Anyons in quantum mechanics with a minimal length

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Abstract

The existence of anyons, i.e. quantum states with an arbitrary spin, is a generic feature of standard quantum mechanics in (2 + 1)–dimensional Minkowski spacetime. Here it is shown that relativistic anyons may exist also in quantum theories where a minimal length is present. The interplay between minimal length and arbitrary spin effects are discussed.

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I. INTRODUCTION

The existence of a minimal observable length in Nature is suggested by string theory and quantum gravity, see e.g. [1]. An economical way of introducing such a minimal length at the level of nonrelativistic quantum mechanics is to use a suitably modified Heisenberg algebra [2–4]. This proposition has been extended to covariant quantum mechanics in [5] and further investigated in many relevant cases including the Klein-Gordon [6] and Dirac [7] equations.

Although the study of quantum theories characterized by a minimal length has become an active field of research, some comments remain to be done about the case of a (2 + 1)−dimensional spacetime with a minimal length. Indeed, it is well-known that spin is no longer quantized in 2 + 1 dimensions, allowing the appearance of anyons, i.e. particles with arbitrary spin. To our knowledge, the question of the existence (or not) of anyons in a (2 + 1)−dimensional spacetime with a minimal length has never been addressed. It is the topic of the present paper.

II. POINCARÉ ALGEBRA IN 2 + 1 DIMENSIONS

A. Generalities

The Poincaré algebra in 2 + 1 dimensions can be written under the form [8]

\[ [J^\mu, J^\nu] = -i \varepsilon^{\mu \nu \rho} J^\rho, \quad [J^\mu, P^\nu] = -i \varepsilon^{\mu \nu \rho} P^\rho; \quad [P^\mu, P^\nu] = 0, \]

(1)

where \( P^\mu \) are the translation generators and where

\[ J^\mu = \frac{1}{2} \epsilon^{\mu \nu \rho} L_{\nu \rho}, \]

(2)

\( L_{\mu \nu} \) being the Lorentz generators. The convention \( \epsilon^{012} = 1 \) is used, as well as the Minkowski metric \( \eta = \text{diag}(+,-,-) \). The two Casimir operators associated to (1) read, for massive representations,

\[ M^2 = P^2, \quad s = \frac{P \cdot J}{M}. \]

(3)

They represent the squared mass and the spin of a given state respectively, while the notation \( X = (X^0, X^1, X^2) \) for a 3−vector.
It is useful to note that, in the rest frame of a given state, \( s \) reduces to \( J^0 = \pm L_{12} \), that is the generator of spatial rotations. One recovers then a maybe more intuitive definition of spin as the phase factor associated to rotations.

The unitary irreducible representations of the Lorentz algebra in 2 + 1 dimensions – the first commutator of \( \{1\} \) – have been built in [8], while the unitary irreducible representations of the full Poincaré algebra have been obtained in [9]. The conclusions shared by both studies are identical in what concerns the spin of the representations: It can be an arbitrary real number. This is a peculiar feature of quantum mechanics in 2 + 1 dimensions and strongly differs from the (3 + 1)– dimensional case, in which the spin of a state can only be integer of half-integer. The properties of such particles with arbitrary spin, called anyons, have been studied in a considerable amount of works. The interested reader may find relevant informations in the pioneering works [10] and [11, 12], as well as in the reviews [13, 14].

It is known that finite-dimensional representations of the Lorentz algebra are non-unitary in 2 + 1 dimensions. However, this does not imply that bosons and fermions are forbidden. It is indeed worth mentioning that bosonic and fermionic states can be built in 2+1 dimensions from the tensor product of non-unitary infinite-dimensional representations of the Lorentz algebra. That subtle issue has been studied in [15], to which we refer the interested reader.

