On Uncertainty Measure Issues in Rough Set Theory

JIANGGUO TANG, JIANGHUA WANG, CHUNLING WU, AND GUOJIAN OU
Artificial Intelligence and Big Data College, Chongqing College of Electronic Engineering, Chongqing 401331, China

Corresponding author: Jianguo Tang (tjguo@126.com)

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ABSTRACT
Rough set theory is a tool for dealing with uncertainty problems. How to measure the uncertainty of a knowledge is an important issue in the theory. However, the existing uncertainty measures may not accurately reflect the uncertainty degree. This study analyzes the causes of it and explores a reasonable solution to it. Firstly, the existing accuracy models only focuses on some factors related to the target set while neglecting its own important influence on the model. Secondly, since no one gives a clear definition of knowledge uncertainty in approximation space, it is difficult to evaluate the accuracy and rationality of a knowledge uncertainty measure. Thirdly, most uncertain measures of knowledge are constructed based on the structure of knowledge itself, while neglecting other factors in the approximation model. In view of these, we first propose a new accuracy model which fully considers the important role of the target set itself. Second, two definitions of accuracy measure of knowledge are proposed to explain what the uncertainty of a knowledge is. And then, two uncertainty measures of knowledge are proposed and a method for quickly calculating them is designed. At last, an uncertain entropy is constructed for more conveniently calculating of knowledge uncertainty.

INDEX TERMS
Accuracy measure, approximation accuracy, approximation quality, rough sets, uncertainty measure.

I. INTRODUCTION
Rough set theory, developed by Pawlak [1], [2], is a popular method for dealing with uncertainty in information system and has been widely applied to areas such as feature selection, data mining, decision support and knowledge discovery [3]–[13], and so on. In order to better deal with the uncertainty problem with rough sets, many scholars have carried out in-depth research on how to measure uncertainty in information and have proposed a variety of uncertain measures [14]–[25]. These work help people to understand and observe the uncertainty in information with rough sets.

As one of the most important issue in rough sets, uncertainty has been extensively mentioned and studied in Pawlak’s works [1], [2], [26], [27]. He proposed accuracy measure and roughness measure to evaluate the quality of knowledge in approximating a target set. Furthermore, in order to study the uncertainty in simultaneously approximating many target sets with a knowledge, Pawlak proposed the concept of approximation accuracy. These concepts provide intuitive and understandable explanations for the uncertainty phenomenons in the approximation space. However, there are two shortcomings which need to be improved in these works. The first is that the Pawlak accuracy measure can not reflect the difference between two different sets when they have the same lower and upper approximations with respect to a knowledge. Yao [28] proposed a new accuracy model which is more reasonable than Pawlak model, but there is still the similar problem, that is, when different sets have the same boundaries on a knowledge, the model can not reflect the differences between them. The second is that, in Pawlak’s seminal book, he did not give a clear way to evaluate the uncertainty in a knowledge. Many researchers have noticed this problem and carried out research on it. Miao and Wang [29], Düntsch and Gediga [15], Liang and Qian [30] used Shannon Entropy to study the uncertainty measure of knowledge. Yao [28], [31] proposed a measure of granularity of a partition to illustrate...
the phenomenon that coarser knowledge should result in greater uncertainty. Liang et al. [32] proposed $GK(P)$ and $E(P)$ (introduced in Subsection II-B) to evaluate the uncertainty in information system. Qian and Liang [33] developed the combination entropy to measure the knowledge content of a knowledge. Although these uncertainty measures provide ways to quantify the degree of uncertainty in knowledge, they have common shortcomings of neglecting that approximation sets in rough set model should be taken into account when measuring the uncertainty of a knowledge.

Klir [34] claimed that each formalization of uncertainty in a problem-solving situation is a mathematical model of the situation and when we commit ourselves to a particular mathematical theory, our modeling becomes necessarily limited by the constrains of the theory. So, in this sense, each uncertainty measure of knowledge in approximation space should be limited by the approximation model of rough sets. As Pawlak pointed out, approximations are the fundamental concepts used to express the imprecise of knowledge and inexactness of a set is due to the existence of borderline region which is the difference between the lower and the upper approximations. Beaubouef et al. [14] believed that Rough set theory inherently models two types of uncertainty. The first type of uncertainty arises from the indiscernibility relation on the universe. The second type of uncertainty is modeled through the approximation regions of rough sets. These viewpoint indicates that the lower and the upper approximations should play an important role in the construction of uncertain measures of a knowledge in approximation space. Unfortunately, the uncertainty measure models discussed above neglect the important influence of the lower and the upper approximations. They only examine the degree of uncertainty in terms of the structure of knowledge itself. Fortunately, Zhu and Wen [35] realized this problem and construct a new pair of information theoretic entropy and co-entropy functions associated to partitions and approximations to measure the uncertainty of knowledge. Nevertheless, in this new model, it is very complicated and difficult to obtain the value of uncertainty of knowledge. Therefore, in the approximation space, how to develop an uncertainty measure which is closely related to the approximation operator and is calculated easily becomes an important issue of rough set theory.

In addition, we find that there is another problem in the existing uncertainty measures, that is, no one is to explain what the uncertainty of a knowledge in the approximate space is. Harmanec [36] believed that before we can measure uncertainty or information, we have to be clear what exactly we are trying to measure. Therefore, when we intend to measure the uncertainty of a knowledge in approximation space, we first need to determine what the uncertainty is. Unfortunately, no researcher has provided a clear definition of this. Researchers usually only evaluate the uncertainty from the perspective of knowledge structure, which ignores the description of uncertainty in rough set model. As a result, it is difficult to evaluate and compare the rationality and correctness of different uncertainty measures. So we need to provide a definition of uncertainty of knowledge in the approximation space.

Focus on these problems, this paper proposes some methods to solve them. First, we propose a new accuracy measure, noted as $\alpha_P(X)$, which takes into account the important role of the target set itself more fully than the existing models. Second, based on some existing concepts in rough sets, we develop two concepts, noted as $AMK(P)$ and $AMK'(P)$, to define the uncertainty of a knowledge in approximation space. However, it is very difficult to calculate them directly according to the definitions of them. Therefore, by means of combinatorial mathematics, we develop a method to calculate the value of them easily. Third, in order to facilitate the comparison of uncertainties of different knowledge, we propose an uncertainty entropy.

The rest of this paper is organized as follows. Section II reviews some basic concepts in rough sets and some existing uncertainty measures of knowledge. Section III analyzes the shortcomings of existing accuracy measures. And then we propose a new definition of accuracy measure based on positive and negative regions to improve these shortcomings. Section IV proposes two definitions of accuracy measure of knowledge to represent the uncertainty of a knowledge in approximation space and provides some methods to compute them. And the inaccuracy of the existing uncertainty measures of knowledge are illustrated by using these two definitions. Section V develops an uncertainty entropy to measure the uncertainty of knowledge and some properties of it are discussed.

II. THE EXISTING UNCERTAINTY MEASURES IN ROUGH SETS

In this section, some basic concepts of uncertainty measures in rough sets are reviewed. For convenience, in the rest of this paper, we suppose that $U$ is a finite and non-empty set of objects and the cardinality of $U$ is equal to $n$, $R \subseteq U \times U$ is an equivalence relation on $U$, $P = U / R = \{X_1, X_2, \ldots, X_m\}$ is a partition of $U$ induced by $R$, where $X_i \subseteq U$ is called an equivalence class which satisfies $X_i \neq \emptyset$ and has an empty intersection with any other equivalence classes, and $\cup P = U$. Usually, a partition $P$ is also called a knowledge $P$, and $\omega = \{U\}$ and $\varphi = \{\{X_i\} | X_i \in U\}$ denote the coarsest and the finest knowledge, respectively.

A. ACCURACY AND ROUGHNESS MEASURES

Definition 1 [1]: For a nonempty subset $X \subseteq U$, the lower and the upper approximations of $X$ induced by $P$ are respectively defined by:

$$LA_P(X) = \bigcup \{X_i \in P : X_i \subseteq X\}$$

(1)

$$UA_P(X) = \bigcup \{X_i \in P : X_i \cap X \neq \emptyset\}$$

(2)

Based on the two approximations, the positive, boundary and negative regions are defined by:

$$Pos_P(X) = LA_P(X)$$

(3)

$$Bnd_P(X) = UA_P(X) - LA_P(X)$$

(4)

$$Neg_P(X) = U - UA_P(X)$$

(5)
As Yao [28] pointed out that elements in the positive region are certainly in set $X$, while elements in the negative region are certainly not in set $X$, elements in the boundary region are both possibly in set $X$ and not in set $X$. The positive region and the negative region of set $X$ can be regarded as the certain information obtained from the knowledge $P$. Correspondingly, the boundary region represents the uncertain information.

