BRAID GROUP REPRESENTATIONS FROM TWISTED
QUANTUM DOUBLES OF FINITE GROUPS

PAVEL ETINGOF, ERIC ROWELL, AND SARAH WITHERSPOON

Abstract. We investigate the braid group representations arising from cate-
gories of representations of twisted quantum doubles of finite groups. For these
categories, we show that the resulting braid group representations always factor
through finite groups, in contrast to the categories associated with quantum
groups at roots of unity. We also show that in the case of p-groups, the cor-
responding pure braid group representations factor through a finite p-group,
which answers a question asked of the first author by V. Drinfeld.

1. Introduction

Any braided tensor category \( \mathcal{C} \) gives rise to finite dimensional representations
of the braid group \( \mathcal{B}_n \). A natural problem is to determine the image of these
representations. This has been carried out to some extent for the braided tensor
categories coming from quantum groups and polynomial link invariants at roots of
unity \([7, 8, 9, 10, 11, 12, 13]\). A basic question in this direction is: \textit{Is the image of
the representation of } \( \mathcal{B}_n \) \textit{a finite group?} In the aforementioned papers the answer
is typically “no”: Finite groups appear only in a few cases when the degree of the
root of unity is small.

In this paper we consider the braid group representations associated to the
(braided tensor) categories \( \text{Mod-} D^\omega(G) \), where \( D^\omega(G) \) is the twisted quantum
double of the finite group \( G \). We show (Theorem 4.2) that the braid group images
are \textit{always} finite. We also answer in the affirmative (Theorem 4.5) a question of
Drinfeld: \textit{If } \( G \) \textit{is a p-group, is the image of the pure braid group } \( \mathcal{P}_n \) \textit{also a p-group?}

The contents of the paper are as follows. In Section 2 we record some definitions
and basic results on braided categories, and Section 3 is dedicated to the needed
facts about \( D^\omega(G) \). Then we prove our main results in Section 4. The last section
describes some open problems suggested by our work.

Acknowledgments. P.E. is grateful to V. Drinfeld for a useful discussion, and
raising the question answered by Theorem 4.5. The work of P.E. was partially sup-
ported by the NSF grant DMS-0504847. The work of S.W. was partially supported
by the NSA grant H98230-07-1-0038.

Date: March 8, 2007.
2. BRAIDED CATEGORIES AND BRAID GROUPS

In this section we recall some facts about braided categories and derive some basic consequences. For more complete definitions the reader is referred to either [2] or [14].

The braid group $B_n$ is defined by generators $\beta_1, \ldots, \beta_{n-1}$ satisfying the relations:

(B1) $\beta_i\beta_{i+1}\beta_i = \beta_{i+1}\beta_i\beta_{i+1}$ for $1 \leq i \leq n-2$,

(B2) $\beta_i\beta_j = \beta_j\beta_i$ if $|i-j| \geq 2$.

The kernel of the surjective homomorphism from $B_n$ to the symmetric group $S_n$ given by $\beta_i \mapsto (i, i+1)$ is the pure braid group $P_n$, and is generated by all conjugates of $\beta_1^2$.

Let $C$ be a $k$-linear braided category over an algebraically closed field $k$ of arbitrary characteristic. The braiding structure affords us representations of $B_n$ as follows. For any object $X$ in $C$ we have braiding isomorphisms $c_{X,X} \in \text{End}(X \otimes X)$ so that defining $\check{R}_i := \text{Id}_X \otimes c_{X,X} \otimes \text{Id}_X \otimes c_{X,X} \otimes \text{Id}_X \otimes \text{Id}_X$ we obtain a representation $\phi_X^\beta$ of $B_n$ by automorphisms of $X \otimes n$ by

$$\phi_X^\beta(\beta_i) = \check{R}_i.$$ 

Similarly, for any collection of objects $\{X_i\}_{i=1}^n$, one has representations of $P_n$ on $X_1 \otimes \cdots \otimes X_n$. Throughout the paper, when we refer to representations of $P_n$ and $B_n$ arising from tensor products of objects in a braided category, these are the representations we mean.

