THE COMPLEMENTARY ROLES OF INTERFEROMETRY AND ASTEROSEISMOLOGY IN DETERMINING THE MASS OF SOLAR-TYPE STARS

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ABSTRACT

How important is an independent diameter measurement for the determination of stellar parameters of solar-type stars? When coupled with seismic observables, how well can we determine the stellar mass? If we can determine the radius of the star to between 1% and 4%, how does this affect the theoretical uncertainties? Interferometry can provide an independent radius determination, and it has been suggested that we should expect at least a 4% precision on such a measurement for nearby solar-type stars. This study aims to provide both qualitative and quantitative answers to these questions for a star, such as our Sun, where seismic information will be available. We show that the importance of an independent radius measurement depends on the combination of observables available and the size of the measurement errors. It is important for determining all stellar parameters and in particular the mass, where a good radius measurement can even allow us to determine the mass with a precision better than 2%. Our results also show that measuring the small frequency separation $\nu_s$ significantly improves the determination of the evolutionary stage $\tau$ and the mixing-length parameter $\alpha$.

Subject headings: methods: numerical — stars: fundamental parameters (mass, age, initial hydrogen abundance, initial metal abundance, mixing-length parameter) — stars: interiors — stars: oscillations — techniques: interferometric

1. INTRODUCTION

Asteroseismology is the interpretation of a star’s oscillation frequency spectrum in order to characterize its internal structure. By probing its interior, we are testing our knowledge of fundamental physics. A star with oscillations, such as our Sun (solar-type star), presents a frequency spectrum with a range of excited modes. Because most of the proper modes are excited, mode identification for each frequency is considerably easier than for other types of pulsating stars (Kjeldsen & Bedding 2004; Bedding et al. 2004; Kjeldsen et al. 2005).

Modeling these stars can be difficult, as there are often multiple sets of parameters that fit the observations. Also, if we use the frequencies to determine the mass of the star, then any solution we find may be incorrect, as it depends on the fit mass. If we could determine the mass of the star through some other means, then modeling the frequencies would be easier, and they could also then be used solely to characterize the internal structure of the star.

Unfortunately the precise determination of mass is normally possible only for detached components of similar mass in spectroscopic binary systems. For single stars we use observables, such as magnitudes and colors (apart from oscillation frequencies), to estimate the mass. However, now that there are various possibilities of measuring a radius for single stars through interferometry (Paresce et al. 2003; Glindemann et al. 2003; Kervella et al. 2003; Aufdenberg et al. 2006), the question then arises: Can radius be used to determine the mass of the star?

Recent studies have shown that interferometry has determined the radius of a star with a precision of about 1% for the brightest stars, and it has been suggested that we might obtain a precision of at least 4% for most nearby solar-type stars whose diameters we could expect to measure (Pijpers et al. 2003; Thévenin et al. 2005; Di Folco et al. 2004). Our interest is exploring how this independent measurement complements oscillation frequency information for solar-type stars. Can it determine the mass of the star? And if so, can we then use the frequencies to probe the physics of the star? How important is the radius for determining other stellar parameters, such as age and chemical composition? What is the impact on the expected uncertainties? Now that we are expecting frequency errors less than about 1.0 $\mu$Hz with the launch of COROT (Baglin et al. 2000) and Kepler (9 (Borucki et al. 1997, 2004; Basri 2004), does this have an effect on our results?

While the objective of this study is mainly to determine stellar mass, we also discuss results for other stellar parameters. Section 2 describes the mathematical background, the physics of the models, and the observables we use for this study. Section 3 discusses some theoretical results indicating the importance of a radius measurement. Section 4 presents results of simulations, which support the theoretical predictions.

2. DESCRIPTION OF THE PROBLEM

2.1. Mathematical Solution

The techniques incorporated for this study follow that of Brown et al. (1994) and Miglio & Montalbán (2005). For a more elaborate description we refer the readers to Brown et al. (1994) or Press et al. (1992) and summarize here the main concepts for the purposes of understanding our work.
2.1.1. Finding the Parameters

Taylor’s theorem allows us to approximate any differentiable function near a point \( x_0 \) by a polynomial that depends only on the derivatives of the function at that point. To first order this can be written as

\[
f(x) = f(x_0) + f'(x_0)(x - x_0),
\]

where \( x = \{x_i\}_{i=1}^{N} \) are the \( N \) parameters defining the system (“input parameters”) and \( f = \{f_i\}_{i=1}^{M} \) are the \( M \) expected outputs of the system that depend on \( x \). These are the expected measurements or “observables.” To distinguish between the expected observables and real observations we denote the latter by \( O = \{O_i\}_{i=1}^{M} \).

We measure \( f \) and would like to find the set \( x \) that produces these measurements. This problem is a typical inverse problem whose solution \( x \) (in the pure linear case) falls neatly out from

\[
x = x_0 + VW^{-1}U^T \delta f,
\]

where

\[
\delta f = \frac{O - f(x_0)}{\epsilon},
\]

\[
f'(x_0)\epsilon^{-1} = D = UWV^T
\]

is the singular value decomposition (SVD) of the derivative matrix (\( f' \), \( \epsilon \), and \( \delta f \)) and the measurement errors.

