Generalized Schröder paths and Young tableaux with skew shapes

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Abstract

A generalized Schröder path is a lattice path with steps (1,0), (1,1) and (0,1), and never goes above the diagonal line \( y = x \). In this paper, we firstly give the distribution of the major index over generalized Schröder paths. Then by providing a bijection between generalized Schröder paths and row-increasing tableaux of skew shapes with two rows, we obtain the distribution of the major index and the amajor index over these tableaux. We also generalize a result of Pechenik, and give the distribution of the major index over increasing tableaux of skew shapes with two rows. Especially, a bijection from row-increasing tableaux with shape \((n, m)\) and maximal value \(n + m - k\) to standard Young tableaux with a skew shape is obtained.

Keywords: major index, generalized Schröder path, row-increasing tableau, increasing tableau, jeu de taquin

1 Introduction

In this paper, we use \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \) to denote the partition with parts \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \). Let \( \lambda \) and \( \mu \) be partitions with \( \lambda \supseteq \mu \). A semistandard Young tableau (SSYT) of shape \( \lambda/\mu \) is defined as an array of positive integers
of shape $\lambda/\mu$ that is strictly increasing in rows and weakly increasing in columns. A standard Young tableau (SYT) of shape $\lambda/\mu$ is an SSYT of shape $\lambda/\mu$ with the set of its entries equal to $\{1, 2, \ldots, |\lambda/\mu|\}$. We denote by $\text{SYT}(\lambda/\mu)$ the set of all standard Young tableaux of shape $\lambda/\mu$.

Define a descent of an SSYT $T$ to be an integer $i$ such that $i + 1$ appears in a lower row of $T$ than $i$, and define the descent set $D(T)$ to be the set of all descents of $T$. The major index of $T$ is defined by $\text{maj}(T) = \sum_{i \in D(T)} i$. Similarly, an ascent of an SSYT $T$ is defined to be an integer $i$ such that $i + 1$ appears in a higher row of $T$ than $i$, and the ascent set $A(T)$ is defined to be the set of all ascents of $T$. The major index of $T$ is defined by $\text{amaj}(T) = \sum_{i \in A(T)} i$.

For given positive integer $n$, we denote by $[n] = 1 + q + \cdots + q^{n-1}$ and $[n]! = [1][2]\cdots[n]$, where we assume $[0] = 0$ and $[0]! = 1$. We use $\begin{bmatrix} n \end{bmatrix}$ to denote the $q$–binomial coefficient $\frac{[n]!}{[m]![n-m]!}$ for $n \geq m$, and define $\begin{bmatrix} n \end{bmatrix} = 0$ for $n < m$. The following $q$–hook length formula gives the distribution of the major index over standard Young tableaux with normal shapes.

**Lemma 1.1.** [1] For any partition $\lambda$ of a positive integer $n$, we have

$$\sum_{T \in \text{SYT}(\lambda)} q^{\text{maj}(T)} = \frac{q^{b(\lambda)}[n]!}{\prod_{u \in \lambda} h(u)},$$

where $b(\lambda) = \sum_{i \geq 1} (i - 1)\lambda_i$, and $h(u)$ is the hook length of $u$.

A reverse tableau of shape $\mu$ is an array of positive integers of shape $\mu$ that is weakly decreasing in rows and strictly decreasing in columns. Let $\text{RT}(\mu, n)$ denote the set of all reverse tableaux of shape $\mu$ whose entries belong to $\{1, 2, \ldots, n\}$. Figure 1 shows all tableaux of $\text{RT}((2, 2), 3)$.

$$
\begin{array}{cccc}
3 & 3 & 3 & 2 \\
2 & 2 & 2 & 1 \\
\end{array}
\begin{array}{cccc}
3 & 3 & 3 & 2 \\
1 & 1 & 1 & 1 \\
\end{array}
\begin{array}{cccc}
3 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 \\
\end{array}
$$

Figure 1: All tableaux of $\text{RT}((2, 2), 3)$.

In [2], we studied the major index of standard Young tableaux with skew shapes, and gave a formula for

$$s_{\lambda/\mu}(1, q, q^2, \ldots)/s_\lambda(1, q, q^2, \ldots),$$

which is equivalent to the following result.
Lemma 1.2. [2] Let $\lambda$ and $\mu$ be partitions with $\lambda \supseteq \mu$ and $n \in \mathbb{N}$ such that $l(\mu) \leq l(\lambda) \leq n$. Then we have

$$\frac{\sum_{T \in \text{SYT}(\lambda/\mu)} q^{\text{maj}(T)}}{\sum_{T \in \text{SYT}(\lambda)} q^{\text{maj}(T)}} = \frac{[\lambda/\mu]!}{[\lambda]!} \sum_{S \in \text{RT}(\mu,n)} \prod_{u \in \mu} q^{1-S(u)} [\lambda S(u) - c(u)],$$

where $c(u) = j - i$ for $u = (i,j)$.

A row-increasing tableau is an SSYT such that the set of its entries is an initial segment of $\mathbb{Z}_{>0}$, and an increasing tableau is a row-increasing tableau with its columns strictly increasing. We denote by $\text{RInc}_k(\lambda/\mu)$ the set of row-increasing tableaux with shape $\lambda/\mu$ and maximal value $|\lambda/\mu| - k$, and denote by $\text{Inc}_k(\lambda/\mu)$ the set of increasing tableaux with shape $\lambda/\mu$ and maximal value $|\lambda/\mu| - k$. Figure 2 shows a row-increasing tableau $T_1 \in \text{RInc}_2((4,3)/(1))$ and an increasing tableau $T_2 \in \text{Inc}_2((4,3)/(1))$.

$$T_1 : \begin{array}{cccc}
2 & 3 & 4 \\
1 & 2 & 3 
\end{array} \quad T_2 : \begin{array}{ccc}
1 & 2 & 3 \\
3 & 4 
\end{array}$$

Figure 2: A row-increasing tableau $T_1 \in \text{RInc}_2((4,3)/(1))$, and an increasing tableau $T_2 \in \text{Inc}_2((4,3)/(1))$.

Pechenik studied the major index for $\text{Inc}_k((n,n))$, and gave the following result.

