AN ALTERNATIVE APPROACH TO COHERENT CHOICE FUNCTIONS

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ABSTRACT. Choice functions constitute a simple, direct and very general mathematical framework for modeling choice under uncertainty. In particular, they are able to represent the set-valued choices that appear in imprecise-probabilistic decision making. We provide these choice functions with a clear interpretation in terms of desirability, use this interpretation to derive a set of basic coherence axioms, and show that this notion of coherence leads to a representation in terms of sets of strict preference orders. By imposing additional properties such as totality, the mixing property and Archimedeanity, we obtain representation in terms of sets of strict total orders, lexicographic probability systems, coherent lower previsions or linear previsions.

1. INTRODUCTION

Choice functions provide an elegant unifying mathematical framework for studying set-valued choice: when presented with a set of options, they generally return a subset of them. If this subset is a singleton, it provides a unique optimal choice or decision. But if the answer contains multiple options, these are incomparable and no decision is made between them. Such set-valued choices are a typical feature of decision criteria based on imprecise-probabilistic uncertainty models, which aim to make reliable decisions in the face of severe uncertainty. Maximality and E-admissibility are well-known examples. When working with a choice function, however, it is immaterial whether it is based on such a decision criterion. The primitive objects on this approach are simply the set-valued choices themselves, and the choice function that represents all these choices serves as an uncertainty model in and by itself.

The seminal work by Seidenfeld et al. [17] has shown that a strong advantage of working with choice functions is that they allow us to impose axioms on choices, aimed at characterising what it means for choices to be rational and internally consistent. This is also what we want to do here, but we believe our angle of approach to be novel and unique: rather than think of choice intuitively, we provide it with a concrete interpretation in terms of desirability [4, 8, 9, 26] or binary preference [15]. Another important feature of our approach is that we consider a very general setting, where the options form an abstract real vector space; horse lotteries and gambles correspond to special cases.

The basic structure of our paper is as follows. We start in Section 2 by introducing choice functions and our interpretation for them. Next, in Section 3, we develop an alternative but equivalent way of describing these choice functions: sets of desirable option sets. We use our interpretation to suggest and motivate a number of rationality, or coherence, axioms for such sets of desirable option sets, and show in Section 4 what are the corresponding coherence axioms for choice (or rejection) functions. Section 5 deals with the special case of binary choice, and its relation to the theory of sets of desirable options [4, 8, 9, 26] and binary preference. This is important because our main result in Section 6 shows that any coherent choice model can be represented in terms of sets of such binary choice models. In the remaining Sections 7–9, we consider additional axioms or properties, such as totality, the mixing property, and an Archimedean property, and prove corresponding representation results. This includes representations in terms of sets of strict total orders, sets of lexicographic probability systems, sets of coherent lower previsions or linear previsions.
previsions and sets of linear previsions. To facilitate the reading, proofs and intermediate results have been relegated to the Appendix.

2. Choice functions and their interpretation

A choice function $C$ is a set-valued operator on sets of options. In particular, for any set of options $A$, the corresponding value of $C$ is a subset $C(A)$ of $A$. The options themselves are typically actions amongst which a subject wishes to choose. We here follow a very general approach where these options constitute an abstract real vector space $\mathcal{V}$ provided with a vector ordering $\succeq$ and a strict version $\succ$. The elements $u$ of $\mathcal{V}$ are called options and $\mathcal{V}$ is therefore called the option space. We let $\mathcal{V}_{>0} := \{ u \in \mathcal{V} : u \succ 0 \}$. The purpose of a choice function is to represent our subject’s choices between such options.

Our motivation for adopting this general framework where options are elements of abstract vector spaces, rather than the more familiar one that focuses on choice between, say, horse lotteries [2, 3, 12, 15, 17], is its applicability to various contexts. A typical set-up that is customary in decision theory, for example, is one where every option has a corresponding reward that depends on the state of a variable $X$, about which the subject is typically uncertain. Hence, the reward is uncertain too. As a special case, therefore, we can consider that the variable $X$ takes values $x$ in some set of states $\mathcal{X}$. The reward that corresponds to a given option is then a function $u$ on $\mathcal{X}$. If we assume that this reward can be expressed in terms of a real-valued linear utility scale, this allows us to identify every act with a real-valued map on $\mathcal{X}$. These these maps are often taken to be bounded and are then called gambles on $X$. In that context, we can consider the different gambles on $X$ as our options, and the vector space $\mathcal{V}$ as the set of all such gambles. Two popular vector orderings on $\mathcal{V}$ then correspond to choosing

$$\mathcal{V}_{\geq} := \{ u \in \mathcal{V} : u \geq 0 \text{ and } u \neq 0 \} \quad \text{or} \quad \mathcal{V}_{>0} := \{ u \in \mathcal{V} : \inf u > 0 \},$$

where $\geq$ represents the point-wise ordering of gambles.

A more general framework, which allows us to dispense with the linearity assumption of the utility scale, consists in considering as option space the linear space of all bounded real-valued maps on the set $\mathcal{X} \times \mathcal{R}$, where $\mathcal{R}$ is a (countable) set of rewards. Zaffalon and Miranda [27] have shown that, in a context of binary preference relations, this leads to a theory that is essentially equivalent to the classical horse lottery approach. It tends, however, to be more elegant, because a linear space is typically easier to work with than a convex set of horse lotteries. Van Camp [22] has shown that this idea can be straightforwardly extended from binary preference relations to the more general context of choice functions. We follow his lead in focusing on linear spaces of options here.

In both of the above-mentioned cases, the options are still bounded real-valued maps. In fairly recent work, Van Camp et al. [22, 23] have shown that a notion of indifference can be associated with choice functions quite easily, by moving from the original option space to its quotient space with respect to the linear subspace of all options that are assessed to be equivalent to the zero option. Even when the original options are real-valued maps, the elements of the quotient space will be equivalence classes of such maps—affine subspaces of the original option space—which can no longer be straightforwardly identified with real-valued maps. This provides even more incentives for considering options to be vectors in some abstract linear space $\mathcal{V}$.

Having introduced and motivated our abstract option space $\mathcal{V}$, sets of options can now be identified with subsets of $\mathcal{V}$, which we call option sets. We restrict our attention here to finite option sets and will use $\mathcal{Q}$ to denote the set of all such finite subsets of $\mathcal{V}$, including the empty set.

**Definition 1** (Choice function). A choice function $C$ is a map from $\mathcal{Q}$ to $\mathcal{Q}$ such that $C(A) \subseteq A$ for every $A \in \mathcal{Q}$. 
Options in $A$ that do not belong to $C(A)$ are said to be rejected. This leads to an alternative but equivalent representation in terms of rejection functions: the rejection function $R_C$ corresponding to a choice function $C$ is a map from $\mathcal{D}$ to $\mathcal{D}$, defined by $R_C(A) := A \setminus C(A)$ for all $A \subseteq \mathcal{D}$.

Alternatively, a rejection function $R$ can also be independently defined as a map from $\mathcal{D}$ to $\mathcal{D}$ such that $R(A) \subseteq A$ for all $A \subseteq \mathcal{D}$. The corresponding choice function $C_R$ is then clearly defined by $C_R(A) := A \setminus R(A)$ for all $A \subseteq \mathcal{D}$. Since a choice function is completely determined by its rejection function, any interpretation for rejection functions automatically implies an interpretation for choice functions. This allows us to focus on the former.

Our interpretation for rejection functions—and therefore also for choice functions—now goes as follows. Consider a subject whose uncertainty is represented by a rejection function $R$, or equivalently, by a choice function $C_R$. Then for a given option set $A \subseteq \mathcal{D}$, the statement that an option $u \in A$ is rejected from $A$—that is, that $u \in R(A)$—is taken to mean that there is at least one option $v$ in $A$ that our subject strictly prefers over $u$.

If we denote the strict preference of one option $v$ over another option $u$ by $v \triangleright u$, this can be written succinctly as

$$\exists v \in A \forall u \in A : (u \in R(A) \iff (\exists v \in A : v \triangleright u)). \quad (1)$$

The only requirements that we impose on the preference ordering $\triangleright$ is that it should be a strict partial order that extends $\succ$ and is compatible with the vector space operations on $\mathcal{Y}$:

- $\triangleright_0$. $\triangleright$ is irreflexive: for all $u \in \mathcal{Y}$, $u \not\triangleright u$;
- $\triangleright_1$. $\triangleright$ is transitive: for all $u, v, w \in \mathcal{Y}$, $u \triangleright v$ and $v \triangleright w$ imply that also $u \triangleright w$;
- $\triangleright_2$. for all $u, v \in \mathcal{Y}$, $u \triangleright v$ implies that $u \succ v$;
- $\triangleright_3$. for all $u, v, w \in \mathcal{Y}$, $u \triangleright v$ implies that—so is equivalent with—$u + w \triangleright v + w$;
- $\triangleright_4$. for all $u, v \in \mathcal{Y}$ and all $\lambda > 0$, $u \triangleright v$ implies that—so is equivalent with—$\lambda u \triangleright \lambda v$.

We then call such a preference ordering $\triangleright$ coherent. It follows from Axioms $\triangleright_0$ and $\triangleright_3$ that we can rewrite Equation (1) as

$$\forall A \subseteq \mathcal{D} \forall u \in A : (u \in R(A) \iff (\exists v \in A : v \triangleright u) \iff (\exists v \in A \setminus \{u\} : v \triangleright u) \iff (\exists v \in A \setminus \{u\} : v \triangleright 0)) \quad (2)$$

where we use Axiom $\triangleright_3$ for the first equivalence, and Axiom $\triangleright_0$ for the second. Both equivalences can be conveniently turned into a single one if we no longer require that $u$ should belong to $A$ and consider statements of the form $u \in R(A \cup \{u\})$. Equation (2) then turns into

$$\forall u \in \mathcal{Y} \forall A \subseteq \mathcal{D} : (u \in R(A \cup \{u\}) \iff (\exists v \in A : v \triangleright u \triangleright 0)) \quad (3)$$

So, according to our interpretation, the statement that $u$ is rejected from $A \cup \{u\}$ is taken to mean that the option set

$$A - u := \{v - u : v \in A\} \quad (4)$$

contains at least one option that, according to $\triangleright$, is strictly preferred to the zero option 0.

### 3. Coherent sets of desirable option sets

A crucial observation at this point is that our interpretation for rejection functions does not require our subject to completely specify the strict ordering $\triangleright$. Instead, all that is needed is for her to specify those option sets $A \subseteq \mathcal{D}$ that—to her—contain at least one option that is strictly preferred to the zero option 0. Options that are strictly preferred to zero—so options $u$ for which $u \triangleright 0$—are also called desirable, which is why we will call such option sets desirable option sets and collect them in a set of desirable option sets $K \subseteq \mathcal{D}$. Our interpretation therefore allows a modeller to specify her beliefs by specifying a set of desirable option sets $K \subseteq \mathcal{D}$.

As can be seen from Equations (3) and (4), such a set of desirable option sets $K$ completely determines a rejection function $R$ and its corresponding choice function $C_R$:

$$\forall u \in \mathcal{Y} \forall A \subseteq \mathcal{D} : (u \in R(A \cup \{u\}) \iff A - u \in K). \quad (5)$$
Our interpretation, together with the basic Axioms $\triangleright_0$ and $\triangleright_4$, therefore allows the study of rejection and choice functions to be reduced to the study of sets of desirable option sets.

We let $K$ denote the set of all sets of desirable option sets $K \subseteq \mathcal{D}$, and consider any such $K \in K$. The first question to address is when to call $K$ coherent: which properties should we impose on a set of desirable option sets in order for it to reflect a rational subject’s beliefs? We propose the following axiomatisation, using $(\lambda, \mu) > 0$ as a shorthand notation for ‘$\lambda \geq 0$, $\mu \geq 0$ and $\lambda + \mu > 0$’.

Definition 2 (Coherence for sets of desirable option sets). A set of desirable option sets $K \subseteq \mathcal{D}$ is called coherent if it satisfies the following axioms:

- $K_0$. if $A \in K$ then also $A \setminus \{0\} \in K$, for all $A \in \mathcal{D}$;
- $K_1$. $\{0\} \notin K$;
- $K_2$. $\{u\} \in K$, for all $u \in \mathcal{V}_0$;
- $K_3$. if $A_1, A_2 \in K$ and if, for all $u \in A_1$ and $v \in A_2$, $(\lambda_{u,v}, \mu_{u,v}) > 0$, then also
  \[ \{\lambda_{u,v} u + \mu_{u,v} v : u \in A_1, v \in A_2\} \subseteq K; \]
- $K_4$. if $A_1 \in K$ and $A_1 \subseteq A_2$, then also $A_2 \in K$, for all $A_1, A_2 \in \mathcal{D}$.

We denote the set of all coherent sets of desirable option sets by $K$.

This axiomatisation is entirely based on our interpretation and the following three axioms for desirability:

- $d_1$. $0$ is not desirable;
- $d_2$. all $u \in \mathcal{V}_0$ are desirable;
- $d_3$. if $u, v$ are desirable and $(\lambda, \mu) > 0$, then $\lambda u + \mu v$ is desirable.

Each of these three axioms follows trivially from our assumptions on the preference relation $\triangleright$. $d_1$ follows from $\triangleright_0$, $d_2$ follows from $\triangleright_2$ and $d_3$ follows from $\triangleright_1$ and $\triangleright_4$.\footnotemark

\footnotetext{Conversely, under Axiom $\triangleright_3$ for the preference relation $\triangleright$, these three Axioms $d_1$–$d_3$ imply the remaining Axioms $\triangleright_0$–$\triangleright_2$ and $\triangleright_4$.}

That the coherence Axioms $K_0$–$K_4$ are implied by our rationality requirements $d_1$–$d_3$ for the concept of desirability, can now be seen as follows. Since a desirable option set is by definition a set of options that contains at least one desirable option, Axiom $K_4$ is immediate. Axioms $K_0$ and $K_1$ follow naturally from $d_1$, and Axiom $K_2$ is an immediate consequence of $d_2$. The argument for Axiom $K_3$ is more subtle. Since $A_1$ and $A_2$ are two desirable option sets, there must be at least one desirable option $u \in A_1$ and one desirable option $v \in A_2$. Since for these two options, the positive linear combination $\lambda_{u,v} u + \mu_{u,v} v$ is again desirable by $d_3$, at least one of the elements of $\{\lambda_{u,v} u + \mu_{u,v} v : u \in A_1, v \in A_2\}$ must be a desirable option. Hence, it must be a desirable option set.

4. COHERENT REJECTION FUNCTIONS

Now that we have formulated our basic rationality requirements $K_0$–$K_4$ for sets of desirable option sets $K$, we are in a position to use their correspondence (5) with rejection functions $R$ to derive the equivalent rationality requirements for the latter.

Equation (5) already allows us to derive a first and very basic axiom for rejection functions—and a very similar one for choice functions, left implicit here—without imposing any requirements on sets of desirable option sets $K$:

- $R_0$. for all $A \in \mathcal{D}$ and $u \in A$, $u \in R(A)$ if and only if $0 \in R(A - u)$.

Alternatively, we can also consider a slightly different—but clearly equivalent—version that perhaps displays better the invariance of rejection functions under vector addition:

- $R_0'$. for all $A \in \mathcal{D}$, $u \in A$ and $v \in \mathcal{V}$, $u \in R(A)$ if and only if $u + v \in R(A + v)$.

When we do impose requirements on sets of desirable option sets $K$, Equation (5) allows us to turn them into requirements for rejection (and hence also choice) functions. In particular, we will see in Proposition 4 below that our Axioms $K_0$–$K_4$ imply that
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R₁. \( R(\emptyset) = \emptyset \), and \( R(A) \neq A \) for all \( A \subseteq \mathcal{C} \);  
R₂. \( 0 \in R(\{0,u\}) \), for all \( u \in \mathcal{C} \);  
R₃. If \( A₁, A₂ \subseteq \mathcal{C} \) such that \( 0 \in R(A₁ \cup \{0\}) \) and \( 0 \in R(A₂ \cup \{0\}) \) and \( (\lambda_{u,v}, \mu_{u,v}) > 0 \) for all \( u \in A₁ \) and \( v \in A₂ \), then also  
\( 0 \in R(\{\lambda_{u,v}u + \mu_{u,v}v : u \in A₁, v \in A₂ \} \cup \{0\}) \);  
R₄. If \( A₁ \subseteq A₂ \) then also \( R(A₁) \subseteq R(A₂) \), for all \( A₁, A₂ \subseteq \mathcal{C} \).

Axiom R₄ is Sen’s condition \( \alpha \) [18, 19]. Arthur Van Camp (private communication) has proved in a direct manner that Aizermann’s condition [1] can be derived from our Axioms K₀–K₄ as well. Indirectly, this can also be inferred from our representation results further on; see Theorem 9 in Section 6, and the discussion following it.

We will call coherent any rejection function that satisfies the five properties above.

Definition 3 (Coherence for rejection and choice functions). A rejection function \( R \) is called coherent if it satisfies the Axioms R₀–R₄. A choice function \( C \) is called coherent if the associated rejection function \( R_C \) is.

Our next result establishes that these notions of coherence are perfectly compatible with the coherence for sets of desirable option sets that we introduced in Section 3.

Proposition 4. Consider any set of desirable option sets \( K \subseteq \mathcal{C} \) and any rejection function \( R \) that are connected by Equation (5). Then \( K \) is coherent if and only if \( R \) is.

We will from now on work directly with (coherent) sets of desirable option sets and will use the collective term (coherent) choice models for (coherent) choice functions, rejection functions, and sets of desirable option sets. Of course, our primary motivation for studying coherent sets of desirable option sets is their connection with the other two choice models. This being said, it should however also be clear that our results do not depend on this connection. The theory of sets of desirable option sets that we are about to develop can therefore be used independently as well.

5. THE SPECIAL CASE OF BINARY CHOICE

According to our interpretation, the statement that \( A \) belongs to a set of desirable option sets \( K \) is taken to mean that \( A \) contains at least one desirable option. This implies that singletons play a special role: for any \( u \in \mathcal{C} \), stating that \( \{u\} \in K \) is equivalent to stating that \( u \) is desirable. For any set of desirable option sets \( K \), these singleton assessments are captured completely by the set of options

\[ D_K := \{u \in \mathcal{C} : \{u\} \in K\} \]  

that, according to \( K \), are definitely desirable—preferred to \( 0 \). A set of desirable option sets \( K \subseteq \mathcal{C} \) that is completely determined by such singleton assessments is called binary.

Definition 5 (Binary set of desirable option sets). We call a set of desirable option sets \( K \) binary if

\[ A \in K \iff (\exists u \in A) \{u\} \in K, \text{ for all } A \subseteq \mathcal{C}. \]  

In order to explain how any binary set of desirable option sets \( K \) is indeed completely determined by \( D_K \), we need a way to associate a rejection function with sets of options such as \( D_K \). To that end, we consider the notion of a set of desirable options: a subset \( D \) of \( \mathcal{C} \) whose interpretation will be that it consists of the options \( u \in \mathcal{C} \) that our subject considers desirable. We denote the set of all such sets of desirable options \( D \subseteq \mathcal{C} \) by \( \mathcal{D} \).

With any \( D \in \mathcal{D} \), our interpretation for rejection functions in Section 2 inspires us to associate a set of desirable option sets \( K_D \), defined by

\[ K_D := \{A \subseteq \mathcal{C} : A \cap D \neq \emptyset\}. \]  

It turns out that a set of desirable options sets \( K \) is binary if and only if it has the form \( K_D \), and the unique representing \( D \) is then given by \( D_K \).
Proposition 6. A set of desirable options sets $K \in \mathcal{K}$ is binary if and only if there is some $D \in \mathcal{D}$ such that $K = K_D$. This $D$ is then necessarily unique, and equal to $D_K$.

Just like we did for sets of desirable option sets in Section 3, we can use the basic rationality principles $d_1$–$d_3$ for the notion of desirability—or binary preference—to infer basic rationality criteria for sets of desirable options. When they do, we call them coherent.

Definition 7 (Coherence for sets of desirable options). A set of desirable options $D \in \mathcal{D}$ is called coherent if it satisfies the following axioms:

1. $0 \notin D$;
2. $\forall \lambda \geq 0 \subseteq D$;
3. if $u, v \in D$ and $(\lambda, \mu) > 0$, then $\lambda u + \mu v \in D$.

We denote the set of all coherent sets of desirable options by $\mathcal{D}$.

So a coherent set of desirable options is a convex cone [Axiom $D_3$] in $\mathcal{V}$ that does not contain 0 [Axiom $D_1$] and includes $\mathcal{V}_{\geq 0}$ [Axiom $D_2$]. Sets of desirable options are an abstract version of the sets of desirable gambles that have an important part in the literature on imprecise probability models [4, 9, 13, 26]. This abstraction was first introduced and studied in great detail in [8, 14].

Our next result shows that the coherence of a binary set of desirable option sets is completely determined by the coherence of its corresponding set of desirable options.

Proposition 8. Consider any binary set of desirable option sets $K \in \mathcal{K}$ and let $D_K \in \mathcal{D}$ be its corresponding set of desirable options. Then $K$ is coherent if and only if $D_K$ is. Conversely, consider any set of desirable options $D \in \mathcal{D}$ and let $K_D$ be its corresponding binary set of desirable option sets, then $K_D$ is coherent if and only if $D$ is.

So the binary coherent sets of desirable option sets are given by $\{K_D : D \in \mathcal{D}\}$, allowing us to call any coherent set of desirable option sets in $\mathcal{K} \setminus \{K_D : D \in \mathcal{D}\}$ non-binary.

What makes coherent sets of desirable options $D \in \mathcal{D}$—and hence also coherent binary sets of desirable option sets—particularly interesting is that they induce a binary preference order $\triangleright_D$—a strict vector ordering—on $\mathcal{V}$, defined by $u \triangleright_D v \iff u - v \in D$, for all $u, v \in \mathcal{V}$. The preference order $\triangleright_D$ is coherent—satisfies Axioms $\triangleright_0$, $\triangleright_4$—and furthermore fully characterises $D$: one can easily see that $u \in D$ if and only if $u \triangleright_D 0$. Hence, coherent sets of desirable options and coherent binary sets of desirable option sets are completely determined by a single binary strict preference order between options. This is of course the reason why we reserve the moniker binary for choice models that are essentially based on singleton assessments.

