Structure of globally hyperbolic spacetimes with timelike boundary

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Abstract
Globally hyperbolic spacetimes with timelike boundary \((\mathcal{M} = M \cup \partial M, g)\) are the natural class of spacetimes where regular boundary conditions (eventually asymptotic, if \(\partial M\) is obtained by means of a conformal embedding) can be posed. \(\partial M\) represents the naked singularities and can be identified with a part of the intrinsic causal boundary. Apart from general properties of \(\partial M\), the splitting of any globally hyperbolic \((\mathcal{M}, g)\) as an orthogonal product \(\mathbb{R} \times \bar{\Sigma}\) with Cauchy slices with boundary \(\{t\} \times \bar{\Sigma}\) is proved. This is obtained by constructing a Cauchy temporal function \(\tau\) with gradient \(\nabla \tau\) tangent to \(\partial M\) on the boundary. To construct such a \(\tau\), results on stability of both, global hyperbolicity and Cauchy temporal functions are obtained. Apart from having their own interest, these results allow us to circumvent technical difficulties introduced by \(\partial M\). As a consequence, the interior \(M\) both, splits orthogonally and can be embedded isometrically in \(\mathbb{L}^N\), extending so properties of globally spacetimes without boundary to a class of causally continuous ones.

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1 Introduction

The celebrated results by Choquet-Bruhat [16] and Choquet-Bruhat and Geroch [17] ensure that the metric of a globally hyperbolic spacetime \((M, g)\) is determined by Cauchy data on a Cauchy hypersurface \(\Sigma\). Such data (a Riemannian metric, the a posteriori second fundamental form) are subject to some constraints (Gauss and Codazzi) and, once \(g\) is prescribed, the result is extended to other fields satisfying hyperbolic equations. The existence of a smooth and spacelike Cauchy hypersurface \(\Sigma\) for any such \((M, g)\) ensures the well-posedness of the Cauchy problem; moreover, the existence of a splitting for the full spacetime \(M\) as an orthogonal product \((\mathbb{R} \times \Sigma, g = -\Lambda d\tau^2 + g_\tau)\), where \(\tau\) is a Cauchy temporal function, shows the consistency of the notion of predictability from each “instant” of time to another (apart from implying practical advantages).

These two existence results were obtained at a topological level by Geroch [32] and at the differentiable and metric levels in [6, 7]; a full and self-contained development of the Cauchy problem has been carried out recently in the painstaking book by Ringström [46]. In the present article, such a splitting as well as other causal properties, will be extended to the case of globally hyperbolic spacetimes with smooth, conformal timelike boundary. The Cauchy problem in this class of spacetimes was studied by Friedrich and Nagy [29] and the class of spacetimes was systematically studied in Solís’ thesis [45]. Globally hyperbolic spacetimes (without boundary) can be characterized as those strongly causal spacetimes whose intrinsic causal boundary points are not naked singularities. In a natural way, the timelike boundary \(\partial M\) of our class of spacetimes is composed by all the naked singularities (see the Appendix); so, the these singularities become the natural place to impose boundary conditions. Chrusciel, Galloway and Solís [18] proved a version of topological censorship for this class of spacetimes; notably, such a class includes asymptotically anti-de Sitter ones.

More precisely, we will show that any globally hyperbolic spacetime with timelike boundary \((\overline{M}, g)\) admits a Cauchy splitting \(\mathbb{R} \times \Sigma\) where \(\Sigma = \Sigma \cup \partial \Sigma\) is a Cauchy hypersurface with boundary \(\partial \Sigma\). From the PDE viewpoint, this suggests that the Cauchy problem will be consistently well-posed in this class of spacetimes as a mixed Cauchy problem for a symmetric positive linear differential equation [29, 30]. Now, not only the Cauchy data on \(\Sigma \cong \{0\} \times \Sigma\) but also boundary data on \(\partial \overline{M} \equiv \mathbb{R} \times \partial \Sigma\) (under suitable compatibility constraints at \(\{0\} \times \partial \Sigma\)) must be provided, resembling the behavior of the elementary wave equation. Background on this mixed problems has been developed recently by Valiente-Kroon and Carranza [47, 13, 14]; see also the article by Enciso and Kamran [23] (including its expanded version [24]). Applications to wave equations and quantum field theory on curved spacetimes were obtained by Lupo [36, 37] in the general case and by Dappiaggi, Drago and Ferreira [20] in the static case, so extending works by Bär, Ginoux and Pfaiffle [2], among others.

From the technical viewpoint, the splitting will be obtained by constructing a Cauchy temporal function \(\tau\) such that \(\nabla \tau\) is tangent to the boundary (everywhere on \(\partial \overline{M}\)). This technical condition will turn out essential to obtain a \(\partial \overline{M}\)-orthogonal splitting by flowing through the integral curves of a suitable normalization of the gradient \(\nabla \tau\). Such a \(\tau\) will be constructed by the following procedure: (a) check the \(C^0\)-stability of Cauchy temporal functions in the set of all the Lorentzian metrics on \(\overline{M}\) (this will make irrelevant any \(C^2\) details of \(g\) on \(\partial \overline{M}\)), (b) reduce the problem to the simplified case of a metric \(g^*\) with a simple product structure close to \(\partial \overline{M}\), (c) extend symmetrically \(g^*\) to the double
manifold $\overline{M}^d$ (which becomes then a globally hyperbolic spacetime without boundary) and (d) obtain a Cauchy temporal function for $\overline{M}^d$ which is invariant by a reflection on $\partial M$. This last problem can be regarded as a simple case of a result by Müller [40] about Cauchy temporal functions invariant by a compact conformal group; the question (a) will be studied in Section 4 and the others in Section 5.

For comparisons with the case without boundary, we will consider the approach by using locally defined smooth temporal functions in the original papers [6, 7], which has shown to be very flexible for a variety of questions [8, 41, 40]. However, some different smoothability procedures with many other applications have been developed since then, namely: Fathi and Siconolfi [25], using methods inspired by weak-KAM theory, applicable to cone structures; Chrusciel, Grant and Minguzzi [19], inspired by Seifert’s approach to smoothability in spacetimes [44]; and Bernard and Suhr [10], inspired by Conley theory, applicable to possibly non-continuous closed cone structures. Similar conclusions seem to hold if any of these alternative approaches were considered.

Summing up, quite a few properties about causality and the causal ladder will be revisited for spacetimes with timelike boundary, with the following main aim:

**Theorem 1.1.** Any globally hyperbolic $n$-spacetime with timelike boundary $(\overline{M}, g)$ admits a Cauchy temporal function $\tau$ whose gradient $\nabla \tau$ is tangent to $\partial M$.

As consequence, $\overline{M}$ splits smoothly as a product $\mathbb{R} \times \overline{\Sigma}$, where $\overline{\Sigma}$ is a $(n - 1)$-manifold with boundary, the metric can be written (with natural abuses of notation) as a parametrized orthogonal product

$$g = -\Lambda d\tau^2 + g_\tau,$$

(1)

where $\Lambda : \mathbb{R} \times \overline{\Sigma} \to \mathbb{R}$ is a positive function, $g_\tau$ is a Riemannian metric on each slice $\{\tau\} \times \overline{\Sigma}$ varying smoothly with $\tau$, and these slices are spacelike Cauchy hypersurfaces with boundary. Moreover, $\overline{M}$ can be isometrically embedded in Lorentz-Minkowski $\mathbb{L}^N$ for some $N \in \mathbb{N}$, the interior $M$ of $\overline{M}$ is always causally continuous, the boundary $\partial M$ is a (possibly non-connected) globally hyperbolic spacetime without boundary, and:

(a) the restriction of $\tau$ to $M$ extends the known orthogonal splitting of globally hyperbolic spacetimes to this class of causally continuous spacetimes without boundary,

(b) the restriction of $\tau$ to $\partial M$ provides a Cauchy temporal function for the boundary whose levels are acausal in $\overline{M}$.

It is worth pointing out that all the procedures to be used will be conformally invariant, so, the result will be also applicable to the case of a conformal boundary. Moreover, the splitting of those spacetimes without boundary which can be seen as the interior $M$ of a globally hyperbolic one with boundary $\overline{M}$, has its own interest. In particular, it permits to extend results about linking and causality by Chernov and coworkers to a bigger class of spacetimes, recall [15].

The paper is organized as follows.

In Section 2, some basic preliminaries are introduced. Many of them are known or expected from standard techniques; anyway, they will become relevant later. In Subsect. 2.1, the double manifold is used for extensions of the spacetime with boundary, and relations between the time-orientations of $\overline{M}, \partial M$ and $M$ are pointed out. Gaussian coordinates are introduced and shown to yield a local version of the splitting (1) (Cor. 2.5).

In Subsect. 2.2, the causal ladder is introduced. Our choices of the definitions allow us

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1 The global hyperbolicity of $\overline{M}$ implies directly the global hyperbolicity of $\partial M$ and, so, the existence of a Cauchy temporal function in each connected component of $\partial M$. However, the global splitting ensures that the different connected components can be synchronized in an acausal one when looked at $\overline{M}$. Such a property becomes natural from the PDE viewpoint commented above (notice that different connected components of $\partial M$ are permitted, even if $\overline{M}$ will be assumed connected with no loss of generality).
a reasonably self-contained development. In particular, the basic properties of the lower levels of the ladder (until stably causal) are quickly checked there. In Subsect. 2.3, causal continuous curves are discussed and an approach following [11, Appendix A] is introduced. This permits to obtain intrinsically limit curves within the same class of curves (Prop. 2.16) and will circumvent subtleties which appear for other notions of causal continuous curves (recall Remark 2.17).

In Section 3, the framework of Geroch’s topological splitting is revisited in order to include boundaries. Even though most of the properties here are transplanted from the case without boundary, we make a fast review to emphasize some differences and provide a reasonably self-contained study. In Subsect. 3.1, after checking that the role of admissible measures can be extended to the case with boundary, we reconstruct the higher levels of the causal ladder (Thm. 3.8), determine the causal properties inherited by the boundary \( \partial M \) and the interior \( M \) at each level, and provide the necessary (counter-) examples (Remark 3.9). Technically, Prop. 3.5 and 3.6 summarize the main properties which can be transplanted directly from the case without boundary; for the remainder, short proofs are provided. In Subsect. 3.2, we go over Geroch’s technique to find the required Cauchy time function (Thm. 3.14). With this aim, some properties of achronal sets and hypersurfaces with boundary are revisited (recall Defn. 3.11 and, then, Prop. 3.12).

In Section 4, the stability of both global hyperbolicity and Cauchy temporal functions is proved in the case with boundary. This question has interest in its own right, and will be used to simplify the proof of the existence of orthogonal splitting in the case with boundary. Recall that, in the case without boundary, the stability of a globally hyperbolic metric \( g \) on \( M \) was also studied by Geroch [32]. This question becomes equivalent to show that there exists a metric \( g' \) with strictly bigger cones (i.e. \( g < g' \)) which is globally hyperbolic (as all the metrics \( g'' \) with \( g'' \leq g' \) will be globally hyperbolic too). As Geroch’s time function \( t \) may be a non-time function for any \( g' > g \) (recall that even in the smooth case the levels of \( t \) may be degenerate hypersurfaces), this question was non-trivial at that moment; however, the problem was widely simplified when a Cauchy temporal function \( \tau \) was proved to exist (see also Section 3 of [5], arxiv version). Indeed, a stronger result holds because \( \tau \) becomes stable as a Cauchy temporal function (i.e. \( \tau \) is also Cauchy temporal for some \( g' > g \) and, thus, any \( g'' < g' \)) and the stability of global hyperbolicity becomes a direct consequence, as will be checked here (see Remark 5.7). Anyway, different proofs of the stability of \( g \) with interest in its own right were found by Benavides and Minguzzi [5] and by Fathi and Siconolfi [25].

As explained in Subsect. 4.1, we will prove stability of both, global hyperbolicity (by means of a direct proof for the sake of completeness) and Cauchy temporal functions (assuming that they have been constructed with no restriction on \( \partial M \) as in [7]), i.e.:

For a spacetime with timelike boundary \((
\overline{M}, g)\), global hyperbolicity is a stable property. What is more, for any Cauchy temporal function \( \tau \) of \((
\overline{M}, g)\) there exists a globally hyperbolic metric with wider cones \( g' > g \) such that \( \tau \) is Cauchy temporal for \( g' \) and, thus, for any other Lorentz metric (necessarily globally hyperbolic) \( g'' \leq g' \).

Apart from the interest in its own, stability will simplify widely the procedure to obtain the tangency of \( \nabla \tau \) to \( \partial M \) in Thm. 1.1. Indeed, our procedure will stress that possible problems associated to, say, the bending of \( \partial M \) or its non-convexity, will have a “higher order” than the requirements for \( \tau \) (and, thus, will be negligible). In Subsect. 4.2 we will see how to construct globally hyperbolic perturbations of \( g \) (eventually, maintaining the Cauchy temporal character of a prescribed one \( \tau \) for \( g \)) and in Subsect. 4.3 such perturbations are shown to yield the stability results.
In Section 5, after an overall explanation of the simplified procedure, Thm. 1.1 is proved. We end with a discussion on the type of problems which can be proven transplanting directly the techniques for the case without boundary (Remark 5.6, 5.7).

Finally, the Appendix justifies rigourously that, in a globally hyperbolic spacetime with timelike boundary, the boundary $\partial M$ is composed by all the naked singularities of the spacetime (Thm. 6.6). As these singularities are regarded naturally as a subset of the intrinsic causal boundary $\partial^c M$ (the pairs $(P, F)$ with $P \neq \emptyset \neq F$, see Remark 6.2), the essential ingredients of $\partial^c M$ are reviewed first for the sake of completeness.

2 Preliminaries on spacetimes with timelike boundary

2.1 Generalities: boundaries, time-orientation and coordinates

**Manifolds with boundary.** In what follows $\overline{M}$ will denote a connected $C^r$ $n$-manifold with boundary, being $n \geq 2$. Any function or tensor field will be smooth when it is as differentiable as possible (compatible with the $C^r$ character of $\overline{M}$); along the paper, $C^1$ will be enough for the metric $g$, and the elements to be obtained (as the Cauchy temporal function) will maintain the maximum differentiability. $\overline{M}$ is then locally diffeomorphic to (open subsets of) a closed half-space of $\mathbb{R}^n$; $\overline{M}$ will denote its interior and $\partial M$ its boundary. For any $p \in \overline{M}$, $T_p \overline{M}$ will denote its $n$-dimensional tangent space while for $p \in \partial M$, $T_p \partial M$ is the $(n-1)$-dimensional tangent space to the boundary. Such a $\overline{M}$ can be regarded as a closed subset of the so-called double manifold $\overline{M}^d$, a $C^r$ $n$-manifold (without boundary) obtained by taking two copies of $\overline{M}$ and identifying homologous boundary points (in particular, partitions of unity for $\overline{M}$ can be constructed from $\overline{M}^d \setminus \overline{M}$ as an extra open subset). Lorentzian and Riemannian metrics on $\overline{M}$ are particular cases of semi-Riemannian metrics, i.e. non-degenerate metric tensors (of constant index). Background on manifolds with boundary can be seen in [35, Section 9], for example. Next, we emphasize a basic property (see [1, Prop. 3.1] for a proof).

**Proposition 2.1.** Regarding $\overline{M}$ as a closed subset of $\overline{M}^d$, any semi-Riemannian metric $g$ on $\overline{M}$ can be extended to some open subset $\overline{M} \subset \overline{M}^d$ with $\overline{M} \subset \overline{M}^d$.

**Remark 2.2.** Recall: (a) in general, the metric defined on two copies of $(\overline{M}, g)$ cannot be extended as a smooth metric on $\overline{M}^d$, (b) in the Riemannian case, $g$ can be extended to the whole $\overline{M}^d$, but in the Lorentzian case this may be non-possible (for example, if $\overline{M}$ is an even-dimensional closed half-sphere, $\overline{M}^d$ admits no Lorentzian metric).

**Time-orientation and spacetimes.** Let us recall some basic notions for spacetimes with timelike boundary. Usual notions for Lorentzian manifolds without boundary such as causal or timelike vectors (here, following conventions in [42, 39]) are extended to the case with boundary with no further mention (see [31, 45] for further background).

**Definition 2.3.** A Lorentzian manifold with timelike boundary $(\overline{M}, g)$, $\overline{M} = M \cup \partial M$, is a Lorentzian manifold with boundary such that the pullback $i^* g$, with $i : \partial M \hookrightarrow M$ the natural inclusion, defines a Lorentzian metric on the boundary. A spacetime with timelike boundary is a time-oriented Lorentzian manifold with timelike boundary.

By *time-oriented* we mean that a time cone has been chosen continuously (i.e., locally selected by a continuous timelike vector field $X$) on all $\overline{M}$. The pull-back $i^*$ will be dropped when there is no possibility of confusion and the time-orientation is assumed implicitly. If $g, g'$ are two Lorentzian metrics on $\overline{M}$, the notation $g < g'$ (resp. $g \leq g'$)
means that any future-directed causal vector for $g$ is future-directed timelike (resp.
causal) for $g'$. The following result ensures that no additional issue on time-orientations
appears because of the presence of the boundary (its proof uses standard background
for the case without boundary, see \cite[Lemma 5.32, Prop. 5.37]{footnote}).

**Proposition 2.4.** The following properties are equivalent for any Lorentzian manifold
with timelike boundary $(\overline{M}, g)$:

(i) $(\overline{M}, g)$ is time-orientable.

(ii) $(M, g)$ and $(\partial M, g)$ are time-orientable.

(iii) There exists a timelike vector field $T$ on all $\overline{M}$ tangent to $T_p\partial M$ at each $p \in \partial M$.

(iv) There exists a timelike vector field $T$ on all $\overline{M}$.

Therefore, for any spacetime with timelike boundary, $\partial M$ is naturally a spacetime$^2$.

**Proof.** (i) $\Rightarrow$ (ii) Notice that if $X$ selects the time-orientation on some neighborhood
$U \subset \overline{M}$, then its orthogonal projection on $T_p\partial M$ for each $p \in U \cap \partial M$ selects continuously
a time orientation on $U \cap \partial M$.

