Grassmann tensor renormalization group for the one-flavor lattice Gross–Neveu model with finite chemical potential

Shinji Takeda* and Yusuke Yoshimura*

Institute for Theoretical Physics, Kanazawa University, Kanazawa 920-1192, Japan
*E-mail: takeda@hep.s.kanazawa-u.ac.jp, yoshimura@hep.s.kanazawa-u.ac.jp

Received December 26, 2014; Revised January 26, 2015; Accepted January 27, 2015; Published April 1, 2015

We apply the Grassmann tensor renormalization group (GTRG) to the one-flavor lattice Gross–Neveu model in the presence of a chemical potential. We compute the fermion number density and its susceptibility and confirm the validity of GTRG for the finite-density system. We introduce a method analogous to the reweighting method for the Monte Carlo method and test it for some parameters.

Subject Index B64

1. Introduction

The tensor renormalization group (TRG) is a numerical renormalization method. It was originally introduced by Levin and Nave in the triangular lattice Ising model [1], and has since been applied to bosonic models on square lattices: the $X$–$Y$ model [2], the $O(3)$ model [3], and $\phi^4$ theory [4]. Recently, Xie et al. developed the second TRG [5] to improve this method by using global instead of local optimization. An extension to the higher-dimensional system, named higher-order TRG, was also introduced by Xie et al. and was examined in the 3D Ising model [6]. Furthermore, a generalization to fermion systems called the Grassmann tensor renormalization group (GTRG) was proposed by Gu et al. [7,8]. This has been applied to 2D quantum electrodynamics (QED) [9], which is a gauge-fermion system; a study including a $\theta$-term [10] has also been undertaken by Shimizu and Kuramashi.

An advantage of the TRG is that this method can be applied to any system suffering from the sign problem with the Monte Carlo method. The sign problem occurs in, e.g., finite-fermion-density systems, $\theta$-term-included systems, lattice chiral gauge theories, and so on. The purpose of this paper is to apply the GTRG to a simple finite-fermion-density system on the lattice, namely, the Gross–Neveu model [11] containing four-fermion interaction in the presence of a chemical potential with the Wilson fermion lattice formulation. This model is known to share important properties—asymptotic freedom and spontaneous symmetry breaking—with quantum chromodynamics (QCD) and is considered to be its toy model. Usually, large $N$-expansion is used to analyze the model. In this paper, nevertheless, we restrict ourselves to one flavor for simplicity, although generalization to many flavors is straightforward. This work provides a benchmark for future study of complicated and higher-dimensional models, e.g., eventually QCD with finite density.
This paper is organized as follows. In Sect. 2, after defining the lattice Gross–Neveu model with chemical potential, we review the derivation of the tensor network representation for the model and explain the GTRG procedure as well as how to implement the anti-periodic boundary condition in this representation. Numerical results are presented in Sect. 3. In Sect. 4, we propose a method analogous to the reweighting method in the Monte Carlo method. Section 5 is devoted to the summary and outlook.

2. TRG for the lattice Gross–Neveu model

2.1. Gross–Neveu model

The Lagrangian density for the Gross–Neveu model \[11\] in 2D Euclidean continuum space-time is given by

\[
\mathcal{L}^{\text{GN}} = \bar{\psi}(\not\!\partial + m)\psi - \frac{g^2}{2N} \left[ (\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2 \right],
\]

(1)

where \(\psi = (\psi_1, \psi_2)^T\) is a 2-component spinor field with \(N\) different flavors and \(m, g^2\) denote the mass and the coupling constant, respectively.

The lattice version of the Lagrangian density is defined by

\[
\mathcal{L}^{\text{GN}}_{\text{Lat}} = \sum_{n'} \bar{\psi}_n D_{n,n'} \psi_{n'} + \frac{g^2}{2N} \left[ (\bar{\psi}_n \psi_n)^2 + (\bar{\psi}_n i\gamma_5 \psi_n)^2 \right],
\]

(2)

where \(n = (n_1, n_2)\) is a lattice site. The Wilson–Dirac operator \[12–14\] \(D_{n,n'}\) including the chemical potential \[15,16\] is explicitly given by

\[
D_{n,n'} = (m + 2)\delta_{n,n'} - \frac{1}{2} \sum_{\nu,\pm} e^{\mp\mu\delta_{\nu,2}} (1 \pm \gamma_\nu) \delta_{n,n'\pm\nu}.
\]

(3)

In the following, we consider a 2D lattice box \(N_1 \times N_2\).

2.2. Tensor network representation

Let us first express the partition function in terms of the tensor network representation. A general procedure to derive the tensor network representation is 1): expanding the integrand (Boltzmann weight); the accompanying discrete variables describing the ordering of expansion then become new degrees of freedom (index of tensor), 2) integrating out the original degree of freedom (\(\psi\) in this case); the elements of the tensor are then determined. Although the derivation has already been given in a previous work \[9\], we re-derive it in a slightly different way to make this paper self-contained and hope that this is useful for readers. In this and subsequent subsections, we temporarily consider a system where the periodic boundary condition is imposed for all directions, while the anti-periodic boundary condition will be discussed in Sect. 2.4.

[1] As discussed in Refs. \[13,14\], the coupling constants for each four-fermion interaction \((\bar{\psi}\psi)^2\) and \((\bar{\psi}i\gamma_5\psi)^2\) should be treated independently for the Wilson fermion formulation but, as we will consider only one-flavor theory where any kind of four-fermion interaction terms give a unique form, we do not distinguish between them in this paper.
In the following, we restrict ourselves to $N = 1$ and choose the representation of the gamma matrices:
\begin{equation}
\gamma_1 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_5 = i \gamma_1 \gamma_2 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.
\end{equation}

