Notes on un-oriented D-brane instantons

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Abstract

In the first lecture, we discuss basic aspects of worldsheet and penta-brane instantons as well as (unoriented) D-brane instantons, which is our main focus here, and threshold corrections to BPS-saturated couplings. The second lecture is devoted to non-perturbative superpotentials generated by ‘gauge’ and ‘exotic’ instantons living on D3-branes at orientifold singularities. In the third lecture we discuss the interplay between worldsheet and D-string instantons on $T^4/Z_2$. We focus on a 4-fermi amplitude, give Heterotic and perturbative Type I descriptions, and offer a multi D-string instanton interpretation. We conclude with possible interesting developments.

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Introduction

Aim of these lectures is introducing the interested reader to the fascinating subject of non-perturbative effects generated by unoriented D-brane instantons. We assume basic knowledge of D-branes, Yang-Mills instantons and CFT techniques on the string world-sheet. Due to lack of time and space we cannot but mention recent developments such as wall crossing and localization.

In the first lecture, after a very short reminder of Yang-Mills instantons and the ADHM construction, we discuss basic aspects of worldsheet and D-brane instantons and their original applications to threshold corrections.

The second lecture is devoted to non-perturbative superpotentials generated by ‘gauge’ and ‘exotic’ instantons living on D3-branes at orientifold singularities.

In the third lecture we discuss the interplay between worldsheet and D-brane instantons on $T^4/Z_2$. We focus on a specific 4-hyperini amplitude, give Heterotic and perturbative Type I descriptions, and offer a multi D-string instanton interpretation. We conclude with proposing new perspectives and drawing lines for future investigation.

Several good reviews are already available on the subject [1], that is also covered in textbooks [2]. We hope our short presentation could offer a complementary view onto such an active research field.

1 First Lecture: Instantons from Fields to Strings

1.1 Yang-Mills Instantons: a reminder

Instantons (anti-instantons) are self-dual (anti-self-dual) classical solutions of the equations of motions of pure Yang-Mills theory in Euclidean space-time.

\[ F_{\mu\nu} = \pm \tilde{F}_{\mu\nu} \]  (1)

with $\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}$. In quantum theory they can be thought of as gauge configurations bridging quantum tunnelling among topologically distinct vacua. It is remarkable that self-dual (anti-self-dual) gauge fields automatically satisfy YM equations in vacuo as a result of the Bianchi identities. These solutions are classified by a topological charge:

\[ K = \frac{g^2}{32\pi^2} \int d^4x F^a_{\mu\nu} \tilde{F}^a_{\mu\nu} \]  (2)

an integer, which computes how many times an $SU(2)$ subgroup of the gauge group is wrapped by the classical solution while its space-time location spans the $S_3$-sphere at
infinity. The action of a self-dual (or anti-self-dual) instanton configuration turns out to be

\[ S_I = \frac{8\pi^2}{g^2} |K| \]

### 1.1.1 ADHM construction

An elegant algebro-geometric construction of YM instantons was elaborated by Atiyah, Drinfeld, Hitchin and Manin and goes under the name of ADHM construction [3].

For \( SU(N) \) groups, the ADHM ansatz for a self-dual gauge field with topological charge \( K \), written as a traceless hermitean \( N \times N \) matrix, reads

\[
(A_\mu)_{uv}(x) = g^{-1}\bar{U}_\lambda^\mu \partial_\mu U^\lambda_{uv},
\]

where \( U_{\lambda u}(x) \) with \( u = 1, ..., N \) and \( \lambda = 1, ..., N+2K \) are \((N+2K)\times N\) complex ‘matrices’ whose columns are the basis ortho-normal vectors for the \( N \) dimensional null-space of a complex \( 2K \times (N + 2K) \) ‘matrix’ \( \bar{\Delta}(x) \), i.e. satisfy

\[
\bar{\Delta}^{\lambda \dot{\lambda}}_i U_{\lambda u} = 0 = \bar{U}^\lambda_i \Delta_{\lambda \dot{\lambda}}
\]

for \( i = 1, ..., K, \alpha, \dot{\alpha} = 1, 2 \). Remarkably, \( \Delta_{\lambda \dot{\lambda}}(x) \) turns out to be at most linear in \( x \). In quaternionic notation\(^2\) for \( x \),

\[
\Delta_{\lambda \dot{\lambda}}(x) = a_{\lambda \dot{\lambda}} + b_{\lambda \dot{\lambda} x_{\alpha \dot{\alpha}}}, \quad \bar{\Delta}^{\lambda \dot{\lambda}}_i(x) = \bar{a}^{\lambda \dot{\lambda}}_i + \bar{x}^{\alpha \dot{\alpha}}_i b_{\lambda \dot{\lambda}} \equiv (\Delta_{\lambda \dot{\lambda}})^*.
\]

The complex constant ‘matrices’ \( a \) and \( b \) form a redundant set of collective coordinates that include the moduli space \( \mathcal{M}_K \). Decomposing the index \( \lambda \) as \( \lambda = u + i\alpha \), with no loss of generality, one can choose a simple canonical form for \( b \)

\[
\begin{bmatrix}
0 \\
\delta_{\dot{\alpha}}^{\alpha} \delta_{ij}
\end{bmatrix}, \quad \bar{b}^{\lambda}_{\dot{\lambda} j} = \bar{b}^{(u+i\alpha)}_{\dot{\lambda} j} = \begin{bmatrix}
0 \\
\delta_{\dot{\alpha}}^{\alpha} \delta_{ij}
\end{bmatrix}
\]

One can also split \( a \) in a similar way as:

\[
a_{\alpha \dot{\alpha}} = a_{(u+i\alpha)j \dot{\alpha}} = \begin{bmatrix}
w_{u j \dot{\alpha}} \\
(X_{\alpha \dot{\alpha}})_{ij}
\end{bmatrix}, \quad \bar{a}^{\dot{\alpha} (u+i\alpha)}_j = \bar{a}^{(u+i\alpha)}_j = \begin{bmatrix}
\bar{w}^{\dot{\alpha} u} \\
(\bar{X}^{\dot{\alpha} \alpha})_{ji}
\end{bmatrix}
\]

In order to ensure self-duality of the connection, the ‘ADHM data’ \( \{w, \bar{w}, X, \bar{X}\} \) with \( X^i = X_\mu \) must satisfy algebraic constraints, known as the ADHM equations, that can be written in the form

\[
w^{(u \dot{\alpha})}_{\dot{\beta} j} \bar{w}^{\dot{\beta} u} + \eta^{\alpha \mu}_{\nu} [X^\mu, X^\nu]_{ij} = 0
\]

\(^2\)Any real 4-vector \( V^\mu_\mu \) can be written as a ‘real’ quaternion \( V_{\alpha \dot{\alpha}} = V^\mu_\mu \sigma^{\mu}_{\alpha \dot{\alpha}} \) with \( \sigma^\mu = \{1, -i\sigma^a\} \).
for later comparison with the D-brane construction. Note the $U(K)$ invariance of the above $3K \times K$ equations. For a recent review of supersymmetric instanton calculus see [4].

The ADHM construction for unitary groups can be generalized to orthogonal and symplectic groups. It is quite remarkable how the rather abstract ADHM construction can be made very intuitive using D-branes and $\Omega$-planes [5] as we will see later on.

1.2 Instantons in String Theory

1.2.1 Worldsheet instantons

World-sheet instantons in Heterotic and Type II theories correspond to Euclidean fundamental string world-sheets wrapping topologically non-trivial internal cycles of the compactification space and produce effects that scale as $e^{-R^2/\alpha'}$ [6]. Depending on the number of supersymmetries (thus on the number of fermionic zero modes), they can correct the two-derivative effective action or they can contribute to threshold corrections to higher derivative (BPS saturated) couplings [7]. For Type II compactifications on CY three-folds, preserving $\mathcal{N} = 2$ supersymmetry in $D = 4$, holomorphic worldsheet instantons ($\bar{\partial}X = 0$) correct the special Kähler geometry of vector multiplets (Type IIA) or the dual quaternionic geometry of hypermultiplets (Type IIB). For heterotic compactifications with standard embedding of the holonomy group in the gauge group, complex structure deformations are governed by the same special Kähler geometry as in Type IIB on the same CY three-fold, that is not corrected by worldsheet instantons. Complexified Kähler deformations are governed by the same special Kähler geometry as in Type IIA on the same CY three-fold, that is corrected by worldsheet instantons, or equivalently, as a result of mirror symmetry, by the same special Kähler geometry as in Type IIB on the mirror CY three-fold that is tree level exact. For standard embedding, the Kähler metrics of charged supermultiplets in the 27 and $27^*$ representations of the surviving/visible $E_6$ are simply determined by the ones of the neutral moduli of the same kind by a rescaling [8]. For non-standard embeddings the situation is much subtler.

1.2.2 Brane instantons

Euclidean NS5-branes (EN5-branes) wrapping the 6-dimensional compactification manifold produce non-perturbative effects in $e^{-c/g_s^2}$ (reflecting the NS5-brane tension) that qualitatively correspond to ‘standard’ gauge and gravitational instantons. Euclidean Dp-brane wrapping $(p+1)$-cycles produce instanton effects that scale as $e^{-c_p/g_s}$ (reflecting the EDp-brane tension) [9]. In Type IIB on CY three-fold, ED(-1), ED1-, ED3- and ED5-brane
instantons, obtained by wrapping holomorphic submanifolds, correct dual quaternionic geometry in combination with world-sheet (EF1-) and EN5-instantons. In Type IIA on CY three-folds, ED2-instantons (D-‘membrane’ instantons) wrapping special Lagrangian submanifolds, correct the dual quaternionic geometry, in combination with EN5-instantons. In both cases, the dilaton belongs to the universal hypermultiplet.

