COMPARING DIAGONALS ON THE ASSOCIAHEDRA

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Abstract. We prove that the formula for the diagonal approximation \( \Delta_K \) on J. Stasheff’s \( n \)-dimensional associahedron \( K_{n+2} \) derived by the current authors in \cite{7} agrees with the “magical formula” for the diagonal approximation \( \Delta_K' \) derived by Markl and Shnider in \cite{5}, by J.-L. Loday in \cite{4}, and more recently by Masuda, Thomas, Tonks, and Vallette in \cite{6}.

Dedicated to the memory of Jean-Louis Loday

1. Introduction

Recently there has been renewed interest in explicit combinatorial diagonal approximations on J. Stasheff’s \( n \)-dimensional associahedron \( K_{n+2} \) \cite{8}. Markl and Shnider (M-S) in \cite{5}, J.-L. Loday in \cite{4}, and more recently Masuda, Thomas, Tonks, and Vallette (MTTV) in \cite{6} constructed a diagonal \( \Delta_K' \) on \( K_{n+2} \) whose components are “matching pairs” of faces, which in the words of Jean-Louis Loday, are “pairs of cells of matching dimensions and comparable under the Tamari order.” By definition, every component of the combinatorial diagonal \( \Delta_K \) on \( K_{n+2} \) constructed by the current authors (S-U) in \cite{7} is a matching pair. In this paper we prove that every matching pair is a component of \( \Delta_K \). Thus the S-U formula for \( \Delta_K \) and the “magical formula” for \( \Delta_K' \) agree (see Definitions \( 2.3 \) and \( 3.1 \)).

Historically, S-U were the first to derive a cellular combinatorial/differential graded formula for \( \Delta_K \), M-S were the first to prove the magical formula for \( \Delta_K' \), and MTTV were the first to construct a point-set topological diagonal map, which descends to the magical formula at the cellular level.

Using the geometric methods of MTTV, Laplante-Anfossi created a general framework for studying diagonals on any polytope in \cite{3}. In this framework, a choice of diagonal on the \( n \)-dimensional permutahedron \( P_{n+1} \) is given by a choice of chambers in its fundamental hyperplane arrangement (\cite{3}, Def. 1.18). While the specific diagonal \( \Delta'_P \) on \( P_{n+1} \) studied in \cite{3} differs from the S-U diagonal \( \Delta_P \), the diagonal \( \Delta'_K \) on \( K_{n+2} \) induced by \( \Delta'_P \) agrees with \( \Delta_K \).

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2. Diagonals Induced by $\Delta_P$

Let $S_n$ be the symmetric group on the finite set $\mathbb{A} = \{1, 2, \ldots, n\}$. The permutahedron $P_n$ is the convex hull of $n!$ vertices $\{(\sigma(1), \ldots, \sigma(n)) : \sigma \in S_n\} \subseteq \mathbb{R}^n$. As a cellular complex, $P_n$ is an $(n-1)$-dimensional convex polytope whose $(n-p)$-faces are indexed by (ordered) partitions $A_1, \ldots, A_p$ of $\mathbb{A}$, $1 \leq p \leq n$. Denoting the set of ordered partitions of $\mathbb{A}$ by $P(n)$, the faces of $P_n$ are identified with elements of $P(n)$ in the standard way.

Let $X$ be an $n$-dimensional polytope that admits a (surjective) cellular projection map $p : P_{n+1} \to X$ and a realization as a subdivision of the $n$-cube $I^n$, i.e., for $0 \leq k \leq n$, each $k$-cell (k-subcube) of $I^n$ is a union of $k$-cells of $X$, any two of which intersect along their boundaries.

