On highly degenerate supersymmetric ground states of the fermion lattice model by Nicolai

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Abstract

We study a spinless fermion lattice model that has $\mathcal{N} = 2$ supersymmetry given by Hermann Nicolai [J. Phys. A. Math. Gen. 9(1976)]. We show that high degeneracy of supersymmetric ground states of the Nicolai model is associated to breakdown of local fermion symmetries hidden in the model.

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1 Introduction

Supersymmetry is a hypothesis of high energy physics that may give a clue to understand several fundamental issues beyond the standard model e.g. the hierarchy problem [10]. The essential idea of supersymmetry is to combine bosons and fermions in the same representation, and its characteristic structure produces several far-reaching consequences. From theoretical interests one may consider supersymmetry in much lower energy scales as well. Actually, as a simplest supersymmetric quantum field theory, supersymmetric quantum mechanics (SUSY QM) has been studied [11, 12].

In this note, we consider a fermion lattice model given by Nicolai [6]. It is made by fermions with no boson, however the exactly same algebraic relation as $\mathcal{N} = 2$ supersymmetry is satisfied. This supersymmetric many-body model will be called the Nicolai model. The Nicolai model is another (not well-known) pioneering work of non-relativistic supersymmetry, see [3] for some historical remark.

The Nicolai model is shown to have highly degenerate vacua (supersymmetric ground states). We show that the degeneracy of the vacua of the Nicolai model is related to breakdown of certain local fermion symmetries hidden in the model. In particular, we classify all classical supersymmetric vacua of the Nicolai model. The degenerate classical supersymmetric vacua of the Nicolai model are connected to each other by the action of broken local fermion charges. It would be interesting to compare the Nicolai model with other supersymmetric models that have a similar property: The supersymmetric fermion lattice model by [2] and certain Wess-Zumino supersymmetric quantum mechanical models have infinitely many degenerate vacua [9] [1].

1.1 Dynamical supersymmetry and fermion symmetries

We shall provide basis of supersymmetry necessary for the present investigation. We also fix some terminologies as well.

We are given some abstract algebra $\mathcal{A}$ to denote our quantum system. We will provide $\mathcal{A}$ with supersymmetric structure. Let $N$ be a positive operator whose eigenvalues are non-negative integers. It abstractly denotes a fermion number operator. Let us consider the grading automorphism $\Theta := \text{Ad}(-1)^N$. 

The natural grading structure is given to \( \mathcal{A} \) by this automorphism \( \Theta \):

\[
\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-, \quad \mathcal{A}_+ = \{ A \in \mathcal{A} \mid \Theta(A) = A \}, \quad \mathcal{A}_- = \{ A \in \mathcal{A} \mid \Theta(A) = -A \}.
\]

(1.1)

Define the graded commutator \([ \cdot, \cdot]_\Theta\) on \( \mathcal{A} \) by the mixture of the commutator \([ \cdot, \cdot]\) and the anti-commutator \(\{ \cdot, \cdot\}\) as

\[
[A_+, B]_\Theta = [A_+, B] \equiv A_+B - BA_+ \text{ for } A_+ \in \mathcal{A}_+, B \in \mathcal{A},
\]

\[
[A, B_+]_\Theta = [A, B_+] \equiv AB_+ - B_+A \text{ for } A \in \mathcal{A}, B_+ \in \mathcal{A}_+,
\]

\[
[A_-, B_-]_\Theta = \{A_-, B_-\} \equiv A_-B_- + B_-A_- \text{ for } A_-, B_- \in \mathcal{A}_-.
\]

(1.2)

Consider a conjugate pair of operators \( Q \) and \( Q^\ast \), where the asterisk \( \ast \) denotes the adjoint of linear operators. Assume that those are fermion operators:

\[
\{(-1)^N, Q\} = \{(-1)^N, Q^\ast\} = 0.
\]

(1.3)

Assume further that \( Q \) is nilpotent:

\[
Q^2 = 0.
\]

(1.4)

This implies that its adjoint is nilpotent as well:

\[
Q^{\ast 2} = 0.
\]

(1.5)

We define a supersymmetric Hamiltonian by

\[
H_{\text{susy}} := \{Q, Q^\ast\}.
\]

(1.6)

The pair of nilpotent fermion operators \( Q \) and \( Q^\ast \) are called supercharges, and \( H_{\text{susy}} \) is called the supersymmetric Hamiltonian associated to \( Q \) and \( Q^\ast \).

From (1.6) (1.3) it follows that

\[
[(-1)^N, H_{\text{susy}}] = 0.
\]

(1.7)

From (1.6) (1.4) (1.5) it follows that

\[
[H_{\text{susy}}, Q] = [H_{\text{susy}}, Q^\ast] = 0.
\]

(1.8)

Thus the (super-)transformation generated by the supercharges \( Q \) and \( Q^\ast \) is actually a symmetry (a constant of motion) for the Hamiltonian \( H_{\text{susy}} \).
call the set of algebraic relations (1.3) (1.4) (1.5) (1.6) as \( \mathcal{N} = 2 \) dynamical supersymmetry as in the supersymmetric theory, see e.g. [10]. It should be emphasized, however, that boson fields and fermion fields (satisfying CCRs and CARs, respectively) are not required here.

Now we shall consider the action of supercharges on the algebra \( \mathcal{A} \). Consider the superderivation generated by the nilpotent supercharge \( Q \):

\[
\delta_Q(A) := [Q, A]_\Theta \quad \text{for} \quad A \in \mathcal{A}.
\]

(1.9)

Strictly speaking, \( \delta_Q \) can be defined on \( A \in \mathcal{A} \) only when the right-hand side is well defined in some sense. We immediately see that \( \delta_Q \) is a linear map that anticommutes with the grading:

\[
\delta_Q \cdot \Theta = -\Theta \cdot \delta_Q.
\]

(1.10)

and that \( \delta_Q \) satisfies the graded Leibniz rule:

\[
\delta_Q(AB) = \delta_Q(A)B + \Theta(A)\delta_Q(B) \quad \text{for} \quad A, B \in \mathcal{A}.
\]

(1.11)

The conjugate superderivation is given by

\[
\delta_Q^*(A) := \delta_Q^*(A) \equiv [Q^*, A]_\Theta \quad \text{for} \quad A \in \mathcal{A}.
\]

(1.12)

We see that the nilpotent condition is satisfied:

\[
\delta_Q \cdot \delta_Q = 0 = \delta_Q^* \cdot \delta_Q^*.
\]

(1.13)

It is easy to see that

\[
d_{\text{H,susy}} = \delta_Q^* \cdot \delta_Q + \delta_Q \cdot \delta_Q^*,
\]

(1.14)

where \( d_{\text{H,susy}} \) denotes the derivation that gives an infinitesimal temporal translation:

\[
d_{\text{H,susy}}(A) := [H_{\text{susy}}, A] \quad \text{for} \quad A \in \mathcal{A}.
\]

(1.15)

Note that the domain of \( d_{\text{H,susy}} \) consists of the elements in \( \mathcal{A} \) for which the right-hand side can be defined and that \( d_{\text{H,susy}} \) satisfies the (usual) Leibniz rule.

We now specify meanings of “supersymmetric states” and “broken-unbroken supersymmetry”.
Definition 1.1. Assume dynamical supersymmetry generated by a pair of
nilpotent supercharges $Q$ and $Q^*$. If a state (normalized positive functional)
$\varphi$ on $\mathcal{A}$ is invariant under the superderivation $\delta_Q$, then $\varphi$ is called a supersymmetric ground state. If there exists such supersymmetric ground state, then the dynamical supersymmetry is called unbroken. Otherwise, the dynamical supersymmetry is called spontaneously broken.

The following familiar criterion can be derived from the above definition. See [4].

Proposition 1.2. Assume dynamical supersymmetry generated by a pair of
nilpotent supercharges $Q$ and $Q^*$. If $\varphi$ is a supersymmetric ground state of $\mathcal{A}$, then its vacuum vector $\Omega_\varphi$ (given by the GNS representation of $\varphi$ on $\mathcal{A}$) is annihilated by each of the supercharge operators $Q$ and $Q^*$. Here $Q$ and $Q^*$ are densely defined closed fermionic operators that implement $\delta_Q$ and $\delta_{Q^*}$, respectively.

Next we shall introduce a more general notion of fermion symmetries. Let
$\tilde{Q}$ denote a linear operator on a Hilbert space. Suppose that $\tilde{Q}$ is a fermion
operator:

$$\{(−1)^N, \tilde{Q}\} = 0.$$  \hfill (1.16)

Let $H$ denote a Hamiltonian operator. If

$$[H, \tilde{Q}] = [H, \tilde{Q}^*] = 0,$$ \hfill (1.17)

then it is said that $\tilde{Q}$ and its conjugate $\tilde{Q}^*$ generate a fermion symmetry. If $\tilde{Q}$ (and accordingly $Q^*$) is nilpotent:

$$\tilde{Q}^2 = 0, \quad \tilde{Q}^{*2} = 0,$$ \hfill (1.18)

then the fermion symmetry is referred to as a (hidden) supersymmetry. The dynamical supersymmetry is a special fermion symmetry. We will later discuss local fermion symmetries that are hidden in the Nicolai model.

We may consider broken-unbroken symmetry for such additional fermion symmetries. For the present purpose, we restrict ourselves to the following case only. (One may naturally consider other situations. For example, $\varphi$ is a thermal state.)

Definition 1.3. Suppose that the model has dynamical supersymmetry generated by a pair of nilpotent supercharges $Q$ and $Q^*$. Suppose that it has
another fermion symmetry generated by $\tilde{Q}$ and $\tilde{Q}^*$ as in (1.16) and (1.17).

