WEINGARTEN AND LINEAR WEINGARTEN
CANAL SURFACES

YILMAZ TUNCER, DAE WON YOON, AND MURAT KEMAL KARACAN

Abstract. In this study, we investigated the \((K,H)\), \((K,K_{II})\), \((H,K_{II})\)−Weingarten and \((K,H),(K,K_{II}),(H,K_{II})\) and \((K,H,K_{II})\)−linear Weingarten canal surfaces in \(\mathbb{R}^3\).

1. Introduction

In 1863, Julius Weingarten was able to make a major step forward in the topic when he gave a class of surfaces isometric to a given surface of revolution. Surface for which there is a definite functional relation between the principal curvatures (which called curvature diagram) and also between the Gaussian and the mean curvatures is called Weingarten surface. The knowledge of first fundamental form I and second fundamental form II of a surface facilitates the analysis and the classification of surface shape. Especially recent years, the geometry of the second fundamental form II has become an important issue in terms of investigating intrinsic and extrinsic geometric properties of the surfaces. Very recent results concerning the curvature properties associated to II and other variational aspects can be found in [19]. One may associate to such a surface \(M\) geometrical objects measured by means of its second fundamental form, as second Gaussian curvature \(K_{II}\), respectively. We are able to compute \(K_{II}\) of a surface by replacing the components of the first fundamental form \(E, F, G\) by the components of the second fundamental form \(e, f, g\) respectively in Brioschi formula which is given by Francesco Brioschi in the years of 1800’s. Identification of the curvatures related to the second fundamental form of a surface opened a door to research the new classes of Weingarten surfaces. Since the middle of the last century, several geometers have studied Weingarten surfaces and linear Weingarten surfaces and obtained many interesting and valuable results [3, 4, 9, 14, 16, 18]. For study of these surfaces, W. Kühnel and G.Stamou investigate ruled \((X,Y)\)-Weingarten surface in Euclidean 3-space \(E^3\) [14, 15]. Also, C.Baikoussis and Th. Koufogiorgos studied helicoidal \((H,K_{II})\)-Weingarten surfaces [4]. F.Dillen and W. Kühnel and F.Dillen and W. Sodsiri gave a classification of ruled \((X,Y)\)-Weingarten surface in Minkowski 3-space \(E^3_1\), where \(X,Y\in\{K,H,K_{II}\}\) [3, 4, 5]. D. Koutroufiotis and Th. Koufogiorgos and T. Hasanis investigate closed ovaloid \((X,Y)\)-linear Weingarten surface in \(E^3_1\) [12, 13]. D. W. Yoon and D.E.Blair and Th. Koufogiorgos classified ruled \((X,Y)\)-linear Weingarten surface in \(E^3_1\) [2, 23]. D. W. Yoon and J.S.Ro studied tubes in Euclidean 3-space which are

Date: June 16, 2011.
2000 Mathematics Subject Classification. 53A05.
Key words and phrases. Canal surface, Gaussian curvatures, Mean curvature, Second mean curvature, Weingarten and Linear Weingarten surfaces.
This paper is in final form and no version of it will be submitted for publication elsewhere.
(K,H),(K,K_{II}),(H,K_{II})-Weingarten and linear Weingarten tubes [17]. D. W. Yoon also studied the Weingarten and linear Weingarten types translation surfaces in Euclidean 3-space.

Surface theory has been a popular topic for many researchers in many aspects. Besides the using curves and surfaces, canal surfaces are the most popular in computer aided geometric design such that designing models of internal and external organs, preparing of terrain-infrastructures, constructing of blending surfaces, reconstructing of shape, robotic path planning, etc. (see, [6, 20, 21]).

In this study, we investigated the (K,H), (K,K_{II}), (H,K_{II})-Weingarten and (K,H),(K,K_{II}),(H,K_{II}) and (K,H,K_{II})-linear Weingarten canal surfaces in IR^3 by using the definition of general canal surfaces. During the study, we faced a very large equations. It was not possible to give them all of course. So we had to make our processes via a computer time to time.

Let \( f \) and \( g \) be smooth functions on a surface \( M \) in Euclidean 3-space \( E^3 \). The Jacobi function \( \Phi (f,g) \) formed with \( f, g \) is defined by

\[
\Phi (f,g) = f_s g_t - f_t g_s
\]

where \( f_s = \frac{\partial f}{\partial s} \) and \( f_t = \frac{\partial f}{\partial t} \). In particular, a surface satisfying the Jacobi equation \( \Phi (K,H) = 0 \) with respect to the Gaussian curvature \( K \) and the mean curvature \( H \) on a surface \( M \) is called a Weingarten surface. Also, if a surface satisfies a linear equation with respect to \( K \) and \( H \), that is, \( aK + bH = c \) \((a, b, c \in \mathbb{R}, (a, b, c) \neq (0, 0, 0))\), then it is said to be a linear Weingarten surface [17].

