BMO $\varepsilon$-REGULARITY RESULTS FOR SOLUTIONS TO LEGENDRE-HADAMARD ELLIPTIC SYSTEMS

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Abstract. We will establish an $\varepsilon$-regularity result for weak solutions to Legendre-Hadamard elliptic systems, under the a-priori assumption that the gradient $\nabla u$ is small in BMO. Focusing on the case of Euler-Lagrange systems to simplify the exposition, regularity results will be obtained up to the boundary, and global consequences will be explored. Extensions to general quasilinear elliptic systems and higher-order integrands is also discussed.

Contents

1. Introduction 1
2. Interior regularity for $F$-extremals 5
3. Preliminaries for boundary regularity 10
4. Regularity up to the boundary for $F$-extremals 16
5. Extensions 23
References 28

1. Introduction

In this paper we study the regularity of weak solutions to the Euler-Lagrange system

\begin{equation}
- \text{div } F'(\nabla u) = 0
\end{equation}

in $\Omega \subset \mathbb{R}^n$ where $u: \Omega \to \mathbb{R}^N$ is a vector-valued mapping, that is $u$ satisfies

\begin{equation}
\int_{\Omega} F'(\nabla u) : \nabla \varphi \, dx = 0
\end{equation}

for all $\varphi \in C^\infty_c(\Omega, \mathbb{R}^N)$. Henceforth referred to as $F$-extremals, solutions to (1.1) are critical points to the functional

\begin{equation}
F(w) = \int_{\Omega} F(\nabla w(x)) \, dx.
\end{equation}

There is a considerable literature studying the partial regularity theory for minimisers of such functionals, under a suitably strict version of the quasiconvexity condition introduced by Morrey [36]. A striking feature of the vectorial $(n, N \geq 2)$ setting is that minimisers need not be everywhere regular (see for instance [11, 40] and also [35]), so the best we can hope for are partial regularity results. In the quasiconvex setting the first result in this direction was due to Evans in [13] which has been extended considerably since; we refer the interested reader to the monograph of Giusti [19] and the references therein.

\begin{flushleft}
Date: September 16, 2021.
2020 Mathematics Subject Classification. 35J47, 35J50, 42B37.
Key words and phrases. Legendre-Hadamard ellipticity, epsilon-regularity, bounded mean oscillation.
Funding: The author was supported by the EPSRC [EP/L015811/1].
\end{flushleft}
For arbitrary weak solutions of the above equation however, the work of Müller & Šverák [39] shows that we cannot hope for improved regularity results. Developing the theory of convex integration for Lipschitz mappings they constructed highly irregular solutions to (1.1), including Lipschitz solutions that fail to be $C^1$ in any open subset and compactly supported solutions whose gradient is $L^q$-integrable if and only if $q \leq 2$. These results have been extended by Kristensen & Taheri [30] for weak local minimisers, and by Székelyhidi [46] for strongly polyconvex integrands.

However it is well-known that if $u$ is suitably regular, we can infer higher regularity by a bootstrap argument. This follows for instance using the classical Schauder estimates, where if the integrand $F$ is smooth and suitably convex, any $C^{1,\alpha}$ solution for $\alpha \in (0,1)$ can be shown to be smooth. A natural question is to ask whether this a-priori Hölder condition can be further relaxed. This was studied by Moser in the preprint [38], who proved it was sufficient to assume that $u$ was Lipschitz such that $\nabla u$ lies in the space VMO of functions of vanishing mean oscillation as introduced by Sarason [41]. This condition was motivated from related regularity results for linear elliptic systems, where the work of Chiarenza, Frasca, & Longo [8] established $W^{2,p}$ estimates for linear uniformly elliptic equations where the coefficient matrix $A$ was assumed to be in VMO. A similar statement was established by Campos Cordero [7] for quasiconvex integrands through different means, noting also an inconsistency in the proof in [38]. In this paper we will extend these results, establishing regularity up to the boundary in a more general setting.

While we focus on the case of $F$-extremals to illustrate the main ideas, it turns out the arguments do not make use of the variational structure and extend to more general Legendre-Hadamard elliptic systems. We will sketch this extension in Section 5, where higher-order equations are also considered.

1.1. Setup and main results. We will study the following class of integrands; we refer the reader to Section 1.3 for the precise notational conventions.

**Hypotheses 1.1.** For $n \geq 2$ and $N \geq 1$, let $F: \mathbb{R}^{Nn} \to \mathbb{R}$ satisfy the following.

(H0) $F$ is of class $C^2$.

(H1) There is $q \geq 2$ such that $F$ satisfies the natural growth condition

$$|F(z)| \leq K (1 + |z|)^q$$

for all $z \in \mathbb{R}^{Nn}$.

(H2) $F''$ satisfies a strict Legendre-Hadamard condition, namely for all $z \in \mathbb{R}^{Nn}$ we have

$$F''(z_0)(\xi \otimes \eta) : (\xi \otimes \eta) \geq 0$$

for all $\xi \in \mathbb{R}^N$ and $\eta \in \mathbb{R}^n$, with equality if and only if $\xi \otimes \eta = 0$.

A key feature of our results is that we only need to assume a strict Legendre-Hadamard condition which is closely related to rank-one convexity of $F$, and as the construction of Šverák [45] illustrates rank-one convexity is strictly weaker than the quasiconvexity condition of Morrey. We also highlight that we do not require control in the $L^q$ scales from below, so if $F$ satisfies the above hypotheses for some $1 \leq q < 2$ instead, it will hold with $q = 2$.

The key ideas are contained in the following interior regularity theorem, which we will prove in Section 2. For the precise definition of BMO functions we adopt in the text we refer the reader to Section 3.1.

**Theorem 1.2** (BMO $\varepsilon$-regularity theorem). Suppose $F$ satisfies Hypotheses 1.1. Then for all $M > 0$ and $\alpha \in (0,1)$, there is $\varepsilon = \varepsilon(M, F, \alpha) > 0$ such that for any ball $B_R(x_0) \subset \mathbb{R}^n$ if $u$ is
F-extremal in $B_R(x_0)$ with $|\langle \nabla u \rangle_{B_R(x_0)}| \leq M$ and
\begin{equation}
\langle \nabla u \rangle_{\text{BMO}(B_R)} \leq \varepsilon,
\end{equation}
we have $u$ is $C^{1,\alpha}$ on $\overline{B_{R/2}(x_0)}$.

We will follow a similar strategy to the partial regularity theory for minimisers, which traces back to the works of Morrey [37] and Giusti & Miranda [20] in the variational setting. This will involve establishing a suitable Caccioppoli inequality and a harmonic approximation result, which are combined in a final iteration argument. For the former we will use a modification of an estimate which appeared in Moser [38], and for the latter we will follow a recent approach of Gmeineder & Kristensen [21] adapted to our setting.

We will also establish an analogous result up to the boundary, using ideas from Kronz [32] and Campos Cordero [7]. We will prove this in Section 4, and will rely on technical results established in Section 3. Here we denote $\Omega_R(x_0) = \Omega \cap B_R(x_0)$.

**Theorem 1.3** (Boundary BMO $\varepsilon$-regularity theorem). Suppose $F$ satisfies Hypotheses 1.1, $\Omega \subset \mathbb{R}^n$ is a $C^{1,\beta}$ domain for some $\beta \in (0, 1)$, $g \in C^{1,\beta}(\overline{\Omega}, \mathbb{R}^N)$, and let $u \in W^{1,q}(\Omega, \mathbb{R}^N)$ be $F$-extremal. Then for each $\alpha \in (0, \beta)$ and $M > 0$ there is $\varepsilon = \varepsilon(M,F,\Omega,g,\beta,\alpha) > 0$ and $R_0 = R_0(M,F,\Omega,g,\beta,\alpha) > 0$ such that if $x_0 \in \partial \Omega$ and $0 < R < R_0$ with $|\langle \nabla u \rangle_{\Omega_R(x_0)}| \leq M$ and
\begin{equation}
|\langle \nabla u \rangle_{\text{BMO}(\Omega_R(x_0))}| \leq \varepsilon,
\end{equation}
we have $u$ is $C^{1,\alpha}$ on $\overline{\Omega_R/2(x_0)}$.

Patching these local regularity results we can infer global consequences, for which we will need some notation. Following [34], we define the infinitesimal mean oscillation of $f \in \text{BMO}(\Omega, \mathbb{R}^N)$ as
\begin{equation}
\{f\}_{\text{osc}(\Omega)} = \limsup_{R \to 0} \sup_{0 < r < R} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f - \langle f \rangle_{B_r(x)}| \, dx.
\end{equation}
Note that $\{f\}_{\text{osc}(\Omega)} = 0$ if and only if $f \in \text{VMO}(\Omega, \mathbb{R}^N)$.

**Corollary 1.4** (Regularity of almost VMO Lipschitz solutions). Suppose $F$ satisfies Hypotheses 1.1, let $\Omega \subset \mathbb{R}^n$ be a $C^{1,\beta}$ domain for some $\beta \in (0, 1)$ and $g \in C^{1,\beta}(\overline{\Omega}, \mathbb{R}^N)$. Then for each $M > 0$ and $\alpha \in (0, \beta)$, there is $\varepsilon = \varepsilon(M,F,\Omega,g,\beta,\alpha) > 0$ such that if $u \in W^{1,\infty}_g(\Omega, \mathbb{R}^N)$ is $F$-extremal such that $|\langle \nabla u \rangle|_{L^\infty(\Omega)} \leq M$, and
\begin{equation}
\{\nabla u\}_{\text{osc}(\Omega)} \leq \varepsilon,
\end{equation}
then $u \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^N)$.

It is unclear if the Lipschitz assumption can be removed; the infinitesimal mean oscillation assumption requires us to consider balls of arbitrarily small radius, which in turn requires a uniform bound on all averages $|\langle \nabla u \rangle_{\Omega_R(x_0)}|$ for all $x_0 \in \overline{\Omega}$ and $R > 0$ small. However this is equivalent to assuming $\nabla u$ is bounded by the Lebesgue differentiation theorem.

We will later show the Lipschitz assumption can be relaxed under controlled quadratic growth conditions in Theorem 4.4, but for the moment we will record several other straightforward consequences.

**Corollary 1.5** (Partial regularity of VMO solutions). Suppose $F$ satisfies Hypotheses 1.1, let $\Omega \subset \mathbb{R}^n$ be a $C^{1,\beta}$ domain for some $\beta \in (0, 1)$ and $g \in C^{1,\beta}(\overline{\Omega}, \mathbb{R}^N)$. Then if $u \in W^{1,q}_g(\Omega, \mathbb{R}^N)$ is $F$-extremal such that $\nabla u \in \text{VMO}(\Omega, \mathbb{R}^N)$, letting
\begin{equation}
\mathcal{R}_\mathcal{M} = \left\{ x \in \overline{\Omega} : \limsup_{R \to 0} |\langle \nabla u \rangle_{\Omega_R(x_0)}| < \infty \right\},
\end{equation}

...
we have $\mathcal{R}_{\mathcal{T}} \subset \overline{\mathcal{T}}$ is a relatively open subset of full measure and $u$ is $C^{1,\alpha}$ on $\mathcal{R}_{\mathcal{T}}$ for all $\alpha \in (0, \beta)$.

We can also obtain a global regularity result if we assume $\nabla u$ is suitably small in both $L^1$ and BMO. The $L^1$ smallness condition allows us to cover $\overline{\mathcal{U}}$ by balls finitely many balls $B_{R_k}(x_k)$ such that each $|(\nabla u)_{\Omega B_k(x_k)}| \leq 1 + |g|_{L^\infty(\Omega)}$, on which we can apply our $\varepsilon$-regularity result to obtain the following.

**Corollary 1.6** (Regularity of BMO-small solutions). Suppose $F$ satisfies Hypotheses 1.1, let $\Omega \subset \mathbb{R}^n$ be a $C^{1,\beta}$ domain for some $\beta \in (0, 1)$ and $g \in C^{1,\beta}(\overline{\Omega}, \mathbb{R}^N)$. Then for each $\alpha \in (0, \beta)$ there is $\varepsilon = \varepsilon(F, \Omega, g, \beta, \alpha) > 0$ such that if $u \in W^{1,q}(\Omega, \mathbb{R}^N)$ is $F$-extremal in $\Omega$ with $\nabla u \in \text{BMO}(\Omega, \mathbb{R}^N)$ satisfying

\begin{equation}
\|\nabla u - \nabla g\|_{L^1(\Omega)} + [\nabla u]_{\text{BMO}(\Omega)} \leq \varepsilon,
\end{equation}

then $u \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^N)$.

1.2. **Connection to minimisers and quasiconvexity.** In the context of strictly quasiconvex integrands, there is a close connection between sufficiency results (whether extremals are minimising) and regularity of the extremal. One of the early results in the quasiconvex setting is due to Zhang [47], who showed that a $C^2$ extremal is absolutely minimising on small balls $B \subset \Omega$. In the opposite direction, it was shown by Kristensen & Taheri in [30, Theorem 4.1] that if $u \in W^{1,p} \cap W^{1,q}_{\text{loc}}$ is a $W^{1,q}_{\text{loc}}$-local minimiser for some $1 \leq q \leq \infty$, then we can establish a partial regularity theorem (we refer the reader to the aforementioned paper for the precise terminology and results).

It was moreover established in [30, Theorems 6.1, 7.1] that if $u$ is a Lipschitz extremal with strictly positive second variation (it is a weak local minimiser) then it is minimising among perturbations such that $[\nabla \varphi]_{\text{BMO}(\Omega)}$ is small, but that this is too weak to infer improved regularity though counterexamples. The former statement uses a modular version of the Fefferman-Stein inequality which we also use (see Section 3.1), and the latter follows by adapting the construction of Müller & Šverák [39]. For Lipschitz weak local minimisers however, it was shown by Campos Cordero [7] that we can infer global regularity if we additionally assume that $\nabla u \in \text{VMO}(\Omega)$. The proof loosely follows the compensated compactness argument used in [30, Section 4].

We see that Corollary 1.4 generalises the above result in [7], by removing the condition on the second variation and allowing $F$ to merely satisfy a strict Legendre-Hadamard condition (H2). Here the Legendre-Hadamard condition can be seen to be a natural relaxation in the following sense: it is proved by Kristensen [28] that (H2) implies that $F$ is locally quasiconvex in the sense that for each $x_0 \in \mathbb{R}^N$ there exists a quasiconvex function $G$ such that $F = G$ in a neighbourhood of $x_0$. Our argument, which builds upon ideas of Moser [38], streamlines this process by establishing regularity directly. In particular we note that the same Fefferman-Stein estimate used for the BMO-sufficiency result in [30] is crucially used to obtain a Caccioppoli-type inequality in [38] and Section 2.2.

