CONTROLLABILITY OF HILFER FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS OF SOBOLEV-TYPE WITH A NONLOCAL CONDITION IN A BANACH SPACE

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Abstract. This paper aims to establish sufficient conditions for the exact controllability of the nonlocal Hilfer fractional integro-differential system of Sobolev-type using the theory of propagation family \{P(t), t \geq 0\} generated by the operators A and R. For proving the main result we do not impose any condition on the relation between the domain of the operators A and R. We also do not assume that the operator R has necessarily a bounded inverse. The main tools applied in our analysis are the theory of measure of noncompactness, fractional calculus, and Sadovskii’s fixed point theorem. Finally, we provide an example to show the application of our main result.

1. Introduction. In recent years, many researchers have paid their attention to the fractional calculus as it has plenteous applications in the field of viscoelasticity, porous media, control, electromagnetic, and so on. It is more suitable for the description of many physical phenomena arising in economics, science, and engineering as compared to classical derivatives and integrals. For more details on it, we refer to [1, 3–5, 10–12, 20, 22, 23, 28, 35] and references cited therein. In 2000, Hilfer [16] introduced the new definition of fractional derivative (known as Hilfer fractional derivative) which is a generalization of Riemann-Liouville fractional derivative as well as an interpolation between Riemann-Liouville and Caputo’s fractional derivative. This derivative appears in the theoretical simulation of dielectric relaxation in glass-forming materials, rheological constitute modeling, polymer science, etc., see, for instance, [7,12–14,30]. In [34], Yang and Wang studied the existence and uniqueness of a mild solution of Hilfer fractional evolution equations. Gu and Trujillo [14] studied a Hilfer fractional differential equation using Mittag-Leffler function and Fox’s \(H\)-function.

The concept of controllability was firstly introduced by Kalman in 1960. It is now widely used not only in control theory but in numerous fields such as quantum systems theory, control of electric bulk power systems, reactor control, chemical process control, aerospace engineering, etc. The controllability problem means that...
we have to find an appropriate control function such that the state of the dynamical system is steered to the desired final state. In [31], Wang and Zhou have built sufficient conditions for the complete controllability of a nonlinear fractional system in a Banach space. There are some interesting and important controllability results [6,11,17–19,21,27,29,31,32] concerning semi-linear or nonlinear differential systems in which authors used semigroup theory and some fixed point theorems.

Moreover, the Sobolev-type differential equations frequently appear in the mathematical modeling of many physical phenomena such as in thermodynamics, the flow of fluid through fissured rocks, soil mechanics, shear in second-order fluids, etc. (for more details, readers refer to [2, 4–6, 13, 27]). Li et al. [23] investigated the existence of a mild solution for Sobolev-type fractional evolution equations with nonlocal conditions. Gou and Li [13] studied nonlinear Sobolev-type delay Hilfer fractional integro-differential equations using the measure of non-compactness theory and Sadovskii’s fixed point theorem in our analysis. The exact controllability in the next section. We also use fractional calculus theory, noncompactness measure in a Banach space. There are some interesting and important controllability results concerning semi-linear or nonlinear differential systems in which authors used semigroup theory and some fixed point theorems.

In this paper, we consider following Hilfer fractional integro-differential equations of Sobolev-type with a nonlocal condition in a Banach space $Z$:  

\[
\begin{align*}
D^{\beta,\nu}_{0+} Rz(t) &= Az(t) + Rf(t, z(t), \int_0^t g(t, r, z(r)) \, dr) + RBw(t), \\
I_{0+}^{1-\gamma(1-\nu)} Rz(0) &= R[z_0 + h(z)],
\end{align*}
\]

where $D^{\beta,\nu}$ is the Hilfer fractional derivative of order $\beta$, $(0 < \beta < 1)$ and type $\nu$, $(0 \leq \nu \leq 1)$; $z(\cdot)$ is a state function which takes the value in Banach space $Z$; $I_{0+}^{\gamma}$ is the Riemann-Liouville fractional integral of order $\gamma > 0$; $A$ and $R$ are closed linear operators (not necessarily bounded) with domains $D(A)$ and $D(R)$ respectively contained in $Z$; $w(\cdot) \in L^2(J, W)$, a Banach space of all admissible control functions, $W$ is another Banach space; $B$ is a bounded linear operator from $W$ into $D(R)$; the nonlinear functions $f: J \times Z \times Z \to D(R)$ and $g: \Sigma \times Z \to Z$ are to be specified later, $\Sigma = \{(t, r) \mid 0 \leq r \leq t \leq b\}$; $z_0 \in D(R)$, and $h: \mathcal{F} \to D(R)$ is a nonlinear function, here $\mathcal{F} = \{z \in C(J', Z) : \lim_{t\to0} t^{(1-\gamma)(1-\nu)} z(t) \text{ exists and finite}\}$.

Motivated from the papers [13,25], we aim to establish sufficient conditions for the exact controllability of the system (1). We prove our main result using the theory of propagation family $\{P(t), t \geq 0\}$ generated by the operators $A$ and $R$ without imposing any condition on the operator $R$ (i.e., the operator $R$ has not necessarily a bounded inverse) as well as without considering any relation between $D(A)$ and $D(R)$. The definition of the propagation family of operators $A$ and $R$ will be given in the next section. We also use fractional calculus theory, noncompactness measure theory and Sadovskii’s fixed point theorem in our analysis. The exact controllability of Sobolev-type fractional integro-differential system (1) with a nonlocal condition in a Banach space $Z$ has not yet been studied in any of the research paper.

We organize the rest of the paper as follows: In section 2, we introduce some notations, hypotheses, basic definitions, and preliminary results. In section 3, we prove the exact controllability of nonlinear Hilfer fractional integro-differential system (1) of Sobolev-type. Finally, in section 4, we provide an example to show the application of our main result.
2. Preliminaries. Let $J' = (0, b]$ and
\[ \mathcal{Z} = \{ z \in C(J', Z) : \lim_{t \to 0} t^{(1-\nu)(1-\varrho)} z(t) \text{ exists and finite} \} . \]
Obviously $\mathcal{Z}$ is a Banach space with norm $\| \cdot \|_\mathcal{Z}$ defined by
\[ \| z \|_\mathcal{Z} = \sup_{t \in J'} \| t^{(1-\nu)(1-\varrho)} z(t) \| . \]

**Definition 2.1** (see [28]). The Riemann-Liouville fractional integral of order $\varrho > 0$ for a function $f \in L^1(J, Z)$ is defined as
\[ I_{0^+}^\varrho f(t) = \frac{1}{\Gamma(\varrho)} \int_0^t (t - \varsigma)^{\varrho-1} f(\varsigma) d\varsigma, \quad t > 0, \]
where $\Gamma$ is the gamma function.

