Resolving domination number of helm graph and it’s operation

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Abstract. Let \( G \) be a connected graph. Dominating set is a set of vertices which each vertex \( D \) has at least one neighbor in \( G \). The minimum cardinality of \( D \) is called the domination number \( G (\gamma(G)) \). The metric dimension of \( G \) is the minimum cardinality of a series of vertices so that each vertex \( G \) is uniquely. It is determined by the distance of vector to the selected vertices. A dominating metric dimension set is a set of vertices has a dominating set \( D \) which has condition of metric dimension. The minimum cardinality is called the resolving domination number of \( G \), \( (Dom_{Dim}(G)) \). We analyze the resolving domination number of helm graph and it’s operation. We study combine the existence concept of dominating set and metric dimension. We have obtained the minimum cardinality of dominating number.

1. Introduction

Let \( G \) be a graph with vertex set \( V (G) \) and edge set \( E(G) \). Let the vertex set of \( G \) be \( W = \{s_1, s_2, \ldots, s_k\} \). Representation of metric dimension is \( r(v|W) = (d(v, s_1), d(v, s_2), \ldots, d(v, s_k)) \). The \( k \)-vector \( r(v|W) = (d(v, s_1), d(v, s_2), \ldots, d(v, s_k)) \). The minimum cardinality of resolving set \( W \) of \( G \) is denoted by \( \text{dim}(G) \). It is called the metric dimension of \( G \) \[13\]. The metric dimension, \( \text{dim}(G) \) of a graph \( G \) is the minimum cardinality of a set of vertices such that every vertex of \( G \) is uniquely. It is determined by the distances of vector to the every choosen vertex. A dominating set \( D \) is a set of vertices such that each vertex of \( G \) has at least one neighbor in \( G \). The minimum cardinality of the domination number of \( G \) is \( (\gamma(G)) \). Based on these two concept dominating set and metric dimension, we initiate to study the combination of those two studies namely \( (Dom_{Dim}(G)) \). We analyze the \( (Dom_{Dim}(G)) \) of helm graph. These are some definition and preposition which is usefull to proof some theorems. We are develop a graph that will become a new graph (line graph, middle graph and total graph). A line graph from a simple graph namely of \( G \) denoted by \( L(G) \) is obtained by associating vertices with each edge of the graph and connecting two vertices with edges if the corresponding edges of \( G \) have the same node \[11\]. In \[18\] the middle graph denoted by \( M(G) \) of the connected graph \( G \) is a graph whose node-set is \( V(G) \cup E(G) \) where two vertices are close together if they are edges which border \( G \) or one is the node of \( G \) and the other is an edge incident with node. The total graph denoted
by $T(G)$ of the connected graph $G$ is a graph whose node-set is $V(G) \cup E(G)$ and two adjacent vertices each time that border or events in $G$ [7].

**Definition 1.1.** The Helm graph $(H_m)$ is a simple graph obtained from the $n$-wheel $W_m$ graph next to the edge of the pendant at each vertex of the $C_m$ cycle.

**Proposition 1.** Let $H_m$ be a helm graph, the domination number of graph $H_m$ is $m$, $\gamma(H_m) = m$.

We will explained about the operation resolving domination number of line graph, middle graph, and total graph. The minimum cardinality is called the resolving domination number $G, (\text{DomDim}(G))$. In this study, resolving domination number will be examined on fan graphs, line graphs, star graphs, and complete graphs. In addition, the development of the graph will become a new graphs (line graph, middle graph and total graph). Line graph $L (G)$ (also called adjoin, cover, derivative, derived, edge, edge-to-vertex point, interchange, representative, or $\theta$-obrazom graph) of a simple graph $G$ is obtained by associating a vertex with each edge of the graph and connecting two vertices with an edge if the corresponding edges of $G$ have a vertex in common[17]. The total graph $T (G)$ from the graph $G$ is a graph whose set of vertices is $V (G) \cup E (G)$ and two vertices side by side each time it borders or occurs in $G$. The middle graph of a connected graph $G$ denoted by $M(G)$ is the graph whose vertex set is $V (G) \cup E(G)$ where two vertices are adjacent if adjacent edges of $G$ or one is a vertex of $G$ and the other is an edge incident with it.[19]

