An entwinement of algebraic topological and variational method to study the \textit{Prandtl Batchelor} problem

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Abstract

In this paper we study the existence of nontrivial weak solution to a \textit{Prandtl-Batchelor} type free boundary value elliptic problem driven by a power nonlinearity. Topics from algebraic topology will be used to establish the existence of a solution to the approximating problem, whereas, the variational technique will be used to fix the claim of existence of a solution to the main problem. In the process, a couple of classical results were also improved to suit the purpose of establishing the existence of a nontrivial solution.

\textbf{Keywords}: Dirichlet free boundary value problem, Sobolev space, Morse relation, cohomology group.

\textbf{AMS Classification}: 35J35, 35J60.

1. Introduction

We will investigate the existence of solution to the following \textit{free boundary value} problem.

\begin{equation}
\begin{aligned}
-\Delta_p u &= \lambda \chi_{\{u>1\}} (u-1)^{q-1}, \text{ in } \Omega \setminus H(u), \\
|\nabla u^+|^p - |\nabla u^-|^p &= \frac{p}{p-1}, \text{ in } H(u) \\
u &= 0, \text{ on } \partial \Omega. 
\end{aligned}
\end{equation}

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Here, \( \lambda > 0 \) is a parameter, \((u - 1)_+ = \max\{u - 1, 0\}\) and 
\[ H(u) = \partial\{u > 1\}. \]
Also \( \nabla u^\pm \) are the limits of \( \nabla u \) from the sets \( \{u > 1\} \) and \( \{u \leq 1\} \) respectively. The domain \( \Omega \subset \mathbb{R}^N (N \geq 2) \) is bounded with a sufficiently smooth boundary \( \partial \Omega \) with the exterior sphere condition. The relation between the exponents are assumed in the order \( 2 \leq p \leq q - 1 \), with \( q < p^* = \frac{Np}{N-p} \). The solution(s) satisfy the free boundary condition in the following sense: for all \( \vec{\phi} \in C^1_0(\mathbb{R}^N) \) such that \( u \neq 1 \) a.e. on the support of \( \vec{\phi} \),
\[
\lim_{\epsilon^+ \to 0} \int_{u=1+\epsilon^+} \left( \frac{p}{p-1} - |\nabla u|^p \right) \vec{\phi} \cdot \hat{n} dS - \lim_{\epsilon^- \to 0} \int_{u=1-\epsilon^-} |\nabla u|^p \vec{\phi} \cdot \hat{n} dS = 0, \tag{1.2}
\]
where \( \hat{n} \) is the outward drawn normal to \( \{1 - \epsilon^- < u < 1 + \epsilon^+\} \). Note that the sets \( \{u = 1 \pm \epsilon^\pm\} \) are smooth hypersurfaces for almost all \( \epsilon^\pm > 0 \) by the Sard’s theorem. The limit above in (1.2) is taken by running such \( \epsilon^\pm > 0 \) towards zero.

The readers should note that while attempting this problem we have encountered many theorems that are still open to attempt and hence the proposed problem has not been entirely solved for all \( p \). A major reason behind this failure is the absence of \( C^2 \) regularity of solutions to PDEs involving \( p \)-Laplacian operators. However, we have improved the results for the case of \( p \geq 2 \) wherever it was possible to prove them. For instance, the derivation of the free boundary condition has been done for \( p \geq 2 \) whereas the proof of the monotonicity lemma in Lemma 4.2 has been done from the case of \( p = 2 \). Further, for the interested readers we have conjectured in Conjecture 4.3 that the monotonicity result may hold if the monotonicity function is chosen in a particular way.

The trend of applying the topics from algebraic topology in elliptic PDEs is not very old. We refer the readers to the works of \[7, 10, 12, 22, 23\] and the references therein. A rich literature survey has been done in the book due to Perera et al. \[15\] where the author has discussed problems of several variety involving the \( p \)-Laplacian operators which could be studied using the Morse theory. The motivation for the current work has been drawn from the work due to Perera \[17\] who has considered a sublinear problem. We also refer the readers to the latest work due to Choudhuri-Repovš \[8\]. The treatment used to address the existence of at least one (or two) solution(s) to the approximating problem may be classical (section 3, Theorems 3.3 and 3.5) but the result concerning the regularity of the free boundary is very new and the question of existence of solution to the problem (1.1) has not been answered till now (section 4, Lemma 4.1), to the best of my knowledge. A result due to Alt-Caffarelli \[1\] (section 4, Lemma 4.2) were improved to the best possible extent to suit the purpose of the problem in this paper. Most importantly we have included a short subsection 2.1 on the fundamentals of algebraic topology for the benefit of the readers.
1.1 A physical motivation

Consider the problem

\[-\Delta u = \lambda \chi_{\{u>1\}}(x), \text{ in } \Omega \setminus H(u),\]
\[|\nabla u^+|^2 - |\nabla u^-|^2 = 2, \text{ in } H(u)\]
\[u = 0, \text{ on } \partial \Omega.\]  \hspace{1cm} (1.3)

This is the well known Prandtl-Batchelor free boundary value problem, where the phase \(\{u > 1\}\) is a representation of the vortex patch bounded by the vortex line \(u = 1\) in a steady fluid flow for \(N = 2\) (refer Batchelor [2, 3]). Thus the current problem defined in (1.1) is a more generalized version of (1.3). For a more physical application to this problem we direct the reader’s attention to the work due to Caflisch [4], Eli- crat and Miller [9].

Another instance of occurrence of such a phenomena is in the non-equilibrium system of melting of ice. In a given block of ice, the heat equation can be solved with a given set of appropriate initial/boundary conditions in order to determine the temperature. However, if there exists a region of ice in which the temperature is greater than the melting point of ice, this subdomain will be filled with water. The boundary thus formed due to the ice-water interface is controlled by the solution of the heat equation. Thus encountering a free boundary in the nature is not unnatural. The problem in this paper is a large enough generalization to this physical phenomena which besides being a new addition to the literature can also serve as a note to bridge the problems in elliptic PDEs with algebraic topology.

2. Preliminaries

We begin by giving the relevant definitions and results besides defining the function space which will be used very frequently in the article.

2.1 Basics of algebraic topology

We begin by referring to the readers the fact that certain topological objects can be understood better by associating a free abelian group known as the Homology groups. As an illustration, we will discuss the homology groups of a Torus.
2.2 Homology group explained through a simple and well known topological space - torus

To make an honest simplical complex out of a torus means having to divide the torus $\mathbb{T}$ to fairly many triangles as in the Figure 1 below. Instead, we think more combinatorially by opting for the following diagram in Figure 2 below represented in a planar form with a square and its edges that is suitably identified. The vertices are all identified by $x$, the directed edges by $a, b$, the oriented triangles by $A, B$. By a $k$-simplex in $\mathbb{R}^l$, we will mean the convex-hull of an ordered set of $(k + 1)$ vectors. Furthermore, by a group $G$ we will always mean $\mathbb{R}$ or $\mathbb{Z}$. We now set up the chain complexes which is as
follows:
\[ 0 \xrightarrow{\partial_3} C_2(T) \xrightarrow{\partial_2} C_1(T) \xrightarrow{\partial_1} C_0(T) \xrightarrow{\partial_0} 0 \]

where

\( C_0 \): talks about the 0 simplexes. In this case it is just the free abelian group generated by the vertex \( x \), i.e. \( \langle x \rangle = \mathbb{Z} \).

\( C_1 \): talks about the 1 simplexes. In this case it is just the free abelian group generated by the edges \( a, b, c \), i.e. \( \langle a, b, c \rangle = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \).

\( C_2 \): talks about the 2 simplexes. In this case it is just the free abelian group generated by the two triangles \( A, B \), i.e. \( \langle A, B \rangle = \mathbb{Z} \oplus \mathbb{Z} \).

Also \( \partial_i \)s are the boundary maps which are also homomorphisms between the abelian groups corresponding to the simplices defined as (i) \( \partial_0 = 0 \); (ii) \( \partial_1 a = \partial_1 b = \partial_1 c = 0 \); (iii) \( \partial_2 A = \partial_2 B = a + b + c \); (iv) \( \partial_3 = 0 \). For simplicity we drop the indices \( i \) and will represent it as just \( \partial \).

**Generation of the Homology groups:**

(i) The **zero homology group** is denoted by \( H_0(T) \) and is defined as \( \frac{Z_0}{B_0} \), where \( Z_0 \) is called the 0-cycle and is the ker \( \partial_0 \), \( B_0 \) is called the 0-boundary and is the Im \( \partial_1 \). Hence \( H_0(T) \simeq \langle x \rangle / \langle 0 \rangle \simeq \mathbb{Z} \). Note that we have used \( \partial_0 \).

(ii) The **first homology group** is denoted by \( H_1(T) \) which is defined as \( \frac{Z_1}{B_1} \). Similar to the case of zero homology, \( Z_1 \) is called the 1-cycle and is the ker \( \partial_1 \), \( B_1 \) is called the 1-boundary and is the Im \( \partial_2 \). Hence \( H_1(T) \simeq \langle a, b, c \rangle / \langle a + b + c \rangle \simeq \langle a + b + c, b, c \rangle / \langle a + b + c \rangle \simeq \langle b, c \rangle \simeq \mathbb{Z}^2 \).