1.2.3 Unoriented D-brane instantons

In Type I, the presence of Ω9-planes severely restricts the possible homologically nontrivial instanton configurations. Only ED1- and ED5-branes are homologically stable. Other (Euclidean) branes may be associated to instanton with torsion (K-theory) charges. For other un-oriented strings the situation is similar and can be deduced by means of T-duality: e.g. for intersecting D6-branes one has two different kinds of ED2-branes (ED0- and/or ED4-brane instanton require $b_{1,5} \neq 0$), for intersecting D3- and D7-branes one has ED(-1) and ED3-branes. There are two classes of unoriented D-brane instantons depending on the stack of branes under consideration.

- ‘Gauge’ instantons correspond to EDp-branes wrapping the same cycle $C$ as a stack of background D(p+4)-branes. The prototype is the D3, D(-1) system \cite{[10]} that has 4 N-D directions. The EDp-branes behave as instantons inside D(p+4)-branes

$$F = \tilde{F}$$

and produce effects whose strength, given by

$$e^{-W_{p+1}(C)/g_s e^{p+1}} = e^{-1/g_5^{YM}},$$

is precisely the one expected from ‘gauge’ instantons in the effective field-theory.

- ‘Exotic’ instantons arise from EDp'-branes wrap a cycle $C'$ which is not wrapped by any stack of background D(p+4)-branes. The prototype is the D9, ED1 system with 8 N-D directions and only a chiral fermion at the intersection. In this case

$$F \neq \tilde{F}$$

and the strength is given by

$$e^{-W_{p'+1}(C')/g_s e^{p'+1}} \neq e^{-1/g_5^{YM}}$$

‘Exotic’ instantons may eventually enjoy a field theory description in terms of octonionic instantons or hyper-instantons with $F \wedge F = *_8 F \wedge F$. 

4
Let us now list possible effects generated by (un)oriented D-brane instantons in diverse string compactifications.

- In $\mathcal{N} = 8$ theories (e.g. toroidal compactifications of oriented Type II A/B) D-brane instantons produce threshold corrections to $R^4$ terms and other 1/2 BPS (higher derivative) terms.

- In $\mathcal{N} = 4$ theories (e.g. toroidal compactifications of Type I / Heterotic) D-brane instantons produce threshold corrections to $F^4$ terms and other 1/2 BPS (higher derivative) terms.

- In $\mathcal{N} = 2$ theories (e.g. toroidal orbifolds with $\Gamma \subset SU(2)$) D-brane instantons produce threshold corrections to $F^2$ terms and other 1/2 BPS terms.

- In $\mathcal{N} = 1$ theories (e.g. toroidal orbifolds with $\Gamma \subset SU(3)$) D-brane instantons produce threshold corrections and superpotential terms.

### 1.3.1 Thresholds in toroidal compactifications

We have not much to add to the vast literature on threshold corrections to $R^4$ terms in $\mathcal{N} = 8$ theories which are induced by oriented D-brane as well as world-sheet instantons.\(^3\) We would only like to argue that in unoriented Type I strings and alike these corrections should survive as functions of the unprojected closed string moduli despite some of the corresponding D-brane or worldsheet instantons be not BPS. These and lower derivative ($R^2$) couplings may receive further perturbative corrections from surfaces with boundaries and crosscaps. Viz: $\mathcal{L}_{\text{II}} \approx R^4 f_{\text{II}}(\phi, \chi) \rightarrow \mathcal{L}_{\text{I}} \approx R^4 [f_{\text{II}}(\phi, \chi = 0) + f_I(\phi)]$.

The original application of unoriented D-brane instanton was in the context of threshold corrections to $F^4$ terms in toroidal compactifications of Type I strings \(^1\). These are closely related to the threshold corrections to $F^4$ terms for heterotic strings on $T^d$. For later use, let us briefly summarize the structure of the latter. After

- Computing the one-loop correlation function of 4 gauge boson vertex operators $V_{(0)} = A^a_\mu (\partial X^\mu + ipq\phi^a)\tilde{J}_a e^{ipx}$

- Taking the limit of zero momentum in the exponential factors \(i.e.\) neglecting the factor $\prod_{i,j} e^{\alpha' p_i \cdot p_j G(z_{ij})} \rightarrow 1$

or, equivalently,

\(^3\)In $D = 4$ and lower Euclidean NS5-branes can also contribute.
computing the character-valued partition function in a constant field-strength background $\nu$

taking the fourth derivative wrt $\nu$

one arrives at the integral over the one-loop moduli space that receives contribution only from BPS states and schematically reads

$$I_{d}[\Phi] = \mathcal{V}_{d} \int_{\mathcal{F}} \frac{d^{2}\tau}{\tau_{2}^{2}} \sum_{M} e^{2\pi i T(M)} e^{-\frac{\pi \text{Im} T(M)}{\tau_{2} \text{det} G(M)}} e^{\tau - \text{Im} U(M)}^{2} \Phi(\tau)$$

where $M = (\vec{n}, \vec{m})$ represent the embedding of the world-sheet torus in the target $T^{d}$, $\Phi(\tau)$ is some modular form. The induced Kahler $T(M)$ and complex $U(M)$ structures are given by

$$T(M) = B_{12} + i \sqrt{\text{det} G}, \quad U(M) = \frac{1}{G_{11}} (B_{12} + i \sqrt{\text{det} G})$$

with $G = M'GM$, $B = M'BM$ induced metric and $B$-field [14]. The integral can be decomposed into three terms

$$I_{d}[\Phi] = I_{d}^{\text{triv}}[\Phi] + I_{d}^{\text{deg}}[\Phi] + I_{d}^{\text{ndeg}}[\Phi].$$

The three different orbits are classified as follows: the orbit of $M = 0$ (trivial orbit), degenerate orbits with $\det(M_{i,j}) = 0$ and non-degenerate orbits with some $\det(M_{i,j}) \neq 0$. Let us consider the various contributions.

- **Trivial orbit**: $M = 0$,

$$T_{d,d}^{\text{triv}}[\Phi] = \mathcal{V}_{d} \int_{\mathcal{F}} \frac{d^{2}\tau}{\tau_{2}^{2}} \Phi(\tau) \rightarrow T_{d,d}^{\text{triv}}[1] = \frac{\pi^{2}}{3} \mathcal{V}_{d}$$

- **Degenerate orbits**: $M \neq 0$, $\det(M_{i,j}) = n^{i}m^{j} - n^{j}m^{i} = 0 \ \forall i, j$. One can choose $\vec{n} = 0$ representative and unfold $\mathcal{F}$ to the strip $\mathcal{S} = \{ |\tau_{1}| < 1/2, \tau_{2} > 0 \}$, then

$$T_{d,d}^{\text{deg}}[\Phi] = \mathcal{V}_{d} \int_{\mathcal{S}} \frac{d^{2}\tau}{\tau_{2}^{2}} \sum_{\vec{m} \neq \vec{0}} e^{-\frac{\pi^{2}}{\tau_{2}} G \vec{m} \vec{n}} \Phi(\tau) \rightarrow T_{d,d}^{\text{deg}}[1] = \mathcal{V}_{d} \mathcal{E}_{d}^{\text{SL}(d)}(G).$$

- **Non degenerate orbits**: at least one $\det(M_{i,j}) = n^{i}m^{j} - n^{j}m^{i} \neq 0$. The representative for these orbits may be chosen to be $\vec{n}^{\alpha} = 0$ for $\alpha = 1, \ldots, k$, $m^{\alpha} \neq 0$, $n^{\alpha} > m^{\alpha} \geq 0$ and enlarging the region of integration $\mathcal{F}$ to the full upper half plane $\mathcal{H}^{+}$ one finds:

$$T_{d,d}^{\text{ndeg}}[\Phi] = \mathcal{V}_{d} \int_{\mathcal{H}^{+}} \frac{d^{2}\tau}{\tau_{2}^{2}} \sum_{(n^{a}, 0, m^{\alpha}, m^{\alpha})} e^{2\pi i T(M)} e^{-\frac{\pi \text{Im} T(M)}{\tau_{2} \text{det} G(M)}} e^{\tau - \text{Im} U(M)}^{2} \Phi(\tau)$$

$$\rightarrow T_{d,d}^{\text{deg}}[1] = \mathcal{V}_{d} \mathcal{E}_{\mathcal{V}, s=1}^{\text{SO}(d,d)}(G, B) \quad (\text{generalized Eisenstein series}).$$
Thanks to Type I / Heterotic duality, heterotic worldsheet instantons are mapped into ED-string instantons. Since $F^4$ terms are 1/2 BPS saturated, matching the spectrum of excitations, including their charges, was believed to be sufficient to match the threshold corrections even in the presence of (non)commuting Wilson lines [12, 15] or after T-duality [16, 17]. More recently, thanks to powerful localization techniques, a perfect match between threshold corrections in Heterotic and Type I' (with D7-branes) has been found on $T^2$ for the specific choice of commuting Wilson lines breaking $SO(32)$ to $SO(8)^4$ [18]. The somewhat unsatisfactory results of [19] for different breaking patterns with orthogonal or symplectic groups can be either interpreted as a failure of localization or as the need to include higher order terms. Notice that only for $SO(8)$, ‘exotic’ string instantons should admit a field theory interpretation in terms of ‘octonionic’ instantons. It would be nice to further explore this issue in this or closely related context of $\mathcal{N} = 1, 2$ theories in D=4 where heterotic worldsheet instantons correcting the gauge kinetic function should be dual to ED-string (or other ED-brane) instantons [20]. A short review of the strategy to compute similar threshold corrections will be presented later on when we discuss Heterotic / Type I duality on $T^4/Z_2$.

