Well-posedness of martingale problem for SBM with interacting branching

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Abstract

In this paper a martingale problem for super-Brownian motion with interactive branching is derived. The uniqueness of the solution to the martingale problem is obtained by using the pathwise uniqueness of the solution to a corresponding system of SPDEs with proper boundary conditions. The existence of the solution to the martingale problem and the Hölder continuity of the density process are also studied.

Keywords. super-Brownian motion; interacting branching; function-valued process; stochastic partial differential equation.

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1 Introduction and main results

Let $M(\mathbb{R})$ be the collection of all finite Borel measures on $\mathbb{R}$. Let $C^n_b(\mathbb{R})$ be the collection of all bounded continuous functions on $\mathbb{R}$ with bounded derivatives up to $n$th order. We consider a $M(\mathbb{R})$-valued process $(X_t)_{t \geq 0}$ satisfying the following martingale problem (MP):

for $\phi \in C^n_b(\mathbb{R})$, the process

$$M_t(\phi) \equiv \langle X_t, \phi \rangle - \langle X_0, \phi \rangle - \int_0^t \langle X_s, \frac{1}{2} \phi'' \rangle \, ds$$

is a continuous martingale with

$$\langle M(\phi) \rangle_t = \int_0^t \langle X_s, \gamma(\mu_s, \cdot) \phi^2 \rangle \, ds,$$

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where $\gamma$ is the interacting branching rate depending on the density process. The notation $\langle \nu, \phi \rangle$ denotes the integral of the function $\phi$ with respect to the measure $\nu$. In this paper we assume $X_0(dx) = \mu_0(x)dx$ with $\mu_0 \in L^2(\mathbb{R})$, where $L^2(\mathbb{R})$ is the space of functions $f$ such that $\int_{\mathbb{R}} f(x)^2 dx < \infty$. When $\gamma$ is a constant, the process $(X_t)_{t \geq 0}$ is a super-Brownian motion. In this case the well-posedness of the MP (1.1, 1.2) was established by the nonlinear partial differential equation satisfied by its log-Laplace transform. Moreover, a new approach for the well-posedness of the MP was suggested by Xiong [21], in which a relationship between the super-Brownian motion and a stochastic partial differential equation (SPDE) satisfied by its corresponding distribution-function-valued process was established. The weak uniqueness of the solution to the MP (1.1, 1.2) was also obtained by the strong uniqueness of the solution to the corresponding SPDE in [21]. See He et. al [9] for the case of super-Lévy process.

While the superprocesses with interaction are more natural since the branching and spatial motion for many species depend on the processes themselves. When the spatial motion is interactive, the well-posedness of the martingale problem was studied by Donnelly and Kurtz [4], see also Perkins [15, Theorem V.5.1] and Li et. al [12]. Uniqueness for the historical superprocesses with certain interaction was investigated by Perkins [15]. Furthermore, the superprocesses with interactive immigration were studied by [2, 7, 17], see also Li [11, Section 10]. The well-posedness of the martingale problem for the interactive immigration process was solved by Mytnik and Xiong [14]. See also [22] for the well-posedness of the martingale problem for a superprocess with location-dependent branching, interactive immigration mechanism and spatial motion.

However, the hard case for the superprocess with interactive branching was rarely investigated. We are interested in the case of $\gamma(\mu_s, x) = \gamma(\mu_s(x))$, i.e., the interactive branching mechanism depending on the density process $(\mu_t(x))_{t \geq 0, x \in \mathbb{R}}$. However, in this case the well-posedness of the MP (1.1, 1.2) is still an open problem. The weak uniqueness of the solution to the MP (1.1, 1.2) is very difficult to prove. As a first step, throughout this paper we assume $\gamma$ satisfies the following condition:

**Condition 1.1.** Fixed integer $n \geq 0$. For $-\infty = a_0 < a_1 < \cdots < a_n < a_{n+1} = \infty$, let

$$
\gamma(\mu_s, x) = \begin{cases} 
g_i^2(\mu_s(a_{i+1})), & a_i \leq x < a_{i+1}, \ i = 0, \cdots, n-1, 
g_n^2 \left( \int_{a_n}^{\infty} \mu_s(y) dy \right), & a_n \leq x < \infty,
\end{cases}
$$

where $g_i$, $i = 0, \cdots, n$ are positive continuous bounded functions from $\mathbb{R}_+$ to $\mathbb{R}_+$.

The existence and Hölder continuity of the density process $(\mu_t(x))_{t \geq 0, x \in \mathbb{R}}$ are investigated in this paper, which satisfies the following SPDE:

$$
\frac{\partial}{\partial t} \mu_t(x) = \frac{1}{2} \Delta \mu_t(x) + \sqrt{\mu_t(x) \gamma(\mu_t, x)} \dot{W}(t, x),
$$

(1.3)

where $\Delta$ denotes the one-dimensional Laplacian operator and $\{W(t, x) : t \geq 0, x \in \mathbb{R}\}$ is a time-space Gaussian white noise based on the Lebesgue measure and the dot denotes the derivative in distribution sense. The existence of the solution to the MP (1.1, 1.2) is given by showing the existence of the solution to (1.3). Moreover, we show the weak
uniqueness of the solution to the MP \( (1.1)\) in Theorem \(1.4\). The main idea is to relate the MP with a system of SPDEs, which is satisfied by a sequence of corresponding function-valued processes on intervals. The weak uniqueness of solution to the MP follows from the pathwise uniqueness of the solution to the system of SPDEs, see Section 3. Throughout this paper we always assume that all random variables defined on the same filtered probability space \( (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\). Let \(\mathbb{E}\) denote the corresponding expectation.

The main results of this paper without special explanation are stated as below.

**Theorem 1.2.** (Existence) There exists a \(M(\mathbb{R})\)-valued continuous process \((X_t)_{t \geq 0}\) satisfying the MP \( (1.1)\). Moreover, \(X_t(dx)\) is absolutely continuous with respect to \(dx\) with density \(\mu_t(x)\) satisfying (1.3).

**Theorem 1.3.** (Joint Hölder continuity) Suppose that \((\mu_t(x))_{t \geq 0, x \in \mathbb{R}}\) satisfies (1.3) and \(\mu_0\) is Hölder continuous with exponent \(0 < \lambda < \frac{1}{2}\). Then \([0, T] \times \mathbb{R} \ni (t, x) \mapsto \mu_t(x)\) is Hölder continuous with exponent \(\lambda_1/2\) in time variable and with exponent \(\lambda_2\) in space variable, where \(\lambda_1, \lambda_2 \in (0, 1/2)\). Namely, there exists a random variable \(K \geq 0\) only depending on \(\lambda_1\) and \(\lambda_2\) such that

\[
|\mu_t(x) - \mu_r(y)| \leq K(|t - r|^{\lambda_1/2} + |x - y|^{\lambda_2}), \quad t, r \in [0, T], \ x, y \in \mathbb{R}.
\]

**Theorem 1.4.** (Uniqueness) Assume \(g_n\) is \(\beta\)-Hölder continuous with \(\frac{1}{2} \leq \beta \leq 1\), i.e.,

\[
|g_n(x) - g_n(y)| \leq K|x - y|^\beta, \quad x, y \geq 0
\]

for some constant \(K\). Then the weak uniqueness of the solution to the MP \((1.1)\) holds.

**Remark 1.5.** Taking \(n = 0\) in Condition \((1.7)\) the branching rate depends on the total mass process, i.e., the quadratic variation process of the martingale defined by \((1.1)\) is

\[
\langle M(\phi) \rangle_t = \int_0^t \gamma(\langle X_s, 1 \rangle \langle X_s, \phi^2 \rangle) ds.
\]

The well-posedness of the MP \((1.1)\) is a corollary of Theorem 1.2 and 1.4.