Let $\varphi$ be a supersymmetric ground state of $\mathcal{A}$. If its vacuum vector $\Omega_\varphi$ is annihilated by each of $\tilde{Q}$ and $\tilde{Q}^*$, then the fermion symmetry is said to be unbroken for the state $\varphi$. Otherwise, the fermion symmetry is said to be broken for the state $\varphi$.

Note that this definition is state-dependent: Some fermion symmetry is broken for certain ground state, but it may be unbroken for other ground state.

**Remark 1.4.** In [7] another model with hidden supersymmetry (kinematical supersymmetry) is studied.

### 1.2 Supersymmetric fermion lattice model by Nicolai

We introduce a spinless fermion lattice model given by Nicolai [6]. By definition it has no boson. However, it has “dynamical supersymmetry”.

First we provide a general frame of spinless fermion lattice systems. For the present purpose, we consider one-dimensional integer lattice $\mathbb{Z}$. For each site $i \in \mathbb{Z}$ let $c_i$ and $c_i^*$ denote the annihilation and the creation of a spinless fermion at $i$. The canonical anticommutation relations (CARs) are satisfied.

For $i, j \in \mathbb{Z}$

$$
\{c_i^*, c_j\} = \delta_{i,j} 1,
\{c_i^*, c_j^*\} = \{c_i, c_j\} = 0.
$$

(1.19)

Let $|1\rangle_i$ and $|0\rangle_i$ denote the occupied and empty vectors of the spinless fermion at site $i$, respectively. For each $i \in \mathbb{Z}$

$$
c_i|1\rangle_i = |0\rangle_i, \quad c_i^*|1\rangle_i = 0, \quad c_i^*|0\rangle_i = |1\rangle_i, \quad c_i|0\rangle_i = 0.
$$

Sometimes we will omit the site from the subscript writing simply $|1\rangle$ and $|0\rangle$.

For each finite $I \subseteq \mathbb{Z}$, $\mathcal{A}(I)$ denotes the finite-dimensional algebra generated by $\{c_i^*, c_i; i \in I\}$, where the notation ‘$I \subseteq \mathbb{Z}$’ means that $I \subset \mathbb{Z}$ and $|I| < \infty$. Namely $\mathcal{A}(I)$ is the subsystem that consists of all the fermions in $I$.

By taking the union of all these $\mathcal{A}(I)$ let us define the local algebra:

$$
\mathcal{A} := \bigcup_{I \in \mathbb{Z}} \mathcal{A}(I).
$$

(1.20)
The norm completion of the local algebra $\mathcal{A}_o$ (called the CAR algebra) gives a total system and it is denoted by $\mathcal{A}$. The total system $\mathcal{A}$ together with its subsystems represents a infinitely many fermion system over $\mathbb{Z}$.

Let $\Theta$ denote the fermion grading automorphism on $\mathcal{A}$ given as:

$$
\Theta(c_i) = -c_i, \quad \Theta(c_i^*) = -c_i^*, \quad \forall i \in \mathbb{Z}.
$$

(1.21)

The fermion system $\mathcal{A}$ is decomposed into the even part $\mathcal{A}_+$ and the odd part $\mathcal{A}_-$ as

$$
\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-, \quad \mathcal{A}_+ = \{ A \in \mathcal{A} | \Theta(A) = A \}, \quad \mathcal{A}_- = \{ A \in \mathcal{A} | \Theta(A) = -A \},
$$

(1.22)

A general operator in $\mathcal{A}_+$ is a linear some of even monomials of fermion field operators, while a general operator in $\mathcal{A}_-$ is a linear some of odd monomials of fermion field operators. Similarly, for each $I \subseteq \mathbb{Z}$,

$$
\mathcal{A}(I) = \mathcal{A}(I)_+ \oplus \mathcal{A}(I)_-, \quad \mathcal{A}(I)_+ := \mathcal{A}(I) \cap \mathcal{A}_+, \quad \mathcal{A}(I)_- := \mathcal{A}(I) \cap \mathcal{A}_-,
$$

(1.23)

and for the local algebra

$$
\mathcal{A}_o = \mathcal{A}_o^+ \oplus \mathcal{A}_o^-, \quad \mathcal{A}_o^+ := \mathcal{A}_o \cap \mathcal{A}_+, \quad \mathcal{A}_o^- := \mathcal{A}_o \cap \mathcal{A}_-.
$$

(1.24)

We now provide the supersymmetric fermion lattice model by Nicolai [6].

Let

$$
Q_{\text{Nic}} = \sum_{i \in \mathbb{Z}} q_{2i}, \quad \text{where } q_{2i} := c_{2i+1}c^*_2c_{2i-1}.
$$

(1.25)

Thus we have

$$
Q_{\text{Nic}}^* = \sum_{i \in \mathbb{Z}} q_{2i}^* = \sum_{i \in \mathbb{Z}} -c_{2i+1}c^*_2c_{2i-1} = \sum_{i \in \mathbb{Z}} c^*_{2i-1}c_{2i}c^*_{2i+1}.
$$

(1.26)

Since we only discuss the Nicolai model, we omit the subscript “Nic”.

We see that $Q$ and $Q^*$ are fermion operators:

$$
\{(-1)^N, \ Q\} = \{(-1)^N, \ Q^*\} = 0,
$$

(1.27)

and that $Q$ and its adjoint are nilpotent:

$$
Q^2 = 0 = Q^{*2}.
$$

(1.28)

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1In [6 Sect.3] the half-sided lattice $\mathbb{N}$ is considered instead of the lattice $\mathbb{Z}$.

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The Hamiltonian is given by the following supersymmetric form:

\[ H := \{ Q, Q^\ast \}. \] (1.29)

The pair of supercharges \( Q, Q^\ast \) and the supersymmetric Hamiltonian \( H \) provide an \( \mathcal{N} = 2 \) supersymmetry model, although there is no boson involved in the model. The explicit form of \( H \) can be easily computed as

\[
H = \sum_{i \in \mathbb{Z}} \left\{ c_{2i}^* c_{2i+1}^* c_{2i+2} + c_{2i-1} c_{2i} c_{2i+3} c_{2i+2}^* \right. \\
\left. + c_{2i}^* c_{2i+1} c_{2i+1}^* + c_{2i-1} c_{2i} c_{2i}^* - c_{2i-1} c_{2i-1} c_{2i+1} c_{2i+1}^* \right\}. \] (1.30)

Let us note some obvious symmetries of the Nicolai model. The global \( U(1) \)-symmetry group \( \gamma_\theta (\theta \in [0, 2\pi]) \) is defined on \( \mathcal{A} \) by

\[
\gamma_\theta (c_i) = e^{-i\theta} c_i, \quad \gamma_\theta (c_i^*) = e^{i\theta} c_i^*, \quad \forall i \in \mathbb{Z}. \] (1.31)

The generator of \( U(1) \)-symmetry is the total fermion number over \( \mathbb{Z} \) which is heuristically written as

\[
N := \sum_{i \in \mathbb{Z}} c_i^* c_i. \] (1.32)

The particle-hole transformation is given by the \( \mathbb{Z}_2 \) action:

\[
\rho(c_i) = c_i^*, \quad \rho(c_i^*) = c_i, \quad \forall i \in \mathbb{Z}. \] (1.33)

Let \( \sigma \) denote the shift-translation automorphism group on \( \mathcal{A} \) defined by

\[
\sigma_k (c_i) = c_{i+k}, \quad \sigma_k (c_i^*) = c_{i+k}^*, \quad \forall i \in \mathbb{Z}, \text{ for each } k \in \mathbb{Z}. \] (1.34)

The Hamiltonian \( H \) (1.30) is invariant under the global \( U(1) \)-symmetry \( \gamma_\theta \) (\( \theta \in [0, 2\pi] \)). As \( \rho(H) = H \), it has the particle-hole symmetry. Note that the Nicolai model has 2-periodic translation symmetry: For any \( k \in \mathbb{Z}, \sigma_{2k}(H) = H \). However, the complete translation symmetry is explicitly broken: \( \sigma_{2k+1}(H) \neq H \). In addition to the above, we will see in §3 that the Nicolai model possesses other local fermion symmetries.

**Remark 1.5.** The Nicolai model has unbroken dynamical supersymmetry. Its variant considered in [8] breaks its dynamical supersymmetry.
2 Classical supersymmetric states of the Nicolai model

In this section we classify classical supersymmetric ground states of the Nicolai model. This section is essentially excerpted from [5].

2.1 Ground-state configurations

First we give some preliminaries. We shall express general (not necessarily supersymmetric) classical states on the fermion lattice system by classical configurations on the Fock state.

Definition 2.1. Let \( g(n) \) denote an arbitrary \( \{0, 1\} \)-valued function over \( \mathbb{Z} \). It is called a classical configuration over \( \mathbb{Z} \). For any classical configuration \( g(n) \) define

\[
|g(n)_{n \in \mathbb{Z}}\rangle := \cdots \otimes |g(i - 1)\rangle_{i-1} \otimes |g(i)\rangle_i \otimes |g(i + 1)\rangle_{i+1} \otimes \cdots \tag{2.1}
\]

This infinite product vector determines a state \( \psi_{g(n)} \) on the fermion system \( \mathcal{A} \) which will be called the classical state associated to the configuration \( g(n) \) over \( \mathbb{Z} \). Let \( \iota_0(n) := 0 \ \forall n \in \mathbb{Z} \). Then

\[
\Omega_0 := |\iota_0(n)_{n \in \mathbb{Z}}\rangle = \cdots \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \cdots . \tag{2.2}
\]

The above \( \Omega_0 \) is called the Fock vector, and its associated translation-invariant state \( \psi_0 \) on \( \mathcal{A} \) is called the Fock state. Similarly let \( \iota_1(n) := 1 \ \forall n \in \mathbb{Z} \). Then

\[
\Omega_1 := |\iota_1(n)_{n \in \mathbb{Z}}\rangle = \cdots \otimes |1\rangle \otimes |1\rangle \otimes |1\rangle \otimes |1\rangle \otimes |1\rangle \cdots . \tag{2.3}
\]

The above \( \Omega_1 \) is called the fully-occupied vector, and its associated translation-invariant state \( \psi_1 \) on \( \mathcal{A} \) is called the fully-occupied state.

