Research Article

Type II Half Logistic Kumaraswamy Distribution with Applications

Ramadan A. ZeinEldin,1,2 Muhammad Ahsan ul Haq,3,4 Sharqa Hashmi,4,5 Mahmoud Elsehety6,7 and M. Elgarhy7

1Deanship of Scientific Research, King Abdulaziz University, Saudi Arabia
2Faculty of Graduate Studies for Statistical Research, Cairo University, Egypt
3College of Statistical and Actuarial Sciences, University of the Punjab, Lahore, Pakistan
4Quality Enhancement Cell, National College of Arts, Lahore, Pakistan
5Lahore College for Women University (LCWU), Lahore, Pakistan
6King Abdulaziz University, Saudi Arabia
7Valley High Institute for Management Finance and Information Systems, Obour, Qaliubia 11828, Egypt

Correspondence should be addressed to Muhammad Ahsan ul Haq; ahsanshani36@gmail.com

Received 27 March 2020; Accepted 5 June 2020; Published 1 August 2020

Academic Editor: Hugo Leiva

Copyright © 2020 Ramadan A. ZeinEldin et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this paper, a new distribution with a unit interval named type II half logistic Kumaraswamy (TIIHLKw) distribution is proposed. Its density and distribution functions are presented using alternate expressions. This distribution is obtained by adding an extra parameter in the existing model to rise its ability fitting complex data sets. Some important statistical properties of TIIHLKw distribution are derived. The estimation of the parameters is obtained by numerous well-recognized approaches and simulation study confirmed the efficiencies of estimates such obtained. We apply the related model to practical datasets, and it is concluded that the proposed model is the best by model selection criteria than other competitive models.

1. Introduction

Over the last few years, inspired by the increasing demand of probability distributions in many fields, many generalized distributions have been studied. Most of them are proposed by the addition of more parameters to well-known probability distributions that exist in literature to make them flexible. For instance, Haq et al. [1] proposed and studied the generalized odd Burr III (GOBIII) family of distributions along with its important characterizations. Ahmed [2] proposed a new model derived by the transformation of the baseline model. Different shapes of failure function can be formed such as increasing and bathtub. Some mathematical properties are obtained. So many research works have been done for proposing more flexible generalized probability distributions such as Alzaatreh et al. [3], Haq et al. [4], Hashmi et al. [5], Elgarhy et al. [6], and ZeinEldin et al. [7]. Alshenawy [8] suggested a new one-parameter distribution. Several statistical properties and characteristics of the proposed distribution are derived along with estimation under Type II censoring. The research is concluded on the basis of a simulation study and real data analysis.

The Kumaraswamy (Kw) distribution was introduced and studied by Kumaraswamy [9] with unit interval, denoted by $Kw(a, b)$, with cumulative distribution function (cdf) is

$$G(x) = 1 - (1 - x^a)^b, \quad (1)$$

and its related probability density function (pdf) is

$$g(x) = abx^{a-1}(1 - x^a)^{b-1}, \quad x, a, b > 0. \quad (2)$$
The \( Kw \) density has one of these shapes depending upon parameter values, unimodal \((a, b > 1)\), bathtub \((a, b < 1)\), increasing \((a > 1 \text{ and } b \leq 1)\), decreasing \((D) (a \leq 1 \text{ and } b > 1)\), or steady \((\text{both } a \text{ and } b \text{ equal to } 1)\).

The behavior of \( Kw \) distribution is analogous to the beta distribution but simpler due to closed-form of both its pdf and cdf. Boundary behavior and the major special models are also the same in both Beta and \( Kw \) distributions. This distribution could be a good substitute in situations wherein reality the bounds are finite i.e., \((0, 1)\).

The \( Kw \) is originally developed as a lifetime distribution. Many authors studied and developed the generalizations of \( Kw \) distribution such as exponentiated \( Kw \) distribution studied by Lemonte et al. [10], El-Sherpieny, and Ahmed [11] proposed \( Kw \) distribution, transmuted \( Kw \) distribution studied by Khan et al. [12], Sharma and Chakraborty [13] studied size biased \( Kw \) distribution, George and Thobias [14] introduced Marshall-Olkin \( Kw \) distribution, exponentiated generalized \( Kw \) distribution studied by Elgarhy et al. [15], type II Topp Leone inverted \( Kw \) by ZeinEldin et al. [16], type I half logistic inverted \( Kw \) by ZeinEldin et al. [17], truncated inverted \( Kw \) by Bantan et al. [18], and Ghosh [19] introduced bivariate and multivariate weighted \( Kw \) distributions.

Hassan et al. [19] proposed Type II half logistic-G (TIIHL-G). The TIIHL-G distribution is expressed by its cdf given by

\[
F(x; \lambda, \zeta) = 1 - \frac{2\lambda e^{-\lambda t}}{(1 + e^{-\lambda t})^2} dt + \frac{2[G(x; \zeta)]^a}{1 + [G(x; \zeta)]^b}, \quad x > 0, \lambda > 0,
\]

where \( G(x; \zeta) \) is cdf of baseline model with parameter vector \( \zeta \) and \( F(x; \lambda, \zeta) \) is cdf derived by the T-X generator proposed by Alzaatreh [3]. The pdf of the TIIHL-G family is given as

\[
f(x; \lambda, \zeta) = \frac{2\lambda g(x; \zeta)[G(x; \zeta)]^{a-1}}{[1 + [G(x; \zeta)]^b]} x > 0, \lambda > 0. \tag{4}
\]

This article is dedicated to both its mathematical and application features. A significant portion is kept for the estimation of the parameters through various methods including maximum likelihood estimation (MLE), least-square estimation (LSE), weighted least square estimation (WLSE), percentiles estimation (PCE), and Cramer-von Mises estimation (CVE).

The core purpose of this research is to suggest a simpler and more flexible model called Type II Half Logistic \( Kw \) (TIIHL\( Kw \)) distribution. This article is organized in the following manner: Section 2 is dedicated for the proposition of type II half logistic-\( Kw \) (TIIHL\( Kw \)) distribution. Section 3 deals with the leading statistical properties of this model. Important binomial expansions of density and distribution functions are presented which involve binomial expansions. In Section 4, an extensive study of five different methods of estimation is carried out, with all derivations and detailed discussions. A comprehensive simulation study is conducted to compute the biases and efficiency for parameters and compare the performances of five estimation approaches stated above in the next section. Section 6 is devoted to the real data application of TIIHL\( Kw \) distribution. Lastly, the conclusion is given in Section 7.

