HOMOGENIZATION IN BV OF A MODEL FOR LAYERED COMPOSITES IN FINITE CRYSTAL PLASTICITY

ELISA DAVOLI, RITA FERREIRA, AND CAROLIN KREISBECK

Abstract. In this work, we study the effective behavior of a two-dimensional variational model within finite crystal plasticity for high-contrast bilayered composites. Precisely, we consider materials arranged into periodically alternating thin horizontal strips of an elastically rigid component and a softer one with one active slip system. The energies arising from these modeling assumptions are of integral form, featuring linear growth and non-convex differential constraints. We approach this non-standard homogenization problem via Gamma-convergence. A crucial first step in the asymptotic analysis is the characterization of rigidity properties of limits of admissible deformations in the space $\text{BV}$ of functions of bounded variation. In particular, we prove that, under suitable assumptions, the two-dimensional body may split horizontally into finitely many pieces, each of which undergoes shear deformation and global rotation. This allows us to identify a potential candidate for the homogenized limit energy, which we show to be a lower bound on the Gamma-limit. In the framework of non-simple materials, we present a complete Gamma-convergence result, including an explicit homogenization formula, for a regularized model with an anisotropic penalization in the layer direction.

MSC (2010): 49J45 (primary); 74Q05, 74C15, 26B30
Keywords: homogenization, $\Gamma$-convergence, linear growth, composites, finite crystal plasticity, non-simple materials.
Date: February 1, 2019.

1. Introduction

Metamaterials are artificially engineered composites whose heterogeneities are optimized to improve structural performances. Due to their special mechanical properties, arising as a result of complex microstructures, metamaterials play a key role in industrial applications and are an increasingly active field of research. Two natural questions when dealing with composite materials are how the effective material response is influenced by the geometric distribution of its components, and how the mechanical properties of the components impact the overall macroscopic behavior of the metamaterial.

In what follows, we investigate these questions for a special class of metamaterials with two characteristic features that are of relevance in a number of applications: (i) the material consists of two components arranged in a highly anisotropic way into periodically alternating layers, and (ii) the (elasto)plastic properties of the two components exhibit strong differences, in the sense that one is rigid, while the other one is considerably softer, allowing for large (elasto)plastic deformations. The analysis of variational models for such layered high-contrast materials was initiated in [13]. There, the authors derive a macroscopic description for a two-dimensional model in the context of geometrically nonlinear but rigid elasticity, assuming that the softer component can be deformed along a single active slip system with linear self-hardening.

These results have been extended to general dimensions, to energy densities with $p$-growth for $1 < p < +\infty$, and to the case with non-trivial elastic energies, which allows treating very stiff (but not necessarily rigid) layers, see [14, 12].

In this paper, we carry the ideas of [13] forward to a model for plastic composites without linear hardening, in the spirit of [18]. This change turns the variational problem in [13], having quadratic growth (cf. also [15, 16]), into one with energy densities that grow merely linearly.

The main novelty lies in the fact that the homogenization analysis must be performed in the class $\text{BV}$ of functions of bounded variation (see [2]) to account for concentration phenomena. This gives rise to conceptual mathematical difficulties: on the one hand, the standard convolution techniques commonly used for density arguments in $\text{BV}$ or $\text{SBV}$ cannot be directly applied because they do not preserve the intrinsic constraints of the problem; on the other hand, constraint-preserving approximations in this weaker setting of $\text{BV}$ are rather challenging, as one needs to simultaneously regularize the absolutely continuous part of the distributional derivative of the functions and accommodate their jump sets.
To state our results precisely, we first introduce the relevant model with its main modeling hypotheses. Throughout the article, we analyze two versions of the model, namely with and without regularization.

Let $e_1$ and $e_2$ be the standard unit vectors in $\mathbb{R}^2$, and let $x = (x_1, x_2)$ denote a generic point in $\mathbb{R}^2$. Unless specified otherwise, $\Omega \subset \mathbb{R}^2$ is an $x_1$-connected, bounded domain with Lipschitz boundary, that is, an open set whose slices in the $x_1$-direction are (possibly empty) open intervals (see Subsection 2.4 for the precise definition). For such a domain $\Omega$, we set

$$a_\Omega := \inf_{x \in \Omega} x_2 \quad \text{and} \quad b_\Omega := \sup_{x \in \Omega} x_2,$$

(1.1)

as well as

$$c_\Omega := \inf_{x \in \Omega} x_1 \quad \text{and} \quad d_\Omega := \sup_{x \in \Omega} x_1.$$

(1.2)

Assume that $\Omega$ is the reference configuration of a body with heterogeneities in the form of periodically alternating thin horizontal layers. To describe the bilayered structure mathematically, consider the periodicity cell $Y := [0,1)^2$, which we subdivide into $Y = Y_{\text{soft}} \cup Y_{\text{rig}}$ with $Y_{\text{soft}} := [0,1) \times [0, \lambda)$ for $\lambda \in (0,1)$ and $Y_{\text{rig}} := Y \setminus Y_{\text{soft}}$. All sets are extended by periodicity to $\mathbb{R}^2$. The (small) parameter $\varepsilon > 0$ describes the thickness of a pair (one rigid, one softer) of fine layers, and can be viewed as the intrinsic length scale of the system. The collections of all rigid and soft layers in $\Omega$ can be expressed as $\varepsilon Y_{\text{rig}} \cap \Omega$ and $\varepsilon Y_{\text{soft}} \cap \Omega$, respectively. For an illustration of the geometrical assumptions, see Figure 1.

Following the classical theory of elastoplasticity at finite strains (see, e.g., [31] for an overview), we assume that the gradient of any deformation $u : \Omega \rightarrow \mathbb{R}^2$ decomposes into the product of an elastic strain, $F_{\text{el}}$, and a plastic one, $F_{\text{pl}}$. In the literature, different models of finite plasticity have been proposed (see, e.g., [3, 22, 29, 30, 37]), as well as alternative descriptions via the theory of structured deformations (see [10, 11, 24, 6] and the references therein). Here, we adopt the classical model by Lee on finite crystal plasticity introduced in [33, 35, 34], according to which the deformation gradients satisfy

$$\nabla u = F_{\text{el}} F_{\text{pl}}.$$  

(1.3)

In addition, we suppose that the elastic behavior of the body is purely rigid, meaning that

$$F_{\text{el}} \in SO(2) \text{ almost everywhere in } \Omega,$$

(1.4)

and that the plastic part satisfies

$$F_{\text{pl}} = I + \gamma s \otimes m,$$

(1.5)

where $s \in \mathbb{R}^2$ with $|s| = 1$ is the slip direction of the slip system, $m = s^\perp$ is the normal to the slip plane, and the map $\gamma$ measures the amount of slip. Denoting by $\mathcal{M}_s$ the set

$$\mathcal{M}_s := \{F \in \mathbb{R}^{2 \times 2} : \det F = 1 \text{ and } |Fs| = 1\},$$

(1.6)
the multiplicative decomposition (1.3) (under assumptions (1.4) and (1.5)) is equivalent to $\nabla u \in \mathcal{M}_s$ almost everywhere in $\Omega$. Whereas the material is free to glide along the slip system in the softer phase, it is required that $\gamma$ vanishes on the layers consisting of a rigid material, i.e., $\gamma = 0$ in $\varepsilon Y_{rig} \cap \Omega$.

Collecting the previous modeling assumptions, we define, for $\varepsilon > 0$, the class $\mathcal{A}_\varepsilon$ of admissible layered deformations by

$$\mathcal{A}_\varepsilon := \{ u \in W^{1,1}(\Omega; \mathbb{R}^2) : \nabla u \in \mathcal{M}_s \text{ a.e. in } \Omega, \nabla u \in SO(2) \text{ a.e. in } \varepsilon Y_{rig} \cap \Omega \}$$

$$= \{ u \in W^{1,1}(\Omega; \mathbb{R}^2) : \nabla u = R(\| + \gamma s \otimes m) \text{ a.e. in } \Omega, R \in L^\infty(\Omega; SO(2)) \text{ and } \gamma \in L^1(\Omega) \text{ with } \gamma = 0 \text{ a.e. in } \varepsilon Y_{rig} \cap \Omega \}.$$  \hfill (1.6)

The elastoplastic energy of a deformation $u \in L^1(\Omega; \mathbb{R}^2) := \{ u \in L^1(\Omega; \mathbb{R}^2) : \int_{\Omega} u \, dx = 0 \}$, given by

$$E_\varepsilon(u) = \begin{cases} \int_{\Omega} |\gamma| \, dx & \text{for } u \in \mathcal{A}_\varepsilon, \\ \infty & \text{otherwise in } L^1(\Omega; \mathbb{R}^2), \end{cases}$$ \hfill (1.7)

represents the internal energy contribution of the system during a single incremental step in a time-discrete variational description. This way of modeling excludes preexistent plastic distortions, and can be considered a reasonable assumption for the first time step of a deformation process. The elastoplastic energy can be complemented with terms modeling the work done by external body or surface forces.

The limit behavior of sequences $(u_\varepsilon)$ of low energy states for $(E_\varepsilon)_\varepsilon$ gives information about the macroscopic material response of the layered composites. In the following, we focus the analysis of this asymptotic behavior on the $s = e_1$ case, when the slip direction is parallel to the orientation of the layers, cf. also Figure 1. Note that different slip directions can be treated similarly, but the arguments are technically more involved. In fact, for $s \notin \{e_1, e_2\}$, small-scale laminate microstructures on the softer layers need to be taken into account, which requires an extra relaxation step. We refer to [18] for the relaxation mechanism and to [13] for the strategy of how to apply it to layered structures.

An important first step towards identifying the limit behavior of the energies $(E_\varepsilon)_\varepsilon$ (in the sense of $\Gamma$-convergence) is the proof of a general statement of asymptotic rigidity for layered structures in the context of functions of bounded variation. The following result characterizes the weak* limits in $BV$ of deformations whose gradients coincide pointwise with rotations on the rigid layers of the material. Note that no additional constraints are imposed on the softer components at this point.

**Theorem 1.1 (Asymptotic rigidity of layered structures in $BV$).** Let $\Omega \subset \mathbb{R}^2$ be an $x_1$-connected domain. Assume that $(u_\varepsilon)_\varepsilon \subset W^{1,1}(\Omega; \mathbb{R}^2)$ is a sequence satisfying

$$\nabla u_\varepsilon \in SO(2) \text{ a.e. in } \varepsilon Y_{rig} \cap \Omega \text{ for all } \varepsilon,$$ \hfill (1.8)

and that $u_\varepsilon \overset{*}{\rightharpoonup} u$ in $BV(\Omega; \mathbb{R}^2)$ for some $u \in BV(\Omega; \mathbb{R}^2)$ as $\varepsilon \to 0$. Then,

$$u(x) = R(x_2)x + \psi(x_2) \text{ for } \mathcal{L}^2\text{- a.e. } x \in \Omega,$$ \hfill (1.9)

where $R \in BV(a_0, b_3; SO(2))$ and $\psi \in BV(a_0, b_3; \mathbb{R}^2)$ (cf. (1.1)).

Conversely, any function $u \in BV(\Omega; \mathbb{R}^2)$ as in (1.9) can be attained as weak* limit in $BV(\Omega; \mathbb{R}^2)$ of a sequence $(u_\varepsilon)_\varepsilon \subset W^{1,1}(\Omega; \mathbb{R}^2)$ satisfying (1.8).

To prove the first part of Theorem 1.1, we adapt the arguments in [13] to the $BV$-setting. The second assertion follows from a tailored one-dimensional density result in $BV$, which involves approximating functions that are constant on the rigid layers (see Lemma 3.3 below). Up to minor adaptations, analogous statements hold in higher dimensions. We refer to Remark 3.4 for the specific assumptions on the geometry of the set $\Omega$ under which a higher-dimensional counterpart of Theorem 1.1 can be proved.

A natural potential candidate for the limiting behavior of $(E_\varepsilon)_\varepsilon$ in the sense of $\Gamma$-convergence (see [8, 20] for an introduction, as well as the references therein) is the functional $E : L^1(\Omega; \mathbb{R}^2) \to [0, \infty]$, given by

$$E(u) = \begin{cases} \int_{\Omega} |\psi' \cdot Re_1| \, dx + |D^s u| (\Omega) & \text{if } u \in \mathcal{A}, \\ \infty & \text{otherwise,} \end{cases}$$ \hfill (1.10)
for an alternative representation of the functional $E$.

We refer to Remark 5.1 for an alternative representation of the functional $E$.

The next theorem states that $E$ provides indeed a lower bound for our homogenization problem.

**Theorem 1.2 (Lower bound on the $\Gamma$-limit of $(E_\varepsilon)_\varepsilon$).** Let $\Omega \subset \mathbb{R}^2$ be an $x_1$-connected domain, and let $E_\varepsilon$ and $E$ be the functionals introduced in (1.7) and (1.10), respectively. Then, every sequence $(u_\varepsilon)_\varepsilon \subset L^1_0(\Omega;\mathbb{R}^2)$ with uniformly bounded energies, $\sup_\varepsilon E_\varepsilon(u_\varepsilon) < \infty$, has a subsequence that converges weakly* in $BV(\Omega;\mathbb{R}^2)$ to some $u \in \mathcal{A} \cap L^1_0(\Omega;\mathbb{R}^2)$. Additionally,

$$\Gamma(L^1)-\liminf_{\varepsilon \to 0} E_\varepsilon \geq E.$$  

The proof of the first assertion is given in Proposition 4.3. It relies on Theorem 1.1 in combination with a technical argument about the weak continuity properties of Jacobian determinants (see Lemma 4.2). In Section 5, we exhibit two different proofs of (1.12): A first one relying on a Reshetnyak’s lower semicontinuity theorem (see, e.g., [2, Theorem 2.38]), and an alternative one exploiting the properties of the admissible layered deformations. The identification of $E$ as the $\Gamma$-limit of the sequence $(E_\varepsilon)_\varepsilon$, though, remains an open problem. Indeed, verifying the optimality of the lower bound in Theorem 1.2 is rather challenging, as it requires to approximate elements of $\mathcal{A}$ by means of sequences in $\mathcal{A}_e$ at least in the sense of the strict convergence in $BV$. We refer to Remark 5.2 for a detailed discussion of the main difficulties. Even if the requirement on the convergence of the energies is dropped, recovering the jumps of maps in the effective domain of $E$ under consideration of the non-standard differential inclusions in $\mathcal{A}_e$ is by itself another challenging problem. Solving this problem requires delicate geometrical constructions, which are currently not available for all elements in $\mathcal{A}$.

Yet, there are two subclasses of physically relevant deformations in $\mathcal{A}$ for which we can find suitable approximations by sequences of admissible layered deformations. The precise statement is given in Theorem 1.3 below.

The first of these two subclasses is $\mathcal{A} \cap SBV_\infty(\Omega;\mathbb{R}^2)$ (we refer to Subsection 2.3 for the definition of the set $SBV_\infty$) whose jump sets are given by a union of finitely many lines. Heuristically, this subclass describes deformations that break $\Omega$ horizontally into a finite number of pieces, which may get sheared and rotated individually.

The second subclass is

$$\mathcal{A}^e := \{ u \in BV(\Omega;\mathbb{R}^2) : u(x) = Rx + \vartheta(x_2)Re_1 + c \text{ a.e. } x \in \Omega \text{ with } \begin{align*} R &\in SO(2), \\ \vartheta &\in BV(a_1, b_1), \text{ and } c \in \mathbb{R}^2 \}. \quad (1.13)$$

In comparison with $\mathcal{A}$, functions in $\mathcal{A}^e$ satisfy two additional constraints, namely the fact that the rotation $R$ is constant and that the jumps of functions in $\mathcal{A}^e$ are parallel to $Re_1$. With the notation $\mathcal{A}^e$, we intend to highlight the second feature. The intuition behind maps in $\mathcal{A}^e$ are non-trivial macroscopic deformations that (up to a global rotation) may make the material break along finite or infinitely many horizontal lines, induce sliding of the pieces relative to each other, and cause horizontal shearing within each individual piece. For an illustration of the two subclasses, see Figure 2.

**Theorem 1.3 (Approximation of maps in $(\mathcal{A} \cap SBV_\infty) \cup \mathcal{A}^e$).** Let $\Omega \subset \mathbb{R}^2$ be an $x_1$-connected domain and $u \in (\mathcal{A} \cap SBV_\infty(\Omega;\mathbb{R}^2)) \cup \mathcal{A}^e$. Then, there exists a sequence $(u_\varepsilon)_\varepsilon \subset W^{1,1}(\Omega;\mathbb{R}^2)$ such that $u_\varepsilon \in \mathcal{A}_e$ for every $\varepsilon$, and $u_\varepsilon \rightharpoonup u$ in $BV(\Omega;\mathbb{R}^2)$.

As a first step towards proving Theorem 1.3, we establish an admissible piecewise affine approximation for limiting deformations with a single jump line (see Lemma 4.5). The construction relies on the character-ization of rank-one connections in $\mathcal{M}_e$, proved in [13, Lemma 3.1], with transition lines stretching over the full width of $\Omega$ to avoid triple junctions (see Remark 4.6). In Propositions 4.7 and 4.9, we extend the arguments to $\mathcal{A} \cap SBV_\infty(\Omega;\mathbb{R}^2)$ and $\mathcal{A}^e$, respectively.
Problems in finite crystal plasticity without additional regularizations are generally known to be challenging because of the oscillations of minimizing sequences arising as a byproduct of relaxation mechanisms in the slip systems. This phenomenon is one of the main reasons why a full relaxation theory in finite crystal plasticity is still missing (see [17, Remark 3.2]). In our setting, it hampers the full characterization of weak limits of sequences with uniformly bounded energies. The observation that regularizations can help overcome the above compensated-compactness issue (see also Remark 6.2) motivates the introduction of a penalized version of our problem. After a higher-order penalization of the energy in the layer direction, we obtain the following $\Gamma$-convergence result. The attained limit deformations are given by the class $A^\parallel$.

**Theorem 1.4 ($\Gamma$-convergence of the regularized energies).** Let $\Omega \subset \mathbb{R}^2$ be an $x_1$-connected domain and $A_\varepsilon$ the set introduced in (1.6). Fix $p > 2$ and $\delta > 0$. For each $\varepsilon > 0$, let $E^\varepsilon : L^1(\Omega; \mathbb{R}^2) \to [0, \infty]$ be the functional defined by

$$E^\varepsilon(u) := \begin{cases} \int_\Omega |\gamma| \ dx + \delta \|\partial_1 u\|_{W^{1,p}(\Omega, \mathbb{R}^2)}^p & \text{for } u \in A_\varepsilon, \\ \infty & \text{otherwise.} \end{cases}$$

Then, the family $(E^\varepsilon)_\varepsilon$ $\Gamma$-converges with respect to the strong $L^1$-topology to the functional $E^\parallel : L^1(\Omega; \mathbb{R}^2) \to [0, \infty]$ given by

$$E^\parallel(u) := \begin{cases} \int_\Omega |\vartheta'(x_2)| \ dx + |D^su|(\Omega) + \delta|\Omega| & \text{for } u \in A^\parallel, \\ \infty & \text{otherwise,} \end{cases}$$

where $\vartheta'$ denotes the approximate differential of $\vartheta$ (cf. Section 2.2).

The penalization in (1.14) can be viewed in the spirit of non-simple materials [39, 40]. Working with stored energy densities that depend on the Hessian of the deformations has proved successful in overcoming lack of compactness in a variety of applications; see, e.g., [5, 21, 27, 36, 38]. Very recently, there has been an effort towards weakening higher-order regularizations: it is shown in [7] that the full norm of the Hessian can be replaced by a control of its minors (gradient polyconvexity) in the context of locking materials; for solid-solid phase transitions, an anisotropic second-order penalization is considered in [23]. Along these lines, we introduce the regularized energies in (1.13) that penalize the variation of deformations only in the layer direction. This is enough to deduce that the limiting rotation (as $\varepsilon \to 0$) is global and that it determines the direction of the limiting jump. In Section 6, we provide two alternative proofs of this result: A first one relying on Alberti’s rank one theorem (see Section 2.1) in combination with the
approximation result in Theorem 1.3, and a second one based on separate regularizations of the regular and the singular part of the limiting maps, and inspired by [19, Lemma 3.2].

This paper is organized as follows. In Section 2.1, we collect a few preliminaries, including some background on (special) functions of bounded variation. Section 3 is devoted to the analysis of asymptotic rigidity for layered structures in the setting of BV-functions. A characterization of limits of admissible layered deformations is provided in Section 4. Eventually, Sections 5 and 6 contain the proof of a lower bound for the homogenization problem without regularization (Theorem 1.2) and the full Γ-convergence analysis of the regularized problem (Theorem 1.4), respectively.

