THE BIHAMILTIONAN STRUCTURES OF THE DR/DZ HIERARCHIES AT THE APPROXIMATION UP TO GENUS ONE

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Abstract. In a recent paper, giving an arbitrary homogeneous cohomological field theory (CohFT), Rossi, Shadrin, and the first author proposed a simple formula for a bracket on the space local functionals that conjecturally gives a second Hamiltonian structure for the double ramification hierarchy associated to the CohFT. In this paper we prove this conjecture at the approximation up to genus 1 and relate this bracket to the second Poisson bracket of the Dubrovin–Zhang hierarchy by an explicit Miura transformation.

1. Introduction

The appearance of integrable systems of PDEs in the intersection theory of the moduli spaces \( \overline{M}_{g,n} \) of stable algebraic curves of genus \( g \) with \( n \) marked points was first manifested by the Kontsevich–Witten theorem [Wit91, Kon92], which states that the generating series of integrals over \( \overline{M}_{g,n} \) of monomials in psi classes (the first Chern classes of tautological line bundles) is controlled by a special solution of the Korteweg–de Vries hierarchy. Various versions of Witten’s conjecture were proposed (the two most famous are in the Gromov–Witten theory of \( \mathbb{CP}^1 \) [DZ04, OP06] and in the \( r \)-spin theory [Wit93, FSZ10]), when it was realized that integrable systems appear in a very general context, where the central role is played by the notion of a cohomological field theory (CohFT). CohFTs are systems of cohomology classes on the moduli spaces \( \overline{M}_{g,n} \) that are compatible with natural morphisms between the moduli spaces. They were introduced by Kontsevich and Manin in [KM94] to axiomatize the properties of Gromov–Witten classes of a given target variety. A correlator of a CohFT is the integral over \( \overline{M}_{g,n} \) of a monomial in the psi classes multiplied by a cohomology class forming the CohFT.

In [DZ98] Dubrovin and Zhang constructed a Hamiltonian hierarchy controlling the correlators of an arbitrary CohFT at the approximation up to genus 1 and proved the polynomiality of the Hamiltonians and of the Poisson bracket. Moreover, in the case of a homogeneous CohFT they endowed the hierarchy with a polynomial bihamiltonian structure (also at the approximation up to genus 1). In the subsequent paper [DZ01] Dubrovin and Zhang presented a construction of a bihamiltonian hierarchy, called now the Dubrovin–Zhang (DZ) hierarchy or the hierarchy of topological type, controlling the correlators (in all genera) of an arbitrary semisimple homogeneous CohFT. However, the polynomiality of the Hamiltonians and of the two Poisson brackets was left as an open problem.

In [BPS12] the authors extended the construction of the DZ hierarchy to an arbitrary, not necessarily semisimple or homogeneous, CohFT and proved the polynomiality of the Hamiltonians and of the Poisson structure in the semisimple case (a simpler proof was obtained in [BPS12a]). In the case of a homogeneous CohFT the hierarchy is endowed with a second Hamiltonian structure whose polynomiality remains an important unproven feature of the DZ hierarchy.

In [Bur15] a new construction of a Hamiltonian hierarchy associated to an arbitrary, not necessarily semisimple, CohFT was introduced. This construction is also based on the intersection theory on \( \overline{M}_{g,n} \), but it employs different tautological classes, notably the double ramification cycle (an appropriate compactification of the locus of smooth curves whose marked points support a principal divisor), which explains why this hierarchy was called the double ramification
(DR) hierarchy. By the construction, the Hamiltonians of the DR hierarchy are polynomial, and the Poisson bracket is very simple, moreover, as opposed to the one for the DZ hierarchy, it does not essentially depend on the underlying CohFT. The two hierarchies coincide in the dispersionless (genus 0) limit and, by a conjecture from [Bur15] called the DR/DZ equivalence conjecture, they are related by a Miura transformation, which was completely identified in [BDGR18]. Although still unproved, the DR/DZ equivalence conjecture has accumulated a remarkable amount of evidence and verifications (see, e.g., [BR16, BG16, BDGR18, BDGR20, BGR19, DR19]). In particular, the DR/DZ equivalence conjecture is proved at the approximation up to genus 1 [BDGR18].

Remark 1.1. Formally speaking, the semisimplicity assumption is present in the statement of Theorem 8.4 in [BDGR18] claiming that the DR/DZ equivalence conjecture is true at the approximation up genus 1. However, this assumption is never used in the proof. So the DR/DZ equivalence conjecture is true for an arbitrary CohFT at the approximation up genus 1.

In [BRS21], giving an arbitrary homogeneous CohFT, the authors proposed a simple formula for a bracket on the space of local functionals and conjectured that it is Poisson and gives a second Hamiltonian structure for the DR hierarchy. These (conjecturally) Poisson brackets depend on the homogeneous CohFT under consideration in a remarkably explicit way. In this paper we prove the conjecture from [BRS21] at the approximation up to genus 1. Moreover, we study a relation with the second Poisson bracket of the DZ hierarchy. The fact that the Hamiltonians and the first Poisson brackets of the DR and the DZ hierarchies are related by the Miura transformation described in [BDGR18] was proved in [BDGR18] at the approximation up to genus 1. Here we check that this Miura transformation also relates the second Poisson brackets (at the approximation up to genus 1).

Notations and conventions.
• Throughout the text we use the Einstein summation convention for repeated upper and lower Greek indices.
• When it doesn’t lead to a confusion, we use the symbol * to indicate any value, in the appropriate range, of a sub- or superscript.
• For a topological space $X$ let $H^*(X)$ denote the cohomology ring of $X$ with coefficients in $\mathbb{C}$.
• For an integer $n \geq 1$ let $[n] := \{1, 2, \ldots, n\}$.

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2. Cohomological field theories

Let $\overline{M}_{g,n}$ be the Deligne–Mumford moduli space of stable algebraic curves of genus $g$ with $n$ marked points, $g \geq 0$, $n \geq 0$, $2g - 2 + n > 0$. Note that $\overline{M}_{0,3} = \text{pt}$, and we will use the identification $H^*(\overline{M}_{0,3}) = \mathbb{C}$. Recall the following system of standard maps between these spaces:

• $\pi: \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ is the map that forgets the last marked point.

• $\text{gl}_{I_1, I_2}: \overline{M}_{g_1,n_1+1} \times \overline{M}_{g_2,n_2+1} \to \overline{M}_{g_1+g_2,n_1+n_2}$, is the gluing map that identifies the last marked points of curves of genus $g_1$ and $g_2$ and turns them into a node. The sets $I_1$ and $I_2$ of cardinality $n_1$ and $n_2$, $I_1 \sqcup I_2 = [n_1 + n_2]$, keep track of the relabelling of the remaining marked points.

• $\text{gl}_{n+2}: \overline{M}_{g,n+2} \to \overline{M}_{g+1,n}$ is the gluing map that identifies the last two marked points and turns them into a node.
Let $V$ be a finite dimensional vector space of dimension $N$ with a distinguished vector $e \in V$, called the unit, and a symmetric nondegenerate bilinear form $(\cdot, \cdot)$ on $V$, called the metric. We fix a basis $e_1, \ldots, e_N$ in $V$ and let $\eta = (\eta_{\alpha\beta})$ denote the matrix of the metric in this basis, $\eta_{\alpha\beta} := (e_\alpha, e_\beta)$, and let $A_\alpha$ denote the coordinates of $e$ in this basis, $e = A^\alpha e_\alpha$. As usual, $\eta^{\alpha\beta}$ denotes the entries of the inverse matrix, $(\eta^{\alpha\beta}) := (\eta_{\alpha\beta})^{-1}$.

