Instability of breathers in the topological discrete sine-Gordon system

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Abstract

It is demonstrated that the breather solutions recently discovered in the weakly coupled topological discrete sine-Gordon system are spectrally unstable. This is in contrast with more conventional spatially discrete systems, whose breathers are always spectrally stable at sufficiently weak coupling.

1 Introduction

There has long been interest in solitonic field theories defined on spacetimes where time is continuous but space is a discrete lattice. The motivation for this scenario varies from physical reality (condensed matter and biophysics, see [1] and references therein), through mathematical convenience (integrable quantum field theory [2]) to computational necessity (when numerically simulating soliton dynamics on a computer). One particularly interesting programme, whose motivation is of the last type, is that of Ward. The aim here is to find spatially discrete versions of Bogomol’nyi field theories which preserve the Bogomol’nyi properties of their continuum counterparts: a topological lower bound on soliton energy saturated by solutions of a first order “self-duality” equation, these solutions occurring in continuous families, or “moduli spaces.” Such systems appear to capture the qualitative features of the continuum soliton dynamics even on very coarse lattices, making them ideal candidates for computer simulations. The programme is essentially complete for one spatial dimension, that is, such a “topological” discretization has been constructed for any kink-bearing nonlinear Klein-Gordon model [3]. Results in higher dimensions are more partial [4, 5, 6].

The most heavily studied example is the topological discrete sine-Gordon (TDSG) system [7] and its variants [8]. Here one has an explicit, continuous one-dimensional moduli space of static kink solutions, despite the discrete nature of space, and the kink dynamics is similar to continuum kink dynamics even when the kink structure covers only 2 or 3 lattice sites. This should be contrasted with the conventional discretization, the Frenkel-Kontorova model, in which the kink dynamics is extremely complicated [9].

Kinks and antikinks are not the only solitons of the continuum SG equation. It also supports breathers, that is, spatially localized, time periodic, oscillatory solutions. It was
recently proved [10] that the TDSG system also supports breathers, provided the lattice is sufficiently coarse. So the question arises: to what extent are these breathers similar to their continuum counterparts? In this note we will demonstrate one crucial difference: discrete breathers are spectrally unstable.

2 Breathers in the TDSG system

The TDSG system is defined on spacetime \( \mathbb{Z} \times \mathbb{R} \). Space is a regular discrete lattice of spacing \( h \in (0,2] \) and spatial position will be denoted by \( n \in \mathbb{Z} \). Time is continuous, denoted \( t \). The field \( \psi_n(t) \) evolves according to the differential-difference equation

\[
\ddot{\psi}_n = \frac{4 - h^2}{4h^2} \cos \psi_n (\sin \psi_{n+1} + \sin \psi_{n-1}) - \frac{4 + h^2}{4h^2} \sin \psi_n (\cos \psi_{n+1} + \cos \psi_{n-1}).
\]

Note that this tends, as \( h \to 0 \), to the usual sine-Gordon PDE (in laboratory coordinates \( x = nh, t \)) for \( \phi = 2\psi \). Equation (1) supports a continuous translation orbit of static kink solutions,

\[
\psi_n = 2 \tan^{-1} e^{a(n-b)}, \quad b \in \mathbb{R}
\]

where \( a = \ln[(2+h)/(2-h)] \). The conserved energy associated with (1), call it \( E \), satisfies \( E \geq 1 \) for all \( \psi \) satisfying kink boundary conditions \( (\psi_n \to 0, n \to -\infty; \psi_n \to \pi, n \to \infty) \), and the static solutions (2) saturate this bound. Note that as \( h \to 2 \) the kinks tend to step functions, which is why one insists \( h \leq 2 \).