(ii) $\Rightarrow$ (iii) As $(M, g)$ has no boundary, it admits a smooth timelike vector field
$T_M$ and each connected part $C$ of $\partial M$ will admit also a smooth timelike vector field
$T_C$. For each $p \in C$, consider a coordinate chart $(U_p, (x^0, x^1, \ldots, s))$ adapted to the boundary, i.e. $s^{-1}(0) = U_p \cap C$, and extend the (restricted) vector field $T_C|_{U_p \cap C}$ to the coordinate chart $U_p$ by making the components of the vector field independent of the $s$ coordinate. Let $T_C[p]$ be such an extension. As the set of points $p \in C$ for which
the time orientation determined by $T_C[p]$ and $T_M$ agree on $U_p \cap \partial M$ is both, open and
closed in $C$, we can choose $T_C$ so that both agree for all $p \in C$. Repeating this for all
the connected components of $\partial M$, considering the covering of $\overline{M}$ provided by $M$ and
all $U_p, p \in \partial M$, and taking a partition of unity subordinate to this covering, one gets a
timelike vector field $T^0$ defined on some neighbourhood $U$ of $\partial M$ which is also tangent
to $\partial M$. So, if $\{\mu, 1 - \mu\}$ is a partition of the unity of $\overline{M}$ subordinate to the covering
$\{U, M\}$, the required vector field is just $T = \mu T^0 + (1 - \mu)T_M$.

The implications (iii) $\Rightarrow$ (iv), (iv) $\Rightarrow$ (i) and the last assertion are trivial. $\blacksquare$

**Gaussian coordinates.** The following coordinates specially well adapted to the boundary
will be useful. Let $\hat{p} \in \partial M$ and take a chart in the boundary $(\hat{U}, (x^0, x^1, \ldots, x^{n-2}),$
with $\hat{U} \subset \partial M$ connected and relatively compact, being $g(\partial_0, \partial_0) = -1, \partial_0$ future-directed
on $\hat{U}$, and $\{\partial_0, \ldots, \partial_{n-2}\}$ an orthonormal basis at $T_{\hat{p}}\partial M$. Since $\partial M$ is timelike, there
exists a unitary spacelike vector field $N$ on $\partial M$ which is orthogonal to the boundary
and points out into $M$. Extend the previous coordinate system to a chart of $\hat{p}$ in $\overline{M}$
by using the geodesics with initial data $(q, N_q), q \in \hat{U}$, that is, consider the geodesic
$\gamma_q(s) = \exp_q(s \cdot N_q), s \geq 0$, and regard its affine parameter $s$ as a transverse coordinate.
This provides the required coordinate system $(\hat{U} \times [0, s_+), (x^0, x^1, \ldots, x^{n-2}, x^{n-1} = s))$
of $\overline{M}$ for some $s_+ > 0$ small enough. Since $\partial_0|_{\partial M} = N$ is orthogonal to the boundary,
$g(\partial_i, \partial_j) = 0$ on $U := \hat{U} \times [0, s_+]$ for all $j = 0, \ldots, n-2$ (see for example \cite[pp. 42-43]{footnote}),
and the metric $g$ can be written as

$$g = \sum_{i,j=0}^{n-2} g_{ij}(\hat{x}, s)dx^idx^j + ds^2, \quad \text{where } \hat{x} = (x^0, x^1, \ldots, x^{n-2}). \quad (2)$$

Moreover, $\hat{U}$ and $s_+$ are taken small enough so that the gradient $\nabla x^0$ is timelike (i.e.,
the slices of $x^0$ are spacelike) on $U$. Any coordinate system constructed as above will be
called a Gaussian chart adapted to the boundary, or just Gaussian coordinates. When

$^2$Without boundary and possibly non-connected, consistently with footnote 1.
Corollary 2.5. For each $\hat{p} \in \partial M$ there exists a product neighborhood $V = (-\epsilon, \epsilon) \times \hat{V}_0$, where $\hat{V}_0$ is a spacelike embedded hypersurface with boundary, such that both factors are $g$-orthogonal and $g$ can be written as a parametrized product

$$g = -\Lambda dt^2 + g_r, \quad \tau : (-\epsilon, \epsilon) \times \hat{V}_0 \to (-\epsilon, \epsilon) \text{ (natural projection)},$$

where $\Lambda = -1/(\nabla \tau, \nabla \tau)$ is a function on $V$ and $g_r$ is a Riemannian metric on $\{\tau\} \times \hat{V}_0$ depending smoothly on $\tau$.

Proof. Take any Gaussian coordinates of $\hat{p}$, put $\tau := \alpha^0$, $\hat{V}_0 = \tau^{-1}(0)$ and recall that, by (2), $\nabla \tau$ must be tangent to $\partial M$ on the boundary. So (taken a smaller hypersurface $\hat{V}_0$ and neighborhood) the flow of the vector field $-\nabla \tau /|\nabla \tau|^2$ is well defined and moves $\hat{V}_0$ yielding the product neighborhood $V = (-\epsilon, \epsilon) \times \hat{V}_0$ (the expression of the metric becomes then standard, see the end of the proof of Prop. 2.4 in [7]).

2.2 Conditions on causality and lower levels of the causal ladder

For any spacetime with timelike boundary $(\overline{M}, g)$, the usual notation $\ll, \leq, I^\pm(p), J^\pm(p)$, will be used for the chronological and causal relations and the chronological and causal future/past of any $p \in \overline{M}$; so, say, $I^+(p, U)$ will denote the chronological future obtained by using curves entirely contained in the subset $U \subset \overline{M}$. In what follows $\cl$ will denote closure.

Proposition 2.6. (a) The binary relation $\ll$ is open (in particular, $I^\pm(p)$ are open in $\overline{M}$). (b) For any $p, q, r \in \overline{M}$, $p \ll q \leq r \Rightarrow p \ll r$, $p \leq q \ll r \Rightarrow p \ll r$. (c) $J^\pm(p) \subset \cl(I^\pm(p))$. (d) $I^\pm(p, M) = I^\pm(p) \cap M$ for all $p \in M$.

Proof. Properties (a), (b), (c) are easy to check (see [45, Prop. 3.5, 3.6, 3.7]). To prove $I^+(p, M) = I^+(p) \cap M$, the inclusion $\subset$ is trivial. So, let $q \in I^+(p) \cap M$, and take some (piecewise smooth) future-directed timelike curve $\gamma : [0, 1] \to \overline{M}$ with $\gamma(0) = p, \gamma(1) = q \in M$. Consider any vector field $N \in \mathfrak{X}(\overline{M})$ which extends the pointing-inward unit normal on $\partial M$ (this can be always done, as $\partial M$ is closed), any smooth function $f : [0, 1] \to \mathbb{R}^+$ vanishing only at 0, 1, and the vector field $V$ on $\gamma$ defined by $V(t) = f(t)N_{\gamma(t)}$ for all $t \in [0, 1]$. For the fixed-endpoints variation of $\gamma$ corresponding to the variational vector $V$, longitudinal curves close to $\gamma$ are still timelike and cannot touch $\partial M$. In conclusion, $q \in I^+(p, M)$.

Remark 2.7. Property (d) can be naturally extended to points at the boundary as follows: for any $p \in \overline{M}$, $I^\pm(p) \cap M$ is the set of $q \in M$ such that there exists a future/past-directed timelike $\gamma$ with $\text{Im}(\gamma) \setminus \{p\} \subset M$ joining $p$ with $q$.

In the case of spacetimes without boundary, there is a well-known causal ladder of spacetimes, each step admitting several characterizations (see [4, 39]). Most of the ladder and characterizations can be transplanted directly to the case of spacetime with timelike boundary. Here, we will focus just on globally hyperbolic spacetimes, postponing the systematic study of other causal subtleties for future work. So, we will make a fast summary of the standard steps of the ladder just making simple choices on the definitions and properties to be used in a self-contained way.

Definition 2.8. A spacetime with timelike boundary $(\overline{M}, g)$ is:
• chronological (resp. causal) if it does not contain closed timelike (resp. causal) curves,

• future (resp. past) distinguishing if the equality $I^+(p) = I^+(q)$ (resp. $I^-(p) = I^-(q)$) implies $p = q$, that is, if the set-valued map $I^+: \mathcal{M} \to P(\mathcal{M})$ (resp. $I^-: \mathcal{M} \to P(\mathcal{M})$), where $P(\mathcal{M})$ is the power set of $\mathcal{M}$, is one-to-one. It is distinguishing, when it is both, future and past distinguishing.

• strongly causal if for all $p \in \mathcal{M}$ and any neighborhood $U \ni p$ there exists another neighborhood $V \subset U$, $p \in V$, such that any causal curve with endpoints in $V$ is entirely contained in $U$.

**Definition 2.9.** A subset $W$ of a spacetime with timelike boundary $(\mathcal{M}, g)$ is causally convex if $J^+(x) \cap J^-(y) \subset W$ for any $x, y \in W$ (equivalently, if any causal curve with endpoints in $W$ must remain in $W$).

Given an open neighborhood $U \subset \mathcal{M}$, a subset $W \subset U$ is causally convex in $U$ when $W$ is causally convex as a subset of $U$, regarding $U$ as a spacetime with timelike boundary.

**Lemma 2.10.** Let $(\mathcal{M}, g)$ be a spacetime with timelike boundary. For any $p \in \mathcal{M}$ and any neighborhood $V \ni p$ there exists a sequence of nested neighborhoods $\{W_m\} \subset V$, $W_{m+1} \subset W_m$, $\{p\} = \cap_m W_m$, such that all $W_m$ are causally convex in $V$.

**Proof.** This can be proved as in the case without boundary [39, Thm. 2.14, Lemma 2.13] (now requiring the nested neighborhoods just to be causally convex instead of globally hyperbolic). Namely, given $V$, one takes (Gaussian) coordinates centered at $p$, $(\mathcal{V}', x')$, $\mathcal{V}' \subset V$, a standard flat metric $g^+$ in these coordinates with $g < g^+$ (say, $g^+ = -\epsilon(dx^0)^2 + \sum_i (dx^i)^2 + ds^2$, with $\epsilon > 0$ small and, eventually, choosing a smaller $\mathcal{V}'$) and $W_m := I^+(x^0 = -1/m, 0, \ldots, 0) \cap I^-(x^0 = 1/m, 0, \ldots, 0)$ for large $m$.

**Proposition 2.11.** $(\mathcal{M}, g)$ is strongly causal if and only if each $p \in \mathcal{M}$ admits arbitrarily small causally convex neighbourhoods, that is, for any neighbourhood $U \ni p$ there exists a causally convex neighbourhood $W \ni p$ contained in $U$.

In this case, causal curves are not partially imprisoned on compact sets, that is, for any future-directed causal curve $\gamma: [a, b) \to \mathcal{M}$, $a < b \leq \infty$ which cannot be extended continuously to $b$, and any compact set $K \subset \mathcal{M}$, there exists some $s_0 \in [a, b)$ such that $\gamma(s) \notin K$ for all $s \geq s_0$.

**Proof.** We will focus on the implication to the right (to the left is trivial). Since $(\mathcal{M}, g)$ is strongly causal, given an open neighborhood $U$ of $p$, there exists a smaller neighborhood $V \subset U$, $p \in V$, such that any closed causal curve with extreme points in $V$ is totally contained in $U$. From Lemma 2.10, there exists some neighborhood $W \subset V$ of $p$ which is causally convex in $V$. The property above satisfied by $V$ ensures that $W$ must be causally convex (in $\mathcal{M}$) as well. Finally, the last assertion follows as in the case without boundary (see [4, Prop. 3.13] or [42, Lemma 14.13]; see also [39, Sect. 3.6.2]), full details are written in [1, Prop. 3.7].

**Proposition 2.12.** In any spacetime with timelike boundary: strongly causal $\Rightarrow$ distinguishing, and future or past distinguishing $\Rightarrow$ causal $\Rightarrow$ chronological.

**Proof.** The first implication follows as in the case without boundary [39, Lemma 3.10] (see [1, Prop. 3.7(a)] for full details); this also happens for the second one (a contradiction follows easily by applying the transitivity relations in Prop. 2.6), and the last one is trivial.
Following the ladder, \((\overline{M}, g)\) is stably causal, when it admits a time function, i.e., a continuous function \(\tau\) which increases strictly on all future-directed causal curves. This step admits some classical characterizations which are stated next only for the sake of completeness. The equivalence among these characterizations can be done transplanting the techniques in the case without boundary, as will be explained at the end, Remark 5.6. So, the proofs are only sketched here (the reader might prefer to come back at this point after finishing the article and read Remarks 5.6, 5.7).

**Proposition 2.13.** For a spacetime with timelike boundary \((\overline{M}, g)\) they are equivalent:

1. It admits a time function (i.e., the spacetime is stably causal).
2. It admits a temporal function \(\tau\) (i.e., \(\tau\) is smooth with timelike past-directed \(\nabla\tau\)).
3. There exists a strongly causal metric \(g'\) with \(g < g'\).

In this case, the spacetime is also strongly causal.

**Proof.**

1 \(\Rightarrow\) 2. The same smoothing procedure as in the case without boundary \([7, 43]\) holds, as no additional condition is required for \(\nabla\tau\) on the boundary (see Remark 5.6).

2 \(\Rightarrow\) 3. Even if \(\overline{M}\) does not split globally as a product, the temporal function \(\tau\) still allows to write \(g\) as \(-\Lambda d\tau^2 + g_{\tau}\) where \(\Lambda > 0\) is a function on \(\overline{M}\) and \(g_{\tau}\) a Riemannian metric on the bundle \(\text{Ker}(d\tau)\). So, for any positive function \(\alpha > 0\) the metric \(g_{\alpha} = -(\Lambda + \alpha)d\tau^2 + g_{\tau}\) satisfies \(g_{\alpha} > g\) and it is also stably causal. Indeed, \(\tau\) is a temporal function also for \(g_{\alpha}\) because the gradients of \(\tau\) for \(g\) and \(g_{\alpha}\) are pointwise proportional and their \(g\) and \(g_{\alpha}\)-orthogonal bundles are equal to \(\text{Ker}(d\tau)\), that is, positive definite (see also Remark 4.10).

3 \(\Rightarrow\) 1. Hawking’s proof for the case without boundary \([34, \text{Prop. 6.4.9}]\) also works here, because it is based on the integration of chronological futures and pasts type I\(^+\) and I\(^-\) by using and admissible measure \(m\) independent of the metric such that both \(\partial M\) and the boundaries of I\(^\pm\) have zero measure (these are the same reasons why Geroch’s construction of Cauchy time functions also work in the case with boundary, as detailed in Section 3).

The last assertion is trivial from the assertion 3; however, a direct proof from 1 is easy to obtain (see [1, Remark 3.9(b)]).

The higher levels of the ladder (related to Geroch’s proof of the splitting) will be revisited in Section 3. Its definitions (as optimized in [9, 39] for the case without boundary) are the following.

**Definition 2.14.** A spacetime with timelike boundary \((\overline{M}, g)\) is:

- causally continuous, when the set valued functions \(I^\pm : \overline{M} \to P(\overline{M})\) are both, one to one (that is, the spacetime is distinguishing) and continuous (for the natural topology in \(P(\overline{M})\) which admits as a basis the sets \(\{U_K : K \subset \overline{M}\text{ is compact}\}\), where \(U_K = \{A \subset \overline{M} : A \cap K = \emptyset\}\); see \([39, \text{Def. 3.37 to Prop. 3.38}]\));
- causally simple, when it is causal and all \(J^+(p), J^-(p), p \in \overline{M}\) are closed;
- globally hyperbolic, when it is causal and all \(J^+(p) \cap J^-(q), p, q \in \overline{M}\) are compact.

**2.3 Continuous vs Lipschitz/\(H^1\) causal curves**

Even though the basic definitions in Lorentzian Geometry are carried out with smooth elements (in particular, causal curves are regarded as piecewise smooth), continuous causal curves are required for relevant purposes. Indeed, a key result is the limit curve
Theorem [4, Prop. 3.31] which, under some hypotheses, ensures the existence of a limit curve to a sequence of causal ones, being the limit only continuous causal (even if the causal curves in the sequence are smooth). In the case of distinguishing spacetimes (without boundary), a continuous future-directed causal curve \( \gamma : I \subset \mathbb{R} \to M \) is any continuous curve that preserves the causal relation, that is, satisfying: \( t, t' \in I \) and \( t < t' \) implies \( \gamma(t) < \gamma(t') \), see [11, Prop. 3.19]; for non-distinguishing spacetimes, this property is required to be satisfied locally in arbitrarily small neighbourhoods, see [41, Sect. 3.5]. Continuous causal curves in a spacetime (without boundary) are known to satisfy a locally Lipschitz condition. This condition allows to identify these curves (when conveniently reparametrized) with \( H^1 \) curves; the latter are relevant for results on convergence of (parametrized) curves [11, 38]. More precisely, recall that a curve \( \gamma : J \to \mathbb{R}^n \) defined on a compact interval \( J \) is \( H^1 \) when it is absolutely continuous (equally, it satisfies both, differentiability almost everywhere and the Fundamental Theorem of Calculus, \( \gamma(t) = \gamma(0) + \int_0^t \gamma'(s)ds, \ t \in J \) and \( \gamma' \) is \( L^2 \) integrable; the set of all such curves is the Sobolev space \( H^1(J, \mathbb{R}^n) \). As Lipschitz curves are \( H^1 \) and any \( H^1 \) curve \( \gamma \) with \( |\gamma'| \) bounded a.e. by a constant is Lipschitz, both conditions will be interchangeable for continuous causal curves. The following is a natural extension to any interval \( I \) and manifold with boundary (see [1, §3.1.3] for further details).

**Definition 2.15.** Let \( \overline{M} \) be an \( n \)-manifold with boundary and \( I \subset \mathbb{R} \) any interval. A continuous curve \( \gamma : I \to \overline{M} \) is a \( H^1 \)-curve if, for any local chart \((U, \varphi)\), \( \varphi \circ \gamma \big|_J \) belongs to \( H^1(J, \mathbb{R}^n) \) for all compact intervals \( J \subset \gamma^{-1}(U) \). The space of \( H^1 \)-curves from \( I \) to \( \overline{M} \) will be denoted \( H^1(I, \overline{M}) \).

In the case that \((\overline{M}, g)\) is a spacetime with timelike boundary, \( \gamma \) is called future (resp. past) -directed \( H^1 \)-causal if it is \( H^1 \) and its a.e. derivative is future (resp. past) -directed \( H^1 \)-causal; \( \gamma \) is \( H^1 \)-causal if it is either future or past-directed \( H^1 \)-causal.

For manifolds without boundary, it is proven in [11, Appendix A] that a curve is \( H^1 \)-causal if and only if it is continuous causal (in the sense described above) up to a reparametrization; moreover, from the proof it is also clear that the reparametrization can be always carried out locally by using any temporal function (in this case, \( \gamma \) can be regarded as a Lipschitz function, according to the discussion above Defn. 2.15). The classical theorem of limit curves is then also valid in the framework of spacetimes with timelike boundary and \( H^1 \)-causal curves, see [4, Prop. 3.31] and [45, Lemma 3.23], namely, just extending \((\overline{M}, g)\) to a manifold without boundary (Prop. 2.1) and applying the results in this case (see [1, Prop. 3.16] for details), i.e.:

**Proposition 2.16.** Let \((\overline{M}, g)\) be a spacetime with timelike boundary and \( \{\gamma_m\}_m \) a sequence of future-directed inextensible \( H^1 \)-causal curves. If \( p \in \overline{M} \) is an accumulation point, then there exists a limit curve \( \gamma \) of \( \{\gamma_m\}_m \) which is a future-directed inextensible \( H^1 \)-causal curve which crosses \( p \).

