1. Introduction

1.1. Aim of the paper. This paper can be viewed as part of a series of papers \[1, 6, 7, 8\] devoted to developing an arithmetic analogue of classical differential geometry \[10\]; for the convenience of the reader, however, the present paper is written so as to be entirely self-contained. Expressed in a naive form, the main idea in this series of papers is to replace functions on smooth manifolds by integer numbers, to replace coordinates by prime numbers, and to replace differentiation acting on functions by “arithmetic derivative” operators acting on numbers. In this setting the “arithmetic derivative” of an integer \(n \in \mathbb{Z}\) with respect to a prime \(p\) is taken to be equal to the Fermat quotient \(\frac{n^p - n}{p}\). In order to make this idea work one needs to “geometrize” it in the same sense in which Lie and Cartan “geometrized” differential equations. We refer to the monographs \[4\] and \[5\] for a comprehensive introduction to this program. Cf. also \[2, 3\] for some purely arithmetic applications of this theory. This whole line of research is, of course, part of the general, well established, effort to unveil and exploit the analogies between numbers and functions.

The papers \[1, 6, 7, 8\] were mainly concerned with an arithmetic analogue of Chern connection. On the other hand, in the monograph \[5\], first steps were taken, in a special case, towards developing an arithmetic analogue of the Levi-Cività connection. The Levi-Cività story in \[5\] has, however, at least two limitations. First, if one fixes a prime \(p\) then the story in \[5\] only deals, in some sense, with an analogue of metrics of cohomogeneity one: indeed, for \(p\) fixed, there is only one “arithmetic derivative” there acting on the coefficients of the metric. Second, for varying \(p\), the story in \[5\] only deals with “metrics with constant coefficients”. Although, in arithmetic, such metrics still lead to non-zero curvature, restricting attention to such metrics is a drastic limitation. In the present paper we would like to revisit from scratch the arithmetic Levi-Cività story in \[5\] by putting it in a more general context: in this context both limitations referred to above will disappear. The first limitation (where \(p\) is fixed) will be overcome by considering several “arithmetic derivatives” corresponding to the primes dividing \(p\) in a number field. The second limitation (where \(p\) varies) will be overcome by constructing algebraizing correspondences for our Levi-Cività connection (in the same sense in which we constructed algebraizing correspondences for the Chern connection in \[8, 5\]).

1.2. Organization of the paper. We start, in Section 2, by introducing the main definitions and stating our main results on Levi-Cività connections. Section 3 contains the main definitions and statements of results on curvature. Section 4 contains the proofs of our results. Section 5 is an Appendix devoted to revisiting (from a
somewhat non-standard viewpoint) the classical differential geometric setting; this Appendix is not logically necessary for the understanding of the paper; rather, it offers the main motivation/blueprint for the arithmetic story.

1.3. General conventions and notation. For background on schemes and formal schemes we refer to Chapter 2 in [9]. For background on local fields and number fields we refer to Chapter 1 of [12].

Unless otherwise stated all rings are commutative and unital. When commutativity is not assumed we will talk about not necessarily commutative rings; in this case homomorphisms and antihomomorphisms will be unital and, to simplify notation and terminology, antihomomorphisms will often be referred to, again, as homomorphisms. Also, in this case, we will often use the same letter to denote a ring and its opposite; the context will always indicate the precise meaning of our notation. By a Lie ring we understand a Lie \( \mathbb{Z} \)-algebra. Any (not necessarily commutative) ring can be viewed as a Lie ring with respect to the commutator. For any (not necessarily commutative) monoid \( M \) we denote by \( \mathbb{Z}M \) the (not necessarily commutative) monoid ring on \( M \). For a ring \( A \) we denote by \( \mathfrak{gl}_n(A) \) the (not necessarily commutative) ring of \( n \times n \) matrices with coefficients in \( A \) and by \( GL_n(A) \) the group of invertible elements of \( \mathfrak{gl}_n(A) \). For any rings \( A \) and \( B \) and any set theoretic map \( f : A \to B \) we still denote by \( f : \mathfrak{gl}_n(A) \to \mathfrak{gl}_n(B) \) the induced map; so for any \( n \times n \) matrix \( a = (a_{ij}) \) with entries \( a_{ij} \in A \) we let \( f(a) = (f(a_{ij})) \) the \( n \times n \) matrix with entries \( f(a_{ij}) \). For a matrix \( a = (a_{ij}) \in \mathfrak{gl}_n(A) \) we denote by \( a^t \) the transpose of \( a \); if in addition \( p \in \mathbb{Z} \) is a prime then we denote by \( a^{(p)} = (a_{ij}^p) \) the matrix with entries \( a_{ij}^p \).

All schemes and formal schemes are assumed separated; formal schemes are assumed Noetherian.

For \( A \) a ring (or \( X \) a Noetherian scheme) and a fixed prime \( p \in \mathbb{Z} \) (always assumed given in our context) we always denote by \( \hat{A} \) (respectively \( \hat{X} \)) the \( p \)-adic completion of \( A \) (respectively \( X \)). As a rule ring homomorphisms \( A \to B \) and the corresponding morphisms between their spectra \( Spec \, A \to Spec \, B \), or between the formal spectra \( Spf \, \hat{B} \to Spf \, \hat{A} \), will be denoted by the same letters.

1.4. Acknowledgements. The author is indebted to Yu.I.Manin for inspiring conversations. The present work was partially supported by the Max Planck Institute for Mathematics in Bonn and by the Simons Foundation (award 311773).

2. Connections

We start by recalling some basic terminology from [4, 5].

2.1. \( p \)-adic connections. Let \( A \) be a ring and let \( p \in \mathbb{Z} \) be an odd prime. By a Frobenius lift on \( A \) we understand a ring endomorphism \( \phi = \phi^A : A \to A \) reducing \( \mod p \) to the \( p \)-power Frobenius \( A/pA \to A/pA \). By a \( p \)-derivation we understand a map of sets \( \delta = \delta^A : A \to A \) such that for all \( a, b \in A \),

1) \( \delta(a + b) = \delta a + \delta b + \sum_{k=1}^{p-1} \binom{p}{k} a^k b^{p-k} \),

2) \( \delta(ab) = a^p \delta b + b^p \delta a + p(\delta a)(\delta b) \).

If \( \delta : A \to A \) is a \( p \)-derivation then the map \( \phi = \phi^A : A \to A \) defined by

\[\phi(a) = a^p + p\delta a\] (2.1)
is a Frobenius lift and we say that \( \phi \) and \( \delta \) are attached to each other. Conversely if \( A \) is \( p \)-torsion free (i.e., \( p \) is a non-zero divisor in \( A \)) then any Frobenius lift \( \phi: A \to A \) defines a unique \( p \)-derivation \( \delta: A \to A \) by the formula (2.1).

There is a subtle difference between the roles of Frobenius lifts and \( p \)-derivations in the theory of arithmetic differential equations [4]: roughly, enlarging formal \( p \)-adic geometry by “adjoining \( \delta \)” is equivalent to enlarging the same geometry by “adjoining \( \phi \) plus certain divergent power series”. However this difference will not be relevant in the present paper.

Assume \( X \) is a Noetherian scheme. A Frobenius lift on \( X \) (respectively on \( \hat{X} \)) will mean a scheme (respectively a formal scheme) endomorphism whose reduction mod \( p \) is the \( p \)-power Frobenius. A \( p \)-derivation on \( \hat{X} \) will mean a map of sheaves of sets \( \delta^\hat{X}: \mathcal{O}_{\hat{X}} \to \mathcal{O}_X \) which is a \( p \)-derivation on each open set. We usually denote by \( \phi^\hat{X}: \hat{X} \to \hat{X} \) the attached Frobenius lift. We sometimes write \( \delta^X, \phi^X \), or even \( \delta, \phi \), instead of \( \delta^\hat{X}, \phi^\hat{X} \). If \( X \) is affine to give a \( p \)-derivation on \( \hat{X} \) is the same as to give a \( p \)-derivation on \( \mathcal{O}(\hat{X}) \). If \( \mathcal{O}_X \) is \( p \)-torsion free then to give a \( p \)-derivation on \( \hat{X} \) is the same as to give a Frobenius lift on \( \hat{X} \).

If \( X \) and \( \delta^X \) are as above and \( Y \subset X \) is a closed subscheme we say that \( Y \) is \( \delta^X \)-horizontal (respectively \( \phi^X \)-horizontal) if the ideal defining \( \hat{Y} \) in \( \hat{X} \) is sent into itself by \( \delta^\hat{X} \) (respectively by \( \phi^\hat{X} \)). If \( Y \) is \( \delta^X \)-horizontal we have an induced \( p \)-derivation \( \delta^Y \) on \( \hat{Y} \). If \( \mathcal{O}_Y \) is \( p \)-torsion free then \( Y \) is \( \delta^X \)-horizontal if and only if it is \( \phi^X \)-horizontal.

**Definition 2.1.** Let \( X \) be a scheme of finite type over a Noetherian ring \( \mathcal{O} \) and let \( \delta^\mathcal{O} \) be a \( p \)-derivation on \( \mathcal{O} \). A \( p \)-adic connection on \( X \) is a \( p \)-derivation \( \delta = \delta^X = \delta^\hat{X} \) on \( \hat{X} \) which extends the \( p \)-derivation \( \delta = \delta^\mathcal{O} \) on \( \mathcal{O} \).

As a rule, in this paper, we will only be interested in \( p \)-adic connections \( \delta^G \) on smooth group schemes \( G \) over \( \mathcal{O} \). They should be viewed as arithmetic analogues of connections in principal bundles in the sense of classical differential geometry. The case when a subgroup scheme \( H \subset G \) is \( \delta^G \)-horizontal should be viewed as an analogue of reduction of the structure group to a subgroup in the classical case. As noted in [5], asking, as in classical differential geometry, that our \( p \)-adic connections be “translation invariant” is a condition that is almost never satisfied in the arithmetic theory. But asking from our \( p \)-adic connections that they be “metric” or “torsion free” (cf. the definitions below) is reasonable and leads to an interesting theory. We shall follow this path in what follows.

### 2.2. Metric connections.

Let \( p \in \mathbb{Z} \) be an odd prime and let \( \mathcal{O} \) be any complete discrete valuation ring with maximal ideal generated by \( p \) and perfect residue field. Such a ring possesses a unique Frobenius lift \( \phi = \phi^\mathcal{O} \) and hence a unique \( p \)-derivation \( \delta = \delta^\mathcal{O} \). If such an \( \mathcal{O} \) is given we will say we are in the local situation.

Assume now we are in this local situation.

Let \( x = (x_{ij}) \) be an \( n \times n \) matrix of indeterminates. We may consider the group scheme

\[
G = GL_n = \text{Spec } \mathcal{O}[x, \det(x)^{-1}];
\]

so for the ring of global functions we have \( \mathcal{O}(G) = \mathcal{O}[x, \det(x)^{-1}] \). According to our terminology a \( p \)-adic connection on \( G \) is a \( p \)-derivation \( \delta^G \) on \( \hat{G} \) (equivalently on \( \mathcal{O}(\hat{G}) \)) extending the \( p \)-derivation \( \delta^\mathcal{O} \).
An example of \( p \)-adic connection is the trivial \( p \)-adic connection, \( \delta^G_0 \), defined by \( \delta^G_0 x = 0 \); i.e., \( \delta^G_0 x_{ij} = 0 \). Its attached Frobenius lift \( \phi^G_0 \) satisfies \( \phi^G_0(x) = x^{(p)} \); i.e., \( \phi^G_0(x_{ij}) = x_{ij}^p \).

Consider next a symmetric matrix \( q \in G(\mathcal{O}) = GL_n(\mathcal{O}) \), i.e., \( q^t = q \). We view \( q \) as an arithmetic analogue of a metric. To \( q \) one can attach the map of schemes over \( \mathcal{O} \),
\[
\mathcal{H}_q : G \to G,
\]
defined on the level of rings by the ring homomorphism (still denoted by)
\[
\mathcal{H}_q : O(G) \to O(G), \quad \mathcal{H}_q(x) := x^t q x,
\]
i.e.,
\[
\mathcal{H}_q(x_{ij}) = q_{kl} x_{ki} x_{kj},
\]
with the repeated indices \( k, l \) summed over. Note that here (and later) we adopt the Einstein summation notation for indices that are not necessarily appearing both in upper and lower positions; no confusion should arise from this. We continue to denote by
\[
\mathcal{H}_q : \hat{G} \to \hat{G}
\]
the induced map of formal schemes. The identity matrix in \( GL_n(\mathcal{O}) \) will always be denoted by \( 1_n \), or simply by \( 1 \) if \( n \) is understood from context.

**Definition 2.2.** A \( p \)-adic connection \( \delta^G \) on \( G \) with attached Frobenius lift \( \phi^G \) is said to be **metric** with respect to \( q \) if the following diagram is commutative:
\[
\begin{array}{ccc}
\hat{G} & \xrightarrow{\phi^G} & \hat{G} \\
\mathcal{H}_q \downarrow & & \downarrow \mathcal{H}_q \\
\hat{G} & \xrightarrow{\phi^G_0} & \hat{G}
\end{array}
\]
Alternatively, following [6, 5], we say that \( \phi^G \) is \( \mathcal{H}_q \)-horizontal with respect to the trivial \( p \)-adic connection \( \phi^G_0 \).

The above condition should be viewed as an arithmetic analogue of the classical concept of **metric connection**; cf. 5.6 in our Appendix.

Explicitly, if we set
\[
\delta^G x = \Delta, \quad \phi^G(x) = x^{(p)} \Lambda = x^{(p)} + p \Delta,
\]
where \( \Delta \) and \( \Lambda \) are matrices with entries in \( O(\hat{G}) \), with
\[
\Lambda \equiv 1 \mod p,
\]
and if we set
\[
A = x^{(p)t} \cdot \phi(q) \cdot x^{(p)}, \quad B = (x^t q x)^{(p)}
\]
then the commutativity of (2.2) is easily seen to be equivalent to the matrix equality
\[
(2.3) \quad \Lambda^t A \Lambda = B.
\]

**Remark 2.3.** Metric connections satisfy congruences that are reminiscent of identities in classical differential geometry. We explain this in what follows. Let \( Z \) be the center of \( G \) and let \( T \) be the diagonal maximal torus of \( G \). Consider an ideal \( J \subset O(\hat{G}) \) and assume one of the following 2 situations:

1) \( J \) is the ideal defining \( Z \);
2) \( J \) is the ideal defining \( T \) and \( q \in T(\mathcal{O}) \).
In situation 1) $J$ is of course generated by 
\[
\{x_{ii} - x_{jj}, \ x_{ij} ; \ i, j = 1, ..., n, \ i \neq j \}
\]
while in situation 2) $J$ is generated by 
\[
\{x_{ij} ; \ i, j = 1, ..., n, \ i \neq j \}.
\]
Taking determinants in 2.3 we immediately get that, in either of the situations 1) or 2) above,
\[(2.4) \ \det(\Lambda) \equiv \{\det(1_n + p(q^{(p)})^{-1} \cdot \delta q)\}^{-1/2} \mod J,
\]
where the $-1/2$ root is the one that is $\equiv 1 \mod p$. Explicitly if
\[
\eta := \frac{1}{p} \{\det(1_n + p(q^{(p)})^{-1} \cdot \delta q) - 1\},
\]
then
\[
\det(\Lambda) \equiv \sum_{k=0}^{\infty} \left(\frac{-1/2}{k}\right) p^k \eta^k \mod J.
\]
In particular, since $\Lambda = 1 + p(x^{(p)})^{-1} \Delta$, and since one has 
\[
\det(1 + pM) \equiv 1 + p \cdot \text{tr}(M) \mod p^2
\]
for any matrix $M$ with coefficients in any ring, we get, in our case,
\[(2.5) \ \text{tr}((x^{(p)})^{-1} \Delta) \equiv -\frac{1}{2} \cdot \text{tr}((q^{(p)})^{-1} \cdot \delta q) \mod (p, J).
\]
This congruence is analogous to an identity in classical differential geometry; cf. 5.8.

Assume now we are in situation 2) above and, in addition $q$ has entries in the ring $\mathbb{Z}_p$ of $p$-adic integers. Then we get the congruence
\[(2.6) \ \det(\Lambda) \equiv \left(\frac{\det(q)}{p}\right) \cdot \det(q)^{\frac{p-1}{2}} \mod J,
\]
where $\left(\frac{p}{\cdot}\right)$ is the Legendre symbol.

2.3. **Torsion free connections.** The concept of metric connection was introduced for $\mathcal{O}$ in the local situation. To introduce torsion freeness we need to consider a global situation.

Let $F$ be a number field which is Galois over $\mathbb{Q}$ and let $M \in \mathbb{Z}$ be an even integer divisible by the discriminant of $F$. The giving of the data $F, M$ will be referred to as the global situation. In such a situation we let $\mathcal{O}_F$ be the ring of integers of $F$ and set
\[
\mathcal{O} := \mathcal{O}_{F,M} := \mathcal{O}_F[1/M].
\]
One can consider the Galois group $\mathcal{G}(F)$ of $F/\mathbb{Q}$ and the natural map
\[
\text{Spec} \ \mathcal{O} \to \mathcal{G}(F)
\]
sending any prime $\mathfrak{p} \in \text{Spec} \ \mathcal{O}$ into the Frobenius element $\phi_{\mathfrak{p}} \in \mathcal{G}(F)$ at $\mathfrak{p}$.

Assume now we are in this global situation.

**Definition 2.4.** By a frame we will understand the following data:

1) A map $\{1, ..., n\} \to \mathcal{G}(F), \ i \mapsto \sigma_i, \ \sigma_1 = \text{id}$;
2) A non-empty set $\mathcal{V}$ of primes in $\mathbb{Z}$ that do not divide $M$;
3) A map $\mathcal{V} \to \text{Spec} \ \mathcal{O}, \ p \mapsto \mathfrak{p}(p)$ such that $\mathfrak{p}(p)$ lies above $p$ for all $p \in \mathcal{V}$.
Remark 2.5. The maps in 1) and 3) above are not a priori subject to any compatibility. One can impose such a compatibility in our theory by asking for one more piece of data. Indeed let us call a soldering any bijection \( \{1,\ldots,n\} \to \mathcal{V} \) making the following diagram commutative

\[
\begin{array}{ccc}
\{1,\ldots,n\} & \longrightarrow & \mathcal{G}(F) \\
\downarrow & & \uparrow \\
\mathcal{V} & \longrightarrow & \text{Spec} \mathcal{O}
\end{array}
\]  

(2.7)

Such a condition can be viewed as an analogue of Cartan’s classical concept of soldering which, roughly, postulates an identification between the vertical directions and the horizontal directions in a vector bundle of rank equal to the dimension of the base manifold. We will not pursue this soldering concept in our paper, though.

Assume a frame is given. For \( p \in \mathcal{V} \) and \( \mathfrak{P} = \mathfrak{P}(p) \) we set

\[ \mathfrak{P}_i = \sigma_i \mathfrak{P}, \]

so \( \mathfrak{P}_1 = \mathfrak{P} \). Note that the \( \mathfrak{P}_i \)'s are not necessarily distinct. Let \( \mathcal{O}_{\mathfrak{P}_i} \) be the localization of \( \mathcal{O}_F \) at \( \mathfrak{P}_i \) and let, as usual, \( \widehat{\mathcal{O}}_{\mathfrak{P}_i} \) denote the \( p \)-adic completion of this localization. So \( \widehat{\mathcal{O}}_{\mathfrak{P}_i} \) is in the local situation considered previously. In particular \( \widehat{\mathcal{O}}_{\mathfrak{P}_i} \) has a unique Frobenius lift \( \phi_{\mathfrak{P}_i} \), which, for simplicity, we denote by \( \phi^i \). Consequently \( \mathcal{O}_{\mathfrak{P}_i} \) has a unique \( p \)-derivation \( \delta_{\mathfrak{P}_i} \), which, for simplicity, we denote by \( \delta^i \). Clearly \( \phi^i \) sends \( \mathcal{O}_F \) into itself and the restriction of \( \phi^i \) to \( \mathcal{O}_F \), further extended to an automorphism of \( F \), is the usual Frobenius element \( \phi_{\mathfrak{P}_i} \in \mathcal{G}(F) \), which we continue to denote by \( \phi^i \). Of course, \( \phi^i \) induces a Frobenius lift on \( \mathcal{O}_{\mathfrak{P}_i} \), but does not generally induce a Frobenius lift on \( \mathcal{O}_F \). If \( F \) is abelian over \( \mathbb{Q} \) then, of course, \( \phi^i \) does induce a Frobenius lift on \( \mathcal{O}_F \).