To estimate the quality of a knowledge in approximating a target set, Pawlak [1] proposed the accuracy measure $\alpha_P(X)$ and the roughness measure $\rho_P(X)$ as follows:

$$\alpha_P(X) = \frac{|\text{Pos}_P(X)|}{|\text{UA}_P(X)|}, \quad (6)$$

$$\rho_P(X) = \frac{|\text{Bnd}_P(X)|}{|\text{UA}_P(X)|}, \quad (7)$$

where $| \cdot |$ represents the cardinality of a set, and the relation between $\alpha_P(X)$ and $\rho_P(X)$ is formulated as:

$$\rho_P(X) = 1 - \alpha_P(X). \quad (8)$$

$\alpha_P(X)$ and $\rho_P(X)$ reflect the quality of a knowledge in approximating target sets, or we can say that they reflect the uncertainty degree of a knowledge in approximating target set $X$ from two different perspectives. That is the smaller the $\alpha_P(X)$, the higher the uncertainty degree, or the greater the $\rho_P(X)$, the higher the uncertainty degree.

As we can see that, in Equations (6) and (7), the negative region is neglected, and although the upper approximation is considered, the role of it is ignored when the lower approximation is empty. Yao [28] made a detailed analysis of these problems by using the following five properties:

- (p1) $\alpha_P(X) = 1 \iff \text{LA}_P(X) = \text{UA}_P(X)$,
- (p2) $\alpha_P(X) = 0 \iff \text{LA}_P(X) = \emptyset, \text{UA}_P(X) = U$,
- (p3) for a fixed $\text{UA}_P(X)$, $\alpha_P(X)$ strictly monotonically increases with $|\text{LA}_P(X)|$,
- (p4) for a fixed $\text{LA}_P(X)$, $\alpha_P(X)$ strictly monotonically decreases with $|\text{UA}_P(X)|$,
- (p5) $P \preceq Q \implies \alpha_P(X) \geq \alpha_Q(X), \quad (10)$

where $P$ and $Q$ are two different knowledge on $U$ and the symbol $\preceq$ means that each equivalence class in $P$ is a subset of some equivalence class in $Q$. We can find that $\alpha_P(X)$ does not satisfy the property (p2). Then Yao proposed new measures to evaluate the “goodness” or “fitness” of knowledge in approximating $X$ with a knowledge $P$. That is:

$$\alpha'_P(X) = \frac{|\text{Pos}_P(X)| + |\text{Neg}_P(X)|}{n}, \quad (9)$$

$$\rho'_P(X) = 1 - \alpha'_P(X) = \frac{|\text{Bnd}_P(X)|}{n}. \quad (10)$$

Compared with $\alpha_P(X)$ and $\rho_P(X)$, $\alpha'_P(X)$ and $\rho'_P(X)$ satisfy all the five properties and are more reasonable than the formers.

**B. UNCERTAINTY MEASURES OF KNOWLEDGE**

To measure the uncertainty degree of a knowledge is another very important issue in rough sets. Many researchers studied this issue and proposed some different uncertainty measures of knowledge. In this subsection, some important and representative uncertainty measures of knowledge are introduced.

In order to measure quantitatively how much information is produced by a random process, Shannon [37] proposed the concept of information entropy which was used to measure the uncertainty of knowledge by Miao and Wang [29], Düntsch and Gediga [15], Liang and Qian [30] and defined as follows:

**Definition 2:** Suppose $p_i = |X_i|/n$, where $X_i \in P$, then

$$H(P) = -\sum_{i=1}^{m} p_i \log p_i. \quad (11)$$

is called the information entropy of knowledge $P$.

For any knowledge $P$ on $U$, $H(\omega) = 0 \leq H(P) \leq H(\varphi) = \log n$, and if there are two knowledge with $P \preceq Q$ then $H(P) \geq H(Q)$. Therefore, for the information entropy, we can say that the finer/coarser the knowledge, the greater/smaller the information entropy of it. As a measure of uncertainty of knowledge, we can also say that the greater/smaller the information entropy, the smaller/greater the uncertainty of knowledge.

Usually, one may believe that coarser knowledge results in greater uncertainty. Semantically, Yao [28], [31] developed a new measure, called a measure of granularity of a partition, to illustrate this phenomenon. That is

$$G(P) = \sum_{i=1}^{m} \frac{|X_i|}{n} \log |X_i|. \quad (12)$$

Thus, for two knowledge with $P \preceq Q$, we have $G(P) \leq G(Q)$.

Liang and Shi [38] had also studied the measure $G(P)$ and denoted it as the rough entropy of $P$.

Liang et al. [38] proposed the following equation to measure the knowledge granularity.

$$GK(P) = \frac{1}{n^2} \sum_{i=1}^{m} |X_i|^2. \quad (13)$$

Equation (13) shows that the greater/smaller the knowledge granularity, the greater/smaller the uncertainty of the knowledge. Taking account of the complement of target set, Liang et al. [32] proposed a complementary entropy to measure the uncertainty of knowledge, which is defined as:

$$E(P) = \frac{1}{n^2} \sum_{i=1}^{m} |X_i||X_i^C|. \quad (14)$$

Qian and Liang [33] introduced the combination entropy to measure the knowledge content of a knowledge. The combination entropy defined as:

$$CE(P) = \sum_{i=1}^{m} \frac{|X_i|}{n} (1 - \frac{D_{2,i}^2}{D_n^2}). \quad (15)$$

where $D_{2,i}^2 = \frac{1}{2} |X_i|(|X_i| - 1)$. Equation (15) shows that the more/less the information content of a knowledge, the smaller/greater the uncertainty of the knowledge.
TABLE 1. Uncertainty measures of knowledge.

| Measures | Min Value | Max Value | \( P \leq Q \) | Trend of increase and decrease |
|----------|-----------|-----------|--------------|--------------------------------|
| \( H(\cdot) \) | \( H(\varphi) = 0 \) | \( H(\varphi) = \log n \) | \( H(P) \geq H(Q) \) | If \( H(\cdot) \) then UN(\cdot) ↓; Else, UN(\cdot) ↑. |
| \( G(\cdot) \) | \( G(\varphi) = 0 \) | \( G(\varphi) = \log n \) | \( G(P) \leq G(Q) \) | If \( G(\cdot) \) then UN(\cdot) ↑; Else, UN(\cdot) ↓. |
| \( GK(\cdot) \) | \( GK(\varphi) = \frac{1}{n} \) | \( GK(\varphi) = 1 \) | \( GK(P) \leq GK(Q) \) | If \( GK(\cdot) \) then UN(\cdot) ↑; Else, UN(\cdot) ↓. |
| \( E(\cdot) \) | \( E(\varphi) = 0 \) | \( E(\varphi) = 1 - \frac{1}{n} \) | \( E(P) \geq E(Q) \) | If \( E(\cdot) \) then UN(\cdot) ↓; Else, UN(\cdot) ↑. |
| \( CE(\cdot) \) | \( CE(\varphi) = 0 \) | \( CE(\varphi) = 1 \) | \( CE(P) \geq CE(Q) \) | If \( CE(\cdot) \) then UN(\cdot) ↓; Else, UN(\cdot) ↑. |
| \( IE(\cdot) \) | \( IE(\varphi) = n - \frac{2n-2\log(2n) - 2}{2n} \) | \( IE(\varphi) = n \) | \( IE(P) \geq IE(Q) \) | If \( IE(\cdot) \) then UN(\cdot) ↓; Else, UN(\cdot) ↑. |

* where \( \cdot \) denotes a knowledge on \( U \), \( \uparrow \) " represents the increasing trend, while \( \downarrow \" the decreasing trend. UN(\cdot) represents the uncertainty degree of \( \cdot \). *

TABLE 2. Approximations of \( X_1 \) and \( X_2 \) WRT \( P_1 \), \( P_2 \) and \( P_3 \).

| \( X_i \) | \( LA_{P_2}(X_i) \) | \( UA_{P_2}(X_i) \) | \( LA_{P_3}(X_i) \) | \( UA_{P_3}(X_i) \) |
|---------|----------------|----------------|----------------|----------------|
| \( X_1 \) | \( \{x_1, x_2, x_3\} \) | \( \{x_1, x_2, x_3, x_4, x_5, x_6\} \) | \( \{x_1, x_2, x_3\} \) | \( \{x_1, x_2, x_3, x_4, x_5, x_6\} \) |
| \( X_2 \) | \( \{x_3, x_4, x_5, x_6\} \) | \( \{x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\} \) | \( \{x_3, x_4, x_5, x_6, x_7, x_8, x_9\} \) | \( \{x_4, x_7, x_8\} \) |

