Cosmology in the Einstein-Electroweak Theory and Magnetic Fields

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Abstract

In the $SU(2)_L \times U(1)_Y$ standard electroweak theory coupled with the Einstein gravity, new topological configurations naturally emerge, if the spatial section of the universe is globally a three-sphere ($S^3$) with a small radius. The $SU(2)_L$ gauge fields and Higgs fields wrap the space nontrivially, residing at or near a local minimum of the potential. As the universe expands, however, the shape of the potential rapidly changes and the local minimum eventually disappears. The fields then start to roll down towards the absolute minimum. In the absence of the $U(1)_Y$ gauge interaction the resulting space is a homogeneous and isotropic $S^3$, but the $U(1)_Y$ gauge interaction necessarily induces anisotropy while preserving the homogeneity of the space. Large magnetic fields are generically produced over a substantial period of the rolling-over transition. The magnetic field configuration is characterized by the Hopf map.

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1. Introduction

It has been well known that the standard electroweak theory does not admit classical lump solutions such as monopoles. There has been, however, unflinching interest in non-perturbative phenomena in the electroweak theory such as baryon number violation by anomaly [1] or sphalerons [2] and dumbbell solutions [3, 4]. Such non-perturbative effects are expected to play a key role in particle physics phenomenology and cosmology.

Not so well investigated are cosmological configurations in the electroweak theory, or more generally in non-Abelian gauge theory. Many years ago, a classical exact solution in the $SU(2)$ Yang-Mills theory coupled to the Einstein gravity was found [5, 6] which describes a closed Friedmann-Robertson-Walker universe with large time-dependent magnetic fields filling the space. It was noticed that there appears a natural map between the gauge group $SU(2)$ and the space $S^3$. The extension to more general gauge group and spacetime has been made [7]. Dynamics of fermions are also investigated [8]. The analysis has been extended to a semiclassical theory in the Euclidean signature in which solutions are wormholes describing quantum transitions to another universe [9, 10].

In the standard electroweak theory the gauge group is not $SU(2)$, but $SU(2)_L \times U(1)_Y$. Furthermore the symmetry is spontaneously broken to $U(1)_{EM}$ by the Higgs fields having a nonvanishing expectation value. We shall show that, in spite of these intricate features, the electroweak theory leads to a new type of cosmological solutions. The presence of the Higgs fields stabilizes topological configurations of the $SU(2)_L$ gauge fields when the size of the universe is sufficiently small. The $U(1)_Y$ gauge interaction deforms the map between the $SU(2)_L$ gauge group and the space, giving rise to anisotropy in the space. The resulting space is a deformed $S^3$ which is an anisotropic, but homogeneous compact manifold [11]. It may be recalled that such minimal distortion of symmetry arises in sphaleron solutions as well; sphalerons are spherically symmetric in the $SU(2)_L$ theory, whereas they become only axially symmetric in the $SU(2)_L \times U(1)_Y$ theory [2]. In our case this deformation is related to the nontrivial Hopf map on $S^3$.

The gravity plays many vital roles in field theory. Quantum effects, for instance, lead to the Hawking effect around black holes which shall prompt unification of thermal or statistical character with gravitational one [12]. Novel effects emerge even in classical theory. Due to attractive nature of gravitational force soliton-like objects become possible.
even when such things are strictly forbidden in flat space. \[13, 14, 15\] In particular, magnetic monopoles in pure Yang-Mills theory become possible in the Einstein gravity with a negative cosmological constant. \[16\] Scalar field theory with a potential of the double well type admits cosmic shells in asymptotically de Sitter space. \[17, 18, 19\] In this paper we investigate another classical interplay of gravity and field theory in the context of the cosmological evolution of the universe.

The paper is organized as follows. After giving general topological consideration of Higgs and gauge field configurations in Section 2, we give in Section 3 our ansätze for the solutions to the equations of motion, whereby switching on and off the $U(1)_Y$ gauge interaction. The shape of the potential perceived by the Higgs and gauge fields is discussed in detail in Sect. 4. We demonstrate in Section 5 that the Hopf mapping is embodied in our gauge field configuration, which makes it possible to distribute vector fields of constant magnitude all over the three-sphere. Time evolution of field configurations is evaluated numerically in Section 6 for various input parameters and initial conditions. It is shown in Section 7 that $U(1)_{EM}$ magnetic field survives the cosmological evolution for a substantial period of time for values of the parameters in a wide range. Section 8 is devoted to summary and discussions.

\section{2. Topology in the Higgs and gauge fields}

In the present paper we discuss only the bosonic part of the standard electroweak theory in the Einstein gravity whose action is given by

$$I = \int d^4x \sqrt{-g} \left\{ \frac{1}{16\pi G} (R - 2\Lambda) - \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{4} G_{\mu\nu} G^{\mu\nu} \right.$$

$$\left. - (D_{\mu} \Phi)^{\dagger} (D^\mu \Phi) - \lambda \left( \Phi^{\dagger} \Phi - \frac{v^2}{2} \right)^2 \right\} ,$$

$$D_{\mu} \Phi = (\partial_{\mu} - i \frac{g}{2} \tau^a A_{\mu}^a - i \frac{g'}{2} B_{\mu}) \Phi ,$$

(2.1)

where $F^a_{\mu\nu}$ and $G_{\mu\nu}$ denote the field strengths of the $SU(2)_L$ and $U(1)_Y$ gauge fields $A_{\mu}^a$ and $B_{\mu}$, respectively. $\Phi$ is a doublet Higgs field which develops a nonvanishing expectation value. We employ the natural unit $\hbar = c = 1$, and $G$ is the gravitational constant.

We shall investigate time evolution of classical field configurations in a Robertson-Walker spacetime with a spatial section $S^3$ or deformed $S^3$. To understand why nontrivial
topological configurations appear in such spacetime, it is instructive to first examine the
topology of the electroweak theory on the fixed space \( S^3 \).

A three-sphere \( S^3 \) with a radius \( a \) is a hypersurface defined by
\[ z_1^2 + z_2^2 + z_3^2 + z_4^2 = a^2 \]
in the four-dimensional Euclidean space with Cartesian coordinates \( \{ z_i \} \). If the gauge
interactions are switched off, there appears a natural map between the Higgs vacuum and
\( S^3 \). For later convenience we parameterize the Higgs field as
\[
\Phi = \frac{1}{\sqrt{2}} \left( \phi_2 + i\phi_1 \right) \\
\phi_4 - i\phi_3
\]
(2.2)
where \( \phi_i \)'s are all real fields. The manifold of the Higgs vacuum, namely a manifold defined
by minima of the Higgs potential, is \( \phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 = v_0^2 \), namely \( S^3 \) in the space of the
Higgs fields. There appears a natural map from the space \( S^3 \) to the Higgs vacuum \( S^3 \);
\[
\phi_j = v_0 \, y_j \ , \quad y_j = \frac{z_j}{a} .
\]
(2.3)
The Higgs fields wrap the space nontrivially.

Now let us switch on the \( SU(2)_L \) gauge interaction. By a gauge transformation, the
Higgs configuration (2.2) can be smoothly unwrapped. In fact by a gauge transformation
with a gauge potential
\[
\Omega = y_4 + i\vec{y} \cdot \vec{r} \in SU(2),
\]
(2.4)
the Higgs and gauge fields undergo, respectively, transformation
\[
\Phi \quad \rightarrow \quad \Phi' = \Omega^{-1} \Phi = \Omega^{-1} \frac{v_0}{\sqrt{2}} \left( y_2 + i y_1 \right) = \begin{pmatrix}
0 \\
v_0/\sqrt{2}
\end{pmatrix},
\]
\[
A = 0 \quad \rightarrow \quad A' = -\frac{i}{g} \Omega^{-1} d\Omega \neq 0 .
\]
(2.5)
The Higgs fields are brought into the standard form. This, however, does not imply that
the effect of the wrapping (2.2) of the Higgs field has gone. The topological information of
the wrapping is encoded in the \( SU(2)_L \) gauge field (2.5) whose form is now far from trivial.
There appears an energy barrier between (2.5) and the trivial configuration. Less clear is
the classical stability of the configuration. There is no topological index which guarantees
the stability of the configuration.

