A SHUFFLING THEOREM FOR LOZENGE TILINGS OF DOUBLY-DENTED HEXAGONS

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Abstract. MacMahon’s theorem on plane partitions yields a simple product formula for tiling number of a hexagon, and Cohn, Larsen and Propp’s theorem provides an explicit enumeration for tilings of a dented semihexagon via semi-strict Gelfand–Tsetlin patterns. In this paper, we prove a natural hybrid of the two theorems for hexagons with an arbitrary set of unit triangles removed along the horizontal axis. In particular, we show that the ‘shuffling’ of removed unit triangles only changes the tiling number of the region by a simple multiplicative factor. Our main result generalizes a number of known enumerations and asymptotic enumerations of tilings. We also reveal connections of the main result to the study of symmetric functions and \( q \)-series.

1. Introduction

MacMahon’s classical theorem \([1]\) on plane partition fitting in a given box is equivalent to the fact that the number of lozenge tilings of a centrally symmetric hexagon \( H(a, b, c) \) of side-lengths \( a, b, c, a, b, c \) (in this cyclic order) is given by the simple product:

\[
\text{PP}(a, b, c) := \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i + j + k - 1}{i + j + k - 2}.
\]

This formula was generalized by Cohn, Larsen and Propp \([2, Proposition 2.1]\) when they presented a correspondence between lozenge tilings of a semihexagon with unit triangles removed on the base and semi-strict Gelfand-Tsetlin patterns. In particular, the dented semihexagon \( S_{a,b}(s_1, s_2, \ldots, s_a) \) is the region obtained from the upper half of the symmetric hexagon of side-lengths \( b, a, a, b, a, a \) (in clockwise order, starting from the north side) by removing \( a \) unit triangles along the base at the positions \( s_1, s_2, \ldots, s_a \) from left to right. The number of lozenge tilings of the dented semihexagon is given by

\[
\text{M}(S_{a,b}(s_1, s_2, \ldots, s_a)) = \prod_{1 \leq i < j \leq a} \frac{s_j - s_i}{j - i},
\]

where we use the notation \( \text{M}(R) \) for the number of lozenge tilings of the region \( R \).

In this paper, we consider a hybrid object between MacMahon’s hexagon and Cohn–Larsen–Propp’s dented semihexagon. Our region is a hexagon on the triangular lattice, as in the case of MacMahon’s theorem, with an arbitrary set of unit triangles removed along a horizontal axis, like the dents in Cohn–Larsen–Propp’s theorem (see Fig. 2.1 A). In general, the tiling numbers of such regions are not given by simple product formula. However, we show that their tiling number only changes by a simple multiplicative factor when we shuffle the positions of up- and down-pointing removed triangles (Theorem 2.1).

Our main theorem implies a number of known tiling enumerations of regions with ‘holes’ (e.g. \([3, 4]\)). Here, a hole is a portion removed from a region. We also show that our main theorem can be used to obtain new results in asymptotic enumeration of tilings, including the enumeration of the so-called ‘doubly-dented hexagon’ in \([4]\), the main results of the first author about hexagon with three arrays.
of triangles removed in [5] Theorems 2.11 and 2.12 and Ciucu’s main results about $F$-cored hexagons in [3] Theorems 1.1 and 2.1 (see Corollary 3.1).

2. A SHUFFLING THEOREM

Let $x, y, n, u, d$ be nonnegative integers, such that $u, d \leq n$. Consider a symmetric hexagon of side-lengths $x + n - u, y + u, y + d, x + n - d, y + d, y + u$ in clockwise order, starting from the north side. We remove $u + d$ arbitrary unit triangles along the lattice line $l$ that contains the west and the east vertices of the hexagon. Assume further that, among these $u + d$ removed triangles, there are $u$ up-pointing ones and $d$ down-pointing ones. Let $U = \{s_1, s_2, \ldots, s_u\}$ and $D = \{t_1, t_2, \ldots, t_d\}$ be, respectively, the sets of positions of the up-pointing and down-pointing removed unit triangles (ordered from left to right), such that $|U \cup D| = n$ (i.e., $U, D \subseteq [x + y + n]$, $U$ and $D$ are not necessarily disjoint). Denote by $H_{x,y}(U; D)$ the resulting region (see Fig. 2.1 A for an example of such a region and Fig. 2.1 B for a sample tiling). We ignore the two horizontal unit “barriers” at the positions 6 and 13 on $l$ at the moment.