When the constant \( b = 0 \), a linear Weingarten surface \( M \) reduces to a surface with constant Gaussian curvature. When the constant \( a = 0 \), a linear Weingarten surface \( M \) reduces to a surface with constant mean curvature. In such a sense, the linear Weingarten surfaces can be regarded as a natural generalization of surfaces with constant Gaussian curvature or with constant mean curvature [17].

If the second fundamental form of a surface \( M \) in \( E^3 \) is non-degenerate, then it is regarded as a new pseudo-Riemannian metric. Therefore, the Gaussian curvature \( K_{II} \) of non-degenerate second fundamental form can be defined formally on the Riemannian or pseudo-Riemannian manifold \((M,II)\). We call the curvature \( K_{II} \) the second Gaussian curvature on \( M \) [17].

Following the Jacobi equation and the linear equation with respect to the Gaussian curvature \( K \), the mean curvature \( H \) and the second Gaussian curvature \( K_{II} \) an interesting geometric question is raised: Classify all surfaces in Euclidean 3-space satisfying the conditions

\[
\Phi (X, Y) = 0
\]

\[
aX + bY = c
\]

where \((X,Y) \in \{K,H,K_{II}\}, X \neq Y \) and \((a, b, c) \neq (0, 0, 0)\). Let \( M \) be a surface immersed in Euclidean 3-space, the first fundamental form of the surface \( M \) is defined by

\[
I = Edu^2 + 2Fdu dv + Gdv^2
\]

where \( E = < M_s, M_s >, F = < M_s, M_t >, G = < M_t, M_t > \) are the coefficients of \( I \). A surface is called degenerate if it has the degenerate first fundamental form. The second fundamental form of \( M \) is given by

\[
II = edu^2 + 2fdu dv + gdv^2
\]
where \( e = \langle M_{ss}, n \rangle \), \( f = \langle M_{st}, n \rangle \), \( g = \langle M_{tt}, n \rangle \) and \( n \) is the unit normal of \( M \). The Gaussian curvature \( K \) and the mean curvature \( H \) are given by, respectively

\[
K = \frac{eg - f^2}{EG - F^2},
\]

\[
H = \frac{Eq - 2Ff + Ge}{2(EG - F^2)}.
\]

A regular surface is flat if and only if its Gaussian curvature vanishes identically. A minimal surface in \( \mathbb{R}^3 \) is a regular surface for which mean curvature vanishes identically\(^7\).

Furthermore, the second Gaussian curvature \( K_{II} \) of a surface is defined by

\[
K_{II} = \frac{1}{(|eg| - f^2)^2} \begin{vmatrix}
-\frac{1}{2}e_{tt} + f_{s\bar{t}} - \frac{1}{2}g_{ss} & \frac{1}{2}e_s & f_s - \frac{1}{2}e_{t} & 0 \\
-f_s & e & f & 0 \\
\frac{1}{2}e_{t} & f & g & 0 \\
\frac{1}{2}g_{s} & f & g & 0
\end{vmatrix}.
\]

A surface is called II-flat if the second Gaussian curvature vanishes identically\(^2\).

A canal surface is an envelope of a 1-parameter family of spheres in \( \mathbb{R}^3 \) is called a canal surface\(^7\). The curve formed by the centers of the spheres is called center curve of the canal surface. The radius of canal surface is the function \( r \) such that \( r(s) \) is the radius of the sphere \( S^2(s) \). Suppose that the center curve of a canal surface is a unit speed curve \( \alpha : I \rightarrow \mathbb{R}^3 \). Then the canal surface can be parametrized by the formula

\[
C(s, t) = \alpha(s) - R(s)T - Q(s)\cos(t)N + Q(s)\sin(t)B
\]

where

\[
R(s) = r(s)r'(s)
\]

\[
Q(s) = \pm r(s)\sqrt{1 - r'(s)^2}.
\]

All the tubes and the surfaces of revolution are subclass of the canal surface.

**Theorem 1.** The center curve of a canal surface \( M \) is a straight line if and only if \( M \) is a surface of revolution for which no normal line to the surface is parallel to the axis of revolution\(^7\).

