WEIGHTED HARDY-LITTLEWOOD AVERAGE OPERATORS
ON BILATERAL GRAND LEBESGUE SPACES

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Abstract.

We obtain in this short article the non-asymptotic exact estimations for the norm of (generalized) weighted Hardy-Littlewood average integral operator in the so-called Bilateral Grand Lebesgue Spaces. We also give examples to show the sharpness of these inequalities.

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1. Introduction. Statement of problem.

"In 1984, Carton-Lebrun, Fosset [26] defined the so-called weighted Hardy-Littlewood average operator $U\phi$. Let $\phi : [0, 1] \to [0, \infty)$ be a non-negative non-zero measurable function. Let $f$ be a measurable real (or complex) valued function on $\mathbb{R}^d$, $d = 1, 2, \ldots$; one then defines the weighted Hardy-Littlewood average $U\phi[f](\cdot)$ as follows:

$$
U\phi[f](x) = \int_0^1 f(tx) \phi(t) \, dt, \quad x \in \mathbb{R}^d.
$$

(1.1)

In this view, we think it may be suitable to call the operator $U\phi$ the generalized Hardy operator.

In [27] J.Xiao obtained that the generalized Hardy operator $U\phi$ is bounded on $L_p(\mathbb{R}^d), 1 \leq p \leq \infty$, if and only if

$$
\theta(p) = \theta_d(p) \overset{def}{=} \int_0^1 t^{-d/p} \phi(t) \, dt < \infty.
$$

(1.2)

Moreover,

$$
||U\phi||\{L(p) \to L(p)\} = \theta(p).
$$

(1.3)

It was investigated also the boundedness of this operator in the BMO spaces etc." [28]. Here as usually
The results seems to be of interest as it is related closely to the Hardy integral inequality. For example, if $\phi = 1$ and $d = 1$, then $U_\phi$ coincides to the classical Hardy operator.

**Our aim is to extend the results of J.Xiao about boundedness of Hardy-Littlewood operator on the so-called (bilateral) Grand Lebesgue Spaces (GLS).**

We intend to obtain the exact value of correspondent multiplicative constants.

The norm estimates for different operators acting in GLS see, e.g. in the works [10], [6], [13], [14], [15], [19], [20], [22], [23].

Some application in the theory of non-linear PDE are described in [21], [24].

We use symbols $C(X,Y)$, $C(p,q; \psi)$, etc., to denote positive constants along with parameters they depend on, or at least dependence on which is essential in our study. To distinguish between two different constants depending on the same parameters we will additionally enumerate them, like $C_1(X,Y)$ and $C_2(X,Y)$. The relation $g(\cdot) \asymp h(\cdot)$, $p \in (A, B)$, where $g = g(p)$, $h = h(p)$, $g, h : (A, B) \to \mathbb{R}_+$, denotes as usually

$$0 < \inf_{p \in (A,B)} h(p)/g(p) \leq \sup_{p \in (A,B)} h(p)/g(p) < \infty.$$  

The symbol $\sim$ will denote usual equivalence in the limit sense.

We will denote as ordinary the indicator function

$$I(x \in A) = 1, x \in A, \ I(x \notin A) = 0, x \notin A;$$

here $A$ is a measurable set.

All the passing to the limit in this article may be grounded by means of Lebesgue dominated convergence theorem.

### 2. Grand Lebesgue Spaces (GLS).

We recall first of all here for reader conventions some definitions and facts from the theory of GLS spaces.

Recently, see [3], [4], [5], [6], [7], [8], [9], [10], [12], [13] etc. appear the so-called Grand Lebesgue Spaces GLS

$$G(\psi) = G = G(\psi; A; B); \ A; B = \text{const}; \ A \geq 1, \ B \leq \infty$$

spaces consisting on all the measurable functions $f : X \to R$ with finite norms

$$||f||G(\psi) \overset{\text{def}}{=} \sup_{p \in (A,B)} \left[\frac{|f|^p}{\psi(p)}\right].$$  

(2.1)

Here $\psi = \psi(p)$, $p \in (A, B)$ is some continuous positive on the open interval $(A; B)$ function such that

$$\inf_{p \in (A;B)} \psi(p) > 0.$$  

(2.2)
We will denote
\[ \text{supp}(\psi) \overset{\text{def}}{=} (A; B). \]
The set of all such a functions with support \( \text{supp}(\psi) = (A; B) \) will be denoted by \( \Psi(A; B) \).