As a matter of notation we will sometimes simply write \( \delta \) and \( \phi \) instead of \( \delta^1 \) and \( \phi^1 \); but we will never abbreviate \( \delta^i, \phi^i \) by \( \delta, \phi \) if \( i \neq 1 \).

Consider the general linear groups over \( \mathcal{O}_{F,M} \) and \( \widehat{\mathcal{O}}_{\mathfrak{P}_i} \), respectively:

\[
G = GL_n = \text{Spec} \mathcal{O}_{F,M}[x, \det(x)^{-1}]
\]

\[
G^i = GL_n \otimes \widehat{\mathcal{O}}_{\mathfrak{P}_i} = \text{Spec} \widehat{\mathcal{O}}_{\mathfrak{P}_i}[x, \det(x)^{-1}].
\]

Note that the \( G^i \)'s are not necessarily distinct. We have induced isomorphisms (still denoted by) \( \sigma_i : \widehat{\mathcal{O}}_{\mathfrak{P}_1} \to \widehat{\mathcal{O}}_{\mathfrak{P}_i} \) extending uniquely to isomorphisms (still denoted by)

\[
\sigma_i : \mathcal{O}(\widehat{G}^1) \to \mathcal{O}(\widehat{G}^i), \quad \sigma_i(x) = x.
\]

(2.8)

For the rest of this section we will concentrate on one fixed prime \( p \in \mathcal{V} \). Varying \( p \) in \( \mathcal{V} \) will not play a role until we discuss “algebraization by correspondences”; cf. Definition 3.11.

Definition 2.6. A \( p \)-adic connection on \( G \) is an \( n \)-tuple \( (\delta^1, \ldots, \delta^n) \) where \( \delta^i \) is a \( p \)-adic connection on \( G^i \). The Christoffel symbols of the first kind of a \( p \)-adic connection are the matrices

\[
\gamma_1, \ldots, \gamma_n \in \mathfrak{gl}_n(\mathcal{O}(\widehat{G}^i))
\]

defined by

\[
\gamma_i = \delta^ix^i \cdot \phi^i(q) \cdot x^{(p)}.
\]

(2.9)
Write $\gamma_i = (\gamma_{ijk})$. We say that the $n$-tuple $(\delta^1, \ldots, \delta^n)$ is torsion free if for all $i, j, k = 1, \ldots, n$ we have the following equalities in $O(\mathbb{G})$:

\begin{equation}
\sigma^{-1}_i \gamma_{ijk} = \sigma^{-1}_j \gamma_{ijk}.
\end{equation}

Remark 2.7.

1) If $F$ is abelian our notion of $p$-adic connection above coincides with the one in [5].

2) If we define

\begin{equation}
\gamma'_i = \delta^i x^t \cdot \phi^i(q), \quad \gamma'_i = (\gamma'_{ijk}),
\end{equation}

then (2.10) holds if and only if

\begin{equation}
\sigma^{-1}_i \gamma'_{ijk} = \sigma^{-1}_j \gamma'_{ijk}.
\end{equation}

3) Our definition of Christoffel symbols of the first kind is analogous to the classical one; cf. 5.5 in the Appendix: right multiplication by $\phi^i(q)$ in (2.10) plays the role of “lowering the indices” in the classical setting 5.5. The symmetry (2.10) is an analogue of the symmetry 5.5 in the definition of classical torsion freeness.

4) We will usually denote by $(\phi^1, \ldots, \phi^n)$ the Frobenius lifts attached to $(\delta^1, \ldots, \delta^n)$. So $\phi^i : O(\mathbb{G}) \to O(\mathbb{G})$ is a Frobenius lift, not to be mixed up with one of the maps $\sigma_i$ in (2.9) which are never Frobenius lifts. A confusion in notation may arise if $\phi^i = \sigma_j$ as elements in $G(F)$; in order to avoid this confusion, when using the letter $\phi$ we will always mean a Frobenius lift and not one of the maps in (2.8).

2.4. Levi-Civitá connection. Assume we are in the global situation, we are given a frame, we fix a prime $p \in \mathcal{V}$, we set $\mathcal{P} = \mathcal{P}(p)$, and we set $\mathcal{P}_i = \sigma_i \mathcal{P}_i$, as usual.

Here is our first main result; it is an analogue of the “Fundamental Theorem of Riemannian Geometry”; cf. Theorem 5.1 in the Appendix.

Theorem 2.8. Assume $q \in GL_n(O_{F,M})$, $q^t = q$. Then there exists a unique $p$-adic connection $(\delta^1, \ldots, \delta^n)$ on $G$ such that the following hold:

1) $\delta^i$ is metric with respect to $q$ for all $i$;
2) $(\delta^1, \ldots, \delta^n)$ is torsion free.

Definition 2.9. The $p$-adic connection $(\delta^1, \ldots, \delta^n)$ in Theorem 2.8 is called the Levi-Civitá connection attached to $q$.

Assume the notation of Theorem 2.8 and consider the matrices

\begin{equation}
C_i := -x^{(p)t} \cdot \sigma^{-1}_i \delta^k q \cdot x^{(p)} + \frac{1}{p} \{(x^t \cdot \sigma^{-1}_i q \cdot x)^{(p)} - x^{(p)t} \cdot \sigma^{-1}_i q^{(p)} \cdot x^{(p)}\}.
\end{equation}

If $C_i = (C_{ijk})$ then, clearly,

\begin{equation}
C_{ijk} = C_{ikj}.
\end{equation}

We will show:

Proposition 2.10. The following congruences hold in $O(\mathbb{G})$:

\begin{equation}
\sigma^{-1}_i \gamma_{ijk} \equiv \frac{1}{2}(C_{ijk} + C_{jik} - C_{kij}) \mod p,
\end{equation}

\begin{equation}
\sigma^{-1}_i \gamma_{ijk} \equiv -\frac{1}{2}(\sigma^{-1}_i \delta^k q_{jk} + \sigma^{-1}_j \delta^i q_{ik} - \sigma^{-1}_k \delta^i q_{ij}) \mod (p, x - 1).
\end{equation}

Recall that 1 is the identity matrix so $(p, x - 1)$ is the ideal generated by $p$, $x_{ii} - 1$, and $x_{ij}$ for $i \neq j$. The formula (2.14) should be viewed as an analogue of the classical expression for the Levi-Civitá connection in classical Riemannian geometry; cf. 5.10 in the Appendix.
2.5. **Local Levi Civit`a connection.** We will (directly) deduce Theorem 2.11 and Proposition 2.10 from corresponding local results; cf. Theorem 2.11 and Proposition 2.14 below. We need some notation.

Assume in what follows that we are in the local situation; so \( \mathcal{O} \) is a complete discrete valuation ring with maximal ideal generated by \( p \) and perfect residue field, viewed as equipped with its unique \( p \)-derivation \( \delta = \delta^\mathcal{O} \).

Set, in this situation,

\[ G = GL_n = \text{Spec} \mathcal{O}[x, \det(x)^{-1}] \]

We will prove the following:

**Theorem 2.11.** Assume \( q_1, \ldots, q_n \in GL_n(\mathcal{O}) \), \( q_i^t = q_i \). Then there exists a unique \( n \)-tuple \( (\delta_{i}^G, \ldots, \delta_n^G) \) of \( p \)-adic connections \( \delta_i^G \) on \( G \) such that, for

\[ \Gamma_i := \delta_i^G x^t \cdot \phi(q_i) \cdot x(p), \quad \Gamma_i = \Gamma_{ijk}, \]

the following hold:
1) \( \delta_i^G \) is metric with respect to \( q_i \) for all \( i \);
2) \( \Gamma_{ijk} = \Gamma_{jik} \) for all \( i,j,k \).

**Definition 2.12.** The tuple \( (\delta_1^G, \ldots, \delta_n^G) \) is called the local Levi-Civit`a connection on \( G \) over \( \mathcal{O} \) attached to the tuple \( (q_1, \ldots, q_n) \).

**Remark 2.13.** The Frobenius lifts \( \phi_{1}^G, \ldots, \phi_n^G \) attached to the local Levi-Civit`a connection do not commute in general; this will be seen when we discuss curvature, cf. Remark 3.7.

Assume the situation in Theorem 2.11 and consider the matrices

\[ C_i := -x(p)^t \cdot \delta q_i \cdot x(p) + \frac{1}{p} \{(x^t q_i x)^{(p)} - x(p)^t q_i^t x(p)\}. \]

If \( C_i = (C_{ijk}) \) then, clearly,

\[ C_{ijk} = C_{ikj}. \]

We will show:

**Proposition 2.14.** The following congruences hold in \( \mathcal{O}(\hat{G}) \):

\[ \Gamma_{ijk} \equiv \frac{1}{2}(C_{ijk} + C_{jik} - C_{kij}) \mod p \]

\[ \Gamma_{ijk} \equiv -\frac{1}{2}(\delta q_{ijk} + \delta q_{jik} - \delta q_{kij}) \mod (p, x-1). \]

**Remark 2.15.** Let \( \delta_i^G x =: \Delta_i \) let \( \phi_{i}^G \) be the corresponding Frobenius lifts, and write

\[ \phi_{i}^G(x) = \Phi_i = x(p)^t + p \Delta_i = x(p)^t \Lambda_i, \quad A_i = x(p)^t \phi(q_i)x(p), \]

\[ \Delta_i = (\Delta_{ijk}), \quad \Lambda_i = (\Lambda_{ijk}). \]

Then the following hold:
1) Condition 2) in Theorem 2.11 is equivalent to the condition

\[ (\phi(q_i) \cdot \Delta_i)_{kj} = (\phi(q_j) \cdot \Delta_j)_{ki} \]

and also to the condition

\[ (A_i(\Lambda_i - 1))_{kj} = (A_j(\Lambda_j - 1))_{ki}. \]

2) If \( q_1 = \ldots = q_n \) then condition 2) in Theorem 2.11 is equivalent to the condition

\[ \Delta_{ikj} = \Delta_{jki}. \]
and also to the condition

\[(\Lambda_i - 1_n)_{kj} = (\Lambda_j - 1_n)_{ki}.\]

3) If all \(q_i\) are scalar matrices, \(q_i = d_i \cdot 1_n\) then condition 2) in Theorem 2.11 is equivalent to the condition

\[\phi(d_i) \cdot (\Lambda_i - 1_n)_{kj} = \phi(d_j) \cdot (\Lambda_j - 1_n)_{ki}.\]

2.6. Case \(n = 1\).

If in Theorem 2.11 we assume \(n = 1\) then condition 2) in that theorem is, of course, automatically satisfied. Also \(x\) is one indeterminate and we can write \(\delta^G = \delta^G, q_{11} = d\). Then condition 1) is trivially seen to be equivalent to the condition that the Frobenius lift

\[\phi^G : \mathcal{O}[x, x^{-1}] \rightarrow \mathcal{O}[x, x^{-1}]\]

attached to the \(p\)-adic connection \(\delta^G\) satisfy

\[(2.16)\]

\[\phi^G(x) = \left(\frac{d^p}{\phi(d)}\right)^{1/2} \cdot x^p,\]

where the square root is chosen to be \(\equiv 1 \mod p\), i.e.,

\[(2.17)\]

\[\left(\frac{d^p}{\phi(d)}\right)^{1/2} = \left(1 + p\frac{\delta_d}{d^p}\right)^{-1/2} := \sum_{k=0}^{\infty} \left(-\frac{1}{2}\right) \frac{p^k}{k!} \left(\frac{\delta_d}{d^p}\right)^k.\]

So in case \(d \in \mathbb{Z}_p\times\) we have the formula

\[(2.18)\]

\[\phi^G(x) = \left(\frac{d}{p}\right) \cdot d^{\frac{x-1}{2}} \cdot x^p,\]

where \(\left(\frac{p}{\cdot}\right)\) is the Legendre symbol.

Note that in case \(n = 1\) the local Levi-Civita connection introduced in Definition 2.12 coincides with the real Chern connection introduced in [5], Introduction (or Definition 4.25). On the other hand, if \(n \geq 2\), the Levi-Civita and the real Chern connection are different objects.

2.7. Case \(n = 2\). In this case there is an analog of “compatibility with complex structure” and “conformal coordinates” which we discuss next.

We start by considering group schemes over \(\mathcal{O}\),

\[G'' \subset G' \subset G\]

as follows.

We let, as usual,

\[G = GL_2 = \text{Spec } \mathcal{O}[x, \det(x)^{-1}],\]

with \(x\) a \(2 \times 2\) matrix of indeterminates.

Then we let \(G' := GL_1^c\) be the centralizer subgroup scheme in \(G\) of the matrix

\[(2.19)\]

\[c = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.\]

The matrix \(c\) can be viewed as an analogue of “complex structure” and \(GL_1^c\) can be viewed as the “complexified \(GL_1\)”. One has

\[G' = \text{Spec } \mathcal{O}[\alpha, \beta, (\alpha^2 + \beta^2)^{-1}]\]
for $\alpha, \beta$ two indeterminates, with $G'$ embedded into $G$ via

$$x \mapsto \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$ 

Note that a $2 \times 2$ matrix $q$ that is symmetric and in $GL^c_1$ must be scalar; such a matrix can be viewed as an analogue of a “conformal metric”.

Finally we consider the kernel

$$(2.20) \quad G'' := U^c_1 := \text{Ker}(\det : GL^c_1 \to GL_1)$$

which can be viewed as the “complexified unitary group in dimension 1.” Of course,

$$G'' = \text{Spec } \mathcal{O}[\alpha, \beta] \left(\alpha^2 + \beta^2 - 1\right).$$

We will prove:

**Proposition 2.16.** Assume

$$d_1, d_2 \in \mathcal{O}^\times, \quad q_1 = d_1 \cdot 1_2 \in G(\mathcal{O}), \quad q_2 = d_2 \cdot 1_2 \in G(\mathcal{O})$$

and let $(\delta^G_1, \delta^G_2)$ be the local Levi-Civitè connection on $G$ over $\mathcal{O}$ attached to $(q_1, q_2)$. Then:

1) $G'$ is $\delta^G_i$-horizontal for $i = 1, 2$.
2) $G''$ is $\delta^G_i$-horizontal for $i = 1, 2$ if and only if $\delta d_1 = \delta d_2 = 0$.
3) Assume $d_1 = d_2 =: d \in \mathbb{Z}, \ d \not\equiv \pm 1, \ d \not\equiv 0 \mod p$. Then there is no closed connected proper subgroup scheme of $G'$ that is $\delta^G_i$-horizontal for $i = 1, 2$.

**Definition 2.17.** The pair $(\delta^G_1, \delta^G_2)$ of $p$-derivations on $G'$ induced by $(\delta^G_1, \delta^G_2)$ (which exist by assertion 1 in Proposition 2.16) is called the local Levi-Civitè con-

**Remark 2.18.** The Frobenius lifts $\phi^G_1, \phi^G_2$ attached to the local Levi-Civitè connection on $G'$ do not commute in general; this will be seen when we discuss curvature, cf. Remark 3.7. Also note that assertion 3 in Proposition 2.16 intuitively says that the Levi-Civitè connection on $G'$, induced from that on $G$, does not induce, in its turn, a connection on any connected proper subgroup of $G'$; this can be viewed as an “irreducibility” (or a “transitivity”) statement.

The objects in Proposition 2.16 can be described explicitly. Indeed assume the situation and notation in that Proposition and let $(\phi^G_1, \phi^G_2)$ be the corresponding Frobenius lifts on $\hat{G}'$. Let

$$\epsilon := \phi \left( \frac{d_2}{d_1} \right) \in \mathcal{O}^\times,$$

and set

$$\theta_i := \frac{d_i^p(\alpha^2 + \beta^2)^p}{\phi(d_i)(\alpha^{2p} + \beta^{2p})} \in \mathcal{O}(\hat{G'})^\times, \quad i = 1, 2.$$

Then the system

$$x_1^2 - 2\epsilon x_2 + \epsilon^2 x_2^2 = \theta_1 - 1$$

$$(2.21) \quad x_1^2 + 2\epsilon x_1 + \epsilon^2 x_2^2 = \epsilon^2(\theta_2 - 1)$$

with unknowns $x_1, x_2$ is trivially seen to have a unique solution

$$(2.22) \quad (v_1, v_2)$$
in the set
\[ p\mathcal{O}(\hat{G}') \times p\mathcal{O}(\hat{G}'). \]
The solution can be computed explicitly, of course, the way one finds the intersection of two circles in analytic geometry: one takes the difference of the equations in (2.21) which is a linear equation,
\[ 2\epsilon(x_1 + x_2) = \epsilon^2(\theta_2 - 1) - (\theta_1 - 1), \]
one solves the latter for \( x_2 \), one substitutes in one of the equations (2.21) and one solves the resulting quadratic equation by the quadratic formula; the radical involved needs to be expressed as a \( p \)-adic series. Define now
(2.23)
\[ u_2 := 1 + \epsilon^{-1}v_1, \quad u_1 := 1 - \epsilon v_2 \in \mathcal{O}(\hat{G}'). \]

**Proposition 2.19.** We have the following equality of matrices with coefficients in \( \mathcal{O}(\hat{G}') \):
(2.24)
\[ \phi_i^{G'} \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} = \begin{pmatrix} \alpha^p & \beta^p \\ -\beta^p & \alpha^p \end{pmatrix} \cdot \begin{pmatrix} u_i & v_i \\ -v_i & u_i \end{pmatrix}, \quad i = 1, 2. \]

**Remark 2.20.** Assume that in the above discussion we have \( d_1 = d_2 \). Then the formulas simplify as follows. One has
\[ \theta_1 = \theta_2, \quad u_1 = u_2, \quad v_1 = -v_2 \]
and if
\[ u := u_1, \quad v := v_1, \quad \theta := \theta_1, \quad d := d_1, \quad \eta := (\theta - 1)/p \]
then
\[ u = 1 + v, \quad u^2 + v^2 = \theta, \quad 2v^2 + 2v + (1 - \theta) = 0, \]
and
\[ v = -\frac{1}{2} + \frac{1}{2}(2\theta - 1)^{1/2} := -\frac{1}{2} + \frac{1}{2} \sum_{k=0}^{\infty} \left( \frac{1}{2} \right)_{2} 2^k p^k \eta^k. \]
Moreover let us set
\[ G''' = GL_1 = \text{Spec} \mathcal{O}[z, z^{-1}] \]
and consider the natural map
\[ \det : G' \to G''' \]
induced by
\[ z \mapsto \alpha^2 + \beta^2. \]
Furthermore let us consider the Frobenius lift
\[ \phi^{G'''} : \hat{G}''' \to \hat{G}''' \]
defined by
(2.25)
\[ \phi^{G'''}(z) = \frac{dp}{\phi(d)} \cdot z^p. \]
Then taking determinants in (2.24) one trivially gets that the following diagrams are commutative:
(2.26)
\[ \begin{array}{ccc}
\hat{G}' & \xrightarrow{\phi^{G'}} & \hat{G}' \\
\downarrow \det & & \downarrow \det \\
\hat{G}''' & \xrightarrow{\phi^{G'''}} & \hat{G}''' 
\end{array} \]
Note that the Frobenius lift $\phi^{G''}$ considered in [2.25] coincides with the Frobenius lift attached to what in [5], Introduction (or Definition 4.33) was called the complex Chern connection on $GL_1$ attached to $q$. So what we have in [2.26] is a compatibility between the Levi-Civita connection on $G' = GL_1^c$ in the present paper and the complex Chern connection on $G'' = GL_1$ in [5].