Lemma 1.3. [3] For any positive integer $n$, and $0 \leq k \leq n$, we have

$$\sum_{T \in \text{Inc}_k((n,n))} q^{\text{maj}(T)} = q^{n+k(k+1)/2} \left[ \begin{array}{c} n - 1 \\ k \end{array} \right] \left[ \begin{array}{c} 2n - k \\ n \end{array} \right].$$

Du, Fan and Zhao obtained the following formulas for the distribution of the major index and the amajor index over $\text{RInc}_k((n,n))$.

Lemma 1.4. [4] For any positive integer $n$, and $0 \leq k \leq n$, we have

$$\sum_{T \in \text{RInc}_k((n,n))} q^{\text{maj}(T)} = q^{n+k(k-3)/2} \left[ \begin{array}{c} 2n - k \\ k \end{array} \right] \left[ \begin{array}{c} 2n - 2k \\ n - k \end{array} \right],$$

and

$$\sum_{T \in \text{RInc}_k((n,n))} q^{\text{amaj}(T)} = q^{k(k-1)/2} \left[ \begin{array}{c} 2n - k \\ k \end{array} \right] \left[ \begin{array}{c} 2n - 2k \\ n - k \end{array} \right].$$
Increasing tableaux and row-increasing tableaux are closely related to
generalized Schröder paths. Here a generalized Schröder path is a lattice
path with steps (1,0), (1,1) and (0,1), and never goes above the diagonal line
\( y = x \). We use \( \text{Sch}_k(r, n, m) \) to denote the set of generalized Schröder paths
from \((r, 0)\) to \((n, m)\) with \(k\) (1,1) steps. Especially, \( \text{Sch}_k(0, n, n) \) is the set of
Schröder \( n \)–paths with \(k\) (1,1) steps. If a generalized Schröder path
doesn’t contain the step (1,1), then we call it a generalized Catalan path,
and we use \( \text{Cat}(r, n, m) \) denote the set of all generalized Catalan paths from \((r, 0)\) to
\((n, m)\).

There is a nature bijection from \( \text{RInc}_k((n, m)/(r)) \) to \( \text{Sch}_k(r, n, m) \): given
\( T \in \text{RInc}_k((n, m)/(r)) \), read the entries of \( T \) from 1 to \( n+m-r-k \)
in increasing order. Entries just appearing in the first (resp. second) row correspond
to the (1,0) (resp. (0,1)) step, and entries appearing in both rows correspond
to the (1,1) step. See Figure 3 for an example of the above bijection. Especially,
by restricting the above mapping to \( \text{SYT}((n, m)/(r)) \), we then obtain
a bijection from \( \text{SYT}((n, m)/(r)) \) to \( \text{Cat}(r, n, m) \).

In the following, we use \( E \) (East), \( D \) (Diagonal) and \( N \) (North) to denote
the three steps (1,0), (1,1) and (0,1) respectively. In this way, we can repre-
sent a generalized Schröder path as a word over the alphabet set \( \{E, D, N\} \),
and define its major index as follows. Given a word \( P = p_1 p_2 \cdots p_t \) which is
a permutation of a multiset whose elements are totally ordered, we say that
\( i \) is a descent of \( P \) if \( p_i > p_{i+1} \). The descent set \( D(P) \) is the collection of all
descents of \( P \). The major index of \( P \) is defined by \( \text{maj}(P) := \sum_{i \in D(P)} i \).

For given linear ordering of \( \{E, D, N\} \), Bonin, Shapiro and Simion gave
the distribution of the major index over \( \text{Sch}_k(0, n, n) \).

**Lemma 1.5.** [5] For given positive integers \( n, k \) and linear ordering of
\{E, D, N\}, the distribution of the major index over Sch_{k}(0, n, n) is

$$\sum_{P \in \text{Sch}_{k}(0, n, n)} q^{\text{maj}(P)} = \frac{q^{n-k}}{n-k+1} \left[\begin{array}{c} 2n-k \\ k \end{array}\right] \left[\begin{array}{c} 2n-2k \\ n-k \end{array}\right], \text{ if } E > N,$$

and

$$\sum_{P \in \text{Sch}_{k}(0, n, n)} q^{\text{maj}(P)} = \frac{1}{n-k+1} \left[\begin{array}{c} 2n-k \\ k \end{array}\right] \left[\begin{array}{c} 2n-2k \\ n-k \end{array}\right], \text{ if } E < N.$$

Note that the above equations differ from equations of Lemma 1.4 only by the factor $q^{k(k-1)/2}$. Thus it would be interesting to find some simple explanation on these relations, which we will show in section 3.

Motivated by the above results about Young tableaux and lattice paths, this paper will study the major index of generalized Schröder paths, row-increasing tableaux and increasing tableaux. In section 2, we give the distribution of the major index over Sch_{k}(r, n, m). In section 3, by studying generalized Schröder paths with a special labelling, we obtain the distribution of the major index and the amajor index over RInc_{k}((n, m)/(r)). In section 4, we give a bijection between Inc_{k}((n, m)/(r)) and the union of SYT((n-k, m-k, 1^{k})/(r)) and SYT((n-k, m-k+1, 1^{k-1})/(r)), and obtain the distribution of the major index over Inc_{k}((n, m)/(r)) as a corollary. For the special case when $r = 0$, we give a bijection between RInc_{k}((n, m)) and the union of Inc_{k}((n, m)) and Inc_{k-1}((n, m-1)), which implies a bijection between RInc_{k}((n, m)) and SYT((n-k+1, m-k+1, 1^{k})/(1^{2})).

## 2 The major index for Sch_{k}(r, n, m)

About the $q$–binomial coefficients, it is well known that

$$\left[\begin{array}{c} n \\ k \end{array}\right] = \left[\begin{array}{c} n-1 \\ k \end{array}\right] + q^{n-k} \left[\begin{array}{c} n-1 \\ k-1 \end{array}\right], \quad (2.1)$$

see [6] for instance. With the above recurrence formula, we obtain the following result, which can be proved by simple computation.