2. Preliminaries

2.1. Notation. In this section, unless mentioned otherwise, Ω is a bounded domain in \( \mathbb{R}^N \) with \( N \in \mathbb{N} \). Throughout the rest of the paper, we assume mostly that \( N = 2 \).

We represent by \( L^N \) the \( N \)-dimensional Lebesgue measure and by \( \mathcal{H}^{N-1} \) the \( (N-1) \)-dimensional Hausdorff measure. Whenever we write “a.e. in \( \Omega \)”, we mean “almost everywhere in \( \Omega \)” with respect to \( L^N(\Omega) \). To simplify the notation, we often omit the expression “a.e. in \( \Omega \)” in mathematical relations involving Lebesgue measurable functions. Given a Lebesgue measurable set \( B \subset \mathbb{R}^N \), we also use the shorter notation \( |B| = L^N(B) \) for the Lebesgue measure of \( B \), while the characteristic function of \( B \) in \( \mathbb{R}^N \) is denoted by \( \mathbb{1}_B \) and takes values 0 and 1.

The set \( SO(N) := \{R \in \mathbb{R}^{N \times N} : RR^T = I, \det R = 1 \} \) where \( I \) is the identity matrix in \( \mathbb{R}^{N \times N} \), consists of all proper rotations. We recall that for \( N = 2, R \in SO(2) \) if and only if there is \( \theta \in [-\pi, \pi) \) such that

\[
R = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}.
\]

For two vectors \( a, b \in \mathbb{R}^d \), \( a \otimes b := ab^T \) stands for their tensor product. If \( a = (a_1, a_2)^T \in \mathbb{R}^2 \), we set \( a^\perp := (-a_2, a_1)^T \).

We use the standard notation for spaces of vector-valued functions; namely, \( L^p(\Omega; \mathbb{R}^d) \) with \( p \in [1, \infty] \) and a positive measure \( \mu \) for \( L^p \)-spaces, \( W^{1,p}(\Omega; \mathbb{R}^d) \) with \( p \in [1, \infty] \) for Sobolev spaces, \( C(\Omega; \mathbb{R}^d) \) for the space of continuous functions, \( C^\infty(\Omega; \mathbb{R}^d) \) and \( C^\infty_c(\Omega; \mathbb{R}^d) \) for the spaces of smooth functions without and with compact support, and \( C^{0,\alpha}(\Omega; \mathbb{R}^d) \) with \( \alpha \in [0, 1] \) for Hölder spaces. We denote by \( C_0(\Omega; \mathbb{R}^d) \) the space of continuous functions that vanish on the boundary of \( \Omega \). Moreover, \( M(\Omega; \mathbb{R}^d) \) is the space of finite vector-valued Radon measures. In the case of scalar-valued functions and measures, we omit the codomain; for instance, we write \( L^1(\Omega) \) instead of \( L^1(\Omega; \mathbb{R}) \).

The duality pairing between \( C_0(\Omega; \mathbb{R}^d) \) and \( M(\Omega; \mathbb{R}^d) \) is represented by \( \langle \mu, \zeta \rangle := \int_\Omega \zeta \, d\mu \), and \( \mu \otimes \nu \) denotes the product measure of two measures \( \mu \) and \( \nu \).

Throughout this manuscript, \( \varepsilon \) stands for a small (positive) parameter, and is usually thought of as taking values on a positive sequence converging to zero.

2.2. Functions of bounded variation. We adopt the standard notations for the space \( BV(\Omega; \mathbb{R}^d) \) of vector-valued functions of bounded variation, and refer the reader to [2] for a thorough treatment of this space. Here, we only recall some of its basic properties.

A function \( u \in L^1(\Omega; \mathbb{R}^d) \) is called a function of bounded variation, written \( u \in BV(\Omega; \mathbb{R}^d) \), if its distributional derivative \( Du \) satisfies \( Du \in M(\Omega; \mathbb{R}^{d \times N}) \). The space \( BV(\Omega; \mathbb{R}^d) \) is a Banach space when endowed with the norm \( \|u\|_{BV(\Omega; \mathbb{R}^d)} := \|u\|_{L^1(\Omega; \mathbb{R}^d)} + |Du|(\Omega) \), where \( |Du| \in M(\Omega) \) is the total variation of \( Du \).

Let \( D^u \) and \( D^u \) denote the absolutely continuous and the singular part of the Radon–Nikodym decomposition of \( Du \) with respect to \( L^N(\Omega) \), and let \( D^u \) and \( D^u \) be the jump and Cantor parts of \( Du \). The following chain of equalities holds:

\[
Du = D^u + D^u = \nabla u L^N + D^u = \nabla u L^N + D^u + D^u
\]

\[
= \nabla u L^N + (u^+ - u^-) \otimes \nu_u \mathcal{H}^{N-1} u_J + D^u, \tag{2.1}
\]

where \( \nabla u \) is the approximate differential of \( u \) (that is, the density of \( D^u \)), \( u^+ \) and \( u^- \) are the approximate one-sided limits at the jump points, \( J_u \) is the jump set of \( u \), and \( \nu_u \) is the normal to \( J_u \) (cf. [2, Chapter 3]).

Following [2, p. 186], we can exploit the polar decomposition of a measure and the fact that all parts of the derivative of \( u \) in (2.1) are mutually singular to write \( Du = g_u|Du| \) with a map \( g_u \in L^1_{|Du|}(\Omega; \mathbb{R}^{d \times N}) \).
satisfying $|g_u| = 1$ for $|Du|$-a.e. $x \in \Omega$ and

$$D^a u = g_u |D^a u|, \quad D^s u = g_u |D^s u|, \quad D^i u = g_u |D^i u|, \quad D^c u = g_u |D^c u|.$$ 

Note that

$$g_u(x) = \frac{\nabla u(x)}{\|\nabla u(x)\|} \text{ for } \mathcal{L}^N \text{-a.e. } x \in \Omega \text{ such that } |\nabla u(x)| \neq 0,$$

$$g_u(x) = \frac{u(x^+) - u(x^-)}{|u(x^+) - u(x^-)|} \otimes \nu_u(x) \text{ for } \mathcal{H}^{N-1} \text{-a.e. } x \in J_u,$$

$$g_u(x) = \bar{g}_u(x) \otimes n_u(x) \text{ for } |D^u|\text{-a.e. } x \in \Omega \text{ with suitable Borel maps } \bar{g}_u : \Omega \to \mathbb{R}^d, n_u : \Omega \to \mathbb{R}^N.$$ 

(2.2)

The last equality relies on Alberti’s rank-one theorem (see [1]).

Let $u \in BV(\Omega; \mathbb{R}^d)$ and $(u_j)_{j \in \mathbb{N}} \subset BV(\Omega; \mathbb{R}^d)$ be a sequence. One says that $(u_j)_{j \in \mathbb{N}}$ weakly* converges to $u$ in $BV(\Omega; \mathbb{R}^d)$, written $u_j \rightharpoonup u$ in $BV(\Omega; \mathbb{R}^d)$, if $u_j \to u$ in $L^1(\Omega; \mathbb{R}^d)$ and $Du_j \rightharpoonup Du$ in $\mathcal{M}(\Omega; \mathbb{R}^{d \times N})$. The sequence $(u_j)_{j \in \mathbb{N}}$ is said to converge strictly to $u$ in $BV(\Omega; \mathbb{R}^d)$, written $u_j \rightharpoonup u$ in $BV(\Omega; \mathbb{R}^d)$, if $u_j \to u$ in $L^1(\Omega; \mathbb{R}^d)$ and $|Du_j|((\Omega)) \to |Du|((\Omega))$. We recall that strict convergence in $BV(\Omega; \mathbb{R}^d)$ implies weak* convergence in $BV(\Omega; \mathbb{R}^d)$. Moreover, from every bounded sequence in $BV(\Omega; \mathbb{R}^d)$ one can extract a weakly* convergent subsequence (see [2, Theorem 3.23]).

In the one-dimensional setting, i.e., for $\varphi \in BV(a, b; \mathbb{R}^d)$ with $\Omega = (a, b) \subset \mathbb{R}^N$ and $N = 1$, we write $\varphi'$ in place of $\nabla \varphi$ to denote the approximate differential of $\varphi$. Accordingly, we use the notation $Du = \varphi' \mathcal{L}^d + D^s \varphi$ for the decomposition of the distributional derivative of $\varphi$ with respect to the Lebesgue measure.

A function $\varphi \in BV(a, b; \mathbb{R}^d)$ is called a jump or Cantor function if $D\varphi = D^i \varphi$ or $D\varphi = D^c \varphi$, respectively. We denote the sets of all jump and Cantor functions by $BV^j(a, b; \mathbb{R}^d)$ and $BV^c(a, b; \mathbb{R}^d)$, respectively. As shown in [2, Corollary 3.33], it is a special property of the one-dimensional setting that

$$BV(a, b; \mathbb{R}^d) = W^{1,1}(a, b; \mathbb{R}^d) + BV^j(a, b; \mathbb{R}^d) + BV^c(a, b; \mathbb{R}^d). \quad (2.4)$$

Throughout this paper, two-dimensional functions of the form

$$u(x) = R(x_2) x + \psi(x_2) \quad (2.5)$$

with $x = (x_1, x_2) \in \Omega = Q := (c, d) \times (a, b) \subset \mathbb{R}^2$, where $R \in BV(a, b; SO(2))$ and $\psi \in BV(a, b; \mathbb{R}^d)$, play a fundamental role. Maps $u$ as in (2.5) satisfy $u \in BV(\Omega; \mathbb{R}^2)$. Denoting by $D_1 u := Du \otimes e_1$ and $D_2 u := Du \otimes e_2$, the first and second columns of $Du$, respectively, we have for all $\zeta \in C_0(\Omega)$ that

$$\langle D_1 u, \zeta \rangle = \int_\Omega \zeta(x) R(x_2) e_1 \, dx_1 \, dx_2,$$

$$\langle D_2 u, \zeta \rangle = \int_\Omega \left( \zeta(x) R(x_2) e_2 + R'(x_2) x + \psi'(x_2) \right) \, dx_1 \, dx_2$$

$$+ \int_\Omega \zeta(x) x_1 \, dx_1 \, dD^s R(x_2) e_1 + \int_\Omega \zeta(x) x_2 \, dx_1 \, dD^s R(x_2) e_2 + \int_\Omega \zeta(x) \, dx_1 \, dD^c \psi(x_2).$$

Hence, $Du = D^a u + D^s u$ with

$$D^a u = \left( R + (R' x + \psi') \otimes e_2 \right) \mathcal{L}^2 |\Omega|,$$

$$D^s u = \left( (x^T \mathcal{L}^1 [(c, d) \otimes D^s \psi])^T + \mathcal{L}^1 [(c, d) \otimes D^s \psi] \right) \otimes e_2,$$

(2.6)

where $\mathcal{L}^1 [(c, d) \otimes D^s \psi]$ and $\mathcal{L}^1 [(c, d) \otimes D^s \psi]$ denote the restrictions to the Borel $\sigma$-algebra on $\Omega = Q$ of the product measures between $\mathcal{L}^1 [(c, d) \otimes D^s \psi]$ and $D^s \psi$, respectively.

We observe further that there exists $\theta \in BV(a, b; [-\pi, \pi])$ such that

$$R = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \quad \text{and} \quad R' = \theta' \begin{bmatrix} -\sin \theta & -\cos \theta \\ \cos \theta & -\sin \theta \end{bmatrix}, \quad (2.7)$$

where the representation of $R'$ follows from the chain rule in BV; see, e.g., [2, Theorem 3.96].
2.3. Special functions of bounded variation. A function \( u \in BV(\Omega; \mathbb{R}^d) \) is said to be a special function of bounded variation, written \( u \in SBV(\Omega; \mathbb{R}^d) \), if the Cantor part of its distributional derivative satisfies
\[
D^c u = 0.
\]
In particular, it holds for every \( u \in SBV(\Omega; \mathbb{R}^d) \) that
\[
Du = \nabla u \mathcal{L}^N[\Omega + (u^+ - u^-)] \otimes \nu_q \mathcal{H}^{N-1}|J_u.
\]
The space \( SBV(\Omega; \mathbb{R}^d) \) is a proper subspace of \( BV(\Omega; \mathbb{R}^d) \) (c.f. [2, Corollary 4.3]).

Next, we recall the definition of the space \( SBV_\infty(\Omega; \mathbb{R}^d) \) of special functions of bounded variation with bounded gradient and jump length, which is given by
\[
SBV_\infty(\Omega; \mathbb{R}^d) := \{ u \in SBV(\Omega; \mathbb{R}^d) : \nabla u \in L^\infty(\Omega; \mathbb{R}^{d \times N}) \text{ and } \mathcal{H}^{N-1}(J_u) < +\infty \}.
\]
It is shown in [9] that the distributional curl of \( \nabla u \) for \( u \in SBV_\infty(\Omega; \mathbb{R}^d) \) is a measure concentrated on \( J_u \).

Finally, we introduce the space
\[
PC(a, b; \mathbb{R}^d) = SBV_\infty(a, b; \mathbb{R}^d) \cap \{ u \in BV(a, b; \mathbb{R}^d) : D^a u = 0 \},
\]
which contains piecewise constant one-dimensional functions with values in \( \mathbb{R}^d \).

2.4. Geometry of the domain. In this section, we specify our main assumptions on the geometry of \( \Omega \), which, as mentioned in the Introduction, will mostly be a bounded Lipschitz domain in \( \mathbb{R}^2 \). Let us first recall from [14, Section 3] the definitions of locally one-dimensional and one-dimensional functions.

**Definition 2.1 (Locally one-dimensional functions in the \( e_2 \)-direction).** Let \( \Omega \subset \mathbb{R}^2 \) be open. A function \( f : \Omega \to \mathbb{R}^d \) is locally one-dimensional in the \( e_2 \)-direction if for every \( x \in \Omega \), there exists an open cuboid \( Q_x \subset \Omega \), containing \( x \) and with sides parallel to the standard coordinate axes, such that for all \( y = (y_1, y_2), z = (z_1, z_2) \in Q_x \),
\[
f(y) = f(z) \quad \text{if } y_2 = z_2.
\]
We say that \( f \) is (globally) one-dimensional in the \( e_2 \)-direction if (2.9) holds for every \( y, z \in \Omega \).

Analogous arguments to those in [14, Section 3] show that a function \( f \in BV(\Omega; \mathbb{R}^d) \) satisfying \( D_y f = 0 \) is locally one-dimensional in the \( e_2 \)-direction. The following geometrical requirement is the counterpart of [14, Definitions 3.6 and 3.7] in our setting.

**Definition 2.2 (\( x_1 \)-connectedness).** We say that an open set \( \Omega \subset \mathbb{R}^2 \) is \( x_1 \)-connected if for every \( t \in \mathbb{R} \), the set \( \{ x_2 = t \} \cap \Omega \) is a (possibly empty) interval.

In what follows, we always assume that the set \( \Omega \subset \mathbb{R}^2 \) is an \( x_1 \)-connected domain. Under this geometrical assumption, the notions of locally and globally one-dimensional functions in the \( e_2 \)-direction coincide. We refer to [14, Section 3] for an extended discussion on the topic, as well as for some explicit geometrical examples.

3. Asymptotic rigidity of layered structures in \( BV \)

In this section, we prove Theorem 1.1, which characterizes the asymptotic behavior of deformations of bilayered materials that correspond to rigid body motions on the stiff layers, but do not experience any further structural constraints on the softer layers. This qualitative result is not just limited to applications in crystal plasticity, but can be useful for a larger class of layered composites where fracture may occur.

We start by introducing some notation. Assume that \( \Omega \subset \mathbb{R}^2 \) is an \( x_1 \)-connected domain. For \( \varepsilon > 0 \), let
\[
B_{\varepsilon} := \{ u \in W^{1,1}(\Omega; \mathbb{R}^2) : \nabla u \in SO(2) \text{ in } \varepsilon Y_{\text{rig}} \cap \Omega \}
\]
represent the class of layered deformations with rigid components, and let
\[
B_0 := \{ u \in BV(\Omega; \mathbb{R}^2) : \text{ there exists } (\varepsilon u)_\varepsilon \subset W^{1,1}(\Omega; \mathbb{R}^2) \text{ with } u_\varepsilon \in B_{\varepsilon} \text{ for all } \varepsilon \text{ such that } u_\varepsilon \rightharpoonup u \text{ in } BV(\Omega; \mathbb{R}^2) \}
\]
be the associated set of asymptotically attainable deformations.
We aim at proving that $B_0$ coincides with the set of asymptotically rigid deformations given by
\[
B := \{ u \in BV(\Omega; \mathbb{R}^2) : u(x) = R(x_3)x + \psi(x_2) \text{ for a.e. } x \in \Omega \}
\]
with $R \in BV(\mathcal{A}, \mathcal{B}; SO(2))$ and $\psi \in BV(\mathcal{A}, \mathcal{B}; \mathbb{R}^2)$, \hspace{1cm} (3.3)
cf. (1.1). This identity will be a consequence of Propositions 3.1 and 3.2 below.

**Proposition 3.1 (Limiting behavior of maps in $B_\varepsilon$).** Let $\Omega = (0, 1) \times (-1, 1)$. Then,
\[
B_0 \subset B,
\] (3.4)
where $B_0$ and $B$ are the sets introduced in (3.2) and (3.3), respectively.

**Proof.** The proof is inspired by and generalizes ideas from [13, Proposition 2.1]. Let $u \in B_0$. Then, there exists a sequence $(u_\varepsilon) \subset W^{1,1}(\Omega; \mathbb{R}^2)$ satisfying $\nabla u_\varepsilon \in SO(2)$ a.e. in $\varepsilon \text{Yrig} \cap \Omega$ for all $\varepsilon$, and $u_\varepsilon \rightharpoonup u$ in $BV(\Omega; \mathbb{R}^2)$.

Fix $0 < \varepsilon < 1$, and let $I_\varepsilon := \{ i \in \mathbb{Z} : (\mathbb{R} \times \varepsilon(i-1,i)) \cap \Omega \neq \emptyset \}$. For each $i \in I_\varepsilon$, we define a strip, $P^i_\varepsilon$, by setting
\[
P^i_\varepsilon := (\mathbb{R} \times \varepsilon(i-1,i)) \cap \Omega.
\]

Note that if $i \in \mathbb{Z}$ is such that $|i| > 1 + [\frac{1}{\varepsilon}]$, then $i \notin I_\varepsilon$. Moreover, defining $i^+_\varepsilon := \max I_\varepsilon$ and $i^-_\varepsilon := \min I_\varepsilon$, then

i) for $i^+_\varepsilon < i < i^-_\varepsilon$, $P^i_\varepsilon$ is the union of two neighboring connected components of $\varepsilon \text{Yrig} \cap \Omega$ and $\varepsilon \text{Ysoft} \cap \Omega$;

ii) we may have $\varepsilon \text{Ysoft} \cap P^i_\varepsilon = \emptyset$ or $\varepsilon \text{Yrig} \cap P^i_\varepsilon = \emptyset$.

From Reshetnyak's theorem, we infer that on each nonempty rigid layer $\varepsilon \text{Yrig} \cap P^i_\varepsilon$ with $i \in I_\varepsilon$, the gradient $\nabla u_\varepsilon$ is constant and coincides with a rotation $R^i_\varepsilon \in SO(2)$. Moreover, there exists $b^i_\varepsilon \in \mathbb{R}^2$ such that $u_\varepsilon(x) = R^i_\varepsilon x + b^i_\varepsilon$ in $\varepsilon \text{Yrig} \cap I^i_\varepsilon$.