We now consider ‘shuffling’ the up- and down-pointing unit triangles in $U \cup D$ to obtain new position sets $U'$ and $D'$ for the up-pointing and down-pointing removed triangles. The following theorem shows that the shuffling of removed triangles only changes the tiling number by a simple multiplicative factor. Moreover, the factor can be written in a similar form to Cohn–Larsen–Propp’s formula (i.e. the product on the right-hand side of Eq. 1.2).

**Theorem 2.1** (Shuffling Theorem). For nonnegative integers $x, y, n, u, d$ ($u, d \leq n$) and four ordered subsets $U = \{s_1, s_2, \ldots, s_u\}$, $D = \{t_1, t_2, \ldots, t_d\}$, $U' = \{s'_1, s'_2, \ldots, s'_u\}$, and $D' = \{t'_1, t'_2, \ldots, t'_d\}$ of $[x + y + n]$ such that $U \cup D = U' \cup D'$, and $U \cap D = U' \cap D'$. Then

$$
\frac{M(H_{x,y}(U; D))}{M(H_{x,y}(U'; D'))} = \prod_{1 \leq i < j \leq u} \frac{s_j - s_i}{s'_j - s'_i} \prod_{1 \leq j \leq d} \frac{t_j - t_i}{t'_j - t'_i}.
$$

We would like to emphasize that, in general, the numbers of tilings of two regions on the left-hand side of Eq. 2.1 are not given by simple product formulas.

**Remark 2.2** (A geometrical interpretation). By Cohn–Larson–Propp’s theorem (see Eq. 1.2), Eq. 2.1 in Theorem 2.1 can be written in terms of tiling numbers as

$$
\frac{M(H_{x,y}(U; D))}{M(H_{x,y}(U'; D'))} = \frac{M(S_{u,x+y+n-u}(U))}{M(S_{u,x+y+n-u}(U'))} \frac{M(S_{d,x+y+n-d}(D))}{M(S_{d,x+y+n-d}(D'))}.
$$

One readily sees that the two dented semihexagons in the numerator of the right-hand side are obtained by dividing the region $H_{x+y,0}(U; D)$ along the horizontal axis $l$. Similarly, the two dented semihexagons

![Figure 2.1](image-url)
in the denominator are obtained by dividing $H_{x+y,0}(U';D')$ along the horizontal axis. It would be interesting to have a combinatorial explanation for this.

The special case of Shuffling Theorem 2.4 in which $U \cap D = \emptyset$ has many applications in the study of a new type of hole, called a ‘fern’ (see e.g. [3–5]). The detailed applications will be discussed in the next sections.

3. AN ASYMPTOTIC ENUMERATION

Ciucu and Krattenthaler in [3] proved a counterpart of MacMahon’s theorem (Eq. 1.1) by obtaining the asymptotic tiling number of the exterior of a concave polygon with an arbitrary number sides (see the contour in Fig. 3.1 C). Recently, the first author generalized the asymptotic result of Ciucu to the union of three polygons [5]. In Corollary 3.1 we will show a multi-parameter generalization of the latter two asymptotic results.

We now assume that the set of removed unit triangles is partitioned into $k$ separated clusters (i.e. chains of contiguous unit triangles). Denote these clusters by $C_1, C_2, \ldots, C_k$ and the distances between them by $d_1, d_2, \ldots, d_k$ ($d_i > 0$), as they appear from left to right. For the sake of convenience, we assume that $C_1$ is attached to the west vertex of the hexagon, that $C_k$ is attached to the east vertex of the hexagon, and that $C_1$ and $C_k$ may be empty. We use the notation $H_{x,y}(C_1, \ldots, C_k; d_1, \ldots, d_k)$ for these regions (see Fig. 3.3 for an example; the black unit triangles indicate the ones removed). For each cluster $C_i$, we use the notations $U_i$ and $D_i$ for the index sets of its up-pointing and down-pointing triangles. For each cluster $C_i$, we can shuffle unit triangles at the positions in $U_i \Delta D_i$ to obtain a new cluster $C'_i$, for $i = 1, 2, \ldots, k$. The index sets of triangles in $C'_i$ are denoted by $U'_i$ and $D'_i$. Assume that $|U_i| = u_i$, $D_i = d_i$, $|U'_i| = u'_i$, $|D'_i| = d'_i$ and $|U_i \cup D_i| = |U'_i \cup D'_i| = f_i$. We call $f_i$ the length of the cluster $C_i$ (and also the length of $C'_i$).

We now consider the behavior of the tiling number of the region when the side-lengths of the outer hexagon and the distances between two consecutive clusters get large.