**Theorem 2.** The following conditions are equivalent for a canal surface \( M \):

i. \( M \) is a tube parametrized by \((1.4)\);

ii. the radius of \( M \) is constant;

iii. the radius vector of each sphere in family that defines the canal surface \( M \) meets the center curve orthogonally\(^7\).

Coefficients of first and second fundamental forms of canal surface are

\[
E = Q^2\kappa^2 \cos^2(t) + (2Q - 2QR' + 2Q'R)\kappa \cos(t) + 2QR\kappa \tau \sin(t) + Q^2\tau^2 + (Q')^2 + R^2\kappa^2 + 1 - 2R' + (R')^2
\]

\[
F = -Q(R\kappa \sin(t) + Q\tau)
\]

\[
G = Q^2
\]
\[ EG - F^2 = Q^2 \left\{ \kappa^2 (R^2 + Q^2) \cos^2(t) + 2\kappa (Q'R - QR' + Q) \cos(t) \right\}, \]
\[ (1.7) \]

and
\[ e = -\frac{1}{r} \left\{ Q^2\kappa^2 \cos^2(t) + (2QR' - 2QR' + Q) \kappa \cos(t) \right\} \]
\[ f = \frac{1}{r} \{ Q(R\kappa \sin(t) + Qr) \} \]
\[ g = -\frac{Q^2}{r^2}. \]

If \( Q(s) = 0 \) then first and second fundamental forms are degenerate so the canal surface is degenerate surface and the radius is \( r(s) = \pm s + c \). Furthermore, in the case \( \kappa(s) = 0 \) and \( 1 - 2R' + (R')^2 + (Q')^2 = 0 \), the radius is
\[ r(s) = \sqrt{s^2 - 2c_1 s + 2c_2}. \]

Let center curve be \( \alpha(s) = (s, 0, 0) \) then \( T = e_1, \, N = e_2 \) and \( B = e_3 \). Thus \( R(s) = s - c_1 \) so \( C(s, t) \) is the curve in the plane \( x = c_1 \). The necessary conditions are \( r(s) \neq \pm s + c \) and \( r(s) \neq \sqrt{s^2 - 2c_1 s + 2c_2} \) for \( \kappa(s) = 0 \) to define a surface of the equation \( (1.9) \). At this point, we can write the following theorem.

**Theorem 3.** Let \( M \) be a canal surface with the center curve \( \alpha(s) \) and the radius \( r(s) \). Then \( M \) is a regular surface in \( IR^3 \) if the radius is \( r(s) \neq \pm s + c \) and \( r(s) \neq \sqrt{s^2 - 2c_1 s + 2c_2} \) for \( \kappa(s) = 0 \).

Additionally, if \( eg - f^2 = 0 \) then \( M \) has degenerate second fundamental form. A canal surface has degenerate second fundamental form if canal surface is a surface of revolution and \( r(s) = \sqrt{s^2 - 2c_1 s + 2c_2} \) or \( r(s) = c_1 s + c_2 \). From \( (1.1) \), \( (1.2) \) and \( (1.3) \) we obtained the Gauss curvature, mean curvature and second Gauss curvature a polynomial expressions in \( \cos(t) \) such that
\[ K(s, t) = \frac{Q^2}{r^2 (EG - F^2)} \left\{ \kappa^2 (R^2 + Q^2) \cos^2(t) + \kappa (2Q'R - QR' + Q) \cos(t) \right\} \]
\[ (1.8) \]
\[ H(s, t) = \frac{-Q^2}{2r (EG - F^2)} \left\{ 2\kappa^2 (Q^2 + R^2) \cos^2(t) + \kappa (4 (Q'R - QR') + 3Q) \cos(t) \right\} \]
\[ (1.9) \]

and
\[ K_{\Pi}(s, t) = \frac{Q^3}{4r^5 (eg - f^2)} \sum_{i=0}^{4} n_i \cos^i(t) \]
\[ (1.10) \]
where the coefficients $n_i$ are

\[
n_0 = \begin{cases} 
-2\kappa^2 r^2 + 4Q'R\kappa^2 r^2 - 4Q(R')^2 \kappa^2 r^2 - 2QQ'R\kappa^2 r^2 - Q^2 Q''r' \\
+2QQ'Q''r^2 + 4Q'Q''r^2 - 2QQ'R''r^2 + 2Q^2 Q''r' + 2Q'Q''r'\kappa^2 \\
+2Q'Q'R''r^2 + 2Q'R''r^2 - 4QQ''R''r^2 - QR'r''r^2 + 4Q'Q'R''r^2 \\
+2QQ'R''r^2 + 2QQ'Q''r^2 - 4Q(Q')^2 r^2 + 2(Q')^2 Q''r^2 + 2QQ'R''r' \\
-2QQ''(r')^2 + QQ'Q''r'r' - QR'R''r' - QR''r'r' - 4QQ''r^2 \\
+\kappa r \left\{ 4QQ'Q''R + 4Q^3 R'' + 2QQ' + 4(Q')^2 R - 4Q^2 Q' \right\} \sin(t) 
\end{cases}
\]  
(1.11)

