Angular Momentum Transport in Simulations of Accretion Disks

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Abstract. In this paper we briefly discuss the ways in which angular momentum transport is included in simulations of non-self-gravitating accretion disks, concentrating on disks in close binaries. Numerical approaches fall in two basic categories; particle based Lagrangian schemes, and grid based Eulerian techniques. Underlying the choice of numerical technique are assumptions that are made about disk physics, in particular about the angular momentum transport mechanism. Grid-based simulations have generally been of hot, relatively inviscid disks whereas particle-based simulations are more commonly of cool, viscous disks. Calculations of the latter type have been instrumental in developing a model for the superhump phenomenon. We describe how we use an artificial viscosity term to introduce angular momentum transport into our smoothed particle hydrodynamics (SPH) disk code.

1. Preamble

The principal difficulty in modelling accretion disks stems from our ignorance of the mechanism by which angular momentum is transported through the disk. This process (or processes) must transfer angular momentum from the inner disk to the outer disk, and also convert kinetic energy to thermal energy in order to allow a mass flux inwards through the disk. Our ignorance is usually reduced to a single shear viscosity term in the Navier-Stokes equation, where the kinematic shear viscosity

\[ \nu = \alpha c H. \]

In writing equation 1, Shakura & Sunyaev (1973) assumed the unknown mechanism to be turbulent viscosity, in which case the scale length of the largest turbulent eddies had to be less than the pressure scale height of the disk \( H \), and the eddy velocity had to be less than the sound speed \( c \). Thus the dimensionless parameter \( \alpha \), that measures the efficiency of the angular momentum transport, is expected to be less than unity. The accretion disks in outbursting dwarf novae for example, are observed to have \( \alpha \approx 0.2 \).

The corresponding shear viscosity is several orders of magnitude too large to be molecular.

A requirement of our simulations then, is that they reproduce observed rates of angular momentum transport. But does the transport occur via disk scale structures such as nonlinear density waves, that need to be resolved in the simulations, or via very small scale, sub-grid, phenomena (e.g. turbulence) that
we can’t explicitly include in the simulations but whose effects must be allowed for in the equations of motion? In the past, authors investigating the former possibility have used high order, ‘inviscid’, grid based hydrodynamics algorithms, whilst the latter has been investigated with ‘artificially viscous’ particle schemes.

Sawada et al. (1986) simulated an accretion disk in a close binary system using the second order Osher upwind scheme, a grid based technique that does not require an artificial viscosity to resolve shocks. In a two dimensional calculation set on the equatorial plane of the binary, they found that the secondary star launched spiral density waves at the outer edge of the disk. These waves propagate inwards, carrying a deficit of angular momentum which they transmit to the general flow when they damp. As a result there is a net outwards flux of angular momentum and a mass flux inwards. A self-similar analytic solution due to Spruit (1987) showed that a disk in which spiral shocks provided the only mechanism for angular momentum transport would have an effective $\alpha \lesssim 0.01$. Calculations by Savonije et al. (1994) found that spiral shocks were most efficient for hot disks with Mach numbers $\mathcal{M} = v_\phi/c \leq 10$. Here $v_\phi$ is the azimuthal velocity of the gas. Disks in cataclysmic variables have Mach numbers closer to 30. Spiral shocks then are not the most important means for angular momentum transport in these objects.

From the above discussion we conclude that there is another mechanism besides spiral shocks at work. At this stage the most likely candidate is a weak field magnetohydrodynamic shear instability discussed by Chandrasekhar (1960), but applied to the accretion disk problem by Balbus & Hawley (1991). Three dimensional simulations of small regions of the disk (in the shearing sheet approximation) by Stone et al. (1996) showed that magnetohydrodynamic turbulence generated by the instability transported angular momentum with an estimated efficiency $\alpha \lesssim 0.01$, similar to the maximum efficiency of spiral shocks. This is disturbing as it is an order of magnitude less than observational estimates of $\alpha$, which are obtained by fitting axisymmetric viscous disk solutions to the decay in luminosity of dwarf novae after outburst (Cannizzo, 1993). Dwarf novae in quiescence though, are thought to have an effective $\alpha$ more in line with the Stone et al. result (although in this case the observational results are less well constrained).

MHD turbulence is obviously a phenomenon that we cannot hope to capture in a global disk simulation, however we can simply introduce a shear viscosity term into our numerical scheme to simulate the action of the weak field instability. Stone et al. suggest that the shear stress due to the instability is consistent with the Shakura & Sunyaev parametrisation (Equation 1).

