain
\[ \frac{1}{4} \|U(t)\|_H^2 \leq E(0) + K, \quad \forall t \in [0, T_{\text{max}}). \] (71)
Therefore, it follows from [43, Chap. 6, Theorems 1.4] that 
\[ T_{\text{max}} = +\infty, \]
which proves (i).

(ii) Since \( U_1 \) and \( U_2 \) are mild solutions of (9), we have
\[ \|U_1(t) - U_2(t)\|_H = \left\| e^{tA}(U_1(0) - U_2(0)) - \int_0^t e^{(t-s)A}(F(U_1(s)) - F(U_2(s)))ds \right\|_H. \]
Being \( e^{tA} \) a semigroup of contractions, we have
\[ \|U_1(t) - U_2(t)\|_H \leq \|U_1(0) - U_2(0)\|_H + \int_0^t \|F(U_1(s)) - F(U_2(s))\|_H ds. \]
From Lemma 2.2 and (49), there exists a positive constant \( C_0 \) depending on \( U_1(0) \) and \( U_2(0) \) such that
\[ \|U_1(t) - U_2(t)\|_H \leq \|U_1(0) - U_2(0)\|_H + C_0 \int_0^t \|U_1(s) - U_2(s)\|_H ds, \quad \forall t \in [0, T]. \]
Applying the Gronwall inequality we get (47). This completes the proof of Theorem 2.5.

3. Long-time dynamics. From Theorem 2.5 we can define the dynamical system 
\( (\mathcal{H}, S(t)) \), associated with the problem (3)-(8), where \( \mathcal{H} \) was defined in (12) and 
\( S(t) \) is semigroup (evolution operator) given by
\[ S(t)U_0 = U(t), \quad \forall t \geq 0, \quad U_0 = (u_0, u_1, \phi_0, \phi_1, f_0) \in \mathcal{H}, \] (72)
where \( U(t) \) is mild solution of (9). Key concepts as well as main results related to 
dynamical systems can be found, among others, in [7, 11, 12, 13, 24, 46, 52]. Here we 
exploit some of these preliminary concepts and apply the concept of quasi-stability 
given in Chueshov and Lasiecka [12, 13].

3.1. Some concepts and results related to dynamical systems. In this subsection, we will outline some concepts and results related to dynamical systems that 
will be important for this work. In the sequence, \( H \) will represent a generic Banach 
space and \( S(t) \) a strongly continuous evolution operator.

Definition 3.1. We say that a set \( B \subset H \) is absorbing for \( (H, S(t)) \), if for any 
bounded set \( D \subset H \), there exists \( t_D \) such that
\[ S(t)D \subset B, \quad \text{for all} \quad t > t_D. \] (73)
A dynamical system \( (H, S(t)) \) is called dissipative, if it has a bounded absorbing 
set.

Definition 3.2. A closed and bounded set \( \mathfrak{A} \subset H \), is a global attractor for \( (H, S(t)) \)
if:
\[ \text{a):} \quad \mathfrak{A} \text{ is an invariant set, that is} \]
\[ S(t)\mathfrak{A} = \mathfrak{A}, \quad \forall t \geq 0; \]
b): The set $\mathfrak{A}$ uniformly attracts bounded sets, that is, for every bounded set $B \subset H$, we have
$$
\lim_{t \to \infty} d_H \{S(t)B|\mathfrak{A}\} = 0,
$$
where $d_H \{A|B\}$ is Hausdorff’s semi-distance between sets $A$ and $B$, that is
$$
d_H \{A, B\} = \sup_{x \in A} \text{dist}(x, B).
$$

Remark 2. It is clear that if a dynamical system $(H, S(t))$ has a global attractor, then it will be dissipative (cf.[52]), in this case we call radius of dissipativity the value $R > 0$, such that $B \subset \{x \in H; \|x\|_H \leq R\}$ where $B$ is an absorbing bounded set for $(H, S(t))$.

Definition 3.3. A dynamical system $(H, S(t))$ is said asymptotically smooth if for any positively invariant bounded set $D \subset H$, that is, $S(t)D \subset D$ for all $t > 0$, there exists a compact set $K \subset D$, where $\overline{D}$ is the closure of $D$, such that
$$
\lim_{t \to +\infty} d_H \{S(t)D|K\} = 0,
$$

Definition 3.4. Let $\mathcal{N}$ be the sets of stationary points of $(H, S(t))$, that is
$$
\mathcal{N} = \{h \in H; S(t)h = h, \forall t \geq 0\},
$$
an unstable manifold emanating from $\mathcal{N}$, represented by $\mathcal{M}^u(\mathcal{N})$, is the set of all $h \in H$ such that there is a full trajectory $\gamma = \{u(t); t \in \mathbb{R}\}$ satisfying
$$
u(0) = h \quad \text{and} \quad \lim_{t \to -\infty} \text{dist}_H(u(t), \mathcal{N}) = 0.
$$
It is clear that $\mathcal{M}^u(\mathcal{N})$ is an invariant set for $(H, S(t))$ and if $\mathfrak{A} \subset H$ is global attractor for $(H, S(t))$, then $\mathcal{M}^u(\mathcal{N}) \subset \mathfrak{A}$ (cf. [7, 13]).

Definition 3.5. The dynamical system $(H, S(t))$ is called gradient, if there exists a strict Lyapunov function on $H$, that is, there exists a continuous function $\Phi$ such that $t \mapsto \Phi(S(t)y)$ is non-increasing for any $y \in H$, and if $\Phi(S(t)y_0) = \Phi(y_0)$ for all $t > 0$ and some $y_0 \in H$, then $y_0$ is a stationary point of $(H, S(t))$.

