Strong converse exponent for classical-quantum channel coding

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We determine the exact strong converse exponent of classical-quantum channel coding, for every rate above the Holevo capacity. Our form of the exponent is an exact analogue of Arimoto’s, given as a transform of the Rényi capacities with parameters $\alpha > 1$. It is important to note that, unlike in the classical case, there are many inequivalent ways to define the Rényi divergence of states, and hence the Rényi capacities of channels. Our exponent is in terms of the Rényi capacities corresponding to a version of the Rényi divergences that has been introduced recently in [Müller-Lennert, Dupuis, Szehr, Fehr and Tomamichel, J. Math. Phys. 54, 122203, (2013)], and [Wilde, Winter, Yang, Commun. Math. Phys., 331, (2014)]. Our result adds to the growing body of evidence that this new version is the natural definition for the purposes of strong converse problems.

I. INTRODUCTION

Reliable transmission of information through a noisy channel is one of the central problems in both classical and quantum information theory. In quantum information theory, a memoryless classical-quantum channel is a map that assigns to every input signal from an input alphabet a state of a quantum system, and repeated use of the channel maps every sequence of input signals into the tensor product of the output states corresponding to the elements of the sequence. This is a direct analogue of a memoryless classical channel, where the outputs are probability distributions on some output alphabet; in fact, classical channels can be seen as a special subclass of classical-quantum channels where all possible output states commute with each other.

To transmit information through $n$ uses of the channel, the sender and the receiver have to agree on a code, i.e., an assignment of a sequence of input signals and a measurement operator on the output system to each possible message, such that the measurement operators form a valid quantum measurement, normally described by a POVM (positive operator valued measure). The maximum rate (the logarithm of the number of messages divided by the number of channel uses) that can be reached by such coding schemes in the asymptotics of large $n$, with an asymptotically vanishing probability of erroneous decoding, is the capacity of the channel. The classical-quantum channel coding theorem, due to Holevo [34] and Schumacher and Westmoreland [57], identifies this operational notion of capacity with an entropic quantity, called the Holevo capacity, that is the maximum mutual information in a classical-quantum state between the input and the output of the channel that can be obtained from some probability distribution on the input through the action of the channel. This is one of the cornerstones of quantum information theory, and is a direct analogue of Shannon’s classic channel coding theorem, which in turn can be considered as the starting point of modern information theory.

Clearly, there is a trade-off between the coding rate and the error probability, and the Holevo-Schumacher-Westmoreland (HSW) theorem identifies a special point on this trade-off curve, marked by the Holevo capacity of the channel. The direct part of the theorem [34, 57] states that for any rate below the Holevo capacity, a sequence of codes with asymptotically vanishing error probability exists, while the converse part (also known as Holevo bound [32, 33]) says that for any rate above the Holevo capacity, the error probability inevitably goes to one asymptotically. This is known as the strong converse theorem, and it is due to Wolfowitz [68, 69]. Winter’s proof follows Wolfowitz’s approach, based on the method of types, while the proof of Ogawa

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and Nagaoka follows Arimoto’s proof for classical channels [3]. A much simplified approach has been found later by Nagaoka [44], based on the monotonicity of Rényi divergences.

Thus, if one plots the optimal asymptotic error against the coding rate, one sees a sharp jump from zero below the Holevo capacity to one above the Holevo capacity. However, to understand the trade-off between the error and the coding rate, one has to plot also the error on the logarithmic scale. Indeed, it is known that in the direct domain, i.e., for any rate below the Holevo capacity, the optimal error probability vanishes with an exponential speed (see, e.g., [28] for the classical-quantum case), and in the converse domain, i.e., for rates above the capacity, the convergence of the optimal success probability to zero is also exponential [44, 49, 67]. The value of these exponents as a function of the coding rate gives a quantification of the trade-off between the error rate and the coding rate. In the direct domain, it is called the error exponent, and its value is only known for classical channels and large enough rates. In the converse domain, it is called the strong converse exponent, and a lower bound on its value has been given in Arimoto’s work [5]. Dueck and Körner [16] obtained an upper bound on the strong converse exponent in the form of a variational expression with the classical relative entropy, and they suggested that their bound coincides with Arimoto’s. Thus, the works of Arimoto and Dueck and Körner give the exact strong converse exponent for classical channels.

In this paper we determine the exact strong converse exponent for classical-quantum channels. Our form of the exponent is an exact analogue of Arimoto’s, given as a transform of the Rényi capacities with parameters $\alpha > 1$. It is important to note that, unlike in the classical case, there are many inequivalent ways to define the Rényi divergence of states, and hence the Rényi capacities of channels. Our exponent is in terms of the Rényi capacities corresponding to a version of the Rényi divergences that has been introduced recently in [43] and [66]. These divergences have already been connected to the strong converse exponent of hypothesis testing [41] and channel discrimination [12], and the corresponding mutual information appears in the strong converse exponent of a composite hypothesis testing problem [26].

II. PRELIMINARIES

A. Notations and basic lemmas

For a finite-dimensional Hilbert space $\mathcal{H}$, we use the notation $\mathcal{L}(\mathcal{H})$ for the set of linear operators on $\mathcal{H}$, and we denote by $\mathcal{L}(\mathcal{H})_+$ and $\mathcal{L}(\mathcal{H})_{++}$ the set of non-zero positive semidefinite and positive definite linear operators on $\mathcal{H}$, respectively. The set of density operators on $\mathcal{H}$ is denoted by $S(\mathcal{H})$, i.e.,

$$S(\mathcal{H}) = \{ \tau \in \mathcal{L}(\mathcal{H})_+ \mid \text{Tr} \tau = 1 \} ,$$

and $S(\mathcal{H})_{++}$ stands for the set of invertible density operators. For any $\varrho \in \mathcal{L}(\mathcal{H})_+$, we use the notation

$$S_\varrho(\mathcal{H}) = \{ \tau \in S(\mathcal{H}) \mid \text{supp} \tau \leq \text{supp} \varrho \} .$$

We use the notation $\text{SU}(\mathcal{H})$ for the special unitary group on $\mathcal{H}$.

For a self-adjoint operator $A \in \mathcal{L}(\mathcal{H})$, let $\{ A \geq 0 \}$ denote the spectral projection of $A$ corresponding to all non-negative eigenvalues. The notations $\{ A > 0 \}$, $\{ A \leq 0 \}$ and $\{ A < 0 \}$ are defined similarly. The positive part $A_+$ of $A$ is then defined as

$$A_+ := A\{ A > 0 \} .$$

It is easy to see that for any $D \in \mathcal{L}(\mathcal{H})_+$ such that $D \leq I$, we have

$$\text{Tr} A_+ \geq \text{Tr} AD . \tag{1}$$

We will use the convention that powers of a positive semidefinite operator are only taken on its support and defined to be 0 on the orthocomplement of its support. That is, if $a_1, \ldots, a_r$ are the eigenvalues of $A \in \mathcal{L}(\mathcal{H})_+$, with corresponding eigenprojections $P_1, \ldots, P_r$, then $A^p := \sum_{i} a_i^p P_i$ for any $p \in \mathbb{R}$. In particular, $A^0$ is the projection onto the support of $A$.

Measurements with finitely many outcomes on a quantum system with Hilbert space $\mathcal{H}$ can be identified with (completely) positive trace-preserving maps $M : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K})$ with some finite-dimensional Hilbert space $\mathcal{K}$, such that $M(\mathcal{L}(\mathcal{H}))$ is commutative. We denote the set of all such maps by $\mathcal{M}(\mathcal{H})$. 
For an operator $\sigma \in \mathcal{L}(\mathcal{H})$, we denote by $v(\sigma)$ the number of different eigenvalues of $\sigma$. If $\sigma$ is self-adjoint with spectral projections $P_1, \ldots, P_r$, then the pinching by $\sigma$ is the map $\mathcal{E}_\sigma : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$, defined as

$$\mathcal{E}_\sigma : X \mapsto \sum_{i=1}^r P_i XP_i, \quad X \in \mathcal{L}(\mathcal{H}).$$

(2)

The pinching inequality [21, 22] tells that if $X$ is positive semidefinite then

$$X \leq v(\sigma)\mathcal{E}_\sigma(X).$$

(3)

We denote the natural logarithm by $\log$, and use the conventions

$$\log 0 := -\infty, \quad \log +\infty := +\infty.$$  

We will also use the following extension of the logarithm function:

$$\log_0 : [0, +\infty) \mapsto \mathbb{R}, \quad \log_0(x) := \log x, \quad x \in (0, +\infty), \quad \text{and} \quad \log_0 0 := 0.$$  

For self-adjoint operators $A,B \in \mathcal{L}(\mathcal{H})$, $A \leq B$ is understood in the sense of positive semidefinite (PSD) ordering, i.e., it means that $B - A$ is positive semidefinite. The following lemma is standard.

**Lemma II.1** Let $f : J \to \mathbb{R}$ be a monotone function, where $J$ is some interval in $\mathbb{R}$, and let $\mathcal{L}(\mathcal{H})_{sa,J}$ be the set of self-adjoint operators with their spectra in $J$.

If $f$ is monotone then $A \mapsto \text{Tr} f(A)$ is monotone on $\mathcal{L}(\mathcal{H})_{sa,J}$, (4)

if $f$ is convex then $A \mapsto \text{Tr} f(A)$ is convex on $\mathcal{L}(\mathcal{H})_{sa,J}$.

(5)

We say that a function $f : (0, +\infty) \to \mathbb{R}$ is operator monotone increasing if $A,B \in \mathcal{L}(\mathcal{H})_+$, $A \geq B$ implies that $f(A) \geq f(B)$. We say that $f$ is operator monotone decreasing if $-f$ is operator monotone increasing. The following lemma is from [3, Proposition 1.1]:

**Lemma II.2** Let $f$ be a nonnegative operator monotone decreasing function on $(0, +\infty)$, and $\omega$ be a positive linear functional on $\mathcal{L}(\mathcal{H})$. Then the functional

$$A \mapsto \log \omega(f(A))$$

is convex on $\mathcal{L}(\mathcal{H})_+$.  

B. Convexity

We will use the following lemma, and its equivalent version for concavity, without further notice:

**Lemma II.3** Let $X$ be a convex set in a vector space and $Y$ be an arbitrary set, and let $f : X \times Y \to \mathbb{R}$ be such that for every $y \in Y$, $x \mapsto f(x,y)$ is convex. Then

$$x \mapsto \sup_{y \in Y} f(x,y)$$

is convex.  

(6)

If, moreover, $Y$ is a convex set in a vector space, and $(x,y) \mapsto f(x,y)$ is convex, then

$$x \mapsto \inf_{y \in Y} f(x,y)$$

is convex.  

(7)

**Proof** The assertion in (6) is trivial from the definition of convexity. The proof of (7) is also quite straightforward; see, e.g., [11, Section 3.2.5].

**Definition II.4** A multi-variable function on the product of convex sets is called jointly convex.
C. Minimax theorems

Let $X, Y$ be non-empty sets and $f : X \times Y \to \mathbb{R} \cup \{-\infty, +\infty\}$ be a function. Minimax theorems provide sufficient conditions under which

$$\inf_{x \in X} \sup_{y \in Y} f(x, y) = \sup_{y \in Y} \inf_{x \in X} f(x, y). \quad (8)$$

The following minimax theorem is from [40, Corollary A.2].

**Lemma II.5** Let $X$ be a compact topological space, $Y$ be an ordered set, and let $f : X \times Y \to \mathbb{R} \cup \{-\infty, +\infty\}$ be a function. Assume that

- $(i)$ $f(., y)$ is lower semicontinuous for every $y \in Y$ and
- $(ii)$ $f(x, .)$ is monotonic increasing for every $x \in X$, or $f(x, .)$ is monotonic decreasing for every $x \in X$.

Then $(8)$ holds, and the infima in $(8)$ can be replaced by minima.

The following lemma combines a special version of the minimax theorems due to Kneser [35] and Fan [17] (with conditions $(i)$ and $(ii)$), and Sion’s minimax theorem [60] (with conditions $(i')$ and $(ii')$). Recall that a function $f : C \to \mathbb{R}$ on a convex set $C$ is quasi-convex if

$$f(tx_1 + (1-t)x_2) \leq \max\{f(x_1), f(x_2)\}, \quad x_1, x_2 \in C, \; t \in (0, 1),$$

and it is quasi-concave if $-f$ is quasi-convex.

**Lemma II.6** Let $X$ be a compact convex set in a topological vector space $V$ and $Y$ be a convex subset of a vector space $W$. Let $f : X \times Y \to \mathbb{R}$ be such that

- $(i)$ $f(x, .)$ is concave on $Y$ for each $x \in X$, and
- $(ii)$ $f(., y)$ is convex and lower semi-continuous on $X$ for each $y \in Y$.

or

- $(i')$ $f(x, .)$ is quasi-concave and upper semi-continuous on $Y$ for each $x \in X$, and
- $(ii')$ $f(., y)$ is quasi-convex and lower semi-continuous on $X$ for each $y \in Y$.

Then $(8)$ holds, and the infima in $(8)$ can be replaced by minima.

D. Universal symmetric states

For every $n \in \mathbb{N}$, let $\mathcal{S}_n$ denote the symmetric group, i.e., the group of permutations of $n$ elements. For every finite-dimensional Hilbert space $\mathcal{H}$, $\mathcal{S}_n$ has a natural unitary representation on $\mathcal{H}^{\otimes n}$, defined by

$$\pi_n : |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle \longmapsto |\psi_{\pi^{-1}(1)}\rangle \otimes \cdots \otimes |\psi_{\pi^{-1}(n)}\rangle \quad |\psi_i\rangle \in \mathcal{H}, \; \pi \in \mathcal{S}_n.$$ 

Let $\mathcal{L}_{\text{sym}}(\mathcal{H}^{\otimes n})$ denote the set of symmetric, or permutation-invariant, operators, i.e.,

$$\mathcal{L}_{\text{sym}}(\mathcal{H}^{\otimes n}) := \{ A \in \mathcal{L}(\mathcal{H}^{\otimes n}) : \pi_n A = A \pi_n \; \forall \pi \in \mathcal{S}_n \} = \{ \pi_n A | \pi \in \mathcal{S}_n \}',$$

where for $A \subset \mathcal{L}(\mathcal{K})$, $A'$ denotes the commutant of $A$. Likewise, we denote by $\mathcal{S}_{\text{sym}}(\mathcal{H}^{\otimes n})$ the set of symmetric states, i.e., $\mathcal{S}_{\text{sym}}(\mathcal{H}^{\otimes n}) := \mathcal{L}_{\text{sym}}(\mathcal{H}^{\otimes n}) \cap \mathcal{S}(\mathcal{H}^{\otimes n})$.

**Lemma II.7** For every finite-dimensional Hilbert space $\mathcal{H}$ and every $n \in \mathbb{N}$, there exists a symmetric state $\sigma_{u,n} \in \mathcal{S}_{\text{sym}}(\mathcal{H}^{\otimes n}) \cap \mathcal{S}_{\text{sym}}(\mathcal{H}^{\otimes n})'$ such that every symmetric state $\omega \in \mathcal{S}_{\text{sym}}(\mathcal{H}^{\otimes n})$ is dominated as

$$\omega \leq v_{n,d} \sigma_{u,n}, \quad v_{n,d} \leq (n + 1)^{\frac{(d+2)(d-1)}{2}}, \quad (9)$$

where $d = \dim \mathcal{H}$. Moreover, the number of different eigenvalues of $\sigma_{u,n}$ is upper bounded by $v_{n,d}$. We call every such state $\sigma_{u,n}$ a universal symmetric state.
A construction for a universal symmetric state has been given in [24], which we briefly review in Appendix A for readers’ convenience. See also [26, Lemma 1] for a different argument for the existence of a universal symmetric state, with \((n + 1)^d - 1\) in place of \((n + 1)^{(d+2)(d-1)}\) in (9).

The crucial property for us is that
\[
\lim_{n \to +\infty} \frac{1}{n} \log v_{n,d} = 0. \tag{10}
\]

### E. Classical-quantum channels

By a classical-quantum channel (or channel, for short) we mean a map

\[
W : \mathcal{X} \to \mathcal{S}(\mathcal{H}),
\]

where \(\mathcal{X}\) is an arbitrary set (called the input alphabet), and \(\mathcal{H}\) is a finite-dimensional Hilbert space. That is, \(W\) maps input signals in \(\mathcal{X}\) into quantum states on \(\mathcal{H}\). We denote the set of classical-quantum channels with input space \(\mathcal{X}\) and output Hilbert space \(\mathcal{H}\) by \(\mathcal{C}(\mathcal{H}|\mathcal{X})\).

For every channel \(W \in \mathcal{C}(\mathcal{H}|\mathcal{X})\), we define the lifted channel

\[
\hat{W} : \mathcal{X} \to \mathcal{S}(\mathcal{H}_X \otimes \mathcal{H}), \quad \hat{W}(x) := |x\rangle \langle x| \otimes W(x).
\]

Here, \(\mathcal{H}_X\) is an auxiliary Hilbert space, and \(\{|x\rangle : x \in \mathcal{X}\}\) is an orthonormal basis in it. As a canonical choice, one can use \(\mathcal{H}_X = L^2(\mathcal{X})\), the \(L^2\)-space on \(\mathcal{X}\) with respect to the counting measure, and choose \(|x\rangle\) to be the characteristic function (indicator function) of the singleton \(\{x\}\).

Let \(\mathcal{P}_f(\mathcal{X})\) denote the set of finitely supported probability measures on \(\mathcal{X}\). We identify every \(P \in \mathcal{P}_f(\mathcal{X})\) with the corresponding probability mass function, and hence write \(P(x)\) instead of \(P(\{x\})\) for every \(x \in \mathcal{X}\). We can redefine every channel \(W\) with input alphabet \(\mathcal{X}\) as a channel on the set of Dirac measures \(\{\delta_x : x \in \mathcal{X}\} \subset \mathcal{P}_f(\mathcal{X})\) by defining \(W(\delta_x) := W(x)\). \(W\) then admits a natural affine extension to \(\mathcal{P}_f(\mathcal{X})\), given by

\[
W(P) := \sum_{x \in \mathcal{X}} P(x)W(x).
\]

In particular, the extension of the lifted channel \(\hat{W}\) outputs classical-quantum states of the form

\[
\hat{W}(P) = \sum_{x \in \mathcal{X}} P(x)|x\rangle \langle x| \otimes W(x).
\]

Note that the marginals of \(\hat{W}(P)\) are

\[
\text{Tr}_{\mathcal{H}} \hat{W}(P) = \sum_{x \in \mathcal{X}} P(x)|x\rangle \langle x|, \quad \text{Tr}_{\mathcal{H}_X} \hat{W}(P) = \sum_{x \in \mathcal{X}} P(x)W(x) = W(P). \tag{11}
\]

With a slight abuse of notation, we will also denote \(\sum_{x \in \mathcal{X}} P(x)|x\rangle \langle x|\) by \(P\).

