A probabilistic approach to the $\Phi$-variation of classical fractal functions with critical roughness

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Abstract

We consider Weierstraß and Takagi-van der Waerden functions with critical degree of roughness. In this case, the functions have vanishing $p^{th}$ variation for all $p > 1$ but are also nowhere differentiable and hence not of bounded variation either. We resolve this apparent puzzle by showing that these functions have finite, nonzero, and linear Wiener–Young $\Phi$-variation along the sequence of $b$-adic partitions, where $\Phi(x) = x/\sqrt{-\log x}$. For the Weierstraß functions, our proof is based on the martingale central limit theorem (CLT). For the Takagi–van der Waerden functions, we use the CLT for Markov chains if a certain parameter $b$ is odd, and the standard CLT for $b$ even.

Key words: Weierstraß function, Takagi-van der Waerden functions, Wiener–Young $\Phi$-variation, martingale central limit theorem, Markov chain central limit theorem

1 Introduction and statement of results

We consider a base function $\varphi : \mathbb{R} \to \mathbb{R}$ that is periodic with period 1 and Lipschitz continuous. Our aim is to study the function

$$f(t) := \sum_{m=0}^{\infty} \alpha^m \varphi(b^m t), \quad t \in [0,1],$$

(1.1)

where $b \in \{2,3,\ldots\}$ and $\alpha \in (-1,1)$. Then the series on the right-hand side converges absolutely and uniformly in $t \in [0,1]$, so that $f$ is indeed a well defined continuous function. If $\varphi(t) = \nu \sin(2\pi t) + \rho \cos(2\pi t)$ for real constants $\nu$ and $\rho$, then $f$ is a Weierstraß function. If $\varphi(t) = \min_{z \in \mathbb{Z}} |z - t|$ is the tent map, then $f$ is a Takagi–van der Waerden function. It was shown in [15] that, under some mild conditions on $\varphi$, the function $f$ is of bounded variation for $|\alpha| < 1/b$, whereas for $|\alpha| > 1/b$ and $p := -\log_{|\alpha|} b$ it has nontrivial and linear $p^{th}$ variation along the sequence

$$T_n := \{kb^{-n} : k = 0,\ldots,b^n\}, \quad n \in \mathbb{N},$$

(1.2)

of $b$-adic partitions of $[0,1]$. That is, for all $t \in (0,1)$,

$$\langle f \rangle_t^{(q)} := \lim_{n \to \infty} \frac{1}{|b^n|} \sum_{k=0}^{|b^n|} \left| f((k+1)b^{-n}) - f(kb^{-n}) \right|^q = \begin{cases} 0 & \text{if } q > p, \\
 t \cdot \mathbb{E}[|Z|] & \text{if } q = p, \\
 +\infty & \text{if } q < p. \end{cases}$$

(1.3)
Here, $Z$ is a certain random variable, whose law is known in some special cases. For instance, if $\varphi$ is the tent map and $b$ is even, then the law of $abZ$ is the infinite Bernoulli convolution with parameter $1/(\alpha|b|$ (see also \cite{9,13,14} for earlier results in this special setup). Clearly, the parameter $p = -\log|\alpha|/b$ can be regarded as a measure for the “roughness” of the function $f$. As a matter of fact, it is well known that a typical sample path $t \mapsto B_{H}(t)$ of a fractional Brownian motion has linear $p^{th}$ variation $\langle B_{H}^{(p)} \rangle_{t} = t \cdot E[|B_{H}^{(1)}|^{p}]$ for $p = 1/H$.

**Remark 1.1** (On the connection with pathwise Itô calculus). Our interest in the $p^{th}$ variation of fractal functions is motivated by its connection to pathwise Itô calculus. For instance, if $|\alpha| = 1/\sqrt{b}$, we have $p = 2$ and the limit in (1.3) is just the usual quadratic variation of the function $f$, taken along the partition sequence $\{\mathcal{T}_{n}\}_{n \in \mathbb{N}}$. It was observed by Föllmer \cite{6} that the existence of this limit is sufficient for the validity of Itô’s formula with integrator $f$, and this is the key to a rich theory of pathwise Itô calculus with applications to robust finance; see, e.g., \cite{7} for a discussion. Recently, Cont and Perkowski \cite{3} extended Föllmer’s Itô formula to functions with finite $p^{th}$ variation, which has led to a substantial increase in the interest in corresponding “rough” trajectories with $p > 2$.