3. Curvature

In what follows we would like to define the curvature of Levi-Civita connection in two contexts that we refer to as the vertical context and the mixed context. The vertical context refers to the case when we fix a prime $p$ and we “vary” the primes of $F$ above $p$; in this context the definition of curvature is straightforward and we will derive some basic congruences for its components that are reminiscent of formulae from classical differential geometry. The mixed context refers to the case when the prime $p$ is allowed to vary “horizontally” while we still allow a “vertical” variation of primes of $F$ above $p$; in this context the definition of curvature is more subtle: it is based on “algebraization by correspondences” in a sense similar to [5]. The two pictures corresponding to the two contexts above turn out to be different in general.

3.1. Vertical context. Assume we are in the global situation with data $F, M$, assume we are given a frame, fix a prime $p \in \mathcal{V}$, set $\mathfrak{P} = \mathfrak{P}(p)$, and $\mathfrak{P}_i = \sigma_i \mathfrak{P}$.

**Definition 3.1.** Let $q \in GL_n(\mathcal{O}_{F,M})$, let $(\delta^1, ..., \delta^n)$ be the Levi-Civita connection attached to $q$, and let $(\tilde{\phi}^1, ..., \tilde{\phi}^p)$ be the attached Frobenius lifts. The curvature of the Levi-Civita connection is the family $(\varphi_{ij})$ where $i,j = 1,...,n$ and $\varphi_{ij} : \mathcal{O}(\widehat{G^1}) \rightarrow \mathcal{O}(\widehat{G^1})$ are the (additive) maps

(3.1) \[ \varphi_{ij} := \frac{1}{p} \{ \sigma_i^{-1} \tilde{\phi}^i \sigma_j \sigma_j^{-1} \tilde{\phi}^j \sigma_j \sigma_j^{-1} \tilde{\phi}^j \sigma_j \}. \]

Our definition [3.1] of curvature is analogous to the classical definition of curvature; cf. [5.2] in the Appendix.

Let $C_i = (C_{ij})$ be as in (2.13), let $q^j$ be the inverse of the matrix $q = (q_{ij})$, and let $(x^i)$ be the inverse of the matrix $x = (x_{ij})$. Set $\Phi_{ij} := \varphi_{ij}(x)$ and let $\Phi_{ijmk}$ be the entries of the matrices $\Phi_{ij}$, so

$\Phi_{ij} = (\Phi_{ijmk})$.

We will prove:

**Proposition 3.2.** The following congruences hold in $\mathcal{O}(\widehat{G^1})$:

\[
\Phi_{ijmk} \equiv \frac{1}{2} (\sigma_i^{-1} q^{ms} p^2 (x^rs) p^2 (C_{jkr} + C_{kjr} - C_{rjk})^p - \frac{1}{2} (\sigma_i^{-1} q^{ms} p^2 (x^rs) p^2 (C_{ikr} + C_{kir} - C_{rik})^p mod \ p, (3.2)
\]

\[
\Phi_{ijmk} \equiv \frac{1}{2} (\sigma_j^{-1} q^{mr} p^2 (\sigma_i^{-1} \delta^i q_{kr} + \sigma_k^{-1} \delta^k q_{ir} - \sigma_r^{-1} \delta^r q_{ik})^p - \frac{1}{2} (\sigma_j^{-1} q^{mr} p^2 (\sigma_j^{-1} \delta^j q_{kr} + \sigma_k^{-1} \delta^k q_{jr} - \sigma_r^{-1} \delta^r q_{jk})^p mod \ (p,x - 1),
\]

\]
where the repeated indices \( r, s \) are summed over.

It is worth noting that only the coefficients of the “metric” and their first “arithmetic derivatives” occur in these congruences. If, instead of congruences, one is interested in equalities then \( \Phi_{ijmk} \) will “involve” the “arithmetic derivatives up to order 2 of the metric” as in the case of classical Riemannian geometry.

A significant simplification occurs if one assumes

\[
\sigma_i(q) = q, \quad i = 1, \ldots, n,
\]

in which case we say \( q \) is \( \sigma \)-invariant. So in this case, with \( \delta = \delta^1 \), we have

\[
\sigma_i^{-1} \delta^i q = \sigma_i^{-1} \delta^i \sigma_i^{-1} q = \delta^i q = \delta q.
\]

Define

\[
C := -x^{(p)t} \cdot \delta q \cdot x^{(p)} + \frac{1}{p} \left( (x^t \cdot q \cdot x)^{(p)} - x^{(p)t} \cdot q^{(p)} \cdot x^{(p)} \right), \quad C = (C_{jk})
\]

and note that

\[
C_{jk} = C_{kj}.
\]

Then, recalling \( C_i \) from (2.13) we have \( C_i = C \) and \( C_{ijk} = C_{jik} \) for all \( i, j, k \); also

\[
C_{jk} \equiv -\delta q_{jk} \mod (p, x-1)
\]

for all \( j, k \). Define \( R_{ijmk} \) as the \((m,k)\) entry of the matrix

\[
x^{(p^2)t} q^{(p^2)} \Phi_{ij} ;
\]

in other words,

\[
x^{(p^2)t} q^{(p^2)} \Phi_{ij} = (R_{ijmk}).
\]

We view \( R_{ijmk} \) as an analogue of the classical covariant Riemann tensor; cf. 5.11 in the Appendix. Then the congruences acquire the following simpler form.

**Corollary 3.3.** Assume \( q \) is \( \sigma \)-invariant. Then the following congruences hold in \( O(\hat{G^1}) \):

\[
R_{ijmk} \equiv \frac{1}{2} (C_{ik} + C_{jm} - C_{jk} - C_{im})^p \mod p
\]

\[\tag{3.6}
R_{ijmk} \equiv \frac{1}{2} (\delta q_{jk} + \delta q_{im} - \delta q_{ik} - \delta q_{jm}) \mod (p, x-1).
\]

The congruences \( \mod (p, x-1) \) in (3.6) are analogous to the formulae for the covariant Riemannian tensor in “normal coordinates”; cf. 5.15 in the Appendix. Note however that, in arithmetic, one “loses” one derivative; so, in some sense, the case when \( q \) is rational behaves (modulo the “loss of one derivative”) as if “coordinates are already normal at \( p \).”

As a consequence of (3.6) we get:
Corollary 3.4. Assume $q$ is $\sigma$-invariant. Then the following congruences hold in $\mathcal{O}(\widehat{G}^1)$:

\[ R_{ijkm} \equiv -R_{ijmk} \mod p, \]
\[ R_{ijkm} \equiv -R_{jikm} \mod p, \]
\[ R_{mijk} + R_{mjki} + R_{mkij} \equiv 0 \mod p, \]
\[ R_{ijkm} \equiv R_{kmij} \mod p. \] (3.7)

The congruences (3.7) are, of course, analogous to the classical symmetries of the covariant Riemann tensor; cf. 5.12 in the Appendix. The first three congruences in (3.7) follow directly from 3.6 while the fourth is a well known formal consequence of the first three.

If, in addition, we define
\[ \Psi_{ij} := (x(p^2))^{-1}\Phi_{ij}, \quad R_{ik} = \Psi_{ijk}, \] (3.8)
with the repeated index $j$ summed over then one gets:

Corollary 3.5. Assume $q$ is $\sigma$-invariant. Then the following congruences hold in $\mathcal{O}(\widehat{G}^1)$:

\[ R_{ik} \equiv R_{ki} \mod p. \] (3.9)

One can regard $R_{ik}$ as an analogue of the Ricci tensor (cf. 5.14 in the Appendix).

As in the case of the classical Ricci tensor, Corollary 3.5 follows by noting that if one sets
\[ u := (x(p^2))^{-1}(q(p^2))^{-1}(x(p^2)t)^{-1} \]
and if one uses the last symmetry in (3.7) one gets
\[ R_{ik} = ((x(p^2))^{-1}\Phi_{ji})_{jk} = u_{jm}R_{jimk} = u_{mj}R_{mkji} = R_{ki} \mod p, \]
where the repeated indices $j, m$ are summed over.

Here is what formula (3.6) gives for $n = 2$ and “conformal coordinates”:

Corollary 3.6. Assume $n = 2$ and $q = d \cdot 12, \ d \in \mathbb{Z}[1/M]^{\times}$. Then
\[ \phi_{12} \equiv \left( \begin{array}{cc} 0 & \frac{\delta d}{dp} \\ -\frac{\delta d}{dp} & 0 \end{array} \right) \mod (p, x - 1). \]

Remark 3.7. Consider the situation:
\[ n = 2, \quad q = d \cdot 12, \quad d \in \mathbb{Z}[1/M]^{\times}, \quad \delta d \equiv 0 \mod p. \]

Then Corollary 3.6 implies
\[ \phi_{12} \not\equiv 0 \]
so the Frobenius lifts $\phi_1^G, \phi_2^G$ on $\widehat{G} = \widehat{GL}_2$ attached to the local Levi-Civita connection on $G$ do not commute; cf. Remark 2.13. Actually the Corollary implies the stronger condition,
\[ \phi_{12} \not\equiv 0 \mod (x_{11} - x_{22}, x_{12} + x_{21}), \]
which shows that already the Frobenius lifts $\phi_1^{G'}, \phi_2^{G'}$ on $\widehat{G'} = \widehat{GL}_2^c$ attached to the local Levi-Civita connection on $G'$ do not commute; cf. Remark 2.18.
3.2. **Mixed context.** This context will involve *correspondences.* We recall some terminology from [5, 8].

**Definition 3.8.**

1) By a *correspondence* (on a scheme $X$) we understand a diagram $\Gamma$ of schemes

$$
\begin{array}{ccc}
Y & \xleftarrow{\varphi} & X \\
\pi & \searrow & \\
& & X
\end{array}
$$

We also write

$$\Gamma = (Y, \pi, \varphi).$$

2) A *morphism* $\Gamma' \to \Gamma$ from a correspondence $\Gamma' = (Y', \pi', \varphi')$ on $X'$ to a correspondence $\Gamma = (Y, \pi, \varphi)$ on $X$ is a pair of morphisms $(u, v)$,

$$u : X' \to X, \quad v : Y' \to Y$$

such that

$$u \circ \varphi' = \varphi \circ v, \quad u \circ \pi' = \pi \circ v.$$

3) For any correspondence $\Gamma = (Y, \pi, \varphi)$ on $X$ and any morphism of schemes $u : X' \to X$ one can consider the *pull back correspondence* on $X'$,

$$\Gamma' := \Gamma \times_X X' := (Y', \pi', \varphi')$$

declared by letting

$$Y' := (X' \times_{u, X, \pi} Y) \times_{pr_2, Y, pr_1} (Y \times_{\varphi, X, u} X'),$$

where the subscripts indicate the maps used to construct the fiber products, $pr_1, pr_2$ are the obvious first and second projections to $Y$, and $\pi', \varphi'$ are defined by the obvious projections to $X'$. There is a naturally induced morphism $\Gamma \times_X X' \to \Gamma$ and for any morphism $u : X' \to X$ one can consider the *pull back correspondence* on $X'$,

$$\Gamma' := \Gamma \times_X X' := (Y', \pi', \varphi')$$

declared by letting

$$Y' := (X' \times_{u, X, \pi} Y) \times_{pr_2, Y, pr_1} (Y \times_{\varphi, X, u} X'),$$

where the subscripts indicate the maps used to construct the fiber products, $pr_1, pr_2$ are the obvious first and second projections to $Y$, and $\pi', \varphi'$ are defined by the obvious projections to $X'$. There is a naturally induced morphism of correspondences $\Gamma' \to \Gamma \times_X X'$.

Furthermore for any further morphism $X'' \to X'$ we have a natural isomorphism

$$(\Gamma \times_X X') \times_{X'} X'' \simeq \Gamma \times_X X''.$$

4) If $\Gamma = (Y, \pi, \varphi)$, $\Gamma' = (Y', \pi', \varphi')$ are two correspondences on $X$ their *composition* is the correspondence

$$\Gamma' \circ \Gamma := (Y_{\Gamma \circ \Gamma}, \pi \circ \pi', \varphi' \circ \varphi_2)$$

where the above data are defined by the following diagram in which the square is cartesian:

$$
\begin{array}{ccc}
Y_{\Gamma \circ \Gamma} & \to & Y' \\
\downarrow \pi' \downarrow & & \downarrow \pi' \downarrow \\
Y_{\Gamma} & \to & X \\
\downarrow \pi \downarrow & & \downarrow \\
& & X
\end{array}
$$

5) A correspondence $(Y, \pi, \varphi)$ is called *strictly symmetric* if $\pi = \varphi$. 


Assume now that we are in the global situation and we are given a frame. For \( p \in \mathcal{V} \) and \( \mathfrak{P} = \mathfrak{P}(p) \) we denote by \( X_{\mathfrak{P}}, A_{\mathfrak{P}} \), the \( \mathfrak{P} \)-adic completions of schemes \( X \) or rings \( A \) over \( \mathcal{O}_{F,M} \) and we continue to denote by \( \hat{X}, \hat{A} \) their \( p \)-adic completions. Recall that \( \mathfrak{P}_1 = \mathfrak{P} \). Set

\[
\begin{align*}
G &= GL_n = \text{Spec} \, \mathcal{O}_{F,M}[x, \det(x)^{-1}], \\
G_{\mathfrak{P}} := G \otimes \hat{\mathcal{O}}_{\mathfrak{P}} &= \text{Spec} \, \hat{\mathcal{O}}_{\mathfrak{P}}[x, \det(x)^{-1}], \\
\hat{G}_{\mathfrak{P}} &= G_{\mathfrak{P}} = \text{Spf} \, \hat{\mathcal{O}}_{\mathfrak{P}}[x, \det(x)^{-1}]^\wedge.
\end{align*}
\]

In our previous notation, of course, \( G_{\mathfrak{P}} = G_1 \). Then we will prove that our Levi-Civit\`{a} connection admits an algebraization by correspondences in the following sense.

**Theorem 3.9.** Let \( q_1, ..., q_n \in GL_n(\mathcal{O}_{F,M}) \), \( q_1' = q_i \). Let \( (\delta_{\mathfrak{P}}^{G_1}, ..., \delta_{\mathfrak{P}}^{G_n}) \) be the local Levi-Civit\`{a} connection on \( G_{\mathfrak{P}} \) over \( \hat{\mathcal{O}}_{\mathfrak{P}} \) attached to \( (q_1, ..., q_n) \) and let \( (\phi_{\mathfrak{P}}^{G_1}, ..., \phi_{\mathfrak{P}}^{G_n}) \) be the attached Frobenius lifts on \( \hat{G}_{\mathfrak{P}} \). Then there exists an \( n \)-tuple of correspondences on \( G \),

\[
\begin{align*}
Y_{p/G} &\xleftarrow{\pi_p} G \\
&\xrightarrow{\phi_{\mathfrak{P}}^{G_i}} G
\end{align*}
\]

where \( i = 1, ..., n \), such that the following hold:

1) The map \( \pi_p : Y_{p/G} \rightarrow G \) is étale and \( Y_{p/G} \) is affine and irreducible.

2) There is a connected component \( Y_{p/G} \) of \( \hat{Y}_{p/G} \) such that the induced map \( \pi_{\mathfrak{P}} : Y_{p/G} \rightarrow \hat{G}_{\mathfrak{P}} \) is an isomorphism.

3) For each \( i \) the following induced diagram is commutative:

\[
\begin{align*}
Y_{p/G} &\xleftarrow{\pi_{\mathfrak{P}}} \hat{Y}_{p/G} \\
&\xrightarrow{\phi_{\mathfrak{P}}^{G_i}} \hat{G}_{\mathfrak{P}}
\end{align*}
\]

Intuitively the correspondences \( \mathcal{E}_i \) give an algebraization of our Frobenius lifts \( \phi_{\mathfrak{P}}^{G_i} \). For a given \( q \) the \( n \)-tuple of correspondences \( \mathcal{E}_i \) with properties 1, 2, 3 in the theorem is, of course, far from being unique. However, for any given frame, the proof of the theorem will provide, for any \( p \) and \( q \), a canonical construction for such an \( n \)-tuple of correspondences \( \mathcal{E}_i \) on \( G \). Once we have at our disposal such a canonical \( n \)-tuple of correspondences on \( G \) there is a general recipe to define curvature as a family of elements in the ring of correspondences on the field \( E = F(x) \) of rational functions of \( G \); cf. [8, 5]. We quickly review in what follows this recipe; we will also add some new terminology, constructions, and notation.

**Definition 3.10.**

In the discussion below we fix a field \( E \) of characteristic zero.
1) We define a category as follows. The objects of the category are correspondences

\[
\begin{array}{c}
\xymatrix{Y \\
\Spec E \ar@{<->}[r]_-{\pi} & \Spec E \\
\Spec E \ar@{<->}[r]^-{\varphi} & Y}
\end{array}
\]

where \( \pi \) is étale (equivalently \( Y \) is the spectrum of a finite product of fields that are finite extensions of \( E \) via \( \pi \)) and \( \varphi \) is a finite morphism of schemes. The finiteness of \( \varphi \) is automatic if \( E \) is finitely generated over \( \mathbb{Q} \) which will always be the case in our applications. If \( Y \) is irreducible we say that the correspondence 3.13 is irreducible.

A morphism in the category is a morphism of correspondences \((u,v)\) with \( u = \text{id} \) and \( v \) surjective. If the two correspondences are irreducible then the degree of the morphism \((u,v)\) is defined as the degree of \( v \).

2) We denote by \( C^+ = C^+_+(E) \) the set of isomorphism classes of objects in the category introduced in 1) to which we add one more element, denoted by 0. Also we denote by 1 the class of the identity correspondence (with \( Y = \Spec E \) and maps \( \pi, \varphi \) equal to the identity). Then \( C^+ = C^+_+(E) \) comes equipped with the following operations:

- transposition (coming from interchanging \( \pi \) and \( \varphi \));
- addition (coming from disjoint union of the \( Y \)'s);
- multiplication (coming from composition of correspondences).

With respect to these operations \( C^+_+(E) \) becomes a semiring with involution in the sense that

- addition and multiplication are associative and addition is commutative;
- multiplication is left and right distributive with respect to addition;
- transposition is an anti-involution;
- 0 is a neutral element for addition, \( 0^t = 0, \ 0 \cdot x = 0 \) for all \( x \);
- 1 is a neutral element for composition and \( 1^t = 1 \).

3) The semiring \( C^+_+ \) has the additive cancellation property so it can be canonically embedded into the (associative, not necessarily commutative) ring \( C = C(E) \),

\[
C := (C^+_+ \times C^+_+)/\sim
\]

where

\[
(c_1, c_2) \sim (c_3, c_4) \ \text{if and only if} \ \ c_1 + c_4 = c_2 + c_3.
\]

The ring \( C(E) \) is called the ring of correspondences on \( E \). We often view \( C(E) \) as a Lie ring with respect to the commutator,

\[
[c_1, c_2] := c_1 c_2 - c_2 c_1 \in C(E), \ c_1, c_2 \in C(E).
\]

The involution \( c \mapsto c^t \) on \( C^+_+ \) induces an involution \( c \mapsto c^t \) on the ring \( C \). Also \( C \) has a structure of ordered ring with set of positive elements the set \( C_+ \setminus \{0\} \).

4) A non-zero element of \( C(E) \) is called irreducible if it is in \( C^+_+ \) and it cannot be written as a sum of two non-zero elements of \( C^+_+ \). Of course the class of a correspondence 3.13 is irreducible if and only if the correspondence 3.13 is irreducible. So any non-zero element in \( C^+_+ \) can be written uniquely as a \( \mathbb{Z} \)-linear combination with positive coefficients of irreducible elements.
5) Consider the following inclusions
\[ G(E) \subset E(E) \subset S(E) \subset L(E) \]
where \( G(E) \) is the group of field automorphisms of \( E \), \( E(E) \) is the monoid of field endomorphisms of \( E \), \( L(E) \) is the (not necessarily commutative) ring of additive group endomorphisms of \( E \), and \( S(E) \) has the following description. Let us denote by \( \mathcal{X}(E) \) the set of all elements \( \chi \in L(E) \) with the following property: there exists an integer \( n \geq 1 \) and a ring homomorphism \( \rho : E \to \mathfrak{gl}_n(E) \) such that
\[ \chi(a) = \text{tr}(\rho(a)), \quad a \in E. \]
The subset \( \mathcal{X}(E) \) of \( L(E) \) is a multiplicative submonoid. So the \( \mathbb{Z} \)-linear span of \( \mathcal{X}(E) \) in \( L(E) \) is a subring which we denote by \( S(E) \). Of course, \( S(E) \) is the image of the natural homomorphism \( \mathbb{Z}\mathcal{X}(E) \to L(E) \).