The hopping terms for the 2-direction are diagonal in spinor space,
\begin{align}
\frac{1}{2} e^{-\mu} \bar{\psi}_n (1 + \gamma_2) \psi_{n - \frac{1}{2}} &= e^{-\mu} \bar{\psi}_{n, 1} \psi_{n, -\frac{1}{2}, 1}, \\
\frac{1}{2} e^{-\mu} \bar{\psi}_n (1 - \gamma_2) \psi_{n + \frac{1}{2}} &= e^{\mu} \bar{\psi}_{n, 2} \psi_{n + \frac{1}{2}, 2},
\end{align}

while the hopping terms for the 1-direction,
\begin{equation}
\bar{\psi}_n (1 \pm \gamma_1) \psi_{n + \frac{1}{2}} = \bar{\psi}_n \begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix} \psi_{n + \frac{1}{2}},
\end{equation}

are not diagonalized. One, however, can make them diagonal by introducing another basis:
\begin{align}
\chi_{n, 1} &= \frac{1}{\sqrt{2}} (\psi_{n, 1} + \psi_{n, 2}), \quad \chi_{n, 2} = \frac{1}{\sqrt{2}} (\psi_{n, 1} - \psi_{n, 2}), \\
\tilde{\chi}_{n, 1} &= \frac{1}{\sqrt{2}} (\bar{\psi}_{n, 1} + \bar{\psi}_{n, 2}), \quad \tilde{\chi}_{n, 2} = \frac{1}{\sqrt{2}} (\bar{\psi}_{n, 1} - \bar{\psi}_{n, 2}),
\end{align}

which yields
\begin{align}
\frac{1}{2} \bar{\psi}_n (1 + \gamma_1) \psi_{n - \frac{1}{2}} &= \tilde{\chi}_{n, 1} \chi_{n - \frac{1}{2}, 1}, \\
\frac{1}{2} \bar{\psi}_n (1 - \gamma_1) \psi_{n + \frac{1}{2}} &= \tilde{\chi}_{n, 2} \chi_{n + \frac{1}{2}, 2}.
\end{align}

Anticommutation relations then hold:
\begin{equation}
\{ \chi_{n, i}, \tilde{\chi}_{n', j} \} = \{ \chi_{n, i}, \chi_{n', j} \} = \{ \tilde{\chi}_{n, i}, \tilde{\chi}_{n', j} \} = 0, \quad \forall n, n', i, j.
\end{equation}

After the change of variable for the 1-direction hopping term, the Lagrangian density is written as
\begin{equation}
\mathcal{L}_{\text{Lat}}^{\text{GN}} = (m + 2) \bar{\psi}_n \psi_n - 2g^2 \bar{\psi}_{n, 1} \psi_{n, 1} \bar{\psi}_{n, 2} \psi_{n, 2} - \tilde{\chi}_{n, 1} \chi_{n - 1, 1} - \tilde{\chi}_{n, 2} \chi_{n + 1, 2} - e^{-\mu} \bar{\psi}_{n, 1} \psi_{n - \frac{1}{2}, 1} - e^{\mu} \bar{\psi}_{n, 2} \psi_{n + \frac{1}{2}, 2},
\end{equation}

and, correspondingly, the partition function with periodic boundary condition is given by
\begin{equation}
Z_P = \int D\psi D\bar{\psi} \exp \left( -\sum_n \mathcal{L}_{\text{Lat}}^{\text{GN}} \right)
= \int D\psi D\bar{\psi} \prod_n e^{-(m + 2)\bar{\psi}_{n, 1} \psi_{n, 1} e^{-(m + 2)\bar{\psi}_{n, 2} \psi_{n, 2}} e^{2g^2 \bar{\psi}_{n, 1} \psi_{n, 1} \bar{\psi}_{n, 2} \psi_{n, 2}} \\
\cdot e^{\tilde{\chi}_{n + 1, 1} \chi_{n, 1} e^{\tilde{\chi}_{n, 2} \chi_{n + 1, 2} e^{\tilde{\chi}_{n + 1, 2} \chi_{n, 2} e^{\tilde{\chi}_{n, 1} \chi_{n + 1, 1}}} e^{\tilde{\chi}_{n, 2} \chi_{n + 1, 2}}} e^{\tilde{\chi}_{n + 1, 2} \chi_{n, 2} e^{\tilde{\chi}_{n, 1} \chi_{n + 1, 1}}} e^{\tilde{\chi}_{n, 2} \chi_{n + 1, 2}}},
\end{equation}

where we have shown the spinor components explicitly. When expanding each exponential factor in the integrand, terms of order 0 and 1 make a non-vanishing contribution due to their Grassmann
nature,

\[
Z_p = \sum_{\{s, t, x=0,1\}} \int D\psi D\bar{\psi} \prod_n \left[ - (m + 2) \bar{\psi}_{n,1} \psi_{n,1} \right]^{s_{n,1}} \times \left[ - (m + 2) \bar{\psi}_{n,2} \psi_{n,2} \right]^{s_{n,2}} \left( 2 g^2 \bar{\psi}_{n,1} \psi_{n,1} \bar{\psi}_{n,2} \psi_{n,2} \right)^{s_{n,3}} \times \left( \tilde{\chi}_{n+1,1} \chi_{n,1} \right)^{s_{n,1}} \left( \tilde{\chi}_{n,2} \chi_{n+1,2} \right)^{s_{n,2}} \left( e^{-\mu} \bar{\psi}_{n+2,1} \psi_{n,1} \right)^{t_{n,1}} \left( e^{\mu} \bar{\psi}_{n,2} \psi_{n+2,2} \right)^{t_{n,2}}, \tag{14}
\]

where the accompanying discrete variables \(\{s, t, x\}\) that describe the ordering of expansion could be a candidate for a new degree of freedom\(^2\).

The next step is to integrate out the original degree of freedom \(\psi, \bar{\psi}\) and obtain the tensor network representation. When performing the integration, it is better to keep the fermion pair structure, which has a common exponent for each fermion field in the pair, to control sign factors originating from the anticommutation relations. For that purpose, we introduce new Grassmann variables \(\{\xi, \xi, \eta, \bar{\eta}\}\); for instance, when integrating \(\bar{\psi}_{n}, \psi_{n}\) on a site \(n\),

\[
\left( \tilde{\chi}_{n+1,1} \chi_{n,1} \right)^{s_{n,1}} = \left( \chi_{n,1} d \eta_{n,1} \right)^{s_{n,1}} \left( \tilde{\chi}_{n+1,1} \eta_{n,1} \right)^{s_{n,1}},
\]

\[
\left( \tilde{\chi}_{n,2} \chi_{n+1,2} \right)^{s_{n,2}} = \left( \tilde{\chi}_{n,2} d \tilde{\eta}_{n,2} \right)^{s_{n,2}} \left( \tilde{\eta}_{n,2} \chi_{n+1,2} \right)^{s_{n,2}},
\]

\[
\left( e^{-\mu} \bar{\psi}_{n+2,1} \psi_{n,1} \right)^{t_{n,1}} = \left( e^{-\mu} \bar{\psi}_{n,1} d \xi_{n,1} \right)^{t_{n,1}} \left( e^{-\mu} \bar{\psi}_{n+2,1} \xi_{n,1} \right)^{t_{n,1}},
\]

\[
\left( e^{\mu} \bar{\psi}_{n,2} \psi_{n+2,2} \right)^{t_{n,2}} = \left( e^{\mu} \bar{\psi}_{n,1} d \xi_{n,2} \right)^{t_{n,2}} \left( e^{\mu} \bar{\psi}_{n+2,2} \xi_{n,2} \right)^{t_{n,2}}.
\]