We introduce some notation. Let \(\mathcal{H}_0\) be the Hilbert space consisting of all functions \(f\) such that

\[
\|f\|_0^2 := \int_{\mathbb{R}} f(x)^2 e^{-|x|} dx < \infty.
\]

We denote the corresponding inner product by \(\langle \cdot, \cdot \rangle_0\). Let \(\mathcal{B}(\mathbb{R})\) (resp. \(\mathcal{B}(a_1, a_2)\)) denote the Borel \(\sigma\)-algebra on \(\mathbb{R}\) (resp. \((a_1, a_2)\)). Let \(B(\mathbb{R})^+\) (resp. \(C_b(\mathbb{R})^+\)) be the collection of all bounded positive (resp. bounded positive continuous) functions on \(\mathbb{R}\). Let \(B[a_1, a_2]\) be the Banach space of bounded measurable functions on \([a_1, a_2]\). For \(f, g \in B[a_1, a_2]\) let \(\langle f, g \rangle = \int_{a_1}^{a_2} f(x)g(x)dx\). Define \(C_b[a_1, a_2]\) to be the set of bounded continuous functions on \([a_1, a_2]\). For any integer \(n \geq 0\), let \(C^n_b[a_1, a_2]\) be the subset of \(C_b[a_1, a_2]\) of functions with bounded continuous derivatives up to the \(n\)th order. Let \(C^n_c(a_1, a_2)\) denote the subset of \(C^n_b[a_1, a_2]\) of functions with compact supports in \((a_1, a_2)\).
The rest of the paper is organized as follows. In Section 2, we give the proofs of Theorem 1.2 and 1.3. The weak uniqueness of the solution to the MP (1.1, 1.2), i.e., the proof of Theorem 1.4, is given in Section 3. Throughout the paper we use $\nabla$ to be the first order spatial differential operator, use $K$ to denote a non-negative constant whose value may change from line to line. In the integrals, we make convention that, for $a \leq b \in \mathbb{R}$,

$$
\int_{a}^{b} = \int_{(a,b]} \quad \text{and} \quad \int_{a}^{\infty} = \int_{(a,\infty)}.
$$

## 2 Proofs of Theorem 1.2 and 1.3

In this section, we give the proofs of Theorem 1.2 and 1.3. The existence of the solution to MP (1.1, 1.2) is obtained by the existence of the corresponding density process. Moreover, the Hölder continuity of density process $(\mu_t(x))_{t \geq 0, x \in \mathbb{R}}$ is given by a standard argument.

**Lemma 2.1.** The martingale defined in (1.1,1.2) induces a $(\mathcal{F}_t)$-martingale measure \{\$M_t(B) : t \geq 0, B \in \mathcal{B}(\mathbb{R})\}$ satisfying

$$
M_t(\phi) = \int_{0}^{t} \int_{\mathbb{R}} \phi(x)M(ds, dx), \quad t \geq 0, \ \phi \in C^2_b(\mathbb{R}),
$$

where $M(ds, dx)$ is an orthogonal martingale measure on $\mathbb{R}_+ \times \mathbb{R}$ with covariance measure $ds \int_{\mathbb{R}} [\gamma(X_s, z)\delta_z(dx)\delta_z(dy)]X_s(dz)$.

**Proof.** Notice that $\gamma$ is bounded by Condition 1.1. By the MP (1.1, 1.2), one can see that $E[\langle \mu_t, 1 \rangle] = \langle \mu_0, 1 \rangle < \infty$. For each $n \geq 1$ we define the measure $\Gamma_n \in M(\mathbb{R})$ by

$$
\Gamma_n(\phi) = E\left[ \int_{0}^{n} ds \int_{\mathbb{R}} \gamma(\mu_s, x)\phi(x)X_s(dx) \right]
$$

with $\phi \in B(\mathbb{R})^\dagger$. Then $\Gamma_n(\phi) \leq K \int_{0}^{n} E[\langle X_s, 1 \rangle]ds$ is bounded for each $n \geq 1$. The rest proof follows by changing $c(z) = \gamma(\mu_s, z)$ and $H(z, d\nu) = 0$ in the proof of Li [11, Theorem 7.25]. We omit it here.

Let $T_t(x, dy)$ be the semigroup generated by $\Delta/2$, which is absolutely continuous with respect to Lebesgue measure $dy$ with density $p_t(x, y)$ satisfying

$$
p_t(x, y) = p_t(x - y) = \frac{1}{\sqrt{2\pi t}}e^{-|x-y|^2/(2t)}.
$$

**Lemma 2.2.** Suppose that $(\mu_t(x))_{t \geq 0, x \in \mathbb{R}}$ is a solution to (1.3) with $\mu_0 \in L^2(\mathbb{R})$. Then for every $t \geq 0$, we have $E[\langle \mu_t, 1 \rangle] < \infty$.

**Proof.** By (1.3) we have

$$
\langle \mu_t, 1 \rangle = \langle \mu_0, 1 \rangle + \int_{0}^{t} \int_{\mathbb{R}} \sqrt{\mu_s(x)\gamma(\mu_s, x)}W(ds, dx).
$$
Notice that $\sqrt{x} \leq x + 1$ for any $x \geq 0$. By Hölder’s inequality one can check that

\[
\mathbb{E}[\langle \mu_t, 1 \rangle] \leq \langle \mu_0, 1 \rangle + \mathbb{E} \left| \int_0^t \int_\mathbb{R} \sqrt{\mu_s(x)} \gamma(\mu_s, x) W(ds, dx) \right|
\]

\[
\leq \langle \mu_0, 1 \rangle + K \left[ \mathbb{E} \left( \int_0^t \int_\mathbb{R} \mu_s(x) dsdx \right) \right]^{1/2}
\]

\[
\leq K + K \int_0^t \mathbb{E}[\langle \mu_s, 1 \rangle] ds.
\]

The result follows by Gronwall’s inequality.

Similar with Li [11, Theorem 7.26, Theorem 7.28], we have the following result.

**Proposition 2.3.** Suppose that $(X_t)_{t \geq 0}$ is a solution to the MP (1.1, 2.2). Then for each $t \geq 0$ the random measure $X_t(dx)$ is absolutely continuous respect to $dx$ with density $\mu_t(x)$ satisfying (1.3). Conversely, assume that $(\mu_t(x))_{t \geq 0, x \in \mathbb{R}}$ is a solution to (1.3) with $\mu_0 \in L^2(\mathbb{R})$. Then $(X_t)_{t \geq 0}$ satisfies the MP (1.1, 2.2).

