GERSTENHABER BRACKETS ON HOCHSCHILD COHOMOLOGY OF QUANTUM SYMMETRIC ALGEBRAS AND THEIR GROUP EXTENSIONS

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We construct chain maps between the bar and Koszul resolutions for a quantum symmetric algebra (skew polynomial ring). This construction uses a recursive technique involving explicit formulae for contracting homotopies. We use these chain maps to compute the Gerstenhaber bracket, obtaining a quantum version of the Schouten–Nijenhuis bracket on a symmetric algebra (polynomial ring). We compute brackets also in some cases for skew group algebras arising as group extensions of quantum symmetric algebras.

1. Introduction

Hochschild [1945] introduced homology and cohomology for algebras. Gerstenhaber [1963] studied extensively the algebraic structure of Hochschild cohomology — its cup product and graded Lie bracket (or Gerstenhaber bracket) — and consequently algebras with such structure are generally termed Gerstenhaber algebras. Many mathematicians have since investigated Hochschild cohomology for various types of algebras, and it has proven useful in many settings, including algebraic deformation theory [Gerstenhaber 1964] and support variety theory [Erdmann et al. 2004; Snashall and Solberg 2004].

The graded Lie bracket on Hochschild cohomology remains elusive in contrast to the cup product. The latter may be defined via any convenient projective resolution. The former is defined on the bar resolution, which is useful theoretically but not computationally, and one typically computes graded Lie brackets by translating to another more convenient resolution via explicit chain maps. Such
chain maps are not always easy to find. One would like to define the graded Lie structure directly on another resolution or to find efficient techniques for producing chain maps.

In this paper, we begin in Section 2 by promoting a recursive technique for constructing chain maps. The technique is not new; for example it appears in a book of Mac Lane [1975]. See also [Le and Zhou ≥ 2016] for a more general setting. We first use this technique to construct chain maps between the bar and Koszul resolutions for symmetric algebras, reproducing in Theorem 3.5 the chain maps of [Shepler and Witherspoon 2011] that had been obtained via ad hoc methods. We then construct new chain maps more generally for quantum symmetric algebras (skew polynomial rings) in Theorem 4.6. We generalize an alternative description, due to Carqueville and Murfet [2016], of these chain maps for symmetric algebras to quantum symmetric algebras in (4.8). We use these chain maps to compute the Gerstenhaber bracket on quantum symmetric algebras, generalizing the Schouten–Nijenhuis bracket on the Hochschild cohomology of polynomial rings (Theorem 5.1). We then investigate the Hochschild cohomology of a group extension of a quantum symmetric algebra, obtaining results on brackets in the special cases that the action is diagonal (Theorem 7.1) or that the Hochschild cocycles have minimal degree as maps on tensor powers of the algebra (Corollary 7.4). In the latter case, we thereby obtain a new proof that all such Hochschild 2-cocycles are noncommutative Poisson structures (cf. [Naidu and Witherspoon 2016], in which algebraic deformation theory was used instead). Some results on brackets for group extensions of polynomial rings were given in [Halbout and Tang 2010] and [Shepler and Witherspoon 2012].

Let \( \mathbb{k} \) be a field. All algebras will be associative \( \mathbb{k} \)-algebras with unity and tensor products will be taken over \( \mathbb{k} \) unless otherwise indicated.

2. Construction of comparison morphisms

Let \( A \) be a ring and let \( M \) and \( N \) be two left \( A \)-modules. Let \( P_* \) (respectively, \( Q_* \)) be a projective resolution of \( M \) (respectively, \( N \)). It is well known that given a homomorphism of \( A \)-modules \( f : M \to N \), there exists a chain map \( f_* : P_* \to Q_* \) lifting \( f \) (and different lifts are equivalent up to homotopy). Sometimes in practice we need an explicit construction of such a chain map, called a comparison morphism, to perform computations. In this section, we recall a method to construct chain maps under the condition that \( P_* \) is a free resolution (see [Mac Lane 1975, Chapter IX, Theorem 6.2]). A method for arbitrary projective resolutions will be presented in [Le and Zhou ≥ 2016].

Let us fix some notation and assumptions. Suppose that

\[
\cdots \rightarrow P_n \xrightarrow{d^n_P} P_{n-1} \xrightarrow{d^{n-1}_P} \cdots \xrightarrow{d^2_P} P_1 \xrightarrow{d^1_P} P_0 \xrightarrow{d^0_P} M \rightarrow 0
\]
is a free resolution of $M$, that is, for each $n \geq 0$, $P_n = A^{(X_n)}$ for some set $X_n$. (The module $A^{(X_n)}$ is a direct sum of copies of $A$ indexed by $X_n$. We identify each element of $X_n$ with the identity $1_A$ in the copy of $A$ indexed by that element.) Suppose that a projective resolution of $N$,

\[ \cdots \longrightarrow Q_n \overset{d_n^0}{\longrightarrow} Q_{n-1} \overset{d_{n-1}^0}{\longrightarrow} \cdots \overset{d_1^0}{\longrightarrow} Q_0 \overset{d_0^0}{\longrightarrow} N \longrightarrow 0, \]

comes equipped with a *chain contraction*: a collection of set maps $t_n : Q_n \rightarrow Q_{n+1}$ for each $n \geq 0$ and $t_{-1} : N \rightarrow Q_0$ such that for $n \geq 0$, we have $t_{n-1}d_n^Q + d_{n+1}^Q t_n = \text{Id}_{Q_n}$ and $d_0^Q t_{-1} = \text{Id}_N$. We use these next to construct a chain map, $f_n : P_n \rightarrow Q_n$ for $n \geq 0$, lifting $f_{-1} := f$. As $P_n$ is free, we need only specify the values of $f_n$ on elements of $X_n$, the generating set of $P_n$.

At first glance, it may appear that $f_n$ defined below will be the zero map, since it is defined recursively by applying the differential more than once. However, the maps $t_n$ are not in general $A$-module homomorphisms. The formula (2.1) is used only to define $f_n$ on free basis elements, and $f_n$ is then extended to an $A$-module map. In our examples the maps $t_n$ will be $k$-linear, but for the construction, they are only required to be maps of sets, since we apply them only to basis elements. In this weaker setting, such a collection of maps may be called a *weak self-homotopy* as in [Bian et al. 2009].

For $n = 0$, given $x \in X_0$, define $f_0(x) = t_{-1}fd_0^P(x)$. Then $d_0^Q f_0(x) = d_0^Q t_{-1}fd_0^P(x) = fd_0^P(x)$.

Suppose that we have constructed $f_0, \ldots, f_{n-1}$ such that for $0 \leq i \leq n-1$, $d_i^Q f_i = f_{i-1}d_i^P$. For $x \in X_n$, define

(2.1) \[ f_n(x) = t_{n-1} f_{n-1} d_n^P(x). \]

Then

\[ d_n^Q f_n(x) = d_n^Q t_{n-1} f_{n-1} d_n^P(x) \]
\[ = f_{n-1} d_n^P(x) - t_{n-2} d_{n-1}^Q f_{n-1} d_n^P(x) \]
\[ = f_{n-1} d_n^P(x) - t_{n-2} f_{n-2} d_{n-1}^P d_n^P(x) \]
\[ = f_{n-1} d_n^P(x). \]

This proves the following.

**Proposition 2.2.** The maps $f_n$ defined in (2.1) form a chain map from $P_\bullet$ to $Q_\bullet$, lifting $f : M \rightarrow N$.

In the next two sections, we use this formula (2.1) to find explicit chain maps for symmetric and quantum symmetric algebras, and in the rest of this article we use the chain maps thus found in computations of Gerstenhaber brackets for these algebras and their group extensions.
3. Chain contractions and comparison maps for polynomial algebras

Let $N$ be a positive integer. Let $V$ be a vector space over the field $\mathbb{k}$ with basis $x_1, \ldots, x_N$, and let

$$S(V) := \mathbb{k}[x_1, \ldots, x_N]$$

be the polynomial algebra in $N$ indeterminates. This is a Koszul algebra, so there is a standard complex $K_*(S(V))$ that is a free resolution of $A := S(V)$ as an $A$-bimodule (equivalently as an $A^e$-module where $A^e = A \otimes A^{\text{op}}$). We recall this complex next: for each $p$, let $\wedge^p(V)$ denote the $p$-th exterior power of $V$. Then $K_*(S(V))$ is the complex

$$\cdots \rightarrow A \otimes \wedge^2(V) \otimes A \xrightarrow{d_2} A \otimes \wedge^1(V) \otimes A \xrightarrow{d_1} A \otimes A \xrightarrow{d_0} A \rightarrow 0;$$

that is, for $0 \leq p \leq N$, the degree $p$ term is $K_p(S(V)) := A \otimes \wedge^p(V) \otimes A$. The differential $d_p$ is defined by

$$d_p(1 \otimes (x_{j_1} \wedge \cdots \wedge x_{j_p})) \otimes 1)$$

$$= \sum_{i=1}^{p} (-1)^i+1 x_{j_i} \otimes (x_{j_1} \wedge \cdots \wedge \hat{x}_{j_i} \wedge \cdots \wedge x_{j_p}) \otimes 1$$

$$- \sum_{i=1}^{p} (-1)^i \otimes (x_{j_1} \wedge \cdots \wedge \hat{x}_{j_i} \wedge \cdots \wedge x_{j_p}) \otimes x_{j_i}$$

whenever $1 \leq j_1 < \cdots < j_p \leq N$ and $p > 0$; the notation $\hat{x}_{j_i}$ indicates that the factor $x_{j_i}$ is deleted. The map $d_0$ is multiplication.

From now on, we will write $\underline{\ell} = (\ell_1, \ldots, \ell_N)$, an $N$-tuple of nonnegative integers, $\underline{\chi} = (x_1, \ldots, x_N)$ and $\underline{\chi}^{\underline{\ell}} = x_1^{\ell_1} \cdots x_N^{\ell_N}$. We shall give a chain contraction of $K_*(S(V))$ consisting of maps $t_{-1} : A \rightarrow A \otimes A$ and

$$t_p : A \otimes \wedge^p(V) \otimes A \rightarrow A \otimes \wedge^{p+1}(V) \otimes A$$

for $p \geq 0$. These maps will be left $A$-module homomorphisms, and thus we need only define them on choices of free basis elements of these free left $A$-modules.

To define $t_{-1}$, it suffices to specify $t_{-1}(1) = 1 \otimes 1$ and extend it $A$-linearly. If $p = 0$ and $\underline{\ell} \in \mathbb{N}^N$, define

$$t_0(1 \otimes \underline{\chi}^{\underline{\ell}}) = - \sum_{j=1}^{N} \sum_{r=1}^{\ell_j} (x_j^{\ell_j-r} x_{j+1}^{\ell_{j+1}} \cdots x_N^{\ell_N}) \otimes x_j \otimes (x_1^{\ell_1} \cdots x_{j-1}^{\ell_{j-1}} x_j^{r-1}).$$

If $p \geq 1$, it suffices to give

$$t_p(1 \otimes (x_{j_1} \wedge \cdots \wedge x_{j_p}) \otimes \underline{\chi}^{\underline{\ell}})$$
for $\ell \in \mathbb{N}^N$ and $1 \leq j_1 < \cdots < j_p \leq N$, and we set
\[
(t_p(1 \otimes (x_{j_1} \wedge \cdots \wedge x_{j_p}) \otimes x_{\ell}) = (-1)^{p+1} \sum_{j_{p+1}=j_p+1}^N \sum_{r=1}^{\ell_{j_p+1}} (x_{j_{p+1}}^{\ell_{j_p+1}-r} x_{j_{p+1}+1} \cdots x_N^r) \otimes (x_{j_1} \wedge \cdots \wedge x_{j_{p+1}}) \otimes (x_1^{\ell_{j_p+1}-1} x_{j_{p+1}+1}^{r-1}).
\]
In the case $j_p = N$, the sum is empty, and so the value of $t_p$ on such an element is 0.

**Proposition 3.1.** The above-defined maps $t_p$, $p \geq -1$, form a chain contraction for the resolution $K_*(S(V))$.

**Proof.** It is easy to verify that $d_0 t_{-1} = \mathrm{Id}$. We need to show that for $p \geq 0$, $t_{p-1} d_p + d_{p+1} t_p = \mathrm{Id}$. We first let $p = 0$, and show that $t_{-1} d_0 + d_1 t_0 = \mathrm{Id}$.

For $\ell \in \mathbb{N}^N$, we have $t_{-1} d_0 (1 \otimes x_{\ell}) = t_{-1} (x_{\ell}) = x_{\ell} \otimes 1$, and
\[
d_1 t_0 (1 \otimes x_{\ell}) = d_1 \left( -\sum_{j=1}^N \sum_{r=1}^{\ell_j} x_j^{\ell_j-r} x_{j+1}^{\ell_{j+1}} \cdots x_N^{\ell_N} \otimes x_j x_1^{\ell_1} \cdots x_{j-1}^{\ell_{j-1}} x_j^{r-1} \right)
\]
\[
= -\sum_{j=1}^N \sum_{r=0}^{\ell_j-1} x_j^{\ell_j-r} x_{j+1}^{\ell_{j+1}} \cdots x_N^{\ell_N} \otimes x_1^{\ell_1} \cdots x_{j-1}^{\ell_{j-1}} x_j^{r} + \sum_{j=1}^N \sum_{r=1}^{\ell_j} x_j^{\ell_j-r} x_{j+1}^{\ell_{j+1}} \cdots x_N^{\ell_N} \otimes x_1^{\ell_1} \cdots x_{j-1}^{\ell_{j-1}} x_j^{r}
\]
\[
= -\sum_{j=1}^N x_j^{\ell_j} x_{j+1}^{\ell_{j+1}} \cdots x_N^{\ell_N} \otimes x_1^{\ell_1} \cdots x_{j-1}^{\ell_{j-1}} + \sum_{j=1}^N x_j^{\ell_j} x_{j+1}^{\ell_{j+1}} \cdots x_N^{\ell_N} \otimes x_1^{\ell_1} \cdots x_{j-1}^{\ell_{j-1}} x_j^{\ell_j}
\]
\[
= -\sum_{j=1}^N x_j^{\ell_j} x_{j+1}^{\ell_{j+1}} \cdots x_N^{\ell_N} \otimes x_1^{\ell_1} \cdots x_{j-1}^{\ell_{j-1}} + \sum_{j=2}^{N+1} x_j^{\ell_j} x_{j+1}^{\ell_{j+1}} \cdots x_N^{\ell_N} \otimes x_1^{\ell_1} \cdots x_{j-1}^{\ell_{j-1}} x_j^{\ell_j}
\]
\[
= -x_{\ell} \otimes 1 + 1 \otimes x_{\ell}.\]
We thus obtain \((t_1 d_0 + d_1 t_0)(1 \otimes x^\ell) = x^\ell \otimes 1 - x^\ell \otimes 1 + 1 \otimes x^\ell = 1 \otimes x^\ell\) and therefore confirm the equality. Note that in the above proof, there are many terms which cancel one another.

