A RESONANCE INTERACTION OF SEISMOGRAVITATIONAL MODES ON TECTONIC PLATES.

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\textit{In memory of our colleague and friend}

Boris Sergeevich Pavlov

Abstract

This paper discusses resonance effects to advance a classical earthquake model, namely the celebrated M8 global test algorithm. This algorithm gives high confidence levels for prediction of Time Intervals of Increased Probability (TIP) of an earthquake. It is based on observation that almost 80\% of earthquakes occur due to the stress accumulated from previous earthquakes at the location and stored in form of displacements against gravity and static elastic deformations of the plates. Nevertheless the M8 global test algorithm fails to predict some powerful earthquakes. In this paper we suggest the additional possibility of considering the dynamical storage of the elastic energy on the tectonic plates due to resonance beats of seismo-gravitational oscillations (SGO) modes of the plates. We make sure that the tangential compression in the middle plane of an “active zone” of a tectonic plate may tune its SGO modes to the resonance condition of coincidence the frequencies of the corresponding localized modes with the delocalized SGO modes of the complement. We also consider the beats arising between the modes under a small perturbations of the plates, and, assuming that the discord between the perturbed and unperturbed resonance modes is strongly dominated by the discord between the non-resonance modes estimate the energy transfer coefficient.

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1 Exterior and interior dynamics of tectonic plates.

The “M8 Global test algorithm” (see [12] and references) for earthquake prediction was designed in 1984 at the International Institute of Earthquake Prediction and Mathematical Geophysics (Moscow) based on the observation that almost 80% of actual events at the selected location arise due to the stress built up thanks to previous events at the corresponding Earthquake-prone (active) zone. J. K. Gardner and L. Knopoff observed a sequence of earthquakes in Southern California, with aftershocks removed, was Poissonian, [7]. Since then a mix of statistical and analytical techniques through the results of a global 20-year long experiment give indirect confirmation of common features of both the predictability and the diverse behaviour of the Earth’s naturally fractal lithosphere. The statistics achieved to date prove (remarkably with confidence above 99%) the rather high efficiency of the M8 and M8-MSc predictions limited to intermediate-term middle- and narrow-range accuracy. These models also adaptable, see eg [25]. There are other models, we note those found in the works Turcotte and Schubert, [35] and Dahlen and Tromp [5] and our ideas may have implications for these as well, though we do not give specific details here.

The analytical and mechanical arguments used to derive the various models are based on the assumption, that for the most part, the energy at the active zone is stored in the form of static elastic deformation and the displacement of tectonic plates in the gravitational field.

Though both M8 and the improved MSc algorithms are extremely efficient for prediction of the Time intervals of Increased Probability (TIP) of earthquakes, some highly dangerous events, such as the recent Tohoku earthquake (Japan, March 11, 2011) were not predicted. In Tohoku the “black box” constructed based on the above algorithms, removed the TIP warning from the list of expected earthquakes at the Tohoku location 70 days before the earthquake, see the retrospective analysis of the Global Test effectiveness given by Kossobokov in [14].

The mechanical arguments for these algorithms are derived from the idea of quasi-statical (adiabatic) variation of the potential energy of the plates during the periods between earthquakes while the tectonicplates participate in the “exterior” dynamics, such as floating of (fragments of) the plates down the slopes formed by previous earthquakes, or responding to the hydrodynamical oscillations in the resonance cavities at the earthquake-prone (active) zones filled with magma, see for instance, [31].

On the other hand, earthquakes do arise within a background formed by oscillations of the planet. Many seismologists study these typical oscillations with amplitudes within $\sim 0.2 \sim 0.5$ cm, and periods of circa a few minutes. However interesting anomalies with periods of circa 10 minutes were noticed by G.A.Sobolev and A.A.Lyubushkin, see [32], in the course of their analysis of seismological data preceding the Sumatra earthquake on December 26, 2004, recorded in the remote zone of the earthquake. Moreover, some decaying periodic patterns, see Fig. 4 in [32], were noticed on the relevant spectral-temporal
(time-spectral) cards, see below, with the periods $\sim 100$ min. On diagram 7 of the same paper one can see decaying patterns with even greater periods $\sim 2400$ min arising prior the major earthquake. Unfortunately the authors of [32] did not consider these as important details and did not develop any extended analysis of them in their paper.

Approximately a decade before this, see [18], long periodic oscillation patterns, with periods $\sim 40 - 70$ minutes, were registered by E.M. Linkov using a “vertical pendulum”, constructed especially for studying Seismogravitational Oscillations (SGO) of the Earth. These oscillations have been intensely monitored during the last decade, see for instance [27, 28], as an important component of the “interior dynamic” of the plates. They form a natural dynamical background of catastrophic events such as Earthquakes, Tsunami and Volcanic Eruptions. Monitoring of SGO confirmed the hypothesis [26, 29] of their spectral nature. According to [29], the SGO should be interpreted as decaying flexural (vertical) eigen-modes of large tectonic plates with linear size up to few thousands kilometers.

The typical energy of the mode may be estimated based on spectral (frequency), physical properties such as density and Young’s modulus and geometric characteristics of the plate. For instance, the elastic energy stored in a single SGO mode with frequency $200 \mu$Hz and amplitude $2 \times 10^{-3}$ m on a tectonic plate with area circa $10^{14}$ m$^2$, thickness $10^5$ m and density 3380 kg m$^{-3}$ is estimated as $54 \times 10^9$ joules. This is almost equivalent to the seismic moment (“full energy”) of the 4M earthquake in Johannesburg (South Africa) November 18, 2013.

While discussing the inner dynamics of tectonic plates in our recent publications, see [21, 6] and similarly [10], we proposed to take into account the migration of elastic energy between regions of tectonic plates, caused by beating of the resonance spectral modes localized on the regions. For an active zone, already unstable under statical stress, the migration of energy defined by the resonance beating might be sufficient to trigger an earthquake.

We therefore suggest that the modelling of tectonic processes with regard of resonance migration of energy would probably help in developing more realistic theoretical scenarios for an earthquake. In the simplest case of two resonance SGO modes, localized on neighbouring regions $\Omega_\varepsilon, \Omega_c$, the beating pattern is periodic and the amount of elastic energy transferred from one location to another on each period is defined by the corresponding transfer coefficient. The transfer coefficient may be large for exact tuning of the corresponding frequencies $|\nu_\varepsilon - \nu_c| << \bar{\nu} \equiv (|\nu_\varepsilon + \nu_c|)/2$. Thus we study the influence of resonance effects for this geophysical phenomenon and how $t$ can be used to inform advances in the classical model and give examples which demonstrate that resonances can be a reason for earthquake. However we are not able to present a full model for the phenomenon. Some discussions pertain to one-dimensional systems and cannot present a completely realistic model, but they demonstrate the possibility of such mechanism in seismic phenomena as quite natural values of parameters are used. We hope the interested reader can follow the construction of the model.
and come to understanding of how full implementation might be realised. The examples we offer can be considered as benchmarks for future, more realistic models, based on these suggested ideas.

In this paper we aim at estimating the transfer coefficient, while considering the problem in frames of perturbation analysis, depending on the spectral characteristics of the unperturbed modes on disjoint regions and the type of interaction imposed. Usually the SGO modes are presented on time-spectral cards obtained from the corresponding seismograms via averaging of the SGO amplitudes, with certain frequency, on a step-wise system of time - windows, obtained by shifting an initial window by certain interval of time on each step. The boundaries of the spectral -time domains on the cards, where the averaged amplitude of the SGO mode exceeds the given value \( A \), form a system of isolines in the frequency/time coordinates \( \nu, T \). The horizontal axis for time, is graded in hours, the vertical axis, for the frequencies, is graded in \( \mu \text{Hz} \). Doctor L. Petrova provided us with some time-spectral cards from her private collection and shared with us some useful and interesting comments concerning the interpretation of the cards in terms of SGO dynamics, which we referred in our previous publication [6]. Find below one of her cards which was obtained by her from seismograms recorded on SSB station, France, during the period preceding the powerful earthquake on 26 September 2004 in Peru. The averaging of the amplitudes of the seismo-gravitational oscillations, with certain frequency, was done, after appropriate filtration, on a system of 20 hours time - windows, obtained by shifting an initial window by 30 minutes on each step. The relief of the window-averaged squared amplitude on the cards is graded by the isolines, with the step \( \delta A^2 = \frac{1}{10} \left[ A^2_{\text{max}} - A^2_{\text{min}} \right] \), and is painted accordingly between the isolines with shades depending on the square amplitude \( A^2 \): dull grey for the background value \( A^2_{\text{min}} \) and white for the maximal value \( A^2_{\text{max}} \). See more comments in [6].
A time - spectral card

The most interesting kind of SGO was represented by pulsations, which were observed by Linkov’s team as intense short (4-6 hours) and sometimes repeated in 30-100 hours, pulses of SGO with large amplitudes. Pulsations have been registered before 95% of powerful earthquakes and may be considered as natural precursors of them, see [18]. One may hope that a deeper analysis of the microseismic data in [32] would reveal a connection between the above “anomalies” with SGO and pulsation patterns studied by Linkov et al. Petrova attracted pointed out to us a peculiar detail on the above SSB card above, consisting of two groups of stationary modes with almost equal frequencies and visually similar relief in $\Delta_{2}^{SSB} = (190, 200) \times (55, 65)$ and in $\Delta_{3}^{SSB} = (200, 210) \times (145, 165)$. The pair was interpreted by her as a typical “seismo-gravitational pulsation”.

In [6] pulsations are interpreted as beatings of spectral modes on the tectonic plates, arising due to resonance interaction of the SGO modes, with close frequencies, while some of them are localized on active zone $\Omega_{\varepsilon}$ and others - on the complement $\Omega_{c}$. According to classical mechanics, [15], the resonance
between two “oscillators” $\Omega_\varepsilon, \Omega_c$ with close frequencies $\omega_\varepsilon, \omega_c$ and precise tuning, the beating of a pair of modes defines the periodic energy migration between the regions $\Omega_\varepsilon, \Omega_c$: $E_\varepsilon(t) = \bar{E}_\varepsilon + \delta E(t)$, $E_c(t) = \bar{E}_c + \delta E(t)$. Here the migrating part $\delta E(t)$ of the total energy arises, with opposite phases, in both locations, with total energy conserved $E = \bar{E}_\varepsilon + \bar{E}_c$.