Note that if \( \chi, \rho \) are as above then \( \rho \) restricted to \( E^\times \) gives a representation \( \rho^\times : E^\times \to GL_n(E) \) of the group \( E^\times \) over the field \( E \) whose character is the restriction \( \chi^\times : E^\times \to E \) of \( \chi \). But, of course, not all characters of representations of the group \( E^\times \) over the field \( E \) occur in this way.

Note on the other hand that there are natural ring (anti)homomorphisms
\[ (3.14) \quad \mathbb{Z}\mathcal{E}(E) \to C(E) \to L(E). \]
We will usually drop the prefix \( \text{anti} \) in what follows. The first homomorphism sends a field endomorphism \( \sigma : E \to E \) into the correspondence \( 3.13 \) with \( \pi \) the identity and \( \varphi \) induced by \( \sigma \). The second homomorphism sends the class
\[ c \in C(E) \]
of a correspondence \( 3.13 \) into the group homomorphism
\[ c^* \in L(E), \quad c^* : E \xrightarrow{\varphi} O(Y) \xrightarrow{\text{tr}_\pi} E \]
where \( \varphi : E \to O(Y) \) is induced by the map \( \varphi \) and \( \text{tr}_\pi : O(Y) \to E \) is the trace of the map \( \pi : E \to O(Y) \) induced by \( \pi \). The composition \( 3.14 \) is the natural map induced by the inclusion \( E(E) \subset L(E) \) and note that \( 3.14 \) is injective by the “linear independence of characters”. For \( \sigma \) a field automorphism of \( E \) the images \( c \) and \( c^{-1} \) of \( \sigma \) and \( \sigma^{-1} \) in \( C(E) \) satisfy \( c^{-1} = c^* \). Also clearly the image of the homomorphism \( C(E) \to L(E) \) is contained in \( S(E) \).

6) An element \( c \in C_+(E) \) is called strictly symmetric if it can be represented by a strictly symmetric correspondence. Two elements \( c_1, c_2 \in C_+(E) \) are called compatible if one can write
\[ c_1c_2^t = c_3 + c_4, \]
with \( c_3, c_4 \in C_+(E) \) and \( c_3 \) strictly symmetric. The relation of compatibility is, of course, symmetric, and trivially seen to be reflexive; it is not transitive in general.

7) There are unique ring homomorphisms (the left and right degree maps)
\[ \deg_l, \deg_r : C(E) \to \mathbb{Z} \]
given by attaching to a correspondence \((Y, \pi, \varphi)\) the positive integers \( \deg(\pi) \) and \( \deg(\varphi) \) respectively.

8) Consider the ideal \( J \) in \( C(E) \) generated by all the elements of the form
\[ c' \cdot d \cdot c \]
where \( c \) and \( c' \) are the classes of two irreducible correspondences \( \Gamma \) and \( \Gamma' \) between which there is a morphism \( \Gamma' \to \Gamma \) of degree \( d \). Define the reduced ring of correspondences by

\[
R(E) := C(E)/J
\]

and denote by \( c \mapsto \tilde{c} \) the projection \( C(E) \to R(E) \). (Here the word reduced has nothing to do with nilpotents but rather to reduction mod \( J \).) Then the ring homomorphism \( C(E) \to L(E), c \mapsto c^* \), is easily seen to factor through a homomorphism

\[
(3.15) \quad R(E) \to L(E)
\]

and in particular we still have an injective ring homomorphism

\[
(3.16) \quad \mathbb{Z}E(E) \to R(E).
\]

Also all the elements of \( J \) have left and right degree 0 so we have induced ring homomorphisms

\[
(3.17) \quad \deg_l, \deg_r : R(E) \to \mathbb{Z}.
\]

Now since \( J^f = J \) the ring \( R(E) \) has an involution induced by \( c \mapsto c' \). We claim that \( R(E) \) has a structure of ordered ring with set of positive elements the image of the set \( C_+ \setminus \{0\} \). Indeed it is enough to check that if \( c = c_1 + j_1 \) and \(-c = c_2 + j_2\) with \( c_1, c_2 \in C_+ \) and \( j_1, j_2 \in J \) then \( c \in J \). But adding the above two equations one gets \( c_1 + c_2 + j_1 + j_2 = 0 \). Taking left degrees we get that \( c_1 \) has left degree 0 hence \( c_1 = 0 \) hence \( c \in J \) and we are done. By the way if \( c \in C_+ \) is strictly symmetric then its image \( \tilde{c} \in R(E) \) lies in \( \mathbb{Z}_{>0} \). So if \( c_1, c_2 \in C(E), c_1, c_2 > 0 \) are compatible then their images \( \tilde{c}_1, \tilde{c}_2 \in R(E) \) satisfy

\[
\tilde{c}_1 \tilde{c}_2^t \geq 1.
\]

We summarize the various rings and ring homomorphisms that we have attached to any field \( E \) of characteristic zero in the following diagram:

\[
\begin{array}{ccccccc}
\mathbb{Z}G(E) & \to & \mathbb{Z}E(E) & \to & C(E) & \to & R(E) & \to & S(E) & \to & L(E) \\
\downarrow & & \downarrow & & \uparrow & & \uparrow & & \\
\mathbb{Z} & & \mathbb{Z}X(E)
\end{array}
\]

9) If \( E \) is a finite Galois extension of \( \mathbb{Q} \) then, of course, \( E(E) = G(E) \).

We claim that, in this case, the injective ring homomorphism \( \mathbb{Z}G(E) \to R(E) \) is also surjective, so an isomorphism. The claim follows from the fact that, due to the normality of \( E/\mathbb{Q} \), for any two field homomorphisms \( \pi, \varphi : E \to L \) we must have \( \pi(E) = \varphi(E) \).

Finally we claim that the injective homomorphism

\[
\mathbb{Z}G(E) \to S(E)
\]

is also surjective, so an isomorphism. Indeed let \( \rho : E \to \mathfrak{g}l_n(E) \) be a ring homomorphism and let \( \chi = tr \circ \rho \). Since any \( a \in E \) is a root of a separable polynomial with coefficients in \( \mathbb{Q} \) the same is true for \( \rho(a) \), hence the minimal polynomials of \( \rho(a) \) are separable, hence \( \rho(a) \) are diagonalizable in \( \mathfrak{g}l_n(E) \), where \( E \) is an algebraic
closure of $E$. Since the family matrices $\{\rho(a); \ a \in E\}$ is commuting it is simultaneously diagonalizable so there exists $U \in GL_n(E)$ and maps $\lambda_1, \ldots, \lambda_n : E \to E$ such that

$$\rho(a) = U^{-1} \cdot \text{diag}(\lambda_1(a), \ldots, \lambda_n(a)) \cdot U, \quad a \in E.$$ 

One immediately gets that the $\lambda_i$'s are ring homomorphisms. Since $F$ is Galois $\lambda_i$ come from elements of $G(E)$. But

$$\chi(a) = \lambda_1(a) + \ldots + \lambda_n(a), \quad a \in E.$$ 

So $\chi$ is in the image of $ZG(E)$.

So we see that in case $E$ is a Galois number field the 3 rings $ZG(E), \mathcal{R}(E), S(E)$ are naturally isomorphic. Hence, for an arbitrary field $E$ of characteristic zero either of the rings $\mathcal{R}(E), S(E)$ could be viewed as an analogue of the group ring of the Galois group. Of these two rings the ring $\mathcal{R}(E)$ has the advantage of being equipped with a structure of ordered ring so it is the ring $\mathcal{R}(E)$ that we will view as the most natural generalization of the group ring of the Galois group.

10) Consider an arbitrary field $E$ of characteristic zero and let $F \subset E$ be a subfield which is a finite Galois extension of $\mathbb{Q}$. Consider the unique additive group homomorphisms

$$\text{res}_l, \text{res}_r : C(E) \to ZG(F) = \mathcal{R}(F)$$

sending the class of any irreducible correspondence $(\text{Spec } L, \pi, \varphi)$, where

$$\pi, \varphi : E \to L$$

are field homomorphisms, into the elements

$$\text{deg}_l(\pi) \cdot \sigma, \quad \text{deg}_r \cdot \sigma,$$

respectively, where $\sigma \in G(F)$ is the composition

$$F \xrightarrow{\varphi} \varphi(F) = \pi(F) \xrightarrow{\pi^{-1}} F.$$ 

It is trivial to check that (3.19) are ring homomorphisms. Then clearly the homomorphisms in (3.19) factor through ring homomorphisms

$$\text{res}_l, \text{res}_r : \mathcal{R}(E) \to \mathcal{R}(F)$$

The next definition relates general correspondences to our Levi-Civita context. Assume again we are in the global situation and a frame is given.

**Definition 3.11.** Let

$$q \in GL_n(O), \quad O = O_{F,M}, \quad q^l = q, \quad q_i = \sigma^{-1}_i(q).$$

Fix $p \in V$, set $\mathfrak{P} := \mathfrak{P}(p)$, let

$$\left(\delta^G_{q_1}, \ldots, \delta^G_{q_n}\right)$$

be the local Levi-Civita connection on $G_{\mathfrak{P}}$ over $O_{\mathfrak{P}}$ attached to $(q_1, \ldots, q_n)$ and let

$$\left(\phi^G_{q_1}, \ldots, \phi^G_{q_n}\right)$$
be the attached Frobenius lifts on \( \hat{G} \). Consider an \( n \)-tuple of correspondences \( 3.11 \) satisfying conditions 1, 2, 3 in Theorem \( 3.9 \). Let \( E = F(x) \) be the field of rational functions on \( G \) and let

\[
\begin{array}{ccc}
\mathbf{Spec} E & & \mathbf{Spec} E \\
\pi_{p/E} & & \varphi_{pi/E} \\
Y_{p/E} & & Y_{pi/E}
\end{array}
\]

be the pull back of \( 3.11 \) via \( \mathbf{Spec} E \to G \). We will prove that \( Y_{p/E} \) is the spectrum of a field and \( \pi_{p/E} \) is finite; cf. Remark \( 3.12 \) below. Also \( \varphi_{pi/E} \) are finite because \( E \) is finitely generated over \( \mathbb{Q} \). So we can consider the classes

\[
c_{pi} \in C(E)
\]

of the correspondences \( 3.21 \); these classes are therefore irreducible. Finally we can define the (mixed) curvature of the Levi-Civit\'a connection attached to \( q \) as the family of commutators

\[
\varphi_{pp'ii'} := [c_{pi}, c_{p'i'}] \in C(E),
\]

where \( i, i' \) run through 1, \ldots, \( n \) and \( p, p' \) run through \( \mathcal{V} \). This family induces a family

\[
\bar{\varphi}_{pp'ii'} = [\bar{c}_{pi}, \bar{c}_{p'i'}] \in \mathcal{R}(E),
\]

in the reduced ring of correspondences, which could be called the (mixed) reduced curvature and finally a family

\[
\varphi_{pp'ii'}^* = [c_{pi}^*, c_{p'i'}^*] \in \mathcal{S}(E),
\]

which we refer to as the (mixed) \( * \)-curvature.

Of the 3 rings \( C(E), \mathcal{R}(E), \mathcal{S}(E) \) the most natural choice for a recipient ring of our curvature is, probably, \( \mathcal{R}(E) \) which can be viewed as the generalization of the group ring of a Galois group; however in order to simplify our discussion we will concentrate in what follows on curvature with values in the other two rings which are easier to analyze.

**Remark 3.12.**

1) We earlier made the claim that \( Y_{p/E} \) is the spectrum of a field and \( \pi_{p/E} \) are finite. Let us check this claim. Note first that, since \( \pi_p \) is étale and \( G \) is an integral regular scheme, \( Y_{p/G} \) is a disjoint union of integral regular schemes. Since \( Y_{p/G} \) is irreducible it is an integral scheme. Hence

\[
\mathbf{Spec} E \times_{G, \pi_p} Y_{p/G}
\]

is the spectrum of a field \( L \) which is a finite extension of \( E \) via \( \pi_p \). So \( Y_{p/E} \) is the spectrum of a ring of fractions of \( L \); hence either \( Y_{p/E} \) is empty or equal to \( \mathbf{Spec} L \). So we are left with proving that \( Y_{p/E} \) is non-empty; we check this in what follows. Denote by an upper bar tensorization with \( \mathcal{O} := \mathcal{O}/\mathfrak{p} \) over \( \mathcal{O} \). Let \( Y \) be an affine open subset of \( Y_{p/G} \) meeting \( Y_{p/G} \) but not meeting any other connected component of \( Y_{p/G} \). By Krull’s intersection theorem \( \mathcal{O}(X) \) and \( \mathcal{O}(Y) \) (being domains in which \( p \) is non-invertible) embed into \( \mathcal{O}(X^{\bar{p}}) \) and \( \mathcal{O}(Y^{\bar{p}}) \) respectively. On the other hand the map

\[
\varphi_{pi} : \mathcal{O}(X^{\bar{p}}) \to \mathcal{O}(Y^{\bar{p}})
\]
is injective (because its reduction mod $p$ is injective and $p$ is a non-zero divisor in both rings). So the map
\[ \varphi_{pi} : \mathcal{O}(X) \rightarrow \mathcal{O}(Y) \]
is injective. So the generic point of $Y_{p/G}$ is mapped by $\varphi_{pi}$ to the generic point of $G$; this implies that $Y_{p/E}$ is non-empty.

2) Setting $q_i = \sigma_i^{-1}(q)$ in Definition 3.11 is (a posteriori) justified by the way the global Theorem [2.8] and the local Theorem 2.11 will turn out to be related; this will become clear after we provide the proofs of these theorems.

3) As already mentioned the proof of Theorem 3.9 will provide, for a given frame, a canonical construction of correspondences 3.11 on $G$ so, for a fixed frame there is a canonical way to construct correspondences 3.21 associated to our Levi-Civit\`a connection and, in particular, the mixed curvature can be canonically attached to $q$ and the frame. We will not make the canonicity of 3.11 explicit here; it will become clear once the proof of Theorem 3.9 will be presented.

On the other hand let us note that if one considers two choices of correspondences 3.11 satisfying the conclusions 1, 2, 3 of Theorem 3.9 and if one denotes by
\[ (3.27) \quad c_{pi}^{(1)} \quad \text{and} \quad c_{pi}^{(2)} \]
the classes in 3.22 corresponding to these two choices then one can show that for each $p$ and $i$ the classes 3.27 are compatible; cf. the argument in the proof of [5], Lemma 3.97. We will not use this compatibility in what follows.

4) There is a discrepancy between the arithmetic formalism we just introduced in Definition 3.11 and the formalism of classical Riemannian geometry (cf. the Appendix). In classical Riemannian geometry curvature can be viewed as a family of $n \times n$ matrices indexed by 2 indices each of which runs through 1, ..., $n$. In the arithmetic case the curvature 3.23 is a family of correspondences on the generic point of $GL_n$ (rather than a family of $n \times n$ matrices) indexed by 4 indices (rather than 2 indices): 2 of the 4 indices still run through 1, ..., $n$ while the other two run through $V$. In order to reduce the 4 indices to 2 indices one could attempt to fix, in some natural way, a bijection $\{1, ..., n\} \rightarrow V, i \mapsto p_i$; then one could attach to any pair of indices $i, j$ the quadruple of indices $p_i, p_j, i, j$. One can choose, for instance, in our situation, a soldering, i.e., a bijection $\{1, ..., n\} \rightarrow V$ making the diagram 2.7 commute. We will not pursue this in what follows.

5) Note that there is a natural embedding
\[ G(F) \rightarrow G(E) \]
sending any $\sigma : F \rightarrow F$ into the unique automorphism (still denoted by) $\sigma : E \rightarrow E$ that extends $\sigma$ on $F$ and satisfies $\sigma(x) = x$. So we get a natural injective ring homomorphism
\[ \mathbb{Z}G(F) \rightarrow \mathbb{Z}G(E). \]
Composing with the natural injective ring homomorphism $\mathbb{Z}G(E) \rightarrow C(E)$ we get an injective ring homomorphism
\[ \mathcal{R}(F) \rightarrow C(E), \]
hence a ring homomorphism
\[ (3.28) \quad \mathcal{R}(F) \rightarrow \mathcal{R}(E). \]
The latter composed with each of the two the homomorphisms 3.20 equals the identity of \( R(F) \) so in particular 3.28 is still injective.

From the above discussion we see that Galois theory (encoded into \( R(F) \)) and curvature (encoded in the correspondence classes \( c_{pi} \)) coexist in \( C(E) \) and hence in \( R(E) \) and can be made to interact. Indeed \( G(F) \) acts by ring automorphisms on \( C(E) \) via conjugation. Also \( R(F) \), viewed as a Lie ring acts by derivations on \( C(E) \), viewed as a Lie ring, via the commutator bracket. These actions descent to actions of \( G(F) \) and \( R(F) \) on \( R(E) \).

Of course, these actions can be made entirely explicit. We explain this in what follows by concentrating on the actions on \( C(E) \). So assume \( c \in C(E) \) is the class of an irreducible correspondence 3.13, \( Y = \text{Spec } L \), \( L \) a field. We can assume

\[
L = \frac{F(x)[y]}{(f(x,y))},
\]

where \( y \) is a family of indeterminates, \( f \) is a vector of polynomials in \( F[x,y] \), and \( \pi \) is the natural inclusion of \( E = F(x) \) into \( L \). (Of course one can always assume \( y \) is a single variable and \( f \) is a single polynomial but in applications one is sometimes presented with several variables and polynomials.) Since \( F \) is Galois over \( Q \), \( \varphi : E \rightarrow L \) restricted to \( F \) is given by an automorphism \( \phi \) of \( F \) followed by the inclusion \( F \subset L \). Let \( \sigma \in G(F) \) and let \( g \in F(x,y) \) be any rational function in \( x, y \) with coefficients in \( F \). Then we denote by \( g^\sigma(x,y) \in F(x,y) \) the rational function obtained from \( g(x,y) \) by applying \( \sigma \) to the coefficients of \( g(x,y) \). Finally let us see \( \sigma \) as an element of \( C(E) \). Then the product \( \sigma \cdot c \in C(E) \) is the class of the correspondence

\[
(3.29) \quad \text{Spec } L_1 \quad \text{Spec } E \quad \text{Spec } E
\]

where \( L_1 = L \), \( \pi_1 = \pi \), \( \varphi_1 \) restricted to \( F \) is given by \( \phi \circ \sigma \), and \( \varphi_1 \) acts on \( x \) as \( \varphi_1(x) = \varphi(x) \). On the other hand the product \( c \cdot \sigma \in C(E) \) is the class of the correspondence

\[
(3.30) \quad \text{Spec } L_2 \quad \text{Spec } E \quad \text{Spec } E
\]

where

\[
L_2 = \frac{F(x)[y]}{(f^\sigma(x,y))},
\]

\( \pi_2 \) is the natural inclusion of \( E = F(x) \) into the right hand side of 3.31, \( \varphi_2 \) restricted to \( F \) is given by \( \sigma \circ \phi \), and if \( \varphi(x) \) is the class of some \( g(x,y) \in F(x)[y] \) then \( \varphi_2 \) acts on \( x \) by

\[
\varphi_2(x) = \text{class}(g^\sigma(x,y)).
\]

As a consequence the product \( \sigma \cdot c \cdot \sigma^{-1} \in C(E) \) is the class of the correspondence

\[
(3.32) \quad \text{Spec } L_3 \quad \text{Spec } E \quad \text{Spec } E
\]
where

\[ L_3 = \frac{F(x)[y]}{(f^{3-1}(x,y))}, \]

\( \pi_3 \) is the natural inclusion of \( E = F(x) \) into the right hand side of (3.33) \( \varphi_3 \) restricted to \( E \) is given by \( \sigma^{-1} \circ \phi \circ \sigma \), and if \( \varphi(x) \) is the class of some \( g(x,y) \in F(x)[y] \) then \( \varphi_3 \) acts on \( x \) by

\[ \varphi_3(x) = \text{class}(g^{\sigma^{-1}}(x,y)). \]

In particular the products \( \sigma \cdot c, c \cdot \sigma \), and \( \sigma \cdot c \cdot \sigma^{-1} \) are irreducible.

