Asymptotic properties of wall-induced chaotic mixing in point vortex pairs.

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The purpose of this work is to analyze the flow due to a potential point vortex pair in the vicinity of a symmetry line (or "wall"), in order to understand why the presence of the wall, even far from the vortices, accelerates fluid mixing around the vortex pair. An asymptotic analysis, in the limit of large distances to the wall, allows to approximate the wall effect as a constant translation of the vortex pair parallel to the wall, plus a straining flow which induces a natural blinking vortex mechanism with period half the rotation period. A Melnikov analysis of lagrangian particles, in the frame translating and rotating with the vortices, shows that a homoclinic bifurcation indeed occurs, so that the various separatrices located near the vortex pair (and rotating with it) do not survive when a wall is present. The thickness of the resulting inner stochastic layer is estimated by using the separatrix map method, and is shown to scale like the inverse of the squared distance to the wall. This estimation provides a lower-bound to the numerical thickness measured from either Poincaré sections or simulations of lagrangian particles transported by the exact potential velocity field in the laboratory frame. In addition, it is shown that the outer homoclinic cycle, separating the vortices from the external (open) flow, is also perturbed from inside by the rotation of the vortex pair. As a consequence, a stochastic layer is shown to exist also in the vicinity of this cycle, allowing fluid exchange between the vortices and the outer flow. However, the thickness of this outer stochastic zone is observed to be much smaller than the one of the inner stochastic zone near vortices, as soon as the distance to the wall is large enough.

Key-words: vortex flow, inviscid fluid, chaotic advection, homoclinic bifurcation.

I. INTRODUCTION

Vortical flows have remarkable transport properties which have been investigated in various contexts in the past decades. Indeed, vortices are often referred to as "violent" or "singular" structures which induce large local velocity gradients which in turn significantly influence the transport of scalar or vector fields. For example, steady two-dimensional isolated fixed vortices, whether pointwise or with a viscous core, have been shown to significantly influence the transport of passive scalars (Flohr & Vassilicos [1]) or magnetic fields (Bajer [2]). If, in addition, the vortex oscillates, then the velocity field is unsteady, and this unsteadiness is likely to influence scalar mixing through some chaotic advection phenomenon (Aref [3, 4], Beigie et al. [5]). One of the simplest unsteady vortex of this kind is the periodically translating isolated vortex, which can be thought of as a variation of the blinking vortex, and which has been shown to induce fast mixing in chaotic regions (Wohnas & Vassilicos [6]). In the case of vortex pairs the flow is also unsteady, since both vortices rotate around each other (provided the total strength is non-zero), but this unsteadiness is not sufficient to induce chaotic mixing. Indeed, in the frame rotating with the vortex pair the flow is steady, and the motion of fluid points is an autonomous dynamical system with only 1 degree-of-freedom: their motion cannot be chaotic.

If, in addition, the two vortices move in the vicinity of a flat wall, then elementary vortex dynamics shows that both the distance between the vortices and their rotation velocity are no longer constant. Fluid point trajectories are therefore much more complex. It has been shown that chaotic advection indeed occurs in the case of leapfrogging vortex pairs (Pentek et al. [7]), which can be thought of as a vortex pair near a wall simulated by using the mirror method. In the present paper we will investigate this phenomenon analytically to show that chaos persists when the vortices are far from the wall, and to quantify this effect by calculating the thickness of the resulting stochastic layers.

We will use an asymptotic method to show that the basic ingredient of chaos in the work by Rom-Kedar et al. [8], namely the artificial unsteady straining flow applied to a counter-rotating vortex pair, is also at work in the case of a co-rotating vortex pair near a wall. These authors show that counter-rotating vortex pairs can induce chaotic advection, provided they are superposed to an oscillating straining flow. They observe that in the absence of the oscillating straining flow "the pair carries a constant body of fluid", whereas the oscillating straining flow forces the fluid to be "entrained and detrained from the neighbourhood of the vortices".

The goal of the present paper is to show that, for co-rotating vortex pairs, a wall-induced straining flow can make the two vortices oscillate to-and-fro with a period half the natural rotation period of the pair, and that this variation of the blinking vortex phenomenon triggers chaotic mixing. This result will be derived by using an asymptotic approach, in the limit \( d_0 \ll L_0 \), where \( d_0 \) is the order-of-magnitude of the distance between vortices, and \( L_0 \) is the distance to the wall. This "inner" asymptotic velocity will approximate the flow in the vicinity of vortices, i.e. at distance \( O(d_0) \) from them. Wall-induced chaotic mixing will then be ana-
lyzed by considering the breaking of the corresponding "inner" homoclinic cycles.

In addition, it will be shown that vortex rotation also perturbs, from inside, the outer homoclinic cycle separating the vortex pair from the external open flow. This effect will be analysed by means of an "outer" asymptotic velocity field (in the same limit constructed to approximate the flow at distance $L_0$ from vortices.

The inner asymptotic model is depicted in section III. Chaotic advection and mixing in this flow are investigated and discussed in section III. We then focus on fluid exchange with the outer open flow by using the outer approximation (section IV). Because $d_0 \ll L_0$, the motion of vortices will be assumed to be regular throughout the paper (see Acheson [9]).

