Abstract

In this paper, we initiate the study of fair clustering that ensures distributional similarity among similar individuals. In response to improving fairness in machine learning, recent papers have investigated fairness in clustering algorithms and have focused on the paradigm of statistical parity/group fairness. These efforts attempt to minimize bias against some protected groups in the population. However, to the best of our knowledge, the alternative viewpoint of individual fairness, introduced by Dwork et al. (ITCS 2012) in the context of classification, has not been considered for clustering so far. Similar to Dwork et al., we adopt the individual fairness notion which mandates that similar individuals should be treated similarly for clustering problems. We use the notion of $f$-divergence as a measure of statistical similarity that significantly generalizes the ones used by Dwork et al. We introduce a framework for assigning individuals, embedded in a metric space, to probability distributions over a bounded number of cluster centers. The objective is to ensure (a) low cost of clustering in expectation and (b) individuals that are close to each other in a given fairness space are mapped to statistically similar distributions.

We provide an algorithm for clustering with $p$-norm objective ($k$-center, $k$-means are special cases) and individual fairness constraints with provable approximation guarantee. We extend this framework to include both group fairness and individual fairness inside the protected groups. Finally, we observe conditions under which individual fairness implies group fairness. We present extensive experimental evidence that justifies the effectiveness of our approach.

1 Introduction

Increasing deployment of machine learning based systems in decision making tasks such as targeted ad placement [48], issuing home loans [6], predicting recidivism [4, 16], and gender inequality at workplace [18, 40] mandates that such algorithms are fair to individuals or groups in a population. An increasing body of research over the last decade has attempted to define various notions of fairness in such systems and design efficient learning algorithms that respect these fairness constraints (see the excellent survey by Mehrabi et al. [39]).

Clustering is a classical unsupervised learning technique with wide applications in domains such as recommender systems [46], customer segmentation [12], feature generation [36, 29], targeted advertisement [1], etc. The seminal work of Chierichetti et al. [14] initiated the study of group fairness (also called statistical fairness) in clustering. Group fairness requires that the representation of various protected groups in all the clusters should be balanced. The work of [14] was immediately
followed up by several researchers [44, 9, 5, 8, 2, 23] leading to efficient algorithms for a wide variety of clustering problems under group fairness constraints.

In this paper, we consider the alternate viewpoint of individual fairness introduced in the influential work of Dwork et al. [21] in the context of classification problems. To the best of our knowledge, this particular notion of individual fairness has not been previously studied for clustering problems. Our main motivation is to address the possibility of standard clustering algorithms or clustering algorithms enforcing group fairness being unfair to ‘similar’ individuals, as illustrated by Figure 1 and Figure 2. Taking Figure 2 for example, group fairness demands that, in each cluster, roughly one-third of the points must be circles (red). Let $R_L$ and $B_L$ be the sets of red and blue points on the left respectively. Naturally, two of the points from the set $R_L$, marked with oval, needs to be assigned to the cluster $C_R$ on the right. However, this would violate individual fairness between the points inside the oval and the remaining points in $R_L \cup B_L$. In fact, it has been shown that forcing group fairness can lead to disparate treatment of similar individuals or open up the possibility of gerrymandering by unfairly targeting a subgroup of a protected group — see [27, 28].

![Figure 1: A simple problem to illustrate the possibility of harming the individuals (on the boundaries of three clusters) via off-the-shelf clustering method.](image1)

![Figure 2: Group fairness might affect Individual Fairness. Moving the red points within the oval from $R_L$ to the right cluster $C_R$ would violate individual fairness constraints between these points and the remaining points in $R_L$.](image2)

**Our notion: Individual fairness in $k$-clustering.** In $k$-clustering problems ($k$-means, $k$-median, $k$-center, etc.), the input consists of a set of points $V$ embedded in a known metric space. The goal is to partition the points into $k$ clusters while minimizing some distance-based objective function. We propose a randomized assignment of points to centers as part of our solution concept. Inspired by ideas from Dwork et al. [21], our algorithm produces a set of $k$ centers denoted by $C$, and a mapping of each point $x \in V$ to a distribution over the $k$ centers, while minimizing the expected clustering cost. Note that this is related to probabilistic clustering solutions such as soft $k$-means [20] or fuzzy $k$-means [29, 10]. However, we show in our experiments that these solutions can be unfair to individuals.

We enforce individual fairness between points through distributional similarity. We assume a fairness similarity measure $F : V \times V \to \mathbb{R}_{\geq 0}$ (not necessarily a metric) that maps every pair of points in the population to some non-negative real number. We require the statistical distance between the output distributions of two points in $V$, measured by $f$-divergence [17, 41, 3], to be upper bounded by their $F$-measure. This is analogous to the definition of individual fairness in classification by Dwork et al. [21], where they utilize the special cases of $f$-divergence, namely, total variational distance and relative $\ell_\infty$ metric. However, in classification, either one has to assume the knowledge of a similarity measure as side information, or face the non-trivial task of computing [30] or learning the same [52]. On the other hand, in clustering problems, the distance metric $d$ provided by the feature space can be considered as a natural choice of the fairness similarity measure. However, we emphasize that all our results hold for any arbitrary choice of fairness similarity measure.

1.1 Our Contribution

Our main contributions can be summarized as follows:
• **Distributional Individual Fairness for Clustering**: We introduce distributional individual fairness for $\ell_p$-norm clustering problems using a general family of divergence functions.

• **Approximation algorithms for Individually Fair Clustering**: We provide a generic solution template that adapts any algorithm for $\ell_p$-norm clustering objective to an individually fair solution. In particular, we give an algorithm for the individually fair $\ell_p$-norm $k$-clustering problem that achieves a constant factor approximation guarantee (Theorem 5).

• **Algorithms for Combined Fairness**: We show connections between individual fairness and group fairness, and extend our solution to combine the two paradigms. One interesting aspect of this result is that we enforce individual fairness only among the individuals belonging to the same protected group. We justify this relaxation in Appendix B by demonstrating that the more stringent requirement of individual fairness across every pair of points can lead to trivial and expensive solutions. Our framework can be seamlessly combined with ideas developed in [8] to give a constant factor approximation algorithm that guarantees both group fairness (in expectation) and individual fairness among members of the same group (Theorem 8).

We provide extensive empirical evidence to support the effectiveness of our method.\(^1\) Experiments show that our method achieves objective cost much better than predicted by our theoretical analysis while respecting individual fairness. Our solution is probabilistic. A single realization according to the distribution that our algorithm produces, might still be unfair to a pair of similar individuals. However, when the clustering algorithm is used upon repeated trials (e.g., profiling a customer for a sequence of different product recommendations), they would be assigned to the clusters with similar empirical distributions. This is the scenario our solution focuses on and tries to address.

1.2 Related Work

Fairness in machine learning is a fast-evolving topic — see [39] for a comprehensive survey of recent advances in this area. Our work mainly concerns with individual fairness, a concept introduced by Dwork et al. [21]. Subsequently in [52, 34, 35], the authors proposed methodologies to learn the similarity measure in order to achieve individual fairness. [11, 49, 26] also explored the direction of implicitly learning the similarity measure in the context of ranking and classification problems. The approach of combining individual fairness and group fairness has been initiated in [21] and further explored in [35, 49]. However, none of these works consider the important case of clustering.

For clustering problems, in a seminal work, Chierichetti et al. [14] initiated the study of fairness. Their notion of fairness is defined at a group level — the population is partitioned into two protected groups and each group required to be well-represented in each cluster. Subsequently, this notion has been greatly generalized to include more than two protected groups [44, 9, 8, 2], and the groups are even allowed to be overlapping [8]. The fairness notion advocated by these works operate within the ambit of disparate impact doctrine [22] — each protected group must be almost equally represented in the outcome of any algorithm. [47, 5, 23] focused on designing scalable algorithms achieving group fairness. Few other notions of fairness have been considered in the clustering domain such as proportionally fair clustering [13], fair selection of cluster centers [31, 15] and fair spectral clustering [32]. None of these works address the question of individual fairness and are orthogonal to the direction we take in this paper. Recently, [25, 37] consider a notion of individual fairness which requires every point $j$ to have a center within a distance of $r_j$ where $r_j$ is the minimum radius ball centered at $j$ that contains at least $n/k$ points. Our notion of individual fairness differs significantly from this notion and is not directly comparable. However, in our experiments, we consider a fairness similarity measure inspired by these works.

2 Problem Definitions and Preliminaries

We begin with the definition of statistical similarity between two distributions used in formulating individual fairness in clustering.

**Definition 1 ($f$-divergence)** Let $P, Q$ be two probability measures on a discrete space $X$. Then for any function $f : [0, \infty) \rightarrow \mathbb{R}$, where $f$ is strictly convex at 1 and $f(1) = 0$, the $f$-divergence between $P$ and $Q$ is defined as $D_f(P||Q) = \sum_{x \in X} f\left(\frac{P(x)}{Q(x)}\right)Q(x)$

\(^1\)We are contributing our code to the community.
The above definition requires the following two assumptions for completeness: (1) \( 0 \cdot f(\frac{0}{0}) = 0 \), (2) \( 0 \cdot f(\frac{c}{d}) = \lim_{x \to 0^+} x f(\frac{x}{d}) \). Some popular instances of \( f \)-divergence include total variation distance \( D_{TV} \) \( f(t) = \frac{1}{2}|t-1| \) and KL-divergence \( f(t) = t \log t \).