In this note, we study the case of critical roughness, $\alpha = -1/b$ or $\alpha = 1/b$, in which $p = 1$. For this case, it was shown in \cite{15} that $\langle f \rangle_{t}^{(q)} = 0$ for all $q > 1$ and $t \in [0,1]$. This, however, does not imply that $f$ is of bounded variation. For instance, if $\varphi$ is the tent map, $b = 2$, and $\alpha = 1/2$, then $f$ is the classical Takagi function, which is nowhere differentiable and hence cannot be of bounded variation; a very short proof of this fact was given by de Rham \cite{4} and later rediscovered by Billingsley \cite{2}. For the Weierstraß function, the proof of nowhere differentiability for all $\alpha \in [1/b, 1)$ is more difficult. Starting from Weierstraß’s original work, it attracted numerous authors until a definite result was given by Hardy \cite{10}.

It is therefore apparent that, in the critical case $|\alpha| = 1/b$, power variation $\langle f \rangle^{(q)}$ is insufficient to capture the exact degree of roughness of the function $f$. To give a precise result on the roughness of the function $f$ in the critical case, we take a strictly increasing function $\Phi : [0,1) \to [0,\infty)$ and investigate the limit

$$\langle f \rangle_{t}^{\Phi} := \lim_{n \uparrow \infty} \sum_{k=0}^{n|b^{\alpha}|} \Phi(|f((k+1)b^{-n}) - f(kb^{-n})|),$$

which can be regarded as the Wiener–Young $\Phi$-variation of $f$ (see, e.g., \cite{11}), restricted to the sequence of $b$-adic partitions (1.2). Our main results will show that the correct choice for $\Phi$ is the function

$$\Phi(x) = \frac{x}{\sqrt{-\log x}} \quad \text{for} \ x \in (0,1) \quad \text{and} \quad \Phi(0) := 0.$$

We fix this function $\Phi$ throughout the remainder of this paper. Our first result establishes the $\Phi$-variation of $f$ from (1.1) for the class of Takagi–van der Waerden functions.

**Theorem 1.2.** Let $\varphi(t) = \min_{z \in \mathbb{Z}} |z - t|$ be the tent map, $b \in \{2,3,\ldots\}$, and $|\alpha| = 1/b$. Then the $\Phi$-variation of the Takagi–van der Waerden function $f$ exists along $\{\mathcal{T}_{n}\}_{n \in \mathbb{N}}$. If $b$ is even, then it is given by

$$\langle f \rangle_{t}^{\Phi} = t \cdot \frac{2}{\pi \log b}, \quad t \in [0,1].$$

If $b$ is odd, then

$$\langle f \rangle_{t}^{\Phi} = t \cdot \frac{2(b + \text{sgn}(\alpha))}{\pi(b - \text{sgn}(\alpha)) \log b}, \quad t \in [0,1].$$

Our results will be consequences of suitable central limit theorems (CLTs). In the preceding theorem, the case of $b$ even will be settled by the standard CLT, whereas the case of $b$ odd will require the use of a CLT for Markov chains. For establishing the $\Phi$-variation of the critical Weierstraß functions, as stated in the following theorem, we rely on the martingale CLT. A loosely related CLT for the classical Takagi function was proved by Gamkrelidze \cite{8}.
Therefore, of the Weierstraß function \( f \) exists along \( \{ T_n \}_{n \in \mathbb{N}} \) and is given by

\[
\langle f \rangle^\Phi_t = t \cdot 2 \sqrt{\frac{\pi (\nu^2 + \mu^2)}{\log b}}, \quad t \in [0, 1].
\]