**Proof.** Suppose that $(X_t)_{t \geq 0}$ satisfies the MP (1.1, 2.2). By Lemma 2.1 and the proof of [11, Theorem 7.26], we have

\[
\langle X_t, \phi \rangle = \langle X_0, T_t \phi \rangle + \int_0^t \int_\mathbb{R} T_{t-s} \phi(x) M(ds, dx),
\]

where $M(ds, dx)$ is an orthogonal martingale measure defined in Lemma 2.1. Recall that $\gamma$ is bounded by Condition 1.1. Let $\phi \in C_b(\mathbb{R})$ and set $p_t(x, z) = 0$ for all $t \leq 0$. For any $n \geq 1$ we have

\[
\mathbb{E} \left[ \int_\mathbb{R} \phi(z) dz \int_0^t \int_\mathbb{R} p_{t-s}(x-z) ^2 \gamma(\mu_s, x) X_s(dx) ds \right] \leq K \mathbb{E} \left[ \int_\mathbb{R} \phi(z) dz \int_0^t \int_\mathbb{R} p_{t-s}(x-z) ^2 X_s(dx) ds \right] \leq K \sup_{z \in \mathbb{R}} \phi(z) \int_0^t \frac{\mathbb{E}[\langle X_s, 1 \rangle]}{\sqrt{2\pi(t-s)}} ds = K \sup_{z \in \mathbb{R}} \phi(z) \sqrt{t} < \infty.
\]

Then by stochastic Fubini’s theorem (e.g., see Li [11, Theorem 7.24]), we get $\langle X_t, \phi \rangle = \int_\mathbb{R} \mu_t(x) \phi(x) dx$ with

\[
\mu_t(x) = \int_\mathbb{R} p_t(x-z) \mu_0(z) dz + \int_0^t \int_\mathbb{R} p_{t-s}(x-z) M(ds, dz).
\]

By El Karoui and Méleard [5, Theorem III-6], on some extension of the probability space one can define a white noise $W(ds, dz)$ on $\mathbb{R}_+ \times \mathbb{R}$ based on $dsdz$ such that the following holds:

\[
\mu_t(x) = \langle \mu_0, p_t(x-\cdot) \rangle + \int_0^t \int_\mathbb{R} \sqrt{\mu_s(z)} \gamma(\mu_s, z) p_{t-s}(x-z) W(ds, dz),
\]

which implies (1.3).
Conversely, suppose that $\mu_t(x)$ satisfies \cite{13} with $\mu_0 \in L^2(\mathbb{R})$ and denote $X_t(dx) = \mu_t(x)dx$. It follows from Lemma 2.2 that $X_t \in M(\mathbb{R})$ almost surely for every $t \geq 0$. For any $\phi \in C_0^\infty(\mathbb{R})$ one can check that

$$
\langle X_t, \phi \rangle = \int_{\mathbb{R}} \mu_t(x)\phi(x)dx
= \int_{\mathbb{R}} \mu_0(x)\phi(x)dx + \frac{1}{2} \int_0^t \int_{\mathbb{R}} \mu_s(x)\phi''(x)dxds
+ \int_0^t \int_{\mathbb{R}} \phi(x)\sqrt{\mu_s(x)\gamma(\mu_s, x)}W(ds, dx)
= \langle X_0, \phi \rangle + \frac{1}{2} \int_0^t \langle X_s, \phi'' \rangle ds + M_t(\phi)
$$

with $\langle M(\phi) \rangle_t$ satisfying (1.2), which completes the proof. \hfill \Box

Now we show the existence of the solution to (1.3). For any $\phi$ is a Lipschitz function on $[0, x]$ where $\{\mathcal{F}_t = k\} \mu_{2m}(x)$ of Mitoma \cite{13}. We may and will replace $e_{\phi}$ space $B$ where ($F$) conditioned on $\{\mathcal{F}_t = k\}$ $\mu_{2m}(x)$ one can check that $\langle \mathcal{G}_t, x \rangle = \int_{\mathbb{R}} p_{m-1}(x-y)\sqrt{|y| \land m)dy}$ is a Lipschitz function on $[0, \infty)$ for fixed $m \geq 1$ and $\lim_{m \to \infty} G_m(x) = \sqrt{x}$ for all $x \geq 0$. For all $m \geq 1$ and $x \geq 0$, one can check that there is a constant $K > 0$ such that

$$
G_m(x) \leq \int_{\mathbb{R}} p_{m-1}(x-y)(1+|y|)dy = 1 + \mathbb{E}[|B_{m-1}^x|] \leq 1 + \left[\mathbb{E} (|B_{m-1}^x|)^2 \right]^{1/2}
\leq 1 + (1 + x^2)^{1/2} \leq K(x+1)
$$

(2.3)

where $(B_t^x)_{t \geq 0}$ is a Brownian motion with initial value $x$. For any $k = 1, \ldots, m$, conditioned on $\mathcal{F}_t = k$, it is well known that there is a strong unique non-negative solution $\{\mu_t^m(x) : t \in [t_{k-1}, t_k), x \in \mathbb{R}\}$ to

$$
\mu_t^m(x) = \mu_{t_{k-1}}^m(x) + \frac{1}{2} \int_{t_{k-1}}^t \Delta \mu_s^m(x)ds + \sqrt{\gamma(\mu_s^{m-1}, x)} \int_{t_{k-1}}^t G_m(\mu_s^{m-1}(x))\dot{W}(s, x)ds,
$$

see \cite{11} Theorem 5.1. Therefore, the process $(\mu_t^m(x))_{t \geq 0, x \in \mathbb{R}}$ is the unique strong solution to (2.2).

Let $J(x) = \int_{\mathbb{R}} e^{-|y|/\rho(x)}\rho(x)dy$, where $\rho$ is the mollifier given by $\rho(x) = C \exp\{-1/(1-x^2)\}1_{|x|<1}$, and $C$ is a constant such that $\int_{\mathbb{R}} \rho(x)dx = 1$. Then for any $n \geq 1$, there are constants $c_n$ and $C_n$ such that

$$
c_n e^{-|x|} \leq J^{(n)}(x) \leq C_n e^{-|x|}, \quad \forall \ x \in \mathbb{R},
$$

(2.4)

see (2.1) of Mitoma \cite{13}. We may and will replace $e^{-|x|}$ by $J(x)$ in the definition of the space $X_0$, and define $\langle f, g \rangle_0 = \int_{\mathbb{R}} f(x)g(x)J(x)dx$ for any $f, g \in C_0^\infty(\mathbb{R})$. 

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Lemma 2.4. Assume that $\mu_0$ satisfies $\int_\mathbb{R} \mu_0(x)^{2p}J(x)dx < \infty$ for some $p \geq 1$. Then for every $T > 0$, we have

$$\sup_{0 \leq t \leq T, m \geq 1} \mathbb{E} \left[ \int_\mathbb{R} \mu_t^m(x)^{2p} J(x)dx \right] < \infty.$$ 

Proof. Using the convolution form, the solution $\mu_t^m(x)$ to (2.2) can be represented as

$$\mu_t^m(x) = \langle \mu_0, p_t(x - \cdot) \rangle + \sum_{k=1}^m \int_{t_{k-1}}^{t_k} \int_\mathbb{R} \sqrt{\gamma(\mu_t^m, z)G_m(\mu_t^m(z))p_{t-s}(x-z)W(ds,dz)}.$$  (2.5)

By Hölder’s inequality we have

$$\left| \int_0^t ds \int_\mathbb{R} \mu_s^m(z)^2 p_{t-s}(x-z)^2 dz \right|^p \leq \int_0^t \frac{1}{\sqrt{2\pi(t-s)}} ds \int_\mathbb{R} \mu_s^m(z)^2 p_{t-s}(x-z) dz \leq K \int_0^t \frac{1}{\sqrt{2\pi(t-s)}} ds \int_\mathbb{R} \mu_s^m(z)^2 p_{t-s}(x-z) dz \leq K \int_0^t \frac{1}{\sqrt{2\pi(t-s)}} ds \int_\mathbb{R} \mu_s^m(z)^2 p_{t-s}(x-z) dz$$

with $1/p + 1/q = 1$ and $p, q \geq 1$. By the above inequality, (2.3) and Burkholder-Davis-Gundy’s inequality one can see that

$$\mathbb{E} \left[ \int_\mathbb{R} J(x)dx \sum_{k=1}^m \int_{t_{k-1}}^{t_k} \int_\mathbb{R} \sqrt{\gamma(\mu_t^m, z)G_m(\mu_t^m(z))p_{t-s}(x-z)W(ds,dz)} \right]^{2p}$$

$$\leq K \mathbb{E} \left[ \int_\mathbb{R} J(x)dx \left| \int_0^t \int_\mathbb{R} G_m(\mu_s^m(z))^2 p_{t-s}(x-z) ds dz \right|^p \right]$$

$$\leq K \mathbb{E} \left[ \int_\mathbb{R} J(x)dx \left| \int_0^t ds \int_\mathbb{R} (\mu_s^m(z)^2 + 1)p_{t-s}(x-z)^2 dz \right|^p \right]$$

$$\leq K + K \mathbb{E} \left[ \int_0^t \frac{1}{\sqrt{2\pi(t-s)}} ds \int_\mathbb{R} J(x)dx \int_\mathbb{R} \mu_s^m(z)^2 p_{t-s}(x-z) dz \right]$$

$$\leq K + K \mathbb{E} \left[ \int_0^t \frac{1}{\sqrt{2\pi(t-s)}} ds \int_\mathbb{R} \mu_s^m(z)^2 p_{t-s}(x-z) dz \int_\mathbb{R} J(u + z)p_{t-s}(u) du \right]$$

$$\leq K + K \int_0^t \frac{1}{\sqrt{t-s}} \mathbb{E} \left[ \int_\mathbb{R} \mu_s^m(z)^2 J(z) dz \right] ds.$$ 

