THE MATRICIAL RANGE OF $E_{21}$

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Abstract. The matricial range of the $2 \times 2$ matrix $E_{21}$ (i.e., the $2 \times 2$ unilateral shift) is described very simply: it consists of all matrices with numerical radius at most $1/2$. The known proofs of this simple statement, however, are far from trivial and they depend on subtle results on dilations. We offer here a brief introduction to the matricial range and a recap of those two proofs, following independent work of Arveson and Ando in the early 1970s.

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1. Introduction

In 1969, W. Arveson published a striking paper in Acta Mathematica, called Subalgebras of C*-algebras [2]. In this paper he developed a non-commutative analog of the Choquet theory for function spaces. His paper is a wonderful mixture of technical prowess and deep thinking about how to rightly generalize certain ideas about function spaces to the non-commutative setting. A key feature of his paper was the use of complete positivity as a non-commutative replacement for the role that positivity has in the commutative case.

A few years later he published an equally remarkable paper [3]. Besides containing his essential Boundary Theorem, this paper defined the matricial range of an operator. As a consequence of an analysis of nilpotent dilations, he was able to explicitly characterize the matricial range of the $2 \times 2$ matrix unit $E_{21}$. This is one of the very few non-trivial (that is, non-normal) cases
where the matricial range has been determined (the other significant one is the unilateral shift on an infinite-dimensional Hilbert space, which is more or less straightforward).

Almost concurrently, Ando published his results characterizing the numerical range \([1]\). As a direct byproduct of his results one recovers the characterization of the matricial range of \(E_{21}\).

The goal of this article is to describe Arveson and Ando’s techniques, together with basic characterizations of the matricial range. The results we offer follow closely the originals, but several of the proofs are new. In the case of Ando, we have also strived to fill in the details from his very condensed arguments.

2. Preliminaries

Throughout, \(H\) will be a Hilbert space, with inner product \(\langle \cdot, \cdot \rangle\). We use \(B(H)\) to denote the (\(C^*\), von Neumann) algebra of bounded operators on \(H\); and \(K(H)\) for the compact operators. When \(\dim H = n\), we canonically identify \(H\) with \(\mathbb{C}^n\) and \(B(H)\) with \(M_n(\mathbb{C})\), the set of \(n \times n\) complex matrices. This is done by fixing an orthonormal basis \(\{\xi_j\} \subset H\) and considering the rank-one operators

\[E_{kj} \xi = \langle \xi, \xi_j \rangle \xi_k, \quad \xi \in H.\]

These are called matrix units and they are characterized up to unitary equivalence (i.e., choice of the orthonormal basis) by the relations

\[(2.1) \quad E_{kj} E_{h\ell} = \delta_{jh} E_{k\ell}, \quad E_{kj}^* = E_{jk}, \quad \sum_{k=1}^n E_{kk} = I.\]

We write \(T = \{z \in \mathbb{C} : |z| = 1\}\), and \(D = \{z \in \mathbb{C} : |z| < 1\}\) for the unit circle and unit disk respectively. When needed for clarity, we will write \(E_{kj}^n\) to emphasize that \(E_{kj}^n \in M_n(\mathbb{C})\). Of particular importance will be, for each \(n\), the unilateral shift:

\[S_n = \sum_{k=1}^{n-1} E_{k+1,k} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}.\]

In the infinite-dimensional case, when \(\{\xi_j\}_{j \in \mathbb{N}}\) is an orthonormal basis of \(H\), the associated shift operator \(S\) is the linear operator induced by \(S : \xi_j \mapsto \xi_{j+1}\).

An element \(x\) of a normed space is said to be contractive if \(\|x\| \leq 1\). A linear map \(\phi : A \to B\) between two \(C^*\)-algebras is positive if it maps positive elements to positive elements; it is completely positive if \(\phi^{(n)}\) is positive for all \(n \in \mathbb{N}\), with \(\phi^{(n)}\) the \(n\)th amplification \(\phi^{(n)} : M_n(B(H)) \to M_n(B(K))\), given by \(\phi^{(n)}(A) = [\phi(A_{kj})]_{kj}\). We will mostly consider completely positive maps which are also unital; these are commonly named ucp (unital, completely
positive). The basics of completely positive and completely contractive maps are covered in many texts. We refer the reader to the following canonical three: \cite{5,15,16}. We mention one explicit result that we will use:

**Proposition 2.1 (Choi).** Let \(\phi : M_n(\mathbb{C}) \to B(H)\) be a linear map. Then \(\phi\) is completely positive if and only if

\[
\phi^{(n)} \left( \begin{bmatrix} E_{11} & \cdots & E_{1n} \\ \vdots & \ddots & \vdots \\ E_{n1} & \cdots & E_{nn} \end{bmatrix} \right) \geq 0.
\]

The operator system generated by \(T \in B(H)\) is the space \(\mathcal{OS}(T) = \text{span}\{I,T,T^*\}\). More generally, an operator system is a unital selfadjoint subspace \(S\) of \(B(H)\). When one considers ucp maps as morphisms, an operator system \(S\) can be characterized by its sequence of positive cones \(M_n(S)_+\). Arveson's Extension Theorem \cite[Theorem 1.2.3]{2} guarantees that if \(S \subset B(H)\) is an operator system and \(\phi : S \to B(K)\) is completely positive, there exists a completely positive extension \(\tilde{\phi} : B(H) \to B(K)\). For any fixed operator system \(S\), the set of ucp maps \(S \to B(K)\) is BW-compact, where the BW-topology is that given by pointwise weak-operator convergence. Given \(T \in B(H)\), its numerical range is the set

\[\mathbb{W}_1(T) = \{f(T) : f \text{ is a state} = \{\text{Tr}(HT) : H \succeq 0, \ \text{Tr}(H) = 1}\}.\]

We note that the equality above is not entirely obvious, since the right-hand-side only accounts for the normal states. But since the normal states are the predual of \(B(H)\), any state is a weak* (that is, pointwise) limit of normal states; so, as \(\mathbb{W}_1(T)\) is closed—by an easy application of Banach–Alaoglu—, the set of all \(f(T)\) where \(f\) runs over all the states, is the same as the set of all \(f(T)\) where \(f\) runs over all the normal states.

The numerical range is always compact and convex. The numerical radius of \(T\) is the number

\[w(T) = \sup\{|\lambda| : \lambda \in \mathbb{W}_1(T)\}.\]

**Remark 2.2.** The numerical range is classically defined as

\[W(T) = \{\langle T \xi, \xi \rangle : \xi \in H\}.\]

It turns out that \(W(T)\) is always dense in \(\mathbb{W}_1(T)\). It is also convex, as proven in the Toeplitz–Hausdorff Theorem. The fact that \(\mathbb{W}_1(T)\) is convex, on the other hand, follows from a straightforward computation.

As it is common—although not standard—we will refer by “strong” convergence of a net, to convergence in the strong operator topology; and by “weak” convergence, to convergence in the weak operator topology.
3. The Matricial Range

The matricial range of $T \in B(H)$ is the sequence
\[ \mathcal{W}(T) = \{ \mathcal{W}_n(T) : n \in \mathbb{N} \}, \]
where
\[ \mathcal{W}_n(T) = \{ \varphi(T) : \varphi : \mathcal{O}(T) \rightarrow M_n(\mathbb{C}) \text{ is ucp} \}. \]

In light of Arveson’s Extension Theorem, the matricial range of $T$ does not change if we consider $C^*(T)$ or even $B(H)$ as the domain of the ucp maps in the definition of $\mathcal{W}_n(T)$. A classic survey on the topic is [12].

One is tempted to include the set
\[ \mathcal{W}_\infty(T) = \{ \phi(T) : \phi : \mathcal{O}(T) \rightarrow B(\ell^2(\mathbb{N})) \text{ is ucp } \} \]
(or even higher-dimensional versions in the non-separable case) in the list \{ $\mathcal{W}_n(T) : n \in \mathbb{N}$ \}. But we have the following:

**Proposition 3.1.** Let $S \in B(H)$, $T \in B(K)$. The following statements are equivalent:

1. $\mathcal{W}_n(S) = \mathcal{W}_n(T)$ for all $n \in \mathbb{N}$;
2. $\mathcal{W}_\infty(S) = \mathcal{W}_\infty(T)$.

**Proof.** Assume first that $\mathcal{W}_n(S) = \mathcal{W}_n(T)$ for all $n \in \mathbb{N}$. Let $X \in \mathcal{W}_\infty(S)$. So $X = \phi(S) \in B(\ell^2(\mathbb{N}))$ for some ucp map $\phi$. Let $\{ P_j \}$ be an increasing net of finite-dimensional projections with $P_j \rightarrow I$ strongly. Let $k(j)$ be the rank of $P_j$. We can think of $P_j XP_j \in M_{k(j)}(\mathbb{C})$. So $P_j XP_j = P_j \phi(S) P_j \in \mathcal{W}_{k(j)}(S) = \mathcal{W}_{k(j)}(T)$. Then there exists a ucp map $\psi_j : B(K) \rightarrow M_{k(j)}(\mathbb{C})$ with $\psi_j(T) = P_j XP_j$. Let $\psi$ be a BW-cluster point of the net $\{ \psi_j \}$. Then $\psi(T) = \lim_j \psi_j(T) = \lim_j P_j XP_j = X$, so $X \in \mathcal{W}_\infty(T)$. We have proven that $\mathcal{W}_\infty(S) \subset \mathcal{W}_\infty(T)$, and exchanging roles we get the equality.

Conversely, assume now that $\mathcal{W}_\infty(S) = \mathcal{W}_\infty(T)$. Fix $n \in \mathbb{N}$. Let $X \in \mathcal{W}_n(S)$. By identifying $M_n(\mathbb{C})$ with the “upper left corner” of $B(\ell^2(\mathbb{N}))$, we may assume $X \in \mathcal{W}_\infty(S) = \mathcal{W}_\infty(T)$. Then there exists a ucp map $\psi : B(K) \rightarrow B(\ell^2(\mathbb{N}))$ with $\psi(T) = X$. If $P$ is the projection of rank $n$ such that $X = PXP$, then $P \psi P$ can be seen as a ucp map $B(K) \rightarrow M_n(\mathbb{C})$. So $X \in \mathcal{W}_n(T)$. We have proven that $\mathcal{W}_n(S) \subset \mathcal{W}_n(T)$, and now reversing roles we get $\mathcal{W}_n(S) = \mathcal{W}_n(T)$. \qed

We will also consider briefly the spatial matricial range
\[ \mathcal{W}^s(T) = \{ \mathcal{W}_n^s(T) : n \in \mathbb{N} \}, \]
where
\[ \mathcal{W}_n^s(T) = \{ V^*TV : V : \mathbb{C}^n \rightarrow H \text{ isometry} \}. \]

The following result is due to Bunce–Salinas [7, Theorem 2.5]. The form we use is taken from [4] pp. 335–336; the proof follows mostly [8] Lemma
II.5.2], but we use a slightly sharper version of Glimm’s Lemma than the one used by Davidson.

**Lemma 3.2 (Bunce–Salinas).** Let \( \phi : B(H) \to M_n(\mathbb{C}) \) be ucp and such that \( \phi(L) = 0 \) for every compact operator \( L \), and let \( A \subset B(H) \) be a separable \( C^* \)-algebra. Then there exists a sequence of isometries \( V_k : \mathbb{C}^n \to H \) such that \( V_k \to 0 \) weakly and

\[
\| \phi(T) - V_k^*TV_k \| \to 0, \quad T \in A.
\]

**Proof.** For a fixed orthonormal basis \( \xi_1, \ldots, \xi_n \) of \( \mathbb{C}^n \), consider the map \( \Phi : M_n(A) \to \mathbb{C} \) given by

\[
\Phi(A) = \langle (\phi^n)(A)\xi, \tilde{\xi} \rangle,
\]

where \( \tilde{\xi} = \frac{1}{\sqrt{n}} [\xi_1 \cdots \xi_n]^\top \in (\mathbb{C}^n)^n \). The fact that \( \phi \) is ucp makes \( \Phi \) a state. By construction, \( \Phi(A) = 0 \) if all entries of \( A \) are compact; and the compact operators of \( M_n(A) \) are precisely the matrices where all entries are compact. Thus Glimm’s Lemma (see [8, Lemma II.5.1]), but here we use the exact form of [6, Lemma 1.4.11]) applies to the \( C^* \)-algebra \( M_n(A) \) and the state \( \Phi \), and we get an orthonormal sequence vectors \( \{\tilde{\eta}^k\} \subset H^n \), where \( \tilde{\eta}^k = [\eta^k_1 \cdots \eta^k_n]^\top \in H^n \) and

\[
(3.1) \quad \Phi(A) = \lim_{k \to \infty} \langle A\tilde{\eta}^k, \tilde{\eta}^k \rangle.
\]

Asymptotically, \( \{\sqrt{n}\eta^k_1, \ldots, \sqrt{n}\eta^k_n\} \) is orthonormal; indeed, using that \( \phi \) is unital and so \( \phi^n(I \otimes E_{h,j}) = I \otimes E_{h,j} \),

\[
\lim_{k \to \infty} \langle \sqrt{n}\eta^k_j, \sqrt{n}\eta^k_h \rangle = n \lim_{k \to \infty} \langle (I \otimes E_{h,j})\tilde{\eta}^k, \tilde{\eta}^k \rangle = n\Phi(I \otimes E_{h,j})
\]

\[
= n \langle (\phi^n)(I \otimes E_{h,j})\tilde{\xi}, \tilde{\xi} \rangle = \langle \xi_j, \xi_h \rangle = \delta_{j,h}.
\]

Define linear maps \( X_k : \mathbb{C}^n \to H \) by \( X_k\xi_j = \sqrt{n}\eta^k_j \). For \( \xi, \eta \in \mathbb{C}^n, \varepsilon > 0 \), and \( k \) big enough so that \( |\langle \sqrt{n}\eta^k_h, \sqrt{n}\eta^k_j \rangle - \delta_{h,j} | < \varepsilon \),

\[
|\langle (X_k^*X_k - I_n)\xi, \eta \rangle| = |\langle X_k\xi, X_k\eta \rangle - \langle \xi, \eta \rangle|
\]

\[
= \left| \sum_{h,j} \langle \xi, \xi_h \rangle \langle \eta, \xi_j \rangle \langle \sqrt{n}\eta^k_h, \sqrt{n}\eta^k_j \rangle \delta_{h,j} \right|
\]

\[
\leq \varepsilon \left| \sum_{h,j} \langle \xi, \xi_h \rangle \langle \eta, \xi_j \rangle \right| \leq \varepsilon n^2 \|\xi\| \|\eta\|.
\]

It follows that \( \|X_k^*X_k - I_n\| < \varepsilon n^2 \), so \( \|X_k^*X_k - I_n\| \to 0 \) as \( k \to \infty \). Now, for each \( k \), let \( X_k = V_k(X_k^*X_k)^{1/2} \) be the polar decomposition. For \( k \) big enough, \( X_k^*X_k \) is invertible, so for such \( k \),

\[
V_k^*V_k = (X_k^*X_k)^{-1/2}X_k^*X_k(X_k^*X_k)^{-1/2} = I_n.
\]
Hence $V_k : \mathbb{C}^n \to H$ is an isometry. Also, $\|V_k - X_k\| = \|X_k(\|X_k^*X_k\|^{-1/2} - I_n)\| \to 0$. And $V_k \to 0$ weakly since the range of $V_k$ is contained in the range of $X_k$, and $\{\tilde{\eta}_k\}$ is orthonormal. We have, by (3.1),

$$\langle (\phi(T) - X_k^*TX_k)\xi_h, \xi_j \rangle = n \langle \Phi(T \otimes E_{jh})\tilde{\xi}, \tilde{\xi} \rangle - n \langle (T \otimes E_{jh})\tilde{\eta}_k, \tilde{\eta}_k \rangle \xrightarrow{k \to \infty} 0.$$ 

As the above convergence is in $M_n(\mathbb{C})$, it also holds in norm. Thus

$$\lim_{k \to \infty} \|\phi(T) - V_k^*TV_k\| = \lim_{k \to \infty} \|\phi(T) - X_k^*TX_k\| \xrightarrow{k \to \infty} 0. \quad \square$$

**Remark 3.3.** Another set considered by Bunce–Salinas [7] is the essential matricial range. This would be

$$\mathcal{W}_n^e(T) = \{\phi(T) : \phi : B(H) \to M_n(\mathbb{C}) \text{ ucp}, \phi|_{K(H)} = 0\}.$$ 

By Lemma 3.2, $\mathcal{W}_n(T) \subset \mathcal{W}_n^e(T)$.