In the case of a nonlinear physical model, the method for finding the true values of \( x \) is to use a modified version of equation (2) iteratively. Incorporating a goodness-of-fit test, such as a \( \chi^2 \) function

\[
\chi^2 = \sum_{i=1}^{M} \frac{|f_i(x) - O_i|^2}{\epsilon^2},
\]

allows us to find the best set of parameters \( x \) by minimizing \( \chi^2 \). There are many reliable algorithms available for performing such minimizations (Press et al. 1992). The algorithm implemented in this work is the Levenberg-Marquardt algorithm, because it is known to be robust and incorporates derivative information. It also usually converges within two or three iterations to its minimum.

2.1.2. Singular Value Decomposition

SVD is the factorization of any \( M \times N \) matrix \( D \) into three components \( U, V^T, \) and \( W \) (eq. [4]). The matrix \( V^T \) is the transpose of \( V \), which is an \( N \times N \) orthogonal matrix that contains the input basis vectors for \( D \), or the vectors associated with the parameter space. The matrix \( U \) is an \( M \times N \) orthogonal matrix that contains the output basis vectors for \( D \), or the vectors associated with the observable space. The matrix \( W \) is a diagonal matrix that contains the singular values of \( D \).

We use SVD in our analysis because it provides a method to investigate the information content of our observables and their impact on each of the parameters. Equations (3) and (4) were defined in terms of the observed or expected error, so SVD can be used to study various properties of \( D \):

1. The expected uncertainties in each of the parameters via the covariance matrix

\[
C_{jk} = \sum_{i=1}^{N} \frac{V_{ji}V_{ki}}{W_i^2}.
\]

2. The significance of each of the observables for the determination of the parameter solution

\[
S_i = \left( \sum_{j=1}^{N} U_{ij}^2 \right)^{1/2}.
\]

2.2. Parameters and Observables

For this study we need to distinguish clearly between our parameters and observables. The “parameters” are the input ingredients \( x \) to our physical model. For example, in a stellar code the parameters include mass, age, and chemical composition. The “observables” \( f \) are then the outputs of the models given these \( x \) and are those things that we can normally measure, such as magnitudes, effective temperature, and oscillation frequencies. To refrain from ambiguity between the observable errors and the uncertainties in the parameters, we denote the former by \( \epsilon \) and the latter by \( \sigma \).

2.2.1. Parameters and Models

We describe a solar-type star with five adjustable parameters. These are mass \( M \), age (or evolutionary stage) \( \tau \), chemical composition given by two of \( X, Y, \) and \( Z \), where \( X + Y + Z = 1 \) and \( X, Y, \) and \( Z \) are the initial hydrogen, helium, and metal abundance, respectively, and the mixing-length parameter \( \alpha \), which describes convection in the outer envelope of the star.

We use the Aarhus stellar evolution code (ASTEC) for stellar structure and evolution and the adiabatic pulsation code (ADIPLS) to calculate nonradial oscillation frequencies (Christensen-Dalsgaard 1982). For each model evolved with the parameters \( M, \tau, X, Y, \) and \( Z \), ASTEC produces stellar quantities, which are then used to calculate the oscillation frequencies with ADIPLS. The outputs from the models implemented in this work are the global structure parameters, such as radius \( R_* \), effective temperature \( T_{\text{eff}} \), and a set of oscillation frequencies \( f_i \), \( \log g \), and \( [Z/X] \) are used to obtain spectra, which are then integrated in a range of wavelengths to produce magnitudes.

We fixed the physics of the stellar evolution models. The equation of state (EOS) is that of Eggleton et al. (1973). The opacities are the OPAL 1995 tables (Iglesias & Rogers 1996), supplemented by Kurucz opacities at low temperatures. Convection is described by the classical “mixing-length theory” (Bohm-Vitense 1958), where the mixing-length \( \ell \) is defined as \( \ell = \alpha H_p \), where \( \alpha \) is the free “mixing-length parameter,” and \( H_p \) is the local pressure scale height. Diffusion is not included, so we allow \( [M/H] \sim [Fe/H] \) to estimate \( [Z/X] \), where we are assuming a near-solar composition. We ignore overshoot effects and do not include Coulomb corrections. This study concentrates on one main-sequence model star with solar-type oscillations whose parameters are given in Table 1, but is extendable to a solar-type star with similar characteristics.

2.2.2. Observables

Typical observables for such a star come from various sources of observations. Spectroscopy provides effective temperature \( T_{\text{eff}} \), gravity \( \log g \), and metallicity \([M/H]\) measurements. With photometry we obtain magnitudes, e.g., \( V \) and colors, e.g., \((U-V)\). Oscillation frequencies are obtained using time-series observations of photometric and/or spectroscopic origin. Interferometry
measures a limb-darkened\textsuperscript{10} stellar angular diameter in milli-arcseconds, and this coupled with a parallax gives an absolute diameter value (we use the radius $R_*$ value for this work).

The observables $O_i$ and expected errors $\epsilon_i$ for our model star are given in Table 2. The measurement errors reflect what is most currently quoted in the literature. However, we do realize that some of these values are optimistic and are unlikely to be improved. In this way, we can investigate how much further interferometry and asteroseismology can take us for determining stellar mass.