The point here is that we can separate the original degrees of freedom at different sites in a different fermion pair. By collecting all contributions related to \(\psi_{n}\) and \(\bar{\psi}_{n}\) in the partition function, the integration for this part is given by

\[
\int d\psi_{n,1} d\bar{\psi}_{n,1} d\psi_{n,2} d\bar{\psi}_{n,2} \times \sum_{s_{n,1}, s_{n,2}, s_{n,3}} \left[ - (m + 2) \bar{\psi}_{n,1} \psi_{n,1} \right]^{s_{n,1}} \left[ - (m + 2) \bar{\psi}_{n,2} \psi_{n,2} \right]^{s_{n,2}} \left( 2 g^2 \bar{\psi}_{n,1} \psi_{n,1} \bar{\psi}_{n,2} \psi_{n,2} \right)^{s_{n,3}} \times \left( \chi_{n,1} d \eta_{n,1} \right)^{s_{n,1}} \left( \tilde{\chi}_{n,2} d \tilde{\eta}_{n,2} \right)^{s_{n,2}} \left( e^{-\mu} \bar{\psi}_{n,1} d \xi_{n,1} \right)^{t_{n,1}} \left( e^{\mu} \bar{\psi}_{n,2} d \tilde{\xi}_{n,2} \right)^{t_{n,2}} \times \left( \tilde{\chi}_{n,1} d \tilde{\eta}_{n,1} \right)^{-s_{n,1}} \left( \chi_{n,2} d \eta_{n,2} \right)^{-s_{n,2}} \left( e^{-\mu} \bar{\psi}_{n,1} d \xi_{n,1} \right)^{t_{n,1}} \left( e^{\mu} \bar{\psi}_{n,2} d \xi_{n,2} \right)^{t_{n,2}}. \tag{23}
\]

\(^2\)Actually, the discrete variables \(\{s\}\) will not be a new degree of freedom since they will be integrated out, as we will see later.
Note that there is no original fermion field at other sites $n' \neq n$. This integration can be done manually\textsuperscript{3} and the result depends on the configuration of exponents in an abbreviated form:

$$x_n = (x_{n,1}, x_{n,2}), \quad t_n = (t_{n,1}, t_{n,2}), \quad x_{n-1} = (x_{n-1,1}, x_{n-1,2}), \quad t_{n-1} = (t_{n-1,1}, t_{n-1,2}).$$ \hspace{1cm} (24)

We rewrite Eq. (23) and define the bosonic part $T_{n \rightarrow x_{n-1} \cdot t_{n-2}}$ as follows:

$$T_{n \rightarrow x_{n-1} \cdot t_{n-2}} = \int d\tilde{D} \frac{\tilde{Z}}{Z} = \prod_{n} T_{n} t_{n-1} \cdot t_{n-2}.$$ \hspace{1cm} (25)

The explicit form of $T_{n \rightarrow x_{n-1} \cdot t_{n-2}}$ is given in Appendix A. By repeating this operation for all other sites, the partition function is finally written in the tensor network representation as

$$Z_P = \sum_{\{x,t\}} \int \prod_{n} T_{n} t_{n-1} \cdot t_{n-2},$$ \hspace{1cm} (26)

where the total tensor is given by

$$T_{n \rightarrow x_{n-1} \cdot t_{n-2}} = U^{1}_{x_{n-1} \rightarrow x_{n-1} \cdot t_{n-2}} S^{3}_{x_{n-1} \rightarrow x_{n-1} \cdot t_{n-2}},$$ \hspace{1cm} (27)

\hspace{1cm} (28)

where $U^{1,3}$ are a unitary matrix, $\sigma^{13}$ is a singular value, and $n^*$ is the coarse-grained lattice site with unit vectors $\hat{1} = \hat{1} + \hat{2}$, $\hat{2} = \hat{1} - \hat{2}$. See Fig. 1 for a graphical representation of this decomposition.

For the Grassmann part, we decompose the bosonic part on a site $n$ by the singular value decomposition (SVD) and truncate at $D_{\text{cut}}$:

$$T_{n \rightarrow x_{n-1} \cdot t_{n-2}} \approx \sum_{\sigma^{13}} \left( U^{1}_{x_{n-1} \rightarrow x_{n-1} \cdot t_{n-2}} \sigma^{13} U^{3*}_{x_{n-1} \rightarrow x_{n-1} \cdot t_{n-2}} \right),$$ \hspace{1cm} (29)

where $D^{1}$ and $D^{3}$ are defined by

$$D^{1}_{x_{n-1} \rightarrow t_{n-2}} = \int d\tilde{D} \frac{\tilde{Z}}{Z} = \prod_{n} T_{n} t_{n-1} \cdot t_{n-2},$$ \hspace{1cm} (30)

$$D^{3}_{x_{n-1} \rightarrow t_{n-2}} = \int d\tilde{D} \frac{\tilde{Z}}{Z} = \prod_{n} T_{n} t_{n-1} \cdot t_{n-2}.$$

\textsuperscript{3}In this integration, of course, one has to break the pair structure and, thus, sign factors appear, but this is still tolerable.
Fig. 1. The decomposition of the tensor. The horizontal (vertical) axis corresponds to the 1-direction (2-direction).

and a new exponent $x_{n^*\cdot i,f}$ with one-component is introduced with constraints:

$$x_{n^*\cdot i,f} = \sum_i x_{n,i} + t_{n,i} \mod 2 = \sum_i (x_{n-1,i} + t_{n-2,i}) \mod 2.$$  (32)