\[
n_1 = \begin{cases} 
Q^{2}rr'\kappa' + 4R^{2}R'r^2\kappa^3 - 4(Q')^2 R\kappa^2 r^2 - 2QQ'R'^2\kappa^2 - 2R^2\kappa^3 r^2 \\
-4Q^2 R'(r')^2 \kappa - 2Q^2 R'r'^2 \kappa + 4R^2 R''\kappa^2 - 2R^2\kappa^3 r^2 - 2(Q')^2 \kappa^2 \\
+2Q^2 (r')^2 \kappa - QQ'rr'\kappa + QQ'Q'(r')^2 \kappa - 2QQ'R'r'r'\kappa + 4QQ'R'r'r'\kappa \\
+2QQ'R'r'r'\kappa + 2Q^2 R''r'r'\kappa + 2QQ''R'r^2 \kappa - 6RR'R''r^2 \kappa \\
+4QQ'R''r^2 + QQ''R'r^2 - 2Q^2 R''r^2 + 2Q^2 R''r^2 + 2RR''r^2 \kappa \\
+6QQ''R''r^2 - 16QQ''R'r^2 - 4QQ''R'^2 - 6Q''R''r^2 + 4Q''R''r^2 \\
-2Q^2 R''r'^2 \kappa - 4QQ'R'r'^2 \kappa - 4QQ'R'r'^2 \kappa + 4QQ''R''r^2 + 4QQ''R''r^2 \\
+\kappa r \left\{ 2Q^2 r^2 (2QQ'R^2 + 2RR' - R) - 2Q (R^2 + Q^2) r^2 \right\} \sin(t) 
\end{cases}
\]  
(1.12)

\[
n_2 = \kappa \begin{cases} 
12Q^2 Q''R''r' + 8QR'R''r^2 - 12Q(R')^2 R''r^2 - 4Q(Q')^2 R''r^2 + 2Q^2 (r')^2 \kappa \\
+2Q^3 R'^2 - 2QQ'R''r' - 2Q^2 R''r'^2 + 2Q^3 Qr'r' - QK^2 + 2Q^3 (r')^2 \kappa \\
-4QR'R''r' + 12QR'R''r'^2 - 8Q^2 R''r'^2 - 4QQ'R''r'^2 + 4QQ'R''r'^2 \\
-4QQ''Q'r'^2 - 4Q^2 R''r'^2 + 4Q'R''r'^2 - 2QR''r'^2 + 4QQ'R''r'^2 
\end{cases}
\]  
(1.13)

\[
n_3 = 2Q^{2}r^3 \left\{ 8QR' - 8Q'R - 3Q \right\}
\]

and

\[
n_4 = -4Q^4 r^2 (Q^2 + R^2) .
\]

2. Weingarten Type Canal Surfaces

Let $M$ be a canal surface with the center curve $a(s)$ and the radius $r(s)$. The existence of a Weingarten relation $\Phi(H,K) = 0$ means that curvatures $H$ and $K$ are functionally related, and since $H$ and $K$ are differentiable functions depending on $s$ and $t$, this implies the Jacobian condition $\Phi(H,K) = 0$. More precisely the following condition

\[
H_t (K)_s - H_s (K)_t = 0
\]

needs to be satisfied. By using equations (1.8) and (1.9) we get

\[
H_t (K)_s - H_s (K)_t = \frac{1}{U} \sum_{i=0}^{5} b_i \cos^i (t)
\]

where

\[
U = 4r^6 \left\{ 1 - 2R' + (R')^2 + (Q')^2 + 2\kappa (Q - QR' + Q'R) \cos (t) \right\}^4
\]
For $i = 5$ coefficient $h_5$ is

\[ h_5 = -2Q^2\kappa^6r^2r' \{2Q^2R^2 + R^4 + Q^4\} \sin(t). \]

The Jacobian condition yields the relation

\[(2.3) \quad 2Q^2\kappa^6r^2r' \{2Q^2R^2 + R^4 + Q^4\} = 0.\]