In analogy to the fact that the classical spatial numerical range is dense in the numerical range—minus the fact that $\mathcal{W}_n^e(T)$ is often not convex—we have the following result. The proof we provide does not follow the original argument.

**Proposition 3.4 (Bunce–Salinas [7]).** Let $T \in B(H)$. Then

$$\mathcal{W}_n(T) = \{\sum_k A_k^*XAk : X \in \mathcal{W}_n(T), A_1, A_2, \ldots \in M_n(\mathbb{C}), \sum_k A_k^*Ak = I_n\}.$$ 

In other words, the $C^*$-convex hull of the closure of the $n^{th}$ spatial matricial range of $T$ equals the $n^{th}$ matricial range of $T$. When $T$ is compact, the closure of $\mathcal{W}_n(T)$ is not needed. If $C^*(T) \cap K(H) = \{0\}$, then $\mathcal{W}_n(T) = \mathcal{W}_n(T)$.

**Proof.** Fix $n \in \mathbb{N}$. It is clear that $\mathcal{W}_n(T) \subset \mathcal{W}_n(T)$. Now consider a ucp map $\phi : B(H) \to M_n(\mathbb{C})$, and write $\phi = V^*\pi V$ for a Stinespring dilation, where $V : \mathbb{C}^n \to K$ is an isometry and $\pi : B(H) \to B(K)$ a representation. We want to show that $\phi(T)$ is of the form $\sum_k A_k^*XAk$ with $X \in \mathcal{W}_n^e(T)$, and $\sum_k A_k^*Ak = I_n$.

Since $\pi$ is bounded, its kernel is a closed two-sided ideal of $B(H)$. Thus there are just two possibilities: either $\pi$ is an isometry, or $ker \pi = K(H)$. Assume first that $\pi$ is an isometry. Then we can identify $\pi(T)$ with $T \otimes I$: indeed, it is not hard to show that there exist a Hilbert space $H_0$ and a unitary $U : K \to H \otimes H_0$ such that $\pi(T) = U^*(T \otimes I)U$. Let $\{F_{st}\}$ denote a set of matrix units for $H_0$, corresponding to the canonical basis $\{f_n\}$ (note that $H_0$ may be finite or infinite-dimensional). Then, with $W : H \otimes \mathbb{C}f_1 \to H$ the isometry $W(\xi \otimes \lambda f_1) = \lambda \xi$,

$$\langle W^*TW(\xi \otimes \lambda f_1), \eta \otimes \mu f_1 \rangle = \lambda \pi(T)\xi, \eta \rangle = \langle (T \otimes F_{11})(\xi \otimes \lambda f_1), \eta \otimes \mu f_1 \rangle.$$ 

For each $s$, let $H_s \subset H$ be the subspace $H_s = W(I \otimes F_{ks})U\mathbb{C}f_1$. We obviously have dim $H_s \leq n$. Let $R_s : \mathbb{C}^n \to H$ be an isometry that contains
$H_s$ in its range. So $R_s^* R_s = I_n$, and $R_s R_s^*$ is a projection that contains $H_s$ in its range. We have

$$\phi(T) = V^* U^* (T \otimes I) U V = V^* U^* \left( \sum_s T \otimes F_{ss} \right) U V$$

$$= V^* U^* \left( \sum_s (I \otimes F_{s1})(T \otimes F_{11})(I \otimes F_{1s}) \right) U V$$

$$= \sum_s V^* U^* (I \otimes F_{1s})^* W^* TW (I \otimes F_{1s}) U V$$

$$= \sum_s V^* U^* (I \otimes F_{1s})^* W^* R_s (R_s^* TR_s) R_s^* W (I \otimes F_{1s}) U V.$$

The convergence of the sum $\sum_s T \otimes F_{ss}$ is strong, but our last sum occurs in $M_n(\mathbb{C})$, so the convergence is in norm. Also,

$$\sum_s V^* U^* (I \otimes F_{1s})^* W^* R_s R_s^* W (I \otimes F_{1s}) U V = V^* U^* \left( \sum_s I \otimes F_{ss} \right) U V$$

$$= V^* U^* UV = I_n.$$

As $R_s^* TR_s \in \mathbb{W}_n^*(T)$ for all $s$, and $R_s^* W (I \otimes F_{1s}) U V \in M_n(\mathbb{C})$, we have shown that

$$\phi(T) \in \{ \sum_k A_k^* X A_k : X \in \mathbb{W}_n^*(T), A_1, A_2, \ldots \in M_n(\mathbb{C}), \sum_k A_k^* A_k = I_n \}$$

(note the lack of closure of the spatial matricial range).

In the case where $\pi = 0$ on $K(H)$, we have the same property for $\phi$, and we may apply \textbf{Lemma 3.2} to the separable C*-algebra $C^*(T)$; that way, we obtain isometries $V_k : \mathbb{C}^n \to H$ with $V_k^* TV_k \to \phi(T)$. So $\phi(T) \in \mathbb{W}_n^*(T)$.

When $T$ is compact, this last case does not apply, and so the closure of $\mathbb{W}_n^*(T)$ is not needed.

When $C^*(T) \cap K(H) = \{0\}$, the quotient map $\rho : B(H) \to B(H)/K(H)$ is isometric on $C^*(T)$. Thus the map $\phi \circ \rho^{-1} : \rho(C^*(T)) \to M_n(\mathbb{C})$ is ucp and maps $\rho(T)$ to $\phi(T)$. By Arveson’s Extension Theorem, there exists $\hat{\phi} : B(H)/K(H) \to M_n(\mathbb{C})$, ucp, that extends $\phi \circ \rho^{-1}$. Now $\hat{\phi} \circ \rho : B(H) \to M_n(\mathbb{C})$ is a ucp map that annihilates $K(H)$ and such that $\hat{\phi}(\rho(T)) = \phi(T)$. By \textbf{Lemma 3.2}, $\phi(T) \in \mathbb{W}_n^*(T)$. \hfill \square

The matricial range was initially defined and studied by Arveson \cite{Arveson}. It is straightforward to check that each $\mathbb{W}_n(T)$ is compact and C*-convex (the latter in the sense of \textbf{2b} in \textbf{Theorem 3.5}), and that $\mathbb{W}_n(X) \subset \mathbb{W}_m(T)$ for all $X \in \mathbb{W}_n(T)$. He mentions, after the definition, that “it is not hard” to see that the aforementioned properties characterize the matricial range. The proof we know and write below (\textbf{Theorem 3.5}) is not “very hard”, but it is not trivial either since it depends on \textbf{Proposition 3.4} that itself depends
on Glimm’s Lemma. Besides Arveson’s characterization—(2) below—we include a slightly more explicit characterization in terms of finite C*-convex combinations.

**Theorem 3.5.** Let \( \{X_n : n \in \mathbb{N}\} \) be a sequence of sets \( X_n \subset M_n(\mathbb{C}) \), and \( c > 0 \). Then the following statements are equivalent:

1. there exists a Hilbert space \( H \) and \( T \in B(H) \) such that \( ||T|| \leq c \) and \( X_n = W_n(T) \) for each \( n \in \mathbb{N} \);

2. the sequence \( \{X_n\} \) satisfies the following properties:
   a) for each \( n \), the set \( X_n \) is compact, and contained in the ball of radius \( c \);
   b) for each \( n \), if \( X_1, X_2, \ldots \subset X_n \) and \( A_1, A_2, \ldots \in M_n(\mathbb{C}) \) with \( \sum_k A_k^* A_k = I_n \), then \( \sum_k A_k^* X_k A_k \in X_n \);
   c) for each \( m, n \in \mathbb{N} \), \( W_m(X_n) \subset X_m \).

3. the sequence \( \{X_n\} \) satisfies the following properties:
   a) for each \( n \), the set \( X_n \) is compact, and contained in the ball of radius \( c \);
   b) for each \( n \), if \( X_1, X_2, \ldots, X_r \subset X_n \) and \( A_1, A_2, \ldots, A_r \in M_{n \times m}(\mathbb{C}) \) with \( \sum_k A_k^* A_k = I_m \), then \( \sum_k A_k^* X_k A_k \in X_m \).

**Proof.** (1) \( \Rightarrow \) (3) If \( X_n = W_n(T) \), as \( ||\phi(T)|| \leq ||T|| \leq c \) for any ucp map \( \phi \), it follows that \( X_n \) is contained in the ball of radius \( c \). Pointwise-norm limits of ucp maps are ucp; so \( X_n \) is the image of the BW-compact set \( \{\phi : B(H) \to M_n(\mathbb{C}), \text{ucp}\} \) under the (continuous) evaluation map \( \phi \mapsto \phi(T) \), and thus compact. If we have a sequence \( X_1, X_2, \ldots, X_r \subset W_n(T) \), there exist ucp maps \( \phi_k \) with \( X_k = \phi_k(T) \). For a sequence \( A_1, A_2, \ldots, A_r \in M_{n \times m}(\mathbb{C}) \) with \( \sum_k A_k^* A_k = I_m \), the map \( \phi := \sum_k A_k^* \phi_n(\cdot) A_k \) is ucp, and so \( \sum_k A_k^* X_k A_k = \phi(T) \in W_m(T) \in X_m \).

(3) \( \Rightarrow \) (2) Fix \( n \), \( X_1, X_2, \ldots \subset X_n \) and matrices \( A_1, A_2, \ldots \in M_n(\mathbb{C}) \) with \( \sum_k A_k^* A_k = I_n \). If only finitely many \( A_k \) are nonzero, then we get directly that \( \sum_k A_k^* X_k A_k \in X_n \). So assume that infinitely many \( A_k \) are nonzero. Choose \( \ell_0 \) such that \( ||I - \sum_{k=1}^\ell A_k^* A_k|| < 1 \) for all \( \ell > \ell_0 \). Then, for \( \ell > \ell_0 \),

\[
R_\ell = \sum_{k=1}^\ell (A_k R_\ell^{-1/2})^* X_k A_k R_\ell^{-1/2} \in X_n
\]

for all \( \ell > \ell_0 \). As \( R_\ell \to I_n \), we also have \( R_\ell^{-1/2} \to I_n \) and so, since \( X_n \) is closed, \( \sum_{k=1}^\infty A_k^* X_k A_k \in X_n \). To check that \( W_m(X_n) \subset X_m \), let \( Y \in W_m(X_n) \); so there exist \( X \in X_n \) and \( \psi : M_n(\mathbb{C}) \to M_m(\mathbb{C}) \), ucp, with \( Y = \psi(X) \). By the Kraus Decomposition, we may write \( \psi = \sum_k A_k^* \cdot A_k \), with \( A_1, \ldots, A_r \in M_{m \times n}(\mathbb{C}) \). Then \( Y = \sum_{k=1}^r A_k^* X A_k \in X_m \).

(2) \( \Rightarrow \) (1) For each \( n \in \mathbb{N} \), let \( \{Q_{n,k}\}_{k \in \mathbb{N}} \) be a countable dense subset of \( X_n \). Let \( H = l^2(\mathbb{N}) \), where we index the canonical orthonormal basis as

\[
\{\xi_{n,k,j} : n, k \in \mathbb{N}, j = 1, \ldots, n\}.
\]

For each \( n, k \in \mathbb{N} \) we denote the canonical basis in \( \mathbb{C}^n \) by \( \delta_1, \ldots, \delta_n \), and we define linear isometries \( W_{n,k} : \mathbb{C}^n \to H \) by
\[ W_{n,k} \delta_j = \xi_{n,k,j}, \quad j = 1, \ldots, n. \]

We have
\[
\langle W_{n,k}^* \xi_{m,\ell,h}, \delta_j \rangle = \langle \xi_{m,\ell,h}, W_{n,k} \delta_j \rangle = \langle \xi_{m,\ell,h}, \xi_{n,k,j} \rangle = \delta_{m,n} \delta_{\ell,k} \delta_{h,j} = \delta_{m,n} \delta_{\ell,k} \langle \delta_h, \delta_j \rangle,
\]

so
\[
W_{n,k} W_{n,k}^* \xi_{m,\ell,h} = \delta_{m,n} \delta_{\ell,k} W_{n,k} \delta_h = \delta_{m,n} \delta_{\ell,k} \xi_{n,k,h}.
\]

It follows that \( W_{n,k} W_{n,k}^* \) is the orthogonal projection onto the span of the vectors \( \xi_{n,k,1}, \ldots, \xi_{n,k,n} \). Also,
\[
W_{n,k}^* W_{n,k'} \delta_j = W_{n,k}^* \xi_{n',k',j} = \delta_{n,n'} \delta_{k,k'} \delta_j
\]
so \( W_{n,k}^* W_{n,k} = I_n \) and \( W_{n,k}^* W_{n,k'} = 0 \) if \( n \neq n' \) or \( k \neq k' \). Now define
\[
T = \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} W_{n,k} Q_{n,k} W_{n,k}^*.
\]

The series is well-defined—via strong convergence—because the projections \( \{ W_{n,k} W_{n,k}^* \} \) add to the identity of \( H \). It is clear that \( \|T\| \leq \max\{\|Q_{n,k}\| : n, k \} < c \). For any isometry \( V : \mathbb{C}^m \to H \), we have
\[
V^*TV = \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} V^* W_{n,k} Q_{n,k} W_{n,k}^* V.
\]

**Claim:** If \( X_j \in \mathcal{X}_{n(j)}, A_j \in M_{n(j) \times m}(\mathbb{C}), j \in \mathbb{N} \), then \( \sum_j A_j^* X_j A_j \in \mathcal{X}_m \).

As
\[
\sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} V^* W_{n,k} W_{n,k}^* V = V^* \left( \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} W_{n,k} W_{n,k}^* \right) V = V^* IV = V^* V = I_m
\]
and \( W_{n,k}^* V \in M_{n \times m}(\mathbb{C}) \) for each \( n \), the Claim above implies that \( V^*TV \in \mathcal{X}_m \).

It follows that \( W_n(T) \subset \mathcal{X}_n \). Then the C*-convexity \( [2b] \), compactness, and \( \boxed{\text{Proposition 3.4}} \) give us \( W_n(T) \subset \mathcal{X}_n \). From \( Q_{n,k} = W_{n,k}^* TW_{n,k} \) we obtain \( Q_{n,k} \in \mathcal{W}_n(T) \) for all \( k \in \mathbb{N} \), since the map \( X \mapsto W_{n,k}^* X W_{n,k} \) is ucp \( B(H) \to M_n(\mathbb{C}) \); the density of \( \{ Q_{n,k} \} \) then shows that \( \mathcal{X}_n \subset \mathcal{W}_n(T) \). Thus \( \mathcal{W}_n(T) = \mathcal{X}_n \) for all \( n \in \mathbb{N} \).

It remains to prove the Claim. We will prove the statement for a finite number of matrices, say \( 1 \leq j \leq r \); if we prove that, then an argument with an \( R_{\ell} \) like in the proof of \( [2] \to [3] \) shows the general result. So fix \( j \in \mathbb{N} \), with \( 1 \leq j \leq r \). Consider the ucp map \( \psi_j : M_{n(j)}(\mathbb{C}) \to M_{n(1)+\cdots+n(r)}(\mathbb{C}) \) given by
\[
\psi_j(X) = \begin{bmatrix}
\psi_{j1}(X) \\
\psi_{j2}(X) \\
\vdots \\
\psi_{jr}(X)
\end{bmatrix}
\]
where each $\psi_{jk} : M_{n(j)}(\mathbb{C}) \to M_{n(k)}(\mathbb{C})$ is some ucp map, and $\psi_{jj}$ is the identity, so $\psi_{jj}(X_j) = X_j$. By construction $\psi_{jj}$ is ucp, so \([2c]\) implies that $\psi_{j}(X_j) \in \mathcal{X}_{n(1)+\ldots+n(r)}$ for each $j = 1, \ldots, r$. This implies, by \([2b]\), that

\[
\begin{bmatrix}
X_1 \\
X_2 \\
\vdots \\
X_r
\end{bmatrix}
= \begin{bmatrix}
I_{n(1)} \\
0 \\
\vdots \\
0
\end{bmatrix} \psi_1(X_1) \begin{bmatrix}
I_{n(1)} \\
0 \\
\vdots \\
0
\end{bmatrix}
+ \cdots
+ \begin{bmatrix}
0 \\
\vdots \\
0 \\
I_{n(r)}
\end{bmatrix} \psi_r(X_r) \begin{bmatrix}
0 \\
\vdots \\
0 \\
I_{n(r)}
\end{bmatrix}
\in \mathcal{X}_{n(1)+\ldots+n(r)}.
\]

Now, by \([2c]\),

\[
\sum_{j=1}^{r} A_j^* X_j A_j = \begin{bmatrix}
A_1^* \\
A_2^* \\
\vdots \\
A_r^*
\end{bmatrix} X_1 \begin{bmatrix}
A_1 \\
A_2 \\
\vdots \\
A_r
\end{bmatrix} \in \mathcal{X}_m,
\]

since conjugation by $[A_1 \ldots A_r]^*$ is ucp.