**Remark 2.17.** For any strongly causal spacetime with timelike boundary the following alternative notion of continuous causal curve was introduced in Solis’ Ph.D. Thesis, see [45, Def. 3.19 and below]: a continuous curve \( \gamma : I \to \overline{M} \) is future-directed causal if for any \( t_0 \in I \) there exists a \( \overline{M} \)-convex neighbourhood \( U_0 \) around \( \gamma(t_0) \) (where \((\overline{M}, \overline{g})\) is a spacetime without boundary that extends \((\overline{M}, g)\), see Prop. 2.1) and an interval \([a, b] \subset I \) such that for all \( s, t \in [a, b] \) with \( s \leq t_0 \leq t \), one has \( \gamma(t) \in J^+(\gamma(s), U_0 \cap \overline{M}) \). This definition is shown to be independent of the chosen \((\overline{M}, \overline{g})\), and causality relations in \((\overline{M}, g)\) defined by using such continuous curves become equivalent to the classical ones with piecewise smooth ones, see [45, Remarks 3.20 and 3.21]. However, it is not obvious from the proof in [45, Lemma 3.23], whether the limit curve theorem for such curves will yield a limit curve which is also continuous causal according to previous definition.
This question will be clarified in a work in progress, anyway, it can be circumvented here just by assuming the following:

**Convention.** All the causal futures and pasts are computed always with $H^1$-curves and all the corresponding causal definitions are carried out accordingly. The consistency of this convention comes from: (a) $H^1$-causal curves are equivalent to classical continuous causal curves in manifolds without boundary, (b) they are intrinsic (extensions $\overline{M}$ are not required), and (c) they are preserved for limit curves as in the case without boundary.

Reasoning again as in [45] (but now taking into account previous convention), we deduce the following extensions to the case with boundary of the corresponding classical limit curves results [4, Prop. 3.34, Cor. 3.32]:

**Proposition 2.18.** (1) Let $(\overline{M},g)$ be a strongly causal spacetime with timelike boundary. Suppose that $\gamma_n$ is a sequence of causal curves defined on $[a,b]$ such that $\gamma_n(a) \to p$, $\gamma_n(b) \to q$. A causal curve $\gamma : [a,b] \to \overline{M}$ with $\gamma(a) = p$ and $\gamma(b) = q$ is a limit curve of $\{\gamma_n\}$ iff there is a subsequence $\{\gamma_m\}$ of $\{\gamma_n\}$ which converges to $\gamma$ in the $C^0$ topology on curves.

(2) Let $(\overline{M},g)$ be a globally hyperbolic spacetime with timelike boundary. Suppose that $\{p_n\}$ and $\{q_n\}$ are sequences in $\overline{M}$ converging to $p$ and $q$ in $\overline{M}$, resp., with $p \neq q$, and $p_n \leq q_n$ for each $n$. Let $\gamma_n$ be a future-directed causal curve from $p_n$ to $q_n$ for each $n$. Then there exists a future-directed causal limit curve $\gamma$ which joins $p$ to $q$.

One could still wonder if the causal future $J^+_{cl}(p)$ of any $p \in \overline{M}$ computed with $H^1$-causal curves coincides with the causal future $J^+_{ps}(p)$ computed with piecewise smooth ones (as defined primarily), and analogously for the past. Even though this question will be also clarified in a work in progress, it can be circumvented again for globally hyperbolic spacetimes, as follows.

**Proposition 2.19.** Let $(\overline{M},g)$ be a spacetime with timelike boundary. If it satisfies the definition of global hyperbolicity computing the causal futures and pasts by using piecewise smooth causal curves, then $J^+_{ps}(p) = J^+_{cl}(p)$ for all $p \in \overline{M}$. So, in this case, $(\overline{M},g)$ also satisfies global hyperbolicity by using $H^1$-causal curves.

**Proof.** Clearly, the last assertion suffices. Reasoning for the future, it reduces to prove

$$J^+_{cl}(p) \subset \text{cl}(I^+(p)) \quad \forall p \in \overline{M}$$

(3)

because of the chain of inclusions $J^+_{ps}(p) \subset J^+_{cl}(p) \subset \text{cl}(I^+(p)) = \text{cl}(J^+_{ps}(p)) = J^+_{ps}(p)$ would hold then (recall Prop. 2.6 and, for the last equality, notice that the causal simplicity of the spacetime follows easily, as in Prop. 3.7 below). Consider first the local version of (3): for any $p \in \overline{M}$ there exists some open neighborhood $V \subset \overline{M}$ such that

$$J^+_{cl}(q,V) \subset \text{cl}(I^+(q,V)) \quad \forall q \in V.$$

(4)

If $p \in M$, any convex $V$ suffices, as $J^+_{cl}(q,V) = J^+_{ps}(q,V)$ [11, Appendix A]. For $\hat{p} \in \partial M$ consider a neighborhood $V = (\mathcal{\epsilon}, \mathcal{\epsilon}) \times V_0$ with compact closure included in a neighborhood as in Cor. 2.5. Reducing $V_0$ if necessary, choose a product coordinate chart $(\tau, y^1, \ldots, y^{n-2}, y^{n-1}, s)$ defined on a cube, where the last coordinate satisfies $s = 0$ and $V \cap \partial M$. As causality is conformally invariant, we can also assume $\Lambda \equiv 1$, that is, $g = -d\tau^2 + g_r$. In order to compute $H^1$-norms and distances, consider just the natural Euclidean metrics $| \cdot |$ in these coordinates for both, $V$ and $V_0$, as well as for the tangent vectors. Recall that there exists $k_1, k_2 > 0$ such that $k_1| \cdot |^2 \leq g_r(\cdot, \cdot) \leq k_2| \cdot |^2$ for all $\tau \in (\mathcal{\epsilon}, \mathcal{\epsilon})$, and these inequalities also remain for the induced distances on $V_0$. 11
Let \( q_1 \in J^+(g,V)_{H^1}\), and let \( \gamma(\tau) = (\tau, y^1(\tau), \ldots, y^{n-1}(\tau), \tilde{s}(\tau)) \equiv (\tau, y(\tau))\), \( \tau \in [\tau_0, \tau_1] \) be the \( \tau \)-reparametrization of a \( H^1 \)-causal curve from \( q = (\tau_0, y_0) \) to \( q_1 = (\tau_1, y_1)\); thus, \( y(\tau) \) is absolutely continuous with \( g_{\tau}(y'(\tau), y'(\tau)) \leq 1 \) a.e. Following [11, Appendix, Lemma A.2], we will prove \((\tau_1, y_1) \in cl(I^+(\{(\tau_0, y_0), V\}))\), by showing that, for any \( \epsilon > 0 \), there exists some \( C^1 \) curve \( x_\epsilon(\tau) \) with \( x_\epsilon(\tau_0) = y_0, x_\epsilon(\tau_1) = y_1, \tilde{s}(x_\epsilon(\tau)) \geq 0 \) and \( g_{\tau}(x'_\epsilon, x''_\epsilon) < (1 + A \epsilon)^2 \), where \( A > 0 \) is independent of \( \epsilon \). Indeed, in this case the curve \( \gamma(\tau) = ((1 + A \epsilon)(\tau - \tau_0) + \tau_0, x_\epsilon(\tau)) \) is a \( C^1 \)-timelike curve between \( q = (\tau_0, y_0) \) and \( q_1 = ((1 + A \epsilon)(\tau_1 - \tau_0) + \tau_0, y_1) \), where \( \lim_{\epsilon \to 0} q_{1+\epsilon} = q_1 \).

To avoid the boundary for the smoothed curves \( x_\epsilon \), first \( y(\tau) \) will be perturbed into some \( y_\epsilon(\tau) \) as follows. For each \( 0 < \epsilon < 1 \) choose a smooth function \( \alpha_\epsilon(\tau) \), satisfying: (i) \( \alpha_\epsilon(\tau) \geq 0 \) with equality only at \( \tau_0, \tau_1 \), (ii) \( \alpha_\epsilon'(\tau_0) > 0, \alpha_\epsilon'(\tau_1) < 0 \), (iii) \( \alpha_\epsilon(\tau), |\alpha_\epsilon'(\tau)| < \epsilon \). Let \( y_\epsilon(\tau) := (y^1(\tau), \ldots, y^{n-1}(\tau), \tilde{s}(\tau) + \alpha_\epsilon(\tau)) \) for all \( \tau \in [\tau_0, \tau_1] \). Using (iii),

\[
g_{\tau}(y_\epsilon'(\tau), y_\epsilon''(\tau)) = g_{\tau}(y'(\tau) + \alpha_\epsilon'(\tau) \partial_3, y''(\tau) + \alpha_\epsilon''(\tau) \partial_3) = g_{\tau}(y'(\tau), y''(\tau)) + 2g_{\tau}(\alpha_\epsilon'(\tau) \partial_3, \alpha_\epsilon''(\tau) \partial_3) + g_{\tau}(\alpha_\epsilon'(\tau) \partial_3, \alpha_\epsilon'(\tau) \partial_3) < (1 + \epsilon A/2)^2,
\]

for some constant \( A > 1 \); in particular, \( y_\epsilon \) is an \( H^1 \)-curve from \( y_0 \) to \( y_1 \).

Now, notice that the requirement (i) for \( \alpha_\epsilon(\tau) \) implies that \( y_\epsilon \) has at most two points in the boundary (its endpoints); indeed, \( \tilde{s}(y_\epsilon(\tau)) > 0 \) for all \( \tau \in (\tau_0, \tau_1) \). Moreover, the requirement (iii) implies that the derivative of \( \tilde{s} \circ y_\epsilon \) has a definite sign a.e. close to the endpoints. Considering the extensions of the metric \( g \) and coordinates \((\tau, y)\) to some neighborhood \( \tilde{V} \) of the double manifold (extending consistently \( V_0 \) in some \( \tilde{V}_0 \)), the density of the space \( C^\infty([\tau_0, \tau_1], \tilde{V}_0) \) in \( H^1([\tau_0, \tau_1], \tilde{V}_0) \) can be used. Then, any \( C^\infty \) curve \( x_\epsilon \) with the same endpoints as \( y_\epsilon \), close enough to \( y_\epsilon \) in the \( H^1 \) norm will satisfy the required properties, namely: (a) \( x_\epsilon(\tau) \) remains in \( V_0 \) (and, moreover, it does not touch the boundary except at its endpoints). Indeed, for all enough \( H^1 \)-close \( C^\infty \) curves, the convergence of the functions implied by the \( H^1 \) norm (in addition to (i)) yields the required property outside any arbitrarily small neighborhood of the endpoints. Moreover, the convergence of the derivatives (in addition to (ii)) yields the property also in some small neighborhood of both endpoints, (b) \( g_{\tau}(x'_\epsilon, x''_\epsilon) < (1 + \epsilon A)^2 \), (by a straightforward use of the a.e. convergence of both, the functions and their derivatives).

To go from the local inclusion (4) to the global one (3), a standard procedure is followed. Take any \( q \in J^+_H(p) \) and let \( \gamma: I \to \overline{M} \) be some future-directed \( H^1 \) causal curve joining \( p \) with \( q \). From the local result, for any \( \gamma(r), r \in I \), there exists some open neighbourhood \( V_{\gamma(r)} \) such that \( J^+(q', V_{\gamma(r)}) \subset cl(I^+((q', V_{\gamma(r)}))) \) for all \( q' \in V_{\gamma(r)} \). So, take a Lebesgue number \( \delta > 0 \) for the open covering \( \{\gamma^{-1}(U_{\gamma(r)}), r \in I\}\) of \( I = [a, b] \), and choose a partition \( \{r_0 = a < r_1, \ldots, r_{l-1} < r_l = b\} \) with diameter smaller than \( \delta \). The case \( l = 1 \) is trivial, and assume by induction that (3) holds for \( l - 1 \). Let \( r \) be so that \( \gamma([r_{l-1}, r_{l}]) \subset V_{\gamma(r)} \) and, thus, \( \gamma(b) \in cl(I^+(\gamma(r_{l-1}), V_{\gamma(r)})) \). So, there is a sequence \( \{q_m\} \to q \) such that \( \gamma(r_{l-1}) \in I^-(q_m, V_{\gamma(r)}) \) for all \( m \). Therefore, for each \( q_m \) all the points in some neighborhood \( U_m \ni \gamma(r_{l-1}) \) lie in \( I^-((q_m, V_{\gamma(r)})) \). By the hypothesis of induction, some \( r \in U_m \) belongs to \( I^+(p) \) and, so, \( p \ll r \ll q_m \) for all \( m \), as required. \( \square \)

**Notation.** According to the convention in Remark 2.17, we will use the notation \( J^\pm_H(p) \) instead of \( J^\pm_H(p) \) in the remainder of the article. At any case if this notation had been adopted from the beginning of the article, everything would have been consistent.

### 3 Topological splitting and Geroch’s equivalence

#### 3.1 Higher steps of the causal ladder.

Geroch’s proof of the topological splitting of any globally hyperbolic spacetime (without boundary) \((M, g)\) is based on the existence of a time function constructed by computing
certain volumes using an appropriate measure \( m \). The conditions to be satisfied by \( m \) are very mild; indeed, they are satisfied by the measure associated to any semi-Riemannian metric \( g^* \) such that the total volume of the manifold is finite (so, one can choose \( g^* \) conformal to the original Lorentzian metric \( g \)). However, following Dieckmann ([21]; see also [39, section 3.7]) the abstract properties required for a measure on \( \overline{M} \) will be recalled first.

Given the spacetime with boundary \((\overline{M}, g)\), \( \overline{M} = M \cup \partial M \), consider the \( \sigma \)-algebra \( \mathcal{A}(\tau_{\overline{M}} \cup \Sigma) \) generated by the topology \( \tau_{\overline{M}} \) of \( \overline{M} \) in addition to the set \( \Sigma \) containing the zero-measure sets of \( \overline{M} \). Since \( M \) is an open subset of \( \overline{M} \), the \( \sigma \)-algebra \( \mathcal{A}(\tau_M \cup \Sigma) \) of \( M \) coincides with the induced \( \sigma \)-algebra of \( \mathcal{A}(\tau_{\overline{M}} \cup \Sigma) \) over \( M \), that is, with the set \( \{ E \cap M \mid E \in \mathcal{A}(\tau_{\overline{M}} \cup \Sigma) \} \). In a natural way, the measures on the previous \( \sigma \)-algebras will be called just measures on \( \overline{M} \) or \( M \), consistently.

It is straightforward to check that if \( m \) is a measure on \( M \) then it induces naturally a measure \( \overline{m} \) on \( \overline{M} \) just imposing \( \overline{m}(\partial M) = 0 \), that is,

\[
\overline{m}(A) := m(A \cap M) \quad \text{for any } A \in \mathcal{A}(\tau_{\overline{M}} \cup \Sigma).
\]

**Definition 3.1.** A measure \( \overline{m} \) on \( \overline{M} \) is admissible when it satisfies:

1. \( \overline{m}(\overline{M}) < \infty \);
2. \( \overline{m}(U) > 0 \) for any open subset \( U \subset \overline{M} \);
3. \( \overline{m}(\partial_+(p)) = 0 \) (\( \partial_+(p) \) denotes the topological boundary of \( \partial_+(p) \) in \( \overline{M} \)), \( \forall p \in \overline{M} \);
4. for any open subset \( U \subset \overline{M} \) there exists a sequence \( \{K_n\}_n \subset M \) of compact subsets such that \( K_n \subset K_{n+1}, K_n \subset U \) for all \( n \) and \( \overline{m}(U) = \lim_n \overline{m}(K_n) \).

**Proposition 3.2.** If \( m \) is an admissible measure on \( M \) then the induced measure \( \overline{m} \) on \( \overline{M} \) is also admissible.

**Proof.** Properties 1, 2, 4 in Def. 3.1 are straightforward. To prove 3, it is enough to check, say, \( m(\partial_-(p) \cap M) = 0 \); this holds because it is an achronal edgeless subset of \( M \) (\( \partial_-(p) \) can be written locally as a graph by using Cor. 2.5)\(^3\).

**Remark 3.3.** An alternative way to define an admissible measure \( \overline{m} \) on \( \overline{M} \) is to consider any admissible measure on an extension \( \hat{M} \) of \( (\overline{M}, g) \) (as in Prop. 2.1) and taking the restriction to \( \overline{M} \).

In what follows, an admissible measure \( \overline{m} \) is fixed on \( \overline{M} \).

**Definition 3.4.** The function \( t^{-}(p) := \overline{m}(\partial^{-}(p)) \) (resp. \( t^{+}(p) := -\overline{m}(\partial^{+}(p)) \)) is the past (resp. future) volume function associated to \( \overline{m} \).

Trivially, the volume functions are non-decreasing on any future-directed causal, but they are constant on any closed causal curve. The next two propositions hold as in the case without boundary (we refer to [1, Sect. 3.2.1] for detailed proofs).

**Proposition 3.5.** If \( (\overline{M}, g) \) is past (resp. future) distinguishing, the volume function \( t^{-} \) (resp. \( t^{+} \)) is (strictly) increasing over any future-directed causal curve \( \gamma \).

**Proposition 3.6.** For any spacetime with timelike boundary \((\overline{M}, g)\) the following properties are equivalent:

\(^3\)Notice also the inclusions \( \partial^{+}(p) \subset \text{cl}(\partial^{+}(p)) \subset \text{cl}(J^{+}(p)) = \text{cl}(I^{+}(p)) \), so the measure of \( \partial^{+}(p) \) coincides with the measure of \( \text{cl}(J^{+}(p)) \).
1. The set valued function $I^+$ (resp. $I^-$) is continuous on $\overline{M}$.

2. The volume function $t^+$ (resp. $t^-$) is continuous on $\overline{M}$.

3. $(\overline{M}, g)$ is past (resp. future) reflecting, that is, for all $p, q \in \overline{M}$:
   \[ q \in \text{cl}(I^+(p)) \Rightarrow p \in \text{cl}(I^-(q)) \quad \text{(resp. } p \in \text{cl}(I^-(q)) \Rightarrow q \in \text{cl}(I^+(p))) \]

So, a distinguishing spacetime with timelike boundary is causally continuous if and only if some/any of the previous equivalent properties (for the future and the past) hold, or equivalently, if and only if both volume functions $t^+, t^-$ are time functions.

The following result will allow to complete the implications of the ladder, taking into account the optimized definitions of global hyperbolicity and causal simplicity used here (consistent with [9]).

