Abstract

We consider a model of 2D gravity with the coefficient of the Euler characteristic having an imaginary part \( \pi/2 \). This is equivalent to introduce a \( \Theta \)-vacuum structure in the genus expansion whose effect is to convert the expansion into a series of alternating signs, presumably Borel summable. We show that the specific heat of the model has a physical behaviour. It can be represented nonperturbatively as a series in terms of integrals over moduli spaces of punctured spheres and the sum of the series can be rewritten as a unique integral over a suitable moduli space of infinitely punctured spheres. This is an explicit realization à la Friedan-Shenker of 2D quantum gravity. We conjecture that the expansion in terms of punctures and the genus expansion can be derived using the Duistermaat-Heckman theorem. We briefly analyze expansions in terms of punctured spheres also for multicritical models.
1. The partition function of 2D quantum gravity is formally given by

\[ Z_+ = \sum_{h=0}^{\infty} \int_{\text{Met}_h} \mathcal{D}g e^{-S(g)}, \quad S(g) = \int_{\Sigma_h} \left( \lambda_1 \sqrt{g} + \frac{\lambda_2}{2\pi} \sqrt{g} R \right), \tag{1} \]

where \( \text{Met}_h \) denotes the space of metrics on compact Riemann surfaces \( \Sigma_h \) of genus \( h \) and \( \lambda_1, \lambda_2 \) are bare parameters. In this paper we discuss a generalization of (1) based on the analogy with the theory of \( \Theta \)-vacua in 2D Yang-Mills theory. In Abelian 2D gauge theories \( \Theta \)-vacua can be introduced in the path-integral formalism by adding the term \( i \Theta \int F \) to the action where \( F \) is the curvature of the \( U(1) \) gauge connection \( A \). The invariance of the gauge action without \( \Theta \)-term under the transformation \( A \to -A \) guarantees the reality of the partition function for any real value of \( \Theta \). In 2D gravity the analogue of the \( \Theta \)-term is

\[ i \frac{\Theta}{2\pi} \int_{\Sigma_h} R \sqrt{g} = i \Theta \chi(\Sigma_h), \tag{2} \]

where \( \chi(\Sigma_h) = 2 - 2h \) denotes the Euler characteristic. In 2D gravity, however, we have not an analogue of the symmetry under \( A \to -A \), but, since \( \chi(\Sigma_h) \) is even, it follows that for

\[ \Theta = k \frac{\pi}{2}, \quad k \in \mathbb{Z}, \tag{3} \]

the reality of the partition function is preserved.

Here we study a model of 2D quantum gravity with \( \Theta = \pi/2 \). The partition function of our model is given by

\[ Z_- = \sum_{h=0}^{\infty} \int_{\text{Met}_h} \mathcal{D}g e^{-S(g) + i \frac{\Theta}{2\pi} \int_{\Sigma_h} R \sqrt{g}} = \sum_{h=0}^{\infty} (-1)^{1-h} \int_{\text{Met}_h} \mathcal{D}g e^{-S(g)}, \quad \Theta = \frac{\pi}{2}. \tag{4} \]

Expressions (1) and (4) are purely formal in two respect. First the integration measure on \( \text{Met}_h \) has no precise mathematical meaning: measures on infinite dimensional spaces involve distributional completion of smooth configuration spaces and it is not completely under control how to define a distributional completion of \( \text{Met}_h \). Second the series appearing in (1) and (4) could diverge. These are standard problems in QFT. The first one is faced with renormalization theory. According to (1) one introduces a cut-off \( \epsilon \) by replacing the integration over \( \text{Met}_h \) by a sum of triangulations of \( \Sigma_h \) by equal size \( k \)-gons (of area \( \epsilon \)) and then taking the \( \epsilon \to 0 \) limit. A crucial point is that by Euler formula it follows that under discretization the value of the \( \Theta \)-term (2) in the action does not change and this guarantees the reality of the discrete approximation of the partition function for \( \Theta = k\pi/2, \quad k \in \mathbb{Z} \).
Let $\lambda_1(\epsilon)$ and $\lambda_2(\epsilon)$ denote the bare parameters in the regularized theory. Under renormalization $\Theta$ is unchanged whereas the renormalized cosmological constant $\lambda_1^R$ and the string coupling constant $g_s$ are given by

