THE CONDITIONAL CONVERGENCE OF THE DIRICHLET SERIES OF AN L-FUNCTION

M. O. RUBINSTEIN

PURE MATHEMATICS
UNIVERSITY OF WATERLOO
200 UNIVERSITY AVE W
WATERLOO, ON, CANADA
N2L 3G1

ABSTRACT. The Dirichlet divisor problem is used as a model to give a conjecture concerning the conditional convergence of the Dirichlet series of an $L$-function.

1. Introduction

Let

$$L(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s},$$

be a Dirichlet series and let $\Re s = \sigma$. A classical summation by parts gives

$$\sum_{n \leq X} \frac{b(n)}{n^s} = \frac{1}{X^s} \sum_{n \leq X} b(n) + s \int_{1}^{X} \sum_{n \leq x} b(n) \frac{dx}{x^{s+1}}.$$  \hfill (1.2)

Say that

$$\sum_{n \leq X} b(n) = O(X^{\sigma_0})$$

for some $\sigma_0 \in \mathbb{R}$. Then, for $\sigma > \sigma_0$, letting $X \to \infty$, (1.2) converges and becomes

$$\sum_{n=1}^{\infty} \frac{b(n)}{n^s} = s \int_{1}^{\infty} \sum_{n \leq x} b(n) \frac{dx}{x^{s+1}}.$$  \hfill (1.4)

Subtracting (1.2) from (1.4) and using (1.3) gives a rate of convergence:

$$\sum_{n > X} \frac{b(n)}{n^s} = -\frac{1}{X^s} \sum_{n \leq X} b(n) + s \int_{X}^{\infty} \sum_{n \leq x} b(n) \frac{dx}{x^{s+1}}$$

$$= O_s(X^{\sigma_0 - \sigma}).$$  \hfill (1.5)

It is therefore natural to ask, for a given $L(s)$, how small can we take $\sigma_0$, i.e. in what half-plane does the Dirichlet series for $L(s)$ converge.

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In the case of the Riemann zeta function or Dirichlet $L$-functions, the answer is immediate. For the Riemann zeta function, $b(n) = 1$, and so $\sigma_0 = 1$ is the best possible.

However, a well known trick allows one to take $\sigma_0 = 0$ and evaluate $\zeta(s)$ in the half plane $\sigma > 0$ by writing
\[
\zeta(s) \left(1 - \frac{2}{2^s}\right) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \ldots
\] (1.6)
so that $b(n) = (-1)^{n-1}$ and $\sum_{n \leq X} b(n) = O(1)$.

For a non-trivial Dirichlet character, $\chi(n)$, of modulus $q$, we can take $\sigma_0 = 0$ for the Dirichlet series of $L(s, \chi)$ because $\sum_{n \leq X} \chi(n) = O(q(1))$.

For higher degree $L$-functions, however, the problem of obtaining a bound for the truncated sum of the Dirichlet coefficients is very difficult, and the best known bounds seem to be far from the truth. For example, let
\[
f(z) = \sum_{1}^{\infty} a(n)q^n, \quad q = \exp(2\pi iz)
\] (1.7)
be a cusp form of weight $l$ and level $N$, and
\[
L(f)(s) = \sum_{1}^{\infty} \frac{a(n)}{n^{(l-1)/2}} \frac{1}{n^s}
\] (1.8)
be its corresponding $L$-function. Each $a(n)$ is normalized by $n^{(l-1)/2}$ so that the critical line is $\Re s = 1/2$. Hecke [H] proved that
\[
\sum_{n \leq X} \frac{a(n)}{n^{(l-1)/2}} = O_f(X^{1/2})
\] (1.9)
giving $\sigma_0 = 1/2$. This was improved to $\sigma_0 = 11/24 + \epsilon$ by Walfisz [W], and when combined with the Ramanujan conjecture proved by Deligne [D] one can get $\sigma_0 = 1/3 + \epsilon$.

However, this seems to be far from the truth. To see what might be a reasonable value for $\sigma_0$ we consider $\zeta(s)^k$, $k$ a positive integer, which is, in some sense, the simplest degree $k$ $L$-function. However, this is not a typical $L$-function in that its Dirichlet coefficients are all positive and no cancellation occurs when they are summed. This differs from the behaviour of entire $L$-function where one expects $\sum_{n \leq X} b(n)$ to cancel. Once one removes the contribution from the order $k$ pole of $\zeta(s)^k$, we conjecture that the $k$-divisor problem provides a good model for entire $L$-functions of degree $k$.

