ON THE CLASSIFICATION OF QUASI-HOMOGENEOUS CURVES

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Abstract. We apply techniques of Holomorphic Foliations in the description
of the analytic invariants associated to germs of quasi-homogeneous curves in
\((\mathbb{C}^2, 0)\). As a consequence, we obtain an effective method to determine whether
two quasihomogeneous curves are analytically equivalent.

1. Introduction

The problem of the classification of germs of analytic plane curves has been
addressed by several authors since the XVII\textsuperscript{th} century with different methods (see
for instance \cite{1}, \cite{2}, \cite{7}). In the present work, we study the problem of the analytic
classification of germs of singular curves with many branches from the viewpoint
of Holomorphic Foliations. This allows the use of geometrical techniques including
the blow-up and holonomy which are related to the study of normal forms for
quasi-homogeneous polynomials in two variables carried out in \cite{3}.

Next, we use the standard resolutions of theses singularities in order to stratify
them and thus identify the moduli space of each stratum. As a consequence, our
method provides an effective way to identify if two curves are equivalent. Finally,
we would like to remark that the analytic type of a quasi-homogeneous curve is one
of the invariants which determine the analytic type of a foliation having such a curve
as separatrix (cf. \cite{3}). Thus the present classification completes the classification
of such germs of complex analytic foliations.

2. Preliminaries

Let \(C\) be a singular curve, \(\pi : (\mathcal{M}, D) \rightarrow (\mathbb{C}^2, 0)\) its standard resolution, i.e. the
minimal resolution of \(C\) whose \textit{strict transform} \(\tilde{C} := \pi^{-1}(C) - D\) is transversal to
the exceptional divisor \(D = \pi^{-1}(0)\). A germ of holomorphic function \(f \in \mathbb{C}\{x, y\}\)
is said to be \textit{quasi-homogeneous} if there is a local system of coordinates in which \(f\)
can be represented by a quasi-homogeneous polynomial, i.e.
\[f(x, y) = \sum_{a_i + b_j = d} a_{ij}x^i y^j\]
where \(a, b, d \in \mathbb{N}\). Let \(M\) be a manifold and \(\Delta(n) := \{(x_1, \ldots, x_n) \in \mathbb{M}^n : x_i \neq x_j\}
for all \(i \neq j\). Let \(S_n\) denote the group of permutations of \(n\) elements and consider
its action in \(\Delta(n)\) given by \((\sigma, \lambda) \mapsto \sigma \cdot \lambda = (\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(n)})\). The quotient space
induced by this action is denoted by \(Symm(\Delta(n))\). Now suppose a Lie group \(G\)
acts in \(M\) and let \(G\) act in \(\Delta(n)\) in the natural way \((g, \lambda) = (g \cdot \lambda_1, \ldots, g \cdot \lambda_n)\)
for every \(\lambda \in \Delta(n)\). Then the actions of \(G\) and \(S_n\) in \(\Delta(n)\) commute. Thus

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one obtains a natural action of $G$ in $\text{Symm}(M_\Delta(n))$. Given $\lambda \in M_\Delta(n)$, denote its equivalence class in $\text{Symm}(M_\Delta(n))/G$ by $[\lambda]$. Let $C$ be a quasi-homogeneous curve determined by $f = 0$, where $f$ is a reduced polynomial. Then Lemma 3 says that $f$ can be (uniquely) written in the form

$$f(x, y) = x^m y^k \prod_{j=1}^{n} (y^p - \lambda_j x^q)$$

where $m, k \in \mathbb{Z}_2$, $p, q \in \mathbb{Z}_+$, $p \leq q$, $\gcd(p, q) = 1$, and $\lambda_j \in \mathbb{C}^*$ are pairwise distinct. In particular $C$ has $n + m + k$ distinct branches. Since the exceptional divisor of the standard resolution and the number of irreducible components are analytic invariants of a germ of curve, then Lemmas 4 and 5 ensure that the triple $(p, q, n)$ is an analytic invariant of the curve. Thus we have to consider the following three distinct cases:

i) $f(x, y) = x^m \prod_{j=1}^{n} (y - \lambda_j x)$ where $m \in \mathbb{Z}_2$, and $\lambda_j \in \mathbb{C}$.

ii) $f(x, y) = x^m \prod_{j=1}^{n} (y - \lambda_j x^q)$ where $m \in \mathbb{Z}_2$, $q \in \mathbb{Z}_+$, $q \geq 2$ and $\lambda_j \in \mathbb{C}$.

iii) $f(x, y) = x^m y^k \prod_{j=1}^{n} (y^p - \lambda_j x^q)$ where $m, k \in \mathbb{Z}_2$, $p, q \in \mathbb{Z}_+$, $2 \leq p < q$, $\gcd(p, q) = 1$, and $\lambda_j \in \mathbb{C}^*$.