1.3. **Basic notation.** We will briefly fix some notation that will be used throughout the text. We will equip $\mathbb{R}^n$ with the Lebesgue measure $\mathcal{L}^n$, and if $A \subset \mathbb{R}^n$ is non-empty and open such that $\mathcal{L}^n(A) < \infty$, for any $f \in L^1(A, \mathbb{R}^k)$ with $k \geq 1$ we define

\begin{equation}
(f)_A := \frac{1}{\mathcal{L}^n(A)} \int_A f \, dx.
\end{equation}

We also denote by $B_R(x_0)$ the open ball in $\mathbb{R}^n$ centred at $x_0$ with radius $R$, and for $\Omega \subset \mathbb{R}^n$ open write $\Omega_R(x_0) = \Omega \cap B_R(x_0)$. We may write $B_R$, $\Omega_R$ respectively if the centre point $x_0$ is clear from context.
We will denote by $\mathbb{R}^{Nn}$ the space of $N \times n$ real matrices, which we equip with the inner product $z : w = \text{tr}(z^*w)$ and $\ell^2$-norm $|z|^2 = z : z$ for $z, w \in \mathbb{R}^{Nn}$. For a differentiable map $F: \mathbb{R}^{Nn} \rightarrow \mathbb{R}$ we define its derivative $F': \mathbb{R}^{Nn} \rightarrow \mathbb{R}^{Nn}$ as

$$F'(z)w = \frac{d}{dt} \bigg|_{t=0} F(z + tw),$$

and for a differentiable map $A: \mathbb{R}^{Nn} \rightarrow \mathbb{R}^{Nn}$ its derivative $A'(z)$ will be a linear map $\mathbb{R}^{Nn} \rightarrow \mathbb{R}^{Nn}$ at each $z \in \mathbb{R}^{Nn}$, defined by

$$A'(z)w = \frac{d}{dt} \bigg|_{t=0} A(z + tw).$$

If $F$ is $C^2$ this allows us to define $F''$, which satisfies $F''(z)v : w = F''(z)w : v$ for all $z, v, w \in \mathbb{R}^{Nn}$.

Additionally $C$ will denote a constant that may change from line to line, and if not specified in proofs they will depend only on the parameters the resulting estimate depends on.

2. Interior regularity for $F$-extremals

We begin by considering the interior regularity theory for solutions to the Euler-Lagrange system. While the techniques extend to the general case, we will present a detailed proof in this simplified setting first to illustrate the key ideas. We will refer to Section 3 for some auxiliary results, but since we only apply them on balls $B$ they can be obtained through simpler means.

2.1. Estimates for $F$. We will consider $F: \mathbb{R}^{Nn} \rightarrow \mathbb{R}$ satisfying Hypotheses 1.1, and fix $M > 0$. Since $F''(z)$ is uniformly continuous on compact subsets, there is $\Lambda_M > 0$ and a modulus of continuity function $\omega_M: [0, \infty) \rightarrow [0, 1]$ such that

$$|F''(z)| \leq \Lambda_M, \quad |F''(z) - F''(w)| \leq \Lambda_M \omega_M(|z - w|)$$

for all $z, w \in \mathbb{R}^{Nn}$ with $|z|, |w| \leq M + 1$. Here $\omega_M$ can be chosen to be a non-decreasing, continuous, and concave function such that $\omega_M(0) = 0$. Also since the strict Legendre-Hadamard condition holds uniformly on compact subsets, there is $\lambda_M > 0$ such that for all $z \in \mathbb{R}^{Nn}$ with $|z| \leq M$ we have

$$F''(z)(\xi \otimes \eta) : (\xi \otimes \eta) \geq \lambda_M |\xi|^2 |\eta|^2$$

for all $\xi \in \mathbb{R}^N$ and $\eta \in \mathbb{R}^n$. Now for $w \in \mathbb{R}^{Nn}$ with $|w| \leq M$, following Acerbi & Fusco [1] consider the shifted integrand

$$F_w(z) = F(z + w) - F(w) - F'(w)z.$$

Since $F''$ satisfies a Legendre-Hadamard condition, we infer $F$ is rank-one convex and so its derivative satisfies $|F'(z)| \leq \sqrt{NnK}(1 + |z|)^{q-1}$. Hence $F_w$ satisfies the growth conditions

$$|F_w(z)| \leq K_M(|z|^2 + |z|^q),$$

$$|F'_w(z)| \leq K_M(|z| + |z|^q - 1)$$

where

$$K_M = \Lambda_M + 2\sqrt{NnK},$$

using the mean value theorem and distinguishing between the cases when $|z| \leq 1$ and $|z| > 1$. A similar argument gives the comparison estimate

$$|F''_w(0)z - F''_w(z)| \leq K_M \omega_M(|z|)(|z| + |z|^q - 1).$$
2.2. Caccioppoli-type inequality. We now prove the following weakening of the Caccioppoli inequality of the second kind introduced by Evans in [13], which is a staple for many partial regularity proofs in the quasiconvex setting. The following estimate was essentially proved by Moser in [38], and involves applying the modular version of the estimate of Fefferman & Stein [14] established in Section 3.1 (see also Remark 2.2 at the end of this subsection).

Lemma 2.1 (Caccioppoli-type inequality). Suppose $F$ satisfies Hypotheses 1.1, and let $M > 0$. Then if $u$ is $F$-extremal in some ball $B_R(x_0) \subset \Omega$ such that $\nabla u \in \text{BMO}(B_R, \mathbb{R}^N)$ and $|[\nabla u]_{B_R}| \leq M$, then setting

\begin{equation}
(a(x) = (u)_{B_R} + (\nabla u)_{B_R} \cdot (x - x_0),
\end{equation}

there is $C = C(n, N, q, K_M/\lambda_M) > 0$ such that

\begin{equation}
\int_{B_{R/2}} |\nabla u - \nabla a|^2 \, dx \leq C \gamma \left( \int_{B_R} |\nabla u|_{\text{BMO}(B_R)} \right) \int_{B_R} |\nabla u - \nabla a|^2 \, dx + \frac{C}{R^2} \left( \int_{B_R} |u - a|^2 \, dx, \right.
\end{equation}

with $\gamma(t) : [0, \infty) \to [0, 1]$ a non-decreasing, continuous function such that $\gamma(0) = 0$, depending on $\omega_M$ and $\eta$ only.

Proof of Lemma 2.1. Set $\tilde{F}(z) = F_{\gamma_\eta}(z)$ as in (2.4), and fix a cutoff $\eta \in C_c^\infty(B_R)$ such that $\eta_{B_{R/2}} \leq \eta \leq 1_{B_R}$ and $|\nabla \eta| \leq \frac{C}{R}$. Putting $w = u - a$ we have $w$ is $\tilde{F}$-extremal since $u$ is $F$-extremal, and so testing the equation against $\phi = \eta^2 w$ gives

\begin{equation}
0 = \int_{B_R} \tilde{F}'(\nabla w) : \nabla (\eta^2 w) \, dx.
\end{equation}

Also since $\tilde{F}''(0) = F''(\nabla a)$ satisfies the strict Legendre-Hadamard condition (2.3) with $|\nabla a| \leq M$, applying this to $\eta w \in W^{1,2}_0(\Omega, \mathbb{R}^N)$ gives (see for instance [19, Theorem 10.1]),

\begin{equation}
\lambda_M \int_{B_R} (|\nabla (\eta w)|)^2 \, dx \leq \int_{B_R} \tilde{F}''(0)(\nabla (\eta w)) : \nabla (\eta w) \, dx.
\end{equation}

Taking the difference of (2.11), (2.12) and rearranging we get

\begin{align}
\lambda_M & \int_{B_R} (|\nabla (\eta w)|)^2 \, dx \\
\leq & \int_{B_R} \eta^2 \left( \tilde{F}''(0)(\nabla w) - \tilde{F}'(\nabla w) \right) : \nabla w \, dx \\
& + \int_{B_R} \tilde{F}''(0)(\nabla (\eta w)) : (2\nabla (\eta w) - w \nabla \eta) \, dx - \int_{B_R} \tilde{F}'(\nabla w) : (2\eta w \nabla \eta) \, dx \\
\leq & K_M \int_{B_R} \omega_M(|\nabla w|) \left( |\nabla w|^2 + |\nabla w|^q \right) \, dx \\
& + 4K_M \int_{B_R} |w \nabla \eta| \left( |w \nabla \eta| + |\nabla (\eta w)| + \eta |\nabla w|^{q-1} \right) \, dx,
\end{align}

where we have used the comparison estimate (2.8) to control the first term along with the fact that $\eta^2 \leq 1$, and the growth estimates (2.1), (2.6) for the additional terms. We apply the modular Fefferman-Stein estimate (Corollary 3.4) to the first term, noting that $\nabla w = \nabla u = (\nabla u)_{B_R}$ so

\begin{equation}
\int_{B_R} \omega_M(|\nabla u|) \left( |\nabla w|^2 + |\nabla w|^q \right) \, dx \leq C \omega_M \left( |\nabla u|_{\text{BMO}(B_R)} \right) \int_{B_R} |\nabla w|^2 + |\nabla w|^q \, dx.
\end{equation}
Combining these with the earlier estimate and using Young’s inequality to absorb the $|\nabla(\eta w)|^2$ term we arrive at

$$\int_{B_{R/2}} |\nabla w|^2 \, dx \leq C_M \omega_M \left( \lfloor |\nabla u|_{\text{BMO}(B_{R})} \rfloor \right) \int_{B_{R}} |\nabla w|^2 + |\nabla w|^q \, dx + \frac{C}{R^2} \int_{B_{R}} |w|^2 \, dx + C \int_{B_{R}} |\nabla w|^{2(q-1)} \, dx. \tag{2.15}$$

Note that if $q = 2$, we do not get the $|\nabla w|^{2(q-1)}$ term. Otherwise by the John-Nirenberg inequality (Proposition 3.2) we can bound

$$\int_{B_{R}} |\nabla w|^q \, dx \leq C \left\lfloor |\nabla u|_{\text{BMO}(B_{R})} \right\rfloor \int_{B_{R}} |\nabla w|^2 \, dx, \tag{2.16}$$

$$\int_{B_{R}} |\nabla w|^{2(q-1)} \, dx \leq C \left\lfloor |\nabla u|_{\text{BMO}(B_{R})} \right\rfloor \int_{B_{R}} |\nabla w|^2 \, dx. \tag{2.17}$$

Therefore if we let $\gamma(t) = \min \{1, (\omega_M(t)(1 + t^{q-2}) + t^{2(q-2)})\}$ (omitting the $t^{q-2}$ terms if $q = 2$) we deduce that

$$\int_{B_{R/2}} |\nabla w|^2 \, dx \leq C \gamma \left( \left\lfloor |\nabla u|_{\text{BMO}(B_{R})} \right\rfloor \right) \int_{B_{R}} |\nabla w|^2 \, dx + \frac{C}{R^2} \int_{B_{R}} |w|^2 \, dx, \tag{2.18}$$
as required. \qed

**Remark 2.2.** We have referred to Section 3.1 for the John-Nirenberg and modular Fefferman-Stein estimates, however in the interior case they can also be deduced from the corresponding statements in the full space using a cutoff argument. We will omit the details, but the argument is similar to that found in [38]; in this case the modular estimate can be proved more simply via a good-$\lambda$ estimate (see [30, Lemma 6.2]).

### 2.3. Harmonic approximation and interior regularity

Our second ingredient is a comparison estimate for solutions to the linearised system. The following duality argument is an adaptation of the estimate proved in [21]. The linear theory we need will straightforwardly follow from the strict Legendre-Hadamard condition satisfied by $F''(\nabla a)$, and we will refer to [19, Chapter 10] for details.

**Lemma 2.3** (Interior harmonic approximation). Suppose $F$ satisfies Hypotheses 1.1, and let $M > 0$. Then if $u$ is $F$-extremal in some ball $B_R(x_0) \subset \Omega$ such that $\nabla u \in \text{BMO}(B_R, \mathbb{R}^N)$ and $|\nabla u|_{B_R} \leq M$, defining $a$ affine by (2.9) we have the unique solution $h \in W^{1,2}(B_R, \mathbb{R}^N)$ to the problem

$$\begin{cases} - \text{div} F''(\nabla a) \nabla h = 0 & \text{in } B_R, \\ h = u - a & \text{on } \partial B_R, \end{cases} \tag{2.19}$$
satisfies the $L^2$ estimate

$$\int_{B_{R}} |\nabla h|^2 \, dx \leq C \int_{B_{R}} |\nabla u - \nabla a|^2 \, dx \tag{2.20}$$

with $C = C(n, N, \lambda_M, M) > 0$, and further the comparison estimate

$$\frac{1}{R^2} \int_{B_{R}} |u - a - h|^2 \, dx \leq C \gamma \left( \left\lfloor |\nabla u|_{\text{BMO}(B_{R})} \right\rfloor \right) \int_{B_{R}} |\nabla u - \nabla a|^2 \, dx, \tag{2.21}$$

with $C = C(n, N, q, K_M, \lambda_M)$ and some $\gamma : [0, \infty) \to [0, 1]$ increasing and continuous such that $\gamma(0) = 0$, depending on $\omega_M$ and $q$ only.
Proof. Put \( w = u - a \) and \( \tilde{F} = F_{\nabla a} \). Then the existence of a unique \( h \in W^{1,2}_w(B, \mathbb{R}^N) \) follows from \( L^2 \)-coercivity of \( \tilde{F}''(0) \) (see [19, Theorem 10.1]) which gives (2.20). Then for any \( \phi \in W^{1,2}_0(B, \mathbb{R}^N) \) we have
\[
\int_{B_R} \tilde{F}''(0)(\nabla w - \nabla h) : \nabla \phi \, dx = \int_{B_R} \left( \tilde{F}''(0)\nabla w - \tilde{F}'(\nabla w) \right) : \nabla \phi \, dx \leq K_M \int_{B_R} \omega_M(|\nabla w|) \left( |\nabla w| + |\nabla w|^{q-1} \right) |\nabla \phi| \, dx,
\]
where we have used the fact that \( w \) is \( \tilde{F} \)-extremal the comparison estimate (2.8). Now choose \( \phi \) to be the unique solution in \( W^{1,2}_0 \cap W^{2,2}_0(B, \mathbb{R}^N) \) to the problem
\[
- \text{div} \tilde{F}''(0)\nabla \phi = w - h
\]
in \( B_R \) (see [19, Theorem 10.3]), so in particular by symmetry of \( \tilde{F}''(0) \) this satisfies
\[
\int_{B_R} \tilde{F}''(0)(\nabla w - \nabla h) : \nabla \phi \, dx = \int_{B_R} |w - h|^2 \, dx.
\]
Moreover \( \phi \) satisfies a \( W^{2,2} \) estimate which combined with the Poincaré-Sobolev inequality (noting \( (\nabla \phi)_{B_R} = 0 \)) gives
\[
||\nabla \phi||_{L^2(B_R)} \leq C(n) ||\nabla^2 \phi||_{L^2(B_R)} \leq C \|w - h\|_{L^2(B_R)}
\]
where \( 2^* = \frac{2n}{n-2} \) provided \( n > 2 \). For this choice of \( \phi \), applying Hölder’s inequality and rearranging (2.22) using (2.25) we get
\[
\int_{B_R} |w - h|^2 \, dx \leq \frac{C}{R^2} \omega_M((|\nabla w|)) ||\nabla w||_{L^n(B_R)}^2 \int_{B_R} |\nabla w|^2 + |\nabla w|^{2(q-1)} \, dx.
\]
If \( n = 2 \) we use the fact that \( ||\nabla \phi||_{L^4(B_R)} \leq CR\|w - h\|_{L^2(B_R)} \) to get the slightly modified estimate
\[
\int_{B_R} |w - h|^2 \, dx = \frac{C}{R} \omega_M((|\nabla w|)) ||\nabla w||_{L^4(B_R)}^2 \int_{B_R} |\nabla w|^2 + |\nabla w|^{2(q-1)} \, dx.
\]
In both cases since \( \omega_M \leq 1 \) is concave by Jensen’s inequality we have
\[
\omega_M((|\nabla w|)) ||\nabla w||_{L^p(B_R)} \leq R^{\frac{p}{q-2}} \left( \int_{B_R} \omega_M((|\nabla w|)) \, dx \right)^{\frac{q-2}{q}} \leq R^{\frac{p}{q-2}} \omega_M\left( (|\nabla w|)_{\text{BMO}(B_R)} \right)^{\frac{q-2}{q}},
\]
and by the John-Nirenberg estimate (Proposition 3.2) we can also estimate
\[
\int_{B_R} |\nabla w|^{2(q-1)} \, dx \leq C \|u\|_{\text{BMO}(B_R)}^{2(q-2)} \int_{B_R} |\nabla w|^2 \, dx.
\]
Putting everything together the result follows by taking \( \gamma(t) = \min\{1, \omega_M(t)^{\frac{q}{q-2}} (1 + t^{2(q-2)})\} \), modified suitably if \( n = 2 \). \( \Box \)

From here Theorem 1.2 follows by combining the above estimate to get a suitable decay estimate, which can be applied iteratively. This approach is standard among many partial regularity proofs, and we follow a similar argument to that found in [21].