### 2 Proofs

We first consider only the \( \Phi \)-variation \( \langle f \rangle^\Phi_t \) for \( t = 1 \). The case \( t < 1 \) will be discussed at the end of this section, simultaneously for both theorems. We fix \( b \in \{2, 3, \ldots \} \) and \( \alpha \in \{-1/b, +1/b\} \). Following [15], we let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space supporting an independent sequence \( U_1, U_2, \ldots \) of random variables with a uniform distribution on \( \{0, 1, \ldots, b - 1\} \) and define the stochastic process \( R_m := \sum_{i=1}^m U_i b^{-1} \). Note that \( R_m \) has a uniform distribution on \( \{0, \ldots, b^m - 1\} \). Therefore, for \( n \in \mathbb{N} \) such that all increments \( |f((k + 1)b^{-n}) - f(kb^{-n})| \) are less than 1,

\[
V_n := \sum_{k=0}^{b^n - 1} \Phi(|f((k + 1)b^{-n}) - f(kb^{-n})|) = b^n \mathbb{E} \left[ \Phi\left( |f((R_n + 1)b^{-n}) - f(R_n b^{-n})| \right) \right].
\]

To simplify the expectation on the right, let the \( n \)-th truncation of \( f \) be given by \( f_n(t) = \sum_{m=0}^{n-1} \alpha^m \varphi(b^n t) \). The periodicity of \( \varphi \) implies that

\[
f((R_n + 1)b^{-n}) - f(R_n b^{-n}) = f_n((R_n + 1)b^{-n}) - f_n(R_n b^{-n})
\]

\[
= b^{-n} \text{sgn}(\alpha)^n \sum_{m=1}^n \text{sgn}(\alpha)^m \varphi((R_n + 1)b^{-m}) - \varphi(R_n b^{-m}).
\]

The periodicity of \( \varphi \) implies moreover that for \( m \leq n \),

\[
\varphi(x + R_n b^{-m}) = \varphi\left(x + \sum_{i=1}^n U_i b^{i-1-m}\right) = \varphi\left(x + \sum_{i=1}^m U_i b^{i-1-m}\right) = \varphi(x + R_m b^{-m}).
\]

Therefore,

\[
\text{sgn}(\alpha)^m \varphi((R_n + 1)b^{-m}) - \varphi(R_n b^{-m}) = \text{sgn}(\alpha)^m \varphi((R_m + 1)b^{-m}) - \varphi(R_m b^{-m})
\]

\( =: Y_m \).

It follows that

\[
V_n = b^n \mathbb{E} \left[ \Phi\left(|b^{-n} \sum_{m=1}^n Y_m|\right) \right].
\]

**Lemma 2.1.** Suppose that \( Z_0, Z_1, Z_2, \ldots \) is a sequence of random variables with \( Z_0 = 0 \) and uniformly bounded increments such that the laws of \( \frac{1}{\sqrt{n}} Z_n \) converge weakly to some normal distribution \( N(0, \sigma^2) \) with \( \sigma^2 > 0 \) and that the expression \( \frac{1}{n} \mathbb{E}[Z_n^2] \) is bounded. Then

\[
b^n \mathbb{E} \left[ \Phi\left(|b^{-n} Z_n|\right) \right] \rightarrow \sqrt{\frac{2\sigma^2}{\pi \log b}}.
\]

**Proof.** The fact that \( \frac{1}{n} \mathbb{E}[Z_n^2] \) is bounded implies together with standard arguments that for every nondegenerate interval \( I \subset [0, \infty) \),

\[
\lim_{n \uparrow \infty} \mathbb{E} \left[ \mathbbm{1}_{\{ |\frac{1}{\sqrt{n}} Z_n| \in I \} } \left\{ \frac{1}{\sqrt{n}} Z_n \right\} \right] = \frac{1}{\sqrt{2\pi \sigma^2}} \int_{\{z \in I\}} |z| e^{-z^2/(2\sigma^2)} \, dz.
\]
We have
\[ b^n \mathbb{E} \left[ \Phi(b^{-n}|Z_n|) \right] = \mathbb{E} \left[ \frac{|Z_n|}{\sqrt{n \log b - \log |Z_n|}} 1\{|Z_n|>0\} \right]. \]