1.3.2 Phenomenological considerations

Despite some success in embedding (MS)SM in vacuum configurations with open and unoriented strings, there are few hampering properties at the perturbative level:

- Forbidden Yukawas in $U(5)$ (susy) GUT’s

  $$H^d_{5,+1} F^c_{5,-1} A_{10,+2} \text{ OK but } H^u_{5,+1} A_{10,+2} A_{10,+2} \text{ KO}$$

  forbidden by (global, anomalous) $U(1)$ invariance, though compatible with $SU(5)$ (yet no way $e^{abcde}$ from Chan-Paton)

- R-handed (s)neutrino masses $W_M = M_R N N$ forbidden by e.g. $U(1)_{B-L}$ in Pati-Salam like models $SO(6) \times SO(4) \rightarrow SU(3) \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$

- $\mu$-term in MSSM $W_\mu = \mu H_1 H_2$ typically forbidden by extra (anomalous) $U(1)$’s

All the above couplings can be generated by ‘stringy’ instantons after integrating over the ‘non-dynamical’ moduli living on the world-volume of the EDp’-branes under consideration. These effects scale as $e^{-T_{EDp'}} V_{EDp'}$ and are non-perturbative in $g_s$, since $T_{EDp'} \approx 1/g_s (\alpha')^{p+1/2}$. Yet a priori they depend on different moduli (through the dependence of $V_{EDp'}$ on various $Z$’s) from the ones appearing in the gauge kinetic function(s) of background Dp’-brane, so they cannot in general be identified with the standard ‘gauge’
instantons. Relying on the $g_s$ power counting introduced in [10], the relevant are disks with insertions of the non-dynamical vertex operators $V_{\Theta}$ (connecting EDP'-EDP') and $V_{\lambda}$ (connecting EDP-DP') with or without insertions of dynamical vertex operators $V_{A}$ etc, which correspond to the massless excitations of the vacuum configuration of (intersecting/magnetized) unoriented Dp-branes [21]. Disks without dynamical insertions yield the ‘instanton action’, with one dynamical vertex they produce classical profiles for $A$ etc. Disks with more insertions contribute to higher-order corrections. One loop diagrams with no insertions produce running couplings and subtle numerical prefactor that can cancel a given type of non-perturbative F-terms [22, 23].

1.3.3 Anomalous $U(1)$’s and gauged PQ symmetries

In general, a ‘naked’ chiral field $Z$ whose pseudoscalar axionic components $\zeta = \text{Im} Z$ shift under some local anomalous $U(1)$ cannot appear in a (super)potential term if not dressed with other chiral fields charged under $U(1)$. $U(1)$ invariance puts tight constraints on the form of the possible superpotential terms. Since the axionic shift is gauged it must be a symmetry of the kinetic term. This is only possible when no non-perturbative (world-sheet or D-brane instanton) corrections spoil the tree level (in fact perturbative) PQ symmetry. This means that the gauging procedure corresponds to turning on fluxes such that the potential instanton corrections in $Z$ are in fact disallowed. In practice, this means the corresponding wrapped brane is either anomalous (à la Freed-Witten) [24] or destabilized due to the flux [25].

Moreover, background fluxes (for both open and closed strings) can lift fermionic zero-modes. Various ‘perturbative’ studies have been carried out [26].

2 Second Lecture: Unoriented D-brane Instantons

2.1 ADHM from branes within branes

As already mentioned, the ADHM construction has a rather intuitive description in open string theory, whereby the gauge theory is realized on a stack of Dp-branes. D(p-4)-branes which are localized within the previous stack of branes behave as instantons [5].

Indeed, the WZ couplings on the Dp-brane worldvolume schematically reads

$$ S_{WZ} = \int C_{p+1} + \int C_{p-1} \wedge Tr(F) + \int C_{p-3} \wedge Tr(F \wedge F) + ... $$ (3)

In particular a localized source of $C_{p-3}$ within a Dp-brane behaves like an instanton
density $Tr(F \wedge F)$. Moreover, the ADHM data are nothing but the massless modes of open strings connecting the D(p-4)-branes with one another or with the background Dp-branes.

Let us take $p = 3$ for definiteness. The low-energy effective theory on the world-volume of $N$ parallel D3-branes is $N = 4$ supersymmetric Yang-Mills theory with gauge group $U(N)$. Instanton moduli are described by the massless modes of open strings with at least one end on the D(-1)-brane stack. In this system of D3 and D(-1) branes there are three sectors of the open string spectrum to be considered. $U(N)$ gauge fields and their superpartners are provided by the strings that start and end on D3-branes. Strings stretching between two D(-1)-branes give rise to $U(K)$ non-dynamical gauge fields and their superpartners. These ‘fields’ represent part of the (super) ADHM data. The remaining (super) ADHM data are provided by the strings with one end on the D3-branes and the other one on the D(-1)-branes and vice-versa.

In the presence of D3-branes, (Euclidean) Lorentz symmetry is broken $SO(10) \to SO(4) \times SO(6)$ and it is convenient to split ten “gauge bosons”, $A_M$, into four gauge bosons $a_\mu$, and six real “scalars”, $\chi_i$. Similarly the $d = 10$ gauginos produce four non-dynamical Weyl “gauginos”, $\Theta^A_\alpha$ as well as their antiparticles $\bar{\Theta}^A_{\dot{\alpha}}$.

Introducing, for later convenience, three auxiliary fields $D^c$, the D(-1)-D(-1) $U(K)$ ‘geometric’ supermoduli are given by

$$a_\mu, \chi_i; \Theta^A_\alpha, \bar{\Theta}^A_{\dot{\alpha}}; D^c$$

while the $4KN$ D(-1)-D(3) ‘gauge’ supermoduli are

$$w^{u}_{\alpha i}, \bar{w}^i_{\alpha u}; \nu^A u_i, \bar{\nu}^A_i$$

with $\mu = 1, \ldots, 4$, $\alpha, \dot{\alpha} = 1, 2$ vector and spinor indices of $SO(4)$, $i = 1, \ldots, 6$, $A = 1, \ldots, 4$ are vector and spinor indices of $SO(6)$ respectively and $c = 1, 2, 3$. The matrices $a_\mu, \chi_i$ describe the position of the instanton along the longitudinal and transverse directions to the D3-brane respectively. $\Theta^A_\alpha$ and $\bar{\Theta}^A_{\dot{\alpha}}$ are their superpartners. $w_{\alpha i}, \bar{w}_{\dot{\alpha} i}$ represent the D3-D(-1) open string in the NS sector, accounting for instanton sizes and orientations, and $\nu^A, \bar{\nu}^A$ are their fermionic superpartners.

### 2.2 The D3-D(-1) action

By computing scattering amplitudes on the disk, one can determine the complete action that governs the dynamics of the light modes (or moduli) of the system of D(-1) branes in the presence of D3-branes. It schematically reads

$$S_{K,N} = Tr_k \left[ \frac{1}{g_0^2} S_G + S_K + S_D \right]$$
with
\[
S_{G} = -[\chi_{i}, \chi_{j}]^2 + i\Theta_{\dot{a}A}[\chi_{AB}, \Theta_{B}] - D^{c}D^{c}
\]
\[
S_{K} = -[\chi_{i}, a_{\mu}]^2 + \chi_{i}^{\dagger}w_{\dot{a}}\chi_{i} - i\Theta^{\alpha A}[\chi_{AB}, \Theta_{A}^{B}] + 2i\chi_{AB}\nu^{A}B
\]
\[
S_{D} = i\left(-[a_{\alpha}, \Theta^{A}] + \nu^{A}w_{\dot{a}} + \tilde{w}_{\dot{a}}\nu^{A}\right)\tilde{\Theta}_{\dot{a}}^{A} + D^{c}\left(\tilde{w}\sigma^{c}w - i\tilde{\eta}_{A}[a^{a}, a^{\alpha}]\right)
\]
where \(\chi_{AB} \equiv \frac{1}{2}\Sigma_{i}a_{\alpha}^{i}\) and \(\Sigma_{i}^{a} = (\eta_{\alpha}^{AB}, i\tilde{\eta}_{\dot{a}A}^{AB})\) are given in terms of t'Hooft symbols and \(g_{0}^2 = 4\pi(4\pi^2\alpha')^{-2}g_s\). Note that the action \(S_{K,N}\) arises from the dimensional reduction of the D5-D9 action in six dimensions down to zero dimension. If there are v.e.v. for the six \(U(N)\)-adjoint scalars \(\varphi_{a}\) belonging to the D3-D3 open string sector one has to add the term
\[
S_{\varphi} = \text{tr}\left[\tilde{w}(\varphi_{i}^{\dagger}\varphi_{i} + 2\chi_{i}^{\dagger}\varphi_{i})w_{\dot{a}} + 2i\nu^{A}\varphi_{AB}\nu^{B}\right]
\]
to the action \(S_{K,N}\). In the limit \(g_{0} \sim (\alpha')^{-1} \rightarrow \infty (g_{0} \text{ fixed})\) gravity decouples from the gauge theory and there are no contributions coming from \(S_{G}\). \(\tilde{\Theta}_{\dot{a}A}\) and \(D^{c}\) fields become Lagrange multipliers for the super ADHM constraints:
\[
D^{a}: [a_{\mu}, a_{\nu}] = w\nu_{a} = 0 \quad \text{ADHM Eqs}
\]
\[
\tilde{\Theta}_{\dot{a}}^{A}: [a_{\mu}, \Theta^{A}] = \sigma^{\mu} + \nu^{A}\tilde{w} = 0 \quad \text{super ADHM Eqs}
\]
In this limit the multi-instanton ‘partition function’ becomes
\[
Z_{k,N} = \int_{M} e^{-S_{k,N}-S_{\varphi}} = \frac{1}{\text{Vol}U(k)} \int_{M} d\chi dD d\theta d\tilde{\theta} dw d\nu e^{-S_{k,N}-S_{\varphi}}.
\]

### 2.3 Vertex operators

Classical actions, (super)instanton profiles and non-perturbative contributions to scattering amplitudes can be derived by computing disk amplitudes with insertions of vertex operators for non-dynamical moduli \(V_{a}, V_{\chi}, V_{w}, V_{w^{\dagger}}\) (ADHM data) and their superpartners [10].