The \(n\)-fold i.i.d. extension of a channel \(W : \mathcal{X} \to \mathcal{S}(\mathcal{H})\) is defined as \(W^{\otimes n} : \mathcal{X}^n \to \mathcal{S}(\mathcal{H}^\otimes n)\),

\[
W^{\otimes n}(\underline{x}) := W(x_1) \otimes \ldots \otimes W(x_n), \quad \underline{x} = x_1 \ldots x_n \in \mathcal{X}^n.
\]

Given \(\mathcal{X}\), we will always choose the auxiliary Hilbert space \(\mathcal{H}_{\mathcal{X}^n}\) to be \(\mathcal{H}_{\mathcal{X}^n}^{\otimes n}\) and \(|\underline{x}\rangle := |x_1\rangle \otimes \ldots \otimes |x_n\rangle\), \(\underline{x} = x_1 \ldots x_n \in \mathcal{X}^n\). With this convention, the lifted channel of \(W^{\otimes n}\) is equal to \(\hat{W}^{\otimes n}\). Moreover, for every probability distribution \(P \in \mathcal{P}_f(\mathcal{X})\),

\[
W^{\otimes n}(P^{\otimes n}) = W(P)^{\otimes n} \quad \text{and} \quad \hat{W}^{\otimes n}(P^{\otimes n}) = \hat{W}(P)^{\otimes n},
\]

where \(P^{\otimes n} \in \mathcal{P}(\mathcal{X}^n)\), \(P^{\otimes n}(\underline{x}) := P(x_1) \ldots P(x_n), \underline{x} = x_1 \ldots x_n \in \mathcal{X}^n\), denotes the \(n\)-th i.i.d. extension of \(P\).
III. QUANTUM RÉNYI DIVERGENCES

A. Definitions and basic properties

For classical probability distributions \( p, q \) on a finite set \( X \), their Rényi divergence with parameter \( \alpha \in [0, +\infty) \setminus \{1\} \) is defined as

\[
D_\alpha(p\|q) := \frac{1}{\alpha - 1} \log Q_\alpha(p\|q), \quad Q_\alpha(p\|q) := \sum_{x \in X} p(x)^\alpha q(x)^{1-\alpha},
\]

when \( \text{supp} \, p \subseteq \text{supp} \, q \) or \( \alpha \in [0, 1) \), and it is defined to be \(+\infty\) otherwise. For non-commuting states, various inequivalent generalizations of the Rényi divergences have been proposed. Here we consider the following quantities, defined for every pair of positive definite operators \( \varrho, \sigma \in \mathcal{L}(\mathcal{H})_+ \) and every \( \alpha \in (0, +\infty) \):

\[
Q_\alpha(\varrho\|\sigma) := \text{Tr} \, \varrho^\alpha \sigma^{1-\alpha}, \quad Q^\ast_\alpha(\varrho\|\sigma) := \text{Tr} \left( \varrho^\frac{1}{\alpha} \sigma^{1-\frac{1}{\alpha}} \right)^\alpha,
\]

\[
Q^b_\alpha(\varrho\|\sigma) := \text{Tr} \, e^{\alpha \log \epsilon^+(1-\alpha) \log \sigma}.
\]

The expression in (12) is a quantum f-divergence or quasi-entropy, corresponding to the power function \( x^\alpha \) \([27, 53]\). Its concavity \([37]\) and convexity \([2]\) properties are of central importance to quantum information theory \([38, 47]\), and the corresponding Rényi divergences have an operational significance in the direct part of binary quantum state discrimination as quantifiers of the trade-off between the two types of error probabilities \([6, 23, 45]\). The Rényi divergences corresponding to \([13]\) have been introduced recently in \([43]\) and \([66]\); in the latter paper, they were named “sandwiched Rényi relative entropy”. They have been shown to have an operational significance in the converse part of various discrimination problems as quantifiers of the trade-off between the type I success and the type II error probability \([12, 24, 41, 42]\). \( Q^\ast_\alpha \) has been shown to have an operational significance in information geometry \([1]\), and its logarithm appeared in \([31]\) in connection to the Golden-Thompson inequality. It is the natural quantity appearing in classical divergence-sphere optimization representations of various information quantities, as pointed out in \([50, \text{Section VI}], [48, \text{Section V}], [22]\) and \([45, \text{Remark 1}]\). The corresponding Rényi divergence was shown to be a limiting case of a two-parameter family of Rényi divergences in \([3]\). A closely related quantity appears as a free energy functional in quantum statistical physics \([51]\). Note that for commuting \( \varrho \) and \( \sigma \), all three definitions \([12, 13, 14]\) coincide, and are equal to the classical expression \( \sum_x \varrho(x)^\alpha \sigma(x)^{1-\alpha} \), where \( \varrho(x) \) and \( \sigma(x) \) are the diagonal elements of \( \varrho \) and \( \sigma \), respectively, in a joint eigen-basis.

We extend the above definitions for general, not necessarily invertible positive semidefinite operators \( \varrho, \sigma \in \mathcal{L}(\mathcal{H})_+ \) as

\[
Q^{(t)}_\alpha(\varrho\|\sigma) := \text{lim}_{\varepsilon \searrow 0} Q^{(t)}_\alpha(\varrho + \varepsilon I\|\sigma + \varepsilon I)
\]

\[
= \text{lim}_{\varepsilon \searrow 0} Q^{(t)}_\alpha(\varrho + \varepsilon(I - \varrho^0)\|\sigma + \varepsilon(I - \sigma^0)).
\]

Here and henceforth \((t)\) stands for one of the three possible values \((t) = \{}\), \((t) = * \) or \((t) = \flat)\), where \(\{}\) denotes the empty string, i.e., \(Q^{(t)}_\alpha\) with \((t) = \{}\) is simply \(Q_\alpha\).

Lemma III.1 For every \(\varrho, \sigma \in \mathcal{L}(\mathcal{H})_+\), and every \(\alpha \in (0, +\infty) \setminus \{1\}\), the limits in (15) and (16) exist and are equal to each other. Moreover, if \(\alpha \in (0, 1)\) or \(\varrho^0 \leq \sigma^0\),

\[
Q_\alpha(\varrho\|\sigma) = \text{Tr} \, \varrho^\alpha \sigma^{1-\alpha},
\]

\[
Q^\ast_\alpha(\varrho\|\sigma) = \text{Tr} \left( \varrho^\frac{1}{\alpha} \sigma^{1-\frac{1}{\alpha}} \right)^\alpha,
\]

\[
Q^b_\alpha(\varrho\|\sigma) = \text{Tr} \, P e^{\alpha P(\log \varrho^0)P+(1-\alpha)P(\log \sigma^0)P},
\]

where \(P := \varrho^0 \land \sigma^0\) is the projection onto the intersection of the supports of \(\varrho\) and \(\sigma\), and for all three values of \(t\),

\[
Q^{(t)}_1(\varrho\|\sigma) = \text{Tr} \, \varrho,
\]
and

\[ Q^{(t)}_\alpha(\varrho\|\sigma) = +\infty \quad \text{when} \quad \alpha > 1 \text{ and } \varrho^0 \not\leq \sigma^0. \]

In particular, the extension in (15)–(16) is consistent in the sense that for invertible \( \varrho \) and \( \sigma \) we recover the formulas in (12)–(14).

**Proof** We only prove the assertions for \( Q^\alpha_\varrho \), as the proofs for the other quantities follow similar lines, and are simpler. For \( \alpha \in (0, 1) \), (19) has been proved in [31, Lemma 4.1]. Next, assume that \( \alpha > 1 \) and \( \varrho^0 \leq \sigma^0 \). Then we can assume without loss of generality that \( \sigma \) is invertible. Hence,

\[ Q^\alpha_\varrho(\varrho + \varepsilon (I - \varrho^0)\sigma + \varepsilon (I - \sigma^0)) = Q^\alpha_\varrho(\varrho + \varepsilon (I - \varrho^0)) = \text{Tr} \exp \left( \alpha \log(\varrho + \varepsilon (I - \varrho^0)) + (\alpha - 1) \log(\sigma - 1) \right), \]

and applying again [31, Lemma 4.1], we see that the limit as \( \varepsilon \to 0 \) is equal to

\[ \text{Tr} \exp \left( \alpha P(\log_0 \varrho)P + (\alpha - 1)P(\log_0 \sigma^{-1})P \right) = \text{Tr} \exp \left( \alpha P(\log_0 \varrho)P + (\alpha - 1)P(\log_0 \varrho)P \right). \]

This shows that the limit in (16) exists and is equal to (19). Showing that the limit in (15) also exists, and is equal to (19), follows by a trivial modification.

Hence, we are left to prove the case where \( \alpha > 1 \) and \( \varrho^0 \not\leq \sigma^0 \). By the latter assumption, there exists an eigenvector \( \psi \), with eigenvalue \( \lambda > 0 \) such that \( c := \langle \psi, (I - \sigma^0)\psi \rangle > 0 \). Then we have

\[
\text{Tr} \exp \left( \alpha \log(\varrho + \varepsilon I) + (1 - \alpha) \log(\sigma + \varepsilon I) \right) \\
\geq \langle \psi, \exp(\alpha \log(\varrho + \varepsilon I) + (1 - \alpha) \log(\sigma + \varepsilon I)) \psi \rangle \\
\geq \langle \alpha \psi, \log(\varrho + \varepsilon I) \psi \rangle + (1 - \alpha) \langle \psi, \log(\sigma + \varepsilon I) \psi \rangle \\
= \exp(\alpha \log(\lambda + \varepsilon) + (1 - \alpha) \langle \psi, \log_0(\sigma + \varepsilon^0) \psi \rangle + (1 - \alpha) \log(\varepsilon) \langle \psi, (I - \sigma^0)\psi \rangle) \\
= \varepsilon^{(1 - \alpha)c} \langle \lambda + \varepsilon \rangle \exp((1 - \alpha) \langle \psi, \log_0(\sigma + \varepsilon^0) \psi \rangle),
\]

where the first inequality is obvious, and the second one is due to the convexity of the exponential function. The expression in (20) goes to \( +\infty \) as \( \varepsilon \to 0 \), and hence the limit in (16) is equal to \( +\infty \), as required. The proof for the limit in (15) goes the same way. \( \square \)

**Remark III.2** When \( \varrho^0 \leq \sigma^0 \), (19) can be written as

\[
Q^\alpha_\varrho(\varrho\|\sigma) = \text{Tr} \varrho^0 e^{\alpha \log_0 e^{(1 - \alpha)} e^{(\log_0 \sigma)}} e^0 = \text{Tr} e^{\alpha \log_0 e^{(1 - \alpha)} e^{(\log_0 \sigma)}} e^0 - \text{Tr}(I - \varrho^0).
\]

The quantum Rényi divergences corresponding to the \( Q \) quantities are defined as

\[
D^{(t)}_\alpha(\varrho\|\sigma) := \frac{1}{\alpha - 1} \log Q^{(t)}_\alpha(\varrho\|\sigma) - \frac{1}{\alpha - 1} \log \text{Tr} \varrho = \frac{\psi^{(t)}_\alpha(\varrho\|\sigma) - \psi^{(t)}_1(\varrho\|\sigma)}{\alpha - 1},
\]

for any \( \alpha \in (0, +\infty) \setminus \{1\} \), where \( t \) is any of the three possible values, and we use the notation

\[
\psi^{(t)}_\alpha(\varrho\|\sigma) := \log Q^{(t)}_\alpha(\varrho\|\sigma), \quad \alpha \in (0, +\infty).
\]

By definition,

\[
\psi^{(t)}_\alpha(\varrho\|\sigma) = \lim_{\varepsilon \to 0} \psi^{(t)}(\varrho + \varepsilon I\|\sigma + \varepsilon I), \quad \alpha \in (0, +\infty).
\]

With these definitions we have

**Lemma III.3** If \( \varrho^0 \leq \sigma^0 \) then \( \psi^{(t)}_\alpha(\varrho\|\sigma) \) is continuous in \( \alpha \) on \( (0, +\infty) \). If \( \varrho^0 \not\leq \sigma^0 \) then \( \psi^{(t)}_\alpha(\varrho\|\sigma) \) is continuous in \( \alpha \) on \( (0, 1) \), it has a jump at \( 1 \) as

\[
\lim_{\alpha \to 1} \psi^{(t)}_\alpha(\varrho\|\sigma) < \log \text{Tr} \varrho = \psi^{(t)}_1(\varrho\|\sigma),
\]

and it is \( +\infty \) on \( (1, +\infty) \).
The case \( (\psi, \sigma) \in \mathcal{L}(\mathcal{H})^2 \) for \( t = 1 \) can be seen the following way. Let \( r_{\min} \) denote the smallest positive eigenvalue of \( \varrho \), let \( \tilde{\varrho} := \varrho / r_{\min} \), let \( P := \varrho^0 \wedge \sigma^0 \) and \( P^\perp := I - P \). If \( P = 0 \) then \( \psi^\alpha_\varrho(\varrho | \sigma) = -\infty \) for all \( \alpha \in (0, 1) \), from which \( \psi^\alpha_\varrho(\varrho | \sigma) \) is immediate. Assume now that \( 0 \neq P \neq \varrho^0 \). Then
\[
\lim_{\alpha \to 1} Q^\alpha_\varrho(\varrho | \sigma) = \text{Tr} P e^{P \log_0(\log_0 \tilde{\varrho})} P = \text{Tr} P e^{P \log_0(\log_0 \tilde{\varrho})} P + \text{Tr} P e^{P \log_0(\log_0 \tilde{\varrho})} - \text{Tr}(I - P)
\leq r_{\min} \left[ \text{Tr} e^{P \log_0(\log_0 \tilde{\varrho})} - \text{Tr}(I - P) \right] = r_{\min} \left[ \text{Tr} \tilde{\varrho} + \text{Tr}(I - \varrho^0) - \text{Tr}(I - P) \right]
\leq \text{Tr} \tilde{\varrho} - r_{\min} \text{Tr}(\varrho^0 - P) < \text{Tr} \tilde{\varrho},
\]
where the first inequality is due to \( \ref{4} \), and the second inequality is due to \( \ref{5} \) and the fact that the pinching by \( (P, P^\perp) \) can be written as a convex combination of unitaries \( \ref{3} \), Problem II.5.4. Taking the logarithm gives \( \ref{28} \). \( \blacksquare \)

The relative entropy of a pair of positive semidefinite operators \( \varrho, \sigma \in \mathcal{L}(\mathcal{H})^+ \) is defined \( \ref{55} \) as
\[
D(\varrho | \sigma) := \text{Tr} \varrho (\log_0 \varrho - \log_0 \sigma)
\]
when \( \varrho^0 \leq \sigma^0 \), and \( +\infty \) otherwise. It is easy to verify that
\[
D(\varrho | \sigma) = \lim_{\varepsilon \searrow 0} D(\varrho + \varepsilon I | \sigma + \varepsilon I).
\]
For any \( \varrho \in \mathcal{L}(\mathcal{H})^+ \), its von Neumann entropy \( H(\varrho) \) is defined as
\[
H(\varrho) := -D(\varrho | I) = -\text{Tr} \varrho \log_0 \varrho.
\]
The same way as in \( \ref{61} \) Lemma 2.1 (see also \( \ref{54} \), Proposition 3)), we see that
\[
D(\varrho | \sigma) - \text{Tr}(\varrho - \sigma) \geq \frac{1}{2 \max\{\|\varrho\|, \|\sigma\|\}} \text{Tr}(\varrho - \sigma)^2 \geq 0
\]
for any \( \varrho, \sigma \in \mathcal{L}(\mathcal{H})^+ \). As it was pointed out in \( \ref{64} \), Lemma 6, this implies that for any \( A \in \mathcal{L}(\mathcal{H})^+ \),
\[
\text{Tr} A = \max_{\tau \in \mathcal{L}(\mathcal{H})^+} \{\text{Tr} \tau - D(\tau | A)\},
\]
and the maximum is attained uniquely at \( \varrho = A \). Strictly speaking, \( \ref{28} \) and \( \ref{29} \) were shown in the above references for invertible operators; they can be obtained in the general case by using \( \ref{27} \).

Lemma III.4 For every \( \varrho, \sigma \in \mathcal{L}(\mathcal{H})^+ \), and all three possible values of \( t \),
\[
D^{(t)}_1(\varrho | \sigma) := \lim_{\alpha \to 1} D^{(t)}_\alpha(\varrho | \sigma) = D_1(\varrho | \sigma) := \frac{1}{\text{Tr} \varrho} D(\varrho | \sigma).
\]

Proof The case \( (t) = \{ \} \) follows by a straightforward computation, and the case \( (t) = * \) have been proved by various methods in \( \ref{30} \), \( \ref{43} \), \( \ref{66} \). Hence, we only have to prove the case \( (t) = b \). Assume first that \( \varrho^0 \leq \sigma^0 \). Then \( \ref{21} \) holds for every \( \alpha \in (0, +\infty) \), and thus
\[
\lim_{\alpha \to 1} D^{(b)}_\alpha(\varrho | \sigma) = \lim_{\alpha \to 1} \frac{\psi^\alpha_\varrho(\varrho | \sigma) - \psi^1_\varrho(\varrho | \sigma)}{\alpha - 1} = \frac{d}{d\alpha} \bigg|_{\alpha = 1} \psi^\alpha_\varrho(\varrho | \sigma)
\leq \frac{1}{Q^\alpha_\varrho(\varrho | \sigma)} \text{Tr} \varrho e^{\varrho^0 \log_0(\log_0 \varrho) + (1 - \alpha) \varrho^0} \left[ \log_0 \varrho - (\varrho^0 \log_0(\log_0 \varrho) \varrho^0) \right] \bigg|_{\alpha = 1}
\leq \frac{1}{\text{Tr} \varrho} \text{Tr} \varrho (\log_0 \varrho - \log_0 \sigma),
\]
as required. Now assume that $\varrho^0 \not\in \sigma^0$. Then $\lim_{\alpha \searrow 1} \psi^\flat_\alpha(\varrho\|\sigma) < \psi^\flat_1(\varrho\|\sigma)$ by \ref{eq:32b}, and thus

$$\lim_{\alpha \searrow 1} D^\flat_\alpha(\varrho\|\sigma) = \lim_{\alpha \searrow 1} \frac{\psi^\flat_\alpha(\varrho\|\sigma) - \psi^\flat_1(\varrho\|\sigma)}{\alpha - 1} = +\infty = \frac{1}{\Tr \varrho} D(\varrho\|\sigma),$$

while $D^\flat_\alpha(\varrho\|\sigma) = +\infty$, $\alpha > 1$, which implies

$$\lim_{\alpha \searrow 1} D^\flat_\alpha(\varrho\|\sigma) = \lim_{\alpha \searrow 1} +\infty = +\infty = \frac{1}{\Tr \varrho} D(\varrho\|\sigma).$$

By \ref{eq:24} and \ref{eq:27} we have, for every $\varrho, \sigma \in \mathcal{L}(\mathcal{H})_+$ and all three values of $t$,

$$D^{(t)}_\alpha(\varrho\|\sigma) = \lim_{\varepsilon \searrow 0} D_\alpha(\varrho + \varepsilon I\|\sigma + \varepsilon I), \quad \alpha \in (0, +\infty). \quad (31)$$

The importance of the $\flat$ quantities stems from the following variational representations in Theorem \ref{thm:III.5}. Let

$$s(\alpha) := \begin{cases} 1, & \alpha \geq 1, \\ -1, & \alpha < 1. \end{cases} \quad (32)$$

**Theorem III.5** For every $\varrho, \sigma \in \mathcal{L}(\mathcal{H})_+$ such that $P := \varrho^0 \land \sigma^0 \neq 0$, and for every $\alpha \in (0, +\infty) \setminus \{1\}$,

$$Q^\flat_\alpha(\varrho\|\sigma) = \max_{\tau \in \mathcal{L}(\mathcal{H})_+, \tau^0 \preceq \varrho^0} \{\Tr \tau - \alpha D(\tau\|\varrho) - (1 - \alpha) D(\tau\|\sigma)\}, \quad \text{\( (33)\)}$$

$$\psi^\flat_\alpha(\varrho\|\sigma) = -\min_{\tau \in \mathcal{S}_\varrho(\mathcal{H})} \{\alpha D(\tau\|\varrho) + (1 - \alpha) D(\tau\|\sigma)\}, \quad \text{\( (34)\)}$$

$$D^\flat_\alpha(\varrho\|\sigma) = s(\alpha) \max_{\tau \in \mathcal{S}_\varrho(\mathcal{H})} s(\alpha) \left\{ D(\tau\|\sigma) - \frac{\alpha}{\alpha - 1} D(\tau\|\varrho) \right\}. \quad \text{\( (35)\)}$$

Moreover, \ref{eq:33}–\ref{eq:35} are valid even if $\varrho^0 \land \sigma^0 = 0$ and $\alpha \in (0, 1)$, and \ref{eq:33}–\ref{eq:35} hold also for $\alpha = 1$. When $D^\flat_\alpha(\varrho\|\sigma)$ is finite and $\alpha \in (0, +\infty) \setminus \{1\}$, the optima in \ref{eq:33}–\ref{eq:35} are reached at the unique state

$$\tau_\alpha := P e^{\alpha P(\log_\varrho \varrho) + (1 - \alpha) P(\log_\sigma \sigma) P}/Q^\flat_\alpha(\varrho\|\sigma), \quad \text{\( (36)\)}$$

and at $Q^\flat_\alpha(\varrho\|\sigma)\tau_\alpha$ in \ref{eq:33}.