Let $d_k(n)$ be the Dirichlet coefficients of
\[
\zeta(s)^k = \sum_{1}^{\infty} \frac{d_k(n)}{n^s},
\] (1.10)
and define
\[
D_k(X) = \sum_{n \leq X} d_k(n).
\] (1.11)

The Dirichlet coefficient $d_k(n)$ is equal to the number of ways of writing $n$ as a product of $k$ factors.
Assume now that, in (1.2), \( b(n) = O(n^\epsilon) \). Perron’s formula \([M][pg 67]\) states, that:
\[
\sum_{n<X} b(n) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} L(s) \frac{X^s}{s} ds + O\left( \frac{X^{c+\epsilon}}{T} \right),
\] (1.12)
where \( c > 1 \). In the case of \( L(s) = \zeta(s)^k \), one proceeds by shifting the line integral to the left and estimating the integral along the four sides of the resulting rectangle. This gives
\[
D_k(X) = X P_k(\log X) + \Delta_k(X)
\] (1.13)
with \( P_k \) being a polynomial of degree \( k-1 \) coming from the residue of the \( k \)-th order pole at \( s = 1 \), and \( \Delta_k(X) \) denoting the remainder term.

The \( k \) divisor problem states that the true order of magnitude for \( \Delta_k(X) \) is:
\[
\Delta_k(X) = O\left( X^{(k-1)/2k+\epsilon} \right).
\] (1.14)
When \( k = 2 \), the traditional Dirichlet divisor problem is
\[
D_2(X) = X \log X + (2\gamma - 1)X + \Delta_2(X),
\] (1.15)
with a conjectured remainder
\[
\Delta_2(X) = O\left( X^{1/4+\epsilon} \right).
\] (1.16)

The estimate (1.14) for the remainder term \( \Delta_k(X) \) is based on expected cancelation in Voronoi-type formulas for \( \Delta_k(X) \) (such as (12.4.4) and (12.4.6) described in \([T]\)), and also on estimates for the mean square of \( \Delta_k \). For example, it is known for \( k = 2, 3 \) \([C]\) \([To]\) and conjectured for \( k \geq 4 \), that
\[
\frac{1}{X} \int_0^X \Delta_k^2(y) dy \sim c_k X^{(k-1)/k}
\] (1.17)
where \( c_k > 0 \) is constant. For \( k = 4 \), Heath-Brown obtained a slightly weaker upper bound, \( O(X^{3/4+\epsilon}) \) rather than the asymptotic \([HB]\). This asymptotic is known to be equivalent to the Lindelöf hypothesis. For a discussion on the Dirichlet divisor problem, see Titchmarsh \([T]\)[Chapters XII,XIII].

By analogy with the \( k \) divisor problem, it seems reasonable to conjecture:

**Conjecture 1.1.** Let \( L(s) = \sum b(n)/n^s \) be an entire \( L \)-function of degree \( k \), normalized so the critical line is through \( \Re s = 1/2 \). Then
\[
\sum_{n \leq X} b(n) = O\left( X^{(k-1)/2k+\epsilon} \right).
\] (1.18)

More generally, let \( L(s) \) be meromorphic with its only pole being at \( s = 1 \) of order \( r \). Then
\[
\sum_{n \leq X} b(n) = XP_L(\log X) + O\left( X^{(k-1)/2k+\epsilon} \right)
\] (1.19)
where \( P_L \) is a polynomial of degree \( r - 1 \).

This conjecture has been stated in various specific cases \([IMT][F]\), but the author could not find a reference that mentions this conjecture for a general \( L \)-function.

Notice that \( (k-1)/2k < 1/2 \), hence, if this conjecture is true, then the Dirichlet series of an entire \( L \)-function can be used to evaluate it at any \( s \in \mathbb{C} \), by summing the terms for \( \sigma \geq 1/2 \), and by applying the functional equation for \( \sigma < 1/2 \).
To mention just two examples, the Dirichlet series of a cusp form, $L_f(s)$ in (1.8), is expected to converge for $\sigma > 1/4$, and the Dirichlet series of its symmetric square $L$-function, which has degree 3, should converge for $\sigma > 1/3$.

The rate of convergence, however, makes this not very practical for smaller values of $\sigma$. For instance, when $k = 3$, summing $10^6$ terms of the Dirichlet series gives, using (1.5), less than four decimal places accuracy at $s = 1$, while at $s = 1 + 100i$ one expects about two digits accuracy. To get 16 digits at $s = 1$ would be impossible from a practical point of view, requiring roughly $10^{24}$ terms of the Dirichlet series.

Nonetheless, it is interesting to know, in principle and also as a way to double check more sophisticated algorithms for computing $L$-functions, that the Dirichlet series of an entire $L$-function does converge up to and slightly beyond its critical line.

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