To each classical configuration over \( \mathbb{Z} \) we assign an operator by the following rule.

Definition 2.2. For each \( i \in \mathbb{Z} \) let \( \hat{\kappa}_i \) denote the map from \( \{0, 1\} \) into \( \mathcal{A} \{i\} \) given as

\[
\hat{\kappa}_i(0) := 1, \quad \hat{\kappa}_i(1) := c_i^*. \tag{2.4}
\]
For each classical configuration $g(n)$ over $\mathbb{Z}$ define the infinite-product of fermion field operators:

$$\hat{O}(g) := \prod_{i \in \mathbb{Z}} \hat{\kappa}_i (g(i)) = \cdots \hat{\kappa}_{i-1} (g(i-1)) \hat{\kappa}_i (g(i)) \hat{\kappa}_{i+1} (g(i+1)) \cdots , \quad (2.5)$$

where the multiplication is taken in the increasing order. If $g(n)$ has a compact support, then

$$\hat{O}(g) \in \mathcal{A}_0. \quad (2.6)$$

Otherwise $\hat{O}(g)$ denotes a formal operator which does not belong to $\mathcal{A}$.

We have the following obvious correspondence between Definition 2.1 (product vectors) and Definition 2.2 (product operators) via the Fock representation.

**Proposition 2.3.** Let $\Omega_0$ denote the Fock vector given in (2.2). For any classical configuration $g(n)$ over $\mathbb{Z}$, the following identity holds:

$$\hat{O}(g) \Omega_0 = |g(n)_{n \in \mathbb{Z}}\rangle. \quad (2.7)$$

**Proof.** This directly follows from Definition 2.1 and Definition 2.2. \qed

It is easy to see that the Fock state $\psi_0$ and the fully-occupied state $\psi_1$ are supersymmetric ground state for the Nicolai model. Our next purpose is to specify all classical supersymmetric ground states of the Nicolai model. We introduce the following class of classical configurations.

**Definition 2.4.** Consider three consequent sites $\{2i-1, 2i, 2i+1\}$ centered at an even site $2i$ ($i \in \mathbb{Z}$). There are $2^3$ configurations (i.e. eight $\{0, 1\}$-valued functions) on $\{2i-1, 2i, 2i+1\}$. Let “0, 1, 0” and “1, 0, 1” be called forbidden triplets. If a classical configuration $g(n)$ ($n \in \mathbb{Z}$) does not include any of such forbidden triplets over $\mathbb{Z}$, then it is called a ground-state configuration over $\mathbb{Z}$ (for the Nicolai model). The set of all ground-state configurations over $\mathbb{Z}$ is denoted by $\Upsilon$. The set of all ground-state configurations whose support is included in some finite region is denoted by $\Upsilon_0$. The set of all ground-state configurations whose support is included in a finite region $I \subseteq \mathbb{Z}$ is denoted by $\Upsilon_I$. 

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By using Definition 2.4 we can classify all the classical supersymmetric ground states as follows. This statement will justify our nomenclature “ground-state configurations” given in Definition 2.4.

**Theorem 2.5.** A classical state on the fermion lattice system $\mathcal{A}$ is supersymmetric for the Nicolai model if and only if its associated configuration $g(n)$ over $\mathbb{Z}$ is a ground-state configuration for the Nicolai model as stated in Definition 2.4, namely, if and only if $g(n) \in \Upsilon$.

**Proof.** From some straightforward computation based on the canonical anticommutation relations (1.19) the statement follows. See [5] for the detail. □

### 2.2 Supersymmetric states on subsystems

We shall discuss supersymmetric states on finite subsystems. First, we specify finite regions that we will consider. For some purpose we mainly consider finite intervals of $\mathbb{Z}$ whose edges are both even. For each $k, l \in \mathbb{Z}$ such that $k < l$ let us take

$$I_{k,l} \equiv [2k, 2k + 1, 2(k + 1), \cdots , 2l - 1, 2l - 1, 2l]. \quad (2.8)$$

By definition $I_{k,l}$ consists of $2(l - k) + 1$ sites.

Occasionally we will use the following intervals

$$J_{k,l} \equiv [2k - 1, 2k, 2k + 1, \cdots , 2l - 1, 2l, 2l + 1]. \quad (2.9)$$

This finite interval $J_{k,l}$ $(k, l \in \mathbb{Z} k \leq l)$ is centered at an even site, and its edges are odd integers. We note the following obvious relationship between $I_{k,l}$ and $J_{k,l}$

$$J_{k,l} \supseteq I_{k,l}, \quad J_{k,l} = \{2k - 1\} \cup I_{k,l} \cup \{2l + 1\}, \quad |J_{k,l}| = 2(l - k + 1) + 1(\geq 3) \quad (2.10)$$

How to define supersymmetric states on subsystems seems not obvious a priori. In the following we specify meaning of supersymmetric states upon $I_{k,l}$.

**Definition 2.6.** Consider any finite interval $I_{k,l} \equiv [2k, 2k + 1, 2(k + 1), \cdots , 2l - 1, 2l - 1, 2l]$ $(k, l \in \mathbb{Z} k < l)$. Let

$$Q[k,l] \equiv \sum_{i=k}^{l} q_{2i} \in \mathcal{A} \{(2k - 1) \cup I_{k,l} \cup \{2l + 1\}\}, \quad (2.11)$$
where \( q_{2i} \equiv -c_{2i-1}c_{2i}^*c_{2i+1} \) as defined in (1.25). If \( k + 1 < l \), define

\[
Q[k + 1, l - 1] = \sum_{i=k+1}^{l-1} q_{2i} \in \mathcal{A}(I_{k,l} \setminus \{2k, 2l\})_*.
\]  

(2.12)

A state on \( \mathcal{A}(I_{k,l}) \) is said to be **open-edge supersymmetric** if its arbitrary state-extension to \( \mathcal{A}(\{2k-1\} \cup I_{k,l} \cup \{2l+1\}) = \mathcal{A}(J_{k,l}) \) is invariant under \( \delta_Q[k,l] \), equivalently, if its cyclic GNS cyclic vector is annihilated by both \( Q[k,l] \in \mathcal{A}(J_{k,l}) \) and its adjoint \( Q[k,l]^* \in \mathcal{A}(J_{k,l}) \). A state on \( \mathcal{A}(I_{k,l}) \) is said to be **close-edge supersymmetric** if its cyclic GNS vector is annihilated by both \( Q[k+1,l-1] \in \mathcal{A}(I_{k,l}) \) and its adjoint \( Q[k+1,l-1]^* \in \mathcal{A}(I_{k,l}) \).

In the above definition, we note that

\[
\delta_{Q[k,l]} = \delta_Q \text{ on } \mathcal{A}(I_{k,l}),
\]  

(2.13)

where \( Q = \sum_{i \in \mathbb{Z}} q_{2i} \) as defined in (1.25). Namely \( Q[k,l] \) generates a genuine action of the supercharge \( Q \) on the total system when being restricted to the smaller subsystem \( \mathcal{A}(I_{k,l}) \).

**Remark 2.7.** Definition 2.6 is defined for arbitrary states, not restricted to classical states. As we will see later in Proposition 2.9, the open-edge supersymmetric condition is stronger than the close-edge supersymmetric condition for classical states on \( I_{k,l} \). For the general case, we can not expect such simple implication due to edge effects of quantum states.

We now introduce a subclass of \( \Upsilon \) of Definition 2.4 requiring some boundary condition:

**Definition 2.8.** Take any segment region \( I_{k,l} \equiv [2k, 2k+1, 2(k+1), \ldots, 2(l-1), 2l-1, 2l] \) with \( k, l \in \mathbb{Z} \) \( k < l \). Assume that \( g(n) \in \Upsilon_{k,l} \equiv \Upsilon_{I_{k,l}} \) should be constant on each of the two-site edges:

\[
\begin{align*}
g(2k) &= g(2k+1) \quad \text{and} \quad g(2l-1) = g(2l).
\end{align*}
\]  

(2.14)

The above requirements will be called the open SUSY boundary condition. The set of all \( g(n) \in \Upsilon_{k,l} \) satisfying the open SUSY boundary condition is denoted by \( \hat{\Upsilon}_{k,l} \).

**Theorem 2.5** gives one-to-one correspondence between classical supersymmetric ground states and ground-state configurations. We can show analogous correspondence for finite regions \( I_{k,l} \) as follows.
Proposition 2.9. Let \( k, l \) be any integers such that \( k < l \). A classical state on the finite system \( \mathcal{A}(I_{k,l}) \) is open-edge supersymmetric (Definition 2.6) if and only if its associated configuration \( g(n) \) on \( I_{k,l} \) belongs to \( \hat{\Upsilon}_{k,l} \) (Definition 2.8). Any classical state on \( \mathcal{A}(I_{k,l}) \) is close-edge supersymmetric if it is open-edge supersymmetric.

**Proof.** The correspondence between classical open-edge supersymmetric states and the configurations spaces as stated can be verified in much the same way as in Theorem 2.5.