Next, we define various clustering problems that we shall consider in subsequent sections. Let \( V \) be a set of points embedded in some metric space \((X, d)\). We use \([n]\) to denote the set \( \{1, 2, \cdots, n\} \).

**Definition 2 (VANILLA \((k, p)\)-CLUSTERING)** The VANILLA \((k, p)\)-CLUSTERING asks for (I) a set of cluster centers \( C \subseteq V \) of size at most \( k \) and (2) an assignment \( \varphi : V \to C \) of every point in \( V \) to a center in \( C \). The objective is to minimize the \( \ell_p \)-norm distance, \( L_p(\varphi, C) = \left( \sum_{j \in V} d(j, \varphi(j))^p \right)^{1/p} \).

Some of the much-studied special cases are \( k\)-center \((p = \infty)\), \( k\)-median \((p = 1)\), and \( k\)-means \((p = 2)\). Note that, for vanilla clustering, the assignment \( \varphi \) maps each point in \( V \) to its closest center in \( C \) and hence fully determined by \( C \). We next define the individually fair clustering problem. Let \( \mathcal{F} : V \times V \to \mathbb{R}_{\geq 0} \) be a non-negative fair similarity measure defined over all pair of points in \( V \). Note that \( \mathcal{F} \) may not be a metric.

**Definition 3 (INDIVIDUALLY FAIR \((k, p, f, \mathcal{F})\)-CLUSTERING)** Assume we are given a function \( f \) as in Definition 1. Then, INDIVIDUALLY FAIR \((k, p, f, \mathcal{F})\)-CLUSTERING asks for (1) a set of cluster centers \( C \subseteq V \) of size at most \( k \) and (2) a distribution \( \mu_j \) over \( C \) for each point \( j \in V \), such that

\[
D_f(\mu_j, \mu_{j'}) \leq \mathcal{F}(j, j'), \forall j, j' \in V
\]

The objective is to minimize \( L_p(\mu, C) := \left( \sum_{j \in V} \mathbb{E}_{c \sim \mu_j} (d(j, c)^p) \right)^{1/p} \).

The definition of individually fair \( k\)-center is not precisely captured by the above definition. We treat that separately in Appendix A. We denote the optimal cost of any instance \( I \) of VANILLA \((k, p)\)-CLUSTERING as \( \text{OPT}_{k,p}(I) \) and that of any instance \( J \) of INDIVIDUALLY FAIR \((k, p, f, \mathcal{F})\)-CLUSTERING as \( \text{OPT}_{k,p,f,\mathcal{F}}(J) \).

We now define a problem that ensures both statistical and individual fairness. Note that, in this definition, we only enforce individual fairness among individuals that belong to the same protected group (see Appendix B).

**Definition 4 (COMBINED FAIR \((k, p, f, \mathcal{F})\)-CLUSTERING)** Assume we are given an instance of the INDIVIDUALLY FAIR \((k, p, f, \mathcal{F})\)-CLUSTERING problem. Additionally, we are given \( \ell \)-many (possibly overlapping) protected groups \( G_1, G_2, \ldots, G_\ell \) and for each such group we are given two input group fairness parameters \( \alpha_i \) and \( \beta_i \). The goal and the objective remain the same. The output distributions \( \mu_j, \forall j \in V \) must satisfy the following two constraints.

1. For each cluster, the expected fraction of the points from group \( G_i \) lies between \( \beta_i \) and \( \alpha_i \),
2. \( D_f(\mu_{j_1}, \mu_{j_2}) \leq \mathcal{F}(j_1, j_2) \) for each pair of points \( j_1, j_2 \in G_p \) for all \( p \in [\ell] \).

We remark here that there exists a trivial and potentially very expensive feasible solution to both the individual and combined fair clustering problems — simply assign a uniform distribution to each point (for the combined fair clustering, this assumes that the instance is feasible with respect to group fairness parameters \( \alpha \) and \( \beta \)). See Appendix B for a discussion on the feasibility question.

## 3 Algorithm for Individually Fair Clustering

In this section, we present our main theoretical result. We give an algorithmic framework for solving the individually fair clustering problem (Algorithm 1). Theorem 5 captures its theoretical guarantees. Suppose we are given an instance \( \mathcal{I} = (V, d, f, \mathcal{F}) \) for INDIVIDUALLY FAIR \((k, p, f, \mathcal{F})\)-CLUSTERING. We first disregard \( f \) and \( \mathcal{F} \), and use any existing algorithm for the VANILLA \((k,p)\)-CLUSTERING problem to obtain a set of cluster centers \( C \). We then create a constrained optimization problem FAIR-ASSGN on the instance \( \mathcal{J} = (V, C, f, \mathcal{F}) \), as given in Equations (2) to (5),
We first claim the following structural property of the mapping \( \varphi \).

We now discuss the problem. For each \( j \in V \) and \( c \in C \), let \( x_{cj} \) be the probability that the client \( j \) is assigned to the center \( c \). Hence, \( x_j \) will give the desired distribution \( \mu_j \) corresponding to \( j \) over the set of centers \( C \). The first constraint ensures that each client is assigned a distribution and the second one enforces the individual fairness constraints (1). Clearly, any solution to FAIR-ASSGN is also a feasible solution to INDIVIDUALLY FAIR \((k, p, f, \mathcal{F})\)-CLUSTERING.

Note that the computational complexity of solving the above constrained optimization depends on the constraints (4). For example, if the LHS of these constraints are convex functions of \( x \), then we can solve this in polynomial time. Indeed, that is the case for many common choices of \( D_f \) (\( D_{TV}, \) KL-divergence, etc.). Let \( A_1 \) be a \( \rho \)-approximate algorithm for VANILLA \((k, p)\)-CLUSTERING with running time \( T(A_1) \) and \( A_2 \) be an optimal solver for the FAIR-ASSGN problem with running time \( T(A_2) \). Then, our main result is the following theorem.

**Theorem 5** Given an instance \( \mathcal{I} \) to INDIVIDUALLY FAIR \((k, p, f, \mathcal{F})\)-CLUSTERING, let \((C, \varphi)\) be a \( \rho \)-approximate solution of VANILLA \((k, p)\)-CLUSTERING on \( \mathcal{I} \). Then, Algorithm 1 produces distributions \( \mu_j, \forall j \in V \), such that \( \mathcal{L}_p(\mu, C) \leq 3(1 - \frac{1}{\rho})(\rho + 2) \cdot \text{OPT}_{k,p,\mathcal{F}}(\mathcal{I}) \) and it runs in time \( O(T(A_1) + T(A_2)) \).

In the remainder of this section, we prove Theorem 5. We state and use several lemmas in this section whose proofs we defer to the Appendix A. We emphasize that the cost guarantee of our algorithm is with respect to \( \text{OPT}_{k,p,\mathcal{F}}(\mathcal{I}) \) and not with respect to \( \text{OPT}_{k,p}(\mathcal{I}) \). It is indeed possible that \( \text{OPT}_{k,p,\mathcal{F}}(\mathcal{I}) \) is much larger than \( \text{OPT}_{k,p}(\mathcal{I}) \), and hence the clustering cost of our algorithm could be much larger compared to \( \text{OPT}_{k,p}(\mathcal{I}) \). The cost of achieving fairness depends on the fairness measure \( \mathcal{F} \) and we discuss it in the experiment section (Section 5).

Assume \((C^*, x^*)\) is an optimal solution to instance \( \mathcal{I} \) of INDIVIDUALLY FAIR \((k, p, f, \mathcal{F})\)-CLUSTERING and ALG-IF \((\mathcal{I})\) returns \((C, \mu)\). We construct a feasible solution \( x \) to FAIR-ASSGN \((\mathcal{I} = (V, C, f, \mathcal{F}))\) using \( C^* \) and \( x^* \). ALG-IF outputs the optimal solution to FAIR-ASSGN, hence, \( \mathcal{L}_p(\mu, C) \leq \mathcal{L}_p(x, C) \).

Let \( \varphi : C^* \to C \) be a function that maps each center in \( C^* \) to its closest center in \( C \): \( \varphi(c^*) = \arg \min_{c \in C} d(c, c^*) \), breaking ties arbitrarily. Let \( \varphi^{-1}(c) \) denote the set of centers mapped to \( c \in C \): \( \varphi^{-1}(c) = \{ c^* \in C^* : \varphi(c^*) = c \} \). Note that \( \varphi^{-1}(c) \) can be empty for some \( c \in C \). For each \( j \in V \) and each \( c \in C \), set \( x_{cj} = \sum_{c^* \in \varphi^{-1}(c)} x_{cj}^{c^*} \). In words, for a fixed point \( j \in V \) and a fixed center \( c \in C \), we look at the centers in the optimal solution that are mapped to \( c \) by \( \varphi \), and sum the corresponding probabilities to get \( x_{cj} \).

We first claim the following structural property of the mapping \( \varphi \). This claim bounds the distance between a point \( j \) in \( V \) and a center \( c \) in \( C \) in terms of the distance between \( j \) and its closest center in \( C \) and the distance between \( j \) and any optimal center \( c^* \) that is mapped to \( c \) by \( \varphi \).

**Claim 6** Assume \( c \in C \) be a center such that \( \varphi^{-1}(c) \) is non-empty. For a point \( j \in V \), let \( c_j \) be its closest center in \( C \): \( c_j = \arg \min \{ d(j, c) : c \in C \} \). Then, for each \( c^* \in \varphi^{-1}(c) \) and for each \( j \in V \),
we have
\[ d(j, c)^p \leq 3^{p-1} (2d(j, c^*)^p + d(j, c_j)^p) \]

Using Claim 6, we show that \( x \) is a low cost solution to the FAIR-ASSGN \((J)\) problem in Lemma 7. Theorem 5 then follows immediately from Lemma 7.