We used frequency separations $\Delta \nu_{n,l}$ and $\delta \nu_{n,0}$ instead of individual frequencies $\nu_{n,l}$, where

\begin{align}
\Delta \nu_{n,l} &= \nu_{n,l} - \nu_{n-1,l}, \\
\delta \nu_{n,0} &= \nu_{n+1,0} - \nu_{n,2}.
\end{align}

While we realize that we are throwing away important information, the reasons for doing this are justified: the high sensitivity of the frequency values to changes in the parameters (nonlinearity of the problem) and the unreliability of the treatment of the stellar surface layers and hence absolute values of $\nu_{n,l}$.

We also assume that we can identify the $(l, n)$ quantum numbers. Rotation will induce a small effect (solar-type star), so we can assume $(l, m) \sim (l, m = 0)$. The degree $l$ can be interpreted by using echelle diagrams (see Fig. 6 in Kjeldsen et al. [2005]). It is only the radial order $n$ that is then difficult to identify. This number is model dependant, and there is no direct way of observing it. However, with a rich frequency spectrum expected of these stars, the relative $n$ position of each mode of degree $l$ can be identified and then also the average frequency separations $\Delta \nu$ and $\delta \nu$ (independent of $n$). These observables are used for the initial fitting process, and an inspection of the $\nu_{n,l}$ that come from the models allows an $n$ identification to $\pm 1$.

3. VARIOUS ROLES OF THE OBSERVABLES

Why should a diameter measurement, such as that expected from interferometry, be important? This section highlights the importance that a radius measurement has when coupled with various observables and different measurement errors.

3.1. Significance

The observables play very different roles in the determination of the parameter solution depending on which combination of observables ($O_i$) are available and the size of their measurement errors ($\epsilon_i$). Using equation (7) we calculated the significance of some observables, $S(O_i)$, using different observable combinations.

The top panel of Figure 1 illustrates the importance of some typical observables in the absence of a radius measurement and oscillation frequencies. The $\epsilon_i$ are those quoted from Table 2. While this panel is mainly for comparison with the bottom panel, we highlight a few points. The photometric observables appear to supply more information than the spectroscopic—this is contrary to what is often believed. For example, the colors are more significant than $T_{\text{eff}}$. The significance $S([M/H]) \sim 0.85$, where we may have assumed that most information about chemical composition comes from $[M/H]$, so it should be $\sim 1$. Also note the high values of both $S(\log g)$ and $S(V)$. They are responsible for determining both the radius and the mass of the star.

\begin{table}[h]
\centering
\caption{System Observables}
\begin{tabular}{lll}
\hline
Measurement & Value & Error (\epsilon_i) \\
\hline
$R_*$ ($R_\odot$) & 0.946 & \ldots \\
$T_{\text{eff}}$ (K) & 5421 & 50 \\
$\log g$ & 4.5 & 0.3 \\
$[\text{M/H}]$ & 0.00 & 0.05 \\
$M_\star$ (mag) & 4.50 & 0.05 \\
$(U-V)$ (mag) & 0.633 & 0.005 \\
$(V-R)$ (mag) & 0.400 & 0.005 \\
$\Delta \nu$ (\mu Hz) & 148.3 & \ldots \\
$\delta \nu$ (\mu Hz) & 14.6 & \ldots \\
$\nu_{12}$ (\mu Hz) & 1968.9 & \ldots \\
\hline
\end{tabular}
\end{table}

\begin{table}[h]
\centering
\caption{System Parameters}
\begin{tabular}{ll}
\hline
Parameter & Value \\
\hline
$M_\star$ ($M_\odot$) & 1.030 \\
$\tau$ (Gyr) & 1.00 \\
$X$ & 0.740 \\
$Z$ & 0.018 \\
$\alpha$ & 1.50 \\
\hline
\end{tabular}
\end{table}

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{fig1}
\caption{Top: Significance of the observables without oscillation information and radius measurements. Bottom: Same as the top panel, but including oscillation and radius measurements with $\epsilon(R) = 0.01 \, R_*$ and $\epsilon(\nu) = 1.3 \, \mu$Hz.}
\end{figure}

\textsuperscript{10} Information regarding the light intensity profile across the star is provided by the interferometric observations of the “second lobe of the visibility function.” In the case where the star is not fully resolved, i.e., we only get some visibility points on the first lobe, we require a limb-darkened model for the star, rendering $R_*$ model-dependent. The limb-darkened profile also varies if we use one- or three-dimensional atmospheric models. Both Aulendenberg et al. (2005) and Bigot et al. (2006) provide detailed and quantitative descriptions on this matter and also confirm the inadequacy of the one-dimensional models in reproducing these second-lobe measurements.
We then included \( R_* \), two \( \Delta \nu_{n,l} \), and one \( \delta \nu_{n,0} \) in the previous set of observables. Figure 1 (bottom) illustrates how the relative information content in each \( O_i \) changes. There are two notable changes: (1) \( \log g \) and \( V \) contain almost no information, while both the observed \( R_* \) and \( \Delta \nu_{n,l} \) become responsible for determining the true radius (and mass) of the star; and (2) \( S(V - R) \) and \( S([M/H]) \) decrease by a small amount, indicating that the information from colors and metallicity is not contained in the new observables. We believe this information is primarily the chemical composition. This implies that either colors or a metallicity are important observables to have when complementing seismic and radius measurements. We also note that while \( S(V - R) \) decreases by >25%, \( S(U - V) \) remains the same.