2. **Resolving Domination Number**

The metric dimension of graph $G$ is the minimum cardinality of a set of vertices so that each $G$ is uniquely determined by its distance vector to the selected vertex. The set of vertices which is both and dominating is called resolving dominating set. Let we denoted the resolving dominating set as $D_r$. The minimum cardinality of resolving dominating set is called the resolving domination number denoted by $\gamma_r(G)$. Let we take an example in $H_4$. We can verify a resolving dominating sets such as $D_r = \{x_1, x_2, x_3, x_4\}$. In order to proof $D_r$ is resolving dominating set, we let see the neighborhood $V - D_r$. The set of vertices which is not included in dominator resolving set is $V - D_r = \{y_1, y_2, y_3, y_4, y_5\}$. We can see that $x_1$ dominated $(y_1, x_2, y_5, x_4)$, $x_2$ dominated $(x_1, y_2, y_5, x_3)$, $x_3$ dominated $(x_2, y_3, x_4, y_5)$, $x_4$ dominated $(x_1, x_3, y_4, y_5)$. The representation of $v \in V(H_4)$ connect to $D_r$ are: $r(y_1|D_r) = \{1, 2, 3, 2\}$, $r(y_2|D_r) = \{2, 1, 2, 3\}$, $r(y_3|D_r) = \{3, 2, 1, 2\}$, $r(y_4|D_r) = \{2, 3, 3, 1\}$, $r(y_5|D_r) = \{1, 1, 1, 1\}$, $r(x_1|D_r) = \{0, 1, 1, 2\}$, $r(x_2|D_r) = \{1, 0, 1, 2\}$, $r(x_3|D_r) = \{2, 1, 0, 1\}$, $r(x_4|D_r) = \{1, 2, 1, 0\}$.

Based on the results, the resolving dominating sets dominated all vertices in $H_4$ and also the representation of vertices in $H_4$ respect to $D_r$ are distinct. We can conclude that $D_r$ is resolving dominating set. 4 is the minimum cardinality of resolving dominating set. It can be concluded that $\gamma_r(H_4) = 4$. The illustration of resolving dominating set of $H_4$ can be seen in figure 1.

3. **Results**

We are show the results of resolving domination number of $H_m$

**Theorem 3.1.** Given that $H_m$ be a helm graph with $m \geq 3$, resolving domination number of $H_m$ is $\gamma_r(H_m) = m$.

**Proof.** The $H_m$ is a graph on $2m + 1$ vertex and $3m$ edge with vertex set $V(H_m) = \{a_i : 1 \leq i \leq m\} \cup \{b_i : 1 \leq i \leq m\} \cup \{C\}$ and edge set $E(H_m) = \{a_i b_i : 1 \leq i \leq m\} \cup \{b_i b_{i+1} : 1 \leq i \leq m-1\} \cup \{b_i C : 1 \leq i \leq m\}$.

For to prove the resolving domination number of helm graph $H_m$, we will prove the lower bound of resolving domination number is $m$ thus $\gamma_r(H_m) \geq m$ and the upper bound of resolving domination number is $m$ which is $\gamma_r(H_m) \leq m$. 4
Figure 1. Resolving Dominating Set of $H_4$

Table 1. The Representation of Resolving Domination Number

| $v$  | $r(v) | D_r$ | condition            |
|------|---------|---------------------|
| $x_i$ | $(3, \ldots, 3), 1, 3, \ldots, 3$ | $3 \leq i \leq m - 1$ |
| $x_i$ | $(1, 2, 3, \ldots, 3), 2$ | $i = 1$ |
| $x_i$ | $(2, 1, 2, 3, \ldots, 3), 2$ | $i = 2$ |
| $x_i$ | $(2, 3, \ldots, 3), 2, 1$ | $i = 3$ |