**Lemma 7** \( x \) is a feasible solution to FAIR-ASSGN \((J)\) with cost \( \mathcal{L}_p(x, C) \leq 3^{1 - \frac{1}{p}} (\rho + 2) \cdot \text{OPT}_{k,p,f,J}(I) \).

**Remark 1** The individually fair \( k \)-center \((p = \infty)\) problem is not handled directly by Algorithm 1. In particular, stating the FAIR-ASSGN optimization problem (Equation (2)) with \( p = \infty \) requires the standard technique of “guess the optimal value”. See Appendix A for details.

### 4 Individual Fairness and Group Fairness

In this section, we consider the combined FAIR \((k, p, f, \mathcal{F})\)-CLUSTERING problem. At a high level, our algorithmic strategy remains the same — we first solve the VANILLA \((k, p)\)-CLUSTERING to find the cluster centers, and then solve a suitable constrained optimization program to find the distribution corresponding to each point. We describe in Appendix B the constrained optimization problem analogous to the FAIR-ASSGN problem given in Section 3. Reusing notation, assume \( \text{OPT}_{k,p,f,J}(I) \) denote the optimal cost of the instance \( I \). We then prove the following theorem in Appendix B.

**Theorem 8** Given an instance \( I \) to combined FAIR \((k, p, f, \mathcal{F})\)-CLUSTERING, let \( C \) be a \( \rho \)-approximate solution for the corresponding VANILLA \((k, p)\)-CLUSTERING on \( I \). Then, there exists an algorithm which produces feasible distributions \( \mu_j, \forall j \in V \), such that \( \mathcal{L}_p(\mu, C) \leq 3^{1 - \frac{1}{p}} (\rho + 2) \cdot \text{OPT}_{k,p,f,J}(I) \).

Note that, the case of \( p = \infty \) \((k-center)\) requires special case (see Remark 1) — we handle this in Appendix B. Finally, we consider the special case of \( \mathcal{F} = d \), that is when the fairness similarity measure is given by the underlying distance metric, and observe the conditions under which individually fair clustering solutions guarantees group fairness. Our characterization is similar to the one discussed in the work of [21] and given in Appendix C.

### 5 Experimental Evaluation

In this section, we present extensive empirical evaluations of our algorithms. We implement our algorithms in Python 3.6 and simulate on Intel Xeon CPU E5-2670 v2 @ 2.50GHz 20 cores and 96 GB 1333 MHz DDR3 memory. We use IBM CPLEX for solving linear programs. \(^2\)

Although our algorithmic framework can handle any \( \ell_p \)-norm based objective, we focus on the widely popular \( k \)-means clustering for demonstration. We measure individual fairness against total variation norm, \( \mathcal{D}_{TV}(\mu_x||\mu_y) = \frac{1}{2} \sum_{c \in C} |\mu_x(c) - \mu_y(c)| \), a widely used \( f \)-divergence measure. Based on our experiments, we report the following key findings. (1) Variants of \( k \)-means and other clustering algorithms that guarantee group fairness are largely unfair to individuals. (2) Our algorithms provide individual fairness by paying at most 1.08 times more than the optimal cost. (3) Unlike group fairness, individual fairness comes at a higher cost when compared against vanilla \( k \)-means.

**Datasets.** We use five datasets from UCI Machine Learning Repository [19]. \(^3\) (1) Bank - 4,521 points 42] (2) Adult - 32,561 points [33] (3) Creditcard - 30,000 points [51] (4) Census1990 - 2,458,285 points [38] (5) Diabetes - 101,766 points [50]. We remark that most of the previous works on fairness in clustering [14, 8, 5, 23] focused on these datasets.

**Algorithms.** We use Lloyd’s algorithm [43] to solve vanilla \( k \)-means and approximate the centers by its nearest neighbour in \( V \). HKM denotes hard \( k \)-means (binary assignment of points to centers) and SKM denotes soft \( k \)-means [7, 20]. SKM outputs a set of \( k \) centers \( \{c_1, c_2, \cdots c_k\} \) and for a

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\(^2\)https://github.com/nihesh/distributional_individual_fairness_in_clustering
\(^3\)https://archive.ics.uci.edu/ml/datasets/
fixed stiffness parameter $\beta$, assigns $x \in V$ to a center $c$ with probability $\frac{e^{-\beta d(c,x)^2}}{\sum_{i \in V} e^{-\beta d(c,i)^2}}$. ALG-IF denotes the algorithm for individual fairness from Section 3 and ALG-CF denotes the algorithm for combined fairness from Section 4. GF denotes the algorithm for group fairness from [8]. OPT-IF and OPT-CF denote the optimal solution to the natural LP relaxation (allowing the fractional opening of centers) for INDIVIDUALLY FAIR $(k, p, f, F)$-CLUSTERING and COMBINED FAIR $(k, p, f, F)$-CLUSTERING respectively. They provide lower bounds to the cost of the optimal solution of the corresponding problems.

**Fairness Similarity Measures.** We consider two different fairness similarity measures $F_1$ and $F_2$. Both the measures are defined using the underlying distance metric $d$ in the given feature space. We choose $F_1 = d$, scaled linearly so that $F_1([j_1, j_2]) \in [0, 1]$ $\forall j_1, j_2 \in V$. In order to lower the computational requirement, we enforce $F_1$ constraints only between every $i \in V$ and its $m$ nearest neighbors. $F_2$ is defined in a more local way. For each $i \in V$, we consider the smallest ball $B_i$ of radius $r_i$ centered at $i$, such that $B_i$ contains at least $\left\lceil |V|/k \right\rceil$ points. Then, we define $F_2(i, j) = d(i, j)/r_i, \forall j \in B_i$ and $F_2(i, j) = 1$, otherwise. The motivation behind $F_2$ is inspired by the individual fairness notion in [25, 37]. More specifically, in $F_2$, each point is required to be treated similarly to its closest $\left\lceil |V|/k \right\rceil$ neighbors. For combined fairness, we enforce $F_1$ and $F_2$ only within protected groups.

**Implementation Details.** We subsample the datasets to 1000 points selected uniformly at random and run the experiments on a subset of numerical attributes. The numerical attributes are normalized to zero mean and unit variance. We choose two protected attributes for each dataset, set $\delta = 0.2$ (measure of tightness of group fairness constraints, introduced in [8]) and set $m = 250$. We run the algorithms for $k = 2, 4, 6, 8, 10$. This configuration of parameters is used in all the simulations unless mentioned otherwise.

Due to space constraints, we present a subset of our results here — further results, including runtime of our algorithm on variable dataset sizes, are given in Appendix D.

| (a) Fairness similarity $F_1$ | (b) Fairness similarity $F_2$ |
|-----------------------------|-----------------------------|
| Clusters $(k)$             | Clusters $(k)$             |
| Adult                      | 4                           | 6                           | 8                           | 10                          |
| 8                          | 84                          | 94                          | 98                          | 99                          |
| Creditcard                 | 61                          | 76                          | 83                          | 85                          |
| Census1990                 | 25                          | 34                          | 44                          | 50                          |
| Adult                      | 4                           | 5                           | 7                           | 8                           |
| Creditcard                 | 6                           | 5                           | 6                           | 6                           |
| Census1990                 | 7                           | 11                          | 13                          | 11                          |

**Unfairness of SKM.** In Table 1, we demonstrate the unfairness of soft $k$-means. Note that the output of SKM depends on the stiffness parameter $\beta$. We experimentally choose $\beta$ such that the cost of SKM is equal to the cost of ALG-IF. Table 1a shows the percentage of individual fairness constraints violated, with respect to $F_1$ and Table 1b shows the same with $F_2$. Observe that $F_2$ is a much relaxed fairness measure compared to $F_1$: for each point, similarity is measured locally, with respect to its $\left\lceil |V|/k \right\rceil$ nearest neighbors. Even with such relaxations, SKM exhibits unfair treatment of similar points. Our solution does not violate any individual fairness constraints.

**Unfairness of GF.** In Figure 3, we show that group fairness does not imply individual fairness. We observe the percentage of individual fairness constraints violated by GF for different values of $k$ and infer that, for $k \geq 4$, at least 25% of the constraints are violated in the best case, and violations increase monotonically as $k$ increases (as expected).

**Cost Analysis of Our Algorithms.** In this section, we compare the cost of ALG-IF and ALG-CF against OPT-IF and OPT-CF, respectively. Since OPT-IF and OPT-CF are computationally expensive, we reduce the size of the dataset to 80 points chosen uniformly at random, and set $m = 20$. We present the plots for two datasets here, and the rest are in Appendix D (similar trend).

In Figure 4, we compare the cost of ALG-IF and OPT-IF using fairness similarity $F_1$ and $F_2$. We observe that the approximation ratio is at most 1.08, which is significantly better than the bound given in Theorem 5.
In Figure 5, we compare the cost of ALG-CF and OPT-CF using fairness similarity $F_1$ and $F_2$. Similar to ALG-IF, we observe that the approximation ratio is at most 1.06, which is significantly better than the bound given in Theorem 8.

**Price of Individual Fairness.** Figure 4 and Figure 5 shows that the cost of OPT-IF and OPT-CF can be at most 1.5 times larger as compared to HKM. In contrast, [8] showed that group fairness can be achieved by paying at most 1.15 times the HKM cost (for all the datasets). It suggests that individual fairness comes at a higher price. We elaborate on this further in Appendix A.

