ON ALMOST EVERYWHERE CONVERGENCE OF TENSOR PRODUCT SPLINE PROJECTIONS

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ABSTRACT. Let $d$ be a positive integer and $f$ be a function in the Orlicz class $L((\log^+ L)^{d-1})$ defined on the unit cube $[0,1]^d$ in $\mathbb{R}^d$. Given partitions $\Delta_1, \ldots, \Delta_d$ of the interval $[0,1]$, we prove that the orthogonal projection $P(\Delta_1, \ldots, \Delta_d) f$ of $f$ onto the space of tensor product splines with arbitrary orders $(k_1, \ldots, k_d)$ and knots $\Delta_1, \ldots, \Delta_d$ converges to $f$ almost everywhere (a.e.) as all the mesh diameters $|\Delta_1|, \ldots, |\Delta_d|$ tend to zero. This extends the one-dimensional result in [3] to arbitrary dimensions.

On the other hand, we also show that this result is sharp, i.e. given any “bigger” Orlicz class $X = \sigma(L)L((\log^+ L)^{d-1})$ with an arbitrary function $\sigma$ tending to zero at infinity, there exists a function $\varphi \in X$ and partitions of the unit cube such that the orthogonal projections of $\varphi$ do not converge a.e.

1. Introduction and Definitions

Let $d \geq 1$ be a positive integer and, for $\mu = 1, \ldots, d$, let $k_\mu$ be the order of polynomials in the direction of the $\mu$-th unit vector. Moreover, for each $\mu$, we define a partition of the interval $[0,1]$ as

$$\Delta_\mu = (t^{(\mu)}_i)_{i=1}^{n_\mu + k_\mu},$$

satisfying

$$t^{(\mu)}_1 = \cdots = t^{(\mu)}_{n_\mu} = 0, \quad 1 = t^{(\mu)}_{n_\mu + 1} = \cdots = t^{(\mu)}_{n_\mu + k_\mu}.$$

A boldface letter always denotes a vector of $d$ entries and its components are denoted by the same letter, for instance $\mathbf{n} = (n_1, \ldots, n_d)$, $\mathbf{k} = (k_1, \ldots, k_d)$ or $\mathbf{\Delta} = (\Delta_1, \ldots, \Delta_d)$. We let $(N_i^{(\mu)})_{i=1}^{n_\mu}$ be the sequence of $L^\infty$-normalized B-splines of order $k_\mu$ on the partition $\Delta_\mu$ with the properties

$$\text{supp} N_i^{(\mu)} = [t_i^{(\mu)}, t_{i+k_\mu}^{(\mu)}], \quad N_i^{(\mu)} \geq 0, \quad \sum_{i=1}^{n_\mu} N_i^{(\mu)} = 1.$$

The tensor product B-splines are defined as

$$N_i(x_1, \ldots, x_d) := N_i^{(1)}(x_1) \cdots N_i^{(d)}(x_d), \quad 1 \leq i \leq \mathbf{n},$$

where $\mathbf{1}$ is the $d$-dimensional vector consisting of $d$ ones and we say that $i \leq \mathbf{n}$, provided $i_\mu \leq n_\mu$ for all $\mu = 1, \ldots, d$. Furthermore, $P_\mathbf{\Delta}$ is defined to be the orthogonal projection operator from $L^2([0,1]^d)$ onto the linear span of the functions $(N_i)_{1 \leq i \leq \mathbf{n}}$. For $\mu = 1, \ldots, d$, we define $|\Delta_\mu| := \max_{1 \leq i \leq d} |t_i^{(\mu)} - t_{i+1}^{(\mu)}|$ and

$$|\Delta| := \max_{1 \leq \mu \leq d} |\Delta_\mu|.$$
We say that a function $f$, defined on the unit cube $[0,1]^d$, belongs to the Orlicz class $L(\log^+ L)^{d-1}$, provided $|f|(\log^+ |f|)^{d}$ is integrable over $[0,1]^d$, where we employed the standard notation $\log^+ := \max(0, \log)$.