#### 2.3.1 Vertex operators for ‘gauge’ instantons

Let us first start considering the vertex operator for a non dynamical gauge boson \(a_{\mu}\) along the four D-D space-time directions. The vertex operator reads
\[
V_{a} = a_{\mu}e^{-\varphi}\psi^{\mu}T_{K\times K}
\]
where \(\varphi\) arises from the bosonization of the \(\beta, \gamma\) worldsheet super-ghosts, \(\psi\) are the worldsheet fermions and \(T_{K\times K}\) are \(U(K)\) Chan-Paton matrices. For the non dynamical transverse scalars \(\chi_{i}\) along the six internal D-D directions, the vertex operator reads
\[
V_{\chi} = \chi_{i}e^{-\varphi}\psi^{i}T_{K\times K}
\]
Similarly

\[ V_A = \Theta^a(p) S_a e^{-\varphi/2} T_{K \times K} \]

with \( a = 1, 16 \), produces four non-dynamical Weyl “gauginos”, \( \Theta^A \), and their antiparticles, \( \tilde{\Theta}^A \).

Bosonic vertex operators for low-lying D(p-4)-Dp strings, with multiplicity \( K \times N \) and their conjugates are given by

\[ V_w = \sqrt{\frac{g_s}{v_{p-3}}} w_a e^{-\varphi} \prod_{\mu} \sigma_{\mu} S^\alpha T_{K \times N} \]

with \( S^\alpha \) an \( SO(4) \) spin field of worldsheet scaling dimension 1/4. \( \sigma_{\mu} \) are \( Z_2 \) bosonic twist fields along the 4 relatively transverse N-D directions. \( T_{K,N} \) denote the \( K \times N \) Chan-Paton ‘matrices’. The super-partners of \( w_a \) are represented by vertex operators of the form

\[ V_\nu = \sqrt{\frac{g_s}{v_{p-3}}} \nu_A e^{-\varphi/2} \prod_{\mu} \sigma_{\mu} S^A T_{K \times N} \]

where \( S^A \) is an \( SO(6) \) spin field of dimension 3/8. Note that the overall normalization \( \sqrt{g_s/v_{p-3}} \) is crucial for the correct field theory limit (\( \alpha' \to 0 \)).

### 2.3.2 Vertex operators for ‘stringy’ instantons

Let us now consider ‘stringy’ instantons. The prototype is the D9, D1 system which has 8 N-D directions. The multi-(instanton) configuration of this system was first analyzed in [12]. The lowest lying modes of an open string stretched between N D9 and K D1 branes are massless fermions with a given chirality (say Right) along the two common N-N directions. For Type I strings there are 32 such chiral fermions \( \lambda^A \) that precisely reproduce the gauge degrees of freedom of the ‘dual’ heterotic string [13]. In addition, in the \( \mathcal{N} = (8, 0) \) theory on the D1 world-sheet with \( SO(8) \) R-symmetry group, there are 8 transverse bosons \( X^I \) in the \( 8_v \) and as many Green-Schwarz type fermions \( S^a \) of opposite chirality (say Left) in the \( 8_s \) of \( SO(8) \). The 32 massless right-moving \( \lambda^A \) are inert under the left-moving susy \( Q_{\bar{a}} \) in the \( 8_c \).

After compactification to \( D = 4 \) on a manifold with non-trivial holonomy some of the global supersymmetries are broken and the corresponding D1 world-sheet theory changes accordingly. In particular \( SO(8) \) breaks to some subgroup.

### 2.4 D-branes at Orbifolds

A particularly promising class of configurations with nice phenomenological perspectives that also allow explicit non-perturbative computations are unoriented D-branes at singu-
larities. Let us consider a stack of D3-branes at the orbifold singularity $T^d/\Gamma \approx R^d/\Gamma$ (locally), and let us take $\Gamma = Z_n$ for simplicity. At the singularity $N$ D3-branes group into stacks of $N_i$ ‘fractional’ branes, that cannot move away from the singularity, with $i = 0, 1, 2, \ldots$ labelling the conjugacy classes of $Z_n$. The gauge group $U(N)$ decomposes as $\Pi_i U(N_i)$.

\[(Z_1, Z_2, Z_3) \approx (\omega^{k_1}Z_1, \omega^{k_2}Z_2, \omega^{k_3}Z_3)\]

for simplicity we take $k_1 + k_2 + k_3 = 0 \pmod{n}$ that generically preserves $\mathcal{N} = 1$ supersymmetry.

The action on Chan-Paton factors is given by

\[\rho(Z_n) = \rho_0(1_{N_0}, \omega^11_{N_1}, \omega^21_{N_2}, \ldots, \omega^{n-1}1_{N_{n-1}})\]

For $\alpha' \approx 0$, keeping only invariant components under (4), the resulting theory turns out to be an $\mathcal{N} = 1$ quiver gauge theory, in which vector multiplets $V$ are in the $N_i\bar{N}_i$ representation while chiral multiplets $\Phi_i$ are in the $N_j\bar{N}_l$ representation with $k_i + j - l = 0 \pmod{n}$ [27].

Twisted RR tadpole cancellation in sectors with non vanishing Witten index can be written as $\text{tr} \rho(Z_n) = 0$ that ensures the cancellation of chiral non-abelian anomalies [28].

### 2.4.1 Unoriented projection

Possible unoriented projections depend on the parity of $n$ and the charge of the $\Omega$-plane. For $n$ odd there is only one possibility

\[N_0 = \bar{N}_0 \quad , \quad N_i = \bar{N}_{n-i}\]

For $n$ even there are two possibilities

\[N_0 = \bar{N}_0 \quad , \quad N_i = \bar{N}_{n-i} \quad , \quad N_{n/2} = \bar{N}_{n/2}\]

\[N_0 = \bar{N}_{n/2} \quad , \quad N_i = \bar{N}_{n/2-i}\]

One should also impose the twisted RR tadpole cancellation condition (non vanishing Witten index) $\text{tr} \rho(Z_n) = \pm q_\Omega$ which from the field theory point of view is just the chiral anomaly cancellation [29].

Let us focus on the very rich and instructive case of $T^6/Z_3 \approx R^6/Z_3$. 

12
2.5 Unoriented $R^6/Z_3$ projection

In the remaining part of this Section, for illustrative purposes, we will discuss unoriented D-brane instantons on a stack of D3-branes located at an unoriented $R^6/Z_3$ orbifold singularity.

Since $n = 3$ is odd, there is only one possible embedding in the Chan-Paton group up to the charge of the $\Omega 3^\pm$ planes. Introduction of $\Omega 3^-$-plane combined with local R-R tadpole cancellation leads to a theory with gauge group $G = SO(N_0) \times U(N_0 + 4) \times H_{\text{reg}}$, where $H_{\text{reg}}$ accounts for the Chan-Paton group of the ‘regular’ branes that can move into the bulk. We will henceforth assume that regular branes are far from the singularity and essentially decoupled from the local quiver theory. For $N_0 = 0$, we have $U(4)$ gauge group with 3 chirals in $6_{-2}$. In the presence of $\Omega 3^+$-plane we get a theory with $G = Sp(2N_0) \times U(2N_0 - 4) \times H_{\text{reg}}$ gauge group, e.g. for $2N_0 = 6$, we have $Sp(6) \times U(2)$ gauge group with 3 chirals in $(6, 2_{+1}) + (1, 3_{-2})$.

In both cases the anomalous $U(1)$ mixes with the twisted RR axion $\zeta$ in a closed string chiral (linear) multiplet $Z$ (gauging of axionic shift)

\[ \delta A = d\alpha, \quad \zeta = - M\alpha \]

\[ L_{\text{ax}} = (d\zeta - M\alpha)^2 + \frac{1}{f\zeta} F \wedge F \]

Anomaly cancellation $\delta_{\alpha}[L_{\text{ax}} + L_{1\text{-loop}}] = 0 \leftrightarrow M/A/\zeta = t_3 = Tr Q^3$.

2.5.1 Field Theory analysis

As already mentioned ‘gauge’ instantons are expected to generate VY-ADS-like superpotentials. Neglecting U(1)’s for the time being, the two choices of $\Omega$-planes and, thus, of gauge group lead to superpotentials of the form

\[ \begin{align*}
SU(4) & : \quad W = \frac{\Lambda^9}{\det_{I,J}(\epsilon_{abcd}A^a_I A^b_J)} \\
Sp(6) \times SU(2) & : \quad W = \frac{\Lambda^9}{\det_{6 \times 6}(\Phi_{ai}^i)}
\end{align*} \]

In string theory, $\Lambda^\beta = M^3 e^{-\frac{9}{8} - \frac{2}{f\zeta}}$ ($\beta = 9$ here), shift of $Z$ compensates the $U(1)$ charge of the denominator! The “thumb rule” is that in each case there are two exact/unlifted

\[ ^{4}\text{More complicated cases with several (non)anomalous U(1)’s, generalized Chern-Simons terms } \mathcal{L}_{\text{GCS}} = E_{[ijkl} A^i \wedge A^j \wedge F^k \text{ are needed in the low-energy effective theory with non-trivial phenomenological consequences} [30]. \]
fermionic zero-modes \( n(\lambda_0) - n(\psi_0) = 2 \). The rest is lifted by Yukawa interaction \( Y_g = g\phi^i\psi\lambda \). We now pass to describe the explicit computations with unoriented D-instantons

\[
U(4)_{D3} \rightarrow U(K)_{D(-1)} \quad , \quad Sp(6)_{D3} \rightarrow O(K)_{D(-1)}
\]

In both cases, there are two exact un-lifted fermionic zero-modes for \( K = 1 \).