**Definition 2.1 (KM94).** A cohomological field theory (CohFT) is a system of linear maps $c_{g,n}: V^{\otimes n} \to H^{even}(\overline{M}_{g,n}), \quad 2g - 2 + n > 0$, such that the following axioms are satisfied:

1. The maps $c_{g,n}$ are equivariant with respect to the $S_n$-action permuting the $n$ copies of $V$ in $V^{\otimes n}$ and the $n$ marked points on curves from $\overline{M}_{g,n}$, respectively;
2. $\pi^*c_{g,n}(\otimes_{i=1}^n e_{\alpha_i}) = c_{g,n+1}(\otimes_{i=1}^n e_{\alpha_i} \otimes e)$ and $c_{0,3}(e_{\alpha_1} \otimes e_{\alpha_2} \otimes e) = \eta_{\alpha_1\alpha_2}$;
3. $g_{g_1}^{\alpha_1,\beta_1;\gamma_1} c_{g_1+1}(\otimes_{i=1}^{n_1} e_{\alpha_i}) + g_{g_2}^{\alpha_2,\beta_2;\gamma_2} c_{g_2+1}(\otimes_{i=1}^{n_2} e_{\alpha_i}) = 2 c_{g_1+g_2+1}(\otimes_{i=1}^{n_1+n_2} e_{\alpha_i})$;
4. $(g_{g,n+2}^{irr})^* c_{g+1,n}(\otimes_{i=1}^n e_{\alpha_i}) = c_{g,n+2}(\otimes_{i=1}^n e_{\alpha_i} \otimes e_{\mu} \otimes e_{\nu}) \eta^{\mu\nu}$.

Let us assume now that $V$ is a graded vector space and the basis $e_1, \ldots, e_N$ is homogeneous with deg $e_\alpha = q_\alpha, \alpha = 1, \ldots, N$. Assume also that deg $e = 0$. By $\text{Deg}: H^*(\overline{M}_{g,n}) \to H^*(\overline{M}_{g,n})$ we denote the operator that acts on $H^*(\overline{M}_{g,n})$ by multiplication by $\frac{1}{2}$.

**Definition 2.2.** A CohFT $\{c_{g,n}\}$ is called homogeneous, or conformal, if there exist complex constants $r^\alpha, \alpha = 1, \ldots, N$, and $\delta$ such that

$$(2.1) \quad \text{Deg} c_{g,n}(\otimes_{i=1}^n e_{\alpha_i}) + \pi^* c_{g,n+1}(\otimes_{i=1}^n e_{\alpha_i} \otimes r^\alpha e_\gamma) = \left( \sum_{i=1}^n q_{\alpha_i} + \delta (g - 1) \right) c_{g,n}(\otimes_{i=1}^n e_{\alpha_i}).$$

The constant $\delta$ is called the conformal dimension of CohFT.

For an arbitrary homogeneous CohFT let us introduce the formal power series

$$F = F(t^1, \ldots, t^N) := \sum_{n \geq 3} \frac{1}{n!} \sum_{1 \leq \alpha_1, \ldots, \alpha_n \leq N} \left( \int_{\overline{M}_{g,n}} c_{0,n}(\otimes_{i=1}^n e_{\alpha_i}) \right) \prod_{i=1}^n t^{c_{\alpha_i}}$$

and define $C_{\alpha \beta \gamma}^{\alpha \gamma} := \eta^{\alpha\nu} \frac{\partial^2 F}{\partial t^\alpha \partial t^\beta \partial t^\gamma}$.

The structure constants $C_{\alpha \beta \gamma}^{\alpha \gamma}$ define a formal family of commutative associative algebras with the unit $\frac{\partial}{\partial t^\alpha} := A^\alpha \frac{\partial}{\partial t^\alpha}$, and moreover

$$((1 - q_\alpha) t^\alpha + r^\alpha) \frac{\partial F}{\partial t^\alpha} = (3 - \delta) F + \frac{1}{2} A_{\alpha\beta} t^\alpha t^\beta, \quad \text{where} \quad A_{\alpha\beta} := r^\nu c_{0,3}(e_\alpha \otimes e_\beta \otimes e_\mu),$$

which means that the formal power series $F$ defines the structure of a homogeneous Dubrovin–Frobenius manifold on a formal neighbourhood of 0 in $V$ with the Euler vector field given by $E = E^\alpha \frac{\partial}{\partial t^\alpha} := ((1 - q_\alpha) t^\alpha + r^\alpha) \frac{\partial}{\partial t^\alpha}$. In particular, we have the following properties:

$$C_{\beta \gamma}^{\alpha \gamma} C_{\delta \theta}^{\beta \gamma} = C_{\delta \theta}^{\alpha \gamma} C_{\beta \gamma}^{\alpha \gamma}; \quad (\mu_\alpha + \mu_\beta) \eta_{\alpha\beta} = 0.$$

**Conjecture 2.3.** We will systematically raise and lower indices in tensors using the metric $\eta$.

For example, $C_{\beta \gamma}^{\alpha \gamma} = \eta^{\alpha\nu} C_{\nu \gamma}^{\beta \gamma}$.

3. The Dubrovnik–Zhang and the double ramification hierarchies

3.1. Differential polynomials, Poisson operators, and Hamiltonian hierarchies. Let $u^1, \ldots, u^N$ be formal variables. Let us very briefly recall the main notions and notations in the formal theory of evolutionary PDEs with one spatial variable (and refer a reader, for example, to BRS21 for details):

- To the formal variables $u^\alpha$ we attach formal variables $u^\alpha_0$ with $d \geq 0$ and introduce the ring of differential polynomials $A_u := \mathbb{C}[[u^*]][[u^*]]_{>1}$ (in BRS21 it is denoted by $A^0_u$). We identify $u^\alpha_0 = u^\alpha$ and also denote $u^\alpha_+ := u^\alpha_1, u^\alpha_{xx} := u^\alpha_2, \ldots$. 

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The space \( \Lambda_u := \mathcal{A}_u / (\mathbb{C} \oplus \text{Im} \partial_x) \) is called the space of local functionals (in \([BRS21]\) it is denoted by \( \Lambda^0_u \)).

\( \mathcal{A}_{u,d} \subset \mathcal{A}_u \) and \( \Lambda_{u,d} \subset \Lambda_u \) are the homogeneous components of (differential) degree \( d \), where \( \deg u^\alpha_i := i \).

The extended spaces of differential polynomials and local functionals are defined by \( \widehat{\mathcal{A}}_u := \mathcal{A}_u[[\varepsilon]] \) and \( \widehat{\Lambda}_u := \Lambda_u[[\varepsilon]] \). Let \( \widehat{\mathcal{A}}_{u,k} \subset \widehat{\mathcal{A}}_u \) and \( \widehat{\Lambda}_{u,k} \subset \widehat{\Lambda}_u \) be the subspaces of degree \( k \), where \( \deg \varepsilon := -1 \).

We associate with \( f \in \widehat{\mathcal{A}}_u \) the sequence of differential operators \( L^k_\alpha(f) := \sum_{i \geq k} \left( \frac{\partial f}{\partial u^\beta_j} \right) \partial^{-k}_x \), \( \alpha = 1, \ldots, N, k \geq 0 \). We denote \( L_\alpha(f) := L^0_\alpha(f) \).

Given an \( N \times N \) matrix \( K = (K^{\mu \nu}) \) of differential operators of the form \( K^{\mu \nu} = \sum_{j \geq 0} K^{\mu \nu}_j \partial_x^j \), where \( K^{\mu \nu}_j \in \mathcal{A}_{u,-j+1} \), a bracket of degree 1 on the space \( \widehat{\mathcal{A}}_u \) is defined by \( \{ f, g \}_K := \int \left( \frac{\delta f}{\delta u^\mu} K^{\mu \nu} \frac{\delta g}{\delta u^\nu} \right) dx \).

An operator \( K \) is called Poisson, if the bracket \( \{ \cdot, \cdot \}_K \) is skewsymmetric and satisfies the Jacobi identity. The space of Poisson operators will be denoted by \( \mathcal{PO}_u \).

Two Poisson operators \( K_1 \) and \( K_2 \) are said to be compatible if the linear combination \( K_2 - \lambda K_1 \) is a Poisson operator for any \( \lambda \in \mathbb{C} \).

A Miura transformation is a change of variables \( u^\alpha \mapsto \bar{u}^\alpha(u^*_\alpha, \varepsilon) \) of the form \( \bar{u}^\alpha(u^*_\alpha, \varepsilon) = u^\alpha + \varepsilon f^\alpha(u^*_\alpha, \varepsilon) \), where \( f^\alpha \in \widehat{\mathcal{A}}_{u,1} \). A Poisson operator \( K \) rewritten in the new variables \( \bar{u}^\alpha \) will be denoted by \( K_{\bar{u}} \).

For a scalar operator \( A = \sum_m A_m \partial_x^m \), \( A_m \in \mathcal{A}_u \) (the sum is finite), let \( A^\dagger := \sum_m (-\partial_x)^m \circ A_m \).

