Auslander-Reiten theory revisited
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Abstract. We recall several results in Auslander-Reiten theory for finite-dimensional algebras over fields and orders over complete local rings. Then we introduce \textit{n}-cluster tilting subcategories and higher theory of almost split sequences and Auslander algebras there. Several examples are explained.

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Introduction

After three fundamental papers

\begin{enumerate}
    \item M. Auslander, \textit{Coherent functors}, 1966,
    \item M. Auslander and M. Bridger, \textit{Stable module theory}, 1969,
    \item M. Auslander, \textit{Representation dimension of artin algebras}, 1971,
\end{enumerate}

M. Auslander and I. Reiten established their theory in series of papers containing \cite{8, 9, 10, 15, 16, 17, 18, 19}. Their exciting application of abstract homological algebra opened a right way to understand the vast world of non-commutative algebras. The aim of this article is not to give a survey of Auslander-Reiten theory. We recall only several results from Auslander-Reiten theory focusing on the concept of Auslander algebras. Then we give an introduction to a higher theory of Auslander algebras and almost split sequences as discussed in \cite{85, 86}.

Our starting point is ‘Auslander correspondence’ given in \cite{7}, which is a bijection between representation-finite algebras and algebras satisfying \(\text{gl.\ dim } \Gamma \leq 2 \leq \text{dom.\ dim } \Gamma\), called \textit{Auslander algebras}. This was a milestone in modern representation theory of algebras leading to later Auslander-Reiten theory. Since Auslander algebras have ‘representation theoretic realization’ \cite{84} as endomorphism algebras of additive generators of module categories, some aspects of Auslander-Reiten theory can be regarded as a generalization of properties of Auslander algebras to general module categories without additive generators by using coherent functors \cite{6} and stable module categories \cite{13}. Our hope is the following.

\begin{quote}
    Certain classes of nice algebras should have ‘representation theoretic realization’ as endomorphism algebras of additive generators of certain classes of nice categories.
\end{quote}
The article is divided into the following sections and subsections.

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In Section 2 we consider finite-dimensional algebras satisfying \( \text{gl. dim} \Gamma \leq n + 1 \leq \text{dom. dim} \Gamma \), called \( n \)-Auslander algebras. We introduce \( n \)-cluster tilting subcategories (maximal \( (n-1) \)-orthogonal subcategories) of module categories. We show that they give ‘representation theoretic realization’ of \( n \)-Auslander algebras. As a generalization of properties of \( n \)-Auslander algebras, we find a certain analogue of Auslander-Reiten theory for \( n \)-cluster tilting subcategories, namely the \( n \)-Auslander-Reiten translation, the \( n \)-Auslander-Reiten duality and \( n \)-almost split sequences.

In Section 3 we study a large class of nice algebras, containing finite-dimensional algebras, called orders, and their nice modules called Cohen-Macaulay modules. We recall results from Auslander-Reiten theory for the category of Cohen-Macaulay modules over orders [10, 11]. The theory is particularly nice for Krull dimension two since there exist fundamental sequences connecting projective modules and injective modules and having properties similar to almost split sequences. We also introduce \( n \)-Auslander-Reiten theory for \( n \)-cluster tilting subcategories of the category of Cohen-Macaulay modules over orders. This is particularly nice for Krull dimension \( n + 1 \) since there exist \( n \)-fundamental sequences connecting projective modules and injective modules and having properties similar to \( n \)-almost split sequences. Consequently \( n \)-Auslander algebras for Krull dimension \( n + 1 \) enjoy particularly nice properties, and they form \( (n+1) \)-Calabi-Yau algebras in certain cases. We also compare them with non-commutative crepant resolutions introduced by Van den Bergh. Although in this article we could not deal with an exciting aspect of 2-cluster tilting theory on categorification of Fomin-Zelevinsky cluster algebras, a few results will be explained in Subsection 3.4.
We give several examples of \(n\)-cluster tilting subcategories. In Subsection 2.4 we give an inductive construction of \((n-1)\)-Auslander algebras with \(n\)-cluster tilting subcategories [87]. In Subsection 4.1 we construct a family of finite-dimensional factor algebras of preprojective algebras parametrized by elements \(w\) in the corresponding Coxeter group. We show that the categories of their syzygy modules contain 2-cluster tilting objects given by reduced expressions of \(w\) [37]. In Subsection 4.2 we give a necessary and sufficient condition for one-dimensional hypersurface singularities to have 2-cluster tilting objects, and classify 2-cluster tilting objects [42] by applying results in previous sections and the theory of tilting mutation due to Riedtmann-Schofield and Happel-Unger.

Conventions. Throughout this paper all modules are usually right modules. For a noetherian ring \(\Lambda\), we denote by \(\text{mod}\ \Lambda\) the category of finitely generated \(\Lambda\)-modules, and by \(\text{f.} \text{l.} \ \Lambda\) the category of \(\Lambda\)-modules with finite length. The composition \(fg\) of morphisms \(f\) and \(g\) means first \(g\), then \(f\). For a module \(M \in \text{mod}\ \Lambda\), we denote by \(\text{add}\ M\) the full subcategory of \(\text{mod}\ \Lambda\) consisting of direct summands of finite direct sums of copies of \(M\). For example, \(\text{add}\ \Lambda\) is the category of finitely generated projective \(\Lambda\)-modules. We denote by \(J\ \Lambda\) the Jacobson radical of \(\Lambda\).

Let \(\mathcal{X}\) be an additive category. We call \(\mathcal{X}\) Krull-Schmidt if any object in \(\mathcal{X}\) is isomorphic to a finite direct sum of objects whose endomorphism rings are local. We call an object in \(\mathcal{X}\) basic if all indecomposable direct summands are mutually non-isomorphic. We say that \(I\) is an ideal of \(\mathcal{X}\) if we have a subgroup \(I(X,Y)\) of \(\text{Hom}_\mathcal{X}(X,Y)\) for any \(X,Y \in \mathcal{X}\) and

\[
(Y, Z) \cdot I(X,Y) \cdot (W, X) \subset I(W, Z)
\]

holds for any \(W, X, Y, Z \in \mathcal{X}\). There exists an ideal \(J_X\) called the Jacobson radical of \(\mathcal{X}\) such that \(J_X(X,X)\) coincides with the Jacobson radical \(J_{\text{End}_\mathcal{X}(X)}\) of the endomorphism ring \(\text{End}_\mathcal{X}(X)\) for any \(X \in \mathcal{X}\). For any object \(M \in \mathcal{X}\), we have an ideal \([M]\) of \(\mathcal{X}\) defined by

\[
[M](X,Y) := \{ f \in (X,Y) \mid f \text{ factors through } M^\ell \text{ for some } \ell \}.
\]

Let \(\mathcal{X}\) be an extension-closed subcategory of an abelian category \(\mathcal{A}\). We denote by

\[
\mathcal{P} := \{ X \in \mathcal{X} \mid \text{Ext}_\mathcal{A}^i(X, \mathcal{X}) = 0 \text{ for } i > 0 \},
\]

\[
\mathcal{I} := \{ X \in \mathcal{X} \mid \text{Ext}_\mathcal{A}^i(\mathcal{X}, X) = 0 \text{ for } i > 0 \},
\]

the categories of projective objects and injective objects in \(\mathcal{X}\). We say that \(\mathcal{X}\) has enough projectives if, for any \(X \in \mathcal{X}\), there exists an exact sequence \(0 \to Y \to P \to X \to 0\) in \(\mathcal{A}\) with \(Y \in \mathcal{X}\) and \(P \in \mathcal{P}\). Dually, we define enough injectives.

1. From Auslander-Reiten theory

In this section we recall several results in Auslander-Reiten theory.
1.1. Auslander algebras. A finite-dimensional algebra $\Lambda$ is called representation-finite if there are only finitely many isoclasses of indecomposable $\Lambda$-modules. In this case there exists $M \in \text{mod}\Lambda$ such that $\text{add} M = \text{mod}\Lambda$, and we have a finite-dimensional algebra $\text{End}_\Lambda(M)$ with a categorical equivalence

$$\text{Hom}_\Lambda(M, -) : \text{mod}\Lambda \to \text{add}\text{End}_\Lambda(M).$$

The representation theory of $\Lambda$ is encoded in the structure of the finite-dimensional algebra $\text{End}_\Lambda(M)$. To study $\text{End}_\Lambda(M)$ we introduce a certain class of ‘Auslander-type conditions’ on injective resolutions of noetherian rings (see [21, 75]), whose prototype is a property of commutative Gorenstein rings.

**Definition 1.1.** Let $\Gamma$ be a noetherian ring with a minimal injective resolution

$$0 \to \Gamma \to I_0 \to I_1 \to \cdots$$

of the $\Gamma$-module $\Gamma$. Let $n$ and $m$ be positive integers. Following Tachikawa [115] and Hoshino [73], we write

$$\text{dom}. \dim \Gamma \geq n$$

if $I_i$ is a flat $\Gamma$-module for any $0 \leq i < n$. More generally, we say that $\Gamma$ satisfies the $(m,n)$-condition [83, 74] if $\text{fd} I_i < m$ holds for any $0 \leq i < n$. Notice that when $\Gamma$ is a finite-dimensional algebra, we have $\text{fd} I_i = \text{pd} I_i$.

While the dominant dimension is known to be left-right symmetric in the sense that $\text{dom}. \dim \Gamma = \text{dom}. \dim \Gamma^{\text{op}}$ holds, easy examples show that the $(m,n)$-condition is not left-right symmetric. We say that $\Gamma$ satisfies the two-sided $(m,n)$-condition if $\Gamma$ and $\Gamma^{\text{op}}$ satisfies the $(m,n)$-condition.

We call a finite-dimensional algebra $\Gamma$ an Auslander algebra if

$$\text{gl}. \dim \Gamma \leq 2 \leq \text{dom}. \dim \Gamma.$$

We have the following classical Auslander correspondence [7, 23]. In Theorem 2.6 we shall give a proof for a more general statement.

**Theorem 1.2.** (a) Let $\Lambda$ be a representation-finite finite-dimensional algebra and $\text{add} M = \text{mod}\Lambda$. Then $\text{End}_\Lambda(M)$ is an Auslander algebra.

(b) Any Auslander algebra can be obtained in this way. This gives a bijection between the Morita-equivalence classes of representation-finite finite-dimensional algebras and those of Auslander algebras.

Auslander algebras $\Gamma$ enjoy a lot of interesting properties, which come from the representation theory of $\Lambda$. Let us mention one of them.

**Definition 1.3.** Let $\Gamma$ be a noetherian ring with $\text{gl}. \dim \Gamma = n < \infty$. We say that $\Gamma$ satisfies the Gorenstein condition if any simple $\Gamma$-module $S$ satisfies

$$\text{Ext}_\Gamma^i(S, \Gamma) = \begin{cases} 0 & \text{if } i \neq n, \\ \text{a simple } \Gamma^{\text{op}}\text{-module} & \text{if } i = n, \end{cases}$$

and the same condition holds for the simple $\Gamma^{\text{op}}$-modules. More generally, we say that $\Gamma$ satisfies the restricted Gorenstein condition if (2) holds for any simple $\Gamma$-module and $\Gamma^{\text{op}}$-module $S$ with $\text{pd} S = n$. 


The condition (2) is satisfied by commutative local Gorenstein rings \(100\), and the Gorenstein condition is used in the definition of Artin-Schelter regular algebra \(4\). The following observation \(83\) shows the relationship between Gorenstein condition and Auslander-type condition.

**Proposition 1.4.** A noetherian ring \(\Gamma\) with \(\text{gl. dim } \Gamma = n < \infty\) satisfies the restricted Gorenstein condition if and only if \(\Gamma\) satisfies the two-sided \((n, n)\)-condition.

In particular, any Auslander algebra \(\Gamma\) satisfies the restricted Gorenstein condition. In the next subsection we shall interpret this property in terms of the representation theory of the corresponding representation-finite algebra \(\Lambda\). It is the existence of almost split sequences.

### 1.2. Almost split sequences.

Let us recall the following observation \(3, 23\).

**Proposition 1.5.** Let \(\Lambda\) be a finite-dimensional algebra, and let

\[
0 \to Z \xrightarrow{g} Y \xrightarrow{f} X \to 0
\]

be an exact sequence in \(\text{mod } \Lambda\) with \(f, g \in \text{J}_{\text{mod } \Lambda}\). Then

\[
0 \to (-, Z) \xrightarrow{g} (-, Y) \xrightarrow{f} \text{J}_{\text{mod } \Lambda} (-, X) \to 0
\]

is exact if and only if

\[
0 \to (X, -) \xrightarrow{f} (Y, -) \xrightarrow{g} \text{J}_{\text{mod } \Lambda} (Z, -) \to 0
\]

is exact. In this case \(X\) is indecomposable if and only if \(Z\) is indecomposable.

When these conditions are satisfied, we call \((3)\) an almost split sequence.

Now we give a relationship between almost split sequences and the restricted Gorenstein conditions. Assume that \(\Lambda\) is a representation-finite finite-dimensional algebra with an additive generator \(M\) of \(\text{mod } \Lambda\). Let \(\Gamma := \text{End}_{\Lambda}(M)\) be the corresponding Auslander algebra. Then we have categorical equivalences

\[
P^- := (M, -) : \text{mod } \Lambda \xrightarrow{\sim} \text{add } \Gamma, \\
P^- := (-, M) : \text{mod } \Lambda \xrightarrow{\sim} \text{add } \Gamma^\text{op},
\]

such that \(P^- \simeq \text{Hom}_{\Gamma}(P^-, \Gamma)\). We also have the functors

\[
S^- := (M, -)/\text{J}_{X}(M, -) : \text{mod } \Lambda \to \text{mod } \Gamma, \\
S^- := (-, M)/\text{J}_{X}(-, M) : \text{mod } \Lambda \to \text{mod } \Gamma^\text{op},
\]

whose images are semisimple \(\Gamma\)-modules and \(\Gamma^\text{op}\)-modules. For an almost split sequence \(3\), we have projective resolutions

\[
0 \to P_{Z} \xrightarrow{P^z} P_{Y} \xrightarrow{P_{X}} P_{X} \to S_{X} \to 0, \\
0 \to P_{X} \xrightarrow{P^f} P_{Y} \xrightarrow{P^g} P_{Z} \to S_{Z} \to 0,
\]
of a simple $\Gamma$-module $S_X$ and a simple $\Gamma^{op}$-module $S_Z$. Applying $\text{Hom}_\Gamma(-, \Gamma)$ to the first sequence and comparing with the second sequence, we have

$$\text{Ext}_\Gamma^i(S_X, \Gamma) = \begin{cases} 0 & i \neq 2, \\ S_Z & i = 2. \end{cases}$$

This implies that $\Gamma$ satisfies the restricted Gorenstein condition. We also observe that almost split sequences in $\text{mod } \Lambda$ correspond to projective resolutions of simple $\Gamma$-modules.

If a finite-dimensional algebra $\Lambda$ is not representation-finite, then one can not consider its Auslander algebra directly. Instead, Auslander and Reiten dealt with the functor category consisting of additive functors from the category $\text{mod } \Lambda$ to the category of abelian groups, and developed an exciting duality theory in functor categories $[6, 15]$ leading to their theory of almost split sequences $[16, 17, 18, 19, 10]$.

Let us recall a construction from stable module theory $[13]$. Let $\Lambda$ be a noetherian ring in general. We have the functor

$$(-)^* := \text{Hom}_\Lambda(-, \Lambda) : \text{mod } \Lambda \leftrightarrow \text{mod } \Lambda^{op}$$

which induces a duality $(-)^* : \text{add } \Lambda \leftrightarrow \text{add } \Lambda^{op}$. For $X \in \text{mod } \Lambda$, take a projective presentation

$$P_1 \xrightarrow{g} P_0 \xrightarrow{f} X \rightarrow 0.$$ 

Define a $\Lambda^{op}$-module $\text{Tr} X$ by an exact sequence

$$P_0^* \xrightarrow{g^*} P_1^* \rightarrow \text{Tr} X \rightarrow 0.$$ 

We notice that $\text{Tr} X$ depends on a choice of projective presentation of $X$. To make $\text{Tr}$ functorial, we need the stable category of $\Lambda$ defined by

$$\text{mod } \Lambda := (\text{mod } \Lambda)/[\Lambda],$$

where the ideal $[\Lambda]$ of $\text{mod } \Lambda$ is defined by $[13]$. Then we have a fundamental duality

$$\text{Tr} : \text{mod } \Lambda \leftrightarrow \text{mod } \Lambda^{op},$$

called the Auslander-Bridger transpose $[13]$.

