Abstract

We study adversary-resilient stochastic distributed optimization, in which $m$ machines can independently compute stochastic gradients, and cooperate to jointly optimize over their local objective functions. However, an $\alpha$-fraction of the machines are Byzantine, in that they may behave in arbitrary, adversarial ways. We consider a variant of this procedure in the challenging non-convex case. Our main result is a new algorithm SafeguardSGD which can provably escape saddle points and find approximate local minima of the non-convex objective. The algorithm is based on a new concentration filtering technique, and its sample and time complexity bounds match the best known theoretical bounds in the stochastic, distributed setting when no Byzantine machines are present. Our algorithm is practical: it improves upon the performance of prior methods when training deep neural networks, it is relatively lightweight, and is the first method to withstand two recently-proposed Byzantine attacks.

1 Introduction

Motivated by the pervasiveness of large-scale distributed machine learning, there has recently been significant interest in providing distributed optimization algorithms with strong fault-tolerance guarantees [7, 12, 26, 29, 30]. In this context, the strongest, most stringent fault model is that of Byzantine faults [19]: given $m$ machines, each having access to private data, at most an $\alpha$ fraction of the machines can behave in arbitrary, possibly adversarial ways, with the goal of breaking or at least slowing down the algorithm. Although extremely harsh, this fault model is the “gold standard” in distributed computing [9, 19, 21], as algorithms proven to be correct in this setting are guaranteed to converge under arbitrary system behaviour.

A setting of particular interest in this context has been that of distributed stochastic optimization. Here, the task is to minimize some stochastic function $f(x) = E_{s \sim D}[f_s(x)]$ over a distribution $D$, where $f_s(\cdot)$ can be viewed as the loss function for sample $s \sim D$. We assume there are $m$ machines (workers) and an honest master, and $\alpha < 1/2$ fraction of the workers may be Byzantine. In each iteration $t$, each worker has access to a version of the global iterate $x_t$, which is maintained by the master. The worker can independently sample $s \sim D$, compute $\nabla f_s(x_t)$, and then synchronously send this stochastic gradient to the master. The master aggregates the workers’ messages, and sends an updated iterate $x_{t+1}$ to all the workers. Eventually, the master has to output an approximate minimizer of $f$. Clearly, the above description only applies to honest workers; Byzantine workers may deviate arbitrarily and return adversarial “gradient” vectors to the master in every iteration.

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This distributed framework is quite general and well studied. One of the first references in this setting studied distributed PCA and regression [12]. Other early approaches [7, 10, 25, 26, 29] relied on defining generalizations of the geometric median, i.e. finding a vector minimizing the sum distance to the set of stochastic gradients received in one iteration. These approaches can withstand up to half of the nodes being malicious, but can have relatively high local computational cost $\Omega(m^2d)$ [7, 10], where $m$ is the number of nodes and $d$ is the problem dimension, and usually have sub-optimal sample and iteration complexities.

Follow-up work resolved this last issue when the objective $f(\cdot)$ is convex, leading to tight sample complexity bounds. Specifically, Yin et al. [33] provided bounds for gradient descent-type algorithms (based on coordinate-wise median and mean trimming), and showed that the bounds are tight when the dimension is constant. Alistarh et al. [1] provided a stochastic gradient descent (SGD) type algorithm (based on martingale concentration trimming) and showed that its sample and time complexities are asymptotically optimal even when the dimension is large. Both approaches match the sample and time complexity of SGD as $\alpha$ approaches zero.

Non-convex Byzantine-resilient stochastic optimization. In this paper, we focus on the more challenging non-convex setting, and shoot for the strong goal of finding approximate local minima (a.k.a. second-order critical points). In a nutshell, our main result is the following. Fix $d$ to denote the dimension, and let the objective $f : \mathbb{R}^d \to \mathbb{R}$ be Lipschitz smooth and second-order smooth. We have $m$ worker machines, each having access to unbiased, bounded estimators of the gradient of $f$. Given an initial point $x_0$, the SafeguardSGD algorithm ensures that, even if at most $\alpha < 1/2$ fraction of the machines are Byzantine, after

$$T = \tilde{O}\left((\alpha^2 + \frac{1}{m}) \frac{d(f(x_0) - \min f(x))}{\epsilon^2}\right)$$

parallel iterations,

for at least a constant fraction of the indices $t \in [T]$, the following hold:

$$\|\nabla f(x_t)\| \leq \epsilon \text{ and } \nabla^2 f(x_t) \succeq -\sqrt{\epsilon}I.\]

If the goal is simply $\|\nabla f(x_t)\| \leq \epsilon$, then $T = \tilde{O}\left((\alpha^2 + \frac{1}{m}) \frac{f(x_0) - \min f(x)}{\epsilon^2}\right)$ iterations suffice. Here, the $\tilde{O}$ notation serves to hide logarithmic factors for readability. We spell out these factors in the detailed analysis.

- When $\alpha < 1/\sqrt{m}$, our sample complexity (= $mT$) matches the best known result in the non-Byzantine case [17] without additional assumptions, and enjoys linear parallel speed-up: with $m$ workers of which $< \sqrt{m}$ are Byzantine, the parallel speedup is $\tilde{O}(m)^{1}$

- For $\alpha \in [1/\sqrt{m}, 1/2)$, our parallel time complexity is $\tilde{O}(\alpha^2)$ times that needed when no parallelism is used. This still gives parallel speedup. This $\alpha^2$ factor appears in convex Byzantine distributed optimization, where it is tight [1, 33].

- The Lipschitz smoothness and second-order smoothness assumptions are the minimal assumptions needed to derive convergence rates for finding second-order critical points [17].

Comparison with prior bounds. The closest known bounds are by Yin et al. [34], who derived three gradient descent-type of algorithms (based on median, mean, and iterative filtering) to find a weaker type of approximate local minima. Since it relies on full gradients, their algorithm is arguably less practical, and their time complexities are generally higher than ours (see Section 2.1).

Other prior works consider a weaker goal: to find approximate stationary points $\|\nabla f(x)\| \leq \epsilon$ only [7, 8, 29, 30, 32, 33]. Bulusu et al. [8] additionally assumed there is a guaranteed good (i.e. non-Byzantine) worker known by the master, Xie et al. [30] gave a practical algorithm when the Byzantine attackers have no information about the loss function or its gradient, Blanchard et al. [7], Xie et al. [29], Yang et al. [32] derived eventual convergence without an explicit complexity bound, and the non-convex result obtained in [33] is subsumed by [34], discussed above.

