Kinematics of a Spacetime with an Infinite Cosmological Constant

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A solution of the sourceless Einstein’s equation with an infinite value for the cosmological constant $\Lambda$ is discussed by using Inönü–Wigner contractions of the de Sitter groups and spaces. When $\Lambda \to \infty$, spacetime becomes a four–dimensional cone, dual to Minkowski space by a spacetime inversion. This inversion relates the four–cone vertex to the infinity of Minkowski space, and the four–cone infinity to the Minkowski light–cone. The non-relativistic limit $c \to \infty$ is further considered, the kinematical group in this case being a modified Galilei group in which the space and time translations are replaced by the non-relativistic limits of the corresponding proper conformal transformations. This group presents the same abstract Lie algebra as the Galilei group and can be named the conformal Galilei group. The results may be of interest to the early Universe Cosmology.

Keywords: Kinematical Group, Cosmological Constant, Cosmology

1. INTRODUCTION

A kinematical group, whichever it may be, will always have a subgroup accounting for both the isotropy of space (rotation group) and the equivalence of inertial frames (boosts). The remaining transformations, generically called translations, can be either commutative or not, and are responsible for the homogeneity of space and time. This holds of course for usual special–relativistic kinematics, but also for Galilean and other conceivable non–relativistic kinematics, which differ from each other precisely by being grounded on different kinematical groups.

The best known example is the Poincaré group $\mathcal{P}$, a group naturally associated to Minkowski spacetime $M$ as its group of motions. It contains, in the form of a semi–direct product, the Lorentz group $\mathcal{L} = SO(3,1)$ and the translation group $\mathcal{T}$. The latter acts transitively on $M$ and its manifold is just $M$. Indeed, Minkowski spacetime is a homogeneous space under $\mathcal{P}$, actually the quotient $M \equiv \mathcal{T} = \mathcal{P}/\mathcal{L}$. The invariance of $M$ under the transformations of $\mathcal{P}$ reflects its uniformity. The Lorentz subgroup provides an isotropy around a given point of $M$, and translation invariance enforces this isotropy around any other point. This is the usual meaning of “uniformity”, in which $\mathcal{T}$ is responsible for the equivalence of all points of spacetime.

The concept of group contraction, on the other hand, was first introduced to formalize and generalize the well known fact that the Galilei group can be obtained from the Poincaré group in the non-relativistic limit $c \to \infty$. The general procedure of group contraction involves always a preliminary choice of convenient coordinates, in terms of which a certain parameter is made explicit. A new kinematics can then be obtained by taking this parameter to an appropriate limit. In the specific case of the contraction of the Poincaré to the Galilei group, the parameter is the speed of light $c$, and the limit is achieved by taking $c$ to infinity.

Another well known example of group contraction is that by which the Poincaré group is obtained from any of the two de Sitter groups through a non-cosmological limit. In that case, the contraction parameter is the de Sitter pseudo–radius $R$. The contraction is achieved by taking the limit $R \to \infty$, under which the de Sitter “translations” reduce to the Poincaré space and time translations. The de Sitter spaces are solutions of Einstein’s equation for an empty space with a nonvanishing cosmological constant $\Lambda = R/4$, with $R$ the scalar curvature of the de Sitter spaces. Since $R \propto R^{-2}$ [see Eq. (18) below], this limit is equivalent to that in which $\Lambda \to 0$.

Inflationary cosmological models, on the other hand, suppose a very high value of the cosmological constant $\Lambda$ at the early stages of the Universe. The relation between its present time value $\Lambda_0$ and its value $\Lambda_{GUT}$ when the breakdown of the grand unification symmetry has taken place is

$$\Lambda_{GUT} \approx 10^{108} \Lambda_0. \quad (1)$$

Subsequently, at the breakdown of the electroweak symmetry, there has been another phase transition with energy–scale of the order