In the rest of this subsection let $\Lambda$ be a finite-dimensional algebra over a field $k$. Then we have a duality

$$D := \text{Hom}_k(-, k) : \text{mod } \Lambda \leftrightarrow \text{mod } \Lambda^{op}.$$ 

Define the costable category of $\Lambda$ by

$$\text{mod } \Lambda := (\text{mod } \Lambda)/[D\Lambda],$$
where the ideal \([DA]\) of mod \(\Lambda\) is defined by \((\text{I})\). The duality \(D : \text{mod} \Lambda \leftrightarrow \text{mod} \Lambda^{\text{op}}\) induces a duality \(\text{mod} \Lambda \leftrightarrow \text{mod} \Lambda^{\text{op}}\), and we have mutually quasi-inverse equivalences

\[
\tau := D \text{Tr} : \text{mod} \Lambda \xrightarrow{\text{Tr}} \text{mod} \Lambda^{\text{op}} \xrightarrow{D} \text{mod} \Lambda,
\]

\[
\tau^{-} := \text{Tr} D : \text{mod} \Lambda^{\text{op}} \xrightarrow{\text{Tr}} \text{mod} \Lambda \xrightarrow{D} \text{mod} \Lambda^{\text{op}},
\]

called the \textit{Auslander-Reiten translations}. Especially, \(\tau\) gives a bijection from isoclasses of indecomposable non-projective \(\Lambda\)-modules to isoclasses of indecomposable non-injective \(\Lambda\)-modules. We have the following functorial isomorphisms called the \textit{Auslander-Reiten duality} \([5, 23]\).

\textbf{Theorem 1.6.} There exist the following functorial isomorphisms for any modules \(X, Y\) in mod \(\Lambda\)

\[
\text{Hom}_\Lambda(\tau^{-}Y, X) \simeq D \text{Ext}^1_\Lambda(X, Y) \simeq \overline{\text{Hom}}_\Lambda(Y, \tau X).
\]

In particular, we have an isomorphism \(\text{Ext}^1_\Lambda(X, \tau X) \simeq D \text{End}_\Lambda(X)\) for any \(X \in \text{mod} \Lambda\), which gives an element of \(\text{Ext}^1_\Lambda(X, \tau X)\). Thus we have the following existence theorem of almost split sequences \([5, 23]\).

\textbf{Theorem 1.7.} (a) For any indecomposable non-projective module \(X\) in mod \(\Lambda\), there exists an almost split sequence \([3]\) such that \(Z \simeq \tau X\).

(b) For any indecomposable non-injective module \(X\) in mod \(\Lambda\), there exists an almost split sequence \([3]\) such that \(X \simeq \tau^{-}Z\).

This is fundamental in the study of the category mod \(\Lambda\). For example, the structure theory of Auslander algebras using mesh categories of translation quivers was developed by Riedtmann \([106]\), Bongartz-Gabriel \([34]\) and Igusa-Todorov \([76, 77]\) (see also \([81, 82, 83]\)).

We end this subsection with the following exercise on further properties of Auslander algebras, which is closely related to combinatorial characterizations of finite Auslander-Reiten quivers given by Brenner \([35]\) and Igusa-Todorov \([76, 77]\) (see also \([84]\)).

\textbf{Exercise.} Let \(\Gamma\) be an Auslander algebra, and let \(S\) be a simple \(\Gamma\)-module with \(\text{pd} S \leq 1\). Give a proof of the following statements by using the representation theory of the corresponding representation-finite algebra \(\Lambda\).

(a) Any composition factor \(T\) of the \(\Gamma^{\text{op}}\)-module \(\text{Ext}'_{\Gamma}(S, \Gamma)\) satisfies \(\text{pd} T = 2\).

(b) The socle \(S'\) of the \(\Gamma^{\text{op}}\)-module \(\text{Hom}_\Gamma(S, \Gamma)\) is simple and satisfies \(\text{pd} S' \leq 1\). Any composition factor \(T\) of the \(\Gamma^{\text{op}}\)-module \(\text{Hom}_\Gamma(S, \Gamma)/S'\) satisfies \(\text{pd} T = 2\).

Also give a direct proof of the above statements by using homological algebra on \(\Gamma\).
1.3. Representation dimension. For a representation-finite finite-dimensional algebra $\Lambda$, a ‘model’ of the category $\text{mod} \Lambda$ is given by the Auslander algebra $\Gamma$ of $\Lambda$. In other words an Auslander algebra $\Gamma$ of $\Lambda$ has a ‘representation theoretic realization’ $\text{mod} \Lambda$. It is natural to ask whether the correspondence $\Gamma \leftrightarrow \text{mod} \Lambda$ can be generalized to other classes of algebras and categories. One satisfactory answer will be given in Theorem 2.6 which gives a bijection between ‘$n$-Auslander algebras’ and ‘$n$-cluster tilting subcategories’. Let us recall another approach given by Auslander [7]. The following result [7] due to Morita and Tachikawa deals with a wider class of algebras than considered in Theorem 1.2.

- Let $\Lambda$ be a finite-dimensional algebra and $M$ a generator-cogenerator in $\text{mod} \Lambda$. Then $\Gamma := \text{End}_\Lambda(M)$ satisfies $\text{dom.dim} \Gamma \geq 2$.

- Any finite-dimensional algebra $\Gamma$ satisfying $\text{dom.dim} \Gamma \geq 2$ is obtained in this way.

In view of this correspondence, Auslander defined the representation dimension of a finite-dimensional algebra $\Lambda$ by

$$\text{rep.dim} \Lambda := \inf \{ \text{gl.dim} \text{End}_\Lambda(M) \mid M \text{ is a generator-cogenerator in } \text{mod} \Lambda \}.$$  

Then he proved that $\Lambda$ is representation-finite if and only if $\text{rep.dim} \Lambda \leq 2$. In this sense the representation dimension measures how far an algebra is from being representation-finite.

A lot of interesting results are known recently (see [120, 57]). It was shown in [80] that $\text{rep.dim} \Lambda$ is always finite. Igusa and Todorov [78] proved that algebras with $\text{rep.dim} \Lambda \leq 3$ have finite finitistic dimension. Rouquier [110] proved that the exterior algebra $\Lambda$ of an $n$-dimensional vector space satisfies $\text{rep.dim} \Lambda = n + 1$. Oppermann [102] gave an effective method to give a lower bound of representation dimension.

2. $n$-Auslander-Reiten theory

2.1. $n$-cluster tilting subcategories. The concept of functorially finite subcategories was introduced by Auslander and Smalø [25, 26] and studied mainly by the school of Auslander. Today it turns out to be one of the fundamental concepts in representation theory of algebras, especially tilting theory [20, 14].

Let $\mathcal{X}$ be an additive category and $\mathcal{C}$ a subcategory of $\mathcal{X}$. We call $\mathcal{C}$ contravariantly finite if, for any $X \in \mathcal{X}$, there exists a morphism $f \in (C, X)$ with $C \in \mathcal{C}$ such that

$$\langle -, C \rangle \xrightarrow{f} \langle -, X \rangle \rightarrow 0$$

is exact. Such $f$ is called a right $\mathcal{C}$-approximation of $X$. Dually, a covariantly finite subcategory and a left $\mathcal{C}$-approximation are defined. A contravariantly and covariantly finite subcategory is called functorially finite.

A class of functorially finite subcategories is introduced in [85, 86] (see also [74]).
**Definition 2.1.** Let $\mathcal{X}$ be an extension-closed subcategory of an abelian category $\mathcal{A}$. We call a subcategory $\mathcal{C}$ of $\mathcal{X}$ $n$-rigid if
\[ \text{Ext}^i_{\mathcal{A}}(\mathcal{C}, \mathcal{C}) = 0 \]
for any $0 < i < n$. We call a subcategory $\mathcal{C}$ of $\mathcal{X}$ $n$-cluster tilting (or maximal $(n-1)$-orthogonal) if it is functorially finite and
\[ \mathcal{C} = \{ X \in \mathcal{X} | \text{Ext}^i_{\mathcal{A}}(\mathcal{C}, X) = 0 \text{ for } 0 < i < n \} \]
\[ \mathcal{C} = \{ X \in \mathcal{X} | \text{Ext}^i_{\mathcal{A}}(X, \mathcal{C}) = 0 \text{ for } 0 < i < n \} \].

An object $M \in \mathcal{X}$ is called $n$-cluster tilting (respectively, $n$-rigid) if so is add $M$.

Clearly $\mathcal{X}$ is a unique 1-cluster tilting subcategory of $\mathcal{X}$, and 2-cluster tilting subcategories are often called cluster tilting. If an $n$-cluster tilting subcategory $\mathcal{C}$ is contained in an $n$-rigid subcategory $\mathcal{D}$, then we have $\mathcal{C} = \mathcal{D}$.

Let us give a few properties of $n$-cluster tilting subcategories [85, 86].

**Proposition 2.2.** Assume that an additive category $\mathcal{X}$ has enough projectives and enough injectives. Let $\mathcal{C}$ be a functorially finite subcategory of $\mathcal{X}$. Then the following conditions are equivalent.

(a) $\mathcal{C}$ is an $n$-cluster tilting subcategory of $\mathcal{X}$.

(b) $\mathcal{C} = \{ X \in \mathcal{X} | \text{Ext}^i_{\mathcal{A}}(\mathcal{C}, X) = 0 \text{ for } 0 < i < n \}$ and $\mathcal{C}$ contains all projective objects in $\mathcal{X}$.

(c) $\mathcal{C} = \{ X \in \mathcal{X} | \text{Ext}^i_{\mathcal{A}}(X, \mathcal{C}) = 0 \text{ for } 0 < i < n \}$ and $\mathcal{C}$ contains all injective objects in $\mathcal{X}$.

The following result shows that any object in $\mathcal{X}$ has a finite sequence of right (respectively, left) $\mathcal{C}$-approximations. There is a nice interpretation of this property in terms of relative homological algebra of Auslander-Solberg [27, 28, 29]. See also [89].

**Proposition 2.3.** Assume that $\mathcal{X}$ has enough projectives and enough injectives. Let $\mathcal{C}$ be an $n$-cluster tilting subcategory of $\mathcal{X}$. For any $X \in \mathcal{X}$, there exist exact sequences
\[ 0 \to C_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} C_1 \xrightarrow{f_1} C_0 \xrightarrow{f_0} X \to 0, \]
\[ 0 \to X \xrightarrow{f_0'} C_0' \xrightarrow{f_1'} C_1' \xrightarrow{f_2'} \cdots \xrightarrow{f_{n-2}'} C_{n-1}' \to 0, \]
in $\mathcal{A}$ with terms in $\mathcal{C}$ such that the following sequences are exact on $\mathcal{C}$
\[ 0 \to (-, C_{n-1}) \xrightarrow{f_{n-1}} (-, C_1) \xrightarrow{f_1} (-, C_0) \xrightarrow{f_0} (-, X) \to 0, \]
\[ 0 \to (X, -) \xrightarrow{f_0'} (C_0', -) \xrightarrow{f_1'} (C_1', -) \xrightarrow{f_2'} \cdots \xrightarrow{f_{n-2}'} (C_{n-1}', -) \to 0. \]

In the rest of this subsection we give a few examples.
**Examples 2.4.** (a) Let $\Lambda_n$ be a finite-dimensional algebra given by the quiver

$$
0 \xrightarrow{a_0} 1 \xrightarrow{a_1} 2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-2}} n-1 \xrightarrow{a_{n-1}} n
$$

with relations $a_ia_{i+1} = 0$ for any $0 \leq i \leq n-2$. Then the Auslander-Reiten quiver of $\Lambda_n$ is the following, where $P_i$ and $S_i$ are projective and simple modules associated with the vertex $i$, respectively.

One can easily check that $\text{add}(\Lambda_n \oplus S_0)$ is a unique $n$-cluster tilting subcategory of $\text{mod} \Lambda_n$ such that $\text{End}_{\Lambda_n}(\Lambda_n \oplus S_0) \simeq \Lambda_{n+1}$.

(b) Let $\Lambda$ be a representation-finite selfinjective algebra such that the tree type of the stable Auslander-Reiten quiver of $\Lambda$ is either $A_m$, $B_m$, $C_m$ or $D_m$. Then a combinatorial criterion for existence of $n$-cluster tilting subcategories in $\text{mod} \Lambda$ is given in [85]. This is related to the example (d) below.

(c) Another example is given by Geiss, Leclerc and Schr"{o}er [59, 60]. See Subsection 4.1 for the definition of preprojective algebras.

**Theorem 2.5.** Let $\Lambda$ be a preprojective algebra of Dynkin type. Then there exists a 2-cluster tilting object in $\text{mod} \Lambda$.

In fact, each reduced expression of the longest element in the corresponding Weyl group gives a 2-cluster tilting object in $\text{mod} \Lambda$.

We shall explain this in Theorem 4.6 below, in a general setting.

(d) An interesting example of categories with $n$-cluster tilting subcategories is given by $(n-1)$-cluster categories ([41, 43] for $n = 2$, and [116, 122, 30, 119] for $n \geq 2$), which are $n$-Calabi-Yau triangulated categories constructed from the derived categories of path algebras [92]. (Notice that our $n$-cluster tilting objects are sometimes called "$(n-1)$-(cluster) tilting" in this setting.)

### 2.2. $n$-Auslander algebras

Throughout this subsection we fix a positive integer $n$. We call a finite-dimensional algebra $\Gamma$ an $n$-Auslander algebra if

$$\text{gl. dim } \Gamma \leq n + 1 \leq \text{dom. dim } \Gamma. \quad (4)$$

It is easily checked that any $n$-Auslander algebra $\Gamma$ is either semisimple or satisfies $\text{gl. dim } \Gamma = n + 1 = \text{dom. dim } \Gamma$. We have the following $n$-Auslander correspondence [86], where we call an additive category $n$-cluster tilting if it is equivalent to an $n$-cluster tilting subcategory of $\text{mod} \Lambda$ for some finite-dimensional algebra $\Lambda$.

**Theorem 2.6.** (a) Let $\Lambda$ be a finite-dimensional algebra and $M$ an $n$-cluster tilting object in $\text{mod} \Lambda$. Then $\text{End}_\Lambda(M)$ is an $n$-Auslander algebra.

(b) Any $n$-Auslander algebra $\Gamma$ can be obtained in this way. This gives a bijection between the sets of equivalence classes of $n$-cluster tilting categories with additive generators and Morita-equivalence classes of $n$-Auslander algebras.
Proof. Although this is shown in [86] in a general setting, we give a complete proof for our setting here for the convenience of the reader.

(a) We have an equivalence

\[ P\sim = \text{Hom}_\Lambda(M, -) : \text{add } M \xrightarrow{\sim} \text{add } \Gamma. \]

For any \( X \in \text{mod } \Gamma \), we take a projective resolution

\[ P_1 \xrightarrow{f} P_0 \rightarrow X \rightarrow 0. \]

Using the equivalence \( P\sim \), there exists a morphism \( g \in \text{Hom}_\Lambda(M_1, M_0) \) in \( \text{add } M \) such that \( f = P g \). Applying Proposition 2.3 to \( \text{Ker } g \), we have an exact sequence

\[ 0 \rightarrow M_{n+1} \rightarrow \cdots \rightarrow M_2 \rightarrow \text{Ker } g \rightarrow 0 \]

such that

\[ 0 \rightarrow P_{M_{n+1}} \rightarrow \cdots \rightarrow P_{M_2} \rightarrow P_{\text{Ker } g} \rightarrow 0 \]

is exact. Since \( P_{\text{Ker } g} = \text{Ker } f \), we have \( \text{pd } X \leq n + 1 \), and so \( \text{gl. dim } \Gamma \leq n + 1 \).

Now we shall show \( \text{dom. dim } \Gamma \geq n + 1 \). Take an injective resolution

\[ 0 \rightarrow M \rightarrow I_0 \rightarrow I_1 \rightarrow \cdots \rightarrow I_n \]

of the \( \Lambda \)-module \( X \). Applying \( P\sim \), we have an exact sequence

\[ 0 \rightarrow \Gamma \rightarrow P_{I_0} \rightarrow P_{I_1} \rightarrow \cdots \rightarrow P_{I_n} \]

since \( M \) is \( n \)-rigid. Since \( D\Lambda \in \text{add } M \), we have that \( P_{I_i} \) is a projective \( \Gamma \)-module. Moreover \( P_{D\Lambda} = D\Lambda = D\text{Hom}_\Lambda(\Lambda, M) \) holds. Since \( \Lambda \in \text{add } M \), we have that \( P_{I_i} \) is an injective \( \Gamma \)-module. Thus we have \( \text{dom. dim } \Gamma \geq n + 1 \).