**Proof** First, note that \ref{eq:33} and \ref{eq:34} only differ in a constant multiplier, and hence we only prove \ref{eq:33}. For $\alpha = 1$ we have

$$\max_{\tau \in \mathcal{L}(\mathcal{H})_+, \tau^0 \preceq \varrho^0} \{\Tr \tau - D(\tau\|\varrho)\} = \Tr \varrho = Q^\flat_1(\varrho\|\sigma),$$

due to \ref{eq:24}, and

$$\min_{\tau \in \mathcal{S}_\varrho(\mathcal{H})} \{D(\tau\|\varrho)\} = \min_{\tau \in \mathcal{S}_\varrho(\mathcal{H})} \{D(\tau\|\varrho)/\Tr \varrho - \log \Tr \varrho\} = -\log \Tr \varrho = -\psi^\flat_1(\varrho\|\sigma).$$

Next, consider the case where $\alpha > 1$ and $\varrho^0 \not\in \sigma^0$. Then the choice $\tau = \tilde{\varrho} := \varrho/\Tr \varrho \in \mathcal{S}_\varrho(\mathcal{H})$ yields

$$\Tr \tau - \alpha D(\tilde{\varrho}\|\varrho) - (1 - \alpha) D(\tilde{\varrho}\|\sigma) = 1 + \alpha \log \Tr \varrho - (1 - \alpha) \cdot (+\infty) = +\infty = Q^\flat_\alpha(\varrho\|\sigma),$$

$$\alpha D(\tilde{\varrho}\|\varrho) + (1 - \alpha) D(\tilde{\varrho}\|\sigma) = -\alpha \log \Tr \varrho + (1 - \alpha) \cdot (+\infty) = -\infty = -\psi^\flat_\alpha(\varrho\|\sigma),$$

proving \ref{eq:33}–\ref{eq:34}. If $\alpha \in (0, 1)$ and $\varrho^0 \land \sigma^0 = 0$ then for any state $\tau$, $D(\tau\|\varrho)$ or $D(\tau\|\sigma)$ is equal to $+\infty$, and thus

$$\min_{\tau \in \mathcal{S}_\varrho(\mathcal{H})} \{(1 - \alpha) D(\tau\|\sigma) + \alpha D(\tau\|\varrho)\} = +\infty = -\psi^\flat_\alpha(\varrho\|\sigma).$$
Hence, for the rest we assume that \( P = \rho^0 \wedge \sigma^0 \neq 0 \), and \( \alpha \in (0,1) \) or \( \rho^0 \leq \sigma^0 \), in which case we can use (19). Note that if \( \rho^0 \leq \sigma^0 \) then \( P = \rho^0 \), and if \( \alpha \in (0,1) \) and \( \sigma^0 \nleq \sigma^0 \) then \( (1 - \alpha)D(\tau||\sigma) + \alpha D(\tau||\rho) = +\infty \). Hence, in both cases the optimization can be restricted to \( \mathcal{S}_P(\mathcal{H}) \), i.e., we have to prove that

\[
Q_\alpha^b(\rho||\sigma) = \max_{\tau \in \mathcal{L}(\mathcal{H})_+, \tau \leq P} \left\{ \alpha D(\tau||\rho) + (1 - \alpha)D(\tau||\sigma) \right\},
\]

\[
\psi_\alpha^b(\rho||\sigma) = -\min_{\tau \in \mathcal{S}_P(\mathcal{H})} \left\{ \alpha D(\tau||\rho) + (1 - \alpha)D(\tau||\sigma) \right\}.
\]

For every \( \tau \in \mathcal{L}(\mathcal{H})_+ \), \( \tau_0 \leq P \), let \( c(\tau) := \alpha D(\tau||\rho) + (1 - \alpha)D(\tau||\sigma) \), and \( \tilde{\tau} := \tau / \text{Tr} \tau \). Then

\[
\max_{\tau \in \mathcal{L}(\mathcal{H})_+, \tau \leq P} \left\{ \alpha D(\tau||\rho) + (1 - \alpha)D(\tau||\sigma) \right\} = \max_{\tau \in \mathcal{S}_P(\mathcal{H})} \left\{ \alpha D(\tau||\rho) + (1 - \alpha)D(\tau||\sigma) \right\}
\]

\[
= \max\left\{ \text{Tr} \tau - t \log \text{Tr} \tau \right\} = \exp\left( -\min_{\tau \in \mathcal{S}_P(\mathcal{H})} c(\tau) \right)
\]

i.e., (37) and (38) are equivalent to each other. A straightforward computation shows that (19) and (29) yield (37), proving (33), (35). The assertion about the unique optimizers then follows from the uniqueness of the optimizer in (29). \( \square \)

**Remark III.6** The family of states in (36) is a quantum generalization of the Hellinger arc.

Next, we give a refinement of (38), which also yields an alternative proof of (38).

**Proposition III.7** Let \( \rho, \sigma \in \mathcal{L}(\mathcal{H})_+ \) be such that \( P := \rho^0 \wedge \sigma^0 \neq 0 \), and let \( \tau_\alpha \) be as in (36). For every \( \tau \in \mathcal{S}_P(\mathcal{H}) \), and every \( \alpha \in (0, +\infty) \setminus \{1\} \),

\[
\alpha D(\tau||\rho) + (1 - \alpha)D(\tau||\sigma) = D(\tau||\tau_\alpha) - \psi_\alpha(\rho||\sigma),
\]

or equivalently,

\[
D_\alpha(\rho||\sigma) = \frac{\alpha}{1 - \alpha} D(\tau||\rho) + D(\tau||\sigma) - \frac{1}{1 - \alpha} D(\tau||\tau_\alpha).
\]

In particular, (38) holds, with \( \tau_\alpha \) being the unique minimizer, and

\[
D_\alpha(\rho||\sigma) = \frac{\alpha}{1 - \alpha} D(\tau_\alpha||\rho) + D(\tau_\alpha||\sigma).
\]

**Proof** Let \( \alpha \in (0, +\infty) \setminus \{1\} \). The quantum relative entropy admits the following simple identity, related to the triangular relation in information geometry [1], Theorems 3.7, 7.1: for any \( \tau, s \in \mathcal{S}(\mathcal{H}) \) and any \( t \in \mathcal{L}(\mathcal{H})_+ \) such that \( \rho^0 \leq \sigma^0 \leq t^0 \),

\[
D(\tau||t) = D(\tau||s) + D(s||t) + \text{Tr}(r - s)(\log_0 s - \log_0 t).
\]

Hence for any \( \tau \in \mathcal{S}_P(\mathcal{H}) \), we have

\[
D(\tau||\rho) = D(\tau||\tau_\alpha) + D(\tau_\alpha||\rho) + \text{Tr}(\tau - \tau_\alpha)(\log_0 \tau_\alpha - \log_0 \rho)
\]

\[
= D(\tau||\tau_\alpha) + D(\tau_\alpha||\rho) + \alpha - 1 \text{Tr}(\tau - \tau_\alpha)(\log_0 \rho - \log_0 \sigma),
\]

\[
D(\tau||\sigma) = D(\tau||\tau_\alpha) + D(\tau_\alpha||\sigma) + \text{Tr}(\tau - \tau_\alpha)(\log_0 \tau_\alpha - \log_0 \sigma)
\]

\[
= D(\tau||\tau_\alpha) + D(\tau_\alpha||\sigma) + \alpha \text{Tr}(\tau - \tau_\alpha)(\log_0 \rho - \log_0 \sigma).
\]

Combining these relations, it holds for any \( \tau \in \mathcal{S}_P(\mathcal{H}) \) that

\[
\alpha D(\tau||\rho) + (1 - \alpha)D(\tau||\sigma) = D(\tau||\tau_\alpha) + D(\tau_\alpha||\rho) + (1 - \alpha)D(\tau_\alpha||\sigma)
\]

By the definition of \( \tau_\alpha \),

\[
\log_0 \tau_\alpha = \alpha P(\log_0 \rho)P + (1 - \alpha)P(\log_0 \sigma)P - \psi_\alpha(\rho||\sigma),
\]
and thus
\[ D(\tau_\alpha | \varrho) = \text{Tr} \tau_\alpha (\log_\alpha \tau_\alpha - \log_\alpha \varrho) = (\alpha - 1) \text{Tr} \tau_\alpha (\log_\alpha \varrho - \log_\alpha \sigma) - \psi_\alpha(\varrho | \sigma), \]
\[ D(\tau_\alpha \| \sigma) = \text{Tr} \tau_\alpha (\log_\alpha \tau_\alpha - \log_\alpha \sigma) = \alpha \text{Tr} \tau_\alpha (\log_\alpha \varrho - \log_\alpha \sigma) - \psi_\alpha(\varrho \| \sigma). \]

Hence,\[
\alpha D(\tau_\alpha \| \varrho) + (1 - \alpha) D(\tau_\alpha \| \sigma) = -\psi_\alpha(\varrho \| \sigma). \tag{47}
\]

Combining this with (45) yields (42). By the strict positivity of the relative entropy, (42) yields (38) and the monotonicity of \( D_{\text{post-measurement R\'enyi divergence}} \) in the asymptotics of many copies, and it follows immediately from Remark III.8.

**Remark III.8** The following variational formulas were shown in [64] and [31], respectively: For any self-adjoint operator \( H \), and any positive definite operator \( A \),
\[ \text{Tr} e^{H + \log A} = \max_{\tau \in \mathcal{L}(\mathcal{H})_{++}} \{ \text{Tr} \tau + \text{Tr} \tau H - D(\tau \| A) \}, \tag{48} \]
\[ \log \text{Tr} e^{H + \log A} = \max_{\tau \in S(\mathcal{H})_{++}} \{ \text{Tr} \tau H - D(\tau \| A) \}. \tag{49} \]

With the substitution \( H := \alpha \log \varrho, A := \sigma^{1-\alpha} \), we can recover (38) and (39) for invertible \( \varrho \) and \( \sigma \). What is new in Theorem III.5, apart from extending the variational representations for non-invertible \( \varrho \) and \( \sigma \), is making the connection between the variational expressions (38) and (39) (equivalently, between (15) and (19)) in [38]—[41]. This shows that proving either of (38) or (39) yields immediately the other variational expression as well. In the proof of Theorem III.5 we followed Tropp’s argument [64] based on (29) to obtain (38), and from it (39). The proof based on Proposition III.7 proceeds the other way around: we first prove (38), which then yields (39). Note that this alternative proof gives a new proof of (15) and (19) through the choice \( \varrho := e^{H/\alpha}, \sigma := A^{1-\alpha} \).

**Remark III.9** For classical random variables (corresponding to commuting density operators), the expression (44) seems to have first appeared in [14]. This yields the variational expressions (31) and (35) for classical random variables; an alternative proof for these have appeared in [54].

Most of the relevant properties of \( D^*_\alpha \) can be derived from the variational formula in Theorem III.5.

**Theorem III.5**

The following Lemma has the same importance for \( D^*_\alpha \):

**Lemma III.10** For any \( \varrho, \sigma \in \mathcal{L}(\mathcal{H})_{++} \), we have
\[
D^*_\alpha(\varrho \| \sigma) = \lim_{n \to \infty} \frac{1}{n} D_\alpha(\mathcal{E}_{\sigma^\otimes n} \varrho^\otimes n \| \sigma^\otimes n), \quad \alpha \in (0, +\infty), \tag{50}
\]
\[
D^*_\alpha(\sigma \| \varrho) = \lim_{n \to \infty} \frac{1}{n} \max_{M_n \in M(\mathcal{H}^\otimes n)} D_\alpha(M_n(\varrho^\otimes n) \| M_n(\sigma^\otimes n)), \quad \alpha \in [1/2, +\infty). \tag{51}
\]

where \( \mathcal{E}_{\sigma^\otimes n} \) is the pinching \( \mathcal{E} \) by \( \sigma^\otimes n \), and the maximization in the second line is over finite-outcome measurements on \( \mathcal{H}^\otimes n \) (see section II A.).

Both (50) and (51) tells that \( D^*_\alpha \) can be recovered as the limit of the R\'enyi divergences of commuting operators. The first identity (50) was proved in [11] Corollary III.8 for \( \alpha \in (1, +\infty) \), and later extended to \( \alpha \in (0, +\infty) \) in [20, Corollary 3]. The second identity (51) tells that \( D^*_\alpha \) can be recovered as the largest post-measurement R\'enyi divergence in the asymptotics of many copies, and it follows immediately from (50) and the monotonicity of \( D^*_\alpha \) under measurements for \( \alpha \in [1/2, +\infty) \) [15].

**Lemma III.11** Let \( \varrho, \sigma \in \mathcal{L}(\mathcal{H})_{++} \), and \( t \) be any of the three possible values. Then the functions
\[
\alpha \mapsto \psi^{(t)}_\alpha(\varrho \| \sigma) \quad \text{and} \quad \alpha \mapsto Q^{(t)}_\alpha(\varrho \| \sigma) \quad \text{are convex on} \quad (0, +\infty), \tag{52}
\]
and the function
\[
\alpha \mapsto D^{(t)}_\alpha(\varrho \| \sigma) \quad \text{is monotone increasing on} \quad (0, +\infty). \tag{53}
\]
Theorem 5. The case (56), \(D\) when \(\sigma\) and \(\kappa\) exist. For (53) follows immediately from the convexity of \(\alpha\mapsto \psi^{(t)}_{\alpha}(\varrho\|\sigma)\). Thus, \(\alpha \mapsto \psi^{(t)}_{\alpha}(\varrho\|\sigma)\) is the supremum of convex functions in \(\alpha\), and hence is itself convex. Since \(Q^{(t)}_{\alpha}(\varrho\|\sigma) = \exp(\psi^{(t)}_{\alpha}(\varrho\|\sigma))\), and the exponential function is monotone increasing and convex, convexity of \(\alpha \mapsto Q^{(t)}_{\alpha}(\varrho\|\sigma)\) follows from the above. Since

\[
D^{(t)}_{\alpha}(\varrho\|\sigma) = \frac{\psi^{(t)}_{\alpha}(\varrho\|\sigma) - \psi^{(t)}_{1}(\varrho\|\sigma)}{\alpha - 1},
\]

(54) follows immediately from the convexity of \(\alpha \mapsto \psi^{(t)}_{\alpha}(\varrho\|\sigma)\).

Monotonicity in \(\alpha\) ensures that the limits

\[
D^{(t)}_{0}(\varrho\|\sigma) := \lim_{\alpha \searrow 0} D^{(t)}_{\alpha}(\varrho\|\sigma) = \inf_{\alpha > 1} D^{(t)}_{\alpha}(\varrho\|\sigma),
\]

\[
D^{(t)}_{\infty}(\varrho\|\sigma) := \lim_{\alpha \to +\infty} D^{(t)}_{\alpha}(\varrho\|\sigma) = \sup_{\alpha > 1} D^{(t)}_{\alpha}(\varrho\|\sigma)
\]

exist. For \(\alpha = 0\), a straightforward computation verifies that

\[
D_{0}(\varrho\|\sigma) = \log \text{Tr} \varrho - \log \text{Tr} \varrho^{0}\sigma\quad \text{and} \quad D^{b}_{0}(\varrho\|\sigma) = \log \text{Tr} \varrho - \log \text{Tr} P e^{P (\log_{0} \varrho) P},
\]

where \(P = \varrho^{0} \land \sigma^{0}\). For \((t) = *\), a procedure to compute \(D^{*}_{0}(\varrho\|\sigma)\) was given in [7, Section 5] for the case \(\varrho^{0} \leq \sigma^{0}\).

For \(\alpha = +\infty\), we get

\[
D_{\infty}(\varrho\|\sigma) = \log \max \left\{ \varrho^{0} : \text{Tr} P_{s} Q_{s} > 0 \right\},
\]

(55)

\[
D^{*}_{\infty}(\varrho\|\sigma) = D_{\max}(\varrho\|\sigma) := \log \inf \left\{ \lambda : \varrho \leq \lambda \varrho \right\},
\]

(56)

\[
D_{\infty}(\varrho\|\sigma) = \log \inf \left\{ \lambda : \log_{0} \varrho \leq \varrho^{0} (\log_{0} (\lambda \varrho)) \varrho^{0} \right\}
\]

(57)

when \(\varrho^{0} \leq \sigma^{0}\), and \(D^{\infty}_{\alpha}(\varrho\|\sigma) = +\infty\) otherwise. In [50], \(P_{s}\) and \(Q_{s}\) denote the spectral projections of \(\varrho\) and \(\sigma\), corresponding to the eigenvalues \(r\) and \(s\), respectively, and the equality follows by a straightforward computation. In [50], \(D_{\max}\) is the max-relative entropy [13, 54], and the equality has been shown in [13, Theorem 5]. The case \((t) = b\) follows from Theorem [11][5] as when \(\varrho^{0} \leq \sigma^{0}\),

\[
D^{b}_{\infty}(\varrho\|\sigma) = \sup_{\alpha > 1} \sup_{\tau \in S(\mathcal{H})} \left\{ D(\tau\|\sigma) - \frac{\alpha}{\alpha - 1} D(\tau\|\varrho) \right\}
\]

\[
= \sup_{\tau \in S(\mathcal{H})} \sup_{\alpha > 1} \left\{ D(\tau\|\sigma) - \frac{\alpha}{\alpha - 1} D(\tau\|\varrho) \right\}
\]

\[
= \sup_{\tau \in S(\mathcal{H})} \{ \text{Tr} \tau (\log_{0} \varrho - \log_{0} \sigma) \}
\]

\[
= \inf \left\{ \kappa : \log_{0} \varrho - \varrho^{0} (\log_{0} \sigma) \varrho^{0} \leq \kappa \varrho^{0} \right\}
\]

\[
= \inf \left\{ \kappa : \log_{0} \varrho \leq \varrho^{0} (\log_{0} (e^{\kappa} \sigma)) \varrho^{0} \right\}
\]

Note that (58) is an extension of (35) to \(\alpha = +\infty\).
B. Convexity and monotonicity

It is easy to see that when $\varrho$ and $\sigma$ commute, all the quantum Rényi divergences $D_{\alpha}^{(t)}$ with $(t) = \{ \}$, $(t) = *$ and $(t) = b$ coincide, and are equal to the classical Rényi divergence of the eigenvalues of $\varrho$ and $\sigma$. The properties of the classical Rényi divergences (monotonicity under stochastic maps, joint convexity, etc.) are very well understood, and are fairly easy to prove. Corresponding properties of the various quantum generalizations are typically much harder to verify and need not hold for every value of $t$ and $\alpha$. The cases $(t) = \{ \}$ and $(t) = *$ are by now quite well understood, too; see, e.g., [27, 40, 53] for $(t) = \{ \}$ and the recent papers [8, 18, 26, 41–43, 66] for $(t) = *$. Hence, we will focus on the so far less studied $D_{\alpha}^{(t)}$ below, and prove most of the claims only for this version, but state the various properties for all three values of $t$ for completeness and for comparison.

We say that $D_{\alpha}^{(t)}$ is monotone for a fixed $\alpha$, if for all finite-dimensional Hilbert spaces $H, K$, every $\varrho, \sigma \in \mathcal{L}(H)_+$ and every linear completely positive trace-preserving map $\Phi : \mathcal{L}(H) \to \mathcal{L}(K)$, we have $D_{\alpha}^{(t)}(\Phi(\varrho)\|\Phi(\sigma)) \leq D_{\alpha}^{(t)}(\varrho\|\sigma)$. Similarly, we say that $s(\alpha)Q_{\alpha}^{(t)}$ is jointly convex, if for every finite-dimensional Hilbert space $H$, the map $(\varrho, \sigma) \mapsto s(\alpha)Q_{\alpha}^{(t)}(\varrho\|\sigma)$ is convex on $\mathcal{L}(H)_+ \times \mathcal{L}(H)_+$, where $s(\alpha)$ is given by (32). By a standard argument, for any $\alpha \in (0, +\infty) \setminus \{1\}$ and any value of $(t)$, monotonicity of $D_{\alpha}^{(t)}$ is equivalent to the joint convexity of $s(\alpha)Q_{\alpha}^{(t)}$. We have the following:

**Theorem III.12** The maximal interval of $\alpha$ for which $s(\alpha)Q_{\alpha}^{(t)}$ is jointly convex is

\[ [0, 2] \text{ for } (t) = \{ \}, \quad [1/2, +\infty) \text{ for } (t) = *, \quad \text{and } \quad [0, 1] \text{ for } (t) = b. \]

These are also the maximal intervals in $\mathbb{R}$ for which $D_{\alpha}^{(t)}$ is monotone.

The case $(t) = \{ \}$ was proved in [2, 37]; see also [53]. The case $(t) = *$ was proved in [18]; see also [43, 66] ($\alpha \in (1, 2)$) and [8, 41] ($\alpha > 1$). Either of these cases yield the monotonicity of the relative entropy under CPTP maps ($\alpha = 1$), which is again equivalent to its joint convexity. Joint convexity for $(t) = b$ and $\alpha \in (0, 1)$ follows immediately from (33) and the joint convexity of the relative entropy. An alternative proof can be obtained from (i) of [29, Theorem 1.1] by taking $A = \varrho, B = \sigma, \Phi = \Psi = \text{id}$, $p = \alpha/z, q = (1 - \alpha)/z$, taking the limit $z \to +\infty$, and using (27) from [7]. Failure of joint convexity for $(t) = \{ \}$ and $\alpha > 2$, and $(t) = *$ and $\alpha < 1/2$, was pointed out in [43]; see also [41, Appendix A] for the former. We are only left to prove the failure of joint convexity of $Q_{\alpha}^{(t)}$ (monotonicity of $D_{\alpha}^{(t)}$) for $\alpha > 1$:

**Lemma III.13** $Q_{\alpha}^{(t)}$ is not monotone under CPTP maps for any $\alpha > 1$. In fact, it is not even monotone under pinching by the reference operator; that is, for every $\alpha > 1$, there exist $\varrho, \sigma \in \mathcal{L}(H)_+$ such that

\[ Q_{\alpha}^{(t)}(\varrho\|\sigma) < Q_{\alpha}^{(t)}(\mathcal{E}_{\sigma}\varrho\|\sigma). \]

As a consequence, $Q_{\alpha}^{(t)}$ is not jointly convex for $\alpha > 1$.