II. INNER ASYMPTOTIC VELOCITY FIELD NEAR VORTICES

![Diagram of vortex pair](image)

We consider a pair of point vortices $(A_1, A_2)$ with equal strength $\Gamma$ moving in the plane $(O, x, y)$. In the absence of any solid boundary both vortices rotate around their centre point $I$ with an angular velocity $\Omega_0 = \Gamma/4\pi d_0^2$, where $2d_0 = |A_1 A_2|$ is the (constant) distance between the vortices in this case. In the presence of a wall, located at $y = 0$, the streamfunction is (inviscid fluid):

$$\psi(x, y, t) = \sum_{i=1}^{2} \frac{\Gamma}{4\pi} \log [(x-x_i)^2 + (y-y_i)^2] + \sum_{i=1}^{2} \frac{\Gamma}{4\pi} \log [(x-x_i)^2 + (y+y_i)^2],$$

(1)

where $(x_1, y_1)$ are the coordinates of point $A_1$, and $(x_2, y_2)$ are the coordinates of point $A_2$. The latter sum is the wall effect, and will be next simplified in the limit where the wall is far from the vortices (see for example Appendix A of Rom-Kedar et al. [8]). Elementary vortex dynamics shows that:

$$\dot{x}_I = \frac{\Gamma L_0}{4\pi} \left( \frac{1}{y_1 y_2} + \frac{4}{|z_1 - \bar{z}_2|^2} \right),$$

(2)

$$y_I = \text{constant} = L_0,$$

in all cases, with $z_k = x_k + iy_k$. By setting $y_k = y_I + \bar{y}_k$, with $\bar{y}_k = O(d_0) \ll L_0$, and expanding (2) we get:

$$\dot{x}_I = \dot{x}_I^0 \left( 1 + O(\varepsilon^2) \right),$$

with $\dot{x}_I^0 = \Gamma/2\pi L_0$, and $\varepsilon = d_0/L_0 \ll 1$. This shows that, to leading order, $I$ moves under the effect of a single mirror vortex with strength $-2\Gamma$. If non-dimensionalized by $\Omega_0$ and $d_0$, the $x$ velocity of $I$ reads:

$$\frac{\dot{x}_I}{\Omega_0 d_0} = 2\varepsilon + O(\varepsilon^3),$$

(3)

so that the vortex pair moves at constant speed up to order 3. In order to exploit further the fact that $d_0 \ll L_0$, we consider positions $(x, y)$ located in the vicinity of the vortex pair, and normalize the variables as: $x^* = (x - x^0_I)/d_0$, $y^* = (y - y_I)/d_0$, and $x_i^* = (x_i - x^0_I)/d_0$, $y_i^* = (y_i - y_I)/d_0$, where $x_i^0 = \Gamma t / 2\pi L_0$. Also, we set $t^* = \Omega_0 t$ and define the non-dimensional streamfunction in the frame translating at the velocity of $I$:

$$\psi^*(x^*, y^*, t^*) = \frac{\psi - \dot{x}_I^0 (y - y_I)}{\Omega_0 d_0^2}.$$  

(4)

Then, by expanding the streamfunction in the limit $\varepsilon \to 0$, we get (removing the stars):

$$\psi(x, y, t) = \sum_{i=1}^{2} -\frac{\Gamma}{4\pi} \log [(x-x_i)^2 + (y-y_i)^2] + \frac{\varepsilon^2}{2} (x^2 - y^2) + O(\varepsilon^3).$$

(5)

Terms of order $O(\varepsilon)$ have cancelled-out with $\dot{x}_I^0$, as expected. Note that this flow does not correspond to a uniform flow $-\dot{x}_I^0$ at infinity (in contrast with Eq. (1)), as it is only an approximation of the streamfunction in the vicinity of the vortices.
A. Approximate vortex motion

We therefore observe that the effect of the wall can be approximated by a stretching flow with axes $\vec{e}_x \pm \vec{e}_y$ (where $\vec{e}_x$ and $\vec{e}_y$ are the unit vectors attached to axes $x$ and $y$ respectively). Note that this flow is very close to the one investigated by Carton et al. \(^{10}\) and Maze et al. \(^{11}\). The line $A_1A_2$ joining the two vortices is stretched and compressed periodically during the motion. It will be shown later that the period of this stretching/compression is half the period of the vortices without wall. The dynamics of vortex $A_1 = (x_1, y_1)$ in this simplified flow is:

$$
\begin{align*}
\dot{x}_1 &= -\frac{y_1 - y_2}{4r^2} - \frac{\varepsilon^2}{2}(y_1 - y_2), \quad (6) \\
\dot{y}_1 &= \frac{x_1 - x_2}{4r^2} - \frac{\varepsilon^2}{2}(x_1 - x_2), \quad (7)
\end{align*}
$$

with $2r(t) = |A_1A_2|$. By writing the motion equation of $A_2 = (x_2, y_2)$ we are led to: $(\dot{x}_1 + \dot{x}_2)/2 = 0 = (y_1 + y_2)/2$. This shows that, to this order, the centre point $I$ is indeed fixed, in agreement with equation (3). The latter equation was also expected: indeed, as noticed in the introduction, $y_I$ is known to remain constant whatever the distance to the wall. The motion of the vortices can then be investigated further by setting $x_1(t) = r(t)\cos\theta(t)$, $y_1(t) = r(t)\sin\theta(t)$, so that equations (6) and (7), with $x_2 = -x_1$ and $y_2 = -y_1$ lead to:

$$
\begin{align*}
\dot{r} &= -\varepsilon^2 r \sin 2\theta, \quad (8) \\
\dot{\theta} &= \frac{1}{r^2} - \varepsilon^2 \cos 2\theta. \quad (9)
\end{align*}
$$

Looking for an asymptotic solution in powers of $\varepsilon$, we get (setting $r(0) = 1 + \varepsilon^2/2$ and $\theta(0) = 0$):

$$
\begin{align*}
r(t) &= 1 + \frac{\varepsilon^2}{2} \cos 2t, \quad (10) \\
\theta(t) &= t - \varepsilon^2 \sin 2t, \quad (11)
\end{align*}
$$

and the vortex coordinates in the reference frame translating with $I$ then read:

$$
\begin{align*}
x_1(t) &= \cos t + \varepsilon^2 \left( \frac{\cos t}{2} \cos 2t + \sin t \sin 2t \right), \quad (12) \\
y_1(t) &= \sin t + \varepsilon^2 \left( \frac{\sin t}{2} \cos 2t - \cos t \sin 2t \right). \quad (13)
\end{align*}
$$

We therefore observe that the wall, when far from the vortex pair, creates an oscillation of the distance between the two vortices with period $\pi$ (half the period of the isolated vortex pair). Also, the rotation velocity of the vortices around $I$ is affected by a perturbation with period $\pi$. This periodic forcing corresponds to the effect of the straining flow $\varepsilon^2(-y, -x)$ appearing in the velocity field, which stretches and compresses the vortex pair twice during each rotation (see Fig. 2).