The proof of the equality \(t_{p-1} d_p + d_{p+1} t_p = \text{Id}\) for \(p \geq 1\) is similar to the above case \(p = 0\), but is much more complicated. As in the case \(p = 0\), for the cases \(p \geq 1\) we must change indices several times in order to cancel many terms.

Now we can use the chain contraction of Proposition 3.1 to give formulae for comparison morphisms between the normalized bar resolution and the Koszul resolution. Such comparison morphisms were found in [Shepler and Witherspoon 2011] by ad hoc methods.

For any \(\mathbb{k}\)-algebra \(A\) associative with unity, denote by \(\overline{A} = A/(\mathbb{k} \cdot 1)\) a \(\mathbb{k}\)-vector space. The normalized bar resolution of \(A\) has \(p\)-th term \(B^p = A \otimes A^\otimes p \otimes A\) and differentials \(\delta_p : A \otimes A^\otimes p \otimes A \to A \otimes A^\otimes (p-1) \otimes A\) given by

\[
\delta_p(a_0 \otimes \overline{a}_1 \otimes \cdots \otimes \overline{a}_p \otimes a_{p+1}) = \sum_{i=0}^{p} (-1)^i a_0 \otimes \cdots \otimes \overline{a}_i \overline{a}_{i+1} \otimes \cdots \otimes a_{p+1}
\]

for \(a_0, \ldots, a_{p+1} \in A\), where an overline indicates an image in \(\overline{A}\). We shall see that this resolution is suitable for computation using the method from Section 2.

There is a standard chain contraction of the normalized bar resolution,

\[
s_p : A \otimes A^\otimes p \otimes A \to A \otimes A^\otimes (p+1) \otimes A,
\]

given by

\[
(3.2) \quad s_p(1 \otimes \overline{a}_1 \otimes \cdots \otimes \overline{a}_p \otimes a_{p+1}) = (-1)^{p+1} \otimes \overline{a}_1 \otimes \cdots \otimes \overline{a}_p \otimes \overline{a}_{p+1} \otimes 1.
\]

Each \(s_p\) is then extended to a left \(A\)-module homomorphism. For convenience, we shall from now on abuse notation and write \(a_i\) in place of \(\overline{a}_i\).

A chain map from the Koszul resolution to the normalized bar resolution is given by the standard embedding: for \(p \geq 0\), define

\[
\Phi_p : A \otimes \wedge^p (V) \otimes A \to A \otimes A^\otimes p \otimes A
\]

by

\[
(3.3) \quad \Phi_p(1 \otimes (x_{j_1} \wedge \cdots \wedge x_{j_p}) \otimes 1) = \sum_{\pi \in \text{Sym}_p} \text{sgn} \pi \otimes x_{j_{\pi(1)}} \otimes \cdots \otimes x_{j_{\pi(p)}} \otimes 1
\]

for \(1 \leq j_1 < \cdots < j_p \leq N\), where \(\text{Sym}_p\) denotes the symmetric group on \(p\) symbols. The other direction is much more complicated. We shall define \(\Psi_p : A \otimes A^\otimes p \otimes A \to A \otimes \wedge^p (V) \otimes A\) for each \(p \geq 0\). Let \(\Psi_0\) be the identity map. For \(p \geq 1\),
where, as in [Shepler and Witherspoon 2011], the $N$-tuple $\hat{Q}^{(\xi_1,\ldots,\xi_p;\ell_1,\ldots,\ell_p)}$ is defined by

$$\hat{Q}^{(\xi_1,\ldots,\xi_p;\ell_1,\ldots,\ell_p)}(r_1,\ldots,r_p) = (\ell_1 + \cdots + \ell_s - 1, r_2, \ldots, r_s)$$

and where the $N$-tuple $\hat{Q}^{(\xi_1,\ldots,\xi_p;\ell_1,\ldots,\ell_p)}$ is defined to be complementary to $\hat{Q}^{(\xi_1,\ldots,\xi_p;\ell_1,\ldots,\ell_p)}$ in the sense that

$$\hat{Q}(r_1,\ldots,r_p) - \hat{Q}(r_1,\ldots,r_p) = \hat{Q}(r_1,\ldots,r_p).$$

Theorem 3.5 [Shepler and Witherspoon 2011]. Let $\Phi$ and $\Psi$ be as defined in (3.3) and (3.4). Then

(i) the map $\Phi$ is a chain map from the Koszul resolution to the normalized bar resolution;

(ii) the map $\Psi$ is a chain map from the normalized bar resolution to the Koszul resolution;

(iii) the composition $\Psi \circ \Phi$ is the identity map.

Proof. (i) We check that this standard map follows from the method in Section 2, in order to illustrate the method. We proceed by induction, applying (2.1) to the chain contraction $s_\Phi$ of the normalized bar resolution defined in (3.2).

The case $p = 0$ is trivial. Now suppose that for $p \geq 0$, $\Phi_p : A \otimes \Lambda^p(V) \otimes A \to A \otimes \Lambda^\otimes A \otimes A$ is given by (3.3). We compute $\Phi_{p+1}(1 \otimes (x_{j_1} \wedge \cdots \wedge x_{j_{p+1}}) \otimes 1)$, where $\Phi_{p+1}$ is defined by (2.1) in terms of $\Phi_p$. We have

$$\Phi_{p+1}(1 \otimes (x_{j_1} \wedge \cdots \wedge x_{j_{p+1}}) \otimes 1)$$

$$= s_p \Phi_p d_{p+1}(1 \otimes (x_{j_1} \wedge \cdots \wedge x_{j_{p+1}}) \otimes 1)$$

$$= s_p \Phi_p \left( \sum_{i=1}^{p+1} (-1)^{i+1} x_{j_i} \otimes (x_{j_1} \wedge \cdots \wedge \hat{x}_{j_i} \cdots \wedge x_{j_{p+1}}) \otimes 1 \right)$$

$$- s_p \Phi_p \left( \sum_{i=1}^{p+1} (-1)^{i+1} \otimes (x_{j_1} \wedge \cdots \wedge \hat{x}_{j_i} \cdots \wedge x_{j_{p+1}}) \otimes x_{j_i} \right).$$
Notice that the value of $s_p$ on
\[
\Phi_p \left( \sum_{i=1}^{p+1} (-1)^{i+1} x_{j_i} \otimes (x_{j_1} \wedge \cdots \wedge \hat{x}_{j_i} \wedge \cdots \wedge x_{j_{p+1}}) \otimes 1 \right)
\]
is 0, since the rightmost tensor factor is 1, and we work with the normalized bar resolution. For a permutation $\pi \in \text{Sym}_p$ that fixes some letter $i$, $1 \leq i \leq p+1$, consider the permutation $\hat{\pi}$ of the set $\{1, \ldots, i-1, i, i+1, \ldots, p+1\}$ corresponding to $\pi$ via the bijection
\[
\{1, \ldots, i-1, i, i+1, \ldots, p\} \simeq \{1, \ldots, i-1, \hat{i}, i+1, \ldots, p+1\}
\]
sending $j$ to $j$ for $1 \leq j \leq i-1$ and to $j+1$ for $i \leq j \leq p$.

Define a new permutation $\tilde{\pi} \in S_{p+1}$ by imposing
\[
\tilde{\pi}(j) = \begin{cases} 
\hat{\pi}(j) & \text{for } j < i, \\
\hat{\pi}(j+1) & \text{for } i \leq j < p+1, \\
i & \text{for } j = p+1.
\end{cases}
\]
Then we have $\text{sgn} \tilde{\pi} = (-1)^{p-i+1} \text{sgn} \pi$, and so
\[
\Phi_{p+1}(1 \otimes (x_{j_1} \wedge \cdots \wedge x_{j_{p+1}}) \otimes 1)
\]
\[
= -s_p \Phi_p \left( \sum_{i=1}^{p+1} (-1)^{i+1} \otimes (x_{j_1} \wedge \cdots \wedge \hat{x}_{j_i} \wedge \cdots \wedge x_{j_{p+1}}) \otimes x_{j_i} \right)
\]
\[
= -s_p \left( \sum_{i=1}^{p+1} (-1)^{i+1} \sum_{\tilde{\pi} \in S_{p+1} \atop \tilde{\pi}(p+1)=i} (-1)^{p-i+1} \text{sgn} \tilde{\pi} \otimes x_{j_{\tilde{\pi}(1)}} \otimes \cdots \otimes x_{j_{\tilde{\pi}(p)}} \otimes x_{j_{\tilde{\pi}(p+1)}} \right)
\]
\[
= -(-1)^{p+1} \sum_{i=1}^{p+1} (-1)^{i+1} \sum_{\tilde{\pi} \in S_{p+1} \atop \tilde{\pi}(p+1)=i} (-1)^{p-i+1} \text{sgn} \tilde{\pi} \otimes x_{j_{\tilde{\pi}(1)}} \otimes \cdots \otimes x_{j_{\tilde{\pi}(p)}} \otimes x_{j_{\tilde{\pi}(p+1)}} \otimes 1
\]
\[
= \sum_{\tilde{\pi} \in S_{p+1} \atop \tilde{\pi}(p+1)=i} \text{sgn} \tilde{\pi} \otimes x_{j_{\tilde{\pi}(1)}} \otimes \cdots \otimes x_{j_{\tilde{\pi}(p)}} \otimes x_{j_{\tilde{\pi}(p+1)}} \otimes 1.
\]
This completes the proof of (i).

(ii) As in (i), we apply the method in Section 2 to the chain contraction $t_\ast$ of Proposition 3.1 to show that $\Psi_\ast$ as defined in (3.4) is indeed the resulting chain map. We proceed by induction on $p$. 
Suppose that $\Psi_p$ is given by (3.4). Let us apply (2.1) and show that $\Psi_{p+1}$ results. First notice that we can write

$$t_p(1 \otimes (x_{j_1} \land \cdots \land x_{j_p}) \otimes x^{\ell})$$

$$= (-1)^{p+1} \sum_{j_{p+1} = j_p + 1}^{N} \sum_{r=1}^{\ell_{j_{p+1}}} x^{Q_r^{(\ell; j_{p+1})}} \otimes x_{j_1} \land \cdots \land x_{j_p} \land x_{j_{p+1}} \otimes x^{Q_r^{(\ell; j_{p+1})}}.$$

We have

$$d_{p+1}(1 \otimes x^{\ell_1} \otimes \cdots \otimes x^{\ell_{p+1}} \otimes 1)$$

$$= x^{\ell_1} \otimes x^{\ell_2} \otimes \cdots \otimes x^{\ell_{p+1}} \otimes 1 + \sum_{i=1}^{p} (-1)^p x^{\ell_1} \otimes \cdots \otimes x^{\ell_i + \ell_{i+1}} \otimes \cdots \otimes x^{\ell_{p+1}} \otimes 1$$

$$+ (-1)^{p+1} \otimes x^{\ell_1} \otimes \cdots \otimes x^{\ell_p} \otimes x^{\ell_{p+1}}.$$

Now consider

$$\Psi_p(x^{\ell_1} \otimes x^{\ell_2} \otimes \cdots \otimes x^{\ell_{p+1}} \otimes 1)$$

$$= \sum_{1 \leq j_1 < \cdots < j_p \leq N} \sum_{1 \leq r_s \leq \ell_{j_s}, 1 \leq s \leq p} x^{Q_r^{(\ell_2, \ldots, \ell_{p+1}; j_1, \ldots, j_p)}} \otimes x_{j_1} \land \cdots \land x_{j_p} \otimes x^{Q_r^{(\ell_2, \ldots, \ell_{p+1}; j_1, \ldots, j_p)}}.$$

However, $\hat{Q}_r^{(\ell_2, \ldots, \ell_{p+1}; j_1, \ldots, j_p)}$, by definition, has no terms of the form $x_u^v$ with $u > j_p$. Thus, we have $t_p \Psi_p(x^{\ell_1} \otimes x^{\ell_2} \otimes \cdots \otimes x^{\ell_{p+1}} \otimes 1) = 0$.

Similarly we can prove that for $1 \leq i \leq p$,

$$t_p \Psi_p(1 \otimes x^{\ell_1} \otimes \cdots \otimes x^{\ell_i + \ell_{i+1}} \otimes \cdots \otimes x^{\ell_{p+1}} \otimes 1) = 0.$$

The only term left is $t_p \Psi_p((-1)^{p+1} x^{\ell_1} \otimes x^{\ell_2} \otimes \cdots \otimes x^{\ell_p} \otimes x^{\ell_{p+1}})$. We obtain

$$t_p \Psi_p((-1)^{p+1} x^{\ell_1} \otimes x^{\ell_2} \otimes \cdots \otimes x^{\ell_p} \otimes x^{\ell_{p+1}})$$

$$= (-1)^{p+1} \sum_{1 \leq j_1 < \cdots < j_p \leq N} \sum_{1 \leq r_s \leq \ell_{j_s}, 1 \leq s \leq p} t_p \left(x^{Q_r^{(\ell_1, \ldots, \ell_p; j_1, \ldots, j_p)}} \otimes x_{j_1} \land \cdots \land x_{j_p} \right.$$}

$$\left. \otimes x^{Q_r^{(\ell_1, \ldots, \ell_p; j_1, \ldots, j_p)}} \right) \otimes x^{\ell_{p+1}}.$$
\[
\sum_{1 \leq j_1 < \cdots < j_p \leq N} \sum_{1 \leq r_s \leq \ell_{j_s}} \sum_{r_p+1}^{N} \mathcal{O}^{(\ell_1, \ldots, \ell_p; j_1, \ldots, j_p)}_{(r_1, \ldots, r_p)} \otimes x_{j_1} \wedge \cdots \wedge x_{j_p+1} \otimes \mathcal{O}^{(\ell; j_p+1)}_{r_p+1},
\]

where
\[
\ell = \hat{\mathcal{O}}^{(\ell_1, \ldots, \ell_p; j_1, \ldots, j_p)}_{(r_1, \ldots, r_p)} + \ell_{p+1}.
\]

Now notice that
\[
\mathcal{O}^{(\ell_1, \ldots, \ell_p; j_1, \ldots, j_p)}_{(r_1, \ldots, r_p)} + \mathcal{O}^{(\ell; j_p+1)}_{r_p+1} = \mathcal{O}^{(\ell_1, \ldots, \ell_{p+1}; j_1, \ldots, j_p+1)}_{(r_1, \ldots, r_p+1)}
\]
and
\[
\hat{\mathcal{O}}^{(\ell; j_p+1)}_{r_p+1} = \hat{\mathcal{O}}^{(\ell_1, \ldots, \ell_{p+1}; j_1, \ldots, j_p+1)}_{(r_1, \ldots, r_p+1)}.
\]