**Lemma 3.7.** Let $(\overline{M}, g)$ be a spacetime with timelike boundary.

(a) If it is causally simple then it is causally continuous.

(b) If it is globally hyperbolic then it is causally simple.

**Proof:** (a) We have to prove that is both, distinguishing and (by Prop. 3.6) future and past reflecting. For the former, following [9], if $p \neq q$ but, say, $I^+(p) = I^+(q)$ then choose any sequence $\{q_n\} \rightarrow q$ with $q \ll q_n$ and, thus, $p \ll q_n$. Then $q \in \text{cl}(I^+(p)) = \text{cl}(I^+(p)) = J^+(p)$ (the first equality by Prop. 2.6 and the second by hypothesis). Analogously, $p \in J^+(q)$ and there is a closed causal curve with endpoints at $p$ crossing $q$. For the latter property, causal simplicity implies $J^+(p) = \text{cl}(I^+(p))$ and, thus, the reflectivity becomes equivalent to the trivially true property $q \in J^+(p) \iff p \in J^-(q)$.

(b) Following [45, Props. 3.16, 3.17]), let us check that, say, $J^+(p)$ is closed. Let $r = \lim_m r_m$ with $r_m \in J^+(p)$. Since $\partial M$ is timelike, there exists some $q \in I^+(r)$ and $r_m \in I^-(q)$ for large $m$. Then, $r \in \text{cl}(J^+(p) \cap J^-(q)) = J^+(p) \cap J^-(q)$ (the latter by global hyperbolicity), and thus, $r \in J^+(p)$. ■

**Theorem 3.8.** Let $(\overline{M}, g)$ be a spacetime with timelike boundary.

1. If $(\overline{M}, g)$ is causally continuous then it is stably causal. Moreover, $(M, g|_M)$ is causally continuous and $(\partial M, g|_{\partial M})$ is stably causal.

2. If $(\overline{M}, g)$ is causally simple then it is causally continuous.

3. If $(\overline{M}, g)$ is globally hyperbolic then it is causally simple. Moreover, $(\partial M, g|_{\partial M})$ is globally hyperbolic too.

**Proof.** 1. The first assertion follows because any of the volume functions $t^+, t^-$ provides the required time function (recall Prop. 3.6). Moreover, $\partial M$ is also stably causal because, trivially, the restrictions of these functions to $\partial M$ are also time functions. The causal continuity of the interior $M$ is again a consequence of Prop. 3.6 taking into account that $t^\pm$ are both continuous on all $\overline{M}$ and their restrictions on $\partial M$ are also time functions. This is just Lemma 3.7 (a).

2. This first assertion is just Lemma 3.7 (b). The last assertion follows from [45, Prop. 3.15], we include the proof for completeness. As $\partial M$ is strongly causal, from previous items 1 and 2 it is enough to check that $\text{cl}(J^+(p, \partial M) \cap J^-(q, \partial M))$ is compact (see [4, Lemma 4.29]). Now, any sequence in this subset admits a subsequence converging to some $r \in J^+(p) \cap J^-(q)$ and, as $\partial M$ is closed in $\overline{M}$, $r \in \text{cl}(J^+(p, \partial M) \cap J^-(q, \partial M))$.  

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Remark 3.9. Thm. 3.8 and Prop. 2.12 allow us to reobtain the strict ordering of all the steps in the classical causal ladder in the case with boundary. The following examples show, in particular, that the inherited properties for $M$ and $\partial M$ are optimal.

(1) Start with the closed half space of Lorentz-Minkowski 3-space $\{(t, x, y) \in \mathbb{L}^3 : y \geq 0\}$ and remove the line $L = \{(0, 0, x) \mid x \geq 0\}$. Since the interior $M$ is causally continuous and the continuous extension of the $M$-volume functions to the whole $\overline{M}$ agree with the volume functions on $M$, the spacetime $\overline{M}$ is causally continuous (recall Prop. 3.6), but, clearly, $\partial M$ is not.

(2) Consider now $\overline{M} = \mathbb{L}^3 \setminus C$, where $C$ is the timelike cylinder $\mathbb{R} \times D$, being $D$ the disk $\{x^2 + y^2 < 1\} \subset \mathbb{R}^2$. Clearly, $J^+(p, M)$ is not closed whenever there exists a future-directed lightlike half-line $l$ starting at $p \in M$ and tangent to $C$ at some point $\hat{q} \in C$; indeed, the points in $l$ beyond $\hat{q}$ will lie in $\text{cl}(J^+(p, M)) \setminus J^+(p, M)$. That is, in general $M$ is not causally simple, even if $\overline{M}$ is globally hyperbolic with timelike boundary.

(3) Finally, consider the closure of the previous cylinder, $\overline{C} = \mathbb{R} \times \overline{D} \subset \mathbb{L}^3$, take the arc $A = \{(0, \cos \theta, \sin \theta) : 0 \leq \theta \leq \pi/4\}$ and consider the spacetime $\overline{M} = \overline{C} \setminus A$. Clearly, $\partial M$ is not causally continuous. However, $\overline{M}$ (and also $M$) is causally simple. Indeed, for each $p \in M$, it is obvious not only that $J^+(p, M)$ is closed but also that so is $J^\pm(p)$. This happens even if $p \in \partial M$ because any $q(\neq p)$ in the boundary of $J^\pm(p)$ can be joined with $p$ by means of a lightlike segment included in $M$ up to the endpoints$^4$.

3.2 Cauchy hypersurfaces and extended Geroch’s proof.

Formally, our notion of Cauchy hypersurface for a spacetime with timelike boundary is equal to the minimal one developed in [42] for the case without boundary.

Definition 3.10. Let $(\overline{M}, g)$ be a spacetime with timelike boundary. An achronal set $\Sigma \subset \overline{M}$ is a Cauchy hypersurface if it is intersected exactly once by every inextensible timelike curve.

Next, let us check some properties of Cauchy hypersurfaces (in particular, that they are truly hypersurfaces) by adapting the approach in [42, pp. 413-415].

Definition 3.11. Let $(\overline{M}, g)$ be a spacetime with timelike boundary of dimension $n$.

(1) A subset $\bar{S} \subset \overline{M}$ is a (embedded) topological hypersurface transverse to $\partial M$ if for any $p \in \bar{S}$ there exists an open neighborhood $V \subset \overline{M}$ of $p$ and a homeomorphism $\phi : V \to (0, \epsilon) \times N$, where $N$ is an open subset of $\mathbb{R}^{n-2} \times [0, \epsilon)$, such that $\phi(V \cap \bar{S}) = \{0\} \times N$. In this case, $\bar{S}$ is locally Lipschitz if the previous homeomorphism can be chosen Lipschitz when written in smooth coordinates.

(2) The edge of an achronal set $A \subset \overline{M}$ is the set of all the points $p \in \text{cl}(A)$ such that, for every open neighborhood $U \subset \overline{M}$ of $p$, there exists a timelike curve contained in $U$ from $I^-(p, U)$ to $I^+(p, U)$ that does not intersect $A$.

Proposition 3.12. Let $(\overline{M}, g)$ be a spacetime with timelike boundary. An achronal set $A$ with $A \cap \text{edge}(A) = \emptyset$ is a locally Lipschitz hypersurface transverse to $\partial M$.

Proof. The condition on edge($A$) implies that the achronal set $A \cap M$ has no edge points, thus, from [42, Prop. 14.25]) $A \cap M$ is a topological hypersurface without boundary in $(M, g)$. So, let $\hat{p} \in A \cap \partial M \neq \emptyset$ and let us construct a Lipschitz topological chart $(V, \phi)$ centered at $\hat{p} \in V \subset \overline{M}$ as in Defin. 3.11 (1). Since $A \cap \text{edge}(A) = \emptyset$, there exists some open neighbourhood $U$ of $\hat{p}$ such that any timelike curve contained in $U$ going from $I^-(\hat{p}, U)$ to $I^+(\hat{p}, U)$ intersects $A$. Without loss of generality, we can assume that

$^4$From a more general viewpoint, this property happens because $\partial M$ is strongly light-convex (see [12, Sect. 3.2 and Thm. 3.5] for further background and results)
\((U, \psi = (x^0, \ldots, x^{n-1}))\) is a Gaussian chart centered at \(\hat{p}\) such that \(\psi(U)\) is a cube. Consider the subset \((-\varepsilon, \varepsilon) \times N_0 \subset \psi(U)\), where \(N_0 = \{y \in \mathbb{R}^{n-1}_+ : (0, y) \in \psi(U)\}\). The required neighbourhood \(V \ni \hat{p}\) will be \(V = \psi^{-1}((-\varepsilon, \varepsilon) \times N)\), where \(N \subset N_0\) is any open ball centered at 0 for the natural Euclidean metric \(| \cdot |\) in the coordinates \(\hat{x} := (x^1, \ldots, x^{n-1})\), with small radius so that \(N\) is relatively compact in \(N_0\) and the slices \(x^0 = \pm \varepsilon/2\) satisfy:

\[
\{q \in V : x^0(q) = -\varepsilon/2\} \subset I^- (\hat{p}, U) \quad \text{and} \quad \{q \in V : x^0(q) = \varepsilon/2\} \subset I^+ (\hat{p}, U)
\]

(this can be achieved trivially because \(\partial_0\) is timelike). Now, the integral curve of \(\partial_0\) starting at any \((0, y) \in \{0\} \times N\) must intersect both, \(I^- (\hat{p}, U)\) and \(I^+ (\hat{p}, U)\). Thus, it must intersect \(A\) (recall \(U \supset V\) and \(\hat{p} \notin \text{edge}(A)\)) in a point \(y_q\), which is unique by the achronality of \(A\). So, \(A \cap V\) can be regarded as the graph of the function \(h : N \to (-\varepsilon, \varepsilon)\), \(h(q) := x^0(y_q)\). To show that \(h\) is Lipschitzian will be enough because, in this case, the desired chart \(\phi\) on \(V\) is just:

\[
\phi(\psi^{-1}(x^0, \ldots, x^{n-1})) = (x^0 - h(x^1, \ldots, x^{n-1}), x^2, \ldots, x^{n-1})
\]

The Lipschitz condition will be checked with respect to the distance \(| \cdot |\) induced in \(V\) by \(\psi\) from the Euclidean one. Recall first that, as \(cl(V)\) is compact and \(\partial_0\) is timelike, there exists some small \(c > 0\) such that the flat Lorentzian metric \(g_c = -c^2(dx^0)^2 + \sum_{i=1}^{n-1} (dx^i)^2\) satisfies \(g_c < g\). The Lipschitz condition \(|h(x) - h(y)| < c^{-1}|x - y|\) must hold for all \(x, y \in N\) because, otherwise, the points \((h(x), x), (h(y), y)\) would be (future or past) causally related for \(g_c\) and, thus, for \(g\) (in contradiction with the achronality of \(A\)). □

Other properties of achronal sets (including a converse to Prop. 3.12) can be found in [1, Sect. 3.2.3]. Next, we focus on consequences for Cauchy hypersurfaces.

**Corollary 3.13.** (1) An achronal set \(A\) with \(\text{edge}(A) = \emptyset\) is a closed (as a subset of \(M\)) locally Lipschitz hypersurface transverse to \(\partial M\).

(2) Let \(F \neq \emptyset, \overline{M}\) be a future set i.e \(I^+(F) = F\). Then, its topological boundary \(F\) is an achronal closed locally Lipschitz hypersurface transverse to \(\partial M\).

(3) Any Cauchy hypersurface \(\Sigma\) of \((\overline{M}, g)\) is an achronal closed locally Lipschitz hypersurface with boundary transverse to \(\partial M\).

**Proof.** (1) Prop. 3.12 implies that \(A\) is a locally Lipschitz hypersurface transverse to \(\partial M\). Closedness follows directly from the general inclusion \(cl(A) \subset A \subset \text{edge}(A)\), which happens because, if \(q \in cl(A) \setminus A\), no timelike curve through \(q\) can intersect \(A\) (\(A\) achronal implies \(cl(A)\) achronal, as the chronological relation is also open in the case with boundary; recall Prop. 2.6 (a)) and, so, \(q \in \text{edge}(A)\).

(2) Taking into account elementary properties of transitivity (Prop. 2.6),

\[
I^+(\hat{F}) \subset \text{interior}(\hat{F}), \quad I^-(\hat{F}) \subset \overline{M} \setminus \text{cl}(\hat{F}), \quad \text{in particular,} \quad I^+(\hat{F}) \cap I^-(\hat{F}) = \emptyset.
\]

From this last equality \(\hat{F}\) is achronal. From the part (1), to prove \(\text{edge}(\hat{F}) = \emptyset\) suffices and, since \(\hat{F}\) is closed, \(\text{edge}(\hat{F}) \subset \hat{F}\), that is, to prove \(\text{edge}(\hat{F}) \cap \hat{F} = \emptyset\) suffices too. Assuming \(p \in \text{edge}(\hat{F}) \cap \hat{F}\), there exists a timelike curve starting at \(I^-(p)\) (thus, in \(\overline{M} \setminus \text{cl}(\hat{F})\)) and ending at \(I^+(p)\) (in \(\text{int}(\hat{F})\)) without crossing \(\hat{F}\), a contradiction.

(3) Clearly, \(\overline{M}\) is the disjoint union \(\overline{M} = I^+(\Sigma) \cup \Sigma \cup I^-(\Sigma)\). So, \(\Sigma\) is the boundary of the future set \(\hat{F} = \overline{F} = I^+(\Sigma)\) and the part (2) applies. □

Next, let us re-take the existence of a Cauchy time function. Let

\[
t : \overline{M} \to \mathbb{R}, \quad t(p) := \ln \left( \frac{I^- (p)}{I^+ (p)} \right) \quad (5)
\]
be a Geroch function. From Thm. 3.8 and Prop. 3.6, this function is continuous for any globally hyperbolic spacetime with timelike boundary. Thus, we have the elements to extend Geroch’s equivalence to the case with boundary.

**Theorem 3.14.** For any globally hyperbolic spacetime with timelike boundary \((\mathcal{M}, g)\), Geroch function \(t\) in (5) is a (acausal) Cauchy time function, that is, \(t\) is a time function and all its levels are acausal Cauchy hypersurfaces. Thus, if \(\Sigma_0 = t^{-1}(0)\) then \(\mathcal{M}\) is homeomorphic to \(\mathbb{R} \times \Sigma_0\), and any other Cauchy hypersurface \(\Sigma\) is homeomorphic to \(\Sigma_0\).

Conversely, any spacetime with timelike boundary admitting a Cauchy hypersurface is globally hyperbolic.

**Proof.** As the sum \(t\) of the time functions \(\ln t^-\) and \(\ln(-t^+)\) is also a time function, its levels are acausal, and \(t\) will be Cauchy if, for any inextendible past-directed causal curve \(\gamma : (a, b) \to \mathcal{M}\),

\[
\lim_{s \to a} t^-(\gamma(s)) = 0, \quad \text{and analogously } \lim_{s \to b} t^+(\gamma(s)) = 0,
\]

so that \(\lim_{s \to a} t(\gamma(s)) = -\infty\) and \(\lim_{s \to b} t(\gamma(s)) = \infty\) (as in the case without boundary). To check \(\lim_{s \to a} t^-(\gamma(s)) = 0\), recall that the measure of \(\mathcal{M}\) can be approximated by compact subsets (Prop. 3.2). So, for any \(\epsilon > 0\) there exists some compact \(K \subset \mathcal{M}\) with \(\overline{\mathcal{M}} \setminus K < \epsilon\) and one has just to show that there exists \(s_0 \in (a, b)\) such that \(I^- \gamma(s_0) \cap K = \emptyset\) (as this implies \(t^-\gamma(s)) = \overline{\mathcal{M}} \setminus \mathcal{M} \setminus m(K) < \epsilon\), for all \(s \leq s_0\). Assuming by contradiction the existence of a sequence \(\{s_m\} \to a\) with \(I^-\gamma(s_m) \cap K \neq \emptyset\) for all \(m\), there exists a sequence \(\{r_m\} \subset K\) with \(r_m \in I^-\gamma(s_m)\); by the compactness of \(K\), \(\{r_m\} \to r \in K\) up to a subsequence. Taking \(p \in I^-r\) and some fixed \(q = \gamma(c)\), one has \(p \ll r_m \ll \gamma(s_m) \ll q\) for large \(m\). This implies \(\gamma(a, c) \subset J^+(p) \cap J^-(q)\), that is, a past inextendible causal curve is imprisoned in the compact subset \(J^+(p) \cap J^-(q)\), which contradicts Prop. 2.11 (b).

To obtain the homeomorphism, Prop. 2.4 ensures the existence of a future-directed timelike vector field \(T \in \mathfrak{X}(\mathcal{M})\) whose restriction to \(\partial\mathcal{M}\) is tangent to \(\partial\mathcal{M}\). With no loss of generality, \(T\) can be chosen unitary for some auxiliary complete Riemannian metric \(g_R\) on \(\mathcal{M}\). Then, \(T\) is complete and, so, its integral curves are inextendible timelike curves in \(\mathcal{M}\). From the part (a), \(t\) diverges along the integral curves of \(T\), and each integral curve of \(T\) intersects \(\Sigma_0 = t^{-1}(0)\) at a unique point \(x \in \Sigma_0\), which will be regarded as the initial point of each integral curve \(\gamma_x\) of \(T\). So, every \(p \in \mathcal{M}\) can be written univocally as \(\gamma_x(t)\) for some \(x \in \Sigma_0\) and \(t \in \mathbb{R}\). Therefore, the map \(\Psi : \mathcal{M} \to \mathbb{R} \times \Sigma_0\), \(p \mapsto (t, x)\) is continuous, bijective and maps boundaries into boundaries. For the continuity of \(\Psi^{-1}\), recall that, in the case without boundary, it is straightforward by the theorem of invariance of the domain. In the case with boundary just apply it to the double manifold \(\mathcal{M}'\). That is, notice that the identification of homologous points of the two copies of \(\mathcal{M}\) yields also an extended hypersurface without boundary \(\Sigma_0^d\) (recall that Cor. 3.13 (3) ensures that the boundary of \(\Sigma_0\) is included in \(\partial\mathcal{M}\)); so, extend naturally \(\Psi\) to a continuous bijective map \(\Psi^d : \mathcal{M}' \to \mathbb{R} \times \Sigma_0^d\), which will be then a homeomorphism.

Moreover, for any Cauchy hypersurface \(\Sigma\) clearly \(\Psi(\Sigma)\) can be regarded as the graph of a locally Lipschitz function \(h : \Sigma_0 \to \mathbb{R}\). So, the continuos map \(\Sigma_0 \ni x \mapsto (h(x), x) \in \Psi(\Sigma)\) is the required homeomorphism.