In the conditions of Theorem 3.5 the radius of $\mathbb{W}_n(T)$ is actually $\|T\|$ for all $n \geq 2$. For $n = 1$, it is well-known that the radius could be $\frac{1}{2} \|T\|$ (as is the case when $T = E_{21}$). Concretely, define

$\nu_n(T) = \sup\{\|X\| : X \in \mathbb{W}_n(T)\}$.

**Proposition 3.6 (Smith–Ward [17]).** Let $T \in B(H)$. Then $\nu_n(T) = \|T\|$ for all $n \geq 2$.

**Proof.** For any ucp map $\phi : B(H) \to M_n(\mathbb{C})$, we have $\|\phi(T)\| \leq \|T\|$, so $\nu_n(T) \leq \|T\|$. Conversely, fix $\varepsilon > 0$. Choose $\xi \in H$ such that $\|\xi\| = 1$ and $\|T\xi\| > \|T\| - \varepsilon$. Let $H_0$ be an $n$-dimensional subspace of $H$ that contains $\xi$ and $T\xi$. We define $\psi : B(H) \to M_n(\mathbb{C})$ by $\psi(L) = P_{H_0} LP_{H_0}$ (with the usual identification $P_{H_0} B(H) P_{H_0} \simeq M_n(\mathbb{C})$). The map $\psi$ is ucp, and

$\nu_n(T) \geq \|\psi(T)\| = \|P_{H_0} TP_{H_0}\| \geq \|T\xi\| > \|T\| - \varepsilon.$

As $\varepsilon$ was arbitrary, we get $\nu_n(T) = \|T\|$.

The family of examples where $\mathbb{W}(T)$ can be found explicitly is fairly small. The most notable example is $\mathbb{W}(E_{21})$, as will be established independently in Corollaries 5.6 and 7.2.
is considered in \cite{19}. As could be expected, though, the case of normal operators is not hard.

**Proposition 3.7.** Let \( T \in B(H) \) be normal, and \( n \in \mathbb{N} \). Then

\[
\mathcal{W}_n(T) = \left\{ \sum_{j=1}^{k} \lambda_j H_j : k \in \mathbb{N}, H_j \geq 0, \lambda_j \in \sigma(T), \sum_{j} H_j = I_n \right\}.
\]

**Proof.** Since \( T \) is normal, we can identify \( C^*(T) \) with \( C(\sigma(T)) \). Given any decomposition \( \sum_{j=1}^{k} \lambda_j H_j \) as above, we can find positive linear functionals \( f_j \) (characters, actually) with \( f_j(T) = \lambda_j \). The map \( \psi : X \mapsto \sum_{j=1}^{k} f_j(X)H_j \) is a unital positive linear map \( C^*(T) \to M_n(\mathbb{C}) \). As the domain is abelian, \( \psi \) is completely positive \cite[Theorems 3.9 and 3.11]{15}. This shows the inclusion \( \supset \) above. Conversely, let \( \phi : B(H) \to M_n(\mathbb{C}) \) be ucp. Fix \( \varepsilon > 0 \). By the Spectral Theorem, we can find projections \( P_1, \ldots, P_k \in B(H) \) with \( \sum_{j} P_j = I \) and \( \lambda_1 \ldots, \lambda_k \in \sigma(T) \) with \( \|T - \sum_{j} \lambda_j P_j\| < \varepsilon \). Then

\[
\|\phi(T) - \sum_{j} \lambda_j \phi(P_j)\| = \|\phi(T) - \sum_{j} \lambda_j P_j\| < \varepsilon.
\]

As we can do this for each \( \varepsilon > 0 \), we have shown that \( \phi(T) \) is a limit of matrices of the form \( \sum_{j} \lambda_j H_j \) as above. \( \square \)

We continue with another of Arveson’s gems from \cite{3}. Given \( T \in B(H), S \in B(K) \), we say that \( S \) is a **compression** of \( T \) if there exists a projection \( P \in B(H) \) such that \( S \) is unitarily equivalent to \( PT|_{PH} \in B(PH) \). This definition agrees with the usual use of the word “compression” or “corner”, but it is important to emphasize the restriction aspect: for instance, in \( M_2(\mathbb{C}) \) the projection \( E_{11} \) is not a compression of \( I_2 \) in the above sense; or, for another example, the matrix unit \( E_{11} \in M_3(\mathbb{C}) \) is a compression of \( E_{11} \in M_3(\mathbb{C}) \), but not viceversa. The important thing to note is that the unitary implementing the unitary equivalence maps \( K \) onto \( PH \).

**Theorem 3.8 (Arveson).** Let \( T \in B(H), S \in B(K) \). The following statements are equivalent:

1. \( \mathcal{W}_n(S) \subset \mathcal{W}_n(T) \) for all \( n \in \mathbb{N} \);
2. for each \( n \in \mathbb{N} \) and \( A, B \in M_n(\mathbb{C}) \),
\[
\|A \otimes I + B \otimes S\| \leq \|A \otimes I + B \otimes T\|;
\]
3. for each finite-dimensional projection \( P \in B(K) \), there exists a \(*\)-representation \( \pi : C^*(T) \to B(H_\pi) \) such that \( PS|_{PK} \in B(PK) \) is unitarily equivalent to a compression of \( \pi(T) \);
4. there exists a \(*\)-representation \( \pi : C^*(T) \to B(H_\pi) \) such that \( S \) is a compression of \( \pi(T) \).

Moreover,
(5) When $S$ is normal, the above conditions are equivalent to $\sigma(S) \subset \mathcal{W}_1(T)$;

(6) when $T$ is compact and irreducible, the above conditions are equivalent to $S$ being unitarily equivalent to a compression of $T \otimes I$.

**Proof.** (1) $\implies$ (2) By repeating the argument in the first paragraph of the proof of Proposition 3.1, we can get a ucp map $\phi : OS(T) \to B(K)$ with $\phi(T) = S$. Then, for any $A, B \in M_n(\mathbb{C})$,

$$\|A \otimes I_K + B \otimes S\| = \|A \otimes \phi(I_H) + B \otimes \phi(T)\| = \|\phi^n(A \otimes I_H + B \otimes T)\| \leq \|A \otimes I_H + B \otimes T\|.$$

(2) $\implies$ (3) Define a linear map $\phi : \text{span}\{I_H, T\} \to \text{span}\{I_K, S\}$ by $\phi(I_H) = I_k, \phi(T) = S$. The condition (3.2) now says that $\phi$ is completely contractive. By Proposition 3.5, $\phi$ extends to a ucp map $\phi : \text{span}\{I_H, T, T^*\} \to B(K)$, and by Arveson’s Extension Theorem we may enlarge the domain of $\phi$ to be $B(H)$. Consider a Stinespring Dilation $\phi = PK \pi|_K$, where $\pi : B(H) \to B(K_{\pi})$ is a representation and $K \subset K_\pi$. For any projection $P \in B(K)$, since $PK \subset K = PK_{\pi}$,

$$PS|_{PK} = PP_K S|_{PK} = PP_K \pi(T)|_{PK} = P \pi(T)|_{PK}.$$ 

(3) $\implies$ (4) Let $\{P_j\} \subset B(K)$ be an increasing net of finite-rank projections that converges strongly to $I$. By hypothesis, for each $j$ there exist a unitary $V_j : P_j K \to Q_j H_j$, a projection $Q_j \in B(H_j)$, and a representation $\pi_j : C^*(T) \to B(H_j)$ such that

$$P_j S|_{P_j K} = V_j^* Q_j \pi_j(T)|_{Q_j H_j} V_j.$$ 

Fix a state $f$ of $C^*(T)$. Then the maps $\psi_j : C^*(T) \to B(K)$ given by

$$\psi_j(X) = V_j^* Q_j \pi_j(X)|_{Q_j H_j} V_j + f(X) (I - P_j)$$

are ucp. Let $\psi : C^*(T) \to B(K)$ be a BW-cluster point of the net $\{\psi_j\}$. It is clear that $\psi(T) = S$. Now a Stinespring decomposition of $\psi$ gives $S$ as a compression of $T$.

(4) $\implies$ (1) By hypothesis, there is a ucp map $\psi$ with $S = \psi(T)$. Given any $X \in \mathbb{W}_n(S)$, there exists a ucp map $\phi$ with $\phi(S) = X$. Then

$$X = \phi(S) = \phi \circ \psi(T) \in \mathbb{W}_n(T).$$

(5) Assume $S$ is normal. If $\sigma(S) \subset \mathbb{W}_1(T)$, then $\mathbb{W}_1(S) \subset \mathbb{W}_1(T)$ since $\mathbb{W}_1(S)$ is the closed convex hull of $\sigma(S)$. This allows us to show that the unital map $\psi : OS(T) \to OS(S)$ with $\psi(T) = S, \psi(T^*) = S^*$ is positive: indeed, if $\alpha I + \beta T + \gamma T^* \geq 0$ then $\mathbb{W}_1(\alpha I + \beta T + \gamma T^*) \subset [0, \infty)$ and, as

$$\sigma(\alpha I + \beta S + \gamma S^*) = \{\alpha + \beta \lambda + \gamma \bar{\lambda} : \lambda \in \sigma(S)\} \subset \{\alpha + \beta \lambda + \gamma \bar{\lambda} : \lambda \in \mathbb{W}_1(T)\} = \{f(\alpha I + \beta T + \gamma T^*) : \text{ state} \} \subset [0, \infty)$$
we obtain that $\alpha I + \beta S + \gamma S^* \geq 0$ (here we use that $S$ is normal to characterize positivity by its spectrum, and also for the first equality above). So $\psi$ is a unital positive map; as $C^*(S)$ is abelian, $\psi$ is ucp. Now we can use this $\psi$ to check that (2) or (4) hold. Conversely, if the equivalent conditions hold, we have $\mathcal{W}_1(S) \subset \mathcal{W}_1(T)$, and so $\sigma(S) \subset \mathcal{W}_1(S) \subset \mathcal{W}_1(T)$.

When $T$ is compact and (4) holds, the representation $\pi$ is nonzero on $T$, and so it has to be isometric on $C^*(T)$—as the only possible kernel is $K(H)$. Because $T$ is irreducible, $K(H) \subset C^*(T)$ (this is a more or less straightforward consequence of Kadison’s Transitivity Theorem; see [8, Theorem I.10.4]). So $\pi$ is isometric on $B(H)$. It is well-known that in this situation $\pi(T)$ is unitarily equivalent to $T \otimes I$. Conversely, if $S$ is unitarily equivalent to a compression of $T \otimes I$, we can recover (4).

It was proven by Hamana [13] (see [9] for a bit of history and a proof within Arveson’s framework) that every operator system admits a $C^*$-envelope. That is, given an operator system $S$, there exists a $C^*$-algebra $A$ and a complete isometry $j : S \to A$ such that for any $C^*$-algebra $B$ and any complete isometry $\psi : S \to B$, there exists a $C^*$-epimorphism $\pi : C^*(\psi(S)) \to A$ with $\pi \circ \psi = j$. That is, the following diagram commutes:

$$
\begin{array}{ccc}
S & \xrightarrow{\psi} & C^*(\psi(S)) \\
\downarrow j & & \downarrow \pi \text{ epimorphism} \\
A & & \\
\end{array}
$$

It is straightforward to prove that for a given operator system $S$ the $C^*$-algebra $A$ above is determined up to isomorphism, and so one denotes it by $C^*_e(S)$ and names it the $C^*$-envelope of $S$. By taking the quotient by the kernel of $\pi$ (the Šilov ideal, in Arveson’s terminology), we always have that $C^*_e(T)$ is a quotient of $C^*(T)$. In the particular case where an operator system $OS(T) \subset B(H)$ has the property that $\pi$ has trivial kernel (so, it is isometric), we say that $T$ is first order (this was Arveson’s original terminology; in later years he used the word reduced).

Let us now draw some consequences from Arveson’s result.

**Corollary 3.9.** Let $T \in B(H)$, $S \in B(K)$. The following statements are equivalent:

1. $\mathcal{W}(S) = \mathcal{W}(T)$;
2. for all $n \in \mathbb{N}$, for any $A, B \in M_n(\mathbb{C})$, $\|A \otimes I + B \otimes T\| = \|A \otimes I + B \otimes S\|$;
3. $OS(S) \simeq OS(T)$ via a complete isometry $\phi$ with $\phi(S) = T$.

If both $S, T$ are irreducible, first order, and both $C^*(S)$ and $C^*(T)$ contain a nonzero compact operator, then the above statements are also equivalent to

4. $S$ and $T$ are unitarily equivalent.
Proof. The equivalences (1) \iff (2) \iff (3) follow directly from Theorem 3.8. The implication (4) \iff (3) is trivial. So assume that \( S,T \) are irreducible, first order, that their \( C^* \)-algebras contain nonzero compact operators, and that there exists a complete isometry \( \phi : C^*(T) \rightarrow C^*(S) \) with \( \phi(T) = S \).

Considering the diagram (3.3) for \( S = OS(S) \) and \( \psi = \phi^{-1} \), and for \( S = OS(T) \) and \( \psi = \phi \) respectively, one deduces that \( C^*_e(T) \simeq C^*_e(S) \) via an isomorphism \( \pi \) with \( \pi(T) = S \). As \( T \) is irreducible and \( C^*(T) \) contains a compact operator, it follows that \( K(H) \subset C^*(T) \) (as mentioned above, see [8, Corollary I.10.4]). Similarly, \( C^*(S) \) contains all compacts of \( B(K) \). Recall that we are assuming that both \( T \) and \( S \) are first order, so \( C^*(T) = C^*_e(T) \) and \( C^*(S) = C^*_e(S) \).

Because \( \pi \) is an irreducible representation of \( C^*(T) \) and \( J = K(H) \subset C^*(T) \) with \( \pi|_J \neq 0 \), we have that \( \pi|_J \) is irreducible. Indeed, let \( \xi \in K \), and consider the subspace \( \pi(J)\xi \subset K \). Since \( J \) is an ideal, \( \pi(J)\xi \) is invariant for \( \pi(C^*(T)) = C^*(S) \); as \( C^*(S) \) is irreducible, it follows that \( \pi(J)\xi = K \) or \( 0 \). If it were 0, we would have \( \xi \in [\pi(J)K]^- \). But then \( \pi(J)K \subset K \) and it is invariant for \( C^*(S) \)—which is irreducible—so \( \pi(J) = 0 \), a contradiction. Thus \( \pi(J)\xi = K \) for all \( \xi \), and so \( \pi(J) \) has no reducing subspaces.

Because both the domain and the range of \( \pi \) contain their respective compact operators, \( \pi \) necessarily maps rank-one projections to rank-one projections. We will show that this implies that \( \pi \) is implemented by unitary conjugation. Fix an orthonormal basis \( \{\xi_j\} \) of \( H \). Choose a unit vector \( \eta_1 \in \pi(\xi_1 \otimes \xi_1)H \), and define

\[ \eta_j = \pi(\xi_j \otimes \xi_1)\eta_1. \]

One then checks easily that \( \{\eta_j\} \) is orthonormal; and it has to be a basis, because a rank-one projection corresponding to a vector orthogonal to \( \{\eta_j\} \) would get brought back by \( \pi \) to a rank-one projection with range orthogonal to \( \{\xi_j\} \), an impossibility. Now define a unitary \( U : H \rightarrow K \) by

\[ U\xi_j = \eta_j. \]

Then

\[ U^*\pi(\xi_j \otimes \xi_k)U\xi_\ell = U^*\pi(\xi_j \otimes \xi_k)\eta_\ell = U^*\pi(\xi_j \otimes \xi_k)\pi(\xi_\ell \otimes \xi_1)\eta_1 \]

\[ = \delta_{\ell,k} U^*\pi(\xi_j \otimes \xi_1)\eta_1 = \delta_{\ell,k} U^*\xi_j = \delta_{\ell,k} \xi_j = (\xi_j \otimes \xi_k)\xi_\ell. \]

Thus \( U^*\pi U \) is the identity on all rank-one operators; by linearity and continuity, it is the identity on all of \( K(H) \). For an arbitrary \( X \in C^*(T) \) and \( \xi \in H \), let \( P \) be the rank-one projection with \( \pi(X) \xi = \xi \). Then

\[ U^*\pi(X)U\xi = U^*\pi(X)UP\xi = U^*\pi(X)UU^*\pi(P)U\xi \]

\[ = U^*\pi(XP)U\xi = X\pi(P) = X\xi. \]

So \( U^*\pi(X)U = X \), that is \( \pi(X) = UXU^* \) for all \( X \in C^*(T) \). In particular,

\[ S = \pi(T) = UTU^*. \]
Remark 3.10. The requirement in (3) above that \( \phi(S) = T \) cannot be relaxed. For example, with
\[
S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},
\]
we have \( \mathcal{OS}(S) = \mathcal{OS}(T) \), but \( \mathcal{W}_1(S) = [0, 1] \) and \( \mathcal{W}_1(T) = [1, 2] \). Also, we refer to Corollary 3.15 for examples of operators with the same matricial range but very far from unitarily equivalent.