Using these rotations $R^i_\varepsilon$, we define a piecewise constant function, $\Sigma_\varepsilon : (-1, 1) \to \mathbb{R}^{2 \times 2}$, by setting $\Sigma_\varepsilon(t) = \sum_{i \in I_\varepsilon} R^i_\varepsilon 1_{[i(i-1), i]}(t)$ for $t \in (-1, 1)$, where $R^i_\varepsilon := R^i_\varepsilon^{-1}$ if $\varepsilon \text{Yrig} \cap P^i_\varepsilon = \emptyset$. We claim that there exist a subsequence of $(\Sigma_\varepsilon)_\varepsilon$, which we do not relabel, and a function $R \in BV(-1, 1; SO(2))$ such that
\[
\Sigma_\varepsilon \to R \quad \text{in} \quad L^1(-1, 1; \mathbb{R}^{2 \times 2}).
\] (3.5)

To prove (3.5), we first observe that the total variation of the one-dimensional function $\Sigma_\varepsilon$ coincides with its pointwise variation, and can be calculated to be
\[
|D \Sigma_\varepsilon|_{(-1,1)} = \sum_{i \in I_\varepsilon \setminus \{i^+_\varepsilon\}} |R^i_\varepsilon - R^{i^-}_\varepsilon| = \sqrt{2} \sum_{i \in I_\varepsilon \setminus \{i^+_\varepsilon\}} |R^i_\varepsilon e_1 - R^{i^-}_\varepsilon e_1|.
\] (3.6)

Next, we show that the right-hand side of (3.6) is uniformly bounded. By linear interpolation in the $x_2$-direction on the softer layers, it follows for all $i \notin I_\varepsilon \setminus \{i^+_\varepsilon\}$ if $\varepsilon \text{Yrig} \cap P^i_\varepsilon \neq \emptyset$ and $i \in I_\varepsilon \setminus \{i^+_\varepsilon\}$ if $\varepsilon \text{Yrig} \cap P^i_\varepsilon = \emptyset$ that
\[
\int_{\varepsilon \text{Yrig} \cap P^i_\varepsilon} |\nabla u_\varepsilon e_2| \, dx = \int_0^1 \int_{\varepsilon(i-1)}^{\varepsilon(i+1)} |\partial_2 u_\varepsilon(x, 1, x_2)| \, dx_2 \, dx_1
\]
\[
\geq \int_0^1 |u_\varepsilon(x_1, \varepsilon(i-1)) - u_\varepsilon(x_1, \varepsilon(i+1))| \, dx_1
\]
\[
= \int_0^1 |(R^i_\varepsilon e_1 - R^{i^-}_\varepsilon e_1)x_1 + b^i_\varepsilon - b^{i^-}_\varepsilon| \, dx_1 \geq \frac{1}{4} |R^i_\varepsilon e_1 - R^{i^-}_\varepsilon e_1|.
\] (3.7)

The first estimate is a consequence of Jensen’s inequality, and optimization over translations yields the second one. To be more precise, the last estimate in (3.7) is based on the observation that for any given $a \in \mathbb{R} \setminus \{0\}$,
\[
\min_{b \in \mathbb{R}^2} \int_0^1 |ta + b| \, dt = \min_{a, \beta \in \mathbb{R}} \int_0^1 |(t + \alpha)a + \beta a^+| \, dt = |a| \min_{a \in \mathbb{R}} \int_0^1 |t + \alpha| \, dt = \frac{|a|}{4}.
\]
From (3.6) and (3.7), since \((u_\varepsilon)_\varepsilon \subset W^{1,1}(\Omega; \mathbb{R}^2)\) as a weakly* converging sequence is uniformly bounded in \(BV(\Omega; \mathbb{R}^2)\), and recalling that \(R_{\varepsilon}^{i\varepsilon} = R_{\varepsilon}^{i\varepsilon-1}\) if \(\varepsilon Y_\text{rig} \cap P_{\varepsilon}^{i\varepsilon} = \emptyset\), we conclude that
\[
|D\Sigma_\varepsilon|(-1,1) \leq 4\sqrt{2} \int_\Omega |\nabla u_\varepsilon| \, dx \leq C. \tag{3.8}
\]

The convergence in (3.5) follows now from the weak* relative compactness of bounded sequences in \(BV(-1,1; \mathbb{R}^{2x2})\) (see Section 2.2), together with the fact that strong \(L^1\)-convergence is length and angle preserving. The latter guarantees that the limit function \(R \in BV(-1,1; \mathbb{R}^{2x2})\) takes values only in \(SO(2)\).

Next, we show that there is \(\psi \in BV(-1,1; \mathbb{R}^2)\) such that
\[
u(x) = R(x_2)x + \psi(x_2) \tag{3.9}
\]
for a.e. \(x \in \Omega\), which implies that \(u \in \mathcal{B}\) and concludes the proof. To this end, we define auxiliary functions \(\sigma_\varepsilon, b_\varepsilon \in L^\infty(\Omega; \mathbb{R}^2)\) for \(\varepsilon > 0\) by setting
\[
\sigma_\varepsilon(x) = \sum_{i \in I_\varepsilon}(R_{\varepsilon}^i x) \mathbb{1}_{P_i}(x) \quad \text{and} \quad b_\varepsilon(x) = \sum_{i \in I_\varepsilon} b_{\varepsilon}^i \mathbb{1}_{P_i}(x)
\]
for \(x \in \Omega\), where \(R_{\varepsilon}^i := R_{\varepsilon}^{i-1}\) and \(b_{\varepsilon}^i := b_{\varepsilon}^{i-1}\) if \(\varepsilon Y_\text{rig} \cap P_{\varepsilon}^i = \emptyset\). Further, let \(w_\varepsilon := \sigma_\varepsilon + b_\varepsilon\).

By Poincaré’s inequality applied in the \(x_2\)-direction, we obtain
\[
\int_\Omega |u_\varepsilon - w_\varepsilon| \, dx = \sum_{i \in I_\varepsilon} \int_{\varepsilon Y_\text{rig} \cap P_i \neq \emptyset} \int_0^1 \min(\varepsilon(1+y),1) \int_0^{\max(\varepsilon(1-y),-1)} |u_\varepsilon - w_\varepsilon| \, dx_2 \, dx_1 \leq \varepsilon \lambda \sum_{i \in I_\varepsilon} \int_{\varepsilon Y_\text{rig} \cap P_i} |\partial_2 u_\varepsilon - R_{\varepsilon}^i x_2| \, dx \leq \varepsilon \lambda \|u_\varepsilon\|_{W^{1,1}(\Omega; \mathbb{R}^2)} + |\Omega| \leq C \varepsilon.
\]
Consequently,
\[
w_\varepsilon \to u \quad \text{in} \quad L^1(\Omega; \mathbb{R}^2). \tag{3.10}
\]

Moreover, for \(x \in \Omega\),
\[
|\sigma_\varepsilon(x) - R(x_2)x| \leq \sum_{i \in I_\varepsilon} |(R_{\varepsilon}^i - R(x_2)) \mathbb{1}_{P_i}(x)| |x| \leq \sqrt{2}\|\Sigma_\varepsilon(x_2) - R(x_2)\|,
\]
which, together with (3.5), proves that
\[
\sigma_\varepsilon \to \sigma \quad \text{in} \quad L^1(\Omega; \mathbb{R}^2), \tag{3.11}
\]
where \(\sigma(x) = R(x_2)x \in BV(\Omega; \mathbb{R}^2)\).

Finally, exploiting (3.10) and (3.11), we conclude that there exists \(b \in BV(\Omega; \mathbb{R}^2)\) such that \(b_\varepsilon \to b\) in \(L^1(\Omega; \mathbb{R}^2)\). In view of the one-dimensional character of the stripes \(P_i^\varepsilon\), we infer that \(\partial_1 b = 0\). Eventually, identifying \(b\) with a function \(\psi \in BV(-1,1; \mathbb{R}^2)\) yields (3.9).

Next, we prove that the converse inclusion of (3.4) holds. In the following, let \(I_\text{rig}\) be the projection of \(Y_\text{rig}\) onto the second component; that is, \(I_\text{rig}\) corresponds to the 1-periodic extension of the interval \([\lambda,1)\). Analogously, we write \(I_\text{soft}\) for the 1-periodic extension of \([0,\lambda)\).

**Proposition 3.2 (Approximation of maps in \(\mathcal{B}\)).** Let \(\Omega = (0,1) \times (-1,1)\). Then,
\[
\mathcal{B}_0 \supset \mathcal{B}. \tag{3.12}
\]
Here, \(\mathcal{B}_0\) and \(\mathcal{B}\) are the sets from (3.2) and (3.3), respectively.

**Proof.** Let \(u \in \mathcal{B}\), and let \(R \in BV(-1,1; SO(2))\) and \(\psi \in BV(-1,1; \mathbb{R}^2)\) be such that
\[
u(x) = R(x_2)x + \psi(x_2)
\]
for a.e. \(x \in \Omega\). Using Lemma 3.3 below, as well as the fact that strict convergence implies weak* convergence in \(BV\), we construct sequences \((R_\varepsilon)_\varepsilon \subset W^{1,\infty}(-1,1; SO(2))\) and \((\psi_\varepsilon)_\varepsilon \subset W^{1,\infty}(-1,1; \mathbb{R}^2)\) such that
\[
R_\varepsilon' = 0 \quad \text{and} \quad \psi_\varepsilon' = 0 \quad \text{on} \quad \varepsilon I_\text{rig} \cap (-1,1), \tag{3.13}
\]
\[ R_\varepsilon \rightharpoonup^* R \text{ in } BV(-1,1;\mathbb{R}^{2 \times 2}) \quad \text{and} \quad \psi_\varepsilon \rightharpoonup^* \psi \text{ in } BV(-1,1;\mathbb{R}^2). \] (3.14)

Define \( u_\varepsilon(x) := R_\varepsilon(x_2)x + \psi_\varepsilon(x_2) \) for \( x \in \Omega \). Then, \( u_\varepsilon \in W^{1,\infty}(\Omega;\mathbb{R}^2) \) for every \( \varepsilon \), with
\[
\nabla u_\varepsilon(x) = R_\varepsilon(x_2) + R_\varepsilon'(x_2)x \otimes e_2 + \psi_\varepsilon(x_2) \otimes e_2
\]
for a.e. \( x \in \Omega \). In particular, \( \nabla u_\varepsilon = R_\varepsilon \in SO(2) \) a.e. in \( \varepsilon Y_{\text{rig}} \cap \Omega \) by (3.13); hence, \( u_\varepsilon \in \mathcal{B}_\varepsilon \). Moreover, \( \sup_{\varepsilon} \| \nabla u_\varepsilon \|_{L^1(\Omega;\mathbb{R}^{2 \times 2})} < \infty \) and \( u_\varepsilon \rightarrow u \) in \( L^1(\Omega;\mathbb{R}^2) \) by (3.14), from which we conclude that \( u_\varepsilon \rightharpoonup^* u \) in \( BV(\Omega;\mathbb{R}^2) \). This completes the proof. \( \square \)

The next lemma states a one-dimensional approximation result of \( BV \)-maps by Lipschitz functions that are constant on \( \varepsilon Y_{\text{rig}} \), which was an important ingredient in the previous proof.

**Lemma 3.3 (1D-approximation by maps constant on \( \varepsilon Y_{\text{rig}} \)).** Let \( I = (a,b) \subset \mathbb{R} \) and \( w \in BV(I;\mathbb{R}^d) \). Then, there exists a sequence \((w_\varepsilon)_\varepsilon \subset W^{1,\infty}(I;\mathbb{R}^d)\) with the following three properties:

- (i) \( w_\varepsilon \rightarrow w \) in \( L^1(I;\mathbb{R}^d) \);
- (ii) \( \int_I |w_\varepsilon'| \, dt \rightarrow |Dw|(I) \);
- (iii) \( w_\varepsilon \rightharpoonup^* 0 \) on \( \varepsilon Y_{\text{rig}} \cap I \).

Moreover, if \( w \) takes values in \( SO(2) \) and \( w \in BV(I;SO(2)) \), then each \( w_\varepsilon \) may be taken in \( W^{1,\infty}(I;SO(2)) \).

**Proof.** Let \( w \in BV(I;\mathbb{R}^d) \). By [2, Theorem 3.9, Remark 3.22], \( w \) can be approximated by a sequence of smooth functions \((v_\varepsilon)_\varepsilon \subset C^\infty(I;\mathbb{R}^d)\) in the sense of strict convergence in \( BV \); that is,
\[
v_\varepsilon \rightarrow w \text{ in } L^1(I;\mathbb{R}^d) \quad \text{and} \quad \int_I |v_\varepsilon'| \, dt \rightarrow |Dw|(I)
\]
as \( \varepsilon \rightarrow 0 \). To obtain property (iii), we will reparametrize \( v_\varepsilon \) so that it is stopped on the set \( \varepsilon Y_{\text{rig}} \) and accelerated otherwise, and eventually apply a diagonalization argument.

We start by introducing for every \( \varepsilon > 0 \) a Lipschitz function \( \varphi_\varepsilon : \mathbb{R} \rightarrow \mathbb{R} \) defined by
\[
\varphi_\varepsilon(t) := \begin{cases} 
\frac{1}{\lambda}(t - \varepsilon i) & \text{if } \varepsilon i \leq t \leq \varepsilon i + \lambda, \\
(i + 1)\varepsilon & \text{if } \varepsilon i + \lambda \leq t < \varepsilon(i + 1),
\end{cases}
\]
for each \( i \in \mathbb{Z} \) and \( t \in \varepsilon(i,i + 1) \). For all \( t \in \mathbb{R} \), we have \( t \leq \varphi_\varepsilon(t) \leq t + \varepsilon(1 - \lambda) \) and \( \varphi_\varepsilon'(t) = \psi(t) \), where \( \psi \) is the 1-periodic function such that \( \psi(t) = \frac{1}{\lambda} \) if \( 0 \leq t \leq \lambda \), and \( \psi(t) = 0 \) if \( \lambda < t < 1 \). By the Riemann–Lebesgue lemma on weak convergence of periodically oscillating sequences, it follows that \( \psi(t) \rightharpoonup^* 1 \) in \( L^\infty(\mathbb{R}) \). Thus, \( \varphi_\varepsilon \rightharpoonup^* \varphi \) in \( W^{1,\infty}_{\text{loc}}(\mathbb{R}) \), where \( \varphi(t) := t \). In particular, \( \varphi_\varepsilon \) converges uniformly to \( \varphi \) on every compact set \( K \subset \mathbb{R} \).

Next, we define for \( \varepsilon > 0 \) a Lipschitz function \( \tilde{\varphi}_\varepsilon : \tilde{I} \rightarrow \tilde{I} \) by setting
\[
\tilde{\varphi}_\varepsilon(t) := \begin{cases} 
\varphi_\varepsilon(t) & \text{if } a \leq t \leq b_\varepsilon, \\
b & \text{if } b_\varepsilon < t \leq b,
\end{cases}
\]
where \( b_\varepsilon \in (a,b) \) is such that \( \varphi_\varepsilon(b_\varepsilon) = b \). Note that by definition of \( \varphi_\varepsilon \), there exists at least one such \( b_\varepsilon \). We claim that \( b_\varepsilon \rightarrow b \) as \( \varepsilon \rightarrow 0 \). In fact, extracting a subsequence if necessary, we have \( b_\varepsilon \rightarrow c \) for some \( c \in [a,b] \). Then,
\[
|b - c| = |\varphi_\varepsilon(b_\varepsilon) - \varphi(c)| \leq |\varphi_\varepsilon(b_\varepsilon) - \varphi_\varepsilon(c)| + |\varphi_\varepsilon(c) - \varphi(c)| \leq \frac{1}{\lambda}|b_\varepsilon - c| + |\varphi_\varepsilon(c) - \varphi(c)|,
\]
from which we infer that \( b = c \) by letting \( \varepsilon \rightarrow 0 \). Because the limit does not depend on the subsequence, the whole sequence \((b_\varepsilon)_\varepsilon\) converges to \( b \). Consequently, \( \tilde{\varphi}_\varepsilon(t) \rightarrow \varphi(t) = t \) for all \( t \in \tilde{I} \), and since also \( \|\tilde{\varphi}_\varepsilon\|_{W^{1,\infty}(I)} = O(1) \) as \( \varepsilon \rightarrow 0 \), we deduce that
\[
\tilde{\varphi}_\varepsilon \rightharpoonup^* \varphi \text{ in } W^{1,\infty}(I) \quad \text{and} \quad \|\tilde{\varphi}_\varepsilon - \varphi\|_{L^\infty(I)} \rightarrow 0.
\] (3.16)

Finally, we set \( w_{\varepsilon,\delta} := v_\delta \circ \tilde{\varphi}_\varepsilon \in W^{1,\infty}(I;\mathbb{R}^d) \), and observe that
\[
\|w_{\varepsilon,\delta} - w\|_{L^1(I;\mathbb{R}^d)} \leq \|v_\delta \circ \tilde{\varphi}_\varepsilon - v_\delta\|_{L^1(I;\mathbb{R}^d)} + \|v_\delta - w\|_{L^1(I;\mathbb{R}^d)} \quad \text{and} \quad \int_I |w_{\varepsilon,\delta}'| \, dt = \int_I |v_\delta'| \circ \tilde{\varphi}_\varepsilon |\tilde{\varphi}_\varepsilon'| \, dt.
\]
Hence, by (3.15), (3.16), the boundedness of each \( v_\delta \) and \( v'_\delta \), and a weak-strong convergence argument, it follows that
\[
\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \| w_{\varepsilon,\delta} - w \|_{L^1(I;\mathbb{R}^d)} = 0, \tag{3.17}
\]
\[
\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \int_I |w_{\varepsilon,\delta}'|\, dt = \lim_{\delta \to 0} \int_I |v'_\delta| = \lim_{\delta \to 0} \int_I |v'_{\delta,\varepsilon}|\, dt = |Du|(I). \tag{3.18}
\]

In view of (3.17) and (3.18), we apply Attouch’s diagonalization lemma [4] to find a sequence \( (w_{\varepsilon})_\varepsilon \subset W^{1,1}(I;\mathbb{R}^d) \) with \( w_{\varepsilon} := w_{\varepsilon,\delta(\varepsilon)} \) satisfying (i) and (ii). We observe further that each \( w_{\varepsilon} \) satisfies (iii) by construction.

To conclude, we address the issue of constraint-preserving approximations for \( w \in BV(I;SO(2)) \). In this case, we argue as above, but replace the density argument leading to (3.15) by its analogue for \( BV \) functions with values on manifolds, see [28, Theorem 1.2]. This allows us to assume that \( v_\delta \subset C^\infty(I;SO(2)) \), and eventually yields \( w_{\varepsilon} \subset W^{1,\infty}(I;SO(2)) \).

We are now in a position to prove Theorem 1.1.

**Proof of Theorem 1.1.** In view of the discussion on locally and globally one-dimensional functions in Section 2.4, it suffices to prove the statement on rectangles with sides parallel to the axes. A simple modification of the proofs of Propositions 3.1 and 3.2 shows that these results hold for any such rectangle. Then, Theorem 1.1 follows by extension and exhaustion arguments in the spirit of [14, Lemma A.2].

**Remark 3.4 (The higher dimensional setting).** We point out that the results of Theorem 1.1 continue to hold for domains \( \Omega \subset \mathbb{R}^N, N \in \mathbb{N} \), satisfying the flatness and cross-connectedness assumptions in [14, Definitions 3.6 and 3.7]. We omit the proof here as it follows from that of Theorem 1.1 up to minor adaptations. Notice in particular that [13, Lemma A1] provides a higher-dimensional version of (3.7).

We conclude this section by characterizing two special subsets of \( \mathcal{B} \) (see (3.3)), which will be useful in the following. Using (2.6), it can be checked that
\[
\mathcal{B} \cap W^{1,1}(\Omega;\mathbb{R}^2) = \{ u \in W^{1,1}(\Omega;\mathbb{R}^2) : u(x) = R(x_2)x + \psi(x_2) \text{ for a.e. } x \in \Omega, \\
\quad \text{ with } R \in W^{1,1}(a_\Omega,b_\Omega;SO(2)) \text{ and } \psi \in W^{1,1}(a_\Omega,b_\Omega;\mathbb{R}^2) \}, \tag{3.19}
\]
and
\[
\mathcal{B} \cap SBV(\Omega;\mathbb{R}^2) = \{ u \in SBV(\Omega;\mathbb{R}^2) : u(x) = R(x_2)x + \psi(x_2) \text{ for a.e. } x \in \Omega, \\
\quad \text{ with } R \in SBV(a_\Omega,b_\Omega;SO(2)) \text{ and } \psi \in SBV(a_\Omega,b_\Omega;\mathbb{R}^2) \}. \tag{3.20}
\]

By definition, and accounting for the fact that \( R \) takes values in \( SO(2) \), the jump set of \( u \in \mathcal{B} \cap SBV(\Omega;\mathbb{R}^2) \) is related to the jump sets of \( R \) and \( \psi \) via
\[
J_u = [(c_\Omega,d_\Omega) \times (J_R \cup J_\psi)] \cap \Omega, 
\]
cf. (1.2).

4. Asymptotic behavior of admissible layered deformations

In this section, we prove Theorem 1.3, which characterizes the asymptotic behavior of deformations of bilayered materials that coincide with rigid body rotations on the stiffer layers, and are subject to a single slip constraint on the softer layers. The latter is described with the help of the set
\[
\mathcal{M}_{c_1} = \{ F \in \mathbb{R}^{2 \times 2} : \det F = 1 \text{ and } |Fe_1| = 1 \} \\
= \{ F \in \mathbb{R}^{2 \times 2} : F = R(1 + \gamma e_1 \otimes e_2) \text{ with } R \in SO(2) \text{ and } \gamma \in \mathbb{R} \}. \tag{4.1}
\]
As in the previous section, we consider \( \Omega = (0,1) \times (-1,1) \) for simplicity. The results for general \( x_1 \)-connected domains follow as in the proof of Theorem 1.1.