In the following lines we write $\vec{x} = (x, y)$, and the vortex position is also written in vector form: $\vec{x}_i(t) = \vec{x}_i^{(0)} + \varepsilon^2 \vec{x}_i^{(2)}$, with the components of the various vectors given by (12)-13). Then the streamfunction in the frame translating with $I$, written in powers of $\varepsilon$, reads:

$$
\psi_{\varepsilon}(\vec{x}, t) = \sum_{i=1}^{2} \psi_0 \left( \vec{x} - \vec{x}_i(t) \right) + \varepsilon^2 \psi_2(\vec{x}), \quad (14)
$$

with $\psi_0(\vec{x}) = -\log(x^2 + y^2)$ and $\psi_2(\vec{x}) = (x^2 - y^2)/2$. It can be expanded as:

$$
\psi_{\varepsilon}(\vec{x}, t) = \sum_{i=1}^{2} \left[ \psi_0 \left( \vec{x} - \vec{x}_i^{(0)}(t) \right) - \varepsilon^2 \vec{x}_i^{(2)} \cdot \nabla \psi_0 |_{\vec{x} - \vec{x}_i^{(0)}} \right] + \varepsilon^2 \psi_2(\vec{x}). \quad (15)
$$

This streamfunction corresponds to the flow induced by the two vortices and the wall, in the reference frame translating with $I$, and is valid only in the vicinity of the vortex pair $A_1A_2$.

B. Flow in the rotating frame

It is very convenient to investigate the motion of lagrangian particles in the reference frame translating with $I$ and rotating at speed $\Omega_0$, that is the angular velocity of the vortex pair in the absence of a wall. Let $\vec{e}_X, \vec{e}_Y$ denote an orthogonal basis attached to this frame, with $X, Y$ the corresponding coordinates. Assuming $\vec{e}_X = \vec{e}_x$ and $\vec{e}_Y = \vec{e}_y$ at $t = 0$ we have:
\(x = X \cos t - Y \sin t\) and \(y = X \sin t + Y \cos t\). Then, in this reference frame, and with these coordinates, the streamfunction reads

\[
\psi_r(X, Y, t) = \psi_a(x(X, Y, t), y(X, Y, t), t) + \frac{1}{2}(X^2 + Y^2),
\]

where the last term corresponds to the entrainment velocity of the rotating frame. By using (15) we obtain, after some algebra:

\[
\psi_r(X, Y, t) = \psi_{r0}(X, Y) + \varepsilon^2 \psi_{r2}(X, Y, t),
\]

where

\[
\psi_{r0}(X, Y, t) = -\log|Z^2 - 1|^2 + \frac{1}{2}|Z|^2,
\]

and

\[
\psi_{r2}(X, Y, t) = \frac{2}{|Z^2 - 1|^2}[(Y^2 - X^2 + 1) \cos 2t + 4XY \sin 2t] + \frac{1}{2} \left[(X^2 - Y^2) \cos 2t - 2XY \sin 2t\right],
\]

where \(Z = X + iY\). The leading order streamfunction corresponds to the steady flow in the rotating frame in the absence of wall, the \(\varepsilon^2\) terms are the wall effect.

### III. WALL-INDUCED CHAOTIC ADVECTION NEAR VORTICES

#### A. Splitting of the inner homoclinic cycle

In order to investigate the occurrence of chaotic advection in the flow (16) we consider fluid points, the dynamics of which reads:

\[
\frac{d\vec{X}}{dt} = \frac{\partial \psi_r}{\partial Y} \vec{e}_X - \frac{\partial \psi_r}{\partial X} \vec{e}_Y,
\]

where \(\vec{X}(t) = (X, Y)\) is the position of the fluid point. This is a perturbed hamiltonian system, the phase portrait of which is, to leading order, the same as the one of Fig. 2a of Angilella [12], which is reproduced here (Fig. 3) for clarity. This widely used phase portrait has two homoclinic cycles (\(\Sigma_1 \cup \Sigma_2\) and its symmetric in the lower half-plane), and two homoclinic trajectories (\(\Sigma_0\) and its symmetric in the left half-plane). Let \(\vec{q}_i(t)\) be a solution of the leading-order fluid points dynamics:

\[
\frac{d\vec{q}_i}{dt} = \frac{\partial \psi_{r0}}{\partial Y} \vec{e}_X - \frac{\partial \psi_{r0}}{\partial X} \vec{e}_Y,
\]

with initial condition on a separatrix \(\Sigma_i\) \((i = 0, 1, 2)\). Then the occurrence of chaos in the fluid point dynamics under the effect of the \(\varepsilon^2\) terms is related to the occurrence of simple zeros in the Melnikov function of the separatrix \(\Sigma_i\):

\[
M_i(t_0) = \int_{-\infty}^{\infty} [\psi_{r0}(\vec{q}_i(t)), \psi_{r2}(\vec{q}_i(t), t + t_0)] dt,
\]

