EQUIVARIANT EXTENSIONS OF $G_a$-TORSORS OVER PUNCTURED SURFACES

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Abstract. Motivated by the study of the structure of algebraic actions the additive group on affine threefolds $X$, we consider a special class of such varieties whose algebraic quotient morphisms $X \to X//G_a$ restrict to principal homogeneous bundles over the complement of a smooth point of the quotient. We establish basic general properties of these varieties and construct families of examples illustrating their rich geometry. In particular, we give a complete classification of a natural subclass consisting of threefolds $X$ endowed with proper $G_a$-actions, whose algebraic quotient morphisms $\pi : X \to X//G_a$ are surjective with only isolated degenerate fibers, all isomorphic to the affine plane $\mathbb{A}^2$ when equipped with their reduced structures.

Introduction

Algebraic actions of the complex additive group $\mathbb{G}_a = \mathbb{G}_{a, \mathbb{C}}$ on normal complex affine surfaces $S$ are essentially fully understood: the ring of invariants $\mathcal{O}(S)^{\mathbb{G}_a, \mathbb{C}}$ is a finitely generated algebra whose spectrum is a smooth affine curve $C = S//\mathbb{G}_a$, and the inclusion $\mathcal{O}(S)^{\mathbb{G}_a} \subset \mathcal{O}(S)$ defines a surjective morphism $\pi : S \to C$ whose general fibers coincide with general orbits of the action, hence are isomorphic to the affine line $\mathbb{A}^1$ on which $\mathbb{G}_a$ acts by translations. The degenerate fibers of such $\mathbb{A}^1$-fibrations are known to consist of finite disjoint unions of smooth affine curves isomorphic to $\mathbb{A}^2$ when equipped with their reduced structure. A complete description of isomorphism classes of germs of invariant open neighborhoods of irreducible components of such fibers was established by Fieseler [8].

In contrast, very little is known so far about the structure of $G_a$-actions on complex normal affine threefolds. For such a threefold $X$, the ring of invariants $\mathcal{O}(X)^{G_a}$ is again finitely generated [13] and the morphism $\pi : X \to S$ induced by the inclusion $\mathcal{O}(X)^{G_a} \subset \mathcal{O}(X)$ is an $\mathbb{A}^1$-fibration over a normal affine surface $S$. But in general, $\pi$ is neither surjective nor equidimensional. Furthermore, it can have degenerate fibers over closed subsets of pure codimension 1 as well as of codimension 2. All of these possible degeneration are illustrated by the following example:

The restriction of the projection $pr_{x,Y}$ to the smooth threefold $X = \{ x^2(x-1)v + yu^2 - x = 0 \}$ in $\mathbb{A}^4$ is an $\mathbb{A}^1$-fibration $\pi : X \to \mathbb{A}^2$ which coincides with the algebraic quotient morphism of the $G_a$-action on $X$ associated to the locally nilpotent derivation $\partial = x^2(x-1)\partial_x - 2yu\partial_y$ of its coordinate ring. The restriction of $\pi$ over the principal open subset $x^2(x-1) \neq 0$ of $\mathbb{A}^2$ is a trivial principal $G_a$-bundle, but the fibers of $\pi$ over the points $(1,0)$ and $(0,0)$ are respectively empty and isomorphic to $\mathbb{A}^2$. Furthermore, for every $y_0 \neq 0$, the inverse images under $\pi$ of the points $(0, y_0)$ and $(1, y_0)$ are respectively isomorphic to $\mathbb{A}^1$ but with multiplicity 2, and to the disjoint union of two reduced copies of $\mathbb{A}^1$.

Partial results concerning the structure of one-dimensional degenerate fibers of $G_a$-quotient $\mathbb{A}^1$-fibrations were obtained by Gurjar-Masuda-Miyanishi [9]. In the present article, as a step towards the understanding of the structure of two-dimensional degenerate fibers, we consider a particular type of non equidimensional surjective $G_a$-quotient $\mathbb{A}^1$-fibrations $\pi : X \to S$ which have the property that they restrict to $G_a$-torsors over the complement of a finite set of smooth points in $S$. These are simpler than the general case illustrated in the previous example since they do not admit additional degeneration of their fibers over curves in $S$ passing through the given points. The local and global study of some classes of such fibrations was initiated by the second author [10]. He constructed in particular many examples of $G_a$-quotient $\mathbb{A}^1$-fibrations on smooth affine threefolds $X$ with image $\mathbb{A}^2$ whose restrictions over the complement of the
origin are isomorphic to the geometric quotient \( \text{SL}_2 \to \text{SL}_2/\mathbb{G}_a \) of \( \text{SL}_2 \) by the action of unitary upper triangular matrices.

One of the simplest examples of this type is the smooth threefold \( X_0 \subset \mathbb{A}^5_{x,y,p,q,r} \) defined by the equations

\[
\begin{align*}
    xr - yq &= 0, \\
    yp - x(q - 1) &= 0, \\
    pr - q(q - 1) &= 0,
\end{align*}
\]

and equipped with the \( \mathbb{G}_a \)-action associated to the locally nilpotent \( \mathbb{C}[x,y] \)-derivation \( x^2\partial_p + xy\partial_q + y^2\partial_r \) of its coordinate ring. The equivariant open embedding \( \text{SL}_2 = \{ xv = yu = 1 \} \to X_0 \) is given by \( (x, y, u, v) \mapsto (x, y, xu, xv, yv) \). The \( \mathbb{G}_a \)-quotient morphism coincides with the surjective \( \mathbb{A}^1 \)-fibration \( \pi_0 : \text{pr}_{x,y} : X_0 \to \mathbb{A}^2 \). Its restriction over \( \mathbb{A}^2 \setminus \{ (0, 0) \} \) is isomorphic to the quotient morphism \( \text{SL}_2 \to \text{SL}_2/\mathbb{G}_a \), while its fiber over \( (0, 0) \) is the smooth quadric \( \{ pr - q(q - 1) = 0 \} \subset \mathbb{A}^3_{p,q,r} \), isomorphic to the quotient \( \text{SL}_2/\mathbb{G}_m \) of \( \text{SL}_2 \) by the action of its diagonal torus (see Example 2.1). A noteworthy property of this example is that the \( \mathbb{G}_a \)-quotient morphism \( \pi : X_0 \to \mathbb{A}^2 \) factors through a locally trivial \( \mathbb{A}^1 \)-bundle \( p : X_0 \to \mathbb{A}^2 \) over the blow-up \( \tau : \mathbb{A}^2 \to \mathbb{A}^2 \) of the origin.

It is a general fact that every irreducible component of a degenerate fiber of pure codimension one of a \( \mathbb{G}_a \)-quotient \( \mathbb{A}^1 \)-fibration \( \pi : X \to S \) on a smooth affine threefold is an \( \mathbb{A}^1 \)-uniruled affine surface (see Proposition 1.3). We do not know whether every \( \mathbb{A}^1 \)-uniruled surface can be realized as an irreducible component of the degenerate fiber of a \( \mathbb{G}_a \)-extension. But besides the smooth affine quadric \( \text{SL}_2/\mathbb{G}_m \) appearing in the previous example, the following one confirms that the affine plane \( \mathbb{A}^2 \) can also be realized (see also Examples 1.4 and 1.5 for other types of surfaces that can be realized): Let \( X_1 \subset \mathbb{A}^5_{x,y,z_1,z_2,w} \) be the smooth affine threefold defined by the equations

\[
\begin{align*}
    xw - y(z_1 + 1) &= 0, \\
    xz_2 - z_1(yz_1 + 1) &= 0, \\
    z_1w - yz_2 &= 0,
\end{align*}
\]

equipped with the \( \mathbb{G}_a \)-action associated to the locally nilpotent \( \mathbb{C}[x,y] \)-derivation \( x\partial_{z_1} + (2yz_1 + 1)\partial_{z_2} + y^2\partial_w \) of its coordinate ring. The morphism \( \text{SL}_2 \to X_1 \) given by \( (x, y, u, v) \mapsto (x, y, u, w, yv) \) is equivariant open embedding. The \( \mathbb{G}_a \)-quotient morphism coincides with the surjective \( \mathbb{A}^1 \)-fibration \( \pi_1 : \text{pr}_{x,y} : X_1 \to \mathbb{A}^2 \), whose fiber over the origin is the affine plane \( \mathbb{A}^2 = \text{Spec}(\mathbb{C}[z_2, w]) \) and whose restriction over \( \mathbb{A}^2 \setminus \{ (0, 0) \} \) is again isomorphic to the quotient morphism \( \text{SL}_2 \to \text{SL}_2/\mathbb{G}_a \). A special additional feature is that the \( \mathbb{G}_a \)-action on \( X_1 \) extending that on \( \text{SL}_2 \) is not only fixed point free but actually proper: its geometric quotient \( X_1/\mathbb{G}_a \) is separated. One can indeed check that \( X_1/\mathbb{G}_a \) is isomorphic to the complement \( \mathbb{A}^2 \setminus \{ o_1 \} \) of a point \( o_1 \) supported on the exceptional divisor \( E \) of the blow-up \( \mathbb{A}^2 \) of \( \mathbb{A}^2 \) at the origin (see Example 3.2).

Relaxing the hypothesis that the \( \mathbb{A}^1 \)-fibration \( \pi : X \to S \) arises as the quotient of a \( \mathbb{G}_a \)-action on an affine threefold \( X \) to consider the broader problem of describing the geometry of degeneration of \( \mathbb{A}^1 \)-fibrations over irreducible closed subsets of pure codimension two of their base, we are led to the following more general notion:

**Definition.** Let \( (S, o) \) be a pair consisting of a normal separated 2-dimensional scheme \( S \) essentially of finite type over a field \( k \) of characteristic zero and of a closed point \( o \) contained in the smooth locus of \( S \). A \( \mathbb{G}_a \)-extension of a \( \mathbb{G}_a \)-torsor \( p : P \to S \setminus \{ o \} \) is a \( \mathbb{G}_a \)-equivariant open embedding \( j : P \hookrightarrow X \) into an integral scheme \( X \) equipped with a surjective morphism \( \pi : X \to S \) of finite type and a \( \mathbb{G}_a, S \)-action, such that the commutative diagram

\[
\begin{array}{ccc}
P & \xrightarrow{j} & X \\
\rho \downarrow & & \downarrow \pi \\
S \setminus \{ o \} & \longrightarrow & S
\end{array}
\]

is cartesian.

The examples \( X_0 \) and \( X_1 \) above provide motivation to study the following natural classes of \( \mathbb{G}_a \)-extensions \( \pi : X \to S \) of a \( \mathbb{G}_a \)-torsor \( p : P \to S \setminus \{ o \} \), which are arguably the simplest possible types of \( \mathbb{G}_a \)-extensions from the viewpoints of their global geometry and of the properties of their \( \mathbb{G}_a \)-actions:
- (Type I) Extensions for which $\pi$ factors through a locally trivial $\mathbb{A}^1$-bundle over the blow-up $\tau : \tilde{S} \to S$ of the point $o$, the fiber $\pi^{-1}(o)$ being then the total space of a locally trivial $\mathbb{A}^1$-bundle over the exceptional divisor of $\tau$.

- (Type II) Extensions for which $\pi^{-1}(o)_{\text{red}}$ is isomorphic to the affine plane $\mathbb{A}^2_\kappa$ over the residue field $\kappa$ of $S$ at $o$, $X$ is smooth along $\pi^{-1}(o)$ and the $\mathbb{G}_a,S$-action on $X$ is proper.

The first main result of this article, Proposition 2.3 and Theorem 2.5, is a complete description of $\mathbb{G}_a$-extensions of Type I together with an effective characterization of which among them have the additional property that the morphism $\pi : X \to S$ is affine. Our second main result, Theorem 3.7, consists of a classification of $\mathbb{G}_a$-extensions of Type II, under the additional assumption that the morphism $\pi : X \to S$ is quasi-projective. More precisely, given a $\mathbb{G}_a$-torsor $\rho : P \to S \setminus \{o\}$ and a $\mathbb{G}_a$-extension $\pi : X \to S$ with proper $\mathbb{G}_a,S$-action and reduced fiber $\pi^{-1}(o)_{\text{red}}$ isomorphic to $\mathbb{A}^2_\kappa$, we establish that the possible geometric quotients $S' = X/\mathbb{G}_a$ belong to a very special class of surfaces isomorphic to open subsets of blow-ups of $S$ with centers over $o$ which we fully describe in § 3.1. We show conversely that every such surface is indeed the geometric quotient of a $\mathbb{G}_a$-extension of $\rho : P \to S \setminus \{o\}$ with the desired properties.

In a second step, we tackle the question of existence of $\mathbb{G}_a$-extensions $\pi : X \to S$ of Type II for which the structure morphism $\pi$ is not only quasi-projective but affine. Our method to produce extensions with this property is inspired by the observation that the threefolds $X_0$ and $X_1$ above are not only birational to each other due to the property that they both contain $\text{SL}_2$ as open subset, but in fact that the birational morphism

$$\eta : X_1 \to X_0, \quad (x, y, z_1, z_2, w) \mapsto (x, x, y, p, q, r) = (x, y, xz_1, yz_1 + 1, w)$$

expresses $X_1$ as a $\mathbb{G}_a$-equivariant affine modification of $X_0$ in the sense of Kaliman and Zaidenberg [11]. This suggests that extensions of Type II for which $X$ is affine over $S$ could be obtained as equivariant affine modification in a suitable generalized sense from extensions of Type I with the same property. Using this technique, we are able to show in Theorem 3.8 that for each possible geometric quotient $S'$ above, there exist $\mathbb{G}_a$-extensions $\pi : X \to S$ of $\rho : P \to S \setminus \{o\}$ with geometric quotient $X/\mathbb{G}_a = S'$ such that $\pi$ is an affine morphism.

As an application towards the initial question of the structure $\mathbb{G}_a$-quotient $\mathbb{A}^1$-fibrations on affine threefolds, we in particular derive from this construction the existence of uncountably many pairwise non isomorphic smooth affine threefolds $X$ endowed with proper $\mathbb{G}_a$-actions, containing $\text{SL}_2$ as an invariant open subset with complement $\mathbb{A}^2$, whose geometric quotients are smooth quasi-projective surfaces which are not quasi-affine, and whose algebraic quotients are all isomorphic to $\mathbb{A}^2$.

The scheme of the article is the following. The first section begins with a review of general properties of $\mathbb{G}_a$-extensions. We then set up the basic tools which will be used through all the article: locally trivial $\mathbb{A}^1$-bundles with additive group actions and equivariant affine birational morphisms between these. In section two, we study $\mathbb{G}_a$-extensions of Type I. The last section is devoted to the classification of quasi-projective $\mathbb{G}_a$-extensions of Type II.

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1. Preliminaries

**Notation 1.1.** In the rest of the article, the term surface refers to a normal separated 2-dimensional scheme essentially of finite type over a field \( k \) of characteristic zero. A punctured surface \( S_o = S \setminus \{ o \} \) is the complement of a closed point \( o \) contained in the smooth locus of a surface \( S \). We denote by \( \kappa \) the residue field of \( S \) at \( o \).

**Remark 1.2.** We do not require that the residue field \( \kappa \) of \( S \) at \( o \) is an algebraic extension of \( k \). For instance, \( S \) can very well be the spectrum of the local ring \( \mathcal{O}_{X, Z} \) of an arbitrary smooth \( k \)-variety \( X \) at an irreducible closed subvariety \( Z \) of codimension two in \( X \) and \( o \) its unique closed point, in which case the residue field \( \kappa \) is isomorphic to the field of rational functions on \( Z \).

In this section, we first review basic geometric properties of equivariant extensions of \( \mathbb{G}_a \)-torsors over punctured surfaces. We then collect various technical results on additive group actions on affine-linear bundles of rank one and their behavior under equivariant affine modifications.

**1.1. Equivariant extensions of \( \mathbb{G}_a \)-torsors.** A \( \mathbb{G}_a \)-torsor over punctured surface \( S_o = S \setminus \{ o \} \) is an \( S_o \)-scheme \( \rho : P \to S_o \) equipped with an \( \mathbb{G}_a \)-action \( \mu : \mathbb{G}_a \times_{S_o} P \to P \) for which there exists a Zariski open cover \( f : Y \to S_o \) of \( S_o \) such that \( P \times_{S_o} Y \) is equivariantly isomorphic to \( \mathbb{G}_a \times_P Y \) acting on itself by translations. In the present article, we primarily focus on \( \mathbb{G}_a \)-torsors \( \rho : P \to S_o \) whose restrictions \( P \times_{S_o} U \to U \setminus \{ o \} \) over every Zariski open neighborhood \( U \) of \( o \) in \( S \) are nontrivial. Since in this case the total space of \( P \) is affine over \( S \) (see e.g. [4, Proposition 1.2] whose proof carries over verbatim to our more general situation), it follows that for every \( \mathbb{G}_a \)-extension \( j : P \to X \) the fiber \( \pi^{-1}(o) \subset X \) of the surjective morphism \( \pi : X \to S \) has pure codimension one in \( X \). Two important families of examples of non trivial normal \( \mathbb{G}_a \)-extensions \( j : \text{SL}_2 \to X \) of the \( \mathbb{G}_a \)-torsor \( \rho : \text{SL}_2 \to \text{SL}_2/\mathbb{G}_a \simeq \mathbb{A}^2 \setminus \{(0,0)\} \), where \( \mathbb{G}_a \) acts on \( \text{SL}_2 \) via left multiplication by upper triangular unipotent matrices, were constructed in [10, Section 5 and 6]. Various other extensions were obtained from these by performing suitable equivariant affine modifications. One can observe that for all these extensions, the fiber \( \pi^{-1}((0,0)) \) is an \( \mathbb{A}^1 \)-ruled surface, a property which is a consequence of the following more general fact:

**Proposition 1.3.** Let \( \rho : P \to S_o \) be a non trivial \( \mathbb{G}_a \)-torsor over the punctured spectrum \( S \setminus \{ o \} \) of a regular local ring of dimension 2 over an algebraically closed field \( k \) and with residue field \( \kappa(o) = k \), and let \( \pi : X \to S \) be an \( \mathbb{G}_a \)-extension of \( P \). If \( X \) is smooth along \( \pi^{-1}(o) \), then every irreducible component \( F \) of \( \pi^{-1}(o)_{\text{red}} \) is an uniruled surface. Furthermore, if \( X \) is affine then \( F \) is an \( \mathbb{A}^1 \)-uniruled, hence \( \mathbb{A}^1 \)-ruled when it is normal.