Then the tensor $T_{x_{n^*\cdot i-1}f}x_{n-1}f$ is decomposed and approximated as

$$T_{x_{n^*\cdot i-1}f}x_{n-1}f \simeq \sum_{x_{n^*\cdot i-1,n-1}f} \sum_{x_{n^*\cdot i-1}} \int S^{1}_{n^*\cdot i-1,n-1} d\eta_{n^*\cdot i-1,f} \times \delta \sum_i (x_{n,i} + t_{n,i}) \mod 2 x_{n^*\cdot i,f} \delta \sum_i (x_{n-1,i} + t_{n-2,i}) \mod 2 x_{n^*\cdot i,f}.$$  (33)

where $x_{n^*} = (x_{n^*,b}, x_{n^*,f})$ and

$$S^{1}_{n^*\cdot i-1,n-1} = S^{1}_{n^*\cdot i-1,n-1} D^{1}_{n^*\cdot i-1,n-1} d\eta_{n^*\cdot i-1,f},$$  (34)

$$S^{3}_{n^*\cdot i-1,n-1} = S^{3}_{n^*\cdot i-1,n-1} D^{3}_{n^*\cdot i-1,n-1} d\eta_{n^*\cdot i-1,f}.$$  (35)

For another decomposition rotated 90 degrees (see Fig. 1), by introducing new variables $\xi_{n^*}$ and $\xi_{n^*\cdot \bar{2}^*}$, it is similarly given by

$$T_{x_{n+1/2^*n+2^*n^*\cdot i+\bar{2}}^{n*\cdot \bar{2}}} \simeq \sum_{t_{n+1/2^*n+2^*n^*\cdot i+\bar{2}}} \sum_{t_{n+1/2^*n+2^*n^*\cdot i+\bar{2}}} \int S^{2}_{n+1/2^*n+2^*n^*\cdot i+\bar{2}} d\xi_{n^*\cdot \bar{2}} \times \delta \sum_i (t_{n,i} + x_{n,i+\bar{2}}) \mod 2 x_{n^*\cdot \bar{2},f} \delta \sum_i (t_{n+2,i} + x_{n+2,i}) \mod 2 t_{n+2^*\cdot i}.$$  (36)

where $S^{2}$ and $S^{4}$ are defined by

$$S^{2}_{n+1/2^*n+2^*n^*\cdot i+\bar{2}} = S^{2}_{n+1/2^*n+2^*n^*\cdot i+\bar{2}} D^{2}_{n+1/2^*n+2^*n^*\cdot i+\bar{2}} d\xi_{n^*\cdot \bar{2}},$$  (37)

$$S^{4}_{n+1/2^*n+2^*n^*\cdot i+\bar{2}} = S^{4}_{n+1/2^*n+2^*n^*\cdot i+\bar{2}} D^{4}_{n+1/2^*n+2^*n^*\cdot i+\bar{2}} d\xi_{n^*\cdot \bar{2}}.$$  (38)
with

\[ D_{t_n^2 x_{n+2}}^2 = d\xi_{n+2}^1 d\xi_{n+2}^2 d\eta_{n+2}^1 d\eta_{n+2}^2 \left( \eta_{n+2}^1 \eta_{n+2}^2 \right)^{x_{n+2}} \]
\[ D_{t_n^2 x_{n-1}+2}^3 = d\xi_{n-1}^1 d\xi_{n-1}^2 d\eta_{n-1}^1 d\eta_{n-1}^2 \left( \eta_{n-1}^1 \eta_{n-1}^2 \right)^{x_{n-1}+2} \]
\[ \times \left( \bar{\eta}_{n+1}^1 \eta_{n+1}^2 \eta_{n+2}^1 \eta_{n+2}^2 \right)^{x_{n+2}}, \]
\[ \times \left( \bar{\xi}_{n-2}^1 \xi_{n-2}^2 \xi_{n+2}^1 \xi_{n+2}^2 \right)^{x_{n-1}+2}. \]

(39)

(40)

For the bosonic part, \( S^2 \) and \( S^4 \) are determined by the SVD as follows:

\[ M_{t_n^2 x_{n+2}+2 t_n^2 x_{n-1}+2} = (-1)^{t_n^1 + t_n^2} T_{x_{n}^2 x_{n+2} x_{n-1}^2}, \]
\[ = \sum_{t_n^* \neq \pm 2} S^2_{x_{n}^2 x_{n+2} x_{n-1}^2} + S^4_{x_{n}^2 x_{n+2} x_{n-1}^2}. \]

(41)

A coarse-grained tensor is obtained by

\[ T_{x_n^* t_n^* x_{n-1}^* t_n^* x_{n-2}^*} = \int \sum_{\{x_n, t_n\}} S^1_{x_n t_n x_{n-1}^*} S^2_{x_n^2 x_{n+2} x_{n-1}^2} S^3_{x_n^2 x_{n+2} x_{n-1}^2} S^4_{x_n^2 x_{n+2} x_{n-1}^2} \]
\[ \times \left( \bar{\eta}_{n}^1 \eta_{n+1}^2 \xi_{n-1}^1 \xi_{n-1}^2 \right)^{x_n^* \pm 2} \xi_{n+2}^1 \xi_{n+2}^2 \]
\[ \times \delta \left( \sum_{t_{n, i}^+ + t_{n, i}^*} \mod 2, x_{n, i}^* \pm 1 \right) \delta \left( \sum_{t_n^1 + t_n^2} \mod 2, t_n^* \pm 2 \right) \]
\[ \times \delta \left( \sum_{t_n^1 + t_n^2} \mod 2, x_n^* \pm t_n^* \right) \delta \left( \sum_{t_n^1 + t_n^2} \mod 2, t_n^* \right), \]
\[ = T_{x_n^* t_n^* x_{n-1}^* t_n^* x_{n-2}^*} d\eta_{x_n^* \pm f} d\xi_{x_n^* \pm f} d\eta_{x_n^* \pm 2} d\xi_{x_n^* \pm 2} \]
\[ \times \left( \bar{\eta}_{n}^1 \eta_{n+1}^2 \xi_{n-1}^1 \xi_{n-1}^2 \right)^{x_n^* \pm 2} \xi_{n+2}^1 \xi_{n+2}^2. \]

(42)

Note that the constraint \( \delta_{0, x_n^* \pm f + t_n^* \pm f + x_n^* \pm t_n^* + t_n^* \pm 2} \mod 2 \) is imposed for the coarse-grained tensor. Figure 2 shows the contraction for the original indices in this renormalization step. We repeat this renormalization step until the number of lattice points reaches \( 2 \times 2 \), namely, four reduced tensors. From these tensors, the partition function is computed by full index contractions.

The computational costs of a standard SVD routine are proportional to the third power of the matrix size, thus the cost of the decomposition of a tensor is of order \( D_{\text{cut}}^6 \). On the other hand, the cost of the contraction is of order \( D_{\text{cut}}^6 \). Therefore, the total cost of GTRG is proportional to \( D_{\text{cut}}^6 \).

