Weak field approximation
in a model of de Sitter gravity:
Schwarzschild-de Sitter solutions

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Abstract

The weak field approximation in a model of de Sitter gravity is investigated in the static and spherically symmetric case, under the assumption that the vacuum spacetime without perturbations from matter fields is a torsion-free de Sitter spacetime. It is shown on one hand that any solution should be singular at the center of the matter field, if the exterior is described by a Schwarzschild-de Sitter spacetime and is smoothly connected to the interior. On the other, all the regular solutions are obtained, which might be used to explain the galactic rotation curves without involving dark matter.

1 Introduction

Many works have been done on the gauge theories of gravity \cite{1}, motivated by the similarities between Einstein’s general relativity (GR) and Yang-Mills gauge theory. A variety of models with different dynamics have been studied, among which there is a model of de Sitter (dS) gravity \cite{2–4, 6, 7} with a gauge-like action constructed by a dS algebra-valued connection. The Einstein term and a cosmological term can be deduced from the gauge-like action besides for two quadratic terms of the curvature and torsion. If the coefficient of the Einstein term is required to be much larger than that of the quadratic curvature term, the cosmological constant deduced from the gauge-like action should be very large. The large cosmological constant may be canceled out by the vacuum energy density, leaving a small cosmological constant \cite{6}. But it is difficult to explain why the large cosmological constant and the vacuum energy density are so close, but not exactly equal, to each other. On the other hand, if the cosmological constant deduced from the gauge-like action is required to be small \cite{2–4, 7}, the coefficient of the Einstein term should be much smaller than that of the quadratic curvature term. The model under this case, hereafter referred to as the dS gravity model or simply as the model, might be very different from GR. But fortunately, it has been shown \cite{8, 9} that all torsion-free vacuum solutions of the model are the vacuum solutions of GR with the same cosmological constant, and vice versa. Moreover, it has been shown \cite{10, 11} that the model may explain the accelerating expansion of the universe and supply a natural transit from decelerating

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expansion to accelerating expansion without the help of dark energy. Of course, further investigations are needed to check whether the model could explain various experimental observations consistently.

In GR with a cosmological constant, the Schwarzschild-de Sitter solution (S-dS) [12], instead of the Schwarzschild solution in GR without cosmological constant, is used to explain the solar-system-scale observations, (such as, the precession of Mercury perihelion, light deflection, gravitational redshift, radar echo,) when the cosmological constant is small enough [13]. Although the vacuum Schwarzschild or the vacuum S-dS solution is sufficient and the details of the internal solution are not necessary for the explanation of these tests, a gravitational theory, as a fundamental theory, should permit regular, reasonable sources to generate such a kind of the field. In other words, a Schwarzschild or S-dS solution should be able to join with regular and reasonable internal solution under suitable junction conditions. It is well known that in GR there exist the regular internal solutions joining with the Schwarzschild or S-dS exterior. Moreover, any alternative, reasonable gravitational theory should be able to reduce to the Newtonian law of gravity in the weak field approximation in order to explain the daily-life gravitational phenomena, which also requires that the vacuum Schwarzschild or the vacuum S-dS solution can be connected to the internal solution with a non-uniform distribution of matter.

The S-dS spacetime, as one of the vacuum solutions of the dS gravity model, is also important for the model to explain the observations within the solar system scale, for the test particles without spin current, including light, move along the same geodesics in the S-dS spacetime obtained in different theories [1, 3, 5]. However, the existence of regular internal solution with the smooth junction to the S-dS solution is still not clear in the dS gravity model. It has been shown that the energy-momentum-stress tensor of a spinless fluid should be with constant trace in the torsion-free case of the model [10]. For the general case, the trace of the energy-momentum-stress tensor is not a constant. It shows that the torsion-free condition is not so reasonable in the general case. This property together with the smooth junction condition may stimulate the search for S-dS solutions with nonzero torsion. Although the different dS spacetimes with nonzero torsion in the model have been obtained in Refs. [14, 15], they are still not the S-dS solutions. For many quadratic models of gauge theory of gravity, S-dS solutions with spherically symmetric torsion have been given [16], but our model does not fall into those cases.

One may firstly check the existence of the smooth junction in the weak field approximation. For GR with a cosmological constant, it is easy to shown that there exist the regular internal solutions with the smooth junction to the S-dS exterior, in the static, spherically symmetric weak field approximation. But for the dS gravity model, more efforts should be taken on the weak field approximation. The Newtonian limit of the general quadratic models without cosmological term has been analyzed [17, 18] in the 1980’s. In those papers, the vacuum spacetime without perturbations from matter fields is the Minkowski spacetime. But in the theory with a cosmological term, the vacuum spacetime without perturbations should be a dS spacetime. The perturbations of a dS spacetime are much more complicated than those of the Minkowski spacetime. A naive way to simplify the computation is to let the cosmological constant $\Lambda \to 0$ directly in the field equations. This way is equivalent to directly set $\Lambda \to 0$ in the action. It is, however, not suitable for the model for the following reasons. The coefficient of the quadratic curvature term is much larger than those of the other terms in the action. When $\Lambda \to 0$, only the quadratic curvature term will survive in the action. As a result, in the Einstein-like equation (see
Eq. (7) in the following), only the symmetric and trace-free part will survive when $\Lambda \to 0$. Hence, the information of the trace part and the antisymmetric part of the Einstein-like equation is lost when $\Lambda \to 0$ is directly set in that equation.