A quasi-homogeneous curve is said to be of type $(1,1,n)$, $(1,q,n)$, and $(p,q,n)$ respectively in cases i), ii), and iii).

**Theorem 1.** The analytic moduli space of germs of quasi-homogeneous curves of type $(p,q,n)$ are given respectively by

i) $\frac{\text{Symm}(\mathbb{C}^1(n))}{\text{PSL}(2,\mathbb{C})}$ if $(p,q) = (1,1)$;

ii) $\mathbb{Z}_2 \times \frac{\text{Symm}(\mathbb{C}^2(n))}{\text{Aff}(\mathbb{C})}$ if $p = 1$ and $q > 1$;

iii) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \frac{\text{Symm}(\mathbb{C}^3(n))}{\text{GL}(1,\mathbb{C})}$ if $1 < p < q$.

3. QUASIHOMOGENEOUS POLYNOMIALS

3.1. Normal forms. A quasi-homogeneous polynomial $f \in \mathbb{C}[x,y]$ is called commode if its Newton polygon intersects both coordinate axis. Further, notice that a polynomial in two variables $P \in \mathbb{C}[x,y]$ may be considered as a polynomial in the variable $y$ with coefficients in $\mathbb{C}[x]$, i.e. $P \in (\mathbb{C}[x])[y]$. Let $\text{ord}_y P$ be the order of $P$ as a polynomial in $(\mathbb{C}[x])[y]$. Similarly let $\text{ord}_x P$ be the order of $P$ as an element of $(\mathbb{C}[y])[x]$. Therefore, a quasi-homogeneous polynomial $P \in \mathbb{C}[x,y]$ is commode if and only if $\text{ord}_y P = \text{ord}_x P = 0$. Next, we recall the general behavior of a quasi-homogeneous polynomial.

**Lemma 1.** Let $P \in \mathbb{C}[x,y]$ be a quasi-homogeneous polynomial, then it has a unique decomposition in the form

$$P(x,y) = x^m y^n P_0(x,y)$$

where $m,n \in \mathbb{N}$, $\lambda \in \mathbb{C}$, and $P_0$ is a commode quasi-homogeneous polynomial.
Lemma 3. Let \( P \in \mathbb{C}[x, y] \) be a quasi-homogeneous polynomial. Then \( P \) can be written, uniquely, in the form

\[
P(x, y) = \mu x^m y^n \prod_{\ell=1}^{k} (y^p - \lambda_{\ell} x^q)
\]

where \( m, n, p, q \in \mathbb{N} \), \( \mu, \lambda_{\ell} \in \mathbb{C}^* \), and \( \gcd(p, q) = 1 \).

Proof. In view of Lemma 1 and Lemma 2 it is enough to remark that any commode quasi-homogeneous polynomial \( P \in \mathbb{C}[x, y] \) can be written uniquely as \( P = \mu P_0 \) where \( P_0 \) is monic in \( y \).

3.2. Resolution. We recall the geometry of the exceptional divisor of the minimal resolution of a germ of quasi-homogeneous curve.

A tree of projective lines is an embedding of a connected and simply connected chain of projective lines intersecting transversely in a complex surface (two dimensional complex analytic manifold) with two projective lines in each intersection. In fact, it consists of a pasting of Hopf bundles whose zero sections are the projective lines themselves. A tree of points is any tree of projective lines in which a finite
number of points is discriminated. The above nomenclature has a natural motivation. In fact, it is well known that one can assign to each projective line a point and to each intersection an edge in other to form the weighted dual graph. Two trees of points are called isomorphic if their weighted dual graph are isomorphic (as graphs). It is well known that any germ of analytic curve \( C \) in \((\mathbb{C}^2, 0)\) has a standard resolution, which we denote by \( \tilde{C} \). If the exceptional divisor of \( \tilde{C} \) has just one projective line containing three or more singular points of \( \tilde{C} \), then it is called the principal projective line of \( \tilde{C} \) and denoted by \( D_{pr}(\tilde{C}) \). A tree of projective lines is called a linear chain if each of its projective lines intersects at most other two projective lines of the tree. A projective line of a linear chain is called an end if it intersects just another one projective line of the chain.