Breather solutions to (1) of any period \( T > 2\pi \) where \( T/2\pi \notin \mathbb{Z} \), arise as follows. Consider the case \( h = 2 \). Then (1) reduces to

\[
\ddot{\psi}_n = -\frac{1}{2} \sin \psi_n (\cos \psi_{n+1} + \cos \psi_{n-1}),
\]

which supports the “one-site breather”

\[
\Psi_n(t) = \begin{cases} 0 & n \neq 0, \\
\vartheta(t) & n = 0,
\end{cases}
\]

where \( \vartheta(t) \) is any \( T \)-periodic solution of the pendulum equation

\[
\ddot{\vartheta} + \sin \vartheta = 0.
\]

By applying an implicit function theorem argument, in the manner of MacKay and Aubry [11], one obtains the following [10]: For all \( T > 2\pi \), \( T \notin 2\pi \mathbb{Z} \), there exists a continuous deformation of the one-site breather (1) through breather solutions of (1) with lattice spacing \( h \in [2-\epsilon, 2] \), where \( \epsilon > 0 \) is sufficiently small. So breather solutions exist for all \( h \) sufficiently close to 2, provided the breather period is nonresonant (not a multiple of \( 2\pi \)).

By truncating the system to \( N = 2m + 1 \) sites \( (-N \leq n \leq N) \) and applying a Newton-Raphson scheme to \( P_T - \text{Id} \), where \( P_T : \mathbb{R}^{2N} \to \mathbb{R}^{2N} \) is the period \( T \) Poincaré return map for system (1), one can construct these breather solutions numerically [12]. One starts at \( h = 2 - \delta h \), \( \delta h \) small, with the one-site breather as an initial guess, then iterates the NR scheme until \( P_T - \text{Id} \) converges (within some tolerance) to 0. Repeating this for \( h = 2 - 2\delta h \) with the newly found breather as initial guess, one works piecemeal away from the \( h = 2 \) limit. In this way a portrait of the existence domain of breathers in the \((T,h)\) parameter space has been built up [14].
3 Spectral instability

One says that a period $T$ solution $q(t)$ of a dynamical system on phase space $M$ is spectrally stable if the spectrum of the associated Floquet map $Fl : T_{q(0)}M \to T_{q(T)}M = T_{q(0)}M$ is contained within the closed unit disk $D \subset \mathbb{C}$. Here $Fl = D P_T$, that is the period $T$ return map of the linearized flow about $q$. Spectral stability is necessary, but not sufficient, for linear and hence practical stability. We shall show that the one-site breather (4) is a spectrally unstable solution of (3). Since the spectrum of $Fl$ varies continuously under continuous perturbation, it follows that all breathers, for $h$ sufficiently close to 2, are unstable.

Let $\psi_n(t)$ be a solution of (1), and $\delta \psi_n(t)$ be a solution of the linearized flow about $\psi_n(t)$. Then

$$\ddot{\delta \psi}_n = -\frac{4 - h^2}{4h^2} \sin \psi_n (\sin(n+1) + \sin(n-1)) + \frac{4 + h^2}{4h^2} \cos \psi_n (\cos(n+1) + \cos(n-1)) \dot{\delta \psi}_n$$

$$+ \frac{4 - h^2}{4h^2} \cos \psi_n (\cos(n+1) \delta \psi_{n+1} + \cos(n-1) \delta \psi_{n-1})$$

$$+ \frac{4 + h^2}{4h^2} \sin \psi_n (\sin(n+1) \delta \psi_{n+1} + \sin(n-1) \delta \psi_{n-1}).$$

(6)

In particular, if $h = 2$ and $\psi_n(t)$ is the one-site breather (4), one finds that

$$\ddot{\delta \psi}_n = \begin{cases} -\delta \psi_n & |n| > 1 \\ -\frac{1}{2}(1 + \cos \vartheta) \delta \psi_n & |n| = 1 \\ -\cos \vartheta \delta \psi_n & n = 0. \end{cases}$$

(7)

Truncating the lattice to size $N = 2m + 1$ (our result is independent of $m$), the Floquet map takes the initial data of $\delta \psi$ to the final data (at $t = T$),

$$Fl : \begin{bmatrix} \delta \psi_{-m}(0) \\ \delta \psi_{-m}(0) \\ \vdots \\ \delta \psi_m(0) \\ \delta \psi_m(0) \end{bmatrix} \mapsto \begin{bmatrix} \delta \psi_{-m}(T) \\ \delta \psi_{-m}(T) \\ \vdots \\ \delta \psi_m(T) \\ \delta \psi_m(T) \end{bmatrix}$$

(8)

Of course, $Fl$ is linear by linearity of (4). In fact, since (4) defines a Hamiltonian flow, $Fl$ is symplectic with respect to the natural symplectic structure on $\mathbb{R}^{2N}$, that is

$$Fl^T \Omega Fl = \Omega,$$

(9)

where $\Omega$ is the block diagonal matrix

$$\Omega = \text{diag}(J, \ldots, J), \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$ 