Proof of Theorem 1.2. We will begin by establishing the following decay estimate for the excess energy
\[
E(x, r) = \int_{B_r(x)} |\nabla u - (\nabla u)_{B_r(x)}|^2 \, dy.
\]

Claim: For any \( B_r(x) \subset B_R(x_0) \) and \( \sigma \in (0, \frac{1}{4}) \) for which \( ||(\nabla u)_{B_{2r}(x)}|| \leq ||(\nabla u)_{B_r(x)}|| \leq 2^{n+1}M \), we have
\[
E(x, \sigma r) \leq C \left( \sigma^2 + \sigma^{-(n+2)} \gamma \left( (\nabla u)_{\text{BMO}(B_r(x))} \right) \right) E(x, r),
\]
where $C = C(n, N, q, K_M, \lambda_M) > 0$ and $\gamma$ satisfies both Lemmas 2.1 and 2.3 with $2^{n+1}M$ in place of $M$.

Indeed this follows by applying the harmonic approximation result (Lemma 2.3) in $B_r(x)$ to find $h \in W^{1,2}_{u-a_0} (B_r(x), \mathbb{R}^N)$ solving
\begin{equation}
- \text{div} F'(\nabla a_0) \nabla h = 0
\end{equation}
in $B_r(x)$ with $a_0(y) = (u)_{B_r(x)} + (\nabla u)_{B_r(x)} \cdot (y-x)$, which satisfies
\begin{equation}
\frac{1}{r^2} \int_{B_r(x)} |u - a_0 - h|^2 \, dy \leq \gamma \left( [\nabla u]_{\text{BMO}(B_{r}(x_0))} \right) E(x, r).
\end{equation}

Now using the estimates (2.36) \text{and} (2.37) we can estimate
\begin{equation}
E(x, \sigma r) \leq \frac{1}{\sigma^{n+2}\gamma} \int_{B_{\sigma r}(x)} |u - a_{\sigma}|^2 \, dy + C \left( [\nabla u]_{\text{BMO}(B_r(x_0))} \right) E(x, 2\sigma r).
\end{equation}

So the claim follows by combining the above two estimates.

We now iteratively apply the claim for suitably chosen parameters. Since $|(\nabla u)|_{B_R(x_0)} \leq M$, for all $x \in B_{R/2}(x_0)$ we have $|(\nabla u)|_{B_{R/2}(x)} \leq 2^n M$ and so
\begin{equation}
|(\nabla u)|_{B_{R/2}(x)} \leq |(\nabla u)|_{B_R(x)} + |(\nabla u)|_{B_{R/2}(x)} \leq 2^n M + \sigma^{-n} E(x, R/2).
\end{equation}

Iteratively applying this therefore gives
\begin{equation}
|(\nabla u)|_{B_{R/2}(x)} \leq 2^n M + \sigma^{-n} \sum_{j=0}^{k-1} E(x, \sigma^j R/2).
\end{equation}

Since $E(x, r) \leq [\nabla u]^2_{\text{BMO}(B_{r}(x_0))} \leq \varepsilon^2$ for all $r < R/2$, see that if $\sigma^{-n} \varepsilon^2 \leq 2^n M$, we can apply the claimed decay estimate (2.31) to obtain
\begin{equation}
E(x, \sigma R) \leq C \left( \sigma^2 + \sigma^{-n-2} \gamma(\varepsilon) \right) E(x, R/2).
\end{equation}
Fix $\alpha \in (0, 1)$, and choose $\sigma \leq \frac{1}{2}$ such that $C\sigma^2 \leq \frac{1}{2} \sigma^{2\alpha}$. Then we can take $\varepsilon > 0$ small enough so $C\sigma^{-(n+2)}\gamma(\varepsilon) \leq \frac{1}{2} \sigma^{2\alpha}$ and $\sigma^{-n}\varepsilon^2 \sum_j \sigma^{\alpha j} \leq 2^n M$. Then we can inductively check that (2.31) gives
\begin{equation}
E(x, \sigma^k R/2) \leq \sigma^{2\alpha k} E(x, R/2),
\end{equation}
and by (2.39) we can ensure
\begin{equation}
|\nabla u|_{B_{\sigma^k R/2}(x)} \leq 2^{n+1} M
\end{equation}
for each $k \geq 1$. Hence for each $r \in (0, R/2)$, choosing $k$ such that $\sigma^k R/2 \leq r < \sigma^{k-1} R/2$ we deduce that
\begin{equation}
E(x, r) \leq C \left( \frac{r}{R} \right)^{2\alpha} E(x_0, R).
\end{equation}
This verifies the Campanato-Meyers characterisation of Hölder continuity (see for instance [19, Theorem 2.9]), allowing us to conclude that $u \in C^{1,\alpha}(\overline{B_{R/2}(x_0)}, \mathbb{R}^N)$ as required. \hfill \Box

\section{Preliminaries for boundary regularity}
Before we consider the boundary case, we will collect some technical results which will be used in our subsequent regularity proofs. While these results are largely known, some care was needed in keeping track of the associated constants.

\subsection{BMO in domains.}
We will review some preliminary results about BMO functions and fix our conventions. For any $D \subset \mathbb{R}^n$ open, we define the \textit{Fefferman-Stein maximal function} associated to $f \in L^1_{\text{loc}}(D, \mathbb{R}^N)$ as
\begin{equation}
\mathcal{M}^D f(x) = \sup_{x \in B \subset D} \int_B |f - (f)_B| \, dy,
\end{equation}
where we are taking the supremum over balls $B$. Using this we can define the John-Nirenberg space $\text{BMO}(D, \mathbb{R}^N)$ of functions of bounded mean oscillation in $D$ as the space of $f \in L^1_{\text{loc}}(D, \mathbb{R}^N)$ for which $\mathcal{M}^D f \in L^\infty(D)$. We equip this space with the seminorm $[f]_{\text{BMO}(D)} = \|M^D f\|_{L^\infty(D)}$.

While we wish to apply the results in this section to domains which are piecewise $C^{1,\beta}$, in order to understand the dependence of constants on the domain $D$ it will be convenient to work with \textit{John domains}; these were first introduced by JOHN [24] and later named by MARTIO & SARVAS [33]. The definition given here is slightly different to what appeared in the original papers, but can be found for instance in [42].

\begin{definition}
For $\delta \in (0, 1)$ we say bounded domain $D \subset \mathbb{R}^n$ is a $\delta$-\textit{John domain} if there exists $x_0 \in D$, called the \textit{John centre}, such that for all $x \in D$ there is a rectifiable curve $\gamma : [0, d] \to D$ parametrised by arclength such that $\gamma(0) = x$, $\gamma(d) = x_0$, and
\begin{equation}
\text{dist}(\gamma(t), \partial D) \geq \delta t
\end{equation}
for all $t \in [0, d]$.
\end{definition}

This can be viewed as a \textit{twisted cone condition}, and since bounded Lipschitz domains satisfy a uniform cone condition (see for instance [2, Section 4.4]) it follows that they are John domains. Moreover if $\Omega$ is a bounded Lipschitz domain, one can verify that there is $R_0 > 0$ and $\delta > 0$ such that $\Omega \cap B_R(x_0)$ is a $\delta$-John domain for all $x \in \overline{\Omega}$ and $0 < R < R_0$.

The first result we need is a global version of the John-Nirenberg inequality, which was proved in greater generality by Smith & Stegenga [43] and Hurri-Syrjänen [22]. We will sketch the proof to clarify the dependence of constants.
Proposition 3.2 (Global John-Nirenberg estimate [43, 22]). Suppose $D$ is a bounded $\delta$-John domain, and $f \in \text{BMO}(D, \mathbb{R}^{Nn})$. Then for all $1 \leq p < \infty$, there is $C = C(n, p, \delta) > 0$ such that

$$
\left( \int_D |f - (f)_D|^p \, dx \right)^{1/p} \leq C |f|_{\text{BMO}(D)}.
$$

Proof sketch. The strategy is to take a Whitney decomposition $W = \{Q_j\}$ of $D$ as given in [44, Section VI.1], and apply the John-Nirenberg inequality on each $Q_j$, which is easily adapted from the original argument in [25] (see also [19, Corollary 2.2]). To patch these local estimates we can use Whitney chains following [26] to show that

$$
\int_D |f - (f)_D|^p \, dx \leq C(n, p) \left( \mathcal{L}^n(D) + \int_D k_D(x_0, x)^p \, dx \right) |f|_{\text{BMO}(D)}^p,
$$

for a distinguished point $x_0 \in D$, where $k_D$ is the quasi-hyperbolic distance introduced in [16] defined by

$$
k_D(x_1, x_2) = \inf_{\gamma} \int_0^1 \frac{1}{\text{dist}(x, \partial\Omega)} \, dt,
$$

taking the infimum over all rectifiable curves $\gamma$ connecting $x_1, x_2 \in D$. To verify the integrability of $k_D(x_0, \cdot)^p$, letting $x_0$ be the John centre it is shown in [15] that for all $x \in D$,

$$
k_D(x_0, x) \leq \frac{1}{\delta} \log \frac{\text{dist}(x_0, \partial\Omega)}{\text{dist}(x, \partial\Omega)} + \frac{1}{\delta} \left( 1 + \log (1 + \delta^{-1}) \right).
$$

Using this and keeping track of constants in the proof of [42, Theorem 4] we have $k_D$ satisfies the integrability condition

$$
\int_D k_D(x_0, x)^p \, dx \leq C(n, p, \delta) \mathcal{L}^n(D),
$$

from which the result follows. \qed

We will also need an modular version of the Fefferman-Stein theorem [14, Theorem 5] that holds up to the boundary. This estimate in the full space appeared in the work of Kristensen & Taheri [30] where is was proven by means of a good-$\lambda$ estimate, however to obtain estimates up to the boundary we will need a more refined approach using the extrapolation results of Cruz-Uribe, Martell & Pérez [9]. We will briefly recall the notions of $N$-functions considered in [9]; these are mappings $\Phi : [0, \infty) \to [0, \infty)$ which are continuous, convex, and strictly increasing such that

$$
\lim_{t \to 0^+} \frac{\Phi(t)}{t} = 0, \quad \lim_{t \to \infty} \frac{\Phi(t)}{t} = \infty.
$$

For such a $\Phi$ we can associate a conjugate function $\overline{\Phi}(t) = \sup_{s > 0} \{s t - \Phi(s)\}$, which can be shown to also be an $N$-function. We say $\Phi \in \Delta_2$ if there is $C > 0$ such that the doubling property $\Phi(2t) \leq C \Phi(t)$ holds, in which case the minimal $C$ will be denoted by $\Delta_2(\Phi)$. We also say $\Phi \in \nabla_2$ if $\Phi \in \Delta_2$ and write $\nabla_2(\Phi) = \Delta_2(\Phi)$; note this holds if there is $r > 0$ such that $\Phi(rt) \geq 2r \Phi(t)$ for all $t \geq 0$.

Proposition 3.3 (Modular Fefferman-Stein estimate). Let $D \subset \mathbb{R}^n$ be a bounded $\delta$-John domain, and $\Phi$ an $N$-function such that $\Phi \in \Delta_2 \cap \nabla_2$. Then there is $C = C(n, \delta, \Delta_2(\Phi), \nabla_2(\Phi)) > 0$ such that

$$
\int_D \Phi(|f - (f)_D|) \, dx \leq C \int_D \Phi \left( M^D_D f \right) \, dx
$$

for all $f \in L^1_{\text{loc}}(D, \mathbb{R}^{Nn})$ such that both sides are finite.
This result is essentially proved in the work of Diening, Růžička, & Schumacher [12] in greater generality, and we will sketch the main ideas behind their work. The key idea is to establish the Fefferman-Stein estimate in the weighted $L^p$ scales using ideas of Roman [4], and extend to the modular scales via extrapolation.

