Study of heat equations with boundary differential equations

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Abstract. Solutions of heat or diffusion equations with the boundary conditions which is a dynamic random field are discussed. This kind of method can be used to obtain the description of heat equations or diffusion equations based on observed physical reality, i.e. ordinary differential equations, representing heat or diffusion propagation, with a boundary condition that satisfies stochastic differential equations. The heat or diffusion equations obtained from the method are compared to the heat equation or the stochastic diffusion. The comparison is emphasized on the existence and properties of Green functions.

1. Introduction

Stochastic process become a new “tool” in explaining physical phenomena since the initiation of Robert Brown when observing pollen in water. Supported by the measure theory, functional analysis, and some other branch of mathematic, the initiation then evolved one of them towards a new concept known as stochastic solid and can explain many things in the field of physics. The development to become a theory in mathematics is even better since concepts such as Ito integrals [1] and Statonovich [2] integrals are found.

Although in mathematics the stochastic theory evolved very rapidly and barely encountered obstacles, slightly different in the study of physics. In physics, one demands conformity between the physical phenomena and the mathematical device used to explain it. The problem arises when the physical phenomenon is seen as a random motion itself. There are two views when considering the random motion of a physical system. First ontological random motion and epistemological random motion. The ontological random motion means randomness of motion due to the system itself, whereas the epistemological randomness of motion does not originate from the system itself but from the ‘reluctance’ of humans to trace one by one the of each part of the mechanical system under consideration. In other words, if each part of the system is followed it will find a deterministic motion of the system. Ontological random motion is commonly found in the realm of quantum physics [3] while epistemological ones are common in the realm of classical physics.

In the realm of classical physics, it can be said that all physical events are deterministic. However, in some cases, for example, every particle of a material that moves at any moment, too much effort is required to observe it one by one. Therefore, the stochastic concept is used to explain the above mentioned classical phenomena such as the motion of a school of fish [4], sea and ocean surface temperature [5], ground motion phenomenon during the earthquake [6], and random cell motion [7].

On the issue of heat dissipation, the heat equation has been discussed extensively using stochastic theory which began in the 1980s. The obtained equation is known as the stochastic heat equation [8]. Other in-depth assessments such as stochastic heat equations with random coefficient [9], optimal control for stochastic heat equations accompanied by noise and control limits [10], and nonlinear stochastic heat equations has been reviewed [11]. However, of all these studies, all are based on an
extended concept of the Langevin equation [12]. The Langevin equation is a differential equation which extends the concept of the Newton law’s motion. A part which representing the force in the Newton equation is then developed into a part that causes randomness in the system. Furthermore, from the result of this expansion will be obtained stochastic differential equation. This equation is then widely used for a variety of cases as well as on the heat transfer.

However, if a system is observed to a certain extent, it can be seen that at the boundary of the system there is a stochastic process, and the problem of heat transfer. Heat transfer in a system satisfies the deterministic heat equation. Now suppose that the system has a boundary which has random behaviour, so it can be used as a boundary condition for the deterministic heat equation. The preamble of such a system will be discussed in this article.

In this article we will discuss heat transfer in the 1-D case, e.g. a thin bar. The heat dissipation on the rod is stated with a heat equation. As long as the rod is assumed there is an isolator so that there is no exchange of energy entering or leaving from the side along the rod. Meanwhile, at the ends of the rod is the limit of the system in which both experience random events. Such random events are expressed by equations that satisfy stochastic differential equations. The equations at the ends of the rod are considered equal and meet the Brownian motion. These two equations are then used as the boundary conditions for the heat equation on the rod. It is also assumed that in this case, an initial condition that satisfies the Dirac equation is determined.

2. Heat Equation with Stochastic Boundary Problems

One method to solve the heat conduction problem is to construct the simplest function, say \( w(x, t) \). This function is required to meet non-homogeneous and time-dependent boundary conditions. The \( u(x, t) \) sought solution is then merged into a \( w(x, t) \) and other functions \( v(x, t) \), which satisfies the homogeneous boundary. When the two functions are substituted to the heat equation, it is found that \( v(x, t) \) must recognize the heat equation which can be time-dependent. This method will be used in the following below.