In order to analyze the weak field approximation of the model, we have presented a way to tackle the above problem [19], in which the $\Lambda \to 0$ limit is set after the Einstein-like equation is decomposed into its trace part, symmetric trace-free part and antisymmetric part. In this way, the weak field approximation for the static and spherically symmetric case has been studied [19]. It is found that there exist the solutions describing the weak Schwarzschild fields with nonzero torsion and the smooth connection to regular internal solutions obeying the Newtonian gravitational law. The existence of such solutions would determine the value of the coupling constant, which is different from that given by Refs. [2–4, 7]. Moreover, there exist the solutions that could deduce the galactic rotation curves without invoking dark matter.

The purpose of this paper is to further study the weak field approximation of the model, taking more effects of $\Lambda$ into consideration. Although there exist dS spacetimes with nonzero torsion in the model [14, 15], we assume in this paper that the base spacetime is a torsion-free dS spacetime for simplicity, where the base spacetime is short for the vacuum spacetime without perturbations from matter fields. Suppose that the deviation from the base spacetime due to the presence of the matter could be characterized by a parameter $s$, called the matter parameter. The metric and torsion fields are required to be differentiable with respect to $s$ in a neighborhood of the point $s = 0$. As was mentioned above, the computation of the deviation is complicated, since the base spacetime is not the Minkowski spacetime, but a dS spacetime. In this paper, the computation is simplified by two means within two cases, respectively.

In the first case (Case A), the metric and torsion fields are assumed to be differentiable with respect to $s$ and $\Lambda$ in a neighborhood of the point $(s, \Lambda) = 0$. The weak field approximation is analyzed by means of expanding the geometry and matter fields at the point $(s, \Lambda) = 0$. All the solutions of the first-order metric and torsion fields with respect to $s$ and $\Lambda$ are surveyed. Unfortunately, any solution will be singular at the center ($r = 0$) of the matter field, if it describes the weak S-dS field with the smooth junction to an internal solution.

In the second case (Case B), the metric and torsion fields are not required to be differentiable with respect to $\Lambda$. The reason to relax the differentiability condition is that there exist indeed the dS spacetimes with torsion, where the torsion fields are not differentiable with respect to $\Lambda$ at the point $\Lambda = 0$ [14]. In this case, the computation is simplified by means of setting $\Lambda \to 0$ in the weak field approximate equations. Interestingly enough, even if the $\Lambda \to 0$ limit is set, the effects of $\Lambda$ show up in the equations and the corresponding solutions. Again, the smooth junction between the weak Schwarzschild fields and regular internal solutions does not exist in this case. The result shows that the $\Lambda \to 0$ limit of the weak field approximate equations is not equivalent to the weak field approximation of the $\Lambda \to 0$ limit of the field equations, even when the Einstein-like equation is decomposed into its trace part, symmetric trace-free part and antisymmetric part, as discussed in Ref. [19].

In addition, all the static and spherically symmetric solutions which are regular at $r = 0$ are presented. These regular solutions might be used to explain the galactic rotation curves without involving dark matter. In GR with a cosmological constant, although the cosmological constant would have some important effects in the galactic-scale physics [20],
the effects are too small to explain the galactic rotation curves [29]. On the contrary, the possible explanation for the galactic rotation curves is based on the torsion effects in this paper.

The paper is organized as follows. We first give a brief review to the model of the dS gravity in section 2. In two subsections of section 3, the static, spherically symmetric weak field approximations of the model are calculated by two means for the two cases, respectively. Finally, we end with some remarks in section 4.

2 A model of dS gravity

A model of dS gravity has been proposed with a gauge-like action [2–4, 6, 7]

\[ S_G = \int L_G = \int \kappa [-\text{tr}(F_{ab}F^{ab})] \]

\[ = \int \kappa [R_{abcd}R^{abcd} - \frac{4}{l^2}(R - \frac{6}{l^2}) + \frac{2}{l^2}S_{abc}S^{abc}] \]  

(1)

in the units of \( \hbar = c = 1 \), where \( \kappa \) is a dimensionless coupling constant, \( l \) is the radius of the internal dS space and is related to a small cosmological constant \( \Lambda = \frac{3}{l^2} \) [2–4, 7], and

\[ F_{ab} = (dA + \frac{1}{2}[A, A])_{ab} \]  

(2)

or explicitly

\[ F^A_{Bab} = (dA^A_{B})_{ab} + A^A_{Ca} \wedge A^C_{Bb} \]

\[ = \begin{pmatrix} R_{ab}^\alpha \beta - l^{-2}e^\alpha_a \wedge e^\beta_b & l^{-1}S^\alpha_{ab} \\ -l^{-1}S^\beta_{ab} & 0 \end{pmatrix} \]  

(3)

is a dS algebra-valued 2-form and

\[ A^A_{Ba} = \begin{pmatrix} \Gamma^\alpha_{\beta a} & l^{-1}e^\alpha_a \\ -l^{-1}e^\beta_a & 0 \end{pmatrix} \]  

(4)

is a dS algebra-valued 1-form. Here \( A, B \ldots = 0, 1, 2, 3, 4 \) stand for matrix indices (internal indices) and the trace in Eq. (1) is taken for those indices. In addition, \( \{e^\alpha_a\} \) is some local orthonormal frame field on the spacetime manifold and \( \Gamma^\alpha_{\beta a} \) is the connection 1-form in this frame field, where \( a, b \ldots \) stand for abstract indices [21, 22] and \( \alpha, \beta \ldots = 0, 1, 2, 3 \). The curvature 2-form \( R^\alpha_{\beta a} \) and torsion 2-form \( S^\alpha_{ab} \) are related to the connection 1-form \( \Gamma^\alpha_{\beta a} \) as follows:

\[ R^\alpha_{\beta a} = (d\Gamma^\alpha_{\beta})_{ab} + \Gamma^\alpha_{\gamma a} \wedge \Gamma^\gamma_{\beta b}, \]

\[ S^\alpha_{ab} = (de^\alpha)_{ab} + \Gamma^\alpha_{\beta a} \wedge e^\beta_b. \]

