SOME LOWER BOUNDS FOR DISTANCES OF ROOTS OF CLASSICAL ORTHOGONAL POLYNOMIALS

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Abstract. Let \((P_N)_{N \geq 0}\) one of the classical sequences of orthogonal polynomials, i.e., Hermite, Laguerre or Jacobi polynomials. For the roots \(z_{1,N}, \ldots, z_{N,N}\) of \(P_N\) we derive lower estimates for \(\min_{i \neq j} |z_{i,N} - z_{j,N}|\) and the distances from the boundary of the orthogonality intervals. The proofs are based on recent results on the eigenvalues of the covariance matrices in central limit theorems for associated \(\beta\)-random matrix ensembles where these entities appear as entries, and where the eigenvalues of these matrices are known.

1. Introduction

Let \((P_N)_{N \geq 0}\) one of the classical sequences of orthogonal polynomials, i.e., Hermite polynomials \((H_N)_{N \geq 0}\), Laguerre polynomials \(L_{N}^{(\nu-1)}\) with \(\nu > 0\), or Jacobi polynomials \((P_N^{(\alpha,\beta)})_{N \geq 0}\) with \(\alpha, \beta > -1\). We derive lower estimates for the minimal distance \(M_N := \min_{1 \leq i < j \leq N} (z_{i,N} - z_{j,N})\) for the ordered roots \(z_{1,N}, \ldots, z_{N,N}\) of \(P_N\) depending on \(N\) and the other parameters \(\nu, \alpha, \beta\). The proof will be different from other estimates for \(M_N\) in the literature where often methods like Sturm’s comparison theorem are used; see e.g. [DJ, JT, K1, K2, H] and Ch. 6 of [S].

We here discuss the following approach: For all three classes above, the roots of \(P_N\) appear for \(\beta \to \infty\) in limit formulas for classical \(\beta\)-random matrix ensembles (where this \(\beta > 0\) has no connection to the \(\beta\) of the Jacobi polynomials) where \(N\) is a fixed dimension parameter. For the general background on random matrix ensembles we refer to [D, Mc, F]. For these classical ensembles there exist associated central limit theorems (CLTs) in the freezing regime with explicit formulas for the covariance matrices \(\Sigma_N\) and their inverses \(\Sigma_N^{-1}\) for the limit Gaussian distributions of dimension \(N\); see [AHV, AV, DE2, GK, V] in the Hermite and Laguerre case and [AHV, HV] in the Jacobi case. In particular, by [AV, HV, V], the entries of \(\Sigma_N^{-1}\) are given in terms of the distances \(z_{i,N} - z_{j,N}\), and the eigenvalues of \(\Sigma_N^{-1}\) have a simple form. This allows to compute the traces \(\text{tr}(\Sigma_N^{-k})\) of powers of \(\Sigma_N^{-1}\). Some simple computations then will lead to the estimates for the minimal distances \(M_N\) and for the distances of the roots from the boundary of the orthogonality measure. In the Laguerre case, some connection between the roots of the polynomials and the associated ensembles also appear in [CD, K1, K2].

We mention that, depending on relations between the parameters and the order \(N\) of the Laguerre and Jacobi polynomials, our estimates are better or worse than
those in the literature mentioned above. For the details see Remarks 3.4, 3.6, 4.5, and 4.8 below. Furthermore, in the Hermite case, our results are always worse than the estimate
\[ z_{i,N} - z_{i+1,N} \geq 2/\sqrt{N} \quad (i = 1, \ldots, N) \]
which follows from [K2]. We thus point out that the complicated inequalities in Proposition 2.3, Lemma 3.2, and Lemma 4.3 below should be seen as the main results of this paper where our lower estimates for the minimal distances and the distances from the boundary are just corollaries from these inequalities.

This paper is organized as follows: Section 2 contains the Hermite case, Section 3 is devoted to the Laguerre case, and the Jacobi case is treated in Section 4.

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2. The Hermite case

We first recapitulate some facts on $\beta$-Hermite ensembles.

Let $\beta = 2k > 0$ be a constant (both parameters are used in the literature; see e.g. [AV, D, DE1, DE2, F, G, K, M, V]). Define the associated Hermite ensemble as a random vector $X_{k,N}$ with values in the closed Weyl chamber $C_{A,N} := \{ x \in \mathbb{R}^N : x_1 \geq x_2 \geq \ldots \geq x_N \}$ with Lebesgue density
\[ c_A^k e^{-\|x\|^2/2} \prod_{1 \leq i < j \leq N} (x_i - x_j)^{2k} \]
with the well-known normalization (see e.g. the survey [FW])
\[ c_A^k := \left( \int_{C_{A,N}} e^{-\|y\|^2/2} \prod_{i<j} (y_i - y_j)^{2k} dy \right)^{-1} = \frac{N!}{(2\pi)^{N/2}} \cdot \prod_{j=1}^N \Gamma(1 + jk) . \]

We now compare $X_{k,N}$ with $\sqrt{2k} \cdot z_N \in C_{A,N}$, where the entries of
\[ z_N = (z_{1,N}, \ldots, z_{N,N}) \in C_{A,N} \]
are the ordered zeros of the classical Hermite polynomial $H_N$ where, as usual (see e.g. [S]), $(H_N)_{N \geq 0}$ is orthogonal w.r.t. the density $e^{-x^2}$. The basis for our estimations for the distances of the roots will be the following CLT:

**Theorem 2.1.** For each $N \geq 2$, the random variables $X_{k,N} - \sqrt{2k} \cdot z_N$ converge for $k \to \infty$ to the $N$-dimensional centered normal distribution $N(0, \Sigma_N)$ with the regular covariance matrix $\Sigma_N$ with $\Sigma_N^{-1} = S_N = (s_{i,j})_{i,j=1}^N$ and
\[ s_{i,j} := \begin{cases} 1 + \sum_{l \neq i} (z_{i,N} - z_{l,N})^{-2} & \text{for } i = j \\ - (z_{i,N} - z_{j,N})^{-2} & \text{for } i \neq j . \end{cases} \]
The matrix $S_N$ has the eigenvalues $1, 2, \ldots, N$.

**Remarks on the proof.** The CLT was first derived by Dumitriu, Edelman [DE2] by using their tridiagonal random matrix models in [DE1] with explicit formulas for $\Sigma_N$. It was then reproved with different methods in [V] with Eq. (2.3) for $\Sigma_N^{-1}$. The eigenvalues of $\Sigma_N^{-1}$ were given in [AV]. We remark that in [AHV] the duality of finite orthogonal polynomials in the sense of de Boor and Saff (see [BS, I, VZ]) was used to compute the entries of $\Sigma_N$ from (2.3). As a curiosity, the equations for the entries of $\Sigma_N$ in [AHV] look different from those in [DE2]. Numerical computation for small $N$ show that the entries in both cases are equal, but it seems to be difficult to verify these identities for all $N$ as in both representations complicated formulas...
depending on the \( z_{i,N} \) appear. Furthermore, \( \text{[GK]} \) gives a third proof of the CLT with the same formulas for \( \Sigma_N \) as in \( \text{[AHV]} \). □

For our estimates for the distances of the roots we need the following fact:

**Lemma 2.2.** Let \( B = (b_{i,j})_{i,j=1,...,N} \in \mathbb{R}^{N,N} \) be a symmetric matrix. Then, for all integers \( r \geq 0 \),

\[
\text{tr}(B^{2^r}) \geq \sum_{i=1}^{N} b_{i,i}^{2^r}. \tag{2.4}
\]

Moreover, if in addition \( B \) has the eigenvalue 0 with the eigenvector \( (1,\ldots,1)^T \), then, for all integers \( r \geq 0 \),

\[
\text{tr}(B^{2^r}) \geq \left( \frac{N}{N-1} \right)^{2^r-1} \sum_{i=1}^{N} b_{i,i}^{2^r}. \tag{2.5}
\]

**Proof.** Let \( B^{2^r} =: (b_{i,j}^{(r)})_{i,j=1,...,N} \). Then for \( r \geq 0 \) and \( i = 1,\ldots,N \),

\[
b_{i,i}^{(r+1)} = \sum_{j=1}^{N} (b_{i,j}^{(r)})^2 \geq (b_{i,i}^{(r)})^2. \]

Hence, by induction, \( b_{i,i}^{(r)} \geq b_{i,i}^{2^r} \). Summation over \( i \) then yields (2.4).

Now assume in addition that \( (1,\ldots,1)^T \) is an eigenvector of \( B \) with eigenvalue 0. Then this also holds for the matrices \( B^{2^r} \), i.e., for all \( i \),

\[
b_{i,i}^{(r)} = - \sum_{j:j \neq i} b_{i,j}^{(r)}. \]

A simple argument with the Cauchy-Schwarz inequality now shows that

\[
b_{i,i}^{(r+1)} = \sum_{i=1}^{N} (b_{i,i}^{(r)})^2 = (b_{i,i}^{(r)})^2 + \sum_{j:j \neq i} (b_{i,j}^{(r)})^2 \\
\geq \left( 1 + \frac{1}{N-1} \right) (b_{i,i}^{(r)})^2 = \frac{N}{N-1} (b_{i,i}^{(r)})^2.
\]

This and induction imply that

\[
b_{i,i}^{(r)} \geq \left( \frac{N}{N-1} \right)^{2^r-1} b_{i,i}^{2^r}.
\]

Summation then yields (2.5). □

With these ingredients we now derive the following result:

**Proposition 2.3.** For all \( N \geq 2 \) and \( i = 1,\ldots,N \),

\[
(\sum_{l,l \neq i} \frac{1}{(z_{i,N} - z_{l,N})^2})^2 + \sum_{l,l \neq i} \frac{1}{(z_{i,N} - z_{l,N})^4} \leq \frac{(N-1)^3}{N}. \tag{2.6}
\]