where \(t_0\) is the starting time of the Poincaré section of the perturbed tracer dynamics, and \([., .]\) is the Poisson bracket. By choosing \(\vec{q}_i(0)\) on the "middle-point" of \(\Sigma_i\) (i.e. the intersection between the separatrix and \(OY\) for \(\Sigma_1\) and \(\Sigma_2\), and \(OX\), \(X \neq 0\), for \(\Sigma_0\)), the coordinates of \(\vec{q}_i(t)\) are either odd or even functions, and by using the various symmetries of the velocity field, one can get rid of the integral of odd functions of \(t\) in (20). We are led to \(M_i(t_0) = \alpha_i \sin(2t_0)\), where the amplitudes of the Melnikov functions \(\alpha_i\) are purely numerical constants. They are calculated by solving \(\vec{q}_i(t)\)'s numerically on the three separatrices (in the upper half-plane for \(\Sigma_{1,2}\) and in the right half-plane for \(\Sigma_0\): \(\alpha_0 \approx -0.58, \alpha_1 \approx -0.89, \alpha_2 \approx 7.31\). We therefore conclude that the two homoclinic cycles and the two homoclinic trajectories of Fig. 3 do not survive when a wall is present. This is confirmed by the Poincaré sections shown in Fig. 4. A stochastic zone indeed exists in the vicinity of the separatrices, and grows with \(\varepsilon\). It will be called "inner" stochastic zone in the following, as it exists in the very vicinity of the vortices (i.e. at distance \(O(d_0)\) from them). The typical scale of this zone, and its boundaries, are investigated in the next section.

![FIG. 3: Streamlines of the leading-order flow in the rotating frame, in the vicinity of the vortices (inner asymptotic field).](image-url)
B. Characterization of the inner stochastic zone

In order to quantify mixing in the vicinity of the vortices we derive a semi-analytic expression for the equation of the border of the stochastic layer. The thickness of this zone grows with \( \varepsilon \), and can be calculated from the above Melnikov functions (Chirikov [13], Weiss & Knobloch [14], Rom-Kedar [15], Kuznetsov & Zaslavsky [16], Trueba & Baltanas [17], Balasuriya [18]). We build the separatrix map of \( \Sigma_i \), for \( i \in \{1, 2\} \) (Fig. 5) by considering a fluid point \( \vec{X}(t) \) running under the effect of the perturbed flow (10). This point crosses the axis \( IY \) at discrete times \( t_n \). In addition, following Weiss & Knobloch [14], we define the set \( S \) composed of the hyperbolic points \( A \) and \( B \), and of segments forming two symmetric stars near \( A \) and \( B \) (Fig. 5). These stars have two branches parallel to \( IX \), and two other branches lying between \( \Sigma_1 \) and \( \Sigma_2 \). The detailed shape of the \( S \) is of no importance to construct the separatrix map, provided the trajectory of \( \vec{X}(t) \) remains close to the separatrices. In addition we call \( \tau_n \) the time at which the trajectory of \( \vec{X}(t) \) intersects \( S \) just prior time \( t_n \), that is: \( \tau_n < t_n < \tau_{n+1} \).

Let \( H_n \) denote the value of the unperturbed streamfunction \( \psi_0 \) at \( \vec{X}(\tau_n) \). The variation of "energy" \( H_n \) between two consecutive crossings of \( S \) is:

\[
\Delta H_n = H_{n+1} - H_n = \varepsilon^2 \int_{\tau_n}^{\tau_{n+1}} [\psi_r(\vec{X}(t)), \psi_2(\vec{X}(t), t)] \, dt.
\]

If the trajectory is close enough to a separatrix (Chirikov [13]), say \( \Sigma_i \), then we have \( \vec{X}(t) \approx \vec{q}_i(t-t_n) \), where \( \vec{q}_i \), parametrizing \( \Sigma_i \), has been introduced in the previous section, and is such that \( \vec{q}_i(0) = 0 \) belongs to the vertical axis \( IY \). Also, by noticing that \( \tau_n \ll t_n \ll \tau_{n+1} \), we obtain:

\[
\Delta H_n \approx \varepsilon^2 M_i(t_n), \tag{21}
\]

where \( M_i \) are the Melnikov functions of separatrices \( \Sigma_i \) calculated in the above section. The time lag \( t_{n+1} - t_n \) can be approximated by the half-period of the unperturbed orbit \([13] \psi_0 = H_{n+1}, \) so that we set:

\[
t_{n+1} = t_n + T(H_{n+1})/2. \tag{22}
\]

Equations (21) and (22) approximate the separatrix map in the vicinity of the homoclinic cycle \( \Sigma_1 \cup \Sigma_2 \). To our knowledge, no exact analytical expression for the period \( T(H) \) is known. However, following Kuznetsov & Zaslavsky [16], one can notice that the dynamics of fluid points is very slow in the vicinity of point \( A \), so that an approximate expression of \( T(H) \) can be found by expanding the streamfunction in the vicinity of \( A \). We then obtain hyperbolic streamlines for this simplified dynamics, and \( T(H)/2 \) is approximated as the time required for a fluid point running on an hyperbolic streamline to pass over \( A \). We are led to (see Appendix A):

\[
T(H) \simeq -\beta \ln|H_A - H|, \tag{23}
\]

with \( \beta = 4/\sqrt{5} \).

The amplitude of the rate of change of the time interval \( dt(t_{n+1} - t_n)/dt_n \) can then be calculated from equations (21) and (22):

\[
\max \left| \frac{dt_{n+1}}{dt_n} - 1 \right| = \varepsilon^2 |\alpha_i|/|H_A - H_{n+1}|. \tag{24}
\]
By writing that this quantity is equal to unity on the border of the stochastic layer (Rom-Kedar [15]) we obtain the energy $H_s$ corresponding to this boundary:

$$|H_s - H_A| = \beta|\alpha_1| \varepsilon^2. \quad (25)$$

The boundary of the stochastic layer around $\Sigma_2$ can then be obtained by setting $i = 2$ in the above formula. The curve $\psi_{t_0} = H_s$ has been plotted on the numerical Poincaré sections of Fig. 4 for comparison (thick black line), and we observe that, in spite of the rough approximations of the model, this curve corresponds to the border of the stochastic zone with an acceptable accuracy.