We included more than one \( \Delta \nu_{n,l} \) to highlight how when we change the measurement errors, the information contained in one observable is passed to another, in particular, between \( R_* \) and \( \Delta \nu_{n,l} \). If we had included just one \( \Delta \nu_{n,l} \), then \( S(\Delta \nu_{n,l}) \approx S(\delta \nu_{n,0}) \approx 1 \). When there are two observables contributing similar (but not the same) information, the content gets spread among them. Similarly, with three \( \Delta \nu_{n,l} \), \( S(\Delta \nu_{n,l}) \approx 0.55 \). Figure 2 illustrates this; \( S(\Delta \nu_{n,l}) \) (bold dashed line) and \( S(R_*) \) (bold solid line) are shown as a function of \( \epsilon(R) \) with \( \epsilon(\nu) = 0.5 \mu Hz \). At the smallest \( \epsilon(R) \), \( S(R_*) \approx 1 \). Increasing \( \epsilon(R) \) causes \( S(R_*) \) to decrease, while \( S(\Delta \nu_{n,l}) \) increases because it is taking over its role. The non-bold lines show the same, but for \( \epsilon(\nu) = 1.3 \) and 2.5 \( \mu Hz \). There is a similar but slower trend, because the radius remains more important than \( \Delta \nu_{n,l} \) even for higher values of \( \epsilon(R) \). Using only one \( \Delta \nu_{n,l} \) would not show an increase in \( S(\Delta \nu_{n,l}) \), because it would already near 1, but \( S(R_*) \) would decrease more slowly, while observables such as \( \log g \) and \( V \) would increase in importance.

Both Figures 1 and 2 show that the relative importance of observables depends not only on the individual \( \epsilon_i \) but also on the combination of \( O_i \) available, and that there is no straightforward relationship between a set of \( O_i \) and the parameters they are responsible for constraining.

### 3.2. Parameter Correlations

The combination of observables we use is important for the independence of the parameters. In general, there should be a unique set of frequencies for every possible combination of \( M, \tau, X, Z, \) and \( \alpha \). However, we are not using individual \( \nu_{n,l} \) but frequency separations for the reasons explained in §2.2.2. This creates some dependencies among the parameters, such as that between \( M \) and \( \tau, M \) and \( \alpha \), and \( M \) and \( X \). With the addition of measurement errors, this degeneracy between different parameter combinations becomes worse.

To illustrate some of the dependencies among the parameters, we calculated \( \chi^2 \) surfaces for two parameters \( p \) and \( q \), while keeping the other parameters fixed at their correct values. We evaluate 68.3%, 90%, and 99% confidence levels as defined for a two-dimensional \( \chi^2 \), i.e., \( \Delta \chi^2 = \chi^2_{\text{min}} - \chi^2_{\text{p}} = 2.31, 4.61, \) and 9.21, respectively. We used \( \epsilon(R) = 0.02 R_* \) and \( \epsilon(\nu) = 1.3 \mu Hz \) and defined \( \chi^2 = \chi^2/(\text{No. of observables} - 2) \).

Using a set of \( \nu_{n,l} \) as observables, the 99% confidence region encircled the correct parameter values to within less than 1% of each of the corresponding parameters (it produced a dot almost centered on its true value), supporting the idea that each set of parameters should produce a unique set of \( \nu_{n,l} \).

To show an example of the dependencies introduced by using the frequency separations, Figure 3 illustrates the correlated \( \chi^2 \) surface for mass and \( \alpha \) while holding \( \tau, X, Z \) at their true values using OS3 (see §3.3). Clearly for any value of \( M \) there is a corresponding \( \alpha \) that will give a minimum \( \chi^2 \) value. We hope that a minimization will lead to accurate values, but we may need additional observables to help constrain the solution.

#### 3.3. Propagated Errors

Can we get a precise determination of the mass using a radius measurement? How well do we need to know our observables? Will the effort of obtaining an interferometric diameter be outweighed by the benefits, or can some other observable produce a similar result?

To answer these questions, we investigated the expected parameter uncertainties through equation (6) for various observables with different errors. For the remainder of this paper we discuss three sets of observables,

\[
\begin{align*}
\text{OS1} &= \{ R_*, T_{\text{eff}}, [M/H], \Delta \nu_{n,l}, \delta \nu_{n,0} \}, \\
\text{OS2} &= \{ R_*, T_{\text{eff}}, [M/H], \Delta \nu_{n,l} \}, \\
\text{OS3} &= \{ R_*, T_{\text{eff}}, [M/H], \Delta \nu_{n,l}, \delta \nu_{n,0} \},
\end{align*}
\]

for \( n = 9, 10, \ldots, 27 \) and \( l = 0, 1, 2 \). For OS2 and OS3 we include a total of 21 large frequency separations, where we have assumed that we cannot identify every consecutive mode for each \( l \) and thus have even fewer frequency separations.