**Proof** Let $\varrho := \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\sigma := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $a, b > 0$. A straightforward computation shows that

\[ Q_{\alpha}^{(t)}(\varrho\|\sigma) = (ab)^{1-\alpha/2}. \]

If we take $a \neq b$ then $\mathcal{E}_{\varrho}\sigma = \frac{1}{2} I$, and

\[ Q_{\alpha}^{(t)}(\mathcal{E}_{\varrho}\varrho\|\sigma) = \frac{a^{1-\alpha} + b^{1-\alpha}}{2^\alpha} \]

for every $\alpha > 1$. Thus, our aim is to find $a$ and $b$ such that

\[ \sqrt{a^{1-\alpha}b^{1-\alpha}} < \frac{a^{1-\alpha} + b^{1-\alpha}}{2^\alpha}, \quad \text{or equivalently,} \quad c \leq \frac{1 + c^2}{2^\alpha}, \quad \text{where} \quad c := \sqrt{b/a}^{1-\alpha}. \]

It is easy to see that this latter inequality has positive solutions, providing examples such that $Q_{\alpha}^{(t)}(\varrho\|\sigma) < Q_{\alpha}^{(t)}(\mathcal{E}_{\sigma}\varrho\|\sigma)$. \qed
Since the log is concave and increasing, $s(\alpha)\psi_\alpha^{(t)}$ is jointly convex for any $\alpha \in [0,1)$ for which $s(\alpha)Q_\alpha^{(t)}$ is jointly convex. On the other hand, it is well-known and easy to verify that $s(\alpha)\psi_\alpha^{(t)}$ is not jointly convex for $\alpha > 1$ even for classical probability distributions. However, we have the following partial convexity properties:

**Proposition III.14** For every $\varrho \in \mathcal{L} (\mathcal{H})_+$, the functions

$$
\sigma \mapsto s(\alpha)Q_\alpha^{(t)}(\varrho \| \sigma), \quad \sigma \mapsto s(\alpha)\psi_\alpha^{(t)}(\varrho \| \sigma), \quad \sigma \mapsto D_\alpha^{(t)}(\varrho \| \sigma)
$$

are convex on $\mathcal{L} (\mathcal{H})_+$ for $(t) = b$ and $\alpha \in [0,\infty)$, for $(t) = *$ and $\alpha \in [1/2,\infty)$, and for $(t) = \{\}$ and $\alpha \in [0,2]$. Moreover, $\sigma \mapsto D_\alpha^{(t)}(\varrho \| \sigma)$ and $\sigma \mapsto D_\alpha^{(t)}(\varrho \| \sigma)$ are also convex.

**Proof** When $\alpha < 1$, the assertions follow immediately from Theorem III.12. The assertions about $Q_1^{(t)}$ and $\psi_1^{(t)}$ are obvious from their definitions, and the assertion about $D_1^{(t)}$ follows from the joint convexity of the relative entropy. Hence for the rest we assume that $\alpha > 1$. Note that convexity of $\psi_\alpha^{(t)}(\varrho \| .)$ is equivalent to the log-convexity of $Q_\alpha^{(t)}(\varrho \| .)$, which is stronger than convexity. Moreover, since $\psi_\alpha^{(t)}(\varrho \| .)$ and $D_\alpha^{(t)}(\varrho \| .)$ only differ in a positive constant multiplier, it is enough to show the convexity of $D_\alpha^{(t)}(\varrho \| .)$.

Thus, we have to show that for any $\varrho, \sigma_1, \sigma_2 \in \mathcal{L} (\mathcal{H})_+$ and any $\lambda \in (0,1)$,

$$
D_\alpha^{(t)}(\varrho \| (1 - \lambda)\sigma_1 + \lambda \sigma_2) \leq (1 - \lambda)D_\alpha^{(t)}(\varrho \| \sigma_1) + \lambda D_\alpha^{(t)}(\varrho \| \sigma_2).
$$

By (61), we may and will assume that $\varrho, \sigma_1, \sigma_2$ are all invertible.

We first consider the case $(t) = b$. By Theorem III.5, we have

$$
D_\alpha^b(\varrho \| (1 - \lambda)\sigma_1 + \lambda \sigma_2)
\leq \sup_{\tau \in \mathcal{S} (\mathcal{H})} \left\{ D(\tau \| (1 - \lambda)\sigma_1 + \lambda \sigma_2) - \frac{\alpha}{\alpha - 1}D(\tau \| \varrho) \right\}
\leq \sup_{\tau \in \mathcal{S} (\mathcal{H})} \left\{ (1 - \lambda)D(\tau \| \sigma_1) + \lambda D(\tau \| \sigma_2) - \frac{\alpha}{\alpha - 1}D(\tau \| \varrho) \right\}
\leq \sup_{\tau_1, \tau_2 \in \mathcal{S} (\mathcal{H})} \left\{ (1 - \lambda) \left( D(\tau_1 \| \sigma_1) - \frac{\alpha}{\alpha - 1}D(\tau_1 \| \varrho) \right) + \lambda \left( D(\tau_2 \| \sigma_2) - \frac{\alpha}{\alpha - 1}D(\tau_2 \| \varrho) \right) \right\}
\leq (1 - \lambda)D_\alpha^b(\varrho \| \sigma_1) + \lambda D_\alpha^b(\varrho \| \sigma_2),
$$

where the first inequality is due to the convexity of the relative entropy in its second argument, the second inequality is obvious, and the last line is again due to Theorem III.5. Convexity of $\sigma \mapsto D_\alpha^\infty(\varrho \| \sigma)$ follows by taking the limit $\alpha \to +\infty$.

Next, we consider $(t) = *$. By [13, Lemma 12], we have

$$
D_\alpha^*(\varrho \| \sigma) = \frac{\alpha}{\alpha - 1} \sup_{\tau \in \mathcal{L} (\mathcal{H})_+, \text{Tr} \tau \leq 1} \log \omega_\tau(f_\alpha(\sigma)),
$$

where $f_\alpha(x) = x^{(1 - \alpha)/\alpha}$, $x \in (0, +\infty)$, and $\omega_\tau(X) := \text{Tr}(X \varrho^{1/2} \tau^{(\alpha - 1)/\alpha} \varrho^{1/2})$, $X \in \mathcal{L} (\mathcal{H})$. Obviously, $\omega_\tau$ is a positive linear functional, and for $\alpha > 1$, $f_\alpha$ is operator monotone decreasing [9]. Hence, by Lemma II.2, $\sigma \mapsto D_\alpha^*(\varrho \| \sigma)$ is the supremum of convex functions, and hence is itself convex.

The case $(t) = \{\}$ follows by a similar argument; see [10, Theorem II.1] for details. \qed

### C. Further properties and relations

Here we establish some further properties of the Rényi divergences that are going to be useful later in the paper. The following Lemma is easy to verify:

**Lemma III.15** Let $\varrho, \sigma \in \mathcal{L} (\mathcal{H})_+$, $\alpha \in (0, +\infty) \setminus \{1\}$, and $t$ be any of the three possible values.

1. The $Q$ quantities are multiplicative, and hence the corresponding Rényi divergences are additive in the sense that for every $n \in \mathbb{N}$,

$$
Q_\alpha^{(t)}(\varrho^\otimes n \| \sigma^\otimes n) = Q_\alpha^{(t)}(\varrho \| \sigma)^n,
\quad D_\alpha^{(t)}(\varrho^\otimes n \| \sigma^\otimes n) = nD_\alpha^{(t)}(\varrho \| \sigma).
$$

(61)
Proposition III.18 For any $\varrho, \sigma \in \mathcal{L}(\mathcal{H})_+$, we have
\[
D^\star_\alpha(\varrho||\sigma) \leq D_\alpha(\varrho||\sigma) \leq D^\star_\alpha(\sigma||\varrho), \quad \alpha \in [0, 1),
\]
\[
D^\star_\alpha(\sigma||\varrho) \leq D^\star_\alpha(\varrho||\sigma) \leq D_\alpha(\varrho||\sigma), \quad \alpha \in (1, +\infty].
\]

Proof It is enough to prove the inequalities for positive definite $\varrho$ and $\sigma$, as the general case then follows by \[15\]. The inequality $D^\star_\alpha(\varrho||\sigma) \leq D_\alpha(\varrho||\sigma)$ is equivalent to the Araki-Lieb-Thirring inequality \[4, 36\]. By the Golden-Thompson inequality \[19, 62, 63\], $\text{Tr} e^{A+B} \leq \text{Tr} e^{A} e^{B}$ for any self-adjoint $A, B$. This yields that $D_\alpha(\varrho||\sigma) \leq D^\star_\alpha(\varrho||\sigma)$ for $\alpha \in (0, 1)$, and $D_\alpha(\varrho||\sigma) \geq D^\star_\alpha(\varrho||\sigma)$ for $\alpha > 1$. (Vice versa, the inequality $s(\alpha)D_\alpha(\varrho||\sigma) \geq s(\alpha)D^\star_\alpha(\varrho||\sigma)$ for a fixed $\alpha \in (0, +\infty) \setminus \{1\}$ and every $\varrho, \sigma \in \mathcal{L}(\mathcal{H})_+$ implies the Golden-Thompson inequality.) Hence, we are left to prove the first inequality in \[63\].

Let us fix $\varrho, \sigma \in \mathcal{L}(\mathcal{H})_+$ and $\alpha \in (1, +\infty)$, and for every $n \in \mathbb{N}$, let $\varrho_n := \varrho^{\otimes n}, \sigma_n := \sigma^{\otimes n}$. Then
\[
Q^\alpha_n(\varrho||\sigma)^n = Q^\alpha_n(\varrho^{\otimes n}||\sigma^{\otimes n}) = \text{Tr} e^{\alpha \log \varrho_n + (1-\alpha) \log \sigma_n} \\
\leq \text{Tr} e^{\alpha \log \varrho_n + \alpha \log \sigma_n} + (1-\alpha) \log \sigma_n \\
= (\varrho_n)^\alpha (\sigma_n)^{1-\alpha} = (\varrho_n)^\alpha Q^\alpha_n(\varrho_n||\sigma_n) \\
\leq (\varrho_n)^\alpha Q^\alpha_n(\varrho_n||\sigma_n) = (\varrho_n)^\alpha Q^\alpha_n(\varrho||\sigma)^n,
\]
where the first and the last identities are due to \[61\], and the first inequality is due to the pinching inequality \[53\], $\varrho_n \leq (\varrho_n)^{\alpha} \varrho_n$, the operator monotonicity of the logarithm, and Lemma \[11\]. The equalities in the third line are due to the fact that $\varrho_n$ and $\varrho_n$ commute, and the last inequality is due to the monotonicity of $D^\star_\alpha$ under pinching \[13, Proposition 14\]. Taking now the $n$-th root and then the limit $n \to +\infty$, and using that $v(\sigma_n) \leq (n+1)^{d-1}$, we get the desired inequality. \hfill \qed

Remark III.17 It is known that equality in the inequality $D^\star_\alpha(\varrho||\sigma) \leq D_\alpha(\varrho||\sigma)$ holds if and only if $\alpha = 1$ or $\varrho$ and $\sigma$ commute with each other \[24\]. It is also easy to see that the other inequalities don’t hold with equality in general, either. Indeed, choosing $\varrho := \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\sigma := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, a, b > 0$, a straightforward computation shows that
\[
Q^\alpha_\star(\varrho||\sigma) = (ab)^{\frac{1-\alpha}{2}}, \quad Q^\alpha_\star(\varrho||\sigma) = \left(\frac{a^{1-\alpha} + b^{1-\alpha}}{2}\right)^\alpha, \quad Q_\alpha(\varrho||\sigma) = \frac{a^{1-\alpha} + b^{1-\alpha}}{2},
\]
which are not equal to each other for general $a$ and $b$.

All the Rényi divergences are strictly positive on pairs of states:

Proposition III.18 For every $\alpha > 0$, and all three values of $t$,
\[
D^{(t)}_\alpha(\varrho||\sigma) \geq \log \text{Tr} \varrho - \log \text{Tr} \varrho, \quad \varrho, \sigma \in \mathcal{L}(\mathcal{H})_+,
\]
with equality if and only if $\varrho$ is a constant multiple of $\sigma$, or equivalently,
\[
D^{(t)}_\alpha(\varrho||\sigma) \geq 0, \quad \varrho, \sigma \in \mathcal{S}(\mathcal{H}),
\]
with equality if and only if $\varrho = \sigma$.

Proof The equivalence of \[41\] and \[65\] is immediate from the scaling properties in Lemma \[11, 15\], and hence we only prove \[65\]. Moreover, by the ordering in \[62\] and the monotonicity in \[53\], it is enough to consider $D^\star_\alpha$ and $\alpha \in (0, 1)$. By the monotonicity under pinching \[42\] Proposition 14) and the classical Hélder inequality, we have $Q_\alpha^\star(\varrho||\sigma) \leq Q_\alpha^\star(\varrho||\sigma) \leq (\text{Tr} \mathcal{E}_\alpha(\varrho||\sigma))^{\alpha} (\text{Tr} \sigma)^{1-\alpha} \leq 1$, which yields \[65\]. As a consequence, $\psi_\alpha^\star(\varrho||\sigma) \leq 0$ for every $\alpha \in (0, 1)$. Now if $D^\star_\alpha(\varrho||\sigma) = 0$ for some $\alpha \in (0, 1)$ then $\psi_\alpha^\star(\varrho||\sigma) = 0$. By the convexity of $\psi_\alpha^\star(\varrho||\sigma)$ in $\alpha$, (Lemma \[11\], this is only possible if $\psi_\alpha^\star(\varrho||\sigma) = 0$ for every $\alpha \in (0, 1)$). Hence, $D^\star_\alpha(\varrho||\sigma) = 0$, $\alpha \in (0, 1)$, and taking the limit $\alpha \searrow 1$ yields $D(\varrho||\sigma) = 0$. By \[28\] this implies $\varrho = \sigma$. \hfill \qed
Remark III.19 An alternative proof for the case \((t) = *\) has been given in [8, Theorem 5].

Lemma III.20 For every \(\varrho \in \mathcal{L}(\mathcal{H})_+\), we have

\[
\varrho' \geq \sigma \implies D_{\alpha}^{(t)}(\varrho \| \sigma') \leq D_{\alpha}^{(t)}(\varrho \| \sigma)
\]

for \((t) = \{\}\) and \(\alpha \in (0,1]\), for \((t) = *\) and \(\alpha \in [1/2, +\infty]\), and for \((t) = b\) and \(\alpha \in [0, +\infty]\).

Proof By (61), we can assume without loss of generality that \(\varrho, \sigma, \sigma'\) are invertible. The assertions then follow immediately from the following: For \((t) = \{\}\) from the fact that \(x \mapsto x^\alpha, x \in (0, +\infty), \) is operator monotone increasing for \(\alpha \in (0,1]\); for \((t) = *\) from the fact that \(x \mapsto \frac{x^\alpha}{\alpha}\) is operator monotone increasing for \(\alpha \in (1, +\infty)\), and from (4); and for \((t) = b\) from the fact that the logarithm is operator monotone increasing. When necessary, the cases \(\alpha = 0, 1, +\infty\) can be obtained by taking the appropriate limit in \(\alpha\). \(\square\)

Remark III.21 The case \((t) = *\) has been shown in [13, Proposition 4], in a slightly different way.

Recall the definition of \(s(\alpha)\) from (32).

Lemma III.22 For any of the three possible values of \(t\), define

\[
\mathcal{A} := \mathcal{A}' := (0, +\infty) \setminus \{1\}, \quad \mathcal{A} := (1, +\infty).
\]

Let \(\varrho, \sigma \in \mathcal{L}(\mathcal{H})_+\), and let \(\alpha \in \mathcal{A}^{(t)}\).

1. The function \(\varepsilon \mapsto s(\alpha)Q_{\alpha}^{(t)}(\varrho \| \sigma + \varepsilon I)\) is monotone decreasing in \(\varepsilon \in (0, +\infty)\), and

\[
s(\alpha)Q_{\alpha}^{(t)}(\varrho \| \sigma) = \inf_{\varepsilon > 0} s(\alpha)Q_{\alpha}^{(t)}(\varrho \| \sigma + \varepsilon I) = \sup_{\varepsilon \geq 0} s(\alpha)Q_{\alpha}^{(t)}(\varrho \| \sigma + \varepsilon I).
\]

The same holds for \(s(\alpha)\psi_{\alpha}^{(t)}\) and \(D_{\alpha}^{(t)}\) in place of \(s(\alpha)Q_{\alpha}^{(t)}\), and also for \(D_{\infty}^{(t)}\).

2. If \(\sigma\) is invertible then the function \(\varepsilon \mapsto Q_{\alpha}^{(t)}(\varrho + \varepsilon I \| \sigma)\) is monotone increasing in \(\varepsilon \in (0, +\infty)\), and

\[
Q_{\alpha}^{(t)}(\varrho \| \sigma) = \lim_{\varepsilon \to 0} Q_{\alpha}^{(t)}(\varrho + \varepsilon I \| \sigma) = \inf_{\varepsilon > 0} Q_{\alpha}^{(t)}(\varrho + \varepsilon I \| \sigma).
\]

The same hold for \(\psi_{\alpha}^{(t)}\) and \(s(\alpha)D_{\alpha}^{(t)}\) in place of \(Q_{\alpha}^{(t)}\). Moreover, these relations are valid also when \((t) = b\) and \(\alpha \in (0,1)\).

3. We have

\[
Q_{\alpha}^{(t)}(\varrho \| \sigma) = \lim_{\varepsilon \to 0, \delta \to 0} Q_{\alpha}^{(t)}(\varrho + \delta I \| \sigma + \varepsilon I) = \begin{cases} \inf_{\varepsilon > 0} \inf_{\delta > 0} Q_{\alpha}^{(t)}(\varrho + \delta I \| \sigma + \varepsilon I), & \alpha \in (0,1), \\ \sup_{\varepsilon > 0} \inf_{\delta > 0} Q_{\alpha}^{(t)}(\varrho + \delta I \| \sigma + \varepsilon I), & \alpha > 1. \end{cases}
\]

and same hold for \(\psi_{\alpha}^{(t)}\) and \(s(\alpha)D_{\alpha}^{(t)}\) in place of \(Q_{\alpha}^{(t)}\).

Proof Note that (69) is immediate from (67) and (68), and the claims about the monotonicity are trivial to verify, and hence the second identities in (67) and (68) follow if we can prove the first identities. It is easy to see that for an invertible \(\sigma\),

\[
\lim_{\varepsilon \to 0} Q_{\alpha}^{(t)}(\varrho + \varepsilon I \| \sigma) = \lim_{\varepsilon \to 0} Q_{\alpha}^{(t)}(\varrho + \varepsilon I - \varrho^0 \| \sigma),
\]

and thus (68) follows from (10) and Lemma III.11.

We only prove (67) for \((t) = b\), as the other cases follow by very similar, and slightly simpler arguments. Let \(P_s\) denote the spectral projection of \(\sigma\) corresponding to \(s \in \mathbb{R}\); if \(s\) is not an eigenvalue of \(\sigma\) then \(P_s = 0\). Then

\[
\varrho^{0}(\log_{\alpha}(\sigma + \varepsilon I))\varrho^{0} = \sum_{s > 0} \varrho^{0} P_s \varrho^{0} \log(s + \varepsilon) + \varrho^{0}(I - \sigma^{0})\varrho^{0} \log \varepsilon.
\]
If \( q^0 \leq \sigma^0 \) then \( q^0(I - \sigma^0)q^0 = 0 \), and (67) follows trivially. Assume next that \( q^0 \not\leq \sigma^0 \). Then there exists some \( c > 0 \) such that \( q^0(I - \sigma^0)q^0 \geq cQ \), where \( Q := (q^0(I - \sigma^0)q^0)^0 \neq 0 \), and \( Q \leq q^0 \). Hence, for every \( \varepsilon \in (0, 1) \),

\[
q^0(\log_0(\sigma + \varepsilon I))q^0 \leq \kappa_\varepsilon q^0 + (\log \varepsilon) cQ,
\]

where \( \kappa_\varepsilon := \max\{0, \log(\|\sigma\| + \varepsilon)\} \). Let \( \varrho_{\min} \) denote the smallest non-zero eigenvalue of \( q \). By the above,

\[
\text{Tr} \, q^0 e^{\alpha \log_0 q + (1 - \alpha) q^0(\log_0(\sigma + \varepsilon I))q^0} \geq \text{Tr} \, q^0 e^{\alpha \log_0 \varrho_{\min} q^0 + (1 - \alpha)\kappa_\varepsilon q^0 + (1 - \alpha)(\log \varepsilon) cQ}
\]

\[
= \varrho_{\min}^{\alpha}\kappa_\varepsilon(1 - \alpha) \text{Tr} \left[ e^{(1 - \alpha)Q + q^0 - Q} \right],
\]

and the last quantity goes to \(+\infty = Q^0_\alpha(q\|\sigma)\) as \( \varepsilon \searrow 0 \).