Note that if \( \sigma \cdot c = c \cdot \sigma \) then necessarily \( \phi \circ \sigma = \sigma \circ \phi \).

But even if \( \phi \circ \sigma = \sigma \circ \phi \) we generally have that \( \sigma \cdot c \neq c \cdot \sigma \). Indeed take, for instance, as a toy example, \( F = \mathbb{Q}(\sqrt{-1}) \), take \( \sigma : F \to F \) to be complex conjugation, let \( f = y^2 - (1 + \sqrt{-1}) \), let \( \phi \) be arbitrary (i.e., \( \sigma \) or id), and let \( \psi \) be \( \phi \) on \( F \) and arbitrary on \( x \). Then \( \phi \circ \sigma = \sigma \circ \phi \); but an equality \( \sigma \cdot c = c \cdot \sigma \) would imply that \( L_2 \) is isomorphic to \( L_1 \) over \( E \) which is not the case.

Finally note that, in general, if \( \phi \circ \sigma = \sigma \circ \phi \), \( f^\sigma = f \), and \( g^\sigma = g \) then \( \sigma \cdot c = c \cdot \sigma \).

6) The above constructions lead to a natural context for holonomy in the correspondence setting. Indeed, assume we are in the setting of Definition 3.11. Then define

\( \mathfrak{hol}_E \subset \mathcal{R}(E) \)

to be the \( \mathbb{Z} \)-linear span of the set of iterated commutators

\[ [\hat{e}_{pi}, [\hat{e}_{p'i'}, [\hat{e}_{p''i''}, [...]]]] \]

of length \( \geq 2 \) (i.e. involving at least 2 elements). Then \( \mathfrak{hol}_E \) is a Lie subring of \( \mathcal{R}(E) \). We can refer to \( \mathfrak{hol}_E \) as the holonomy ring of the Levi-Civita connection attached to \( q \). It is a “correspondence version” of the holonomy ring \( \mathfrak{hol} \) introduced in [5]. Now let \( \mathfrak{hol}_E^0 \) and \( \mathfrak{hol}_F \) be the kernel and the image of the composition

\[ \mathfrak{hol}_E \to \mathcal{R}(E) \xrightarrow{\text{res}} \mathcal{R}(F). \]

(A similar discussion can be made for \( \text{res}_r \).) We get an exact sequence of Lie rings

\[ 0 \to \mathfrak{hol}_E^0 \to \mathfrak{hol}_E \to \mathfrak{hol}_F \to 0 \]

and hence a Lie ring homomorphism

\[ \mathfrak{hol}_F \to \text{Out}(\mathfrak{hol}_E^0) \]

where Out stands for the Lie ring of outer derivations, i.e., derivations modulo inner derivations. Such a construction was conjectured in [5] and could be viewed as an analogue of the classical presentation of the monodromy group of a connection as the quotient of the holonomy group by its identity component [10].

We turn now to the case \( n = 2 \) of the above correspondence story and show it is “compatible with complex structure”. Recall our notation from [3,10]. In analogy with [5,10] let us introduce, again, 2 variables \( \alpha, \beta \) and set

\[ G' = GL_1 \]

\[ G'_q := G' \otimes \mathcal{O}_q = \text{Spec} \mathcal{O}_{F,M}[[\alpha, \beta, (\alpha^2 + \beta^2)^{-1}]], \]

\[ \widehat{G'_q} = (G')^{\hat{q}} = \text{Spec} \widehat{\mathcal{O}_q}[[\alpha, \beta, (\alpha^2 + \beta^2)^{-1}]]. \]
We have a closed embedding \( G' \to G \) defined by
\[
x \mapsto \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.
\]
Then we will prove:

**Theorem 3.13.** Let \( d_1, d_2 \in O_{P,M}^\times, \quad q_1 = d_1 \cdot 1_2, \quad q_2 = d_2 \cdot 1_2 \in GL_2(O_{F,M}). \)

Let \((\delta'_1, \delta'_2)\) be the local Levi-Civita connection on \( G'_\mathfrak{p} \) over \( \hat{O}_\mathfrak{p} \) attached to \((q_1, q_2)\) and let \((\phi'_1, \phi'_2)\) be the attached Frobenius lifts on \( \hat{G}'_\mathfrak{p} \). Then there exists a pair of correspondences on \( G' \),
\[
(3.35) \quad \xymatrix{ Y'_p/G' \ar@{^{(}->}[r]^{\pi'_p} \ar@{_{(}->}[d]_{\phi'_p} & G' \ar@{^{(}->}[l]^{(G')^{\mathfrak{p}}}_{{\phi'}_p} }
\]
where \( i = 1, 2 \), such that the following hold:

1) The map \( \pi'_p : Y'_p/G' \to G' \) is étale and \( Y'_p/G' \) is affine and irreducible.

2) There is a connected component \( Y'_p/G' \) of \( (Y'_p/G')^{\mathfrak{p}} \) such that the induced map \((\pi'_p)^{\mathfrak{p}} : Y'_p/G' \to (G')^{\mathfrak{p}}\) is an isomorphism.

3) For each \( i \) the following induced diagram is commutative:
\[
(3.36) \quad \xymatrix{ Y'_p/G' \ar@{^{(}->}[r]^{(\pi'_p)^{\mathfrak{p}}} \ar@{_{(}->}[d]_{(\phi'_p)^{\mathfrak{p}}} & (G')^{\mathfrak{p}} \ar@{^{(}->}[l]^{\phi'_p} }
\]
In addition the inclusion map \( G' \to G \) lifts to a (natural) morphism from the correspondence \((3.36)\) to the correspondence \((3.11)\) inducing a morphism from the diagram \((3.36)\) to the diagram \((3.12)\).

**Remark 3.14.** Assume we are in a global situation \( F, M \) and that a frame with \( n = 2 \) is given. Let
\[
d \in O^\times, \quad O := O_{F,M}, \quad q := d \cdot 1_2 \in GL_2(O), \quad d_i = \sigma_i^{-1}d.
\]
This situation could be viewed as an analogue of the case of “conformal coordinates for metrics on surfaces” in classical differential geometry. Fix \( p \in V \) and set \( \mathfrak{P} = \mathfrak{P}(p) \). A formal argument as in Remark 3.12 1), shows that, if \( E' \) is the field of rational functions on \( G' \) then the pull-backs of the diagrams \((3.35)\) via \( \text{Spec } E' \to G' \) yield correspondences
\[
(3.37) \quad \xymatrix{ Y'_p/E' \ar@{^{(}->}[r]^{\pi'_p} \ar@{_{(}->}[d]_{\psi'_p} & \text{Spec } E' \ar@{^{(}->}[l]^{\phi'_p} }
\]
whose isomorphism classes are irreducible elements of \( C(E') \):
\[
(3.38) \quad c'_p \in C(E').
\]
So, if we are in the setting of Definition 3.11 with \( q = d \cdot 12, d \in \mathcal{O}^*_{F,M}, \) then we get a well defined curvature \( \langle \varphi_{pp'}^{i'i'} \rangle \) attached to \( q \) with components

\[
\varphi_{pp'}^{i'i'} = [c_{pi}'; c_{p'i'}'] \in C(E').
\]

The latter induces a family of elements

\[
\varphi_{pp'}^{i'i'} = [c_{pi}'; c_{p'i'}'] \in \mathcal{R}(E')
\]

and a family of group endomorphisms

\[
(\varphi_{pp'}^{i'i'})^* = [(c_{pi}')^*, (c_{p'i'})^*] \in \mathcal{S}(E'),
\]

which we refer to as the \( * \)-curvature.

Our discussion of the action of \( \mathcal{G}(E) \) on \( C(E) \) in Remark 3.12 applies without any change to the case of the action of \( \mathcal{G}(E') \) on \( C(E') \).

We have the following explicit description of the correspondence 3.37 in case \( d \) is \( \sigma \)-invariant. In the statement below we fix \( p \) and denote by \( \phi_p = \phi_p \in \mathcal{G}(F) \) the Frobenius element corresponding to \( \mathfrak{P} := \mathfrak{P}(p) \).

**Proposition 3.15.** Assume \( d \) is \( \sigma \)-invariant and set

\[
\theta_p = \frac{d^p(\alpha^2 + \beta^2)^p}{\phi_p(d)(\alpha^{2p} + \beta^{2p})} \in E'.
\]

Then \( V_{p/E'} \) in 3.37 is isomorphic to the spectrum of a field \( L_p' \), which, viewed as an extension of \( E' \) via \( \pi_p' \), is generated by a root \( v_p \) of the quadratic polynomial

\[
2z^2 + 2z + 1 - \theta_p.
\]

On the other hand, for \( u_p = 1 + v_p \), the homomorphisms \( \varphi_{p1}', \varphi_{p2}' \) in 3.37 correspond to the homomorphisms (still denoted by)

\[
\varphi_{p1}', \varphi_{p2}' : E' \to L_p'
\]

that act on \( F \) via \( \phi \) and act on \( \alpha, \beta \) via the formulae

\[
\begin{align*}
\varphi_{p1}' & \left( \begin{array}{cc} \alpha & \beta \\ -\beta & \alpha \end{array} \right) = \left( \begin{array}{cc} \alpha^p & \beta^p \\ -\beta^p & \alpha^p \end{array} \right) \left( \begin{array}{cc} u_p & v_p \\ -v_p & u_p \end{array} \right), \\
\varphi_{p2}' & \left( \begin{array}{cc} \alpha & \beta \\ -\beta & \alpha \end{array} \right) = \left( \begin{array}{cc} \alpha^p & \beta^p \\ -\beta^p & \alpha^p \end{array} \right) \left( \begin{array}{cc} u_p & -v_p \\ v_p & u_p \end{array} \right).
\end{align*}
\]

**Remark 3.16.** Assume the hypotheses of Proposition 3.15 and \( d \in \mathbb{Z}[1/M]^* \). Note that the trace of \( v_p \) in the extension \( E' \subset L_p' \) is the trace of the matrix

\[
(3.39) \quad V_p = \begin{pmatrix} 0 & 1 \\ \frac{\theta_p - 1}{2} & -1 \end{pmatrix}.
\]

One immediately gets that for any (not necessarily distinct) primes \( p, p' \), the following formulae hold:

\[
\begin{align*}
(3.40) \quad (\varphi_{pp'}^{i'i'})^*(\alpha) &= -\text{tr}\{ (\beta^{p'} + (\beta^{p'} - \alpha^{p'})V_{p'p})^p \} - \text{tr}\{ (\alpha^p + (\alpha^p - \beta^p)V_p)^p \}, \\
(3.41) \quad (\varphi_{pp'}^{i'i'})^*(\beta) &= \text{tr}\{ (\alpha^{p'} + (\alpha^{p'} + \beta^{p'})V_{p'p})^p \} + \text{tr}\{ (\alpha^p + (\alpha^p - \beta^p)V_p)^p \}.
\end{align*}
\]

Note on the other hand that the matrix \( V_p \) in (3.39) has entries in the ring

\[
(3.42) \quad \mathbb{Z}[1/M][\alpha, \beta, (\alpha^{2p} + \beta^{2p})^{-1}]
\]
hence the right hand sides of 3.40 and 3.41 are elements of 3.42. Since raising matrices to power \( p \) and taking trace of matrices commute modulo \( p \) we immediately get

\[ \varphi'_{pp'}(\alpha) \equiv -2(\alpha^{pp'} + \beta^{pp'}) \mod p \]

in the ring 3.42. Similarly one gets, of course,

\[ \varphi'_{pp'}(\alpha) \equiv -2(\alpha^{pp'} + \beta^{pp'}) \mod p'. \]

So we get that for all (not necessarily distinct) \( p, p' \), the *-curvature satisfies

\[ (\varphi'_{pp'})^* \neq 0 \text{ in } S(E'). \]

In particular, for all (not necessarily distinct) \( p, p' \), we have

\[ \tilde{\varphi}_{pp'} \neq 0 \text{ in } R(E'), \text{ hence } \varphi_{pp'} \neq 0 \text{ in } C(E'). \]

It is interesting to note a contrast between formula 3.43 and the formula in Corollary 3.6. The former tells the "correspondence story" (that involves the trace) whereas the latter is the \( p \)-adic story (that does not involve the trace); the two stories turn out to be different even in this simple example. In particular the reduction mod \( p \) of \((\varphi'_{pp'})^* \) in 3.43 does not depend on \( d \) whereas the reduction mod \( p \) of \( \varphi_{12} \) in Corollary 3.6 does depend on \( d \). Note, on the other hand that \((\varphi'_{pp'})^* \) itself (not reduced mod \( p \)) still depends on \( d \).

Remark 3.17. One can explicitly compute the action of \( G(F) \) on the classes 3.38; cf. Remark 4.12.

4. Proofs

4.1. Levi-Civit\`a connections. We begin by proving our results about the existence and uniqueness of Levi-Civit\`a connections.

Proof of Theorem 2.11

We first prove the existence of the tuple \((\delta^G_1, \ldots, \delta^G_n)\). The argument is an extension of the argument in the proof of Theorem 4.38 in [5].

Consider the matrices

\[ A_i = x^{(p)t} \cdot \phi(q_i) \cdot x^{(p)}, \quad B_i = (x^t q_i x)^{(p)}. \]

Note that for any \( n \)-tuple \((\delta^G_1, \ldots, \delta^G_n)\) of \( p \)-adic connections on \( G \), if

\[ \Delta_i := \delta^G_i x, \quad \Gamma_i := \Delta_i^t \cdot \phi(q_i) \cdot x^{(p)} \]

then

\[ (4.1) \quad \Gamma_i = \Delta_i^t \cdot (x^{(p)t})^{-1} \cdot A_i. \]

We will construct by induction a sequence of \( n \)-tuples

\[ (4.2) \quad (\Lambda^1, \ldots, \Lambda^n), \quad \nu \geq 1 \]

of \( n \times n \) matrices with entries in \( O(\widehat{G}) \) such that if

\[ \Delta_i^{(\nu)} := \frac{1}{p} x^{(p)} (\Lambda_i^{(\nu)} - 1), \quad \Gamma_i^{(\nu)} := \Delta_i^{(\nu)t} \cdot (x^{(p)t})^{-1} \cdot A_i, \quad \Gamma_i^{(\nu)} = (\Gamma_i^{(\nu)})_{i,j,k}, \]

then the following properties hold:

i) \( \Lambda_i^{(1)} = 1 \),
\[
\begin{align*}
n ii) \quad & \Lambda_{i(\nu+1)}^{(\nu)} \equiv \Lambda_{i(\nu)}^{(\nu)} \mod p^\nu, \\
n iii) \quad & \Lambda_{i(\nu)}^{(\nu)} A_i \Lambda_{i(\nu)}^{(\nu)} \equiv B_i \mod p^\nu, \\
n iv) \quad & \Gamma^{(\nu)}_{ijk} = \Gamma^{(\nu)}_{jik}, \text{ equivalently } (A_i(\Lambda_{i(\nu)}^{(\nu)} - 1))_{kj} = (A_j(\Lambda_{j(\nu)}^{(\nu)} - 1))_{ki}.
\end{align*}
\]

We claim this ends the proof of the existence of the tuple \((\delta_i^G, ..., \delta_n^G)\) in our theorem. Indeed one can then set
\[
\Lambda_i := \lim_{\nu \to \infty} \Lambda_{i(\nu)}^{(\nu)}
\]
and one can define \(\phi_i^G\) by setting
\[
\phi_i^G(x) := x(p)^\Lambda_i.
\]

By the way, with these definitions if
\[
\Delta_i := \lim_{\nu \to \infty} \Delta_{i(\nu)}^{(\nu)}
\]
then
\[
\delta_i^G x = \Delta_i, \quad \Gamma_i = \lim_{\nu \to \infty} \Gamma_{i(\nu)}^{(\nu)}.
\]

Now condition iii) above implies
\[
\Lambda_{i(\nu)}^{(\nu)} A_i \Lambda_{i(\nu)}^{(\nu)} = B_i
\]
which is equivalent to assertion 1 of the Theorem; also condition iv) above implies assertion 2 of the Theorem, which ends our proof.

To construct our sequence of \(n\)-tuples \((4.2)\) define the \(n\)-tuple for \(\nu = 1\) by condition i), assume the \(n\)-tuple \((4.2)\) was constructed for some \(\nu\) and seek the \(n\)-tuple \((4.2)\) corresponding to \(\nu + 1\) in the form
\[
(4.3) \quad \Lambda_{i(\nu+1)}^{(\nu)} = \Lambda_{i(\nu)}^{(\nu)} + p^\nu Z_i.
\]
Write
\[
(4.4) \quad \Lambda_{i(\nu)}^{(\nu)} A_i \Lambda_{i(\nu)}^{(\nu)} = B_i - p^\nu C_{i(\nu)}^{(\nu)}.
\]
Then
\[
(4.5) \quad \Lambda_{i(\nu+1)}^{(\nu+1)} = \Lambda_{i(\nu)}^{(\nu)} A_i \Lambda_{i(\nu)}^{(\nu)} + p^\nu (\Lambda_{i(\nu)}^{(\nu)} A_i Z_i + Z_i^t A_i \Lambda_{i(\nu)}^{(\nu)} \mod p^{\nu+1}) \equiv B_i + p^\nu (-C_{i(\nu)}^{(\nu)} + A_i Z_i + Z_i^t A_i) \mod p^{\nu+1}.
\]

Now \(A_i^t = A_i\) and \(B_i^t = B_i\) so \(C_{i(\nu)}^{(\nu)} = C_{i(\nu)}^{(\nu)}\). So if \(C_{i(\nu)}^{(\nu)} = (C_{ijk}^{(\nu)})\) then \(C_{i(\nu)}^{(\nu)} = C_{i(\nu)}^{(\nu)}\).

Define
\[
(4.6) \quad D_{ijk}^{(\nu)} := \frac{1}{2}(C_{ij}^{(\nu)} + C_{jk}^{(\nu)} - C_{ki}^{(\nu)}).
\]
Then
\[
(4.7) \quad D_{ijk}^{(\nu)} = D_{jik}^{(\nu)}
\]
and
\[
(4.8) \quad D_{ijk}^{(\nu)} + D_{ikj}^{(\nu)} = C_{ijk}^{(\nu)}.
\]
So if we define the matrices
\[
D_i^{(\nu)} = (D_{ijk}^{(\nu)})
\]
we have:

\[ D_i^{(\nu)} + D_i^{(\nu)t} = C_i^{(\nu)}. \]

Setting

\[ (4.9) \quad Z_i := A_i^{-1} D_i^{(\nu)t} \]

we get

\[ D_i^{(\nu)t} = A_i Z_i, \quad D_i^{(\nu)} = Z_i^t A_i. \]

So, by \(4.5\)

\[ \Lambda_i^{(\nu+1)t} A_i \Lambda_i^{(\nu+1)} \equiv B_i \mod p^{\nu+1}, \]

hence condition iii) holds for \( \Lambda_i^{(\nu+1)} \mod p^{\nu+1} \).

To check condition iv) for \( \nu \) replaced by \( \nu + 1 \) note that

\[ \begin{align*}
(A_i(\Lambda_i^{(\nu+1)} - 1) )_{k_j} &= (A_i(\Lambda_i^{(\nu)} + p^\nu Z_i - 1))_{k_j} \\
&= (A_i(\Lambda_i^{(\nu)} - 1))_{k_j} + p^\nu D_{ijk}^{(\nu)} \\
&= (A_j(\Lambda_j^{(\nu)} - 1))_{k_i} + p^\nu D_{ijk}^{(\nu)} \\
&= (A_j(\Lambda_j^{(\nu+1)} - 1))_{k_i}.
\end{align*} \]

This ends the proof of the existence part of our Theorem.