2.4 Boundary condition

From now on, we consider a system where the anti-periodic (periodic) boundary condition is imposed for the 2-direction (1-direction). This is taken into account by modifying the partition function:

\[ Z = \sum_{\{x, t, t'\}} \int \prod_{t_n} T_{x_n t_n x_n t_n'^*} B_{t_n^* t_n'^*}, \]

(43)

where the full tensor \( T \) is the same as before and the new matrix \( B \) is given by

\[ B_{t_n^* t_n'^*} = \begin{cases} (-1)^{t_n^1 + t_n^2} & \text{if } n_2 = 0, \\ \delta_{t_n^1 + t_n^2} & \text{else.} \end{cases} \]

(44)
As a result, a coarse-grained tensor contracted on a site with \( n_2 = N_2 - 1 \) is modified by

\[
A_{x_n x_{n+1}^+ x_{n+2}^+ t_{n+1}^+ t_{n+2}^+} = \int \sum_{\{x_n t_n\}} (-1)^{t_{n+1}^+ + t_{n+2}^+ + t_{n+1}^+ + t_{n+2}^+} S_1^{x_n t_{n+1}^+ t_{n+2}^+} S_2^{x_n t_{n+1}^+ x_{n+2}^+} S_3^{x_n t_{n+2}^+ x_{n+1}^+} S_4^{x_n + x_{n+1}^+ x_{n+2}^+} \\
\times \left( \tilde{\eta}_{n+1}^{x_n^+} \tilde{\eta}_{n+2}^{x_n^+} \xi_{n+1}^{x_n^+} \xi_{n+2}^{x_n^+} \right) t_{n+1}^+, t_{n+2}^+ \\
\times \delta \sum_i (x_{n+1,i} t_{n+1}^+ + t_{n+2}^+ \xi_{n+1}^{x_n^+} \xi_{n+2}^{x_n^+} \xi_{n+1}^{x_n^+} \xi_{n+2}^{x_n^+} \xi_{n+1}^{x_n^+} \xi_{n+2}^{x_n^+} \xi_{n+1}^{x_n^+} \xi_{n+2}^{x_n^+} \xi_{n+1}^{x_n^+} \xi_{n+2}^{x_n^+} \xi_{n+1}^{x_n^+} \xi_{n+2}^{x_n^+} \xi_{n+1}^{x_n^+} \xi_{n+2}^{x_n^+} \xi_{n+1}^{x_n^+} \xi_{n+2}^{x_n^+} ) \delta \sum_i (x_{n+1,i} t_{n+1}^+ + t_{n+2}^+ \xi_{n+1}^{x_n^+} \xi_{n+2}^{x_n^+} \xi_{n+1}^{x_n^+} \xi_{n+2}^{x_n^+} \xi_{n+1}^{x_n^+} \xi_{n+2}^{x_n^+} \xi_{n+1}^{x_n^+} \xi_{n+2}^{x_n^+} \xi_{n+1}^{x_n^+} \xi_{n+2}^{x_n^+} \xi_{n+1}^{x_n^+} \xi_{n+2}^{x_n^+} ) \delta \sum_i (x_{n+1,i} t_{n+1}^+ + t_{n+2}^+ \xi_{n+1}^{x_n^+} \xi_{n+2}^{x_n^+} \xi_{n+1}^{x_n^+} \xi_{n+2}^{x_n^+} \xi_{n+1}^{x_n^+} \xi_{n+2}^{x_n^+} \xi_{n+1}^{x_n^+} \xi_{n+2}^{x_n^+} \xi_{n+1}^{x_n^+} \xi_{n+2}^{x_n^+} ) \\
= \left( -1 \right)^{x_n^+, t_{n+1}^+, t_{n+2}^+, t_{n+1}^+, t_{n+2}^+, t_{n+1}^+, t_{n+2}^+, t_{n+1}^+, t_{n+2}^+} T_{x_n x_{n+1}^+ x_{n+2}^+ t_{n+1}^+ t_{n+2}^+} d\eta_{x_n^+} d\xi_{x_n^+} d\eta_{x_n^+} d\xi_{x_n^+} d\eta_{x_n^+} d\xi_{x_n^+} d\eta_{x_n^+} d\xi_{x_n^+} d\eta_{x_n^+} d\xi_{x_n^+} \\
\times \left( \tilde{\eta}_{n+1}^{x_n^+} \tilde{\eta}_{n+2}^{x_n^+} \xi_{n+1}^{x_n^+} \xi_{n+2}^{x_n^+} \right) t_{n+1}^+, t_{n+2}^+. \tag{45}
\]

Therefore, the once-renormalized partition function is obtained by

\[
Z^{(1)} = \sum_{\{x,t\}} \int \prod_{n_{:1} \neq n_2} T_{x_n t_{n+1}^+ t_{n+2}^+ x_{n+1}^+ x_{n+2}^+ t_{n+1}^+ t_{n+2}^+} \prod_{n_{:1} = n_2} A_{x_n^+ t_{n+1}^+ t_{n+2}^+ x_{n+1}^+ x_{n+2}^+ t_{n+1}^+ t_{n+2}^+} = \sum_{\{x,t\}} \int \prod_{n} T_{x_n t_{n+1}^+ t_{n+2}^+ x_{n+1}^+ x_{n+2}^+ t_{n+1}^+ t_{n+2}^+} B_{x_n^+ t_{n+1}^+ t_{n+2}^+} \tag{46}
\]

where the site indices are replaced by \( n^* \rightarrow n \) for readability and the other boundary matrices are given by

\[
B_{x_n^+ t_{n+1}^+ t_{n+2}^+} = \begin{cases} (-1)^{x_n^+, t_{n+1}^+, t_{n+2}^+} \delta_{x_n^+, x_{n+1}^+} & \text{if } n_{:1} = n_2, \\
\delta_{x_n^+, x_{n+1}^+} & \text{else}. \end{cases} \tag{47}
\]
\[
\begin{align*}
B_{n,n-2}^2 &= \begin{cases} 
(-1)^{\nu_n \cdot f} & \text{if } n_1 = n_2, \\
\delta_{n_n',n_{n-2}} & \text{else.}
\end{cases}
\end{align*}
\]