For example, $X = P_n$ can be realized as a subdivision of $I^{n-1}$ inductively as follows: Identify $P_1$ with $1 \in P(1)$. If $P_{n-1}$ has been constructed and $a = A_1 \cdots A_p \in P(n-1)$ is a face, let $a_0 = 0$, $a_j = \# (A_{p-j+1} \cup \cdots \cup A_p)$ for $0 < j < p$, $a_p = \infty$, and define $\frac{1}{A_j} := 0$. Let $I(a) := I_1 \cup I_2 \cup \cdots \cup I_p$, where $I_j := [1 - \frac{1}{2^{j-1}}, 1 - \frac{1}{2^j}]$; then $P_n = \bigcup_{a \in P(n-1)} a \times I(a)$, where the identification of faces with partitions is given by $\frac{a \times I(a)}{\text{Face of } a \times I(a)} \quad \text{Partition in } P(n)$

\begin{align*}
   a \times 0 & \quad A_1 \cdots |A_p|n \\
   a \times (I_j \cap I_{j+1}) & \quad A_1 \cdots |A_{p-j}|n|A_{p-j+1}| \cdots |A_p|, \quad 1 \leq j \leq p-1 \\
   a \times 1 & \quad n|A_1| \cdots |A_p|, \\
   a \times I_j & \quad A_1|A_{p-j+1} \cup n| \cdots |A_p|, \quad 1 \leq j \leq p
\end{align*}

(see Figures 1 and 2). We refer to a vertex common to $P_n$ and $I^{n-1}$ as a cubical vertex. Thus $a$ is a cubical vertex of $P_n$ if and only if $a|n$ and $n|a$ are cubical vertices of $P_{n+1}$. Indeed, a cubical vertex has the form $a = a_1 \cdots |a_{i-1}|1|a_{i+1}| \cdots |a_n$, where $a_1 > \cdots > a_{i-1}$ and $a_i+1 < a_i \cdots < a_n$.

We begin with a review of the diagonal $\Delta_P$ and the diagonal $\Delta_X$ induced by the projection $p$; then $\Delta_X$ is obtained by setting $X = K_{n+2}$. Whereas the vertices of $P_{n+1}$ are identified with the permutations in $S_{n+1}$, the weak order on $S_{n+1}$ given by $\cdots |x_i|x_{i+1}| \cdots < \cdots |x_i|x_{i+1}| \cdots$ if $x_i < x_{i+1}$ extends to a partial order (p-o) and the associated Hasse diagram orients the 1-skeleton of $P_{n+1}$ \[1\]. Denote the minimal and maximal vertices of a face $e$ of $P_{n+1}$ by $\min e$ and $\max e$, respectively, and define $0 < e' \leq e$ if there exists an oriented edge-path in $P_{n+1}$ from $e$ to $e'$. Then $p$ induces a p-o on the cells of $X$. For example, when the faces of $P_{n+1}$ are indexed by planar leveled trees (PLTs) with $n + 2$ leaves and the faces of $K_{n+2}$ are indexed by planar rooted trees (PRTs) with $n + 2$ leaves (without levels), Tonks’ projection $p = \theta$ given by forgetting levels \[9\] induces the Tamari order on the faces $\{\theta(T_i)\}$ of $K_{n+2}$ given by $\theta(T_i) \leq \theta(T_j)$ if $T_i \preceq T_j$. In particular, the vertices of $K_{n+1}$ form a subset of the vertices of $P_n$ and the Tamari order restricted to this subset agrees with the weak order.