6. REPRESENTATION IN TERMS OF SETS OF DESIRABLE OPTIONS

It should be clear—and it should be stressed—at this point that making a direct desirability assessment for an option $u$ typically requires more of a subject than making a typical desirability assessment for an option set $A$: the former requires that our subject should state that $u$ is desirable, while the latter only requires the subject to state that some option in $A$ is desirable, but not to specify which. It is this difference—this greater latitude in making assessments—that guarantees that our account of choice is much richer than one that is purely based on binary preference. In the framework of sets of desirable option sets, it is for instance possible to express the belief that at least one of two options $u$ or $v$ is desirable, while remaining undecided about which of them actually is; in order to express this belief, it suffices to state that $\{u, v\} \in K$. This is not possible in the framework of sets

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2The Axioms $D_1$–$D_3$ for sets of desirable options should not be confused with the rationality criteria $d_1$–$d_3$ for our primitive notion of desirability—or binary preference. Like the Axioms $K_0$–$K_4$, they are only derived from these primitive assumptions on the basis of their interpretation.
of desirable options. Sets of desirable option sets therefore constitute a much more general uncertainty framework than sets of desirable options.

So while it is nice that there are sets of desirable option sets \( K_D \) that are completely determined by a set of desirable options \( D \), such binary choice models are typically not what we are interested in here: using \( K_D \) is equivalent to using \( D \) here, so there is no benefit in using the more convoluted model \( K_D \) to represent choice. No, it is the non-binary coherent choice models that we have in our sights. If we replace such a non-binary coherent set of desirable option sets \( K \) by its binary version \( D_K \), we lose information, because then necessarily \( K_{D_K} \subset K \). These non-binary choice models are therefore more expressive than sets of desirable options. But it turns out that our coherence axioms lead to a representation result that allows us to still use sets of desirable options, or rather, sets of them, to completely characterise any coherent choice model.

**Theorem 9.** A set of desirable option sets \( K \in \mathbf{K} \) is coherent if and only if there is a non-empty set \( D \subseteq \overline{\mathbf{D}} \) of coherent sets of desirable options such that \( K = \bigcap \{K_D : D \in D\} \). The largest such set \( D \) is then \( \overline{\mathbf{D}}(K) := \{D \in \overline{\mathbf{D}} : K \subseteq K_D\} \).

Due to the one-to-one correspondence between coherent sets of desirable options \( D \) and coherent preference orders \( \succ_D \), this representation result tells us that working with a coherent set of desirable option sets \( K \) is equivalent to working with the set of those coherent preference orders \( \succ_D \) for which \( K \subseteq K_D \). For the rejection function \( R \) that corresponds to \( K \) through Equation (5), \( u \in R(A) \) means that \( u \) is dominated in \( A \) for all these representing coherent preference orders \( \succ_D \). Similarly, \( u \in C_R(A) \) means that \( u \) is undominated according to at least one of these representing coherent preference orders \( \succ_D \). This effectively tells us that our coherence axioms \( K_0-K_4 \) for choice models characterise a generalised type of choice under Levi’s notion of E-admissibility [10, 20, 23], but with representing preference orders \( \succ_D \) that need not be total orders based on comparing expectations.

Interestingly, any potential property of sets of desirable option sets that is preserved under taking arbitrary intersections, and that the binary choice models satisfy, is inherited from the binary models through the representation result of Theorem 9. It is easy to see that this applies in particular to Aizermann’s condition [1].

**7. Imposing Totality**

We have just shown that every coherent choice model \( K \) can be represented by a collection of coherent sets of desirable options \( D \). This leads us to wonder whether it is possible to achieve representation using only particular types of coherent \( D \), and, if yes, for which types of coherent sets of desirable option sets \( K \)—and hence for which types of rejection functions \( R \) and choice functions \( C \)—this is possible. In this section, we clear the air by starting with a rather simple case, where we restrict attention to total sets of desirable options \( D \), corresponding to total preference orders \( \succ_D \).

**Definition 10** (Totality for sets of desirable options). We call a set of desirable options \( D \in \mathbf{D} \) total if it is coherent and

\[
D \in \mathbf{D}_T \quad \text{for all } u \in \mathbf{U} \setminus \{0\}, \text{ either } u \in D \text{ or } -u \in D.
\]

The set of all total sets of desirable options is denoted by \( \overline{\mathbf{D}}_T \).

That the binary preference order \( \succ_D \) corresponding to a total set of desirable options \( D \) is indeed a total order can be seen as follows. For all \( u, v \in \mathbf{U} \) such that \( u \neq v \), the property \( D_T \) implies that either \( u - v \in D \) or \( v - u \in D \). Hence, for all \( u, v \in \mathbf{U} \), we have that either \( u = v \), \( u \succ_D v \) or \( v \succ_D u \), which indeed makes \( \succ_D \) a total order.

It was shown in [5, 9] that what we call total sets of desirable options here, are precisely the maximal or undominated coherent ones, i.e. those coherent \( D \in \overline{\mathbf{D}} \) that are not included in any other coherent set of desirable option sets: \( (\forall D' \in \overline{\mathbf{D}})(D \subseteq D' \Rightarrow D = D') \). The
The set of all total sets of desirable options is denoted by $\overline{K}_T$.

**Proposition 12.** For any set of desirable options $D \in D$, $D$ is total if and only if $K_D$ is, so $K_D \in \overline{K}_T \iff D \in \overline{D}_T$.

So a binary $K$ is total if and only if its corresponding $D_K$ is. For general total sets of desirable option sets $K \in \overline{K}_T$, which are not necessarily binary, we nevertheless still have representation in terms of total binary ones.

**Theorem 13.** A set of desirable option sets $K \in \overline{K}$ is total if and only if there is a non-empty set $\mathcal{D} \subseteq \overline{D}_T$ of total sets of desirable options such that $K = \bigcap \{K_D : D \in \mathcal{D}\}$. The largest such set $\mathcal{D}$ is then $\overline{D}_T(K) := \{D \in \overline{D}_T : K \subseteq K_D\}$.

This representation result shows that our total choice models correspond a generalised type of choice under Levi’s notion of E-admissibility [10, 20], but with representing preference orders $\succeq$ that are now maximal, or undominated. They correspond what Van Camp et al. [23, Section 4] have called $M$-admissible choice models. Our discussion above provides an axiomatic characterisation for such choice models.

We conclude our study of totality by characterising what it means for a rejection function to be total.

**Proposition 14.** Consider any set of desirable option sets $K \in \overline{K}$ and any rejection function $R$ that are connected by Equation (5). Then $K$ is total if and only if $R$ is coherent and satisfies}

$R_T: 0 \in R(\{0, u, -u\}), \text{ for all } u \in \mathcal{V} \setminus \{0\}$.

### 8. Imposing the Mixing Property

Totality is, of course, a very strong requirement, and it leads to a very special and restrictive type of representation. We therefore now turn to weaker requirements, and their consequences for representation. One such additional property, which sometimes pops up in the literature about choice and rejection functions, is the following mixing property [17, 22], which asserts that an option that is rejected continues to be rejected if one removes mixed options—convex combinations of other options in the option set:

$R_M$: if $A \subseteq B \subseteq \text{conv}(A)$ then also $R(B) \cap A \subseteq R(A)$, for all $A, B \in \mathcal{D}$,

where $\text{conv}(\cdot)$ is the convex hull operator, defined by

$$\text{conv}(V) := \left\{ \sum_{k=1}^{n} \lambda_k u_k : n \in \mathbb{N}, \lambda_k \in \mathbb{R}_{>0}, \sum_{k=1}^{n} \lambda_k = 1, u_k \in V \right\} \text{ for all } V \subseteq \mathcal{V}. \quad (9)$$

$\mathbb{N}$ is the set of natural numbers, or in other words all positive integers, excluding 0, and $\mathbb{R}_{>0}$ is the set of all (strictly) positive reals. A rejection function that satisfies this mixing property is called mixing.

The following result characterises the mixing property in terms of the corresponding set of desirable option sets. We provide two equivalent conditions: one in terms of the convex hull operator, and one in terms of the $\text{posi}(\cdot)$ operator, which, for any subset $V$ of $\mathcal{V}$, returns the set of all positive linear combinations of its elements:

$$\text{posi}(V) := \left\{ \sum_{k=1}^{n} \lambda_k u_k : n \in \mathbb{N}, \lambda_k \in \mathbb{R}_{>0}, u_k \in V \right\}. \quad (10)$$

Van Camp [22] refers to this property as ‘convexity’, but we prefer to stick to the original name suggested by Seidenfeld et al. [17].
Proposition 15. Consider any set of desirable option sets \( K \in K \) and any rejection function \( R \) that are connected by Equation (5). Then \( R \) is coherent and mixing if and only if \( K \) is coherent and satisfies any—and hence both—of the following conditions:

\( K_M \): if \( B \in K \) and \( A \subseteq B \subseteq \text{posi}(A) \), then also \( A \in K \), for all \( A, B \in \mathcal{D} \);

\( K_M \): if \( B \in K \) and \( A \subseteq B \subseteq \text{conv}(A) \), then also \( A \in K \), for all \( A, B \in \mathcal{D} \).

In the context of sets of desirable options in linear spaces, we prefer to use the \( \text{posi}(\cdot) \) operator, and we therefore adopt \( K_M \) as our definition for mixingness.

Definition 16 (Mixing property for sets of desirable option sets). We call a set of desirable option sets \( K \in K \) mixing if it is coherent and satisfies \( K_M \). The set of all mixing sets of desirable options is denoted by \( \overline{K}_M \).

We now proceed to show that these mixing sets of desirable option sets allow for a representation in terms of sets of desirable options that are themselves mixing, in the following sense.

Definition 17 (Mixing property for sets of desirable options). We call a set of desirable options \( D \in \mathcal{D} \) mixing if it is coherent and

\( D_M \): for all \( A \in \mathcal{D} \), if \( \text{posi}(A) \cap D \neq \emptyset \), then also \( A \cap D \neq \emptyset \).

We denote the set of all mixing sets of desirable options by \( \overline{D}_M \).

The binary elements of \( \overline{K}_M \) are precisely the ones based on such a mixing set of desirable options; they can be represented by a single element of \( \overline{D}_M \).

Proposition 18. For any set of desirable options \( D \in \mathcal{D} \), \( K_D \) is mixing if and only if \( D \) is, so \( K_D \in \overline{K}_M \iff D \in \overline{D}_M \).

For general mixing sets of desirable option sets that are not necessarily binary, we nevertheless still obtain a representation theorem analogous to Theorems 9 and 13, where the representing sets of desirable options are now mixing.

Theorem 19. A set of desirable option sets \( K \in K \) is mixing if and only if there is a non-empty set \( \mathcal{D} \subseteq \overline{D}_M \) of mixing sets of desirable options such that \( K = \bigcap \{ K_D: D \in \mathcal{D} \} \). The largest such set \( \mathcal{D} \) is then \( \overline{D}_M(K) := \{ D \in \overline{D}_M: K \subseteq K_D \} \).

This representation result is akin to the one proved by Seidenfeld et al [17], but without the additional condition of weak Archimedeanity they impose. In order to better explain this, and to provide this result with some extra intuition, we take a closer look at the mixing sets of desirable options that make up our representation. The following result is an equivalent characterisation of such sets.

Proposition 20. Consider any set of desirable options \( D \in \overline{D} \) and let \( D^c := \mathcal{Y} \setminus D \). Then \( D \) is mixing if and only if \( \text{posi}(D^c) = D^c \).

So we see that the coherent sets of desirable options that are also mixing are precisely those whose complement is again a convex cone. They are therefore identical to the lexicographic sets of desirable options sets introduced by Van Camp et al. [22, 24]. What makes this particularly relevant and interesting is that these authors have shown that when \( \mathcal{Y} \) is the set of all gambles on some finite set \( \mathcal{X} \) and \( \mathcal{Y} \cap \mathcal{Y}_{\geq 0} = \{ u \in \mathcal{Y}: u \geq 0 \text{ and } u \neq 0 \} \), then the sets of desirable options in \( \overline{D} \) that are lexicographic—and therefore mixing—are exactly the ones that are representable by some lexicographic probability system that has no non-trivial Savage-null events. This is, of course, the reason why they decided to call such coherent sets of desirable options lexicographic. Because of this connection, it follows that in their setting, Theorem 19 implies that every mixing choice model can be represented by a set of lexicographic probability systems.

Due to the equivalence between coherent lexicographic sets of desirable options and mixing ones on the one hand, and between total sets of desirable options and maximal...
coherent ones on the other, the following proposition is an immediate consequence of a similar result by Van Camp et al. [22, 24]. It shows that the total sets of desirable options constitute a subclass of the mixing ones: mixingness is a weaker requirement than totality.

**Proposition 21.** Every total set of desirable options is mixing: \( \overline{D}_T \subseteq \overline{D}_M \).

By combining this result with Theorems 13 and 19, it follows that every total set of desirable options sets is mixing, and similarly for rejection and choice functions. So mixingness is implied by totality for non-binary choice models as well. Since totality is arguably the more intuitive of the two, one might therefore be inclined to discard the mixing property in favour of totality. We have nevertheless studied the mixing property in some detail, because it can be combined with other properties, such as the notions of Archimedeanity studied in the next section. As we will see, this combination leads to very intuitive representation, where the role of lexicographic probability systems is taken over by expectation operators—called linear previsions.

### 9. IMPOSING ARCHIMEDEANITY

There are a number of ways a notion of Archimedeanity can be introduced for preference relations and choice models [2, 3, 12, 15, 17]. Its aim is always to guarantee that the real number system is expressive enough, or more precisely, that the preferences expressed by the models can be represented by (sets of) real-valued probabilities and utilities, rather than, say, probabilities and utilities expressed using hyper-reals. Here, we consider a notion of Archimedeanity that is close in spirit to an idea explored by Walley [25, 26] in his discussion of so-called strict desirability.

For the sake of simplicity, we will restrict ourselves to a particular case of our abstract framework, where \( \mathcal{V} := \mathcal{L}(\mathcal{X}) \) is the set of all gambles on a set of states \( \mathcal{X} \) and \( \mathcal{V}_{>0} = \{ u \in \mathcal{L}(\mathcal{X}) : \inf u > 0 \} \). We identify every real number \( \mu \in \mathbb{R} \) with the constant gamble that takes the value \( \mu \), and then define Archimedeanity as follows.

**Definition 22 (Archimedean set of desirable options).** We call a set of desirable options \( D \in \mathcal{D} \) Archimedean if it is coherent and satisfies the following openness condition:

\[
\text{D}_A : \text{for all } u \in D, \text{ there is some } \epsilon \in \mathbb{R}_{>0} \text{ such that } u - \epsilon \notin D.
\]

We denote the set of all Archimedean sets of desirable options by \( \overline{D}_A \), and let \( \overline{D}_{M,A} \) be the set of all Archimedean sets of desirable options that are also mixing.

What makes Archimedean and mixing Archimedean sets of desirable options particularly interesting, is that they are in a one-to-one correspondence with coherent lower previsions and linear previsions [25], respectively.

**Definition 23 (Coherent lower prevision and linear prevision).** A coherent lower prevision \( P \) on \( \mathcal{L}(\mathcal{X}) \) is a real-valued map on \( \mathcal{L}(\mathcal{X}) \) that satisfies

1. **LP1.** \( P(u) \geq \inf u \) for all \( u \in \mathcal{L}(\mathcal{X}) \);
2. **LP2.** \( P(\lambda u) = \lambda P(u) \) for all \( u \in \mathcal{L}(\mathcal{X}) \) and \( \lambda \in \mathbb{R}_{>0} \);
3. **LP3.** \( P(u + v) \geq P(u) + P(v) \) for all \( u, v \in \mathcal{L}(\mathcal{X}) \);

A linear prevision \( P \) on \( \mathcal{L}(\mathcal{X}) \) is a coherent lower prevision that additionally satisfies

\[
P_3. \quad P(u + v) = P(u) + P(v) \text{ for all } u, v \in \mathcal{L}(\mathcal{X});
\]

We denote the set of all coherent lower previsions on \( \mathcal{L}(\mathcal{X}) \) by \( \mathcal{P} \) and let \( \mathcal{P} \) be the subset of all linear previsions.

\[4\]It is possible to introduce a version of our notion of Archimedeanity in our general framework as well, but explaining how this works would take up much more space than we are allowed in this conference paper.
In order to make the above-mentioned one-to-one correspondences explicit, we introduce the following maps. With any set of desirable options $D$ in $\mathbb{D}$, we associate a (possibly extended) real functional $P_D$ on $\mathcal{L}(\mathcal{X})$, defined by

$$P_D(u) := \sup\{\mu \in \mathbb{R} : u - \mu \in D\}, \text{ for all } u \in \mathcal{L}(\mathcal{X}).$$  \hfill (11)

Conversely, with any (possibly extended) real functional $\mathcal{P}$ on $\mathcal{L}(\mathcal{X})$, we associate a set of desirable options

$$D_{\mathcal{P}} := \{ u \in \mathcal{L}(\mathcal{X}) : \mathcal{P}(u) > 0 \}.$$  \hfill (12)

Our next result shows that these two maps lead to an isomorphism between $\mathbb{D}_A$ and $\mathcal{P}$, and similarly for $\mathbb{D}_{M,A}$ and $\mathcal{P}$.

**Proposition 24.** For any Archimedean set of desirable options $D$, $P_D$ is a coherent lower prevision on $\mathcal{L}(\mathcal{X})$ and $D_{P_D} = D$. If $D$ is moreover mixing, then $P_D$ is a linear prevision. Conversely, for any coherent lower prevision $\mathcal{P}$ on $\mathcal{L}(\mathcal{X})$, $D_{\mathcal{P}}$ is an Archimedean set of desirable options and $P_{D_{\mathcal{P}}} = \mathcal{P}$. If $\mathcal{P}$ is furthermore a linear prevision, then $D_{\mathcal{P}}$ is mixing.

The import of these correspondences is that any representation in terms of sets of Archimedean (mixing) sets of desirable options is effectively a representation in terms of sets of coherent lower (or linear) previsions. As we will see, these kinds of representations can be obtained for sets of desirable option sets—and hence also rejection and choice functions—that are themselves Archimedean in the following sense.

**Definition 25** (Archimedean set of desirable option sets). We call a set of desirable option sets $K \in \mathbb{K}$ Archimedean if it is coherent and satisfies

$$K_A, \text{ for all } A \in K, \text{ there is some } \varepsilon \in \mathbb{R}_{>0} \text{ such that } A - \varepsilon \in K.$$ \hfill (13)

We denote the set of all Archimedean sets of desirable option sets by $\mathbb{K}_A$, and let $\mathbb{K}_{M,A}$ be the set of all Archimedean sets of desirable options that are also mixing.

This notion easily translates from sets of desirable option sets to rejection functions.

**Proposition 26.** Consider any set of desirable option sets $K \in \mathbb{K}$ and any rejection function $R$ that are connected by Equation (5). Then $K$ is Archimedean if and only if $R$ is coherent and satisfies

$$R_A, \text{ for all } A \in \mathcal{D} \text{ and } u \in \mathcal{Y} \text{ such that } u \in R(A \cup \{u\}), \text{ there is some } \varepsilon \in \mathbb{R}_{>0} \text{ such that } u \in R((A - \varepsilon) \cup \{u\}).$$

A first and basic result is that our notion of Archimedeanity for sets of desirable option sets is compatible with that for sets of desirable options.

**Proposition 27.** For any set of desirable options $D \in \mathbb{D}$, $K_D$ is Archimedean (and mixing) if and only if $D$ is, so $K_D \in \mathbb{K}_A \iff D \in \mathbb{D}_A$ and $K_D \in \mathbb{K}_{M,A} \iff D \in \mathbb{D}_{M,A}$.

In order to state our representation results for Archimedean choice models that are not necessarily binary, we require a final piece of machinery: a topology on $\mathbb{D}_A$ and $\mathbb{D}_{M,A}$, or equivalently, a notion of closedness. We do this by defining the convergence of a sequence of Archimedean sets of desirable options $\{D_n\}_{n \in \mathbb{N}}$ in terms of the point-wise convergence of the corresponding sequence of coherent lower previsions:

$$\lim_{n \to +\infty} D_n = D \iff (\forall u \in \mathcal{L}(\mathcal{X})) \lim_{n \to +\infty} P_{D_n}(u) = P_D(u).$$  \hfill (13)

A set $\mathcal{D} \subseteq \mathbb{D}_A$ of Archimedean sets of desirable options is then called closed if it contains all of its limit points, or equivalently, if the corresponding set of coherent lower previsions—or linear previsions when $\mathcal{D} \subseteq \mathbb{D}_{M,A}$—is closed with respect to point-wise convergence.

Our final representation results state that a set of desirable option sets $K$ is Archimedean if and only if it can be represented by such a closed set, and if $K$ is moreover mixing, the elements of the representing closed set are as well.
Theorem 28 (Representation for Archimedean choice functions). A set of desirable option sets \( K \subseteq K \) is Archimedean if and only if there is some non-empty closed set \( D \subseteq D_A \) of Archimedean sets of desirable options such that \( K = \bigcap \{ K_D : D \in D \} \). The largest such set \( D \) is then \( D_A(K) := \{ D \in D_A : K \subseteq K_D \} \).

Theorem 29 (Representation for Archimedean mixing choice functions). A set of desirable option sets \( K \subseteq K \) is mixing and Archimedean if and only if there is some non-empty closed set \( D \subseteq D_M,A \) of mixing and Archimedean sets of desirable options such that \( K = \bigcap \{ K_D : D \in D \} \). The largest such set \( D \) is then \( D_M,A(K) := \{ D \in D_M,A : K \subseteq K_D \} \).

If we combine Theorem 29 with the correspondence result of Proposition 24, we see that Axioms \( K_0-K_4 \) together with \( K_M \) and \( K_A \) characterise exactly those choice models that are based on E-admissibility with respect to a closed—but not necessarily convex—set of linear previsions. In much the same way, Theorem 28 can be seen to characterise a generalised notion of E-admissibility, where the representing objects are coherent lower previsions. Walley–Sen maximality [20, 25] can be regarded as a special case of this generalised notion, where only a single representing coherent lower prevision is needed.

