Quantum dynamics of non-relativistic particles and isometric embeddings

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Abstract

It is considered, in the framework of constrained systems, the quantum dynamics of non-relativistic particles moving on a $d$-dimensional Riemannian manifold $\mathcal{M}$ isometrically embedded in $\mathbb{R}^{d+n}$. This generalizes recent investigations where $\mathcal{M}$ has been assumed to be a hypersurface of $\mathbb{R}^{d+1}$. We show, contrary to recent claims, that constrained systems theory does not contribute to the elimination of the ambiguities present in the canonical and path integral formulations of the problem. These discrepancies with recent works are discussed.

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I. INTRODUCTION

Some recent works have been devoted to the study of non-relativistic particles in curved spaces in the framework of constrained systems \([1,2,3]\). The idea is that the theory of constrained systems \([4,5]\) might shed some light in the long standing problem of the quantization of non-relativistic particles in curved spaces \([6]\). Such a problem consists in a non-relativistic point particle of mass \(M\) moving on a \(d\)-dimensional Riemannian manifold \(\mathcal{M}\) and is described by the Lagrangian \(L = \frac{M}{2} g_{\alpha\beta} \dot{q}^\alpha \dot{q}^\beta\). The tangent vectors \(\dot{q}^\alpha = \frac{d}{dt} q^\alpha\) are the velocities of the particle and \(g_{\alpha\beta}\) is the metric tensor of \(\mathcal{M}\). The Lagrangian \(L\) is non-singular, in contrast to the relativistic case, and the Hamiltonian equations of motion can be immediately obtained

\[
\dot{q}^\alpha = \{q^\alpha, H\},
\]

\[
\dot{p}_\alpha = \{p_\alpha, H\},
\]

\[
H = \frac{1}{2M} g_{\alpha\beta}(q) p^\alpha p^\beta,
\] (1)

where \(p_\alpha\) are the momenta canonically conjugated to \(q^\alpha\). It is clear that the determination of \(\hat{H}\), the quantum counterpart of \(H\) necessary to define the Schrödinger equation, is plagued with severe ordering ambiguities, which represent the greatest difficulty of the problem \([4]\). From the correspondence principle, the unitarity of the quantum evolution, and dimensional analysis, one concludes that \(\hat{H}\) must have the form

\[
\hat{H} = -\frac{\hbar^2}{2M} \frac{1}{\sqrt{g}} \partial_\alpha \sqrt{g} g^{\alpha\beta} \partial_\beta + \kappa \frac{\hbar^2}{M} R,
\] (2)

where \(R\) is the scalar of curvature of \(\mathcal{M}\). There is no consensus in the literature about the dimensionless constant \(\kappa\). Different approaches have led to different values for \(\kappa\) (Reference \([6]\), for instance, contains a recent survey on the subject.). The path integral formulation does not solve these ambiguities. It is possible to read off the operator \(\hat{H}\) directly from the phase space path integral of the quantum evolution operator \(K\) of the problem \([4]\), but then the value of \(\kappa\) will depend on the parameterization used to evaluate the \(K\) path integral.
In the works [1,2,5,6] it is considered the case where $\mathcal{M}$ is a hypersurface of $\mathbb{R}^{d+1}$. The motion of the particle in $\mathbb{R}^{d+1}$ is enforced to take place on $\mathcal{M}$ by the introduction of a Lagrange multiplier. $\hat{H}$ is then obtained in the canonical and path integral formulations by following the standard steps of the theory of constrained systems [7,8]. Although it is well known that the Hamiltonian operators obtained from the canonical and path integral formulations may not coincide in general [11], it is claimed that $\kappa$ could be set unambiguously for each formulation in the constrained systems framework.

The case where $\mathcal{M}$ is a hypersurface of $\mathbb{R}^{d+1}$ is an especial and convenient one because of the existence of an unique (up to a sign) normal vector in all points of $\mathcal{M}$. However, we know that even locally the hypersurfaces of $\mathbb{R}^{d+1}$ correspond only to a small portion of the $d$-dimensional manifolds for $d > 2$. In order to see it, consider the embedding of a $d$-dimensional Riemannian manifold $\mathcal{M}$ in $\mathbb{R}^{d+1}$. $\mathcal{M}$ can be described parametrically by the equations

$$x^i = x^i(q^\alpha), \quad \{i = 1, ..., d + 1; \alpha = 1, ..., d\}$$

such that the rank of the matrix $B^i_\alpha = \frac{\partial x^i}{\partial q^\alpha}$ is maximal. From (3) we have that, for $dx^i$ tangent to $\mathcal{M}$, $ds^2 = \eta_{ij}dx^i dx^j = \eta_{ij}B^i_\alpha B^j_\beta dq^\alpha dq^\beta$, where $\eta_{ij} = \eta^{ij} = \text{diag}(1, ..., 1)$. The embedding is called isometric if the metric tensor $g_{\alpha\beta}$ of $\mathcal{M}$ is given by