The above consideration suggests that there are nontrivial topological configurations in
the standard electroweak theory. To be realistic we have to include the \( U(1)_Y \) interaction
and the dynamics of the curved space must be incorporated. It shall turn out that the configuration (2.5) is stable only if \( g a v_0 \) is small enough, and that the Einstein equations dictate \( a \) to expand so that the configuration can be stable only for a short period. In the following sections we shall give thorough discussions both in the absence and presence of the \( U(1)_Y \) interaction.

3. Configurations in the closed universe

In the Einstein gravity, the existence of matter fields is the source of distortion of the spacetime geometry. In the electroweak theory (2.1) the energy-momentum tensor is given by

\[
T_{\mu \nu} = F^a_{\mu \rho} F^a_{\nu \rho} - g_{\mu \nu} \frac{1}{4} F^a_{\rho \sigma} F^a_{\rho \sigma} + G_{\mu \rho} G_{\nu}^{\rho} - g_{\mu \nu} \frac{1}{4} G_{\rho \sigma} G^{\rho \sigma} + (D_\mu \Phi)^\dagger (D_\nu \Phi) + (D_\nu \Phi)^\dagger (D_\mu \Phi) - g_{\mu \nu} \left\{ (D_\rho \Phi)^\dagger (D^\rho \Phi) + \lambda \left( \Phi^\dagger \Phi - \frac{v_0^2}{2} \right) \right\}.
\]

An ansatz for the metric must be consistently made with a configuration of the gauge and Higgs fields. To facilitate our discussions in switching on or off the \( U(1)_Y \) interactions, it is most convenient to use differential forms. We in particular write the metric of the \( S^3 \) or \( SU(2) \) manifold, using the Maurer-Cartan forms. Being equipped with them, we can easily go over to a deformed \( S^3 \) manifold which is necessary to describe the \( SU(2)_L \times U(1)_Y \) electroweak theory in gravity.

3.1 The Robertson-Walker spacetime \( R^1 \times S^3 \)

The Maurer-Cartan 1-forms, \( \sigma_j \)'s, are expressed in terms of \( \Omega \) in (2.4), by

\[
\sigma_j^i \tau^j = -i \Omega^{-1} d\Omega , \quad d\sigma^i = e^{i\ell \sigma^j} \wedge \sigma^\ell ,
\]

where \( \tau_j \)'s are Pauli matrices. In terms of \( y_j \)

\[
\sigma_j = e^{i\ell \sigma^j} y_k d y_{\ell} + y_4 d y_j - y_j d y_4 .
\]

The metric of a unit three-sphere is written as

\[
d\Omega_3^2 = \sigma^1 \otimes \sigma^1 + \sigma^2 \otimes \sigma^2 + \sigma^3 \otimes \sigma^3 = \sigma^a_j \sigma^a_k d x^j d x^k ,
\]
which reduces in the spherical coordinates \((\chi, \theta, \phi)\) to the standard form \(d\Omega_3^2 = d\chi^2 + \sin^2 \chi(d\theta^2 + \sin^2 \theta d\phi^2)\).

The metric and tetrads of the Robertson-Walker spacetime with a spatial section \(S^3\) is given by

\[
\begin{align*}
\text{ds}^2 &= -N(t)^2 dt^2 + a(t)^2 d\Omega_3^2, \\
e^0 &= N(t) dt, \quad e^j = a(t) \sigma^j \quad (j = 1, 2, 3). 
\end{align*}
\]

The lapse function \(N(t)\) has been included for later convenience.

The curvature 2-forms in the tetrad basis are given by

\[
\begin{align*}
\mathcal{R}_{jk} &= A \ e^j \wedge e^k, \quad A = \frac{1}{a^2} \left\{ 1 + \left( \frac{\dot{a}}{N} \right)^2 \right\}, \quad (j, k = 1, 2, 3), \\
\mathcal{R}_{0j} &= B \ e^0 \wedge e^j, \quad B = -\frac{1}{a N} \frac{d}{dt} \left( \frac{\dot{a}}{N} \right) \quad (j = 1, 2, 3). 
\end{align*}
\]

The Ricci tensor \(R_{ab}\) is diagonal. The Einstein tensor is given by

\[
\begin{pmatrix}
3A \\
-A + 2B \\
-A + 2B \\
-A + 2B
\end{pmatrix}
\]

where \(\eta_{ab} = \text{diag}(-1, 1, 1, 1)\). The space is homogeneous and isotropic.

### 3.2 In the \(\theta_W = 0\) theory

Suppose that the \(U(1)_Y\) gauge interaction is absent, i.e. the weak mixing angle vanishes; \(\theta_W = \tan^{-1}(g'/g) = 0\). There is a natural map between the space \(S^3\) and \(SU(2)_L\) gauge field configurations. Following [4], we start from the following ansatz:

\[
\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 \\
v(t)
\end{pmatrix}
\]

\[
A = A^j \frac{\tau^j}{2} dx^\mu = \frac{1}{2g} f(t) \sigma^j \tau^j = -\frac{i}{2g} f(t) \Omega^{-1} d\Omega
\]

\[
B = B_\mu dx^\mu = 0 .
\]

Note that \(\Phi'\) and \(A'\) in (2.3) corresponds to \(v(t) = v_0\) and \(f(t) = 2\) in (3.8); the ansatz (3.8) incorporates non-trivial wrapping of configurations. We remark that each component of
the gauge fields is, in spite of its simple appearance in (3.8), endowed with nontrivial dependence in space, since the orientation of the tetrad basis \( \{e^a\} \) varies in space. Nevertheless the configuration (3.8) leads to a self-consistent closed set of equations of motion.

The \( SU(2)_L \) field strength is

\[
F = dA + igA \wedge A
\]

\[
= \left\{ \frac{\dot{j}}{aN} e^0 \wedge e^\ell + \frac{f(2-f)}{2a^2} e^m \wedge \epsilon^{\ell mn} e^m \wedge e^n \right\} \frac{\tau^\ell}{2g}.
\] (3.9)

Note that \( f(t) = 2 \) and \( f(t) = 0 \) correspond to pure gauge configurations. However, they are physically distinct and are separated by an energy-barrier when the Higgs fields are nonvanishing.

Insertion of (3.8) into (3.1) gives energy-momentum tensors in the tetrad basis, \( T_{ab} = e_a \mu e_b \nu T_{\mu \nu} \). Off-diagonal components identically vanish. Diagonal components are

\[
T_{00} = \frac{3}{2} \frac{\dot{j}^2}{g^2 a^2 N^2} + \frac{\dot{v}^2}{2N^2} + V_{\theta = 0}(v, f; a),
\]

\[
T_{11} = T_{22} = T_{33} = p(t),
\] (3.10)

where the potential \( V_{\theta = 0} \) and the pressure \( p(t) \) are given, respectively, by

\[
V_{\theta = 0}(v, f; a) = \frac{\lambda}{4} (v^2 - v_0^2)^2 + \frac{3}{8} \frac{v^2 f^2}{a^2} + \frac{3}{2} \frac{f^2(f - 2)^2}{g^2 a^4}.
\] (3.11)

\[
p(t) = \frac{1}{2} \frac{\dot{j}^2}{g^2 a^2 N^2} + \frac{1}{2} \frac{f^2(f - 2)^2}{g^2 a^4} + \frac{\dot{v}^2}{2N^2} - \frac{\lambda}{4} (v^2 - v_0^2)^2 - \frac{1}{8} \frac{v^2 f^2}{a^2}.
\] (3.12)

Observe that \( (v, f) = (v_0, 0) \) and \( (v_0, 2) \) yield distinct \( T_{ab} \)’s. Each component of \( T_{ab} \) does not depend on spatial coordinates \( x^j \)’s. Further they preserve the rotational symmetry; \( T_{0k} = 0 \) and \( T_{jk} = p(t) \delta_{jk} \). The pressure is the same in all spatial directions.