For a matrix operator \( K = (K^{\alpha \beta}) \), \( K^{\alpha \beta} = \sum_m K^{\alpha \beta}_m \partial_x^m \), \( K^{\alpha \beta}_m \in \mathcal{A}_u \) (the sum is finite), let \( K^\dagger = (K^{\dagger \alpha \beta}) \), where \( K^{\dagger \alpha \beta} := \sum_m (-\partial_x)^m \circ K^{\beta \alpha}_m \).

**Definition 3.1.** A Hamiltonian hierarchy of PDEs is a system of the form

\[
\frac{\partial u^\alpha}{\partial \tau_i} = K^{\alpha \mu} \frac{\partial \bar{h}_i}{\partial u^\mu}, \quad 1 \leq \alpha \leq N, \quad i \geq 1,
\]

where \( \bar{h}_i \in \widehat{\mathcal{A}}_{u,0} \), \( K = (K^{\mu \nu}) \) is a Poisson operator, and \( \{ \bar{h}_i, \bar{h}_j \}_K = 0, i, j \geq 1 \). The local functionals \( \bar{h}_i \) are called the Hamiltonians.

**Definition 3.2.** A Hamiltonian hierarchy of the form

\[
\frac{\partial u^\alpha}{\partial \tau_q} = K^{\alpha \beta}_1 \frac{\partial \bar{h}_{\beta,q}}{\partial u^\mu}, \quad 1 \leq \alpha, \beta \leq N, \quad q \geq 0,
\]
equipped additionally with \( N \) linearly independent Casimirs \( \bar{h}_{\alpha,-1} \), \( 1 \leq \alpha \leq N \), of the Poisson bracket \( \{ \cdot , \cdot \}_K^1 \), is said to be bihamiltonian if it is endowed with a Poisson operator \( K_2 \) compatible with the operator \( K_1 \) and such that

\[
\{ \cdot , \bar{h}_{\alpha,i-1} \}_K^2 = \sum_{j=0}^{i} R^{\alpha \beta}_{i,j} \{ \cdot , \bar{h}^{\beta,j}_{\beta,i-j} \}_K^1, \quad 1 \leq \alpha \leq N, \quad i \geq 0,
\]

where \( R^j_i = (R^{j \beta}_{i,\alpha})^j, 0 \leq j \leq i \), are constant \( N \times N \) matrices. The relation \( \{ \cdot , \cdot \}_K^2 \) is called a bihamiltonian recursion.
3.2. The double ramification hierarchy. Denote by $\psi_i \in H^2(\overline{M}_{g,n})$ the first Chern class of the line bundle over $\overline{M}_{g,n}$ formed by the cotangent lines at the $i$-th marked point of stable curves. Denote by $E$ the rank $g$ Hodge vector bundle over $\overline{M}_{g,n}$ whose fibers are the spaces of holomorphic one-forms on stable curves. Let $\lambda_j := c_j(E) \in H^{2j}(\overline{M}_{g,n})$.

For any $a_1, \ldots, a_n \in \mathbb{Z}$, $\sum_{i=1}^n a_i = 0$, denote by $\text{DR}_g(a_1, \ldots, a_n) \in H^{2g}(\overline{M}_{g,n})$ the double ramification (DR) cycle. We refer the reader, for example, to [BSSZ15] for the definition of the DR cycle on $\overline{M}_{g,n}$, which is based on the notion of a stable map to $\mathbb{CP}^1$ relative to 0 and $\infty$. If not all the multiplicities $a_i$ are equal to zero, then one can think of the class $\text{DR}_g(a_1, \ldots, a_n)$ as the Poincaré dual to a compactification in $\overline{M}_{g,n}$ of the locus of pointed smooth curves $(C; p_1, \ldots, p_n)$ satisfying $O_C(\sum_{i=1}^n a_ip_i) \cong O_C$.

The crucial property of the DR cycle is that for any cohomology class $\theta \in H^*(\overline{M}_{g,n})$ the integral $\int_{\overline{M}_{g,n+1}} \lambda_g \text{DR}_g (\sum a_i, \ldots, a_n) \theta$ is a homogeneous polynomial in $a_1, \ldots, a_n$ of degree $2g$ (see, e.g., [Bur15]). Therefore, for a given CohFT $\{c_{g,n} : V^\otimes n \to H^{\text{even}}(\overline{M}_{g,n})\}$, define differential polynomials $g_{a,d} \in \hat{A}_{u,0}$, $1 \leq \alpha \leq N$, $d \geq 0$, as follows:

$$g_{a,d} := \sum_{g,n \geq 0 > n > 0} \frac{\varepsilon^{2g}}{n!} \sum_{b_1, \ldots, b_n \geq 0} b_1 \cdots b_n u^{a_1}_{b_1} \cdots u^{a_n}_{b_n} \times$$

$$\times \text{Coef}_{a_1 \cdots a_n} \int_{\overline{M}_{g,n+1}} \text{DR}_g \left( - \sum a_1, \ldots, a_n \right) \lambda_g \psi^d_1 c_{g,n+1}(e_{\alpha} \otimes \otimes_{i=1}^n e_{a_i}).$$

It is proved in [Bur15] that the local functionals $\overline{g}_{a,d} := \int g_{a,d} dx$ mutually commute with respect to the bracket $\{\cdot, \cdot\}_{\eta^{-1}\partial_x}$.

**Definition 3.3.** The Hamiltonian hierarchy

$$\frac{\partial u^\alpha}{\partial q^\beta} = \eta^{\alpha\mu} \partial_x \frac{\partial g_{\beta\mu}}{\partial u^\mu}, \quad 1 \leq \alpha, \beta \leq N, \quad q \geq 0,$$

is called the double ramification hierarchy.

Let us equip the DR hierarchy with the following $N$ linearly independent Casimirs of its Poisson bracket $\{\cdot, \cdot\}_{\eta^{-1}\partial_x}$: $\overline{g}_{a,-1} := \int \eta_{a\beta} u^\beta dx$, $1 \leq \alpha \leq N$. Another important object related to the DR hierarchy is the local functional $\overline{g} \in \hat{A}_{u,0}$ determined by the relation [Bur15 Section 4.2.5]

$$\overline{g}_{1,1} = (D - 2)\overline{g}, \quad \text{where} \quad D := \sum_{n \geq 0} (n + 1)u^\alpha_n \frac{\partial}{\partial u^\alpha_n},$$

and $\overline{g}_{1,1} := A^\alpha \overline{g}_{a,1}$. Note that $\frac{\partial g_{\alpha \beta}}{\partial u^\alpha} = g_{a,0}$. Also, the local functional $\overline{g}$ has the following explicit expression at the approximation up to $\varepsilon^2$ ([BDGR13 Lemma 8.1]):

$$\overline{g} = \int f dx - \frac{\varepsilon^2}{48} \sum_{n \geq 0} c_{n}^{\epsilon} \epsilon_{a\beta\gamma} u^\alpha_1 u^\beta dx + O(\varepsilon^4),$$

where $f := F|_{t^* = u^*}$, and $c_{\beta\gamma} := C_{\beta\gamma} |_{t^* = u^*}$.

**Conjecture 1** ([BRS21]). Consider a homogeneous CohFT and the associated DR hierarchy.

1. The operator $K^\text{DR}_{g} = \left( K^\text{DR}_{\alpha\beta} \right)$ defined by

$$K^\text{DR}_{\alpha\beta} := \eta^{\alpha\mu} \eta^{\beta\nu} \left( \left( \frac{1}{2} - \mu_\beta \right) \partial_x \circ L_\nu(g_{\mu,0}) + \left( \frac{1}{2} - \mu_\alpha \right) L_\nu(g_{\mu,0}) \circ \partial_x ight)$$

$$+ \lambda_{\mu\nu} \partial_x + \partial_x \circ L_1(g_{\mu,0}) \circ \partial_x$$

is Poisson and is compatible with the operator $K^\text{DR}_1 := \eta^{-1} \partial_x$. Here $\mu_\alpha := q_\alpha - \frac{\delta}{2}$.
The Poisson brackets \( \{ \cdot, \cdot \}_{K^{PR}} \) and \( \{ \cdot, \cdot \}_{K^{DR}} \) give a bihamiltonian structure for the DR hierarchy with the following bihamiltonian recursion:

\[
\{ \cdot, \mathfrak{g}_{\alpha,d} \}_{K^{PR}} = \left( d + \frac{3}{2} + \mu_\alpha \right) \{ \cdot, \mathfrak{g}_{\alpha,d+1} \}_{K^{DR}} + A_\alpha^\beta \{ \cdot, \mathfrak{g}_{\beta,d} \}_{K^{PR}}, \quad d \geq -1,
\]

where \( A_\alpha^\beta := \eta^{\alpha \nu} A_{\nu \beta} \).

