A Note on the Representation Power of GHHs

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Abstract
In this note we prove a sharp lower bound on the necessary number of nestings of nested absolute-value functions of generalized hinging hyperplanes (GHH) to represent arbitrary CPWL functions. Previous upper bound states that $n+1$ nestings is sufficient for GHH to achieve universal representation power, but the corresponding lower bound was unknown. We prove that $n$ nestings is necessary for universal representation power, which provides an almost tight lower bound. We also show that one-hidden-layer neural networks don’t have universal approximation power over the whole domain. The analysis is based on a key lemma showing that any finite sum of periodic functions is either non-integrable or the zero function, which might be of independent interest.

1 Introduction
We consider the complexity of representing continuous piecewise linear functions using the generalized hinging hyperplane model Wang and Sun [2005]. We begin with a short review on these two notions.

1.1 Continuous Piecewise Linear (CPWL) Functions
Continuous piecewise linear (CPWL) functions play an important role in non-linear function approximation, such as nonlinear circuit or neural networks. We introduce the definition of CPWL functions borrowed from Chua and Deng [1988].

Definition 1.1 (CPWL function). A function $f(x) : R^n \rightarrow R$ is said to be a CPWL function iff it satisfies:

1): The domain space $R^n$ is divided into a finite number of polyhedral regions by a finite number of disjunct boundaries. Each boundary is a subset of a hyperplane and takes non-zero measure (standard lebesgue measure) on the hyperplane (as $R^{n-1}$).

2): The restriction of $f(x)$ on each polyhedral region is an affine function.

3): $f(x)$ is continuous on $R^n$.

1.2 Generalized Hinging Hyperplanes (GHH)
The model of hinging hyperplanes (HH) is a sum of hinges like

$$\pm \max\{w_1^\top x + b_1, w_2^\top x + b_2\}$$

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where $w_1, w_2 \in \mathbb{R}^n$ and $b_1, b_2 \in \mathbb{R}$ are parameters. The HH model (in fact equivalent to a one hidden-layer ReLU network) can approximate any continuous function over a compact domain to arbitrary precision as the number of hinges go infinity Breiman [1993].

However, this model can’t exactly represent all CPWL function as pointed out in He et al. [2018], which brings doubt on its approximation efficiency. To overcome this problem, Wang and Sun [2005] first proposed a generalization of HH model, called GHH which allows more than 2 affine functions within the nested maximum operator:

**Definition 1.2** ($n$-order hinge). A $n$-order hinge is a function of the following form:

$$\pm \max\{w_1^\top x + b_1, w_2^\top x + b_2, \cdots, w_{n+1}^\top x + b_{n+1}\}$$

(2)

where $w_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$ are parameters.

A linear combination of a finite number of $n$-order hinges is called a $n$-order hinging hyperplane ($n$-HH) model. Such model has universal representation power over all CPWL functions, as formalized in the theorem below:

**Theorem 1.3** (Theorem 1 in Wang and Sun [2005]). For any positive integer $n$ and CPWL function $f(x) : \mathbb{R}^n \to \mathbb{R}$, there exists a $n$-HH which exactly represents $f(x)$.

The question is whether we can give a sharp lower bound on the necessary number of affine functions within the nested maximum operator. Wang and Sun [2005] conjected that $(n-1)$-HH can’t represent all CPWL functions, but this open problem is left unanswered for more than a decade. In the following section we will prove our main result that $(n-2)$-HH can’t represent all CPWL functions, yielding an almost tight lower bound.

## 2 Main Result

Observe that any $(n-2)$-order hinge depends on only $n-1$ affine transforms of $x$, thus there always exists a direction in which the value of the $(n-2)$-order hinge remains the same. We make such observation precise by introducing the definition of low-dimensional and periodic functions.

**Definition 2.1** (Low-dimensional/periodic function). A function $f(x) : \mathbb{R}^n \to \mathbb{R}$ is said to be low-dimensional, if there exists a vector $v \neq 0$, such that for any $x \in \mathbb{R}^n$ and $c \in \mathbb{R}$, we have that $f(x) = f(x + cv)$. If we have only $f(x) = f(x + v)$ then $f$ is said to be periodic (a weaker notion). $v$ is called an invariant vector of $f$.

Any $(n-2)$-order hinge is a low-dimensional function on $\mathbb{R}^n$, so our problem is reduced to proving the class of finite sum of low-dimensional functions has limited representation power. The following key lemma actually proves (a stronger result) that finite sum of periodic functions can’t represent any non-trivial integrable functions.