The Einstein equation

\[
R_{ab} - \frac{1}{2} \eta_{ab}(R - 2\Lambda) = 8\pi G T_{ab},
\] (3.13)

reduces to two equations.

\[
\frac{3}{a^2} \left\{ 1 + \left( \frac{\dot{a}}{N} \right)^2 \right\} - \Lambda = 8\pi G T_{00}
\] (3.14)
\[-\frac{2}{aN} \frac{d}{dt} \left( \frac{\dot{a}}{N} \right) - \frac{1}{a^2} \left\{ 1 + \left( \frac{\dot{a}}{N} \right)^2 \right\} + \Lambda = 8\pi G p(t) \, . \tag{3.15}\]

The equations of motion of the gauge and Higgs fields are simplified to

\[ a \frac{d}{N} \left( \frac{a}{N} \frac{df}{dt} \right) + 2f(f-1)(f-2) + \frac{1}{4} (gva)^2 f = 0 \, , \tag{3.16} \]

\[ \frac{1}{a^3} \frac{d}{dt} \left( a^3 \frac{dv}{dt} \right) + \left\{ \lambda(v^2 - v_0^2) + \frac{3}{4} f^2 \right\} v = 0 \, . \tag{3.17} \]

Not all of the four equations (3.14) \sim (3.17) are independent. Eq. (3.13) follows from the other three. The lapse function \( N(t) \), which may be taken at will, is chosen to be \( N(t) = 1 \) in the following discussions. We have thus three independent equations for three unknown functions, i.e., \( a(t) \), \( f(t) \) and \( v(t) \). We comment that Eqs. (3.14) \sim (3.17) can also be obtained by first inserting the ansatz into the action (2.1), and the n by varying the action with respect to \( N, a, f, \) and \( v \).

It is extremely intriguing that the potential \( V_{\theta_W=0}(v, f, a) \) in (3.13) has a nontrivial minimum in \((v, f)\) space in addition to the trivial one \((v, f) = (v_0, 0)\), when the scale factor \( a \) is small enough. (A more detailed account will be given in subsections 4.1 and 4.2.) The static configuration at the minimum in \((v, f)\) space would be a solution to (3.16) and (3.17), provided the time evolution of the scale factor \( a \) could be frozen. As time develops, however, the scale factor \( a \) necessarily evolves subject to the Einstein equations and the shape of the potential accordingly changes. It is interesting to investigate the time development in the \((v, f)\) space starting from the local minimum by solving (3.14), (3.16) and (3.17). Before jumping into such enterprise, however, we have to refine the ansatz to include the \( U(1)_Y \) gauge interaction.

3.3 In the \( \theta_W \neq 0 \) theory

In our real world, there is the \( U(1)_Y \) gauge interaction whose presence gives an effect on the symmetry and structure of the universe. We shall see that the resulting universe is homogeneous but anisotropic.

We need to generalize the ansatz (3.8). The spatial component of the \( U(1)_Y \) gauge field necessarily picks one particular direction on \( S^3 \), giving rise to anisotropy. This in turn affects and deforms the \( SU(2)_L \) symmetry as well. We fix the Higgs field in the standard form given in (3.8), employing an \( SU(2)_L \) gauge transformation. With this choice the
$U(1)_Y$ gauge field $B$ must be proportional to $e^3$ and the asymmetry in the $SU(2)_L$ gauge fields must be aligned along this direction. This is confirmed a posteriori by computing energy-momentum tensors of the configuration.

The ansatz for the fields is given by

$$\Phi = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 0 \\ v(t) \end{array} \right),$$

$$A = \frac{1}{2g} \left\{ f_1(t)(\sigma^1 \tau^1 + \sigma^2 \tau^2) + f_3(t)\sigma^3 \tau^3 \right\},$$

$$B = h(t)\sigma^3.$$

The resulting space is a deformed three-sphere. The metric and tetrads of the spacetime are

$$ds^2 = -N(t)^2 dt^2 + a_1(t)^2(\sigma^1 \otimes \sigma^1 + \sigma^2 \otimes \sigma^2) + a_3(t)^2\sigma^3 \otimes \sigma^3,$$

$$e^0 = Ndt, \quad e^1 = a_1 \sigma^1, \quad e^2 = a_1 \sigma^2, \quad e^3 = a_3 \sigma^3.$$  

(3.18)

In Appendix A the Riemann curvature and Ricci tensors are summarized for the more general metric $ds^2 = -N^2 dt^2 + \sum_{j=1}^3 (a_j)^2 \sigma^j \otimes \sigma^j$. It is shown there that even in this general metric the Ricci tensors are diagonal, depending only on time $t$. It gives an anisotropic, but homogeneous space.

Non-vanishing components of the energy-momentum tensors for the configuration (3.18) are

$$T_{00} = \frac{1}{2g^2 N^2} \left( \frac{\dot{f}_1^2}{a_1^2} + \frac{\dot{f}_3^2}{a_3^2} \right) + \frac{\dot{h}^2}{2N^2 a_3^2} + \frac{\dot{v}^2}{2N^2} + V(v, f_1, f_3, h; a_1, a_3),$$

$$T_{11} = T_{22} = \frac{1}{2g^2 N^2} \left( \frac{\dot{f}_3^2}{a_3^2} + \frac{\dot{h}^2}{2N^2 a_3^2} + \frac{\dot{v}^2}{2N^2} - \frac{\lambda}{4} (v^2 - v_0^2)^2 \right) + \frac{\lambda}{8} (f_3 - g'h)^2 + \frac{1}{2g^2} \left( \frac{2f_3 - f_1^2}{a_1^2} \right)^2 + \frac{2h^2}{a_1^4},$$

$$T_{33} = \frac{1}{2g^2 N^2} \left( \frac{\dot{f}_1^2}{a_1^2} - \frac{\dot{f}_3^2}{a_3^2} \right) - \frac{\dot{h}^2}{2N^2 a_3^2} - \frac{\dot{v}^2}{2N^2} - \frac{\lambda}{4} (v^2 - v_0^2)^2 - \frac{\lambda}{8} \left( \frac{2f_1^2 - (f_3 - g'h)^2}{a_1^2} \right) + \frac{1}{2g^2} \left( \frac{2f_3^2 (2 - f_3)}{a_1^2 a_3^2} + \frac{(2f_2 - f_3)^2}{a_1^4} \right) - \frac{2h^2}{a_1^4}. \quad (3.20)
In showing that $T_{ab} = 0$ for $a \neq b$, the alignment of $B$ and $\Phi$ is crucial.

The Einstein equations are

$$\frac{1}{N^2} \left( \frac{a_3^2}{a_1^2} + 2 \frac{a_1 a_3}{a_1^2} \right) + \frac{1}{a_1^3} \left( 4 - \frac{a_3^2}{a_1^2} \right) - \Lambda = 8\pi G T_{00} \ ,$$  \hspace{1cm} (3.22)

$$\frac{1}{N a_1} \frac{d}{dt} \left( \frac{a_3}{N} \right) + \frac{1}{N a_3} \frac{d}{dt} \left( \frac{a_1}{N} \right) + \frac{a_1 a_3}{N^2 a_1 a_3} + \frac{a_1^2}{a_1^3} - \Lambda = -8\pi G T_{11} \ ,$$  \hspace{1cm} (3.23)

$$\frac{2}{N a_1} \frac{d}{dt} \left( \frac{a_1}{N} \right) + \frac{a_1^2}{N^2 a_1^2} + \frac{4}{a_1^2} - \frac{3 a_3^2}{a_1^2} - \Lambda = -8\pi G T_{33} \ .$$  \hspace{1cm} (3.24)

All off-diagonal components identically vanish.

Equations for the gauge fields and Higgs field are

$$d(*F) - ig(A \wedge *F - *F \wedge A) = -* j^{SU(2)} \ ,$$

$$d(*B) = -* j^{U(1)} \ ,$$

$$\frac{1}{\sqrt{-g}} D_\mu \left\{ \sqrt{-g} g^{\mu \nu} D_\nu \Phi \right\} = \lambda \left( 2 \Phi^\dagger \Phi - g_0^2 \right) \Phi \ ,$$  \hspace{1cm} (3.25)

where $*$ denotes Hodge dual, and the currents are given by

$$* j^{SU(2)} = \frac{\delta L^{Higgs}}{\delta A^{a\mu}} \tau^a d x^\mu \ , \hspace{1cm} * j^{U(1)} = \frac{\delta L^{Higgs}}{\delta B^\mu} d x^\mu .$$  \hspace{1cm} (3.26)