2.5.2 Non-perturbative superpotential for \( Sp(6) \times U(2) \)

After the projection, in the \( D(-1)\)-\( D(-1) \) sector one has geometric supermoduli: \( a_{\mu} \) (instanton position) and \( \Theta_{\alpha}^0 \) (Grassman coordinate), which yield the \( \mathcal{N} = 1 \) superspace measure. There is no room for \( D^c \) and \( \bar{\Theta}_{\dot{\alpha}}^0 \) in the present case, since the relevant instanton is an \( O(1) \) instanton in the \( Sp(6) \) group and as such there are no super ADHM constraints.

In the \( D(-1)\)-\( D3 \) sector the gauge super-moduli are \( w^u_a, \nu^0_u, \nu^I_a \) with \( u = 1, \ldots 6 \) \( Sp(6) \), \( a = 1, 2 \) \( U(2) \), and \( I = 1, 2, 3 \) \( SU(3) \) ‘pseudo’ flavor indices. Both \( \tilde{\Theta}_{\dot{\alpha}}^0 \) and \( D^c \) are projected out. Taking into account the interactions with the \( D3 \)-\( D3 \) excitations \( \Phi_{Iua} = \phi_{Iua} + ... \), the instanton action can be reduced to the form

\[
S_{D(-1)-D(3)} = \bar{w}^u_{\alpha} \bar{\phi}_{Iua} \phi^{Iva} w_{v\dot{\alpha}} + \nu^0_u \nu^I_a \bar{\phi}_{Iua}
\]

Integrations over gauge super-moduli are gaussian and the final result can be written as

\[
\int d^6w d^6\nu d^6\nu' e^{S_{D(-1)-D(3)}} = \frac{\det(\bar{\phi}_{Iua})}{\det(\phi^{Iva}\phi_{Iua})} = \frac{1}{\det(\phi_{Iva})}
\]

Including \( D(-1)\)-\( D(-1) \) action and one-loop contribution, up to a non vanishing numerical constant, we get

\[
\int d^4\alpha d^2\Theta \mu^3 e^{2\pi i r(\mu)} = \int d^4x d^2\theta \frac{\Lambda^9}{\det(\phi_{Iva})}
\]

to a non-zero numerical constant.

2.5.3 Non-perturbative superpotential for \( U(4) \)

As explained in [27] for \( U(4) \) gauge theory with three chiral multiplets in the \( 6 \), the \( D(-1)\)-\( D(-1) \) geometric ‘supermoduli’ are \( a_{\mu}^{(0)} \) (instanton position), \( \bar{\chi}_{I(-2)} \), \( \chi_{(+2)} \) (internal) and \( \Theta_{\alpha}^{(0)}, \bar{\Theta}_{\dot{\alpha}(0)}, \bar{\Theta}_{I(\dot{\alpha}(-1))} \) (Grassman coordinates), which give \( \mathcal{N} = 1 \) superspace measure. \( D(-1)\)-\( D3 \) gauge ‘supermoduli’ are \( w^u_{\alpha(-1)}, \nu^0_{\alpha(-1)}, \nu^I_a w^{Iu} \) with \( u = 1, \ldots 4 \) \( U(4) \) and \( I = 1, 2, 3 \) \( SU(3) \) ‘pseudo’ flavor respectively. Notice that the subscript in parentheses represents the charge under \( U(1)_{k1} \). Taking into account the interactions with \( D3 \)-\( D3 \) excitations \( \Phi^{Iuv} = \phi^{Iuv} + ... \), the fermionic integration will lead to the determinant

\[
\Delta_F = \rho^8 e^{w_1 w_2 w_3 w_4} e^{u_1 u_2 u_3 u_4} e^{v_1 v_2 v_3 v_4} X_{u_1 u_2 v_1 v_2} X_{u_3 u_4 v_3 v_4} Y_{w_1 w_2} Y_{w_3 w_4}
\]

\[5\text{In string theory, SU(3) is an accidental symmetry of the two-derivative effective action}\]
with \( X = \epsilon^{IJK} \bar{\chi}_I \phi_J \phi_K \), \( Y_{uv} = \mathcal{U}^o_{uv} \mathcal{U}^\alpha_{\bar{\alpha}} \) and \( \rho, \mathcal{U} \) are defined by \( w_{u\bar{\alpha}} = \rho \mathcal{U}_{u\bar{\alpha}}, \bar{w}^{u\bar{\alpha}} = \rho \mathcal{U}^{u\bar{\alpha}} \), \( \bar{\mathcal{U}}^{u\bar{\alpha}} \mathcal{U}^\beta_{\bar{\beta}} = \delta^\beta_{\bar{\beta}} \). Integration over bosonic 'moduli' is more involved. For arbitrary choices of the v.e.v.'s \( \phi^{Iu} \) and \( \phi^{Iuv} \), even along the flat directions, integration over \( \mathcal{U} \) represents a difficult task. Fortunately for the choice \( \phi^{Iuv} = \eta^{Iuv} \), the full \( \phi \)-dependence can be factorized. And after rescaling \( \rho^2 \rightarrow \rho^{2/\mathcal{U}^{u\bar{\alpha}} \mathcal{U}^\beta_{\bar{\alpha}}}, \bar{\chi}_I \rightarrow \bar{\eta} \bar{\chi}_I \), \( \chi \rightarrow \bar{\eta} \bar{\chi}_I \), \( X_{u_1 v_1} \rightarrow \epsilon^{I_1 I_2 I_3} \bar{\chi}_I \bar{\eta}^{I_1 I_2 I_3} \), the \( \phi \)-independent integral \( I_B \) becomes

\[
I_B = \int d\rho d^2 U d^3 \bar{\chi} d^3 \chi \Delta F e^{-\tilde{S}_B}
\]

where \( \tilde{S}_B = -\rho^2 (1 + \eta^{Iuv} Y_{uv} \chi + \bar{\eta}^{Iuv} \bar{Y}^{Iuv} \bar{\chi} + \bar{\chi} I \chi I) \). Restoring the \( SU(4) \) gauge and \( SU(3) \) 'flavor' invariance the superpotential follows after promoting \( \phi \rightarrow \Phi \):

\[
S_W = c \int d^4 a d^2 \Theta \Phi^{I} e^{2\pi i (\mu)} I_B = \int d^4 x d^2 \theta \frac{\Lambda^9}{\det_3 \chi_3^{u_1 \ldots u_4} \Phi^{I_1 u_1 v_2} \Phi^{I_2 u_2 v_3}}
\]

up to a non-zero numerical constant.

### 2.6 Exotic/Stringy instantons

EDp-branes on unoccupied nodes of the quiver produce exotic instanton effects. The gauge theory on EDp' is of the same kind as on EDp (like 8 N-D directions, periodic sector).

Grassmann integration over chiral fermions \( \nu \)'s at intersections produces positive powers of \( \Phi \). The resulting non-perturbative superpotential can grow at large VEV's, which is incompatible with field theory intuition (asymptotic freedom) for standard 'gauge' instantons. Yet it is compatible with gauge invariance and 'exotic' scaling

\[
e^{-A(C')/\ell_s^{\prime+1}} \neq e^{-1/g_{YM}^2}
\]

For generic \( K \), there are many unlifted fermionic zero-modes and one gets higher derivative F-terms, threshold corrections, ... or dangerous bosonic zero-modes. For specific \( K \), there are only two unlifted zero-modes \( (d^2 \theta) \) and one gets superpotential terms. For ED1, the relevant \( \nu \)'s are in the direction of the worldsheet.

#### 2.6.1 \( U(4) \) model: non-perturbative masses

Let us consider "our" \( U(4) \) model, \( \Theta_\alpha^0, X_\mu \) plus 4 \( \nu^\mu \) that couple to one complex component \( \phi_{uv} \) (related to \( C \)) through

\[
S_{D3-ED3} = \phi_{uv} \nu^\mu \nu^\nu + ...
\]
The superpotential generated by ED-strings wrapping 2-cycles $C$ passing through the singularity schematically reads

$$W(\Phi) = \sum_C M_s e^{-A(C)/g_s\alpha'} \Phi_C^2$$

and thus represents a mass terms for $\Phi \approx A$.

Effect of multi-instanton are hard to evaluate ... Heterotic / Type I duality may help clarifying the procedure.

3 Third Lecture: Worldsheet vs D-brane instantons

3.1 Heterotic-Type I duality in $D \leq 10$

Perturbatively different string theories may be shown to be equivalent once non-perturbative effects are taken into account. Heterotic and Type I string theories with gauge group $SO(32)$, were conjectured in [13, 31], to be equivalent. In fact, up to field redefinitions, they share the same low-energy effective field theory. It was shown in [32] that for the equivalence to work, the strong coupling limit of one should correspond to the weak coupling limit of the other. In $D = 10$ the strong - weak coupling duality takes the following form [13, 31]

$$g_s^H = 1/g_s^I, \quad \alpha'_H = g_s^I \alpha'_I$$

where $g_s^H$, $\alpha'_H$ and $g_s^I$, $\alpha'_I$ are the heterotic and Type I coupling constants and tensions respectively. The simple strong-weak coupling duality $\phi_I = -\phi_H$ in $D = 10$ changes significantly in lower dimensions. Indeed, since the dilaton belongs to the universal sector of the compactification, the relation between the heterotic and Type I dilatons in $D$ dimensions is determined by dimensional reduction to be [33], [34]

$$\phi_I^{(D)} = \frac{(6 - D)}{4} \phi_H^{(D)} - \frac{(D - 2)}{16} \log \det G_H^{(10-D)}$$

where $G_H^{(10-D)}$ is the internal metric in the heterotic-string frame, and there is a crucial sign change at $D = 6$ where $\phi_H$ and $\phi_I$ are independent [35]. It is well known that Type I models exist with different number of tensor multiplets in $D = 6$ [36, 37]. This does not have an analogue in perturbative heterotic compactifications on $K3$. In $D = 6$, the Type I dilaton belongs to a hypermultiplet to be identified with one of the moduli of the $K3$ compactification on the heterotic side. In four dimensional $\mathcal{N} = 1$ models on both sides the dilaton appears in a linear multiplet, and heterotic-type I duality is related to

\[\text{Similar situations in which the strong coupling limit of one string theory is the weak coupling of another 'dual' string theory were discussed earlier by Duff [38].}\]
chiral-linear duality. The presence of anomalous $U(1)$’s under which R-R axions shift suggests that the latter correspond to changed scalars on the heterotic side.