Let $e$ be a cell of $X$ and let $|e|$ denote its dimension. A $k$-subdivision cube of $e$ is a set of faces of $e$ whose union is a $k$-subcube of $I^n$ for some $k \leq n$. For example, when $e$ is the top dimensional cell of $P_3$, the facets in $\{2|34, 24|13\}$ and $\{2|34, 24|13, 23|14, 23|14\}$ form 2-subdivision cubes of $e$, but any three in the latter do not (see Figure 2). Denote the set of vertices of $e$ by $V_e$ (when $e = X$ we suppress...
the subscript $e$). Given a vertex $v \in V_c$, let $I_{v,1}^{k_1}$ and $I_{v,2}^{k_2}$ be $k_i$-subdivision cubes of $e$ such that $\max I_{v,1}^{k_1} = \min I_{v,2}^{k_2} = v$ and $k_1 + k_2 = |e|$; then $\left( I_{v,1}^{k_1}, I_{v,2}^{k_2} \right)$ is a pair of $(k_1, k_2)$-subdivision cubes of $e$. Denote the set of all such pairs by $e_v$ and let $(I_{v,1}^{k_1}, I_{v,2}^{k_2})_e$ denote its unique maximal element; then $(I_{v,3}^{k_3}, I_{v,4}^{k_4}) \subseteq (I_{v,1}^{k_1}, I_{v,2}^{k_2})_e$ for all $(I_{v,3}^{k_3}, I_{v,4}^{k_4}) \in e_v$. For example, when $e$ is the top dimensional cell of $P_4$ and $v = 4|2|3|1$, we have $(I_{v,1}^{k_1}, I_{v,2}^{k_2})_e = \left( \{2|134, 24|13\}, \{4|23|1\} \right)$. For an explicit description of $(I_{v,1}^{k_1}, I_{v,2}^{k_2})_e$ when $e \subseteq P_n$ see [2.3] below.

![Figure 1: $P_3$ as a subdivision of $P_2 \times I$.](image1)

![Figure 2: The facets of $P_4$ as a subdivision of $I^3$.](image2)

If in addition, the cellular projection $p : P_{n+1} \rightarrow X$ preserves maximal pairs of $(k_1, k_2)$-subdivision cubes, i.e., for every cell $e$ of $P_{n+1}$ we have

$$p(I_{v,1}^{k_1}, I_{v,2}^{k_2})_e = \left( I_{p(v),1}^{k_1}, I_{p(v),2}^{k_2} \right)_{p(e)},$$

the components of the induced diagonal $\Delta_X$ on a cell $f \subseteq X$ form the set of product cells

$$\Delta_X(f) := \bigcup_{(e^{k_1}, e^{k_2}) \in (I_{v,1}^{k_1}, I_{v,2}^{k_2})_f} \{e^{k_1} \times e^{k_2}\}.$$
In particular, $p = \theta$ preserves maximal pairs of $(k_1, k_2)$-subdivision cubes and $\Delta_K(e)$ is given by setting $X = K_{n+2}$ (see (2.4) below). Note that $(e^{k_1}, e^{k_2}) \in \left( T_{v_1}^{k_1}, T_{v_2}^{k_2} \right)$ implies $e^{k_1} \leq e^{k_2}$. Thus $e^{k_1} \times e^{k_2}$ is a “matching pair” in the sense of MTTV (see Definition 2). Furthermore, since $f = p(e)$ for some $e = P_{n_1} \times \cdots \times P_{n_s}$ and $p(e) = p(P_{n_1}) \times \cdots \times p(P_{n_s})$, the diagonal $\Delta_X(f)$ is automatically the comultiplicative extension of its values on the factors of $f$, i.e.,

$$\Delta_X(f) = \Delta_X(p(P_{n_1})) \times \cdots \times \Delta_X(p(P_{n_s})).$$

The subset $\mathcal{V}_e \subseteq S_n$ determines the components of $\Delta_P(e)$ in the following way: Let $\sigma = x_1, \ldots, x_n \in \mathcal{V}_e$. Reading $\sigma$ from left-to-right and from right-to-left, construct the partitions $\overline{\sigma}_1, \ldots, \overline{\sigma}_p$ and $\overline{\sigma}_q, \ldots, \overline{\sigma}_1$ of maximal decreasing subsets and form the Strong Complementary Pair (SCP)

$$a_\sigma \times b_\sigma := \overline{\sigma}_1 \times \cdots \overline{\sigma}_p \times \overline{\sigma}_q, \ldots, \overline{\sigma}_1 \in P(n) \times P(n).$$

Then

$$\sigma = \max a_\sigma = \min b_\sigma, \; \min \overline{\sigma}_j < \max \overline{\sigma}_{j+1} \text{ for all } j < p, \text{ and } \min \overline{\sigma}_j < \max \overline{\sigma}_{j+1} \text{ for all } i < q.$$ 

Thus, for $\sigma = 2[1|3|5|4]$ we have $\overline{\sigma}_1 \overline{\sigma}_2 | \overline{\sigma}_3 = 21|3|54$ and $\overline{\sigma}_3 | \overline{\sigma}_2 | \overline{\sigma}_1 = 2|135|4$ so that $a_\sigma \times b_\sigma = 21|3|54 \times 2|135|4$.