Let \( C \) be an almost sure uniform bound for \(|Z_{k+1} - Z_k|\). Hence, for all \( \beta \in (0, \log b) \) there exists \( n_0 \in \mathbb{N} \) such that \( n \beta < n \log b - \log(C n) \) for all \( n \geq n_0 \). Hence,
\[
\sqrt{n \log b - \log |Z_n|} \geq \sqrt{n \beta} \quad \text{for } n \geq n_0, \tag{2.4}
\]
and taking \( I := (0, \infty) \) in (2.3) gives
\[
\limsup_{n \to \infty} b^n \mathbb{E} \left[ \Phi(b^{-n}|Z_n|) \right] \leq \frac{1}{\sqrt{2 \pi \sigma^2}} \int_{\{z \in I\}} |z| e^{-z^2/(2 \sigma^2)} dz = \sqrt{\frac{2 \sigma^2}{\pi \beta}}.
\]

To get a lower bound, observe that for every \( \epsilon > 0 \) and \( n \geq 1/\epsilon^2 \),
\[
\mathbb{1}_{\{|\frac{1}{\sqrt{n}} Z_n| \geq \epsilon\}} \sqrt{n \log b - \log |Z_n|} \leq \mathbb{1}_{\{|\frac{1}{\sqrt{n}} Z_n| \geq \epsilon\}} \sqrt{n \log b}.
\]

Hence, we get from (2.3) that
\[
\liminf_{n \to \infty} b^n \mathbb{E} \left[ \Phi(b^{-n}|Z_n|) \right] \geq \frac{1}{\sqrt{2 \pi \sigma^2 \log b}} \int_{\{z \geq \epsilon\}} |z| e^{-z^2/(2 \sigma^2)} dz.
\]

Sending \( \epsilon \downarrow 0 \) and \( \beta \uparrow \log b \) gives the result. \( \square \)

*Proof of Theorem 1.2 for \( t = 1 \).* For \( b \) even, [15, Proposition 3.2 (a)] states that \( Y_1, Y_2, \ldots \) is an i.i.d. sequence of symmetric \( \{-1, +1\} \)-valued Bernoulli random variables. Therefore, (2.2), the classical CLT, and Lemma 2.1 give \( V_n \to \sqrt{2/(\pi \log b)} \). If \( b \) is odd, then [15, Proposition 3.2 (b)] states that the random variables \( \text{sgn}(\alpha)^n Y_n \) form a time-homogeneous Markov chain on \( \{-1, 0, +1\} \) with initial distribution \( \mu_1 = (\frac{b-1}{2b}, \frac{1}{2b}, \frac{b-1}{2b}) \) and transition matrix \( P_+ \), where
\[
P_+ := \frac{1}{2b} \begin{pmatrix} b+1 & 0 & b-1 \\ b-1 & 2 & b-1 \\ b+1 & 0 & b+1 \end{pmatrix}.
\]

It follows that \( Y_1, Y_2, \ldots \) also form a time-homogeneous Markov chain with initial distribution \( \mu_1 \) and transition matrix \( P_+ \) for \( \alpha > 0 \) and \( P_- \) for \( \alpha < 0 \). Since 0 is a transient state, we can clearly consider only the restriction of the Markov chain to its positive recurrent states, \(-1\) and \(+1\). Let \( \overline{P} \) be the \( 2 \times 2 \)-matrix obtained from \( P_\pm \) by deleting the second row and second column from \( P \), and define \( \overline{\mu}_1 = (1/2, 1/2) \). Then \( \overline{\mu}_1 \) is the unique stationary distribution for \( \overline{P} \). Moreover,
\[
\overline{P}^n = \frac{1}{2} \left( \begin{array}{cc} 1 + (\pm b)^{-n} & 1 - (\pm b)^{-n} \\ 1 - (\pm b)^{-n} & 1 + (\pm b)^{-n} \end{array} \right).
\]

For the state-constraint Markov chain \( \overline{Y}_1, \overline{Y}_2, \ldots \) with initial distribution \( \overline{\mu}_1 \) and transition matrix \( \overline{P} \), we thus have \( \text{var}(\overline{Y}_1) = 1 \) and
\[
\text{cov}(\overline{Y}_1, \overline{Y}_{n+1}) = \sum_{y_1, y_{n+1} \in \{-1, +1\}} \overline{\mu}_1(y_1) \overline{P}^n(y_1, y_{n+1}) y_1 y_{n+1} = (\pm b)^{-n}.
\]