2. Particle Methods and the Explanation for Superhump

Lin & Pringle (1976) developed a particle based scheme that included an artificial method for ‘viscously’ dissipating energy, in order to study the evolution of disks in close binaries. In their ‘sticky particle method’, gas pressure is neglected and particles move as test particles in the restricted three body problem. Then at the end of each time step a grid is placed over the computational domain. Particle velocities are adjusted in order to minimise the kinetic energy in each cell, whilst conserving linear and angular momentum. Lucy (1977) cited the
sticky particle method as being a progenitor of smoothed particle hydrodynamics. Having neglected pressure forces, Lin & Pringle excluded the possibility of angular momentum transport via density waves. They simulated the evolution of accretion disks under steady mass transfer from the companion star for three different binary mass ratios, and in each obtained a steady state disk that was comparable in size to the accreting star’s Roche lobe.

Whitehurst (1988 a and b) developed a completely Lagrangian scheme that included pressure forces, and subsequently found that in close binaries with extreme mass ratios the secondary star is able to excite a large eccentricity in the disk. This was particularly exciting as some extreme mass ratio cataclysmic variables exhibited a periodic luminosity variation known as superhump that was thought to be the signature of an eccentric disk. The superhump period was always a few percent larger than the orbital period. In Whitehurst’s simulation, the eccentric disk (as seen in the inertial frame) exhibited a slow prograde precession. In his model the superhump signal was due to the periodic stressing of the eccentric disk by the tidal field, and the superhump period was the beat period between the disk precession period and the binary period.

Lubow (1991) showed analytically that eccentricity growth occurred when the disk was large enough to encompass an eccentric inner Lindblad resonance. This could only happen when the mass ratio $q = M_s/M_p \lesssim 0.25$, where $M_p$ is the mass of the accreting star and $M_s$ is the mass of the companion.

Whitehurst’s result was verified and expanded upon by Hirose & Osaki (1990) using the original sticky particle method, and later by Murray (1996) using SPH. The three particle techniques introduced viscous dissipation in different ways, and the resulting kinematic viscosities had different forms. In the sticky particle and SPH simulations $\nu$ was constant, whereas Whitehurst’s default choice was $\nu \propto r^{-3/2}$ (Whitehurst also ran a simulation with or $\nu \propto r^{1/2}$). Thus, the detailed functional dependence of the viscous dissipation has not proven to be critical. What is important and was common to the three sets of simulations was a large $\nu$ at large radii which allowed the disk to spread out to the 3 : 1 resonance. Heemskerk (1994) studied the phenomenon using an Eulerian, grid-based scheme for solving the inviscid equations of hydrodynamics. Heemskerk performed some simulations using only the $m = 3$ component of the tidal field (that being the term responsible for the Lindblad resonance), and found that the disk became eccentric. When however he used the full tidal potential, the accretion disk was unable to maintain contact with the resonance, even though they had initially overlapped, and significant eccentricity growth did not occur.

### 3. Artificial viscosity in our SPH algorithm

Smoothed particle hydrodynamics is a completely Lagrangian numerical scheme, which uses an ensemble of particles to model a fluid (see Monaghan 1992). The fluid equations are replaced by a set of equations for the evolution of the particles. For example, the fluid momentum equation becomes, in the SPH scheme, a set of equations for the forces on each particle with spatial derivatives estimated by interpolating quantities from neighbouring particles. The ‘standard’ SPH form
of the momentum equation for particle $a$, neglecting gravitational forces, is

$$\frac{dv_a}{dt} = -\sum_b m_b \left( \frac{P_a}{\rho_a^2} + \frac{P_b}{\rho_b^2} + \frac{\beta \mu_{ab}^2 - \zeta \overline{\rho_{ab}} \mu_{ab}}{\rho_{ab}} \right) \nabla_a W(r_a - r_b, h).$$

Here $r_a$, $v_a$ and $m_a$ are the position, velocity and mass of particle $a$. $P_a$, $\rho_a$ and $c_a$ are the pressure, density and sound speed of the fluid evaluated at $r_a$. $\rho$ is obtained by interpolation, and then $P$ and $c$ are obtained using the chosen equation of state. The $\overline{X_{ab}}$ notation indicates the arithmetic mean of quantity $X$ evaluated at $r_a$, and $r_b$. $\zeta$ and $\beta$ are the coefficients of the linear and non-linear artificial viscosity terms. $W(r, h)$ is the interpolation kernel.