$$\lambda_1^R = \lambda_1(\epsilon) - c/\epsilon, \quad \lambda_2^R = \lambda_2(\epsilon) - \frac{5}{4} \log \epsilon, \quad \log g_s \equiv \lambda_2^R.$$ 

In terms of the renormalized parameters we have

$$Z_\pm = \sum_{h=0}^{\infty} \left( g_s^{-1} e^{i\Theta_\pm} \right)^{2-2h} \left( \lambda_1^R \right)^{\frac{5}{2}(1-h)} Z_h, \quad \Theta_+ = 0, \quad \theta_- = \pi/2.$$ 

Let us define $t = \lambda_1^R \cdot g_s^{-4/5}$. It has been conjectured in [2] that there is a nonperturbative definition of $Z_+$ through the matrix models such that the “specific heat”

$$Z_+(t) \equiv -Z_+''(t),$$

satisfies the Painlevé I (PI). It has been proved in [3] that by a suitable modification the conjecture is true, although the precise formulation of the matrix model is not completely satisfactory (see [4] [5] [3] [6]).

Let us briefly recall some basic facts about the solutions of PI [7]

$$f^2(z) - \frac{1}{3} f''(z) = z.$$ 

Its solutions can be characterized by five monodromy parameters $s_l, l = 1, \ldots, 5$, satisfying

$$s_{k+5} = s_k, \quad s_{k+5} = i(1 + s_{2+k}s_{3+k}), \quad k \in \mathbb{Z}.$$

There are five one parameter families of solutions, $f_l(z)$, called “simply truncated solutions”, characterized by the vanishing of the monodromy parameter $s_{5-2l}$. In the sector

$$\Omega_l \equiv \left\{ -\frac{2\pi}{5} + \frac{2\pi}{5} l < \arg z < \frac{2\pi}{5} + \frac{2\pi}{5} l \right\},$$

these solutions are characterized by the following asymptotic

$$f_l(z) = \sqrt{|z|} e^{i\left(\frac{1}{2}\arg z + \pi l\right)} \sum_{n=0}^{\infty} b_n \left( \sqrt{|z|} e^{i\left(\frac{1}{2}\arg z + \pi l\right)} \right)^{-5n} + f_l^{n,p}(z),$$

where the coefficients $b_n$ satisfy the recursion relations

$$b_0 = 1 \quad b_{n+1} = \frac{25n^2 - 1}{24} b_n - \frac{1}{2} \sum_{k=1}^{n} b_{n-k+1} b_k.$$ 

and $f_{l}^{n.p.}(z)$ is a nonperturbative (“instanton”) contribution. Along the rays $-\frac{2\pi}{5} + \frac{2\pi}{5} l$ and $\frac{2\pi}{5} + \frac{2\pi}{5} l$, the nonperturbative contributions behave asymptotically as

$$f_{l}^{n.p.} \sim c_{+}|z|^{-\frac{1}{8}} s_{4-2l} e^{i|z|^{\frac{5}{4}}}, \quad f_{l}^{n.p.} \sim -c_{-}|z|^{-\frac{1}{8}} s_{1-2l} e^{i|z|^{\frac{5}{4}}},$$

(9)

respectively, where $c_{+}$, $c_{-}$ and $c$ are nonvanishing real constants given in [7]. In any family of simply truncated solutions there is a special solution characterized by $s_{5} - 2l = 0 = s_{1-2l}$, called the “triply truncated solution”. It has poles only in the sector $\Omega_{l-1}$. Finally there is a symmetry among the solutions of PI: if $f(z)$ is a solution then also

$$\hat{f}_{k}(z) \equiv e^{-\frac{4\pi i k}{5}} f \left( e^{-\frac{2\pi i}{5} z} \right), \quad k \in \mathbb{Z}/5\mathbb{Z},$$

(10)

is a solution. In terms of monodromy parameters the transformation (10) corresponds to $s_{l} \rightarrow s_{l-2k}$.