The fractal dimension of a compact set $M$ in $H$ is defined by
$$
\dim_H^f M = \lim_{\varepsilon \to 0} \sup \frac{\ln n(M, \varepsilon)}{\ln(1/\varepsilon)},
$$
where $n(M, \varepsilon)$ is the minimal number of closed balls of radius $\varepsilon$ which covers $M$.

The proof of the following Theorem can be found in [13] p. 360.

Theorem 3.6. Let $(H, S(t))$ be a gradient and asymptotically smooth dynamical system. Assume that the Lyapunov function $\Phi(y)$ of $(H, S(t))$ is bounded from above on any bounded subset of $H$ and the set $\Phi_R = \{y; \Phi(y) \leq R\}$ is bounded for every $R$. If the set $\mathcal{N}$ of stationary points of $(H, S(t))$ is bounded, then $(H, S(t))$ possesses a compact global attractor $\mathfrak{A} = \mathcal{M}^u(\mathcal{N})$.

Let $X$, $Y$ and $Z$ be reflexives Banach spaces with $X$ compactly embedded in $Y$. We consider the space $H = X \times Y \times Z$, with norm
$$
\|h\|_H := \|\pi_0\|_X^2 + \|\pi_1\|_Y^2 + \|\eta_0\|_Z^2, \quad h = (\pi_0, \pi_1, \eta_0) \in H,
$$
and the dynamical system $(H, S(t))$ given by an evolution operator
$$
S(t)h_0 = (\pi(t), \pi_t(\eta(t))) \quad t \geq 0, \quad h_0 = (\pi(0), \pi_t(0), \eta(0)) \in H,
$$

(74)
where the functions $\pi(t)$ and $\eta(t)$ possess the properties

$$\pi \in C(\mathbb{R}_+, X) \cap C^1(\mathbb{R}_+, Y), \quad \eta \in C(\mathbb{R}_+, Z). \quad (76)$$

The dynamical system $(H, S(t))$ is called quasi-stable on a set $B \subset H$ if there exist a compact seminorm $\eta_X(\cdot)$ on the space $X$ and nonnegative scalar functions $a(t)$, $b(t)$ and $c(t)$ on $\mathbb{R}_+$ such that

QS1): $a(t)$ and $c(t)$ are locally bounded on $[0, \infty)$.
QS2): $b(t) \in L^1(\mathbb{R}_+)$ possesses the property

$$\lim_{t \to \infty} b(t) = 0, \quad (77)$$

QS3): for every $h_1, h_2 \in B$ and $t > 0$ the following relations

$$\|S(t)h_1 - S(t)h_2\|_H^2 \leq a(t)\|h_1 - h_2\|_H^2 \quad (78)$$

and

$$\|S(t)h_1 - S(t)h_2\|_H^2 \leq b(t)\|h_1 - h_2\|_H^2 + c(t) \sup_{0 \leq s \leq t} [\eta_X(\pi^1(s) - \pi^2(s))]^2 \quad (79)$$

hold. Here we denote $S(t)h_i = (\pi^i(t), \pi^i_1(t), \pi^i_2(t)), i = 1, 2$.

The following two results can be found in [13, Chapter 7], show us how strong the property of quasi-stability is for a dynamical system. The first, relates the quasi-stability to the asymptotically smooth and the second relates the quasi-stability to the fractal dimension of an attractor.

**Theorem 3.7.** Let $(H, S(t))$ be a dynamical system with the evolution operator of the form (4). Assume that $(H, S(t))$ is quasi-stable over bounded forward invariant set $B \subset H$. Then, $(H, S(t))$ is asymptotically smooth.

**Theorem 3.8.** Suppose $(H, S(t))$ be a dynamical system with the evolution operator of the form (4). Assume that $(H, S(t))$ possesses a compact global attractor $A$ and is quasi-stable on $A$. Then the fractal dimension of $A$ is finite.

**Definition 3.9.** A compact set $A_{\exp} \subset H$ is called a fractal exponential attractor if it has finite fractal dimension, is positively invariant, and for any bounded set $B \subset H$, there exist constants $t_B, C_B > 0$ and $\gamma_B > 0$ such that for all $t \geq t_B$,

$$\text{dist}_H(S(t)B, A_{\exp}) \leq C_B \exp(-\gamma_B(t-t_B)).$$

In some cases one can prove the existence of an exponential attractor whose dimension is finite in some extended space $\tilde{H} \supset H$ only. We frequently call this exponentially attracting set a generalized exponential attractor.

**3.2. Main results.** In the following we give our main results on long-time dynamics of the problem.

**Theorem 3.10.** Suppose that $\mu_2 < \mu_1$ and (A1)-(A3) are valid. we have:

I. The dynamical system $(H, S(t))$ given in (1) is quasi-stable on any bounded positively invariant set $B \subset H$.

II. The dynamical system $(H, S(t))$ possesses a unique compact global attractor $A \subset H$, which is characterized by the unstable manifold $\mathfrak{N} = M^u(N)$, emanating from the set $N = \{U = (u, 0, 0, 0) \in H; AU + F(U) = 0\}$ of stationary solutions.
III. Every trajectory stabilizes to the set $\mathcal{N}$, namely, for any $U \in \mathcal{H}$ one has
$$\lim_{t \to +\infty} \text{dist}_\mathcal{N}(S(t)U, \mathcal{N}) = 0.$$  
In particular, there exists a global minimal attractor $\mathfrak{A}_{\text{min}}$ given by $\mathfrak{A}_{\text{min}} = \mathcal{N}$.