**B. Minimal length representation of the Poincaré algebra**

It has been shown in [5] that, for a \((D+1)\)–dimensional spacetime with metric \( \eta = \text{diag}(+,-,\ldots,-) \), the Poincaré algebra is represented by the generators

\[
P_\mu = (1 - \beta \hat{P}^2)^{-1} \hat{P}_\mu, \quad L_{\mu\nu} = (1 - \beta \hat{P}^2)^{-1} (\hat{P}_\mu \hat{X}_\nu - \hat{P}_\nu \hat{X}_\mu)
\]

provided that

\[
\left[ \hat{X}_\mu, \hat{P}_\nu \right] = -i \left( (1 - \beta \hat{P}^2) \eta^{\mu\nu} - \beta' \hat{P}^\mu \hat{P}^\nu \right),
\]

\[
\left[ \hat{X}_\mu, \hat{X}_\nu \right] = -i \left[ (2\beta - \beta') - (2\beta + \beta')\beta \hat{P}^2 \right] \eta^{\mu\nu},
\]

\[
\left[ \hat{P}_\mu, \hat{P}_\nu \right] = 0,
\]

where it is assumed that \( \beta, \beta' \in \mathbb{R}^+ \). The modified Heisenberg algebra [5] leads to a minimal uncertainty on the measurement of a position that can be computed to be, assuming isotropy [5],

\[
\Delta X_{\text{min}} = \sqrt{(D\beta + \beta')(1 - \beta((P^0)^2))},
\]
and is usually referred to as “minimal length”.

III. RELATIVISTIC ANYONS WITH A MINIMAL LENGTH

It follows from the general results recalled in the previous section that, in \((2 + 1)\) dimensions, quantum theories which are both Poincaré invariant and formulated in a spacetime with minimal length are allowed. As a consequence, anyons may exist when a minimal length is present too. Wave equations describing an anyon \(|\psi\rangle\) have been proposed in \([16]\) in a form that is convenient regarding to our framework. They read

\[
V_\mu |\psi\rangle = 0, \quad V_\mu = sP_\mu - i\epsilon_{\mu\nu\rho}P^\nu J^\rho + MJ_\mu. \tag{7}
\]

They are valid whatever the representation chosen for \(P^\mu\) and \(J^\mu\), so they can be used with the minimal length representation \((4), (5)\). An extensive discussions of the solutions of \((7)\) can be found in \([16]\).

In view of explicit computations, it is convenient to note that the algebra \((5)\) can be represented by the operators \([5]\)

\[
\hat{P}^\mu = p^\mu, \\
\hat{X}^\mu = \left(1 - \beta p^2\right)x^\mu - \beta' p^\mu p \cdot x + i\gamma p^\mu, \tag{8}
\]

where \(x_\mu = -i\frac{\partial}{\partial p^\mu}\) and \(p\) are standard position and momentum operators, \(i.e. \ [x^\mu, p^\nu] = -i\eta^\mu{}^\nu\). A straightforward computation shows that, using this representation, the Lorentz generators take a more familiar form

\[
L_{\mu\nu} = i(p_\mu \frac{\partial}{\partial p^\nu} - p_\nu \frac{\partial}{\partial p^\mu}). \tag{9}
\]

Using the representation \((8)\) in polar coordinates, namely \(p^\mu = (p^0, p, \theta, \phi)\), one quickly shows that

\[
J^0 = L_{12} = i \frac{\partial}{\partial \theta}, \tag{10}
\]

recovering a standard form for \(J^0\) even if a modified Heisenberg algebra is considered.

A peculiarity of algebra \((5)\) is that \(\left[\hat{X}^\mu, \hat{X}^\nu\right] \propto L^{\mu\nu}\). Hence, the noncommutativity of spatial coordinates is such that \(\left[\hat{X}^1, \hat{X}^2\right] \propto J^0\). Let us indeed consider an anyonic state of
squared mass $M^2$ and spin $s$ in its rest frame and denote $|\psi; M^2, s, j\rangle$ such a state. Then one is led to the following uncertainty relation on the spatial coordinates:

$$\Delta X^1 \Delta X^2 \geq \frac{1}{2} |(2\beta - \beta') - (2\beta + \beta')\beta M^2| s. \quad (11)$$

Anyons with larger spins then “feel” a larger spatial noncommutativity. Note however that, even for small values of $s$, one has the lower bound $\Delta X^1 \Delta X^2 \geq \Delta X^2_{\text{min}}$ implied by the modified Heisenberg algebra.