## 6 Conclusion

In this work, we initiate the study of individual fairness in clustering, inspired by the notion of Dwork et al. [21] in the context of classification. We discuss and demonstrate the limitations of group fairness alone. We give a general framework for handling individual fairness and combined fairness for a variety of clustering objectives as well as statistical distance measures. Empirically, we demonstrate the effectiveness of our approach. One caveat of our generic framework is that we rely on an efficient solver for a convex optimization problem. We leave the problem of designing more efficient and scalable algorithms for specific instances of $f$-divergence as an interesting future research direction.
Figure 5: Clustering cost vs number of clusters for ALG-CF, OPT-CF and HKM.

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A Missing Details from Section 3

In this section, we fill out the details of various items that we have omitted in the main body due to lack of space. In Appendix A.1, we complete the proof of Theorem 5. In Appendix A.2 we take up the case of $k$-center and discuss how to modify our algorithm to get the same result as given in Theorem 5. Finally, in Appendix A.3, we discuss the price of achieving individual fairness by comparing the cost of a fair clustering solution against the corresponding vanilla clustering solution.

A.1 Proof of Theorem 5

In this section, we present the proofs of various lemmas and claims that are used in proving Theorem 5. We use Jensen’s Inequality in the proof, and for the sake of completeness, we include it here.

**Lemma 9 (Jensen’s inequality [24])** Let $g$ be a real-valued convex function and $p$ be a distribution over finite discrete space $X$. Then,

$$g\left(\sum_{i \in X} p_i x_i\right) \leq \sum_{i \in X} p_i g(x_i).$$

We now restate the claim regarding the structural property of the mapping $\varphi$ and prove it.

**Claim 6** Assume $c \in C$ be a center such that $\varphi^{-1}(c)$ is non-empty. For a point $j \in V$, let $c_j$ be its closest center in $C$: $c_j = \arg \min \{d(j, c) : c \in C\}$. Then, for each $c^* \in \varphi^{-1}(c)$ and for each $j \in V$, we have

$$d(j, c)^p \leq 3^{p-1} (2d(j, c^*)^p + d(j, c_j)^p)$$

**Proof:** We begin the proof by first considering $p = 1$. In this case, $\psi = 1$. This was implicitly proved in [8]. For completeness, we present a proof here as well. The proof follows by application of triangle inequality and definition of $\varphi$.

$$d(j, c)$$

$$\leq d(j, c^*) + d(c^*, c), \quad \text{(triangle inequality)}$$

$$\leq d(j, c^*) + d(c^*, c_j), \quad \text{(since $c^* \in \varphi^{-1}(c)$)}$$

$$\leq d(j, c^*) + d(c^*, j) + d(j, c_j), \quad \text{(triangle inequality)}$$

$$= 2d(j, c^*) + d(j, c_j). \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quart, 2014:

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Lemma 7 $x$ is a feasible solution to FAIR-ASSGN ($\mathcal{J}$) with cost $\mathcal{L}_p(x, C) \leq 3^{(1 - \frac{1}{p})}(\rho + 2) \cdot \text{OPT}_{k,p,f,\mathcal{J}}(\mathcal{I})$.

Proof: We first prove that $x$ is a feasible solution to FAIR-ASSGN ($\mathcal{J}$). First, we show that $x^*_j$ is a probability distribution. Clearly, $0 \leq x^*_{cj} \leq 1$ for all $c$:

$$x^*_{cj} = \sum_{c^* \in \varphi^{-1}(c)} x^*_{c^*j} \leq \sum_{c^* \in C^*} x^*_{c^*j} \leq 1.$$ 

We next show that $\sum_{c \in C} x^*_{cj} = 1$ for all $j \in V$.

$$\sum_{c \in C} x^*_{cj} = \sum_{c \in C} \sum_{c^* \in \varphi^{-1}(c)} x^*_{c^*j} = \sum_{c^* \in C^*} x^*_{c^*j} = 1,$$

where the second last equality follows since $\varphi^{-1}$ forms a partition of $C^*$ and the final equality follows from feasibility of $x^*$.

We now show that $x$ satisfies (4). Fix two points $j_1$ and $j_2$ in $V$. Recall the Definition 1 of $f$-divergence between $x^*_{j_1}$ and $x^*_{j_2}$:

$$D_f(x^*_{j_1} || x^*_{j_2}) = \sum_{c \in C} x^*_{cj_1} f \left( \frac{x^*_{cj_1}}{x^*_{cj_2}} \right) = \sum_{c \in C} \left( \sum_{c^* \in \varphi^{-1}(c)} x^*_{c^*j_1} \right) \left( \frac{\sum_{c^* \in \varphi^{-1}(c)} x^*_{c^*j_1}}{\sum_{c^* \in \varphi^{-1}(c)} x^*_{c^*j_2}} \right).$$

Observe that, for any center $c$ with $\varphi^{-1}(c) = \emptyset$, $x^*_{cj} = 0$ for each $j \in V$. We call such centers empty. Hence, assuming $0f(\frac{0}{0})$ is well-defined, we can disregard any empty center $c$. Fix a center $c \in C$ that is non-empty. For ease exposition, let $B = \sum_{c^* \in \varphi^{-1}(c)} x^*_{c^*j_2}$. Since $f$ is convex, by applying Jensen’s inequality, we derive the following:

$$f \left( \frac{\sum_{c^* \in \varphi^{-1}(c)} x^*_{c^*j_1}}{B} \right) \leq \sum_{c^* \in \varphi^{-1}(c)} \frac{x^*_{c^*j_1}}{B} f \left( \frac{x^*_{c^*j_1}}{x^*_{c^*j_2}} \right).$$

Plugging this in the above equations, we derive:

$$D_f(x^*_{j_1} || x^*_{j_2}) \leq \sum_{c \in C} \sum_{c^* \in \varphi^{-1}(c)} x^*_{c^*j_2} f \left( \frac{x^*_{c^*j_1}}{x^*_{c^*j_2}} \right),$$

$$= \sum_{c^* \in C^*} x^*_{c^*j_2} f \left( \frac{x^*_{c^*j_1}}{x^*_{c^*j_2}} \right),$$

$$= D_f(x^*_{j_1} || x^*_{j_2}),$$

$$\leq \mathcal{F}(x, y),$$

where (a) the first equality follows since $\varphi^{-1}$ partitions the set $C^*$, (b) the second equality follows by definition of $D_f$, and (c) the last inequality follows since $x^*$ is a feasible solution. This completes the proof of the lemma.

We now prove the second part of the lemma. Fix a point $j \in V$. Let $d^*(j)$ and $d(j)$ denote the expected cost paid by the point $j$ in the optimal solution $x^*$ and our constructed solution $x$, respectively. Formally,

$$d^*(j) = \sum_{c \in C^*} x^*_{cj} d(j, c^*)^p,$$  \hspace{1cm} (7)

$$d(j) = \sum_{c \in C} x_{cj} d(c, j)^p.$$  \hspace{1cm} (8)

Recall that in VANILLA $(k, p)$-CLUSTERING, $j$ is assigned to its closest cluster center in $C$. Assume $c_j$ is the closest center to $j$ in $C$: $c_j = \arg \min \{ d(j, c) : c \in C \}$. Then, $\left( \sum_{j \in V} d(j, c_j)^p \right)^{1/p} \leq \text{OPT}_{k,p,f,\mathcal{J}}(\mathcal{I})$.\]
\( \rho \cdot OPT_{k,p}(I) \). Further, \( OPT_{k,p}(I) \leq OPT_{k,p,f,(F)}(I) \), since any solution to INDIVIDUALLY FAIR \((k,p,f,F)\)-CLUSTERING is also a feasible solution to VANILLA \((k,p)\)-CLUSTERING.

We now bound \( d(j) \) in terms of \( d^*(j) \) and \( d(j,c_j) \). Assume \( \psi = 3p-1 \).

\[
d(j) = \sum_{c \in C} x_{cj} \cdot d(j,c)^p, \\
= \sum_{c \in C} \sum_{c' \in \varphi^{-1}(c)} x_{c'j} \cdot d(j,c)^p, \\
\leq \psi \sum_{c \in C} \sum_{c' \in \varphi^{-1}(c)} x_{c'j} \cdot (2d(j,c^*)^p + d(j,c_j)^p), \quad \text{(using Claim 6)} \\
= 2\psi \sum_{c' \in C} x_{c'j} \cdot d(j,c^*)^p + \psi d(j,c_j)^p \sum_{c \in C} \sum_{c' \in \varphi^{-1}(c)} x_{c'j}, \\
= 2\psi d^*(j) + \psi d(j,c_j)^p,
\]

where (1) the second last equality follows since \( \varphi \) is a function, and (2) the last equality follows from the definition of \( d^*(j) \) and uses the fact that \( \sum_{c' \in C} x_{c'j} = 1 \).

Taking a sum over all the points in \( V \), we get

\[
\sum_{j \in V} d(j) \leq 2\psi \sum_{j \in V} d^*(j) + \psi \sum_{j \in V} d(j,c_j)^p \\
\leq 3p-1 \left( 2 \sum_{j \in V} d^*(j) + \sum_{j \in V} d(j,c_j)^p \right) \\
\leq 3p-1 (\rho + 2)^p \sum_{j \in V} d^*(j)^p \\
= 3p-1 (\rho + 2) OPT_{k,p,f,(F)}(I)^p
\]

Taking the \( p \)-th root on both sides gives us the lemma.