$$g_{\alpha\beta} = \eta_{ij}B^i_\alpha B^j_\beta.$$  

A metric tensor in a $d$-dimensional manifold has $\frac{d(d+1)}{2}$ independent components, and we see the right hand side of (4) has only $d + 1$ independent components. By this simple argument, we have that for $d > 2$ equation (4) defines only a restricted class of metrics. The solution is to raise the number of independent components of the right hand side by raising the dimension of the Euclidean space where $\mathcal{M}$ is embedded. Indeed, it is possible to embed isometrically any orientable Riemannian manifold in an Euclidean space of sufficiently high dimension [12]. In the references [3,4], the canonical quantization for the case of $\mathcal{M}$ isometrically embedded in a higher dimensional Euclidean space was considered.
It is quite surprising that for the case of $\mathcal{M}$ isometrically embedded, one gets for $\hat{H}$ an extra contribution proportional to the square of the extrinsic mean curvature of $\mathcal{M}$ \cite{1}. Such a geometrical quantity is not intrinsic to $\mathcal{M}$ and it also appears when the particle is enforced to move on $\mathcal{M}$ due to external potentials (See \cite{13} for a recent approach and for references).

The purpose of the present work is to show that constrained systems theory cannot contribute to the solution of the ambiguities of the problem, and in particular, it does not give a definite answer about the value of $\kappa$. We perform the Hamiltonian analysis of a particle moving in a Riemannian manifold $\mathcal{M}$ isometrically embedded in a high-dimensional Euclidean space. Such a system has a set of second–class constraints, and we show that there is a canonical transformations casting the dynamical equations for the physical variables in the form (I). The canonical quantization of theories with second–class constraints in different canonical variables leads to equivalent physical theories \cite{7}, and thus all ordering ambiguities inherent to (I) are also present in the constrained system framework, and the difficulties to solve them are essentially the same ones of the usual analysis. We will see that although the physical Hamiltonian will be apparently free of ordering ambiguities, they will be hidden in the constraints of the problem. By using the same canonical transformation, we will show that also the constrained path integral formulation has exactly the same ambiguity problems of the usual analysis. Our results are compatible with the idea that Hamiltonian and Lagrangian descriptions of singular systems are fully equivalent \cite{7}.

The next section is devoted to the canonical formulation of the problem. In Sect. II we discuss the path integral approach, and the last section is left to some concluding remarks.

\section*{II. CANONICAL FORMULATION}

Consider a particle of mass $M$ moving in $R^{d+n}$, $n > 0$. We can enforce the motion of the particle to take place on a $d$-dimensional submanifold $\mathcal{M}$ of $R^{d+n}$ by imposing $n$ constraints $f^A(x) = 0$. Greek indices run over $(1, ..., d)$, lower case roman ones over $(1, ..., d + n)$, and
upper case ones over \((1,...,n)\). The Lagrangian of the particle moving on \(\mathcal{M}\) is
\[
L = \frac{M}{2} \eta_{ij} \dot{x}^i \dot{x}^j + \lambda_A f^A(x),
\]
where \(\lambda_A\) are \(n\) Lagrange multipliers. The functions \(f^A(x)\) must be non-degenerated in the sense that the \(n\) covariant vectors \(\frac{\partial f^A}{\partial x^i}\) are linearly independents for all points of \(\mathcal{M}\). Such a condition assures that the tangent space of \(\mathcal{M}\) has \((d+n)-n = d\) dimensions for all points of \(\mathcal{M}\). The Lagrangian \(L\) is singular and when going to the Hamiltonian formalism we get constraints. In this case the first-stage constraints are
\[
\Phi_1^A = P^A = 0,
\]
where \(P^A\) are the momenta canonically conjugated to the variables \(\lambda_A\). We can now construct the hamiltonian \(H^{(1)}\)
\[
H^{(1)} = \frac{1}{2M} \eta^{ij} \pi_i \pi_j - \lambda_A f^A(x) + \xi_A \Phi_1^A,
\]
where \((x^i; \pi_j)\) are canonical variables and \(\xi_A\) another multipliers. By following Dirac procedure, we check for the existence of \((m+1)\)-stage constraints by verifying the conservation in time of the \(m\)-stage ones. We have the following new constraints
\[
\Phi_2^A = \Phi_1^A = f^A(x) = 0,
\]
\[
\Phi_3^A = M \Phi_2^A = \pi^i \partial_i f^A = 0,
\]
\[
\Phi_4^A = M \Phi_3^A = \pi^i \pi^j \partial_i \partial_j f^A + M \lambda_B S^{BA} = 0,
\]
where \(S^{BA} = \eta^{ij} \partial_i f^B \partial_j f^A\). The condition \(\Phi_1 = 0\) determines \(\xi_A\) and no more constraints arise. One can check by using elementary properties of determinants that \(S^{BA}\) is non-singular if \(f^A\) are non-degenerated. The constraints (8) and (9) form a set of second-class ones, in contrast to the relativistic case, where it is well known that due to the reparameterization invariance we also have first-class constraints. Furthermore, \(\Phi_1\) and \(\Phi_4\) are constraints of special form (8), and they can be used to eliminate the variables \(\lambda_A\) and \(P^A\). There remain \(2(d+n)\) variables and \(2n\) constraints, what shows that in fact the system has \(2d\) degrees of freedom. The classical equations
\[ \dot{x}^i = \{ x^i, \bar{H} \}, \]
\[ \dot{\pi}_i = \{ \pi_i, \bar{H} \}, \]
\[ \Phi_2^A = f^A(x) = 0, \]
\[ \Phi_3^A = \pi^i \partial_i f^A(x) = 0, \]
\[ \text{(9)} \]

