Approximation theorems for Banach-valued almost periodic and semi-almost periodic holomorphic functions

Alexander Brudnyi*
Department of Mathematics and Statistics
University of Calgary, Calgary
Canada

Damir Kinzebulatov
Department of Mathematics
University of Toronto, Toronto
Canada

Abstract
The paper studies semi-almost periodic holomorphic functions on a polydisk which have, in a sense, the weakest possible discontinuities on the boundary torus. The basic result used in the proofs is an extension of the classical Bohr approximation theorem for almost periodic holomorphic functions on a strip to the case of Banach-valued almost periodic holomorphic functions.

1 Introduction and Formulation of Main Results

1.1. Let $B$ be a complex Banach space and $C^B_b(U)$ be the space of bounded continuous $B$-valued functions $f$ on a closed subset $U \subset \mathbb{C}$ endowed with sup-norm $\|f\| := \sup_{t \in U} \|f(t)\|_B$.

*Research supported in part by NSERC.

2000 Mathematics Subject Classification. Primary 30H05, Secondary 46J20.

Key words and phrases. Banach-valued holomorphic functions, almost periodic functions, approximation property
Definition 1.1. A function $f \in C^B_b(R)$ is said to be almost periodic if the family of shifts $\{S_\tau f\}_{\tau \in R}$, $S_\tau f(t) := f(t + \tau)$, $t \in R$, is relatively compact in $C^B_b(R)$.

By $AP_B(R) \subset C^B_b(R)$ we denote the Banach space of $B$-valued almost periodic functions on $R$ endowed with sup-norm.

The following result due to H. Bohr, commonly called the approximation theorem, is a cornerstone of the theory of Banach-valued almost periodic functions.

Theorem 1.2 (H. Bohr). The vector subspace of $C^B_b(R)$ spanned over $B$ by functions $t \mapsto e^{i\lambda t}$, $\lambda \in R$, is dense in $AP_B(R)$.

Similarly one defines $B$-valued holomorphic almost periodic functions on the strip $\Sigma := \{ z \in C : 0 \leq \text{Im}(z) \leq \pi \}$.

Definition 1.3. A function $f \in C^B_b(\Sigma)$ is called holomorphic almost periodic on $\Sigma$ if it is holomorphic in the interior of $\Sigma$ and the family of shifts $\{S_x f\}_{x \in R}$ is relatively compact in $C^B_b(\Sigma)$.

By $APH_B(\Sigma) \subset C^B_b(\Sigma)$ we denote the Banach space of $B$-valued almost periodic holomorphic functions on $\Sigma$ endowed with sup-norm.

Our first result is an analog of Theorem 1.2 for functions from $APH_B(\Sigma)$.

Theorem 1.4. The complex vector subspace of $C^B_b(\Sigma)$ spanned over $B$ by functions $z \mapsto e^{i\lambda z}$, $\lambda \in R$, is dense in $APH_B(\Sigma)$.

This result is equivalent to the fact that the Banach space $APH(\Sigma) := APH_C(\Sigma)$ of holomorphic almost periodic functions on $\Sigma$ has the approximation property along with the scalar version of Theorem 1.4, see Grothendieck [5, Section 5.1]. (Similarly Theorem 1.2 can be derived from the fact that $AP(\mathbb{R}) := AP_R(\mathbb{R})$ has the approximation property.) Let us recall

Definition 1.5. A Banach space $B$ is said to have the approximation property, if, for every compact set $K \subset B$ and every $\varepsilon > 0$, there is an operator $T : B \to B$ of finite rank so that $\|Tx - x\|_B \leq \varepsilon$ for every $x \in K$. If such $T$ has norm at most 1, then $B$ is said to have metric approximation property.

It is easy to see that every space $B$ with a Schauder basis has the approximation property. However, there are Banach spaces without this property, the first such example was constructed by Enflo [3]. In Proposition 3.1 we will show that $APH(\Sigma)$ has metric approximation property.

1.2. We apply Theorem 1.4 to study the algebra of bounded holomorphic functions on the unit polydisk $D^n \subset C^n$ with semi-almost periodic boundary values. Originally
the algebra of semi-almost periodic functions on the real line \( \mathbb{R} \) was introduced and studied by Sarason [6] in connection with some important problems of operator theory. By the definition it is a closed subalgebra of \( C_b(\mathbb{R}) \) generated by algebras \( AP(\mathbb{R}) \) and \( PC_\infty \) (the algebra of continuous functions on \( \mathbb{R} \) that have finite limits at \( +\infty \) and at \( -\infty \)). In [2] we introduced the algebra \( SAP(\partial \mathbb{D}) \) of semi-almost periodic functions on the unit circle \( \partial \mathbb{D} \) which generalizes the Sarason algebra. Then we studied the algebra \( H^\infty(\mathbb{D}) \cap SAP(\partial \mathbb{D}) \) of bounded holomorphic functions on the unit disk \( \mathbb{D} \) having boundary values in \( SAP(\partial \mathbb{D}) \) which we called the algebra of semi-almost periodic holomorphic functions. (Recall that for a function \( f \in H^\infty(\mathbb{D}) \), one defines its boundary values a.e. on \( \partial \mathbb{D} \) by taking limits in non-tangential directions, so that \( f|_{\partial \mathbb{D}} \in L^\infty(\partial \mathbb{D}) \).

Our interest in algebra \( H^\infty(\mathbb{D}) \cap SAP(\partial \mathbb{D}) \) is motivated by the following problem:

\[ \text{Find a closed subalgebra } A \subset H^\infty(\mathbb{D}) \text{ whose elements have, in a sense, the weakest possible discontinuities at the boundary } \partial \mathbb{D}. \]

An obvious candidate would be the subalgebra of bounded holomorphic functions having discontinuities of at most first kind at the boundary. Unfortunately, as it follows from the Lindelöf Theorem, see [4], any such function must be continuous on \( \overline{\mathbb{D}} \). Moreover, the same result holds if we consider bounded holomorphic functions with first kind discontinuous on \( \partial \mathbb{D} \) of their real or imaginary parts.

Another way to measure singularities of a function from \( H^\infty(\mathbb{D}) \) is to consider discontinuities of its modulus on \( \partial \mathbb{D} \). Assuming that \( A \subset H^\infty(\mathbb{D}) \) consists of functions \( f \) such that \( |f|_{\partial \mathbb{D}} \in L^\infty(\partial \mathbb{D}) \) has first-kind discontinuous only, we obtain that \( A \) contains all inner functions. Then by the Marshall theorem, see [4], \( A = H^\infty(\mathbb{D}) \).

Thus to get a nontrivial answer we correct the above problem restricting ourselves to the case of algebras \( A \subset H^\infty(\mathbb{D}) \) generating by subgroups \( G_A \) of invertible elements of \( H^\infty(\mathbb{D}) \) such that for each \( f \in G_A \) the function \( |f| \) (or, equivalently, the harmonic function \( \ln |f| \)) has finitely many discontinuities of at most first kind on \( \partial \mathbb{D} \). The main result in [2] (Theorem 1.8) establishes a connection between such algebras \( A \) and certain subalgebras of \( H^\infty(\mathbb{D}) \cap SAP(\partial \mathbb{D}) \).

The purpose of the present paper is to define semi-almost periodic holomorphic functions on \( \mathbb{D}^n \) and to extend the results of [2] to this class of functions.

First we recall some definitions from [2].

