Any $l$-state improved quasi-exact analytical solutions of the spatially dependent mass Klein–Gordon equation for the scalar and vector Hulthén potentials

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Abstract
We present a new approximation scheme for the centrifugal term to obtain a quasi-exact analytical bound state solution within the framework of the position-dependent effective mass radial Klein–Gordon equation with the scalar and vector Hulthén potentials in any arbitrary $D$ dimension and orbital angular momentum quantum numbers $l$. The Nikiforov–Uvarov (NU) method is used in the calculations. The relativistic real energy levels and corresponding eigenfunctions for the bound states with different screening parameters have been given in a closed form. It is found that the solutions in the case of constant mass and in the case of s-wave ($l = 0$) are identical with the ones obtained in the literature.

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(Some figures in this article are in colour only in the electronic version.)

1. Introduction

The bound and scattering states of the s- and $l$-waves for any interaction system have raised great interest in both non-relativistic and relativistic quantum mechanics [1–3]. The exact solution of the wave equation is very important since the wavefunction contains all the necessary information regarding the quantum system under consideration. A number of methods have been used to solve the wave equations exactly or quasi-exactly for nonzero angular momentum quantum number ($l \neq 0$) by means of a given potential. The bound state eigenvalues were solved numerically [4, 5] and quasi-analytically using the variational [4, 6], perturbation [7], shifted $1/N$ expansion [8, 9], (NU) [10, 11], supersymmetric quantum mechanics (SUSYQM) [12–14] and asymptotic iteration method (AIM) [15] methods.

The Hulthén potential [10, 12, 13, 15, 16] is one of the most important short-range potentials in physics and it has been applied to a number of areas such as nuclear and particle physics [17], atomic physics [18, 19], molecular physics [20, 21] and chemical physics [22]. Therefore, it would be interesting and important to solve the relativistic equation for this potential for the $l \neq 0$ case, since it has been extensively used to describe the bound and continuum states of interaction systems. Recently, the exact solutions for the bound and scattering states of the s-wave Schrödinger [16, 23], Klein–Gordon [1–3] and Dirac equations [24, 25] with the scalar and vector Hulthén potentials have been investigated.

Relativistic effects with the scalar plus vector Hulthén-type potential [1, 2] in three- and $D$ dimensions and harmonic oscillator-type potential [26, 27] have also been discussed in the literature. The bound states of the Dirac and Klein–Gordon equations with the Coulomb-like scalar plus vector potentials have been studied in arbitrary dimension [28–32]. Furthermore, the exact results for
the scattering states of the Klein–Gordon equation with Coulomb-like scalar plus vector potentials have been investigated in arbitrary dimension [33]. This equation has been exactly solved for a larger class of linear, exponential and linear plus Coulomb potentials to determine the bound state energy spectrum using two semiclassical methods with the following relationship between the scalar and vector potentials: \( V(r) = V_0 + \beta S(r) \), \( S(r) > V(r) \), where \( V_0 \) and \( \beta \) are arbitrary constants [34]. In particular, inserting the constants \( V_0 = 0 \) and \( \beta = \pm 1 \) provides the equal scalar and vector potential case \( V(r) = \pm S(r) \).

Also, the position-dependent mass solutions of the non-relativistic and relativistic systems have received much attention recently. Many authors have used different methods to study both the partially exactly solvable and the exactly solvable Schrödinger, Klein–Gordon and Dirac equations in the presence of variable mass having a suitable mass distribution function in 1D, 3D and/or any dimension \( D \) cases for different potentials, such as the exponential-type potential [35], the Coulomb potential [36], the Lorentz scalar interactions [37], the hyperbolic-type potential [38], the Morse potential [39], the Pöschl–Teller (PT) potential [40], the Coulomb and harmonic potentials [41], the modified Kratzer-type, rotationally corrected Morse potential [42] and the Mie-type and pseudoharmonic potentials [43]. Recently, the point canonical transformation (PCT) has also been employed to solve the \( D \)-dimensional position-dependent effective mass Schrödinger equation for some molecular potentials to get the exact bound-state solutions including the energy spectrum and corresponding wavefunctions [41–43].

A new method to obtain the exactly solvable PT-symmetric potentials within the framework of the variable mass Dirac equation with the vector potential coupling scheme in \((1 + 1)\) dimensions is described in [38]. Three \( PT \)-symmetric potentials are produced: the \( PT \)-symmetric harmonic oscillator-like potential, the \( PT \)-symmetric linear plus inversely linear potential and the \( PT \)-symmetric kink-like potential. The SUSYQM formalism and the function analysis method are used to obtain the real energy levels and corresponding spinor components for the bound states.

Further, the position-dependent effective mass Dirac equation with the \( PT \)-symmetric hyperbolic cosecant potential can be mapped onto the Schrödinger-like equation with the exactly solvable modified PT potential [38]. The real relativistic energy levels and corresponding spinor wavefunctions for the bound states have been given in a closed form.

The \( NU \) method [44] and other methods have also been used to solve the \( D \)-dimensional Schrödinger equation [45] and the relativistic \( D \)-dimensional Klein–Gordon [46], Dirac [47] and the spinless Salpeter equations [48].