The first main result of this article is a.e. convergence of $P_\Delta f$ to $f$ for the Orlicz class $L(\log^+ L)^{d-1}$:

**Theorem 1.1.** Let $f \in L(\log^+ L)^{d-1}$. Then,

$$P_\Delta f \to f \quad \text{a.e. as } |\Delta| \to 0.$$ 

On the other hand, given any “bigger” function class than $L(\log^+ L)^{d-1}$, we prove the following negative result:

**Theorem 1.2.** Given an arbitrary positive function $\sigma$ on the interval $[0, \infty)$ with the property $\lim \inf_{t \to \infty} \sigma(t) = 0$, there exists a non-negative function $\varphi$ on $[0,1]^d$ such that

(i) the function $\sigma(|\varphi|) \cdot |\varphi| \cdot (\log^+ |\varphi|)^{d-1}$ is integrable,

(ii) there exists a subset $B \subset [0,1]^d$ of positive measure such that for all $x \in B$,

$$\lim \sup_{\text{diam } I \to 0, I \ni x} |P_I \varphi(x)| = \infty,$$

where $P_I$ is the orthogonal projection operator onto the space of $d$-variate polynomials on $I$ with orders $(k_1, \ldots, k_d)$ and $\lim sup$ is taken over all $d$-dimensional rectangles $I$ parallel to the coordinate axes.

The proof of Theorem 1.1 is based on the standard approach of verifying two conditions that imply a.e. convergence of $P_\Delta f$ for $f \in L(\log^+ L)^{d-1}$ (cf. [3, pp. 3-4]):

(a) there is a dense subset $F$ of $L(\log^+ L)^{d-1}$ where a.e. convergence takes place,

(b) the maximal operator $P^* f := \sup |P_\Delta f|$ satisfies some weak type inequality.

We remark that this proof is similar to the corresponding proof for $d = 1$ in [4] and proceed by discussing the two conditions [a] and [b] in the following. Concerning [a], we first note that for $d = 1$, A. Shadrin [7] proved that the one-dimensional projection operator $P_{\Delta_1}$ is uniformly bounded on $L^\infty$ for any spline order $k_1$, i.e.,

$$\|P_{\Delta_1}\|_\infty \leq c_{k_1},$$

where the constant $c_{k_1}$ depends only on $k_1$ and not on the partition $\Delta_1$. A direct corollary of this result and the tensor structure of the underlying operator $P_\Delta$ is that this assertion also holds in higher dimensions $d$:

**Corollary 1.3.** For any $d \geq 1$, there exists a constant $c_{d,k}$ that depends only on $d$ and $k$ such that

$$\|P_\Delta\|_\infty \leq c_{d,k}.$$ 

In particular, $c_{d,k}$ is independent of the partitions $\Delta$.

This can easily be used to prove uniform convergence of $P_\Delta g$ to $g$ for continuous functions $g$, provided $|\Delta|$ tends to zero. We give a short proof of this fact in Section 2. Therefore, we choose $F$ to be the space of continuous functions, which is dense in $L(\log^+ L)^{d-1}$. 
We now turn to the discussion of condition (b) and define the strong maximal function $M_S f$ of $f \in L^1([0,1]^d)$ to be
\[
M_S f(x) := \sup_{I \ni x} |I|^{-1} \int_I |f(y)| \, dy, \quad x \in [0,1]^d,
\]
where the supremum is taken over all $d$-dimensional rectangles $I \subset [0,1]^d$ parallel to the coordinate axes that contain the point $x$. The strong maximal function satisfies the weak type inequality (cf. [2] and for more information about the strong maximal function cf. [8, Chapter 17])
\[
|\{ x : M_S f(x) > \lambda \}| \leq c_M \int_{[0,1]^d} \frac{|f(x)|}{\lambda} \left(1 + \log^+ \frac{|f(x)|}{\lambda} \right)^{d-1} \, dx,
\]
where $|A|$ denotes the $d$-dimensional Lebesgue measure of the set $A$ and $c_M$ is a constant independent of $f$ and $\lambda$. In order to get that kind of weak type inequality for the maximal operator $P^* f$, we prove the following pointwise estimate for $P_\Delta f$ by the strong maximal function:

**Theorem 1.4.** Let $f$ be an integrable function. Then, there exists a constant $c$ that depends only on the dimension $d$ and the spline orders $k$ such that
\[
|P_\Delta f(x)| \leq c \cdot M_S f(x), \quad x \in [0,1]^d,
\]
for all partitions $\Delta$.

This theorem is essentially a consequence of a geometric decay inequality involving the inverse Gram matrix of B-spline functions proved in [4].

Next, we discuss the negative result Theorem 1.2. In [5], S. Saks proved the following

**Theorem 1.5.** Given a positive function $\sigma$ on the interval $[0, \infty)$ with the property $\liminf_{t \to \infty} \sigma(t) = 0$, there exists a non-negative function $\phi$ on $[0,1]^d$ such that

1. the function $\sigma(|\phi|) \cdot |\phi| \cdot (\log^+ |\phi|)^{d-1}$ is integrable,
2. for all $x \in [0,1]^d$,
\[
\limsup_{\text{diam } I \to 0, I \ni x} \frac{1}{|I|} \int_I \phi(y) \, dy = \infty,
\]
where $\limsup$ is taken over all $d$-dimensional rectangles $I$ parallel to the coordinate axes containing the point $x$.