$$\Lambda_{EW} \approx 10^{57} \Lambda_0. \quad (2)$$

Thus, according to this scheme, the cosmological constant was very high at the beginning of the Universe, changing later to $\Lambda_{GUT}$, then to $\Lambda_{EW}$, and finally to $\Lambda_0$. How $\Lambda$ managed to change from a very large initial value to its present value is still an open question.
Now, for symmetry reasons, it is appealing to assume an infinite primordial $\Lambda$ at some initial time, to be followed in the succeeding epochs by a decaying (either through a time dependence or through phase transitions) but still large cosmological term which would drive inflation. All matter (energy) becomes negligible in the presence of an infinite cosmological constant, so that at that initial stage the Universe spacetime should be a limit solution of the sourceless Einstein’s equation with an infinite $\Lambda$. By using Inönü–Wigner contractions\(^{(3)}\) of the de Sitter groups and spaces, it has already been shown that this solution is a four–dimensional cone–spacetime, whose corresponding kinematical group is the conformal Poincaré group.\(^{(4)}\) On this cone–spacetime, the metric is singular everywhere except on a subspace, the three–dimensional light–cone. Rather surprisingly, this singular character of the metric does not prevent the existence of well–defined Levi–Civita connection and Riemann curvature tensor. It makes of the four–dimensional cone, nonetheless, a very strange spacetime. Distance can be defined only on the light–cone, where it is zero between any two points. In particular, no distance is defined on the spacelike sections. Space sections can, however, recover a notion of distance in the non–relativistic limit of infinite speed of light $c$. In that limit, the cone lends its limit metric to the space section. Our aim in this paper will be to study, by using Inönü–Wigner contractions, the geometrical properties and the kinematical group of such a speculative spacetime, on which both the cosmological constant and the speed of light are infinite. We start with a brief review of the de Sitter groups and spaces.

2. THE DE SITTER GROUPS AND SPACES

The de Sitter spaces are the only possible uniformly curved four–dimensional metric spacetimes.\(^{(5)}\) There are two kinds of them,\(^{(6)}\) one with positive, and one with negative curvature. They can be defined as hypersurfaces in the pseudo–Euclidean spaces $\mathbb{E}^{4,1}$ and $\mathbb{E}^{3,2}$, inclusions whose points in Cartesian coordinates $(\xi^A) = (\xi^0, \xi^1, \xi^2, \xi^3, \xi^4)$ satisfy, respectively,

$$
\eta_{AB} \xi^A \xi^B = -(\xi^0)^2 + (\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 + (\xi^4)^2 = R^2 ; 
$$

(3)

$$
\eta_{AB} \xi^A \xi^B = -(\xi^0)^2 + (\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 - (\xi^4)^2 = -R^2 .
$$

(4)

We use $\eta_{ab} (a,b = 0, 1, 2, 3)$ for the Lorentz metric $\eta = \text{diag} (-1, 1, 1, 1)$, and the notation $\epsilon = \eta_{44}$ to put both the above conditions together as

$$
\epsilon \eta_{ab} \xi^a \xi^b + (\xi^4)^2 = R^2 .
$$

(5)

Defining the scaled coordinate $\xi^4 = \xi^4 / R$, one has

$$
\frac{\epsilon}{R^2} \eta_{ab} \xi^a \xi^b + (\xi^4)^2 = 1 ,
$$

(6)

where $\epsilon / R^2$ represents the Gaussian curvature.

The de Sitter space $dS(4,1)$, whose metric is derived from the pseudo–Euclidean metric $\eta_{AB} = (-1, +1, +1, +1, +1)$, has the pseudo–orthogonal group $SO(4,1)$ as group of motions. The other, which comes from $\eta_{AB} = (-1, +1, +1, +1, -1)$, is frequently called anti–de Sitter space and is denoted $dS(3,2)$ because its group of motions is $SO(3,2)$. The de Sitter spaces are both homogeneous spaces:

$$
dS(4,1) = SO(4,1)/SO(3,1) \quad \text{and} \quad dS(3,2) = SO(3,2)/SO(3,1) .
$$

(7)

The manifold of each de Sitter group is a bundle with the corresponding de Sitter space as base space and $\mathcal{L} = SO(3,1)$ as fiber.\(^{(7)}\)

In the Cartesian coordinates $\xi^A$, the generators of the infinitesimal de Sitter transformations are given by

$$
J_{AB} = \eta_{AC} \xi^C \frac{\partial}{\partial \xi^B} - \eta_{BC} \xi^C \frac{\partial}{\partial \xi^A} ,
$$

(8)

which satisfy the commutation relations

$$
[J_{AB}, J_{CD}] = \eta_{BC} J_{AD} + \eta_{AD} J_{BC} - \eta_{BD} J_{AC} - \eta_{AC} J_{BD} .
$$

(9)

The four–dimensional stereographic coordinates $x^\mu$ are defined by\(^{(8)}\)

$$
\xi^a = n(x) \delta^a_\mu x^\mu \equiv h^a_\mu x^\mu \quad \text{and} \quad \xi^4 = -\mathcal{R} n(x) \left( 1 - \epsilon \frac{\sigma^2}{4R^2} \right) ,
$$