(b) Let \( I \) be a direct sum of all indecomposable projective-injective \( \Gamma \)-modules. Define a projective \( \Gamma \)-module by

\[ Q := \text{Hom}_\Gamma(DI, \Gamma). \]

Put \( \Lambda := \text{End}_\Gamma(Q) \) and \( M := \text{Hom}_\Gamma(Q, \Gamma) \). We have the functors

\[ P\sim := \text{Hom}_\Lambda(M, -) : \text{mod } \Lambda \rightarrow \text{mod } \Gamma, \]

\[ Q\sim := \text{Hom}_\Gamma(Q, -) : \text{mod } \Gamma \rightarrow \text{mod } \Lambda, \]

such that \( Q P\sim \simeq \text{id}_{\text{mod } \Lambda} \). It is easily checked that we have mutually quasi-inverse equivalences

\[ P\sim : \text{add } (D\Lambda) \rightarrow \text{add } \Gamma \quad \text{and} \quad Q\sim : \text{add } \Gamma \rightarrow \text{add } (D\Lambda) \]. \hspace{1cm} (5)

Take a minimal injective resolution

\[ 0 \rightarrow \Gamma \rightarrow I_0 \rightarrow \cdots \rightarrow I_n \]

of the \( \Gamma \)-module \( \Gamma \). Applying \( Q\sim \), we have an exact sequence

\[ 0 \rightarrow M \rightarrow Q_{I_0} \rightarrow \cdots \rightarrow Q_{I_n} \]

with injective \( \Lambda \)-modules \( Q_{I_i} \). Applying \( P\sim \) to (7) and using (5), we get the exact sequence (6). This implies that \( M \) is an \( n \)-rigid \( \Lambda \)-module, and that \( \text{End}_\Lambda(M) \simeq \Gamma \).
We shall show that $M$ is an $n$-cluster tilting object in $\mod\Lambda$. Since $M$ is a generator in $\mod\Lambda$, it is enough, by Proposition 2.2, to show that any $X \in \mod\Lambda$ satisfying $\Ext^i_\Lambda(M, X) = 0$ for any $0 < i < n$ belongs to $\add M$. Take an injective resolution

$$0 \to X \to I'_0 \to \cdots \to I'_n.$$  

Applying $P_-$, we have an exact sequence

$$0 \to P_X \to P_{I'_0} \to \cdots \to P_{I'_n}$$

with projective $\Gamma$-modules $P_{I'_i}$ since we assumed $\Ext^i_\Lambda(M, X) = 0$ for any $0 < i < n$. Since $\gldim \Gamma \leq n + 1$, we have $P_X \in \add \Gamma$. Thus $X = QP_X \in \add Q\Gamma = \add M$.

We also have the relative version \[84, 86\] of Theorem 2.6. For a tilting $\Lambda$-module $T$ with $\pd T < \infty$ \[67, 101\], define a full subcategory $T^\perp$ of $\mod\Lambda$ by

$$T^\perp := \{ X \in \mod\Lambda \mid \Ext^i_\Lambda(T, X) = 0 \text{ for } i > 0 \}.$$  

Then $T^\perp$ forms an extension-closed subcategory of $\mod\Lambda$ with enough projectives $\add T$ and enough injectives $\add D\Lambda$.

Let $m$ and $n$ be integers with $0 \leq m \leq n$ and $1 \leq n$. We call an additive category $m$-relative $n$-cluster tilting if it is equivalent to an $n$-cluster tilting subcategory of $T^\perp$ for some finite-dimensional algebra $\Lambda$ and some tilting $\Lambda$-module $T$ with $\pd T \leq m$. On the other hand, we say that a finite-dimensional algebra is an $m$-relative $n$-Auslander algebra if $\Gamma$ satisfies $\gldim \Gamma \leq n + 1$ and the two-sided $(m + 1, n + 1)$-condition in Definition 1.1.

**Theorem 2.7.** Let $m$ and $n$ be integers with $0 \leq m \leq n$ and $1 \leq n$.

(a) Let $\Lambda$ be a finite-dimensional algebra and $T$ a tilting $\Lambda$-module with $\pd T \leq m$. Let $M$ be an $n$-cluster tilting object in $T^\perp$. Then $\End_\Lambda(M)$ is an $m$-relative $n$-Auslander algebra.

(b) Any $m$-relative $n$-Auslander algebra is obtained in this way. This gives a bijection between the sets of equivalence classes of $m$-relative $n$-cluster tilting categories with additive generators and Morita-equivalence classes of $m$-relative $n$-Auslander algebras.

By Proposition\[1,4\] any $m$-relative $n$-Auslander algebra $\Gamma$ with $\gldim \Gamma = n + 1$ satisfies the Gorenstein condition. From the viewpoint of Subsection 1.2 it is natural to consider an analogous theory of almost split sequences for $n$-cluster tilting subcategories, which is the subject in the next subsection.

### 2.3. $n$-almost split sequences

In this subsection we introduce an analogous theory for $n$-cluster tilting subcategories as discussed in the paper \[55\].

Let $\Lambda$ be a finite-dimensional algebra, and let $\mathcal{C}$ be an $n$-cluster tilting subcategory of $\mod\Lambda$ with $n \geq 1$. Let

$$\tau = D\Tr : \mod\Lambda \to \mod\Lambda \text{ and } \tau^- = \Tr D : \mod\Lambda \to \mod\Lambda$$

be the Auslander-Reiten translate and its inverse. Then $\tau$ induces an equivalence $\mod\Lambda/\add M \to \mod\Lambda/\add N$. We call $\tau$ the shift by $M$.
be the Auslander-Reiten translations defined in Subsection 1.2. We denote by
\[ \Omega : \text{mod} \Lambda \to \text{mod} \Lambda \quad \text{and} \quad \Omega^- : \text{mod} \Lambda \to \text{mod} \Lambda \]
the syzygy and the cosyzygy functors. Consider the functors
\[ \tau_n := \tau \Omega^{n-1} : \text{mod} \Lambda \to \text{mod} \Lambda, \]
\[ \tau_n^- := \tau^- \Omega^{- (n-1)} : \text{mod} \Lambda \to \text{mod} \Lambda. \]
These functors \( \tau_n \) and \( \tau_n^- \) are not equivalences in general though \( (\tau_n^- , \tau_n) \) forms an adjoint pair. We denote by \( \mathcal{C} \) (respectively, \( \mathcal{C}^- \)) the corresponding subcategory of \( \text{mod} \Lambda \) (respectively, \( \text{mod} \Lambda \)). The following result shows that it is fundamental in the study of \( n \)-cluster tilting subcategories.

**Theorem 2.8.** The functors \( \tau_n \) and \( \tau_n^- \) induce mutually quasi-inverse equivalences \( \tau_n : \mathcal{C} \to \mathcal{C} \) and \( \tau_n^- : \mathcal{C} \to \mathcal{C}^- \).

Especially, \( \tau_n \) gives a bijection from isoclasses of indecomposable non-projective objects in \( \mathcal{C} \) to isoclasses of indecomposable non-injective objects in \( \mathcal{C} \). We call \( \tau_n \) and \( \tau_n^- \) the \( n \)-\textit{Auslander-Reiten translations}. For \( n \)-cluster tilting subcategories \( \mathcal{C} \), we have the following \( n \)-\textit{Auslander-Reiten duality} which is analogous to the Auslander-Reiten duality.

**Theorem 2.9.** There exist the following functorial isomorphisms for any modules \( X, Y \) in \( \mathcal{C} \)
\[ \text{Hom}_\Lambda(\tau_n^- Y, X) \simeq D \text{Ext}_\Lambda^n(X, Y) \simeq \text{Hom}_\Lambda(Y, \tau_n X). \]

In particular, we have an isomorphism \( \text{Ext}_\Lambda^n(X, \tau_n X) \simeq D \text{End}_\Lambda(X) \) for any \( X \in \mathcal{C} \). Thus one can get a long exact sequence given by a class of \( \text{Ext}_\Lambda^n(X, \tau_n X) \) as in the case of Auslander-Reiten theory. Let us start with the following preliminary observation which is analogous to Proposition 1.5.

**Proposition 2.10.** Let
\[ 0 \to Y \xrightarrow{f_{n+1}} C_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} C_1 \xrightarrow{f_1} X \to 0 \quad (8) \]
be an exact sequence with terms in \( \mathcal{C} \) and \( f_i \in \text{J}_\mathcal{C} \) for any \( i \). Then
\[ 0 \to (-, Y) \xrightarrow{f_{n+1}} (-, C_n) \xrightarrow{f_n} \cdots \xrightarrow{f_2} (-, C_1) \xrightarrow{f_1} \text{J}_\mathcal{C}(-, X) \to 0 \]
is exact on \( \mathcal{C} \) if and only if
\[ 0 \to (X, -) \xrightarrow{f_1} (C_1, -) \xrightarrow{f_2} \cdots \xrightarrow{f_n} (C_n, -) \xrightarrow{f_{n+1}} \text{J}_\mathcal{C}(Y, -) \to 0 \]
is exact on \( \mathcal{C} \). In this case, \( X \) is indecomposable if and only if \( Y \) is indecomposable.

When these conditions are satisfied, the sequence \( (8) \) is called an \textit{n-almost split sequence}. We have the following existence theorem of \textit{n-almost split sequences}. 
Theorem 2.11.  (a) For any indecomposable non-projective module \(X\) in \(C\), there exists an \(n\)-almost split sequence \([\mathbf{5}]\) such that \(Y \simeq \tau_n X\).

(b) For any indecomposable non-injective module \(X\) in \(C\), there exists an \(n\)-almost split sequence \([\mathbf{5}]\) such that \(X \simeq \tau_{-n} Y\).

It is interesting to have a structure theory of \(n\)-Auslander algebras which is similar to the structure theory of Auslander algebras using translation quivers and mesh relations \([106, 34, 78, 80, 81, 82, 83]\). For \(n = 2\), one can hope that it is given by some generalization of Jacobian algebras of quivers with potentials \([50]\) like 3-CY algebras \([45, 64, 93]\), from the viewpoint of Proposition 2.24 below.

We end this subsection with the following useful characterization \([86]\) of \(n\)-cluster tilting objects, which we shall use later.

Lemma 2.12. Let \(\Lambda\) be a finite-dimensional algebra, and \(T\) a tilting \(\Lambda\)-module with \(\text{pd} T \leq n + 2\). Let \(M\) be an \(n\)-rigid generator-cogenerator in \(T^\perp\). Then the following conditions are equivalent.

(a) \(M\) is an \(n\)-cluster tilting object in \(T^\perp\).

(b) \(\text{gl. dim } \text{End}_\Lambda(M) \leq n + 1\).

(c) For any indecomposable object \(X \in \text{add } M\), there exists an exact sequence

\[
0 \to C_{n+1} \xrightarrow{f_{n+1}} C_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} C_1 \xrightarrow{f_1} X
\]

with terms in \(\text{add } M\) such that the following sequence is exact

\[
0 \to (M, C_{n+1}) \xrightarrow{f_{n+1}} (M, C_n) \xrightarrow{f_n} \cdots \xrightarrow{f_2} (M, C_1) \xrightarrow{f_1} J_{\mathcal{C}}(M, X) \to 0.
\]

The condition (b) is a property of relative \(n\)-Auslander algebras. The sequence in the condition (c) is given by an \(n\)-almost split sequences if \(X\) is non-projective.

2.4. Example: \(n\)-cluster tilting for \((n - 1)\)-Auslander algebras. In this subsection we give a series of examples of finite-dimensional algebras with \(n\)-cluster tilting objects from \([87]\). Let us start with general results on \(n\)-cluster tilting subcategories of \(\text{mod } \Lambda\). For a finite-dimensional algebra \(\Lambda\), we define full subcategories of \(\text{mod } \Lambda\) by

\[
\mathcal{M} = \mathcal{M}_n(\Lambda) := \text{add}\{\tau_n(D\Lambda) \mid i \geq 0\} \quad \text{and} \quad \mathcal{M}' = \mathcal{M}'_n(\Lambda) := \text{add}\{\tau_{-i}\Lambda \mid i \geq 0\}.
\]

Then the following result is an immediate consequence of Theorem 2.8.

Proposition 2.13.  (a) Any \(n\)-cluster tilting subcategory \(\mathcal{C}\) of \(\text{mod } \Lambda\) contains \(\mathcal{M}\) and \(\mathcal{M}'\). In particular, if \(\text{mod } \Lambda\) contains an \(n\)-cluster tilting subcategory, then \(X \oplus Y\) is \(n\)-rigid for any \(X \in \mathcal{M}\) and \(Y \in \mathcal{M}'\).

(b) If \(\text{mod } \Lambda\) contains an \(n\)-cluster tilting object, then we have a bijection between indecomposable projective \(\Lambda\)-modules \(I\) and indecomposable injective \(\Lambda\)-modules \(I\). It is given by \(I \mapsto \tau_\ell^I I\), where \(\ell\) is a maximal natural number \(\ell\) satisfying \(\tau_n^\ell I \neq 0\) and \(\tau_{n+1}^\ell I = 0\).
In this subsection we construct a family of finite-dimensional algebras \( \Lambda \) such that \( \mathcal{M} \) itself forms an \( n \)-cluster tilting subcategory of \( \text{mod} \Lambda \) (or \( T^\perp \) for some tilting \( \Lambda \)-module \( T \)). Let us start with the following typical examples.

**Example 2.14.** Let \( \Lambda_1 \) and \( \Lambda_1' \) be the path algebras of the quivers
\[
\bullet \rightarrow \bullet \rightarrow \bullet \quad \text{and} \quad \bullet \rightarrow \bullet \rightarrow \bullet ,
\]
respectively, over a field \( k \). Then we have \( \mathcal{M}_1(\Lambda_1) = \text{mod} \Lambda_1 \) and \( \mathcal{M}_1(\Lambda_1') = \text{mod} \Lambda_1' \). We denote by \( \Lambda_2 \) and \( \Lambda_2' \) the Auslander algebras of \( \Lambda_1 \) and \( \Lambda_1' \) respectively, so they are given by the translation quivers
\[
\begin{array}{c}
\begin{array}{c}
6 \longrightarrow 5 \longrightarrow 4 \longrightarrow 3 \longrightarrow 2 \longrightarrow 1
\end{array}
\end{array}
\quad \quad \quad
\begin{array}{c}
\begin{array}{c}
5 \longrightarrow 6 \longrightarrow 4 \longrightarrow 3 \longrightarrow 2 \longrightarrow 1
\end{array}
\end{array}
\]
with the mesh relations. We have
\[
\begin{align*}
DA_2 &= \left( I + 1 \oplus 1 \oplus 1 \oplus 1 \oplus 1 \oplus 1 \oplus 1 \right), \\
\tau_2DA_2 &= \left( 4 \oplus 4 \oplus 5 \oplus 6 \right), \\
\tau_2^2DA_2 &= 0,
\end{align*}
\]
\[
\begin{align*}
DA_2' &= \left( I + 2 \oplus 1 \oplus 1 \oplus 1 \oplus 1 \oplus 1 \right), \\
\tau_2DA_2' &= \left( 4 \oplus 5 \oplus 4 \right), \\
\tau_2^2DA_2' &= 0.
\end{align*}
\]
The quivers of \( \mathcal{M}_2(\Lambda_2) \) and \( \mathcal{M}_2(\Lambda_2') \) are
\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
6 \rightarrow 5 \rightarrow 4 \rightarrow 3 \rightarrow 2 \rightarrow 1
\end{array}
\end{array}
\end{array}
\quad \quad \quad
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
5 \rightarrow 6 \rightarrow 4 \rightarrow 3 \rightarrow 2 \rightarrow 1
\end{array}
\end{array}
\end{array}
\]
where the dotted arrows indicate \( \tau_2 \). Using Lemma 2.12 one can verify that \( \mathcal{M}_2(\Lambda_2) \) is a 2-cluster tilting subcategory of \( \text{mod} \Lambda_2 \), and that \( \mathcal{M}_2(\Lambda_2') \) is a 2-cluster tilting subcategory of \( T_2^\perp \) for the tilting \( \Lambda_2' \)-module
\[
T_2 = \left( I + 1 \oplus 3 \oplus 4 \oplus 5 \oplus 6 \oplus 7 \oplus 8 \oplus 9 \right)
\]
with \( \text{pd} T_2 = 1 \). Now let \( \Lambda_3 \) and \( \Lambda_3' \) be the endomorphism algebras of the above 2-cluster tilting objects. Again one can verify that \( \mathcal{M}_3(\Lambda_3) \) is a 3-cluster tilting subcategory of \( \text{mod} \Lambda_3 \), and that \( \mathcal{M}_3(\Lambda_3') \) is a 3-cluster tilting subcategory of \( T_3^\perp \) for some tilting \( \Lambda_3' \)-module \( T_3 \) with \( \text{pd} T_3 = 2 \).

Moreover our inductive construction of \( n \)-cluster tilting objects continues infinitely!
Let us formalize the above example.

**Definition 2.15.** Let $\Lambda$ be a finite-dimensional algebra and $n$ a positive integer. Define subcategories of $\mathcal{M} = \mathcal{M}_n(\Lambda)$ by

$$
\mathcal{P}(\mathcal{M}) := \{ X \in \mathcal{M} \mid \text{pd} X < n \},
$$

$$
\mathcal{M}_P := \{ X \in \mathcal{M} \mid X \text{ has no non-zero summand in } \mathcal{P}(\mathcal{M}) \}.
$$

We call $\Lambda$ $n$-complete if the following conditions are satisfied:

(a) There exists a tilting $\Lambda$-module $T$ satisfying $\text{pd} T < n$ and $\mathcal{P}(\mathcal{M}) = \text{add} T$;

(b) there exists an $n$-cluster tilting object $M$ of $T^\perp$ satisfying $\mathcal{M} = \text{add} M$;

(c) $\text{gl. dim } \Lambda \leq n$;

(d) $\text{Ext}_\Lambda^i(\mathcal{M}_P, \Lambda) = 0$ for any $0 < i < n$.