This spaces are rearrangement invariant; and are used, for example, in the theory of Probability, theory of Partial Differential Equations, Functional Analysis, theory of Fourier series, Martingales, Mathematical Statistics, theory of Approximation etc.

Notice that the classical Lebesgue-Riesz spaces \( L^p \) are extremal particular case of Grand Lebesgue Spaces, see [13], [14].

3. Main result: upper and lower estimations for Hardy - Littlewood average operator

Let the function \( \phi = \phi(t) \) with it the function \( \theta = \theta(p) \) be a given. Let also \( \psi = \psi(p) \) be any function described below. Assume that

\[ \text{supp} \psi \cap \text{supp} \theta =: (A, B) \neq \emptyset, \]
and denote
\[ \psi_\theta(p) = \psi(p) \cdot \theta(p), \; p \in (A, B). \]

**Theorem 3.1.**

\[ ||U_\phi[f]||_{G\psi \theta} \leq 1 \cdot ||f||_{G\psi}, \]
where the constant "1" is the best possible.

**Proof.** The upper estimate may be proved very simple. Indeed, let \( f \in G\psi, \; f \neq 0 \). Then we have by definition of the \( G\psi \) norm \( ||f||_{G\psi} \leq ||f||_{G\psi} \cdot \psi(p) \). We deduce using Xiao inequality:

\[ ||U_\phi[f]||_{G\psi} \leq ||f||_{G\psi} \cdot \theta(p) \leq ||f||_{G\psi} \cdot \psi(p) \theta(p) = ||f||_{G\psi} \cdot \psi_\theta(p), \]
which implies (3.2).

**Lower bound.** Let the value \( \epsilon \) and the function \( \phi \) be a fix. Denote \( \Delta = \frac{1}{\epsilon}, \)

\[ \theta_\epsilon(p) = \int_\epsilon^1 t^{d/p-\epsilon} \phi(t) \; dt, \; \epsilon \in (0, 1/2), \]
\[ K = \sup_{\psi} \sup_{f \in G\psi, f \neq 0} \left[ \frac{||U_\phi[f]||_{G\psi \theta}}{||f||_{G\psi}} \right]; \]
it remains to prove \( K \geq 1. \)

Notice that

\[ K = \sup_{\psi} \sup_{f \in G\psi, f \neq 0} \frac{\left[ \sup_p ||U_\phi[f]|_{\psi(p)} / (\theta(p)\psi(p)) \right]}{\sup_p [||f||_{p}/\psi(p)]}, \]

We conclude choosing \( \psi(p) = ||f||_{p} \)
\[ K \geq \sup_{f \in G, f \neq 0} \left\{ \frac{|U_\phi[f]|_p}{\theta(p) |f|_p} \right\}. \tag{3.5} \]

We have after some calculations using at the same example as in [27]

\[ f_\epsilon(x) = I(|x| \geq 1) \cdot |x|^{-d/p-\epsilon}, \quad |x| = \sqrt{x_1^2 + \cdots + x_d^2}; \tag{3.6} \]

\[ |f_\epsilon|^p = \frac{c(d)}{p\epsilon}, \quad c(d) = \frac{d\pi^{d/2}}{\Gamma(1 + d/2)}; \tag{3.7} \]

\[ U_\phi[f_\epsilon] = I(|x| > 1) \cdot |x|^{-d/p-\epsilon} \cdot \int_{1/|x|}^{1} t^{-d/p-\epsilon} \phi(t) \, dt; \]

\[ |U_\phi[f_\epsilon]|_p^p = \int_{|x| \geq 1} \left( |x|^{-d/p-\epsilon} \cdot \int_{1/|x|}^{1} t^{-d/p-\epsilon} \phi(t) \, dt \right)^p \, dx \geq \]

\[ |f_\epsilon|^p \cdot \left( \Delta^{-\epsilon} \int_{1/\Delta}^{1} t^{-d/p} \phi(t) \, dt \right)^p; \tag{3.8} \]