Result (26) can also be used to determine the distance $r_C$ between the border of the stochastic layer and point $C$, which can be thought of as the half-thickness of the stochastic layer at $C$. Indeed, by writing that $\psi_{t_0}(X_C + r_C \hat{e}_Y) = H_s$, and using a first-order Taylor expansion (assuming that $r_C$ is small):

$$r_C = \frac{\beta\alpha_2}{\gamma} \varepsilon^2, \quad (26)$$

where $\gamma = \partial \psi_{t_0}/\partial Y(C) \simeq 1.67$. The thickness of the stochastic layer of $\Sigma_2$ at $C$ is therefore: $\Delta_C = 2r_C \simeq 15.8 \varepsilon^2$. This formula is compared to measurements of the thickness $\Delta_C$ obtained from numerical computations (Fig. 5). Three kinds of computations are used here: Poincaré sections, particle clouds computed by using the asymptotic velocity in the rotating frame, and particle clouds computed by using the exact potential velocity $\hat{e}_X$ in the laboratory frame. Point $C$ has been chosen for these measurements because the thickness is well-defined there. We observe that the order-of-magnitude of $\Delta_C$ is well predicted by the theory, especially if $\varepsilon$ is small. For larger $\varepsilon$ discrepancies of about 100% appear, and these errors are known to be due to resonances occurring in the fluid point dynamics (Luo & Han [19], Trueba & Baltanas [17]) and to regular islands embedded in the stochastic layer. In spite of these discrepancies we can conclude that the theoretical prediction obtained from the separatrix map gives an acceptable lower bound to the thickness of the stochastic zone.

The typical size of the stochastic layer can also be estimated in the vicinity of the hyperbolic points $A$ (and $B$), by writing that $\psi_{t_0}(X_A + r_A \hat{e}_X) = H_s$, where $r_A$ is the distance from $A$ to the external border of the stochastic zone, towards $\hat{e}_X$. To obtain $r_A$ we then use a second-order Taylor expansion (assuming that $r_A$ is small). Indeed, because $A$ is a stagnation point of the unperturbed flow, the gradient of $\psi_{t_0}$ vanished there, and we are left with:

$$r_A = 2 \left( \frac{\beta\alpha_2}{5} \right)^{1/2} \varepsilon. \quad (27)$$

The typical size of the stochastic layer is therefore larger in the vicinity of $A$ (and $B$) than anywhere else, in the limit where $\varepsilon$ is small. This is a rather general
result which is a consequence of the fact that $A$ and $B$ are stagnation points. Here also we expect to provide only a lower bound of the actual $r_A$, since resonances might occur and make the thickness even bigger.

C. Comparison with transport by the exact potential velocity field

The asymptotic model presented in the above sections shows that some chaotic mixing must occur in the vicinity of a vortex pair as soon as a wall is present in the surroundings of the two vortices. In order to check the reliability of these calculations, which are valid only if the wall is far from the vortices, we have performed some calculations of fluid points transport by using the exact potential velocity field (set dimensionless by using $\Omega_0$ and $d_0$), in the laboratory frame.

Fig. 7 shows two particle clouds corresponding to $L_0 = 10d_0$ and $5d_0$ (that is $\varepsilon = 0.1$ and 0.2, respectively). Because our goal is to visualize fluid point mixing in the stochastic zone, the initial cloud is taken to be a small spot of size $0.1 \times 0.1$ centered at $I$ (which is initially located at $(0,0)$). The initial positions of the vortices are $(x_1,y_1) = (1 + \varepsilon^2/2, 1/\varepsilon)$ and $(x_2,y_2) = (-1 - \varepsilon^2/2, 1/\varepsilon)$. The final time of the three simulations is $t = 50\pi$, and the final location of the centre point is very close to the asymptotic value $(x_I,y_I) \approx (2\varepsilon t, L_0)$ in both cases. We observe that some particles collect on a structure which is very close to the separatrices described above, as a consequence of the existence of a stochastic zone in the vicinity of these separatrices. The thickness of the stochastic zone is of the order of the theoretical one, as shown in Fig. 6 (filled circles).

IV. MIXING WITH THE OUTER OPEN FLOW

The results of the above sections show that chaotic advection can occur in the vicinity of the vortices, due to the presence of the wall which induces a periodic unsteadiness in the vortex motion. However, the inner asymptotic model (valid only at distance $O(d_0)$ from vortices) does not bring any information about fluid mixing with the "external flow" which carries fluid elements from the far field. Fig. 8(a) shows the complete flow (solid lines), obtained from the exact potential solution in the frame translating with the vortices. Both the real and mirror vortices are shown. Lengths and velocities have been normalized by $L_0$ and $\Gamma/L_0$ respectively, which are relevant scales for the external flow, and which will be used to derive an outer asymptotic velocity field below. The vortex system is separated from the rest of the flow by a homoclinic cycle which can be thought of as the boundary between closed and open streamlines. The breaking of this homoclinic cycle is obviously an important mechanism for fluid mixing between the vortex system and the fluid outside (Pentek et al. [7]). This homoclinic cycle will be called the "outer cycle" in the following, as opposed to the so-called "inner cycle" $\Sigma_2 \cup \Sigma_1$.