Let $a = A_1 \times \cdots \times A_p \in P(n)$. For $1 \leq j < p$, let $M_j \subseteq \{A_j \setminus \{\min A_j\}\}$ such that $\min \pi M_j > \max A_{j+1}$ when $M_j \neq \emptyset$. Define the right-shift $M_j$ action

$$R_{M_j}(a) := \left\{ A_1 \times \cdots A_j \setminus M_j A_{j+1} \cup M_j A_{j+1} \cdots A_k, \; M_j \neq \emptyset, \; M_j = \emptyset \right\}.$$ 

Let $M := (M_1, M_2, \ldots, M_{p-1})$ and denote the composition $R_{M_{p-1}} \cdots R_{M_2} R_{M_1}(a)$ by $R_M(a)$.

Dually, let $b = B_0 \cdots B_1 \in P(n)$. For $1 \leq i < q$, let $N_i \subseteq (B_i \setminus \{\min B_i\})$ such that $\min N_i > \max B_{i+1}$ when $N_i \neq \emptyset$. Define the left-shift $N_i$ action

$$L_{N_i}(b) := \left\{ B_0 \cdots B_{i+1} \cup N_i B_i \cup N_i B_i \cdots B_1, \; N_i \neq \emptyset, \; N_i = \emptyset \right\}.$$ 

Let $N := (N_1, N_2, \ldots, N_{q-1})$ and denote the composition $L_{N_{q-1}} \cdots L_{N_2} L_{N_1}(b)$ by $L_N(b)$.

Now given $\sigma \in \mathcal{V}_e$ and the SCP $a_\sigma \times b_\sigma$, the pair $R_M(a_\sigma) \times L_N(b_\sigma)$ is a Complimentary Pair (CP) on $a_\sigma \times b_\sigma$. Define

$$A_\sigma \times B_\sigma := \bigcup_{M,N} \{ R_M(a_\sigma) \times L_N(b_\sigma) \}$$

and

$$\Delta_P(e) := \bigcup_{\sigma \in \mathcal{V}_e} A_\sigma \times B_\sigma.$$ 

**Example 1.** On the top dimensional cell $e^2 \subseteq P_3$, $\Delta_P(e^2)$ is the union of

$\begin{align*}
A_{1|2|2} \times B_{1|2|3} &= \{ 1|2|3 \times 123 \}, & A_{1|3|2} \times B_{3|3|2} &= \{ 1|32 \times 132 \}, \\
A_{2|1|3} \times B_{2|1|3} &= \{ 2|3 \times 213, 21|3 \times 231 \}, & A_{2|3|1} \times B_{3|2|1} &= \{ 2|31 \times 231 \}, \\
A_{3|1|2} \times B_{3|1|2} &= \{ 31|2 \times 3|12, 1|32 \times 3|12 \}, & A_{3|2|1} \times B_{3|2|1} &= \{ 321 \times 3|21 \}.
\end{align*}$
Remark 1. Note that the matrix representation of a CP introduced in [7] conveniently organizes and systematizes the combinatorial calculation of $\Delta_P$. An SCP is represented by a step matrix and a general CP is represented by a derived matrix, given by left-shift and down-shift actions on a step matrix.