While two weakly connected oscillators in resonance yield a periodic beating, a larger group of oscillators, under resonance conditions $|\omega_i - \omega_k| \equiv \delta_{ik} << \Omega$ with respect to the average frequency $\bar{\omega} \equiv \frac{1}{N} \sum_{k=1}^{N} \omega_k$ reveal a quasi-chaotic beating phenomenon, if the difference frequencies $\delta_{ik}$ are non-co-measurable.

The problem of estimation of the energy transfer associated with beats in the system of several connected oscillators can be reduced to the similar problem for a single oscillator under almost periodic resonance force. This is treated in Landau’s book [15], Section 22. Indeed, we will consider two 1D oscillators, with masses $m, M$ attached to springs $v, V$ and connected by an hermitian pair of elastic bonds $\gamma, \Gamma$ constrained by the Hermitian requirement $\gamma m - 1 = \Gamma M - 1 = \varepsilon$. For instance, a pair of oscillators the dynamics is described by the equations:

$$x'' + \frac{v}{m} x + \gamma m^{-1} X = 0,$$

$$X'' + \frac{V}{M} X + \Gamma M^{-1} x = 0. \quad (1.1)$$

While the elastic bonds are neglected, the eigenfrequencies of the oscillators can be calculated to be $\omega_m = \sqrt{v m^{-1}} \equiv \lambda_m$, $\omega_M = \sqrt{V M^{-1}} \equiv \lambda_M$, but with regard of the bonds are found from the quadratic equation for $\lambda = \omega^2$

$$\lambda_{\pm} = \frac{\lambda_m + \lambda_M}{2} \pm \sqrt{\left(\frac{\lambda_m + \lambda_M}{2}\right)^2 - \lambda_m \lambda_M + \varepsilon^2} = \frac{\lambda_m + \lambda_M}{2} \pm \sqrt{\left(\frac{\lambda_m - \lambda_M}{2}\right)^2 + \varepsilon^2} \quad (1.2)$$

Hereafter we will assume that the bonds are relatively weak so that we may calculate the frequencies $\omega_{\pm} \equiv \sqrt{\lambda_{\pm}}$ of the normal modes of the pair approximately based on $\varepsilon << \left[\frac{\lambda_m - \lambda_M}{2}\right] \equiv \delta > 0$, at least up to second order with respect to $\varepsilon^2 [2\delta]^{-1}$:

$$\lambda_{\pm} \approx \omega_m^2 + \varepsilon^2 [2\delta]^{-1}, \quad \lambda = \omega_M - \varepsilon^2 [2\delta]^{-1}$$

and allows to calculate the complex normal modes with $\omega_{\pm} = \sqrt{\lambda_{\pm}}$

$$\left(\begin{array}{c} e^+ \\ E_+ \end{array}\right) e^{\pm i \omega_\pm t}, \quad \left(\begin{array}{c} e^- \\ E_- \end{array}\right) e^{\pm i \omega_\pm t}. \quad (1.4)$$

The eigenvectors $\left(\begin{array}{c} e^+ \\ E_+ \end{array}\right)$ and $\left(\begin{array}{c} e^- \\ E_- \end{array}\right)$ are found from the homogeneous equations

$$\left(\begin{array}{cc} -\delta & \varepsilon \\ \varepsilon & -\delta - \varepsilon^2 \end{array}\right) \left(\begin{array}{c} e^+ \\ E_+ \end{array}\right) = 0, \quad \left(\begin{array}{cc} \delta + \varepsilon^2 & \varepsilon \\ \varepsilon & \delta + \varepsilon^2 \end{array}\right) \left(\begin{array}{c} e^- \\ E_- \end{array}\right) = 0. \quad (1.5)$$
which yield for the normalized eigenvectors, up to $O(\varepsilon^2/\delta^2)$:

\begin{equation}
\begin{pmatrix}
  e_+ \\
  E_+
\end{pmatrix} = \begin{pmatrix}
  1 \\
  \frac{\varepsilon}{2\delta}
\end{pmatrix}, \quad \begin{pmatrix}
  e_- \\
  E_-
\end{pmatrix} = \begin{pmatrix}
  -\frac{\varepsilon}{2\delta} \\
  1
\end{pmatrix}.
\end{equation}

(1.6)

In [15] the beats phenomenon is considered for a harmonic external force. Hereafter we consider above system of two oscillators $x, X$, governed by the equations (1.1), under initial condition $x(0) = 1, x'(0) = 0, X(0) = \frac{\varepsilon}{2\delta}, X'(0) = 0$. These initial conditions correspond to an excitation of the normal mode of above the pair oscillators, with second order terms $\varepsilon^2\delta^2$ neglected:

\begin{equation}
\begin{pmatrix}
  x(t) \\
  X(t)
\end{pmatrix} = \begin{pmatrix}
  1 \\
  \frac{\varepsilon}{2\delta}
\end{pmatrix} \cos \omega t + \int_0^t \frac{\varepsilon}{2\delta} \cos \omega t e^{i\omega_m t} d\tau.
\end{equation}

(1.7)

Then the dynamics if the first oscillators may be considered under an exterior force:

\begin{equation}
x'' + \omega_m^2 x + \frac{\varepsilon^2}{2\delta} \cos \omega t = 0, \quad \text{with} \quad \omega_+ = \omega_m + \frac{\varepsilon^2}{2\delta},
\end{equation}

(1.8)

under the initial conditions $x(0) = 1, x'(0) = 0$. Following [15] for the real solution of the above equation (1.7) we introduce a complex characteristic

\begin{equation}
\xi = x' + i\omega_m x,
\end{equation}

(1.9)

and rewrite the above equation (1.8) for $\xi$ as

\begin{equation}
\frac{d\xi}{dt} = -i\omega_m + \frac{\varepsilon^2}{2\delta} \cos \omega t,
\end{equation}

(1.10)

and obtain the solution $\xi(t)$ as in [15]:

\begin{equation}
\xi(t) = e^{i\omega_m t} \left[ \int_0^t \frac{\varepsilon}{2\delta} \cos \omega t e^{i\omega_m \tau} \cos \omega \tau d\tau + i\omega_m \right]
\end{equation}

(1.11)

The energy $E(x)$ of the oscillator $x$ is can be calculated as per [15] again, in terms of this characteristic as

\begin{equation}
\frac{m}{2} |\xi(t)|^2 = \frac{m}{2} \left[ |x'|^2 + \omega_m^2 |x|^2 \right] = E(x).
\end{equation}

(1.12)

For two connected oscillators $\delta E(t)$, the migrating part of the total energy, is close to zero if the difference in frequency is relatively large. Vice versa, it may be close to the full energy if the tuning is very sharp, that is difference frequency is relatively small, $|\omega_i - \omega_k| \equiv \delta_{ik} << \omega$. An example of this is the celebrated Wilberforce pendulum, [36, 37, 2]. We posit that, in the case of tectonic plates, beating of the resonance SGO modes implies (depending on conditions) migration of an essential part $\max \delta E(t) = k E$ of the total energy $E$, with the transfer coefficient $k = k(\delta \omega/\omega)$, depending on the relative variation of the frequencies. Then the total energy of the active zone $\Omega_\varepsilon$ at some moment $T_0$ in time may exceed the destruction limit $E_\varepsilon(T) = E_\varepsilon + \delta E(T_0) > E_\varepsilon^d$,
causing the destruction of some structure in the active zone and thus triggering an earthquake.

Exactly this scenario was considered in [6] as a resonance mechanism for an earthquake. Comparison of the full energy of a single SGO mode with the seismic moment requires an estimation of the transfer coefficient $k$. This becomes an important question in analytic modelling of the resonance mechanism for an earthquake.

To obtain this estimation of the transfer coefficient $k$, in the next Section 2 we sketch the basics of the Kirchhoff model of the thin plate, which is used hereafter as the tectonic plate. We leave the matter of the hydrodynamical component of the dynamics and the dissipation to future work, and concentrate our attention here on the effect of the compressing (tangential) tension in the middle plane of the plate $\Omega$, causing a lowering of the eigenfrequencies of the active zone to the resonances with the SGO modes of the complement. Then in Section 3 we consider the resonance condition for the circular active zone, with the prescribed frequency $200 \mu Hz$, choosing typical physical and geometrical parameters of a pair $\Omega, \Omega_c$ of the disjoint plates. We consider the simplest explicitly solvable example of two unperturbed circular disjoint plates, and make sure that the resonance condition is fulfilled for the active zone $\Omega_\varepsilon$, under the compressing tension, with the parameters properly chosen, and a circular plate $\Omega_c'$, while the tangential compression on the large plate is neglected in the resonance interval of frequencies. The circular large plate $\Omega_c'$ in Section 3 will differ from the actual complement $\Omega_c$ of the circular active zone $\Omega_\varepsilon$ in minor ways, dominated by the typical wavelengths of lower SGO modes on the plate. Hence so are the resonance conditions $\omega' \approx 200 \mu Hz \approx \omega_c$. A more accurate analysis of the ring-like complement will be postponed to an appendix, where the Neumann-to-Dirichlet map of the ring is calculated in terms of Bessel functions. Similar arguments will work for the sectorial boundary active zone.

2 Modelling the resonance interaction between SGO and beating phenomena.

This article investigates theoretical aspects of the resonance interaction of SGO modes based on the Kirchhoff model for a thin tectonic plate $\Omega$, see [30], and the dynamics of the plate described by a perturbed biharmonic wave equation for vertical displacement $u(x, t)$

$$H \rho u_{tt} + D \Delta^2 u + \nabla Q \nabla u = 0,$$

(2.1)

We formulate the appropriate boundary conditions on $\partial \Omega \equiv \Gamma$ which are derived from the corresponding Hamiltonian.

In (2.1) the flexural rigidity is

$$D = \frac{H^3 E}{12(1 - \sigma^2)} \equiv D_H = D_1 \times H^3 = 1.56 \times 10^{10} \frac{kg m^2}{sec^2}$$
is defined via

- the Young modulus \( E = 17.28 \times 10^{10} \frac{kg}{m^2 sec^2} \),
- the Poisson coefficient \( \sigma = 0.28 \),
- the thickness \( H \sim 3 \times 10^4 m - 10^5 m \), and
- the density \( \rho = 3380 \frac{kg}{m^3} \) of the plate.

The tangential tension in the middle plane of the plate is modelled by the symmetric elliptic operator

\[ \nabla Q \nabla u \equiv HTu \equiv \nabla QH \nabla u = H [T_x u_{xx} + 2T_{xy} u_{xy} + T_y u_{yy}] = H \nabla Q_1 \nabla u, \]

as in [23, Chapter 4], and constrained in our case by the maximal non-destructing estimate from above \( Q_1 \leq 0.3 \times 10^{10} \), see [6].