It is easy to see that $d(*A) = 0$ and $d(*B) = 0$. Upon the insertion of the ansatz, these equations are reduced to

$$\frac{1}{N a_3} \frac{d}{dt} \left( \frac{a_3}{N} \right) = \frac{1}{2 g^2 a_1^2} \frac{\partial V}{\partial f_1}$$

$$= - \left\{ \frac{1}{a_3^2} (2 - f_3)^2 - \frac{1}{a_1^2} (2 f_3 - f_1^2) + \frac{1}{4} (g v)^2 \right\} f_1 \ ,$$  \hspace{1cm} (3.27)

$$\frac{a_3}{N a_1^2} \frac{d}{dt} \left( \frac{a_1^2}{N a_3} f_3 \right) = - g^2 a_3^2 \frac{\partial V}{\partial f_3}$$

$$= - \frac{2 a_3^2}{a_1^2} (2 f_3 - f_1^2) + \frac{2}{a_1^2} f_1^2 (2 - f_3) - \frac{1}{4} (g v)^2 (f_3 - g' h) \ ,$$  \hspace{1cm} (3.28)

$$\frac{a_3}{N a_1^2} \frac{d}{dt} \left( \frac{a_1^2}{N a_3} h \right) = - a_3^2 \frac{\partial V}{\partial h}$$
\[ I = 2\pi^2 \cdot \frac{1}{16\pi G} \int dt a_1^2 a_3 \left\{ \frac{d}{a_1} \left( \frac{\dot{a}_1}{N} \right) + \frac{2}{a_3} \frac{d}{dt} \left( \frac{\dot{a}_3}{N} \right) + \frac{2}{N} \left( \frac{\dot{a}_1}{a_1} \right)^2 + \frac{2}{N} \left( \frac{\dot{a}_3}{a_3} \right)^2 \right\} \]

All of the equations, (3.22)-(3.24) and (3.27)-(3.30), have been obtained by inserting the ansatz into the equations of motion. They can be obtained by first putting the ansatz into the action and then varying the action (3.31) with respect to \( N, a_1, a_3, f_1, f_3, h \) and \( v \).

4. Potential in the fixed metric

Before examining the time evolution of the field configurations, it is most appropriate to understand the shape of the potential, supposing that the background metric is fixed. The emergence of a new local minimum in the potential is a crucial ingredient in our scenario in order to make it plausible to suppose that the universe once assumes a topologically non-trivial field configuration. We examine the \( \theta_W = 0 \) case first for which the location of the minima of the potential can be analytically determined, and then proceed to the \( \theta_W \neq 0 \) case.

4.1 In the \( \theta_W = 0 \) theory

In this case the potential \( V_{\theta_W=0} \) in (3.11) depends on the two variables \( v(\geq 0) \) and \( f \). We write it in the form

\[ V_{\theta_W=0} = \lambda v_0^4 \left\{ \frac{1}{4} \left( \frac{v^2}{v_0^2} - 1 \right)^2 + \frac{3\alpha}{8\beta^2 v_0^2} f^2 + \frac{3\alpha}{2\beta^3} f^2(f - 2)^2 \right\} , \]
\[ \alpha = \frac{g^2}{\lambda} \quad \text{and} \quad \beta = g v_0. \quad (4.1) \]

This shows that the shape of the potential depends on two dimensionless parameters \( \alpha \) and \( \beta \). In the standard model \( \alpha = \mathcal{O}(1) \). \( \beta \) depends on the scale factor \( a \).

If \( \beta \gg 1 \), the first term dominates over the rest in (4.1) so that there appears only one minimum at \( v \sim v_0 \) and \( f = 0 \). Less trivial is the case in which \( \beta \) becomes \( \mathcal{O}(1) \) or smaller. The conditions for extrema are

\[ f \left( f^2 - 3f + 2 + \frac{1}{8} \beta^2 \frac{v^2}{v_0^2} \right) = 0, \]
\[ \frac{v}{v_0} \left( \frac{v^2}{v_0^2} - 1 + \frac{3\alpha}{4\beta^2} f^2 \right) = 0. \quad (4.2) \]

Nontrivial extrema appear for

\[ \alpha < \frac{32}{3}, \quad \beta < \beta_c = \left( 2 \cdot \frac{1 + \frac{3}{32} \alpha}{1 - \frac{3}{32} \alpha} \right)^{1/2}. \quad (4.3) \]

Let us define

\[ f_\pm = \frac{1}{2(1 - \frac{3}{32} \alpha)} \left\{ 3 \pm \sqrt{9 - \left( 1 - \frac{3}{32} \alpha \right) \left( 8 + \frac{1}{2} \beta^2 \right)} \right\}, \quad (4.4) \]
\[ v_\pm = v_0 \sqrt{1 - \frac{3\alpha}{4\beta^2} f_\pm^2}. \quad (4.5) \]

A local minimum is located at

\[ (v, f) = \begin{cases} (v_+, f_+) & \text{for } \sqrt{3}\alpha < \beta < \beta_c, \\ (0, 2) & \text{for } \beta < \sqrt{3}\alpha. \end{cases} \quad (4.6) \]

The global minimum is always located at \( (v, f) = (v_0, 0) \). For \( \sqrt{3}\alpha/2 < \beta < \beta_c \), \( (v_-, f_-) \) is a saddle point. The local minimum is separated from the global minimum by a barrier. An example of the potential is depicted in Figure 1.

The location of the local minimum varies as \( \beta \). With a given value of \( \alpha \), \( f_+ (\beta) \) monotonically decreases from 2 to \( \frac{3}{2} (1 - \frac{3}{32} \alpha)^{-1} \) as \( \beta \) varies from \( \sqrt{3}\alpha \) to \( \beta_c \). The \( \beta \)-dependence of the local minimum \( (v, f) \) is depicted in Figures 2 and 3. Here and hereafter we use values

\[ v_0 \approx 246 \text{GeV}, \quad g \approx 0.653, \quad g' \approx 0.358 \quad (4.7) \]
Figure 1: SU(2) gauge-Higgs potential (4.1) as a function of $f$ and $v/v_0$ ($\alpha = 0.426$ and $\beta = 1.14$) as the electroweak parameters. As to $\lambda$ we put

$$\lambda \approx 1$$

as a generic value.

In the above considerations, our analysis of Eq. (4.2) has been restricted to the case $\alpha < 32/3$ in accordance with our choice of the parameters, (4.7) and (4.8). For completeness we give a short remark as to the case $\alpha > 32/3$. Eq. (4.2) gives us obviously solutions $(v, f) = (v_0, 0)$, and $(0, 2)$ in this case as well. Apparently, $(v_0, 0)$ is the absolute minimum of the potential. On the other hand $(0, 2)$ is a local minimum for $\beta < \sqrt{3\alpha}$ and is a saddle point for $\beta > \sqrt{3\alpha}$. In addition to these solutions, there exists another solution $(v_-, f_-)$, provided that $\sqrt{3\alpha}/2 < \beta < \sqrt{3\alpha}$. This solution is always a saddle point. In any way, we do not consider the case $\alpha > 32/3$ hereafter.

4.2 In the $\theta_W \neq 0$ theory

The potential $V$ defined in (3.21) may be split into three terms according to the power behavior with respect to the scales $a_1$ and $a_3$:

$$V(v, f_1, f_3, h; a_1, a_3) = V_0 + V_2 + V_4$$

$$V_0 = \frac{\lambda}{4} (v^2 - v_0^2)^2$$
Figure 2: Location of the extrema $f_\pm$ (eq. (4.4)) as a function of $\beta$ for $\alpha = 0.426$. $f_+$ and $f_-$ correspond to the local minimum and saddle point, respectively. $P_1$, $P_2$, and $P_3$ have $\beta = \beta_c = 1.658$, $\sqrt{3} \alpha = 1.131$, and $\sqrt{3} \alpha / 2 = 0.5655$ respectively.

Figure 3: Location of the extrema $v_\pm$ (eq. (4.5)) as a function of $\beta$. $v_+$ and $v_-$ correspond to the local minimum and saddle point, respectively. At $P_3$, the local minimum and saddle point merge.
\[ V_2 = \frac{v^2}{8} \left\{ \frac{2f_1^2}{a_1^2} + \frac{(f_3 - g'h)^2}{a_3^2} \right\} \]
\[ V_4 = \frac{1}{2g^2} \left\{ \frac{2f_1^2(2 - f_3)^2}{a_1^2a_3^2} + \frac{(2f_3 - f_1^2)^2}{a_1^4} \right\} + \frac{2h^2}{a_1^4}. \] (4.9)

It is a function of four variables, \( v, f_1, f_3 \) and \( h \). It also depends on the values of the two scale factors \( a_1 \) and \( a_3 \). As we shall see below, the difference between \( a_1 \) and \( a_3 \) remains relatively small in the cosmological evolution. The global minimum is located at \((v, f_1, f_3, h) = (v_0, 0, 0, 0)\).