(10)
where
\[ n(x) = \frac{1}{1 + \epsilon \sigma^2 / 4R^2} \] (11)
and
\[ \sigma^2 = \eta_{\mu\nu} x^\mu x^\nu, \] (12)
with \( \eta_{\mu\nu} = \delta^a_\mu \delta^b_\nu \eta_{ab} \). The \( h^a_\mu \) introduced in (10) are the components of a tetrad field, actually of the 1-form basis members \( \omega^a = h^a_\mu dx^\mu = n(x) \delta^a_\mu dx^\mu \).

In these coordinates, the line element
\[ ds^2 = \eta_{AB} d\xi^A d\xi^B \] (13)
is found to be
\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \]
with
\[ g_{\mu\nu} = h^a_\mu h^b_\nu \eta_{ab} \equiv n^2(x) \eta_{\mu\nu} \] (14)
the corresponding metric tensor. The de Sitter spaces, therefore, are conformally flat, with the conformal factor given by \( n^2(x) \). We could have written simply \( \xi^\mu = n(x) x^\mu \), but we are carefully using the Latin alphabet for the algebra (and flat space) indices, and the Greek alphabet for the homogeneous space fields and cofields. As usual with changes from flat tangent–space to spacetime, letters of the two kinds are interchanged with the help of the tetrad field. This is true for all tensor indices. Connections, which are vectors only in the last (1-form) index, will gain an extra “vacuum” term.

The Christoffel symbol corresponding to the metric \( g_{\mu\nu} \) is
\[ \Gamma^\lambda_{\mu\nu} = \left[ \delta^\lambda_\mu \delta^\sigma_\nu + \delta^\lambda_\nu \delta^\sigma_\mu - \eta_{\mu\nu} \eta^{\lambda\sigma} \right] \partial_\sigma [\ln n(x)]. \] (15)
The corresponding Riemann tensor components,
\[ R^{\nu\rho\sigma}_{\mu} = \partial_\mu \Gamma^{\nu}_{\rho\sigma} - \partial_\sigma \Gamma^{\nu}_{\rho\mu} + \Gamma^{\mu}_{\epsilon\sigma} \Gamma^{\epsilon}_{\rho\nu} - \Gamma^{\mu}_{\epsilon\nu} \Gamma^{\epsilon}_{\rho\sigma}, \] (16)
are found to be
\[ R^{\nu\rho\sigma}_{\mu} = \epsilon \frac{1}{R^2} \left[ \delta^\mu_\rho g_{\nu\sigma} - \delta^\mu_\sigma g_{\nu\rho} \right]. \] (17)
The Ricci tensor and the scalar curvature are, consequently,
\[ R_{\mu\nu} = \epsilon \frac{3}{R^2} g_{\mu\nu} \quad \text{and} \quad R = \epsilon \frac{12}{R^2}. \] (18)

In terms of the coordinates \( \{x^\mu\} \), the generators \( [8] \) of the infinitesimal de Sitter transformations are given by
\[ J_{ab} = \delta_a^\mu \delta_b^\nu ( \eta_{\mu\nu} x^P P_\nu - \eta_{\mu\nu} x^P P_\mu ) \] (19)
and
\[ J_{a4} = \epsilon \delta_a^\mu \left( R P_\mu + \frac{\epsilon}{4R} K_\mu \right), \] (20)
where
\[ P_\mu = \frac{\partial}{\partial x^\mu} \quad \text{and} \quad K_\mu = (2\eta_{\mu\lambda} x^\lambda x^\rho - \sigma^2 \delta^\mu_\rho) P_\rho \] (21)
are respectively the generators of translations and proper conformal transformations. For \( \epsilon = +1 \), we get the generators of the de Sitter group \( SO(4,1) \). For \( \epsilon = -1 \), we get the generators of the de Sitter group \( SO(3,2) \). The \( J_{ab} \)'s of (21) behave as translation generators on the corresponding homogeneous spaces, while the \( J_{a4} \)'s span the Lorentz subgroup \( SO(3,1) \).