For \( \alpha \in \mathcal{A}^{(t)} \), the assertions about \( \psi^{(t)}_\alpha \) and \( D^{(t)}_\alpha \) follow trivially from (67) and (68). Finally, \( \varepsilon \mapsto D^{(t)}_\infty(q\|\sigma + \varepsilon I) \) is the pointwise limit of monotone functions, and hence is itself monotone, and

\[
\lim_{\varepsilon \searrow 0} D^{(t)}_\infty(q\|\sigma + \varepsilon I) = \sup_{\varepsilon > 0} D^{(t)}_\infty(q\|\sigma + \varepsilon I) = \sup_{\varepsilon > 0, \alpha \in (1, +\infty)} D^{(t)}_\alpha(q\|\sigma + \varepsilon I) = \sup_{\alpha \in (1, +\infty)} \sup_{\varepsilon > 0} D^{(t)}_\alpha(q\|\sigma + \varepsilon I) = \sup_{\alpha \in (1, +\infty)} D^{(t)}_\alpha(q\|\sigma) = D^{(t)}_\infty(q\|\sigma).
\]

\[
\Box
\]

**Corollary III.23** Let \( t \) be any of the three possible values, and let \( \alpha \in \mathcal{A}^{(t)} \), where \( \mathcal{A}^{(t)} \) is given in (66). For every \( \varrho \in \mathcal{L}(\mathcal{H})_+ \), the function

\[
\sigma \mapsto s(\alpha)Q^{(t)}_\alpha(\varrho\|\sigma)
\]

is lower semicontinuous on \( \mathcal{L}(\mathcal{H})_+ \), and for every \( \sigma \in \mathcal{L}(\mathcal{H})_+ \), the function

\[
\varrho \mapsto Q^{(t)}_\alpha(\varrho\|\sigma)
\]

is upper semicontinuous on \( \mathcal{L}(\mathcal{H})_+ \).

The same hold for \( \psi^{(t)}_\alpha \) and \( s(\alpha)D^{(t)}_\alpha \) in place of \( Q^{(t)}_\alpha \). Moreover, \( \sigma \mapsto D^{(t)}_\infty(\varrho\|\sigma) \) is also lower semicontinuous on \( \mathcal{L}(\mathcal{H})_+ \). The assertions about upper semicontinuity are also valid for \( (t) = b \) and \( \alpha \in (0, 1) \).

**Proof** Let \( \alpha \in \mathcal{A}^{(t)} \) be fixed. For every \( \varepsilon > 0 \), \( \sigma \mapsto s(\alpha)Q^{(t)}_\alpha(\varrho\|\sigma + \varepsilon I) \) is continuous. Hence, by Lemma III.22 the function \( \sigma \mapsto s(\alpha)Q^{(t)}_\alpha(\varrho\|\sigma) \) is the supremum of continuous functions, and thus is itself lower semicontinuous. Similarly, if \( \sigma \in \mathcal{L}(\mathcal{H})_+ \), then \( \varrho \mapsto Q^{(t)}_\alpha(\varrho + \varepsilon I\|\sigma) \) is continuous for every \( \varepsilon > 0 \), and hence, by Lemma III.22 the function \( \varrho \mapsto Q^{(t)}_\alpha(\varrho\|\sigma) \) is the infimum of continuous functions and thus upper semicontinuous. The assertions about \( \psi^{(t)}_\alpha \) and \( s(\alpha)D^{(t)}_\alpha \) follow immediately. In particular, \( \sigma \mapsto D^{(t)}_\infty(\varrho\|\sigma) \) is the supremum of lower semicontinuous functions in \( \sigma \), and hence is itself lower semicontinuous.

\[
\Box
\]

**IV. RÉNYI CAPACITIES**

**A. Equivalent definitions**

For a quantum channel \( W : \mathcal{X} \to \mathcal{S}(\mathcal{H}) \) and a finitely supported probability distribution \( P \in \mathcal{P}_f(\mathcal{X}) \), we define the *generalized Holevo quantities*, corresponding to each Rényi \( \alpha \)-divergence \( D^{(t)}_\alpha \), as

\[
\chi^{(t)}_{\alpha,1}(W, P) := \inf_{\sigma \in \mathcal{S}(\mathcal{H})} D^{(t)}_\alpha(W\|P \otimes \sigma), \quad \alpha \in (0, +\infty].
\]

Recall that in \( P \otimes \sigma \) in (70), \( P \) stands for \( \sum_{x \in \mathcal{X}} P(x)|x\rangle\langle x| \), the first marginal of \( W(P) \) (11). A straightforward computation verifies that for all \( \alpha \),

\[
Q^{(t)}_\alpha(W\|P \otimes \sigma) = \sum_{x \in \mathcal{X}} P(x)Q^{(t)}_\alpha(W(x)\|\sigma),
\]
and hence
\[ \chi^{(t)}_{\alpha,1}(W, P) = \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \frac{1}{\alpha - 1} \log \sum_{x \in \mathcal{X}} P(x)Q^{(t)}_{\alpha}(W(x)\|\sigma). \] (71)

We also define the following variant:
\[ \chi^{(t)}_{\alpha,2}(W, P) := \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \sum_{x \in \mathcal{X}} P(x)D^{(t)}_{\alpha}(W(x)\|\sigma), \quad \alpha \in (0, +\infty]. \] (72)

Note that for \( \alpha > 1 \), the infima in (70) and (72) can be replaced with minima, due to Corollary [11.23].

We define the Rényi capacities of a channel \( W \), corresponding to the Rényi \( \alpha \)-divergence \( D^{(t)}_{\alpha} \), as
\[ \chi^{(t)}_{\alpha}(W) := \sup_{P \in \mathcal{P}(\mathcal{X})} \chi^{(t)}_{\alpha,1}(W, P). \] (73)

Note that for \( \alpha = 1 \), (70) is the mutual information in the classical-quantum state \( \mathcal{W}(P) \), called the Holevo quantity, and it can be written in the equivalent forms
\[ \chi(W, P) := \chi_1(W, P) := \inf_{\sigma \in \mathcal{S}(\mathcal{H})} D(\mathcal{W}(P)\|P \otimes \sigma) = D(\mathcal{W}(P)\|P \otimes W(P)) \] (74)

\[ = \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \sum_{x \in \mathcal{X}} P(x)D(W(x)\|\sigma) = \sum_{x \in \mathcal{X}} P(x)D(W(x)\|W(P)). \] (75)

As a consequence, we have the following equivalent forms for the Holevo capacity \( \chi(W) \):
\[ \chi(W) := \sup_{P \in \mathcal{P}(\mathcal{X})} \chi(W, P) = \sup_{P \in \mathcal{P}(\mathcal{X})} \inf_{\sigma \in \mathcal{S}(\mathcal{H})} D(\mathcal{W}(P)\|P \otimes \sigma) \] (76)
\[ = \sup_{P \in \mathcal{P}(\mathcal{X})} \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \sum_{x \in \mathcal{X}} P(x)D(W(x)\|\sigma). \] (77)

Moreover, it has been shown in [52, 57] that
\[ \chi(W) = R(W) := \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \sup_{x \in \mathcal{X}} D(W(x)\|\sigma), \] (78)

where \( R(W) \) is called the divergence radius of \( W \). Although in general the identities in (74)–(76) don’t extend to the Rényi quantities with \( \alpha \neq 1 \), we will show in Proposition [IV.1] below that the Rényi generalizations of (70)–(78) coincide with each other, where the corresponding quantity to (78) is defined as
\[ R^{(t)}_{\alpha}(W) := \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \sup_{x \in \mathcal{X}} D^{(t)}_{\alpha}(W(x)\|\sigma). \]

**Proposition IV.1** We have
\[ R^{(t)}_{\alpha}(W) = \sup_{\varepsilon > 0} \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \sup_{x \in \mathcal{X}} D^{(t)}_{\alpha}(W(x)\|\sigma + \varepsilon I) \] (79)
\[ = \sup_{P \in \mathcal{P}(\mathcal{X})} \chi^{(t)}_{\alpha,2}(W, P), \] (80)
\[ = \sup_{P \in \mathcal{P}(\mathcal{X})} \chi^{(t)}_{\alpha,1}(W, P) \] (81)
\[ = \chi^{(t)}_{\alpha}(W). \] (82)

For \( (t) = \{ \} \) and \( \alpha \in (0, 2] \), for \( (t) = * \) and \( \alpha \in [1/2, +\infty) \), and for \( (t) = \flat \) and \( \alpha \in (1, +\infty) \). Moreover, the expressions in (30) and (31) are also equal to each other for \( (t) = \flat \) and \( \alpha \in (0, 1) \).

**Proof** Let us fix a matching pair \( t \) and \( \alpha \) as in the statement of the Theorem. We assume that \( \alpha \neq 1 \), since that case is already known, as pointed out before. By definition,
\[ R^{(t)}_{\alpha}(W) = \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \sup_{x \in \mathcal{X}} D^{(t)}_{\alpha}(W(x)\|\sigma) = \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \sup_{x \in \mathcal{X}} \sup_{\varepsilon > 0} D^{(t)}_{\alpha}(W(x)\|\sigma + \varepsilon I) \] (83)
\[ = \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \sup_{x \in \mathcal{X}} D^{(t)}_{\alpha}(W(x)\|\sigma + \varepsilon I), \]
where the last expression in (83) follows from Lemma III.22. Note that \( \varepsilon \mapsto \sup_{x \in \mathcal{X}} D_{\alpha}^{(t)}(W(x)\|\sigma + \varepsilon I) \) is monotone decreasing for every \( \sigma \in \mathcal{S}(\mathcal{H}) \), due to Lemma III.22. On the other hand, for every \( \varepsilon > 0 \) and \( x \in \mathcal{X} \), \( \sigma \mapsto D_{\alpha}^{(t)}(W(x)\|\sigma + \varepsilon I) \) is continuous, and hence \( \sigma \mapsto \sup_{x \in \mathcal{X}} D_{\alpha}^{(t)}(W(x)\|\sigma + \varepsilon I) \) is lower semi-continuous on the compact set \( \mathcal{S}(\mathcal{H}) \). Hence, by Lemma III.3

\[
R_{\alpha}^{(t)}(W) = \sup_{\varepsilon > 0} \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \sup_{x \in \mathcal{X}} D_{\alpha}^{(t)}(W(x)\|\sigma + \varepsilon I),
\]

proving (79).

By Lemma III.22

\[
\sup_{P \in \mathcal{P}_f(\mathcal{X})} \chi_{\alpha,1}^{(t)}(W, P) = \frac{1}{\alpha - 1} \log s(\alpha) \sup_{P \in \mathcal{P}_f(\mathcal{X})} \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \sup_{x \in \mathcal{X}} P(x)Q_{\alpha}^{(t)}(W(x)\|\sigma + \varepsilon I),
\]

where \( s(\alpha) \) is given in (32). Note that \( s(\alpha) \sum_{x \in \mathcal{X}} P(x)Q_{\alpha}^{(t)}(W(x)\|\sigma + \varepsilon I) \) is monotone decreasing in \( \varepsilon \) and continuous in \( \sigma \), and hence, by Lemma III.3

\[
\sup_{P \in \mathcal{P}_f(\mathcal{X})} \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \sup_{x \in \mathcal{X}} P(x)Q_{\alpha}^{(t)}(W(x)\|\sigma + \varepsilon I) = \sup_{\varepsilon > 0} \sup_{P \in \mathcal{P}_f(\mathcal{X})} \inf_{\sigma \in \mathcal{S}(\mathcal{H})} s(\alpha) \sum_{x \in \mathcal{X}} P(x)Q_{\alpha}^{(t)}(W(x)\|\sigma + \varepsilon I) = \sup_{\varepsilon > 0} \sup_{P \in \mathcal{P}_f(\mathcal{X})} \inf_{\sigma \in \mathcal{S}(\mathcal{H})} s(\alpha) \sum_{x \in \mathcal{X}} P(x)Q_{\alpha}^{(t)}(W(x)\|\sigma + \varepsilon I),
\]

where the second equality is trivial. For every \( \varepsilon > 0 \), \( s(\alpha) \sum_{x \in \mathcal{X}} P(x)Q_{\alpha}^{(t)}(W(x)\|\sigma + \varepsilon I) \) is convex and continuous in \( \sigma \) due to Proposition III.14, and it is affine (and thus concave) in \( P \). Hence, by Lemma II.6

\[
\sup_{P \in \mathcal{P}_f(\mathcal{X})} \inf_{\sigma \in \mathcal{S}(\mathcal{H})} s(\alpha) \sum_{x \in \mathcal{X}} P(x)Q_{\alpha}^{(t)}(W(x)\|\sigma + \varepsilon I) = \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \sup_{P \in \mathcal{P}_f(\mathcal{X})} s(\alpha) \sum_{x \in \mathcal{X}} P(x)Q_{\alpha}^{(t)}(W(x)\|\sigma + \varepsilon I) = \inf_{\sigma \in \mathcal{S}(\mathcal{H})} s(\alpha) \sum_{x \in \mathcal{X}} P(x)Q_{\alpha}^{(t)}(W(x)\|\sigma + \varepsilon I),
\]

where the second equality is trivial. This proves the equality of (79) and (81), and the equality of (81) and (82) is by definition.

Finally, the expression in (81) can be written as

\[
\sup_{\varepsilon > 0} \sup_{P \in \mathcal{P}_f(\mathcal{X})} \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \sum_{x \in \mathcal{X}} P(x)D_{\alpha}^{(t)}(W(x)\|\sigma + \varepsilon I) = \sup_{P \in \mathcal{P}_f(\mathcal{X})} \sup_{\varepsilon > 0} \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \sum_{x \in \mathcal{X}} P(x)D_{\alpha}^{(t)}(W(x)\|\sigma + \varepsilon I) = \sup_{\varepsilon > 0} \inf_{P \in \mathcal{P}_f(\mathcal{X})} \sum_{x \in \mathcal{X}} P(x)D_{\alpha}^{(t)}(W(x)\|\sigma + \varepsilon I),
\]

where the first expression is due to Lemma III.22. The second expression follows from the continuity of \( D_{\alpha}^{(t)}(W(x)\|\sigma + \varepsilon I) \) in \( \sigma \) and its monotonicity in \( \varepsilon \), due to Lemma II.5. The third expression follows trivially from the second. Using now the convexity of \( D_{\alpha}^{(t)}(W(x)\|\sigma + \varepsilon I) \) in \( \sigma \), due to Proposition III.14 and following the same argument as in the previous paragraph, we see that the last expression above is equal to (79). \( \square \)

**Corollary IV.2** For \( (t) = * \) and \( (t) = \hat{\nu} \),

\[
\chi_{\infty}^{(t)}(W) = \sup_{P \in \mathcal{P}_f(\mathcal{X})} \chi_{\infty,1}^{(t)}(W, P) = \sup_{P \in \mathcal{P}_f(\mathcal{X})} \chi_{\infty,2}^{(t)}(W, P) = \sup_{\alpha \in (1, \infty)} \chi_{\alpha}^{(t)}(W).
\]

(84)
Proof The first equality in \( (84) \) is by definition, and
\[
\sup_{P \in \mathcal{P}_f(X)} \chi_{\infty,1}^{(t)}(W, P) = \sup_{P \in \mathcal{P}_f(X)} \sup_{\alpha \in (1, +\infty)} \chi_{\alpha,1}^{(t)}(W, P) = \sup_{\alpha \in (1, +\infty)} \sup_{P \in \mathcal{P}_f(X)} \chi_{\alpha,1}^{(t)}(W, P) = \sup_{\alpha \in (1, +\infty)} \chi_{\alpha}^{(t)}(W) = \sup_{\alpha \in (1, +\infty)} \chi_{\alpha,2}^{(t)}(W, P) = \sup_{P \in \mathcal{P}_f(X)} \sup_{\alpha \in (1, +\infty)} \chi_{\alpha,2}^{(t)}(W, P) = \sup_{P \in \mathcal{P}_f(X)} \chi_{\infty,2}^{(t)}(W, P),
\]
where we used Proposition \([\text{IV.1}]\). □

Lemma IV.3 For \((t) = \ast\) and \((t) = b\), any \(P \in \mathcal{P}_f(X)\), and \(i = 1, 2, \alpha \mapsto \chi_{\alpha,i}^{(t)}(W, P)\) is monotone increasing on \([1, +\infty)\), and
\[
\chi(W, P) = \lim_{\alpha \searrow 1} \chi_{\alpha,i}^{(t)}(W, P) = \inf_{\alpha > 1} \chi_{\alpha,i}^{(t)}(W, P), \tag{85}
\]
\[
\chi_{\infty,i}^{(t)}(W, P) = \lim_{\alpha \searrow +\infty} \chi_{\alpha,i}^{(t)}(W, P) = \sup_{1 < \alpha < +\infty} \chi_{\alpha,i}^{(t)}(W, P). \tag{86}
\]
Similarly, \(\alpha \mapsto \chi_{\alpha}^{(t)}(W)\) is monotone increasing on \([1, +\infty)\), and
\[
\chi_{\infty}^{(t)}(W) = \lim_{\alpha \searrow +\infty} \chi_{\alpha}^{(t)}(W) = \sup_{1 < \alpha < +\infty} \chi_{\alpha}^{(t)}(W). \tag{87}
\]

Proof The assertions about the monotonicity are obvious from Lemma \([\text{III.11}]\). Let \(f_{\alpha,i}^{(t)}(W, P, \sigma) := D_{\alpha}^{(t)}(\mathbb{P}(P)\|P \otimes \sigma)\), and \(f_{\alpha,2}^{(t)}(W, P, \sigma) := \mathbb{P}(P)D_{\alpha}^{(t)}(W(P)\|\sigma)\). By Lemma \([\text{III.4}]\) and \([\text{IV.4}]\)–\([\text{IV.5}]\),
\[
\lim_{\alpha \searrow 1} \chi_{\alpha,i}^{(t)}(W, P) = \inf_{\alpha > 1} \chi_{\alpha,i}^{(t)}(W, P) = \inf_{\alpha > 1, \sigma \in \mathcal{S}(\mathcal{H})} f_{\alpha,i}^{(t)}(W, P, \sigma) = \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \inf_{\alpha > 1} f_{\alpha,i}^{(t)}(W, P, \sigma) = \inf_{\sigma \in \mathcal{S}(\mathcal{H})} D(\mathbb{P}(P)\|P \otimes \sigma) = \chi(W, P).
\]

By Corollary \([\text{II.23}]\) both \(f_{\alpha,1}\) and \(f_{\alpha,2}\) are lower semicontinuous in \(\sigma\) on \(\mathcal{S}(\mathcal{H})\), and hence, by Lemma \([\text{II.5}]\),
\[
\lim_{\alpha \searrow +\infty} \chi_{\alpha,i}^{(t)}(W, P) = \sup_{\alpha > 1} \chi_{\alpha,i}^{(t)}(W, P) = \sup_{\alpha > 1, \sigma \in \mathcal{S}(\mathcal{H})} f_{\alpha,i}^{(t)}(W, P, \sigma) = \sup_{\sigma \in \mathcal{S}(\mathcal{H})} \sup_{\alpha > 1} f_{\alpha,i}^{(t)}(W, P, \sigma) = \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \left\{ D_{\alpha}^{(t)}(\mathbb{P}(P)\|P \otimes \sigma), \quad \sum_{x \in X} P(x)D_{\alpha}^{(t)}(W(P)\|\sigma), \quad i = 1, \quad i = 2, \right\} = \chi_{\infty,i}^{(t)}(W, P).
\]

The identities in \([87]\) are immediate from Corollary \([\text{IV.2}]\). □

B. Rényi capacities of pinched channels

For the rest of the section, we fix a channel \(W : X \rightarrow \mathcal{S}(\mathcal{H})\). For every \(n \in \mathbb{N}\), let \(\sigma_{u,n}\) be a universal symmetric state on \(\mathcal{H}^\otimes n\) as in Lemma \([\text{II.7}]\). We denote by \(\mathcal{E}_n\) the pinching by \(\sigma_{u,n}\). If we use the construction from Appendix \([3]\) then \(\mathcal{E}_n\) can be explicitly written as
\[
\mathcal{E}_n(X) = \sum_{\lambda \in Y_{n,d}} (I_{U_{\lambda}} \otimes I_{U_{\lambda}})X(I_{U_{\lambda}} \otimes I_{U_{\lambda}}), \quad X \in \mathcal{L}(\mathcal{H}^\otimes n),
\]
where \(d := \dim \mathcal{H}\). By the pinching inequality \([3]\) and Lemma \([\text{II.7}]\),
\[
X \leq v(\sigma_{u,n})\mathcal{E}_n(X) \leq v_{n,d}\mathcal{E}_n(X), \quad X \in \mathcal{L}(\mathcal{H}^\otimes n),
\]
where \( v_{n,d} \leq (n + 1)^{(d+2)(d-1)} \).