IV. The attractor $\mathfrak{A}$ has finite fractal and Hausdorff dimension $\dim^f \mathfrak{A}$.

V. For any full trajectory
$$\left(u(t), \phi(t), u_t(t), \phi_t(t), z(t)\right) \text{ in } \mathfrak{A}$$
has further regularity
$$\left(u_t, \phi_t, u_{tt}, \phi_{tt}, z_t\right) \in L^\infty(\mathbb{R}, \mathcal{H}).$$
Moreover, there exists $R > 0$ such that
$$\|(u_t(t), \phi_t(t))\|_{(H^2_0(\Omega, L))}^2 + \|(u_{tt}(t), \phi_{tt}(t))\|_{(L^2(\Omega, L))}^2 + \|z_t(t)\|_{L^2((0,1) \times (0,1))}^2 \leq R^2, \quad \forall t \in \mathbb{R}. \quad (82)$$

VI. The dynamical system $(\mathcal{H}, S(t))$ possesses a generalized exponential attractor representing $\mathfrak{A}_{\exp} \subset \mathcal{H}$ with finite dimension in the extended space $\mathcal{H}_{-\delta} := L^2(0,1) \times H^{-1}(0,1) \times L^2(0,1) \times H^{-1}(0,1) \times L^2((0,1) \times (0,1))$, which is isomorphic to space $L^2(0,1) \times L^2(0,1) \times H^{-1}(0,1) \times H^{-1}(0,1) \times L^2((0,1) \times (0,1))$. In addition, from the interpolation theorem, for all $0 < \delta < 1$ there exists a generalized fractal exponential attractor whose fractal dimension is finite in the extended space $\mathcal{H}_{-\delta}$, where $\mathcal{H}_0 := \mathcal{H}$, and $\mathcal{H} \subset \mathcal{H}_{-\delta} \subset \mathcal{H}_{-1}$.

3.3. Proofs of main results.

3.3.1. Technical lemmas. To prove Theorem 3.10, we need the following technical lemmas.

Lemma 3.11. Suppose that $\mu_2 < \mu_1$ and $(A1)-(A3)$ hold. Then the dynamical system $(\mathcal{H}, S(t))$ is gradient. In addition there exists a Lyapunov functional $\Phi$ defined in $\mathcal{H}$ such that

a): the Lyapunov functional $\Phi$ is bounded from above on any bounded subset of $\mathcal{H}$;

b): the set $\Phi_R = \{U \in \mathcal{H} \mid \Phi(U) \leq R\}$ is bounded in $\mathcal{H}$ for every $R > 0$.

Proof. Let us consider the functional energy defined in (36) as the Lyapunov function, that is, $\Phi \equiv E$. Thus, given $U_0 = (u_0, u_1, \phi_0, \phi_1, z_0) \in \mathcal{H}$, it follows from the Lemma 2.4 that the function $t \mapsto \Phi(S(t)U_0)$ is non-increasing and

$$\Phi(S(t)U_0) + \int_0^t \left[\|u(s)\|_2^2 + \left(\mu_1 - \frac{\mu_2}{2}\right)\|\phi_t(s)\|_2^2\right]ds + \left(\frac{\mu_2}{2}\right)\int_0^t \|z_1(s)\|_2^2 ds \leq \Phi(U_0), \quad \forall t \geq 0, \quad (84)$$

with
$$\mu_1 - \frac{\xi}{2\tau} - \frac{\mu_2}{2} > 0 \quad \text{and} \quad \frac{\xi}{2\tau} - \frac{\mu_2}{2} > 0. \quad (85)$$

If $\Phi(S(t)U_0) = \Phi(U_0)$ for all $t \geq 0$ then, from (13), we have
$$u_t(t) = 0, \quad \phi_t(t) = 0, \quad z_1(t) = 0, \quad \text{a.e. in } (0,1), \forall t \geq 0, \quad (86)$$

...
from (7) we have $f_0 = 0$ and from (1) we obtain

$$z(x, y, t) = \phi_1(x, t - \tau y) = 0, \quad (x, y) \in (0, 1) \times (0, 1), \quad t > 0,$$

(87) that implies

$$u(t) = u_0, \quad \phi(t) = \phi_0, \quad \text{and} \quad z(t) = 0 \quad \forall t \geq 0.$$

(88) This gives us $U(t) = S(t)U_0 = (u_0, 0, \phi_0, 0, 0)$ for all $t \geq 0$, that is, $U_0$ is a stationary point of $(\mathcal{H}, S(t))$, thus proving that $\Phi$ is a strict Lyapunov function of $(\mathcal{H}, S(t))$ and therefore, the dynamical system is gradient.

It is easy to see from (13) that $\Phi$ is bounded from above on bounded subsets of $\mathcal{H}$, which proves (a). Given $W_0 \in \Phi_R$, consider $W(t)$ the mild solution corresponding to $W_0$, from the inequalities (46) and (13) we have

$$\|W(t)\|_\mathcal{H} \leq 4 \Phi(S(t)W_0) + 4K_{E_1} \leq 4\Phi(W_0) + 4K_{E_1}, \quad t \geq 0,$$

for $t = 0$, we obtain

$$\|W_0\|_\mathcal{H} \leq 4R + 4K_{E_1},$$

showing thus $\Phi_R$ is a bounded set of $\mathcal{H}$, which proves (b) and completes the proof of the Lemma 3.11.

Lemma 3.12. The set $\mathcal{N} = \{U = (u, 0, \phi, 0, 0) \in \mathcal{H}; AU + F(U) = 0\}$ of stationary solutions is bounded in $\mathcal{H}$.