When $X = P_{n+1}$, Formulas (2.1) and (2.2) are equivalent. The maximal $(k_1, k_2)$-subdivision pair with respect to a vertex $\sigma$ of $P_{n+1}$ is

$$
(2.3) \quad \left(1_{S_{\sigma, 1}}^{k_1}, 1_{S_{\sigma, 2}}^{k_2}\right) = \left(\bigcup_{e_1 \in A_\sigma} e_1, \bigcup_{e_2 \in B_\sigma} e_2\right)
$$

Definition 1. A positive dimensional face $e$ of $P_n$ is non-degenerate if $|\theta(e)| = |e|$. A positive dimensional partition $a = A_1|\cdots|A_p \in P(n)$ is degenerate if for some $j$ and some $k > 0$, there exist $x, z \in A_j$ and $y \in A_{j+k}$ such that $x < y < z$; otherwise $a$ is non-degenerate. A CP $\alpha \times \beta$ is non-degenerate if $\alpha$ and $\beta$ are non-degenerate.

Define $\Delta_K(K_{n+1}) = \Delta_K(\theta(P_n)) := (\theta \times \theta)\Delta_P(P_n)$; then

$$
(2.4) \quad \Delta_K(e^{n-1}) = \bigcup_{\text{non-degenerate CPs} \, \alpha \times \beta \in A_{\sigma} \times B_{\sigma}} \{\theta(\alpha) \times \theta(\beta)\}.
$$

3. Agreement of $\Delta_K$ and $\Delta_K'$

Definition 2. A pair of faces $a \times b \subseteq K_{n+1} \times K_{n+1}$ is a Matching Pair (MP) if $a \leq b$ and $|a| + |b| = n - 1$.

The “magical formula” derived in [5] and [9] is

$$
(3.1) \quad \Delta_K'(e^{n-1}) = \bigcup_{\text{faces} \, a \times b \subseteq K_{n+1} \times K_{n+1}} \{a \times b\}.
$$

Tonks’ projection $\theta$ sends every non-degenerate CP to an MP. The converse is our main result: Every MP is the image of a unique non-degenerate CP under $\theta$; thus $\Delta_K'$ and $\Delta_K$ agree. Our proof of this fact views $P_n$ as a subdivision of $K_{n+1}$.

Definition 3. Let $0 \leq k < n$. An associahedral $k$-cell of $P_n$ is a $k$-cell of $K_{n+1}$. A subdivision $k$-cell of $P_n$ is a $k$-cell of some associahedral $k$-cell of $P_n$. The maximal (resp. minimal) subdivision $k$-cell of an associahedral $k$-cell $a$, denoted by $a_{\text{max}}$ (resp. $a_{\text{min}}$), satisfies $\max a_{\text{max}} = \max a$ (resp. $\min a_{\text{min}} = \min a$). A non-degenerate vertex of $P_n$ is an associahedral vertex.

Thus a subdivision $k$-cell of $P_n$ has the form $A_1|\cdots|A_{n-k}$. In fact, a vertex $v$ of $P_n$ is associahedral if and only if the $(n - q)$-cell $\vec{v}_q|\cdots|\vec{v}_1$ is non-degenerate, in which case $\min \vec{v}_q > \cdots > \min \vec{v}_1$. If $k > 0$, an associahedral $k$-cell $a$ is a subdivision $k$-cell if and only if $a = a_{\text{min}}$.

Proposition 1. If $a$ is an associahedral $k$-cell and $u$ is a subdivision $k$-cell of $a$, then

(i) $a_{\text{min}}$ is non-degenerate.
(ii) If $u \neq a_{\text{min}}$, then $u$ is degenerate and $u = L_N(a_{\text{min}})$ for some $N$.
(iii) $a_{\text{min}} = R_M(a_{\text{max}})$ for some $M$. 
Proof. Set $p = n - k$ and consider an associahedral $k$-cell $a$ of $P_n$. If $a$ is also a subdivision $k$-cell, then $a = a_{\min} = \theta(a)$ is non-degenerate and $M = \emptyset$. Otherwise, conclusions (i) and (ii) follow from the construction of $P_n$ as a subdivision of $K_{n+1}$. For part (iii), given a subdivision $k$-cell $u = A_1 \cdots A_p$ of $a$, let

$$N_p := \{x \in A_p \setminus \{\min A_p\} : x > \max A_{p-1}\}.$$ 

Inductively, if $1 < i < p$ and $N_{i+1}$ has been constructed, let $A_i' := A_i \cup N_{i+1}$ and let

$$N_i := A_i' \setminus \{x \in A_i' \setminus \{\min A_i'\} : x > \max A_{i-1}\}.$$ 