**Proof.** Since \( \pi^{-1}(o) \) has pure codimension one in \( X \) and \( X \) is smooth along \( \pi^{-1}(o) \), every irreducible component of \( \pi^{-1}(o) \) is a \( \mathbb{G}_a \)-invariant Cartier divisor on \( X \). The complement \( X' \) of \( X \) in all but one irreducible component of \( \pi^{-1}(o) \) is thus again a \( \mathbb{G}_a \)-extension of \( P \), and we may therefore assume without loss of generality that \( F = \pi^{-1}(o)_{\text{red}} \) is irreducible. Let \( x \in F \) be a closed point in the regular locus of \( F \). Since \( F \) and \( X \) are smooth at \( x \) and \( X \) is connected, there exists a curve \( C \subset X \), smooth at \( x \) and intersecting \( F \) transversally at \( x \). The image \( \pi(C) \) of \( C \) is a curve on \( S \) passing through \( o \), and the closure \( B \) of \( \pi^{-1}(\pi(C) \cap S_o) \) in \( X \) is a surface containing \( C \). Since \( \rho : P \to S_o \) is a \( \mathbb{G}_a \)-torsor, the restriction of \( \pi \) to \( B \cap P \) is a trivial \( \mathbb{G}_a \)-torsor over the affine curve \( \pi(C) \). So \( \pi|_B : B \to \pi(C) \) is an \( \mathbb{A}^1 \)-fibration. Let \( \nu : \tilde{C} \to \pi(C) \) be the normalization of \( \pi(C) \). Then \( \pi|_B \) lifts to an \( \mathbb{A}^1 \)-fibration \( \theta : \tilde{B} \to \tilde{C} \) on the normalization \( \tilde{B} \) of \( B \). The fiber of \( \theta \) over every point in \( \nu^{-1}(o) \) is a union of rational curves. Since the normalization morphism \( \mu : \tilde{B} \to B \) is surjective, one of the irreducible components of \( \nu^{-1}(o) \) is mapped by \( \mu \) onto a rational curve in \( F \) passing through \( x \). This shows that for every smooth closed point \( x \) of \( F \), there exists a non constant rational map \( h : \mathbb{P}^1 \to F \) such that \( x \in h(\mathbb{P}^1) \). Thus \( F \) is uniruled. If \( X \) is in addition affine, then \( B \) and \( \tilde{B} \) are affine surfaces, and the fibers of the \( \mathbb{A}^1 \)-fibration \( \theta : \tilde{B} \to \tilde{C} \) consist of disjoint union of curves isomorphic to \( \mathbb{A}^1 \) when equipped with their reduced structure. This implies that \( F \) is not only uniruled but actually \( \mathbb{A}^1 \)-uniruled.

□
Example 1.4. Let $X$ be the smooth affine threefold in $\mathbb{A}^2 \times \mathbb{A}^4 = \text{Spec}(k[x,y][c,d,e,f])$ defined by the equations

$$\begin{align*}
xd - y(c + 1) &= 0 \\
xc^2 - y^2e &= 0 \\
yf - c(c + 1) &= 0 \\
x^2f - (c + 1)^2e &= 0 \\
de - cf &= 0
\end{align*}$$

equipped with the $G_a$-action induced by the locally nilpotent $k[x,y]$-derivation

$$xy\partial_x + y^2\partial_y + x(2c + 1)\partial_f + (2x^2f - 2xye)\partial_e$$

of its coordinate ring. The morphism $j : \text{SL}_2 = \{xv - yu = 1\} \to X$ defined by $(x, y, u, v) \mapsto (x, y, yu, yv, xu^2, xuv)$ is an open embedding of $\text{SL}_2$ in $X$ as the complement of the fiber over $o = (0, 0)$ of the projection $\pi = \text{pr}_{x,y} : X \to \mathbb{A}^2$. So $j : \text{SL}_2 \to X$ is an affine $G_a$-extension of the $G_a$-torsor $\rho : \text{SL}_2 \to \text{SL}_2/G_a = \mathbb{A}^2 \setminus \{0\}$, for which $\pi^{-1}(0)$ consists of the disjoint union of two copies $D_1 = \{x = y = c = 0\} \simeq \text{Spec}(k[d,f])$ and $D_2 = \{x = y = c + 1 = 0\} \simeq \text{Spec}(k[d,e])$ of $\mathbb{A}^2$. Note that the induced $G_a$-action on each of these is the trivial one.

Example 1.5. Let $X$ be the affine $G_a$-extension constructed in the previous example and let $C \subset D_1$ be any smooth affine curve. Let $\tau : \tilde{X} \to X$ be the blow-up of $X$ along $C$, let $i : X' \to \tilde{X}$ be the open immersion of the complement of the proper transform of $D_1 \cup D_2$ in $\tilde{X}$ and let $\pi' = \pi \circ \tau \circ i : X' \to \mathbb{A}^2$. Since $C$ and $D_1 \cup D_2$ are $G_a$-invariant, the $G_a$-action on $X$ lifts to a $G_a$-action on $X'$ which restricts in turn to $X'$. By construction, $\pi'$ is surjective, with fiber $\pi'^{-1}(o)$ isomorphic to $C \times \mathbb{A}^1$ and $\tau \circ i : X' \to X$ restricts to an equivariant isomorphism between $X' \setminus \pi'^{-1}(o)$ and $X \setminus \pi^{-1}(0) \simeq \text{SL}_2$. So $\pi' : X' \to \mathbb{A}^2$ is a $G_a$-extension of the $G_a$-torsor $\rho : \text{SL}_2 \to \text{SL}_2/G_a = \mathbb{A}^2 \setminus \{0\}$.

1.2. Recollection on affine-linear bundles. Affine-linear bundles of rank one over a scheme are natural generalization of $G_a$-torsors. To fix the notation, we briefly recall their basic definitions and properties.

By a line bundle on a scheme $S$, we mean the relative spectrum $p : M = \text{Spec}(	ext{Sym} M^\vee) \to S$ of the symmetric algebra of the dual of an invertible sheaf of $\mathcal{O}_S$-module $M$. Such a line bundle $M$ can be viewed as a locally constant group scheme over $S$ for the group law $m : M \times_S M \to M$ whose co-morphism

$$m^\#: \text{Sym} M^\vee \to \text{Sym} M^\vee \otimes \text{Sym} M^\vee \simeq \text{Sym} (M^\vee \oplus M^\vee)$$

is induced by the diagonal homomorphism $M^\vee \to M^\vee \oplus M^\vee$. An $M$-torsor is then an $S$-scheme $\theta : W \to S$ equipped with an action $\mu : M \times_S W \to W$ which is Zariski locally over $S$ isomorphic to $M$ acting on itself by translations.

This is the case precisely when there exists a Zariski open cover $f : Y \to S$ and an $\mathcal{O}_Y$-algebra isomorphism $\psi : f^*\mathcal{A} \to \text{Sym} f^*M^\vee$ such that over $Y' = Y \times_S Y$ the automorphism $p_1^\psi \circ p_2^\psi^{-1}$ is $\text{Sym} M^\vee_{Y'} \to \text{Sym} M^\vee_{Y'}$. Of the symmetric algebra of $M^\vee_{Y'}$, $p_1^\psi \circ p_2^\psi = p_1^\psi$ is affine-linear, i.e. induced by an $\mathcal{O}_{Y'}$-module homomorphism $M^\vee_{Y'} \to \text{Sym} M^\vee_{Y'}$, of the form

$$\beta \oplus \text{id} : M^\vee_{Y'} \to \mathcal{O}_{Y'} \oplus M^\vee_{Y'} \to \bigoplus_{n \geq 0} (M^\vee_{Y'})^\otimes n = \text{Sym} M^\vee_{Y'},$$

for some $\beta \in \text{Hom}_{Y'}(M^\vee_{Y'}, \mathcal{O}_{Y'}) \simeq H^0(Y', M^\vee_{Y'})$, which is a Čech 1-cocycle with values in $M$ for the Zariski open cover $f : Y \to S$. Standard arguments show that the isomorphism class of $\theta : W \to S$ depends only on the class of $\beta$ in the Čech cohomology group $H^1(S, M)$, and one eventually gets a one-to-one correspondence between isomorphism classes of $M$-torsors over $S$ and elements of the cohomology group $H^1(S, M) = H^1(S, M) \simeq H^1(S, M)$ with the zero element corresponding to the trivial torsor $p : M \to S$.

It is classical that every locally trivial $\mathbb{A}^1$-bundle $\theta : W \to S$ over a reduced scheme $S$ can be equipped with the additional structure of a torsor under a uniquely determined line bundle $M$ on $S$. The existence of this additional structure will be frequently used in the sequel, and we now quickly review its construction (see also e.g. [2, § 2.3 and § 2.4]). Letting $\mathcal{A} = \theta_*\mathcal{O}_Y$, there exists by definition a Zariski open cover $f : Y \to S$ and a quasi-coherent $\mathcal{O}_Y$-algebra isomorphism $\varphi : f^*\mathcal{A} \to \mathcal{O}_Y[u]$. Over $Y' = Y \times_S Y$ equipped with the two projections $p_1$ and $p_2$ to $Y$, the $\mathcal{O}_{Y'}$-algebra isomorphism $\Phi = p_1^\varphi \circ p_2^\varphi^{-1}$ has the form

$$\Phi : \mathcal{O}_{Y'}[u] \to \mathcal{O}_{Y'}[u], \quad u \mapsto au + b$$
for some $a \in \Gamma(Y',\mathcal{O}_{Y'}^*)$ and $b \in \Gamma(Y',\mathcal{O}_{Y'})$ whose pull back over $Y'' = Y \times_S Y \times_S Y$ by the three projections $p_{12},p_{23},p_{13} : Y'' \rightarrow Y'$ satisfy the cocycle relations $p_{13}^*a = p_{23}^*a \cdot p_{12}^*a$ and $p_{13}^*b = p_{23}^*a \cdot p_{12}^*b + p_{23}^*b$ in $\Gamma(Y''\mathcal{O}_{Y''})$ and $\Gamma(Y''\mathcal{O}_{Y''})$ respectively. The first one says that $a$ is a Čech 1-cocycle with values in $\mathcal{O}_S$ for the cover $f : Y \rightarrow S$, which thus determines, via the isomorphism $H^1(S,\mathcal{O}_S^*) \simeq \text{Pic}(S)$, a unique invertible sheaf $\mathcal{M}$ on $S$ together with an $\mathcal{O}_S$-module isomorphism $\alpha : f^*\mathcal{M}' \rightarrow \mathcal{O}_S$ such that $p_{13}^*\alpha \circ p_{23}^*\alpha^{-1} : \mathcal{O}_Y \rightarrow \mathcal{O}_Y$ is the multiplication by $a$. The second one can be equivalently reinterpreted as the fact that $\beta = p_{23}^*(\alpha)(b) \in \Gamma(Y',\mathcal{M}_Y)$ is a Čech 1-cocycle with values in $\mathcal{M}$ for the Zariski open cover $f : Y \rightarrow S$. Letting $\text{Sym}(\alpha) : \text{Sym}f^*\mathcal{M}' \rightarrow \mathcal{O}_Y[u]$ be the graded $\mathcal{O}_Y$-algebra isomorphism induced by $\alpha$, the isomorphism $\psi = \text{Sym}(\alpha^{-1}) \circ \phi : f^*\mathcal{A} \rightarrow \text{Sym}f^*\mathcal{M}'$ has the property that $p_{13}^*\psi \circ p_{23}^*\psi^{-1}$ is affine-linear, induced by the homomorphism $\beta \circ \text{id} : \mathcal{M}_Y \rightarrow \mathcal{O}_Y \oplus \mathcal{M}_Y$. So $\theta : W \rightarrow S$ is a torsor under the line bundle $\mathcal{M} = \text{Spec}(\text{Sym}\mathcal{M}')$, with isomorphism class in $H^1(S,\mathcal{M})$ equal to the cohomology class of the cocycle $\beta$. Summing up, we obtain:

**Proposition 1.6.** Let $\theta : W \rightarrow S$ be a locally trivial $\mathbb{A}^1$-bundle. Then there exists a unique pair $(\mathcal{M},g)$ consisting of a line bundle $\mathcal{M}$ on $S$ and a class $g \in H^1(S,\mathcal{M})$ such that $\theta : W \rightarrow S$ is an $\mathcal{M}$-torsor with isomorphism class $g$.

1.3. **Additive group actions on affine-linear bundles of rank one.** Given a locally trivial $\mathbb{A}^1$-bundle $\theta : W \rightarrow S$, which we view as an $\mathcal{M}$-torsor for a line bundle $\mathcal{M} = \text{Spec}(\text{Sym} \mathcal{M}') \rightarrow S$ on $S$, with corresponding action $\mu : M \times_S W \rightarrow W$, every nonzero group scheme homomorphism $\xi : G_{a,S} \rightarrow M$ induces a nontrivial $G_{a,S}$-action $\nu = \mu \circ (\xi \times \text{id}) : G_{a,S} \times_S W \rightarrow W$ on $W$. A nonzero group scheme homomorphism $\xi : G_{a,S} = \text{Spec}(\mathcal{O}_S[t]) \rightarrow M = \text{Spec}(\text{Sym} \mathcal{M}')$ is uniquely determined by a nonzero $\mathcal{O}_S$-module homomorphism $\mathcal{M}' \rightarrow O_S$, equivalently by a nonzero global section $s \in \Gamma(S,\mathcal{M})$. The following proposition asserts conversely that every nontrivial $G_{a,S}$-action on an $\mathcal{M}$-torsor $\theta : W \rightarrow S$ uniquely arises from such a section.

**Proposition 1.7.** ([1, Chapter 3]) Let $\theta : W \rightarrow S$ be a torsor under the action $\mu : M \times_S W \rightarrow W$ of a line bundle $\mathcal{M} = \text{Spec}(\text{Sym} \mathcal{M}') \rightarrow S$ on $S$ and let $\nu : G_{a,S} \times_S W \rightarrow W$ be a non trivial $G_{a,S}$-action on $W$. Then there exists a non zero global section $s \in \Gamma(S,\mathcal{M})$ such that $\nu = \mu \circ (\xi \times \text{id})$ where $\xi : G_{a,S} \rightarrow M$ is the group scheme homomorphism induced by $s$.

**Proof.** Let $\mathcal{A} = \theta_*\mathcal{O}_W$ and let $f : Y \rightarrow S$ be a Zariski open cover such that there exists an $\mathcal{O}_Y$-algebra isomorphism $\varphi : f^*\mathcal{A} \rightarrow \mathcal{O}_Y[u]$, and let

$$\Phi = p_{13}^*\varphi \circ p_{23}^*\varphi^{-1} : \mathcal{O}_Y[u] \rightarrow \mathcal{O}_Y[u], \quad u \mapsto au + b$$

be as in (1.2) above. Since $\theta : W \rightarrow S$ is an $\mathcal{M}$-torsor, $\varphi$ also determines an $\mathcal{O}_Y$-module isomorphism $\alpha : f^*\mathcal{M}' \rightarrow \mathcal{O}_Y$ such that $p_{13}^*\alpha \circ p_{23}^*\alpha^{-1} : \mathcal{O}_Y \rightarrow \mathcal{O}_Y$ is the multiplication by $a$. The $G_{a,S}$-action $\nu$ on $W$ pulls back to a $G_{a,Y}$-action $\nu \times \text{id}$ on $W \times_S Y$. The co-morphism $\eta : \mathcal{O}_Y[u] \rightarrow \mathcal{O}_Y[u] \otimes \mathcal{O}_Y[t]$ of the nontrivial $G_{a,Y}$-action $\varphi \circ (\nu \times \text{id})$ $\circ (\text{id} \otimes \varphi^{-1})$ on $\text{Spec}(\mathcal{O}_Y[u])$ has the form $u \mapsto u \otimes 1 + 1 \otimes \gamma t$ for some nonzero $\gamma \in \Gamma(Y,\mathcal{O}_Y)$. Letting $I = \gamma : \mathcal{O}_Y$ be the ideal sheaf generated by $\gamma$, $\eta$ factors as

$$\eta = (\text{id} \otimes j) \circ \tilde{\eta} : \mathcal{O}_Y[u] \rightarrow \mathcal{O}_Y[u] \otimes \text{Sym} I \rightarrow \mathcal{O}_Y[u] \otimes \mathcal{O}_Y[t]$$

where $\tilde{\eta}$ is the co-morphism of an action of the line bundle $\text{Spec}(\text{Sym} I) \rightarrow Y$ on $\mathbb{A}_S^1 \times_S Y \simeq W \times_S Y$ and $j : \text{Sym} I \rightarrow \mathcal{O}_Y[t]$ is the homomorphism induced by the inclusion $I \subset \mathcal{O}_Y$. Pulling back to $Y'$, we find that $p_2^*\gamma = a \cdot p_1^*\gamma$, which implies that $f^*\gamma \in \Gamma(Y,f^*\mathcal{M})$ is the pull-back of $f^*s$ to $Y$ of a nonzero global section $s \in \Gamma(S,\mathcal{M})$. Letting $D = \text{div}_0(s)$ be the divisors of zeros of $s$, we have $\mathcal{M}' \simeq \mathcal{O}_S(-D) \subset \mathcal{O}_S$ and $f^*\mathcal{M}' \simeq \mathcal{O}_Y(-f^*D) \subset \mathcal{O}_Y$. We can thus rewrite $\eta$ in the form

$$\eta = (\text{id} \otimes \text{Sym} f^*s) \circ \tilde{\eta} : \mathcal{O}_Y[u] \rightarrow \mathcal{O}_Y[u] \otimes \text{Sym} f^*\mathcal{M}' \rightarrow \mathcal{O}_Y[u] \otimes \mathcal{O}_Y[t].$$

By construction $\tilde{\eta} = (\varphi \otimes \text{id}) \circ f^*\mu^\theta \circ \varphi^{-1}$ where $f^*\mu^\theta$ is the pull-back of the co-morphism $\mu^\theta : \mathcal{A} \rightarrow \mathcal{A} \otimes \text{Sym} \mathcal{M}'$ of the action $\mu : M \times_S W \rightarrow W$ of $M$ on $W$. It follows that the pull-back $f^*\mu^\theta$ of the co-morphism of the action $\nu : G_{a,S} \times W \rightarrow W$ factors as

$$f^*\nu^\theta = (\text{id} \otimes \text{Sym} f^*s) \circ f^*\mu^\theta = f^*\mathcal{A} \rightarrow f^*\mathcal{A} \otimes \text{Sym} f^*\mathcal{M}' \rightarrow f^*\mathcal{A} \otimes \mathcal{O}_Y[t].$$

This in turn implies that $\nu^\theta$ factors as $(\text{id} \otimes \text{Sym} s) \circ \mu^\theta : \mathcal{A} \rightarrow \mathcal{A} \otimes \text{Sym} \mathcal{M}' \rightarrow \mathcal{A} \otimes \mathcal{O}_Y[t]$ as desired. □
Remark 1.8. In the setting of Proposition 1.7, let $U \subset S$ be the complement of the zero locus of $s$, the morphism $\xi$ restricts to an isomorphism of group schemes $\xi|_U : \mathbb{G}_a,U \to M|_U$ for which $W|_U$ equipped with the $\mathbb{G}_a,U$-action $\nu|_U : \mathbb{G}_a,U \times_U W|_U \to W|_U$ is a $\mathbb{G}_a,U$-torse. This isomorphism class in $H^1(U,\mathcal{O}_U)$ of this $\mathbb{G}_a,U$-torse coincides with the image of the isomorphism class $g \in H^1(S,M)$ of $W$ by the composition of the restriction homomorphism res : $H^1(S,M) \to H^1(U,M|_U)$ with the inverse of the isomorphism $H^1(U,\mathcal{O}_U) \to H^1(U,M|_U)$ induced by $s|_U$.