We next prove the uniqueness of the tuple

\[ (\delta_G^1, \ldots, \delta_G^n) \]

in our theorem.

Assume we have two such tuples which we denote by

\[ (\delta_1, \ldots, \delta_n) \quad \text{and} \quad (\delta_1', \ldots, \delta_n'). \]

Let \( \phi_i \) and \( \phi_i' \) be the corresponding Frobenis lifts on \( \hat{G} \), write

\[ \phi_i(x) = x^{(p)} \Lambda_i, \quad \phi_i'(x) = x^{(p)} \Lambda_i' \]

for matrices \( \Lambda_i, \Lambda_i' \), and let

\[ \delta_i x = \Delta_i, \quad \delta_i' x = \Delta_i', \quad \Gamma_i = \Delta_i' \cdot (x^{(p)t})^{-1} \cdot A_i, \quad \Gamma_i' = (\Delta_i')^{-1} \cdot (x^{(p)t})^{-1} \cdot A_i. \]

We have

\[ \begin{align*}
(4.10) \quad &\Lambda_i' A_i A_i = B_i, \quad (\Lambda_i')^t A_i A_i = B_i \\
(4.11) \quad &\Gamma_{ijk} = \Gamma_{ijk}, \quad \Gamma_{ijk}' = \Gamma_{ijk}'.
\end{align*} \]

We will prove that

\[ \begin{align*}
(4.12) \quad &\Lambda_i \equiv \Lambda_i' \mod p^\nu \\
\end{align*} \]

by induction on \( \nu \) and this will end the proof. The case \( \nu = 1 \) is clear. Assume \(4.12\) holds for some \( \nu \geq 1 \) and write

\[ \begin{align*}
(4.13) \quad &\Lambda_i' = \Lambda_i + p^\nu Z_i. \\
\end{align*} \]

From \(4.10\) we get

\[ B_i \equiv B_i + p^\nu A_i Z_i + p^\nu Z_i^t A_i \mod p^{\nu+1}, \]
hence, setting 
\[ E_i = Z_i^t A_i = (E_{ijk}) \]
we get 
\[ E_i + E_i^t \equiv 0 \mod p, \]
hence
\[ (4.14) \quad E_{ijk} = -E_{ikj} \mod p. \]
On the other hand, from 4.13 we get
\[ \Gamma'_i = \Gamma_i + p^\nu - 1 E_i \]
hence, by 4.11
\[ (4.15) \quad E_{ijk} = E_{jik}. \]
Combining 4.14 and 4.15 we get
\[ (4.16) \quad E_{ijk} \equiv E_{jik} \equiv -E_{jki} \mod p. \]
Applying 4.16 three times we get
\[ E_{ijk} \equiv -E_{jki} \equiv E_{kij} \equiv -E_{ijk} \mod p, \]
and our induction step is proved. \(\square\)

**Proof of Proposition 2.14** Assume the notation in the proof of Theorem 2.11. Then \(C_i\) in the statement of Proposition 2.14 equals \(C^{(1)}_i\) in the proof of Theorem 2.11. The proof of Theorem 2.11 shows that
\[ (4.17) \quad \Gamma^{(\nu+1)}_i = \Gamma^{(\nu)}_i + p^\nu - 1 D^{(\nu)}_i. \]
Since \(\Lambda^{(1)}_i = 1\) we have \(\Gamma^{(1)}_i = 0\) hence, by 4.17
\[ \Gamma_{ijk} \equiv \Gamma^{(2)}_{ijk} \equiv D^{(1)}_{ijk} \mod p \]
and we are done by 4.16. \(\square\)

**Proof of Theorem 2.8** Consider any index \(i\). Consider the bijection \(\delta^i \mapsto \delta_i := \sigma_i^{-1} \circ \delta^i \circ \sigma_i\) between \(p\)-adic connections on \(G^i\) and \(p\)-adic connections on \(G^1\). The Frobenius lifts \(\phi^i\) and \(\phi^1_i\) attached to \(\delta^i\) and \(\delta^1_i\) are then related by 
\[ \phi^1_i := \sigma_i^{-1} \circ \phi^i \circ \sigma_i. \]
Consider, on the other hand, the matrices 
\[ \alpha_i = x^{(p)t} \cdot \phi^i(q) \cdot x^{(p)}, \quad \beta = (x^t qx)^{(p)} \]
with entries in $O(\hat{G}^i)$. Set
\[ \phi^i(x) = x^{(p)} \lambda_i. \]
The condition that $\phi^i$ be $H_q$-horizontal with respect to the trivial connection $\phi_0^i$ on $G^i$ is equivalent to
\[ (4.18) \quad \lambda_i^1 \alpha_i \lambda_i = \beta. \]
Consider the matrices
\[ A_i = \sigma_i^{-1} \alpha_i, \quad B_i = \sigma_i^{-1} \beta, \quad \Lambda_i = \sigma_i^{-1} \lambda_i \]
with entries in $O(\hat{G}^1)$ and set
\[ q_i = \sigma_i^{-1}(q). \]
So $[4.18]$ is equivalent to
\[ (4.19) \quad \Lambda_i^1 A_i \Lambda_i = B_i. \]
On the other hand we have
\[ A_i = (x^{(p)\ell} \cdot \phi_1^i(\sigma_i^{-1}(q)) \cdot x^{(p)}) = x^{(p)\ell} \cdot \phi_1^i(q_i) \cdot x^{(p)}; \]
\[ B_i = (\sigma_i^{-1}(x^i q x^i))(p) = (x^i \sigma_i^{-1}(q)x^i)(p) = (x^i q x^i)(p); \]
\[ \phi_G^i(x) = (\sigma_i^{-1} \circ \phi \circ \sigma_i)(x) = \sigma_i^{-1}(\phi(x)) = \sigma_i^{-1}(x^{(p)} \lambda_i) = x^{(p)} \Lambda_i. \]
So $[4.19]$ is equivalent to $\phi_G^i$ being $H_q$-horizontal with respect to $\phi_0^i$; hence the latter condition is equivalent to the condition that $\phi^i$ be $H_q$-horizontal with respect to $\phi_0^i$.

To tackle torsion freeness consider the matrices
\[ \gamma_i := \delta^i x^\ell \cdot \phi_i(q) \cdot x^{(p)}, \quad \Gamma_i := \sigma_i^{-1} \gamma_i. \]
Note that
\[ \Gamma_{ijk} = \sigma_i^{-1} \gamma_{ijk} \]
hence $(\delta^1, ..., \delta^n)$ is torsion free if and only if $\Gamma_{ijk} = \Gamma_{jik}$. But on the other hand we have
\[ \Gamma_i = \sigma_i^{-1}(\delta^i(x^\ell)) \cdot \sigma_i^{-1}(\phi^i(q)) \cdot x^{(p)} \]
\[ = \delta^i(\sigma_i^{-1}(x^\ell)) \cdot \phi^i(\sigma_i^{-1}(q)) \cdot x^{(p)} \]
\[ = \delta^i x^\ell \cdot \phi^i(q_i) \cdot x^{(p)}. \]
At this point it is clear that Theorem 2.8 follows from Theorem 2.11 applied to $\mathcal{O} = \hat{\mathcal{O}}_{\varphi_1}$ and $G = G^1$. \qed

**Proof of Proposition 2.10** Assume the notation in the proof of Theorem 2.8
Note that
\[ C_i = -x^{(p)\ell} \cdot \sigma_i^{-1} \delta^i q \cdot x^{(p)} + \frac{1}{p} \{(x^{\ell} \cdot \sigma_i^{-1} q \cdot x)(p) - x^{(p)\ell} \cdot \sigma_i^{-1} q(p) \cdot x(p)\} \]
\[ = -x^{(p)\ell} \cdot \delta_i^1 \sigma_i^{-1} q \cdot x^{(p)} + \frac{1}{p} \{(x^{\ell} \cdot \sigma_i^{-1} q \cdot x)(p) - x^{(p)\ell} \cdot \sigma_i^{-1} q(p) \cdot x(p)\} \]
\[ = -x^{(p)\ell} \cdot \delta_i^1 q_i \cdot x^{(p)} + \frac{1}{p} \{(x^{\ell} \cdot q_i \cdot x)(p) - x^{(p)\ell} \cdot q_i(p) \cdot x(p)\}. \]
So Proposition 2.10 follows directly from Proposition 2.11 applied to $\mathcal{O} = \hat{\mathcal{O}}_{\varphi_1}$ and $G = G^1$. \qed
Proof of assertion 1 in Proposition 2.16. With the notation in the proof of Theorem 2.11 view

\[ A_i, B_i, \Lambda_i \]
as matrices with coefficients in \( \widehat{B} = \mathcal{O}(\widehat{G}) \), where

\[ G = GL_2 = Spec \, B, \quad B = \mathcal{O}[x, \det(x)^{-1}] \]

Let \( \alpha, \beta \) be two variables and recall that we view

\[ GL_2^c = Spec \, B', \quad B' = \mathcal{O}[\alpha, \beta, (\alpha^2 + \beta^2)^{-1}] \]

embedded into \( G \) via the map

\[ (4.20) \quad B \to B', \quad b \mapsto b' \]
defined by

\[ x \mapsto x' := \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \]

Recall that we denoted by \( gl_2 \) the functor that attaches to any ring the algebra of \( 2 \times 2 \) matrices with coefficients in that ring; let \( gl_2^c \) the functor that attaches to any ring the commutator of \( c \) in \( gl_2 \) applied to that ring. We still denote by

\[ gl_2^c(B) \to gl_2^c(B'), \quad M \mapsto M' \]

the map induced by (4.20). It is enough to prove that for \( i = 1, 2 \),

\[ \Lambda'_i \in GL_2^c(\widehat{B}) \]

It is then enough to prove, by induction on \( \nu \), that

\[ (4.21) \quad (\Lambda_i^{(\nu)})' \in GL_2^c(\widehat{B}) \]

This is clearly true for \( \nu = 1 \). Assume (4.21) for some \( \nu \). Now clearly

\[ A'_i, B'_i \in GL_2^c(\widehat{B}) \]

By (4.13) we get

\[ C'_i := (C_i^{(\nu)})' \in gl_2^c(\widehat{B}) \]

Set

\[ D'_{ijk} := \frac{1}{2}(C'_{ijk} + C'_{jik} - C'_{kij}) \]

We have

\[ D'_{i11} = \frac{1}{2}C'_{i11} = \frac{1}{2}C''_{i22} = D''_{i11} \]

and

\[ D'_{i12} = \frac{1}{2}(C'_{i12} + C'_{1i2} - C'_{2i1}) = \frac{1}{2}(-C''_{i21} + C''_{1i2} - C''_{2i1}) = -D''_{i21} \]

So, with notation as in (4.9)

\[ D'_i = (D_i^{(\nu)})' \in gl_2^c(\widehat{B}) \]

Hence, with the notation in (4.9) we have

\[ Z'_i \in gl_2^c(\widehat{B}) \]

and hence, by (4.3)

\[ (\Lambda_i^{(\nu+1)})' \in gl_2^c(\widehat{B}) \]

which ends our induction. \( \square \)
Proof of Proposition 2.19 With the notation in the proof of Theorem 2.11 set \( \Phi_i = \varphi(p) \Lambda_i \) and view \( \Phi_i, A_i, B_i, \Lambda_i \) as matrices with coefficients in \( \mathcal{O}(\hat{G}) \). Let \( \Phi'_i, A'_i, B'_i, \Lambda'_i \) be the images of the corresponding matrices in the ring of matrices with coefficients in \( \mathcal{O}(\hat{G}') \). We then have

\[
A'_i = \varphi(d_i) \cdot (\alpha^2 + \beta^2)^p \cdot 1_2, \quad B'_i = d^p \cdot (\alpha^2 + \beta^2)^p \cdot 1_2
\]

and recall that we defined

\[
(4.22) \quad \theta_i := \frac{d^p(\alpha^2 + \beta^2)^p}{\varphi(d_i)(\alpha^2 + \beta^2)^{p}}.
\]

Now, by the proof of Theorem 2.11 and by Remark 2.15, the following hold in \( \mathcal{O}(\hat{G}) \) for all \( i, j, k \in \{1, 2\} \):

1) \( \Lambda_i \equiv 1_2 \mod p \);
2) \( \Lambda'_i A_i \Lambda_i = B_i \);
3) \( \varphi(d_i)(\Lambda_i - 1_2)_{kj} = \varphi(d_j)(\Lambda_j - 1_2)_{ki} \).

We get that the following hold in \( \mathcal{O}(\hat{G}') \):

1’) \( \Lambda'_i \equiv 1_2 \mod p \);
2’) \( (\Lambda'_i)^t A'_i \Lambda'_i = B'^i \);
3’) \( \varphi(d_i)(\Lambda'_i - 1_2)_{kj} = \varphi(d_j)(\Lambda'_j - 1_2)_{ki} \).

Of course 3’) only needs to be checked for \( (i, j) = (1, 2) \). Now an argument similar to the one proving uniqueness in Theorem 2.11 shows that the conditions 1’, 2’, 3’ uniquely determine

\[
\Lambda'_1, \Lambda'_2 \in GL_2(\mathcal{O}(\hat{G}')).
\]

So in order to conclude our proof it is enough to show that 1’, 2’, 3’ hold if one replaces \( \Lambda'_1 \) and \( \Lambda'_2 \) by the matrices

\[
(4.23) \quad \begin{pmatrix} u_1 & v_1 \\ -v_1 & u_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} u_2 & v_2 \\ -v_2 & u_2 \end{pmatrix},
\]

respectively, where \( u_i, v_i \) are as in (2.23) and (2.22) respectively. The conditions 1’, 2’, 3’ for the matrices (4.23) translate into the following conditions:

1’) \( u_i \equiv 1 \) and \( v_i \equiv 0 \mod p \),
2’) \( u^2_i + v^2_i = \theta_i \),
3’) \( \varphi(d_1)v_1 = \varphi(d_2)(u_2 - 1) \) and \( \varphi(d_1)(u_1 - 1) = -\varphi(d_2)v_2 \).

Checking 1’, 2’, 3’ is a trivial exercise left to the reader. \( \square \)

Remark 4.1. The argument in the proof of Proposition 2.19 can be used to give an alternative proof of (the already proved) assertion 1 in Proposition 2.16.
**Proof of assertions 2 and 3 in Proposition 2.16**

We start with assertion 2. Taking determinants in 2.24, using condition 2') in the proof of Proposition 2.19 and finally using 4.22, we get

$$
\phi_i^{G'}(\alpha^2 + \beta^2) = (\alpha^{2p} + \beta^{2p})(u_i^2 + v_i^2) = (\alpha^{2p} + \beta^{2p}) \cdot \theta_i = \frac{d_i^p}{\phi(d_i)} \cdot (\alpha^2 + \beta^2)^p.
$$

It is then trivial to check that

$$
\phi_i^{G'}(\alpha^2 + \beta^2 - 1)
$$

is in the ideal generated by

$$
\alpha^2 + \beta^2 - 1
$$

if and only if

$$
\frac{d_i^p}{\phi(d_i)} = 1
$$

hence if and only if

$$
\delta d_i = 0,
$$

which ends the proof of assertion 2.

We next address assertion 3, so assume

$$
d_1 = d_2 =: d \in \mathbb{Z}, \quad d \neq \pm 1, \quad d \neq 0 \mod p.
$$

Without loss of generality we may assume $\sqrt{-1} \in \mathcal{O}$. So $GL_1^c$ is isomorphic to $GL_1 \times GL_1$ where the isomorphism is defined on points by

$$
\begin{pmatrix}
a & b \\
-b & a
\end{pmatrix} \mapsto (a + \sqrt{-1}b, a - \sqrt{-1}b).
$$

Since the closed connected subgroup schemes of $GL_1 \times GL_1$ are all kernels of characters it follows that any connected closed subgroup scheme of $GL_1^c$ is of the form $T_{k_1,k_2}$ where the latter is given schematically by the equations

$$
\begin{pmatrix}
\alpha & \beta \\
-\beta & \alpha
\end{pmatrix}^{k_1} \begin{pmatrix}
\alpha & -\beta \\
\beta & \alpha
\end{pmatrix}^{k_2} = 1_2,
$$

and $k_1,k_2 \in \mathbb{Z}$ are coprime. Equivalently $T_{k_1,k_2}$ is given schematically by the equations

$$
\begin{pmatrix}
\alpha & \beta \\
-\beta & \alpha
\end{pmatrix}^k = (\alpha^2 + \beta^2)^l \cdot 1_2
$$

where $k = k_2 - k_1$, $l = k_2$.

Assume now $T_{k_1,k_2}$ is $\delta_i^{G'}$-horizontal for some $k_1,k_2$ for $i = 1,2$; we will derive a contradiction.

If $k = 0$ then $l = \pm 1$ so $T_{k_1,k_2} = U_i^c$ and we are done by assertion 2 of the Proposition.

Assume now $k \neq 0$. Applying $\phi_i^{G'}$ to 4.24 and using Proposition 2.19 and Remark 2.20 we get

$$
\begin{pmatrix}
\alpha^p & \beta^p \\
-\beta^p & \alpha^p
\end{pmatrix}^k \cdot \begin{pmatrix}
u & u \\
v & -u
\end{pmatrix}^k = S + M,
$$
where \( u = v + 1 \), \( v \) is the root \( \equiv 0 \mod p \) of the equation

\[
2v^2 + 2v = d^{p-1}(\alpha^2 + \beta^2)^{p^2} + 1
\]

in the ring

\[
\mathcal{O}(\hat{G}) = \mathcal{O}[\alpha, \beta, (\alpha^2 + \beta^2)^{-1}]
\]

the matrix \( S \) is scalar,

\[
S \in \mathcal{O}(\hat{G}) \cdot 1_2,
\]

and the matrix

\[
M \in \mathfrak{gl}_n(\mathcal{O}(\hat{G}))
\]

has entries in the ideal defining \( T_{k_1, k_2} \). Let \( u_0, v_0 \in \mathcal{O} \) be obtained from \( u, v \) by setting \( \alpha = 1 \) and \( \beta = 0 \). Then from 4.25 and 4.26 we get

\[
(u_0 v_0 - v_0 u_0)^k \in \mathcal{O} \cdot 1_2
\]

and

\[
2v_0^2 + 2v_0 + 1 - d^{p-1} = 0.
\]

Set

\[
J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

and write

\[
(u_0 v_0 - v_0 u_0)^k = (u_0 \cdot 1_2 + v_0 \cdot J)^k.
\]

Using the binomial formula to expand and looking at the upper right corner entry of the matrix in 4.27 we get that

\[
\binom{k}{1} u_0^{k-1} v_0 - \binom{k}{3} u_0^{k-3} v_0^3 + \binom{k}{5} u_0^{k-5} v_0^5 - \ldots = 0.
\]

Now set \( e = |d^{(p-1)/2}| \), where \( | \cdot | \) is the Archimedean absolute value, and note that the discriminant \( 2e^2 - 1 \) of the polynomial in 4.28 is a positive rational number. So we can and will choose an embedding of \( \mathbb{Q}(v_0) \) into \( \mathbb{C} \) such that, in this embedding,

\[
v_0 = -1 + \sqrt{2e^2 - 1}, \quad u_0 = 1 + \sqrt{2e^2 - 1}
\]

where the square root is the real positive one. In particular \( u_0 \) and \( v_0 \) are real in this embedding. Then by 4.30 we have that the complex number

\[
(u_0 + \sqrt{-1} \cdot v_0)^k
\]

is real. So the complex number

\[
\zeta := \frac{u_0 + \sqrt{-1} \cdot v_0}{|u_0 + \sqrt{-1} \cdot v_0|} = \frac{u_0 + \sqrt{-1} \cdot v_0}{\sqrt{u_0^2 + v_0^2}} = \frac{u_0 + \sqrt{-1} \cdot v_0}{e}
\]

is a root of unity. Since

\[
\zeta \in \mathbb{Q}(\sqrt{-1}, \sqrt{2e^2 - 1})
\]

and the latter field has degree a divisor of 4 the order \( N \) of \( \zeta \) must satisfy

\[
\varphi(N) \in \{1, 2, 4\}
\]
where \( \varphi \) is the Euler function. So the only possibilities for \( N \) are:

\[
N \in \{1, 2, 3, 4, 5, 6, 8, 10, 12\}.
\]

Since the Galois group over \( \mathbb{Q} \) of the field in 4.32 cannot be cyclic of order 4 it follows that \( N \) cannot be 5 or 10. Now equation 4.31 gives

\[
(4.33) \quad \text{Re}\, \zeta = \frac{u_0}{e} = \frac{1 + \sqrt{2}e^2 - 1}{2e},
\]

so in particular \( \text{Re}\, \zeta > 0 \). Hence the only possibilities for \( \text{Re}\, \zeta \) are

\[
(4.34) \quad \text{Re}\, \zeta \in \{1, \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}\}.
\]

The case \( \text{Re}\, \zeta = 1 \) of equation 4.33 yields \( e = 1 \), hence \( d = \pm 1 \), a contradiction. For the other 3 values of \( \text{Re}\, \zeta \) in 4.34 equation 4.33 gives values of \( e \) that are not in \( \mathbb{Q} \), which is again a contradiction. This ends the proof.