3.3. The Dubrovin–Zhang hierarchy. Consider an arbitrary homogeneous CohFT \( \{ c_{g,n} \} \). Let \( t^\alpha_0, 1 \leq \alpha \leq N, a \geq 0, \) be formal variables, where we identify \( t^\alpha_0 = t^\alpha \). The potential of our CohFT is defined by

\[
\mathcal{F}(t^*_\alpha, \varepsilon) = \sum_{g \geq 0} \varepsilon^{2g} \mathcal{F}_g(t^*_\alpha) := \sum_{g,n \geq 0} \varepsilon^{2g} \sum_{\nu_1, \ldots, \nu_n \leq N} \left( \int_{\mathcal{M}_{g,n}} c_{g,n}(\otimes_{i=1}^n e_{\alpha_i}) \prod_{i=1}^n \psi_i^{d_i} \right) \prod_{i=1}^n t^{\alpha_i}_i \in \mathbb{C}[\{ t^*_\alpha, \varepsilon \}],
\]

and introduce also the formal power series \( w^{\text{top}, \alpha} := \eta^{\alpha \nu} \frac{\partial^2 \mathcal{F}}{\partial \delta^a \partial \delta^b} \) and \( w^{\text{top}, \alpha} := \frac{\partial \psi}{\partial w^{\text{top}, \alpha}} \), where \( 1 \leq \alpha \leq N \) and \( n \geq 0 \).

Conjecture 2 ([DZ01]). Consider the ring \( \hat{\mathcal{A}}_w \) of differential polynomials in variables \( w^1, \ldots, w^N \).

1. For any \( 1 \leq \alpha, \beta \leq N \) and \( a, b \geq 0 \) there exists a differential polynomial \( \Omega_{\alpha,a;\beta,b} \in \hat{\mathcal{A}}_{w^0} \) such that

\[
\frac{\partial^2 \mathcal{F}}{\partial \delta^a \partial \delta^b} = \Omega_{\alpha,a;\beta,b} \bigl|_{w^\alpha = w^{\text{top}, \gamma}},
\]

2. There exists a Poisson operator \( K^{DZ}_1 = (K^{DZ}_1, \alpha \beta) \), for which the local functionals \( \Omega_{\alpha,-1} := \int \eta^{\alpha \nu} w^\nu \partial t^\mu \) are Casimirs, such that

\[
\eta^{\alpha \nu} \partial_t \Omega_{\mu,0;\beta,b} = K^{DZ}_{1,\alpha \nu} \frac{\delta \Omega_{\beta,b}}{\delta w^\nu},
\]

where \( \Omega_{\beta,b} := \int \Omega_{1,0;\beta,b+1} \partial t^\mu, 1 \leq \alpha, \beta \leq N, b \geq 0 \).

3. There exists a Poisson operator \( K^{DZ}_2 = (K^{DZ}_2, \alpha \beta) \) such that the following relations are satisfied:

\[
\{ \cdot, \mathfrak{h}_{\alpha,d} \}_{K^{DZ}_2} = \left( d + \frac{3}{2} + \mu_\alpha \right) \{ \cdot, \mathfrak{h}_{\alpha,d+1} \}_{K^{DZ}_1} + A_\alpha^\beta \{ \cdot, \mathfrak{h}_{\beta,d} \}_{K^{DZ}_1}, \quad 1 \leq \alpha \leq N, \quad d \geq -1.
\]

If differential polynomials from Part 1 of the conjecture exist, then they are unique (see, e.g., [BDGR13, Section 7.1]). Moreover, if Poisson operators from Parts 2 and 3 exist, then they are also unique (see, e.g., [BPS12a, Section 6]). Part 2 of the conjecture implies that the local functionals \( \mathfrak{h}_{\alpha,d} \) mutually commute with respect to the bracket \( \{ \cdot, \cdot \}_{K^{DZ}_1} \). The resulting bihamiltonian hierarchy (if the conjecture is true)

\[
\frac{\partial w^\alpha}{\partial t^\beta_q} = K^{DZ}_{1,\alpha \nu} \frac{\delta \Omega_{\beta,b}}{\delta w^\nu}, \quad 1 \leq \alpha, \beta \leq N, \quad q \geq 0,
\]

is called the Dubrovin–Zhang (DZ) hierarchy. The \( N \)-tuple of formal power series \( w^{\text{top}, \alpha} \) is a solution of the hierarchy, where we identify the derivative \( \partial_t \) with \( \frac{\partial}{\partial t^\beta} \). This solution is called the topological solution.
Conjecture 3 is proved at the approximation up to genus 1 \([\text{BDGR18}]\). In particular,

\[
\Omega_{\alpha,\alpha;\beta,b} = \left. \frac{\partial^2 F_0}{\partial t_\gamma^\beta \partial t_b^\beta} \right|_{t_\gamma^\alpha = \delta_{\alpha,0} \omega^\gamma} + O(\varepsilon^2),
\]

\[
K_1^{\text{DZ};\alpha\beta} = \eta^\alpha \partial_x + O(\varepsilon^2),
\]

\[
K_2^{\text{DZ};\alpha\beta} = \left. (E^\gamma C_\gamma^{\alpha\beta}) \right|_{t^\gamma = w^\gamma} \partial_x + \left( \frac{1}{2} - \mu_\beta \right) \left. (C_\gamma^{\alpha\beta}) \right|_{t^\gamma = w^\gamma} w_x^\gamma + O(\varepsilon^2).
\]

Parts 1 and 2 of the conjecture are proved for an arbitrary semisimple, not necessarily homogeneous, CohFT \([\text{BDGR20}]\) (a considerably simplified proof of Part 2 is presented in \([\text{BPS12a}]\)).

### 3.4. The DR/DZ hierarchy equivalence conjecture

We again consider an arbitrary homogeneous CohFT.

The normal coordinates of the DR hierarchy are defined by \(\tilde{u}^\alpha(u^*_\alpha, \varepsilon) := \eta^\alpha \partial_x + \varepsilon^2 G(u^1, \ldots, u^N) + O(\varepsilon^4)\), where \(\varepsilon^2 (t^1, \ldots, t^N) := F |_{t^\alpha = 0}\). The power series \(F^{\text{red}} = F + \mathcal{P} |_{w^\gamma = w^\gamma, t^\alpha = 0}\) satisfies the following vanishing property:

\[
\text{Coef}_{\varepsilon^2} \left. \frac{\partial^n F^{\text{red}}}{\partial t_d^1 \cdots \partial t_d^n} \right|_{t^\alpha = 0} = 0, \quad \text{if} \quad \sum_{i=1}^n d_i \leq 2g - 2.
\]

The differential polynomial \(\mathcal{P}\) has the following form: \(\mathcal{P} = -\varepsilon^2 G(u^1, \ldots, u^N) + O(\varepsilon^4)\), where \(G(t^1, \ldots, t^N) := F |_{t^\alpha = 0}\). The power series \(F^{\text{red}}\) is called the reduced potential of our CohFT.

Let us relate the variables \(\tilde{u}^\alpha\) to the variables \(w^\alpha\) by the following Miura transformation:

\[
\tilde{u}^\alpha(u^*_\alpha, \varepsilon) = u^\alpha + \eta^\alpha \partial_x \mathcal{P}, \tilde{h}_\nu, 0 \right)_{\mathcal{P}^{K^{\text{red}}}.
\]

**Conjecture 3** (\([\text{BDGR18}]\) and \([\text{BRS21}]\)). Assuming that Conjectures 1 and 2 are true, the DR and the DZ hierarchies, together with their bihamiltonian structures, coincide when we rewrite them in the coordinates \(\tilde{u}^\alpha\).

Our main result is the following theorem.