**Proof.** We write the matrix \( S_N \) from Theorem 2.1 as \( S_N = I_N + A_N \) with the identity matrix \( I_N \) where, by Theorem 2.1, \( A_N \) has the eigenvalues 0,1,\ldots,N − 1. Therefore, for all integers \( r \geq 0 \),

\[
\text{tr}(A_N^{2^r}) = 0 + 1^{2^r} + 2^{2^r} + \ldots + (N-1)^{2^r} \leq (N-1)^{2^r+1}.
\]
On the other hand, if we write \( A^2_N = (a^{(2)}_{i,j})_{i,j=1,\ldots,N} \), we obtain from (2.3) that
\[
a^{(2)}_{i,i} = \left( \sum_{l \neq i} \frac{1}{(z_{i,N} - z_{l,N})^2} \right)^2 + \sum_{l \neq i} \frac{1}{(z_{i,N} - z_{l,N})^4}
\]
for \( i = 1, \ldots, N \). We now apply (2.5) in Lemma 2.2 to the matrix \( B := A^2_N \) (notice that the assumptions are satisfied!) and conclude that
\[
\text{tr}(A^2_N) \geq \left( \frac{N}{N-1} \right)^{2r-1} \sum_{i=1}^{N} (a^{(2)}_{i,i})^{2r-1}.
\]
(2.7)

Therefore, for all integers \( r \geq 0 \),
\[
\frac{N-1}{N} \left( \frac{N}{N-1} \right)^2 \sum_{i=1}^{N} (a^{(2)}_{i,i})^{2r} \leq (N-1) \cdot ((N-1)^2)^r.
\]
As the condition \( 0 \leq x^{2r} \leq C \) for all \( r \in \mathbb{N} \) with some constant \( C > 0 \) implies that \( x \leq 1 \), we conclude from the terms with power \( 2^r \) that for all \( i \),
\[
\frac{N}{N-1} a^{(2)}_{i,i} \leq (N-1)^2.
\]
(2.8)
The equation for \( a^{(2)}_{i,i} \) above now yields the first inequality. The second inequality of the proposition is trivial. □

Proposition 2.3 has the following obvious consequences:

**Corollary 2.4.** For all \( i = 1, \ldots, N \),
\[
\sum_{l \neq i} (z_{i,N} - z_{l,N})^{-4} \leq \frac{(N-1)^3}{2N} \leq \frac{(N-1)^2}{2}
\]
and
\[
\sum_{l \neq i} (z_{i,N} - z_{l,N})^{-2} \leq \frac{(N-1)^{3/2}}{N^{1/2}} \leq N - 1.
\]
(2.9)

(2.10)

In particular, for all \( i < N \),
\[
z_{i,N} - z_{i+1,N} \geq \frac{(2N)^{1/4}}{(N-1)^{3/4}} \geq \frac{2^{1/4}}{(N-1)^{1/2}}.
\]
(2.11)

**Remark 2.5.**

(1) It is well known that the order \( O(N^{-1/2}) \) in (2.11) and thus in (2.9) and (2.10), is sharp. However, the constant in (2.11) is not optimal. For instance, it can be shown in an elementary way that \( z_{i,N} - z_{i+1,N} \geq 2/\sqrt{N} \) for all \( i = 1, \ldots, N \); see [12]. On the other hand, for \( N = 2 \), \( z_{1,2} - z_{2,2} = \sqrt{2} \) (see (5.5.4) in [5]), i.e., (2.11) here is an equality.

(2) If we sum the left hand sides of (2.9) and (2.10) over \( i \), we obtain
\[
\sum_{i=1}^{N} (z_{i,N} - z_{l,N})^{-2} = N(N-1)/2
\]
and
\[
\sum_{i=1}^{N} \left( \frac{1}{(z_{i,N} - z_{l,N})^2} \right)^2 + \sum_{i,l \neq i} \frac{1}{(z_{i,N} - z_{l,N})^4} = 0^2 + 1^2 + \ldots + (N-1)^2 = \frac{N(N-1)(2N-1)}{6}.
\]
(2.12)
(2.13)
In fact, by Theorem 2.1 both sides of (2.12) are equal to $\text{tr}(S_N - I_N)$, and both sides of (2.13) are equal to $\text{tr}((S_N - I_N)^2)$. (2.12) and (2.13) show that (2.10) and (2.6) are sharp up to a factor of size 2 and 3 respectively.

3. THE LAGUERRE CASE

We start with some facts on $\beta$-Laguerre ensembles. Let $k_1, k_2 > 0$ be constants and $k = (k_1, k_2)$. Define the associated Laguerre ensemble as random vector $X_{k,N}$ with values in the closed Weyl chamber 

$$C_N^B := \{ x \in \mathbb{R}^N : x_1 \geq x_2 \geq \ldots \geq x_N \geq 0 \}$$

with the Lebesgue density

$$c_k B e^{-\|x\|_2^2/2} \prod_{i<j} (x_i^2 - x_j^2)^{k_2} \prod_{i=1}^N x_i^{2k_1}$$

(3.1)

with the well-known normalization

$$c_k B := \left( \int_{C_N^B} e^{-\|x\|_2^2/2} \prod_{i<j} (x_i^2 - x_j^2)^{k_2} \prod_{i=1}^N x_i^{2k_1} \, dy \right)^{-1}$$

(3.2)

$$= \frac{N!}{2^N (k_1 + (N-1)k_2 - 1)!} \prod_{j=1}^N \frac{\Gamma(1 + k_2)}{\Gamma(1 + jk_2) \Gamma(\frac{k_1}{2} + k_1 + (j-1)k_2)}.$$