1.4. $\mathbb{G}_a$-equivariant affine modifications of affine-linear bundles of rank one. Recall [3] that given an integral scheme $X$ with sheaf of rational functions $K_X$, an effective Cartier divisor $D$ on $X$ and a closed subscheme $Z \subset X$ whose ideal sheaf $I \subset \mathcal{O}_X$ contains $\mathcal{O}_X(-D)$, the affine modification of $X$ with center $(I,D)$ is the affine $X$-scheme $\sigma : X' = \text{Spec}(\mathcal{O}_X(I,D)) \to X$ where $\mathcal{O}_X(I,D)$ denotes the quotient of the Rees algebra

$$\mathcal{O}_X[(I \otimes \mathcal{O}_X(D))] = \bigoplus_{n \geq 0} (I \otimes \mathcal{O}_X(D))^n t^n \subset K_X[t]$$

of the fractional ideal $I \otimes \mathcal{O}_X(D) \subset K_X$ by the ideal generated by $1 - t$. In the case where $X = \text{Spec}(A)$ is affine, $D = \text{div}(f)$ is principal and $Z$ is defined by an ideal $I \subset A$ containing $f$ then $X'$ is isomorphic to the affine modification $X'$ of $\text{Spec}(A[I/f])$ of $X$ with center $(I,f)$ in the sense of [11].

Now let $S$ be an integral scheme and let $\theta : W \to S$ be a locally trivial $\mathbb{A}^1$-bundle. Let $C \subset S$ be an integral Cartier divisor, let $D = \theta^{-1}(C)$ be its inverse image in $W$ and let $Z \subset D$ be a non empty integral closed subscheme of $D$ on which $\theta$ restricts to an open embedding $\theta|_Z : Z \hookrightarrow C$. Equivalently, $Z$ is the closure of $D$ in the image $\alpha(U)$ of a rational section $\alpha : C \to D$ of the locally trivial $\mathbb{A}^1$-bundle $\theta|_D : D \to C$ defined over a non empty open subset $U$ of $C$. The complement $\bar{F}$ of $\theta(Z)$ in $C$ is a closed subset of $C$ hence of $S$. Letting $i : S \setminus \bar{F} \hookrightarrow S$ be the natural open embedding, we have the following result:

**Lemma 1.9.** Let $\sigma : W' \to W$ be the affine modification of $W$ with center $(I_Z,D)$. Then the composition $\theta \circ \sigma : W' \to S$ factors through a locally trivial $\mathbb{A}^1$-bundle $\theta' : W' \to S \setminus \bar{F}$ in such a way that we have a cartesian diagram

$$\begin{array}{ccc}
W' & \overset{\sigma}{\longrightarrow} & W \\
\downarrow{\theta'} & & \downarrow{\theta} \\
S \setminus \bar{F} & \overset{i}{\longrightarrow} & S.
\end{array}$$

**Proof.** The question being local with respect to a Zariski open cover of $S$ over which $\theta : W \to S$ becomes trivial, we can assume without loss of generality that $S = \text{Spec}(A)$, $W = \text{Spec}(A[x])$, $C = \text{div}(f)$ for some non zero element $f \in A$. The integral closed subscheme $Z \subset D$ is then defined by an ideal $I$ of the form $(f,g)$ where $g(x) \in A[x]$ is an element whose image in $(A/f)[x]$ is a polynomial of degree one in $t$. So $g(x) = a_0 + a_1x + x^2 fR(x)$ where $a_0 \in A$, $a_1 \in A$ has non zero residue class in $A/f$ and $R(x) \in A[x]$. The condition that $\theta|_Z : Z \to C$ is an open embedding implies further that the residue classes $\overline{a_0}$ and $\overline{a_1}$ of $a_0$ and $a_1$ in $A/f$ generate the unit ideal. The complement $\bar{F}$ of the image of $\theta(Z)$ in $C$ is then equal to the closed subscheme of $C$ with defining ideal $(\overline{a_1}) \subset A/f$, hence to the closed subscheme of $S$ with defining ideal $(fa_1) \subset A$. The algebra $A[t]/(I/f)$ is isomorphic to

$$A[x][u]/(g-fu) \cong A[x][u-x^2 R(x)]/(a_0 + a_1x - f(u-t^2 R(x))) \cong A[x][v]/(a_0 + a_1x - fu).$$

One deduces from this presentation that the morphism $\theta \circ \sigma : W' = \text{Spec}(A[I/f]) \to \text{Spec}(A)$ corresponding to the inclusion $A \to A[I/f]$ factors through a locally trivial $\mathbb{A}^1$-bundle $\theta' : W' \to S \setminus \bar{F}$ over the complement of $F$. Namely, since $\overline{a_0}$ and $\overline{a_1}$ generate the unit ideal in $A/f$, it follows that $a_1$ and $f$ generate the unit ideal in $A[x][u]/(g-fu)$. So $W'$ is covered by the two principal affine open subsets

$$W'_{a_1} \cong \text{Spec}(A_{a_1}[x][v]/(a_0 + a_1x - fu)) \cong \text{Spec}(A_{a_1}[v]) \cong S_{a_1} \times \mathbb{A}^1$$

$$W'_{f} \cong \text{Spec}(A_{f}[x][v]/(a_0 + a_1x - fu)) \cong \text{Spec}(A_{f}[x]) \cong S_{f} \times \mathbb{A}^1$$

on which $\theta'$ restricts to the projection onto the first factor.

With the notation above, $\theta : W \to S$ and $\theta' : W' \to S \setminus \bar{F}$ are torsors under the action of line bundles $M = \text{Spec}(\text{Sym}^\vee \mathcal{M}^\vee)$ and $M' = \text{Spec}(\text{Sym}^\vee \mathcal{M}'^\vee)$ for certain uniquely determined invertible sheaves $\mathcal{M}$ and $\mathcal{M}'$ on $S$ and $S \setminus \bar{F}$ respectively.
Lemma 1.10. ([1, §4.3]) Let $\sigma : W' \to W$ be the affine modification of $W$ with center $(I_Z, D)$ as is Lemma 1.9. Then $M' = M \otimes_{O_S} O_S(-C)|_{S \setminus \tau}$ and the cartesian diagram of Lemma 1.9 is equivariant for the group scheme homomorphism $\xi : M' \to M$ induced by the homomorphism $M \otimes_{O_S} O_S(-C) \to M$ obtained by tensoring the inclusion $O_S(-C) \to O_S$ by $M$.

**Proof.** Since $M$ and $M'$ are uniquely determined, the question is again local with respect to a Zariski open cover of $S$ over which $\theta : W \to S$, hence $M'$, becomes trivial. We can thus assume as in the proof of Lemma 1.9 that $S = \text{Spec}(A)$, $W = \text{Spec}(A[x])$, that $C = \text{div}(f)$ for some non zero element $f \in A$ and that $Z \subset D$ is defined by the ideal $(f,g)$ for some $g = a_0 + a_1 x + f x^2 R(x) \in A[x]$. Furthermore, the action of $M \simeq G_{a,S} = \text{Spec}(A[t])$ on $W \simeq S \times \mathbb{A}^1$ is the one by translations $x \mapsto x + t$ on the second factor. Let $N = \text{Spec}(\text{Sym}(O_S(C))) \simeq \text{Spec}(\text{Sym}(f^{-1}A))$ where $f^{-1}A$ denotes the free sub-$A$-module of the field of fractions Frac($A$) of $A$ generated by $f^{-1}$. As in the proof of Proposition 1.7, the inclusion $O_S(-C) = f \cdot O_S \hookrightarrow O_S$ induces a group-scheme homomorphism $\xi : N \to M$ whose co-morphism $\xi^t$ coincides with the inclusion $A[S] \subset \text{Sym}(f^{-1}A) = A[(f^{-1}t)]$. The co-morphism of the corresponding action of $N$ on $W'$ is given by

$$A[x] \to A[x] \otimes A[f^{-1}t], \; x \mapsto x \otimes 1 + t \otimes x + f \otimes f^{-1}t.$$ 

This action lifts on $W' \simeq \text{Spec}(A[x]/(a_0 + a_1 x - f v))$ to an action $\nu : N \times_S W' \to W'$ whose co-morphism $A[x]/(a_0 + a_1 x - f v) \to A[x]/(a_0 + a_1 x - f v) \otimes A[f^{-1}t]$ is given by $x \mapsto x \otimes 1 + t \otimes x - f \nu \otimes f^{-1}t$. By construction, the principal open subsets $W'_i \simeq \text{Spec}(A[x]/(a_i)) \simeq \text{Spec}(A[x]/(a_i))$ and $W'_j \simeq \text{Spec}(A[x]/(f)) \simeq \text{Spec}(A[x]/(f))$ of $W'$ equipped with the induced actions of $N|_{S_{a_i}}$ and $N|_{S_{a_j}}$, respectively, are equivariantly isomorphic to $N|_{S_{a_i}}$ and $N|_{S_{a_j}}$ acting on themselves by translations. So $\theta' : W' \to S\setminus F$ is an $N|_{S\setminus F}$-torsor, showing that $M' = M \otimes_{O_S} O_S(-C)|_{S\setminus F}$ as desired.

2. Extensions of $\mathbb{G}_a$-torsors of Type I: locally trivial bundles over the blow-up of a point

Given a surface $S$ and a locally trivial $\mathbb{A}^1$-bundle $\theta : W \to \tilde{S}$ over the blow-up $\tau : \tilde{S} \to S$ of a closed point $o$ in the smooth locus of $S$, the restriction of $W$ over the complement $S \setminus E$ of the exceptional divisor $E$ of $\tau$ is a locally trivial $\mathbb{A}^1$-bundle $\tau \circ \theta : W \mid_{S \setminus E} \to \tilde{S} \setminus E \simeq S \setminus \{o\}$. This observation combined with the following re-interpretation of an example constructed in [10] suggests that locally trivial $\mathbb{A}^1$-bundles over the blow-up of closed point $o$ in the smooth locus of a surface $S$ form a natural class of schemes in which to search for nontrivial $\mathbb{G}_a$-extension of $G_a$-bundles over punctured surfaces.

**Example 2.1.** Let $o = V(x,y)$ be a global scheme-theoretic complete intersection closed point in the smooth locus of a surface $S$. Let $\rho : P = S \setminus \{o\}$ and $\pi_0 : X_0 \to S$ be the affine $S$-schemes with defining sheaves of ideals $(xv - yu - 1)$ and $(x - yq, y - x(q - 1), pr - q(q - 1))$ in $O_S[u,v]$ and $O_S[p,q,r]$ respectively. The morphism of $S$-schemes $j_0 : P \to X_0$ defined by $(x, y, u, v) \mapsto (x, y, xu, xv, yv)$ is an open embedding, equivariant for the $G_{a,S}$-actions on $P$ and $X_0$ associated with the locally nilpotent $O_S$-derivatives $x\partial_x + y\partial_y$ and $x^2\partial_x + xy\partial_y + y^2\partial_y$ of $\rho_o \circ \sigma_0$ and $(\pi_0)_* O_{X_0}$ respectively. It is straightforward to check that $\rho : P \to S \setminus \{o\}$ is a $G_{a,S}$-torsor and that $\pi_0 : X_0 \to S$ is a $G_{a}$-extension of $P$ whose fiber over $o$ is isomorphic to the smooth affine quadric $\{pr - q(q - 1) = 0\} \subset \mathbb{A}^2_\mathbb{C}$. Viewing the blow-up $\tilde{S}$ of $o$ as the closed subscheme of $S \times_k \text{Proj}(k[u_0, u_1])$ with equation $xu_1 - yu_0 = 0$, the morphism of $S$-schemes $\theta : X_0 \to \tilde{S}$ defined by

$$(x, y, p, q, r) \mapsto ((x, y), [x : y]) = ((x, y), [q : r]) = ((x, y), [p : q - 1])$$

is a locally trivial $\mathbb{A}^1$-bundle, actually a torsor under the line bundle corresponding to the invertible sheaf $O_{\tilde{S}}(-2E)$, where $E \simeq \mathbb{P}^1_k$ denotes the exceptional divisor of the blow-up.

**Notation 2.2.** Given a surface $S$ and a closed point $o$ in the smooth locus of $S$, with residue field $\kappa$, we denote by $\tau : \tilde{S} \to S$ be the blow-up of $o$, with exceptional divisor $E \simeq \mathbb{P}^1_k$. We identify $\tilde{S} \setminus E$ and $S_o = S \setminus \{o\}$ by the isomorphism induced by $\tau$. For every $\ell \in \mathbb{Z}$, we denote by $M(\ell) = \text{Spec}(\text{Sym}(O_{\tilde{S}}(\ell E)))$ the line bundle on $\tilde{S}$ corresponding to the invertible sheaf $O_{\tilde{S}}(\ell E)$.
The aim of this section is to give a classification of all possible $\mathbb{G}_a$-equivariant extensions of Type I of a given $\mathbb{G}_a$-torsor $\rho : P \to S_*$, that is $\mathbb{G}_a$-extensions $\pi : W \to S$ that factor through locally trivial $\mathbb{A}^1$-bundles $\theta : W \to \tilde{S}$.

2.1. Existence of $\mathbb{G}_a$-extensions of Type I. By virtue of Propositions 1.6 and 1.7, there exists a one-to-one correspondence between $\mathbb{G}_a$-equivariant extensions of a $\mathbb{G}_a$-torsor $\rho : P \to S_*$ that factor through a locally trivial $\mathbb{A}^1$-bundle $\theta : W \to \tilde{S}$ and pairs $(M, \xi)$ consisting of an $M$-torsor $\theta : W \to \tilde{S}$ for some line bundle $M$ on $\tilde{S}$ and a group scheme homomorphism $\xi : \mathbb{G}_a, S \to M$ restricting to an isomorphism over $\tilde{S} \setminus E$, such that $W$ equipped with the $\mathbb{G}_a, S$-action deduced by composition with $\xi$ restricts on $S_* = \tilde{S} \setminus E$ to a $\mathbb{G}_a, S$-torsor $\theta : S_* \to S_*$ isomorphic to $\rho : P \to S_*$. The condition that $\xi : \mathbb{G}_a, S \to M$ restricts to an isomorphism outside $E$ implies that $M \cong M(\ell)$ for some $\ell$, which is necessarily non negative, and that $\xi$ is induced by the canonical global section of $\mathcal{O}_S(\ell E)$ with divisor $\ell E$.

Proposition 2.3. Let $\rho : P \to S_*$ be a $\mathbb{G}_a, S_*$-torsor. Then there exists an integer $\ell_0 \geq 0$ depending on $P$ only such that for every $\ell \geq \ell_0$, $P$ admits a $\mathbb{G}_a, \ell_0$-extension to a uniquely determined $M(\ell)$-torsor $\theta_\ell : W(P, \ell) \to \tilde{S}$ equipped with the $\mathbb{G}_a, \tilde{S}$-action induced by the canonical global section $s_\ell = \Gamma(\tilde{S}, \mathcal{O}_S(\ell E))$ with divisor $\ell E$.

Proof. The invertible sheaves $\mathcal{O}_S(nE)$, $n \geq 0$, form an inductive system of sub-$\mathcal{O}_S$-modules of the sheaf $\mathcal{K}_S$ of rational function on $\tilde{S}$, where for each $n$, the injective transition homomorphism $j_{n,n+1} : \mathcal{O}_S(nE) \to \mathcal{O}_S((n+1)E)$ is obtained by tensoring the canonical section $\mathcal{O}_S(0) \to \mathcal{O}_S(nE)$ with divisor $E$ with $\mathcal{O}_S(nE)$. Let $i : S_* = \tilde{S} \setminus E \hookrightarrow \tilde{S}$ be the open inclusion. Since $E$ is a Cartier divisor, it follows from [6, Théorème 9.3.1] that $i_*\mathcal{O}_{S_*} \simeq \operatorname{colim}_{n \geq 0} \mathcal{O}_S(nE)$. Furthermore, since $E \simeq \mathbb{P}^1_\kappa$ is the exceptional divisor of $\tau : \tilde{S} \to S$, we have $\mathcal{O}_S(\ell E)|_E \simeq \mathcal{O}_{\mathbb{P}^1_\kappa}(-1)$, and the long exact sequence of cohomology for the short exact sequence

\begin{equation}
0 \to \mathcal{O}_S(nE) \to \mathcal{O}_S((n+1)E) \to \mathcal{O}_S((n+1)E)|_E \to 0, \quad n \geq 0,
\end{equation}

combined with the vanishing of $H^0(\mathbb{P}^1_\kappa, \mathcal{O}_{\mathbb{P}^1_\kappa}(-n-1))$ for every $n \geq 0$ implies that the transition homomorphisms

\begin{equation}
H^1(j_{n,n+1} : \mathcal{O}_S(nE) \to \mathcal{O}_S((n+1)E)), \quad n \geq 0,
\end{equation}

are all injective. By assumption, $S$ whence $\tilde{S}$ is noetherian, and $i : S_* \to \tilde{S}$ is an affine morphism as $E$ is a Cartier divisor on $\tilde{S}$. We thus deduce from [12, Theorem 8] and [7, Corollaire 1.3.3] that the canonical homomorphism

\begin{equation}
\psi : \operatorname{colim}_{n \geq 0} H^1(\tilde{S}, \mathcal{O}_S(nE)) \to H^1(S_*, \mathcal{O}_{S_*})
\end{equation}

obtained as the composition of the canonical homomorphisms

\begin{equation}
\operatorname{colim}_{n \geq 0} H^1(\tilde{S}, \mathcal{O}_S(nE)) \to H^1(\tilde{S}, \mathcal{O}_S(nE)) = H^1(\tilde{S}, i_*\mathcal{O}_{S_*})
\end{equation}

and $H^1(\tilde{S}, i_*\mathcal{O}_{S_*}) \to H^1(S_*, \mathcal{O}_{S_*})$ is an isomorphism.