(iii) The **second homology group** is denoted by \( H_2(T) \) which is defined as \( \frac{Z_2}{B_2} \). As above, \( Z_2 \) is called the 2-cycle and is the ker \( \partial_2 \), \( B_2 \) is called the 2-boundary and is the Im \( \partial_3 \). Note that a prototype member of \( C_2 \) is of the form \( \alpha A + \beta B \) for \( \alpha, \beta \in \mathbb{Z} \). Therefore, \( 0 = \partial(\alpha A + \beta B) = \alpha \partial A + \beta \partial B = \alpha(a + b + c) + \beta(a + b + c) \). Thus \( \alpha = -\beta \) and hence \( Z_2 = \langle A - B \rangle \). Hence \( H_2(T) \simeq \langle A - B \rangle / \langle 0 \rangle \simeq \mathbb{Z} \).

Therefore we have

(i) \( H_0(T) = \mathbb{Z} \), which corresponds to the fact that \( \mathbb{Z} \) is a connected topological space.

(ii) \( H_1(T) = \mathbb{Z} \oplus \mathbb{Z} \), representing to independent circles that generates the torus.

(iii) \( H_2(T) = \mathbb{Z} \), representing the fact that there exists a two dimensional hole.

(iv) \( H_m(T) = 0 \), for \( m \geq 3 \).
With this short information on homology, we now move on to recall some of the results from algebraic topology that will be used to study the considered problem (1.1). Throughout this discourse we let $X$ to be a topological space and $A \subset X$ be a topological subspace. A fundamental tool that will be used to work with, namely the deformation retraction (refer Definition 5.3.2) and the homology theory (refer Definition 6.1.12), can be found in the book by Papageorgiou et al. [21]. We now recall the definition of deformation retraction and Homology group on a pair of topological spaces.

**Definition 2.1.** A continuous map $F : X \times [0, 1] \to X$ is a deformation retraction of a space $X$ onto a subspace $A$ if, for every $x \in X$ and $a \in A$, $F(x, 0) = x$, $F(x, 1) \in A$, and $F(a, 1) = a$.

**Definition 2.2.** A homology group on a family of pairs of spaces $(X, A)$ consists of:

1. A sequence $\{H_k(X, A)\}_{k \in \mathbb{N}_0}$ of abelian groups is known as homology group for the pair $(X, A)$ (note that for the pair $(X, \phi)$, we write $H_k(X)$, $k \in \mathbb{N}_0$). Here $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

2. To every map of pairs $\varphi : (X, A) \to (Y, B)$ is associated a homomorphism $\varphi^* : H_k(X, A) \to H_k(Y, B)$ for all $k \in \mathbb{N}_0$.

3. To every $k \in \mathbb{N}_0$ and every pair $(X, A)$ is associated a homomorphism $\partial : H_k(X, A) \to H_{k-1}(A)$ for all $k \in \mathbb{N}_0$.

These items satisfy the following axioms.

(A1) If $\varphi = id_X$, then $\varphi_* = id|_{H_k(X, A)}$.

(A2) If $\varphi : (X, A) \to (Y, B)$ and $\psi : (Y, B) \to (Z, C)$ are maps of pairs, then $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$.

(A3) If $\varphi : (X, A) \to (Y, B)$ is a map of pairs, then $\partial \circ \varphi_* = (\varphi|_A)_* \circ \partial$.

(A4) If $i : A \to X$ and $j : (X, \phi) \to (X, A)$ are inclusion maps, then the following sequence is exact

$$\ldots \to H_k(A) \xrightarrow{i_*} H_k(X) \xrightarrow{j_*} H_k(X, A) \xrightarrow{\partial} H_{k-1}(A) \to \ldots$$

Recall that a chain $\ldots \to C_k(X) \xrightarrow{\partial_{k+1}} C_{k+1}(X) \xrightarrow{\partial_k} C_k(X) \xrightarrow{\partial_{k-1}} C_{k-1}(X) \xrightarrow{\partial_{k-2}} \ldots$ is said to be exact if $im(\partial_{k+1}) = ker(\partial_k)$ for each $k \in \mathbb{N}_0$.

(A5) If $\varphi, \psi : (X, A) \to (Y, B)$ are homotopic maps of pairs, then $\varphi_* = \psi_*$.
(A_6) (Excision): If \( U \subseteq X \) is an open set with \( \bar{U} \subseteq \text{int}(A) \) and \( i : (X \setminus U, A \setminus U) \rightarrow (X, A) \) is the inclusion map, then \( i_* : H_k(X \setminus U, A \setminus U) \rightarrow H_k(X, A) \) is an isomorphism.

(A_7) If \( X = \{\ast\} \), then \( H_k(\ast) = 0 \) for all \( k \in \mathbb{N} \).

The deformation lemma which will be quintessential in computing the homology groups can be found in the Lemma 5.5.1, [18]. We will be using the *Palais-Smale* condition which is a special type of compactness that can be referred to the Definition 5.5.1 in the book [18]. Following is the definition of Morse index which will be used in the subsequent sections.

**Definition 2.3.** Morse index of a functional \( J : V \rightarrow \mathbb{R} \) is defined to be the maximum subspace of \( V \) such that \( J'' \), the second Fréchet derivative, is negative definite on it.

### 2.3 Space description

We begin by defining the standard Lebesgue space \( L^p(\Omega) \) for \( 1 \leq p < \infty \) as

\[
L^p(\Omega) = \left\{ u : \Omega \rightarrow \mathbb{R} : u \text{ is measurable and } \int_\Omega |u|^p dx < \infty \right\}
\]

endowed with the norm \( ||u||_p = (\int_\Omega |u|^p dx)^{\frac{1}{p}} \). We will define the Sobolev space as

\[
W^{1,p}(\Omega) = \{ u \in L^p(\Omega) : \nabla u \in (L^p(\Omega))^N \}
\]

with the norm \( ||u||^{p}_{1,p} = ||u||_p + ||\nabla u||_p \). We further define

\[
W^{1,p}_0(\Omega) = \{ u \in W^{1,p}(\Omega) : u = 0 \text{ on } \partial \Omega \}.
\]

The associated norm to the space will be denoted by \( ||u|| = ||\nabla u||_p \). With these norms, \( L^p(\Omega), W^{1,p}(\Omega) \) and \( W^{1,p}_0(\Omega) \) are separable, reflexive Banach spaces ([18]). The embedding results pertaining to the Sobolev spaces can be referred to the Lemma 2.4.1 in [18].

### 3. The way to tackle the problem using Morse theory

We at first define an energy functional associated to the problem in [11] which is as follows.

\[
I(u) = \int_\Omega |\nabla u|^p dx + \int_\Omega \chi_{\{u>1\}}(x)dx - \lambda \int_\Omega \frac{(u-1)^q}{q}dx.
\]
However, this functional is not even differentiable and hence poses serious issues as far as the application of variational theorems are concerned. Thus we approximate $I$ using the following functionals that varies with respect to a parameter $\alpha > 0$. This method is adapted from the work of Jerison-Perera [14]. We define a smooth function $g : \mathbb{R} \to [0, 2]$ as follows:

$$g(t) = \begin{cases} 0, & \text{if } t \leq 0 \\ g(t) > 0, & \text{if } 0 < t < 1 \\ 0, & \text{if } t \geq 1 \end{cases}$$

and $\int_0^1 g(t) dt = 1$. We further let $G(t) = \int_0^t g(t) dt$. Clearly, $G$ is smooth and nondecreasing function such that

$$G(t) = \begin{cases} 0, & \text{if } t \leq 0 \\ 0 < G(t) < 1, & \text{if } 0 < t < 1 \\ 1, & \text{if } t \geq 1 \end{cases}$$

We thus define

$$I_\alpha(u) = \int_\Omega \frac{|\nabla u|^p}{p} dx + \int_\Omega G \left( \frac{u - 1}{\alpha} \right) dx - \lambda \int_\Omega \left( \frac{u - 1}{q} \right)^q dx.$$

This functional $I_\alpha$, is of at least $C^2$ class and hence

$$\langle I''_\alpha(u)v, w \rangle = \int_\Omega [||\nabla u|^{p-2}\nabla v \cdot \nabla w + (p-2)|\nabla u|^{p-4}(\nabla u \cdot \nabla v)(\nabla u \cdot \nabla w)] dx$$

$$+ \int_\Omega g' \left( \frac{u - 1}{\alpha} \right) vwdx - \lambda \int_\Omega \left( \frac{u - 1}{q} \right)^{q-2} vwdx.$$

Following is an important result in Morse theory which explains the effect of the associated Homology groups on the set $K_{J,(-\infty,a]} = \{ x \in V : J(x) \leq a \}$.

**Theorem 3.1.** Let $J \in C^2(V, \mathbb{R})$ satisfy the Palais-Smale condition and let ‘$a$’ be a regular value of $J$. Then if, $H_*(V, J^a) \neq 0$, implies that $K_{J,(-\infty,a]} \neq \emptyset$.

**Remark 3.2.** Before we apply the Morse lemma we recall that for a Morse function the following holds

1. $H_*(J^*, J^c \setminus \text{Crit}(J,c)) = \bigoplus_j H_*(J^c \cap N_j, J^c \cap N_j \setminus \{x_j\})$,

where $\text{Crit}(J,c) = \{ x \in V : J(x) = c, J'(x) = 0 \}$, $N_j$ is a neighbourhood of $x_j$.  