**Corollary 3.1.** For nonnegative integers $x$ and $y$,

$$
\lim_{N \to \infty} \frac{M(H_{x,y,N}(C_1, \ldots, C_k; Nd_1, \ldots, Nd_k))}{M(H_{x,y,N}(C'_1, \ldots, C'_k; Nd_1, \ldots, Nd_k))}
= \prod_{i=1}^{k} \frac{s^+(C_i)s^-(C'_i)}{s^+(C'_i)s^-(C_i)}
$$

(3.1)

where $s^+(C_i) = M(S_{u_i,f_i-u_i}(U_i))$ and $s^-(C_i) = M(S_{d_i,f_i-d_i}(D_i))$ are respectively the tiling numbers of the dented semihexagons whose dents are defined by the up-pointing triangles and down-pointing triangles in the cluster $C_i$, and where $s^+(C'_i)$ and $s^-(C'_i)$ are defined similarly with respect to $C'_i$. 

![Figure 3.1. Three contours: (A) the contour in MacMahon’s theorem, (B) the contour in [6], and (C) the contour in [3].](image)
Corollary 3.1 can be visualized as in Fig. 3.2 for $k = 3$. The dented semiheaxagons corresponding to $s^+(C_i)$ and $s^-(C_i)$ are the upper and lower halves of the ‘numerator hexagon’ in the $i$th fraction on the right-hand side; the dented semiheaxagons corresponding to $s^+(C_i')$ and $s^-(C_i')$ are the upper and lower halves of the ‘denominator hexagon’ in the $i$th fraction, for $i = 1, 2, 3$.

**Sketch of the proof.** Assume that $U_i = \{a_{1i}^{(i)}, \ldots, a_{ui}^{(i)}\}$ and $U'_i = \{c_{1i}^{(i)}, \ldots, c_{ui}^{(i)}\}$. Applying Theorem 2.1 to the regions $H_{Nx,Ny}(U; D)$ and $H_{Nx,Ny}(U'; D')$, for $U := \bigcup_i U_i$, $D := \bigcup_j D_j$, $U' := \bigcup_i U'_i$, $D' := \bigcup_j D'_j$, we get

$$\begin{align*}
\frac{M(H_{Nx,Ny}(C_1, \ldots, C_k; Nd_1, \ldots, Nd_k-1))}{M(H_{Nx,Ny}(C'_1, \ldots, C'_k; Nd_1, \ldots, Nd_k-1))} = \frac{\Delta(U)\Delta(D)}{\Delta(U')\Delta(D')}.
\end{align*}
$$

where the operation $\Delta$ is defined as $\Delta(S) := \prod_{1 \leq i < j \leq n} |s_j - s_i|$ for an ordered set $S = \{s_1 < s_2 < \cdots < s_n\}$. However, one would find it more convenient to use the equivalent definition $\Delta(S) := \prod_{1 \leq i \neq j \leq n} |s_j - s_i|$ here.

We have $\frac{\Delta(U)}{\Delta(U')} = \prod_{i,j,p,q} \frac{|a_{ij}^{(i)} - a_{ij}^{(j)}|}{|\hat{e}_p^{(i)} - \hat{e}_q^{(j)}|}$ (where $p \neq q$ if $i \neq j$). It is easy to see that, if $i \neq j$, the fraction $\frac{|a_{ij}^{(i)} - a_{ij}^{(j)}|}{|\hat{e}_p^{(i)} - \hat{e}_q^{(j)}|}$ tends to 1, as $N$ gets large (for any $p, q$). Thus $\frac{\Delta(U)}{\Delta(U')}$ tends to $\prod_{i=1}^k \prod_{p \neq q} \frac{|a_{ij}^{(i)} - a_{ij}^{(j)}|}{|\hat{e}_p^{(i)} - \hat{e}_q^{(j)}|} = \prod_{i=1}^k s^+(F_i)$. Similarly, we see that $\frac{\Delta(D)}{\Delta(D')}$ tends to $\prod_{i=1}^k s^-(F_i)$, completing the proof. \hfill \Box

A **forced lozenge** in a region $R$ is a lozenge that appears in every tiling of $R$. The removal of one or more forced lozenges does not change the number of tilings of the region. If a region admits a lozenge tiling, then it must have the same number of up-pointing and down-pointing unit triangles.