In this case, we call $\text{End}_\Lambda(M)$ the cone of $\Lambda$.

It is easy to check that a finite-dimensional algebra is 1-complete if and only if it is representation-finite and hereditary. Our inductive construction is given by the following result.

**Theorem 2.16.** The cone of an $n$-complete algebra is $(n + 1)$-complete.

A special case is the following result, which explains the above examples.

**Corollary 2.17.** Let $Q$ be a Dynkin quiver and $\Lambda_1 = kQ$ its path algebra over a field $k$. For any $n \geq 1$, there exists an $n$-complete algebra $\Lambda_n$ with the cone $\Lambda_{n+1}$.

Thus, for each Dynkin quiver $Q$, we have an infinite series of $(n - 1)$-relative $n$-Auslander algebras $\Lambda_n$ starting from the path algebra $\Lambda_1 = kQ$. If $Q$ is of Dynkin type $A_2$, then we have Example 2.4 (a). The following case is especially interesting.

**Corollary 2.18.** Let $\Lambda_1$ be the $m \times m$ triangular matrix algebra over a field $k$ for some $m \geq 1$. For any $n \geq 1$, there exist an algebra $\Lambda_n$ and an $n$-cluster tilting object $M_n$ in $\text{mod } \Lambda_n$ such that $\Lambda_{n+1} = \text{End}_{\Lambda_n}(M_n)$.

A lot of $n$-complete algebras will be constructed in a joint work [88] with Oppermann by using ‘$n$-APR tilting modules’.

In the rest of this subsection we describe the quiver of $\Lambda_n$ in Corollary 2.17. Let $I_x$ be the indecomposable injective $\Lambda_1$-module corresponding to the vertex $x \in Q_0$ and

$$
\ell_x := \sup\{ \ell \geq 0 \mid \tau^\ell I_x \neq 0 \}.
$$

Since $Q$ is a Dynkin quiver, we have $\ell_x < \infty$ for any $x \in Q_0$. We put

$$
\Delta_x := \{ (\ell_1, \cdots, \ell_n) \in \mathbb{Z}^n \mid \ell_1, \cdots, \ell_n \geq 0, \ell_1 + \cdots + \ell_n \leq \ell_x \}.
$$

For $1 \leq i \leq n$, we put

$$
e_i := \begin{cases} 1 & i = 1, \\ (0, \cdots, 0, 1, 0, \cdots, 0) & i = n \end{cases} \in \mathbb{Z}^n,
$$

$$
v_i := \begin{cases} -e_1 & i = 1, \\ e_{i-1} - e_i & 1 < i \leq n.
$$

Thus, for each $n$, $\Lambda_n$ is a $\Delta_n$-relative $n$-complete algebra, and we have $(n + 1)$-complete algebras $\Lambda_{n+1}$ with $\text{End}_{\Lambda_n}(M_n)$ as cones.

**Example 2.4 (a).** Let $Q$ be a Dynkin quiver and $\Lambda_1 = kQ$ its path algebra over a field $k$. For any $n \geq 1$, there exists an $n$-complete algebra $\Lambda_n$ with the cone $\Lambda_{n+1}$.

Thus, for each Dynkin quiver $Q$, we have an infinite series of $(n - 1)$-relative $n$-Auslander algebras $\Lambda_n$ starting from the path algebra $\Lambda_1 = kQ$. If $Q$ is of Dynkin type $A_2$, then we have Example 2.4 (a). The following case is especially interesting.

**Corollary 2.18.** Let $\Lambda_1$ be the $m \times m$ triangular matrix algebra over a field $k$ for some $m \geq 1$. For any $n \geq 1$, there exist an algebra $\Lambda_n$ and an $n$-cluster tilting object $M_n$ in $\text{mod } \Lambda_n$ such that $\Lambda_{n+1} = \text{End}_{\Lambda_n}(M_n)$.

A lot of $n$-complete algebras will be constructed in a joint work [88] with Oppermann by using ‘$n$-APR tilting modules’.

In the rest of this subsection we describe the quiver of $\Lambda_n$ in Corollary 2.17. Let $I_x$ be the indecomposable injective $\Lambda_1$-module corresponding to the vertex $x \in Q_0$ and

$$
\ell_x := \sup\{ \ell \geq 0 \mid \tau^\ell I_x \neq 0 \}.
$$

Since $Q$ is a Dynkin quiver, we have $\ell_x < \infty$ for any $x \in Q_0$. We put

$$
\Delta_x := \{ (\ell_1, \cdots, \ell_n) \in \mathbb{Z}^n \mid \ell_1, \cdots, \ell_n \geq 0, \ell_1 + \cdots + \ell_n \leq \ell_x \}.
$$

For $1 \leq i \leq n$, we put

$$
e_i := \begin{cases} 1 & i = 1, \\ (0, \cdots, 0, 1, 0, \cdots, 0) & i = n \end{cases} \in \mathbb{Z}^n,
$$

$$
v_i := \begin{cases} -e_1 & i = 1, \\ e_{i-1} - e_i & 1 < i \leq n.
$$

Thus, for each $n$, $\Lambda_n$ is a $\Delta_n$-relative $n$-complete algebra, and we have $(n + 1)$-complete algebras $\Lambda_{n+1}$ with $\text{End}_{\Lambda_n}(M_n)$ as cones.
Definition 2.19. For a Dynkin quiver $Q$ and $n \geq 1$, we define a quiver $Q^{(n)} = (Q_0^{(n)}, Q_1^{(n)})$ as follows: We put

$$Q_0^{(n)} := \{(x, \ell) \mid x \in Q_0, \ell \in \Delta_x\}.$$ 

There are the following $(n+1)$ kinds of arrows in $Q_1^{(n)}$:

- For any arrow $a : w \to x$ in $Q$, we have an arrow $(a^*, \ell) : (x, \ell) \to (w, \ell)$,
- for any arrow $b : x \to y$ in $Q$, we have an arrow $(b, \ell) : (x, \ell) \to (y, \ell + v_1)$,
- for any $1 < i \leq n$, we have an arrow $(x, \ell)_i : (x, \ell) \to (x, \ell + v_i)$,

if both the tail and the head belong to $Q_0^{(n)}$.

It is helpful to regard $x$ as a 0-th entry of $(x, \ell)$, above $w$ as ‘$x - 1$’, and above $y$ as ‘$x + 1$’.

Theorem 2.20. Under the circumstances in Corollary 2.17, the quiver of $\Lambda_n^{\text{op}}$ is given by $Q^{(n)}$.

We end this subsection with the following example. For simplicity, we denote by

$$x \ell_1 \cdots \ell_n$$

(respectively, $x \ell_1 \cdots \ell_i \ell_{i+1} \cdots \ell_n$, $a^* \ell_1 \cdots \ell_n$, $b \ell_1 \cdots \ell_n$)

the vertex $(x, \ell) \in Q_0^{(n)}$ (respectively, the arrow $(x, \ell)_i$, $(a^*, \ell)$, $(b, \ell) \in Q_1^{(n)}$) for $\ell = (\ell_1, \ldots, \ell_n)$.

Example 2.21. Let $Q$ be the quiver $1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{c} 4$ of Dynkin type $A_4$. Then the quiver of $\Lambda_2^{\text{op}}$ is the following

The quiver of $\Lambda_3^{\text{op}}$ is the following
The quiver of $\Lambda_{op}^{n}$ is the following

\[
\begin{array}{c}
\text{•} \\
\text{↓↓↓} \\
\text{•} \\
\text{↑↑↑} \\
\text{•} \\
\text{↓↓↓} \\
\text{•} \\
\text{→→→} \\
\text{•} \\
\text{↑↑↑} \\
\text{•} \\
\text{↓↓↓} \\
\text{•} \\
\text{→→→} \\
\text{•} \\
\text{↑↑↑} \\
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\text{•} \\
\text{→→→} \\
\text{•} \\
\text{↑↑↑} \\
\text{•} \\
\text{↓↓↓} \\
\text{•} \\
\text{→→→} \\
\text{•} \\
\text{↑↑↑} \\
\text{•} \\
\text{↓↓↓} \\
\text{•} \\
\text{→→→} \\
\text{•} \\
\text{↑↑↑} \\
\text{•} \\
\text{↓↓↓} \\
\text{•} \\
\text{→→→} \\
\text{•} \\
\text{↑↑↑} \\
\end{array}
\]

In general, the shape of the quiver of $\Lambda_{op}^{n}$ looks like an $n$-simplex.

2.5. Questions. In this subsection we shall pose a few questions on $n$-cluster tilting subcategories. Throughout let $\Lambda$ be a finite-dimensional algebra. By definition $\Lambda$ contains a 1-cluster tilting object if and only if $\Lambda$ is representation-finite. For $n \geq 2$, the following question is not understood well.

- When does $\text{mod } \Lambda$ contain an $n$-cluster tilting object (respectively, subcategory)?

A necessary conditions is given by Proposition 2.13. Let us give another necessary condition. The complexity of a $\Lambda$-module $M$ with a minimal projective resolution $\cdots \to P_1 \to P_0 \to M \to 0$ is defined by

$$\inf\{ \ell \geq 0 \mid \text{there exists } c > 0 \text{ such that } \dim_k P_i \leq ct^{\ell-1} \text{ for any } i \}.$$ 

Erdmann and Holm [56] proved the following result.

**Theorem 2.22.** Let $\Lambda$ be a selfinjective algebra and $n \geq 1$. If there exists an $n$-cluster tilting object in $\text{mod } \Lambda$, then the complexity of any $\Lambda$-module is at most one.

The following question is also interesting.

- Does any $n$-cluster tilting subcategory $C$ of $\text{mod } \Lambda$ with $n \geq 2$ have an additive generator?

This is related to the following question due to Auslander and Smalø [26]:

- Let $\mathcal{X}$ be a functorially finite extension-closed subcategory of $\text{mod } \Lambda$. Are there only finitely many isoclasses of indecomposable projective (respectively, injective) objects in $\mathcal{X}$?

If this is true, then the previous question is also true since any $n$-cluster tilting subcategory satisfies the conditions and all objects are projective.

It seems that answer to the following simple question is also unknown.
Is there a finite-dimensional algebra $\Lambda$ with an infinite set $\{X_i \mid i \in I\}$ of isoc-classes of indecomposable $\Lambda$-modules satisfying $\text{Ext}^1_\Lambda(X_i, X_j) = 0$ for any $i, j \in I$?

It is shown in [86] that if $\text{rep.dim} \Lambda \leq 3$, then there is no such an infinite set. The following result [87] is interesting also from the viewpoint of the above question.

**Proposition 2.23.** For any finite-dimensional algebra $\Lambda$, the subcategories $\mathcal{M}_2(\Lambda)$ and $\mathcal{M}'_2(\Lambda)$ are 2-rigid.

In Theorem 3.15 we shall show that the numbers of indecomposable direct summands of all basic 2-cluster tilting objects in $\text{mod} \Lambda$ are equal. Thus the following question is natural.

• Are the numbers of indecomposable direct summands of all basic $n$-cluster tilting objects in $\text{mod} \Lambda$ equal for a fixed $n$?

## 3. Orders and Cohen-Macaulay modules

In this section we consider a large class of algebras, containing the finite-dimensional algebras, which are called orders over complete local rings with Krull dimension $d$. We briefly recall the Auslander-Reiten theory for the category of (maximal) Cohen-Macaulay modules over orders. The case $d = 0$ is what we recalled in Section 1. The case $d = 1$ was developed by many authors including Drozd, Kirichenko, Reiner, Roggenkamp, Simson, Hiyikata, Nishida (see [72, 83, 84, 108, 109, 111, 112, 113, 71, 72]). The case $d \geq 2$ is studied mainly in the context of commutative ring theory (see the references in [121]) or non-commutative algebraic geometry (see [2, 3, 4, 65, 66, 104, 117, 118]). An especially nice phenomenon appears for the case $d = 2$. In this case, the category $\text{CM}(\Lambda)$ has fundamental sequences connecting indecomposable projective modules and indecomposable injective modules in $\text{CM}(\Lambda)$ which behave like almost split sequences.

In Subsection 3.2 we introduce $n$-Auslander-Reiten theory on $n$-cluster tilting subcategories $\mathcal{C}$ of $\text{CM}(\Lambda)$ by constructing the $n$-Auslander-Reiten translation, the $n$-Auslander-Reiten duality and $n$-almost split sequences. For the case $n = d - 1$ we shall show the existence of $(d - 1)$-fundamental sequences connecting indecomposable projective objects in $\mathcal{C}$ and indecomposable injective objects in $\mathcal{C}$. This implies that the $(d - 1)$-Auslander algebras $\text{End}_\Lambda(M)$ of $(d - 1)$-cluster tilting objects $M \in \text{CM}(\Lambda)$ are especially nice. In fact they are $R$-orders with global dimension $d$. If in addition $\Lambda$ is a symmetric $R$-order, then $\text{End}_\Lambda(M)$ is a $d$-Calabi-Yau algebra. In Subsections 3.3 and 3.4 we recall several results on Calabi-Yau algebras and Calabi-Yau triangulated categories given by orders.

Our basic references in this section are [36, 100] for commutative rings, [10, 11, 121] for the representation theory of orders, and [48, 66] for module-finite algebras.

### 3.1. From Auslander-Reiten theory

In this section we denote by $R$ a noetherian complete local ring of Krull dimension $d$. For $X \in \text{mod} R$, let us introduce
the invariants
\[ \dim X := \dim(R/ \text{ann} X), \]
\[ \text{depth } X := \inf\{i \geq 0 \mid \Ext^i_R(R/J_R, X) \neq 0\}, \]
called the \textit{dimension} and the \textit{depth}. We always have an inequality \( \text{depth } X \leq \dim X \), and an important class of \( R \)-modules is the category
\[
\text{CM}_i(R) := \{X \in \mod\Lambda \mid X = 0 \text{ or } \dim X = i = \text{depth } X\}
\]
of \textit{Cohen-Macaulay modules of Krull dimension} \( i \). We are interested in the category
\[
\text{CM}(R) := \text{CM}_d(R)
\]
of \textit{(maximal) Cohen-Macaulay} \( R \)-modules. We call \( R \) a \textit{Cohen-Macaulay ring} if \( R \in \text{CM}(R) \).

In the rest of this section we assume that \( R \) is a \textit{Gorenstein ring}, that is the injective dimension \( \text{id } R \) of the \( R \)-module \( R \) is finite, or equivalently \( \text{id } R = d \). Any Gorenstein ring is Cohen-Macaulay. An important class of Gorenstein rings is given by \textit{regular rings} \( R \), for which \( \text{gl.dim } R < \infty \), or equivalently \( \text{gl.dim } R = d \). A typical example of regular rings is the formal power series ring \( k[[t_1, \cdots, t_d]] \) over a field \( k \). Conversely, any complete regular local ring containing a field must be a formal power series ring over a field by Cohen’s structure theorem. Since we have the Auslander-Buchsbaum formula
\[
\text{depth } X + \text{pd } X = \text{depth } R \text{ if pd } X < \infty, \quad (9)
\]
in general, we have \( \text{CM}(R) = \text{add } R \) if \( R \) is regular.

We have a nice duality theory for Gorenstein rings \( R \). In particular we have a duality
\[
D_i := \Ext^d_R(-, R) : \text{CM}_i(R) \cong \text{CM}_i(R).
\]
The category \( \text{CM}_0(R) \) coincides with the category \( \text{f.l. } R \) of finite length \( R \)-modules, and the duality
\[
D := D_0 = \Ext^d_R(-, R) : \text{f.l. } R \cong \text{f.l. } R
\]
is called the \textit{Matlis duality}.

Let us introduce the algebras which we are interested in. Let \( \Lambda \) be a \textit{module-finite} \( R \)-algebra, namely an \( R \)-algebra which is finitely generated as an \( R \)-module (and not necessarily commutative). Then \( \Lambda \) is clearly noetherian, and the category \( \mod\Lambda \) is Krull-Schmidt in the sense of Introduction since we assumed that the base ring \( R \) is complete \cite{[18]}. We put
\[
\text{CM}(\Lambda) := \{X \in \mod\Lambda \mid X \in \text{CM}(R) \text{ as an } R\text{-module}\}.
\]
The category \( \text{CM}(\Lambda) \) is independent of the choice of the central subring \( R \) of \( \Lambda \) since so are \( \dim X \) and \( \text{depth } X \).
Definition 3.1. We call $\Lambda$ an $R$-order if $\Lambda \in \text{CM}(\Lambda)$.