2. 
\[ H_k(J^c \cap N, J^c \cap N \setminus \{x\}) = \begin{cases} \mathbb{R}, & k = m(x) \\ 0, & \text{otherwise} \end{cases} \]

where \( m(x) \) is a Morse index of \( x \), a critical point of \( J \).

3. Further
\[ H_k(J^a, J^b) = \bigoplus \{ i : m(x_i) = k \} \mathbb{R} = \mathbb{R}^{m_k(a,b)} \]
where \( m_k(a,b) = n(\{ i : m(x_i) = k, x_i \in K_{J(a,b)} \}) \). Here \( n(S) \) denotes the number of elements present in the set \( S \).

4. Morse relation
\[ \sum_{u \in K_{J(a,b)}} \sum_{k \geq 0} \dim(C_k(J, u))t^k = \sum_{k \geq 0} \dim(H_k(J^a, J^b))t^k + (1 + t)Q_t \]
for all \( t \in \mathbb{R} \). Here \( Q_t \) is a nonnegative polynomial in \( \mathbb{N}_0[t] \).

**Theorem 3.3.** The functional \( I_\alpha \) has at least one nontrivial critical point when \( 0 < \lambda \leq \lambda_1 \), \( \lambda_1 \) being the first eigenvalue of \( (-\Delta_p) \).

**Proof.** We observe that \( I_\alpha(tu) \to -\infty \) as \( t \to \infty \). Furthermore, a key observation here is that there exists \( (r, \lambda) \) sufficiently small positive numbers such that
\[ I_\alpha(u) \geq A > 0 \]
whenever \( \|u\| = r \). We choose \( \epsilon > 0 \) such that \( c = \epsilon \) is a regular value of \( I_\alpha \). Thus, \( I_\alpha^c \) is not path connected since it has at least two path connected components namely in the form of a neighbourhood of 0 and a set \( \{ u : \|u\| \geq R \} \) for \( R \) sufficiently large. Therefore, from the theory of homology groups we get that \( \dim(H_0(I_\alpha^c)) \geq 2 \); ‘dim’ denoting the dimension of the Homology group. From the Definition 2.2-(A4) of homology group, let us consider the following exact sequence
\[ \ldots \to H_1(W_0^{1,p}(\Omega), I_\alpha^c) \xrightarrow{\partial_1} H_0(I_\alpha^c, \emptyset) \xrightarrow{\text{in}} H_0(W_0^{1,p}(\Omega), \emptyset) \to \ldots \]
We know that \( \dim(H_0(W_0^{1,p}(\Omega), \emptyset)) = 1 \) and \( \dim(H_0(I_\alpha^c)) \geq 2 \). Moreover, due to the exactness of the sequence we conclude that \( \dim H_1(W_0^{1,p}(\Omega), I_\alpha^c) \geq 1 \). Thus by the Theorem 3.1 we have \( K_{I_\alpha,(-\infty,\epsilon)} \neq \emptyset \).

Suppose that the only critical point to \( \text{(1.1)} \) is \( u = 0 \) at which the energy of the functional \( I_\alpha \) is also 0. Thus from the discussion above and the Remark (3.2)-(4) we have from the Morse relation the following identity over \( \mathbb{R} \)
\[ 1 = t + P(t) + (1 + t)Q_t, \]
\( P \) being a power series in any \( t \in \mathbb{R} \), \( Q_t \geq 0 \). This leads to a contradiction since a nonzero polynomial can never be an identity on \( \mathbb{R} \). Thus there exists at least one \( u \neq 0 \) which is a critical point to \( I_\alpha \) whenever \( \lambda \leq \lambda_1 \).

**Remark 3.4.** Without any demonstration we quote that a standard application of the Moser iteration technique leads to the conclusion that a solution \( u \) of (1.1) is in \( L^\infty(\Omega) \).

We refer the reader to the Definition 1.3 in [10] for the definition of *Krasnoselskii genus.* To each closed and symmetric subsets \( M \) of \( W^{1,p}_0(\Omega) \) with the Krasnoselskii genus \( \gamma(M) \geq k \), define

\[
\lambda_k = \inf_{M \in \mathcal{F}_k} \sup_{u \in M} I_\alpha(u).
\]

Here \( \mathcal{F}_k = \{ M \subset W^{1,p}_0(\Omega), \text{ closed and symmetric } : \gamma(M) \geq k \} \). A natural question at this point will be to ask if the same conclusion as in Theorem 3.3 can be drawn when \( \lambda_k < \lambda \leq \lambda_{k+1} \). We will define \( \lambda_0 = 0 \). The next theorem answers this question.

**Theorem 3.5.** The problem in (1.1) has at least one nontrivial solution when \( \lambda_i < \lambda \leq \lambda_j, \lambda_i, \lambda_j \) being as defined above, for some \( i, j \).

**Proof.** We at first show that \( H_k(W^{1,p}_0(\Omega), I_\alpha^{-a}) = 0 \) for all \( k \geq 0 \). Towards this we pick \( u \in \{ v : \| v \| = 1 \} = \partial B^\infty \), where \( B^\infty = \{ v : \| v \| \leq 1 \} \). Therefore there exists \( t_0 > 0 \) such that \( I_\alpha(tu) = \int_\Omega \frac{|v(tu)|^p}{p} dx + \int_\Omega G \left( \frac{|u-1|}{\alpha} \right) dx - \lambda \int_\Omega \frac{(tu-1)^q}{q} dx < -a < 0 \) for all \( t \geq t_0 \). We observe from the Remark 3.4 that for \( t > 0 \) small enough, the sign of \( tu - 1 \) becomes negative, whence, there exists \( \tilde{t} > 0 \) such that \( I_\alpha'(\tilde{tu}) > 0 \). Thus, there exists \( t(u) \) such that \( I_\alpha'(t(u)u) = 0 \) by the continuity of \( I_\alpha' \). We can thus say that there exists a \( C^1 \)-function \( T : W^{1,p}_0(\Omega) \setminus \{ 0 \} \to \mathbb{R}^+ \). We now define a standard deformation retract \( \eta \) of \( W^{1,p}_0(\Omega) \setminus B_{B^R}(0) \) into \( I_\alpha^{-a} \) as follows:

\[
\eta(s, u) = \begin{cases} 
(1-s)u + sT \left( \frac{u}{\| u \|} \right) \frac{u}{\| u \|}, & \| u \| \geq R, I_\alpha(u) \geq -a \\
u, & I_\alpha(u) \leq -a.
\end{cases}
\]

It is not difficult to see that \( \eta \) is a \( C^1 \) function over \( [0, 1] \times W^{1,p}_0(\Omega) \setminus B_{B^R}(0) \). On using the map \( \Theta(s, u) = \frac{u}{\| u \|} \), for \( u \in W^{1,p}_0(\Omega) \setminus B_{B^R}(0) \) we claim that \( H_k(W^{1,p}_0(\Omega), W^{1,p}_0(\Omega) \setminus B_{s}(0)) = H_k(\mathbb{B}^\infty, S^\infty) \) for all \( k \geq 0 \). This is because, \( H_k(\mathbb{B}^\infty, S^\infty) \cong H_k(\ast, 0) \). From the elementary computation of homology groups with two 0-dimensional simplices it is easy to see that \( H_k(\ast, 0) = \{ 0 \} \) for each \( k \geq 0 \). Therefore, from the Morse relation in the Remark (3.2) and the result above, we have for \( b > 0 \)

\[
\sum_{u \in K_{[s, -a, \infty)}} \sum_{k \geq 0} \dim(C_k(I, u)) t^k = tm(u) + p(t)
\]
where $m(u)$ is the Morse index of $u$ and $\mathcal{P}(t)$ contains the rest of the powers of $t$ corresponding to the other critical points, if any. The Morse index is finite because of the following reason: The mountain pass geometry around 0 allows in establishing a maxima (say $u_0$) along with the assumption $\lambda < C^{−q\frac{p}{q-1}}\|u\|^{p-q}$. Owing to $u_0$ being a maxima, we have $I′′_{\alpha}(u_0) < 0$ which necessarily requires $\lambda > C^{−q\frac{p}{q-1}}\|u\|^{p-q}$. Thus we have

$$C^{−q\frac{p}{q-1}}\|u\|^{p-q} < \lambda < C^{−q\frac{p}{q-1}}\|u\|^{p-q}.$$ 

This implies that $\lambda_i < \lambda < \lambda_j$ for some $i, j \in \mathbb{N}_0$. On further using the Morse relation we obtain

$$t^{m(u)} + \mathcal{P}(t) = (1 + t)Q_t.$$ 

(3.2)

This is because the $H_k$s are all trivial groups. Hence, $Q_t$ either contains $t^{m(u)}$ or $t^{m(u)−1}$ or both. Thus there exists at least one nontrivial $u \in K_{I_{\alpha},[−a,∞)}$ with $m(u) < \infty$. □

Remark 3.6. As a note if $0 < \lambda \leq \lambda_{k+1}$, then there exists at least $k$ solutions to the equation [1.1].

Remark 3.7 (see Iannizzotto et al. [13]). Suppose $D$ is a bounded domain with a sufficiently smooth boundary, then the problem

$$−\Delta_p u = f \text{ in } D$$

(3.3)

has a $C^{1,a}_{\text{loc}}$ solution whenever $f$ is bounded and nothing more can be expected, irrespective of the smoothness of $f$.