Similarly, a twice-coarse-grained tensor contracted on \( n_1 = n_2 \) is obtained by

\[
A_{x_n t_n x_{n-1} t_{n-2}} = \sum_{x_{n',t_n}} \left(-1\right)^{y_{n',t_n} + 4 t_n} S_{x_{n'+1} x_{n-1} t_{n-2}} S_{x_n t_{n+2} x_{n-1} t_{n-2}} S_{x_n t_{n+2} x_{n-1} t_{n-2}} S_{x_{n'+1} x_{n-1} t_{n-2}}
\]

\[
\times \left( \frac{\eta_{n'+1} \cdot \eta_n}{\xi_{n'+2} \cdot \xi_n} \right) ^ {x_{n',t_n} + 4 t_n} \delta (x_{n',t_n} + 4 t_n) \mod 2, x_n {\cdot} t_{n-1} \mod 2, t_{n-1} - 2 \cdot f
\]

\[
\times \delta (x_{n+2}, t_n) \mod 2, x_n {\cdot} t_{n+1} \mod 2, t_{n+1} = \left(-1\right)^{y_{n+2} - 2 \cdot f} T_{x_n t_{n+2} x_{n-1} t_{n-2} + 2} d^2 \xi_{n',t_n} + 2 d^2 \eta_{n',t_n} + 2 d^2 \eta_{n',t_n} + 2 \mod 2, t_{n+1} - 2 \cdot f
\]

\[
\times \left( \frac{\eta_{n'+1} \cdot \eta_n}{\xi_{n'+2} \cdot \xi_n} \right) ^ {x_{n'+1} \cdot f} \delta (x_{n'+1} \cdot f) \mod 2, t_{n+1} - 2 \cdot f
\]

\[
\times \delta (x_{n+2}, t_{n+1}) \mod 2, x_n {\cdot} t_{n+1} \mod 2, t_{n+1} = \left(-1\right)^{y_{n+2} - 2 \cdot f} T_{x_n t_{n+2} x_{n-1} t_{n-2} + 2} d^2 \xi_{n',t_n} + 2 d^2 \eta_{n',t_n} + 2 d^2 \eta_{n',t_n} + 2 \mod 2, t_{n+1} - 2 \cdot f
\]

\[
A_{x_n t_n x_{n-1} t_{n-2}} = \sum_{x_{n',t_n}} \int \prod \left[ T_{x_n t_n x_{n-1} t_{n-2}} \right] A_{x_n t_n x_{n-1} t_{n-2}} = \sum_{x_{n',t_n}} \int \prod \left[ T_{x_n t_n x_{n-1} t_{n-2}} \right] B_{n',n-2}
\]

where

\[
B_{n',n-2} = \begin{cases} 
(-1)^{\nu_n \cdot f} & \text{if } n_2 = 0, \\
\delta_{n'_n,n_{n-2}} & \text{else.}
\end{cases}
\]

Therefore, in this formulation, the boundary condition returns to the original one every 2 renormalization steps.

### 3. Numerical results

First, we compare the numerical results of \( \ln Z \) with the exact value \( \ln Z_{\text{exact}} \) in the free massless case. Figure 3 shows the relative deviation

\[
\delta (D_{\text{cut}}) = \frac{\ln Z(D_{\text{cut}}) - \ln Z_{\text{exact}}}{\ln Z_{\text{exact}}}
\]

as a function of \( D_{\text{cut}} \). The convergence behavior is roughly observed although it is not so smooth. The convergence rate at \( \mu = 1 \) is slower than that of \( \mu = 2 \). For \( \mu = 2 \), lattice volume dependence is not seen while, for \( \mu = 1 \), a larger volume is strongly affected by truncation error. To see the convergence issue in more detail, we investigate the spectrum of a bosonic tensor in Fig. 4. Clear hierarchy is observed for \( \mu = 2 \) while nearly degenerate structure is seen for \( \mu = 1 \), especially after several iterations. Figure 5 shows the relative deviation as a function of \( \mu \) with fixed \( D_{\text{cut}} = 64 \). The deviation rapidly increases around \( \mu \approx 0.3 \) and 1, where transition-like behavior is actually observed, as shown later.

Next, we compute the fermion number density, defined as

\[
n = \frac{1}{N_1 N_2} \frac{\partial \ln Z}{\partial \mu}.
\]

Figure 6 plots the fermion number density as a function of \( \mu \) for some non-trivial sets of parameters. Since the model is in a 2D system with one flavor, the saturation density for the fermion number is one. We observe that the fermion density saturates to this value for larger chemical potentials.
Fig. 3. The relative deviation \( \delta \) as a function of \( D_{\text{cut}} \) for the free massless case.

Fig. 4. Spectrum of the bosonic tensor for the free massless case.

Finally, we perform a finite-size scaling analysis for the quark number susceptibility, defined as

\[
\chi = \frac{1}{N_1 N_2} \frac{\partial^2 \ln Z}{\partial \mu^2}.
\]  

(54)

The susceptibility as a function of \( \mu \) is shown in Fig. 7 for various spatial volumes with two sets of parameter \( (m, g) = (0, 0) \) and \( (0, 0.7) \). For both cases, we observe that there is a peak around \( \mu = 1 \) and the peak height shows no volume dependence; therefore, we conclude that this transition is a crossover. For lower \( \mu \lesssim 0.6 \), the TRG results develop some peaks for both couplings. In order to check whether these peaks are fake or not, we compare them with the exact results at \( g = 0 \), shown as curves for each volume \( N_1 = 32, 64, 96 \), where for larger volumes the peaks disappear in the lower-\( \mu \) region. From the comparison, we find that the TRG results at \( g = 0 \), shown as dots, tend to deviate...
Fig. 5. The relative deviation $\delta$ as a function of $\mu$ with fixed $D_{\text{cut}} = 64$ for the free massless case. Around $\mu = 0.3$ and 1, the deviation becomes large. For $N_1 = N_2 = 4$, the TRG result becomes exact; thus, the relative deviation is exactly zero up to machine precision. This therefore shows the validity of our calculation.

Fig. 6. Fermion number density $n$ as a function of $\mu$ with fixed $N_1 = N_2 = 32$ and $D_{\text{cut}} = 64$. For larger chemical potentials, the number density for all cases of $(m, g)$ that we investigated saturates to unity as expected.

from these curves for larger volumes. Thus we conclude that these peaks at $g = 0$ in the TRG results, especially at larger volumes, are fake. For $g = 0.7$, since we cannot make a direct comparison with the exact results, we are content with comparing two results obtained by different resolutions of the chemical potential in the numerical derivative. The difference is then barely seen; thus, we expect that the peak around $\mu = 0.4$ for $g = 0.7$ is not a fake but, of course, further study is required to solidify our expectation. For the free massless case, around the peak positions ($\mu \approx 1$), the relative deviation in Fig. 5 becomes large. It is known that the approximation for TRG gets worse near a critical point,
Fig. 7. Finite-size scaling of the fermion number susceptibility for fixed \((m, N_2, D_{\text{cut}}) = (0, 32, 64)\). \(N_1\) dependence is not observed even in the presence of interaction, \(g = 0.7\).