We rewrite $k$ as $(k_1, k_2) = (\nu \cdot \beta, \beta)$ with $\nu, \beta > 0$. For $\nu > 0$ fixed and $\beta \to \infty$ we shall compare $X_{k,N}$ with the vector $\sqrt{\beta} \cdot r_N \in C_N^B$ where $r_N = (r_1, \ldots, r_N)$ satisfies $(r_1^2, \ldots, r_N^2) = 2z_N$ where the entries of $z_N = \{z_{1,N}, \ldots, z_{N,N}\} \in C_N^B$ are the ordered zeros of the classical Laguerre polynomial $L_N^{(\nu-1)}$. Recapitulate that, as usual, the $L_N^{(\nu-1)}$ are orthogonal w.r.t. the density $e^{-x} \cdot x^{\nu-1}$ on $[0, \infty[$ for $\nu > 0$. The following CLT will be crucial for our estimates:

**Theorem 3.1.** Let $N \geq 1$ be an integer and $\nu > 0$. Then $X_{k,N} - \sqrt{\beta} \cdot r_N$ converges for $\beta \to \infty$ to the centered $N$-dimensional normal distribution $N(0, \Sigma_N)$ with the regular covariance matrix $\Sigma_N$ with $\Sigma_N^{-1} = S_N = (s_{i,j})_{i,j=1,\ldots,N}$ with

$$s_{i,j} := \begin{cases} 1 + \frac{\nu}{z_{i,N}} + 2 \sum_{t \neq i} \frac{z_{i,N} + z_{t,N}}{(z_{i,N} - z_{t,N})^2} & \text{for } i = j, \\ -\frac{2 \sqrt{z_{i,N} z_{j,N}}}{(z_{i,N} - z_{j,N})^2} & \text{for } i \neq j. \end{cases}$$

(3.3)

The matrix $S_N$ has the eigenvalues $2, 4, \ldots, 2N$.

Remarks on the proof. This CLT was first derived by Dumitriu, Edelman [DE2] via their tridiagonal random matrix models [DE1] with explicit formulas for $\Sigma_N$. The CLT was then reproved in a different way in [V] with Eq. (2.23) for $\Sigma_N^{-1}$ where the $s_{i,j}$ there are expressed in terms of $r_N$ instead of $z_N$ by using

$$\frac{z_{i,N} + z_{j,N}}{(z_{i,N} - z_{j,N})^2} = (r_i - r_l)^{-2} + (r_i + r_l)^{-2},$$

$$-2 \frac{\sqrt{z_{i,N} z_{j,N}}}{(z_{i,N} - z_{j,N})^2} = (r_i + r_j)^{-2} - (r_i - r_j)^{-2}.$$  

(3.4)

The eigenvalues were determined in [AV], and in [AHV] the duality of finite orthogonal polynomials was used to compute the entries of $\Sigma_N$ from (2.23). Again, the equations for the entries of $\Sigma_N$ in [AHV] look different from those in [DE2]. □
We now use Theorem 3.1 as in Section 2 to derive estimates. This approach leads to preliminary results.

**Lemma 3.2.** Let \( \nu > 0 \). For all \( N \geq 1 \), the ordered roots \( z_{1,N} > \ldots > z_{N,N} > 0 \) of \( L_N^{(\nu)} \) satisfy

\[
\left( \frac{\nu}{z_i,N} + 2 \sum_{l \neq i} \frac{z_i,N + z_l,N}{(z_i,N - z_l,N)^2} \right)^2 + 16 \sum_{l \neq i} \frac{z_i,N z_l,N}{(z_i,N - z_l,N)^4} \leq (2N - 1)^2. \tag{3.5}
\]

In particular

\[
z_{N,N} \geq \frac{\nu}{2N - 1}. \tag{3.6}
\]

**Proof.** We proceed as in the proof of Theorem 2.3 and write

\[
A = \begin{pmatrix} A_{i,i} & \cdots & A_{i,N} \\ \vdots & \ddots & \vdots \\ A_{N,i} & \cdots & A_{N,N} \end{pmatrix}
\]

and

\[
A = \begin{pmatrix} A_{i,i} & \cdots & A_{i,N} \\ \vdots & \ddots & \vdots \\ A_{N,i} & \cdots & A_{N,N} \end{pmatrix}
\]

for all integers \( r \geq 0 \),

\[
tr(A_N^r) = 1^{2r} + 3^{2r} + 5^{2r} + \ldots + (2N - 1)^{2r} \leq N(2N - 1)^{2r}. \tag{3.7}
\]

On the other hand, if we write \( A^2 =: (a_{i,j})_{i,j=1,\ldots,N} \), we obtain from (3.3) that

\[
a_{i,i}^{(2)} = \left( \frac{\nu}{z_i,N} + 2 \sum_{l \neq i} \frac{z_i,N + z_l,N}{(z_i,N - z_l,N)^2} \right)^2 + 16 \sum_{l \neq i} \frac{z_i,N z_l,N}{(z_i,N - z_l,N)^4}
\]

for \( i = 1, \ldots, N \). Moreover, (3.7) and Lemma 2.2 for \( B := A_N^2 \) lead to

\[
N((2N - 1)^2)^{2r-1} \geq tr(A_N^r) \geq \sum_{i=1}^N (a_{i,i}^{(2)})^{2r-1} \tag{3.8}
\]

for all integers \( r \geq 1 \), which implies that \( a_{i,i}^{(2)} \leq (2N - 1)^2 \) for all \( i \). This proves (3.8) and also (3.6).