III. NEW ACCURACY AND ROUGHNESS MEASURES

A. ANALYSIS OF EXISTING ACCURACY MEASURES

Pawlak [1] and Yao [28] respectively defined different accuracy measures formulated with Equations (6) and (9) to measure the uncertainty of knowledge in approximating objects. From Equation (6), we can find that the \( \alpha_P(X) \) does not pay enough attention to the negative region of \( X \) as it pays to the positive region. However, both the positive region and the negative region are the certain and important information about \( X \) and should be considered at the same time. The absence of negative region results in an undesirable consequence that \( \alpha_P(X) \) dose not satisfy property (p2) listed in subsection II-A. Although the negative region of \( X \) was considered enough as the positive region in \( \alpha'_{\varphi}(X) \), the set \( X \) itself was neglected. The set \( X \) does not be involved directly both in \( \alpha_P(X) \) and in \( \alpha'_{\varphi}(X) \) and just appears indirectly in them. And this causes the two accuracy measures are insensitive in many cases. Specifically, in these cases, both \( \alpha_P(X) \) and \( \alpha'_{\varphi}(X) \) can not distinguish the difference among them. We will illustrate it by the following example.

Example 1: Given \( U = \{u_1, u_2, \ldots, u_{15}\} \), \( X_1, X_2, X_3 \) and \( X_4 \) are subsets of \( U \) and \( X_1 = \{u_1, u_2, u_3, u_4, u_5\} \), \( X_2 = \{u_3, u_4, u_5, u_6, u_7, u_8\} \), \( X_3 = \{u_4, u_7, u_9, u_{10}, u_{11}\} \), \( X_4 = \{u_7, u_8, u_9, u_{10}, u_{11}, u_{13}\} \). \( P_1, P_2 \) and \( P_3 \) are different knowledge on \( U \) and \( P_1 = \{u_1, u_2\}, \{u_3\}, \{u_4, u_5, u_6\}, \{u_7, u_8, u_9, u_{10}\}, \{u_{11}, u_{12}, u_{13}, u_{14}, u_{15}\} \), \( P_2 = \{u_1, u_2\}, \{u_3\}, \{u_4, u_5, u_6\}, \{u_7, u_8, u_9\}, \{u_{10}, u_{11}, u_{12}\}, \{u_{13}, u_{14}\}, \{u_{15}\} \), \( P_3 = \{u_1, u_2, u_3\}, \{u_4, u_7, u_8\}, \{u_5, u_6, u_9\}, \{u_{10}\}, \{u_{11}, u_{12}, u_{13}, u_{14}, u_{15}\} \).

According to Definition 1 and Equations (6) and (9), we can get the results listed in the following Tables 2 and 3.

TABLE 3. \( \alpha_P(X) \) and \( \alpha'_{\varphi}(X) \) of \( X_1 \) and \( X_2 \) WRT \( P_1, P_2 \) and \( P_3 \).

| \( X_i \) | \( \alpha_{P_1}(X_i) \) | \( \alpha'_{P_1}(X_i) \) | \( \alpha_{P_2}(X_i) \) | \( \alpha'_{P_2}(X_i) \) | \( \alpha_{P_3}(X_i) \) | \( \alpha'_{P_3}(X_i) \) |
|---------|----------------|----------------|----------------|----------------|----------------|----------------|
| \( X_1 \) | 0.5000 | 0.8000 | 0.5000 | 0.8000 | 0.3333 | 0.6000 |
| \( X_2 \) | 0.5000 | 0.7333 | 0.5714 | 0.8000 | 0.3333 | 0.6000 |
equal, this means the \( \alpha P(X) \) is more sensitive than \( \alpha' P(X) \) in this case. Furthermore, both \( \alpha P(X) \) and \( \alpha' P(X) \) of \( X_1 \) and \( X_2 \) with respect to \( P_3 \) are equal, this means \( \alpha P(X) \) and \( \alpha' P(X) \) can not reflect the difference in approximating \( X_1 \) and \( X_2 \) with knowledge \( P_3 \).

In Tables 4 and 5, we are concerned about the sensitivities of \( \alpha P(X) \) and \( \alpha' P(X) \) in approximating the same target set with different knowledge. In Table 5, we can find that the \( \alpha P(X) \) of \( X_3 \) with respect to \( P_1 \) and \( P_2 \) are equal, while the \( \alpha' P(X) \) of \( X_3 \) with respect to \( P_1 \) and \( P_2 \) are not equal, this means the \( \alpha' P(X) \) is more sensitive than \( \alpha P(X) \) in this case. On the contrary, the \( \alpha P(X) \) of \( X_4 \) with respect to \( P_1 \) and \( P_2 \) are not equal, while the \( \alpha' P(X) \) of \( X_4 \) with respect to \( P_1 \) and \( P_2 \) are equal, this means the \( \alpha P(X) \) is more sensitive than \( \alpha' P(X) \) in this case.

\[
\begin{array}{ccccc}
X_i & \alpha P_1(X_i) & \alpha P_2(X_i) & \alpha' P_1(X_i) & \alpha' P_2(X_i) \\
X_3 & 0.3333 & 0.3333 & 0.4667 & 0.6000 \\
X_4 & 0.4444 & 0.3750 & 0.6667 & 0.6667 \\
\end{array}
\]

This example shows that although \( \alpha' P(X) \) takes full account of the negative region and satisfies the properties introduced in Subsection II-A, it still has the same shortcoming as the \( \alpha P(X) \) that they can not provide accurate measures in many cases. The reason, for \( \alpha P(X) \), is that it is computed just by using approximations of the target set. As shown in the following Fig. 1, although \( X \) and \( Y \) are very different, the \( \alpha P(X) \) of them are equal as long as they have the same lower and upper approximations, or the ratios of their lower and upper approximations are equal.

For \( \alpha' P(X) \), according to Equation (10), different target sets have the same \( \alpha' P(X) \) as long as the boundary regions of them are equal. Furthermore, as shown in Fig. 2, although we are intuitive to think that the uncertainty of knowledge in approximating \( X \) is obviously smaller than it in approximating \( Y \), the fact is that the \( \alpha P(X) \) of \( X \) is smaller than \( Y \) because the boundary region of \( X \) is greater than \( Y \).

These reasons reflect the fact that both \( \alpha P(X) \) and \( \alpha' P(X) \) do not take account of the target sets themselves directly and sufficiently. Hence, it is necessary to develop a more accurate and comprehensive accuracy measure for rough sets.

B. A NEW TYPE OF ACCURACY MEASURE

We know that, for any subset \( X \) of \( U \), there is a partition on \( U \) induced by \( X \), that is \( P_X = \{X, X^C\} \), where \( X^C \) is equal to \( U - X \). According to the knowledge \( P_X \), elements in \( X \) are certainly belong to \( X \) and elements in \( X^C \) are certainly do not belong to \( X \). Similarly, elements in \( Pos P(X) \) are certainly belong to \( X \) and elements in \( Neg P(X) \) are certainly do not belong to \( X \). We realized these “certain” relationships between these sets are very important. Therefore, based on these relationships, we define a new type of accuracy measure as follows.

Definition 3: A new type of accuracy measure of \( X \) with respect to \( P \) is defined by:

\[
\alpha'' P(X) = \frac{1}{2} \left( \frac{|Pos P(X)|}{|X|} + \frac{|Neg P(X)|}{|X^C|} \right), \quad X^C \neq \emptyset
\]

(17)

Correspond to the accuracy measure \( \alpha'' P(X) \), the roughness measure is formulated by:

\[
r'' P(X) = 1 - \alpha'' P(X).
\]

(18)
Furthermore, according to Equatinions (1), (2), (3) and (5), for any \( X \subseteq U \), \( \text{Neg}_P(X) = \text{Pos}_P(X^C) \). Therefore, we can get:

\[
\alpha''_P(X) = \frac{1}{2} \left( \left| \text{Pos}_P(X) \right| \frac{|X|}{|X|} + \left| \text{Pos}_P(X^C) \right| \frac{|X^C|}{|X^C|} \right).
\]  

(19)

In Equation (17), we can find that the target set \( X \) itself and its complement \( X^C \) are taken account into \( \alpha''_P(X) \). The following Example 2 shows that \( \alpha''_P(X) \) can provide a more sensitive and comprehensive measure than \( \alpha_P(X) \) and \( \alpha_P'(X) \) in approximating the target set.