**Definition 2.2** (see [28]). The Riemann-Liouville fractional derivative of order $\varrho$, $0 \leq n - 1 < \varrho < n$, is defined as
\[ D_{0^+}^\varrho f(t) = \frac{1}{\Gamma(n-\varrho)} \frac{d^n}{dt^n} \int_0^t \frac{f(\varsigma)}{(t - \varsigma)^{n+1-\varrho}} d\varsigma, \quad t > 0, \]
where $f$ is an $n$-times continuous differentiable function.

If $f$ is an abstract function with values in a Banach space $Z$, then integrals which appear in Definition 2.1 and 2.2 are taken in Bochner’s sense.

**Definition 2.3** (see [16]). The Hilfer fractional derivative of the type $\nu \in [0, 1]$ and of order $\varrho \in (0, 1)$ with lower limit 0 for a function $f$ is defined as
\[ D_{0^+}^{\varrho,\nu} f(t) = \left( I_{0^+}^{\nu(1-\varrho)} \frac{d}{dt} (I_{0^+}^{1-\mu}(f)) \right)(t) = \left( I_{0^+}^{\nu(1-\varrho)} (D_{0^+}^\mu f) \right)(t), \]
where $\mu = \varrho + \nu - \varrho \nu$, provided that the right hand side exists.

**Remark 1.** If $\nu = 0$, $0 < \varrho < 1$, then Hilfer fractional derivative is Riemann-Liouville fractional derivative of order $\varrho$.

Consider the abstract degenerate Cauchy problem:
\[
\begin{align*}
\frac{d}{dt} Rz(t) &= Az(t), \quad t \in J, \\
Rz(0) &= Rz_0.
\end{align*}
\]

**Definition 2.4** (see [25]). A strongly continuous operators family $\{ P(t) \}_{t \geq 0}$ from $D(R)$ to $Z$, satisfying that there are constants $c > 0$, $M > 0$ such that
\[ \| P(t) z \| \leq M e^{ct} \| z \|, \quad t \geq 0 \text{ and } z \in D(R), \]
(3) is said to be an exponentially bounded propagation family for the system (2) if
\[ (\lambda R - A)^{-1} Rz = \int_0^\infty e^{-\lambda t} P(t) z \, dt, \quad \text{for} \quad \lambda > c \text{ and } z \in D(R). \]
(4)

If the equation (4) holds, then one can say that the pair $(A, R)$ is a generator of the exponentially bounded propagation family $\{ P(t) \}_{t \geq 0}$.

**Definition 2.5** (see [13, 14]). A function $z : [0, b] \to Z$ is said to be a mild solution of the system (1) if the function $z|_{[0, b]}$ is continuous and it satisfies the following fractional integral equation:
\[ z(t) = S_{\varrho,\nu}(t)[z_0 + h(z)] \]
\[ + \int_0^t K_\varphi(t-c) \left[ f(\zeta, z(\zeta), \int_0^\zeta g(\zeta, r, z(r)) \, dr) + Bw(c) \right] \, d\zeta, \]  

where  
\[ S_\varphi(\nu)(\varphi) = I^\nu(1-\varphi)K_\varphi(t)\varphi, \quad K_\varphi(t)\varphi = t^{\nu-1}Q_\varphi(t)\varphi, \]  
\[ Q_\varphi(t)\varphi = \varphi \int_0^\infty \theta \psi_\varphi(\theta)P(t^\varphi\theta)\theta \, d\theta, \quad \psi_\varphi(\theta) = \frac{1}{\theta} \theta^{-1-1/\varphi} \xi_\varphi(\theta^{-1/\varphi}), \]  
\[ \xi_\varphi(\theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \theta^{-n} \frac{\Gamma(n+1)}{n!} \sin(n\pi \theta), \quad \theta \in (0, \infty), \quad y \in D(R). \]

**Lemma 2.6** (see [14]). If \( \{ P(t) \}_{t \geq 0} \) is norm continuous and \( \| P(t) \| \leq M \) for some \( M \geq 1 \) and any \( t > 0 \), then \( K_\varphi(t)\varphi \) and \( S_\varphi(\nu)(t)\varphi \) are strongly continuous linear operators, and for any \( z \in Z \) and \( t > 0 \) we have  
\[ \| K_\varphi(t)z \| \leq \frac{Mt^{\nu-1}}{\Gamma(\varphi)} \| z \| \quad \text{and} \quad \| S_\varphi(\nu)(t)z \| \leq \frac{M(t^{\nu-1})(1-\varphi)}{\Gamma(\nu(1-\varphi)+\varphi)} \| z \|. \]

Let’s recall Kuratowski’s measure of noncompactness and its properties that will be used in Section 3.

**Definition 2.7** (see [9, 15]). Let \( \mathcal{B}(Z) \) be a collection of bounded subsets of \( Z \). The function \( \alpha: \mathcal{B}(Z) \rightarrow \mathbb{R}^+ \) defined by  
\[ \alpha(D) = \inf \{ \varepsilon > 0 : D \subset \bigcup_{j=1}^{n} D_j, \text{diam}(D_j) < \varepsilon (j = 1, 2, \ldots, n \in \mathbb{N}) \}, \quad D \in \mathcal{B}(Z), \]  
is called the Kuratowski’s measure of noncompactness.

**Lemma 2.8** (see [9, 15]). If \( E_1, E_2 \) and \( E \) are bounded subsets of a Banach space \( Z \), then the following statements are true:

(i) \( E \) is relatively compact set in \( Z \) if and only if \( \alpha(E) = 0 \).
(ii) \( \alpha(E_1) \leq \alpha(E_2) \) if \( E_1 \subset E_2 \).
(iii) \( \alpha(E_1 + E_2) \leq \alpha(E_1) + \alpha(E_2) \).
(iv) \( \alpha(cE) \leq |c| \alpha(E) \) for any \( c \in \mathbb{R} \).