Proof. We know that $u$ and $\phi$ satisfy

$$-\mu u_{xx} - b\phi_x + f_1(u, \phi) = h_1 \quad \text{in} \quad (0, 1),$$

(89)

$$-\delta \phi_{xx} + bu_x + \xi \phi + f_2(u, \phi) = h_2 \quad \text{in} \quad (0, 1).$$

(90)

Multiplying (18) by $u$ and (19) by $\phi$, integrating over $[0, 1]$ and adding the results, we obtain

$$\delta \|\phi_x\|_2^2 + \chi \|u_x\|_2^2 + \left\| \frac{b}{\sqrt{\xi}} u_x + \sqrt{\xi} \phi \right\|_2^2 + \int_0^1 [f_1(u, \phi)u + f_2(u, \phi)\phi]dx = \int_0^1 [h_1u + h_2\phi]dx.$$  

(91)

From (19)-(20), we have

$$\int_0^1 [f_1(u, \phi)u + f_2(u, \phi)\phi]dx \geq -2\theta_2 - 2\theta_1\|u\|_2^2 - 2\alpha_1\|\phi\|_2^2 \geq -2\theta_2 - 2\theta_1\|u_x\|_2^2 - 2\alpha_1\|\phi_x\|_2^2.$$  

(92)

By Hölder’s and Poincaré’s inequalities, we have

$$\int_0^1 [h_1u + h_2\phi]dx \leq \|h_1\|_2\|u\|_2 + \|h_2\|_2\|\phi\|_2$$

$$\leq \|h_1\|_2\|u_x\|_2 + \|h_2\|_2\|\phi_x\|_2$$

$$\leq \frac{\alpha}{2}\|u_x\|_2^2 + \frac{\delta}{2}\|\phi_x\|_2^2 + \frac{1}{2\chi}\|h_1\|_2^2 + \frac{1}{2\delta}\|h_2\|_2^2.$$  

(93)

From (20)-(22) and (21) we obtain

$$\frac{1}{4}\|U\|_\mathcal{H}^2 = \frac{\delta}{4}\|\phi_x\|_2^2 + \frac{\chi}{4}\|u_x\|_2^2 + \frac{b}{4\sqrt{\xi}} u_x + \sqrt{\xi} \phi \right\|_2^2 \leq \frac{1}{2\chi}\|h_1\|_2^2 + \frac{1}{2\delta}\|h_2\|_2^2 + 2\theta_2,$$

showing that $\mathcal{N}$ is bounded in $\mathcal{H}$, this completes the proof of the Lemma 3.12. □
The following lemma will be crucial to obtain the existence of a global attractor for the dynamical system \((\mathcal{H}, S(t))\) and its properties. We usually called stabilizability inequality.

**Lemma 3.13 (Stabilizability Inequality).** Suppose the assumptions of Theorem 3.10 hold. Let \(S(t)U_i = (u^i(t), u^i_1(t), \phi^i(t), \phi^i_1(t), z^i(t)) \ (i = 1, 2)\) be the mild solutions of problem (3)-(8) with initial data \(U_i\) lying in a bounded set \(B \subset \mathcal{H}\). Then there exist positive constants \(\gamma, \zeta\) and \(C_B\) such that for any \(t \geq 0\),

\[
\|S(t)U_1 - S(t)U_2\|_\mathcal{H}^2 \leq \zeta e^{-\gamma t}\|U_1 - U_2\|_\mathcal{H}^2 + C_B\int_0^t e^{-\gamma(t-s)}\left[\|p(s)\|_2^2 + \|q(s)\|_2^2\right]ds,
\]

where \(p = u^1 - u^2\) and \(q = \phi^1 - \phi^2\).

**Proof.** Consider the representation

\[
U(t) = S(t)U_1 - S(t)U_2 = (p(t), p_1(t), q(t), q_1(t), w(t)), \quad t \geq 0
\]

where \(w = z^1 - z^2\). Thus \(U(t)\), in the sense of mild solution, solves the following system

\[
\rho p_{tt} - \rho p_{xx} - b q_x + p_t + f_1(u^1, \phi^1) - f_1(u^2, \phi^2) = 0,
\]

\[
J q_{tt} - \delta q_{xx} + b p_x + \xi q + \mu_1 q_t + \mu_2 w(x, t) + f_2(u^1, \phi^1) - f_2(u^2, \phi^2) = 0,
\]

\[
\tau w_t + w_y = 0.
\]

Multiplying (24) by \(p_t\) and (25) by \(q_t\), integrating with respect to \(x\) over \([0, 1]\) and adding the results, we obtain

\[
\frac{1}{2} \frac{d}{dt} \left( \rho \|p_t\|_2^2 + J \|q_t\|_2^2 + \delta \|q_x\|_2^2 + \chi \|p_x\|_2^2 + \left( \frac{b}{\sqrt{\xi}} p_x + \sqrt{\xi} q \right)^2 \right) =
\]

\[
- \int_0^1 (\Delta f_1) p_t dx - \int_0^1 (\Delta f_2) q_t dx - \|p_t\|_2^2 - \mu_1 \|q_t\|_2^2 - \mu_2 \int_0^1 w_1 q_t dx,
\]

where \(w_1(x, t) = w(x, 1, t)\) for all \((x, t) \in (0, 1) \times (0, \infty)\) e

\[
\Delta f_i = f_i(u^1, \phi^1) - f_i(u^2, \phi^2)
\]

\[
= [f_i(u^1, \phi^1) - f_i(u^2, \phi^2)] + [f_i(u^1, \phi^1) - f_i(u^2, \phi^2)].
\]

Multiplying (26) by \(\frac{\partial}{\partial y} w\) and integrating with respect to \(x\) and \(y\) over \([0, 1] \times [0, 1]\) we obtain

\[
\frac{\partial}{\partial t} \frac{d}{dt} \int_0^1 \int_0^1 w^2 dy dx = - \frac{\partial}{\partial y} \int_0^1 \int_0^1 \frac{\partial}{\partial y} w^2 dy dx
\]

\[
= - \frac{\partial}{\partial y} \int_0^1 w^2|y=1 dx
\]

\[
= \frac{\partial}{\partial y} \|q_t\|_2^2 - \frac{\xi}{2\tau} \|w_1\|_2^2.
\]