In what follows, we consider \( \partial \mathbb{D} \) with the counterclockwise orientation. For \( t_0 \in \mathbb{R} \) let

\[ \gamma_{t_0}^k(s) := e^{i(t_0 + ks)} : 0 \leq t < s < 2\pi, \quad k \in \{-1, 1\}, \]

be two open arcs having \( e^{it_0} \) as the right and the left endpoints, respectively.
Definition 1.6 ([2]). A function $f \in L^\infty(\partial \mathbb{D})$ is called semi-almost periodic on $\partial \mathbb{D}$ if for any $t_0 \in [0, 2\pi)$ and any $\varepsilon > 0$ there exist a number $s = s(t_0, \varepsilon) \in (0, \pi)$ and functions $f_k : \gamma^k_{t_0}(s) \to \mathbb{C}$, $k \in \{-1, 1\}$, such that functions

$$t \mapsto f_k(e^{i(t_0 + kse^t)}), \quad -\infty < t < 0, \quad k \in \{-1, 1\},$$

are restrictions of some almost periodic functions from $AP(\mathbb{R})$, and

$$\sup_{z \in \gamma^k_{t_0}(s)} |f(z) - f_k(z)| < \varepsilon, \quad k \in \{-1, 1\}.$$

We denote by $SAP(\partial \mathbb{D})$ the Banach algebra of semi-almost periodic functions on $\partial \mathbb{D}$ endowed with sup-norm. For $S$ a closed subset of $\partial \mathbb{D}$ we denote by $SAP(S)$ the Banach algebra of semi-almost periodic functions on $\partial \mathbb{D}$ that are continuous on $\partial \mathbb{D} \setminus S$. (Note the the Sarason algebra is isomorphic to $SAP\{z_0\}$ for any $z_0 \in \partial \mathbb{D}$.)

Next, let $A(\mathbb{D})$ be the algebra consisting of holomorphic functions in $H^\infty(\mathbb{D})$ that are continuous on $\bar{\mathbb{D}}$. Suppose that $S$ contains at least two points. By $A_S$ we denote the closure in $L^\infty(\partial \mathbb{D})$ of the algebra generated by $A(\mathbb{D})$ and holomorphic functions of the form $e^{if}$, where $Re(f)|_{\partial \mathbb{D}}$ is a finite linear combination (over $\mathbb{R}$) of characteristic functions of closed arcs in $\partial \mathbb{D}$ whose endpoints belong to $S$. If $S$ consists of a single point, then we define $A_S$ to be the closure of the algebra generated by $A(\mathbb{D})$ and functions $ge^{if}$, where $Re(f)|_{\partial \mathbb{D}}$ is the characteristic function of a closed arc with an endpoint in $S$ and $g \in A(\mathbb{D})$ is a function such that $ge^{if}$ has discontinuity on $S$ only.

Now, Theorem 1.8 of [2] describes the structure of the algebra of semi-almost periodic holomorphic functions:

**Theorem 1.7 ([2]).** $H^\infty(\mathbb{D}) \cap SAP(S) = A_S$.

(Here and below we identify the elements of algebra $H^\infty(\mathbb{D}^n)$ with their boundary values defined on $(\partial \mathbb{D})^n$.)

**Remark 1.8.** Suppose that $S \subset \partial \mathbb{D}$ contains at least 2 points. Let $e^{if} \in A_S$, $\lambda \in \mathbb{R}$, where $Re(f)$ is the characteristic function of an arc $[x, y]$ with $x, y \in S$. Let $\phi_{x,y} : \mathbb{D} \to \mathbb{H}_+$ be a bilinear map onto the upper half-plane that maps $x$ to $0$, the midpoint of the arc $[x, y]$ to $1$ and $y$ to $\infty$. Then there is a constant $C$ such that

$$e^{if(z)} = e^{-\frac{\lambda}{\pi} \text{Log} \phi_{x,y}(z) + \lambda C}, \quad z \in \mathbb{D},$$

where Log is the principal branch of the logarithmic function. Thus Theorem 1.7 implies that the algebra $H^\infty(\mathbb{D}) \cap SAP(S)$ is the uniform closure of the algebra generated by $A(\mathbb{D})$ and the family of functions $e^{i\lambda(\text{Log} \phi_{x,y})}$, $\lambda \in \mathbb{R}$, $x, y \in S$. 

Let us define now multi-dimensional analogues of algebras $SAP(S)$ and $A_S$. Namely, if $S_k \subset \partial \mathbb{D}$ are closed sets, $1 \leq k \leq n$, and $S = \prod_{k=1}^{n} S_k \subset (\partial \mathbb{D})^n$, then we define

$$SAP^n(S) := \bigotimes_{k=1}^{n} SAP(S_k), \quad A_S^n := \bigotimes_{k=1}^{n} A_{S_k}.$$ 

Here $\bigotimes$ stands for completion of symmetric tensor product of the corresponding algebras. In particular, $SAP^n(S)$ and $A_S^n$ are uniform closures of the algebras of complex polynomials in variables $f_1(z_1), \ldots, f_n(z_n)$ with $f_k(z_k) \in SAP(S_k)$ and $f_k(z_k) \in A_{S_k}$, $1 \leq k \leq n$, respectively, where $z = (z_1, \ldots, z_n) \in (\partial \mathbb{D})^n$.

Now the extension of Theorem 1.7 is read as follows.

**Theorem 1.9.** $H^\infty(\mathbb{D}^n) \cap SAP^n(S) = A_S^n$.

**Remark 1.10.** We apply Theorem 1.4 and some arguments from [2] to show that the algebra $H^\infty(\mathbb{D}) \cap SAP(S)$ has the approximation property. This will imply Theorem 1.9.

Let $M_S$ be the maximal ideal space of the algebra $H^\infty(\mathbb{D}^n) \cap SAP^n(S)$ and $M_{S_k}$ be the maximal ideal space of the algebra $H^\infty(\mathbb{D}) \cap SAP(S_k)$, $1 \leq k \leq n$. As a corollary of Theorem 1.9 we obtain

**Theorem 1.11.** $M_S$ is homeomorphic to $M_{S_1} \times \cdots \times M_{S_n}$. Moreover, the Corona Theorem is valid for $H^\infty(\mathbb{D}^n) \cap SAP^n(S)$, i.e., $\mathbb{D}^n$ is dense in $M_S$ in the Gelfand topology.

**Remark 1.12.** The structure of $M_{S_k}$ is described in [2], Theorems 1.7 and 1.14. Let us recall this result. In what follows $M(A)$ stands for the maximal ideal space of the Banach algebra $A$.

Since $A(\mathbb{D}) \hookrightarrow A_{S_k}$, there is a continuous surjection of the maximal ideal spaces

$$a_{S_k} : M_{S_k} \to M(A(\mathbb{D})) \cong \mathbb{D}.$$ 

Recall that the Šilov boundary of $A_{S_k}$ is the smallest compact subset $K \subset M_{S_k}$ such that for each $f \in A_{S_k}$

$$\sup_{z \in M_{S_k}} |f(z)| = \sup_{\xi \in K} |f(\xi)|.$$ 

Here we assume that every $f \in A_{S_k}$ is also defined on $M_{S_k}$ where its extension to $M_{S_k} \setminus \mathbb{D}$ is given by the Gelfand transform: $f(\xi) := \xi(f)$, $\xi \in M_{S_k}$.

**Theorem 1.13** ([2]). (1) $a_{S_k} : M_{S_k} \setminus a_{S_k}^{-1}(S_k) \to \mathbb{D} \setminus S_k$ is a homeomorphism.
(2) The Šilov boundary $K_{S_k}$ of $A_{S_k}$ is naturally homeomorphic to $M(SAP(S_k))$. Moreover, $K_{S_k} \setminus a_{S_k}^{-1}(S_k) = \partial \mathbb{D} \setminus S_k$ and $K_{S_k} \cap a_{S_k}^{-1}(z)$, $z \in S_k$, is homeomorphic to the disjoint union $b \mathbb{R} \sqcup b \mathbb{R}$ of the Bohr compactifications $b \mathbb{R}$ of $\mathbb{R}$.