4. Existence of solution to the main problem [1.1] and smoothness of the boundary $\partial\{u > 1\}$

The following lemma has been proved for $p = 2$.

**Lemma 4.1.** Let $\alpha_j \to 0$ ($\alpha_j > 0$) as $j \to \infty$ and $u_j$ be a critical point of $I_{\alpha_j}$. If $(u_j)$ is bounded in $W^{1,2}_0(\Omega) \cap L^\infty(\Omega)$, then there exists $u$, a Lipschitz continuous function, on $\bar{\Omega}$ such that $u \in W^{1,2}_0(\Omega) \cap C^{1,a}(\bar{\Omega} \setminus H(u))$ and a subsequence (still denoted by $(u_j)$) such that

(i) $u_j \to u$ uniformly over $\bar{\Omega}$,

(ii) $u_j \to u$ locally in $C^1(\bar{\Omega} \setminus \{u = 1\})$,

(iii) $u_j \to u$ strongly in $W^{1,2}_0(\Omega)$,
\( I(u) \leq \lim \inf I_{\alpha_j}(u_j) \leq \lim \sup I_{\alpha_j}(u_j) \leq I(u) + |\{u = 1\}|, \) i.e. \( u \) is a nontrivial function if \( \lim \inf I_{\alpha_j}(u_j) < 0 \) or \( \lim \sup I_{\alpha_j}(u_j) > 0. \)

Furthermore, \( u \) is a \( C^{1,\alpha} \)-weak solution of
\[
-\Delta u = \lambda \chi_{\{u > 1\}}(x)(u - 1)^{q-1}
\]
in \( \Omega \setminus H(u) \), the free boundary condition is satisfied in the generalized sense and vanishes continuously on \( \partial \Omega \). In the case of \( u \) being nontrivial, then \( u > 0 \) in \( \Omega \), the set \( \{u < 1\} \) is connected and the set \( \{u > 1\} \) is nonempty.

An important result that will be used to pass the limit in the proof of the Lemma 4.1 is the following theorem which is in line to the theorem due to CAFFARELLI ET AL. in [6, Theorem 5.1].

**Lemma 4.2.** Let \( u \) be a Lipschitz continuous function on the unit ball \( B_1(0) \subset \mathbb{R}^N \) satisfying the distributional inequalities
\[
\pm \Delta u \leq \left( \frac{1}{\alpha} \chi_{\{|u-1|<\alpha\}}(x) \mathcal{H}(|\nabla u|^2) + A \right)
\]
for constants \( A > 0, 0 < \alpha \leq 1 \) and \( \mathcal{H} \) is a continuous function such that \( \mathcal{H}(t) = o(t^\rho) \) near infinity. Then there exists a constant \( C > 0 \) depending on \( N, A \) and \( B = \int_{B_1(0)} u^2 dx \), but not on \( \alpha \), such that
\[
\max_{x \in B_1(0)} \{|\nabla u(x)|\} \leq C.
\]

**Proof.** In order to prove the result, we first prove a couple of results that will be applied.

We first observe that if \( w_\pm \geq 0 \), continuous function on \( B_1 \), and \( \Delta w_\pm \geq -1 \) distributionally, with \( w^+(x)w^-(x) = 0 \) for all \( x \in B_1 \), then the monotonicity function
\[
\Phi(r) := \left( \frac{1}{r^2} \int_{B_r} \frac{|\nabla(w^+)|^2}{|x|^{N-2}} dx \right) \left( \frac{1}{r^2} \int_{B_r} \frac{|\nabla(w^-)|^2}{|x|^{N-2}} dx \right)
\]
obeyes the following estimate:
\[
\Phi(r) \leq C \left( 1 + \int_{B_1} \frac{|\nabla(w^+)|^2}{|x|^{N-2}} dx + \int_{B_1} \frac{|\nabla(w^-)|^2}{|x|^{N-2}} dx \right)^2,
\]
for each \( 0 < r \leq 1 \) if \( \Delta w_\pm \geq -1 \). This can be concluded from ALT-CAFFARELLI [1].

for each \( 0 < r \leq 1 \). We now prove that if \( w \geq 0 \) is a continuous function, \( \Delta w \geq -1 \) on
then there exists $C > 0$ such that $\int_{B_1} \frac{|\nabla w|^2}{|x|^{N-2}} \, dx \leq C + C \int_{B_2} w^2 \, dx$.
Let $\varphi \in C_0^\infty (B_{3/2})$ such that $0 \leq \varphi \leq 1$ in $B_2$ and $\varphi \equiv 1$ in $B_1$. Then
\[
\int \frac{\varphi}{|x|^{N-2}} |\nabla w|^2 \, dx \leq \int \frac{\varphi}{|x|^{N-2}} (\Delta w^2 + 2w) \, dx
\]
\[
= \int 2u \frac{\varphi}{|x|^{N-2}} + w^2 \Delta \left( \frac{\varphi}{|x|^{N-2}} \right) \, dx
\]
\[
\leq C + C \int_{B_2 \setminus B_1} w \, dx,
\]
where $C > 0$ is a dimensional constant that also depends on $\text{esssup}_{B_2} |\nabla w|$. Note that the domain being a ball, it automatically obeys the exterior ball condition thus yielding that $w \in C^2 (B_2)$ (see Theorem 15.19, [11]).
From the given inequation (4.1), we note that $\Delta (u - \alpha)^+ \geq -A$ and $\Delta (u + \alpha)^- \geq -A$.
Choose $x_0 \in B_{3/4}$ such that $|u(x_0)| < \alpha$. Such a point exists, since if no such points exists then $|\Delta u| \leq A$ on $B_{3/4}$. By the remark 3.7, one can guarantee that $u \in C^{1,a}_{\text{loc}}$ and the result follows. By (4.2) and (4.3) we have
\[
\int_{B_{\alpha/15}(x_0)} |x - x_0|^{2-N} |\nabla u|^2 \, dx \leq A_1 B(\alpha/15)^2.
\]
We further use an application of the Green’s identity and the mean value theorem (see Ruzhansky-Surugan [24, Propositions 2.2, 2.9] for the Green’s first and the second identity on a continuous function for which $\Delta \psi$ is bounded we obtain
\[
\int_{B_r} \psi \, dx \leq C \psi(0) + \int_{B_r} |x|^{2-N} (\Delta \psi)^+ \, dx
\]
for any radius $r$, $1/100 \leq r \leq 1$.
Given that $u$ is a Lipschitz continuous function on the unit ball $B_1(0) \subset \mathbb{R}^N$, so $u$ is also bounded in the unit ball say by a constant $M_0$. In addition, $u$ is also differentiable a.e. in $B_1(0)$. We will prove the result stated in the lemma for $u_+$, as the proof with $u_-$ will follow suit. For $\alpha > 0$, denote $v(x) = \frac{15}{\alpha} u_+ (x_0 + \alpha x/15)$ and
\[
v_1 = v + \max_{B_{3/4}} \{v^-\}.
\]
The main goal is to establish that $|\nabla v|$ is bounded on a small ball around the origin.

\textbf{Step 1}: We first prove that $v^+$ has an $L^\infty$ bound. Define
\[
v_1 = (v - 15 - A|x|^2)^+.
\]
We note from (4.1) that $\Delta v_1 \leq 0$ on $B_1$ on which $v_1 > 0$ and $v_1$ is bounded. Therefore,
\[
\Delta v_1^2 = 2 \left[ |\nabla v_1|^2 + v_1 \Delta v_1 \right]
\]
\[
\leq 2|\nabla v_1|^2 \leq 4|\nabla (v - 15)|^2 + 4A_0^2.
\]
On taking $\psi = v_1^2$ we obtain
\[
\int_{B_1} v_1^2 \, dx \leq A_2 B_1 + A_2 \tag{4.7}
\]
for some constant $A_2$ depending on $A, A_1, N$. However, from (4.1) we get $\Delta_p v_1 \geq -(2N+1)A$. Therefore,
\[
\max_{B_{1/2}} v_1 \leq \int_{B_1} v_1 \, dx + A_3 \leq A_4 B_1^{1/2} + A_4.
\]

Step 2: We now prove that $v^-$ has an $L^\infty$ bound.