![Figure 2](image2.png)

**Fig. 2.** Change in \( S(R_*) \) and \( S(\Delta \nu_{n,l}) \) as we vary \( \epsilon(R) \). Note that \( \Delta \nu_{n,l} \) increases in significance as \( R_* \) becomes more poorly constrained. Bold lines indicate \( \epsilon(\nu) = 0.5 \mu Hz \); lighter lines indicate \( \epsilon(\nu) = 1.3 \) and 2.5 \( \mu Hz \).

![Figure 3](image3.png)

**Fig. 3.** Contour plot of \( \chi^2 \) for \( M \) and \( \alpha \), where \( X, Z, \) and \( \tau \) are fixed at their correct value. We used OS3 (see §3.3) with \( \epsilon(R) = 0.02 R_* \) and \( \epsilon(\nu) = 1.3 \mu Hz \) to calculate \( \chi^2 \).
we also include five small frequency separations. We remind readers that the parameter uncertainty is denoted by $\sigma$, while the measurement errors are denoted by $\epsilon$.

For ease of reading we refer to $\epsilon(\nu) = 0.5$ $\mu$Hz as LN and 1.3 $\mu$Hz as HN. Figure 4 shows the theoretical uncertainties on each of the parameters as a function of $\epsilon(R)$. We show results using LN (black lines) and HN (gray lines). The dashed lines show the results for OS1. The solid lines show OS2, and the dotted lines show OS3.

For all parameters we find that $\sigma(P)$ (parameter uncertainty) is a function of $\epsilon(R)$ for all combinations of observables. We also see that for $M$, $X$, and $Z$ the different observable combinations and errors do not impact significantly the theoretical uncertainties for $\epsilon(R) \leq 3\%$. Only for $\tau$ and $\alpha$, using OS2 instead of OS1 may not always be beneficial, because $\delta \nu$ is excluded from OS2, and this observable is essential for determining $\tau$ and, as it turns out, $\alpha$ too.

**Mass.**—Whatever combination we choose, we find that for $\epsilon(R) \leq 3\%$, $\sigma(M)$ does not change. A radius measurement will always be more important than seismic information once we reach a precision of this order. As the radius error begins to increase, its relative weight for mass determination decreases, while the relative weight of the seismic information increases. The quantitative differences between using LN and HN for each observable combination for $\epsilon(R) \geq 3\%$ supports the idea that seismic information is the dominant contributor to the reduction of the uncertainty in mass at higher $\epsilon(R)$.

**Age.**—Either a reduction in frequency error or an increase in observables significantly affects $\sigma(\tau)$. OS1 and OS3 do not benefit significantly from the radius measurement, but both produce better constraints on the age than OS2. This is precisely because they contain $\delta \nu$, which is fundamental in constraining this parameter. However, for OS2 a precise radius measurement reduces $\sigma(\tau)$ by $200\%$ for LN and HN. The lack of the observable $\delta \nu$ forces $R$ to constrain the age. Not surprisingly, we find that if we use LN instead of HN for all observable combinations, there is a large improvement in $\sigma(\tau)$.

**$X$.**—Adding observables and obtaining smaller frequency errors leads to subtle differences in $\sigma(X)$ for low radius error. OS1 with either LN or HN produce very similar constraints on $X$. Using OS2 does not change $\sigma(X)$ very much for $\epsilon(R) \leq 3\%$. At larger errors, we find that the additional large frequency separations (especially for LN) constrain $X$ to about $10\%$, an improvement of a factor of 2 over HN. OS3+HN and OS2+LN produce similar results, indicating that it is not the presence of $\delta \nu$ specifically that is important, just the addition of (seismic) observables.

**$Z$.**—This is one of the parameters that is least influenced by either radius or seismic observables. It is mainly constrained by [M/H], but because this implies knowing $X$, other observables play a role in constraining this parameter. Radius is most important for $Z$ when we use OS1, because we have less information available. Using OS2 and OS3 decreases the dependence of $Z$ on $R$. In fact, with OS3+LN, we find that the radius has almost no role in constraining $Z$. For $Z$, the radius is important only when there is not enough information available. As we increase the number of seismic observables, this information supersedes the role of the radius. We also find that whatever combination we use, $\sigma(Z)$ never decreases below $10\%$ even using LN.

**$\alpha$.**—The determination of $\alpha$ seems to be more complex than the determination of any other parameter. In general, the addition of observables (in number) coupled with better frequency errors leads to lower parameter uncertainties. This is not always the case for $\alpha$. OS2 is least influenced by a reduction in radius error. In fact, we find for HN, OS1 gives better constraints on $\alpha$ than OS2 for $\epsilon(R) \leq 4\%$. Only when an observable set contains the small frequency separation and for small values of radius error do we get a significant decrease in the theoretical uncertainty in $\alpha$. Using LN instead of HN always affects the determination of $\sigma(\alpha)$, but more so for OS2, when the radius information is less important than the frequency information.

The role of the radius sometimes depends on what other information is available. For example, if we seem to be missing information, then improving the radius error will lead to a corresponding reduction in the uncertainty of some parameter, like $Z$. The determination of $\alpha$ shows an interesting trend. A precise radius determination complements precise frequencies. Either of these alone...
does improve the uncertainty, but both together have a larger impact on \( \sigma(\alpha) \). Clearly for \( \epsilon(R) < 3\% \) the choice of observable set does not change \( \sigma(M) \); it is \( R_s \) that dominates the mass determination.