**Remark 3.3.** Similar to the identities (2.12) and (2.13) in the Hermite case, we have in the Laguerre case

\[
\frac{\nu}{z_i,N} + 2 \sum_{l \neq i} \frac{z_i,N + z_l,N}{(z_i,N - z_l,N)^2} = 2 + 4 + \ldots + 2(N - 1) = N(N - 1) \tag{3.9}
\]

and

\[
\sum_{i=1}^N \left( \frac{\nu}{z_i,N} + 2 \sum_{l \neq i} \frac{z_i,N + z_l,N}{(z_i,N - z_l,N)^2} \right)^2 + 16 \sum_{l \neq i} \frac{z_i,N z_l,N}{(z_i,N - z_l,N)^4}
\]

\[
= 1^2 + 3^2 + \ldots + (2N - 1)^2 = \frac{N(2N - 1)(2N + 1)}{3}. \tag{3.10}
\]

In fact, both sides of (3.9) and (3.10) are equal to \( tr(S_N - I_N) \) and \( tr((S_N - I_N)^2) \) respectively. (3.10) in particular shows that (3.5) is sharp up to a factor 3 in the worst case.

**Remark 3.4.** We briefly discuss (3.6), which is a byproduct of (3.5):

1. The order \( O(N^{-1}) \) in (3.6) is sharp by classical upper bounds for \( z_{N,N} \) in [H] or Eq. (6.31.12) in [S].
2. For \( N = 1 \), we have equality in (3.6) by (5.1.6) in [S].
(3) By Eq. (6.31.12) in [S], $z_{N,N}$ can be estimated in terms of the first zero of the Bessel function $J_{\nu-1}$. If one combines this estimate with Eq. (5) in Section 15.3 of Watson [W] on these zeroes of $J_{\nu-1}$, one obtains

$$z_{N,N} \geq \frac{\nu^2 - 1}{4(N + \nu/2)}. \quad (3.11)$$

For large $\nu$, (3.11) is clearly better than (3.6). However, for small $\nu > 0$, (3.11) is worse than (3.6). The same holds for similar bounds in Theorem 1 of [K3].

(4) We finally notice that 18.16.12 in [NIST] implies for $\nu \to \infty$ and $\nu \gg N$ that

$$z_{N,N} \geq \nu + o(\nu). \quad (3.12)$$

We next use Lemma 3.2 to derive estimates for the distances of roots.

**Proposition 3.5.** Let $\nu > 0$. For all $N \geq 2$, the ordered roots $z_{1,N} > \ldots > z_{N,N} > 0$ of $L_N^{(\nu-1)}$ satisfy

$$z_{i,N} - z_{i+1,N} \geq \frac{\sqrt{2(1 + \sqrt{1 + 8\nu^2})}}{2N - 1} \geq \frac{2 \cdot 2^{1/4} \sqrt{\nu}}{2N - 1} \quad (3.12)$$

and

$$z_{i,N} - z_{i+1,N} \geq \frac{\sqrt{2}}{2N - 1} \sqrt{2 + \sqrt{2} \cdot \sqrt{2 + \frac{(2N - 1)^2(\nu^2 - 1)}{(N + \nu/2)^2}}} \geq \frac{2^{3/4} \cdot \sqrt{\nu^2 - 1}}{\sqrt{(2N - 1)(N + \nu/2)}} \quad (3.13)$$

for $i = 1, \ldots, N - 1$.

**Proof.** (3.5) and (3.6) imply

$$(2N - 1)^2 \geq 4 \frac{(z_{i,N} + z_{i+1,N})^2}{(z_{i,N} - z_{i+1,N})^4} + 16 \frac{z_{i,N} z_{i+1,N}}{(z_{i,N} - z_{i+1,N})^4} \quad (3.14)$$

$$= 4 \frac{(z_{i,N} - z_{i+1,N})^2 + 8z_{i,N} z_{i+1,N}}{(z_{i,N} - z_{i+1,N})^4} \geq 4 \frac{(z_{i,N} - z_{i+1,N})^2 + 8\nu^2}{(z_{i,N} - z_{i+1,N})^4} \quad (3.14)$$

On the other hand, elementary calculus shows that for $w := (z_{i,N} - z_{i+1,N})^2 > 0$ the inequality $(2N - 1)^2 w^2 \geq 4w + \frac{32\nu^2}{(2N - 1)^2}$ implies that

$$w \geq \frac{2(1 + \sqrt{1 + 8\nu^2})}{(2N - 1)^2}$$

which completes the proof of the first $\geq$ in (3.12) while the second one is trivial. The proof of (3.13) is analog by using (3.11) in the last $\geq$ of (3.14) instead of (3.6). \hfill \Box

**Remark 3.6.** (1) The order $O(N^{-1})$ in (3.12) and (3.13) is sharp for fixed $\nu$.

(2) Clearly, (3.12) is better than (3.13) for large $\nu$, while (3.12) is the better for small $\nu$. 