Moreover,

\[ R_{abc}^d = R_{aba}^\beta e^\alpha_a e^\beta_b e^\delta_c, \quad S^c_{ab} = S^\alpha_{ab} e^\alpha_c, \]

\[ R_{ab} = g^{cd}R_{abc}, \quad R = g^{ab}R_{ab}, \]

where the indices \( \alpha, \beta \) are raised or lowered by \( \eta^{\alpha\beta} \) or \( \eta_{\alpha\beta} \), respectively. The signature is such chosen that \( \eta^{00} = \eta_{00} = -1, \eta^{ii} = \eta_{ii} = 1 \), where \( i = 1, 2, 3 \). In fact, if the
spacetime is an umbilical submanifold of some (1+4)-dimensional (5d) ambient manifold, then \( \mathcal{A}_a \) and \( \mathcal{F}_{ab} \) could be viewed as the connection 1-form and curvature 2-form (in a dS-Lorentz frame) of the ambient manifold restricted to the spacetime [7]. Here, an umbilical submanifold means a submanifold with constant normal curvature, such as a dS spacetime can be seen as an umbilical submanifold of the 5d Minkowski spacetime. \( \mathcal{A}_a \) can also be seen as the pullback of the Cartan connection by a local section of the frame bundle [23]. A principle bundle equipped with a Cartan connection is a Cartan geometry which could be seen as a generalization of a homogenous space, and one may refer to Ref. [23] for more details.

The total action is \( S = S_M + S_G \), where \( S_M \) is the action of the matter fields, and the field equations can be given via the variational principle with respect to \( e^\alpha_a, \Gamma^a_{\beta a} \):

\[
\frac{8}{l^2} (G_{ab} + \Lambda g_{ab}) + |R|^2 g_{ab} - 4 R_{acde} R_b^{cde} + \frac{2}{l^2} |S|^2 g_{ab} \\
- \frac{8}{l^2} S_{cda} S^{cda} + \frac{8}{l^2} \nabla_c S_{ab} + \frac{4}{l^2} S_{acda} T_b^{dca} + \frac{1}{\kappa} \Sigma_{ab} = 0, \\
- \frac{4}{l^2} T_a^{bc} - 4 \nabla_a R^{da}_\text{bc} + 2 T_a^{de}_{\text{debc}} - \frac{8}{l^2} S_{[bc]}^a + \frac{1}{\kappa} \tau_a^c = 0,
\]

where

\[
G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}, \quad T^c_{ab} = S^c_{ab} + 2 \delta^c_{[a} S^{bd]}_{b}, \\
|R|^2 = R_{abcd} R^{abcd}, \quad |S|^2 = S_{abc} S^{abc}, \\
\Sigma^a = \delta S_M/\delta e^\alpha_a, \quad \Sigma^a_b = \delta S_M/\delta e^a_b, \\
\tau^a_{\beta a} = \delta S_M/\delta \Gamma^a_{\beta a}, \quad \tau^c_{a\beta} = \tau^a_{\beta a} e^b_{c\beta} e^c_b,
\]

\( \nabla_a \) is the covariant derivative operator, which satisfies \( \nabla_a g_{bc} = 0 \) and is related to the connection 1-form by

\[ \Gamma^a_{\beta a} = e^\beta_b \nabla_a e^a_b, \]

and the variational derivatives are defined as follows: if

\[ \delta S_M = \int (X^a_\alpha \delta e^a_a + Y_{\alpha \beta} \delta \Gamma^a_{\beta a} ), \]

then

\[ \delta S_M/\delta e^a_a = X^a_\alpha, \quad \delta S_M/\delta \Gamma^a_{\beta a} = Y_{[\alpha \beta]}^a. \]

## 3 Weak field approximation

In this section, we focus on the weak field approximation of the model, which represents the linear deviation of the spacetime from the base spacetime. As mentioned in Introduction, the base spacetime is chosen to be a torsion-free dS spacetime. In a dS spacetime, there exist Beltrami coordinate systems which play the role of inertial coordinate systems [24], where timelike/null geodesics are all coordinate straight lines. The components of the metric field of a dS spacetime in a Beltrami system can be written down as follows:

\[
g_{\mu\nu} = \frac{1}{\sigma} \eta_{\mu\nu} - \frac{1}{\sigma^2 l^2} x^\mu x^\nu = \eta_{\mu\nu} - (\eta_{\mu\nu} x^\sigma x_\sigma + x^\mu x^\nu) \frac{1}{l^2} + o(\frac{1}{l^2}),
\]

where \( \eta_{\mu\nu} = \text{diag}(-1,1,1,1) \), \( \sigma = 1 + x^\mu x^\mu/l^2 \), \( x^\mu = \eta_{\mu\nu} x^\nu \) and \( \mu, \nu = 0,1,2,3 \). The Beltrami coordinates \( x^\mu \) are confined by the domain condition \( \sigma > 0 \).
3.1 Case A