In the rest of this subsection we assume that $\Lambda$ is an $R$-order. The duality $D_d : \text{CM}(R) \to \text{CM}(R)$ induces a duality

$$D_d = \text{Hom}_R(-, R) : \text{CM}(\Lambda) \leftrightarrow \text{CM}(\Lambda^{\text{op}}).$$

Thus $\text{CM}(\Lambda)$ forms an extension-closed subcategory of $\text{mod } \Lambda$ with enough projectives $\text{add } \Lambda$ and enough injectives $\text{add } D_d \Lambda$. We have the Nakayama functors

$$\nu := D_d(-)^* : \text{mod } \Lambda \to \text{mod } \Lambda \quad \text{and} \quad \nu^- := (-)^* D_d : \text{mod } \Lambda \to \text{mod } \Lambda,$$

which induce mutually quasi-inverse equivalences

$$\nu : \text{add } \Lambda \sim \to \text{add } D_d \Lambda \quad \text{and} \quad \nu^- : \text{add } D_d \Lambda \sim \to \text{add } \Lambda.$$

Examples 3.2. (a) An $R$-order of Krull dimension $d = 0$ is nothing but an artin $R$-algebra, and we have $\text{mod } \Lambda = \text{CM}(\Lambda)$ in this case.

(b) Let $S$ be a noetherian commutative complete local ring containing a field. Then $S$ always contains a complete regular local subring $R$ such that $S$ is a module-finite $R$-algebra. In this case, $S$ is an $R$-order if and only if $S$ is a Cohen-Macaulay ring. Moreover, an $S$-module is Cohen-Macaulay as an $S$-module if and only if it is Cohen-Macaulay as an $R$-module.

Since $R$ is Goresntein, we have

$$\dim X = d - \inf \{i \geq 0 \mid \text{Ext}^i_R(X, R) \neq 0\}$$
$$\text{depth } X = d - \inf \{i \geq 0 \mid \text{Ext}^i_R(X, R) \neq 0\},$$

In particular, we have

$$\text{CM}(\Lambda) = \{X \in \text{mod } \Lambda \mid \text{Ext}^i_{\Lambda}(X, D_d \Lambda) = 0 \text{ for } i > 0\}.$$

Thus the study of Cohen-Macaulay modules is a special case of cotilting theory with respect to $D_d \Lambda$ [20, 14].

For an $R$-order $\Lambda$, we always have an inequality $\text{gl. dim } \Lambda \geq d$.

Definition 3.3. An $R$-order $\Lambda$ is called non-singular [11] if $\text{gl. dim } \Lambda = d$.

An $R$-order $\Lambda$ is called an isolated singularity [11] if

$$\text{gl. dim}(\Lambda \otimes_R R_p) = \dim R_p$$

for any non-maximal prime ideal $p$ of $R$. For example, non-singular $R$-orders are isolated singularities.
In the rest of this subsection we assume that \( \Lambda \) is an \( R \)-order which is an isolated singularity. For example, an \( R \)-order of Krull dimension \( d = 0 \) is always an isolated singularity, and an \( R \)-order of Krull dimension \( d = 1 \) is an isolated singularity if and only if \( \Lambda \otimes_R K \) is a semisimple \( K \)-algebra for the quotient field \( K \) of \( R \). As for \( \text{mod} \ \Lambda \) for finite-dimensional algebras \( \Lambda \), many results in Auslander-Reiten theory hold in \( \text{CM}(\Lambda) \) for an \( R \)-order \( \Lambda \) which is an isolated singularity. We put
\[
\overline{\text{CM}}(\Lambda) := \text{CM}(\Lambda)/[\Lambda] \quad \text{and} \quad \overline{\text{CM}}(\Lambda) := \text{CM}(\Lambda)/[D_d\Lambda],
\]
where \([\Lambda]\) and \([D_d\Lambda]\) are ideals of \( \text{CM}(\Lambda) \) defined in Introduction. We notice that isolated singularities are characterized in terms of stable categories as follows [11].

- An \( R \)-order \( \Lambda \) is non-singular if and only if \( \text{CM}(\Lambda) = \text{add} \ \Lambda \) if and only if \( \overline{\text{CM}}(\Lambda) = 0 \).
- An \( R \)-order \( \Lambda \) is an isolated singularity if and only if the morphism set \( \text{Hom}_\Lambda(X,Y) \) in \( \overline{\text{CM}}(\Lambda) \) belongs to \( \text{f.1} \ R \) for any \( X,Y \in \text{CM}(\Lambda) \).

We denote by
\[
\Omega : \text{CM}(\Lambda) \to \overline{\text{CM}}(\Lambda) \quad \text{and} \quad \Omega^\sim : \overline{\text{CM}}(\Lambda) \to \text{CM}(\Lambda)
\]
the syzygy and the cosyzygy functors. A key observation is the following, which gives the Auslander-Bridger transpose duality for \( \text{CM}(\Lambda) \) [10] (see [86, 1.1.1, 1.3.1] for a short proof).

**Theorem 3.4.** We have a duality
\[
\Omega^d \text{Tr} : \text{CM}(\Lambda) \xrightarrow{\sim} \text{CM}(\Lambda^{\text{op}})
\]
satisfying \((\Omega^d \text{Tr})^2 \simeq 1_{\text{CM}(\Lambda)}\).

Since the duality \( D_d : \text{CM}(\Lambda) \leftrightarrow \text{CM}(\Lambda^{\text{op}}) \) induces a duality \( D_d : \overline{\text{CM}}(\Lambda) \leftrightarrow \overline{\text{CM}}(\Lambda^{\text{op}}) \), we have mutually quasi-inverse equivalences
\[
\tau := D_d \Omega^d \text{Tr} : \overline{\text{CM}}(\Lambda) \xrightarrow{\Omega^d \text{Tr}} \text{CM}(\Lambda^{\text{op}}) \xrightarrow{D_d} \overline{\text{CM}}(\Lambda),
\]
\[
\tau^\sim := \Omega^d \text{Tr} D_d : \text{CM}(\Lambda) \xrightarrow{D_d} \overline{\text{CM}}(\Lambda^{\text{op}}) \xrightarrow{\Omega^d \text{Tr}} \text{CM}(\Lambda),
\]
called the *Auslander-Reiten translations*. As in the case of finite-dimensional algebras, \( \tau \) gives a bijection from isoclasses of indecomposable non-projective objects in \( \text{CM}(\Lambda) \) to isoclasses of indecomposable non-injective objects in \( \text{CM}(\Lambda) \). Moreover, we have the following results.

**Theorem 3.5.** (a) There exist the following functorial isomorphisms for any \( X,Y \in \text{CM}(\Lambda) \) (called the Auslander-Reiten duality)
\[
\text{Hom}_\Lambda(\tau^{-1} Y, X) \simeq D \text{Ext}_\Lambda^1(X,Y) \simeq \overline{\text{Hom}}_\Lambda(Y,\tau X).
\]
(b) For any indecomposable non-projective $X \in \text{CM}(\Lambda)$ (respectively, non-injective $Z \in \text{CM}(\Lambda)$), there exists an exact sequence (called an almost split sequence)

$$0 \to Z \xrightarrow{g} Y \xrightarrow{f} X \to 0$$

with terms in $\text{CM}(\Lambda)$ which induces the following exact sequences on $\text{CM}(\Lambda)$

$$0 \to (-, Z) \xrightarrow{g} (-, Y) \xrightarrow{f} J_{\text{CM}(\Lambda)}(-, X) \to 0,$$
$$0 \to (X, -) \xrightarrow{f} (Y, -) \xrightarrow{g} J_{\text{CM}(\Lambda)}(Z, -) \to 0.$$

Moreover, we have $Z \simeq \tau X$ (respectively, $X \simeq \tau^{-} Z$).

The category $\text{CM}(\Lambda)$ is especially nice for the case $d = 2$.

**Theorem 3.6.** If $d = 2$, then we have the following.

(a) The Nakayama functor induces an equivalence $\nu : \text{CM}(\Lambda) \to \text{CM}(\Lambda)$ which makes the following diagram commutative

$$\begin{array}{ccc}
\text{CM}(\Lambda) & \xrightarrow{\nu} & \text{CM}(\Lambda) \\
\downarrow & & \downarrow \\
\text{CM}(\Lambda) & \xrightarrow{\tau} & \text{CM}(\Lambda)
\end{array}$$

(b) For any projective $P \in \text{CM}(\Lambda)$, there exists an exact sequence (called a fundamental sequence)

$$0 \to \nu P \xrightarrow{g} C \xrightarrow{f} P \to P/PJ_\Lambda \to 0$$

which induces the following exact sequences on $\text{CM}(\Lambda)$

$$0 \to (-, \nu P) \xrightarrow{g} (-, C) \xrightarrow{f} J_{\text{CM}(\Lambda)}(-, P) \to 0,$$
$$0 \to (P, -) \xrightarrow{f} (C, -) \xrightarrow{g} J_{\text{CM}(\Lambda)}(\nu P, -) \to 0.$$

The existence of fundamental sequences plays a crucial role in the classification theory of representation-finite orders of Krull dimension $d = 2$ due to Reiten and Van den Bergh [104]. In this case, the structure theory of $\text{CM}(\Lambda)$ is much nicer than in other dimensions $d \neq 2$ (including $d = 0, 1$) since fundamental sequences play the role of almost split sequences for projective $\Lambda$-modules (see [81, 82, 83]), and the whole Auslander-Reiten quiver of $\text{CM}(\Lambda)$ always can be regarded as a stable translation quiver.

### 3.2. $n$-Auslander-Reiten theory.

In this subsection we consider $n$-cluster tilting subcategories of $\text{CM}(\Lambda)$. As in the case of finite-dimensional algebras, we consider the functors

$$\tau_n := \tau \Omega^{n-1} : \text{CM}(\Lambda) \xrightarrow{\Omega^{n-1}} \text{CM}(\Lambda) \xrightarrow{\tau} \text{CM}(\Lambda),$$
$$\tau_n^{-} := \tau^{-} \Omega^{-(n-1)} : \text{CM}(\Lambda) \xrightarrow{\Omega^{-(n-1)}} \text{CM}(\Lambda) \xrightarrow{\tau^{-}} \text{CM}(\Lambda).$$

As in the case of finite-dimensional algebras, we have the following results.
Theorem 3.7. Let $C$ be an $n$-cluster tilting subcategory of $\text{CM}(\Lambda)$.

(a) $\tau_n$ and $\tau_n^-$ induce mutually quasi-inverse equivalences $\tau_n : \mathcal{C} \to \mathcal{C}'$ and $\tau_n^- : \mathcal{C}' \to \mathcal{C}$ (called the $n$-Auslander-Reiten translations). In particular, $\tau_n$ gives a bijection from isoclasses of indecomposable non-projective objects in $C$ to isoclasses of indecomposable non-injective objects in $C$.

(b) There exist the following functorial isomorphisms for any $X,Y \in C$ (called the $n$-Auslander-Reiten duality)

$$\text{Hom}_\Lambda(\tau_n^{-} Y, X) \simeq D \text{Ext}_\Lambda^n(X,Y) \simeq \text{Hom}_\Lambda(Y, \tau_n X).$$

(c) Any indecomposable non-projective $X \in C$ (respectively, non-injective $Y \in C$) has an exact sequence (called an $n$-almost split sequence)

$$0 \to Y \xrightarrow{f_{n+1}} C_n \xrightarrow{f_n} \cdots \xrightarrow{f_3} C_1 \xrightarrow{f_2} X \to 0$$

with terms in $C$ such that $f_i \in J_C$ for any $i$ and the following sequences are exact on $C$

$$0 \to (-,Y) \xrightarrow{f_{n+1}} (-,C_n) \xrightarrow{f_n} \cdots \xrightarrow{f_3} (-,C_1) \xrightarrow{f_2} J_C(-,X) \to 0,$n+1 \to J_C(Y,-) \to 0.$$

Moreover, we have $Y \simeq \tau_n X$ (respectively, $X \simeq \tau_n^{-} Y$).

The case $n = d - 1$ is especially nice and we have the following analogous result to Theorem 3.6.

Theorem 3.8. Let $C$ be a $(d-1)$-cluster tilting subcategory in $\text{CM}(\Lambda)$.

(a) The Nakayama functor induces an equivalence $\nu : \mathcal{C} \to \mathcal{C}$ which makes the following diagram commutative

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\nu} & \mathcal{C} \\
\downarrow & & \downarrow \\
\mathcal{C} & \xrightarrow{\tau_{d-1}} & \mathcal{C}
\end{array}$$

(b) For any indecomposable projective $P \in \mathcal{C}$, there exists an exact sequence (called a $(d-1)$-fundamental sequence)

$$0 \to \nu P \xrightarrow{f_{n+1}} C_n \xrightarrow{f_n} \cdots \xrightarrow{f_3} C_1 \xrightarrow{f_2} P \to P/P\J_\Lambda \to 0$$

with terms in $C$ such that $f_i \in J_C$ for any $i$ and the following sequences are exact on $C$

$$0 \to (-,\nu P) \xrightarrow{f_{n+1}} (-,C_n) \xrightarrow{f_n} \cdots \xrightarrow{f_3} (-,C_1) \xrightarrow{f_2} J_C(-,P) \to 0,$n+1 \to J_C(\nu P,-) \to 0.$$
Let us give an example of \((d-1)\)-cluster tilting object.

**Example 3.9.** Let \(k\) be a field of characteristic zero, and \(S = k[[x_1, \ldots, x_d]]\) the formal power series ring. Let \(G\) be a finite subgroup of \(\text{GL}_d(k)\). Then \(G\) naturally acts on the \(k\)-vector space \(V := k^d\), and the action extends to the completion \(S\) of the symmetric algebra of \(V\). Let

\[
\Lambda := S^G = \{ f \in S \mid f^\sigma = f \text{ for any } \sigma \in G \}
\]

be the invariant subring of \(S\). Assume that \(\Lambda\) satisfies the following conditions:

- \(\Lambda\) is an isolated singularity;
- if \(\sigma \in G\) is a pseudo-reflection (namely \(\text{rank}(\sigma - 1) \leq 1\)), then \(\sigma = 1\).

The second condition implies that \(\text{End}_\Lambda(S)\) is naturally isomorphic to the skew group ring \(S^*G\) [12, 121] (the assumption \(d = 2\) there was not used). Then we have the following result [85, 86].

**Theorem 3.10.**

(a) \(S\) is a \((d-1)\)-cluster tilting object in \(\text{CM}(\Lambda)\).

(b) The Koszul complex

\[
0 \rightarrow S \otimes_k (\wedge^d V) \rightarrow S \otimes_k (\wedge^{d-1} V) \rightarrow \cdots \rightarrow S \otimes_k V \rightarrow S \rightarrow k \rightarrow 0
\]

is a direct sum of \((d-1)\)-almost split sequences and \((d-1)\)-fundamental sequences in the \((d-1)\)-cluster tilting subcategory \(\text{add} S_\Lambda\) in \(\text{CM}(\Lambda)\).

(c) The quiver of the category \(\text{add} S_\Lambda\) coincides with the McKay quiver of \(G\).

This generalizes famous results due to Herzog [70] and Auslander [12] for Krull dimension \(d = 2\). In this case, \(S\) is a 1-cluster tilting object in \(\text{CM}(\Lambda)\), so we have \(\text{add} S = \text{CM}(\Lambda)\). In particular, \(\Lambda\) is representation-finite.

It is interesting to classify all \((d-1)\)-cluster tilting objects in \(\text{CM}(\Lambda)\). We have the following result [80, 85, 86].

**Theorem 3.11.**

(a) Let \(G\) be a cyclic subgroup of \(\text{GL}_3(k)\) generated by \(\sigma = \text{diag}(\omega, \omega, \omega)\) with the primitive third root of unity \(\omega\). Put

\[
S_i := \{ f \in S \mid f^\sigma = \omega^i f \}
\]

for \(i \in \mathbb{Z}/3\mathbb{Z}\). Then all basic 2-cluster tilting objects in \(\text{CM}(\Lambda)\) are given by \(S_0 \oplus \Omega^i S_1 \oplus \Omega^i S_2\) and \(S_0 \oplus \Omega^{i+1} S_1 \oplus \Omega^i S_2\), for \(i \in \mathbb{Z}\).

(b) Let \(G\) be a cyclic subgroup of \(\text{GL}_4(k)\) generated by \(\sigma = \text{diag}(-1, -1, -1, -1)\). Put

\[
S_i := \{ f \in S \mid f^\sigma = (-1)^i f \}
\]

for \(i \in \mathbb{Z}/2\mathbb{Z}\). Then all basic 3-cluster tilting objects in \(\text{CM}(\Lambda)\) are given by \(S_0 \oplus \Omega^i S_1\), for \(i \in \mathbb{Z}\).
In these cases, any 2-rigid object in $\text{CM}(\Lambda)$ is a direct summand of a direct sum of copies of some $(d - 1)$-cluster tilting object. This is not the case in general.

In the rest of this subsection we study endomorphism algebras of $n$-cluster tilting objects in $\text{CM}(\Lambda)$ \cite{24, 86}. We have $\text{Hom}_\Lambda(X, Y) \in \text{CM}(R)$ for any $X, Y \in \text{CM}(\Lambda)$ for $d \leq 2$, but this is not the case for $d \geq 3$ (see \cite{85, 2.5.1}). Thus the general characterization of endomorphism algebras of $n$-cluster tilting objects in $\text{CM}(\Lambda)$ becomes rather complicated. Nevertheless the case $d \leq n$ is quite nice.