Let
\[
C_1 := \max_{B_{1/2}} v; \ v_2 := C_1 - v.
\]

Apparently the value of $C_1$ depends on $B$ from Step 1 and $v_2 \geq 0$. Let $A_5 > 2A_2$ such that
\[
H(t) \leq A_5(1 + t^2/2),
\]
\[
\pm \Delta v_2 \leq \frac{A_5}{2}(1 + |\nabla v_2|^2)\chi_{\{v_2 < C_1 + 10\}} + A. \tag{4.8}
\]

Define
\[
\phi(t) := \begin{cases} 
1 - e^{-A_5 t}, & \text{if } 0 \leq t \leq C_1 + 10 \\
 at + b, & \text{if } C_1 + 10 \leq t < \infty \end{cases} \tag{4.9}
\]

The choice of $a, b$ has been made in such a way that $\phi, \phi'$ are continuous. This implies that
\[
a(C_1 + 10) + b = 1 - e^{-A_5(C_1 + 10)}, \ A_5 e^{-A_5(C_1 + 10)} = a.
\]

If $v_2 < C_1 + 10$, then we have
\[
\Delta \phi(v_2) = \phi'(v_2)\Delta v_2 + \phi'(v_2)\phi''(v_2)|\nabla v_2|^2
= A_5 e^{-A_5 v_2} \Delta v_2 - A_5^2 e^{-A_5 v_2} |\nabla v_2|^2
\leq 2A_5 e^{-A_5 v_2} \left( \frac{A_5}{2}(1 + |\nabla v_2|^2) + A \right) - A_5^2 e^{-A_5 v_2} |\nabla v_2|^2 \tag{4.10}
\]
\[
\leq A_5^2 e^{-A_5 v_2} + 2A_5 A e^{-A_5 v_2}.
\]

Similarly, if $v_2 > C_1 + 10$, then
\[
\Delta \phi(v_2) = \phi'(v_2)\Delta_p v_2 + \phi'(v_2)\phi''(v_2)|\nabla v_2|^2
= a \Delta v_2 \leq aA \leq A_5 A. \tag{4.11}
\]
Moreover, \( \phi(v_2(0)) = \phi(0) = 1 \). Thus on using (4.5) with \( \psi = \phi(v_2) \) we obtain

\[
\int_{B_{1/2}} \phi(v_2) dx \leq A_6
\]

where \( A_6 \) depends on the constants with lesser suffixes. Therefore,

\[
\int_{B_{1/2}} v_2 dx \leq A_5 A_6 e^{A_5(M_1 + 10)} + (M_1 + 10)\text{vol}B_{1/2}.
\]

Finally, from (4.8) since \( \Delta(v_2 - C_1 - 10)^+ \geq -A \), we get

\[
\sup_{B_{1/4}} v_2 \leq M_2
\]

where \( M_2 \) depends on \( N, A \) and \( C_1 \).

**Step 3:** Then \( 0 \leq v_1 \leq C \). Furthermore, for any \( \beta \in (0, 1] \) we have a positive finite number \( A(\beta) \) such that

\[
H(t) \leq A(\beta) + \beta t^2.
\] (4.12)

Therefore, \( 0 \leq v_1 \leq M_1 \). Let us choose a function \( \eta \in C_0^\infty(B_{1/4}) \) which is such that \( 0 \leq \eta \leq 1 \) in \( B_{1/4} \) and \( \eta = 1 \) in \( B_{1/8} \). Thus on testing

\[
\pm \Delta u \leq \left( \frac{1}{\alpha} \chi_{\{|u| < \alpha\}}(x) H(|\nabla u|^2) + A \right)
\]

with \( \eta^2 v_1 \) we have

\[
\int_{B_{1/2}} \eta^2 |\nabla v_1|^2 = -\int_{B_{1/2}} (2v_1 \eta (\nabla v_1 \cdot \nabla \eta) + \eta^2 v_1 \Delta v_1) dx
\]
\[
\leq \frac{1}{2} \int_{B_{1/2}} \eta^2 |\nabla v_1|^2 dx + 2 \int_{B_{1/2}} v_1^2 |\nabla \eta|^2 dx
\]
\[
+ A M_1 \int_{B_{1/2}} \eta^2 (A(\beta) + \beta |\nabla v_1|^2) dx
\] (4.13)
\[
\leq \frac{1}{2} \int_{B_{1/2}} \eta^2 |\nabla v_1|^2 dx + p M_1^2 \int_{B_{1/2}} |\nabla \eta|^2 dx
\]
\[
+ M_1 \int_{B_{1/2}} \eta^2 (\beta |\nabla v_1|^2 + A(\beta)) dx.
\]

It is now established that

\[
\frac{1}{2} \int_{B_{1/2}} |\nabla v_1|^2 dx \leq M_2.
\] (4.14)
We define the maximal operator as
\[
\mathcal{M} f(x) = \sup_{0 < r < \frac{1}{100}} \frac{1}{|B_r(x)|} \int_{B_r(x)} f(y) dy.
\]
(4.15)

We further denote
\[
S_\mu = \{ x \in B_{1/32} : \mathcal{M}(|\nabla v_1|^p)(x) > \mu \}.
\]

We now prove the following for \( p \geq 2 \):

There exists constant \( C_1 \) such that for any \( \epsilon > 0 \) there exists a finite and positive number \( \mu_0 \) such that for any \( \mu \geq \mu_0 \) then

1. \( |S_\mu \cap Q_0| \leq |S_{\mu_0} \cap Q_0| < \epsilon |Q_0| \). Here \( Q_0 \) is a cube with length of each side being \( 2^{-10-10N} \) and \( Q_0 \cap B_{1/32} \neq \emptyset \).

2. Suppose \( Q \) is a dyadic subcube of \( Q_0 \) for which \( |S_{C_1 \mu} \cap Q| \geq \epsilon |Q| \), then \( Q \subset Q^* \subset S_\mu \), where \( Q \) is an immediate dyadic subcube of \( Q^* \).

**Proof:** We only sketch the proof as the ideas are borrowed from [6]. The claim in 1. follows from the argument given in [6].

Suppose 2. fails to hold. Then one can find a cube \( Q \) such that \( |S_\mu \cap Q| \geq \epsilon |Q| \) and \( y \in Q^* \) but \( \mathcal{M}(|\nabla v_1|^p)(y) \leq \mu \). Let \( \rho = 2^{6N} \) times the length of the sides of \( Q \) and consider \( \mathcal{M}_{\rho/4}(|\nabla v_1|^p)(0) \), with the supremum taken over \((0, \rho/4)\). Since \( \mathcal{M}(|\nabla v_1|^p)(y) \leq \mu \), there exists a constant \( C_2 \) such that for any \( x \in Q \),
\[
\mathcal{M}(|\nabla v_1|^p)(x) \leq \max\{ \mathcal{M}_{\rho/4}(|\nabla v_1|^p)(x), C_2 \mu \}.
\]
(4.16)

Let \( \phi \) be such that
\[
-\Delta_p \phi = 0 \text{ in } B_{\rho}(y)
\]
\[
\phi = v_1 \text{ on } \partial B_{\rho}(y).
\]
(4.17)

Since \( \phi \) is a minimizer of the functional \( \frac{1}{2} \int_{B_{\rho}(y)} |\nabla \phi|^p dx \), we have
\[
\int_{B_{\rho}(y)} |\nabla \phi|^p dx \leq \int_{B_{\rho}(y)} |\nabla v_1|^p dx \leq \mu |B_{\rho}(y)|.
\]
(4.18)

Of course, we have the mean value theorem (refer Theorem 1 in LINDQVIST-MANFREDI [19]) at our disposal to guarantee that
\[
\sup_{B_{\rho/2}(y)} \{|\nabla \phi|^p\} \leq C_3 \mu.
\]
(4.19)

On choosing \( C_1 = 15 \max\{C_2, C_3 \mu \} \) we have
\[
\mathcal{A} := \{ x \in Q : \mathcal{M}_{\rho/4}(|\nabla v_1|^p)(x) > C_1 \mu \} = \{ x \in Q : \mathcal{M}(|\nabla v_1|^p)(x) > C_1 \mu \} =: \mathcal{B}.
\]
(4.20)
If \( x \in A \), then it is easy see that \( x \in B \). Thus \( A \subset B \). Suppose \( x \in B \), then \( \mathcal{M}(\|\nabla v_1\|^p)(x) > C_1 \mu \). However, by (4.16) and by the choice of \( C_1 \) we have that \( x \in A \). Also observe that \( \{x \in Q : \mathcal{M}_{\rho/4}(\|\nabla \phi\|^p) > C_1 \mu/4\} = \emptyset \). For if not, then there exists \( x \in Q \) such that \( \mathcal{M}(\|\nabla \phi\|^p)(x) > C_\mu/4 \). One may thus produce \( r \in (0, \rho/4) \) such that

\[
\frac{C_\mu}{4} < \frac{1}{|B_r(x)|} \int_{B_r(x)} |\nabla \phi|^p dy \leq \frac{C_\mu}{15}.
\]

This is a contradiction since this leads to an absurdity \( 4 > 15 \).

Therefore,

\[
\{x \in Q : \mathcal{M}_{\rho/4}(\|\nabla v_1\|^p) > C_1 \mu\}
\leq \{x \in Q : \mathcal{M}_{\rho/4}(\|\nabla (v_1 - \phi)\|^p) + \mathcal{M}_{\rho/4}(\|\nabla \phi\|^p) > C_1 \mu/2\}
\leq \{x \in Q : \mathcal{M}_{\rho/4}(\|\nabla (v_1 - \phi)\|^p) > C_1 \mu/4\} + \{\mathcal{M}_{\rho/4}(\|\nabla \phi\|^p) > C_1 \mu/4\}
\leq \{x \in Q : \mathcal{M}_{\rho/4}(\|\nabla (v_1 - \phi)\|^p) > C_1 \mu/4\}.
\]

Thus there exists a constant \( C_4 \), which follows by the weak \((1,1)\) inequality of \( \mathcal{M} \), such that

\[
C_4 \mu^{-1} \int_{B_\rho(y)} |\nabla (v_1 - \phi)|^p dx \geq \{x \in Q : \mathcal{M}_{\rho/4}(\|\nabla (v_1 - \phi)\|^p) > C_1 \mu/4\}.
\]

Furthermore, by the maximum principle we have \( |v_1 - \phi| \leq C \) on the ball \( B_\rho(y) \). By the weak formulation of the problem (4.17), we have

\[
0 = \int_{B_\rho(y)} |\nabla \phi|^{p-2} \nabla \phi \cdot \nabla (v_1 - \phi) dx.
\]