In this section we proposed the proof of lower and upper bound of resolving domination number on helm graph. We will prove that the lower bound of of resolving domination number is $m$, which is $\gamma_r(H_m) \geq m$. It has been explained that the resolving dominating set of helm graph fulfill the requirement of dominating set and resolving set. Assume that $\gamma_r(H_m) < m$. We take $|D_r| = m - 1$ where $w_i \in D_r$. Based on preposition 1, we know that $\gamma_r(H_m) = m$. If we have $|D_r| = m - 1$, it means that there will be a set of vertices which is not dominated by $D_r$. Thus, it is a contradiction. It contradict with the definition of resolving domination number which should fulfill the requirement of dominating set and resolving set. Furthermore, we should have $m$ vertices to dominate all of the vertices in helm graph $H_m$. Thus, we should have minimum $m$ vertices to be the resolving domination number. Then, we should check whether the representation of all vertices respect to $D_r$ are distinct or not to check the upper bound.

Furthermore, we will prove that the upper bound of resolving dominating set of $H_m$ is $m$, which is $\gamma_r(H_m) \leq m$. Choose the edge metric generator $S = \{x_i, 1 \leq i \leq m\}$ so the representation of all vertices $v \in V(H_m)$ respect to $D_r$ can be seen in the Table 1. Based on Table 1, all vertices representation of $H_m$ with respect to $D_r$ are distinct, so $D_r$ is the dominating metric generator of $H_m$ with the cardinality of $D_r$ namely is $|D_r| = m$. So, the upper bound of the resolving domination number of $H_m$ is $\gamma_r(H_m) \leq m$. It has been proved that the lower bound of resolving domination number is $m$ and the upper bound of resolving domination number is $n$. It can be concluded that $\gamma_r(H_m) = m$.

**Theorem 3.2.** Let $L(H_m)$ be a line $(H_m)$ with $m \geq 3$, resolving domination number of $L(H_m)$ is $\gamma_r(L(H_m)) = m$
Figure 2. Line Helm Graph of $H_4$

Proof. The $(H_m)$ is a graph on $3m$ vertices and $5m + 6$ edges with vertex set $V(H_m) = \{a_i : 1 \leq i \leq m\} \cup \{b_i : 1 \leq i \leq m\} \cup \{C_i, C_{i+1}\}$ and edge set $E(LH_m) = \{a_i b_i : 1 \leq i \leq m\} \cup \{b_i b_{i+1} : 1 \leq i \leq m-1\} \cup \{b_m b_1\} \cup \{b_i C : 1 \leq i \leq m\}$.

For to prove the resolving domination number of helm graph $H_m$ is $\gamma_r(H_m) = m$, we will prove the lower bound of resolving domination number is $m$ thus $\gamma_r(H_m) \geq m$ and the upper bound of resolving domination number is $m$ which is $\gamma_r(H_m) \leq m$.

In this section we proposed the proof of lower and upper bound of resolving domination number on helm graph. We will prove that the lower bound of of resolving domination number is $m$, which is $\gamma_r(H_m) \geq m$. It has been explained that the resolving dominating set of helm graph fulfill the requirement of dominating set and resolving set. Assume that $\gamma_r(H_m) < m$.

We take $|D_r| = m - 1$ where $w_i \in D_r$. Based on preposition 1, we know that $\gamma_l(H_m) = m$. If we have $|D_r| = m - 1$, it means that there will be a set of vertices which is not dominated by $D_r$. Thus, it is a contradiction. It contradict with the definition of resolving domination number which should fulfill the requirement of dominating set and resolving set. Furthermore, we should have $m$ vertices to dominate all of the vertices in helm graph $H_m$. Thus, we should have minimum $m$ vertices to be the resolving domination number. Then, we should check whether the representation of all vertices respect to $D_r$ are distinct or not to check the upper bound.