In strong coupling cases, it is crucial to understand relativistic effects on a moving particle in a potential field. In a non-relativistic case, the Schrödinger equation with the Hulthén potential [10, 12, 13, 15] was solved using the usual existing approximation, \( (1/r^2) \approx \alpha^2 [e^{\alpha r} / (e^{\alpha r} − 1)^2] \), for the centrifugal potential, which was found to be consistent with the results of other methods [4, 8, 13, 15]. Unfortunately, this approximation is valid only for small values of the screening parameter \( \alpha \), but the agreement becomes poor in the high-screening region [10, 15]. Hence, there is sufficient need to improve the analytical results for the Schrödinger equation with the Hulthén potential by means of a new approximation scheme. Recently, Haouat and Chetouani [49] have solved the Klein–Gordon and Dirac equations in the presence of the Hulthén potential, where the energy spectrum and the scattering wavefunctions are obtained for spin-0 and spin-\( \frac{1}{2} \) particles, using a more general approximation scheme, \( (1/r^2) \approx \alpha^2 [e^{\gamma \alpha r} / (1 − e^{-\gamma \alpha r})^2] \), where \( \gamma \) is a dimensionless parameter \((\gamma = 0, 1 \) and 2\) for the centrifugal potential. They found a good approximation, however, when the screening parameter \( \alpha \) and the dimensionless parameter \( \gamma \) are taken as \( \alpha = 0.1 \) and \( \gamma = 1 \), respectively, which is simply the case of the usual approximation [10, 12, 13, 15]. Also, Jia et al. [50] have recently proposed an alternative approximation scheme, \[ \frac{1}{r^2} \approx \alpha^2 [\omega (\omega r - 1) + \frac{1}{(\omega r - 1)^2}] \], where \( \omega \) is a dimensionless parameter \((\omega = 1.030)\), for the centrifugal potential to solve the Schrödinger equation with the Hulthén potential. Taking \( \omega = 1 \), their approximation can be reduced to the usual approximation [10, 12, 13, 15]. However, the accuracy of their numerical results [50] is found to be in poor agreement with other numerical methods such as the integration and variational methods [4, 5]. In order to improve the accuracy of the used approximation, we propose and apply an alternative shifted approximation scheme to approximate the centrifugal term given by [51, 52]

\[ \frac{1}{r^2} = \lim_{a \to 0} \alpha^2 \left[ c_0 + \frac{e^{\alpha r}}{(\omega a r - 1)^2} \right] . \]  

where \( c_0 \) is a shifting dimensionless parameter. The approximation scheme (1) emerged as a quite successful formalism to study the Schrödinger equation with the Manning–Rosen, hyperbolic and Hulthén potentials in calculating the energy eigenvalues within the framework of the \( NU \) method [51–53]. The accuracy our results is significantly improved and seems to be in high agreement with all other existing literature approximation schemes [4, 13, 15, 50, 52]. With extremely high accuracy, we have obtained the numerical energy eigenvalues, similarly to those obtained by the numerical integration [4, 5, 53] and variational [4] methods and also by a MATHEMATICA package programmed by Lucha and Schöberl [54].

The purpose of the present work is to employ the approximation scheme given in equation (1) to solve the position-dependent mass radial Klein–Gordon equation with any orbital angular quantum number \( l \) for the scalar and vector Hulthén potentials in \( D \) dimensions. This offers a simple, accurate and efficient scheme for the exponential-type potential models in quantum mechanics.

The present paper is organized as follows. In section 2, we review the \( NU \) method. In section 3, we present a brief derivation to find the shifting parameter \( c_0 \). Then, the analytical solution of the position-dependent mass Klein–Gordon equation with the scalar and vector Hulthén potentials is obtained for any \( l \)-state by means of the \( NU \) method. Section 4 contains the summary and conclusions.
2. The NU method

The NU method is briefly outlined here and the details can be found in [44]. This method is proposed to solve the second-order differential equation of hypergeometric type:

\[ \psi''_n(z) + \frac{\tilde{\tau}(z)}{\sigma(z)} \psi'_n(z) + \frac{\tilde{\sigma}(z)}{\sigma^2(z)} \psi_n(z) = 0, \]  
(2)

where \( \sigma(z) \) and \( \tilde{\sigma}(z) \) are polynomials, at most, of second-degree, and \( \tilde{\tau}(z) \) is a first-degree polynomial. In order to find a particular solution for equation (2), let us decompose the wavefunction \( \psi_n(z) \) as follows:

\[ \psi_n(z) = \phi_n(z) y_n(z). \]  
(3)

We can reduce equation (2) into the form

\[ \sigma(z) y''_n(z) + \tau(z) y'_n(z) + \lambda y_n(z) = 0, \]  
(4)

with

\[ \tau(z) = \tilde{\tau}(z) + 2\pi(z), \quad \tau'(z) < 0, \]  
(5)

where \( \tau'(z) = (d\tau(z)/dz) \) is the derivative. Also, \( \lambda \) is a constant given in the form

\[ \lambda = \lambda_n = -n\tau'(z) - \frac{1}{2} (n - 1) \sigma''(z), \]  
(6)

where

\[ \lambda = k + \pi'(z). \]  
(7)

The \( y_n(z) \) can be written in terms of the Rodrigues relation

\[ y_n(z) = \frac{B_n}{\rho(z)} \frac{d^n}{dz^n} \left[ \sigma^n(z) \rho(z) \right], \]  
(8)

where \( B_n \) is the normalization constant and the weight function \( \rho(z) \) satisfies

\[ \sigma(z) \rho'(z) + (\sigma'(z) - \tau(z)) \rho(z) = 0. \]  
(9)

The other wavefunction in the solution is defined by

\[ \sigma(z) \phi'(z) - \pi(z) \phi(z) = 0. \]  
(10)

Further, to find the weight function in equation (8), we need to obtain the following polynomial:

\[ \pi(z) = \frac{1}{2} \left[ \sigma'(z) - \tilde{\tau}(z) \right] \pm \sqrt{\frac{1}{4} \left[ \sigma'(z) - \tilde{\tau}(z) \right]^2 - \tilde{\sigma}(z) + k \sigma(z)} \]  
(11)

The expression under the square root sign in equation (11) can be arranged as the square of a polynomial. This is possible only if its discriminant is zero. In this regard, an equation for \( k \) is being obtained. After solving such an equation, the determined values of \( k \) are included in the NU method.

3. Bound-state solutions

3.1. An improved shifted approximation scheme

The approximation is based on the expansion of the centrifugal term in a series of exponentials depending on the intermolecular distance \( r \). Therefore, instead of using the usual approximation existing in the literature, let us, instead, take the following exponential-type potential to approximate the centrifugal potential:

\[ \frac{1}{r^2} \approx \alpha^2 \left[ c_0 + v(r) + v^2(r) \right], \quad v(r) = e^{\alpha r} - 1. \]  
(12)

In the low-screening region, \( 0.4 \alpha r \ll 1.2 \) [15] (i.e. a small screening parameter \( \alpha \)), equation (12) is a very good approximation to the centrifugal potential, and the Schrödinger equation for such an approximation can easily be solved analytically. In figure 1, a plot of the variation of the centrifugal potential and its approximation given in equation (12) versus \( \alpha r \) is shown. It shows that the approximation (12) and \( 1/r^2 \) are similar and coincide in both the high-screening and the low-screening regions.