Furthermore,

\[
- \int_{B_\rho(y)} \Delta_p v_1 (v_1 - \phi) dx = - \int_{B_\rho(y)} (\Delta_p v_1 - \Delta_p \phi)(v_1 - \phi) dx
\]

Thus by the Simon’s inequality (see (4.6) [7]) we have

\[
C_5 \int_{B_\rho(y)} |\nabla (v_1 - \phi)|^p dx \leq \int_{B_\rho(y)} (|\nabla v_1|^{p-2} \nabla v_1 - |\nabla \phi|^{p-2} \nabla \phi) \cdot \nabla (v_1 - \phi) dx
\leq - \int_{B_\rho(y)} (\Delta_p v_1 - \Delta_p \phi)(v_1 - \phi) dx
= - \int_{B_\rho(y)} (\Delta_p v_1)(v_1 - \phi) dx
\leq \int_{B_\rho(y)} C (\beta |v_1|^p + A(\beta)) dx.
\]
Using the above inequality (4.22) obtained, we get
\[ |\{ x \in Q : \mathcal{M}_{\rho/4}(|\nabla v_1|^p) > C_1 \mu \}| \leq C_6 \left( \beta + \frac{A(\beta)}{\mu} \right) |Q|. \] (4.26)

Thus, for a sufficiently small \( \delta > 0 \) and large \( \mu > 0 \) we have \( C_6 \delta < \epsilon/3 \) and \( C_6 A(\beta)/\mu < \epsilon/3 \). Therefore
\[ \{ x \in Q : \mathcal{M}(|\nabla v_1|^p) > C_1 \mu \} < \epsilon |Q| \]
which indeed is a contradiction to the hypothesis. Therefore, 2. holds.

One may now follow verbatim of [6] to conclude that the conclusion of 2. leads to
\[ |S_{C_7} \cap Q_0| \leq \epsilon^{k+1} |Q_0|. \] (4.27)

We now note from (4.27) that for any \( 1 < \theta < \infty \) a sufficiently small \( \epsilon > 0 \) can be chosen so that \( \mathcal{M}(|\nabla v_1|^p) \) is bounded in \( L^\theta(B_{1/16}) \), i.e.
\[ \int_{B_{1/16}} |\nabla v_1|^\theta \, dx \leq C_7, \] (4.28)
where \( C_7 \) is a uniform constant that depends on \( \theta, A, \mathcal{H} \). On choosing \( \theta = N \) we have \( 2\theta > N \). Hence from (4.12) we obtain
\[ \sup_{B_{1/32}} \{|\nabla v_1|\} \leq C_8. \] (4.29)

Reverting back to the variables in terms of \( u \) we get
\[ \sup_{B_{\alpha/320}(x)} \{|\nabla u|\} \leq C_8 \text{ for any } x \in B_{1/4} \text{ such that } |\{ u(x) < \alpha \}|. \] (4.30)

To finally arrive at the conclusion
\[ \sup_{B_{r/4}(0)} \{|\nabla u|\} \leq C_9. \] (4.31)
we follow the proof of [6] again, however with the choice of \( w(x) = A_0r^{N-p}|x|^{p-N-1} + A(|x|^p - r^p) + O(\alpha) \). Therefore \( \sup_{B_{r/2}} \{|\nabla u|\} < \infty \).

\[ \square \]

**Proof of Lemma 4.1.** Let \( 0 < \alpha_j < 1 \). Consider the problem sequence \((P_j)\)
\[ -\Delta u_j = -\frac{1}{\alpha_j} g \left( \frac{(u_j - 1)_+}{\alpha_j} \right) + \lambda(u - 1)^{q-1} \text{ in } \Omega \]
\[ u_j > 0 \text{ in } \Omega \]
\[ u_j = 0 \text{ on } \partial \Omega. \] (4.32)
The nature of the problem being a sublinear one allows us to conclude by an iterative technique that the sequence \((u_j)\) is bounded in \(L^\infty(\Omega)\). Therefore, there exists \(C_0\) such that \(0 \leq g \left( \frac{(u_j - 1)_+}{\alpha_j} \right) (u - 1)^{q-1}_+ \leq C_0\). Let \(\varphi_0\) be a solution of

\[
-\Delta \varphi_0 = \lambda C_0 \text{ in } \Omega \\
\varphi_0 = 0 \text{ on } \partial \Omega. \tag{4.33}
\]

Now since \(g \geq 0\), we have that \(-\Delta u_j \leq \lambda C_0 = -\Delta \varphi_0 \text{ in } \Omega\). Therefore by the maximum principle,

\[
0 \leq u_j(x) \leq \varphi_0(x) \text{ for all } x \in \Omega. \tag{4.34}
\]

Since \(\{u_j \geq 1\} \subset \{\varphi_0 \geq 1\}\), hence \(\varphi_0\) gives a uniform lower bound, say \(d_0\), on the distance from the set \(\{u_j \geq 1\}\) to \(\partial \Omega\). Thus \((u_j)\) is bounded with respect to the \(C^{1,\alpha}\)-norm. Therefore, it has a convergent subsequence in the \(C^{1,\alpha}\)-norm in a \(\frac{d_0}{2}\) neighbourhood of the boundary \(\partial \Omega\). Obviously \(0 \leq g \leq 2\chi_{(-1,1)}\) and hence

\[
\pm \Delta u_j = \pm \frac{1}{\alpha_j} g \left( \frac{(u_j - 1)_+}{\alpha_j} \right) \mp \lambda(u_j - 1)^{q-1}_+ \leq \frac{2}{\alpha_j} \chi_{\{|u_j - 1| < \alpha_j\}}(x) + \lambda C_0. \tag{4.35}
\]

Since, \((u_j)\) is bounded in \(L^2(\Omega)\) and by Lemma 4.2 it follows that there exists \(A > 0\) such that

\[
\operatorname{esssup}_{x \in B_{r/2}(x_0)} \{|\nabla u_j(x)|\} \leq \frac{A}{r} \tag{4.36}
\]

for a suitable \(r > 0\) such that \(B_r(0) \subset \Omega\). However, since \((u_j)\) is a sequence of Lipschitz continuous functions that are also \(C^1\), therefore

\[
\sup_{x \in B_{r/2}(x_0)} \{|\nabla u_j(x)|\} \leq \frac{A}{r}. \tag{4.37}
\]

Thus \((u_j)\) is uniformly Lipschitz continuous on the compact subsets of \(\Omega\) such that its distance from the boundary \(\partial \Omega\) is at least \(\frac{d_0}{2}\) units.

Thus by the Ascoli-Arzelà theorem applied to \((u_j)\) we have a subsequence, still named the same, such that it converges uniformly to a Lipschitz continuous function \(u\) in \(\Omega\) with zero boundary values and with a \(C^1\) convergence on a \(\frac{d_0}{2}\)-neighbourhood of \(\partial \Omega\). By the Eberlein-Šmulian theorem we conclude that \(u_j \rightharpoonup u\) in \(W^{1,2}_0(\Omega)\).

We now prove that \(u\) satisfies

\[
-\Delta u = \alpha \chi_{\{u > 1\}}(x)(u - 1)^{q-1}_+ \tag{4.38}
\]
in the set \{u \neq 1\}. Let \(\varphi \in C_0^\infty(\{u > 1\})\) and therefore \(u \geq 1 + 2\delta\) on the support of \(\varphi\) for some \(\delta > 0\). On using the convergence of \(u_j\) to \(u\) uniformly on \(\Omega\) we have \(|u_j - u| < \delta\) for any sufficiently large \(j, \delta_j < \delta\). So \(u_j \geq 1 + \delta_j\) on the support of \(\varphi\). On testing (4.38) with \(\varphi\) yields

\[
\int_\Omega \nabla u_j \cdot \nabla \varphi \, dx = \lambda \int_\Omega (u_j - 1)^{q-1}_+ \varphi \, dx.
\]

(4.39)

On passing the limit \(j \to \infty\) to (4.38), we get

\[
\int_\Omega \nabla u \cdot \nabla \varphi \, dx = \lambda \int_\Omega (u - 1)^{q-1}_+ \varphi \, dx.
\]

(4.40)

To arrive at (4.40) we have used the weak convergence of \(u_j\) to \(u\) in \(W_0^{1,2}(\Omega)\) and the uniform convergence of the same in \(\Omega\). Hence \(u\) is a weak solution of \(-\Delta u = \lambda(u - 1)^{q-1}_+\) in \(\{u > 1\}\). Since \(u\) is a Lipschitz continuous function, hence it is also a solution of \(-\Delta u = \lambda(u - 1)^{q-1}_+\) in \(\{u > 1\}\). Similarly on choosing \(\varphi \in C_0^\infty(\{u < 1\})\) one can find \(\delta > 0\) such that \(u \leq 1 - 2\delta\). Therefore, \(u < 1 - \delta\).