**Lemma 2.9** (see [24]). If \( S \subset Z \) is bounded, there is a countable set \( E \subset S \) such that \( \alpha(S) \leq 2\alpha(E) \).

**Lemma 2.10** (see [9, 15]). Let \( b_1, b_2 \) be any positive numbers such that \( b_1 < b_2 \). If \( E \) is a bounded subset in \( C([b_1, b_2], Z) \), then \( E(t) \) is bounded in \( Z \), and \( \alpha(E(t)) \leq \alpha(E) \). Further, if \( E \) is also equicontinuous on \([b_1, b_2]\), then \( \alpha(E(t)) \) is continuous for \( t \in [b_1, b_2] \) and  
\[ \alpha(E) = \sup \{ \alpha(E(t)) : t \in [b_1, b_2] \}, \quad \text{where} \quad E(t) = \{ z(t) : z \in E \} \subset E. \]

**Lemma 2.11** (see [26]). Let \( \{ g_m \}_{m=1}^\infty \) be a sequence of functions in \( L^1([0, \sigma], \mathbb{R}^+) \), and suppose that there are \( \zeta, \gamma \in L^1([0, \sigma], \mathbb{R}^+) \) satisfying \( \sup_{m \geq 1} \| g_m(r_1) \| \leq \zeta(r_1) \) and \( \alpha(\{ g_m \}_{m=1}^\infty) \leq \gamma(r_1) \) a.e. \( r_1 \in [0, \sigma] \), then for each \( r_1 \in [0, \sigma] \), we get  
\[ \alpha \left( \left\{ \int_0^{r_1} g_m(r) \, dr : m \geq 1 \right\} \right) \leq 2 \int_0^{r_1} \gamma(r) \, dr. \]

**Definition 2.12.** Let \( Z \) and \( Y \) be two Banach spaces. A continuous map \( F : \Omega \subset Z \rightarrow Y \) is said to be a condensing map if \( F \) takes the bounded sets into bounded sets and \( \alpha(F(S)) < \alpha(S) \) for all bounded sets \( S \subset \Omega \) with \( \alpha(S) \neq 0 \).
Theorem 2.13 (Sadovskii’s fixed point theorem). Let $Z$ be a Banach space and $\Omega$ be a closed convex bounded subset in $Z$. If $F: \Omega \to \Omega$ is a condensing map. Then $F$ has a fixed point in $\Omega$.

3. Main result. In this section, we prove the controllability result of problem (1). First, we take the following assumptions:

(A0) The propagation family $\{P(t)\}_{t \geq 0}$ is norm continuous and uniformly bounded, i.e. $\|P(t)\| \leq M$ for some $M \geq 1$ and any $t \geq 0$.

(A1) The function $f: J \times Z \times Z \to D(R)$ satisfies the Carathéodory condition, i.e. the function $f(t, \cdot, \cdot): Z \times Z \to Z$ is continuous for each $t \in J$ and $f(\cdot, z_1, z_2): J \to Z$ is strongly measurable for any $z_1, z_2 \in Z$. For each $t \in J$ and $k > 0$ there are $l_{k,1}, l_{k,2} \in L^1([0, t], \mathbb{R}^+)$, $i = 1, 2$ such that

$$\sup \{\|f(t, z_1, z_2)\|: \|t^{(1-\nu)(1-\rho)}z_1\| \leq k\} \leq l_{k,1}(t) + l_{k,2}(t)\|z_2\| \quad \text{for a.e. } t \in J,$$

$$\lim_{k \to \infty} \inf_{t \in J} \frac{1}{k} \int_0^t \frac{l_{k,1}(s)}{(t-s)^{1-\nu}} = l_1 < +\infty, \quad \text{and}$$

$$\lim_{k \to \infty} \int_0^t \frac{l_{k,2}(s)}{(t-s)^{1-\rho}} = l_2 < +\infty.$$

(A2) The function $g: \Sigma \times Z \to Z$ satisfies that the function $g(t, \cdot, \cdot): Z \to Z$ is continuous for each $(t, \varsigma) \in \Sigma$ and the function $g(\cdot, \cdot, y): \Sigma \to Z$ is strongly measurable for each $y \in Z$. There exists a function $m \in L^1(\Sigma, \mathbb{R}^+)$ such that

$$\|g(t, \varsigma, y)\| \leq m(t, \varsigma)\|\varsigma^{(1-\nu)(1-\rho)}y\|.$$

Take $m^* = \max_{t \in J} \int_0^t m(t, \varsigma)d\varsigma$.

(A3) The map $h: \mathfrak{z} \to D(R)$ is such that

$$\|h(y) - h(z)\| \leq l_k\|y - z\|_{\mathfrak{z}}, \quad \forall \ y, z \in \mathfrak{z}$$

for some $l_k > 0$.

(A4) There exist a function $\eta: [0, b] \to \mathbb{R}$ with the condition $(t - \cdot)^{1-\eta} \eta(\cdot) \in L^1([0, t], \mathbb{R}^+)$ and an integrable function $\rho: \Sigma \to [0, \infty)$ such that

$$\alpha(f(t, E, S)) \leq \eta(t) \left[\alpha(t^{(1-\nu)(1-\rho)}E) + \alpha(S)\right]$$

and

$$\alpha(g(t, \varsigma, E)) \leq \rho(t, \varsigma)\alpha(\varsigma^{(1-\nu)(1-\rho)}E)$$

for a.e. $t \in J$, and $E, S \subset Z$. Also suppose that $\eta^* = \max_{t \in J} \int_0^t (t - \varsigma)^{1-\eta}\rho(\varsigma)d\varsigma$ and $\rho^* = \max_{t \in J} \int_0^t \rho(t, \varsigma)d\varsigma$.