We briefly compare (3.13) with other bounds in the literature which are of particular interest for \( \nu \gg N \). Theorem 3.1 of [CD] implies
\[
\sqrt{z_{i,N} - z_{i+1,N}} \geq \frac{\nu - 1}{\sqrt{(N + \nu - 1)N}},
\]
and results in [K2] lead to
\[
\sqrt{z_{i,N} - z_{i+1,N}} \geq \frac{2\sqrt{2} \cdot \nu}{\sqrt{(N + \nu)N}},
\]
(3.15), (3.16), and (3.15) have the same order \( \sqrt{\nu/\sqrt{N}} \) for \( \nu \gg N \gg 1 \), where the asymptotic constant in (3.16) is the best one.

On the other hand, Theorem 3.1 of [JT] contains the bound
\[
\sqrt{z_{i,N} - z_{i+1,N}} \geq \frac{\pi \sqrt{2}}{2\nu N + \nu + 2N^2}
\]
which is better than (3.12) and (3.13) for small \( \nu \), but worse for large \( \nu \).

Notice that our estimates for \( z_{i,N} - z_{i+1,N} \) depend heavily on good estimates for \( z_{i,N}, z_{i+1,N} \). In this way, one may use Theorem 6.3 in [S] (see also 18.16.10 in [NIST]) in combination with estimates on the zeroes of Bessel functions in Section 15.3 in [W] to derive better bounds from (3.5) than in (3.12) which then depend on \( \nu \).

The dependence of Proposition 3.5 on good estimates for \( z_{i,N}, z_{i+1,N} \) is the motivation to compare the distances of consecutive roots on a different scale by studying the vectors \( r_N \) above. In this way, Theorem 3.1 leads to the following estimate which is independent from \( \nu \):

**Theorem 3.7.** Let \( \nu > 0 \). For all \( N \geq 2 \), the ordered roots \( z_{1,N} > \ldots > z_{N,N} > 0 \) of \( L_{N}^{(\nu - 1)} \) satisfy
\[
\sqrt{z_{i,N} - z_{i+1,N}} \geq \frac{1}{\sqrt{2N - 1}} \quad (i = 1, \ldots, N - 1).
\]

**Proof.** We use the numbers \( r_i = \sqrt{2z_{i,N}} \) from the beginning of this section: By (3.4), the matrix \( S_N = (s_{i,j})_{i,j=1,\ldots,N} \) from Theorem 3.1 then satisfies
\[
s_{i,i} = 1 + \frac{2\nu}{r_i^2} + 2 \sum_{l \neq i} \left( \frac{1}{(r_i - r_l)^2} + \frac{1}{(r_i + r_l)^2} \right)
\]
for all \( i \); see also [V]. If we apply the first statement of Lemma 2.2 to \( B := S_N - I_N \) with the eigenvalues 1, 3, \ldots, 2N - 1, we obtain from the methods of the proof of Theorem 2.4 that for all \( i \neq l \),
\[
\frac{2}{(r_i - r_l)^2} \leq 2N - 1
\]
and thus the claim.

**Remark 3.8.** The Hermite and Laguerre polynomials are related by
\[
H_{2N}(x) = c_N L_{N}^{(1/2)}(x^2), \quad H_{2N+1}(x) = d_N x \cdot L_{N}^{(1/2)}(x^2)
\]
with some constants \( c_N, d_N \) (see (5.6.1) of [S]). If we compare Theorem 3.7 (for \( \nu = 1/2, 3/2 \) with (2.11), we see that (2.11) is better by the factor \( 2^{1/2} \). We are not able to explain this discrepancy precisely, but there should exist some connection,
as the covariance matrices (written in terms of the $r_i$ as in [V] in the Laguerre case) and their eigenvalues in these two cases admit obvious similarities.

4. The Jacobi case

We first recapitulate some facts on $\beta$-Jacobi ensembles; see e.g. [F, KN, Me, HV]. It turns out that it is convenient here to study these ensembles in a trigonometric form as in [HV]. For this we fix $k = (k_1, k_2, k_3) \in [0, \infty]^3$ and consider a random vector $X_{k,N}$ on the trigonometric alcove

$$A_N := \{ t \in \mathbb{R}^N \mid \frac{\pi}{2} \geq t_1 \geq \ldots \geq t_N \geq 0 \}$$

with the Lebesgue density

$$\tilde{c}_k \cdot \prod_{1 \leq i < j \leq N} (\cos(2t_i) - \cos(2t_j))^{k_3} \prod_{i=1}^N \left( \sin^{k_1} t_i \sin^{k_2} (2t_i) \right)$$

with a suitable Selberg normalization $\tilde{c}_k > 0$ for parameters $k = (k_1, k_2, k_3) \in [0, \infty]^3$; see [FW] for explicit formulas for $\tilde{c}_k > 0$. Following [HV], we write

$$(k_1, k_2, k_3) = \kappa \cdot (a, b, 1),$$

where $a \geq 0, b > 0$ are fixed, and where $\kappa$ tends to infinity. By [HV], the limit of the $X_{k,N}$ for $\kappa \to \infty$ can be described via the ordered zeros of the Jacobi polynomials $P^{(\alpha, \beta)}_N$ with

$$\alpha := a + b - 1 > -1, \quad \beta = b - 1 > -1. \quad (4.2)$$

Please notice that the $\beta$ in (4.2) is different from the $\beta$ in the preceding sections, and that the $\beta$ there is now called $\kappa$. We denote the ordered zeros of $P^{(\alpha, \beta)}_N$ by

$$z_N := (z_{1,N}, \ldots, z_{N,N}) \quad \text{and} \quad \tilde{z}_N := \left( \frac{1}{2} \arccos z_{1,N}, \ldots, \frac{1}{2} \arccos z_{N,N} \right).$$