Using the representations of \( \mathcal{M}_{c_1} \) in (4.1) and recalling the sets \( \mathcal{B}_\varepsilon \) introduced in (3.1), the sets of admissible layered deformations defined in (1.6) admit the equivalent representations
\[
\mathcal{A}_\varepsilon = \mathcal{B}_\varepsilon \cap \{ u \in W^{1,1}(\Omega;\mathbb{R}^2) : \nabla u \in \mathcal{M}_{c_1} \text{ a.e. in } \Omega \} \\
= \{ u \in W^{1,1}(\Omega;\mathbb{R}^2) : \nabla u = R(1 + \gamma e_1 \otimes e_2) \text{ with } R \in L^\infty(\Omega;SO(2)) \text{ and } \gamma \in L^1(\Omega) \text{ such that } \gamma = 0 \in \varepsilon Y_{rig} \cap \Omega \}. \tag{4.2}
\]
In the sequel, according to the context, we will always adopt the most convenient representation.

In analogy with $\mathcal{B}_0$ defined in (3.2), we introduce the set

$$\mathcal{A}_0 := \{u \in BV(\Omega; \mathbb{R}^2) : \text{there exists } (u_\varepsilon)_\varepsilon \subset W^{1,1}(\Omega; \mathbb{R}^2) \text{ with } u_\varepsilon \in \mathcal{A}_\varepsilon \text{ for all } \varepsilon \}$$

(4.3)

such that $u_\varepsilon \rightharpoonup u$ in $BV(\Omega; \mathbb{R}^2)$

of asymptotically admissible deformations. We aim at characterizing $\mathcal{A}_0$, or suitable subclasses thereof, in terms of the set $\mathcal{A}$ introduced in (1.11). Note that

$$\mathcal{A} = \mathcal{B} \cap \{u \in BV(\Omega; \mathbb{R}^2) : \det \nabla u = 1 \text{ a.e. in } \Omega\},$$

(4.4)

where $\mathcal{B}$ is given by (3.3). Moreover, recalling the notation for the distributional derivative of one-dimensional $BV$-functions discussed in Section 2.2, we can equivalently express $\mathcal{A}$ as follows.

**Proposition 4.1.** Let $\Omega = (0,1) \times (-1,1)$. Then, $\mathcal{A}$ from (1.11) admits these two alternative representations:

$$\mathcal{A} = \{u \in BV(\Omega; \mathbb{R}^2) : \nabla u(x) = R(x_2)(\mathbb{I} + \gamma(x_2)e_1 \otimes e_2) \text{ for a.e. } x \in \Omega, \text{ with }$$

$$R \in BV((-1,1; SO(2)), \gamma \in L^1((-1,1), \text{ and } (D^s u)e_1 = 0\}$$

(4.5)

and

$$\mathcal{A} = \{u \in BV(\Omega; \mathbb{R}^2) : u(x) = R(x_2)x + \psi(x_2) \text{ for a.e. } x \in \Omega, \text{ with } R \in BV((-1,1; SO(2))$$

$$\text{and } \psi \in BV((-1,1; \mathbb{R}^2) \text{ such that } \psi' \cdot Re_2 = 0 \text{ and } R' = 0 \text{ a.e. in } (-1,1)\}.$$  

(4.6)

**Proof.** Let $\hat{\mathcal{A}}$ and $\tilde{\mathcal{A}}$ denote the sets on the right-hand side of (4.5) and (4.6), respectively. We will show that $\mathcal{A} \subset \hat{\mathcal{A}} \cap \tilde{\mathcal{A}} \subset \mathcal{A}$, and $\tilde{\mathcal{A}} \subset \hat{\mathcal{A}}$, from which (4.5) and (4.6) follow.

We start by proving that $\mathcal{A} \subset \hat{\mathcal{A}} \cap \tilde{\mathcal{A}}$. Fix $u \in \mathcal{A}$. Due to (2.6), we have $(D^s u)e_1 = 0$ and

$$\nabla u = R + (R'x + \psi') \otimes e_2 = R(\mathbb{I} + R^T(R'x + \psi') \otimes e_2).$$

(4.7)

We first observe that the condition $\det \nabla u = 1$ becomes $1 + R^T(R'x + \psi') \cdot e_2 = 1$ or, equivalently, $(R'x + \psi') \cdot Re_2 = 0$. This condition, together with the independence of $R$, $R'$, and $\psi'$ on $x_1$, yields

$$R'e_1 \cdot Re_2 = 0 \text{ and } (x_2R'e_2 + \psi') \cdot Re_2 = 0.$$ 

(4.8)

Let $\theta \in BV((-1,1; [-\pi, \pi])$ be as in (2.7). Then, the first condition in (4.8) gives $\theta' = 0$; consequently, also $R' = 0$. Thus, the second equation in (4.8) becomes $\psi' \cdot Re_2 = 0$, which shows that $u \in \hat{\mathcal{A}}$. Moreover, $\psi' \cdot Re_2 = 0$ is equivalent to $R^T\psi' \cdot e_2 = 0$; hence, $u \in \tilde{\mathcal{A}}$ with $\gamma := Re_1 \cdot \psi'$. Thus, $\mathcal{A} \subset \hat{\mathcal{A}} \cap \tilde{\mathcal{A}}$.

Next, we observe that if $u \in \hat{\mathcal{A}}$, then, using (4.7), we have

$$\det \nabla u = 1 + R^T(R'x + \psi') \cdot e_2 = 1 + R^T\psi' \cdot e_2 = 1 + \psi' \cdot Re_2 = 1.$$ 

Hence, $u \in \mathcal{A}$, which shows that $\hat{\mathcal{A}} \subset \mathcal{A}$.

Finally, we prove that $\tilde{\mathcal{A}} \subset \mathcal{A}$. Let $u \in \tilde{\mathcal{A}}$. Then, $(Du)e_1 = (\nabla u)e_1L^2[\Omega] + (D^s u)e_1 = Re_1L^2[\Omega]$. By this identity and the Du Bois-Reymond lemma (see [32], for instance), we can find $\phi \in BV(-1,1; \mathbb{R}^2)$ such that

$$u(x) = R(x_2)x_1e_1 + \phi(x_2).$$

In particular, $\nabla u(x) = R(x_2)e_1 \otimes e_1 + (R'(x_2)x_1e_1 + \phi'(x_2)) \otimes e_1$. Consequently, using the expression for $\nabla u$ given by the definition of $\tilde{\mathcal{A}}$, together with the independence of $R$, $R'$, $\gamma$, and $\phi'$ on $x_1$, we conclude that

$$R' = 0 \text{ and } \phi' = Re_2 + \gamma Re_1.$$ 

Finally, set $\psi(x_2) := \phi(x_2) - R(x_2)x_2e_2$ for $x_2 \in (-1,1)$. Then, we have $\psi \in BV(-1,1; \mathbb{R}^2)$, which satisfies $\psi' \cdot Re_2 = \gamma Re_1 \cdot Re_2 = 0$, because $R \in SO(2)$ in $(-1,1)$, and also $u(x) = R(x_2)x + \psi(x_2)$. Thus, $u \in \tilde{\mathcal{A}}$, which implies $\tilde{\mathcal{A}} \subset \mathcal{A}$. □

The following lemma on weak continuity of Jacobian determinants for gradients in $W^{1,1}(\Omega; \mathbb{R}^2)$ with suitable additional properties will be instrumental in the proof of the inclusion $\mathcal{A}_0 \subset \mathcal{A}$.
Lemma 4.2 (Weak continuity properties of Jacobian determinants). Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain, and let $(u_\varepsilon)_{\varepsilon} \subset W^{1,1}(\Omega; \mathbb{R}^2)$ be a uniformly bounded sequence satisfying $\det \nabla u_\varepsilon = 1$ a.e. in $\Omega$ for all $\varepsilon$ and

$$\|\partial_1 u_\varepsilon\|_{L^\infty(\Omega; \mathbb{R}^2)} \leq C,$$

where $C$ is a positive constant independent of $\varepsilon$. If $u_\varepsilon \rightarrow u$ in $L^1(\Omega; \mathbb{R}^2)$ for some $u \in BV(\Omega; \mathbb{R}^2)$, then $\det \nabla u = 1$ a.e. in $\Omega$.

Proof. The claim in Lemma 4.2 would be an immediate consequence of [26, Theorem 2] if in place of (4.9), we required

$$(\text{adj} \nabla u_\varepsilon)_{\varepsilon} \subset L^2(\Omega; \mathbb{R}^{2\times 2}),$$

which, because of the structure of the adjoint matrix in this two-dimensional setting, is equivalent to $\nabla u_\varepsilon \in L^2(\Omega; \mathbb{R}^{2\times 2})$ for all $\varepsilon$. Even though we are not assuming this here, it is still possible to validate the arguments of [26, Proof of Theorem 2] in our context, as we detail next.

Since $|\text{adj} \nabla u_\varepsilon| = |\nabla u_\varepsilon|$, it can be checked that in order to mimic the proof of [26, Theorem 2] with $N = 2$, we are only left to prove the following: If $(\varphi_j)_{j \in \mathbb{N}}$ is a sequence of standard mollifiers and $\Omega'$ is an arbitrary open set compactly contained in $\Omega$, then $(\det \nabla u_{\varepsilon,j})_{j \in \mathbb{N}}$ converges to $\det \nabla u_j$ in $L^1(\Omega')$ as $j \rightarrow \infty$ for all $\varepsilon$, where $u_{\varepsilon,j} := \varphi_j * u_\varepsilon$.

In Step 4 of the proof of [26, Theorem 2], this convergence is a consequence of the Vitali-Lebesgue lemma using (4.10), the bound $|\det A| \leq |\text{adj} A|^2$ for all $A \in \mathbb{R}^{2\times 2}$ (see [26, (7)]), and well-known properties of mollifiers.

Here, similar arguments can be invoked, but instead of the estimate $|\det A| \leq |\text{adj} A|^2$ for $A \in \mathbb{R}^{2\times 2}$, we use the fact that (4.9) yields

$$|\det u_{\varepsilon,j}| = |(\partial_1 u_{\varepsilon,j})^\perp \partial_2 u_{\varepsilon,j}| \leq C \|\partial_2 u_{\varepsilon,j}\| \leq C \|\nabla u_{\varepsilon,j}\|$$

a.e. in $\Omega$. Hence, since $u_{\varepsilon,j} \rightarrow u_\varepsilon$ in $W^{1,1}(\Omega; \mathbb{R}^2)$ and pointwise a.e. in $\Omega$ as $j \rightarrow \infty$, we conclude that $(\det \nabla u_{\varepsilon,j})_{j \in \mathbb{N}}$ converges to $\det \nabla u_j$ in $L^1(\Omega')$ as $j \rightarrow \infty$ for all $\varepsilon$ by the Vitali-Lebesgue lemma.

We obtain from the following proposition that weak* limits of sequences in $A_\varepsilon$ belong to $A$.

Proposition 4.3 (Asymptotic behavior of sequences in $A_\varepsilon$). Let $\Omega = (0, 1) \times (-1, 1)$. Then,

$$A_0 \subset A,$$

where $A_0$ and $A$ are the sets introduced in (4.3) and (1.11), respectively.

Proof. The statement follows from the inclusion $A_\varepsilon \subset B_\varepsilon$ (see (4.2)) and the identity (4.4) in conjunction with Proposition 3.1 and Lemma 4.2, observing that the condition $\nabla u_\varepsilon \in \mathcal{M}_{\varepsilon,1}$ a.e. in $\Omega$ guarantees $|\partial_1 u_\varepsilon| = |\nabla u_{\varepsilon,1}| = 1$ a.e. in $\Omega$, and hence $|\partial_1 u_\varepsilon|_{L^\infty(\Omega; \mathbb{R}^2)} = 1$ for any $\varepsilon$.

The question whether the set $A$ can be further identified as limiting set for sequences in $A_\varepsilon$, namely, whether the equality $A_0 = A$ is true, cannot be answered at this point. However, as stated in Theorem 1.3, the inclusions $A_0 \supset A \cap SBV_\varepsilon(\Omega; \mathbb{R}^2)$ and $A_0 \supset A^1$ hold. Before proving these inclusions, we discuss a further characterization of some special subsets of $A$.

Remark 4.4 (Structure of subsets of $A$). Similarly to (3.19) and (3.20), using fine properties of one-dimensional $BV$ functions, the sets $A \cap W^{1,1}(\Omega; \mathbb{R}^2)$, $A \cap SBV(\Omega; \mathbb{R}^2)$, and $A \cap SBV_\varepsilon(\Omega; \mathbb{R}^2)$ can be characterized as follows.

(a) In view of (2.6) and (1.6), one observes that

$$A \cap W^{1,1}(\Omega; \mathbb{R}^2) = \{ u \in W^{1,1}(\Omega; \mathbb{R}^2) : u(x) = Rx + \theta(x_2)Re_1 + c \text{ for a.e. } x \in \Omega, \text{ with } R \in SO(2), \theta \in W^{1,1}(-1,1), c \in \mathbb{R} \}$$

$$= \{ u \in W^{1,1}(\Omega; \mathbb{R}^2) : \nabla u(x) = R[I + \gamma(x_2)e_1 \otimes e_2] \text{ for a.e. } x \in \Omega, \text{ with } R \in SO(2), \gamma \in L^1(-1,1) \}.$$
such that $u_\varepsilon \rightharpoonup u$ in $W^{1,1}(\Omega; \mathbb{R}^2)$.

(b) Using (2.6) and (4.6) once more, we have

$$
\mathcal{A} \cap \text{SBV}(\Omega; \mathbb{R}^2) = \{ u \in \text{SBV}(\Omega; \mathbb{R}^2) : u(x) = R(x_2)x + \psi(x_2) \text{ for a.e. } x \in \Omega,
\text{ with } R \in \text{SBV}(-1,1; SO(2)) \text{ and } \psi \in \text{SBV}(-1,1; \mathbb{R}^2)
\text{ such that } R' = 0 \text{ and } \psi' \cdot Re_2 = 0 \text{ a.e. in } (-1,1) \}.
$$

Note that both $J_R$ and $J_\psi$ are given by an at most countable union of points in $(-1,1)$, which implies that $J_u$ consists of at most countably many segments parallel to $e_1$. It is not possible to conclude that the functions $R$ are piecewise constant according to [2, Definition 4.21], as we have, a priori, no control on $H^0(J_R)$ (cf. [2, Example 4.24]).

(c) With (b) and [2, Theorem 4.23], and recalling (2.8), it follows that

$$
\mathcal{A} \cap \text{SBV}_\infty(\Omega; \mathbb{R}^2) = \{ u \in \text{SBV}_\infty(\Omega; \mathbb{R}^2) : u(x) = R(x_2)x + \psi(x_2) \text{ for a.e. } x \in \Omega,
\text{ with } R \in PC(-1,1; SO(2)) \text{ and } \psi \in \text{SBV}_\infty(-1,1; \mathbb{R}^2)
\text{ such that } \psi' \cdot Re_2 = 0 \text{ a.e. in } (-1,1) \}.
$$

Here, both $J_R$ and $J_\psi$ are finite sets of points in $(-1,1)$, and $J_u$ is given by a finite union of segments parallel to $e_1$. Alternatively, one can express $\mathcal{A} \cap \text{SBV}_\infty(\Omega; \mathbb{R}^2)$ with the help of a Caccioppoli partition of $\Omega$ into finitely many horizontal strips; precisely,

$$
\mathcal{A} \cap \text{SBV}_\infty(\Omega; \mathbb{R}^2) = \{ u \in \text{SBV}_\infty(\Omega; \mathbb{R}^2) : \nabla u|_{E_i} = R_i(\| + \gamma_i e_1 \otimes e_2), \text{ with } \{ E_i \}_{i=1}^n \text{ a partition of } \Omega
\text{ such that } E_i = (\mathbb{R} \times I_i) \cap \Omega \text{ with } I_i \subset (-1,1) \text{ for } i = 1, \ldots, n,
R_i \in SO(2) \text{ and } \gamma_i \in L^1(E_i) \text{ with } \partial_1 \gamma_i = 0 \text{ for } i = 1, \ldots, n \}.
$$

In the following lemma, we construct an admissible piecewise affine approximation for basic limit deformations in $\mathcal{A} \cap \text{SBV}_\infty(\Omega; \mathbb{R}^2)$ with a non-trivial jump along the horizontal line at $x_2 = 0$. Based on this construction, we will then establish the inclusion $\mathcal{A}_0 \supset \mathcal{A} \cap \text{SBV}_\infty(\Omega; \mathbb{R}^2)$ in Proposition 4.7 below.

**Lemma 4.5 (Approximation of maps in $\mathcal{A} \cap \text{SBV}_\infty$ with a single jump).** Let $\Omega = (0,1) \times (-1,1)$, and let $u \in \mathcal{A} \cap \text{SBV}_\infty(\Omega; \mathbb{R}^2)$ be such that $u(x) = R(x_2)x + \psi(x_2)$ for a.e. $x \in \Omega$, where

$$
R(t) := \begin{cases} 
R^+ & \text{if } t \in [0,1), \\
R^- & \text{if } t \in (-1,0),
\end{cases}
$$

and $\psi(t) := \begin{cases} 
\psi^+ & \text{if } t \in [0,1), \\
\psi^- & \text{if } t \in (-1,0),
\end{cases}$ for $t \in (-1,1)$,

with some $R^\pm \in SO(2)$ and $\psi^\pm \in \mathbb{R}^2$. Then, there exists a sequence $(u_\varepsilon)_\varepsilon \subset W^{1,1}(\Omega; \mathbb{R}^2)$ with $\int_\Omega u_\varepsilon \, dx = \int_\Omega u \, dx$ and $u_\varepsilon \in \mathcal{A}_\varepsilon$ for all $\varepsilon$, and such that $u_\varepsilon \rightharpoonup u$ in $BV(\Omega; \mathbb{R}^2)$.

**Proof.** We start by observing that for $u$ as in the statement of the lemma, there holds

$$
Du = R L^2[\Omega + [(R^+ - R^-)e_1 x_1 + (\psi^+ - \psi^-)] \otimes e_2] H^1((0,1) \times \{0\}).
$$

Let $S \in SO(2)$ be such that (i) $S \neq R^\pm$; (ii) $S e_1$ and $R^\pm e_1$ are linearly independent; (iii) $\theta^\pm \in (-\pi, \pi) \setminus \{0\}$ is the rotation angle of $S T R^\pm$, cf. (2.7). Due to (ii), there exist $\alpha, \beta \in \mathbb{R}$ such that

$$
\psi^+ - \psi^- = \alpha R^\pm e_1 + \beta S e_1.
$$

For each $\varepsilon > 0$, set

$$
\gamma^+_\varepsilon := \frac{4\alpha}{\varepsilon \lambda^2}, \, \gamma^-_\varepsilon := \frac{4\beta}{\varepsilon \lambda^2}, \, \mu^+_\varepsilon := \pm \frac{4}{\varepsilon \lambda} + \tan \left(\frac{\theta^\pm}{2}\right), \, \mu^-_\varepsilon := \pm \frac{4}{\varepsilon \lambda} - \tan \left(\frac{\theta^\pm}{2}\right),
$$

where

$$
\lambda := \max \{ |R^+|, |R^-|, |\psi^+ - \psi^-| \}.
$$
and let $V_\varepsilon \in L^1(\Omega; \mathbb{R}^{2\times 2})$ be the function defined by

$$
V_\varepsilon(x) = \begin{cases} 
R^+ & \text{if } x \in (0, 1) \times (\varepsilon \lambda, 1), \\
R^+(I + \gamma_\varepsilon e_1 \otimes e_2) & \text{if } x \in (0, 1) \times \left(\frac{3\varepsilon \lambda}{4}, \varepsilon \lambda\right), \\
R^+(I + \mu_\varepsilon^2 e_1 \otimes e_2) & \text{if } x_1 \in (0, 1) \text{ and } x_2 \in \left(-\frac{\varepsilon}{4}, -\frac{\varepsilon}{4}\right) x_1 + \frac{3\varepsilon \lambda}{4}, \\
S(I + \mu_\varepsilon^2 e_1 \otimes e_2) & \text{if } x_1 \in (0, 1) \text{ and } x_2 \in \left(\frac{\varepsilon}{4}, \frac{\varepsilon}{4}\right) x_1 + \frac{3\varepsilon \lambda}{4}, \\
S(I + \gamma_\varepsilon e_1 \otimes e_2) & \text{if } x \in (0, 1) \times \left(\frac{3\varepsilon \lambda}{4}, \frac{\varepsilon}{4}\right), \\
R^-(I + \mu_\varepsilon^2 e_1 \otimes e_2) & \text{if } x \in (0, 1) \text{ and } x_2 \in (0, \frac{\varepsilon}{4} x_1), \\
R^- & \text{if } x \in (0, 1) \times (-1, 0),
\end{cases}
$$

(4.15)

see Figure 3.