\( \square \)

### A.2 Individually Fair \( k \)-Center

In this section, we revisit the individually fair \( k \)-center problem. As alluded in Remark 1, we need to be careful when dealing with \( p = \infty \). As such, the same theorem still holds, but the algorithmic details are slightly different. We first define the problem in the following way.

**Definition 10** Assume we are given a function \( f \) as in Definition 1 and a fair similarity measure \( F \). Then, INDIVIDUALLY FAIR \((f,F)\) \( k \)-CENTER asks for the minimum distance \( R \) along with (1) a set of cluster centers \( C \subseteq V \) of size at most \( k \) and (2) a distribution \( \mu_j \) over \( C \) for each point \( j \in V \), such that any center \( c \in C \) that lies in the support of \( \mu_j \) satisfies \( d(c,j) \leq R \). Further, the following individual fairness constraints need to be satisfied by the output distributions.

\[
D_f(\mu_j,||\mu_{j_2}) \leq F(j_1,j_2), \forall j_1,j_2 \in V
\]

**Algorithmic Details.** The algorithm follows exactly the same template as described for INDIVIDUALLY FAIR \((k,p,F)\)-CLUSTERING in Section 3. We first use a standard 2-approximation algorithm for VANILLA \( k \)-CENTER to determine the set \( C \). Next we define the constrained problem FAIR-ASSIGN-\( k \)-KC which is analogous to INDIVIDUALLY FAIR \( p \)-ASSIGNMENT in Section 3. As is standard for \( k \)-center problems, suppose we make the correct ‘guess’ for the optimal radius for the INDIVIDUALLY FAIR \((f,F)\) \( k \)-CENTER problem - call it \( R^* \). For any client \( j \), define \( B_j \) to be the ball with center at \( j \) and radius \( 4R^* \). We define the following feasibility mathematical program. A
variable $x_{cj}$ is defined if and only if $c \in B_j$, for all $c \in \mathcal{C}, j \in V$.

\[
\text{FAIR-ASSGN-KC} : \sum_{c \in \mathcal{C} \cap B_j} x_{cj} = 1 \quad \forall j \in V \tag{10}
\]

\[
D_f(\bar{x}_{j1}, |\bar{x}_{j2}|) \leq F(j_1, j_2) \quad \forall j_1, j_2 \in V \tag{11}
\]

\[
0 \leq x_{cj} \leq 1 \quad \forall j \in V, c \in B_j \cap \mathcal{C} \tag{12}
\]

We return any feasible solution $x$ to the above constrained program as our final solution. In the remainder of the section, we prove that such a solution exists. Let $x^*$ be an optimal solution to INDIVIDUALLY FAIR $(f, \mathcal{F})$ $k$-CENTER with radius $R^*$. We again define the mapping $\varphi$ from the centers in the support of $x^*$ to those in $\mathcal{C}$ and a potential solution $x$ to the above LP, exactly in the same way as done in Section 3 and subsequently used in Claim 6. We define $\text{supp}(x^*, j)$ as the set of open centers in the support of $x^*_j$ for any solution $x'$ to INDIVIDUALLY FAIR $(f, \mathcal{F})$ $k$-CENTER

**Claim 11** For any point $j \in V$, consider any center $c \in \text{supp}(x, j)$. Let $c_j$ be the closest center to $j \in \mathcal{C}$. Then for each $c^* \in \varphi^{-1}(c) \cap \text{supp}(x^*, j)$, we have

\[d(j, c) \leq 2d(j, c^*) + d(j, c_j)\]

The proof is immediate from the first part of the proof for Claim 6 and we skip that to avoid repetition. This claim will now give the following lemma.

**Lemma 12** $x$ is a feasible solution to FAIR-ASSGN-KC.

**Proof:** The proof that $x$ satisfies the individual fairness constraints (11) is exactly the same as done in the proof of Lemma 7.

However, we also need to prove that $x$ satisfies the constraints (10). Consider any point $j \in V$. Let $c_j$ be the closest center to $j$ in $\mathcal{C}$. Recall that $x^*$ is an optimal solution to INDIVIDUALLY FAIR $(f, \mathcal{F})$ $k$-CENTER. Clearly $d(c^*, j) \leq R^*$ for any $c^* \in \text{supp}(x^*, j)$. Also, by definition of the mapping $\varphi$, $\sum_{c \in \text{supp}(x, j)} x_{cj} = \sum_{c^* \in \text{supp}(x^*, j) \cap \varphi^{-1}(c)} x^*_{c, j} = 1$, by feasibility of $x^*$. Now consider any $c \in \text{supp}(x, j)$. By Claim 11, $d(c, j) \leq 2d(j, c^*) + d(j, c_j)$ for any $c^* \in \text{supp}(x^*, j) \cap \varphi^{-1}(c)$. We use the following three facts — (1) $\mathcal{C}$ is a set of centers for a 2-approximate solution to VANILLA $k$-CENTER, (2) an optimal solution to INDIVIDUALLY FAIR $(f, \mathcal{F})$ $k$-CENTER is a feasible solution to VANILLA $k$-CENTER, and (3) $x^*$ is an optimal solution to INDIVIDUALLY FAIR $(f, \mathcal{F})$ $k$-CENTER with radius $R^*$. This gives us $d(c, j) \leq 4R^*$ and we are done.

Combining all of the above, we have the following theorem.

**Theorem 13** There exists a 4-approximation algorithm for INDIVIDUALLY FAIR $(f, \mathcal{F})$ $k$-CENTER.

**Hardness of Individually Fair $k$-Center.** The NP-hardness of INDIVIDUALLY FAIR $(f, \mathcal{F})$ $k$-CENTER follows almost immediately from the hardness of VANILLA $k$-CENTER. Suppose $D_{TV}$ is the choice for $f$-divergence and the fairness similarity measure is $\mathcal{F} = d$, the underlying metric. It is a standard fact that the hard instances of VANILLA $k$-CENTER arise from a metric defined by $d(v, v') = 1$ or $d(v, v') = \infty$ (here $\infty$ is a very large number). Now suppose $\mathcal{I}$ is such an instance of VANILLA $k$-CENTER. Then we have the following lemma.

**Lemma 14** The instance $\mathcal{I}$ has a solution with $k$ centers and radius 1 if and only if the corresponding INDIVIDUALLY FAIR $(f, \mathcal{F})$ $k$-CENTER instance has a solution with radius 1.

**Proof:** Suppose $\mathcal{I}$ is a ‘yes’ instances to VANILLA $k$-CENTER with radius 1 and suppose $\psi(j)$ be the center to which $j$ has been assigned in such a solution. Now consider the solution to INDIVIDUALLY FAIR $(f, \mathcal{F})$ $k$-CENTER where for $j$, we return the distribution with $\psi(j)$ as the only center in its support. Since $D_{TV}$ can take a value of at most 1 and all distances are either 1 or $\infty$, this solution is individually fair.

Conversely, if there exists a solution to INDIVIDUALLY FAIR $(f, \mathcal{F})$ $k$-CENTER with radius 1, then trivially, there exists a solution to VANILLA $k$-CENTER with radius 1. \(\square\)
A.3 On the Price of Achieving Individual Fairness

In this section, we discuss the price associated with achieving individual fairness. More specifically, we call the ratio of the optimal cost of an individually fair clustering instance, to that of the clustering instance without the fairness constraints, as the price of achieving fairness. We give a simple example to show that, perhaps unsurprisingly, the price of achieving fairness can be arbitrarily large depending on the underlying fairness measure $\mathcal{F}$.

Recall that, for an instance $I$ of VANILLA $(k, p)$-CLUSTERING and INDIVIDUALLY FAIR $(k, p, f, \mathcal{F})$-CLUSTERING, we denote the corresponding optimal costs as $\text{OPT}_{k,p}(I)$ and $\text{OPT}_{k,p,f,\mathcal{F}}(I)$, respectively. We show that the ratio of $\text{OPT}_{k,p,f,\mathcal{F}}(I)$ to that of $\text{OPT}_{k,p}(I)$ can be arbitrarily large, depending on the fairness measure $\mathcal{F}$. In Figure 6, the input instance $I$ consists of data points on a line and assume $d(u_2, v_1) \geq R >> r \geq d(u_1, u_2) \approx d(v_1, v_2)$. Further, assume $k = 2$ and $p = 1$ (k-median). Then, $\text{OPT}_{k,p}(I) = O(r)$. Now let $\mathcal{F}(u, v) = \varepsilon$ for some small positive constant $\varepsilon$, and the measure of the individual fairness is the total variation norm $D_{TV}$. Then, in any solution to the INDIVIDUALLY FAIR $(k, p, f, \mathcal{F})$-CLUSTERING instance, $D_{TV}(\mu_u||\mu_v) \leq \varepsilon$. This implies, $\text{OPT}_{k,p,f,\mathcal{F}}(I) = O(R) >> \text{OPT}_{k,p}(I)$. A similar argument is true for the case of $k$-means ($p = 2$) and $k$-center ($p = \infty$) as well.

![Figure 6: The price of achieving individual fairness.](image)

Figure 6: The price of achieving individual fairness. Let $k = 2$ and $p = 1$. Assume $d(u_1, u_2) \approx d(v_1, v_2) \leq r$, $d(u_2, v_1) \geq R$, and $R >> r$. Then, $\text{OPT}_{k,p} = O(r)$. Now assume $\mathcal{F}(u, v) = \varepsilon$. Then, $\text{OPT}_{k,p,f,\mathcal{F}} = O(R) >> \text{OPT}_{k,p}$.