From the inclusion \( \hat{\Upsilon}_{k,l} \subset \Upsilon_{k,l} \) it follows that every classical open-edge supersymmetric state on \( \mathcal{A}(I_{k,l}) \) is close-edge supersymmetric. \( \square \)

3 Hidden local fermion symmetries

We will show that there are infinitely many local fermion symmetries hidden in the Nicolai model. We need some preparation.

**Definition 3.1.** Take any \( k, l \in \mathbb{Z} \) such that \( k < l \) and the finite interval \( I_{k,l} \) defined in (2.8). Let \( f \) be a \( \{-1, +1\} \)-valued sequence on the finite interval \( I_{k,l} \). For any consequent triplet \( \{2i - 1, 2i, 2i + 1\} \subset I_{k,l} \) \( (i \in \mathbb{Z}) \) assume that neither

\[
\begin{align*}
f(2i - 1) &= -1, \quad f(2i) = +1, \quad f(2i + 1) = -1, \quad (3.1) \\
nor \quad f(2i - 1) &= +1, \quad f(2i) = -1, \quad f(2i + 1) = +1, \quad (3.2)
\end{align*}
\]

is satisfied. Furthermore assume that \( f \) is constant on the left-end pair sites \( \{2k, 2k + 1\} \) and on the right-end pair sites \( \{2l - 1, 2l\} \). Namely, on the left-end pair

\[
f(2k) = f(2k + 1) = +1 \quad \text{or} \quad f(2k) = f(2k + 1) = -1 \quad (3.3)
\]

and on the right-end pair

\[
f(2l - 1) = f(2l) = +1 \quad \text{or} \quad f(2l - 1) = f(2l) = -1. \quad (3.4)
\]
The set of all \{-1, +1\}-valued sequences on \(I_{k,l}\) satisfying all the above conditions is denoted by \(\hat{\Xi}_{k,l}\). The union of \(\hat{\Xi}_{k,l}\) over all \(k, l \in \mathbb{Z} \ (k < l)\) is denoted by \(\hat{\Xi}\):

\[
\hat{\Xi} := \bigcup_{k, l \in \mathbb{Z} \ (k < l)} \hat{\Xi}_{k,l}.
\]  

(3.5)

Take any \(p, q \in \mathbb{Z} \ (p < q)\). Let

\[
\hat{\Xi}(p, q) := \bigcup_{k, l \in \mathbb{Z} ; \ p \leq k < l \leq q} \hat{\Xi}_{k, l}.
\]  

(3.6)

Each \(f \in \hat{\Xi}\) is called a local \{-1, +1\}-sequence of conservation for the Nicolai model.

Remark 3.2. The requirements (3.3) (3.4) on the edges of \(I_{k,l}\) are essential to make conservation laws for the Nicolai model.

Remark 3.3. By crude estimate we can see that the number of local \{-1, +1\}-sequences of conservation in \(\hat{\Xi}_{k,l}\) is roughly \(\left(\frac{3^3-2}{2}\right)^{(l-k)} = 3^{(l-k)} = 3^{m/2}\), where \(m = 2(l - k)\) denotes approximately the size of the system (i.e. the number of sites in \(I_{k,l}\)).

It is convenient to consider the following subclasses of \(\hat{\Xi}\).

Definition 3.4. For each \(k, l \in \mathbb{Z} \ (k < l)\) let \(r^+_{[2k,2l]} \in \hat{\Xi}_{k,l}\) and \(r^-_{[2k,2l]} \in \hat{\Xi}_{k,l}\) denote the constants over \(I_{k,l}\) as

\[
r^+_{[2k,2l]}(i) = +1 \ \forall i \in I_{k,l}, \quad r^-_{[2k,2l]}(i) = -1 \ \forall i \in I_{k,l}.
\]  

(3.7)

The set \(\left\{r^+_{[2k,2l]}\right\}\) over all \(k, l \in \mathbb{Z} \ (k < l)\) will be denoted as \(\hat{\Xi}^{+1}_{\text{const.}}\), and the set \(\left\{r^-_{[2k,2l]}\right\}\) over all \(k, l \in \mathbb{Z} \ (k < l)\) will be denoted as \(\hat{\Xi}^{-1}_{\text{const.}}\). Let \(\hat{\Xi}^{\text{const.}} := \hat{\Xi}^{+1}_{\text{const.}} \cup \hat{\Xi}^{-1}_{\text{const.}}\).

We shall give a rule to assign a local fermion operator for every local \{-1, +1\}-sequence of conservation in \(\hat{\Xi}\) of Definition 3.1.

Definition 3.5. For each \(i \in \mathbb{Z}\) let \(\zeta_i\) denote the assignment from \{-1, +1\} into the fermion field at \(i\) given as

\[
\zeta_i(-1) := c_i, \quad \zeta_i(+1) := c_i^*.
\]  

(3.8)
Take any pair of integers \( k, l \in \mathbb{Z} \) such that \( k < l \). For each \( f \in \hat{\Xi}_{k,l} \), set
\[
\mathcal{Q}(f) := \prod_{i=2k}^{2l} \zeta_i(f(i))
\]
\[
\equiv \zeta_{2k}(f(2k)) \zeta_{2k+1}(f(2k+1)) \cdots \zeta_{2l-1}(f(2l-1)) \zeta_{2l}(f(2l)) \in \mathcal{A}(I_{k,l}),
\]
where the multiplication is taken in the increasing order as above. The formulas (3.9) for all \( k, l \in \mathbb{Z} \) \((k < l)\) yield a unique assignment \( \mathcal{Q} \) from \( \hat{\Xi} \) to \( \mathcal{A} \).

By Definition 3.5 the following local fermion operators are assigned to \( \pm \)-characters supported on the segment \( I_{k,l} \) of Definition 3.4. For \( k, l \in \mathbb{Z} \) \((k < l)\)
\[
\mathcal{Q}(r_{[2k,2l]}^+) := c_{2k}^* c_{2k+1}^* \cdots c_{2l-1}^* c_{2l}^* \in \mathcal{A}(I_{k,l}),
\]
\[
\mathcal{Q}(r_{[2k,2l]}^-) := c_{2k} c_{2k+1} \cdots c_{2l-1} c_{2l} \in \mathcal{A}(I_{k,l}).
\]

The constant of motion or symmetry is given as an operator which is elementwise invariant under the Heisenberg time evolution. If it is a local operator, then it is called a local constant of motion, or a local symmetry. The following is the main result of this section.

**Theorem 3.6.** For every \( f \in \hat{\Xi} \)
\[
[H, \mathcal{Q}(f)] = 0 = [H, \mathcal{Q}(f)^*],
\]
where \( H \) denotes the Hamiltonian of the Nicolai model over \( \mathbb{Z} \).

**Proof.** This theorem is established as \( C^* \)-dynamics in [5]. Because of its importance and the reader’s convenience, we will provide a more formal derivation below.

It suffices to show that
\[
\{Q, \mathcal{Q}(f)\} = \{Q^*, \mathcal{Q}(f)^*\} = 0,
\]
and that
\[
\{Q, \mathcal{Q}(f)^*\} = \{Q^*, \mathcal{Q}(f)^*\} = 0.
\]
as the former implies \([H, \mathcal{Q}(f)] = 0\) and the latter implies \([H, \mathcal{Q}(f)^*] = 0\) by the graded Leibniz rule of superderivations (1.11). Recall \(Q = \sum_{i \in \mathbb{Z}} q_{2i}\) and \(q_{2i} \equiv c_{2i+1} c_{2i}^* c_{2i-1}\) defined in (1.25). By Definitions 3.1-3.5 we have
\[
\mathcal{Q}(f) q_{2i} = 0 = q_{2i} \mathcal{Q}(f), \quad \mathcal{Q}(f) q_{2i}^* = 0 = q_{2i}^* \mathcal{Q}(f) \quad \text{for all } i \in \mathbb{Z}.
\]
From the above we obtain (3.12) and (3.13).

Theorem 3.6 tells that the Nicolai model has infinitely many local fermion symmetries. Below we introduce terminologies relevant to this theorem.

**Definition 3.7.** For each local \([-1, +1]\)-sequence of conservation \(f \in \hat{\Xi}\), \(\mathcal{Q}(f)\) is called the local fermion constant of motion associated to \(f\), and the pair \(\{\mathcal{Q}(f), \mathcal{Q}(f)^*\}\) is called the local fermion charge associated to \(f\).

**Remark 3.8.** Any element of the algebra generated by \(\{\mathcal{Q}(f) \in A_0 | f \in \hat{\Xi}\}\) is a local constant of motion. It includes many self-adjoint bosonic operators that generate (bosonic) symmetries for the Nicolai model.

## 4 Degenerate classical supersymmetric states and broken local fermion symmetries

The purpose of this section is to relate the high-degeneracy of ground states shown in §3 to the existence of many local fermion symmetries shown in §3. We aim to establish that every classical supersymmetric ground state can be constructed from (broken) local fermion symmetries. We need some care on the choice of local subsystems.

**Theorem 4.1.** Take any segment \(I_{k,l}\) indexed by \(k, l \in \mathbb{Z} (k < l)\) as in (2.8). Any classical open-edge supersymmetric state on \(A(I_{k,l})\) (Definition 2.6) can be constructed by some finitely many actions of operators \(\mathcal{Q}(f)\) (and \(\mathcal{Q}(f)^*\)) with \(f \in \hat{\Xi}(k,l)\) (Definition 3.1) upon the Fock vector \(\Omega_0\) (2.2). Similarly, any classical open-edge supersymmetric state on \(A(I_{k,l})\) can be constructed by some finitely many actions of operators \(\mathcal{Q}(f)\) (and \(\mathcal{Q}(f)^*\)) with \(f \in \hat{\Xi}(k,l)\) upon the fully-occupied vector \(\Omega_1\) (2.3).