(3) For each $z \in S_k$ preimage $a_{S_k}^{-1}(z)$ is homeomorphic to the maximal ideal space of the algebra $APH(\Sigma)$ of holomorphic almost periodic functions on the strip $\Sigma := \{z \in \mathbb{C} : \text{Im}(z) \in [0, \pi]\}$.

Acknowledgment. We are grateful to S. Favorov for useful discussions and valuable comments improving the presentation.

2 Auxiliary Results

2.1. In our proofs we use some results on Bohr’s compactifications.

First recall that the algebra of almost periodic functions $AP(\mathbb{R}) := AP_\mathbb{C}(\mathbb{R})$ is naturally isomorphic to the algebra $C(b\mathbb{R})$ of complex continuous functions on the Bohr compactification $b\mathbb{R}$ of $\mathbb{R}$ (the maximal ideal space of $AP(\mathbb{R})$). That is, a complex continuous function on $\mathbb{R}$ is almost periodic if and only if it admits a continuous extension to $b\mathbb{R}$ by means of the Gelfand transform. (Recall that the maximal ideal space $b\mathbb{R}$ is defined as the space of continuous non-zero characters $AP(\mathbb{R}) \hookrightarrow \mathbb{C}$ endowed with the Gelfand (i.e. weak*) topology. Then $b\mathbb{R}$ is a compact abelian group, $\mathbb{R}$ is naturally embedded into $b\mathbb{R}$ as a dense subset, so that the action of $\mathbb{R}$ on itself by translations extends uniquely to the continuous action of $\mathbb{R}$ on $b\mathbb{R}$.)

Let us describe now the maximal ideal space of the algebra $APH(\Sigma)$.

Let $R = \{z \in \mathbb{C} : e^{-2\pi^2} \leq |z| \leq 1\}$. Then $\pi(z) := e^{2\pi iz}$, $z \in \mathbb{C}$, determines a projection of the strip $\Sigma$ onto the annulus $R$, so that the triple $(\Sigma, R, \pi)$ forms a principal bundle on $R$ with fibre $\mathbb{Z}$. Suppose that $U_1$ and $U_2$ are compact simply connected subsets of $R$ which cover $R$. Then we can represent $\Sigma$ as a quotient space of $(U_1 \times \mathbb{Z}) \sqcup (U_2 \times \mathbb{Z})$ by the equivalence relation $\sim$ determined by a locally constant function $c_{12} : U_1 \cap U_2 \hookrightarrow \mathbb{Z}$ such that $U_1 \times \mathbb{Z} \ni (z, n) \sim (z, n + c_{12}(z)) \in U_2 \times \mathbb{Z}$ for all $z \in U_1 \cap U_2$, $n \in \mathbb{Z}$.

Let $b\mathbb{Z}$ be the Bohr compactification of $\mathbb{Z}$. Then $b\mathbb{Z}$ is a compact abelian group with $\mathbb{Z} \subset b\mathbb{Z}$ acting continuously on $b\mathbb{Z}$. In particular, given $z \in R$ we have a continuous mapping $b\mathbb{Z} \hookrightarrow b\mathbb{Z}$ determined by the formula $\xi \mapsto \xi + c_{12}(z)$, $\xi \in b\mathbb{Z}$. So, we can define

$$b\Sigma := (U_1 \times b\mathbb{Z}) \sqcup (U_2 \times b\mathbb{Z}) / \sim$$

where, by definition, $U_1 \times b\mathbb{Z} \ni (z, \xi) \sim (z, \xi + c_{12}(z)) \in U_2 \times b\mathbb{Z}$ for all $z \in U_1 \cap U_2$, $\xi \in b\mathbb{Z}$. The local embeddings $U_k \times \mathbb{Z} \hookrightarrow U_k \times b\mathbb{Z}$ $(k = 1, 2)$ determine an embedding
ι₀ : Σ ↪ bΣ. Similarly, an embedding ιξ : Σ ↪ bΣ, where ξ ∈ bZ, is determined by the local embedding (z, n) ↦ (z, n + ξ). Since Z is dense in bZ, ιξ(Σ) is dense in bΣ for each ξ ∈ bΣ. Furthermore, the sets ιξ(Σ) are mutually disjoint and cover bΣ.

We say that a function f : bΣ ↪ C is holomorphic if its pullback ιξ⁎f is holomorphic on Σ for each ξ ∈ bZ. The space of functions holomorphic on bΣ is denoted by O(bΣ). Now, we have

**Proposition 2.1** ([2]). APH(Σ) is naturally isomorphic to O(bΣ).

The result states that O(bΣ)|Σ = APH(Σ). Moreover, as it is shown in [2], if ιξ⁎f is holomorphic on Σ for a certain ξ ∈ bΣ, then f is holomorphic on bΣ. This gives another definition of a holomorphic almost periodic function (cf. Definition 1.3).

**Proposition 2.2** ([2]). bΣ is homeomorphic to the maximal ideal space of the algebra APH(Σ).

2.2. In the proof of Theorem 1.4 we use the following equivalent definition of B-valued holomorphic functions on Σ (cf. Definition 1.3).

**Proposition 2.3.** A continuous function f : Σ ↪ B is holomorphic almost periodic on Σ if and only if it is holomorphic in the interior of Σ and admits a continuous extension to bΣ.

*Proof.* Assume that f is a B-valued continuous almost periodic function on Σ. Since g ◦ f ∈ APH(Σ) for every continuous functional g ∈ B⁎, the function f admits a continuous extension ˆf : bΣ → B**; here B** is equipped with weak* topology. Next, due to Definition 1.3 the closure ˆf(Σ) of f(Σ) in the strong topology of B is compact. We naturally identify B with a subspace of B**. Since any compact subset of B is also compact in the weak* topology of B**, ˆf(bΣ) = ˆf(Σ) ⊂ B. Hence ˆf maps bΣ into B and is continuous.

Conversely, assume that f : bΣ → B is continuous and f|Σ is holomorphic. Since f is uniformly continuous on bΣ and the natural action of group ℝ on Σ by shifts is extended to a continuous action of bℝ on bΣ, the family of shifts {Sₓf}ₓ∈ℝ is relatively compact in C₀ᵇ(bΣ) ⊂ C₀ᵇ(Σ).

Now, suppose that f ∈ APH_B(Σ). According to Proposition 2.3 there exists a continuous extension ˆf of f to bΣ. For each continuous functional g ∈ B⁎ the function g ◦ ˆf belongs to APH(Σ). By Proposition 2.1 g ◦ ˆf admits a continuous extension ˆg ◦ ˆf ∈ O(bΣ). Since Σ is dense in bΣ, the identity ˆg ◦ ˆf = ˆg ◦ ˆf is valid for all g ∈ B⁎. This implies that for each ξ ∈ bZ, the function g ◦ ˆf ◦ ιξ ∈ APH(Σ) for all g ∈ B*. Therefore the continuous B-valued function ˆf ◦ ιξ is holomorphic in the interior of Σ, i.e., it belongs to APH_B(Σ).
3 Proof of Theorem 1.4

Proposition 3.1. \( \text{AP} H(\Sigma) \) has metric approximation property.

Proof. We refer to the book of Besicovich [1] for the corresponding definitions and facts from the theory of almost periodic functions.