Finally, the converse follows as in the case without boundary (see [48, Th.8.3.10], [42, Cor. 14.39] for the proof) just taking into account that the limit curve theorem (which is the essential tool for that case) still holds here (Prop. 2.16).
4 Stability

4.1 Results for global hyperbolicity and Cauchy temporal functions

Next, our aim is to prove the following two theorems.

**Theorem 4.1** (Stability of global hyperbolicity). Let $(\overline{M}, g)$ be a globally hyperbolic spacetime with timelike boundary. Then, there exists a metric $g' > g$ (thus, necessarily with timelike boundary) which is also globally hyperbolic.

This result will be used to prove that such a spacetime $(\overline{M}, g)$ admits a Cauchy temporal function $\tau$ with gradient $\nabla \tau$ tangent to $\partial M$ by reducing the problem to the simple case when $\partial M$ is totally geodesic. However, once the existence of such functions is established, the following stronger stability result for the restrictive properties of the obtained $\tau$ will become interesting in its own right.

**Theorem 4.2** (Stability of Cauchy temporal functions). If $\tau$ is a Cauchy temporal function for a globally hyperbolic metric $g$ with $\nabla \tau$ tangent to $\partial M$, then there exists $g'$ with wider timecones, $g < g'$, such that $\tau$ is also a Cauchy temporal function for $g'$ with $g'$-gradient tangent to $\partial M$.

Moreover, $\tau$ is Cauchy temporal for any $g'' \leq g'$ (with $g''$-gradient not necessarily tangent to $\partial M$). In particular, its level sets $\Sigma_{\tau_0}$, $\tau_0 \in \mathbb{R}$, are spacelike Cauchy hypersurfaces for any such $g''$.

**Remark 4.3.** (1) As in the case without boundary, the last assertion of Thm. 4.2 does not hold for an arbitrary Geroch time function $t$ (even if $t$ is smooth) because its levels may be degenerate. That is, independently of the presence of the boundary, we can say that a smooth time function $\tau$ for $g$ is also a smooth time function for some $g' > g$ if and only if $\tau$ is a temporal function for $g$.

(2) We will give a direct complete proof of Thm. 4.1 for the sake of completeness, which may have interest to compare with previous techniques in [32, 5, 25]. However, it is worth emphasizing that most of this proof can be skipped. Indeed, from the hypotheses of Thm. 4.1 one can construct a Cauchy temporal function $\tau_0$ on $\overline{M}$ with no restriction on $\partial M$ just by working exactly as in the case without boundary (say, as in [7] or [41], see also Remark 5.6). From here, one can reproduce the proof of Thm. 4.1 developed in the following subsections, but skipping the subtle details associated to working there with a Geroch time function instead of a temporal one (see Remark 5.7 for further details).

Summing up, Thms. 4.1 and 4.2 (with Remark 5.7) yield two proofs of stability in both, the case with and without boundary, one of them direct (and more technical) and the other one by using an auxiliary Cauchy temporal function\(^5\) $\tau_0$.

4.2 Techniques to perturb $g$ maintaining Cauchy hypersurfaces

To prove Thm. 4.1, let $t$ be a prescribed Geroch’s time function on $(\overline{M}, g)$, so that, topologically $\overline{M} \equiv \mathbb{R} \times \Sigma$ with $\Sigma$ Cauchy and acausal. We will also consider a complete Riemannian metric $g_t$ on $\overline{M}$ with associated distance $d_R$. For any two non-empty subsets $A, B \subset \overline{M}$ we denote $J^+(A, B) := J^+(A) \cap J^-(B)$. Moreover, $d_R(A, B)$ will denote the $d_R$-distance between $A, B$ (i.e., the infimum of the set $\{d_R(x, y) : x \in A, y \in B\}$) and $d_H(A, B)$ the Hausdorff distance associated with $d_R$ between the two sets (i.e., the infimum of the $r \geq 0$ such that $d_R(x, B) \leq r$ for all $x \in A$ and $d_R(A, y) \leq r$ for all $y \in B$). The latter will be applied to relatively compact subsets $A, B \subset \overline{M}$ (recall that $d_H$ becomes a true distance in the set of compact non-empty subsets). When $A = \{p\}, p \in \overline{M}$, the notation will be simplified to $A = p$.

\(^5\) Compare with [32, Sect. 6] and the Section 3 in [5] (arxiv version).
Let $\omega$ be the 1-form metrically associated with any timelike vector field on $\overline{M}$. For any function $\alpha \geq 0$ on $\overline{M}$, consider the metric:

$$g_\alpha = g - \alpha \cdot \omega^2. \quad (6)$$

Whenever $\alpha > 0$, one has $g < g_\alpha$. We will add the subscript $\alpha$ to denote elements computed with $g_\alpha$ (for example, $J_\alpha^+(p)$ is the $g_\alpha$-causal future of $p$). Our final aim is to prove that some $\alpha > 0$ can be chosen small enough so that $g_\alpha$ remains causal with compact $J_\alpha(p,q)$ for all $p,q$, and thus, $(\overline{M}, g_\alpha)$ is globally hyperbolic with timelike boundary. Let us introduce a working definition.

**Definition 4.4.** Fix an open neighborhood $U_0 \subset \overline{M}$ bounded by two $t$-levels, i.e. $U_0 \subset J(\Sigma_{t_-}, \Sigma_{t_+})$, for some $t_-, t_+$, which will be assumed to be optimal\(^6\). Let $C_0 \subset \overline{M}$ be a closed subset included in $U_0$, and choose $\epsilon_0 > 0$.

A $t$-perturbation of $g$ with wider timecones on $C_0$, support in $U_0$ and $d_H$-distance smaller than $\epsilon_0$, is any smooth function $\alpha \geq 0$ satisfying:

(i) $\alpha > 0$ on $C_0$ (thus, $g < g_\alpha$ on $C_0$),

(ii) $\alpha \equiv 0$ outside $U_0$ (i.e., $g \equiv g_\alpha$ on $\overline{M}\setminus U_0$),

(iii) $g_\alpha$ is globally hyperbolic and it admits $\Sigma_{t_-}, \Sigma_{t_+}$ (and, thus, all $\Sigma_t$, $t \in \mathbb{R}\setminus [t_-, t_+]$) as Cauchy hypersurfaces.

(iv) the following bounds hold for all $p \in J(\Sigma_{t_-}, \Sigma_{t_+})$:

$$d_H(J(p, \Sigma_{t_-}', p), J_\alpha(p, \Sigma_{t_-}') < \epsilon_0 \quad \text{if} \ t_-' \in [t(p), t_+],$$

$$d_H(J(\Sigma_{t_-}', p), J_\alpha(\Sigma_{t_-}', p)) < \epsilon_0 \quad \text{if} \ t_- \in [t_-, t(p)],$$

$$d_H(p, J_\alpha(p, \Sigma_{t_-}')) < \epsilon_0 \quad \text{if} \ t_- \in [t_-', t(p)] \text{ and } J_\alpha(p, \Sigma_{t_-}') \neq \emptyset,$$

$$d_H(p, J_\alpha(\Sigma_{t_-}', p)) < \epsilon_0 \quad \text{if} \ t_- \in [t(p), t_+] \text{ and } J_\alpha(\Sigma_{t_-}', p) \neq \emptyset.$$  

**Remark 4.5.** The conditions in this definition imply strong restrictions on the sets type $J_\alpha(p, \Sigma_{t_-}')$ such as compactness. Indeed, $d_H(J(p, \Sigma_{t_-}'), J_\alpha(p, \Sigma_{t_-}')) < \epsilon_0$ implies that $J_\alpha(p) \cap \Sigma_{t_-}'$ lies in a compact set $D$ of $\Sigma_{t_-}'$. Hence

$$J_\alpha(p, \Sigma_{t_-}') = J_\alpha(p, J_\alpha(p) \cap \Sigma_{t_-}') = J_\alpha(p, D). \quad (7)$$

As the global hyperbolicity of $g_\alpha$ implies that $J_\alpha(A, B)$ is compact for any compact $A, B \subset \overline{M}$, the set $J_\alpha(p, \Sigma_{t_-}') = J_\alpha(p, D)$ in (7) is compact too. In particular, $J_\alpha(p) \cap \Sigma_{t_-}' = J_\alpha(p, \Sigma_{t_-}') \cap \Sigma_{t_-}'$ is also compact.

Notice also that, if $d_H(J(p, \Sigma_{t_-}'), J_\alpha(p, \Sigma_{t_-}')) < \epsilon_0$, then for any $\alpha'$ with $0 \leq \alpha' \leq \alpha$ one has $J(p, \Sigma_{t_-}') \subset J_{\alpha'}(p, \Sigma_{t_-}') \subset J_\alpha(p, \Sigma_{t_-}')$, and thus, $d_H(J(p, \Sigma_{t_-}'), J_{\alpha'}(p, \Sigma_{t_-}')) < \epsilon_0$.

Finally, recall also that if $\alpha$ is such a perturbation then so is $\alpha/N$ for any $N > 1$.

**Lemma 4.6.** (Existence of a $t$-perturbation for a compact set). For any compact $(C_0 = )K_0 \subset \overline{M}$, any open neighborhood $U_0 \subset J(\Sigma_{t_-}, \Sigma_{t_+})$ of $K_0$ bounded by two $t$-levels $t_\pm$ and any $\epsilon_0 > 0$, there exists a $t$-perturbation of $g$ as in Defn. 4.4.

**Proof.** With no loss of generality, we can assume that the closure $\text{cl}(U_0)$ is compact (otherwise, take any compact $K_0'$ with $K_0 \subset \text{int}(K_0')$, redefine $U_0$ as $U_0 \cap \text{int}(K_0')$ and, eventually, take a bigger $t_-$ and smaller $t_+$ to ensure that they remain optimal). Let $\alpha \geq 0$ be any function with support in $U_0$ and $\alpha > 0$ on $K_0$. We will check that the required properties hold for some $\alpha_m := \alpha/m, m \in \mathbb{N}$ with large $m$. Recall that, necessarily, $K := J_\alpha(\text{cl}(U_0), \Sigma_{t_-}) \cup J_\alpha(\Sigma_{t_-}, \text{cl}(U_0))$ is equal to $J(\text{cl}(U_0), \Sigma_{t_-}) \cup J(\Sigma_{t_-}, \text{cl}(U_0))$ and, so, it is compact.

\(^6\)That is, with no loss of generality we will always assume that no $t_- \geq t_-, t_+ \leq t_+$ satisfy $U_0 \subset J(\Sigma_{t_-}, \Sigma_{t_+}')$ if some of the two inequalities is strict.
Let us see first that $g_{\alpha m}$ is strongly causal for large $m$. Assume by contradiction that the strong causality of each $g_{\alpha m}$ is violated at some $p_m \in M$. Necessarily, $p_m \in K$ and, up to a subsequence, it will converge to some $p \in K$. Around $p$, choose a small neighborhood $W$ with $(W, g_o)$ intrinsically strongly causal, and a smaller one $W \subset W$ which is $g_o$-causally convex in $\overline{W}$. Up to a finite number, $p_m \in W$ and, thus, there exists some $g_{\alpha m}$-causal curve $\rho_m$ with endpoints in $W$ not included in $\overline{W}$: indeed, all $\rho_m$ must escape $\overline{W}$ and come back. Necessarily, $\rho_m$ is entirely contained in $K$ (recall footnote 7). As all $\rho_m$ are $g_o$-causal, the sequence $\{\rho_m\}$ will have a limit curve $\rho$ (also included in $K$) starting at $p$. The constructive method of the limit curve implies that $\rho$ is $H^1$-causal for all $g_{\alpha m}$, which implies that $\rho$ is $H^1$-causal for $g$ too. Indeed, the velocity of $\rho$ (whenever it is differentiable) must be causal for all $g_m$ and, as $\{g_{\alpha m}\} \rightarrow g$ (uniformly on all $\overline{M}$ by construction), it is causal for $g$. Moreover, since $\rho$ remains in the compact set $K$, a contradiction with the strong causality of $g$ appears: either $\rho$ is closed or it is inextensible but imprisoned in a compact subset.

Now, let us check that, for any $m$ such that $g_{\alpha m}$ is strongly causal, the property (iii) holds. Indeed, no inextensible future-directed $g_{\alpha m}$-causal curve $\gamma : \mathbb{R} \rightarrow \overline{M}$ can be partially imprisoned to the future (resp. past) in the compact set $K$. So, either $\gamma$ does not intersect $K$ (and, trivially, will cross $\Sigma_\pm$, $\Sigma_{\pm\pm}$ once) or there is a last point $\gamma(s_0^\pm)$ (resp. $\gamma(s_0^-)$) such that $\gamma\big|_{(s_0^+, \infty)}$ (resp. $\gamma\big|_{(-\infty, s_0^-)}$) does not intersect $K$. In this case, as $\gamma(s_0^+) \in J^- (\Sigma_{\pm})$ (resp. $\gamma(s_0^-) \in J^+ (\Sigma_{\pm})$), then $\gamma\big|_{(s_0^+, \infty)}$ (resp. $\gamma\big|_{(-\infty, s_0^-)}$) touches $\Sigma_{t+}$ (resp. $\Sigma_{t-}$) exactly once. Clearly, $\gamma\big|_{(s_0^+, s_0^+)}$ cannot touch $\Sigma_{t+}$ (resp. $\Sigma_{t-}$) at some $\gamma(s_0)$ with $s_0 \in (s_0^-, s_0^+)$ because in this case $\gamma$ would be contained in $I^+ (\Sigma_{t+})$ (resp. $I^- (\Sigma_{t-})$) beyond $s_0$ towards $+\infty$ (resp. $-\infty$).

To prove that the property (iv) can be also ensured, let us reason by contradiction (taking always values of $m$ bigger than the one required for strong causality and, thus, global hyperbolicity of $g_{\alpha m}$). Consider first the case that there exists $t_m^+ \in [t_-, t(q))]$ (so, $J(q_m, \Sigma_{t_m^+}) = \emptyset$) such that

$$d_H(q_m, J_{\alpha m}(q_m, \Sigma_{t_m^+})) \geq \epsilon_0, \quad \forall m \in \mathbb{N}.$$  

Up to subsequences, $\{q_m\}$ converges to some $q \in K$ and $\{t_m^+\} \rightarrow t^+ (\in [t_-, t(q)])$. Take a point $r_m' \in J_{\alpha m}(q_m, \Sigma_{t_m^+}))$ which realizes the Hausdorff distance to $q_m$, thus $d_R(q_m, r_m') \geq \epsilon_0$. Choose any second point $r_m'' \in J_{\alpha m}(r_m', \Sigma_{t_m^+}) \cap \Sigma_{t_m^+}$ (which must exist from the definition of $J_{\alpha m}(q_m, \Sigma_{t_m^+})$). Up to subsequences, $\{r_m'\} \rightarrow r' \in K$ and $\{r_m''\} \rightarrow r'' \in \Sigma_{t_m^+} \cap K$; moreover,

$$d_R(q, r') \geq \epsilon_0 > 0. \quad (8)$$

Now (reasoning as for strong causality), the sequence $\{\gamma_m\}$ of future-directed $g_{\alpha m}$-causal curves from $q_m$ to $r_m''$ through $r_m'$ must have a limit curve $\gamma$ which is $H^1$-causal for $g$. As $g$ is globally hyperbolic, $\gamma$ goes from $q$ to $r''$ through $r'$. Moreover, $t$ is a $g$-time function, $t(q) \geq t(r'')$ and $\Sigma_{t(q)}$ is $g$-causal; thus, $\gamma$ is a constant and $q = r' = r''$, in contradiction with (8).

Consider now the case that, say, there exists $q_m \in J(\Sigma_{t-}, \Sigma_{t+})$ and $t_m^+ \in [t(q_m), t_+]$:

$$d_H(J(q_m, \Sigma_{t_m^+}), J_{\alpha m}(q_m, \Sigma_{t_m^+})) \geq \epsilon_0, \quad \forall m \in \mathbb{N}.$$  

\footnote{Otherwise, trivially, $q_m \in J(\Sigma_{t-}, \Sigma_{t+})$ and $p_m$ would admit neighborhood $V$ in $J(\Sigma_{t-}, \Sigma_{t+}) \setminus K$ where strong causality is violated. However, no causal curve starting at some $q \in V$ would cross $U_0$ (as $q \in K$ otherwise). So, strong causality would be violated in $(M, g)$, a contradiction.}

\footnote{Here, a small subtlety appears. In a strongly causal spacetime, any limit curve of causal curves is $H^1$-causal. However, if the spacetime is not strongly causal, a limit curve may be non-causal. Anyway, the constructive procedure of a limit curve by using Arzela’s theorem [4, Prop. 3.31] implies the existence of a limit curve for $g_o$ and, clearly, this curve will be $g_{\alpha m}$-causal for all $m$ as $g_{\alpha m+1} < g_{\alpha m}$ for all $m$.}
Necessarily, $q_m \in K$ and there exists $r_m \in J(q_m, \Sigma_m^n)$, $r'_m \in J_{\alpha_m}(q_m, \Sigma_m^n)$ such that $d_R(r_m, r'_m) = d_H(J(q_m, \Sigma_m^n), J_{\alpha_m}(q_m, \Sigma_m^n)) \geq \epsilon_0$ for all $m$. Up to subsequences, we can assume that $(q_m), (r_m), (r'_m)$ converge to some $q, r, r' \in K$, resp., and $r_m^{\alpha} \to t_+^\epsilon + \{ t(q), t_+ \}$. Necessarily, $d_H(r, r') \geq \epsilon_0$ and, thus, $r' \notin J(q, \Sigma_m^n)$. However, now the $g_{\alpha_m}$-causal curves $\gamma_m$ from $q_m$ to $r'_m$ will have a $g$-causal limit curve $\gamma$ starting at $q$ and arriving at $r'$, that is, $r' \in J^+(q)$. To check $t(r') \leq t'_+^\epsilon$ (and, thus, the contradiction $r' \in J^+(q, \Sigma_m^n)$), note that otherwise, $d_H(r', J_{\alpha_m}(r', \Sigma_{t'}^n) \geq d_H(r', \Sigma_{t'}^n \cap K) > 0$ for all $m$, in contradiction with the previous case.

**Proposition 4.7. (Existence of a $t$-perturbation for a strip)** Let $C$ be the closed strip $J(\Sigma_1, \Sigma_2)$, $t_1 < t_2$, let $U = (t_-, t_+) \times \Sigma$ be a neighborhood of $C$ (i.e. $t_- < t_1, t_2 < t_+$), and choose $\epsilon > 0$. Then, there exists a $t$-perturbation of $g$ for $C, U$ and $\epsilon$ as in Defn. 4.4.