For every \( n \in \mathbb{N} \), we define the pinched channel \( \mathcal{E}_n \mathcal{W}^{\otimes n} : \mathcal{X}^n \to \mathcal{S}(\mathcal{H}^{\otimes n}) \) as

\[
(\mathcal{E}_n \mathcal{W}^{\otimes n})(\underline{x}) := \mathcal{E}_n(W^{\otimes n}(\underline{x})), \quad \underline{x} \in \mathcal{X}^n.
\]

We use the shorthand notation \( \mathcal{E}_n \mathcal{W}^{\otimes n} \) for its lifted channel, i.e.,

\[
(\mathcal{E}_n \mathcal{W}^{\otimes n})(\underline{x}) := ((\text{id} \otimes \mathcal{E}_n)\mathcal{W}^{\otimes n})(\underline{x}) = |\underline{x}\rangle \langle \underline{x}| \otimes \mathcal{E}_n(W^{\otimes n}(\underline{x})), \quad \underline{x} \in \mathcal{X}^n.
\]

Our aim in the rest of the section is to relate the \( \chi_{\alpha,1} \)-quantity for the pinched channel \( \mathcal{E}_n \mathcal{W}^{\otimes n} \) to the \( \chi_{\alpha,1}^* \)-quantity of the original channel \( W^{\otimes n} \). We obtain such a relation in Corollary IV.7, which will be a key technical tool to determine the strong converse exponent of \( W \) in Section V.

We will benefit from the following additivity properties:

**Lemma IV.4** For every \( P \in \mathcal{P}(\mathcal{X}) \) and every \( \alpha > 1 \),

\[
\chi_{\alpha,1}(W^{\otimes n}, P^{\otimes n}) = n \chi_{\alpha,1}(W, P), \quad \chi_{\alpha,1}^*(W^{\otimes n}, P^{\otimes n}) = n \chi_{\alpha,1}^*(W, P), \quad n \in \mathbb{N}.
\]

**Proof** In the case of \( \chi_{\alpha,1} \), the unique minimizer state in (70) can be determined explicitly due to the quantum Sibson’s identity, and one can observe that the minimizer for a general \( n \) is the \( n \)-th tensor power of the minimizer for \( n = 1 \); see, e.g., [39, Section 4.4] for details. The additivity of \( \chi_{\alpha,1}^* \) is a special case of [8, Theorem 11].

For every \( \pi \in \mathcal{S}_n \), we denote its natural action on \( \mathcal{X}^n \) by the same symbol \( \pi \), i.e.,

\[
\pi(x_1, \ldots, x_n) := (x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)}), \quad x_1, \ldots, x_n \in \mathcal{X}.
\]

We say that a probability density \( P_n \in \mathcal{P}(\mathcal{X}^n) \) is symmetric if \( P_n \circ \pi = P_n \) for every \( \pi \in \mathcal{S}_n \).

**Lemma IV.5** Let \( P_n \in \mathcal{P}(\mathcal{X}^n) \) be a symmetric probability density on \( \mathcal{X}^n \). Then for any \( \alpha > 1 \) and \( (t) = \circ \) or \( (t) = \ast \),

\[
\chi^{(t)}_{\alpha,1}(\mathcal{E}_n W^{\otimes n}, P_n) = \min_{\sigma_n \in \mathcal{S}_n(\mathcal{H}^{\otimes n})} D^{(t)}_{\alpha}(\mathcal{E}_n W^{\otimes n}(P_n)||P_n \otimes \sigma_n),
\]

\[
\chi^{(t)}_{\alpha,2}(\mathcal{E}_n W^{\otimes n}, P_n) = \min_{\sigma_n \in \mathcal{S}_n(\mathcal{H}^{\otimes n})} \sum_{\underline{x} \in \mathcal{X}^n} P_n(\underline{x}) D^{(t)}_{\alpha}(\mathcal{E}_n W^{\otimes n}(\underline{x})||\sigma_n),
\]

i.e., the minimizations in (70) and (72) can be restricted to symmetric states. Moreover, the minima in (88)–(89) can be replaced with infima over \( \mathcal{S}_n(\mathcal{H}^{\otimes n})_{++} \), the set of invertible symmetric states.

The same hold for \( (t) = \circ \) and \( \alpha \in (1, 2) \).

**Proof** Let us fix \( \alpha > 1 \). We only prove (88) and for \( (t) = \circ \), as the proofs for the other cases follow completely similar lines. Thus, with the shorthand notation

\[
f(\sigma_n) := Q_{\alpha}^n(\mathcal{E}_n W^{\otimes n}(P_n)||P_n \otimes \sigma_n) = \sum_{\underline{x} \in \mathcal{X}^n} P_n(\underline{x}) Q_{\alpha}^n(\mathcal{E}_n W^{\otimes n}(\underline{x})||\sigma_n),
\]

our aim is to show that

\[
\min_{\sigma_n \in \mathcal{S}(\mathcal{H}^{\otimes n})} f(\sigma_n) = \min_{\sigma_n \in \mathcal{S}_n(\mathcal{H}^{\otimes n})} f(\sigma_n),
\]

which follows immediately if we can show that for any \( \sigma_n \in \mathcal{S}(\mathcal{H}^{\otimes n}) \),

\[
f\left(\frac{1}{n!} \sum_{\pi \in \mathcal{S}_n} \pi^\ast_H \sigma_n \pi_H\right) \leq \max_{\pi \in \mathcal{S}_n} f(\pi^\ast_H \sigma_n \pi_H) = f(\sigma_n).
\]

Note that the minima in (90) exist because of the lower semicontinuity established in Corollary III.23 and the inequality in (91) follows from the convexity of \( Q_{\alpha}^n \) in its second argument, Proposition III.14.
Hence, the assertion follows if we can prove the permutation invariance of \( f \), i.e., that \( f(\pi^*_H \sigma_n \pi_H) = f(\sigma_n) \) for any \( \sigma_n \in S(H^{\otimes n}) \) and any \( \pi \in \mathcal{G}_n \).

Let us introduce the shorthand notation \( \varrho^0_{x\pi} := E_n(W^{\otimes n}(x)), \ x \in X^n \). Then
\[
f(\pi^*_H \sigma_n \pi_H + \varepsilon I) = \sum_{x \in X^n} P_n(x) \text{Tr} \varrho^0_{x\pi \pi} \exp \left( \alpha \log_0 \varrho_{x\pi}^0 + (1 - \alpha) \varrho^0_{x} \log(\pi^*_H (\sigma_n + \varepsilon I) \pi_H) \right) \varepsilon_{x\pi}^0.
\]
Note that the spectral projections of \( \sigma_{u,n} \) commute with all \( \pi_H, \pi \in \mathcal{G}_n \), thus
\[
\pi_H \varrho^0_{x\pi} \pi^*_H = \pi_H E_n(W^{\otimes n}(x)) \pi^*_H = E_n(\pi_H W^{\otimes n}(x) \pi_H) = E_n(\pi(x)), \ \pi \in \mathcal{G}_n.
\]
As a consequence,
\[
\varrho^0_{x}(\pi^*_H (\sigma_n + \varepsilon I) \pi_H) \varrho^0_{x} = \pi_H \varrho^0_{x\pi} \pi^*_H = \pi_H \varrho^0_{x\pi \pi} \varrho^0_{x\pi \pi} \pi_H = \log_0(\pi^*_H \sigma_n \pi_H),
\]
and
\[
\log_0 \varrho_{x \pi} = \log(\pi^*_H \varrho_{x \pi \pi} \pi_H) = \pi_H \log_0(\varrho_{x \pi \pi} \pi_H).
\]
Putting it together, we get
\[
f(\pi^*_H \sigma_n \pi_H + \varepsilon I) = \sum_{x \in X^n} P_n(x) \text{Tr} \varrho^0_{x\pi \pi} \exp \left( \alpha \log_0 \varrho_{x\pi}^0 (\pi^*_H \sigma_n \pi_H) + (1 - \alpha) \varrho^0_{x \pi \pi} \log(\pi^*_H (\sigma_n + \varepsilon I) \pi_H) \right) \varepsilon_{x\pi}^0 \pi_H
\]
where in the third line we used that \( P_n \) is symmetric. Taking the limit \( \varepsilon \to 0 \) proves the desired permutation invariance, due to Lemma III.22.

To show the assertion that the minimization can be restricted to invertible states, let \( \sigma_n \) be a state where the minimum in (38) is attained, and for every \( \varepsilon \in (0, 1) \), let \( \sigma_{n,\varepsilon} := (1 - \varepsilon) \sigma_n + \varepsilon (I - \sigma_n^0) / \text{Tr}(I - \sigma_n^0) \). Note that \( \sigma_{n,\varepsilon} \) is symmetric and invertible for every \( \varepsilon \in (0, 1) \). Since \( \sigma_n \) is a minimizer, we have \( \varrho^0_{x \pi \pi} = (E_n W^{\otimes n}(x))^0 \leq \sigma_n^0 \) for every \( x \in X^n \) such that \( P_n(x) > 0 \), and thus
\[
\varrho^0_{x \pi \pi} (\log_0 \sigma_{n,\varepsilon}) \varrho^0_{x \pi \pi} = \varrho^0_{x \pi \pi} (\log_0 (1 - \varepsilon) \sigma_n) \varrho^0_{x \pi \pi} = \varrho^0_{x \pi \pi} (\log_0 \sigma_n) \varrho^0_{x \pi \pi} + \varrho^0_{x \pi \pi} \log(1 - \varepsilon),
\]
which in turn yields
\[
f(\sigma_{n,\varepsilon}) = (1 - \varepsilon)^{1-\alpha} f(\sigma_n) \rightarrow f(\sigma_n).
\]
Hence,
\[
f(\sigma_n) = \inf_{\varepsilon \in (0,1)} f(\sigma_{n,\varepsilon}) \geq \inf_{\sigma_n \in S_{\text{sym}}(H^{\otimes n})_{++}} f(\sigma_n),
\]
and the converse inequality is obvious. \( \square \)

**Lemma IV.6** For every \( P \in \mathcal{P}(X) \), every \( \alpha > 1 \), and \( i = 1, 2 \),
\[
\chi^b_{\alpha,i}(E_n W^{\otimes n}, P^{\otimes n}) \geq \begin{cases} 
\chi_{\alpha,i}(E_n W^{\otimes n}, P^{\otimes n}) - \log v_{n,d}, \\
\chi_{\alpha,i}(W^{\otimes n}, P^{\otimes n}) - 3 \log v_{n,d}.
\end{cases}
\]

**Proof** We prove the assertion only for \( i = 2 \), since the other case is completely similar. Let \( \sigma_n \in S_{\text{sym}}(H^{\otimes n})_{++} \) be an invertible symmetric state. Since \( \sigma_{u,n} \) is a universal symmetric state, we have \( \sigma_n \leq v_{n,d} \sigma_{u,n} \), and thus for every \( x \in X^n \),
\[
D^b_{\alpha}(E_n W^{\otimes n}(x) \| \sigma_n) \geq D^b_{\alpha}(E_n W^{\otimes n}(x) \| \sigma_{u,n}) - \log v_{n,d} = D_{\alpha}(E_n W^{\otimes n}(x) \| \sigma_{u,n}) - \log v_{n,d},
\]
where the inequality is due to Lemma [11, 20] and Lemma [13, 15] and the equality is due to the fact that 
\( \mathcal{E}_nW^{\otimes n}(x) \) and \( \sigma_{u,n} \) commute with each other. Hence,

\[
\chi_{\alpha,2}(\mathcal{E}_nW^{\otimes n}, P^{\otimes n}) = \inf_{\sigma_n \in \mathcal{S}(\mathcal{H}^{\otimes n})_{++}} \sum_{x \in \mathcal{X}^n} P^{\otimes n}(x) D_{\alpha}(\mathcal{E}_nW^{\otimes n}(x) \| \sigma_n) \\
\geq \sum_{x \in \mathcal{X}^n} P^{\otimes n}(x) D_{\alpha}(\mathcal{E}_nW^{\otimes n}(x) \| \sigma_{u,n}) - \log v_{n,d} \\
\geq \chi_{\alpha,2}(\mathcal{E}_nW^{\otimes n}, P^{\otimes n}) - \log v_{n,d},
\]

where the first inequality is due to Lemma [14, 15], the first inequality is due to (93), and the last inequality is due to the definition (72). This proves the first bound in (92).

By [14, Lemma 2], we have

\[
D_{\alpha}(\mathcal{E}_nW^{\otimes n}(x) \| \sigma_{u,n}) \geq D_{\alpha}(W^{\otimes n}(x) \| \sigma_{u,n}) - 2 \log v_{n,d}.
\]

Plugging it into (92), we get

\[
\chi_{\alpha,2}(\mathcal{E}_nW^{\otimes n}, P^{\otimes n}) \geq \sum_{x \in \mathcal{X}^n} P^{\otimes n}(x) D_{\alpha}(W^{\otimes n}(x) \| \sigma_{u,n}) - 3 \log v_{n,d} \geq \chi_{\alpha,2}(W^{\otimes n}, P^{\otimes n}) - 3 \log v_{n,d},
\]

proving the second bound in (92).

\[\square\]

**Corollary IV.7** For every \( P \in \mathcal{P}_f(\mathcal{X}) \) and every \( \alpha > 1 \),

\[
\lim_{n \to +\infty} \frac{1}{n} \chi_{\alpha,1}(\mathcal{E}_nW^{\otimes n}, P^{\otimes n}) = \lim_{n \to +\infty} \frac{1}{n} \chi_{\alpha,1}(\mathcal{E}_nW^{\otimes n}, P^{\otimes n}) = \chi_{\alpha,1}(W, P).
\]

\[\text{Proof}\] We have

\[
\chi_{\alpha,1}(W^{\otimes n}, P^{\otimes n}) - 3 \log v_{n,d} \leq \chi_{\alpha,1}(\mathcal{E}_nW^{\otimes n}, P^{\otimes n}) \leq \chi_{\alpha,1}(\mathcal{E}_nW^{\otimes n}, P^{\otimes n}) \leq \chi_{\alpha,1}(W^{\otimes n}, P^{\otimes n}),
\]

where the first inequality is due to Lemma [14, 6] the second one is due to Proposition [11, 16] and the last one follows from the monotonicity of \( D_{\alpha}^* \) under pinching [14, Proposition 14]. By Lemma [14, 4], \( \chi_{\alpha,1}(W^{\otimes n}, P^{\otimes n}) = n \chi_{\alpha,1}^*(W, P) \). Thus, dividing the above chain of inequalities by \( n \), taking the limit \( n \to +\infty \), and using [10], we obtain

\[
\lim_{n \to +\infty} \frac{1}{n} \chi_{\alpha,1}(\mathcal{E}_nW^{\otimes n}, P^{\otimes n}) = \chi_{\alpha,1}^*(W, P).
\]

Next, we use

\[
\chi_{\alpha,1}(\mathcal{E}_nW^{\otimes n}, P^{\otimes n}) \leq \chi_{\alpha,1}(\mathcal{E}_nW^{\otimes n}, P^{\otimes n}) \leq \chi_{\alpha,1}(\mathcal{E}_nW^{\otimes n}, P^{\otimes n}) + \log v_{n,d},
\]

where the first inequality is due to Proposition [11, 16] and the second one is due to Lemma [14, 6]. Combining with (96), we get

\[
\lim_{n \to +\infty} \frac{1}{n} \chi_{\alpha,1}(\mathcal{E}_nW^{\otimes n}, P^{\otimes n}) = \chi_{\alpha,1}(W, P).
\]

\[\square\]

**V. THE STRONG CONVERSE EXPONENT FOR CLASSICAL-QUANTUM CHANNELS**

A. Classical-quantum channel coding and the strong converse exponent

Let \( W : \mathcal{X} \to \mathcal{S}(\mathcal{H}) \) be a classical-quantum channel, as described in Section [10]. The encoding and decoding process of message transmission over the \( n \)-fold extension of the channel is described as follows. Each message \( k \in \{1, 2, \ldots, M_n\} \) is encoded to a codeword by an encoder \( \phi_n \):

\[
\phi_n : k \in \{1, 2, \ldots, M_n\} \mapsto \phi_n(k) = x_{k,1}, x_{k,2}, \ldots, x_{k,n} \in \mathcal{X}^n
\]
and is mapped to
\[ W^\otimes n(\phi_n(k)) = W(x_{k,1}) \otimes W(x_{k,2}) \otimes \cdots \otimes W(x_{k,n}) \in S(\mathcal{H}^\otimes n). \]

The set \( \{\phi_n(k)\}_{k=1}^{M_n} \subset \mathcal{X}^n \) is called a codebook, which is agreed upon by the sender and the receiver in advance. The decoding process, called the decoder, is described by a POVM \( D_n = \{ D_n(k) \}_{k=1}^{M_n} \) on \( \mathcal{H}^\otimes n \), where the outcomes \( 1, 2, \ldots, M_n \) indicate decoded messages. The pair \( C_n = (\phi_n, D_n) \) is called a code with cardinality \( |C_n| := M_n \).

When a message \( k \) was sent, the probability of obtaining the outcome \( l \) is given by
\[ P(l|k) = \text{Tr} W^\otimes n(\phi_n(k))D_n(l). \]

The average error probability of the code \( C_n \) is then given by
\[ P_e(W^\otimes n, C_n) = 1 - \frac{1}{M_n} \sum_{k=1}^{M_n} \text{Tr} W^\otimes n(\phi_n(k))D_n(l), \]
which is required to vanish asymptotically for reliable communication. At the same time, the aim of classical-quantum channel coding is to make the transmission rate \( \liminf_{n \to \infty} \frac{1}{n} \log |C_n| \) as large as possible. The channel capacity \( C(W) \) is defined as the supremum of achievable rates with asymptotically vanishing error probabilities, i.e.,
\[ C(W) = \sup \left\{ R \mid \exists \{C_n\}_{n=1}^\infty \text{ such that } \liminf_{n \to \infty} \frac{1}{n} \log |C_n| \geq R \text{ and } \lim_{n \to \infty} P_e(C_n, W^\otimes n) = 0 \right\}. \]

According to the Holevo-Schumacher-Westmoreland theorem \[34, 57\],
\[ C(W) = \chi(W), \tag{97} \]
where \( \chi(W) \) is the Holevo capacity from \[70\].

By the definition of \( C(W) \), \( \tag{97} \) means that for any rate \( R \) below the Holevo capacity, there exists a sequence of codes with rate \( R \) and asymptotically vanishing error probability. Moreover, it is known that the error probability can be made to vanish with an exponential speed \[23\]. On the other hand, the strong converse theorem of classical-quantum channel coding \[44, 67\] tells that for any sequence of codes with a rate above the Holevo capacity, the error probability inevitably goes to 1, with an exponential speed, or equivalently, the success probability
\[ P_s(W^\otimes n, C_n) := 1 - P_e(W^\otimes n, C_n) \]
decays to zero exponentially fast. The optimal achievable exponent of this decay for a given rate \( R \) is called the strong converse exponent \( sc(R, W) \):

**Definition V.1 (strong converse exponent)** The success rate \( r \) is said to be \( R \)-achievable, if there exists a sequence of codes \( \{C_n\}_{n=1}^\infty \) such that
\[ \liminf_{n \to \infty} \frac{1}{n} \log |C_n| \geq R \text{ and } \liminf_{n \to \infty} \frac{1}{n} \log P_s(C_n, W^\otimes n) \geq -r. \tag{98} \]

The strong converse exponent corresponding to the rate \( R \) is the infimum of all \( R \)-achievable rates:
\[ sc(R, W) = \inf \{ r \mid r \text{ is } R \text{-achievable} \}. \]

Alternatively, the strong converse exponent can be expressed as
\[ sc(R, W) = \inf \left\{ \left. -\liminf_{n \to \infty} \frac{1}{n} \log P_s(C_n, W^\otimes n) \mid \liminf_{n \to \infty} \frac{1}{n} \log |C_n| \geq R \right\}, \]
where the infimum is taken over all sequences of codes \( \{C_n\}_{n \in \mathbb{N}} \). Note that we take the infimum here, as our aim is to make the success probability vanish as slow as possible.