We now consider the special case when $U \cup D = \emptyset$ (i.e. in each cluster $C_i$ we have $U_i \cap D_i = \emptyset$). Each cluster $C_i$ can be partitioned into intervals of triangles of the same orientation (we call each of these intervals an ‘up-interval’ or a ‘down-interval’ if it consists of up-pointing triangles or down-pointing triangles, respectively). For each cluster $C_i$, we can remove forced vertical lozenges above each up-interval and below each down-interval comprised of two or more unit triangles. We obtain a new region with the same tiling number in which each cluster is replaced by a chain of removed equilateral triangles of alternating orientations (see Fig. 3.3); the forced lozenges are colored white. Each such chain of triangles is called a ‘fern’ (see e.g. 5); the side-lengths of triangles in a fern are equal to the lengths of the intervals of unit triangles of the same orientation in the corresponding cluster. Denote by $E_{x,y}(F_1, \ldots, F_k; d_1, \ldots, d_k-1)$ the corresponding hexagon with ferns removed (the fern $F_i$ corresponds to the cluster $C_i$; and the fern $F'_i$ corresponds to the cluster $C'_i$). By setting $k = 3$, $d_1 = d_2$ or $d_1 = d_2 - 1$, and specifying that the cluster $C_i'$ have all their up-pointing triangles on the left and all down-pointing triangles on the right, for $i = 1, 2, 3$ (equivalently, the fern $F'_i$ consists

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**Figure 3.2.** Illustrating Corollary 3.1
A SHUFFLING THEOREM FOR LOZENGE TILINGS OF DOUBLY-DENTED HEXAGONS

Figure 3.3. Obtained a hexagon with ferns removed from the region $H_{x,y}(C_1, \ldots, C_k; d_1, \ldots, d_{k-1})$.

of two triangles, an up-pointing triangles followed by a down-pointing triangles), our Corollary 3.1 implies the first author’s work in [5, Theorem 2.11]. Similarly, we can recover the work of Ciucu in [3, Theorem 1.1] by further specialization $C_1 = C_3 = \emptyset$ (i.e. we actually have only a non-empty fern $F_2$ in the center of the region).

4. PROOF OF THE MAIN THEOREM

Our proofs are based on the following powerful graphical condensation lemma first introduced by Kuo [7]:

Lemma 4.1. Let $G = (V_1, V_2, E)$ be a planar bipartite graph with the two vertex classes $V_1$ and $V_2$ such that $|V_1| = |V_2| + 1$. Assume that $u, v, w, s$ are four vertices appearing in this cyclic order around a face of $G$, such that $u, v, w \in V_1$ and $s \in V_2$. Then

$$M(G-v)M(G-\{u, w, s\}) = M(G-u)M(G-\{v, w, s\}) + M(G-w)M(G-\{u, v, s\}).$$

We call such a region balanced. The following lemma allows us to decompose a region into smaller regions when enumerating tilings in certain situations.

Lemma 4.2 (Region-splitting Lemma [8, 9]). Let $R$ be a balanced region on the triangular lattice. Assume that a balanced sub-region $Q$ of $R$ satisfies the condition that the unit triangles in $Q$ that are adjacent to some unit triangle of $R - Q$ have the same orientation. Then $M(R) = M(Q)M(R - Q)$.

Proof of Theorem 2.1. We need to show that

$$M(H_{x,y}(U; D)) = \frac{\Delta(U)\Delta(D)}{\Delta(U')\Delta(D')} M(H_{x,y}(U'; D')).$$

We prove Eq. (1.2) by induction on $x + y$. The base cases are the situations in which $x = 0$ or $y = 0$.

If $y = 0$, then we apply Lemma 4.2 to the region $R = H_{x,0}(U; D)$ with the subregion $Q$ the portion above the horizontal axis $l$. The subregion $Q$ and its complement $R - Q$ are the dented semihexagons $S_{u, x+n-u}(U)$ and $S_{d, x+n-d}(D)$ (see Fig. 4.1 A), and by Cohn–Larsen–Propp’s formula (Eq. 1.2)

$$M(H_{x,0}(U; D)) = \prod_{1 \leq i < j \leq u} \frac{s_j - s_i}{j - i} \prod_{1 \leq i < j \leq d} \frac{t_j - t_i}{j - i}. \quad (4.3)$$

For the region $R' = H_{x,0}(U'; D')$, we similarly obtain

$$M(H_{x,0}(U'; D')) = \prod_{1 \leq i < j \leq u'} \frac{s'_j - s'_i}{j - i} \prod_{1 \leq i < j \leq d'} \frac{t'_j - t'_i}{j - i}, \quad (4.4)$$

and Eq. (1.2) follows.