By Definition 2.1 we can identify any classical supersymmetric state \(\psi_{g(n)}\) on \(A\) with its corresponding classical configuration \(g(n)\) over \(\mathbb{Z}\), and vice versa. By Proposition 2.9 we can identify the set of all classical open-edge
supersymmetric states on $\mathcal{A}(I_{k,l})$ (Definition 2.6) with $\hat{\Upsilon}_{k,l}$ (Definition 2.8). We will frequently use these identifications in the following.

The following lemma implies the latter part (using $\Omega_1$) of Theorem 4.1 once the former part (using $\Omega_0$) is proved. Furthermore, it will be used in the main part of the proof.

**Lemma 4.2.** For any $n \in \mathbb{N}$, if $g \in \hat{\Upsilon}_{0,n}$ can be constructed by some finitely many actions of local fermion charges within $I_{0,n}$ upon the Fock vector $\Omega_0$. Then it can be constructed by some actions of local fermion charges within $I_{0,n}$ upon the fully-occupied state $\Omega_1$.

**Proof.** For any $f \in \hat{\Xi}(0,n)$, $-f \in \hat{\Xi}(0,n)$ by definition. From (3.9) in Definition 3.5

$$Q(-f) = \rho(Q(f)),$$

where $\rho$ is the particle-hole translation defined in (1.33). Thus by using the particle-hole translation, we can use $\Omega_0$ and $\Omega_1$ interchangeably. 

Obviously it is enough to show Theorem 4.1 by setting $k = 0$ and $l = \forall n \in \mathbb{N}$ by shift-translations. Thus we will prove the following.

**Proposition 4.3.** For any $n \in \mathbb{N}$, every $g \in \hat{\Upsilon}_{0,n}$ can be constructed by some actions of local fermion charges within $I_{0,n}$:

$$\left\{ \mathcal{D}(f); f \in \hat{\Xi}(0,n) \equiv \bigcup_{m \in \{1,2,\ldots,n\}} \hat{\Xi}_{0,m} \right\}$$

upon the Fock vector $\Omega_0$ (2.2).

**Proof.** We need concrete forms of elements in $\hat{\Xi}$ which are listed in §A.1. First, let us consider the case $n = 1$. $\hat{\Upsilon}_{0,1}$ consists of the following two sequences on $I_{0,1}$:

| $\hat{\Upsilon}_{0,1}$ | 0 | 1 | 2 |
|------------------------|---|---|---|
| $g^{\circ}_{[0,2]}$   | 0 | 0 | 0 |
| $g^{\bullet}_{[0,2]}$ | 1 | 1 | 1 |
The classical configuration $g^0_{[0,2]}$ corresponds to the Fock vector $\Omega_0$ (restricted to the local region $[0,1,2]$). We shall write simply $g^0_{[0,2]} = \Omega_0$, and this identification will be used hereafter. On the other hand, $g^4_{[0,2]}$ corresponds to $P(r^+_{[0,2]})\Omega_0$ which is the fully-occupied state on $I_{0,1}$, where $r^+_{[0,2]} \in \hat{\Xi}_{0,1}$. Thus we obtain $g^4_{[0,2]} = r^+_{[0,2]}\Omega_0$ which is the desired formula.

Second, let us consider the case $n = 2$. $\hat{\Upsilon}_{0,2}$ consists of the following 6 sequences on $I_{0,2}$:

| $\hat{\Upsilon}_{0,2}$ | 0 | 1 | 2 | 3 | 4 |
|------------------------|---|---|---|---|---|
| $g^0_{[0,4]}$          | 0 | 0 | 0 | 0 | 0 |
| $g^1_{[0,4]}$          | 0 | 0 | 0 | 1 | 1 |
| $g^2_{[0,4]}$          | 0 | 0 | 1 | 1 | 1 |
| $g^3_{[0,4]}$          | 1 | 1 | 1 | 0 | 0 |
| $g^4_{[0,4]}$          | 1 | 1 | 0 | 0 | 0 |
| $g^3_{[0,4]}$          | 1 | 1 | 1 | 1 | 1 |

We have $g^0_{[0,4]} = \Omega_0$ and $g^1_{[0,4]} = r^+_{[0,4]}\Omega_0 = c_0^* c_1^* c_2^* c_3^* c_4^* \Omega_0$ (the fully-occupied state on $I_{0,2}$) according to (4.3). We have

\[
\begin{align*}
  g^0_{[0,4]} &= r^+_{[0,2]}\Omega_0 = c_0^* c_1^* c_2^* \Omega_0, \quad r^+_{[0,2]} \in \hat{\Xi}_{0,1} \\
  g^1_{[0,4]} &= r^+_{[2,4]}\Omega_0 = c_2^* c_3^* c_4^* \Omega_0, \quad r^+_{[2,4]} \in \hat{\Xi}_{1,2}.
\end{align*}
\]

To get $g^1_{[0,4]}$ and $g^4_{[0,4]}$ we have to use ‘double’ actions. Precisely

\[
\begin{align*}
  g^1_{[0,4]} &= r^-_{[0,2]} g^0_{[0,4]} = r^-_{[0,2]} r^+_{[0,4]} \Omega_0 = c_3^* c_4^* \Omega_0, \quad r^-_{[0,2]} \in \hat{\Xi}_{0,1}, \quad r^+_{[0,4]} \in \hat{\Xi}_{0,2}, \quad (4.3)
\end{align*}
\]

and similarly

\[
\begin{align*}
  g^4_{[0,4]} &= r^-_{[2,4]} g^0_{[0,4]} = r^-_{[2,4]} r^+_{[0,4]} \Omega_0 = c_0^* c_1^* \Omega_0, \quad r^-_{[2,4]} \in \hat{\Xi}_{1,2}, \quad r^+_{[0,4]} \in \hat{\Xi}_{0,2}. \quad (4.4)
\end{align*}
\]

We have derived all the elements of $\hat{\Upsilon}_{0,2}$ and accordingly all the classical open-edge supersymmetric states on $A(I_{0,2})$ from the Fock vector $\Omega_0$. Let us note that $g^1_{[0,4]}$ and $g^4_{[0,4]}$ are mapped to each other by the particle-hole translation, and so are $g^0_{[0,4]}$ and $g^4_{[0,4]}$. However, as shown above, we do not need to use the particle-hole translation.

By an analogous manner, we can get all the elements of $\hat{\Upsilon}_{0,2}$ (all the classical open-edge supersymmetric states on $A(I_{0,2})$) from the fully-occupied

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vector $\Omega_1$ in place of $\Omega_0$. This fact is important.

Let us consider the case $n = 3$. $\hat{\Upsilon}_{0,3}$ consists of the following 18 sequences on $I_{0,3}$:

| $\hat{\Upsilon}_{0,3}$ | 0 1 2 3 4 5 6 |
|------------------------|---------------|
| $g_{0,6}^{0}$          | 0 0 0 0 0 0 0 |
| $g_{0,6}^{1}$          | 0 0 0 1 1 0 0 |
| $g_{0,6}^{2}$          | 0 0 1 1 0 0 0 |
| $g_{0,6}^{3}$          | 0 0 0 1 0 0 0 |
| $g_{0,6}^{4}$          | 0 0 1 1 1 0 0 |
| $g_{0,6}^{5}$          | 0 0 0 0 0 1 1 |
| $g_{0,6}^{6}$          | 0 0 0 0 1 1 1 |
| $g_{0,6}^{7}$          | 0 0 0 1 1 1 1 |
| $g_{0,6}^{8}$          | 1 1 1 1 1 0 0 |
| $g_{0,6}^{9}$          | 1 1 1 1 0 0 0 |
| $g_{0,6}^{10}$         | 1 1 1 0 0 0 0 |
| $g_{0,6}^{11}$         | 1 1 0 0 0 0 0 |
| $g_{0,6}^{12}$         | 1 1 1 1 1 1 1 |
| $g_{0,6}^{13}$         | 1 1 1 1 0 0 1 |
| $g_{0,6}^{14}$         | 1 1 0 0 1 1 1 |
| $g_{0,6}^{15}$         | 1 1 1 0 1 1 1 |
| $g_{0,6}^{16}$         | 1 1 0 0 0 1 1 |

We will generate all the above $g_{0,6}^{*}$. Obviously $g_{0,6}^{0} = \Omega_0$, and $g_{0,6}^{*} = r_{0,6}^{\frac{1}{n}} \Omega_0 = c_0^* c_1^* c_2^* c_3^* c_4^* c_5^* c_6^* \Omega_0$ which is the fully-occupied vector $\Omega_1$ restricted to $I_{0,3}$.

The restriction of $g_{0,6}^{0}$ to $[0, 4]$ is $g_{0,6}^{0}$, and the restriction of $g_{0,6}^{8}$ to $[0, 4]$ is $g_{0,4}^2$. The restriction of $g_{0,6}^{2}$ to $[2, 6]$ is $g_{0,6}^{4}$, which is the translate of $g_{0,4}^4$ used before. Thus each of $g_{0,6}^{0}, g_{0,6}^{4}$ and $g_{0,6}^{2}$ can be given by local actions upon $\Omega_0$ from the case $n = 2$.