As in the \( \theta_W = 0 \) case there appears a new local minimum when \( ga_1v_0 \) and \( ga_3v_0 \) are small enough. To pin down the location of the local minimum we again utilize the stationary conditions (3.28) - (3.30), ignoring the time dependence. For \( v \neq 0 \) we have
\[ g'h = -\frac{g'^2}{2g^2} \left\{ 2f_3 - f_1^2 + \frac{a_1^2}{a_3^2}f_1^2(f_3 - 2) \right\}, \] (4.10)
\[ v^2 = v_0^2 - \frac{1}{4\lambda} \left\{ \frac{2f_1^2}{a_1^2} + \frac{(f_3 - g'h)^2}{a_3^2} \right\}. \] (4.11)

Insertion of (4.10) and (4.11) into (4.9) yields a potential \( \hat{V} \) as a function of \( f_1 \) and \( f_3 \). We look for extrema of \( \hat{V} \) under the condition that the right hand side of (4.11) be positive.

The location of the new local minimum is not altered so much by the presence of the \( U(1)_Y \) gauge interaction. As can be seen from (4.10), the value of \( g'h \) is very small for \( f_1, f_3 \sim 2 \). However, this does not necessarily mean that the \( U(1)_Y \) gauge interaction is unimportant. In the course of expansion of the universe, the local minimum disappears. One would then ask if the fields roll down towards the global minimum. We shall see in Section 6 that in a wide range of the parameters in the theory field configurations never reach the global minimum. As \( a_1 \) and \( a_3 \) become large, the \( V_4 \) part of the potential \( V \) becomes irrelevant. The relevant part of the potential \( V_0 + V_2 \) has a flat direction along \( v = v_0, f_1 = f_3 - g'h = 0 \). Neither \( f_3 \) nor \( h \) approaches zero. In such cases the \( U(1)_{\text{EM}} \) fields play an important role in a substantial period of the expansion of the universe.

5. \( U(1)_Y \) gauge fields and the Hopf map

As explained in Section 3 the presence of the \( U(1)_Y \) gauge interaction alters the symmetry of the space. The \( U(1)_Y \) field strengths, both electric and magnetic, pick a preferred
direction at each space point, thus breaking the isotropy of the space. The homogeneity of
the space is more subtle, depending on the configuration. We have found that the configu-
ration (3.18) breaks the isotropy of the space, but maintains its homogeneity. How can it
be possible to have nonvanishing $U(1)_Y$ field strengths all over the compact space without
spoiling the homogeneity of the space?

To address the issue more precisely, we recall the $U(1)_Y$ field strengths are given by

$$dB = \frac{\dot{h}}{a_3} e^0 \wedge e^3 + \frac{2h}{a_1^2} e^1 \wedge e^2 \quad (5.1)$$

Both the electric and magnetic fields point in the $e^3$-direction at each point $\vec{x}$. The direc-
tion varies in space as $\sigma_3(\vec{x})$ is $\vec{x}$-dependent. The magnitudes of the fields, however, are
independent of $\vec{x}$. In other words we have a vector field $\vec{K}(\vec{x})$ defined over an entire com-
 pact space topologically equivalent to $S^3$. One may wonder what is ensuring such vector
configuration on $S^3$.

It would be helpful to contrast the situation with a two-dimensional vector field on
$S^2$. Placing a vector field $\vec{K}(\vec{x})$ with $|\vec{K}| =$ constant necessarily induces sources/sinks
or vortices where the vector field is ill defined. It is thus impossible to have a constant
two-dimensional vector field on $S^2$ without introducing singularities. In three dimensions,
however, such a vector field can be smoothly defined on $S^3$. With the condition $|\vec{K}| =$
constant the vector field defines a map from the space $S^3$ to the $\vec{K}$ space $S^2$. There exists a
nontrivial map called the Hopf map in mathematical literatures $[20]$. It is straightforward
to check that $K = k_0 \sigma^3$ is a Hopf map.

A Hopf map is realized in electromagnetic $U(1)_{EM}$ field as well. In the standard
electroweak theory the nonvanishing Higgs field breaks the symmetry $SU(2)_L \times U(1)_Y$
down to the electromagnetic $U(1)_{EM}$. Let us define

$$A_{EM} = \frac{1}{\sqrt{g^2 + g'^2}} \left\{ gh + g \frac{f_3}{g} \right\} \sigma^3 \equiv h_{EM} \sigma^3,$$

$$Z = \frac{1}{\sqrt{g^2 + g'^2}} \left\{ -g'h + f_3 \right\} \sigma^3 \equiv h_Z \sigma^3 \quad (5.2)$$

where $h_{EM}$ and $h_Z$ correspond to the electromagnetic and $Z$-field, respectively. The elec-
 tromagnetic field strengths are

$$F_{EM} = \frac{\dot{h}_{EM}}{a_3} e^0 \wedge e^3 + \frac{2h_{EM}}{a_1^2} e^1 \wedge e^2 \quad (5.3)$$
As we see above, both electric and magnetic fields point in the $e^3$ direction. Their magnitude depends on time, but is independent of space position. These vector fields thus realize a nontrivial Hopf map.

The electromagnetic current one-form is connected to the gauge field by $d(*dA_{EM}) = -*j_{EM}$:

$$j_{EM} = \frac{a_3}{a_1} \left\{ \frac{a_1^2}{Na_3} \frac{d}{dt} (\frac{a_1^2}{Na_3} h_{EM}) + 4h_{EM} \right\} e^3 .$$

(5.4)

Making use of Eqs. (3.28) and (3.29) or

$$\frac{a_1^2}{Na_3} \frac{d}{dt} (\frac{a_1^2}{Na_3} h_{EM}) = -4h_{EM} + \frac{2 \sin \theta W}{g} f_1^2 \left\{ 1 + \frac{a_2^2}{a_3} (2 - f_3) \right\}$$

(5.5)

in (5.4), one finds

$$j_{EM} = \frac{2 \sin \theta W}{ga_1 a_3} f_1^2 \left\{ 2 + \frac{a_2^2}{a_3^2} - f_3 \right\} e^3 .$$

(5.6)

The magnitude of the current is space-position independent. The current also realizes a non-trivial Hopf map.

6. Cosmological evolution

We have seen in the preceding sections that a nontrivial local minimum of the potential appears when the size of the universe is sufficiently small. In such a case we can expect interesting phenomena in the history of the universe. Suppose that our system starts from the local minimum at some stage of the universe. The configuration at the minimum cannot be stationary, however. The Einstein equations dictate that the universe expand. We need to solve the Einstein and field equations simultaneously to find precisely how the configuration evolves. Without going into detailed calculation, however, we may surmise the followings. As the universe expands, the barrier separating the local minimum from the global minimum disappears. If the expansion rate of the universe is slow enough, the field configuration could reach the global minimum or its vicinity within finite time. If the universe expands very fast, however, the potential $V$ becomes so flat along the flat direction of $V_0 + V_2$, the field configuration would never reach the global minimum.