Our algorithm and techniques. The structure of our algorithm is deceptively simple. The master node keeps track of the sum of gradients produced by each worker across time. It labels (allegedly) good workers as those whose sum of gradients “concentrate” well with respect to a surrogate of the median vector, and labels bad workers otherwise. Once a worker is labelled bad, it is removed from consideration forever. The

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1By parallel speedup we mean the reduction in wall-clock time due to sampling gradients in parallel among the $m$ nodes. In each time step, the algorithm generates $m$ new gradients, although some may be corrupted.
The idea of using concentration (for each worker across time) to filter out Byzantine machines traces back to the convex setting \cite{1}. However, the quantities used in \cite{1} to check for concentration are necessarily different from this paper, and our analysis is completely new, as deriving non-convex rates is known to be much more delicate and challenging, compared to convex ones \cite{13, 16, 17}. Recently, Bulusu et al. \cite{8} used similar concentration filters to Alistarh et al. \cite{1} in the non-convex setting, but under stronger assumptions, and for the simpler task of finding stationary points.

Many other algorithms do not rely on concentration filters. In each iteration, they ask each worker to compute a batch of stochastic gradients, and then use coordinate-wise median or mean over the batch average (e.g. \cite{32-34}) or iterative filtering (e.g. \cite{27, 34}) by the master to derive a “robust mean.” These works fundamentally rely on each iteration to calculate an almost precise full gradient, so that they can apply a surrogate of full gradient descent. Such algorithms can introduce higher sample and time complexities (see \cite{1} and our Section 2 for comparisons in the convex and non-convex settings respectively), are less practical than stochastic gradient schemes, require additional restrictions on the resilience factor \(\alpha\), e.g. \(\alpha < 1/4\) \cite{27}, and, critically, have been shown to be vulnerable to recent attacks \cite{6, 31}.

**Attack resilience and experimental validation.** There is a growing literature on customized attacks against Byzantine-resilient algorithms, showing that many defenses can be entirely circumvented in real-world scenarios \cite{6, 31}. Our algorithm is provably correct against these attacks, a fact we also validate experimentally. Specifically, we implemented SafeguardSGD to examine its practical performance against a range of prior works \cite{7, 10, 30, 33, 34}, and against recent attacks on the distributed task of training deep neural networks. Our experiments show that SafeguardSGD generally outperforms previous methods in convergence speed and final accuracy, sometimes by a wide accuracy margin. This is true not only against known Byzantine attacks, but also against attack variants we fine-crafted to specifically slow down our algorithm, and against transient node failures.

## 2 Statement of Our Theoretical Result

We denote by \(\|\cdot\|\) the Euclidean norm and \([n] := \{1, 2, \ldots, n\}\). Given symmetric matrices \(A, B\), we let \(\|A\|_2\) denote the spectral norm of \(A\). We use \(\succeq\) to denote Loewner ordering, i.e. \(A \succeq B\) if \(A - B\) is positive semi-definite. We denote by \(\lambda_{\text{min}}(A)\) the minimum eigenvalue of matrix \(A\).

We consider arbitrary \(d\)-dimensional non-convex functions \(f : \mathbb{R}^d \to \mathbb{R}\) satisfying the following:

- \(f(x)\) is \(L\)-Lipschitz smooth: meaning \(\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|\) for any \(x, y \in \mathbb{R}^d\);
- \(f(x)\) is \(L_2\)-second-order smooth: \(\|\nabla^2 f(x) - \nabla^2 f(y)\|_2 \leq L_2 \cdot \|x - y\|\) for any \(x, y \in \mathbb{R}^d\);

For notational simplicity of the proofs, we assume \(L = L_2 = \mathcal{V} = 1\).\footnote{In the literature of convergence analysis for non-convex optimization, the final complexity bounds naturally and polynomially depend on these parameters \(L, L_2, \mathcal{V}\), and the way the dependence goes is typically unique \cite{2-5, 11, 15, 17, 20, 22, 24, 28}. This is why it suffices to ignore their appearance and only compare the polynomial dependence on \(\varepsilon\) and \(d\). One can carefully derive the bounds in this paper to make \(L, L_2, \mathcal{V}\) show up; we decide not to include that version to make this paper as concise as possible.} Note that we have also assumed the domain of \(f\) is the entire space \(\mathbb{R}^d\). If instead there is a compact domain \(\mathcal{X} \subset \mathbb{R}^d\), then one can use projected SGD and re-derive similar results of this paper. We choose to present our result in the simplest setting to convey our main ideas.
Our goal is to find an $\varepsilon$-approximate local minima (a.k.a. second-order critical point) of this function $f(x)$, satisfying:

$$\|\nabla f(x)\| \leq \varepsilon \quad \text{and} \quad \nabla^2 f(x) \succeq -\sqrt{\varepsilon} \mathbf{I}$$

We quickly point out that, if a method is capable of finding approximate local minima, then it necessarily has the ability to escape (approximate) saddle points [13].

**Byzantine non-convex stochastic distributed optimization.** We let $m$ be the number of worker machines and assume at most an $\alpha$ fraction of them are Byzantine for $\alpha \in \left[0, \frac{1}{3}\right)$. We denote by $\text{good} \subseteq [m]$ the set of good (i.e. non-Byzantine) machines, and the algorithm does not know $\text{good}$.

**Assumption 2.1.** In each iteration $t$, the algorithm (on the master) is allowed to specify a point $x_t$ and query $m$ machines. Each machine $i \in [m]$ gives back a vector $\nabla_{t,i} \in \mathbb{R}^d$ satisfying

- If $i \in \text{good}$, the stochastic gradient $\nabla_{t,i}$ satisfies $\mathbb{E}[\nabla_{t,i}] = \nabla f(x_t)$ and $\|\nabla f(x_t) - \nabla_{t,i}\| \leq \mathcal{V}$.
- If $i \in [m] \setminus \text{good}$, then $\nabla_{t,i}$ can be arbitrary (w.l.o.g. we assume $\|\nabla f(x_t) - \nabla_{t,i}\| \leq \mathcal{V}$).\(^3\)

**Remark 2.2.** For each $t$ and $i \notin \text{good}$, the vector $\nabla_{t,i}$ can be adversarially chosen and may depend on $\{\nabla_{t,i}\}_{t \leq t, i \in [m]}$. In particular, the Byzantine machines can even collude during an iteration.

### 2.1 Our Algorithm and Theorem

Our algorithm is based on arguably the simplest possible method for achieving this goal, (*perturbed*) stochastic gradient descent (SGD) [13].\(^4\) Our techniques more broadly apply to more complicated methods (e.g. at least to [3, 4]), but we choose to analyze the simplest variant of SGD, since it is the most widely applied method in modern non-convex machine learning.