On testing (4.38) with any nonnegative function and passing the limit \(j \to \infty\) and using the fact that \(g \geq 0, G \leq 1\) we can show that \(u\) satisfies

\[-\Delta u \leq \lambda(u - 1)^{q-1}_+ \text{ in } \Omega\]

(4.41)

in the distributional sense. We note here that the set \(\{u < 1\}\) is of nonzero Lebesgue measure since \(u\) is continuous. Furthermore, we claim that \(\Delta(u - 1)_- \geq 0\) in the distributional sense. the proof is as follows:

We follow the proof due to ALT-CAFFARELLI [1]. Choose \(\delta > 0\) and a test function \(\varphi^2 \chi_{\{u < 1 - \delta\}}\) where \(\varphi \in C_0^\infty(\Omega)\). Therefore,

\[
0 = \int_\Omega \nabla u \cdot \nabla (\varphi^2 \min\{u - 1 + \delta, 0\}) \, dx
= \int_{\Omega \cap \{u < 1 - \delta\}} \nabla u \cdot \nabla (\varphi^2 \min\{u - 1 + \delta, 0\}) \, dx
= \int_{\Omega \cap \{u < 1 - \delta\}} |\nabla u|^2 \varphi^2 \, dx + 2 \int_{\Omega \cap \{u < 1 - \delta\}} \varphi(u - 1 + \delta) \nabla u \cdot \nabla \varphi \, dx,
\]

(4.42)

and so by Caccioppoli like estimate we have

\[
\int_{\Omega \cap \{u < 1 - \delta\}} |\nabla u|^2 \varphi^2 \, dx = -2 \int_{\Omega \cap \{u < 1 - \delta\}} \varphi(u - 1 + \delta) \nabla u \cdot \nabla \varphi \, dx
\leq c \int_\Omega u^2 |\nabla \varphi|^2 \, dx.
\]

(4.43)
Since $\int_{\Omega} |u|^2 dx < \infty$, therefore on passing the limit $\delta \to 0$ we conclude that $u \in W^{1,2}_{\text{loc}}(\Omega)$. Furthermore, for a nonnegative $\zeta \in C_0^\infty(\Omega)$ and $\delta > 0$ we have

$$- \int_\Omega \nabla (u - 1)_- \cdot \nabla \zeta dx
\geq - \int_{\Omega \cap \{1 - \delta < u < 1\}} |\nabla (u - 1)_- \cdot \nabla \zeta| dx. \tag{4.44}$$

Note that by the Hölder’s inequality we have

$$0 \leq \int_{\Omega \cap \{1 - 2\delta < u < 1\}} |\nabla u \cdot \nabla \zeta| dx \leq \left( \int_{\Omega \cap \{1 - 2\delta < u < 1\}} |\nabla u|^2 dx \right)^{1/2} \times \left( \int_{\Omega \cap \{1 - 2\delta < u < 1\}} |\nabla \zeta|^2 dx \right)^{1/2} \to 0 \text{ as } \delta \to 0. \tag{4.45}$$

Thus $\Delta (u - 1)_- \geq 0$ in the distributional sense on $\Omega \cap \partial\{u < 1\}$. Hence by Theorems 2.3 – 2.4 in [16] there exists a nonnegative Radon measure $\mu$ (say) such that $\mu =: \Delta (u - 1)_- \text{ in } \Omega \cap \partial\{u < 1\}$.

From (4.41), the positivity of the Radon measure $\mu$ and the usage of Section 9.4 in Gilbarg-Trudinger [11] we conclude that $u \in W^{2,q}_{\text{loc}}(\{u \leq 1\})$. Thus $\mu$ is supported on $\Omega \cap \partial\{u < 1\} \cap \partial\{u > 1\}$ and $u$ satisfies $-\Delta u = 0$ in the set $\{u \leq 1\}$. In order to prove (ii), we will show that $u_j \to u$ locally in $C^1(\Omega \setminus \{u = 1\})$. Note that we have already proved that $u_j \to u$ in the $C^1$ norm in a neighbourhood of $\partial \Omega$ of $\Omega$.

Suppose $M \subset \subset \{u > 1\}$. In this set $M$ we have $u \geq 1 + \delta_j$ for some $\delta_j > 0$. Thus for sufficiently large $j$, with $\delta_j < \delta$, we have $|u_j - u| < \delta$ in $\Omega$ and hence $u_j \geq 1 + \delta_j$ in $M$. From (4.32) we have

$$-\Delta u_j = \lambda (u_j - 1)^{q_j - 1}_+ \text{ in } M.$$  

Clearly, $(u_j - 1)^{q_j - 1}_+ \to (u - 1)^{q - 1}_+$ in $L^q(\Omega)$ for $1 < q < \infty$ and $u_j \to u$ uniformly in $\Omega$. This analysis says something more stronger – since $-\Delta u_j = \lambda (u_j - 1)^{q_j - 1}_+$ in $M$, we have that $u_j \to u$ in $W^{2,q}(M)$. By the embedding $W^{2,q}(M) \hookrightarrow C^1(M)$ for $2 < q$, we have $u_j \to u$ in $C^1(M)$. This shows that $u_j \to u$ in $C^1(\{u > 1\})$. Working on similar lines we can also show that $u_j \to u$ in $C^1(\{u < 1\})$.

We will now prove (iii). Since $u_j \to u$ in $W^{1,2}_0(\Omega)$, we have that by the weak lower semicontinuity of the norm $\| \cdot \|$ that

$$\|u\| \leq \lim \inf \|u_j\|.$$  

It is sufficient to prove that $\lim sup \|u_j\| \leq \|u\|$. To achieve this, we multiply (4.32) with $(u_j - 1)$ and then integrate by parts. We will also use the fact that $tg\left(\frac{t}{\delta_j}\right) \geq 0$
for any $t \in \mathbb{R}$. This gives,
\[
\int_{\Omega} |\nabla u_j|^2 \, dx \leq \lambda \int_{\Omega} f(u_j - 1)^q \, dx - \int_{\partial \Omega} \frac{\partial u_j}{\partial n} \, dS \\
\rightarrow \lambda \int_{\Omega} (u - 1)^q \, dx - \int_{\partial \Omega} \frac{\partial u}{\partial n} \, dS \tag{4.46}
\]
as $j \to \infty$. Here $\hat{n}$ is the outward drawn normal to $\partial \Omega$.

**Conjecture 4.3.** We leave the following as an open conjecture that if one considers the function
\[
\Phi(r) := \left( \frac{1}{r^p} \int_{B_r} \frac{\nabla(|\nabla w^+|^{p-2} w^+(p-1)^2) \cdot \nabla w^+}{|x|^{N-p}} \, dx \right) \left( \frac{1}{r^p} \int_{B_r} \frac{\nabla(|\nabla w^-|^{p-2} w^-+(p-1)^2) \cdot \nabla w^-}{|x|^{N-p}} \, dx \right),
\]
then the estimate
\[
\Phi(r) \leq C \left( 1 + \int_{B_1} \frac{\nabla(|\nabla w^+|^{p-2} w^+(p-1)^2) \cdot \nabla w^+}{|x|^{N-p}} \, dx + \int_{B_1} \frac{\nabla(|\nabla w^-|^{p-2} w^-+(p-1)^2) \cdot \nabla w^-}{|x|^{N-p}} \, dx \right)^2,
\]
for each $0 < r \leq 1$ if $\Delta_p w_\pm \geq -1$ may be obtained. However, a big drawback in the case of $p$-Laplacian operators is that we do not have $C^2$ regularity of solutions.

We now establish the free boundary condition for the case of $p > 2$ (for the case of $p = 2$, the readers may refer to Perera [17]). Towards this we choose $\tilde{\varphi} \in C_0^1(\Omega, \mathbb{R}^N)$ such that $u \neq 1$ a.e. on the support of $\tilde{\varphi}$. On multiplying $\nabla u_n \cdot \tilde{\varphi}$ to the weak formulation of (4.32) and integrating over the set $\{1 - \epsilon^- < u_n < 1 + \epsilon^+\}$ gives
\[
\int_{\{1-\epsilon^-<u_n<1+\epsilon^+\}} \left[ -\Delta_p u_n + \frac{1}{\alpha_n} g \left( \frac{u_n - 1}{\alpha_n} \right) \right] \nabla u_n \cdot \tilde{\varphi} \, dx
\]
\[
= \int_{\{1-\epsilon^-<u_n<1+\epsilon^+\}} (u_n - 1)^{q-1} \nabla u_n \cdot \tilde{\varphi} \, dx. \tag{4.47}
\]
The term on the left hand side of (4.47) can be expressed as follows:
\[
\nabla \cdot \left( \frac{1}{p} |\nabla u_n|^p \tilde{\varphi} - (\nabla u_n \cdot \tilde{\varphi}) |\nabla u_n|^{p-2} \nabla u_n \right) + (\nabla \tilde{\varphi} \cdot \nabla u_n) \cdot \nabla u_n |\nabla u_n|^{p-2} - \frac{1}{p} |\nabla u_n|^p \nabla \cdot \tilde{\varphi}
\]
\[
+ \nabla G \left( \frac{u_n - 1}{\alpha_n} \right) \cdot \tilde{\varphi}. \tag{4.48}
\]
Using (4.48) and on integrating by parts we obtain

\[
\int_{\{u_n=1+\epsilon\} \cup \{u_n=1-\epsilon\}} \left[ \frac{1}{p} \left| \nabla u_n \right|^p \tilde{\varphi} - (\nabla u_n \cdot \tilde{\varphi}) |\nabla u_n|^{p-2} \nabla u_n + G \left( \frac{u_n-1}{\alpha_j} \right) \right] \cdot \hat{n} dS \\
= \int_{\{1-\epsilon<u_n<1+\epsilon\}} \left( \frac{1}{p} |\nabla u_n|^p \nabla \cdot (\nabla \tilde{\varphi} \cdot \nabla u_n) |\nabla u_n|^{p-2} \nabla u_n \right) dS + \int_{\{1-\epsilon<u_n<1+\epsilon\}} \left[ G \left( \frac{u_n-1}{\alpha_n} \right) \nabla \cdot \tilde{\varphi} + \lambda (u_n-1)^{2-1} (\nabla u_n \cdot \tilde{\varphi}) \right] dS.
\]