Let $g \in H^1(S_*, \mathcal{O}_{S_*})$ be the isomorphism class of the $\mathbb{G}_a, S_*$-torsor $\rho : P \to S_*$. If $g = 0$, then since $\psi$ is an isomorphism, we have $\psi^{-1}(g) = 0$, and, since the homomorphisms $H^1(j_{n,n+1})$ are injective, it follows that $\psi^{-1}(g)$ is represented by the zero sequence $(0)_n \in H^1(\tilde{S}, \mathcal{O}_S(nE))$, $n \geq 0$. Consequently, the only $\mathbb{G}_a$-extensions of $P$ are the line bundles $W(P, \ell) = M(\ell)$, $\ell \geq 0$, each equipped with the $\mathbb{G}_a, S_*$-action induced by its canonical global section $s_\ell = \Gamma(\tilde{S}, \mathcal{O}_S(\ell E))$.

Otherwise, if $g \neq 0$, then $h = \psi^{-1}(g) \neq 0$, and since the homomorphisms $H^1(j_{n,n+1})$, $n \geq 0$ are injective, it follows that there exists a unique minimal integer $\ell_0$ such that $h$ is represented by the sequence

\begin{equation}
h_n = H^1(j_{n-1,n}) \circ \cdots \circ H^1(j_{\ell_0,\ell_0+1}(h_{\ell_0}) \in H^1(\tilde{S}, \mathcal{O}_S(nE))
\end{equation}

for some non zero $h_{\ell_0} \in H^1(\tilde{S}, \mathcal{O}_S(\ell_0 E))$. It then follows from Proposition 1.7 that for every $\ell \geq \ell_0$, the $M(\ell)$-torsor $\theta_\ell : W(P, \ell) \to \tilde{S}$ with isomorphism class $h_\ell$ equipped with the $\mathbb{G}_a, \tilde{S}$-action induced by the canonical global section $s_\ell = \Gamma(\tilde{S}, \mathcal{O}_S(\ell E))$, it follows from Proposition 1.7 again that the
image of the isomorphism class $h_\ell \in H^1(\tilde{S}, \mathcal{O}_S(\ell E))$ of $W$ in $H^1(\tilde{S} \setminus E, \mathcal{O}_S(\ell E)|_{\tilde{S} \setminus E}) \simeq H^1(S, \mathcal{O}_S)$ is equal to $g$. Letting $h \in \text{colim}_{\ell \geq 0}H^1(\tilde{S}, \mathcal{O}_S(nE))$ be the element represented by the sequence

$$h_n = (H^1(j_{n-1,n} \circ \cdots \circ j_{\ell,\ell+1})(h_\ell))_{n \geq \ell} \in H^1(\tilde{S}, \mathcal{O}_S(nE)), \ n \geq \ell$$

we have $\psi(h) = g$ and since $\psi$ is an isomorphism, we conclude that $W \simeq W(P, \ell)$.

2.2. $G_a$-extensions with affine total spaces. The extensions $\theta : W \to \tilde{S}$ we get from Proposition 2.3 are not necessarily affine over $S$. In this subsection we establish a criterion for affineness which we then use to characterize all extensions $\theta : W \to \tilde{S}$ of a $G_a$-torsor $\rho : P \to S_*$ whose total spaces $W$ are affine over $S$.

Lemma 2.4. Let $S = \text{Spec}(A)$ be an affine surface and let $o = V(x, y)$ be a global scheme-theoretic complete intersection point in the smooth locus of $S$. Let $\tau : \tilde{S} \to S$ be the blow-up of $o$ with exceptional divisor $E$ and let $\theta : W \to \tilde{S}$ be an $M(\ell)$-torsor for some $\ell \geq 0$. Then the following hold:

a) $H^1(W, \omega_W) = 0$.
b) If $H^1(W, \theta^*\mathcal{O}_S(\ell E)) = 0$ for some $\ell \geq 2$ then $W$ is an affine scheme.

Proof. Since $o$ is a scheme-theoretic complete intersection, we can identify $\tilde{S}$ with the closed subvariety of $S \times_k \mathbb{P}^1_k = S \times_k \text{Proj}(k[t_0, t_1])$ defined by the equation $xt_1 - yt_0 = 0$. The restriction $p : \tilde{S} \to \mathbb{P}^1_k$ of the projection to the second factor is an affine morphism. More precisely, letting $U_0 = \mathbb{P}^1_k \setminus \{1 : 0\} \simeq \text{Spec}(k[z])$ and $U_\infty = \mathbb{P}^1_k \setminus \{0 : 1\} \simeq \text{Spec}(k[z])$ be the standard affine open cover of $\mathbb{P}^1_k$, we have $p^{-1}(U_0) \simeq \text{Spec}(A[z]/(x - yz))$ and $p^{-1}(U_\infty) \simeq \text{Spec}(A[z]/(y - xz))$. The exceptional divisor $E \simeq \mathbb{P}^1_k$ of $\tau : \tilde{S} \to S$ is a flat quasi-section of $p$ with local equations $y = 0$ and $x = 0$ in the affine charts $p^{-1}(U_0)$ and $p^{-1}(U_\infty)$ respectively. Every $M(\ell)$-torsor $\theta : W \to \tilde{S}$ for some $\ell \geq 0$ is isomorphic to the scheme obtained by gluing $W_0 = p^{-1}(U_0) \times \text{Spec}(k[u])$ with $W_\infty = p^{-1}(U_\infty) \times \text{Spec}(k[u])$ over $U_0 \cap U_\infty$ by an isomorphism induced by a $k$-algebra isomorphism of the form

$$A[z]((z')^{\pm 1})/(y - xz')\{u'\} \ni (z', u') \mapsto (z^{-1}, z^\ell u + p) \in A[z^{\pm 1}]/(x - yz)[u]$$

for some $p \in A[z^{\pm 1}]/(x - yz)$. Since $H^1(W, \mathcal{O}_W) \simeq H^1(W, \mathcal{O}_W) \simeq H^1(W_0, \mathcal{O}_{W_0}, \mathcal{O}_{W_0})$, it is enough in order to prove a) to check that every Čech 1-cocycle $g$ with value in $\mathcal{O}_W$ for the covering of $W$ by the affine open subsets $W_0$ and $W_\infty$ is a coboundary. Viewing $g$ as an element $g = g(z^{\pm 1}, u) \in A[z^{\pm 1}]/(x - yz)[u]$, it is enough to show that every monomial $g_h = h z^{r} u^s$ where $h \in A$, $r \in \mathbb{Z}$ and $s \in \mathbb{Z}_{\geq 0}$ is a coboundary, which is the case if and only if there exist $a(z, u) \in A[z]/(f - gz)[u]$ and $b(z, u') \in A[z']/(y - xz')[u']$ such that $g = b(z^{-1}, z^\ell u + p) - b(z, u)$. If $r \geq 0$ then $g \in A[z]/(x - yz)[u]$ is a coboundary. We thus assume from now on that $r < 0$. Suppose that $s > 0$. Then we can write $u^s = z^{-\ell s}(z^\ell u + p) - R(u)$ where $R \in A[z^{\pm 1}]/(x - yz)[u]$ is polynomial whose degree in $u$ is strictly less than $s$. Then since $r < 0$,

$$hz^{r} u^s = h z^{-\ell s}(z^\ell u + p) - h z^{r} R(u)$$

$$= b(z^{-1}, z^\ell u + p) - b(z^\ell u + p) - h z^{r} R(u)$$

where $b(z', u') = h(z')^{-r + \ell s}(u')^s \in A[z']/(y - xz')[u']$. So $g_a$ is a coboundary if and only if $-h z^{r} R(u)$ is.

By induction, we only need to check that every monomial $g_0 = h z^{\ell} \in A[z^{\pm 1}]/(x - yz)[u]$ of degree 0 in $u$ is a coboundary. But such a cocycle is simply the pull-back to $W$ of a Čech 1-cocycle $h_0$ with value in $\mathcal{O}_S$ for the covering of $\tilde{S}$ by the affine open subsets $p^{-1}(U_0)$ and $p^{-1}(U_\infty)$. Since the canonical homomorphism

$$H^1(S, \mathcal{O}_S) = H^1(S, \tau_* \mathcal{O}_S) \to H^1(\tilde{S}, \mathcal{O}_S) \simeq \tilde{H}^1(\{p^{-1}(U_0), p^{-1}(U_\infty)\}, \mathcal{O}_S)$$

is an isomorphism and $H^1(S, \mathcal{O}_S) = 0$ as $S$ is affine, we conclude that $h_0$ is a coboundary, hence that $g_0$ is a coboundary too. This proves a).
Now suppose that $H^1(W, \theta^*O_S(\ell(E))) = 0$ for some $\ell \geq 2$. Let $\eta : V \to \mathbb{P}^1_k$ be a non trivial $O_{\mathbb{P}^1_k}(-\ell)$-torsor and consider the fiber product $W \times_{\mathbb{P}^1_k, \eta} \eta V:

\[
\begin{array}{c}
W \\
| p\theta \\
\mathbb{P}^1_k \\
\eta \\
V
\end{array}
\]

By virtue of [5, Proposition 3.1], $V$ is an affine surface. Since $p \circ \theta : W \to \mathbb{P}^1_k$ is an affine morphism, so is $pr_V : W \times_{\mathbb{P}^1_k} V \to V$ and hence, $W \times_{\mathbb{P}^1_k} V$ is an affine scheme. On the other hand, since $p^*O_{\mathbb{P}^1_k}(-1) \simeq O_S(\ell)$, the projection $pr_W : W \times_{\mathbb{P}^1_k} V \to W$ is a $\theta^*M(\ell)$-torsor, hence is isomorphic to the trivial one $q : \theta^*M(\ell) \to W$ by hypothesis. So $W$ is isomorphic to the zero section of $\theta^*M(\ell)$, which is a closed subscheme of the affine scheme $W \times_{\mathbb{P}^1_k} V$, hence an affine scheme.

We are now ready to prove the following characterization:

**Theorem 2.5.** A $G_{a,S_*}$-torsor $\rho : P \to S_*$ admits a $G_a$-extension to a locally trivial $\mathbb{A}^1$-bundle whose total space is affine over $S$ if and only if for every Zariski open neighborhood $U$ of $o$, $P \times_{S_*} U \to U_* = U \setminus \{o\}$ is a non trivial $G_{a,U_*}$-torsor.

When it exists, the corresponding locally trivial $\mathbb{A}^1$-bundle $\theta : W \to \tilde{S}$ is unique and is an $M(\ell_0)$-torsor for some $\ell_0 \geq 2$, whose restriction to $E \simeq \mathbb{P}^1_k$ is a non trivial $O_{\mathbb{P}^1_k}(-\ell_0)$-torsor.

**Proof.** The scheme $W$ is affine over $S$ if and only if its restriction $W|_E$ over $E \subset \tilde{S}$ is a nontrivial torsor. Indeed, if $W|_E$ is a trivial torsor then it is a line bundle over $\mathbb{P}^1_k$. Its zero section is then a proper curve contained in the fiber of $\pi = \tau \circ \theta : W \to S$, which prevents $\pi$ from being an affine morphism. Conversely, if $W|_E$ is nontrivial, then it is a torsor under a uniquely determined line bundle $O_{\mathbb{P}^1_k}(-m)$ for some $m \geq 2$ necessarily. Since by construction $\pi$ restricts over $S_*$ to $\rho : P \to S_*$ which is an affine morphism, $\pi$ is affine if and only if there exists an open neighborhood $U$ of $o$ in $S$ such that $\pi^{-1}(U)$ is affine. Replacing $S$ by a suitable affine open neighborhood of $o$, we can therefore assume without loss of generality that $S = \text{Spec}(A)$ is affine and that $o$ is a scheme-theoretic complete intersection $o = V(x,y)$ for some elements $x, y \in A$. By virtue of [5, Proposition 3.1] every nontrivial $O_{\mathbb{P}^1_k}(-m)$-torsor, $m \geq 2$, has affine total space. The Cartier divisor $D = W|_E$ in $W$ is thus an affine surface, and so $H^1(D, O_W((n+1)D)|_D) = 0$ for every $n \in \mathbb{Z}$. By a) in Lemma 2.4, $H^1(W, O_W) = 0$, and we deduce successively from the long exact sequence of cohomology for the short exact sequence

$$0 \to O_W(nD) \to O_W((n+1)D) \to O_W((n+1)D)|_D \to 0$$

in the case $n = 0$ and then $n = 1$ that $H^1(W, O_W(D)) = H^1(W, O_W(2D)) = 0$. Since $O_W(2D) \simeq \theta^*O_S(2E)$, we conclude from b) in the same lemma that $W$ is affine.

The condition that $P \times_{S_*} U \to U_*$ is nontrivial for every open neighborhood $U$ of $o$ is necessary for the existence of an extension $\theta : W \to \tilde{S}$ of $P$ for which $W|_E$ is a nontrivial torsor. Indeed, if there exists a Zariski open neighborhood $U$ of $o$ such that the restriction of $P$ over $U_*$ is the trivial $G_{a,U_*}$-torsor, then the image in $H^1(U_*, O_{U_*})$ of the isomorphism class $g$ of $P$ is zero and so, arguing as in the proof of Proposition 2.3, every $G_{a,U_*}$-extension $\theta : W \to \tilde{S}$ restricts on $\tau^{-1}(U)$ to the trivial $M(\ell)(\tau^{-1}(U))$-torsor $M(\ell)|_{\tau^{-1}(U)} \to \tau^{-1}(U)$, hence to a trivial torsor on $E \subset \tau^{-1}(U)$.

Now suppose that $P \to S_*$ is a $G_{a,S_*}$-torsor with isomorphism class $g \in H^1(S_*, O_{S_*})$ such that $P \times_{S_*} U \to U_*$ is non trivial for every open neighborhood $U$ of $o$. The inverse image $h = \psi^{-1}(g) \in \text{colim}_{n \geq 0} H^1(\tilde{S}, O_{\tilde{S}}(nE))$ of $g$ by the isomorphism (2.2) is represented by a sequence of nonzero elements $h_n \in H^1(\tilde{S}, O_{\tilde{S}}(nE))$ as in (2.3) above. By the long exact sequence of cohomology of the short exact sequence (2.1), the image $\overline{h}_n$ of $h_n$ in $H^1(E, O_S(nE)|_E) \simeq H^1(\mathbb{P}^1_k, O_{\mathbb{P}^1_k}(-n))$ is nonzero if and only if $h_n$ is not in the image of the injective homomorphism $H^1(J_{n,n-1})$. Since $\overline{h}_n$ coincides with the isomorphism class of the restriction $W|_E$ of an $M(n)$-torsor $\theta_n : W_n \to \tilde{S}$ with isomorphism class $h_n$, we conclude
that there exists a unique \( \ell_0 \geq 2 \) such that the restriction to \( E \) of an \( M(\ell_0) \)-torsor \( \theta_{\ell_0} : W_{\ell_0} \to \tilde{S} \) with isomorphism class \( h_{\ell_0} \in H^1(S, \mathcal{O}_{\tilde{S}}(\ell_0)E) \) is nontrivial \( \mathcal{O}_{\mathbb{P}^1}(-\ell_0) \)-torsor. \( \square \)

2.3. Examples. In this subsection, we consider \( \mathbb{G}_a \)-torsors of the punctured affine plane. So \( S = \mathbb{A}^2 = \text{Spec}(k[x, y]), \) \( o = (0, 0) \) and \( \mathbb{A}_x^2 = \mathbb{A}^2 \setminus \{o\} \). We let \( \tau : \mathbb{A}^2 \to \mathbb{A}^2 \) be the blow-up of \( o \), with exceptional divisor \( E \simeq \mathbb{P}^1 \) and we let \( i : \mathbb{A}_x^2 \to \mathbb{A}^2 \) be the immersion of \( \mathbb{A}_x^2 \) as the open subset \( \mathbb{A}_x^2 \setminus E \). We further identify \( \mathbb{A}_x^2 \) with the total space \( f : \mathbb{A}_x^2 \to \mathbb{P}^1 \) of the line bundle \( \mathcal{O}_{\mathbb{P}^1}(-1) \) in such a way that \( E \) corresponds to the zero section of this line bundle.