We first consider the case where the metric and torsion fields are differentiable with respect to \( s \) and \( \Lambda \) in a neighborhood of the point \((s, \Lambda) = 0\). Precisely speaking, it is assumed that

\[
\begin{align*}
g_{ab} &= \eta_{ab} + \dot{g}_{ab}^{(1)} + \gamma_{ab} + \mathcal{O}(s^2 + (l_0/l)^4), \quad S^c_{\ ab} = \mathcal{O}(s^2 + (l_0/l)^4), \\
g_{ab}|_{s=0} &= \dot{g}_{ab}, \quad S^c_{\ ab}|_{s=0} = 0, \\
\Sigma_{ab} &= \mathcal{O}(s), \quad \tau_{ab}^c = \mathcal{O}(s),
\end{align*}
\]

(10)

where \( s \) is a dimensionless parameter, called the matter parameter, which describes the effects of the existence of matter fields, \( l_0 \) is a fixed constant with dimension of length, which could be chosen as the scale of the system under consideration, \( \gamma_{ab} \propto s \) is independent of \( l \), \( \eta_{ab} = \eta_{\mu\nu}(dx^\mu)_a(dx^\nu)_b \), \( \dot{g}_{ab} = \dot{g}_{\mu\nu}(dx^\mu)_a(dx^\nu)_b \), and

\[
\dot{g}_{ab}^{(1)} = -\left(\eta_{\mu\nu}x^\sigma x_\sigma + x_\mu x_\nu\right)\frac{1}{l^2}(dx^\mu)_a(dx^\nu)_b,
\]

(13)

where \( x^\mu \) are local coordinates of the spacetime, which will come back to the Beltrami coordinates of the dS spacetime when \( s \to 0 \). With respect to the assumption (10), the terms independent of \( s \) and \( l \) are regarded as the zeroth-order terms, the terms proportional to \( s \) or \( 1/l^2 \) are regarded as the first-order terms, the terms proportional to \( s^2 \), \( s/l^2 \) or \( 1/l^4 \) are regarded as the second-order terms, and so on. In this viewpoint, the metric field of the base spacetime contains not only the zeroth-order term but also the higher-order terms, as was shown in Eq. (9). Let \( \Gamma^c_{\ ab} = \Gamma_{\ mu}(\partial_\sigma)^c(dx^\mu)_a(dx^\nu)_b \), where \( \Gamma_{\ mu} = (\partial_\nu)^a(dx^\sigma)_b \nabla_{\ a}(\partial_\mu)b \) is the connection coefficient in \( \{x^\mu\} \). \( \Gamma^c_{\ ab} \) and the curvature tensor have the following first-order approximate expressions:

\[
\Gamma^c_{\ ab} = \dot{\Gamma}^{(1)c}_{\ ab} + \frac{1}{2}(\partial_a \gamma^c_b + \partial_b \gamma^c_a - \partial^c \gamma_{ab}) - K^c_{\ ab},
\]

(14)

\[
R^d_{\ abc} = \dot{R}^{(1)d}_{\ abc} - \frac{1}{l^2}(\partial_a \partial^d \gamma_{b[c]} + \partial^d \partial_{[a} \gamma_{b]c}) + 2\partial^d (\partial_{[a} K^c_{\ b]c}),
\]

(15)

where

\[
\dot{\Gamma}^{(1)c}_{\ ab} = -\frac{2}{l^2}\delta^c_{\ (\mu x^\nu)}(\partial_\sigma)^c(dx^\mu)_a(dx^\nu)_b,
\]

(16)

\[
\dot{R}^{(1)d}_{\ abc} = \frac{2}{l^2}\eta_{\ [c} \delta^d_{\ b]}.
\]

(17)

are from the dS metric field and

\[
K^c_{\ ab} = \frac{1}{2}(S^c_{\ ab} + S_{\ ad}^c + S_{\ bc}^c)\]

(18)

is the contorsion tensor.

The coupling constant \( \kappa \) is set to be \(-l^2/64\pi G\) in Refs. [2–4, 7] and \( l^2/32\pi G \) in Ref. [19]. In this paper, \( \kappa \) is to be determined. It has been pointed [19] out that when \( l \to \infty \), \( l^2/\kappa \) should tend to a finite value. Therefore, \( 1/\kappa = \mathcal{O}(1/l^2) \), and there does not exist first-order term in Eq. (7). The second-order approximation of Eq. (7) and the first-order approximation of Eq. (8) are as follows:

\[
\frac{8}{l^2}(G_{ab} + \Lambda \eta_{ab}) + |R|^2 \eta_{ab} - 4R_{acd}R_{b}^{\ cde} + \frac{8}{l^2} \partial_c S_{ab}^c + \frac{1}{\kappa} \Sigma_{ab} = 0,
\]

(19)
\[ \partial_d R^{da}{}_{bc} = 0. \]  

They are both equations for the first-order approximate metric and torsion fields. The second-order approximation of Eq. (8) would not be considered, for the reason that it involves the second-order approximate metric and torsion fields.