Changing the \( r \) coordinate to \( x \) by shifting the parameters as \( x = (r - r_0) / r_0 \) to avoid singularities [55], we obtain

\[ \frac{1}{r_0^2} (1 + x)^{-2} = \alpha^2 \left[ c_0 + \frac{1}{e^{\alpha (1 + x)} - 1} + \frac{1}{(e^{\alpha (1 + x)} - 1)^2} \right], \]  
(13)

\[ \gamma = \alpha r_0, \]  
and expanding equation (13) around \( r = r_0 \) \( (x = 0) \), we obtain the following expansion:

\[ 1 - 2x + O(x^2) = \gamma^2 \left[ c_0 + \frac{1}{e^{\gamma} - 1} + \frac{1}{(e^{\gamma} - 1)^2} \right] \]  
\[ - \gamma^3 \left( \frac{1}{e^{\gamma} - 1} + \frac{3}{(e^{\gamma} - 1)^2} + \frac{2}{(e^{\gamma} - 1)^3} \right) x \]  
\[ + O(x^2), \]  
(14)

and consequently

\[ \gamma^2 \left[ c_0 + \frac{1}{e^{\gamma} - 1} + \frac{1}{(e^{\gamma} - 1)^2} \right] = 1, \]  
(15)

\[ \gamma^3 \left( \frac{1}{e^{\gamma} - 1} + \frac{3}{(e^{\gamma} - 1)^2} + \frac{2}{(e^{\gamma} - 1)^3} \right) = 2. \]  

By solving equations (14) and (15) for the dimensionless parameter \( c_0 \), we obtain

\[ c_0 = \frac{1}{\gamma^2} - \frac{1}{e^{\gamma} - 1} - \frac{1}{(e^{\gamma} - 1)^2} = 0.0823058167837972, \]  
(16)

where \( e = 2.718281828459045 \) is the base of the natural logarithms and the parameter \( \gamma = 0.4990429999. \)
Therefore, the centrifugal potential takes the form
\[
\lim_{\alpha \to 0} \left( \frac{1}{r^2} - \frac{1}{e^{-\alpha r} - 1} - \frac{1}{(e^{-\alpha r} - 1)^2} \right)
+ \frac{e^{-\alpha r}}{1 - e^{-\alpha r}} + \left( \frac{e^{-\alpha r}}{1 - e^{-\alpha r}} \right)^2 = \frac{1}{r^2}.
\]

(17)

Let us remark at the end of this analysis that the approximation used in many papers in the literature \[10, 12, 13, 15\] is a special case of equation (12) if \(c_0\) is set as zero.

### 3.2. Quasi-exactly energy eigenvalues and eigenfunctions

The \(D\)-dimensional time-independent radial position-dependent mass Klein–Gordon equation with scalar and vector potentials \(S(r)\) and \(V(r)\), respectively, \(r = |\mathbf{r}|\), and position-dependent mass \(m(r)\) describing a spin-zero particle takes the general form \[3, 46\]
\[
\nabla_D^2 \psi_{l_1, l_2}^{(l, m_l)}(\mathbf{x}) + \frac{1}{\hbar^2 c^2} \left( [E_{nl} - V(r)]^2 - [m(r)c^2 + S(r)]^2 \right) \psi_{l_1, l_2}^{(l, m_l)}(\mathbf{x}) = 0,
\]
\[
\nabla_D^2 = \sum_{j=1}^{D} \frac{\partial^2}{\partial x_j^2},
\]
\[
\psi_{l_1, l_2}^{(l, m_l)}(\mathbf{x}) = R_l(r) Y_{l_1, l_2}^{(l, m_l)}(\theta_1, \theta_2, \ldots, \theta_{D-1}),
\]

then, we reduce equation (18) into the following \(D\)-dimensional radial position-dependent effective mass Schrödinger-like equation:
\[
\frac{d^2 g(r)}{dr^2} + \frac{1}{\hbar^2 c^2} \left\{ \frac{(E_{nl} - V(r))^2}{m(r)c^2 + S(r)} - \frac{(D + 2l - 1)(D + 2l - 3)\hbar^2 c^2}{4r^2} \right\} g(r) = 0.
\]

(20)

Further, taking the vector and scalar potentials as the Hulthén potentials
\[
V(r) = - \frac{V_0}{1 - e^{-\alpha r}},
\]
\[
S(r) = - \frac{S_0}{1 - e^{-\alpha r}},
\]
\[
\alpha = r_0^{-1}
\]

and choosing the following mass function:
\[
m(r) = m_0 + \frac{m_1 e^{-\alpha r}}{1 - e^{-\alpha r}},
\]

we can rewrite equation (20) as
\[
\frac{d^2 g(r)}{dr^2} + \frac{1}{\hbar^2 c^2} \left\{ \frac{2\left[ m_0 c^2 (S_0 - m_1 c^2) + E_{nl} V_0 \right] e^{-\alpha r}}{1 - e^{-\alpha r}} 
+ \frac{\left[ V_0^2 - (S_0 - m_1 c^2)^2 \right] e^{-2\alpha r} - \frac{\hbar^2 c^2}{r^2} (D + 2l - 1)(D + 2l - 3) e^{-\alpha r}}{(1 - e^{-\alpha r})^2} \right\} g(r) = 0,
\]
\[
g(0) = 0,
\]