We say that $Y$ is a subobject of $Z$ if there exists a monomorphism $q \in \text{Hom}_C(Y, Z)$, and $W$ is a quotient object of $Z$ if there exists an epimorphism $p \in \text{Hom}_C(Z, W)$. Because of the functoriality of the braiding, we have the following obvious lemma, which will be used in Section 4.

Lemma 2.1. (i) If $Y$ is a quotient object or a subobject of $Z$, then $\phi_Y^\beta(B_n)$ is a quotient group of $\phi_Z^\beta(B_n)$ and similarly for the restrictions of these representations to $P_n$.

(ii) Let $S$ be a finite set of objects of a braided tensor category $C$ for which the image of the representation of $P_n$ in $\text{End}(X_1 \otimes \cdots \otimes X_n)$ is finite for all $X_1, \ldots, X_n \in S$. Let $X$ be the direct sum of finitely many objects taken from $S$. Then the image of the representation of $B_n$ in $\text{End}(X \otimes n)$ is finite.

3. THE TWISTED QUANTUM DOUBLE OF A FINITE GROUP

In this section we define the twisted quantum double of a finite group, and give some basic results that we need. For more details, see for example [3, 5, 16].

Let $k$ be an algebraically closed field of arbitrary characteristic $\ell$. Let $G$ be a finite group with identity element $e$, $kG$ the corresponding group algebra, and
(kG)* the dual algebra of linear functions from kG to k, under pointwise multiplication. There is a basis of (kG)* consisting of the dual functions δg (g ∈ G), defined by δg(h) = δg,h (g, h ∈ G). Let ω : G × G → k× be a 3-cocycle, that is

ω(a, b, c)ω(a, bc, d)ω(b, c, d) = ω(ab, c, d)ω(a, b, cd)

for all a, b, c, d ∈ G. The twisted quantum (or Drinfeld) double Dω(G) is a quasi-Hopf algebra whose underlying vector space is (kG)* ⊗ kG. We abbreviate the basis element δx ⊗ g of Dω(G) by δxg (x, g ∈ G). Multiplication on Dω(G) is defined by

(δxg)(δy) = θx(y, h)δxg,γγ−1δyγh,

where

θx(y, h) = ω(x, y, h)ω(h, h, h−1g−1xgh)

ω(g, g−1xg, h).

As an algebra, Dω(G) is semisimple if and only if the characteristic ℓ of k does not divide the order of G [16].

The quasi-coassociative coproduct ∆ : Dω(G) → Dω(G) ⊗ Dω(G) is defined by

∆(δxg) = ∑ y,z∈G,γ=1 γg(y, z)δyγ ⊗ δxγ,

where

γg(y, z) = ω(y, z, g)ω(g, g−1yg, g−1zg)

ω(y, g, g−1zg).

The quasi-Hopf algebra Dω(G) is quasitriangular with

R = ∑ g∈G δg ⊗ γg and R−1 = ∑ g,h∈G θgh g−1δg ⊗ δhg−1.

In particular R∆(a)R−1 = σ(∆(a)) for all a ∈ Dω(G), where σ is the transposition map. If X and Y are Dω(G)-modules, then R = σ ◦ R provides a Dω(G)-module isomorphism from X ⊗ Y to Y ⊗ X. Let cX,Y be this action by R. Then the category Mod-Dω(G) of finite dimensional Dω(G)-modules is a braided category with braiding c.

4. The images of Bn and Pn

In this section we fix a finite group G and a 3-cocycle ω, and prove that the image of Bn in EndDω(G)(V ⊗n) is finite for any positive integer n and any finite dimensional Dω(G)-module V. In case G is a p-group, we prove that the image of Pn in EndDω(G)(V ⊗n) is also a p-group.