Letting
\[
\sigma^2 := \text{var}(\overline{Y}_1) + 2 \sum_{n=1}^{\infty} \text{cov}(\overline{Y}_1, \overline{Y}_{n+1}) = \frac{b+1}{b-1},
\]

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the central limit theorem for Markov chains (see, e.g., [11]) implies that \( \frac{1}{\sqrt{n}} \sum_{k=1}^{n} Y_k \) converges in law to \( N(0, \sigma^2) \). Due to the stationarity of the Markov chain, we have moreover

\[
\mathbb{E} \left[ \left( \frac{1}{\sqrt{n}} \sum_{k=1}^{n} Y_k \right)^2 \right] = \frac{1}{n} \sum_{k=1}^{n} \text{var}(Y_k) + \frac{2}{n} \sum_{k=1}^{n-1} \sum_{\ell=k+1}^{n} \text{cov}(Y_k, Y_\ell)
\]

\[
= 1 + \frac{2}{n} \sum_{k=1}^{n-1} \sum_{\ell=k+1}^{n} (\pm b)^{k-\ell} \leq 1 + \frac{2}{n} \cdot \frac{b^{1-n} + bn + b - n}{(b - 1)^2},
\]

which is uniformly bounded in \( n \). Therefore, Lemma 2.1 and (2.2) give \( V_n \to \sqrt{2(b \pm 1)/(\pi(b \mp 1) \log b)} \). \( \square \)

Now we prepare for the proof of Theorem 1.3 for \( t = 1 \). Let \( \mathcal{F}_0 = \{ \emptyset, \Omega \} \) and \( \mathcal{F}_n := \sigma(U_1, \ldots, U_n) \) for \( n \in \mathbb{N} \). Then each \( Y_n \) is \( \mathcal{F}_n \)-measurable. Since \( U_1, \ldots, U_n \) can be recovered from \( R_n \), we have \( \mathcal{F}_n = \sigma(R_n) \) for \( n \geq 1 \). We define \( Z_0 := 0 \) and \( Z_n := \sum_{k=1}^{n} Y_k \) for \( n \in \mathbb{N} \).

**Lemma 2.2.** If \( \varphi(t) = \nu \sin(2\pi t) + \rho \cos(2\pi t) \), then \( \{Z_n\}_{n \in \mathbb{N}_0} \) is a martingale with respect to \( \{ \mathcal{F}_n \}_{n \in \mathbb{N}_0} \).

**Proof.** We must show that \( \mathbb{E}[Y_n|R_{n-1}] = 0 \) \( \mathbb{P} \)-a.s. for \( n \geq 1 \). To this end, we use that \( R_n = R_{n-1} + U_n b^{n-1} \), where \( R_0 := 0 \) and \( U_n \) is independent of \( R_n \). Therefore,

\[
\mathbb{E}[Y_n|R_{n-1} = r] = (\text{sgn}(\alpha))^{n} \mathbb{E} \left[ \varphi \left( \frac{r + U_n b^{n-1} + 1) b^{-n}}{b^{-n}} \right) - \varphi \left( \frac{r + U_n b^{n-1} - 1) b^{-n}}{b^{-n}} \right) \right] = (\text{sgn}(\alpha) b) \sum_{k=0}^{b-1} \left( \varphi \left( \frac{r + 1) b^{-n} + k/b}{b^{-n}} \right) - \varphi \left( r b^{-n} + k/b \right) \right).
\]

If \( n = 1 \), then \( r \) must be zero, and the sum in (2.5) is a telescopic sum with value \( \varphi(1) - \varphi(0) = 0 \). Now consider the case \( n \geq 2 \). Then, for all \( x \in \mathbb{R} \), \( i = \sqrt{-1} \), and \( \Re \) denoting the real part of a complex number,

\[
\sum_{k=0}^{b-1} \varphi(x + k/b) = \Re \left( (\rho - iv) \sum_{k=0}^{b-1} e^{2\pi i(x+k/b)} \right) = \Re \left( (\rho - iv) e^{2\pi i x} \cdot \frac{e^{2\pi ib/b} - 1}{e^{2\pi i/b} - 1} \right) = 0.
\]

