Sign changes of a product of Dirichlet characters and Fourier coefficients of Hecke eigenforms

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Abstract Let \( f \in S_k(\Gamma_0(N)) \) be a normalized Hecke eigenform of even integral weight \( k \) and level \( N \). Let \( j \geq 1 \) be a positive integer. We prove that for almost all primes \( p, p \nmid N \), and for all characters \( \chi_0 = \pm 1 \pmod{N} \), the sequence \((\chi_0(p^{nj})a(p^{nj}))_{n \in \mathbb{N}}\) has infinitely many sign changes. We also obtain a similar result for the sequence \((a(p^{j(1+2nj)}))_{n \in \mathbb{N}}\) when \( j \) is odd.

Keywords Sign change · Fourier coefficients · Cusp forms · Dirichlet series

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1 Introduction

Let \( k, N \in \mathbb{N} \) be integers. Throughout the paper, \( S_k(\Gamma_0(N), \chi) \) denotes the space of cusp forms of weight \( k \) and level \( N \), with Dirichlet character \( \chi \pmod{N} \). When \( k \) is even and \( \chi = 1 \), the trivial character modulo \( N \), we denote \( S_k(\Gamma_0(N), 1) = S_k(\Gamma_0(N)) \). If in addition \( N = 1 \), we abbreviate notation with \( S_k \).

In [5], it has been shown that for every normalized Hecke eigenform \( f \) of even integral weight \( k \) on the modular group \( SL_2(\mathbb{Z}) \) with Fourier coefficients \( a(n) \) \( (n \geq 1) \), each sequence \((a(n^j))_{n \geq 1} \) for \( j \in \{2, 3, 4\} \) has infinitely many sign changes. The proof of this uses suitable estimates of the sums

\[
\sum_{n \leq x} \lambda(n^j) \text{ and } \sum_{n \leq x} \lambda^2(n^j),
\]

where \( \lambda(n) \) is given by \( \lambda(n) = \frac{a(n)}{\phi(n)} \).

Recently, Kohnen and Martin showed, in [3], that if \( j \) is a positive integer then for almost all primes \( p \) the sequence \((a(p^{nj}))_{n \geq 0}\) has infinitely many sign changes.

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The proof requires the use of Landau’s theorem and suitable computations applied to the Dirichlet series
\[ \sum_{n \geq 0} a(p^j n) p^{-jn^2}. \]

In this work we extend the results of [4] to normalized Hecke eigenforms of even integral weight \( k \) and level \( N \). Furthermore, we will show that the sequence \((a(p^{j(1+2n)}))_{n \geq 0}\) has infinitely many sign changes. More precisely, our first main theorem is the following.

**Theorem 1** Let \( f \in S_k(\Gamma_0(N), \chi) \) be a normalized Hecke eigenform of even integral weight \( k \) and level \( N \), with Dirichlet character \( \chi \). Let \( \chi_0 \pmod{\chi} = \chi \). Let
\[
f(z) = \sum_{n \geq 1} a(n)e(nz),
\]
be the Fourier expansion of \( f \) at \( \infty \). Let \( j \geq 1 \) be an integer. Then for almost all primes \( p \), \( p \nmid N \), the sequence \((a(p^{j}(n+2n)))_{n \in \mathbb{N}}\) has infinitely many sign changes.

This result extends [4, Theorem 2.1]. Indeed, when \( \chi = 1 \), we get the following result.

**Corollary 1** Let \( f \in S_k(\Gamma_0(N)) \) be a normalized Hecke eigenform of even integral weight \( k \) and level \( N \). Let \( j \geq 1 \) be a positive integer. Then for almost all primes \( p \), \( p \nmid N \), and for all characters \( \chi_0 = \pm 1 \pmod{\chi} \), the sequence \((\chi_0(p^{j}n)a(p^{j}n))_{n \in \mathbb{N}}\) has infinitely many sign changes.

Our second main theorem shows that the subsequence of \((a(p^{j}(n)))_{n \in \mathbb{N}}\), with odd indices, has infinitely many sign changes.

**Theorem 2** Let \( f \in S_k(\Gamma_0(N)) \) be a normalized Hecke eigenform of even integral weight \( k \) and level \( N \). Let \( j \geq 1 \) be a positive integer such that \( 2 \nmid j \). Then for almost all primes \( p \), \( p \nmid N \), the sequence \((a(p^{j(1+2n)})_{n \in \mathbb{N}}\) has infinitely many sign changes.