Remark 4.1. It follows from a theorem of C. Vafa (see [2, Theorem 3.1.19]) and the so-called balancing axioms that for braided fusion categories over C, the images of the braid group generators βi in the above representations of Bn always have finite order. This is far from enough to conclude that the image of Bn is
finite; Coxeter [4] has shown that the quotient of $\mathcal{B}_n$ by the normal closure of the subgroup generated by $\{\beta_i^k : 1 \leq i \leq n - 1\}$ is finite if and only if $\frac{1}{n} + \frac{1}{k} > \frac{1}{2}$.

The case of general finite groups. Let $r$ and $m$ be positive integers. The full monomial group $G(r, 1, m)$ is the multiplicative group consisting of the $m \times m$ matrices having exactly one nonzero entry in each row and column, all of whose nonzero entries are $r$th roots of unity. It is one of the irreducible complex reflection groups.

Let $r = |G|_{\ell'}$ be the part of $|G|$ not divisible by the characteristic $\ell$ of $k$ (i.e. $|G| = r\ell^s$ and $(r, \ell) = 1$).

Theorem 4.2. Let $V$ be a finite dimensional $D^w(G)$-module. Then the image of $\mathcal{B}_n$ in $\text{End}(V^\otimes n)$ is finite. More specifically, this image is a quotient of a subgroup of $G(r, 1, m)$ for $m = |G|^{2n}$.

Proof. We will need the following well known lemma, which follows from [15, Theorem 6.5.8]. Let $\mu_r \subset k^\times$ be the set of $r$-th roots of unity.

Lemma 4.3. The natural map $H^i(G, \mu_r) \to H^i(G, k^\times)$ is surjective. In particular, any element in $H^i(G, k^\times)$ may be represented by a cocycle taking values in $\mu_r$.

Now we turn to the proof of the theorem. As any finite dimensional $D^w(G)$-module is finitely generated, and therefore is a quotient of a finite rank free module, by Lemma 2.1 (i), it suffices to prove the statement when $V$ is a finite rank free module. By Lemma 2.1 (ii), we need only consider the case $V = D^w(G)$, the left regular module.

Assume first that $n = 2$. Let $x, y, a, b \in G$. The action of $\bar{R}$ on the basis element $\delta_x \bar{a} \otimes \delta_y \bar{b}$ of $D^w(G) \otimes D^w(G)$ is

$$\bar{R}(\delta_x \bar{a} \otimes \delta_y \bar{b}) = \sigma(\sum_{g \in G} \delta_g \otimes \bar{g})(\delta_x \bar{a} \otimes \delta_y \bar{b})$$

$$= \sigma(\theta_{xy^{-1}}(x, b)\delta_x \bar{a} \otimes \delta_{xy^{-1}}x\bar{b})$$

$$= \theta_{xy^{-1}}(x, b)\delta_x \bar{a} \otimes \delta_{xy^{-1}}x\bar{b} \otimes \delta_x \bar{a}.$$

If $n > 2$, similar calculations show that each $\bar{R}_i$ permutes the chosen basis of $D^w(G)$ up to scalar multiples of the form $\theta_{xy^{-1}}(x, b)$. By Lemma 1.3 who may assume that $\omega$ and hence $\theta$ takes values in the $r$-th roots of unity. This implies that the image of $\mathcal{B}_n$ in $\text{End}(D^w(G)^{\otimes n})$ is contained in $G(r, 1, m)$. □

Corollary 4.4. Let $C$ be a braided fusion category that is group-theoretical in the sense of [6]. Let $V$ be any object of $C$. Then the image of $\mathcal{B}_n$ in $\text{End}(V^\otimes n)$ is finite.

Proof. Let $Z(C)$ be the Drinfeld center of $C$. Since $C$ is braided, we have a canonical braided tensor functor $F : C \to Z(C)$. Thus it suffices to show the result holds for the category $Z(C)$. Since $C$ is group-theoretical, $Z(C)$ is equivalent to $\text{Mod-}D^w(G)$ for some $G, \omega$. Thus the desired result follows from Theorem 4.2. □
The case of $p$-groups.