**Lemma 2.1.** Given positive integers $n$ and $k$ with $n \geq k$, we have

$$\sum_{s=k}^{n} q^{s-k} \left[\begin{array}{c} s \\ k \end{array}\right] = \left[\begin{array}{c} n+1 \\ k+1 \end{array}\right].$$
For $E > N$, we have the following generalization of a classical result of MacMahon [7] about the distribution of the major index over $\text{Cat}(0, n, n)$.

**Lemma 2.2.** Let $r$, $n$ and $m$ be positive integers with $r \leq n$ and $m \leq n$. Then for $E > N$, we have

$$\sum_{P \in \text{Cat}(r,n,m)} q^{\text{maj}(P)} = \left[\begin{array}{c} n + m - r \\ n - r \end{array}\right] - \left[\begin{array}{c} n + m - r \\ n + 1 \end{array}\right].$$

**Proof.** Let $\varphi$ denote the bijection from $\text{SYT}((n,m)/(r))$ to $\text{Cat}(r,n,m)$ given in the introduction. It is not difficult to verify that $\varphi$ preserves the descent set for $E > N$. Thus we have

$$\sum_{P \in \text{Cat}(r,n,m)} q^{\text{maj}(P)} = \sum_{T \in \text{SYT}((n,m)/(r))} q^{\text{maj}(T)}. \quad (2.2)$$

Let $\lambda$ denote the partition $(n,m)$. Then we obtain from Lemma 1.2 that

$$\sum_{T \in \text{SYT}(\lambda/(r))} q^{\text{maj}(T)} \sum_{T \in \text{RT}((r),2)} \prod_{u \in \mu} q^{1-S(u)}(\lambda - \mu) = [n + m - r] \cdot [n + m] \cdot [n + m - r]. \quad (2.3)$$

where the last equality is derived from Lemma 2.1.

By Lemma 1.1, we have

$$\sum_{T \in \text{SYT}(n,m)} q^{\text{maj}(T)} = q^m \left[\begin{array}{c} n + m + 1 \\ n + 1 \end{array}\right]. \quad (2.4)$$

Combining Equation (2.2)~(2.4) together, we then obtain Lemma 2.2. \qed
To obtain the distribution of the major index over $\text{Cat}(r, n, m)$ for $E < N$, we firstly consider the major index for $\overline{\text{SYT}}((n, m)/(r))$, where

$$\overline{\text{SYT}}((n, m)/(r)) := \{ T \in \text{SYT}((n, m)/(r)) | T(2, m) = n + m - r \}.$$ 

**Lemma 2.3.** Let $r$, $n$ and $m$ be positive integers with $r \leq n$ and $m \leq n$. Then we have

$$\sum_{T \in \overline{\text{SYT}}((n, m)/(r))} q^{\text{amaj}(T)} = \left[ \frac{n + m - r - 1}{n - r} \right] - q^{r+1} \left[ \frac{n + m - r - 1}{n + 1} \right].$$

**Proof.** For $T \in \overline{\text{SYT}}((n, m)/(r))$, let

$$D(T) = \{ x_1, x_2, \ldots, x_k \} \leq.$$ 

We define $\hat{A}(T) = A(T) \cup \{0\}$ if $x_1 = 1$, and $\hat{A}(T) = A(T)$ otherwise. Let

$$\hat{A}(T) = \{ y_1, y_2, \ldots, y_k \} \leq.$$

Assume $j_i$ to be the column number of $x_i$ in $T$ for $1 \leq i \leq k$ and set $j_0 = r$. Then we have

$$x_i - y_i = j_i - j_{i-1},$$

which implies that

$$\text{maj}(T) - \text{amaj}(T) = \sum_{i=1}^{k} (x_i - y_i) = j_k - j_0 = n - r.$$ 

Thus we have

$$\sum_{T \in \overline{\text{SYT}}((n, m)/(r))} q^{\text{amaj}(T)} = q^{r-n} \sum_{T \in \overline{\text{SYT}}((n, m)/(r))} q^{\text{maj}(T)}. \quad (2.5)$$

Therefore it is enough to consider the distribution of the major index over $\overline{\text{SYT}}((n, m)/(r))$. For any $T \in \overline{\text{SYT}}((n, m)/(r))$, the tableau $\psi(T)$ is produced by the following algorithm. If $n + m - r - 1$ is a descent of $T$, then $\psi(T) \in \text{SYT}((n-1, m-1)/(r))$ is obtained from $T$ by deleting the entries $n + m - r - 1$ and $n + m - r$. Otherwise, we obtain $\psi(T) \in \overline{\text{SYT}}((n, m-1)/(r))$ from $T$ by deleting the entry $n + m - r$. It is not hard to see that $\psi$ gives a
bijection from $\text{SYT}((n,m)/(r))$ to the union of $\text{SYT}((n-1,m-1)/(r))$ and $\text{SYT}((n,m-1)/(r))$. Thus we obtain from Lemma 2.2 that

$$
\sum_{T \in \text{SYT}((n,m)/(r))} q^{\text{maj}(T)} - \sum_{T \in \text{SYT}((n-1,m)/(r))} q^{\text{maj}(T)} = q^{n+m-r-1} \sum_{T \in \text{SYT}((n-1,m-1)/(r))} q^{\text{maj}(T)}
$$

(2.6)

$$
= q^{n+m-r-1} \left\{ \left[ \frac{n + m - r - 2}{n - r - 1} \right] - \left[ \frac{n + m - r - 2}{n} \right] \right\}.
$$

Since $\sum_{T \in \text{SYT}((n,0)/(r))} q^{\text{maj}(T)} = 0$, Equation (2.6) implies that

$$
\sum_{T \in \text{SYT}((n,m)/(r))} q^{\text{maj}(T)} = q^{n+m-r-1} \sum_{T \in \text{SYT}((n-1,m-1)/(r))} q^{\text{maj}(T)}
$$

(2.7)

where the last equality is derived from Lemma 2.1. Combining Equation (2.5) and (2.7) together, we then obtain Lemma 2.3.