It should be noted that the proofs of these two theorems rely on Landau’s theorem applied to the suitable Dirichlet series and Deligne’s bound for the Fourier coefficients \( a(n) \).

Let \( f \in S_k(\Gamma_0(N), \chi) \) be a cusp form with Fourier coefficients \( a(n), n \geq 1 \). Let \( j \geq 1 \) be any non-negative integer and \( p \) a prime number. In order to state the following theorem, we define the operator \( T_j(p) \) acting on \( S_k(\Gamma_0(N), \chi) \) by
\[
T_j(p)f(z) = \sum_{n \geq 1} \left( a(p^j n) + p^{j(k-1)} \chi^j(p) a \left( \frac{n}{p^j} \right) \right) e(nz),
\]
with the convention \( a(n/p^j) = 0 \) if \( p^j \) does not divides \( n \). Notice that \( T_0(p) = 2 \) and \( T_1(p) = T(p) \) where \( T(p) \) is the \( p \)-th classical Hecke operator. When \( f \in S_k \), these operators are the same as those defined in [4], and it was shown in this case that the characteristic polynomial \( P(T_j(p), X) \) of \( T_j(p) \) on \( S_k \) has rational coefficients.
Theorem 3 Suppose that $P(T_j(p), X)$ is irreducible over $\mathbb{Q}$. Assume further that there are no different eigenvalues $\lambda_1$ and $\lambda_2$ of $T_j(p)$ such that $\lambda_1 + \lambda_2 = 0$. Let $f \in S_k$ be a non zero cusp form of even integral weight $k$ with Fourier coefficients $a(n), n \geq 1$. Let $j \geq 1$ be a positive integer such that $2 \nmid j$. Then for almost all primes $p$, $p \nmid N$, the sequence $\left(a(p^{j(1+2n)})\right)_{n \in \mathbb{N}}$ has infinitely many sign changes.

Notice that when $j = 1$ and $T_1(p) = T(p)$, the conjecture of Maeda says that $P(T(p), X)$ is irreducible over $\mathbb{Q}$. This conjecture is supported by some numerical results [1–3].

2 Proof of Theorem 1

In this subsection, we prove Theorem 1. We begin with the following lemma.

Lemma 1 Let $p$ be a prime number and $j \geq 1$ an integer. The following assertions hold.

1. $T_j(p)$ is a monic polynomial in $T(p)$ of degree $j$.
2. If $f \in S_k(\Gamma_0(N), \chi)$ is an eigenfunction of $T_j(p)$ with eigenvalue $\lambda_j(p)$, then

$$\sum_{n \geq 0} a(p^{jn}) \chi_0(p^{jn}) X^n = \frac{1}{1 - \frac{\lambda_j(p)}{\chi_0(p)} X + p^{(k-1)}X^2}$$

where $a(n)$ denotes the $n$-th Fourier coefficient of $f$.

Proof (Proof of Lemma 1)

1. We see easily from (1) that for all $j \geq 1$ one has

$$T_{j+1}(p) = T_j(p)T(p) - p^{k-1}\chi(p)T_{j-1}(p),$$

hence the result follows by recurrence on $j$.

2. Let $n \in \mathbb{N}$. We have

$$a(p^{j(n+1)}) = \lambda_j(p)a(p^{jn}) - p^{j(k-1)}\chi_j(p)a(p^{j(n-1)}),$$

for all $j \geq 1$, which can be deduced from (1). Therefore,

$$S = \sum_{n \geq 0} a(p^{jn}) \chi_0(p^{jn}) X^n$$

$$= a(1) + \frac{a(p^j)}{\chi_0(p^j)} X + \sum_{n \geq 0} \frac{a(p^{j(n+2)})}{\chi_0(p^{j(n+2)})} X^{n+2}$$

$$= a(1) + \frac{a(p^j)}{\chi_0(p^j)} X + \sum_{n \geq 0} \frac{\lambda_j(p)}{\chi_0(p^j)} \frac{a(p^{j(n+1)})}{\chi_0(p^{j(n+1)})} X^{n+2}$$