In this toy example, we have shown that the choice $\mathcal{F}$ plays an important role in determining the price of achieving fairness. In our experiments (Section 5), we demonstrate a similar effect in real-world scenarios. We consider two different fairness similarity measures. The first one, $\mathcal{F}_1$, is simply the underlying metric feature space $d$. The second one, $\mathcal{F}_2$, is an asymmetric notion where $\mathcal{F}(i, j)$ is decided based on the “small” local neighborhood information of $i$ in the feature space (see Section 5 for exact details). In Figure 4, we compare the cost of $\text{OPT}_{k,p,f,\mathcal{F}}$ vs HKM (which approximates $\text{OPT}_{k,p}$). We observe that for $\mathcal{F}_1$, the price of achieving fairness is quite large. In comparison, for $\mathcal{F}_2$, we can achieve fairness by almost paying the same cost as that of vanilla solutions.

We emphasize that this discussion is not be confused with the theoretical guarantees of our algorithm (Algorithm 1) — there we bound the cost of our solution with respect to $\text{OPT}_{k,p,f,\mathcal{F}}(I)$. The current discussion, on the other hand, studies the value of $\text{OPT}_{k,p,f,\mathcal{F}}(I)$ itself and highlights the impact of the fairness measure in determining the price of achieving fairness.

B Missing Details from Section 4

In this section, we revisit the COMBINED FAIR $(k, p, f, \mathcal{F})$-CLUSTERING problem and spell out the missing details from Section 4. For the sake of completeness, we first restate the problem definition.

**Definition 15 (COMBINED FAIR $(k, p, f, \mathcal{F})$-CLUSTERING)** Assume we are given an instance of the INDIVIDUALLY FAIR $(k, p, f, \mathcal{F})$-CLUSTERING problem. Additionally, we are given $\ell$-many (possibly overlapping) protected groups $G_1, G_2, \ldots, G_\ell$ and for each such group, we are given two input group fairness parameters $\alpha_i$ and $\beta_i$. The goal is to output (1) a set of cluster centers $\mathcal{C} \subseteq V$ of size at most $k$ and (2) a distribution $\mu_j$ over $\mathcal{C}$ for each point $j \in V$, such that

1. For each cluster, the expected fraction of the points from group $G_i$ lies between $\beta_i$ and $\alpha_i$.
2. $D_{\mathcal{F}}(\mu_{j_1}||\mu_{j_2}) \leq \mathcal{F}(j_1, j_2)$ for each pair of points $j_1, j_2 \in G_p$, for all $p \in [\ell]$.

The objective is to minimize $\mathcal{L}_p(\mu, \mathcal{C}) := \left( \sum_{j \in V} \mathbb{E}_{\mathcal{C} \sim \mu_j}(d(j, c)^p) \right)^{\frac{1}{p}}$.

**Why is Individual Fairness Enforced only Inside Protected Groups?**

In Fig 7, suppose the entire population of size $N$ is partitioned into two protected groups $A$ and $B$ according to some protected attribute. Let $P_A$ and $P_B$ be two sets of closely packed points separated
We now prove the “else if” direction. Let \( A \) as before, due to statistical fairness conditions, the distributions of each cluster to be formed with points from \( B \) and 90% from \( A \). If we impose individual fairness only inside \( B \) and \( A \), then a reasonable solution would be to assign all points in \( A_0 \) to centroid of \( P_0 \) and \( A_1 \) to that of \( P_1 \), each with probability 1. Further, we can assign each point in \( B \) to each of the centers with probability 0.5.

On the other hand, imposing individual fairness across every pair of points requires that points in \( B \) and \( A \) have roughly the same distribution, since the radius of \( P_0 \) is very small compared to \( r \). As before, due to statistical fairness conditions, the distributions of \( A_0 \) and \( A_1 \) also needs to be approximately the same. This will result in a trivial solution where each point is assigned to each centroid with roughly probability 0.5. The discussion above closely follows a similar discussion in the paper by Dwork et al. [21] where they term the notion of combined fairness as fair affirmative action.

On the Feasibility of a Combined Fair Clustering Instance. Before describing an algorithm for computing an approximate solution, we address the question of finding a feasible solution to the COMBINED FAIR \((k, p, f, \mathcal{F})\)-CLUSTERING problem instance. Note that in the absence of the group fairness constraints, it is trivial to construct a feasible solution to the INDIVIDUALLY FAIR \((k, p, f, \mathcal{F})\)-CLUSTERING problem. Indeed, we can simply assign to each point \( j \) in \( V \) a uniform distribution \( \mu_j \) over any arbitrary set of clusters centers \( C (|C| \leq k) \). By definition of \( \mathcal{F} \)-divergence, \( D_{\mathcal{F}}(\mu_{j_1} || \mu_{j_2}) = 0 \) for all pair of points \( j_1, j_2 \in V \). Since, the fair similarity measure \( \mathcal{F} \) is non-negative, this satisfies the individual fairness constraints (Equation (1)). Can we verify the feasibility of a COMBINED FAIR \((k, p, f, \mathcal{F})\)-CLUSTERING instance efficiently? We answer this question in affirmative. In fact, we give a simple condition in the following claim for the existence of a feasible solution. We remark that such a claim holds true for the group fairness problem considered in [8, 9] as well.

**Claim 16** Given an instance \( I = (V, k, f, \mathcal{F}, \alpha, \beta) \) to COMBINED FAIR \((k, p, f, \mathcal{F})\)-CLUSTERING, there exists a feasible solution to \( I \), iff the following condition is true:

\[
\beta_r |V| \leq |G_r| \leq \alpha_r |V| \quad \forall r \in [\ell] \tag{13}
\]

**Proof:** We first prove the “if” direction. For each point \( j \in V \), let \( \mu_j \) be a uniform distribution over an arbitrary set of \( k \) centers \( C \subseteq V \). Then, \( \mu_j \) is a feasible solution to \( I \). We have already argued above that such uniform distributions trivially satisfy individual fairness constraints between each pair of points in \( V \), and hence for each pair of points inside each protected group as well. Now, fix a cluster \( c \in C \) and a protected group \( G_r \), for \( r \in [\ell] \). The expected number of points assigned to the cluster \( c \) from the group \( G_r \) is \( |G_r|/|C| \). The expected size of the cluster with cluster center \( c \) is \(|V|/|C|\). Then, the condition in Equation (13) immediately implies group fairness.

We now prove the “else if” direction. Let \((C, \{\mu_j\}_{j \in V})\) be some feasible solution to the instance. Let \( \mu_j[c] \) denote the probability of assigning \( j \) to the cluster center \( c \). We then use the sub-additive
We now prove Theorem 8 that captures our main result on the Combined Fair Clustering Problem. We now discuss our algorithm for solving the problem. This completes the proof of the claim.

Property of the group fairness constraint to argue that Equation (13) must hold. More formally, group fairness implies for each center \( c \in C \) and for each \( r \in [\ell] \), we have

\[
\beta_r \sum_{j \in V} \mu_j[c] \leq \sum_{j \in G_r} \mu_j[c] \leq \alpha_r \sum_{j \in V} \mu_j[c].
\]

Summing over all \( c \in C \) and rearranging, we get

\[
\beta_r \sum_{c \in C} \sum_{j \in V} \mu_j[c] \leq \sum_{c \in C} \sum_{j \in G_r} \mu_j[c] \leq \alpha_r \sum_{c \in C} \sum_{j \in V} \mu_j[c],
\]

\[
\Rightarrow \beta_r \sum_{j \in V} \sum_{c \in C} \mu_j[c] \leq \sum_{j \in G_r} \sum_{c \in C} \mu_j[c] \leq \alpha_r \sum_{j \in V} \sum_{c \in C} \mu_j[c],
\]

\[
\Rightarrow \beta_r |V| \leq |G_r| \leq \alpha_r |V|.
\]

This completes the proof of the claim.

**Algorithm for the Combined Fair Clustering Problem.** We now discuss our algorithm for solving the Combined Fair \((k, p, f, F)\)-CLUSTERING problem. Recall that, our algorithmic strategy is to first solve the Vanilla \((k, p)\)-CLUSTERING problem on the input instance to find the cluster centers and then use these cluster centers to solve a fair assignment problem. For completeness we present it formally in Algorithm 2. We describe the fair assignment problem as an optimization problem below and denote it as the COMBINED-FAIR-ASSGN problem.

\[
\min \sum_{j \in V} \sum_{c \in C} x_{cj} d(i, j)^p
\]

s.t.
- \( \sum_{c \in C} x_{cj} = 1 \ \forall j \in V \)
- \( D_f(x_{j_1}^r, x_{j_2}^r) \leq F(j_1, j_2) \ \forall r \in [\ell], j_1, j_2 \in G_r \)
- \( \beta_r \sum_{j \in V} x_{cj} \leq \sum_{j \in G_r} x_{cj} \ \forall c \in C, r \in [\ell] \)
- \( 0 \leq x_{ij} \leq 1 \)

The second constraint enforces individual fairness between points in the same protected group and the third constraint ensures group fairness on the solution.

We now prove Theorem 8 that captures our main result on the Combined Fair \((k, p, f, F)\)-CLUSTERING problem. For completeness, we restate the theorem here.

**Theorem 8** Given an instance \( I \) to Combined Fair \((k, p, f, F)\)-CLUSTERING, let \( C \) be a \( \rho \)-approximate solution for the corresponding Vanilla \((k, p)\)-CLUSTERING on \( I \). Then, there exists an algorithm which produces feasible distributions \( \mu_j, \forall j \in V \), such that \( L_p(\mu, C) \leq 3^{1-\frac{\rho}{p}}(\rho + 2) \cdot \text{OPT}_{k,p,f,F}(I) \).