As illustrated in Algorithm 1, in each iteration $t = 0, 1, \ldots, T - 1$, we maintain a set of (allegedly) good machines $\text{good}_t \subseteq [m]$. We begin with $\text{good}_0 = [m]$ and start to detect malicious machines and remove them from the set. We choose a learning rate $\eta > 0$, and perform the SGD update

$$x_{t+1} = x_t + \xi_t - \eta [\text{good}_t] \sum_{i \in \text{good}_t} \nabla_{t,i}$$

where $\xi_t \sim \mathcal{N}(0, \nu^2 \mathbf{I})$ is a random Gaussian perturbation that we introduce.

For each machine $i \in [m]$, we keep track of the history of its stochastic gradients up to *two windows*. Namely, $A_i \leftarrow \sum_{k = \text{last}_1}^{t} \nabla_{k,i}/|\text{good}_k|$ and $B_i \leftarrow \sum_{k = \text{last}_0}^{t} \nabla_{k,i}/|\text{good}_k|$, for windows sizes $T_0 \leq T_1 \leq T$. We compare among remaining machines in $\text{good}_t$, and kick out those ones whose $A_i$ or $B_i$ deviate “more than usual” to construct $\text{good}_{t+1}$. Conceptually, we view these two as *safe guards*.

Our theory makes sure that, when the “window sizes” and the thresholds for “more than usual” are defined properly, then $\text{good}_t$ shall always include $\text{good}$, and the algorithm shall proceed to find approximate local minima. Formally, we have (letting the $\tilde{O}$ notion to hide polylogarithmic factors)

**Theorem 2.3.** Let $C_3 = \alpha^2 + \frac{1}{m}$. Suppose we choose $\nu^2 = \tilde{O}(C_3)$, $\eta = \tilde{O}(\frac{\varepsilon}{\alpha^2 C_3})$, $T_0 = \tilde{O}(\frac{1}{\eta})$, $T_1 = \tilde{O}(\frac{1}{\sqrt{\varepsilon}})$, $\xi_0 = \tilde{O}(\sqrt{T_0})$, and $\xi_1 = \tilde{O}(\sqrt{T_1})$, then after

$$T = \tilde{O} \left( \frac{(f(x_0) - \min_{x \in \mathcal{X}} f(x)) d}{\varepsilon^4} (\alpha^2 + \frac{1}{m}) \right)$$

iterations, with high probability, for at least constant fraction of the indices $t \in [T]$, they satisfy

$$\|\nabla f(x_t)\| \leq \varepsilon \quad \text{and} \quad \nabla^2 f(x_t) \succeq -\sqrt{\varepsilon} \mathbf{I} .$$

**Remark 2.4.** If one only wishes to achieve a *significantly simpler goal* — finding first-order critical points $\|\nabla f(x_t)\| \leq \varepsilon$ — the analysis becomes much easier (see Section 3.1). In particular, having one safe guard without perturbation (i.e. $\nu = 0$) suffices, and the iteration complexity reduces to $T = \tilde{O} \left( \frac{f(x_0) - \min_{x \in \mathcal{X}} f(x)}{\varepsilon^2} (\alpha^2 + \frac{1}{m}) \right)$.

\(^3\)This requirement $\|\nabla f(x_t) - \nabla_{t,i}\| \leq \mathcal{V}$ is “without loss of generality” because it is trivial for the algorithm to catch bad machines if they output $\nabla_{t,i}$ that are more than $2\mathcal{V}$ away from the majorities. One can add another safe guard for this purpose, see [1].

\(^4\)Theoretically, adding random perturbation is a necessary step to escape saddle points [13, 16]; although in practice, the noise from SGD usually suffices.
Algorithm 1 SafeSafeguardSGD: perturbed SGD with double safe guard

Input: point $x_0 \in \mathbb{R}^d$, rate $\eta > 0$, lengths $T \geq T_1 \geq T_0 \geq 1$, threshold $\mathcal{F}_1 > \mathcal{F}_0 > 0$;
1: $\text{good}_0 \leftarrow [m]$;
2: for $t \leftarrow 0$ to $T - 1$ do
3: \hspace{1em} last$_1 \leftarrow \max\{t_1 \in [T]: t_1$ is a multiple of $T_1\}$;
4: \hspace{1em} last$_0 \leftarrow \max\{t_0 \in [T]: t_0$ is a multiple of $T_0\}$
5: \hspace{1em} for each $i \in \text{good}_t$ do
6: \hspace{2em} receive $\nabla_{t,i} \in \mathbb{R}^d$ from machine $i$;
7: \hspace{2em} $A_i \leftarrow \sum_{k=\text{last}_1}^{t} \frac{\nabla_{k,i}}{|\text{good}_k|}$ and $B_i \leftarrow \sum_{k=\text{last}_0}^{t} \frac{\nabla_{k,i}}{|\text{good}_k|}$;
8: \hspace{1em} end for
9: $A_{\text{med}} \leftarrow A_i$ where $i \in \text{good}_t$ is any machine s.t. $\{j \in \text{good}_t : \|A_j - A_i\| \leq \mathcal{F}_1\} > m/2$.
10: $B_{\text{med}} \leftarrow B_i$ where $i \in \text{good}_t$ is any machine s.t. $\{j \in \text{good}_t : \|B_j - B_i\| \leq \mathcal{F}_0\} > m/2$.
11: $\text{good}_{t+1} \leftarrow \{i \in \text{good}_t : \|A_i - A_{\text{med}}\| \leq 2\mathcal{F}_1 \land \|B_i - B_{\text{med}}\| \leq 2\mathcal{F}_0\}$;
12: $x_{t+1} = x_t - \eta \left(\xi_t + \frac{1}{|\text{good}_t|} \sum_{i \in \text{good}_t} \nabla_{t,i}\right)$: \hspace{1em} \quad \Delta \text{ Gaussian noise } \xi_t \sim \mathcal{N}(0, \nu^2I)$
13: end for

Recent work [8] achieves this easier goal but requires an additional assumption: there is one guaranteed good worker known by the master (see Remark 3.5). In this paper we do not make such assumption.

Our contribution. We reiterate our theoretical contributions from three perspectives. 1) When $\alpha < 1/\sqrt{m}$, our algorithm requires $mT = \tilde{O}(\frac{(f(x_0)-\min f(x))d^4}{\epsilon})$ stochastic gradient computations. This matches the best known result [17] under our minimal assumptions of the non-convex objective. (There exist other works in the stochastic setting that break the $\epsilon^{-4}$ barrier and get rid of the dimension dependence $d$ under stronger assumptions.[5])
2) When $\alpha < 1/\sqrt{m}$, our algorithm enjoys linear parallel speed-up: the parallel time complexity reduces by a factor of $\Theta(m)$. When $\alpha \in [1/\sqrt{m}, 1/2)$, our parallel time complexity is $\tilde{O}(\alpha^2)$ times that needed when no parallelism is used, still giving noticeable speedup. The $\alpha^2$ factor also appeared in convex Byzantine distributed optimization (and is known to be tight there) [1, 33].