The boundary conditions for the biharmonic wave equation (2.1) are derived based on the Hamiltonian

\[ \mathcal{E}(u) = \frac{1}{2} \int_\omega [H \rho u_t^2 + D|\Delta u|^2 + 2D(1-\sigma) (|u_{xy}|^2 - u_{xx} u_{yy}) + \langle \nabla u, Q \nabla u \rangle] d\Omega + \frac{1}{2} \int_\Gamma |\frac{\partial u}{\partial n}|^2 d\Gamma, \]

see our Appendix 1, and a more detailed discussion in [23, Chapters 4,8], as well as [3], with regard to the boundary bending defined by an elastic bond \( \beta \frac{kgm}{sec^2} \).

The expression (2.2) can be transformed, under the Dirichlet boundary condition \( u \bigg|_\Gamma = 0 \) into the following equation.

\[ \mathcal{E}_D(u) = \frac{1}{2} \int_\omega [H \rho u_t^2 + D|\Delta u|^2 + \langle \nabla u, Q \nabla u \rangle] d\Omega + \frac{1}{2} \int_\Gamma [\beta - D \frac{1-\sigma}{r}] |\frac{\partial u}{\partial n}|^2 d\Gamma, \]

(2.2)

where \( r \) is the curvature radius of the boundary (positive or negative depending on the position of the center of the curvature), see for instance [3].

Minimizing of the spacial part of the Hamiltonian (2.2) leads to the corresponding “natural” boundary conditions:

\[ [\beta - D \frac{1-\sigma}{r}] \frac{\partial u}{\partial n} + D \Delta u \bigg|_\Gamma = 0. \]

(2.3)

Mikhlin proved, see [23], that the thin Kirchhoff plate is stable if \( [\beta - D \frac{1-\sigma}{r}] \geq 0 \) and \( Q \geq 0 \), corresponding to stretching in the middle plane. He also considered the contracting tension of the middle plane and found sufficient conditions for stability, see [23, Chapters 5,8]. Later Heisim, [9], actually noticed in an experiment that that plates of ice are unstable with respect to certain contracting
tension in the middle plane. In Section 3 we shall consider an example of a cir-
cular plates with centrally symmetric boundary conditions. The corresponding
wave equations in the active zone $\Omega_\varepsilon$ are

$$H_\varepsilon \rho w_{tt} + D_\varepsilon \Delta^2 w + Q_\varepsilon \Delta w = 0, \quad (2.4)$$

and on the complement $\Omega_c$. Here the tangent contraction in the middle plane
is neglected ($Q_c = 0$), so that

$$H_c \rho w_{tt} + D_c \Delta^2 w + Q_c \Delta w = 0. \quad (2.5)$$

With properly selected parameters $H_\varepsilon = 3 \times 10^4 m$, $H_c = 10^5 m$, $D_\varepsilon = D_1 \times H_\varepsilon^3$, $D_c = D_1 \times H_c^3$..., solutions admit a spectral representation constructed with the use of Bessel functions.

In next section we ensure that the resonance condition $\lambda_\varepsilon = \lambda_c$ is satisfied
for a pair of circular tectonic plates. In Section 4 we impose a weak bond onto a
family of oscillators with a multiple eigenvalue and observe the dynamics of the
relevant perturbed system. It is this that exposes the beat phenomena
involving the energy transfer between oscillators originally constrained by the
resonance condition.

Ultimately we will estimate the transition coefficient for the energy transfer for the simplest solvable model of disjoint oscillators with relatively small
masses and close frequencies $[\omega]$, perturbed by imposing a bond of them with an oscillator with larger mass $M$ and frequency $\Omega \approx \omega$.

This leaves the challenging problem of realising this program for the calculation the energy transfer coefficient for oscillator's to the more interesting
system of tectonic plates, based on fitted zero-range model. We may discuss
this elsewhere.

3 Example : A circular active zone.

Consider a thin circular plate $\Omega$ divided by a crack $\Gamma_\varepsilon$ into two complementary
parts: the circular active zone $\Omega_\varepsilon$ and the ring-like complement $\Omega_c$, centered at
$\Omega_\varepsilon$.

We begin with the case of non-connected parts, imposing on both sides
of $\Gamma_\varepsilon$ the kinematic boundary condition $w_\varepsilon \big|_\Gamma = w_c \big|_\Gamma = 0$ and independent
free-reclining boundary conditions, (see Appendix 1), or Neumann boundary
conditions imposed on elements of the domain of the generators $L_\varepsilon, L_c$ for the
relevant biharmonic wave equations:

$$H \rho w_{tt} + D_c \Delta^2 w + Q_c \Delta w = H \rho w_{tt} + L_c w = 0, \quad (3.1)$$
Figure 1: The small circular tectonic plate $\Omega_\varepsilon$ (the active zone) contacts the complementary large circular tectonic plate $\Omega_c$ along the boundary $\Gamma_\varepsilon$, while the large plate is loaded and covers the small plate on the contact line $\Gamma_\varepsilon$. Because of the load and the special geometry of the contact, the large plate develops a normal stress (vertical arrow) resulting in bending of the plate and a corresponding storage of elastic energy.

with tangential compression $Q_\varepsilon \Delta \equiv T$, $Q_\varepsilon > 0$, characterized by the positive scalar $Q_\varepsilon$ and an elastic bond $\beta$ applied on the boundary, as in (2.3):

$$w \bigg|_{\Gamma_\varepsilon} = 0, \quad \beta_\varepsilon \frac{\partial w}{\partial n} + D_\varepsilon \Delta w \bigg|_{\Gamma_\varepsilon} - \frac{D_\varepsilon}{\varepsilon} \frac{\partial w}{\partial n} \bigg|_{\Gamma_\varepsilon} = 0. \quad (3.2)$$

On the complement $\Omega_c$ we neglect the tangential compression, $Q_c = 0$, but do keep the bending and the elastic bond $\rho H w_{tt} + D_c \Delta^2 w = \rho H w_{tt} + L_c w = 0,$

for the boundary condition on the outer side of the crack $\Gamma_\varepsilon$

$$w \bigg|_{\Gamma_\varepsilon} = 0, \quad \beta_c \frac{\partial w}{\partial n} + D_c \Delta w \bigg|_{\Gamma_\varepsilon} - \frac{D_c}{\varepsilon} \frac{\partial w}{\partial n} \bigg|_{\Gamma_\varepsilon} = 0, \quad (3.4)$$

and for the boundary condition on the remote part $\Gamma_a$ of the boundary of ring-like complement $\Omega_c$ we have

$$\Psi_c \bigg|_{\Gamma_a} = 0, \quad \left[D_c \frac{1 - \sigma}{r} - \beta_c \right] \frac{\partial \Psi_c}{\partial n} - D_c \Delta \Psi_c \bigg|_{\Gamma_a} = 0. \quad (3.5)$$

The spectral characteristics of the generators $L_\varepsilon$ and $L_c$ of the wave dynamics on both $\Omega_\varepsilon, \Omega_c$, for given geometrical and physical parameters can be recovered by separation of variables.

We attempt to select the parameters $D_\varepsilon = H^3_r \times D_1, \varepsilon, Q_\varepsilon = H_s \times Q_1$ with regard of the resonance frequency $\nu_0 = 2 \times 10^{-4}$ Hz and later choose the outer
radius $a$ of the complement $\Omega_c$ such that one of ground frequencies of the biharmonic generator $L_c$ also coincides with $\nu_0 = 2 \times 10^{-4}$ Hz. Then the so constructed pair of operators $L_c, L_c$ can be perturbed by the connecting boundary condition on the crack, such that the multiple eigenvalue $\omega_0^2 = 4\pi^2 \nu_0^2$ would split into a starlet $\omega_0^2 \rightarrow \omega_0^2 (1 + \delta[\alpha])$, $[\alpha] = [\alpha_1, \alpha_2]$ as described below, implementing the beating of corresponding spectral modes on $Q_c, Q_c$.

All data, except the radius $\varepsilon$ of the active zone and the outer radius of the complement, are selected as in the previous section, but we may assume the freedom of an adiabatic change, with time, of the tangent tension in the middle plane as $Q_1 = qH \times 10^9$, $1 \leq q < 3$ below the destruction limit of the active zone $\Omega_\varepsilon$, implying the change of eigenfrequencies $\omega_\varepsilon(q)$ of the active zone depending on the compressing tension.

Removing the common factor $H_\varepsilon$ from the coefficients of the wave equation on the active zone $Q_\varepsilon$ and from the boundary conditions we obtain an equivalent form of (3.1) and the spectral problem on $\Omega_\varepsilon$ with $H = H_\varepsilon$:

$$D_1 H^2 \Delta^2 w + Q_1 \Delta w = \omega^2 \rho w, \quad \frac{\partial w}{\partial n} \Bigr|_{\Gamma_\varepsilon} = u \Bigr|_{\Gamma_\varepsilon} = 0. \quad (3.6)$$

In our earlier work [6] we estimated the small eigenvalues of the above equations neglecting the boundary effects. This is acceptable for large plates, see [6], but we notice that improved results can be obtained based on a more accurate spectral analysis of the biharmonic generator $L_\varepsilon$, using a factorization of the above equation with Dirichlet-Neumann boundary conditions at the crack:

$$0 = \left[ -\sqrt{D_1} H \Delta - \frac{Q_1}{2\sqrt{D_1} H} - \sqrt{\Omega^2 \rho + \frac{Q_1^2}{4D_1 H^2}} \right]$$
$$\times \left[ -\sqrt{D_1} H \Delta - \frac{Q_1}{2\sqrt{D_1} H} + \sqrt{\Omega^2 \rho + \frac{Q_1^2}{4D_1 H^2}} \right] \quad (3.7)$$

For the circular active zone $\Omega_\varepsilon$ and centrally symmetric $w = \Psi_\varepsilon$, $\Delta \equiv \Delta_0$ is the corresponding radial Laplacian, hence the above equation [3.7] has a solution vanishing on $\Gamma_\varepsilon : r = \varepsilon$ presented as a linear combinations of Bessel functions and modified Bessel functions with appropriate arguments:

$$\Psi_\varepsilon(r) = J_0 \left( \frac{\omega^2 \rho}{H^2 D_1} e^{2\Theta} \right)^{1/4} r - J_0 \left( \frac{\omega^2 \rho}{H^2 D_1} e^{-2\Theta} \right)^{1/4} r$$

$$J_0 \left( \frac{\omega^2 \rho}{H^2 D_1} e^{2\Theta} \right)^{1/4} \varepsilon - J_0 \left( \frac{\omega^2 \rho}{H^2 D_1} e^{-2\Theta} \right)^{1/4} \varepsilon. \quad (3.8)$$