Heterotic-Type I duality requires that the heterotic fundamental string and the Type I D-string be identified. The massless fluctuations of a Type I D-string are eight bosons and eight negative chirality fermions in the D1-D1 sector together with 32 positive chirality fermions in the D1-D9 sector. Thus, the world-sheet of the D-string exactly matches the world-sheet of the Heterotic fundamental string. By the same token, the Type I D5-brane should be identified with the heterotic NS5-brane. The latter is a soliton of the effective low-energy heterotic action and its microscopic description is not fully understood. The tensions agree in the two descriptions since $T_{\text{NS5}} = 1/g_H^2(\alpha'_H)^3 \equiv 1/g_I(\alpha'_I)^3 = T_{D5}$.

$SO(32)$ Heterotic / Type I duality has been well tested in $D = 10$ and in toroidal compactifications. In $D = 10$ BPS-saturated terms, like $F^4$, $F^2R^2$ and $R^4$, are anomaly related and match in the two theories as a consequence of supersymmetry and absence of anomaly. In toroidal compactifications, the comparison of BPS-saturated terms becomes more involved. The spectra of BPS states become richer and differ on the two sides at the perturbative level.

Non-perturbative corrections to $F^4$, $F^2R^2$ and $R^4$ terms are due to instantons that preserve half of the supersymmetry. In the heterotic string they get perturbative corrections at one loop only and the NS5-brane is the only relevant non-perturbative configuration in $D \leq 4$. Instanton configurations can be provided by taking the world-volume of the NS5-brane to be Euclidean and to wrap supersymmetrically around a compact manifold, so as to keep finite the classical action. This requires at least six-dimensional compact manifold. Therefore, BPS-saturated terms do not receive non-perturbative corrections for toroidal compactifications with more than four non-compact directions. Thus, the full heterotic result arises from tree level and one loop for $D > 4$. In the Type I string both D1- and D5-branes can provide instanton configurations after Euclideanization. D5-brane will contribute in four or less noncompact dimensions, D1-brane can contribute in eight or fewer noncompact dimensions. Thus, in nine dimensions the two theories can be compared in perturbation theory. In eight dimensions the perturbative heterotic result at one-loop corresponds to perturbative as well as nonperturbative Type I contributions coming from the D1-instanton via duality. The heterotic results can be expanded and the Type I instanton terms can be identified. The classical action can be written straightforwardly and it matches with the heterotic result. The determinants and multi instanton summation can also be performed in the Type I theory. In general, world-sheet instantons in heterotic string duals of Type I models help clarifying the rules for multi-instanton calculus with unoriented D-branes. Two prototypical examples are the $T^4/Z_3$ orbifold to $D = 4$, that we have already encountered [33], and the $T^6/Z_2$ orbifold to $D = 6$ [36], that we are going
to discuss in the following.

### 3.2 Compactification on $T^4/Z_2$ to $D = 6$

#### 3.2.1 Type I description

The Type I theory is an un-oriented projection of the Type IIB theory. Upon compactification on $T^4/Z_2$ to $D = 6$, the Type IIB theory has $\mathcal{N} = (2,0)$ spacetime supersymmetry with 16 supercharges, i.e. those satisfying $Q = RQ$, where $R$ denotes the inversion of the four coordinates of $T^4$. The $\Omega$ projection preserves only the sum of left- and right-moving supersymmetries $Q_\alpha + \tilde{Q}_\alpha$. The $\Omega R$ projection preserves the same linear combination since $Q_\alpha + R\tilde{Q}_\alpha \equiv Q_\alpha + \tilde{Q}_\alpha$. The massless little group in six dimensions is $SO(4) = SU(2) \times SU(2)$. The massless bosonic content of the unoriented closed string spectrum contains in untwisted NS-NS sector $(3,3)^{+11}(1,1)$, in the untwisted R-R sector $(3,1)^{(1,3)}+6(1,1)$, in the twisted NS-NS sector there are 48 $(1,1)$ and in the twisted R-R sector 16 $(1,1)$. This is exactly the bosonic content of the $D = 6 \mathcal{N} = (1,0)$ supergravity coupled to one tensor and 20 hypermultiplets.

Let us now discuss the unoriented open string spectrum. Tadpole cancellation conditions imply that the total Chan-Paton dimensionalities of twisted and untwisted sectors both equal to 32. The $U(16)_9 \times U(16)_5$ model, which arises at the maximally symmetric point, where all the D5-branes are on top of an $\Omega_5$-plane and no Wilson lines are turned on the D9-branes, is of particular interest. This model was first discussed by Bianchi and Sagnotti and later by Gimon and Polchinski in [36]. The D9-D9 sector contributes a vector multiplet in the adjoint of $U(16)_9$ and hypermultiplets in the $120^+ + 120^-$. The D5-D5 gives a vector multiplet in the adjoint of $U(16)_5$ and the hypermultiplet in the $120^+ + 120^-$. In the D5-D9 ‘twisted’ spectrum there are half-hypers in the $(16^+_{+1}, \bar{16}^-_{-1}) + (16^+_{-1}, \bar{16}^-_{+1})$ of $U(16)_9 \times U(16)_5$.

#### 3.2.2 Compactification on $T^4/Z_2$ to $D = 6$: Heterotic description

The Type I model corresponds to a compactification without vector structure [39], [40]:

$$\tilde{\omega}^{SW}_{2,YM} \neq 0 \approx B^{NS-NS}_2 = 1/2 (\text{mod } 1)$$

where $\tilde{\omega}^{SW}_{2,YM}$ is modified second Stieffel-Whitney class (obstruction to vector structure).

The $Z_2$ orbifold (besides its geometrical action) acts on the 32 heterotic fermions as $\lambda^{A}_{ws} \rightarrow (i)\lambda^u_{ws}, (-i)\lambda^{\bar{u}}_{ws}$, which breaks the gauge group $SO(32)$ to $U(16)$. The resulting

---

\[\text{In the sense that the 4 N-D directions have half-integer bosonic modes.}\]
massless spectrum is as follows. In the untwisted sector we have four neutral hypers, charged hypers in $120 + 120^\ast - 2$, vector in the adjoint, one tensor and the $\mathcal{N} = (1, 0)$ supergravity multiplet.

The twisted sector (16 fixed points) does not contain neutral hypermultiplets, it has charged half hypermultiplets in $16 - 3 + 16^\ast + 3$.

### 3.2.3 Matching the spectrum

In order to match the massless spectrums of the two descriptions one has to distribute one ‘fractional’ D5-brane per each fixed point, thus breaking the D5-brane gauge group $U(16)_5 \rightarrow U(1)^{16}$ [40].

In six dimensions the full gauge plus gravitational anomaly can be written as

$$\mathcal{I}_8 = \sum_i \left( X_i^i \wedge X_i^6 + X_i^4 \wedge \tilde{X}_i^4 \right)$$

The GSS counterterm reads as $L_{GSS} = C_2^{RR} X_4 + \sum_f C_{0,f}^{RR} X_f^4$, so that Type I photons become massive by eating twisted RR axions: $\partial C_{0,f}^{RR} \rightarrow DC_{0,f}^{RR} = \partial C_{0,f}^{RR} + 4 A_{(9)}^{(0)} + A_{(5)}^{(9)}$. The Type I combination $A_I = A_{(9)} - 4 \sum_f A_{f}^{(5)}$ decouples from twisted closed string scalars and matches with the heterotic photon $A^H$. The vector multiplets get massive by eating neutral closed string hypers. Thus we have a supersymmetric Higgs-like mechanism: full hypers are eaten.

### 3.3 Duality and dynamics in $D = 6$

In order to further test the correspondence and gain new insights into multi D-brane instantons, we are going to consider a four-hyperini Fermi type interaction that is generated by instantons and corresponds to a ‘chiral’ (1/2 BPS) coupling in the $\mathcal{N} = (1, 0)$ low energy effective action. If the four hyperini are localized at four different fixed points, this coupling is absent to any order in perturbation theory. This is so, because twisted fields at different fixed points do not interact perturbatively. ED1-brane or worldsheet instantons which connect the four fixed points can generate such a term. The contributions will be exponentially suppressed with the area of the cycle wrapped by the instanton.