Adding (27) and (29), we get

\[
\frac{1}{2} \frac{d}{dt} \left( \rho \|p_t\|_2^2 + J \|q_t\|_2^2 + \delta \|q_x\|_2^2 + \chi \|p_x\|_2^2 + \left( \frac{b}{\sqrt{\xi}} p_x + \sqrt{\xi} q \right)^2 + \|w\|_2^2 \right) =
\]

\[
- \int_0^1 (\Delta f_1) p_t dx - \int_0^1 (\Delta f_2) q_t dx - \|p_t\|_2^2 - \left( \mu_1 - \frac{\partial}{\partial y} \right) \|q_t\|_2^2 - \mu_2 \int_0^1 w_1 q_t dx - \frac{\partial}{\partial y} \|w_1\|_2^2.
\]
Considering now the functional $\mathcal{L}$ given by

$$
\mathcal{L}(t) := \rho \|p(t)\|_2^2 + J \|q(t)\|_2^2 + \delta \|q_x(t)\|_2^2 + \chi \|p_x(t)\|_2^2 + \frac{b}{\sqrt[k]{t}} |p_x(t) + \sqrt[k]{q(t)}|_2^2 + \vartheta \|w(t)\|_2^2 \equiv \|U(t)\|_{\mathcal{H}}^2
$$

(102)

Let’s now estimate the right side of (31). Since $\mathcal{B}$ is bounded, it follows from (46)-(47) the existence of a constant $K_{B_1}$ depending on $\mathcal{B}$ such that

$$
\|S(t)U_1\|_{\mathcal{H}}, \|S(t)U_2\|_{\mathcal{H}} \leq K_{B_1}, \forall t \geq 0.
$$

(103)

Since $\sigma_i$ is continuous and $H_0^1(0, 1) \hookrightarrow L^\infty(0, 1)$, there exists a constant $K_{B_2} > 0$ depending on $\mathcal{B}$ such that

$$
\sigma_i(|u^i|), \sigma_i(|\phi^j|) \leq K_{B_2} \text{ a.e in } (0, 1) \times (0, \infty), i, j = 1, 2.
$$

(104)

From (A2), (32)-(33) and Hölder’s inequality we obtain

$$
\left| \int_0^1 (\Delta f_1)p(t) \, dt \right| \leq \int_0^1 \sigma_1(|u^1(t)|)(1 + |\phi^1(t)|^{\gamma_1} + |\phi^2(t)|^{\gamma_1}) \|q(t)\|p(t) \, dt
$$

$$
+ \int_0^1 \sigma_1(|\phi^2(t)|)(1 + |u^1(t)|^{\gamma_1} + |u^2(t)|^{\gamma_1}) \|p(t)\|p(t) \, dt
$$

$$
\leq K_{B_3} (1 + \|\phi^1(t)\|_{X_1}^{\gamma_1} + \|\phi^2(t)\|_{X_2}^{\gamma_1}) \int_0^1 \|q(t)\|p(t) \, dt
$$

$$
+ K_{B_3} (1 + \|u^1(t)\|_{X_1}^{\gamma_1} + \|u^2(t)\|_{X_2}^{\gamma_1}) \int_0^1 \|p(t)\|p(t) \, dt
$$

$$
\leq K_{B_3} \|q(t)\|_2 \|p(t)\|_2 + K_{B_3} \|p(t)\|_2 \|p(t)\|_2,
$$

(105)

for some constant $K_{B_3}$ depending on $\mathcal{B}$. Applying Young’s inequality with $\varepsilon = \frac{\beta_1}{4} > 0$, there exists a constant $K_{B_4} > 0$ such that

$$
\left| \int_0^1 (\Delta f_1)p(t) \, dt \right| \leq K_{B_4} (\|p(t)\|_2^2 + \|q(t)\|_2^2) + \frac{\beta_1}{2} \|p(t)\|_2^2.
$$

(106)

In a similar way we can obtain a constant $K_{B_5} > 0$ depending on $\mathcal{B}$ such that

$$
\left| \int_0^1 (\Delta f_2)q(t) \, dt \right| \leq K_{B_5} (\|p(t)\|_2^2 + \|q(t)\|_2^2) + \frac{\beta_2}{2} \|q(t)\|_2^2.
$$

(107)

From Young’s inequality, we have

$$
\mu_2 \int_0^1 w_1(t)q(t) \, dt \leq \frac{\mu_2}{2} \|q(t)\|_2^2 + \frac{\mu_2}{2} \|w_1(t)\|_2^2.
$$

(108)

Combining (35)-(37), we arrive at

$$
\frac{1}{2} \frac{d}{dt} \mathcal{L}(t) \leq K_{B_6} (\|p(t)\|_2^2 + \|q(t)\|_2^2) - \left(1 - \frac{\beta_1}{2}\right) \|p(t)\|_2^2
$$

$$
- \left(\mu_1 - \frac{\nu_1}{2\tau} - \frac{\mu_2}{2} - \frac{\beta_2}{2}\right) \|q(t)\|_2^2 - \left(\frac{\vartheta_1}{2\tau} - \frac{\mu_2}{2}\right) \|w_1(t)\|_2^2.
$$

(109)

where $K_{B_6} = K_{B_4} + K_{B_5}$. Considering now

$$
\beta_1 = 1 \quad \text{and} \quad \beta_2 = \mu_1 - \frac{\vartheta_1}{2\tau} - \frac{\mu_2}{2} > 0,
$$

(110)
we obtain
\[
\frac{d}{dt} L(t) \leq 2K_B(\|p(t)\|_2^2 + \|q(t)\|_2^2) - \|p_t(t)\|_2^2 - \left(\mu_1 - \frac{\theta}{2\tau} - \frac{\mu_2}{2}\right)\|q(t)\|_2^2 \tag{111}
\]