Here sinh $\Theta = \frac{Q_1}{2\omega H \sqrt{D_1 \rho}}$ reveals the dependence of the resonance on the
tangential tension $Q$. Besides

$$-\Delta J_0 \left[ \frac{\omega^2 \rho}{D} e^{2\Theta} \right]^{1/4} r = \left[ \frac{\omega^2 \rho}{D_1 H^2} e^{2\Theta} \right]^{1/2} J_0 \left( \frac{\omega^2 \rho}{D} e^{2\Theta} \right)^{1/4} r,$$

$$\Delta I_0 \left[ \frac{\omega^2 \rho}{D} e^{-2\Theta} \right]^{1/4} r = \left[ \frac{\omega^2 \rho}{D_1 H^2} e^{-2\Theta} \right]^{1/2} I_0 \left( \frac{\omega^2 \rho}{D} e^{2\Theta} \right)^{1/4} r.$$ 

We are interested in small positive eigenvalues $\omega^2$ of the equation (3.6) which may be in resonance with the lower eigenmodes of the complementary part $\Omega_c$ of the plate. To recover an algebraic equation for the eigenfrequencies we substitute the spectral parameter $\omega$ by a new spectral parameter $\Theta$ connected to $\omega$ by the equation

$$\sinh \Theta = \frac{Q}{2\omega\sqrt{D\rho}}$$

or

$$\omega = \frac{Q}{2\sinh \Theta \sqrt{D\rho}}.$$ 

The unperturbed spectral problem with the new spectral parameter is defined by the Dirichlet-Neumann boundary condition while constructed of Bessel functions as

$$\Psi_\varepsilon(r) = \frac{J_0 \left( e^{\Theta/2} \sqrt{D_1 H^2} e^{-\Theta} r \right)}{J_0 \left( e^{\Theta/2} \sqrt{D_1 H^2} e^{-\Theta} r \right)} - \frac{I_0 \left( e^{-\Theta/2} \sqrt{D_1 H^2} e^{-\Theta} r \right)}{I_0 \left( e^{-\Theta/2} \sqrt{D_1 H^2} e^{-\Theta} r \right)}, \quad (3.9)$$

and satisfies zero boundary condition identically on the inner side of the crack $\Gamma_\varepsilon$. The Neumann boundary condition on the inner side of the crack

$$\frac{\partial \Psi_\varepsilon \varepsilon}{\partial n} = e^{\Theta/2} \sqrt{D_1 H^2} e^{-\Theta} \frac{J_0 \left( e^{\Theta/2} \sqrt{D_1 H^2} e^{-\Theta} \varepsilon \right)}{J_0 \left( e^{\Theta/2} \sqrt{D_1 H^2} e^{-\Theta} \varepsilon \right)} - e^{-\Theta/2} \sqrt{D_1 H^2} e^{-\Theta} \frac{I_0 \left( e^{-\Theta/2} \sqrt{D_1 H^2} e^{-\Theta} \varepsilon \right)}{I_0 \left( e^{-\Theta/2} \sqrt{D_1 H^2} e^{-\Theta} \varepsilon \right)}, \quad (3.10)$$

and defines the spectrum of the unperturbed Dirichlet-Neumann problem for selected values of the parameters involved.

We select the typical geometrical and physical parameters of the active zone $\Omega_\varepsilon$ as

- $D_1 \times H^2 = 1.56 \times 10^{10} \times H^2 kgm^{2} sec^{-2}$,
- $H = H_\varepsilon \sim 3 \times 10^4 m,$
\( \rho = 3380 \frac{kg}{m^3}, \sigma = 0.28, Q \approx 3 \times 10^9 \frac{kg}{m \sec}, \) and
\( \varepsilon = \text{radius} \Omega \varepsilon \sim 2.6 \times 10^5 m \)

(with a little bit of accurate tuning). Then we are able to substitute the Bessel functions \( J_0 \) and the corresponding derivatives by the asymptotics for “large” values of the argument, and \( I_0 \) and the corresponding derivatives by the asymptotics for small ones, with regard of the factor \( e^{-\Theta} \), calculated for the selected resonance frequency \( \nu = 2 \times 10^{-4} \) Hz of the lower eigenmodes of large tectonic plates.

\[
\sinh \Theta = \frac{Q}{4\pi\nu\sqrt{\rho D}} = \frac{3 \times 10^9}{12.56 \times 2 \times 10^{-4} \times 3 \times 10^4 \times 1.26 \times 10^5} = 5.5, \quad e^\Theta = 11.
\]

Then the arguments of the Bessel functions and their derivatives can be calculated to be

\[
J_0(\omega^{1/2} \left( \frac{\rho}{D} \right)^{1/4} e^{\Theta/2} \varepsilon) = J_0 \left( \frac{12.6 \times 10^{-4} \times 58 \times 11 \times 6.76 \times 10^{10}}{1.26 \times 10^5 \times 3 \times 10^4} \right)^{1/2}
\]
\[
\approx J_0 \left( \frac{\sqrt{1505}}{10} \right) = J_0(3.9), \tag{3.11}
\]

\[
I_0'(\omega^{1/2} \left( \frac{\rho}{D} \right)^{1/4} e^{\Theta/2} \varepsilon) = I_0 \left( \frac{12.6 \times 10^{-4} \times 58 \times 11 \times 6.76 \times 10^{10}}{1.26 \times 10^5 \times 3 \times 10^4} \right)^{1/2}
\]
\[
\approx I_0 \left( \frac{\sqrt{61}}{10} \right) \approx I_0(0.8) \tag{3.12}
\]

Hereafter we use standard Taylor asymptotics of the modified Bessel function \( I_0 \) for small values of argument \( (<1) \) and the exponential asymptotics for “large” arguments \( (\geq 3.9) \) in \( J_0 \),

\[
J_0(z) \approx \cos(z - \pi/4) + O(1/z), \quad J_0'(z) \approx -\sin(z - \pi/4) + O(1/z)
\]
\[
I_0(z) \approx \cosh z + O(1/z), \quad I_0'(z) \approx \frac{\sinh z + O(1/z)}{\sqrt{\pi} \, 2}, \tag{3.13}
\]

Then we notice that the equation (3.10) is satisfied, with data, selected above, up to an error \( \sim 0.1 \). This means that our guess concerning the magnitude of the frequency was reasonably accurate for an active zone with radius \( 2.6 \times 10^5 \) with standard physical characteristics and circular shape.

A more profound correspondence between the interval 150\( \mu \)Hz – 250\( \mu \)Hz of typical frequencies and various shapes of the active zone requires a further analysis of the corresponding dispersion equation (analog of [3.10]) and is postponed.
4 Resonance conditions for circular plates.

To reveal the resonance condition for the circular plate divided into two parts: the circular active zone \( \Omega_\varepsilon \) and the ring-like complement \( \Omega_c = \{ \varepsilon < r < a \} \), which is centered at \( \Omega_\varepsilon \) and elastically disconnected from the active zone due to independent Dirichlet-Neumann conditions on both sides of the common boundary \( \Gamma_\varepsilon \), we must select the geometrical parameters of the complement such that the disconnected spectral problem has a multiple eigenvalue \( \omega^2_0 = 4\pi^2\nu_0\epsilon_2 \).

Then the perturbation of the disconnected spectral problem defined by replacing the disconnecting boundary conditions by an interactive condition would reveal a splitting of the multiple eigenvalue, and, eventually, the resonance beating of SGO modes, localized on \( \Omega_\varepsilon, \Omega_c \). This would imply migration of energy between the locations. It is technically convenient to consider a circular plate \( \Omega'_c \) with the same outer radius \( a \), but without a hole reserved for \( \Omega_\varepsilon \) at the center. The perturbation obtained by replacement \( \Omega_c \to \Omega'_c \) should not affect the part of spectrum corresponding to the standing waves in which lengths exceed the geometric size of the details affected by the change, in our case the radius \( 2.6 \times 10^5 \) m of the hole \( \Omega_\varepsilon \) (the active zone). This condition is obviously satisfied for ground SGO modes on the disc \( \Omega'_c \) with \( R \approx 5000 \) m. Now estimation of the eigenfrequencies of the complement for an equivalent circular plate \( \Omega'_c \), with radius \( a \approx 5 \times 10^5 \) m, neglecting relatively small terms, compared with the typical ground flexural wavelengths \( 2a \approx 5 \times 10^6 \) m, the hole radius \( \varepsilon \). We assume for \( \Omega_\varepsilon, \Omega'_c \) : \( D_a = 1.56 \times 10^{10} \times H_a^2 \frac{kgm}{sec^2}, H_a = H_\varepsilon \sim 10^5 m, Q_\varepsilon \approx 0 \) and \( \rho = 3380 \frac{kg}{m^3} \).

Using the asymptotics of Bessel functions for large arguments on the remote end of the boundary.

Using the asymptotics of Bessel functions for large arguments on the remote
part of the boundary \( r = a \) we find that

\[
\frac{I_0\left(\sqrt{\omega} \left(\frac{\rho}{D_1 H_2 a}\right)^{1/4} a\right)}{I_0\left(\sqrt{\omega} \left(\frac{\rho}{D_1 H_2 a}\right)^{1/4} a\right)} \approx 1 = -\tan\pi\left(\frac{l}{4}\right), \quad \frac{J_0\left(\sqrt{\omega} \left(\frac{\rho}{D_1 H_2 a}\right)^{1/4} a\right)}{J_0\left(\sqrt{\omega} \left(\frac{\rho}{D_1 H_2 a}\right)^{1/4} a\right)}
\approx -\tan\left(0.6a \times 10^{-7} - \frac{\pi}{4}\right).
\]

This results in \( 0.6a \times 10^{-7} - \pi/4 = \pi l - \pi/4 \) and hence \( a \approx 5 \times 10^6 m \), for \( l = 1 \) in agreement with our preliminary guess.