**Theorem 4.5.** Suppose that $G$ is a finite $p$-group and $V$ is a finite dimensional $D^\omega(G)$-module. Then the image of $P_n$ in $\text{End}(V^{\otimes n})$ is also a $p$-group.

The rest of the subsection is occupied by the proof of Theorem 4.5. We will need a technical lemma:

**Lemma 4.6.** Let $H$ be a group with normal subgroups $H = H_0 \supset H_1 \supset \ldots \supset H_N = 1$, such that $H_i/H_{i+1}$ is abelian, and $[H_i, H_j] \subset H_{i+j}$, and let $I$ be a subgroup of $	ext{Aut}(H)$ that preserves this filtration and acts trivially on the associated graded group. Then $I$ is nilpotent of class at most $N - 1$.

**Proof.** Let $L_1(I) = I$, $L_2(I) = [I, I]$, $L_3(I) = [[I, I], I], \ldots$, be the lower central series of $I$. We must show $L_N(I) = 1$.

We prove by induction on $n$ that for any $f \in L_n(I)$ and $h \in H_m$, $f(h) = ha(h)$, where $a(h) \in H_{n+m}$.

The case $n = 1$ is clear, since $f \in I$ acts trivially on $H_m/H_{m+1}$. Suppose the statement is true for $n$. Take $g \in I, f \in L_n(I)$ and $h \in H_m$ so that: $f(h) = ha(h)$, $g(h) = hb(h)$, where $a(h) \in H_{n+m}$ and $b(h) \in H_{m+1}$. Then $fg(h) = f(h)f(b(h)) = ha(h)hb(h)ab(h))$, while $gf(h) = hb(h)ab(h))$.

Since $g$ acts trivially on the associated graded group, $b(a(h)) \in H_{n+m+1}$. Also $a(b(h)) \in H_{n+m+1}$ since $b(h) \in H_{m+1}$, by the induction assumption. Moreover, $a(b(h))b(h) = b(h)a(h)$ modulo $H_{n+m+1}$ since $[H_i, H_j] \subset H_{i+j}$. Thus, $fg(h) = gf(h)$ in $H/H_{n+m+1}$, and thus $[f, g](h) = h$ in $H/H_{n+m+1}$, which is what we needed to show.

Taking $m = 0$ and $n = N - 1$, any $[f, g] \in L_N(I)$ is the identity on $H = H/H_N$, and the lemma is proved. \[ \square \]

Now we are ready to prove the theorem. Any finite dimensional $D^\omega(G)$-module is a quotient of a multiple of the left regular $D^\omega(G)$-module $H = D^\omega(G)$. By Lemma 2.1, it suffices to show that the image of $P_n$ in $\text{End}(H^{\otimes n})$ is a $p$-group. By Theorem 4.2, the image $K$ of $P_n$ is a subgroup of the full monomial group $G(r, 1, m)$, where $r = p^t$ for some $t$, and $m = |G|^{2n}$. The normal subgroup of diagonal matrices in $K$ is thus a $p$-group, so it is enough to show that $K$ modulo the diagonal matrices is a $p$-group. Thus it suffices to assume that $\omega = 1$ and $H = D(G)$.