**Proof:** The proof of this theorem follows along the line of the proof of Theorem 5.

**Algorithm 2** Algorithm for Combined Fair \((k, p, f, F)\)-CLUSTERING

1: ALG-CF(I)
2: Use a \( \rho \)-approximation algorithm for Vanilla \((k, p)\)-CLUSTERING on \( I \) — let \( C \) be the set of centers.
3: Solve the COMBINED-FAIR-ASSGN problem on instance \( J = (V, C, f, \alpha, \beta) \) — let \( \mu \) be the solution.
4: return \( (C, \mu) \)

Assume \((C^*, x^*)\) be an optimal solution to the instance \( I \) of Combined Fair \((k, p, f, F)\)-CLUSTERING and ALG-CF(I) returns \((C, \mu)\). We construct a feasible solution \( x \) to the COMBINED-FAIR-ASSGN \((J = (V, C, f, F, \alpha, \beta))\) using \( C^* \) and \( x^* \). ALG-CF outputs the optimal solution.
to COMBINED-FAIR-ASSGN, hence \( L_p(\mu, C) \leq L_p(x, C) \). So, as in the proof of Theorem 5, it is sufficient to bound \( L_p(x, C) \) to prove the approximation ratio of ALG-CF.

Recall the definition of the nearest function \( \varphi \) and its inverse: \( \varphi(c^*) = \arg\min_{c \in \mathcal{C}} d(c, c^*) \) for each \( c^* \in \mathcal{C}^* \), and \( \varphi^{-1}(c) = \{ c^* \in \mathcal{C}^* : \varphi(c^*) = c \} \) for each \( c \in \mathcal{C} \). For each \( j \in \mathcal{V} \) and each \( c \in \mathcal{C} \), set \( x_{cj} = \sum_{c^* \in \varphi^{-1}(c)} x^*_{c^*j} \). In words, for a fixed point \( j \in \mathcal{V} \) and a fixed center \( c \in \mathcal{C} \), we look at the centers in the optimal solution that are mapped to \( c \) by \( \varphi \), and sum the corresponding probabilities to get \( x_{cj} \). In the remaining, we prove that \( x \) is a feasible solution to COMBINED-FAIR-ASSGN \((\mathcal{J})\) and bound its cost.

**Lemma 17** \( x \) is feasible to COMBINED-FAIR-ASSGN \((\mathcal{J})\).

**Proof:** It follows from the proof of first part of Lemma 7 that \( x \) satisfies all the constraints in the COMBINED-FAIR-ASSGN LP (eq. (14)) barring the group fairness constraints. The group fairness follows by the sub-additivity of the constraints. We show this formally below. For any center \( c \in \mathcal{C} \), if \( \varphi^{-1}(c) = \emptyset \), then the corresponding group fairness constraints are trivially satisfied. Now assume \( \varphi^{-1}(c) \neq \emptyset \). Fix a group \( \mathcal{G}_r \).

\[
\sum_{j \in \mathcal{G}_r} x_{cj} = \sum_{j \in \mathcal{G}_r} \sum_{c^* \in \varphi^{-1}(c)} x^*_{c^*j},
\]

\[
= \sum_{c^* \in \varphi^{-1}(c)} \sum_{j \in \mathcal{G}_r} x^*_{c^*j},
\]

\[
\leq \sum_{c^* \in \varphi^{-1}(c)} \alpha_r \sum_{j \in \mathcal{V}} x^*_{c^*j}, \quad \text{(by optimality of } x^*)
\]

\[
= \alpha_r \sum_{j \in \mathcal{V}} x_{cj}.
\]

Similarly, we can show that \( \sum_{j \in \mathcal{G}_r} x_{cj} \geq \beta_r \sum_{j \in \mathcal{V}} x_{cj} \), proving that \( x \) is a feasible solution to the COMBINED-FAIR-ASSGN \((\mathcal{J})\) LP. \( \square \)

We next bound the cost of the solution \( x \).

**Lemma 18** \( L_p(x, C) \leq 3(1 - \frac{1}{\beta}) (\rho + 2) \cdot L_p(x^*, C^*) \).

The proof of this lemma is identical to the proof of second part of Lemma 7. Together Lemmas 17 and 18 prove the claim in the theorem. \( \square \)

**Combined Fair k-Center Problem.** We remark here that the Combine Fair k-center problem needs to be treated slightly differently, as we discussed in Appendix A.2. The details are analogous, and we refrain from repeating them here.

### C Individual Fairness to Group Fairness when \( \mathcal{F} = d \)

In this section, we explore the connection between the notion of individual fairness and group fairness for the special case of \( \mathcal{F} = d \). In particular, we are interested in finding conditions under which individually fair clustering solutions guarantees group fairness. Such connections are well-known in the context of classification problems [21]. We show that similar connections exist in the clustering context.

Before we discuss the technical details, it is perhaps imperative to discuss the apparent tension between the two notions of fairness in the context of clustering. Individual fairness is modeled after the concept of “equality of treatment” whereas group fairness is modeled after “equality of outcome”. In clustering, to ensure the later, some point \( v \) might be assigned to a cluster center that is not the closest center to \( v \). However, such assignments might be unfair to \( v \) if its close neighbors are assigned to a center that was in fact closest to \( v \) as well. Indeed, we demonstrate this aspect in Figure 2. Nevertheless, if the “spread” of the points from each protected group are “similar” to each other in
We define a quantity \textit{maximum additive violation}, and denote it as \text{max-violation}, which captures the unfairness of the clusters for each protected group. This quantity helps us establish a connection between group fairness and individual fairness. Recall that, a solution to the INDIVIDUALLY FAIR ($k, p, f, F$)-CLUSTERING instance assigns to each point $x \in V$ a distribution $\mu_x$ over the set of cluster centers $C$. For each protected groups $G_1, G_2, \ldots, G_r$, let $p_r = |G_r|/|V|$ for each $r \in [\ell]$. Then, for the protected group $G_r$, we define \text{max-violation}_r as follows:

$$\text{max-violation}_r = \max_{i \in C} \left| \sum_{j \in G_r} \mu_x(i) - p_r \sum_{j \in V} \mu_x(i) \right|$$  \hspace{1cm} (15)

To elucidate further on this definition, note that in the entire population, $p_r$ fraction of the points belong to group $G_r$, and hence, we expect that each cluster will have the same proportional representation for group $G_r$. The notion of \text{max-violation}_r captures the additive deviation from this expected number. We remark here that our notion of \text{max-violation}_r is consistent with the most general group fairness constraints defined in prior works [8, 9]. Indeed, the $(\alpha, \beta)$-group fairness formulation [8] aims at providing a desired bound on the quantity \text{max-violation}_r. For each protected group $G_r$, we define a distribution $\nu_{G_r}$ as a uniform distribution over the set of points in $G_r$. In particular, $\nu_{G_r}(x) = 1/|G_r|$ if $x \in G_r$, and 0 otherwise. Let $\nu_V$ be the uniform distribution over the set of all points: $\nu_V(x) = 1/|V|$ for all $x \in V$. Let $d_{EM}(S, T)$ be the Earthmover’s distance between the distribution $S$ and $T$, introduced formally in [45]. Our main result of this section is the following lemma.

\textbf{Lemma 19} Let $(C, \mu)$ be any feasible solution to the INDIVIDUALLY FAIR ($k, p, f, F$)-CLUSTERING problem instance with $D_f$ as the statistical similarity measure. Further suppose the $f$-divergence $D_f(\mu_x||\mu_y) \geq D_{TV}(\mu_x, \mu_y)$ for all pair of points $x, y \in V$. Then, For each group $G_r$, \text{max-violation}_r \leq |G_r| \cdot d_{EM}(\nu_{G_r}, \nu_V)$

\textbf{Proof:} For a fixed group $G_r$, we first define the notion of bias of the group, which is essentially an upper bound on the quantity \text{max-violation}_r. For convenience, let us extend the notation of \text{max-violation}_r to \text{max-violation}_r(\mu, D_f) to include the underlying $f$-divergence function and the distributions $\mu_x$.

Let $\mu'_x$ be a bi-point distribution defined over the set of all points $x \in V$ that satisfy the individual fairness constraints. Then,

$$\text{bias}(r, D_f) = \max_{\mu'} \text{max-violation}_r(\mu', D_f)$$

Note that we have restricted the definition of $\text{bias}(r, D_f)$ with respect to all possible bi-point distributions. In the next claim, we justify this. We remark here that a similar observation is made by [21] in the context of classification problems.