The restriction of $g_{0,6}^{13}$ to $[0, 4]$ is $g_{0,4}^{3}$, the restriction of $g_{0,6}^{14}$ to $[2, 6]$ is $g_{0,6}^{2}$, the restriction of $g_{0,6}^{16}$ to $[0, 4]$ is $g_{0,4}^0$ (also the restriction of $g_{0,6}^{16}$ to $[2, 6]$ is $g_{2,6}^4$). Hence each of $g_{0,6}^{13}, g_{0,6}^{14}$ and $g_{0,6}^{16}$ can be given by local actions upon $\Omega_1$ from the case $n = 2$. Note that $\Omega_1 |_{0,6} = r_{0,6}^{\frac{1}{n}} \Omega_0 |_{0,6}$ with
\( r_{[0,6]}^+ \in \mathcal{A}(I_{0,3}) \). Therefore each of \( g_{[0,6]}^{13} \), \( g_{[0,6]}^{14} \) and \( g_{[0,6]}^{16} \) can be generated by local supercharges in \([0, 6]\) upon \( \Omega_0 \).

We have

\[
g_{[0,6]}^3 = r_{[0,2]}^r r_{[4,6]}^r \Omega_1 = r_{[0,2]}^r r_{[4,6]}^r r_{[0,6]}^+ \Omega_0 = c_3^r \Omega_0,
\]

and

\[
g_{[0,6]}^{15} = r_{[0,2]}^r r_{[4,6]}^r \Omega_0 = c_0^r c_2^r c_3^r c_5^r c_6^r \Omega_0.
\]

We have

\[
g_{[0,6]}^5 = r_{[0,4]}^r \Omega_1 = r_{[0,4]}^r r_{[0,6]}^r \Omega_0 = c_5^r \Omega_0,
\]

\[
g_{[0,6]}^6 = r_{[4,6]}^r \Omega_0 = c_4^r c_6^r \Omega_0,
\]

\[
g_{[0,6]}^7 = r_{[0,2]}^r \Omega_1 = r_{[0,2]}^r r_{[0,6]}^r \Omega_0 = c_3^r c_5^r \Omega_0,
\]

\[
g_{[0,6]}^8 = r_{[2,6]}^r \Omega_0 = c_2^r c_3^r c_4^r c_6^r \Omega_0,
\]

and similarly

\[
g_{[0,6]}^9 = r_{[0,4]}^r \Omega_0 = c_0^r c_1^r c_2^r c_3^r \Omega_0,
\]

\[
g_{[0,6]}^{10} = r_{[4,6]}^r \Omega_1 = r_{[4,6]}^r r_{[0,6]}^r \Omega_0 = c_0^r c_2^r c_3^r \Omega_0,
\]

\[
g_{[0,6]}^{11} = r_{[0,2]}^r \Omega_0 = c_0^r c_1^r c_2^r \Omega_0,
\]

\[
g_{[0,6]}^{12} = r_{[2,6]}^r \Omega_1 = r_{[2,6]}^r r_{[0,6]}^r \Omega_0 = c_0^r \Omega_0.
\]

We have now derived all the sequences of \( \hat{\Upsilon}_{0,3} \), i.e. all the classical open-edge supersymmetric states on \( \mathcal{A}(I_{0,3}) \).

We will start the argument of induction. We have verified the statement for \( n = 1, 2, 3 \). Now let us assume that the statement holds for any integer from \( 1 \in \mathbb{N} \) up to \( n \in \mathbb{N} \). We are going to show that the statement holds for \( n + 1 \in \mathbb{N} \). Concretely, we will construct \( \hat{\Upsilon}_{0,n+1} \) from \( \hat{\Upsilon}_{p,q} \) \((0 \leq p < q \leq n+1)\) where \( 0 < p \) or \( q < n + 1 \).

We divide \( \hat{\Upsilon}_{0,n+1} \) into four cases (Case I-IV) as below. We shall indicate how the induction argument can be applied to each of them.
Case I:
We deal with all \( g \in \hat{\Upsilon}_{0,n+1} \) whose left and right ends are
\[
g(0) = g(1) = 0, \quad g(2n + 1) = g(2(n + 1)) = 0.
\] (4.5)

\[
\begin{array}{cccccccc}
\hat{\Upsilon}_{0,n+1} & 0 & 1 & 2 & 3 & \cdots & \cdots & \cdots & 2n-1 & 2n & 2n+1 & 2(n+1) \\
I-1 & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & 0 \\
I-2 & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 1 & 1 & 0 & 0 \\
I-3 & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 1 & 0 & 0 & 0 \\
I-4 & 0 & 0 & 1 & 1 & \cdots & \cdots & \cdots & 0 & 0 & 0 & 0 \\
I-5 & 0 & 0 & 1 & 1 & \cdots & \cdots & \cdots & 1 & 1 & 0 & 0 \\
I-6 & 0 & 0 & 1 & 1 & \cdots & \cdots & \cdots & 1 & 0 & 0 & 0 \\
I-7 & 0 & 0 & 0 & 1 & \cdots & \cdots & \cdots & 0 & 0 & 0 & 0 \\
I-8 & 0 & 0 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 1 & 0 & 0 \\
I-9 & 0 & 0 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 0 & 0 & 0 \\
\end{array}
\]

Note that ‘\(*\)’s in the middle mean some appropriate sequences of 0, 1 so that the sequence belongs to \( \hat{\Upsilon}_{0,n+1} \), not being arbitrary.

All the above elements in \( \hat{\Upsilon}_{0,n+1} \) except I-9 belong to \( \hat{\Upsilon}_{1,n+1} \) or to \( \hat{\Upsilon}_{0,n} \) when being restricted to \([2, 2(n + 1)]\) or to \([0, 2n]\), respectively. By acting \( r_{[0,2]} \) upon I-9, we get

\[
\begin{array}{cccccccc}
\hat{\Upsilon}_{0,n+1} & 0 & 1 & 2 & 3 & \cdots & \cdots & \cdots & 2n-1 & 2n & 2n+1 & 2(n+1) \\
\text{New I-9} & 1 & 1 & 1 & 1 & \cdots & \cdots & \cdots & 1 & 0 & 0 & 0 \\
\end{array}
\]

“New I-9” above belongs to \( \hat{\Upsilon}_{1,n+1} \) when being restricted to \([2, 2(n + 1)]\).
Therefore we can obtain New I-9 by acting some local supercharges in \([2, 2(n + 1)]\) upon \( \Omega_1 \) (not \( \Omega_0 \) here). Note that \( \Omega_1 = r_{[0,2(n+1)]}^+ \Omega_0 \) on the segment \([0, 2(n + 1)]\) as noted in Lemma 4.2. Hence we can construct I-9 by acting some local supercharges in \([0, 2(n + 1)]\) upon \( \Omega_0 \). We have made all the configurations of Case I by the specified rule.

By acting \( r_{[2n,2(n+1)]}^+ \) upon I-9, we get

\[
\begin{array}{cccccccc}
\hat{\Upsilon}_{0,n+1} & 0 & 1 & 2 & 3 & \cdots & \cdots & \cdots & 2n-1 & 2n & 2n+1 & 2(n+1) \\
\text{New I-9(2)} & 0 & 0 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 1 & 1 & 1 \\
\end{array}
\]

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“New I-9(2)” above belongs to \( \hat{\Upsilon}_{0,n} \) when being restricted to \([0, 2n]\). Therefore we can obtain New I-9(2) by acting some local supercharges in \([0, 2n]\) upon \(\Omega_1\) (not \(\Omega_0\) here). By noting Lemma 4.2 we can construct I-9 by acting some local supercharges in \([0, 2(n + 1)]\) upon \(\Omega_0\).

Case II:
We deal with all \( g \in \hat{\Upsilon}_{0,n+1} \) whose left and right ends are
\[
g(0) = g(1) = 1, \quad g(2n + 1) = g(2(n + 1)) = 1.
\]
(4.6)
The proof for Case II can be done in the same way as done for Case I.

Case III:
We deal with all \( g \in \hat{\Upsilon}_{0,n+1} \) whose left and right ends are
\[
g(0) = g(1) = 0, \quad g(2n + 1) = g(2(n + 1)) = 1.
\]
(4.7)

| \(\hat{\Upsilon}_{0,n+1}\) | 0 | 1 | 2 | 3 | \ldots | \ldots | \ldots | 2n − 1 | 2n | 2n + 1 | 2(n + 1) |
|----------------|---|---|---|---|-------|-------|-------|--------|----|--------|--------|
| III-1          | 0 | 0 | 0 | 0 | ***   | ***   | ***   | 0       | 0  | 1      | 1      |
| III-2          | 0 | 0 | 0 | 0 | ***   | ***   | ***   | 1       | 1  | 1      | 1      |
| III-3          | 0 | 0 | 0 | 0 | ***   | ***   | ***   | 0       | 1  | 1      | 1      |
| III-4          | 0 | 0 | 1 | 1 | ***   | ***   | ***   | 0       | 0  | 1      | 1      |
| III-5          | 0 | 0 | 1 | 1 | ***   | ***   | ***   | 1       | 1  | 1      | 1      |
| III-6          | 0 | 0 | 1 | 1 | ***   | ***   | ***   | 0       | 1  | 1      | 1      |
| III-7          | 0 | 0 | 1 | 1 | ***   | ***   | ***   | 0       | 0  | 1      | 1      |
| III-8          | 0 | 0 | 1 | 1 | ***   | ***   | ***   | 1       | 1  | 1      | 1      |
| III-9          | 0 | 0 | 1 | 1 | ***   | ***   | ***   | 0       | 1  | 1      | 1      |

All the above elements in \( \hat{\Upsilon}_{0,n+1} \) except III-9 belong to \( \hat{\Upsilon}_{1,n+1} \) or to \( \hat{\Upsilon}_{0,n} \) when being restricted to \([2, 2(n + 1)]\) or to \([0, 2n]\), respectively. By acting \( r^+_{[0,2]} \) upon III-9, we get

| \(\hat{\Upsilon}_{0,n+1}\) | 0 | 1 | 2 | 3 | \ldots | \ldots | \ldots | 2n − 1 | 2n | 2n + 1 | 2(n + 1) |
|----------------|---|---|---|---|-------|-------|-------|--------|----|--------|--------|
| New III-9      | 1 | 1 | 1 | 1 | ***   | ***   | ***   | 0       | 1  | 1      | 1      |

“New III-9” above belongs to \( \hat{\Upsilon}_{1,n+1} \) when being restricted to \([2, 2(n + 1)]\). Therefore we can obtain New III-9 by acting some local supercharges in
[2, 2(n + 1)] upon Ω_1 (not Ω_0 here). Note that Ω_1 can be constructed from Ω_0 on [0, 2(n + 1)] by using local supercharges on [0, 2(n + 1)]. Thus we can construct I-9 by acting some local supercharges in [0, 2(n + 1)] upon Ω_0. We have completed the assertion for Case III.