$$\mu_{ab} = \frac{(v_a - v_b) \cdot (r_a - r_b)}{(r_a - r_b)^2 + \eta^2},$$

with $\eta$ being a softening parameter usually set equal to one tenth the smoothing length $h$. Artificial viscosity was originally added to the SPH equations in order to improve the resolution of shocks, and is usually used with a switch that sets it to zero when $(v_a - v_b) \cdot (r_a - r_b) > 0$. This ensures that dissipation only occurs for compressive flows. For disk simulations however, we want viscous dissipation to occur wherever there is velocity shear so we use only the linear viscosity term with the switch disabled. By letting the number of particles $n \to \infty$, the smoothing length $h \to 0$, and approximating the summation with an integral, we can obtain the continuum equivalent of the linear artificial viscosity term. Following Pongracic (1988) and Meglicki et al. (1993), we find that the ‘viscous’ force per unit mass

$$a_{\text{visc}} = \frac{\zeta h \kappa}{2 \rho} (\nabla \cdot (c \rho S) + \nabla (c \rho \nabla \cdot v)),$$

where the deformation tensor

$$S_{ab} = \frac{\partial v_a}{\partial x_b} + \frac{\partial v_b}{\partial x_a},$$

$\kappa$, a constant dependant only upon the smoothing kernel used, is $1/4$ for the standard cubic spline kernel (in two dimensions). If we assume the density and sound speed vary on much longer length-scales than the velocity, we have

$$a_v = \frac{\zeta h c}{8} (\nabla^2 v + 2 \nabla (\nabla \cdot v)).$$

In other words the linear artificial viscosity term generates both shear and bulk viscosity in a fixed ratio. In the interior of the disk where $\nabla \cdot v \approx 0$, we simply have a kinematic viscosity

$$\nu = \frac{1}{8} \zeta ch.$$ 

In order to check the accuracy of equation 7 we simulated the viscous spreading of an axisymmetric disk with a Gaussian initial surface density profile

$$\Sigma(r, t = 0) = \Sigma_0 e^{\frac{(r-r_0)^2}{2r^2}}.$$
Figure 1. Evolution of an axisymmetric ring with a Gaussian initial density profile. The solid lines denote the analytic solution at the times shown, and the heavy points show the SPH solution. In this calculation we use \( r_0 = 0.85, l = 0.025, \Sigma_0 = 1.0 \) and \( \nu = 2.5 \times 10^{-4} \) (see equation 8). Units have been scaled so \( GM = 1 \), where \( M \) is the mass of the central star. We used a fixed smoothing length \( h = 0.01 \).

For this (two dimensional) calculation we laid down 22,400 particles in 21 equally spaced concentric rings, varying the particle mass to generate the Gaussian density profile. Gas pressure forces were switched off, and a constant smoothing length was used. We find that the actual shear viscosity generated by the linear artificial viscosity term was within \( \pm 10\% \) of the value given by equation 7. Figure 1 shows the close agreement between the analytic solution for a Gaussian ring with viscosity given by equation 7, and our SPH simulation. The test is described in more detail in Maddison et al. (1996) along with a study of the stability of a narrow ring of viscously interacting particles.

It is possible to derive SPH interpolant estimates of \( \nabla^2 \nu \), and thus to directly include shear viscosity in the SPH equations. Flebbe et al. (1994) and Watkins et al. (1996) introduced formulations and tested them with ring spreading calculations similar to the one above, obtaining reasonable agreement between predicted and actual values for \( \nu \). However, their results are clearly more noisy than those shown in figure 1. Both Flebbe et al., and Watkins et al. must interpolate twice in order to obtain \( \nabla^2 \nu \) which may increase the susceptibility of the calculations to the single ring instability discussed in Maddison et al.. Most importantly, in the Flebbe et al., and Watkins et al. formulations the forces between two particles are not antisymmetric and along the particles’ line of centers. Therefore they do not exactly conserve linear and angular momen-
tum. With our term, linear and angular momentum is conserved to machine accuracy.

4. Conclusions

We described above that superhump simulations completed with a low viscosity code gave results that were qualitatively different from more viscous calculations. Clearly, we must quantify the effective shear viscosity of accretion disk simulations if we are to make meaningful comparisons with observations. The ring-spreading calculation is a simple way to do this. There are several ways to introduce a known shear viscosity into SPH disk calculations. We described in detail a method which conserves linear and angular momentum. Momentum conservation is important in disk calculations that may run for thousands of dynamical time scales. We showed that the effective \( \nu \) introduced by our term could be accurately estimated with equation 7.

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