Remark 3.11. The conditions in Corollary 3.9 do not imply the equality of the spatial matricial ranges \( \mathcal{W}^s(S) \) and \( \mathcal{W}^s(T) \). The equality \( \mathcal{W}^s(S) = \mathcal{W}^s(T) \) implies \( \mathcal{W}(S) = \mathcal{W}(T) \) by Proposition 3.4, but the converse is not true. For instance consider \( H = K = \ell^2(\mathbb{N}) \), take \( S \) to be the unilateral shift with respect to the canonical basis \( \{ \xi_k \} \), and let \( T \) be the unitary given by \( T\xi_k = \gamma_k \xi_k \), where \( \{ \gamma_k \} \) is a dense sequence in \( \mathbb{T} \). By Corollary 3.15 below, we have \( \mathcal{W}(S) = \mathcal{W}(T) \). Any isometry \( V : \mathbb{C}^2 \to H \) is given by \( Ve_1 = x, Ve_2 = y \), where \( \{ x, y \} \subset H \) is orthonormal; we will write \( V_{x,y} \) for such an isometry. It is not hard to check that, for any \( R \in B(H) \),
\[
V_{x,y}^*RV_{x,y} = \begin{bmatrix} \langle x,Rx \rangle & \langle y,Rx \rangle \\ \langle x,Ry \rangle & \langle y,Ry \rangle \end{bmatrix}.
\]
By choosing the sequence \( \{ \gamma_k \} \) with \( \gamma_1 = \gamma_2 = 1 \) and taking \( x = \xi_1, y = \xi_2 \), we get
\[
I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = V_{x,y}^*TV_{x,y} \in \mathcal{W}^s_2(T).
\]
But, while \( I_2 \in \mathcal{W}^s_2(S) \), we have \( I_2 \not\in \mathcal{W}^s_2(S) \). Indeed, if we had \( I_2 = V_{x,y}^*SV_{x,y} \) for orthonormal \( x, y \in H \), then \( V_{x,y}V_{x,y}^* = P \), the orthogonal projection onto the span of \( \{ x, y \} \). So
\[
P = V_{x,y}V_{x,y}^* = V_{x,y}I_2V_{x,y}^* = V_{x,y}V_{x,y}^*SV_{x,y}V_{x,y}^* = PSP.
\]
This equality cannot hold, because we would have
\[
PS^*SP = P = PS^*PSP,
\]
which implies that \( PS^*(I-P)SP = 0 \), and so \( (I-P)SP = 0 \), from where \( SP = PSP = P \). This would make \( x \) and \( y \) eigenvectors for \( S \), a contradiction. It is easy to see, on the other hand, that \( I_2 \not\in \mathcal{W}^s_2(S) \), so it is not clear at first sight whether \( \mathcal{W}^s_2(T) \neq \mathcal{W}^s_2(S) \) or not.

Let us now specialize the above result to the case of matrices. We mention [11] for a different and detailed proof of the equivalence (3) \( \iff \) (5) in Corollary 3.12 below.

Corollary 3.12. Let \( n \in \mathbb{N} \) and \( S, T \in M_n(\mathbb{C}) \). The following statements are equivalent:

1. \( \mathcal{W}(S) = \mathcal{W}(T) \);
2. \( \mathcal{W}(S) = \mathcal{W}_n(T) \);
If both $A, B \in M_n(\mathbb{C})$, \[\|A \otimes I + B \otimes T\| = \|A \otimes I + B \otimes S\|;\]

(4) $\mathcal{OS}(S) \simeq \mathcal{OS}(T)$ via a complete isometry $\phi$ with $\phi(T) = S$.

If both $S, T$ are irreducible, the above statements are also equivalent to

(5) $S$ and $T$ are unitarily equivalent.

Proof. The equivalences (1) $\iff$ (3) $\iff$ (4) $\iff$ (5) follow directly from

Corollary 3.9 (note that $M_n(\mathbb{C})$ is simple, so the irreducibility of $S$ and $T$

imply that they are first order). The implication (1) $\implies$ (2) is trivial, so all

that remains is to prove (2) $\implies$ (1). Assume that $\mathcal{W}_n(S) = \mathcal{W}_n(T)$, and let

$X \in \mathcal{W}_m(S)$. So there exists a ucp map $\varphi : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ with $\varphi(S) = X$. As $S \in \mathcal{W}_m(S) = \mathcal{W}_n(T)$, there exists a ucp map $\psi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$

with $\psi(T) = S$. Then

$X = \phi(S) = \phi(\psi(T)) \in \mathcal{W}_m(T)$.

It follows that $\mathcal{W}_m(S) \subset \mathcal{W}_m(T)$, and by reversing the roles of $S$ and $T$ we get equality.

Remark 3.13. The reason one requires irreducibility for unitary equivalence in

Corollary 3.12 is multiplicity: for an easy example, we can take $S = E_{11}$,

$T = E_{11} + E_{22}$ in $M_3(\mathbb{C})$ and then $\mathcal{W}(S) = \mathcal{W}(T)$ but they are obviously not

unitarily equivalent.

We will later use some sophisticated ideas—mainly by Arveson and by

Ando both building on ideas related to dilations—to calculate the matricial

range of the $2 \times 2$ unilateral shift (Corollaries 5.6 and 7.2). The matricial

range of the $n \times n$ unilateral shift is unknown for $n \geq 3$, but the infinite-

dimensional unilateral shift (and, a posteriori, proper isometries) can be

tackled with a rather direct approach. We are grateful to D. Farenick for a

simplification of our original argument.

Proposition 3.14. Let $T \in B(K)$ with $\|T\| \leq 1$, and $S \in B(H)$ the

unilateral shift. Then there exists a ucp map $\psi : B(H) \to B(K)$ with $\psi(S) = T$.

Proof. Since $T$ is a contraction, we can construct a unitary

\[U_0 = \begin{bmatrix} T & (I - TT^*)^{1/2} \\ (I - T^*T)^{1/2} & -T^* \end{bmatrix} \in M_2(B(K)).\]

If $U$ is a universal unitary (that is, $\sigma(U) = \mathbb{T}$), then $C^*(U) = C(\mathbb{T})$. As

$\sigma(U_0)$ is a compact subset of $\mathbb{T}$, there is a $*$-epimorphism (onto by Tietze’s

Extension Theorem) acting by restriction:

$\pi : C^*(U) = C(\mathbb{T}) \to C(\sigma(U_0)) = C^*(U_0)$,

with $\pi(U) = U_0$. Let $\rho : B(H) \to B(H)/K(H)$ be the quotient map. As

$S$ becomes a universal unitary in the Calkin algebra, $C^*(\rho(S)) \simeq C(\mathbb{T}) \simeq C^*(U)$. Let $\gamma : C^*(\rho(S)) \to C^*(U)$ be a $*$-isomorphism with $\gamma(\rho(S)) = U$.

Then $\pi \circ \gamma : C^*(\rho(S)) \to C^*(U_0)$ is a $*$-epimorphism with $\pi \circ \gamma(\rho(S)) = U_0$;
in particular, ucp. Let \( \phi : M_2(B(K)) \to B(K) \) be the compression to the 1,1 entry. Then \( \psi = \phi \circ \pi \circ \rho : B(H) \to B(K) \) is a ucp map with \( \phi \circ \pi \circ \rho(S) = T \).

**Corollary 3.15.** Let \( S \in B(H) \) be a proper isometry, or a unitary with full spectrum \( T \), and let \( T \in B(K) \). Then the following statements are equivalent:

1. there exists a ucp map \( \phi : B(H) \to B(K) \) such that \( \phi(S) = T \);
2. \( \| T \| \leq 1 \).

In other words, \( \mathcal{W}_\infty(S) = \{ T : \| T \| \leq 1 \} \), and so for all \( n \in \mathbb{N} \), \( \mathcal{W}_n(S) = \{ A \in M_n(\mathbb{C}) : \| A \| \leq 1 \} \).

**Proof.** (1) \( \Rightarrow \) (2) Since \( \| S \| = 1 \) and \( \phi \) is ucp, we get that \( \| T \| = \| \phi(S) \| \leq 1 \).

(2) \( \Rightarrow \) (1) Let \( S_0 \) be the unilateral shift. By Proposition 3.14 there exists a ucp map \( \psi \) with \( \psi(S_0) = T \). If \( S \) is a proper isometry, by the Wold Decomposition there exist unitaries \( U \) and \( W \) such that \( S = W^*(U \oplus \bigoplus_j S_0)W \). Then

\[
S \mapsto WSW^* = U \oplus \bigoplus_j S_0 \mapsto T
\]

is a ucp map. If \( S \) is a unitary with full spectrum, we can proceed as in the proof of Proposition 3.14 to get a ucp map \( \phi \) with \( \phi(S) = T \). \( \square \)

## 4. Unitary Dilations, Numerical Radius, and Matricial Range

Advances in operator theory—concretely, about dilations—gave between in late 1960s and the early 1970s several striking characterizations of the numerical radius and, as a byproduct, a characterization of the matricial range of the \( 2 \times 2 \) unilateral shift. We will visit, in Sections 5 and 7 respectively, Ando’s and Arveson’s techniques built upon these theories.

Given a group \( G \), a function \( T : G \to B(H) \) is said to be positive-definite if

\[
\sum_{s \in G} \sum_{t \in G} \langle T(t^{-1}s)\xi(s),\xi(t) \rangle \geq 0
\]

for all functions \( \xi : G \to H \) of finite support. It is not hard to see that (4.1) implies that \( T(s^{-1}) = T(s)^* \) for all \( s \in G \).

A particular case of a positive-definite function is given by a unitary representation. That is, a function \( U : G \to B(H) \) such that \( U(e) = I \), \( U(s) \) is a unitary for all \( s \in G \), and \( U(st) = U(s)U(t) \) for all \( s, t \in G \). It turns out that one can do a kind of GNS representation for a positive-definite function, and so all positive-definite functions arise from unitary representations. We will only need the particular case where \( G = \mathbb{Z} \).

**Theorem 4.1 (Sz.Nagy–Foiaș).** Let \( H \) be a Hilbert space, and \( T : \mathbb{Z} \to B(H) \) with \( T(0) = I \). The following statements are equivalent:
There exists a Hilbert space $K \supset H$ and a unitary $U \in B(K)$ such that $T(n) = P_H U^n |_H$ for all $N \in \mathbb{Z}$.

(2) $T$ is positive-definite.

Proof. This is [18, Theorem 7.1].

Remark 4.2. In the case where $H = \mathbb{C}^2$ and $T = T(1) = E_{21}$, $T(n) = 0$ for all $n \geq 2$, the unitary $U$ can be obtained explicitly as the bilateral shift. Concretely, we take $K = \ell_2(\mathbb{Z})$, and define $U$ on the canonical basis by $Ue_k = e_{k+1}$, extended by linearity and continuity (since $U$ is isometric). If we identify $H = \mathbb{C}^2$ with $\text{span}\{e_0, e_1\}$, then $P_H U^n |_H = E_{12}$ for all $n \in \mathbb{N}$ (which simply means that $P_H U^1 |_H = E_{12}, P_H U^2 |_H = 0$).

Most considerations of the numerical radius will use the following elementary characterization:

Proposition 4.3. Let $T \in B(H)$ with $\|T\| \leq 1$. The following statements are equivalent:

(1) $w(T) \leq 1$;
(2) for all $\lambda \in \mathbb{T}$, $I + \text{Re} \lambda T \geq 0$;
(3) for all $\lambda \in \mathbb{T}$, $\text{Re} \lambda T \leq I$;
(4) for all $z \in \mathbb{D}$, $\text{Re} zT \leq I$.

Proof. (1) $\Rightarrow$ (2) If $w(T) \leq 1$, then for any $\lambda \in \mathbb{T}$ and $\xi \in H$ with $\|\xi\| = 1$,

$$(- \text{Re} \lambda T \xi, \xi) = \text{Re}(((-\lambda)T \xi, \xi) \leq |<(-\lambda)T \xi, \xi>| = |<T \xi, \xi>| \leq 1 = \langle \xi, \xi \rangle.$$ 

Thus $\langle (I + \text{Re} \lambda T) \xi, \xi \rangle \geq 0$.

(2) $\Rightarrow$ (3) This is a direct consequence of the fact that $\mathbb{T} = -\mathbb{T}$.

(3) $\Rightarrow$ (4) If $z \in \mathbb{D}$, then $z = r\lambda$ with $0 \leq r < 1$ and $\lambda \in \mathbb{T}$. So

$$\text{Re} zT = r \text{Re} \lambda T \leq rI \leq I.$$ 

(4) $\Rightarrow$ (1) Given $\xi \in H$ with $\|\xi\| = 1$, let $\lambda \in \mathbb{T}$ such that $|<T \xi, \xi>| = \lambda |<T \xi, \xi|$. Then

$$|<T \xi, \xi>| = \text{Re} |<T \xi, \xi| = \text{Re} \lambda |<T \xi, \xi|$$

$$= (\text{Re} \lambda T \xi, \xi) = \lim_{r \to 1} (\text{Re} r \lambda T \xi, \xi) \leq |<\xi, \xi>| = 1.$$ 

Thus $w(T) \leq 1$.

We state and prove a version of a characterization of unitary dilations, due to Sz.-Nagy and Foias [18, Theorem 11.1]. In their terminology, we only consider 2-dilations.

Theorem 4.4 (Sz.Nagy-Foias). Let $T \in B(H)$. The following statements are equivalent:

(1) There exists a Hilbert space $K \supset H$ and a unitary $U \in B(K)$ such that $T(n) = P_H U^n |_H$ for all $N \in \mathbb{Z}$.

(2) $T$ is positive-definite.

Proof. This is [18, Theorem 7.1].
there exists a Hilbert space \( K \supset H \), and a unitary \( U \in B(K) \) such that
\[
T^n = 2 P_H U^n |_H \quad \text{for all } n \in \mathbb{N};
\]
for all \( n \in \mathbb{N} \).

(2) \( w(T) \leq 1 \).

Proof. (1) \( \implies \) (2) Fix \( z \in \mathbb{D} \). Since \( |z| < 1 \), the series below converges and we can manipulate it as follows:
\[
I + 2 \sum_{k=1}^{\infty} z^k U^k = -I + 2 \sum_{k=0}^{\infty} z^k U^k = -I + 2(I - zU)^{-1} = (I + zU)(I - zU)^{-1}.
\]
Then
\[
(I_H - zT)^{-1} = I_H + \sum_{k=1}^{\infty} z^k T^k = P_H \left( I_H + 2 \sum_{k=1}^{\infty} z^k U^k \right)|_H \tag{4.3}
\]
\[
= P_H (I + zU)(I - zU)^{-1}|_H.
\]

For any \( \xi \in K \),
\[
\text{Re}(\langle (I + zU)\xi, (I - zU)\xi \rangle) = (1 - |z|^2)\|\xi\|^2 \geq 0.
\]

In particular, with \( \xi = (I - zU)^{-1}\eta \), we obtain
\[
(\text{Re}(I_H - zT)^{-1}\xi, \xi) = \text{Re}(\langle (I_H - zT)^{-1}\xi, \xi \rangle)
\]
\[
= \text{Re}(P_H (I + zU)(I - zU)^{-1}\xi, \xi)
\]
\[
= \text{Re}((I + zU)(I - zU)^{-1}\xi, \xi) \geq 0.
\]

Replacing \( \xi \) with \( (I_H - zT)\xi \), the above becomes
\[
\langle \xi, \text{Re}(I_H - zT)\xi \rangle \geq 0,
\]
and so \( \text{Re}(I_H - zT) \geq 0 \). Now [Proposition 4.3](#) gives \( w(T) \leq 1 \).