**Lemma 4.** Let \( C \) be a commode quasi-homogeneous curve. Then its standard resolution tree is a linear chain and its standard resolution \( \tilde{C} \) intersects just one projective line of \( D \), i.e. \( C \) has one of the following diagrams of resolution:

\[
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\circ
\end{array}
\]

Proof. From Lemma 2 there is a local system of coordinates \((x, y)\) such that \( C = f^{-1}(0) \) where \( f(x, y) = \prod_{i=1}^{k} (y^p - \lambda_i x^q) \) with \( p < q \) and \( \gcd(p, q) = 1 \). Since each irreducible curve \( y^p - \lambda_i x^q = 0 \) is a generic fiber of the fibration \( \frac{t^p}{x^q} \equiv \text{const} \), then it is resolved together with the fibration. After one blowup we obtain:

\[
\begin{align*}
t^p/x^q & \equiv \text{const}, \\
v^q y^{q-p} & \equiv \text{const}.
\end{align*}
\]

Since \( p < q \), we have a singularity with holomorphic first integral at infinity and a meromorphic first integral at the origin (as before). Going on with this process, Euclid’s algorithm assures that the resolution ends after the blowup of a radial foliation. In particular, if \( p = 1 \), then it is easy to see that the principal projective line is transversal to just one projective line of the divisor. Otherwise (i.e. if \( p \neq 1 \)) the singularity with meromorphic first integral “moves” to the “infinity”, i.e. it will appear in a corner singularity. Then the principal projective line intersects exactly two projective lines of the divisor. \( \square \)

Let \( \#irred(\tilde{C}) \) denote the number of irreducible components of \( \tilde{C} \).

**Lemma 5.** Let \( C \) be a non-commode quasi-homogeneous curve. Then its minimal resolution tree is a linear chain having a principal projective line such that \( \#(\tilde{C} \cap D_{pr}(\tilde{C})) \leq \#irred(\tilde{C}) - 1 \). Further \( \tilde{C} \cap D_j = \emptyset \) whenever \( D_j \) is neither the principal projective line nor an end; i.e. \( C \) has one of the following diagrams of resolution:

\[
\begin{array}{c}
\circ \\
\circ \\
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\circ \\
\circ \\
\circ
\end{array}
\]
we define $f$ λ-particular it is resolved after one blowup. Thus, given for all $j$ from Lemma 3, there is a local system of coordinates $(x, y)$ such that $C = f^{-1}(0)$ where $f(x, y) = \mu x^m y^n \prod_{j=1}^{k}(y^p - \lambda_j x^q)$, $p < q$, and $\gcd(p, q) = 1$. Since $\mu x^m y^n$ is resolved after one blowup, then $f(x, y)$ is resolved together with the fibration $\frac{y^p}{x^q} = \text{const}$, as before. Then the result follows from Lemma 4. □

4. QUASI-HOMOGENEOUS CURVES

We consider each case separately and prove Theorem 1 in a series of lemmas.

4.1. Curves of type $(1, 1, n)$. In this case the curve is given as the zero set of a polynomial of the form $f(x, y) = x^m \prod_{j=1}^{n}(y - \lambda_j x)$ where $m \in \mathbb{Z}_2$, and $\lambda_j \in \mathbb{C}$; in particular it is resolved after one blowup. Thus, given $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{P}_\Delta^1(n)$ we define $f_\lambda(x, y) = x \prod_{j \neq i}(y - \lambda_j x)$ if $\lambda_i = \infty$ or $f_\lambda(x, y) = \prod_{j=1}^{n}(y - \lambda_j x)$ if $\lambda_j \neq \infty$ for all $j = 1, \ldots, k$. We denote the curve $f_\lambda = 0$ by $C_\lambda$. Recall that the natural action of $PSL(2, \mathbb{C})$ in $\mathbb{P}_\Delta^1$ as the group of homographies induces a natural action of $PSL(2, \mathbb{C})$ in $\text{Symm}(\mathbb{P}_\Delta^1(n))$. Further, recall that the equivalence class of $\lambda \in \mathbb{P}_\Delta^1(n)$ in $\text{Symm}(\mathbb{P}_\Delta^1(n))/PSL(2, \mathbb{C})$ is denoted by $[\lambda]$.