The integral on the left of equation (4.49) converges to

\[
\int_{\{u=1+\epsilon\} \cup \{u=1-\epsilon\}} \left( \frac{1}{p} |\nabla u|^p \tilde{\varphi} - (\nabla u \cdot \tilde{\varphi}) |\nabla u|^{p-2} \nabla u \right) \cdot \hat{n} dS + \int_{\{u=1+\epsilon\}} \tilde{\varphi} \cdot \hat{n} dS = \int_{\{u=1+\epsilon\}} \left[ 1 - \left( \frac{p-1}{p} \right) |\nabla u|^p \right] \tilde{\varphi} \cdot \hat{n} dS - \int_{\{u=1-\epsilon\}} \left( \frac{p-1}{p} \right) |\nabla u|^p \tilde{\varphi} \cdot \hat{n} dS.
\]

Thus the equation (4.50) under the limit \( \epsilon \to 0 \) becomes

\[
0 = \lim_{\epsilon \to 0} \int_{\{u=1+\epsilon\}} \left[ \left( \frac{p-1}{p} \right) - |\nabla u|^p \right] \tilde{\varphi} \cdot \hat{n} dS - \lim_{\epsilon \to 0} \int_{\{u=1-\epsilon\}} |\nabla u|^p \tilde{\varphi} \cdot \hat{n} dS.
\]

This is because \( \hat{n} = \pm \frac{\nabla u}{|\nabla u|} \) on the set \( \{u = 1 + \epsilon\} \cup \{u = 1 - \epsilon\} \). This proves that \( u \) satisfies the free boundary condition. The solution cannot be trivial as it satisfies the free boundary condition. Thus a solution to (1.1) exists that obeys the free boundary condition besides the Dirichlet boundary condition.

\textbf{Remark 4.4.} We note that \( I_\alpha \) obeys the Palais-Smale (PS) condition, for \( p \geq 2 \). In order to prove this, we define

\[
u_n^+(x) := \max\{u_n(x), 0\}, \quad u^+ + u^- := (u - 1)_+ + [1 - (u - 1)_-] = u.
\]

We further note that

\[
I_\alpha(u_n) \geq p^{-1} \|u_n\|^p - \frac{\lambda}{q} \int_{\Omega} (u_n)_+^q dx,
\]

\[
\langle I'_\alpha(u_n), u_n \rangle \leq \|u_n\|^p - \lambda \int_{\Omega} (u_n)_+^q dx + \frac{2}{\alpha} |\Omega|.
\]

Consider \( c \in \mathbb{R} \) and

\[
c + \sigma \|u_n\| + o(1) \geq I_\alpha(u_n) - \frac{1}{q} \langle I'_\alpha(u_n), u_n \rangle \geq (p^{-1} - q^{-1}) \|u_n\|^p - \frac{2}{\alpha} |\Omega|. \tag{4.53}
\]
This implies that \((u_n)\) is bounded in \(W^{1,p}_0(\Omega)\). This implies that there exists a subsequence of \((u_n)\) such that \(u_n \to u\) in \(W^{1,p}_0(\Omega)\), \(u_n \to u\) in \(L^2(\Omega)\) and \(u_n(x) \to u(x)\) a.e. in \(\Omega\). Since \(\langle I'_\alpha(u_n), v \rangle \to 0\) as \(n \to \infty\) we have

\[
\lim_{n \to \infty} \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla v dx = \lim_{n \to \infty} \left[ \int_{\Omega} \frac{1}{\alpha} g \left( \frac{u_n - 1}{\alpha} \right) v dx + \lambda \int_{\Omega} (u_n - 1)^{q-1} v dx \right]
\]

(4.54)

for all \(v \in W^{1,p}_0(\Omega)\). In particular on choosing \(v = u_n - u\) in (4.54) we obtain

\[
\lim_{n \to \infty} \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla (u_n - u) dx = \lim_{n \to \infty} \left[ \int_{\Omega} \frac{1}{\alpha} g \left( \frac{u_n - 1}{\alpha} \right) (u_n - u) dx + \lambda \int_{\Omega} (u_n - 1)^{q-1} (u_n - u) dx \right] = 0.
\]

(4.55)

Hence \(u_n \to u\) in \(W^{1,p}_0(\Omega)\). Therefore the functional \(I_\alpha\) satisfies the (PS) condition.

Before we prove the existence of a solution to the problem (1.1), we sharpen a few tools which that be used in the proof. We observe that

\[
I_\alpha(u) \leq I(u) \quad \text{in} \quad W^{1,p}_0(\Omega).
\]

Furthermore, we have

\[
I_\alpha(u) \geq \frac{1}{p} \|u\|^p - \frac{\lambda}{q} \int_{\Omega} (u^+)^q dx \geq \frac{1}{p} \|u\|^p - \frac{C\lambda}{q} \|u\|^q
\]

(4.56)

by Sobolev embeddings. Therefore, there exists \(\rho_0 = \rho_0(\nu, \lambda) > 0\) such that

\[
I_\alpha(u) \geq \frac{1}{2p} \|u\|^p
\]

(4.57)

for \(\|u\| \leq \rho_0\). Furthermore, for a fixed nonzero \(u\) we have \(I_\alpha(tu) \to -\infty\) as \(t \to \infty\) and hence there exists a function \(v_0\) such that \(I_\alpha(v_0) < 0 = I_\alpha(0)\). This indicates that the set

\[
\Lambda_\alpha := \{\xi \in C([0, 1]; W^{1,p}_0(\Omega)) : \xi(0) = 0, I_\alpha(\xi(1)) < 0\}
\]

is nonempty. Hence by the Mountain pass theorem we have

\[
c_\alpha := \inf_{\xi \in \Lambda_\alpha} \max_{u \in \xi([0,1])} I_\alpha(u).
\]

(4.58)

Since by the definition of the set \(\Lambda_\alpha\) we have \(\Lambda \subset \Lambda_\alpha\) and

\[
c_\alpha \leq \max_{u \in \xi([0,1])} I_\alpha(u) \leq \max_{u \in \xi([0,1])} I(u)
\]

(4.59)

for all \(\xi \in \Lambda\). This implies that \(c_\alpha \leq c\).
Remark 4.5. Let \( \phi_1 \) be the first eigenfunction pertaining to the first eigenvalue \( \lambda_1 \) of \( (-\Delta_p) \). We observe that

\[
I(t\phi) \to -\infty \text{ as } t \to \infty.
\]

Thus there exists \( t_* \) such that \( I(t_*\phi_1) < 0 \). Consider the path which is defined by \( \xi(t) = t\phi_1 \) for \( t \in [0, t_*] \). Then \( \xi \) yields a path from \( \Lambda \) on which

\[
I(t\phi_1) \leq D := \sup_{t \geq 0} \int_{\Omega} \left( \frac{\lambda_1 t^p \phi_1 + 1}{p} \right) dx.
\]

Therefore \( c \leq D \).

Proof. From the Remark 4.5 we conclude that \( c_\alpha \leq c \leq D \). Since \( I_\alpha \) obeys the (PS) condition, hence a limit of the (PS) sequence, say \( u_\alpha \), can be proved to be a critical point of \( I_\alpha \). Thus we have \( I_\alpha(u_\alpha) = c_\alpha \). Now consider a sequence \( \alpha_n \) which converges to zero and name \( u_{\alpha_n} \) as \( u_n \), \( c_{\alpha_n} \) as \( c_n \). By the Lemma 4.1 \((i) - (ii)\) we extract a subsequence of \((u_n)\), still denoted by the same name, converges uniformly in \( \tilde{\Omega} \), locally in \( C^1(\tilde{\Omega} \setminus \{u = 1\}) \) and strongly in \( W^{1,p}_0(\Omega) \), to a locally Lipschitz function \( u \in W^{1,p}_0(\Omega) \cap C^{1,\alpha}(\Omega \setminus H(u)) \). Moreover, from \( (4.57) \) in Remark 4.4 we have \( \lim \sup \alpha_n(u_n) = \lim \sup c_n \geq \frac{r_0}{4} > 0 \). This indicates that one of the limit conditions \( \lim \sup I_{\alpha_n}(u_n) > 0 \) or \( \lim \inf I_{\alpha_n}(u_n) < 0 \) in Lemma 4.1 indeed holds. Hence by the paragraph succeeding Lemma 4.1 \((iv)\) we conclude that \( u \) is nontrivial. Furthermore, by Lemma 4.1 we have that \( u \) is a \( C^2(\Omega) \) weak solution of \( -\Delta_p u = \lambda(u - 1)^q \) in \( \Omega \setminus \partial\{u > 1\} \) and the free boundary condition \( |\nabla u + |p - |\nabla u - |p = \frac{p}{p-1} \) in sense of \( (4.51) \) in addition to vanishing on the boundary \( \partial\Omega \).

Remar 4.6. We note that the Lemma 4.1 and 4.2 are still open questions to be answered for problems that are superlinear in nature and for \( p \neq 2 \).

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Data availability statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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