Then $a_{\max} = L(N_1, \ldots, N_2)(a_{\min})$. Set $M = (M_1, \ldots, M_{p-1}) := (N_2, \ldots, N_p)$; then $a_{\min} = R_M(a_{\max})$. \hfill $\square$

**Example 2.** Consider the associahedral facet $a = 1|234 \cup 13|24 \cup 14|23 \cup 134|2$; then $a_{\min} = 1|234$ is non-degenerate, $13|24 = L(3)(a_{\min})$, $14|23 = L(4)(a_{\min})$, and $a_{\max} = 134|2 = L(3,4)(a_{\min})$. Furthermore, $a_{\min} = 1|234 = R(3,4)(134|2)$.

**Proposition 2.** Let $v$ be an associahedral vertex of $P_n$ and let $a = \tau_1| \cdots |\tau_1$. If $b$ is a non-degenerate cell of $P_n$, such that $|b| = |a|$ and $\min a \leq \min b$, then $b = L_N(a)$ for some $N$.

**Proof.** Let $a = A_{n-k} \cdots |A_1$ and let $r_i = \min A_i$. Since $v$ is associahedral, it follows that $r_n \cdots > \cdots > r_1$. Since $\min a \leq \min b$, there is a product of $p$-o increasing transpositions $\tau = \tau_1 \cdots \tau_2 \tau_1$ such that $\tau(\min a) = \min b$ and $\tau_i$ preserves the inequality $r_j > r_{j+1}$ for $1 \leq i \leq t$ and $1 \leq j \leq n - k$. Define $\tau_0 := 1d$ and consider the (possibly degenerate) cell $u_i := \tau_i \cdots \tau_1 \tau_0(\min a)$ for each $1 \leq i \leq t$. For each $i$, there is the (possibly degenerate) cell $u_i := \tau_i \cdots \tau_1 \tau_0(\min a)$, where $q \in \{n - k, n - k + 1\}$. Thus there is the sequence $\{a = u_0, u_1, \ldots, u_t = b\}$ and its subsequence of $k$-cells $\{a = u_i, u_{i+1}, \ldots, u_{i+1}, u_i = b\}$. By construction, for $1 \leq j \leq s$, there exists $n_j \in N$ such that $u_i = L_{n_j}(u_i)$. For $1 \leq i < s$, let

$$N_i = \{u_j \in A_i \cup N_1 \cup \cdots \cup N_{i-1} : u_j = L_{n_j}(u_{i-1}) \text{ for some } j\}$$

and form the sequence of sets $N := (N_1, \ldots, N)$. Since $b$ is non-degenerate, the action $L_N(a)$ is defined and $L_N(a) = b$.

Identify a $k$-face $F \subset K_{n+1}$ with its corresponding associahedral cell of $P_n$ and label $F$ with its minimal subdivision $k$-cell $F_{\min}$; then $\theta(F_{\min}) = F$ (compare Figures 2 and 3).

**Example 3.** Consider the associahedral vertex $v = 5|3|1|2|4|6$, the associated 3-cell $a = \tau_1|\tau_2|\tau_1 = 5|3|1246$ and the non-degenerate 3-cell $b = 56|34|12$. Then

$$\min a = 5|3|1|2|4|6 < 5|6|3|4|1|2 = \min b,$$

and there is the product of $p$-o increasing transpositions

$$\tau = \tau_6 \cdots \tau_1 := (3,6)(4,6)(1,6)(2,6)(1,4)(2,4)$$

such that

$$\{v_1 = \tau_1(\min a) = 5|3|1|2|4|6, v_2 = \tau_2(v_1) = 5|3|4|1|2|6, v_3 = \tau_3(v_2) = 5|3|4|1|6|2, v_4 = \tau_4(v_3) = 5|3|4|6|1|2, v_5 = \tau_5(v_4) = 5|3|6|4|1|2, v_6 = \tau_6(v_5) = 5|6|3|4|1|2\}.$$