While we observe that such a behavior occurs even for the crossover case. Needless to say, with other parameters \((N_1, N_2, m, g)\), real phase transitions can occur and the strength of the transition may change; thus, the source of the loss of accuracy that we observed here could be a remnant of the real phase transition.

4. Reweighting method

4.1. Formulation

In the TRG calculation, one usually computes the partition function at several parameter points (mesh). Then the numerical derivative of the partition function with respect to the parameter is found by using a few points; one needs a fine mesh to reduce the discretization error. To reduce the computational time, we propose a method to obtain an approximated coarse-grained tensor at one parameter by using another set of singular values at a different parameter. Using an analogy from the Monte Carlo method, we refer to this method as the reweighting method.

Let the bosonic part be \(T_{x_n l_n x_{n-1} l_{n-2}}\) at the original parameter; its SVD is given by

\[
T_{x_n l_n x_{n-1} l_{n-2}} = \sum_{x_n^* \rightarrow \hat{1}_b, x_{n-1}^* \rightarrow \hat{1}_b} \sum_{U_{x_n l_n, x_{n-1} l_{n-2}}} \sigma_{13} U_{x_n l_n, x_{n-1} l_{n-2}}^{13} \delta_{x_n^* \rightarrow \hat{1}_b} \delta_{x_{n-1}^* \rightarrow \hat{1}_b} U_{x_n^* l_n^*, x_{n-1}^* l_{n-2}^*}^{3*},
\]

Another tensor at a different parameter \(T'_{x_n l_n x_{n-1} l_{n-2}}\) can be written as

\[
T'_{x_n l_n x_{n-1} l_{n-2}} = \sum_{x_n^* \rightarrow \hat{1}_b, x_{n-1}^* \rightarrow \hat{1}_b} \left( U_{x_n l_n, x_{n-1} l_{n-2}}^{1*} \right) \left( U_{x_n l_n, x_{n-1} l_{n-2}}^{1*} \right)
\]

\[
\times T_{x_n l_n x_{n-1} l_{n-2}} \left( U_{x_n^* l_n^*, x_{n-1}^* l_{n-2}^*}^{3*} \right) \left( U_{x_n^* l_n^*, x_{n-1}^* l_{n-2}^*}^{3*} \right)
\]

\[
= \sum_{x_n^* \rightarrow \hat{1}_b, x_{n-1}^* \rightarrow \hat{1}_b} \sum_{U_{x_n l_n, x_{n-1} l_{n-2}}} \sigma_{13} U_{x_n l_n, x_{n-1} l_{n-2}}^{13} \delta_{x_n^* \rightarrow \hat{1}_b} \delta_{x_{n-1}^* \rightarrow \hat{1}_b} U_{x_n^* l_n^*, x_{n-1}^* l_{n-2}^*}^{3*},
\]
where the new matrix $\Sigma^{13}$ is given by
\[
\Sigma^{13}_{x_{n^*+2a}^*b'x_{n^*+2a}^*b'} = \sum_{x_{n^*+2a}^*b'} U_{x_{n^*+2a}^*b'}^{\delta s'} T'_{x_{n^*+2a}^*b'} U_{x_{n^*+2a}^*b'}^{\delta s'}. 
\] (57)

By truncating the indices $x_{n^*+2a}^*b', x_{n^*+2a}^*b'$ at $D_{cut}$, the decomposition of $T'$ can be formally defined as
\[
T'_{x_{n^*+2a}^*b'} \simeq \sum_{x_{n^*+2a}^*b'} D_{cut} U_{x_{n^*+2a}^*b'}^{\delta s'} \sum_{x_{n^*+2a}^*b'} U_{x_{n^*+2a}^*b'}^{\delta s'} U_{x_{n^*+2a}^*b'}^{\delta s'}. 
\] (58)

Similarly, for another decomposition
\[
M_{x_{n^*+2a}^*b'}^{\delta s'} = \sum_{x_{n^*+2a}^*b'} U_{x_{n^*+2a}^*b'}^{\delta s'} T'_{x_{n^*+2a}^*b'} U_{x_{n^*+2a}^*b'}^{\delta s'}. 
\] (59)

the new matrix $\Sigma^{24}$ is defined by
\[
\Sigma^{24}_{x_{n^*+2a}^*b'} = \sum_{x_{n^*+2a}^*b'} (-1)^{l_{n^*+2a}^*b'} U_{x_{n^*+2a}^*b'}^{\delta s'} T'_{x_{n^*+2a}^*b'} U_{x_{n^*+2a}^*b'}^{\delta s'}. 
\] (60)

By using the singular vectors $U^{1,2,3,4}$ at the original parameter, an intermediate tensor is defined by
\[
\tilde{T}_{x_{n^*+2a}^*b'}^{\delta s'} = \int \sum_{x_{n^*+2a}^*b'} U_{x_{n^*+2a}^*b'}^{\delta s'} U_{x_{n^*+2a}^*b'}^{\delta s'} U_{x_{n^*+2a}^*b'}^{\delta s'} U_{x_{n^*+2a}^*b'}^{\delta s'} \times (\tilde{\eta}_{n^*+2a}^* \tilde{\xi}_{n^*})^{\delta s',f} \times \delta_a^{\text{(2)}}(x_{n^*+2a}^*b', \tilde{\eta}_{n^*+2a}^* \tilde{\xi}_{n^*})^{\delta s',f} \times \delta_a^{\text{(2)}}(x_{n^*+2a}^*b', \tilde{\eta}_{n^*+2a}^* \tilde{\xi}_{n^*})^{\delta s',f} \times \delta_a^{\text{(2)}}(x_{n^*+2a}^*b', \tilde{\eta}_{n^*+2a}^* \tilde{\xi}_{n^*})^{\delta s',f} 
\] (61)

where
\[
U_1^{\delta s'} = U_{x_{n^*+2a}^*b'}^{\delta s'} D_{x_{n^*+2a}^*b'}^{\delta s'}, 
\] (62)
\[
U_2^{\delta s'} = U_{x_{n^*+2a}^*b'}^{\delta s'} D_{x_{n^*+2a}^*b'}^{\delta s'}, 
\] (63)
\[
U_3^{\delta s'} = U_{x_{n^*+2a}^*b'}^{\delta s'} D_{x_{n^*+2a}^*b'}^{\delta s'}, 
\] (64)
\[
U_4^{\delta s'} = U_{x_{n^*+2a}^*b'}^{\delta s'} D_{x_{n^*+2a}^*b'}^{\delta s'}. 
\] (65)