(23)
with the shift energy $\Delta E_i = h^2 c^2 a^2 (D + 2l - 1)(D + 2l - 3)\epsilon_n/4$. On account of the wavefunction $g(r)$ satisfying the standard bound-state condition (real values), i.e. $g(r \to \infty) \to 0$, if we rewrite equation (23) by using a new variable of the form $z = e^{-\sigma r}$ ($r \in [0, \infty], z \in [0, 1]$), we get
\[
d^2g(z) + \frac{1 - z}{z(1 - z)} \frac{dg(z)}{dz} + \frac{1}{[z(1 - z)]^2} \times \left\{ -\varepsilon_{nl}^2 + (\beta_1 - \beta_4 - \gamma + 2\varepsilon_{nl}^2)z \\
- (\beta_1 + \beta_2 + \beta_3 - \beta_4 + \varepsilon_{nl}^2)z^2 \right\} g(z) = 0, 
\]
where the following definitions of parameters are used:
\[
\varepsilon_{nl} = \sqrt{(m_0 c^2)^2 - \left(\frac{\Delta E_i}{Q}\right)^2}, \\
\beta_1 = \frac{2(m_0 c^2 S_0 + E_{nl} V_0)}{Q^2}, \\
\beta_2 = \frac{S_0^2 - V_0^2}{Q^2}, \\
\beta_3 = \frac{m_1 c^2 (m_1 c^2 - 2S_0)}{Q^2}, \\
\beta_4 = \frac{2m_0 m_1 c^4}{Q^2}, \\
\gamma = \frac{(D + 2l - 1)(D + 2l - 3)}{4}, \\
Q = \hbar ca. 
\]

For bound-state solutions, we require that $V_0 \leq (S_0 - m_1 c^2)$ and $E_{nl} \leq \sqrt{(m_0 c^2)^2 + \Delta E_i}$. In order to solve equation (24) by means of the NU method, we should compare it with equation (2). The following values for parameters are found:
\[
\tilde{\tau}(z) = 1 - z, \quad \sigma(z) = z - z^2, \\
\tilde{\sigma}(z) = -\varepsilon_{nl}^2 + (\beta_1 - \beta_4 - \gamma + 2\varepsilon_{nl}^2)z \\
- (\beta_1 + \beta_2 + \beta_3 - \beta_4 + \varepsilon_{nl}^2)z^2. 
\]

If we insert these values of parameters into equation (11), with $\sigma'(z) = 1 - 2z$, the following linear function is obtained:
\[
\pi(z) = -\frac{z}{2} \pm \frac{1}{2} \sqrt{\left[1 + 4(\beta_1 + \beta_2 + \beta_3 - \beta_4 + \varepsilon_{nl}^2 - k)\right]z^2 + \left[4(k - \beta_1 + \beta_4 + \gamma - 2\varepsilon_{nl}^2)z + 4\varepsilon_{nl}^2\right].} 
\]

The discriminant of the square root must be set as zero, that is, $\Delta = (k - \beta_1 + \beta_4 + \gamma - 2\varepsilon_{nl}^2)^2 - \varepsilon_{nl}^2[1 + 4(\beta_1 + \beta_2 + \beta_3 - \beta_4 + \varepsilon_{nl}^2 - k)] = 0$. Thus, the constant $k$ is found to be
\[
k = \beta_1 - \beta_4 - \gamma \pm \varepsilon_{nl} \sqrt{1 + 4(\beta_2 + \beta_3 + \gamma)}. 
\]
In this regard, we can find the four possible functions for $\pi(z)$ as
\[
\pi(z) = -\frac{z}{2} \pm \frac{1}{2} \sqrt{\left[1 + 4(\beta_2 + \beta_3 + \gamma)\right]z}, \quad \text{for} \quad k_1 = d + \varepsilon_{nl} \sqrt{1 + 4b}, \\
\pi(z) = \frac{z}{2} - \frac{1}{2} \sqrt{\left[1 + 4(\beta_2 + \beta_3 + \gamma)\right]z}, \quad \text{for} \quad k_2 = d - \varepsilon_{nl} \sqrt{1 + 4b}. 
\]
where $b = \beta_2 + \beta_3 + \gamma$ and $d = \beta_1 - \beta_4 - \gamma$. Thus, taking the following values:
\[
k = \beta_1 - \beta_4 - \gamma - \varepsilon_{nl} \sqrt{1 + 4(\beta_2 + \beta_3 + \gamma)} \quad \text{(30)}
\]
and
\[
\pi(z) = -\frac{z}{2} + \varepsilon_{nl} - \left[\varepsilon_{nl} - \frac{1}{2} \sqrt{1 + 4(\beta_2 + \beta_3 + \gamma)}\right] z, \quad \text{(31)}
\]
they give
\[
\tau(z) = 1 + 2\varepsilon_{nl} - 2 \left[1 + \varepsilon_{nl} + \frac{1}{2} \sqrt{1 + 4(\beta_2 + \beta_3 + \gamma)}\right] z, \quad \text{(32)}
\]
Using equations (30)–(32) together with the assignments given in equation (26), the following expressions for $\lambda$ are obtained:
\[
\lambda_n = \lambda = n^2 \left[1 + 2\varepsilon_{nl} + \sqrt{1 + 4(\beta_2 + \beta_3 + \gamma)}\right] n, 
\]
where $n$ is the radial quantum number. Let us define
\[
\delta = \frac{1}{2} \left(1 + \sqrt{1 + 4(\beta_2 + \beta_3 + \gamma)}\right), 
\]
where $\beta_2 + \beta_3 = \delta^2 - \delta - \gamma$. With the aid of equation (35), we can easily obtain the energy eigenvalue equation of the Hulthén potential by solving equations (33) and (34):
where $\tilde{S}_0 = S_0 - m_1 c^2$ is the modified scalar potential. Solving the last equation for the energy eigenvalues $E_{nl}^\pm$, we obtain

$$E_{nl}^\pm = \frac{V_0}{2} \left[ 1 - \frac{4\tilde{S}_0(\tilde{S}_0 + 2m_0 c^2)}{4V_0^2 + \kappa_{nl}^2} \right] \pm \frac{\kappa_{nl}}{2} \sqrt{\xi - 1 \left[ 1 - \frac{4\tilde{S}_0(\tilde{S}_0 + 2m_0 c^2)}{4V_0^2 + \kappa_{nl}^2} \right]^2},$$

$$\xi = \left(2m_0 c^2\right)^2 + \hbar^2 c^2 \alpha^2 (D + 2l - 1)(D + 2l - 3)c_0 \frac{V_0}{4V_0^2 + \kappa_{nl}^2},$$