10. Conclusion

The main conclusion of this paper is that the language of desirability is capable of representing non-binary choice models, provided we extend it with a notion of disjunction, allowing statements such as ‘at least one of these two options is desirable’. When we do so, the resulting framework of sets of desirable options turns out to be a very flexible and elegant tool for representing set-valued choice. Not only does it include E-admissibility and maximality, it also opens up a range of other types of choice functions that have so far received little to no attention. All of these can be represented in terms of sets of strict preference orders or—if additional properties are imposed—in terms of sets of strict total orders, sets of lexicographic probability systems, sets of coherent lower previsions or sets of linear previsions.

In our future work on this topic, we intend to investigate how we can let go of the closedness condition in Theorems 28 and 29. We expect to have to turn to other types of Archimedeanity; variations on Seidenfeld et al.’s weak Archimedeanity [6, 17] come to mind. Finally, we also intend to further develop conservative inference methods for coherent choice functions, by extending our earlier natural extension results [7] to the more general setting that we have considered here.

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This work owes a large intellectual debt to Teddy Seidenfeld, who introduced us to the topic of choice functions. His insistence that we ought to pay more attention to non-binary choice if we wanted to take imprecise probabilities seriously, is what eventually led to this work.

The discussion in Arthur Van Camp’s PhD thesis [22] was the direct inspiration for our work here, and we would like to thank Arthur for providing a pair of strong shoulders to stand on.

As with most of our joint work, there is no telling, after a while, which of us two had what idea, or did what, exactly. We have both contributed equally to this paper. But since a paper must have a first author, we decided it should be the one who took the first significant steps: Jasper, in this case.

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**APPENDIX A. PROOFS AND INTERMEDIATE RESULTS**

A.1. Terminology and notation used only in the appendix. For any subset \( V \) of \( \mathcal{Y} \) we consider its set of linear combinations, or linear span

\[
\operatorname{span}(V) := \left\{ \sum_{k=1}^{n} \lambda_k u_k : n \in \mathbb{N}, \lambda_k \in \mathbb{R}, u_k \in V \right\}.
\]

We also consider several operators on—transformations of—the set \( K \) of all sets of desirable option sets. The first is denoted by \( R_n(\cdot) \), and allows us to add smaller option sets by removing from any option set elements of \( \mathcal{Y}_{\leq 0} := \{ u \in \mathcal{Y} : u \leq 0 \} \):

\[
R_n(K) := \{ A \in \mathcal{P} : (\exists B \in K) B \setminus \mathcal{Y}_{\leq 0} \subseteq A \subseteq B \}, \text{ for all } K \in K.
\]

The second is denoted by \( R_s(\cdot) \), and allows us to add smaller option sets by removing from any option set positive combinations from some of its other elements:

\[
R_p(K) := \{ A \in \mathcal{P} : (\exists B \in K) A \subseteq B \subseteq \operatorname{posi}(A) \}, \text{ for all } K \in K.
\]

The final one is denoted by \( \operatorname{Posi}(\cdot) \)—not to be confused with \( \operatorname{posi}(\cdot) \)—and defined for all \( K \in K \) by

\[
\operatorname{Posi}(K) := \left\{ \left\{ \sum_{k=1}^{n} \lambda_k^{u_1,k} u_k : u_1,k \in \times_{k=1}^{n} A_k \right\} : n \in \mathbb{N}, (A_1, \ldots, A_n) \in K^n, \right.
\]

\[
\left. \left( \forall u_{1,n} \in \times_{k=1}^{n} A_k \right) \lambda_k^{u_1,n} > 0 \right\}, \quad (14)
\]

where we use the notations \( u_{1,n} \) and \( \lambda_k^{u_1,n} \) for \( n \)-tuples of options \( u_k \) and real numbers \( \lambda_k^{u_1,n} \), \( k \in \{1, \ldots, n\} \), so \( u_{1,n} \in \mathcal{Y}^n \) and \( \lambda_k^{u_1,n} \in \mathbb{R}^n \). We also use \( \lambda_k^{u_1,n} > 0 \) as a shorthand for \( \lambda_k^{u_1,n} \geq 0 \) for all \( k \in \{1, \ldots, n\} \) and \( \sum_{k=1}^{n} \lambda_k^{u_1,n} > 0 \).

A.2. Proofs and Intermediate Results for Section 5.

**Proof of Proposition 6.** First assume that there is some \( D \in \mathcal{D} \) such that \( K = K_D \). Then for all \( A \in \mathcal{P} \):

\[
A \in K \iff A \in K_D \iff A \cap D \neq \emptyset \iff (\exists u \in A) \{ u \} \cap D \neq \emptyset
\]

\[
\iff (\exists u \in A) \{ u \} \in K_D \iff (\exists u \in A) \{ u \} \in K.
\]

It therefore follows from Definition 5 that \( K \) is binary.

Furthermore, for any \( u \in \mathcal{Y} \), we find that

\[
u \in D \iff \{ u \} \cap D \neq \emptyset \iff \{ u \} \in K_D \iff \{ u \} \in K \iff u \in D_K.
\]

So we find that \( D \) is equal to \( D_K \), and therefore necessarily unique.

Finally, we assume that \( K \) is binary. Let \( D := D_K \). Then for all \( A \in \mathcal{P} \):

\[
A \in K \iff (\exists u \in A) \{ u \} \in K \iff (\exists u \in A) u \in D_K \iff A \cap D_K \neq \emptyset \iff A \cap D \neq \emptyset \iff A \in K_D,
\]

where the first equivalence follows from Definition 5 and the fact that \( K \) is binary. Hence, we find that \( K = K_D \).

\( \square \)
Corollary 30. A set of desirable option sets $K \subseteq K$ is binary if and only if $K_{D_K} = K$.

Proof. Immediate consequence of Proposition 6.

Proposition 31. For any coherent set of desirable option sets $K$, $D_K$ is a coherent set of desirable options, and $K_{D_K} \subseteq K$.

Proof. We first prove that $D_K$ is coherent, or equivalently, that it satisfies Axioms $D_1$–$D_3$. For Axiom $D_1$, observe that $0 \in D_K$ implies that $\{0\} \in K$, contradicting Axiom $K_1$. For Axiom $D_2$, observe that for any $u \in \mathcal{U}$, $u \in D_K$ is equivalent to $\{u\} \in K$, and take into account Axiom $K_2$. And, finally, for Axiom $D_3$, observe that $u, v \in D_K$ implies that $\{u\}, \{v\} \in K$, and that Axiom $K_3$ then implies that $\{u + \mu v\} \in K$, or equivalently, that $\lambda u + \mu v \in D_K$, for any choice of $(\lambda, \mu) > 0$.

For the last statement, consider any $A \in K_{D_K}$, meaning that $A \cap D_K \neq \emptyset$. Consider any $u \in A \cap D_K$, then on the one hand $u \in D_K$, so $\{u\} \in K$. But since on the other hand also $u \in A$, we see that $\{u\} \subseteq A$, and therefore Axiom $K_4$ guarantees that $A \in K$.

Proposition 32. For any set of desirable options $D \in D$, $D = D_{K_D}$. If, moreover, $D$ is coherent, then $K_D$ is a coherent set of desirable option sets.

Proof. For the first statement, simply observe that

$$u \in D_{K_D} \iff \{u\} \in K_D \iff \{u\} \cap D \neq \emptyset \iff u \in D,$$

for all $u \in \mathcal{U}$.

For the second statement, assume that $D$ is coherent, then we need to prove that $K_D$ is coherent, or equivalently, that it satisfies Axioms $K_0$–$K_4$. For Axiom $K_0$, observe that $A \cap D \neq \emptyset$ implies that $(A \setminus \{0\}) \cap D \neq \emptyset$ because we know from the coherence of $D$ [Axiom $D_1$] that $0 \notin D$. For Axiom $K_1$, observe that Equation (8) implies that $\{0\} \in K_{D_K}$, and use Axiom $D_1$. For Axiom $K_2$, observe that $\{u\} \in K_{D_K}$ is equivalent to $u \in D \in K$ for all $u \in \mathcal{U}$, and take into account the coherence of $D$ [Axiom $D_2$]. For Axiom $K_3$, consider any $A_1, A_2 \in K_{D_K}$, and let $A := \{a_{\alpha} u + \mu_{\alpha} v : u \in A_1, v \in A_2\}$ for any particular choice of the $(\lambda_{\alpha}, \mu_{\alpha}, v)$, for any particular choice of $(\lambda_{\alpha}, \mu_{\alpha}, v)$.

Next assume that Equation (15) holds. Because of Corollary 30, it suffices to show that $K_{D_K} = K$. We infer from Proposition 31 that $D_K$ is a coherent set of desirable options, and that $K_{D_K} \subseteq K$. Assume ex absurdo that $K_{D_K} \subset K$, so there is some $A \in K$ such that $A \notin K_{D_K}$, or equivalently, such that $A \cap D_K = \emptyset$. But then we must have that $|A| \geq 2$,
Axiom K
removes option sets from a set of desirable option sets, so the option sets
\{K \in \mathbf{K} \mid u \in B \subseteq \mathbf{V} \}
that there is some

There is some

To prove that Rn
then it follows from Equation (15) that there is some A1 := A \setminus \{u_1\} ∈ K. We can now repeat the same argument with A1 instead of A to find that it must be that |A1| ≥ 2, so there is some u2 ∈ A1 such that A2 := A1 \setminus \{u_2\} ∈ K and A2 \not\in K. Repeating the same argument over and over again will eventually lead to a contradiction with |A_n| ≥ 2. Hence it must be that K_D = K.

A.3. Proofs and Intermediate Results for Section 6.

Lemma 34. Consider any set of desirable option sets \( K ∈ \mathbf{K} \) that satisfies Axioms \( K_2 \) and \( K_3 \). Consider any \( A ∈ K \). Then for any \( v ∈ A \) and any \( v′ ∈ \mathbf{V} \) such that \( v \preceq v′ \), the option set \( B := \{v′\} \cup (A \setminus \{v\}) \) obtained by replacing \( v \) in \( A \) with the dominating option \( v′ \) still belongs to \( K: B ∈ K \).

Proof. We may assume without loss of generality that \( A \neq \emptyset \) and that \( v′ \neq v \). Let \( v' := v' - v \), then \( v' ∈ \mathbf{V}_{>0} \), and therefore Axiom \( K_2 \) implies that \( \{v'\} ∈ K \). Applying Axiom \( K_3 \) for A and \( \{v′\} \) allows us to infer that \( \{\lambda_u u + \mu_v v′: u ∈ A\} ∈ K \) for all possible choices of \( (\lambda_u, \mu_v) > 0 \). Choosing \( (\lambda_u, \mu_v) := (1,0) \) for all \( u ∈ A \setminus \{v\} \) and \( (\lambda_v, \mu_v) := (1,1) \) yields in particular that \( B = \{v′\} \cup (A \setminus \{v\}) \subseteq K \).

Lemma 35. Consider any set of desirable option sets \( K ∈ \mathbf{K} \) that satisfies Axioms \( K_2 \) and \( K_3 \). Consider any \( A ∈ K \) such that \( A ∩ \mathbf{V}_{<0} ≠ \emptyset \) and any \( v ∈ A ∩ \mathbf{V}_{<0} \), and construct the option set \( B := \{0\} ∪ (A \setminus \{v\}) \) by replacing \( v \) with \( 0 \). Then still \( B ∈ K \).

Proof. Immediate consequence of Lemma 34.

Proposition 36. \( R_n(K) = K \) for any coherent set of desirable option sets \( K ∈ \mathbf{K} \).

Proof. That \( K ⊆ R_n(K) \) is an immediate consequence of the definition of the Rn operator. To prove that \( R_n(K) ⊆ K \), consider any \( A ∈ R_n(K) \), which means that there is some \( B ∈ K \) such that \( B \setminus \mathbf{V}_{<0} ⊆ A ⊆ B \). We need to prove that \( A ∈ K \). Since \( K \) satisfies Axiom \( K_4 \), it suffices to prove that \( B \setminus \mathbf{V}_{<0} ∈ K \).

If \( B ∩ \mathbf{V}_{<0} = \emptyset \), then \( B \setminus \mathbf{V}_{<0} = B ∈ K \). Therefore, without loss of generality, we may assume that \( B ∩ \mathbf{V}_{<0} ≠ \emptyset \). For any \( u ∈ B ∩ \mathbf{V}_{<0} \), Lemma 35 implies that we may replace \( u \) by \( 0 \) and still be guaranteed that the resulting set belongs to \( K \). Hence, we can replace all elements of \( B ∩ \mathbf{V}_{<0} \) with \( 0 \) and still be guaranteed that the result \( B' := \{0\} ∪ (B \setminus \mathbf{V}_{<0}) \) belongs to \( K \). Applying Axiom \( K_0 \) now guarantees that, indeed, \( B' \setminus \mathbf{V}_{<0} = B' \setminus \{0\} ∈ K \).

Proposition 37. Consider any set of desirable option sets \( K ∈ \mathbf{K} \). Then \( R_n(K) \) satisfies Axiom \( K_0 \). Moreover, if \( K \) satisfies Axioms \( K_1, K_2, K_3 \) and \( K_4 \) and does not contain \( \emptyset \), then so does \( R_n(K) \).

Proof. The proof of the first statement is trivial. For the second statement, assume that \( K \) does not contain \( \emptyset \) and satisfies Axioms \( K_1, K_2, K_3 \) and \( K_4 \).

To prove that \( R_n(K) \) satisfies Axiom \( K_1 \) and does not contain \( \emptyset \), assume ex absurdo that \( \emptyset ∈ R_n(K) \) or \( \{0\} ∈ R_n(K) \). We then find that there is some \( B ∈ K \) such that \( B \setminus \mathbf{V}_{<0} ⊆ \emptyset ⊆ B \) or that there is some \( B ∈ K \) such that \( B \setminus \mathbf{V}_{<0} ⊆ \{0\} ⊆ B \). In both cases, it follows that \( B ⊆ \mathbf{V}_{<0} \). If \( B = \emptyset \), this contradicts our assumption that \( K \) does not contain \( \emptyset \). If \( B ≠ \emptyset \), it follows from Lemma 35 that we can replace every \( u ∈ B \) by \( 0 \) and still be guaranteed that the resulting option set \( \{0\} \) belongs to \( K \), contradicting our assumption that \( K \) satisfies Axiom \( K_1 \).

To prove that \( R_n(K) \) satisfies Axiom \( K_2 \), simply observe that the operator \( R_n \) never removes option sets from a set of desirable option sets, so the option sets \( \{u\}, u ∈ \mathbf{V}_{<0} \), which belong to \( K \) by Axiom \( K_2 \), will also belong to the larger \( R_n(K) \).
To prove that $Rn(K)$ satisfies Axiom $K_3$, consider any $A_1, A_2 \subseteq Rn(K)$, meaning that there are $B_1, B_2 \subseteq K$ such that $B_1 \setminus \mathcal{Y}_{\leq 0} \subseteq A_1 \subseteq B_1$ and $B_2 \setminus \mathcal{Y}_{\leq 0} \subseteq A_2 \subseteq B_2$. For any $u \in A_1$ and $v \in A_2$, we choose $(\lambda_{u,v}, \mu_{u,v}) > 0$, and let

$$A := \{\lambda_{u,v}u + \mu_{u,v}v : u \in A_1, v \in A_2\}.$$ 

Then we have to prove that $A \in Rn(K)$. Since $K$ satisfies Axiom $K_3$, we infer from $B_1, B_2 \subseteq K$ that

$$C := \{\lambda_{u,v}u + \mu_{u,v}v : u \in A_1, v \in A_2\}$$





belongs to $K$ as well. Furthermore, since $B_1 \setminus \mathcal{Y}_{\leq 0} \subseteq A_1$ and $B_2 \setminus \mathcal{Y}_{\leq 0} \subseteq A_2$ imply that $B_1 \setminus A_1 \subseteq \mathcal{Y}_{\leq 0}$ and $B_2 \setminus A_2 \subseteq \mathcal{Y}_{\leq 0}$, we see that

$$\{u : u \in B_1 \setminus A_1, v \in B_2\} \cup \{v : u \in A_1, v \in B_2 \setminus A_2\}$$

contains $\lambda_{o,v}o$.

Hence, $C \setminus \mathcal{Y}_{\leq 0} \subseteq A \subseteq C$. Since $C \in K$, this implies that, indeed, $A \in Rn(K)$.

Finally, to prove that $Rn(K)$ satisfies Axiom $K_4$, consider any $A_1 \in Rn(K)$ and any $A_2 \subseteq \mathcal{Q}$ such that $A_1 \subseteq A_2$. We need to prove that $A_2 \subseteq K$. That $A_1 \in Rn(K)$ implies that there is some $B_1 \in K$ such that $B_1 \setminus \mathcal{Y}_{\leq 0} \subseteq A_1 \subseteq B_1$. Let $B_2 := B_1 \cup (A_2 \setminus A_1)$, then $B_1 \subseteq B_2$ and therefore also $B_2 \subseteq K$, because $K$ satisfies Axiom $K_4$. We now infer from $B_1 \setminus \mathcal{Y}_{\leq 0} \subseteq A_1 \subseteq B_1$ that

$$B_2 \setminus \mathcal{Y}_{\leq 0} \subseteq (B_1 \setminus \mathcal{Y}_{\leq 0}) \cup (A_2 \setminus A_1) \subseteq A_1 \cup (A_2 \setminus A_1) \subseteq B_1 \cup (A_2 \setminus A_1).$$

Since $A_1 \cup (A_2 \setminus A_1) = A_2$, this allows us to conclude that $B_2 \setminus \mathcal{Y}_{\leq 0} \subseteq A_2 \subseteq B_2$, and therefore, since $B_2 \in K$, that, indeed, $A_2 \in Rn(K)$. □

**Lemma 38.** Let $A, B \in \mathcal{Q}$ be option sets, and consider any non-zero $u_o \in \mathcal{Y}$. Then there are $\alpha_{u}, u \in A$ such that $u - \alpha_{u}u_o \notin B$ and $u - \alpha_{u}u_o \notin B$ for all $u, v \in A$.

**Proof.** Partition the finite set $A$ into the finite number of disjoint subsets $A_k$ of options $u$ belonging to the same affine space $\{u + \beta_{u_o} : \beta \in B\} = u + \text{span}\{u_o\}$ parallel to $\text{span}\{u_o\}$. When $A_k$ has $n_k$ elements, choose $n_k$ different options in the corresponding affine space that are not in the finite $B$, which is always possible. □

**Lemma 39.** Consider a coherent set of desirable option sets $K \subseteq \mathcal{K}$ and any $A_o \in K$ such that $|A_o| \geq 2$ and $A_o \setminus \{u\} \notin K$ for all $u \in A_o$. Choose any $u_o \in A_o$ and let

$$K^* := \{\lambda u + \mu u_o : u \in B, (\forall v \in B)(\lambda u, \mu) > 0\}. \tag{16}$$

Then $K^* := Rn(K^*)$ is a coherent set of desirable option sets that is a superset of $K$ and contains $\{u_o\}$. Furthermore, $\{u_o\} \notin K$ and $u_o \notin K$.

**Proof of Lemma 39.** To prove that $\{u_o\} \notin K$, assume ex absurdo that $\{u_o\} \in K$. Since $|A_o \setminus \{u_o\}| \geq 1$, we can pick any element $v \in A_o \setminus \{u_o\}$, and then $\{u_o\} \subseteq A_o \setminus \{v\}$ and therefore $A_o \setminus \{v\} \in K$ by Axiom $K_4$, contradicting the assumptions. To prove that $u_o \notin K$, assume ex absurdo that $u_o \in \mathcal{Y}_{\leq 0}$, then we infer that also $A_o \setminus \{u_o\} \in K$ [use Proposition 36 and the coherence of $K_1$], contradicting the assumptions. To prove that $\{u_o\} \in K^*$, it suffices to notice that $\{u_o\} = \{0 + 1u_o : u \in A_o\} \in K^*$, whence also $\{u_o\} \in K^*$. Similarly, since $K^*$ is clearly a superset of $K$, the same is true for $K^*$.

It only remains to prove, therefore, that $K^*$ is coherent. To this end, we intend to show that the set of desirable option sets $K^*$ satisfies Axioms $K_1$, $K_2$, $K_3$ and $K_4$ and that $\emptyset \notin K^*$. The coherence of $K^*$ will then be an immediate consequence of Proposition 37.

For Axiom $K_1$, assume ex absurdo that $\emptyset \in K^*$, meaning that there is some $B \in K$ and, for all $u \in B$, some choice of $(\lambda_u, \mu_u) > 0$, such that $\{\lambda u + \mu u_o : u \in B\} = \emptyset$. Hence, $B \neq \emptyset$ and $\lambda u + \mu u_o = 0$ for all $u \in B$. 

Recall that we already know that \( u_o \neq 0 \). For any \( v \in B \), \( \lambda_o v + \mu_v u_o = 0 \) implies that \( \lambda_o > 0 \), because otherwise, since \( \langle \lambda_o, \mu_v \rangle > 0 \), \( \lambda_o = 0 \) would imply that \( \mu_v > 0 \) and therefore \( u_o = 0 \), a contradiction. Hence, for all \( v \in B \), \( v = -\delta_v u_o \) with \( \delta_v := \frac{\mu_v}{\lambda_o} \geq 0 \). Now let \( (k_{u,v}, \rho_{u,v}) := (1, 0) \) for all \( u \in A_o \setminus \{ u_o \} \) and \( v \in B \), and let \( (k_{u,v}, \rho_{u,v}) := (\delta_v, 1) \) for all \( v \in B \). Then

\[
\{ k_{u,v} u + \rho_{u,v} v : u \in A_o, v \in B \} = \{ u : u \in A_o \setminus \{ u_o \}, v \in B \} \cup \{ \delta_v u_o + v : v \in B \}
= \{ u : u \in A_o \setminus \{ u_o \}, v \in B \} \cup \{ 0 : v \in B \}
= \{ 0 \} \cup (A_o \setminus \{ u_o \}),
\]

where the last equality follows from \( B \neq \emptyset \). However, since \( A_o \in K \) and \( B \in K \), the coherence of \( K \) [Axiom K3] implies that \( \{ k_{u,v} u + \rho_{u,v} v : u \in A_o, v \in B \} \in K \). We therefore find that \( \{ 0 \} \cup (A_o \setminus \{ u_o \}) \in K \). The coherence of \( K \) now guarantees that \( A_o \setminus \{ u_o \} \in K \) [use Axiom K0 if \( \emptyset \not\in A_o \setminus \{ u_o \} \)], contradicting the assumptions.