Now we restrict ourselves to the static and \( O(3) \)-symmetric case, with the static spherical coordinate system \( \{ T, \varrho, \theta, \varphi \} \). Let \( \{ \xi^i \} \) be related to \( \{ \varrho, \theta, \varphi \} \) as follows:

\[ \xi^1 = \varrho \sin \theta \cos \varphi, \quad \xi^2 = \varrho \sin \theta \sin \varphi, \quad \xi^3 = \varrho \cos \theta. \]  

Suppose that the coordinate systems \( \{ x^\mu \} \) and \( \{ T, \xi^i \} \) be related by:

\[ t \equiv x^0 = l \tanh(T/l) = T(1 - \frac{1}{3} T^2/l^2) + o(\frac{1}{l^2}), \]  

\[ x^i = \xi^i [\cosh(T/l)]^{\frac{1}{2}}[1 - \frac{1}{2} T^2/l^2 + \frac{1}{3} \frac{1}{2} T^2/l^2 + o(\frac{1}{l^2})]. \]  

The above relation is just the same as that of the Beltrami coordinates and the static spherical coordinates in the \( dS \) spacetime [25]. In the following discussion, the difference between \( t \) and \( T \), and the difference between \( x^i \) and \( \xi^i \) will turn out to be negligible. In the static and \( O(3) \)-symmetric case, \( g_{ab} \) and \( S^c{}_{ab} \) have only these independent components in \( \{ T, \varrho, \theta, \varphi \} \) [26]:

\[ g_{TT} = g_{TT}(\varrho), \quad g_{\varrho \varrho} = g_{\varrho \varrho}(\varrho), \quad g_{\vartheta \vartheta} = \varrho^2, \quad g_{\varphi \varphi} = \varrho^2 \sin^2 \theta, \]  

\[ \left\{ \begin{array}{ll}
S^T T^b = f(\varrho), & S^\varrho T^\varrho = S^\varphi T^\varphi = g(\varrho), \\
S^\vartheta T^\vartheta = h(\varrho), & S^\varphi T^\varphi = -k(\varrho).
\end{array} \right. \]  

By Eqs. (10), (21)—(25), it can be checked that the components of \( \gamma_{ab} \) and the first-order approximate \( S^c{}_{ab} \) in \( \{ x^\mu \} \) are as follows:

\[ \gamma_{00} = -2 \psi, \quad \gamma_{0i} = 0, \quad \gamma_{ij} = (2 \psi) x_i x_j/r^2, \]  

\[ \left\{ \begin{array}{ll}
S^0_{0i} = f x_i /r, & S^0_{ij} = 0, \\
S^i_{0j} = (h + k) x^i x_j/r^2 - k \delta^i_j, & S^i_{jk} = (-g/r)(\delta^i_j x_k - \delta^i_k x_j),
\end{array} \right. \]  

where \( \psi, \phi \) are functions of \( r = \sqrt{\sum_{i=1}^3 (x^i)^2} \), and \( f, g, h, k \) are the first-order approximations of \( f(\varrho), g(\varrho), h(\varrho), k(\varrho) \) with \( l \to \infty \). From Eqs. (21), (22) and (23), when \( l \to \infty, \varrho \to r \). So \( f, g, h, k \) are functions of \( r \). For the case of (27), the contorsion tensor is related to the torsion tensor by

\[ K_{abc} = S_{cda}. \]  

The equation of motion for a free spinless particle is assumed to be the Riemannian geodesic equation:

\[ U^b \partial_b U^a + \{ a \}_{bc} U^b U^c = 0, \]  

where \( U^a \) is the normalized tangent vector field of the particle’s world line and

\[ \{ a \}_{bc} = \frac{1}{2} g^{ad} (\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}). \]
is the Riemannian part of the connection $\Gamma^e_{ab}$. Suppose that the particle’s world line could be described by the functions $x^i(t)$ in $\{x^\mu\}$. Let $u^i = dx^i/dt = O(\sqrt{\epsilon^2 + s + (l_0/l)^2}$, where $\epsilon$ is a dimensionless parameter, called the low velocity parameter, which describes the effects coming from the initial velocity $u^i(0)$. Under this assumption, those terms proportional to $\epsilon$, $\sqrt{s}$ or $1/l$ are regarded as the $\frac{1}{2}$th-order terms. Both the zeroth-order and the $\frac{1}{2}$th-order approximations of Eq. (29) will give $a^i = 0$, and the first-order approximation of Eq. (29) will give $a^i = -\partial^i \phi$, where $a^i = \partial^2 x^i/dt^2$. Therefore, $\phi$ may play the role of the Newtonian gravitational potential. The equation $a^i = -\partial^i \phi$ contains the solution which describes a circular motion with uniform speed:

$$u^2 = \frac{\partial \phi}{\partial r} \bigg|_{r=r(0)},$$

where $u^2 = \sum_{i=1}^3 (u^i)^2$, and $r(0) = \sqrt{\sum_{i=1}^3 [x^i(0)]^2}$ is the radius of the circular orbit.

Applying Eq. (15) to Eq. (20), we have

$$\partial_a(\partial^d \partial^c x^b_a) + \partial^d K^a_{bc} = 0,$$

which gives [19]

$$\phi' + f = C r + D/r^2,$$

$$h + k + r k' = 0,$$

$$\psi/r^2 - g/r = B/r^3 + A,$$

where $A, B, C, D$ are constants of integration.