Proof. We first need a weighted $L^p$ estimate in $D$, so let $1 < p < \infty$ and $w \in A_p$. Then for any cube $Q$ we have (see for instance [12, Lemma 7.2])

$$
(3.10) \int_Q |f - (f)_Q|^p w \, dx \leq C \left( n, p, [w]_{A_p} \right) \int_Q |\mathcal{M}^p(f)|^p w \, dx
$$

for all $f \in L^p(Q, w, \mathbb{R}^{Nn})$, that is $f : Q \to \mathbb{R}^{Nn}$ such that $|f|^p w$ is integrable on $Q$. To extend this to John domains we can apply [23, Theorem 3]; note it is proved in [4, Lemma 2.1] that a $\delta$-John domain $D$ is a $F(\sigma, N)$-domain as in [23], where $\sigma = \min \left\{ \frac{m}{N}, \frac{n+1}{n} \right\}$ and $N = N(n, \delta)$. Hence the argument shows that

$$
(3.11) \int_D |f - (f)_{Q_0}|^p w \, dx \leq C \left( n, p, [w]_{A_p}, \delta \right) \int_D |\mathcal{M}^p_D(f)|^p w \, dx
$$

for all $1 < p < \infty$ and $w \in A_p$, for a distinguished cube $Q_0 \subset D$. Then applying the modular extrapolation theorem in [9] (for a detailed proof see [10, Chapter 4]) to the family of pairs $((f - f_{Q_0})_D, |\mathcal{M}^p_D(f)|_D)$ we obtain

$$
(3.12) \int_D \Phi(|f - (f)_{Q_0}|) \, dx \leq C(n, \delta, \Delta_2(\Phi), \nabla_2(\Phi)) \int_D \Phi \left( |\mathcal{M}^p_D(f)| \right) \, dx.
$$

Replacing the average $(f)_{Q_0}$ by $(f)_D$ using the doubling property and convexity of $\Phi$, the results follows. \[ \square \]

We wish to apply this result to $\Phi(t) = \omega(t)^p$ with $p > 1$, where $\omega : [0, \infty) \to [0, 1]$ is a continuous, non-decreasing, concave function such that $\omega(0) = 0$ as in Section 2.1. A technical complication arises as this need not be convex in general, but we can work with a modified $\tilde{\Phi}$ which is convex instead. This follows from the works of Kokilashvili & Krbec [27]; since $\omega$ is increasing we have $\Phi(t) \leq \frac{1}{2a} \Phi(at)$ with $a = 2^{\frac{1}{p-1}}$, and so by [27, Lemmas 1.1.1, 1.2.3] we get

$$
(3.13) \tilde{\Phi}(t) = \frac{1}{a} \int_0^t \sup_{0 < \tau < s} \left( \omega(\tau)^{p-1} \right) \, ds
$$

is convex and increasing on $[0, \infty)$ satisfying

$$
(3.14) \tilde{\Phi}(t) \leq \Phi(t) \leq \tilde{\Phi}(2at)
$$

for all $t \geq 0$. Further since $\Phi$ satisfies $\Phi(2t) \leq 2^{p+1} \Phi(t)$ and $\Phi(at) \geq 2a \Phi(t)$ we can infer that $\tilde{\Phi}(2t) \leq 2^{p+1} \tilde{\Phi}(t)$ and $\tilde{\Phi}(at) \geq 2a \tilde{\Phi}(t)$ also, so $\tilde{\Phi} \in \Delta_2 \cap \nabla_2$ and the associated constants can be chosen to depend on $p$ only. Applying the above result to $\tilde{\Phi}$ and using (3.14) we deduce the following.

**Corollary 3.4.** Suppose $D \subset \mathbb{R}^n$ is a bounded $\delta$-John domain, $1 < p < \infty$, and $\omega : [0, \infty) \to [0, 1]$ is non-decreasing, continuous, concave with $\omega(0) = 0$. Then if $f \in \text{BMO}(D, \mathbb{R}^{Nn})$, for each $1 < p < \infty$ there is $C = C(n, p, \delta) > 0$ such that

$$
(3.15) \int_D \omega(|f - (f)_D|) |f - (f)_D|^p \, dx \leq C \omega \left( [f]_{\text{BMO}(D)} \right) \int_D |f - (f)_D|^p \, dx.
$$
3.2. Localisation near the boundary. For the Caccioppoli-type estimate in the interior (Lemma 2.1), our strategy involved testing the equation against $\phi = \eta(u-a)$ with $\eta$ a cutoff and $a$ an affine approximation to $u$ in a ball. This will need to be modified for the boundary case to ensure our test function $\phi$ vanishes on $\partial\Omega$. In this section we collect the necessary technical ingredients to construct a suitable replacement function, using ideas of Kronz [32] along with the refinements of Campos Cordero [7].

Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,\beta}$ domain, that is, $\partial\Omega$ can locally be written as the graph of a $C^{1,\beta}$ function in the following sense; for all $x_0 \in \partial\Omega$, there is $R_0 > 0$ and a unit vector $\nu_{x_0} \in \mathbb{R}^n$ such that letting $T_{x_0} = (\nu_{x_0})^\perp$ denote the orthogonal complement, there is a map

$$\gamma: T_{x_0} \cap B_{R_0} \to \mathbb{R}$$

which is of class $C^{1,\beta}$ such that we have $\nabla\gamma(0) = 0$ and

$$\Omega \cap B_{R_0}(x_0) = B_{R_0}(x_0) \cap \{x_0 + y + \lambda \nu : y \in T_{x_0} \cap B_{R_0}, \lambda \leq \gamma(y)\},$$
$$\partial\Omega \cap B_{R_0}(x_0) = B_{R_0}(x_0) \cap \{x_0 + y + \gamma(y)\nu : y \in T_{x_0} \cap B_{R_0}\}.$$  

This implies there is an outward facing unit normal $\nu_{\partial\Omega}$ given by $\nu_{\partial\Omega}(x_0) = \nu_{x_0}$ at each $x_0 \in \partial\Omega$. This also allows us to construct a defining function $\rho = \rho_\Omega \in C^{1,\beta}(\mathbb{R}^n)$ with the property that

$$\Omega = \{x \in \mathbb{R}^n : \rho(x) < 0\}, \quad \mathbb{R}^n \setminus \overline{\Omega} = \{x \in \mathbb{R}^n : \rho(x) > 0\},$$

and such that $\nabla\rho(x) \neq 0$ in $\partial\Omega$, by locally defining $\rho(x) = (|x - x_0|, \nu_{\partial\Omega}(x_0) - \gamma(x - x_0) \in B_{R_0}(x_0))$ and patching using a partition of unity. Note that $\nabla\rho(x)$ is normal to $\partial\Omega$ at each $x \in \partial\Omega$, so we have $\nu_{\partial\Omega}(x) = \frac{\nabla\rho(x)}{|\nabla\rho(x)|}$. We also define the associated $C^{1,\beta}$-constant of $\Omega$ as

$$\|\Omega\|_{C^{1,\beta}} = \inf \left\{ \sup_{1 \leq j \leq N} \|\nabla\gamma_j\|_{C^{0,\beta}(T_{x_j} \cap B_{R_j}(x_j))} \right\},$$

where the infimum is taken over collections $\{\gamma_j, x_j, R_j\}_{j=1}^N$ where $\{B_{R_j}(x_j)\}$ covers $\partial\Omega$ and each $\Omega \cap B_{R_j}(x_j)$ is represented as the graph of the $C^{1,\beta}$ function $\gamma_j$.

The idea is to use this defining function $\rho$ as a replacement for the affine approximation, considering maps of the form

$$a(x) = \xi \frac{\rho(x)}{|\nabla\rho(x)|},$$

with $\xi \in \mathbb{R}^N$. Since $\nabla a = \xi \otimes \frac{\nabla\rho(x)}{|\nabla\rho(x)|}$ which is close to $\xi \otimes \nu_{x_0}$, however, taking $\xi = (\nabla v \cdot \nu_{x_0})\Omega_{R}(x_0)$ only allows us to control the normal component compared to the full derivative $\nabla a = (\nabla u)_{B_{R_0}(x_0)}$ from the interior case. It turns out this is sufficient however; this is illustrated by the following result, which is an adaptation of an observation of Campos Cordero [7].

Lemma 3.5. Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,\beta}$ domain and let $p > \frac{2}{\beta}$. There is $R_0 > 0$ and $C > 0$ such that for all $x_0 \in \partial\Omega$ and $0 < R < R_0$, for all $v \in W^{1,p}(\Omega_R(x_0), \mathbb{R}^N)$ such that $v = 0$ on $\partial\Omega \cap B_{R}(x_0)$ we have

$$\left(\int_{\Omega_{R}(x_0)} |\nabla v - (\nabla v \cdot \nu_{x_0})\Omega_{R}(x_0) \otimes \nu_{x_0}|^p \, dx\right)^{\frac{1}{p}} \leq C \left(\int_{\Omega_{R}(x_0)} |\nabla v|^p \, dx\right)^{\frac{1}{p}} + C(|\nabla v|_{\Omega_{R}(x_0)})R^\beta.$$
Proof. Fix \( x_0 \in \partial \Omega \), then by translating and rotating we can assume \( x_0 = 0 \) and \( \nu(x_0) = e_n \), and take \( R_0 > 0 \) small enough so we can write \( \Omega_{R_0}(x_0) \) as the graph of some \( \gamma \). We have

\[
\left( \int_{\Omega} |\nabla v - (\nabla_n v)_{\Omega_R} \otimes e_n|^p \, dx \right)^{\frac{1}{p}} \leq C \left( \int_{\Omega} |\nabla v - (\nabla_n v)_{\Omega_R}|^p \, dx \right)^{\frac{1}{p}} + \sum_{i=1}^{n-1} |(\partial_i v)_{\Omega_R}|,
\]

where we write \( \nabla_j v = \nabla v \cdot e_j \), so we need to estimate the tangential derivatives. We proceed analogously to [7, Lemma 5.6] with minor modifications to account for the curved boundary, so letting \( \rho \) be the defining function for \( \Omega \) as above we consider

\[
\tilde{v}(x) = v(x) - (\nabla_n v)_{\Omega_R} \frac{\rho(x)}{\sqrt{\rho(0)}}.
\]

Note that \( \tilde{v} \) still vanishes on \( \partial \Omega \cap B_R \), so writing \( x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \), so a similar argument to [7] gives

\[
\int_{\Omega} \nabla \tilde{v} \, dx = \int_{\Omega \cap \partial B_R} \tilde{v}(x) \frac{2}{R} (\mathcal{H}^{n-1}(x)) = \int_{\Omega} \nabla_n v(x) \frac{x_1}{(R^2 - |x'|^2)^\frac{3}{2}} \, dx,
\]

where the only difference is that \( \tilde{v} \) vanishes at \( (x', \gamma(x')) \) writing \( x = (x', x_n) \). This can then be estimated using Hölder’s inequality as in [7], and using the fact that \( \rho \) is of class \( C^{1,\beta} \) we deduce that

\[
\left| \int_{\Omega} \nabla_i v \, dx \right| \leq \left| \int_{\Omega} \nabla_i \tilde{v} \, dx \right| + |(\nabla_n v)_{\Omega_R} | \frac{|(\nabla_i \rho)_{\Omega_R}|}{|\rho(0)|} \leq C(n, p) \left( \int_{\Omega} |\nabla_n v - (\nabla_n v)_{\Omega_R}|^p \, dx \right)^{\frac{1}{p}} + C(n, p |\Omega|) |(\nabla_n v)_{\Omega_R} | R^2,
\]

from which the result follows. \( \square \)

3.3. Reference estimates up to the boundary. We will also need some reference estimates for linear elliptic systems for the harmonic approximation step. We consider a linear mapping \( \Lambda : \mathbb{R}^{n} \rightarrow \mathbb{R}^{n} \) which is symmetric in the sense that \( v : \Lambda w = \Lambda v : w \), satisfying the uniform Legendre-Hadamard ellipticity condition

\[
\lambda |\xi|^2 |\eta|^2 \leq \Lambda(\xi \otimes \eta) : (\xi \otimes \eta) \leq \Lambda |\xi|^2 |\eta|^2
\]

holds for all \( \xi \in \mathbb{R}^N, \eta \in \mathbb{R}^n \) with \( \lambda > 0 \). By means of the Fourier transform one can infer that for any \( \Omega \subset \mathbb{R}^n \) open the estimate

\[
\int_{\Omega} |\nabla \phi|^2 \, dx \leq \frac{1}{\lambda} \int_{\Omega} \Lambda \nabla \phi : \nabla \phi \, dx
\]

holds for all \( \phi \in W^{1,2}_0(\Omega, \mathbb{R}^N) \), so the Lax-Milgram lemma gives the associated operator \( - \text{div}(\Lambda \nabla \cdot) : W^{1,2}_0(\Omega, \mathbb{R}^N) \rightarrow W^{-1,2}(\Omega, \mathbb{R}^N) \) is an isomorphism.

In the interior case we considered the same setting, but we used uniform and \( W^{2,2} \) estimates which could be found in many sources such as [19]. For boundary regularity we wish to establish analogous estimates for \( \Omega_R(x_0) \), however such domains are merely piecewise \( C^{1,\beta} \) which is too weak to expect estimates in those scales. To circumvent this we will need to replace \( \Omega_R \) by a suitably regular domain following an argument used by Kristensen & Mingione in [29], and obtain weakened estimates which will be sufficient for our purposes.

Lemma 3.6. Let \( \Omega \subset \mathbb{R}^n \) be a bounded \( C^{1,\beta} \) domain and let \( \Lambda \) be symmetric and uniformly Legendre-Hadamard elliptic as above. Then there is \( R_0 > 0 \) such that for each \( x_0 \in \partial \Omega \) and \( 0 < R < R_0 \), there exists a \( C^{1,\beta} \) domain \( \overline{\Omega}_R(x_0) = \Omega_R \) such that

\[
\overline{\Omega}_{R/2}(x_0) \subset \overline{\Omega}_R \subset \Omega_R(x_0),
\]
on which the following solvability results hold.
(i) If \( v \in W^{1,2}(\tilde{\Omega}_R(x_0)) \) such that \( v = 0 \) on \( \partial \Omega \cap \partial \tilde{\Omega}_R(x_0) \), the unique \( h \in W^{1,2}_v(\tilde{\Omega}_R(x_0)) \) solving

\[
\begin{cases}
- \text{div}(A \nabla h) = 0 & \text{in } \tilde{\Omega}_R(x_0), \\
h = v & \text{on } \partial \tilde{\Omega}_R(x_0),
\end{cases}
\]

is of class \( C^{1,\beta} \) in \( \tilde{\Omega}_R \cup (\partial \Omega \cap \partial \tilde{\Omega}_R(x_0)) \) with the associated estimate

\[
[\nabla h]_{C^{1,\beta}((\Omega_{R/2}(x_0)}) \leq C(n, N, \Lambda, \beta, \|\Omega\|_{C^{1,\beta}}) \left( \int_{\tilde{\Omega}_R} |\nabla h|^2 \, dx \right)^{1/2}.
\]

(ii) If \( 2 \leq p < \infty \) and \( F \in L^p(\Omega, \mathbb{R}^N) \), then there is a unique \( u \in W^{1,p}_0(\Omega, \mathbb{R}^N) \) solving

\[
\begin{cases}
- \text{div}(A \nabla u) = - \text{div} F & \text{in } \tilde{\Omega}_R(x_0), \\
u = 0 & \text{on } \partial \tilde{\Omega}_R(x_0),
\end{cases}
\]

which satisfies the estimate

\[
\int_{\tilde{\Omega}_R} |\nabla u|^p \, dx \leq C(n, N, p, \Lambda, \beta, \|\Omega\|_{C^{1,\beta}}) \int_{\tilde{\Omega}_R} |F|^p \, dx.
\]

**Proof.** Fix a \( C^{1,\beta} \) domain \( A \subset \mathbb{R}^n \) such that \( \mathbb{B}^{25}_A(0)^+ \subset A \subset B_1(0)^+ \). Using the graph representation above we can construct a diffeomorphism \( \psi: B_{R_0}(x_0) \to U \subset B_1(0) \) such that \( A \subset U \), \( \psi(B_{R_0} \cap \Omega) = U \cap \mathbb{R}^n_+ \), and such that \( D \psi(x_0) \) is orthogonal. Hence by shrinking \( R_0 \) if necessary we can assume that

\[
B_{\frac{3 R_0}{4}}(0) \subset \psi(B_{R_0}(x_0)) \subset B_{\frac{25}{24} R_0},
\]

for all \( R \in (0, R_0) \). Hence if we let \( \tilde{\Omega}_R = \psi^{-1}\left( \frac{18 R}{25 R_0} A \right) \) this satisfies,

\[
\tilde{\Omega}_{R/2} \subset \psi^{-1}\left( \frac{B_{\frac{3 R_0}{4}}(0)}{2^{5/4} R_0} \right) \subset \tilde{\Omega}_R \subset \psi^{-1}\left( \frac{B_{\frac{18 R}{25 R_0}}(0)^+}{2} \right) \subset \Omega_R,
\]

as claimed. Now if \( \varphi \in W^{1,2}(\tilde{\Omega}_R, \mathbb{R}^N) \), setting \( \tilde{\varphi} = \varphi \circ \psi^{-1} \) we have for \( \psi(y) = x \) that

\[
- \text{div}(A \nabla \varphi) = - \text{div}(A \nabla \tilde{\varphi})
\]

where we define

\[
\tilde{A}(y)v : w = |\det(\nabla \psi(y))|^{-1} A(\nabla \psi(x)v) : (\nabla \psi(x)w)
\]

for \( y = \psi(x) \) and all \( v, w \in \mathbb{R}^N \). We can check \( \tilde{A} \) is Legendre-Hadamard elliptic and \( \beta \)-Hölder continuous with constants depending on \( n, \lambda, \Lambda \) and \( \|\Omega\|_{C^{1,\beta}} \), noting \( \nabla \psi \in C^{0,\beta} \) with bounded inverse. Hence (i) and (ii) follow by analogous estimates on \( A_R := \frac{18 R}{25 R_0} A \) applying the classical Schauder and Calderón-Zygmund estimates respectively; see for instance Theorems 10.12, 10.17 in [19] for details.