Let \( K \subset \text{AP} H(\Sigma) \) be compact. Given \( \varepsilon > 0 \) consider an \( \frac{\varepsilon}{3} \)-net \( \{f_1, \ldots, f_l\} \subset K \).

Let
\[
K(t) := \sum_{|\nu_1| \leq n_1, \ldots, |\nu_r| \leq n_r} \left(1 - \frac{\nu_1}{n_1}\right) \ldots \left(1 - \frac{\nu_r}{n_r}\right) e^{-i\left(\frac{\nu_1}{n_1}\beta_1 + \cdots + \frac{\nu_r}{n_r}\beta_r\right)t},
\]
be a Bochner-Fejer kernel such that for all \( 1 \leq k \leq l \)
\[
\sup_{z \in \Sigma} |f_k(z) - M_t\{f_k(z + t)K(t)\}| \leq \frac{\varepsilon}{3}.
\]
(H3.1)
Here \( \beta_1, \ldots, \beta_r \) are elements of a basis over \( \mathbb{Q} \) of the union of spectra of functions \( f_1, \ldots, f_l, \nu_1, \ldots, \nu_r \in \mathbb{Z}, n_1, \ldots, n_r \in \mathbb{N} \) and

\[
M_t\{f_k(z + t)K(t)\} := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f_k(z + t)K(t) \, dt
\]
are the corresponding holomorphic Bochner-Fejer polynomials.

We define an operator \( T : \text{AP} H(\Sigma) \to \text{AP} H(\Sigma) \) from Definition 1.5 by the formula
\[
(Tf)(z) := M_t\{f(z + t)K(t)\}, \quad f \in \text{AP} H(\Sigma).
\]
(H3.2)
Then \( T \) is a linear projection onto a finite-dimensional space generated by functions
\[ e^{i\left(\frac{\nu_1}{n_1}\beta_1 + \cdots + \frac{\nu_r}{n_r}\beta_r\right)z}, \quad |\nu_1| \leq n_1, \ldots, |\nu_r| \leq n_r. \]
Moreover, since \( K(t) \geq 0 \) for all \( t \in \mathbb{R} \) and \( M_t\{K(t)\} = 1 \), the norm of \( T \) is 1. Finally, given \( f \in K \) choose \( k \) such that
\[
\|f - f_k\|_{\text{AP} H(\Sigma)} \leq \frac{\varepsilon}{3}. \]
Then we have by (3.1)
\[
\|Tf - f\|_{\text{AP} H(\Sigma)} \leq \|T(f - f_k)\|_{\text{AP} H(\Sigma)} + \|Tf_k - f_k\|_{\text{AP} H(\Sigma)} + \|f_k - f\|_{\text{AP} H(\Sigma)} < \varepsilon.
\]

This completes the proof of the proposition. \(\square\)

Let us prove now Theorem 1.4. According to Section 2.2 each \( B \)-valued almost periodic function \( f \) on \( \Sigma \) admits a continuous extension \( \hat{f} \in C_b^B(b\Sigma) \). Also, for each \( g \in B^* \) the function \( g \circ \hat{f} \in \mathcal{O}(b\Sigma) \). Since \( \mathcal{O}(b\Sigma) \cong \text{AP} H(\Sigma) \) has the approximation property by Proposition 3.1 and since \( b\Sigma \) is compact, the results of Section 5.1 of [3] imply that each \( \hat{f} \) belongs to the closure in \( C_b^B(b\Sigma) \) of the symmetric tensor product of \( \text{AP} H(\Sigma) \) and \( B \). Finally, the classical Bohr theorem, see, e.g., [1], asserts that each element of \( \text{AP} H(\Sigma) \) is the uniform limit of finite linear combinations of functions \( e^{i\lambda z}, \lambda \in \mathbb{R} \). These facts complete the proof of the theorem. \(\square\)
4 Proofs of Theorems 1.9 and 1.11

4.1 Semi-almost periodic holomorphic functions on $\mathbb{D}$ with values in a Banach space

To prove Theorem 1.9 we generalize results of [2] to the Banach space of semi-almost periodic holomorphic functions on the unit disk $\mathbb{D}$ with values in a Banach space $B$.

4.1.1. Let $L^\infty_B(\partial \mathbb{D})$ be the Banach space of $B$-valued bounded measurable functions on $\partial \mathbb{D}$ equipped with sup-norm.

Definition 4.1. A function $f \in L^\infty_B(\partial \mathbb{D})$ is called semi-almost periodic on $\partial \mathbb{D}$ if for any $t_0 \in [0, 2\pi)$ and any $\varepsilon > 0$ there exist a number $s = s(t_0, \varepsilon) \in (0, \pi)$ and functions $f_k : \gamma^k_{t_0}(s) \to B$, $\gamma^k_{t_0}(s) := \{e^{it_0 + kt} : 0 \leq t < s, 0 \leq k \leq 2\pi\}$, $k \in \{-1, 1\}$, such that functions

$$
t \mapsto f_k(e^{it_0 + kse^t}), \quad -\infty < t < 0, \quad k \in \{-1, 1\},
$$

are restrictions of some $B$-valued almost periodic functions from $AP_B(\mathbb{R})$ and

$$
\sup_{z \in \gamma^k_{t_0}(s)} \|f(z) - f_k(z)\|_B < \varepsilon, \quad k \in \{-1, 1\}.
$$

Analogously, we denote by $SAP_B(S) \subset L^\infty_B(S)$ the Banach space of semi-almost periodic functions that are continuous on $\partial \mathbb{D} \setminus S$.

Using the Poisson integral formula we can extend each function from $SAP_B(S)$ to a bounded $B$-valued harmonic function on $\mathbb{D}$ having the same sup-norm. Below we identify $SAP_B(S)$ with its harmonic extension.

Suppose that $S$ contains at least two points. Recall that $A_S$ is the closure in $H^\infty(\mathbb{D})$ of the algebra generated by $A(\mathbb{D})$ and holomorphic functions of the form $e^f$, where $\text{Re}(f)|_{\partial \mathbb{D}}$ is a finite linear combination (over $\mathbb{R}$) of characteristic functions of closed arcs in $\partial \mathbb{D}$ whose endpoints belong to $S$. If $S$ consists of a single point, then $A_S$ is the closure of the algebra generated by $A(\mathbb{D})$ and functions $ge^f$, where $\text{Re}(f)|_{\partial \mathbb{D}}$ is the characteristic function of a closed arc with an endpoint in $S$ and $g \in A(\mathbb{D})$ is a function such that $ge^f$ has discontinuity on $S$ only. (Here $A(\mathbb{D}) \subset H^\infty(\mathbb{D})$ is the disk-algebra, see Section 1.)

We define the Banach space $A^B_S$ as

$$
A^B_S = A_S \otimes B.
$$

Here $\otimes$ stands for completion of symmetric tensor product with respect to norm

$$
\left\| \sum_{k=1}^m f_k b_k \right\| := \sup_{z \in \partial \mathbb{D}} \left\| \sum_{k=1}^m f_k(z) b_k \right\|_B \quad \text{with} \quad f_k \in A_S, \ b_k \in B. \quad (4.1)
$$
Let $H^\infty_B(\mathbb{D})$ be the Banach space of bounded $B$-valued holomorphic functions on $\mathbb{D}$ equipped with sup-norm. The Banach-valued analogue of Theorems 1.9 is now formulated as follows.

**Theorem 4.2.** $SAP_B(S) \cap H^\infty_B(\mathbb{D}) = A^B_S$.

4.1.2. For the proof we require some auxiliary results.

Let $APC(\Sigma)$ be the Banach algebra of functions $f : \Sigma \mapsto \mathbb{C}$ uniformly continuous on $\Sigma := \{ z \in \mathbb{C} : \text{Im}(z) \in [0, \pi] \}$ and almost periodic on each horizontal line.