**Example 2:** According to Tables 2 and 4 in Example 1, we can get the results listed in the following Tables 6 and 7.

**TABLE 6.** \( \alpha''_P(X) \) of \( X_1 \) and \( X_2 \) WRT \( P_1, P_2 \) and \( P_3 \).

| \( X_i \) | \( \alpha''_P(X_i) \) | \( \alpha''_P'(X_i) \) | \( \alpha''_P''(X_i) \) |
|---|---|---|---|
| \( X_1 \) | 0.7500 | 0.7500 | 0.6000 |
| \( X_2 \) | 0.7222 | 0.7778 | 0.5833 |

**TABLE 7.** \( \alpha''_P(X) \) of \( X_3 \) and \( X_4 \) WRT \( P_1 \) and \( P_2 \).

| \( X_i \) | \( \alpha''_P(X_i) \) | \( \alpha''_P'(X_i) \) |
|---|---|---|
| \( X_3 \) | 0.5000 | 0.5833 |
| \( X_4 \) | 0.6667 | 0.6389 |

Table 6 shows that the \( \alpha''_P(X) \) of \( X_1 \) with respect to \( P_1, P_2 \) and \( P_3 \) are different from the \( \alpha''_P'(X) \) of \( X_2 \) with respect to \( P_1, P_2 \) and \( P_3 \), respectively.

Table 7 shows that the \( \alpha''_P(X) \) of \( X_3 \) with respect to \( P_1 \) and \( P_2 \) are different, and the \( \alpha''_P(X) \) of \( X_4 \) with respect to \( P_1 \) and \( P_2 \) are different as well.

**C. PROPERTIES OF \( \alpha''_P(X) \)**

Yao [28] provided a set of axioms for a measure of the uncertainty of knowledge in approximating target sets, which are listed in Subsection II-A as properties (p1) ~ (p5). In this subsection, we will examine whether \( \alpha''_P(X) \) satisfies these properties. Moreover, we will employ some new properties of \( \alpha''_P(X) \).

**Proposition 1:** For a non-empty subset \( X \) of \( U \), \( \alpha''_P(X) = 0 \) if and only if \( \text{LA}_P(X) = \emptyset \) and \( \text{UA}_P(X) = U \).

Proof (\( \Rightarrow \)) According to Definition 3, if \( \alpha''_P(X) = 0 \), then \( |\text{Pos}_P(X)| = 0 \) and \( |\text{Neg}_P(X)| = 0 \). Therefore, \( \text{LA}_P(X) = \emptyset \) and \( \text{Neg}_P(X) = \emptyset \), and then \( U = \text{UA}_P(X) = U \). Thus, \( \text{UA}_P(X) = U \).

(\( \Leftarrow \)) It is obvious that sufficiency is established.

**Proposition 2:** For a non-empty subset \( X \) of \( U \), \( \alpha''_P(X) = 1 \) if and only if \( \text{Pos}_P(X) = X \) or \( \text{Neg}_P(X) = X^C \).

Proof (\( \Rightarrow \)) According to Definition 3, if \( \alpha''_P(X) = 1 \), then \( \frac{|\text{Pos}_P(X)|}{|X|} = 1 \). Therefore, \( \text{Pos}_P(X) = X \). Similarly, if \( \text{Neg}_P(X) = X^C \), then \( \frac{|\text{Neg}_P(X)|}{|X^C|} = 1 \). Therefore, \( \text{Neg}_P(X) = X^C \).

(\( \Leftarrow \)) If \( \text{Pos}_P(X) = X \), then \( \text{UA}_P(X) = X \); i.e., \( \text{Neg}_P(X) = U - \text{UA}_P(X) = U - X = X^C \). Thus, \( \alpha''_P(X) = 1 \).

**Proposition 3:** For a non-empty subset \( X \) of \( U \), \( \alpha''_P(X) \) strictly monotonically increases with \( |\text{LA}_P(X)| \).

Proof: If \( \text{LA}_P(X) = X \) is fixed, then \( \alpha''_P(X) \) strictly monotonically increases with \( |\text{LA}_P(X)| \).

**Proposition 4:** For a non-empty subset \( X \) of \( U \), \( \alpha''_P(X) \) strictly monotonically decreases with \( |\text{UA}_P(X)| \).

Proof: If \( \text{UA}_P(X) = X \) is fixed, then \( \alpha''_P(X) \) strictly monotonically decreases with \( |\text{UA}_P(X)| \).

**Proposition 5:** Suppose that \( P \) and \( Q \) are two knowledge on \( U \), \( X \) is a non-empty subset of \( U \). If \( P \prec Q \) then \( \alpha''_P(X) \geq \alpha''_Q(X) \).

Proof: If \( P \prec Q \), then \( \text{LA}_Q(X) \subseteq \text{LA}_P(X) \). Therefore, \( \alpha''_P(X) \geq \alpha''_Q(X) \).

**Proposition 6:** For a non-empty subset \( X \) of \( U \), \( \alpha''_P(X) = \alpha''_P(X^C) \).

Proof: According to Definitions 1 and 3, and Equa-
tions (3) and (5), \( \alpha''_P(X) \) is not usually satisfied with the property because the ratios of cardinalities of approximations of \( X \) and \( X^C \) are usually not fixed. Therefore, \( \alpha''_P(X) \neq \alpha''_P(X^C) \).

**Corollary:** For a non-empty subset \( X \) of \( U \), \( 0 \leq \alpha''_P(X) \leq 1 \).

**IV. UNCERTAINTY MEASURES OF KNOWLEDGE IN ROUGH SETS**

Accuracy and roughness measures provide a quantitative interpretation in approximating target sets with a knowledge.
Another important uncertainty issue in rough sets is how to measure the uncertainty of knowledge itself. Many researchers are very concerned about this issue and constructed some different measures to measure it [15], [28], [29], [32], [33], [38]–[40]. Although these studies provide many good properties of these measures, they neglect two important problems.

The first is that the concept of uncertainty of knowledge was not defined explicitly in these studies. As a result, it is hard to verify the reasonability and correctness of these measures and to compare their performance theoretically and realistically. Some properties of these measures only show that they can meet some intuitive impression and understanding of people to uncertainty of knowledge. For instance, the uncertainty measure of the coarsest knowledge would reaches the maximum, while the finest knowledge would have the minimum uncertainty measure, and the finer knowledge has the smaller uncertainty measure than the coarser one. But for any two different knowledge, in most cases, we can not evaluate the uncertainty degree of them with these measures because there is a lack of evaluation criteria. In other words, we do not know what the uncertainty of knowledge in rough sets is because no one gives a explicit definition about it.

The second, these measures usually do not take full account of the situation of rough sets. Klir [34] claimed that the theory constrains should be considered when people construct the uncertainty model based on a particular theory. Nevertheless, for most existing uncertainty measures of knowledge, the constrains of rough sets are neglected, which can not provide reasonable reflection to the uncertainty degree of a knowledge.

In the rest of this section, we will explore how to define the uncertainty of knowledge in approximation space and provide some methods to calculate it easily. Furthermore, by the definition, we will verify the existing uncertainty measures.

### A. UNCERTAINTY OF KNOWLEDGE IN ROUGH SETS

Although Pawlak, in his seminal book [2], had not discussed explicitly and directly about the issue of how to measure the uncertainty of a knowledge, there are still enough information in the book for us to produce some definitions to characterize the uncertainty degree of a knowledge. Let us first review some basic knowledge related to uncertainty measures in rough sets. The accuracy and roughness measures defined by Equations (6) and (7) provide a quantitative method to compute the uncertainty degree of a knowledge in approximating a target set. Subsequently, based on the accuracy measure, Pawlak proposed the concept of approximation accuracy to describe inexactness of approximate classification, which was defined as

$$\alpha_P(F) = \frac{\sum_{i=1}^{n} |LA_P(G_i)|}{\sum_{i=1}^{n} |UA_P(G_i)|},$$  \hspace{1cm} (20)

$\alpha_P(F)$ is called the approximation accuracy of $F$ by $P$, where $F = \{G_1, G_2, \ldots, G_l\}$ is a set family consisted of some non-empty subsets of universe $U$.

Obviously, Equation (20) can be viewed as a measure to evaluate the uncertainty of knowledge $P$ in approximating the set family $F$, but it can not be viewed as the measure of uncertainty of knowledge $P$. The reason is that $F$ is just composed of part non-empty subsets of $U$ but are not the whole non-empty subsets of $U$. If $F$ is extended to the family which consists of all non-empty subsets of $U$, then $\alpha_P(F)$ can be used to represent the uncertainty of knowledge $P$. Therefore, we produce the definition of accuracy measure of knowledge as follows.