(10)

Note that $Fl$ is continuously connected through $GL(2N, \mathbb{R})$ to $I_{2N}$ (by the time evolution (4)), so (3) implies $\det Fl = 1$. Symplectomorphicity imposes many more constraints on the spectrum of $Fl$ [13], but we shall not need these.
Since system (7) is decoupled, $F_l$ is block diagonal with one symplectic $2 \times 2$ block $F_l_n$ per lattice site. To demonstrate instability, therefore, it suffices to exhibit one block with an eigenvalue outside $D = \{ z \in \mathbb{C} : |z| \leq 1 \}$. The offending block is $F_l_1$ (or $F_l_{-1}$):

$$F_l_1 = \begin{bmatrix} y_1(T) & y_2(T) \\ \dot{y}_1(T) & \dot{y}_2(T) \end{bmatrix}$$

where $y_1, y_2$ form a basis of solutions of

$$\ddot{y} + \frac{1}{2}(1 + \cos \vartheta(t))y = 0$$

with $y_1(0) = \dot{y}_2(0) = 1$, $\dot{y}_1(0) = y_2(0) = 0$. Note that, as argued above, $\text{det} F_l_1 = 1$, so if $|\text{tr} F_l_1| > 2$, the eigenvalues are a real conjugate pair $\{ \lambda, 1/\lambda \}$ with $|\lambda| > 1$, and $\Psi_n(t)$ is spectrally unstable.

It is numerically trivial to solve (12) (coupled to the pendulum equation (5) for $\vartheta(t)$) over $[0,T]$ and hence compute $\text{tr} F_l_1$. The results are shown in figure 1. Clearly $\text{tr} F_l_1 > 2$ for all $T$ within our range (the limit $T \to \infty$ is numerically inaccessible), so both $F_l_1$ and $F_l_{-1}$ have a real eigenvalue exceeding 1, confirming our claim that the breathers are unstable for $h$ close to 2.

It is possible, of course, that as $h$ decreases substantially below 2, the two offending eigenvalues move into $D$, so that spectral stability is recovered. To eliminate this possibility, we have computed the spectrum of $F_l$ at a sample of points within the breather existence domain. This domain, whose construction is described in detail in [10], is depicted in $(\omega, h)$ space ($\omega = 2\pi/T$ being breather frequency) in figure 2, with the sampled points marked by crosses. Direct computation of the spectrum of $F_l$ is certainly a much cruder approach to numerical spectral stability analysis than that of Aubry [13]. However, since we seek only to confirm instability, rather than examine bifurcations between stability and instability, a simple approach is justified.

At each point sampled, $F_l$ was constructed by solving (3) approximately using a 4th order Runge-Kutta scheme on an 11 site lattice. Note that no decoupling occurs now that $h \neq 2$. Numerical accuracy of $F_l$ was tested by measuring the deviation of $F_l$ from symplectomorphicity by comparing the euclidean norm $||\text{Err}||$ of

$$\text{Err} = F_l^T \Omega F_l - \Omega, \quad ||M|| := (\sum_{i,j} M_{ij}^2)^{\frac{1}{2}},$$

with $||F_l||$. In each case it was found that the largest eigenvalue of $F_l$ substantially exceeds 1. The results are summarized in table 4.

### 4 Conclusion

This paper reinforces the point that breathers in the weakly coupled TDSG system are really quite different from continuum sine-Gordon breathers. Their domain of existence does not extend to the continuum limit, their initial profiles do not resemble continuum breathers sampled on a lattice [10], and they are spectrally unstable. In this last property, they are also different from breathers constructed in conventional discrete systems (oscillator chains),
Table 1: The largest eigenvalue of $F_l$ for various breathers.