We have the desired result:
\[
t_p \Psi_p d_{p+1} (1 \otimes x^{\ell_1} \otimes \cdots \otimes x^{\ell_{p+1}} \otimes 1)
= t_p \Psi_p ((-1)^{p+1} x^{\ell_1} \otimes \cdots \otimes x^{\ell_p} x^{\ell_{p+1}})
\]
\[
= \sum_{1 \leq j_1 < \cdots < j_{p+1} \leq N} \sum_{1 \leq s \leq \ell_{j_s}} \sum_{r_{p+1}}^{N} \mathcal{O}^{(\ell_1, \ldots, \ell_{p+1}; j_1, \ldots, j_{p+1})}_{(r_1, \ldots, r_{p+1})} \otimes x_{j_1} \wedge \cdots \wedge x_{j_{p+1}} \otimes \mathcal{O}^{(\ell_1, \ldots, \ell_{p+1}; j_1, \ldots, j_{p+1})}_{(r_1, \ldots, r_{p+1})}
\]
\[
= \Psi_{p+1} (1 \otimes x^{\ell_1} \otimes \cdots \otimes x^{\ell_{p+1}} \otimes 1).
\]

(iii) For \(1 \leq i_1 < \cdots < i_p \leq N\), we have
\[
\Psi_p \Phi_p (1 \otimes (x_{i_1} \wedge \cdots \wedge x_{i_p}) \otimes 1)
\]
\[
= \Psi_p \left( \sum_{\pi \in \text{Sym}_p} \text{sgn} \pi \otimes x_{i_{\pi(1)}} \otimes \cdots \otimes x_{i_{\pi(p)}} \otimes 1 \right)
\]
\[
= \sum_{\pi \in \text{Sym}_p} \text{sgn} \pi \sum_{1 \leq j_1 < \cdots < j_{p} \leq N} \sum_{0 \leq r_s \leq (e_i_{\pi(s)})_{j_s-1}} \sum_{s=1, \ldots, p}^{N} \mathcal{O}^{(e_{\pi(1)}, \ldots, e_{\pi(p)}; j_1, \ldots, j_p)}_{(r_1, \ldots, r_p)} \otimes x_{j_1} \wedge \cdots \wedge x_{j_p} \otimes \hat{\mathcal{O}}
\]

where \(e_u\) is the \(u\)-th canonical basis vector \((0, \ldots, 0, 1, 0, \ldots, 0)\), the 1 in the \(u\)-th position, and
\[
\hat{\mathcal{O}} = \hat{\mathcal{O}}^{(e_{\pi(1)}, \ldots, e_{\pi(p)}; j_1, \ldots, j_p)}_{(r_1, \ldots, r_p)}.
\]
Notice that $Q^{(e_i\pi(1), \ldots, e_i\pi(p); j_1, \ldots, j_p)}_{(r_1, \ldots, r_p)}$ occurs in the sum only if $(i_\pi(1), \ldots, i_\pi(p)) = (j_1, \ldots, j_p)$. Here, $\pi$ is the identity, $r_1 = \cdots = r_p = 0$ and $Q^{(e_i\pi(1), \ldots, e_i\pi(p); j_1, \ldots, j_p)}_{(r_1, \ldots, r_p)}$ is the zero vector. Therefore,

$$\Psi_p \Phi_p (1 \otimes x_{i_1} \land \cdots \land x_{i_p} \otimes 1) = 1 \otimes x_{i_1} \land \cdots \land x_{i_p} \otimes 1. \quad \Box$$

For comparison, we give an alternative description of the maps $\Psi_p$ due to Carqueville and Murfet [2016]: for each $i$, let $\tau_i : S(V)^e \to S(V)^e$ be the $k$-linear map that is defined on monomials as follows. (We denote application of the map $\tau_i$ by a left superscript.)

$$\tau_i(x_{j_1} \cdots x_{j_N} \otimes x_{l_1} \cdots x_{l_N}) = x_{j_1} \cdots x_{i-1} x_{i+1} \cdots x_{j_N} \otimes x_{l_1} \cdots x_{i-1} x_{i+1} \cdots x_{l_N}.$$  

Define difference quotient operators $\partial[i] : S(V) \to S(V)^e$ for each $i$, $1 \leq i \leq N$, as in [Carqueville and Murfet 2016, (2.12)] by

$$\partial[i](f) := \frac{\tau_{1 \cdots i-1}(f \otimes 1) - \tau_{1 \cdots i}(f \otimes 1)}{x_i \otimes 1 - 1 \otimes x_i}.$$  

For example, $\tau_1(x_1^2 x_2 \otimes 1) = x_2 \otimes x_1^2$, so that

$$\partial[1](x_1^2 x_2) = \frac{x_1^2 x_2 \otimes 1 - x_2 \otimes x_1^2}{x_1 \otimes 1 - 1 \otimes x_1} = x_1 x_2 \otimes 1 + x_2 \otimes x_1.$$  

Similarly, $\partial[2](x_1^2 x_2) = 1 \otimes x_1^2$.

Identify elements in $S(V)^e \otimes \wedge^p(V)$ with elements in $S(V) \otimes \wedge^p(V) \otimes S(V)$ via the canonical isomorphism between these two spaces. Then $\Psi_p$ may be expressed as in [Carqueville and Murfet 2016, (2.22)]:

$$\Psi_p (1 \otimes x_{\ell_1} \otimes \cdots \otimes x_{\ell_p} \otimes 1) = \sum_{1 \leq j_1 < \cdots < j_p \leq N} \left( \prod_{s=1}^p \partial[j_s](x_{\ell_s}) \right) \otimes x_{j_1} \land \cdots \land x_{j_p}.$$  

For example, if $N = 2$, then

$$\Psi_2 (1 \otimes x_1^2 x_2 \otimes 1) = x_1 x_2 \otimes 1 \otimes x_1 + x_2 \otimes x_1 \otimes x_1 + 1 \otimes x_1^2 \otimes x_2.$$  

We may similarly express the chain contraction $t_p$ as

$$t_p (1 \otimes x_{j_1} \land \cdots \land x_{j_p} \otimes x_{\ell}) = (-1)^{p+1} \sum_{j_{p+1} = j_{p+1}}^{N} \partial[j_{p+1}](x_{\ell}) \otimes x_{j_1} \land \cdots \land x_{j_{p+1}}.$$  

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4. Chain contractions and comparison maps for quantum symmetric algebras

Let $N$ be a positive integer, and for each pair $i, j \in \{1, 2, \ldots, N\}$, let $q_{i,j}$ be a nonzero scalar in the field $\mathbb{k}$ such that $q_{i,i} = 1$ and $q_{j,i} = q_{i,j}^{-1}$ for all $i, j$. Denote by $q$ the corresponding tuple of scalars, $q := (q_{i,j})_{1 \leq i, j \leq N}$. Let $V$ be a vector space with basis $x_1, \ldots, x_N$, and let

\[ S_q(V) := k \langle x_1, \ldots, x_N \mid x_i x_j = q_{i,j} x_j x_i \text{ for all } 1 \leq i, j \leq N \rangle, \]

be the quantum symmetric algebra determined by $q$. This is a Koszul algebra, and there is a standard complex $K_*(S_q(V))$ that is a free resolution of $S_q(V)$ as an $S_q(V)$-bimodule (see, e.g., [Wambst 1993, Proposition 4.1(c)]). Setting $A = S_q(V)$, the complex is

\[ \cdots \to A \otimes \wedge^2(V) \otimes A \xrightarrow{d_2} A \otimes \wedge^1(V) \otimes A \xrightarrow{d_1} A \otimes A \xrightarrow{(d_0, A \to 0)}, \]

with differential $d_p$ defined by

\[ d_p(1 \otimes (x_{j_1} \wedge \cdots \wedge x_{j_p}) \otimes 1) = \sum_{i=1}^{p} (-1)^{i+1} \left( \prod_{s=1}^{i} q_{j_s,j_i} \right) x_{j_i} \otimes (x_{j_1} \wedge \cdots \wedge \hat{x}_{j_i} \wedge \cdots \wedge x_{j_p}) \otimes 1 \]

\[ - \sum_{i=1}^{p} (-1)^{i+1} \left( \prod_{s=i}^{p} q_{j_i,j_s} \right) \otimes (x_{j_1} \wedge \cdots \wedge \hat{x}_{j_i} \wedge \cdots \wedge x_{j_p}) \otimes x_{j_i} \]

whenever $1 \leq j_1 < \cdots < j_p \leq N$ and $p > 0$; the map $d_0$ is multiplication.

As in the previous section, we write $\ell = (\ell_1, \ldots, \ell_N)$, $\chi = (x_1, \ldots, x_N)$ and $\chi^\ell = x_1^{\ell_1} \cdots x_N^{\ell_N}$. We shall give a chain contraction of $K_*(S_q(V))$,

\[ t_p : A \otimes \wedge^p(V) \otimes A \to A \otimes \wedge^{p+1}(V) \otimes A \]

for $p \geq 0$ and $t_{-1} : A \to A \otimes A$, which are moreover left $A$-module homomorphisms (cf. [Wambst 1993]).

Let $t_{-1}(1) = 1 \otimes 1$ and extend $t_{-1}$ to be left $A$-linear. For $p \geq 0$, $\ell \in \mathbb{N}^N$, and $1 \leq j_1 < \cdots < j_p \leq N$, let

\[ t_p(1 \otimes (x_{j_1} \wedge \cdots \wedge x_{j_p}) \otimes \chi^\ell) = (-1)^{p+1} \sum_{j_{p+1}=j_{p}+1}^{N} \sum_{r=1}^{\ell_{j_{p+1}}} \lambda_{j_{p+1}, r}^{j_{j_1}, \ldots, j_{j_{p+1}}} x_{j_{p+1}+1}^{\ell_{j_{p+1}}+1} \cdots x_N^{\ell_N} \]

\[ \otimes x_{j_1} \wedge \cdots \wedge x_{j_{p+1}} \otimes x_1^{\ell_1} \cdots x_{j_{p+1}-1}^{\ell_{j_{p+1}-1}} x_{j_{p+1}}, \]
where

\[ \lambda_{j_{p+1},r}^{(\ell; j_1, \ldots, j_p)} = \left( \prod_{s=1}^{j_{p+1}-1} \sum_{t=j_{p+1}}^{N} q_{s,t} \right) \left( \prod_{t=1}^{N} q_{j_{p+1},t} \right) \left( \prod_{t=1}^{p} \frac{\ell_{j_{p+1},r}}{\ell_{j_{p+1},t}} \right) \left( \prod_{s=1}^{p+1} q_{j_{s+1},t} \right) \]

Compared with the maps in the previous section for polynomial algebras, the only difference is that now there is a new coefficient. This (rather complicated) coefficient \( \lambda_{j_{p+1},r}^{(\ell; j_1, \ldots, j_p)} \) can be obtained as follows: in the right side of the formula for \( t_p \), in comparison to its argument \( 1 \otimes x_{j_1} \wedge \cdots \wedge x_{j_p} \otimes x_{j}^{\ell} \) on the left side, whenever a factor \( x_i \) of \( x_{j}^{\ell} \) has changed positions so that it is now to the left of a factor \( x_j \) with \( i > j \) (including factors of the exterior product), one should include one factor of \( q_{j,i} \). One can verify easily that \( \lambda_{j_{p+1},r}^{(\ell; j_1, \ldots, j_p)} \) has the given form. We shall call this rule the twisting principle and we use it several times later.

**Proposition 4.2.** The above-defined maps \( t_p, p \geq -1 \), form a chain contraction over the resolution \( K_*(S_q(V)) \).

**Proof.** One needs to verify that for \( n \geq 0 \), we have \( t_{n-1}d_n + d_{n+1}t_n = \text{Id} \) and \( d_0t_{-1} = \text{Id} \). Notice that the computation used in the above equalities is the same as that for polynomial algebras, except that now for quantum symmetric algebras, we have some extra coefficients. One needs to show that these extra coefficients do not cause any problem.

Recall that in the proof of Proposition 3.1, the concrete computation is simplified by many terms which cancel one another. For example, this occurs in the verification of the equation \( t_{-1}d_0 + d_1t_0 = \text{Id} \) in the proof of Proposition 3.1. For polynomial algebras, the proof works due to these cancelling terms.

For quantum symmetric algebras, things are not so easy. However, the twisting principle always holds; that is, when we apply a differential or chain contraction, once we produce a monomial (always in lexicographical order) or tensor of monomials, we need to include a coefficient before this monomial according to the twisting principle. Thus, if two terms cancel each other for polynomial algebras, as we have included the same coefficient, they still cancel for quantum symmetric algebras.

Now we can use (2.1) and the chain contraction of Proposition 4.2 to give formulae for comparison morphisms between the normalized bar resolution and the Koszul resolution. A chain map from the Koszul resolution to the normalized bar resolution is induced from the standard embedding of the Koszul resolution into the (unnormalized) bar resolution. See also [Wambst 1993, Lemma 5.3 and Theorem 5.4] for a more general setting. We give the formula as it appears in [Naidu et al. 2011,


§2.2(3)]. For \( p \geq 0 \), we define

\[
\Phi_p : A \otimes \bigwedge^p (V) \otimes A \to A \otimes \overline{A}^\otimes p \otimes A
\]

by

\[
(4.3) \quad \Phi_p (1 \otimes (x_{j_1} \wedge \cdots \wedge x_{j_p}) \otimes 1) = \sum_{\pi \in \text{Sym}_p} \sgn \pi q_{\pi}^{j_1, \ldots, j_p} \otimes x_{j_{\pi(1)}} \otimes \cdots \otimes x_{j_{\pi(p)}} \otimes 1
\]

for \( 1 \leq j_1 < \cdots < j_p \leq N \). In the above formula, the coefficients \( q_{\pi}^{j_1, \ldots, j_p} \) are the scalars obtained from the twisting principle, that is,

\[
(4.4) \quad q_{\pi}^{j_1, \ldots, j_p} x_{j_{\pi(1)}} \cdots x_{j_{\pi(p)}} = x_{j_1} \cdots x_{j_p}.
\]

The other direction is much more complicated. We shall see that for quantum symmetric algebras, the comparison morphism is a twisted version of that for a polynomial ring given in the previous section, with certain coefficients included according to the twisting principle.