**Definition 4**: Suppose $F'$ is the set family consisting of all non-empty subsets of $U$, then

$$AMK(P) = \frac{\sum_{i=1}^{2^n-1} |LA_P(K_i)|}{\sum_{i=1}^{2^n-1} |UA_P(K_i)|}$$  \hspace{1cm} (21)

is called the accuracy measure of knowledge $P$, where $K_i \in F'$.

In a sense, $AMK(P)$ is a special case of $\alpha_P(F)$ with the set family $F'$ is equal to $F'$. In Equation (21), $2^n - 1$ represents the number of sets in $F'$. Because $F'$ includes all non-empty subsets of $U$, according to Equation (20), we think Definition 4 is reasonable and comprehensive for measuring the uncertainty of knowledge. Definition 4 shows that the greater/smaller the $AMK(P)$, the less/more the uncertainty of the knowledge $P$. We can also use a variety of $AMK(P)$ to evaluate the uncertainty of knowledge $P$, which is defined as

$$RMK(P) = 1 - AMK(P),$$  \hspace{1cm} (22)

and referred to as the roughness measure of knowledge $P$. And then we can say that the greater/smaller the $RMK(P)$, the more/less the uncertainty of the knowledge $P$. It is easy to find that $AMK(P)$ satisfies the following properties:

(a). $AMK(P)$ reaches the maximum value 1 if $P = \varphi$,
(b). $AMK(P)$ reaches the minimum value $\frac{1}{2^n - 1}$ if $P = \omega$,
(c). for any two knowledge $P$ and $Q$, if $P \preceq Q$ then $AMK(P) \geq AMK(Q)$.

According to Equations 20 and 21, we can find that they are approximate description of knowledge uncertainty based on accuracy measure $\alpha_P(X)$. In order to facilitate the follow-up discussion, we use $\gamma(P)$ defined in the following Equation 23 to represent the accurate description of knowledge uncertainty based on $\alpha_P(X)$.

$$\gamma(P) = \frac{1}{2^n - 1} \sum_{i=1}^{2^n-1} |LA_P(K_i)|$$  \hspace{1cm} (23)

Similarly, based on the accuracy measure proposed by Yao in Equation (9), we give a new accuracy measure of knowledge to evaluate the knowledge uncertainty, which is defined as follows.

$$AMK'(P) = \frac{\sum_{i=1}^{2^n-1} (|Pos_P(K_i)| + |Neg_P(K_i)|)}{(2^n - 1)n}$$  \hspace{1cm} (24)
And the variety of AMK′(P) is defined as:

\[ RMK'(P) = 1 - AMK'(P) = \frac{\sum_{i=1}^{2^n-1} |Bnd_p(K_i)|}{(2^n - 1)n} \]  

(25)

Obviously, AMK′(P) satisfies the properties (a), (b) and (c).

For the convenience to the follow-up discussion, similarly with \( \gamma(P) \), we define \( \gamma'(P) \) to represent the accurate measure of knowledge uncertainty based on \( \alpha'_p(X) \).

\[ \gamma'(P) = \frac{1}{2^n - 1} \sum_{i=1}^{2^n-1} |pos_p(K_i)| + |neg_p(K_i)| \]  

(26)

Although AMK′(P) and AMK′(P) can be used to evaluate quantitatively the uncertainty of knowledge, there is an urgent need to be solved at first. That is how to get the values of them. Since \( F' \) is usually large, calculating AMK′(P) and AMK′(P) will become very difficult or even impossible. Therefore, in order to solve this problem, we need to develop a novel and effective way.

**B. WAY FOR CALCULATING AMK(P) AND AMK′(P)**

According to the formulas of AMK′(P) and AMK′(P), it is very difficult to get the accurate measures of knowledge directly. In this subsection, we will propose a novel way to solve this problem.

Equation (21) shows that if we want to get AMK′(P) then we need to get the sums of cardinalities of the lower and the upper approximations of all non-empty subset of \( U \), respectively. Equations (24) and (25) shows that if we want to get AMK′(P) then we need to get the sum of cardinalities of boundary region of all non-empty subset of \( U \). Furthermore, according to Equations (3),(4) and (5), we just need to get the sums of cardinalities of the lower approximation and the boundary region of each non-subset of \( U \), respectively.

Firstly, we will discuss how to calculate the the sums of cardinalities of lower approximations of all non-subsets of \( U \). We know that, for each non-subset \( X \) of \( U \), the lower approximation of \( X \) with respect to \( P \) is composed of some equivalence classes in \( P \). Therefore, if we know that how many lower approximations of non-empty subsets of \( U \) include an equivalence class \( X_i \) in \( P \), then we know how many times the equivalence class \( X_i \) is put into the lower approximations of non-empty subsets of \( U \). In doing so, we can get the number of times of that each equivalence class in \( P \) is put into the lower approximations of non-empty subsets of \( U \), respectively. Thus, we can get the sum of cardinalities of lower approximations of all non-subsets of \( U \). Similarly, we can get the sum of cardinalities of boundary regions of all non-empty subsets of \( U \).

Suppose the sums of cardinalities of lower approximations and boundary regions of non-empty subsets of \( U \) are denoted by SCLA(P) and SCBR(P), respectively. For each equivalence class \( X_i \in P, |X_i| = n - |X_i| \). Y for \( X^C_i \), \( X_i \subseteq Y \subseteq X_i \), and then \( X_i \subseteq LA_p(Y \cup X_i) \). As the cardinality of the power set on \( X^C_i \) is equal to \( 2^{n-|X_i|} \), we can get how many non-subsets of \( U \) contain \( X_i \), i.e., the number is equal to \( 2^{n-|X_i|} \). Then, for an equivalence class \( X_i \) in \( P \), the total number of elements which belong to \( X_i \) and are included in SCLA(P) is expressed by

\[ TNLA_p(X_i) = 2^{n-|X_i|} |X_i| \]  

(27)

Therefore, for knowledge \( P \),

\[ SCLA(P) = \sum_{i=1}^{m} TNLA_p(X_i). \]  

(28)

SCLA(P) is the sum of cardinalities of lower approximations of all non-empty subsets of \( U \), that is

\[ SCLA(P) = \sum_{i=1}^{2^n-1} |LA_p(K_i)|. \]  

(29)

**Proposition 6:** For any two equivalence classes \( X \) and \( Y \) in \( P; |X| \geq |Y| \) if and only if \( TNLA_p(X) \leq TNLA_p(Y) \).

**Proof:** Suppose that \( |X| - |Y| = k \). Therefore \( |X| = |Y| + k \) and \( TNLA_p(X) = 2^{n-|Y|} = \frac{1}{2^k} (|Y| + k) \).

(1)(\( \Rightarrow \)): \( TNLA_p(X) = TNLA_p(Y) = 2^{n-|Y|} = \frac{1}{2^k} (|Y| + k) - |Y| \), so we just need to proof that \( d = \frac{1}{2^k} (|Y| + k) - |Y| \leq 0 \).

1. If \( k = 0 \) then \( d = 0 \);
2. If \( k = 1 \) and \( |Y| = 1 \) then \( d = 0 \), and if \( k = 1 \) and \( |Y| > 1 \) then \( d < 0 \);
3. If \( k > 1 \), then \( d = \frac{|Y| + 1 - 2^k |Y|}{2^k} < 0 \).

Thus, \( d \leq 0 \). That is \( TNLA_p(X) \leq TNLA_p(Y) \).

(2)(\( \Leftarrow \)): As \( TNLA_p(X) \leq TNLA_p(Y) \), \( d = \frac{|Y| + 1 - 2^k |Y|}{2^k} \leq 0 \).

Therefore, \( k \geq 0 \). That is \( |X| \geq |Y| \).

**Corollary 2:** For any two equivalence classes \( X \) and \( Y \) in \( P; |X| > |Y| \), \( TNLA_p(X) = TNLA_p(Y) \) if and only if \( |X| = 2 \) and \( |Y| = 1 \).

Next, we will discuss how to compute the SCBR(P). We know that, for an equivalence class \( X_i \) in \( P \), if the intersection between \( X_i \) and a non-empty subset \( Y \) of \( U \) is a non-empty proper subset of \( X_i \), then \( X_i \) is included in the boundary region of \( Y \). Obviously, \( Y \) should contain two parts. First, \( Y \) contains a non-empty proper subset of \( X_i \), Second, \( Y \) contains a subset of \( X^C_i \). So the number of such \( Y \) is equal to \( \sum_{j=1}^{2^n-|X_i|} C_j^{|X_i|} \), where \( C_j^{|X_i|} \) is a combination operation which represents the optional number of selecting \( j \) elements from \( X_i \). Then the total number of elements which belong to \( X_i \) and are included in SCBR(P) is expressed by

\[ TNBR_p(X_i) = \sum_{j=1}^{2^n-|X_i|} 2^{n-|X_i|} C_j^{|X_i|} |X_i|. \]  

(30)

where \( TNBR_p(X_i) = 0 \) if \( |X_i| = 1 \).