2.3.1. A simple case: homogeneous \( \mathbb{G}_a \)-torsors. Following [4, §1.3], we say that a non trivial \( \mathbb{G}_a, \mathbb{A}_x^2 \)-torsor \( \rho : P \to \mathbb{A}_x^2 \) is homogeneous if it admits a lift of the \( \mathbb{G}_m \)-action \( \lambda \cdot (x, y) = (\lambda x, \lambda y) \) on \( \mathbb{A}_x^2 \) which is locally linear on the fibers of \( \rho \). By [4, Proposition 1.6], this is the case if and only if the isomorphism class \( g \) of \( P \) in \( H^1(\mathbb{A}_x^2, \mathcal{O}_{\mathbb{A}_x^2}) \) can be represented on the open covering of \( \mathbb{A}_x^2 \) by the principal open subsets \( \mathbb{A}_x^2 \) and \( \mathbb{A}_y^2 \) by a \( \check{C}ech \) 1-cocycle of the form \( x^{-m}y^{-n}p(x, y) \) where \( m, n \geq 0 \) and \( p(x, y) \in k[x, y] \) is a homogeneous polynomial of degree \( r \leq m + n - 2 \). Equivalently, \( P \) is isomorphic the \( \mathbb{G}_a, \mathbb{A}_x^2 \)-torsor

\[
\rho = \text{pr}_{x,y} : P_{m,n,r} = \{ x^m v - y^n u = p(x, y) \} \setminus \{ x = y = 0 \} \to \mathbb{A}_x^2,
\]

which admits an obvious lift \( \lambda \cdot (x, y, u, v) = (\lambda x, \lambda y, \lambda^{m-d} u, \lambda^{n-d} v) \), where \( d = m + n - r \), of the \( \mathbb{G}_m \)-action on \( \mathbb{A}_x^2 \). Let \( q : \mathbb{A}_x^2 \to \mathbb{A}_x^2 / \mathbb{G}_m = \mathbb{P}^1 \) be the quotient morphism of the aforementioned \( \mathbb{G}_m \)-action on \( \mathbb{A}_x^2 \). Then it follows from [4, Example 1.8] that the inverse image by the canonical isomorphism

\[
\bigoplus_{k \in \mathbb{Z}} H^1(\mathbb{P}^1, \mathcal{O}_\mathbb{P}^1(k)) \simeq H^1(\mathbb{P}^1, q_* \mathcal{O}_{\mathbb{A}_x^2}) \to H^1(\mathbb{A}_x^2, \mathcal{O}_{\mathbb{A}_x^2})
\]

of the isomorphism class \( g \) of such an homogeneous torsor is an element \( h \) of \( H^1(\mathbb{P}^1, \mathcal{O}_\mathbb{P}^1(-d)) \). Furthermore, the \( \mathbb{G}_m \)-equivariant morphism \( \rho : P \to \mathbb{A}_x^2 \) descends to a locally trivial \( \mathbb{A}^1 \)-bundle \( \overline{\rho} : P / \mathbb{G}_m \to \mathbb{P}^1 = \mathbb{A}_x^2 / \mathbb{G}_m \) which is an \( \mathcal{O}_{\mathbb{P}^1}(-d) \)-torsor with isomorphism class \( h \in H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-d)) \).

Since \( f^* \mathcal{O}_{\mathbb{P}^1}(-d) \simeq \mathcal{O}_{\mathbb{A}_x^2}(dE) \), the fiber product \( W(P, d) = \mathbb{A}_x^2 \times_{\mathbb{P}^1} P / \mathbb{G}_m \) is equipped via the restriction of the first projection with the structure of an \( M(d) \)-torsor \( \theta : W(P, d) \to \mathbb{A}_x^2 \) with isomorphism class \( f^* h \in H^1(\mathbb{A}_x^2, \mathcal{O}_{\mathbb{A}_x^2}(dE)) \). On the other hand, \( W(P, d) \) is a line bundle over \( P / \mathbb{G}_m \) via the second projection, hence is an affine threefold as \( P / \mathbb{G}_m \) is affine. By construction, we have a commutative diagram

\[
\begin{array}{ccc}
W(P, d) & \xrightarrow{\rho} & P / \mathbb{G}_m \\
\downarrow j & & \downarrow \theta \\
\mathbb{A}_x^2 & \xrightarrow{f} & \mathbb{P}^1 \\
\downarrow i & & \downarrow q \\
\mathbb{A}_y^2 & \xrightarrow{\overline{\rho}} & \mathbb{P}^1
\end{array}
\]

in which each square is cartesian. In other words, \( W(P, d) \) is obtained from the \( \mathbb{G}_m \)-torsor \( P \to \mathbb{A}_x^2 \) by “adding the zero section”. The open embedding \( j : P \to W(P, d) \) is equivariant for the \( \mathbb{G}_a \)-action on \( W(P, d) \) induced by the canonical global section of \( \mathcal{O}_{\mathbb{A}_x^2}(dE) \) with divisor \( dE \) (see Proposition 1.7). By Theorem 2.5, \( \theta : W(P, d) \to \mathbb{A}_x^2 \) is the unique \( \mathbb{G}_a \)-extension of \( \rho : P \to \mathbb{A}_x^2 \) with affine total space.

In the simplest case \( d = 2 \), the unique homogeneous \( \mathbb{G}_a, \mathbb{A}_x^2 \)-torsor is the geometric quotient \( \text{SL}_2 \to \text{SL}_2 / \mathbb{G}_a \) of the group \( \text{SL}_2 \) by the action of its subgroup of upper triangular unipotent matrices equipped with the diagonal \( \mathbb{G}_m \)-action, and we recover Example 2.1.

2.3.2. General case. Here, given an arbitrary non trivial \( \mathbb{G}_a \)-torsor \( \rho : P \to \mathbb{A}_x^2 \), we describe a procedure to explicitly determine the unique \( \mathbb{G}_a \)-extension \( \theta : W \to \mathbb{A}_x^2 \) of \( P \) with affine total space \( W \) from a \( \check{C}ech \) 1-cocycle \( x^{-m}y^{-n}p(x, y) \), where \( m, n \geq 0 \) and \( p(x, y) \in k[x, y] \) is a non zero polynomial of degree \( r \leq m + n - 2 \), representing the isomorphism class \( g \in H^1(\mathbb{A}_x^2, \mathcal{O}_{\mathbb{A}_x^2}) \) of \( P \) on the open covering of \( \mathbb{A}_x^2 \) by the principal open subsets \( \mathbb{A}_x^2 \) and \( \mathbb{A}_y^2 \).
Write $p(x, y) = p_d + p_{d+1} + \cdots + p_r$ where the $p_i \in k[x, y]$ are the homogeneous components of $p,$ and $p_d \neq 0.$ In the decomposition
\[
H^1(\mathbb{A}_k^2, \mathcal{O}_{\mathbb{A}_k^2}) \cong H^1(\mathbb{P}^1, q_*\mathcal{O}_{\mathbb{A}_k^2}) \cong \bigoplus_{s \in \mathbb{Z}} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(s))
\]
a non zero homogeneous component $x^{-m}y^{-n}p_i$ of $x^{-m}y^{-n}p(x, y)$ corresponds to a non zero element of $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-m-n+i)).$ On the other hand, since for every $\ell \in \mathbb{Z}, \mathcal{O}_{\mathbb{A}_k^2}(\ell E) = f^*\mathcal{O}_{\mathbb{P}^1}(-\ell)$ and $f : \hat{A}^2 \rightarrow \mathbb{P}^1$ is the total space of the line bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$, it follows from the projection formula that
\[
H^1(\hat{A}^2, \mathcal{O}_{\hat{A}^2}(\ell E)) \cong H^1(\mathbb{P}^1, f_*\mathcal{O}_{\hat{A}^2} \otimes \mathcal{O}_{\mathbb{P}^1}(-\ell)) \cong \bigoplus_{t \geq 0} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(t-\ell)).
\]
The image of $x^{-m}y^{-n}p(x, y)$ in $\bigoplus_{t \geq 0} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(s))$ belongs to $\bigoplus_{t \geq 0} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(t-\ell))$ if and only if $\ell \geq \ell_0 = m+n-d \geq 2.$

The image of $x^{-m}y^{-n}p(x, y)$ in $\bigoplus_{t \geq 0} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(s))$ belongs to $\bigoplus_{t \geq 0} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(t-\ell))$ if and only if $\ell \geq \ell_0 = m+n-d \geq 2.$

The image of $x^{-m}y^{-n}p(x, y)$ in $\bigoplus_{t \geq 0} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(s))$ belongs to $\bigoplus_{t \geq 0} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(t-\ell))$ if and only if $\ell \geq \ell_0 = m+n-d \geq 2.$ Given such an $\ell$, the image $(h_1, \ell) \in \bigoplus_{t \geq 0} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(t-\ell))$ of $x^{-m}y^{-n}p(x, y)$ then defines a unique $M(\ell)$-torsor $\theta_\ell : W(P, \ell) \rightarrow \hat{A}^2$ whose restriction over the complement of $E$ is isomorphic to $\rho : P \rightarrow \mathbb{A}_k^2$ when equipped with the action $G_w$-action induced by the canonical section of $O_{\mathbb{A}_k^2}(\ell E)$ with divisor $E$. On the other hand, the restriction of $W|_E \rightarrow E$ over $E$ is an $\mathcal{O}_{\mathbb{P}^1}(-\ell)$-torsor with isomorphism class $h_0 \in H^1(P, \mathcal{O}_{\mathbb{P}^1}(-\ell)).$ By definition, $h_0$ is non zero if and only if $\ell = \ell_0$, and we conclude from Theorem 2.5 that $\theta_{\ell_0} : W(P, \ell_0) \rightarrow \hat{A}^2$ is the unique $G_w$-extension of $\rho : P \rightarrow \mathbb{A}_k^2$ with affine total space.

3. Quasi-projective $G_w$-Extensions of Type II

In this section we consider the following subclass of extensions of Type II of a $G_w$-torsor over a punctured surface.

**Definition 3.1.** A $G_w$-extension $\pi : X \rightarrow S$ of a $G_w$-torsor $\rho : P \rightarrow S$, over a punctured surface $S, S = S \setminus \{0\}$ is said to be a quasi-projective extension of Type II if it satisfies the following properties

i) $X$ is quasi-projective over $S$ and the $G_w$-action on $X$ is proper,

ii) $X$ is smooth along $\pi^{-1}(0)$ and $\pi^{-1}(0)_{red} \cong \mathbb{A}^2_{K}$.

**Example 3.2.** Let $o = V(x, y)$ be a global scheme-theoretic complete intersection closed point in the smooth locus of a surface $S$ and let $\rho : P \rightarrow S \setminus \{0\}$ be the $G_w$-torsor with $\rho$ defines certain $\rho$-ideals $(xv - yu - 1) \subset \mathcal{O}_S[u, v]$ as in Example 2.1. Let $\pi_1 : X_1 \rightarrow S$ be the affine $S$-scheme with $\pi_1$-ideals $(xw - y(z_1 + 1), xz_2 - z_1, yz_1 - z_2, z_1, z_2, u, w)$. The morphism of $S$-schemes $j_1 : P \rightarrow X_1$ defined by $(x, y, u, v) \mapsto (x, y, u, v, yu)$ is an open embedding, equivariant for the $G_w$-action on $X_1$ associated with the locally nilpotent $\mathcal{O}_S$-derivation $x\partial_{z_1} + (2yz_1 + 1)\partial_{z_2} + y\partial_w$ of $\mathcal{O}_{X_1}.$

The fiber $\pi^{-1}(0)$ of $\pi_{1}$-extension is isomorphic to $\mathbb{A}^2_{K}$ isomorphism to $\mathbb{A}^2_{K} = \text{Spec}(K[z_2, w])$ on which the $G_w$-action restricts to $G_w$-action by translations associated to the derivation $\partial_{z_2}$ of $\mathcal{O}_S$. It is straightforward to check that $X_1$ is smooth along $\pi^{-1}(0).$ We claim that the geometric quotient of the $G_w$-action on $X_1$ is isomorphic to the complement of a $\kappa$-rational point $o_1$ in the blow-up $\tau : \tilde{S} \rightarrow S$ of $S$. Such a surface being in particular separated, the $G_w$-action on $X_1$ is proper, implying that $j_1 : P \rightarrow X_1$ is a quasi-projective extension of $P$ of Type II.

Indeed, let us identify $\tilde{S}$ with the closed subvariety of $S \times_k \text{Proj}(k[u_0, u_1])$ with equation $xu_1 - yu_0 = 0$ in such a way that $\tau$ coincides with the restriction of the first projection. The morphism $f : X_1 \rightarrow \tilde{S}$ defined by
\[
(x, y, z, u, v) \mapsto ((x, y), [x : y]) = ((x, y), [yz_1 + 1 : w])
\]
is $G_w$-invariant and maps $\pi^{-1}(0)$ dominantly onto the exceptional divisor $E \cong \text{Spec}(K[z_2, w])$ of $\tau$. The induced morphism
\[
f|_{\pi^{-1}(0)} : \pi^{-1}(0) = \text{Spec}(K[z_2, w]) \rightarrow E, \quad (z_2, w) \mapsto [1 : w]
\]
factors as the composition of the geometric quotient $\pi^{-1}(0) \rightarrow \pi^{-1}(0)/G_w \cong \text{Spec}(K[w])$ with the open immersion $\pi^{-1}(0)/G_w \rightarrow E$ of $\pi^{-1}(0)/G_w$ as the complement of the $\kappa$-rational point $o_1 = ((0, 0), [0 : 1]) \in E$. On the other hand, the composition
\[
\tau \circ f \circ j_1 : P \xrightarrow{\sigma} X_1 \setminus \pi^{-1}(0) \rightarrow \tilde{S} \setminus S \setminus \{0\}
\]
coincides with the geometric quotient morphism $\rho : P \to S \setminus \{ o \}$. So $f : X_1 \to \tilde{S}$ factors through a surjective morphism $q : X_1 \to \tilde{S} \setminus \{ o_1 \}$ whose fibers all consist of precisely one $G_a$-orbit. Since $q$ is a smooth morphism, $q$ is a $G_a$-torsor which implies that $X_1/G_a \simeq \tilde{S} \setminus \{ o_1 \}$.

The scheme of the classification of quasi-projective extensions of Type II of a given $G_a$-torsor $\rho : P \to S_*$ which we give below is as follows: we first construct in §3.1 families of such extensions, in the form of $G_a$-torsors $q : X \to S'$ over quasi-projective $S$-schemes $\tau : S' \to S$ such that $\tau^{-1}(o)_{\text{red}}$ is isomorphic to $\mathbb{A}^1_k \setminus \{ o \}$, $S'$ is smooth along $\tau^{-1}(o)$, and $\pi : S' \setminus \tau^{-1}(o) \to S_*$ is an isomorphism. We then show in §3.2 that for quasi-projective $G_a$-extension $\pi : X \to S$ of Type II of a given $G_a$-torsor $\rho : P \to S_*$, the structure morphism $\pi : X \to S$ factors through a $G_a$-torsor $q : X \to S'$ over one of these $S$-schemes $S'$. In the last subsection, we focus on the special case where $\pi : X \to S$ has the stronger property of being an affine morphism.

3.1. A family of $G_a$-extensions over quasi-projective $S$-schemes.

Let again $(S,o)$ be a pair consisting of a surface and a closed point $o$ contained in the smooth locus of $S$, with residue field $\kappa$. We let $\tau_1 : \overline{S}_1 \to S$ be the blow-up of $o$, with exceptional divisor $\overline{E}_1 \simeq \mathbb{P}^1_\kappa$. Then for every $n \geq 2$, we let $\tau_n : \overline{S}_n = \overline{S}_n(o_1, \ldots, o_{n-1}) \to \overline{S}_1$ be the scheme obtained from $\overline{S}_1$ by performing the following sequence of blow-ups of $\kappa$-rational points:

a) The first step $\tau_{2,1} : \overline{S}_2(o_1) \to \overline{S}_1$ is the blow-up of a $\kappa$-rational point $o_1 \in \overline{E}_1$ with exceptional divisor $\overline{E}_2 \simeq \mathbb{P}^1_\kappa$.

b) Then for every $2 \leq i \leq n - 2$, we let $\tau_{i+1} : \overline{S}_i = \overline{S}_i(o_1, \ldots, o_i) \to \overline{S}_k(o_1, \ldots, o_i)$ be the blow-up of a $\kappa$-rational point $o_i \in \overline{E}_i$, with exceptional divisor $\overline{E}_{i+1} \simeq \mathbb{P}^1_\kappa$.

c) Finally, we let $\tau_{n,n-1} : \overline{S}_n(o_1, \ldots, o_{n-1}) \to \overline{S}_n(o_1, \ldots, o_{n-2})$ be the blow-up of a $\kappa$-rational point $o_{n-1} \in \overline{E}_n$ which is the smooth point of the reduced total transform of $\overline{E}_1$ by $\tau_1 \circ \cdots \circ \tau_{n-1,n-2}$.

We let $\overline{E}_n \simeq \mathbb{P}^1_\kappa$ be the exceptional divisor of $\tau_{n,n-1}$ and we let

\[ \tau_{n,n-1} = \tau_{2,1} \circ \cdots \circ \tau_{n,n-1} : \overline{S}_n(o_1, \ldots, o_{n-1}) \to \overline{S}_1. \]

The inverse image of $o$ in $\overline{S}_n(o_1, \ldots, o_{n-1})$ by $\tau_1 \circ \tau_{n,n-1}$ is a tree of $\kappa$-rational curves in which $\overline{E}_n$ intersects the reduced proper transform of $\overline{E}_1 \cup \cdots \cup \overline{E}_{n-1}$ in $\overline{S}_n(o_1, \ldots, o_{n-1})$ transversally in a unique $\kappa$-rational point.

\[ \begin{array}{ccccccc} 
\overline{E}_1 & \overline{E}_1 & \overline{E}_1 & \overline{E}_1 & \overline{E}_1 & \overline{E}_1 & \overline{E}_1 \\
\tau_{2,1} & \tau_{3,2} & \tau_{4,3} & \tau_{5,4} \\
\overline{S}_2(o_1) & \overline{S}_3(o_1,o_2) & \overline{S}_4(o_1,o_2,o_3) & \overline{S}_5(o_1,o_2,o_3,o_4) \\
\end{array} \]

\textbf{Figure 3.1.} The successive total transforms of $\overline{E}_1$ in a possible construction of a surface of the form $\overline{S}_5(o_1, \ldots, o_4)$ over a $k$-rational point $o$. The integers indicate the self-intersections of the corresponding curves.

\textbf{Notation 3.3.} For every $\kappa$-rational point $o_1 \in \overline{E}_1$, we let $S_1(o_1) = \overline{S}_1 \setminus \{ o_1 \}$, $E_1 = \overline{E}_1 \cap S_1 \simeq \mathbb{A}^1_\kappa$ and we let $\tau_1 : S_1(o_1) \to S$ be the restriction of $\tau_1$.

For $n \geq 2$, we let $S_n(o_1, \ldots, o_{n-1}) = \overline{S}_n(o_1, \ldots, o_{n-1}) \setminus (\overline{E}_1 \cup \cdots \cup \overline{E}_{n-1})$ and $E_n = S_n(o_1, \ldots, o_{n-1}) \cap \overline{E}_n \simeq \mathbb{A}^1_\kappa$. We denote by $\tau_{n,1} : S_n(o_1, \ldots, o_{n-1}) \to \overline{S}_1$ the birational morphism induced by $\tau_{n,1}$ and we let $\tau_n = \tau_1 \circ \tau_{n,1} : S_n(o_1, \ldots, o_{n-1}) \to S$.