**Remark 3.7.** The second estimate (ii) replaces \( W^{2,2} \) estimates by weaker bounds in \( W^{1,p} \), which suffices for our application. We will apply this with \( f \in L^2(\tilde{\Omega}_R(x_0), \mathbb{R}^N) \) by using the Newtonian potential to define

\[
F = -\frac{1}{n \omega_n} \int_{\tilde{\Omega}_R(x_0)} f(y) \frac{x - y}{|x - y|^n} \, dx,
\]

which satisfies \( - \text{div} F = f \chi_{\tilde{\Omega}_R(x_0)} \) in \( \mathbb{R}^n \). By standard potential estimates (see for instance Lemmas 7.12, 7.14, and Theorem 9.9 in [18]) we have \( C = C(n, p) \) such that

\[
\|F\|_{L^p(\tilde{\Omega}_R(x_0))} \leq CL^n \left( \tilde{\Omega}(x_0) \right)^{\frac{1}{2} + \frac{1}{p} - \frac{1}{2} (\|\tilde{f}\|_{L^2(\tilde{\Omega}_R(x_0))})^2,
\]

provided \( \frac{1}{2} - \frac{1}{p} \leq \frac{1}{n} \) with \( 1 \leq p < \infty \), which puts us in the setting of the above lemma.
Finally we conclude by stating a Poincaré inequality we will use extensively later. For the case of the modified domain, this follows by flattening the boundary and rescaling $A_1$.

**Lemma 3.8 (Poincaré inequality).** Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,\beta}$ domain and let $R_0 > 0$, $\Omega_R(x_0)$ as in Lemma 3.6 above. Then for all $x_0 \in \partial \Omega$, $0 < R < R_0$, $1 < p < \infty$, for $u \in W^{1,p}(\Omega_R(x_0), \mathbb{R}^N)$ such that $u = 0$ on $\partial \Omega \cap B_R(x_0)$ in the trace sense we have

\[
R^{\frac{n}{p} - \frac{n}{q} - 1} \|u\|_{L^q(\Omega_R)} \leq C \|\nabla u\|_{L^p(\Omega_R)}
\]

for all $1 \leq q < \infty$ such that $\frac{1}{p} - \frac{1}{q} \leq \frac{\beta}{n}$, with $C = C(n, p, q, \|\Omega\|_{C^{1,\beta}}) > 0$. Also the same conclusion holds for $\Omega_R(x_0)$ in place of $\Omega_R(x_0)$.

4. Regularity up to the boundary for $F$-extremals

We now use the results from the previous section to prove Theorem 1.3. The framework will be analogous to the interior regularity theory, involving establishing a Caccioppoli-type inequality and a harmonic approximation result.

We will continue to use the notation introduced in Section 2.1. Additionally, given a bounded $C^{1,\beta}$ domain $\Omega \subset \mathbb{R}^n$, we will fix $R_0 > 0$ and $\delta \in (0, 1)$ such that $\Omega_R(x_0)$ is a $\delta$-John domain for all $x_0 \in \partial \Omega$, $0 < R < R_0$, and given $\rho$ as above we will also assume that we have $\mathcal{L}^n(\Omega_R(x_0)) \geq 4^{-n} \mathcal{L}^n(B_R(x_0))$ and

\[
C(n)^{-1} R^2 \leq \int_{\Omega_R(x_0)} \frac{\rho(x)^2}{\|\nabla \rho(x_0)\|^2} \, dx \leq C(n) R^2,
\]

for all $R < R_0$. Shrinking $R_0$ further if necessary, we will moreover assume Lemmas 3.5, 3.6, and 3.8 from the previous section hold with this choice of $R_0$.

4.1. Boundary Caccioppoli-type inequality.

**Lemma 4.1 (Boundary Caccioppoli-type inequality).** Suppose $F$ satisfies Hypotheses 1.1, let $M \geq 1$, and suppose $\Omega \subset \mathbb{R}^n$ is a bounded $C^{1,\beta}$ domain for some $\beta \in (0, 1)$. Given $g \in C^{1,\beta}(\Omega, \mathbb{R}^N)$, there is $R_0 = R_0(n, \Omega) > 0$ such that the following holds. Suppose $x_0 \in \partial \Omega$, $0 < R < R_0$, and $u \in W^{1,q}(\Omega, \mathbb{R}^N)$ is $F$-extremal in $\Omega_R(x_0)$ such that $\nabla u \in \BMO(\Omega_R(x_0), \mathbb{R}^N)$, $|\nabla u|_{\BMO(\Omega_R(x_0))} \leq 1$, and $(|\nabla u|_{\Omega_R}) \leq M$. Then if we define

\[
a_R(x) = \xi_R \frac{\rho(x)}{|\nabla \rho(x)|} = \frac{(u - g)\rho|_{\Omega_R}}{\rho^2|_{\Omega_R}} \rho(x),
\]

with $\rho$ the defining function for $\Omega$ as in Section 3.2, we have the estimate

\[
\int_{\Omega_{R/2}} |\nabla u - (\nabla u)|_{\Omega_{R/2}}^2 \, dx \leq C \left( |\nabla u|_{\BMO(\Omega_{R/2})} \right) \int_{\Omega_R} |\nabla u - (\nabla u)|_{\Omega_R}^2 \, dx + \frac{C}{R^2} \int_{\Omega_R} |u - g|_{\Omega_R}^2 \, dx + CM^{2(q-1)}R^{2\beta},
\]

where setting $\bar{M} = C(n, \beta, \|\Omega\|_{C^{1,\beta}}, |\nabla u|_{C^{0,\beta}(\Omega)})M$, $\gamma: [0, \infty) \to [0, 1]$ is a non-decreasing continuous function satisfying $\gamma(0) = 0$ depending on $\bar{M}$ and $q$ only, and

\[
C = C(n, N, q, K_N, \lambda_K, \delta, \|\Omega\|_{C^{1,\beta}}, R_0, |\nabla g|_{C^{0,\beta}(\Omega)} > 0.
\]

The main technical obstruction is that we need a suitable test function $\phi$ vanishing on $\partial \Omega \cap B_R(x_0)$ in our coercivity estimates. We will achieve this without flattening the boundary, using ideas from CAMPOS CORDERO [6, Chapter 4] and results from Section 3.2.
Remark 4.2. The significance of this choice of $a_R(x)$ in (4.2) is that the quantity

$$
\xi \mapsto \int_{\Omega_R} \left| u - g - \xi \frac{\rho}{|\nabla \rho(x_0)|} \right|^2 \, dx
$$

is minimised in $\xi \in \mathbb{R}^N$ when $\xi = \xi_R$, as noted by KRONZ [32]. While this form will be useful for the final iteration argument, to establish this lemma it will be more convenient to consider $\xi = (\nabla(u - g) \cdot \nu(x_0))_{\Omega_R}$. These can be compared through the estimate

$$
|\xi_R - (\nabla(u - g) \cdot \nu(x_0))_{\Omega_R}| \leq C \left( \int_{\Omega_R} \left| \nabla(u - g) - (\nabla(u - g) \cdot \nu(x_0))_{\Omega_R} \otimes \frac{\nabla \rho}{|\nabla \rho(x_0)|} \right|^2 \, dx \right)^{1/2} + C M R^3,
$$

where $C = C(n, \beta, \|\Omega\|_{C^1, \alpha}) > 0$. This is proved in [32, Lemma 2(ii)], relying on the Poincaré inequality (Lemma 3.8) and (4.1).

Proof. Let $R_0 > 0$ as in the beginning of this section, and define

$$
w(x) = u(x) - g(x) - a_R(x),
$$

noting that $w = 0$ on $\partial \Omega \cap B_R(x_0)$. We also fix a cutoff $\eta \in C_0^\infty(B_R(x_0))$ such that $1_{B_{R/2}(x_0)} \leq \eta \leq 1_{B_R(x_0)}$ and $|\nabla \eta| \leq \frac{C}{R}$, and consider the shifted functional $\tilde{F}(z) = F_{z_R}(z)$ as in (2.4) where

$$
z_R = \xi_R \otimes x_0 + (\nabla g)_{\Omega_R},
$$

with $\xi_R$ as in (4.2). Using the Poincaré inequality (Lemma 3.8), we can choose $\tilde{M}$ so that

$$
|z_R| \leq C(n, \beta, \|\Omega\|_{C^1, \alpha}) \left( \int_{\Omega_R} |\nabla u - \nabla g|^2 \, dx \right)^{1/2} + (|\nabla g|_{\Omega_R}) \leq \tilde{M}.
$$

Now by the strict Legendre-Hadamard condition applied to $\eta w$ and testing the equation (1.1) against $\eta^2 w$ we have

$$
\lambda \tilde{M} \int_{\Omega_R} |\nabla (\eta u)|^2 \, dx \leq \int_{\Omega_R} \tilde{F}'(0)(\nabla (\eta u) : \nabla (\eta u) - \int_{\Omega_R} \tilde{F}'(\nabla u - z_R) : \nabla (\eta^2 w) \, dx
$$

$$
= \int_{\Omega_R} \eta \left( \tilde{F}'(0)(\nabla u - z_R) - \tilde{F}'(\nabla u - z_R) \right) : \nabla (\eta w) \, dx
$$

$$
+ \int_{\Omega_R} \eta \tilde{F}'(0)(z_R - \nabla a_R - \nabla g) : \nabla (\eta w) \, dx
$$

$$
+ \int_{\Omega_R} \eta \tilde{F}'(0)(\nabla u - z_R) \, \nabla (\eta w) \, dx - \int_{\Omega_R} \eta w \tilde{F}'(\nabla u - z_R) \, \nabla \eta \, dx.
$$

We can absorb the $\nabla (\eta w)$ terms using Cauchy-Schwarz and Young’s inequality; for the last term we can use the growth estimate (2.6) for $\tilde{F}'$ to estimate

$$
\int_{\Omega_R} \eta w \tilde{F}'(\nabla u - z_R) \, \nabla \eta \, dx
$$

$$
\leq K \tilde{M} \int_{\Omega_R} \eta w |\nabla \eta| \left( |\nabla (\nabla u - z_R)| + \eta |\nabla u - z_R|^{q-1} \right) \, dx
$$

$$
\leq CK \tilde{M} \int_{\Omega_R} \eta w |\nabla \eta|^2 \, dx + \lambda \tilde{M} \int_{\Omega_R} |\nabla (\eta u)|^2 + \eta^2 |\nabla w|^2 \, dx
$$

$$
+ C \lambda \tilde{M} \int_{\Omega_R} \eta^2 |z_R - a_R - \nabla g|^2 + \eta^2 |z_R - a_R - \nabla g|^{2(q-1)} \, dx.
$$
Hence since $\eta^2 \leq 1$ we deduce that

$$\frac{1}{2} \int_{\Omega_n} |\nabla (\eta w)|^2 \, dx \leq \frac{4}{\lambda_{M}} \int_{\Omega_n} \left| \tilde{F}'(0)(\nabla u - z_R) - \tilde{F}'(\nabla u - z_R) \right|^2 \, dx$$

(4.12)

$$+ C \int_{\Omega_n} \left( |z_R - \nabla a_R - \nabla g|^2 + |z_R - \nabla a_R - \nabla g|^2(q-1) \right) \, dx$$

$$+ \frac{C}{R^2} \int_{\Omega_n} |\omega|^2 \, dx + C \int_{\Omega_n} |\nabla \omega|^{2(q-1)} \, dx,$$

where the final term can be omitted if $q = 2$. For the second term we note that since $g, \rho$ are $C^{1, \beta}$ we have

$$|z_R - \nabla a_R - \nabla g| \leq |\xi_R \otimes \nu_{x_o} \frac{|\nabla \rho(x) - \nabla \rho(x_0)|}{|\nabla \rho(x_0)|} + |\nabla g - (\nabla g)_{\Omega_R}| \leq CMR^3$$

(4.13)

in $\Omega_R$, where $C = C(\|\Omega\|_{C^{1, \beta}}, \|\nabla g\|_{C^{1, \beta}}) > 0$. For the first term we apply the comparison estimate (2.8); writing $\Phi(t) = \omega_{M}^{-1}(t)(t^2 + t^{2(q-1)})$ this gives

$$\int_{\Omega_n} \left| \tilde{F}'(0)(\nabla u - z_R) - \tilde{F}'(\nabla u - z_R) \right|^2 \, dx \leq K_{\tilde{M}} \int_{\Omega_R} \Phi(|\nabla u - z_R|) \, dx,$$

noting that $\omega_{M}(t) \leq 1$. Now we estimate

$$|\nabla u - z_R| \leq |\nabla u - (\nabla u)_{\Omega_R}| + |\xi_R - (|\nabla (u - g)| \cdot \nu_{x_o})_{\Omega_R}| + |(\nabla (u - g))_{\Omega_B} - ((\nabla (u - g)) \cdot \nu_{x_o})_{\Omega_R} - ((\nabla (u - g)) \cdot \nu_{x_o})_{\Omega_R} \otimes \nu_{x_o}|.$$

(4.15)