If we compare (ii) of this theorem with (ii) of Theorem 1.2 we note that
\[
\frac{1}{|I|} \int_I \phi(y) \, dy = P_I \phi(x), \quad x \in I
\]
for the choice $k = 1$. It turns out that the same function $\phi$ that is constructed in the proof of the above theorem also has the properties stated in Theorem 1.2.

2. **Almost everywhere convergence**

In this section, we prove Theorem 1.1 about a.e. convergence of $P_\Delta f$ for $f \in L((\log^+ L)^{d-1})$. We first recall the inequality about the geometric decay of inverses of B-spline Gram matrix proved in [4]. In order to state this inequality in our setting, we need some more notations. For $\mu = 1, \ldots, d$, let $(a_{ij}^{(\mu)})$ be the inverse of the Gram matrix $\left(\langle N_i^{(\mu)}, N_j^{(\mu)} \rangle \right)_{i,j=1}^{n_{\mu}}$ of B-spline functions and $a_{ij} :=
Proof of Theorem 1.4. Let \( a^{(1)}_{i_1,j_1} \cdots a^{(d)}_{i_d,j_d} \). We shall use the following abbreviations for important grid point intervals:

\[
I_i^{(\mu)} := [t_{\min(i)}, t_{\max(i)+k}] \\
E_{i,j}^{(\mu)} := [t_{\min(i,j)}, t_{\max(i,j)+k}] \\
\]

and for their \( d \)-dimensional counterparts:

\[
I_i := \prod_{\mu=1}^{d} I_{i_{\mu}}^{(\mu)}, \quad E_{ij} := \prod_{\mu=1}^{d} E_{i_{\mu}j_{\mu}}^{(\mu)}.
\]

The above mentioned inequality from [4] is of the following form in \( d \) dimensions:

**Proposition 2.1.** For the expressions \( a_{ij} \), we have the inequality

\[
|a_{ij}| \leq K \gamma^{|i-j|} |E_{ij}|^{-1}, \quad 1 \leq i, j \leq n,
\]

where \( K > 0 \) and \( \gamma \in (0,1) \) are constants that depend only on \( d \) and \( k \) and \( |y|_1 := \sum_{m=1}^{d} |y_m| \) denotes the \( \ell^1 \)-norm of the \( d \)-dimensional vector \( y \).

We next use this result to deduce an inequality for the Dirichlet kernel \( \Delta \) of \( P_{\Delta}f \), which is defined by the relation

\[
P_{\Delta}f(x) = \int_{[0,1]^d} K_{\Delta}(x,y)f(y) \, dy, \quad f \in L^1([0,1]^d), \quad x \in [0,1]^d,
\]

and satisfies

\[
K_{\Delta}(x,y) = \sum_{1 \leq i,j \leq n} a_{ij} N_i(x) N_j(y), \quad x,y \in [0,1]^d.
\]

We will use the abbreviations

\[
I_{i,j}^{(\mu)} := \text{conv}(I_i, I_j) = [t_{\min(i,j)}, t_{\max(i,j)+k}], \quad I_{ij} := \prod_{\mu=1}^{d} I_{i_{\mu}j_{\mu}}^{(\mu)},
\]

where \( \text{conv}(A,B) \) denotes the convex hull of the sets \( A \) and \( B \). Then, Proposition 2.1 and the properties of B-splines imply the inequality

\[
|K_{\Delta}(x,y)| \leq C \gamma^{|i-j|} |I_{ij}|^{-1}, \quad x \in I_i, \quad y \in I_j,
\]

where \( C \) is some constant depending only on \( d \) and \( k \) and \( \gamma \) is the same as in Proposition 2.1. We note that \( K_{\Delta} \) is the product of the one-dimensional Dirichlet kernels, given by

\[
K_{\Delta \mu}(x, y) := \sum_{1 \leq i,j \leq n_{\mu}} a_{ij}^{(\mu)} N_i^{(\mu)}(x) N_j^{(\mu)}(y),
\]

and thus \( P_{\Delta} \equiv P_{\Delta_1} \cdots P_{\Delta_d} \), where \( P_{\Delta \mu} \) is the integral operator with kernel \( K_{\Delta \mu} \) and \( P_{\Delta \mu} \) acts on the \( \mu \)-th variable.

In order to get a.e. convergence of \( P_{\Delta}f \) for \( f \in L(\log^+ L)^{d-1} \), we next prove the pointwise bound of \( P_{\Delta}f \) by the strong maximal function \( M_Sf \) employing the upper bound (2.4) for the Dirichlet kernel \( K_{\Delta} \).