Our main result is the following expression for the strong converse exponent, which is an exact analogue of the Arimoto-Dueck-Körner exponent for classical channels.
Theorem V.2 Let $W : \mathcal{X} \rightarrow \mathcal{S}(\mathcal{H})$ be a classical-quantum channel. For any rate $R \geq 0$,

$$sc(R, W) = \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \{ R - \chi_\alpha^*(W) \} =: H_{R,c}^*(W). \tag{99}$$

The proof follows from Lemma V.4 and Theorem V.13 below.

Remark V.3 In binary state discrimination, the strong converse exponent is given by a similar transform of the Rényi divergences, known as the Hoeffding anti-divergence [22, 41]; for two states $\varphi$ and $\sigma$ and a rate $R$ it is defined as

$$H_{R,c}^*(\varphi\|\sigma) := \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \{ R - D_\alpha^*(\varphi\|\sigma) \}.$$\hspace{1cm} (98)

The expression in (98) is a direct analogue of this for capacities, which we can also extend to other Rényi capacities as

$$H_{R,c}^{(t)}(W) := \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \left\{ R - \chi_\alpha^{(t)}(W) \right\}. \tag{100}$$

We call these quantities converse Hoeffding capacities. Theorem V.2 shows that it is the converse Hoeffding capacity corresponding to $D_\alpha^*$ that is operationally relevant for the strong converse of c-q channel coding, (just as in state discrimination), but in our proof in Section IV the quantity corresponding to $D_\alpha^*$ also plays an important role.

B. Lower bound for the strong converse exponent

Applying the method developed in [44, 55, 58] to the new Renyi relative entropies, we have the following lemma.

Lemma V.4 For any classical-quantum channel $W \in C(\mathcal{H}|\mathcal{X})$ and $R \geq 0$, we have

$$sc(R, W) \geq \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \{ R - \chi_\alpha^*(W) \}. \tag{101}$$

Proof Suppose that $r$ is $R$-achievable. Then there exists a sequence of codes $\mathcal{C}_n = (\phi_n, D_n)$, $n \in \mathbb{N}$, such that (88) holds. Let $\sigma \in \mathcal{S}(\mathcal{H})$, and define density operators on $\mathcal{K}_n := \bigoplus_{k=1}^{M_n} \mathcal{H}^{\otimes n}$ by

$$R_n := \frac{1}{M_n} \bigoplus_{k=1}^{M_n} W^{\otimes n}(\phi_n(k)), \quad S_n := \frac{1}{M_n} \bigoplus_{k=1}^{M_n} \sigma^{\otimes n},$$

where $M_n := |\mathcal{C}_n|$. Note that $T_n := \bigoplus_{k=1}^{M_n} D_n(k)$ and $T_n^\perp := I_{\mathcal{K}_n} - T_n$ form a two-valued POVM \{ $T_n, T_n^\perp$ \} on $\bigoplus_{k=1}^{M_n} \mathcal{H}^{\otimes n}$, and we have $\text{Tr} R_n T_n = P_\alpha(W^{\otimes n}, \mathcal{C}_n)$ and $\text{Tr} S_n T_n = \frac{1}{M_n}$. With the notation $\phi_n(k) = x_{k,1}, x_{k,2}, \ldots, x_{k,n}$, the monotonicity of $Q^*_\alpha$ yields that for $\alpha \geq 1$,

$$P_\alpha(W^{\otimes n}, \mathcal{C}_n)^\alpha \frac{1}{M_n^{1-\alpha}} (\text{Tr} R_n T_n)^{1-\alpha} \leq (\text{Tr} R_n T_n)^\alpha (\text{Tr} S_n T_n)^{1-\alpha} + (\text{Tr} R_n T_n^\perp)^\alpha (\text{Tr} S_n T_n^\perp)^{1-\alpha} \leq Q^*_\alpha(\text{Tr} S_n) \leq \frac{1}{M_n} \sum_{k=1}^{M_n} \prod_{i=1}^{n} Q^*_\alpha(W(x_{k,i})\|\sigma) \leq \left( \sup_{x \in \mathcal{X}} Q^*_\alpha(W(x)\|\sigma) \right)^n, \tag{102}$$
where the last equality follows from the multiplicativity of $Q_\alpha^*$. Since this holds for every $\sigma \in \mathcal{S}(\mathcal{H})$, we get
\[
\alpha \log \frac{1}{n} P_s(W^{\otimes n},C_n) + \frac{\alpha - 1}{n} \log M_n \leq \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \sup_{x \in \mathcal{X}} \log Q_\alpha^*(W(x)||\sigma),
\]
or equivalently,
\[
\frac{1}{n} \log P_s(W^{\otimes n},C_n) \leq -\frac{\alpha - 1}{\alpha} \left\{ \frac{1}{n} \log M_n - \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \sup_{x \in \mathcal{X}} D_\alpha^*(W(x)||\sigma) \right\}.
\]
Using Proposition [IV.1] and taking the limsup in $n$, we get
\[
-\frac{\alpha - 1}{\alpha} \{ R - \chi_\alpha^*(W) \} = \limsup_{n \to +\infty} \frac{1}{n} \log P_s(W^{\otimes n},C_n) \geq \liminf_{n \to +\infty} \frac{1}{n} \log P_s(W^{\otimes n},C_n) \geq -r.
\]
Since this is true for every $\alpha > 1$, the assertion follows.

C. Dueck-Körner exponent

In this section we show the following weak converse to Lemma [V.4]

**Theorem V.5** For every $R > 0$,
\[
\mathrm{sc}(R,W) \leq \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \{ R - \chi_\alpha^*(W) \}.
\]
This will follow immediately from Theorems [V.6] and [V.11] and Lemma [V.12]. We give the proof at the end of the section.

Note that for classical channels, the right-hand sides of the bounds in (101) and (103) coincide, and the two bounds together give Theorem [V.2]. For quantum channels, however, they need not be the same. In Section [V.1] we will combine the bound in (103) with block pinching to obtain an upper bound on $\mathrm{sc}(R,W)$ that matches (101).

Given classical-quantum channels $V, W \in C(\mathcal{H}|\mathcal{X})$ and a finitely supported probability distribution $P \in \mathcal{P}_f(\mathcal{X})$, the quantum relative entropy between the outputs of the extended channels, $V(P)$ and $W(P)$, is written as
\[
D(V(P)||W(P)) = \sum_{x \in \mathcal{X}} P(x) D(V(x)||W(x)) =: D(V||W|P),
\]
which is called the conditional quantum relative entropy.

For every $P \in \mathcal{P}_f(\mathcal{X})$ and $R \geq 0$, let
\[
F(P, R, W) := \inf_{V \in C(\mathcal{H}|\mathcal{X})} \left\{ D(V||W|P) + |R - \chi(V, P)|^+ \right\},
\]
where
\[
|x|^+ = \max\{0,x\}, \quad x \in \mathbb{R}.
\]
Note that for $D(V||W|P)$ and $\chi(V, P)$, only the values of $V$ and $W$ on the support of $P$ are relevant, and therefore we can replace $\mathcal{X}$ with $\text{supp}P$ without loss of generality. Moreover, $D(V||W|P) = +\infty$ if there exists an $x \in \text{supp}P$ such that $V(x)^0 \not\in W(x)^0$. Hence, we can restrict the infimum to
\[
C_{W,P} := \{ V \in C(\mathcal{H}) : V(x)^0 \leq W(x)^0, x \in \text{supp}P \}.
\]
Let us introduce the norm $\|F\| := \sum_{x \in \text{supp}P} \|F(x)\|_1$ on $\{ F : \text{supp}P \to \mathcal{L}(\mathcal{H}) \}$. Then $C_{W,P}$ is compact w.r.t. this norm, and it is easy to see that $V \mapsto D(V||W|P) + |R - \chi(V, P)|^+$ is continuous on $C_{W,P}(\mathcal{H}|\mathcal{X})$. Hence, we have
\[
F(P, R, W) = \min_{V \in C_{W,P}} \left\{ D(V||W|P) + |R - \chi(V, P)|^+ \right\}.
\]

The following is a direct analogue of the Dueck-Körner upper bound [10]:
**Theorem V.6** For any rate $R > 0$, and any $P \in \mathcal{P}_f(\mathcal{X})$, 
\[ sc(R, W) \leq F(P, R, W). \]

**Proof** Let 
\[ F_1(P, R, W) := \inf_{V: \chi(V, P) > R} D(V\|W|P), \]
\[ F_2(P, R, W) := \inf_{V: \chi(V, P) \leq R} \{ D(V\|W|P) + R - \chi(V, P) \}. \]

Then it is easy to see that 
\[ F(P, R, W) = \min\{ F_1(P, R, W), F_2(P, R, W) \}, \]
and hence the assertion follows from Lemmas V.9 and V.10 below. \hfill \square

We will need the following two lemmas to prove Lemma V.9. The first one is a key tool in the information spectrum method [10, 46], and is a consequence of the quantum Stein’s lemma [30, 50].

**Lemma V.7** For any states $\varrho, \sigma \in \mathcal{S}(\mathcal{H})$, we have 
\[ \lim_{n \to \infty} \text{Tr}(\varrho^{\otimes n} - e^{na} \sigma^{\otimes n})_+ = \begin{cases} 1, & a < D(\varrho\|\sigma), \\ 0, & a > D(\varrho\|\sigma). \end{cases} \]

The second one is the key observation behind the dummy channel technique:

**Lemma V.8** Let $W \in C(\mathcal{H}|\mathcal{X})$ be a classical-quantum channel. For any code $C_n = (\phi_n, D_n)$, any $a \in \mathbb{R}$, and any classical-quantum channel $V \in C(\mathcal{H}|\mathcal{X})$ (the dummy channel), we have 
\[ P_s(W^{\otimes n}, C_n) \geq e^{-na} \left\{ P_s(V^{\otimes n}, C_n) - \frac{1}{M_n} \sum_{k=1}^{M_n} \text{Tr} \left( V^{\otimes n}(\phi_n(k)) - e^{na} W^{\otimes n}(\phi_n(k)) \right)_+ \right\}. \]

**Proof** Let $C_n = (\phi_n, D_n)$ be a code with $|C_n| = M_n$. According to (1), we have 
\[ \text{Tr} \left( V^{\otimes n}(\phi_n(k)) - e^{na} W^{\otimes n}(\phi_n(k)) \right)_+ \geq \text{Tr} \left( V^{\otimes n}(\phi_n(k)) - e^{na} W^{\otimes n}(\phi_n(k)) \right) D_n(k), \]
and hence 
\[ \text{Tr} W^{\otimes n}(\phi_n(k)) D_n(k) \geq e^{-na} \text{Tr} V^{\otimes n}(\phi_n(k)) D_n(k) - e^{-na} \text{Tr} \left( V^{\otimes n}(\phi_n(k)) - e^{na} W^{\otimes n}(\phi_n(k)) \right)_+ \]
for every $k \in \{1, \ldots, M_n\}$. Summing over $k$ and dividing by $M_n$ yields (107). \square

**Lemma V.9** In the setting of Theorem V.6, we have 
\[ sc(R, W) \leq F_1(P, R, W). \]

**Proof** We show that any rate $r$ satisfying $r > F_1(P, R, W)$ is $R$-achievable. By the definition (105), there exists a classical-quantum channel $V$ such that 
\[ r > D(V\|W|P), \]
\[ R < \chi(V, P). \]

By Lemma V.8, we have 
\[ P_s(W^{\otimes n}, C_n) \geq e^{-nr} \left\{ P_s(V^{\otimes n}, C_n) - \frac{1}{M_n} \sum_{k=1}^{M_n} \text{Tr} \left( V^{\otimes n}(\phi_n(k)) - e^{nr} W^{\otimes n}(\phi_n(k)) \right)_+ \right\} \]
for any code $C_n = (\phi_n, D_n)$. Now we apply the random coding argument and choose codewords $\phi_n(k) \in \mathcal{X}^{n}$, $k = 1, 2, \ldots, M_n = \lfloor e^{nr} \rfloor$, independently and identically according to $P^{\otimes n}$. For the decoder, we choose the Hayashi-Nagaoka decoder [25].
Let $E[\cdot]$ denote the expectation w.r.t. the random coding ensemble. Taking the expectation of both sides of (111) w.r.t. $E$, we get

$$E [P_s(W^{\otimes n}, C_n)] \geq e^{-nr} \left\{ E [P_s(V^{\otimes n}, C_n)] - \sum_{\bar{x} \in \mathcal{X}^n} P^n(\bar{x}) \text{Tr} (V^{\otimes n}(\bar{x}) - e^{nr} W^{\otimes n}(\bar{x}))_+ \right\}$$

$$= e^{-nr} \left\{ E [P_s(V^{\otimes n}, C_n)] - \text{Tr} \left( \mathbb{V}^{\otimes n} - e^{nr} \mathbb{W}^{\otimes n} \right)_+ \right\}.$$

Since $R < \chi(V, P)$, the results of (25) yield

$$\lim_{n \to \infty} E [P_s(V^{\otimes n}, C_n)] = 1,$$

and, since $r > D(V\|W|P) = D(\mathbb{V}(P)\|\mathbb{W}(P))$, Lemma V.7 yields

$$\lim_{n \to +\infty} \text{Tr} \left( \mathbb{V}^{\otimes n} - e^{nr} \mathbb{W}^{\otimes n} \right)_+ = 0.$$

Hence,

$$\liminf_{n \to \infty} \frac{1}{n} \log \max_{C_n} P_s(W^{\otimes n}, C_n) \geq \liminf_{n \to \infty} \frac{1}{n} \log E [P_s(W^{\otimes n}, C_n)]$$

$$\geq -r + \liminf_{n \to \infty} -\frac{1}{n} \log \left\{ E [P_s(V^{\otimes n}, C_n)] - \sum_{\bar{x} \in \mathcal{X}^n} P^n(\bar{x}) \text{Tr} (V^{\otimes n}(\bar{x}) - e^{nr} W^{\otimes n}(\bar{x}))_+ \right\}$$

$$= -r,$$

where the maximum in the first line is taken over all codes with cardinality $|C_n| = [e^{nR}]$. Thus, $r$ is R-achievable.

**Lemma V.10** In the setting of Theorem V.6 we have $sc(R, W) \leq F_2(P, R, W)$.

**Proof** First note that if $P$ is supported on one single point $x_0 \in \mathcal{X}$ then $\chi(V, P) = 0$ for every channel $V$, and hence $F_2(P, R, W) = R$. It is easy to see that the trivial encoding $\phi_n(k) := x_0$, $i \in \{1, \ldots, n\}$, to every message $k \in \{1, \ldots, [e^{nR}]\}$, together with any decoding yields a code $C_n$ with transmission rate $R$ and success rate $R$, showing that $sc(R, W) \leq R = F_2(P, R, W)$. Hence for the rest we will assume that $|\text{supp } P| > 2$.

To prove the assertion, it is enough to show that any rate $r$ satisfying $r > F_2(P, R, W)$ is R-achievable. By the definition (110), there exists a classical-quantum channel $V$ such that

$$r > D(V\|W|P) + R - \chi(V, P),$$

$$R \geq \chi(V, P).$$

Using the assumption that $|\text{supp } P| > 2$, and the continuity of $V \mapsto \chi(V, P)$ and $V \mapsto D(V\|W|P)$, we can assume that $\chi(V, P) > 0$, while (112) and (113) still hold. Let $\delta > 0$ be such that $r > D(V\|W|P) + R - \chi(V, P) + \delta$ and $R_1 := \chi(V, P) - \delta > 0$. Then we have

$$r_1 := r - R + \chi(V, P) - \delta > D(V\|W|P),$$

$$R_1 < \chi(V, P).$$

Since (110) and (113) are satisfied for $r_1$ and $R_1$, we can see that $r_1$ is $R_1$-achievable from Lemma V.9 i.e., there exists a sequence of codes $\Psi_n = (\psi_n, Y_n)$ such that

$$\liminf_{n \to \infty} \frac{1}{n} \log |\Psi_n| \geq -R_1,$$

$$\liminf_{n \to \infty} \frac{1}{n} \log P_s(W^{\otimes n}, \Psi_n) \geq -r_1.$$

Let $N_n = [e^{nR_1}]$ and $M_n = [e^{nR}]$. Since $N_n \leq M_n$ holds, we can expand the code $\Psi_n = (\psi_n, Y_n)$ to construct a code $C_n = (\phi_n, D_n)$ with the rate $R$ by

$$\phi_n(k) := \begin{cases} \psi_n(k) & (1 \leq k \leq N_n) \\ \psi_n(1) & (N_n < k \leq M_n) \end{cases},$$

$$D_n(k) := \begin{cases} Y_n(k) & (1 \leq k \leq N_n) \\ 0 & (N_n < k \leq M_n). \end{cases}$$
Then we have
\[ P_n(W^\otimes n, C_n) = \frac{1}{M_n} \sum_{k=1}^{M_n} \text{Tr} \; W^\otimes n(\phi_n(k)) D_n(k) = \frac{N_n}{M_n} \frac{1}{N_n} \sum_{k=1}^{N_n} \text{Tr} \; W^\otimes n(\psi_n(k)) Y_n(k) = \frac{N_n}{M_n} P_n(W^\otimes n, \Psi_n), \]
and hence,
\[ \liminf_{n \to \infty} \frac{1}{n} \log P_n(W^\otimes n, C_n) \geq \liminf_{n \to \infty} \frac{1}{n} \log P_n(W^\otimes n, \Psi_n) + \liminf_{n \to \infty} \frac{1}{n} \log \frac{N_n}{M_n} \geq -r_1 + R_1 - R \]
\[ = -(r - R + \chi(V, P) - \delta) + \chi(V, P) - \delta - R = -r, \]
proving that \( r \) is \( R \)-achievable. \( \square \)

Our next step is deriving another representation for \( F(P, R, W) \) defined in (104).

**Theorem V.11** Given \( W \in C(C|X) \) and \( R \geq 0 \), for any \( P \in \mathcal{P}(X) \), we have
\[ F(P, R, W) = \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \left\{ R - \chi_{\alpha,2}(W, P) \right\}. \] (114)

**Proof** Since \( \sup_{0 < \delta < 1} \delta(a - b) = |a - b|^+ \) holds for any \( a, b \in \mathbb{R} \), we have
\[ F(P, R, W) = \min_{V \in C(W, P)} \{ D(V || W | P) + |R - \chi(V, P)|^+ \} \]
\[ = \min_{V \in C(W, P)} \sup_{0 < \delta < 1} \{ D(V || W | P) + \delta(R - \chi(V, P)) \}. \] (115)

Note that \( \chi(V, P) = \inf_{\sigma \in S(C|P)} D(V || \sigma | P) \), where \( D(V || \sigma | P) \) denotes, with a slight abuse of notation, the conditional relative entropy of \( V \) with respect to the constant classical-quantum channel \( x \in X \mapsto \sigma \in S(H) \). Thus, we can rewrite (115) as
\[ F(P, R, W) = \min_{V \in C(W, P)} \sup_{0 < \delta < 1} \sup_{\sigma \in S(C|P)} G(P, \delta, \sigma, V), \] (116)
where for every \( \delta \in (0, 1) \), \( V \in C(W, P) \) and \( \sigma \in S(H)_{++} \),
\[ G(P, \delta, \sigma, V) := D(V || W | P) + \delta(R - D(V || \sigma | P)) \]
\[ = \delta R - (1 - \delta) \sum_{x \in X} P(x) H(V(x)) \]
\[ - \sum_{x \in X} P(x) \text{Tr} \; V(x) \log_0 \; W(x) + \delta \sum_{x \in X} P(x) \text{Tr} \; V(x) \log \sigma, \] (118)

In the above, \( H(\varrho) := -\text{Tr} \; \varrho \log \varrho \) stands for the von Neumann entropy of a state \( \varrho \in S(H) \). Moreover,
\[ \sup_{\sigma \in S(H)_{++}} G(P, \delta, \sigma, V) = D(V || W | P) + \delta(R - \chi(V, P)) \]
\[ = \sum_{x \in X} P(x) D(V(x) || W(x)) + \delta R - \delta H \left( \sum_{x \in X} P(x) V(x) \right) + \delta \sum_{x \in X} P(x) H(V(x)). \] (119)

The following properties of \( G \) are easy to verify:
(i) \( G(P, \delta, \sigma, V) \) is concave and continuous with respect to \( \sigma \in S(H)_{++} \),
(ii) \( G(P, \delta, \sigma, V) \) is convex and continuous with respect to \( V \in C(W, P) \).
(iii) \( \sup_{\sigma \in \mathcal{S}(\mathcal{H})_{++}} G(P, \delta, \sigma, V) \) is convex and continuous with respect to \( V \in C_{W,P} \).