Notice that the above expressions for the generators involve quantities appearing also in the metric and the curvature. Geometry and algebra are deeply mixed, as a result of the quotient character (7) of spacetime. De Sitter spacetimes are actually imbedded in the manifolds of the de Sitter groups. We shall now proceed to deform the algebras and, consequently, the groups. The imbedded spacetimes will follow these deformations, changing accordingly.
3. INFINITE COSMOLOGICAL–CONSTANT CONTRACTION

Let us consider now the limit \( R \to 0 \). First, we rewrite Eqs. (19) and (20) in the form

\[
J_{ab} \equiv \delta_a^\mu \delta_b^\nu L_{\mu \nu} \tag{22}
\]

\[
J_{a4} \equiv R^{-1} \delta_a^\mu \kappa_\mu \tag{23}
\]

where

\[
L_{\mu \nu} = (\eta_{\rho \mu} x^\rho P_\nu - \eta_{\rho \nu} x^\rho P_\mu) \tag{24}
\]

are the generators of the Lorentz group, and

\[
\kappa_\mu = \frac{1}{4} K_\mu + \epsilon R^2 P_\mu \tag{25}
\]

In terms of these generators, the commutation relations (9) become

\[
[L_{\mu \nu}, L_{\lambda \rho}] = \eta_{\nu \lambda} L_{\mu \rho} + \eta_{\mu \rho} L_{\nu \lambda} - \eta_{\nu \rho} L_{\mu \lambda} - \eta_{\mu \lambda} L_{\nu \rho} \tag{26}
\]

\[
[\kappa_\mu, L_{\lambda \rho}] = \eta_{\mu \lambda} \kappa_\rho - \eta_{\mu \rho} \kappa_\lambda \tag{27}
\]

\[
[\kappa_\mu, \kappa_\lambda] = -R^2 L_{\mu \lambda} \tag{28}
\]

Now, in the contraction limit \( R \to 0 \), one can see that

\[
\lim_{R \to 0} L_{\mu \nu} = L_{\mu \nu} ; \quad \lim_{R \to 0} \kappa_\mu = \frac{1}{4} K_\mu \tag{29}
\]

and consequently the de Sitter algebra contracts to

\[
[L_{\mu \nu}, L_{\lambda \rho}] = \eta_{\nu \lambda} L_{\mu \rho} + \eta_{\mu \rho} L_{\nu \lambda} - \eta_{\nu \rho} L_{\mu \lambda} - \eta_{\mu \lambda} L_{\nu \rho} \tag{30}
\]

\[
[K_\mu, L_{\lambda \rho}] = \eta_{\mu \lambda} K_\rho - \eta_{\mu \rho} K_\lambda \tag{31}
\]

\[
[K_\mu, K_\lambda] = 0 \tag{32}
\]

These commutation relations coincide with those of the Poincaré group Lie algebra. However, the Lie group corresponding to this algebra, denoted by \( Q \) and formed by a semi–direct product of Lorentz and proper conformal transformations, is completely different from the usual Poincaré group. It is the group ruling the local kinematics of high–\( \Lambda \) spaces and has been called the second or conformal Poincaré group\(^7\). Though the first terminology may have more appeal to a physicist, the second is more precise\(^8\).

From Eq. (6) one sees that in the limit \( R \to \infty \), \((\xi^4)^2 = 1\), the curvature vanishes, and one obtains the flat Minkowski space, a 4-dimensional hyperplane in the 5-dimensional linear ambient space. In terms of the variable \( \xi^4 \), Eq. (5) shows that the contraction limit \( R \to 0 \) leads both de Sitter spaces to a four–dimensional cone–space, denoted by \( N \), in which \( ds = 0 \). It presents a geometry gravitationally related to an infinite cosmological constant, and its kinematical group of motions is \( Q \). Analogously to the Minkowski case, \( N \) is also a homogeneous space, but under the kinematical group \( Q \), that is, \( N = Q/L \). The point–set of \( N \) is the point–set of the proper conformal transformations. The kinematical group \( Q \), like the Poincaré group, has the Lorentz group \( L \) as the subgroup accounting for the isotropy of the cone–space \( N \). However, the proper conformal transformations introduce a new kind of homogeneity. In fact, instead of ordinary translations, all points of \( N \) are equivalent through proper conformal transformations.