2. The TIIHL\( Kw \) Distribution

In this section, we examine the usefulness, and flexibility of a new associate of type II half-logistic-G family having \((0, 1)\), using \( Kw \) distribution as the baseline. The pdf and cdf of \( Kw \) distribution \((2 \text{ shape parameters: } a, b > 0)\) are given as follows

\[
g(x; a, b) = abx^{a-1}(1 - x^b)^{-1}, 0 < x < 1, \tag{5}
\]

\[
G(x; a, b) = 1 - (1 - x^b)^a, 0 < x < 1. \tag{6}
\]

The random variable \((r.v.) X \) follows TIIHL\( Kw \) distribution if its cdf is obtained by inserting \( (6) \) in \((3)\)

\[
F(x; \lambda, a, b) = \frac{2[1 - (1 - x^b)^a]}{1 + [1 - (1 - x^b)^a]^2}, \quad a, b, \lambda > 0, \quad 0 < x < 1. \tag{7}
\]

The pdf of TIIHL\( Kw \) distribution is as follows

\[
f(x; \lambda, a, b) = \frac{2\lambda abx^{a-1}(1 - x^b)^{-1}(1 - (1 - x^b)^a)^{-1}}{\left(1 + (1 - x^b)^a\right)^2}, \quad a, b, \lambda > 0, 0 < x < 1. \tag{8}
\]

The survival function of TIIHL\( Kw \) distribution is

\[
\tilde{F}(x; \lambda, a, b) = \frac{1 - \left[1 - (1 - x^b)^a\right]^\lambda}{1 + \left[1 - (1 - x^b)^a\right]^\lambda} = \frac{2}{1 + (1 - x^b)^a} - 1. \tag{9}
\]

The failure rate function of TIIHL\( Kw \)D is

\[
h(x; a, b) = \frac{2\lambda abx^{a-1}(1 - x^b)^{-1}(1 - (1 - x^b)^a)^{-1}}{1 - (1 - x^b)^a} \cdot \frac{1}{2\lambda}. \tag{10}
\]

The flexibility of TIIHL\( Kw \) distribution can be illustrated in Figures 1, 2, and 3. The pdf plots for the TIIHL\( Kw \)
distribution are given in Figures 1 and 2, and plots of hrf are given in Figure 3.

3. Some Mathematical Properties

The properties of TIIHLKwD are derived here. After this, we consider a r.v. $X$ follows the pdf (8) and cdf (7).

3.1. Quantile Function.

The quantile function of $X$ is denoted by $Q(u)$, defined as

$$Q(u) = G^{-1} \left[ \frac{u}{2} \right], \quad u \in (0, 1),$$

the inverse function of $G(.)$ is $G^{-1}(.)$, given in (6).

Figure 1: Pdf plot of TIIHLKwD.

Figure 2: Plots of some pdfs of TIIHLKwD.
Also,

\[ x_u = \left( 1 - \left( 1 - \frac{u}{u - 2} \right)^{1/b} \right)^{1/a}. \]  

(12)

Simulated values from TIIHLKwD can be utilized in the simulation study. The r. v. \( u \) follows rectangular distribution on the interval 0 to 1, i.e., \( X_u = Q(u) \) follows TIIHLKw distribution. In particular, the first three quartiles are obtained by putting \( u = 1/4, 1/2, \) and \( 3/4 \), respectively, in (11) and (12).

On differentiation, we can have the density of quantile function

\[
q(u) = \frac{1}{ab\lambda} \left( \frac{u}{2 - u} \right)^{-1+1/\lambda} \left[ \frac{1}{2 - u} + \frac{u}{(u - 2)^2} \right]^{-1+1/b} \left[ 1 - \left( \frac{u}{2 - u} \right)^{1/\lambda} \right]^{-1+1/a}.
\]

(13)

3.2. Alternate Representation. Here, we present the expansion of the pdf and cdf for TIIHLKw distribution for further mathematical manipulation.

The binomial theorem, for \( \beta > 0 \) and \( |z| < 1 \), can be expressed as

\[
(1 - z)^{\beta - 1} = \sum_{i=0}^{\infty} (-1)^i \binom{\beta - 1}{i} z^i.
\]

(14)

Then, by applying (13) in (8), the pdf of TIIHLKwD becomes

\[
f(x; \lambda, a, b) = 2\lambda abx^{a-1} \left( 1 + \left( 1 - x^b \right)^{\lambda} \right)^{-2} \cdot \sum_{i=0}^{\infty} (-1)^i \binom{\lambda - 1}{i} \left( 1 - x^a \right)^{b(i+1)-1},
\]

(15)

for \( a, \lambda, b > 0, 0 < x < 1 \).

Another form of pdf (8) can be obtained by means of the following expansion

\[
(1 + z)^{-\beta} = \sum_{i=0}^{\infty} (-1)^i \binom{\beta + i - 1}{i} z^i
\]

(16)

for \( |z| < 1 \) and \( \beta > 0 \). By applying (16) in pdf (8), we get

\[
f(x) = \sum_{i=0}^{\infty} \eta_i x^{a-1} (1 - x^a)^{b(i+1)-1} \left[ 1 - (1 - x^a)^b \right]^{\lambda(i+1)-1},
\]

(17)

where \( \eta_i = 2ab\lambda(-1)^i(i+1) \). Now considering (13), we have

\[
\left[ 1 - (1 - x^a)^b \right]^{\lambda(i+1)-1} = \sum_{j=0}^{\infty} (-1)^j \binom{\lambda(i+1) - 1}{j} (1 - x^a)^{bj}.
\]

(18)
Inserting this expansion in (17), we have pdf of the following form
\[ f(x) = \sum_{i,j=0}^{\infty} \eta_{i,j} x^{a-1} (1 - x^a)^{b(j+1)-1}, \] (19)
where \( \eta_{i,j} = (-1)^j \left( \lambda (i+1) - 1 \right) \). 

Another formula can be formed from pdf (19), which is given in (20) using an infinite linear combination
\[ f(x) = \sum_{i,j=0}^{\infty} \eta_{i,j} x^{a(k+1)-1}, \] (20)

where \( \eta_{i,j} = (-1)^j \left( \frac{b(j+1) - 1}{k} \right) \eta_{i,j} \).

**Proposition 1.** Let “\( h \)” be a positive integer. The expansion of cdf can be written in following form:
\[ F(x)^h = \sum_{i=0}^{\infty} \sum_{m=0}^{\infty} \eta_{i,m} (1 - x^a)^{b(m)}, \] (22)
where \( \eta_{i,m} = (2)^h (-1)^{i+m} \left( \frac{h+l+1}{l} \right) \lambda (l+i) \).