where $n = 0, 1, 2, \ldots$ and $l = 0, 1, 2, \ldots$ signify the usual radial and angular momentum quantum numbers, respectively, and

$$(\hbar c \alpha)^2 (D + 2l - 2)^2 + 4\tilde{S}_0^2 \geq 4V_0^2,$$  

are constraints over the strength of the potential coupling parameters. In the above equation, let us remark that it is not difficult to conclude that all bound states appear in pairs; two energy solutions are valid for the particle $E_0^\nu = E_{nl}^+ \nu$ and the second one corresponds to the anti-particle energy $E_0^\alpha = E_{nl}^- \nu$ in the Hulthén field. When we take the scalar and vector potentials as $\tilde{S}_0 = 0$ (i.e. $S_0 = m_1 c^2$) and $V_0 \neq 0$, the energy equation (37) becomes

$$E_{nl}^\pm = \frac{V_0}{2} \left[ 1 - \frac{4\tilde{S}_0(\tilde{S}_0 + 2m_0 c^2)}{4V_0^2 + \kappa_{nl}^2} \right] \pm \frac{\kappa_{nl}}{2} \sqrt{\xi - 1 \left[ 1 - \frac{4\tilde{S}_0(\tilde{S}_0 + 2m_0 c^2)}{4V_0^2 + \kappa_{nl}^2} \right]^2},$$

$$\kappa_{nl} = \hbar c \alpha (2n + 1) + \sqrt{4(\tilde{S}_0^2 - V_0^2)} + (\hbar c \alpha)^2 (D + 2l - 2)^2,$$

and the following constraints on the coupling parameter of the vector potential:

$$(\hbar c \alpha)^2 (D + 2l - 2)^2 \geq 4V_0^2$$

must be fulfilled for real eigenvalues.

Therefore, having solved the $D$-dimensional position-dependent mass Klein–Gordon equation for the scalar and vector usual Hulthén potentials, we should make some useful remarks.

(i) For the s-wave ($l = 0$), the exact energy eigenvalues of the 1D Klein–Gordon equation become

$$E_{nl}^\pm = \frac{V_0}{2} \left( 1 - \frac{4\tilde{S}_0(\tilde{S}_0 + 2m_0 c^2)}{4V_0^2 + \kappa_{nl}^2} \right) \pm \frac{\kappa_{nl}}{2} \sqrt{\frac{m_0^2 c^4}{4V_0^2 + \kappa_{nl}^2} - \frac{1}{16} \left( 1 - \frac{4\tilde{S}_0(\tilde{S}_0 + 2m_0 c^2)}{4V_0^2 + \kappa_{nl}^2} \right)^2},$$

$$\kappa_{nl} = \hbar c \alpha (2n + 1) + \sqrt{(\hbar c \alpha)^2 + 4(\tilde{S}_0^2 - V_0^2)}.$$  

In order that at least one level might exist, it is necessary that the inequalities

$$\hbar^2 c^2 \alpha^2 + 4\tilde{S}_0^2 \geq 4V_0^2,$$

$$\frac{16m_0^2 c^4}{4V_0^2 + \kappa_{nl}^2} \geq \left( 1 - \frac{4\tilde{S}_0(\tilde{S}_0 + 2m_0 c^2)}{4V_0^2 + \kappa_{nl}^2} \right)^2,$$

are fulfilled. In the case of $\tilde{S}_0 = 0$, $V_0 \neq 0$, the energy spectrum (in units where $\hbar = c = 1$) is

$$E_{nl}^\pm = \frac{V_0}{2} \left[ \alpha (2n + 1) + \sqrt{\alpha^2 - 4V_0^2} \right] \times \frac{m_0^2}{4V_0^2 + \alpha (2n + 1) + \sqrt{\alpha^2 - 4V_0^2}} \geq \frac{1}{16},$$

and the following constraints on the potential coupling constant:

$$16m_0^2 \geq 4V_0^2 + \sqrt{\alpha^2 - 4V_0^2} + (2n + 1)^2, \quad \alpha \geq 2V_0,$$

are fulfilled for bound-state solutions. We notice that the result given in equation (43) is identical to equation (31) of [56]. As can be seen from equation (43), there are only two lower-lying states ($n = 0, 1$) for a Klein–Gordon particle of rest mass $m_0 = 1$ and screening parameter $\alpha = 1$ with vector coupling strength $V_0 \leq 1/2$. As an example, one may calculate the ground-state energy for the vector coupling strength $V_0 = \alpha/2$ as

$$E_0^\pm = \frac{V_0}{2} \left[ 1 + \sqrt{\frac{m_0^2}{V_0^2} - 1} \right].$$

Further, in the case of the pure scalar potential ($V_0 = 0$, $S_0 = m_1 c^2$), the energy spectrum is

$$E_{nl}^\pm = \pm \frac{m_0^2 c^4}{4} (\hbar c \alpha)^2 (n + 1)^2,$$

$$4m_0^2 c^4 \geq (\hbar c \alpha)^2 (n + 1)^2.$$

Since the Klein–Gordon equation is independent of the sign of $E_{nl}^\pm$ for scalar potentials, the wavefunctions become the same.
for both energy values. If the range parameter $\alpha$ is chosen to be $\alpha = 1/\lambda_c$, where $\lambda_c = \hbar / m_0 c = 1/m_0$ denotes the Compton wavelength of the Klein–Gordon particle, it can be easily seen that while $S_0 \to mc^2$ in the ground state ($n = 0$), all energy eigenvalues tend to the value $E_0 \approx 0.866 m_0$.