It has been shown that the nonperturbative definition of the specific heat $Z_{+} = -Z''_{+}$ obtained in the matrix model is in the family of simply truncated solutions characterized by $s_{5} = 0$, so that (as one can check also by (9))

$$Z_{+}(t) \rightarrow +\sqrt{t}, \quad \text{as} \quad t \rightarrow +\infty,$$

and the coefficients of its asymptotic expansion in $t$ are identified with the $b_{n}$ in (8). Substituting $g_{s}$ by $e^{\pm \pi i/2} g_{s}$ implies $t \rightarrow e^{\pm 2\pi i/5} t$ and perturbatively

$$Z_{-}(t) = Z_{+} \left( e^{\pm \frac{4\pi i}{5}} t \right),$$

(11)

so that $\partial_{t}^{2} Z_{-}(t) = e^{\pm \frac{4\pi i}{5}} \partial_{t}^{2} Z_{+}(t')$, $t' = e^{\pm \frac{2\pi i}{5}} t$. Given a solution $Z_{+}$ of PI we set

$$Z_{-}^{(1)}(t) \equiv e^{\frac{4\pi i}{5}} Z_{+} \left( e^{\frac{2\pi i}{5}} t \right), \quad Z_{-}^{(2)}(t) \equiv e^{-\frac{4\pi i}{5}} Z_{+} \left( e^{-\frac{2\pi i}{5}} t \right).$$

(12)

By (11) $Z_{-}^{(1)}$ and $Z_{-}^{(2)}$ are simply truncated solutions of PI characterized by the vanishing of the monodromy parameters $s_{2}$ and $s_{3}$ respectively. Their leading asymptotic behaviour is given by

$$Z_{-}^{(k)}(t) \rightarrow -\sqrt{t}, \quad \text{as} \quad t \rightarrow +\infty, \quad k = 1, 2.$$  

(13)

If one further imposes the physical reality condition for the “specific heat” $Z$ of 2D gravity with $\Theta = \pi/2$, then by (10) and (9) we find that $Z_{-}^{(1)} = Z_{-}^{(2)} \equiv Z_{-}$ is the triply truncated solution characterized by $s_{2} = s_{3} = 0$. We expect that this solution is the unique real one on the positive real axis without “instanton” contributions $f_{l}^{n.p.}$.  

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2. In this section we show that $Z_-$ is strictly related to the theory of moduli space of punctured Riemann spheres compactified in the sense of Deligne-Knudsen-Mumford (DKM) [8]. In some sense, identifying $g_s$ as the coupling constant, we can think about the genus expansion as a low-temperature expansion labelled by the number of handles added to the sphere, whereas the expansion we will describe can be thought of as a high-temperature expansion labelled by the number of punctures inserted in the sphere.

It is not totally clear to us how exactly one can derive an expansion in terms of punctures from the genus expansion (see later for comments on this point). A general reason for this relation is nevertheless clear: the (compactified) moduli space of genus $h$ compact Riemann surfaces $\overline{M}_h \equiv \overline{M}_{h,0}$ (we denote by $M_{h,n}$ the moduli space of $n$-punctured Riemann surfaces of genus $h$) has a boundary, in the DKM compactification, given as a sum of contributions of moduli spaces with one or two punctures

$$\partial \overline{M}_h = c_{h,0} \overline{M}_{h-1,2} + \sum_{k=1}^{[h/2]} c_{h,k} \overline{M}_{h-k,1} \times \overline{M}_{k,1}, \quad (14)$$

