THE WINNING PROPERTY
OF MIXED BADLY APPROXIMABLE NUMBERS

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Abstract. For any pair of real numbers $(i, j)$ with $0 < i, j < 1$ and $i + j = 1$, we prove that the set of $p$-adic mixed $(i, j)$-badly approximable numbers $\text{Bad}_p(i, j)$ is $1/2$-winning in the sense of Schmidt’s game. This improves a recent result of Badziahin, Levesley, and Velani on mixed Schmidt conjecture.

1. Introduction

Given a pair of real numbers $(i, j)$ such that

$$0 < i, j < 1 \quad \text{and} \quad i + j = 1.$$  \hfill (1.1)

Let $\text{Bad}(i, j)$ denote the set of $(i, j)$-badly approximable vectors in $\mathbb{R}^2$, that is,

$$\text{Bad}(i, j) := \left\{ (x, y) \in \mathbb{R}^2 : \exists c(x, y) > 0 \text{ such that } \max\{\|qx\|^{1/i}, \|qy\|^{1/j}\} > \frac{c(x, y)}{q}, \forall q \in \mathbb{N} \right\},$$  \hfill (1.2)

where $\|\cdot\|$ denotes the distance of a number to its nearest integer. The Schmidt conjecture says that for any two pairs of real numbers $(i_1, j_1)$ and $(i_2, j_2)$ satisfying (1.1), we have

$$\text{Bad}(i_1, j_1) \cap \text{Bad}(i_2, j_2) \neq \emptyset.$$

Recently, Badziahin, Pollington, and Velani proved Schmidt conjecture affirmatively in [3] by showing that the intersection of countably many $\text{Bad}(i_n, j_n)$ is of full Hausdorff dimension. In two recent papers, An improved BPV’s theorem by showing that every $\text{Bad}(i, j)$ is a winning set in the sense of Schmidt’s game, see [1, 2]. Recall that any countable intersection of $\alpha$-winning sets is also $\alpha$-winning, and an $\alpha$-winning set is of full Hausdorff dimension, see [7, 8]. Hence An indeed improved BPV’s theorem.

We now consider the case in dimension one, the set (1.2) is then reduced to the classical set of badly approximable numbers.

$$\text{Bad} := \left\{ x \in \mathbb{R} : \exists c(x) > 0 \text{ such that } \|qx\| > \frac{c(x)}{q}, \forall q \in \mathbb{N} \right\}.$$  \hfill (1.3)

Given $0 < \alpha, \beta < 1$, let $\gamma := 1 - 2\alpha + \alpha\beta$, we say the pair $(\alpha, \beta)$ is admissible if $\gamma > 0$. A classical result of Schmidt says that $\text{Bad}$ is $(\alpha, \beta)$-winning for all admissible $(\alpha, \beta)$. Observe that $(\frac{1}{2}, \beta)$ is always admissible for every $0 < \beta < 1$, so in particular, $\text{Bad}$ is $1/2$-winning.

In [5], the $D$-adic mixed Diophantine problems were studied. Let $D$ be a bounded sequence of positive integers $(d_k)_{k=1}^\infty$, where every $d_k \geq 2$. Let $D_0 := 1$, $D_n := \prod_{k=1}^n d_k$. For every natural number $q \in \mathbb{N}$, define the $D$-adic pseudo absolute value as follows,

$$|q|_D := \inf\left\{ \frac{1}{D_n} : q \in D_n\mathbb{Z} \right\}.$$
The $\mathcal{D}$-adic pseudo absolute value reduces to the usual $p$-adic norm if we let $\mathcal{D}$ be the constant sequence consisting of a prime number $p$. Recently, Badziahin, Levesley, and Velani initiated the study of mixed Schmidt conjecture in [4]. Let

$$\text{Bad}_\mathcal{D}(i, j) := \left\{ x \in \mathbb{R} : \exists c(x) > 0 \text{ such that } \max \left\{ \frac{\| q x \|^{i/j}}{q}, \forall q \in \mathbb{N} \right\} \geq \frac{c(x)}{q} \right\}.$$ (1.4)

We call this set as the set of mixed $(i, j)$-badly approximable numbers. The mixed Schmidt conjecture is then stated as follows: for any two pairs of real numbers $(i_1, j_1)$ and $(i_2, j_2)$ satisfying (1.1), we have

$$\text{Bad}_\mathcal{D}(i_1, j_1) \cap \text{Bad}_\mathcal{D}(i_2, j_2) \neq \emptyset.$$ In [4], they proved $\text{Bad}_\mathcal{D}(i, j)$ is $1/4$-winning, thus resolved the mixed Schmidt conjecture affirmatively.