We define the maps

\[
\Psi_p : A \otimes \overline{A}^\otimes p \otimes A \to A \otimes \bigwedge^p (V) \otimes A
\]

as follows. Let \( \Psi_0 \) be the identity map. For \( p \geq 1 \), define \( \Psi_p \) by

\[
(4.5) \quad \Psi_p (1 \otimes \sum_{s=1}^{\ell^1} \otimes \cdots \otimes \sum_{s=1}^{\ell^p} 1)
\]

\[
= \sum_{1 \leq j_1 < \cdots < j_p \leq N} \sum_{s=1}^{\ell^s} \mu_{(r_1, \ldots, r_p)} (\ell^1, \ldots, \ell^p ; j_1, \ldots, j_p) \times Q_{(r_1, \ldots, r_p)}^{(\ell^1, \ldots, \ell^p ; j_1, \ldots, j_p)}
\]

\[
\otimes x_{j_1} \wedge \cdots \wedge x_{j_p} \otimes \sum_{s=1}^{\ell^s} \mu_{(r_1, \ldots, r_p)} (\ell^1, \ldots, \ell^p ; j_1, \ldots, j_p) \times \hat{Q}_{(r_1, \ldots, r_p)}^{(\ell^1, \ldots, \ell^p ; j_1, \ldots, j_p)},
\]

where, as before, we define the \( N \)-tuple \( Q_{(r_1, \ldots, r_p)}^{(\ell^1, \ldots, \ell^p ; j_1, \ldots, j_p)} \) by

\[
(Q_{(r_1, \ldots, r_p)}^{(\ell^1, \ldots, \ell^p ; j_1, \ldots, j_p)})_j = \begin{cases} r_j + \ell_1^j + \cdots + \ell_{s-1}^j & \text{if } j = j_s, \\ \ell_1^j + \cdots + \ell_s^j & \text{if } j_s < j < j_{s+1}, \end{cases}
\]

and where the \( N \)-tuple \( \hat{Q}_{(r_1, \ldots, r_p)}^{(\ell^1, \ldots, \ell^p ; j_1, \ldots, j_p)} \) and scalar \( \mu_{(r_1, \ldots, r_p)} (\ell^1, \ldots, \ell^p ; j_1, \ldots, j_p) \) are defined (uniquely) by

\[
\mu_{(r_1, \ldots, r_p)} (\ell^1, \ldots, \ell^p ; j_1, \ldots, j_p) \times Q_{(r_1, \ldots, r_p)}^{(\ell^1, \ldots, \ell^p ; j_1, \ldots, j_p)} x_{j_1} \cdots x_{j_p} \times \hat{Q}_{(r_1, \ldots, r_p)}^{(\ell^1, \ldots, \ell^p ; j_1, \ldots, j_p)} = \sum_{s=1}^{\ell^s} \mu_{(r_1, \ldots, r_p)} (\ell^1, \ldots, \ell^p ; j_1, \ldots, j_p) \times \hat{Q}_{(r_1, \ldots, r_p)}^{(\ell^1, \ldots, \ell^p ; j_1, \ldots, j_p)} \in S_q (V).
\]

The coefficient \( \mu_{(r_1, \ldots, r_p)} (\ell^1, \ldots, \ell^p ; j_1, \ldots, j_p) \) is obtained using the twisting principle in the right side of the formula for \( \Psi_p \), and \( Q_{(r_1, \ldots, r_p)}^{(\ell^1, \ldots, \ell^p ; j_1, \ldots, j_p)} \) and \( \hat{Q}_{(r_1, \ldots, r_p)}^{(\ell^1, \ldots, \ell^p ; j_1, \ldots, j_p)} \) are the same as in the case of the polynomial algebra \( k[x_1, \ldots, x_n] \). For comparison, we note that Wambst [1993, Lemma 6.7] gave such a chain map in degree 1.
Theorem 4.6. Let $\Phi_*$ and $\Psi_*$ be as defined in (4.3) and (4.5). Then

(i) the map $\Phi_*$ is a chain map from the Koszul resolution to the normalized bar resolution;

(ii) the map $\Psi_*$ is a chain map from the normalized bar resolution to the Koszul resolution;

(iii) the composition $\Psi_* \circ \Phi_*$ is the identity map.

Proof. (i) One direct proof was given in [Naidu et al. 2011, Lemma 2.3]. (The characteristic of $k$ was assumed to be 0 in that paper; however, this assumption is not needed in that proof.) Another proof can be given by applying (2.1) to a chain contraction $s_*$ over the normalized bar resolution as in the proof of Theorem 3.5(i). The twisting principle gives the coefficients.

(ii) One direct computational proof can be given by applying (2.1) to the chain contraction $t_*$ of Proposition 4.2, as in the proof of Theorem 3.5(ii). Thus the same proof as that of Theorem 3.5(ii) works, taking care with the coefficients, by the twisting principle.

(iii) The same proof as that of Theorem 3.5(iii) works; by the twisting principle, the coefficients on both sides of the equation coincide.

We now give alternative descriptions of the maps $t_p$ and $\Psi_p$ in this case of a quantum symmetric algebra. The description of $\Psi_p$ will generalize that of Carqueville and Murfet [2016] from $S(V)$ to $S_q(V)$. To this end, it is convenient to replace each term $S_q(V) \otimes \wedge^p(V) \otimes S_q(V)$ of the Koszul resolution by $S_q(V) \otimes S_q(V) \otimes \wedge^p(V)$, using the canonical isomorphism

$$\sigma_p : S_q(V) \otimes S_q(V) \otimes \wedge^p(V) \to S_q(V) \otimes \wedge^p(V) \otimes S_q(V)$$

in which coefficients are inserted according to the twisting principle. For example, for $x^\ell \in S_q(V)$ and $1 \leq j_1 < \cdots < j_p \leq N$,

$$\sigma_p(1 \otimes x^\ell \otimes x_{j_1} \wedge \cdots \wedge x_{j_p}) = \left( \prod_{s=1}^{N} \prod_{t=1}^{p} q_{s,j_t}^{\ell_s} \right) x_{j_1} \wedge \cdots \wedge x_{j_p} \otimes x^\ell.$$

Via this isomorphism, consider $t_p$ as a map from $S_q(V) \otimes S_q(V) \otimes \wedge^p(V)$ to $S_q(V) \otimes S_q(V) \otimes \wedge^{p+1}(V)$. By abuse of notation, we still denote by $t_p$ this new map; the same rule applies to $\Psi_p$.

For $1 \leq j \leq N$, define $\tau_j : S_q(V)^e \to S_q(V)^e$ to be the operator that replaces all factors of the form $x_j \otimes 1$ with $1 \otimes x_j$, but with coefficient inserted according to the twisting principle. For example, if $x^\ell \in S_q(V)$, then

$$\tau_j(x^\ell \otimes 1) = \left( \prod_{s=j+1}^{N} q_{j,s}^{\ell_j \ell_s} \right) x_1^{\ell_1} \cdots x_{j-1}^{\ell_{j-1}} x_{j+1}^{\ell_{j+1}} \cdots x_N^{\ell_N} \otimes x^\ell_j.$$
It is not difficult to see that for \( 1 \leq i \neq j \leq N \), \( \tau_i \tau_j = \tau_j \tau_i \). Define quantum difference quotient operators \( \partial_{[i]} : S_q(V) \to S_q(V) \otimes S_q(V) \) for each \( i, 1 \leq i \leq N \), by

\[
(4.7) \quad \partial_{[i]}(f) := (x_i \otimes 1 - 1 \otimes x_i)^{-1} (\tau_1 \cdots \tau_{i-1}(f \otimes 1) - \tau_1 \cdots \tau_{i}(f \otimes 1)). \]

This definition should be understood as follows: by writing \( f \) as a linear combination of monomials, it suffices to define \( \partial_{[i]} \) on each monomial \( x_i^{\ell} \). The difference \( \tau_1 \cdots \tau_{i-1}(x_i^{\ell} \otimes 1) - \tau_1 \cdots \tau_{i}(x_i^{\ell} \otimes 1) \) may be divided by \( x_i \otimes 1 - 1 \otimes x_i \) on the left, by first factoring out \( x_i^{\ell_i} \otimes 1 - 1 \otimes x_i^{\ell_i} \) on the left. Applying the twisting principle, one sees that this is indeed always a factor. One must include a coefficient given by the twisting principle, then use the identity

\[
(x_i \otimes 1 - 1 \otimes x_i)^{-1} (x_i^{\ell_i} \otimes 1 - 1 \otimes x_i^{\ell_i}) = \sum_{r=1}^{\ell_i} x_i^{\ell_i - r} \otimes x_i^{-r}. \]

For example, for \( f = x_1 x_2^2 \), let us compute \( \partial_{[2]}(f) \). We have

\[
\tau_1(x_1 x_2^2 \otimes 1) = q_{1,2}^2 x_2^2 \otimes x_1 = q_{1,2}^2 (x_2^2 \otimes 1)(1 \otimes x_1),
\]

\[
\tau_1 \tau_2(x_1 x_2^2 \otimes 1) = 1 \otimes x_1 x_2^2 = q_{1,2}^2 (1 \otimes x_2^2)(1 \otimes x_1),
\]

and so

\[
\tau_1(x_1 x_2^2 \otimes 1) - \tau_1 \tau_2(x_1 x_2^2 \otimes 1) = q_{1,2}^2 (x_2^2 \otimes 1 - 1 \otimes x_2^2)(1 \otimes x_1).
\]

We obtain thus

\[
\partial_{[2]}(f) = (x_2 \otimes 1 - 1 \otimes x_2)^{-1} (\tau_1(x_1 x_2^2 \otimes 1) - \tau_1 \tau_2(x_1 x_2^2 \otimes 1))
\]

\[
= (x_2 \otimes 1 - 1 \otimes x_2)^{-1} (q_{1,2}^2 (x_2^2 \otimes 1 - 1 \otimes x_2^2)(1 \otimes x_1))
\]

\[
= q_{1,2}^2 (x_2 \otimes 1 - 1 \otimes x_2)(1 \otimes x_1)
\]

\[
= q_{1,2}^2 x_2 \otimes x_1 + q_{1,2} \otimes x_1 x_2.
\]

In general, we have

\[
\partial_{[j]}(x_i^{\ell}) = \left( \prod_{s=1}^{j-1} q_{s,j}^{\ell_s} \right) \sum_{r=1}^{\ell_j} \left( \prod_{s=1}^{j-1} q_{s,t}^{\ell_s} \prod_{t=j+1}^{N} q_{s,t}^{\ell_s} \right) \times \left( \prod_{t=j+1}^{N} q_{j,t}^{\ell_j(r-1)} \right) x_j^{\ell_j-r} x_{j+1}^{\ell_{j+1}} \cdots x_N^{\ell_N} \otimes x_1^{\ell_1} \cdots x_{j-1}^{\ell_{j-1}} x_j^{r-1}.
\]

That is, one has an extra coefficient \( \left( \prod_{s=1}^{j-1} q_{s,j}^{\ell_s} \right) \) as well as the coefficient included according to the twisting principle.
The chain contraction

\[ t_p : S_q(V) \otimes S_q(V) \otimes \wedge^p(V) \to S_q(V) \otimes S_q(V) \otimes \wedge^{p+1}(V) \]

may be expressed as

\[ t_p(1 \otimes x^\ell \otimes x_{j_1} \wedge \cdots \wedge x_{j_p}) = (-1)^{p+1} \sum_{j_{p+1} = j_{p+1}}^N \left( \prod_{t=1}^{p+1} q_{j_{p+1},t} \right) \left( \prod_{t=1}^p q_{j_{p+1},t} \right) \partial_{[j_{p+1}]}(x^\ell) \otimes x_{j_1} \wedge \cdots \wedge x_{j_{p+1}}. \]

This is justified by the fact that the coefficient in \( \partial_{[j_{p+1}]}(x^\ell) \) is nearly the coefficient needed by the twisting principle. The discrepancy is that \( \partial_{[j_{p+1}]}(x^\ell) \) has an extra factor \( \prod_{t=1}^{p+1-1} q_{j_{p+1},t} \), and we still need to insert \( \prod_{t=1}^N q_{j_{p+1},t} \) and \( \prod_{t=1}^p q_{j_{p+1},t} \) because the last factor in \( x_{j_1} \wedge \cdots \wedge x_{j_{p+1}} \) lies to the right of \( x_{j_1} \wedge \cdots \wedge x_{j_{p+1}} \) and of \( x_{j_{p+1}+1} \wedge \cdots \wedge x_{N} \) in \( \partial_{[j_{p+1}]}(x^\ell) \). Altogether then, we need to include an extra factor of \( \left( \prod_{t=1}^{p+1-1} q_{j_{p+1},t} \right) \left( \prod_{t=1}^p q_{j_{p+1},t} \right) \) in the coefficient in \( \partial_{[j_{p+1}]}(x^\ell) \).

The chain map \( \Psi_p : S_q(V) \otimes S_q(V) \otimes \overline{S_q(V)^{\otimes p}} \to S_q(V) \otimes S_q(V) \otimes \wedge^p(V) \)

may be expressed as

\[ \Psi_p(1 \otimes 1 \otimes x^{\ell_1} \otimes \cdots \otimes x^{\ell_p}) = \sum_{1 \leq j_1 < \cdots < j_p \leq N} \mu_{\ell_1, \ldots, \ell_p}^{(j_1, \ldots, j_p)} \left( \prod_{s=1}^p \partial_{[j_s]}(x^{\ell_s}) \right) \otimes x_{j_1} \wedge \cdots \wedge x_{j_p}, \]

where the scalar is defined according to the twisting principle by

\[ x^{\ell_1} \cdots x^{\ell_p} = \mu_{(j_1, \ldots, j_p)}^{(\ell_1, \ldots, \ell_p)} \left( \prod_{s=1}^p \partial_{[j_s]}(x^{\ell_s}) \right) \partial_{[j_1]}(x^{\ell_1}) \cdots x_{j_p} \in S_q(V). \]

Here the factor \( \left( \prod_{s=1}^p \partial_{[j_s]}(x^{\ell_s}) \right)' \) is understood as follows: if \( \partial_{[j_s]}(x^{\ell_s}) = a_s \otimes b_s \) (symbolically), then the product \( \left( \prod_{s=1}^p \partial_{[j_s]}(x^{\ell_s}) \right)' \) is \( (\prod_s a_s)(\prod_s b_s) \in A \).