Case IV:
We deal with all \( g \in \hat{\Upsilon}_{0,n+1} \) whose left and right ends are

\[
g(0) = g(1) = 1, \quad g(2n + 1) = g(2(n + 1)) = 0. \tag{4.8}
\]

The proof for Case IV is similar to that for Case III given above.

In conclusion, for all the cases (Case I-IV) we have generated all the elements of \( \hat{\Upsilon}_{0,n+1} \) from \( \hat{\Upsilon}_{0,n} \) and \( \hat{\Upsilon}_{1,n+1} \). Hence by the induction, we have shown the statement. \( \square \)

The number of classical supersymmetric states can be computed explicitly.

**Proposition 4.4.** The number of classical open-edge supersymmetric states on \( I_{0,n} \) \( (n \in \mathbb{N}) \) is \( 2 \cdot 3^{n-1} \).

**Proof.** This computation is given by the transfer-matrix method. We first divide \( I_{0,n} \) into \( n \)-sequential pairs as

\[
I_{0,n} = \{0, 1, 2\} \cup \{3, 4\} \cdots \cup \{2k - 1, 2k\} \cup \cdots \cup \{2n - 1, 2n\},
\]

where the first group exceptionally consists of three sites \( \{0, 1, 2\} \). On each \( \{2k - 1, 2k\} \) all classical configurations are possible. However, to connect \( \{2k - 1, 2k\} \) and \( \{2k + 1, 2(k + 1)\} \) we have to avoid the forbidden triplets: \( \{0, 1, 0\} \) \( \{1, 0, 1\} \) on \( \{2k - 1, 2k, 2k + 1\} \). So the transfer matrix should be

\[
T := \begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1
\end{bmatrix}.
\tag{4.9}
\]

By taking the edge condition \[2.13\] into account, the possible configurations
are any of

\begin{align*}
0, 0, 0, \ldots , 0, 0, \\
0, 0, 0, \ldots , 1, 1, \\
0, 0, 1, \ldots , 0, 0, \\
0, 0, 1, \ldots , 1, 1, \\
1, 1, 0, \ldots , 0, 0, \\
1, 1, 0, \ldots , 1, 1, \\
1, 1, 1, \ldots , 0, 0, \\
1, 1, 1, \ldots , 1, 1,
\end{align*}

which correspond to \((1, 1), (1, 4), (2, 1), (2, 4), (3, 1), (3, 4), (4, 1)\) and \((4, 4)\) elements of \(T^{n-1}\), respectively. Those amount to \(2 \cdot 3^{n-1}\).

One may argue that our choice of subregions \((I_{k,l})\) and the boundary condition (the open-edge supersymmetric condition) are artificial. However, as long as we consider classical states there is no loss of generality.

**Proposition 4.5.** Given any finite subset \(\Lambda\) of \(\mathbb{Z}\). Any supersymmetric classical state on \(\Lambda\) can be given by restriction of some open-edge supersymmetric classical state on some \(I_{k,l}\) such that \(I_{k,l} \supset \Lambda\).

**Proof.** First recall the one-to-one correspondence between the set of classical supersymmetric states on \(\Lambda\) and \(\Upsilon_{\Lambda}\) by Theorem 2.5. Recall the one-to-one correspondence between the set of classical open-edge supersymmetric states on \(I_{k,l}\) and \(\hat{\Upsilon}_{k,l}\) by Proposition 2.9. Thus any classical supersymmetric state on \(\Lambda\) can be extended to at least one classical open-edge supersymmetric state on \(I_{k,l}\) that includes \(\Lambda\).

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A Appendix

A.1 Forms of \( \hat{\Xi} \)

We will give concrete examples for local \( \{-1, +1\}\)-sequences of conservation of Definition 3.1 and their associated local fermion operators of Definition 3.5. First we see that \( \hat{\Xi}_{0,1} \) on \( I_{0,1} \equiv [0, 1, 2] \) consists of two \( \pm \)-characters only.
By (3.9) of Definition 3.5 the corresponding local fermion operators are

\[ Q(r^-_{[0,2]}) = c_0 c_1 c_2 \in A(I_{0,1})_-, \]
\[ Q(r^+_{[0,2]}) = c_0^* c_1^* c_2^* \in A(I_{0,1})_- \] (A.1)

We consider a next smallest segment \( I_{0,2} \equiv [0, 1, 2, 3, 4] \) by setting \( k = 0 \) and \( l = 2 \). The space \( \hat{\Xi}_{0,2} \) on \( I_{0,2} \) consists of the following five \( \{−1, +1\} \) sequences:

| \( \hat{\Xi}_{0,2} \) | 0 | 1 | 2 | 3 | 4 |
|----------------------|---|---|---|---|---|
| \( r^-_{[0,4]} \)    | −1| −1| −1| −1| −1|
| \( u^i_{[0,4]} \)    | −1| −1| −1| +1| +1|
| \( u^{ii}_{[0,4]} \)| −1| −1| +1| +1| +1|
| \( v^i_{[0,4]} \)    | +1| +1| +1| −1| −1|
| \( v^{ii}_{[0,4]} \)| +1| +1| −1| −1| −1|
| \( r^+_{[0,4]} \)    | +1| +1| +1| +1| +1|

Note that

\[ r^-_{[0,4]} = −r^+_{[0,4]}, \quad u^i_{[0,4]} = −u^{ii}_{[0,4]}, \quad u^{ii}_{[0,4]} = −u^i_{[0,4]} \] (A.2)

By (3.9) of Definition 3.5 we have

\[ Q(r^-_{[0,4]}) = c_0 c_1 c_2 c_3 c_4 \in A(I_{0,2})_-, \]
\[ Q(u^i_{[0,4]}) = c_0 c_1 c_2 c_3^* c_4^* \in A(I_{0,2})_- \]
\[ Q(u^{ii}_{[0,4]}) = c_0^* c_1^* c_2^* c_3 c_4 \in A(I_{0,2})_- \]
\[ Q(v^i_{[0,4]}) = c_0^* c_1 c_2 c_3^* c_4 \in A(I_{0,2})_- \]
\[ Q(v^{ii}_{[0,4]}) = c_0 c_1^* c_2^* c_3 c_4 \in A(I_{0,2})_- \] (A.3)

We then consider the segment \( I_{0,3} \equiv [0, 1, 2, 3, 4, 5, 6] \) taking \( k = 0 \) and \( l = 3 \). By definition it consists of \( 5 + 4 + 4 + 5 = 18 \) \( \{−1, +1\} \)-sequences:
Note that $s_{[0,6]}^\circ \equiv r_{[0,6]}^-$ and $t_{[0,6]}^\bullet \equiv r_{[0,6]}^+$ and that

$$
\begin{align*}
    s_{[0,6]}^{\circ} &= -t_{[0,6]}^\bullet, \quad s_{[0,6]}^{\circ k} = -t_{[0,6]}^{k}, \quad \forall k \in \{i, ii, iii, iv\} \\
    u_{[0,6]}^{k} &= -v_{[0,6]}^{k}, \quad \forall k \in \{i, ii, iii, iv\}. \quad \text{(A.4)}
\end{align*}
$$

According to the rule we obtain the following list of 18 fermion operators

| $\Xi_{0,3}$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|-----------|---|---|---|---|---|---|---|
| $s_{[0,6]}^{\circ}$ | -1 | -1 | -1 | -1 | -1 | -1 | -1 |
| $s_{[0,6]}^i$ | -1 | -1 | -1 | +1 | +1 | -1 | -1 |
| $s_{[0,6]}^{ii}$ | -1 | -1 | +1 | +1 | -1 | -1 | -1 |
| $s_{[0,6]}^{iii}$ | -1 | -1 | -1 | +1 | -1 | -1 | -1 |
| $s_{[0,6]}^{iv}$ | -1 | -1 | +1 | +1 | +1 | -1 | -1 |
| $u_{[0,6]}^i$ | -1 | -1 | -1 | -1 | -1 | +1 | +1 |
| $u_{[0,6]}^{ii}$ | -1 | -1 | -1 | -1 | +1 | +1 | +1 |
| $u_{[0,6]}^{iii}$ | -1 | -1 | -1 | +1 | +1 | +1 | +1 |
| $u_{[0,6]}^{iv}$ | -1 | -1 | +1 | +1 | +1 | +1 | +1 |
| $v_{[0,6]}^i$ | +1 | +1 | +1 | +1 | +1 | -1 | -1 |
| $v_{[0,6]}^{ii}$ | +1 | +1 | +1 | +1 | -1 | -1 | -1 |
| $v_{[0,6]}^{iii}$ | +1 | +1 | +1 | -1 | +1 | +1 | +1 |
| $v_{[0,6]}^{iv}$ | +1 | +1 | +1 | -1 | +1 | +1 | +1 |
| $t_{[0,6]}^i$ | +1 | +1 | +1 | +1 | +1 | +1 | +1 |
| $t_{[0,6]}^{ii}$ | +1 | +1 | +1 | +1 | +1 | +1 | +1 |
| $t_{[0,6]}^{iii}$ | +1 | +1 | +1 | -1 | -1 | +1 | +1 |
| $t_{[0,6]}^{iv}$ | +1 | +1 | -1 | -1 | -1 | +1 | +1 |
associated to $\hat{\Xi}_{0,3}$:

\[
\mathcal{D}(\xi^0_{[0,6]}) = c_0c_1c_2c_3c_4c_5c_6 \in \mathcal{A}(I_{0,3})_-, \\
\mathcal{D}(\xi^i_{[0,6]}) = c_0c_1c_2c_3c_4c_5c_6 \in \mathcal{A}(I_{0,3})_-, \\
\mathcal{D}(\xi^{ii}_{[0,6]}) = c_0c_1c_2c_3c_4c_5c_6 \in \mathcal{A}(I_{0,3})_-, \\
\mathcal{D}(\xi^{iii}_{[0,6]}) = c_0c_1c_2c_3c_4c_5c_6 \in \mathcal{A}(I_{0,3})_-, \\
\mathcal{D}(\xi^{iv}_{[0,6]}) = c_0c_1c_2c_3c_4c_5c_6 \in \mathcal{A}(I_{0,3})_-, \\
\mathcal{D}(\xi^{v}_{[0,6]}) = c_0c_1c_2c_3c_4c_5c_6 \in \mathcal{A}(I_{0,3})_-, \\
\mathcal{D}(\xi^{vi}_{[0,6]}) = c_0c_1c_2c_3c_4c_5c_6 \in \mathcal{A}(I_{0,3})_-, \\
\mathcal{D}(\xi^{vii}_{[0,6]}) = c_0c_1c_2c_3c_4c_5c_6 \in \mathcal{A}(I_{0,3})_-, \\
\mathcal{D}(\xi^{viii}_{[0,6]}) = c_0c_1c_2c_3c_4c_5c_6 \in \mathcal{A}(I_{0,3})_-. \\
\mathcal{D}(\xi^{ix}_{[0,6]}) = c_0c_1c_2c_3c_4c_5c_6 \in \mathcal{A}(I_{0,3})_-. \\
\mathcal{D}(\xi^{x}_{[0,6]}) = c_0c_1c_2c_3c_4c_5c_6 \in \mathcal{A}(I_{0,3})_-. \\
\mathcal{D}(\xi^{xi}_{[0,6]}) = c_0c_1c_2c_3c_4c_5c_6 \in \mathcal{A}(I_{0,3})_-. \\
\mathcal{D}(\xi^{xii}_{[0,6]}) = c_0c_1c_2c_3c_4c_5c_6 \in \mathcal{A}(I_{0,3})_-. \\
\mathcal{D}(\xi^{xiii}_{[0,6]}) = c_0c_1c_2c_3c_4c_5c_6 \in \mathcal{A}(I_{0,3})_.
\]

(A.5)

Finally we consider the segment $I_{0,4} \equiv [0, 1, 2, 3, 4, 5, 6, 7, 8]$ by setting $k = 0$ and $l = 4$. It consists of the following $54 (= 13 \cdot 2 + 14 \cdot 2)$ $\{−1, +1\}$-sequences.
| $\Xi_{0.4}$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $s_{[0,8]}^0$ | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 |
| $s_{[0,8]}^1$ | -1 | -1 | -1 | +1 | -1 | -1 | -1 | -1 | -1 |
| $s_{[0,8]}^{ii}$ | -1 | -1 | -1 | -1 | -1 | +1 | -1 | -1 | -1 |
| $s_{[0,8]}^{iii}$ | -1 | -1 | -1 | +1 | +1 | -1 | -1 | -1 | -1 |
| $s_{[0,8]}^{iv}$ | -1 | -1 | -1 | +1 | +1 | +1 | -1 | -1 | -1 |
| $s_{[0,8]}^{v}$ | -1 | -1 | -1 | -1 | -1 | +1 | -1 | -1 | -1 |
| $s_{[0,8]}^{vi}$ | -1 | -1 | -1 | +1 | +1 | +1 | +1 | -1 | -1 |
| $s_{[0,8]}^{vii}$ | -1 | -1 | +1 | +1 | +1 | +1 | +1 | -1 | -1 |
| $s_{[0,8]}^{viii}$ | -1 | -1 | -1 | +1 | +1 | +1 | +1 | -1 | -1 |
| $s_{[0,8]}^{ix}$ | -1 | -1 | -1 | +1 | +1 | +1 | +1 | -1 | -1 |
| $s_{[0,8]}^{x}$ | -1 | -1 | +1 | +1 | +1 | +1 | +1 | -1 | -1 |
| $s_{[0,8]}^{xi}$ | -1 | -1 | +1 | +1 | +1 | +1 | +1 | -1 | -1 |
| $s_{[0,8]}^{xii}$ | -1 | -1 | +1 | +1 | +1 | +1 | +1 | -1 | -1 |

| $t_{[0,8]}^{*}$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $t_{[0,8]}^{1}$ | +1 | +1 | +1 | +1 | +1 | +1 | +1 | +1 | +1 |
| $t_{[0,8]}^{ii}$ | +1 | +1 | +1 | -1 | +1 | +1 | +1 | +1 | +1 |
| $t_{[0,8]}^{iii}$ | +1 | +1 | +1 | -1 | -1 | +1 | +1 | +1 | +1 |
| $t_{[0,8]}^{iv}$ | +1 | +1 | +1 | +1 | +1 | -1 | -1 | +1 | +1 |
| $t_{[0,8]}^{v}$ | +1 | +1 | +1 | -1 | -1 | -1 | +1 | +1 | +1 |
| $t_{[0,8]}^{vi}$ | +1 | +1 | +1 | +1 | +1 | -1 | -1 | -1 | +1 |
| $t_{[0,8]}^{vii}$ | +1 | +1 | +1 | -1 | -1 | -1 | -1 | +1 | +1 |
| $t_{[0,8]}^{viii}$ | +1 | +1 | +1 | -1 | -1 | -1 | -1 | +1 | +1 |
| $t_{[0,8]}^{ix}$ | +1 | +1 | +1 | -1 | -1 | -1 | -1 | +1 | +1 |
| $t_{[0,8]}^{x}$ | +1 | +1 | +1 | -1 | -1 | -1 | -1 | +1 | +1 |
| $t_{[0,8]}^{xi}$ | +1 | +1 | +1 | -1 | -1 | -1 | -1 | +1 | +1 |
| $t_{[0,8]}^{xii}$ | +1 | +1 | +1 | -1 | -1 | -1 | -1 | +1 | +1 |
| $\Xi_{0,4}$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|-------------|----|----|----|----|----|----|----|----|----|
| $u^1_{[0,8]}$ | $-1$ | $-1$ | $-1$ | $-1$ | $-1$ | $-1$ | $-1$ | $+1$ | $+1$ |
| $u^2_{[0,8]}$ | $-1$ | $-1$ | $-1$ | $+1$ | $-1$ | $-1$ | $-1$ | $+1$ | $+1$ |
| $u^3_{[0,8]}$ | $-1$ | $-1$ | $-1$ | $-1$ | $-1$ | $-1$ | $+1$ | $+1$ |
| $u^4_{[0,8]}$ | $-1$ | $-1$ | $+1$ | $+1$ | $-1$ | $-1$ | $-1$ | $+1$ | $+1$ |
| $u^5_{[0,8]}$ | $-1$ | $-1$ | $-1$ | $+1$ | $+1$ | $-1$ | $-1$ | $+1$ | $+1$ |
| $u^6_{[0,8]}$ | $-1$ | $-1$ | $-1$ | $-1$ | $+1$ | $+1$ | $+1$ | $+1$ |
| $u^7_{[0,8]}$ | $-1$ | $-1$ | $-1$ | $-1$ | $+1$ | $+1$ | $+1$ |

| $\Xi_{0,4}$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|-------------|----|----|----|----|----|----|----|----|----|
| $\tilde{v}^1_{[0,8]}$ | $+1$ | $+1$ | $+1$ | $+1$ | $+1$ | $+1$ | $-1$ | $-1$ |
| $\tilde{v}^2_{[0,8]}$ | $+1$ | $+1$ | $+1$ | $-1$ | $+1$ | $+1$ | $-1$ | $-1$ |
| $\tilde{v}^3_{[0,8]}$ | $+1$ | $+1$ | $+1$ | $+1$ | $+1$ | $-1$ | $-1$ | $-1$ |
| $\tilde{v}^4_{[0,8]}$ | $+1$ | $+1$ | $-1$ | $-1$ | $+1$ | $+1$ | $-1$ | $-1$ |
| $\tilde{v}^5_{[0,8]}$ | $+1$ | $+1$ | $+1$ | $-1$ | $-1$ | $+1$ | $-1$ | $-1$ |
| $\tilde{v}^6_{[0,8]}$ | $+1$ | $+1$ | $+1$ | $+1$ | $+1$ | $+1$ | $-1$ | $-1$ |
| $\tilde{v}^7_{[0,8]}$ | $+1$ | $+1$ | $+1$ | $+1$ | $+1$ | $+1$ |

We will omit listing of the fermion operators associated to $\hat{\Xi}_{0,4}$ given above.