(ii) For $D = 3$, the mixed scalar and vector Hulthén potentials, the energy eigenvalues for $l \neq 0$ are given by

$$E_{nl}^\pm = \frac{V_0}{2} \left( 1 - \frac{4\tilde{S}_0(\tilde{S}_0 + 2mc^2)}{4V_0^2 + \kappa_{nl}^2} \right) \pm \kappa_{nl} \sqrt{\frac{1}{4} \left( 1 - \frac{4\tilde{S}_0(\tilde{S}_0 + 2mc^2)}{4V_0^2 + \kappa_{nl}^2} \right)^2},$$

$$(47)$$

Further, in order that at least one real eigenvalue might exist, it is necessary that the inequality

$$(h\alpha)^2(2l+1)^2 + 4\tilde{S}_0^2 \geq 4V_0^2,$$

$$(48)$$

be fulfilled. For the case where $\tilde{S}_0 = 0$ in the spatial-dependent mass ($S_0 = 0$, in the constant mass case) [46], the energy eigenvalues turn out to be

$$E_{nl}^\pm = \frac{V_0}{2} \left( 1 - \frac{4\tilde{S}_0(\tilde{S}_0 + 2mc^2)}{4V_0^2 + \kappa_{nl}^2} \right) \pm \kappa_{nl} \frac{(m_0 c^2)^2 + \hbar^2 c^2 \alpha^2 l(l+1)c_0}{4V_0^2 + \kappa_{nl}^2},$$

$$(49)$$

with the following constraint over the potential parameters:

$$(4m_0 c^2)^2 + 16\hbar^2 c^2 \alpha^2 l(l+1)c_0 \geq 4V_0^2,$$

$$(50)$$

(iii) When $D = 3$ and $l = 0$, the centrifugal term

$$\frac{(D + 2l - 1)(D + 2l - 3)c_0}{4r^2} = 0,$$

and consequently the approximation term

$$\frac{(D + 2l - 1)(D + 2l - 3)c_0}{4} \left[ c_0 + \frac{e^{-\alpha r}}{(1 - e^{-\alpha r})^2} \right] = 0,$$

too. Thus, the energy eigenvalues turn out to be

$$\sqrt{(m_0 c^2)^2 - E_{nl}^2} = \frac{2[m_0 c \tilde{S}_0 + E_{nl}^2 V_0] + \tilde{S}_0^2 - V_0^2}{2h\alpha(n + \delta)} - h\alpha \left( \frac{n + \delta}{2} \right),$$

$$(51)$$

$$(n = 0, 1, 2, 3, \ldots),$$

which gives

$$E_{nl}^\pm = \frac{V_0}{2} \left( 1 - \frac{4\tilde{S}_0(\tilde{S}_0 + 2mc^2)}{4V_0^2 + \xi_{nl}^2} \right) \pm \xi_{nl} \frac{(m_0 c^2)^2 + \hbar^2 c^2 \alpha^2 l(l+1)c_0}{4V_0^2 + \xi_{nl}^2},$$

$$(52)$$

(iv) For equal scalar and vector usual Hulthén potentials (i.e. $S_0 = V_0$), equation (36) with the aid of equation (25) can be reduced to the relativistic energy equation (in the conventional atomic units $\hbar = c = 1$):

$$\sqrt{m_0^2 + \frac{(D + 2l - 1)(D + 2l - 3)c_0}{4r_0^2}} = E_r^2,$$

$$= \frac{2r_0 V_0 [m_0 + E_R - m_1] + r_0(m_1 - 2m_0)m_1}{2(n + \delta)} - n + \delta,$$

$$(53)$$

which is equation (22) of [58] if the perturbed mass $m_1 = 0$ and shifting parameter $c_0 = 0$.

(v) We discuss the non-relativistic limit of the energy equation (53). When $V_0 = S_0$, equation (23) reduces to a Schrödinger-like equation for the potential $2V(r)$. In other words, the non-relativistic limit is the Schrödinger equation for the potential $-2V_0 e^{-\alpha r}/[1 - e^{-\alpha r}]$. $r_0 = \alpha^{-1}$. After making the parameter replacements $m_0 + E_R \to 2m_0$ and $m_0 - E_R \to -E_{NR}$ in equation (53) [58], it reduces to the
non-relativistic energy equation of references [10, 12, 13, 15, 57, 59]:

$$E_{NR} = \alpha^2(D + 2l - 1)(D + 2l - 3)c_0 \frac{8m_0}{8m_0\alpha^2} \left[ \frac{(2V_0 - m_1)(2m_0 - m_1) - \alpha^2(n + \delta)^2}{(n + \delta)} \right]^2,$$

$$\delta = \frac{1}{2} \left[ 1 + \frac{1}{\alpha} \sqrt{\alpha^2(D + 2l - 2)^2 + (2m_1c^2)^2 - 8V_0m_1c^2} \right],
(n = l = 0, 1, 2, 3, \ldots), \tag{54}$$

which is equation (23) of [57] when \(c_0\) and \(m_1\) are set as zero. It is noted that equation (54) is identical to equation (59) of [56] for the s-wave in 1D when the potential is \(2V(r)\), when \(\alpha\) becomes pure imaginary, i.e. \(\alpha \rightarrow i\alpha\), and when we set \(m_0 = 0\), \(m_1 = 0\) and \(c_0 = 0\). Equation (54) can be reduced to the constant mass \((m_1 = 0)\) case in the 3D Schrödinger equation:

$$E_{NR} = \alpha^2 \frac{2}{2m_0} \left[ l(l + 1)c_0 - \frac{(2V_0m_0}{\alpha^2(n + l + 1)} - \frac{(n + l + 1)}{2} \right]^2,$$

which is identical to the expressions given in [50, 52] when the vector potential is taken as \(2V(r)\), \(c_0 = 0\) and \(\omega = 1\) in [50]. The numerical approximation to the energy eigenvalues in [50] for the last energy equation was found to be more efficient than the approximation given in equation (19) of [50]. Taking \(V_0 = Zae^2\) as in [50], we obtain

$$E_{NR} = \alpha^2 \frac{2}{2m_0} \left[ l(l + 1)c_0 - \frac{(2V_0m_0}{\alpha^2(n + l + 1)} - \frac{(n + l + 1)}{2} \right]^2,$$

For the s-wave \((l = 0)\), the above energy spectrum is identical to the factorization method [23], the SUSYQM method [12, 13] and the NU [46] method. Expanding the energy equation (53) under the weak coupling conditions \([(n + \delta)/m\alpha c^2] \ll 1\) and \([V_0g(r)/(n + \delta)] \ll 1\), retaining only the terms containing the power of \((1/m\alpha c^2)\) and \((r_0V_0)^2\), we obtain the relativistic energy equation

$$E_R \approx E_{NR} + m_0 + 2(2m_0 - m_1) \frac{(2V_0 - m_1)}{2\alpha(n + \delta)}^4,$$ \tag{55}

which is simply equation (24) of [57], where \(\delta\) is given in equation (54). The first term is the non-relativistic energy and the third term is the relativistic approximation to energy.