**Lemma 2.2.** Any finite sum of periodic functions is either non-integrable or the zero function, i.e. given periodic functions $f_i(x), i = 1, \ldots, m$, then $f(x) = \sum_{i=1}^m f_i(x)$ satisfies

$$\int_{\mathbb{R}^n} |f| = \infty \quad \text{or} \quad f \equiv 0$$

(3)

**Proof.** We prove Lemma 2.2 by induction. Suppose each $f_i$ has an invariant vector $v_i$, base case $m = 1$ is trivial since if we denote the orthogonal hyperplane $H_i = \{x | x^\top v_i = 0\}$, we have

$$\int_{\mathbb{R}^n} |f| = \int_{\mathbb{R}} \int_{H_i} |f|$$

(4)
thus \( \int_{R^n} |f| < \infty \) if and only if \( \int_{H} |f| = 0 \). Assume \( f = \sum_{i=1}^{m} f_i \) is integrable, then \( g(x) \triangleq f(x + v_m) - f(x) \) is also integrable. We make the following decomposition of \( g \):

\[
g(x) = \sum_{i=1}^{m} f_i(x + v_m) - f_i(x) = \sum_{i=1}^{m-1} f_i(x + v_m) - f_i(x)
\]

where each \( f_i(x + v_m) - f_i(x) \) is periodic (with invariant vector \( v_i \)) as well. By induction we have \( g \equiv 0 \) and \( f \) is also a periodic function (with invariant vector \( v_m \)). Using the base case on \( f \) again concludes our proof.

Our main result is a direct corollary of Lemma 2.2, as stated below:

**Theorem 2.3.** For any positive integer \( n \geq 2 \), there exists a CPWL function \( g(x) : R^n \rightarrow R \), such that no \((n-2)\)-HH can exactly represent \( g(x) \).

**Proof.** Let \( g(x) \triangleq \max\{0, 1 - ||x||_{\infty}\} \). It’s straightforward to check that \( g(x) \) is a CPWL function with at most \( 2^{n+1} \) affine polyhedral regions, and meanwhile is an integrable function with positive integral. As any \((n-2)\)-HH can be written as a finite sum of low-dimensional functions, it can’t represent \( g(x) \) by Lemma 2.2.

Theorem 2.3 implies that in order to achieve universal representation power over all CPWL functions, a \((n-1)\)-HH model is necessary which provides an almost tight lower bound corresponding to the upper bound in Theorem 1.3.

3 Implications on Universal Approximation of ANNs

Traditional universal approximation theorems of artificial neural networks (ANN) Cybenko [1989], Hornik et al. [1989], Barron [1994] typically states that an ANN with one hidden layer and unbounded width can approximate any measurable function with arbitrary precision on a compact set. Our result demonstrates that the compact set assumption is indeed necessary for ANNs with traditional activation (composition of an affine transform and a fixed univariate function \( \sigma \)):

**Corollary 3.1.** Given an integrable function \( f \) on \( R^n \) \((n \geq 2)\), for any one-hidden-layer neural network \( g \) with traditional activation \( \sigma(w^\top x + b) \), we have that

\[
\int_{R^n} |f - g| = \infty \quad \text{or} \quad \int_{R^n} |f - g| = \int_{R^n} |f|
\]

**Proof.** Any unit \( \sigma(w^\top x + b) \) is obviously a low-dimensional function when \( n \geq 2 \), thus by Lemma 2.2 we finish our proof.

Corollary 3.1 reveals a fundamental gap of representation power between one-hidden layer neural networks and deeper ones, as Theorem 1.3 indicates a neural network with \( \lceil \log_2(n + 1) \rceil \) hidden layers can represent any CPWL function He et al. [2018], showing the benefits of depth in universal approximation Lu et al. [2017].
4 Conclusion

In this note we give a sharp lower bound on the necessary number of nestings of nested absolute-value functions of generalized hinging hyperplanes (GHH) to represent arbitrary CPWL functions, which is the first non-trivial lower bound to the best of our knowledge. Our results fully characterizes the representation power (and limit) of the GHH model.

Our result also has implications on ANNs, a much more popular model in machine learning. It shows that one-hidden-layer neural networks with traditional activation can’t control the approximation error on the whole domain despite existing universal approximation theorems, a fundamental gap between one-hidden-layer networks and deeper ones. We conject similar depth-separation results should hold for deeper networks and the \( \lceil \log_2(n+1) \rceil \) bound should be tight in representing CPWL functions. Instead of low-dimensional (periodic), other properties need to be discovered for deeper networks.

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