By Remark 4.2 the second term can be estimated as

$$|\xi_R - (\nabla (u - g) \cdot \nu_{x_o})_{\Omega_R}|$$

(4.16)

$$\leq C \left( \int_{\Omega_n} |\nabla u - \nabla g - (\nabla (u - g) \cdot \nu_{x_o})_{\Omega_R} \otimes \nu_{x_o}|^2 \, dx \right)^{\frac{1}{2}} + CMR^3$$

$$\leq \frac{C}{\lambda_{M}} (|\nabla (u - g)|_{\Omega_R} - ((\nabla (u - g)) \cdot \nu_{x_o})_{\Omega_R} \otimes \nu_{x_o})$$

$$+ C \left( \int_{\Omega_n} |\nabla u - \nabla g - (\nabla (u - g))_{\Omega_R}|^2 \, dx \right)^{\frac{1}{2}} + CMR^3,$$

and applying CAMPOS CORDERO’S trick (Lemma 3.5) followed by the John-Nirenberg estimate (Proposition 3.2) we have

$$|(\nabla u - \nabla g)_{\Omega_R} - ((\nabla u - \nabla g) \cdot \nu_{x_o})_{\Omega_R} \otimes \nu_{x_o}|$$

(4.17)

$$\leq C \left( \int_{\Omega_n} |\nabla u - \nabla g - (\nabla u - \nabla g)_{\Omega_B}|^p \, dx \right)^{\frac{1}{p}} + CMR^3$$

$$\leq C |\nabla u - \nabla g|_{\text{BMO}(\Omega_R)} + CMR^3,$$

for $p \in \{2, q\}$. Also applying the modular Fefferman-Stein estimate (Corollary 3.4) we can bound

$$\int_{\Omega_n} \Phi \left( |\nabla u - (\nabla u)_{\Omega_R}| \right) \, dx$$

(4.18)

$$\leq C \omega_{M} \left( |\nabla u|_{\text{BMO}(\Omega_R)} \right) \int_{\Omega_n} |\nabla u - (\nabla u)_{\Omega_R}|^2 + |\nabla u - (\nabla u)_{\Omega_R}|^{2(q-1)} \, dx.$$
Now since $|\nabla g|_{\text{BMO}(\Omega_R)} \leq CR^3$ and $\Phi(R^3) \leq \left(1 + R_0^{2(q-2)}\right) R^{2\beta}$, we can combine the above using the doubling property of $\Phi$ to get

$$\int_{\Omega_R} \Phi(|\nabla u - z_0|) \, dx \leq CM^{2(q-1)} R^{2\beta}$$

(4.19)

$$+ C \omega_{\tilde{M}} \left(|\nabla u|_{\text{BMO}(\Omega_R)}\right) \int_{\Omega_R} |\nabla u - (\nabla u)_{\Omega_R}|^2 + |\nabla u - (\nabla u)_{\Omega_R}|^{2(q-1)} \, dx.$$

To complete the estimate, note by the John-Nirenberg inequality (Proposition 3.2) that

$$\int_{\Omega_R} |\nabla u - (\nabla u)_{\Omega_R}|^{2(q-1)} \, dx \leq C |\nabla u|_{\text{BMO}(\Omega_R)}^{2(q-2)} \int_{\Omega_R} |\nabla u - (\nabla u)_{\Omega_R}|^2 \, dx,$$

and similarly

$$\int_{\Omega_R} |\nabla w|^{2(q-1)} \, dx \leq C |\nabla w|_{\text{BMO}(\Omega_R)}^{2(q-2)} \int_{\Omega_R} |\nabla u - (\nabla u)_{\Omega_R}|^2 \, dx.$$

Hence putting everything together gives

$$\int_{\Omega_R} |\nabla(uw)|^2 \, dx \leq C \omega_{\tilde{M}} \left(|\nabla u|_{\text{BMO}(\Omega_R)}\right) \left(1 + |\nabla u|_{\text{BMO}(\Omega_R)}^{2(q-2)}\right) \int_{\Omega_R} |\nabla u - (\nabla u)_{\Omega_R}|^2$$

$$+ C \frac{R^2}{R_0} \int_{\Omega_R} |w|^2 \, dx + C |\nabla u|_{\text{BMO}(\Omega_R)}^{2(q-2)} \int_{\Omega_R} |\nabla u - (\nabla u)_{\Omega_R}|^2 \, dx$$

$$+ CM^{2(q-1)} R^{2\beta},$$

from which the result follows taking $\gamma(t) = \min\{1, \omega_{\tilde{M}}(t)\left(1 + t^{2(q-2)}\right) + t^{2(q-2)}\}$, omitting the $t^{2(q-2)}$ terms if $q = 2$. \(\square\)

4.2. Boundary harmonic approximation.

**Lemma 4.3** (Boundary harmonic approximation). Suppose $F$ satisfies Hypotheses 1.1, let $M \geq 1$, and suppose $\Omega \subset \mathbb{R}^n$ is a bounded $C^{1,\beta}$ domain and $g \in C^{1,\beta}(\overline{\Omega}, \mathbb{R}^N)$, for some $\beta \in (0, 1)$. Suppose $x_0 \in \partial \Omega$, $0 < R < R_0$ with $R_0 = R_0(n, \Omega) > 0$ and $u \in W^{1,2}(\Omega_R, \mathbb{R}^N)$ is $F$-extremal in $\Omega_R(x_0)$ with $\nabla u \in \text{BMO}(\Omega_R, \mathbb{R}^N)$, $|\nabla u|_{\text{BMO}(\Omega_R)} \leq 1$, and $|\nabla(u)|_{\Omega_R} \leq M$.

Then letting $\Omega_R$ as in Lemma 3.6, the unique solution $h \in W^{1,2}(\overline{\Omega_R}, \mathbb{R}^N)$ to the Dirichlet problem

$$\begin{cases}
-d \text{div} F''(z_R) \nabla h = 0 & \text{in } \tilde{\Omega}_R, \\
h = u - g - a_R & \text{on } \partial \Omega_R,
\end{cases}$$

with $z_R, a_R$ as in (4.2), (4.8) respectively satisfies

$$\int_{\tilde{\Omega}_R} |\nabla h|^2 \, dx \leq C \int_{\tilde{\Omega}_R} |\nabla(u - g - a_R)|^2 \, dx,$$

where $C = C\left(n, \Lambda_{\tilde{M}}/\Lambda_{\tilde{M}}\right) > 0$ with $\tilde{M} = C\left(n, \beta, \|\nabla g\|_{C^{1,\beta}}(\Omega)\right)$. Moreover we have the remainder estimate

$$\frac{1}{R^2} \int_{\Omega_R} |u - g - a - h|^2 \, dx \leq C \gamma \left(|\nabla u|_{\text{BMO}(\Omega_R)}\right) \int_{\Omega_R} |\nabla u - (\nabla u)_{\Omega_R}|^2 \, dx + CM^2 R^{2\beta},$$

where $C = C\left(n, N, q, K_{\tilde{M}}, \Lambda_{\tilde{M}}, \|\nabla g\|_{C^{1,\beta}}, \|\nabla g\|_{C^{0,\beta}(\Omega)}\right) > 0$ and $\gamma: [0, \infty) \to [0, 1]$ non-decreasing continuous such that $\gamma(0) = 0$, depending on $n, q$ and $\omega_{\tilde{M}}$ only.

**Proof.** We will assume $n \geq 3$ so Sobolev embedding applies, taking similar modifications as in the interior case if $n = 2$. Additionally we will use similar arguments used in the proof of Lemma 4.1 which we will not reproduce in detail, in particular choosing $R_0, \tilde{M}$ in the same.
way. Letting $\tilde{F} = F_{z_R}$ be the shifted functional with $z_R$ as in (4.8) and setting $w = u - g - a_R$, note for $\phi \in W^{1,2}_0(\tilde{\Omega}_R, \mathbb{R}^N)$ we have
\[
\int_{\tilde{\Omega}_R} \tilde{F}''(0)(\nabla w - \nabla h) : \nabla \phi \, dx \\
= \int_{\tilde{\Omega}_R} \left( \tilde{F}''(0)\nabla w - \tilde{F}'(\nabla u - z_R) \right) : \nabla \phi \, dx \\
\leq K\tilde{M} \int_{\tilde{\Omega}_R} \omega_M(|\nabla u - z_R|) \left( |\nabla u - z_R| + |\nabla u - z_R|^{q-1} \right) |\nabla \phi| \, dx \\
+ K\tilde{M} \int_{\tilde{\Omega}_R} |\nabla a_R - \nabla g - z_R| |\nabla \phi| \, dx,
\]
(4.26)
where we used the comparison estimate (2.8). We now choose $\phi$ to be the unique solution to the Dirichlet problem
\[
\begin{cases}
-\text{div} \tilde{F}''(0)\nabla \phi = w - h & \text{in } \tilde{\Omega}_R, \\
\phi = 0 & \text{on } \partial \tilde{\Omega}_R.
\end{cases}
\]
(4.27)
Since $w - h \in L^2(\tilde{\Omega}_R) \hookrightarrow W^{-1,2^*}(\tilde{\Omega}_R)$ by Remark 3.7, by Lemma 3.6(ii) with $p = 2^*$ we obtain the estimate $\|\nabla \phi\|_{L^{2^*}(\tilde{\Omega}_R)} \leq C \|w - h\|_{L^2(\tilde{\Omega}_R)}$. Therefore for this choice of $\phi$ we get
\[
\int_{\tilde{\Omega}_R} |w - h|^2 \, dx \leq C \tilde{M} \int_{\tilde{\Omega}_R} |\nabla u - z_R| \, dx \\
+ C \int_{\tilde{\Omega}_R} |\nabla a_R - \nabla g - z_R| \, dx,
\]
(4.28)
where we have used Hölder and Jensen’s inequalities (here $2^* = \frac{2n}{n+2}$), and absorbed the $\int_{\tilde{\Omega}_R} |w - h|^2 \, dx$ term on the right-hand side. Arguing by splitting $|\nabla u - z_R|$ as in (4.15) from the previous section (proof of Lemma 4.1) we arrive at the estimate
\[
\int_{\tilde{\Omega}_R} |w - h|^2 \, dx \leq C\gamma \left( |\nabla u|_{\text{BMO}(\tilde{\Omega}_R)} \right) \int_{\tilde{\Omega}_R} |\nabla u - (\nabla u)_{\tilde{\Omega}_R}|^2 \, dx + CM^2R^{2\beta}
\]
with $\gamma(t) = \min\{1, \omega_{\tilde{M}}(t)\frac{1}{2}(1 + t^{2(q-2)})\}$, as required. \qed

4.3. Boundary $\varepsilon$-regularity and a global result. We now combine the estimates from the previous sections to conclude as in the interior case.

Proof of Theorem 1.3. For $B_r(x) \subset B_{R_0}(x_0)$ with $x \in \overline{\Omega}$ we consider the excess energy
\[
E(x, r) = \int_{\Omega_v(x)} |\nabla u - (\nabla u)_{\Omega_v(x)}|^2 \, dy,
\]
(4.30)
so by assumption and Proposition 3.2 there is $C_1 = C_1(n, \delta) > 0$ such that $E(x, r) \leq C_1\varepsilon^2$, which we can assume is less than 1.

Claim: If $x \in \partial \Omega$ and $r > \delta$ so that $\Omega_v(x) \subset \Omega_R(x_0)$ and $\sigma \in (0, \frac{r}{2})$ for which
\[
|\nabla u|_{\Omega_v(x)}|, |(\nabla u)_{\Omega_v(x)}| \leq 2^{3n+1}M,
\]
(4.31)
we have
\[
E(x, \sigma r) \leq C \left( \sigma^{2\beta} + \sigma^{-(n+2)} \gamma \left( |\nabla u|_{\text{BMO}(\Omega_v(x))} \right) \right) E(x, r) + CM^{2(q-1)}\sigma^{-(n+2)}\sigma^{2\beta},
\]
(4.32)
where $\gamma$ is as in Lemmas 4.1 and 4.3 with $2^{3n+1}M$ in place of $M$, and
\[
C = C \left( n, N, q, K\tilde{M}, \lambda\tilde{M}, \delta, ||\Omega||_{C^1,\beta}, R_0, ||\nabla g||_{C^{0,\beta}(\Omega)} \right) > 0.
\]
(4.33)
Proof of claim: Applying the Caccioppoli-type inequality (Lemma 4.1) we have
\[
E(x, \sigma r) \leq C \gamma \left( \| \nabla u \|_{\text{BMO}(\Omega_{2\sigma r}(x))} \right) E(x, 2\sigma r) + \frac{1}{\sigma^2 r^2} \int_{\Omega_{2\sigma r}(x)} |u - g - a_{2\sigma r}|^2 \, dy + CM^{2(q-1)}(\sigma r)^{2\beta},
\] (4.34)
where \(a_{2\sigma r}\) is given by (4.2) in \(\Omega_{2\sigma r}(x)\). Also by the boundary harmonic approximation (Lemma 4.3) in \(\Omega_r(x)\) the unique solution \(h \in W^{1,2}(\tilde{\Omega}_r(x), \mathbb{R}^N)\) solving
\[
\begin{cases}
- \text{div} F'(z_r) \nabla h = 0 & \text{in } \tilde{\Omega}_r(x), \\
h = u - g - a_r & \text{on } \partial \tilde{\Omega}_r(x),
\end{cases}
\] (4.35)
satisfies
\[
\frac{1}{\sigma^2 r^2} \int_{\Omega_{r/2}(x)} |u - g - a_r - h|^2 \, dy \leq C \gamma \left( \| \nabla u \|_{\text{BMO}(\Omega_r(x))} \right) E(x, r) + CM^{2}\sigma^{2\beta},
\] (4.36)
noting that \(\Omega_{r/2}(x) \subset \tilde{\Omega}_r(x) \subset \Omega_r(x)\). Now by Remark 4.2 we have
\[
\frac{1}{\sigma^2 r^2} \int_{\Omega_{2\sigma r}(x)} |u - g - a_x|^2 \, dy \leq \frac{1}{\sigma^2 r^2} \int_{\Omega_{2\sigma r}(x)} \left| u - g - \frac{\rho}{|\nabla \rho(x)|} \right|^2 \, dy
\] (4.37)
for all \(\xi \in \mathbb{R}^N\), so taking \(\xi = \xi_r + (\nabla h \cdot \nu_x)_{\Omega_{2\sigma r}(x)}\) we can split
\[
\frac{1}{\sigma^2 r^2} \int_{\Omega_{2\sigma r}(x)} |u - g - a_x|^2 \, dy \leq \frac{1}{\sigma^2 r^2} \int_{\Omega_{2\sigma r}(x)} \left| h - (\nabla h \cdot \nu_x)_{\Omega_{2\sigma r}(x)} + \frac{\rho}{|\nabla \rho(x)|} \right|^2 \, dy + C\sigma^{-(n+2)} \gamma \left( \| \nabla u \|_{\text{BMO}(\Omega_r(x))} \right) E(x, r) + CM^{2(q-1)}\sigma^{-(n+2)}r^{2\beta}.
\] (4.38)
For the second term we use the Poincaré inequality (Lemma 3.8) and Lemma 3.5 to estimate
\[
\frac{1}{\sigma^2 r^2} \int_{\Omega_{2\sigma r}(x)} \left| h - (\nabla h \cdot \nu_x)_{\Omega_{2\sigma r}(x)} \right|^2 \, dy \\
\leq C \int_{\Omega_{2\sigma r}(x)} \left| \nabla h - (\nabla h \cdot \nu_x)_{\Omega_{2\sigma r}(x)} \right|^2 \, dy + CM^2(\sigma r)^{2\beta}
\] (4.39)
where we have used the bound \(|(\nabla h)_{\Omega_{2\sigma r}(x)}| \nu_x|^2 \leq CM^2\sigma^{-n}\). Now as \(h\) vanishes on \(\partial \Omega \cap \partial \Omega_r(x)\), using (3.31) from Lemma 3.6(ii) we have the estimate
\[
[\nabla h]_{C^{0,\beta}(\Omega_{r/2}(x))} \leq C \int_{\tilde{\Omega}_r(x)} |\nabla(u - g - a_R)| \, dy \leq CE(x, r) + CM^2 r^{2\beta},
\] (4.40)
where the last line is obtained by arguing as in the proof of Lemma 4.1. Hence it follows that
\[
\frac{1}{\sigma^2 r^2} \int_{\Omega_{2\sigma r}(x)} \left| h - (\nabla h \cdot \nu_x)_{\Omega_{2\sigma r}(x)} \right|^2 \, dy \leq C\sigma^{2\beta}E(x, r) + CM^2\sigma^{-n}r^{2\beta},
\] (4.41)
so the claim follows by putting everything together.
We now argue analogously as in the interior case; note for $x \in \partial \Omega \cap B_{R/2}(x_0)$ we have $|(|\nabla u|_{B_{sr/2}(x)})| \leq 2^{3n}M$, and so $|(|\nabla u|_{B_{sr/2}(x)})| \leq 2^{3n}M + C_1 \sigma^{-n} \varepsilon \leq 2^{3n+1}M$ for $\varepsilon > 0$ sufficiently small. Hence applying the claim gives