Comparison to [34]. Yin et al. [34] derived three gradient descent-type algorithms to find points with a weaker (and less standard) guarantee: $\|\nabla f(x)\| \leq \epsilon$ and $\nabla^2 f(x) \succeq -(\epsilon^2d)^{1/3}I$. Despite practical differences (namely, gradient descent may be less favorable comparing to stochastic gradient descent especially in deep learning applications), the parallel time complexities derived from their result are also generally larger than ours.

Their paper focuses on bounding the number of sampled stochastic functions, as opposed to the number of stochastic gradient evaluations like we do in this paper. The later notion is more connected to time complexity. When translated to our language, each of the workers in their setting needs to evaluate $T$ stochastic gradients, where (1) $T = \tilde{O}(\frac{\alpha^2d^2}{\epsilon^2} + \frac{\epsilon^2}{\epsilon^2m})$ if using coordinate-wise median, (2) $T = \tilde{O}(\frac{\alpha^2d^2}{\epsilon^2} + \frac{\epsilon^2}{\epsilon^2m})$ if using trimmed mean, and (3) $T = \tilde{O}(\frac{d^2}{\epsilon^2} + \frac{\epsilon^2}{\epsilon^2m})$ if using iterative filtering. The complexities (1) and (2) are larger than ours (also with a weaker guarantee): the complexity (3) seems incomparable to ours, but when translating to the more standard $(\epsilon, \sqrt{d})$ guarantee, becomes $T = \tilde{O}(\frac{\alpha^2d^2}{\epsilon^2} + \frac{\epsilon^2}{\epsilon^2m})$ so is also larger than ours. It is worth noting that (3) requires $\alpha < 1/4$ so cannot withstand half of the machines being Byzantine.

Tolerating transient failures. Our algorithm can also withstand transient node failures, by simply resetting the set of good nodes, along with resetting the detection metrics at the window boundaries. More precisely, every max($T_0,T_1$) steps, the algorithm can reset $\text{good}_i$ to include all nodes, and may also reset the detection metrics $A_i$ and $B_i$. The algorithm then proceeds as usual. The key observation behind this relaxation is the fact that our analysis only requires that the attack conditions hold inside the current window. (Please see the Theorem B.1 for details.) We validate this statement experimentally in Section 5.

Resilience against practical attacks. Our algorithm’s filtering technique is based upon tracking $B_i$, the stochastic gradients of each machine $i$ averaged over a window of $T_0$ iterations. This is a departure from previous defenses, most of which are history-less, and enables us to be provably Byzantine-resilient against recent state-of-the-art attacks [6, 31].

5Works such as [3, 4, 11, 20, 22, 28] require $f(x) = \mathbb{E}_{s \sim D}[f_s(x)]$ where each $f_s(x)$ is second-order smooth and/or Lipschitz smooth. This requirement may be too strong for certain practical applications.
Definition 3.1. We make the following definition to simplify notations: let \( w \) be any point.

Algorithm 2 Perturbed SGD with single safe guard (for analysis purpose only)

**Input:** point \( w_0 \in \mathbb{R}^d \), set \( \text{good}_0 \supseteq \text{good} \), rate \( \eta > 0 \), length \( T \geq 1 \), threshold \( \mathbb{T} > 0 \);

1:  \( t \leftarrow 0 \) to \( T-1 \) do
2:     for each \( i \in \text{good}_t \) do
3:         receive \( \nabla_{t,i} \in \mathbb{R}^d \) from machine \( i \);
4:         \( B_i \leftarrow \sum_{k=0}^{t-1} \nabla_{k,i} \) \( \) \( \) \( |	ext{good}_t| \);
5:     end for
6:     \( B_{\text{med}} \leftarrow B_i \) where \( i \in \text{good}_t \) is any machine s.t. \( \left| \left\{ j \in \text{good}_t : \|B_j - B_i\| \leq \mathbb{T} \right\} \right| > m/2 \).
7:     \( \text{good}_{t+1} \leftarrow \left\{ i \in \text{good}_t : \|B_i - B_{\text{med}}\| \leq 2\mathbb{T} \right\} \);
8:     \( w_{t+1} = w_t - \eta \left( \xi_t + \frac{1}{|\text{good}_t|} \sum_{i \in \text{good}_t} \nabla_{t,i} \right) \).
9:  end for

Specifically, in Baruch et al. [6], Byzantine workers collude to shift the gradient mean by a factor \( \beta \) times the standard deviation of the (true stochastic) gradient, while staying within population variance. They noticed \( \beta \) can be quite large in practice, since stochastic gradients tend to have large variance, especially in neural network training. Their attack can circumvent existing defenses because those defense algorithms are “historyless”, while their attack is statistically indistinguishable from an honest execution in a single iteration. However, our algorithm can provably defend against this attack since it has memory: Byzantine workers following their strategy will progressively diverge from the (honest) “median” \( B_{\text{med}} \) (by an amount proportional to \( \Omega(T) \) in \( T \) iterations), and thus be marked as malicious by our algorithm (since our safeguard threshold is proportional to \( O(\sqrt{T}) \)). Alternatively, if the Byzantine workers attempt to disrupt the mean while staying within our algorithm’s thresholds, then we prove their influence on convergence must be negligible. In [31], Byzantine workers collude to deviate in the negative direction of the gradient. Similarly to the previous attack, to avoid being caught by our algorithm, the maximum “magnitude” of this attack again has to stay within our thresholds. We implemented both attacks and showed our algorithm’s robustness experimentally.

Finally, we note that prior “historyless” schemes, such as Krum or median-based schemes, could be thought of as providing stronger guarantees, as they in theory allow Byzantine nodes to change IDs during the computation: such schemes only require an upper bound on the number of Byzantine agents in each round. However, the attack of Baruch et al. [6] essentially shows that all such schemes are vulnerable to variance attacks, and that such attacks are eminently plausible in practice. Thus, this suggests that the use of historical information, which requires that Byzantine nodes cannot change their IDs during the execution, may be necessary for Byzantine resilience.

3 Warmup: Single Safe Guard

As a warmup, let us first analyze the behavior of perturbed SGD with a single safe guard. Consider Algorithm 2, where we start with a point \( w_0 \), a set \( \text{good}_0 \supseteq \text{good} \), and perform \( T \) steps of perturbed SGD. (We use the \( w_t \) sequence instead of the \( x_t \) sequence to emphasize that we are in Algorithm 2.)