More detailed estimation of the radius of the ring - like complement \( \Omega_c \equiv (\varepsilon < r < a) \), centered at the active zone \( \Omega_\varepsilon \) may be derived from an explicit construction of the basic Bessel solutions and the Neumann-to-Dirichlet map of the biharmonic d’Alambert equation (4.1) on the ring with and appropriate symmetry, compatible with typical standing waves on the complement.

\[
J_p(z) \approx \frac{\cos(z - \frac{p\pi}{2} - \frac{\pi}{4})}{\sqrt{\pi z/2}} + O(\frac{1}{z})e^{i|\Omega z|},
\]

\[
J_p'(z) \approx \frac{-\sin(z - \frac{p\pi}{2} - \frac{\pi}{4})}{\sqrt{\pi z/2}} + O(\frac{1}{z})e^{i|\Omega z|}
\]

\[
I_p(z) \approx \frac{e^z}{\sqrt{\pi z/2}} + O(e^{i|\Omega z|}),
\]

\[
I_p'(z) \approx \frac{e^z}{\sqrt{\pi z/2}} + O(e^{i|\Omega z|})
\]  

(4.4)

(4.5)

One can also consider a circular active zone on an arbitrary plate with smooth boundary, see [3], or derive the formulae for the ND -map based on integral equation techniques, see [3].

Henceforth we wish to consider sectorial circular active zones which admit a separation of variable along with solutions of the relevant perturbed biharmonic equation

\[
D\Delta^2 u + Q\Delta u = \omega^2 \rho u
\]

(4.6)

combined on the sector \( \Omega_\varepsilon^p : 0 \leq r < \varepsilon, \ 0 < \varphi < \pi/p \) of above Bessel functions with index \( p \) and the corresponding circular harmonics \( \cos p\varphi, \sin p\varphi \) on the sector, as above.

The roles of basic regular solutions of the biharmonic d’Alambert equation play the products of circular harmonics \( \sin p\varphi, \cos p\varphi \), with relevant Bessel functions \( J_p(r\sqrt{\omega} \left(\frac{\rho}{D}\right)^{-1/4} e^{\Theta/2}) \), and the modified Bessel functions

\[
I_p(r\sqrt{\omega} \left(\frac{\rho}{D}\right)^{-1/4} e^{-\Theta/2}).
\]

The parameter \( \Theta \) is derived from the factorization of the d’Alambert equation as at (3.7). Then, for \( p \geq 1 \) the only continuous at \( r = 0 \) solution are square
integrable. Hence continuous solutions of the d’Alambert equation, vanishing on the circular part \( \Gamma_\varepsilon \) of the boundary can be obtained as linear combinations.

\[
\Psi_s(r) = \sin p\phi \left[ J_p \left( \frac{\omega^2 \rho e^{2\Theta}}{D} r^{1/4} \right) - \varepsilon I_p \left( \frac{\omega^2 \rho e^{-2\Theta}}{D} r^{1/4} \right) \right],
\]

\[
\Psi_c(r) = \cos p\phi \left[ J_p \left( \frac{\omega^2 \rho e^{2\Theta}}{D} r^{1/4} \right) - \varepsilon I_p \left( \frac{\omega^2 \rho e^{-2\Theta}}{D} r^{1/4} \right) \right].
\]

They satisfy the Dirichlet boundary conditions \( u |_{\Gamma} = \Delta u |_{\Gamma} = 0 \) for \( \Psi_s \) and the Neumann boundary condition \( \frac{\partial u}{\partial n} |_{\Gamma} = \frac{\partial \Delta u}{\partial n} |_{\Gamma} = 0 \) respectively.

The spectral problem with Dirichlet-Neumann boundary condition \( u |_{\Gamma_\varepsilon} = \frac{\partial u}{\partial n} |_{\Gamma_\varepsilon} = \frac{\partial \Delta u}{\partial n} |_{\Gamma_\varepsilon} = 0 \) on the circular part of the boundary is given by linear combinations (4.7, 4.8), and which satisfy the corresponding dispersion equations.

\[
0 = \frac{\partial \Psi_s(\varepsilon)}{\partial n} \sin \gamma \phi \left[ e^{\Theta/2} \sqrt{Q} \frac{\sqrt{Q} \sqrt{D} e^{\Theta/2}}{\sqrt{D} e^{\Theta/2} - e^{-\Theta/2}} J'_\gamma \left( e^{\Theta/2} \frac{\sqrt{Q} \sqrt{D} e^{\Theta/2}}{\sqrt{D} e^{\Theta/2} - e^{-\Theta/2}} \right) \right.
\]

\[
\left. - e^{-\Theta/2} \frac{\sqrt{Q} \sqrt{D} e^{\Theta/2}}{\sqrt{D} e^{\Theta/2} - e^{-\Theta/2}} I'_\gamma \left( e^{-\Theta/2} \frac{\sqrt{Q} \sqrt{D} e^{\Theta/2}}{\sqrt{D} e^{\Theta/2} - e^{-\Theta/2}} \right) \right].
\]

or a similar equation for \( \Psi_c \).

For sectors characterized by \( p \in (0, 1) \) there are singular square-integrable solutions \( J_{-p}(z) \), \( I_{-p}(z) \) of the biharmonic d’Alambert equation, see [4]. Then we are able to consider the singular (discontinuous) square integrable solutions of the biharmonic d’Alambert equation, considering them as elements of the corresponding defect, see [4], and construct self-adjoint extensions of the corresponding biharmonic operator in the space of square-integrable functions in a sectorial active zone on the boundary. The corresponding operator extension machinery can be developed with use of extension procedure, similar to one developed above for the inner zero-range active zone.

When varying the tension parameter \( Q \), we will come to the moment when the tangent compression in the middle plane is large enough for the minimal eigenvalue on the active zone of an unperturbed problem to coincide with an
eigenvalue of the biharmonic spectral problem \((Q = 0)\) on the complement. At this point substituting the asymptotics of the Bessel functions \(J_p, I_p\) for large and small values of the arguments, one can obtain an estimation for the contracting tension \(Q\), this leads to the resonance conditions for the operators \(L_{\varepsilon}, L_c\) on \(\Omega_{\varepsilon}, \Omega_c\), without an interaction between them. We now explore this in the simplest case.

5 A simple model of alternation.

Now we consider a simple universal interaction depending on a small parameter between similar operators constructed for the zero-range active zone. Based on this construction, we observe the relevant beating phenomenon and estimate the transfer coefficient.

The system of two tectonic plates considered in the previous section is a special case of a decoupled oscillator’s system under the resonance conditions. The interaction between the plates could be introduced by imposing free reclining or natural boundary conditions.

We now make some simplifying assumptions: we consider a weakly coupled oscillatory system obtained by attaching one (supposedly “large”) multi-dimensional oscillator \(X = (X_1, X_2, X_3, \ldots X_\mu) \in C_\mu\) to a 1D (supposedly “small”) oscillator characterized by the coordinate \(x = x_1 \in C_1\).

The dynamics of the resulting system is defined by the system of linear equations

\[
\begin{align*}
m x_{tt} + v x + b^+ X &= 0, \\
M X_{tt} + V X + b x &= 0.
\end{align*}
\]  

Here \(m = m_1, v = v_1, M = [M_1, M_2, M_3 \ldots M_\mu], V = [V_1, V_2, V_3 \ldots V_\mu]\), are positive diagonal matrices acting in the Hilbert spaces \(C_1, C_\mu, C_1 \rightarrow C_{\mu}, C_{\mu} \rightarrow C_1\). Interaction of the oscillators is introduced by the Hermitian matrix \(B : C_1 \oplus C_\mu \rightarrow C_1 \oplus C_\mu \equiv K\)

\[
B = \begin{pmatrix} 0 & b^+ \\ b & 0 \end{pmatrix} = \text{antidiag} \ (b^+, b),
\]

which plays the role of bonds imposed on the boundary data of solutions of the biharmonic equation on the border of the tectonic plate. Separating time, we obtain the spectral problem corresponding to the above wave equation (5.1)

\[
\begin{align*}
v x + b^+ X &= m \lambda x \\
V X + b x &= M \lambda X.
\end{align*}
\]

To make this consistent with the above model of tectonic plates supplied with natural structure of active zones we assume that the unperturbed multidimensional oscillator has a family of unperturbed eigenvalues (square frequencies) \([\omega^0]^2 = v/m, [\Omega^0]^2 = V_s/M_s, s = 1, 2, \ldots \mu\). For non-zero interaction \(B \neq 0\)
the eigenvalues $\lambda^b = (\omega^b)^2$ (and the corresponding eigenfrequencies) of the perturbed selfadjoint spectral problem are found from the algebraic equation

$$\left[\lambda_m - v\right]a + b^+ \frac{I}{V - M\lambda}ba \equiv M(\lambda)a = 0,$$

(5.4)

obtained via the elimination of $X$ from the second equation in (5.3). In particular, for the 1D case, $\mu = 1$, we have two unperturbed frequencies

$$v/m = \omega_0^1 \equiv (\omega_0)^2, \ V/M = \Omega_0^1 \equiv (\omega_0^2)^2$$

and a quadratic equation for the perturbed eigenvalues

$$\lambda^b_r = (\omega^b_r)^2, \ r = 1, 2,$$

while the unperturbed are $\lambda^0_1 = (\omega^0)^2$, $\lambda^0_2 = (\Omega^0)^2$. Denoting $\tilde{\lambda}^0 = (\omega^0)^2 + (\Omega^0)^2$, $\delta \lambda^0 = \frac{(\omega^0)^2 - (\Omega^0)^2}{2}$, we find the perturbed frequencies/eigenvalues $\lambda^b_r \equiv (\omega^b_r)^2$ from the quadratic equation

$$\lambda^2 - 2\delta \lambda^0 + \lambda^0_1 \lambda^0_2 = \frac{b^2}{MM}.$$

This gives

$$\lambda^b_r = \tilde{\lambda}^0 \pm h, \ \text{with} \ h^2 = (\delta \lambda)^2 + \frac{b^2}{MM},$$

with $m_1 = m$, $m_2 = M$. We further assume that $\frac{b^2}{m_1 m_2} \ll \delta^2 \lambda$. This assumption allows us to calculate the perturbed eigenvalues $\lambda^b_{1,2}$ with $h = \delta \lambda + \frac{b^2}{2m_1 m_2 \delta \lambda}$ approximately. These turn out to be

$$\lambda^b_{1,2} = \lambda^0_{1,2} \pm \frac{b^2}{2m_1 m_2 \delta \lambda}, \ \omega^b_{1,2} = \omega^0_{1,2} \pm \frac{b^2}{4m_1 m_2 \delta \lambda \omega^0_{1,2}}$$

$$= \omega^0_{1,2} \left[ 1 \pm \frac{b^2}{4m_1 m_2 \delta \lambda^2} \right] \frac{\delta \lambda}{\lambda^0_{1,2}} \approx \omega^0_{1,2} \frac{\delta \lambda}{\lambda^0_{1,2}}, \ \text{(5.5)}$$

Generally, for $\mu \geq 1$, the eigenvalues of the ultimate spectral problem $\lambda = \lambda^b_1, \lambda^b_2, \ldots, \lambda^b_s, \ldots, \lambda^b_{1+\mu}$ may be found from the corresponding determinant condition. The components of the corresponding eigenvectors $\{a, \Psi^b_s\} \equiv \Psi^b_s$ in $K, C_\mu$ are calculated from the equation

$$\Psi^b_s = (V - M\lambda^b_s)^{-1}ba,$$

(5.6)

with an appropriate normalization $m|a|^2 \left[ 1 + \langle M \hat{b}, V - M\lambda^b \rangle b \right] = 1$. If the interaction $b$ is real and the initial data of the relevant Cauchy problem, see below (5.9), are real, then the corresponding solution is real too. Similarly the solution of the corresponding inhomogeneous equation, with a real function in the right side, is real as well.
In the more general case where $K = C_1 \oplus C_\mu$, $\mu \geq 1$, the eigenvalues $\lambda_b^s = (\omega_b^s)^2$ are found as graphical solutions based on the diagram, corresponding to the algebraic equation (5.4), see [13].