**Proposition 4.3** ([2]). $M(APH(\Sigma)) = M(APC(\Sigma))$.

We define $APC_B(\Sigma) = APC(\Sigma) \otimes B$.

**Lemma 4.4.** Suppose that $f_1 \in AP_B(\mathbb{R})$, $f_2 \in AP_B(\mathbb{R} + i\pi)$. Then there exists a function $F \in APC_B(\Sigma)$ which is harmonic in the interior of $\Sigma$ whose boundary values are $f_1$ and $f_2$. Moreover, $F$ admits a continuous extension to the maximal ideal space $M(APH(\Sigma))$.

**Proof.** By definition $f_1$ and $f_2$ can be approximated on $\mathbb{R}$ and $\mathbb{R} + i\pi$ by functions of the form

$$q_1 = \sum_{l=0}^{k_1} b_le^{i\lambda lt}, \quad q_2 = \sum_{l=0}^{k_2} c_le^{i\mu lt},$$

respectively, where $b_l, c_l \in B$, $\lambda_l, \mu_l \in \mathbb{R}$. Using the Poisson integral formula we extend $q_1$ and $q_2$ from the boundary to a function $H \in APC_B(\Sigma)$ harmonic in the interior of $\Sigma$, see, e.g., [2], Lemma 4.3 for similar arguments. Also, by the definition of the algebra $APC_B(\Sigma)$ and by Proposition 4.3, $H$ admits a continuous extension to $M(APH(\Sigma))$. Now, by the maximum principle for harmonic functions the sequence of functions $H$ converges in $APC_B(\Sigma)$ to a certain function $F$ harmonic in the interior of $\Sigma$ which satisfies the required properties of the lemma. □

Let $\phi_{z_0} : \mathbb{D} \to \mathbb{H}_+$,

$$\phi_{z_0}(z) := \frac{2i(z_0 - z)}{z_0 + z}, \quad z \in \mathbb{D},$$

be a conformal map of $\mathbb{D}$ onto the upper half-plane $\mathbb{H}_+$. Then $\phi_{z_0}$ is also continuous on $\partial \mathbb{D} \setminus \{-z_0\}$ and maps it diffeomorphically onto $\mathbb{R}$ (the boundary of $\mathbb{H}_+$) so that $\phi_{z_0}(z_0) = 0$. Let $\Sigma_0$ be the interior of the strip $\Sigma$. Consider the conformal map $\text{Log} : \mathbb{H}_+ \to \Sigma_0$, $z \mapsto \text{Log}(z) := \ln |z| + i\text{Arg}(z)$, where $\text{Arg} : \mathbb{C} \setminus \mathbb{R}_- \to (-\pi, \pi)$ is the principal branch of the multi-function arg, the argument of a complex number. The function $\text{Log}$ is extended to a homeomorphism of $\overline{\mathbb{H}}_+ \setminus \{0\}$ onto $\Sigma$; here $\overline{\mathbb{H}}_+$ stands for the closure of $\mathbb{H}_+$. 
The proof of the next statement uses Lemma 4.4 and is very similar to the proof of Lemma 4.2 (for $B = \mathbb{C}$) in [2], so we omit it.

Suppose that $z_0 = e^{i\theta_0}$. For $s \in (0, \pi)$ we set $\gamma_1(z_0, s) := \log(\phi_{z_0}(\gamma_0^1(s))) \subset \mathbb{R}$ and $\gamma_{-1}(z_0, s) := \log(\phi_{z_0}(\gamma_0^{-1}(s))) \subset \mathbb{R} + i\pi$.

**Lemma 4.5.** Let $z_0 \in S$, suppose that $f \in SAP_B(\{-z_0, z_0\})$. We put $f_k = f|_{\gamma_0^1(\pi)}$, and define on arc $\gamma_k(z_0, s)$

$$h_k = f_k \circ \varphi_{z_0}^{-1} \circ \log^{-1}, \quad k \in \{-1, 1\}.$$ 

Then for any $\varepsilon > 0$ there exist a number $s_\varepsilon \in (0, s)$ and a function $H \in APC_B(\Sigma)$ harmonic on $\Sigma_0$ such that

$$\sup_{z \in \gamma_k(z_0, s_\varepsilon)} \|h_k(z) - H(z)\|_B < \varepsilon, \quad k \in \{-1, 1\}.$$ 

Let $z_0 \in \partial \mathbb{D}$ and $U_{z_0}$ be the intersection of an open disk of radius $\leq 1$ centered at $z_0$ with $\mathbb{D} \setminus z_0$. We call such $U_{z_0}$ a *circular neighbourhood* of $z_0$.

We say that a bounded continuous function $f : \mathbb{D} \mapsto B$ is *almost-periodic near* $z_0$ if there exist a circular neighbourhood $U_{z_0}$, and a function $\hat{f} \in APC_B(\Sigma)$ such that

$$f(z) = \hat{f}(\log(\varphi_{z_0}(z))), \quad z \in U_{z_0}.$$ 

Let $M_S$ be the maximal ideal space of $H^\infty(\mathbb{D}) \cap SAP(S)$ and $a_S : M_S \rightarrow \overline{\mathbb{D}}$ be the natural continuous surjection, see Remark 1.12. Recall that for $z_0 \in S$, $a_S^{-1}(z_0)$ can be naturally identified with $b\Sigma := M(APH(\Sigma))$. Therefore the algebra $O(a_S^{-1}(z_0))$ of holomorphic functions on $a_S^{-1}(z_0)$ can be defined similarly to $O(b\Sigma)$ from Section 2 (by means of this identification), see [2] for details.

In the proof of Theorem 1.8 of [2] (see Lemmas 4.4, 4.6 there) we established

1. Any scalar harmonic function $f$ on $\mathbb{D}$ almost periodic near $z_0$ admits a continuous extension to $a_S^{-1}(\overline{U}_{z_0}) \subset M_S$ for some circular neighbourhood $U_{z_0}$.

2. For any holomorphic function $f \in O(a_S^{-1}(z_0))$, $z_0 \in S$, there is a bounded holomorphic function $\hat{f}$ on $\mathbb{D}$ of the same sup-norm almost periodic near $z_0$ such that its extension to $a_S^{-1}(z_0)$ coincides with $f$.

Similarly to $O(b\Sigma)$ we define the Banach space $O_B(b\Sigma)$ of $B$-valued holomorphic functions on $b\Sigma$, cf. Section 2. Then by Theorem 1.4 $O_B(b\Sigma) := O(b\Sigma) \otimes B$. This together with the $B$-valued Bohr approximation theorem for the subalgebra of harmonic functions in $APC_B(\Sigma)$ (see Lemma 4.4) and statements (1), (2) imply

3. Any $B$-valued harmonic function $f$ on $\mathbb{D}$ almost periodic near $z_0$ admits a continuous extension to $a_S^{-1}(\overline{U}_{z_0}) \subset M_S$ for some circular neighbourhood $U_{z_0}$. 

11
(4) For any holomorphic function \( f \in \mathcal{O}_B(a_s^{-1}(z_0)) \), \( z_0 \in S \), there is a bounded \( B \)-valued holomorphic function \( \hat{f} \) on \( \mathbb{D} \) of the same sup-norm almost periodic near \( z_0 \) such that its extension to \( a_s^{-1}(z_0) \) coincides with \( f \).