The following lemma summarizes some basic properties of the so-constructed $S$-schemes:

\textbf{Lemma 3.4.} For every $n \geq 1$, the following hold for $S_n = S_n(o_1, \ldots, o_{n-1})$:

a) $\tau_{n,1} : S_n \to S$ is quasi-projective and restricts to an isomorphism over $S_*$ while $\tau_{n,1}^{-1}(o)_{\text{red}} = E_n$,

b) $S_n$ is smooth along $\tau_{n,1}^{-1}(o)$. 
c) \( \tau^n_n : \Gamma(S, O_S) \to \Gamma(S_n, O_{S_n}) \) is an isomorphism. Moreover for \( n \geq 2 \), the morphism \( \tau_{n,1} : S_n \to \overline{S}_1 \) is affine.

**Proof.** The first three properties are straightforward consequences of the construction. For the last one, let \( D = \overline{E}_1 + \sum_{i=2}^{n-1} a_i \overline{E}_i \) where \( a_i \) is a sequence of positive rational numbers and let \( m \geq 1 \) be so that \( mD \) is a Cartier divisor on \( S_n \). Then a direct computation shows that the restriction of \( O_{\overline{S}_1}(mD) \) to \( \overline{\tau}^{-1}(S_1) = \bigcup_{i=1}^{n-1} \overline{E}_i \) is an ample invertible sheaf provided that the sequence \( (a_i)_{i=2}^{n-1} \) decreases rapidly enough with respect to the distance of \( \overline{E}_i \) to \( \overline{E}_1 \) in the dual graph of \( \overline{E}_1 \cup \cdots \cup \overline{E}_{n-1} \). Since \( \tau_{n,1} \) restricts to an isomorphism over \( \overline{S}_1 \setminus \{o_1\} \), it follows from [7, Théorème 4.7.1] that \( O_{\overline{S}_1}(mD) \) is \( \tau_{n,1} \)-ample on \( S_n \). Since by definition \( \tau_{n,1} \) is the restriction of the projective morphism \( \tau_{n,1} : S_n \to \overline{S}_1 \) to \( S_n = \overline{S}_n \setminus \overline{E}_1 \cup \cdots \cup \overline{E}_{n-1} = \overline{S}_1 \setminus \text{Supp}(D) \), we conclude that \( \tau_{n,1} \) is an affine morphism. \( \square \)

**Remark 3.5.** By construction, \( \tau^{-1}_1(o) = E_1 \) in \( S_1(o_1) \), but for \( n \geq 2 \), we have \( \tau^{-1}_n(o) = mE_n \) for some integer \( m \geq 1 \) which depends on the sequence of \( \kappa \)-rational points \( o_1, \ldots, o_{n-1} \) blown-up to construct \( S_n(o_1, \ldots, o_{n-1}) \). For instance, it is straightforward to check that \( m = 1 \) if and only if for every \( i \geq 1 \), \( o_i \in \overline{E}_1 \) is a smooth point of the reduced total transform of \( \overline{E}_1 \) in \( \overline{S}_1(o_1, \ldots, o_{n-1}) \).

The structure morphism of a \( G_a \)-torsor being affine, hence quasi-projective, the total space of any \( G_a \)-torsor \( q : X \to S_n \) over an \( S \)-scheme \( \tau_n : S_n = (S_n(o_1, \ldots, o_n)) \to S \) is a quasi-projective \( S \)-scheme equipped with a proper \( G_a \)-action. Furthermore \( \pi -1(\mathcal{O}_{E_n}) = \mathcal{O}_E \simeq E_n \times E_n \) and \( X \) is smooth along \( \pi^{-1}(a) \) as \( S_n \) is smooth along \( E_n \). On the other hand, \( \pi : X \to S \) is a construction a \( G_a \)-extension of its restriction \( \rho : P \to S_n \setminus E_n \simeq S \) over \( S_n \setminus E_n \), hence is a quasi-projective \( G_a \)-extension of \( P \) of Type II. The following proposition shows conversely that every \( G_a \)-torsor \( \rho : P \to S_n \) admits a quasi-projective \( G_a \)-extension of Type II into a \( G_a \)-torsor \( q : X \to S_n \).

**Proposition 3.6.** Let \( \rho : P \to S_n \) be a \( G_a \)-torsor. Then for every \( n \geq 1 \) and every \( S \)-scheme \( \tau_n : S_n(o_1, \ldots, o_{n-1}) \to S \) as in Notation 3.3 there exists a \( G_a \)-torsor \( q : X \to S_n(o_1, \ldots, o_{n-1}) \) and an equivariant open embedding \( j : P \to X \) such that in the following diagram

\[
\begin{array}{ccc}
P & \xrightarrow{j} & X \\
\downarrow{\rho} & & \downarrow{q} \\
S_n(o_1, \ldots, o_{n-1}) \setminus E_n & \xrightarrow{j} & S_n(o_1, \ldots, o_{n-1}) \\
\tau_n \downarrow & & \tau_n \\
S_n & \to & S
\end{array}
\]

all squares are cartesian. In particular, \( j : P \to X \) is a quasi-projective \( G_a \)-extension of \( P \) of Type II.

**Proof.** Letting \( S_n = S_n(o_1, \ldots, o_n) \), we have to prove that every \( G_a \)-torsor \( \rho : P \to S_n \setminus E_n \simeq S_n \) is the restriction of a \( G_a \)-torsor \( q : X \to S_n \), or equivalently that the restriction homomorphism \( H^1(S_n, O_{S_n}) \to H^1(S_n \setminus E_n, O_{S_n \setminus E_n}) \) is surjective. It is enough to show that there exists a Zariski open neighborhood \( U \) of \( E_n \) in \( S_n \) and a \( G_a \)-torsor \( q : Y \to U \) such that \( Y \mid U \setminus E_n \simeq P \mid U \setminus E_n \). Indeed, if so then a \( G_a \)-torsor \( q : X \to S_n \) with the desired property is obtained by gluing \( P \) and \( Y \) over \( U \setminus E_n \) by the isomorphism \( Y \mid U \setminus E_n \simeq P \mid U \setminus E_n \). In particular, we can replace \( S_n \) by the inverse image by \( \tau_n : S_n \to S \) of any Zariski open neighborhood of \( o \) in \( S \). We can thus assume from the very beginning that \( S = \text{Spec}(\mathcal{A}) \) is affine and that \( o = V(f, g) \) is a scheme-theoretic intersection for some \( f, g \in A \). Up to replacing \( f \) and \( g \) by other generators of the maximal ideal of \( o \) in \( A \), we can assume that the proper transform \( L_1 = \overline{S}_1 \setminus S \) of the curve \( L = V(f) \subset S \) intersects \( \overline{E}_1 \) in \( o_1 \). We denote by \( M_1 \subset \overline{S}_1 \) the proper transform of the curve \( M = V(g) \subset S \).

We first treat the case \( n = 1 \). The open subset \( U_1 = \overline{S}_1 \setminus L \) of \( \overline{S}_1 \) is then affine and contained in \( S_1 \). Furthermore \( U_1 \setminus E_1 = \overline{S}_1 \setminus \overline{\tau}^{-1}(L) \simeq S \setminus L \) is also affine. The Mayer-Vietoris long exact sequence of cohomology of \( O_{S_1} \) for the open covering of \( S_1 \) by \( S_1 \setminus E_1 \) and \( U_1 \) then reads

\[
0 \to H^0(S_1, O_{S_1}) \to H^0(U_1, O_{S_1}) \oplus H^0(S_1 \setminus E_1, O_{S_1}) \to H^0(U_1 \setminus E_1, O_{S_1}) \to \cdots \\
\cdots \to H^1(S_1, O_{S_1}) \to H^1(U_1, O_{U_1}) \oplus H^1(S_1 \setminus E_1, O_{S_1}) \to H^1(U_1 \setminus E_1, O_{U_1}) \to \cdots
\]
Since $U_1 \setminus E_1$ is affine, $H^1(U_1 \setminus E_1, \mathcal{O}_{U_1}) = 0$ and so, the homomorphism $H^1(S_1, \mathcal{O}_{S_1}) \to H^1(S_1 \setminus E_1, \mathcal{O}_{S_1})$ is surjective as desired.

In the case where $n \geq 2$, the open subset $V_1 = \overline{S_1} \setminus M_1$ of $\overline{S_1}$ is affine and it contains $o_1$ since $M_1$ intersects $\overline{E_1}$ in a point distinct from $o_1$. Since $\tau_{n,1} : S_n \to \overline{S_1}$ is an affine morphism by Lemma 3.4, $U_n = \tau_{n,1}(U_1)$ is an affine open neighborhood of $E_n$ in $S_n$. By construction, $S_n$ is then covered by the two open subset $U_n$ and $S_n \setminus E_n$ which intersect along the affine open subset $U_n \cap S_n \setminus E_n = U_n \setminus E_n = \tau_{n,1}(S_1 \setminus \tau_{n,1}^{-1}(M))$ of $S_n$. The conclusion then follows from the Mayer-Vietoris long exact sequence of cohomology of $\mathcal{O}_{S_n}$ for the open covering of $S_n$ by $S_n \setminus E_n$ and $U_n$. □

3.2. Classification.

The following theorem shows that every quasi-projective $\mathbb{G}_a$-extension of Type II of a given $\mathbb{G}_a$-torsor $\rho : P \to S$, is isomorphic to one of the schemes $q : X \to S$ constructed in § 3.1.

**Theorem 3.7.** Let $\rho : P \to S$ be a $\mathbb{G}_a$-torsor and let

$$
\begin{array}{ccc}
P & \xrightarrow{j} & X \\
\rho \downarrow & & \downarrow \pi \\
S & \xrightarrow{\delta} & S
\end{array}
$$

be a quasi-projective $\mathbb{G}_a$-extension of $P$ of Type II. Then there exists an integer $n \geq 1$ and a scheme $\tau_n : S_n(o_1, \ldots, o_{n-1}) \to S$ such that $X$ is a $\mathbb{G}_a$-torsor $q : X \to S_n(o_1, \ldots, o_{n-1}) \simeq X/\mathbb{G}_a$ and $\rho : P \to S$ coincides with the restriction of $q$ to $S_n(o_1, \ldots, o_{n-1}) \setminus E_n$.

**Proof.** Since the $\mathbb{G}_a,S$-action on $X$ is proper, the geometric quotient $X/\mathbb{G}_a,S$ exists in the form of a separated algebraic $S$-space $\delta : X/\mathbb{G}_a,S \to S$. Furthermore, since by definition of an extension $\pi^{-1}(S_1) \simeq P$, we have $\pi^{-1}(S_1)/\mathbb{G}_a,S \simeq P/\mathbb{G}_a,S \simeq S$, and so $\delta$ restricts to an isomorphism over $S$. On the other hand, $\pi^{-1}(o) \simeq \mathbb{A}^2_G$ is equipped with the induced proper $\mathbb{G}_a,S$-action, whose geometric quotient $\mathbb{A}^2_G/\mathbb{G}_a,S$ is isomorphic to $\mathbb{A}^1_G$. It follows from the universal property of geometric quotient that $\delta^{-1}(o) = \mathbb{A}^2_G/\mathbb{G}_a,S = \mathbb{A}^1_G$.

Since $X$ is smooth in a neighborhood of $\pi^{-1}(o)$, $X/\mathbb{G}_a,S$ is smooth in neighborhood of $\delta^{-1}(o)$. In particular, $\pi^{-1}(o)$ and $\delta^{-1}(o)$ are Cartier divisors on $X$ and $X/\mathbb{G}_a,S$ respectively. Let $\tau_1 : \overline{S_1} \to S$ be the blow-up of $o$. Then by the universal property of blow-ups [14, Tag 085P], the morphisms $\pi : X \to S$ and $\delta : X/\mathbb{G}_a,S \to S$ lift to morphisms $\pi_1 : X \to \overline{S_1}$ and $\delta_1 : X/\mathbb{G}_a,S \to \overline{S_1}$ respectively, and we have a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\pi_1} & \overline{S_1} \\
\downarrow \delta_1 & & \downarrow \tau_1 \\
X/\mathbb{G}_a,S & \xrightarrow{\delta} & S
\end{array}
$$

Furthermore, since $\delta : X/\mathbb{G}_a,S \to S$ and $\tau_1 : \overline{S_1} \to S$ are separated, it follows that $\delta_1 : X/\mathbb{G}_a,S \to \overline{S_1}$ is separated. By construction, the image of $\pi^{-1}(o)/\mathbb{G}_a,S$ by $\delta_1$ is contained in $\overline{E_1}$.

If $\delta_1$ is not constant on $\pi^{-1}(o)/\mathbb{G}_a,S$, then $\delta_1$ is a separated quasi-finite birational morphism. Since $\overline{S_1}$ is normal, $\delta_1$ is thus an open immersion by virtue of Zariski Main Theorem for algebraic spaces [14, Tag 05W7]. Since $\pi^{-1}(o)/\mathbb{G}_a,S \simeq \mathbb{A}^1_G$, the only possibility is that $\overline{S_1} \setminus \delta_1(X/\mathbb{G}_a,S)$ consists of a unique $\kappa$-rational point $o_1 \in \overline{E_1}$ and $\delta_1 : X/\mathbb{G}_a,S \to S_1(o_1) = \overline{S_1} \setminus \{o_1\}$ is an isomorphism. So $\pi_1 : X \to S_1(o_1)$ is a $\mathbb{G}_a$-torsor whose restriction to $S_1(o_1) \setminus E_1 \simeq S$ coincides with $\rho : P \to S$.

Otherwise, if $\delta_1$ is constant on $\pi^{-1}(o)/\mathbb{G}_a,S$, then its image consists of a unique $\kappa$-rational point $o_1 \in \overline{E_1}$. The same argument as above implies that $\pi_1 : X \to \overline{S_1}$ and $\delta_1 : X/\mathbb{G}_a,S \to \overline{S_1}$ lift to a $\mathbb{G}_a,S$-invariant morphism $\pi_2 : X \to \overline{S_2}(o_1)$ and a separated morphism $\delta_2 : X/\mathbb{G}_a,S \to \overline{S_2}(o_1)$ to the blow-up $\overline{\tau_2} : \overline{S_2}(o_1) \to \overline{S_1}$ of $\overline{S_1}$ at $o_1$, with exceptional divisor $\overline{E_2}$. If the restriction of $\delta_2$ to $\pi^{-1}(o)/\mathbb{G}_a,S$ is not constant then $\delta_2$ is an open immersion and the image of $\pi^{-1}(o)/\mathbb{G}_a,S$ is an open subset of $\overline{E_2}$ isomorphic to $\mathbb{A}^1_G$. The only possibility is that $\overline{\delta_2}(\pi^{-1}(o)/\mathbb{G}_a,S) = \overline{E_2} \setminus \overline{E_1}$. Indeed, otherwise $\overline{S_2} \setminus \overline{\delta_2}(X/\mathbb{G}_a,S)$ would consist of the disjoint union of a point in $\overline{E_2} \setminus (\overline{E_1} \cap \overline{E_2})$ and of the curve $\overline{E_1} \setminus (\overline{E_1} \cap \overline{E_2})$ which is not closed in $\overline{S_2}$, in contradiction to the fact that $\delta_2$ is an open immersion. Summing up, $\delta_2 : X/\mathbb{G}_a,S \to S_2(o_1) = \overline{S_2}(o_1) \setminus \overline{E_1}$ is an isomorphism mapping $\pi^{-1}(o)/\mathbb{G}_a,S$ isomorphically onto $\overline{E_2}$. So $\pi_2 : X \to S_2(o_1)$ is a $\mathbb{G}_a$-torsor whose restriction to $S_2(o_1) \setminus \overline{E_2} \simeq S$ coincides with $\rho : P \to S$.
Otherwise, if $\delta_2$ is constant on $\pi^{-1}(o)_{\text{red}}/\mathbb{G}_{a,\kappa}$, then $\delta_2(\pi^{-1}(o)/\mathbb{G}_{a,\kappa})$ is a $\kappa$-rational point $o_2 \in \mathbb{F}_2$, and there exists a unique minimal sequence of blow-ups $\mathbb{F}_{k+1:k} : \mathbb{F}_{k+1}(0_1, \ldots, 0_k) \to \mathbb{F}_k(0_1, \ldots, 0_{k-1}), k = 2, \ldots, m-1$ of successive $\kappa$-rational points $o_k \in \mathbb{F}_k \subset \mathbb{F}_{k+1}(0_1, \ldots, 0_{k-1})$, with exceptional divisor $\mathbb{F}_{k+1} \subset \mathbb{F}_{k+1}(0_1, \ldots, 0_k)$ such that $\pi_2 : X \to \mathbb{F}_2(o_1)$ and $\delta_2 : X/\mathbb{G}_{a,\kappa} \to \mathbb{F}_2(o_1)$ lift respectively to a $\mathbb{G}_{a,S}$-invariant morphism $\pi_2 : X \to \mathbb{F}_2(o_1, \ldots, o_{m-1})$ and a separated morphism $\delta_2 : X/\mathbb{G}_{a,S} \to \mathbb{F}_2(o_1, \ldots, o_{m-1})$ with the property that the restriction of $\delta_2$ to $\pi_2^{-1}(o_1)_{\text{red}}/\mathbb{G}_{a,\kappa}$ is non constant. By Zariski Main Theorem [14, Tag 05W7] again, we conclude that $\delta_2$ is an open immersion, mapping $\pi^{-1}(o)_{\text{red}}/\mathbb{G}_{a,\kappa} \simeq \mathbb{A}^1_{\mathbb{K}}$ isomorphically onto an open subset of $\mathbb{F}_m \simeq \mathbb{P}^1_{\mathbb{K}}$. As in the previous case, the image of $\pi^{-1}(o)_{\text{red}}/\mathbb{G}_{a,\kappa}$ in $\mathbb{F}_m$ must be equal to the complement of the intersection of $\mathbb{F}_m$ with the proper transform of $\mathbb{F}_1 \cup \cdots \cup \mathbb{F}_{m-1}$ in $\mathbb{F}_m(o_1, \ldots, o_{m-1})$ since otherwise $\mathbb{F}_m(o_1, \ldots, o_{m-1}) \setminus \delta_2(M/X_{\mathbb{G}_{a,S}})$ would not be closed in $\mathbb{F}_m(o_1, \ldots, o_{m-1})$. Since $\pi^{-1}(o)_{\text{red}}/\mathbb{G}_{a,\kappa} \simeq \mathbb{A}^1_{\mathbb{K}}$, it follows that $\mathbb{F}_m$ intersects the proper transform of $\mathbb{F}_1 \cup \cdots \cup \mathbb{F}_{m-1}$ in a unique $\kappa$-rational point, implying in turn that $\pi_2^{-1}(o_1)_{\text{red}}/\mathbb{G}_{a,\kappa}$ is a smooth $\kappa$-rational point of the reduced total transform $\mathbb{F}_1 \cup \cdots \cup \mathbb{F}_{m-1}$ of $\mathbb{F}_1$ in $\mathbb{F}_{m-1}(o_1, \ldots, o_{m-2})$. Summing up,

$$\delta_2 : X/\mathbb{G}_{a,S} \to \mathbb{F}_m(o_1, \ldots, o_{m-1}) \setminus \mathbb{F}_1 \cup \cdots \cup \mathbb{F}_{m-1}$$

is an isomorphism with an $S$-scheme of the form $\mathbb{F}_m(o_1, \ldots, o_{m-1})$ as constructed in §3.1, mapping $\pi^{-1}(o)_{\text{red}}/\mathbb{G}_{a,\kappa}$ isomorphically onto $E_m = \mathbb{F}_m(o_1, \ldots, o_{m-1}) \cap \mathbb{F}_m$. It follows in turn that $\pi_m : X \to \mathbb{F}_m(o_1, \ldots, o_{m-1})$ is a $\mathbb{G}_{a,S}$-torsor whose restriction to $\mathbb{F}_m(o_1, \ldots, o_{m-1}) \setminus \mathbb{F}_m \simeq S$ coincides with $\rho : P \to S$. This completes the proof. \hfill $\square$