Lemma 6. Two homogeneous curves $C_\lambda$ and $C_\mu$ are analytically equivalent if and only if $[\lambda] = [\mu] \in \text{Symm}(\mathbb{P}_\Delta^1(n))/PSL(2, \mathbb{C})$.

Proof. Suppose $C_\lambda$ and $C_\mu$ are analytically equivalent and let $\Phi \in \text{Diff}(\mathbb{C}^2, 0)$ take $C_\lambda$ into $C_\mu$. Let $\tilde{\Phi}$ be the blowup of $\Phi$, then it takes the strict transform of $C_\lambda$ into the strict transform of $C_\mu$. Blowing up $f_\lambda$ and $f_\mu$ we obtain at once that the first tangent cones of $C_\lambda$ and $C_\mu$ are respectively given by $\{\lambda_1, \ldots, \lambda_n\}$ and $\{\mu_1, \ldots, \mu_n\}$. Therefore, there is $\sigma \in S_n$ such that the Möbius transformation $\varphi = \tilde{\Phi}|_{\mathbb{P}_1}$ satisfies $\mu_{\sigma(j)} = \varphi(\lambda_j)$ for all $j = 1, \ldots, n$. In other words $[\lambda] = [\mu]$. Conversely, suppose $[\lambda] = [\mu]$. Reordering the indexes of $\{\mu_1, \ldots, \mu_n\}$ we may suppose, without loss of generality, that there is a Möbius transformation $\varphi(z) = \frac{az + b}{cz + d}$ with $ad - bc = 1$, such that $\mu_j = \varphi(\lambda_j)$ for all $j = 1, \ldots, n$. Now consider the linear transformation $T(x, y) = (dx + cy, bx + ay)$ with inverse $T^{-1}(x, y) = (ax - cy, -bx + dy)$. Then a straightforward calculation shows that $f_\lambda = \alpha \cdot T^* f_\mu$ where $\alpha \in \mathbb{C}^*$. Thus $C_\lambda$ is analytically equivalent to $C_\mu$, as desired. □

Remark 1. Recall that for any three distinct points $\{\lambda_1, \lambda_2, \lambda_3\} \subset \mathbb{P}_1$ there is a Möbius transformation $\varphi$ such that $\varphi(0) = \lambda_1$, $\varphi(1) = \lambda_2$ and $\varphi(\infty) = \lambda_3$.

As a straightforward consequence of Lemma 6 and Remark 1 one has:

Corollary 1. Let $\lambda, \mu \in \mathbb{P}_\Delta^1(n)$ with $n \leq 3$. Then $C_\lambda$ and $C_\mu$ are analytically equivalent.
4.2. **Curves of type** $(1, q, n)$, $q \geq 2$. In this case, the curve is given as the zero set of a polynomial of the form $f_{m, \lambda}(x, y) = x^m \prod_{j=1}^{n} (y - \lambda_j x^q)$ where $m \in \mathbb{Z}_2$, $q \in \mathbb{Z}_+$, $q \geq 2$, and $\lambda_j \in \mathbb{C}$. Thus given $m \in \mathbb{Z}_2$ and $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{C}_\Delta(n)$, we denote a curve of type $(1, q, n)$ by $C_{m, \lambda}$ if it is given as the zero set of $f_{m, \lambda}$. Recall that the group of affine transformations of $\mathbb{C}$, denoted by $\text{Aff}(\mathbb{C})$, acts in a natural way in $\text{Symm}(\mathbb{C}_\Delta(n))$. Further, recall that the equivalence class of $\lambda \in \mathbb{C}_\Delta(n)$ in $\text{Symm}(\mathbb{C}_\Delta(n))/\text{Aff}(\mathbb{C})$ is denoted by $[\lambda]$.

**Lemma 7.** Two homogeneous curves $C_{m, \lambda}$ and $C_{m, \mu}$ are analytically equivalent if and only if $[\lambda] = [\mu] \in \text{Symm}(\mathbb{C}_\Delta(n))/\text{Aff}(\mathbb{C})$.