**Theorem 3:** For an equivalence class \( X \) in \( P; TNBR_p(X) = (2^{|X|} - 2)TNLA_p(X) \).

**Proof:** According to Equations (27) and (30),

\[ TNBR_p(X) = TNLA_p(X) \sum_{j=1}^{2^n-|X|} C_j^{|X|}. \]  

Therefore, we just need to prove that

\[ \sum_{j=1}^{2^n-|X|} C_j^{|X|} = 2^{|X|} - 2. \]  

(31)
(1) Obviously, Equation (31) is established when $|X| = 2$.
(2) Suppose that Equation (31) is established when $|X| = k$. That is
\[
\sum_{j=1}^{k-1} C_j^k = k + \frac{1}{2!}k(k-1) + \cdots + \frac{1}{(k-1)!}k(k-1)
\times \cdots \times 2 = 2^k - 2.
\]
(3) when $|X| = k + 1$, we can get that
\[
\sum_{j=1}^{k+1} C_j^{k+1} = k + 1 + \frac{1}{2!}(k+1)k
+ \cdots + \frac{1}{(k-1)!}(k+1)k(k-1) \times \cdots \times (k-3)
+ \frac{1}{k!}(k+1)k(k-1) \times \cdots \times 2
= k + 1 + \frac{1}{2!}k(k-1) + k + \cdots + \frac{1}{(k-1)!}k(k-1)
\times \cdots \times 2 + \frac{1}{(k-2)!}k(k-1) \times \cdots \times 3
+ \frac{1}{k!}k(k-1) \times \cdots \times 1 + \frac{1}{(k-1)!}k(k-1)
\times \cdots \times 2
= 2 \sum_{j=1}^{k-1} C_j^k + 1 + \frac{1}{k!}k(k-1) \times \cdots \times 1
= 2^{k+1} - 4 + 1 + 1
= 2^{k+1} - 2.
\]
That is, Equation (31) is established. And then Theorem 1 is established.

**Corollary 3:** For an equivalence class $X$ in $P$, $\sum_{i=1}^{|X|-1} C_i^{|X|} = 2^{|X|} - 2$.

**Proposition 7:** For any two equivalence classes $X$ and $Y$ in $P$, $|X| > |Y|$ if and only if $\text{TNBR}_P(X) > \text{TNBR}_P(Y)$.

**Proof:** According to Theorem 3, we can get
\[
\text{TNBR}_P(X) - \text{TNBR}_P(Y) = (2^{|X|} - 2)\text{TNLA}_P(X) - (2^{|Y|} - 2)\text{TNLA}_P(Y).
\]
Suppose that $|X| - |Y| = k$.

(1)$(\Longrightarrow)$:
\[
\text{TNBR}_P(X) - \text{TNBR}_P(Y) = (2^{|X|} - 2)2^{n-|X|}|X| - (2^{|Y|} - 2)2^{n-|Y|}|Y|
= (2^{|X|}-1)2^{n-|X|}|X| + (2^{|Y|} - 1)2^{n-|Y|}|Y|
= 2^{n-|Y|-k}(k2^{|Y|} + |X|)2^{k+1} - 2|Y| - 2k).
\]
As $|X| > |Y|$, $k > 0$ and then $k2^{|Y|+k} + |X|2^{k+1} - 2|Y| - 2k > 0$.
That is, $\text{TNBR}_P(X) > \text{TNBR}_P(Y)$.

(2)$(\Longleftarrow)$: Because $\text{TNBR}_P(X) > \text{TNBR}_P(Y)$, $\text{TNBR}_P(X) - \text{TNBR}_P(Y) = 2^{n-|Y|-k}(k2^{|Y|} + |Y|)2^{k+1} - 2|Y| - 2k > 0$.
That is, $k > 0$ and $|X| > |Y|$.

Therefore, for knowledge $P$,
\[
\text{SCBR}(P) = \sum_{i=1}^m \text{TNBR}_P(X_i).
\]

$\text{SCBR}(P)$ is the sum of cardinalities of boundary regions of all non-empty subsets of $U$, that is
\[
\text{SCBR}(P) = \sum_{i=1}^{2^n-1} |\text{Bnd}_P(K_i)|.
\]

**Theorem 4:** $2^n n = 2\text{SCLA}(P) + \text{SCBR}(P)$.

**Proof:** According to Equations (27), (28), (32) and Theorem 3, we can get that

\[
2\text{SCLA}(P) + \text{SCBR}(P)
= 2 \sum_{i=1}^m 2^{n-|X_i|}|X_i| + \sum_{i=1}^m (2^{|X_i|} - 2)2^{n-|X_i|}|X_i|
= 2^n \sum_{i=1}^m |X_i| + 2^n \sum_{i=1}^m (2^{|X_i|} - 2)|X_i|/2|X_i|
= 2^n \sum_{i=1}^m |X_i| + (2^{|X_i|} - 2)|X_i|/2|X_i|
= 2^n \sum_{i=1}^m |X_i|
= 2^n n.
\]

**Corollary 4:** $2^n n = \text{SCLA}(P) + \text{SCUA}(P)$.

**Corollary 5:** For any two knowledge $P$ and $Q$ on $U$, $\text{SCLA}(P) \geq \text{SCLA}(Q)$ if and only if $\text{SCBR}(P) \leq \text{SCBR}(Q)$.

**Corollary 6:** For any two knowledge $P$ and $Q$ on $U$, $\text{SCLA}(P) \geq \text{SCLA}(Q)$ if and only if $\text{SCUA}(P) \leq \text{SCUA}(Q)$.

Suppose $\text{SCUA}(P)$ and $\text{SCNR}(P)$ represent the sums of cardinalities of upper approximations and negative regions of all non-empty subsets of $U$ with respect to $P$, respectively. According to Equations (3),(4) and (5), we can get

\[
\text{SCUA}(P) = \text{SCLA}(P) + \text{SCBR}(P),
\]

and

\[
\text{SCNR}(P) = (2^n - 1)n - \text{SCUA}(P).
\]

Finally, we can get
\[
\text{AMK}(P) = \frac{\text{SCLA}(P)}{\text{SCUA}(P)},
\]
\[
\text{RMK}(P) = 1 - \text{AMK}(P) = \frac{\text{SCBR}(P)}{\text{SCUA}(P)},
\]
\[
\text{AMK}'(P) = \frac{\text{SCLA}(P) + \text{SCNR}(P)}{(2^n - 1)n},
\]
and
\[
\text{RMK}'(P) = \frac{\text{SCBR}(P)}{(2^n - 1)n}.
\]

The last four Equations (36) to (39) show that the values of $\text{AMK}(P)$ and $\text{AMK}'(P)$ can be obtained easily.

**Theorem 5:** For two knowledge $P$ and $Q$ on $U$, $\text{AMK}(P) \geq \text{AMK}(Q)$ if and only if $\text{SCLA}(P) \geq \text{SCLA}(Q)$.
Proof: According to Equation (37), we know that
\[ \text{AMK}(P) = \frac{\text{SCLA}(P)}{\text{SCUA}(P)} = \frac{\text{SCLA}(P)}{\text{SCLA}(P) + \text{SCBR}(P)}. \]
Therefore, according to Corollary 5, \( \text{AMK}(P) \geq \text{AMK}(Q) \) if and only if \( \text{SCLA}(P) \geq \text{SCLA}(Q) \).

Theorem 6: For two knowledge \( P \) and \( Q \) on \( U \), \( \text{AMK}'(P) \geq \text{AMK}'(Q) \) if and only if \( \text{SCLA}(P) \geq \text{SCLA}(Q) \).

Proof: According to Equation (39) and Corollary 5, it is obviously that \( \text{AMK}'(P) \geq \text{AMK}'(Q) \) if and only if \( \text{SCLA}(P) \geq \text{SCLA}(Q) \).

C. INACCURACY ANALYSIS BETWEEN DIFFERENT UNCERTAINTY MEASURES OF KNOWLEDGE

Klir [34] asserted that there are various manifestations of information deficiency pertaining to the system within which the situation is conceptualized and these various information deficiencies determine the type of the associated uncertainty. The information deficiency may be, for example, incomplete, imprecise, fragmentary, unreliable, vague, or contradictory. In rough sets, the situation refers to the lower and upper approximations and the information deficiency refer to the inexactness or roughness of knowledge. Therefore, \( \text{AMK}(P) \) and \( \text{RMK}(P) \) or \( \text{AMK}'(P) \) and \( \text{RMK}'(P) \) would be appropriate ways to express quantitatively the uncertainty of knowledge in rough sets.