In the cases $(r' \neq 0, \kappa = 0), (r' = 0, \kappa \neq 0)$ and $(r' = 0, \kappa = 0)$, all of the coefficients $h_i$ in (2.2) are zero so the condition (2.4) is satisfy. If $r' = 0$ and $\kappa \neq 0$ then $r(s) = c \neq 0$ and (1.4) turns to a tubular surface such that

\[(2.4) \quad C(s,t) = \alpha(s) + c \cos(t) N \pm c \sin(t) B.\]

If $\kappa = 0$ then let assume that the center curve is the $x-$axis (1.4) turns to a surface of revolution and a cylinder such that

\[(2.5) \quad C(s,t) = (s - r(s)r'(s), \mp r(s)\sqrt{1 - r'(s)^2} \cos(t), \pm r(s)\sqrt{1 - r'(s)^2} \sin(t))\]

and

\[(2.6) \quad C(s,t) = (s, \mp c \cos(t), \pm c \sin(t))\]

respectively. Thus we can state the following theorem.

**Theorem 4.** All the tubular surfaces and the surfaces of revolution in $\mathbb{R}^3$ are $(H,K)$-Weingarten canal surfaces.

From (1.9) and (1.10), the Jacobi function $\Phi(H,K)$ is obtained a polynomial expressions in $\cos(t)$ such that

\[(2.7) \quad H_t(K_{11})_s - H_s(K_{11})_t = \frac{1}{V_1V_2} \sum_{i=0}^{9} g_i \cos^i(t)\]

where

\[
V_1 = 64r^8 \left\{1 - 2R' + (R')^2 + (Q')^2 + 2\kappa(Q - QR' + Q'R) \cos(t)\right\}^2 + \left\{Q^2 + R^2\right\} \kappa^2 \cos^2(t)
\]

\[
V_2 = \left\{-RR'' - QQ'' + (2Q'R - 2QR' + Q) \kappa \cos(t) + (Q^2 + R^2) \kappa^2 \cos^2(t)\right\}^2.
\]

For $i = 9$, $g_9$ is

\[ g_9 = 128Q^2\kappa^6r^5r' \{4Q^2R^2(Q^2 + R^4) + 6Q^4R^4 + R^8 + Q^8\} \sin(t). \]

The Jacobian condition $\Phi(H,K_{11})=0$ requires the solutions $(r' \neq 0, \kappa = 0), (r' = 0, \kappa \neq 0)$ and $(r' = 0, \kappa = 0)$. In the cases $(r' = 0, \kappa = 0)$ and $(r' \neq 0, \kappa = 0)$, $g_i = 0$ is satisfies automatically for all $i$, and (1.4) turns to (2.6) and (2.5), respectively. In another case $(r' = 0, \kappa \neq 0)$, $g_i = 0$ is satisfies for all $i$ except $i = 1, 2, 3, 4$. Thus we get following equations for $r' = 0$

\[
g_1 = 16\kappa^3r^10r' \sin(t) \\
g_2 = 3\kappa r g_1, \quad g_3 = 3\kappa^2r^2 g_1, \quad g_4 = \kappa^3r^3 g_1
\]

and one gets $\kappa = c \neq 0$, so (1.4) turns to (2.4), we have the following theorem.

**Theorem 5.** All the tubular surfaces and the surfaces of revolution in $\mathbb{R}^3$ are $(H,K_{11})$-Weingarten canal surfaces.
Jacobi function $\Phi(K, K_{11})$ is obtained a polynomial expressions in $\cos(t)$ by using \eqref{1.8} and \eqref{1.10} as follows.

\[(2.8) \quad K_1(K_{11})_s - K_3(K_{11})_t = \frac{1}{W_1 W_2} \sum_{i=0}^{9} f_i \cos^i(t)\]

where

\[
W_1 = 16r^{10} \left\{ 1 - 2R' + (R')^2 + (Q')^2 + 2\kappa (Q - QR' + Q'R) \cos(t) \right\}^2
\]

\[
W_2 = \left\{ RR'' + QQ'' + (2QR' - 2Q'R - Q) \kappa \cos(t) - (Q^2 + R^2) \kappa^2 \cos^2(t) \right\}^2.
\]

For $i = 9$, $f_9$ is

\[
f_9 = 64Q^2 \kappa^{10} r^6 \{ 4Q^2 R^2 (Q^4 + R^4) + R^8 + 6Q^4 R^4 + Q^8 \} \sin(t).
\]

In the cases $(r' \neq 0, \kappa = 0), (r' = 0, \kappa \neq 0)$ and $(r' = 0, \kappa = 0), f_9$ is zero. For the case $(r' = 0, \kappa \neq 0), f_0, f_4, f_6, f_7$, and $f_8$ are zero and $f_1, f_2, f_3, f_4$ are

\[
f_1 = 8\kappa^{3} r^{11} \kappa' \sin(t), \quad f_2 = 3\kappa^{2} r^{3} f_1, \quad f_3 = 3\kappa r f_1
\]

and one gets $\kappa = c \neq 0$, thus $\eqref{1.4}$ turns to $\eqref{2.4}$. For the other cases $(r' \neq 0, \kappa = 0)$ and $(r' = 0, \kappa = 0), f_1 = 0$ is satisfies automatically for all $i$, and $\eqref{1.4}$ turns to $\eqref{2.6}$ and $\eqref{2.5}$ thus we can write the following theorem.