The following CLT is shown in [HV]:

**Theorem 4.1.** Let $X_{k,N}$ be $A_N$-valued random variables as described above. Then, for $\kappa \to \infty$,

$$\sqrt{\kappa} (X_{N,k} - \tilde{z}_N)$$

converges in distribution to $N(0, \Sigma_N)$ where the inverse of the covariance matrix $\Sigma_N$ is given by $\Sigma^{-1}_N := S_N = (s_{i,j})_{i,j=1,\ldots,N}$ with

$$s_{i,j} = \begin{cases} 4 \sum_{l \neq j} \frac{(1-z_{l,N}^2)(1-z_{l,N})^2}{(1-z_{l,N}^2)(1-z_{l,N})^2} & \text{for } i = j \\ -4 \frac{(1-z_{i,N}^2)(1-z_{j,N}^2)}{(1-z_{i,N}^2)(1-z_{j,N}^2)} & \text{for } i \neq j \end{cases}$$

(4.3)

The matrix $S_N$ has the eigenvalues $\lambda_j = 2j(2N + \alpha + \beta + 1 - j) > 0$ ($j = 1, \ldots, N$).

In order to apply the methods of Sections 2 and 3 in the Jacobi case, we need the maximum of the eigenvalues $\lambda_j$, i.e the spectral radius. For this define

$$M := M(\alpha, \beta, N) := \max_{j=1,\ldots,N} 2j(2N + \alpha + \beta + 1) \quad (4.4)$$

Elementary calculus yields the following facts:
Lemma 4.2. For all \( \alpha, \beta > -1 \),
\[
M \leq 2 \left( N + \frac{\alpha + \beta + 1}{2} \right)^2.
\] (4.5)

Moreover, for \( \alpha + \beta + 1 \geq 0 \),
\[
M = \lambda_N = 2N(N + \alpha + \beta + 1).
\] (4.6)

Furthermore, for all \( \alpha, \beta > -1 \),
\[
M(\alpha, \beta, 1) = 2(\alpha + \beta + 2).
\]

Proof. By the definition of \( M \), we have for all integers \( r \geq 0 \),
\[
\operatorname{tr}(S_N^{2^r}) \leq N \cdot M^{2^r}.
\] (4.8)

On the other hand, if we write \( S_N^2 =: (s_{i,j}^{(2)})_{i,j=1,...,N} \), we obtain from Theorem 4.1 that for \( i = 1, \ldots, N \), \( s_{i,i}^{(2)} \) is equal to the left hand side of (4.7). Moreover, (4.8) and Lemma 2.2 for \( B := S_N^2 \) lead to
\[
N \cdot (M^2)^{2^{r-1}} \geq \operatorname{tr}(S_N^{2^r}) \geq \sum_{i=1}^{N} (s_{i,i}^{(2)})^{2^{r-1}}
\] (4.9)

for all integers \( r \geq 1 \), which implies that \( s_{i,i}^{(2)} \leq M^2 \) for all \( i \) as claimed. \( \square \)

Lemma 4.3 has the following consequences:

Corollary 4.4. For \( \alpha, \beta > -1 \) and \( N \geq 1 \),
\[
1 - z_{N,N} \geq \frac{8(\alpha + 1)}{M + 4(\alpha + 1) + \sqrt{M^2 - 16(\alpha + 1)(\beta + 1)}} \geq \frac{4(\alpha + 1)}{M + 2(\alpha + 1)}
\] (4.10)

and
\[
1 + z_{1,N} \geq \frac{8(\beta + 1)}{M + 4(\beta + 1) + \sqrt{M^2 - 16(\alpha + 1)(\beta + 1)}} \geq \frac{4(\beta + 1)}{M + 2(\beta + 1)}.
\] (4.11)

Proof. Lemma 4.3 implies that
\[
2(\alpha + 1) \frac{1 + z_{N,N}}{1 - z_{N,N}} + 2(\beta + 1) \frac{1 - z_{i,N}}{1 + z_{i,N}} \leq M.
\] (4.12)

Thus \( x := 1 - z_{1,N} \in [0,2] \) satisfies
\[
2(\alpha + 1) \frac{2 - x}{x} + 2(\beta + 1) \frac{x}{2 - x} \leq M
\]

which is, by elementary calculus, equivalent to
\[
(2(\alpha + \beta + 2) + M)x^2 - 2(4(\alpha + 1) + M)x + 8(\alpha + 1) \leq 0.
\]
This yields that
\[ x \in [x_-, x_+] \quad \text{for} \quad x = \frac{M + 4(\alpha + 1) \pm \sqrt{M^2 - 16(\alpha + 1)(\beta + 1)}}{M + 2(\alpha + \beta + 2)}. \]