with \( \bar{H} = \frac{1}{2M} \eta_{ij} \pi^i \pi^j - \lambda^A f_A \), are dynamically equivalent to (1). Note that due to such a choice for \( \bar{H} \), the symplectic structure of the phase space \( (x^i; \pi_j) \) is just the usual Poisson brackets. The Hamiltonian \( \bar{H} \big|_{\Phi=0} \) appears to be free of ordering ambiguities, and thus a good starting point for canonical quantization. However, it indeed has the ambiguities entangled to those ones of the constraints. In order to see it, we will perform a canonical transformation such that the constraints will be free of ordering ambiguities.

Our canonical transformation will be essentially a coordinate transformation. In order to construct it, first note that due to the assumption that the vectors \( \partial_i f^A \) are linearly independent, the neighbourhood of \( \mathcal{M} \) is spanned by the coordinates \( (q^\alpha, f^A) \), where \( q^\alpha \) are coordinates in the submanifold \( \mathcal{M}_W \) of \( \mathbb{R}^{d+n} \) defined by \( f^A \) constants, \textit{i.e.}

\[ f^A(x(q)) = W^A \]
\[ \text{(10)} \]

identically for constants \( W^A \). The vectors \( B^i_\alpha = \frac{\partial x^i}{\partial q^\alpha} \) and \( \partial_i f^A \) span respectively the tangent and normal spaces of \( \mathcal{M}_W \). The canonically conjugated variables \( (x^i; \pi_j) \) and \( (q^\alpha, f^A; p_\beta, \theta_B) \) are related by the following canonical transformation

\[ x^i = x^i(q, f), \]
\[ \pi_i = \frac{\partial q^\alpha}{\partial x^i} p_\alpha + \frac{\partial f^A}{\partial x^i} \theta_A, \]
\[ \pi^i = B^i_\alpha p_\alpha + \frac{\partial x^i}{\partial f^A} \theta^A. \]
\[ \text{(11)} \]

In the new canonical variables \( (q^\alpha, f^A; p_\beta, \theta_B) \), the constraints \( \Phi_2 \) and \( \Phi_3 \) are given by the following ambiguity-free expressions

\[ \Phi_2^A = f^A = 0, \]
\[ \Phi_3^A = \theta^A = 0. \]
\[ \text{(12)} \]
On the other hand, \((q^\alpha, p_\beta)\) obey the usual equations \((\text{I})\). This finally shows that the canonical quantization based in \((\text{II})\) and in \((\text{I})\) are plagued by the same ordering ambiguities. In the equations \((\text{I})\), the ordering ambiguities are clearly in the constraint \(\Phi_3\), which involves \(\pi_i\) and a function of \(x^i\). The constraint \(\Phi_3\) has many inequivalent and equally acceptable operatorial representations. In the references \([1,3,6]\), for instance, it is used a symmetric expression for the operatorial representation of \(\Phi_3\), and it is obtained \(\kappa = 0\) and a contribution proportional to extrinsic geometrical quantities. In the other hand, in the reference \([2]\) another choice for the operators lead to \(\kappa = 0\) and no extrinsic contributions. We could yet choose another representation compatible with the requirement of hermiticity of the theory. The constraint \(\Phi_3\) is used explicitly in order to determine \(\hat{H}\), and different representations would lead to distinct \(\hat{H}\).