Now we recall the notion of winning dimension, which is introduced in [7]. The definition of $(\alpha, \beta)$-winning and $\alpha$-winning will be reviewed in Section 2. Let $S \subset \mathbb{R}^n$, the winning dimension of $S$, denoted by $\text{windim} S$, is defined as follows

$$\text{windim} S = \sup \{ 0 < \alpha < 1 : S \text{ is } \alpha\text{-winning} \}.$$ A result in [7] says that if $S$ is a nontrivial subset, then $0 \leq \text{windim} S \leq \frac{1}{2}$. For example, $\text{windim} \text{Bad} = \frac{1}{2}$. BLV’s theorem says that $\text{windim} \text{Bad}_\mathcal{D}(i, j) \geq \frac{1}{4}$.

Moshchevitin asked whether $\text{windim} \text{Bad}_\mathcal{D}(i, j) = \frac{1}{2}$ in his recent survey on Diophantine problems, see [6]. This paper answers this question affirmatively. In fact, our theorem is a natural generalization of Schmidt’s classical result on the set $\text{Bad}$.

**Theorem 1.** The set $\text{Bad}_\mathcal{D}(i, j)$ is $(\alpha, \beta)$-winning for all admissible $(\alpha, \beta)$, in particular, $\text{Bad}_\mathcal{D}(i, j)$ is $1/2$-winning.

As a result, $\text{windim} \text{Bad}_\mathcal{D}(i, j) = \frac{1}{2}$. A result in [8] says a set is $(\alpha, \beta)$-winning for $\gamma \leq 0$ if and only if this set is the whole set, accordingly, the winning property of $\text{Bad}_\mathcal{D}(i, j)$ is the best possible, so we give the best improvement of BLV’s result in the sense of winning dimension.

Given a prime number $p$, we use $\text{Bad}_p(i, j)$ to denote the set defined by (1.4) where the $\mathcal{D}$-adic is replaced by $p$-adic. Theorem 1 gives the following corollary.

**Corollary 1.** For any two different prime numbers $p, q$, for any two pairs of real numbers $(i_1, j_1)$ and $(i_2, j_2)$ satisfying (1.1), the set $\text{Bad}_p(i_1, j_1) \cap \text{Bad}_q(i_2, j_2)$ is $1/2$-winning. In particular,

$$\text{Bad}_p(i_1, j_1) \cap \text{Bad}_q(i_2, j_2) \neq \emptyset.$$ Note that the “In particular” part could also be deduced from BLV’s result.

The rest of the paper is organized as follows: in Section 2, we introduce the notion of Schmidt’s game and establish some notations, then we give two useful lemmas; the proof of theorem 1 will be given in Section 3.

## 2. Schmidt’s Game and Two Lemmas

First we recall the notion of Schmidt’s game, for details see [7, 8]. In this paper we only consider Schmidt’s game on $\mathbb{R}$, so we restrict our description only in this situation. Given a set $S \subset \mathbb{R}$, given two real numbers $0 < \alpha, \beta < 1$, two players, say Alice and Bob, will play the game. The game is played as follows, Bob starts the game by choosing a closed interval $B_1 \subset \mathbb{R}$, then Alice chooses an closed interval $A_1$ such that $A_1 \subset B_1$ and $\rho(A_1) = \alpha \rho(B_1)$, then Bob chooses another closed interval $B_2$ such that $B_2 \subset A_1$.
and $\rho(B_2) = \beta \rho(A_1)$, then Alice chooses another closed interval $A_2$ such that $A_2 \subset B_2$ and $\rho(A_2) = \alpha \rho(B_2)$, and so on. Here $\rho(A) = \frac{1}{2}|A|$, where $|A|$ denotes the length of the interval $A$. We can see that intervals appearing in the game obey the following relation, $B_1 \supset A_1 \supset B_2 \supset A_2 \supset \ldots$. We say Alice wins the game if she can play such that the single point in $\cap_{k=1}^{\infty} A_k = \cap_{k=1}^{\infty} B_k$ lies in $S$, otherwise Bob wins. We say $S$ is $(\alpha, \beta)$-winning if Alice can always win the game no matter how Bob plays, and $S$ is $\alpha$-winning if it is $(\alpha, \beta)$-winning for every $0 < \beta < 1$.

Let $\rho_k := \rho(B_k)$, then $\rho_{k+t} = (\alpha \beta)^t \rho_k$. We now give the first lemma.