From (3) and (4) we obtain

**Lemma 4.6.** Let \( f \in SAP_B(S) \cap H_c^\infty(\mathbb{D}) \) and \( z_0 \in \partial \mathbb{D} \). There is a bounded \( B \)-valued holomorphic function \( \hat{f} \) on \( \mathbb{D} \) almost periodic near \( z_0 \) such that for any \( \varepsilon > 0 \) there is a circular neighbourhood \( U_{z_0;\varepsilon} \) of \( z_0 \) so that

\[
\sup_{z \in U_{z_0;\varepsilon}} \| f(z) - \hat{f}(z) \|_B < \varepsilon.
\]

**Proof.** Assume, first, that \( z_0 \in S \). By Lemma 4.5 for any \( n \in \mathbb{N} \) there exist a number \( s_n \in (0, s) \) and a function \( H_n \in APC_B(\Sigma) \) harmonic on \( \Sigma_0 \) such that

\[
\sup_{z \in \gamma_k(z_0, s_n)} \| f_k(z) - H_n(z) \|_B < \frac{1}{n}, \quad k \in \{-1, 1\}. \tag{4.3}
\]

Using the Poisson integral formula for the bounded \( B \)-valued harmonic function \( f - H_n \) on \( \mathbb{D} \) we easily obtain from (4.3) that there is a circular neighbourhood \( V_{z_0;n} \) of \( z_0 \) such that

\[
\sup_{z \in V_{z_0;n}} \| f(z) - H_n(z) \|_B < \frac{2}{n}. \tag{4.4}
\]

According to (3) each \( H_n \) admits a continuous extension \( \hat{H}_n \) to \( a_s^{-1}(z_0) \cong b\Sigma \). Moreover, (4.4) implies that the restriction of the sequence \( \{ \hat{H}_n \}_{n \in \mathbb{N}} \) to \( a_s^{-1}(z_0) \) forms a Cauchy sequence in \( C_B(a_s^{-1}(z_0)) \). Let \( \hat{H} \in C_B(a_s^{-1}(z_0)) \) be the limit of this sequence.

Further, for any functional \( \phi \in B^* \) the function \( \phi \circ f \in SAP(S) \cap H_c^\infty(\mathbb{D}) \) and therefore admits a continuous extension \( f_\phi \) to \( a_s^{-1}(z_0) \) such that on \( a_s^{-1}(z_0) \) the extended function belongs to \( \mathcal{O}(a_s^{-1}(z_0)) \). Now, (4.4) implies directly that \( f_\phi = \phi \circ \hat{H} \) for any \( \phi \in B^* \). Then from the definition of \( \mathcal{O}_B(a_s^{-1}(z_0)) \), see Section 2, follows that \( \hat{H} \in \mathcal{O}_B(a_s^{-1}(z_0)) \). Thus by (4) we find a bounded \( B \)-valued holomorphic function \( \hat{f} \) on \( \mathbb{D} \) of the same sup-norm as \( \hat{H} \) almost periodic near \( z_0 \) such that its extension to \( a_s^{-1}(z_0) \) coincides with \( \hat{H} \). Now by the definition of the topology of \( M_s \), see [2], Lemma 4.4 (a), we obtain that for any \( \varepsilon > 0 \) there is a number \( N \in \mathbb{N} \) such that for all \( n \geq N \),

\[
\sup_{z \in V_{z_0;n}} \| \hat{f}(z) - H_n(z) \|_B < \frac{\varepsilon}{2}.
\]

Finally, choose \( n \geq N \) in (4.4) such that the right-hand side there \( < \frac{\varepsilon}{2} \). For this \( n \) we set \( U_{z_0;\varepsilon} := V_{z_0;n} \). Then the previous inequality and (4.4) imply the required

\[
\sup_{z \in U_{z_0;\varepsilon}} \| f(z) - \hat{f}(z) \|_B < \varepsilon.
\]
If, now, \( z_0 \notin S \), then, by definition, \( f|_{\partial D} \) is continuous at \( z_0 \). In this case as the function \( \hat{f} \) we can choose the constant \( B \)-valued function equal to \( f(z_0) \) on \( D \). Then the required result follows from the Poisson integral formula for \( f - \hat{f} \). We leave the details to the reader. □

4.1.3. We are now ready to prove Theorem 4.2

Proof. As it was shown in [2], \( A_S \subset SAP(S) \cap H^\infty(D) \). So \( A^B_S \subset SAP_B(S) \cap H^\infty_B(D) \).

Let us prove the opposite inclusion.

(A) Consider first the case \( S = F \), where \( F = \{z_i\}_{i=1}^m \) is a finite subset of \( \partial D \).

Let \( f \in SAP_B(F) \cap H^\infty_D \). Then according to Lemma 4.6, there exists a function \( f_{z_1} \in APH_B(\Sigma) \) such that the bounded \( B \)-valued holomorphic function \( g_{z_1} = f - f_{z_1} \), \( g_{z_1} := f_{z_1} \circ \Log \circ \varphi_{z_0} \), on \( D \) is continuous and equals 0 at \( z_1 \).

Let us show that \( g_{z_1} \in A^B_{\{z_1,-z_1\}} \). Indeed, since \( f_{z_1} \in APH_B(\Sigma) \), by Theorem 1.4, it can be approximated in \( APH_B(\Sigma) \) by finite sums of functions \( be^{i\lambda z} \), \( b \in B \), \( \lambda \in \mathbb{R} \), \( z \in \Sigma \). In turn, \( g_{z_1} \) can be approximated by finite sums of functions \( be^{i\lambda \Log \circ \varphi_{z_1}} \). As it was shown in [2], \( e^{i\lambda \Log \circ \varphi_{z_1}} \in A_{\{z_1,-z_1\}} \). Hence, \( g_{z_1} \in A^B_{\{z_1,-z_1\}} \).

We define

\[
\hat{g}_{z_1} = \frac{g_{z_1}(z)(z + z_1)}{2z_1}.
\]

Then, since the function \( z \mapsto (z + z_1)/(2z_1) \in A(D) \) and equals 0 at \( -z_1 \), and \( g_{z_1} \in A^B_{\{z_1,-z_1\}} \), the function \( \hat{g}_{z_1} \in A^B_{\{z_1\}} \). Moreover, by the definitions of \( g_{z_1} \) and \( \hat{g}_{z_1} \), the function \( \hat{g}_{z_1} - f \) is continuous and equal to zero at \( z_1 \). Thus,

\[
\hat{g}_{z_1} - f \in SAP(F \setminus \{z_1\}) \cap H^\infty_B(D).
\]

We proceed in this way to get functions \( \hat{g}_{z_k} \in A^B_{\{z_k\}} \) such that

\[
f - \sum_{k=1}^m \hat{g}_{z_k} \in A_B(D).
\]

Here \( A_B(D) \) is the Banach space of \( B \)-valued bounded holomorphic functions on \( D \) continuous up to the boundary. As in the scalar case using the Taylor expansion at 0 of functions from \( A_B(D) \) one can easily show that \( A_B(D) = A(D) \otimes B \). This and the above implication imply that \( f \in A^B \) := \( A \otimes B \), as required.

(B) Let us consider the general case of \( S \subset \partial D \) an arbitrary closed set.

Let \( f \in SAP_B(S) \cap H^\infty_B(D) \). As it follows from Lemma 4.6 and the arguments presented in part (A), given an \( \varepsilon > 0 \) there exist points \( z_k \in \partial D \), functions \( f_k \in A^B_{\{z_k\}} \) and circular neighbourhoods \( U_{z_k} \) \( (1 \leq k \leq m) \) such that \( \{U_k\}_{k=1}^m \) forms an open cover of set \( \partial D \setminus \{z_k\}_{k=1}^m \) and

\[
\|f(z) - f_k(z)\|_B < \varepsilon \quad \text{on} \quad U_{z_k}, \quad 1 \leq k \leq m.
\]
Since $S$ is closed, for $z_k \notin S$ we may assume that $f_k$ is continuous in $\bar{U}_{z_k}$.