**Theorem 3.4.** Conjectures 1 and 3 are true at the approximation up to genus 1.

The proof will be given in Section 5. Together with Conjecture 2, which was proved in \([\text{BDGR18}]\) at the approximation up to genus 1, the theorem gives a full understanding of the bihamiltonian structures of the DR and the DZ hierarchies and their relation at the approximation up to genus 1 for an arbitrary homogeneous CohFT.

### 4. Extending the space of differential polynomials by tame rational functions

Before proving Theorem 3.4 in the next section, let us present several technical lemmas.

**Conjecture 2** is not proved at the moment, but a weaker version is true if we extend the space of differential polynomials. Following \([\text{BDGR20}]\) Section 7.3 consider formal variables \(v^1, \ldots, v^N\) and, for \(d \in \mathbb{Z}\), denote by \(\mathcal{A}_{vd}^d\) the vector space spanned by expressions of the form

\[
\sum_{i \geq m} P_i(v^*_i) \left( \frac{\partial x}{v^*_i} \right)^i,
\]

where \(m \in \mathbb{Z}\), \(P_i \in \mathcal{A}_{vd+i}\) and \(\frac{\partial F}{\partial v^*_i} = 0\). Let \(\mathcal{A}_{vd}^d := \bigoplus_{d \in \mathbb{Z}} \mathcal{A}_{vd}^d\). Define also the extended space \(\tilde{\mathcal{A}}_{vd}^d := \mathcal{A}_{vd}^d[[\varepsilon]]\).
A rational function \( (4.1) \) is called tame if there exists a nonnegative integer \( C \) such that \( \frac{\partial v}{\partial x_k} = 0 \) for \( k > C \). The subspace of tame elements in \( A^{t^*} \) will be denoted by \( A^{t^*,t} \subset A^{t^*} \).

We also introduce the extended space \( \hat{A}^{t^*,t} := \tilde{A}^{t^*,t}[\varepsilon] \). A rational Miura transformation is a change of variables \( v^\alpha \mapsto \tilde{v}^\alpha(v^*_\epsilon, \epsilon) \) of the form \( \tilde{v}^\alpha(v^*_\epsilon, \epsilon) = v^\alpha + \epsilon f^\alpha(v^*_\epsilon, \epsilon) \), where \( f^\alpha \in \hat{A}^{t^*,t} \).

Introduce formal power series \( v^{top,\alpha} := \eta^{\alpha\beta} \frac{\partial^2 F_0}{\partial v^\alpha \partial v^{\top,\beta}} \) and \( v^{top,n} := \left( \frac{\partial^n}{\partial v^\alpha_0} \right)^n v^{top,\alpha} \). Note that the map \( \hat{A}^{t^*,t} \to \mathbb{C}[[t^*_\epsilon, \epsilon]] \) given by

\[
\hat{A}^{t^*,t} \ni f \mapsto f|_{v^*_\epsilon = v^{top,\alpha}} \in \mathbb{C}[[t^*_\epsilon, \epsilon]]
\]
is injective [BDGR20, Section 7.3]. By [BDGR20, Proposition 7.6] there exists a unique tame rational function of Proposition 7.6 in [BDGR20] also gives that there exists a unique tame rational function \( \sum \alpha \epsilon \), which gives

\[
\partial v^\alpha \partial v^{\top,\beta} \partial \epsilon
\]

respectively, by the rational Miura transformation \( v^\alpha \mapsto w^\alpha(v^*_\epsilon, \epsilon) \). With this definition, the relations (7.6) and (7.7) are true (see, e.g., [BDGR20, Section 7.3]).

Therefore, equivalently, Conjecture 2 says that \( \Omega_{\alpha,\beta,\epsilon} \) \( \hat{A}^{t^*,t_0} \) and \( K^{DZ}_1, K^{DZ}_2 \in \mathcal{P} \mathcal{O}_w \) be Poisson operators obtained from the operators

\[
\eta^{\alpha\beta} \partial \epsilon \quad \text{and} \quad \left( E^\gamma C^\gamma \right) |_{t^* = v^*, \epsilon} \partial \epsilon + \left( \frac{1}{2} - \mu_\beta \right) \left( C^\gamma \right) |_{t^* = v^*, \epsilon} \partial \epsilon,
\]

respectively, by the rational Miura transformation \( v^\alpha \mapsto w^\alpha(v^*_\epsilon, \epsilon) \). With this definition, the relations (3.5) and (3.6) are true (see, e.g., [BDGR20, Section 7.3]).

Lemma 4.1. Consider a Poisson operator \( K \in \mathcal{P} \mathcal{O}_w \) and a rational Miura transformation \( v^\alpha \mapsto \tilde{v}^\alpha(v^*_\epsilon, \epsilon) \) such that \( \tilde{v}^\alpha(v^*_\epsilon, \epsilon) - v^\alpha \in \mathrm{Im} \partial \epsilon \). Then we have \( K^{\alpha\beta} = \sum_{m \geq 0} \frac{\partial v^\alpha}{\partial v^\beta_m} K_{0^\beta} \).

Proof. We compute

\[
K^{\alpha\beta} = \frac{\partial v^\alpha}{\partial v^\beta} = \frac{\partial v^\alpha}{\partial v^\beta} \left( \sum_{m, n \geq 0} \frac{\partial \tilde{v}^\alpha}{\partial v^\beta_m} \partial \epsilon \circ K^{\beta\theta} \circ (\partial \epsilon)^n \frac{\partial \tilde{v}^\beta}{\partial v^\beta_n} \right) = \frac{\partial v^\alpha}{\partial v^\beta} \left( \sum_{m \geq 0} \frac{\partial \tilde{v}^\alpha}{\partial v^\beta_m} \partial \epsilon \circ K^{\beta\theta} \circ (\partial \epsilon)^n \frac{\partial \tilde{v}^\beta}{\partial v^\beta_n} \right) = \frac{\partial v^\alpha}{\partial v^\beta} \left( \sum_{m \geq 0} \frac{\partial \tilde{v}^\alpha}{\partial v^\beta_m} \partial \epsilon \circ K^{\beta\theta} \right) = \sum_{m \geq 0} \frac{\partial \tilde{v}^\alpha}{\partial v^\beta_m} \partial \epsilon \circ K^{\beta\theta} = \sum_{m \geq 0} \frac{\partial \tilde{v}^\alpha}{\partial v^\beta_m} \partial \epsilon \circ K^{\beta\theta}.
\]

Lemma 4.2. We have \( K^{DZ,\alpha\beta}_{2,0} = \left( \frac{1}{2} - \mu_\beta \right) \eta^{\alpha\gamma} \eta^{\beta\nu} \partial \epsilon \Omega_{\theta,0;\nu,0} \).

Proof. By Lemma 4.1 we have \( K^{DZ,\alpha\beta}_{2,0} = \left( \frac{1}{2} - \mu_\beta \right) \sum_{m \geq 0} \frac{\partial \tilde{v}^\alpha}{\partial v^\beta_m} \partial \epsilon \left( C^\gamma \right) |_{t^* = v^*, \epsilon} \partial \epsilon \). Note that \( \left( C^\gamma \right) |_{t^* = v^*, \epsilon} \partial \epsilon = \eta^{\beta\nu} \partial \epsilon \left( C^{\beta\nu} \right) |_{t^* = v^*, \epsilon} \partial \epsilon \). Therefore,

\[
\sum_{m \geq 0} \frac{\partial \tilde{v}^\alpha}{\partial v^\beta_m} \partial \epsilon \left( C^\gamma \right) |_{t^* = v^*, \epsilon} \partial \epsilon = \eta^{\beta\nu} \partial \epsilon \left( C^{\beta\nu} \right) |_{t^* = v^*, \epsilon} \partial \epsilon = \eta^{\alpha\gamma} \eta^{\beta\nu} \partial \epsilon \Omega_{\theta,0;\nu,0} \left|_{t^* = v^*, \epsilon} \right.,
\]

which gives \( \eta^{\alpha\gamma} \eta^{\beta\nu} \partial \epsilon \Omega_{\theta,0;\nu,0} \), as required.
Remark 4.3. The lemma, in particular, implies that the constant term of the operator $K^{DZ}_2$ is a differential polynomial if $\Omega_{\bar{\theta},\bar{\nu},0}$ is a differential polynomial, which is true in the semisimple case. This is also noticed in [HS21, Theorem 4.11].