Computing, we have:

$$
\tilde{R}(\pi \delta_x \otimes \overline{b} \delta_y) = \sigma(\sum_{g \in G} \delta_g \otimes \gamma)(\pi \delta_x \otimes \overline{b} \delta_y) = axa^{-1}b \delta_y \otimes \pi \delta_x
$$

for all $a, b, x, y \in G$. Denote by $(g, x)$ the element $\overline{y} \delta_x$ so that a basis of $H^{\otimes n}$ is:

$$(g_1, x_1) \otimes \cdots \otimes (g_n, x_n)$$
with $g_i, x_i \in G$. The braid generator $\beta_i$ fixes all factors other than the $i$th and $(i+1)$st, and on these it acts by:

$$
(g_i, x_i) \mapsto (g_i x_i g_i^{-1} x_{i+1}, x_i) \otimes (g_i, x_i),
$$

where $[a, b]$ denotes the group commutator. This action induces a homomorphism $\psi : B_n \to \text{Aut}(\text{Fr}_{2n})$ where $\text{Fr}_{2n}$ is the free group on $2n$ generators. Explicitly, $\psi(\beta_i)$ is the automorphism defined on generators $\{g_i, x_i\}_{i=1}^n$ of $\text{Fr}_{2n}$ by:

- $x_j \mapsto x_j$, $g_j \mapsto g_j$ for $j \not\in \{i, i+1\}$
- $x_i \mapsto x_{i+1}$, $x_{i+1} \mapsto x_i$
- $g_i \mapsto [g_i, x_i] x_i g_{i+1}$, $g_{i+1} \mapsto g_i$.

Since $G$ is a $p$-group, it is nilpotent of class, say, $N - 1$. Note that $\psi$ descends to a homomorphism $\psi_N : B_n \to \text{Aut}(\text{Fr}_{2n} / L_N(\text{Fr}_{2n}))$ where $L_N(\text{Fr}_{2n})$ denotes the $N$th term of the lower central series of $\text{Fr}_{2n}$. Since $G$ is nilpotent of class $N - 1$, the action of $B_n$ on the set $G^{2n}$ defined above factors through $\psi_N$. Thus, setting $I = \psi_N(P_n)$, one sees that the action of $P_n$ on $H^{\otimes n}$ factors through $I$, that is, $K$ is a quotient of $I$.

Let us now show that $I$ is nilpotent. Define a descending filtration on $M = \text{Fr}_{2n} / L_N(\text{Fr}_{2n})$ by positive integers as follows. Let $M_1 = M$. Define degrees on the generators by $\deg(g_i) = 1$ and $\deg(x_i) = 2$ for all $i$, and define $M_j$ for $j \geq 2$ to be the normal closure of the group generated by $[M_k, M_{j-k}]$ for all $0 \leq k \leq j$ together with the generators of degree at least $j$. Since $M$ is nilpotent, this filtration is finite. Further, $I$ preserves this filtration and acts trivially on the quotients $M_i/M_{i+1}$. By Lemma 4.6, $I$ is nilpotent.

It follows that the finite group $K$ is nilpotent. However, $K$ is generated by conjugates of $\beta_1^2$, and we claim that $\beta_1^2$ is an element of order a power of $p$. Indeed, this follows from the fact that if the ground field is $\mathbb{C}$ (which may be assumed without loss of generality, since the double of $G$ is defined over the integers), then the eigenvalues of $c_{X,Y} c_{Y,X}$ for any objects $X, Y$ are ratios of twists, which are computed from characters of $G$ (in [2]), and hence are roots of unity of degree a power of $p$. Therefore, $K$ is a finite $p$-group. The theorem is proved.

5. Questions

We mention some directions for further investigation suggested by these (and other) results. We refer the reader to [6] and [14] for the relevant definitions.

(1) Suppose $G$ is a $p$-group. Theorem 4.5 shows that the image of the associated representation of $P_n$ is also a $p$-group. What is its nilpotency class relative to that of $G$? Some upper bounds can be obtained from the proof of Theorem 4.5, but it is not clear how tight they are.
(2) The finite groups that appear as images of representations of $B_n$ associated to quantum groups and link invariants at roots of unity (see [7, 9, 10, 11, 12, 13]) basically fall into two classes: symplectic groups and extensions of $p$-groups by the symmetric group $S_n$. Does this hold for the representations of $B_n$ associated with $\text{Mod} - D^\omega(G)$? In general, is there a relationship between the image of $P_n$ and $G$?

