On a theorem of J. Shallit concerning Fibonacci partitions

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Abstract. In this note, I prove a claim on determinants of some special tridiagonal matrices. Together with my result about Fibonacci partitions (https://arxiv.org/pdf/math/0307150.pdf), this claim allows one to prove one (slightly strengthened) Shallit’s result (https://arxiv.org/pdf/2007.14930.pdf) about such partitions.

1 Introduction

Let $f_1 = 1, f_2 = 2$ and $f_i = f_{i-1} + f_{i-2}$ for $i > 2$ be the sequence of Fibonacci numbers. Observe that the “conventional” definition of Fibonacci numbers is different, see https://en.wikipedia.org/wiki/Fibonacci_number.

A Fibonacci partition of a positive integer $n$ is a representation of $n$ as an unordered sum of distinct Fibonacci numbers, which are referred to as the parts of the Fibonacci partition.

Let $\Phi_h(n)$ be the quantity (the cardinality of the set) of Fibonacci partitions of $n$ with $h$ parts. J. Shallit has established the following interesting property of the function $\Phi_h(n)$: for integers $n > 0$, $d \geq 2$ and $i$, let $r_{d,i}(n)$ be the quantity of all Fibonacci partitions of $n$ with number of parts $\equiv i \pmod{d}$. Then, (see [3, Th. 2])

$$|r_{3,i}(n) - r_{3,i+1}(n)| \leq 1.$$  

To prove this inequality, J. Shallit used a technique of automata theory.

Set

$$\Phi(n; t) := \sum_{h>0} \Phi_h(n)t^h.$$  

In [4], I obtained a formula which expresses $\Phi(n; t)$ as determinant of a tridiagonal matrix depending on $n$. In §2 of this note, I establish Theorem 2.6 on a property of such determinants.

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In §3, I explain (Theorem 3.1) how the mentioned (see [4]) formula for $\Phi(n; t)$ together with Theorem 2.6 imply not only Shallit’s result, but also the formula

$$(r_{3,0}(n) - r_{3,1}(n)) \cdot (r_{3,0}(n) - r_{3,2}(n)) \cdot (r_{3,1}(n) - r_{3,2}(n)) = 0.$$  

\section{3–special polynomials}

Let $d \geq 2$ be an integer number. For any $g(t) = \sum_{h \geq 0} a_h t^h \in \mathbb{Z}[t]$, define

$$\|g(t)\| := \sum_{h \geq 0} a_h, \quad R_i(g(t)) := \sum_{h \equiv i \mod d} a_h, \text{ where } i \in \{0, 1, \ldots, d - 1\}.$$  

Let $K_d[T] := \mathbb{Z}[T]/(T^d - 1)$. Define a map $R^{(d)} : \mathbb{Z}[t] \rightarrow K_d[T]$ by the formula

$$R^{(d)}(g(t)) := R_0(g(t)) + R_1(g(t))T + \cdots + R_{d-1}(g(t))T^{d-1}.$$  

The following Lemma is subject to easy direct verification.

\begin{lemma}
The map $R^{(d)} : \mathbb{Z}[t] \rightarrow K_d[T]$ is a homomorphism of $\mathbb{Z}$–algebras.
\end{lemma}

In this Section, I consider only the case $d = 3$. For brevity, set $K := K_3[T]$ and $R := R^{(3)}$.

For any $g(t) \in \mathbb{Z}[t]$, we obviously have

$$R((1 + t + t^2) \cdot g(t)) = \|g(t)\| \cdot \varphi(T), \text{ where } \varphi(T) := 1 + T + T^2. \quad (1)$$  

\begin{definition}
We say that $a + bT + cT^2 \in K$ is a \textit{special element} if either $a = b = c$, or $|a - b| + |a - c| + |b - c| = 2$.
\end{definition}

Formula (1) easily implies

\begin{lemma}
An element $A[T] \in K$ is special if and only if

$$A[T] \cdot (T - 1) \in M[T] := \{0, \pm (T - 1), \pm T(T - 1), \pm T^2(T - 1)\}.$$  

\end{lemma}

\begin{corollary}
Any product of special elements is a special element.
\end{corollary}

\begin{definition}
We say that $g(t) \in \mathbb{Z}[t]$ is a 3–\textit{special polynomial} if $R(g(t))$ is a special element.
\end{definition}