Because the flow is two-dimensional, the only candidate for the splitting of this outer cycle is the rotation of the vortices around each other, which induces an unsteady perturbation to the homoclinic cycle $S_1S_2$. This bifurcation is investigated in the next paragraphs.
A. Asymptotic analysis of the outer homoclinic cycle

As noticed above, \( L_0 \) and \( \Gamma \) being the relevant scales of the external flow, we non-dimensionalize the streamfunction (1) by using \( L_0 \) for lengths and \( \Gamma/4\pi L_0 \) for velocities. Also, we consider the reference frame translating at the velocity \( x_1^0 = \Gamma/2\pi L_0 \) of a vortex with strength \( 2\Gamma \), placed at distance \( L_0 \) from the wall. The non-dimensional streamfunction of the flow induced by the four vortices with strengths \( \Gamma, \Gamma, -\Gamma, -\Gamma \), in this frame, is:

\[
\psi_*(\vec{X}_s) = \frac{\psi - x_1^0 y}{\Gamma/(4\pi)}.
\]

where \( \psi(x,y,t) \) is the streamfunction of the flow observed in the laboratory frame, and stars indicate non-dimensional variables. Still assuming that \( \varepsilon = \frac{d_0}{L_0} \ll 1 \), and using Eqs. (12) and (13) for the vortex motion, the streamfunction can be expanded as:

\[
\psi_*(\vec{X}_s) \simeq \psi_{0*}(\vec{X}_s) + \varepsilon^2 \psi_{2*}(\vec{X}_s) \cos \frac{2t_s}{\varepsilon^2} + \varepsilon^2 \psi_{2*}(\vec{X}_s) \sin \frac{2t_s}{\varepsilon^2} + O(\varepsilon^3),
\]

where \( \psi_{0*}(x_s, y_s) = 2 \log[\frac{x_1^2 + y_1^2 +1+2y_s}{x_1^2 + y_1^2 +1-2y_s}] - 2y_s \) is the streamfunction of a simple dipole centered at \((0,0)\) (i.e. single vortex plus its mirror vortex, see Fig. 8(b)), and the \( \varepsilon^2 \) terms manifest the fact that we have a vortex pair with strength \( \Gamma + \Gamma \), instead of a single vortex with strength \( 2\Gamma \), so that the flow is unsteady. The streamfunctions \( \psi_{2*} \) and \( \psi_{2*} \) are steady, rational functions of \( x_s \) and \( y_s \) only.

With the time scale used here, the rotation of the vortex pair is fast, so that the flow (29) is a rapidly perturbed dipolar flow. Fig. 8(a) shows this streamfunction when \( \varepsilon = 0.3 \) (dashed line), together with the exact four-vortex streamfunction already discussed (solid line). The two flows are very close, except in the vicinity of both pairs, as expected. The \( O(\varepsilon^3) \) terms in (29) are identically zero (for symmetry reasons), and this is why the agreement is good even if \( \varepsilon \) is not very small here. For \( \varepsilon \leq 0.2 \) dashed lines and solid lines (not shown here) are indistinguishable, except in the very vicinity of the vortices.

Particle motion can therefore be approximated by a perturbed Hamiltonian system, with Hamiltonian \( \psi_{0*} \). The Melnikov function attached to the cycle \( S_1S_2 \) reads (removing the stars):

\[
M_{ext}(t_0) = \int_{-\infty}^{\infty} \left[ \psi_0(\vec{q}(t)), \psi_{2*}(\vec{q}(t)) \right] \cos(\frac{2t + t_0}{\varepsilon^2}) dt
\]

\[+ \int_{-\infty}^{\infty} \left[ \psi_0(\vec{q}(t)), \psi_{2*}(\vec{q}(t)) \right] \sin(\frac{2t + t_0}{\varepsilon^2}) dt,
\]

where \( \vec{q}(t) \) runs on the unperturbed cycle, and is chosen such that \( \vec{q}(0) \) coincides with point C of Fig. 8(b). This Melnikov function approximates (to order \( \varepsilon^2 \)) the "energy" difference \( \psi_0(\vec{X}(\tau_2)) - \psi_0(\vec{X}(\tau_1)) \), where \( \vec{X}(t) \) is the trajectory of a fluid point advected by the perturbed flow, with a trajectory very close the separatrix \( S_1S_2 \). The times \( \tau_1 \) and \( \tau_2 \) correspond to the particle passing near the saddle points \( S_1 \) and \( S_2 \). Because our perturbation is rapid, the amplitude of the Melnikov function will now depend on \( \varepsilon \) (Gelfreich...
The integrands in the above integrals are unknown analytically, they are however localized around zero and decay exponentially. We have chosen to approximate them with a mixed polynomial/gaussian expression of the form $P(t)/\cosh^2(\delta t)$, where $P$ is a fitted polynomial and $\delta$ is a fitted constant. The integrals can then be obtained analytically, and we are led to $M_{ext}(t_0) = A(\varepsilon) \sin(2t_0/\varepsilon^2)$, with:

$$A(\varepsilon) = \frac{\gamma}{\varepsilon^2} e^{-\pi/\delta^2},$$

where $\delta$ and $\gamma$ are purely numerical constants depending on the fitting procedure: $\gamma \approx 18$, $\delta \approx 5$. Note that this formula depends on the form chosen to approximate our integrands. This is therefore a numerical result, rather than a truly analytical one, being underestimated our integrands. This is therefore a numerical result, rather than a truly analytical one, being underestimated.

Because $A(\varepsilon) > 0$ for all $\varepsilon$, and $M_{ext}(t_0)$ always has simple zeros when $t_0$ varies, portions of fluid will be exchanged between the interior of the cycle $S_1S_2$, and the fluid outside. However, the amplitude is so small as soon as $\varepsilon \leq 0.2$ (i.e. $A(0.2) < 7 \times 10^{-5}$), that mixing through the cycle $S_1S_2$ should be hardly visible for such small $\varepsilon$’s. In contrast, when $\varepsilon$ increases above 0.2, $A(\varepsilon)$ quickly reaches values of order unity. By constructing the separatrix map of $S_1S_2$, one can derive the theoretical thickness $\Delta$ of the stochastic layer surrounding the outer homoclinic cycle. In the vicinity of point C of Fig. 8(b), we are led to $\Delta_C \approx 0.5A(\varepsilon)$. This quantity has been plotted in Fig. 9 (dashed line).