![Figure 3. Construction of $V_\varepsilon$.](image)

By construction, each function $V_\varepsilon$ takes values only in $\mathcal{M}_{\varepsilon_1}$, and its piecewise definition is chosen such that neighboring matrices in Figure 3 are rank-one-connected along their separating lines according to [13, Lemma 3.1]. Hence, there exists a Lipschitz function $u_\varepsilon \in W^{1,\infty}(\Omega; \mathbb{R}^2)$ such that $\nabla u_\varepsilon = V_\varepsilon$. By adding a suitable constant, we may assume that $\int_\Omega u_\varepsilon \, dx = \int_\Omega u \, dx$. In view of the Poincaré–Wirtinger inequality and (4.15), $(u_\varepsilon)_\varepsilon$ is a uniformly bounded sequence in $W^{1,1}(\Omega; \mathbb{R}^2)$ satisfying $u_\varepsilon \in \mathcal{A}_\varepsilon$ for all $\varepsilon$ (cf. (4.2)).

To prove that $u_\varepsilon \rightharpoonup u$ in $BV(\Omega; \mathbb{R}^2)$, it suffices to show that

$$
Du_\varepsilon \rightharpoonup Du \text{ in } \mathcal{M}(\Omega; \mathbb{R}^{2\times 2}),
$$

(4.16)

or, equivalently, in view of (4.12), that for every $\varphi \in C_0(\Omega; \mathbb{R}^2),

$$
\lim_{\varepsilon \to 0} \int_\Omega \nabla u_\varepsilon(x) \varphi(x) \, dx = \int_\Omega R(x_2) \varphi(x) \, dx + \int_0^1 \left( \int_{\Omega} (\nabla u_\varepsilon \cdot \nabla \varphi) \, dx \right) \, dx.
$$

(4.17)

Clearly,

$$
\lim_{\varepsilon \to 0} \int_{(0,1) \times ((-1,0) \cup (\varepsilon \lambda, 1))} \nabla u_\varepsilon(x) \varphi(x) \, dx = \lim_{\varepsilon \to 0} \int_{(0,1) \times ((-1,0) \cup (\varepsilon \lambda, 1))} R(x_2) \varphi(x) \, dx
$$

$$
= \int_\Omega R(x_2) \varphi(x) \, dx.
$$

(4.18)

Moreover, using (4.14), a change of variables, and Lebesgue’s dominated convergence theorem together with the continuity and boundedness of $\varphi$, we have

$$
\lim_{\varepsilon \to 0} \int_{(0,1) \times (0, \frac{\varepsilon}{4} x_1)} \nabla u_\varepsilon(x) \varphi(x) \, dx
$$

...
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^2} \nabla u_\varepsilon(x) \varphi(x) \, dx
\]
\[
= \lim_{\varepsilon \to 0} \int_0^1 \int_0^1 R^{-}(\varepsilon + \tan \theta) e_1 \otimes e_2 - \frac{1}{\varepsilon} e_1 \otimes e_2) \varphi(x) \, dx_2 \, dx_1
\]
\[
= \lim_{\varepsilon \to 0} \int_0^1 \int_0^1 R^{-}(\varepsilon \varepsilon^{2}) \tan \theta) e_1 \otimes e_2 - e_1 \otimes e_2) \varphi(x_1, \varepsilon \varepsilon^{2}) \, dz \, dx_1
\]
\[
= - \int_0^1 \int_0^1 R^{-} e_1 \otimes e_2 \varphi(x_1, 0) \, dz \, dx_1 = - \int_0^1 x_1 R^{-} e_1 \otimes e_2 \varphi(x_1, 0) \, dx_1.
\] (4.19)

Similarly,
\[
\lim_{\varepsilon \to 0} \int_{(0,1) \times (\varepsilon \varepsilon^{2}, \varepsilon \varepsilon^{2})} \nabla u_\varepsilon(x) \varphi(x) \, dx
\]
\[
= \lim_{\varepsilon \to 0} \int_0^1 \int_0^1 \int_{x_1}^1 \int_{x_2}^1 S(\varepsilon \varepsilon^{2}) \tan \theta) e_1 \otimes e_2 - e_1 \otimes e_2) \varphi(x_1, \varepsilon \varepsilon^{2}) \, dz \, dx_1
\]
\[
= \int_0^1 (x_1 - 1) e_1 \otimes e_2 \varphi(x_1, 0) \, dx_1,
\] (4.20)

and
\[
\lim_{\varepsilon \to 0} \int_{(0,1) \times (-\varepsilon \varepsilon^{2}, \varepsilon \varepsilon^{2})} \nabla u_\varepsilon(x) \varphi(x) \, dx
\]
\[
= \lim_{\varepsilon \to 0} \int_0^1 \int_0^1 \int_{x_1}^1 \int_{x_2}^1 \int_{x_3}^1 R^{+}(\varepsilon \varepsilon^{2}) \tan \theta) e_1 \otimes e_2 + e_1 \otimes e_2) \varphi(x_1, \varepsilon \varepsilon^{2}) \, dz \, dx_1
\]
\[
= \int_0^1 x_1 R^{+} e_1 \otimes e_2 \varphi(x_1, 0) \, dx_1,
\] (4.23)

and
\[
\lim_{\varepsilon \to 0} \int_{(0,1) \times \varepsilon \varepsilon^{2}, \varepsilon \varepsilon^{2}} \nabla u_\varepsilon(x) \varphi(x) \, dx
\]
\[
= \lim_{\varepsilon \to 0} \int_0^1 \int_3^1 \int_3^1 R^{+}(\varepsilon \varepsilon^{2}) + \alpha e_1 \otimes e_2) \varphi(x_1, \varepsilon \varepsilon^{2}) \, dz \, dx_1 = \int_0^1 \alpha R^{+} e_1 \otimes e_2 \varphi(x_1, 0) \, dx_1.
\] (4.24)

Combining (4.18)–(4.24) and (4.13), we finally obtain (4.17). \qed

**Remark 4.6 (On the construction in Lemma 4.5).** Notice that the main idea of the construction in the proof of Lemma 4.5 for dealing with jumps is to use piecewise affine functions that are as simple as possible to accommodate them. Since triple junctions where two of the three angles add up to \(\pi\) are not compatible (compare with [13, Lemma 3.1]), we work with inclined interfaces that stretch over the full width of \(\Omega\).

Let \(u \in A \cap SBV_{\infty}(\Omega; \mathbb{R}^2)\) be as in Lemma 4.5, and assume that either \(R^+ \neq \pm R^-\) or \(R^+ = R^-\). In these cases, we can simplify the construction of \(u_\varepsilon\) in the previous proof. We focus here on stating the counterparts of Figure 3 and (4.14), and omit the detailed calculations, which are very similar to (4.18)–(4.24). Note further that these constructions are not just simpler, but also energetically more favorable, see Remark 5.2 below for more details.

(i) If \(R^+ \neq \pm R^-\), we may replace the construction depicted in Figure 3 by:
(ii) If $R$ is constant, i.e., $R^+ = R^-$, and $\psi^+ - \psi^-$ is not parallel to $Re_1$, the construction in Figure 3 can be replaced by:

Figure 5. Alternative construction of $V_\varepsilon$ if $R^+ \neq \pm R^-$. 

(iii) If $R$ is constant, i.e., $R^+ = R^-$, and $\psi^+ - \psi^-$ is parallel to $Re_1$, then we can use the following construction in place of Figure 3:

Figure 6. Alternative construction of $V_\varepsilon$ if $R$ is constant and $\psi^+ - \psi^-$ is parallel to $Re_1$.

Note that in case (i), the slope $\rho$ of the interfaces can attain any value between 0 and 1, while in (ii), $\rho$ is determined by the value of $\beta$. In terms of the energies, the construction in case (iii) provides an optimal approximation, which will be detailed in Section 6.

We proceed by extending Lemma 4.5 to arbitrary functions $u \in A \cap SBV_\infty(\Omega; \mathbb{R}^2)$. 

\[
\psi^+ - \psi^- = \alpha R^+ e_1 + \beta R^- e_1
\]

$\theta \in (-\pi, \pi) \setminus \{0\}$ rotation angle of $(R^-)^T R^+$

$\rho \in (0, 1)$, 

$\gamma_\epsilon^+ := \frac{\epsilon \lambda_\epsilon}{\rho + \tan(\frac{\epsilon \lambda_\epsilon}{\rho})}$, 

$\gamma_\epsilon^- := \frac{\epsilon \lambda_\epsilon}{\rho - \tan(\frac{\epsilon \lambda_\epsilon}{\rho})}$

$\gamma_\epsilon^+$ satisfies $\alpha = \lim_{\epsilon \to 0} \gamma_\epsilon^+ (\epsilon \lambda - \epsilon h_\epsilon - \rho \epsilon \lambda)$

$\gamma_\epsilon^-$ satisfies $\beta - 1 = \lim_{\epsilon \to 0} \gamma_\epsilon^- \epsilon h_\epsilon$

$S \in SO(2)$: $Re_1$ and $Se_1$ are linearly independent

$\psi^+ - \psi^- = \alpha Re_1 + \beta Se_1$, $\beta \neq 0$, $\iota := \sign(\beta)$

$\theta \in (-\pi, \pi) \setminus \{0\}$ rotation angle of $R^T S$

$\rho := \frac{|\alpha|}{2}$, $\epsilon h_\epsilon := \frac{\epsilon \lambda_\epsilon - \rho \lambda_\epsilon}{2}$

$\gamma_\epsilon^+ := \frac{\epsilon |\alpha|}{\rho + \tan(\frac{\epsilon |\alpha|}{\rho})}$, 

$\gamma_\epsilon^- := \frac{\epsilon |\alpha|}{\rho - \tan(\frac{\epsilon |\alpha|}{\rho})}$

$\gamma_\epsilon$ satisfies $\alpha - \iota = \lim_{\epsilon \to 0} \gamma_\epsilon h_\epsilon$

$\psi^+ - \psi^- = \alpha Re_1$

$\alpha = \iota |\psi^+ - \psi^-|$, $\iota := \sign((\psi^+ - \psi^-) \cdot Re_1)$
Proposition 4.7. Let \( \Omega = (0,1) \times (-1,1) \). Then, for every \( u \in A \cap SBV_\infty(\Omega;\mathbb{R}^2) \), there exists a sequence \( (u_\varepsilon)_\varepsilon \subset W^{1,1}(\Omega;\mathbb{R}^2) \) with \( \int_\Omega u_\varepsilon \, dx = \int_\Omega u \, dx \) and \( u_\varepsilon \in A_\varepsilon \) for all \( \varepsilon \), and such that \( u_\varepsilon \rightharpoonup u \) in \( BV(\Omega;\mathbb{R}^2) \) or, in other words,

\[
A \cap SBV_\infty(\Omega;\mathbb{R}^2) \subset A_0,
\]

cf. (4.3).

Proof. In view of Remark 4.4 (c), it holds that \( J_u = \bigcup_{i=1}^\ell \{ a_i \} \) for some \( \ell \in \mathbb{N} \) and \( a_i \in (-1,1) \) with \( a_1 < a_2 < \cdots < a_\ell \), and setting \( a_0 := -1 \) and \( a_{\ell+1} := 1 \), gives

\[
Du = \sum_{i=0}^\ell R_i(1 + \gamma e_i \otimes e_2) \mathcal{L}^2 \left( (0,1) \times \{ a_i, a_{i+1} \} \right)
\]

\[
+ \sum_{i=1}^\ell \left( (R_i - R_{i-1}) x_1 e_1 + (R_i a_{i+2} + \psi_1^+ - R_{i-1} a_i e_2 - \psi_1^-) \right) \otimes e_2 \mathcal{H}^1 \left( (0,1) \times \{ a_i \} \right),
\]

(4.25)

where \( \gamma \in L^1(-1,1) \), and \( R_i \in SO(2) \) and \( \psi_1 \in \mathbb{R} \) for \( i = 0,\ldots,\ell \).

We now perform a similar construction as in Lemma 4.5 in a convenient softer layer near each \( a_i \), accounting for the possibility that one or more of the jump lines may not intersect \( \varepsilon Y_{\text{soft}} \cap \Omega \), and replacing \( R^+ \) by \( R_i^+ \), \( R^- \) by \( R_{i-1}^- \), \( \psi^+ \) by \( R_i a_{i+2} + \psi_1^+ \) and \( \psi^- \) by \( R_{i-1} a_i e_2 + \psi_1^- \).

To be precise, fix \( \varepsilon > 0 \) and \( i \in \{1,\ldots,\ell\} \). Let \( S_i \in SO(2) \) be such that (i) \( S_i \notin \{R_{i-1}, R_i\} \); (ii) \( S_i e_1 \) and \( R_i e_1 \) are linearly independent; (iii) \( \theta_1^-, \theta_1^+ \in (-\pi, \pi) \setminus \{0\} \) are the rotation angles of \( S_i^T R_{i-1} \) and \( S_i^T R_i \), respectively. By (ii), there exist \( \alpha_i, \beta_i \in \mathbb{R} \) such that

\[
R_i a_{i+2} + \psi_1^+ - R_{i-1} a_i e_2 - \psi_1^- = \alpha_R i e_1 + \beta_S e_1.
\]

(4.26)

Moreover, we set

\[
\gamma_{\varepsilon,i} := \frac{4\alpha_i}{\varepsilon \lambda}, \quad \gamma_{\varepsilon,i} := \frac{4\beta_i}{\varepsilon \lambda}, \quad \mu_{\varepsilon,i}^\pm := \pm \frac{4}{\varepsilon \lambda} + \tan \left( \frac{\theta_1^\pm}{2} \right), \quad \tilde{\mu}_{\varepsilon,i}^\pm := \pm \frac{4}{\varepsilon \lambda} - \tan \left( \frac{\theta_1^\pm}{2} \right),
\]

and let \( \kappa_0^\varepsilon \subset \mathbb{Z} \) be the unique integer such that \( a_i \in [\kappa_0^\varepsilon, \kappa_0^\varepsilon + 1] \). Observing that \( a_i \neq a_j \) for \( i, j \in \{1,\ldots,\ell\} \) with \( i \neq j \) and \( a_i \in (-1,1) \) for all \( i \in \{1,\ldots,\ell\} \), we may assume that the sets \( \{ \varepsilon \kappa_i^\varepsilon, \varepsilon \kappa_i^\varepsilon + 1 \} \) are pairwise disjoint, and that \( \bigcup_{i=1}^\ell [\varepsilon \kappa_i^\varepsilon, \varepsilon \kappa_i^\varepsilon + 1] \subset (-1,1) \) (this is true for sufficiently small \( \varepsilon > 0 \)). Finally, with \( \kappa_0^\varepsilon := -\lambda - \frac{1}{\varepsilon} \) and \( \kappa_0^\varepsilon + 1 := \frac{1}{\varepsilon} \), let \( V_\varepsilon \in L^1(\Omega;\mathbb{R}^{2 \times 2}) \) be the function defined by

\[
V_\varepsilon(x) := \begin{cases} 
R_i(1 + \frac{\varepsilon}{\lambda} V_{\text{soft}} e_1 \otimes e_2) & \text{if } x \in \{(0,1) \times (\varepsilon \lambda + \varepsilon \kappa_i^\varepsilon, \varepsilon \kappa_{i+1}^\varepsilon) \}, \\
R_i(1 + \gamma_{\varepsilon,i} e_1 \otimes e_2) & \text{if } x \in \{(0,1) \times (\varepsilon \lambda + \varepsilon \kappa_i^\varepsilon, \varepsilon \lambda + \varepsilon \kappa_{i+1}^\varepsilon) \}, \\
R_i(1 + \mu_{\varepsilon,i}^- e_1 \otimes e_2) & \text{if } x \in \{(0,1) \times (-\varepsilon \lambda + \varepsilon \kappa_i^\varepsilon, \varepsilon \lambda - \varepsilon \kappa_i^\varepsilon + \varepsilon \kappa_{i+1}^\varepsilon) \} \\
S_i(1 + \tilde{\mu}_{\varepsilon,i}^- e_1 \otimes e_2) & \text{if } x \in \{(0,1) \times (\varepsilon \lambda - \varepsilon \kappa_i^\varepsilon, \varepsilon \lambda - \varepsilon \kappa_i^\varepsilon) \} \\
S_i(1 + \tilde{\mu}_{\varepsilon,i}^+ e_1 \otimes e_2) & \text{if } x \in \{(0,1) \times (\varepsilon \lambda + \varepsilon \kappa_i^\varepsilon, \varepsilon \lambda + \varepsilon \kappa_i^\varepsilon + \varepsilon \kappa_{i+1}^\varepsilon) \}
\end{cases}
\]

for some \( i \in \{1,\ldots,\ell\} \),

\[
S_i(1 + \gamma_{\varepsilon,i} e_1 \otimes e_2) & \text{if } x \in \{(0,1) \times (\varepsilon \lambda + \varepsilon \kappa_i^\varepsilon, \varepsilon \lambda + \varepsilon \kappa_i^\varepsilon + \varepsilon \kappa_{i+1}^\varepsilon) \} \\
S_i(1 + \gamma_{\varepsilon,i} e_1 \otimes e_2) & \text{if } x \in \{(0,1) \times (\varepsilon \lambda - \varepsilon \kappa_i^\varepsilon, \varepsilon \lambda - \varepsilon \kappa_i^\varepsilon) \}
\end{cases}
\]

for some \( i \in \{1,\ldots,\ell\} \),

and \( V_\varepsilon \) is a gradient field, meaning that there is \( u_\varepsilon \in W^{1,\infty}(\Omega;\mathbb{R}^2) \) such that \( \nabla u_\varepsilon = V_\varepsilon \). Adding a suitable constant allows us to assume that \( \int_\Omega u_\varepsilon \, dx = \int_\Omega u \, dx \) by construction. \( u_\varepsilon(x) \) is a uniformly bounded sequence in \( W^{1,1}(\Omega;\mathbb{R}^2) \) such that \( u_\varepsilon \in A_\varepsilon \) for all \( \varepsilon \) (see (4.2)). To prove that \( u_\varepsilon \rightharpoonup u \) in \( BV(\Omega;\mathbb{R}^2) \), it suffices to show that

\[
Du_\varepsilon \rightharpoonup Du \text{ in } M(\Omega;\mathbb{R}^{2 \times 2}).
\]

(4.27)

The proof of (4.27) follows along the lines of (4.16). For this reason, we only highlight the main differences. First, note that the conditions \( \varepsilon \kappa_0^\varepsilon = -\varepsilon \lambda - 1 = -\varepsilon \lambda + a_0, \varepsilon \kappa_0^\varepsilon + 1 = 1 = a_{\ell+1} \), and \( \varepsilon \kappa_i^\varepsilon \leq a_i \leq \varepsilon (\kappa_i^\varepsilon + 1) \)

\[
\lim_{\varepsilon \to 0} \varepsilon \kappa_i^\varepsilon = a_i \quad \text{for all } i \in \{0,\ldots,\ell+1\}.
\]
Hence, \( \mathbb{I}_{(0,1) \times (\varepsilon_0 + \varepsilon K_1, \varepsilon K_1 + \varepsilon)} \to \mathbb{I}_{(0,1) \times (a, a+1)} \) and \( \gamma \mathbb{I}_{(0,1) \times (\varepsilon_0 + \varepsilon K_1, \varepsilon K_1 + \varepsilon)} \to \gamma \mathbb{I}_{(0,1) \times (a, a+1)} \) in \( L^1(\Omega) \) for \( i \in \{0, \ldots, \ell + 1\} \). On the other hand, by the Riemann–Lebesgue lemma, we have \( \mathbb{I}_{\varepsilon Y_{\min}} \xrightarrow{\varepsilon \to 0} \lambda \) in \( L^\infty(\mathbb{R}^2) \); thus,
\[
\lim_{\varepsilon \to 0} \int_{(0,1) \times (\varepsilon_0 + \varepsilon K_1, \varepsilon K_1 + \varepsilon)} \nabla u_\varepsilon(x) \varphi(x) \, dx = \lim_{\varepsilon \to 0} \int_\Omega R_i(\mathbb{I} + \mathbb{I}_{\varepsilon Y_{\min}} e_1 \otimes e_2) \mathbb{I}_{(0,1) \times (\varepsilon_0 + \varepsilon K_1, \varepsilon K_1 + \varepsilon)} \varphi(x) \, dx
\]
\[
= \int_{(0,1) \times (a, a+1)} R_i(\mathbb{I} + \gamma(x_2) e_1 \otimes e_2) \varphi(x) \, dx
\]
for all \( i \in \{0, \ldots, \ell\} \) and \( \varphi \in C_0(\Omega) \). Arguing as in (4.19) with the change of variables \( z = \frac{1}{x_2 - \varepsilon K_1} \), leads to
\[
\lim_{\varepsilon \to 0} \int_{(0,1) \times (\varepsilon_0, \varepsilon K_1 + \varepsilon)} \nabla u_\varepsilon(x) \varphi(x) \, dx
\]
\[
= \lim_{\varepsilon \to 0} \int_0^1 \int_{\varepsilon x_1 + \varepsilon K_1}^1 R_{i-1}(\mathbb{I} + \tan \left( \frac{\varepsilon_0}{\varepsilon} \right) e_1 \otimes e_2 - \frac{4}{\varepsilon} e_1 \otimes e_2) \varphi(x_2) \, dx_2 \, dx_1
\]
\[
= \lim_{\varepsilon \to 0} \int_0^1 \int_0^{x_1} R_{i-1}(\frac{4}{\varepsilon} \mathbb{I} + \frac{2}{\varepsilon} \tan \left( \frac{\varepsilon_0}{\varepsilon} \right) e_1 \otimes e_2 - e_1 \otimes e_2) \varphi(x_1, \frac{4}{\varepsilon} z + \varepsilon K_1) \, dz \, dx_1
\]
\[
= - \int_0^1 \int_0^{x_1} R_{i-1} e_1 \otimes e_2 \varphi(x_1, a_i) \, dz \, dx_1 = - \int_0^1 R_{i-1} x_1 e_1 \otimes e_2 \varphi(x_1, a_i) \, dx_1
\]
for all \( i \in \{1, \ldots, \ell\} \) and \( \varphi \in C_0(\Omega) \). Similarly, one can calculate the counterparts to (4.20)–(4.24) in the present setting. In view of (4.25) and (4.26), we deduce (4.27), which ends the proof.