In the simplest situation where $\mu = 1, K = C_1 \oplus C_1$ the equation (5.3) for $(u, U) = u_1, u_2$ and the unperturbed frequencies $\omega^0 = \omega_1^0, \Omega^0 = \omega_2^0$ and

\begin{align*}
\dot{\lambda} &= \frac{\lambda_1^0 + \lambda_2^0}{2}, \\
\delta \lambda &= \frac{\lambda_1^0 - \lambda_2^0}{2}
\end{align*}

the equation (5.3) can be represented as

\begin{align*}
\begin{pmatrix}
m_1 \lambda_1^0 & b^+ \\
b & m_2 \lambda_2^0
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2
\end{pmatrix}
= \lambda
\begin{pmatrix}
m_1 & 0 \\
0 & m_2
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2
\end{pmatrix},
\end{align*}

(5.7)

and the eigenvectors

\begin{align*}
\Psi^b_1 &= \begin{pmatrix}
a_1 \\
a_1 bm_1^{-1} - \delta \lambda - \kappa
\end{pmatrix}, \\
\Psi^b_2 &= \begin{pmatrix}
a_2 \\
a_2 bm_2^{-1} - \delta \lambda + \kappa
\end{pmatrix}
\end{align*}

(5.8)

are orthogonal with respect to diag $(m_1, m_2)$ and normalized in $l_2(m_1, m_2)$ with $a_{1,2} = \sqrt{\frac{b + \delta \lambda}{2\kappa}}$. When discussing the general case $\mu \geq 1$ we may assume that the eigenfunctions $\Psi^b_s = (\psi^b_s, \Psi^b_s)$ of the perturbed spectral problem (5.3) correspond to simple eigenvalues and are orthogonal and normalized. Using the eigenvectors, the normal modes of the wave equation

\begin{align*}
mu_{tt} + \nu u + b^+ U = 0, \\
MU_{tt} + VU + bu = 0, \\
(u, U) \equiv U, \ U(0) = U_0, \ \frac{dU}{dt}(0) = U'_0.
\end{align*}

(5.9)
can be constructed, and subsequently the solutions of this Cauchy problem (5.9) are obtained as linear combinations

\[ U(t) = \sum_s U^s \Psi_s \cos(\omega^b_s t + \varphi_s) = \Re \sum_s U^s \Psi_s e^{i(\omega^b_s t + \varphi_s)} \]

of the eigenmodes. But we also can reduce the above homogeneous equation (5.9) to a pair of unperturbed formally inhomogeneous equations,

\[ m u_{tt} + v u + \sum_s U^s b^+ \Psi_s \cos(\omega^b_s t + \varphi_s) \equiv m u_{tt} + v u + f, \quad (5.10) \]

\[ M U_{tt} + V U + \sum_s U^s b \, \psi^s \cos(\omega^b_s t + \varphi_s) \equiv M U_{tt} + V U + F = 0, \quad (5.11) \]

or a similar complex equation

\[ m \vec{u}_{tt} + v \vec{u} + \sum_s U^s b^+ \Psi_s e^{i(\omega^b_s t + \varphi_s)} \equiv m \vec{u}_{tt} + v \vec{u} + \vec{f}, \quad (5.12) \]

\[ M \vec{U}_{tt} + V \vec{U} + \sum_s U^s b \, \psi^s e^{i(\omega^b_s t + \varphi_s)} \equiv M \vec{U}_{tt} + V \vec{U} + \vec{F} = 0, \quad (5.13) \]

with inhomogeneities \( \vec{f} = b^+ U, \vec{F} = b \vec{u} \) obtained via substitution, for \( u, U \), the corresponding components of the solution \( U(t) = (u, U) \) of the Cauchy problem (5.9) for the original oscillators system:

\[ u_f(t) = \sum_s b^+ \Psi_s U^s \left[ \frac{e^{i(\omega^b_s t + \varphi_s)}}{m \lambda^b_s - v} \right], \quad u_f(t) = \sum_s b^+ \Psi_s U^s \left[ \frac{\cos(\omega^b_s t + \varphi_s)}{m \lambda^b_s - v} \right], \quad (5.14) \]

\[ U_f(t) = \sum_s b \psi^s U^s \left[ \frac{e^{i(\omega^b_s t + \varphi_s)}}{M \lambda^b_s - V} \right], \quad U_F(t) = \sum_s b \psi^s U^s \left[ \frac{\cos(\omega^b_s t + \varphi_s)}{M \lambda^b_s - V} \right], \quad (5.15) \]

General solutions of the equations (5.10)-(5.13) are obtained by adding general solutions of the corresponding homogeneous equations \( m u_{tt} + v u = 0, \ M U_{tt} + V U = 0, \) to these solutions of the inhomogeneous equations.

The last formulas allow us to calculate partial values of energy of the “small” and “large” oscillators depending on time, for instance.

\[ E_f(U) = \frac{m |u_f|^2 + v |u_f|^2}{2}, \quad E_F(U) = \frac{M |U_F|^2 + V |U_F|^2}{2} \quad (5.16) \]

under formally “exterior” forces \( f, F \), defined by a linear combination of the perturbed normal modes \( U \) of the perturbed oscillator’s system.

The partial values of energy of each oscillator \( u_f, U_F \) do not remain constant in the course of evolution, but depend on time, exposing beats while the oscillators exchange energy due to the bond \( B = \text{antidiag}(b^+, b) \). Beats are calculated using the 1D theory developed for a periodic exterior force as
in [15, Section 22]. To study the above (formally) complex version of the inhomogeneous equation (5.13) we follow Landua [15], and introduce the data

\[ \vec{\xi}_f = \sqrt{m} u'_f + i\sqrt{v} u_f, \quad \xi_f = \sqrt{m} u'_f + i\sqrt{v} u_f \]

and re-write the last equation (5.12) as

\[ \sqrt{m} \vec{\xi} - i\sqrt{v} \vec{\xi} = \vec{f}(t), \quad \sqrt{m} \xi' - i\sqrt{v} \xi = f(t), \quad \sqrt{v} = \omega_0 \sqrt{m}. \quad (5.17) \]

If the Cauchy problem is solved, with an initial condition \( \xi(0) = \xi_0 \), then the energy (5.16) of the small oscillator for the real solution \( U \) of the total problem, is calculated as

\[ \bar{\xi}_f(t) \xi_f(t) = [\sqrt{mu'} + i\sqrt{uv}] [\sqrt{mu'} + i\sqrt{uv}] = 2E_f(u). \quad (5.18) \]

The last formula allows us to estimate the energy transfer “between the modes of unperturbed oscillators”. It is sufficient to be able to monitor the time-dependent energy of the small oscillator.

We now suggest a 1D version of the corresponding analysis, assuming that \( \mu = 1, \ K = C_1 \oplus C_1 \). Back to discussion of the 1D oscillators, with \( \lambda_{1,2} = \lambda + h \equiv \lambda_{\pm} \), assume, that an approximate resonance condition is satisfied,

\[ \omega_{b_{1,2}} = \omega_{1,2} + \delta \omega_{1,2}, \]

and assume that the solution of the original Cauchy problem (5.9) is given as a linear combination of the perturbed modes

\[ U = \begin{pmatrix} u \\ U \end{pmatrix} = \sum_{s=1}^{2} \Psi_s^b U^s \cos(\omega_s^b t + \varphi_s). \]

Then the above inhomogeneous evolution equation for the small oscillator is either

\[ m_1 u'' + m(\omega^2)^2 u_1 + \sum_{s=1}^{2} b^+ \Psi_s^b U^s \cos(\omega_s^b t + \varphi_s) = 0, \]

or

\[ \sqrt{m} [\xi' - i\omega_1^0 \xi] + \sum_{s=1}^{2} b^+ \Psi_s^b U^s \cos(\omega_s^b t + \varphi_s) = 0, \]

and has a partial solution

\[ u_1 = \sum_{s=1}^{2} b^+ \Psi_s^b U^s \frac{\cos(\omega_s^b t + \varphi_s)}{m_1 (\omega_s^b)^2 - (\omega_1^0)^2}. \]