From Eqs. (15), (26) and (27), the leading components of $R_{abcd}$ in $\{x^\mu\}$ could be attained as follows:

$$\begin{cases} R_{00ij} = -\delta_{ij}/l^2 + (\phi'' + f')x_i x_j/r^2 + (\phi' + f)(\delta_{ij} r^2 - x_i x_j)/r^3, \\ R_{0ijk} = 0, \\ R_{ijkl} = (2/r^2)(h + k + r k')x_i x_j x_k x_l, \\ R_{ijkl} = 2\delta_{ik}\delta_{jl}/l^2 + (\psi/r^2 - g/r)'(4/r)x_i x_j x_k x_l - 4(\psi/r^2 - g/r)\delta_{ik}\delta_{jl}, \end{cases}$$

Substituting Eqs. (33), (34) and (35) into the above equation, we obtain

$$\begin{cases} R_{00ij} = -\delta_{ij}/l^2 + (C r + D/r^2)'x_i x_j/r^2 + (C r + D/r^2)(\delta_{ij} r^2 - x_i x_j)/r^3, \\ R_{0ijk} = 0, \\ R_{ijk0} = 0, \\ R_{ijkl} = 2\delta_{ik}\delta_{jl}/l^2 + (B/r^3 + A)'(4/r)x_i x_j x_k x_l - 4(B/r^3 + A)\delta_{ik}\delta_{jl}. \end{cases}$$

Then

$$\begin{align*}
|R|^2 \eta_{00} - 4R_{oade} R^o_{cde} &= (-24/l^2)(C - 2A) + 12(C^2 - 4A^2) + 24(D^2 - B^2)/r^6, \\
|R|^2 \eta_{ii} - 4R_{ioed} R^i_{cde} &= 0, \\
|R|^2 \eta_{ij} - 4R_{ioed} R^i_{jcd} &= (-8/l^2)(C - 2A - 2B/r^3 - 2D/r^3)\delta_{ij} \\
&+ (-48/l^2)(B/r^3 + D/r^3)x_i x_j/r^2 \\
&+ (4C^2 - 16A^2 - 16CD/r^3 - 32AB/r^3 + 16D^2/r^6 - 16B^2/r^6)\delta_{ij} \\
&+ (4CD/r^3 + 96AB/r^3 - 24D^2/r^6 + 24B^2/r^6)x_i x_j/r^2.
\end{align*}$$
\begin{align*}
G_{00} + \Lambda \eta_{00} &= -6A, \\
G_{0i} + \Lambda \eta_{0i} &= G_{i0} + \Lambda \eta_{i0} = 0, \\
G_{ij} + \Lambda \eta_{ij} &= (2C + 2A - D/r^3 - B/r^3)\delta_{ij} + [3(B + D)/r^3]x_ix_j/r^2. \\
\end{align*}

From Eq. (27), the leading components of $\partial_c S_{ab}^c$ could be given as follows:

\begin{align*}
\partial_c S_{00}^c &= -f' - 2f/r, \\
\partial_c S_{0i}^c &= 0, \\
\partial_c S_{i0}^c &= x_i[h'/r + 2(h + k)/r^2], \\
\partial_c S_{ij}^c &= [-2(g/r) - r(g/r)'\delta_{ij} + (g/r)'x_ix_j/r]. \\
\end{align*}

Substituting Eqs. (33) and (35) into the above equation, we have

\begin{align*}
\partial_c S_{00}^c &= \Delta \phi - 3C, \\
\partial_c S_{0i}^c &= 0, \\
\partial_c S_{i0}^c &= x_i[h'/r + 2(h + k)/r^2], \\
\partial_c S_{ij}^c &= (-B/r^3 + 2A - \psi'/r)\delta_{ij} + [r(\psi/r^2)' + 3B/r^3]x_ix_j/r^2, \\
\end{align*}

where $\Delta \phi = \phi'' + (2/r)\phi'$.

With the help of Eqs. (38), (39) and (41), Eq. (19) can be solved. Note that from Eqs. (10) and (11), $S^c_{ab} \propto s$ in the first-order approximation, and so all of the constants $A$, $B$, $C$ and $D$ are proportional to $s$. Eq. (19) splits into two equations, one for those terms proportional to $s^2$ and the other one for those terms proportional to $s/l^2$, while those terms proportional to $1/l^2$ add up to zero. The $s^2$ equation yields the following relations:

\begin{align*}
C &= \pm 2A, \quad B = \pm D, \quad CD + 2AB = 0. \\
\end{align*}

Suppose that $\Sigma_{ab} = \rho(dt)_a(dt)_b$. Then the 00 component of the $s/l^2$ equation is

\begin{equation}
\Delta \phi - 6C + (\dot{\rho}^2/8\kappa)\rho = 0, 
\end{equation}

the 0i component is an identity and the i0 component gives

\begin{equation}
h'/r + 2(h + k)/r^2 = 0 
\end{equation}

with the solution

\begin{equation}
h = 2k + C_1, \quad h + k = C_2/r^3, 
\end{equation}

where Eq. (34) has been used. Finally, the ij component gives

\begin{equation}
(C + 6A + D/r^3 - \psi'/r)\delta_{ij} + [r(\psi/r^2)' - 3D/r^3]x_ix_j/r^2 = 0 
\end{equation}

with the solution

\begin{equation}
\psi = \frac{1}{2} (C + 6A)r^2 - D/r. 
\end{equation}

For the S-dS solution, no matter with or without torsion, there should be $\phi = \psi = -GM/r$ and so $D = GM$. If it has a smooth connection to an internal solution, then the integration constant $D = GM$ should also hold for that internal solution. As a result, the internal solution will be singular at $r = 0$. Such a singular solution may represent a strong field near $r = 0$. It should be stressed that it is the solution of the first-order gravitational field under the assumptions (10)–(12). The solution shows that if the interior is smoothly
connected to the S-dS exterior, its first-order term should be singular at \( r = 0 \). Therefore, the interior in the full theory should be singular at \( r = 0 \), too.