**Remark 4.8** (On the construction in Proposition 4.7). We observe that the sequence of Lipschitz functions \( (u_\varepsilon)_\varepsilon \) constructed in Proposition 4.7 to approximate a given \( u \in \mathcal{A} \cap SBV_{\infty}(\Omega; \mathbb{R}^2) \) is such that
\[
\lim_{\varepsilon \to 0} \int_{\Omega} |\nabla u_\varepsilon| \, dx \sim |Du| + 2\ell.
\]
In other words, the asymptotic behavior of the total variation of \( (u_\varepsilon)_\varepsilon \) incorporates a positive term that is proportional to the number of jumps of the limit function. This fact prevents us from bootstrapping the argument in Proposition 4.7 to generalize it to an arbitrary function in \( \mathcal{A} \cap SBV(\Omega; \mathbb{R}^2) \).

An analogous statement to Proposition 4.7 holds in \( \mathcal{A}_0 \).

**Proposition 4.9.** Let \( \Omega = (0,1) \times (-1,1) \). If \( u \in \mathcal{A}_1 \), then there exists a sequence \( (u_\varepsilon)_\varepsilon \subset W^{1,1}(\Omega; \mathbb{R}^2) \) such that \( u_\varepsilon \in \mathcal{A}_0 \) for all \( \varepsilon \) and \( u_\varepsilon \rightharpoonup^* u \) in \( BV(\Omega; \mathbb{R}^2) \); that is,
\[
\mathcal{A}_1 \subset \mathcal{A}_0.
\]

**Proof.** Let \( u \in \mathcal{A}_1 \). Based on (1.13) and (2.4), we can split \( u \) into \( u = v + w \), where
\[
v(x) := Rx + \partial_\alpha(x_2) Re_1 + c \quad \text{and} \quad w(x) := \partial_\alpha(x_2) Re_1 \quad \text{for } x \in \Omega,
\]
with \( R \in SO(2), c \in \mathbb{R}^2, \partial_\alpha \in W^{1,1}(-1,1), \) and \( \partial_\alpha \in BV(-1,1) \) such that \( \partial_\alpha = 0 \). By construction, we have that \( v \in W^{1,1}(\Omega; \mathbb{R}^2) \) with \( \nabla v(x) = R(\mathbb{I} + \partial_\alpha(x_2) e_1 \otimes e_2) \).

For every \( \varepsilon > 0 \), let \( v_\varepsilon \in W^{1,1}(\Omega; \mathbb{R}^2) \) be the function satisfying \( \int_{\Omega} v_\varepsilon \, dx = \int_{\Omega} v \, dx \) and
\[
\nabla v_\varepsilon(x) = R(\mathbb{I} + \frac{\partial_\alpha(x_2)}{\varepsilon} \mathbb{I}_{\varepsilon Y_{\min}}(x_1) e_1 \otimes e_2).
\]

By the Riemann–Lebesgue lemma,
\[
v_\varepsilon \rightharpoonup^* v \quad \text{in } W^{1,1}(\Omega; \mathbb{R}^{2 \times 2}).
\]

On the other hand, applying Lemma 3.3 to \( \partial_\alpha \), we can find a sequence \( (\partial_\alpha)_\varepsilon \subset W^{1,\infty}(-1,1) \) such that \( \partial_\alpha_\varepsilon \rightharpoonup \partial_\alpha \) in \( BV(-1,1) \) and \( \partial_\alpha_\varepsilon = 0 \) on \( \varepsilon I_{16} \cap (-1,1) \). Then, setting \( w_\varepsilon(x) := \partial_\alpha(x_2) Re_1 + \int_\Omega (w - \partial_\alpha(x_2) Re_1) \, dx \) yields
\[
\nabla w_\varepsilon(x) = \partial_\alpha(x_2) Re_1 \otimes e_2 = \partial_\alpha(x_2) \mathbb{I}_{\varepsilon Y_{\min}} Re_1 \otimes e_2
\]
(4.31)
and
\[ w_\varepsilon \rightharpoonup w \quad \text{in } BV(\Omega; \mathbb{R}^2). \tag{4.32} \]

We define the maps \( u_\varepsilon := v_\varepsilon + w_\varepsilon \) in \( W^{1,1}(\Omega; \mathbb{R}^2) \) for every \( \varepsilon \), and infer from (4.29) and (4.31) that
\[ \nabla u_\varepsilon = R(1 + \gamma_\varepsilon e_1 \otimes e_2), \]
where \( \gamma_\varepsilon(x) := \left( \frac{\vartheta'(x_2)}{\lambda} + \vartheta'(x_2) \right) \mathbb{1}_{\varepsilon Y_{\text{ext}}}(x) \) is a function in \( L^1(\Omega) \) satisfying \( \gamma_\varepsilon = 0 \) in \( \varepsilon Y_{\text{rig}} \cap \Omega \). In particular, \( u_\varepsilon \in A_\varepsilon \) for all \( \varepsilon \).

Combining (4.30) and (4.32) shows that \( u_\varepsilon \rightharpoonup v + w = u \) in \( BV(\Omega; \mathbb{R}^2) \), which finishes the proof. \( \square \)

Finally, we prove Theorem 1.3.

**Proof of Theorem 1.3.** In view of the discussion in Section 2.4, it suffices to prove the statement on a rectangle of the form \((c_0, d_0) \times (a_0, b_0)\), where we recall (1.1) and (1.2). A simple modification of the proofs of Propositions 4.3, 4.7, and 4.9 shows that these results hold for any such rectangles, from which Theorem 1.3 follows. \( \square \)

## 5. A LOWER BOUND ON THE HOMOGENIZED ENERGY

In this section, we present partial results for the homogenization problem for layered composites with rigid components discussed in the Introduction. More precisely, we establish a lower bound estimate on \( E \), assuming that \((\varepsilon)\).

**Step 1: Compactness.** Assume that \((u_\varepsilon) \in L^1_0(\Omega; \mathbb{R}^2) \) is such that \( \sup_\varepsilon E_\varepsilon(u_\varepsilon) < \infty \). Then, \( u_\varepsilon \in A_\varepsilon \) and \( \sup_\varepsilon \|\nabla u_\varepsilon\|_{L^1(\Omega; \mathbb{R}^{2\times 2})} < \infty \). Hence, using the Poincaré–Wirtinger inequality, there exist a subsequence \((u_{\varepsilon_j})\) and \( u \in L^1_0(\Omega; \mathbb{R}^2) \cap BV(\Omega; \mathbb{R}^2) \) such that \( u_{\varepsilon_j} \rightharpoonup u \) in \( BV(\Omega; \mathbb{R}^2) \). By Proposition 4.3, we conclude that \( u \in L^1_0(\Omega; \mathbb{R}^2) \cap A \).

**Step 2: Lower bound.** Let \((u_\varepsilon) \in L^1_0(\Omega; \mathbb{R}^2) \) and \( u \in L^1_0(\Omega; \mathbb{R}^2) \) be such that \( u_\varepsilon \rightharpoonup u \) in \( L^1(\Omega; \mathbb{R}^2) \). We want to show that
\[ \liminf_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon) \geq E(u). \tag{5.1} \]

**Remark 5.1 (Equivalent formulations for \( E_\varepsilon \) and \( E \)).** In view of the definition of \( A_\varepsilon \) (see (1.6)), it is straightforward to check that the functional \( E_\varepsilon \) in (1.7) satisfies
\[ E_\varepsilon(u) = \begin{cases} \int_\Omega \sqrt{\det|\nabla u|^2 - 1} \, dx & \text{if } u \in A_\varepsilon, \\ \int_\Omega \sqrt{\det|\nabla u|^2 - 2 \det \nabla u} \, dx & \text{if } u \in A_\varepsilon, \\ \int_\Omega \sqrt{\det|\nabla u|^2} \, dx & \text{otherwise,} \end{cases} \]
for \( u \in L^1_0(\Omega; \mathbb{R}^2) \). Similarly, according to Proposition 4.1, the functional \( E \) from (1.10) can be expressed as
\[ E(u) = \begin{cases} \int_\Omega |\nabla u|^2 + |D^s u|(|\nabla u|) \, dx & \text{if } u \in A, \\ \int_\Omega |\nabla u|^2 \, dx & \text{otherwise}, \end{cases} \]
for \( u \in L^1_0(\Omega; \mathbb{R}^2) \).

We can now provide a bound from below on \( \Gamma\liminf_{\varepsilon \to 0} E_\varepsilon \) and prove Theorem 1.2.

**Proof of Theorem 1.2.** For clarity, we subdivide the proof into two steps. In the first one, we establish the compactness property. In the second step, we provide two alternative proofs of (1.12). The first proof is based on a Reshetnyak’s lower semicontinuity result, while the second version is more elementary, relying on the weak* lower semicontinuity of the total variation of a measure. Either of the arguments highlights a different feature of the representation of \( \mathcal{A} \).

**Step 1: Compactness.** Assume that \((u_\varepsilon) \subseteq L^1_0(\Omega; \mathbb{R}^2) \) is such that \( \sup_\varepsilon E_\varepsilon(u_\varepsilon) < \infty \). Then, \( u_\varepsilon \in A_\varepsilon \) and \( \sup_\varepsilon \|\nabla u_\varepsilon\|_{L^1(\Omega; \mathbb{R}^{2\times 2})} < \infty \). Hence, using the Poincaré–Wirtinger inequality, there exist a subsequence \((u_{\varepsilon_j})\) and \( u \in L^1_0(\Omega; \mathbb{R}^2) \cap BV(\Omega; \mathbb{R}^2) \) such that \( u_{\varepsilon_j} \rightharpoonup u \) in \( BV(\Omega; \mathbb{R}^2) \). By Proposition 4.3, we conclude that \( u \in L^1_0(\Omega; \mathbb{R}^2) \cap A \).

**Step 2: Lower bound.** Let \((u_\varepsilon) \subseteq L^1_0(\Omega; \mathbb{R}^2) \) and \( u \in L^1_0(\Omega; \mathbb{R}^2) \) be such that \( u_\varepsilon \rightharpoonup u \) in \( L^1(\Omega; \mathbb{R}^2) \). We want to show that
\[ \liminf_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon) \geq E(u). \tag{5.1} \]
To prove (5.1), one may assume without loss of generality that the limit inferior on the right-hand side of (5.1) is actually a limit and that this limit is finite. Then, $u_\varepsilon \in A_\varepsilon$ and $E_\varepsilon(u_\varepsilon) < C$ for all $\varepsilon$, where $C > 0$ is a constant independent of $\varepsilon$. Hence, by Step 1, $u_\varepsilon \rightharpoonup u$ in $BV(\Omega; \mathbb{R}^2)$ and $u \in A$.

**Step 2a: Version I.** We observe that the map $\mathbb{R}^{2 \times 2} \ni F \mapsto \sqrt{|F|^2 - 2\det F}$ is convex (see [18]) and one-homogeneous. Consequently, it follows from Remark 5.1 and Reshetnyak’s lower semicontinuity theorem (see [2, Theorem 2.38]), under consideration of our notation for the polar decomposition $Du = g_u|Du|$ introduced in Section 2.2, that

$$\liminf_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon) = \liminf_{\varepsilon \to 0} \int_\Omega \sqrt{|\nabla u_\varepsilon|^2 - 2\det \nabla u_\varepsilon} \, dx \geq \int_\Omega \sqrt{|g_u|^2 - 2\det g_u} \, d|Du|. \tag{5.2}$$

Since $\nabla u = R(I + \gamma e_1 \otimes e_2)$ with $R \in BV(\Omega; SO(2))$ and $(D^u)e_1 = 0$ (see (4.5)), we have $|\nabla u|^2 - 2\det \nabla u = |\gamma|^2$ for $L^2$-a.e. in $\Omega$ and $\det g_u = 0$ for $|D^u u|$-a.e. in $\Omega$. Thus,

$$\begin{align*}
\int_\Omega \sqrt{|g_u|^2 - 2\det g_u} \, d|Du| &= \int_\Omega \sqrt{|\nabla u|^2 - 2\det \nabla u} \, dx + \int_\Omega \sqrt{|g_u|^2 - 2\det g_u} \, d|D^u u| \\
&= \int_\Omega |\gamma| \, dx + |D^u u|(\Omega) = E(u), \tag{5.3}
\end{align*}$$

where we also used that the relation $|g_u| = 1$ holds $|D^u u|$-a.e. in $\Omega$.

From (5.2) and (5.3), we deduce (5.1).

**Step 2b: Version II.** By the definition of $A_\varepsilon$ and (4.1),

$$\nabla u_\varepsilon = R_\varepsilon + \gamma_\varepsilon R_\varepsilon e_1 \otimes e_2$$

with $R_\varepsilon \in L^\infty(\Omega; SO(2))$ and $\gamma_\varepsilon \in L^1(\Omega)$. Since $|\gamma_\varepsilon R_\varepsilon e_1 \otimes e_2| = |\gamma_\varepsilon|$ due to $|R_\varepsilon e_1| = 1$, the estimate $E_\varepsilon(u_\varepsilon) = \int_\Omega |\gamma_\varepsilon| \, dx < C$ implies that $(\gamma_\varepsilon R_\varepsilon e_1 \otimes e_2)$ is uniformly bounded in $L^1(\Omega; \mathbb{R}^{2 \times 2})$. Hence, after extracting a subsequence if necessary (not relabeled),

$$(\gamma_\varepsilon R_\varepsilon e_1 \otimes e_2) L^2(\Omega) \rightharpoonup \nu \quad \text{in} \quad \mathcal{M}(\Omega; \mathbb{R}^{2 \times 2})$$

for some $\nu \in \mathcal{M}(\Omega; \mathbb{R}^{2 \times 2})$. Note further that the convergence $\nabla u_\varepsilon L^2(\Omega) \rightharpoonup Du$ in $\mathcal{M}(\Omega; \mathbb{R}^{2 \times 2})$ along with (4.5) yields also $R_\varepsilon \rightharpoonup R$ in $L^\infty(\Omega; \mathbb{R}^{2 \times 2})$, where $R \in L^\infty(\Omega; SO(2))$ satisfies in particular that $(\nabla u)e_1 = R e_1$. Hence, we have

$$\nu = Du - RL^2(\Omega) - (\gamma R e_1 \otimes e_2) L^2(\Omega) + D^u u,$$

where the last equality follows again from (4.5), and by the lower semicontinuity of the total variation,

$$\liminf_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon) = \liminf_{\varepsilon \to 0} \int_\Omega |\gamma_\varepsilon| \, dx \leq \int_\Omega |\gamma R e_1 \otimes e_2| \, dx \geq |\nu|(\Omega) = \int_\Omega |\gamma R e_1 \otimes e_2| \, dx + |D^u u|(\Omega) = \int_\Omega |\gamma| \, dx + |D^u u|(\Omega) = E(u). \tag*{□}
$$

**Remark 5.2 (Discussion regarding optimality of the lower bound).** (a) The lower bound (1.12) is optimal in $A \cap W^{1,1}(\Omega; \mathbb{R}^2) \cap L^1(\Omega; \mathbb{R}^2)$ and, more generally (cf. also Remark 4.4), in the set $A^l \cap L^1(\Omega; \mathbb{R}^2)$ introduced in (1.13). Precisely, we have

$$\Gamma(L^1) \cdot \liminf_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon) = E(u) \tag{5.4}$$

for all $u \in A^l \cap L^1(\Omega; \mathbb{R}^2)$. In view of (1.12), the proof of (5.4) is directly related to the ability to construct a recovery sequence. We detail two alternative constructions for $u \in A^l$ in Section 6 below. For illustration, we treat here the simpler special case where $u \in A \cap W^{1,1}(\Omega; \mathbb{R}^2) \cap L^1(\Omega; \mathbb{R}^2)$.

If $u \in A \cap W^{1,1}(\Omega; \mathbb{R}^2) \cap L^1(\Omega; \mathbb{R}^2)$, then $\nabla u = R(I + \gamma e_1 \otimes e_2)$ for some $R \in SO(2)$ and $\gamma \in L^1(\Omega)$ such that $\partial_1 \gamma = 0$ (see Remark 4.4 (a)). As in the proof of Proposition 4.9, we take $(u_\varepsilon)_\varepsilon \subset W^{1,1}(\Omega; \mathbb{R}^2) \cap L^1(\Omega; \mathbb{R}^2)$ such that $\nabla u_\varepsilon = R(\mathbb{I} + \frac{1}{2\varepsilon} \mathbb{E} e_1 \otimes e_2)$ for all $\varepsilon$. Then, by the Riemann–Lebesgue lemma, $u_\varepsilon \rightharpoonup u$ in $BV(\Omega; \mathbb{R}^2)$ and $\lim_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon) = E(u)$

(b) The question whether (5.4) holds for a larger class than $A^l$ is open at this point. We observe that the gradient-based constructions in Lemma 4.5, Remark 4.6 (i)–(ii), and Proposition 4.7 yield upper...
bounds on the $\Gamma$-lim sup, which, however, do not match the lower bound of Theorem 1.2. This indicates that, in general, a more tailored approach will be necessary.