Therefore, the sum in (2.5) vanishes. \( \square \)

**Lemma 2.3.** With \( \delta_x \) denoting the Dirac measure in \( x \in \mathbb{R} \) and \( \lambda \) denoting the Lebesgue measure on \([0,1]\), we have \( \mathbb{P} \)-a.s., \( \frac{1}{n} \sum_{k=1}^{n} \delta_{b^{-k} R_k} \to \lambda \) weakly as \( n \uparrow \infty \).

**Proof.** Without loss of generality, we can extend the sequence \( \{U_i\}_{i \in \mathbb{N}} \) to a two-sided sequence \( \{U_i\}_{i \in \mathbb{Z}} \) of i.i.d. random variables with a uniform distribution on \( \{0, \ldots, b-1\} \). Then we define \( X_n := \sum_{j=1}^{\infty} U_{n-1} b^{-j-1} = \sum_{j=0}^{\infty} U_{n-j} b^{-(j+1)} \) for \( n \in \mathbb{Z} \). Each \( X_n \) is uniformly distributed on \([0,1]\), i.e., has law \( \lambda \). Moreover, in comparison with \( X_n \), the random variable \( X_{n+1} \) is obtained by shifting the sequence \( \{U_i\}_{i \in \mathbb{Z}} \) one step to the right. It is well-known that the dynamical system corresponding to such a two-sided Bernoulli shift is mixing and hence ergodic (for a proof, see, e.g., Example 20.26 in [12]). By Birkhoff’s ergodic theorem, we thus have \( \frac{1}{n} \sum_{k=1}^{n} f(X_k) \to \int_0^1 f \, d\lambda \) \( \mathbb{P} \)-a.s. for each bounded Borel-measurable function on \([0,1]\). Since \( |b^{-n} R_n - X_n| \leq b^{-n} \), we hence obtain \( \frac{1}{n} \sum_{k=1}^{n} f(b^{-k} R_k) \to \int_0^1 f \, d\lambda \) \( \mathbb{P} \)-a.s. for each (uniformly) continuous function on \([0,1]\). Since \( C[0,1] \) is separable, the result follows. \( \square \)

**Proof of Theorem 1.3** for \( t = 1 \). Let \( \langle Z \rangle_n := \sum_{k=1}^{n} \mathbb{E}[Y_k^2 | \mathcal{F}_{k-1}] \) be the predictable quadratic variation of the martingale \( \{Z_n\}_{n \in \mathbb{N}_0} \). We define \( \psi_n(x) := (\varphi(x + b^{-n}) - \varphi(x))/b^{-n} \). Then \( \psi_n(x) \to \varphi'(x) \) uniformly in \( x \).

By arguing as in (2.3), we see that \( \mathbb{E}[Y_k^2 | \mathcal{F}_{k-1}] = \frac{1}{b} \sum_{\ell=0}^{b-1} (\psi_k(b^{-k} R_{k-1} + \ell/b))^2 \). We therefore conclude from Lemma 2.3 that

\[
\frac{1}{n} \langle Z \rangle_n = \frac{1}{b} \sum_{\ell=0}^{b-1} \sum_{k=1}^{n} (\psi_k(b^{-k} R_{k-1} + \ell/b))^2 \longrightarrow \int_0^1 (\varphi'(t))^2 \, dt = 2\pi^2 (\nu^2 + \rho^2) =: \sigma^2.
\]
Analogously, one sees easily that \( s_n^2 := \mathbb{E}[\langle Z \rangle_n] \) satisfies \( \frac{1}{n} s_n^2 \to \sigma^2 \). Since the increments \( Y_k \) are uniformly bounded, the Lindeberg condition,

\[
\frac{1}{n} \sum_{k=1}^{n} \mathbb{E} \left[ Y_k^2 \mathbb{1}_{\{ Y_k^2 \geq \varepsilon n \}} \right] |_{\mathcal{F}_{k-1}} \to 0 \quad \mathbb{P}\text{-a.s. for all } \varepsilon > 0,
\]

is clearly satisfied. Therefore, the martingale central limit theorem in the form of [5, (7.4) in Chapter 7] yields that the laws of \( \frac{1}{\sqrt{n}} Z_n \) converge weakly to \( N(0, \sigma^2) \). Lemma 2.1 hence gives

\[
V_n \to 2 \sqrt{\frac{\pi (v^2 + \rho^2)}{\log b}}.
\]