in the sense of cycles on orbifolds. The coefficients $c_{h,j}$’s are combinatorial factors. Since each term in the RHS (14) has itself a boundary given in terms of compactified moduli spaces of punctured surfaces of lower genus and higher number of punctures, we can think to iterate this procedure ending up with only compactified moduli spaces of punctured spheres $\overline{M}_{0,n}$. In turn, $\overline{M}_{0,n}$ has a boundary given in terms of compactified moduli spaces of spheres with a lower number of punctures according to the recursion relation

$$\partial \overline{M}_{0,n} = \sum_{k=1}^{[n/2]-1} c_{0,k}^{(n)} \overline{M}_{0,n-k} \times \overline{M}_{0,k+2}. \quad (15)$$

The coefficients $b_h$ of the asymptotic expansion of $Z_\pm$ satisfy the recursion relations (8) and these relations appear to be related to the decomposition (14) of the boundary of $\overline{M}_h$. If one conjectures an analogous relation with (15) for the coefficients $Z_k$ of the expansion of $Z(t)$ in terms of punctured spheres, then one is led to the Ansatz considered in [9]

$$Z_n = d(n) \sum_{k=1}^{n-3} Z_{k+2} Z_{n-k}. \quad (16)$$

As shown in [9] the solution of the PI corresponding to (16) is given in a neighborhood of $t = 0$ by the series

$$Z(t) = \sum_{k=3}^\infty Z_k t^{5k-12}, \quad (17)$$
with
\[ d(n) = \frac{3}{(12 - 5n)(13 - 5n)}, \quad Z_3 = -1/2. \] (18)

One can easily verify that such series converges in a disk of finite radius around the origin. Furthermore the corresponding solution of PI is characterized by the initial conditions
\[ Z(0) = 0 = Z'(0), \] (19)
and a computer simulation shows that \( Z(t) \sim -\sqrt{t} \) as \( t \to +\infty \) (see figures) and it exhibits poles in the negative real axis. General arguments on the localization of poles of solutions of the PI combined with bounds of the coefficients of (17) prove that the Borel sum of the series (16)-(18) extends analytically the solution of PI to a neighborhood of the whole positive real axis. Reality of the solution on the real axis together with the asymptotic behaviour implies that the solution (17) can be indeed identified with the triply truncated solution \( Z_-(t) \) \((s_2 = s_3 = 0)\) considered above. Hence one can finally assert that the solution of the PI given on the positive real axis by the Borel sum
\[ Z_-(t) = t^3 \int_0^\infty dx e^{-x} \sum_{k=0}^\infty Z_{k+3}(t^5 x)^k \frac{1}{k!}, \] (20)
with the coefficients \( Z_k \)'s given by (16) and (18) is exactly the “specific heat” of the 2D quantum gravity with \( \Theta = \pi/2 \). In particular the \( h^{th} \) coefficient of its asymptotic expansion as \( t \to +\infty \) is the partition function of such a model of 2D gravity at genus \( h \).

If we apply to \( Z_-(t) \) the transformation (10) corresponding to \( t \to e^{2\pi i/5} t \), then we obtain a solution \( Z_+(t) \) of PI given by the triply truncated solution \( s_5 = s_4 = 0 \) considered by [2]. As shown in [3] this solution is not real on the real axis. According to standard thermodynamics, if one defines (as done here following [2]) the “specific heat” as the second derivative of the free energy, it should be negative. One can check that \( Z_-(t) \) is negative for all \( t > 0 \) (see figures). Notice instead that \( Z_+(t) \) is always positive for sufficiently large \( t \) which is an unphysical behaviour.

The asymptotic expansion (as \( t \to +\infty \)) of \( Z_- \) is presumably Borel summable. If this is the case then one can circumvent the problem of Borel summability of the asymptotic expansion in [2] by using the relationship \( Z_+(t) = e^{\frac{2\pi i}{5}} Z_- (e^{\frac{2\pi i}{5}} t) \).