(A5) The linear operator $T: L^2(J, W) \to Z$ defined by

$$Tw = \int_0^b K_\theta(b - \varsigma)Bw(\varsigma)d\varsigma$$

has an induced inverse operator $\tilde{T}^{-1}$ which takes values in $L^2(J, W)/\ker T$ and there exists a constant $C_T > 0$ such that

$$\|\tilde{T}^{-1}\|_{L^2} \leq C_T$$

and also there is an integrable function $\Phi: J \to \mathbb{R}^+$ such that for every bounded set $E \subset Z$,

$$\alpha(\tilde{T}^{-1}E(t)) \leq \Phi(t)\alpha(E) \quad \text{and}$$

$$\Phi^* = \max_{t \in J} \int_0^t (t - \varsigma)^{1-\Phi(\varsigma)}d\varsigma.$$
(A6) Take
\[ N = M \left[ \frac{l_h}{\Gamma(\nu(1 - \theta) + \varrho)} + \frac{2b(1 - \nu)(1 - \varrho)(1 + 2\rho^*)}{\Gamma(\varrho)} \right] \left[ 1 + \frac{2M\|B\|\Phi^*}{\Gamma(\varrho)} \right] < \frac{1}{2}. \]

Using hypothesis (A5), for each \( z_b \in Z \) we define the control
\[ w_z(t) = \bar{T}^{-1} \left[ z_b - S_{\varrho,\nu}(b)[z_0 + h(z)] \right. \]
\[ - \int_0^b K_\varphi(b - \varsigma)f(\varsigma, z(\varsigma), \int_0^\varsigma g(\varsigma, r, z(r)) dr) d\varsigma \right] (t), \quad (6) \]
where \( z(\cdot) \in C(J', Z) \). It is clear that \( C(J, Z) \) is a Banach space with sup-norm \( \| \cdot \| \). We now define the sets \( E_k \) and \( E^3_k \) as
\[ E_k = \{ y(\cdot) \in C(J, Z) : \| y \|_\infty \leq k \} \quad \text{and} \quad E_k^3 = \{ z \in 3 : \| z \|_3 \leq k \}. \]

**Theorem 3.1.** Assume that the hypothesis (A0)-(A6) hold. Then the system (1) is controllable on \( J \) if the condition
\[ M \left\{ 1 + \frac{Mb}{\Gamma(\nu(1 - \theta) + \varrho)} \right\} \left\{ \frac{l_h}{\Gamma(\nu(1 - \theta) + \varrho)} + \frac{b(1 - \nu)(1 - \varrho)}{\Gamma(\varrho)} \right\} \left\{ l_1 + l_2m^* \right\} < 1 \quad (7) \]
is satisfied.

**Proof.** Using the control (6), we define an operator \( F : \mathcal{F} \rightarrow \mathcal{F} \) as
\[ (Fz)(t) = S_{\varrho, \nu}(t)[z_0 + h(z)] \]
\[ + \int_0^t K_\varphi(t - \varsigma)f(\varsigma, z(\varsigma), \int_0^\varsigma g(\varsigma, r, z(r)) dr) + Bw_z(\varsigma) d\varsigma \quad (8) \]
For any \( y \in C(J, Z) \), \( z(t) = t^{(\nu - 1)(1 - \varrho)}y(t), \ t \in J' \). We define a map \( \tilde{\mathcal{F}} \) as
\[ (\tilde{\mathcal{F}}y)(t) = \begin{cases} t^{(1 - \nu)(1 - \varrho)}(Fz)(t), & t \in (0, b], \\ \frac{z_0 + h(z)}{\Gamma(\nu(1 - \theta) + \varrho)}, & t = 0. \end{cases} \quad (9) \]
It is obvious that \( y \) is any fixed point of map \( \tilde{\mathcal{F}} \) if and only if \( z \) is a fixed point of \( F \), which is further equivalent to say that \( z \) is the mild solution of (1). If \( z \) is the mild solution of (1) with the control (6), then \( z(b) = z_b \).

Let \( y \in E_k \) and \( z(t) = t^{(\nu - 1)(1 - \varrho)}y(t), \ t \in J' \). Therefore \( z \in E^3_k \). We now obtain from (6) and Lemma 2.6 that
\[ \| Bw_z(t) \| \leq \| B \| C_T \left[ \| z_b \| + \frac{M}{\Gamma(\nu(1 - \theta) + \varrho)} t^{(\nu - 1)(1 - \varrho)} \right] \| z_0 + h(z) \| \]
\[ \left. + \frac{M}{\Gamma(\varrho)} \int_0^b \| z_b \| \left[ l_1 + l_2m^* \right] \left\{ \int_0^\varsigma g(\varsigma, r, z(r)) dr \right\} d\varsigma \right] \]
\[ \leq \| B \| C_T \left[ \| z_b \| + \frac{M}{\Gamma(\nu(1 - \theta) + \varrho)} t^{(\nu - 1)(1 - \varrho)} \| z_0 \| + l_h \| \| z \|_3 + \| h(0) \| \right] \]
\[ + \frac{M}{\Gamma(\varrho)} \int_0^b \| z_b \| \left[ l_{k, 1}(\varsigma) + l_{k, 2}(\varsigma) \right] \left\{ \int_0^\varsigma g(\varsigma, r, z(r)) dr \right\} d\varsigma \]
\[ \leq \| B \| C_T \left[ \| z_b \| + \frac{M}{\Gamma(\nu(1 - \theta) + \varrho)} t^{(\nu - 1)(1 - \varrho)} \| z_0 \| + l_h \| \| z \|_3 + \| h(0) \| \right] \]
\[
+ \frac{M}{\Gamma(\varrho)} \int_0^b (b - \varsigma)^{\varrho - 1} \{ l_{k,1}(\varsigma) + m^* l_{k,2}(\varsigma) \| z \|_3 \} \, d\varsigma
\]

\[
\leq \|B\| C_T \left[ \| z_0 \| + \frac{M}{\Gamma(\nu(1 - \varrho) + \varrho)} \int_0^{t_k} (t_k - \varsigma)^{\varrho - 1} \right] \left[ \| z_0 \| + l_k h + \| h(0) \| \right]
\]

\[
+ \frac{M}{\Gamma(\varrho)} \int_0^b (b - \varsigma)^{\varrho - 1} \{ l_{k,1}(\varsigma) + km^* l_{k,2}(\varsigma) \} \, d\varsigma
\]

\[
= M^* \text{ (say).} \quad (10)
\]