IV. NONRELATIVISTIC LIMIT

The modified Heisenberg algebra initially proposed by Kempf in a nonrelativistic version [2] can be recovered from algebra (5) and from (4) by neglecting the terms in $\beta (P^0)^2$, i.e. by assuming that the typical energy of the system under study is negligible compared to $1/\sqrt{\beta}$, and by only considering the spatial commutators. The only nonvanishing commutators in the case of two spatial dimensions are then

$$[\hat{X}^i, \hat{P}^j] = i \left[(1 + \beta \hat{P}^2)\delta^{ij} + \beta' \hat{P}^i \hat{P}^j\right],$$

$$[\hat{X}^1, \hat{X}^2] = -i \left[(2\beta - \beta') + (2\beta + \beta')\beta \hat{P}^2\right] J^0, \quad (12)$$

where $\hat{P}^2 = (P^1)^2 + (P^2)^2$. The momentum representation (8) has the nonrelativistic counterpart

$$\hat{P}^j = p^j,$$

$$\hat{X}^j = i(1 + \beta p^2) \frac{\partial}{\partial p^j} + i\beta' p^j \vec{p} \cdot \vec{\nabla} p + i\gamma p^j. \quad (13)$$

It can be check from the above representation that any Hamiltonian of the form $T(\hat{P}^2) + V(\hat{X}^2)$, being spherically symmetric in momentum representation, will commute with $J^0$, according to (11). Moreover, in the rest frame of the eigenstate $|\chi\rangle$ under study, $J^0$ is identified with the spin; its eigenvalue $s$ can then be arbitrary. It follows

$$\langle \vec{p}|\chi\rangle = e^{is\theta p} \psi(p), \quad (14)$$

which fixes the angular dependence of the state. This last relation also shows that, if $\vec{P}$ and $\vec{X}$ are the relative momentum and position of a two-body system made of two identical bodies, the wave function acquires a phase factor $e^{is\pi}$ under permutation of the two bodies. This
exotic phase factor is the signature of a braid statistics, in agreement with the generalized spin-statistics theorem [17].

As an illustration, let us consider the Hamiltonian $\hat{H} = \frac{\mu^2}{2\pi} + \frac{1}{2} \mu \omega^2 \hat{X}^2$, whose energy spectrum is analytically known in $D$ spatial dimensions [18]. When $D = 2$, the spin is arbitrary and, using the results of this last reference, the energy spectrum is given by

$$E = \omega \sqrt{2n + |s| + 1} \left[ 1 + \left( \frac{\beta^2 s^2 + \beta'^2}{4} \right) \mu^2 \omega^2 ight]^{1/2}$$

$$\left[ (\beta + \beta')(2n + |s| + 1)^2 + (\beta - \beta')(s^2 + 1) + \beta' \right] \frac{\mu \omega}{2},$$

which is a generalization of the anyonic harmonic oscillator, that can be recovered with $\beta = \beta' = 0$, see e.g. [14, 19]. The main effect of the modified Heisenberg algebra is to break to well-known degeneracy in $2n + |s| + 1$.

It is worth pointing out that there exists a deformation of the Heisenberg algebra which is inequivalent to (12) but which can also be related to anyons. It reads

$$[a^-, a^+] = 1 + \nu K,$$

$$\{K, a^\pm\} = 0, \quad K^2 = 1,$$

where the real parameter $\nu$ and the Kleinian $K$ are responsible for the deformation. $a^\pm$ are creation and annihilation operators. As shown in [20], this algebra can be used to describe in an elegant way relativistic quantum states with arbitrary spin. In the nonrelativistic limit, anyons built from such a formalism reduce to free massive particles in the noncommutative plane [21]: As in Eq. (11), states with arbitrary spin are indeed associated to noncommutative spatial coordinates.

V. SUMMARY AND OUTLOOK

Anyons may exist in $2 + 1$ dimensions as representations of the Poincaré algebra. It follows that they may be present in any Poincaré invariant spacetime, including those where a minimal length is present. In such spacetimes, an anyon wave equation can be defined.