**Proof.** The cdf \([F(x)]^h\) is obtained, for ’\( h \)’ an integer by using (16).

\[ [F(x)]^h = \frac{2^h \left[ 1 - (1 - x^a)^b \right]^{h k}}{\left[ 1 + (1 - x^a)^b \right]^h}. \]

\[ [F(x)]^h = \sum_{i=0}^{\infty} (2)^h (-1)^i \left( \frac{h+l+1}{l} \right) \left( 1 - (1 - x^a)^b \right)^{\lambda (l+i)} \lambda (i+1). \] (23)

Once more binomial expansion is applied to \([F(x)]^h\) and it can be written as
\[ [F(x)]^h = \sum_{i=0}^{\infty} \sum_{m=0}^{\infty} (2)^h (-1)^{i+m} \left( \frac{h+l+1}{l} \right) \lambda (l+i) \left( 1 - x^a \right)^{b(m)}. \] (24)

The required result is obtained by combing some expression together, completing the proof.

**3.3. Probability Weighted Moments:** The probability-weighted moments (PWMs) are used to study some more characteristics of the probability distribution. Under the specified setting discussed above, PWMs are denoted by \( \tau_{r,s} \), can be defined as
\[ \tau_{r,s} = E[X^r (F(x))^s] = \int_{-\infty}^{\infty} x^r (F(x))^s f(x) dx. \] (25)

The PWMs of TIIHLKw are obtained by substituting (21) and (22) into (25), as follows
\[ \tau_{r,s} = \sum_{i,j,k=0}^{\infty} \eta_{i,j,k} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \eta_{l,m,k} x^{(k+1)l-r-1} (1 - x^a)^{b(m)} dx. \] (26)

Then,
\[ \tau_{r,s} = \sum_{i,j,k=0}^{\infty} \eta_{i,j,k} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \eta_{l,m,k} \left( 1 + bm \right) \Gamma \left( 1 + k + \frac{r}{a} \right). \] (27)

where \( \Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt, s > 0 \). Also, \( \beta(a, b) \) is beta function.

**3.4. Moments.** The moments have a vital role in the study of the distribution and real data applications. Now, we get the \( r \)th moment for the TIIHLKwD. Under the specified assumptions, the \( r \)th moment is obtained as
\[ \mu_r = \int_{-\infty}^{\infty} x^r f(x) dx = \int_{0}^{\infty} \sum_{i,j=0}^{\infty} \eta_{i,j} x^{r+i-1} (1 - x^a)^{b(j+1)-1} dx, \] (28)

\[ \mu_r = \sum_{i,j=0}^{\infty} \eta_{i,j} \beta( r + a, b(j+1) ) ; r = 1, 2, 3, \ldots, \] (29)

where \( \eta_{i,j} = (-1)^j \left( \frac{\lambda (i+1) - 1}{j} \right) \eta_{i,j} \).

The mean \( (\mu_1) \) and variance \( (\text{var}) \) of TIIHLKw distribution can be derived as
\[ \mu_1 = \sum_{i,j=0}^{\infty} \eta_{i,j} \beta( 1 + a, b(j+1) ), \]
\[ \text{var}(X) = \sum_{i,j=0}^{\infty} \eta_{i,j} \beta( 2 + a, b(j+1) ) - \left( \sum_{i,j=0}^{\infty} \eta_{i,j} \beta( 1 + a, b(j+1) ) \right)^2. \] (30)

Also, the coefficients of skewness and kurtosis of TIIHLKw distribution are given by
\[ S_k = \frac{\mu_3}{\mu_2^{3/2}}, \quad K = \frac{\mu_4}{\mu_2^2}. \] (31)
The summary measures, mean, variance, skewness ($S_k$) and kurtosis ($K$) values are presented in Table 1. The plots of mean and var for the TIHLKwD are given in Figure 4 and graphs of skewness and kurtosis are presented in Figure 5 for different parameter ranges.

We see a monotonic variation in these measures caused by variation in parameters $a$, $b$, and $\lambda$.

### 3.5. Moment-Generating Function

The moment-generating function of $X$, using moments about the origin (29), is obtained as

$$M_X(t) = \frac{\Gamma(\eta)}{\Gamma(\eta + t)} \sum_{i,j=0}^{\infty} \frac{\eta_{ij} \beta[(r + a, b(j + 1))]}{\Gamma(r + \eta + t)}.$$  (32)

### 3.6. Incomplete Moments

The $r^{th}$ incomplete moment of TIHLKwD can be obtained by using (19) is

$$E_r(t) = \int_0^\infty x^r f(x) dx = \sum_{i,j=0}^{\infty} \eta_{ij} \beta((r + a, b(j + 1))).$$  (33)

where $\beta(\cdot, \cdot)$ is incomplete beta function.

### 3.7. The Mean Deviation

Following are expressions used to get mean deviation about mean and median, respectively

$$M_1(X) = 2\mu F(\mu) - 2T(\mu)$$ and

$$M_2(X) = \mu - 2T(M_d).$$  (34)

where $M_d$ is Median of $X$ and $T(q) = \int_0^q x f(x) dx$ is the initial incomplete moment. Now

$$T(\mu) = \int_0^\mu x f(x) dx = \sum_{i,j=0}^{\infty} \eta_{ij} \beta(1 + a, b(j + 1)).$$

$$T(M_d) = \int_0^{M_d} x f(x) dx = \sum_{i,j=0}^{\infty} \eta_{ij} \beta(M_d(1 + a, b(j + 1))).$$  (35)

### 3.8. Bonferroni and Lorenz Curves

Bonferroni and Lorenz curves are important applications of the first incomplete moment. These curves are mostly used in different fields of life such as economics, reliability analysis, demographic studies, life testing, life insurance, and medical technology. The Lorenz and Bonferroni curves are obtained, respectively, as follows

$$L_F(x) = \frac{1}{E(X)} \int_0^x t f(t) dt = \frac{\sum_{i,j=0}^{\infty} \eta_{ij} \beta(1 + a, b(j + 1))}{\sum_{i,j=0}^{\infty} \eta_{ij} \beta(1 + a, b(j + 1))},$$

and,

$$B_F(x) = \frac{L_F(x)}{F(x)} = 1 + \left[1 - (1 - x^a)^b\right]^{1/\lambda} \left(\frac{\sum_{i,j=0}^{\infty} \eta_{ij} \beta(1 + a, b(j + 1))/(\sum_{i,j=0}^{\infty} \eta_{ij} \beta(1 + a, b(j + 1)))}{2 \left[1 - (1 - x^a)^b\right]^{1/\lambda}}\right).$$  (37)