Furthermore, we will prove that the upper bound of resolving dominating set of $H_m$ is $m$, which is $\gamma_r(H_m) \leq m$. Choose the edge metric generator $S = \{x_i, 1 \leq i \leq m\}$ so the representation of all vertices $v \in V(H_m)$ respect to $D_r$ can be seen in the Table 2. Based on Table 2, all vertices representation of $H_m$ with respect to $D_r$ are distinct, so $D_r$ is the dominating metric generator of $H_m$ with the cardinality of $D_r$ namely is $|D_r| = m$. So, the upper bound of the resolving domination number of $H_m$ is $\gamma_r(H_m) \leq m$. It has been proved that the lower bound of resolving domination number is $m$ and the upper bound of resolving domination number is $m$. It can be concluded that $\gamma_r(H_m) = m$.

Theorem 3.3. Let $M(H_m)$ be a middle $(H_m)$ with $m \geq 3$, resolving domination number of $M(H_m)$ is $\gamma_r(M(H_m)) = m + 1$

Proof. The middle $(H_m)$ is a graph on $5m + 1$ vertex and $3m$ edge. For to prove resolving
The Representation of Resolving Domination Number

| v       | r(v)[D_r] | condition          |
|---------|-----------|--------------------|
| x_i    | (3, ..., 3), 1, 3, ..., 3 | 3 ≤ i ≤ m - 1     |
| x_i    | (1, 2, 3, ..., 3), 2   | i = 1             |
| x_i    | (2, 1, 2, 3, ..., 3), 2 | i = 2             |
| x_i    | (2, 3, ..., 3), 2, 1   | i = 3             |

domination number of helm graph \( M(H_m) \) is \( \gamma_r(M(H_m)) = m + 1 \). In here will prove that lower bound of resolving domination number is \( m + 1 \) so \( \gamma_r(M(H_m)) \geq n + 1 \) and upper bound of dominating metric dimension number is \( m \) which is \( \gamma_r(M(H_m)) \leq m + 1 \).

For to prove the resolving domination number of helm graph \( H_m \) is \( \gamma_r(H_m) = m + 1 \), we will prove the lower bound of resolving domination number is \( m + 1 \) thus \( \gamma_r(H_m) \geq m + 1 \) and the upper bound of resolving domination number is \( m \) which is \( \gamma_r(H_m) \leq m + 1 \).

In this section we proposed the proof of lower and upper bound of resolving domination number on helm graph. We will prove that the lower bound of of resolving domination number is \( m + 1 \), which is \( \gamma_r(H_m) \geq m + 1 \). It has been explained that the resolving dominating set of helm graph fulfill the requirement of dominating set and resolving set. Assume that \( \gamma_r(H_m) < m + 1 \). We take \( |D_r| = m - 1 \) where \( w_i \in D_r \). Based on preposition 1, we know that \( \gamma(H_m) = m + 1 \). If we have \( |D_r| = m - 1 \), it means that there will be a set of vertices which is not dominated by \( D_r \). Thus, it is a contradiction. It contradict with the definition of resolving domination number which should fulfill the requirement of dominating set and resolving set. Furthermore, we should have \( m + 1 \) vertices to dominate all of the vertices in helm graph \( H_m \). Thus, we should have minimum \( m + 1 \) vertices to be the resolving domination number. Then, we should check whether the representation of all vertices respect to \( D_r \) are distinct or not to check the upper bound.

Then, we will prove that the upper bound of resolving dominating set of \( H_m \) is \( m + 1 \), which is \( \gamma_r(H_m) \leq m + 1 \) and \( \gamma_r(H_m) \leq m + 1 \). Choose the edge metric generator \( S = \{ x_i, 1 \leq i \leq m \} \) so the representation of all vertices \( v \in V(H_m) \) respect to \( D_r \) can be seen in the Table 3. Based on Table 3, all vertices representation of \( H_m \) with respect to \( D_r \) are distinct, so \( D_r \) is the dominating metric generator of \( H_m \) with the cardinality of \( D_r \) namely is \( |D_r| = m + 1 \). So, the upper bound of the resolving domination number of \( H_m \) is \( \gamma_r(H_m) \leq m + 1 \). It has been proved that the lower bound of resolving domination number is \( m + 1 \) and the upper bound of resolving domination number is \( n \). It can be concluded that \( \gamma_r(H_m) = m + 1 \).