We indeed observe that the thickness is exponentially small when $\varepsilon \leq 0.2$ ($L_0 \geq 5d_0$), and quickly reaches values of order unity as soon as $\varepsilon$ is larger than about 0.25. [Note that, even though the diagram suggests a sharp increase of $\Delta_C$, there is no critical $\varepsilon$ here: the Melnikov function has simple zeros for all $\varepsilon$.] Fig. 9 suggests that the vortex pair might be apparently separated from the outer open flow when $L_0 \geq 5d_0$, since the whole vortex system will be surrounded by an extremely thin stochastic zone. Also, the thickness of this stochastic zone should quickly reach a value of order unity as soon as $L_0 \leq 3d_0$. These two points are checked numerically in the next paragraph.

B. Numerical verification

Computations of particle clouds in the laboratory frame, using the exact four-vortex potential solution (9), have been performed. Particles have been injected at $t = 0$ near the vortices (in box A of Fig. 8(b)). Also, to check whether portions of fluid could enter within the cycle $S_1S_2$, we have injected another set of particles outside the cycle $S_1S_2$ (in box B of Fig. 8(b)). These simulations confirm the significant difference between the thickness of the inner cycle and the one of the outer cycle, as suggested by the theoretical curves of Fig. 9.

![FIG. 9: The role of the inner and outer separatrices in chaotic mixing. The four graphs show particle clouds in the laboratory frame at $t = 100\pi$, obtained by using the exact four-vortex potential solution. Particles marked with black dots have been injected near the upper vortices at $t = 0$ (in box A of Fig. 8(b)). Those marked with empty circles have been injected outside the cycle $S_1S_2$, ahead of vortices (in box B of Fig. 8(b)). Graph (a): $\varepsilon \approx 0.2$ ($L_0/d_0 = 5$). Graph (b): $\varepsilon \approx 0.25$ ($L_0/d_0 = 4$). Graph (c): $\varepsilon \approx 0.286$ ($L_0/d_0 = 3.5$). Graph (d): $\varepsilon \approx 0.3125$ ($L_0/d_0 = 3.2$). Solid lines are streamlines.](image-url)
Indeed, Fig. (4a) shows the two particle clouds at $t = 100\pi$, when $L_0/d_0 = 5$ ($\varepsilon = 0.2$). Units are the same as in the simulations of section III, that is: $d_0$ for lengths and $\Omega_0$ for times. Particles marked with black dots are those which have been injected in box A (they are also indicated with a (A)), and those marked with empty circles have been injected in box B (they are also indicated with a (B)). Even though these computations have been conducted in the laboratory frame, the homoclinic cycles appearing in the rotating frame (section III) and in the translating frame (section IV), naturally appear here. Particles injected near vortices wander in the vicinity of the inner cycle $\Sigma_2 \cup \Sigma_4$ investigated in section III whereas those injected ahead of the vortices move in the vicinity of the outer cycle $S_1S_2$. Some particles penetrate within this cycle, in agreement with the fact that the Melnikov function has simple zeros. When $L_0/d_0 = 4$ (Fig. (4b)) the thickness of the inner stochastic zone around the cycle $\Sigma_2$ is larger, whereas the thickness of the outer stochastic zone around the cycle $S_1S_2$ does not appear to be very different from case (a), in agreement with the theoretical thickness calculated in the previous section (dashed line of Fig. 6). When $L_0/d_0 = 3.5$ (Fig. (4c)) the thickness of the outer stochastic zone is slightly larger than in case (b), but the two clouds are still separated. For $L_0/d_0 = 3.2$ ($\varepsilon = 0.3125$, Fig. (4d)) the two particle clouds mix significantly, in agreement with the fact that the amplitude of the Melnikov function is of order unity there. However, the thickness of the outer stochastic zone cannot be measured from these graphs and compared to the theoretical value of $\Delta_C$: this is due to the fact that particles injected in box B now mix with those injected in box A. These simulations confirm, however, that the outer stochastic zone remains negligible compared to the inner one as long as $L_0/d_0$ is larger than about 4, so that the vortex pair might exchange very little fluid with the outer flow in this case.

V. CONCLUSION

We have analyzed how point vortex pairs, which are closed flow structures in the absence of walls, exchange portions of fluid with their surroundings as soon as a wall, even far from the vortices, is present. This is not a new result, since Pentek et al. [7] had shown that chaotic advection is present in leapfrogging vortex pairs, but the asymptotic approach used here shows that the mechanism at work is a combination of the stretching flow induced by the wall and of the unsteadiness due to vortex rotation. Our work has therefore some common features with the one of Rom-Kedar et al. [8], even though the flow is different. Relevant quantities, like the thickness of stochastic layers, have been calculated by using Melnikov functions, as done in previous works for other flows (e.g. Kuznetsov & Zaslavsky [10], Trueba & Baltanas [17]).

The breaking of the inner separatrices, located in the very vicinity of the vortex pair and rotating with them, induces a stochastic layer the thickness of which scales like the squared inverse distance to the wall. The rotation of vortices around each other can also affect the outer homoclinic cycle $S_1S_2$ separating the vortex system and the rest of the flow domain. The thickness of this outer stochastic layer has been shown to be negligible as long as $\varepsilon \leq 0.25$, so that the vortex pairs might be apparently separated from the open flow zone when $L_0 \geq 4d_0$. However, this thickness rapidly jumps to unity for larger $\varepsilon$'s. As a consequence, significant portions of fluid coming from infinity can penetrate within this cycle and mix with the fluid located near vortices.

One could argue that the values of $\varepsilon$ considered in the various computations presented here are not very small, so that the comparison with the asymptotic results might be done with care. However, the asymptotic analyses show that the leading-order wall effect scales like $\varepsilon^2$. The approximation is especially accurate in the case of the outer separatrix (section IV), since the velocity truncation error scales like $\varepsilon^4$ there.