\[ |U_\phi[f_\epsilon]|_p \geq |f_\epsilon|_p \cdot \theta_\epsilon(p); \tag{3.9} \]

\[ K \geq \sup_{\epsilon \in (0,1/2)} \sup_p \left[ \frac{\theta_\epsilon(p)}{\theta(p)} \frac{|f_\epsilon|_p}{|f|_p} \right] \geq \sup_p \lim_{\epsilon \to 0^+} \frac{\theta_\epsilon(p)}{\theta(p)} = 1. \tag{3.10} \]

\[ \square \]

4. Multidimensional case

We recall here the definition of the so-called anisotropic Lebesgue (Lebesgue-Riesz) spaces, which appear in the famous article belonging to Benedek A. and Panzone R. [1]. More detail information about this spaces see in the books of Besov O.V., Ilin V.P., Nikolskii S.M. [2], chapter 16,17; Leoni G. [11], chapter 11.

Let \((X_j, A_j, \mu_j), \quad j = 1, 2, \ldots, d\) be measurable spaces with sigma-finite non-trivial measures \(j\). It is clear that in this article \(X_j = \mathbb{R}^{d_j}\) and \(\mu_j\) is ordinary Lebesgue measure.

Let \(p = \vec{p} = (p_1, p_2, \ldots, p_d)\) be \(d\) dimensional vector such that \(1 \leq p_j \leq \infty\). Recall that the anisotropic Lebesgue space \(L(\vec{p})\) consists on all the total measurable real valued function \(f = f(x_1, x_2, \ldots, x_d) = f(x) = f(\vec{x}), \quad x_j \in X_j\) with finite norm \(|f|_{\vec{p}} \overset{df}{=} \]

\[ \left( \int_{X_d} \mu_d(dx_d) \left( \int_{X_{d-1}} \mu_{d-1}(dx_{d-1}) \cdots \left( \int_{X_1} \mu_1(dx_1) |f(x_1, x_2, \ldots, x_d)|^{p_1}_d \right)^{p_2/p_1} \right)^{p_3/p_2} \cdots \right)^{1/p_d}. \]

Note that in general case \(|f|_{p_1, p_2} \neq |f|_{p_2, p_1}\), but \(|f|_{p_3, p_2} = |f|_{p_2, p_1}\). Observe also that if \(f(x_1, x_2) = g_1(x_1)g_2(x_2)\), (condition of factorization), then \(|f|_{p_1, p_2} = |g_1|_{p_1}|g_2|_{p_2}\), (formula of factorization).

It is obvious that in the multidimensional (anisotropic) case the inequality of Xiao look as follows:
\[
|U_\phi[f]|_{\vec{p}} \leq \prod_{k=1}^{l} \theta_{d_k}(p_k) \cdot |f|_{\vec{p}}. \tag{4.1}
\]

The last estimate is exact, for instance, for factorable function:

\[
f(\vec{x}) = \prod_{k=1}^{k} f_k(\vec{x}_k).
\]

**Anisotropic Grand Lebesgue-Riesz spaces.**

Let \( Q \) be convex (bounded or not) subset of the set \( \otimes_{j=1}^{l}[1, \infty] \). Let \( \psi = \psi(\vec{p}) \) be continuous in an interior \( Q^0 \) of the set \( Q \) strictly positive function such that

\[
\inf_{\vec{p} \in Q^0} \psi(\vec{p}) > 0; \quad \inf_{\vec{p} \notin Q^0} \psi(\vec{p}) = \infty.
\]

We denote the set all of such a functions as \( \Psi(Q) \).

The Anisotropic Grand Lebesgue Spaces \( AGLS = AGLS(\psi) \) space consists on all the measurable functions

\[
f : \otimes_{j=1}^{l} X_j \to R
\]

with finite (mixed) norms

\[
||f||_{AGL \psi} = \sup_{\vec{p} \in Q^0} \left[ \frac{|f|_{\vec{p}}}{\psi(\vec{p})} \right]. \tag{4.2}
\]

An application into the theory of multiple Fourier transform of these spaces see in articles \[1\] and \[14\], where are considered some problems of boundedness of singular operators in (weight) Grand Lebesgue Spaces and in Anisotropic Grand Lebesgue Spaces.