As it turns out, interesting phenomena take place when a non-vanishing cosmological constant $\Lambda$ drives the universe to reasonably fast inflation. We are not going to ask the
origin of inflation or the source of $\Lambda$. Instead we limit ourselves to ask how large the cosmological constant should be for an initial nontrivial field configuration to lead to observable effects. We suppose that at an initial time $t_0$ the sizes of the universe, $a_1(t_0)$ and $a_3(t_0)$, are small. To get an idea of the magnitude of $\Lambda$ for a nontrivial configuration to exist, we look at one of the Einstein equations (6.22), supposing $a_1 \approx a_3 (\equiv a)$. In the $N = 1$ gauge it reads

$$3 \left( \frac{\dot{a}^2}{a^2} + \frac{1}{a^2} \right) - \Lambda \approx 8\pi G T_{00}.$$

(6.1)

In the $\theta_W = 0$ case the condition $g a(t_0) v_0 \lesssim \beta_c \sim 1.658$ must be satisfied to have a nontrivial local minimum as discussed in Section 4.1 with $g \sim 0.653$. In $\theta_W \neq 0$ theory the condition is modified. A little numerical computation shows that the condition for the existence of a nontrivial local minimum in the potential remains almost unaltered. $T_{00}$ is either $O(v_0^4)$ or $O(a^{-4})$ and the right hand side of (6.1) is much smaller than $a^{-2}$ for $g a v_0 \lesssim 1.658$. In other words

$$a(t) = \sqrt{\frac{3}{\Lambda}} \cosh \sqrt{\frac{\Lambda}{3}} (t - t_1)$$

(6.2)

at $t \sim t_0$. As $a(t) \gtrsim (3/\Lambda)^{1/2}$, the cosmological constant must be larger than $3(g v_0/\beta_c)^2$ for a nontrivial configuration to exist. It is therefore reasonable to discriminate the following three cases:

**Case I:** $\Lambda \gg (g v_0)^2$ \quad [$\rho \gg (10^{11} \text{ GeV})^4$],

**Case II:** $\Lambda \sim (g v_0)^2$ \quad [$\rho \sim (10^{11} \text{ GeV})^4$],

**Case III:** $\Lambda \ll (g v_0)^2$ \quad [$\rho \ll (10^{11} \text{ GeV})^4$].

Here $\rho$ is the energy density corresponding to $\Lambda = 8\pi G \rho$. The GUT energy scale $\rho \sim (10^{15} \text{ GeV})^4$ corresponds to Case I and the electroweak scale $\rho \sim (v_0)^4$ to Case III. Cases I and II both encompass those of nontrivial local minimum being present, while Case III does not.

The behavior of the various fields can be understood in general terms. The field $f_1$, $h_z$ and $v$ share common behavior; they approach the values at the global minimum of the potential $V$. Take $f_1$, as an example. Eq. (3.27) becomes

$$\ddot{f}_1 + \frac{\dot{a}_3}{a_3} \dot{f}_1 + \frac{1}{4} \frac{a_1^2}{a_3^2} g^2 v^2 f_1 = - \left\{ \frac{1}{a_3^3} (2 - f_3)^2 + \frac{1}{a_1^2} (2 f_3 - f_1^2) \right\} f_1.$$

(6.3)
As $a_j$ becomes large, the right hand side becomes negligibly small. The second term of the left hand side of (6.3) may be regarded as a friction term. In the inflationary phase $\dot{a}_j/a_j \sim (\Lambda/3)^{1/2}$. As justified a posteriori, $a_1 \sim a_3 (\equiv a)$. Eq. (6.3) is approximated by

$$\ddot{f}_1 + \sqrt{\frac{\Lambda}{3}} \dot{f}_1 + \frac{1}{4} g^2 v^2 f_1 = 0$$ (6.4)

whose solution is given by

$$f_1 \sim f_1(t_1) e^{-c(\Lambda/3)^{1/2}(t-t_1)} \sim f_1(t_1) \left[ \frac{a(t_1)}{a(t)} \right]^{1/2},$$

$$c = \frac{1}{2} \left\{ 1 - \left( 1 - \frac{3g^2 v_0^2}{\Lambda} \right)^{1/2} \right\}.$$

(6.5)

$f_1$ eventually approaches zero.

The situation is qualitatively different for $h_{EM}$. Eq. (5.3) reads

$$\ddot{h}_{EM} + \left( \frac{2\dot{a}_1}{a_1} - \frac{\dot{a}_3}{a_3} \right) \dot{h}_{EM} = \frac{a_3^2}{a_1^2} \left\{ -4h_{EM} + \frac{2 \sin \theta_W}{g} f_1^2 \left[ 1 + \frac{a_1^2}{a_3^2} (2 - f_3) \right] \right\}.$$ (6.6)

which is approximated by

$$\ddot{h}_{EM} + \sqrt{\frac{\Lambda}{3}} \dot{h}_{EM} = 0.$$ (6.7)

The solution is

$$h_{EM}(t) = h_{EM}^\infty + c' e^{-\sqrt{\Lambda/3} t} = h_{EM}^\infty + \frac{c''}{a(t)}.$$ (6.8)

$h_{EM}$ approaches an asymptotic value $h_{EM}^\infty$ whose magnitude depends on the initial conditions.

In the following we shall solve Eqs. (3.23), (3.24), (3.27) – (3.30) numerically. We use the numerical values given in (4.7) and (4.8) for $g$, $g'$, $v_0$ and $\lambda$. For convenience we set $\dot{f}_1 = \dot{f}_3 = \dot{h} = \dot{v} = 0$, $a_1 = a_3$, and $\dot{a}_1 = \dot{a}_3$ at $t = t_0$. We start from a field configuration at or near the local minimum of the potential (3.21). Initial values for $a_1 = a_3$, $f_1$, $f_3$, $h$, and $v$ are chosen to be consistent with Eq. (3.22) with a given $\Lambda$. It is satisfied at $t = t_0$ if the potential on the right hand side of (3.22) and the scalar curvature and cosmological constant on the left are balanced at the local minimum. The initial value for $\dot{a}_1 = \dot{a}_3$ is determined consistently. We have checked that Eq. (3.22) is satisfied and $a_1 \sim a_3$ in the subsequent evolution.

**Case I.** $\Lambda \gg (gv_0)^2$, $\rho \gg (10^{11} \text{GeV})^4$
This includes the GUT inflation where \( \rho \sim (10^{15} \text{GeV})^4 \). The universe expands so fast that the fields can change only slowly. It takes long time before the fields start to significantly evolve to their asymptotic values. The approach to the asymptotic values is governed by (6.3) and (6.8). As \((gv_0)^2/\Lambda \ll 1\),

\[
c = \frac{3g^2v_0^2}{4\Lambda}
\]

in (6.7). One example is shown in Fig. [4]

![Graph](image)

Figure 4: The evolution of \( f_1, f_3, h, h_{EM} \) when the configuration started from the local minimum of the potential. In this example, \( \Lambda = 1.0 \times 10^7 \text{GeV}^2 \) and \( a_1(t_0) = a_3(t_0) = \sqrt{3/\Lambda} \). In the transition region \( f_1 \) is well approximated by (6.3) with \( c = 0.001935 \).

Let us denote by \( a_{trans} \) the scale factor \( a \) at which the transition in the fields takes place. The value of \( a_{trans} \) depends on \( \Lambda \). We have explored it numerically up to \( \Lambda = 10^8 \text{GeV}^2 \) to find that \( a_{trans} \) is proportional to \( \Lambda \);

\[
a_{trans} \sim 1530 \cdot \frac{\Lambda}{(gv_0)^3}.
\]

See Fig. [4]. It is of great interest to know why (5.10) holds. It is also confirmed that the asymptotic value for \( h \) or \( h_{EM} \) depends on the initial value \( h(t_0) \), but depends little on \( \Lambda \).

**Case II.** \( \Lambda \sim (gv_0)^2 \), \( \rho \sim (10^{11} \text{GeV})^4 \)

This is the most interesting case. Suppose that the initial configuration is at or very close to the local minimum of the potential, and the corresponding \( \beta \) in (4.1) is well below the critical value \( \beta_c \). In this case the field configuration stays near the minimum for a
Figure 5: For large $\Lambda$ the transition scale $a_{\text{trans}}$ linearly depends on $\Lambda$. In the plot $a_{\text{trans}}$ is defined by $f_1 = 1$ at $a = a_{\text{trans}}$. Points + correspond to the evolution starting from the local minimum, while points $\times$ correspond to the evolution starting from the configuration $h = 2$. Little difference is seen. The line $a_{\text{trans}} = 0.00037\Lambda$ is drawn for visual guide.

while before starting to roll down the hill of the potential. If the field configuration is away from the minimum, the fields quickly start to roll down toward the asymptotic values. In case the initial $\beta$ is close to the critical value $\beta_c$, the fields quickly undergo transition, irrespective of whether the initial configuration is near or away from the local minimum.

The approach to the asymptotic values of the fields is governed by (6.5) again. $f_1$ approaches zero in two to five ten-fold growth of $a_j$, depending on the value of $\Lambda/g^2v_0^2$. The way of approaching zero depends on the initial values. We illustrate it by taking the following two examples:

**Case IIa**: $a_1(t_0) = a_3(t_0) = 9.0 \times 10^{-3}\text{GeV}^{-1}$, $\Lambda = 1.0 \times 10^5 \text{GeV}^2$, $\beta(t_0) = 1.446$

**Case IIb**: $a_1(t_0) = a_3(t_0) = 6.0 \times 10^{-3}\text{GeV}^{-1}$, $\Lambda = 1.0 \times 10^5 \text{GeV}^2$, $\beta(t_0) = 0.964$.