| $\omega$ | $h$ | $||\text{Err}||$ | $||F_l||$ | $|\lambda_{\text{max}}|$ | $\omega$ | $h$ | $||\text{Err}||$ | $||F_l||$ | $|\lambda_{\text{max}}|$ |
|--------|----|----------------|---------|----------------|--------|----|----------------|---------|----------------|
| 0.40   | 1.83| 0.0743         | 28.6358 | 45.2109        | 0.65   | 1.89| 0.0062         | 4.4025  | 4.8864 |
| 0.40   | 1.90| 0.0854         | 22.3127 | 30.8483        | 0.70   | 1.79| 0.0053         | 3.9849  | 3.9052 |
| 0.45   | 1.87| 0.0916         | 15.7065 | 21.7614        | 0.70   | 1.92| 0.0004         | 3.9330  | 3.5739 |
| 0.45   | 1.95| 0.0084         | 10.6190 | 14.7902        | 0.75   | 1.83| 0.0031         | 3.1875  | 2.9886 |
| 0.55   | 1.85| 0.0011         | 7.6868  | 10.1095        | 0.75   | 1.90| 0.0013         | 3.0531  | 2.8147 |
| 0.60   | 1.75| 0.0040         | 5.5526  | 6.7439         | 0.80   | 1.88| 0.0008         | 2.1615  | 2.2982 |
| 0.60   | 1.82| 0.0144         | 5.5808  | 7.3534         | 0.80   | 1.96| 0.0006         | 2.1080  | 2.1469 |
| 0.65   | 1.78| 0.0040         | 4.4930  | 4.8864         | 0.90   | 1.95| 0.0002         | 2.6793  | 1.4589 |

which are ubiquitously stable, at least on sufficiently coarse lattices\cite{13}. We conclude that, while the TDSG system captures the dynamics of kinks and antikinks quite faithfully even for large $h$, the same is not true of kink-antikink bound states, that is, breathers.

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References

[1] R. Boesch, C.R. Willis and M. El-Batanouny, “Spontaneous emission of radiation from a discrete sine-Gordon kink”\textit{Phys. Rev. B40} (1989) 2284.

[2] L.D. Faddeev and L.A. Takhtajan, \textit{Hamiltonian Methods in the Theory of Solitons} (Springer, London, 1987).

[3] J.M. Speight, “Topological discrete kinks”\textit{Nonlinearity} 12 (1999) 1373.

[4] R.S. Ward, “Bogomol’nyi bounds for two-dimensional lattice systems”\textit{Commun. Math. Phys.} 184 (1997) 397.

[5] T. Ioannidou, “Soliton dynamics in a novel discrete $O(3)$ sigma model in (2 + 1) dimensions”\textit{Nonlinearity} 10 (1997) 1357.

[6] R. Leese, “Discrete Bogomol’nyi equations for the nonlinear $O(3)$ sigma model in (2+1) dimensions”\textit{Phys. Rev. D40} (1989) 2004.

[7] J.M. Speight and R.S. Ward, “Kink dynamics in a novel discrete sine-Gordon system”\textit{Nonlinearity} 7 (1994) 475.

[8] W.J. Zakrzewski, “A modified discrete sine-Gordon system”\textit{Nonlinearity} 8 (1995) 517.

[9] M. Peyrard and M.D. Kruskal, “Kink dynamics in the highly discrete sine-Gordon system”\textit{Physica D14} (1984) 88.
[10] M. Haskins and J.M. Speight, “Breathers in the weakly coupled topological discrete sine-Gordon system” *Nonlinearity* **11** (1998) 1651.

[11] R.S. MacKay and S. Aubry, “Proof of existence of breathers for time-Reversible or Hamiltonian networks of weakly coupled oscillators” *Nonlinearity* **7** (1994) 1623.

[12] J.L. Marín and S. Aubry, “Breathers in nonlinear lattices: numerical calculation from the anticontinuous limit” *Nonlinearity* **9** (1996) 1501.

[13] S. Aubry, “Breathers in nonlinear lattices: existence, stability and quantization” *Physica D* **103** (1997) 201.
Figure captions

Figure 1: Plot of \(\text{tr}F_{l_1}\) against breather period \(T\). Note that \(\text{tr}F_{l_1} > 2\) for all \(T\).

Figure 2: Domain of existence of discrete breathers in \((\omega, h)\) space. The jagged dotted curve is the theoretical lower boundary obtained by a phonon resonance argument. The solid curve is the actual lower boundary, generated numerically as in [10]. The small \(\omega\) region (effectively \(\omega < 1/3\)) is numerically inaccessible. Crosses mark those breathers whose spectral instability was confirmed numerically (see table 1).
Figure 1