First we would like to show that \( \widehat{\mathcal{G}}(E_k) \subseteq E_k \) for some \( k > 0 \). On contrary, we suppose that this is not true for each \( k > 0 \). Thus there exists \( y_k \in E_k \) for each \( k > 0 \) such that \( \| \widehat{\mathcal{G}}(y_k)(t_k) \| \) is greater for some \( t_k \in J \). Let \( z_k(t) = t^{(\nu - 1)(\nu - 1)} y_k(t) \), \( t \in J' \). We obtain from assumptions (A0)-(A5) and Lemma 2.6 that \( k < \| \widehat{\mathcal{G}}(y_k)(t_k) \| \)

\[
\leq \frac{M}{\Gamma(\nu(1 - \varrho) + \varrho)} \| z_0 \| + \frac{M l_{k,1}(\nu(1 - \varrho))}{\Gamma(\varrho)} \int_0^{t_k} (t_k - \varsigma)^{\varrho - 1}
\]

\[
\times \left[ f \left( \varsigma, z(\varsigma), \int_0^\varsigma g(\varsigma, r, z(r)) \, dr \right) + Bw(\varsigma) \right] \, d\varsigma
\]

\[
\leq \frac{M}{\Gamma(\nu(1 - \varrho) + \varrho)} \| z_0 \| + l_k h + \| h(0) \| + \frac{M l_{k,1}(\nu(1 - \varrho))}{\Gamma(\varrho)} \int_0^{t_k} (t_k - \varsigma)^{\varrho - 1}
\]

\[
\times \left[ l_{k,1}(\varsigma) + km^* l_{k,2}(\varsigma) \right] \| Bw(\varsigma) \| \, d\varsigma.
\]

Dividing both sides of (11) by \( k \) and then taking \( k \to \infty \), we obtain

\[
1 < M \left\{ 1 + \frac{M b^\varrho}{\Gamma(\varrho + 1)} \right\} \left\{ \int_{S_{\varrho, \nu}(b)} (h(\varsigma_n) - h(\varsigma)) \, d\varsigma \right\}
\]

This gives us a contradiction. Thus there is a \( k > 0 \) such that \( \widehat{\mathcal{G}}(E_k) \subseteq E_k \).

We shall now show that \( \mathcal{F} \) is continuous on \( E_k \). Let \( \{ y_n \} \subseteq E_k \) with \( y_n \to y \in E_k \) as \( n \to \infty \). Take \( z_n(t) = t^{(\nu - 1)(\nu - 1)} y_n(t) \), \( t \in J' \). Then we obtain from the assumptions (A1)-(A3) and the equation (10) that

(i) \( \| g(t, \varsigma, z_n(\varsigma)) \| - g(t, \varsigma, z(\varsigma)) \| \leq 2m(t, \varsigma) \| \varsigma^{(\nu - 1)(\nu - 1)} z_n(\varsigma) \| + \| \varsigma^{(\nu - 1)(\nu - 1)} z(\varsigma) \| \leq 2km(t, \varsigma), 0 < \varsigma \leq t. \)

(ii) \( \| g(t, \varsigma, z_n(\varsigma)) \| - g(t, \varsigma, z(\varsigma)) \| \to 0 \) as \( n \to \infty \), \( 0 < \varsigma \leq t. \)

(iii) \( \int_{\varsigma_n(t)}^{t_k} g(t, \varsigma, z_n(\varsigma)) \, d\varsigma - f \left( t, z(t), \int_0^t g(t, \varsigma, z(\varsigma)) \, d\varsigma \right) \leq 2l_{k,1}(t) + m^* l_{k,2}(t)b. \)

(iv) \( \int_{\varsigma_n(t)}^{t_k} g(t, \varsigma, z_n(\varsigma)) \, d\varsigma - f \left( t, z(t), \int_0^t g(t, \varsigma, z(\varsigma)) \, d\varsigma \right) \to 0 \) as \( n \to \infty \).

These together with Lebesgue dominated convergence theorem, we get

\[
\| Bw_{z_n}(t) - Bw_z(t) \| \leq \| B \| C_T \left( \| S_{\varrho, \nu}(b) (h(\varsigma_n) - h(\varsigma)) \| \right)
\]
This shows that

\[ \begin{align*}
&\|f(z_n) - z\|_3 \\
&\leq \|B\|C_T \left[ \frac{Ml_h}{\Gamma(\nu(1-\theta) + \theta)} h^{(\nu-1)(1-\theta)} \right] \|z_n - z\|_3 \\
&\quad + \frac{M}{\Gamma(\theta)} \int_0^b (b - \xi)^{\theta-1} \left[ f\left(\xi, z_n(\xi), \int_0^\xi g(\xi, r, z(r)) dr\right) - f\left(\xi, z(\xi), \int_0^\xi g(\xi, r, z(r)) dr\right) \right] d\xi \\
&\quad \to 0 \text{ as } n \to \infty.
\end{align*} \]

Therefore, we obtain for \( t \in J' \) that

\[ \|\mathfrak{g}_n(t) - \mathfrak{g}(t)\| \leq \frac{Ml_h}{\Gamma(\nu(1-\theta) + \theta)} + \frac{M}{\Gamma(\theta)} \int_0^t (t - \xi)^{\theta-1} \]

\[ \times \left[ \left\| f\left(\xi, z_n(\xi), \int_0^\xi g(\xi, r, z(r)) dr\right) - f\left(\xi, z(\xi), \int_0^\xi g(\xi, r, z(r)) dr\right) \right\| \\
\quad + \left\| Bw_{z_n}(\xi) - Bw_{z}(\xi) \right\| \right] d\xi \]

\[ \to 0 \text{ as } n \to \infty \text{ independent of } t. \]

This shows that \( F \) is continuous on \( E_k \).

Next, we shall prove that \( F(E_k) \) is equicontinuous on \( J \). Let \( y \in E_k \) and \( z(t) = t^{(\nu-1)(1-\theta)} y(t), t \in J' \). For any \( t_1, t_2 \in J' \) with \( t_1 < t_2 \) and \( z \in E_k \), we have

\[ \|\mathfrak{g}(t_2) - \mathfrak{g}(t_1)\| \leq \left\| t_2^{(\nu-1)(1-\theta)} S_{\theta, \nu}(t_2)[z_0 + h(z)] - t_1^{(\nu-1)(1-\theta)} S_{\theta, \nu}(t_1)[z_0 + h(z)] \right\| \\
\quad + \left\| \int_0^{t_1} \left[ t_2^{(\nu-1)(1-\theta)} K_\theta(t_2 - \eta) - t_1^{(\nu-1)(1-\theta)} K_\theta(t_1 - \eta) \right] \right\| \\
\quad \times \left[ f\left(\eta, z(\eta), \int_0^\eta g(\eta, r, z(r)) dr\right) + Bw_{z}(\eta) \right] d\eta \\
\quad + \frac{M}{\Gamma(\theta)} \int_{t_1}^{t_2} (t_2 - \xi)^{\theta-1} \left[ I_{k,1}(\xi) + km^*l_{k,2}(\xi) + M^* \right] d\xi \]