Finally, we show how the preceding results can be extended to the case \( 0 \leq t < 1 \). Writing \( Z_n \) for \( \sum_{k=1}^{n} Y_k \), the \( \Phi \)-variation over the interval \( [0, t] \) is equal to

\[
V_{n,t} := \sum_{k=0}^{b^{n-1}} \Phi \left( f((k+1)b^{-n}) - f(kb^{-n}) \right) \mathbb{1}_{[0,t]}(kb^{-n}) = b^n \mathbb{E} \left[ \Phi \left( f((R_n+1)b^{-n}) - f(R_nb^{-n}) \right) \mathbb{1}_{\{b^{-n} R_n \leq t\}} \right] = b^n \mathbb{E} \left[ \Phi \left( b^{-n} \sum_{m=1}^{n} Y_m \right) \mathbb{1}_{\{b^{-n} R_n \leq t\}} \right] = \mathbb{E} \left[ \frac{|Z_n|}{\sqrt{n \log b - \log |Z_n|}} \mathbb{1}_{\{|Z_n| > 0\}} \mathbb{1}_{\{b^{-n} R_n \leq t\}} \right].
\]

Let \( \delta > 0 \) be given and pick \( m \in \mathbb{N} \) such that \( b^{-m} \leq \delta \). Clearly, \( \{b^{-n} R_n \leq t\} \subset \{b^{-n} R_{m,n} \leq t\} \), where \( R_{m,n} := R_n - R_{n-m} = \sum_{k=n-m+1}^{n} k \). In addition, we argue as in the proof of Lemma 2.1 and take \( \beta \in (0, \log b) \) and \( n_0 \in \mathbb{N} \) such that \( n \beta < n \log b - \log(C n) \) for all \( n \geq n_0 \) and (2.4) holds. Therefore, for \( n \geq m \vee n_0 \),

\[
V_{n,t} \leq \frac{1}{\sqrt{n \beta}} \mathbb{E} \left[ |Z_n| \mathbb{1}_{\{b^{-n} R_{m,n} \leq t\}} \right] \leq \frac{1}{\sqrt{n \beta}} \mathbb{E} \left[ |Z_{n-m}| \mathbb{1}_{\{b^{-n} R_{m,n} \leq t\}} \right] + \frac{1}{\sqrt{n \beta}} \mathbb{E} \left[ \left| \sum_{k=n-m+1}^{n} Y_k \right| \right].
\]

Clearly, the rightmost term converges to zero as \( n \uparrow \infty \). Moreover, \( Z_{n-m} \) and \( R_{m,n} \) are independent, and so

\[
\limsup_{n \uparrow \infty} V_{n,t} \leq \frac{2 \sigma^2}{\pi \beta} \limsup_{n \uparrow \infty} \mathbb{P}[b^{-n} R_{m,n} \leq t] \leq \frac{2 \sigma^2}{\pi \beta} \limsup_{n \uparrow \infty} \mathbb{P}[b^{-n} R_n \leq t + \delta] = \frac{2 \sigma^2}{\pi \beta} (t + \delta),
\]

where the second inequality follows from the fact that \( b^{-n} R_{m,n} \geq b^{-n} R_n - \delta \) for \( n > m \). Sending \( \beta \uparrow \log b \) and \( \delta \downarrow 0 \) gives the desired upper bound.