$$- \sum_{n \geq 0} p^{j(k-1)} \chi_j(p) \frac{a(p^{jn})}{\chi_0(p^j) \chi_0(p^{jn})} X^{n+2}.$$
Since $\chi = \chi_0^2$, then
\[
S = a(1) + \frac{a(p^j)}{\chi_0(p^j)}X + \frac{\lambda_j(p)}{\chi_0(p^j)}(S - a(1))X - p^{j(k-1)}SX^2.
\]
Hence
\[
S = a(1) + \frac{\left(\frac{a(p^j) - a(1)\lambda_j(p)}{\chi_0(p^j)}\right)X}{1 - \frac{\lambda_j(p)}{\chi_0(p^j)}X + p^{j(k-1)}X^2}.
\]
Replacing $n = 0$ in (3), we obtain $a(p^j) = a(1)\lambda_j(p) = \lambda_j(p)$. This proves the Lemma.

**Proof (Proof of Theorem [1])** Let $f \in S_k(G_0(N))$, $\chi$ be a normalized Hecke eigenform of even integral weight $k$ and level $N$, with Dirichlet character $\chi$. Let $\chi_0 \pmod{N}$ be a Dirichlet character such that $\chi_0^2 = \chi$. Let $j$ be an integer. It is well known that $\forall n \in \mathbb{N}$, $a(n) = \chi(n)a(n) = \chi_0^2(n)a(n)$. Let $p$ be a prime, $p \nmid N$. Then $\chi_0(p^j) \neq 0$ and the above equation implies $a(p^j) = \chi_0(p^j)a(p^j)$. Hence
\[
\frac{a(p^j)}{\chi_0(p^j)} = \frac{\overline{a(p^j)}}{\chi_0(p^j)},
\]
from which we obtain $a(p^j)/\chi_0(p^j) \in \mathbb{R}$. Suppose that the sequence $\left(\frac{a(p^n)}{\chi_0(a(p^n))}\right)_{n \geq 0}$ does not have infinitely many sign changes.

Applying Landau’s theorem, we deduce that the Dirichlet series
\[
\sum_{n \geq 0} \frac{a(p^n)}{\chi_0(p^n)}p^{-sn}(\Re(s) \gg 1),
\]
either has a pole on the real point of its line of convergence or must converges for all $s \in \mathbb{C}$. We will disprove the both assertions when $p$ is large.

We start by considering the first case. Since $f$ is a normalized Hecke eigenform, we have $a(p^n) = a(p)a(p^{n-1}) - \chi(p)p^{k-1}a(p^{n-1})$ for all integers $n \in \mathbb{N}$. Taking this and applying the similar computations of Lemma [1] we get
\[
P(X) = \sum_{n \geq 0} \frac{a(p^n)}{\chi_0(p^n)}X^n = \frac{1}{1 - \frac{a(p)}{\chi_0(p)}X + p^{k-1}X^2}.
\]
The denominator of the right-hand side of (5) factorizes as
\[
1 - \frac{a(p)}{\chi_0(p)}X + p^{k-1}X^2 = (1 - \alpha_p X)(1 - \beta_p X),
\]
where
\[
\alpha_p, \beta_p = \frac{a(p)}{\chi_0(p)} \pm \sqrt{\left(\frac{a(p)}{\chi_0(p)}\right)^2 - 4p^{k-1}}.
\]
Applying Deligne’s bound, $\left(\frac{a(p)}{\chi_0(p)}\right)^2 \leq |a(p)|^2 \leq 4p^{k-1}$, since $\left(\frac{a(p)}{\chi_0(p)}\right) \in \mathbb{R}$. We deduce that $\alpha_p$ and $\beta_p$ are complex conjugates numbers $\beta_p = \overline{\alpha_p}$. 
Let $\zeta := e^{2\pi i/j}$ be a primitive $j$-th root of unity and let $\nu \in \mathbb{Z}$. The following orthogonality relation
\[
\sum_{\mu=0}^{j-1} \zeta^{\mu \ell} = \begin{cases} j, & \text{if } \ell \equiv 0 \pmod{j}, \\ 0, & \text{if } \ell \not\equiv 0 \pmod{j}, \end{cases}
\]
implies
\[
\sum_{n \geq 0} a(p^{nj}) \chi_0(p^{nj}) = \frac{1}{j} \sum_{\mu=0}^{j-1} P(\zeta^{\mu} X).
\]
Replacing $X = p^{-s}$ ($s \in \mathbb{C}$), we get
\[
\sum_{n \geq 0} a(p^{nj}) \chi_0(p^{nj}) p^{-jn s} = \frac{1}{j} \sum_{\mu=0}^{j-1} \frac{1}{(1 - \zeta^\mu \alpha_p p^{-s})(1 - \zeta^\mu \beta_p p^{-s})} \quad (\Re(s) \gg 1). \quad (8)
\]
Notice that using (8), the Dirichlet series
\[
\sum_{n \geq 0} a(p^{nj}) \chi_0(p^{nj}) p^{-jn s}
\]
can be meromorphically extended to the whole complex plane $\mathbb{C}$.