(2) \( \implies \) (1) Since \( w(T) \leq 1 \), we have \( \sigma(T) \subset \mathcal{W}_1(T) \subset \mathbb{D} \). Then \( I - zT = z(I - T) = z(T - I) \) is invertible for all \( z \in \mathbb{D} \) (when \( z = 0 \), \( I - zT = I \)). For any \( z \in \mathbb{D} \),
\[
\|z^n T^n\|^{1/n} \to \text{spr} (zT) = |z| \text{spr} (T) \leq |z| < 1.
\]

In particular, for \( \varepsilon < 1 - |z| \), there exists \( n_0 \) such that \( \|z^n T^n\| \leq (|z| + \varepsilon)^n < 1 \) for all \( n \geq n_0 \). Thus the series \( \sum_{k=0}^{\infty} z^n T^n \) is norm convergent and equal to \((I - zT)^{-1}\). Let \( \xi \in H \), and put \( \eta = (I - zT)^{-1}\xi \). By [Proposition 4.3](#) we have \( \text{Re}(I - zT) \geq 0 \) for all \( z \in \mathbb{D} \); thus
\[
0 \leq \text{Re}(\eta, (I - zT)\eta) = \text{Re}(\langle (I - zT)^{-1}\xi, (I - zT)(I - zT)^{-1}\xi \rangle)
\]
\[ Q(r, t) = I + \frac{1}{2} \sum_{k=1}^{\infty} r^k (e^{ikt}T^k + e^{-ikt}T^*k). \]

This converges in norm by the argument with the spectral radius we just used above. Also, with \( z = re^{it} \),

\[ \langle Q(r, t) \xi, \xi \rangle = \| \xi \|^2 + \frac{1}{2} \text{Re} \left( \sum_{k=0}^{\infty} z^k T^k \xi, \xi \right) = \| \xi \|^2 + \frac{1}{2} \text{Re} \langle (I - zT)^{-1} \xi, \xi \rangle \geq 0. \]

Given a sequence \( \{\xi_n\}_{n \in Z} \) with finite support, let us form \( \xi(t) = \sum_{n \in Z} e^{-int} \xi_n. \)

Then (recall that the sum over \( n \) has finitely many nonzero terms, so the exchange with the integral is not an issue; for the sum over \( k \), the convergence is uniform and the exchange is again possible)

\[
0 \leq \frac{1}{2\pi} \int_0^{2\pi} \langle Q(r, t) \xi(t), \xi(t) \rangle \, dt \\
= \sum_{m,n \in Z} \frac{1}{2\pi} \int_0^{2\pi} e^{-i(n-m)t} \langle \xi_n + \frac{1}{2} \sum_{k=1}^{\infty} r^k e^{ikt}T^k \xi_n + r^k e^{-ikt}T^*k \xi_n, \xi_m \rangle \, dt \\
= \sum_{n \in Z} \langle \xi_n, \xi_n \rangle + \frac{1}{2} \sum_{n>m} r^{n-m} \langle T^{n-m} \xi_n, \xi_m \rangle + \frac{1}{2} \sum_{n<m} r^{m-n} \langle T^{m-n} \xi_n, \xi_m \rangle
\]

(for the last equality, note that the integrals will be nonzero only when \(-n + m + k = 0\) and \(-n + m - k = 0\); this fixes \( k \), and we also have the restriction that \( k \geq 1 \), which makes the case \( n = m \) vanish). Now define a function \( T: \mathbb{Z} \to B(H) \) by

\[
T(0) = I, \quad T(n) = \frac{1}{2} T^n, \quad T(-n) = \frac{1}{2} T^{*n}, \quad n \geq 1.
\]

We can rewrite the inequality above as

\[
0 \leq \sum_{n \in Z} \langle T(n-n) \xi_n, \xi_n \rangle + \sum_{n>m} r^{n-m} \langle T(n-m) \xi_n, \xi_m \rangle + \sum_{n<m} r^{m-n} \langle T(n-m) \xi_n, \xi_m \rangle.
\]

Noting that all sums are finite by hypothesis, we may take \( r \not\to 1 \), and then

\[
0 \leq \sum_{n,m \in Z} \langle T(n-m) \xi_n, \xi_m \rangle,
\]
which shows that the function \( n \mapsto T(n) \) is positive-definite. By Theorem 4.1 there exists a Hilbert space \( K \supset H \) and a unitary \( U \in B(K) \) such that \( T(n) = P_H U^n |_H \) for all \( n \in \mathbb{Z} \). In particular,

\[
T^n = 2^n T(n) = 2^n P_H U^n |_H, \quad n \in \mathbb{N}.
\]

\[\square\]

5. Ando’s Characterizations of the Numerical Radius

The following surprising characterization is [1, Lemma 1]. We do not think that “lemma” is a fair word to describe it. The proof follows closely that of Ando, but we have tried to make it a bit clearer.

**Theorem 5.1.** Let \( T \in B(H) \) with \( w(T) \leq 1 \). Then there exists \( X \in B(H)^+ \), contractive (i.e., \( 0 \leq X \leq I \)) such that for all \( \xi \in H \)

\[
\langle X \xi, \xi \rangle = \inf \left\{ \left\langle \left[ \begin{array}{cc} I_{\frac{1}{2}} T & \frac{1}{2} T^n \\ I_{\frac{1}{2}} X & \frac{1}{2} X \\
\end{array} \right] \left[ \begin{array}{c} \xi \\ \eta \\
\end{array} \right], \left[ \begin{array}{c} \xi \\ \eta \\
\end{array} \right] \right\rangle : \eta \in H \right\}.
\]

Such \( X \) satisfies

\[
X = \max \left\{ Y \in B(H) : 0 \leq Y \leq I, \left[ \begin{array}{cc} I_{\frac{1}{2}} T & \frac{1}{2} T^n \\ I_{\frac{1}{2}} Y & \frac{1}{2} Y \\
\end{array} \right] \geq 0 \right\}.
\]

**Proof.** We assume that \( w(T) \leq 1 \). By Theorem 4.4 there exist a Hilbert space \( K \supset H \) and a unitary \( U \) in \( B(K) \) such that \( T^n = 2 P_H U^n |_H, \; k \in \mathbb{N} \). We will define a sequence \( \{X_n\} \subset B(H) \) in the following way: we start with \( X_0 = I \), and put \( X_n = P_H (I - Q_n) P_H \), where \( Q_n \) is the projection onto \( \overline{\text{span}} \bigcup_{k=1}^n U^k H \). Since by construction these subspaces are increasing on \( n \), we have

\[
I = X_0 \geq X_1 \geq X_2 \geq \cdots \geq 0.
\]

So the sequence converges strongly to a positive operator \( X \in B(H) \)—note that we could have defined \( X \) directly, but it provides no obvious benefit to the proof. For \( \xi \in H \), we have

\[
\langle U^k \xi, \xi \rangle = \langle U^k \xi, P_H \xi \rangle = \langle P_H U^k |_H \xi, \xi \rangle = \frac{1}{2} \langle T^k \xi, \xi \rangle, \quad k \in \mathbb{N}.
\]

Define \( A \in M_{n+1}(B(H)) \) by

\[
A_{kj} = \begin{cases} I, & j = k \\ \frac{1}{2} T^{(j-k)}, & k < j \\ \frac{1}{2} T^{(k-j)}, & k > j \\
\end{cases}
\]

We can write, for \( \xi \in H \), and using \( \xi_0 = \xi \),

\[
\langle X_n \xi, \xi \rangle = \langle (I - Q_n) \xi, \xi \rangle = \| (I - Q_n) \xi \|^2 = \| \xi - Q_n \xi \|^2 = \text{dist} (\xi, Q_n H)^2
\]

\[
= \inf \left\{ \left\| \xi + \sum_{k=1}^n U^k \xi_k \right\|^2 : \xi_1, \ldots, \xi_n \in H \right\}
\]
We can write the matrix $A$ as

$$
A = \begin{pmatrix}
I & T^* & \cdots & \cdots & T^{*(n-1)} & \frac{1}{2} T^n \\
0 & I & \cdots & \cdots & 0 & \frac{1}{2} T \ T^* \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \cdots & I & \frac{1}{2} T^* & \vdots \\
0 & 0 & \cdots & 0 & I & \frac{1}{2} T \ I
\end{pmatrix}
$$

As the invertible triangular block matrix preserves the first entry of the $n$-tuple vector, and as the infimum is taken over all $n$-tuples in $H$, we obtain (5.3)

$$
\langle X_n \xi, \xi \rangle = \inf \left\{ \left\| \sum_{k=0}^{n} U^{*k} \xi_k \right\|^2 : \xi_1, \ldots, \xi_n \in H \right\}
$$

$$
= \inf \left\{ \sum_{k,j=0}^{n} \langle U^{*(k-j)} \xi_k, \xi_j \rangle : \xi_1, \ldots, \xi_n \in H \right\}
$$

$$
= \inf_{\xi_j \in H} \left\{ \sum_{k=0}^{n} \langle \xi_k, \xi_k \rangle + \sum_{j<k} \langle \xi_k, \xi_j \rangle + \sum_{k<j} \langle \xi_k, \xi_j \rangle \right\}
$$

$$
= \inf_{\xi_j \in H} \left\{ \sum_{k,j=0}^{n} \langle \xi_k, \xi_j \rangle + \sum_{j<k} \frac{1}{2} \langle T^{*(k-j)} \xi_k, \xi_j \rangle + \sum_{k<j} \frac{1}{2} \langle T^{*(j-k)} \xi_k, \xi_j \rangle \right\}
$$

$$
= \inf_{\xi_j \in H} \left\{ \sum_{k,j=0}^{n} \langle A_{jk} \xi_k, \xi_j \rangle : \xi_1, \ldots, \xi_n \in H \right\}
$$

$$
= \inf_{\xi_j \in H} \left\{ \left\langle \begin{bmatrix} I & \frac{1}{2} T^* & \cdots & \cdots & \frac{1}{2} T^{*(n-1)} & \frac{1}{2} T^n \\
\frac{1}{2} T & I & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \cdots & I & \frac{1}{2} T^* & \vdots \\
0 & 0 & \cdots & 0 & I & \frac{1}{2} T \ I
\end{bmatrix} \begin{bmatrix} \xi_1 \\
\xi_1 \\
\vdots \\
\vdots \\
\vdots \\
\xi_n
\end{bmatrix}, \begin{bmatrix} \xi_1 \\
\xi_1 \\
\vdots \\
\vdots \\
\vdots \\
\xi_n
\end{bmatrix} \right\} : \xi_1, \ldots, \xi_n \in H \right\}
$$

$$
= \inf \left\{ \langle R_n \tilde{\xi}, \tilde{\xi} \rangle : \tilde{\xi} = [\xi_1 \ \cdots \ \xi_n]^T, \xi_1, \ldots, \xi_n \in H \right\},
$$
where we use $R_n$ to denote the $(n + 1) \times (n + 1)$ tri-diagonal block-matrix from the previous line. We were able to remove the negative signs because the infimum is taken over all $n$-tuples $\xi_1, \ldots, \xi_n \in H$; in particular, we can use $\xi_1, -\xi_2, \xi_3, -\xi_4, \ldots, (-1)^{n+1}\xi_n$.

Now we will use to our advantage the fact that each $R_n$ contains $R_{n-1}$ in its lower right corner. We have

\begin{equation}
\langle R_n \hat{\xi}, \hat{\xi} \rangle = \langle \xi, \xi \rangle + \text{Re}(T \xi, \xi_1) + \langle R_{n-1} \hat{\xi}, \hat{\xi} \rangle,
\end{equation}

with $\hat{\xi} = [\xi_1 \ \cdots \ \xi_n]^T$. Fix $\varepsilon > 0$; by (5.3), applied to $n-1$ and $\xi_1$, we can choose $\xi_2, \ldots, \xi_n \in H$ so that $\langle R_{n-1} \hat{\xi}, \hat{\xi} \rangle \leq \langle X_{n-1} \xi_1, \xi_1 \rangle + \varepsilon$. Thus,

\begin{align*}
\langle \left[ \begin{array}{cc}
I & \frac{1}{2} T^* \\
\frac{1}{2} T & X_{n-1}
\end{array} \right] \xi & , \xi \rangle \\
& = \langle \xi, \xi \rangle + \text{Re}(T \xi, \xi_1) + \langle X_{n-1} \xi_1, \xi_1 \rangle \\
& \geq \langle \xi, \xi \rangle + \text{Re}(T \xi, \xi_1) + \langle R_{n-1} \hat{\xi}, \hat{\xi} \rangle - \varepsilon \\
& = \langle R_n \hat{\xi}, \hat{\xi} \rangle - \varepsilon \geq \langle X_n \xi, \xi \rangle - \varepsilon.
\end{align*}

For an arbitrary $\xi_1, \ldots, \xi_n$,

\begin{align*}
\langle \left[ \begin{array}{cc}
I & \frac{1}{2} T^* \\
\frac{1}{2} T & X_{n-1}
\end{array} \right] \xi & , \xi \rangle \\
& = \langle \xi, \xi \rangle + \text{Re}(T \xi, \xi_1) + \langle X_{n-1} \xi_1, \xi_1 \rangle \\
& \leq \langle \xi, \xi \rangle + \text{Re}(T \xi, \xi_1) + \langle R_{n-1} \hat{\xi}, \hat{\xi} \rangle \\
& = \langle R_n \hat{\xi}, \hat{\xi} \rangle
\end{align*}

It follows from (5.3) and the last two estimates (writing $\eta$ instead of $\xi_1$), that

\begin{equation}
\langle X_n \xi, \xi \rangle = \inf \left\{ \langle \left[ \begin{array}{cc}
I & \frac{1}{2} T^* \\
\frac{1}{2} T & X_{n-1}
\end{array} \right] \xi & , \eta \rangle : \eta \in H \right\}.
\end{equation}

Taking limit in (5.5),

\begin{equation}
\langle X \xi, \xi \rangle = \inf \left\{ \langle \left[ \begin{array}{cc}
I & \frac{1}{2} T^* \\
\frac{1}{2} T & X
\end{array} \right] \xi & , \eta \rangle : \eta \in H \right\},
\end{equation}

as claimed in (5.1).

We have, for all $\xi, \eta \in H$,

\begin{equation}
\langle \left[ \begin{array}{cc}
X & 0 \\
0 & 0
\end{array} \right] \xi & , \xi \rangle = \langle X \xi, \xi \rangle \leq \langle \left[ \begin{array}{cc}
I & \frac{1}{2} T^* \\
\frac{1}{2} T & X
\end{array} \right] \xi & , \xi \rangle.
\end{equation}

Thus

\begin{align*}
\left[ \begin{array}{cc}
I - X & \frac{1}{2} T^* \\
\frac{1}{2} T & X
\end{array} \right] \geq 0,
\end{align*}
and \( X \) belongs to the set in (5.2). Now assume that \( 0 \leq Y \leq I \) and that \[
\begin{bmatrix}
I - Y & \frac{1}{2} T^* \\
\frac{1}{2} T & Y
\end{bmatrix} \geq 0; \]
this we can write as \[
\begin{bmatrix}
I & \frac{1}{2} T^* \\
\frac{1}{2} T & Y
\end{bmatrix} \geq \begin{bmatrix}
Y & 0 \\
0 & 0
\end{bmatrix}.
\] By assumption, \( X_0 = I \geq Y \). Suppose that \( X_{n-1} \geq Y \). Then, for each \( \xi \in H \),
\[
\langle X_n \xi, \xi \rangle = \inf \left\{ \langle \begin{bmatrix}
I & \frac{1}{2} T^* \\
\frac{1}{2} T & X_{n-1}
\end{bmatrix} \begin{bmatrix}
\xi \\
\eta
\end{bmatrix}, \begin{bmatrix}
\xi \\
\eta
\end{bmatrix} \rangle : \eta \in H \right\}
= \inf \{ \langle \xi, \xi \rangle + \text{Re} \langle T \xi, \eta \rangle + \langle X_{n-1} \eta, \eta \rangle : \eta \in H \}
\geq \inf \{ \langle \xi, \xi \rangle + \text{Re} \langle T \xi, \eta \rangle + \langle \begin{bmatrix}
Y & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
\xi \\
\eta
\end{bmatrix}, \begin{bmatrix}
\xi \\
\eta
\end{bmatrix} \rangle : \eta \in H \}
\geq \inf \{ \langle \begin{bmatrix}
Y & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
\xi \\
\eta
\end{bmatrix}, \begin{bmatrix}
\xi \\
\eta
\end{bmatrix} \rangle : \eta \in H \}
= \langle Y \xi, \xi \rangle.
\]
It follows by induction that \( X_n \geq Y \) for all \( n \), and thus \( X \geq Y \). \( \square \)

If the manipulations in the proof of Theorem 5.1 were not impressive enough, Ando keeps going at it, with the following striking characterization of the numerical radius:

**Theorem 5.2 (Ando [1]).** Let \( T \in B(H) \). Then the following statements are equivalent:

1. \( \omega(T) \leq 1; \)
2. there exist \( Y, Z \in B(H) \) with \( Y \) selfadjoint, \( \|Y\| \leq 1 \), and \( \|Z\| \leq 1 \) such that
\[
T = (I + Y)^{1/2} Z (I - Y)^{1/2}.
\]

When the above conditions are satisfied, the set
\[
\{ Y = Y^* : \exists Z, \|Z\| \leq 1 \text{ and } T = (I + Y)^{1/2} Z (I - Y)^{1/2} \}
\]
admits a maximum \( Y_{\text{max}} \) and a minimum \( Y_{\text{min}} \). The corresponding \( Z_{\text{max}} \) is isometric on the range of \( I - Y_{\text{max}} \), and \( Z_{\text{min}} \) is isometric on the range of \( I + Y_{\text{min}} \).