**Proof of Theorem 1.4.** Let \( x \in [0,1]^d \) and \( i \) be such that \( x \in I_i \) and \( |I_i| > 0 \). By equation (2.2),

\[
|P_{\Delta}f(x)| = \left| \int_{[0,1]^d} K_{\Delta}(x,y) f(y) \, dy \right|,
\]

and thus \( P_{\Delta}f \) is bounded by the strong maximal function \( M_Sf \) outside the set \( I_i \).
where $K_{\Delta}$ is the Dirichlet kernel given by (2.3). Using estimate (2.4), we obtain

$$|P_{\Delta}f(x)| \leq C \sum_{1 \leq j \leq n} \frac{\gamma^{j-1}1}{I_{ij}} \int_{I_j} |f(y)| dy.$$  

Since $I_j \subset I_{ij}$ and $x \in I_k \subset I_{ij}$, we conclude

$$|P_{\Delta}f(x)| \leq C \sum_{1 \leq j \leq n} \gamma^{j-1} M_\delta f(x),$$

which proves the theorem. \hfill \square

We now give a short proof of uniform convergence of $P_{\Delta}g$ to $g$ for a dense subset $F$ of $L(\log^+ L)^{d-1}$. In our case, $F$ is the space of continuous functions. Clearly, uniform convergence in particular implies a.e. convergence stated in condition (ii) of Section 1 on page 2.

**Proposition 2.2.** Let $g \in C([0,1]^d)$. Then

$$\|P_{\Delta}g - g\|_{\infty} \to 0 \quad \text{as } |\Delta| \to 0.$$  

**Proof.** Since $P_{\Delta}$ is a projection operator, we get by the Lebesgue inequality and Corollary 1.3

$$\|P_{\Delta}g - g\|_{\infty} \leq (1 + c_{d,1}) \cdot E_{\Delta}(g),$$

where $E_{\Delta}(g)$ is the error of best approximation of $g$ by splines in the span of tensor product B-splines $(g_N)_{1 \leq N \leq n}$. It is known that (cf. [6] Chapter 12)

$$E_{\Delta}(g) \leq c \cdot \sum_{\mu=1}^d \sup_{h_{\mu} \leq |\Delta_{\mu}|} \sup_{x} |(D_{h_{\mu}}g_{\mu}(x))|,$$

where $g_{\mu}(x) := g(x_1, \ldots, x_{\mu-1}, s, x_{\mu+1}, \ldots, x_d)$ and $D_{h_{\mu}}$ is the forward difference operator with step size $h_{\mu}$. This is the sum of moduli of smoothness in each direction $\mu$ of the function $g$ with respect to the mesh diameters $|\Delta_1|, \ldots, |\Delta_d|$, respectively. As these diameters tend to zero, the right hand side of the above display also tends to zero since $g$ is continuous. This, together with (2.5), proves the assertion of the proposition. \hfill \square

Now we are able to prove a.e. convergence of $P_{\Delta}f$ for $f \in L(\log^+ L)^{d-1}$.

**Proof of Theorem 1.1.** Let $f \in L(\log^+ L)^{d-1}$. Then we define

$$R(f,x) := \limsup_{|\Delta| \to 0} P_{\Delta}f(x) - \liminf_{|\Delta| \to 0} P_{\Delta}f(x).$$

Let $g \in C([0,1]^d)$. Since by Proposition 2.2 $R(g,x) \equiv 0$ for continuous functions $g$ and $P_{\Delta}$ is a linear operator,

$$R(f,x) \leq R(f-g,x) + R(g,x) = R(f-g,x).$$

Let $\delta > 0$ be an arbitrary positive number, then we have by Theorem 1.4

$$\{|x : R(f,x) > \delta| \| \leq \{|x : R(f-g,x) > \delta| \| \leq \{|x : 2c \cdot M_\delta f(x) > \delta| \|.$$  

Now we employ the weak type inequality (1.1) of $M_\delta$ to find

$$\{|x : R(f,x) > \delta| \| \leq c_M \int_{[0,1]^d} \frac{2c \cdot |(f-g)(x)|}{\delta} \left(1 + \log^+ \frac{2c \cdot |(f-g)(x)|}{\delta} \right)^{d-1} dx.$$
By assumption, the expression on the right hand side of the latter display is finite. Choosing a suitable sequence of continuous functions \( g_n \) (first approximate \( f \) by a bounded function and then apply Lusin’s theorem), the above expression tends to zero and we obtain
\[
|\{x : R(f, x) > \delta\}| = 0.
\]
Since \( \delta > 0 \) was chosen arbitrarily, \( R(f, x) = 0 \) for a.e. \( x \in [0, 1]^d \). This means that \( P_\Delta f \) converges almost everywhere as \( |\Delta| \to 0 \). It remains to show that this limit equals \( f \) a.e., but this is obtained by a similar argument as above replacing \( R(f, x) \) by \( |\lim_{|\Delta| \to 0} P_\Delta f(x) - f(x)| \).