3.3. Affine $\mathbb{G}_{a,S}$-extensions of Type II. In this subsection, given a $\mathbb{G}_{a}$-torsor $\rho : P \to S$, we consider the existence of quasi-projective $\mathbb{G}_{a}$-extensions of Type II

$$\begin{array}{ccc}
P & \xrightarrow{\rho} & X \\
\downarrow & & \downarrow \pi \\
S & \longrightarrow & S
\end{array}$$

with the additional for which $X$ is affine over $S$. As in the case of extension to $\mathbb{A}^1$-bundles over the blow-up of $o$ treated in § 2.2, a necessary condition for the existence of such extensions is that the restriction of $P$ over every open neighborhood of the closed point $o$ in $S$ is nontrivial. Indeed, if there exists an affine open neighborhood $U$ of $o$ over which $P$ is trivial, then $P \simeq U \setminus \{o\} \times \mathbb{A}^1_{\mathbb{K}}$ is strictly affine-hence, cannot be the complement of a Cartier divisor $\pi^{-1}(o)$ is any affine $U$-scheme $X|_U$. The next theorem shows that this condition is actually sufficient:

**Theorem 3.8.** Let $\rho : P \to S$ be a $\mathbb{G}_{a}$-torsor such that for every open neighborhood $U$ of $o$ in $S$, the restriction $P \times_S U \to U \setminus \{o\}$ is non trivial. Then for every $n \geq 1$ and every $S$-scheme $\tau_n : S_n(o_1, \ldots, o_{n-1}) \to S$ as in Notation 3.3 there exists a quasi-projective $\mathbb{G}_{a}$-extension of $P$ of Type II into the total space of a $\mathbb{G}_{a}$-torsor $\pi : X \to S_n(o_1, \ldots, o_{n-1})$ for which $\pi = \tau_n \circ q : X \to S$ is an affine morphism.

The following example illustrates the strategy of the proof given below, which consists in constructing such affine extensions $\pi : X \to S$ by performing a well-chosen equivariant affine modification of extensions of $\rho : P \to S$, into locally trivial $\mathbb{A}^1$-bundles $\theta : W(P) \to S$ over the blow-up $\tau : S \to S$ of the point $o$.

**Example 3.9.** Let again $X_0$ and $X_1$ be the $\mathbb{G}_{a,S}$-extensions of $\rho : P = \{xy-y=1\} \to S \setminus \{o\}$ considered in Example 2.1 and 3.2. Recall that $X_0$ and $X_1$ are the affine $S$-schemes in $\mathbb{A}_{\mathbb{K}}^3$ defined respectively by the equations

$$X_0 : \begin{cases} \quad x\tau - y\tau &= 0 \\ \quad y\tau - x\tau &= 0 \end{cases} \quad \text{and} \quad X_1 : \begin{cases} \quad x\tau - y\tau &= 0 \\ \quad y\tau - x\tau &= 0 \quad \text{and} \quad x\tau - y\tau &= 0 \end{cases}$$

equipped with the $\mathbb{G}_{a,S}$-actions associated with the locally nilpotent $S$-derivations $\partial_0 = x^2\partial_x + xy\partial_y + y^2\partial_y$, and $\partial_1 = x\partial_z + (2yz_1 + 1)\partial_{z_2} + y^2\partial_y$ respectively.

The morphism $\pi_0 : X_0 \to S$ factors through the structure morphism $\theta : X_0 \to S$ of a torsor under a line bundle on the blow-up $\tau : S \to S$ of the origin, with the property that the restriction of $X_0$ to exceptional divisor $E = \mathbb{P}^1_{\mathbb{K}}$ of $\tau$ is a nontrivial torsor under the total space of the line bundle $\mathcal{O}_{\mathbb{P}^1_{\mathbb{K}}}(-2)$. 
The $G_{a,S}$-action on $X_0$ restricts to the trivial one on $X_0|_E = \pi_0^{-1}(o)$. More precisely, $\theta_0$ is a global section of the sheaf $\mathcal{T}_{X_0} \otimes \mathcal{O}_{X_0}(-2X_0|_E)$ of vector fields on $X_0$ that vanish at order 2 along $X_0|_E$. One way to obtain from $X_0$ a $G_{a,S}$-extension $\pi : X \to S$ of $\rho : P \to S \setminus \{o\}$ is to perform an equivariant affine modification which simultaneously replaces $X_0|_E$ by a copy of $\mathbb{A}^2$ and decreases the “fixed point order of $\theta_0$ along $X_0|_E$.” Typically a modification with divisor $D$ equal to $X_0|_E$ and whose center $Z \subset X_0|_E$ is supported by a curve isomorphic to $\mathbb{A}^1$ which is mapped isomorphically onto its image by the restriction of $\theta$. The birational $S$-morphism

$$\eta : X_1 \to X_0, \quad (x, y, z_1, z_2, w) \mapsto (x, y, xz_1, yz_1 + 1, w)$$

is equivariant for the $G_{a,S}$-actions on $X_0$ and $X_1$ and corresponds to an equivariant affine modification of this type: it restricts to an isomorphism outside the fibers of $\pi_0$ and $\pi_1$ over $o$, and it contracts $\pi_1^{-1}(o) = \text{Spec}(k[z_2, w])$ onto the curve $\{p = q - 1 = 0\} \subset \pi_0^{-1}(o) = \{pr - q(q - 1) = 0\}$. This curve is isomorphic to $\mathbb{A}^1 = \text{Spec}(k[\tau])$ and it is mapped by the restriction

$$\theta_{|\pi_0^{-1}(o)} : \pi_0^{-1}(o) \cong \{pr - q(q - 1) = 0\} \to E = \mathbb{P}^1, \quad (p, q, r) \mapsto [p : q - 1] = [q : r]$$

of $\theta$ isomorphically onto the complement of the $\kappa$-rational point $[0 : 1] \in \mathbb{P}^1$. 

Proof of Theorem 3.8. By virtue of Theorem 2.5, there exists a unique integer $\ell_0 \geq 2$ such that $\rho : P \to S_s$ is the restriction of a $\tau_0 : W_1 \to \mathbb{F}^1$ under the line bundle $M_1(\ell_0) = \text{Spec}(\text{Sym} \, \mathcal{O}_{\mathbb{F}^1}(-\ell_0 \mathbb{F}_1)) \to \mathbb{F}^1$ whose total space $W_1$ is affine over $\mathbb{F}^1$. We now treat the case of $S_1(o_1)$ and $S_n(o_1, \ldots, o_{n-1})$, $n \geq 2$ separately. 

Given a $\kappa$-rational point $o_1 \in \mathbb{F}^1$, the restriction of $W_1$ over $E_1 = \mathbb{F}^1 \setminus \{o_1\} \cong \mathbb{A}^1$ is the trivial $\mathbb{A}^1$-bundle $E_1 \times \mathbb{A}^1$. Since on the other hand the restriction $\theta_{|\mathbb{F}^1 \setminus \{o_1\}} : W_1|_{\mathbb{F}^1 \setminus \{o_1\}} \to \mathbb{F}^1$ is a non trivial $\mathbb{O}_{\mathbb{F}^1}(-\ell_0)$-torsor (see Theorem 2.5), it follows that for every section $s : \mathbb{F}^1 \to W_1|_{\mathbb{F}^1 \setminus \{o_1\}}$ the image $Z_1$ of $W_1$ in $W_1|_{\mathbb{F}^1}$ is a closed curve isomorphic to $E_1$. Indeed, otherwise if $Z_1$ is not closed in $W_1|_{\mathbb{F}^1}$ then its closure $\overline{Z_1}$ is a section of $\theta_{|\mathbb{F}^1 \setminus \{o_1\}}$ in contradiction with the fact that $\theta_{|\mathbb{F}^1 \setminus \{o_1\}} : W_1|_{\mathbb{F}^1 \setminus \{o_1\}} \to \mathbb{F}^1$ is a non trivial $\mathbb{O}_{\mathbb{F}^1}(-\ell_0)$-torsor. 

Let $D_1 = \theta_1^{-1}(\mathbb{F}^1)$ and let $\sigma_1 : W_1' \to W_1$ be the affine modification of $W_1$ with center $(Z_2, D_1)$. By virtue of Lemmas 1.9 and 1.10, $\tau_1 \circ \sigma_1 : W_1' \to \mathbb{F}^1$ factors through a $\tau_1' : W_1' \to \mathbb{F}^1$ under the line bundle

$$M_1'(\ell_0 - 1) = \text{Spec}(\text{Sym} \, \mathcal{O}_{S_1(o_1)}((-\ell_0 + 1)E_1)) \to S_1(o_1).$$

Now since $E_1 \cong \mathbb{A}^1$ is affine, the restriction of $\theta_1'$ over $E_1 \subset S_1(o_1)$ is the trivial $M_1'(\ell_0 - 1)|_{E_1}$-torsor. Letting $D_2 = \theta_1'^{-1}(E_1)$ and $Z_2 \subset D_2$ be any section of $\theta_1'|_{D_2} : D_2 \to E_1$, the affine modification $\sigma_2 : W_2' \to W_1'$ with center $(Z_2, D_2)$ is then an $M_1'(\ell_0 - 2)$-torsor $\theta_2' : W_2' \to S_1(o_1)$. Iterating this construction $\ell_0 - 1$ times, we reach a $G_{a,S_1(o_1)}$-torsor $q = \theta_{\ell_0 + 1} : X = W_{\ell_0 + 1} \to S_1(o_1)$. Since $\sigma_1 : W_1' \to W_1$ and each $\sigma_i : W_i' \to W_{i-1}'$, $i \geq 2$, restricts to an isomorphism over the complement of $E_1$, the restriction of $q : X \to S_1(o_1)$ over $S_1(o_1) \setminus E_1 \cong S_s$ is isomorphic to $\rho : P \to S_s$. Furthermore, since the morphisms $\sigma_i$, $i = 1, \ldots, \ell_0 + 1$ are affine and $\tau_1 \circ \theta : W_1 \to S$ is an affine morphism, it follows that

$$\tau_1 \circ q = \tau_1 \circ \theta \circ \sigma_1 \circ \cdots \sigma_{\ell_0 + 1} : X \to S$$

is an affine morphism. So $q : X \to S_1(o_1)$ is a $G_{a,S}$-extension of $\rho : P \to S_s$ with the desired property. 

Now suppose that $n \geq 2$. It follows from the construction of the morphism $\tau_{n, 1} : S_n = S_n(o_1, \ldots, o_{n-1}) \to \mathbb{F}_n$ given in subsection 3.1 that $\tau_{n, 1}^* \mathcal{O}_{\mathbb{F}_n}(\ell_0 \mathbb{F}_1) \cong \mathcal{O}_{S_n}(mE_n)$ for some $m \geq 2$. The fiber product $W_n = W_1 \times_{\mathbb{F}^1} S_n$ is thus a torsor $\theta_n : W_n \to S_n$ under the line bundle

$$M_n(m) = \text{Spec}(\text{Sym} \, \mathcal{O}_{S_n}(-mE_n)) \to S_n$$

whose restriction to $S_n \setminus E_n \cong S_s$ is isomorphic to $\rho : P \to S_s$. Furthermore, since $\tau_{n, 1}$ is an affine morphism by virtue of Lemma 3.4, so is the projection $p_{W_n} : W_n \to W_1$. Since $\tau_1 \circ \theta : W_1 \to S$ is an affine morphism, we conclude that $\tau_n \circ \theta_n = \tau_1 \circ \tau_{n, 1} \circ \theta = \tau_1 \circ \theta \circ p_{W_1} : W_n \to S$ is an affine morphism as well. Since $E_n \cong \mathbb{A}^1$, the restriction of $\theta_n$ over $E_n$ is the trivial $M_n(m)|_{E_n}$-torsor. The desired $G_{a,S_n}$-torsor $q : X \to S_n$ extending $\rho : P \to S_n$ is then obtained from $\theta_n : W_n \to S_n$ by performing a sequence of $m$ successive affine modifications similar to those applied in the previous case. 

Remark 3.10. In the case where $S$ is affine, the total spaces $X$ of the varieties $q : X \to S_n(o_1, \ldots, o_{n-1})$ of Theorem 3.8 are all affine. To our knowledge, these are the first instances of smooth affine threefolds
equipped with proper $\mathbb{G}_a$-actions whose geometric quotients are smooth quasi-projective surfaces which are not quasi-affine.

We do not know in general if under the conditions of Theorem 3.8 every quasi-projective $\mathbb{G}_a$-extensions of $P$ of Type II into the total space of a $\mathbb{G}_a$-torsor $q : X \to S_n(o_1, \ldots, o_{n-1})$ has the property that $\pi = \tau_n \circ q : X \to S$ is an affine morphism. In particular, we ask the following:

**Question 3.11.** Is the total space $X$ of a quasi-projective $\mathbb{G}_a$-extension $\pi : X \to \mathbb{A}^2$ of $\rho = \text{pr}_{x,y} : SL_2 = \{xv - yu = 1\} \to \mathbb{A}^2$ of Type II always an affine variety?

**3.4. Examples.** In the next paragraphs, we construct two countable families of quasi-projective $\mathbb{G}_a$-extensions of the $\mathbb{G}_a$-torsor $SL_2 \to SL_2/\mathbb{G}_a \simeq \mathbb{A}^2 \setminus \{(0,0)\}$ of Type II with affine total spaces. As a consequence of [10, Section 3], for any nontrivial $\mathbb{G}_a$-torsor $\rho : P \to S_\ast$ over a local punctured surface $S_\ast$, these provide, by suitable base changes, families of examples of $\mathbb{G}_a$-extensions of $P$ whose total spaces are all affine over $S$.

3.4.1. A family of $\mathbb{G}_a$-extensions of $SL_2$ of “Type II-A”. Let $S = \mathbb{A}^2 = \text{Spec}(k[x,y])$ and let $X_n \subset \mathbb{A}^{n+2}_S = \text{Spec}(k[x,y][z_1,z_2,y_1,\ldots,y_n])$, $n \geq 1$, be the smooth threefold defined by the system of equations

$$
\begin{aligned}
    y_i y_j - y_k y_l &= 0, \quad i,j,k,l = 0, \ldots, n, \quad i + j = k + l \\
    z_2 y_i - z_1 y_{i+1} &= 0, \quad i = 0, \ldots, n - 1 \\
    x y_{i+1} - y_i (y_i + 1) &= 0, \quad i = 0, \ldots, n - 1 \\
    x z_2 - z_1 (y_i + 1) &= 0.
\end{aligned}
$$

The threefold $X_n$ can be endowed with a fixed point free $\mathbb{G}_a,S$-action induced by the locally nilpotent $k[x,y]$-derivation

$$
x \partial_{z_1} + (2y_0 + 1) \partial_{z_2} + \sum_{i=1}^n i y_i \partial_{y_{i-1}} \partial_{y_i}
$$

of its coordinate ring. The scheme-theoretic fiber over $o = \{(0,0)\}$ of the $\mathbb{G}_a$-invariant morphism $\pi_n = \text{pr}_{x,y} : X_n \to S$ is isomorphic $\mathbb{A}^2 = \text{Spec}(k[z_2,y_n])$, on which the induced $\mathbb{G}_a$-action is a translation induced by the derivation $\partial_{z_2}$ of $k[z_2,y_n]$. On the other hand, the morphism $j : SL_2 = \{xv - y_0 u = 1\} \to X_n$ defined by

$$(x,y,u,v) \mapsto (x,u,v,y,yv^2,\ldots,yv^n)$$

is an equivariant open embedding of $SL_2$ equipped with the $\mathbb{G}_a$-action induced by the locally nilpotent derivation $x \partial_{x} + y_0 \partial_y$ of its coordinate ring into $X_n$ with image equal to $\pi^{-1}(\mathbb{A}^2 \setminus \{o\})$. So $j : SL_2 \hookrightarrow X_n$ is a quasi-projective $\mathbb{G}_a$-extension of $SL_2$ into the affine variety $X_n$, with $\pi_n^{-1}(o) \simeq \mathbb{A}^2$. The restrictions of the projection $\mathbb{A}^{n+2}_S \to \mathbb{A}^{n+2}_S$ onto the first $n + 2$ variables induce a sequence of $\mathbb{G}_a$-equivariant birational morphisms $\sigma_{n+1,n} : X_{n+1} \to X_n$. The threefolds $X_n$ thus form a countable tower of $\mathbb{G}_a$-equivariant affine modifications of $X_1$. It follows from Example 3.2 that $X_1$ is a quasi-projective extension of $SL_2$ of Type II with geometric quotient isomorphic to a quasi-projective surface of the form $S_1(o_1)$. More generally, we have the following result.

**Proposition 3.12.** For every $n \geq 2$, the morphism $j : SL_2 \hookrightarrow X_n$ is a quasi-projective $\mathbb{G}_a$-extension of Type II. The geometric quotient $X_n/\mathbb{G}_a$ is isomorphic to a quasi-projective surface $S_n = S_n(o_1,\ldots,o_n)$ as in § 3.1 for which $\mathbb{G}_n(o_1,\ldots,o_{n-1}) \setminus S_n$ consists of a chain of $n - 1$ smooth rational curves with self-intersection $-2$, i.e. the exceptional set of the minimal resolution of a surface singularity of type $A_{n-1}$.