\textbf{Claim 20} Suppose we have distributions $\mu_x$ defined over a set of $k$ points for all $x \in V$, where $k \geq 2$, where the distributions $\mu$ satisfy the individual fairness constraints (1). Then,

$$\max_{\mu} \text{max-violation}_r(\mu, D_f) \leq \text{bias}(r, D_f)$$

\textbf{Proof:} Suppose we are given the distributions $\mu_x, x \in V$ over a set of centers $C$. We define corresponding bi-point distributions $\mu'_x$ over a set $C' = \{c_1, c_2\}$ on two centers. Let $C_1 = \{i \in C : \mu_{G_r}(i) > p_r \mu_V(i)\}$ and $C_2 = C \setminus C_1$. Assign $\mu'_x(c_1) = \mu_x(c_1), \mu'_x(c_2) = \mu_x(c_2)$. First we claim that $\mu'_x, \forall x \in V$ satisfies the individual fairness constraints (1). The proof is exactly the same as that in Lemma 7.
Next we prove that \( \max_{\mu} \text{max-violation}_r(\mu, D_f) \leq \text{bias}(r, D_f) \). This follows using the definitions.

\[
\text{max-violation}_r(\mu, D_f) = \max_{i \in \mathbb{C}} |\mu_G_r(i) - p_r \cdot \mu_V(i)|
\]

\[
= \max \left\{ \max_{i' \in C_1} (\mu_G_r(i') - p_r \cdot \mu_V(i')), \max_{i' \in C_2} (\mu_G_r(i') - p_r \cdot \mu_V(i')) \right\}
\]

\[
\leq \max \left\{ \sum_{i' \in C_1} (\mu_G_r(i') - p_r \cdot \mu_V(i')), \sum_{i' \in C_2} (p_r \cdot \mu_V(i') - \mu_G_r(i')) \right\}
\]

\[
= \max_{i \in \{c_1, c_2\}} |\mu_G_r(i) - p_r \cdot \mu_V(i)|
\]

\[
\leq \text{bias}(r, D_f)
\]

\[\square\]

We first show that if the \( f \)-divergence function is indeed \( D_{TV} \), then the above lemma holds. The proof follows a framework similar to that in [21]. However, we need to make non-trivial modifications to handle our definition of \( \text{max-violation}_r \).

We show how to upper bound the quantity \( \mu'_G_r(c_1) - p_r \cdot \mu'_V(c_1) \). An analogous proof can be done for \( c_2 \). The high-level idea of the proof is as follows. We write a maximization linear program that finds the bi-point distributions \( \mu'_x \) for all \( x \in V \) that satisfy individual fairness with respect to \( D_{TV} \). The dual to a relaxation of this program will turn out to be the minimization linear program, whose solution gives exactly the the Earthmover’s distance between \( \nu_{G_r}, \nu_V \), up to a scaling factor of \( |G_r| \). The claim then follows from weak duality.

**LP-BIAS** : \( \max \sum_{x \in V} \nu_{G_r}(x) \cdot \mu'_x(c_1) \)

\[
- |G_r| \cdot \sum_{x \in V} \nu_V(x) \cdot \mu'_x(c_1)
\]

s.t. \( \mu'_x(c_1) + \mu'_x(c_2) = 1 \)

\( \mu'_x(c_1) - \mu'_y(c_1) \leq d(x, y), \forall x, y \in V \)

\( \mu'_x(c_1) \geq 0, \forall x \in V \)

Here \( \mu'_x(c_1), \mu'_x(c_2), \forall x \in V \) are the variables. The first constraint ensures that they form a distribution while the second one enforces the individual fairness constraints with respect to \( D_{TV} \). Here we are using the fact that \( D_{TV}(\mu'_x, \mu'_y) \leq d(x, y) \) is equivalent to \( |\mu'_x(c_1) - \mu'_y(c_1)| \leq d(x, y) \) since the distribution is bi-point. Note that we have to write this constraint for every ordered pair \( x, y \in V \).

We relax the above LP by removing the first set of constraints and take the dual.

**LP-BIAS-DUAL** : \( \min \sum_{x, y \in V} \lambda(x, y)d(x, y) \)

\[
\sum_{y \in V} \lambda(x, y) \geq \sum_{y \in V} \lambda(y, x)
\]

\[
+ \nu_{G_r}(x) - |G_r|\nu_V(x), \forall x \in V
\]

\( \lambda(x, y) \geq 0, \forall x, y \in V \)

Finally, recall that the Earthmover’s distance between the distributions \( \nu_{G_r}, \nu_V \) is given by the following LP.

**LP-EM** : \( \min \sum_{x, y \in V} \lambda(x, y)d(x, y) \)

s.t. \( \sum_{y \in V} \lambda(x, y) = \nu_{G_r}(x), \forall x \in V \)

\[
\sum_{y \in V} \lambda(y, x) = \nu_V(x) \forall x \in V
\]

\( \lambda(x, y) \geq 0, \forall x, y \in V \)
Now, for any feasible solution $\lambda^*$ to LP-EM, we can create a feasible solution to LP-BIAS-DUAL as follows. For $\lambda^*(x, y)$ appearing in the first set of constraints, we define the corresponding $\hat{\lambda}(x, y)$ for LP-BIAS-DUAL to be the same. For $\lambda^*(x, y)$ appearing in the second set of constraints, we set $\hat{\lambda}(y, x) = |G_r|\lambda^*(y, x)$. It is straightforward to observe that $\hat{\lambda}$ is a feasible solution to LP-BIAS-DUAL. Putting everything together and using weak duality, we can conclude that the optimal solution to LP-BIAS is upper bounded by $|G_r|$ times the Earthmover’s LP optimal, and we are done.

Finally, if $D_f(\mu_x || \mu_y) \geq D_{TV}(\mu_x, \mu_y)$, then any set of distributions which satisfies individual fairness with respect to $D_f$ will also form a feasible solution to LP-BIAS. Hence, we have the lemma. \qed

\section{Additional Experiments}

In this section, we present additional experiments and the plots mentioned in Section 5, for all the datasets. We also show the practical running time of ALG-IF.

Table 2: Running time of ALG-IF for $k = 4$, $m = 250$, enforcing $F_1$, on creditcard dataset for different sample sizes

| Number of sampled points | 500  | 1000 | 2000 | 3000 | 4000 |
|--------------------------|------|------|------|------|------|
| Time (in seconds)        | 80   | 436  | 2901 | 10113 | 32896 |

**Running time.** In this paper, we provide a generic framework and do not emphasize on running time optimization. Table 2 shows the running time of ALG-IF on creditcard dataset for $k = 4$ and $m = 250$, enforcing fairness similarity $F_1$. Although we solve a linear program with around 10,000,000 constraints and variables, we observe that CPLEX solves it in around 9 hours.

Table 3: Percentage of individual fairness constraint violations of SKM when SKM and ALG-IF incur the same clustering cost.

| Clusters ($k$)       | Bank | Adult | Creditcard | Census1990 | Diabetes |
|----------------------|------|-------|------------|------------|----------|
|                      | 4    | 6     | 8          | 10         |
| Fairness similarity $F_1$ |
| Bank                 | 95   | 98    | 99         | 99         |
| Adult                | 88   | 94    | 98         | 99         |
| Creditcard           | 61   | 76    | 83         | 85         |
| Census1990           | 25   | 34    | 44         | 50         |
| Diabetes             | 53   | 68    | 63         | 82         |
| Clusters ($k$)       | Bank | Adult | Creditcard | Census1990 | Diabetes |
|                      | 4    | 6     | 8          | 10         |
| Fairness similarity $F_2$ |
| Bank                 | 4    | 4     | 4          | 4          |
| Adult                | 4    | 5     | 7          | 8          |
| Creditcard           | 6    | 5     | 6          | 6          |
| Census1990           | 7    | 11    | 13         | 11         |
| Diabetes             | 4    | 4     | 7          | 7          |

**Unfairness of SKM.** The output of SKM depends on stiffness parameter $\beta$ introduced in [7]. More specifically, when $\beta = 0$, we get a uniform distribution over the centers, which guarantees individual fairness at a very high cost. On the other hand, when $\beta \to \infty$, we get a low cost HKM solution, which is unfair to individuals. In Figure 8, we show the variation of clustering cost and percentage of individual fairness constraints violated ($F_1$ and $F_2$) by SKM for different values of $\beta$. In Table 3 (extension of Table 1), we find $\beta$ at which SKM and ALG-IF incur the same clustering cost and observe the percentage of individual fairness constraints violated ($F_1$ and $F_2$). Note that $F_2$ is a much relaxed fairness measure compared to $F_1$: for each point, similarity is measured locally, with respect to its $|V|/k$ nearest neighbors. Even with such relaxations, SKM exhibits unfair treatment of similar points. Our solution does not violate any individual fairness constraints.

**Cost Analysis.** In this section, we present the plots for all the datasets, comparing the cost of ALG-IF and ALG-CF against OPT-IF and OPT-CF, respectively, as shown in Figure 9 and Figure 10 (extension of Figure 4 and Figure 5).

**Individual Fairness to Group Fairness under $F_1$.** In this experiment, we use $F_1$ as fairness similarity measure and $D_{TV}$ as statistical distance measure. Let $r = \arg\max_{G_r} \frac{\text{max-violation}_{G_r}}{|G_r|}$, we plot statistical bias defined by $\frac{\text{max-violation}_{G_r}}{|G_r|}$ and the corresponding earth-mover distance $d_{EM}(\mu_{G_r}, \nu_V)$. 

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Figure 8: Percentage of individual fairness constraint violations and relative clustering cost of SKM vs stiffness parameter $\beta$ for $k = 8$. The vertical black line shows the $\beta$ at which SKM and ALG-IF incur the same clustering cost.

as shown in Figure 11. As Lemma 19 suggests, we observe that $\max_{r \in [0, 1]} \max_{|G_r|} \max_{n_r} \nu_r \leq d_{EM}(\nu_G, \nu_V)$. Moreover, the gap between statistical bias and earth-mover distance is tight in practice.
Figure 9: Clustering cost vs number of clusters for ALG-IF, OPT-IF and HKM.
Figure 10: Clustering cost vs number of clusters for ALG-CF, OPT-CF and HKM.

Figure 11: Variation of statistical bias and earth-mover distance vs number of clusters ($k$). We observe that statistical bias is upper bounded by earth-mover distance as suggested by Lemma 19 and the gap is tight in practice.