**Proof.** Suppose $\Phi \in \text{Diff}(\mathbb{C}^2, 0)$ is an equivalence between $C_{m, \lambda}$ and $C_{m, \mu}$. From the proof of Lemma 4, both curves are resolved after $q$ blowups. Further, after $q - 1$ blowups $\Phi$ will be lifted to a local conjugacy $\Phi^{(q-1)}$ between the germs of curves given in local coordinates $(x, y)$ respectively by $p_\lambda(x, y) = x \prod_{j=1}^{n} (y - \lambda_j x)$ and $p_\mu(x, y) = x \prod_{j=1}^{n} (y - \mu_j x)$ where $(x = 0)$ is the local equation of the exceptional divisor $D^{(q-1)}$. Let $\pi$ denote a further blowup given in local coordinates by $\pi(t, x) = (x, tx)$ and $\pi(u, y) = (u, uy)$, and $\Phi^{(q)}$ be the map obtained by the lifting of $\Phi^{(q-1)}$ by $\pi$. Further, let $\varphi = \Phi^{(q)} \big|_{D_q}$ where $D_q = \pi^{-1}(0)$. Since $\Phi^{(q)}$ preserves the irreducible components of $\pi^*(D^{(q-1)})$, then $\varphi(t) = \Phi^{(q)}(t, 0)$ is a homography fixing $\infty$ and conjugating the first tangent cones of $p_\lambda = 0$ and $p_\mu = 0$ respectively. Thus $[\lambda] = [\mu] \in \text{Symm}(\mathbb{C}_\Delta(n))/\text{Aff}(\mathbb{C})$. Conversely, (reordering the indexes of $\mu$, if necessary) suppose there is $\varphi(z) = az + b \in \text{Aff}(\mathbb{C})$ such that $\mu_j = \varphi(\lambda_j)$ for all $j = 1, \ldots, n$, and let $T(x, y) = (x, ay + bx^q)$. Then a straightforward calculation shows that $f_{m, \lambda} = \alpha : T^* f_{m, \mu}$ where $\alpha \in \mathbb{C}^*$. Thus $C_{m, \lambda}$ and $C_{m, \mu}$ are analytically equivalent, as desired. \hfill \-box

As a straightforward consequence of Lemma 7 and Remark 1 one has:

**Corollary 2.** Let $\lambda, \mu \in \mathbb{C}_\Delta(n)$ with $n \leq 2$. Then $C_{m, \lambda}$ and $C_{m, \mu}$ are analytically equivalent.

4.3. **Curves of type** $(p, q, n)$, $2 \leq p < q$. In this case, the curve is given as the zero set of a polynomial of the form $f_{m, k, \lambda}(x, y) = x^m y^k \prod_{j=1}^{n} (y^p - \lambda_j x^q)$ where $m, k = 0, 1, p, q \in \mathbb{Z}_+$, $2 \leq p < q$, and $\lambda_j \in \mathbb{C}^*$. Thus given $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{C}_\Delta(n)$ we denote a curve of type $(p, q, n)$ by $C_{m, k, \lambda}$ if it is given as the zero set of $f_{m, k, \lambda}(x, y)$. Recall that the group of linear transformations of $\mathbb{C}$, denoted by $GL(1, \mathbb{C})$, acts in a natural way in $\text{Symm}(\mathbb{C}_\Delta(n))$. Further, recall that the equivalence class of $\lambda \in \mathbb{C}_\Delta(n)$ in $\text{Symm}(\mathbb{C}_\Delta(n))/GL(1, \mathbb{C})$ is denoted by $[\lambda]$.

**Lemma 8.** Two homogeneous curves $C_{m, k, \lambda}$ and $C_{m, k, \mu}$ are analytically equivalent if and only if $[\lambda] = [\mu] \in \text{Symm}(\mathbb{C}_\Delta(n))/GL(1, \mathbb{C})$.

**Proof.** First recall from the proof of Lemma 4 that $C_{m, k, \lambda}$ is resolved after $N$ blowups, where $N$ depends on the Euclid’s division algorithm between $q$ and $p$. 


