No hair theorem for bound-state massless static scalar fields outside horizonless Neumann compact stars

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Abstract

We study no-hair theorem for horizonless objects, being subject to Neumann boundary conditions. For massive scalar fields, a no hair theorem for Neumann compact stars was proved by us in a previous paper, where the nonzero scalar field mass condition is essential in the proof. In the present work, for massless scalar fields, we prove a no hair theorem, which claims that bound-state massless static scalar fields cannot exist outside asymptotically flat horizonless neutral Neumann compact stars.

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I. INTRODUCTION

Black holes are theoretical predictions of the general relativity gravity [1]. And direct detections of gravitational waves provided mounting evidence that black holes indeed exist in the universe [2]. Recently, the first ever image of a black hole has been captured by a network of eight radio telescopes around the world [3]. These discoveries open up hope to directly test various black hole theories from astronomical aspects. One remarkable property of classical black holes is the famous no hair theorem [4]–[11]. If generically true, it would signify that black hole solutions are very simple and uniquely determined by three parameters: mass M, charge Q and angular momentum J, see recent references [12]–[38] and reviews [39, 40].

Interestingly, it was recently shown that such no hair theorem are not restricted to black hole spacetimes. In the horizonless gravity, Hod firstly proved no static scalar hair theorem for asymptotically flat neutral reflecting compact stars [41]. When including a positive cosmological constant, it was found that massive scalar, vector and tensor field hairs cannot exist outside neutral horizonless reflecting compact stars [42]. In the charged background, reflecting shells can exclude scalar field hairs when the shell radius is large enough [43–45]. Large charged reflecting compact stars also cannot support the existence of scalar field hairs [46–51]. With nonminimal field-curvature couplings, no hair theorem could still appear in the horizonless gravity [52–54]. In addition, such no scalar hair theorem was further extended to horizonless compact stars with Dirichlet surface boundary conditions [55].

For massive scalar fields, no hair theorem was also proved for horizonless compact stars with Neumann boundary conditions [56]. The proof is based on the scalar field equation \( \psi \psi'' + \frac{1}{2} (\frac{4}{r} + \nu' - \lambda') \psi \psi' - m^2 e^\lambda \psi^2 = 0 \), where \( \psi \) is the scalar field with nonzero mass \( m \) and \( \nu, \lambda \) correspond to metric solutions. At the extremum point of \( \psi \), there are relations \( \psi \psi'' \leq 0 \), \( \psi' = 0 \) and \( m^2 e^\lambda \psi^2 < 0 \), which are in contradiction with the scalar field equation. This contradiction leads to the no hair theorem. However, the nonzero scalar field mass assumption \( m^2 \neq 0 \) is essential in the proof. As is known, scalar field mass usually plays an important role in scalar condensations. So it is meaningful to examine whether there is no hair theorem for massless scalar fields in the background of Neumann compact stars.

In the following, we introduce the system of massless scalar fields in the background of asymptotically flat neutral horizonless Neumann compact stars. We prove a no hair theorem that bound-state massless scalar fields cannot exist outside Neumann compact stars. We give conclusions in the last section.
II. NO MASSLESS SCALAR FIELD HAIR OUTSIDE NEUMANN COMPACT STARS

We shall analyze the physical and mathematical properties of an asymptotically flat gravity system, which is constructed by a central compact object coupled to a static massless scalar field. In Schwarzschild coordinates $(t, r, \theta, \phi)$, the line element of the external spherically symmetric curved spacetime can be expressed in the form

$$ds^2 = -g(t) dt^2 + \frac{dr^2}{g} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (1)$$

where $g$ and $\chi$ are functions only depending on the radial coordinate $r$. In order to recover the flat spacetime at the infinity, we impose asymptotical behaviors $g(r \to \infty) \sim 1 + O(r^{-1})$ and $\chi(r \to \infty) \sim O(r^{-1})$.

The Lagrange density for a massless scalar field in the asymptotically flat spacetime is

$$\mathcal{L} = R - (\partial_\mu \psi)^2, \quad (2)$$

where $R$ corresponds to the Ricci scalar curvature and $\psi$ is the scalar field.

We consider the scalar field $\psi$ as a functions of $r$ only. This leads to the scalar field equation

$$\psi'' + \left( \frac{2}{r} - \frac{\chi'}{2} + \frac{g'}{g} \right) \psi' = 0. \quad (3)$$

In the proof of the uniqueness of the Kerr solution (Carter-Hawking-Robinson theorem), metric solutions were assumed to be real analytic. In this work, we also assume that metric solutions are real analytic. That is to say metric solutions can be locally expressed with power series. We define the radial coordinate $r = r_s$ as the star surface radius. For every $r_\nu \in [r_s, R]$ there is a positive constant $\delta_\nu$ and in the range $(r_\nu - \delta_\nu, r_\nu + \delta_\nu)$, metric solutions can be expanded with power series. We pay attention to properties in the range $[r_s, R]$. We choose

$$(r_1 - \delta_1, r_1 + \delta_1), (r_2 - \delta_2, r_2 + \delta_2), \ldots, (r_N - \delta_N, r_N + \delta_N),$$

which also satisfies

$$r_1 = r_s, \quad r_N = R \quad and \quad r_i < r_{i+1} - \delta_{i+1} < r_i + \delta_i < r_{i+1} \quad for \quad 1 \leq i \leq N - 1. \quad (5)$$