3.4. Magnitude Versus Radius

Are these uncertainty values worth the observing time? We compared the propagated error using OS2 with \( M_V \) (magnitude) instead of \( R_s \) (radius) to see whether \( M_V \) could produce results similar to \( R_s \); \( M_V \) is the only other measurement that might constrain the mass as \( R_s \) does.

Figure 5 shows \( \sigma(M) \) as a function of both \( \epsilon(R) \) (solid lines) and \( \epsilon(M_V) \) (dashed lines) for \( \epsilon(T_{\text{eff}}) = 50 \) and 230 K for LN (left) and HN (right). We have attempted to scale the expected errors on both observables so that the quoted errors are comparable in terms of measurement difficulty. We chose \( \epsilon(R) = 0.01 \) \( R_s \) (\(~1\%) to be similar to \( \epsilon(M_V) \) = 0.05 mag. For both cases \( \sigma(M) \) using \( M_V \) will never reach the precision that a radius measurement allows. The smallest mass uncertainty using \( M_V \) is \(~5\%) while if we use \( R_s \), it reaches below 1%. Also \( M_V \) needs to be complemented with other good observables, such as \( T_{\text{eff}} \) and LN, to get this precision, whereas \( R_s \) is a completely independent measurement that does not need the help of other observables. These are two very clear reasons for choosing to obtain an interferometric measurement of diameter. For HN (Fig. 5, right) it is even clearer that \( R_s \) is most important.

Even if our scaling is incorrect for the comparison of magnitude and radius errors, it is unlikely that an \( M_V \) measurement will have a precision of better than 0.05 mag, in which case the smallest mass uncertainty is really above 7%.

4. SIMULATIONS

To test the validity of the theoretical results presented in § 3.3 we performed simulations and tried to fit the observables to recover the input parameters. Part of this work also involved finding a reliable method to apply to any set of observables so that we could successfully estimate the parameters of the star within the theoretical uncertainties. We use two methods: direct inversions and minimizations. The direct inversions provide a test for the linear approximation of equation (1) by using an initial guess of the parameters \( x_0 \) and equation (2). We did this only for OS1. Minimizations iteratively incorporate a modified version of equation (2) (Levenberg-Marquardt algorithm) and were performed with OS1, OS2, and OS3.

The observables \( \mathbf{O} = \{ R_o, T_o, [M/H], \nu_{(o),o} \} \) were simulated as

\[
R_o = R_{re} + r_g \epsilon(R),
\]

\[
T_o = T_{re} + r_g \epsilon(T),
\]

\[
[M/H]_o = [M/H]_{re} + r_g \epsilon([M/H]),
\]

\[
\nu_{io} = \nu_{i, re} + r_g \epsilon(\nu) + \left( \frac{\nu_i}{\nu_0} \right)^2,
\]

where \( R = R_o, T = T_{\text{eff}}, \) the subscript “o” denotes the simulated observed value, the subscript “re” denotes the real value that comes directly from the stellar code, the \( \nu_i \) are individual frequencies, \( r_g \) is a random Gaussian error for each observable with \( \sigma = 1 \), and \( \nu_0 \) is an arbitrary reference frequency value, which was chosen to be 6000 \( \mu \)Hz. Using these simulated frequencies we calculated each of the \( \Delta \nu_{n,i} \) and \( \delta \nu_{n,0} \) from equations (8) and (9). Table 2 provides the errors, and \( \epsilon(\Delta \nu_{n,i}) = \epsilon(\delta \nu_{n,0}) = \left[ 2 \epsilon(\nu_i) \right]^{1/2} \).

4.1. Linear Approximation Versus Minimizations for OS1

Direct inversions allow us to investigate the natural weight that \( R_s \) and its error have on the determination of \( M \), because we do not use a minimization algorithm but just the linear approximation and equation (2). For each radius and frequency error, we generated a total of 100 simulations of the observables. We obtained a list of \( x_0 \) from a grid of models (§ 4.2.1) and chose the first set from this list to be the initial guess. For most of the simulations, an initial mass of \(~0.9 \) \( M_\odot \) was used (the real value =1.03). The system is relatively linear in these parameters and observables; hence, the inversion results should be independent of the initial values. (This was tested, and the results varied to less than 1%). The simulations were repeated for various frequency and radius errors.

Figure 6 shows the mean and standard deviation of the inverted (fit) mass from the simulations as a function of the radius error. The dotted lines show the real mass value. Each panel corresponds to a different \( \epsilon(\nu) \). The top panel shows LN, the middle
panel HN, and the bottom panel has $\epsilon(\nu) = 2.5 \, \mu Hz$. The dashed lines show the theoretical uncertainties.

The top and middle panels of Figure 6 show that the mass uncertainty is a clear linear function of the error in radius. In fact, $\epsilon(\nu)$ has a very modest effect on both the precision and the accuracy of the mass. For these two upper panels, there is no apparent difference, indicating that the radius primarily determines the mass. However, at $\epsilon(\nu) = 2.5 \, \mu Hz$ we do see an effect on both the precision and the accuracy of the results. This appears to be a side effect of poorly constraining the other parameters, which indirectly leads to incorrect mass estimates.