(c) The upper bounds on the $\Gamma$-lim sup of $(E_\varepsilon)^{\varepsilon}$ resulting from Lemma 4.5, Remark 4.6 (i)–(ii), and Proposition 4.7 can be quantified. As previously mentioned, the constructions in Remark 4.6 (iii) and Proposition 4.9 are even recovery sequences. This is not the case for the general construction in Lemma 4.5 and for those highlighted in Remark 4.6 (i)–(ii). In the following, we suppose that $u \in A \cap SBV_\infty(\Omega; \mathbb{R}^2) \cap L^1_0(\Omega; \mathbb{R}^2)$ has a single jump as in the statement of Lemma 4.5; i.e.,

$$u(x) = \mathbb{I}_{(0,1) \times (0,1)}(x)(R^+(x_2)x + \psi^+(x_2)) + \mathbb{I}_{(0,1) \times (-1,0)}(x)(R^-(x_2)x + \psi^-(x_2))$$

with $R^\pm \in SO(2)$ and $\psi^\pm \in \mathbb{R}^2$. Then,

$$E(u) = \int_0^1 |(R^+ - R^-)e_1x_1 + (\psi^+ - \psi^-)| \, dx_1,$$

which can be estimated from above by

$$E(u) \leq |R^+e_1 - R^-e_1| \int_0^1 x_1 \, dx_1 + |\psi^+ - \psi^-| \leq 1 + |\psi^+ - \psi^-|.$$  \hspace{1cm} (5.5)

For the sequence $(u_\varepsilon)^{\varepsilon}$ constructed in Lemma 4.5 (and Lemma 4.7), we obtain, recalling (4.13), that

$$\lim_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon) = |\alpha| + |\beta| + 2 \geq |\alpha| + |\beta| + 1 \geq E(u).$$

Regarding the construction of $(u_\varepsilon)^{\varepsilon}$ in Remark 4.6 (i), it follows that

$$\lim_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon) = |\alpha| + |\beta - 1| + 1.$$

This limit is strictly greater than $E(u)$ as we will show next. If $|\beta - 1| > |\beta|$ (i.e., if $\beta < \frac{1}{2}$), this is an immediate consequence of (5.5). For $\frac{1}{2} \leq \beta < 1$, we use that $\psi^+ - \psi^- = \alpha R^+e_1 + \beta R^-e_1$ yields

$$E(u) \leq \int_0^1 |x_1 + \alpha \beta - x_1| \, dx_1 + \int_0^1 (x_1 - \beta) \, dx_1 + \int_{x_1 = \beta}^1 (x_1 - \alpha \beta) \, dx_1 \leq 1 + |\alpha| + |\beta - 1| < 1 + |\alpha| + |\beta - 1|.$$

If $\beta \geq 1$, we note that $\lim_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon) = |\alpha| + \beta$, and subdivide the estimate of $E(u)$ into three cases. Recalling the assumption $R^+ \neq \pm R^-$, we set $c := R^+e_1 \cdot R^-e_1 \in (-1, 1)$ to obtain

$$E(u) = \int_0^1 \sqrt{(x_1 + \alpha)^2 + (\beta - x_1)^2 + 2c(x_1 + \alpha)(\beta - x_1)} \, dx_1.$$

Then, we have for $\alpha \geq 0$

$$E(u) < \int_0^1 \sqrt{(x_1 + \alpha)^2 + (\beta - x_1)^2 + 2(x_1 + \alpha)(\beta - x_1)} \, dx_1 = |\alpha + \beta| \leq |\alpha| + |\beta|,$$

for $\alpha \leq -1$

$$E(u) < \int_0^1 \sqrt{(x_1 + \alpha)^2 + (\beta - x_1)^2 - 2(x_1 + \alpha)(\beta - x_1)} \, dx_1 = \int_0^1 (2x_1 - \alpha + \beta) \, dx_1 = -1 + |\alpha + \beta| \leq |\alpha| + |\beta|,$$

and for $-1 < \alpha < 0$

$$E(u) < \int_{-\alpha}^0 (2x_1 - \alpha + \beta) \, dx_1 + \int_{-\alpha}^1 |\alpha + \beta| \, dx_1 = \alpha \beta + \alpha^2 < -\alpha + \beta = |\alpha| + |\beta|.$$

Summing up, we have shown that in the context of Remark 4.6 (i),

$$\lim_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon) > E(u).$$

Finally, we consider the sequence $(u_\varepsilon)^{\varepsilon}$ constructed in Remark 4.6 (ii). Then, we have

$$\lim_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon) = |\alpha - \tilde{u}| + |\beta| + 1,$$

and since $R^+ = R^-$ in this case,

$$E(u) = \sqrt{\alpha^2 + \beta^2 + 2\alpha\beta R e_1 \cdot S e_1}.$$
6. Homogenization of the Regularized Problem

This section is devoted to the proof of our main \( \Gamma \)-convergence result, Theorem 1.4. We first provide an alternative characterization of the class \( \mathcal{A}^\parallel \) of restricted asymptotically admissible deformations introduced in (1.13).

Lemma 6.1. Let \( \Omega = (0,1) \times (-1,1) \). Then, \( \mathcal{A}^\parallel \) as in (1.13) admits the representation

\[
\mathcal{A}^\parallel = \{ u \in BV(\Omega; \mathbb{R}^2) : \nabla u = R(1 + \gamma e_1 \otimes e_2) \text{ with } R \in SO(2), \gamma \in L^1(\Omega) \text{ such that } \partial_1 \gamma = 0, \]

\[
D^\parallel u = (\varrho \otimes e_2)|D^\parallel u| \quad \text{with } \varrho \in L^1_{|D^\parallel u|}(\Omega; \mathbb{R}^2) \text{ such that } |\varrho| = 1 \text{ and } \varrho||Re_1 \text{ for } |D^\parallel u|\text{-a.e. in } \Omega. \tag{6.1}
\]

Proof. Let \( \tilde{\mathcal{A}}^\parallel \) denote the set on the right-hand side of (6.1). Arguing as in the beginning of the proof of Proposition 4.9 (precisely, with the notation of (1.13)), we set \( \gamma(x) = \vartheta'_u(x_2) \) for \( x \in \Omega \), and observe that \( (D^\parallel u)e_2 = \mathcal{L}^1(0,1) \otimes D^\parallel \vartheta_u Re_1 \) and exploiting the polar decomposition of measures (cf. (2.2) and (2.3)) gives rise to \( \mathcal{A}^\parallel \subset \tilde{\mathcal{A}}^\parallel \). Conversely, the inclusion \( \tilde{\mathcal{A}}^\parallel \subset \mathcal{A}^\parallel \), which follows from (4.5), along with (4.6) yields that \( \tilde{\mathcal{A}}^\parallel \subset \mathcal{A}^\parallel \). \( \square \)

We are now in a position to prove the \( \Gamma \)-convergence of the energies \( (E^\parallel_\varepsilon)_\varepsilon \) in (1.14) as \( \varepsilon \to 0 \).

Proof of Theorem 1.4. As before in the proofs of Theorems 1.1 and 1.3, one may assume without loss of generality that \( \Omega = (0,1) \times (-1,1) \). We proceed in three steps.

Step 1: Compactness. Let \( (u_\varepsilon)_\varepsilon \subset W^{1,1}(\Omega; \mathbb{R}^2) \cap L^1_0(\Omega; \mathbb{R}^2) \) be a sequence such that \( E^\parallel_\varepsilon(u_\varepsilon) \leq C \) for all \( \varepsilon > 0 \). Then, because \( u_\varepsilon \in \mathcal{A}_\varepsilon \) for all \( \varepsilon \),

\[
\nabla u_\varepsilon = R_\varepsilon(\mathbb{I} + \gamma_\varepsilon e_1 \otimes e_2) \in L^1(\Omega; \mathbb{R}^{2 \times 2}), \tag{6.2}
\]

and \( \|\gamma_\varepsilon\|_{L^1(\Omega)} \leq C \) for every \( \varepsilon > 0 \). Additionally, since each map \( R_\varepsilon \) takes value in the set of proper rotations, it holds that \( \|R_\varepsilon\|_{L^\infty(\Omega; \mathbb{R}^{2 \times 2})} = 2 \) for all \( \varepsilon > 0 \). Consequently, along with the Poincaré-Wirtinger inequality,

\[
\|u_\varepsilon\|_{W^{1,1}(\Omega; \mathbb{R}^2)} \leq C.
\]

We further know that \( \|\partial_1 u_\varepsilon\|_{W^{1,1}(\Omega; \mathbb{R}^2)} = \|R_\varepsilon e_2\|_{W^{1,1}(\Omega; \mathbb{R}^2)} \leq C/\delta \) for any \( \varepsilon \). Thus, after extracting subsequences if necessary, one can find \( u \in BV(\Omega; \mathbb{R}^2), \gamma \in \mathcal{M}(\Omega), \) and \( R \in W^{1,p}(\Omega; \mathbb{R}^{2 \times 2}) \) such that

\[
u_\varepsilon \rightharpoonup u \quad \text{in } BV(\Omega; \mathbb{R}^2), \tag{6.3}
\]

\[
\gamma_\varepsilon \rightharpoonup \gamma \quad \text{in } \mathcal{M}(\Omega),
\]

\[
R_\varepsilon \rightharpoonup R \quad \text{in } W^{1,p}(\Omega; \mathbb{R}^{2 \times 2}). \tag{6.4}
\]

Recalling the compact embedding \( W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega}) \) for some \( 0 < \alpha < 1 - \frac{2}{p} \), it follows even that \( R \in W^{1,p}(\Omega; SO(2)) \cap C^{0,\alpha}(\overline{\Omega}; \mathbb{R}^{2 \times 2}) \) and

\[
R_\varepsilon \rightarrow R \quad \text{in } L^\infty(\Omega; \mathbb{R}^{2 \times 2}). \tag{6.5}
\]

As a consequence of Proposition 4.3, it holds that \( u \in \mathcal{A} \). From Proposition 4.1 and Alberti’s rank one theorem (cf. Section 2.1), we can further infer that \( R \in SO(2), \gamma \in L^1(\Omega) \) with \( \partial_1 \gamma = 0 \), and that \( Du \) satisfies

\[
\nabla u = R(\mathbb{I} + \gamma e_1 \otimes e_2) \quad \text{and} \quad D^\parallel u = (\varrho \otimes e_2)|D^\parallel u|, \tag{6.6}
\]

where \( \varrho \in L^1_{|D^\parallel u|}(\Omega; \mathbb{R}^2) \) with \( |\varrho| = 1 \) for \( |D^\parallel u|\)-a.e. in \( \Omega \). To conclude that \( u \in \mathcal{A}^\parallel \), in view of Lemma 6.1, it remains to show that

\[
\varrho||Re_1 \ |D^\parallel u|\text{-a.e. in } \Omega. \tag{6.7}
\]

To prove (6.7), we first observe that for every \( \varepsilon \), the identity \( (\nabla u_\varepsilon) e_2 = R_\varepsilon e_2 + \gamma_\varepsilon R_\varepsilon e_1 \), which follows from \( u_\varepsilon \in \mathcal{A}_\varepsilon \), yields

\[
\int_\Omega [(\nabla u_\varepsilon) e_2 \cdot R_\varepsilon e_2 - 1]|\varphi| \, dx = 0 \tag{6.8}
\]
for all $\varphi \in C_c^\infty(\Omega)$. Thus, by (6.3) and (6.5) in combination with a weak-strong convergence argument, taking the limit $\varepsilon \to 0$ in (6.8) leads to
\[
\int_\Omega \varphi \, dx = \int_\Omega \varphi Re_2 \cdot d((Du)e_2) = \int_\Omega \varphi Re_2 \cdot (\nabla u)e_2 \, dx + \int_\Omega \varphi Re_2 \cdot d(D^s u)e_2
\]
for every $\varphi \in C_c^\infty(\Omega)$. Next, we plug in the identities $(\nabla u)e_2 = Re + \gamma Re_1$ and $(D^s u)e_2 = \varepsilon |D^s u|$ (see (6.6)) to derive that
\[
0 = \int_\Omega \varphi Re_2 \cdot d((D^s u)e_2) = \int_\Omega \varphi Re_2 \cdot \varepsilon |D^s u|
\]
for every $\varphi \in C_c^\infty(\Omega)$, which completes the proof of (6.7).

**Step 2: Lower bound.** Let $(u_\varepsilon) \subset L_0^1(\Omega; \mathbb{R}^2)$ and $u \in L_0^1(\Omega; \mathbb{R}^2)$ be such that $u_\varepsilon \to u$ in $L^1(\Omega; \mathbb{R}^2)$. We want to show that
\[
E^\delta(u) \leq \liminf_{\varepsilon \to 0} E^\delta_z(u_\varepsilon). \tag{6.9}
\]
To prove (6.9), we proceed as in the proof of (5.1), observing in addition that
\[
\liminf_{\varepsilon \to 0} \delta \|\partial_1 u_\varepsilon\|_{W^{1,p}((\Omega; \mathbb{R}^2))}^p = \liminf_{\varepsilon \to 0} \delta\|Re_1\|_{W^{1,p}((\Omega; \mathbb{R}^2))}^p = \delta |\Omega|
\]
due to (6.2) and (6.4) with $R \in SO(2)$.

**Step 3: Upper bound.** Let $u \in L_0^1(\Omega; \mathbb{R}^2) \cap A^\parallel$. We want to show that there is a sequence $(u_\varepsilon) \subset L_0^1(\Omega; \mathbb{R}^2)$ such that $u_\varepsilon \to u$ in $L^1(\Omega; \mathbb{R}^2)$, and
\[
E^\delta(u) \geq \limsup_{\varepsilon \to 0} E^\delta_z(u_\varepsilon). \tag{6.10}
\]
Let $(u_\varepsilon) \subset W^{1,1}(\Omega; \mathbb{R}^2) \cap L_0^1(\Omega; \mathbb{R}^2)$ be the sequence constructed in the proof of Proposition 4.9, that is, $u_\varepsilon \in A_\varepsilon$ for every $\varepsilon$ with
\[
\nabla u_\varepsilon(x) = R \left( \mathbb{I} + \left( \frac{\partial'_a(x_2)}{\lambda} + \frac{\partial'_b(x_2)}{\lambda} \right) \mathbb{I} \cdot \gamma_{Y_\varepsilon}(x) \right) e_1 \otimes e_2,
\]
where $(\partial_\varepsilon) \subset W^{1,\infty}(-1, 1)$ satisfies
\[
\lim_{\varepsilon \to 0} \int_{-1}^1 |\partial'_\varepsilon(x)| \, dx = |D^s \vartheta_\varepsilon|(-1, 1) = |D^s u|(\Omega),
\]
and $u_\varepsilon \rightharpoonup u$ in $BV(\Omega; \mathbb{R}^2)$. Recalling that $\vartheta' = \vartheta'_a + \vartheta'_b = \vartheta'_a$, we have
\[
\limsup_{\varepsilon \to 0} E^\delta_z(u_\varepsilon) \leq \lim_{\varepsilon \to 0} \left( \int_\Omega \frac{|\partial'_a(x_2)|}{\lambda} \|e_\varepsilon \gamma_{Y_\varepsilon}(x)\|_{p,1} \, dx + \int_{-1}^1 |\partial'_b(x_2)| \, dx + \varepsilon \|Re_1\|_{W^{1,p}((\Omega; \mathbb{R}^2))} \right)
\]
\[
= \int_\Omega |\partial'(x_2)| \, dx + |D^s u|(\Omega) + \varepsilon |\Omega| = E^\delta(u),
\]
which proves (6.10) and completes the proof of the theorem. \qed

**Remark 6.2 (On compensated compactness).** We point out that if $u_\varepsilon \in A_\varepsilon$, with $\nabla u_\varepsilon = R_\varepsilon \left( \mathbb{I} + \gamma_{e_1} \otimes e_2 \right)$ for $R_\varepsilon \in L^\infty(\Omega; SO(2))$ and $\gamma_{e_1} \in L^1(\Omega)$ with $\gamma_\varepsilon = 0$ on $\varepsilon Y_\varepsilon \cap \Omega$, is such that $u_\varepsilon \rightharpoonup u$ in $BV(\Omega; \mathbb{R}^2)$, and if in addition,
\[
R_\varepsilon \to R \quad \text{in} \quad C(\Omega; \mathbb{R}^{2 \times 2}),
\]
then a weak-strong convergence argument implies that
\[
\gamma_{e_1} \mathcal{L}^2 = \left[ (\nabla u_\varepsilon) \cdot e_2 \cdot R_\varepsilon e_1 \right] \mathcal{L}^2 \rightharpoonup (Du) e_2 \cdot Re_1 \quad \text{in} \quad \mathcal{M}(\Omega).
\]
However, if continuity and uniform convergence of $R_\varepsilon$ fail, the limit representation above may no longer be true in general, even if $R \in C(\Omega; SO(2))$. To see this, let us consider the basic construction in Remark 4.6 (ii). In this case,
\[
\gamma_{e_1} \mathcal{L}^2 \rightharpoonup (\alpha + \beta) \mathcal{H}^1 \left[ (0, 1) \times \{0\} \right] \quad \text{in} \quad \mathcal{M}(\Omega), \tag{6.11}
\]
whereas
\[
(Du)e_2 \cdot Re_1 = \left[ (\psi^+ - \psi^-) \cdot Re_1 \right] \mathcal{H}^1 \left[ (0, 1) \times \{0\} \right]. \tag{6.12}
\]
Recalling that $\psi^+ - \psi^- = \alpha Re_1 + \beta Se_1$, the quantities in (6.11) and (6.12) can only match if $Re_1||Se_1$, which contradicts the assumption that $Re_1$ and $Se_1$ are linearly independent.
The role of the higher-order regularization in (1.14) is exactly that it helps overcome the issue discussed above. In fact, it guarantees the desired compactness properties for sequences of deformations with equibounded energies.

To conclude, we present an alternative construction for the recovery sequence in Step 3 of the proof of Theorem 1.4.

**Alternative proof of Theorem 1.4.** As before, we may assume without loss of generality that \( \Omega = (0,1) \times (-1,1) \). Moreover, the compactness property and lower bound can be studied exactly as in the proof of Theorem 1.4 above.

We are then left to show that given \( u \in L^1_\text{loc}(\Omega;\mathbb{R}^2) \cap A^1 \), there exists a sequence \( (u_\epsilon)_\epsilon \subset L^1_\text{loc}(\Omega;\mathbb{R}^2) \) satisfying \( u_\epsilon \rightharpoonup u \) in \( L^1(\Omega;\mathbb{R}^2) \) and (6.10). We will proceed in three steps, building up complexity.

**Step 1.** We assume first that \( u \in L^1_\text{loc}(\Omega;\mathbb{R}^2) \cap A^1 \) is an SBV-function with a single, constant jump line at \( x_2 = 0 \).

This case can be treated as highlighted in Remark 4.6 (iii). Let \( R \in SO(2) \), \( \gamma \in L^1(\Omega) \) with \( \partial_t \gamma = 0 \), and \( \psi^+, \psi^- \in \mathbb{R}^2 \) with \((\psi^+ - \psi^-)|\text{Re}e_1 \) be such that

\[
D\psi = R(1 + \gamma_1 e_1 \otimes e_2) L^2(\Omega + (\psi^+ - \psi^-) \otimes e_2 \mathcal{H}^1((0,1) \times \{0\})).
\]

Note that setting \( \psi = \gamma \in \mathbb{R}^2 \) at \( x \in \mathbb{R} \) intersect equibounded energies.

**Step 2.** As before, we may assume without loss of generality that \( \Omega = (0,1) \times (0,\lambda x) \), we have \( \psi^+ - \psi^- = \epsilon \psi^+ - \psi^- |\text{Re}e_1 \) and \( |D\psi| = |D^\alpha u|(|\Omega| + |D^\alpha u|(|\Omega|) = \int_\Omega |R(1 + \gamma_1 e_1 \otimes e_2)| \, dx + |\psi^+ - \psi^-|).\]

For each \( \epsilon > 0 \), set \( \tau_\epsilon := \frac{|D^\alpha u|(|\Omega|)}{\epsilon} = \epsilon |\psi^+ - \psi^-| \). Arguing as, for instance, in the proof of Lemma 4.5, we can find \( u_\epsilon \in L^1_\text{loc}(\Omega;\mathbb{R}^2) \cap A_e \) such that

\[
\nabla u_\epsilon = \begin{cases} R(1 + \tau_\epsilon e_1 \otimes e_2) & \text{if } x \in (0,1) \times (0,\lambda x), \\ R(1 + \frac{\epsilon}{\lambda} 1_{[y_{\text{soft}} \cap \Omega]} e_1 \otimes e_2) & \text{otherwise}, \end{cases}
\]

and \( u_\epsilon \rightharpoonup u \) in \( BV(\Omega;\mathbb{R}^2) \). Next, we show that this construction yields convergence of energies. Indeed, we have

\[
\lim_{\epsilon \to 0} E_\epsilon^\delta(u_\epsilon) = \lim_{\epsilon \to 0} \left( \int_{(0,1) \times (0,\lambda x)} |\tau_\epsilon| \, dx + \int_{\Omega \setminus (0,1) \times (0,\lambda x)} \frac{\gamma}{\lambda} \left| 1_{[y_{\text{soft}} \cap \Omega]} \, dx + \delta \|\text{Re}e_1\|_{L^1(\Omega;\mathbb{R}^2)} \right| \right).
\]

As in the proof of Proposition 4.7, the idea is to perform a construction similar to that in Step 1 around each jump line but accounting for the possibility that one or more of the jump lines may not intersect \( y_{\text{soft}} \cap \Omega \).