where

\[ I_1 = \left\| t_2^{(\nu-1)(1-\theta)} S_{\theta, \nu}(t_2)[z_0 + h(z)] - t_1^{(\nu-1)(1-\theta)} S_{\theta, \nu}(t_1)[z_0 + h(z)] \right\|, \]

\[ I_2 = \left\| \int_0^{t_1} \left[ t_2^{(\nu-1)(1-\theta)} K_\theta(t_2 - \eta) - t_1^{(\nu-1)(1-\theta)} K_\theta(t_1 - \eta) \right] \right\| \\
\quad \times \left[ f\left(\eta, z(\eta), \int_0^\eta g(\eta, r, z(r)) dr\right) + Bw_{z}(\eta) \right] d\eta \]

\[ I_3 = \frac{M}{\Gamma(\theta)} \int_{t_1}^{t_2} (t_2 - \xi)^{\theta-1} \left[ I_{k,1}(\xi) + km^*l_{k,2}(\xi) + M^* \right] d\xi. \]
From expression of $I_3$, we can easily see that $I_3$ tends to 0 as $t_2 \to t_1$ independent of $z$.

$$I_1 = \frac{1}{\Gamma(\nu(1-\theta))} \left\| t_2^{(1-\nu)(1-\theta)} \int_0^{t_2} (t_2 - \varsigma) \nu(1-\theta) - 1 \varsigma^{\nu-1} Q_\nu(\varsigma)[z_0 + h(\varsigma)] d\varsigma \right\|$$

$$- t_1^{(1-\nu)(1-\theta)} \int_0^{t_1} (t_1 - \varsigma) \nu(1-\theta) - 1 \varsigma^{\nu-1} Q_\nu(\varsigma)[z_0 + h(\varsigma)] d\varsigma \right\|$$

$$\leq \frac{t_2^{(1-\nu)(1-\theta)}}{\Gamma(\nu(1-\theta))} \int_{t_1}^{t_2} (t_2 - \varsigma) \nu(1-\theta) - 1 \varsigma^{\nu-1} ||Q_\nu(\varsigma)[z_0 + h(\varsigma)]|| d\varsigma$$

$$+ \int_0^{t_1} \left[ t_2^{(1-\nu)(1-\theta)} (t_2 - \varsigma) \nu(1-\theta) - 1 - t_1^{(1-\nu)(1-\theta)} (t_1 - \varsigma) \nu(1-\theta) - 1 \right]$$

$$\times \varsigma^{\nu-1} ||Q_\nu(\varsigma)[z_0 + h(\varsigma)]|| d\varsigma$$

$$\leq \frac{M t_1^{\nu-1} t_2^{(1-\nu)(1-\theta)}}{\Gamma(\nu(1-\theta))} \left[ \left\| z_0 \right\| + \left\| h(0) \right\| + \int_0^{t_1} \left[ t_2^{(1-\nu)(1-\theta)} (t_2 - \varsigma) \nu(1-\theta) - 1 \right]$$

$$\times \varsigma^{\nu-1} ||Q_\nu(\varsigma)[z_0 + h(\varsigma)]|| d\varsigma$$

$$\to 0 \text{ as } t_2 \to t_1 \text{ independent of } z.$$
\[
\left[ f \left( \varsigma, z(\varsigma), \int_0^\varsigma g(\varsigma, r, z(r)) \, dr \right) + Bw_z(\varsigma) \right] \, d\varsigma.
\]

If \( \epsilon \in (0, t_1) \), then we have
\[
I_{2,2} \leq t_1^{(1-\nu)(1-\rho)} \int_0^{t_1-\epsilon} (t_1 - \varsigma)^{\rho-1} \left\| Q_{\rho}(t_2 - \varsigma) - Q_{\rho}(t_1 - \varsigma) \right\| \left[ l_{k,1}(\varsigma) + km^* l_{k,2}(\varsigma) \right] \, d\varsigma + M^* \left\| Q_{\rho}(t_2 - \varsigma) - Q_{\rho}(t_1 - \varsigma) \right\| \left[ l_{k,1}(\varsigma) + km^* l_{k,2}(\varsigma) + M^* \right] \, d\varsigma.
\]

Since \((t - \varsigma)^{\rho-1} l_{k,1}(\varsigma)\) and \((t - \varsigma)^{\rho-1} l_{k,2}(\varsigma)\) are integrable over the interval \([0, t]\), and the continuity of \(Q_{\rho}(t)\) in uniform operator topology for \(t > 0\), it is obvious that \(I_{2,1}\) and \(I_{2,2}\) tend to 0 as \(t_2 \to t_1\) and \(\epsilon \to 0\) independent of \(y\). Therefore \(\left\| \mathcal{F}y(t_2) - \mathcal{F}y(t_1) \right\| \to 0\) as \(t_2 \to t_1\) independent of \(y\). That is, \(\mathcal{F}(E_k)\) is family of equicontinuous functions on \(J'\). Since \(t^{(1-\nu)(1-\rho)} S_{\rho,\nu}(t)\) is uniformly continuous on \(J\), the family \(\mathcal{F}(E_k)\) is equicontinuous on \(J\).

Let \(S\) be any subset of \(E_k\). From Lemma 2.9, we get a countable set \(S_1 = \{y_n\} \subset S\) such that
\[
\alpha(\mathcal{F}(S)) \leq 2\alpha(\mathcal{F}(S_1)).
\]

Let \(z_n(t) = t^{(\nu-1)(1-\rho)} y_n(t)\). Then we have
\[
\alpha(\mathcal{F}(S_1)(t)) \leq \alpha \left( \left\{ t^{(1-\nu)(1-\rho)} S_{\rho,\nu}(\varsigma)[z_0 + h(z_n)] \right\} \right) + \alpha \left( \left\{ t^{(1-\nu)(1-\rho)} \int_0^t K_{\rho}(t - \varsigma) f \left( \varsigma, z_n(\varsigma), \int_0^\varsigma g(\varsigma, r, z_n(r)) \, dr \right) d\varsigma \right\} \right) + \alpha \left( \left\{ t^{(1-\nu)(1-\rho)} \int_0^t K_{\rho}(t - \varsigma) Bw_{z_n}(\varsigma) d\varsigma \right\} \right).
\]