**Proof.** We can assume that $\Sigma$ is not compact (otherwise, the result would follow directly from Lemma 4.6). Let $\{B_m\}$ be an exhaustion of $\Sigma$ by compact subsets, i.e. $B_m \subset \text{int}(B_{m+1})$ (where the interior of $B_{m+1}$ is regarded as a subset of the topological space $\Sigma$) and $\Sigma = \bigcup_m B_m$. By convenience, put $B_0 := B_{-1} := \emptyset$, and let $K_m := [t_1, t_2] \times B_m$ and $U_m := (t_-, t_+) \times \text{int}(B_{m+1})$ for all $m \in \mathbb{N} \cup \{0, -1\}$; recall $K_m \subset U_m$.

Let us construct $\alpha$ inductively as follows. Consider first a perturbation $\alpha_1$ obtained by applying Lemma 4.6 to the compact set $K_1$, its neighborhood $U_1$ and putting $\epsilon_0 := \epsilon_1$ equal to $\epsilon/2$. Assuming inductively that $\alpha_m$ has been defined (and, thus, the corresponding globally hyperbolic metric $g_{\alpha_m}$), let $\sigma(m + 1)$ be the first integer greater than $\sigma(m)$ such that

$$J_{\alpha_m}(\text{cl}(U_{\sigma(m)}), \Sigma_{t_-}) \cup J_{\alpha_m}(\Sigma_{t_-}, \text{cl}(U_{\sigma(m)})) \subset U_{\sigma(m+1)-1} \cup \Sigma_{t_-} \cup \Sigma_{t_+} \quad (10)$$

($\sigma(m + 1)$ exists as the subset of the left-hand side is compact). Now, obtain $\alpha_m$ by applying Lemma 4.6 to the metric $g_{\alpha_m}$, the compact set $K_{\sigma(m+1)} \setminus \text{int}(K_{\sigma(m)})$, its neighborhood $U_{\sigma(m+1)} \setminus \text{cl}(U_{\sigma(m)-2})$ and choosing some positive $\epsilon_0 := \epsilon_2^{m+1}$ small so that, the relations (10) still hold when the causal futures and pasts are computed with $g_{\alpha_{m+1}}$ instead of $g_{\alpha_m}$. Namely, $\epsilon_{m+1}$ is taken also smaller than the minimum of:

$$d_R([t_-, t_+]) \times \hat{B}_{\sigma(m+1)-1}, J_{\alpha_m}(\text{cl}(U_{\sigma(m)}), \Sigma_{t_-}) \cup J_{\alpha_m}(\Sigma_{t_-}, \text{cl}(U_{\sigma(m)})))$$

(11)

$$d_R([t_-, t_+]) \times \hat{B}_{\sigma(m)-1}, J_{\alpha_m}(\text{cl}(U_{\sigma(m)-1}), \Sigma_{t_-}) \cup J_{\alpha_m}(\Sigma_{t_-}, \text{cl}(U_{\sigma(m-1)}))) \quad (12)$$

(where $U$ denotes the topological boundary of the corresponding subset, and the distance in the second line is taken into account only for $m > 1$). Notice that both, (11) and (12) are positive, as both are distances between compact disjoint sets; in particular, the second distance is lower bounded by

$$d_R([t_-, t_+]) \times \hat{B}_{\sigma(m)-1}, J_{\alpha_{m-1}}(\text{cl}(U_{\sigma(m-1)}), \Sigma_{t_-}) \cup J_{\alpha_{m-1}}(\Sigma_{t_-}, \text{cl}(U_{\sigma(m-1)}))) - \epsilon_m > 0$$

These requirements for $\epsilon_{m+1}$ ensure, for all $k \geq 1$:

$$J_{\alpha_m+2}(\text{cl}(U_{\sigma(m)}), \Sigma_{t_-}) = J_{\alpha_m+2+k}(\text{cl}(U_{\sigma(m)}), \Sigma_{t_-}),$$

$$J_{\alpha_m+2}(\Sigma_{t_-}, \text{cl}(U_{\sigma(m)})) = J_{\alpha_m+2+k}(\Sigma_{t_-}, \text{cl}(U_{\sigma(m)})).$$

(13)

[9] Recall that if $A, B, C$ are three compact subsets of $\overline{M}$ then $d_R(A, C) \leq d_R(A, B) + d_R(B, C)$, and apply this to $A = [t_-, t_+] \times \hat{B}_{\sigma(m)-1}, B = J_{\alpha_m}(\text{cl}(U_{\sigma(m-1)}), \Sigma_{t_-}) \cup J_{\alpha_m}(\Sigma_{t_-}, \text{cl}(U_{\sigma(m-1)})))$ and $C = J_{\alpha_{m-1}}(\text{cl}(U_{\sigma(m-1)}), \Sigma_{t_-}) \cup J_{\alpha_{m-1}}(\Sigma_{t_-}, \text{cl}(U_{\sigma(m-1)}))).$

[10] For the first eqn. (13), recall: $J_{\alpha_m+1}(\text{cl}(U_{\sigma(m)}), \Sigma_{t_-}) \subset U_{\sigma(m+1)-1}$ (once determined the value of $\sigma(m + 1)$, by using the bound (11) for $\epsilon_{m+1}$), $J_{\alpha_m+2}(\text{cl}(U_{\sigma(m)}), \Sigma_{t_-}) \subset U_{\sigma(m+1)-1}$ (use (12)) and $J_{\alpha_m+2+k}(\text{cl}(U_{\sigma(m)}), \Sigma_{t_-}) = J_{\alpha_m+2+k}(\text{cl}(U_{\sigma(m)}), \Sigma_{t_-})$ (because $U_{\sigma(m+1)-1} \cap (U_{\sigma(m+3)} \setminus \text{cl}(U_{\sigma(m+2)-2})) = \emptyset$).
Each metric $g_{\alpha_m}$ satisfies the properties (iii) and (iv) in Defn. 4.4 and it can be regarded as a $t$-perturbation of the original metric $g$ with perturbing function $\alpha_1 + \cdots + \alpha_m$. Moreover, clearly, for each compact subset $Z \subset M$ there exists some $m_0 \in \mathbb{N}$ such that (a) $\alpha_m(Z) \equiv 0$ for all $m \geq m_0$ and (b) the property (13) holds if $\mathcal{C}(U_\alpha(m))$ is replaced by $Z$. Assertion (a) allows to define $\alpha$ as the locally finite sum $\alpha := \sum_m \alpha_m$, and let us check that, then, $g_\alpha$ satisfies the properties stated in Defn. 4.4. By construction, $g_\alpha$ satisfies (i) and (ii). For (iii), apply the assertion (b) to $Z = \{\rho\}$, $p \in U_0$ and recall that all $g_{\alpha_m}$ satisfied (iii). Finally, for (iv), putting $g_{m_0} := g$,

$$d_H(J(p, \Sigma_t^+), J_\alpha(p, \Sigma_t^+)) \leq \sum_{k=0}^{m-1} d_H(J_{\alpha_k}(p, \Sigma_t^+), J_{\alpha_{k+1}}(p, \Sigma_t^+)) \leq \sum_{k=0}^{m-1} \epsilon_k < \epsilon$$

and the result follows taking into account the assertion (b) again. \(\square\)

### 4.3 Proofs of the main results

**Proof of Thm. 4.1.** Consider the closed strips $C_k = [k - 1, k + 2] \times \Sigma$ and open ones $U_k = (k - 2, k + 3) \times \Sigma$ for each integer $k \in \mathbb{Z}$, and choose the complete Riemannian metric $g_R$ so that $d_R(\Sigma_{k+k'}/3, \Sigma_{k+(k'+1)/3}) > 1$ for all $k \in \mathbb{Z}$, $k' \in \{0, 1, 2\}$. Construct a $t$-perturbation $\alpha_k$ for each $C_k$, $U_k$ as in Prop. 4.7 with $\epsilon_0 = 1/2$ for all $k$. Now, take any smooth function $\alpha > 0$ on $M$ such that $\alpha < \alpha_k$ on each strip\(^{11}\) $C_k$.

Let us check that $g_\alpha$ can be chosen as the required metric $g'$. Trivially, $g_\alpha > g$. In order to prove that $(M, g_\alpha)$ is globally hyperbolic, consider first the following.

**Claim.** Let $p \in M$ and $q \in \partial \Sigma^-(p)$. Any future-directed $g_\alpha$-causal curve $\rho$ from $p$ to $q$ is included in the strip $(t(p) - 1, t(q) + 1] \times \Sigma$. In particular, $J_\alpha^+(p) \cap \Sigma_t \neq \emptyset$ if $t_0 \leq t(p) - 1$.

Moreover, for each $t_0 \in \mathbb{R}$, $J_\alpha(p, \Sigma_{t_0})$ and $J_\alpha(\Sigma_{t_0}, p)$ are compact.

Assuming the claim, the result follows easily. Indeed, (strong) causality holds at any $p \in M$ because, otherwise, any closed causal loop $\rho$ would be contained in the strip $(t(p) - 1, t(q) + 1] \times \Sigma$. However, this strip is included in the closed strip $C_{k'}$, where $k'$ is the integer part of $t(p)$. As $g_\alpha \leq g_{\alpha_{k'}}$ on $C_{k'}$, the closed $g_\alpha$-causal curve $\rho$ is also causal for $g_{\alpha_{k'}}$, in contradiction with the global hyperbolic character of $g_{\alpha_{k'}}$.

Finally, notice that $J_\alpha^+(p) \cap \Sigma_{t_0}^-$ is the intersection of the compact sets $J_\alpha(p, \Sigma_{t_0+1})$ and $J_\alpha(\Sigma_{t_0}, p)$.

**Proof of the Claim.** For the first assertion, assume by contradiction that there exists a first point $q'$ with $t(q') = t(p) - 1$ crossed by $\rho$ (a similar contradiction would appear at the last point where $\rho$ left the region $t \leq t(q) + 1$). Let $p'$ be the last point before $q'$ crossed by $\rho$ with $t(p') = t(p)$; trivially, $q' \in J_\alpha^+(p')$. The portion of $\rho$ from $p'$ and $q'$ is entirely contained in the strip $C_{k'}$, thus $q' \in J_{\alpha_{k'}}^+(p')$. So, putting $t_0 = t(q') = (t(p)-1)$ and taking into account Defn. 4.4 (iv):

$$d_H(p', J_{\alpha_{k'}}(p', \Sigma_{t_0})) < \epsilon_0 = 1/2.$$  

(14)

This is a contradiction because $q' \in J_{\alpha_{k'}}(p', \Sigma_{t_0})$ and, by our choice of $g_R$, $d_R(p', q') > 1$.

For the last assertion, let us reason for $J_\alpha(p, \Sigma_{t_0})$. If $t_0 \leq t(p)$, then $J_\alpha(p, \Sigma_{t_0})$ is entirely included in the strip $[t(p) - 1, t(p) + 1] \times \Sigma$, which is included in the strip $C_{k'}$. By

\(^{11}\)To construct $\alpha$, consider the (continuous, positive) function $\hat{\alpha}_k$ obtained as the minimum of $\{\alpha_{k-1}, \alpha_k, \alpha_{k+1}\}$ on each strip $[k, k+1] \times \Sigma$, $k \in \mathbb{Z}$. Merge all the functions $\hat{\alpha}_k$’s on the strips into a single function $\hat{\alpha}$ on all $\mathbb{M}$ just by choosing the minimum between $\hat{\alpha}_k$ and $\hat{\alpha}_{k+1}$ on each slice $\Sigma_k$. As $\hat{\alpha}$ is positive and lower continuous, one can choose a smooth $\alpha$ satisfying $0 < \alpha < \hat{\alpha}$ by means of $s$ standard partition of unity argument.
applying Remark 4.5 to \( g_{α_k} \), we deduce that \( J_{α_k}(p, S_{t_0}) \) is compact. Then, taking into account that \( α ≤ α_k \), necessarily \( J_α(p, S_{t_0}) \subset J_{α_k}(p, S_{t_0}) \), and so, \( J_α(p, S_{t_0}) \) is also compact. Assume inductively that compactness hold when \( t_0 ≤ t(p) + k \), and consider \( t(p) + k < t_0 ≤ t(p) + k + 1 \). Then, both \( J_α(p, S_{t(p)+k}) \) and \( D_0 := J^+_α(p) ∩ S_{t(p)+k} \) are compact by hypothesis, and \( J_α(D_0, S_{t_0}) \) must lie in the strip \( C_{k_δ+k} \). So (using analogously the global hyperbolicity of \( g_{α_k+k} \) and \( α ≤ α_k+k \)), \( J_α(D_0, S_{t_0}) \) is compact and the result follows by noticing \( J_α(p, S_{t_0}) = J_α(p, S_{t(p)+k}) \cup J_α(D_0, S_{t_0}) \). ■

**Remark 4.8.** Recall that all the perturbed metrics constructed so far are of the form (6). Indeed, the globally hyperbolic metric has been obtained for some \( α > 0 \). Obviously, any other metric as in (6) constructed with some function \( α' > 0 \) such that \( α' ≤ α \) would be globally hyperbolic too (the Cauchy hypersurfaces for \( g_α \) would be also Cauchy for \( g_{α'} \) as \( g_{α'} ≤ g_α \)).

Next, let us focus on the stability of Cauchy temporal functions.

**Proof of Thm. 4.2.** The proof of Thm. 4.1 will be mimicked by choosing \( ω = drτ \), which is metrically equivalent to the timelike vector field \( −\partial_t/Λ \) in the orthogonal splitting \((\mathbb{R} × \Sigma, g ≡ −Λdτ^2 + g_τ)\) associated with the Cauchy temporal function \( τ \). So,

\[
g_α = g − αdτ^2 = −(Λ + α)dτ^2 + g_τ. \tag{15}
\]

Recall that, now, for all the metrics \( g_α \), \( α ≥ 0 \), the slices \( Σ_τ \) remain spacelike. This is an important difference with Thm. 4.1, where all the slices for Geroch function \( t \) might be non \( g_α \)-acausal for any \( α \). So, in both, Lemma 4.6 and Lemma 4.7, not only the slices \( Σ_t \), \( t ∈ \mathbb{R} \setminus (t^−, t^+) \) are Cauchy but also all the slices with \(^{12} t ∈ (t^−, t^+) \). Therefore, we can follow the proof of Thm. 4.1 but we do not need to replace \( τ \) as was done for the function \( t \) (inductively replaced by some \( t_m \) to construct a Cauchy time \( t_∞ \)): simply, \( τ \) remains equal at all the steps of the proof and becomes a Cauchy temporal function for \( g_α \). Recall also that, as the expression (15) remains valid for any perturbed metric \( g_α \), all the \( g_α \)-gradients of \( τ \) remain tangent to \( ∂M \). □

**Remark 4.9.** (1) Notice that the metrics \( g_α \) in (15) with \( α > 0 \) not only satisfy \( g < g_α \) but also \( −g_α(v, v) > −g(v, v) \) whenever \( v \) is \( g \)-causal.

(2) For any \( α ≥ 0 \), one has:

\[
∇τ = −\frac{∂τ}{Λ}, \quad ∇_{g_α}τ = −\frac{∂τ}{Λ + α}, \quad \text{thus,} \quad −g(∇_{g_α}τ, ∇_{g_α}τ) = −dτ(∇_{g_α}τ) = \frac{1}{Λ + α}.
\]

Therefore, if the Cauchy temporal function \( τ \) is steep for \( g_α \) (i.e., \( Λ + α ≤ 1 \)) then, it is also steep for \( g \) (\( Λ ≤ 1 \)). Conversely, if \( τ \) is steep for \( g \) and \( α \) is chosen bounded \( α ≤ C \) then the Cauchy temporal function \( √(1 + C')τ \) is steep for \( g_α \).

**Remark 4.10.** In connection with the proof of Prop. 2.13 (implication 2 ⇒ 3), it is clear now that, if one considers any stably causal spacetime and \( τ \) is a temporal function for \( g \), then \( τ \) is also temporal for all the metrics with wider cones \( g_α \) constructed in (15), for any \( α ≥ 0 \).

5 The Cauchy orthogonal decomposition.

Along this section, \( (M, g) \) will be globally hyperbolic with timelike boundary.

\(^{12}\)This follows from the proof of those lemmas, but can be also noticed because any closed spacelike hypersurface contained in the region between two disjoint Cauchy hypersurfaces is Cauchy too, see for example [6, Corollary 11].
5.1 Simplification of the problem by using stability

Next, our aim is to show that the problem of existence of $\tau$ in Thm. 1.1 can be reduced to the case of a new metric $g^*$ with a simple product behaviour around $\partial M$. First, let us globalize Gaussian coordinates.

**Lemma 5.1 (Existence of a global tubular neighborhood).** There exists a smooth function $\rho : \partial M \to \mathbb{R}$, $\rho > 0$, such that the orthogonal exponential map

$$\exp^\perp : \{ (\hat{p}, s) \in \partial M \times [0, \infty) : 0 \leq s < \rho(\hat{p}) \} \to \mathbb{M}, \quad (\hat{p}, s) \mapsto \exp_p(s N_{\hat{p}})$$

is a diffeomorphism onto its image $E$, where $N_{\hat{p}}$ is the pointing-inward $\partial M$-orthogonal unit vector at $\hat{p}$. This will be called tubular neighborhood of $\partial M$.

**Proof.** The technique is standard. Start with a Gaussian neighborhood around each $\hat{p} \in \mathbb{M}$ with normal coordinate $s \in [0, \epsilon_p)$, take an exhaustion by compact sets $\{ \mathcal{K}_m \}_{m}$, $\mathcal{K}_m \subset \text{int}(\mathcal{K}_{m+1})$ (even if $\partial M$ has infinitely many connected components, this can be done by intersecting $\partial M$ with closed balls of radius $m$ for a complete Riemannian metric on $\mathbb{M}$ centered in a freely chosen point), determine $\epsilon_{\mathcal{K}_m}$ such that $\exp^\perp$ satisfies the required properties on $\mathcal{K}_m \times [0, \epsilon_{\mathcal{K}_m})$ and choose $\rho > 0$ such that $\rho < \epsilon_{\mathcal{K}_m}$ on $\mathcal{K}_m \setminus \mathcal{K}_{m-1}$. \hfill $\Box$

Elements on $E$ and their preimages by $\exp^\perp$ will be identified with no further mention. In particular, $E$ is endowed with the $(\partial M$-orthogonal, geodesic) vector field $\partial_s$.

**Proposition 5.2.** Let $g' > g$ be globally hyperbolic (as obtained in Thm. 4.1). Then, there exist another Lorentzian metric $g^*$ on $\mathbb{M}$ satisfying:

(a) $g \leq g^* < g'$ on $\mathbb{M}$, and so, $g^*$ is also globally hyperbolic.

(b) On some $g^*$-tubular neighborhood $E$ of $\partial M$, each $p \in E$ admits a neighborhood type $\hat{U} \times [0, \epsilon) \subset E$, where $g^*$ is a product metric $g^* = \hat{g}_0 + ds^2$, being $\hat{g}_0$ a Lorentzian metric on $\hat{U} \subset \partial M$.