We now define the following functional
\[
I(t) = N\mathcal{L}(t) + J(t) + K(t) + MP(t), \tag{112}
\]
where \(N\) and \(M\) are positive constants to be chosen later and
\[
J(t) = \rho \int_0^1 p_t(t)p(t)dx, \quad K(t) = J \int_0^1 q_t(t)q(t)dx \quad \text{and} \quad \mathcal{P}(t) = \tau \int_0^1 \int_0^1 e^{-2\tau y}w(t)dydx. \tag{113}
\]

It is not difficult to check that there exists a constant \(C_{I_1} > 0\) such that
\[
|I(t) - N\mathcal{L}(t)| \leq C_{I_1}\mathcal{L}(t), \quad \forall t \geq 0. \tag{115}
\]

Therefore, for \(N\) large enough, we obtain positive constants \(C_{I_2}\) and \(C_{I_3}\) such that
\[
C_{I_2}\mathcal{L}(t) \leq I(t) \leq C_{I_3}\mathcal{L}(t), \quad \forall t \geq 0. \tag{116}
\]

We will now show that there are positive constants \(N_1\) and \(N_B\), with \(N_B\) depending on \(B\), such that
\[
\frac{d}{dt} I(t) + N_1\mathcal{L}(t) \leq N_B(\|p(t)\|_2^2 + \|q(t)\|_2^2), \quad \forall t > 0. \tag{117}
\]

In fact, taking the derivative of \(J\), we have
\[
\frac{d}{dt} J(t) = \rho \int_0^1 p_t^2(t)dx + \rho \int_0^1 p_{tt}(t)p(t)dx
\]
\[
= \rho\|p(t)\|_2^2 + \int_0^1 [\mu p_{xx}(t) + bq_x(t) - p_t(t) - (\Delta f_1)]p(t)dx
\]
\[
= \rho\|p(t)\|_2^2 - \mu \|p_x(t)\|_2^2 - b \int_0^1 q(t)p_x(t)dx - \int_0^1 p_t(t)p(t)dx
\]
\[
- \int_0^1 (\Delta f_1)p(t)dx. \tag{118}
\]

Now, take derivative of \(K\), we obtain
\[
\frac{d}{dt} K(t) = J \int_0^1 q_t^2(t)dx + J \int_0^1 q_{tt}(t)q(t)dx
\]
\[
= J\|q_t(t)\|_2^2 + \int_0^1 [\delta q_{xx}(t) - bp_x(t) - \xi q(t) - \mu_1 q_t(t) - \mu_2 w(t)]
\]
\[
- \Delta f_2|q(t)dx
\]
\[
= \rho_2\|q(t)\|_2^2 - \delta\|q_x(t)\|_2^2 - b \int_0^1 p_x(t)q(t)dx - \xi\|q^2(t)\|_2
\]
\[
- \mu_1 \int_0^1 q_t(t)q(t)dx - \mu_2 \int_0^1 w(t)q(t)dx - \int_0^1 (\Delta f_2)q(t)dx. \tag{119}
\]
From (47)-(48), we arrived at

\[
\frac{d}{dt}[\mathcal{J}(t) + \mathcal{K}(t)] = \rho \|p(t)\|_2^2 + J \|q(t)\|_2^2 - \delta \|q_x(t)\|_2^2 - \chi \|p_x(t)\|_2^2 \\
- \left\| \frac{b}{\sqrt{\xi}} p_x(t) + \sqrt{\xi} q(t) \right\|_2^2 - \int_0^1 \left[ (\Delta f_1)p(t) + (\Delta f_2)q(t) \right] dx 
\]

(120)

By analogous arguments to (34), we can conclude the existence of a constant \(K_{\mathcal{B}_t} > 0\) depending on \(\mathcal{B}\), such that

\[
\int_0^1 [(\Delta f_1)p(t) + (\Delta f_2)q(t)] dx \leq K_{\mathcal{B}_t}(\|p(t)\|_2^2 + \|q(t)\|_2^2). 
\]

(121)

By using the Young's and Poincaré's inequalities, we have

\[
\left| \int_0^1 [p_t(t)p(t) + \mu_1 q_t(t)q(t)] dx \right| \leq \int_0^1 |p_t(t)p(t)| dx + \mu_1 \int_0^1 |q_t(t)q(t)| dx \\
\leq \|p_t(t)\|_2 \|p_x(t)\|_2 + \mu_1 \|q_x(t)\|_2 \|q_t(t)\|_2 \\
\leq \frac{1}{2\varepsilon_1} \|p_t(t)\|_2^2 + \frac{\varepsilon_1}{2} \|p_x(t)\|_2^2 + \mu_1^2 \|q_t(t)\|_2^2 + \frac{\varepsilon_2}{2} \|q_x(t)\|_2^2 
\]

(122)

and

\[
\mu_2 \int_0^1 w_1(t) q dx \leq \frac{\varepsilon_3}{2} \|q_x\|_2^2 + \frac{\mu_2^2}{2\varepsilon_3} \|w_1(t)\|_2^2 dx. 
\]

(123)

From (48)-(52) we obtain

\[
\frac{d}{dt}[\mathcal{J}(t) + \mathcal{K}(t)] \leq K_{\mathcal{B}_t}(\|p(t)\|_2^2 + \|q(t)\|_2^2) + \left( \rho + \frac{1}{2\varepsilon_1} \right) \|p_t\|_2^2 + \\
\left( J + \frac{\mu_1^2}{2\varepsilon_2} \right) \|q_t\|_2^2 + \left( -\delta + \frac{\varepsilon_2}{2} + \frac{\varepsilon_3}{2} \right) \|q_x\|_2^2 + \\
\left( -\chi + \frac{\varepsilon_1}{2} \right) \|p_x(t)\|_2^2 - \left\| \frac{b}{\sqrt{\xi}} p_x + \sqrt{\xi} q \right\|_2^2 + \frac{\mu_2^2}{2\varepsilon_3} \|w_1(t)\|_2^2. 
\]