As
\[
\begin{align*}
   x_- &= \frac{(M + 4(\alpha + 1))^2 - M^2 + 16(\alpha + 1)(\beta + 1)}{M + 2(\alpha + \beta + 2)} \left( M + 4(\alpha + 1) + \sqrt{M^2 - 16(\alpha + 1)(\beta + 1)} \right) \\
   &= \frac{8(\alpha + 1)}{M + 4(\alpha + 1) + \sqrt{M^2 - 16(\alpha + 1)(\beta + 1)}},
\end{align*}
\]
the first \( \geq \) in (4.10) follows. The second \( t \geq \) there is obvious. Moreover, (4.11) follows in the same way. \( \square \)

**Remark 4.5.**
(1) The first \( \geq \) in (4.10) and (4.11) are equalities for \( N = 1 \) by the explicit form of \( P^N_{\alpha, \beta} \) in (4.21.2) of [S].
(2) We briefly compare (4.10) and (4.11) with other known estimates. We first notice that Theorem 6.3.2 of [S] leads to an estimate for \(-1/2 \leq \alpha, \beta \leq 1/2\), which in most cases is better than (4.10) and (4.11) in this restricted case.

Moreover, the asymptotic result 18.16.8 in [NIST] for fixed \( \alpha, \beta > -1/2 \) and \( N \to \infty \) in combination with the estimate Eq. (5) in Section 15.3 of Watson [W] on the first zeros of the Bessel functions \( J_\alpha \) imply that
\[
1 - z_{N,N} \geq \frac{\alpha(\alpha + 2)}{2(N + (\alpha + \beta + 1)/2)^2} + o(1/N^2). \quad (4.13)
\]
If we compare this with (4.10), we obtain that, for fixed \( \alpha, \beta \) and large \( N \), (4.13) is better than (4.10) for \( \alpha > 1 + \sqrt{5} \), while the converse holds for \(-1/2 < \alpha < 1 + \sqrt{5} \).

The preceding estimates also have the following variant:

**Corollary 4.6.** For all \( i = 1, \ldots, N \),
\[
1 - z_{i,N}^2 \geq 2 \frac{\min(\alpha + 1, \beta + 1)}{M}. \quad (4.14)
\]
Moreover, for \( \alpha = \beta > -1 \),
\[
1 - z_{i,N}^2 \geq \frac{8(\alpha + 1)}{M + 4(\alpha + 1)}. \quad (4.15)
\]

*Proof.* Eq. (4.12) implies
\[
M(1 - z_{i,N}^2) \geq 2(\alpha + 1)(1 + z_{i,N})^2 + 2(\beta + 1)(1 - z_{i,N})^2 \geq 2 \cdot \min(\alpha + 1, \beta + 1)
\]
and thus (4.14). Moreover, for \( \alpha = \beta > -1 \), Eq. (4.12) leads to
\[
4(\alpha + 1)(1 + z_{i,N}^2) \leq M(1 - z_{i,N}^2)
\]
and thus to (4.15). \( \square \)

We next estimate the distances of consecutive roots.

**Theorem 4.7.** Let \( \alpha, \beta > -1 \). For all \( N \geq 2 \), the ordered roots \(-1 < z_{1,N} < \ldots < z_{N,N} < 1 \) of \( P_N^{(\alpha, \beta)} \) satisfy
\[
z_{i+1,N} - z_{i,N} \geq \frac{2^{7/4}}{M} \cdot (\min(\alpha + 1, \beta + 1))^{1/2} \quad (i = 1, \ldots, N - 1). \quad (4.16)
\]
Moreover, for $\alpha = \beta > -1$,
\[
z_{i+1,N} - z_{i,N} \geq \frac{2^{11/4}(\alpha + 1)^{1/2}}{\sqrt{M(M + 4(\alpha + 1))}} \quad (i = 1, \ldots, N - 1).
\]

**Proof.** Lemma 4.3 yields that
\[
16 \sum_{i \neq i'} (1 - z_{i,N}^2)^2 + (1 - z_{i',N}^2)(1 - z_{i,N}^2) \leq M^2.
\]
This and (4.14) lead to
\[
2^{7} \frac{\min(\alpha + 1, \beta + 1)!)^2}{M^2(z_{i,N} - z_{i+1,N})^4} \leq M^2
\]
and thus to (4.19). In the same way, (4.11) leads to (4.17). □

**Remark 4.8.**
(1) If one compares Theorem 4.7 with Theorem 6.3.1 of [S] for $-1/2 \leq \alpha, \beta \leq 1/2$, then in most cases the estimate in [S] is again the better one under this restriction.

(2) If one uses the asymptotic result (4.13) for $\alpha, \beta$ fixed and $N$ large in the proof of the preceding theorem instead of (4.14), one gets an asymptotic modifications of Theorem 4.7 where the asymptotic rate is slightly better than in (4.10) for large $\alpha, \beta$.

(2) In summary, we have the impression that our approach here on Jacobi ensembles and polynomials should be rewritten in some trigonometric form similar to the square-root-form in Theorem 3.7 in the Laguerre case. Unfortunately, we were not able to transform the entries (4.3) in the inverse covariance matrices $S_N$ above in a trigonometric and useful way.

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