Let us define a $B$-valued 1-cocycle $\{c_{kj}\}_{k,j=1}^m$ on intersections of the sets in $\{U_{z_k}\}_{k=1}^m$ by the formula

$$c_{kj}(z) := f_k(z) - f_j(z), \quad z \in U_{z_k} \cap U_{z_j}. \quad (4.6)$$

Then (4.5) implies

$$\sup_{k,j,z} \|c_{kj}(z)\|_B < 2\varepsilon. \quad (4.7)$$

Since $m < \infty$, we may assume, without loss of generality, that none of the intersections $U_{z_k} \cap U_{z_j}$, $k \neq j$, contains points $z_k$. Further, we may choose the above functions and sets such that $c_{kj}$ is holomorphic in $U_{z_k} \cap U_{z_j}$ and continuous in the closure of $U_{z_k} \cap U_{z_j}$. Let $\{\rho\}_{k=1}^m$ be a smooth partition of unity subordinate to the open cover $\{U_{z_k}\}_{k=1}^m$ of an open annulus $A \subset \subset \cup_{k=1}^m U_{z_k}$ with outer boundary $\partial \mathbb{D}$ such that each $\rho_k$ is the restriction of a smooth function defined on a neighbourhood of $\partial \mathbb{D}$ and $\rho_k(z_k) = 1$, $1 \leq k \leq m$. We resolve the cocycle $\{c_{kj}\}_{k,j=0}^m$ using this partition of unity by formulas

$$\tilde{f}_j(z) = \sum_{k=1}^m \rho_k(z)c_{kj}(z), \quad z \in U_{z_j} \cap A, \quad (4.8)$$

so, by definition,

$$c_{kj}(z) = \tilde{f}_k(z) - \tilde{f}_j(z), \quad z \in U_{z_k} \cap U_{z_j} \cap A. \quad (4.9)$$

Now since $c_{kj}$ are $B$-valued holomorphic functions in $U_{z_k} \cap U_{z_j}$ continuous up to the boundary, $\rho_{kj}$ are smooth functions on a neighbourhood of $\partial \mathbb{D}$ and $A \subset \subset \cup_{k=1}^m U_{z_k}$,

$$h(z) := \frac{\partial \tilde{f}_j(z)}{\partial \bar{z}}, \quad z \in U_{z_j} \cap A, \quad 1 \leq j \leq m,$$

is a smooth global $B$-valued function on $A$ continuous up to the boundary.

We define

$$H(z) := \frac{1}{2\pi i} \int_{\zeta \in \mathbb{D}} \frac{h(\zeta)}{\zeta - z} d\zeta \wedge d\bar{\zeta}, \quad z \in \bar{\mathbb{D}}. \quad (4.9)$$

Similarly to the scalar case we use polar coordinates in (4.9) to obtain

$$\sup_{z \in A} \|H(z)\|_B \leq C w(A) \sup_{z \in A} \|h(z)\|_B,$$

where $w(A)$ is width of $A$ and $C > 0$ is a numerical constant. Furthermore, $H$ is continuous on $\bar{A}$ and $\frac{\partial H}{\partial \bar{z}} = h$ on $A$. Without loss of generality (see (4.7) and the definition of functions $\rho_j$) we may assume that $w(A)$ is sufficiently small, so that

$$\sup_{z \in A} \|H(z)\|_B < \varepsilon.$$
Let us define
\[ c_i(z) := \tilde{f}_i(z) - H(z), \quad z \in U_{z_i} \cap \bar{A}. \] (4.10)

Then \( c_i \) is a \( B \)-valued continuous function holomorphic in \( U_{z_i} \cap A \) satisfying (see (4.7))
\[ \sup_{z \in U_{z_i} \cap A} \|c_i(z)\|_B \leq 3\varepsilon. \] (4.11)

By (4.8),
\[ c_i(z) - c_j(z) = c_{ij}(z), \quad z \in U_{z_i} \cap U_{z_j} \cap A. \] (4.12)

Let us determine a function \( f_\varepsilon \) defined on \( \bar{A} \setminus \{z_i\}_{i=1}^m \) by formulas
\[ f_\varepsilon(z) := f_i(z) - c_i(z), \quad z \in U_{z_i} \cap \bar{A}. \]

According to (4.6) and (4.12), \( f_\varepsilon \) is a bounded continuous \( B \)-valued function on \( \bar{A} \setminus \{z_i\}_{i=1}^m \) holomorphic in \( A \). Furthermore, since \( c_i \) is continuous on \( \bar{U}_{z_i} \cap \bar{A} \), and \( f_i \in A_{z_i}^B \), for \( z_i \in S \) and \( f_i \in A_{z_i}(\mathbb{D}) \) otherwise, \( f_\varepsilon|_{\partial \mathbb{D}} \in \text{SAP}_B(F) \) where \( F = \{z_i\}_{i=1}^m \cap S \). Also, from inequalities (4.5) and (4.11) we have
\[ \sup_{z \in A} \|f(z) - f_\varepsilon(z)\|_B < 4\varepsilon. \] (4.13)

Now let \( D' \) be an open disk centered at 0 whose intersection with \( A \) is an annulus of width \( \frac{w(A)}{2} \). Let \( A' \) be the open annulus with the interior boundary coinciding with the interior boundary of \( A \) and with the outer boundary \( \{z \in \mathbb{C} : |z| = 2\} \).

Then \( \{A', D'\} \) forms an open cover of \( \mathbb{D} \). Let us define a 1-cocycle on \( A' \cap D' = A \cap D' \) by the formula
\[ c(z) = f(z) - f_\varepsilon(z), \quad z \in A \cap D'. \]

As it follows from (4.13), \( \|c(z)\|_B < 4\varepsilon \) for all \( z \in D' \cap A' \). Consider a smooth partition of unity subordinate to the cover \( \{A', D'\} \) of \( \mathbb{D} \) which consists of smooth radial functions \( \rho_A, \rho_{D'} \) defined on \( \mathbb{C} \) such that
\[ \max\{\|\nabla \rho_A\|_{L^\infty(\mathbb{C})}, \|\nabla \rho_{D'}\|_{L^\infty(\mathbb{C})}\} \leq \frac{C}{w(D' \cap A')} = \frac{2C}{w(A)} \] (4.14)

where \( C > 0 \) is a numerical constant. We resolve the cocycle \( c \) as follows:
\[ f_{A'}(z) = -\rho_{D'}(z)c(z), \quad z \in A', \]
\[ f_{D'}(z) = \rho_A(z)c(z), \quad z \in D'. \]

Then \( c(z) = f_{D'}(z) - f_{A'}(z), \) \( z \in A' \cap D' \). Since \( c \) is holomorphic in \( A' \cap D' \),
\[ g(z) = \begin{cases} \frac{\partial f_{A'}(z)}{\partial \bar{z}}, & z \in A', \\ \frac{\partial f_{D'}(z)}{\partial \bar{z}}, & z \in D' \end{cases} \] (4.15)
is a bounded smooth $B$-valued function on $\mathbb{D}$ with support in $A$.