By (2.4) we have $J(u + z) \leq J(z)e^{2|u|}$ which implies the last inequality, since

$$\int_\mathbb{R} e^u p_{t-s}(u) du \leq \int_\mathbb{R} e^{T|u|} p_1(u) du = \mathbb{E} \left[ e^{T|B_1|} \right] \leq K$$
for $0 \leq s \leq t \leq T$, where $B_t$ is a standard Brownian motion. Moreover, we have

$$
\int_\mathbb{R} (\mu_0, p_t(x - \cdot))^{2p} J(x) dx = \int_\mathbb{R} J(x) dx \left[ \int_\mathbb{R} p_t(x - z) \mu_0(z) dz \right]^{2p}
\leq \int_\mathbb{R} J(x) dx \int_\mathbb{R} p_t(x - z) \mu_0(z)^{2p} dz
\leq K \int_\mathbb{R} \mu_0(z)^{2p} J(z) dz < \infty.
$$

Combining the above inequality, we have

$$
\mathbb{E} \left[ \int_\mathbb{R} \mu_t^m(x)^{2p} J(x) dx \right] \leq K + K \int_0^t \frac{1}{\sqrt{t - s}} \mathbb{E} \left[ \int_\mathbb{R} \mu_s^m(x)^{2p} J(x) dx \right] ds.
$$

Iterating the above once, one can check that

$$
\mathbb{E} \left[ \int_\mathbb{R} \mu_t^m(x)^{2p} J(x) dx \right] \leq K + K \int_0^t \mathbb{E} \left[ \int_\mathbb{R} \mu_s^m(x)^{2p} J(x) dx \right] dr \int_r^t \frac{1}{\sqrt{(t - s)(s - r)}} ds
\leq K + K \int_0^t \mathbb{E} \left[ \int_\mathbb{R} \mu_s^m(x)^{2p} J(x) dx \right] dr.
$$

The result follows by Gronwall’s inequality.

We proceed to proving the tightness of $(\mu^m)$ in $C([0, T] \times \mathbb{R})$. Denote

$$
\nu_t^m(x) = \sum_{k=1}^{m} \int_{t_{k-1}}^{t_{k}} \int_\mathbb{R} \sqrt{\gamma(\mu_{t_{k-1}}^m, z) G_m(\mu_s^m(z)) p_{t-s}(x - z) W(ds, dz).}
$$

(2.6)

**Lemma 2.5.** Assume that $\mu_0$ satisfies $\int_\mathbb{R} \mu_0(x)^{2p} J(x) dx < \infty$ for some $p \geq 1$. For fixed $0 < \alpha < 1$ and $T > 0$, there is a constant $K$ such that

$$
\mathbb{E}[|\nu_t^m(x) - \nu_r^m(x)|^{2p}] \leq K(t - r)^{\alpha p/2}
$$

for all $0 < r < t \leq T$.

**Proof.** Recall that $\gamma$ is bounded by Condition 1.4. By (2.3), (2.6) and Burkholder-Davis-Gundy’s inequality one can check that

$$
\mathbb{E} \left[ |\nu_t^m(x) - \nu_r^m(x)|^{2p} \right] \leq K \mathbb{E} \left[ \int_0^r \int_\mathbb{R} |p_{r-s}(x - z) - p_{t-s}(x - z)|^{2p} [\mu_s^m(z) + 1] ds dz \right]
+ K \mathbb{E} \left[ \int_r^t \int_\mathbb{R} |\mu_s^m(z) + 1| p_{t-s}(x - z) ds dz \right]
= I_1^m(t, r) + I_2^m(t, r).
$$

By [14] Lemma III4.5, for any $0 < s < r < t \leq T$ and $\delta \in (0, 1/4)$, there is a constant $K = K(T) > 0$ such that

$$
|p_{r-s}(x - z) - p_{t-s}(x - z)|^2
$$
By Lemma 1.4.4, we have
\[ \int_0^r \int_\mathbb{R} [p_{t-s}(x-z) - p_{t-s}(x-z)]^2 dsdz \leq K|t-r|^{1/2} \]
and \( \int_r^t \int_\mathbb{R} p_{t-s}(x-z)^2 dsdz \leq K|t-r|^{1/2} \). From (2.7) and Lemma 2.4, one can obtain that
\[ \mathbb{E} \left[ \int_0^r ds \int_\mathbb{R} [p_{t-s}(x-z) - p_{t-s}(x-z)]^2 |\mu_s^m(z)|^p dz \right] \leq K(t-r)\delta \int_0^r (r-s)^{-3\delta/2} ds \mathbb{E} \left[ \int_\mathbb{R} [p_{t-s}(x-z)^{2-\delta} + p_{t-s}(x-z)^{2-\delta}] |\mu_s^m(z)|^p dz \right] \]
\[ \leq K(t-r)\delta \int_0^r (r-s)^{-3\delta/2} ds \left( \int_\mathbb{R} [p_{t-s}(x-z)^{4-2\delta} + p_{t-s}(x-z)^{4-2\delta}] e^{\frac{1}{\delta}|z|} dz \right)^{1/2} \]
\[ \cdot \mathbb{E} \left[ \left( \int_\mathbb{R} |\mu_s^m(z)|^{2p} J(z) dz \right)^{1/2} \right] \]
\[ \leq K(t-r)\delta \int_0^r (r-s)^{-(3+4\delta)/4} ds + \int_0^r (r-s)^{-3\delta/2} (t-s)^{(3-2\delta)/4} ds \]
\[ \leq K(t-r)\delta. \]

By Hölder’s inequality, it implies that
\[ I_1^m(t,r) \leq K \mathbb{E} \left[ \int_0^r ds \int_\mathbb{R} [p_{t-s}(x-z) - p_{t-s}(x-z)]^2 |\mu_s^m(z)|^p dz \right] \cdot \left[ \int_0^r \int_\mathbb{R} [p_{t-s}(x-z) - p_{t-s}(x-z)]^2 dsdz \right]^{p-1} + K(t-r)^{p/2} \]
\[ \leq K(t-r)\delta^{(p-1)/2} + K(t-r)^{p/2}. \] (2.8)

Similarly, by Hölder’s inequality and Lemma 2.4, one can see that
\[ \mathbb{E} \left[ \int_r^t \int_\mathbb{R} |\mu_s^m(z)|^p p_{t-s}(x-z)^2 dsdz \right] \]
\[ \leq \mathbb{E} \left[ \int_r^t ds \left[ \int_\mathbb{R} |\mu_s^m(z)|^{2p} J(z) dz \right]^{1/2} \left( \int_\mathbb{R} p_{t-s}(x-z)^{4} e^{\frac{1}{\delta}|z|} dz \right)^{1/2} \right] \]
\[ \leq K \int_r^t (t-s)^{-3/4} ds = K(t-r)^{1/4}, \]
which implies that
\[ I_2^m(t,r) \leq K \mathbb{E} \left[ \int_r^t \int_\mathbb{R} |\mu_s^m(z)|^p p_{t-s}(x-z)^2 dsdz \right] \left[ \int_r^t \int_\mathbb{R} p_{t-s}(x-z)^2 dsdz \right]^{p-1} \]
\[ + K(t-r)^{p/2} \]
\[ \leq K(t-r)^{1/4+(p-1)/2} + K(t-r)^{p/2}. \] (2.9)

The result follows.