Definition 3.1. We make the following definition to simplify notations: let \( \Xi_t \overset{\text{def}}{=} \sigma_t + \Delta_t \) where

- \( \sigma_t \overset{\text{def}}{=} \frac{1}{|\text{good}_t|} \sum_{i \in \text{good}_t} (\nabla_{t,i} - \nabla f(w_t)) \)
- \( \Delta_t \overset{\text{def}}{=} \frac{1}{|\text{good}_t|} \sum_{i \in \text{good}_t \setminus \text{good}} (\nabla_{t,i} - \nabla f(w_t)) \)

Therefore, we can re-write the SGD update as \( w_{t+1} = w_t - \eta(\nabla f(w_t) + \xi_t + \Xi_t) \).

The following lemma is fairly immediate to prove:

Lemma 3.2 (single safe guard). In Algorithm 2, suppose we choose \( \mathbb{T} = 8 \sqrt{T \log(16mT/p)} \). Then, with probability at least \( 1 - p/4 \), for every \( t = 0, \ldots, T-1 \),

- \( \text{good}_t \supseteq \text{good} \).
\[
\begin{align*}
\bullet & \|\sigma_t\|^2 \leq O\left(\frac{\log(T/p)}{m}\right) \text{ and } \|\sigma_0 + \cdots + \sigma_{t-1}\|^2 \leq O\left(\frac{T\log(T/p)}{m}\right) \\
\bullet & \|\Delta_t\|^2 \leq \alpha^2 \text{ and } \|\Delta_0 + \cdots + \Delta_{t-1}\|^2 \leq O(\alpha^2 T\log(nT/p)) \\
\bullet & |\langle \nabla f(w_t), \xi_t \rangle| \leq \|\nabla f(w_t)\| \cdot O(\nu\sqrt{\log(T/p)}), \\
\bullet & \|\xi_t\|^2 \leq O(\nu^2 d^2 \log(T/p)), \|\xi_0 + \cdots + \xi_{t-1}\|^2 \leq O(\nu^2 dT \log(T/p))
\end{align*}
\]

We call this probabilistic event \(\text{Event}_T^{\text{single}}(w_0)\) and \(\Pr[\text{Event}_T^{\text{single}}(w_0)] \geq 1 - p/4\).

(The third property above is ensured by our choice of \(T\) and the use of safe guard, and the rest of the properties follow from simple martingale concentration arguments. Details are in Appendix A.1)

### 3.1 Core Technical Lemma 1: Objective Decrease

Our first main technical lemma is the following:

**Lemma 3.3.** Suppose we choose \(T\) as in [Lemma 3.2]. Denote by \(C_1 = \log(T/p)\) and \(C_2 = \alpha^2 \log \frac{mT}{p} + \frac{\log(T/p)}{m}\). Suppose \(\eta \leq 0.01 \min\{1, \frac{1}{C_2}\}\), \(T = \frac{1}{100\eta(1 + \sqrt{C_2})}\) and we start from \(w_0\) and apply Algorithm 2. Under event \(\text{Event}_T^{\text{single}}(w_0)\), it satisfies

\[
f(w_0) - f(w_T) \geq 0.7\eta \left(\sum_{t=0}^{T-1} (\|\nabla f(w_t)\|^2 - \eta \cdot O(C_2 + (C_2)^{1.5}) - O(C_1\nu^2 d + \sqrt{C_2})) \right)
\]

**Lemma 3.3** says after \(T \approx \frac{1}{\eta}\) steps of perturbed SGD, the objective value decreases by, up to some small additive error and up to logarithmic factors, \(f(w_0) - f(w_T) \geq 0.7\eta \sum_{t=0}^{T-1} (\|\nabla f(w_t)\|^2 - \eta C_2)\). This immediately implies, if we choose \(\eta \approx \frac{\alpha^2}{C_2}\), then by repeating this analysis for \(O(C_T^2) = O(\frac{n^2 + 1}{m})\) iterations, we can find approximate critical point \(x\) with \(\|\nabla f(x)\| \leq \varepsilon\).

**Proof sketch of Lemma 3.3.** The full proof is in Appendix A.2 but we illustrate the main idea and difficulties below. After simple manipulations, it is not hard to derive that

\[
f(w_0) - f(w_T) \geq 0.9\eta \sum_{t=0}^{T-1} (\|\nabla f(w_t)\|^2 - \eta) + \eta \sum_{t=0}^{T-1} \langle \nabla f(w_t), \Xi_t \rangle
\]

where recall that \(\Xi_t = \sigma_t + \Delta_t\). When there are no Byzantine machines, we have \(E[\Xi_t] = E[\sigma_t] = 0\) so the remainder terms must be small by martingale concentration. Therefore, the main technical difficulty arises to deal with those Byzantine machines, who can adversarially design their \(\nabla_t\) (even by collusion) so as to negatively correlate with \(\nabla f(w_t)\) to “maximally destroy” the above inequality.

Our main idea is to use second-order smoothness to write \(\nabla f(w_t) \approx \nabla f(w_0) + \nabla^2 f(w_0) \cdot (w_t - w_0)\). To illustrate our idea, let us ignore the constant vector and assume that the Hessian is the identity: that is, imagine as if \(\nabla f(w_t) \approx w_t - w_0\). Using \(w_t - w_0 = -\sum_{k<t} \Xi_t + \xi_t\), we immediately have

\[
-\langle \nabla f(w_t), \Xi_t \rangle \approx -\langle (w_t - w_0), \Xi_t \rangle = \sum_{k<t} \langle \xi_k, \Xi_t \rangle + \sum_{k<t} \langle \xi_k, \Xi_t \rangle
\]

For the first partial sum \(\sum_{k<l} \langle \xi_k, \Xi_l \rangle\) in (3.1), it is easy to bound its magnitude using our safeguard. Indeed, we have \(\sum_{l} (\sum_{k<l} \Xi_k, \Xi_l) \leq \|\sum_{l} \Xi_l\|^2 + \|\sum_{l} \Xi_l\|^2\) so we can apply Lemma 3.2. For the second partial sum \(\sum_{l} \sum_{k<l} \langle \xi_k, \Xi_l \rangle\), we can apply the concentration Proposition 3.4 below.