Suppose now that one of the denominators on the right-hand side of (8) has a real zero, for example $\alpha_p \zeta^\mu \in \mathbb{R}$. Then $\alpha_p \zeta^\mu = \nu \in \mathbb{R}$. This implies $\zeta^\mu \cdot \nu = \nu$, and using (7) we get $\nu^2 = |\alpha_p|^2 = p^{k-1}$. Therefore $\nu = \pm p^{(k-1)/2}$. It follows that
\[
a(p) = (\alpha_p + \beta_p) \chi_0(p) = \pm p^{(k-1)/2} (\zeta^{-\mu} + \zeta^\mu) \chi_0(p).
\]
We get the same result if we start with the condition that $\beta_p \zeta^\mu$ is real.

Suppose, for the sake of contradiction, there are infinitely many primes $p$ for which there are integers $\mu_p \pmod{j}$ such that
\[
a(p) = \pm p^{(k-1)/2} (\zeta^{-\mu} + \zeta^\mu) \chi_0(p). \quad (9)
\]
It is well known that
\[K_f := \mathbb{Q}(\{a(p)\}_p),\]
the subfield of $\mathbb{C}$ generated by all $a(p)$, where $p$ runs on primes, is a number field. Particularly, it is a finite extension of $\mathbb{Q}$. Therefore, the field $K_f(\zeta)$ is also a finite extension of $\mathbb{Q}$. Let $K$ denote the field obtained by adjoining all $\chi_0(n)$, $\forall n \in \mathbb{N}$, to the field $K_f(\zeta)$. The field $K$ is also a number field and particularly, a finite extension of $\mathbb{Q}$. From (9) and since $k$ is even, we see
\[
\sqrt{p} \in K.
\]
By our hypothesis, we conclude that there are infinitely many primes $p_1 < p_2 < p_3 \ldots$ satisfying
\[\mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3}, \ldots) \subset K.
\]
However, it is a classical fact that the degree of the extension
\[\mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3}, \ldots)/\mathbb{Q}\]
is infinite, which gives our contradiction. Consequently, we have proved that, for 
almost all primes $p$, the right-hand side of (8) has no real poles.

It remains to exclude the second case of Landau’s theorem. Suppose that for 
a prime $p$, the series converges everywhere, and particularly, it is an entire 
function in $s$. By (1) of Lemma we see that $f$ is an eigenfunction of $T_j(p)$. Let 
$\lambda_j(p)$ be the corresponding eigenvalue, hence from (2) of Lemma we get

$$
\sum_{n \geq 0} \frac{a(p^{jn})}{\chi_0(p^{jn})} X^{jn} = \frac{1}{1 - \frac{\lambda_j(p)}{\chi_0(p^j)} X^j + p^{(k-1)}X^{2j}}.
$$

The denominator on the right-hand side is a polynomial in $X^j$ of degree 2, hence 
it is non-constant and so has zeros. Setting $X = p^{-s}$ to obtain a contradiction.