| $\lambda$ | $b$ | Mean | Var | $S_k$ | $K$ |
|-----------|-----|------|-----|-------|-----|
| 2.0       | 0.56556 | 0.03908 | -0.09702 | 2.30905 |
| 3.0       | 0.50543 | 0.03430 | 0.04856 | 2.38086 |
| 4.0       | 0.46478 | 0.03053 | 0.13105 | 2.45360 |
| 5.0       | 0.43469 | 0.02758 | 0.18440 | 2.51360 |
| 6.0       | 0.41114 | 0.02522 | 0.22179 | 2.56192 |
| 7.0       | 0.39198 | 0.02330 | 0.24948 | 2.60112 |
| 2.0       | 0.64566 | 0.02830 | -0.26723 | 2.53251 |
| 3.0       | 0.57974 | 0.02601 | -0.08852 | 2.52495 |
| 4.0       | 0.53444 | 0.02373 | 0.01141 | 2.56666 |
| 5.0       | 0.50062 | 0.02176 | 0.07559 | 2.61164 |
| 6.0       | 0.47399 | 0.02011 | 0.12039 | 2.65173 |
| 7.0       | 0.45225 | 0.01871 | 0.15347 | 2.68603 |
| 2.0       | 0.69540 | 0.02170 | -0.35350 | 2.69440 |
| 3.0       | 0.62680 | 0.02080 | -0.15040 | 2.62680 |
| 4.0       | 0.57910 | 0.01930 | -0.03790 | 2.64750 |
| 5.0       | 0.54310 | 0.01800 | 0.03400 | 2.68340 |
| 6.0       | 0.51470 | 0.01680 | 0.08400 | 2.71910 |
| 7.0       | 0.49140 | 0.01570 | 0.12090 | 2.75120 |
| 2.0       | 0.72980 | 0.01740 | -0.40310 | 2.80610 |
| 3.0       | 0.66000 | 0.01720 | -0.18150 | 2.69350 |
| 4.0       | 0.61070 | 0.01640 | -0.05970 | 2.70010 |
| 5.0       | 0.57340 | 0.01540 | 0.01780 | 2.73070 |
| 6.0       | 0.54380 | 0.01440 | 0.07160 | 2.76440 |
| 7.0       | 0.51950 | 0.01360 | 0.11120 | 2.79570 |
| 2.0       | 0.81500 | 0.00830 | -0.48390 | 3.03600 |
| 3.0       | 0.74510 | 0.00940 | -0.21230 | 2.81430 |
| 4.0       | 0.69370 | 0.00950 | -0.06610 | 2.79630 |
| 5.0       | 0.65380 | 0.00930 | 0.02610 | 2.82280 |
| 6.0       | 0.62160 | 0.00900 | 0.08980 | 2.85860 |
| 7.0       | 0.59490 | 0.00860 | 0.13640 | 2.89410 |
3.9. Order Statistics. In statistical theory, order statistics is widely applied and practiced. Let \(X_1, X_2, \ldots, X_n\) be r.v.s. with their corresponding cdfs \(F(x)\). Let \(X_1^{*}, X_2^{*}, \ldots, X_n^{*}\) be the related ordered r. sample of size \(n\), then the density of \(r\)th order statistic is given as

\[
f_{r,n}(x) = K \sum_{v=0}^{n-r} (-1)^v \binom{n-r}{v} f(x) F(x)^{v+r-1},
\]

where \(K = 1/\beta(r, n-r + 1)\) and \(\beta(.,.)\) is the beta function. The pdf of the \(r\)th order statistic of TIIHLKwD is obtained by putting (21) and (22) in (38), changing \(h\) with \(v + k - 1\),

\[
f_{r,n}(x) = K \sum_{v=0}^{n-r} \sum_{k,m=0}^{\infty} \eta_v^* x^{v+(k+1)-1} (1-x^a)^{bm},
\]

where

\[
\eta_v^* = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{v+j+m} (2)^{v+r-1} \binom{\lambda(i+1)-1}{j} \binom{b(j+1)-1}{k} \binom{v+r+l-2}{l} \binom{\lambda(l+v+r-1)}{m} \binom{n-r}{v} \eta_j.
\]

More, the \(s\)th moment of \(r\)th order statistics for TIIHLKwD is given by

\[
E(X_{r,n}^s) = \int_{-\infty}^{\infty} x^s f_{r,n}(x) dx.
\]

By substituting (39) in (41), we have

\[
E(X_{r,n}^s) = \frac{1}{\beta(r, n-r + 1)} \sum_{v=0}^{n-r} \sum_{k,m=0}^{\infty} \eta_v^* \int_0^1 x^{s+v+(k+1)-1} (1-x^a)^{bm} dx.
\]

Then,

\[
E(X_{r,n}^s) = \frac{1}{B(r, n-r + 1)} \sum_{v=0}^{n-r} \sum_{k,m=0}^{\infty} \eta_v^* \beta(s+a(k+1), bm+1).
\]

3.10. Rényi Entropy. Renyi (1961) proposed and used this measure. It can be obtained by

\[
I_\delta(X) = \frac{1}{1-\delta} \log \int_{-\infty}^{\infty} f(x)^\delta dx, \delta > 0 \text{ and } \delta \neq 1.
\]
Applying binomial expansion (16) in (8) then \( f(x) \) can be written as
\[
\begin{align*}
fx(x)\delta &= \sum_{i=0}^{\infty} \frac{\eta^i \delta}{i!} (1-x^a)^{i(b-1)} \left[ 1 - (1-x^a)^b \right]^{i(j+\delta)-\delta} \\
&= \sum_{i=0}^{\infty} \frac{\eta^i \delta}{i!} (1-x^a)^{i(b-1)} (1-x^a)^{i(j+\delta)-\delta} dx.
\end{align*}
\]
(45)

So, the Rényi entropy of TIIHLKw distribution is as follows
\[
I_\delta(X) = \frac{1}{1-\delta} \log \sum_{i=0}^{\infty} \eta^i \delta \left[ 1 - (1-x^a)^b \right]^{i(j+\delta)-\delta} dx \]
\[
= \frac{1}{1-\delta} \log \sum_{i=0}^{\infty} \eta^i \delta \int_0^1 x^{i(a-1)} (1-x^a)^{i(b-1)} dx.
\]
(46)