**Lemma 2.4.** Let $r$, $n$ and $m$ be positive integers with $r \leq n$ and $m \leq n$. Then for $E < N$, we have

$$
\sum_{P \in \text{Cat}(r,n,m)} q^{\text{maj}(P)} = \left[ \frac{n + m - r}{n - r} \right] - q^{r+1} \left[ \frac{n + m - r}{n + 1} \right].
$$

Proof. Let $\varphi$ denote the bijection from $\text{SYT}((n,m)/(r))$ to $\text{Cat}(r,n,m)$ given in the introduction. Then for $E < N$ and $T \in \text{SYT}((n,m)/(r))$, we have $\text{amaj}(T) = \text{maj}(\varphi(T))$, which implies that

$$
\sum_{P \in \text{Cat}(r,n,m)} q^{\text{maj}(P)} = \sum_{T \in \text{SYT}((n,m)/(r))} q^{\text{amaj}(T)}.
$$

(2.8)

We firstly consider the case when $n > m$. Let $T$ be any tableau in $\text{SYT}((n,m)/(r))$. Let $T^+$ be the tableau obtained from $T$ by inserting $n+m-$
It is not hard to see that the above operation gives a bijection from $\text{SYT}((n, m)/(r))$ to $\overline{\text{SYT}}((n, m + 1)/(r))$, and the bijection preserves the ascent set. Thus we have $\text{amaj}(T) = \text{amaj}(T^*)$, which implies that

$$\sum_{T \in \text{SYT}((n, m)/(r))} q^{\text{amaj}(T)} = \sum_{T^+ \in \overline{\text{SYT}}((n, m+1)/(r))} q^{\text{amaj}(T^*)}. \quad (2.9)$$

Combining Equation (2.8), Equation (2.9) and Lemma 2.3 together, we then obtain Lemma 2.4.

If $n = m$, then we have $\text{SYT}((n, m)/(r)) = \overline{\text{SYT}}((n, m)/(r))$. Thus we derive from Lemma (2.3) and Equation (2.8) that

$$\sum_{P \in \text{Cat}(r, n, n)} q^{\text{maj}(P)} = \sum_{T \in \text{SYT}((n, n)/(r))} q^{\text{amaj}(T)} = \left[\frac{2n - r - 1}{n - r}\right] - q^{r+1} \left[\frac{2n - r - 1}{n + 1}\right].$$

where the last equality is derived from Equation (2.1).

A generalized Schröder path of $\text{Sch}_k(r, n, m)$ can be viewed as a shuffle of a generalized Catalan path of $\text{Cat}(r, n - k, m - k)$ and $D^k$, where shuffles of permutations of multisets are defined as follows.

Let $M = \{a_1^{n_1}, a_2^{n_2}, \ldots, a_m^{n_m}\}$ be a multiset containing $n_i$ copies of $a_i$, where $a_1 < a_2 < \cdots < a_m$ and $n = n_1 + n_2 + \cdots + n_m$. We denote by $\sigma_M$ the set of all permutations of $M$. Let $\{\sigma_1, \sigma_2, \ldots, \sigma_k\}$ be a collection of complementary permutations of subsets of $M$, i.e., they are disjoint as subsets and their union equals to $M$. A permutation $\sigma \in \sigma_M$ is called a shuffle of $\{\sigma_1, \sigma_2, \ldots, \sigma_k\}$ if it contains $\sigma_i (1 \leq i \leq k)$ as a subword. We denote by $F(\sigma_1, \sigma_2, \ldots, \sigma_k)$ the set of all shuffles of $\{\sigma_1, \sigma_2, \ldots, \sigma_k\}$.

For the distribution of the major index over shuffles of permutations of normal sets, Garsia and Gessel gave the following remarkable result, which extended a classical result of MacMahon([8]) and Foata([9]).

**Lemma 2.5.** [10] Let $\{\sigma_1, \sigma_2, \ldots, \sigma_k\}$ be a collection of complementary permutations of subsets of $\{1, 2, \ldots, n\}$. Then we have

$$\sum_{\sigma \in F(\sigma_1, \sigma_2, \ldots, \sigma_k)} q^{\text{maj}(\sigma)} = \left[\begin{array}{c} n \\ \mu_1, \mu_2, \ldots, \mu_k \end{array}\right] q^{\text{maj}(\sigma_1) + \cdots + \text{maj}(\sigma_k)},$$

where $\{\mu_1, \mu_2, \ldots, \mu_k\}$ is the partition of $n$. 

where $\mu_i$ is the cardinality of $\sigma_i$.

The above result can be extended to multisets as follows.

**Lemma 2.6.** Let $M = \{a_1^{n_1}, a_2^{n_2}, \ldots, a_m^{n_m}\}$ be a multiset with $a_1 < a_2 < \cdots < a_m$ and $n = n_1 + n_2 + \cdots + n_m$. Let $\{\sigma_1, \sigma_2, \ldots, \sigma_k\}$ be a collection of complementary permutations of subsets of $M$, then

$$\sum_{\sigma \in F(\sigma_1, \sigma_2, \ldots, \sigma_k)} q^{\text{maj}(\sigma)} = \left[ \begin{array}{c} n \\ \mu_1, \mu_2, \ldots, \mu_k \end{array} \right] q^{\text{maj}(\sigma_1) + \cdots + \text{maj}(\sigma_k)}$$

where $\mu_i$ is the cardinality of $\sigma_i$.

**Proof.** For $1 \leq i \leq k$, let $\tilde{\sigma}_i$ be the word obtained from $\sigma_i$ by replacing $a_j (1 \leq j \leq m)$ with $n_1 + \cdots + n_{j-1} + 1, \ldots, n_1 + \cdots + n_j$ in increasing order, where we set $n_0 = 0$. Since $\sigma_1, \sigma_2, \ldots, \sigma_k$ are complementary permutations of subsets of $M$, we know that $\sigma_i \cap \sigma_j = \emptyset$ for $i \neq j$, and $\sigma_i$ contains all of the $n_j$ copies of $a_j$ if $a_j \in \sigma_i$. Thus the above operation is well defined. Moreover, $\{\tilde{\sigma}_1, \tilde{\sigma}_2, \ldots, \tilde{\sigma}_k\}$ is a collection of complementary permutations of subsets of $\{1, 2, \ldots, n\}$.