Lemma 4.4. We have $K^{DR;\alpha\beta}_{2;0} = (\frac{1}{2} - \mu_\beta) \eta^{\alpha\beta} \eta^\beta\nu \partial_{\nu} \frac{\partial g_{\mu,0}}{\partial u^\mu_{xx}}$.

Proof. This directly follows from the definition (3.4).

5. Proof of Theorem 5.2

If we exclude the operators $K^{DZ}_2$ and $K^{DR}_2$ from consideration, then the fact that the DZ hierarchy and the DR hierarchy coincide in the coordinates $\tilde{u}^\alpha$, at the approximation up to genus 1, was already proved in [BDGR18, Theorem 8.4] (see also Remark 4.3). Thus, it is sufficient to prove that $K^{DZ}_{2;\tilde{u}} = K^{DR}_{2;\tilde{u}} + O(\varepsilon^4)$. Since we know that $K^{DZ;[0]}_{2;\tilde{u}} = K^{DR;[0]}_{2;\tilde{u}}$ [BRS21], it remains to check that

$$K^{DZ;[2]}_{2;\tilde{u};\tilde{l}} = K^{DR;[2]}_{2;\tilde{u};\tilde{l}}$$

for $l = 0, 1, 2, 3$.

We split the proof in several steps.

Step 1. Let us check (5.1) for $l = 2$ and $l = 3$. We will do that by direct computation.

We will denote $\partial_\alpha := \frac{\partial}{\partial u^\alpha}$ and also use the notations

$$c^{\alpha\beta}_{\gamma\delta} := \partial_\gamma c^{\alpha\beta}_{\delta}, \quad c^{\alpha\beta}_{\gamma\delta} := \partial_\delta c^{\alpha\beta}_{\gamma}, \quad e^\gamma := E^\gamma|_{\nu = u^\nu} = (1 - q_\gamma)u^\gamma + r^\gamma, \quad g^{\alpha\beta} = e^\gamma c^{\alpha\beta}_\gamma.$$

In [DZ98, Theorem 2] the authors obtained the following formulas:

$$K^{DZ;[2],\alpha\beta}_{2;\tilde{u};2} = h^{\alpha\beta}|_{\tilde{u} = \tilde{u}^*},$$

$$K^{DZ;[2],\alpha\beta}_{2;\tilde{u};2} = \left( 3 \left( \partial_\alpha h^{\alpha\beta} + \frac{1}{24} \left( \frac{3}{2} - \mu_\beta \right) c^{\alpha\nu\beta\mu}_{\gamma\nu} - \frac{1}{24} \left( \frac{3}{2} - \mu_\alpha \right) c^{\beta\nu\alpha\mu}_{\gamma\nu} \right) \right)|_{\tilde{u} = \tilde{u}^*},$$

where

$$h^{\alpha\beta} = \frac{1}{12} \left( \partial_\nu \left( g^{\mu\nu}_{\alpha\beta} \right) + \frac{1}{2} c^{\alpha\beta}_{\mu\nu} \right).$$

On the other hand, since $\tilde{u}^\alpha = u^\alpha + \frac{\varepsilon^2}{24} \partial_x^2 c^{\alpha\lambda}_{\lambda} + O(\varepsilon^4)$ [BDGR18, proof of Theorem 8.4], we have

$$K^{DR;[2],\alpha\beta}_{2;\tilde{u}} = L_\nu \left( u^\alpha + \frac{\varepsilon^2}{24} \partial_x^2 c^{\alpha\lambda}_{\lambda} \right) \odot K^{DR;[2],\alpha\beta}_{2;\tilde{u}} + O(\varepsilon^4) =$$

$$= K^{DR;[2],\alpha\beta}_{2;\tilde{u}} + \frac{\varepsilon^2}{24} \left( \partial_\nu \left( c^{\alpha\lambda}_{\lambda} \right) \odot K^{DR;[2],\alpha\beta}_{2;\tilde{u}} + K^{DR;[2],\alpha\beta}_{2;\tilde{u}} \odot \partial_\nu \left( c^{\beta\delta}_{\nu} \right) \odot \partial_\delta \right) + O(\varepsilon^4) =$$

$$= K^{DR;[2],\alpha\beta}_{2;\tilde{u}} + \frac{\varepsilon^2}{24} \left( \partial_\nu \odot c^{\alpha\nu\lambda}_{\nu} \odot K^{DR;[2],\alpha\beta}_{2;\tilde{u}} + K^{DR;[2],\alpha\beta}_{2;\tilde{u}} \odot c^{\beta\theta}_{\nu} \odot \partial_\delta \right) + O(\varepsilon^4),$$

where $R^{\alpha\beta}_{i} \in A_{u;3-i}$. Considering the expansion $g_{\mu,0} = \sum_{g \geq 0} \varepsilon^{2g} g^{[2g]}_{\mu,0}$, $g^{[2g]}_{\mu,0} \in A_{u;2g}$, from (3.4) we compute

$$K^{DR;[2],\alpha\beta}_{2;3} = (3 - \mu_\alpha - \mu_\beta) \eta^{\alpha\beta} \eta^\beta\nu \frac{\partial g^{[2]}_{\mu,0}}{\partial u^\mu_{xx}},$$

$$K^{DR;[2],\alpha\beta}_{2;2} = \eta^{\alpha\beta} \eta^\beta\nu \left( \frac{1}{2} \frac{\partial g^{[2]}_{\mu,0}}{\partial u^\mu_{xx}} + \frac{5}{2} \frac{\partial g^{[2]}_{\mu,0}}{\partial u^\mu_{xx}} \right).$$
Using (3.3) we then compute

\[
\frac{\partial g_{\mu\rho}^{[2]}}{\partial u^\nu_{xx}} = \frac{1}{24} \epsilon_\theta^\nu c_\mu^\epsilon, \\
\frac{\partial g_{\mu\rho}^{[2]}}{\partial u^\nu_x} = \frac{1}{24} \left[ \partial_\nu \left( \epsilon_\theta^\nu c_{\mu\gamma}^\epsilon \right) + \partial_\gamma \left( \epsilon_\theta^\nu c_{\nu\mu}^\epsilon \right) - \partial_\mu \left( \epsilon_\theta^\nu c_{\nu\gamma}^\epsilon \right) \right] u^\gamma_x,
\]

and finally get

\[
K^{2,3}_{2;\alpha\beta} = \frac{3 - \mu_\alpha - \mu_\beta c_\gamma^\sigma c_\sigma^\beta}{24}, \\
K^{2,2}_{2;\alpha\beta} = \left[ \frac{2 - \mu_\alpha - \mu_\beta}{24} \left( \epsilon_\theta^\mu c_\nu^\alpha c_\gamma^\xi - \epsilon_\theta^\mu c_\nu^\alpha c_\gamma^\xi \right) + \frac{9}{2} - \mu_\alpha - 2\mu_\beta \partial_\gamma \left( \epsilon_\theta^\alpha c_\mu^\beta \right) \right] u^\gamma_x.
\]

Using that \( K^{2,2}_{2;\alpha\beta} = g^{\alpha\beta} \partial_x + \left( \frac{1}{2} - \mu_\beta \right) c_\gamma^\beta u^\gamma_x \), we also compute

\[
R_3^{\alpha\beta} = \frac{1}{24} \left( c_\nu^\lambda g^{\nu\beta} + g^{\alpha\nu} c_\nu^\beta \right), \\
R_2^{\alpha\beta} = \frac{1}{24} \left[ 2\partial_\nu \left( c_\nu^\lambda g^{\nu\beta} \right) + c_\nu^\lambda c_\nu^\beta \left( \frac{1}{2} - \mu_\beta \right) + g^{\alpha\nu} c_\nu^\beta + c_\nu^\gamma \left( \frac{1}{2} - \mu_\nu \right) c_\nu^\beta \right] u^\gamma_x.
\]