For Axiom K2, consider any \( u \in \mathcal{F}_0 \). Then \( \{ u \} \subseteq B \) because \( \{ u \} \subseteq \mathcal{F}_2 \) and \( \mathcal{F}_0 \subseteq \mathcal{F}_2 \). For any \( A \subseteq \mathcal{F}_2 \), let \( A := \{ a \} \) for which \( a \notin A \setminus \{ u \} \) and, for all \( v \in B \), some choice of \( (\lambda_o, \mu_v) > 0 \), such that

\[
A = \{ \lambda_o v + \mu_v u_o : v \in B \}.
\]

For every \( u \in A \setminus \{ u \} \), we now choose some real \( \alpha_u > 0 \) such that \( u - \alpha_u u_o \notin B \) and such that, for all \( u, u' \in A \setminus \{ u \} \), \( v \notin A \setminus \{ u \} \), and \( u_o \neq 0 \) and \( A_1 \) and \( B_1 \) are finite, this is always possible, by Lemma 38. Let

\[
B_2 := B_1 \cup \{ u - \alpha_u u_o : u \in A \setminus \{ u \} \}
\]

and, for each \( v \in B_2 \setminus B_1 \), let \( u_v \) be the unique element of \( A_2 \) for which \( v = u_v - \alpha_u u_o \), and let \( (\lambda_v, \mu_v) := (1, \alpha_u) > 0 \). We then see that

\[
A_2 = A_1 \cup (A_2 \setminus A_1) = \{ \lambda_v v + \mu_v u_o : v \in B_1 \} \cup \{ u - \alpha_u u_o + \alpha_u u_o : u \in A_2 \setminus A_1 \}
= \{ \lambda_v v + \mu_v u_o : v \in B_1 \} \cup \{ v + \alpha_u u_o : v \in B_2 \setminus B_1 \}
= \{ \lambda_v v + \mu_v u_o : v \in B_2 \}.
\]

Furthermore, since \( B_1 \subseteq B \) and \( B_1 \subseteq B_2 \), it follows from the coherence of \( K \) and Axiom K4 that \( B_2 \subseteq K \). Hence, \( A_2 \subseteq K \).

For Axiom K3, consider any \( A_1, A_2 \subseteq K \) and, for all \( u_1 \in A_1 \), \( u_2 \in A_2 \), any choice of \( (\alpha_{u_1, u_2}, \beta_{u_1, u_2}) > 0 \), and let \( (\mu_{u_1, u_2}) > 0 \). We then see that

\[
C := \{ (\alpha_{u_1, u_2} u_1 + \beta_{u_1, u_2} u_2 : u_1 \in A_1, u_2 \in A_2 \} \subseteq K \).
\]

Since \( A_1 \subseteq K \) and \( A_2 \subseteq K \), there are \( B_1, B_2 \subseteq K \) and, for all \( v_1 \in B_1 \) and \( v_2 \in B_2 \), some choices of \( (\lambda_{1, v_1}, \mu_{1, v_1}) > 0 \) and \( (\lambda_{2, v_2}, \mu_{2, v_2}) > 0 \), such that

\[
A_1 = \{ \lambda_{1, v_1} v_1 + \mu_{1, v_1} u_o : v_1 \in B_1 \} \text{ and } A_2 = \{ \lambda_{2, v_2} v_2 + \mu_{2, v_2} u_o : v_2 \in B_2 \}.
\]

Now fix any \( v_1 \in B_1 \) and \( v_2 \in B_2 \), and let \( (\alpha_{1, v_1}' \lambda_{1, v_1} v_1 \mu_{1, v_1} u_o, \beta_{v_1, v_2, v_2} : (\alpha_{v_1, v_2} \lambda_{v_1, v_2} v_2 + \mu_{v_1, v_2} u_o) : v_1 \in B_1, v_2 \in B_2 \}
\]

We consider two cases. If \( \alpha_{v_1, v_2} : \lambda_{v_1, v_2} \lambda_{v_1, v_2} > 0 \) and \( \beta_{v_1, v_2} \mu_{v_1, v_2} > 0 \), we let

\[
(k_{v_1, v_2}, \rho_{v_1, v_2}) := (\alpha_{v_1, v_2} \lambda_{v_1, v_2} + \beta_{v_1, v_2} \mu_{v_1, v_2} v_2 > 0, \beta_{v_1, v_2} \mu_{v_1, v_2} v_2 > 0, \delta_{v_1, v_2} := (1, \alpha_{v_1, v_2} \mu_{v_1, v_2} + \beta_{v_1, v_2} \mu_{v_1, v_2} v_2 > 0, \delta_{v_1, v_2} := (1, 1) > 0, \]

if \( \alpha_{v_1, v_2} : \lambda_{v_1, v_2} v_2 = 0 \), we let

\[
(k_{v_1, v_2}, \rho_{v_1, v_2}) := (1, 1) > 0, \]

and

\[
(k_{v_1, v_2}, \rho_{v_1, v_2}) := (1, 1) > 0, \]

for all \( v_1 \in B_1 \) and \( v_2 \in B_2 \).
\[\gamma_{1,v_2}: \delta_{1,v_2} := (0, \alpha_{1,v_2} \mu_{1,v_1} + \beta_{1,v_2} \mu_{2,v_2}) > 0.\]

In both cases, we find that
\[\gamma_{1,v_2}(\kappa_{1,v_2} v_1 + \rho_{1,v_2} v_2) + \delta_{1,v_2} u_0 = \alpha_{1,v_2} (\lambda_{1,v_1} v_1 + \mu_{1,v_1} u_0) + \beta_{1,v_2} (\lambda_{2,v_2} v_2 + \mu_{2,v_2} u_0) \in C.\] (17)

Now let
\[B := \{ \kappa_{1,v_2} v_1 + \rho_{1,v_2} v_2 : v_1 \in B_1, v_2 \in B_2 \}.\]

Then clearly, for all \(w \in B\), there are \(v_1 \in B_1\) and \(v_2 \in B_2\) such that \(w = \kappa_{1,v_2} v_1 + \rho_{1,v_2} v_2\). However, there could be multiple such pairs. We choose any one such pair and denote its two elements by \(v_{1,w}\) and \(v_{2,w}\), respectively. Using this notation, we now define the set
\[C := \{ \gamma_{1,v_2} w + \delta_{1,v_2} u_0 : w \in B \}.

Since \(B_1, B_2 \in K\), the coherence of \(K\) [Axiom \(K_4\)] implies that \(B \in K\), which in turn implies that \(C \in K^{**}\). Also, since
\[C' = \{ \gamma_{1,v_2} w + \delta_{1,v_2} u_0 : w \in B \}
\]
we infer from Equation (17) that \(C' \subseteq C\). Since we have already proved that \(K^{**}\) satisfies Axiom \(K_4\), this implies that, indeed, \(C \in K^{**}\).

It therefore now only remains to prove that \(\emptyset \notin K^{**}\). Observe that that \(\emptyset \notin K\) because \(K\) is coherent [combine Axioms \(K_1\) and \(K_4\)]. It therefore follows from Equation (16) that, indeed, \(\emptyset \notin K^{**}\).

**Proposition 40.** Any non-binary coherent set of desirable option sets \(K\) is strictly dominated by some other coherent set of desirable option sets.

**Proof.** Consider an arbitrary coherent non-binary set of desirable option sets \(K\). We infer from Lemma 33 that there is some \(A_o \in K\) such that \(|A_o| \geq 2\) and \(A_o \setminus \{u\} \notin K\) for all \(u \in A_o\). Consider any \(u_o \in A_o\) and let \(K^* := \text{Rn}(K^{**})\), with \(K^{**}\) as in Equation (16). It then follows from Lemma 39 that \(K^*\) is a coherent set of desirable option sets that is a superset of \(K\) and contains \(\{u_o\}\), and that \(\{u_o\} \notin K^*\). Hence, \(K \subseteq K^*\).

**Lemma 41.** For any non-empty chain \(\mathcal{X}\) in \(\mathcal{K}\), its union \(K_o := \bigcup \mathcal{X}\) is a coherent set of desirable option sets.

**Proof.** For Axiom \(K_0\), consider any \(A \in K_o\). Then there is some \(K' \in \mathcal{X}\) such that \(A \in K'\), and since \(K'\) is coherent, this implies that \(A \setminus \{0\} \in K' \subseteq K_o\).

For Axiom \(K_1\), simply observe that since \(\{0\}\) belongs to no element of \(\mathcal{X}\) [since they are all coherent], it cannot belong to their union \(K_o\).

For Axiom \(K_2\), consider any \(u \succ 0\) and any \(K \in \mathcal{X}\), then we know that \(\{u\} \in K\) [since \(K\) is coherent], and therefore also \(\{u\} \in K_o\), since \(K \subseteq K_o\).

For Axiom \(K_3\), consider any \(A_1, A_2 \in K_o\) and, for all \(u \in A_1\) and \(v \in A_2\), choose some \((\lambda_{u,v}, \mu_{u,v}) > 0\). Since \(A_1, A_2 \in K_o\), we know that there are \(K_1, K_2 \in \mathcal{X}\) such that \(A_1 \in K_1\) and \(A_2 \in K_2\).

Since \(\mathcal{X}\) is a chain, we can assume without loss of generality that \(K_1 \subseteq K_2\), and therefore \(\{A_1, A_2\} \subseteq K_2\). Since \(K_2\) is coherent, it follows that \(\{\lambda_{u,v} u + \mu_{u,v} v : u \in A_1, v \in A_2\} \in K_2 \subseteq K_o\).

And finally, for Axiom \(K_4\), consider any \(A_1 \in K_o\) and any \(A_2 \in \mathcal{X}\) such that \(A_1 \subseteq A_2\). Then we know that there is some \(K \in \mathcal{X}\) such that \(A_1 \in K\). Since \(K\) is coherent, this implies that also \(A_2 \in K \subseteq K_o\).

**Lemma 42.** For any coherent set of desirable option sets \(K \in \mathcal{K}\) and any set of desirable option sets \(K^* \in \mathcal{K}\) such that \(K \cap K^* = \emptyset\), the partially ordered set
\[\uparrow K := \{K' \in \mathcal{K} : K \subseteq K' \text{ and } K' \cap K^* = \emptyset\}\]
has a maximal element.
Proof. We will use Zorn’s Lemma to establish the existence of a maximal element. So consider any (non-empty) chain $\mathcal{K}$ in $\uparrow K$, then we must prove that $\mathcal{K}$ has an upper bound in $\uparrow K$. Since $K_o := \bigcup \mathcal{K}$ is clearly an upper bound, we are done if we can prove that $K_o \in \uparrow K$.

That $K_o \cap K^* = \emptyset$ follows from the fact that $K' \cap K^* = \emptyset$ for every $K' \in \mathcal{K} \subseteq \uparrow K$. That $K_o$ is a coherent set of desirable option sets follows from Lemma 41. □

Proposition 43. Every coherent set of desirable option sets $K \in \mathcal{K}$ is dominated by some binary coherent set of desirable option sets. 

Proof. Lemma 42 with $K^* = \emptyset$ tells us that the partially ordered set $\{K' : K \subseteq K'\}$ has some maximal element $\hat{K}$. Assume $\text{ex absurdo}$ that $\hat{K}$ is non-binary. It then follows from Proposition 40 that $\hat{K}$ is strictly dominated by a coherent set of desirable option sets, meaning that there is some $K^* \in \mathcal{K}$ such that $\hat{K} \subseteq K^*$, and therefore also $K \subseteq K^*$. Hence also $K^* \in \{K' : K \subseteq K'\}$, which contradicts that $\hat{K}$ is a maximal element of that set. We conclude that $\hat{K}$ is indeed binary. □

Theorem 44. Every coherent set of desirable option sets $K \in \mathcal{K}$ is dominated by at least one binary coherent set of desirable option sets: $\overline{D}(K) := \{D \in \overline{D} : K \subseteq K_D\} \neq \emptyset$. Moreover, $K = \bigcap\{K_D : D \in \overline{D}(K)\}$.

Proof. Let $K_o$ be any coherent set of desirable option sets. We prove that $\overline{D}(K_o) := \{D \in \overline{D} : K_o \subseteq K_D\} \neq \emptyset$ and that $K_o = \bigcap\{K_D : D \in \overline{D}(K_o)\}$.

For the first statement, recall from Proposition 43 that $K_o$ is dominated by a binary coherent set of desirable option sets $\hat{K}$. Proposition 6 therefore implies that $\hat{K} = K_{D_o}$, with $D = D_{K_o}$. Furthermore, because $\hat{K}$ is coherent, Proposition 8 implies that $D$ is coherent, whence $D \in \overline{D}$. Since $K_o \subseteq \hat{K} = K_{D_o}$, we see that $D_{\hat{K}} \in \overline{D}(K_o)$, so $\overline{D}(K_o) \neq \emptyset$. For the second statement, it is obvious that $K_o \subseteq \bigcap\{K_D : D \in \overline{D}(K_o)\}$, so we concentrate on the proof of the converse inclusion. Assume $\text{ex absurdo}$ that $K_o \not\subseteq \bigcap\{K_D : D \in \overline{D}(K_o)\}$, so there is some option set $B_o \in \mathcal{K}$ such that $B_o \not\subseteq K_o$ and $B_o \in K_D$ for all $D \in \overline{D}(K_o)$, so $B_o \neq \emptyset$. Then $B_o \not\subseteq K_o$ [use the coherency of $K_o$ and Axiom $K_4$] and $B_o \not\subseteq K_D$ for all $D \in \overline{D}(K_o)$ [use the coherence of $D_o$—which follows from Proposition 32 and the coherence of $D_o$—and Proposition 36], so we may assume without loss of generality that $B_o$ has no non-positive options: $B_o \cap \forall \ni 0 = \emptyset$.

The partially ordered set $\uparrow K_o^* := \{K \in \mathcal{K} : K_o \subseteq K \land B_o \not\subseteq K\}$ is non-empty because it contains $K_o$. Furthermore, due to Lemma 42 [applied for $K = K_o$ and $K^* = \{B_o\}$], it has at least one maximal element. If we can prove that any such maximal element $\hat{K}$ is binary, then we know from Propositions 6 and 8 that there is some coherent set of desirable options $D_o = D_{\hat{K}}$ such that $K_o \subseteq K_D$, and therefore $D_o \in \overline{D}(K_o)$—and $B_o \not\subseteq K_D$, a contradiction. To prove that all maximal elements of $\uparrow K^*_o$ are binary, it suffices to prove that any non-binary element of $\uparrow K^*_o$ is strictly dominated in that set, which is what we now set out to do.

So consider any non-binary element $K$ of $\uparrow K^*_o$, so in particular $K \in \mathcal{K}$, $K_o \subseteq K$ and $B_o \not\subseteq K$. Since $K$ is non-binary, it follows from Lemma 33 that there is some $A_o \in K$ such that $|A_o| \geq 2$ and $A_o \setminus \{u\} \not\subseteq K$ for all $u \in A_o$. The partially ordered set $\{A \in K : B_o \subseteq A\}$ contains $A_o \cup B_o$ [because $A_o \in K$ and because $K$ satisfies Axiom $K_4$] and therefore has some minimal (non-dominating) element $B^*$ below it, so $B^* \in K$ and $B_o \subseteq B^* \subseteq A_o \cup B_o$.

Let us first summarise what we know about this minimal element $B^*$. It is impossible that $B^* \subseteq B_o$ because otherwise $B_o = B^* \subseteq K$, a contradiction. Hence $B^* \setminus B_o \neq \emptyset$, so we can fix some element $u_0$ in $B^* \setminus B_o \subseteq A_o$. Since $B_o \subseteq B^* \setminus \{u_0\}$ but $B^* \setminus \{u_0\} \subset B^*$, it must be that $B^* \setminus \{u_0\} \not\subseteq K$, by the definition of a minimal element. Observe that $B^* \not\subseteq K$.

Let $K^* := R_n(K^{**})$, with $K^{**}$ as in Equation (16). Since $u_0 \in A_o$, it then follows from Lemma 39 that $K^*$ is a coherent set of desirable option sets that is a superset of $K$—and therefore also of $K_o$—and contains $\{u_0\}$, and $\{u_0\} \not\subseteq K$ and $u_0 \not\subseteq 0$. Hence, it follows
that $K \subset K^*$. If we can now prove that $B_o \notin K^*$ and therefore $K^* \in \uparrow K_o^*$, we are done, because then $K$ is indeed strictly dominated by $K^*$ in $\uparrow K_o^*$.

Assume therefore *ex absurdo* that $B_o \in K^* = \text{Ru}(K^*)$. Taking into account Equation (16), this implies that there are $C \in K$ and $(\gamma, \mu_o) > 0$ for all $v \in C$, such that $\{b_r : v \in C\} \setminus \mathcal{V} \subseteq B_o \subseteq \{b_r : v \in C\}$, where, for all $v \in C$, $b_r := \lambda_v v + \mu_o u_o$. Given our assumption that $B_o \cap \mathcal{V} = \emptyset$, this also implies that $\{b_r : v \in C\} \setminus \mathcal{V} \subseteq \emptyset$. Now let $C_1 := \{v \in C : b_r \in B_o\}$ and $C_2 := \{v \in C : b_r \notin B_o\}$. Then $C_1 \neq \emptyset$ [because $B_o \neq \emptyset$] and $\{b_r : v \in C_1\} = B_o$. Consider now any $v \in C_2$. Then $b_r \notin B_o$. Since $\{b_r : v \in C\} \setminus \mathcal{V} \subseteq \emptyset$, this implies that $b_r = \lambda_v v + \mu_o u_o \leq 0$. Hence, we must have that $\lambda_v > 0$, because otherwise $\mu_v u_o \leq 0$ with $\mu_v > 0$, and therefore also $u_o \leq 0$, contradicting what we inferred earlier from Lemma 39. So we find that

$$v \leq -\frac{\mu_v}{\lambda_v} u_o$$

for all $v \in C_2$.

Consequently, and because $C_1 \cup C_2 = C \in K$, we infer from Lemma 34 that

$$C' := C_1 \cup \left\{ -\frac{\mu_v}{\lambda_v} u_o : v \in C_2 \right\} \in K.$$

Let $C_3 := C' \setminus C_1$. Then for all $v \in C_3$, there is some $\gamma_v \geq 0$ such that $v = -\gamma_v u_o$. Now let $(\alpha_{a,v}, \beta_{a,v}) := (\mu_v, \lambda_v)$ for all $v \in C_1$ and $(\alpha_{a,v}, \beta_{a,v}) := (\gamma_v, 1)$ for all $v \in C_3$ and, for all $u \in B^* \setminus \{u_o\}$ and $v \in C'$, let $(\alpha_{a,v}, \beta_{a,v}) := (1, 0)$. Then

$$\{\alpha_{a,v} u + \beta_{a,v} v : u \in B^*, v \in C'\}$$

$$= \{\mu_v u_o + \lambda_v v : v \in C_1\} \cup \{\gamma_v u_o + v : v \in C_3\} \cup \{u : u \in B^* \setminus \{u_o\}, v \in C'\}$$

$$= \{b_r : v \in C_1\} \cup \{0 : v \in C_3\} \cup \{u : u \in B^* \setminus \{u_o\}\}$$

$$= B_o \cup \{0 : v \in C_3\} \cup \{B^* \setminus \{u_o\}\}$$

$$= (B^* \setminus \{u_o\}) \cup \{0 : v \in C_3\},$$

where the second equality holds because $C' \in K$ and Axioms K1 and K4 imply that $\emptyset \neq C'$, and where the fourth equality holds because $B_o \subset B^* \setminus \{u_o\}$. Since $B^* \in K$ and $C' \in K$, we can now invoke Axiom K3 to find that

$$B^* \setminus \{u_o\} \cup \{0 : v \in C_3\} = \{\alpha_{a,v} u + \beta_{a,v} v : u \in B^*, v \in C'\} \in K.$$

If $C_3 = \emptyset$, we find that $B^* \setminus \{u_o\} \in K$, a contradiction. If $C_3 \neq \emptyset$, we find that $\{0\} \cup B^* \setminus \{u_o\} \in K$. If $0 \in B^* \setminus \{u_o\}$, then we get that $B^* \setminus \{u_o\} \in K$, a contradiction. And if $0 \notin B^* \setminus \{u_o\}$, then we can still derive from Axiom K0 that $B^* \setminus \{u_o\} \in K$, again a contradiction.

**Proof of Theorem 9.** If the set of desirable option sets $K$ is coherent, we infer from Theorem 44 that $\overline{D}(K) := \{D \in \overline{D} : K \subseteq K_D \neq \emptyset\}$ and $K = \cap\{K_D : D \in \overline{D}(K)\}$. This clearly implies that there is at least one non-empty set $\mathcal{D} \subseteq \overline{D}$ of coherent sets of desirable options such that $K = \cap\{K_D : D \in \mathcal{D}\}$, namely the set $\overline{D}(K)$. Furthermore, for any non-empty set $\mathcal{D} \subseteq \overline{D}$ of coherent sets of desirable options such that $K = \cap\{K_D : D \in \mathcal{D}\}$, we clearly have that $K \subseteq K_D$ for all $D \in \mathcal{D}$. Since $\mathcal{D} \subseteq \overline{D}$, this implies that $\mathcal{D} \subseteq \overline{D}(K)$. So $\overline{D}(K)$ is indeed the largest such set.

It remains to prove the ‘if’ part of the statement. So consider any non-empty set $\mathcal{D} \subseteq \overline{D}$ of coherent sets of desirable options such that $K = \cap\{K_D : D \in \mathcal{D}\}$. For any $D \in \mathcal{D} \subseteq \overline{D}$, it then follows from Proposition 32 that $K_D$ is coherent. Because Axioms K0-K4 are trivially preserved under taking arbitrary non-empty intersections, it follows that $K$ is coherent.

**A.4. Proofs and Intermediate Results for Section 4.**

**Proposition 45.** Posi($K$) = $K$ for any coherent set of desirable option sets $K \in K$. 

Proof of Proposition 45. That $K \subseteq \text{Posi}(K)$, is an immediate consequence of the definition of the Posi operator, and holds for any set of desirable option sets, coherent or not. Indeed, consider any $A \in K$, then it is not difficult to see that $A \in \text{Posi}(K)$: choose $n := 1$, $A_1 := A \in K$, and $\lambda_{1:n}^{u_1} := 1$ for all $u_1 \in \times_{k=1}^{n} A_k$ in the definition of the Posi operator.