An important property of the cone–space \( N \) is that its metric tensor is singular everywhere,

\[
\lim_{R \to 0} g_{\mu \nu} = 0 ; \quad \lim_{R \to 0} g^{\mu \nu} \to \infty \tag{33}
\]

except on the points defined by \( \sigma^2 = 0 \), where

\[
g_{\mu \nu} = \eta_{\mu \nu} \tag{34}
\]
In other words, the metric turns out to be defined only on the three-dimensional light-cone subspace of the cone-spacetime $N$. It is singular at every other point. Nevertheless, the Levi-Civita connection is well defined everywhere, and consequently the Riemann curvature tensor is also well defined. An explicit computation shows that, whereas both the Riemann and the Ricci curvature tensors vanish for $R \to 0$, the scalar curvature goes to infinity:

$$\lim_{R \to 0} R \to \infty.$$  \hfill (35)

This is a characteristic property of a spacetime with an infinite cosmological constant. Finally, it is important to mention that the conformal Poincaré group $Q$, which is the group of motion of the cone-space $N$, preserves the light-cone structure.

4. INFINITE SPEED OF LIGHT CONTRACTION

As already mentioned, no distance can be defined on the space-like sections of the cone-space $N$. However, a notion of distance can be recovered in the non-relativistic limit. Let us then proceed to examine the limit in which the speed of light $c$ goes to infinity. It has been emphasized by Bacry & Lévy-Leblond that this limit corresponds to a situation in which not only velocities are small, but also that spacelike intervals are small as compared to timelike intervals. In order to perform this contraction, we need to introduce new coordinates so as to exhibit $c$ explicitly.

Denoting the old coordinates with a bar, we define new coordinates $x^\mu$ according to

$$\bar{x}^\mu = \frac{1}{c} x^\mu.$$  \hfill (36)

In terms of the new coordinates, the generators $L_{\mu\nu}$ and $K_\mu$ of the conformal Poincaré group, given respectively by Eqs. (24) and (21), become ($i = 1, 2, 3$)

$$L_{ij} = \eta_{ik} x^k P_j - \eta_{jk} x^k P_i$$  \hfill (37)

$$L_{i4} = -c B_i$$  \hfill (38)

$$K_i = c T_i$$  \hfill (39)

$$K_4 = T_t,$$  \hfill (40)

where we have used the notation

$$B_i = t P_i - \eta_{ik} \frac{x^k P_t}{c^2},$$  \hfill (41)

$$T_i = 2 \eta_{ik} \frac{x^k x^j}{c^2} P_j + 2 \eta_{ik} \frac{x^k t}{c^2} P_t - t^2 P_t + \frac{r^2}{c^2} P_i,$$  \hfill (42)

$$T_t = 2 t x^i P_i + t^2 P_t + \frac{r^2}{c^2} P_t$$  \hfill (43)

with $P_t = \partial/\partial t$ and $P_i = \partial/\partial x^i$. Notice furthermore that

$$\sigma^2 = -t^2 + \left(\frac{r^2}{c^2}\right),$$  \hfill (44)

with $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$. In terms of the generators $L_{ij}, B_i, T_i$ and $T_t$, the commutation relations (30), (31) and (32) of the conformal Poincaré group can be rewritten in the form:

$$[L_{ij}, L_{kl}] = \eta_{jk} L_{il} + \eta_{il} L_{jk} - \eta_{jl} L_{ik} - \eta_{ik} L_{jl}$$  \hfill (45)

$$[L_{ij}, B_k] = \eta_{jk} B_i - \eta_{ik} B_j$$  \hfill (46)

$$[B_i, B_k] = -\frac{1}{c^2} L_{ik}$$  \hfill (47)
\[ [T_i, B_k] = \frac{1}{c^2} \eta_{ik} T_i \] (48)

\[ [T_i, L_{kl}] = \eta_{ik} T_l - \eta_{il} T_k \] (49)

\[ [T_i, B_k] = T_k \] (50)

\[ [T_i, L_{kl}] = [T_i, T_k] = [T_i, T_t] = 0. \] (51)

Let us now consider the limit \( c \to \infty \). It is easy to see that, in this limit, the generators assume the form

\[ L_{ij} = \eta_{ik} x^k P_j - \eta_{jk} x^k P_i \] (52)

\[ B_i = t P_i \] (53)

\[ T_i = -t^2 P_i \] (54)

\[ T_t = 2te^i P_i + t^2 P_t. \] (55)

The corresponding commutation relations become

\[ [L_{ij}, L_{kl}] = \eta_{jk} L_{il} + \eta_{il} L_{jk} - \eta_{il} L_{ik} - \eta_{jk} L_{jl} \] (56)

\[ [L_{ij}, B_k] = \eta_{jk} B_i - \eta_{ik} B_j \] (57)

\[ [B_i, B_k] = [T_i, B_k] = 0 \] (58)

\[ [T_i, L_{kl}] = \eta_{ik} T_l - \eta_{il} T_k \] (59)

\[ [T_i, B_k] = T_k \] (60)

\[ [T_t, L_{kl}] = [T_i, T_k] = [T_i, T_t] = 0. \] (61)