From the assumption (A3), we obtain
\[
\| h(z_n) - h(z_m) \| \leq l_h \| z_n - z_m \|_3 = l_h \sup_{t \in J'} \| t^{(1-\nu)(1-\rho)} [z_n(t) - z_m(t)] \| = l_h \| y_n - y_m \|_\infty,
\]
where \( z_n(t) = t^{(\nu-1)(1-\varrho)}y_n(t), z_m(t) = t^{(\nu-1)(1-\varrho)}y_m(t), \) and \( y_n, y_m \in S_1. \) By the measure of noncompactness, we get

\[
\alpha(\{h(z_n)\}) \leq h\alpha(\{y_n\}).
\]

Therefore, from Lemma 2.6, we have

\[
\Psi_1 \leq \frac{M}{\Gamma[\nu(1-\varrho) + \varrho]}[\alpha(\{z_0\}) + \alpha(\{h(z_n)\})] \leq \frac{Ml_h}{\Gamma[\nu(1-\varrho) + \varrho]}\alpha(\{y_n\}). \tag{13}
\]

By assumption (A4) and Lemma 2.11, we have

\[
\Psi_2 \leq \frac{2Mb^{(\nu-1)(1-\varrho)}}{\Gamma(\varrho)} \int_0^t (t - \zeta)^{\varrho-1} \alpha \left( \left\{ f \left( \zeta, z_n(\zeta), \int_0^\zeta g(\zeta, r, z_n(r)) \, dr \right) \right\} \right) \, d\zeta
\]

\[
\leq \frac{2Mb^{(\nu-1)(1-\varrho)}}{\Gamma(\varrho)} \int_0^t (t - \zeta)^{\varrho-1} \eta(\zeta) \left[ \alpha \left( \left\{ \zeta^{(1-\nu)(1-\varrho)}z_n(\zeta) \right\} \right) 
\right.
\]

\[
+ \alpha \left( \left\{ \int_0^\zeta g(\zeta, r, z_n(r)) \, dr \right\} \right) \, d\zeta
\]

\[
\leq \frac{2Mb^{(\nu-1)(1-\varrho)}[1 + 2\rho^*]}{\Gamma(\varrho)} \eta^* \sup_{0 \leq \zeta \leq b} \alpha(\{y_n(\zeta)\}). \tag{14}
\]

\[
\Psi_3 \leq \frac{2M \|B\|b^{(\nu-1)(1-\varrho)}}{\Gamma(\varrho)} \int_0^t (t - \zeta)^{\varrho-1} \alpha \left( \left\{ w_{z_n}(\zeta) \right\} \right) \, d\zeta
\]

\[
\leq \frac{2M \|B\|b^{(\nu-1)(1-\varrho)}}{\Gamma(\varrho)} \int_0^t (t - \zeta)^{\varrho-1} \Phi(\zeta) \left[ \frac{Mb^{(\nu-1)(1-\varrho)}}{\Gamma(\nu(1-\varrho) + \varrho)} \alpha(\{z_0 + h(z_n)\}) 
\right.
\]

\[
+ \alpha \left( \left\{ \int_0^b K(\zeta) f \left( \zeta, z_n(\zeta), \int_0^\zeta g(\zeta, r, z_n(r)) \, dr \right) \, d\zeta \right\} \right) \right) \]

\[
\leq \frac{2M \|B\|b^{(\nu-1)(1-\varrho)}}{\Gamma(\varrho)} \Phi^* \left[ \frac{Mb^{(\nu-1)(1-\varrho)}}{\Gamma(\nu(1-\varrho) + \varrho)} l_h + \frac{2M[1 + 2\rho^*]}{\Gamma(\varrho)} \eta^* \right] 
\]

\[
\times \sup_{0 \leq \zeta \leq b} \alpha(\{y_n(\zeta)\}). \tag{15}
\]

Thus we can evaluate from (12), (13), (14) and (15), and from Lemma 2.10 that

\[
\alpha(\mathcal{F}(S_1)(t)) \leq N\alpha(\{y_n\}).
\]

Since \( \mathcal{F}(S_1) \subset \mathcal{F}(E_k) \) is equicontinuous, it follows from Lemma 2.10 that \( \alpha(\mathcal{F}(S_1)) = \sup_{t \in J} \alpha(\mathcal{F}(S_1)(t)). \) Thus from the equicontinuity of \( \mathcal{F}(E_k) \) and the hypothesis (A6), we have

\[
\alpha(\mathcal{F}(S)) \leq 2\alpha(\mathcal{F}(S_1)) \leq 2N\alpha(\{S\}).
\]

Since \( N < \frac{1}{2}, \) the map is a condensing map from \( E_k \) to \( E_k. \) Therefore from Sadovskii’s fixed point theorem the map \( \mathcal{F} \) has a fixed point \( y \) in \( E_k. \) Hence the system (1) has a mild solution \( z(t) = t^{(\nu-1)(1-\varrho)}y(t) \) satisfying that \( z(b) = z_0. \) That is, the system (1) is exactly controllable on \([0, b]).\) This completes the proof. \( \square \)
4. Example. Let \( Z = L^2([0, \pi], \mathbb{R}) \) with norm \( \| \cdot \|_2 \). Consider the following fractional partial differential equations with nonlocal conditions.

\[
\begin{aligned}
D_{0+}^{\alpha, \nu} \left[ z(t, \zeta) - \frac{\partial^2}{\partial t^2} z(t, \zeta) \right] &= \frac{\partial^2}{\partial t^2} z(t, \zeta) \\
\hat{f}(t, z(t, \zeta), \int_0^t g(t, r, z(r, \zeta)) \, dr) &= \hat{B} w(t, \zeta), \quad (t, \zeta) \in [0, b] \times [0, \pi], \\
\hat{f}(t, 0) &= \hat{f}(t, \pi) = 0, \quad t \in [0, b], \\
\hat{f}(t, \zeta) &= \hat{z}_0(\zeta) + \hat{h}(z, \zeta), \zeta \in [0, \pi],
\end{aligned}
\]

(16)

where \( D_{0+}^{\alpha, \nu} \) is a Hilfer fractional partial derivative of order \( \rho \) and type \( \nu \), \( 0 < \rho < 1 \), \( 0 \leq \nu \leq 1 \); \( \hat{z}_0 \in Z \). The functions \( \hat{f} \), \( \hat{h} \) and \( g \) are described below.