**Theorem 3.4.** Let \( T(H_m) \) be a total \( (H_m) \) with \( m \geq 3 \), resolving domination number \( T(H_m) \) is \( \gamma_r(T(H_m)) = m + 1 \)

**Proof.** The middle \( (H_m) \) is a graph on \( 5m + 1 \) vertex and \( 3m \) edge. For to prove resolving domination number of helm graph \( T(H_m) \) is \( \gamma_r(T(H_m)) = m + 1 \). We will prove that lower bound of resolving domination number is \( m + 1 \) so \( \gamma_r(T(H_m)) \geq n + 1 \) and upper bound of resolving domination number is \( n \) which is \( \gamma_r(T(H_m)) \leq m + 1 \).

For to prove the resolving domination number of helm graph \( H_m \) is \( \gamma_r(H_m) = m + 1 \), we will prove the lower bound of resolving domination number is \( m + 1 \) thus \( \gamma_r(H_m) \geq m + 1 \) and the upper bound of resolving domination number is \( n \) which is \( \gamma_r(H_m) \leq m + 1 \).

We are proposed the proof of lower and upper bound of resolving domination number on helm graph. We will prove that the lower bound of of resolving domination number is \( m + 1 \),
which is $\gamma_r(H_m) \geq m + 1$. It has been explained that the resolving dominating set of helm graph fulfill the requirement of dominating set and resolving set. Assume that $\gamma_r(H_m) < m + 1$. We take $|D_r| = m - 1$ where $v_i \in D_r$. Based on preposition 1, we know that $\gamma(H_m) = m + 1$. If we have $|D_r| = m - 1$, it means that there will be a set of vertices which is not dominated by $D_r$. Thus, it is a contradiction. It contradict with the definition of resolving domination number which should fulfill the requirement of dominating set and resolving set. Furthermore, we should have $m + 1$ vertices to dominate all of the vertices in helm graph $H_m$. Thus, we will have minimum $m + 1$ vertices to be the resolving domination number. Then, we should check whether the representation of all vertices respect to $D_r$ are distinct or not to check the upper bound. Furthermore, we will prove that the upper bound of resolving dominating set of $H_m$ is $m + 1$, which is $\gamma_r(H_m) \leq m + 1$. Choose the edge metric generator $S = \{x_i, 1 \leq i \leq m\}$ so the representation of all vertices $v \in V(H_m)$ respect to $D_r$ can be seen in the Table 3. Based on Table 3, all vertices representation of $H_m$ with respect to $D_r$ are distinct, so $D_r$ is the dominating metric generator of $H_m$ with the cardinality of $D_r$ namely is $|D_r| = m + 1$. So, the upper bound of the resolving domination number of $H_m$ is $\gamma_r(H_m) \leq m + 1$. It has been proved that the lower bound of resolving domination number is $m + 1$ and the upper bound of resolving domination number is $n$. It can be concluded that $\gamma_r(H_m) = m + 1$. 

**Table 3. The Representation of Resolving Domination Number**

| $v$          | $r(v|D_r)$ | condition          |
|--------------|------------|--------------------|
| $x_i$        | (3, \ldots, 3), 1, 3, \ldots, 3 | $3 \leq i \leq m - 1$ |
| $x_i$        | (1, 2, 3, \ldots, 3), 2 | $i = 1$ |
| $x_i$        | (2, 1, 2, 3, \ldots, 3), 2 | $i = 2$ |
| $x_i$        | (2, 3, \ldots, 3), 2, 1 | $i = 3$ |