In what follows, $A = (a_1, a_2, \ldots, a_m)$ is either a vector with integer non-negative coordinates if $m > 0$, or the empty set if $m = 0$. Let us define a polynomial

$$\Delta(A; t) := \Delta(a_1, \ldots, a_m; t) \in \mathbb{Z}[t]$$  

by the formulas

$$\Delta(\emptyset; t) := 1, \quad \Delta(0; t) := 0, \quad \Delta(a; t) := t + t^2 + \cdots + t^a \text{ \ for } a > 0,$$
\( \Delta(a_1, \ldots, a_m; t) := \Delta(a_1, \ldots, a_{m-1}; t) \cdot \Delta(a_m; t) - \Delta(a_1, \ldots, a_{m-2}; t) \cdot t^{a_m+1} \) if \( m \geq 2 \). \hspace{1cm} (2)

Obviously, for \( m > 0 \),

\[
\Delta(a_1, a_2, \ldots, a_m; t) = \begin{vmatrix}
\Delta(a_1; t) & t^{a_2+1} & 0 & 0 & \cdots & 0 \\
1 & \Delta(a_2; t) & t^{a_3+1} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 1 & \Delta(a_{m-1}; t) & t^{a_m+1} \\
0 & 0 & \cdots & 0 & 1 & \Delta(a_m; t)
\end{vmatrix}.
\]

The main result of this note is

2.6. Theorem. For any \( A = (a_1, a_2, \ldots, a_m) \), the polynomial \( \Delta(A; t) \) is a 3-special one.

The proof uses the following auxiliary claim.

2.7. Lemma. Let \( \varepsilon(A) := (\varepsilon(a_1), \ldots, \varepsilon(a_m)) \), where \( \varepsilon(a) := a - 3 \lfloor \frac{a}{3} \rfloor \). Then,

\[
R(\Delta(A; t)) = R(\Delta(\varepsilon(A); t)) + k \cdot \varphi(T), \quad \text{where} \quad k = k(A) \in \mathbb{Z}.
\]

Proof. Let us prove by induction on \( m \). For \( m = 1 \) and \( a \geq 1 \), we have

\[
\Delta(a; t) = t \left( 1 + t^3 + \cdots + t^3 \lfloor \frac{a}{3} \rfloor \right) (1 + t + t^2) + t^3 \lfloor \frac{a}{3} \rfloor \cdot \Delta(\varepsilon(a); t).
\]

Applying \( R \) to both sides of this equality we obtain

\[
R(\Delta(a; t)) = R(\Delta(\varepsilon(a); t)) + k \cdot \varphi(T), \quad \text{where} \quad k = 1 + \left\lceil \frac{a}{3} \right\rceil.
\]

For \( m \geq 2 \), let us apply \( R \) to expression (2). The induction hypothesis, Lemma 2.1, formulas (1) and (3), the obvious formula \( R(t^a) = T^{\varepsilon(a)} \), and a short computation yield the required result. \( \square \)

Proof of Theorem 2.6. In view of Lemma 2.7, it suffices to assume that \( a_i \in \{0, 1, 2\} \) for any \( i = 1, 2, \ldots, m \). Keeping Lemma 2.3 in mind, define

\[
S(a_1, \ldots, a_m) := R(\Delta(a_1, \ldots, a_m; t)) \cdot (T - 1) \in K.
\]

The expression (2) and formula \( \varphi(T) \cdot (T-1) = 0 \) easily imply the recurrent formula

\[
S(a_1, \ldots, a_m) = \begin{cases}
-S(a_1, \ldots, a_{m-2}) \cdot T & \text{if } a_m = 0, \\
S(a_1, \ldots, a_{m-1}) \cdot T + S(a_1, \ldots, a_{m-2}) \cdot (T + 1) & \text{if } a_m = 1, \\
-S(a_1, \ldots, a_{m-1}) - S(a_1, \ldots, a_{m-2}) & \text{if } a_m = 2.
\end{cases}
\hspace{1cm} (4)

By Lemma 2.3 it remains to show that \( S(a_1, \ldots, a_m) \in M[T] \).