5. Gerstenhaber brackets for quantum symmetric algebras

The Schouten–Nijenhuis (Gerstenhaber) bracket on Hochschild cohomology of the symmetric algebra \( S(V) \) is well known. In this section, we generalize it to the quantum symmetric algebras \( S_q(V) \). First we recall the definition of the Gerstenhaber bracket on Hochschild cohomology as defined on the normalized bar resolution of any \( \mathbb{k} \)-algebra \( A \) (associative with unity).
Let \( f \in \text{Hom}_{A^e}(A \otimes \widetilde{A}^p \otimes A, A) \) and \( f' \in \text{Hom}_{A^e}(A \otimes \widetilde{A}^q \otimes A, A) \). Define their bracket, \([f, f'] \in \text{Hom}_{A^e}(A \otimes \widetilde{A}^{(p+q-1)} \otimes A, A)\), by

\[
[f, f'] = \sum_{k=1}^{p} (-1)^{(q-1)(k-1)} f \circ_k f' - (-1)^{(p-1)(q-1)} \sum_{k=1}^{q} (-1)^{(p-1)(k-1)} f' \circ_k f,
\]

where

\[
(f \circ_k f')(1 \otimes a_1 \otimes \cdots \otimes a_{p+q-1} \otimes 1) = f(1 \otimes a_1 \otimes \cdots \otimes a_{k-1} \otimes f(1 \otimes a_k \otimes \cdots \otimes a_{k+q-1} \otimes 1) \otimes a_{k+q} \otimes \cdots \otimes a_{p+q-1} \otimes 1).
\]

In the above definition, the image of an element under \( f \) or \( f' \) is understood in \( \widetilde{A} \), whenever required.

Let \( \wedge_q^{-1}(V^*) \) be the quantum exterior algebra defined by the tuple \( q^{-1} \); that is, \( \wedge_q^{-1}(V^*) \) is the algebra generated by the dual basis \( \{dx_1, \ldots, dx_N\} \) of \( V^* \) with respect to the basis \( \{x_1, \ldots, x_N\} \) of \( V \), subject to the relations \( (dx_i)^2 = 0 \) and \( dx_i \cdot dx_j = -q_{i,j}^{-1} \) for all \( i, j \). We denote the product on \( \wedge_q^{-1}(V^*) \) by \( \wedge \). It is convenient to use abbreviated notation for monomials in this algebra: if \( I \) is the \( p \)-tuple \( I = (i_1, \ldots, i_p) \), denote by \( dx_I \) the element \( dx_{i_1} \wedge \cdots \wedge dx_{i_p} \) of \( \wedge_q^{-1}(V^*) \). We also write \( \widetilde{x}^{I} \) for \( x_{i_1} \wedge \cdots \wedge x_{i_p} \). Another notation we shall use is \( dx_b \), defined for any \( b \in \{0, 1\}^N \) to be \( dx_{i_1} \wedge \cdots \wedge dx_{i_p} \), where \( i_1, \ldots, i_p \) are the positions of the entries 1 in \( b \), all other entries being 0. In this case we say the length of \( b \) is \( p \), and write \( |b| = p \).

In [Naidu et al. 2011, Corollary 4.3], the Hochschild cohomology of \( S_q(V) \) is given as the graded vector subspace of \( S_q(V) \otimes \wedge_q^{-1}(V^*) \) that in degree \( m \) is

\[
\text{HH}^m(S_q(V)) = \bigoplus_{b \in \{0, 1\}^N} \bigoplus_{|b| = m} \bigoplus_{a \in \mathbb{N}^N} \text{Span}_k \{\widetilde{x}^a \otimes dx_b\},
\]

where

\[
\gamma = \left\{ \gamma \in (\mathbb{N} \cup \{-1\})^N \mid \text{for each } i \in \{1, \ldots, N\}, \prod_{s=1}^{N} q_{i_s}^{\gamma_i} = 1 \text{ or } \gamma_i = -1 \right\}.
\]

We wish to compute the bracket of two elements

\[
\alpha = \widetilde{x}^a \otimes dx_J \quad \text{and} \quad \beta = \widetilde{x}^b \otimes dx_L,
\]

where \( J = (j_1, \ldots, j_p) \) and \( L = (l_1, \ldots, l_q) \). We fix some notations. We denote by \( J \sqcup L \) the reordered disjoint union of \( J \) and \( L \) (multiplicities counted if there are equal indices), so \( dx_{J \sqcup L} = 0 \) if \( J \cap L \neq \emptyset \) and the entries of \( J \sqcup L \) are in
increasing order. For \( 1 \leq k \leq p \), set
\[
I_k := (j_1, \ldots, j_{k-1}, l_1, \ldots, l_q, j_{k+1}, \ldots, j_p),
\]
although we do not have \( j_1 < \cdots < j_{k-1} < l_1 < \cdots < l_q < j_{k+1} < \cdots < j_p \) in general. So we have \( dx_{I_k} = \text{sgn } \pi \alpha^I \prod_{i,k} dx_{J_k \cup L} \), where \( J_k = (j_1, \ldots, j_{k-1}, j_{k+1}, \ldots, j_p) \).
Similarly for \( 1 \leq k \leq q \), set
\[
I'_k := (l_1, \ldots, l_{k-1}, j_1, \ldots, j_p, l_{k+1}, \ldots, l_q).
\]

Once we know the bracket of two elements of this form, others may be computed by extending bilinearly. The scalars arising in each term from the twisting principle are potentially different, so it is more convenient to express brackets in terms of these basis elements of Hochschild cohomology.

**Theorem 5.1.** The graded Lie bracket of \( \alpha = \chi^a \otimes dx_J \) and \( \beta = \chi^b \otimes dx_L \) is
\[
[\alpha, \beta] = \sum_{1 \leq k \leq p} (-1)^{(q-1)(k-1)} \rho^b_{k,J,L} \left( \partial_{[jk]}(\chi^b) \right) \cdot \chi^a \otimes dx_{J_k \cup L} \]
\[
- (-1)^{(p-1)(q-1)} \sum_{1 \leq k \leq q} (-1)^{(p-1)(k-1)} \rho^a_{k,L,J} \left( \partial_{[jk]}(\chi^a) \right) \cdot \chi^b \otimes dx_{J_k \cup L},
\]
for certain scalars \( \rho^b_{k,J,L} \) and \( \rho^a_{k,L,J} \), where \( \partial_{[jk]}(\chi^b) \) is defined in (4.7) and \( \partial_{[jk]}(\chi^a) \cdot \chi^a \) is given by the \( A^e \)-module structure over \( A \), that is, if \( \partial_{[jk]}(\chi^b) = \sum_i u_i \otimes v_i \in A \otimes A \), then \( \partial_{[jk]}(\chi^a) \cdot \chi^a = \sum_i u_i \chi^a v_i \).

**Proof.** We denote by \( \cdot \) the composition of two maps instead of \( \circ \), in order to avoid confusion with the circle product. We compute the bracket using the formula
\[
[\alpha, \beta] = [\alpha \cdot \Psi_p, \beta \cdot \Psi_q] \cdot \Phi_{p+q-1}.
\]

The element \( \alpha = \chi^a \otimes dx_J \) as a map from \( A \otimes A \otimes \wedge^p(V) \) to \( A \otimes 1 \otimes \chi^a \) to \( \delta_{IJ} \chi^a \) for \( I = (i_1, \ldots, i_p) \); similarly the element \( \beta = \chi^b \otimes dx_L \) as a map from \( A \otimes A \otimes \wedge^q(V) \) to \( A \otimes 1 \otimes \chi^b \) to \( \delta_{IL} \chi^b \). By formula (4.8) for \( \Psi_p \), the map \( \alpha \cdot \Psi_p : A \otimes A \otimes \tilde{A}^{p+1} \rightarrow A \otimes A \otimes \wedge^p(V) \rightarrow A \) is given by
\[
\alpha \cdot \Psi_p(1 \otimes 1 \otimes \chi^{m_1} \otimes \cdots \chi^{m_p}) = \mu(\begin{pmatrix} m_1 \cdots m_p \end{pmatrix}, \begin{pmatrix} \prod_{s=1}^{p} (\partial_{[js]}(\chi^{m_s})) \end{pmatrix}) \cdot \chi^a,
\]
where the scalar coefficient is defined by (4.9). We have a similar formula for \( \beta \cdot \Psi_q \).

For \( 1 \leq k \leq p \), the map \( (\alpha \cdot \Psi_p) \circ_k (\beta \cdot \Psi_q) : A \otimes A \otimes \tilde{A}^{p+q-1} \rightarrow A \) sends \( 1 \otimes 1 \otimes \chi^{m_1} \otimes \cdots \otimes \chi^{m_p+q-1} \) to
\[
\mu_k \mu_L (\begin{pmatrix} m_1 \cdots m_{k-1}, \tilde{m}_k, m_{k+q}, \cdots, m_{p+q-1} \end{pmatrix}) \cdot (\partial_{[j_1]}(\chi^{m_1}) \cdots \partial_{[j_{k-1}]}(\chi^{m_{k-1}}) \partial_{[jk]}(\chi^{\tilde{m}_k}) \partial_{[j_{k+1}]}(\chi^{m_{k+q}}) \cdots \partial_{[j_{p}]}(\chi^{m_{p+q-1}})) \cdot \chi^a,
\]
where $\mu_k$ and $\tilde{m}^k$ are defined by $\mu_k \tilde{m}^k = \left( \prod_{t=1}^q \partial_{[I_t]}(x^{m_{t+1}}) \right) \cdot x^b$.

For $I = (i_1, \ldots, i_{p+q-1})$ with $1 \leq i_1 < \cdots < i_{p+q-1} \leq N$, let us compute $((\alpha \cdot \Psi_p) \circ_k (\beta \cdot \Psi_q)) \cdot \Phi_{p+q-1}(1 \otimes 1 \otimes x^I)$. Indeed, by (4.3) and our identifications,

$$\Phi_{p+q-1}(1 \otimes 1 \otimes x^I) = \sum_{\pi \in \text{Sym}_{p+q-1}} \text{sgn } \pi \ q_I^i \otimes 1 \otimes x_{\pi(1)} \otimes \cdots \otimes x_{\pi(p+q-1)}.$$

Now for a fixed $\pi \in \text{Sym}_{p+q-1}$, as input into the formula of the previous paragraph, we have

$$m^1 = e_{\pi(1)}, \ldots, \ m^{p+q-1} = e_{\pi(p+q-1)},$$

where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, with the 1 in the $i$-th position, and since $\partial_{[j]}(x_i) = \delta_{ij} \otimes 1$, the factor

$$(\partial_{[I_1]}(x^m) \cdots \partial_{[I_{k-1}]}(x^{m_{k-1}}) \partial_{[I_k]}(x^{m_k}) \partial_{[I_{k+1}]}(x^{m_{k+1}}) \cdots \partial_{[I_p]}(x^{m_{p+q-1}})) \cdot x^a$$

vanishes unless

$$j_1 = i_{\pi(1)}, \ldots, j_{k-1} = i_{\pi(k-1)},$$

$$l_1 = i_{\pi(k)}, \ldots, l_q = i_{\pi(k+q-1)},$$

$$j_{k+1} = i_{\pi(k+q)}, \ldots, j_p = i_{\pi(p+q-1)},$$

that is, when $I_k = \pi(I) := (i_{\pi(1)}, \ldots, i_{\pi(p+q-1)})$ or equivalently $I = J_k \sqcup L$. As long as $J_k \cap L = \emptyset$, there exist a unique $I$ and permutation $\pi_k \in \text{Sym}_{p+q-1}$ satisfying this property. In this case,

$$\mu_k \tilde{m}^k = \left( \prod_{t=1}^q \partial_{[I_t]}(x^{m_{t+1}}) \right) \cdot x^b = x^b,$$

so that $\mu_k = 1$ and $\tilde{m}^k = b$. Consequently, the map $((\alpha \cdot \Psi_p) \circ_k (\beta \cdot \Psi_q)) \cdot \Phi_{p+q-1}$ sends $1 \otimes 1 \otimes x^I$ to $\delta_{I,J_k \sqcup L} \rho_k^{b;J,L} \partial_{[jk]}(x^b) \cdot x^a$, where

$$\rho_k^{b;J,L} = \text{sgn } \pi_k \ q_I^i \mu_L^{(e_{j_1}, \ldots, e_{j_{k-1}}, b, e_{j_{k+1}}, \ldots, e_{j_p})} \ (e_{\ell_1}, \ldots, e_{\ell_q})$$

is determined by the permutation $\pi_k$ as described above and the scalars defined by (4.4) and (4.9). Therefore,

$$((\alpha \cdot \Psi_p) \circ_k (\beta \cdot \Psi_q)) \cdot \Phi_{p+q-1} = \rho_k^{b;J,K} \partial_{[jk]}(x^b) \cdot x^a \otimes dx_{J_k \sqcup L}.$$

The formula in the statement can be obtained accordingly. \hfill \square

6. Gerstenhaber brackets for group extensions of quantum symmetric algebras

Let $G$ be a finite group for which $|G| \neq 0$ in $\mathcal{A}$, acting linearly on a finite dimensional vector space $V$, thus inducing an action on the symmetric algebra $S(V)$ by
automorphisms. When the action preserves the relations on the quantum symmetric algebra \( S_q(V) \) as defined by (4.1), there is also an action on this algebra. This is always the case, for example, if \( G \) acts diagonally on the chosen basis \( x_1, \ldots, x_N \) of \( V \). We shall first recall the definition of a group extension, \( S_q(V) \rtimes G \), of \( S_q(V) \), and explain how the Koszul resolution of \( S_q(V) \rtimes G \) is related to that of \( S_q(V) \). In fact this works for an arbitrary Koszul algebra, as we shall explain next. Although this is well known, we include details for completeness.

Let \( R \subseteq V \otimes V \) be a \( G \)-invariant subspace. Let \( T_k(V) \) denote the tensor algebra of \( V \) over \( \mathbb{k} \). Suppose that \( A = T_k(V)/(R) \) is a Koszul algebra over \( \mathbb{k} \), with the induced action of \( G \). That is, the complex \( K_*(A) \) in which \( K_0(A) = A \otimes A \), \( K_1(A) = A \otimes V \otimes A \), and

\[
K_i(A) = \bigcap_{j=0}^{i-2} (A \otimes V^\otimes j \otimes R \otimes V^\otimes (i-2-j) \otimes A)
\]

for \( i \geq 2 \) is a free \( A \)-bimodule resolution of \( A \) under the differential from the bar resolution. In the case \( A = S_q(V) \), this can be shown to be equivalent to the Koszul resolution given in Section 4. The group extension \( A \rtimes G \) of \( A \), or skew group algebra, is the tensor product \( A \otimes \mathbb{k} G \) as a vector space, with multiplication given by \((a \otimes g)(b \otimes h) = a(gb) \otimes gh \) for all \( a, b \in A \) and \( g, h \in G \) (where we have used a left superscript to denote the group action). We shall denote elements of \( A \rtimes G \) by \( a \uparrow g \), in place of \( a \otimes g \), for \( a \in A \) and \( g \in G \), to indicate that they are elements of this skew group algebra. In this section we adapt and generalize the techniques of [Halbout and Tang 2010; Shepler and Witherspoon 2012] from \( S(V) \rtimes G \) to \( S_q(V) \rtimes G \), explaining how to compute the Gerstenhaber bracket via the Koszul resolution and our chain maps from Section 4. In the next section we focus on some special cases to give explicit results.