Let us mention what kind of corrections one expects in the two descriptions before describing the computation. In $D = 6$ Heterotic / Type I duality implies

$$\phi_H = \omega_i \quad , \quad \phi_i = \omega_H$$

\[8\] This is an efficient, not fully exploited mechanism for moduli stabilization even in $D = 4$. The remnant of the $D = 6$ anomaly in $D = 4$ is massive ‘non-anomalous’ $U(1)$’s [12].
where $\phi$ is the dilaton and $\omega$ is the volume modulus. Supersymmetry implies that there are no neutral couplings between vectors and hypers. The gauge couplings can only depend (linearly) on the scalar $\phi_{\mu} = \omega_i$ in the unique tensor multiplet, while $\phi_i = \omega_{\mu}$ belongs to a neutral hyper. For these reasons in the heterotic description the hypermultiplet geometry is tree-level exact, but may get worldsheet instanton corrections $e^{-h(C)/\alpha'}$, where neutral hypers $h$ determine the size of 2-cycles $C$ in $T^4/Z_2$. In the Type I description, hypers receive both perturbative (string loops) and non-perturbative corrections from BPS Euclidean D-string instantons wrapping susy 2-cycles $C$ in $T^4/Z_2$. The Type I gauge couplings are completely determined by disk amplitudes. In the heterotic string, they receive (only) a one-loop correction.

### 3.4 Four-hyperini amplitude

#### 3.4.1 Computational strategy

Let us summarize our strategy:

- Focus on a specific 4-hyperini amplitude

$$A_{\text{4hyper}}^{f_1 f_2 f_3 f_4} = \langle V_{16}^\zeta f_1 V_{16}^\zeta f_2 V_{16}^\zeta f_3 V_{16}^\zeta f_4 \rangle$$

absent at tree level for particular choices of fixed points

- Compute $A_{\text{4hyper}}^{f_1 f_2 f_3 f_4}$ in the limit of vanishing momenta

- Start with heterotic string, where it is tree level exact and extract worldsheet instanton corrections

- Translate into Type I language and interpret the result in terms of perturbative and non-perturbative contributions

- Learn new rules for unoriented multi D-brane instantons

#### 3.4.2 Heterotic description

To compute the four-hyperini Fermi interaction in the heterotic description we need the hyperini vertex operators

$$V_{16/16^*}^\zeta = \zeta_u^{u/\bar{u}}(p)S^a e^{-\varphi/2}(z)\tilde{\Sigma}^{u/\bar{u}}(\bar{z})\sigma_f e^{ipX}(z, \bar{z})$$

where $\sigma_f$ is the bosonic $Z_2$-twist field ($h = 1/4$), $\tilde{\Sigma}^{u/\bar{u}} = e^{\pm i\phi_u} \prod_v e^{\pm i\phi_v/4}$ are twisted ground-states ($h = 3/4$) for heterotic fermions $\lambda_u^{u/\bar{u}}$, $S^a$ are $SO(5,1)$ spin fields, $\varphi$ and $\tilde{\phi}_u$
are the bosonizations of the superghost and $SO(32)$ gauge fermions respectively. One can use $SL(2, C)$ invariance on the sphere to set $z_1 \to \infty$, $z_2 \to 1$, $z_3 \to z$, $z_4 \to 0$ with cross ratio $z = z_{12}z_{34}/z_{13}z_{24}$. Then the string amplitude will depend on the $SL(2, C)$ invariant cross ratio $z$.

The $Z_2$-twist field correlator is given by [13]

$$\langle \prod_{i=1}^{4} \sigma_{f_i}(z_i, \bar{z}_i) \rangle \to |z_\infty|^{-1} \Psi_{qu}(z, \bar{z}) \Lambda_{cl} \left[ \frac{f_{i_2}}{f_{i_3}} \right] (z, \bar{z})$$

The quantum part $\Psi_{qu}$ is independent of the twist-fields locations i.e. of the choice of 4 out 16 fixed points $\bar{f}_i = 1/2(\epsilon_i^1, \epsilon_i^2, \epsilon_i^3, \epsilon_i^4)$ with $\epsilon_i^a = 0, 1$ and in order to get a non-trivial coupling the $\bar{f}_i$ should satisfy $\sum_i \bar{f}_i = 0 \mod \Lambda(T^4)$. $\Lambda_{cl} = \sum e^{-S_{inst}}$ is the classical part accounting for worldsheet instantons depending on the relative positions $\bar{f}_{ij} = \bar{f}_i - \bar{f}_j$. The $Z_2$-twist field correlator can be mapped to the torus doubly covering the sphere with two $Z_2$ branch cuts using the relation between the cross-ratio $z$ and the Teichmüller parameter of the torus $\tau(z)$

$$z = \frac{\partial_4^1(\tau)}{\partial_4^1(\tau)}.$$

The quantum and classical parts of the 4-twist correlator read

$$\Psi_{qu}(z, \bar{z}) = 2^{-8/3} |z(1-z)|^{-1/3} \tau_2^{-2} |\eta(\tau)|^{-8} \Lambda_{cl} \left[ \frac{f_{i_2}}{f_{i_3}} \right] (z, \bar{z})$$

where $G_{ij}$ is the metric and $B_{ij}$ is the antisymmetric tensor of $T^4/Z_2$ (neutral hyps).

Writing the $z$-integral as integral over the torus modulus $\tau$ (for $s, t \to 0$) one finds

$$\mathcal{A}_{u_1u_2u_3u_4}^{f_1, f_2, f_3, f_4} = \mathcal{V}(T^4) \int_{\mathcal{F}_2} \frac{d^2 \tau}{\tau_2} \left( \frac{\partial_4^1}{\partial_3^4} \delta_{u_1u_2} \delta_{u_3u_4} - \frac{\partial_4^1}{\partial_2^4} \delta_{u_1u_4} \delta_{u_3u_2} \right) \Lambda_{cl} \left[ \frac{f_{i_2}}{f_{i_3}} \right].$$

The integral goes over the fundamental domain $\mathcal{F}_2$ of the index 6 subgroup $\Gamma_2$ of $SL(2, Z)$, leaving invariant $\vartheta_{even}$ [14]. The region $\mathcal{F}_2$ can be decomposed into 6 domains each of which is an image of the fundamental domain $\mathcal{F}$ of $SL(2, Z)$ under the action of the 6 elements of $SL(2, Z)/\Gamma_2$

$$\int_{\mathcal{F}_2} \frac{d^2 \tau}{\tau_2} \Phi(\tau, \bar{\tau}) = \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2} \sum_{s=1}^{6} \Phi(\tau_s, \bar{\tau}_s)$$

where $\tau_s = \gamma_s(\tau)$, $\gamma_s = \{1, S, T, TS, ST, TST\}$. For the 4-hyperini amplitude one gets

$$\Phi(\tau, \bar{\tau}) = \left( \frac{\partial_4^1}{\partial_3^4} \delta_{u_1u_2} \delta_{u_3u_4} - \frac{\partial_4^1}{\partial_2^4} \delta_{u_1u_4} \delta_{u_3u_2} \right) \Lambda_{cl} \left[ \frac{f_{i_2}}{f_{i_3}} \right].$$
In the special case when all 4-hyperini are located at the same fixed point \( \vec{f}_{12} = \vec{f}_{13} = (\vec{0}) \), the amplitudes receive contribution only from BPS-like modes as in Type I (see later). The instanton sum \( \Lambda_{cl} \) is modular invariant. Sums over 6 images produce

\[
\sum_{s=1}^{6} \frac{\bar{j}_4^4}{\bar{j}_s^4}(\tau_s) = 3, \quad \sum_{s=1}^{6} \frac{\bar{j}_2^4}{\bar{j}_s^2}(\tau_s) = -3
\]

and the final expression for the amplitude with \( \vec{f}_1 = \vec{f}_2 = \vec{f}_3 = \vec{f}_4 \) is given by

\[
\mathcal{A}_{1113} = 3(\delta_{u_1u_2} + \delta_{u_3u_4}) \mathcal{V}(T^4) \int_{\mathcal{F}} \frac{d^2\tau}{\tau^2} \Lambda_{cl} \left[ \frac{0}{0} \right].
\]

Next consider the case when hyperini are located in pairs at two different fixed point:

- \( \vec{f}_{12} = \vec{f}_{13} = \vec{f} \) for \( \delta_{u_1u_2}\delta_{u_3u_4} \) structure in 4-hyperini amplitude

\[
\mathcal{A}_{1113} = \mathcal{V}(T^4) \int_{\mathcal{F}} \frac{d^2\tau}{\tau^2} \left( \Lambda_{cl} \left[ \frac{f}{0} \right] + \Lambda_{cl} \left[ \frac{0}{0} \right] + \Lambda_{cl} \left[ \frac{0}{0} \right] \right).
\]

- \( \vec{f}_{12} = \vec{0}, \vec{f}_{13} = \vec{h} \) for \( \delta_{u_1u_4}\delta_{u_3u_2} \) structure in 4-hyperini amplitude

\[
\mathcal{A}_{1113} = \mathcal{V}(T^4) \int_{\mathcal{F}} \frac{d^2\tau}{\tau^2} \left( \Lambda_{cl} \left[ \frac{0}{h} \right] + \Lambda_{cl} \left[ \frac{0}{0} \right] + \Lambda_{cl} \left[ \frac{0}{0} \right] \right).
\]

These are the same integrals as for BPS saturated thresholds to \( F^4 \) in \( T^4 \) compactifications (with shifts) \[14\]. Since the pieces proportional to \( \delta_{u_1u_2}\delta_{u_3u_4} \) and \( \delta_{u_1u_4}\delta_{u_3u_2} \) are related by a simple relabeling of the fixed points \( f_i \)'s, we have restricted our attention onto the amplitude with color structure \( \delta_{u_1u_2}\delta_{u_3u_4} \) for the first and the amplitude with color structure \( \delta_{u_1u_4}\delta_{u_3u_2} \) for the second case. Performing Poisson resummation over \( \vec{m} \) in \( \Lambda_{cl} \) one finds

\[
\Lambda_{cl} \left[ \frac{0}{0} \right] = \frac{\tau_2^2}{\mathcal{V}(T^4)} \sum_{k,i} (-)^{2f_k^2} q_k^{3/2} \bar{q}_i^{3/2}
\]

where \( \vec{p}_{L/R} = \frac{1}{\sqrt{2}} (E^{-1}k + E\vec{m}) \), \( EE^t = G \) and we have set \( B = 0 \) for simplicity. One can recognize the shifted orbifold partition function

\[
\Lambda_{cl} \left[ \frac{0}{0} \right] (G) + \Lambda_{cl} \left[ \frac{f}{0} \right] (G) + \Lambda_{cl} \left[ \frac{0}{f} \right] (G) = 2\Lambda_{cl} \left[ \frac{0}{0} \right] (G_{\vec{f}}) - \Lambda_{cl} \left[ \frac{0}{0} \right] (G).
\]