Figure 3. Middle Helm Graph of $H_4$
Table 4. The Representation of Resolving Domination Number

| $v$       | $r(v|D_r)$                                      | Condition       |
|-----------|-----------------------------------------------|-----------------|
| $x_i$     | $(3,\ldots, 3), 1, 3,\ldots, 3$              | $3 \leq i \leq m-1$ |
|           | $^{i-2}_{i-2}$                                | $m-i-1$        |
| $x_i$     | $(1, 2, 3,\ldots, 3), 2$                      | $i = 1$        |
| $x_i$     | $(2, 1, 2, 3,\ldots, 3), 2$                   | $i = 2$        |
|           | $^{m-i-1}_{m-i-1}$                            |                |
| $x_i$     | $(2, 3,\ldots, 3), 2, 1$                     | $i = 3$        |
|           | $^{m-i-1}_{m-i-1}$                            |                |

4. Conclusion

We have given results of line graph, middle graph, and total graph on helm graph. Due to this research is still new, thus many problems must be found.

Open Problem 1. Given that $G$ be a graph, need to prove the dominating metric dimension any graph and its operations.

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References

[1] Adawiyah R, Agustin I H, Dafik, Slamin, and Albirri E R 2018 Related Wheel Graphs and Its Locating Edge Domination Number Journal of Physics: Conf. Series 1022
[2] Adawiyah R, Prihandini R M, Albirri E R, Agustin I H, Alfarisi R 2019 The local multiset dimension of unicyclic graph IOP Conference Series: Earth and Environmental Science 243 p 012075 IOP Publishing
[3] Alfarisi R, Darmaji, and Dafik 2017 On the star partition dimension of comb product of cycle and complete graph Journal of Physics Conference Series 855
[4] Alfarisi R and Darmaji 2017 On the star partition dimension of comb product of cycle and path AIP Conference Proceedings 1867
[5] Chartrand G, Eroh L, Johnson M A, and Oellermann O R 2000 Resolvability in Graphs and the Metric Dimension of a Graph Discrete Appl. Math. 105 pp 99-113
[6] Chartrand G and Lesniak L 2000 Graphs and digraphs 3rd (London: Chapman and Hall)
[7] Capobianco M and Molluzzo J 1978 Examples and Countereamples in Graph Theory (New York: North Holland) 2(3) 274
[8] Dafik, Agustin I H, Surahmat, Syafriyal S y and Alfarisi R 2017 On non-isolated resolving number of some graph operations Far East Journal of Mathematical Sciences 102(2) pp 2473-2492
[9] Dafik, Agustin I H, Hasan M, Adawiyah R, Alfarisi R, Wardani D A R 2018 On the Locating Edge Domination Number of Comb Product of Graphs Journal of Physics: Conf. Series 1022 012003
[10] Darmaji and Alfarisi R 2017 2017 On the partition dimension of comb product of path and complete graph AIP Conference Proceedings 1867
[11] Gross J L and Yellen J 2005 Graph Theory and it’s Application (CRC Press)
[12] Gross J L, Yellen J and Zhang P 2014 Handbook of graph Theory Second Edition (CRC Press Taylor and Francis Group)
[13] Harary F and Melter F A 1976 On The metric dimension of a graph Ars Combin 2 pp 191-195
[14] Harfield N and Ringel G 1994 Pearls in Graph Theory (United Kingdom: Academic Press)
[15] Okamoto F, Phinezy B, Zhang P 2010 The Local Metric Dimension Of A Graph Mathematica Bohemica 135(3) pp 239-255
[16] Slater P J 1975 Leaves of trees Proc. 6th Southeast Conf. Comb., Graph Theory, Comput. Boca Rotan 14 pp 549-559
[17] Simanjuntak R, Vetrik T, and Mulia P B 2017 The multiset dimension of graphs arXiv preprint arXiv 1711.00225
[18] Thenmozhi B and Prabha R 2017 Power Domination of Middle Graph of Path, Cycle and Star International Journal of Pure and Applied Mathematics 114 (5) 13-19
[19] Ullianita E, Estumingsih N, Susilowati L 2014 Dimensi Metrik Lokal pada Graf Hasil Kali Comb dari Graf Siklus dan Graf Lintasan Journal of Mathematics 1(3) pp 24-33