So, in order to prove (5.1) for \( l = 2 \) and \( l = 3 \), we need to check the following two equations:

\[
(5.2) \quad \frac{1}{12} \partial_\nu \left( g^{\mu\nu} c_\mu^{\alpha\beta} \right) + \frac{1}{4} \frac{1}{g^{\nu\mu}} g^{\nu\mu} c_\mu^{\alpha\beta} \left( \frac{1}{2} - \mu_\beta \right) + g^{\alpha\nu} c_\nu^\beta c_\nu^\mu + c_\nu^\gamma \left( \frac{1}{2} - \mu_\nu \right) c_\nu^\beta = 0,
\]

\[
(5.3) \quad \frac{1}{8} \partial_\gamma \left( g^{\mu\nu} c_\mu^{\alpha\beta} \right) + \frac{1}{2} \frac{1}{g^{\nu\mu}} g^{\nu\mu} c_\mu^{\alpha\beta} \left( \frac{1}{2} - \mu_\beta \right) + g^{\alpha\nu} c_\nu^\beta c_\nu^\mu + c_\nu^\gamma \left( \frac{1}{2} - \mu_\nu \right) c_\nu^\beta = 0.
\]

Let us prove (5.2). Collecting the underlined terms together, using that \( g^{\alpha\beta} = e^\gamma c_\gamma^{\alpha\beta} \), and multiplying both sides by 12, we see that (5.2) is equivalent to the following equation:

\[
(1 - q_\nu) c_\mu^{\nu\mu} c_\mu^{\alpha\beta} + \epsilon^\gamma \partial_\nu \left( c_\mu^{\nu\mu} c_\mu^{\alpha\beta} \right) = \frac{2 - \mu_\alpha - \mu_\beta c_\gamma^\sigma c_\sigma^\beta}{2} + \frac{1}{2} e^\gamma \left( c_\nu^\alpha c_\mu^\beta + c_\gamma^\alpha c_\nu^\beta \right).
\]

Collecting the underlined terms together, transforming the boxed term as \( e^\gamma \partial_\nu \left( c_\mu^{\nu\mu} c_\mu^{\alpha\beta} \right) = e^\gamma \partial_\nu \left( c_\mu^{\nu\mu} c_\mu^{\alpha\beta} \right) = e^\gamma \left( c_\mu^{\nu\mu} c_\mu^{\alpha\beta} \right) \), and moving all the terms to the left-hand side, we come to the expression

\[
\frac{\mu_\alpha + \mu_\beta - 2q_\nu}{2} c_\mu^{\nu\mu} c_\mu^{\alpha\beta} + e^\gamma \left( c_\nu^\alpha c_\mu^\beta + c_\nu^\beta c_\mu^\gamma - \frac{1}{2} e^\gamma c_\nu^\alpha c_\mu^\beta \right) = 0.
\]

whose vanishing we have to prove. From the theory of Dubrovin–Frobenius manifolds [Dub96] we know that \( \mathcal{L}_E C_\beta^\alpha = C_\beta^\alpha \), where \( \mathcal{L}_E \) denotes the Lie derivative, which implies that

\[
e^\lambda c_\lambda^{\alpha\beta} = (\delta - q_\alpha - q_\beta + q_\gamma) C_\gamma^{\alpha\beta}.
\]
Applying this to the boxed term, we see that the expression (5.4) is equal to
\[
\frac{q_\alpha - q_\beta + \delta - 2q_\mu c_\beta^\mu c_\gamma^\mu c_\gamma^\lambda}{2} c_\gamma^\mu c_\gamma^\lambda + (\delta - q_\beta - q_\mu + q_\nu)c_\beta^\mu c_\alpha^\mu c_\mu^\nu + e^\gamma \left( \frac{1}{2} c_\gamma^\mu c_\alpha^\nu - \frac{1}{2} c_\alpha^\nu c_\beta^\gamma \right) =
\]
(5.5)
\[
\frac{q_\alpha - q_\beta + \delta - 2q_\mu c_\beta^\mu c_\gamma^\mu c_\gamma^\lambda}{2} + \frac{1}{2} e^\gamma \left( c_\beta^\mu c_\alpha^\nu - c_\alpha^\nu c_\beta^\gamma \right).
\]
We transform the underlined terms as follows:
\[
c_\gamma^\mu c_\gamma^\lambda - c_\alpha^\nu c_\beta^\gamma = \left( \partial_\nu c_\gamma^\mu c_\gamma^\lambda - c_\beta^\mu c_\gamma^\lambda \right) - \left( \partial_\lambda c_\gamma^\mu c_\beta^\gamma - c_\alpha^\nu c_\beta^\gamma \right) = c_\alpha^\nu c_\beta^\gamma - c_\beta^\mu c_\gamma^\lambda,
\]
and therefore the expression (5.5) is equal to
\[
\frac{q_\alpha - q_\beta + \delta - 2q_\mu c_\beta^\mu c_\gamma^\mu c_\gamma^\lambda}{2} + \frac{1}{2} \delta - q_\alpha - q_\nu + q_\mu c_\beta^\mu c_\beta^\nu \frac{\delta - q_\beta - q_\mu + q_\nu c_\beta^\mu c_\gamma^\lambda}{2} =
\]
\[
= - \mu_\nu c_\beta^\nu c_\gamma^\lambda = - \mu_\nu c_\gamma^\nu c_\gamma^\lambda.
\]
It is sufficient to check that \( \mu_\nu c_\beta^\nu = 0 \) for any \( \alpha \). Indeed, we compute \( \gamma = \mu_\nu c_\beta^\nu = \mu_\nu \eta^\nu_\lambda \eta^\lambda_\nu c_\alpha^\beta = - \mu_\nu \eta^\nu_\lambda \eta^\lambda_\nu = - \mu_\lambda c_\alpha^\lambda = - \gamma \), which implies that \( \gamma = 0 \), as required.