The toroidal metric \( G_{\vec{f}} \) is ‘halved’ along the direction \( \vec{v} = 2\vec{f} \) by \( SO(d,d) \) transformation. This is similar to what one gets for the threshold corrections to \( F^4 \) terms in toroidal compactifications, reviewed in Sect. 5.1.
3.4.3 Type I description

As we already noticed, in the Type I description hypers can receive both perturbative and non-perturbative corrections since the dilaton belongs to a hypermultiplet. Some scattering amplitudes may vanish in perturbation theory and receive only contributions from non-perturbative effects. The four-hyperini Fermi interaction term does not get perturbative contributions for \( f_i \) all different from one another or for \( f_1 = f_3 \) (same charge). When all fixed points \( f_i \) are equal or are equal in pairs \( f_1 = f_2 \) or \( f_2 = f_3 \) (opposite charge) there is a perturbative correction which matches with the contribution of the degenerate orbit in the heterotic description.

The open string vertex operators are given by

\[
V_{16}^f = \zeta_a(p) S^a e^{-\varphi/2} \sigma_f e^{ipX} \Lambda_u^f \\
V_{16}^{\xi} = \zeta_a(p) S^a e^{-\varphi/2} \sigma_f e^{ipX} \tilde{\Lambda}_u^f
\]

They involve Chan-Paton matrices \( \Lambda_u^f \) in the bifundamental of \( U(16) \times U(1) \) rather than heterotic fermions \( \lambda \) yielding

\[
Tr(\Lambda_{u_1}^f \Lambda_{u_2}^f \Lambda_{u_3}^f \Lambda_{u_4}^f) = \delta_{f_1 f_2} \delta_{f_3 f_4} \delta^{u_1 u_4} \delta^{u_3 u_2} + \delta_{f_1 f_4} \delta_{f_2 f_3} \delta^{u_1 u_2} \delta^{u_3 u_4}.
\]

Consider \( Z_2 \)-twist field correlator for open strings with 4 N-D boundary conditions \[45\]. The quantum part of the 4-twist correlator, which is independent of location \( f_i \) of twist fields, is given by

\[
\Psi_{qu} = [x(1-x)]^{-1/3} t(x)^{-2} \eta(it)^{-4}
\]

where \( x = \partial_3^4(it)/\partial_4^4(it) \) is \( SL(2, R) \) invariant ratio with \( t \) the modular parameter of the annulus doubly covering the disk. The classical part from exchange of (massive) open string modes stretched between (different) fixed points is

\[
\Lambda_{cl}^{f_{12} f_{13}} = \delta_{f_1 f_2} \delta_{f_3 f_4} \sum_{\tilde{n}} e^{-\pi t(\tilde{n} + \tilde{f}_{12})^t G(\tilde{n} + \tilde{f}_{12}) + \delta_{f_1 f_4} \delta_{f_2 f_3} \sum_{\tilde{n}} e^{-\pi t(\tilde{n} + \tilde{f}_{12})^t G(\tilde{n} + \tilde{f}_{12})}.
\]

Plugging into the open string amplitude and taking the limit \( s, t \to 0 \) one finds a perfect agreement with heterotic degenerate orbits, which is independent of \( B_2^H \approx C_2^{RR} - R \) but only on \( G \) and \( \phi \) (recall \( \omega_H = \phi_I, \phi_H = \omega_I \)). Terms involving \( B_2^H \) have no disk counterpart in the Type I description since the dual \( C_2^{RR} \) couples to (E)D-strings.

3.4.4 ED-string corrections

We then consider non-perturbative corrections to the four-hyperini coupling in the Type I description. When the four hyperini are located at the different fixed points \( f_i \) we have only non perturbative contribution from (regular) ED-strings wrapping supersymmetric
(untwisted) two cycles \( C \approx T^2/Z_2 = S^2 \), passing through the four fixed points. Fractional ED-strings wrapping the 16 collapsed ‘rigid’ 2-cycles (since corresponding moduli are eaten by anomalous \( U(1)^{16}_5 \)) may contribute to amplitudes having also perturbative contributions.

Using by now the well established Heterotic / Type I duality we will deduce the ‘exact’ 4-hyperini amplitude and determine ED-string corrections. Then we interpret these corrections in terms of symmetric orbifold CFT [46].

Let us describe the spectrum of ED-strings. The instanton dynamics is governed by a gauge theory describing the excitations of unoriented strings connecting E1, D5 and D9 branes. Three sectors of open string excitations are

- E1-E1 strings (2 N-N, 8 D-D): \( X^I, S^a, \bar{S}^\dot{a} \) with \( I = 1, \ldots , 8_a, a, \dot{a} = 1, \ldots , 4 \)
- E1-D9 strings (2 N-N, 8 N-D): \( \lambda^u, \bar{\lambda}^\bar{u} \) with \( u, \bar{u} = 1, \ldots , 16 \)
- E1-D5 strings (2 D-D, 8 N-D): \( \mu^f, \bar{\mu}^\bar{f} \) with \( f = f_1, f_2, f_3, f_4 \)

Alternatively, after T-duality along the wrapped 2-cycle we will have E1 \( \rightarrow E(-1) \), D9 \( \rightarrow D7_9 \), D5 \( \rightarrow D7_5 \). The residual (super)symmetry of the spectrum is

\[ \mathcal{N} = (8, 0) \rightarrow \mathcal{N} = (4, 0). \]

And the spacetime symmetry breaks according to

\[ SO(9, 1) \rightarrow SO(5, 1) \times SU(2) \times SU(2) \rightarrow SO(5, 1) \times SO(2)_E \times SO(2). \]

\( \mathcal{N} = (4, 0) \) gauge theory in IR flows to symmetric product CFT

\[ (H^6 \times T^4/Z_2)^k/S_k. \]

ED-string wraps two-cycle \( C \) inside \( T^4/Z_2 \) which is specified by the two vectors \( M_k = (k_1, k_2) \) each made out of four integers with greatest common divisor 1. \( \tilde{k}_{1,2} \) show how many times the two 1-cycles of \( C \) wrap around 1-cycle of \( T^4/Z_2 \).

Heterotic vertex operator can be derived from interaction term with hyperino:

\[ \mathcal{L}_{4F} = (\zeta^a_J)_{\mu_i} \chi^I \lambda^a u^a = V^H_\zeta \]

Only \((\ell)^m\)-twisted sectors (with \( m \ell = k \) and \( Z^s_\ell \) projection) with exactly four fermionic zero modes of \( S^a \) contribute. So, we can fold the \( k \) copies of fields and form a single field on a worldsheet with the following Kahler and complex structures:

\[ T(M) = k T(M_k) \quad , \quad U(M) = \frac{m U(M_k) + s}{\ell} \]

24
where \( M = M_k \begin{pmatrix} l & s \\ 0 & m \end{pmatrix} \). This is in perfect agreement with the heterotic result \( \mathcal{I}^{n_{deg}}_{d,d} \) for the four-hyperino coupling on \( T^4/Z_2 \).

This is the generalized Hecke transform as in the Heterotic computation!

### 3.5 Summary and outlook

Let us summarize the content of our lectures

- There are two classes of unoriented D-brane instantons:
  - ‘Gauge instantons’ may generate a VY-ADS-like superpotential of the form
    \[
    W \approx \frac{\Lambda^\beta}{\phi^{\beta-3}}
    \]
    where \( \beta \) is the one-loop coefficient in the expansion of the \( \beta \) function and \( \Lambda^\beta = M_s^\beta e^{-T(C)} \).
  - ‘Exotic instantons’ may generate a non-perturbative superpotential of the form
    \[
    W \approx M_s^{\beta-n} e^{-S_{E_D,T'}(C')} \phi^n \quad (n = 0, 1, ...)
    \]
    The thumb rule is the existence of exactly two unlifted fermionic zero-modes.
    We illustrated this situation with \( T^6/Z_3 \).

- Combining the two kinds of superpotentials one can achieve (partial) moduli stabilization and SUSY breaking! The same may happen when only one kind of superpotential is generated in the presence of fluxes and other dynamical effects, such as FI terms [47]...

- When extra zero-modes are present, threshold corrections to (higher-derivative) terms may arise. We illustrated this possibility for a compactification to \( D = 6 \) on \( T^6/Z_2 \), where a fully non-perturbative four hyperini amplitude (Fermi interaction) can be computed exploiting Heterotic - Type I duality.

  Threshold corrections to gauge couplings in freely acting orbifolds \( T^6/Z_2 \times Z_2 \) were computing by similar means.

- A by-product of the analysis in \( D = 6 \), an economical mechanism of moduli stabilization can be exploited whereby non-anomalous \( U(1)'s \) in \( D = 4 \) eat would-be hypers due to anomalies in \( D = 6 \).
The behaviour of D-brane instanton effects in the presence of fluxes or under wall crossing and the reformulation of (unoriented) D-brane instanton calculus in terms of localization are extremely active subjects. We hope the interested reader could consult part of the vast literature on the subject \cite{18, 19, 21, 23, 26, 27, 47, 1}.

There is a long way to go ... and a lot to learn on unoriented D-brane instanton.

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