The corresponding \( \xi \) - function is calculated from the equation

\[ \sqrt{m} [\xi'_1 - i\xi_1 \omega_1^0] + \sum_{s=1}^{2} b^+ \Psi_s^b U^s \cos(\omega_s^b t + \varphi_s) = 0, \]
\[ \xi_1(t) = e^{i\omega_1^0 t} \left[ \xi_1(0) - \frac{1}{\sqrt{m_1}} \int_0^t e^{-i\omega_1^0 \tau} \sum_{s=1}^2 b^+ \Psi_s^b U^s \cos(\omega_s^b \tau + \varphi_s) d\tau \right] \]

\[ = e^{i\omega_1^0 t} \left[ \xi_1(0) + \dot{\xi}_1(t) \right], \quad (5.19) \]

with

\[ \xi_1(0) = \sqrt{m_1} \left[ u_1'(0) + i\omega_1^0 u_1^0(0) \right] \]

\[ = \frac{1}{\sqrt{m_1}} \sum_{s=1}^2 b^+ \Psi_s^b U^s \left[ \frac{-\omega_s^b \sin \varphi_s + i\omega_1^0 \cos \varphi_s}{(\omega_s^b)^2 - (\omega_1^0)^2} \right] \]

\[ = \frac{1}{\sqrt{m_1}} \sum_{s=1}^2 b^+ \Psi_s^b U^s \left[ \frac{\sin \varphi_s}{\omega_s^1 + \omega_1^0} + \frac{i\omega_1^0 \cos \varphi_s}{(\omega_s^b + \omega_1^b)(\omega_s^b - \omega_1^b)} \right] \]

and the integral \( \dot{\xi}_1(t) \) in (5.19) is calculated as

\[ \dot{\xi}_1(t) = -\frac{1}{\sqrt{m_1}} \int_0^t e^{-i\omega_1^0 \tau} \sum_{s=1}^2 b^+ \Psi_s^b U^s \cos(\omega_s^b \tau + \varphi_s) d\tau \]

\[ = -\frac{1}{\sqrt{m_1}} \sum_{s=1}^2 b^+ \Psi_s^b U^s \cdot \left[ \frac{e^{i\varphi_s}}{2i(\omega_s^b - \omega_1^0)} \left( e^{i(\omega_s^b - \omega_1^0)t} - 1 \right) - \frac{e^{-i\varphi_s}}{2i(\omega_s^b + \omega_1^0)} \left( e^{-i(\omega_s^b - \omega_1^0)t} - 1 \right) \right] \]

\[ = -\frac{1}{\sqrt{m_1}} \sum_{s=1}^2 b^+ \Psi_s^b U^s \cdot \left[ e^{i\varphi_s} \frac{e^{i(\omega_s^b - \omega_1^0)t}}{\omega_s^b - \omega_1^0} + e^{-i\varphi_s} \frac{e^{-i(\omega_s^b + \omega_1^0)t}}{\omega_s^b + \omega_1^0} \right] \]

To compare the above theoretical estimation of time dependence of the energy of the “small” oscillator of time, we should consider the averaged energy over a stepwise system of windows, such as used for manufacturing the time-spectral cards discussed in Section 1. Set

\[ |\xi|^2(T) = \frac{1}{\Delta} \int_{T-\Delta/2}^{T+\Delta/2} |\xi(t)|^2 dt, \quad (5.20) \]

with an appropriate choice of parameters for the averaging depending on the basic characteristics of \( \lambda_{1,2} \) and \( \bar{\lambda}, \delta \).

Let us first examine the time-dependence of energy of the “small” oscillator in the simplest case of two 1D oscillators, we assume, that there exists only one resonance eigenvalue of the perturbed system, closest to the unperturbed
eigenvalue of the “small” oscillator. Further assuming that $\frac{b^2}{m_1 m_2} < (\delta \lambda)^2$, we may estimate the perturbed eigenvalues and eigenfrequencies of the system as

$$ (\omega_1^0)^2 - (\omega_1^0)^2 \approx \lambda + \sqrt{(\delta \lambda)^2 + \frac{b^2}{m_1 m_2} - \lambda_1^0} = \frac{b^2}{2m_1 m_2 \delta \lambda \lambda_1^0}, $$

$$ (\omega_2^0)^2 - (\omega_1^0)^2 \approx \lambda - \sqrt{(\delta \lambda)^2 + \frac{b^2}{m_1 m_2} - \lambda_1^0} = -\delta \lambda - \frac{b^2}{2m_1 m_2 \delta \lambda \lambda_2^0} (5.21) $$

$$ \omega_1^0 - \omega_1^0 \approx \frac{b^2}{4m_1 m_2 \delta \lambda \omega_1^0}, \quad \omega_1^0 + \omega_1^0 \approx 2\omega_1^0 + \frac{b^2}{4m_1 m_2 \delta \lambda \omega_1^0} $$

$$ \omega_2^0 - \omega_1^0 \approx \omega_2^0 - \omega_1^0 - \frac{b^2}{4m_1 m_2 \delta \lambda \omega_2^0} = \delta \omega - \frac{b^2}{4m_1 m_2 \delta \lambda \omega_2^0} (5.22) $$

Notice the resonance terms with small denominators or/and with slowly oscillating exponents on the window give the crucial contribution to the average [5.20] while the rapidly oscillating and smooth terms may be neglected while integrating over the window. Using the above representation for $\xi(t) = \xi_1(0) + \xi(t)$ we estimate the averaged $\xi$-function $|\xi|^2(T)$

$$ |\xi|^2(T) = \frac{1}{\Delta} \int_{T-\Delta/2}^{T+\Delta/2} |\xi(t)|^2 dt $$

$$ = \frac{1}{m_1} \sum_{r,s=1}^2 b^r \Psi_b^r U^r b^s \Psi_b^s U^s \frac{1}{\Delta} \int_{T-\Delta/2}^{T+\Delta/2} \overline{AB} (5.23) $$

where

$$ A = \sin \phi_s + \frac{i \omega_1^0 e^{i \phi_s} + [e^{i \phi_s} e^{i(\omega_1^0 - \omega_1^0)} + e^{-i \phi_s} e^{-i(\omega_1^0 - \omega_1^0)}]}{\lambda_1^0 - \lambda_1^0} \sin(\omega_1^0 + \omega_1^0) \frac{t}{2} \frac{1}{\omega_1^0 + \omega_1^0} $$

$$ B = \sin \phi_r + \frac{i \omega_1^0 e^{i \phi_r} + [e^{i \phi_r} e^{i(\omega_1^0 - \omega_1^0)} + e^{-i \phi_r} e^{-i(\omega_1^0 - \omega_1^0)}]}{\lambda_1^0 - \lambda_1^0} \sin(\omega_1^0 + \omega_1^0) \frac{t}{2} \frac{1}{\omega_1^0 + \omega_1^0} $$

choosing the window such that the leading resonance term $\frac{\sin(\omega_1^0 - \omega_1^0) t}{\omega_1^0 - \omega_1^0}$ only slightly deviates from a constant on the window $|t - \tau| < \Delta$:

$$ \left| \sin(\omega_1^0 - \omega_1^0) \frac{t}{2} - \sin(\omega_1^0 - \omega_1^0) \tau / 2 \right| \leq \Delta(\omega_1^0 - \omega_1^0) \approx \frac{b^2 \Delta}{4m_1 m - 2\delta \lambda \omega_1^0}. $$

On another hand, the slowest oscillation of non-resonance exponentials in the integrand is defined by the exponent

$$ \frac{1}{\Delta} \int_{T-\Delta/2}^{T+\Delta/2} \cos(\omega_1^0 - \omega_1^0) t dt \approx \frac{1}{\Delta \delta \omega} $$

and this should be small too.

$$ \frac{b^2 \Delta}{4m_1 m_2 \delta \omega_1^0} \approx \frac{1}{\Delta} \delta \omega $$

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This now defines the width of the optimal window giving the dependence on \( \lambda_0^0, \lambda_0^1 \) and the other parameters.

Then, selecting this optimal window and taking into account only leading terms of the integrand, we obtain an approximate estimation for the averaged energy of the “small” oscillator in dependence on time:

\[
|\xi|(T) = \frac{1}{\Delta} \int_{T-\Delta/2}^{T+\Delta/2} |\xi|^2(t) dt \approx |b^+ \Psi^s U^s|^2 dt \left[ \frac{|\omega_1^0|^2}{\lambda_1^2 - \lambda_1^0} + \frac{\sin^2(\omega_1^b - \omega_1^0) \frac{b^2}{2}}{(\omega_1^0 - \omega_1^0)^2} \right]
\]

\[
\approx |b^+ \Psi^s U^s|^2 \left[ \frac{2m_2 \delta \lambda \lambda_1^0}{b^2} + \frac{\sin^2(\frac{b^2}{4m_1 m_2 \delta \lambda \omega_1^0})}{(\frac{b^2}{4m_1 m_2 \delta \lambda \omega_1^0})^2} \right]. \tag{5.24}
\]

Figure 3: Symbolic diagram of the energy content \( E_\varepsilon(T), E_c(T) \) of the components \( u_\varepsilon, u_c \) of the perturbed dynamics \( u \) with respect to the de-localized mode \( \Psi_0^\varepsilon \) on the complement \( \Omega_c \) and one of the localized mode on the active zone \( \Omega_\varepsilon \). Positions \( O_\varepsilon, O_c \) of the minima (and maxima) of the energy content of the localized and the delocalized modes alternate with opposite phases. The dangerous intervals of time at the minima \( O_c \), when the destruction of \( \Omega_\varepsilon \) is expected, are marked with thin rectangles.

6 Appendix 1: Natural boundary conditions and the perturbed biharmonic wave equation.

In our second section we considered a perturbed biharmonic equation (2.1), borrowed from [9, 23], and modelled the normal stress by the boundary condition. Here we give more details describing the “bridge” connecting the model dynamics presented in us with the corresponding chapter of classical mechanics of the plates and shells, see [33, 34, 23] and also [8, 11]. We keep in mind, that the linear theory of small oscillations of tectonic plates will only shed light on an initial phase of the process which may lead to the catastrophic results. But we hope that the mechanical realization of the preliminary small oscillation
model of the resonance process may be able to preview some initial features of
the catastrophic phase of the process. In reality, constructing the mechanical
model of the resonance interaction of the SGO modes of tectonic plates is a
problem to resolve based on experiment, with use of far more detailed mechan-
ical details, see for instance [34], [30], [22]. We only attempt here a first step in
this direction, by considering two thin tectonic plates developing typical kinds
of stresses when colliding under fluctuation of the rotation speed of Earth an/or
the convection flow in the liquid underlay (asthenosphere). To further simplify
our analysis, we assume that both Ωε, Ωc are Kirchhoff plates, see [34], and
endure different kinds of stresses. We assume that the normal pressure and
the corresponding bending dominate the potential energy of the large plate Ωε
and the tangential (shearing) component of the stress dominates the potential
energy of the small plate. The shearing part of the stress is defined by the
tension T in the middle plane with the components Tx, Ty, Txy, satisfying the
equilibrium conditions, see Mikhlin’s book, [23, Chapter 4, Section 28] which
we use as a basic reference in what comes. The equilibrium conditions are
\[
\frac{\partial T_x}{\partial x} + \frac{\partial T_{xy}}{\partial y} = 0, \quad \frac{\partial T_y}{\partial y} + \frac{\partial T_{xy}}{\partial x} = 0,
\]
and the corresponding quadratic form
\[
\langle w, Tw \rangle = \int_{\Omega \varepsilon} [w_x T_x w_x + 2 w_x T_{xy} w_y + w_y T_y w_y] \, d\Omega
\]
defines, with the kinematic boundary condition w \big|_\Gamma = 0, a symmetric operator T
of second order on smooth functions w. The operator T is positive for stretching
tension and negative for compressing tension, see [9].

