A Simple Proof that Major Index and Inversions are Equidistributed

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Abstract

We present a short proof of MacMahon’s classic result that the number of permutations with \( k \) inversions equals the number whose major index (sum of positions at which descents occur) is \( k \).

1 Introduction

Let \( p = p_0p_1 \cdots p_{n-1} \) be a permutation of \( [n] = \{0, 1, \cdots, n-1\} \). A descent of \( p \) is an index \( i \) at which \( p_{i-1} > p_i \), and an inversion of \( p \) is a pair of indices \( i < j \) with \( p_i > p_j \). Define \( \text{inv}(p) \) to be the number of inversions in \( p \), and define \( \text{maj}(p) \), the “major index” of \( p \), to be the sum of all descent positions (so for instance \( \text{maj}(241350) = 2 + 5 = 7 \)). See (\[1, 4\]) for other standard definitions and results regarding permutations.

MacMahon (\[3\]) proved that \( \text{inv} \) and \( \text{maj} \) are equidistributed: the number of length-\( n \) permutations with \( \text{inv}(p) = k \) equals the number of such permutations with \( \text{maj}(p) = k \). This common value is denoted \( b(n,k) \). MacMahon originally proved this by showing that the generating functions coincide, and Foata (\[2\]) gave a bijective proof; in this note we present a simpler proof.

2 Proof of Equidistribution

An inversion table of length \( n \) is an \( n \)-tuple of nonnegative integers \((a_0, a_1, \cdots, a_{n-1})\) such that \( a_j \leq j \) for all \( j \). Clearly there are \( n! \) inversion tables, and each represents a distinct permutation of \( [n] \) as follows: starting with the empty permutation, we repeatedly insert \( j \) so that it will have \( a_j \) items to its right.\(^1\) For instance the inversion table \((0,1,0,3,3)\)

\(^1\)to simplify our presentation we have reversed the usual convention which would have \( a_j \leq n - j \)
yields the permutation 31402, building it as

\[
\begin{array}{c}
0 \\
10 \\
102 \\
3102 \\
34102
\end{array}
\]

The insertion of \( j \) creates \( a_j \) inversions, proving the well-known result that \( b(n,k) \) is the number of inversion tables whose elements sum to \( k \). To prove equidistribution, we reinterpret \( (a_0, a_1, \ldots a_{n-1}) \) as meaning repeated insertion of \( j \) at a position that will increase the major index by \( a_j \). Finding such a position is always possible. For instance (using boldface to emphasize descents) we have \( \text{maj}(24130) = 2 + 5 = 7 \), and the possibilities for insertion of 6 are:

\[
\begin{align*}
\text{maj}(241306) &= 7 + 0 = 2 + 5 \\
\text{maj}(241350) &= 7 + 1 = 2 + 6 \\
\text{maj}(241356) &= 7 + 6 = 2 + 5 + 6 \\
\text{maj}(241635) &= 7 + 5 = 2 + 4 + 6 \\
\text{maj}(246135) &= 7 + 2 = 3 + 6 \\
\text{maj}(264135) &= 7 + 4 = 2 + 3 + 6 \\
\text{maj}(624135) &= 7 + 3 = 1 + 3 + 6
\end{align*}
\]

In general, say the \( \kappa \) inversions of a permutation of \([j]\) occur at positions \( d_\kappa < d_{\kappa-1} < \cdots < d_1 \). Inserting \( j \) at the rightmost position will not change the major index. Insertion at \( d_t \) \((1 \leq t \leq \kappa)\) will create no new descents, but the descents at \( d_t \) through \( d_1 \) will be shifted to positions \( d_t + 1, \cdots d_1 + 1 \), so \( \text{maj} \) will increase by \( t \). Finally, consider inserting \( j \) at the \( r^{\text{th}} \) position (from the left) which is \textit{not} a descent: if there are \( r' \) descents to the left of this position, we create a new descent at \( r + r' \) and shift \( \kappa - r' \) old descents to the right, increasing the major index by \( \kappa + r \). Thus the number of permutations with \( \text{maj}(p) = k \) is again the number of inversion tables with entries summing to \( k \).

### 3 Symmetric Joint Distribution

Our proof of equidistribution is simpler than Foata’s, but the machinery of Foata’s proof can be used to prove the stronger result that \( \text{maj} \) and \( \text{inv} \) have a \textit{symmetric joint distribution} \((2, 4)\): for any pair of integers \( k, k' \) the number of \( p \) with \( \text{inv}(p) = k, \text{maj}(p) = k' \) equals the number with \( \text{inv}(p) = k', \text{maj}(p) = k \). We now note that this result can be stated entirely in terms of inversion tables.

To do this we define another way to interpret an inversion table \((a_0, \cdots a_{n-1})\) as a way to build a permutation, one which makes the relationship between \( \text{inv} \) and \( \text{maj} \) more direct. Now \( a_j \) will mean “put \( j - a_j \) in the rightmost position, and increment all other elements
which are greater than or equal to $j - a_j$. More formally, if $(a_0 \cdots a_{j-1})$ generates the permutation $p_0 \cdots p_{j-1}$ then $(a_0 \cdots a_j)$ generates the permutation $p'$ with $p'_j = j - a_j$ and otherwise

$$p'_k = p_k + [p_k > j - a_j]$$

Here we make use of the “Iverson bracket” notation, where $[S] = 1$ if the statement $S$ is true, 0 if it is false.

In fact this just yields the inverse of the permutation generated by reading $(a_0, \cdots a_{n-1})$ as an inversion table. For instance our previous example of $(0, 1, 0, 3, 3)$ now yields the permutation 32401, building it as

$$0$$

$$10$$

$$102$$

$$2130$$

$$32401$$

At step $j$ we create $a_j$ new inversions; the increments do not change any existing inversions, since a pair $r < s$ is either unchanged or becomes $r + 1 < s + 1$ or $r < s + 1$. Furthermore we create a descent at position $j$ if and only if $a_j > a_{j-1}$ (i.e. if position $j$ is an ascent of $a$), and similarly the increments do not destroy or create any descents. So the resulting permutation $p$ has

$$\text{inv}(p) = \sum a_j$$

$$\text{maj}(p) = \sum_{a_j > a_{j-1}} j$$

Therefore, since inv and maj are eqidistributed over permutations, the “sum of elements” and “sum of ascent positions” are eqidistributed over the set of all inversion tables.

References

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[4] Richard P. Stanley. *Enumerative Combinatorics*, volume 1 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, New York, NY, 2 edition, 2012.