**Proof.** As $g' > g$, we can choose a metric $\hat{g}_0$ on $\partial M$ such that

$$g|\partial M < \hat{g}_0 < g'|\partial M \quad \text{and} \quad -\hat{g}_0(v, v) > -g(v, v) \quad \text{for all } g\text{-causal } v \in T(\partial M),$$

and such that the (locally) product metric

$$g_0 := \hat{g}_0 + ds^2$$

defined in some tubular neighborhood $E'$ of $\partial M$, satisfies

$$g < g_0 < g' \quad \text{on } E' \quad \text{and} \quad -\hat{g}_0(v, v) > -g(v, v) \quad \text{for all } g\text{-causal } v \in TE'.$$

Indeed, notice that $g|\partial M < g'|\partial M$ are globally hyperbolic metrics on the manifold $\partial M$ (Thm. 3.8 (3)) and $\hat{g}_0$ can be constructed as a metric $g_\alpha$, $\alpha > 0$, in Remark 4.9 (1), ensuring both assertions in (17). Moreover, as $\partial_s$ is unit and $g$-orthogonal with both, $g_0$ and $g$, this also ensures $g < g_0$ on all the points of $\partial M$ and (choosing a smaller $\alpha$ if necessary) $g_0 < g'$ on $\partial M$. Thus, reducing $E'$ if necessary, both assertions in (19) also hold.

Now, take any smaller tubular neighborhood $E$ with $\text{cl}(E) \subset E'$ (say, the associated with the function $\rho/2$ in equation (16)). Consider the covering $\{ E', \mathbb{M} \setminus E \}$ of $\mathbb{M}$, choose a subordinate partition of unity $\{ \mu, 1 - \mu \}$ with $\text{supp}(\mu) \subset E'$ and construct

$$g^* = \mu g_0 + (1 - \mu)g.$$  

By the second assertion in (19), $g \leq g^* \leq g_0$; thus (by the first assertion), $g^*$ fulfills (a). The requirements (b) and (c) hold because $g^* = g_0$ on the whole $E$. \hfill $\Box$
Remark 5.3. On $E$, both $g$ and $g^*$ can be written in Gaussian coordinates as in (2); however, the dependence on $s$ of $g_{ij}(\hat{x}, s)$ is dropped for $g^*$.

Lemma 5.4 (Reduction to a local product around $\partial M$). For any $g^*$ as in Prop. 5.2:
(a) if $\tau$ is Cauchy temporal for $g^*$, then so is it for $g$.
(b) if the $g^*$-gradient $\nabla^*\tau$ of $\tau$ is tangent to $\partial M$, then so is the $g$-gradient $\nabla\tau$.

Proof. (a) $g \leq g^*$ implies that $\tau$ is $g$-Cauchy temporal (the $g$-orthogonal to $\nabla\tau$ is tangent to a $\tau$-slice, which is $g^*$-spacelike, and thus, $g$-spacelike).
(b) On $\partial M$, $g(\nabla\tau, \partial_s) = d\tau(\partial_s) = g^*(\nabla^*\tau, \partial_s) = 0$, thus $\nabla\tau$ is tangent to $\partial M$. □

Remark 5.5. As a summary, Thm. 4.1, Prop. 5.2 and Lemma 5.4 yield: in order to obtain a Cauchy temporal function $\tau$ for $g$ with $\nabla\tau$ tangent to $\partial M$ (as required for Thm. 1.1), one can assume, with no loss of generality, that $g$ satisfies the local product property stated for $g^*$ in Prop. 5.2 (c) (otherwise, work with $g^*$ itself).

5.2 Proof of Theorem 1.1
Take two copies of $(\overline{M}, g)$ and identify their homologous points along the boundary $\partial M$ in order to obtain the double manifold $\overline{M}^d$. Now, $\overline{M}^d$ inherits not only a structure of smooth manifold (without boundary) but also a smooth metric $g^d$. Indeed, $g$ is assumed to satisfy the properties stated for $g^*$ in Prop. 5.2 (by Remark 5.5). So, the local product structure of any tubular neighborhood $E$ of $\partial M$ yields directly the smoothness of both, the exponential and the metric around $\overline{\Sigma} = \partial M^d$ (just work in Gaussian coordinates).
Moreover, the natural reflection $i : \overline{M}^d \to \overline{M}^d$ (which maps each point in its homologous one) becomes an isometry for $g^d$. Notice that $g^d$ is globally hyperbolic because any pair of homologous Cauchy hypersurfaces with boundary in each copy merges into a continuous Cauchy hypersurface without boundary of $(\overline{M}, g^d)$. So, the main result in [40] (applied with 0 invariant Cauchy hypersurfaces) ensures the existence of a Cauchy temporal function $\tau^d$ on $\overline{M}^d$ which is invariant by $i$. Therefore, the restriction $\tau$ of $\tau^d$ to $\overline{M}$ becomes a Cauchy temporal function, and its gradient must be tangent to the boundary as $i_*(\nabla\tau) = \nabla\tau$ on the set $\partial M$ (of fixed points for $i$). Once $\tau$ is obtained, the Cauchy orthogonal decomposition (1) is deduced by choosing a slice $\Sigma = \tau^{-1}(0)$ and moving it with the flow of $-\nabla\tau/|\nabla\tau|^2$ as in the case without boundary (see Cor. 2.5 or [7, Prop. 2.4]).

In order to find an isometric embedding in $L^N$, notice that the technique in [41] works whenever a steep Cauchy function $\tilde{\tau}$ is obtained (that is, the requirement of tangency of $\nabla\tau$ to $\partial M$ necessary for the orthogonal splitting can be dropped now). In order to find such a $\tilde{\tau}$, the same procedure as in [41] can be used 13.

The remainder of the assertions in Thm. 1.1 follow just using Thm. 3.8. □

Remark 5.6. As emphasized above, the proof of isometric embeddability in $L^n$ does not require the steep Cauchy temporal function $\tau$ for $g$ having gradient tangent to $\partial M$ and, thus, it can be proven by using the same techniques as in the case without boundary. Analogously, other problems on smoothability can be proved directly as in the case without boundary. For example:
(a) Geroch’s topological splitting (Thm. 3.14) can be improved into a smooth one just noticing that the same procedure as in [6] gives a spacelike Cauchy hypersurface

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13 This is straightforward because the technique in [41] is based in local functions type $j_p(\cdot) := \exp(-1/d(p, \cdot)^2)$, where $d$ is the Lorentzian distance. Moreover, the required local functions can be chosen in a very flexible way (for example, taking the distance associated with a $C^d$-close flat metric $h_0$ with $-h_0(v, v) > -g(v, v)$ on any $g$-causal $v$). So, the constructive technique works in the same way even if, now, $p$ may belong to $\partial M$. 
$\Sigma$, and moving it by using the flow of any timelike vector field $T$ tangent to $\partial M$ (as obtained in Prop. 2.4 (iii), and choosing $T$ complete).

(b) The technique in [7] directly gives also a Cauchy temporal function $\tau_0$ and, this, a smooth foliation of $\overline{M}$ by Cauchy hypersurfaces (however, they are not necessarily orthogonal to $\partial M$).

(c) The extensions of the classical equivalences of the notion of stably causal spacetime to the case with boundary rely also on the case without boundary (as described in Prop. 2.13).

(d) As in the case without boundary (see [41]), stably causal spacetimes with boundary can be conformally embedded in some Lorentz-Minkowski spacetime $\mathbb{L}^N$ for big $N$.

Remark 5.7. Finally, let us revisit how, if one assumed the existence of the Cauchy temporal function $\tau_0$ (as in Remark 5.6 (b) above), the proof of Thm. 4.1 is widely simplified. First, one would take $\omega = d\tau_0$ in (6), so that $g_\alpha$ can be written as in (15). Thus, $\tau_0$ is directly a temporal function for $g_\alpha$. This implies that, in Def. 4.4 (iv), neither the case $t'_+ \in [t_-, t(p)]$ and $J_\alpha(p, \Sigma_{t'_+}) \neq \emptyset$, nor the case $t'_- \in [t(p), t_+]$ and $J_\alpha(\Sigma_{t'_-}, p) \neq \emptyset$ can hold. So, they should not be taken into account in the proof of Lemma 4.6; what is more, in this lemma it is not necessary to prove (strong) causality, and all the slices $\tau_0 = \text{constant}$ become directly spacelike and, then, acausal and Cauchy for $g_\alpha$ (this was not true even for smooth Geroch’s functions). These properties also hold in Prop. 4.7; so, the claim in the proof of Thm. 4.1 is not needed.

6 Appendix: naked singularities and the causal boundary

Easily, all the points of the timelike boundary $\partial M$ of $\overline{M}$ correspond to (conformally invariant) naked singularities of $M$. In this Appendix, globally hyperbolic spacetimes with timelike boundary are characterized as those containing all its naked singularities. Previously, we need to recall some basic notions and properties associated to the causal boundary (c-boundary) of spacetimes without boundary.

6.1 Brief review on the c-completion of spacetimes

We refer to [28, 26] for further details and proofs (see also the original article [22]).

A past set $P \subset M$ (i.e., $P \neq \emptyset$, $I^-(P) = P$) that cannot be written as the union of two proper subsets, both of which are also past sets, is said to be an indecomposable past set (IP). It can be shown that an IP either coincides with the past of some point of the spacetime, i.e., $P = I^-(p)$ for $p \in M$, or else $P = I^-(\gamma)$ for some inextendible future-directed timelike curve $\gamma$. In the former case, $P$ is said to be a proper indecomposable past set (PIP), and in the latter case $P$ is said to be a terminal indecomposable past set (TIP). These two classes of IPs are disjoint.

The common past of a given set $S \subset M$ is defined by

$$\downarrow S := I^- \left( \{ p \in M : \ p \ll q \ \forall q \in S \} \right).$$

The corresponding definitions for future sets, IFS, TIFs, PIFs, common future, etc., are obtained just by interchanging the roles of past and future, and will always be understood.

The set of all IPs constitutes the so-called future c-completion of $(M, g)$, denoted by $\overline{M}$. If $(M, g)$ is strongly causal, then $M$ can naturally be viewed as a subset of $\overline{M}$ by identifying every point $p \in M$ with its respective PIP, namely $I^-(p)$. The future c-boundary $\partial M$ of $(M, g)$ is defined as the set of all its TIPs. Therefore, upon identifying
$M$ with its image in $\hat{M}$ by the natural inclusion as outlined above,

$$\partial M \equiv \hat{M} \setminus M.$$ 

The definitions of past c-completion $\hat{M}$ and past c-boundary $\partial M$ of $(M,g)$ are readily defined in a time-dual fashion using IFs.

Next, we introduce the so-called Szabados relation (or $S$-relation) between IPs and IFs: an IP $P$ and an IF $F$ are $S$-related, denoted $P \sim_i S F$, if $P$ is a maximal IP inside $\downarrow F$ and $F$ is a maximal IF inside $\uparrow P$. In particular, for any $p \in M$, it can be shown that $I^-(p) \sim_i S I^+(p)$.

**Definition 6.1.** The (total) c-completion $\overline{M}^c$ is composed by all the pairs $(P,F)$ formed by $P \in \hat{M} \cup \{\emptyset\}$ and $F \in \hat{M} \cup \{\emptyset\}$ such that either

i) both $P$ and $F$ are non-empty and $P \sim_i S F$; or

ii) $P = \emptyset$, $F \neq \emptyset$ and there is no $P' \neq \emptyset$ such that $P' \sim_i S F$; or

iii) $F = \emptyset$, $P \neq \emptyset$ and there is no $F' \neq \emptyset$ such that $P \sim_i S F'$.

The original manifold $M$ is then identified with the set $\{(I^-(p),I^+(p)) : p \in M\}$, and the c-boundary is defined as $\partial^c M \equiv \overline{M}^c \setminus M$.

**Remark 6.2.** Any pair $(P,F) \in \partial^c M$, with $P \neq \emptyset \neq F$, will be called a naked singularity. Notice that, necessarily $P = I^- (\gamma)$ for some inextendible future-directed timelike curve, and $\gamma$ must lie in the past of any $z \in F$ (a dual assertion follows by interchanging the role of $P$ and $F$). According to a classical physical interpretation, $\gamma$ may represent a particle disappearing of the spacetime, and all this process can be seen at $z$. Conversely, whenever such a $z, \gamma (\gamma \subset I^- (z))$ exist, a pair $(P,F) \in \partial^c M, P \neq \emptyset \neq F$ must appear.

Clearly, such a notion of naked singularity is conformally invariant. However, one expects that the physically relevant representatives of the conformal class will present curvature-related divergences along such $\gamma$’s which make the spacetime inextensible.

Having defined the set structure of the c-completion, the next step is to extend the chronological relation in $(M,g)$ to the c-completion as follows:

$$(P,F) \ll (P',F') \iff F \cap P' \neq \emptyset. \quad (21)$$

Next, let us define the future chronological limit operator $\hat{L}$ on $\hat{M}$ as follows. Given a sequence $\sigma = \{P_n\}_n \subset \hat{M}$ of IPs and $P \in \hat{M}$, we set

$$P \in \hat{L}(\sigma) \iff \left\{ \begin{array}{l} P \subset \overline{LI}(\sigma) \\ P \text{ is a maximal IP in } LS(\sigma). \end{array} \right. \quad (22)$$

Again, by simply interchanging past and future sets we may analogously define the past chronological limit operator $\hat{L}$ on $\hat{M}$. Then, the future (resp. past) chronological topology on $\hat{M}$ (resp. $\hat{M}$) is the derived topology associated to the limit operator $\hat{L}$ (resp. $\hat{L}$), that is, the topology whose closed sets are those subsets $C \subset \hat{M}$ (resp. $C \subset \hat{M}$) such that $\hat{L}(\sigma) \subset C$ (resp. $\hat{L}(\sigma) \subset C$) for any sequence $\sigma$ of elements of $C$.

In order to define the chronological topology on the full c-boundary, first define a limit operator $\hat{L}$ on $\overline{M}^c$ as follows: given a sequence $\sigma = \{(P_n,F_n)\}_n \subset \overline{M}^c$, put

$$(P,F) \in \hat{L}(\sigma) \iff \left\{ \begin{array}{l} P \in \hat{L}(P_n) \text{ if } P \neq \emptyset \\ F \in \hat{L}(F_n) \text{ if } F \neq \emptyset. \end{array} \right. \quad (23)$$

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By definition, the **chronological topology** on $\overline{M}$ is the derived topology $\tau_L$ associated to the limit operator $L$ defined in (23), that is, the topology whose closed sets are those subsets $C \subset \overline{M}$ such that $L(\sigma) \subset C$ for any sequence $\sigma$ of elements of $C$.

The following result (see [28, Thm. 3.27]) summarizes the key properties of the chronological topology.

**Theorem 6.3.** Let $\mathcal{M} = (\mathcal{M}, g)$ be a strongly causal spacetime and consider its associated $c$-completion $\mathcal{M}^c$ endowed with the chronological relations and chronological topology defined in (21) and (23), respectively. Then, the following statements hold.

1. The inclusion $\mathcal{M} \hookrightarrow \mathcal{M}^c$ is continuous, with an open dense image. In particular, $\partial c^\mathcal{M}$ is closed in $\mathcal{M}^c$ and the topology induced on $\mathcal{M}$ by the chronological topology on $\mathcal{M}^c$ coincides with the original manifold topology.

2. The chronological topology is second-countable and $T_1$ (but not necessarily $T_2$).

3. Let $\{x_n\} \subset \mathcal{M}$ be a future (resp. past) chain, i.e., a sequence satisfying that $x_n \ll x_{n+1}$ (resp. $x_{n+1} \ll x_n$) for all $n$. Then,

$$L(\{x_n\}) = \{(P, F) \in \mathcal{M}^c : P = I^-(\{x_n\})\}$$

(resp. $L(\{x_n\}) = \{(P, F) \in \mathcal{M}^c : F = I^+(\{x_n\})\}$).

4. The $c$-completion is complete in the following sense: given any (future or past) chain $\{x_n\} \subset \mathcal{M}$, necessarily $L(\{x_n\}) \neq \emptyset$, i.e. any (future or past) chain converges in $\mathcal{M}^c$.

5. The sets $I^{\pm}(\mathcal{M}, g)$ are open for all $(P, F) \in \mathcal{M}^c$.

**6.2 Main result**

It is worth emphasizing the following result in the case without boundary [28, Cor. 4.34]:

Let $\mathcal{M} = (\mathcal{M}, g)$ be a spacetime which admits a conformal boundary $\partial \mathcal{M}$ such that $\overline{\mathcal{M}} = \mathcal{M} \cup \partial \mathcal{M}$ is $C^1$ and $\partial \mathcal{M}$ chronologically complete (i.e., each inextensible future/past-directed timelike curve in $\mathcal{M}$ has an endpoint in $\partial \mathcal{M}$).

$\mathcal{M}$ is globally hyperbolic if and only if $\partial \mathcal{M}$ does not admit timelike points (i.e., $T_p(\partial \mathcal{M})$ is everywhere either a spacelike or a lightlike hyperplane).

This result suggests that, for a globally hyperbolic spacetime with timelike boundary, the boundary should contain all the naked singularities. The formal statement and proof of this assertion will be given in Th. 6.6 below. Previously, Lemma 6.5 will extend the basic results [3, Lemma 3.4], [33, Thm. 2.3] to the case of a strongly causal spacetime with timelike boundary.

**Remark 6.4.** The hypotheses which ensure that the (conformal) boundary $\partial \mathcal{M}$ of a spacetime with boundary $\overline{\mathcal{M}}$ can be identified with the $c$-boundary $\partial c^\mathcal{M}$ of its interior $\mathcal{M}$ and, moreover, that the $c$-completion $\overline{\mathcal{M}}^c$ agrees with the spacetime with boundary $\overline{\mathcal{M}}$, were analyzed in [28, Section 4], see specially Thm. 4.26, Cor. 4.28 and Thm. 4.32 therein. In particular, it is straightforward to check that, the boundary $\partial \mathcal{M}$ of any spacetime with timelike boundary $\partial \mathcal{M}$ is regularly accessible. Essentially, this means that $\partial \mathcal{M}$ cannot present a variety of pathologies which occur for arbitrary conformal embeddings, and $\partial \mathcal{M}$ can be regarded as a part of the causal boundary. Indeed, $\partial \mathcal{M}$ can be identified with the whole $\partial \mathcal{M}$ (and $\overline{\mathcal{M}}$ with $\overline{\mathcal{M}}^c$) when $\partial \mathcal{M}$ is both, regularly accessible and chronologically complete, see [28, Th. 4.16]. Anyway, here we will make a self-contained development adapted to our purposes.
Taking into account Remark 2.7, we will abuse of notation in the remainder by writing \( I^+(p,M) \) instead of \( I^+(p) \cap M \) for any \( p \in \overline{M} \).