### 3.11. Stress-Strength Reliability
This subsection deals with the stress-strength parameter of TIIHLKw distribution. Let \( X_1 \) be the strength of a structure with a stress \( X_2 \), and if \( X_1 \) follows TIIHLKw(\( \lambda_1, a_1, b_1 \)) and \( X_2 \) follows TIIHLKw(\( \lambda_2, a_2, b_2 \)), provided \( X_1 \) and \( X_2 \) are statistically independent r.vs.,

| Table 2: Estimates and MSEs of parameters of TIIHLKw Distribution for the Set1 (\( a=2, \lambda=2, b=2 \)). |
|---|---|---|---|---|---|---|---|---|---|---|
| \( n \) | MLEs Estimates | MSEs Estimates | LSEs Estimates | MSEs Estimates | WLSEs Estimates | MSEs Estimates | CVEs Estimates | MSEs Estimates | PCEs Estimates | MSEs Estimates |
|---|---|---|---|---|---|---|---|---|---|---|
| 50 | 2.285 | 0.821 | 1.733 | 0.263 | 1.833 | 0.442 | 1.751 | 0.293 | 1.727 | 0.317 |
| 100 | 2.266 | 0.393 | 1.741 | 0.239 | 1.755 | 0.392 | 1.703 | 0.235 | 1.616 | 0.304 |
| 200 | 1.925 | 0.329 | 2.554 | 0.84 | 2.651 | 1.238 | 2.713 | 1.128 | 2.810 | 1.221 |
| 500 | 2.056 | 0.031 | 1.961 | 0.123 | 1.952 | 0.093 | 2.014 | 0.099 | 1.929 | 0.078 |
| 3.11. Stress-Strength Reliability. This subsection deals with the stress-strength parameter of TIIHLKw distribution. Let \( X_1 \) be the strength of a structure with a stress \( X_2 \), and if \( X_1 \) follows TIIHLKw(\( \lambda_1, a_1, b_1 \)) and \( X_2 \) follows TIIHLKw(\( \lambda_2, a_2, b_2 \)), provided \( X_1 \) and \( X_2 \) are statistically independent r.vs.,

| Table 3: Estimates and MSEs of parameters of TIIHLKwD for the Set2 (\( a=2, \lambda=3, b=2 \)). |
|---|---|---|---|---|---|---|---|---|---|---|
| \( n \) | MLEs Estimates | MSEs Estimates | LSEs Estimates | MSEs Estimates | WLSEs Estimates | MSEs Estimates | CVEs Estimates | MSEs Estimates | PCEs Estimates | MSEs Estimates |
|---|---|---|---|---|---|---|---|---|---|---|
| 50 | 2.209 | 0.417 | 1.941 | 0.291 | 1.95 | 0.369 | 2.069 | 0.365 | 2.12 | 0.327 |
| 100 | 2.024 | 0.044 | 2.02 | 0.164 | 2.033 | 0.25 | 2.17 | 0.289 | 2.171 | 0.264 |
| 200 | 2.02 | 0.082 | 1.986 | 0.198 | 1.938 | 0.329 | 1.976 | 0.245 | 1.706 | 0.207 |
| 500 | 2.002 | 0.022 | 2.02 | 0.045 | 2.017 | 0.055 | 2.053 | 0.062 | 2.063 | 0.074 |
| 3.11. Stress-Strength Reliability. This subsection deals with the stress-strength parameter of TIIHLKw distribution. Let \( X_1 \) be the strength of a structure with a stress \( X_2 \), and if \( X_1 \) follows TIIHLKw(\( \lambda_1, a_1, b_1 \)) and \( X_2 \) follows TIIHLKw(\( \lambda_2, a_2, b_2 \)), provided \( X_1 \) and \( X_2 \) are statistically independent r.vs.,

### 8 Journal of Function Spaces
The reliability is defined by
\[ R = \eta^* \beta \left( \frac{a_1(m + 1)}{a_2}, b_2 + 1 \right) \]  \hspace{1cm} (48)

Proof. The reliability is defined by
\[ R = P(X_2 < X_1) = \int_0^\infty f_1(x; \lambda_1, a_1, b_1) F_2(x; \lambda_2, a_2, b_2) dx, \]  \hspace{1cm} (49)

Then, we can write
\[ R = 2 \sum_{i,j,k=0}^{\infty} (-1)^k \eta_{i,j} \int_0^{x^{a_2}} \left( 1 - x^{a_2} \right)^{b_j(y) + 1} \cdot \left( 1 - (1 - x^{a_2}) b_j \right)^{\lambda_i(k+1)} dx. \]  \hspace{1cm} (50)

where
\[ \eta^* = 2 \sum_{i,j,k=0}^{\infty} (-1)^{k+j+m} \frac{\lambda_i(k+1) + 1}{l} \left( b_j(k+1) - 1 \right) \eta_{i,j}, \]  \hspace{1cm} (51)

which completes the proof.

4. Inference

This section is dedicated to estimation aspects of TIIHLKw distribution, assuming that population parameters \((a, b, \lambda)\) are unknown and can be estimated using different methods of estimation including ML, LS, WLS, PC, and CV.

**Proposition 2.** Under the assumption discussed above, we have

\[ R = \eta^* \beta \left( \frac{a_1(m + 1)}{a_2}, b_2 + 1 \right). \]  \hspace{1cm} (48)

**Proof.** The reliability is defined by
\[ R = P(X_2 < X_1) = \int_0^\infty f_1(x; \lambda_1, a_1, b_1) F_2(x; \lambda_2, a_2, b_2) dx, \]  \hspace{1cm} (49)

Then, we can write
\[ R = 2 \sum_{i,j,k=0}^{\infty} (-1)^k \eta_{i,j} \int_0^{x^{a_2}} \left( 1 - x^{a_2} \right)^{b_j(y) + 1} \cdot \left( 1 - (1 - x^{a_2}) b_j \right)^{\lambda_i(k+1)} dx. \]  \hspace{1cm} (50)

where
\[ \eta^* = 2 \sum_{i,j,k=0}^{\infty} (-1)^{k+j+m} \frac{\lambda_i(k+1) + 1}{l} \left( b_j(k+1) - 1 \right) \eta_{i,j}, \]  \hspace{1cm} (51)

which completes the proof.