We know that \( A \rtimes G \) is a Koszul ring over \( \mathbb{k} G \) (see [Beilinson et al. 1996, Definition 1.1.2 and Section 2.6]). In fact let \( V \otimes \mathbb{k} G \) be the \( \mathbb{k} G \)-bimodule under the actions \( g \cdot (v \otimes h) = gv \otimes gh \) and \((v \otimes h) \cdot g = v \otimes hg \) for all \( v \in V \) and \( g, h \in G \). Then there is an algebra isomorphism

\[
T_kG(V \otimes \mathbb{k} G) \simeq T_k(V) \rtimes G
\]

sending \((v_1 \otimes g_1) \otimes \mathbb{k} G \cdots \otimes \mathbb{k} G (v_{m-1} \otimes g_{m-1}) \otimes \mathbb{k} G (v_m \otimes g_m)\) to \((v_1 \otimes g_1 v_2 \otimes \cdots \otimes g_{m-1} v_m) \otimes g_1 \cdots g_m\), and the inverse isomorphism sends \((v_1 \otimes \cdots \otimes v_m) \otimes g \) to \((v_1 \otimes e_G) \otimes \mathbb{k} G \cdots \otimes \mathbb{k} G (v_{m-1} \otimes e_G) \otimes \mathbb{k} G (v_m \otimes g)\), where we write \( e_G \) or \( e \) for the unit element of \( G \). Via this isomorphism, \( R \otimes \mathbb{k} G \) becomes a \( \mathbb{k} G \)-subbimodule of \((V \otimes \mathbb{k} G) \otimes \mathbb{k} G \) \( V \otimes \mathbb{k} G \) \( V \otimes \mathbb{k} G \), and it induces an isomorphism of algebras, \( A \rtimes G \simeq T_kG(V \otimes \mathbb{k} G)/(R \otimes \mathbb{k} G) \).
The Koszul resolution $K_*(A \times G)$ of $A \times G$ as a Koszul ring over $kG$ is related to the Koszul resolution of $A$ as follows:

$$K_0(A \times G) = (A \times G) \otimes_{kG} (A \times G)$$

$$\simeq A \otimes A \otimes kG$$

$$= K_0(A) \otimes kG,$$

$$K_1(A \times G) = (A \times G) \otimes_{kG} (V \otimes kG) \otimes_{kG} (A \times G)$$

$$\simeq A \otimes V \otimes A \otimes kG$$

$$= K_1(A) \otimes kG,$$

and for $i \geq 2$,

$$K_i(A \times G) = (A \times G) \otimes_{kG} \bigcap_{j=0}^{i-2} (V \otimes kG)^{\otimes_{kG} (i-2-j)} \otimes_{kG} (A \times G)$$

$$\simeq (A \times G) \otimes_{kG} \left( \bigcap_{j=0}^{i-2} (V \otimes j \otimes R \otimes V^{\otimes (i-2-j)}) \otimes_{kG} (A \times G) \right)$$

$$\simeq \left( A \otimes \bigcap_{j=0}^{i-2} (V \otimes j \otimes R \otimes V^{\otimes (i-2-j)}) \otimes A \right) \otimes kG$$

$$\simeq K_i(A) \otimes kG.$$ 

Notice that the above isomorphism is induced by the map sending

$$(a_0 \# g_0) \otimes_{kG} ((a_1 \otimes g_1) \otimes_{kG} \cdots \otimes_{kG} (a_p \otimes g_p)) \otimes_{kG} (a_{p+1} \# g_{p+1})$$

to

$$(a_0 \otimes (g_0 a_1 \otimes \cdots \otimes g_0 \cdots g_{p-1} a_p) \otimes g_0 \cdots g_p a_{p+1}) \otimes (g_0 \cdots g_{p+1}).$$

The inverse isomorphism sends $(a_0 \otimes (a_1 \otimes \cdots \otimes a_p) \otimes a_{p+1}) \# g$ to

$$(a_0 \# e) \otimes_{kG} ((a_1 \otimes e) \otimes_{kG} \cdots \otimes_{kG} (a_p \otimes e)) \otimes_{kG} (a_{p+1} \# g).$$

One may check that this isomorphism commutes with the differentials. Therefore as complexes of $A \times G$-bimodules,

$$K_*(A \times G) \simeq K_*(A) \otimes kG.$$
Under this isomorphism, the $A \times G$-bimodule structure of $K_p(A) \otimes \mathbb{k}G$, for each $p \geq 0$, is given by

$$(b \cdot h) \left((a_0 \otimes (a_1 \otimes \cdots \otimes a_p) \otimes a_{p+1}) \otimes g\right) (c \cdot k) = (b^h a_0 \otimes (h^a_1 \otimes \cdots \otimes h^a_p) \otimes h^a_{p+1}^c \otimes h_c k).$$

Similar statements apply to the normalized bar resolution:

$$B_*(A \rtimes G) \simeq B_*(A) \otimes \mathbb{k}G,$$

where the former involves tensor products over $\mathbb{k}G$, and the latter over $\mathbb{k}$.

Now we consider the case of $A := S_q(V)$, under the condition that the action of $G$ on $V$ preserves the relations of $S_q(V)$. The differentials on $K_*(A \rtimes G)$ (respectively, $B_*(A \rtimes G)$) are those induced by the Koszul resolution (respectively, bar resolution) of $S_q(V)$, under the exact functor $- \otimes \mathbb{k}G$. Therefore the contracting homotopy and chain maps for $S_q(V)$ may be extended to the corresponding complexes for $S_q(V) \rtimes G$:

$$\Phi_* \otimes \mathbb{k}G : K_*(A \rtimes G) \simeq K_*(A) \otimes \mathbb{k}G \rightarrow B_*(A) \otimes \mathbb{k}G \simeq B_*(A \rtimes G)$$

and

$$\Psi_* \otimes \mathbb{k}G : B_*(A \rtimes G) \simeq B_*(A) \otimes \mathbb{k}G \rightarrow K_*(A) \otimes \mathbb{k}G \simeq K_*(A \rtimes G).$$

However, since $\Phi_*$ and $\Psi_*$ are in general not $G$-invariant, there is no reason to expect that $\Phi_* \otimes \mathbb{k}G$ and $\Psi_* \otimes \mathbb{k}G$ should be chain maps of complexes of $(A \rtimes G)^e$-modules. Since $|G|$ is invertible in $\mathbb{k}$, we can apply the Reynolds operator (that averages over images of group elements) to obtain chain maps of complexes of $(A \rtimes G)^e$-modules, which are denoted by $\Phi^e_*$ and $\Psi^e_*$ respectively. We have thus quasi-isomorphisms

$$\text{Hom}_{(A \rtimes G)^e}(K_*(A) \otimes \mathbb{k}G, A \rtimes G) \xrightarrow{\Phi^e_*} \text{Hom}_{(A \rtimes G)^e}(B_*(A) \otimes \mathbb{k}G, A \rtimes G).$$

We shall use the complex on the left side to compute Lie brackets, via the chain maps $\Psi^e_*$ and $\Phi^e_*$. Notice that for $A = S_q(V)$, we have

$$\text{Hom}_{(A \rtimes G)^e}(K_*(A) \otimes \mathbb{k}G, A \rtimes G) \simeq \text{Hom}_{\mathbb{k}G^e}(\wedge^e(V) \otimes \mathbb{k}G, A \rtimes G) \simeq \text{Hom}_{\mathbb{k}G}(\wedge^*(V), A \rtimes G) \simeq (A \rtimes G \otimes \wedge^*(V^*))^G.$$

We wish to express the Lie bracket at the chain level, on elements of $(A \rtimes G \otimes \wedge^*(V^*))^G$. The method consists of the following steps (see [Halbout and Tang 2010; Shepler and Witherspoon 2012]).

(i) Compute the cohomology groups of the complexes $((A \rtimes G) \otimes \wedge^*(V^*))^G$. In the case where the action of $G$ on $V$ is diagonal, this computation is done in [Naidu et al. 2011, Section 4].
(ii) Give a precise formula for the chain map \( \Theta \) that is the composition
\[
\Theta : ((A \times G) \otimes \wedge^*(V^*))^G \xrightarrow{\sim} \text{Hom}_{(A \times G)^G}(K_*(A) \otimes \mathbb{k}G, A \times G)
\]
\[
\xrightarrow{\tilde{\Psi}^*} \text{Hom}_{(A \times G)^G}(B_*(A) \otimes \mathbb{k}G, A \times G)
\]
\[
\xrightarrow{\sim} \text{Hom}_{(A \times G)^G}(B_*(A \times G), A \times G).
\]

(iii) Give a precise formula for the chain map \( \Gamma \) that is the composition
\[
\Gamma : \text{Hom}_{(A \times G)^G}(B_*(A \times G), A \times G) \xrightarrow{\sim} \text{Hom}_{(A \times G)^G}(B_*(A) \otimes \mathbb{k}G, A \times G)
\]
\[
\xrightarrow{\tilde{\Phi}^*} \text{Hom}_{(A \times G)^G}(K_*(A) \otimes \mathbb{k}G, A \times G)
\]
\[
\xrightarrow{\sim} ((A \times G) \otimes \wedge^*(V^*))^G.
\]

(iv) Use the formulae in the previous two steps to compute the Lie bracket of two cocycles given by Step (i).

We obtain thus:

**Theorem 6.1.** Let \( \alpha, \beta \in ((A \times G) \otimes \wedge^*(V^*))^G \) be two cocycles. Then the Lie bracket of the two corresponding cohomological classes is represented by the cocycle
\[
[\alpha, \beta] = \Gamma([\Theta(\alpha), \Theta(\beta)]).
\]

We see that the actual computations are rather hard and we shall perform these computations for the diagonal action case in the next section.

7. Diagonal actions

Assume now that \( G \) acts diagonally on the basis \( \{x_1, \ldots, x_N\} \) of \( V \), in which case the action extends to an action of \( G \) on \( S_q(V) \) by automorphisms. Let \( \chi_i : G \to \mathbb{k}^\times \) be the character of \( G \) corresponding to its action on \( x_i \), that is,
\[
g \cdot x_i = \chi_i(g)x_i
\]
for all \( g \in G \) and \( i = 1, \ldots, N \). For \( I = (i_1, \ldots, i_p) \) with \( 1 \leq i_1 < \cdots < i_p \leq N \), define \( \chi_I(g) = \prod_{j=1}^p \chi_{i_j}(g) \), and for \( \ell \in \mathbb{N}^N \), define \( \chi_{\ell}^I(g) = \prod_{1 \leq i \leq N} \chi_{i}^{\ell_i}(g) \) for \( g \in G \).

Let us make precise the action of \( G \) on \( (A \times G) \otimes \wedge^*(V^*) \) occurring in the isomorphism of the previous section,
\[
\text{Hom}_{(A \times G)^G}(K_*(A) \otimes \mathbb{k}G, A \times G) \simeq ((A \times G) \otimes \wedge^*(V^*))^G.
\]

Letting \( g, h \in G, \ell \in \mathbb{N}^N \), and \( I = (i_1 < \cdots < i_p) \), we have
\[
h(\chi_{\ell}^I \otimes g \otimes dx_I) = h(\chi_{\ell}^I) \otimes h(g) \otimes h(dx_I) = \chi_{\ell}^I(h(g)) h^{-1}(dx_I) \otimes h(g) \otimes h(dx_I).
\]
In [Naidu et al. 2011, Section 4], the authors compute homology of this chain complex \((A \times G) \otimes \wedge^*(V^*)\) with the differential
\[
d_p(\text{x}^p \# g \otimes d\text{x}_I) = \sum_{i \neq I} (-1)^{\text{#}(s: i_s < i)} \left( \left( \prod_{s: i_s < i} q_{i_s,i} \right) x_i \text{x}^p \right) \left( \prod_{s: i_s > i} q_{i,i_s} \right) \text{x}_I \# g \otimes d\text{x}_I + e_i,
\]
where \(e_i\) is the \(i\)-th element of the canonical basis of \(\mathbb{N}^N\), and \(I + e_i\) is the sequence of \(p + 1\) integers obtained by inserting \(1\) in the \(i\)-th position. Since the action of \(G\) is diagonal, this differential is \(G\)-equivariant. So the Reynolds operator is a chain map from \((A \times G) \otimes \wedge^*(V^*)\) to \(((A \times G) \otimes \wedge^*(V^*))^G\) which realizes \(((A \times G) \otimes \wedge^*(V^*))^G\) as a direct summand of \((A \times G) \otimes \wedge^*(V^*)\) as complexes. We shall see that in fact, the induced structure of \(((A \times G) \otimes \wedge^*(V^*))^G\), as a complex, is the same as the one induced from the isomorphism
\[
\text{Hom}_{(A \times G)^e}(K_\bullet(A) \otimes \mathbb{k}G, A \times G) \simeq ((A \times G) \otimes \wedge^*(V^*))^G.
\]
We shall prove this fact in the first step below.

We follow the step-by-step outline given towards the end of Section 6. As we shall use the result of the second step in the first one, we begin with the second step.

**Step (ii).** As shown in the previous section, we have a series of isomorphisms:

\[
\text{Hom}_{(A \times G)^e}(K_\bullet(A) \otimes \mathbb{k}G, A \times G) \simeq \text{Hom}_{\mathbb{k}G^e}(\wedge^*(V) \otimes \mathbb{k}G, A \times G) \\
\simeq \text{Hom}_{\mathbb{k}G}(\wedge^*(V), A \times G) \\
\simeq ((A \times G) \otimes \wedge^*(V^*))^G.
\]

A map \(f \in \text{Hom}_{(A \times G)^e}(K_\bullet(A) \otimes \mathbb{k}G, A \times G)\) corresponds to \(f_1 \in \text{Hom}_{\mathbb{k}G^e}(\wedge^p V \otimes \mathbb{k}G, A \times G)\) via
\[
f_1(\text{x}^p \otimes g) = f(1 \otimes \text{x}^p \otimes 1 \otimes g)
\]
and
\[
f(a_0 \otimes \text{x}^p \otimes a_{p+1} \otimes g) = (a_0 \# e) f_1(\text{x}^p \otimes g)(g^{-1} a_{p+1} \# e).
\]
The map \(f_1 \in \text{Hom}_{\mathbb{k}G^e}(\wedge^p V \otimes \mathbb{k}G, A \times G)\) corresponds to \(f_2 \in \text{Hom}_{\mathbb{k}G}(\wedge^p V, A \times G)\) via
\[
f_2(\text{x}^p) = f_1(\text{x}^p \otimes e)
\]
and
\[
f_1(\text{x}^p \otimes g) = f_2(\text{x}^p)(1 \# g).
\]
Finally, \(f_2 \in \text{Hom}_{\mathbb{k}G}(\wedge^p V, A \times G)\) corresponds to \(f_3 \in ((A \times G) \otimes \wedge^p (V^*))^G\) via
\[
f_3 = \sum_{|I|=p} f_2(\text{x}^p) \otimes d\text{x}_I,
\]
and for $f_3 = \sum_{|I|=p} \sum_{g \in G} (a_{J,g} \otimes g) \otimes dx_I \in (A \times G \otimes \bigwedge^p (V^*))^G$, the corresponding $f_2 \in \text{Hom}_k(\bigwedge^p V, A \times G)$ sends $\chi^I$ to $\sum_{g \in G} a_{J,g} \otimes g$.