3 Proof of Theorem

Assume the hypothesis of Theorem 1. We want to compute the following sum

$$
S_1(X) = \sum_{n=0}^{\infty} \frac{a(p^{1+2n})}{\chi_0(p^{1+2n})} X^{1+2n},
$$

By the same reasoning as in (8) we have

$$
S_0(X) = \sum_{n \geq 0} \frac{a(p^{2n})}{\chi_0(p^{2n})} X^{2n} = \frac{1}{2} \sum_{\mu=0}^{1} \frac{1}{(1 - (-1)^\mu \alpha_p X)(1 - (-1)^\mu \beta_p X)}
= \frac{1 + \alpha_p \beta_p X^2}{(1 - \alpha_p^2 X^2)(1 - \beta_p^2 X^2)}.
$$

Since $S_1(X) = P(X) - S_0(X)$, we obtain

$$
S_1(X) = \frac{(\alpha_p + \beta_p)X}{(1 - \alpha_p^2 X^2)(1 - \beta_p^2 X^2)}.
$$

Now, let $j \geq 1$ be an integer. Let $S_{1,j}$ denote the following sum

$$
S_{1,j}(X) = \sum_{n=0}^{\infty} \frac{a(p^{j(1+2n)})}{\chi_0(p^{j(1+2n)})} X^{j(1+2n)}.
$$

Assume further that the integer $j \geq 1$ satisfy $(j,2) = 1$. Once again, let $\zeta := e^{2\pi i/j}$ be a primitive $j$-th root of unity and let $\nu \in \mathbb{Z}$. The orthogonality relation

$$
\sum_{\mu=0}^{j-1} \zeta^{\mu \nu} = \begin{cases} j, & \text{if } \nu \equiv 0 \pmod{j}, \\ 0, & \text{if } \nu \not\equiv 0 \pmod{j}, \end{cases}
$$

implies

$$
S_{1,j}(X) = \frac{1}{j} \sum_{\mu=0}^{j-1} S_1(\zeta^\mu X).
$$
Proof (Proof of Theorem 2)
Assume the hypothesis of Theorem 1 and take , \( j \neq 1 \), \( \chi = 1 \), \( \chi_0 = 1 \). Replacing this in (14) to obtain

\[
S_{1,j}(X) = \sum_{n=0}^{\infty} a(p^{j(1+2n)})X^{j(1+2n)} = \frac{1}{j} \sum_{\mu=0}^{j-1} S_1(\zeta^{j\mu}X),
\]

where

\[
S_1(\zeta^{j\mu}X) = \frac{(\alpha_p + \beta_p)\zeta^{j\mu}X}{(1 - \alpha_p^2\zeta^{2j\mu}X^2)(1 - \beta_p^2\zeta^{2j\mu}X^2)}.
\]

Replacing \( X = p^{-s} \) ( \( s \in \mathbb{C} \) ), we obtain

\[
\sum_{n=0}^{\infty} a(p^{j(1+2n)}) \frac{1}{p^{s(j(1+2n))}} = \frac{\alpha_p + \beta_p}{jp^s} \sum_{\mu=0}^{j-1} \frac{\zeta^{j\mu}}{(1 - \alpha_p^2\zeta^{2j\mu}p^{s(2j)})(1 - \beta_p^2\zeta^{2j\mu}p^{s(2j)})}.
\]

Using this formula, the Dirichlet series

\[
\sum_{n=0}^{\infty} a(p^{j(1+2n)}) \frac{1}{p^{s(j(1+2n))}}
\]

can be meromorphically extended to the whole complex plane \( \mathbb{C} \). Suppose that the sequence \((a(p^{j(1+2n)}))_{n \in \mathbb{N}}\) does not have infinitely many sign changes for infinitely many primes \( p \) and apply once again Landau’s theorem.

Suppose now that one of the denominators on the right-hand side of (18) has a real zero, for example \( \alpha_p\zeta^{j\mu} \in \mathbb{R} \). Then as in the proof of Theorem 1 we find

\[
a(p) = (\alpha_p + \beta_p)\chi_0(p) = \pm p^{(k-1)/2}(\zeta^{-j\mu} + \zeta^{j\mu}).
\]

We repeat the procedure of Theorem 1 to show that the right-hand side of (18) has no real poles, and then the first case of Landau’s theorem is excluded.