Let us prove this by induction on \( m \). For \( m = 1, 2 \), the claim is directly checked. In particular, \( S(0) = 0 \) and \( S(a, 0) = -T(T - 1) \).
For \( a_m = 0 \), the last expressions and formula (4) imply the theorem by induction for any \( m \geq 1 \). Therefore, assume that \( a_m = 1 \) or \( a_m = 2 \). From expressions (4) it is not difficult to obtain the expressions

\[
S(a_1, \ldots, a_m, 1) = \begin{cases}
S(a_1, \ldots, a_{m-2}, 2) \cdot T^2 & \text{if } a_{m-1} = 0, \\
-S(a_1, \ldots, a_{m-2}, 2) \cdot (T + 1) & \text{if } a_{m-1} = 1, \\
S(a_1, \ldots, a_{m-3}, a_{m-2} + 2) \cdot T & \text{if } a_{m-1} = 2,
\end{cases}
\]

\[
S(a_1, \ldots, a_{m-1}, 2) = \begin{cases}
-S(a_1, \ldots, a_{m-3}, a_{m-2} + 2) \cdot T & \text{if } a_{m-1} = 0, \\
S(a_1, \ldots, a_{m-2}, 2) \cdot (T + 1) & \text{if } a_{m-1} = 1, \\
S(a_1, \ldots, a_{m-3}) & \text{if } a_{m-1} = 2.
\end{cases}
\]

Since

\[
(T - 1)(T + 1) = -T^2(T - 1),
\]

these expressions and the induction hypothesis complete the proof. □

3 An application to Fibonacci partitions

In §2 of the article [4], for any positive integer \( n \), a certain sequence is uniquely defined

\[
\alpha(n) = \{\alpha_1(n), \alpha_2(n), \ldots, \alpha_k(n)\}
\]

where \( \alpha_k(n) \) is a vector with positive integer coordinates for any \( k = 1, 2, \ldots, k(n) \), and it is shown ([4, Th.2.11]) that

\[
\Phi(n; t) = \Delta(\alpha_1(n); t) \cdot \Delta(\alpha_2(n); t) \cdots \Delta(\alpha_k(n); t).
\]

By Theorem 2.6 the polynomial \( \Delta(\alpha_k(n); t) \) is a 3–special one for any \( k \). Thus, Lemma 2.1 and Corollary 2.4 imply

3.1. Theorem. For any integer \( n > 0 \), the polynomial \( \Phi(n; t) \) is a 3–special one.

3.2. Remark. Using arguments similar to those in §2 (where \( d = 3 \) is replaced with \( d = 2 \)) and the formula for \( \Phi(n; t) \) one can easily show that \(|r_{2,0}(n) - r_{2,1}(n)| \leq 1\) for any positive integer \( n \). It is obvious that this inequality is equivalent to the analytic identity

\[
\prod_{i=1}^{\infty} (1 - x^{f_i}) = 1 + \sum_{n=1}^{\infty} \chi(n)x^n, \quad \text{where } |\chi(n)| \leq 1.
\]

For other proofs of this identity, see [1],[2] and [4].

In addition to that, an interesting result of Y. Zhao should be mentioned. Namely, Proposition 2 of the article [5] implies the polynomial identity

\[
\prod_{a \leq i \leq b} (1 - x^{f_i}) = 1 + \sum_{n} \chi_{a,b}(n)x^n, \quad \text{where } |\chi_{a,b}(n)| \leq 1,
\]

which is valid for any positive integers \( a \leq b \).
3.3. **Conjecture.** For positive integers \(a \leq b\), let
\[
M(a, b) := \{f_a, f_{a+1}, \ldots, f_b\}.
\]
For integers \(n > 0\) and \(i\), let \(r_{3,i}^{(a,b)}(n)\) be the quantity of Fibonacci partitions of \(n\) with parts from the set \(M(a, b)\) and with number of parts \(\equiv i \mod 3\).
Then, \(\left| r_{3,i}^{(a,b)}(n) - r_{3,j}^{(a,b)}(n) \right| \leq 1\) for any \(i, j \in \{0, 1, 2\}\). Moreover,
\[
\left( r_{3,0}^{(a,b)}(n) - r_{3,1}^{(a,b)}(n) \right) \cdot \left( r_{3,0}^{(a,b)}(n) - r_{3,2}^{(a,b)}(n) \right) \cdot \left( r_{3,1}^{(a,b)}(n) - r_{3,2}^{(a,b)}(n) \right) = 0.
\]

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