As an outlook, let us mention that previously known analytical results about the spectra of Schrödinger-like Hamiltonians with modified Heisenberg algebra could be re-analysed in the particular case of two spatial dimensions to better appraise the interplay between of an arbitrary spin and a minimal length.
An other way of producing anyons is to minimally couple a particle to a vortex-like gauge field [11]. It could be interesting to see what is the generalization of such a gauge field configuration in electrodynamic with a minimal length, which has been proposed recently in [22]; we leave such a task for future works.

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[1] D. J. Gross and P. F. Mende, Nucl. Phys. B 303, 407 (1988); D. Amati, M. Ciafaloni, and G. Veneziano, Phys. Lett. B 216, 41 (1989); M. Maggiore, Phys. Lett. B 319, 83 (1993) [hep-th/9309034]; A. Connes, Commun. Math. Phys. 182, 155 (1996) [hep-th/9603053]; G. Amelino-Camelia, J. Ellis, N. E. Mavromatos, and D. V. Nanopoulos, Mod. Phys. Lett. A 12, 2029 (1997) [hep-th/9701144]; N. Seiberg and E. Witten, JHEP 9909, 032 (1999) [hep-th/9908142], and references therein.

[2] A. Kempf, J. Math. Phys. 35, 4483 (1994) [hep-th/9311147].

[3] A. Kempf, G. Mangano, and R. B. Mann, Phys. Rev. D 52, 1108 (1995) [hep-th/9412167]; H. Hinrichsen and A. Kempf, J. Math. Phys. 37, 2121 (1996) [hep-th/9510144].

[4] A. Kempf, J. Phys. A 30, 2093 (1997) [hep-th/9604045].

[5] C. Quesne and V. M. Tkachuk, J. Phys. A 39, 10909 (2006) [quant-ph/0604118].

[6] S. K. Moayedi, M. R. Setare and H. Moayeri, Int. J. Theor. Phys. 49, 2080 (2010) [arXiv:1004.0563].

[7] S. K. Moayedi, M. R. Setare and H. Moayeri, Int. J. Mod. Phys. A 26, 4981 (2011) [arXiv:1105.1900]; A. Boumali and H. Hassanabadi, Can. J. Phys. 93, no. 5, 542 (2015).

[8] A. O. Barut and C. Fronsdal, Proc. Roy. Soc. London A287, 532-548 (1965).

[9] B. Binegar, J. Math. Phys. 23, 1511 (1982).

[10] J. M. Leinaas and J. Myrheim, Nuovo Cim. B 37, 1 (1977).

[11] F. Wilczek, Phys. Rev. Lett. 49, 957 (1982).
[12] J. Frohlich and P. A. Marchetti, Lett. Math. Phys. 16, 347 (1988); G. W. Semenoff, Phys. Rev. Lett. 61, 517 (1988).

[13] S. Forte, Rev. Mod. Phys. 64, 193 (1992).

[14] A. Khare, Fractional statistics and quantum theory (2005, World Scientific Publishing, 2nd Ed.).

[15] P. A. Horvathy, M. S. Plyushchay and M. Valenzuela, Annals Phys. 325, 1931 (2010) [arXiv:1001.0274].

[16] J. L. Cortes and M. S. Plyushchay, J. Math. Phys. 35, 6049 (1994) [hep-th/9405193].

[17] J. Mund, Commun. Math. Phys. 286, 1159 (2009) [arXiv:0801.3621].

[18] L. N. Chang, D. Minic, N. Okamura and T. Takeuchi, Phys. Rev. D 65, 125027 (2002) [hep-th/0111181].

[19] R. MacKenzie and F. Wilczek, Int. J. Mod. Phys. A 3, 2827 (1988).

[20] M. S. Plyushchay, Phys. Lett. B 320, 91 (1994) [hep-th/9309148]; Annals Phys. 245, 339 (1996) [hep-th/9601116].

[21] P. A. Horvathy and M. S. Plyushchay, Phys. Lett. B 595, 547 (2004) [hep-th/0404137].

[22] S. K. Moayedi, M. R. Setare and B. Khosropour, [arXiv:1303.0100].