**Proof.** (1) \( \Rightarrow \) (2). By Theorem 5.1 there exists a positive contraction \( X \) satisfying (5.1). We can write this as
\[
\langle X \xi, \xi \rangle = \inf \{ \langle \xi, \xi \rangle + \text{Re} \langle T \xi, \eta \rangle + \langle X \eta, \eta \rangle : \eta \in H \}.
\]
Flipping a few terms around, we get
\[
\langle (I - X) \xi, \xi \rangle = \sup \{ - \text{Re} \langle T \xi, \eta \rangle - \|X^{1/2} \eta\|^2 : \eta \in H \}.
\]
Since \( \eta \) moves over all of \( H \), and since the second term is invariant if we replace \( \eta \) with \( \lambda \eta \) for \( \lambda \in \mathbb{T} \), we get
\[
\| (I - X)^{1/2} \xi \|^2 = \sup \{ \|T \xi, \eta\| - \|X^{1/2} \eta\|^2 : \eta \in H \}.
\]
As we can write \( t\eta \) instead of \( \eta \), we have shown that for a fixed \( \eta \)
\[(5.9) \quad \|(I - X)^{1/2}\xi\|^2 - t|\langle T\xi, \eta \rangle| + t^2\|X^{1/2}\eta\|^2 \geq 0, \quad t \in \mathbb{R}.
\]
The discriminant inequality for this quadratic is
\[\|\langle T\xi, \eta \rangle\|^2 \leq 4\|(I - X)^{1/2}\xi\|^2 \|X^{1/2}\eta\|^2,
\]
which we write as
\[(5.10) \quad \frac{1}{2} |\langle T\xi, \eta \rangle| \leq \|(I - X)^{1/2}\xi\| \|X^{1/2}\eta\|.
\]
Because of the supremum in (5.8), for any given \( \xi \), there exists \( \eta \) such that the quadratic in (5.9) is arbitrarily close to zero. Thus the discriminant can be made arbitrarily close to zero by such an \( \eta \), and the inequality in (5.10) can be made arbitrarily close to an equality. So
\[(5.11) \quad \|(I - X)^{1/2}\xi\| = \sup \left\{ \frac{1}{2} |\langle T\xi, \eta \rangle| : \|X^{1/2}\eta\| \neq 0 \right\}.
\]

We now construct a densely-defined sesquilinear form in the following way. Let \( H_0, H_1 \subset H \) be the following dense (due to \( X \) being selfadjoint) linear manifolds:
\[
H_0 = \{ \xi_0 + (I - X)^{1/2}\xi : \xi_0 \in \ker(I - X), \xi \in H \},
\]
\[
H_1 = \{ \eta_0 + X^{1/2}\eta : \eta_0 \in \ker X, \eta \in H \}.
\]
Then we define, on \( H_0 \times H_1 \), a form
\[(5.12) \quad \langle \xi_0 + (I - X)^{1/2}\xi, \eta_0 + X^{1/2}\eta \rangle := \frac{1}{2} \langle T\xi, \eta \rangle.
\]
By (5.10), the above form is well-defined and, since the kernel and range of \( X \) are orthogonal to each other,
\[
\|\xi_0 + (I - X)^{1/2}\xi, \eta_0 + X^{1/2}\eta\| = \frac{1}{2} |\langle T\xi, \eta \rangle| \leq \|(I - X)^{1/2}\xi\| \|X^{1/2}\eta\|
\]
\[
\leq \|\xi_0 + (I - X)^{1/2}\xi\| \|\eta_0 + X^{1/2}\eta\|.
\]
The sesquilinear form is thus bounded with norm at most one: we can then extended it to all of \( H \times H \). By the Riesz Representation Theorem there exists a linear contraction \( Z \in B(H) \), with \( Z|_{\ker(I - X)} = 0 \) and \( Z^*|_{\ker X} = 0 \), and such that
\[(5.13) \quad \frac{1}{2} \langle T\xi, \eta \rangle = \langle Z(I - X)^{1/2}\xi, X^{1/2}\eta \rangle, \quad \xi, \eta \in H.
\]
In particular, \( \frac{1}{2} T = X^{1/2}Z(I - X)^{1/2} \). If we now let \( Y = 2X - I \), then \( Y = Y^* \) and
\[
(I + Y)^{1/2}Z(I - Y)^{1/2} = (2X)^{1/2}Z(2I - 2X)^{1/2} = 2X^{1/2}Z(I - X)^{1/2} = T.
\]
Using (5.11) and (5.13), for a fixed \( \xi \in H \) and \( \epsilon > 0 \) there exists \( \eta \in H \) with
\[
\|(I - X)^{1/2}\xi\| \leq \frac{1}{2} |\langle T\xi, \eta \rangle| + \epsilon = \frac{|\langle Z(I - X)^{1/2}\xi, X^{1/2}\eta \rangle|}{\|X^{1/2}\eta\|} + \epsilon.
\]
\[ \leq \|Z(I - X)^{1/2}\xi\| + \varepsilon. \]

But \( Z \) is a contraction and we can do this for all \( \varepsilon > 0 \), so \( \|Z(I - X)^{1/2}\xi\| = \|(I - X)^{1/2}\xi\| \) for all \( \xi \in H \); a fortiori, as we can replace \( \xi \) with \( (I - X)^{1/2}\xi \), and writing \( Z_{\text{max}} \) for the contraction \( Z \) we constructed, we get that
\[ \|Z_{\text{max}}(I - X)\xi\| = \|(I - X)\xi\|, \quad \text{for all } \xi \in H. \]

From \( I - X = \frac{1}{2}(I - Y) \), we obtain that \( Z_{\text{max}} \) is isometric on the range of \( I - Y \). This \( Y = 2X - I \) we constructed, that we will denote as \( Y_{\text{max}} \), is the maximum of the set in (5.7). Indeed, if \( T = (I + Y_0)^{1/2}Z_0(I - Y_0)^{1/2} \) for selfadjoint \( Y_0 \) and contractive \( Z_0 \), then
\[
\begin{bmatrix}
I - \frac{1}{2}(I + Y_0) & \frac{1}{2}T^* \\
\frac{1}{2}T & \frac{1}{2}(I + Y_0)
\end{bmatrix}
= \frac{1}{2}
\begin{bmatrix}
I - Y_0 & R \\
R^* & I + Y_0
\end{bmatrix}
= M
\begin{bmatrix}
I & Z_0 \\
Z_0^* & I
\end{bmatrix}
M^* \geq 0,
\]

where \( R = (I - Y_0)^{1/2}Z_0^*(I + Y_0)^{1/2} \) and \( M = \frac{1}{\sqrt{2}} \begin{bmatrix} (I - Y_0)^{1/2} & 0 \\ 0 & (I + Y_0)^{1/2} \end{bmatrix} \).

By the maximality of \( X \) in (5.2), we have \( \frac{1}{2}(I + Y_0) \leq X \), which we can write as
\[ Y_0 \leq 2X - I = Y_{\text{max}}. \]

All of the above can be done for \( T^* \), so there is a maximum, say \( Y_* \), corresponding to \( T^* \). By taking the adjoint, we can rewrite any decomposition \( T = (I + Y)^{1/2}Z(I - Y)^{1/2} \) as \( T^* = (I + (-Y))^{1/2}Z^*(I - (-Y))^{1/2} \). It follows that \( -Y \leq Y_* \) for all \( Y \) that give a decomposition of \( T \). In other words, \( -Y_* = Y_{\text{min}} \).

(2) \( \implies \) (1) If \( T = (I + Y)^{1/2}Z(I - Y)^{1/2} \) for a contraction \( Z \), then, using the trivial number inequality \( |ab| \leq \frac{1}{2}(|a|^2 + |b|^2) \),
\[
|\langle T\xi, \xi \rangle| = |\langle (Z(I - Y)^{1/2}\xi, (I + Y)^{1/2}\xi \rangle| \leq \|(I - Y)^{1/2}\xi\| \| (I + Y)^{1/2}\xi \|
\leq \frac{1}{2} \left( \| (I - Y)^{1/2}\xi \|^2 + \| (I + Y)^{1/2}\xi \|^2 \right)
= \frac{1}{2} \left( \| (I - Y)\xi, \xi \| + \| (I + Y)\xi, \xi \| \right) = \langle \xi, \xi \rangle. \]

\[ \square \]

Remark 5.3. Let us find the above decomposition for the case \( T = 2E_{21} \).

We have \( w(T) = 1 \), so the above results apply. In light of Remark 4.2, we may take \( K = \ell^2(\mathbb{Z}) \), \( U \) the bilateral shift, and \( H = \text{span}\{e_1, e_2\} \). Then \( U^*H = \text{span}\{e_0, e_1\} \), and
\[
\text{span} \bigcup_{k=1}^{n} U^{*k}H = \text{span}\{e_{-n+1}, e_{-n+2}, \ldots, e_1\}. \]

So, in Theorem 5.1 \( I - E_n = \sum_{k=n}^{\infty} E_{-k,-k} + \sum_{k=2}^{\infty} E_{kk} \) and \( P_H = E_{11} + E_{22} \).

We get that \( X_n = P_H(I - Q_n)P_H = E_{22} \) for all \( n \). So \( X = E_{22} \). Then
\[ \langle X \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \rangle = |\beta|^2; \text{ and} \]

\[ \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} \quad \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} \rangle = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix}, \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = |\alpha + \delta|^2 + |\beta|^2, \]

with the infimum over \( \gamma, \delta \) being \(|\beta|^2 \) (achieved when \( \delta = -\alpha \)). If \( Y \) satisfies

\[ \begin{bmatrix} I - Y & E_{12} \\ E_{21} & Y \end{bmatrix} \geq 0, \]

we immediately get from diagonal entries that \( 0 \leq Y \leq I \), and considering the \( 2 \times 2 \) matrix formed by the corner entries, we have

\[ \begin{bmatrix} 1 - Y_{11} & 1 \\ 1 & Y_{22} \end{bmatrix} \geq 0. \]

This can only be satisfied if \( Y_{11} = 0, Y_{22} = 1 \). From \( Y_{12} = Y_{21} = 0 \) (a zero in the main diagonal forces its column and row to be zero). Thus \( Y = E_{22} = X \). We have shown that, in this example, the set in (5.2) consists of just \( X \).

Looking into Theorem 5.2, we have \( Y = 2X - I = E_{22} - E_{11} \). The proof in that theorem constructs \( Z \) via the bilinear form on \( H_0 \times H_1 \). In this case, \( (I - X)^{1/2} = E_{11}, X^{1/2} = E_{22} \), so the form is

\[ \frac{1}{2} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \begin{bmatrix} \gamma \\ \delta \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2E_{21} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \end{bmatrix}, \begin{bmatrix} \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} E_{21} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \end{bmatrix}, \begin{bmatrix} \gamma \\ \delta \end{bmatrix} \rangle = \langle E_{21} E_{11} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, E_{22} \begin{bmatrix} \gamma \\ \delta \end{bmatrix} \rangle. \]

Thus \( Z = E_{21} \). The maximal decomposition is then

\[ 2E_{21} = (I + Y)^{1/2} Z (I - Y)^{1/2} = (2E_{22})^{1/2} (2E_{11})^{1/2} \]

and, as we said,

\[ Y_{\text{max}} = 2X - I = E_{22} - E_{11} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \]

The computation that showed that \( 2X - I \) is maximum in the proof of Theorem 5.2 implies that if \( X \) is unique, so is \( Y \). Thus \( Y_{\text{min}} = Y_{\text{max}} \) in this case.

**Remark 5.4.** If instead we consider \( T = E_{21} \), the situation is very different. It seems hard to follow the path all the way from Theorem 4.1 to Theorem 5.1 to find \( X \) explicitly. Still, from Theorem 5.1 we know that we need to find
the maximum of those selfadjoint $X$ such that $0 \leq X \leq I$ and

$$
\begin{bmatrix}
I - X & \frac{1}{2} E_{12} \\
\frac{1}{2} E_{21} & X
\end{bmatrix} = 
\begin{bmatrix}
1 - X_{11} & -X_{12} & 0 & 1/2 \\
-X_{21} & 1 - X_{22} & 0 & 0 \\
0 & 0 & X_{11} & X_{12} \\
1/2 & 0 & X_{21} & X_{22}
\end{bmatrix} \geq 0.
$$

(5.14)

It is easy to check that $X_0 = \frac{3}{4} E_{11} + E_{22}$ satisfies (5.14), and so $X \geq \frac{3}{4} E_{11} + E_{22}$. We get immediately that $X_{11} \geq 3/4$ and $X_{22} = 1$ (since $I - X \geq 0$). Again from $I - X \geq 0$,

$$
\begin{bmatrix}
1 - X_{11} & X_{12} \\
X_{12} & 0
\end{bmatrix} \geq 0,
$$

and so $X_{12} = 0$ by the positivity. If we now look, in (5.14), at the $2 \times 2$ matrix formed by the corner entries, we have

$$
\begin{bmatrix}
1 - X_{11} & X_{12} \\
X_{12} & 0
\end{bmatrix} \geq 0.
$$

Then $1 - X_{11} \geq 1/4$, i.e., $X_{11} \leq 3/4$. So $X_{11} = 3/4$, and $X = \begin{bmatrix} 3/4 & 0 \\ 0 & 1 \end{bmatrix}$.

Now

$$
Y_{\max} = 2X - I = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}.
$$

If we look again at the proof of Theorem 5.2, we have $(I - X)^{1/2} = \frac{1}{2} E_{11}$, $X^{1/2} = \frac{\sqrt{2}}{2} E_{11} + E_{22}$. Writing (5.13) explicitly, we immediately get $Z = E_{21}$.

The maximal decomposition is then

$$
E_{21} = (I + Y_{\max})^{1/2} Z (I - Y_{\max})^{1/2} = \begin{bmatrix} \sqrt{3/2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}.
$$

As opposed to the previous case, though, different decompositions are possible. If we repeat the analysis above for $T^* = E_{12}$, we find that $X_*$ is now $E_{11} + \frac{3}{4} E_{22}$. Its maximum $Y_*$ will be $2X_* - I = E_{11} + \frac{1}{2} E_{22}$. Then, with respect to our original $T = E_{21}$, we have

$$
Y_{\min} = -Y_* = \begin{bmatrix} -1 & 0 \\ 0 & -1/2 \end{bmatrix}.
$$

We can also get decompositions that do not come from $Y_{\max}$ nor $Y_{\min}$. For a trivial one, take $Y_0 = 0$, $Z_0 = E_{21}$. Then, of course, $E_{21} = (I + 0)^{1/2} E_{21} (I - 0)^{1/2}$. Yet another fairly trivial decomposition can be found if $Y = E_{22}$. Then $(I + Y)^{1/2} = E_{11} + \sqrt{2} E_{22}$, and $(I - Y)^{1/2} = E_{11}$. So we get the decomposition

$$
E_{21} = \begin{bmatrix}
1 & 0 \\
0 & \sqrt{2}
\end{bmatrix} \begin{bmatrix}
0 & 0 \\
1/\sqrt{2} & 0
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}.
$$

The following result is a well-known matricial characterization of the numerical radius. It uses Theorem 5.2 in an essential way.
Corollary 5.5. Let $T \in B(H)$. Then the following statements are equivalent:

(1) $w(T) \leq 1/2$;

(2) there exists $A \in B(H)^+$, with $A \leq I$, such that
\[
\begin{bmatrix}
A & T^* \\
T & I - A
\end{bmatrix} \succeq 0.
\]

Proof. If
\[
\begin{bmatrix}
A & T^* \\
T & I - A
\end{bmatrix} \succeq 0,
\]
then for any $\xi, \eta \in H$
\[
0 \leq \left\langle \begin{bmatrix}
A & T^* \\
T & I - A
\end{bmatrix} \begin{bmatrix}
\xi \\
\eta
\end{bmatrix}, \begin{bmatrix}
\xi \\
\eta
\end{bmatrix} \right\rangle = \langle A\xi, \xi \rangle + \langle (I - A)\eta, \eta \rangle + 2 \text{Re} \langle T\xi, \eta \rangle.
\]
Taking $\eta = \lambda \xi$ for $\lambda \in T$ such that $\langle T\xi, \xi \rangle = \lambda |\langle T\xi, \xi \rangle|$, we get
\[
0 \leq \|\xi\|^2 - 2|\langle T\xi, \xi \rangle|,
\]
implying $w(T) \leq 1/2$.