Given $\sigma \in F(\sigma_1, \sigma_2, \ldots, \sigma_k)$, we denote by $f(\sigma)$ the permutation obtained from $\sigma$ by replacing $\sigma$ with $\tilde{\sigma}_i$. It is not difficult to see that $f$ gives a bijection from $F(\sigma_1, \sigma_2, \ldots, \sigma_k)$ to $F(\tilde{\sigma}_1, \tilde{\sigma}_2, \ldots, \tilde{\sigma}_k)$, and the bijection preserves the major index. Since $\text{maj}(\sigma_i) = \text{maj}(\tilde{\sigma}_i)$ and $|\sigma_i| = |\tilde{\sigma}_i|$, we derive from Lemma 2.5 that

$$\sum_{\sigma \in F(\sigma_1, \sigma_2, \ldots, \sigma_k)} q^{\text{maj}(\sigma)} = \sum_{f(\sigma) \in F(\tilde{\sigma}_1, \tilde{\sigma}_2, \ldots, \tilde{\sigma}_k)} q^{\text{maj}(f(\sigma))}$$

$$= \left[ \begin{array}{c} n \\ \mu_1, \mu_2, \ldots, \mu_k \end{array} \right] q^{\text{maj}(\tilde{\sigma}_1) + \cdots + \text{maj}(\tilde{\sigma}_k)}$$

$$= \left[ \begin{array}{c} n \\ \mu_1, \mu_2, \ldots, \mu_k \end{array} \right] q^{\text{maj}(\sigma_1) + \cdots + \text{maj}(\sigma_k)}.$$

**Example 2.7.** For $M = \{1^2, 2^3, 3^2, 4\}$, let $\sigma_1 = 1313$ and $\sigma_2 = 2242$ be complementary permutations of subsets of $M$. Let $\sigma = 12321423$ be a shuffle of $\sigma_1$ and $\sigma_2$. Then $\tilde{\sigma}_1 = 1627$, $\tilde{\sigma}_2 = 3485$, and $f(\sigma) = 13642857$ is a shuffle of $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ with $D(f(\sigma)) = D(\sigma) = \{3, 4, 6\}$. 

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Combining Lemma 2.2, Lemma 2.4 and Lemma 2.6 together, we then obtain the distribution of the major index over generalized Schröder paths for given linear ordering of \( \{E, D, N\} \).

**Theorem 2.8.** Let \( r, n, m \) and \( k \) be positive integers with \( \text{Sch}_k(r, n, m) \neq \emptyset \). If \( E > N \), then we have

\[
\sum_{P \in \text{Sch}_k(r, n, m)} q^{\text{maj}(P)} = \left[ \frac{n + m - r - k}{k} \right] \left\{ \left[ \frac{n + m - r - 2k}{n - r - k} \right] - \left[ \frac{n + m - r - 2k}{n - k + 1} \right] \right\}.
\]

If \( E < N \), then we have

\[
\sum_{P \in \text{Sch}_k(r, n, m)} q^{\text{maj}(P)} = \left[ \frac{n + m - r - k}{k} \right] \left\{ \left[ \frac{n + m - r - 2k}{n - r - k} \right] - q^{r+1} \left[ \frac{n + m - r - 2k}{n - k + 1} \right] \right\}.
\]

### 3 The major index and the amajor index for \( \text{RInc}_k((n, m)/(r)) \)

For given linear ordering of \( \{E, D, N\} \), a labelling

\[ w : \text{Sch}_k(r, n, m) \to \sigma_{1,2,...,n+m-r-k} \]

is said to be diagonal-reverse if for \( P = p_1 p_2 \cdots p_{n+m-r-k} \in \text{Sch}_k(r, n, m) \), we have

\[ w(P)_i > w(P)_j \Leftrightarrow p_i > p_j, \text{ or } p_i = p_j = D \text{ and } i < j. \]

**Lemma 3.1.** Let \( r, n, m \) and \( k \) be positive integers with \( \text{Sch}_k(r, n, m) \neq \emptyset \). For given linear ordering of \( \{E, D, N\} \), let \( w \) be a diagonal-reverse labelling over \( \text{Sch}_k(r, n, m) \). If \( E > D > N \), then we have

\[
\sum_{P \in \text{Sch}_k(r, n, m)} q^{\text{maj}(w(P))} = q^{\frac{k(k-1)}{2}} \left[ \frac{n + m - r - k}{k} \right] \left\{ \left[ \frac{n + m - r - 2k}{n - r - k} \right] - \left[ \frac{n + m - r - 2k}{n - k + 1} \right] \right\}.
\]
If $E < D < N$, then we have
\[
\sum_{P \in \text{Sch}_k(r,n,m)} q^{\text{maj}(w(P))} = q^{\frac{k(k-1)}{2}} \left[ \begin{array}{c} n + m - r - k \\ k \end{array} \right] \left\{ \left[ \begin{array}{c} n + m - r - 2k \\ n - r - k \end{array} \right] - q^{r+1} \left[ \begin{array}{c} n + m - r - 2k \\ n - k + 1 \end{array} \right] \right\}.
\]

Proof. We just give the proof for $E > D > N$, while the proof for $E < D < N$ is almost the same. Let $\sigma = D_k D_{k-1} \cdots D_1$ be a word with $E < D_1 < D_2 < \cdots < D_k < N$. We denote by $\mathcal{F}((\text{Cat}(r,n,m), \sigma))$ the set of all shuffles of $\sigma$ with lattice paths in $\text{Cat}(r,n,m)$.

Given $P \in \text{Sch}_k(r,n,m)$, let $\psi(P)$ denote the word obtained from $P$ by replacing the $i$-th $D$ with $D_{k-i+1}$ for $1 \leq i \leq k$. It is not difficult to see that $\psi$ gives a bijection from $\text{Sch}_k(r,n,m)$ to $\mathcal{F}((\text{Cat}(r,n-m-k,m-k), \sigma))$. Moreover, the bijection $\psi$ satisfies $D(w(P)) = D(\psi(P))$. Thus we obtain from Lemma 2.2 and Lemma 2.6 that
\[
\sum_{P \in \text{Sch}_k(r,n,m)} q^{\text{maj}(w(P))} = q^{\frac{k(k-1)}{2}} \left[ \begin{array}{c} n + m - r - k \\ k \end{array} \right] \left\{ \left[ \begin{array}{c} n + m - r - 2k \\ n - r - k \end{array} \right] - \left[ \begin{array}{c} n + m - r - 2k \\ n - k + 1 \end{array} \right] \right\}.
\]

As a direct corollary of the above result, we then obtain the distribution of the major index and the amajor index over $\text{RInc}_k((n,m)/(r))$.