### III. PATH-INTEGRAL FORMULATION

The derivation of the path integral for system with second-class constraints is straightforward \([8]\). Note first that due to that the symplectic structure of the phase space \((x^i, \pi_j)\) is given by the usual Poisson brackets, the Liouville measure of \((x^i; \pi_j)\) will be simply \(\prod_{i=1}^{d+n} dx^i d\pi_i\). Thus, the quantum evolution operator \(K\) of the system governed by \((\text{I})\) will be given by

\[
K = \int D\lambda DP \, Dx \, D\pi \, \delta(\Phi_1) \delta(\Phi_2) \delta(\Phi_3) \delta(\Phi_4) \exp \left\{ \pi_i \dot{x}^i - \frac{1}{2M} \pi_i \pi^i + \lambda_A f^A - \xi_A \Phi^A_1 \right\}.
\]

(13)

The integration over the variables \(\lambda\) and \(P\) is straightforward and allows us to write \(\lambda\) as a function of the variables \((x^i; \pi_j)\). After doing it, we have

\[
K = \int Dx \, D\pi \, \delta \left( f^A(x) \right) \delta \left( \pi^i \partial_i f^A(x) \right) \exp \left\{ \pi_i \dot{x}^i - \frac{1}{2M} \pi_i \pi^i + \lambda_A f^A \right\}.
\]

(14)

The integration over the momenta and constraints using the midpoint parameterization allows us to get the Weyl ordered Hamiltonian operator directly from \((\text{I})\) \([10]\). This was
done for the case of \( \mathcal{M} \) embedded in \( R^{d+1} \) in [5,6], and it was obtained a non vanishing \( \kappa \) and extra extrinsic terms. However, we can check that in the new variables \((q^\alpha, f^A; p_\beta, \theta_B)\) (14) reads

\[
K = \int Dq \, Dp \, \exp \left\{ p_\alpha \dot{q}^\alpha - \frac{1}{2M} p_\alpha p^\alpha \right\}.
\] (15)

The expression (15) is the same one we would obtain starting from the system described by (1), and thus it has exactly the same ambiguities problems. The results of [5,6] are a consequence of the choice of the phase space variables and thus one cannot claim that the Weyl ordered \( \hat{H} \) can be determined unambiguously in the path integral formulation of the problem.

**IV. FINAL REMARKS**

To summarize, we have shown that constrained systems theory does not contribute to an unambiguous description of the quantum dynamics of non-relativistic particles in curved spaces. This is not a great surprise since we know that constrained systems theory does not solve the analogous ambiguities of the relativistic case. For such a case, we have, besides of the ambiguities described here, the subtle issue of time-depending constraints [14,15]. However, it is possible to avoid the latter by assuming a static spacetime, and recently we have shown that the ambiguities questions remain unsolved in this case [16]. In the works [1–6], the values for \( \kappa \) are obtained by using implicit assumptions about the ordering of \( \Phi_3 \) in (9) and the parameterization used to evaluate (14). Such assumptions are sometimes privileged ones due to the phase space coordinates used, but nevertheless they are arbitrary and not unique. It is clear that canonical and path-integral quantizations do not commute with canonical transformations in the classical phase space.

We finish noticing that the use of \( \dot{f}^A(x) = 0 \) instead of \( f^A(x) = 0 \) as the constraints to enforce the particle to move on \( \mathcal{M} \) does not lead to any improvement in the ordering ambiguities. Such a possibility was first tried for the case of \( R^{d+1} \) in [2]. In our case, starting with
\[ L = \frac{M}{2} \eta_{ij} \dot{x}^i \dot{x}^j + \lambda_A \dot{f}^A \]  \hspace{1cm} (16)

instead of (3), would lead to the following Hamiltonian equations

\[ \dot{x}^i = \{ x^i, H' \}, \]
\[ \dot{\pi}_i = \{ \pi_i, H' \}, \]
\[ \Phi^A_1 = P^A = 0, \]
\[ \Phi^A_2 = \eta^{ij} \partial_i f^A \left( \lambda_B \partial_j f^B - \pi_i \right) = 0, \]  \hspace{1cm} (17)

where \( H' = \frac{\eta^{ij}}{2M} \left( \pi_i - \lambda_A \partial_i f^A \right) \left( \pi_j - \lambda_A \partial_j f^A \right) + \xi_A P^A \), and \( \xi_A \) is determined from the condition \( \dot{\Phi}^A_2 = 0 \). The quantization of the system governed by (17) will be also plagued with severe ordering ambiguities. Although we can use \( \Phi^A_1 \) and \( \Phi^A_2 \) to eliminate the variables \( P^A \) and \( \lambda_A \), the resulting Hamiltonian will have several ordering dependent terms. The path integral formulation is equivalent to the case discussed in the Sect. II.

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