If $x = 0$, consider the subregion $Q$ of $R = H_{0,y}(U; D)$ that is obtained from the portion above the axis $l$ by removing all up-pointing unit triangles in $(U \cup D)^c = [y + n] \setminus (U \cup D)$. $Q$ is the dented

Figure 3.3. Obtained a hexagon with ferns removed from the region $H_{x,y}(C_1, \ldots, C_k; d_1, \ldots, d_{k-1})$.
semihexagon $S_{y+u,n-u}((U \cup D)^c \cup U)$, and its complement, after removing forced lozenges at the positions not in $U \cup D$, is the dented semihexagon $S_{y+d,n-u}((U \cup D)^c \cup D)$ (see Fig. 4.1 B). Therefore, the tiling number of $R$ can be written as a product of the tiling numbers of two dented semihexagons. Similarly, the tiling number of $R' = H_{0,y}(U'; D')$ can also be written as a product of tiling numbers of two dented semihexagons. Then Eq. 4.2 also follows from Cohn–Larsen–Propp’s formula (i.e. Eq. 1.2).

For the induction step, we assume that $x$ and $y$ are both positive, and that Eq. 4.2 holds for any $H$-type regions whose sum of $x$- and $y$-parameters is strictly less than $x + y$. We will use Kuo condensation in Theorem 4.1 to obtain a recurrence for the tiling number on the left-hand side of Eq. 4.2, and we will show that the expression on the right-hand side satisfies the same recurrence. Then Eq. 4.2 follows from the induction principle.

We apply Kuo condensation to the dual graph $G$ of the region $R$ obtained from $H_{x,y}(U; D)$ by adding a layer of unit triangles on the top of the hexagon, with the four vertices $u, v, w, s$ as in Fig. 4.2 (the region restricted by the bold contour is $H_{x,y}(U; D)$). Here the (planar) dual graph of a region on the triangular lattice is the graph whose vertices are the unit triangles inside the region, and whose edges connect vertices corresponding to unit triangles that share an edge. In particular, the vertices $w$ and $u$ correspond to the up-pointing triangles on the horizontal axis at the first and the last positions not in $U \cup D$, the vertex $v$ corresponds to the up-pointing triangle on the northeast corner of the region, and the vertex $s$ corresponds to the down-pointing triangle on the southeast corner of the region.

We consider the region corresponding to $G - v$ (i.e., the region obtained from $R$ by removing the $v$-triangle as in Fig. 4.3 A). The removal of the $v$-triangle yields several forced lozenges along the top of the region. After removing these forced lozenges (this removal does not change the tiling number of the region), we get back the region $H_{x,y}(U; D)$. This means that we have

$$M(G - v) = M(H_{x,y}(U; D)).$$

By considering forced lozenges in the regions corresponding to the graphs $G - \{u, w, s\}$, $G - u$, $G - \{v, w, s\}$, $G - w$, and $G - \{u, v, s\}$ (as shown in Fig. 4.3 B – F, respectively), we get

$$M(G - \{u, w, s\}) = M(H_{x-1,y-1}(\alpha \beta U; D)),$$

$$M(G - u) = M(H_{x-1,y}(\beta U; D)),$$

$$M(G - \{v, w, s\}) = M(H_{x,y-1}(\alpha U; D)),$$

$$M(G - w) = M(H_{x-1,y}(\alpha U; D)).$$

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**Figure 4.1.** Applications of the Region-splitting Lemma 4.2 in the base cases: (A) $y = 0$, (B) $x = 0$. 

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TRI LAI AND RANJAN ROHATGI
A SHUFFLING THEOREM FOR LOZENGE TILINGS OF DOUBLY-DENTED HEXAGONS

(4.10) \[ M(G - \{u, v, s\}) = M(H_{x,y-1}(\beta U; D)), \]
where we use the notations \(\alpha U\), \(\beta U\) and \(\alpha\beta U\) for the unions \(U \cup \{\alpha\}, U \cup \{\beta\}\) and \(U \cup \{\alpha, \beta\}\), respectively. Plugging Eqs. (4.5)–(4.10) into the equation in Kuo’s Lemma 4.1, we get the recurrence

\[
M(H_{x,y}(U; D)) M(H_{x-1,y-1}(\alpha U; D)) = \\
M(H_{x-1,y}(\beta U; D)) M(H_{x,y-1}(\alpha U; D)) \\
+ M(H_{x-1,y}(\alpha U; D)) M(H_{x,y-1}(\beta U; D)).
\]

Next, we show that the expression on the right-hand side of Eq. (4.2) also satisfies recurrence (4.11). We note that the index set of removed down-pointing triangles does not change in the recurrence. By plugging the right-hand side of Eq. (4.2) to the recurrence and cancel out the factor \(\Delta(D)\Delta(D')\), our task is reduced to the verification of the following.