So they combine to cover $[r_s, R]$ as

$$[r_s, R] \subset \bigcup_{i=1}^{i=N} (r_i - \delta_i, r_i + \delta_i). \quad (6)$$
At the star surface, we impose the Neumann boundary condition

\[ \psi'(r_s) = 0. \]  

(7)

According to Cauchy-Kowalevski theorem, in every range \((r_1 - \delta_1, r_1 + \delta_1)\), solutions \(\psi(r)\) of equation (3) uniquely exist and can be expressed with power series \([57–60]\). In the range \((r_1 - \delta_1, r_1 + \delta_1)\), the unique solution satisfying \(\psi'(r_s) = 0\) is

\[ \psi(r) = \psi(r_s). \]  

(8)

In the range \((r_2 - \delta_2, r_2 + \delta_2)\), the scalar field can be expressed as

\[ \psi(r) = \sum_{n=0}^{\infty} a_n (r - r_2)^n \]  

(9)

with \(a_n = \frac{\psi^{(n)}(r_2)}{n!}\). Since \((r_2 - \delta_2, r_1 + \delta_1) \subset (r_1 - \delta_1, r_1 + \delta_1)\), we conclude that \(\psi(r)\) is a constant \(\psi(r_s)\) in \((r_2 - \delta_2, r_1 + \delta_1)\) according to (8). Relation (5) yields \((r_2 - \delta_2, r_1 + \delta_1) \subset (r_2 - \delta_2, r_2 + \delta_2)\). So \(\psi(r)\) can be expressed as (9) in \((r_2 - \delta_2, r_1 + \delta_1)\). Also considering that (9) is a constant in \((r_2 - \delta_2, r_1 + \delta_1)\), we deduce relations

\[ a_0 = \psi(r_s), \]  

(10)

\[ a_n = 0 \quad \text{for all} \quad n \geq 1. \]  

(11)

Then in the larger range \((r_2 - \delta_2, r_2 + \delta_2)\), there is

\[ \psi(r) = a_0 = \psi(r_s). \]  

(12)

Following this analysis, in the range \((r_3 - \delta_3, r_3 + \delta_3)\), the scalar field can be expressed as

\[ \psi(r) = \sum_{n=0}^{\infty} b_n (r - r_3)^n \]  

(13)

with \(b_n = \frac{\psi^{(n)}(r_3)}{n!}\). Since \((r_3 - \delta_3, r_2 + \delta_2) \subset (r_2 - \delta_2, r_2 + \delta_2)\), we deduce that \(\psi(r)\) is a constant \(\psi(r_s)\) in \((r_3 - \delta_3, r_2 + \delta_2)\) according to (12). As (13) is a constant \(\psi(r_s)\) in \((r_3 - \delta_3, r_2 + \delta_2)\), there is

\[ b_0 = \psi(r_s), \]  

(14)

\[ b_n = 0 \quad \text{for all} \quad n \geq 1. \]  

(15)
Then in the range \((r_3 - \delta_3, r_3 + \delta_3)\), there is

\[
\psi(r) = b_0 = \psi(r_s).
\] (16)

Following this method, we can further obtain \(\psi(r) = \psi(r_s)\) in other ranges \((r_4 - \delta_4, r_4 + \delta_4), (r_5 - \delta_5, r_5 + \delta_5), \ldots, (r_N - \delta_N, r_N + \delta_N)\). So in the range \([r_s, R] \subset \bigcup_{i=1}^{N} (r_i - \delta_i, r_i + \delta_i)\), the scalar field solution is

\[
\psi(r) = \psi(r_s).
\] (17)

In particular, (17) yields the relation

\[
\psi(R) = \psi(r_s).
\] (18)

In the far region, the scalar field equation can be approximated by the differential equation

\[
\psi'' + \frac{2}{r} \psi' = 0,
\] (19)

whose general solution is

\[
\psi \sim A + \frac{B}{r},
\] (20)

with \(A\) and \(B\) as integral constants. In this work, we study the bound-state scalar configurations [61]. It means that the scalar field must asymptotically approaches zero at the infinity. So we fix \(A = 0\). Then the scalar field satisfies the asymptotical behavior

\[
\psi \sim \frac{B}{r}.
\] (21)

We can choose a large enough \(R\) satisfying

\[
\psi(R) < \psi(r_s).
\] (22)

Relation (22) is in contradiction with the relation (18), which means that the equation (3) cannot possess no nontrivial solutions. So we conclude that bound-state massless scalar field hairs cannot exist outside neutral horizonless Neumann compact stars in the asymptotically flat spherical background.

### III. CONCLUSIONS

We studied no hair theorem for bound-state static massless scalar fields outside neutral horizonless Neumann compact stars. We chose the asymptotically flat spherically symmetric spacetime. We found that the solution...
should satisfy (18), which is in contradiction with the relation (22). This contradiction leads to the fact that no nontrivial scalar field solution of equation (3) cannot exist. We concluded that neutral horizonless Neumann compact stars cannot support the bound-state massless scalar field hair in the asymptotically flat spherically symmetric background.

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