Figures 6 and 7 correspond to Case IIa and IIb respectively. The evolution of $f_1, f_3, h$ and $h_{\text{EM}}$ is depicted as a function of $a_1(t)$. In fig. 6 the initial fields are located at the local minimum of the potential, $(f_1, f_3, h, v) = (1.985, 1.985, 0.03307, 158.6 \text{GeV})$. $f_1$ and $h_z \propto f_3 - g'h$ approach zero around $a = 10^3\text{GeV}^{-1}$, but $f_3$ and $h$ remain non-vanishing. $h_{\text{EM}}$ asymptotically approaches a non-vanishing value in accordance with our previous argument. The magnetic field $h_{\text{EM}}/a_1^2$ thus produced is decreasing only by the factor $a_1^2$. In Figure 6, the fields initially start a tiny bit off the local minimum $(f_1, f_3, h, v) = (2.0, 2.0, 0, 0.1, \text{GeV})$. The corresponding $\beta < \sqrt{3}\alpha$ in (4.6).
We observe in this case that $f_1$ and $f_2$ stay at the minimum for some time and then decrease. $h$ also stays at 0 for a while and start to move towards the asymptotic value. Here again, we observe that $h_{\text{EM}}$ approaches non-vanishing value.

Figure 6: The evolution of $f_1, f_3, h, h_{\text{EM}}$ when the configuration started from the local minimum of the potential. In this example, $\Lambda = 1.0 \times 10^5 \text{ GeV}^2$ and $a_1(t_0) = a_3(t_0) = 9 \times 10^{-3}\text{GeV}^{-1}$.

One may perhaps wonder what would happen if we start with a vanishing magnetic field $h_{\text{EM}}(t_0) = 0$. It is of interest to start not exactly from the local minimum but away from it, thereby adjusting to $h_{\text{EM}}(t_0) = 0$. An example is displayed in Fig. 8 the initial

Figure 7: The evolution of $f_1, f_3, h, h_{\text{EM}}$ when the configuration started from the local minimum of the potential when $\beta < \sqrt{3}\alpha$. In this example, $\Lambda = 1.0 \times 10^5 \text{ GeV}^2$ and $a_1(t_0) = a_3(t_0) = 6 \times 10^{-3}\text{GeV}^{-1}$.
Figure 8: The evolution of $f_1, f_3, h, h_{EM}$ when the configuration started off the minimum of the potential. We put $\Lambda = 1.0 \times 10^5\text{GeV}^2$ and $a_1(t_0) = a_3(t_0) = 9.0 \times 10^{-3}\text{GeV}^{-1}$

Figure 9: The evolution of $f_1, f_3, h, h_{EM}$ when the configuration started off the minimum of the potential when $\beta < \sqrt{3}\alpha$. We put $\Lambda = 1.0 \times 10^5\text{GeV}^2$ and $a_1(t_0) = a_3(t_0) = 6.0 \times 10^{-3}\text{GeV}^{-1}$
conditions are \((f_1, f_3, h, h_{EM}, v) = (1.958, 1.958, -1.644, -0.009838, 129.8 \text{ GeV})\) at \(t_0\). Note that the fields immediately start to roll down. \(f_1\) and \(h_Z\) quickly approach zero. However, \(h_{EM}\), which started with a vanishing value, quickly gains a nonvanishing value. The final value depends on the initial condition, but the fact that \(h_{EM}\) approaches a nonvanishing value does not. Fig. 8 illustrates this remarkable fact.

The behavior of the Higgs field \(v\) is depicted in Fig. 10. It approaches the value \(v_0\) at the global minimum, much in the same way as the gauge fields \(f_1\) and \(h_Z\). It leads to symmetry breaking as in the usual case.

**Case III. \(\Lambda \ll (gv_0)^2\), \(\rho \ll (10^{11} \text{ GeV})^4\)**

In this case the potential \(V\) in (4.9) does not have a local minimum. Nevertheless it is of great interest to ask what would happen if the universe, at one instant \(t_0\), assumes nonvanishing values for \(f_1, f_3\) and \(h\). The size \(a_j(t_0)\) of the universe has to be very large to be consistent with the Einstein equations, which in turn implies that resultant field strengths of the gauge fields are negligibly small.

As an example we set \(\Lambda = 1.0 \times 10^{-29} \text{ GeV}^2\) corresponding to \(\rho = \Lambda/8\pi G = v_0^4\). We pick up an initial configuration with \(a_j = 4.5 \times 10^{14} \text{ GeV}^{-1}\), \(f_1 = f_3 = 2\), \(h = 0\), and \(v = v_0\). The evolution is plotted in Fig. 12. The gauge fields \(f_1, f_3,\) and \(h\) show oscillatory behavior. They vary in the time scale of 0.05 GeV\(^{-1}\). As \(a_j\) is very large, \(V_4\) is totally irrelevant. \(V_2\) quadratically depends on \(f_1\) and \(h\). As \(a_j\) varies very slowly, its \(t\)-dependence may be ignored. \(f_1\) and \(h_Z\), then, exhibit harmonic oscillation with frequency \(\omega = \frac{1}{2}gv_0\) and \(\frac{1}{2}(g^2 + g'^2)^{1/2}v_0\), respectively. \(h_{EM}\), on the other hand, remains constant.

### 7. Generation of electromagnetic field

One interesting aspect of the cosmological evolution of the nontrivial topological configuration is that electromagnetic fields are produced over a substantial period of the expansion of the universe. In this paper we have examined only the bosonic sector of the electroweak theory, ignoring quarks and leptons. It is expected that once dynamics of quarks and leptons are included, the presence of large electromagnetic fields triggers pair creation of fermions, thus affecting the subsequent evolution of the universe. In this section we would like to see how large electromagnetic fields are, and how they depend on the parameters of the theory.
Figure 10: The evolution of $v$ when the configuration started from and away from the local minimum of the potential. The situation for “from” and “off” the minimum correspond to those of Figure 6 and Figure 8 respectively.

Figure 11: The evolution of $v$ when the configuration started from and away from the local minimum of the potential. The situation for “from” and “off” the minimum when $\beta < \sqrt{3\alpha}$ correspond to those of Figure 7 and Figure 9 respectively.
We recall that the electric and magnetic fields are given by $E_3 = \dot{h}_{\text{EM}}/a_3$ and $B_3 = 2h_{\text{EM}}/a_1^2$. As we have seen, generated $h_{\text{EM}}$ is typically $O(1)$ so that the electromagnetic fields become relevant only when $a_j$'s are sufficiently small. In fig. 13 the evolution of the electromagnetic fields is displayed in which the field configuration starts from the local minimum. (We take the same initial condition as in fig. 6.) One sees that the magnetic field $B_3$ persists to exist for considerable time. [11]

One may wonder if the magnetic field just found is a result of a special initial condition chosen, and if it can be generated even with a more general initial condition. To have a more careful look at this point, another example of the evolution is displayed in fig. 14. Here we have adjusted the initial magnetic field as $B_3 = 0$, choosing the starting point away from the local minimum. (We take the same initial condition as in Fig. 8) We clearly see that the magnetic field is indeed generated for substantially long period. In Fig. 15 the dependence on the initial $h(t_0)$ is plotted with other parameters fixed. Amusingly the linear dependence is observed.

The final value of $h_{\text{EM}}$ depends on the initial condition. In Fig. 16 the dependence of $h_{\text{EM}}$ upon $a_j(t_0)$ or $\beta(t_0)$ is plotted with $\Lambda = 1.0 \times 10^{29}\text{GeV}^2$ given. Here the initial configuration is set at the local minimum of the potential. There exists little $a_1$-dependence for small $\beta$, but has weak dependence near $\beta_c$. We can safely conclude that the existence of the
magnetic field for a substantial time during in the early universe is a generic and quite probable phenomenon.

Another interesting question arises about the $\Lambda$-dependence on the final value of $h_{EM}$. In varying $\Lambda$, we always take $a_j^2 = 3/\Lambda$ initially, the field configuration residing at the local minimum. It turns out the asymptotic values of the fields do not depend on $\Lambda$, though the transition scale shows the dependence (6.10).