It remains to exclude the second case of Landau’s theorem. By Lemma 1 we have

\[
S_{1,j}(X) = \sum_{n=0}^{\infty} a(p^{jn})X^{jn} - \sum_{n=0}^{\infty} a(p^{2jn})X^{2jn}
\]

\[
= \frac{p^{2j(k-1)}X^{4j} - (a(p^{2j}) + p^{j(k-1)})X^{2j} - a(p^{j})X^{j}}{(1 + a(p^{j})X^{j} + p^{j(k-1)}X^{2j}) (1 - a(p^{2j})X^{2j} + p^{2j(k-1)}X^{4j})}
\]

The numerator on the right-hand side is a polynomial of degree \( 4j \) and the denominator is a non constant polynomial of degree \( 6j \), hence the denominator has zeros. Setting \( X = p^{-s} \) to obtain a contradiction.
4 Proof of Theorem 3

Proof The proof is similar to the one of [4 Theorem 2.2], it suffices to make the following change, the set \( V_p \subseteq S_k \) is defined to be the set of all cusp forms \( g \) whose Fourier coefficients \( b(p^{l+2n}) \) satisfy \( b(p^{l+2n}) \ll_{g,c} p^{l+2n} \) for all \( n \geq 0 \) and every \( c \in \mathbb{R} \). The first part of the proof remains unchanged. Now, \( V_p \) is stable under \( T_j(p^2) \), then by the same argument, there is an eigenform \( f_0 \in V_p \) of \( T_j(p^2) \) since this operator is Hermitian.

From this and since \( P(T_j(p), X) \) is irreducible, we deduce that there is \( \lambda \neq 0 \) such that \( T_j(p^2)f_0 = \lambda f_0 \). We should note that \( T_j(p)^h_1 = \sqrt{\lambda h_1} \) and \( T_j(p)^h_2 = -\sqrt{\lambda h_2} \) where \( h_1 = \sqrt{\lambda}f_0 + T_j(p)f_0 \) and \( h_2 = -\sqrt{\lambda}f_0 + T_j(p)f_0 \). Then by our hypothesis, either \( h_1 = 0 \) or \( h_2 = 0 \). Suppose without loss of generality that \( h_2 = 0 \) and \( T_j(p)f_0 = \sqrt{\lambda}f_0 \). We can now proceed as in the proof of [4 Theorem 2.2] to deduce that \( f_0 \) is an eigenfunction of all Hecke operators. Finally we apply (40) to \( f_0 \) and the second case of Landau’s theorem is excluded.

5 Sign changes of the sequence \( \left( \frac{a(p^{l+m} \chi(n))}{\chi(n)} \right)_{n \in \mathbb{N}} \)

Finally, by modifying the method above one can obtain the following result.

Theorem 4 Let \( f \in S_k(G_0(N), \chi) \) be a normalized Hecke eigenform of even integral weight \( k \) and level \( N \), with Dirichlet character \( \chi \). Let \( \chi_0 \) (mod \( N \)) be a Dirichlet character satisfying \( \chi_0^2 = \chi \). Let

\[
f(z) = \sum_{n \geq 1} a(n)e(nz),
\]

be the Fourier expansion of \( f \) at \( \infty \). Consider the primes \( p \) for which the polynomial \( (\beta_p \alpha_p - \alpha_p\beta_p)X^m + (\beta_p^m - \alpha_p^m)X^m+1 + (\alpha_p - \beta_p) \) has no real zero, where \( m_p \) is an integer satisfying \( \chi_0(p)^{m_p} = 1 \). Then for almost all of those primes \( p \), the sequence \( \left( \frac{a(p^{l+m} \chi(n))}{\chi(n)} \right)_{n \in \mathbb{N}} \) has infinitely many sign changes with \( l \) runs through the integers satisfying \( 1 \leq l \leq m_p - 1 \).

Remark 1 Notice that for those sequences, \( \{Re(a(p^{l+m} \chi(n)))\}_{n \in \mathbb{N}} \) (resp. \( \{Im(a(p^{l+m} \chi(n)))\}_{n \in \mathbb{N}} \) ) has infinitely many sign changes when \( \chi_0(p)^l \neq \pm i \) (resp. \( \chi_0(p)^l = \pm i \)).