To get a corresponding lower bound, we choose \( \delta > 0 \) and \( m \) as in the upper bound. In addition, we choose \( \varepsilon > 0 \). For \( n > m \vee 1/\varepsilon^2 \), we then get as in the proof of Lemma 2.1

\[
V_{n,t} \geq \mathbb{E} \left[ \frac{|Z_n|}{\sqrt{n \log b}} \mathbb{1}_{\{|\frac{1}{\sqrt{n \log b}} |Z_n| \geq \varepsilon\}} \mathbb{1}_{\{b^{-n} R_n \leq t\}} \right] \geq \mathbb{E} \left[ \frac{|Z_{n-m}|}{\sqrt{n \log b}} \mathbb{1}_{\{|\frac{1}{\sqrt{n \log b}} |Z_{n-m}| \geq \varepsilon\}} \mathbb{1}_{\{b^{-n} R_{m,n} \leq t-\delta\}} \right].
\]

Now let \( C \) be a uniform upper bound for \( |Y_k| \) and choose \( n_1 \) such that \( mC \leq \varepsilon \sqrt{m} \). Then, for \( n \geq n_1 \vee \sqrt{m} \),

\[
V_{n,t} \geq \mathbb{E} \left[ \frac{|Z_{n-m}|}{\sqrt{n \log b}} \mathbb{1}_{\{|\frac{1}{\sqrt{n \log b}} |Z_{n-m}| \geq 2\varepsilon\}} \mathbb{1}_{\{b^{-n} R_{m,n} \leq t-\delta\}} \right] - \frac{1}{\sqrt{n \log b}} \mathbb{E} \left[ \left| \sum_{k=n-m+1}^{n} Y_k \right| \right].
\]

Again, the second expectation on the right converges to zero. Using as before the independence of \( Z_{n-m} \) and \( R_{m,n} \) now easily gives the desired lower bound. This concludes the proof of Theorems 1.2 and 1.3 for \( 0 \leq t < 1 \).
References

[1] Jürgen Appell, Józef Banaś, and Nelson Merentes. *Bounded variation and around*, volume 17 of De Gruyter Series in Nonlinear Analysis and Applications. De Gruyter, Berlin, 2014.

[2] Patrick Billingsley. Van der Waerden’s continuous nowhere differentiable function. *Am. Math. Mon.*, 89:691, 1982.

[3] Rama Cont and Nicolas Perkowski. Pathwise integration and change of variable formulas for continuous paths with arbitrary regularity. *Trans. Amer. Math. Soc. Ser. B*, 6:161–186, 2019.

[4] Georges de Rham. Sur un exemple de fonction continue sans dérivée. *Enseign. Math*, 3:71–72, 1957.

[5] Rick Durrett. *Probability: theory and examples*. Brooks/Cole—Thomson Learning, Belmont, CA, third edition, 2005.

[6] H. Föllmer. Calcul d’Itô sans probabilités. In *Seminar on Probability, XV (Univ. Strasbourg, Strasbourg, 1979/1980)*, volume 850 of Lecture Notes in Math., pages 143–150. Springer, Berlin, 1981.

[7] Hans Föllmer and Alexander Schied. Probabilistic aspects of finance. *Bernoulli*, 19(4):1306–1326, 2013.

[8] N. G. Gamkrelidze. On a probabilistic properties of Takagi’s function. *J. Math. Kyoto Univ.*, 30(2):227–229, 1990.

[9] Nina Gantert. Self-similarity of Brownian motion and a large deviation principle for random fields on a binary tree. *Probab. Theory Related Fields*, 98(1):7–20, 1994.

[10] G. H. Hardy. Weierstrass’s non-differentiable function. *Trans. Amer. Math. Soc.*, 17(3):301–325, 1916.

[11] Galin L. Jones. On the Markov chain central limit theorem. *Probab. Surv.*, 1:299–320, 2004.

[12] Achim Klenke. *Probability theory*. Universitext. Springer, London, second edition, 2014.

[13] Yuliya Mishura and Alexander Schied. On (signed) Takagi–Landsberg functions: pth variation, maximum, and modulus of continuity. *J. Math. Anal. Appl.*, 473(1):258–272, 2019.

[14] Alexander Schied. On a class of generalized Takagi functions with linear pathwise quadratic variation. *J. Math. Anal. Appl.*, 433:974–990, 2016.

[15] Alexander Schied and Zhenyuan Zhang. On the pth variation of a class of fractal functions. *arXiv:1909.05239, to appear in Proceedings AMS*. 