**Proposition 3.4.** Fix the dimension parameter \(d \geq 1\). Suppose \(\xi_0, \ldots, \xi_{T-1} \in \mathbb{R}^d\) are i.i.d. drawn from \(\mathcal{N}(0, I)\), and that \(\Delta_1, \ldots, \Delta_{T-1}\) are arbitrary vectors in \(\mathbb{R}^d\). Here, each vector \(\Delta_t\) with \(t = 1, \ldots, T-1\) can depend on \(\xi_0, \ldots, \xi_{t-1}\) but not on \(\xi_t, \ldots, \xi_{T-1}\). Suppose that these vectors satisfy \(\|\Delta_1 + \cdots + \Delta_t\|^2 \leq \mathcal{T}\) for every \(t = 1, \ldots, T-1\). Then, with probability at least 1 - \(p\),

\[
\left|\sum_{t=1}^{T-1} \langle \xi_0 + \cdots + \xi_{t-1}, \Delta_t \rangle\right| \leq O(\sqrt{dT^2 \log(T/p)})
\]

**Remark 3.5.** Another way to bound the remainder terms is to introduce another safe guard to directly ensure the remainder terms are small. This seems impossible because the algorithm does not know \(\nabla f(w_t)\). One workaround (as used in [8]) is to assume that the master has access to some guaranteed good worker that can compute a stochastic gradient \(\nabla_{t,0}\) with \(E[\nabla_{t,0}] = \nabla f(w_t)\). This is an additional assumption that we do not wish to make in this paper.
3.2 Core Technical Lemma 2: Randomness Coupling

Our next technical lemma studies that, if run Algorithm 2 from a point \(w_0\) so that the Hessian \(\nabla^2 f(w_0)\) has an eigenvalue which is less than \(-\delta\) (think of \(w_0\) as a saddle point), then with good probability, after sufficiently many iterations, the sequence \(w_1, w_2, \ldots, w_T\) shall escape from \(w_0\) to distance at least \(R\) for some parameter \(R \approx \delta\). To prove this, motivated by [16], we study two executions of Algorithm 2 where their randomness are coupled. We then argue that at least one of them has to escape from \(w_0\). For any vector \(v\), let \([v]_i\) denote the \(i\)-th coordinate of \(v\).

Lemma 3.6. Suppose we choose \(\xi\) as in Lemma 3.2 and \(C_1, C_2\) as in Lemma 3.3. Suppose \(w_0 \in \mathbb{R}^d\) satisfies \(\lambda_{\min}(\nabla^2 f(w_0)) = -\delta\) for some \(\delta \geq 0\). Without loss of generality let \(e_1\) be the eigenvector of \(\nabla^2 f(w_0)\) with smallest eigenvalue. Consider now two executions of Algorithm 2 both starting from \(w_0^0 = w_0^b = w_0\), and suppose their randomness \(\{\xi^a_t\}_t\) and \(\{\xi^b_t\}_t\) are coupled so that \(\xi^a_0 = -\xi^b_0\) but \(\xi^a_t = \xi^b_t\) for \(i > 1\). In words, the randomness is the same orthogonal to \(e_1\), but along \(e_1\), the two have opposite signs. Now, suppose we perform \(T = \Theta(\frac{1}{\delta^2} \log \frac{R^2 \delta}{\eta^2})\) steps of perturbed SGD from \(w_0^a, w_0^b\) respectively using Algorithm 2. Suppose \(R \leq O\left(\frac{\delta}{\sqrt{C_1 \log(R^2 \delta/\eta^2)}}\right)\) and \(\nu^2 \geq \Omega\left(C_2 \log \frac{R^2 \delta}{\eta \nu}\right)\).

Then, under events \(Event^a_T(w_0^a)\) and \(Event^b_T(w_0^b)\), with probability at least 0.98, either \(|w_t^a - w_0^a| > R\) or \(|w_t^b - w_0^b| > R\) for some \(t \in [T]\).

Proof details in Appendix A.4. The main proof difficulty is to analyze a noisy version of the power method, where the noise comes from (1) Gaussian perturbation (which is the good noise), (2) stochastic gradients (which has zero mean), and (3) Byzantine workers (which can be adversarial).

4 From Warmup to Final Theorem with Double Safe Guards

At a high level, Lemma 3.3 ensures that if we keep encountering points with large gradient \(\|\nabla f(w_t)\|\), then the objective value should sufficiently decrease; in contrast, Lemma 3.6 says that if we keep encountering points with negative Hessian directions (i.e., \(\lambda_{\min}(\nabla^2 f(w_t)) < -\delta\)), then the points must move a lot (i.e., by more than \(R\) in \(T\) iterations, which can also lead to sufficient objective decrease, see Lemma B.4). Therefore, at a high level, when the two lemmas are combined, they should tell that we must not encounter points with \(\|\nabla f(x)\|\) being large, or \(\lambda_{\min}(\nabla^2 f(x))\) being very negative, for too many iterations. Therefore, the algorithm can find approximate local minima.

The reason we eventually need two safe guards, is because the number of rounds \(T\) for Lemma 3.3 and Lemma 3.6 differ by a factor. Thus, we need two safe guards with different window sizes to ensure that the two lemmas simultaneously hold. We encourage the reader to examine the full analysis in Appendix B.

5 Experimental Validations

We evaluate the convergence of SafeguardSGD to examine its practical performance against prior works [7, 10, 30, 33, 34]. We perform the non-convex task of training a deep residual network (ResNet-20) [14] on the CIFAR-10 dataset [18]. A full experimental report is given in the Appendix.

We instantiate \(m = 10\) workers and one master executing data-parallel SGD for 200 passes (i.e. epochs) over the training dataset. The results for higher number of workers are similar.

The ideal baseline is gradient mean without attacks, and we compare against (Naive) Mean, Geometric Median [10], Coordinate-wise Median [33, 34], Krum [7], and Zeno [30]. Overall, our experimental setup is very similar to Zeno [30] but with additional attacks. Results are averaged over three repetitions.

We implemented the approach of [32], but found it very sensitive to hyper-parameter values and were not able to make it converge across all attacks even after significant tuning of its \(\gamma\) parameter.

We also implemented the convex algorithm of [1], and executed it in our setting. However, we found that their algorithm can be successfully prevented from converging in the non-convex setting. Specifically, there

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6 As common in the literature, we omit the theoretical noise component to the gradient, and instead rely on the natural noise in the training process.
exists a simple attack, described in Appendix C.5 which causes their algorithm to either mislabel most good workers as Byzantine, or converge to a poor accuracy solution. This is not surprising, since their algorithm is designed for, and only guaranteed to work in, the convex setting. Please see Appendix C.5 for details.

To make the comparison stronger, when implementing SafeguardSGD, we have chosen fixed window sizes $T_0 = 1$ epoch and $T_1 = 6$ epochs across all experiments, and adopted an automated process to select $\alpha_0, \alpha_1$ (by pre-running the experiment for 20 epochs). We have also implemented a single safeguard variant of SafeguardSGD with window size $T = 3$ epochs.

**Attackers.** We set $\alpha = 0.4$ so there are 4 Byzantine workers. (This exceeds the fault-tolerance of Krum, and so we also tested Krum with only 3 Byzantine workers.)