Let us comment on models with real valued \( \Theta \) whose “specific heat” is defined by
\[ Z_\Theta(t) \equiv e^{\frac{2\pi i}{5}} Z_+ (t e^{\frac{2\pi i}{5}}) = -\frac{d^2}{dt^2} Z_+ (t e^{\frac{2\pi i}{5}}). \] (21)
The asymptotic expansion of the corresponding partition function is given by (4) for real \( \Theta \). By definition, \( Z_{\Theta}(t) \) satisfies the string equation with complex parameters

\[
f(t)^2 - \frac{1}{3} f''(t) = te^{i4\Theta},
\]

which reduces to PI (3) only for \( \Theta = \frac{\pi}{2}k, k \in \mathbb{Z} \).

3. We now make explicit a closer connection between \( Z_{-}(t) \) and the structure of \( \overline{M}_{0,n} \) expressing the \( n^{th} \) term in the series (17) as an integral over \( \overline{M}_{0,n} \) and \( Z_{-}(t) \) as an integral on a suitable infinite dimensional moduli space.

In a path integral approach we expect that

\[
Z_n = \int_{\overline{M}_{0,n}} \omega^{(n)n-3} e^{-S(n)}, \quad \omega^{(n)} = \frac{\omega_{WP}^{(n)}}{\pi^2},
\]

where \( S(n) \) is a suitable scalar action and \( \omega_{WP}^{(n)} \) is the Weil-Petersson two-form on \( \overline{M}_{0,n} \). On the other hand it has been shown in [9] that one can write \( Z_n \) as the rational intersection number

\[
Z_n = \int_{\overline{M}_{0,n}} \omega^{(n)n-4} \wedge \omega_{F_0}, \quad n \geq 4,
\]

where \( \omega_{F_0} \) is a closed two-form given in [9] as a linear combination of the Poincaré dual of the divisors of \( \overline{M}_{0,n} \) (we identify \( Z_k \) with \( Z_k^{F_0} \) in [9]).

We now define the “\( q \)-weighted” moduli space\(^1\) of infinitely punctured spheres \( \overline{M}_{0,\infty}(q) \) as follows: let us consider the embedding \( i_n : \overline{M}_{0,n} \to \overline{M}_{0,n+1}, n > 2 \). Then, for \( q \in \mathbb{R}_+ \), we define by inductive limit

\[
\overline{M}_{0,\infty}(q) = \prod_{n=0}^{\infty} \left( \overline{M}_{0,n+3} \times [0, q^n] \right) / \left( \overline{M}_{0,n}, q^n \right) \sim \left( i_n, \overline{M}_{0,n}, 0 \right).
\]

Let \( dy \) denote the Lebesgue measure on \( \mathbb{R} \), assume the form (24) for \( Z_n \) and define the indefinite rank forms

\[
\zeta_{\infty} = \left\{ \frac{1}{2} + \sum_{k=1}^{\infty} \frac{\omega^{(k+3)}k^{-1}}{k!} \wedge \omega_{F_0} \right\} \wedge dy,
\]

\[
S_{\infty} = \left\{ \frac{1}{2} + \sum_{k=1}^{\infty} \omega^{(k+3)}k^{-1} \wedge \omega_{F_0} \right\} \wedge dy.
\]

\(^1\)“\( q \)-weighted” moduli spaces have been considered also in (10).
Then by (17), (20) and (23) it follows that

\[ Z_-(t) = t^3 \int_0^\infty dx e^{-x} \int_{M_{0,\infty}(t^5)} \zeta_\infty, \quad (28) \]

and, in a sufficiently small neighborhood of \( t = 0 \), one can simply write

\[ Z_-(t) = t^3 \int_{M_{0,\infty}(t^5)} S_\infty. \quad (29) \]

Eqs. (28)-(29) express nonperturbative 2D quantum gravity as an integral on an infinite dimensional space which involves all moduli spaces of punctured spheres. The structure of this space and the crucial role of the DKM compactification are at the basis of this explicit realization of the Friedan-Shenker program [11].