Let \( \psi \in \Psi(Q) \); we denote

\[
\psi_{\theta}(\vec{p}) = \prod_{k=1}^{l} \theta_{d_k}(p_k) \cdot \psi(\vec{p}). \tag{4.3}
\]

We deduce analogously to the theorem 3.1:

**Proposition 4.1.**

\[
||U_\phi[f]||_{G \psi} \leq 1 \cdot ||f||_{G \psi}, \tag{4.4}
\]

where the constant "1" is the best possible.

\[\Box\]
5. Concluding remarks

1. Analogously may be investigated the "conjugate" operator of a view

\[ V_\phi[f](x) = \int_0^1 f(x/t) \ t^{-d} \phi(t) \ dt. \]

J.Xiao in [27] proved that

\[ \|V_\phi\|(L(p) \to L(p)) = \zeta(p) \overset{\text{def}}{=} \int_0^1 t^{-d(1-1/p)} \phi(t) \ dt. \]

Define for any function \( \psi = \psi(p) \) a new function

\[ \psi^{(\zeta)}(p) = \psi(p) \zeta(p). \]

We conclude alike to the assertion of theorem 3.1:

\[ \|V_\phi[f]\|G^{\psi^{(\zeta)}} \leq 1 \cdot ||f||G\psi, \]

with exact value of the coefficient "1".

2. Examples.

A. Let

\[ \phi(t) = t^{\alpha-1} (1 - t)^{\beta-1}, \ \alpha, \beta = \text{const} > 0; \]

then

\[ \theta(p) = B(\alpha - d/p, \beta) = \frac{\Gamma(\alpha - d/p) \Gamma(\beta)}{\Gamma(\alpha + \beta - d/p)}, \ p > d/\alpha \]

and \( \theta(p) = \infty \) otherwise. Here as usually \( B(\cdot, \cdot), \Gamma(\cdot) \) denote correspondingly Beta and Gamma functions.

Note that as \( p \to d/\alpha + 0, \ p > d/\alpha \Rightarrow \theta(p) \sim p/(\alpha p - d). \)

B. Let now

\[ \phi(t) = |\log t|^\gamma L(|\log t|), \ \gamma = \text{const} > -1, \]

\( L = L(z), \ z \in (0, \infty) \) is positive continuous slowly varying as \( z \to \infty \) function. We have as \( p \to d + 0, \ p > d \)

\[ \theta(p) = \int_0^1 t^{-d/p} |\log t|^\gamma L(|\log t|) \ dt = \int_0^\infty e^{-y(1-d/p)} y^\gamma L(y)dy = \]

\[ \left[ \frac{p}{p - d} \right]^{-\gamma+1} \int_0^\infty e^{-z}z^\gamma L \left( \frac{pz}{p - d} \right) dz \sim \]

\[ \Gamma(\gamma + 1) \left[ \frac{p}{p - d} \right]^{-\gamma+1} L \left( \frac{p}{p - d} \right). \]

Evidently, \( \theta(p) = \infty, \ p \leq d. \)
C. Multivariate Hardy transform.

In 1976 Faris [25] first gave a definition of Hardy operator in $d-$ dimensional case. In 1995, Christ and Grafakos [17] gave its equivalent version of $d-$ dimensional Hardy operator as follows

$$H_d(f)(x) = \frac{1}{\Omega(d)|x|^d} \int_{y: |y| \leq |x|} f(y) dy, \quad \Omega(d) = \frac{\pi^{d/2}}{\Gamma(1 + d/2)}, \quad x \neq 0.$$ 

and proved that

$$||H_d||_{(L(p) \to L(p))} = \frac{p}{p - 1}, \quad p > 1,$$

as in the one-dimensional case.

Define the following transform for every $\psi-$ function:

$$\psi_1(p) = \psi(p) \frac{p}{p - 1}, \quad p > 1;$$

then

$$||H_d[f]|G_{\psi_1} \leq 1 \cdot ||f||G_{\psi},$$

where the constant "1" is non-improvable.

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