There is the sequence of cells

$$\{u_0 = 5|3|1246, u_1 = 5|3|14|26, u_2 = 5|3|4|26, u_3 = 5|3|1246, u_3 = 5|3|4|16|2\}.$$
Theorem 1. Let \( u_4 = 5\{346\}12, u_5 = 5\{36\}4\{12, u_6 = 56\{34\}12 \)
and its subsequence of 3-cells

\[ \{ u_0 = 5\{3\}1246, u_2 = 5\{34\}126, u_4 = 5\{346\}12, u_6 = 56\{34\}12 \}. \]

Thus

\[
N_1 = \{ n_j \in A_1 : u_{ij} = L_{(n_j)}(u_{ij-1}) \text{ for some } j \} = \{ 4, 6 \}, \text{ and } \\
N_2 = \{ n_j \in A_2 \cup N_1 : u_{ij} = L_{(n_j)}(u_{ij-1}) \text{ for some } j \} = \{ 6 \}.
\]

Conclude that \( 56\{34\}12 = L_{(4,6),(6)}(5\{3\}1246) \).

**Figure 3:** The facets of \( K_5 \) labeled with their minimal subdivision 2-cells in \( P_4 \).

**Theorem 1.** Let \( F \times G \subseteq K_{n+1} \times K_{n+1} \) be an MP. Then \( F_{\min} \times G_{\min} \subseteq P_n \times P_n \) is a CP and \( F \times G = \theta(F_{\min}) \times \theta(G_{\min}) \). Consequently, the diagonals \( \Delta'_{K} \) and \( \Delta_{K} \)
agree.

**Proof.** Let \( \sigma = \max F \); then \( F_{\max} = \overrightarrow{\sigma}_1 \cdots \overrightarrow{\sigma}_p \) for some \( p \) and \( F_{\min} = R_M(F_{\max}) \) for some \( M \) by Proposition 1. Let \( \beta = \overrightarrow{\sigma}_q \cdots \overrightarrow{\sigma}_1 \) and consider the SCP \( F_{\max} \times \beta \). Since \( \sigma \) is an associahedral vertex and \( \min \beta \leq \min G_{\min} \) the hypotheses of Proposition 2 is satisfied; hence \( G_{\min} = L_N(\beta) \) for some \( N \). Therefore \( F_{\min} \times G_{\min} = R_M(F_{\max}) \times L_N(\beta) \) is a CP and \( F \times G = \theta(F_{\min}) \times \theta(G_{\min}) \).

**Example 4.** Consider the diagonal component

\[ F \times G = (\bullet \bullet \bullet) \bullet \times (\bullet \bullet \bullet) \]

of \( \Delta_K(K_5) \). Then \( F = 21\{43\}421\{3 \) is an associahedral 2-cell, \( \sigma = \max F = 4\{2\}1\{3 \) is an associahedral vertex,

\[
F_{\max} = \overrightarrow{\sigma}_1 \overrightarrow{\sigma}_2 = 421\{3, \text{ and } F_{\min} = 21\{43 = R(421\{3 \). 
\]

Furthermore,

\[
\beta = \overrightarrow{\sigma}_3 \overrightarrow{\sigma}_2 \overrightarrow{\sigma}_1 = 4\{2\}13, \min \beta_1 = 4\{2\}1\{3 = \max F, \text{ and } \\
G_{\min} = L(3_3)(4\{2\}13) = 4\{2\}1. 
\]

Thus \( F \times G = \theta(21\{43 \times \theta(4\{2\}1 \).
Addendum. After this paper was written, B. Delcroix-Oger, G. Laplante-Anfossi, V. Pilaud, and K. Stoeckl proved that $\Delta_P$ can be recovered from $\Delta_P'$ by an appropriate choice of chambers in the fundamental hyperplane arrangements of the permutahedra (see [2]). The fact that all known diagonals on the associahedra agree (up to mirror symmetry) follows immediately.

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