4. Let us now consider the multicritical models, that is 2D quantum gravity coupled to conformal matter [2][6]. In this section we will use the notations of [9]. The string equation for the \( m^{th} \)-model is

\[ \frac{m!}{(2m-1)!!} \left\{ -1/2 \partial_t^2 + Z^{F_{m-2}}(t) + \partial_t^{-1} Z^{F_{m-2}}(t) \partial_t \right\}^m \cdot 1 = t, \quad (30) \]

where \( Z^{F_{m-2}} \) denotes the “specific heat”. Consider the eq.(30) with the initial conditions \( \left( \frac{\partial}{\partial t} \right)^k Z^{F_{m-2}}(0) = 0 \), for \( k = 0, \ldots, 2m - 3 \). Since the RHS of the string equation vanishes for this choice of initial conditions it follows that also \( \left( \frac{\partial}{\partial t} \right)^{2m-2} Z^{F_{m-2}}(0) = 0 \).

These conditions imply that around \( t = 0 \)

\[ Z^{F_{m-2}}(t) = t^{-4(m+1)} \sum_{k=3}^{\infty} Z_k^{F_{m-2}} t^{(2m+1)k}, \quad (31) \]

where \( Z_k^{F_{m-2}} \) are coefficients satisfying recursion relations of the form

\[ Z_3^{F_{m-2}} = \frac{(-2)^{m-1}}{(2m-2)!!m!}, \]

\[ Z_n^{F_{m-2}} = d(n, m) \left( \sum_{k=1}^{n-3} c(m; n, k) Z_{n-k}^{F_{m-2}} Z_{k+2}^{F_{m-2}} + \text{higher order terms} \right), \quad n > m + 1, \quad (32) \]

where \( d(n, m) \) and \( c(m; n, k) \) are positive factors. The Taylor series (31) is invariant under the transformation \( t \rightarrow te^{\frac{2\pi i}{2m+1}} \), up to a \( e^{\frac{2\pi i n}{2m+1}} \) factor. For \( m \) even (odd) it has alternating (positive) signs, that is

\[ \text{sgn} Z_k^{F_{m-2}} = (-1)^{(m-1)k}. \quad (33) \]
This follows by the structure of the string equation and eq.(32).

It is instructive to analyze the $m = 3$ model. The string equation then reads

$$t = Z_{F_3}^F - Z_{F_1}^F Z_{F_1}'' - \frac{1}{2} \left( Z_{F_1}' \right)^2 + \frac{1}{10} Z_{F_1}^{(4)}.$$  \hspace{1cm} (34)

The expansion $Z_{F_1}^F(t) = t^{-16} \sum_{k=3}^{\infty} Z_k^F t^k$ converges in a disk of finite radius around $t = 0$. Its coefficients are determined by the recursion relations

$$Z_{F_3}^F = \frac{1}{12}, \quad Z_{F_4}^F = \frac{5 \cdot 13}{2 \cdot 9 \cdot 11 \cdot (12)^3}, \quad Z_{F_n}^F = \frac{10}{(7n - 16)(7n - 17)(7n - 18)(7n - 19)}.$$

$$\left( \frac{1}{2} \sum_{k=1}^{n-3} Z_{F_{n-k}}^F Z_{F_{k+2}}^F (7k + 8)(7n + 7k - 23) - \sum_{k=2}^{n-3} \sum_{j=1}^{k-1} Z_{F_{n-k}}^F Z_{F_{k-j+2}}^F Z_{F_j}^F \right), \quad n > 4. \hspace{1cm} (35)$$

We can fully generalize to multimatrix models the results obtained in [9] for the $m = 2$ case, i.e. one can prove that these are “Liouville $F$-models”. In fact, from eq.(32), it follows that the divisor structure function is

$$F_{m-2}(n, k) = \frac{2(n - 1) d(n, m) (c(m; n, k) Z_{F_{m-k}}^F Z_{F_{k+2}}^{m-2} + \text{higher order terms})}{(n - 4)! a_{n-k} a_{k+2}}, \hspace{1cm} (36)$$

for $n > 1 + m$, while the lower ones are fixed by initial conditions (see [9]).