**Remark 4.2.** Note that the above proof works if one replaces the hypotheses

\[
d \in \mathbb{Z}, \quad d \neq \pm 1, \quad d \not\equiv 0 \mod p
\]

by the hypotheses

\[
d \in \mathbb{Z}_\mathbb{p}^\times, \quad |d| > 1.
\]

**Proof of Proposition 3.2.** Let us place ourselves, in what follows, in the global situation and consider a symmetric matrix \( q \in GL_n(\mathcal{O}_{F,M}) \), the Levi-Civita connection

\[
(\delta^1, \ldots, \delta^n)
\]

attached to \( q \), the attached Frobenius lifts

\[
(\phi^1, \ldots, \phi^n),
\]

and the curvature \( (\varphi_{ij}) \). Let, as before,

\[
\phi^1 = \sigma^{-1}_i \phi^i \sigma_i.
\]

Then we have

\[
(4.35) \quad \varphi_{ij} = \frac{1}{p} (\phi^1_i \phi^j_1 - \phi^j_1 \phi^1_i).
\]

Set

\[
\Phi_{ij} := \varphi_{ij}(x), \quad \Delta_i := \delta^1_i x,
\]

and recall that the Christoffel symbols \( \gamma_i \) of the first kind are given by the equalities

\[
\sigma^{-1}_i \gamma_i = \Delta^t_i \cdot \phi^1(q_i) \cdot x^{(p)}.
\]

We have

\[
\phi^1_i \phi^j_1(x) = \phi^1_i (x^{(p)} + p\Delta_j) = (x^{(p)} + p\Delta_j)^{(p)} + p\phi^1_i (\Delta_j) = x^{(p^2)} + p\Delta_j^{(p)} \mod p,
\]
hence
\[
\Phi_{ij} \equiv \Delta_j^{(p)} - \Delta_i^{(p)} \\
\equiv ((\sigma_j^{-1}(q))^{(p^2)})^{-1}(x^{(p^2)t})^{-1}(\sigma_j^{-1} \gamma_j)(p) t \\
(4.36) -((\sigma_i^{-1}(q))^{(p^2)})^{-1}(x^{(p^2)t})^{-1}(\sigma_i^{-1} \gamma_i)(p) t \\
\equiv ((\sigma_j^{-1}(q))^{(p^2)})^{-1}(\sigma_j^{-1} \gamma_j)(p) t \\
-((\sigma_i^{-1}(q))^{(p^2)})^{-1}(\sigma_i^{-1} \gamma_i)(p) t \\n\mod (p,x - 1).
\]
Combining the congruences 4.36 with the congruences 2.14 one immediately gets the congruences 3.2.

4.2. Construction of an étale cover.

We discuss, in what follows, a construction that will be later used to prove the existence of our correspondences.

Let
\[
y = (y_1, \ldots, y_n), \quad z = (z_1, \ldots, z_n)
\]
be two \(n\)-tuples of matrices of size \(n \times n\) with indeterminates as entries,
\[
y_i = (y_{ijk}), \quad z_i = (z_{ijk}).
\]
Consider the system of linear equations in \(n^3\) unknowns \(z_{ijk}\), with coefficients in the ring \(\mathbb{Z}[y]\),
\[
(y^t z_i)_{jk} + (z^t y_i)_{jk} = 0, \quad i, j, k = 1, \ldots, n, \quad j \leq k,
\]
\[
z_{ikj} - z_{jki} = 0, \quad i, j, k = 1, \ldots, n, \quad i < j.
\]
There are
\[
n^2(n + 1) \quad 2
\]
equations in the first row of 4.37 and
\[
n^2(n - 1) \quad 2
\]
equations in the second row of 4.37 so there are are \(n^3\) equations in all. So the matrix of the system 4.37 is square and one can consider the determinant of this matrix which we denote by
\[
(4.38) \quad D(y) := D(y_1, \ldots, y_n) \in \mathbb{Z}[y].
\]
Of course \(D(y)\) is well defined only up to sign because the order of the variables and the order of the equations has not been specified.

Example 4.3. For \(n = 2\), \(y = (y_1, y_2)\),
\[
y_1 = \begin{pmatrix} y_{111} & y_{112} \\ y_{121} & y_{122} \end{pmatrix}, \quad y_2 = \begin{pmatrix} y_{211} & y_{212} \\ y_{221} & y_{222} \end{pmatrix}, \quad y_{1|2} := \begin{pmatrix} y_{112} & y_{211} \\ y_{122} & y_{221} \end{pmatrix},
\]
one gets
\[
(4.39) \quad D(y_1, y_2) = \pm \det(y_1) \cdot \det(y_2) \cdot \det(y_{1|2}).
\]
Going back to an arbitrary \(n\) and writing, as usual, \(1 = 1_n\) we may consider the integer \(D(1, \ldots, 1) \in \mathbb{Z}\).
Lemma 4.4. For any odd prime $p$ one has:

$$D(1, \ldots, 1) \not\equiv 0 \mod p.$$ 

In other words $D(1, \ldots, 1)$ is $\pm 1$ times (possibly) a power of 2.

Proof. Assume an odd prime $p$ divides $D(1, \ldots, 1)$. Then the system

\begin{align*}
z_{ijk} + z_{ikj} &= 0, \quad i, j, k = 1, \ldots, n, \\
z_{ikj} - z_{jki} &= 0, \quad i, j, k = 1, \ldots, n, \\
\end{align*}

has a zero determinant in $\mathbb{F}_p$ so it has a non-trivial solution $(\zeta_{ijk})$ in $\mathbb{F}_p$. So

$$\zeta_{ijk} = -\zeta_{ikj} = -\zeta_{jki}.$$

Using the latter 3 times one gets

$$2\zeta_{ijk} = 0$$

hence

$$\zeta_{ijk} = 0,$$

a contradiction. \qed

Assume now $\mathcal{B}$ is a Noetherian ring, fix an integer $n \geq 2$, and consider the polynomial $D(y) \in \mathbb{Z}[y]$ in \[4.38\]

Also let

$$A_1, \ldots, A_n, B_1, \ldots, B_n$$

be $n \times n$ symmetric matrices with entries in $\mathcal{B}$, let

$$b := \det(B_1) \cdot \ldots \cdot \det(B_n) \in \mathcal{B}, \quad B_b = B[1/b],$$

and define the ring $\mathcal{C}$ associated to the data $(\mathcal{B}, A, B)$ by the formula

\[4.41\] 

$$\mathcal{C} := \mathcal{C}(\mathcal{B}, A, B) := \frac{\mathcal{B}_b[y, D(y)^{-1}]}{(\langle y_i^t A_i y_i - B_i \rangle_{jk}, \langle A_j(y_j - 1) \rangle_{kj} - (A_j(y_j - 1))_{ki}),}$$

where $A$ is the $n$-tuple $(A_i)$ and $B$ is the $n$-tuple $(B_i)$.

Note that the triples $(\mathcal{B}, A, B)$ are the objects of an obvious category: a morphism

$$(\mathcal{B}, A, B) \to (\mathcal{B}', A', B')$$

is a morphism of rings $\mathcal{B} \to \mathcal{B}'$ which sends the matrices $A, B$ into $A', B'$ respectively. Then we clearly obtain a functor

$$\{ (\mathcal{B}, A, B) \} \to \{ \text{rings} \}, \quad (\mathcal{B}, A, B) \mapsto \mathcal{C}(\mathcal{B}, A, B).$$

For a morphism as above we have

$$\mathcal{C}(\mathcal{B}', A', B') \simeq \mathcal{C}(\mathcal{B}, A, B) \otimes_\mathcal{B} \mathcal{B}'.$$

So for any triple $(\mathcal{B}, A, B)$ we have

$$\mathcal{C}(\mathcal{B}, A, B) \simeq \mathcal{C}(\mathcal{B}^{\text{univ}}, A^{\text{univ}}, B^{\text{univ}}) \otimes_{\mathcal{B}^{\text{univ}}} \mathcal{B},$$

where $A^{\text{univ}}$ and $B^{\text{univ}}$ are two $n$-tuples of symmetric matrices with indeterminate coefficients on and above the diagonal,

$$\mathcal{B}^{\text{univ}} := \mathbb{Z}[A^{\text{univ}}, B^{\text{univ}}],$$

is the polynomial ring in these variables, and $\mathcal{B}^{\text{univ}} \to \mathcal{B}$ is given by $A^{\text{univ}} \mapsto A$, $B^{\text{univ}} \mapsto B$.

For $\mathcal{C} = \mathcal{C}(\mathcal{B}, A, B)$ we have a natural map of schemes

\[4.42\] 

$$\pi : Y := \text{Spec } \mathcal{C} \to X := \text{Spec } \mathcal{B}.$$
Lemma 4.5. The map $\pi : Y \to X$ is étale.

Proof. Consider a diagram of rings

$$
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{\pi} & \mathcal{C} \\
v \downarrow & & \downarrow u \\
\mathcal{D} & \xrightarrow{\rho} & \mathcal{D}/I
\end{array}
$$

where $I \subset \mathcal{D}$ is an ideal with $I^2 = 0$. We need to show that there is a unique map $w : \mathcal{C} \to \mathcal{D}$ such that

$$
\rho \circ w = u, \quad w \circ \pi = v.
$$

Set

$$
v(A_i) = a_i, \quad v(B_i) = b_i, \quad u(y_i) = \rho(\lambda_i),
$$

with $a_i, b_i, \lambda_i$ matrices with entries in $\mathcal{D}$. So we have that

$$
\begin{align*}
\lambda_i^t a_i \lambda_i - b_i &= c_i \\
(a_i(\lambda_i - 1))_{kj} - (a_j(\lambda_j - 1))_{ki} &= f_{ijk}
\end{align*}
$$

for some symmetric matrices $c_i$ with coefficients in $I$ and some elements $f_{ijk} \in I$ with

$$
f_{ijk} = - f_{jik}.
$$

To find $w$ is the same as to find elements

$$
w(y_{ijk}) = \lambda_{ijk} + \epsilon_{ijk},
$$

with $\epsilon_{ijk} \in I$, such that if $\epsilon_i = (\epsilon_{ijk})$ then

$$
\begin{align*}
(\lambda_i^t + \epsilon_i^t) a_i (\lambda_i + \epsilon_i) - b_i &= 0, \\
(a_i(\lambda_i + \epsilon_i - 1))_{kj} - (a_j(\lambda_j + \epsilon_j - 1))_{ki} &= 0.
\end{align*}
$$

In view of (4.43) if we set

$$
\zeta_i = a_i \epsilon_i,
$$

with $\zeta_i = (\zeta_{ijk})$, then the equations (4.44) can be rewritten as

$$
\begin{align*}
(\lambda_i^t \zeta_i)_{jk} + (\epsilon_i^t \lambda_i)_{jk} &= - c_{ijk}, \\
\zeta_{ikj} - \zeta_{jki} &= - f_{ijk},
\end{align*}
$$

where $i, j, k = 1, \ldots, n$. Now the system (4.46) is, of course, equivalent to the system consisting of the same equations but where the indices satisfy, in addition, $j \leq k$ for the equations in the first line and $i < j$ for the equations in the second line of (4.46). Since

$$
D(\lambda_1, \ldots, \lambda_n)
$$

is invertible in $\mathcal{D}$ (because it is invertible mod $I$) the system (4.46) has a unique solution $(\zeta_{ijk})$ with entries in $I$. Since $u(b)$ is invertible in $\mathcal{D}$ (because it is invertible mod $I$) it follows that $\det(b_i)$ and hence $\det(a_i)$ are invertible in $\mathcal{D}$ so the system (4.45) has a unique solution $(\epsilon_{ijk})$ with entries in $I$ and the Lemma is proved. \(\square\)

Assume, in what follows that $F$ is a number field, $0 \neq M \in \mathbb{Z}$ is an even integer, $\mathcal{O} = \mathcal{O}_{F,M} = \mathcal{O}[1/M]$, let $p$ be an odd prime in $\mathbb{Z}$ unramified in $F$ and not dividing $M$, let $\mathfrak{p}$ be a prime in $\mathcal{O}$ above $p$ and let $\overline{\mathcal{O}} := \mathcal{O}/\mathfrak{p}$. Assume furthermore that $X = \text{Spec} \mathcal{B}$ is a smooth connected scheme over $\mathcal{O}$ with geometrically irreducible fibers. Also we denote by an upper bar tensorization over $\mathcal{O}$ with $\overline{\mathcal{O}}$. 

Lemma 4.6. Assume $\bar{b} \in \bar{B}^\times$ and
\[ A_i \equiv B_i \mod \mathfrak{P} \]
for all $i$. Then the map $\bar{\pi} : \bar{Y} \to \bar{X}$ has a section for which the induced map between the corresponding rings pulls back $y_i$ into 1.

In particular the map $\bar{\pi} : \bar{Y} \to \bar{X}$ is surjective, hence an étale cover.

Proof. The map
\[ \mathcal{B}[y] \to \bar{\mathcal{B}}, \quad y_i \mapsto 1 \]
sends
\[ y_i^t A_i y_i - B_i \mapsto 0 \]
\[ (A_i(y_i - 1))_{ij} - (A_j(y_j - 1))_{ki} \mapsto 0 \]
\[ b \mapsto \bar{b} \in \bar{B}^\times \]
\[ D(y) \mapsto (D(1, \ldots, 1) \mod \mathfrak{P}) \in F_p^\times \]
by Lemma 4.4.
So we have an induced map $\mathcal{C} \to \bar{\mathcal{B}}$; the latter induces a section of the projection $\bar{\pi} : \bar{Y} \to \bar{X}$ and we are done. $\square$

Denote now by $\bar{\sigma} : \bar{X} \to \bar{Y}$ the section of $\bar{\pi} : \bar{Y} \to \bar{X}$ constructed in Lemma 4.6. The image of the section $\bar{\sigma}$,
\[ \bar{Y}^1 := \bar{\sigma}(\bar{X}), \]
is a closed subscheme of $\bar{Y}$, so by Lemma 4.3 and dimension considerations it is an irreducible component of $\bar{Y}$; since, again by Lemma 4.3, $\bar{Y}$ is smooth, it follows that $\bar{Y}^1$ is a connected component of $\bar{Y}$. Let
\[ \bar{Y}^2 := \bar{Y} \setminus \bar{Y}^1 \]
and let $\bar{e} \in \mathcal{O}(\bar{Y})$ be the idempotent which is 1 on $\bar{Y}^1$ and 0 on $\bar{Y}^2$. Finally let $e \in \mathcal{O}(Y) = \mathcal{C}$ be any lift of $\bar{e}$, consider the scheme
\[ Y^0 := \text{Spec} \mathcal{C}[1/e], \]
and the open immersion $Y^0 \subset Y$. Also define the formal scheme
\[ \mathcal{Y} := (Y^0)^{\hat{\mathfrak{P}}}. \]
Since
\[ \bar{Y} = \bar{Y}^1 \]
it follows that $\mathcal{Y}$ is a connected component of $Y^{\hat{\mathfrak{P}}}$ where $^{\hat{\mathfrak{P}}}$ means, as usual, $\mathfrak{P}$-adic completion. Clearly

Corollary 4.7. The induced map
\[ \bar{\pi} : \bar{Y}^1 \to \bar{X} \]
is an isomorphism and its inverse pulls back $y_i$ into 1.
Corollary 4.8. The map of formal schemes 
\[ \pi^\hat{P} : \mathcal{Y} = (Y^0)^\hat{P} \to X^\hat{P} \]

is an isomorphism.

Proof. By Corollary 4.7 the map \( \mathcal{B} \to \mathcal{C}[1/e] \) induces an isomorphism after tensorization with \( \mathcal{O} \). Hence the map \( \mathcal{B}^\hat{P} \to \mathcal{C}[1/e]^\hat{P} \) is an isomorphism because \( p \) is a non-zero divisor in \( \mathcal{C}[1/e] \); the latter fact follows from the fact that \( \mathcal{C} \) is étale, hence flat, over \( \mathcal{B} \); cf. Lemma 4.5.

\[ \square \]

4.3. The case \( X = GL_n \).

The aim of this subsection is to prove Theorem 3.9.

We continue to assume \( F \) is a number field, \( 0 \neq M \in \mathbb{Z} \) is an even integer, and \( \mathcal{O} = \mathcal{O}_{F,M} = \mathcal{O}[1/M] \). Let \( p \) be an odd prime in \( \mathbb{Z} \) unramified in \( F \) and not dividing \( M \), let \( \mathcal{P} \) be a prime in \( \mathcal{O} \) above \( p \) and let \( \mathcal{O} := \mathcal{O}/\mathcal{P} \). Furthermore let \( \phi = \phi_{\mathcal{P}} : \mathcal{O} \to \mathcal{O} \) be the Frobenius element attached to \( \mathcal{P} \), let \( q_1, \ldots, q_n \in GL_n(\mathcal{O}) \) be symmetric matrices, and set

(4.50) \[ \mathcal{B} = \mathcal{O}[x, \det(x)^{-1}], \quad A_i = x^{(p)}(\phi(q_i)x^{(p)}), \quad B_i = (x^t q_i x)^{(p)}. \]

With these data the ring \( \mathcal{C} = \mathcal{C}(\mathcal{B}, A, B) \) in (4.41) becomes

(4.51) \[ \mathcal{C} = \frac{\mathcal{O}[x, \det(x)^{-1}, b, y, D(y)^{-1}]}{(y_i^t A_i y_i - B_i)_{ij}, (A_i(y_i - 1))_{kj} - (A_j(y_j - 1))_{ki}} \]

and the map of schemes \( \pi \) in (4.42) becomes the map

(4.52) \[ \pi : Y := \text{Spec} \mathcal{C} \to X := G := GL_n = \text{Spec} \mathcal{B} \]

induced by \( \mathcal{B} \to \mathcal{C}, x \mapsto x \). By Lemma 4.4 the map \( \pi \) is étale.