**Lemma 2.1.** Assume $(\alpha, \beta)$ is admissible. Let $t \in \mathbb{N}$ be such that $(\alpha \beta)^t < \frac{1}{2}\gamma$. Suppose an interval $B_k$ occurs in the $(\alpha, \beta)$ game, and suppose $y \in \mathbb{R}$ is an arbitrary fixed point, then Alice can play, no matter how Bob plays, such that for every point $x \in B_{k+t}$,

$$|x - y| > \frac{1}{2}\gamma \rho_k.$$ 

This lemma is essentially due to Schmidt, we just write it in a slightly different form in order to facilitate the proof of our theorem. See Schmidt’s book [8] p.49 for a complete proof. Here we give only the proof’s main idea.

**Proof.** Without loss of generality, we could assume $y$ be the middle point of $B_k$. Alice adopts the strategy that always selecting the most left possible inscribed interval in each turn. Then after $t$ turns, all points in $B_{k+t}$ will satisfy the property in the lemma. □

To give the next lemma we need some notations from [4], we put them here for completeness. For any real number $c > 0$, let

$$\text{Bad}_D(c; i, j) := \left\{ x \in \mathbb{R} : \max\{ |q|^{1/i}_D, \|q x\|^{1/j} \} > \frac{c}{q}, \forall q \in \mathbb{N} \right\},$$

then we see

$$\text{Bad}_D(i, j) = \bigcup_{c > 0} \text{Bad}_D(c; i, j).$$

Let $C_c := \left\{ \frac{r}{q} \in \mathbb{Q} : (r, q) = 1, q > 0, \text{ and } |q|_D \leq \left( \frac{c}{q} \right)^i \right\}$. Let $P = \frac{r}{q}$, and let

$$\Delta_c(P) := \left\{ x \in \mathbb{R} : |x - P| \leq \frac{cq}{q^{i+j}} \right\},$$

then clearly we have

$$\text{Bad}_D(c; i, j) = \mathbb{R} \setminus \bigcup_{P \in C_c} \Delta_c(P).$$

Let $R \in \mathbb{R}, R > 1$, let $t \in \mathbb{N}$, both of which will be fixed in Section 3. Define

$$C_{c,k} := \left\{ P = \frac{r}{q} \in C_c : R^{k-1} \leq \frac{q^{i+j}}{q^{i+j}} < R^k \right\},$$

then we have

$$C_c = \bigcup_{k=1}^{\infty} C_{c,k},$$

hence

$$\text{Bad}_D(c; i, j) = \mathbb{R} \setminus \bigcup_{k=1}^{\infty} \bigcup_{P \in C_{c,k}} \Delta_c(P). \tag{2.1}$$

It is this relation that will be used in the proof in Section 3.

Now we give the following lemma, the idea of which is already in [4].
Lemma 2.2. For any two different points $P_s = \frac{r_s}{q_s} \in C_{c,k}$, $s = 1, 2$, we have

$$|P_1 - P_2| > c^{-1} R^{2t(k-1) - \frac{1}{t+2} 2tk}.$$  

Proof. By $P = \frac{r}{q} \in C_{c,k} \subset C_c$, we have $|q| \leq (\frac{c}{q})^i$, by the definition of the norm $|q|_D$, there is an appropriate $n \in \mathbb{N}, q^* \in \mathbb{N}$ such that

$$q = D_nq^*, \text{ and } q \notin D_{n+1}\mathbb{Z},$$

hence,

$$D_n \geq \left(\frac{q}{c}\right)^i \geq c^{-1} R^{2t(k-1)}.$$

Now there will be $D_{n_1}$ and $D_{n_2}$ respectively for $P_1$ and $P_2$, and one of them will divide another by the definition of $D_n$, so $(q_1, q_2) \geq \min\{D_{n_1}, D_{n_2}\} \geq c^{-1} R^{2t(k-1)}$. Therefore,

$$|P_1 - P_2| \geq \frac{(q_1, q_2)}{q_1q_2} > c^{-1} R^{2t(k-1) - \frac{1}{t+2} 2tk}.$$ 

\[\square\]

3. Proof of theorem

Now we prove theorem 1. Given $(\alpha, \beta)$ be admissible, then $0 < \gamma < 1$. Fix one $t \in \mathbb{N}$ such that $(\alpha\beta)^t < \frac{1}{2}\gamma$, which is used in lemma 2.1. Our aim is to show that $\text{Bad}_D(i, j)$ is $(\alpha, \beta)$-winning. Without loss of generality we can assume that $\rho_1$ is very small, so we take the following constants,

$$\begin{align*}
R = \frac{1}{\alpha^2} > 1, \\
0 < \rho_1 < \left(\frac{1}{4} R^{-\frac{2t}{t+2}}\right)^j, \\
0 < c < (\frac{c}{\gamma}\rho_1)^{1/j} < \rho_1^{1/j}.
\end{align*}$$

(3.1)

As we pointed out in Section 2 that $\text{Bad}_D(i, j) = \bigcup_{c > 0} \text{Bad}_D(c; i, j)$, hence it suffices to show that for the $c$ satisfying (3.1), $\text{Bad}_D(c; i, j)$ is $(\alpha, \beta)$-winning.