**Lemma 6.5.** Let \((\overline{M},g)\) be a strongly causal spacetime with timelike boundary.

(1) If \( cl(J^+(p) \cap J^-(q)) \) is compact for all \( p,q \in \overline{M} \) then \((\overline{M},g)\) is globally hyperbolic with timelike boundary.

(2) Let \( \hat{x} \in \overline{M} \) and \( \{x_n\} \) be a sequence in \( \overline{M} \) such that \( I^-(x,M) \in L(I^-(x_n,M)) \). Then \( \{x_n\} \) converges to \( \hat{x} \) with the manifold topology of \( \overline{M} \).

**Proof.** (1) Essentially, we will follow the proof of [4, Lemma 4.29]. It suffices to show that \( J^+(p) \cap J^-(q) \subset \overline{M} \) is closed for every \( p,q \in \overline{M} \). By contradiction, suppose that \( r \in cl(J^+(p) \cap J^-(q)) \) \( \setminus J^+(p) \cap J^-(q) \). Then, there exists \( \{r_n\} \subset J^+(p) \cap J^-(q) \) such that \( r_n \rightarrow r \). Let \( \gamma_n : [0,1] \rightarrow \overline{M} \) be inextensible future-directed causal curves such that \( \gamma_n(0) = p \), \( r_n \in \gamma_n \) and \( q \in \gamma_n \) for all \( n \). By Prop. 2.16 there exists a future-directed causal limit curve \( \gamma : [0,1] \rightarrow \overline{M} \) of \( \{\gamma_n\} \) such that \( \gamma(0) = p \). Since \((\overline{M},g)\) is strongly causal, the inextensible causal curve \( \gamma : [0,1] \rightarrow \overline{M} \) is not imprisoned on the compact subset \( cl(J^+(p) \cap J^-(q)) \). Hence, there exists \( x \in Im(\gamma) \) such that \( x \not\in cl(J^+(p) \cap J^-(q)) \).

From the notion of limit curve, any neighborhood of \( x \in Im(\gamma) \) intersects all but finitely many of the \( \gamma_n \)'s. So, we can assume without restriction the existence of a sequence \( x_n \in Im(\gamma_n) \) such that \( \{x_n\} \) converges to \( x \). Since \( x \not\in cl(J^+(p) \cap J^-(q)) \) we also have that \( x_n \not\in cl(J^+(p) \cap J^-(q)) \) for all \( n \) large enough. So taking into account that \( \gamma_n \subset J^+(p) \), it follows that \( x_n \not\in J^-(q) \) for large \( n \). Hence \( q \) lies between the points \( p \) and \( x_n \) on \( \gamma_n \), i.e., \( p \leq r_n \leq q \leq x_n \) for large \( n \). Denote by \( \gamma_{|p,x_n} \) the portion of \( \gamma \) between the point \( p \) and \( x_n \), and by \( \gamma_{|p,x_n} \) the portion of \( \gamma_n \) between the points \( p \) and \( x_n \). From Prop. 2.18 (1), we may assume, by taking a subsequence of \( \{\gamma_n \}_{n \in \mathbb{N}} \) if necessary, that \( \{\gamma_n \}_{n \in \mathbb{N}} \) converges to \( \{\gamma_{|p,x_n} \} \) in the \( C^0 \) topology of curves. Since \( q \in \gamma_{|p,x_n} \), necessarily \( q \in \gamma_{|p,x} \). On the other hand, since \( r_n \leq q \) and \( r_n \rightarrow r \), necessarily \( r \in \gamma_{|p,x} \) and \( r \leq q \). Therefore, \( r \in J^+(p) \cap J^-(q) \), a contradiction.

(2) We will follow the reasoning in the implication to the left of [33, Thm. 2.3]. Assume by contradiction that \( x_n \neq \hat{x} \). Then, there exists a relative compact open neighbourhood \( U \supsetneq \hat{x} (U \subset \overline{M}) \), and a subsequence \( \{x_{n_k}\} \) with \( x_{n_k} \not\in U \) for all \( k \); by strong causality, \( U \) can be assumed causally convex (Prop. 2.11). Consider a future chain \( \{\zeta_n\} \subset \overline{M} \) such that \( z_n \rightarrow \hat{x} \). For \( n \) sufficiently large, \( z_n \in U \). Since \( z_n \neq \hat{x} \), there exists \( K_n \in \mathbb{N} \) such that \( z_n \neq x_{n_k} \) for \( k \geq K_n \). So, there is a timelike curve \( \gamma_{n_k}^n \) in \( \overline{M} \) from \( x_{n_k} \) to \( z_{n_k} \). Since \( x_{n_k} \not\in U \), \( \gamma_{n_k}^n \) exists \( U \) at some point \( y_{n_k} \in U \). For each \( n \), the curves \( \{\gamma_{n_k}^n \}_{n \geq K_n} \) have a future-directed causal limit curve \( \gamma_n^\# \) in \( \overline{M} \) from \( z_n \) to some point \( y_n \in U \). Moreover, the sequence of curves \( \{\gamma_n\} \) has a future-directed causal limit curve \( \gamma \) in \( \overline{M} \) to some \( y \in U \) as \( \gamma \) cannot remain imprisoned in \( cl(U) \).

Let \( \Gamma(y,M) = \Gamma(\gamma,M) \) and note that \( \Gamma(y,M) \subset LI(\{\Gamma(x_{n_k},M)\}) \). In fact, take \( u \in \Gamma(y,M) \), then \( y \in I^+(u,M) \), so, for large \( n \), we have that \( y_{n_k} \in I^+(u,M) \). Therefore, for \( k \) large enough, \( y_{n_k} \neq x_{n_k} \). Since \( y_{n_k} \neq x_{n_k} \), necessarily \( x_{n_k} \in I^+(u,M) \) for large \( k \). Hence, \( \Gamma(\hat{x},M) \subset I^-(y,M) \subset LI(\{\Gamma(x_{n_k},M)\}) \), in contradiction with \( \Gamma(\hat{x},M) \not\subset LI(\{\Gamma(x_{n_k},M)\}) \).

**Theorem 6.6.** Let \((\overline{M},g)\) be a strongly causal spacetime with timelike boundary. The following sentences are equivalent:

(a) \((\overline{M},g)\) is a globally hyperbolic spacetime with timelike boundary.

(b) \( \partial M \) contains all the naked singularities, i.e., for any pair \((P,F)\) in the c-boundary \( \partial^c M \), with \( P \neq \emptyset \neq F \), there exists \( \hat{z} \in \partial M \) such that \((P,F) = (I^-(\hat{z},M),I^+(\hat{z},M)) \).

**Proof.** (a) \( \Rightarrow \) (b). Let \((P,F) \in \partial^c M \) with \( P \neq \emptyset \neq F \). Take chains \( \{p_n\} \), \( \{q_n\} \subset M \) generating \( P \) and \( F \), respectively. Then, for some \( p_0,q_0 \in M \) (which could be included
as the first element of each chain) we have
\[ p_n, q_n \in I^+(p_0, M) \cap I^-(q_0, M) \subset I^+(p_0) \cap I^-(q_0) \subset J^+(p_0) \cap J^-(q_0). \]
The compactness of \( J^+(p_0) \cap J^-(q_0) \) implies the existence of \( \bar{z}, \bar{z}' \in \partial M \) such that \( p_n \to \bar{z} \) and \( q_n \to \bar{z}' \) respectively (since \((M, g)\) is strongly causal we can ensure that previous limits hold for the entire sequences, not only for some subsequences of them).
Assume by contradiction that \( \bar{z} \neq \bar{z}' \). Since \( p_n \leq q_n \), there exists future-directed timelike curves \( \gamma_n : [0, 1] \to M \) joining each \( p_n \) and \( q_n \) with \( \gamma_n(0) = p_n \to \bar{z} \) and \( \gamma_n(1) = q_n \to \bar{z}' \). Since \((M, g)\) is globally hyperbolic, Prop. 2.18 (2) implies the existence of a causal limit curve \( \gamma : [0, 1] \to \overline{M} \) so that \( \gamma(0) = \bar{z} \) and \( \gamma(1) = \bar{z}' \). Take some \( 0 < s_0 < 1 \) such that \( \bar{z} < (\gamma(s_0), s_0) < \bar{z}' \). Note that \( I^-(\gamma(s_0), M) \subset F \). In fact, if \( p \in I^-(\gamma(s_0), M) \) then \( \gamma(s_0) \in I^+(p) \). So, taking into account that \( \gamma \) is a limit curve and \( I^+(p) \) is an open set, there exists a subsequence \( \{\gamma_{s_{n_k}}\} \) and a subsequence \( \{s_{n_k}\} \subset [0, 1] \) such that \( \gamma_{s_{n_k}}(s_{n_k}) \in I^+(p) \). This implies \( p \leq \gamma_{s_{n_k}}(s_{n_k}) \leq q_{n_k} \), and so, \( p \leq q_{n_k} \) for all \( k \), which implies \( I^-(\gamma(s_0), M) \subset F \), as required. In conclusion, to prove \( P \subseteq I^-(\gamma(s_0), M) \subset F \) suffices, as this would contradict \( P \: \sim F \).

First, let us justify the identity \( P = I^-(\bar{z}, M) \) by proving the following

**Claim 1:** If \( \beta : [a, b) \to \overline{M} \) is a future-directed timelike curve such that \( \beta(t) \to p \) (\( \beta \) is continuously extendible to \( p \)) then \( I^-(\beta) = I^-(\beta) \).

**Proof of the Claim 1.** For the inclusion to the right, take any \( w \in I^-(p) \). Then, \( p \in I^+(w) \) and, taking into account that it is an open, essentially \( \beta(t) \in I^+(w) \) for large \( t \). So, \( w \in I^- (\beta) \). For the inclusion to the left, take now \( w \in I^- (\beta) \). Then, \( w \leq \beta(t_0) \) for some \( t_0 \in [a, b) \), and, taking into account that \( \beta \) is continuously extendible to \( p \), necessarily \( w \leq \beta(t_0) \leq p \). So, \( w \in I^- (p) \). \( \square \)

Taking into account the previous claim, to show \( \overline{I^-(\bar{z}, M)} \subset I^- (\gamma(s_0), M) \) suffices. We already know that \( \overline{I^-(\bar{z}, M)} \subset I^- (\gamma(s_0), M) \) (from Prop. 2.12, \((M, g)\) is distinguishing). Assume by contradiction that \( \overline{I^-(\bar{z}, M)} \subset I^- (\gamma(s_0), M) \). Since \( \partial M \) is smooth, we can take future-directed timelike curves \( \sigma_i : [0, 1] \to M \subset \overline{M}, i = 1, 2, \) such that \( \sigma_1(1) = \bar{z} \in \partial M \) and \( \sigma_2(1) = \gamma(s_0) \in \partial M \). Moreover, these curves satisfy \( \overline{I^-(\bar{z}, M)} = \overline{I^- (\gamma(s_0), M)} = \overline{I^- (\sigma_2, M)} \). Since each curve \( \sigma_i, i = 1, 2, \) is future-directed timelike, necessarily \( \sigma_i \subset I^- (\sigma_i, M) \) for \( i = 1, 2 \). Moreover, from the initial assumption, \( \sigma_1 \subset I^- (\sigma_2, M) \) and \( \sigma_2 \subset I^- (\sigma_1, M) \). Hence \( I^- (\sigma_1) \subset I^- (\sigma_2) \) and \( I^- (\sigma_2) \subset I^- (\sigma_1) \), and thus, \( I^- (\sigma_1) = I^- (\sigma_2) \). Indeed, let us prove that, say, \( I^- (\sigma_1) \subset I^- (\sigma_2) \). If \( w_0 \in I^- (\sigma_1) \) then \( w_0 \leq \sigma_1(t) \) (for some \( t \in [0, 1] \). Since \( \sigma_1 \subset I^- (\sigma_2, M) \), necessarily \( \sigma_1(t) \leq \sigma_2(s) \), which implies that \( w_0 \leq \sigma_2(s) \), and thus, \( w_0 \in I^- (\sigma_2) \). The proof of \( I^- (\sigma_2) \subset I^- (\sigma_1) \) is analogous. In conclusion, we have proved that \( I^- (\sigma_2) = I^- (\sigma_1) \) whenever \( \sigma_1(t) \to z \) and \( \sigma_2(t) \to \gamma(s_0) \), which implies \( I^- (z) = I^- (\gamma(s_0)) \), a contradiction.

(a) = (b). From Lemma 6.5 (1), it suffices to show that \( \text{cl}(J^+(p) \cap J^-(q)) \subset \overline{M} \) is (sequentially) compact for any \( p, q \in \overline{M} \). To this aim, consider any sequence \( \{z_n\} \subset \text{cl}(J^+(p) \cap J^-(q)) \). It is not a restriction to assume that \( \{z_n\} \subset J^+(p) \cap J^-(q) \) (otherwise, replace \( \{z_n\} \) by some other sequence \( \{z_n'\} \) in \( J^+(p) \cap J^-(q) \) such that \( d_R(z_n', z_n) < 1/n \) for some auxiliary Riemannian metric \( g_R \) on \( \overline{M} \). In order to prove that \( \{z_n\} \) converges to some \( z \in \text{cl}(J^+(p) \cap J^-(q)) \), first note that every \( z_n \in J^+(p) \cap J^-(q) \) satisfies \( I^- (p, M) \subset I^- (z_n, M) \) and \( I^+(q, M) \subset I^+(z_n, M) \). Hence, [27, Prop. 5.3] ensures the existence of some IP \( P \) and some IF \( F \) containing \( I^- (p, M) \) and \( I^+(q, M) \), respectively, such that, up to a subsequence,
\[
P \in \hat{L}(\{I^- (z_n, M)\}) \quad \text{and} \quad F \in \hat{L}(\{I^+(z_n, M)\}).
\]  
In particular, \( \emptyset \neq P \subset \overline{F} \) and \( \emptyset \neq F \subset \overline{P} \).
In the case that \( P = I^-(z, M) \) for some \( z \in M \) (and analogously for \( F \)), the closedness of \( \partial M \) (see Thm. 6.3) and (22) imply that \( z_n \in M \) for large \( n \) and \( z_n \to z \) with the manifold topology. So, only the case when both \( P \) and \( F \) are terminal (in \( M \)) must be taken into account. In this case, it suffices to show that \( P \sim_S F \), since then, by hypothesis, \( P = I^-(z, M) \) for some \( z \in \partial M \), and Lemma 6.5 (2) gives the result. So, assume by contradiction that \( P \not\sim_S F \). Choose any \( F' \) which is a maximal IF in \( \uparrow P \) containing \( F \). If \( P \sim_S F' \) holds, again by hypothesis we have \( F = I^+(z, M) \) for some \( z \in \partial M \), and Lemma 6.5 (2) implies that \( z_n \to z \) (anyway, this case could not hold because, then, one would also have \( F' \in \mathcal{L}(\{I^+(z_n, M)\}) \), in contradiction with the second expression in (24)). So, the problem is reduced to the following claim.

**Claim 2:** If \( F' \) is a maximal IF in \( \uparrow P \) then \( P \) must be a maximal IF in \( \downarrow F' \). In particular, \( P \sim_S F' \).

**Proof of the Claim 2.** Assume by contradiction that \( P \subsetneq P' \subsetneq F' \). Then, \( (P', F') \in \overline{M} \). Let \( z' \in \overline{M} \) be the point such that \( (P', F') = (I^-(z'), I^+(z')) \) (which exists by hypothesis) and consider future-directed chains \( \{p_i\} \) and \( \{p'_i\} \) generating \( P \) and \( P' \), respectively, such that \( p_i \ll p'_i \) for all \( i \). So, there exists some inextensible past-directed timelike curves \( \gamma_i \) joining \( p'_i \) with \( p_i \) for all \( i \). Since \( \{p'_i\} \) converges to \( z' \), from Prop. 2.16 there exists a past-directed causal limit curve \( \gamma : [0,1) \to \overline{M} \) with \( \gamma(0) = z' \). Next, we consider two excluding cases:

- There exists \( q_0 \in \text{Im}(\gamma) \) and some subsequence \( \{p_{i_k}\} \) such that \( p_{i_k} \to q_0 \). Note that we can directly exclude the case \( q_0 = z' \), since, otherwise, \( p_{i_k} \to z' \), and thus, \( P = I^-(z', M) = P' \) (see Claim 1), a contradiction. In this case, take \( s_0 \in (0,1) \) and \( \{s_{i_k}\} \) such that \( \gamma(s_{i_k}) = q_0 \neq z' \) and \( p_{i_k} = \gamma_{i_k}(s_{i_k}) \) for all \( k \).

- There is no subsequence of \( \{p_i\} \) converging to some point of \( \gamma \); in this case, take any \( s_0 \in (0,1) \), and define \( q_0 := \gamma(s_0) \). Since \( \gamma \) is a limit curve, there exists \( \{s_{i_k}\} \) such that \( \gamma_{i_k}(s_{i_k}) \to q_0 \). Moreover, we can assume additionally \( p_{i_k} \ll \gamma_{i_k}(s_{i_k}) \) for all \( k \). Indeed, otherwise we can assume, up to a subsequence, \( \gamma(s_{i_k}) \ll p_{i_k} \). By Prop. 2.18 (1) \( \{\gamma_{i_k} \mid [0,s_{i_k}]\} \) converges in the \( C^0 \)-topology to \( \gamma \mid [z',q_0] \). Thus, any relatively compact neighborhood of \( \text{Im}(\gamma|_{z',q_0}] \) contains all the points \( p_{i_k} \) up to a subsequence, in contradiction with the hypothesis of non-convergence.

Trivially, \( I^+(z', M) \subset I^+(q_0, M) \) and we will prove \( I^+(q_0, M) \subset \uparrow P \) with a reasoning valid in both cases. If \( w \in I^+(q_0, M) \) then \( q_0 \in I^-(w) \), and taking into account that \( \gamma_{i_k}(s_{i_k}) \to q_0 \), we have that \( \gamma_{i_k}(s_{i_k}) \in I^-(w) \). So, \( p_{i_k} \ll w \). Therefore, \( p \ll p_{i_k} \ll w \) for any \( p \in P \) and any \( k \) large enough, and thus, \( I^+(q_0, M) \subset \uparrow P \). Summarizing, we have proved \( I^+(z', M) \subset I^+(q_0, M) \subset \uparrow P \). On the other hand, since \( z' \neq q_0 \), [28, Lemma 4.6] ensures that \( F' = I^+(z', M) \subset I^+(q_0, M) \), in contradiction with the maximality of \( F' \) into \( \uparrow P \). □

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