**Proof.** To see this, we consider the following sequence of blow-ups: the first one $\tau_1 : \mathbb{F}_1 \to \mathbb{F}_2 \to U_0 = \mathbb{A}^2$ is the blow-up of the origin, with exceptional divisor $\mathbb{E}_1$, and we let $U_1 \simeq \mathbb{A}^2 = \text{Spec}(k[x,w_1])$ be the affine chart of $\mathbb{F}_1$ on which $\tau_1 : \mathbb{F}_1 \to \mathbb{A}^2$ is given by $(x,w_1) \mapsto (x,xw_1)$. Then we let $\tau_{2,1} : \mathbb{F}_2(o_1) \to \mathbb{F}_1$ be the blow-up of the point $o_1 = (0,0) \in U_1 \subset \mathbb{F}_1$ with exceptional divisor $\mathbb{E}_2$, and we let $U_2 \simeq \mathbb{A}^2 = \text{Spec}(k[x,w_2])$ be the affine chart of $\mathbb{F}_2(o_1)$ on which the restriction of $\tau_{2,1} : \mathbb{F}_2(o_1) \to \mathbb{F}_1$ coincides with the morphism $U_2 \to U_1$, $(x,w_2) \mapsto (x,xw_2)$. For every $2 < m \leq n$, we define by induction the blow-up

$$
\tau_{m,m-1} : \mathbb{F}_m(o_1,\ldots,o_{m-1}) \to \mathbb{F}_{m-1}(o_1,\ldots,o_{m-2})
$$
of the point \( o_{m-1} = (0, 0) \in U_{m-1} \subset S_{m-1}(o_1, \ldots, o_{m-2}) \) with exceptional divisor \( E_m \) and we let \( U_m \simeq k^2 = \text{Spec}(k[x, w_m]) \) be the affine chart of \( S_{m}(o_1, \ldots, o_{m-1}) \) on which the restriction of \( \tau_{m,m-1} \) coincides with the morphism \( U_m \to U_{m-1}, (x, w_m) \to (x, xw_m) \). By construction, we have a commutative diagram

\[
\begin{array}{c}
S_n(o_1, \ldots, o_{n-1}) \xrightarrow{\tau_{n-1,n-2}} S_{n-1}(o_1, \ldots, o_{n-2}) \xrightarrow{\tau_{n-2,n-3}} \cdots \xrightarrow{\tau_{2,1}} S_1 \xrightarrow{\tau_1} k^2
\end{array}
\]

The total transform of \( E_1 \) in \( S_n(o_1, \ldots, o_{n-1}) \) is a chain \( E_1 \cup E_2 \cup \cdots \cup E_{n-1} \cup E_n \) is a chain formed of \( n - 1 \) curves with self-intersection \(-2\) and the curve \( E_n \) which has self-intersection \(-1\).

\[
\begin{array}{cccc}
E_1 & E_2 & & E_n \ 
\text{-2} & -2 & & \text{-2} \ 
\end{array}
\]

**Figure 3.2.** Dual graph of the total transform of \( E_1 \) in \( S_n(o_1, \ldots, o_{n}) \).

The morphism \( \pi : X_n \to S \) lifts to a morphism \( \pi_1 : X_n \to S_1 \) defined by

\[
(x, z_1, z_2, y_0, y_1, \ldots, y_n) \mapsto ((x, y_0), [x : y_0]) = ((x, y), [y_0z_1 + 1 : y_1]).
\]

This morphism contracts \( \pi^{-1}(o) \) onto the point \( o_1 = ((0, 0), [1 : 0]) \) of the exceptional divisor \( E_1 \) of \( S_1 \). The induced rational map \( \pi_1 : X_n \dashrightarrow U_1 \) is given by

\[
(x, z_1, z_2, y_0, y_1, \ldots, y_n) \mapsto (x, \frac{y_1}{y_0z_1 + 1})
\]

and it contracts \( \pi^{-1}(o) \) onto the origin \( o_1 = (0, 0) \). So \( \pi_1 \) lifts to a morphism \( \pi_2 : X_n \to S_2(o_1) \), and with our choice of charts, the induced rational map \( \pi_2 : X_n \dashrightarrow U_2 \) is given by

\[
(x, z_1, z_2, y_0, y_1, \ldots, y_n) \mapsto (x, \frac{y_2}{(y_0z_1 + 1)^2})
\]

If \( n = 2 \) then the image of \( \pi^{-1}(o) = \text{Spec}(k[z_2, y_2]) \) by \( \pi_2 \) is equal to \( E_2 \cap U_2 \) and \( \pi^{-1}_2(E_2 \cap U_2) \) is equivariantly isomorphic to \( (E_2 \cap U_2) \times \text{Spec}(k[z_2]) \) on which \( G_a \) acts by translations on the second factor. So \( \pi_2 : X_n \to S_2(o_1) \) factors through a \( G_a \)-bundle \( q_2 : X_2 \to S_2(o_1) = S_2(o_1) \setminus E_1 \) and \( X_2/G_a \simeq S_2(o_1) \). Otherwise, if \( n > 2 \) then \( \pi_2 \) contracts \( \pi^{-1}(o) \) onto the point \( o_2 = (0, 0) \in E_2 \cap U_2 \subset S_2(o_1) \). So \( \pi_2 : X_n \to S_2(o_1) \) lifts to a morphism \( \pi_3 : X_n \to S_3(o_1, o_2) \). With our choice of charts, for each \( 2 < m < n \), the induced rational map \( \pi_m : X_n \dashrightarrow \hat{U}_m \) is given by

\[
(x, z_1, z_2, y_0, y_1, \ldots, y_n) \mapsto (x, \frac{y_m}{(y_0z_1 + 1)^m})
\]

hence contracts \( \pi^{-1}(o) \) onto the point \( o_m = (0, 0) \in U_m \subset S_m(o_1, \ldots, o_{m-1}) \). It thus lifts to a morphism \( \pi_m : X_n \to S_m(o_1, \ldots, o_{m-1}) \). At the last step, the image of \( \pi^{-1}(o) = \text{Spec}(k[z_2, y_n]) \) by the rational map \( \pi_n : X_n \dashrightarrow U_n \) induced by \( \pi_n : X_n \to S_n(o_1, \ldots, o_{n-1}) \) is equal to \( E_n \cap U_n \), and we conclude as above that \( \pi_n : X_n \to S_n(o_1, \ldots, o_{n-1}) \) factors through a \( G_a \)-bundle

\[
q_n : X_n \to S_n(o_1, \ldots, o_{n-1}) = S_n(o_1, \ldots, o_{n-1}) \setminus (E_1 \cup \cdots \cup E_{n-1}),
\]

hence that \( X_n/G_a \) is isomorphic to the quasi-projective surface \( S_n(o_1, \ldots, o_{n-1}) \). \( \square \)

### 3.4.2. A family of \( G_a \)-extensions of \( SL_2 \) of "Type II-D".

To conclude this section, we present an illustration of the proof of Theorem 3.8 another countable family of quasi-projective \( G_a \)-extensions of \( SL_2 \) of Type II with affine total spaces.

Let again \( \pi_1 : S_1 \to S = \mathbb{A}^2 \) be the blow-up of the origin \( o = \{(0,0)\} \in \mathbb{A}^2 = \text{Spec}(k[x, y]) \) with exceptional divisor \( E_1 \simeq \mathbb{P}^1 \), identified with closed subvariety of \( \mathbb{A}^2 \times \mathbb{P}^1 \) with equation \( xu_1 - yv_0 = 0 \) in such a way that \( \tau \) coincides with the restriction of the first projection. The second projection identifies \( S_1 \) with the total space \( p : S_1 \to \mathbb{P}^1 \) of the invertible sheaf \( \mathcal{O}_{\mathbb{P}^1}(-1) \). We fix trivializations \( p^{-1}(U_{\infty}) = \text{Spec}(k[z_{\infty}]|u_{\infty}) \) and \( p^{-1}(U_0) = \text{Spec}(k[z_0]|u_0) \) over the open subsets \( U_{\infty} = \mathbb{P}^1 \setminus \{(0 : 1)\} = \text{Spec}(k[z_{\infty}]) \)

\[
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\]

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and $U_0 = \mathbb{P}^1 \setminus \{[1 : 0]\} = \text{Spec}(k[z_0])$ in such a way that the gluing of $p^{-1}(U_\infty)$ and $p^{-1}(U_0)$ over $U_0 \cap U_\infty$ is given by the isomorphism $(z_0, u_0) \mapsto (z_\infty, u_\infty) = (z_0^{-1}, z_0 u_0)$.

For every $n \geq 1$, we let $S_{2n+3,0} = \text{Spec}(k[z_0, u_0^{n+1}])$,

$$S_{2n+3,\infty} = \text{Spec}(k[z_\infty, u_\infty, v_\infty]/(u_\infty^n v_\infty - z_\infty^2 - u_\infty)), $$

and we let $S_{2n+3}$ be the surface obtained by gluing $S_{2n+3,0}$ and $S_{2n+3,\infty}$ along the open subsets $S_{2n+3,0} \setminus \{z_0 = 0\}$ and $S_{2n+3,\infty} \setminus \{z_\infty = u_\infty = 0\}$ by the isomorphism

$$(z_0, u_0) \mapsto (z_\infty, u_\infty, v_\infty) = (z_0^{-1}, z_0 u_0, (z_0 u_0)^{-n} (z_0^{-2} + z_0 u_0)).$$

The canonical open immersion $S_{2n+3,0} \hookrightarrow p^{-1}(U_0)$ and the projection $p_{z_\infty,u_\infty} : S_{2n+3,\infty} \to p^{-1}(U_\infty)$ glue to a global birational affine morphism $\tau_{2n+3,1} : S_{2n+3} \to \overline{S}_1$ restricting to an isomorphism $S_{2n+3,\infty} \setminus \{z_\infty = u_\infty = 0\} \to \overline{S}_1 \setminus \overline{E}_1$ where we identified the closed subset $E_{2n+3} = \{z_\infty = u_\infty = 0\} \simeq \text{Spec}(k[v_\infty])$ of $S_{2n+3,\infty}$ with its image in $S_{2n+3}$. We leave to the reader to check that with the notation of § 3.1, $S_{2n+3} = \tau_{2n+3}^{-1}(S_{2n+3}(o_1, \ldots, o_{2n+2}))$ for a surface $\tau_{2n+3,1} : S_{2n+3,1}(o_1, \ldots, o_{2n+2}) \to \overline{S}_1$ obtained by first blowing-up the point $o_1 = (0,0) \in p^{-1}(U_\infty)$ with exceptional divisor $E_2$, then the point $o_2 = E_1 \cap E_2$ with exceptional divisor $E_3$, then a point $o_3 = E_3 \setminus (E_1 \cup E_2)$ with exceptional divisor $E_4$ and then a sequence of points $o_i \in E_i \setminus E_{i-1}$ with exceptional divisor $E_{i+1}$, $i = 5, \ldots, 2n+2$ in such a way that the total transform of $E_1$ in $S_{2n+3,1}$ is a tree depicted in Figure 3.3. Letting $\tau_{2n+3} = \tau_1 \circ \tau_{2n+3,1} : S_{2n+3} \to \mathbb{A}^2$, we have $\tau_{2n+3,1}^{-1}(o)_{\text{red}} = E_{2n+3} \simeq \mathbb{A}^1$ and $\tau_{2n+3}^{-1}(o) = 2E_{2n+3}$. 

![Figure 3.3. Dual graph of the total transform of $E_1$ in $S_{2n+3}(o_1, \ldots, o_{2n+2})$.](image)

Now we let $q : X_{2n+3} \to S_{2n+3}$ be the $\mathbb{G}_a$-bundle defined as the gluing of the trivial $\mathbb{G}_a$-bundles $X_{2n+3,0} = S_{2n+3,0} \times \text{Spec}(k[t_0])$ and $X_{2n+3,\infty} = S_{2n+3,\infty} \times \text{Spec}(k[t_\infty])$ over $S_{2n+3,0}$ and $S_{2n+3,\infty}$ respectively along the open subsets $X_{2n+3,0} \setminus \{z_0 = 0\}$ and $X_{2n+3,\infty} \setminus \{z_\infty = u_\infty = 0\}$ by the $\mathbb{G}_a$-equivariant isomorphism

$$(z_0, u_0, t_0) \mapsto (z_\infty, u_\infty, v_\infty, t_\infty) = (z_0^{-1}, z_0 u_0, (z_0 u_0)^{-n} (z_0^{-2} + z_0 u_0), t_0 + z_0^{-1} u_0^{-2}).$$

Let $\pi_{2n+3} = \tau_1 \circ \tau_{2n+3,1} \circ q : X_{2n+3} \to \mathbb{A}^2$.

**Proposition 3.13.** For every $n \geq 1$, the variety $X_{2n+3}$ is affine and there exists a $\mathbb{G}_a$-equivariant open embedding $j : SL_2 \hookrightarrow X_{2n+3}$ which makes $\pi_{2n+3} : X_{2n+3} \to \mathbb{A}^2$ a quasi-projective $\mathbb{G}_a$-extension of $\text{SL}_2$ of Type II, with fiber $\pi_{2n+3}^{-1}(o)$ isomorphic to $\mathbb{A}^2$ of multiplicity two, and geometric quotient $X_{2n+3}/\mathbb{G}_a \simeq S_{2n+3}$.

**Proof.** Let $j_1 : SL_2 \hookrightarrow W = W(SL_2, 2)$ be the $\mathbb{G}_a$-extension of $SL_2$ into a locally trivial $\mathbb{A}^1$-bundle $\theta : W \to \overline{S}_1$ with affine total space constructed in Example 2.1. Recall that the image of $j_1$ coincides with the restriction of $\theta$ to $\overline{S}_1 \setminus \overline{E}_1 = \mathbb{A}^2 \setminus \{0\}$. With our choice of coordinates, the open subsets $W_0 = \theta^{-1}(q^{-1}(U_0))$ and $W_\infty = \theta^{-1}(q^{-1}(U_\infty))$ of $W$ are respectively isomorphic to $p^{-1}(U_0) \times \text{Spec}(k[w_0])$ and $p^{-1}(U_\infty) \times \text{Spec}(k[w_\infty])$ glued over $U_0 \cap U_\infty$ by the isomorphism

$$(z_0, u_0, w_0) \mapsto (z_\infty, u_\infty, w_\infty) = (z_0^{-1}, z_0 u_0, z_0^{-2} w_0 + z_0).$$

The $\mathbb{G}_a$-action on $W_0$ and $W_\infty$ are given respectively by $\alpha \cdot (z_0, u_0, w_0) = (z_0, u_0, w_0 + \alpha u_0^2)$ and $\alpha \cdot (z_\infty, u_\infty, w_\infty) = (z_\infty, u_\infty, w_\infty + \alpha^2 z_\infty)$.

Let $W' = W \times_{\overline{S}_1} S_{2n+3}$, equipped with the natural lift of the $\mathbb{G}_a$-action on $W$. Since $\tau_{2n+3,1} : S_{2n+3} \to \overline{S}_1$ restricts to an isomorphism over $\overline{S}_1 \setminus \overline{E}_1$, the composition $j' = \tau_{2n+3,1}^{-1} \circ j_1 : SL_2 \to W'$ is a $\mathbb{G}_a$-equivariant open embedding. Furthermore, since $W$ is affine and $\tau_{2n+3,1}$ is an affine morphism, it follows
that \( W' \) is affine. By construction, \( W' \) is covered by the two open subsets

\[
\begin{align*}
W'_0 &= W \times_{p^{-1}(U_0)} S_{2n+3,0} \cong S_{2n+3,0} \times \text{Spec}(k[w_0]) \\
W'_\infty &= W \times_{p^{-1}(U_\infty)} S_{2n+3,\infty} \cong S_{2n+3,\infty} \times \text{Spec}(k[w_\infty]).
\end{align*}
\]

The local \( \mathbb{G}_a \)-equivariant morphisms

\[
\begin{align*}
\beta_0 : X_{2n+3,0} &= S_{2n+3,0} \times \text{Spec}(k[t_0]) \to W'_0 \\
\beta_\infty : X_{2n+3,\infty} &= S_{2n+3,\infty} \times \text{Spec}(k[t_\infty]) \to W'_\infty
\end{align*}
\]

of schemes over \( S_{2n+1,0} \) and \( S_{2n+3,\infty} \) respectively defined by \( t_0 \mapsto w_0 = u_0^2 t_0 \) and \( t_\infty \mapsto w_\infty = u_\infty^2 t_\infty \) glue to a global \( \mathbb{G}_a \)-equivariant birational affine morphism \( \beta : X_{2n+3} \to W' \), restricting to an isomorphism over \( S_{2n+3} \setminus E_{2n+3} \cong \mathbb{A}^2 \setminus \{0\} \). Summing up, \( X_{2n+3} \) is affine over \( W' \) hence affine, and the composition \( \beta^{-1} \circ j' : SL_2 \to X_{2n+3} \) is a \( \mathbb{G}_a \)-equivariant open embedding which realizes \( \pi : X_{2n+3} \to \mathbb{A}^2 \) as a \( \mathbb{G}_a \)-extension of \( SL_2 \) of Type II with affine total space. By construction, \( \pi_{2n+3}^{-1}(o) = q^{-1}(2E_{2n+3}) \) is isomorphic to \( \mathbb{A}^2 \), with multiplicity two, while the geometric quotient \( X_{2n+3}/\mathbb{G}_a \) is isomorphic to \( S_{2n+3} \).

\[ \square \]

Remark 3.14. For every \( n \geq 1 \), the birational morphism \( S_{2(n+1)+3,\infty} \to S_{2n+3,\infty} \), \( (z_\infty, u_\infty, v_\infty) \mapsto (z_\infty, u_\infty, v_\infty) \) extends to a birational morphism \( S_{2(n+1)+3} \to S_{2n+3} \) which lifts in turn in a unique way to a \( \mathbb{G}_a \)-equivariant birational morphism \( \gamma_{n+1,n} : X_{2(n+1)+3} \to X_{2n+3} \). So in a similar way as for the family constructed in § 3.4.1, the family of threefolds \( X_{2n+3} \), \( n \geq 1 \), form a tower of \( \mathbb{G}_a \)-equivariant affine modifications of the initial one \( X_5 \).

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