Next,

$$G(z) = \frac{1}{2\pi i} \int_{\zeta \in \mathbb{D}} \frac{g(\zeta)}{\zeta - z} d\zeta \wedge d\bar{\zeta}$$  \hspace{1cm} (4.16)$$

is a smooth $B$-valued function on $\mathbb{D}$, continuous up to the boundary such that $\frac{\partial G}{\partial \bar{z}} = g$ on $\mathbb{D}$. Moreover, by (4.13), (4.14) and the fact that $\text{supp}(g) \subset A$ we obtain from (4.16)

$$\sup_{z \in \mathbb{D}} \|G(z)\|_B \leq C' w(A) \frac{\varepsilon}{w(A)} = C' \varepsilon  \hspace{1cm} (4.17)$$

for a numerical constant $C' > 0$.

Now we define

$$c_A'(z) = f_A'(z) - G(z), \quad z \in A,$$

$$c_{D'}(z) = f_{D'}(z) - G(z), \quad z \in D'.$$

Then $c_A'$ and $c_{D'}$ are $B$-valued holomorphic functions in $A$ and $D'$, respectively. Clearly, we have $c_{D'}(z) - c_A'(z) = c(z)$ for all $z \in D' \cap A'$. Furthermore, as it follows from (4.13), (4.17) there exists a numerical constant $\hat{C} > 0$ such that

$$\|c_A'(z)\|_B < \hat{C} \varepsilon, \quad z \in A, \quad \|c_{D'}(z)\|_B < \hat{C} \varepsilon, \quad z \in D'.$$

Finally, let us define

$$F_\varepsilon(z) := \begin{cases} 
    f(z) - c_{D'}(z), & z \in D', \\
    f_\varepsilon(z) - c_A'(z), & z \in A.
\end{cases}$$

Clearly, $F_\varepsilon$ is a $B$-valued holomorphic function on $\mathbb{D}$ (since for every $z \in D' \cap A$ we have $f(z) - f_\varepsilon(z) - c_{D'}(z) + c_A'(z) = f(z) - f_\varepsilon(z) - c(z) = 0$). Also,

$$\sup_{z \in \mathbb{D}} \|f(z) - F_\varepsilon(z)\|_B < \hat{C} \varepsilon$$

for a numerical constant $\hat{C} > 0$, and by definition $F_\varepsilon \in SAP_B(F) \cap H^\infty_B(\mathbb{D})$, where $F = \{z_1, \ldots, z_m\} \cap S$.

The last inequality and part (A) of the proof show that the complex vector space generated by spaces $A^B_F$ for all possible finite subsets $F \subset S$ is dense in $SAP_B(S) \cap H^\infty_B(\mathbb{D})$. Since the closure of all such $A^B_F$ is $A^S_B$, we obtain the required: $SAP_B(S) \cap H^\infty_B(\mathbb{D}) = A^S_B$. \hfill \square

**Remark 4.7.** From the proof of Theorem 4.2 one obtains also that each function from $SAP_B(S) \cap H^\infty_B(\mathbb{D})$ admits a continuous extension to the maximal ideal space of the algebra $SAP(S) \cap H^\infty(\mathbb{D})$. Then this theorem and the results of Section 5.1 of [5] imply that $SAP(S) \cap H^\infty(\mathbb{D})$ has the approximation property.
4.2 Proofs

Proof of Theorem 1.9. Let $S = \prod_{k=1}^{n} S_{k} \subset (\partial \mathbb{D})^{n}$ where $S_{k} \subset \partial \mathbb{D}$ is a closed set. We must prove that $H^{\infty}(\mathbb{D}^{n}) \cap SAP^{n}(S) = A_{S}^{n}$.

We prove this statement by induction over $n$. For $n = 1$ the identity was already proved. Suppose that it is true for $n - 1$, that is, $H^{\infty}(\mathbb{D}^{n-1}) \cap SAP^{n-1}(S') = A_{S'}^{n-1}$ where $S' := \prod_{k=1}^{n-1} S_{k}$, and prove it for $n$. Let $S = S' \times S_{n}$.

**Lemma 4.8.**

\[ SAP^{n}(S) = SAP_{B}(S_{n}) \]

for $B = SAP^{n-1}(S')$ and

\[ A_{S}^{n} = A_{S_{n}}^{C} \]

for $C = A_{S'}^{n-1}$.

**Proof.** The proof follows directly from the definitions of $SAP^{n}(S)$ and $A_{S}^{n}$. □

Let $f \in H^{\infty}(\mathbb{D}^{n}) \cap SAP^{n}(S)$. According to Lemma 4.8, $f \in SAP_{B}(S_{n})$ for $B = SAP^{n-1}(S')$. Also, by the Poisson integral formula for bounded polyharmonic functions on $\mathbb{D}^{n}$, and the fact that $f \in H^{\infty}(\mathbb{D}^{n})$ we obtain that $f \in H_{B_{S}}^{\infty}(\mathbb{D})$. Thus, in virtue of Theorem 4.2

\[ f \in SAP_{B}(S_{n}) \cap H_{B_{S}}^{\infty}(\mathbb{D}) = A_{S_{n}}^{B}. \tag{4.18} \]

Let $\xi = (\xi_{1}, \ldots, \xi_{n})$ be coordinates on $(\partial \mathbb{D})^{n}$. Applying again the Poisson integral formula for bounded polyharmonic functions on $\mathbb{D}^{n}$ we obtain that the functions $f(\cdot, \ldots, \cdot, \xi_{n}) \in H^{\infty}(\mathbb{D}^{n-1})$ for almost all $\xi_{n} \in \partial \mathbb{D}$. Thus, by the induction hypothesis, for almost all $\xi_{n} \in \mathbb{D}$ we have $f(\cdot, \ldots, \cdot, \xi_{n}) \in H^{\infty}(\mathbb{D}^{n-1}) \cap B = A_{S'}^{n-1}$. From here and (4.18) we get $f \in A_{S_{n}}^{C}$, $C := A_{S'}^{n-1}$. Now, Lemma 4.8 implies that $f \in A_{S}^{n}$. This shows that $H^{\infty}(\mathbb{D}^{n}) \cap SAP^{n}(S) \subset A_{S}^{n}$. The converse embedding $A_{S}^{n} \subset H^{\infty}(\mathbb{D}^{n}) \cap SAP^{n}(S)$ follows immediately from the definition of $A_{S}^{n}$. □

**Proof of Theorem 1.11.** According to Theorem 1.9 and the scalar version of Theorem 4.2, the algebra $H^{\infty}(\mathbb{D}^{n}) \cap SAP^{n}(S)$ is symmetric tensor product of algebras $H^{\infty}(\mathbb{D}) \cap SAP(S_{k})$, $1 \leq k \leq n$. Therefore by a standard result of the theory of Banach algebras we obtain that $M_{S}$ is homeomorphic to $M_{S_{1}} \times \cdots \times M_{S_{n}}$ for the corresponding maximal ideal spaces of these algebras. As it was proved in [2], $\mathbb{D}$ is dense in each $M_{S_{k}}$ in the corresponding Gelfand topology. Therefore $\mathbb{D}^{n}$ is dense in $M_{S}$ in the Gelfand topology. □
References

[1] A. S. Besicovich, *Almost periodic functions*. Dover Publications, 1958.

[2] A. Brudnyi and D. Kinzebulatov, On uniform subalgebras of $L^\infty$ on the unit circle generated by almost periodic functions. Algebra and Analysis 19 (2007), 1–33.

[3] P. Enflo, A counterexample to the approximation property in Banach spaces. Acta Math. 130 (1973), 309–317.

[4] J. Garnett, *Bounded analytic functions*. Academic Press, 1981.

[5] A. Grothendieck, Products tensoriels topologiques et espaces nucléaires. *Memoirs Amer. Math. Society* 16, 1955.

[6] D. Sarason, Toeplitz operators with semi-almost periodic kernels. Duke Math J. 44 (1977), 357–364.