Proof. We prove it by showing the following two facts.

Fact 1. For every $k \geq 1$,

$$\#\{P \in C_{c,k} : \Delta_c(P) \cap B_{t(k-1)+1} \neq \emptyset\} \leq 1.$$  

Fact 2. Suppose Fact 1 holds, then Alice can play, no matter how Bob plays, such that for every $k \geq 1$,

$$\#\{P \in C_{c,k} : \Delta_c(P) \cap B_{tk+1} \neq \emptyset\} = 0,$$

which is equivalent to

$$\Delta_c(P) \cap B_{tk+1} = \emptyset, \quad \forall P \in C_{c,k}.$$  

Notice that the above equation implies

$$\Delta_c(P) \cap B_{tk+1} = \emptyset, \quad \forall P \in C_{c,l}, \quad l = 1, 2, \ldots, k.$$ 

Recall the relation (2.1), then Fact 2 is equivalent to say that Alice can play such that the single point in $\cap_{k=1}^\infty B_{tk+1}$ lies in $\text{Bad}_D(c; i, j)$, so Alice can always win the game and we are done. Hence we are only left to show the two facts.

Now we show Fact 1. Let $z$ be the middle point of $B_{t(k-1)+1}$. For those $P \in C_{c,k}$ satisfying $\Delta_c(P) \cap B_{t(k-1)+1} \neq \emptyset$, let $x \in \Delta_c(P) \cap B_{t(k-1)+1}$, then

$$|P - x| \leq |P - z| + |x - z| \leq \frac{c^j}{q^{1+j}} + \rho_1 R^{-t(k-1)} < 2\rho_1 R^{-t(k-1)}.$$
Assume there are two points $P_1, P_2 \in \mathcal{C}_{c,k}$, $P_1 \neq P_2$, and they satisfy $\Delta_c(P_1) \cap B_{t(k-1)+1} \neq \emptyset$, $\Delta_c(P_2) \cap B_{t(k-1)+1} \neq \emptyset$. By applying lemma 2.2, then
\[
c^{-1}R^{1+\frac{1}{t}2tk} < |P_1 - P_2| \leq |P_1 - z| + |z - P_2| < 4\rho_1 R^{-t(k-1)},
\]
which is equivalent to
\[
4\rho_1 c^j > R^{(1+\frac{1}{t})t(k-1)-\frac{1}{t}2tk} = R^{-\frac{2j}{t+1}}.
\]
Now use our assumption for $c$ in (3.1), then
\[
R^{-\frac{2j}{t+1}} < 4\rho_1 c^j < 4\rho_1^{1+\frac{j}{t}} = 4\rho_1^{\frac{1}{2}},
\]
contradicts to our assumption on $\rho_1$.

Now we show Fact 2. We proceed by induction. The base is quite clear. Suppose that an interval $B_{t(k-1)+1}$ occurs in the game and it has empty intersection with all intervals $\Delta_c(P)$ with $P \in \mathcal{C}_{c,k}$. By Fact 1, there is not more than one “dangerous” point $P \in \mathcal{C}_{c,k}$ such that $\Delta_c(P) \cap B_{t(k-1)+1} \neq \emptyset$. View this point $P$ as the point $y$ in lemma 2.1, by applying that lemma, then Alice can play, no matter how Bob plays, such that for all $x \in \Delta_c(P) \cap B_{tk+1}$, we have
\[
\frac{c^j}{q^{1+j}} \geq |x - P| > \frac{1}{2}\gamma\rho_1 R^{-t(k-1)} = \frac{1}{2}\gamma\rho_1 R^{-t(k-1)},
\]
since $P \in \mathcal{C}_{c,k}$, this gives
\[
c^j R^{-t(k-1)} \geq \frac{c^j}{q^{1+j}} > \frac{1}{2}\gamma\rho_1 R^{-t(k-1)},
\]
which reduces to
\[
c^j > \frac{1}{2}\gamma\rho_1,
\]
contradicts to our assumption on $c$. So $\Delta_c(P) \cap B_{tk+1} = \emptyset$. □

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