Before giving the proof we shall establish some needed formulas in the full generality. Assume the conditions of Theorem 4. Let \( \omega := e^{2\pi i/m} \) be a primitive \( m \)-th root of unity of order \( m \) and \( \chi_0(p)^m = 1 \). We want to compute the following sum

\[
S_l = \sum_{n=0}^{\infty} \frac{a(p^{l+m} \chi(n))}{\chi_0(p^{l+m})} \chi^{l+m},
\]

where \( l \) is an integer satisfying \( 0 \leq l \leq m - 1 \). By (39), we have

\[
a(p^{l+m}) = a(p)a(p^{l-1+m}) - p^{k-1} \chi(p)a(p^{l+2+m}),
\]
this yields

$$S_l = a(p) S_{l-1} X - p^{k-1} X^2 S_{l-2}.$$  \(21\)

On the other hand, we have

$$S_0 + \cdots + S_{m-1} = P = \sum_{n \geq 0} a(p^n) \chi_0^n = \frac{1}{1 - a(p) \chi_0(p) X + p^{k-1} X^2}.$$  \(22\)

From \(21\), we get

$$S_l = (a \alpha_l + b \beta_l) X^l,$$  \(23\)

where \(a\) and \(b\) are terms depending upon \(X\) which will be computed.

By the same reasoning as in \(8\) we have

$$S_0 = \sum_{n \geq 0} a(p^{mn}) \chi_0(p^{mn}) X^{mn} = \frac{1}{m} \sum_{\mu=0}^{m-1} \frac{1}{(1 - \omega^\mu \alpha_p X)(1 - \omega^\mu \beta_p X)}.$$  \(24\)

Hence by \(23\), we have

$$a + b = S_0 = \frac{1}{m} \sum_{\mu=0}^{m-1} \frac{1}{(1 - \omega^\mu \alpha_p X)(1 - \omega^\mu \beta_p X)}.$$  \(25\)

Combine now the equations \(22\) and \(23\) to get

$$S_0 + \cdots + S_{m-1} = a \sum_{l=0}^{m-1} \alpha_l X^l + b \sum_{l=0}^{m-1} \beta_l X^l.$$  \(26\)

$$= a \frac{\alpha^m X^m - 1}{\alpha_p X - 1} + b \frac{\beta^m X^m - 1}{\beta_p X - 1}.$$  \(27\)

$$= P.$$  \(28\)

From this and \(23\) we obtain

$$R(X)a = P - S_0 \frac{\beta^m X^m - 1}{\beta_p X - 1},$$  \(29\)

$$R(X)b = S_0 \frac{\alpha^m X^m - 1}{\alpha_p X - 1} - P.$$  \(30\)

where

$$R(X) = \left( \frac{\alpha^m X^m - 1}{\alpha_p X - 1} \right) - \left( \frac{\beta^m X^m - 1}{\beta_p X - 1} \right).$$

Replacing this in \(23\), then
\[ S_l = \frac{\alpha_p X^l (P - S_0 \beta_p^m X^{m-1}) + \beta_p X^l (S_0 \alpha_p^m X^{m-1} - P)}{R(X)} \]  
(31)

\[ = \frac{\alpha_p X^l (P(\beta_p X - 1) - S_0(\beta_p^m X^{m-1} - 1)) + \beta_p X^l (S_0(\alpha_p^m X^{m-1} - P(\alpha_p X - 1))}{(\beta_p \alpha_p^m - \alpha_p \beta_p^m)X^m + (\beta_p^m - \alpha_p^m)X^{m-1} + (\alpha_p - \beta_p)} \]  
(32)

Notice that using (32), the Dirichlet series
\[ \sum_{n=0}^{\infty} \frac{a(p^l+m\alpha)}{\chi_0(p^l+m\alpha)} p^{-s(l+m\alpha)} \]  

can be meromorphically extended to the whole complex plane \( \mathbb{C} \).

**Proof (Proof of Theorem 4)**

Suppose that the sequence \( \left( \frac{a(p^l+m\alpha)}{\chi_0(p^l+m\alpha)} \right) \) does not have infinitely many sign changes.

Applying Landau’s theorem, we deduce that the Dirichlet series
\[ \sum_{n=0}^{\infty} \frac{a(p^l+m\alpha)}{\chi_0(p^l+m\alpha)} p^{-s(l+m\alpha)} \]  
(33)

either has a pole on the real point of its line of convergence or must converges for all \( s \in \mathbb{C} \). We start by considering the first case.

Since the denominator of (32) has no real pole by hypothesis, then either the denominator of \( P(X) \) or one of the denominators of \( \beta_p \) has real zero. We deduce that in all cases \( \sqrt{p} \in \mathbb{R} \). The contradiction is obtained by the same way as above. Consequently, for almost all primes \( p \) satisfying the hypothesis, the right-hand side of (33) has no real poles. We exclude the second case of Landau’s theorem by using the both equations (32) and (10).

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