Suppose a 1D heat equation with the boundary conditions and initial conditions is stated as follows

\[
\begin{align*}
    u_t &= \alpha^2 u_{xx} \quad \text{for } x \in (0, L) \quad t > 0, \\
    u(0, t) &= \eta_0(t), \\
    u(L, t) &= \eta_1(t), \\
    u(x, 0) &= f(x) = \delta(x_1 - x_0)
\end{align*}
\]

(1)

The boundary conditions are given in the form of general linear stochastic differential equation

\[
    dX_t = d\eta(t) = (a(t)X_t + c(t))dt + (b(t)X_t + d(t))dW_t
\]

(2)

with the solution:

\[
    \eta(t) = \Phi_{t,t_0} X_{t_0} + \int_{t_0}^t \Phi_{t,s}^{-1} c(s) - b(s)d(s)ds + \int_{t_0}^t \Phi_{t,s}^{-1} d(s)dW_s
\]

In order to obtain a solution to the equation (1) with such an initial and boundary condition, a new function is required that meet the original terms and condition. Suppose the new function is

\[
    w(x, t) = \eta_0(t) + \frac{\eta_1(t) - \eta_0(t)}{L}
\]

(3)

Based on this equation, when \( x = 0 \), obtained \( w(0, t) = \eta_0(t) \) and when \( x = L \) obtained \( w(L, t) = \eta_1(t) \). Then, suppose \( v(x, t) \) is a function that qualifies the homogeneous boundary. With this function, the solution to the heat flow equation can be expressed as \( u(x, t) = w(x, t) + v(x, t) \), which can be substituated into (1).

Next enter the boundary \( x = 0, x = L \) and initial condition \( t = 0 \), for the equation. The result of these and substituting on \( u(x, t) = w(x, t) + v(x, t) \), is obtained
\[ v_t = \frac{\partial}{\partial t}(v(x, t)) = \frac{\partial}{\partial t} \left( \sum_{n=1}^{\infty} \hat{v}_n \sin(\lambda_n x) \right) = \sum_{n=1}^{\infty} \dot{\hat{v}}_n \sin(\lambda_n x). \]  

(4)

Meanwhile for \( v_{xx} \) is obtained

\[ v_{xx} = \frac{\partial^2}{\partial x^2}(v(x, t)) = \sum_{n=1}^{\infty} \hat{v}_n [-\lambda_n^2] \sin(\lambda_n x). \]  

(5)

And for \( w_t \) itself it can be considered as

\[ S(x, t) = \sum_{n=1}^{\infty} \xi_n(t) \sin(\lambda_n x). \]  

(6)

Therefore, \( v_t = \lambda^2 v_{xx} - w_t \) can be expressed as follows:

\[ 0 = \sum_{n=1}^{\infty} \left[ \dot{\hat{v}}_n(t) + \alpha^2 \hat{v}_n(t) \lambda_n^2 - \xi_n(t) \right] \sin(\lambda_n x) \]  

(7)

Since the linear function is linearly free, the portion in the [] range is equal to zero. By knowing the integration factor for the first-order differential linear equation, we found

\[ \sum_{n=1}^{\infty} c_n \sin(\lambda_n x) = f(x) - \left( \eta_1(0) - \eta_0(0) \right) \left( \frac{x}{L} \right) + \eta_0(0) \]  

(8)

which is similar to the Fourier sine series. We can find \( c_n \) as below

\[ c_n = \frac{x}{L} \int_0^L \left( f(x) - \left( \eta_1(0) - \eta_0(0) \right) \left( \frac{x}{L} \right) + \eta_0(0) \right) \sin \left( \frac{n\pi x}{L} \right) dx \]  

(9)

and the solution of heat equation with the stochastic boundary condition as follows:

\[ u(x, t) = \eta_0(t) \left( 1 - \frac{x}{L} \right) + \frac{x}{L} \eta_1(t) + \sum_{n=1}^{\infty} e^{\alpha^2 \lambda_n^2 t} \xi_n(t) \int_0^t e^{-\alpha^2 \lambda_n^2 \tau} c_n d\tau + e^{-\alpha^2 \lambda_n^2 t} c_n \sin(\lambda_n x). \]  

(10)

The solution of the 1D heat equation with the stochastic boundary condition is given by equation (10). In the heat equations there is a term representing the existence of a Green function or a fundamental solution to the heat equation. The first equation fulfills the Green function, while the second equation fulfills the Green function as well as the fundamental solution. This is because the Green function solves only the problem with the subset domain, such as \( \Omega \in \mathbb{R}^d \) whereas the fundamental solution for the whole \( \mathbb{R}^d \).

There are two parts in (2), the first and second part on the right. These two parts are derived from a new function that is structured to simplify the search for a solution of heat equation. Therefore, these two terms can be regarded as coupling between deterministic and differential equations boundary conditions.

### 3. Conclusion

Based on these results, it can be concluded that the differential equation (in this case the 1D as the example equation) provided that the stochastic limit meets the criterion for use as well as the differential stochastic equation used for the development of the Langevin equation. However, in fact it is still very hasty to say that the method used in this study in observing the random phenomenon can be used for each case as well as stochastic differential equations that have been established so far.
There are still many things, especially on the mathematical aspects especially involving further mathematical studies such as functional analysis and differential geometry that need to be studied more deeply. Meanwhile, it is also important to review other popular cases such as for the case of Markov, Bernoulli, Poisson, and several others. From computational reviews it is also necessary to be considered to lead to other applied fields. With the results of these studies later, it will be possible to know the method or the kind of way discussed in this study can be used further or not.

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