4.1. Maximum Likelihood Estimation. For a random sample of \(x_1, x_2, x_3, \ldots, x_n\) from the TIIHLKw\((\lambda, a, b)\) distribution, the log-likelihood function for \(\Phi = (\lambda, a, b)\) is
\[ \log L(\Phi) = n \log (2ab\lambda) + (a - 1) \sum_{i=1}^n \log[x_i] + (\lambda - 1) \sum_{i=1}^n \log[1 - x_i^a] + (b - 1) \sum_{i=1}^n \log[1 - x_i^a] + \sum_{i=1}^n \left( 1 + \left( 1 - x_i^a \right)^{b} \right)^{\lambda}. \]  \hspace{1cm} (51)

The members of \(U(\Phi) = (U_\lambda, U_a, U_b)\) are given below
\[ U_\lambda = \frac{n}{\lambda} + \sum_{i=1}^n \log[1 - (1 - x_i^a)^b] \]  \hspace{1cm} (52)

\[ U_a = \frac{n}{a} + \sum_{i=1}^n \log[x_i] + (\lambda - 1) \sum_{i=1}^n b \log[x_i] x_i^a (1 - x_i^a)^{-1+b} \]  \hspace{1cm} (53)

\[ - (b - 1) \sum_{i=1}^n \log[x_i] x_i^a - b \sum_{i=1}^n 2b \log[x_i] x_i^a (1 - x_i^a)^{-1+b} \]  \hspace{1cm} (53)
equating (52), (53), and (54) to zero and solve them simultaneously give the ML estimates (MLEs) \( \Phi = (\lambda, a, b) \) of \( \Phi = (\lambda, a, b) \). The iterative algorithm is used to obtain the numerical solution of these nonlinear equations such as the Newton-Raphson method.

4.2. Ordinary and Weighted LS Estimation (LSEs and WLSEs): The LSEs of \( \lambda, a, b \) can be obtained by minimizing the sum of squares of errors with respect to parameters. Suppose \( X_1, X_2, \ldots, X_n \) is a random sample of size \( n \) from TIIHLKwD and suppose \( X_1, X_2, \ldots, X_n \) be the related ordered sample.

The sum is independent of the unknown parameters.

\[
U_b = \frac{n}{b} + \sum_{i=1}^{n} \log(1 - x_i^a) - (\lambda - 1) \sum_{i=1}^{n} \frac{\log(1 - x_i^a)(1 - x_i^b)}{1 - (1 - x_i^a)b} + \sum_{i=1}^{n} \frac{2\lambda \log(1 - x_i^a)(1 - x_i^b)(1 - (1 - x_i^b)^{1+\lambda})}{1 + (1 - (1 - x_i^a)b)^{1+\lambda}}. 
\]  

(54)

Table 5: Estimates and MSEs of parameters of TIIHLKw Distribution for the Set4 \((a = 1, \lambda = 2, b = 2)\).

| \( N \) | MLEs Estimates | MSEs | LSEs Estimates | MSEs | WLSEs Estimates | MSEs | CVEs Estimates | MSEs | PCEs Estimates | MSEs |
|---|---|---|---|---|---|---|---|---|---|---|
| 50 | 1.251 | 0.2 | 1.321 | 0.614 | 1.353 | 0.817 | 1.331 | 0.645 | 1.471 | 1.469 |
| 100 | 1.674 | 0.157 | 1.182 | 0.195 | 1.229 | 0.5 | 1.226 | 0.333 | 1.341 | 0.501 |
| 500 | 2.131 | 0.069 | 2.145 | 0.294 | 2.116 | 0.181 | 2.247 | 0.506 | 2.307 | 0.403 |

Equivalently, \( \lambda = 2 \), \( a = 2 \), \( b = 2 \). The following function is minimized with respect to \( \lambda, a \) and \( b \) to get WLS estimators of model parameters.

\[
WLS(\lambda, a, b) = \sum_{i=1}^{n} \left[ \frac{2 \left( 1 - \left( 1 - x_i^a \right)^b \right)^{\lambda}}{1 + \left( 1 - \left( 1 - x_i^a \right)^b \right)^{1+\lambda}} - \frac{i}{n + 1} \right]^2. 
\]  

(57)

Table 6: Ranks of all the methods of estimation for different parametric specifications.

| Parameters | \( N \) | ML | LS | WLS | PC | CV |
|---|---|---|---|---|---|---|
| \( a = 2 \) | 50 | 2 | 1 | 4.5 | 3 | 4.5 |
| \( \lambda = 2 \) | 100 | 1 | 3.5 | 5 | 2 | 3.5 |
| \( b = 2 \) | 200 | 1 | 2 | 5 | 3 | 4 |
| 500 | 1 | 4 | 5 | 2 | 3 |
| \( a = 2 \) | 50 | 1 | 2 | 5 | 3 | 4 |
| \( \lambda = 2 \) | 100 | 1 | 3 | 5 | 4 | 2 |
| \( b = 3 \) | 200 | 1 | 3 | 5 | 4 | 2 |
| 500 | 1 | 3 | 5 | 4 | 2 |
| \( a = 1 \) | 50 | 1 | 2 | 4 | 3 | 5 |
| \( \lambda = 2 \) | 100 | 1 | 2.5 | 3.5 | 5 | 3.5 |
| \( b = 2 \) | 200 | 1 | 2 | 5 | 3.5 | 3.5 |
| 500 | 1 | 4 | 5 | 3 | 2 |

\[ \sum \text{Ranks} \quad 18.5 \quad 38 \quad 76.5 \quad 54 \quad 53 \]

Overall rank \( 1 \quad 2 \quad 5 \quad 4 \quad 3 \)
i.e., \( \partial \text{WLS}(\lambda, a, b)/\partial \lambda = 0, \partial \text{WLS}(\lambda, a, b)/\partial a = 0, \) and \( \partial \text{WLS}(\lambda, a, b)/\partial b = 0. \)

4.3. PC Estimators (PCEs). Under the specification defined above including order statistics having relationship \( X_{(1)} < X_{(2)} < \cdots < X_{(n)}. \) In the PC method of estimation, the estimators of \( \lambda, a, \) and \( b \) are derived by minimizing the following expression

\[
\sum_{i=1}^{n} \ln \left( \frac{i}{n+1} \right) - \ln \left( \frac{2 \left( 1 - \left( 1 - x_{(i)}^a \right)^b \right) \lambda}{1 + \left( 1 - \left( 1 - x_{(i)}^a \right)^b \right) \lambda} \right) \]  

(58)

with respect to \( \lambda, a, \) and \( b. \)

4.4. The Cramer-von Mises Minimum Distance Estimation. The CVE is another estimation method based on minimum distance. The CV estimators are obtained by minimizing, with respect to \( \lambda, a, \) and \( b. \)

\[
\text{CV} = \frac{1}{12n} + \sum_{i=1}^{n} \left[ \frac{2 \left( 1 - \left( 1 - x_{(i)}^a \right)^b \right) \lambda}{1 + \left( 1 - \left( 1 - x_{(i)}^a \right)^b \right) \lambda} - \frac{2i - 1}{2n} \right]^2.
\]

(59)

CV minimum distance estimators provide empirical evidence that the bias of this estimator is smaller than the other minimum distance estimators.