(124)
Taking the derivative of $P$, we have

\[
\frac{d}{dt} P(t) = 2\tau \int_0^1 \left( \int_0^1 e^{-2\tau y} w(t) w_y(t) dx \right) dx
\]

\[
= -2 \int_0^1 \left( \int_0^1 e^{-2\tau y} w(t) w_y(t) dx \right) dx
\]

\[
= -\int_0^1 \left( \int_0^1 e^{-2\tau y} \frac{\partial}{\partial y} w^2(t) dx \right)
\]

\[
= \int_0^1 q^2(t) dx - \int_0^1 w^2_1(t) dx - 2\tau \int_0^1 \left( \int_0^1 e^{-2\tau y} w^2(t) dx \right)
\]

\[
= \|q(t)\|^2 - e^{-2\tau} \|w_1(t)\|^2 - 2\tau \int_0^1 e^{-2\tau y} w^2(t) dx.
\]

Therefore, combining (41), (55) and (54) we arrive at

\[
\frac{d}{dt} I(t) \leq (2NK_{B_0} + K_{B_1})(\|p(t)\|^2 + \|q(t)\|^2) + \left( - N + \rho + \frac{1}{2\varepsilon_1} \right) \|p(t)\|^2 + \left[ - N \left( \mu_1 - \frac{\vartheta}{2\tau} - \frac{\mu_2}{2} \right) + \left( J + \frac{\mu_1^2}{2\varepsilon_2} + M \right) \right] \|q(t)\|^2 + \left( - \delta + \frac{\varepsilon_2}{2} + \frac{\varepsilon_1}{2} \right) \|q_x(t)\|^2 + \left( - \chi + \frac{\varepsilon_1}{2} \right) \|p_x(t)\|^2 - \left\| \frac{b}{\sqrt{\varepsilon_2}} p_x(t) + \sqrt{\varepsilon_2} q(t) \right\|_2^2 + \left( -2\tau e^{-2\tau} \|w(t)\|^2 \right) - 2M \tau e^{-2\tau} \|w(t)\|^2.
\]

On the other hand,

\[
N_1 \mathcal{L}(t) := \rho N_1 \|p(t)\|^2 + JN_1 \|q(t)\|^2 + \delta N_1 \|q_x(t)\|^2 + \chi N_1 \|p_x(t)\|^2 +
\]

\[
N_1 \left\| \frac{b}{\sqrt{\varepsilon_2}} p_x(t) + \sqrt{\varepsilon_2} q(t) \right\|_2^2 + N_1 \vartheta \|w(t)\|^2.
\]
Accordingly
\[ \frac{d}{dt}L(t) + N_1 L(t) \leq (2NK_{K_{su}} + K_{Kr})(\|u(t)\|_2^2 + \|v(t)\|_2^2) \]
\[ - \left( N - \rho - \frac{1}{2\varepsilon_1} - \rho N_1 \right) \|p_l(t)\|_2^2 \]
\[ - \left[ N \left( \mu_1 - \frac{\sigma}{2\tau} - \frac{\mu_2}{2} \right) - \left( J + \frac{\mu_2}{2\varepsilon_2} - M \right) - JN_1 \right] \|q_l(t)\|_2^2 \]
\[ - \left( 1 - N_1 \delta - \frac{\varepsilon_2}{2} - \frac{\varepsilon_3}{2} \right) \|q_e(t)\|_2^2 \]
\[ - \left( 1 - N_1 \chi - \frac{\varepsilon_1}{2} \right) \|p_e(t)\|_2^2 - \left( 1 - N_1 \right) \left\| \frac{b}{\sqrt{\xi}} p_x(t) + \sqrt{\xi} q(t) \right\|_2^2 \]
\[ - \left[ N \left( \frac{\sigma}{\tau} - \mu_2 \right) + \left( Me^{-2\tau} - \frac{\mu_2}{2\varepsilon_3} \right) \right] \|w_1(t)\|_2^2 \]
\[ - (2M \tau e^{-2\tau} - N_1 \theta) \|w(t)\|_2^2. \]

We must first consider
\[ 0 < N_1 < \frac{1}{2}, \quad 0 < \varepsilon_2 + \varepsilon_3 < \delta \quad \text{and} \quad 0 < \varepsilon_1 < \chi \]
and after that
\[ M > \max \left\{ \frac{\mu_2 e^{2\tau}}{2\varepsilon_3}, \frac{N_1 \sigma e^{2\tau}}{2\tau} \right\}. \]

Once,
\[ \mu_1 - \frac{\xi}{2\tau} - \frac{\mu_2}{2} > 0 \quad \text{and} \quad \frac{\xi}{\tau} - \mu_2 > 0, \]
just take \( N > 0 \) large enough to get (46). Finally, combining (45) and (46) and using Gronwall’s inequality, we arrived at
\[ L(t) \leq \zeta e^{-\gamma t} L(0) + C B \int_0^t e^{-\gamma(t-s)} (\|u(s)\|_2^2 + \|v(s)\|_2^2) ds, \quad t \geq 0. \]

Recalling that
\[ L(t) = \|U(t)\|_H^2 = \|S(t)U_1 - S(t)U_2\|_H, \quad t \geq 0. \]

The proof of Lemma 3.13 is complete. \( \square \)

**Remark 3.** Since the embedded \( H_0^1(0, 1) \times H_0^1(0, 1) \to L^2(0, 1) \times L^2(0, 1) \) is compact, in order to obtain the quasi-stability for the dynamical system \( (\mathcal{H}, S(t)) \), we will consider the isomorphism \( \mathcal{H} \cong \tilde{\mathcal{H}} \), where
\[ \tilde{\mathcal{H}} := (H_0^1(0, 1) \times H_0^1(0, 1)) \times (L^2(0, 1) \times L^2(0, 1)) \times L^2(0, 1) \times (0, 1). \]