Theorem 3.2. For positive integers $r,n,m$ and $k$ with $\text{RInc}_k((n,m)/(r)) \neq \emptyset$, we have
\[
\sum_{T \in \text{RInc}_k((n,m)/(r))} q^{\text{maj}(T)} = q^{\frac{k(k-1)}{2}} \left[ \begin{array}{c} n + m - r - k \\ k \end{array} \right] \left\{ \left[ \begin{array}{c} n + m - r - 2k \\ n - r - k \end{array} \right] - \left[ \begin{array}{c} n + m - r - 2k \\ n - k + 1 \end{array} \right] \right\},
\]
and
\[
\sum_{T \in \text{RInc}_k((n,m)/(r))} q^{\text{amaj}(T)} = q^{\frac{k(k-1)}{2}} \left[ \begin{array}{c} n + m - r - k \\ k \end{array} \right] \left\{ \left[ \begin{array}{c} n + m - r - 2k \\ n - r - k \end{array} \right] - q^{r+1} \left[ \begin{array}{c} n + m - r - 2k \\ n - k + 1 \end{array} \right] \right\}.
\]
Proof. Let \( \varphi \) be the bijection from \( \text{RInc}_k((n, m)/(r)) \) to \( \text{Sch}_k(r, n, m) \) given in the introduction. For given linear ordering of \( \{E, D, N\} \), let \( w \) be a diagonal-reverse labelling over \( \text{Sch}_k(r, n, m) \). Then for any \( T \in \text{RInc}_k((n, m)/(r)) \), we have

\[
\text{maj}(w(\varphi(T))) = \text{maj}(T), \text{ if } E > D > N,
\]

and

\[
\text{maj}(w(\varphi(T))) = \text{amaj}(T), \text{ if } E < D < N.
\]

In fact, if \( \varphi(T) = p_1p_2 \cdots p_{n+m-r-k} \), then for \( E > D > N \), we have \( i \) a descent of \( w(\varphi(T)) \) if and only if the pair \( (p_i, p_{i+1}) \) equals to \( (D, N) \), \( (E, N) \), \( (E, D) \) or \( (D, D) \). In all cases, \( i \) is a descent of \( T \). The discussion for \( E < D < N \) is similar. Thus Theorem 3.2 is derived from Lemma 3.1. \( \square \)

4 The major index for \( \text{Inc}_k((n, m)/(r)) \)

Pechenik [3] gave a bijection between \( \text{Inc}_k((n, n)) \) and \( \text{SYT}((n-k, n-k, 1^k)) \), which can be generalized as follows.

Lemma 4.1. There is an explicit bijection between \( \text{Inc}_k((n, m)/(r)) \) and the union of \( \text{SYT}((n-k, m-k, 1^k)/(r)) \) and \( \text{SYT}((n-k, m-k+1, 1^{k-1})/(r)) \) that preserves the major index.

Proof. Given \( T \in \text{Inc}_k((n, m)/(r)) \), let \( A \) be the set of numbers that appear twice in \( T \). Let \( B \) be the set of numbers that appear in the second row immediately right of an element of \( A \). Then \( \chi(T) \) is a Young tableau produced by the following algorithm. We firstly delete all elements of \( A \) from the first row of \( T \) and all elements of \( B \) from the second row, and obtain \( \chi(T) \) by appending \( B \) to the first column. Then \( \chi(T) \in \text{SYT}((n-k, m-k, 1^k)) \) if \( T(2, m) \) just appears in the second row, and \( \chi(T) \in \text{SYT}((n-k, m-k+1, 1^{k-1})) \) otherwise.

It is not hard to see that \( \chi \) preserves the descent set and the major index. Thus we just need to show that \( \chi \) is reversible. For any \( S \) in the union of \( \text{SYT}((n-k, m-k, 1^k)/(r)) \) and \( \text{SYT}((n-k, m-k+1, 1^{k-1})/(r)) \), let \( B \) be the set of entries below the second row. By deleting entries below the second row and inserting \( B \) into the second row of \( S \) while maintaining increasingness, we obtain a tableau \( T_1 \). Let \( A \) be the set of numbers immediately left of an element of \( B \) in the second row of \( T_1 \). If \( S \in \text{SYT}((n-k, m-k, 1^k)/(r)) \), then \( \chi^{-1}(S) \) is obtained by inserting \( A \) into the first row of \( T_1 \) while maintaining
increasingness. If \( S \in \text{SYT}((n-k, m-k+1, 1^{k-1})(r)) \), then \( \chi^{-1}(S) \) is obtained by inserting \( A \) and \( T_1(2, m) \) into the first row of \( T_1 \) while maintaining increasingness. \( \square \)

The following result gives the distribution of the major index over standard Young tableaux with shape \((n, m, 1^k)/(r)\).

**Lemma 4.2.** For positive integers \( r, n, m \) and \( k \) with \( \text{SYT}((n, m, 1^k)/(r)) \neq \emptyset \), we have

\[
\sum_{T \in \text{SYT}((n, m, 1^k)/(r))} q^{\text{maj}(T)} = q^{\frac{k(k+1)}{2}} \binom{n+m+k-r}{k} \cdot \left( \frac{[m]}{[m+k]} \binom{n+m-r}{m} - \frac{[n+1]}{[n+k+1]} \binom{n+m-r}{n+1} \right).
\]

**Proof.** Let \( \lambda \) denote the partition \((n, m, 1^k)\). By Lemma 1.2, we have

\[
\frac{\sum_{T \in \text{SYT}(\lambda/(r))} q^{\text{maj}(T)}}{\sum_{T \in \text{SYT}(\lambda)} q^{\text{maj}(T)}} = \frac{[n+m+k-r]}{[n+m+k]} \cdot \sum_{S \in \text{RT}((r), k+2)} \prod_{u \in (r)} q^{1-S(u)}[\lambda_{S(u)} - c(u)]. \tag{4.1}
\]