For the converse inclusion, that $\text{Posi}(K) \subseteq K$, we use the coherence of $K$, and in particular the representation result of Theorem 44, which allows us to write that $K = \bigcap\{K_D : D \in \mathbf{D} \text{ and } K \subseteq K_D\}$.

So, if we fix any $D \in \mathbf{D}$ such that $K \subseteq K_D$, then it clearly suffices to prove that also $\text{Posi}(K) \subseteq K_D$. Consider, therefore, any $A \in \text{Posi}(K)$, meaning that there are $n \in \mathbb{N}$, $(A_1, \ldots, A_n) \in K^n$ and, for all $u_1, \ldots, u_n \in \times_{k=1}^{n} A_k$, some choice of $\lambda_{1:n}^{u_1} > 0$ such that

$$A = \left\{ \sum_{k=1}^{n} \lambda_{1:n}^{u_k} u_k : u_1, \ldots, u_n \in \times_{k=1}^{n} A_k \right\}.$$ 

For any $k \in \{1, \ldots, n\}$, since $A_k \in K \subseteq K_D$, we know that $A_k \cap D \neq \emptyset$, so we can fix some $v_k \in A_k \cap D$. Then, on the one hand, we see that $\sum_{k=1}^{n} \lambda_{k}^{v_k} v_k \in A$. On the other hand, since $\lambda_{1:n}^{u_1} > 0$, we infer from Axiom $D_4$ [by applying it multiple times] that also $\sum_{k=1}^{n} \lambda_{k}^{v_k} v_k \in D$. Therefore, we find that $A \cap D \neq \emptyset$, or equivalently, that $A \in K_D$. Since $A \in \text{Posi}(K)$ was chosen arbitrarily, it follows that, indeed, $\text{Posi}(K) \subseteq K_D$. \qed

Proof of Proposition 4. First, suppose that $K$ satisfies Axioms $K_0$–$K_4$, then we show that $R$ satisfies Axioms $R_0$–$R_4$.

Axiom $R_4$ follows even without Axioms $K_0$–$K_4$, from the chain of equivalences

$$u \in R(A) \iff u \in R(A \cup \{u\}) \iff A - u \in K \iff (A - u) - 0 \in K \iff 0 \in R((A - u) \cup \{0\}) \iff 0 \in R(A - u),$$

where the first and last equivalences follow because $u \in A$ and the second and fourth equivalences follow from Equation (5). We now concentrate on Axioms $R_1$–$R_4$.

$R_1$. It is obvious from $R(\emptyset) \subseteq \emptyset$ that $R(\emptyset) = \emptyset$. For any non-empty option set $A \in \mathcal{D}$, assume ex absurdo that $R(A) = A$. For all $u \in A$, it then follows from Equation (5) that $A - u \in K$. So if we denote $A$ by $\{v_1, \ldots, v_n\}$, with $n \in \mathbb{N}$, and let $v_{ik} := v_l - v_k$, then we find that for all $k \in \{1, \ldots, n\}$

$$A_k := \{v_{ik} : l \in \{1, \ldots, n\}\} \in K.$$ 

Proposition 45 and Equation (14) now tell us that, for any choice of the $\lambda_{1:n}^{u_k} > 0$ in Equation (14), the option set

$$\left\{ \sum_{k=1}^{n} \lambda_{1:n}^{u_k} u_k : u_1, \ldots, u_n \in \times_{k=1}^{n} A_k \right\} \in K.$$ 

So if we can show that for any $u_1, \ldots, u_n \in \times_{k=1}^{n} A_k$ we can always choose the $\lambda_{1:n}^{u_k} > 0$ in such a way that $\sum_{k=1}^{n} \lambda_{1:n}^{u_k} u_k = 0$, we will have that $\{0\} \in K$, contradicting Axiom $K_1$. We now set out to do this.

For any $k \in \{1, \ldots, n\}$, since $u_k \in A_k$, there is a unique $l \in \{1, \ldots, n\}$ such that $u_k = v_{lk}$. Let $\phi(k)$ be this unique index, so $u_k = v_{\phi(k)}$. For the resulting map $\phi : \{1, \ldots, n\} \to \{1, \ldots, n\}$, we now consider the sequence—$\phi$-orbit—in $\{1, \ldots, n\}$:

$$1, \phi(1), \phi^2(1), \ldots, \phi^p(1), \ldots$$

Because $\phi$ can assume at most $n$ different values, this sequence must be periodic, and its fundamental (smallest) period $p$ cannot be larger than $n$, so $1 \leq p \leq n$ and $1 = \phi^p(1)$. Now let $\lambda_{\phi^p(1)} := 1$ for $r = 0, \ldots, p - 1$, and let all other components be zero, then indeed

$$\sum_{k=1}^{n} \lambda_{1:n}^{u_k} u_k = \sum_{r=0}^{p-1} v_{\phi^{r+1}(1)} \phi^r(1) = \sum_{r=0}^{p-1} (v_{\phi^{r+1}(1)} - v_{\phi^r(1)}) = 0.$$
R2. Consider any \( u \in \mathcal{V} \setminus \{0\} \), so \( \{u\} \in K \) [use \( K_2 \)]. Since \( \{u\} = \{u\} - 0 \), Equation (5) guarantees that, indeed, \( 0 \in R(\{0, u\}) \).

R3. Since \( 0 \in R(A_1 \cup \{0\}) \) and \( 0 \in R(A_2 \cup \{0\}) \), it follows from Equation (5) that \( A_1, A_2 \in K \). Axiom \( K_2 \) therefore implies that
\[
\{\lambda_{u,v}u + \mu_{u,v}v : u \in A_1, v \in A_2\} \subseteq K.
\]
A final application of Equation (5) now tells us that, indeed,
\[
0 \in R(\{\lambda_{u,v}u + \mu_{u,v}v : u \in A_1, v \in A_2\} \cup \{0\}).
\]

R4. Consider \( A_1, A_2 \in \mathcal{D} \) and \( u \in A_1 \), and assume that \( u \in R(A_1) \) and \( A_1 \subseteq A_2 \). Since \( u \in \mathcal{A}_1 \), Equation (5) then implies that \( A_1 - u \in K \) and by Axiom \( K_4 \), therefore also \( A_2 - u \in K \), which in turn implies [again using Equation (5)] that \( u \in R(A_2 \cup \{u\}) \), and therefore, since \( u \in A_1 \subseteq A_2 \), that \( u \in R(A_2) \).

Next, we suppose that \( R \) satisfies Axioms R1–R4, and show that \( K \) then satisfies Axioms \( K_0 \)–\( K_4 \). Again, the first axiom [Axiom \( K_0 \) in this case] holds without imposing any conditions on \( R \). To see this, it suffices to consider the following chain of equivalences
\[
A \in K \iff 0 \in R(A \cap \{0\}) \iff 0 \in R((A \setminus \{0\}) \cup \{0\}) \iff A \setminus \{0\} \subseteq K,
\]
for all \( A \subseteq \mathcal{D} \).

A5. Proofs and Intermediate Results for Section 7.

Proof of Proposition 12. Since for all \( u \in \mathcal{V} \setminus \{0\} \),
\[
(u \in D \lor -u \in D) \iff \{u, -u\} \cap D = \emptyset \iff \{u, -u\} \in K_D,
\]
this result is an immediate consequence of Definitions 10 and 11, and Proposition 8. □

Proof of Theorem 13. First assume that \( K \) is total. Since \( K \) is then in particular coherent, we know from Theorem 9 that there is some non-empty set \( \mathcal{D} \subseteq \mathcal{D}_T \) of coherent sets of desirable options such that \( K = \bigcap\{K_D : D \subseteq \mathcal{D}\} \), and that the largest such set \( \mathcal{D} \) is \( \mathcal{D}(K) \) := \{D \in \mathcal{D} : K \subseteq K_D\}. Consider any such set \( \mathcal{D} \). Then for all \( D \in \mathcal{D} \), since \( K \subseteq K_D \), the totality of \( K \) implies that \( K_D \) is total, and therefore, because of Proposition 12, that \( D \) is total as well. Hence, \( \mathcal{D} \subseteq \mathcal{D}_T \) and therefore also \( \mathcal{D}(K) = \mathcal{D}_T(K) \). All this allows us to conclude that there is some non-empty set \( \mathcal{D} \subseteq \mathcal{D}_T \) of total sets of desirable options such that
\[
K = \bigcap\{K_D : D \subseteq \mathcal{D}\},
\]
and that the largest such set \( \mathcal{D} \) is \( \mathcal{D}_T(K) := \{D \in \mathcal{D}_T : K \subseteq K_D\} \).

To prove the \textquoteleft{}if\textquoteright{} part of the statement, consider any non-empty set \( \mathcal{D} \subseteq \mathcal{D}_T \) of total sets of desirable options such that \( K = \bigcap\{K_D : D \subseteq \mathcal{D}\} \). For every \( D \in \mathcal{D} \subseteq \mathcal{D}_T \), it then follows from Proposition 12 that \( K_D \) is total. Because Axioms \( K_0 \)–\( K_4 \) and \( K_T \) are trivially preserved under taking arbitrary non-empty intersections, this implies that \( K \) is total as well. □

Proof of Proposition 14. First suppose that \( K \) is total. Since \( K \) is then in particular coherent, we know from Proposition 4 that \( R \) is coherent. Consider now any \( u \in \mathcal{V} \setminus \{0\} \), so \( \{u, -u\} \in K \) by Axiom \( K_T \). Equation (5) then guarantees that, indeed, \( 0 \in R(\{0, u, -u\}) \).

Next, suppose that \( R \) is coherent and satisfies Axiom \( R_T \). Since \( R \) is coherent, we know from Proposition 4 that \( K \) is coherent too. Consider now any \( u \in \mathcal{V} \setminus \{0\} \), so \( 0 \in R(\{0, u, -u\}) \) by Axiom \( R_T \). Equation (5) then guarantees that, indeed, \( \{u, -u\} \in K \). □
A.6. Proofs and Intermediate Results for Section 8.

Lemma 46. Consider any set of desirable option sets $K \in \mathbf{K}$ that satisfies Axiom $K_3$. Consider any $A \in K$ and, for all $u \in A$, some $\lambda_u > 0$. Then also $\{\lambda_u u \colon u \in A\} \subseteq K$.

Proof. Axiom $K_3$ for $A$ and $A$ allows us to infer that $\{\lambda_{u,v} u + \mu_{u,v} v \colon u, v \in A\} \subseteq K$ for all possible choices of $(\lambda_{u,v}, \mu_{u,v}) > 0$. Choosing $(\lambda_{u,v}, \mu_{u,v}) := (\lambda_u, 0)$ for all $u, v \in A$, yields in particular that, indeed, $\{\lambda_u u \colon u \in A\} \subseteq K$.

Proof of Proposition 15. First assume that $R$ is coherent and mixing. Proposition 4 then tells us that $K$ is coherent. We will prove that $K$ also satisfies Axioms $K_M$ and $K'_M$. Since $K_M$ clearly implies $K_M'$, it suffices to prove the former. So consider any $A, B \in \mathcal{D}$ such that $B \in K$ and $A \subseteq B \subseteq \text{posi}(A)$. For every $u \in B \setminus A$, since $u \in \text{posi}(A)$, it follows from Equations (9) and (10) that there is some $\lambda_u > 0$ such that $\lambda_u u \in \text{conv}(A)$. Furthermore, for every $u \in A$, if we let $\lambda_u := 1$, then also $\lambda_u u \in \text{conv}(A)$. Let $\tilde{B} := \{\lambda_u u \colon u \in B\}$, then, clearly, $A \subseteq \tilde{B} \subseteq \text{conv}(A)$. Furthermore, since $B \in K$, it follows from Lemma 46 that also $\tilde{B} \in K$. In order to prove that then also $A \in K$, observe that $A \subseteq \tilde{B} \subseteq \text{conv}(A)$ implies that $A \cup \{0\} \subseteq \tilde{B} \cup \{0\} \subseteq \text{conv}(A \cup \{0\})$, so Axiom $R_M$ tells us that

$$R(\tilde{B} \cup \{0\}) \cap (A \cup \{0\}) \subseteq R(A \cup \{0\}).$$

(18)

Since $\tilde{B} \in K$ is equivalent to $\tilde{B} \setminus 0 \subseteq K$, we infer from Equation (5) that $0 \in R(\tilde{B} \cup \{0\})$, and since also $0 \in (A \cup \{0\})$, we infer from Equation (18) that $0 \in R(A \cup \{0\})$. Equation (5) then leads us to conclude that, indeed, $A = A \setminus 0 \subseteq K$.

Next, we assume that $K$ is coherent and satisfies $K_M$ or $K_M'$. Since $K_M$ implies $K_M'$, it follows that $K$ is coherent and satisfies $K_M'$. Proposition 4 already tells us that then $R$ is coherent, so we are left to prove that $R$ satisfies $R_M$. So, consider any $A, B \in \mathcal{D}$ such that $A \subseteq B \subseteq \text{conv}(A)$. In order to prove that $R(B \cap A) \subseteq R(A)$, consider any $u \in R(B \cap A)$. Then also $u \in B$, so $B = B \cup \{u\}$, and therefore $u \in R(B \cup \{u\})$, whence $B - u \in K$ by Equation (5). Since it follows from the assumptions that also $A - u \subseteq B - u \subseteq \text{conv}(A - u)$, it follows from Axiom $K_M'$ that $A - u \subseteq K$, so Equation (5) and $A = A \cup \{u\}$ guarantee that, indeed, $u \in R(A)$.

Proof of Proposition 18. First, assume that $D \in \overline{D}_M$. Then $D$ is in particular coherent, so $K_D$ is coherent too, by Proposition 8. To prove that $K_D \in \overline{K}_M$, it therefore suffices to show that $K_D$ satisfies $K_M$. So consider any $A, B \in \mathcal{D}$ such that $B \in K_D$ and $A \subseteq B \subseteq \text{posi}(A)$, then we have to prove that also $A \in K_D$. Since $B \in K_D$, it follows from Equation (8) that $B \cap D \neq \emptyset$ and therefore, since $B \subseteq \text{posi}(A)$, that $\text{posi}(A) \cap D \neq \emptyset$. We can now use the assumption that $D$ satisfies $D_M$ to infer that also $A \cap D \neq \emptyset$, which in turn implies that, indeed, $A \in K_D$, again because of Equation (8).

Conversely, assume that $K_D \in \overline{K}_M$. Then $K_D$ is in particular coherent, and therefore so is $D$, by Proposition 8. To prove that $D \in \overline{D}_M$, it therefore suffices to show that $D$ satisfies $D_M$. Consider, therefore, any $A \in \mathcal{D}$ such that $\text{posi}(A) \cap D \neq \emptyset$, then we have to prove that also $A \cap D \neq \emptyset$. It follows from $\text{posi}(A) \cap D \neq \emptyset$ that there is some $u \in \text{posi}(A) \cap D$. Let $B := A \cup \{u\}$, then clearly $A \subseteq B \subseteq \text{posi}(A)$. Also, $B \cap D \neq \emptyset$, so Equation (8) guarantees that $B \in K_D$. We then infer from the assumption that $K_D$ satisfies $K_M$ that also $A \in K_D$, whence, indeed, $A \cap D \neq \emptyset$, by Equation (8).

Lemma 47. Consider any set of desirable option sets $K \in \mathbf{K}$ that satisfies Axiom $K_3$. Consider any $A \in K$ such that $\{w, \lambda w\} \subseteq A$ for some $w \in \mathcal{V} \setminus \{0\}$ and $\lambda > 0$ such that $\lambda \neq 1$. Then $A \setminus \{w\} \subseteq K$.

Proof. Axiom $K_3$ for $A$ and $A$ allows us to infer that $\{\lambda_{w,v} u + \mu_{w,v} v \colon u, v \in A\} \subseteq K$ for all possible choices of $(\lambda_{w,v}, \mu_{w,v}) > 0$. Choosing $(\lambda_{w,v}, \mu_{w,v}) := (1, 0)$ for all $u \in A \setminus \{w\}$ and $(\lambda_{w,v}, \mu_{w,v}) := (\lambda, 0)$, for all $v \in A$, yields in particular that $A \setminus \{w\} \subseteq K$. \[\square\]
**Proposition 48.** Consider any set of desirable option sets $K \in K$. Then $R_p(K)$ satisfies $K_M$. Moreover, if $K$ satisfies $K_0$, $K_1$, $K_2$, $K_3$ and $K_4$, then so does $R_p(K)$.

**Proof.** We begin with the first statement. Consider any $A_1 \in R_p(K)$ and any $A \in \mathcal{A}$ such that $A \subseteq A_1 \subseteq \text{posi}(A)$. Then, on the one hand, $\text{posi}(A) = \text{posi}(A_1)$, and on the other hand, there is some $B_1 \in K$ such that $A_1 \subseteq B_1 \subseteq \text{posi}(A_1)$. Hence $A \subseteq A_1 \subseteq B_1 \subseteq \text{posi}(A_1) = \text{posi}(A)$, and therefore indeed $A \in R_p(K)$.

For the second statement, assume that $K$ satisfies $K_0$, $K_1$, $K_2$, $K_3$, and $K_4$.

To prove that $R_p(K)$ satisfies $K_0$, consider any $A \in R_p(K)$, meaning that there is some $B \in K$ such that $A \subseteq B \subseteq \text{posi}(A)$. To see that $A \setminus \{0\} \in R_p(K)$, it suffices to show that $B \setminus \{0\} \subseteq \text{posi}(A \setminus \{0\})$, because clearly $A \setminus \{0\} \subseteq B \setminus \{0\}$. Consider any element $u$ of $B \setminus \{0\}$, then it follows from the assumption $B \subseteq \text{posi}(A)$ that $u = \sum_{k=1}^n \lambda_k u_k$, with $n \geq 1$, different $u_k \in A$, and $\lambda_k > 0$. Even if $u_k = 0$ for some $k$, we still find that $u \in \text{posi}(A \setminus \{0\})$.

To prove that $R_p(K)$ satisfies $K_1$, assume ex absurdo that $\{0\} \in R_p(K)$, so there is some $B \in K$ such that $\{0\} \subseteq B \subseteq \text{posi}(\{0\})$, or in other words, such that $B = \{0\}$, contradicting that $K$ satisfies $K_1$.

To prove that $R_p(K)$ satisfies $K_2$, simply observe that the operator $R_p$ never removes option sets from a set of desirable option sets, so the option sets $\{u\}$, $u \succ 0$ that belong to $K$ by $K_2$, will also belong to the larger $R_p(K)$.

To prove that $R_p(K)$ satisfies $K_3$, consider any $A_1, A_2 \in R_p(K)$, meaning that there are $B_1, B_2 \in K$ such that $A_1 \subseteq B_1 \subseteq \text{posi}(A_1)$ and $A_2 \subseteq B_2 \subseteq \text{posi}(A_2)$. This tells us that any $v_1 \in B_1$ can be written as $v_1 = \sum_{u \in A_1} \alpha_{u_1,1} u_1$ with $\alpha_{u_1,1} \succ 0$, and similarly, any $v_2 \in B_2$ can be written as $v_2 = \sum_{u \in A_2} \beta_{u_2,2} u_2$ with $\beta_{u_2,2} \succ 0$.

Choose, for all $u \in A_1$ and $v \in A_2$, $(\lambda_{u_1,v}, \mu_{u_1,v}) > 0$, then we must show that $A := \{\lambda_{u_1,v} u + \mu_{u_1,v} v : u \in A_1, v \in A_2\} \subseteq R_p(K)$,

or in other words that there is some $B \in K$ such that $A \subseteq B \subseteq \text{posi}(A)$. We will show that there is some $B \in \text{Posi}(B_1, B_2)$ that does the job, or in other words that there are suitable choices of $(\kappa_{v_1,v_2}, \rho_{v_1,v_2}) > 0$ for all $v_1 \in B_1$ and $v_2 \in B_2$ such that

$$A \subseteq \{\kappa_{v_1,v_2} v_1 + \rho_{v_1,v_2} v_2 : v_1 \in B_1, v_2 \in B_2\} \subseteq \text{posi}(A).$$ (19)

For a start, if we let $\kappa_{v_1,v_2} := \lambda_{u_1,v_2}$ and $\rho_{v_1,v_2} := \mu_{u_1,v_2}$ for all $v_1 \in A_1$ and all $v_2 \in A_2$, then we are already guaranteed that the first inequality in (19) holds. It now remains to choose the remaining $(\kappa_{v_1,v_2}, \rho_{v_1,v_2}) > 0$ in such a way that the second inequality in (19) will also hold, meaning that $\kappa_{v_1,v_2} v_1 + \rho_{v_1,v_2} v_2 \in \text{posi}(A)$.

We consider three mutually exclusive possibilities, for any fixed remaining $v_1$ and $v_2$. The first possibility is that $v_1 = u_1 \in A_1$ and $v_2 \in B_2 \setminus A_2$. If there is some $u_2 \in A_2$ such that $\mu_{u_1,u_2} = 0$ and therefore $\lambda_{u_1,u_2} = 0$, then we choose $\kappa_{v_1,v_2} := \lambda_{u_1,u_2}$ and $\rho_{v_1,v_2} := 0$, and then indeed

$$\kappa_{v_1,v_2} u_1 + \rho_{v_1,v_2} v_2 = \lambda_{u_1,u_2} u_1 + \mu_{u_1,u_2} v_2 \in A \subseteq \text{posi}(A).$$

If, on the other hand, $\mu_{u_1,u_2} > 0$ for all $u_2 \in A_2$, then we choose $\rho_{v_1,v_2} := 1$ and

$$\kappa_{v_1,v_2} := \sum_{u_2 \in A_2} \frac{\lambda_{u_1,u_2}}{\mu_{u_1,u_2}} \beta_{u_2,u_2},$$

and then indeed

$$\kappa_{v_1,v_2} u_1 + \rho_{v_1,v_2} v_2 = \sum_{u_2 \in A_2} \frac{\lambda_{u_1,u_2}}{\mu_{u_1,u_2}} \beta_{u_2,u_2} u_1 + \sum_{u_2 \in A_2} \beta_{u_2,u_2} u_2 = \sum_{u_2 \in A_2} \frac{\beta_{u_2,u_2}}{\mu_{u_1,u_2}} (\lambda_{u_1,u_2} u_1 + \mu_{u_1,u_2} u_2) \in \text{posi}(A).$$

The second possibility is that $v_1 \in B_1 \setminus A_1$ and $v_2 = u_2 \in A_2$, and this is treated in a similar way as the first.
And the third and last possibility is that \( v_1 \in B_1 \setminus A_2 \) and \( v_2 \in B_2 \setminus A_2 \). We now partition both \( A_1 \) and \( A_2 \) in three disjoint pieces. For \( A_1 \) we let \( A_1^1 := \{ u_1 \in A_1 : \alpha_{v_1,u_1} = 0 \} \), \( A_1^2 := \{ u_1 \in A_1 : (\exists u_2 \in A_2) \mu_{u_1,u_2} > 0 \} \) and \( A_1^3 := \{ u_1 \in A_1 : (\forall u_2 \in A_2) \mu_{u_1,u_2} = 0 \} \). Similarly, for \( A_2 \) we let \( A_2^1 := \{ u_2 \in A_2 : \beta_{v_2,u_2} = 0 \} \), \( A_2^2 := \{ u_2 \in A_2 : (\forall u_1 \in A_1) \lambda_{u_1,u_2} > 0 \} \) and \( A_2^3 := \{ u_2 \in A_2 : (\exists u_1 \in A_1) \lambda_{u_1,u_2} = 0 \} \).