Now, let us find the wavefunction \(\psi_n(r)\), which is the polynomial solution of the hypergeometric-type equation. We multiply equation (4) by the weight function \(\rho(s)\) so that it can be rewritten in self-adjoint form [45, 46]

$$[\alpha^2(s)\psi_n'(s)] + \lambda\rho(s)\psi_n(s) = 0. \tag{56}$$

The weight function \(\rho(s)\) that satisfies equation (9) takes the following form:

$$\rho(z) = z^{2\alpha}(1 - z)^{\beta}, \quad \beta = 2\delta - 1, \tag{57}$$

which gives the Rodrigues relation:

$$\psi_n(z) = B_n z^{-2\alpha}(1 - z)^{-\beta} \left[ e^{2\alpha}(1 - z)^{\beta+\delta} \right] = B_n P_n^{(2\alpha, \beta)}(1 - 2z). \tag{58}$$

On the other hand, inserting the values of \(\sigma(s)\), \(\pi(s)\) and \(\tau(s)\) given in equations (26), (31) and (32) into equation (10), we get the other part of the wavefunction

$$\phi(s) = z^{\alpha}(1 - z)^{\beta}.$$

Hence, the wavefunction \(g_n(z) = \phi_n(z)\psi_n(z)\) becomes

$$g(z) = C_n z^{\alpha}(1 - z)^{\beta} P_n^{(2\alpha, \beta)}(1 - 2z)$$

$$= C_n z^{\alpha}(1 - z)^{\beta} P_n^{(2\alpha, \beta)}(1 - 2z), \quad z \in [1, 0]. \tag{60}$$

Finally, the radial wavefunctions of the Klein–Gordon equation are obtained as

$$R_l(r) = N_n r^{-(D - 1)/2} e^{-\sqrt{m_0^2 - E^2}r} (1 - e^{-\sqrt{m_0^2 - E^2}r}) P_n^{(2\alpha, \beta)}(1 - 2e^{-\sqrt{m_0^2 - E^2}r}), \tag{61}$$

with

$$\varepsilon_{nl}^{(D)} = \frac{1}{\hbar c^2} \sqrt{(m_0c^2)^2 + \hbar^2 c^2 \alpha^2(D + 2l - 1)(D + 2l - 3)c_0 - E_{nl}^2},$$

$$\beta = \frac{1}{\hbar c} \sqrt{4(S_0^2 - V_0^2) + (\hbar \alpha c)^2(D + 2l - 2)^2},$$

$$\delta = \frac{1}{2}(1 + \beta),$$

where \(E_{nl}\) is given in equation (37) and \(N_{nl}\) is the radial normalization factor. The Jacobi polynomials \(P_n^{(2\alpha, \beta)}(1 - 2e^{-\sqrt{m_0^2 - E^2}r})\) [60] in the last result can be written in terms of the hypergeometric function \(F_1(-n, n; e^{-\sqrt{m_0^2 - E^2}r}, \beta + 1, 2e^{-\sqrt{m_0^2 - E^2}r}); e^{-\sqrt{m_0^2 - E^2}r}\), which gives the same result obtained in [57].

(i) The exact radial wavefunctions for the s-wave Klein–Gordon equation in 1D reduce to the following form (in \(\hbar = c = 1\)):

$$R_n(x) = C_n e^{-\sqrt{m_0^2 - E^2}x}(1 - e^{-\sqrt{m_0^2 - E^2}x})^{(1 + \alpha)/2} \times P_n^{(2\alpha, \beta)}(1 - 2e^{-\sqrt{m_0^2 - E^2}x}),$$

$$a = \sqrt{1 + 4x^2(S_0^2 - V_0^2)}, \tag{63}$$

where \(E_n\) is given in equation (41). The last formula is identical to equation (35) of [56] when the modified scalar potential, \(S_0\), is set as zero.

(ii) Choosing the atomic units \(\hbar/2\pi = \hbar = c = 1\), the exact radial wavefunctions for the s-wave Klein–Gordon equation in 3D reduce to the following form:

$$R_n(r) = N_n e^{-\sqrt{m_0^2 - E^2}r}(1 - e^{-\sqrt{m_0^2 - E^2}r})^{(1 + \alpha)/2} \times P_n^{(2\alpha, \beta)}(1 - 2e^{-\sqrt{m_0^2 - E^2}r}),$$

$$\times P_n^{(2\alpha, \beta)}(1 - 2e^{-\sqrt{m_0^2 - E^2}r}),$$

$$\times P_n^{(2\alpha, \beta)}(1 - 2e^{-\sqrt{m_0^2 - E^2}r}),$$

$$= 2F_1\left(n - n + 2r_0 \sqrt{m_0^2 - E^2 + a + 1}, 2r_0 \sqrt{m_0^2 - E^2 + a + 1}, e^{-\sqrt{m_0^2 - E^2}r}\right), \tag{64}$$
Table 1. The energy spectrum of the scalar and vector Hulthén potentials for $m_0 = 1$ and $m_1 = 0$.