- **Sign-flipping attack:** each Byzantine worker sends the negative gradient to the master.
- **Zero-gradient attack:** each Byzantine worker sends zero gradient to the master.
- **Label-flipping attack:** each Byzantine worker computes its gradient based on the cross-entropy loss with flipped labels: for CIFAR-10, label $\ell \in \{0, ..., 9\}$ is flipped to $9 - \ell$.
- **Delayed-gradient attack:** each Byzantine worker sends an old gradient to master. In our experiments, the delay is of $D = 1000$ iterations.
- **Safe-guard attack:** Byzantine workers send negative but re-scaled gradient to the master. The re-scale factor 0.4 is chosen to avoid triggering the safe-guard conditions at the master (we also show results for re-scale factor 0.5 in the Appendix). This attack is an instantiation of the inner-product attack [31], customized specifically to maximally affect our algorithm.

| Method              | Test Accuracy / Attack Type |
|---------------------|-----------------------------|
|                     | Delayed-gradient | Label-flipping | Sign-flipping | Zero-gradient | Safeguard Attack (rescaling-factor = 0.4) | Variance Attack (stdev-factor = 0.3) |
| Ideal Baseline      | 90.8               | 90.8           | 90.8          | 90.8          | 90.8                                       | 90.8                                    |
| Double Safe-guard   | 91.7               | 90.6           | 90.5          | 90.3          | 88.3                                       | 90.8                                    |
| Single Safe-guard   | 91.1               | 85.5           | 90.9          | 90.6          | 87.5                                       | 91.0                                    |
| Best Other Method   | 79.6 (Zeno)        | 85.3 (Zeno)    | 74.4 (Zeno)   | 88.6 (Zeno)   | 87.7 (Zeno)                                | 29.0 (Zeno)                             |

Figure 1: Convergence comparison (CIFAR-10 test accuracy) under different attacks. Figures for the Safeguard and Variance attacks are deferred to the Appendix.
• **Variance attack** [6]: Byzantine workers measure the mean and the standard-deviation of gradients at each round, and collude to move the mean by the largest value which still operates within population variance. (For our parameter settings, this is 0.3 times the standard deviation. We discuss results for additional parameter values in the Appendix.)

**Discussion.** Figure 1 compares the convergence curves, while Table 1 compares the best test accuracy. **SafeguardSGD** generally outperforms the previous methods in test accuracy and convergence, and closely tracks the performance of the ideal baseline, across all attacks. The test accuracy difference can be > 10% between our algorithm and the best prior work (see delayed gradient, sign-flipping, and variance attacks). **SafeguardSGD** slightly outperforms all other algorithms even for the customized safeguard attacks, which were designed to maximally impact its performance. In most cases, the single-safeguard algorithm is close to double-safeguard, except for the label-flipping attack. We conclude that **SafeguardSGD** can be practical, and outperforms previous approaches.

**Other Experiments**

**Attack analysis.** We consider the inner workings of the algorithm in the context of a “delayed” attack, where the Byzantine nodes collude to execute an attack after a specific, given, point in the execution (in this case, the first half-epoch). Figure 2(a) presents the results from the perspective of the value of $\|B_i - B_{med}\|$ registered at the central server, for two nodes, an honest one, and a Byzantine one. The value of $\|B_i - B_{med}\|$ increases for all nodes (at a rate of roughly $\sqrt{T}$), but, once the attack starts, the statistic for the Byzantine node grows linearly, leading to fast detection.

**Transient attacks.** Finally, in Figure 2(b) we analyze the behaviour of the algorithm variant allowing for *transient* node faults. Specifically, the algorithm periodically (every 2 epochs) resets the set of *good* nodes to include all nodes, restarting the detection process from scratch. The faulty nodes execute the Variance attack throughout the execution. This explains the zig-zagging pattern of the accuracy, where drops correspond to epochs where the bad nodes are re-enabled, and therefore temporarily manage to shift the gradient mean. (A milder attack would correspond to a single attack epoch, which is easily handled.) **Safeguard** maintains good accuracy: its best Top-1 validation accuracy is 87%; the best alternative method reaches 29% accuracy under this attack.
APPENDIX

A Missing Proofs for Section 3

A.1 Proof of Lemma 3.2

Recall the following, useful inequality.

**Lemma A.1** (Pinelis’ 1994 inequality [23]). Let $X_1, \ldots, X_T \in \mathbb{R}^d$ be a random process satisfying $\mathbb{E}[X_t | X_1, \ldots, X_{t-1}] = 0$ and $\|X_t\| \leq M$. Then, $\Pr \left[ \|X_1 + \cdots + X_T\|^2 > 2 \log(2/\delta) M^2 T \right] \leq \delta$.

**Lemma 3.2** is in fact a direct corollary of the following claim, whose proof is quite classical. Denote by $C = \log(16mT/p)$. Denote by

$$B_i^{(t)} \overset{\text{def}}{=} \frac{\nabla_{0,i}}{|\text{good}_i|} + \cdots + \frac{\nabla_{t-1,i}}{|\text{good}_{t-1}|} \quad \text{and} \quad B_*^{(t)} \overset{\text{def}}{=} \frac{\nabla f(w_0)}{|\text{good}_0|} + \cdots + \frac{\nabla f(w_{t-1})}{|\text{good}_{t-1}|}.$$ 

Recall at iteration $t - 1$, Algorithm 2 computes $\{B_i^{(t)}, \ldots, B_*^{(t)}\}$ as well as some $B_{med}^{(t)} = B_i^{(t)}$ where $i$ is any machine in good$_{-1}$ such that at least half of $j \in [m]$ satisfies $\|B_j^{(t)} - B_i^{(t)}\| \leq 8\sqrt{tC}/m$.

**Claim A.2.** Let $C = \log(16mT/p)$. Then, with probability at least $1 - p/4$, we have

(a) for all $i \in \text{good}$ and $t \in [T]$, $\|B_i^{(t)} - B_*^{(t)}\| \leq 4\sqrt{tC}/m$.

(b) for all $t \in [T]$, each $i \in \text{good}$ is a valid choice for $B_{med}^{(t)} = B_i^{(t)}$.

(c) for all $i \in \text{good}$ and $t \in [T]$, $\|B_i^{(t)} - B_{med}^{(t)}\| \leq 16\sqrt{tC}/m$ and $\|B_*^{(t)} - B_{med}^{(t)}\| \leq 12\sqrt{tC}/m$.

(d) for all $i \in \text{good}$ and $t \in [T]$, we also have $i \in \text{good}_{t+1}$.

(e) $\|\sum_{i \in \text{good}} (B_i^{(t)} - B_*^{(t)})\| \leq O(\sqrt{t \log(T/p)}/\sqrt{m})$.