For those solutions which are regular at \( r = 0 \), we should let \( B = D = 0 \) and \( C_2 = 0 \) from Eqs. (33), (35) and (44). Then

\[
\psi = \frac{1}{2} (C + 6A) r^2, \tag{46}
\]

\[
f = Cr - \phi', \quad g = \frac{1}{2} (C + 4A) r, \quad h = -k = \frac{1}{3} C_1, \tag{47}
\]

and \( \phi \) is still given by Eq. (43). Generally, the torsion tensor can be decomposed \([27, 28]\) into three irreducible parts with respect to the Lorentz group: the tensor part, trace-vector part, and the axial vector part. For static and \( O(3) \)-symmetric torsion, the axial vector part vanishes automatically, the tensor part satisfies \( f = 2g, h = 2k \) and the trace-vector part satisfies \( f = -g, h = -k \). In the above solutions, if the torsion tensor only contains the trace-vector part, then

\[
\phi' = \frac{1}{2} (3C + 4A) r, \quad \Delta \phi = \frac{3}{2} (3C + 4A).
\]

If the torsion tensor only contains the tensor part, then

\[
\phi' = -4Ar, \quad \Delta \phi = -12A.
\]

Therefore the above two cases can only describe matter fields with constant density.

As a special choice, we may let \( C = A = 0 \) and \( C_1 = 0 \), then

\[
\Delta \phi + (l^2/8\kappa) \rho = 0, \quad \psi = 0, \quad f = -\phi', \quad g = h = k = 0. \tag{48}
\]

In this solution, the curvature tensor is just the same as that of the dS spacetime, and the torsion tensor has the property that both the tensor and trace-vector parts have nonzero values unless \( \rho = 0 \). A comparison between Eq. (48) and \( \Delta \phi = 4\pi G \rho \) would fix the value of the coupling constant:

\[
\kappa = -l^2/32\pi G, \tag{49}
\]

which is different from those of Refs. [2–4, 7, 19]. The choice in Refs. [2–4, 7] is to guarantee the ratio of the coefficient of \( G_{ab} \) to that of \( \Sigma_{ab} \) in Eq. (7) be 1 : \((-8\pi G)\), just like the case in GR. But it has been pointed [19] out that the Einstein term in our model plays a completely different role from that of GR. By Eq. (39), the 00 component of the Einstein term with the \( \Lambda \) term only contributes a constant term, which would be canceled by the quadratic curvature term and does not show up in Eq. (43). In the \( \Lambda \to 0 \) limit in Ref. [19], the choice of the coupling constant is to guarantee the existence of the smooth junction between the weak Schwarzschild fields and regular internal solutions obeying the Newtonian gravitational law. But that smooth junction has no correspondence in the present case with \( \Lambda \to 0 \).

Generally, the regular solutions are given by Eqs. (46), (47) and

\[
\Delta \phi - 6C = 4\pi G \rho, \tag{50}
\]

where Eq. (49) has been used. The integration of Eq. (50) gives

\[
\phi' = Gm(r)/r^2 + 2Cr, \tag{51}
\]
where

\[ m(r) = 4\pi \int_0^r \rho r^2 dr. \]

Substituting Eq. (51) into Eq. (31) results in

\[ u^2(r) = \frac{Gm(r)}{r} + 2Cr^2, \quad (52) \]

which describes the speed of a free particle revolving around the center of the gravitational field. Suppose that the coordinate radius of the matter surface is \( R_s \), and denote \( m(R_s) \) by \( M \). In the region with \( r > R_s \), Eq. (52) can be viewed as the first-order term of

\[ v^2(r) = \frac{GM/r + 2Cr^2}{1 - 2GM/r + 2Cr^2}, \quad (53) \]

which is the square of the rotation speed in GR with a cosmological constant \( \Lambda_0 \), relative to an inertial observer which is instantaneously at rest in the static frame of reference, if \( \Lambda_0 = -6C \). Eq. (53) with \( \Lambda_0 = -10^{-48}\text{m}^{-2} (C \approx 10^{-48}\text{m}^{-2}) \) has been used to fit the flat rotation curves for the galaxies NGC 224, 2841, 2903, etc without involving dark matter [29]. Hence, Eq. (52), as the first order approximation of Eq. (53), might also be used to explain the flat rotation curves in the same way with the same value for \( C \). It is worth mentioning that the constant \( C \) represents the torsion effects here. It may get rid of the inconsistency of the value and the sign of the cosmological constant with cosmological observations in Ref. [29]. In fact, Eqs. (42) and (47) show that \( C \) is related to the torsion component \( g \) by

\[ C = \frac{2g}{(1 \pm 2)r}. \quad (54) \]

Finally, it should be remarked that Eq. (53) would no longer be valid in our model. The higher-order behavior of the rotation speed function will be different from that in GR.

### 3.2 Case B

Now, we turn to the general case where the metric field and torsion field are not required to be differentiable with respect to \( \Lambda \). The assumption on matter fields is still given by Eq. (12), and the geometry is assumed to be

\[ g_{ab} = \hat{g}_{ab} + \gamma_{ab} + o(s), \quad S_{ab} = O(s), \quad (55) \]

where \( \gamma_{ab}, o(s) \) and \( O(s) \) may depend on \( l \). From Eqs. (9), (13) and (55), we have

\[ g_{ab} = \eta_{ab} + \hat{g}^{(1)}_{ab} + \gamma_{ab} + o\left(\frac{1}{l^2}\right) + o(s), \quad (56) \]

where \( o(1/l^2) \) is independent of \( s \). The expressions for the connection and curvature are as follows:

\[ \Gamma^{(1)c}_{ab} = \hat{\Gamma}^{(1)c}_{ab} + \frac{1}{2}(\partial_a \gamma^c_b + \partial_b \gamma^c_a - \partial^c \gamma_{ab}) - K^c_{ab} + o\left(\frac{1}{l^2}\right) + O\left(\frac{1}{l^2}\right)O(s) + o(s), \quad (57) \]

\[ R_{abc}^d = \hat{R}_{abc}^d - (\partial_c \partial_a \gamma^d_b - \partial^d \partial_c \gamma_{ab}) + 2\partial_a K^d_{|c|b} + o\left(\frac{1}{l^2}\right) + O\left(\frac{1}{l^2}\right)O(s) + o(s). \quad (58) \]
For simplicity, we only consider the $\Lambda \to 0$ limit of the weak field approximate equations. With respect to the assumption \((55)\), those terms independent of $s$ are regarded as the zeroth-order terms, those terms proportional to $s$ are regarded as the first-order terms, those terms proportional to $s^2$ are regarded as the second-order terms, and so on. Then the $\Lambda \to 0$ limit of the second-order approximation of Eq. \((7)\) and the first-order approximation of Eq. \((8)\) are:

\[
|R|^2\eta_{ab} - 4R_{acde}R^{cde}_b = 0, \quad \partial_d R^{da}_{\ bc} = 0 \tag{59}
\]

with the solutions given by Eqs. \((33)\), \((34)\), \((35)\) and \((42)\). The $\Lambda \to 0$ limit of the first-order approximation of Eq. \((7)\) is

\[
8G_{ab} + l^2(|R|^2\eta_{ab} - 4R_{acde}R^{cde}_b)|_{l \to \infty} + 8\partial_c S_{ab}c + (l^2/\kappa)|_{l \to \infty}\Sigma_{ab} = 0. \tag{60}
\]

Obviously, the terms $o(1/l^2) + O(1/l^3)O(s)$ in Eq. \((58)\) do not contribute to Eq. \((60)\). The solutions of Eq. \((60)\) would then be given by Eqs. \((43)\), \((44)\) and \((45)\). Therefore, the smooth junction between the weak Schwarzschild fields and regular internal solutions does not exist. It shows that the $\Lambda \to 0$ limit of the weak field approximation is not equivalent to the weak field approximation of the $\Lambda \to 0$ limit discussed in Ref. \([19]\). In addition, all solutions of Eqs. \((59)\) and \((60)\), which are regular in the finite $r$ region, are given by Eqs. \((43)\), \((46)\) and \((47)\). They might be used to explain the galactic rotation curves, as discussed in the last subsection.

\section{Remarks}

The weak field approximation of the dS gravity model is analyzed in the static and spherically symmetric case, under the assumption that the base spacetime is a torsion-free dS spacetime. Concretely, we study the weak field approximation by two means within two cases, respectively. In Case A, the metric and torsion fields are required to be differentiable with respect to the matter parameter $s$ and the cosmological constant $\Lambda$ in a vicinity of the point $(s, \Lambda) = 0$. We expand the field equations with respect to $s$ and $\Lambda$, and calculate the first-order approximate metric and torsion fields. In Case B, the metric and torsion fields are not required to be differentiable with respect to $\Lambda$. We expand the field equations with respect to $s$, and solve the $\Lambda \to 0$ limit of the weak field approximate equations. Unfortunately, for both cases, we reach the conclusion that the S-dS solutions could not be smoothly connected to regular internal solutions.

For those solutions which are regular everywhere in the finite $r$ region, we find that due to the torsion effects, they might be used to deduce the galactic rotation curves without involving dark matter. In addition, in the regular solutions, both the tensor and trace-vector parts of the torsion field should be nonzero in a generic case with inhomogeneous matter density. It shows that the torsion tensor plays an important role in this model, as was pointed out by Ref. \([19]\). On the other hand, the curvature tensor has a special form, for example, given by Eq. \((37)\) in Case A. As a result, the Einstein term in this model plays a completely different role from that of GR. The choice of the coupling constant $\kappa = -l^2/64\pi G$ \([2-4, 7]\) may be problematic, which is due to the comparison between GR and the dS gravity model. In fact, if and only if $\kappa = -l^2/32\pi G$ is set, there exist the solutions which are in accordance with the Newtonian gravitational law in the two
cases of this paper, with $\Lambda \to 0$ and $\Lambda \to 0$, respectively. In the $\Lambda \to 0$ limit discussed in Ref. [19], $\kappa$ is set to be $l^2/32\pi G$, which is to guarantee the existence of the smooth junction between the weak Schwarzschild fields and regular internal solutions obeying the Newtonian gravitational law. But the limit cannot be attained from the two cases in the present paper.

Similar to the Schwarzschild solution in GR, the S-dS solution in the dS gravity model plays an important role in the explanation of the solar-system-scale observations. For the theory consistent with the observations within the solar system, the S-dS solution should be able to link to the regular internal solution smoothly. From the results of this paper, if there exist the S-dS solutions with the smooth junction to regular internal solutions, they would not satisfy Eqs. (12) and (55) or would be singular in the $\Lambda \to 0$ limit. Therefore, the future studies may be based on this fact. Especially, since the torsion tensor plays an important role, one may try to see what would happen if the base spacetime is a dS spacetime with torsion, the examples of which have been obtained in Refs. [14, 15]. If there does not exist the torsional S-dS solution with the smooth junction to a regular internal solution, the remaining way is to study higher order approximation of the gravitational field equations as well as the particle’s motion, which might be the only possible way to make the model consistent with the observations within the solar system.

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