\[\square\]

3. Proof of the negative result

In this section, we prove the negative result Theorem 1.2 and give a function \( \varphi \) with the stated properties. In fact, this function \( \varphi \) is the same as the one constructed by S. Saks [5] and its definition rests on a construction by H. Bohr that appears in [1, pp. 689-691] for dimension \( d = 2 \). We will prove Theorem 1.2 only for \( d = 2 \) as well, since the cases \( d > 2 \) follow from straightforward generalizations of the argument for \( d = 2 \) and begin with recalling briefly Bohr’s construction and Saks’ definition of the function \( \varphi \).

**Bohr’s construction.** We let \( \alpha > 1 \) be a real number and we define \( N \) to be the largest integer not greater than \( \alpha \). Let \( S := [a_1, b_1] \times [a_2, b_2] \) be a rectangle in the plane. We define subsets of this rectangle as follows:
\[
I_j^{(1)} := \left[ a_1, a_1 + \frac{j(b_2 - a_1)}{N} \right] \times \left[ a_2, a_2 + \frac{b_2 - a_2}{j} \right], \quad 1 \leq j \leq N.
\]
The part \( S \setminus \bigcup_{j=1}^N I_j^{(1)} \) consists of \( N - 1 \) disjoint rectangles on which we apply the same splitting as we did with \( S \) (cf. Figure 1). This procedure is carried out until the area of the remainder is less than \(|S|/N^2\). The remainder is again a disjoint union of rectangles \( J_1^{(1)}, \ldots, J^{(r)} \). Thus we obtain a sequence of rectangles whose union is \( S \):
\[
(3.1) \quad I_1^{(1)}, \ldots, I_N^{(1)} ; J_1^{(2)}, \ldots, J_N^{(2)} ; \cdots ; J_1^{(s)}, \ldots, J_N^{(s)} ; J^{(1)}, \ldots, J^{(r)}.
\]

The subsequent discussion (up to and not including Proposition 3.1) follows S. Saks [5]. For the proofs of the indicated results, we refer to [5]. We first set \( \delta^{(i)} := \bigcap_{j=1}^N I_j^{(i)} \) for \( 1 \leq i \leq s \) and define \( \psi_{S, \alpha} \) to be \( \alpha \) times the characteristic function of the set \( \bigcup_{i=1}^s \delta^{(i)} \cup \bigcup_{l=1}^r J^{(l)} \). The function \( \psi_{S, \alpha} \) has the properties

(i) \( \psi_{S, \alpha} \) attains only the two values 0 and \( \alpha \), each of it on a finite union of rectangles,
(ii) \( \int \psi_{S, \alpha} \log^+ \psi_{S, \alpha} \, dy \leq 9|S| \),
(iii) for all rectangles \( I \) in the collection (3.1), we have
\[\int_I \psi_{S, \alpha} \, dy \geq |I|\).

**Remark.** The corners of the rectangles \( I_j^{(1)} \), \( 1 \leq j \leq N \), lie on the curve \( (x-a_1)(y-a_2) = (b_1-a_1)(b_2-a_2)/N = |S|/N \). Given a rectangle \( S := [a_1, b_1] \times \cdots \times [a_d, b_d] \), \( d \geq 2 \), we consider rectangles whose corners lie on the variety \( (x_1-a_1)(x_2-a_2) \cdots (x_d-a_d) = |S|/N^{d-1} \). The total volume of these rectangles is approximately \((\log N/N)^{d-1} \cdot |S|\). This explains the role of the power \( d - 1 \) in both Theorems 1.1 and 1.2.
We continue and define the non-negative function $\psi_i$ to be a suitable finite sum of functions $\psi_{S_j(i)}$, where for all positive integers $i$, $(S_j(i))_{j=1}^{L_i}$ is a finite partition of $[0, 1]^2$ into rectangles with $\text{diam } S_j(i) \leq 1/i$ and $\alpha_j(i) > 1$ are suitable real numbers. Then, there exists a sequence of positive numbers $(\varepsilon_i)$, tending to zero, such that the function

$$\varphi := \sum_{i=1}^{\infty} \frac{\psi_i}{\varepsilon_i}$$

satisfies (i) and (ii) of Theorem 1.5. In particular, for all $i, j$ and all rectangles $I$ in the collection (3.1) of subsets of $S_j(i)$,

$$\int_I \varphi \, dy \geq \frac{1}{\varepsilon_i} \int_I \psi_i \, dy \geq \frac{|I|}{\varepsilon_i}.$$