It is encouraging to find that the simulation results are consistent with theory and that we can safely estimate the mass to within its theoretical uncertainty. Unfortunately for $\epsilon(R) = 5\%$, safely estimating the mass to within 20% may not be very useful. However, we do find that for $\epsilon(R) \leq 2\%$ we are capable of determining the mass better than the theoretical uncertainty and quite accurately (for the two smaller frequency errors). For all simulation results, the true mass value always falls within 1 standard deviation of the fit mass.

For the other parameters, the theoretical predictions are consistent with the simulation results. Figure 7 shows the mean and the standard deviation of the inverted (fit) age. As expected, the size of the frequency error is very important, especially for OS1; $\epsilon(R)$ does have a modest effect and is most notable for LN (Fig. 7, top).

To test whether a minimization algorithm improves the uncertainties on the parameters, we repeated the simulations and then fit for the parameters (using the minimizing method) for a range of radius errors and LN. Figure 8 shows the standard deviations of the fit parameters [for age we show $\sigma(\tau)/4$]. All of the parameter’s real values were within the $\sigma$ shown in this figure.

The precision of the parameters are better than the theoretical ones (Figs. 4 and 8). There is a clear dependence of the uncertainties on the radius error when $\epsilon(R) < 3\%$. Unlike the theoretical uncertainties, for larger $\epsilon(R)$ there is no significant degradation in the precision of the parameters. This uncertainty-error dependence changes drastically for larger $\epsilon(R)$, because the minimization method is forcing a balance between the errors and the observations, unlike when we use the simple linear approximation (eq. [2]).

4.2. Simulations and Minimizations with OS2 and OS3

Since we are introducing more frequency observables in OS2 and OS3, we are effectively giving less weight to the radius...
measurement for constraining the parameters. The fit parameters are thus more sensitive to the initial parameter estimates. It is therefore necessary to use small grids of models to obtain a good initial guess and then let the minimization algorithm fit all of the observables.

4.2.1. Model Grids

We created grids of models that span the five-dimensional model space over a range of values that encompass about 10%–30% on either side of the parameter values given in Table 1. These values covered $0.95 \leq M \leq 1.09 \, M_\odot$, $0.1 \leq \tau \leq 2.0 \, \text{Gyr}$, $0.68 \leq X \leq 0.76$, $0.014 \leq Z \leq 0.022$, and $1.2 \leq \alpha \leq 1.8$. We divided each of the parameter ranges into five (10 for mass), calculated models for each of these points, and then interpolated only along the mass for a much finer grid. We are restricting the range in mass, age, and $Z$, but we assume that we can constrain the parameters within this range from the observables.

We later extended the mass range to be between 0.75 and 1.20 $M_\odot$ to investigate whether our original parameter range was too restricted. Using $\tau = 0.1$, $X = 0.70$, $Z = 0.024$, and $\alpha = 1.5$ and varying mass, we used the grids to see how well we could recover the mass in this extended mass range. These parameters were chosen to test for any biases introduced by selecting our original ranges to be nearly symmetric about the correct values. We tested this for $\epsilon(R) = 0.02 \, R_\odot$ and HN only. There were 100 simulations for each mass.

We searched through the grids to find all of the parameter combinations that reproduced $R_*$ and $\Delta \nu$ to within $\pm \epsilon$. We discarded the combinations with $\chi^2$ larger than 1 and adopted the mean value of each parameter from the remaining combinations to be the fit values. We later used them as the initial parameter values for the minimization. Figure 9 shows the mean of the simulations $\pm 3 \sigma$, where $\sigma$ is the standard deviation of each group of simulations. The dotted line indicates the ideal position for the mean of the simulation results. While there seems to be some systematic offset as a function of mass, in most cases the $3 \sigma$ error bar crosses the correct value. For the other parameters, the fit values are not as well determined (we have not interpolated for them). This appears to be responsible for the offset in fit mass.

4.2.2. Results of Minimizations

Initial estimates of the parameters were obtained from the model grids. We added a constant of 0.02 to the initial estimate of $X$ for two reasons: (1) $X$ had always shown a systematic offset when using the model grids; and (2) if the value of $\chi^2$ from our initial estimates is too low, the minimization algorithm may not search among all possible solutions. We then allowed the code to minimize in all available observables. The solution usually converged within four iterations. We set the $\chi^2$ tolerance level to be 0.1, where $\chi^2$ is calculated from equation (5), but we used $3\sigma$ errors for the process instead of $1\sigma$, and $\chi^2 = \chi^2/n$, where $n$ is the number of degrees of freedom. An ideal value of $\chi^2$ is 1. Anything much larger implies a bad fit, an error in the physics, or incorrectly quoted errors.

The error $\epsilon(R)$ took values from 0.001–0.05 $R_\odot$ (~0.1%–6.0%); $\epsilon(\alpha)$ took values of 0.5 (LN) and 1.3 $\mu$Hz (HN). Figure 10 shows the mean and standard deviation (1 $\sigma$) of the fit mass as a function of $\epsilon(R)$ for OS2+HN. The dotted line is the true mass value. We show only this combination because those for LN11 and those using OS3 did not lead to a significant difference, except for $\tau$ and $\alpha$ whose uncertainties are represented by the dashed lines in Figure 11.