(iv) \( \sup_{\sigma \in \mathcal{S}(\mathcal{H})_{++}} G(P, \delta, \sigma, V) \) is affine with respect to \( 0 < \delta < 1 \).

Indeed, the claim (iv) is obvious from (119), and (i) is immediate from (118) due to the operator concavity of the logarithm. Also by (118) and the concavity of the von Neumann entropy, we get (ii). The convexity property in (iii) is obvious from (11), and the continuity is clear from (120).

Applying now Lemma III.5 to each term in (124) with the choice \( \varrho \) and combining it with (123) yields

\[
F(P, R, W) = \sup_{\alpha > 1} \sup_{\sigma \in \mathcal{S}(\mathcal{H})_{++}} \min_{V \in C_{W,P}} G(P, \frac{\alpha - 1}{\alpha}, \sigma, V). \tag{123}
\]

Note that for every \( \sigma \in \mathcal{S}(\mathcal{H})_{++} \),

\[
\min_{V \in C_{W,P}} G(P, \frac{\alpha - 1}{\alpha}, \sigma, V) = \frac{\alpha - 1}{\alpha} R + \min_{V \in C_{W,P}} \left\{ D(V\|W|P) - \frac{\alpha - 1}{\alpha} D(V\|\sigma|P) \right\} \nonumber
\]

\[= \frac{\alpha - 1}{\alpha} R + \sum_{x \in X} P(x) \inf_{V(x) \in \mathcal{S}_{V(x)}(\mathcal{H})} \left\{ D(V(x)\|W(x)) - \frac{\alpha - 1}{\alpha} D(V(x)\|\sigma) \right\}. \tag{124}
\]

Applying now Lemma III.5 to each term in (124) with the choice \( g := W(x) \), we get

\[
\min_{V \in C_{W,P}} G(P, \frac{\alpha - 1}{\alpha}, \sigma, V) = \frac{\alpha - 1}{\alpha} R - \frac{1}{\alpha} \sum_{x \in X} P(x) \psi_{\alpha}^1(W(x)\|\sigma) \tag{125}
\]

\[= \frac{\alpha - 1}{\alpha} \left\{ R - \sum_{x \in X} P(x) D_\alpha^1(W(x)\|\sigma) \right\}, \]

and combining it with (123) yields

\[
F(P, R, W) = \sup_{\alpha > 1} \left\{ R - \inf_{\sigma \in \mathcal{S}(\mathcal{H})_{++}} \sum_{x \in X} P(x) D_\alpha^1(W(x)\|\sigma) \right\} = \sup_{\alpha > 1} \left\{ R - \chi_{\alpha,2}(W, P) \right\}, \tag{126}
\]

as required.

By Theorems V.6 and V.11 we have

\[
sc(R, W) \leq \inf_{P \in \mathcal{P}_f(X)} \sup_{\alpha > 1} \left\{ R - \chi_{\alpha,2}(W, P) \right\}. \tag{126}
\]

To arrive at the expression in Theorem V.2 we need the following minimax-type lemma (for \( i = 2 \):

**Lemma V.12** For every \( R \geq 0 \),

\[
H_{R,c}^i(W) = \sup_{\alpha > 1} \inf_{P \in \mathcal{P}_f(X)} \left\{ R - \chi_{\alpha,i}^c(W, P) \right\} = \inf_{P \in \mathcal{P}_f(X)} \sup_{\alpha > 1} \left\{ R - \chi_{\alpha,i}^c(W, P) \right\}, \tag{127}
\]

where the equalities hold for both \( i = 1 \) and \( i = 2 \).
Proof The first identity is due to the definition (100), so our aim is to show that the infimum and the supremum in (127) can be interchanged. Let us introduce the notation \( \delta = (\alpha - 1)/\alpha, \alpha \in (1, +\infty) \), and for every \( P \in \mathcal{P}_f(\mathcal{X}) \) and \( \sigma \in S(\mathcal{H})_{++} \), let

\[
G_1(P, \delta, \sigma) := \delta R - (1 - \delta) \psi_{\frac{1}{1-\delta}}(\|W(P)\| P \otimes \sigma),
\]

\[
G_2(P, \delta, \sigma) := \delta R - (1 - \delta) \sum_{x \in \mathcal{X}} P(x) \psi_{\frac{1}{1-\delta}}(W(x)\|\sigma),
\]

and

\[
G_i(P, \delta) := \delta R - \delta \chi_{\frac{1}{1-\delta}, i}(W, P) = \sup_{\sigma \in S(\mathcal{H})_{++}} G_i(P, \delta, \sigma).
\]

Then (127) is equivalent to

\[
\sup_{\delta \in (0, 1)} \inf_{P \in \mathcal{P}_f(\mathcal{X})} G_i(P, \delta) = \inf_{P \in \mathcal{P}_f(\mathcal{X})} \sup_{\delta \in (0, 1)} G_i(P, \delta).
\]

Note that

\[
G_i(P, 0) := \lim_{\delta \searrow 0} G_i(P, \delta) = \lim_{\delta \searrow 0} \left\{ \delta R - \delta \chi_{\frac{1}{1-\delta}, i}(W, P) \right\} = 0,
\]

\[
G_i(P, 1) := \lim_{\delta \nearrow 1} G_i(P, \delta) = R - \chi_{\frac{1}{1-\delta}, i}(W, P)
\]

and

\[
\lim_{\delta \searrow 0} \inf_{P \in \mathcal{P}_f(\mathcal{X})} G_i(P, \delta) = \lim_{\delta \searrow 0} \left\{ \delta R - \delta \chi_{\frac{1}{1-\delta}, i}(W) \right\} = 0 = \inf_{P \in \mathcal{P}_f(\mathcal{X})} G_i(P, 0),
\]

\[
\lim_{\delta \nearrow 1} \inf_{P \in \mathcal{P}_f(\mathcal{X})} G_i(P, \delta) = \lim_{\delta \nearrow 1} \left\{ \delta R - \delta \chi_{\frac{1}{1-\delta}, i}(W) \right\} = R - \chi_{\frac{1}{1-\delta}, i}(W) = \inf_{P \in \mathcal{P}_f(\mathcal{X})} G_i(P, 1).
\]

Hence, (131) can be rewritten as

\[
\sup_{\delta \in [0, 1]} \inf_{P \in \mathcal{P}_f(\mathcal{X})} G_i(P, \delta) = \inf_{P \in \mathcal{P}_f(\mathcal{X})} \sup_{\delta \in (0, 1]} G_i(P, \delta).
\]

This will follow from Lemma (116) if we can show that \( G_i(P, \delta) \) is convex and lower semi-continuous in \( P \) and concave and upper semi-continuous in \( \delta \). By the above construction, it is enough to verify these properties for \( \delta \in (0, 1) \). We will only give a proof for the case \( i = 2 \), since this is what we need for our main result, and because the proof for \( i = 1 \) follows very similar lines.

By (129) and (130), \( G_2(P, \delta) \) is the supremum of convex and continuous functions in \( P \), and hence is itself convex and lower semi-continuous.

By (125) and (117), we have

\[
G_2(P, \delta, \sigma) = \min_{V \in C_{W,P}} G(P, \delta, \sigma, V).
\]

Using properties (i) and (ii) in the proof of Theorem (V.11) and Lemma (116), we get

\[
G_2(P, \delta) = \sup_{\sigma \in S(\mathcal{H})_{++}} \min_{V \in C_{W,P}} G(P, \delta, \sigma, V) = \min_{V \in C_{W,P}} \sup_{\sigma \in S(\mathcal{H})_{++}} G(P, \delta, \sigma, V).
\]

Using (119),

\[
G_2(P, \delta) = \min_{V \in C_{W,P}} \left\{ D(V\|W|P) + \delta \{ R - \chi(V, P) \} \right\},
\]

and hence \( G_2(P, \delta) \) is the infimum of affine functions in \( \delta \), and therefore is itself concave and upper semi-continuous. This finishes the proof.

Proof of Theorem (V.5) We have

\[
sc(R, W) \leq \inf_{P \in \mathcal{P}_f(\mathcal{X})} F(P, R, W) = \inf_{P \in \mathcal{P}_f(\mathcal{X})} \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \left\{ R - \chi_{\alpha, 2}(W, P) \right\}
\]

\[
= \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \left\{ R - \chi_{\alpha}(W) \right\}
\]

where the first line is due to (126), and the second line follows by Lemma (V.12).
D. Strong converse exponent: achievability

Here we prove the following converse to Lemma V.4.

**Theorem V.13** Let $W : \mathcal{X} \to \mathcal{S}(\mathcal{H})$ be a classical-quantum channel. For any $R > 0$, we have

$$sc(R,W) \leq \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \{ R - \chi_\alpha^*(W) \}.$$  \hspace{1cm} (137)

**Proof** For every $m \in \mathbb{N}$, define the pinched channel

$$W_m(x) := (\mathcal{E}_m W^{\otimes m})(x) = \mathcal{E}_m(W(x_1) \otimes \ldots \otimes W(x_m)), \quad x = (x_1, \ldots, x_m) \in \mathcal{X}^m,$$

where $\mathcal{E}_m = \mathcal{E}_{\sigma_{a,m}}$ is the pinching by a universal symmetric state $\sigma_{a,m}$. By Theorem V.4 and [127], for every $R > 0$, there exists a sequence of codes $\mathcal{C}_k^{(m)} = (\phi_k^{(m)}, D_k^{(m)})$, $k \in \mathbb{N}$, such that

$$\liminf_{k \to +\infty} \frac{1}{k} \log |\mathcal{C}_k^{(m)}| \geq R m,$$

$$\liminf_{k \to +\infty} \frac{1}{k} \log P_s (W_{m}^{\otimes k} , C_k^{(m)}) \geq - \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \{ R m - \chi_\alpha^*(\mathcal{E}_m W^{\otimes m}) \}. \hspace{1cm} (139)$$

Note that we can assume that the elements of the decoding POVM $D_k^{(m)} = \{ D_k^{(m)}(i) : i = 1, \ldots, |\mathcal{C}_k^{(m)}| \}$ are invariant under $\mathcal{E}_k^{\otimes k}$, i.e.,

$$\mathcal{E}_k^{\otimes k}(D_k^{(m)}(i)) = D_k^{(m)}(i) \quad \forall i \forall k. \hspace{1cm} (140)$$

Indeed,

$$P_s(W_{m}^{\otimes k} , C_k^{(m)}) = \frac{1}{|\mathcal{C}_k^{(m)}|} \sum_{i=1}^{|\mathcal{C}_k^{(m)}|} \text{Tr} W_{m}^{\otimes k}(\phi_k^{(m)}(i))D_k^{(m)}(i)$$

$$= \frac{1}{|\mathcal{C}_k^{(m)}|} \sum_{i=1}^{|\mathcal{C}_k^{(m)}|} \text{Tr} \mathcal{E}_k^{\otimes k} (W_{m}^{\otimes k}(\phi_k^{(m)}(i)) \right) D_k^{(m)}(i)$$

$$= \frac{1}{|\mathcal{C}_k^{(m)}|} \sum_{i=1}^{|\mathcal{C}_k^{(m)}|} \text{Tr} \mathcal{E}_k^{\otimes k} (W_{m}^{\otimes k}(\phi_k^{(m)}(i)) \mathcal{E}_k^{(m)}(D_k^{(m)}(i))$$

$$= \frac{1}{|\mathcal{C}_k^{(m)}|} \sum_{i=1}^{|\mathcal{C}_k^{(m)}|} \text{Tr} W_{m}^{\otimes k}(\phi_k^{(m)}(i))\mathcal{E}_k^{(m)}(D_k^{(m)}(i)),$$

i.e., the success probability doesn’t change if we replace $\{ D_k^{(m)}(i) : i = 1, \ldots, |\mathcal{C}_k^{(m)}| \}$ with $\{ \mathcal{E}_k^{(m)}(D_k^{(m)}(i)) : i = 1, \ldots, |\mathcal{C}_k^{(m)}| \}$.

Now, from the above code $\mathcal{C}_k^{(m)}$, we construct a code $\mathcal{C}_n$ for $W^{\otimes n}$ for every $n \in \mathbb{N}$. For a given $n \in \mathbb{N}$, let $k \in \mathbb{N}$ be such that $knm \leq n < (k + 1)m$ (we assume that $m > 1$). For every $i \in \{ 1, \ldots, |\mathcal{C}_k^{(m)}| \}$, define $\phi_n(i)$ as any continuation of $\phi_k^{(m)}(i)$ in $\mathcal{X}^n$, and define the decoding POVM elements as $D_n(i) := D_k^{(m)}(i) \otimes I^{\otimes (n - mk)}$. Then $|\mathcal{C}_n| = |\mathcal{C}_k^{(m)}|$, and hence, by [135]

$$\liminf_{n \to +\infty} \frac{1}{n} \log |\mathcal{C}_n| = \frac{1}{m} \liminf_{k \to +\infty} \frac{1}{k} \log |\mathcal{C}_k^{(m)}| \geq R. \hspace{1cm} (141)$$
Moreover,

\[ P_s(W^\otimes n, \mathcal{C}_n) = \frac{1}{|\mathcal{C}_n|} \sum_{i=1}^{|\mathcal{C}_n|} \text{Tr} W^\otimes n(\phi_n(i)) \left( D^{(m)}_k(i) \otimes I^{(n-\mu n)} \right) \]

\[ = \frac{1}{|\mathcal{C}_n|} \sum_{i=1}^{|\mathcal{C}_m|} \text{Tr} W^\otimes \mu n(\phi^{(m)}_k(i)) D^{(m)}_k(i) \]

\[ = \frac{1}{|\mathcal{C}_n|} \sum_{i=1}^{|\mathcal{C}_m|} \text{Tr} W^\otimes \mu n(\phi^{(m)}_k(i)) \mathcal{E}^{\otimes k}_m(D^{(m)}_k(i)) \]

\[ = \frac{1}{|\mathcal{C}_m|} \sum_{i=1}^{|\mathcal{C}_m|} \text{Tr} \mathcal{E}^{\otimes k}_m(W^\otimes \mu n(\phi^{(m)}_k(i))) D^{(m)}_k(i) \]

\[ = P_s(W^\otimes \mu n, \mathcal{C}_m), \]

where in the third line we used (140). Hence, by (139),

\[ \liminf_{n \to +\infty} \frac{1}{n} \log P_s(W^\otimes n, \mathcal{C}_n) = \frac{1}{m} \liminf_{k \to +\infty} \frac{1}{k} \log P_s \left( W^\otimes \mu n, \mathcal{C}_m^{(m)} \right) \]

\[ \geq - \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \left\{ R - \frac{1}{m} \chi_\alpha(W_m) \right\}. \quad (142) \]

Now we have

\[ \chi_\alpha^b(W_m) = \sup_{P_{\mathcal{E}m}(\mathcal{X}^m) \subseteq P_{\mu m}(\mathcal{X}^m)} \chi_\alpha^b(\mathcal{E}m W^\otimes n, P_m) \geq \sup_{P \in \mathcal{P}(\mathcal{X})} \chi_\alpha^b(\mathcal{E}m W^\otimes n, P^\otimes m) \]

\[ \geq \sup_{P \in \mathcal{P}(\mathcal{X})} \chi_\alpha^*(W^\otimes n, P^\otimes m) - 3 \log v_{m,d} = m \sup_{P \in \mathcal{P}(\mathcal{X})} \chi_\alpha^*(W, P) - 3 \log v_{m,d} \]

\[ = m \chi_\alpha^*(W) - 3 \log v_{m,d}, \]

where the first equality is by definition, the first inequality is trivial, the second inequality is due to \textbf{Lemma IV.4}, the following identity is due to \textbf{Lemma IV.3}, and the last identity is again by definition. Thus, by (142),

\[ \liminf_{n \to +\infty} \frac{1}{n} \log P_s(W^\otimes n, \mathcal{C}_n) \geq - \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \left\{ R - \chi_\alpha^*(W) \right\} - 3 \frac{1}{m} \log v_{m,d}. \]

Taking the limit \( m \to +\infty \) and using (10), we obtain

\[ \liminf_{n \to +\infty} \frac{1}{n} \log P_s(W^\otimes n, \mathcal{C}_n) \geq - \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \left\{ R - \chi_\alpha^*(W) \right\}. \quad (143) \]

Combining this with (141), the assertion follows. \( \square \)

\textbf{Appendix A: Universal symmetric states}

For every \( n \in \mathbb{N} \), let \( \mu_{\mathcal{H},n} \) be the \( n \)-th tensor power representation of the identical representation of \( \mathcal{H} \) on \( \mathcal{H} \), i.e., \( \mu_{\mathcal{H},n} : A \in \mathcal{H} \mapsto A^\otimes n \), and let \( \mathcal{L}_{\mu_{\mathcal{H},n}}(\mathcal{H}^\otimes n) := \{ A \in \mathcal{L}(\mathcal{H}^\otimes n) : \mu_{\mathcal{H},n}(U)A = A\mu_{\mathcal{H},n}(U) \quad \forall U \in \mathcal{SU}(\mathcal{H}) \} \) be the commutant algebra of the representation. According to the Schur-Weyl duality (see, e.g., [20, Chapter 9]), \( \mathcal{L}_{\text{sym}}(\mathcal{H}^\otimes n) \) and \( \mathcal{L}_{\mu_{\mathcal{H},n}}(\mathcal{H}^\otimes n) \) are each other’s commutants, i.e.,

\[ \mathcal{L}_{\text{sym}}(\mathcal{H}^\otimes n) = \mathcal{L}_{\mu_{\mathcal{H},n}}(\mathcal{H}^\otimes n)', \quad \mathcal{L}_{\mu_{\mathcal{H},n}}(\mathcal{H}^\otimes n) = \mathcal{L}_{\text{sym}}(\mathcal{H}^\otimes n)' = \{ \pi_{\mathcal{H}} | \pi \in \mathfrak{S}_n \}'. \]

(Note that the double commutant is equal to the algebra generated by the given set.) Moreover, \( \mathcal{H}^\otimes n \) decomposes as

\[ \mathcal{H}^\otimes n \cong \bigoplus_{\lambda \in \mathfrak{S}_n} U_\lambda \otimes V_\lambda, \]
where $U_\lambda$ and $V_\lambda$ carry irreducible representations of $\mathfrak{S}_n$ and $\mathrm{SU}(\mathcal{H})$, respectively, and we have

$$\mathcal{L}_{\text{sym}}(\mathcal{H}^\otimes n) = \bigoplus_{\lambda \in Y_{n,d}} \mathcal{L}(U_\lambda) \otimes I_{V_\lambda}, \quad \mathcal{L}_{\mu_{n,d}}(\mathcal{H}^\otimes n) = \bigoplus_{\lambda \in Y_{n,d}} I_{U_\lambda} \otimes \mathcal{L}(V_\lambda).$$

Here, $Y_{n,d}$ is the set of the Young diagrams up to the depth $d := \dim \mathcal{H}$, defined as

$$Y_{n,d} = \left\{ \lambda = (n_1, n_2, \ldots, n_d) \bigg| n_1 \geq n_2 \geq \cdots \geq n_d \geq 0, \sum_{i=1}^n n_i = n \right\}.$$

In particular, every permutation invariant state $\varrho_n \in \mathcal{L}_{\text{sym}}(\mathcal{H}^\otimes n)$ can be written according to the above decomposition as

$$\varrho_n = \bigoplus_{\lambda \in Y_{n,d}} \frac{p_\lambda}{\dim V_\lambda} \cdot \varrho_\lambda \otimes I_{V_\lambda},$$

where $\varrho_\lambda$ is a density operator acting on $U_\lambda$ and $\{p_\lambda\}$ is a probability function on $Y_{n,d}$. Using inequalities $\varrho_\lambda \leq I_{U_\lambda}$ and $p_\lambda \leq 1$, we have

$$\varrho_n \leq \bigoplus_{\lambda \in Y_{n,d}} \frac{1}{\dim V_\lambda} \cdot I_{U_\lambda} \otimes I_{V_\lambda} \leq \max_\lambda \{\dim U_\lambda\} \cdot |Y_{n,d}| \cdot \sigma_{u,n},$$

where

$$\sigma_{u,n} := \bigoplus_{\lambda \in Y_{n,d}} \frac{1}{|Y_{n,d}|} \cdot \frac{I_{U_\lambda}}{\dim U_\lambda} \otimes \frac{I_{V_\lambda}}{\dim V_\lambda}.$$

It is known that

$$\max_\lambda \{\dim U_\lambda\} \leq (n + 1)^{d(d-1)/2}, \quad |Y_{n,d}| \leq (n + 1)^{d-1},$$

and hence $\sigma_{u,n}$ satisfies the criteria in Lemma II.7.

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