\[
\frac{\Delta(U)}{\Delta(U')} M(H_{x,y}(U'; D')) \frac{\Delta(\alpha U)}{\Delta(\alpha U')} M(H_{x-1,y-1}(\alpha U'; D')) \\
= \frac{\Delta(\beta U)}{\Delta(\beta U')} M(H_{x-1,y}(\beta U'; D')) \frac{\Delta(\alpha U)}{\Delta(\alpha U')} M(H_{x,y-1}(\alpha U'; D')) \\
+ \frac{\Delta(\alpha U)}{\Delta(\alpha U')} M(H_{x-1,y}(\alpha U'; D')) \frac{\Delta(\beta U)}{\Delta(\beta U')} M(H_{x,y-1}(\beta U'; D')).
\]

It is routine to check that

\[
\frac{\Delta(\beta U)}{\Delta(\beta U')} \frac{\Delta(\alpha U)}{\Delta(\alpha U')} = \frac{\Delta(U)}{\Delta(U')} \frac{\Delta(\alpha U)}{\Delta(\alpha U')}.
\]

Indeed, dividing both sides of the above equation by \(\Delta(U')^2\Delta(U')^2\), we see that both are equal to

\[
\prod_{i=1}^{n} |\beta - s_i| \prod_{i=1}^{n} |\alpha - s_i| \\
\prod_{i=1}^{n} |\beta' - s_i'| \prod_{i=1}^{n} |\alpha' - s_i'|.
\]

By Eq. (4.13), we have observe that Eq. (4.12) simplifies to

\[
M(H_{x,y}(U'; D')) M(H_{x-1,y-1}(\alpha U'; D')) = \\
M(H_{x-1,y}(\beta U'; D')) M(H_{x,y-1}(\alpha U'; D')) \\
+ M(H_{x-1,y}(\alpha U'; D')) M(H_{x,y-1}(\beta U'; D')).
\]
Figure 4.3. Obtaining the recurrence for the tiling numbers.

which follows directly from the application of recurrence 4.11 to the region $H_{x,y}(U', D')$. This finishes the proof.

\[\square\]

5. Generalizations, more applications, and future directions

(1) We can generalize our Shuffling Theorem 2.1 by allowing ‘flipping’ unit triangles (from up-pointing to down-pointing, and vice versa) in the symmetric difference $U \Delta D$, besides shuffling the triangles. It is possible, then, that the new position sets of removed up-pointing triangles and removed down-pointing triangles $U'$ and $D'$ may have sizes different from those of $U$ and $D$. 

\[\square\]
The general idea is that we choose suitable position sets this way, our Theorem 5.1 implies a number of known enumerations about regions with ferns removed.

\[ \text{Theorem 5.1 (Generalized Shuffling Theorem 1). For nonnegative integers } x, y, n \text{ and four ordered subsets } U = \{s_1, s_2, \ldots, s_u\}, \ D = \{t_1, t_2, \ldots, t_d\}, \ U' = \{s'_1, s'_2, \ldots, s'_u\}, \text{ and } D' = \{t'_1, t'_2, \ldots, t'_d\} \text{ of } [x + y + n], \text{ such that } U \cup D = U' \cup D' \text{ and } U \cap D = U' \cap D', \]

\[ M(H_{x,y}(U; D)) = \frac{\prod_{1 \leq i < j \leq u} s_j - s_i \prod_{1 \leq i < j \leq d} t_j - t_i \text{ PP}(u, d, y)}{\prod_{1 \leq i < j \leq u'} s'_j - s'_i \prod_{1 \leq i < j \leq d'} t'_j - t'_i \text{ PP}(u' d', y)}. \]

We omit the proof of Theorem 5.1 as it is essentially the same as the proof of Theorem 2.1.

Recall that when \( U \cap D = \emptyset \), our region (written in the ‘cluster form’) \( H_{x,y}(U; D) = H_{x,y}(C_1, \ldots, C_k; d_1, \ldots, d_{k-1}) \) is (tiling-)equinumerous with a hexagon with ferns removed \( E_{x,y}(F_1, \ldots, F_k; d_1, \ldots, d_k) \). Viewed in this way, our Theorem 5.1 implies a number of known enumerations about regions with ferns removed.