8. Summary

In the present paper we have explored the interplay between the $SU(2)_L \times U(1)_Y$ electroweak interactions and gravity, especially in the context of the expanding universe. We have unveiled that in the Einstein-electroweak theory there exists a nontrivial topological configuration of the Higgs and gauge fields which corresponds to a local minimum of the potential in the field space. We have looked into the time evolution and fate of such a nontrivial configuration to discover an interesting conspiracy by the gravity-gauge-Higgs system. Even if the gauge-Higgs system is initially endowed with non-trivial topology, the system cannot maintain the topology perpetually. As the universe expands, the potential of the gauge-Higgs system undergoes a change, the barrier separating the non-trivial and trivial configurations thereby disappearing. The gauge and Higgs fields start to roll
down the hill in the potential toward a configuration with a lower energy. However, if the
universe expands fast enough, say, driven by an effective cosmological constant, then the
fields can never reach the global minimum of the potential. The electromagnetic field $h_{\text{EM}}$
survives in the evolution. It is generated for a wide range of parameters. The space of the
resultant universe is a deformed $S^3$ which is homogeneous but anisotropic.

It is of great interest to apply our findings in the actual history of the universe. In
the standard scenario of the early universe, the temperature effect which we have ne-
glected throughout is important. It modifies the shape of the potential as well as the
time-evolution. What we have in mind as one possible scenario is an era preceding the hot
universe which continuously evolves to the current universe by radiation or matter domi-
nance. We suppose that at one instant the universe was very small and cool, and the gauge
and Higgs fields assumed a nontrivial configuration. Driven by an effective cosmological
constant the universe underwent inflation. In a substantial period in the expansion sizable
electromagnetic fields were generated. Eventually the inflation stopped and the universe
was reheated to the temperature about $(\Lambda/8\pi G)^{1/4}$. The universe continued to expand by
the radiation dominance since then.

It would be very interesting to investigate consequences of strong electromagnetic fields
thus generated. In such strong background of electric and magnetic fields, there could
Figure 15: The dependence of the generated electromagnetic field $h_{EM}$ and $f_3$ on the initial $h_{EM}$ imposed is plotted. $\Lambda = 1 \times 10^6$ GeV$^2$ and $a_j(t_0) = \sqrt{3/\Lambda}$. The linear dependence is observed.

be phenomena such as pair creation of fermions and might affect the relic abundance of various elements. All these problems are left for our future work.

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A. Geometry of $S^3$ and deformed $S^3$

Let us summarize the Riemann, Ricci and scalar curvatures of the deformed sphere $S^3$, whose metric is given by

$$ds^2 = -N(t)^2 dt^2 + \sum_{j=1}^{3} a_j(t)^2 \sigma^j \otimes \sigma^j = \eta_{ab} e^a \otimes e^b$$  \hspace{1cm} (A.1)

where use has been made of the local Lorentz metric $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$ and tetrad bases

$$e^0 = N(t) dt, \hspace{1cm} e^i = a_i(t) \sigma^i, \hspace{1cm} (i = 1, 2, 3).$$ \hspace{1cm} (A.2)
Figure 16: $a_1$-dependence of final $h_{\text{EM}}$. $\Lambda = 1.0 \times 10^5\text{GeV}^2$. Points + and the dotted line correspond to configurations starting from the local minimum of $SU(2)_L \times U(1)_Y$ potential and $(f,v) = (f_+,v_+)$ in (4.4) and (4.5), respectively. Points $p_1, p_2$ and the line $f_+$ correspond to those in figs. 2 and 3.

In the main body of the present paper we have set

$$a_1(t) = a_2(t) = a_3(t) = a(t)$$

(A.3)

for $\theta_W = 0$ case and

$$a_1(t) = a_2(t)$$

(A.4)

for $\theta_W \neq 0$ case. Here, however, we keep our metric (A.1) as general as possible by putting the three scale factors $a_i(t)$’s ($i = 1, 2, 3$) on an equal footing.

We impose conditions for vanishing torsion

$$de^a + \omega^a_{\ b} \wedge e^b = 0, \quad (a, b = 0, 1, 2, 3)$$

(A.5)

and express the connections $\omega_{ab} = -\omega_{ba}$ in terms of the tetrads. Straightforward calculations lead

$$\omega_{0i} = -\frac{\dot{a}_i}{a_i N} e^i, \quad \omega_{ij} = \epsilon_{ijk} e^k \tilde{\omega}^k$$

(A.6)

Here we have introduced notation

$$\tilde{\omega}^k = \frac{a_\ell}{a_\ell a_m} + \frac{a_m}{a_\ell a_\ell} - \frac{a_k}{a_\ell a_m}$$

(A.7)
and indices \((k, \ell, m)\) are cyclic permutations of \((1, 2, 3)\).

Curvature 2-form is defined as

\[
\mathcal{R}^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b = \frac{1}{2} R^a_{bcd} e^c \wedge e^d
\]  

(A.8)

By putting (A.6) into (A.8), we obtain curvature 2-forms as follows:

\[
\mathcal{R}_{0i} = -\mathcal{R}_{i0} = \frac{1}{N a_i} \frac{d}{dt} \left( \frac{\dot{a}_i}{N} \right) e^0 \wedge e^i - \frac{1}{N} \left( \frac{2\dot{a}_i}{a_j a_k} - \frac{\dot{a}_j}{a_j} \tilde{\omega}^k - \frac{\dot{a}_k}{a_k} \tilde{\omega}^j \right) e^j \wedge e^k, \tag{A.9}
\]

\[
\mathcal{R}_{ij} = -\mathcal{R}_{ji} = \frac{1}{N a_k} \frac{d}{dt} \left( a_k \tilde{\omega}^k \right) e^0 \wedge e^k + \left( \frac{\dot{a}_i \dot{a}_j}{N^2 a_i a_j} + \frac{2a_k}{a_i a_j} \tilde{\omega}^k - \tilde{\omega}^i \tilde{\omega}^j \right) e^i \wedge e^j. \tag{A.10}
\]

The indices \((i, j, k)\) are cyclic permutations of \((1, 2, 3)\) and repeated indices are not summed over in (A.9) or (A.10), either. Each component of the Riemann tensors can be easily read off by comparing (A.9) and (A.10) with (A.8).

The Ricci tensor is non-vanishing only for diagonal components:

\[
R_{00} = -\frac{1}{N} \left\{ \frac{1}{a_1} \frac{d}{dt} \left( \frac{\dot{a}_1}{N} \right) + \frac{1}{a_2} \frac{d}{dt} \left( \frac{\dot{a}_2}{N} \right) + \frac{1}{a_3} \frac{d}{dt} \left( \frac{\dot{a}_3}{N} \right) \right\} \tag{A.11}
\]

\[
R_{ii} = \frac{1}{N a_i} \frac{d}{dt} \left( \frac{\dot{a}_i}{N} \right) + \frac{4}{(a_i)^2} + \frac{\dot{a}_i}{N a_i} \left( \frac{\dot{a}_i}{N a_j} + \frac{\dot{a}_k}{N a_k} \right) + 2 \left\{ \left( \frac{a_i}{a_j a_k} \right)^2 - \left( \frac{a_j}{a_i a_k} \right)^2 - \left( \frac{a_k}{a_i a_j} \right)^2 \right\} \tag{A.12}
\]

As before the indices \((i, j, k)\) are cyclic permutations of \((1, 2, 3)\). Notice that the Riemann and Ricci tensors in the tetrads depend on time \(t\), but not on spatial coordinates. The spacetime defined by metric (A.1) is spatially homogeneous, but anisotropic. The scalar curvature is given by

\[
R = -R_{00} + R_{11} + R_{22} + R_{33}
\]

\[
= \sum_{i=1}^{3} \left\{ \frac{2}{N a_i} \frac{d}{dt} \left( \frac{\dot{a}_i}{N} \right) + \frac{4}{(a_i)^2} \right\} + \frac{2}{N^2} \left( \frac{\dot{a}_1 \dot{a}_2}{a_1 a_2} + \frac{\dot{a}_2 \dot{a}_3}{a_2 a_3} + \frac{\dot{a}_3 \dot{a}_1}{a_3 a_1} \right)
\]

\[
-2 \left\{ \left( \frac{a_1}{a_2 a_3} \right)^2 + \left( \frac{a_2}{a_3 a_1} \right)^2 + \left( \frac{a_3}{a_1 a_2} \right)^2 \right\} \tag{A.13}
\]

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