In Subsection II-B, we list some uncertainty measures of knowledge. Because these measures neglect the influence of approximation set on knowledge uncertainty, their characterization of knowledge uncertainty in approximation space may be inaccurate.

The following example will illustrate the inaccuracy in existing uncertainty measures of knowledge.

Example 3: Suppose \( U = \{a, b, c, d, e, f, g, h, i, j\} \), there are twelve partitions on \( U \) listed in Table 8, then we can get the results listed in the following Table 9.

**TABLE 8. Some partitions on \( U \).**

| Partition | Description of partition |
|-----------|--------------------------|
| \( P_1 \) | \{a, b, c, d, e, f, g, h, i, j\} |
| \( P_2 \) | \{a, b, c, d, e\} |
| \( P_3 \) | \{a, b, c\} |
| \( P_4 \) | \{a, b\} |
| \( P_5 \) | \{a\} |
| \( P_6 \) | \{a\} |
| \( P_7 \) | \{a\} |
| \( P_8 \) | \{a\} |
| \( P_9 \) | \{a\} |
| \( P_{10} \) | \{a\} |
| \( P_{11} \) | \{a\} |
| \( P_{12} \) | \{a\} |

In order to better illustrate the laws reflected by the data in Table 9, we provide four maps in Fig. 3. Map (a) shows that the values of \( \gamma(P) \) and \( \gamma'(P) \) with respect to partition \( P_1 \sim P_{12} \) are monotonically increasing. According to the definition of \( \gamma(P) \) and \( \gamma'(P) \), although they are difficult to be calculated, they provide ways to describe the uncertainty of knowledge as accurately as possible. Furthermore, they fully consider the role of the lower and the upper approximations, which is consistent with Klir’s view on uncertainty. That is, different systems have different information deficiencies, so there are different uncertain descriptions. In the rough set model, the uncertainty of knowledge has a close relationship with the lower and the upper approximations, so they should be fully considered when describing the uncertainty in the approximation space. Therefore, we believe that a proper uncertainty measure in approximation space should be consistent with \( \gamma(P) \) and \( \gamma'(P) \) in describing the uncertainty of knowledge.

Map (b) shows that \( \text{AMK}(P) \) and \( \text{AMK}'(P) \) with respect to the twelve partitions exhibit a monotonically increasing state as the subscript of partitions increase, which is consistent with \( \gamma(P) \) and \( \gamma'(P) \). Map (c) and (d) show that the first six measures listed in Table 9 do not have monotonically increasing states for the twelve partitions as \( \gamma(P) \) and \( \gamma'(P) \). This result may be due to the fact that these six uncertainty measure models do not consider the background of rough set theory and neglect the important influence of the upper and the lower approximations on uncertain measurement.

This Example indicates that \( \text{AMK}(P) \) and \( \text{AMK}'(P) \) may be more suitable than other six measure models to characterize the uncertainty of knowledge in the context of rough sets.

In addition, because of ignoring the theoretical background of rough set, the first six uncertainty measures listed in Table 9 lack a criterion to evaluate their accuracy when they are defined. Therefore, for any two knowledge \( P \) and \( Q \) on \( U \) and \( P \preceq Q \), although these six measures have the same trend of uncertainty about \( P \) and \( Q \) with \( \text{AMK}(P) \) and \( \text{AMK}'(P) \), we cannot evaluate their correctness when the knowledge \( P \) and \( Q \) do not have the relation of \( P \preceq Q \). In other words, the values of uncertainty of knowledge \( P \) and \( Q \) obtained by using these six measures are unreliable in general cases because of the lack of criterion estimating the uncertainty of knowledge.

For this reason, we provide two different criterions by defining the accuracy and roughness measures of knowledge and produce two formalizations for them, i.e., \( \text{AMK}(P) \) and \( \text{AMK}'(P) \), respectively. In Example 3, the result shows not only the inaccuracy in existing uncertainty measures of knowledge but also the necessity and significance of such criterions like \( \text{AMK}(P) \) and \( \text{AMK}'(P) \).

D. RELATIONS BETWEEN \( \gamma'(P), \text{AMK}(P) \) AND \( \text{AMK}'(P) \)

In the previous subsections, we provided the methods for calculating \( \text{AMK}(P) \) and \( \text{AMK}'(P) \) and found the problems of existing uncertainty measures of knowledge. In this section, the relation between \( \gamma'(P) \), \( \text{AMK}(P) \) and \( \text{AMK}'(P) \) will be discussed.

Proposition 8: For a knowledge \( P \) on \( U \), \( \gamma'(P) = \text{AMK}'(P) \).
According to Equations 24 and 26, it is obviously valid. This indicates that, based on the definition of $\alpha'_P(X)$, $AMK'(P)$ can be viewed as an accurate uncertainty measure of knowledge. However, based on the definition of $\alpha_P(X)$, $AMK(P)$ just give an approximate description of uncertainty measure of knowledge. Furthermore, it is hard to prove whether the description of knowledge uncertainty by $\gamma(P)$ and $AMK(P)$ are consistent, that is, $AMK(P) \leq AMK(Q) \iff \gamma(P) \leq \gamma(Q)$ may not be valid.

Theorem 7: For two knowledge $P$ and $Q$ on $U$, $AMK(P) \geq AMK(Q)$ if and only if $AMK'(P) \geq AMK'(Q)$.

V. UNCERTAINTY ENTROPY OF KNOWLEDGE

In this section, an uncertainty entropy is provided to measure the uncertainty of knowledge. The relations between the uncertainty entropy, $AMK(P)$ and $AMK'(P)$ are discussed.
A. UNCERTAINTY ENTROPY

Definition 5: Let \( P \) be a partition on \( U, X_i \in P \). Then
\[
UE(P) = 1 - \frac{2}{n} \sum_{i=1}^{m} \frac{|X_i|}{2^{|X_i|}},
\]
is called the uncertainty entropy of \( P \).

In terms of Equation 40 we can find that \( UE(P) \) is more convenient to calculate than AMK(\( P \)) and AMK\(^{2}\)(\( P \)).

Proposition 9: Let \( P \) be a knowledge on \( U \). Then \( 0 \leq UE(P) < 1 \).

Proof: Suppose that \( |U| = n, n = n_1 + n_2 \) and \( n_1 \leq n_2 \). Then \( \frac{2}{n} \leq \frac{n_1}{n_2} + \frac{n_2}{n_2} \Leftrightarrow n_2(2^{n_1} + n_2) \leq 2^n(n_12^{n_1} + n_22^{n_2}) = 2^{n_1 + n_2}(n_12^{n_1} + n_2) = 2^{n_1 + n_2}(n_12^{n_1} + n_2) \Leftrightarrow n < 2^{n_2}(n_12^{n_1} + n_2) \leq 2^{n_1 + n_2}. \) Similarly, if \( n = n_1 + n_2 + \cdots + n_m \), then we can get that \( \frac{n}{2^n} \leq \sum_{i=1}^{m} \frac{|X_i|}{2^{|X_i|}} \). That is \( UE(P) \) reaches the maximum value while \( P = \{U\} \) and \( UE(P) \) reaches the minimum value while \( |K_i| = 1 \) for each \( K_i \in P \). Therefore, \( 0 \leq UE(P) < 1 \).

Proposition 9 shows that the uncertainty entropy of knowledge is nonnegative. With the increase of the cardinality of the universe, the uncertainty of knowledge \( \phi \) tends to be close to \( 1 \).

Proposition 10: Let \( P \) and \( Q \) be two knowledge on \( U \). Then \( P \preceq Q \Rightarrow UE(P) \leq UE(Q) \).

Proof: If \( P \preceq Q \) then for any block \( Y_i \in Q \) there are some blocks \( X_1, \ldots, X_b \in P \) such that \( \cup X_i \subseteq Y_i \). According to the proof process of Proposition 9, we can get that \( \sum_{i=1}^{b} \frac{|X_i|}{2^{|X_i|}} \geq \frac{|Y_i|}{2^{|Y_i|}} \). Therefore, \( P \preceq Q \Rightarrow UE(P) \leq UE(Q) \).

For any two different knowledge \( P \) and \( Q \), will their uncertainty entropy be equal? Or under what circumstance are they equal? Next we will discuss these issues.

Example 4: Let \( P = \{[a, b], [c, d], [e]\} \) and \( Q = \{[a, c], [b, e], [d]\} \) be two knowledge on \( U = \{a, b, c, d, e\} \). Then, according to Equation 40, we can get that \( UE(P) = UE(Q) = 0.4 \).