If \( A_1^1 = \emptyset \) then it cannot be that also \( A_1^2 = \emptyset \), because \( \alpha_{v_1,u_1} > 0 \) for at least one \( u_1 \in A_1 \). So for each \( u_1 \in A_1^i \neq \emptyset \) we choose some \( u_{2,u_1} \in A_2 \) such that \( \mu_{u_1,u_{2,u_1}} > 0 \) and therefore \( \lambda_{u_{1,u_{2,u_1}}} > 0 \) (which is always possible, by the definition of \( A_1^i \)), and we let \( \kappa_{v_1,v_2} := 1 \) and \( \rho_{v_1,v_2} := 0 \). Then indeed

\[
\kappa_{v_1,v_2} v_1 + \rho_{v_1,v_2} v_2 = \sum_{u_1 \in A_1^i} \alpha_{v_1,u_1} u_1 + \sum_{u_1 \in A_1^i} \frac{\alpha_{v_1,u_1}}{\lambda_{u_1,u_{2,u_1}}} (\lambda_{u_1,u_{2,u_1}} u_1 + \mu_{u_1,u_{2,u_1}} v_2) \in \text{posi}(A).
\]

A completely symmetrical argument can be made when \( A_2^3 = \emptyset \), so we may now assume without loss of generality that both \( A_1^2 \neq \emptyset \) and \( A_2^3 \neq \emptyset \). Then, as before, for any \( u_1 \in A_1^2 \) we choose some \( u_{2,u_1} \in A_2 \) such that \( \mu_{u_1,u_{2,u_1}} > 0 \) and therefore \( \lambda_{u_{1,u_{2,u_1}}} > 0 \) (possible by the definition of \( A_1^2 \)), and for any \( u_2 \in A_2^3 \), we choose some \( u_{1,u_2} \in A_1 \) such that \( \lambda_{u_{1,u_2}} = 0 \) and therefore \( \mu_{u_{1,u_2}} > 0 \) (possible by the definition of \( A_2^3 \)). We also let

\[
w_1 := \sum_{u_1 \in A_1^2} \alpha_{v_1,u_1} u_1 \quad \text{and} \quad w'_1 := \sum_{u_1 \in A_1^2} \alpha_{v_1,u_1} u_1, \quad \text{so} \quad v_1 = w_1 + w'_1,
\]

and

\[
w_2 := \sum_{u_2 \in A_2^3} \beta_{v_2,u_2} u_2 \quad \text{and} \quad w'_2 := \sum_{u_2 \in A_2^3} \beta_{v_2,u_2} u_2, \quad \text{so} \quad v_2 = w_2 + w'_2.
\]

Then we already know that, by a similar argument as before:

\[
w'_1 = \sum_{u_1 \in A_1^2} \frac{\alpha_{v_1,u_1}}{\lambda_{u_1,u_{2,u_1}}} (\lambda_{u_1,u_{2,u_1}} u_1 + \mu_{u_1,u_{2,u_1}} u_2) \in \text{posi}(A) \quad (20)
\]

and

\[
w'_2 = \sum_{u_2 \in A_2^3} \beta_{v_2,u_2} u_2 \in \text{posi}(A) \quad (21)
\]

Next, recall that \( \lambda_{u_{1,u_{2,u_1}}} > 0 \) and \( \mu_{u_1,u_{2,u_1}} > 0 \) for any \( u_1 \in A_1 \) and \( u_2 \in A_2 \). We then infer from \( \lambda_{u_{1,u_{2,u_1}}} + \mu_{u_1,u_{2,u_1}} \in A \) for all \( u_1 \in A_1^2 \) and \( u_2 \in A_2^3 \) that

\[
\phi_{v_1,u_1} + w_2 \in \text{posi}(A), \quad \text{with} \quad \phi_{v_1,u_1} := \sum_{u_2 \in A_2^3} \beta_{v_2,u_2} \frac{\lambda_{u_1,u_{2,u_1}}}{\mu_{u_1,u_{2,u_1}}} > 0,
\]

and therefore also

\[
w_1 + \psi_{v_1,v_1} w_2 \in \text{posi}(A), \quad \text{with} \quad \psi_{v_1,v_1} := \sum_{u_1 \in A_1^2} \frac{\alpha_{v_1,u_1}}{\phi_{v_1,u_1}} > 0. \quad (22)
\]

Now let \( \kappa_{v_1,v_2} := 1 \) and \( \rho_{v_1,v_2} := \psi_{v_1,v_2} \), then we infer from (20)–(22) that, indeed,

\[
\kappa_{v_1,v_2} v_1 + \rho_{v_1,v_2} v_2 = v_1 + \psi_{v_1,v_1} v_2 = w_1 + w'_1 + \psi_{v_1,v_2} (w_2 + w'_2) = w_1 + \psi_{v_1,v_2} w_2 + w'_1 + \psi_{v_1,v_2} w'_2 \in \text{posi}(A).
\]

To prove that, finally, \( \text{Rp}(K) \) satisfies \( K_4 \), consider any \( A_1 \in \text{Rp}(K) \) and any \( A_2 \in \mathcal{A} \) such that \( A_1 \subseteq A_2 \). This implies on the one hand that \( \text{posi}(A_1) \subseteq \text{posi}(A_2) \), and on the other hand that there is some \( B_1 \in K \) such that \( A_1 \subseteq B_1 \subseteq \text{posi}(A_1) \), and therefore also \( \text{posi}(A_1) = \text{posi}(B_1) \). But then \( \text{posi}(A_2 \setminus B_1) = \text{posi}(A_2 \cup \text{posi}(B_1)) = \text{posi}(A_2) \) and therefore \( A_2 \subseteq A_2 \cup B_1 \subseteq \text{posi}(A_2) \). Since, moreover, \( B_1 \in K \) and \( B_1 \subseteq A_2 \cup B_1 \), \( K_4 \) guarantees that \( A_2 \cup B_1 \in K \), and therefore indeed \( A_2 \in \text{Rp}(K) \). \( \square \)
Proposition 49. Any non-binary mixing set of desirable option sets \( K \) is strictly dominated by some other mixing set of desirable option sets.

Proof. Consider an arbitrary mixing non-binary set of desirable option sets \( K \). Since \( K \) is non-binary and coherent, Lemma 40 guarantees the existence of a coherent set of desirable option sets \( K' \) such that \( K \subseteq K' \). Let \( K^* := \text{Rp}(K') \). Since \( K' \) is coherent, we know from Proposition 48 that \( K^* \) is mixing. Since \( \text{Rp} \) does not remove option sets, we also know that \( K' \subseteq K^* \). Since \( K \subseteq K^* \), this implies that \( K \) is strictly dominated by the mixing set of desirable option sets \( K^* \).

\[ \□ \]

Lemma 50. For any non-empty chain \( \mathcal{K} \) in \( \mathcal{K}_M \), its union \( K_o := \bigcup \mathcal{K} \) is a mixing set of desirable option sets.

Proof. Since \( \mathcal{K}_M \subseteq \mathcal{K} \), it follows from Lemma 41 that \( K_o \) is coherent. To prove that it is also mixing, we consider any \( A, B \in \mathcal{D} \) such that \( B \in K_o \) and \( A \subseteq B \subseteq \text{posi}(A) \). Since \( B \in K_o \), there is some \( K' \in \mathcal{K} \) such that \( B \in K' \), and since \( K' \) is mixing, this implies that \( A \in K' \subseteq K_o \).

\[ \□ \]

Lemma 51. For any mixing set of desirable option sets \( K \in \mathcal{K}_M \) and any set of desirable option sets \( K^* \in \mathcal{K} \) such that \( K \cap K^* = \emptyset \), the partially ordered set

\[ \uparrow K_M := \{ K' \in \mathcal{K}_M : K \subseteq K' \text{ and } K' \cap K^* = \emptyset \} \]

has a maximal element.

Proof. We will use Zorn’s Lemma to establish the existence of a maximal element. So consider any non-empty chain \( \mathcal{K} \) in \( \uparrow K_M \), then we must prove that \( \mathcal{K} \) has an upper bound in \( \uparrow K_M \). Since \( K_o := \bigcup \mathcal{K} \) is clearly an upper bound, we are done if we can prove that \( K_o \in \uparrow K_M \).

That \( K_o \cap K^* = \emptyset \) follows from the fact that \( K' \cap K^* = \emptyset \) for every \( K' \in \mathcal{K} \subseteq \uparrow K_M \). That \( K_o \) is mixing follows from Lemma 50.

\[ \□ \]

Proposition 52. Every mixing set of desirable option sets \( K \in \mathcal{K}_M \) is dominated by some binary mixing set of desirable option sets.

Proof. Lemma 51 for \( K^* = \emptyset \) guarantees that the partially ordered set \( \{ K' \in \mathcal{K}_M : K \subseteq K' \} \) has a maximal element. Let \( K_o \in \mathcal{K}_M \) be such maximal element. Assume \textit{ex absurdo} that \( K_o \) is non-binary. It then follows from Proposition 49 that \( K_o \) is strictly dominated by some other mixing set of desirable option sets, meaning that there is some \( K^* \in \mathcal{K}_M \) such that \( K_o \subseteq K^* \). Then clearly also \( K^* \in \{ K' \in \mathcal{K}_M : K \subseteq K' \} \), which contradicts the maximal character of \( K_o \). Hence, it must be that \( K_o \) is indeed binary.

\[ \□ \]

Theorem 53 (Representation for mixing choice functions). Any mixing set of desirable option sets \( K \in \mathcal{K}_M \) is dominated by some binary mixing set of desirable option sets:

\[ \mathcal{D}_M(K) := \{ D \in \mathcal{D}_M : K \subseteq K_D \} \neq \emptyset \]. Moreover, \( K = \bigcap \{ K_D : D \in \mathcal{D}_M(K) \} \).

Proof of Theorem 53. Let \( K_o \in \mathcal{K}_M \) be any mixing set of desirable option sets. We prove that \( \mathcal{D}_M(K_o) := \{ D \in \mathcal{D}_M : K_o \subseteq K_D \} \neq \emptyset \) and that \( K_o = \bigcap \{ K_D : D \in \mathcal{D}_M(K_o) \} \).

For the first statement, recall from Proposition 52 that \( K_o \) is dominated by some binary mixing coherent set of desirable option sets \( K \). Since \( K \) is binary, Proposition 6 implies that \( \hat{K} \subseteq \bigcup \mathcal{K} \). Furthermore, because \( K \) is mixing, Proposition 18 implies that \( D \hat{K} \) is mixing too, whence \( D \hat{K} \in \mathcal{D}_M \). Since \( K_o \subseteq \bigcup \mathcal{K} \), we find that \( D \hat{K} \in \mathcal{D}_M(K_o) \).

For the second statement, it is obvious that \( K_o \subseteq \bigcap \{ K_D : D \in \mathcal{D}_M(K_o) \} \), so we concentrate on the converse inclusion. Assume \textit{ex absurdo} that \( K_o \subseteq \bigcap \{ K_D : D \in \mathcal{D}_M(K_o) \} \), so there is some option set \( B_o \in \mathcal{D} \) such that \( B_o \notin K_o \) and \( B_o \in K_D \) for all \( D \in \mathcal{D}_M(K_o) \). Then \( B_o \in \mathcal{Y}_{<o} \subseteq K_o \) [use the coherence of \( K_o \) and \( K_o \)] and \( B_o \in \mathcal{Y}_{<o} \subseteq K_D \) for all \( D \in \mathcal{D}_M(K) \) [use the coherence of \( K_D \)]—which follows from Proposition 32 and the coherence of \( D \)—and
Proposition 36], so we may assume without loss of generality that $B_o$ has no non-positive options: $B_o \cap \mathcal{Y}_{\geq 0} = \emptyset$. Moreover, if $B_o$ contains some option $w$ that is a positive linear combination of other elements of $B_o$, then still $B_o \setminus \{w\} \notin \mathcal{K}_o$ [use the coherence of $K_o$ and $K_4$] and $B_o \setminus \{w\} \in \mathcal{D}_M(K_o)$ [use the coherence of $K_0$ and $K_M$], so we may assume without loss of generality that $B_o$ has no elements that are positive linear combinations of some of its other elements.

Consider the partially ordered set $\uparrow K_o := \{K \in \overline{\mathcal{K}}_M: K_o \subseteq K \text{ and } B_o \notin K\}$, which is non-empty because it contains $K_o$. Due to to Lemma 51 [applied for $K_o$ and $K^* = \{B_o\}$], it has a maximal element. If we can prove that any such maximal element is binary, then it will follow from Propositions 6 and 18 that there is some mixing set of desirable options $D_o \in \overline{\mathcal{D}}_M$ such that $K_o \subseteq K_{D_o}$ and therefore $D_o \in \overline{\mathcal{D}}_M(K_o)$—and therefore $B_o \notin K_{D_o}$, a contradiction.

To prove that all maximal elements of $\uparrow K_o$ are binary, it suffices to prove that any non-binary element of $\uparrow K_o$ is strictly dominated in that set, which is what we now set out to do.

So consider any non-binary mixing coherent element $K$ of $\uparrow K_o$, so $K_o \subseteq K$ and $B_o \notin K$. It follows from Lemma 33 that there is some $A_o \in K$ such that $|A_o| \geq 2$ and $A_o \setminus \{u\} \notin K$ for all $u \in A_o$. The partially ordered set $\{A \in K: B_o \subseteq A\}$ contains $A_o \cup B_o$, and therefore has some minimal (undominating) element $B^*$ below it, so $B^* \in K$ and $B_o \subseteq B^* \subseteq A_o \cup B_o$.

Let us first summarise what we know about this minimal element $B^*$. It is impossible that $B^* \subseteq B_o$, because otherwise $B_o = B^* \in K$, a contradiction. Hence $B^* \setminus B_o \neq \emptyset$, so we can fix any element $u_o$ in $B^* \setminus B_o \subseteq A_o$. Then we know that $\{u_o\} \notin K$, because otherwise, since $A_o$ has at least one other element $v$ [because $|A_o| \geq 2$], we would have that $\{u_o\} \subseteq A \setminus \{v\}$ and therefore $A \setminus \{v\} \subseteq K$ by the coherence of $K$, a contradiction. And since also $B_o \subseteq B^* \setminus \{u_o\}$ but $B^* \setminus \{u_o\} \subseteq B^*$, we can conclude that $B^* \setminus \{u_o\} \notin K$, by the definition of a minimal element. We also know that $B^*$, and therefore also $B_o$, cannot contain any positive multiple $\lambda u_o$ of $u_o$ with $\lambda > 0$ and $\mu \neq 1$, because otherwise we could use Lemma 47 to remove $u_o$ from $B^* \in K$ and still be guaranteed that $B^* \setminus \{u_o\} \in K$. Since we know that $u_o \notin B_o$, we would then still have that $B_o \subseteq B^* \setminus \{u_o\}$, which would contradict the minimality of $B^*$ in the partially ordered set $\{A \in K: B_o \subseteq A\}$. And, finally, we know that $u_o \notin \text{pos}(B_o)$, because otherwise $u_o \in B^*$ would be a positive linear combination of elements of $B_o \subseteq B^*$ different from $u_o$ [because $u_o \notin B_o$], so we could use Proposition 48 and $K_M$ to make sure that still $B^* \setminus \{u_o\} \in K$, again contradicting the minimality of $B^*$ in the partially ordered set $\{A \in K: B_o \subseteq A\}$.

Consider now the set of desirable option sets $K^* := \text{Rp}(K^{**})$, with $K^{**} := \text{Rn}(K^{***})$, where

$$K^{***} := \left\{ \{\lambda_v v + \mu_u u_o: v \in B\} : B \in K, (\forall v \in B)(\lambda_v, \mu_u) > 0 \right\}.$$ 

We know from Lemma 39 [with $K^{***}$ in this proof taking on the role of $K^{**}$ in the statement of Lemma 39, and with $K^{**}$ in this proof taking on the role of $K^*$ in the statement of Lemma 39] that $K^{**}$ is a coherent set of desirable option sets that is a superset of $K$ and contains $\{u_o\}$. Proposition 48 now guarantees that $K^* = \text{Rp}(K^{**})$ satisfies $K_0 - K_4$ and $K_M$, and is therefore mixing. Furthermore, since $\text{Rp}$ never removes option sets, $K^*$ is a superset of $K$ that contains $\{u_o\}$. Since we know that $\{u_o\} \notin K$, this shows that $K \subseteq K^*$. If we can now prove that $B_o \notin K^*$ and therefore $K^* \in \uparrow K_o$, we are done, because then $K$ is strictly dominated by $K^*$ in $\uparrow K_o$.

Assume therefore ex absurd that $B_o \in K^* = \text{Rp}(\text{Rn}(K^{***}))$, meaning that there are $C \in K, (\lambda_v, \mu_u) > 0$ for all $v \in C$ and $B \in \mathcal{S}$ such that

$$\{\lambda_v v + \mu_u u_o: v \in C\} \setminus \mathcal{Y}_{\geq 0} \subseteq B \subseteq \{\lambda_v v + \mu_u u_o: v \in C\} \text{ and } B_o \subseteq B \subseteq \text{pos}(B_o).$$

(23)

We let $C_2 := \{v \in C: \lambda_v v + \mu_u u_o \leq 0\}$, and consider any $v \in C_2$, so $\lambda_v v + \mu_u u_o \leq 0$. But then we must have that $\lambda_v > 0$, because otherwise $\mu_u u_o \leq 0$, with $\mu_v > 0$, and therefore also $u_o \leq 0$. But then Lemma 35 and Axiom $K_0$ would imply that we can remove the non-positive $u_o$ from $A_o \in K$ and guarantee that still $A_o \setminus \{u_o\} \in K$, which contradicts our
assumptions about $A_n$. So we find that
\[ v \preceq \frac{\mu_v}{\lambda_v} b_v \text{ for all } v \in C_2. \]

If we also call $C_1 := \{ v \in C : \lambda_v v + \mu_v u_0 \in B_0 \}$, then we infer from (23) that $C_1 \neq \emptyset$. Also, $C_1 \cap C_2 = \emptyset$, because we know that $B_0 \cap \mathcal{Y}_{\geq 0} = \emptyset$. We complete the partition of $C$ by letting $C_3 := C \setminus (C_1 \cup C_2)$. We know that

for each $b \in B$ there are $v_b \in C$ such that $b = \frac{\lambda_v}{\lambda_v} v_b + \mu_v u_0$. 

This means that for any $v \in C_1 \cup C_2$ there is by construction a necessarily unique $b \in B$ such that $v = v_b$, which we will denote by $b_v$. Then $b_v = \lambda_v v + \mu_v u_0$. But then $\lambda_v > 0$, because otherwise we would have that $\mu_v u_0 = b_v \in B$, with $\mu_v > 0$, so (23) would imply that $u_0 \in \text{posi}(B_0)$, and we have argued above that this is impossible. So we find that

\[ v = \frac{1}{\lambda_v} b_v - \frac{\mu_v}{\lambda_v} u_0 \text{ for all } v \in C_1 \cup C_3. \]

Let $B_3 := \{ b_v : v \in C_3 \}$, then it follows from our construction that $B_3 \subseteq \text{posi}(B_0) \setminus B_0$. 

Since $(C_1 \cup C_3) \cup C_2 = C \in K$, we infer from Lemma 34 that if we construct the option set $C'$ as follows

\[ C' := \left\{ \frac{1}{\lambda_v} b_v - \frac{\mu_v}{\lambda_v} u_0 : v \in C_1 \cup C_3 \right\} \cup \left\{ \mu_v u_0 : v \in C_2 \right\}, \]

then still $C' \in K$. If we recall that also $B^* \in K$, we can invoke $K_3$ to find that $\text{Posi}(B^*, C') \subseteq K$. In other words, we find that

\[ \{ \alpha_{u,v} u + \beta_{u,v} v : u \in B^* \setminus \{ u_0 \} \} \subseteq K \text{ for all choices of } (\alpha_{u,v}, \beta_{u,v}) > 0. \]

If we now choose $(\alpha_{u,v}, \beta_{u,v}) := (\mu_v, \lambda_v)$ and $(\alpha_{u,v}, \beta_{u,v}) := (1, 0)$ for all $u \in B^* \setminus \{ u_0 \}$, for all $v \in C'$, then we find in particular that

\[ \{ b_v : v \in C_1 \cup C_3 \} \cup \{ 0 : v \in C_2 \} \cup \{ u : u \in B^* \setminus \{ u_0 \} \} \]

\[ = B_0 \cup B_3 \cup \{ 0 : v \in C_2 \} \cup B^* \setminus \{ u_0 \} = B_3 \cup B^* \setminus \{ u_0 \} \cup \{ 0 : v \in C_2 \} \]

belongs to $K$. If $C_2 = \emptyset$, we find that $B_3 \cup B^* \setminus \{ u_0 \} \subseteq K$. If $C_2 \neq \emptyset$, we find that $\{ 0 \} \cup B_3 \cup B^* \setminus \{ u_0 \} \subseteq K$, and then we can still derive from $K_0$ that $B_3 \cup B^* \setminus \{ u_0 \} \subseteq K$. Any $b \in B_3$ that does not belong to $B^* \setminus \{ u_0 \}$ is a positive linear combination of elements of $B_0$, and therefore of $B^* \setminus \{ u_0 \}$. Proposition 48 and $K_M$ then tell us that we can remove such $b$ and still be guaranteed that the result belongs to $K$. This tells us that $B^* \setminus \{ u_0 \} \subseteq K$, a contradiction. 

\[ \square \]

Proof of Theorem 19. First assume that $K$ is mixing. It then follows from Theorem 53 that $\mathcal{D}_M(K) := \{ D \in \mathcal{D}_M : K \subseteq K_D \} \neq \emptyset$ and $K = \bigcap\{ K_D : D \in \mathcal{D}_M(K) \}$. This clearly implies that there is at least one non-empty set $\mathcal{D} \subseteq \mathcal{D}_M$ of mixing sets of desirable options such that $K = \bigcap\{ K_D : D \in \mathcal{D} \}$: the set $\mathcal{D}_M(K)$. Furthermore, for any non-empty set $\mathcal{D} \subseteq \mathcal{D}_M$ of mixing coherent sets of desirable options such that $K = \bigcap\{ K_D : D \in \mathcal{D} \}$, we clearly have that $K \subseteq K_D$ for all $D \in \mathcal{D}$. Since $\mathcal{D} \subseteq \mathcal{D}_M$, this implies that $\mathcal{D} \subseteq \mathcal{D}_M(K)$. So $\mathcal{D}_M(K)$ is indeed the largest such set.