\[(4.42) \quad E(x, \sigma R/2) \leq C \left( \sigma^{2\beta} + \sigma^{-(n+2)} \gamma(\varepsilon) \right) E(x, r/2) + C M 2^{(n-1)\sigma^{-(n+2)} R^2/\alpha} R^{2\alpha}.
\]

We choose $\sigma \in (0, \frac{1}{4})$ such that $C \sigma^{2\beta} \leq \frac{1}{4} \sigma^{2\alpha}$, and $\varepsilon > 0$ such that $C \sigma^{-(n+2)} \gamma(\varepsilon) \leq \frac{1}{4} \sigma^{2\alpha}$. We then choose $R_0 > 0$ such that $C M 2^{(n-1)\sigma^{-(n+2)} R^2/\alpha} \leq \kappa \sigma^{2\alpha}$ for $0 < \kappa < 1$ to be chosen to get

\[(4.43) \quad E(x, \sigma R/2) \leq \frac{1}{2} \sigma^{2\alpha} E(x, R/2) + \kappa (\sigma R)^{2\alpha}.
\]

Further shrinking $\varepsilon > 0$ if necessary and taking $\kappa > 0$ small enough so

\[(4.44) \quad \sigma^{-(n+2)} (C_1 \varepsilon + \kappa) \sum_j \sigma^{\alpha j} \leq 3^n M,
\]

we can iteratively argue that for all $k \geq 0$,

\[(4.45) \quad |(|\nabla u|_{B_{sr/2}(x)})| \leq 2^{3n+1}M,
\]

\[(4.46) \quad E(x, \sigma^k R/2) \leq 2^{-k} \sigma^{2\alpha k} E(x, R/2) + (\sigma^k R)^{2\alpha}.
\]

Hence it follows that $E(x, r) \leq C r^{2\alpha}$ for all $r \in (0, R/2)$.

By the interior case we also have $E(x, r) \leq C r^{2\alpha}$ when $B(x, r) \subset \Omega_R(x_0)$ with $x \in B_{R/2}$. We can extend this to all $x \in \Omega_R(x_0)$ and $0 < r < R/2$ by a covering argument (adjusting constants as necessary), so by the Campanato-Meyers characterisation we get $u$ is $C^{1, \alpha}$ in $\Omega_{R/2}(x_0)$, as required.

Looking closely at the dependence of constants in our proofs, we see that we can improve the global regularity results presented in the introduction if we assume stronger control on $F''$. We were only able to obtain these results for the case of quadratic growth, and we would be interested if these results extend to integrands with controlled $p$-growth.

**Theorem 4.4** (Regularity of almost VMO solutions in the uniformly elliptic case). Suppose $F: \mathbb{R}^{2n} \to \mathbb{R}$ is a $C^2$ function with uniformly continuous second derivative satisfying the uniform Legendre-Hadamard ellipticity condition

\[(4.47) \quad \lambda |\xi|^2 \eta^2 \leq F''(\xi \otimes \eta) : (\xi \otimes \eta) \leq \Lambda |\xi|^2 \eta^2
\]

for all $\xi \in \mathbb{R}^n$ and $\xi \in \mathbb{R}$. Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1, \beta}$ domain and $g \in C^{1, \beta}(\Omega; \mathbb{R}^n)$ for some $\beta \in (0, 1)$. Then for each $\alpha \in (0, \beta)$ there is $\varepsilon > 0$ such that if $u \in W^{1, 2}_g(\Omega; \mathbb{R}^n)$ is $F$-extremal in $\Omega$ such that

\[(4.48) \quad \{\nabla u\}_{\text{osc}(\Omega)} \leq \varepsilon,
\]

then $u \in C^{1, \alpha}(\Omega; \mathbb{R}^n)$.

**Proof.** We observe the estimates for the shifted functional $F_{z_0}$ from Section 2.1 hold uniformly without bounds of the form $|z_0| \leq M$, so arguing as in the proofs of Theorems 1.2 and 1.3 we obtain the excess decay estimate

\[(4.49) \quad E(x, \sigma r) \leq C \left( \sigma^{2\beta} + \sigma^{-(n+2)} \gamma \left( |\nabla u|_{\text{BMO}(\Omega)} \right) \right) E(x, r)
\]

\[\quad + C (1 + |(|\nabla u|_{\Omega_{2\sigma r}}) + |(|\nabla u|_{\Omega_{r}})|)^2 \sigma^{-(n+2)} r^{2\beta}.
\]

for all $x \in \Omega$, $R > 0$ such that either $B_R(x) \subset \Omega$ or $x \in \Omega$ and $0 < R < R_0$ (with $R_0 = R_0(n, \Omega) > 0$). Note in the interior case the second term can be omitted.

Fix $\varepsilon > 0$ to be determined. Then there is $0 < R < \frac{M}{2}$ for which there exists a finite covering of $\Omega$ by balls $\{B_{R}(x_j)\}$ where either $B_R(x_j) \subset \Omega$ or $x_j \in \Omega$, and $|\nabla u|_{\text{BMO}(\Omega_{2\sigma R}(x_j))} \leq 2\varepsilon \leq 1$
for each $j$. Let $M > 0$ such that $|\langle \nabla u \rangle_{\Omega^B(x_j)}| \leq M$ for all $j$, then observe that for all $x \in \Omega$ and $0 < r < R$ we have $|\langle \nabla u \rangle_{\Omega^B(x)}| \leq C(n)M (1 + \log(R/r))$. Hence the excess decay estimate becomes

\[
E(x, \sigma r) \leq C \left( a^{2\beta} + \sigma^{-(n+2)\gamma(2\varepsilon)} \right) E(x, r) + CM^2 \sigma^{-(n+2)\gamma(2\alpha)} (1 + \log(R/r))
\]

whenever $0 < r < R$, and modifying constants this holds for all $x \in \Omega$.

Now choose $\sigma \in (0, \frac{1}{2})$ such that $C\sigma^{2\beta} \leq \frac{1}{4}\sigma^{2\alpha}$, and $\varepsilon > 0$ such that $C\sigma^{-(n+2)\gamma(2\varepsilon)} \leq \frac{1}{4}\sigma^{2\alpha}$. Then choose $0 < r_0 < R$ such that $CM^2 \sigma^{-(n+2)\gamma(2\alpha)} (1 + \log(R/r_0)) \leq \sigma^{2\alpha}$. This gives

\[
E(x, \sigma r) \leq \frac{1}{2} \sigma^{2\alpha} E(x, r) + (\sigma r)^{2\alpha},
\]

from which the result follows by iteration as in the proof of Theorem 1.3. \qed

5. Extensions

Up until now we have confined our discussion to the setting of autonomous integrands, however the framework we developed extends to more general elliptic systems. Rather than state the most general case possible, we will aim to highlight the necessary changes to adapt our arguments to these more general situations.

5.1. Quasilinear elliptic systems. While our motivation for this investigation arose from studying the behaviour of extremals, it turns out our arguments do not make use of the variational structure of the equation. We will illustrate this by considering general Legendre-Hadamard elliptic systems, and also show how lower order terms can be handled.

More precisely we consider weak solutions to the equation

\[
- \text{div} A(x, u, \nabla u) + B(x, u, \nabla u) = 0
\]

in $\Omega$, subject to the following conditions.

**Hypotheses 5.1.** Let $n \geq 2$, $N \geq 1$, $\beta \in (0, 1)$, $q \geq 2$ and $\Omega \subset \mathbb{R}^n$ a bounded $C^{1,\beta}$ domain. We consider Carathéodory functions

\[
A: \Omega \times \mathbb{R}^N \times \mathbb{R}^n \rightarrow \mathbb{R}^{nN},
\]

\[
B: \Omega \times \mathbb{R}^N \times \mathbb{R}^n \rightarrow \mathbb{R}^N,
\]

satisfying the following (we use $D_u, D_z$ to denote partial derivatives in $u, z$ respectively).

(\text{H1}) For all $(x, u, z) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^n$ we have

\[
|A(x, u, z)| + |B(x, u, z)| \leq K (1 + |z|^{q-1}).
\]

(\text{H2}) The map $z \mapsto A(x, u, z)$ is continuously differentiable for each $(x, u)$, and for all $M > 0$ there is $\Lambda_M > 0$ and a continuous, non-decreasing concave function $\omega_M: [0, \infty) \rightarrow [0, 1]$ satisfying $\omega_M(0) = 0$ such that

\[
|D_z A(x, u, z_1) - D_z A(x, u, z_2)| \leq \Lambda_M \omega_M(|z_1 - z_2|)
\]

for all $x \in \Omega$, $|u| \leq M$ and $|z_1|, |z_2| \leq M + 1$.

(\text{H3}) For all $M > 0$, for $x \in \Omega$ and $|u|, |z| \leq M$ we have the strong Legendre-Hadamard ellipticity condition

\[
D_z A(x, u, z)(\xi \odot \eta) : (\xi \odot \eta) \geq \lambda_M |\xi|^2 |\eta|^2
\]

for all $\xi \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^n$.

(\text{H4}) For all $x_1, x_2 \in \Omega$, $u_1, u_2 \in \mathbb{R}^N$ and $z \in \mathbb{R}^n$ we have

\[
|A(x_1, u_1, z) - A(x_2, u_2, z)| \leq K (1 + |z|^{q-1}) \beta_3(|x_1 - x_2| + |u_1 - u_2|),
\]

where $\beta_3(t) = \min\{1, t^\beta\}$. 

Remark 5.2. A special case of the above is the Euler-Lagrange system associated to the non-autonomous integrand \( F = F(x, u, z) \). Here the Euler-Lagrange system reads

\[
- \operatorname{div} D_z F(x, u, \nabla u) + D_u F(x, u, \nabla u) = 0,
\]

so we need \( F \) to be \( C^2 \) in \( z \) and \( C^1 \) in \( x \), such that Hypotheses 5.1 are satisfied with \( A(x, u, z) = D_z F(x, u, z) \) and \( B(x, u, z) = D_u F(x, u, z) \).

Theorem 5.3 (BMO \( \varepsilon \)-regularity theorem for elliptic systems). Suppose \( \Omega, A, B \) satisfies Hypotheses 5.1 and suppose \( u \in W^{1,q}_0(\Omega, \mathbb{R}^N) \) solves (5.1) with \( g \in C^{1,\beta}(\overline{\Omega}, \mathbb{R}^N) \). Then for each \( \alpha \in (0, \beta) \) and \( M > 0 \) there is \( \varepsilon > 0 \) and \( R_0 > 0 \) such that if \( x \in \overline{\Omega} \) and \( R \in (0, R_0) \) such that \( |(\nabla u)|_{BMO(\Omega R(x))} \leq \varepsilon \), then \( u \) is \( C^{1,\alpha} \) in \( \Omega R/2(x_0) \).

Our strategy will be similar to before; we fix \( x_0 \in \overline{\Omega} \) and \( R > 0 \) such that either \( B_R(x_0) \subset \Omega \), or \( x_0 \in \partial \Omega \) and \( 0 < R < R_0 \) with \( R_0 > 0 \) as in the start of Section 4. We will then fix \( M > 0 \) such that \( |(\nabla u)|_{BMO(\Omega R(x_0))} \leq M \) and prove a local Caccioppoli-type inequality and harmonic approximation estimate. We will obtain extra perturbation terms arising from the \( x, u \) dependences and the lower-order term \( B \), but these can be easily controlled.

We first observe that we can suppress the \( u \)-dependence; since \( \nabla u \in \text{BMO}(\Omega, \mathbb{R}^N) \) we can use the John-Nirenberg and Sobolev inequalities to obtain \( u \) dependences and the lower-order term \( \text{harmonic approximation estimate.} \) We will obtain extra perturbation terms arising from the \( x, u \) dependences and the lower-order term \( B \), but these can be easily controlled.

We then consider the linearisation

\[
\tilde{A}(z) = A(x_0, z + z_0) - A(x_0, z_0),
\]

which satisfies the growth estimate

\[
|\tilde{A}'(0)z - \tilde{A}(z)| \leq C \omega_\tilde{A}(|z| + |z|^{\beta - 1})
\]

for all \( z \in \mathbb{R}^n \) when \( |z| \leq \tilde{M} \). We also have the coercivity estimate

\[
\lambda_M \int_{\Omega R} |\nabla \phi|^2 \, dx \leq \int_{\Omega R} \tilde{A}'(0) \nabla \phi : \nabla \phi \, dx
\]

for all \( \phi \in W^{1,2}_0(\overline{\Omega}, \mathbb{R}^N) \).

From here one can proceed analogously as in the autonomous case detailed in Sections 2 and 4 replacing \( \tilde{F}' \) with \( \tilde{A}' \), and linearising at \( z_0 = \nabla u(x_0) + (\nabla g)|_{\Omega R} \) as before. Note we get an extra term arising from the \( x \)-dependence we bound using (H4), and the \( B(x, \nabla u) \) term is lower order so it can be controlled using the Poincaré inequality; hence both terms add an extra error term of the form \( CM^{2/\varepsilon} R^2 \).