Let us now prove equation (5.3). Collecting together like terms we come to the equivalent equation
\[
\frac{1}{8} \partial_\gamma \partial_\nu \left( g^{\alpha\nu} c_\gamma^\beta \right) + \frac{\mu_\alpha + \mu_\nu - 1}{24} c_\gamma^\beta c_\mu^\alpha = \frac{3 - \mu_\beta - 2\mu_\beta}{24} \partial_\gamma \left( c_\nu^\alpha c_\gamma^\beta \right) + \frac{1}{24} \left[ 2 \partial_\gamma \left( c_\nu^\alpha g^{\nu\beta} \right) + g^{\alpha\nu} c_\nu^\beta \right],
\]
which is equivalent to
\[
\frac{1 - \gamma}{8} \partial_\gamma \left( g^{\alpha\nu} c_\gamma^\beta \right) + \frac{1}{8} \partial_\gamma \left( e^\beta g^{\alpha\nu} (c_\gamma^\beta c_\mu^\alpha) \right) + \frac{\mu_\alpha + \mu_\nu - 1}{24} c_\gamma^\beta c_\mu^\alpha = \frac{\mu_\alpha + \mu_\nu - 1}{24} c_\gamma^\beta c_\mu^\alpha + \frac{\mu_\alpha + 2\mu_\beta - 3}{24} \partial_\gamma \left( c_\nu^\alpha c_\gamma^\beta \right) - \frac{1}{24} \left[ 2 \partial_\gamma \left( c_\nu^\alpha g^{\nu\beta} \right) + g^{\alpha\nu} c_\nu^\beta \right].
\]
Using that \( \mu_\nu e_\gamma^\nu = 0 \) we see that the left-hand side is equal to
\[
\frac{1}{8} \partial_\gamma \left( e^\beta g^{\alpha\nu} (c_\gamma^\beta c_\mu^\alpha) \right) + \frac{\mu_\alpha + \mu_\nu - 1}{24} c_\gamma^\beta c_\mu^\alpha = \frac{\mu_\alpha + \mu_\nu - 1}{24} c_\gamma^\beta c_\mu^\alpha + \frac{q_\alpha + 2q_\beta - 3\delta}{24} \partial_\gamma \left( c_\nu^\alpha c_\gamma^\beta \right) - \frac{1}{24} \left[ 2 \partial_\gamma \left( c_\nu^\alpha g^{\nu\beta} \right) + g^{\alpha\nu} c_\nu^\beta \right].
\]
Transforming the first term in this expression as
\[
\partial_\gamma \left( e^\beta g^{\alpha\nu} (c_\gamma^\beta c_\mu^\alpha) \right) = \partial_\gamma \left( e^\beta g^{\alpha\nu} (c_\gamma^\beta c_\mu^\alpha) \right) = \partial_\gamma \left( e^\beta g^{\alpha\nu} c_\mu^\alpha \right) + \partial_\gamma \left( e^\beta g^{\alpha\nu} c_\mu^\alpha \right) =
\]
\[
= (\delta - q_\mu - q_\beta + q_\nu) \partial_\gamma \left( c_\mu^\alpha c_\gamma^\lambda \right) + \partial_\gamma \left( g^{\mu\beta} c_\mu^\alpha \right) =
\]
\[
= (\delta - q_\beta) \partial_\gamma \left( c_\mu^\beta c_\gamma^\lambda \right) + \partial_\gamma \left( g^{\mu\beta} c_\mu^\alpha \right),
\]
we come to the expression
\[
\frac{1}{8} \partial_\gamma \left( c_\gamma^\beta c_\mu^\alpha \right) + \frac{1}{8} \partial_\gamma \left( g^{\mu\beta} c_\mu^\alpha \right) + \frac{\mu_\alpha + \mu_\nu - 1}{24} c_\gamma^\beta c_\mu^\alpha = \frac{\mu_\alpha + \mu_\nu - 1}{24} c_\gamma^\beta c_\mu^\alpha + \frac{\mu_\alpha + \mu_\nu - 1}{24} c_\gamma^\beta c_\mu^\alpha + \frac{1}{24} \partial_\gamma \left( g^{\mu\beta} c_\mu^\alpha \right) - \frac{1}{24} \left[ 2 \partial_\gamma \left( c_\nu^\alpha g^{\nu\beta} \right) + g^{\alpha\nu} c_\nu^\beta \right].
\]
Applying to the underlined term the formula
\[
\partial_\gamma g^{\nu\beta} = (1 - q_\nu) c_\gamma^\nu c_\gamma^\nu + e^\beta c_\nu^\beta = (1 - \mu_\nu - \mu_\beta) c_\gamma^\nu,
\]
we obtain
\[
\frac{\mu_\alpha - \mu_\beta}{24} \partial_\gamma \left( c^{\alpha \nu}_\gamma c^{\nu \mu}_\mu \right) + \frac{\mu_\alpha + \mu_\nu - 1}{24} c^{\alpha \nu}_\gamma c^{\beta \mu}_\mu + \frac{1}{24} c^{\alpha \lambda}_\gamma g^{\nu \beta} + \left( 1 - \mu_\nu - \mu_\beta \right) c^{\alpha \lambda}_\gamma c^{\nu \beta}_\gamma - g^{\nu \beta} c^{\alpha \nu}_\gamma c^{\beta \lambda}_\gamma = \]
\[
= \frac{\mu_\alpha - \mu_\beta}{24} \partial_\gamma \left( c^{\alpha \nu}_\nu c^{\nu \mu}_\mu \right) + \frac{\mu_\alpha + \mu_\nu - 1}{24} c^{\alpha \nu}_\nu c^{\beta \mu}_\mu + \frac{1}{24} - \mu_\nu - \mu_\beta \right) c^{\alpha \lambda}_\nu g^{\beta \lambda} - g^{\nu \beta} c^{\alpha \nu}_\nu c^{\beta \lambda}_\nu .
\]
Expressing the underlined terms as follows:
\[
\partial_\lambda \partial_\gamma \left( c^{\alpha \lambda}_\gamma g^{\nu \beta} - g^{\alpha \nu} c^{\nu \beta}_\gamma \right) \left( -c^{\alpha \lambda}_\gamma \partial_\lambda g^{\nu \beta} - c^{\alpha \nu}_\nu \partial_\lambda g^{\nu \beta} - c^{\alpha \nu}_\nu \partial_\gamma g^{\nu \beta} + \partial_\gamma g^{\alpha \nu} c^{\nu \beta}_\lambda + \partial_\lambda \partial_\gamma g^{\alpha \nu} c^{\nu \beta}_\mu = \right.
\]
\[
= (\mu_\beta + \mu_\nu - \mu_\lambda - \mu_\alpha) \partial_\gamma \left( c^{\alpha \nu}_\nu c^{\nu \mu}_\mu \right) + (\mu_\beta + \mu_\nu - 1) c^{\alpha \nu}_\nu c^{\beta \nu}_\lambda + (1 - \mu_\alpha - \mu_\nu) c^{\alpha \nu}_\gamma c^{\nu \beta}_\gamma c^{\beta \lambda}_\nu,
\]
we obtain
\[
\frac{\mu_\alpha - \mu_\beta}{24} \partial_\gamma \left( c^{\alpha \nu}_\nu c^{\nu \mu}_\mu \right) + \frac{\mu_\alpha + \mu_\nu - 1}{24} c^{\alpha \nu}_\nu c^{\beta \mu}_\mu + \frac{1}{24} c^{\alpha \lambda}_\nu g^{\beta \lambda} - g^{\nu \beta} c^{\alpha \nu}_\nu c^{\beta \lambda}_\nu = \]
\[
= \frac{\mu_\alpha - \mu_\lambda}{24} \partial_\gamma \left( c^{\alpha \nu}_\nu c^{\nu \mu}_\mu \right) = 0,
\]
as required.

**Step 2.** Let us prove (5.1) for \( l = 0 \). Note that by the definition \( \tilde{u}^\alpha (w^*_\epsilon, \epsilon) = u^\alpha \in \text{Im} \partial_x \). Therefore, Lemmas 4.3 and 4.2 imply that
\[
K_{2;\tilde{\alpha}}^{\text{DZ}, \alpha \beta} = \left( \frac{1}{2} - \mu_\beta \right) \eta^{\beta \nu} \sum_{m \geq 0} \frac{\partial \tilde{u}^\alpha}{\partial u^\nu_m} \eta^{\nu \theta} \partial_x^{m+1} \Omega_{\theta,0;\epsilon,0} = \left( \frac{1}{2} - \mu_\beta \right) \eta^{\beta \nu} \sum_{m \geq 0} \{ \tilde{u}^\alpha, \Omega_{\nu,0} \} K_1 \text{DZ}.
\]
Lemmas 4.3 and 4.4 together with the fact that \( \tilde{u}^\alpha (w^*_\epsilon, \epsilon) = u^\alpha \in \text{Im} \partial_x \) [BDGR18 Lemma 7.1] imply that
\[
K_{2;\tilde{\alpha}}^{\text{DR}, \alpha \beta} = \left( \frac{1}{2} - \mu_\beta \right) \eta^{\beta \nu} \sum_{m \geq 0} \frac{\partial \tilde{u}^\alpha}{\partial u^\nu_m} \eta^{\nu \theta} \partial_x^{m+1} \delta \Omega_{\nu,0;\epsilon,0} = \left( \frac{1}{2} - \mu_\beta \right) \eta^{\beta \nu} \sum_{m \geq 0} \{ \tilde{u}^\alpha, \delta \Omega_{\nu,0} \} K_1 \text{DR}.
\]
Since, in the coordinates \( \tilde{u}^\alpha \) and at the approximation up to \( \epsilon^2 \), the local functionals \( \tilde{\Omega}_{\alpha,\alpha} \) coincide with the local functionals \( \tilde{\Omega}_{\alpha,\alpha} \) and the Poisson operator \( K_1 \text{DZ} \) coincide with the Poisson operator \( K_1 \text{DR} \), we obtain \( K_{2;\tilde{\alpha}}^{\text{DZ}, \alpha \beta} = K_{2;\tilde{\alpha}}^{\text{DR}, \alpha \beta} + O(\epsilon^4) \), as required.

**Step 3.** Let us finally prove that \( K_{2;\tilde{\alpha}}^{\text{DZ}}[2] = K_{2;\tilde{\alpha}}^{\text{DR}}[2] \). Since we have proved (5.1) for \( l = 0, 2, 3 \), the difference \( K_{2;\tilde{\alpha}}^{\text{DZ}}[2] - K_{2;\tilde{\alpha}}^{\text{DR}}[2] \) has the form \( R \partial_x, R = (R^\alpha \beta) \), where \( R^\alpha \beta \in \mathcal{A}_{1;2} \). Since the operators \( K_{2;\tilde{\alpha}}^{\text{DZ}}[2] \) and \( K_{2;\tilde{\alpha}}^{\text{DR}}[2] \) are skewsymmetric, we have
\[
(R \partial_x)^\dagger = -R \partial_x \Leftrightarrow R^T = R \text{ and } \partial_x R = 0.
\]
The property \( \partial_x R = 0 \) immediately implies that \( R = 0 \), which completes the proof of the theorem.

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