We call an $R$-algebra $\Gamma$ an $n$-Auslander algebra if

- $\Gamma$ is an $R$-order which is an isolated singularity;
- $\Gamma$ satisfies $\text{gl.dim} \, \Gamma \leq n + 1$ and the two-sided $(d + 1, n + 1)$-condition in Definition 1.1.

This is consistent with the definition of $n$-Auslander algebras in Subsection 1.1. We have the following characterization of $n$-Auslander algebras, where we call an additive category $n$-cluster tilting if it is equivalent to an $n$-cluster tilting subcategory of CM$(\Lambda)$ for an $R$-order $\Lambda$ which is an isolated singularity.

**Theorem 3.12.** Assume $d \leq n$.

(a) Let $M \in \text{CM}(\Lambda)$ be an $n$-cluster tilting object. Then $\text{End}_\Lambda(M)$ is an $n$-Auslander algebra.

(b) Any $n$-Auslander algebra is obtained in this way. This gives a bijection between the sets of equivalence classes of $n$-cluster tilting categories with additive generators and Morita-equivalence classes of $n$-Auslander algebras.

Let us explain another converse of (a). Later we shall use the following characterization \cite{86} of $n$-cluster tilting objects, which is analogous to Lemma 2.12.

**Lemma 3.13.** Let $M \in \text{CM}(\Lambda)$ be an $n$-rigid generator-cogenerator in $\text{CM}(\Lambda)$ with $d \leq n + 2$. Then the following conditions are equivalent.

(a) $M$ is an $n$-cluster tilting object in $\text{CM}(\Lambda)$.

(b) $\text{gl.dim} \, \text{End}_\Lambda(M) \leq n + 1$.

(c) For any indecomposable object $X \in \text{add} \, M$, there exists an exact sequence

$$0 \to C_{n+1} \xrightarrow{f_{n+1}} C_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} C_1 \xrightarrow{f_1} X$$

with terms in $\text{add} \, M$ such that the following sequence is exact on $\text{add} \, M$

$$0 \to (M, C_{n+1}) \xrightarrow{f_{n+1}} (M, C_n) \xrightarrow{f_n} \cdots \xrightarrow{f_2} (M, C_1) \xrightarrow{f_1} \text{Jc}(M, X) \to 0.$$

We have observed in Theorem 3.8 that $(d - 1)$-cluster tilting subcategories in $\text{CM}(\Lambda)$ have $(d - 1)$-fundamental sequences. Consequently the endomorphism algebras of $(d - 1)$-cluster tilting objects enjoys especially nice properties.
**Theorem 3.14.** Let $M$ be a $(d-1)$-cluster tilting object in $\text{CM}(\Lambda)$. Then $\text{End}_\Lambda(M)$ is a non-singular $R$-order (Definition 3.3) and satisfies the Gorenstein condition (Definition 3.3).

Conversely, if $\Gamma$ is a non-singular $R$-order, then $\text{CM}(\Gamma) = \text{add} \Gamma$ holds and $\Gamma$ is an $n$-cluster tilting object in $\text{CM}(\Gamma)$ for arbitrary $n$. In particular, Theorem 3.12 holds also for $n = d - 1$, since any $R$-order which is an isolated singularity satisfies the two-sided $(d, d)$-condition by [65].

In general, there are a lot of $n$-cluster tilting subcategories in a fixed category $\text{CM}(\Lambda)$, and it is natural to study their relationship. For the case $n = 2$ (and arbitrary $d$), they are related by derived equivalence [85, 99].

**Theorem 3.15.** Let $M$ and $N$ be 2-cluster tilting objects in $\text{CM}(\Lambda)$ with $\Gamma := \text{End}_\Lambda(M)$ and $\Gamma' := \text{End}_\Lambda(N)$. Then $T := \text{Hom}_\Lambda(M, N)$ is a tilting $\Gamma$-module of projective dimension at most one with $\text{End}_{\Gamma}(T) \cong \Gamma'$. In particular, $\Gamma$ and $\Gamma'$ are derived equivalent.

In particular, the numbers of indecomposable direct summands of all basic 2-cluster tilting objects in $\text{CM}(\Lambda)$ are equal. Moreover, we have a relationship between 2-cluster tilting objects and tilting modules of projective dimension at most one.

Our Theorem 3.15 reminds us of a conjecture of Van den Bergh [117, 118], who introduced a certain class of non-commutative algebras to study a conjecture of Bondal and Orlov [33] in birational geometry. In the rest of this subsection we assume that $R$ is a normal domain. Then reflexive $R$-modules behave nicely, for example they are closed under kernels and extensions.

**Definition 3.16.** Let $\Lambda$ be a module-finite $R$-algebra. We say that $M \in \text{mod} \Lambda$ gives a non-commutative crepant resolution (or NCCR) $\text{End}_\Lambda(M)$ of $\Lambda$ if

(a) $M$ is a reflexive $R$-module and $\text{End}_\Lambda(M)$ is a non-singular $R$-order;

(b) $M_p$ is a generator of $\Lambda_p$ for any height one prime ideal $p$ of $R$.

Notice that the condition (b) is not assumed in [117, 118] where the case $\Lambda = R$ is treated and (b) is automatically satisfied. Van den Bergh conjectured that all (commutative and non-commutative) crepant resolutions of $\Lambda = R$ are derived equivalent, and proved this for 3-dimensional terminal singularities. Easy examples suggest that it is appropriate to impose the condition (b) to treat non-commutative algebras $\Lambda$.

Comparing the above condition (a) and Theorem 3.14, it is natural to hope that NCCR is related to $(d - 1)$-cluster tilting objects. In fact, we have the following result [85].

**Theorem 3.17.** Let $\Lambda$ be an $R$-order which is an isolated singularity. The following conditions are equivalent for $M \in \text{CM}(\Lambda)$.

(a) $M$ is a $(d - 1)$-cluster tilting object in $\text{CM}(\Lambda)$. 

(b) $M$ is a generator-cogenerator in $\text{CM}(\Lambda)$ and gives a NCCR of $\Lambda$.

Since modules giving NCCR are not necessarily Cohen-Macaulay, they are more general than $(d-1)$-cluster tilting objects in $\text{CM}(\Lambda)$. But the case $d = 3$ is especially nice, and we have the following stronger result than Theorem 3.15 [89].

**Theorem 3.18.** Let $d = 3$ and $\Lambda$ be a module-finite $R$-algebra. Assume that $M$ and $N$ give NCCR, $\Gamma := \text{End}_\Lambda(M)$ and $\Gamma' = \text{End}_\Lambda(N)$ of $\Lambda$, respectively. Then $T := \text{Hom}_\Lambda(M, N)$ is a tilting $\Gamma$-module of projective dimension at most one with $\text{End}_\Gamma(T) \simeq \Gamma'$. In particular, $\Gamma$ and $\Gamma'$ are derived equivalent.

In particular, if $d = 3$, then all NCCR of $\Lambda$ are derived equivalent.

We call an $R$-order $\Lambda$ symmetric if $\text{Hom}_R(\Lambda, R) \simeq \Lambda$ as $(\Lambda, \Lambda)$-modules. If $d = 3$ and $\Lambda$ is a symmetric $R$-order which is an isolated singularity, then we have the following nice behaviour of modules giving NCCR of $\Lambda$ [89].

**Theorem 3.19.** Let $d = 3$ and $\Lambda$ be a symmetric $R$-order which is an isolated singularity.

(a) $\text{CM}(\Lambda)$ contains a 2-cluster tilting object if and only if $\Lambda$ has a NCCR.

(b) Let $M$ be a 2-cluster tilting object in $\text{CM}(\Lambda)$. Then $N \mapsto \text{Hom}_\Lambda(M, N)$ gives a bijection between $\Lambda$-modules giving NCCR and tilting $\text{End}_\Lambda(M)$-modules which are reflexive $R$-modules.

### 3.3. Cohen-Macaulay modules and triangulated categories

In this subsection we observe that nice triangulated categories are obtained from the category of Cohen-Macaulay modules.

Again, let $R$ be a complete local Gorenstein ring of Krull dimension $d$. We call an $R$-order $\Lambda$ Gorenstein if $D_d \Lambda$ is a projective $\Lambda$-module, or equivalently $D_d \Lambda$ is a projective $\Lambda^{\text{op}}$-module. For example, Gorenstein $R$-orders of Krull dimension $d = 0$ are nothing but selfinjective artin $R$-algebras. In this case, $\text{CM}(\Lambda)$ forms a Frobenius category whose projective objects are just projective $\Lambda$-modules. By a general result of Happel [67], the stable category $\text{CM}(\Lambda)$ forms a triangulated category with a suspension functor $\Sigma^{-1}$. Let us observe that the category $\text{CM}(\Lambda)$ enjoys a further nice property.

**Definition 3.20.** Let $T$ be an $R$-linear category such that $\text{Hom}_T(X,Y) \in \text{I} \, \text{I} \, \text{I} \, \text{I} \, \text{I} \, \text{I}$ for any $X,Y \in T$. We call an autoequivalence $F$ of $T$ a Serre functor if there exists a functorial isomorphism

$$\text{Hom}_T(X,Y) \simeq D \text{Hom}_T(Y, FX)$$

for any $X,Y \in T$ [32] [105]. A triangulated category $T$ with the suspension functor $\Sigma$ is called $n$-Calabi-Yau (or $n$-CY) for an integer $n$ if $\Sigma^n$ gives a Serre functor of $T$ [98] [93].

A Frobenius category $\mathcal{X}$ is called $n$-CY if the stable category $T := \mathcal{X}$ is an $n$-CY triangulated category.
We have the following result from the Auslander-Reiten duality given in Theorem 3.5.

**Theorem 3.21.** Let $\Lambda$ be a Gorenstein $R$-order which is an isolated singularity. Then $\text{CM}(\Lambda)$ forms a triangulated category with a Serre functor $\Omega^{-1}\tau$.

A special class of Gorenstein $R$-orders is given by symmetric $R$-orders defined in the previous subsection. For a symmetric $R$-order $\Lambda$, we have that the Nakayama functor $\nu : \text{CM}(\Lambda) \to \text{CM}(\Lambda)$ is isomorphic to the identity functor $\text{id}_{\text{CM}(\Lambda)}$. Using this property one can show the following result due to Auslander [10].

**Proposition 3.22.** Let $\Lambda$ be a symmetric $R$-order which is an isolated singularity. Then we have an isomorphism $\tau \cong \Omega^{-2d}$ of functors $\text{CM}(\Lambda) \to \text{CM}(\Lambda)$. In particular $\text{CM}(\Lambda)$ forms a $(d-1)$-CY triangulated category.

For example, if $\Lambda$ is a finite-dimensional symmetric algebra over a field $k$, then the stable category $\text{mod} \Lambda$ forms a $(−1)$-CY triangulated category.

In the rest of this subsection we discuss other triangulated categories with Serre functors. Let $\Lambda$ be a module-finite $R$-algebra. We denote by $D^b(\text{f.l.}\Lambda)$ (respectively, $D^b(\text{f.l.}\Lambda)$) the bounded derived category of $\text{mod} \Lambda$ (respectively, $\text{f.l.} \Lambda$), and by $K^b(\text{add} \Lambda)$ the homotopy category of bounded complexes on $\text{add} \Lambda$. It is not difficult to check the following result [67, 90, 1].

**Proposition 3.23.** (a) If $\text{gl.} \dim \Lambda < \infty$, then $D^b(\text{f.l.}\Lambda)$ forms a triangulated category with a Serre functor $- \otimes \Lambda R \text{Hom}(\Lambda, R)[d]$.

(b) If $\text{id}_\Lambda = \text{id}_\Lambda < \infty$, then $D^b(\text{f.l.}\Lambda) \cap K^b(\text{add} \Lambda)$ forms a triangulated category with a Serre functor $- \otimes \Lambda R \text{Hom}(\Lambda, R)[d]$.

A module-finite $R$-algebra $\Lambda$ is called $n$-CY for an integer $n$ if $D^b(\text{f.l.}\Lambda)$ forms an $n$-CY triangulated category in the sense of Definition 3.20 [89] (see also [25, 64]). Similarly, $\Lambda$ is called perfectly $n$-CY if $D^b(\text{f.l.}\Lambda) \cap K^b(\text{add} \Lambda)$ forms an $n$-CY triangulated category. Immediately from Proposition 3.23 we have that

- any non-singular symmetric $R$-order is $d$-CY,
- any symmetric $R$-order is perfectly $d$-CY.

Notice that a symmetric $R$-order $\Lambda$ is non-singular if and only if $\text{gl.} \dim \Lambda < \infty$. For example, a finite-dimensional symmetric algebra $\Lambda$ over a field is perfectly 0-CY. If a symmetric $R$-order $\Lambda$ has a NCCR $\Gamma$, then it is not difficult to show that $\Gamma$ is a non-singular symmetric $R$-order, hence $\Gamma$ is $d$-CY.

For example, let $k$ be a field of characteristic zero and $S = k[[x_1, \cdots, x_d]]$ the formal power series ring. For a finite subgroup $G$ of $\text{SL}_d(k)$, let $\Lambda$ be the invariant subring $S^G$. Then $S$ gives a NCCR $\text{End}_\Lambda(S) \simeq S * G$ of $\Lambda$ [118], and $S * G$ is a $d$-CY algebra.

In particular, we have the following observation.
Proposition 3.24. Let $\Lambda$ be a symmetric $R$-order which is an isolated singularity. If $M$ is a $(d-1)$-cluster tilting object in $\text{CM}(\Lambda)$, then $\text{End}_\Lambda(M)$ is a non-singular symmetric $R$-order, and hence a $d$-CY algebra.

We end this subsection by the following characterization of $n$-CY algebras [89].

Theorem 3.25. Let $\Lambda$ be a module-finite $R$-algebra which is a faithful $R$-module. Then the following conditions are equivalent.

(a) $\Lambda$ is $n$-CY for some $n$.
(b) $\Lambda$ is $d$-CY.
(c) $\Lambda$ is a non-singular symmetric $R$-order.

3.4. 2-cluster tilting for 2-Calabi-Yau categories. Let $T$ be a triangulated category. We define $n$-cluster tilting subcategories (respectively, objects) and $n$-rigid subcategories (respectively, objects) of $T$ by replacing $\text{Ext}_A^i(-, -)$ in Definition 2.1 by $\text{Hom}_T(-, \Sigma^i -)$.

In this section we briefly recall a few results on 2-cluster tilting subcategories of 2-CY triangulated categories. They are mainly studied from the viewpoint of categorification of Fomin-Zelevinsky cluster algebras [58] using cluster categories [41, 43, 44] and stable categories of preprojective algebras [59, 63, 37]. Especially, the following unique replacement property [41, 90] of indecomposable direct summands of 2-cluster tilting objects is important.

Theorem 3.26. Let $M$ be a basic 2-cluster tilting object in a 2-CY triangulated category $T$. For any indecomposable direct summand $X$ of $M$, we take a decomposition $M = X \oplus N$.

(a) There exists exactly one indecomposable object $Y \in T$ which is not isomorphic to $X$ such that $Y \oplus N$ is a basic 2-cluster tilting object in $T$.
(b) There exist triangles

$$X \xrightarrow{f} N_1 \xrightarrow{L} Y \twoheadrightarrow \Sigma X \quad \text{and} \quad Y \xrightarrow{g} N_0 \xrightarrow{L} X \twoheadrightarrow \Sigma Y$$

in $T$ such that $f$ and $f'$ are right (add $N$)-approximation and $g$ and $g'$ are left (add $N$)-approximation.

In this case, we call $Y \oplus N$ a 2-cluster tilting mutation of $X \oplus N$. This is an analogue of tilting mutations which we shall apply in Subsection 4.2. There are many interesting results on 2-cluster tilting mutations. For example, the quivers of $\text{End}_T(X \oplus N)$ and $\text{End}_T(Y \oplus N)$ are related by Fomin-Zelevinsky mutation [58, 40, 37].

Next we consider the triangulated analogue of 2-Auslander algebras. The endomorphism algebra $\text{End}_T(M)$ of a 2-cluster tilting object $M$ in a 2-CY triangulated category $T$ is called a 2-CY tilted algebra. When $T$ is the stable category $\text{CM}(\Lambda)$ of an $R$-order $\Lambda$, then $\text{End}_T(M)$ is a factor algebra of the 2-Auslander algebra $\text{End}_\Lambda(M)$ which enjoys nice properties (see Theorem 3.14, Proposition 3.24). The following result [89, 94] shows that 2-CY tilted algebras also enjoy nice properties.
**Theorem 3.27.** Let $\Gamma = \text{End}_T(M)$ be a 2-CY tilted algebra.

(a) $\text{id}_\Gamma \Gamma = \text{id}_\Gamma \Gamma \leq 1$ holds.