**Theorem 6.** All the tubular surfaces and the surfaces of revolution in $\mathbb{R}^3$ are $(K, K_{11})$-Weingarten canal surfaces.

3. **Linear Weingarten Type Canal Surfaces**

Let $M$ be a canal surface with the center curve $\alpha(s)$ and the radius $r(s)$ then $M$ is called $(K, H)$—linear Weingarten surface if Gaussian and the mean curvatures of $M$ satisfies a linear equation with the constants $a$, $b$ and $d$ such that

\[aK + bH = d.\]

By using equations \eqref{1.8} and \eqref{1.9} we get

\[
aK + bH - d = \frac{1}{2r^3 \Omega} \sum_{i=0}^{4} u_i \cos^i(t) = 0
\]

where

\[
\Omega = \left\{ 1 - 2R' + (R')^2 + (Q')^2 + 2\kappa (Q - QR' + Q'R) \cos(t) \right\}^2
\]

and for $i = 4$

\[
u_4 = -2\kappa r^4 (R^2 + Q^2)^2 \left\{ -a + rb + r^2 d \right\}.
\]

$u_4 = 0$ requires $\kappa = 0$ or $-a + rb + r^2 d = 0$. For $(\kappa = 0, -a + rb + r^2 d \neq 0), u_1 = \ldots = u_4 = 0$ and $u_0$ is

\[
u_0 = -2r \left( 1 + (Q')^2 - 2R' + (R')^2 \right) \left\{ \frac{a + b}{2} r \right \} \left\{ RR'' + QQ'' \right\} + \frac{a}{2} (2rd + b) \left( 1 + (Q')^2 - 2R' + (R')^2 \right).
\]
If \((Q')^2 - 2R' + (R')^2 = -1\) then from (1.5) and (1.6) following differential equation is obtained
\[
(3.1) \quad \frac{1}{(r')^2 - 1} \left( (r')^2 + rr'' - 1 \right)^2 = 0.
\]
The solution of (3.1) is
\[
(3.2) \quad r = \sqrt{s^2 - 2c_1 s + 2c_2}
\]
but in this case (1.4) is not a surface. If
\[
(3.3) \quad \left( a + \frac{b}{2} r \right) (RR'' + QQ'') + \frac{r}{2} (2rd + b) \left( 1 + (Q')^2 - 2R' + (R')^2 \right) = 0
\]
then we get the following differential equation for \(r\)
\[
(3.4) \quad \frac{1}{(r')^2 - 1} \left( r'' (r^2d + rb - a) + \frac{1}{2} (r')^2 - 1 \right) (2rd + b) = 0.
\]
The solution of (3.4) is the same (3.2). Thus the case \((\kappa = 0, -a + rb + r^2d \neq 0)\) can not occur. If \((\kappa = 0, -a + rb + r^2d = 0)\) then radius is constant such that
\[
(3.5) \quad r = \frac{-b \pm \sqrt{b^2 + 4ad}}{2d}
\]
with the condition \(b^2 \geq -4ad\). From (3.3) we get
\[
(3.6) \quad \frac{r}{2} (2rd + b) = 0 \iff r = \frac{-b}{2d}.
\]
It is easily to see that \(b = \pm 2\sqrt{-ad}\) by comparing (3.6) with the (3.5). Thus we obtained
\[
(3.7) \quad r = \frac{-b}{2d} = \mp \sqrt{-\frac{a}{d}}.
\]
If \((\kappa \neq 0, -a + rb + r^2d = 0)\) then radius is constant such that
\[
(3.8) \quad r = \frac{-b \pm \sqrt{b^2 + 4ad}}{2d}
\]
with the condition \(b^2 \geq -4ad\). Thus \(u_3\) is
\[
u_3 = \frac{- (b \pm \sqrt{b^2 + 4ad}) (b^2 + 4ad \pm b\sqrt{b^2 + 4ad})}{32d^5} = 0
\]
so one has \(b^2 + 4ad = 0\) and \(u_3 = u_2 = u_1 = u_0 = 0\) is satisfies. Thus we can state the following theorem.