(3) As a modular category, $\text{Mod} - D^\omega(G)$ gives rise to (projective) representations of mapping class groups of compact surfaces with boundary. Are the images always finite? It is known to be true for the mapping class groups of the torus and the $n$-punctured sphere (Theorem 4.2). For more general modular categories, the answer is definitely “no,” see [11, Conjecture 2.4].

(4) Let us say that a braided category $C$ has property $\mathcal{F}$ if all braid group representations associated to $C$ have finite images. What class of braided categories have property $\mathcal{F}$? Among braided fusion categories, Corollary 4.4 shows that all braided group-theoretical categories (in the sense of [6]) have property $\mathcal{F}$. Do all braided fusion categories with integer Frobenius-Perron dimension have property $\mathcal{F}$?

References

[1] J. E. Andersen, G. Masbaum, and K. Ueno, Topological Quantum Field Theory and the Nielsen-Thurston classification of $M(0,4)$, Math. Proc. Cambridge Philos. Soc., 141 (2006), no. 3, 477–488.
[2] B. Bakalov and A. Kirillov, Jr., Lectures on Tensor Categories and Modular Functors, University Lecture Series, vol. 21, Amer. Math. Soc., 2001.
[3] V. Chari and A. Pressley, A Guide to Quantum Groups, Cambridge Univ. Press, 1994.
[4] H. S. M. Coxeter, Factor groups of the braid group, Proceedings of the Fourth Can. Math. Cong., Banff 1957, University of Toronto Press (1959), 95–122.
[5] V. G. Drinfeld, Quasi-Hopf algebras, Leningrad Math. J. 1 (1990), 1419–1457.
[6] P. Etingof, D. Nikshych, and V. Ostrik, On fusion categories, Ann. of Math. (2) 162 (2005), no. 2, 581-642.
[7] J. Franko, E. C. Rowell, and Z. Wang, Extraspecial 2-groups and images of braid group representations, J. Knot Theory Ramifications 15 (2006) no. 4, 1–15.
[8] M. H. Freedman, M. J. Larsen, and Z. Wang, The two-eigenvalue problem and density of Jones representation of braid groups, Comm. Math. Phys. 228 (2002), 177-199.
[9] D. M. Goldschmidt and V. F. R. Jones, Metaplectic link invariants. Geom. Dedicata 31 (1989), no. 2, 165–191.
[10] V. F. R. Jones, Braid groups, Hecke algebras and type II$_1$ factors, Geometric methods in operator algebras (Kyoto, 1983), 242–273, Pitman Res. Notes Math. Ser. 123, Longman Sci. Tech., Harlow, 1986.
[11] V. F. F. Jones, On a certain value of the Kauffman polynomial, Comm. Math. Phys. 125 (1989), no. 3, 459–467.
[12] M. J. Larsen and E. C. Rowell, An algebra-level version of a link-polynomial identity of Lickorish, Math. Proc. Cambridge Philos. Soc., to appear.
[13] M. J. Larsen, E. C. Rowell, and Z. Wang, The $N$-eigenvalue problem and two applications, Int. Math. Res. Not. 2005 (2005), no. 64, 3987–4018.
[14] V. G. Turaev, *Quantum Invariants of Knots and 3-Manifolds*, de Gruyter Studies in Mathematics 18, Walter de Gruyter & Co., Berlin, 1994.
[15] Weibel, C., An introduction to Homological algebra, Cambridge Studies in Advanced Mathematics (No. 38), 1995.
[16] S. J. Witherspoon, *The representation ring of the twisted quantum double of a finite group*, Canad. J. Math. 48 (1996), 1324–1338.

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA  
E-mail address: etingof@math.mit.edu

Department of Mathematics, Texas A&M University, College Station, TX 77843, USA  
E-mail address: rowell@math.tamu.edu  
E-mail address: sjw@math.tamu.edu