The generalization to vortices with unequal strengths can be done by using the same asymptotic approach when the wall is far from the vortices, provided the sum of the two strengths is non-zero. Here also we obtain that the center of vorticity $I$ (strength-weighted averaged position) moves at constant speed up to order 3 and that the wall induces a stretching flow, in the frame translating with $I$, which triggers a blinking-vortex mechanism. One can check that the various separatrices do not survive when a wall is present in this case also.

Finally, it should be noted that the "wall" analyzed in this work is in fact a symmetry line, rather than a wall, as the inviscid fluid considered here slips on solid boundaries. The generalization of these results to real fluids, using the Navier-Stokes equations together with no-slip boundary conditions, must be done with care. Indeed, it is well known that vortices, even with a large Reynolds number, induce counter-rotating secondary vortices when a wall is present. This vorticity, which is initially located inside the wall boundary layer, can contaminate the flow and interact with the primary vortices, leading to a complex vortex system, very different from the ideal one investigated here.

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Appendix A: Calculation of the period near the separatrix

In the vicinity of saddle point A the unperturbed streamfunction can be expanded as:

\[ H = \psi_0(\vec{X}) \simeq H_A + \frac{1}{2}(\vec{X} - \vec{X}_A) \cdot \nabla^2 \psi_0(\vec{X} - \vec{X}_A) \]

\[ \simeq H_A - \frac{\gamma^2}{4} + \frac{5}{4}(X - X_A)^2. \quad (A1) \]

It corresponds to open hyperbolic trajectories like the one shown in Fig. 10(a). For such a trajectory the time required to pass near point A is very large if \( \delta H = |H - H_A| \) is small, and this time is a good approximation of the half-period of trajectories around the homoclinic cycle [16]. We therefore write

\[ T(H)/2 \simeq \int_{-a}^{a} \frac{dY}{V_Y} \]

where \( a \) is an arbitrary initial height of the fluid point, which has very little effect on \( T(H) \) when the trajectory is close to the separatrix. The vertical velocity is \( V_Y = -(5/2)(X - X_A) \), where \( X \) is given by (A1), and we get:

\[ T(H) = -\frac{8}{\sqrt{5}} \log \frac{2\delta H^{1/2}}{a + (4\delta H + a^2)^{1/2}}. \]

In the limit \( \delta H \to 0 \), keeping \( a \) fixed, we obtain Eq. (23). Fig. 10(b) shows a plot of Eq. (23) together with a numerical measurement of \( T(H) \), and we observe that the agreement is excellent.

[1] P. FLOHR and J.C. VASSILICOS. Accelerated scalar dissipation in a vortex. J. Fluid Mech., 348:295–317, 1997.
[2] K. BAJER. Flux expulsion by a point vortex. European Journal of Mechanics - B/Fluids, 17(4):653–664, 1998.
[3] H. AREF. Stirring by chaotic advection. J. Fluid Mech., 143:1–21, 1984.
[4] H. AREF. The development of chaotic advection. Phys. Fluids, 14(4):1315–1325, 2002.
[5] D. BEIGIE, A. LEONARD, and S. WIGGINS. Invariant manifold templates for chaotic advection. Chaos, Solitons & Fractals, 4(6):749–868, 1994.
[6] A. WONHAS and VASSILICOS. Mixing in frozen and time-periodic two-dimensional vortical flows. J. Fluid Mech., 442:359–385, 2001.
[7] A. PENTEK, T. TEL, and Z. TOROCZKAI. Chaotic advection in the velocity field of leapfrogging vortex pairs. J. Phys. A: Math. Gen., 28:2191–2216, 1995.
[8] V. ROM-KEDAR, A. LEONARD, and S. WIGGINS. An analytical study of transport, mixing and chaos in an unsteady vortical flow. J. Fluid Mech., 214:347–394, 1990.
[9] D.J. ACHESON. Instability of vortex leapfrogging. Eur. J. Phys., 21:269–273, 2000.
[10] X. CARTON, G. MAZE, and B. LEGRAS. A two-dimensional vortex merger in an external strain field. Journal of Turbulence, 3:045, 2002.
[11] G. MAZE, X. CARTON, and G. LAPEYRE. Dynamics of a 2D vortex doublet under external deformation. Regular and Chaotic Dynamics, 9(4):477–497, 2004.
[12] J. R. ANGILELLA. Dust trapping in vortex pairs. Physica D: Nonlinear Phenomena, 239:1789–1797, 2010.
[13] B.V. CHIRIKOV. A universal instability of many-dimensional oscillator systems. Physics Reports, 52(5):263–379, 1979.
[14] J.B. WEISS and E. KNOBLOCH. Mass transport and mixing by modulated travelling waves. Physical Review A, 40(5):2579–2589, 1989.
[15] V. ROM-KEDAR. Homoclinic tangles - classification and applications. Nonlinearity, 7:441–473, 1994.
[16] L. KUZNETSOV and G. M. ZASLAVSKY. Regular and chaotic advection in the flow field of a three-vortex system. Phys. Rev. E, 58(6):7330–7349, 1998.
[17] J.L. TRUEBA and J.P. BALTANAS. On the estimate of the stochastic layer width for a model of tracers dynamics. Chaos, 13(3):866–873, 2003.
[18] S. BALASURIYA. Optimal perturbation for enhanced chaotic transport. Physica D, 202:155–176, 2005.
[19] A. C. J. LUO and A.P. S. HAN. The resonance theory for stochastic layers in nonlinear dynamic systems. Chaos, Solitons and Fractals, 12:2493–2508, 2001.
[20] V. G. GELFREICH. Melnikov method and exponentially small splitting of separatrices. Physica D., 101:227–248, 1997.
FIG. 10: Graph (a): linearized flow in the vicinity of stagnation point A. Graph (b): comparison between the theoretical period of trajectories near the separatrix (Eq. 23, dashed line) and the numerical one (dots).