It remains to prove the ‘if’ part. So consider any non-empty set $\mathcal{D} \subseteq \mathcal{D}_M$ of mixing sets of desirable options such that $K = \bigcap\{ K_D : D \in \mathcal{D} \}$. For every $D \in \mathcal{D} \subseteq \mathcal{D}_M$, it then follows from Proposition 18 that $K_D$ is mixing. Because Axioms $K_0$-$K_4$ and $K_M$ are trivially preserved under taking arbitrary (non-empty) intersections, this implies that $K$ is, indeed, also mixing.

\[ \square \]

Proof of Proposition 20. First assume that $D$ is mixing. Since $D^c$ is trivially a subset of $\text{posi}(D^c)$, we only need to prove that $\text{posi}(D^c) \subseteq D^c$. So consider any $u \in \text{posi}(D^c)$ and assume ex absurdo that $u \notin D^c$, so $u \in D$. Since $u \in \text{posi}(D^c)$, we know that $u$ is positive linear combination of a finite number of elements of $D^c$. Hence, there is some (finite)
option set \( A \in \mathcal{D} \) such that \( A \subseteq D^* \) and \( u \in \text{posi}(A) \). Since we have assumed that \( u \in D \), this implies that \( \text{posi}(A) \cap D \neq \emptyset \), so \( A \cap D \neq \emptyset \) because \( D \) satisfies \( D_M \) by assumption. Since \( A \subseteq D^* \), we find that \( D^* \cap D \neq \emptyset \), a contradiction. So it must be that \( u \in D^* \), and therefore, indeed, \( \text{posi}(D^*) \subseteq D^* \).

Conversely, assume that \( \text{posi}(D^*) = D^* \), then we need to prove that \( D \) satisfies \( D_M \). So consider any \( A \in \mathcal{D} \) such that \( \text{posi}(A) \cap D \neq \emptyset \) and assume \textit{ex absurdo} that \( A \cap D = \emptyset \). Then \( A \subseteq D^* \) and therefore also \( \text{posi}(A) \subseteq \text{posi}(D^*) = D^* \). So we find that \( \text{posi}(A) \cap D = \emptyset \), a contradiction.

\[ \square \]

**Proof of Proposition 21.** Consider any total set of desirable options \( D \), and any \( A \in \mathcal{D} \) such that \( A \cap D = \emptyset \). If we can show that then also \( \text{posi}(A) \cap D = \emptyset \), if will clearly follow from Definition 17 that \( D \) is indeed mixing.

So consider any \( u \in \text{posi}(A) \), then we need to prove that \( u \notin D \). Since \( u \in \text{posi}(A) \), \( u \) is a finite positive linear combination of elements of \( A \), meaning that \( u = \sum_{i=1}^{n} \lambda_i u_i \), with \( n \in \mathbb{N} \) and, for all \( i \in \{1, \ldots, n\} \), \( \lambda_i > 0 \) and \( u_i \in A \). Let \( I := \{ i \in \{1, \ldots, n\} : u_i \neq 0 \} \). Then for all \( i \in I \), since \( u_i \in A \), \( A \cap D = \emptyset \), and \( u_i \neq 0 \), it follows from the totality of \( D \) that \( -u_i \notin D \). We now consider two cases: \( I = \emptyset \) and \( I \neq \emptyset \). If \( I = \emptyset \) then \( u = 0 \), which, since \( D \) is in particular also coherent [use Axiom \( D_1 \)], implies that, indeed, \( u \notin D \). If \( I \neq \emptyset \) then \(-u = \sum_{i \in I} \lambda_i (-u_i) = \sum_{i \notin I} \lambda_i (-u_i) \) is a finite positive linear combination of elements of \( D \), and therefore follows from the coherence of \( D \{A, D_1\} \) that \( -u \notin D \). This implies that, indeed, \( u \notin D \), because otherwise, it would follow from Axiom \( D_3 \) that \( 0 = u - u \in D \), contradicting Axiom \( D_1 \).

\[ \square \]

### A.7. Proofs and Intermediate Results for Section 9

Various proofs in this section make use of the \( \text{arch}(\cdot) \) operator, defined for any set of desirable options \( D \in \mathbf{D} \) by

\[
\text{arch}(D) := \{ u \in D : (\exists \varepsilon \in \mathbb{R}_{>0}) u - \varepsilon \in D \},
\]

where the identification of \( \varepsilon \in \mathbb{R}_{>0} \) with an option in \( \mathcal{Y} \) is justified by the additional assumptions on \( \mathcal{Y} \) that we impose in Section 9.

**Proof of Proposition 24.** First consider any Archimedean (and hence also coherent) set of desirable options \( D \). We will prove that \( \mathcal{P}_D \) is a coherent lower prevision on \( \mathcal{L}(\mathcal{X}) \), that \( D_{\mathcal{P}_D} = D \), and that if \( D \) is moreover mixing, then \( \mathcal{P}_D \) is a linear prevision.

We begin by showing that \( \mathcal{P}_D \) is a coherent lower prevision on \( \mathcal{L}(\mathcal{X}) \). Consider any \( u \in \mathcal{L}(\mathcal{X}) \) and any \( u \in \mathcal{L}(\mathcal{X}) \) such that \( u < \inf u \). Since \( D \) is coherent and \( \mathcal{Y}_{0} = \mathcal{L} = \{ u \in \mathcal{L}(\mathcal{X}) : \inf u > 0 \} \), we find that \( u - \inf u \notin D \), because \( \inf (u - \inf u) = \inf u - \inf u > 0 \). Since this is true for any \( u \in \mathcal{L}(\mathcal{X}) \) such that \( u < \inf u \), we infer from Equation (11) that \( \mathcal{P}_D(u) \geq \inf u \).

So \( \mathcal{P}_D \) satisfies LP1. Next, we prove that \( \mathcal{P}_D \) is a real-valued map. To that end, consider again any \( u \in \mathcal{L}(\mathcal{X}) \). Since \( u \) is bounded, \( \inf u \) and \( \sup u \) are real. Consider any \( \mu \in \mathbb{R} \) such that \( \mu > \sup u \). Then \( \inf (\mu - u) = \mu - \sup u > 0 \), so \( \mathcal{D}_2 \) implies that \( \mu - u \notin D \). Hence, \( u - \mu \notin D \), because otherwise it would follow from \( D_3 \) that \( 0 = (u - \mu) + (\mu - u) \in D \), which contradicts \( D_1 \). Since this is true for any \( \mu \in \mathbb{R} \) such that \( \mu > \sup u \), Equation (11) implies that \( \mathcal{P}_D(u) \leq \sup u \). On the other hand, we know from LP1 that \( \mathcal{P}_D(u) \geq \inf u \). Since \( \inf u \) and \( \sup u \) are real, we find that \( \mathcal{P}_D(u) \) is a real-valued map. It remains to show that \( \mathcal{P}_D \) satisfies LP2 and LP3. We start with LP2. Consider any \( u \in \mathcal{L}(\mathcal{X}) \) and \( u \in \mathbb{R}_{>0} \). For all \( \mu \in \mathbb{R} \), it then follows from the coherence of \( D \) that \( u - \mu \notin D \) if and only if \( \mu u - \mu u \notin D \). Equation (11) therefore implies that \( \mu \mathcal{P}_D(\mu u) = \mu \mathcal{P}_D(u) \). So \( \mathcal{P}_D \) satisfies LP2. Next, for LP3, consider any \( u, v \in \mathcal{L}(\mathcal{X}) \). Fix any \( \varepsilon \in \mathbb{R}_{>0} \). It then follows from Equation (11) that there is some \( \mu u \in \mathbb{R} \) such that \( \mu u > \mathcal{P}_D(u) - \varepsilon \) and \( u - \mu u \in D \), and similarly, that there is some \( \mu v \in \mathbb{R} \) such that \( \mu v > \mathcal{P}_D(v) - \varepsilon \) and \( u - \mu v \in D \). The coherence of \( D \) now implies that \( u + v - \mu u - \mu v \in D \), so Equation (11) implies that \( \mathcal{P}_D(u + v) \geq \mu u + \mu v > \mathcal{P}_D(u) + \mathcal{P}_D(v) - 2\varepsilon \). Since this is true for every \( \varepsilon \in \mathbb{R}_{>0} \), we infer that \( \mathcal{P}_D(u + v) \geq \mathcal{P}_D(u) + \mathcal{P}_D(v) \). Hence, \( \mathcal{P}_D \) satisfies LP3.
Next, we prove that $D_{P_D} = D$. First consider any $u \in D$. Since $D$ is Archimedean, there is then some $\varepsilon \in \mathbb{R}_{>0}$ such that $u - \varepsilon \in D$. Combined with Equation (11), this implies that $P_D(u) \geq u$. So we infer from Equation (12) that $u \in D_{P_D}$. Hence, $D \subseteq D_{P_D}$.

Consider now any $u \in D_{P_D}$. Equation (12) then implies that $P_D(u) > 0$, so we know from Equation (11) that there is some $\mu \in \mathbb{R}_{>0}$ such that $u - \mu \in D$. Therefore, and because $\mu \in L_{sp}(\mathcal{X})$, it follows from the coherence of $D$ [D[2] and D[3]] that $u = (u - \mu) + \mu \in D$. Hence, $D_{P_D} \subseteq D$. We conclude that $D_{P_D} = D$.

Suppose now that $D$ is also mixing. We will prove that $P_D$ is then a linear prevision. Since we already know that $P_D$ is a coherent lower prevision, it suffices to prove that $P_D$ satisfies P[3]. So consider any $u, v \in L(\mathcal{X})$. We only need to prove that $P_D(u + v) \leq P_D(u) + P_D(v)$ because the converse inequality follows from LP[3]. Fix any $\mu \in \mathbb{R}$ such that $\mu > P_D(u) + P_D(v)$ and let $\varepsilon := \frac{1}{2}(\mu - P_D(u) - P_D(v)) > 0$. It then follows from Equation (11) that $u_\varepsilon := u - (P_D(u) + \varepsilon) \notin D$ and $v_\varepsilon := v - (P_D(v) + \varepsilon) \notin D$. So if we let $A := \{u_\varepsilon, v_\varepsilon\}$, then $A \cap D = \emptyset$. Therefore, and since $A \in \mathcal{E}$ and $\{u_\varepsilon, v_\varepsilon, u_\varepsilon + v_\varepsilon\} \in \text{posi}(A)$, it follows from the mixingness of $D$ that $\{u_\varepsilon, v_\varepsilon, u_\varepsilon + v_\varepsilon\} \cap D = \emptyset$. Hence, we find that

$$u + v - \mu = u + v - P_D(u) - P_D(v) - 2\varepsilon = u_\varepsilon + v_\varepsilon \notin D.$$ 

Since this holds for every $\mu \in \mathbb{R}$ such that $\mu > P_D(u) + P_D(v)$, it follows from Equation (11) that $P_D(u + v) \leq P_D(u) + P_D(v)$, as desired.

For the second part of this proposition, we consider any coherent lower prevision $P$ on $L(\mathcal{X})$. We will prove that $D_P$ is an Archimedean set of desirable options, that $P_{P_{D_P}} = P$, and that if $P$ is furthermore a linear prevision, then $D_P$ is mixing.

We start by proving that $D_P$ is Archimedean, meaning that it satisfies D[1]–D[3] and D[\lambda]. Since LP[3] implies that $P(0) + P(0) \leq P(0)$, and since $P$ is real-valued, we know that $P(0) \leq 0$, so Equation (12) implies that $0 \in D_P$. Hence, $D_P$ satisfies D[1]. For D[2], it suffices to observe that for any $u \in \mathcal{Y}_{>0} = \mathcal{L}_{sp}(\mathcal{X})$, $P(u) \geq \inf u > 0$ because of LP[1], so $u \in D_P$ because of Equation (12). Let us now prove D[3]. So consider any $u, v \in D_P$ and any $(\lambda, \mu) > 0$. It then follows from Equation (12) that $P(u) > 0$ and $P(v) > 0$, and from LP[2] and LP[3], that $P(\lambda u + \mu v) \geq \lambda P(u) + \mu P(v)$. Since $(\lambda, \mu) > 0$, this implies that $P(\lambda u + \mu v) > 0$, which in turn implies that $\lambda u + \mu v \in D_P$ because of Equation (12). So $D_P$ satisfies D[3]. Let us now prove D[\lambda]. To that end, consider any $u \in D_P$, so $P(u) > 0$ because of Equation (12). Let $\varepsilon := \frac{1}{2}P(u) > 0$. Then since $P$ is a coherent lower prevision,

$$P(u - \varepsilon) \geq P(u) + P(-\varepsilon) = P(u) + \inf(-\varepsilon) = P(u) - \varepsilon = 2\varepsilon - \varepsilon = \varepsilon > 0,$$

so $u - \varepsilon \in D_P$ because of Equation (12). Hence, $D_P$ satisfies D[\lambda].

Next, we prove that $D_{P_{D_P}} = P$. Consider any $u \in L(\mathcal{X})$ and any $\mu \in \mathbb{R}$. Then on the one hand, if $\mu < P(u)$, it follows from LP[3] and LP[1] that

$$P(u - \mu) \geq P(u) + P(-\mu) \geq P(u) + \inf(-\mu) = P(u) - \mu > 0,$$

so Equation (12) implies that $u - \mu \in D_P$. On the other hand, if $\mu > P(u)$, it follows from LP[1] and LP[3] that

$$P(u - \mu) = P(u) + P(-\mu) \leq P(u - \mu) + P(\mu) = P(u) + \mu < 0,$$

so Equation (12) implies that $u - \mu \notin D_P$. We conclude from Equation (11) that $P_{D_{P_{D_P}}} = P$.

Suppose now that $P$ is furthermore a linear prevision. We will prove that $D_{P_{D_P}}$ is then mixing. Since we already know that $D_{P_{D_P}}$ is coherent, it suffices to prove that $D_{P_{D_P}}$ satisfies DM[3]. So consider any $A \in \mathcal{E}$ such that $\text{posi}(A) \cap D_{P_{D_P}} \neq \emptyset$, meaning that there is some $u \in \text{posi}(A)$ such that also $u \in D_{P_{D_P}}$. Since $u \in \text{posi}(A)$, it follows from Equation (10) that

$$u = \sum_{k=1}^{n} \lambda k u_k,$$

with $n \in \mathbb{N}$ and, for all $k \in \{1, \ldots, n\}$, $\lambda k \in \mathbb{R}_{>0}$ and $u_k \in A$. It therefore follows from LP[3] and LP[2] that

$$P(u) = P \left( \sum_{k=1}^{n} \lambda k u_k \right) \geq \sum_{k=1}^{n} \lambda k P(u_k).$$
Furthermore, since \( u \in D_F \), we know from Equation (12) that \( P(u) > 0 \). Hence, since \( \lambda_k > 0 \) for all \( k \in \{1, \ldots, n\} \), there must be some \( k^* \in \{1, \ldots, n\} \) such that \( P(u_{k^*}) > 0 \). Equation (12) then implies that \( u_{k^*} \in D_F \). Since \( u_{k^*} \in A \), this in turn implies that \( A \cap D_F \neq \emptyset \). We conclude that \( D_F \) satisfies \( D_M \), as desired.

\( \square \)

**Proof of Proposition 27.** We know from Proposition 8 that \( D \) is coherent if and only if \( K_D \) is, and from Proposition 18 tells us that \( D \) is mixing if and only if \( K_D \) is. So the only thing left to prove is that \( D \) satisfies \( D_A \) if and only if \( K_D \) satisfies \( K_A \).

First assume that \( D \) satisfies \( D_A \). Consider any \( A \in K_D \), meaning that \( A \cap D \neq \emptyset \). Consider any \( u \in A \cap D \). Since \( u \in D \) and \( D \) is Archimedean, we know that there is some \( \varepsilon \in \mathbb{R}_{>0} \) such that \( u - \varepsilon \in D \). Since \( u \in A \), we also know that \( u - \varepsilon \in A - \varepsilon \). We therefore find that \( u \in (A - \varepsilon) \cap D \), which implies that \( (A - \varepsilon) \cap D \neq \emptyset \), or equivalently, that, indeed \( A - \varepsilon \in K_D \).

Conversely, assume that \( K_D \) satisfies \( K_A \). Consider any \( u \in K_D \). Then \( \{u\} \in K_D \) and therefore, since \( K_D \) is Archimedean, there is some \( \varepsilon \in \mathbb{R}_{>0} \) such that \( \{u\} - \varepsilon \in K_D \). Since \( \{u\} - \varepsilon = \{u\} - \varepsilon \), this implies that, indeed, \( u - \varepsilon \in D \).

\( \square \)

**Lemma 54.** For any coherent set of desirable options \( D \in \overline{D} \), arch(\( D \)) is coherent.

**Proof.** Consider any coherent set of desirable options \( D \in \overline{D} \), arch(\( D \)), then we have to prove that arch(\( D \)) satisfies Axioms \( D_1 \)–\( D_3 \).

That arch(\( D \)) satisfies \( D_1 \) follows directly from the fact that \( D \) does, because arch(\( D \)) \( \subseteq \) \( D \). To prove that arch(\( D \)) satisfies \( D_2 \), we consider any \( u \in \mathcal{Y}_D \). Then \( \inf u > 0 \), so there is some \( \varepsilon \in \mathbb{R}_{>0} \) such that \( \inf(u - \varepsilon) > 0 \). Since \( D \) is coherent, this implies that \( u - \varepsilon \in \mathcal{Y}_D \), and therefore that \( u \in \operatorname{arch}(D) \). To prove that arch(\( D \)) satisfies \( D_3 \), we consider any \( u, v \in \operatorname{arch}(D) \) and \( \lambda, \mu \geq 0 \) such that \( \lambda + \mu > 0 \), and prove that \( \lambda u + \mu v \in \operatorname{arch}(D) \).

Since \( u, v \in \operatorname{arch}(D) \), there are \( \varepsilon_u, \varepsilon_v \in \mathbb{R}_{>0} \) such that \( u - \varepsilon_u \in D \) and \( v - \varepsilon_v \in D \), from which, since \( D \) satisfies \( D_3 \), implies that \( \lambda u + \mu v - (\lambda \varepsilon_u + \mu \varepsilon_v) = \lambda (u - \varepsilon_u) + \mu (v - \varepsilon_v) \in D \).

Since \( \lambda \varepsilon_u + \mu \varepsilon_v \in \mathbb{R}_{>0} \), this implies that \( \lambda u + \mu v \in \operatorname{arch}(D) \).

\( \square \)

**Lemma 55.** For any coherent set of desirable options \( D \in \overline{D} \), arch(\( D \)) is Archimedean.

**Proof.** Due to Lemma 54, we already know that arch(\( D \)) is coherent, so it only remains to prove that arch(\( D \)) satisfies \( D_A \). Consider any \( u \in \operatorname{arch}(D) \). This implies that \( u \in D \) and that there is some \( \varepsilon \in \mathbb{R}_{>0} \) such that \( u - \varepsilon \in D \). Since \( u - \varepsilon \in \mathcal{Y}_D \) and \( u - \varepsilon \in D \), the coherence of \( D \) implies that \( u - \frac{1}{\varepsilon} \varepsilon = (u - \varepsilon) + \frac{1}{\varepsilon} \varepsilon \in D \). Since \( u - \frac{1}{\varepsilon} \varepsilon = u - \varepsilon \in D \) and \( \frac{1}{\varepsilon} \varepsilon \in \mathbb{R}_{>0} \), this implies that \( u - \frac{1}{\varepsilon} \varepsilon \in \operatorname{arch}(D) \). Since this is true for every \( u \in \operatorname{arch}(D) \), we conclude that arch(\( D \)) is indeed Archimedean.

\( \square \)

**Lemma 56.** For any mixing set of desirable options \( D \in \overline{D}_M \), arch(\( D \)) is mixing.

**Proof.** Consider any mixing set of desirable options \( D \in \overline{D}_M \). Since \( D \) is in particular coherent, we infer from Lemma 54 that arch(\( D \)) is coherent too, so we only need to prove that arch(\( D \)) satisfies \( D_M \).

So consider any \( A \in \mathcal{D} \) such that \( \operatorname{posi}(A) \cap \operatorname{arch}(D) \neq \emptyset \). Then there is at least one \( u \in \operatorname{posi}(A) \) such that \( u \in \operatorname{arch}(D) \). Fix any such \( u \). Since \( u \in \operatorname{arch}(D) \), there is some \( \varepsilon \in \mathbb{R}_{>0} \) such that \( u - \varepsilon \in D \). Fix any such \( \varepsilon \). Since \( u \in \operatorname{posi}(A) \), we know that \( u = \sum_{i=1}^{n} \lambda_i u_i \), with \( n \in \mathbb{N} \) and, for all \( i \in \{1, \ldots, n\} \), \( \lambda_i > 0 \) and \( u_i \in A \). Let \( \lambda := \sum_{i=1}^{n} \lambda_i > 0 \) and \( \alpha := 1/\lambda > 0 \).