**Theorem 7.** Let \(M\) be a canal surface with the center curve \(\alpha(s)\) and the radius \(r(s)\). Then \(M\) is a \((K, H)\) - linear Weingarten surface in \(\mathbb{R}^3\) if \(M\) is either a tubular surface or a cylinder.

Hence, in the case \((\kappa \neq 0, -a + rb + r^2d = 0)\), (1.4) turns to as in form (2.4) with (3.7). In the case \((\kappa = 0, -a + rb + r^2d = 0)\), (1.4) turns to as in form (2.6) with \(c = \frac{-b}{2d} = \mp \sqrt{-\frac{ad}{d}}\).

A canal surface \(M\) is called \((K,K_{II})\) - linear Weingarten surface if \(K\) and \(K_{II}\) curvatures of \(M\) satisfies a linear equation with the constants \(a, b\) and \(d\) such that
\[
aK + bK_{II} = d.
\]
By using equations (1.8) and (1.10) we get
\[ aK + bK_{II} - d = \frac{-1}{4r^5\Omega} \sum_{i=0}^{6} v_i \cos^i(t) = 0 \]
where
\[ \Gamma = \{ RR'' + QQ'' + (2QR' - Q'R - Q) \kappa \cos(t) - (R^2 + Q^2) \kappa^2 \cos^2(t) \} . \]
For \( i = 6 \), \( v_6 \) is
\[ v_6 = -4r^4bQ^2\kappa^6(R^2 + Q^2)^2 \]
and the solution of \( v_6 = 0 \) is \( \kappa = 0 \). All of the coefficients \( v_i \) are zero for \( \kappa = 0 \) except \( v_0 \). From (1.5) and (1.6) we get
\[ v_0 = \frac{r^5 \left( (r')^2 + rr'' - 1 \right)^2 \left\{ brr' \left( (r')^2 - 1 \right) r''' - 4r''p \right\}}{\left( (r')^2 - 1 \right)^2} \]
where
\[ \omega = br^3 (r'')^3 + \frac{3}{2} br^2 \left( (r')^2 - 1 \right) (r'')^2 + \left\{ \frac{3}{2} br (r')^2 + a + br (r')^4 + \frac{1}{2} br + dr^2 \right\} r'' + \left\{ \frac{1}{4} b(r')^4 + \frac{1}{4} b(r')^2 + dr \right\} (r')^2 - 1 \).
One of the solution of \( v_0 = 0 \) is the same (3.2). In the case \( brr' \left( (r')^2 - 1 \right) r''' - 4r''p \omega = 0 \) : Since \( r \neq \pm s + c \), one has \( r = c \neq 0 \). Hence we have state the following theorem.

**Theorem 8.** Let \( M \) be a canal surface with the center curve \( \alpha(s) \) and the radius \( r(s) \). Then \( M \) is a \((K,K_{II}) \)–linear Weingarten surface in \( IR^3 \) if \( M \) is a cylinder.

A canal surface in \( IR^3 \) parametrized as in (1.14) is called \((H,K_{II}) \)–linear Weingarten surface if mean and the second Gaussian curvatures satisfies a linear equation with the constants \( a, b \) and \( d \) such that
\[ aH + bK_{II} = d. \]
From (1.9) and (1.10) we get
\[ aH + bK_{II} - d = \frac{-1}{8r^4\Omega} \sum_{i=0}^{6} w_i \cos^i(t) = 0. \]
The coefficient \( w_6 \) is
\[ w_6 = -8r^3bQ^2\kappa^6(R^2 + Q^2)^2 \]
and one get \( \kappa = 0 \). Coefficients \( w_1, w_2, w_3, w_4 \) and \( w_5 \) are zero. From (1.5) and (1.6), \( w_0 \) is
\[ w_0 = \frac{2r^4 \left( (r')^2 + rr'' - 1 \right)^2 \left\{ brr' \left( (r')^2 - 1 \right) r''' - 4r''p \right\}}{\left( (r')^2 - 1 \right)^2} \]
where
\[ p = br^3 (r'')^3 + \frac{3}{2} br^2 ( (r')^2 - 1 ) (r'')^2 \]
\[ + \left\{ -\frac{3}{2} b (r')^2 + a + b (r')^4 + \frac{1}{2} b + dr \right\} rr'' \]
\[ + \left\{ -\frac{1}{4} b (r')^4 + \frac{1}{4} b (r')^2 + dr + \frac{1}{2} a \right\} ((r')^2 - 1) . \]

One of the solution of \( w_0 = 0 \) is the same \( w_{32} \). In the case \( br' ( (r')^2 - 1 ) r''' - 4r''p = 0 \) : Since \( r \neq \pm s + c \), one has \( r = c \neq 0 \). Thus we can state the following theorem.