This commutation table coincides with the Lie algebra of the Galilei group. The group, however, is quite distinct from Galilei. The rotation and boost generators, given respectively by \( L_{ij} \) and \( B_k \), are the same as those of the Galilei group. This means that the concept of isotropy of space and the equivalence of inertial frames coincide with that of the Galilei group. Nevertheless, the concepts of time and space homogeneity are completely different. Instead of ordinary time and space translations, homogeneity in space and time are defined respectively by the generators \( T_i \) and \( T_t \), given by the non-relativistic limit of the proper conformal generators. This new group does deserve, for this reason, the name conformal Galilei group.

5. FINAL REMARKS

As is well known, by the process of Inönü–Wigner group contraction with \( R \to \infty \), both de Sitter groups are reduced to the Poincaré group \( \mathcal{P} \), and both de Sitter spacetimes are reduced to the Minkowski space \( \mathcal{M} \). On the other hand, in a similar fashion but taking the limit \( \Lambda \to \infty \), both de Sitter groups are contracted to the conformal Poincaré group \( \mathcal{Q} \), formed by the semi-direct product of Lorentz and proper conformal transformations, and both de Sitter spaces are reduced to the cone–space \( \mathcal{N} \), a spacetime characterized by presenting vanishing Riemann and Ricci curvature tensors, but an infinite scalar curvature.

Minkowski space and the cone–spacetime can be considered as dual to each other in the sense that their geometries are determined, respectively, by a vanishing and an infinite cosmological constant. The same can be said of their
kinematical group of motions: $\mathcal{P}$ is associate to a vanishing cosmological constant, and $\mathcal{Q}$ to an infinite cosmological constant. The duality transformation connecting these two geometries is the spacetime inversion

$$x^\mu \rightarrow -x^\mu \sigma^2.$$  \hfill (62)

Under such a transformation, the Poincaré group $\mathcal{P}$ is transformed into the conformal Poincaré group $\mathcal{Q}$, and the Minkowski space $M$ becomes the four–dimensional cone–space $N$. The points at infinity of $M$ are concentrated in the vertex of the cone–space $N$, and those on the light–cone of $M$ becomes the infinity of $N$.

Now, as we have seen, the metric of $N$ is singular everywhere except on the three–dimensional light–cone, where it coincides with the Minkowski metric $\eta_{\mu\nu}$. Although conceivable from the mathematical point of view, a Universe described by such a spacetime would be quite peculiar. No distance could be defined, except on the light–cone where it would be always vanishing. However, in the non–relativistic limit $c \rightarrow \infty$, the notion of distance is recovered. On the other hand, as is widely known, under this limit, the Poincaré group is contracted to the Galilei group, and the Minkowski spacetime is divided into two disconnected pieces, a three–dimensional Euclidean space and time, which becomes a parameter. Under the same limit, the conformal Poincaré group $\mathcal{Q}$ is contracted to a group that includes the same three–dimensional rotation and boost transformations of the Galilei group, but time and space translations are replaced by the corresponding non-relativistic limit of the proper conformal transformation. Interesting enough, this group presents the same abstract Lie algebra as the Galilei group, and for this reason it has been named the conformal Galilei group. Analogously to what occurs to the Minkowski space, for $c \rightarrow \infty$ the cone–space $N$ is divided into two disconnected parts, a three–dimensional Euclidean space and time, which also in this limit becomes a parameter. Furthermore, as the three–dimensional Euclidean space comes from the light–cone, where the metric is well defined, the metric of the resulting Euclidean space will be well defined also. The group of motions of this space is the conformal Galilei group, whose generators of infinitesimal transformations are those given by Eqs. (13), (14).

It is important to mention finally that the order of the contractions ($\Lambda \rightarrow \infty$ then $c \rightarrow \infty$, or $c \rightarrow \infty$ then $\Lambda \rightarrow \infty$) is not important for the results obtained. The intermediary results, however, would change. In fact, the non-relativistic limit ($c \rightarrow \infty$) of the de Sitter groups and spacetimes leads respectively to the Newton–Hooke group and spacetime. A further infinite cosmological contraction ($\Lambda \rightarrow \infty$) would then lead to the conformal Galilei group.

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