Furthermore we argue that the following asymptotic expansion holds, corresponding to our initial conditions:

$$Z_{F_{m-2}} \sim \sum_{k=0}^{\infty} b_k^{(m)} t^{-\frac{2m+1}{m} k+\frac{1}{m}}, \quad \text{as } t \to e^{\pi i m} \cdot \infty \hspace{1cm} (37)$$

where $b_0^{(m)} = (-1)^{m+1}$.

5. We conclude the paper with some more speculative observations.

i) The $\Theta$-vacuum ($\Theta = \pi/2$) can be introduced in any string theory and it appears to convert a perturbative genus expansion with definite signs for $\Theta = 0$ to a genus expansion with alternating signs. This might improve the Borel summability of the perturbative series and therefore the $\Theta = \pi/2$ string theories could be in general more tractable than the standard ones.

ii) One can hope to implement more concretely the idea that the recursion relations (8) and (16) are related to the decomposition of $\partial \mathcal{M}_h$ and $\partial \mathcal{M}_{0,n}$, if the coefficients of the genus
expansion and the expansion in terms of punctured spheres, when expressed as integrals on moduli spaces, can be evaluated by means of a Duistermaat-Heckman (DH) theorem [12]. Let $\omega$ be a symplectic form in a manifold $X$ of dimension $2n$ and $H$ an Hamiltonian on $X$; then, roughly speaking, DH tells that integrals of the forms

$$\int_X \omega^n e^{-H}$$

only depend on the behaviour of the integrand near the critical points of the flow of the Hamiltonian vector fields. Equation (23) for $X = \mathcal{M}_{0,n}$ or its analogue in the genus expansion for $X = \mathcal{M}_h$ are exactly in the form that can be treated by DH, since the Weil-Petersson 2-form $\omega_{WP}$ is a symplectic form on $X$. Hamiltonian vector fields are proportional to $\omega_{WP}^{-1}$ vanishing at $\partial X$ and this suggests that the critical points of the Hamiltonian vector fields could be given by the boundary of the relative moduli space. An application of DH then would imply that the relevant integrals over moduli spaces reduce to integrals over their boundaries. Then the factorization property of the Mumford forms [13][14] near $\partial \mathcal{M}_h$ would probably be a key ingredient in the derivation of recursion relations (8).

$iii)$ A feature of the genus expansion is that the first two terms are distinguished with respect to the others. The reason is that the Euler characteristic is positive for the Riemann sphere, zero for the torus and negative for $h > 1$. The path integral admits a semiclassical limit ($g_s \to 0$) only for $h > 1$, since the Liouville equation appearing in the classical limit does not admit solutions on positively curved surfaces (see [15] for a detailed discussion). It is interesting to observe that instead only negative curved surfaces appear in the expansion in terms of punctured spheres characterizing the $\Theta = \pi/2$ model. Indeed this expansion precisely starts from the sphere with three punctures, which is the “minimum negatively curved manifold”. This suggests that near the classical limit factorizations related to the DKM compactification together with unitarity requirement could select among all complex values of the coefficient of the Euler characteristic precisely the ones with imaginary part $\Theta = \pi/2$ (mod $\pi$). In this respect it is intriguing to notice that phases naturally appear in the factorization of the Mumford forms near $\partial \mathcal{M}_h$ (see $ii)$).

Finally we note that our construction is strictly related to topological field theories [16][19] and the Kontsevich model [17][18].

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Figure Captions

Painlevé solution with boundary conditions $f(0) = 0$, $f'(0) = 0$. $f$: solid line, $f'$: dotted line, $g(t) = -t^{1/2}$: dashed line.
This figure "fig1-1.png" is available in "png" format from:

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This figure "fig1-2.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9407091v1