Given \( S \in \text{RT}((r), k+2) \), if \( \lambda_{S(u)} < c(u_0) \) for some \( u_0 \in (r) \), then by [2, Equation 2.19], we have

\[
\prod_{u \in (r)} q^{1-S(u)}[\lambda_{S(u)} - c(u)] = 0.
\]
Thus we can assume $S(1, i) \leq 2$ for $2 \leq i \leq r$, and divide the sum of the right-hand side of Equation (4.1) into two parts:

$$
\sum_{S(1,1)=1} \prod_{u \in (r)} q^{1-S(u)}[\lambda_{S(u)} - c(u)] = \frac{[n]!}{[n-r]!}.
$$

(4.2)

and

$$
\sum_{S(1,1) \geq 2} \prod_{u \in (r)} q^{1-S(u)}[\lambda_{S(u)}^1 - c(u)] = (\frac{[m]}{q} + \sum_{i=2}^{k+1} \frac{1}{q^i})
\cdot \left\{ \frac{[m-1]![n-m]!}{q^{m-1}[n-r]!} \cdot \sum_{i=0}^{r-1} \left[ \frac{n-i-1}{n-m} \right] q^{m-i-1} \right\}
\cdot \frac{[m+k]![m-1]![n-m]!}{q^{m+k}[n-r]!} \cdot \left( \left[ \frac{n}{n-m+1} \right] - \left[ \frac{n-r}{n-m+1} \right] \right).
$$

(4.3)

where the last identity is derived from Lemma 2.1.

By Lemma 1.1, we have

$$
\sum_{T \in SYT((n,m,1^k)/(r))} q^{maj(T)} = q^m \cdot \frac{(k+1)}{2} \left[ \frac{n+m+1}{n+k+1} \right] \left[ \frac{m+k-1}{k} \right] \left[ \frac{n+m+k}{n} \right].
$$

(4.4)

Combining Equation (4.1) ~ (4.4) together, we then obtain Lemma 4.2. $\Box$

Applying the above result to Lemma 4.1, we then obtain the distribution of the major index over Inc$_k((n,m)/(r))$.

**Theorem 4.3.** For positive integers $r, n, m$ and $k$ with Inc$_k((n,m)/(r)) \neq \emptyset$, we have

$$
\sum_{T \in Inc_k((n,m)/(r))} q^{maj(T)} = q^\frac{k(k-1)}{2} \left[ \frac{n+m-k-r}{k} \right]
\cdot \left( \left[ \frac{n+m-2k-r}{m-k} \right] - \frac{q^n[k] + [n][m-r]}{[n][n+1]} \left[ \frac{n+m-2k-r}{n-k} \right] \right).
$$

Proof. By Lemma 4.2, we have

$$
\sum_{T \in SYT((n-k,m-k,1^k)/(r))} q^{maj(T)} = q^\frac{k(k+1)}{2} \left[ \frac{n+m-k-r}{k} \right]
\cdot \left( \left[ \frac{m-k}{[m]} \left[ \frac{n+m-2k-r}{m-k} \right] - \frac{[n-k+1]}{[n+1]} \left[ \frac{n+m-2k-r}{n-k+1} \right] \right). \right.
$$

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and
\[ q^{\text{maj}(T)} = q^{\frac{k(k-1)}{2}} \binom{n+m-k-r}{k} \cdot \left( \binom{m + \binom{n + m - 2k - r}{m - k} - \binom{n + m - 2k - r}{n - k}}{k} \right). \]

Therefore, we obtain from Lemma 4.1 that
\[ \sum_{T \in \text{SYT}((n-k,m-k,1^{k-1})/(r))} q^{\text{maj}(T)} = \sum_{T \in \text{Inc}_k((n,m)/(r))} q^{\text{maj}(T)} + \sum_{T \in \text{SYT}((n-k,m-k,1^{k-1})/(r))} q^{\text{maj}(T)} \]
\[ = q^{\frac{k(k-1)}{2}} \binom{n+m-k-r}{k} \cdot \left( \binom{m + \binom{n + m - 2k - r}{m - k} - \binom{n + m - 2k - r}{n - k}}{k} \right). \]

With the above result, we can give another proof of Theorem 3.2 for the special case when \( r = 0 \) by generalizing a bijection given in [4, Theorem 6] as follows.

**Lemma 4.4.** There is an explicit bijection between \( \text{RInc}_k((n,m)) \) and the union of \( \text{Inc}_k((n,m)) \) and \( \text{Inc}_{k-1}((n,m-1)) \) that preserves the major index.

**Proof.** Given \( T \in \text{RInc}_k((n,m)) \), we obtain the tableau \( \rho(T) \) by the following algorithm. If \( T \in \text{Inc}_k((n,m)) \), then \( \rho(T) = T \). Otherwise, let \( i \) be the minimal integer such that \( T(1,i) = T(2,i) \). Then \( \rho(T) \) is produced by first deleting \( T(2,i) \) and then moving \( T(2,j) \) one box to the left for \( i < j \leq m \). For the latter case, since \( i \) is minimal, \( T(2,i) - 1 \) just appears in the second row of \( T \) or \( i = 1 \), which implies that \( T(2,i) - 1 \) is not a descent of \( T \) and \( \rho(T) \). Thus the operation \( \rho \) preserves the descent set and the major index.

\[
T : \begin{array}{cccccc}
1 & 2 & 4 & 5 & 6 \\
2 & 3 & 4 & 6
\end{array} \quad \rho \quad \rho(T) : \begin{array}{cccc}
1 & 2 & 4 & 5 & 6 \\
2 & 3 & 6
\end{array}
\]

Figure 5: An example of \( \rho \) with \( T \in \text{RInc}_3((5,4)) \setminus \text{Inc}_3((5,4)) \).
It is not hard to see that
\[ \rho(T) \in \text{Inc}_k((n, m)) \cup \text{Inc}_{k-1}((n, m-1)). \]

Therefore we just need to show that \( \rho \) is reversible. If \( T \in \text{Inc}_k((n, m)) \), we have \( \rho^{-1}(T) = T \). If \( T \in \text{Inc}_{k-1}((n, m-1)) \), let \( i \) be the maximal integer such that \( T(2, i) = T(1, i + 1) - 1 \), where we assume \( T(2, 0) = 0 \). Then \( \rho^{-1}(T) \) is obtained from \( T \) by firstly moving \( T(2, j) \) one box to the right for \( i < j \leq m - 1 \), and then setting \( T(2, i + 1) = T(1, i + 1) \).