Conversely, if $w(T) \leq 1/2$, by Theorem 5.2 there exist a selfadjoint contraction $Y$ and contraction $Z$ with $2T = (I + Y)^{1/2}Z(I - Y)^{1/2}$. Since $Z$ is contractive, the matrix
\[
\begin{bmatrix}
I & Z^* \\
Z & I
\end{bmatrix}
\]
is positive; then
\[
\begin{bmatrix}
I + Y & 2T \\
2T^* & I - Y
\end{bmatrix} = \begin{bmatrix}
I + Y & (I + Y)^{1/2}Z^*(I - Y)^{1/2} \\
(I - Y)^{1/2}Z(I + Y)^{1/2} & I - Y
\end{bmatrix}
\]
\[
= \begin{bmatrix}
(I + Y)^{1/2} & 0 \\
0 & (I - Y)^{1/2}
\end{bmatrix} \begin{bmatrix}
I & Z^* \\
Z & I
\end{bmatrix} \begin{bmatrix}
(I + Y)^{1/2} & 0 \\
0 & (I - Y)^{1/2}
\end{bmatrix} \succeq 0.
\]
Multiplying by $1/2$ we get
\[
\begin{bmatrix}
A & T^* \\
T & I - A
\end{bmatrix} \succeq 0,
\]
where $A = \frac{1}{2} (I + Y)$. \hfill \Box

Our main use of this result is Corollary 5.6, characterizing the matricial range of $E_{21}$. The use of Corollary 5.5 to characterize $\mathcal{W}(E_{21})$ is a very well-known result, though we are not aware of any published reference.

Corollary 5.6. For any $T \in B(H)$, the following statements are equivalent:

(1) $w(T) \leq 1/2$;

(2) there exists $\varphi : M_2(\mathbb{C}) \to B(H)$, ucp, with $T = \varphi(E_{21})$.

Proof. (1) $\implies$ (2). By Corollary 5.5 there exists $A \in B(H)^+$ with $A \leq I$ and
\[
\begin{bmatrix}
A & T^* \\
T & I - A
\end{bmatrix} \succeq 0.
\]
Now define a linear map $\varphi : M_2(\mathbb{C}) \to B(H)$ by
\[
\varphi(E_{11}) = A, \quad \varphi(E_{21}) = T, \quad \varphi(E_{12}) = T^*, \quad \varphi(E_{22}) = I - A.
\]
Then $\varphi$ is unital, and by Choi’s criterion (Proposition 2.1) it is completely positive, since
\[
\varphi^{(2)} \left( \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} \right) = \begin{bmatrix} A & T^* \\ T & I - A \end{bmatrix} \succeq 0.
\]
(2) $\implies$ (1). Let $A = \varphi(E_{11})$. As $\varphi$ is ucp, Choi’s criterion (Proposition 2.1) implies that
\[
\begin{bmatrix}
A & T^* \\
T & I - A
\end{bmatrix} = \varphi(2) \begin{bmatrix}
E_{11} & E_{12} \\
E_{21} & E_{22}
\end{bmatrix} \succeq 0,
\]
and so by Corollary 5.5 $w(T) \leq 1/2$. \(\square\)

We remark that the proof (2) $\implies$ (1) in Corollary 5.6 can be achieved without appealing to Ando’s nor Choi’s results. Indeed, by using the Stinespring dilation one can show that if $\phi$ is ucp, then $\mathcal{W}_1(\phi(T)) \subset \mathcal{W}_1(T)$.

We finish this section with another characterization due to Ando. The interesting information we find in its proof, is that the operator $C$ below allows us to express a unitary 2-dilation of $T$ explicitly.

**Theorem 5.7 (Ando [1]).** Let $T \in B(H)$. Then the following statements are equivalent:

1. $w(T) \leq 1$;
2. there exists $C \in B(H)$, contractive, such that $T = 2(I - C^*C)^{1/2}C$.

*Proof.* (1) $\implies$ (2) By Theorem 5.2 there exist $Y, Z \in B(H)$, both contractive and $Y$ selfadjoint, with $Z$ isometric on the range of $(I - Y)^{1/2}$, and satisfying $T = (I + Y)^{1/2}Z(I - Y)^{1/2}$. Let $C = \frac{1}{\sqrt{2}} Z(I - Y)^{1/2}$.

The fact that $Z$ is isometric on the range of $(I - Y)^{1/2}$ can be written as $(I - Y)^{1/2}Z^*Z(I - Y)^{1/2} = (I - Y)^{1/2}(I - Y)^{1/2} = I - Y$. Then
\[
I - C^*C = I - \frac{1}{2} (I - Y)^{1/2}Z^*Z(I - Y)^{1/2} = I - \frac{1}{2} (I - Y) = \frac{1}{2} (I + Y).
\]

Thus
\[
2(I - C^*C)^{1/2}C = (I + Y)^{1/2}Z(I - Y)^{1/2} = T.
\]

(2) $\implies$ (1) We will explicitly construct a unitary 2-dilation of $T$, and then the result will follow from Theorem 4.4. We first construct a unitary dilation $W$ of $C$ on $K = \bigoplus_{k \in \mathbb{Z}} H$ as follows:
\[
W = \sum_{k \geq 1} \sum_{k \leq -2} I \otimes E_{k+1,k} + C \otimes E_{0,0} + (I - CC^*)^{1/2} \otimes E_{0,-1}
\]
\[
+ (I - C^*C)^{1/2} \otimes E_{1,0} - C^* \otimes E_{1,-1}.
\]

We encourage the reader to check that this is indeed a unitary; besides a decent amount of patience and care, the only non-trivial (but well-known) manipulation required is to note that $C(I - C^*C)^{1/2} = (I - CC^*)^{1/2}C$.

With $S$ the bilateral shift $S = \sum_{k \in \mathbb{Z}} I \otimes E_{k+1,k}$, we define $U = S^*W^2$. This is again a unitary and it has the form
\[
U = \sum_{k \geq 2} \sum_{k \leq -2} I \otimes E_{k,k-1} + (I - C^*C)^{1/2} \otimes E_{1,0} - C^* \otimes E_{1,-1} + C^2 \otimes E_{-1,0}
\]
We claim that this $U$ is a 2-dilation of $T$; that is, that $(U^n)_{0,0} = \frac{1}{2} T^n$ for all $n \in \mathbb{N}$. We proceed by induction. Assume that, for a fixed $n$ and for all $\ell \geq 1$,

$$(U^n)_{0,0} = \frac{1}{2} T^n, \quad (U^n)_{0,-1} = T^{n-1}(I - C^*C)^{1/2}(I - CC^*)^{1/2}, \quad (U^n)_{0,\ell} = 0.$$ 

This clearly holds for $n = 1$, and so now we show that the above equalities for $n$ imply the corresponding versions for $n+1$. We will use repeatedly the equality $C(I - C^*C)^{1/2} = (I - CC^*)^{1/2}C$. Then (recall that our hypothesis is that $T = 2(I - C^*C)^{1/2}C$ and that $U_{\ell,0} \neq 0$ and $U_{\ell,-1} \neq 0$ only when $\ell \in \{-1,0,1\}$)

$$(U^{n+1})_{0,-1} = (U^n)_{0,-1}U_{-1,-1} + (U^n)_{0,0}U_{0,-1} + (U^n)_{0,1}U_{1,-1}$$

$$= T^{n-1}(I - C^*C)^{1/2}(I - CC^*)^{1/2}C(I - CC^*)^{1/2} + \frac{1}{2} T^n(I - C^*C)^{1/2}(I - CC^*)^{1/2}$$

$$= T^{n-1}(I - C^*C)^{1/2}C(I - C^*C)^{1/2} + \frac{1}{2} T^n(I - C^*C)^{1/2}(I - CC^*)^{1/2}$$

$$= \frac{1}{2} T^{n-1}(I - C^*C)^{1/2}(I - CC^*)^{1/2} + \frac{1}{2} T^n(I - C^*C)^{1/2}(I - CC^*)^{1/2}$$

$$= T^n(I - C^*C)^{1/2}(I - CC^*)^{1/2}.$$ 

Also,

$$(U^{n+1})_{0,0} = (U^n)_{0,-1}U_{-1,0} + (U^n)_{0,0}U_{0,0} + (U^n)_{0,1}U_{1,0}$$

$$= T^{n-1}(I - C^*C)^{1/2}(I - CC^*)^{1/2}C^2 + \frac{1}{2} T^n \frac{1}{2} T^n$$

$$= T^{n-1}(I - C^*C)^{1/2}C(I - C^*C)^{1/2} + \frac{1}{4} T^{n+1}$$

$$= \frac{1}{4} T^{n+1} + \frac{1}{4} T^{n+1} = \frac{1}{2} T^{n+1}.$$ 

And, since for $\ell \geq 1$ the only $k$ such that $U_{k,\ell} \neq 0$ is $k = \ell + 1$ ($U_{\ell+1,\ell} = I$),

$$(U^{n+1})_{\ell,\ell} = (U^n)_{0,\ell+1}U_{\ell+1,\ell} = (U^n)_{0,\ell+1} = 0.$$ 

The induction is then complete: for all $n \in \mathbb{N}$, we have $(U^n)_{0,0} = \frac{1}{2} T^n$. □
6. Toeplitz Matrices

The goal in this section is [Theorem 6.5]. As this is a matricial generalization of the classical [Theorem 6.3] we present first the scalar version to fix ideas.

6.1. Scalar Matrices. The following is [15, Lemma 2.5].

**Lemma 6.1 (Fejer-Riesz).** Let \( \tau \) be a trigonometric polynomial of the form
\[
\tau(\lambda) = \sum_{n=-N}^{N} a_n \lambda^n.
\]
If \( \tau(\lambda) > 0 \) for all \( \lambda \in \mathbb{T} \), then there exists a polynomial
\[
p(z) = \sum_{n=0}^{N} p_n z^n
\]
such that
\[
\tau(\lambda) = |p(\lambda)|^2, \quad \lambda \in \mathbb{T}.
\]

**Proof.** Since \( \tau(\lambda) > 0 \), we have
\[
\sum_{n=-N}^{N} a_n \lambda^n = \text{Re} \left( \sum_{n=-N}^{N} a_n \lambda^n \right)
\]
\[
= \sum_{n=-N}^{N} \text{Re}(a_n \lambda^n) = \text{Re}(a_0) + \sum_{n=1}^{N} \text{Re}(a_n \lambda^n + a_{-n} \lambda^{-n})
\]
\[
= \text{Re}(a_0) + \sum_{n=1}^{N} \frac{(a_n + a_{-n}) \lambda^n + (\overline{a_n} + a_{-n}) \lambda^{-n}}{2}
\]
\[
= \text{Re}(a_0) + \sum_{n=-N}^{N} \frac{a_n + \overline{a_{-n}}}{2} \lambda^n.
\]

It follows that, for all \( n \), \( a_n = \frac{a_n + a_{-n}}{2} \), and then \( a_n = \overline{a_{-n}} \). Also, \( a_0 \in \mathbb{R} \).

If necessary, we may decrease \( N \) so that \( a_{-N} \neq 0 \). Although we consider \( \tau \) as a polynomial on \( \mathbb{T} \), its formula works of course for all \( z \in \mathbb{C} \). Define \( g(z) = z^N \tau(z) \). Note that \( g(0) = a_{-N} \neq 0 \), so all roots of \( g \) are nonzero. Also, for \( \lambda \in \mathbb{T} \), \( g(\lambda) = \lambda^N \tau(\lambda) \neq 0 \), so no zero of \( g \) is in \( \mathbb{T} \). We have
\[
g(1/z) = \frac{1}{z^N} \tau(1/z) = z^{-N} \sum_{n=-N}^{N} a_n z^{-n} = z^{-N} \sum_{n=-N}^{N} a_{-n} z^{-n}
\]
\[
= z^{-N} \sum_{n=-N}^{N} a_n z^n = z^{-2N} g(z).
\]

This implies that \( g(z) = 0 \) if and only if \( g(1/z) = 0 \). Since no zero is in \( \mathbb{T} \), we get that the zeroes of \( g \) are of the form \( z_1, \ldots, z_N, 1/z_1, \ldots, 1/z_N \). Then
\[
g(z) = a_N r(z)s(z),
\]
where \( r, s \) are the polynomials

\[
\begin{align*}
  r(z) &= \prod_{n=1}^{N} (z - z_n), \\
  s(z) &= \prod_{n=1}^{N} (z - 1/z_n).
\end{align*}
\]

These two polynomials are related by

\[
\frac{s(z)}{r(z)} = \frac{(-1)^N z^N r(1/z)}{z_1 \cdots z_N}.
\]

Then, for \( \lambda \in T \),

\[
\tau(\lambda) = |\tau(\lambda)| = |\lambda^{-N} g(\lambda)| = |g(\lambda)| = |a_N| |r(\lambda)| |s(\lambda)|
\]

\[
= |a_N| |r(\lambda)| \left| \frac{(-1)^N \lambda^{-N} r(\lambda)}{z_1 \cdots z_N} \right| = \left| \frac{a_N}{z_1 \cdots z_N} \right| |r(\lambda)|^2.
\]

Thus \( \tau(\lambda) = |p(\lambda)|^2 \), where

\[
p(z) = \left| \frac{a_N}{z_1 \cdots z_N} \right|^{1/2} r(z).
\]

Matricial versions of the Fejer-Riesz Lemma exist—see for instance [10, 14, 20]—but we will not discuss them here. Recall from page 2 that we denote by \( S_n \) the \( n \times n \) unilateral shift.

**Definition 6.2.** An \( n \times n \) Toeplitz matrix is a matrix \( T \) of the form

\[
T = a_0 I_n + \sum_{k=1}^{n-1} a_k S_n^k + \sum_{k=1}^{n-1} a_{-k} S_n^{*k},
\]

where \( a_k \in \mathbb{C} \) for all \( k \). Graphically, this is

\[
T = \begin{bmatrix}
  a_0 & a_{-1} & a_{-2} & \cdots & \cdots & \cdots & \cdots & a_{-n+1} \\
  a_1 & a_0 & a_{-1} & a_{-2} & \cdots & \cdots & \cdots & \cdots \\
  a_2 & a_1 & a_0 & a_{-1} & \cdots & \cdots & \cdots & \cdots \\
  \vdots & a_2 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{-2} \\
  \vdots & \vdots & \cdots & \cdots & \cdots & a_1 & a_0 & a_{-1} \\
  \vdots & \vdots & \cdots & \cdots & a_2 & a_1 & a_0 & a_{-1} \\
  a_{n-1} & \cdots & \cdots & \cdots & \cdots & \cdots & a_2 & a_1 & a_0
\end{bmatrix}
\]

If in particular \( T \) is Hermitian, i.e. \( T = T^* \), then

\[
(6.1) \quad T = a_0 I_n + 2 \text{Re} \sum_{k=1}^{n-1} a_k S_n^k, \quad a_0, a_1, \ldots, a_{n-1} \in \mathbb{C}.
\]

In the case of a Hermitian Toeplitz matrix we will write, when needed, \( a_{-k} = \overline{a_k} \).

The following theorem is based on [15, Theorem 2.14]; we do not need to make use of this theorem, but we will use a matricial generalization,
Theorem 6.5 and so the scalar proof might help some readers. Paulsen considers infinite sequences, which we don’t need here.