Further, in the \((N - 1)^{th}\) step we have to blow up a singularity given in local coordinates \((x, y)\) as the zero set of the polynomial \(g_\lambda(x, y) = xy\prod_{j=1}^{n}(y - \lambda_j x)\). Therefore, if \(\Phi \in Diff(\mathbb{C}^2, 0)\) is an equivalence between \(C_{m,k,\lambda}\) and \(C_{m,k,\mu}\) and \(\Phi^{(N-1)}\) is its lifting to the \((N - 1)^{th}\) step of the resolution, then it conjugates the germs of curves given in local coordinates \((x, y)\) respectively by \(p_\lambda(x, y) = xy\prod_{j=1}^{n}(y - \lambda_j x)\) and \(p_\mu(x, y) = xy\prod_{j=1}^{n}(y - \mu_j x)\) where \((x = 0)\) and \((y = 0)\) are local equations for the exceptional divisor \(D^{(N-1)}\). Let \(\pi\) denote the final blowup of the resolution given in local coordinates by \(\pi(t, x) = (x, tx)\) and \(\pi(u, y) = (u, uy)\), and \(\Phi^{(N)}\) be the map obtained by the lifting of \(\Phi^{(N-1)}\) by \(\pi\). Further let \(\varphi = \Phi^{(N)}|_{D_N}\) where \(D_N = \pi^{-1}(0)\). Since \(\Phi^{(N)}\) preserves the irreducible components of \(\pi^*(D^{(q-1)})\), then \(\varphi(t) = \Phi^{(q)}(t, 0)\) is a homography fixing 0 and \(\infty\), and conjugating the first tangent cones of \(p_\lambda = 0\) and \(p_\mu = 0\) respectively. Thus \([\lambda] = [\mu] \in Symm(C^*_\Delta(n))/GL(1, \mathbb{C})\). Conversely, (reordering the indexes of \(\mu\), if necessary) suppose there is \(\varphi(z) = az \in GL(1, \mathbb{C})\) such that \(\mu_j = \varphi(\lambda_j)\) for all \(j = 1, \ldots, n\), and let \(T(x, y) = (x, \sqrt{\alpha}y)\). Then a straightforward calculation shows that \(f_{m,\lambda} = \alpha \cdot \tau^* f_{m,\mu}\) where \(\alpha \in \mathbb{C}^*\). Thus \(C_{m,\lambda}\) and \(C_{m,\mu}\) are analytically equivalent, as desired.

As a straightforward consequence of Lemma 8 and Remark 1 one has:

**Corollary 3.** Let \(\lambda, \mu \in C^*_\Delta(1)\), then \(C_{m,k,\lambda}\) and \(C_{m,k,\mu}\) are analytically equivalent.

## 5. Resolution and factorization

We study the relationship between the resolution tree and the factorization of a quasi-homogeneous polynomial. We use the resolution in order to study the equivalence between two quasi-homogeneous polynomials.

First recall that a quasi-homogeneous polynomial split uniquely in the form \(P = x^mp^nP_0\) where \(P_0\) is a commode quasi-homogeneous polynomial. In particular \(P\) and \(P_0\) share the same resolution process.

**Corollary 4.** Let \(P \in \mathbb{C}[x, y]\) be a commode quasi-homogeneous polynomial with the weights \((p, q)\), where \(gcd(p, q) = 1\). Let \(q_j = sq_j + r_j\), \(j = 1, \ldots, m\), be the Euclid’s algorithm of \((p, q)\), where \(q_1 := q\), \(p_1 := p\), \(q_{j+1} := p_j\), and \(p_{j+1} := r_j\) for all \(j = 1, \ldots, m - 1\). Then the exceptional divisor of its minimal resolution is given by a linear chain of projective lines, namely \(D = \bigsqcup_{j=1}^{n} D_j\), whose self-intersection numbers are given as follows:

1. If \(m = 1\), then

\[
c_1(D_j) = \begin{cases} 
-1 & \text{if } j = s_1; \\
-2 & \text{otherwise}
\end{cases}
\]