| $V_0 = S_0$, $n$, $l$, $E_{nl}^{a}$, $E_{nl}^{b}$, $E_{nl}^{[61, 62]}$, $E_{nl}^{[61, 62]}$ | $\uparrow$ | $\downarrow$ |
|---|---|---|
| $1$ $1$ $0$ | $1.000000$ | $-0.600000$ | $1.000000$ | $-0.600000$ |
| $1$ $1$ | $-0.707107$ | $-0.707107$ | $-0.707107$ | $-0.707107$ |
| $1$ $1$ | $-0.984171$ | $-0.214941$ | $-0.984171$ | $-0.214941$ |
| $2$ $0$ | $0.984171$ | $-0.214941$ | $0.984171$ | $-0.214941$ |
| $3$ $0$ | $0.302169$ | $-0.763708$ | $0.302169$ | $-0.763708$ |
| $1$ $1$ | $0.911438$ | $-0.411438$ | $-0.911438$ | $-0.411438$ |
| $1$ $2$ | $0.600000$ | $0.600000$ | $0.600000$ | $0.600000$ |
| $3$ $0$ | $0.600000$ | $0.600000$ | $0.600000$ | $0.600000$ |
| $6$ $1$ $0$ | $-0.355501$ | $-0.844949$ | $-0.355501$ | $-0.844949$ |
| $1$ $1$ | $0.235890$ | $-0.635890$ | $-0.235890$ | $-0.635890$ |
| $1$ $2$ | $0.763708$ | $-0.302169$ | $-0.763708$ | $-0.302169$ |
| $1$ $3$ | $0.994273$ | $0.284416$ | $0.994273$ | $0.284416$ |
| $3$ $0$ | $0.763708$ | $-0.302169$ | $0.763708$ | $-0.302169$ |
| $3$ $1$ | $0.994273$ | $0.284416$ | $0.994273$ | $0.284416$ |
| $4$ $0$ | $0.994273$ | $0.284416$ | $0.994273$ | $0.284416$ |

\[a \text{ The present NUC method.}\]

\[b \text{ The results from AIM and SUSY.}\]

where $E_{nl}$ and $a$ are given in equations (52) and (63), respectively. The last formula is identical to equation (22) of [57] when the perturbed mass $m_1$ is set as zero.

(iii) The quasi-exact radial wavefunctions for the $l$-wave Klein–Gordon equation in 3D reduce to the following form (in $\hbar = c = 1$):

\[
R_{nl}(r) = N_{nl} e^{-\sqrt{m_0^2 + l(l+1)} c_0 - E_{nl}^0 r} (1 - e^{-r/r_{\alpha}})^{(1+a)/2} \\
\times \left[ 2n_0 \sqrt{m_0^2 + l(l+1)} c_0 - E_{nl}^0 r \right] (1 - 2e^{-r/r_{\alpha}}),
\]

\[
= 2F_1 \left( -n, n + 2r_0 \sqrt{m_0^2 + l(l+1)c_0 / r_0^2} - E_{nl}^0, a + 1, 2r_0 \sqrt{m_0^2 + l(l+1)c_0 / r_0^2} - E_{nl}^0, e^{-ar} \right),
\]

where $E_{nl}$ is given in equation (43) and $\alpha = r_{\alpha}^{-1}$. It is identical to [57] when $m_1 = 0$. The eigenfunctions in the constant mass

where $n_{l,m}^{(l)}$ and $\beta$ are given in equation (62) and $E_{nl}$ is given in equation (37) [46].
Let us check the accuracy of the resulting analytical expressions. We give a few numerical real eigenvalues for some selected values of the mass \( m_0 \) and \( m_1 \) and the potential parameters \( S_0 \) and \( V_0 \). In tables 1 and 2, taking \( \alpha = 1 \) and \( m_0 = 1 \), we present some numerical values for the energy spectrum of the constant mass Klein–Gordon equations with the condition \( S_0 = V_0 \) for all possible real eigenvalues. To get more real energy eigenvalues in the constant mass
case (e.g. \(m_0 = 1, m_1 = 0\)), the vector parameter \(V_0\) of the Hulthén potential should be increased. As shown in tables 1 and 2, when the parameter \(V_0 = S_0 = 1, 2, 3, 6, 8\) and 20, we obtain \(N = 1, 3, 6, 10, 15\) and 36 real energy eigenvalues, respectively. The numerical solutions of the position-dependent mass case with vector and scalar Hulthén potential parameters satisfying the conditions \(S_0 = \pm V_0\) and \(S_0 > V_0\) are presented in table 3. For example, in table 3, when the Hulthén potential parameter \(V_0 = S_0 = 1, m_0 = 5\) and \(m_1 \neq 0\), we obtain \(N = 46\) real energy eigenvalues. Obviously, the number of real eigenvalues increases more in the solution of the position-dependent case than in the constant mass case where the condition \(S_0 \geq V_0\) must be fulfilled.

4. Conclusions

In summary, we have proposed an alternative approximation scheme for the centrifugal potential similar to the non-relativistic case. This is because the usual approximation [10, 13, 15] for the centrifugal term is valid only for the low-screening region; however, for the high-screening region, as \(\alpha\) increases, the agreement between the old approximation and centrifugal term decreases. Using this approximation scheme, the analytical solutions of the radial Klein–Gordon equation with position-dependent mass for scalar and vector Hulthén potentials can be approximately obtained for any dimension \(D\) and orbital angular momentum quantum number \(l\). It is found that the expressions for the eigenvalues and the corresponding eigenfunctions become complicated and tedious since the eigenvalues are related to the parameters \(m_0, m_1, S_0, V_0, c_0\) and \(\alpha\). We have investigated the possibility of obtaining the bound-state (real) energy spectra with some constraints to be imposed on the parameters and, further, the relationship between the strengths of vector \(V_0\) and scalar \(S_0\) coupling parameters. In 1D and 3D, the special cases for the angular momentum \(l = 0, 1\) are carried out in detail. We find that the analytical expressions of the energy eigenvalues and eigenfunctions are identical with the results obtained by other methods. The analytical energy equation and the unnormalized radial wavefunctions are expressed in terms of hypergeometric polynomials. For the constant mass case \((m_1 = 0)\) and s-wave \((l = 0)\), the results are reduced to the exact solution of bound states of the s-wave Klein–Gordon equation with scalar and vector Hulthén potentials. To test our results, we have also calculated the energy eigenvalues of a particle and antiparticle for the constant mass limit as well as the position-dependent mass case. The case of spatial-dependent mass with scalar potential \(S_0 = m_1 c^2\) is found to be equivalent to the constant mass with scalar potential \(S_0 = 0\) in a pure vector case. Hence, the spectrum is found to be the same.

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