Consider now the maps

(4.53) \[ \varphi_i : Y \to G \]

induced by the ring homomorphisms \( \varphi_i : \mathcal{O}(G) \to \mathcal{O}(Y) \) satisfying

(4.54) \[ \varphi_i(a) = \phi(a), \quad a \in \mathcal{O}, \]

and sending

(4.55) \[ x \mapsto \varphi_{pi}(x) := \text{class}(x^{(p)}y_i) \in \mathcal{O}(Y). \]

One can then consider the induced map between \( \mathfrak{P} \)-adic completions

\[ \varphi_i^\mathfrak{P} : Y^\mathfrak{P} \to G^\mathfrak{P} \]

and the restriction of the latter,

\[ \varphi_i^\mathfrak{P} : Y \to G^\mathfrak{P} \]

where \( Y \) is as in (4.38). Then we have:

Lemma 4.9. For each \( i = 1, \ldots, n \) the map \( \varphi_i^\mathfrak{P} : Y \to G^\mathfrak{P} \) equals the composition

\[ \phi_i^G \circ \pi^\mathfrak{P} : Y \to G^\mathfrak{P} \to G^\mathfrak{P}. \]
Proof. Let $\Lambda_i \in \mathcal{O}(G\hat{\mathcal{P}})$ be the pull back of class $(y_i) \in \mathcal{O}(\mathcal{Y})$ via $(\pi\hat{\mathcal{P}})^{-1}$. Then clearly we have

1) $\Lambda_i \equiv 1 \mod p$ (by Corollary 4.7)
2) $A_i A_i = B_i$;
3) $(A_i(A_i - 1))_{k} = (A_j(A_j - 1))_{k}.

But by the proof of Theorem 2.11 there is a unique tuple $\Lambda_i$ with properties 1, 2, 3 and the Frobenius lifts $\phi_i^{G\hat{\mathcal{P}}}$ corresponding to the local Levi-Civita connection attached to $(q_1, ..., q_n)$ send $x$ into $x^{(p)}\Lambda_i$. This ends our proof. □

Proof of Theorem 3.9. With notation as above we let $Y_{p/G}$ be the connected component of $Y$ containing $\mathcal{Y}$ (which is a regular scheme hence irreducible). Also we let $\pi_{p/G} : Y_{p/G} \to G$ and $\varphi_{p} : Y_{p/G} \to G$ be the restrictions of $\pi : Y \to G$ and $\varphi : Y \to G$ respectively and we let $\mathcal{Y}_{p/G} = \mathcal{Y}$. Then all assertions of Theorem 3.9 are satisfied. □

Remark 4.10.

1) It is clear that the conjunction of Lemma 4.5 and Corollaries 4.8 and 4.9 implies our Theorem 3.9. It is also trivial to see that our arguments in the proof of Theorem 3.9 can also be used to prove the existence part of our Theorem 2.8; however the proof that we already gave for the existence part of Theorem 2.8 has the advantage of also immediately yielding our proof of Proposition 2.14.

2) The construction of $Y_{p/G}$ and of the maps $\pi_{p}, \varphi_{pi}$ in the proof of Theorem 3.9 was entirely canonical/functorial. So our construction of mixed curvature in Definition 3.11 is canonical.

4.4. The case $X = GL_{c}^{2}$.

The aim of this subsection is to prove Theorem 3.13 and Proposition 3.15.

We consider the situation in the previous section with $n = 2$. In addition, we let $d_1, d_2 \in \mathcal{O}^{\times}$, we let $\alpha, \beta$ be 2 indeterminates, and set

$$a_i = (\alpha^{2p} + \beta^{2p}) \cdot \phi(d_i), \quad b_i = (\alpha^2 + \beta^2)p \cdot d_i^p,$$

$$B' := \mathcal{O}[\alpha, \beta, (\alpha^2 + \beta^2)^{-1}], \quad A'_i = a_i \cdot 1_2, \quad B'_i := b_i \cdot 1_2, \quad b' = b'_1 b'_2.$$

As usual, we set

$$G' = GL_c^{2} = \text{Spec } B'$$

viewed as embedded into

$$G = GL_{2} = \text{Spec } B.$$

With

$$A' = (A'_1, A'_2), \quad B' = (B'_1, B'_2)$$

we consider the ring

$$C' := C(B', A', B')$$

associated to the data $(B', A', B')$ as in 4.41. Note that $b_i$ are invertible in $C'$ hence so are $b'$ and $a_i$ hence, setting

$$\theta_i = \frac{b_i}{a_i} \in B'_i a_i a_2,$$

we have

$$C' = \frac{B'_1 a_2 [y, D(y)^{-1}]}{(a_i(y_i - 1)_{j}, (a_i(y_i - 1)_{j}))_{k} - a_j(y_j - 1)_{k}}.$$
By Lemmas 4.5 and 4.6 we have that the map
\[ Z' := Spec \mathcal{C}' \to G' = Spec B' \]
is étale and its reduction mod \( p \) has a section defined by the map
\[ \mathcal{C}' \to \overline{B'}, \; y_i \mapsto 1_2. \]
Now let
\[ t_i := \text{class}(\text{tr}(y_i)) := \text{class}(y_{i11} + y_{i22}) \in \mathcal{C}' \]
and
\[ \tau_i := \text{class}(\text{det}(y_i) + \theta_i) \in \mathcal{C}' \]
Then (4.56) sends \( t_i \) and \( \tau_i \) into \( \mathfrak{g} \in (\mathcal{B}')^\times \).
Setting \( s = t_1 t_2 \tau_1 \tau_2 \) we get an induced map
\[ \overline{C}'_s \to \overline{B}'. \]
Let
\[ Y' := Spec \mathcal{C}'_s \]
and denote by
\[ \pi' : Y' \to G' \]
the induced morphism which is, of course, still étale. We get a section
\[ \sigma : \overline{G}' \to \overline{Y}' \]
of the projection
\[ \overline{\pi'} : \overline{Y}' \to \overline{G}'. \]
Exactly as in the case of \( GL_n \), denoting by \( \mathcal{Y}' \) the connected component of \( (Y')^\hat{\mathcal{U}} \)
containing \( \overline{\sigma(\mathcal{G}')} \) we get an isomorphism
\[ \mathcal{Y}' \to (G')^\hat{\mathcal{U}}. \]
We will next construct for \( i = 1, 2 \) morphisms
\[ \varphi'_i = \varphi'_i : Y' \to G' \]
as follows. We already have at our disposal the morphisms \( \varphi_i : Y \to G \) in 4.5.5.
We want to construct \( \varphi'_i \) so as to be induced by \( \varphi_i \). Note that the canonical map
\( \mathcal{B} \to B' \) sends
\[ A_i \mapsto A'_i, \; B_i \mapsto B'_i \]
so it induces a canonical map
\[ \text{can} : \mathcal{C} \to \mathcal{C}' \to \mathcal{C}'_s. \]
On the other hand we have the following:

**Lemma 4.11.** Let \( J \) be the ideal in \( \mathcal{O}(G) \) defining \( G' \); so \( J \) is generated by
\[ x_{11} - x_{22}, \; x_{12} + x_{21}. \]
Then \( J \) is sent into 0 by the map
\[ \mathcal{B} \xrightarrow{\varphi_i} \mathcal{C} \xrightarrow{\text{can}} \mathcal{C}'_s. \]
In particular the maps \( \varphi_i : Y \to G \) induce maps \( \varphi'_i : Y' \to G' \).
Proof. Recall from the proof of Theorem 3.9 that \( \varphi_i(x) \) was defined as the class of \( x^{(p)} \cdot y_i \) in \( C \). So in order to conclude we need to show that the image \( y'_i \) of \( y_i \) in \( GL_2(C'_s) \) belongs to \( GL_1(C'_s) \). Pick an \( i = 1, 2 \) and write
\[
y'_i = \begin{pmatrix} u & v \\ w & z \end{pmatrix}, \quad u, v, w, z \in C'_s.
\]
The equality
\[
(y'_i)^t y'_i = \theta_i
\]
gives
\[
u^2 + w^2 = \theta_i, \quad uv + wz = 0, \quad v^2 + z^2 = \theta_i.
\]
A formal manipulation of the first 2 equations in (4.58) gives
\[
w(wv - uz) = \theta_i v.
\]
On the other hand taking the determinant in (4.57) we get
\[
(uz - uv)^2 = \theta_i^2
\]
so
\[
(\det(y'_i) + \theta_i)(\det(y'_i) - \theta_i) = 0.
\]
Since \( \det(y'_i) + \theta_i \) is invertible in \( C'_s \) we get
\[
uz - uv = \det(y'_i) = \theta_i.
\]
Combining with (4.59) we get
\[
v = -w.
\]
Subtracting the first and third equations in (4.58) we get
\[
(u + z)(u - z) = 0.
\]
But now \( u + z = \text{tr}(y'_i) \) is invertible in \( C'_s \). So we get
\[
u = z
\]
which ends the proof of the fact that \( y'_i \) belongs to \( GL_1(C'_s) \).
\( \square \)

Proof of Theorem 3.13. With the notation above we let \( Y''_{p/G'} = Y' \) and we let \( Y'_{p/G'} \) be the irreducible component of \( Y' \) that contains \( Y' \). Furthermore we let \( \pi'_{p/G'}, \varphi'_{p/G'} : Y'_{p/G'} \rightarrow G' \) be the restrictions of \( \pi', \varphi_i : Y' \rightarrow G' \). Then, clearly, all the assertions of Theorem 3.13 follow.
\( \square \)

Proof of Proposition 3.15. By our construction and the formula (4.39) the tensor product \( C'_s \otimes_B' E' \) (with \( C' \) over \( B' \) viewed via \( \pi' \)) is isomorphic to
\[
M' := \frac{E'[y, g(y)^{-1}]}{(y'_i y_j - \theta_p)_{ij}, \ y_{112} - y_{211} + 1, \ y_{122} - y_{221} - 1}
\]
where \( y = (y_1, y_2) \) and
\[
g(y) := \det(y_1) \cdot \det(y_2) \cdot \text{tr}(y_1) \cdot \text{tr}(y_2) \cdot (\det(y_1) + \theta_p) \cdot (\det(y_2) + \theta_p).
\]
Set
\[ L' := L'_p := \frac{E'[z]}{(2z^2 + 2z + 1 - \theta_p)} \]
where \( z \) is a variable. The discriminant of \( 2z^2 + 2z + 1 - \theta_p \) is \( 2\theta_p - 1 \) which is not a square in \( E' \) because \( \alpha^{2p} + \beta^{2p} \) is a product of distinct linear factors. So \( L' \) is a quadratic field extension of \( E' \).

We will construct in what follows a natural isomorphism \( L' \simeq M' \).

Let \( v = v_p \in L' \) be the class of \( z \) and let \( u = 1 + v \). Then the homomorphism

\[
E'[y] \to L', \ y_1 \mapsto \begin{pmatrix} u & v \\ -v & u \end{pmatrix}, \ y_2 \mapsto \begin{pmatrix} u & -v \\ v & u \end{pmatrix}
\]

is trivially seen to factor through a homomorphism \( M' \to L' \). We also claim that the homomorphism

\[
E'[z] \to M', \ z \mapsto y_{112} := \text{class}(y_{112})
\]

factors through a homomorphism \( L' \to M' \). This can be seen as follows. By an argument similar to the one in the proof of Lemma 4.11 the classes \( y_i' \) of \( y_i \) in \( M' \) have the form

\[
y_i' = \begin{pmatrix} u_i & v_i \\ -v_i & u_i \end{pmatrix}
\]

with

\[
u_2 = 1 + v_1, \quad u_1 = 1 - v_2.
\]

From the equations

\[
u_1^2 + v_1^2 = \theta_p, \quad u_2^2 + v_2^2 = \theta_p
\]

we get

\[
1 - 2v_2 + v_2^2 + v_1^2 = \theta_p, \quad 1 + 2v_1 + v_1^2 + v_2^2 = \theta_p.
\]

Subtracting the last 2 equations we get \( v_1 = -v_2 \) hence \( u_1 = u_2 \). So \( v_1 = y_{112}' \) is a root of \( 2z^2 + 2z + 1 - \theta_p \) and our claim is proved. Finally, using the above considerations it is trivial to check that the two morphisms (4.61) and (4.62) are inverse to each other. This ends the construction of the isomorphism \( L' \simeq M' \). Since \( M' \) is a field we get, in particular, that \( Y' := \text{Spec} \ C' \) itself is irreducible, so \( Y'_{/G'} = Y' \).

The Proposition now follows easily by using formula (4.55).

\[\square\]

**Remark 4.12.** Assume the hypotheses of Proposition 3.15. Then the proof of our Proposition plus the discussion in Remark 3.12, 5), yield the following. For any \( d \in \mathcal{O}_{F,M} \) and \( \psi_1, \psi_2 \in \mathcal{G}(F) \) let us denote by

\[
c_{pr}'(d, \psi_1, \psi_2) \in C(E')
\]

the class of the correspondence

\[
a' \begin{array}{c}
\pi' \\
\psi' \\
\Spec E'
\end{array} \Spec M' \begin{array}{c}
\Spec E'
\end{array}
\]

where \( M' \) is given by the formula (4.60) with

\[
\theta_p = \frac{d^p(\alpha^2 + \beta^2)p}{\psi_1(d)(\alpha^{2p} + \beta^{2p})}.
\]
\( \pi' \) is given by the inclusion \( E' \subset M' \), \( \varphi'_i \) is \( \psi_2 \) on \( F \) and \( \varphi'_i(x) = x^{(p)}y_i \). Then for any \( \sigma \in \mathcal{G}(F) \) we have the following formulae in \( C(E') \):

\[
\sigma \cdot c'_{pi}(d, \psi_1, \psi_2) = c'_{pi}(d, \psi_1, \psi_2 \circ \sigma)
\]

\[(4.65)\]

\[
c'_{pi}(d, \psi_1, \psi_2) \cdot \sigma = c'_{pi}(\sigma(d), \sigma \circ \psi_1 \circ \sigma^{-1}, \sigma \circ \psi_2)
\]

\[
\sigma^{-1} \cdot c'_{pi}(d, \psi_1, \psi_2) \cdot \sigma = c'_{pi}(\sigma(d), \sigma \circ \psi_1 \circ \sigma^{-1}, \sigma \circ \psi_2 \circ \sigma^{-1}).
\]

On the other hand, by the proof of Proposition 3.15, our classes \( c'_{pi} \) in 3.38 identify with the classes \( c'_{pi}(d, \phi, \phi) \) in 4.63. So the formulae 4.65 describe the action of \( \mathcal{G}(F) \) on our classes 3.38.

5. Appendix: Classical Levi-Civit\(\mathring{a} \) connection revisited

The aim of this Appendix is to quickly revisit the classical theory of the Levi-Civit\(\mathring{a} \) connection [10] with an emphasis on the analogy with the arithmetic case. We are only interested in the algebraic aspects of the classical theory so we place ourselves in the context of differential algebra [11] by considering a ring \( A \) equipped with commuting derivations \( \delta^A_1, \ldots, \delta^A_n \). (Recall that a derivation is an additive map that satisfies the usual Leibniz rule.) For convenience we assume \( A \) contains \( \mathbb{Q} \). (The example we have in mind is, of course, the ring \( A \) of smooth functions on \( \mathbb{R}^n \) equipped with the partial derivations with respect to the coordinates.) Following the Introduction to [4] we consider an \( n \times n \) matrix of indeterminates \( x = (x_{ij}) \) and the ring

\[
B = A[x, \det(x)^{-1}],
\]

By a \textit{connection} (on \( GL_n := \text{Spec } B \)) we mean an \( n \)-tuple

\[
(\delta^B_1, \ldots, \delta^B_n)
\]

of derivations on \( B \) extending the corresponding derivations on \( A \). The \textit{curvature} of the connection is the family \( (\varphi_{ij}) \) of commutators

\[
\varphi_{ij} := [\delta^B_i, \delta^B_j] = \delta^B_i \delta^B_j - \delta^B_j \delta^B_i : B \rightarrow B.
\]

We say that the connection is \textit{linear} if

\[
\delta^B_i x = A_i x
\]

for some \( n \times n \) matrices

\[
A_i = (A_{ijk})
\]

with coefficients in \( A \). For a linear connection the curvature satisfies

\[
\varphi_{ij}(x) = F_{ij} x
\]

where \( F_{ij} \) is the matrix given by the classical formula

\[
F_{ij} := \delta_i A_j - \delta_j A_i - [A_i, A_j];
\]

we still refer to \( (F_{ij}) \) as the curvature of the connection. There is one distinguished connection \( (\delta^B_{01}, \ldots, \delta^B_{0n}) \) called \textit{trivial}, defined by

\[
(5.3) \quad \delta^B_{0i} x = 0.
\]

By a \textit{metric} we understand a symmetric matrix

\[
q = (q_{ij}) \in GL_n(A), \quad q^t = q.
\]
One defines the Christoffel symbols of the second kind by the formulae
\[ \Gamma^k_{ij} = -A_{ikj} \]
and the Christoffel symbols of the first kind by
\[ \Gamma_{ijk} := \Gamma^l_{ij} q_{lk}, \]
where the repeated index \( l \) is summed over. Passing from the \( A_{ijk} \)'s to the \( \Gamma^k_{ij} \)'s (and later passing from the entries of the curvature matrices \( F_{ij} \) to the components of the covariant Riemann tensor \( R_{ijkl} \)) is accounted for by our starting with a connection that is dual to the classical Levi-Civit`a connection; we adopted this approach simply in order to match the conventions in \([5]\).

Consider the unique \( A \)-algebra homomorphism
\[ \mathcal{H}_q : B \to B \]
such that
\[ \mathcal{H}_q(x) = x^t q x. \]
Say that a connection is metric with respect to \( q \) if the following diagrams are commutative:
\[ \begin{array}{ccc}
B & \xrightarrow{\delta^0} & B \\
\mathcal{H}_q \downarrow & & \downarrow \mathcal{H}_q \\
B & \xrightarrow{\delta^0} & B 
\end{array} \]
It is trivial to check that a linear connection is metric with respect to \( q \), in the sense of the above (somewhat non-standard) definition, if and only if the following classical equalities hold:
\[ \delta_i q_{jk} = \Gamma_{ijk} + \Gamma_{ikj}. \]
Note, by the way, that \([5.7]\) implies the following formula
\[ \text{tr}(A_i) = -\frac{1}{2} \text{tr}(q^{-1} \delta_i q). \]
Say that a connection is torsion free if
\[ \Gamma_{ijk} = \Gamma_{jik}. \]
The “Fundamental Theorem of Riemannian Geometry” is the following statement that can be checked by easy algebraic manipulations:

**Theorem 5.1.** Let \( q \) be a metric. Then there is a unique linear connection which is metric with respect to \( q \) and torsion free. It is given by the following formulae:
\[ \Gamma_{ijk} = \frac{1}{2} (\delta_i q_{jk} + \delta_j q_{ki} - \delta_k q_{ij}). \]

For \( (F_{ij}) \) the curvature of the Levi-Civit`a connection attached to a metric \( q = (q_{ij}) \) we set:
\[ F_{ij} = (F_{ijkl}) \text{, } R^k_{ij} := -F_{ijkl}, \text{ } R_{ijkl} := q_{lm} R^m_{jkl}, \]
where the repeated index \( m \) is summed over. One refers to \( R_{ijkl} \) as the covariant Riemann tensor; then one shows by easy algebraic manipulations that:
Proposition 5.2. The covariant Riemann tensor has the following symmetries:
\[
R_{ijkl} = -R_{ijlk},
\]
\[
R_{ijkl} = -R_{jikl},
\]
\[
R_{lijk} + R_{ljki} + R_{lkij} = 0,
\]
\[
R_{ijkl} = R_{klij}.
\]

In particular if one defines the \textit{Ricci tensor} by the formula
\[
R_{ik} := R_{ijk}^l = q^{jl}R_{jilk},
\]
where the repeated indeces \(j,l\) are summed over, then one gets the following formal consequence of 5.11 and 5.12:
\[
R_{jk} = R_{kj}.
\]

We end by recording a version of a classical formula that appears when one considers normal coordinates; its proof is, again, a trivial algebraic manipulation.

Proposition 5.3. Assume that \(J\) is an ideal in \(A\) and we are given a metric \(q = q^t\) such that \(q \equiv 1 \mod J^2\) where 1 is, as usual, the identity matrix. Then the covariant Riemann tensor satisfies the following congruences:
\[
R_{ijkl} \equiv \frac{1}{2}(\delta_j \delta_k q_{il} + \delta_i \delta_l q_{jk} - \delta_i \delta_k q_{jl} - \delta_j \delta_l q_{ik}) \mod J.
\]

As explained in previous sections a number of formulae in the classical setting, especially 5.8, 5.10, 5.12, 5.15 have corresponding arithmetic analogues.

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