We will make the following identification
\[ (v, w, \varphi, \psi, p) \in \mathcal{H} \Longleftrightarrow (v, \varphi, w, \psi, p) \in \tilde{\mathcal{H}}. \]

The inner product and norm in \( \tilde{\mathcal{H}} \) are the same as in (15). The trajectory of the solutions will be given by \( (u(t), \phi(t), u_\tau(t), \phi_\tau(t), z(t)) \). When there is no danger of confusion, we will write \( \mathcal{H} \) instead of \( \tilde{\mathcal{H}} \).
3.3.2. Proof of Theorem 3.10.

Proof. (I). Let $B \subset H$ be a limited and positively invariant set of $(H, S(t))$ and consider $U_1, U_2 \in B$. As already mentioned, we denote to $i = 1, 2$

\[ S(t)U_i = (u^i(t), \phi^i(t), u^i_t(t), \phi^i_t(t), z^i(t)), \quad (p, q) = (u^1 - u^2, \phi^1 - \phi^2). \quad (136) \]

From the Theorem 2.5 (ii), we obtain $a(t) = e^{C_{\text{at}} t} > 0$ which is locally bounded in $[0, \infty)$. We also consider the seminorm $\eta(\cdot)$ in $\mathcal{H} = H^1_0(0, 1) \times H^1_0(0, 1)$ given by

\[ \eta(p, q)^2 = \|p\|_2^2 + \|q\|_2^2, \quad (137) \]

which is compact in $X$, since the embedding $X \hookrightarrow L^2(0, 1) \times L^2(0, 1)$ is compact. It follows from Lemma 3.13 that

\[ \|S(t)U_1 - S(t)U_2\|_H^2 \leq b(t)\|U_1 - U_2\|_H^2 + c(t) \sup_{0 \leq s \leq t} \eta(p, q)^2, \quad (138) \]

where

\[ b(t) = \zeta e^{-\gamma t} \quad \text{and} \quad c(t) = C_B \int_0^t e^{-\gamma (t-s)} ds, \quad t \geq 0. \quad (139) \]

Thus we have $b(t) \in L^1(\mathbb{R}_+)$, with $\lim_{t \to \infty} b(t) = 0$ and $c(\infty) = \sup_{t \in \mathbb{R}_+} c(t) \leq \frac{C_B}{\zeta} < \infty$. Hence (QS1)-(QS3) are satisfied and the $(H, S(t))$ is quasi-stable over any positively invariant set.

(II). It follows from Lemma 3.11 and Theorem 3.8 and (I) that $(\mathcal{H}, S(t))$ is gradient and asymptotically smooth. Thus, the result is readily established by properties (a) and (b) of Lemma 3.11, Theorem 3.6 and Theorem 3.8.

(III). It is an immediate consequence of (II) and [13, Theorem 7.5.10].

(IV) and (V). From the above, $(\mathcal{H}, S(t))$ is quasi-stable on the attractor $\mathfrak{A}$. Thus, as a consequence of Theorem 7.9.6 in [13], it follows that $\mathfrak{A}$ has finite fractal dimension $\dim_H \mathfrak{A}$. Since we have shown that $(\mathcal{H}, S(t))$ is quasi-stable on the global attractor $\mathfrak{A}$ with $c(\infty) = \sup_{t \in \mathbb{R}_+} c(t) < \infty$, then the regularity properties (10) and (11) follows by [13, Theorem 7.9.8].

(VI). Let $\Phi$ be the functional of Lyapunov considered in Lemma 3.11, let us take

\[ \mathfrak{B} = \{ U : \Phi(U) \leq R \}. \]

It is clear that for $R$ large enough, by Remark 2 and Lemma 2.4, the set $\mathfrak{B}$ is absorbing and positively invariant, thus $(\mathcal{H}, S(t))$ is quasi-stable on $\mathfrak{B}$.

From the invariant positivity of $\mathfrak{B}$, there exists a constant $C_{\mathfrak{B}} > 0$ such that, for all $T > 0$ and every solution $U(t) = S(t)U_0 = (u(t), u_t(t), \phi(t), \phi_t(t), z(t))$ with initial data $U_0 \in \mathfrak{B}$, we have

\[ \|S(t)U_0\|_H \leq C_{\mathfrak{B}}, \quad 0 \leq t \leq T \quad (140) \]

From (69) and (3)-(5), we obtain a constant still represented by $C_{\mathfrak{B}} > 0$ such that

\[ \|U_0(t)\|_{L^1} \leq C_{\mathfrak{B}}, \quad 0 \leq t \leq T. \quad (141) \]

Consequently

\[ \|S(t_1)U_0 - S(t_2)U_0\|_{H_{-1}} \leq \int_{t_1}^{t_2} \|U_0(s)\|_{H_{-1}} ds \leq C_{\mathfrak{B}}|t_1 - t_2|, \quad (142) \]

for $0 \leq t_1 \leq t_2 \leq T$. Therefore, the application $t \mapsto S(t)U_0$ is H"older continuous on space extending $\mathcal{H}_{-1}$ with exponent $\delta = 1$ for every $U \in \mathfrak{B}$. Thus, based on [13,
Theorem 7.9.9] the system \((H, S(t))\) possesses a generalized exponential attractor with finite fractal dimension in generalized space \(\tilde{H}^{-1}\).

Using an analogous argument to that found in \([8, 35]\) we can show the existence of exponential attractor with finite fractal dimension in the generalized space \(H^{-\delta}\) with \(\delta \in (0, 1)\).

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