Then

\[
    u - \varepsilon = \sum_{i=1}^{n} \lambda_i u_i - \alpha \left( \sum_{i=1}^{n} \lambda_i \right) \varepsilon = \sum_{i=1}^{n} \lambda_i (u_i - \alpha \varepsilon),
\]

so \( u - \varepsilon \in \operatorname{posi}(A - \alpha \varepsilon) \). Since also \( u - \varepsilon \in D \), it follows that \( \operatorname{posi}(A - \alpha \varepsilon) \cap D \neq \emptyset \).

Because \( D \) is by assumption mixing, we find that \( (A - \alpha \varepsilon) \cap D \neq \emptyset \), meaning that there is some \( \tilde{u} \in A \) such that \( \tilde{u} - \alpha \varepsilon \in D \). Since \( \alpha \varepsilon \in \mathbb{R}_{>0} \), we conclude that \( \tilde{u} \in \operatorname{arch}(D) \), so, indeed, \( A \cap \operatorname{arch}(D) \neq \emptyset \).

\( \square \)
Lemma 57. For any Archimedean set of desirable options \( D \in \mathbf{D}_A \), \( \text{arch}(D) = D \).

Proof. Since \( \text{arch}(D) \) is trivially a subset of \( D \), we only need to prove that \( D \subseteq \text{arch}(D) \). So consider any \( u \in D \). Then since \( D \) is Archimedean, there is some \( \varepsilon \in \mathbb{R}_{>0} \) such that \( u - \varepsilon \in D \), which implies that \( u \in \text{arch}(D) \). Since this is true for every \( u \in D \), we conclude that \( D \subseteq \text{arch}(D) \). \( \square \)

Lemma 58. Consider any non-empty set \( \mathcal{D} \subseteq \mathbf{D} \) of coherent sets of desirable options such that \( K := \bigcap \{ K_D : D \in \mathcal{D} \} \) is Archimedean. Then also \( K = \bigcap \{ \text{arch}(D) : D \in \mathcal{D} \} \).

Proof. For any \( D \in \mathcal{D} \), since \( \text{arch}(D) \subseteq D \), it follows that \( \text{arch}(D) \subseteq K_D \). Hence, \( \bigcap \{ \text{arch}(D) : D \in \mathcal{D} \} \subseteq \bigcap \{ K_D : D \in \mathcal{D} \} = K \), so it remains to prove the converse set inclusion. To this end, consider any \( D \in \mathcal{D} \) and any \( A \in K \). Then since \( K \) is Archimedean, we know that there is some \( \varepsilon \in \mathbb{R}_{>0} \) such that \( A - \varepsilon \in K \subseteq K_D \). Fix any such \( \varepsilon \). \( A - \varepsilon \in K_D \) implies that there is some \( u \in A \) such that \( u - \varepsilon \in D \), by Equation (8). Since \( D \) is coherent [use Axiom D1] and also \( e \in D \) [use \( e \in \mathcal{D} \) and Axiom D2], this in turn implies that \( u \in D \). Since also \( u - \varepsilon \in D \), it follows that \( u \in \text{arch}(D) \) and therefore, that \( A \cap \text{arch}(D) \neq \emptyset \). Hence, \( A \in \text{arch}(D) \), again by Equation (8). Since this is true for every \( D \in \mathcal{D} \) and any \( A \in K \), we find that, as desired, \( K \subseteq \bigcap \{ \text{arch}(D) : D \in \mathcal{D} \} \). \( \square \)

Proof of Proposition 26. First assume that \( K \) is Archimedean. Then \( K \) is in particular coherent, and therefore so is \( R \), by Proposition 4. It therefore remains to prove that \( R \) satisfies \( R_A \). Consider, to this end, any \( A \in \mathcal{D} \) and \( u \in \mathcal{A} \) such that \( u \in R(A \cup \{ u \}) \). Equation (5) then tells us that \( A - u \in K \). Since \( K \) is Archimedean, this implies that there is some \( \varepsilon \in \mathbb{R}_{>0} \) such that \( A - u - \varepsilon \in K \), so it follows from Equation (5) that, indeed, \( u \in R((A - \varepsilon) \cup \{ 0 \}) \).

Next, assume that \( R \) is coherent and satisfies \( R_A \). Then it already follows from Proposition 4 that \( K \) is coherent, so we only need to prove that \( K \) satisfies \( K_A \). Consider, to this end, any \( A \in K \). Equation (5) then tells us that \( 0 \in R(A \cup \{ 0 \}) \). Since \( R \) satisfies \( R_A \), this implies that there is some \( \varepsilon \in \mathbb{R}_{>0} \) such that \( 0 \in R((A - \varepsilon) \cup \{ 0 \}) \). Hence, it follows from Equation (5) that \( A - \varepsilon \in K \), as desired. \( \square \)

Proof of Theorem 28. First assume that \( K \) is Archimedean. It then follows from Theorem 9 that \( \overrightarrow{A}(K) := \{ D \in \mathbf{D}_A : K \subseteq K_D \} \neq \emptyset \) and \( K = \bigcap \{ K_D : D \in \mathbf{D}(K) \} \). Let \( \overrightarrow{A}(K) := \{ \text{arch}(D) : D \in \mathbf{D}(K) \} \neq \emptyset \), then Lemma 58 already guarantees that \( K = \bigcap \{ K_D : D \in \mathbf{D}_A(K) \} \).

We first prove that \( \overrightarrow{A}(K) = \overrightarrow{A}(K) \). To this end, first consider any \( D \in \mathbf{D}_A(K) \). Then \( K \subseteq K_D \) because \( K = \bigcap \{ K_D : D \in \mathbf{D}(K) \} \). Furthermore, it follows from Lemmas 54 and 55 that \( D \in \mathbf{D}_A(K) \). Hence \( D \in \mathbf{D}_A(K) \), and therefore \( \overrightarrow{A}(K) \subseteq \overrightarrow{A}(K) \). Conversely, consider any \( D \in \mathbf{D}_A(K) \). Since \( D \) is Archimedean, it follows from Lemma 57 that \( \text{arch}(D) = D \), and because also \( D \in \mathbf{D}_A(K) \), we find that \( D = \text{arch}(D) \in \mathbf{D}_A(K) \). So \( \overrightarrow{A}(K) \subseteq \overrightarrow{A}(K) \), and therefore, indeed, \( \overrightarrow{A}(K) = \overrightarrow{A}(K) \).

Next, we prove that \( \overrightarrow{A}(K) \) is closed. So consider any convergent sequence \( \{ D_n \}_{n \in \mathbb{N}} \) in \( \overrightarrow{A}(K) \) and let \( D_\infty \) be its limit. Then we need to prove that \( D_\infty \in \overrightarrow{A}(K) \). Equation (13) and Proposition 24 tell us that \( D_\infty = D_{\overrightarrow{P}} \), where \( P \) is the point-wise limit of the corresponding convergent sequence of coherent lower previsions \( \{ P_{D_n} \}_{n \in \mathbb{N}} \). Indeed, since for any \( n \in \mathbb{N} \), \( D_n \) is Archimedean, we know from Proposition 24 that \( P_{D_n} \) is a coherent lower prevision. Therefore, and because \( P \) is closed and compact in the (weak*) topology induced by point-wise convergence [11, 21], the limit \( P \) is a coherent lower prevision as well, so we infer from Proposition 24 that \( D_\infty = D_{\overrightarrow{P}} \) is Archimedean, so \( D_\infty \in \overrightarrow{A}(K) \). It therefore remains to prove that \( K \subseteq K_{D_\infty} \). Consider, to this end, any \( A \in K \). Since \( K \) is Archimedean, there is some \( \varepsilon \in \mathbb{R}_{>0} \) such that \( A - \varepsilon \in K \). Fix any \( n \in \mathbb{N} \). Then since \( D_n \in \mathbf{D}_A(K) \), we have that \( K \subseteq K_{D_n} \). Therefore, and because \( A - \varepsilon \in K \), we infer from Equation (8) that there is
some \( u_n \in A \) such that \( u_n - \epsilon \in D_n \) and therefore also \( P_{D_n}(u_n) \geq \epsilon \). So for all \( n \in \mathbb{N} \), we have some \( u_n \in A \) such that \( P_{D_n}(u_n) \geq \epsilon \). Since \( A \) is finite, this implies that there is some \( u' \in A \) and some subsequence \( \{D_n\}_{n \in \mathbb{N}} \) such that \( P_{D_n}(u') \geq \epsilon \) for all \( n \in \mathbb{N} \). Since \( P \) is the point-wise limit of \( \{P_{D_n}\}_{n \in \mathbb{N}} \) and therefore also the point-wise limit of the subsequence \( \{P_{D_n}\}_{n \in \mathbb{N}} \), this implies that \( P(u') \geq \epsilon > 0 \), so \( u' \in D_{\epsilon} = D_{\infty} \). Since \( u' \in A \), this implies that \( A \in K_{D_{\infty}} \), again by Equation (8). Since this is true for every \( A \in K \), we find that, indeed, \( K \subseteq K_{D_{\infty}} \), so \( D_{\infty} = \overline{D}_A(K) \) and \( \overline{D}_A(K) \) is therefore indeed closed.

What we have found so far is that there is at least one non-empty closed set \( \mathcal{D} \subseteq \overline{D}_A \) such that \( K = \bigcap \{K_D : D \in \mathcal{D}\} \), because \( \overline{D}_A(K) = \overline{D}_A(K) \neq \emptyset \) satisfies all these properties. Furthermore, for any other non-empty closed set \( \mathcal{D} \subseteq \overline{D}_A \) such that \( K = \bigcap \{K_D : D \in \mathcal{D}\} \), we clearly have that \( K \subseteq K_D \) for all \( D \in \mathcal{D} \). Since \( \mathcal{D} \subseteq \overline{D}_A \), this implies that \( \mathcal{D} \subseteq \overline{D}_A(K) \), so \( \overline{D}_A(K) \) is indeed the largest such set.

On to the ‘if’ part. Consider any non-empty closed set \( \mathcal{D} \subseteq \overline{D}_A \) of Archimedean sets of desirable options such that \( K = \bigcap \{K_D : D \in \mathcal{D}\} \). For every \( D \in \mathcal{D} \subseteq \overline{D}_A \), it then follows from Proposition 27 that \( K_D \) is Archimedean. Because Axioms \( K_0-K_4 \) are trivially preserved under taking arbitrary (non-empty) intersections, it already follows that \( K \) is coherent. It therefore only remains to prove that \( K \) is Archimedean as well, or in other words that \( K \) satisfies \( K_A \).

To this end, consider any \( A \in K \) and assume ex absurdo that \( A - \epsilon \notin K \) for all \( \epsilon \in \mathbb{R}_{>0} \). Fix any \( n \in \mathbb{N} \) and let \( \epsilon_n := 1/n > 0 \), so by assumption \( A - \epsilon_n \notin K \). Hence, since \( K = \bigcap \{K_D : D \in \mathcal{D}\} \), there is some \( D_n \in \mathcal{D} \) such that \( A - \epsilon_n \notin K_{D_n} \). This allows us to consider the sequence \( \{P_{D_n}\}_{n \in \mathbb{N}} \), which, because of Proposition 24, belongs to \( \overline{P} \). Since we already know that \( \overline{P} \) is compact under the (weak*) topology induced by point-wise convergence, this sequence has some subsequence \( \{P_{D_{n_i}}\}_{i \in \mathbb{N}} \) that converges point-wise to a limit \( P \in \overline{P} \). Since \( P \in \overline{P} \), we know from Proposition 24 that \( P = P_{D_{n_i}} \), with \( D_{n_i} := D_{n_i} \) an Archimedean set of desirable options. Equation (13) tells us that also \( D_{n_i} = \lim_{i \to +\infty} D_{n_i} \), and since \( \mathcal{D} \) was assumed to be closed, this implies that \( D_{n_i} \in \mathcal{D} \). Since \( A \in K = \bigcap \{K_D : D \in \mathcal{D}\} \), this in turn implies that \( A \in K_{D_{n_i}} \), so there is some \( u \in A \) such that \( u \in D_{n_i} \), by Equation (8). Since \( D_{n_i} = D_{\epsilon} \), this implies that \( P(u) > 0 \). Fix any \( n \in \mathbb{N} \). Since by our construction \( A - \epsilon_n \notin K_{D_n} \), we know that \((A - \epsilon_n) \cap D_n = \emptyset \), and therefore, since \( u \in A \), that \( u - \epsilon_n \notin D_n \), again by Equation (8). Since we know from Proposition 24 that \( D_{n_i} = D_{D_{n_i}} \), this implies that \( P_{D_{n_i}}(u - \epsilon_n) \leq 0 \). Because Proposition 24 also tells us that \( P_{D_{n_i}} \) is a coherent lower prevision, this in turn implies that

\[
0 \geq P_{D_{n_i}}(u - \epsilon_n) \geq P_{D_{n_i}}(u) + P_{D_{n_i}}(-\epsilon_n) \geq P_{D_{n_i}}(u) - \epsilon_n,
\]

whence \( P_{D_{n_i}}(u) \leq \epsilon_n = 1/n \). Since this is true for every \( n \in \mathbb{N} \), we find that

\[
P(u) = \lim_{i \to +\infty} P_{D_{n_i}}(u) \leq \lim_{i \to +\infty} \frac{1}{i} = 0.
\]

However, we had already found that \( P(u) > 0 \), a contradiction. Hence, there must be some \( \epsilon \in \mathbb{R}_{>0} \) such that \( A - \epsilon \notin K \). Since this is true for every \( A \in K \), we find that \( K \) is indeed Archimedean.

\[\square\]

**Proof of Theorem 29.** First assume that \( K \) is mixing and Archimedean. It then follows from Theorem 53 that \( \overline{D}_M(K) := \{D \in \overline{D}_M : K \subseteq K_D\} \neq \emptyset \) and \( K = \bigcap \{K_D : D \in \overline{D}_M(K)\} \). Let \( \overline{D}_M(A)(K) := \{\text{arch}(D) : D \in \overline{D}_M(K)\} \neq \emptyset \), then Lemma 58 already guarantees that \( K = \bigcap \{K_D : D \in \overline{D}_M(A)(K)\} \).

We first prove that \( \overline{D}_M(A)(K) = \overline{D}_M(A)(K) \). To this end, first consider any \( D \in \overline{D}_M(A)(K) \). Then \( K \subseteq K_D \) because \( K = \bigcap \{K_D : D \in \overline{D}_M(A)(K)\} \). Furthermore, it follows from Lemmas 54–56 that \( D \in \overline{D}_M(A) \). Hence \( D \in \overline{D}_M(A)(K) \), and therefore \( \overline{D}_M(A)(K) \subseteq \overline{D}_M(A)(K) \). Conversely, consider any \( D \in \overline{D}_M(A)(K) \). Since \( D \) is Archimedean, it follows from Lemma 57
that arch(D) = D, and because also D ∈ D_{\text{MA}}(K) ⊆ D_{\text{M}}(K), we find that D = arch(D) ∈ D_{\text{MA}}(K). So D_{\text{MA}}(K) ⊆ D_{\text{MA}}(K), and therefore, indeed, D_{\text{MA}}(K) = D_{\text{MA}}(K).

Next, we prove that D_{\text{MA}}(K) is closed. Consider any convergent sequence \{D_n\}_{n \in \mathbb{N}} in D_{\text{MA}}(K) and let D_∞ be its limit. Then we need to prove that D_∞ ∈ D_{\text{MA}}(K). Equation (13) and Proposition 24 tell us that D_∞ = D_P, where P is the point-wise limit of the corresponding convergent sequence of coherent lower previsions \{P_{D_n}\}_{n \in \mathbb{N}}. Since for any \(n \in \mathbb{N}, D_n\) is mixing and Archimedean, we know from Proposition 24 that P_{D_n} is a linear prevision. Therefore, and because P is closed and compact in the (weak\(^*)\) topology induced by point-wise convergence [11, 21, 25], the limit P is a linear prevision as well, so we infer from Proposition 24 that D_∞ = D_P is mixing and Archimedean, so D_∞ ∈ D_{\text{MA}}.

It therefore remains to prove that K ⊆ K_{D_∞}. Consider, to this end, any A ∈ K. Since K is Archimedean, there is some \(\varepsilon \in \mathbb{R} \setminus \mathbb{Q}\) such that \(A - \varepsilon \in K\). Fix any \(n \in \mathbb{N}\). Then since D_n ∈ D_{\text{MA}}(K), we have that K ⊆ K_{D_n}. Therefore, and because \(A - \varepsilon \in K\), we infer from Equation (8) that there is some \(u_n \in A\) such that \(u_n = \varepsilon \in D_n\) and therefore also \(P_{D_n}(u_n) \geq \varepsilon\).

So for all n ∈ \mathbb{N}, we have some \(u_n \in A\) such that \(P_{D_n}(u_n) \geq \varepsilon\). Since A is finite, this implies that there is some \(u' \in A\) and some subsequence \(\{D_{n_i}\}_{i \in \mathbb{N}}\) such that \(P_{D_{n_i}}(u') \geq \varepsilon\) for all \(i \in \mathbb{N}\). Since P is the point-wise limit of \(\{P_{D_{n_i}}\}_{i \in \mathbb{N}}\) and therefore also the point-wise limit of the subsequence \(\{P_{D_{n_i}}\}_{i \in \mathbb{N}}\), this implies that \(P(u') \geq \varepsilon > 0\), so \(u' \in D_P = D_∞\). Since \(u' \in A\), this implies that \(A \subseteq K_{D_∞}\), again by Equation (8). Since this is true for every A ∈ K, we find that, indeed, \(K \subseteq K_{D_∞}\), so D_∞ ∈ D_{\text{MA}}(K) and D_{\text{MA}}(K) is therefore indeed closed.

What we have found so far is that there is a least one non-empty closed set \(\mathcal{D} \subseteq D_{\text{MA}}\) such that each Archimedean and mixing sets of desirable options such that \(K = \bigcap\{K_D : D \subseteq \mathcal{D}\}\), because D_{\text{MA}}(K) = D_{\text{MA}}(K) \neq \emptyset satisfies all these properties. Furthermore, for any other non-empty closed set \(\mathcal{D} \subseteq D_{\text{MA}}\) such that \(K = \bigcap\{K_D : D \subseteq \mathcal{D}\}\), we clearly have that \(K \subseteq K_D\) for all \(D \subseteq \mathcal{D}\). Since \(\mathcal{D} \subseteq D_{\text{MA}}\), this implies that \(\mathcal{D} \subseteq D_{\text{MA}}(K)\), so D_{\text{MA}}(K) is indeed the largest such set.

We now turn to the ‘if’ part. So consider any non-empty closed set \(\mathcal{D} \subseteq D_{\text{MA}}\) of Archimedean and mixing sets of desirable options such that \(K = \bigcap\{K_D : D \subseteq \mathcal{D}\}\). For every \(D \subseteq \mathcal{D} \subseteq D_{\text{MA}}\), it then follows from Proposition 27 that \(K_D\) is Archimedean and mixing. Because Axioms K₀-K₄ and K₉ are trivially preserved under taking arbitrary (non-empty) intersections, it already follows that \(K = \bigcap\{K_D : D \subseteq \mathcal{D}\}\) is mixing. It therefore only remains to prove that \(K\) is Archimedean as well, or in other words—because we already know that \(K\) is Archimedean because mixing—that \(K\) satisfies K₄.

To this end, consider any \(A \in K\) and assume ex absurdo that \(A - \varepsilon \notin K\) for all \(\varepsilon \in \mathbb{R} \setminus \mathbb{Q}\). Fix any \(n \in \mathbb{N}\) and let \(\varepsilon_n = \frac{1}{n} > 0\), so by assumption \(A - \varepsilon_n \notin K\). Hence, since \(K = \bigcap\{K_D : D \subseteq \mathcal{D}\}\), there is some \(D_n \in \mathcal{D}\) such that \(A - \varepsilon_n \notin K_{D_n}\). This allows us to consider the sequence \(\{D_{n_k}\}_{k \in \mathbb{N}}\), which, because of Proposition 24, belongs to \(\mathcal{D}\). Since we already know that \(\mathcal{D}\) is compact under the (weak\(^*)\) topology induced by point-wise convergence, this sequence has some subsequence \(\{D_{n_k}\}_{k \in \mathbb{N}}\) that converges point-wise to a limit \(P \in \mathcal{D}\). Since \(P \in \mathcal{D}\), we know from Proposition 24 that \(P = P_{D_{n_k}}\), with \(D_{n_k} := D_P\) a mixing and Archimedean set of desirable options. Equation (13) tells us that also \(D_∞ = \lim_{n \to \infty} D_{n_k}\), and since \(\mathcal{D}\) was assumed to be closed, this implies that \(D_∞ \subseteq \mathcal{D}\). Since \(A \in K = \bigcap\{K_D : D \subseteq \mathcal{D}\}\), this in turn implies that \(A \in K_{D_∞}\), so there is some \(u \in A\) such that \(u \in D_∞\), by Equation (8). Since \(D_∞ = D_P\), this implies that \(P(u) > 0\). Fix any \(n \in \mathbb{N}\). Since by our construction \(A - \varepsilon_n \notin K_{D_n}\), we know that \(A - \varepsilon_n \in \mathcal{D}\), and therefore, since \(u \in A\), that \(u - \varepsilon_n \notin D_n\), again by Equation (8). Since we know from Proposition 24 that \(D_n = D_{\mathcal{D}_n}\), this implies that \(P_{D_n}(u - \varepsilon_n) \leq 0\). Because Proposition 24 also tells us that \(P_{D_n}\) is a coherent lower prevision, this in turn implies that

\[0 \geq P_{D_n}(u - \varepsilon_n) \geq P_{D_n}(u) + P_{D_n}(-\varepsilon_n) \geq P_{D_n}(u) - \varepsilon_n.\]
whence $P_{D_n}(u) \leq \varepsilon_n = 1/n$. Since this is true for every $n \in \mathbb{N}$, we find that
\[ P(u) = \lim_{i \to +\infty} P_{D_{n_i}}(u) \leq \lim_{i \to +\infty} \frac{1}{i} = 0. \]

However, we had already found that $P(u) > 0$, a contradiction. Hence, there must be some $\varepsilon \in \mathbb{R}_{>0}$ such that $A - \varepsilon \in K$. Since this is true for every $A \in K$, we find that $K$ is indeed Archimedean.

\[ \square \]

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