Altogether then, $f \in \text{Hom}_{(A \times G)^e} (K_p(A) \otimes k, A \times G)$ corresponds to $f_3 \in (A \times G \otimes \bigwedge^p V^*))^G$ via

$$f_3 = \sum_{|I|=p} f(1 \otimes \chi^I \otimes 1 \otimes e) \otimes dx_I,$$

and for $f_3 = \sum_{|I|=p} \sum_{g \in G} a_{J,g} \otimes g \otimes dx_I \in (A \times G \otimes \bigwedge^p (V^*))^G$,

$$f(a_0 \otimes \chi^I \otimes a_{p+1} \otimes g) = \sum_{h \in G} (a_0 \otimes e)(a_{I,h} \otimes h)(1 \otimes g)(g^{-1} \otimes a_{p+1} \otimes e)$$

$$= \sum_{h \in G} a_0 a_{I,h} (a_{p+1} \otimes h g).$$

Now for $\alpha = a \otimes g \otimes dx_I \in A \times G \otimes \bigwedge^p (V^*)$, the Reynolds operator

$$\mathcal{R} : A \times G \otimes \bigwedge^p (V^*) \to (A \times G \otimes \bigwedge^p (V^*))^G$$

gives

$$f_3 = \frac{1}{|G|} \sum_{h \in G} \chi_J(h^{-1}) a \otimes h g h^{-1} \otimes dx_I,$$

and thus $\alpha$ corresponds to the map $f \in \text{Hom}_{(A \times G)^e} (K_p(A) \otimes k, A \times G)$ sending $a_0 \otimes \chi^I \otimes a_{p+1} \otimes k$ to

$$\delta_{IJ} \frac{1}{|G|} \sum_{h \in G} \chi_J(h^{-1}) a_0(h a)(h g h^{-1} a_{p+1} \otimes h g h^{-1} k).$$

We shall compute $\Theta \mathcal{R}(\alpha) \in \text{Hom}_k((A \times G)^{\otimes p}, A \times G)$ corresponding to $f$ with $a = \chi_{\ell}$, which is the composition

$$\chi_{\ell^1} \otimes g_1 \otimes \cdots \otimes \chi_{\ell^p} \otimes g_p$$

$$\mapsto \chi_{\ell} \otimes g_1(\chi_{\ell^2}) \otimes \cdots \otimes g_1^{\cdots g_{p-1}}(\chi_{\ell^p}) \otimes g_1 \cdots g_p$$

$$= \chi_{\ell^2}(g_1) \cdots \chi_{\ell^p}(g_1 \cdots g_{p-1}) \chi_{\ell^1} \otimes \cdots \otimes \chi_{\ell^p} \otimes g_1 \cdots g_p$$

$$\mapsto \chi_{\ell^2}(g_1) \cdots \chi_{\ell^p}(g_1 \cdots g_{p-1}) \sum_{|I|=p} \sum_{0 \leq r_s \leq \ell_{s-1} \ell} \mu \chi_{\ell} \otimes \chi^I \otimes \chi_{\ell^p} \otimes g_1 \cdots g_p$$

(use $\Psi_*$)

$$\mapsto \frac{1}{|G|} \chi_{\ell^2}(g_1) \cdots \chi_{\ell^p}(g_1 \cdots g_{p-1}) \sum_{h \in G} \sum_{0 \leq r_s \leq \ell_{s-1} \ell} \chi_{\ell} \otimes \chi_{\ell^p} \otimes g_1 \cdots g_p$$

$$\times \chi_J(h^{-1}) \chi_{\ell}(h) \chi_{\ell^p}(h g h^{-1}) \chi_{\ell^1}^{\ell^{\cdots} \ell^p + \ell - J} \otimes h g h^{-1} g_1 \cdots g_p,$$
where, as in (4.5),
\[
\mu = \mu^{(\ell_1, \ldots, \ell_p; j_1, \ldots, j_p)}{r_1, \ldots, r_p}, \quad Q = Q^{(\ell_1, \ldots, \ell_p; j_1, \ldots, j_p)}{(r_1, \ldots, r_p)} , \quad \hat{Q} = \hat{Q}^{(\ell_1, \ldots, \ell_p; j_1, \ldots, j_p)}{(r_1, \ldots, r_p)} , \quad \lambda x Q x \hat{Q} = \chi^{\ell_1 + \ldots + \ell_p + I} \in S_q(V).
\]

This completes the second step.

**Step (i).** We identify the cohomology groups of the complexes \((A \rtimes G \otimes \wedge^*(V^*))^G\) with the computation in [Naidu et al. 2011, Section 4]. It suffices to see that the map
\[
A \rtimes G \otimes \wedge^*(V^*) \overset{\mathcal{R}}{\longrightarrow} (A \rtimes G \otimes \wedge^*(V^*))^G \overset{\cong}{\longrightarrow} \text{Hom}_{(A \rtimes G)^c}(K_*(A) \otimes \mathbb{k} G, A \rtimes G)
\]
is a chain map, where \(A \rtimes G \otimes \wedge^*(V^*)\) is endowed with the differential given in [Naidu et al. 2011, Section 4] and \(\text{Hom}_{(A \rtimes G)^c}(K_*(A) \otimes \mathbb{k} G, A \rtimes G)\) with the differential induced from that of \(K_*(A)\). We shall use the computations in the second step to prove this statement.

In fact, given \(a \# g \otimes dx_I \in A \rtimes G \otimes \wedge^p(V^*)\), by the second step, it corresponds to the map \(f \in \text{Hom}_{(A \rtimes G)^c}(K_p(A) \otimes \mathbb{k} G, A \rtimes G)\) sending \(a_0 \otimes x^{\wedge J} \otimes a_{p+1} \otimes k\) to
\[
\delta_{IJ} \frac{1}{|G|} \sum_{h \in G} \chi_I(h^{-1})a_0(ha)^{\# hgh^{-1}k} a_{p+1}\otimes k.
\]

Now \(df\) is the composition (for \(k \in G\) and \(L = (l_1, \ldots, l_{p+1})\))
\[
1 \otimes x^{\wedge L} \otimes 1 \otimes k \mapsto \sum_{j=1}^{p+1} (-1)^{j-1} \left( \left( \prod_{s=1}^{j} q_{l_s,l_j} \right) x_{l_j} \otimes x^{\wedge(L-e_{l_j})} \otimes 1 \otimes k \right.
\]
\[
- \left( \prod_{s=j}^{p+1} q_{l_s,l_j} \right) 1 \otimes x^{\wedge(L-e_{l_j})} \otimes x_{l_j} \otimes k \right)
\]
\[
\mapsto \frac{1}{|G|} \sum_{h \in G} \sum_{j=1}^{p+1} (-1)^{j-1} \delta_{I,L-e_{l_j}} \chi_I(h^{-1}) \left( \left( \prod_{s=1}^{j} q_{l_s,l_j} \right) x_{l_j} h^a \right.
\]
\[
- \left( \prod_{s=j}^{p+1} q_{l_s,l_j} \right) x_{l_j} (hgh^{-1})^a x_{l_j} \right) \# hgh^{-1}k.
\]
We shall use these notations when expressing the Lie bracket of two cohomological classes. This completes the first step.

**Step (iii).** Now given a map \( f \in \text{Hom}_k((A \rtimes G)^{\otimes*}, A \rtimes G) \), we compute the corresponding \( \Gamma(f) \in ((A \rtimes G) \otimes \wedge^p (V^*))^G \). Direct inspection gives

\[
\Gamma(f) = \sum_{|I|=p} \sum_{\pi \in \text{Sym}_p} \text{sgn} \, \pi \, q^I_{\pi} f(x_{i_{\pi(1)}}, \# e \otimes \cdots \otimes x_{i_{\pi(p)}} \# e) \otimes d\gamma_I,
\]

where \( q^I_{\pi} = q^{I_1} \cdots q^{I_p} \) is defined in (4.4), and \( e \) denotes the identity group element.

**Step (iv).** We can now compute the Lie bracket of two cohomological classes. Let

\[
\alpha = \chi^a \# g \otimes dx_I \quad \text{and} \quad \beta = \chi^b \# h \otimes dx_L
\]

which corresponds to the map sending \( 1 \otimes \chi^A \otimes 1 \otimes k \) to

\[
\frac{1}{|G|} \sum_{h \in G} \sum_{i \in I} (-1)^{\#\{s : i_s < i\}} \left( \prod_{s : i_s < i} q_{i_s} \right) \chi_L(h^{-1}) \delta_{I+e_i} \chi_i(h) \chi_i h a - \left( \prod_{s : i_s > i} q_{i_s} \right) \chi_i(hg)^h a x_i \# hgh^{-1} k.
\]

One sees readily that these two expressions are the same.

Let us recall the result of [Naidu et al. 2011, Section 4]. For \( g \in G \), define

\[
C_g = \left\{ c \in (\mathbb{N} \cup \{-1\})^N \mid \text{for each } i \in \{1, \ldots, N\}, \prod_{s=1}^N q_{i,s}^c = \chi_i(g) \text{ or } c_i = -1 \right\}.
\]

For \( g \in G \) and \( \gamma \in (\mathbb{N} \cup \{-1\})^N \), Naidu et al. introduced certain subcomplexes \( K^*_{g,\gamma} \) of \( (A \rtimes G) \otimes \wedge^p (V^*) \) with \( (A \rtimes G) \otimes \wedge^p (V^*) = \bigoplus_{g,\gamma} K^*_{g,\gamma} \). They also proved that if \( \gamma \in C_g \), the subcomplex \( K^*_{g,\gamma} \) has zero differential, and if \( \gamma \notin C_g \), the subcomplex \( K^*_{g,\gamma} \) is acyclic. (We do not define \( K^*_{g,\gamma} \) here as we shall not need the details.) Using this information, for \( m \in \mathbb{N} \), [Naidu et al. 2011, Theorem 4.1] gives

\[
H^m((A \rtimes G) \otimes \wedge^p (V^*)) \simeq HH^m(A, A \rtimes G)
\]

We shall use these notations when expressing the Lie bracket of two cohomological classes. This completes the first step.
for some group elements $g, h \in G$, where $J = (j_1, \ldots, j_p)$ and $L = (l_1, \ldots, l_q)$ and such that $a - J \in C_g$ and $b - K \in C_h$. Then $\alpha$ and $\beta$ are cocycles for the complex $A \cong G \otimes \wedge(V^*)$, because the subcomplex $K_{\gamma}^\bullet$ of $\text{Hom}_{A^e}(K_{\gamma}^\bullet(A), A \cong G)$ is a complex with zero differential whenever $\gamma \in C_g$ (for details, see [Naidu et al. 2011, Section 4]). Consequently, $\mathcal{R}\alpha$ and $\mathcal{R}\beta$ are $G$-invariant cocycles where, as before, $\mathcal{R}$ is the Reynolds operator. The bracket operation on Hochschild cohomology is determined by its values on cocycles of this form.

**Theorem 7.1.** In the case where $G$ acts diagonally on the basis $x_1, \ldots, x_N$, the graded Lie bracket of $\mathcal{R}\alpha$ and $\mathcal{R}\beta$, where $\alpha = \frac{a}{2} \notag g \otimes dx_J$ and $\beta = \frac{b}{2} \notag h \otimes dx_L$, is

$$[\mathcal{R}\alpha, \mathcal{R}\beta] = \sum_{1 \leq s \leq p} (-1)^{(q-1)(s-1)} \frac{1}{|G|^2} \sum_{k, \ell \in G} \rho^\alpha_{s, \ell} \partial_{[j_s]}(\frac{a}{2} \notag g \otimes k g k^{-1} \ell h \ell^{-1} \otimes dx_{J_s \cup L}

- (-1)^{(p-1)(q-1)} \sum_{1 \leq s \leq q} (-1)^{(p-1)(s-1)} \frac{1}{|G|^2} \times \sum_{k, \ell \in G} \rho^\beta_{s, \ell} \partial_{[i_s]}(\frac{a}{2} \notag h \otimes k g k^{-1} \ell h \ell^{-1} \otimes dx_{J \cup L_s})$$

for certain coefficients $\rho^\alpha_{s, \ell}$ and $\rho^\beta_{s, \ell}$.

**Remark 7.2.** This formula generalizes Theorem 5.1 (the case $G = 1$) and [Shepler and Witherspoon 2012, Corollary 7.3] (the case $q_{i, j} = 1$ for all $i, j$).

**Proof.** We may compute $[\mathcal{R}(\alpha), \mathcal{R}(\beta)]$ as $\Gamma([\Theta \mathcal{R}(\alpha), \Theta \mathcal{R}(\beta)])$.

Now by the third step,

$$\Gamma([\Theta \mathcal{R}(\alpha), \Theta \mathcal{R}(\beta)]) = \sum_{|I| = p+q-1} \sum_{\pi \in \text{Sym}_{p+q-1}} \text{sgn } \pi q^I_{\pi}[\Theta(\mathcal{R}(\alpha)), \Theta(\mathcal{R}(\beta))] \times (x_{i(1)} \notag e \otimes \cdots \otimes x_{i(p+q-1)} \notag e) \otimes dx_I.$$

Note that $\Psi_p$, when applied to an element of the form $1 \otimes x_{c_1} \otimes \cdots \otimes x_{c_p} \otimes 1$ if $1 \leq c_1 < \cdots < c_p \leq N$, and is 0 otherwise. This simplifies considerably the computation of $[\Theta \mathcal{R}(\alpha), \Theta \mathcal{R}(\beta)](x_{i(1)} \notag e \otimes \cdots \otimes x_{i(p+q-1)} \notag e)$.