5.2. Higher order integrands. We will also sketch how analogous results can be obtained for \( k^{\text{th}} \) order problems. For this fix \( k \geq 1 \), and let \( \mathbb{M}_k = \text{Sym}_k(\mathbb{R}^n, \mathbb{R}^N) \) denote the space of symmetric \( k \)-linear maps \( (\mathbb{R}^n)^k \to \mathbb{R}^N \). If \( \xi \in \mathbb{R}^N \) and \( \eta \in \mathbb{R}^n \), we write \( \eta^\xi = \eta \otimes \ldots \otimes \eta \) to denote the \( k \)-fold tensor product and identify elements \( \xi \otimes \eta^\xi \in \mathbb{M}_k \) to send \((x_1, \ldots, x_k) \to \xi \sum_{|\alpha| = k} \eta^\alpha \). Similarly in the case when \( k = 1 \), for \( z, w \in \mathbb{M}_k \) we write \( z : w = \sum_{|\alpha| = k} z^\alpha \cdot w(e^\alpha) \), where we take tensor powers of the standard orthonormal basis \( \{e_i\} \) for \( \mathbb{R}^n \). This defines an inner product and hence an associated norm \( | \cdot | \) on \( \mathbb{M}_k \).

We will consider extremals of the integrand

\[
\mathcal{F}(w) = \int_{\Omega} F(\nabla^k w(x)) \, dx,
\]
where $F: \mathbb{R}^k \to \mathbb{R}$ and $\nabla^k u$ denotes the $k^{th}$ order partial derivatives of $u$. These satisfy the Euler-Lagrange equation

$$(-1)^k \nabla^k : F'(\nabla^k u) = 0$$

weakly in $\Omega$ in the sense that

$$\int_\Omega F'(\nabla^k u) : \nabla^k \varphi \, dx = 0$$

for all $\varphi \in C^\infty(\Omega, \mathbb{R}^N)$. The minimising case has been studied for instance in [17, 31].

**Hypotheses 5.4.** For $n \geq 2$, $N, k \geq 1$, let $F: \mathbb{R}^k \to \mathbb{R}$ be a $C^2$ integrand satisfying the natural growth condition

$$|F(z)| \leq K(1 + |z|)^q$$

for all $z \in \mathbb{R}^k$ with $q \geq 2$, and the strict Legendre-Hadamard condition

$$F''(z_0)(\xi \otimes \eta^k) : (\xi \otimes \eta^k) \geq 0$$

for all $z_0$ and all $\xi, \eta \in \mathbb{R}^n$, with equality if and only if $\xi \otimes \eta^k = 0$.

**Theorem 5.5** (Higher order BMO $\varepsilon$-regularity theorem). Suppose $F$ satisfies Hypotheses 5.4, $\Omega$ is a bounded $C_1^\alpha$ domain for some $\beta \in (0, 1)$, and $g \in C^{k, \beta}(\overline{\Omega}, \mathbb{R}^N)$. Then for each $\alpha \in (0, \beta)$ and $\lambda > 0$, there is $\varepsilon > 0$ and $\tilde{R}_0 > 0$ such that if $x \in \overline{\Omega}$ and $0 < R < \tilde{R}_0$ such that if $u \in W^{k, \beta}(\Omega, \mathbb{R}^N)$ is $F$-extremal in $\Omega_R(x_0)$ such that $|\nabla^k u(\Omega_R(x_0))| \leq \lambda$ and

$$[\nabla^k u]_{\text{BMO}(\Omega_R(x_0))} \leq \varepsilon,$$

we have $u$ is $C^{k, \alpha}$ in $\overline{\Omega}_{R/2}(x_0)$.

Similarly as in Section 2.1 for each $M > 0$ there is $K_M, \lambda_M > 0$ and a non-decreasing continuous and concave function $\omega_M: [0, \infty) \to [0, 1]$ satisfying $\omega_M(0) = 0$ for which the following holds. If for $z_0 \in \mathbb{R}^n$ such that $|z_0| \leq M$ we define

$$F_{z_0}(z) = F(z_0 + z) - F(z_0) - F'(z_0)z,$$

which satisfies similar growth and perturbation estimates as in (2.6), (2.8), along with the coercivity estimate

$$\int_{\mathbb{R}^n} F''_{z_0}(0)\nabla^k \varphi : \nabla^k \varphi \, dx \geq \lambda_M \int_{\mathbb{R}^n} |\nabla^k \varphi|^2 \, dx$$

for all $\varphi \in C^\infty(\mathbb{R}^n, \mathbb{R}^N)$.

We will also need the following extension of CAMPOS CORDERO’s estimate (Lemma 3.5).

**Lemma 5.6.** Suppose $\Omega$ is a bounded $C^{k, \beta}$ domain for some $\beta \in (0, 1)$ and $p > \frac{k}{2}$, then there is $R_0 > 0$ such that for all $x_0 \in \partial \Omega$, $0 < R < R_0$ and $v \in W^{k, p}(\Omega_R(x_0), \mathbb{R}^N)$ such that $\nabla^j v = \nabla^j v \cdot \nu = 0$ on $\partial \Omega \cap B_R(x_0)$ for each $0 \leq j \leq k - 1$, we have the estimate

$$\left(\int_{\Omega_R(x_0)} |\nabla^k v - (\nabla^k v \cdot \nu_{x_0})_{\Omega_R(x_0)} \otimes \nu_{x_0}|^p \, dx\right)^{\frac{1}{p}} \leq \left(\int_{\Omega_R(x_0)} |\nabla^k v - (\nabla^k v)_{\Omega_R(x_0)}|^p \, dx\right)^{\frac{1}{p}} + C|\nabla^k v|_{\Omega_R(x_0)} R^\beta,$$

with $C = C(n, k, \beta, p, \Omega)$. 
Proof sketch. We can argue similarly to the proof of Lemma 3.5; assuming $\nu_k = c_n$ we need to estimate

\[
\left| \int_{\Omega} \nabla^a \tilde{v} \, dx \right| \leq C \left( \int_{\Omega} |\nabla^k \tilde{v}|^p \, dx \right)^{\frac{1}{p}}
\]

whenever $|\alpha| = k$, where

\[
\tilde{v}(x) = v(x) - (\nabla^k v \cdot x_{k0}) |_{\Omega_n(x_0)} \frac{\rho(x)^k}{|\nabla \rho(x_0)|^k}.
\]

To see this note the divergence theorem gives $\int_{\Omega} \nabla \cdot \nabla_j \tilde{v} \, dx = 0$ if $i \neq j$, so the estimate trivially holds unless $\alpha = k e_i$ for some $1 \leq i \leq n - 1$. Similarly $\int_{\Omega} \nabla^k \tilde{v} \, dx = 0$ if $k \geq 3$, so we are reduced to the case when $k = 2$. In this case we can show that

\[
\int_{\Omega} \nabla^2 \tilde{v} \, dx = \int_{\Omega} \nabla^2 \tilde{v}(x) \frac{x_n}{(R^2 - |x|^2)^{\frac{3}{2}}} \, dx,
\]

which can be estimated analogously to before.}

With this technical estimate in hand, we can turn to the proof of Theorem 5.5. We fix $x_0 \in \Omega$ and chose $R > 0$ such that either $B_R(x_0) \subset \Omega$, or $x_0 \in \partial \Omega$ and $R > 0$ is sufficiently small. In the interior case we take

\[
a(x) = \sum_{|\alpha| \leq k} \frac{x^\alpha}{\alpha!} (\nabla^\alpha u)_{B_R(x_0)},
\]

and in the boundary case we take

\[
a(x) = \xi_R \frac{\rho(x)^k}{|\nabla \rho(x_0)|^k} = \frac{((u - g) \rho^k)_{\Omega_n(x_0)}}{(\rho^k)_{\Omega_n(x_0)}} \rho(x)^k.
\]

We then set $w = u - g - a$ and $z_0 = \nabla^k a(x_0) + (\nabla^k g)_{\Omega_n(x_0)}$ omitting the $g$-terms in the interior case. Since $\rho$ vanishes at $x_0$ we note that $\nabla^k (\rho^k)(x_0) = (\nabla \rho(x_0))^k$, and so $\nabla^k a(x_0) = \xi_R \rho u_{\Omega_n(x_0)}^k$ in the boundary case. We assume $|\nabla^k u|_{\Omega_n(x_0)} \leq M$, so then $|z_0| \leq M = CM$.

For the Caccioppoli-type estimate we will need a slight modification to account for intermediate derivatives. Fix $0 < t < s < R$ and let $\eta \in C_0^\infty(B_R(x_0))$ such that $1_{B_t} \leq \eta \leq 1_{B_s}$, with $|\nabla^j \eta| \leq C(s - t)^{-j}$ for each $0 \leq j \leq k$. Then applying the coercivity estimate (5.16) to $\eta u$ and testing the equation (5.11) against $\eta^2 w$, using the growth estimates for $F$ and arguing as in the proof of Lemma 4.1 we get

\[
\int_{\Omega} |\nabla^k (\eta u)|^2 \, dx \leq C \int_{\Omega} \omega_{M}(|\nabla^k u - z_0|) \left( |\nabla^k u - z_0|^2 + |\nabla^k u - z_0|^{2(q-1)} \right) \, dx
\]

\[
+ C \int_{\Omega} \omega_{M}(|\nabla^k u - z_0|) \frac{1}{|s - t|^2 j} \int_{\Omega} |\nabla^{k-j} u|^2 \, dx.
\]

For the last term we use the interpolation estimate to bound the intermediate derivatives $|\nabla^{k-j} u|_{L^2(\Omega)}$, using for instance in [2, Lemma 5.6] (applied in $B_s(x_0)$ after extending by zero). Applying this for the terms we can bound

\[
C \sum_{j=0}^{k-1} \frac{1}{(s - t)^{2j}} \int_{\Omega_s} |\nabla^{k-j} u|^2 \, dx \leq \frac{1}{2} \int_{\Omega} |\nabla^k u|^2 \, dx + \frac{C}{(s - t)^{2k}} \int_{\Omega_s} |w|^2 \, dx
\]
so then we can absorb the $\nabla^k w$ term by a standard iteration argument (for instance [19, Lemma 6.1]). For the remaining terms we can bound $|\nabla^k u - \nabla^k g| \leq CR^3$ and for the $|\nabla^k u - z_0|$ term we note that
\begin{equation}
|\xi_R - (\nabla^k (u - g) \cdot \nu_{z_0})|_{\Omega_R} 
\leq C \left(\int_{\Omega_R} |\nabla^k (u - g) - (\nabla^k (u - g) \cdot \nu_{z_0}) \otimes \nu_{z_0}|^2 \, dx\right)^{1/2} + CR^3,
\end{equation}
which generalises the estimate of KRONZ [32] used in Remark 4.2. This involves noting that $\int_{\Omega_R} \rho^2 |\nabla \rho(x_0)|^{-2k} \, dx \sim R^{2k}$ for $R > 0$ sufficiently small and applying the Poincaré inequality $k$-times. We can also apply Lemma 5.6 to replace the $(\nabla^k (u - g) \cdot \nu_{z_0}) \otimes \nu_{z_0}$ terms with the full $k$-gradient average, so we are left with terms of the form $|\nabla^k u - (\nabla^k u)_{\Omega_0}|$, along with terms bounded by $CR^{2\beta}$. Hence arguing similarly as in the proofs of Lemmas 2.1, 4.1 we arrive at the estimate
\begin{equation}
\int_{\Omega_{R/2}} |\nabla w|^2 \, dx 
\leq C \gamma \left(|[u]_{\text{BMO}(\Omega_R)}\right) \int_{\Omega_R} |\nabla^k u - (\nabla^k u)_{\Omega_0}|^2 \, dx + \frac{C}{R^{2\beta}} \int_{\Omega_R} |w| \, dx + CR^{2\beta},
\end{equation}
with $\gamma(t) = \min\{1, \omega_M(t)(1 + t^{2(q-2)}) + t^{2(q-2)}\}$.

Now for the harmonic approximation we take the unique $h \in W^{k,2}(\Omega_R, \mathbb{R}^N)$ solving the Dirichlet problem
\begin{equation}
\begin{cases}
(-1)^k \nabla^k : \tilde{F}^{\mu}(0) \nabla^k h = 0 & \text{in } \tilde{\Omega}_R, \\
h = \nabla^j \omega & \text{on } \partial \Omega_R \text{ for all } 0 \leq j \leq k-1,
\end{cases}
\end{equation}
and for the duality argument we also consider the unique $\phi \in W^{k,2^*}(\Omega_R, \mathbb{R}^N)$ to
\begin{equation}
\begin{cases}
(-1)^k \nabla^k : \tilde{F}^{\mu}(0) \nabla^k \phi = w - h & \text{in } \tilde{\Omega}_R, \\
\nabla^j \phi = 0 & \text{on } \partial \tilde{\Omega}_R \text{ for all } 0 \leq j \leq k-1,
\end{cases}
\end{equation}
which satisfies the scaled estimate
\begin{equation}
\|\nabla^k \phi\|_{L^{2^*}(\tilde{\Omega}_R)} \leq R^{k-1} \|w - h\|_{L^2(\tilde{\Omega}_R)}.
\end{equation}
For the excess decay estimate we will also need the Hölder estimate
\begin{equation}
\|\nabla^k h\|_{C^{0,\beta}(\Omega_{R/2})} \leq C \int_{\Omega_{R}} |\nabla^k h|^2 \, dx.
\end{equation}
These results go back to [5] (see also [3]), but they can also be straightforwardly adapted from the second-order case detailed in [19, Chapter 10].

Given these estimates we can argue analogously to the proofs of Lemmas 2.3, 4.3 to show that
\begin{equation}
\frac{1}{R^{2\beta}} \int_{\tilde{\Omega}_R} |w - h|^2 \, dx \leq C \gamma \left(|[\nabla^k u]_{\text{BMO}(\Omega_R)}\right) \int_{\Omega_R} |\nabla^k u - (\nabla^k u)_{\Omega_0}|^2 \, dx + CR^{2\beta},
\end{equation}
with $\gamma(t) = \min\{1, \omega_M(t)(1 + t^{2(q-2)})\}$, suitably modified if $n = 2$.

Finally to conclude we consider the higher-order excess
\begin{equation}
E(x, r) = \int_{\Omega_{r}(x)} |\nabla^k u - (\nabla^k u)_{\Omega_{r}(x)}|^2 \, dy.
\end{equation}
Then assuming $|[\nabla^k u]_{\Omega_{2r}(x)}|, |(\nabla^k u)_{\Omega_{r}(x)}| \leq 2^{3n+1} M$ we can argue that
\begin{equation}
E(x, sr) \leq C \left(\sigma^{2\beta} + \sigma^{-(n+2k)} \gamma \left(\nabla^k u\right)_{\text{BMO}(\Omega_{r}(x))}\right) E(x, r) + C \sigma^{-(n+2k)} r^{2\beta}.
\end{equation}
Now we can iterate in the usual way to conclude.
Acknowledgements. The author would like to thank Jan Kristensen for the many helpful discussions and suggestions.

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