We define operators \( A : Z \rightarrow Z \) and \( R : Z \rightarrow Z \) respectively by \( Av = v' \) and \( Rv = v - v'' \) with domain \( D(A) = D(R) = \{ v \in Z : v' \text{ is absolutely continuous and } v'' \in Z \} \).

Obviously, the operator \( A \) and \( R \) are given by

\[
Av = -\sum_{m=1}^{\infty} m^2 (v, e_m) e_m, \quad v \in D(A),
\]

and

\[
Rv = \sum_{m=1}^{\infty} (m^2 + 1) (v, e_m) e_m, \quad v \in D(R),
\]

respectively, where \( e_m(\zeta) = \sqrt{\frac{2}{\pi}} \sin(m\zeta), m \in \mathbb{N} \). Clearly the set \( \{ e_m : m \in \mathbb{N} \} \) consists of eigenfunction of the operator \( A \) and forms an orthonormal basis for \( Z \).

One can easily obtain that

\[
R^{-1} v = \sum_{m=1}^{\infty} (m^2 + 1)^{-1} (v, e_m) e_m, \quad v \in Z.
\]

We can see from paper [25] that the pair of operators \( A \) and \( R \) generates the propagation family \( P(t) \) of bounded linear operators is given by

\[
P(t) v = \sum_{m=1}^{\infty} \exp \left( -\frac{m^2}{m^2 + 1} t \right) (v, e_m) e_m, \quad v \in Z.
\]

Obviously, \( AR^{-1} \) generates a \( C_0 \)-semigroup \( \{ P(t) \}_{t \geq 0} \) of bounded linear operator with self-adjoint in Hilbert space \( Z \) and \( \| P(t) \| \leq 1 \).

Let \( \hat{f}(t, z(t, \zeta), \int_0^t g(t, r, z(r, \zeta)) \, dr) = Rf \left( t, z(t, \zeta), \int_0^t g(t, r, z(r, \zeta)) \, dr \right), \hat{h}(z, \zeta) = Rh(z, \zeta), \) and \( \hat{B} w(t, \zeta) = RBw(t, \zeta) \). We now take \( g(t, \zeta, z(\zeta)) = g(t, \zeta, z(\zeta)) \zeta = \frac{2}{\pi} (t - \zeta)^{-\frac{1}{2}} \zeta^1 - \nu - \nu^{\nu} e^{\nu} \zeta(\zeta), \) \( f(t, z(t, \zeta), \phi(\zeta)) = f(t, z(t, \zeta), \phi(\zeta)) = f(t, z(t, \zeta)) = L[\sin(z(t, \zeta)) + \phi(\zeta)], h(z, \zeta) = h(z)(\zeta) = \frac{1}{2} \int_0^b \frac{p}{1 + (t, \zeta)} d\zeta, \) here \( L, \) and \( p \) are constants.

The bounded linear control operator \( B : D(R) \rightarrow Z \) is defined by \( Bz = z, z \in D(R) \) and \( w(t)(\zeta) = w(t, \zeta) \). We are now able to rewrite the system (16) as in the form of (1).

It is easy to verify that functions \( f, g \) and \( h \) satisfy the conditions (A1) to (A4) with the fact that \( l_1 = l_2 = |L|, l_1 = 0, m(t, \zeta) = \rho(t, \zeta) = |\sigma| (t - \zeta)^{\frac{1}{2}} \zeta, l_2 = \eta^* = |L|, m^* = \rho^* = |\sigma|, h = |p|, B = 1 \).

The linear operator \( T : L^2(J, D(R)) \rightarrow Z \) is given by

\[
(Tw)(\zeta) = \int_0^b K_{\psi}(b - \zeta)w(\zeta) d\zeta.
\]
Since \( \{P(t)\}_{t \geq 0} \) is self-adjoint in Hilbert space \( Z \), \( K_\varrho(t) \) is also self-adjoint in Hilbert space \( Z \). Let \( v \in Z \) be any element and \( v_m = \langle v, e_m \rangle \). Then \( v = \sum_{m=1}^{\infty} v_m e_m \), and we obtain

\[
\|K_\varrho^*(t)v\|_Z^2 = \|K_\varrho(t)v\|_Z^2 = \langle K_\varrho(t)v, K_\varrho(t)v \rangle
= (te^{-1}Q_\varrho(t)v, te^{-1}Q_\varrho(t)v)
= t^2e^{-2} \varrho^2 \int_0^\infty \int_0^\infty \vartheta \psi_\varrho(\vartheta) \psi_\varrho(\theta) \exp(-b^2(\vartheta + \theta))d\vartheta d\theta
\geq t^2e^{-2} \varrho^2 \int_0^\infty \int_0^\infty \vartheta \psi_\varrho(\vartheta) \psi_\varrho(\theta) \exp(-b^2(\vartheta + \theta))d\vartheta d\theta \left(\sum_{m=1}^{\infty} |v_m|^2\right)
= \lambda_2^2 t^2e^{-2} \|v\|_Z^2,
\]

where \( \lambda_2^2 = \varrho^2 \int_0^\infty \int_0^\infty \vartheta \psi_\varrho(\vartheta) \psi_\varrho(\theta) \exp(-b^2(\vartheta + \theta))d\vartheta d\theta < \infty \). This implies

\[
\int_0^b \|K_\varrho^*(s)v\|_Z^2 ds \geq \lambda_2^2 \|v\|_Z^2, \quad \forall v \in Z,
\]

where \( \lambda_2^2 = \lambda_2^2 b^2 e^{-1} \). Thus we conclude from [8, Theorem 4.1.7] that controllability map \( T \) has induced inverse \( \widetilde{T}^{-1} \) in \( L^2(J, D(R))/\ker T \) and \( \|\widetilde{T}^{-1}\| \leq \frac{1}{\lambda_2} \). Thus \( T \) satisfies the assumption (A5). Since the function \( h(z, \zeta) \) is continuous and compact, we can take \( \ell_h = 0 \) in (A6). If we choose the constants \( |\sigma|, |L|, |p| \) sufficiently small, then the assumption (A6) and the condition (7) are verified. Hence the system (16) satisfies all the assumptions of the Theorem 3.1. That is, the system (16) is exactly controllable on the interval \( J \).

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