For $1 \leq s \leq p$, we have

$$(\Theta \mathcal{R}(\alpha) \circ \Theta \mathcal{R}(\beta))(x_{i(1)} \notag e \otimes \cdots \otimes x_{i(p)} \notag e) = \Theta \mathcal{R}(\alpha)(x_{i(1)} \notag e \otimes \cdots \otimes \Theta \mathcal{R}(\beta)(x_{i(s)} \notag e \otimes \cdots \otimes x_{i(p+s-1)} \notag e) \otimes \cdots \otimes x_{i(p+q-1)} \notag e).$$

By Step (ii), a simple computation shows that $\Theta \mathcal{R}(\beta)(x_{i(s)} \notag e \otimes \cdots \otimes x_{i(p+s-1)} \notag e)$ is nonzero only when

$$i_{\pi(s)} = l_1, \ldots, i_{\pi(s+q-1)} = l_q.$$
in which case it is equal to \( \frac{1}{|G|} \sum_{\ell \in G} \chi_L(\ell^{-1}) \chi_b(\ell) \chi^{b \ell \ell^{-1}} \). Therefore, when
\[
i_{\pi(s)} = l_1, \ldots, \quad i_{\pi(s+q-1)} = l_q,
\]
we have
\[
\Theta \mathcal{R}(\alpha)(x_{i_{\pi(1)}} \# e \otimes \cdots \otimes x_{i_{\pi(s)}} \# e \otimes \cdots \otimes x_{i_{\pi(s+q-1)}} \# e)
\]
\[
= \Theta \mathcal{R}(\alpha) \left( x_{i_{\pi(1)}} \# e \otimes \cdots \otimes \left( \frac{1}{|G|} \sum_{\ell \in G} \chi_L(\ell^{-1}) \chi_b(\ell) \chi^{b \ell \ell^{-1}} \otimes x_{i_{\pi(s+q)}} \# e \right) \right)
\]
\[
= \frac{1}{|G|} \sum_{\ell \in G} \chi_L(\ell^{-1}) \chi_b(\ell) \Theta \mathcal{R}(\alpha) \left( x_{i_{\pi(1)}} \# e \otimes \cdots \otimes \chi^{b \ell \ell^{-1}} \otimes x_{i_{\pi(s+q)}} \# e \right)
\]
Applying Step (ii), in order that the above expression be nonzero, we must have
\[
j_1 = i_{\pi(1)}, \ldots, \quad j_{s-1} = i_{\pi(s-1)}, \quad j_{s+1} = i_{\pi(s+q)}, \ldots, \quad j_p = i_{\pi(p+q-1)}.
\]
When
\[
i_{\pi(s)} = l_1, \ldots, \quad i_{\pi(s+q-1)} = l_q,
\]
\[
j_1 = i_{\pi(1)}, \ldots, \quad j_{s-1} = i_{\pi(s-1)}, \quad j_{s+1} = i_{\pi(s+q)}, \ldots, \quad j_p = i_{\pi(p+q-1)},
\]
we have
\[
(\Theta \mathcal{R}(\alpha) \circ_s \Theta \mathcal{R}(\beta))(x_{i_{\pi(1)}} \# e \otimes \cdots \otimes x_{i_{\pi(p)}} \# e)
\]
\[
= \frac{1}{|G|^2} \sum_{k \in G} \sum_{\ell \in G} \chi_L(\ell^{-1}) \chi_b(\ell) \chi_{js+1}(\ell \ell^{-1}) \chi_{j p}(\ell \ell^{-1})
\]
\[
\times \sum_{0 \leq r \leq b_{js-1}} \lambda \mu \chi_f(k^{-1}) \chi_q(k) \chi_\hat{Q}(k g k^{-1}) \chi^{a+b-e_{js} \otimes \ell k g k^{-1} \ell \ell^{-1}},
\]
where
\[
\chi_\hat{Q} = \chi_{j_1} \cdots \chi_{j_{s-1}} \chi^{b_{js+1} \cdots b_N},
\]
\[
\chi_\hat{Q} = \chi_{j_1} \cdots \chi_{j_{s-1}} \chi^{b_{js} - r + 1},
\]
\[
\mu \chi_\hat{Q} = x_{j_1} \cdots x_{j_{s-1}} x_{j_{s+1}} \cdots x_{j_p} \in S_q(V),
\]
\[
\lambda \chi_\hat{Q} = x^{a+b-e_{js}} \in S_q(V).
\]
We see that in this case we have $I = J_s \sqcup L$. Furthermore, if this is the case, there is a unique permutation $\pi_s \in \text{Sym}_{p+q-1}$ such that

\[
\begin{align*}
j_1 &= i_{\pi_s(1)}, & \ldots & & j_{s-1} &= i_{\pi_s(s-1)}, \\
l_1 &= i_{\pi_s(s)}, & \ldots & & l_q &= i_{\pi_s(s+q-1)}, \\
\end{align*}
\]

that is, $\pi_s(I) = J_s \sqcup L$ as introduced before Theorem 5.1. We obtain that when $I = J_s \sqcup L$ and $\pi = \pi_s$ for $1 \leq s \leq p$,

\[
(\Theta \mathcal{R}(\alpha) \circ \Theta \mathcal{R}(\beta))(x_{i_{\pi_s(1)}} \otimes \cdots \otimes x_{i_{\pi_s(p+q-1)}} \otimes e) = \frac{1}{|G|^2} \sum_{k, \ell \in G} \rho_s^{\alpha, \beta} \partial_{[j_s]}(\chi^b) \cdot \chi^a \| k g k^{-1} \ell h \ell^{-1}
\]

for a certain coefficient $\rho_s^{\alpha, \beta}$ determined by the above data.

Finally

\[
\Gamma([\Theta \mathcal{R}(\alpha), \Theta \mathcal{R}(\beta)]) = \sum_{|I| = p+q-1} \sum_{\pi \in \text{Sym}_{p+q-1}} \text{sgn } \pi \frac{1}{|G|^2} \sum_{1 \leq s \leq p} (-1)^{(q-1)(s-1)} \rho_s^{\alpha, \beta} \partial_{[j_s]}(\chi^b) \cdot \chi^a \| k g k^{-1} \ell h \ell^{-1} \otimes dx_I
\]

In this diagonal case, the following corollary is immediate, since the difference operators in the bracket formula take 1 to 0. It generalizes [Shepler and Witherspoon 2012, Theorem 8.1].

**Corollary 7.3.** Assume $G$ acts diagonally on the chosen basis $x_1, \ldots, x_N$ of $V$, and let $\alpha = 1 \otimes dx_J$ and $\beta = 1 \otimes dx_L$. Then $[\mathcal{R} \alpha, \mathcal{R} \beta] = 0 \in \text{HH}^*(A \ltimes G)$.

In fact, this result can be seen to hold in the nondiagonal case as well, even without an explicit description of Hochschild cocycles in that case. Nonetheless we may still use a general argument for those cocycles having a particular form.

**Corollary 7.4.** Assume $G$ acts on $V$, not necessarily diagonally. Let $\alpha$ and $\beta$ be cocycles in $(A \ltimes G \otimes \wedge^*(V^*))^G$ for which $\alpha$ (respectively, $\beta$) is a linear combination of elements of the form $1 \otimes g \otimes dx_J$ (respectively, $1 \otimes h \otimes dx_L$). Then $[\alpha, \beta] = 0 \in \text{HH}^*(A \ltimes G)$. In particular, if $\alpha$ is a 2-cocycle, then it is a noncommutative Poisson structure.
Proof. The proof is similar to that of Theorem 7.1. However, rather than computing explicitly, we shall only explain why the bracket is 0.

We compute $[\alpha, \beta]$ using Theorem 6.1. Consider $\alpha$ as a homomorphism in $\text{Hom}(A \rtimes G)^e (K, (A) \otimes_k G, A \rtimes G)$; then it maps into $\otimes_k G \subset A \rtimes G$. By Theorem 6.1

$$[\alpha, \beta] = [\alpha \cdot \bar{\Psi}, \beta \cdot \bar{\Psi}] \cdot \bar{\Phi}.$$ 

Here $\bar{\Phi}$ and $\bar{\Psi}$ are chain maps of complexes of $(A \rtimes G)^e$-modules obtained by applying the Reynolds operator (that averages over images of group elements) to $\Phi$ and $\Psi$, respectively. So one needs to consider certain terms like $(\alpha \cdot a \Psi) \circ_k (\beta \cdot b \Psi)$ applied to $c \Phi(1 \otimes 1 \otimes \chi^I)$ for $k \geq 1$, and $a, b, c \in G$.

Recall that, if $I = (i_1, \ldots, i_p)$, then

$$\Phi(1 \otimes 1 \otimes \chi^I) = \sum_{\pi \in \text{Sym}_p} \text{sgn } \pi q_{i_1, \ldots, i_p} \otimes x_{i_{\pi(1)}} \otimes \cdots \otimes x_{i_{\pi(p)}} \otimes 1.$$ 

So $c \Phi(1 \otimes 1 \otimes \chi^I)$ is a linear combination of terms of the form $1 \otimes x_{j_1} \otimes \cdots \otimes x_{j_p} \otimes 1$ for $1 \leq j_1, \ldots, j_p \leq N$. In applying $(\alpha \cdot a \Psi) \circ_k (\beta \cdot b \Psi)$ to each term above, one first applies $b \Psi$ to $1 \otimes x_{j_k} \otimes \cdots \otimes x_{j_{k+m-1}} \otimes 1$, if the degree of $\beta$ is $m$. By (4.5),

$$\Psi_m(1 \otimes x_{j_k} \otimes \cdots \otimes x_{j_{k+m-1}} \otimes 1) = \mu \otimes x_{j_k} \wedge \cdots \wedge x_{j_{k+m-1}} \otimes 1$$

for some scalar $\mu$ and so $b \Psi_m(1 \otimes x_{j_k} \otimes \cdots \otimes x_{j_{k+m-1}} \otimes 1)$ is a linear combination of terms of the form $1 \otimes x_{\ell_1} \wedge \cdots \wedge x_{\ell_m} \otimes 1$ with $1 \leq \ell_1 < \cdots < \ell_m \leq N$.

Applying $\beta$ to the result, we obtain $0$ unless $L = (\ell_1, \ldots, \ell_m)$ for some $L$ for which $1 \circ h \otimes d\chi_L$ has a nonzero coefficient in the expression $\beta$, in which case we obtain a nonzero scalar multiple of $1 \circ h$ for that term. After factoring $h$ to the right, this becomes 0 as an element of the normalized bar resolution. The same argument applies to each term in $[\alpha, \beta]$, and so $[\alpha, \beta] = 0$.

For the last statement, recall that a noncommutative Poisson structure is simply a Hochschild 2-cocycle whose square bracket is a coboundary. \qed

Compare to the proof of [Naidu and Witherspoon 2016, Theorem 4.6], of which the above corollary is a consequence via the alternative route of algebraic deformation theory.

References

[Beilinson et al. 1996] A. Beilinson, V. Ginzburg, and W. Soergel, “Koszul duality patterns in representation theory”, J. Amer. Math. Soc. 9:2 (1996), 473–527. MR 1322847 Zbl 0864.17006

[Bian et al. 2009] N. Bian, G.-L. Zhang, and P. Zhang, “Setwise homotopy category”, Appl. Categ. Structures 17:6 (2009), 561–565. MR 2564122 Zbl 1210.18014

[Carqueville and Murfet 2016] N. Carqueville and D. Murfet, “Adjunctions and defects in Landau–Ginzburg models”, Adv. Math. 289 (2016), 480–566. MR 3439694 Zbl 06530919

[Erdmann et al. 2004] K. Erdmann, M. Holloway, R. Taillefer, N. Snashall, and Ø. Solberg, “Support varieties for selfinjective algebras”, K-Theory 33:1 (2004), 67–87. MR 2199789 Zbl 1116.16007
[Gerstenhaber 1963] M. Gerstenhaber, “The cohomology structure of an associative ring”, *Ann. of Math.* (2) **78** (1963), 267–288. MR 0161898 Zbl 0131.27302

[Gerstenhaber 1964] M. Gerstenhaber, “On the deformation of rings and algebras”, *Ann. of Math.* (2) **79** (1964), 59–103. MR 0171807 Zbl 0123.03101

[Halbout and Tang 2010] G. Halbout and X. Tang, “Noncommutative Poisson structures on orbifolds”, *Trans. Amer. Math. Soc.* **362**:5 (2010), 2249–2277. MR 2584600 Zbl 1269.58002

[Hochschild 1945] G. Hochschild, “On the cohomology groups of an associative algebra”, *Ann. of Math.* (2) **46** (1945), 58–67. MR 0011076 Zbl 0063.02029

[Le and Zhou 2016] J. Le and G. Zhou, “Comparison morphisms and Hochschild cohomology”, in preparation.

[Mac Lane 1975] S. Mac Lane, *Homology*, Grundlehren der Mathematischen Wissenschaften **114**, Springer, Berlin, 1975. Reprinted in 1995. MR 0156879 Zbl 0328.18009

[Naidu and Witherspoon 2016] D. Naidu and S. Witherspoon, “Hochschild cohomology and quantum Drinfeld Hecke algebras”, *Selecta Mathematica* (online publication February 2016).

[Naidu et al. 2011] D. Naidu, P. Shroff, and S. Witherspoon, “Hochschild cohomology of group extensions of quantum symmetric algebras”, *Proc. Amer. Math. Soc.* **139**:5 (2011), 1553–1567. MR 2763745 Zbl 1259.16011

[Shepler and Witherspoon 2011] A. V. Shepler and S. Witherspoon, “Quantum differentiation and chain maps of bimodule complexes”, *Algebra Number Theory* **5**:3 (2011), 339–360. MR 2833794 Zbl 1266.16005

[Shepler and Witherspoon 2012] A. V. Shepler and S. Witherspoon, “Group actions on algebras and the graded Lie structure of Hochschild cohomology”, *J. Algebra* **351** (2012), 350–381. MR 2862214 Zbl 1276.16005

[Snashall and Solberg 2004] N. Snashall and Ø. Solberg, “Support varieties and Hochschild cohomology rings”, *Proc. London Math. Soc.* (3) **88**:3 (2004), 705–732. MR 2044054 Zbl 1067.16010

[Wambst 1993] M. Wambst, “Complexes de Koszul quantiques”, *Ann. Inst. Fourier (Grenoble)* **43**:4 (1993), 1089–1156. MR 1252939 Zbl 0810.16010

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**SARAH WITHERSPOON**

**DEPARTMENT OF MATHEMATICS**

**TEXAS A&M UNIVERSITY**

**MAILSTOP 3368**

**COLLEGE STATION, TX 77843-3368**

**UNITED STATES**

sjw@math.tamu.edu

**GUODONG ZHOU**

**DEPARTMENT OF MATHEMATICS**

**SHANGHAI KEY LABORATORY OF PMMP**

**EAST CHINA NORMAL UNIVERSITY**

**DONG CHUAN ROAD 500**

**SHANGHAI 200241**

**CHINA**

gdzhou@math.ecnu.edu.cn
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