5. Simulation Study

In this section, we present a Monte Carlo (MC) simulation study in order to illustrate the behavior of different estimates. We consider four random sample sizes: \( n = 50, 100, 200, \) and \( 500 \) from the TIIHLKw distribution and the samples are drawn 10,000 times. Four specifications of the parameters are used in this simulation study, given as, \( \text{set1} = (\lambda = 2, b = 0.5, a = 2), \text{set2} = (\lambda = 3, b = 2, a = 2), \text{set3} = (\lambda = 2, b = 3, a = 2), \text{set4} = (\lambda = 2, b = 2, a = 1). \) For sample generated, the MLE, LSE, WLSE, CVE, PCE, and MPSE of estimators are computed numerically. Then, the estimates of all methods and their mean square errors (MSEs) are documented in Tables 1–4.

Estimates of parameters and their corresponding MSEs are calculated using the same sample under different methods of estimation for four sets of parameters mentioned above for small sample size \( (n = 50) \) to sufficient large sample size, that is, \( n = 500. \)

The entries of Table 2 to Table 5 show that estimates are reliable and consistent. MSEs reduce as sample size \( (n) \) increases under each method of estimation. The summary of these four tables is presented in Table 5.

Using the entries of Table 6 for different parametric combinations, we can conclude that the MLE method outperforms than all other estimation methods (with an overall score of 18.5). Therefore, depending on the simulation study, the MLE method performs best for TIIHLKwD.

6. Applications

The TIIHLKw distribution aims at providing an alternative distribution to fit data on the unit interval to other distributions available in the literature. Here, we used the following probability distributions as competitor models;

(i) The Kumaraswamy (Kw) distribution

\[
f(x; a, b) = abx^{a-1}(1-x)^{b-1}, 0 < x < 1.
\]

(ii) Transmuted Kumaraswamy (TKw) distribution

\[
f_{\text{TKw}}(x; a, b) = abx^{a-1}(1-x^b)^{b-1}\left(1 - \lambda + 2\lambda(1-x^b)^b\right).
\]
Table 7: MLEs and goodness-of-fit measures for the first data set.

| Model      | \( \hat{a} \) | \( \hat{b} \) | \( \hat{\lambda} \) | \( L \) | AIC  | BIC  | \( A^* \) | \( W^* \) |
|------------|--------------|-------------|----------------|--------|------|------|--------|--------|
| TIIHLKw    | 8.45118      | 28229.6     | 0.209229       | 57.6664| -109.333 | -104.597  | 0.336229 | 0.014361 |
| Kw         | 2.07740      | 33.1374     | —              | 56.0687| -108.137 | -104.313  | 0.688407 | 0.105252 |
| Tkw        | 2.07740      | 16.5687     | 0.99999        | 56.0687| -106.137 | -100.401  | 0.688407 | 0.105252 |
| SBKw       | 1.44716      | 19.9669     | —              | 55.2067| -106.413 | -102.589  | 0.838017 | 0.127612 |
| Beta       | 2.68257      | 13.8658     | —              | 54.6067| -105.213 | -97.4772  | 0.926850 | 0.155443 |

Table 8: MLEs and goodness-of-fit measures for the second data set.

| Model      | \( \hat{a} \) | \( \hat{b} \) | \( \hat{\lambda} \) | \( L \) | AIC  | BIC  | \( A^* \) | \( W^* \) |
|------------|--------------|-------------|----------------|--------|------|------|--------|--------|
| TIIHLKw    | 0.32008      | 5.84284     | 222.766        | 58.5465| -111.093 | -105.479  | 0.17531 | 0.025823 |
| Kw         | 2.71874      | 44.6604     | —              | 52.4915| -100.983 | -97.2407  | 1.31057 | 0.208116 |
| Tkw        | 2.98542      | 43.9205     | 0.651548       | 53.7972| -101.594 | -95.9808  | 1.02032 | 0.155492 |
| SBKw       | 2.19509      | 32.9024     | —              | 53.6088| -103.218 | -99.4752  | 1.10420 | 0.174441 |
| Beta       | 5.94177      | 21.2057     | —              | 55.6002| -107.200 | -99.5868  | 0.78992 | 0.131428 |

Figure 7: The fitted pdf, cdf, survival, and PP plots of the TIIHLKw distribution for the first data.
(iii) Size biased Kumaraswamy (SBKw) distribution

\[ f_{\text{SBKw}}(x; a, b) = \frac{ax^a(1 - x^a)^{b-1}}{\text{Beta}[1 + (1/a), b]} . \]  

(iv) Beta distribution

\[ f_{\text{Beta}}(x; a, b) = \frac{1}{\text{Beta}[a, b]} x^{a-1}(1 - x)^{b-1}. \]

We use the following accuracy measures for model comparison: Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), log-likelihood, Anderson-Darling (A*), and Cramer–von Mises (W*). R-Language is used for numerical computations.

1st data set. The data set was taken from Dasgupta [20], and considered \( n = 50 \) on burr (in millimeters), with hole diameter and a sheet thickness of 12 mm and 3.15 mm, respectively.

2nd data set. This data set was taken from [21], and considered \( n = 48 \) measurements on petroleum rock samples from a petroleum reservoir.

The total test time (TTT) for both datasets are presented in Figure 6. We can observe that the shape of TTT plots is concave for both datasets, which demonstrates an increasing failure rate.

The MLEs for TIIHLKw distribution along with some adequacy measures are presented in Tables 7 and 8. Hence, it is concluded that the new model provides the better fit. Figures 7 and 8 show the estimated densities, cdfs, estimated survival functions, and PP plots for the considered distributions of both data sets, respectively. We note that the proposed model is more appropriated to fit the data than the other competing models.

7. Conclusion

In this article, a new Type-II Half Logistic Kumaraswamy distribution is proposed. Some characteristics of the TIIHLKw distribution including linear combination expressions for the...
density function, probability weighted moments, moments, incomplete moments, quantile function, mean deviation about mean and about median, Bonferroni and Lorenz curves, order statistics, stress-strength reliability, and Rényi entropy are derived. The ML method is used to estimate model parameters. An extensive simulation study is conducted to compare several well-known estimation methods, including the method of maximum likelihood estimation, methods of least squares and weighted least squares estimation, and method of Cramer-von Mises minimum distance estimation. The simulation study showed the reliability and efficiency of the estimates. Finally, by considering the method of maximum likelihood estimation, the new model is fitted to two practical data sets. The applications on real data sets validated the significance of the new distribution.