**Theorem 9.** Let \( M \) be a canal surface with the center curve \( \alpha(s) \) and the radius \( r(s) \). Then \( M \) is a \((H,K_{II})\)-linear Weingarten surface in \( IR^3 \) if \( M \) is a cylinder.

A canal surface in \( IR^3 \) parametrized as in \( (1.4) \) is called \((K,H,K_{II})\)-linear Weingarten surface if the curvatures \( K \), \( H \) and \( K_{II} \) satisfies a linear equation with the constants \( a, b, c \) and \( d \) such that
\[ aK + bH + cK_{II} = d. \]

From \( (1.8), (1.9) \) and \( (1.10) \) we get
\[ aK + bH + cK_{II} - d = \frac{1}{8r^6 \Omega^2 \Gamma} \sum_{i=0}^{8} z_i \cos^i (t) = 0 \]

For \( i = 8 \), \( z_8 \) is
\[ z_8 = -8r^5 c Q^2 x^5 ( R^2 + Q^2 )^3 \]
and the solution of \( z_8 = 0 \) is \( \kappa = 0 \). The coefficients \( z_i \) are zero for \( i = 1, 2, \ldots, 7 \) for \( \kappa = 0, w_2, w_3, w_4 \) and \( w_5 \) are zero. By using \( (1.5 \) and \( (1.6), z_0 \) is
\[ z_0 = -2r^6 ( (r')^2 + rr'' - 1)^4 \left\{ rr' ( (r')^2 - 1 ) r''' - 4r''q \right\} \]
\[ \left( (r')^2 - 1 \right)^3 \]

where
\[ q = cr^3 (r'')^3 + \frac{3}{2} cr^2 ( (r')^2 - 1 ) (r'')^2 \]
\[ + \left\{ -\frac{3}{2} c (r')^2 + b + c (r')^4 + \frac{1}{2} c + dr \right\} rr'' \]
\[ + \left\{ -\frac{1}{4} c (r')^4 + \frac{1}{4} c (r')^2 + dr + \frac{1}{2} b \right\} ((r')^2 - 1) . \]

One of the solution of \( z_0 = 0 \) is the same \( w_{32} \). In the case \( cr' ( (r')^2 - 1 ) r''' - 4r''q = 0 \) : Since \( r \neq \pm s + \lambda \), one has \( r = \lambda \neq 0 \) where \( \lambda \) is a constant. We can state the following theorem.

**Theorem 10.** Let \( M \) be a canal surface with the center curve \( \alpha(s) \) and the radius \( r(s) \). Then \( M \) is a \((K,H,K_{II})\)-linear Weingarten surface in \( IR^3 \) if \( M \) is a cylinder.
Example 1. Let the center curve be

\[ \alpha(s) = \left( \frac{1}{2} \cos(s), \frac{1}{2} \sin(s), \frac{s \sqrt{3}}{2} \right) \]

and the radius be \( r = 2 \) then canal surface is parametrized as

\[ C(s, t) = (\phi_1, \phi_2, \phi_3) \]

where

\[ \phi_1 = \left( \frac{1}{2} + 2 \cos(t) \right) \cos(s) + \sqrt{3} \sin(s) \sin(s) \]
\[ \phi_2 = \left( \frac{1}{2} + 2 \cos(t) \right) \sin(s) - \sqrt{3} \sin(t) \cos(s) \]
\[ \phi_3 = \frac{s \sqrt{3}}{2} + \sin(t) \].

Thus \( C(s, t) \) is \( (H, K), (H, K_{II}), (K, K_{II}) \)-types Weingarten canal surface. Also for \( a = b = -4 \) and \( d = 1 \), \( C(s, t) \) is \( (H, K) \)-linear Weingarten surface such that \(-4H - 4K = 1\).

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(Yılmaz TUNÇER) UŞAK UNIVERSITY SCIENCE AND ART FACULTY, MATHEMATICS DEPARTMENT UŞAK TURKEY
E-mail address, Corresponding author: yilmaz.tuncer@usak.edu.tr

Current address, Dae Won YOON: Department of Mathematic Education and RINS,Gyeongsang National University, Jinju 660- 701, Republic of Korea
E-mail address, Dae Won YOON: dwyoon@gnu.ac.kr