Example 4 shows that two different knowledge maybe have the same uncertainty entropy.

For convenience of discussion, we use \( C_P(m) = \{K_i \in P : |K_i| = m\} \) to represent the number of equivalence classes in \( P \) which bases are equal to \( m \), where \( m \) is an integer. Example 4 indicates that the following conclusion can be obtained.

Proposition 11: Let \( P \) and \( Q \) be two knowledge on \( U \). For \( j = 1, 2, \ldots, |U|, \) if \( C_P(j) = C_Q(j) \) then \( UE(P) = UE(Q) \).

Proof: According to the definition of \( C_P(m) \), this conclusion is clearly valid.

Proposition 11 gives a sufficient condition for that different knowledge have the same uncertainty entropy.

B. RELATIONS BETWEEN AMK(\( P \)), AMK\(^{2}\)(\( P \)) AND UE(\( P \))

Theorem 8: Let \( P \) be a knowledge on \( U \). \( AMK(P) = 1 - UE(P) \).

Proof: According to Equations (3), (4), (5) and (37), we can get
\[
RMK(\frac{SCBR(P)}{SCUA(P)}) = \frac{SCBR(P)}{SCLA(P) + SCBR(P)} = \frac{1}{1 + \frac{SCLA(P)}{SCBR(P)}}.
\]
According to Theorem 4,
\[
\frac{SCLA(P)}{SCBR(P)} = \frac{2^n - SCBR(P)}{SCBR(P)} = \frac{2^n - SCBR(P)}{2^n - 1} = \frac{n - 1}{\sum_{i=1}^{m} 2^n - 2^n - |X_i| |X_i| - 1} = 1 - \frac{1}{UE(P)} - 1.
\]
According to Equation 22, we can get \( AMK(P) = 1 - UE(P) \).

Corollary 8: Let \( P \) be a knowledge on \( U \). \( RMK(P) = UE(P) \).

Proposition 12: For two knowledge \( P \) and \( Q \) on \( U \), \( AMK^{2}(P) \geq AMK^{2}(Q) \) if and only if \( UE(P) \leq UE(Q) \).

Proof: According to Equations (25), (27), (39) and Theorem 3, we can get
\[
RMK^{2}(P) = \frac{\sum_{i=1}^{m} (2^n - 2^n - |X_i| |X_i|)}{n(2^n - 1)} = 2^n \frac{\sum_{i=1}^{m} (2^n - 2^n - |X_i| |X_i|)}{n(2^n - 1)} = 2^n \frac{1}{2^n - 1} \sum_{i=1}^{m} (|X_i| - 2^n - |X_i| |X_i|) = 2^n \frac{1}{2^n - 1} \sum_{i=1}^{m} (2^n - 2^n - |X_i| |X_i|)
\]
Similarly, we can get
\[
RMK^{2}(Q) = \frac{2^n}{2^n - 1} UE(Q).
\]
Therefore, according to Equation (25), \( AMK^{2}(P) \geq AMK^{2}(Q) \) if and only if \( UE(P) \leq UE(Q) \).

Corollary 9: Let \( P \) be a knowledge on \( U \). \( AMK^{2}(P) = 1 - \frac{2^n}{2^n - 1} UE(P) \).

According to the proof of Proposition 12, we can get this conclusion directly.

C. A VARIANT OF UE(\( P \))

Although \( UE(P) \) can help us to calculate the value of the uncertainty measure of knowledge more quickly, the values from different knowledge are usually tends to 1 and then the difference between them seems to be very small as well, which makes it difficult to distinguish and understand the
In the following Fig. 4, the uncertainty entropies of different rules are usually very big and then the base of each equivalence class in knowledge $P$ is usually large. As a result, the value of $UE(P)$ usually tends to 1. Then we can find that, for different knowledge on $U$, it is hard to tell the difference between the uncertainty entropies of them. For this reason, we provide a variant of $UE(P)$ to improve this problem. The variant is defined as follows:

$$UE'(P) = -\log\left(\frac{2}{n} \sum_{i=1}^{m} \frac{|X_i|}{2^{|X_i|}}\right),$$  \hspace{1cm} (41)

**Proposition 13:** Let $P$ be a partition on $U$, $|U| = n$. Then $0 \leq UE'(P) \leq \log(2^n - 1)$.

It is easy to find that $UE'(P)$ is equal to 0 when $P = \varnothing$ and $\log(2^n - 1)$ when $P = U$. This shows that for a knowledge $P$, the finer the $P$ is, the smaller the $UE'(P)$ is, and vice versa.

**Proposition 14:** Let $P$ and $Q$ be two partitions on $U$. Then $UE(P) \leq UE(Q)$ if and only if $UE'(P) \leq UE'(Q)$.

Proposition 14 shows that the descriptions of uncertainty measures of knowledge $P$ by $UE(P)$ and $UE'(P)$ have the same rules.

In the following table, we list some knowledge noted by $P_1, \ldots, P_6$ and $UE(P_i)$ and $UE'(P_i)$ of them.

| $P_i$ | $UE(P_i)$ | $UE'(P_i)$ |
|------|-----------|-------------|
| $P_1$ | 0.9977361000000 | 8.7870 |
| $P_2$ | 0.9999936483000 | 17.2644 |
| $P_3$ | 0.999994069000 | 20.8769 |
| $P_4$ | 0.9999997023600 | 25.0019 |
| $P_5$ | 0.9999999870170 | 29.5207 |
| $P_6$ | 0.999999987618 | 32.9110 |

In Table 10, almost every $UE(P_i)$ tends to 1. As shown in the following Fig. 4, the uncertainty entropies of $P_2, P_3, P_4, P_5$ and $P_6$ in map (a) seem to be the same, and it is difficult to distinguish the subtle difference between them. However, in map (b), $UE'(P_i)$ for $i = 2, 3, 4, 5, 6$ have obvious differences, and it is easy to distinguish the size relationship between them.

**VI. CONCLUSION**

The uncertainty measure issue in rough sets is discussed in this paper. Some reasons for the irrationalities of existing uncertainty measures have been pointed out. Then we did some work to improve it. First, we proposed a new accuracy measure $\alpha''_{AMK}(X)$ in which the role of the target set itself is fully considered. Compared with the existing accuracy measure models, $\alpha''_{AMK}(X)$ provides a more reasonable description. Second, we proposed two definitions, $AMK(P)$ and $AMK'(P)$, to clarify what the uncertainty of a knowledge refers to in the approximation space. And we developed a method to calculate the value of $AMK(P)$ and $AMK'(P)$ easily. Based on them, we can evaluate the accuracy and rationality of the uncertainty measure of a knowledge objectively and comprehensively. Third, in order to more conveniently represent the uncertainty degree of a knowledge $P$, we construct an uncertain entropy $UE(P)$. We also provide a variant of $UE(P)$ to better distinguish the difference of uncertainty degrees of different knowledge. There is a regret in this study, that is, although $\alpha''_{AMK}(X)$ is more reasonable than $\alpha_{AMK}(X)$ and $\alpha''_{AMK}(X)$, we have not been able to construct the model of approximation accuracy of a knowledge corresponding to $\alpha''_{AMK}(X)$ due to its more complex structure. In the future, we will continue to pay attention to this issue to explore an effective solution.

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JIANGUO TANG received the B.S. degree in computer science from Xinjiang University, Urumqi, China, in 2003, and the Ph.D. degree in computer science from the University of Electronic Science and Technology of China, Chengdu, China, in 2012. He is currently an Associate Professor with the Chongqing College of Electronic Engineering. His current research interests are in rough sets, granular computing, and machine learning.

JIANHUA WANG received the B.S. degree in computer science from Xinjiang Normal University, Urumqi, China, in 2004, and the Ph.D. degree in information science from the University of Chinese Academy of Sciences, Beijing, China, in 2013. She is currently an Associate Professor with the Chongqing College of Electronic Engineering. Her current research interests are in data mining and machine learning.

CHUNLING WU received the B.S. degree from the School of Computer Science, Chongqing University, Chongqing, China, in 2002, and the M.S. degree in software engineering from the School of Software Engineering, Chongqing University, in 2005. He is currently a Professor with the Chongqing College of Electronic Engineering. He has authored more than 30 research articles in journals and conference proceedings. His research interests include information security, cloud computing, and artificial intelligence.

GUOJIAN OU received the Ph.D. degree from Chongqing University, in 2016. He is currently an Associate Professor with the Chongqing College of Electronic Engineering. His current research interests include polynomial phase signal processing and sparse decomposition.