Similar with above, we have the following result.
Lemma 2.6. Assume that $\mu_0$ satisfies $\int_\mathbb{R} \mu_0(x)^{2p} J(x) dx < \infty$ for some $p \geq 1$. For fixed $0 < \beta < 1$, there is a constant $K$ such that
\[
\mathbb{E} \left[ |\nu_t^m(x) - \nu_t^m(y)|^{2p} \right] \leq K|x - y|^{\beta p}
\]
for any $x, y \in \mathbb{R}$.

Proof of Theorem 1.2. By (2.5) one can check that
\[
\mu_t^m(x) = \langle \mu_0, p_t(x \cdot) \rangle + \nu_t^m(x).
\]
By Kolmogorov’s criteria (see [10] Corollary 16.9), Lemma 2.5 and Lemma 2.6, for each fixed $T, K > 0$, the sequence of laws of $\{\mu_t^m(x) : (t, x) \in [0, T] \times [-K, K]\}$ on $C([0, T] \times [-K, K])$ is tight, and hence, has a convergent subsequence. By the standard diagonalization argument, there exists a subsequence $(\mu_t^{mk}, W_t^{mk})$ which converges to $(\mu_t, W_t)$ in law on $C([0, T] \times [-K, K])$ for each $K$ and $T$. Therefore, $(\mu_t^{mk}, W_t^{mk})_{t \geq 0}$ converges in law as $k \to \infty$. Applying Skorokhod’s representation (e.g. [6] Theorem 1.8), on another probability space, there are continuous processes $(\hat{\mu}_t^{mk}, W_t^{mk})_{t \geq 0}$ and $(\hat{\mu}_t, W_t)_{t \geq 0}$ with the same distribution as $(\mu_t^{mk}, W_t^{mk})$ and $(\mu_t, W_t)$, respectively. Moreover,
\[
(\hat{\mu}_t^{mk}, \hat{W}_t^{mk})_{t \geq 0} \to (\hat{\mu}_t, \hat{W}_t)_{t \geq 0}
\]
after $k \to \infty$. Recall that there is a unique strong non-negative solution to (2.2). For any $f \in C^2_b(\mathbb{R})$ with $|f| + |f''| \leq KJ$, we have
\[
\langle \hat{\mu}_t^{mk}, f \rangle = \langle \mu_0, f \rangle + \frac{1}{2} \int_0^t \langle \hat{\mu}_s^{mk}, f'' \rangle ds
\]
\[
+ \sum_{j=1}^{m_k} \int_{t_{j-1}}^{t_j} \int_\mathbb{R} \int_\mathbb{R} \sqrt{\gamma(\hat{\mu}_{t_{j-1}}^{mk}, z) G_{mk}(\hat{\mu}_s^{mk}(z))} f(z) \hat{W}_t^{mk}(ds, dz).
\]
It follows from Lemma 2.7 that for every $t > 0$, we have
\[
\lim_{k \to \infty} \mathbb{E} \left[ \int_\mathbb{R} |\hat{\mu}_t^{mk}(x) - \hat{\mu}_t(x)|^2 J(x) dx \right] = 0 \tag{2.10}
\]
and
\[
\sup_{0 \leq t \leq T, k \geq 1} \mathbb{E} \left[ \int_\mathbb{R} |\hat{\mu}_t(x) + \hat{\mu}_t^{mk}(x)|^2 J(x) dx \right] < \infty, \tag{2.11}
\]
since $\mu_0(x) \in L^2(\mathbb{R})$. Then by the above and dominated convergence theorem, we have
\[
\mathbb{E} \left[ \int_0^t \langle |\hat{\mu}_s^{mk} - \hat{\mu}_s|, f'' \rangle ds \right] \leq \mathbb{E} \left[ \int_0^t ds \int_\mathbb{R} |\hat{\mu}_s^{mk}(x) - \hat{\mu}_s(x)|^2 J(x) dx \right]^{1/2} \to 0
\]
as $k \to \infty$. Recall that $\gamma$ satisfies Condition 1.1. Thus,
\[
\mathbb{E} \left[ \int_0^t \int_\mathbb{R} \sqrt{\gamma(\hat{\mu}_s, z) \hat{\mu}_s(z)} f(z) \hat{W}_t^{mk}(ds, dz) \right]^2
\]
\[
\leq K \mathbb{E} \left[ \int_0^t \int_\mathbb{R} \hat{\mu}_s(z) J(z) ds dz \right] \leq K \left\{ \mathbb{E} \left[ \int_0^t \int_\mathbb{R} \hat{\mu}_s(z)^2 J(z) ds dz \right] \right\}^{1/2} < \infty.
\]
It thus follows from \[23\] Lemma 2.4 that
\[
\langle \mu_t, f \rangle = \langle \mu_0, f \rangle + \frac{1}{2} \int_0^t \langle \hat{\mu}_s, f'' \rangle \, ds + \int_0^t \int_{\mathbb{R}} \sqrt{\gamma(\hat{\mu}_s, z)} \hat{\mu}_s(z) f(z) \, \hat{W}(ds, dz).
\]
That completes the existence of solution to (1.3). By Proposition 2.3 one can see that \(X_t(dx) = \mu_t(x)dx\) satisfies the MP (1.1) (1.2), which implies the conclusion. \(\square\)