Corollary 4.5. For positive integers \( n, m \) and \( k \) with \( \text{RInc}_k((n, m)) \neq \emptyset \), we have

\[ \sum_{T \in \text{RInc}_k((n, m))} q^{\text{maj}(T)} = q^{m + \frac{k(k-3)}{2}} \left[ \frac{n - m + 1}{n - k + 1} \right] \left[ \frac{n + m - k}{k} \right] \left[ \frac{n + m - 2k}{m - k} \right]. \]

Proof. By Theorem 4.3, we have

\[ \sum_{T \in \text{Inc}((n, m))} q^{\text{maj}(T)} = q^{m + \frac{k(k-3)}{2}} \left[ \frac{n + m - k}{k} \right] \left[ \frac{n + m - 2k}{m - k} \right] \cdot q^{k \left[ \frac{n - m}{n} + q^{n-m+k} \right]} \]

and

\[ \sum_{T \in \text{Inc}_{k-1}((n, m-1))} q^{\text{maj}(T)} = q^{m + \frac{k(k-3)}{2}} \left[ \frac{n + m - k}{k} \right] \left[ \frac{n + m - 2k}{m - k} \right] \cdot \frac{q^{k \left[ \frac{n - m + 1}{n} + q^{n-m+k} \right]}}{\left[ \frac{n - k + 1}{n} \right].} \]

Since

\[ q^{k \left[ \frac{n - m}{n - k + 1} \right]} = \frac{q^{k \left[ n - k + 1 \right] \left[ n - m + 1 \right]}}{\left[ n - k + 1 \right]} = \frac{(n + 1 - k) \left[ n - m + 1 \right] + q^{n-m}}{\left[ n - k + 1 \right]} = \frac{n + 1 \left[ n - m \right] + q^{n-m} \left[ n - k + 1 \right]}{\left[ n - k + 1 \right]}, \]

and

\[ q^{n-m+k} \left( q^{k \left[ n - k \right]} + \left[ k \right] \right) = q^{n-m+k} \left[ n \right], \]

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Figure 6: An example of the jdt transformation.

we obtain from Lemma 4.4 that

$$\sum_{T \in RInc_k((n,m))} q^{maj(T)} = q^{m+k(k-3)/2} \cdot \frac{1}{[n-k+1]} \left[ \begin{array}{c} n+m-k \\ k \end{array} \right] \left[ \begin{array}{c} n+m-2k \\ m-k \end{array} \right]$$

$$\cdot \left\{ \frac{[n+1][n-m] + q^{n-m}[k]}{[n+1][n-k+1]} + \frac{q^{n-m}}{[n+1]} \right\}$$

$$= q^{m+k(k-3)/2} \cdot \frac{[n-m+1]}{[n-k+1]} \left[ \begin{array}{c} n+m-k \\ k \end{array} \right] \left[ \begin{array}{c} n+m-2k \\ m-k \end{array} \right].$$

\[\square\]

\textit{Jeu de taquin} (jdt) is a well-known transformation among skew Young tableaux. Readers can see [1, Ch. 7, App. I] for the detailed definition. We show an example of the jdt transformation in Figure 6, where bold numbers denote the entries moved during the transformation. Combining the jdt transformation, Lemma 4.1 and Lemma 4.4 together, we then obtain the following result.

**Corollary 4.6.** There is a bijection between $RInc_k((n,m))$ and $SYT((n-k+1, m-k+1, 1^k)/(1^2))$ that preserves the major index.

\textbf{Proof.} We firstly give a bijection from $SYT((n-k+1, m-k+1, 1^k)/(1^2))$ to the union of $SYT((n-k, m-k, 1^k))$, $SYT((n-k, m-k+1, 1^{k-1}))$, $SYT((n-k+1, m-k, 1^{k-1}))$ and $SYT((n-k+1, m-k+1, 1^{k-2}))$. Given $T \in SYT((n-k+1, m-k+1, 1^k)/(1^2))$, let $a = (1, 1)$ and $b = (2, 1)$ denote the two boxes beside the northwest corner of $T$. Then the tableau $g(T)$ is defined to be $jdt_a(jdt_b(T))$. If $T_1 = jdt_b(T) \in SYT((n-k+1, m-k, 1^k)/(1))$, then we have $T_1(2, i) > T_1(1, i+1)$ for $1 \leq i \leq m-k$, which implies that
$g(T) \in \text{SYT}((n - k, m - k, 1^k))$. Otherwise, $g(T)$ belongs to the union of $	ext{SYT}((n - k, m - k + 1, 1^{k-1}))$, $	ext{SYT}((n - k + 1, m - k, 1^{k-1}))$ and $	ext{SYT}((n - k + 1, m - k + 1, 1^{k-2}))$.

By the definition of jdt, it is easy to check that $g$ preserves the major index. We now construct the reverse of $g$ as follows. Given $S \in \text{SYT}((n - k, m - k, 1^k))$, let $a = (1, n - k + 1)$ and $b = (2, m - k + 1)$. Then we have $g^{-1}(S) = \text{jdt}_b(\text{jdt}_a(S))$. The construction of $g^{-1}$ for other cases is similar.

Combining Lemma 4.1 and Lemma 4.4 together, we know that $\chi \circ \rho$ is a bijection from $\text{RInc}_k((n, m))$ to the union of $\text{SYT}((n - k, m - k, 1^k))$, $\text{SYT}((n - k, m - k + 1, 1^{k-1}))$, $\text{SYT}((n - k + 1, m - k, 1^{k-1}))$ and $\text{SYT}((n - k + 1, m - k + 1, 1^{k-2}))$. Thus $g^{-1} \circ \chi \circ \rho$ gives the required bijection from $\text{RInc}_k((n, m))$ to $\text{SYT}((n - k + 1, m - k + 1, 1^k)/(1^2))$ that preserves the major index.

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