Before we continue, we need a few simple properties of polynomials:

**Proposition 3.1.** Let $0 < \rho < 1$ and $I$ be an interval. Then, there exists a constant $c_{k, \rho}$ only depending on $k$ and $\rho$ such that for every polynomial $Q$ of order $k$ on $I$ and all subsets $A \subset I$ with $|A| \geq \rho |I|$,

$$\max_{t \in I} |Q(t)| \leq c_{k, \rho} \sup_{t \in A} |Q(t)|.$$

**Corollary 3.2.** Let $t$ be a positive real number and $Q$ a polynomial of order $k$ defined on an interval $I$ with $\|Q\|_{L^\infty(I)} \geq t$. Then there exists a constant $c_k$ only depending on $k$ such that $|\{x \in I : |Q(x)| > t/c_k\}| \geq |I|/2$.

**Proof.** If the decreasing function $u : s \mapsto |\{x \in I : |Q(x)| > s\}|$ attains the value $|I|/2$, say for $s_0$, we simply define $A := \{x \in I : |Q(x)| > s_0\}$. If $u$ does not attain the value $|I|/2$, we must have $|\{x \in I : |Q(x)| = s\}| > 0$ for some $s$. But this is impossible for polynomials $Q$ unless $Q$ is constant on $I$, in which case we let $A$ be an arbitrary subset of $I$ with measure $|I|/2$. In either case, we have by definition

$$\inf_{x \in A} |Q(x)| \geq \sup_{x \in A^c} |Q(x)|.$$
Proposition 3.1 implies the existence of a constant $c_k$, only depending on $k$, such that
\[ \sup_{x \in A^c} |Q(x)| > \frac{1}{c_k} \max_{x \in I} |Q(x)|. \]
Since the assumption implies $\max_{x \in I} |Q(x)| \geq t$, we have $Q > t/c_k$ on the set $A$ with measure $|I|/2$. \hfill \Box

Now we are able to show that $|P_I \varphi|$ is big on a large subset of $I$ as long as $\frac{1}{|I|} \int_I \varphi \, dy$ is big. This is the first important step in proving (ii) of Theorem 1.2.

**Theorem 3.3.** Let $I = I_1 \times I_2$ be a rectangle in the plane and $\varphi : I \to \mathbb{R}$ be such that $\frac{1}{|I|} \int_I \varphi(x) \, dx \geq c_k t$ with the constants $c_k, c_{k_2}$ as in Corollary 3.2. Then there exists a (measurable) subset $A \subset I$ with $|A| \geq |I|/4 = |I|/2^d$ and
\[ |P_I \varphi(x)| \geq t, \quad x \in A. \]
Here, $P_I \varphi$ is the orthogonal projection of $\varphi$ onto the space of bivariate polynomials on $I$ with orders $(k_1, k_2)$.

**Proof.** Since $P_I$ is the orthogonal projection operator onto the space of bivariate polynomials of order $(k_1, k_2)$ on $I$, the assumption on $\varphi$ implies
\[ \langle P_I \varphi, 1_I \rangle = \langle \varphi, 1_I \rangle \geq c_k c_{k_2} t |I|. \]
The fact that $I = I_1 \times I_2$ gives us
\[ \frac{1}{|I|} \int_I P_I \varphi \, dx = \frac{1}{|I_1|} \int_{I_1} \frac{1}{|I_2|} \int_{I_2} P_I \varphi(x, y) \, dy \, dx. \]
If we define $Q(x) := \frac{1}{|I_2|} \int_{I_2} P_I \varphi(x, y) \, dy$ on $I_1$, $Q$ is a polynomial of order $k_1$ on $I_1$. Since $\|Q\|_{L^\infty(I_1)} \geq \frac{1}{|I_1|} \int_{I_1} Q(x) \, dx = \frac{1}{|I|} \int_I P_I \varphi \, dx \geq c_k c_{k_2} t$ by (3.3), we apply Corollary 3.2 to find
\[ |U| \geq |I_1|/2, \]
where $U := \{x \in I_1 : |Q(x)| > c_{k_2} t\}$ is an open subset of $I_1$. For $x \in U$,
\[ Q(x) = \frac{1}{|I_2|} \int_{I_2} P_I \varphi(x, y) \, dy \geq c_{k_2} t, \]
hence we apply Corollary 3.2 again to obtain that the set $B_x := \{y \in I_2 : |P_I \varphi(x, y)| > t\}$ satisfies $|B_x| \geq |I_2|/2$ for all $x \in U$. Finally, we estimate the measure of the measurable set $A_I := \{(x, y) \in I : |P_I \varphi(x, y)| > t\}$:
\[ |A_I| = \int_I 1_{A_I} \, dx = \int_{I_1} |B_x| \, dx \geq \int_U |B_x| \, dx \geq \frac{|I_1 \times I_2|}{4}, \]
where the latter inequality follows from (3.4). \hfill \Box