**Theorem 6.3.** Let $T$ be a Hermitian Toeplitz matrix as in (6.1). Then the following statements are equivalent:

1. $T$ is positive;
2. there exists a positive linear functional $\phi$ on $C(\mathbb{T})$ such that $a_k = \phi(z^k)$, $k = 0, 1, \ldots, n - 1$.

**Proof.** $(1) \implies (2)$ Consider the operator system $S = \text{span}\{z^k : k \in \mathbb{Z}\} \subset C(\mathbb{T})$. Define a linear map $\phi : S \to \mathbb{C}$ by $\phi(z^k) = a_k$, $\phi(z^{-k}) = \overline{a_k}$ for $k = 0, \ldots, n - 1$, and $\phi(z^k) = 0$ for $|k| \geq n$. Let $\tau \in S$ be strictly positive, i.e. $\tau(\lambda) > 0$ for all $\lambda \in \mathbb{T}$. By Lemma 6.1, there exist $p_0, p_1, \ldots, p_m$ such that $\tau(\lambda) = \sum_{k,j=0}^{m} p_k p_j \lambda^{k-j}$. Assume, without loss of generality, that $m \geq n$ (we complete the list of $p_k$ with zeroes if it is not the case). Then, with the convention that $a_{-k} = \overline{a_k}$, $a_k = 0$ if $|k| \geq n$, and with $x = (p_0, \ldots, p_m)^\top$, $\phi(\tau) = \sum_{k,j=0}^{m} p_k p_j a_{k-j} = \langle (T \oplus 0)x, x \rangle \geq 0$ by the positivity of $T$. For arbitrary positive $\tau$, we have that for any $\varepsilon > 0$ the function $\tau'(\lambda) = \tau(\lambda) + \varepsilon$ is strictly positive, and so $\phi(\tau) + \varepsilon = \phi(\tau') \geq 0$ for all $\varepsilon > 0$, which implies that $\phi(\tau) \geq 0$. Thus $\phi$ is a positive linear functional on the operator system of the trigonometric polynomials; this implies that it is bounded and we can extend it by density to $C(\mathbb{T})$.

$(2) \implies (1)$ Note that, since $T$ is Hermitian, $a_{-k} = \overline{a_k}$, $a_k = 0$ if $|k| \geq n$, and with $x = (p_0, \ldots, p_n)^\top \in \mathbb{C}^n$,

$$\langle Tx, x \rangle = \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} a_{k-j} p_k p_j = \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} \phi(z^{k-j}) p_k p_j = \phi \left( \sum_{k=0}^{n} z^k p_k \right)^2 \geq 0.$$  

**Remark 6.4.** The proof of $(2) \implies (1)$ in Theorem 6.3 can also be achieved by using that $\phi$ is completely positive (due to the abelian domain) and then noting that

$$T = I_n + 2 \text{Re} \sum_{k=1}^{n-1} \phi(z^k) z_n^k$$

$$= \phi^{(n)} \begin{bmatrix} 1 & z & z^2 & \cdots & z^{n-1} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \ast \begin{bmatrix} 1 & z & z^2 & \cdots & z^{n-1} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \geq 0.$$
6.2. Block Toeplitz Matrices. An \( n \times n \) block Toeplitz matrix is a matrix \( T \) of the form

\[
T = A_0 \otimes I_n + \sum_{k=1}^{n-1} A_k \otimes S^k_n + \sum_{k=1}^{n-1} A_{-k} \otimes S^*_n, 
\]

where \( A_{-(n-1)}, \ldots, A_0, \ldots, A_{n-1} \in \mathcal{A} \) for some \( C^* \)-algebra \( \mathcal{A} \). If in particular \( T \) is Hermitian, i.e. \( T = T^* \), then

\[
(6.2) \quad T = A_0 \otimes I_n + 2 \Re \sum_{k=1}^{n-1} A_k \otimes S^k_0
\]

and we may write \( A_{-k} = A^*_k \).

Theorem 6.5 is the block-matrix version of Theorem 6.3. We will later use it in the proof of Theorem 7.1 with \( \mathcal{A} = M_m(\mathbb{C}) \).

Theorem 6.5. Let \( T \) be a Hermitian block Toeplitz matrix as in (6.2), with coefficients in the \( C^* \)-algebra \( \mathcal{A} \). Then the following statements are equivalent:

1. \( T \) is positive;
2. there exists a (completely) positive map \( \phi : C(\mathbb{T}) \to \mathcal{A} \) such that

\[
(6.3) \quad \phi(z^k) = A_k, \quad k = 0, \ldots, n-1.
\]

Proof. \([1] \implies [2]\) Consider the operator system \( \mathcal{S} = \text{span}\{ z^k : k \in \mathbb{Z} \} \subset C(\mathbb{T}) \). Define a linear map \( \phi : \mathcal{S} \to \mathcal{A} \) by \( \phi(z^k) = A_k \), \( \phi(z^{-k}) = A^*_k \) for \( k = 0, \ldots, n-1 \), and \( \phi(z^k) = 0 \) for \( |k| \geq n \). Let \( \tau \in \mathcal{S} \) be strictly positive, i.e. \( \tau(\lambda) > 0 \) for all \( \lambda \in \mathbb{T} \). By Lemma 6.1 there exist \( p_0, p_1, \ldots, p_m \) such that \( \tau(\lambda) = \sum_{k,j=0}^m p_k p_{kj} \lambda^{k-j} \). Assume, without loss of generality, that \( m \geq n \) (we complete the list of \( p_k \) with zeroes if it is not the case). Then, with the convention that \( A_{-k} = A^*_k \), \( A_k = 0 \) if \( |k| \geq n \),

\[
\phi(\tau) = \sum_{k,j=0}^m p_k p_{kj} A_{k-j} = \begin{bmatrix} p_0 I & T & \vdots & p_m I \\ \vdots & 0_{m-n} & \vdots & \vdots \\ p_m I \end{bmatrix}^* \begin{bmatrix} p_0 I \\ \vdots \\ p_m I \end{bmatrix} \geq 0
\]

by the positivity of \( T \). For arbitrary positive \( \tau \), we have that for any \( \varepsilon > 0 \) the function \( \tau'(\lambda) = \tau(\lambda) + \varepsilon \) is strictly positive, and so \( \phi(\tau) + \varepsilon = \phi(\tau') \geq 0 \) for all \( \varepsilon > 0 \), which implies that \( \phi(\tau) \geq 0 \). Thus \( \phi \) is a positive linear map in the operator system of the trigonometric polynomials; thus it is bounded, and it extends by density to a positive map on \( C(\mathbb{T}) \) with range still contained in \( \mathcal{A} \).

\([2] \implies [1]\) Since \( C(\mathbb{T}) \) is abelian, \( \phi \) is completely positive. Consider a Stinespring dilation of \( \phi \), i.e. \( \phi(f) = V^* \pi(f) V \). Then, for each vector \( x = (\xi_0, \ldots, \xi_{n-1})^T \in H^n \),

\[
\langle Tx, x \rangle = \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} \langle A_{k-j} \xi_k, \xi_j \rangle = \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} \langle V^* \pi(z^{k-j}) V \xi_k, \xi_j \rangle
\]
\[ n-1 \sum_{k=0}^{n-1} \langle \pi(z^k) V \xi_k, V \xi_j \rangle \]
\[ = \left\| \sum_{k=0}^{n-1} \pi(z^k) V \xi_k \right\|^2 \geq 0. \]

7. Nilpotent Dilations and Matricial Range

The following is a significant technical result by Arveson, characterizing those contractions that can be power-dilated to nilpotents. The proof does not follow the original; in particular, the argument that we offer for \(4) \implies \, (1)\) is more algebraic and direct that Arveson’s original.

**Theorem 7.1.** [3, Theorem 1.3.1] Let \( T \in B(H) \) with \( \|T\| \leq 1 \) and let \( n \in \mathbb{N} \) with \( n \geq 2 \). Then the following statements are equivalent:

1. There exists \( \phi : M_n(\mathbb{C}) \to B(H) \), ucp, with \( \phi(S^j_n) = T^j, j = 1, \ldots, n-1 \).

2. There exists a Hilbert space \( K \supseteq H \) and \( N \in B(K) \) such that \( N \) is unitarily equivalent to \( \bigoplus_j S_n \), and \( T^j = P_H N^j|_H, j = 0, 1, \ldots, n-1 \).

3. There exists a Hilbert space \( K \supseteq H \) and \( N \in B(K) \) such that \( \|N\| \leq 1, N^n = 0 \), and \( T^j = P_H N^j|_H, j = 0, 1, \ldots, n-1 \).

4. \( I + 2 \text{Re} \sum_{k=1}^{n-1} \lambda^k T^k \geq 0 \) for all \( \lambda \in \mathbb{T} \).

**Proof.** \((1) \implies (2)\) This is a straightforward consequence of Stinespring’s Dilation Theorem. Indeed, after writing \( \phi(X) = P_H \pi(X)|_H \)—under the usual identification of \( H \) with its range under \( V \)—we can take \( N = \pi(S_n) \) (recall that any representation of \( M_n(\mathbb{C}) \) is of the form \( X \mapsto X \otimes I \)).

\((2) \implies (3)\) Trivial.

\((3) \implies (4)\) We have that \( T^j = P_H N^j|_H \) for \( j = 0, 1, \ldots, n-1 \). As compressions are (completely) positive, it is enough to prove the inequality \( I + 2 \text{Re} \sum_{k=1}^{n-1} \lambda^k N^k \geq 0 \) for all \( \lambda \in \mathbb{T} \). Now we can take advantage of the fact that \( N^n = 0 \). Fix \( z \in \mathbb{C} \) with \( |z| < 1 \). We have

\[
I + 2 \text{Re} \sum_{k=1}^{n-1} z^k N^k = I + 2 \text{Re} \sum_{k=1}^{\infty} z^k N^k = \text{Re} \left( I + 2zN(I - zN)^{-1} \right)
= \text{Re} \left( [(I - zN) + 2zN](I - zN)^{-1} \right)
= \text{Re} \left( I + zN \right)(I - zN)^{-1}
= \frac{1}{2} \left( (I + zN)(I - zN)^{-1} + (I - zN^*)(I + zN^*) \right)
= 2(I - zN^*)^{-1}(I - |z|^2 N^* N)(I - zN)^{-1} \geq 0.
\]
Now given $\lambda \in \mathbb{T}$, we have from above that $I + 2 \text{Re} \sum_{k=1}^{n-1} r^k \chi^k T^k \geq 0$ for all $r \in [0, 1)$. The positivity is preserved as $r \nearrow 1$.

Define a linear map $\psi : OS(I, S_n, \ldots, S_{n-1}^n) \to B(H)$ by $\psi(S_n^k) = T^k$. We will show that $\psi$ is ucp. If we take $A_0, \ldots, A_{n-1} \in M_m(\mathbb{C})$ such that $A_0 \otimes I_n + 2 \text{Re} \sum_{k=1}^{n-1} A_k \otimes S_n^k \geq 0$, then by Theorem 6.5 there exists a completely positive map $\phi : C(\mathbb{T}) \to M_m(\mathbb{C})$ with $\phi(z^k) = A_k$ for $k = 0, \ldots, n - 1$. We have

$$\psi^{(m)}\left( A_0 \otimes I_n + 2 \text{Re} \sum_{k=1}^{n-1} A_k \otimes S_n^k \right) = A_0 \otimes \psi(I_n) + 2 \text{Re} \sum_{k=1}^{n-1} A_k \otimes \psi(S_n^k)$$

$$= \phi(1) \otimes I + 2 \text{Re} \sum_{k=1}^{n-1} \phi(z^k) \otimes T^k$$

$$= (\phi \otimes \text{id}) \left( 1 \otimes I + 2 \text{Re} \sum_{k=1}^{n-1} z^k \otimes T^k \right).$$

The expression inside the brackets is a function $C(\mathbb{T}) \to M_n(\mathbb{C})$. For each $\lambda \in \mathbb{T}$ it evaluates to

$$1 \otimes I + 2 \text{Re} \sum_{k=1}^{n-1} \lambda^k \otimes T^k = 1 \otimes \left( 1 + 2 \text{Re} \sum_{k=1}^{n-1} \lambda^k T^k \right) \geq 0.$$  

As $\phi$ is ucp, $\phi \otimes \text{id}$ is positive (this can be seen quickly by using Stinespring) and so the expression $(\phi \otimes \text{id})(\cdot)$ above is positive; thus $\psi^{(m)}$ is positive for any $m$, i.e., completely positive.

Now we can extend it via Arveson’s Extension Theorem to $M_n(\mathbb{C})$. \hfill $\Box$

In the case $n = 2$, Theorem 7.1 characterizes the matricial range of $S_2 = E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Indeed:

**Corollary 7.2.** The extended matricial range of the $2 \times 2$ unilateral shift consists of the set of all operators $T \in B(H)$ (for any dimension of $H$) such that $w(T) \leq 1/2$.

**Proof.** Note that $w(T) \leq 1/2$ implies that $\|T\| \leq 1$, since $\|T\| \leq 2w(T)$. Now combine the case $n = 2$ in Theorem 7.1 with Proposition 4.3. \hfill $\Box$

We remark that for $n > 2$, Theorem 7.1 does not characterize the matricial range of $S_n$, as one is considering the additional requirement that higher powers of $S_n$ are mapped to the corresponding powers of $T$. Forty five years after the above results, the matricial range of $S_n$, $n \geq 3$, has not been characterized. As far as we can tell, there is not even a conjecture of what it could be.

**Remark 7.3.** Fix $n \in \mathbb{N}$. The map $\gamma : \text{span} \{I, S_n, S_n^2, \ldots, S_n^{n-1}\} \to C(\mathbb{T})$ given by $\gamma(S_n^k) = z^k$, where $z$ is the identity function, is never completely contractive. Indeed, every power of the shift is mapped to a unitary. If $\gamma$
were ucc, we would be able to extend it to a ucp map $M_n(C) \to X$ where $X$ is the injective envelope of $C(T)$. Then
\[
\gamma(E_{11}) = \gamma(S_n^{n-1}S_n^{n-1}) \geq \gamma(S_{n-1}^{n-1})^*\gamma(S_{n-1}^{n-1}) = 1 = \gamma(I).
\]
Positivity then implies that $\gamma(E_{22}) = \cdots = \gamma(E_{nn}) = 0$. So, for any $k = 2, \ldots, n,$
\[
0 \leq \gamma(E_{kk}) \leq \gamma(E_{kk}) = 0,
\]
implying that $\gamma(E_{kk}) = 0$ for $k \geq 2$. Then
\[
\gamma(S_n) = \gamma \left( \sum_{k=1}^{n-1} E_{k+1,k} \right) = 0,
\]
a contradiction.

8. CHARACTERIZATIONS OF THE NUMERICAL RADIUS

The following theorem requires no proof, as it just collects equivalences we proved in previous sections. It is a consequence of very subtle ideas.

**Theorem 8.1.** Let $T \in B(H)$. The following statements are equivalent:

1. $w(T) \leq 1$;
2. $\Re(\lambda T) \leq 1$ for all $\lambda \in \mathbb{T}$;
3. there exists a Hilbert space $K \supset H$, and a unitary $U \in B(K)$ such that $T^n = 2P_H U^n|_H$ for all $n \in \mathbb{N}$;
4. there exists a Hilbert space $K \supset H$, and $N \in B(K)$, unitarily equivalent to $\bigoplus_j E_{21}$, such that $T = 2P_H N|_H$;
5. there exists a Hilbert space $K \supset H$, and $N \in B(K)$ with $\|N\| \leq 1$ and $N^2 = 0$, such that $T = 2P_H N|_H$;
6. there exist contractions $Y, Z \in B(H)$, with $Y = Y^*$, such that $T = (I - Y)^{1/2}Z(I + Y)^{1/2}$;
7. there exists $A \in B(H)$, with $0 \leq A \leq I$, such that $\begin{bmatrix} A & 2T \\ 2T^* & I - A \end{bmatrix} \geq 0$;
8. there exists $A \in B(H)$, with $0 \leq A \leq I$, with $T = 2(I - A^*A)^{1/2}A$;
9. there exists $\varphi : M_2(C) \to B(H)$, ucp, with $\varphi(E_{21}) = \frac{1}{2}T$.

**Proof.** Combine Proposition 4.3 with Theorem 4.4, Corollary 5.5, Theorem 5.7 and Theorem 7.1.

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THE MATRICIAL RANGE OF $E_{21}$

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