**Proof of Theorem 1.3.** Assume that \((\mu_t(x))_{t \geq 0, x \in \mathbb{R}}\) is a solution to SPDE (1.3). Then it also satisfies (2.1) by Proposition 2.3. For any \(0 < r \leq t\) and \(p \geq 1\), we have
\[
\mathbb{E}[|\mu_t(x) - \mu_r(x)|^{2p}] \leq K \left( \int_{\mathbb{R}} |p_t(x-z) - p_r(x-z)| \mu_0(z) \, dz \right)^{2p} + K \mathbb{E} \left[ \int_r^t \int_{\mathbb{R}} |p_t-s(x-z)| \sqrt{\mu_s(z)} \, W(ds, dz) \right]^{2p} + K \mathbb{E} \left[ \int_0^r \int_{\mathbb{R}} |p_r-s(x-z) - p_t-s(x-z)| \sqrt{\mu_s(z)} \, W(ds, dz) \right]^{2p} =: K(I_1 + I_2 + I_3).
\]
Recall that \(\mu_0\) is Hölder continuous with exponent \(\lambda < \frac{1}{2}\). Similar with the proof of \[8\] Theorem 1.1, by Hölder’s inequality one can see that
\[
I_1 = \left| \int_{\mathbb{R}} |p_t(x-z) - p_r(x-z)| \mu_0(z) \, dz \right|^{2p} = \left\| \mathbb{E}[\mu_0(x + B_t) - \mu_0(x + B_r)] \right\|^{2p} \leq K \mathbb{E}[|B_{t-r}|^{2p\lambda}] \leq K \int_{\mathbb{R}} x^{2p\lambda} \frac{1}{\sqrt{2\pi(t-r)}} e^{-\frac{x^2}{2(t-r)}} \, dx \leq K(t-r)^{\lambda p},
\]
where \((B_t)_{t \geq 0}\) is a standard Brownian motion. Moreover, similar with (2.8) and (2.9) one can see that
\[
I_2 \leq K \mathbb{E} \left[ \left( \int_r^t \int_{\mathbb{R}} |p_t-s(x-z)|^{2p} \mu_s(z) \, ds \, dz \right)^{\frac{p}{2}} \right] \leq K(t-r)^{1/4+(p-1)/2},
\]
and
\[
I_3 \leq K \mathbb{E} \left[ \left( \int_0^r \int_{\mathbb{R}} |p_r-s(x-z) - p_t-s(x-z)|^{2p} \mu_s(z) \, ds \, dz \right)^{\frac{p}{2}} \right] \leq K(t-r)^{\delta+(p-1)/2}
\]
with \(\delta \in (0, 1/4)\). Then there exists \(\alpha \in (0, 1)\) such that
\[
\mathbb{E}[|\mu_t(x) - \mu_r(x)|^{2p}] \leq K(t-r)^{\alpha p} + K(t-r)^{1/4+(p-1)/2} + K(t-r)^{\delta+(p-1)/2} \leq K(t-r)^{\alpha p/2}.
\]
(2.12)
Similarly, for any \(x, y \in \mathbb{R}\) we have
\[
\mathbb{E}[|\mu_t(x) - \mu_t(y)|^{2p}] \leq K|x-y|^{2p\lambda} + K|x-y|^{2p}
\]
(2.13)
with \(\beta \in (0, 1)\). For \(t, r \in [0, T]\) and \(x, y \in \mathbb{R}\), notice that
\[
\mathbb{E}[|\mu_t(x) - \mu_t(y)|^{2p}] \leq K \mathbb{E}[|\mu_t(x) - \mu_t(x)|^{2p}] + K \mathbb{E}[|\mu_t(x) - \mu_t(y)|^{2p}] .
\]
Combining with the above, (2.12) and (2.13), the result follows from Kolmogorov’s continuity criteria (see e.g. \[19\] Corollary 1.2). \(\square\)

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3 Proof of Theorem 1.4

In this section we show the weak uniqueness of the solution to the MP (1.1, 1.2) under Condition 1.1. The main idea of uniqueness is to relate the MP (1.1, 1.2) with a system of SPDEs, which is satisfied by a sequence of corresponding function-valued processes \((u_i^t)_{t \geq 0}, i = 0, 1, \ldots, n\). The weak uniqueness of the solution to the MP (1.1, 1.2) follows from the pathwise uniqueness of the solution to the system of SPDEs.

For a solution \((X_t)_{t \geq 0}\) to the MP (1.1, 1.2), we define the function-valued processes as

\[
u_i^t(x) = X_t((a_i, x]), \quad a_i \leq x < a_{i+1}, \quad i = 0, \ldots, n. \tag{3.1}
\]

In Proposition 3.1 we show that (3.1) is a solution to the following system of SPDEs:

\[
\begin{aligned}
u_i^t(x) &= \nu_i^0(x) + \int_0^t \frac{1}{2} \Delta u_i^s(x) ds + \int_0^t \int_0^{u_i^s(x)} g_i(\nabla u_i^s(a_{i+1})) W_i(ds, dz), \\
\nu_n^t(x) &= \nu_n^0(x) + \int_0^t \frac{1}{2} \Delta u_n^s(x) ds + \int_0^t \int_0^{u_n^s(x)} g_n(u_n^s(\infty)) W_n(ds, dz), \\
u(a_i) &= 0, \quad i = 0, 1, \ldots, n, \\
\nabla u_i^{t-1}(a_i) &= \nabla u_i^t(a_i), \quad i = 1, \ldots, n.
\end{aligned} \tag{3.2, 3.3}
\]

where \(u_n^{s}(\infty) := \lim_{x \to \infty} u_n^s(x)\) and \(W_i(ds, dz), \quad i = 0, \ldots, n\) are independent time-space white noises on \(\mathbb{R}_+ \times \mathbb{R}_+\) with intensity \(ds dz\). The pathwise uniqueness of the solution to the above system of SPDEs is obtained in Proposition 3.3.

The system of SPDEs (3.2, 3.3) can be understood in the following form: for any \(\phi_i \in C_b^2[a_i, a_{i+1}]\) with \(\phi_i(a_i) = \phi_i'(a_{i+1}) = 0, \quad i = 0, 1, \ldots, n-1\), and \(\phi_n \in C_b^2[a_n, \infty)\) with \(\phi_n(a_n) = \phi_n(\infty) = 0\) (given \(\phi_n(\infty) := \lim_{x \to \infty} \phi_n(x)\)), we have

\[
\begin{aligned}
\langle \nu_i^t, \phi_i \rangle &= \langle \nu_0^t, \phi_i \rangle + \frac{1}{2} \int_0^t \left[ \langle \nu_s^t, \phi_i^n \rangle + \phi_i(a_{i+1}) \nabla u_i^s(a_{i+1}) \right] ds \\
&\quad + \int_0^t \int_0^{u_{i+1}^s} 1_{\{x \leq u_i^s(x)\}} \phi_i(x) dx g_i(\nabla u_i^s(a_{i+1})) W_i(ds, dz); \\
\langle \nu_n^t, \phi_n \rangle &= \langle \nu_0^t, \phi_n \rangle + \frac{1}{2} \int_0^t \langle \nu_s^t, \phi_n^\infty \rangle ds \\
&\quad + \int_0^t \int_{a_n}^\infty 1_{\{x \leq u_n^s(x)\}} \phi_n(x) dx g_n(u_n^s(\infty)) W_n(ds, dz). \tag{3.5, 3.6}
\end{aligned}
\]