---

4 This decomposition is not optimal since this is not the SVD of $T'$.

5 The corresponding singular values are not included.
Fig. 8. The relative deviation between the reweighting method and the exact value in Eq. (68) as a function of $\mu$ for the free massless case with fixed $D_{\text{cut}} = 64$. From top to bottom, the original value of $\mu$ is given by $\mu = 0, 1,$ and 2, respectively.

From this intermediate tensor, we can obtain not only the original coarse-grained tensor,

$$
T_{\tilde{x}_n \tilde{t}_n, \tilde{x}_n' \tilde{t}_n'} = \sum_{x_n \tilde{x}_n, t_n \tilde{t}_n} \sigma_{x_n \tilde{x}_n, t_n \tilde{t}_n}^{13} \delta_{x_n \tilde{x}_n, t_n \tilde{t}_n} \sigma_{x_n \tilde{x}_n, t_n \tilde{t}_n}^{24} \delta_{x_n \tilde{x}_n, t_n \tilde{t}_n} \tilde{T}_{x_n \tilde{t}_n, x_n' \tilde{t}_n'}.
$$

(66)
but also a coarse-grained tensor at a different parameter,

\[
\mathcal{T}'_{x_n^* t_n^* x_{n-1}^* t_{n-2}^*} = \sum_{x_n^*, t_n^*} \sum_{x_n^*, t_n^*} \delta_{x_n^*, f} \delta_{x_n^*, f} \delta_{t_n^*, f} \delta_{t_n^*, f} \tilde{T}_{x_n^* t_n^* x_{n-1}^* t_{n-2}^*}, \tag{67}
\]

which is not optimal but is approximately fine if the difference between the original and target parameters is small. If the intermediate tensor and the singular vectors at the original parameter have been stored, a coarse-grained tensor for another parameter is obtained by only \( D^5 \text{cut} \)-order computational cost.

4.2. Numerical results

The relative deviation

\[
\delta = \frac{\ln Z_{RW} - \ln Z_{\text{exact}}}{\ln Z_{\text{exact}}} \tag{68}
\]

between \( \ln Z_{RW} \), computed by using the reweighting method and the exact one, is shown in Fig. 8. The relative deviation increases with distance from the original parameters and lattice size. The deviation reweighting from nearly the transition point (\( \mu = 1 \)) quickly increases compared with that off-transition (\( \mu = 0, \mu = 2 \)).

5. Summary and outlook

We have applied the GTRG to the one-flavor lattice Gross–Neveu model with chemical potential in the Wilson fermion formulation. At some non-trivial parameter set at finite density, we found a transition-like behavior; a finite-size scaling study shows that this transition is a crossover but not a real phase transition. Furthermore, we observed that, around the “transition” point, the approximation of TRG gets worse, although this is not a critical point.

We introduced the reweighting method for TRG and demonstrated it for some parameters. As a result, the errors increase with distance from the original parameters and lattice size. Furthermore, we observed that the reweighting from around the “transition” point quickly deteriorates compared with that from the off-transition region.

This is the first application of the GTRG to a finite-density system. We hope that the formulation given in this work is extended to other finite-density systems.

Acknowledgements

We would like to thank Y. Shimizu and D. Satou for helpful advice. S.T. is grateful to Y. Kuramashi for useful conversation. This work is partially supported by Grants-in-Aid for Scientific Research from the Ministry of Education, Culture, Sports, Science and Technology (Nos. 26800130).

Appendix A. Details of the bosonic tensor

In this appendix, we show explicit elements of the bosonic tensor \( T_{x_n t_n x_{n-1} t_{n-2}} \) in Eq. (26).
Note that the four-fermion coupling $g^2$ enters only in the first element.

References

[1] M. Levin and C. P. Nave, Phys. Rev. Lett. 99, 120601 (2007).
[2] J. F. Yu, Z. Y. Xie, Y. Meurice, Y. Liu, A. Denbleyker, H. Zou, M. P. Qin, J. Chen, and T. Xiang, Phys. Rev. E 89, 013308 (2014).
[3] J. Unmuth-Yockey, Y. Meurice, J. Osborn, and H. Zou, [arXiv:1411.4213 [hep-lat]] [Search INSPIRE].
[4] Y. Shimizu, Mod. Phys. Lett. A 27, 1250035 (2012).
[5] Z. Y. Xie, H. C. Jiang, Q. N. Chen, Z. Y. Weng, and T. Xiang, Phys. Rev. Lett. 103, 160601 (2009).
[6] Z. Y. Xie, J. Chen, M. P. Qin, J. W. Zhu, L. P. Yang, and T. Xiang, Phys. Rev. B 86, 045139 (2012).
[7] Z.-C. Gu, F. Verstraete, and X.-G. Wen, [arXiv:1004.2563 [cond-mat.str-el]] [Search INSPIRE].
[8] Z.-C. Gu, Phys. Rev. B 88, 115139 (2013).
[9] Y. Shimizu and Y. Kuramashi, Phys. Rev. D. 90, 014508 (2014).
[10] Y. Shimizu and Y. Kuramashi, Phys. Rev. D. 90, 074503 (2014).
[11] D. J. Gross and A. Neveu, Phys. Rev. D 10, 3235 (1974).
[12] T. Eguchi and R. Nakayama, Phys. Lett. B 126, 89 (1983).
[13] S. Aoki and K. Higashijima, Prog. Theor. Phys. 76, 521 (1986).
[14] T. Izubuchi, J. Noaki, and A. Ukawa, Phys. Rev. D 58, 114507 (1998).
[15] J. B. Kogut, H. Matsuoka, M. Stone, H. W. Wyld, S. Shenker, J. Shigemitsu, and D. K. Sinclair, Nucl. Phys. B 225, 93 (1983).
[16] P. Hasenfratz and F. Karsch, Phys. Lett. B 125, 308 (1983).