The general idea is that we choose suitable position sets \( U, D, U', D' \) so that the region \( H_{x,y}(U; D) \) in Eq. 5.1 is the one that we want enumerate, and the region \( H_{x,y}(U'; D') \) is a known region (up to a removal of forced lozenges), for example the centrally symmetric hexagon in MacMahon’s theorem \( [1] \) or the cored hexagon in Ciucu–Eisenkölbl–Krattenthaler–Zare’s theorem in \( [10] \). In particular, by letting \( k = 3 \), the cluster \( C_1 \) and \( C_3 \) contain the same number of unit triangles, the cluster \( C_2 \) stays evenly between \( C_1 \) and \( C_3 \); we pick \( C'_1 \) consisting of all triangles of the same orientation, \( C'_3 \) consists of triangles with opposite orientation as that in \( C'_1 \), and \( C'_2 \) consists of all up-pointing triangles. In this case, the region \( H_{x,y}(U; D) \) becomes a hexagon with three ferns removed and \( H_{x,y}(U'; D') \) becomes a cored hexagon in \( [10] \), after the removal of forced lozenges; see illustration in Figure 5.1. This way our Theorem 5.1 implies the enumeration hexagons with three ferns removed in \( [5] \) Theorem 2.12. By further specifying that \( C_1 = C'_1 \) and \( C_3 = C'_3 \) consisting of only unit triangles of the same orientation, we imply Ciucu’s Theorem 2.1 in \( [3] \) about \( F \)-cored hexagons (after removing forced lozenges, the region \( H_{x,y}(U; D) \) becomes a \( F \)-cored hexagon and \( H_{x,y}(U'; D') \), as in the previous case, becomes a cored hexagon).

Similarly, by letting \( k = 2 \), and \( C'_1 \) and \( C'_2 \) consist unit triangles of the same orientations like the clusters \( C'_1 \) and \( C'_2 \) in the cases above (however the numbers of triangles in \( C_1 \) and \( C_2 \) may be different), we get Theorem 1.1 in \( [3] \) about hexagons with two ferns removed, called ‘doubly–intruded hexagons’ (the region \( H_{x,y}(U; D) \) is now a doubly–intruded hexagon and \( H_{x,y}(U'; D') \) becomes a hexagon, after removing forced lozenges).

(2) Another remarkable generalization of Shuffling Theorem 2.1 is the following model of tilings with “barriers.” A barrier is a unit horizontal lattice interval which is not allowed to be contained within a lozenge in a tiling. Assume that we have a set of barriers at the positions \( T \subseteq [x+y+n] - U \cap D \) so that vertical lozenges may not appear at the positions in \( T \) and that \( |T| \leq x \) (see the red barriers in

![Figure 5.1. Obtaining cored hexagons from hexagons with three ferns removed.](image-url)
Fig. 5.2. Illustrating Corollary 5.3

Fig. 2.4 $T = \{6, 13\}$ in this case. We now consider the tilings of $H_{x,y}(U; D)$ which are compatible with the set of barriers $T$. Denote by $M_T(R)$ the number of such compatible tilings. By a proof similar to that of Theorem 2.1 we can show the following.

**Theorem 5.2 (Generalized Shuffling Theorem 2).** With the same assumptions as in Theorem 5.1 we have

\[
M_T(H_{x,y}(U, D)) = \prod_{1 \leq i < j \leq u} \frac{s_j - s_i}{j - i} \prod_{1 \leq i < j \leq d} \frac{t_j - t_i}{j - i} \frac{PP(u, d, y)}{PP(u', d', y)}.
\]

It is striking that the right-hand side of Eq. 5.2 does not depend on the barrier set $T$. It would be quite interesting to explain this phenomenon combinatorially. It is not obvious even for the following corollary.

**Corollary 5.3.** Assume that $a, b, x, y$ are nonnegative integers, and $T$ is any subset of $[x+y]$ with $|T| \leq x$. Denote by $H_T(x, y + a, y + b)$ (resp., $H_T(x, y + a + b, y)$) the centrally symmetric hexagon $H(x, y + a, y + b)$ (resp., $H(x, y + a + b, y)$) with a set of barriers at the positions in $T$ along the horizontal lattice line that is a unit below its west vertex. Then

\[
M(H_T(x, y + a, y + b)) = \frac{PP(a, b, y)}{PP(a, b, x + y)}.
\]

The corollary follows directly from Theorem 5.2 by choosing $U = [u] \cup [x+y+n-d+1, x+y+n]$, $D = \emptyset$, $U' = [u]$, $D' = [x+y+n-d+1, x+y+n]$ and letting the barrier set be $a + T$. More precisely, after removing forced lozenges from $H_{x,y}(U; D)$ and $H_{x,y}(U'; D')$, we get the regions $H_T(x, y + a, y + b)$ and $H_T(x, y + a + b, y)$, respectively (see Fig. 5.2 A and B, in which $a = 3$, $b = 2$, $x = 4$, $y = 2$, $T = \{3, 5\}$).