Our global results show that for all of the parameters: (1) The mean values fit to within 1 $\sigma$ of the correct value (Fig. 10 shows this for $M$ only). (2) The parameter uncertainties are smaller than those predicted by theory. This is highlighted in Figure 11. (3) All of the uncertainties show some dependence on $\epsilon(R)$.

Mass is the parameter that is most affected by an improvement in the precision of $R_*$. Figure 10 shows the mean and standard deviation of the fit mass. Note that $\sigma(M)$ is clearly dependent on $\epsilon(R)$ for $<3\sigma$ errors. If we reach 0.1% in radius error, then we can determine the mass to better than 1%. For $\epsilon(R) > 3\%$, $\sigma(M)$ does not change significantly. This can be seen more clearly in Figure 11. For $\epsilon(R) < 3\%$, there is little difference between the uncertainties arising from the minimizations (and more observables) or the linear approximation with five observables (Figs. 6 and 8). This can be understood by considering that for small radius errors, this observable has such a large weight that it determines uniquely the mass value. In this regime, the different combinations of observables do not have any influence, just as the theoretical uncertainties showed (Fig. 4, top). So whether we use a minimization method with 26 observables or a direct inversion with only five observables, having a well-measured radius will uniquely determine the mass. When the relative weight of the radius decreases, i.e., its error increases, other observables begin to play a role. This is why we see a dramatic change in the mass uncertainty between using OS1, OS2, and OS3 when $\epsilon(R) > 3\%$ (Figs. 6 and 10).

11 Extending the radius error to 0.001 $R_\odot$ did produce a small improvement in the parameters at this level for LN.
reaching a precision of about 4% and slowly degrading with $\epsilon(R)$ until $\epsilon(R) \approx 3%$ when it levels off at a $\sigma$-value comparable to the other observable combinations. The uncertainty $\sigma(X)$ shows a dependence on radius error similar to $\sigma(M)$. For $\epsilon(R) \leq 3%$ we generally estimate $X$ to within 2%–3%. For larger radius errors, $\sigma(X)$ does not increase, indicating the redundancy of a radius measurement after $\epsilon(R) = 3%$.

While some parameters may not seem very dependent on $\epsilon(R)$, having a radius observable allows us to estimate some parameters correctly. This in turn leads to a better estimation of the other non-radius-dependent parameters because of the more restricted acceptable range.

These results stem from an automatic method of fitting the observations. Given that there is no human input, we are confident in the results and expect that even more precise results will be possible by paying attention to each individual case and individual observable $\chi^2$ values.

5. CONCLUSIONS

We investigated the role that a precisely measured radius, such as that obtained through interferometry, has on the determination of the mass of a star. We also looked at how radius and oscillation frequencies work together to determine other stellar parameters.

1. We found that the importance of a radius measurement depends on the combination of the available observables and their corresponding errors. For some typical observables, such as $R$, and $\Delta\nu_{i,j}$, we can expect a mass uncertainty of between 1% and 4% for $\epsilon(R)$ between 0.1% and 3% (keeping in mind that these numbers stem from an automatic parameter search and in a real case study we would pay more attention to each observable and can expect better precision). For $\epsilon(R) \leq 3%$ the mass is uniquely determined by the radius observable, allowing us to use the frequencies just to probe the stellar interior.

2. Our simulation errors are consistent with the theoretical errors when we use the linear approximation (Fig. 6). Using a minimization method yields smaller than predicted uncertainties with similar qualitative results for all values of radius error, but particularly when $\epsilon(R) > 3%$. For these values, the weight of the radius measurement decreases, allowing other observables to play a greater role in the determination of the parameters. The minimization method allows an optimal trade-off between all observables and their corresponding errors.

3. We emphasize the importance of understanding how each observable contributes to the determination of each parameter. For example, if we want to get an estimate of $Z$ better than 10% we need a more precise [M/H]. A radius measurement would not be useful. To correctly determine age (or stellar evolutionary stage), it is crucial to observe $\nu$. Even with 21 large frequency spacings, we cannot determine the age to the precision that 1 small frequency spacing will provide. It is also interesting that $\nu$ coupled with a small radius error had a strong impact on the precision of the mixing-length parameter $\alpha$.

4. We know that a discrepancy exists between the observed and the model frequencies, which leads to a systematic offset in mass determination (e.g., Miglio & Montalbán 2005). By allowing the radius to determine the mass, we have the advantage that (1) we have an independent measurement that we can use to try to resolve the discrepancy; and (2) by using the radius to determine mass, we can use the frequencies to probe the stellar interior.

5. One final remark we would like to make is how effective a radius measurement can be to detect some error in the model.
During this study we conducted some hare and hounds\footnote{Searching for model parameters when O. L. C. is unaware of the real values.} tests. In one particular case, we found that all of the observations were fit very well except for the radius, which showed a larger than $3\sigma$ deviation value. This turned out to be the result of an incorrectly quoted metallicity measurement! This just shows how powerful the radius can be to detect an error/flaw, and this could possibly be used to detect a flaw in the physics of the models.

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