(b) The following category forms a 3-CY Frobenius category

$$\text{Sub} \Gamma := \{X \in \text{mod} \Gamma \mid X \text{ is a submodule of } \Gamma^\ell \text{ for some } \ell\}.$$ 

(c) We have the following equivalence of categories

$$\text{Hom}_T(M, -) : T/\Sigma M \rightarrow \text{mod} \Gamma.$$ 

We end this subsection with the following analogue [37] of Bongartz completion for tilting modules of projective dimension at most one [5].

**Theorem 3.28.** Let $\mathcal{X}$ be a 2-CY Frobenius category. Assume that $\mathcal{X}$ has a 2-cluster tilting object. Then any 2-rigid object in $\mathcal{X}$ is a direct summand of some 2-cluster tilting object in $\mathcal{X}$.

### 4. Examples of 2-cluster tilting objects

In this section we give two classes of 2-CY Frobenius categories with 2-cluster tilting objects. One class is constructed from finite-dimensional factor algebras of preprojective algebras of arbitrary quivers without loops. Another class is constructed from one-dimensional hypersurface singularities $k[[x, y]]/(f)$. Although these constructions are rather different, they are similar in the sense that we use factor algebras of 2-CY algebras (preprojective algebras or $k[[x, y]]$).

**4.1. Preprojective algebras.** In this subsection we shall recall the construction of 2-CY Frobenius categories with 2-cluster tilting objects given in joint work with Buan, Reiten and Scott [89, 37]. This is closely related to the work of Geiss, Leclerc and Schröer [62, 63].

To accord with the conventions in [37], all modules in this subsection are left modules. Throughout this subsection we fix a connected quiver $Q$ without loops. Let $\widetilde{Q}$ be the quiver constructed from $Q$ by adding an arrow $a^* : j \rightarrow i$ for each arrow $a : i \rightarrow j$ in $Q$. We denote by $k\widetilde{Q}$ the complete path algebra of $\widetilde{Q}$ over a field $k$. Thus as a $k$-vector space $k\widetilde{Q}$ is a direct product $\prod_{i \geq 0} k\widetilde{Q}_i$ of $k$-vector spaces $k\widetilde{Q}_i$ whose basis is given by all paths in $\widetilde{Q}$ of length $i$, and the multiplication in $k\widetilde{Q}$ is given by the same way as for usual path algebra $kQ$. The complete preprojective algebra of $Q$ is defined as $\Lambda := k\widetilde{Q}/I$, where $I$ is the closure of the ideal

$$\langle \sum_{a \in Q_1} (aa^* - a^*a) \rangle$$

of $k\widetilde{Q}$ with respect to the $J_{k\widetilde{Q}}$-adic topology on $k\widetilde{Q}$. For the case of Dynkin and extended Dynkin quivers, we have the following.
• If $Q$ is a Dynkin quiver if and only if $\Lambda$ is finite-dimensional over $k$. In this case $\Lambda$ forms a selfinjective algebra.

• If $Q$ is an extended Dynkin quiver, then $\Lambda$ is Morita-equivalent to the skew group algebra $k[[x,y]] \ast G$ for a finite subgroup $G$ of $\text{SL}_2(k)$.

We observed in Subsection 3.3 that $k[[x,y]] \ast G$ is a 2-CY algebra. More generally, we have the following result [47, 45, 61].

**Theorem 4.1.** (a) If $Q$ is a Dynkin quiver, then $\text{mod } \Lambda$ forms a 2-CY Frobenius category.

(b) If $Q$ is not a Dynkin quiver, then $\mathcal{D}^b(\text{f.l. } \Lambda)$ forms a 2-CY triangulated category.

Let $Q_0 = \{1, \cdots, n\}$ be the set of vertices of $Q$, and we denote by $e_i$ the idempotent of $\Lambda$ corresponding to $i \in Q_0$. Define a 2-sided ideal of $\Lambda$ by

$$I_i := \Lambda(1 - e_i)\Lambda.$$ 

Since $I_i$ is generated by an idempotent, we have

(i) $I_i^2 = I_i$ for any $i$.

We denote by

$$\langle I_1, \cdots, I_n \rangle$$

the set of two-sided ideals $I_{i_1} \cdots I_{i_i}$, $i_1, \ldots, i_i \in \{1, \ldots, n\}$, of $\Lambda$ obtained by multiplying the ideals $I_1, \cdots, I_n$. This set contains $\Lambda$, too. We call a left ideal $I$ of $\Lambda$ cofinite tilting if $\Lambda/I \in \text{f.l. } \Lambda$ and $I$ is a tilting $\Lambda$-module. Dually, we define cofinite tilting right ideals. If $Q$ is a Dynkin quiver, then any tilting $\Lambda$-module is projective since $\Lambda$ is selfinjective. Otherwise, we have a lot of tilting $\Lambda$-modules by the following result.

**Theorem 4.2.** Assume that $Q$ is not a Dynkin quiver.

(a) $\langle I_1, \cdots, I_n \rangle$ gives a set of cofinite tilting left (respectively, right) ideals of $\Lambda$. They have projective dimension at most one, and two different elements are non-isomorphic as left (respectively, right) $\Lambda$-modules.

(b) If $Q$ is an extended Dynkin quiver, then $\langle I_1, \cdots, I_n \rangle$ gives a set of isoclasses of basic tilting $\Lambda$-modules (respectively, tilting $\Lambda^{\text{op}}$-modules) of projective dimension at most one.

For $i, j \in \{1, \cdots, n\}$ with $i \neq j$, we define the two-sided ideal of $\Lambda$ by $I_{i,j} := \Lambda(1 - e_i - e_j)\Lambda$. Then any ideal of $\Lambda$ obtained by multiplying $I_i$ and $I_j$ any times contains $I_{i,j}$. If there is no arrow between $i$ and $j$, then the factor algebra $\Lambda/I_{i,j}$ is isomorphic to $k \times k$. This implies

(ii) $I_iI_j = I_{i,j}I_i$, if there is no arrow in $Q$ between $i$ and $j$. 
If there is exactly one arrow in $Q$ between $i$ and $j$, then $\Lambda/I_{i,j}$ is isomorphic to the preprojective algebra of type $A_2$. Looking at Loewy series of $\Lambda$, we have

(iii) $I_i I_j I_i = I_{i,j} = I_j I_i I_j$, if there is exactly one arrow in $Q$ between $i$ and $j$.

Using the relations (i), (ii) and (iii), we can describe the set $\langle I_1, \cdots, I_n \rangle$.

**Definition 4.3.** The Coxeter group $W$ of $Q$ is a group presented by generators $s_1, \cdots, s_n$ with relations

(a) $s_i^2 = 1$, for any $i$;

(b) $s_i s_j = s_j s_i$, if there is no arrow in $Q$ between $i$ and $j$;

(c) $s_i s_j s_i = s_j s_i s_j$, if there is exactly one arrow in $Q$ between $i$ and $j$.

The relations (ii) and (iii) coincide with (b) and (c), respectively, but (i) is slightly different from (a). Thus it is natural to compare the set $\langle I_1, \cdots, I_n \rangle$ with $W$. In fact, we have the following result.

**Theorem 4.4.** Let $Q$ be an arbitrary quiver without loops. Then there exists a well-defined bijection $W \sim \langle I_1, \cdots, I_n \rangle$. It is given by $w \mapsto I_w := I_{i_1} \cdots I_{i_\ell}$ for an arbitrary reduced expression $w = s_{i_1} \cdots s_{i_\ell}$ of $w$ in $W$.

As an immediate consequence of Theorems 4.2 and 4.4, we have the following result which is related to work of Ishii and Uehara on McKay correspondence for the minimal resolutions of simple singularities.

**Corollary 4.5.** Let $Q$ be an extended Dynkin quiver. Then basic tilting $\Lambda$-modules correspond bijectively to elements in $W$.

Again, let $Q$ be an arbitrary quiver. For any $w \in W$, we put $\Lambda_w := \Lambda/I_w$, which is a finite-dimensional algebra. It can be shown that $\Lambda_w$ satisfies

$$\text{id}_{\Lambda_w}(\Lambda_w) = \text{id}(\Lambda_w)_{\Lambda_w} \leq 1.$$ 

Thus $\Lambda_w$ is a cotilting $\Lambda_w$-module of injective dimension at most one, and the classical tilting theory implies that the full subcategory

$$\text{Sub} \Lambda_w := \{ X \in \text{f.i.} \Lambda \mid X \text{ is a submodule of } \Lambda_w^\ell \text{ for some } \ell \}$$

of f.i. $\Lambda$ forms an extension-closed subcategory of f.i. $\Lambda$ with enough projectives and enough injectives add $\Lambda_w$. In particular Sub $\Lambda_w$ forms a Frobenius category, and Ext$^1_{\Lambda_w}(X,Y)$ and Ext$^1_{\Lambda_w}(X,Y)$ are isomorphic for any $X,Y \in \text{Sub} \Lambda_w$. By Theorem 4.4, we have a functorial isomorphism

$$\text{Ext}^1_{\Lambda_w}(X,Y) \simeq D\text{Ext}^1_{\Lambda_w}(Y,X)$$

for any $X,Y \in \text{Sub} \Lambda_w$. Consequently Sub $\Lambda_w$ forms a 2-CY Frobenius category. Moreover, it contains 2-cluster tilting objects by the following result.
**Theorem 4.6.** Let \( w \in W \). For any reduced expression \( w = s_{i_1} \cdots s_{i_\ell} \) of \( w \) in \( W \), we have the following 2-cluster tilting object in the 2-CY Frobenius category \( \text{Sub} \Lambda_w \)

\[
T(i_1, \cdots, i_\ell) := \bigoplus_{k=1}^\ell \Lambda_{s_{i_1} \cdots s_{i_k}} = \bigoplus_{k=1}^\ell \Lambda / I_{i_1} \cdots I_{i_\ell}.
\]

A crucial step in the proof of this theorem is to show that the global dimension of the endomorphism algebra \( \text{End}_{\Lambda_w}(T(i_1 \cdots, i_\ell)) \) is at most three by using induction on \( \ell \). Then we can apply Lemma 3.13.

It is easy to show that \( \text{add} T(i_1, \cdots, i_\ell) \) contains exactly one indecomposable object which is not contained in \( \text{add} T(i_1, \cdots, i_{\ell-1}) \). It is shown in [37] that any full subcategory of \( \text{f.l.} \Lambda \) which is closed under extensions and submodules and contains 2-cluster tilting objects has the form \( \text{Sub} \Lambda_w \) for some \( w \in W \).

We have a nice description of the quivers of the endomorphism algebras of these 2-cluster tilting objects. For each reduced expression \( w = s_{i_1} \cdots s_{i_\ell} \), we define a quiver \( Q(i_1, \cdots, i_\ell) \) as follows:

- For each \( i \in \{1, \cdots, n\} \), pick out the expression consisting of the \( i_k \) which are \( i \). We draw an arrow from each \( i \) to the previous \( i \).
- For each arrow \( a : i \rightarrow j \) in \( Q \), pick out the expression consisting of the \( i_k \) which are \( i \) or \( j \). We draw an arrow from the last \( i \) in a connected set of \( i \)’s to the last \( j \) in the next set of \( j \)’s.

Define the quiver \( Q(i_1, \cdots, i_\ell) \) by removing the last \( i \) from \( Q(i_1, \cdots, i_\ell) \) for each \( i \in \{1, \cdots, n\} \).

**Theorem 4.7.** Let \( w = s_{i_1} \cdots s_{i_\ell} \) be a reduced expression, and let \( T := T(i_1, \cdots, i_\ell) \) be the corresponding 2-cluster tilting object in \( \text{Sub} \Lambda_w \). Then the quiver of \( \text{End}_{\Lambda}(T) \) is \( Q(i_1, \cdots, i_\ell) \), and the quiver of \( \text{End}_{\Lambda}(T) \) is \( Q(i_1, \cdots, i_\ell) \).

Moreover, it is shown in [38] that the algebra \( \text{End}_{\Lambda}(T) \) is isomorphic to the Jacobian algebra of a quiver with a potential [50].

**Example 4.8.** Let \( Q \) be the quiver \( 1 \rightarrow 2 \rightarrow 3 \), and let \( w = s_1 s_2 s_1 s_3 s_2 = s_2 s_1 s_2 s_3 s_2 = s_2 s_1 s_3 s_2 s_3 \) be reduced expressions. The first expression gives the quiver

\[
\begin{array}{ccc}
1 & \rightarrow & 2 \\
\downarrow & & \downarrow \\
1 & \rightarrow & 2 \\
\end{array}
\]

the second one gives the quiver

\[
\begin{array}{ccc}
2 & \rightarrow & 1 \\
\downarrow & & \downarrow \\
2 & \rightarrow & 1 \\
\end{array}
\]
and the third one gives the quiver

\[ \begin{array}{cccccccc}
2 & \rightarrow & 1 & \rightarrow & 3 & \rightarrow & 2 & \rightarrow & 3 \\
2 & 1 & 3 & 2 & 1 & 3 & 2 & 1
\end{array} \]

We end this subsection with the following question.

- Do Theorem 4.2(b) and Corollary 4.5 hold for arbitrary quivers which are neither Dynkin nor extended Dynkin?

### 4.2. Hypersurface singularities.

In this subsection we study \( n \)-cluster tilting for isolated hypersurface singularities. Especially, we recall a classification of 2-cluster tilting objects for one-dimensional hypersurface singularity given in a joint work with Burban, Keller and Reiten [42].

Let \( k \) be a field, and \( S = k[[x_0, \cdots, x_d]] \) the formal power series ring of \((d + 1)\) variables. We put \( \Lambda = S/(f) \) for \( f \in (x_0, \cdots, x_d) \). Then \( \Lambda \) is a complete local Gorenstein ring of Krull dimension \( d \). In this subsection we assume that \( \Lambda \) is an isolated singularity. Then the category \( \text{CM}(\Lambda) \) of Cohen-Macaulay \( \Lambda \)-modules forms a \((d - 1)\)-CY Frobenius category by Proposition 3.22.

A remarkable property of hypersurface singularities is given by matrix factorization introduced by Eisenbud [54, 121]. For any \( X \in \text{CM}(\Lambda) \), we have \( \text{pd} \ X_S = 1 \) by Auslander-Buchsbaum formula [40]. Thus there exists a projective resolution

\[ 0 \rightarrow S^\ell \xrightarrow{a} S^\ell \rightarrow X \rightarrow 0, \]

where \( a \) can be regarded as an \( \ell \times \ell \) matrix with entries in \( S \). Using \( fX = 0 \), one can easily check that there exists another \( \ell \times \ell \) matrix \( b \in \text{Hom}_S(S^\ell, S^\ell) \) such that \( ab = f \cdot 1_{S^\ell} \) and \( ba = f \cdot 1_{S^\ell} \). It is easily checked that we have a 2-periodic projective resolution

\[ \cdots \xrightarrow{b} \Lambda^\ell \xrightarrow{a} \Lambda^\ell \xrightarrow{b} \Lambda^\ell \xrightarrow{a} \Lambda^\ell \rightarrow X \rightarrow 0 \quad (10) \]

of the \( \Lambda \)-module \( X \). In particular, we have the following conclusion.

**Theorem 4.9.** We have \( \Omega^2 \simeq \text{id}_{\text{CM}(\Lambda)} \) as functors \( \text{CM}(\Lambda) \rightarrow \text{CM}(\Lambda) \).

Immediately we have the following observation.

- If \( d \) is even, then \( \text{CM}(\Lambda) \) forms a 1-CY Frobenius category. In particular, any 2-rigid object in \( \text{CM}(\Lambda) \) is projective.

- If \( d \) is odd, then \( \text{CM}(\Lambda) \) forms a 2-CY Frobenius category. In particular, any 3-rigid object in \( \text{CM}(\Lambda) \) is projective.
Another important application of matrix factorization is the following equivalence often called Knörrer periodicity \[^{[97,121]}\] (see \[^{[114]}\] for characteristic two).

**Theorem 4.10.** Let \(\Lambda' := \mathbb{k}[x_0, \ldots, x_d, u, v]/(f + uv)\). Then we have a triangulated equivalence

\[
\mathrm{CM}(\Lambda) \to \mathrm{CM}(\Lambda'),
\]

which sends \(X \in \mathrm{CM}(\Lambda)\) with the projective resolution \(^{[10]}\) to \(X' \in \mathrm{CM}(\Lambda')\) with the following projective resolution

\[
\cdots \to \Lambda' \to \Lambda' \to \Lambda' \to \cdots \to X' \to 0.
\]

In the rest of this subsection we concentrate on the case \(d = 1\) and study 2-cluster tilting objects of \(\mathrm{CM}(\Lambda)\). Thus

\[
S = \mathbb{k}[[x,y]] \quad \text{and} \quad \Lambda = \mathbb{k}[[x,y]]/(f).
\]

In this case, \(\mathrm{CM}(\Lambda)\) consists of \(X \in \text{mod } \Lambda\) such that \(\text{soc } X = 0\). For a technical reason we assume that \(k\) is an infinite field. We take a decomposition \(f = f_1 \cdots f_n\) into irreducible factors. Since we assumed that \(\Lambda\) is an isolated singularity, we have that \((f_i) \neq (f_j)\) for any \(i \neq j\). We put

\[
S_i := \mathbb{k}[[x,y]]/(f_1 \cdots f_i)
\]

for any \(1 \leq i \leq n\). We have a projective resolution

\[
\cdots \to \Lambda' \to \Lambda' \to \cdots \to S_i \to 0
\]

of \(S_i\) as the simplest case of matrix factorization. Applying \(\text{Hom}_\Lambda(-, S_j)\), one can easily check that \(\text{Ext}^1_{\Lambda}(S_i, S_j) = 0\) holds for any \(1 \leq i, j \leq n\). We have the following criterion for existence of 2-cluster tilting objects in \(\mathrm{CM}(\Lambda)\).