We note that the above generalized shuffling theorem can be viewed as the enumeration of certain ‘restricted tilings’ in the case when $U \cap D = \emptyset$. Indeed, the tilings of $H_{x,y}(U; D)$ which are compatible with $T$ are in bijection with tilings of $H_{x,y+(a+d-n)}(U \setminus D; D \setminus U)$ which contain a fixed vertical lozenge at each position in $U \cap D$ and which do not contain a vertical lozenge at each position in $T$. These types of tiling enumerations were investigated by Fischer [11] and Fulmek and Krattenthaler in [12,13].

(3) By equation (7.105) of [11], we have

\[
\prod_{1 \leq i < j \leq u} \frac{s_j - s_i}{j - i} = s_N(s_1, \ldots, s_n)^{1^u},
\]

The notation $1^u$ in the argument of a Schur function stands for $n$ arguments equal to 1.
where the partition $\lambda(\{s_1, \ldots, s_u\}) := (s_u - u + 1, \ldots, s_2 - 1, s_1)$. On the other hand, it is not hard to see that we also have
\begin{equation}
M(H_{x,y}(U; D)) = \sum_{|S| = y} s_{\lambda(U \cup S)}(1^{u+y}) s_{\lambda(D \cup S)}(1^{d+y}),
\end{equation}
where the sum runs over all $y$-subsets $S$ of $[x+y+n] - (U \cup D)$. Indeed, this follows from the fact that in each tiling, precisely $y$ of the $x+y$ unit segments on the lattice line from along which we removed the $n$ unit triangles are straddled by vertical lozenges. Hence one can write Eq. 2.1 in Theorem 2.1 as
\begin{equation}
\sum_{|S| = y} s_{\lambda(U \cup S)}(1^{u+y}) s_{\lambda(D \cup S)}(1^{d+y}) = \frac{s_{\lambda(U)}(1^{u}) s_{\lambda(D)}(1^{d})}{s_{\lambda(U')}(1^{u}) s_{\lambda(D')}(1^{d})},
\end{equation}
where the sum is taken over all $y$-subsets $S$ of $[x+y+n] - (U \cup D)$.

It would be interesting to know if the following general sum has a similar simplification:
\begin{equation}
\sum_{|S| = y} s_{\lambda(U \cup S)} X^{m+y} s_{\lambda(D \cup S)}(1^{d+y}) = \frac{s_{\lambda(U')} X^{m+d-u+y} s_{\lambda(D')} (1^{d})}{s_{\lambda(U')} (1^{u}) s_{\lambda(D')} (1^{d})},
\end{equation}
where $X^m$ denotes the sequence of variables $x_1, x_2, \ldots, x_m$.

(4) We can assign to each right-tilting lozenge a weight $q^z$, where $z$ is the distance from its top to the base of the region. The ‘tiling-generating function’ of a region $H_{x,y}(U, D)$ is the sum of the weights of all tilings of the region, where the weight of a tiling is the product of weights of all its constituent lozenges. We denote by this tiling-generating function by $M_q(H_{x,y}(U, D))$. Our data supports the following $q$-analogue of Theorem 1.

**Conjecture 5.4.** With the same assumptions as in Theorem 2.1, we have
\begin{equation}
\frac{M_q(H_{x,y}(U, D))}{M_q(H_{x,y}(U', D'))} = q^C \prod_{1 \leq i < j \leq u} \frac{1 - q^{s_j - s_i}}{1 - q^{s'_j - s'_i}} \prod_{1 \leq i < j \leq d} \frac{1 - q^{j - t_i}}{1 - q^{j' - t_i}},
\end{equation}
where $C$ depends only on $x, y, U, D, U', D'$.

We note that Conjecture 5.4 (if proven) would give supporting evidence for the existence of a Schur function identity behind Shuffling Theorem 2.1, as we have
\begin{equation}
\prod_{1 \leq i < j \leq u} \frac{1 - q^{s_j - s_i}}{1 - q^{s'_j - s'_i}} = q^A s_{\lambda((s_1, \ldots, s_u))}(q, q^2, q^3, \ldots),
\end{equation}
for some constant $A$. This would imply that Eq. 5.3 is still true (up to a $q$-power) when $1^n$ is replaced by the sequence $(q, q^2, q^3, \ldots)$.

(5) Motivated by Stanley’s classical paper [15] on symmetric plane partitions, we would like to investigate symmetric tilings of $H_{x,y}(U; D)$. There are two natural classes of symmetric tilings: the tilings which are invariant under a reflection over a vertical axis, and which are invariant under a $180^\circ$ rotation (these tilings correspond to the transposed-complementary and self-complementary plane partitions). The shuffling theorems for these symmetry classes will be investigated in a separate paper.

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