This may cause an instability of the tectonic plate, see for instance the analysis
of a numerical example for an elliptic plate under a compressive tension in
[23, Chapter 8, Section 72]. The materials composing tectonic plates can’t resist
the stretching tension, so hereafter we assume that the tension T is compressing,
and the corresponding second order operator T is negative.

We begin with considering of the non-interacting plates Ωε, Ωc under normal
bending stress F, and the compressing stress T, assuming that each of them
is elastically fixed at the common boundary Γε, see below. The zero bound-
ary condition is applied on all boundaries, free reclining conditions are applied
on the remote part of boundary of Ωc, and the elastic fixture on both sides
Γε,c of the common boundary Γ is modeled by the corresponding functionals
\[
\int_\Gamma \beta_{\varepsilon,c} \left( \frac{\partial w_{\varepsilon,c}}{\partial n} \right)^2 \, d\Gamma.
\]
The equilibrium deformations wε, wc are found by minimizing of the corre-
spounding quadratic functionals on the subspace of the virtual strains, subject
to the kinematic boundary condition $w\big|_{\Gamma^*} = 0$

$$W(u) = D \int_{\Omega} |\Delta w|^2 d\Omega - 2(1 - \sigma) D \int_{\Omega} (w_{xx} w_{yy} - |u_{xy}|^2) d\Omega$$

$$+ \int_{\Gamma} \beta \left( \frac{\partial w}{\partial n} \right)^2 d\Gamma + H \langle w, Tw \rangle - 2 \int_{\Omega} w F d\Omega.$$  \hspace{1cm} (6.2)

Here $\sigma$ is the Poisson coefficient $0 < \sigma < 1$, $D$ is the flexural rigidity, $D = \frac{H^2 E}{12(1 - \sigma^2)}$, $E$ is the Young modulus, $H$ is the thickness of the plate and $\beta > 0$ is the parameter defining the elastic contact of the plate with environment. The second integral of the Monge-Ampere form (the curvature of the surface $z = w(x, y)$) can be presented as an integral on the boundary, see [3], with regard of the above kinematic boundary condition $w\big|_{\Gamma^*} = 0$:

$$2 \int_{\Omega} (w_{xx} w_{yy} - |w_{xy}|^2) d\Omega = \int_{\Gamma} \frac{\left( \frac{\partial w}{\partial n} \right)^2}{r(\gamma)} d\Gamma$$

For the circular $\Gamma$, we assume $r(\gamma) = \epsilon$ and $H(w, Tw) = H^{-2} D \Delta w$. Calculation of the first variation of the energy functional yields the Euler equation

$$D \Delta^2 w + Tw = F$$  \hspace{1cm} (6.3)

with the kinematic and the natural (under the above elastic $\beta$-bound) condition on the each side of the common boundary, e.g.

$$D \Delta w + \beta \frac{\partial w}{\partial n} - D \frac{1 - \sigma}{r} \frac{\partial w}{\partial n} \bigg|_{\Gamma^*} = 0.$$  \hspace{1cm} (6.4)

To derive an equation for the small oscillation we consider, for each plate, the Lagrangian, associated with the thin plate $\Omega$, subject to the above elastic bound and the kinematic boundary conditions $u\big|_{\Gamma} = 0$

$$\mathcal{L} = \int_0^t \int_{\Omega} D|\Delta w|^2 d\Omega dt - \int_0^t \int_{\Omega} \left[ 2D(1 - \sigma) (w_{xx} w_{yy} - w_{xy}^2) - \rho H w^2 \right] d\Omega dt$$

$$+ 2 \int_0^t \int_{\Gamma} \beta \left( \frac{\partial w}{\partial n} \right)^2 d\Gamma dt + \int_0^T \langle w, Tw \rangle dt,$$  \hspace{1cm} (6.5)

we obtain the perturbed biharmonic wave equation as the Euler equation for the critical points of the Lagrangian:

$$\rho H w_{tt} + D \Delta^2 w + Tw = 0,$$  \hspace{1cm} (6.6)

with the "$\beta$-natural" free reclining boundary conditions on both sides of the common boundary

$$w\big|_{\Gamma} = 0, \quad D \Delta w + \beta \frac{\partial w}{\partial n} - \frac{D(1 - \sigma)}{r} \frac{\partial w}{\partial n} \bigg|_{\Gamma} = 0.$$  \hspace{1cm} (6.7)
and free reclining boundary condition on the complementary part of the boundary of \( \Omega_c \).

While considering a circular plate \( \Omega : 0 < r < a \), assume that the active zone \( \Omega_e \) is centered in \( \Omega \), so that the complement \( \Omega_c \) is a ring \( 0 < r < a \), as described earlier. Generally we have the Lagrangian

\[
L = \int_0^t \int_\Omega D|\Delta w|^2 d\Omega \, dt + \int_0^t \int_{\Gamma_e} \beta \left( \frac{\partial w}{\partial n} \right)^2 \, d\Gamma_e \, dt - \int_0^t \int_\Omega \rho H \left( \frac{\partial w}{\partial t} \right)^2 \, d\Omega \, dt
- 2D(1 - \sigma) \int_0^t (w_{xx}w_{yy} - w_{xy}^2) \, d\Omega \, dt + \int_0^t \langle w, Tw \rangle \, dt,
\]

This gives the perturbed biharmonic wave equation as the Euler equation for the critical points of the Lagrangian on \( \Omega_c \).

\[
\rho H w_{tt} + D\Delta^2 w + Tw = 0,
\]

and

\[
\rho H w_{tt} + D\Delta^2 w + Tw = 0,
\]
on \( \Omega_e \), with \( T < 0 \) and the boundary condition with regard of the elastic bond:

\[
w \bigg|_{\Gamma_e} = 0, \quad \beta \frac{\partial w}{\partial n} D\Delta w \bigg|_{\Gamma_e} - \frac{D(1 - \sigma)}{\varepsilon} \frac{\partial w}{\partial n} \bigg|_{\Gamma_e} = 0.
\]

For the complement we neglect the tangential compression, but keep the bending and the elastic bond, so that the Lagrangian is reduced to

\[
L_c = \int_0^T \int_\Omega D|\Delta w|^2 d\Omega \, dt + \int_0^T \int_{\Gamma_e} \beta \left( \frac{\partial w}{\partial n} \right)^2 \, d\Gamma_e \, dt - \int_0^T \int_\Omega \rho H \left( \frac{\partial w}{\partial t} \right)^2 \, d\Omega \, dt,
\]

we obtain the perturbed biharmonic wave equation as Euler equation for the critical points of the Lagrangian \( L_c \):

\[
\rho H w_{tt} + D\Delta^2 w = 0,
\]

with the boundary condition

\[
w \bigg|_{\Gamma_e} = 0, \quad \beta \frac{\partial w}{\partial n} - D \frac{1 - \sigma}{r} \frac{\partial w}{\partial n} + D\Delta w \bigg|_{\Gamma_e} = 0.
\]

We make now one more simplifying assumption, assuming that the operator \( L_e \) on the active zone is defined on circular harmonics of zero order \( n = 0 \) (independent of the angular variable), and the operator \( L_c \) on the complement is defined on the linear span of circular harmonics of the first order, \( e_1^c = \pi^{-1} \cos \varphi, \ e_1^e = \pi^{-1} \sin \varphi \), so that the corresponding eigenfunctions are spanned by the circular harmonics.

The unperturbed spectral problems, associated with the 1D differential equations

\[
L_c^e w = D_c \Delta^2 w + Tw = \omega^2 H_c \rho w
\]

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have 4 linearly independent solutions $J_1, H_1^1 \equiv H_1, I_1, K_1$ with factors $\sin \varphi, \cos \varphi$.

We denote them by $J_1^{c,s}, H_1^{c,s}, I_1^{c,s}, K_1^{c,s}$.

Selecting appropriate elastic bonds, we may set the parameters $\beta_0, \beta_1^c, \beta_1^s$ so that the unperturbed spectral problems for $L_0 \oplus L_c^c \oplus L_s^s$ has a multiple eigenfrequency $\nu = (2\pi)^{-1} \omega$ for $\delta = 0$. Then the perturbed spectral problems, associated with the unperturbed operator

\[
\begin{bmatrix}
D\Delta_0^2 + Q\Delta & 0 & 0 \\
0 & D\Delta_1^2 & 0 \\
0 & 0 & D\Delta_1^2
\end{bmatrix} \equiv L_0^0 \oplus L_1^c \oplus L_1^s
\]

and separate $\beta$-natural boundary conditions on the common part of the boundary. For the perturbed problem, defined by the same differential expression

\[
\begin{bmatrix}
\beta_0 - D_\varepsilon \frac{\kappa^c}{\tau_c} & \kappa^c e_0^c & \kappa^s e_0^s \\
\kappa^c e_0^c & \beta_1^c - D_\varepsilon \frac{\kappa^c}{\tau_c} & 0 \\
\kappa^s e_0^s & 0 & \beta_1^s - D_\varepsilon \frac{\kappa^s}{\tau_s}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial \Psi_0}{\partial n} \\
\frac{\partial \Psi_1^c}{\partial n} \\
\frac{\partial \Psi_1^s}{\partial n}
\end{bmatrix} = \Delta
\begin{bmatrix}
\Psi_0 \\
\Psi_1^c \\
\Psi_1^s
\end{bmatrix}
\]

(6.13)

Here $\kappa^{c,s}$ are small real parameters. The multiple eigenfrequency is split into the starlet of simple perturbed eigenfrequencies with eigenfunctions constructed as linear combinations $\Psi_0^0$ of $J_0, I_0$ on $\Omega_\varepsilon$, see Section 4, and a linear combination $\Psi_1^{c,s}$ of the $J_1^{c,s}, H_1^{c,s}, I_1^{c,s}, K_1^{c,s}$ with coefficients found from the above analog (6.13) of the free reclining boundary conditions and the unperturbed boundary condition on the complementary part of the boundary.

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