**Theorem 4.11.** If \(f_i \notin (x,y)^2\) for any \(i\), then \(\mathrm{CM}(\Lambda)\) has a 2-cluster tilting object \(M := \bigoplus_{i=1}^n S_i\). The converse holds if \(k\) is an algebraically closed field of characteristic zero.

We give examples for representation-finite cases \[^{[121]}\].

**Example 4.12.** Let \(\Lambda = \mathbb{k}[[x,y]]/(f)\) be a simple singularity, so in characteristic zero \(f\) is one of the following

\[
\begin{align*}
(A_m) & \quad x^2 + y^{m+1} \quad (m \geq 1), \\
(D_m) & \quad x^2y + y^{m-1} \quad (m \geq 4), \\
(E_6) & \quad x^3 + y^4, \\
(E_7) & \quad x^3 + xy^3, \\
(E_8) & \quad x^3 + y^5.
\end{align*}
\]

By our theorem \(\mathrm{CM}(\Lambda)\) has a 2-cluster tilting object if and only if \(\Lambda\) is of type \(A_m\) with odd \(m\) or \(D_m\) with even \(m\).
We give a sketch of the proof of Theorem 4.11. Let us prove the former assertion. For simplicity we assume that \((f_i, f_{i+1}) = (x, y)\) holds for any \(1 \leq i < n\). (Any case can be reduced to this case by the assumption that \(k\) is an infinite field.) Since we already observed that \(M\) is a 2-rigid generator-cogenerator in \(\text{CM}(\Lambda)\), we only have to construct the sequence in Lemma 3.13(c) for each \(X = S_i\) \((1 \leq i \leq n)\). It is given by

\[
0 \rightarrow S_i \xrightarrow{(f_{i+1})} S_{i+1} \oplus S_{i-1} \xrightarrow{(f_i, f_{i+1})} S_{i+1} \oplus S_{i-1} \xrightarrow{(-1)f_i} S_i \rightarrow 0 \quad (1 \leq i < n),
\]

\[
0 \rightarrow S_{n-1} \xrightarrow{(-f_{n+1})} S_n \oplus S_{n-1} \xrightarrow{(f_{n+1}, f_n)} S_n \quad (i = n).
\]

Notice that the upper sequence gives a 2-almost split sequence in \(\text{add}\ M\).

The proof of the converse is indirect and needs results in birational geometry. Thus our assumptions on the base field \(k\) are necessary. Let

\[
\Lambda' := k[[x, y, u, v]]/(f(x, y) + uv).
\]

By Theorem 4.10, the following conditions are equivalent.

(i) \(\text{CM}(\Lambda)\) has a 2-cluster tilting object.

(ii) \(\text{CM}(\Lambda')\) has a 2-cluster tilting object.

By Theorem 3.19, the condition (ii) is equivalent to the following condition.

(iii) \(\Lambda'\) has a non-commutative crepant resolution.

Now let us translate the condition (iii) into a geometric condition. The singularity \(\Lambda'\) is a \(cA_m\)-singularity for \(m := \text{ord}(f) - 1\) in the sense that the intersection of the hypersurface \(f(x, y) + uv = 0\) with a generic hyperplane \(ax + by + cu + dv = 0\) in \(k^4\) is an \(A_m\)-singularity. Since any isolated \(cA_m\)-singularity is a terminal singularity \[103\], the condition (iii) is equivalent to the following condition by results of Van den Bergh \[117, 118\].

(iv) \(\text{Spec}(\Lambda')\) has a crepant resolution.

For isolated \(cA_m\)-singularities, the condition (iv) is equivalent to the following condition by a result of Katz \[91\].

(v) The number of irreducible factors of \(f\) is \(m + 1\).

Clearly (v) is equivalent to that \(f_i \notin (x, y)^2\) for any \(1 \leq i \leq n\). Thus we finish the proof of the converse. Notice that we have also proved the former assertion by a different approach without giving explicit 2-cluster tilting objects in \(\text{CM}(\Lambda)\).

To state our classification result of 2-cluster tilting objects in \(\text{CM}(\Lambda)\), we introduce some notations. We denote by \(S_n\) the symmetric group of degree \(n\). For \(w \in S_n\), we put

\[
S^w_i := S/(f_{w(1)}f_{w(2)} \cdots f_{w(n)}) \quad \text{and} \quad M_w := \bigoplus_{i=1}^n S^w_i.
\]
For a non-empty subset \( I \) of \( \{1, \cdots, n\} \), we put
\[
S_I := S / \left( \prod_{i \in I} f_i \right).
\]

**Theorem 4.13.** Assume that \( f_i \notin (x, y)^2 \) for any \( i \).

(a) There exist exactly \( n! \) basic 2-cluster tilting objects \( M_w \) \( (w \in S_n) \) in \( \text{CM}(\Lambda) \).

(b) There exist exactly \( (2^n - 1) \) indecomposable rigid objects \( S_I \) \( (I \subset \{1, \cdots, n\}, I \neq \emptyset) \) in \( \text{CM}(\Lambda) \).

We shall give a sketch of proof. Thanks to Theorem 3.28, we only have to show the part (a). Fix \( w \in S_n \) and put \( \Gamma := \text{End}_\Lambda(M_w) \). Then \( \Gamma \) forms a \( \Lambda \)-order and we have a functor
\[
P_- := \text{Hom}_\Lambda(M_w, -) : \text{CM}(\Lambda) \to \text{CM}(\Gamma).
\]

Since \( M_w \) contains \( \Lambda \) as a direct summand, we have that the functor \( P_- \) is fully faithful. Moreover, Theorem 3.15 shows that the image \( P_N \) of any basic 2-cluster tilting object \( N \) in \( \text{CM}(\Lambda) \) is a basic tilting \( \Gamma \)-module of projective dimension at most one. Thus we only have to classify all basic Cohen-Macaulay tilting \( \Gamma \)-modules of projective dimension at most one. It is given by the following result.

**Theorem 4.14.** Let \( w \in S_n \). Then there exist exactly \( n! \) basic Cohen-Macaulay tilting \( \text{End}_\Lambda(M_w) \)-modules \( \text{Hom}_\Lambda(M_w, M_{w'}) \) \( (w' \in S_n) \) of projective dimension at most one.

It is interesting to compare with Corollary 4.5 where a bijection between tilting modules of projective dimension at most one and elements in an affine Weyl group is given.

The ingredient of the proof of Theorem 4.14 is the theory of ‘tilting mutation’ developed by Riedtmann-Schofield [107] and Happel-Unger [68, 69]. Although they deal with finite-dimensional algebras, their theory is valid also for our \( \Gamma \). Let us recall several results.

(a) The set of basic tilting modules of projective dimension at most one has a natural partial order defined as follows. For basic tilting \( \Gamma \)-modules \( T \) and \( U \) of projective dimension at most one, we define
\[
T \geq U
\]
if \( \text{Ext}^1_{\Gamma}(T, U) = 0 \) for any \( i > 0 \). Clearly \( \Gamma \) is the unique maximal element.

(b) Let \( T = \bigoplus_{i=1}^n T_i \) be a basic tilting \( \Gamma \)-module of projective dimension at most one. A tilting mutation of \( T \) is a tilting \( \Gamma \)-module \( U \) of projective dimension at most one such that \( T \) and \( U \) have exactly \( (n - 1) \) common indecomposable direct summands. It is shown in [107] that for any \( i \) \( (1 \leq i \leq n) \), there exists at most one tilting mutation of \( T \) obtained by replacing \( T_i \).
(c) Two tilting $\Gamma$-modules $T$ and $U$ of projective dimension at most one are neighbours with respect to the partial order if and only if they are in the relationship of tilting mutation [69].

(d) If $T > U$, then there exists a sequence $T = T_0 > T_1 > T_2 > \cdots > U$ of tilting $\Gamma$-modules of projective dimension at most one satisfying the following conditions

- $T_{i+1}$ is a tilting mutation of $T_i$,
- either $T_i = U$ for some $i$ or the sequence is infinite.

The $\Gamma$-module $P_{\Lambda}$ is projective-injective in $\text{CM}(\Gamma)$ since we have $\text{Hom}_{\Gamma}(P_{\Lambda}, \Lambda) = M_w = \text{Hom}_{\Lambda}(\Lambda, M_w)$. Consequently, any Cohen-Macaulay tilting $\Gamma$-module $T$ of projective dimension at most one has $P_{\Lambda}$ as an indecomposable direct summand, since there exists an exact sequence

$$0 \to P_{\Lambda} \to T_0 \to T_1 \to 0$$

with $T_i \in \text{add } T$, which must split. Thus the above observation (b) implies that

- any Cohen-Macaulay tilting $\Gamma$-module has at most $(n-1)$ tilting mutations which are Cohen-Macaulay.

On the other hand, $M_{w_i'}$ and $M_{w_j'}$ ($1 \leq i \leq n-1$) have exactly $(n-1)$ common indecomposable direct summands. Thus $P_{M_{w_i'}}$ has at least $(n-1)$ tilting mutations $P_{M_{w_j'}}$, with $1 \leq i \leq n-1$. Consequently, we have that

- any Cohen-Macaulay tilting $\Gamma$-module of the form $P_{M_{w_i'}}$ has exactly $(n-1)$ tilting mutations $P_{M_{w_j'}}$ ($1 \leq i \leq n-1$) which are Cohen-Macaulay.

Now we can finish our proof of Theorem 4.14 as follows. Take any basic tilting $\Gamma$-module $U$ of projective dimension at most one. Since $\Gamma > U$, we have a sequence $\Gamma = T_0 > T_1 > T_2 > \cdots > U$ satisfying the properties in (d) above. Since $\Gamma = P_{M_{w_i}}$, the above observation implies that each $T_i$ has the form $T_i = P_{M_{w_i}}$ for some $w_i \in \mathfrak{S}_n$. Since $T_i \not\simeq T_j$ for any $i \neq j$, we have $w_i \neq w_j$ for any $i \neq j$. Moreover, since $\mathfrak{S}_n$ is a finite group, the sequence must be finite. Thus $U = T_1 = P_{M_{w_i}}$ for some $i$, and the proof is completed.

We give a few remarks on the 2-CY tilted algebra $\text{End}_\Lambda(M)$ with $M$ from Theorem 4.14. Since $\text{CM}(\Lambda)$ is 2-CY and $\Omega^2 \simeq 1_{\text{CM}(\Lambda)}$ by Theorem 4.9, we have

$$\text{End}_\Lambda(M) \simeq D \text{Ext}^2_{\Lambda}(M, M) \simeq D \text{Hom}_{\Lambda}(M, M).$$

Thus we have the first assertion in the following result.

**Proposition 4.15.** Assume that $f_i \notin (x, y)^2$ for any $1 \leq i \leq n$. Then $\text{End}_\Lambda(M)$ is a finite-dimensional symmetric algebra satisfying $\tau^2 \simeq \text{id}_{\text{mod}(\text{End}_\Lambda(M))}$. Its quiver is

$$S_1 \overleftarrow{\longrightarrow} S_2 \overleftarrow{\longrightarrow} \cdots \overleftarrow{\longrightarrow} S_{n-2} \overleftarrow{\longrightarrow} S_{n-1}$$

where in addition there is a loop at $S_i$ ($1 \leq i < n$) if and only if $(f_i, f_{i+1}) \neq (x, y)$. 
This result is interesting from the viewpoint of the following conjecture of Crawley-Boevey on finite-dimensional algebras \( \Lambda \) over algebraically closed fields, where ‘only if’ part is proved in [46],

- \( \Lambda \) is representation-tame if and only if all but a finite number of indecomposable modules \( X \) of fixed dimension satisfies \( \tau X \simeq X \).

Since \( \Lambda \) is representation-wild for \( n > 4 \) [51], Theorem 3.27(c) suggests that the algebra \( \text{End}_\Lambda(M) \) should be representation-wild for \( n > 4 \). Note that there are examples of representation-wild self-injective algebras \( \Lambda \) satisfying \( \tau^3 \simeq \text{id}_{\text{mod}\Lambda} \) [22].

**Example 4.16.** We shall give examples of 2-CY tilted algebras (with \( M \) from Theorem 4.11).

(a) Let \( \Lambda \) be a simple singularity of type \( A_{2m-1} \), so

\[
\Lambda = k[[x,y]]/((x - y^m)(x + y^m)).
\]

By Theorem 4.13, there exist exactly two 2-cluster tilting objects in \( \text{CM}(\Lambda) \), and \( \text{End}_\Lambda(M) \) is isomorphic to \( k[x]/(x^m) \).

(b) Let \( \Lambda \) be a simple singularity of type \( D_{2m} \), so

\[
\Lambda = k[[x,y]]/((x - y^{m-1})(x + y^{m-1})y).
\]

By Theorem 4.13, there exist exactly six 2-cluster tilting objects in \( \text{CM}(\Lambda) \), and \( \text{End}_\Lambda(M) \) is given by the quiver

\[
\begin{array}{ccc}
\varphi & \xrightarrow{\alpha} & \beta \\
\downarrow & & \downarrow \\
\alpha & \xrightarrow{\beta} & \varphi
\end{array}
\]

with relations

\[
\varphi^{m-1} = \alpha\beta, \quad \varphi\alpha = \beta\varphi = 0.
\]

(c) Let \( \Lambda \) be a minimally elliptic curve singularity of type \( T_{3,2q+2} \) with \( q \geq 3 \), so

\[
\Lambda = k[[x,y]]/((x - y^q)(x + y^q)(x - y^2)).
\]

By Theorem 4.13, there exist exactly six 2-cluster tilting objects in \( \text{CM}(\Lambda) \), and \( \text{End}_\Lambda(M) \) is given by the quiver

\[
\begin{array}{ccc}
\varphi & \xrightarrow{\alpha} & \beta \\
\downarrow & & \downarrow \\
\alpha & \xrightarrow{\beta} & \varphi
\end{array}
\]

with relations

\[
\alpha\beta = \varphi^2, \quad \beta\alpha = \varphi^q, \quad \alpha\psi = \varphi\alpha, \quad \psi\beta = \beta\varphi.
\]

(d) Let \( \Lambda \) be a minimally elliptic curve singularity of type \( T_{2p+2,2q+2} \) with \( p, q \geq 1 \) and \( (p,q) \neq (1,1) \), so

\[
\Lambda = S/((x - y^q)(x + y^q)(x^p - y)(x^p + y)).
\]
By Theorem 4.13, there exist exactly twenty four 2-cluster tilting objects in CM(Λ), and \( \text{End}_\Lambda(M) \) is given by the quiver

\[
\begin{array}{c}
\bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet \xrightarrow{\gamma} \bullet \xrightarrow{\delta} \bullet
\end{array}
\]

with relations

\[
\alpha \beta = \varphi^p, \quad \delta \gamma = \psi^q, \quad \varphi \alpha = \alpha \gamma \delta, \quad \beta \varphi = \gamma \delta \beta, \quad \psi \delta = \delta \beta \alpha, \quad \gamma \psi = \beta \alpha \gamma.
\]

Note that the algebras in (c) and (d) appear in Erdmann’s list of algebras of quaternion type \[55\].

Finally we give an application to Krull dimension three. Again, let \( \Lambda' := k[[x, y, u, v]]/(f(x, y) + uv) \). For \( w \in S_n \), we put

\[
U^w_i := (u, f^1_w, f^2_w, \ldots, f^n_w) \subset \Lambda' \quad \text{and} \quad M'_w := \bigoplus_{i=1}^n U^w_i.
\]

For a non-empty subset \( I \) of \( \{1, \ldots, n\} \), we put

\[
U_I := (u, \prod_{i \in I} f_i) \subset \Lambda'.
\]

We have the following results from Theorems 4.10, 4.13, 3.19 and 3.15.

**Theorem 4.17.** Assume that \( f_i \notin (x, y)^2 \) for any \( i \).

(a) There exist exactly \( n! \) basic 2-cluster tilting objects \( M'_w \) \( (w \in S_n) \) in CM(\( \Lambda' \)).

(b) There exist exactly \( (2^n - 1) \) indecomposable rigid objects \( U_I \) \((I \subset \{1, \ldots, n\}, I \neq \emptyset) \) in CM(\( \Lambda' \)).

(c) \( \Lambda' \) has non-commutative crepant resolutions \( \text{End}_{\Lambda'}(M'_w) \) \((w \in S_n)\), which are mutually derived equivalent 3-CY algebras.

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