Proposition 3.1. Suppose that \((X_t)_{t \geq 0}\) is a solution to the MP (1.1, 1.2). Then \(\{u_i^t : t \geq 0, x \in [a_i, a_{i+1})\}, i = 0, 1, \ldots, n\) defined as (3.1), solves the group of SPDEs (3.2, 3.3) with boundary condition (3.4).
Proof. For any \( \phi_i \in C^3_c(a_i, a_{i+1}) \) with \( i = 0, 1, \ldots, n - 1 \), by integration by parts, we have
\[
\langle u'_i, \phi'_i \rangle = -\langle X_t, \phi_i \rangle = -M_t(\phi_i) - \langle X_0, \phi_i \rangle - \frac{1}{2} \int_0^t \langle X_s, \phi''_i \rangle \, ds
\]
\[
= -M_t(\phi_i) + \langle u'_0, \phi'_0 \rangle + \frac{1}{2} \int_0^t \langle u'_s, (\phi'_i)^n \rangle \, ds.
\]
Thus
\[
-M_t(\phi_i) = \langle u'_i, \phi'_i \rangle - \langle u'_0, \phi'_0 \rangle - \frac{1}{2} \int_0^t \langle u'_s, (\phi'_i)^n \rangle \, ds
\] (3.7)
is a continuous martingale. By Lemma 2.1 we have
\[
-M_t(\phi_i) = \int_0^t \int_{\mathbb{R}} \phi_i(x) M(ds, dx)
\]
with
\[
\langle -M(\phi_i) \rangle_t = \int_0^t g_i(\nabla u_s^{i}(a_{i+1}))^2 ds \int_{\mathbb{R}} \phi_i(x)^2 X_s(dx)
\]
\[
= \int_0^t g_i(\nabla u_s^{i}(a_{i+1}))^2 ds \int_0^{u_s^{i}(a_{i+1})} \phi_i(u_s^{i}(y)-1)^2 dy
\]
\[
= \int_0^t \int_0^\infty \left( \int_{a_i}^{u_s^{i}(y)} \phi_i'(x) dx \right)^2 g_i(\nabla u_s^{i}(a_{i+1}))^2 ds dy,
\]
where \( u_s^{i}(y)^{-1} \) denotes the generalized inverse of the nondecreasing function \( u_s^{i} \), that is,
\[
u_s^{i}(y)^{-1} = \sup\{x \in [a_i, a_{i+1}) : u_s^{i}(x) \leq y\}.
\]
Moreover, for \( \phi_n \in C^3_c(a_n, \infty) \), one can see that
\[
-M_t(\phi_n) = \langle u_n^n, \phi'_n \rangle - \langle u_0^n, \phi'_0 \rangle - \frac{1}{2} \int_0^t \langle u_n^n, \phi''_n \rangle \, ds
\] (3.8)
is a continuous martingale with
\[
\langle -M(\phi_n) \rangle_t = \int_0^t g_n(\nabla u_s^n(\infty))^2 ds \int_{\mathbb{R}} \phi_n(x)^2 X_s(dx)
\]
\[
= \int_0^t \int_0^\infty \left( \int_{a_n}^{\infty} \phi_n'(x) dx \right)^2 g_n(\nabla u_s^n(\infty))^2 ds dy.
\]
Similar with Lemma 2.1, the family \( \{-M_t(\phi_i) : t \geq 0, \phi_i \in C^3_c(a_i, a_{i+1})\} \) determines a martingale measure \( \{M_t(B) : t \geq 0, B \in \mathcal{B}(a_i, a_{i+1})\} \). Moreover, for \( \phi_i \in C^3_c(a_i, a_{i+1}) \) and \( \phi_j \in C^3_c(a_j, a_{j+1}) \) with \( i \neq j \), we have
\[
\langle -M(\phi_i), -M(\phi_j) \rangle_t = \int_0^t ds \int_{\mathbb{R}} \gamma(X_s, z) X_s(dz) \int_{\mathbb{R}} \phi_i(x) \delta_z(dx) \int_{\mathbb{R}} \phi_j(y) \delta_z(dy)
\]
\[
= \int_0^t ds \int_{\mathbb{R}} \gamma(X_s, z) \phi_i(z) \phi_j(z) X_s(dz) = 0.
\]
By El Karoui and Mélaërd [3, Theorem III-7, Corollary III-8], on some extension of the probability space on can define a sequence independent Gaussian white noise $W_i(ds, du, i = 0, \cdots, n)$ on $(0, \infty)^2$ based on $dsdu$ such that

$$-M_t(\phi_t) = \int_0^t \int_0^\infty \int_{a_i}^{a_{i+1}} 1_{\{z \leq u_i^i(x)\}} \phi_t^i(x) \sqrt{\gamma(X_s, x)} dx W_i(ds, dz)$$

$$= \int_{a_i}^{a_{i+1}} \phi_t^i(x) dx \int_0^t \int_0^\infty g_i(\nabla u_i^i(a_{i+1})) W_i(ds, dz)$$

for any $\phi_t \in C_c^3(a_i, a_{i+1}), i = 0, \cdots, n - 1$, and

$$-M_t(\phi_n) = \int_{a_n}^\infty \phi_n^i(x) dx \int_0^t \int_0^\infty g_n(u_n^i(\infty)) W_n(ds, dz)$$

for any $\phi_n \in C_c^3(a_n, \infty)$. Then (3.5) and (3.6) follow from the approximation method and Lemma 4.2 in Appendix.

**Lemma 3.2.** Suppose that $(u_n^i)_{i \geq 0}$ satisfies (3.3). Then $(u_n^i(\infty))_{i \geq 0}$ satisfies

$$u_n^i(\infty) = u_0^i(\infty) + \int_0^t \int_0^\infty g_n(u_s^i(\infty)) W_n(ds, dz).$$

**Proof.** Recall that $p_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)}$ and let

$$q_t^x(y) := p_t(x + a_n - y) - p_t(x - a_n + y)$$

for $t > 0$ and $x, y \geq a_n$. Then (3.3) can be written into the following mild form:

$$u_t^n(x) = \langle u_0^n, q_t^x \rangle + \int_0^t \int_0^\infty \left[ \int_{a_n}^\infty 1_{\{z \leq u_s^i(y)\}} q_t^{x-s}(y) dy \right] g_n(u_s^n(\infty)) W_n(ds, dz)$$

for any $x \geq a_n$. By a change of variable, we have

$$\langle u_0^n, q_t^x \rangle = \int_{a_n}^\infty u_0^n(y) [p_t(x + a_n - y) - p_t(x - a_n + y)] dy$$

$$= \int_{-\infty}^x u_0^n(x + a_n - z) p_t(z) dz - \int_x^\infty u_0^n(z - x + a_n) p_t(z) dz$$

$$\rightarrow u_0^n(\infty) \int_{-\infty}^\infty p_t(z) dz = u_0^n(\infty)$$

and

$$\int_{a_n}^\infty 1_{\{z \leq u_s^i(y)\}} q_t^{x-s}(y) dy \rightarrow 1_{\{z \leq u_s^n(\infty)\}}$$

as $x \rightarrow \infty$, which ends the proof.
Proposition 3.3. (Pathwise uniqueness) Suppose that \((u_t)_{t \geq 0}\) and \((\tilde{u}_t)_{t \geq 0}\) are two solutions to (3.10) satisfying the boundary conditions (3.4). If \(u_0(x) = \tilde{u}_0(x)\) for all \(x \in \mathbb{R}\), then \(\mathbb{P}\{ u_t(x) = \tilde{u}_t(x) \text{ for all } t \geq 0 \text{ and } x \in \mathbb{R} \} = 1\).

Proof. The pathwise uniqueness of the solution to (3.10) holds by [3, Theorem 2.1]. Moreover, the pathwise uniqueness of the solution \(\{u^n_t(x) : t \geq 0, x \in [a_n, \infty)\}\) holds for (3.3) by [24, Theorem 1.4]. That implies the strong uniqueness of \((\nabla u^n_t(a_n))_{t \geq 0}\). By the proof of [24, Theorem 1.4] and induction method, one can get the pathwise uniqueness of the solution to (3.2) for \(i = 0, 1, \ldots, n - 1\). The proof ends here.

\[
\text{Proof of Theorem 1.4} \text{ The result is a direct conclusion of Proposition 3.3.}
\]

4 Appendix

In this section we give some results about the process \((u_t(x))_{t \geq 0, x \in [0, 1]}\) satisfying the following SPDE:

\[
u_t(x) = u_0(x) + \int_0^t \frac{1}{2} \Delta u_s(x) ds + \int_0^t \int_0^t g(\nabla u_s(1)) W(ds, dz),
\]

where \(g\) is a positive continuous bounded function from \(\mathbb{R}_+\) to \(\mathbb{R}_+\), and \(W(ds, dz)\) is a time-space Gaussian white noise with density \(dsdz\). Let \(\Phi \in C^2_c(0, 1)\) satisfies \(0 \leq \Phi \leq 2\) and \(\int_0^1 \Phi(x)dx = 1\). For \(k \geq 1\) and \(x \in [0, 1]\) let

\[
h_k(x) = \int_0^{kx} \Phi(z)dz \cdot \int_1^1 \Phi(z)dz.
\]

(4.1)

Then \(h_k \in C^2_c(0, 1)\) for all \(k \geq 1\).