Appendix

A.1. Data I
0.04, 0.02, 0.06, 0.12, 0.14, 0.08, 0.22, 0.12, 0.08, 0.26, 0.24, 0.04, 0.14, 0.16, 0.08, 0.26, 0.32, 0.28, 0.14, 0.16, 0.24, 0.22, 0.12, 0.18, 0.24, 0.32, 0.16, 0.14, 0.08, 0.16, 0.24, 0.16, 0.32, 0.18, 0.24, 0.22, 0.16, 0.12, 0.24, 0.06, 0.02, 0.18, 0.22, 0.14, 0.06, 0.04, 0.14, 0.26, 0.18, 0.16.

A.2. Data II
0.0903296, 0.2036540, 0.2043140, 0.2808870, 0.1976530, 0.3286410, 0.1486220, 0.1623940, 0.2627270, 0.1794550, 0.3263550, 0.2300810, 0.1833120, 0.1509440, 0.2000710, 0.1918020, 0.1541920, 0.4641250, 0.1170630, 0.1481410, 0.1448100, 0.1330830, 0.2760160, 0.4204770, 0.1224170, 0.2285950, 0.1138520, 0.2252140, 0.1769690, 0.2007440, 0.1670450, 0.2316230, 0.2910290, 0.3412730, 0.4387120, 0.2626510, 0.1896510, 0.1725670, 0.2400770, 0.3116460, 0.1670450, 0.1824530, 0.1641270, 0.1534810, 0.1618650, 0.2760160, 0.2538320, 0.2004470.

Data Availability
Data is present in appendix.

Conflicts of Interest
The authors declare that they have no conflicts of interest.

Acknowledgments
This work was funded by the Deanship of Scientific Research (DSR), King Abdul Aziz University, Jeddah, under grant No. (DF-287-305-1441). The authors gratefully acknowledge the DSR technical and financial support.

References
[1] M. A. u. Haq, M. Elgarhy, and S. Hashmi, “The generalized odd Burr III family of distributions: properties, applications and characterizations,” Journal of Taibah University for Science, vol. 13, no. 1, pp. 961–971, 2019.
[2] A. M. T. Abd El-Bar and I. E. Ragab, “On weighted exponential-Gompertz distribution: properties and applications,” Journal of Taibah University for Science, vol. 13, no. 1, pp. 616–627, 2019.
[3] A. Alzaatreh, C. Lee, and F. Famoye, “A new method for generating families of continuous distributions,” Metron, vol. 71, no. 1, pp. 63–79, 2013.
[4] M. A. u. Haq, M. Elgarhy, S. Hashmi, G. Ozel, and Q. u. Ain, “Transmuted Weibull power function distribution: its properties and applications,” Journal of Data Science, vol. 397, p. 418, 2018.
[5] S. Hashmi, M. A. u. Haq, and R. M. Usman, “A generalized exponential distribution with increasing, decreasing and constant shape hazard curves,” Electronic Journal of Applied Statistical Analysis, vol. 12, no. 1, pp. 223–244, 2019.
[6] M. Elgarhy, M. A. u. Haq, and Q. u. Ain, “Exponentiated generalized Kumaraswamy distribution with applications,” Annals of Data Science, vol. 5, no. 2, pp. 273–292, 2018.
[7] R. A. ZeinEldin, S. Hashmi, and M. Elgarhy, “Alpha power transformed inverse Lomax distribution with different methods of estimation and applications,” Complexity, vol. 2020, Article ID 1860813, 15 pages, 2020.
[8] R. Alshenawy, “A new one parameter distribution: properties and estimation with applications to complete and type II censored data,” Journal of Taibah University for Science, vol. 14, no. 1, pp. 11–18, 2020.
[9] P. Kumaraswamy, “A generalized probability density function for double-bounded random processes,” Journal of Hydrology, vol. 46, no. 1–2, pp. 79–88, 1980.
[10] A. J. Lemonte, W. B. Souzaa, and G. M. Cordeiro, “The exponentiated Kumaraswamy distribution and its log-transform,” Brazilian Journal of Probability and Statistics, vol. 27, no. 1, pp. 31–53, 2013.
[11] E.-S. El-Sherpieny and M. A. Ahmed, “On the Kumaraswamy Kumaraswamy distribution,” International Journal of Basic and Applied Sciences, vol. 3, no. 4, 2014.
[12] M. S. Khan, R. King, and I. L. Hudson, “Transmuted Kumaraswamy distribution,” Statistics in Transition New Series, vol. 17, no. 2, pp. 183–210, 2016.
[13] D. Sharma and T. K. Chakraborty, “On size biased Kumaraswamy distribution,” Statistics, Optimization & Information Computing, vol. 4, no. 3, 2016.
[14] R. George and S. Thobias, “Marshall-Olkin Kumaraswamy distribution,” International Mathematical Forum, vol. 12, no. 2, pp. 47–69, 2017.
[15] R. A. ZeinEldin, F. Jamal, C. Chesneau, and M. Elgarhy, “Type II Topp-Leone inverted Kumaraswamy distribution with statistical inference and applications,” Symmetry, vol. 11, no. 12, p. 1459, 2019.
[16] R. A. ZeinEldin, C. Chesneau, F. Jamal, and M. Elgarhy, “Statistical properties and different methods of estimation for type I half logistic inverted Kumaraswamy distribution,” Mathematics, vol. 7, no. 10, pp. 1002, 2019.
[17] R. A. Bantan, F. Jamal, C. Chesneau, and M. Elgarhy, “Truncated inverted Kumaraswamy generated family of distributions with applications,” Entropy, vol. 21, no. 11, p. 1089, 2019.
[18] I. Ghosh, “Bivariate and multivariate weighted Kumaraswamy distributions: theory and applications,” Journal of Statistical Theory and Applications, vol. 18, no. 3, p. 198, 2019.
[19] A. S. Hassan, M. Elgarhy, and M. Shakh, “Type II half logistic family of distributions with applications,” Pakistan Journal of
Statistics and Operation Research, vol. 13, no. 2, pp. 245–264, 2017.

[20] R. Dasgupta, “On the distribution of burr with applications,” Sankhya B, vol. 73, no. 1, pp. 1–19, 2011.

[21] G. M. Cordeiro and R. dos Santos Brito, “The beta power distribution,” Brazilian Journal of Probability and Statistics, vol. 26, no. 1, pp. 88–112, 2012.