Fix \( i \in \{1,\ldots,\ell\} \) and \( \epsilon > 0 \), and let \( \kappa^i_\epsilon \in \mathbb{Z} \) be the integer such that \( a_i \in \epsilon [\kappa^i_\epsilon, \kappa^i_\epsilon + 1) \). Since \( a_i \neq a_j \) if \( i \neq j \), we may assume that the sets \( \epsilon [\kappa^i_\epsilon, \kappa^i_\epsilon + 1) \) are pairwise disjoint for all \( \epsilon > 0 \) (this is true for all \( \epsilon > 0 \) sufficiently small). Then, we take \( u_\epsilon \in L^1_\text{loc}(\Omega;\mathbb{R}^2) \cap A_e \) such that

\[
\nabla u_\epsilon = \begin{cases} R(1 + \tau_\epsilon e_1 \otimes e_2) & \text{in } (0,1) \times \epsilon [\kappa^i_\epsilon, \kappa^i_\epsilon + 1), \\ R(1 + \frac{\epsilon}{\lambda} 1_{[y_{\text{soft}} \cap \Omega]} e_1 \otimes e_2) & \text{otherwise}, \end{cases}
\]

and

\[
|D^\alpha u|(|\Omega|) = \sum_{i=1}^{\ell} |\psi^+_i - \psi^-_i|.
\]
where $\gamma^i_\epsilon = \int_\Omega |\psi^+ - \psi^-|\,dx$ with $\xi^i := \text{sign}((\psi^+ - \psi^-) \cdot \nabla e_1) \in \{\pm 1\}$. As in the proof of Proposition 4.7, we obtain that

$$\lim_{\epsilon \to 0} \int_\Omega \nabla u_\epsilon \varphi \,dx = \sum_{i=1}^{\ell} \int_0^1 \xi^i_\epsilon |\psi^+ - \psi^-|(Re_1 \otimes e_2)\varphi(x_1, a_i) \,dx_1 + \int_\Omega R(\nabla + \gamma e_1 \otimes e_2)\varphi \,dx \quad (6.14)$$

for all $\varphi \in C_0(\Omega)$. Recalling (6.13) and the equalities $|\psi^+ - \psi^-| = \xi_\epsilon |\psi^+ - \psi^-| |Re_1|$ for $i \in \{1, \ldots, \ell\}$, (6.14) shows that $Du_\epsilon \rightharpoonup Du$ in $\mathcal{M}(\Omega; \mathbb{R}^{2 \times 2})$. Hence, $u_\epsilon \rightharpoonup u$ in $BV(\Omega; \mathbb{R}^2)$.

Finally, proceeding exactly as in Step 1, we conclude that this construction also yields convergence of the energies. This ends Step 2.

**Step 3.** We consider now the general case $u \in L^1_0(\Omega; \mathbb{R}^2) \cap A^\parallel$.

Similarly to the beginning of the proof of Proposition 4.9 (see (4.28)), we can write

$$u(x) = x_1Re_1 + \phi_\epsilon(x_2) + \phi_s(x_2), \quad x \in \Omega,$$

where $\phi_\epsilon(x_2) := x_2Re_2 + \theta(x_2)Re_1 + c$ and $\phi_s(x_2) := \bar{\theta}(x_2)Re_1$. Note that $\phi_\epsilon \in W^{1,1}(-1, 1; \mathbb{R}^2)$ and $\phi_s \in BV(-1, 1; \mathbb{R}^2)$ is the sum of a jump function and a Cantor function; in particular, $\theta = \theta_s$ and $D\phi_s = D^s\phi_s$ (see (2.4)). Moreover,

$$\nabla u = Re_1 \otimes e_1 + \nabla \phi_\epsilon \otimes e_2 = R(\nabla + \theta_s e_1 \otimes e_2) = R(\nabla + \theta' e_1 \otimes e_2),$$

$$D^s u = L^1[0, 1] \otimes D\phi_s, \quad |D^s u| |\Omega = L^1[0, 1] \otimes |D\phi_s|. \quad (6.15)$$

By Lemma 6.1, there exists $\varrho \in L^1(|D^s u|(-1, 1; \mathbb{R}^2)$ with $|\varrho| = 1$ such that

$$D^s u = (\varrho \otimes e_2)|D^s u| \quad \text{and} \quad \varrho = (\varrho \cdot Re_1)Re_1. \quad (6.16)$$

Let $\varrho_\epsilon \in C^\infty([-1, 1])$ be such that

$$\lim_{\epsilon \to \infty} \int_\Omega |\varrho_\epsilon(x_2) - \varrho(x_2)| \,d|D^s u|(x) = 0. \quad (6.17)$$

Since $|\varrho| = 1$, we can choose such a sequence so that $|\varrho_\epsilon| \leq 1$.

Due to the properties of good representatives (see [2, (3.24)]) and [19, Lemma 3.2], for each $n \in \mathbb{N}$, there exists a piecewise constant function $\phi_n \in BV(-1, 1; \mathbb{R}^2)$, of the form

$$\phi_n = \sum_{i=0}^{\ell_n} b^n_{\chi A^n_i},$$

where $\ell_n \in \mathbb{N}$, $(b^n_i)_{i=0}^{\ell_n} \subset \mathbb{R}^2$, and $(A^n_i)_{i=0}^{\ell_n}$ is a partition of $(-1, 1)$ into intervals with $sup A^n_i = inf A^n_{i+1}$, satisfying

$$J_{\phi_n} = \sum_{i=1}^{\ell_n} \{a^n_i\} \quad \text{with} \quad a^n_i := sup A^n_{i-1},$$

$$\lim_{n \to \infty} \|\phi_n - \phi_n\|_{L^1(-1, 1; \mathbb{R}^2)} = 0, \quad (6.18)$$

$$\lim_{n \to \infty} |D\phi_n|(-1, 1) = \lim_{n \to \infty} |D^s\phi_n|(-1, 1) = |D\phi_n|(-1, 1) = |D^s u||\Omega. \quad (6.19)$$

Indeed, (6.18) and (6.19) mean that $(\phi_n)_{n \in \mathbb{N}}$ converges strictly to $\phi_s$ in $BV(-1, 1; \mathbb{R}^2)$, which implies that

$$|D\phi_n| \rightharpoonup |D\phi_s| \quad \text{in} \quad \mathcal{M}(-1, 1), \quad (6.20)$$

see [2, Proposition 3.5].

Finally, for $n \in \mathbb{N}$, we define

$$u_n(x) := x_1Re_1 + \phi_\epsilon(x_2) + \phi_s(x_2) + c_n, \quad x \in \Omega,$$

where $c_n \in \mathbb{R}^2$ are constants chosen so that $\int_\Omega u_n \,dx = 0$. Note that $c_n \to 0$ as $n \to \infty$ by (6.18). Moreover, for each $n \in \mathbb{N}$, the map $u_n \in L^1_0(\Omega; \mathbb{R}^2)$ has the same structure as in Step 2 apart from the condition $(u_n^+ - u_n^-)|Re_1$ on $J_{u_n}$, which a priori is not satisfied. Choosing $\xi^i_\epsilon := \varrho_\epsilon(a^n_i) \cdot Re_1$, we can
invoke Step 2 up to, and including, (6.14) to construct a sequence \((u^n_{\epsilon,h})_\epsilon \subset L^1(\Omega; \mathbb{R}^2) \cap W^{1,1}(\Omega; \mathbb{R}^2)\) that satisfies for all \(\varphi \in C_0(\Omega),\)

\[
\lim_{\epsilon \to 0} \int_\Omega \nabla u^n_{\epsilon,h} \varphi \, dx = \sum_{i=1}^{\ell_n} \int_0^1 (\varphi_2(a^n_1) \cdot R e_1) |b^n_i - b^{n-1}_i| (Re_1 \otimes e_2) \varphi(x_1, a^n_1) \, dx_1
\]

\[
+ \int_\Omega R(1 + \varphi_1(x_2)e_1 \otimes e_2) \varphi \, dx.
\]

We conclude from (6.15), (6.16), (6.17), (6.18), (6.20), and the Lebesgue dominated convergence theorem that

\[
\lim_{h \to \infty} \lim_{n \to \infty} \sum_{i=1}^{\ell_n} \int_0^1 (\varphi_2(x_2) \cdot R e_1) (Re_1 \otimes e_2) \varphi(x_1, x_2) \, d|D\phi_n|(x_2) \, dx_1
\]

\[
= \lim_{h \to \infty} \int_0^1 \int_0^1 (\varphi_2(x_2) \cdot R e_1) (Re_1 \otimes e_2) \varphi(x_1, x_2) \, d|D\phi_2|(x_2) \, dx_1
\]

\[
= \int_\Omega (\varphi(x_2) \cdot R e_1) (Re_1 \otimes e_2) \varphi \, d|Du| = \int_\Omega (\varphi(x_2) \otimes e_2) \varphi \, d|Du| = \int_\Omega \varphi \, dDu.
\]

Recalling that \(|\varphi_2(a^n_1) \cdot R e_1| \leq 1,\) we can further argue as in Steps 1 and 2 regarding the convergence of the energies to get

\[
\limsup_{\epsilon \to 0} E^\delta_\epsilon(u^n_{\epsilon,h}) \leq E^\delta(u_\epsilon) = \int_\Omega |\varphi_2'(x_2)| \, dx + |D^j \phi_n|(-1,1) + \delta|\Omega|
\]

\[
= \int_\Omega |\varphi_2'(x_2)| \, dx + |D^j \phi_n|(-1,1) + \delta|\Omega|.
\]

Letting \(n \to \infty\) and \(h \to \infty\) in (6.21) and (6.23), from (6.22), (6.19), and (6.15), we conclude that for all \(\varphi \in C_0(\Omega),\)

\[
\lim_{h \to \infty} \lim_{n \to \infty} \int_\Omega \nabla u^n_{\epsilon,h} \varphi \, dx = \int_\Omega \varphi \, dDu,
\]

\[
\limsup_{h \to \infty} \sup_{n \to \infty} \sup_{\epsilon \to 0} E^\delta_\epsilon(u^n_{\epsilon,h}) \leq \int_\Omega |\varphi_2'(x_2)| \, dx + |Du| + \delta|\Omega| = E^\delta(u).
\]

Owing to the separability of \(C_0(\Omega)\) and (6.25)–(6.26), we can use a diagonalization argument as that in [25, proof of Proposition 1.11 (p.449)] to find sequences \((h_\epsilon)\_\epsilon\) and \((n_\epsilon)\_\epsilon\) such that \(h_\epsilon, n_\epsilon \to \infty\) as \(\epsilon \to 0\) and \(u_{\epsilon,h} := u^n_{\epsilon,h} \in L^1(\Omega; \mathbb{R}^2) \cap W^{1,1}(\Omega; \mathbb{R}^2)\) has all the desired properties. \(\square\)

**Acknowledgements.** The work of Elisa Davoli has been funded by the Austrian Science Fund (FWF) project F65 “Taming complexity in partial differential systems”. Carolin Kreisbeck gratefully acknowledges the support by a Westerdijk Fellowship from Utrecht University. The research of Elisa Davoli and Carolin Kreisbeck was supported by the Mathematisches Forschungsinstitut Oberwolfach through the program “Research in Pairs” in 2017. The hospitality of King Abdullah University of Science and Technology, Utrecht University, and of the University of Vienna is acknowledged. All authors are thankful to the Erwin Schrödinger Institute in Vienna, where part of this work was developed during the workshop “New trends in the variational modeling of failure phenomena”.

**References**

[1] G. Alberti. Rank one property for derivatives of functions with bounded variation. *Proc. Roy. Soc. Edinburgh Sect. A*, 123:239–274, 1993.

[2] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 2000.

[3] S. Amstutz and N. Van Goethem. Incompatibility-governed elasto-plasticity for continua with dislocations. *Proc. A.*, 473(2199):20160734, 21, 2017.

[4] H. Attouch. *Variational convergence for functions and operators*. Applicable Mathematics Series. Pitman (Advanced Publishing Program), Boston, MA, 1984.

[5] J. M. Ball, J. C. Currie, and P. J. Olver. Null Lagrangians, weak continuity, and variational problems of arbitrary order. *J. Funct. Anal.*, 41(2):135–174, 1981.
[6] A. C. Barroso, J. Matias, M. Morandotti, and D. R. Owen. Second-order structured deformations: relaxation, integral representation and applications. Arch. Ration. Mech. Anal., 225(3):1025–1072, 2017.

[7] B. Benešová, M. Kružík, and A. Schlömerkemper. A note on locking materials and gradient polyconvexity. Math. Models Methods Appl. Sci., 28(12):2367–2401, 2018.

[8] A. Braides. Gamma-convergence for beginners. Number 22 in Oxford lecture series in mathematics and its applications. Oxford University Press, Oxford, 1. ed edition, 2005.

[9] A. Chambolle, A. Giacomini, and M. Ponsiglione. Piecewise rigidity. J. Funct. Anal., 244:134–153, 2007.

[10] R. Choksi, G. Del Piero, I. Fonseca, and D. Owen. Structured deformations as energy minimizers in models of fracture and hysteresis. Math. Mech. Solids, 4(3):321–356, 1999.

[11] R. Choksi and I. Fonseca. Bulk and interfacial energy densities for structured deformations of continua. Arch. Rational Mech. Anal., 138(1):37–103, 1997.

[12] F. Christowiak. Homogenization of layered materials with stiff components. PhD thesis, Universität Regensburg, 2012.

[13] F. Christowiak and C. Kreisbeck. Homogenization of layered materials with rigid components in single-slip finite crystal plasticity. Calc. Var. Partial Differential Equations, 56(3):Art. 75, 28pp, 2017.

[14] F. Christowiak and C. Kreisbeck. Asymptotic rigidity of layered structures and its application in homogenization theory. Preprint arXiv:1808.10494, 2018.

[15] S. Conti. Relaxation of single-slip single-crystal plasticity with linear hardening. In P. Gumbsch, editor, Multiscale Materials Modeling, pages 30–35. Fraunhofer IRB, Freiburg, 2006.

[16] S. Conti, G. Dolzmann, and C. Kreisbeck. Asymptotic behavior of crystal plasticity with one slip system in the limit of rigid elasticity. SIAM Journal on Mathematical Analysis, 43(5):2337–2353, 2011.

[17] S. Conti, G. Dolzmann, and C. Kreisbeck. Relaxation of a model in finite plasticity with two slip systems. Math. Models Methods Appl. Sci., 23(11):2111–2128, 2013.

[18] S. Conti and F. Theil. Single-slip elastoplastic microstructures. Arch. Ration. Mech. Anal., 178(1):125–148, 2005.

[19] G. Crasta and V. De Cicco. A chain rule formula in the space $BV$. SIAM J. Math. Anal., 43(10):430–456, 2011.

[20] G. Dal Maso. An introduction to gamma-convergence. Number 8 in Progress in nonlinear differential equations and their applications. Birkhäuser, Boston, 1993.

[21] G. Dal Maso, I. Fonseca, G. Leoni, and M. Morini. Higher-order quasiconvexity reduces to quasicontinuity. Arch. Ration. Mech. Anal., 171(1):55–81, 2004.

[22] E. Davoli and G. Francfort. A critical revisiting of finite elastoplasticity. SIAM Journal of Mathematical Analysis, 47:526–565, 2015.

[23] E. Davoli and M. Friedrich. Two-well rigidity and multidimensional sharp-interface limits for solid-solid phase transitions. arXiv:1810.06298, 2018.

[24] G. Del Piero and D. R. Owen. Structured deformations of continua. Arch. Rational Mech. Anal., 124(2):99–155, 1993.

[25] R. Ferreira and I. Fonseca. Characterization of the multiscale limit associated with bounded sequences in $BV$. J. Convex Anal., 19(2):403–452, 2012.

[26] I. Fonseca, G. Leon, and J. Malý. Weak continuity and lower semicontinuity results for determinants. Arch. Ration. Mech. Anal., 178(3):411–448, 2005.

[27] M. Friedrich and M. Kružík. On the passage from non-linear to linearized viscoelasticity. Math. Models Methods Appl. Sci., 28(12):2367–2401, 2018.

[28] A. Braides. Gamma-convergence for beginners. Number 22 in Oxford lecture series in mathematics and its applications. Oxford University Press, Oxford, 1. ed edition, 2005.

[29] A. Chambolle, A. Giacomini, and M. Ponsiglione. Piecewise rigidity. J. Funct. Anal., 244:134–153, 2007.

[30] R. Choksi, G. Del Piero, I. Fonseca, and D. Owen. Structured deformations as energy minimizers in models of fracture and hysteresis. Math. Mech. Solids, 4(3):321–356, 1999.

[31] R. Choksi and I. Fonseca. Bulk and interfacial energy densities for structured deformations of continua. Arch. Rational Mech. Anal., 138(1):37–103, 1997.

[32] F. Christowiak. Homogenization of layered materials with stiff components. PhD thesis, Universität Regensburg, 2012.

[33] F. Christowiak and C. Kreisbeck. Homogenization of layered materials with rigid components in single-slip finite crystal plasticity. Calc. Var. Partial Differential Equations, 56(3):Art. 75, 28pp, 2017.

[34] F. Christowiak and C. Kreisbeck. Asymptotic rigidity of layered structures and its application in homogenization theory. Preprint arXiv:1808.10494, 2018.

[35] S. Conti. Relaxation of single-slip single-crystal plasticity with linear hardening. In P. Gumbsch, editor, Multiscale Materials Modeling, pages 30–35. Fraunhofer IRB, Freiburg, 2006.

[36] S. Conti, G. Dolzmann, and C. Kreisbeck. Asymptotic behavior of crystal plasticity with one slip system in the limit of rigid elasticity. SIAM Journal on Mathematical Analysis, 43(5):2337–2353, 2011.

[37] S. Conti and F. Theil. Single-slip elastoplastic microstructures. Arch. Ration. Mech. Anal., 178(1):125–148, 2005.

[38] G. Crasta and V. De Cicco. A chain rule formula in the space $BV$. SIAM J. Math. Anal., 43(10):430–456, 2011.

[39] G. Dal Maso. An introduction to gamma-convergence. Number 8 in Progress in nonlinear differential equations and their applications. Birkhäuser, Boston, 1993.

[40] G. Dal Maso, I. Fonseca, G. Leoni, and M. Morini. Higher-order quasiconvexity reduces to quasicontinuity. Arch. Ration. Mech. Anal., 171(1):55–81, 2004.

[41] E. Davoli and G. Francfort. A critical revisiting of finite elastoplasticity. SIAM Journal of Mathematical Analysis, 47:526–565, 2015.

[42] E. Davoli and M. Friedrich. Two-well rigidity and multidimensional sharp-interface limits for solid-solid phase transitions. arXiv:1810.06298, 2018.

[43] G. Del Piero and D. R. Owen. Structured deformations of continua. Arch. Rational Mech. Anal., 124(2):99–155, 1993.

[44] R. Ferreira and I. Fonseca. Characterization of the multiscale limit associated with bounded sequences in $BV$. J. Convex Anal., 19(2):403–452, 2012.

[45] I. Fonseca, G. Leon, and J. Malý. Weak continuity and lower semicontinuity results for determinants. Arch. Ration. Mech. Anal., 178(3):411–448, 2005.

[46] M. Friedrich and M. Kružík. On the passage from nonlinear to linearized viscoelasticity. SIAM J. Math. Anal., 50(4):4426–4456, 2018.

[47] M. Giaquinta and D. Mucci. Maps of bounded variation with values into a manifold: total variation and relaxed energy. Pure Appl. Math. Q., 3(2, Special Issue: In honor of Leon Simon. Part 1):513–538, 2007.

[48] A. Mielke. Finite elastoplasticity, lie groups and geodesics on $SL(d)$. In Geometry, Dynamics, and Mechanics, pages 61–90. Springer, New York, 2002.

[49] A. Mielke. Energetic formulation of multiplicative elastoplasticity using dissipation distances. Contin. Mech. Thermodyn., 15:351–382, 2003.

[50] A. Mielke and T. Roubiček. Rate-independent elastoplasticity at finite strains and its numerical approximation. Math. Models Methods Appl. Sci., 26(12):2203–2236, 2016.

[51] P. Niederle. A critical review of the state of finite plasticity. Z. Angew. Math. Phys., 41:315–394, 1990.

[52] F. Podio-Guidugli. Contact interactions, stress, and material symmetry, for nonsimple elastic materials. Theoret. Appl. Mech., 28(2):261–276, 2002.

[53] R. Toupin. Elastic materials with couple stresses. Arch. Ration. Mech. Anal., 11:385–414, 1962.

[54] R. Toupin. Theory of elasticity with couple stress. Arch. Ration. Mech. Anal., 17:85–112, 1964.
University of Vienna, Faculty of Mathematics, Oskar-Morgenstern-Platz 1, 1090 Vienna, Austria
E-mail address: elisa.davoli@univie.ac.at

King Abdullah University of Science and Technology (KAUST), CEMSE Division, Thuwal 23955-6900, Saudi Arabia
E-mail address: rita.ferreira@kaust.edu.sa

Mathematisch Instituut, Universiteit Utrecht, Postbus 80010, 3508 TA Utrecht, The Netherlands
E-mail address: C.Kreisbeck@uu.nl