Lemma 4.1. Suppose that \(f \in C[0, 1]\) with \(f'(1)\) and \(f'(0)\) exist. Then

\[
\lim_{k \to \infty} \langle f, h'_k \rangle = f(0) - f(1), \quad \lim_{k \to \infty} \langle f, h'_k \rangle = f'(1) - f'(0).
\]

Proof. Observe that for each \(n \geq 1\),

\[
h'_k(x) = k\Phi(kx) \int_x^1 \Phi(z)dz - kx^{k-1}\Phi(x^k) \int_0^{kx} \Phi(z)dz, \quad x \in [0, 1].
\]

Then by change of variables and dominated convergence, as \(k \to \infty\),

\[
\langle f, h'_k \rangle = \int_0^1 f(x)k\Phi(kx) \left[ \int_x^1 \Phi(z)dz \right] dx - \int_0^1 f(x)kx^{k-1}\Phi(x^k) \left[ \int_0^{kx} \Phi(z)dz \right] dx
\]

\[
= \int_0^1 f(y/k)\Phi(y) \left[ \int_y^{y/k} \Phi(z)dz \right] dy - \int_0^1 f(y^{1/k})\Phi(y) \left[ \int_0^{ky^{1/k}} \Phi(z)dz \right] dy
\]

converges to \(f(0) - f(1)\), which gives the first assertion.
In the following we prove the second assertion. Observe that

\[ h''_k(x) = k^2 \Phi'(kx) \int_{x^k}^1 \Phi(z) dz - 2k^2 x^{k-1} \Phi(kx) \Phi(x^k) \]

\[ - [k(k-1)x^{k-2} \Phi(x^k) + k^2 x^{2k-2} \Phi'(x^k)] \int_{0}^{kx} \Phi(z) dz \]

\[ =: M_{1,k}(x) - 2M_{2,k}(x) - M_{3,k}(x). \tag{4.2} \]

By change of variables and dominated convergence again, as \( k \to \infty \),

\[ \int_0^1 [f(x) - f(0)] M_{1,k}(x) dx = \int_0^1 k[f(y/k) - f(0)] \Phi'(y) \left[ \int_{y^{k-1}}^{y^k} \Phi(z) dz \right] dy \]

\[ \to f'(0) \int_0^1 y \Phi'(y) dy = -f'(0) \tag{4.3} \]

and

\[ \int_0^1 [f(x) - f(1)] M_{2,k}(x) dx = \int_0^1 \frac{f(y^{1/k}) - f(1)}{y^{1/k} - 1} k(y^{1/k} - 1) \Phi(y^{1/k}) \Phi'(y) dy \]

\[ \to 0. \tag{4.4} \]

Similarly, as \( k \to \infty \),

\[ \int_0^1 [f(x) - f(1)] M_{3,k}(x) dx \]

\[ = \int_0^1 \frac{f(y^{1/k}) - f(1)}{y^{1/k} - 1} k(y^{1/k} - 1) [k^{-1}(k-1)y^{-1/k} \Phi(y) + y^{(k-1)/k} \Phi'(y)] \left[ \int_0^{ky^{1/k}} \Phi(z) dz \right] dy \]

\[ \to f'(1) \int_0^1 \ln y [\Phi(y) + y \Phi'(y)] dy = -f'(1). \tag{4.5} \]

Applying integration by parts and the fact \( 0 \leq \Phi \leq 2 \) and \( \text{supp}(\Phi) \subset (0,1) \),

\[ \int_0^1 M_{1,k}(x) dx = \int_0^1 \left( \int_{x^k}^{1} \Phi(z) dz \right)'' \cdot \left( \int_{x^k}^{1} \Phi(z) dz \right) dx = \int_0^1 M_{2,k}(x) dx \]

\[ = k \int_0^1 \Phi(ky^{1/k}) \Phi(y) dy = k \int_0^{k^{-1}} \Phi(ky^{1/k}) \Phi(y) dy \leq 4k^{1-k} \]

and

\[ \int_0^1 M_{3,k}(x) dx = - \int_0^1 \left( \int_{x^k}^{1} \Phi(z) dz \right)'' \cdot \left( \int_{x^k}^{1} \Phi(z) dz \right) dx \]

\[ = \int_0^1 M_{2,k}(x) dx \leq 4k^{1-k}. \]

Then combining (4.2) with (4.3)-(4.5) one completes the proof. \( \square \)
Lemma 4.2. Suppose that for each \( \phi \in C_c^2(0, 1) \), \( (u_t)_{t \geq 0} \) satisfies

\[
\langle u_t, \phi \rangle = \langle u_0, \phi \rangle + \frac{1}{2} \int_0^t \langle u_s, \phi'' \rangle \, ds + \int_0^t \int_0^\infty g(\nabla u_s(1)) \left[ \int_0^1 1_{\{z \leq u_s(x)\}} \phi(x) \, dx \right] W(ds, dz). \tag{4.6}
\]

Then for each \( \phi \in C_b^2[0, 1] \),

\[
\langle u_t, \phi \rangle = \langle u_0, \phi \rangle + \frac{1}{2} \int_0^t \left[ \langle u_s, \phi'' \rangle + F_s(\phi) \right] \, ds + \int_0^t \int_0^\infty g(\nabla u_s(1)) \left[ \int_0^1 1_{\{z \leq u_s(x)\}} \phi(x) \, dx \right] W(ds, dz), \tag{4.7}
\]

where

\[ F_s(\phi) := [\phi(1)\nabla u_s(1) - \phi(0)\nabla u_s(0)] - [u_s(1)\phi'(1) - u_s(0)\phi'(0)]. \tag{4.8} \]

Proof. Recall \( h_k \) in (4.1). For \( m \geq 1 \) define stopping time \( \tau_m \) by

\[ \tau_m := \inf \left\{ t \geq 0 : \sup_{x \in [0, 1]} |u_t(x)| \geq m \right\} \]

with the convention \( \inf \emptyset = \infty \). Then \( \lim_{m \to \infty} \tau_m = \infty \) almost surely. It follows from (4.6) that

\[
\langle u_{t \wedge \tau_m}, \phi h_k \rangle = \int_0^{t \wedge \tau_m} \int_0^\infty g(\nabla u_s(1)) \left[ \int_0^1 1_{\{z \leq u_s(x)\}} \phi(x) h_k(x) \, dx \right] W(ds, dz)
\]

\[
+ \langle u_0, \phi h_k \rangle + \frac{1}{2} \int_0^{t \wedge \tau_m} \langle u_s, (\phi h_k)'' \rangle \, ds. \tag{4.9}
\]

Notice that

\[
\langle u_s, (\phi h_k)'' \rangle = \langle u_s, \phi'' h_k \rangle + 2\langle u_s, \phi' h_k' \rangle + \langle u_s, \phi h_k'' \rangle.
\]

It follows from Lemma 4.1 that

\[
\lim_{k \to \infty} \langle u_s, (\phi h_k)'' \rangle = \langle u_s, \phi'' \rangle + [u_s(0)\phi'(0) - u_s(1)\phi'(1)] - [\phi(0)\nabla u_s(0) - \phi(1)\nabla u_s(1)]
\]

\[
= \langle u_s, \phi'' \rangle + F_s(\phi).
\]

Thus letting \( k \to \infty \) in (4.9) we obtain

\[
\langle u_{t \wedge \tau_m}, \phi \rangle = \langle u_0, \phi \rangle + \frac{1}{2} \int_0^{t \wedge \tau_m} \left[ \langle u_s, \phi'' \rangle + F_s(\phi) \right] \, ds
\]

\[
+ \int_0^{t \wedge \tau_m} \int_0^\infty g(\nabla u_s(1)) \left[ \int_0^1 1_{\{z \leq u_s(x)\}} \phi(x) \, dx \right] W(ds, dz).
\]

Letting \( m \to \infty \) the result holds. \( \square \)

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