Considering the construction of $\varphi$, in particular inequality (3.2), the above theorem shows that for any index pair $(i, j)$ and any rectangle $I$ in the enumeration (3.1) corresponding to the set $S_{ij}^{(i)}$, there exists a subset $J \subset I$ with measure $\geq |I|/4$ on which $|P_I \varphi| \geq (\varepsilon_i c_{k_1} c_{k_2})^{-1}$. In the following two lemmata, we ensure that the union of those $J$’s still has big measure relatively to the union of the $I$’s.
Lemma 3.4. Let \( N \) be a positive integer and \( I_j := [a_1, a_1 + (b_1 - a_1) \frac{j}{N}] \times [a_2, a_2 + (b_2 - a_2) \frac{j}{N}] \) for \( j = 1, \ldots, N \). Let \( A_j \subset I_j \) with \( |A_j| \geq c|I_j| \). Then we have
\[
|A_n \cap I_\ell| \geq |I_n| \left( c - \frac{\ell}{n} \right), \quad 1 \leq \ell \leq n \leq N.
\]

Proof. Without loss of generality, we assume \( a_1 = a_2 = 0 \) and \( b_1 = b_2 = 1 \). For \( \ell \leq n \), we have
\[
I_n \cap I_\ell = \left[ 0, \frac{\ell}{N} \right] \times \left[ 0, \frac{1}{n} \right],
\]
and thus \( |I_n \cap I_\ell| = \frac{\ell}{nN} = \frac{\ell}{n} |I_n| \). This and the assumption \( |A_n| \geq c|I_n| \) imply
\[
|I_n| c \geq |A_n| = |A_n \cap I_\ell| + |A_n \cap I_\ell^c| \\
\leq |I_n \cap I_\ell| + |A_n \cap I_\ell^c| \\
= \frac{\ell}{n} |I_n| + |A_n \cap I_\ell^c|,
\]
which proves the assertion.

Lemma 3.5. Let \( N \) be a positive integer and \( I_j := [a_1, a_1 + (b_1 - a_1) \frac{j}{N}] \times [a_2, a_2 + (b_2 - a_2) \frac{j}{N}] \) for \( j = 1, \ldots, N \). Let \( A_j \subset I_j \) with \( |A_j| \geq c|I_j| \). Then there exists a constant \( c_1 \) independent of \( N \) such that
\[
\left| \bigcup_{j=1}^{N} A_j \right| \geq c_1 \left| \bigcup_{j=1}^{N} I_j \right|.
\]

Proof. We assume without loss of generality that \( a_1 = a_2 = 0 \), \( b_1 = b_2 = 1 \) and \( N \) is such that \( c - N^{-1} \geq c/2 \). Let \( m \) be the smallest positive integer \( \leq N \) such that
\[
c - \frac{1}{m} \geq \frac{c}{2}.
\]
Then, we denote by \( j_0 \) the biggest index such that \( m^{j_0} \leq N \). Since the sets
\[
I_1, I_1^c \cap I_m, I_1^c \cap I_m^\ell, \ldots, I_1^c \cap I_m^\ell \cap \cdots \cap I_m^{\ell - 1} \cap I_m^{j_0} = I_1^c \cap I_m \cap \cdots \cap I_m^{j_0},
\]
form a partition of the unit square \([0,1]^2\), we obtain
\[
\left| \bigcup_{j=1}^{N} A_j \right| \geq \sum_{j=0}^{j_0} \left| \bigcup_{\ell=0}^{\ell - 1} \left( \bigcup_{j=0}^{j_0} A_m^\ell \right) \cap I_m^{\ell - 1} \right| I_m^{\ell - 1}
\]
\[
\geq \sum_{\ell=0}^{j_0} \left| A_m^{\ell - 1} \cap I_m^{\ell - 1} \right| I_m^{\ell - 1}.
\]
We note that for \( n \geq r \), \( A_n \cap I_r^c = A_n \cap I_r^c \cap \cdots \cap I_r^c \), since due to the fact that \( A_n \subset I_n \), the sets \( A_n \cap I_r^c, j = 1, \ldots, r \) form a decreasing sequence. Thus, we find by Lemma 3.4
\[
\left| \bigcup_{j=1}^{N} A_j \right| \geq \sum_{\ell=0}^{j_0} |A_m^{\ell - 1} \cap I_m^{\ell - 1}| \geq \sum_{\ell=0}^{j_0} (c - m^{\ell - 1}) |I_m^{\ell - 1}|.
\]
Since all rectangles \( I_j \) have the same measure \( N^{-1} \) and the index \( m \) is such that \( c - m^{-1} \geq c/2 \), we conclude
\[
\left| \bigcup_{j=1}^{N} A_j \right| \geq \frac{c}{2N} j_0.
\]
By definition, \( j_0 \) is the largest integer not greater than \( \log N / \log m \). Since the measure of \( \bigcup_{j=1}^{N} I_j \) is approximately \( \log N / N \) and the number \( m \) is independent of \( N \), the proof of the proposition is completed.