2. If \(m = 2\alpha + 1\), \(\alpha \geq 1\), then

\[
c_1(D_j) = \begin{cases} 
-(s_{2k} + 2) & \text{if } j = s_1 + s_4 + \cdots + s_{2k-1}, \; k = 1, \ldots, \alpha; \\
-1 & \text{if } j = s_1 + s_3 + \cdots + s_{2\alpha+1}; \\
-(s_{2k+1} + 2) & \text{if } j = m - (s_2 + s_4 + \cdots + s_{2k}) + 1, \; k = 1, \ldots, \alpha - 1; \\
-(s_{2\alpha+1} + 1) & \text{if } j = m - (s_2 + s_4 + \cdots + s_{2\alpha}) + 1; \\
-2 & \text{otherwise}
\end{cases}
\]
(3) If \( m = 2\alpha, \alpha \geq 1 \), then

\[
c_1(D_j) = \begin{cases} 
-(s_{2k} + 2) & \text{if } j = s_1 + s_3 + \cdots + s_{2k-1}, \; k = 1, \ldots, \alpha - 1; \\
-(s_{2\alpha} + 1) & \text{if } j = s_1 + s_3 + \cdots + s_{2\alpha-1}; \\
-(s_{2k+1} + 2) & \text{if } j = m - (s_2 + s_4 + \cdots + s_{2k}) + 1, \; k = 1, \ldots, \alpha - 1; \\
-1 & \text{if } j = m - (s_2 + s_4 + \cdots + s_{2\alpha}) + 1; \\
-2 & \text{otherwise}.
\end{cases}
\]

Finally, if \( C \) is given by \( f = 0 \) where \( f(x, y) = x^m y^n \prod_{j=1}^{k} (y_p - \lambda_j x^q) \), then a representative of \([c]\) is determined by the intersection of the strict transform of \( C \) with the exceptional divisor \( D \).

**Proof.** The proof shall be performed by induction on \( m \), the length of the Euclidean algorithm. In order to better understand the arguments, the reader have to keep in mind the proof of Lemma 4. From Lemma 2 we may suppose without loss of generality that \( P \) can be written in the form \( P(x, y) = \prod_{j=1}^{k} (y^p - \lambda_j x^q) \). First remark that if \( m = 1 \) then \( p = 1 \). Thus we prove the statement for \( m = 1 \) by induction on \( q \). For \( q = 1 \) the result is easily verified after one blowup. Now suppose the result is true for all \( q \leq q_0 - 1 \). Then after one blowup \( \pi(t, x) = (x, tx), \pi(u, y) = (uy, y) \), \( P \) is transformed into \( \pi^* P(t, x) = \prod_{j=1}^{k} (t - \lambda_j x^{q-1}) \). Thus the result follows for \( m = 1 \) by induction on \( q \). Suppose the result is true for all polynomials whose pair of weights have Euclid’s algorithm length less than \( m \), and let \( (p, q) \) has length \( m \). Since \( p_j = s_j q_j + r_j, j = 1, \ldots, m \), is the Euclid’s algorithm of \( (p, q) \), then \( p_j = s_j q_j + r_j, j = 2, \ldots, m \), is the Euclid’s algorithm of \( (p_2, q_2) \). In particular the Euclid’s algorithm of \( (p_2, q_2) \) has length \( m - 1 \). Reasoning in a similar way as in the case \( m = 1 \), we have after \( s_1 \) blowups a linear chain of projective lines \( \cup s_1 \{ D_j^{(1)} \} \) such that \( c_1(D_j^{(1)}) = -2 \) for all \( j = 1, \ldots, s_1 - 1 \) and \( c_1(D_{s_1}) = -1 \). Further, the strict transform of \( P = 0 \) is given by the zero set of the polynomial

\[
\tilde{P}(t, x) = \prod_{j=1}^{k} (t^{p_j} - \lambda_j x^{q_j}) = \lambda_1 \cdots \lambda_k \prod_{j=1}^{k} (x^{p_j} - \lambda_j t^{q_j})
\]

where the local equation for \( D_{s_1}^{(1)} \) is \( (x = 0) \). The first statement thus follows by the induction hypothesis. The last statement comes immediately from the above reasoning. For the above induction arguments ensure that the strict transform of \( P \) assume the form \( \tilde{P} = 0 \), with \( \tilde{P}(x, y) = \prod_{j=1}^{k} (y - \lambda_j x) \), just before the last blowup. \( \square \)

The above Corollary gives an effective way to compute the relatively prime weights of a quasi-homogeneous polynomials and further an easy way to determine a normal form for a quasi-homogeneous function from the dual weighted tree of its standard resolution. In particular it shows how to split a quasi-homogeneous polynomial into irreducible components from the dual weighted tree of its minimal resolution.
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