Putting together the above facts, we now prove our negative result:

**Proof of Theorem 1.2.** Since Theorem 1.5 proves the integrability condition (i) of Theorem 1.2 we need only prove (ii), i.e. the existence of a set \( B \subset [0,1]^2 \) with positive measure, such that for \( x \in B \), \( \limsup |P_I\varphi(x)| = \infty \) where \( \limsup \) is taken over all rectangles \( I \) containing the point \( x \) with \( \text{diam } I \to 0 \). We let \( i \) be an arbitrary positive integer and define

\[
B_i := \{ x \in [0,1]^2 : \text{there exists a rectangle } I \text{ with } x \in I, \text{diam } I \leq 1/i \text{ and } |P_I\varphi(x)| \geq (\varepsilon_i c_k c_k)^{-1} \},
\]

where \( c_{k_1} \) and \( c_{k_2} \) are the constants from Corollary 3.2 and \( (\varepsilon_i) \) is the sequence from the definition of \( \varphi \). Recall that \( \varepsilon_i \to 0 \) as \( i \to \infty \). We will show the estimate \(|B_i| \geq c > 0\) for all positive integers \( i \) and some constant \( c \), for then the set \( B := \{ B_n \text{ i.o.} \} \) is the desired subset of \([0,1]^2\):

\[
|B| = \{ \{B_n \text{ i.o.} \} \} = \lim_n \left| \bigcup_{m \geq n} B_m \right| \geq \limsup_n |B_n| \geq c > 0.
\]

In order to show the inequality \(|B_i| \geq c\), we consider the partition \((S_j)^{L_i}\) of \([0,1]^2\). For all \( j \) we have

\[
|S_j^{(i)}| = \sum_{\ell=1}^{s} \left| \bigcup_{m=1}^{N_j^{(i)}} I_m^{(\ell)} \right| + \sum_{\ell=1}^{r} |J^{(\ell)}|,
\]

where \( N_j^{(i)} \) is the largest integer not greater than the number \( o_j^{(i)} \) from the construction of \( \psi_i \) and \( I_m^{(\ell)}, J^{(\ell)} \) are the rectangles (3.1) corresponding to \( S_j^{(i)} \). Let \( I \) be one of the sets in this partition. We recall that, by construction, \( \text{diam } I \leq \text{diam } S_j^{(i)} \leq 1/i \). Then, by definition of \( \varphi \) (cf. equation (3.2)),

\[
\frac{1}{|I|} \int_I \varphi \, dx \geq \varepsilon_i^{-1}.
\]

Thus, Theorem 3.3 gives us a set \( A(I) \subset I \) with \( |A(I)| \geq |I|/4 \) and

\[
|P_I\varphi(x)| \geq (\varepsilon_i c_k c_k)^{-1}, \quad x \in A(I).
\]

As a consequence of the latter display, we find

\[
|S_j^{(i)} \cap B_i| \geq \sum_{\ell=1}^{s} \left| \bigcup_{m=1}^{N_j^{(i)}} A(I_m^{(\ell)}) \right| + \sum_{\ell=1}^{r} |A(J^{(\ell)})| \geq c_1 \sum_{\ell=1}^{s} \left| \bigcup_{m=1}^{N_j^{(i)}} I_m^{(\ell)} \right| + \frac{1}{4} \sum_{\ell=1}^{r} |J^{(\ell)}|,
\]

where in the second inequality, we applied Lemma 3.5. Consequently, with the understanding that \( 1/4 \geq c_1 \) and using (3.5),

\[
|B_i| = \sum_{j=1}^{L_i} |S_j^{(i)} \cap B_i| \geq c_1 \sum_{j=1}^{L_i} |S_j^{(i)}| = c_1 |[0,1]^2| = c_1,
\]
concluding the proof of Theorem \[1,2\]. □

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