**Proof of Claim A.2.** We prove by induction. Suppose the statements hold for $t - 1$ and we now move to $t$.

(a) For each $i \in \text{good}$, note $\mathbb{E}[\nabla_{t,i}] = \nabla_i$ and $\|\nabla_{t,i} - \nabla_i\| \leq 1$. Let $X_t = \frac{\nabla_{t,i} - \nabla_i}{|\text{good}_i|}$, so that $\|X_t\| \leq \frac{1}{|\text{good}_i|} \leq \frac{1}{(1 - \alpha)m} \leq \frac{1}{2}$. We can thus apply **Lemma A.1** to the $X_t$ and then take a union bound over all $i \in \text{good}$.

Thus, with probability at least $1 - \frac{2}{p}$ we have $\|B_i^{(t)} - B_*^{(t)}\| \leq 4\sqrt{tC}/m$ for all $i \in \text{good}$. The result follows from a further union bound over $t \in [T]$.

(b) **Claim A.2a** implies for every $i, j \in \text{good}$ we have $\|B_i^{(t)} - B_j^{(t)}\| \leq 8\sqrt{tC}/m$. Therefore each $i \in \text{good}$ is a valid choice for setting $B_{med}^{(t)} = B_i^{(t)}$.

(c) This is a consequence of the previous items and the definition of $B_{med}^{(t)}$.

(d) This is a consequence of the previous item.

(e) We can apply **Lemma A.1** with $\{X_1, X_2, \ldots, X_{[t]}_{\text{good}}\} = \{\nabla_{k,i} - \nabla f(w_{ik})\}_{k \in [t], i \in \text{good}}$. It holds with probability at least $1 - \frac{p}{8T}$ that $\|\sum_{i \in \text{good}} (B_i^{(t)} - B_*^{(t)})\| \leq O(\sqrt{t \log(T/p)}/\sqrt{m})$.

**Proof of Lemma 3.2.** The property good$_{i} \supseteq \text{good}$ is from **Claim A.2d**.

The property $\|\sigma_i\|^2 \leq O(\frac{\log(T/p)}{m})$ is by standard concentration inequalities for sums of bounded random vectors.

The property $\|\sigma_0 + \cdots + \sigma_{t-1}\|^2 \leq O(\frac{T \log(T/p)}{m})$ is from **Claim A.2c**.

The property $\|\Delta_i\| \leq \alpha$ is obvious as we have at most a fraction of the bad machines.

The bound on $\|\Delta_0 + \cdots + \Delta_{t-1}\|^2$ can be derived as follows. For every $i \in [m] \setminus \text{good}$, let $t$ be the last iteration $i$ satisfies $i \in \text{good}_i$. Then, by the triangle inequality,

$$\|B_i^{(t+1)} - B_*^{(t+1)}\| \leq \frac{2}{m} + \|B_i^{(t)} - B_*^{(t)}\|$$
On the other hand, \( t \in \text{good}, \) implies \( \|B^{(t)}_{i} - B^{(t)}_{\text{med}}\| \leq 16\sqrt{tC}/m \) by the algorithm; combining this with \( \|B^{(t)}_{i} - B^{(t)}_{\text{med}}\| \leq 12\sqrt{tC}/m, \) and summing up over all such bad machines \( i \) finishes the proof.

The final two properties follow from standard facts about Gaussian random vectors.

\[ \square \]

### A.2 Proof of Lemma 3.3

**Proof of Lemma 3.3.** Using the Lipschitz smoothness of \( f(\cdot) \), we have

\[
f(w_t) - f(w_{t+1}) \geq \langle \nabla f(w_t), w_t - w_{t+1} \rangle - \frac{1}{2}\|w_t - w_{t+1}\|^2
\]

\[
= \eta \|\nabla f(w_t)\|^2 + \eta \langle \nabla f(w_t), \xi_t \rangle - \frac{1}{2}\|w_t - w_{t+1}\|^2 + \eta \langle \nabla f(w_t), \xi_t \rangle
\]

We first show:

\[
\|w_t - w_{t+1}\|^2 = \eta^2 \|\nabla f(w_t) + \Xi_t - \xi_t\|^2 \leq 3\eta^2 (\|\nabla f(w_t)\|^2 + \|\Xi_t\|^2 + \|\xi_t\|^2)
\]

\[
|\sum_{t=0}^{T-1} \eta \langle \nabla f(w_t), \xi_t \rangle| \leq \eta \left( \sum_{t=0}^{T-1} \|\nabla f(w_t)\|^2 \cdot O(\sqrt{C_1}) \right) \leq \left( 0.05\eta \sum_{t=0}^{T-1} \|\nabla f(w_t)\|^2 \right) + O(C_1\nu^2\eta)
\]

The first follows since \((a+b+c)^2 \leq 3(a^2+b^2+c^2)\) for any \(a, b, c \in \mathbb{R}\), and the second follows from Lemma 3.2 Combining them, and also using that \(\|\Xi_t\|^2 \leq O(C_2), \|\xi_t\|^2 \leq O(d\nu^2C_1)\), and \(\eta \leq 0.01\), we have

\[
f(w_0) - f(w_T) \geq 0.9\eta \sum_{t=0}^{T-1} (\|\nabla f(w_t)\|^2 - O(\eta C_2)) + \eta \sum_{t=0}^{T-1} \langle \nabla f(w_t), \Xi_t \rangle - O(\eta T\nu^2C_1(\eta d + \frac{1}{T})) \quad \text{(A.1)}
\]

For the inner product on the right hand of (A.1), we have that

\[
\eta \sum_{t=0}^{T-1} \langle \nabla f(w_t), \Xi_t \rangle = \eta \sum_{t=0}^{T-1} \left( \sum_{q=0}^{T-1} \langle \nabla f(w_q), \sum_{t=0}^{T-1} \Xi_t \rangle + \frac{\eta}{T} \sum_{q=0}^{T-1} \sum_{t=0}^{T-1} \langle \nabla f(w_t) - \nabla f(w_q), \Xi_t \rangle \right) \quad \text{(A.2)}
\]

For the first term \(\blacklozenge\), we have

\[
|\blacklozenge| \leq \frac{\eta}{T} \sum_{q=0}^{T-1} \left| \sum_{t=0}^{T-1} \langle \nabla f(w_q), \sum_{t=0}^{T-1} \Xi_t \rangle \right| \leq \frac{\eta}{T} \sum_{q=0}^{T-1} \|\nabla f(w_q)\| \cdot \|\sum_{t=0}^{T-1} \Xi_t\|
\]

\[
\leq 0.1\eta \sum_{q=0}^{T-1} \|\nabla f(w_q)\|^2 + \frac{O(\eta)}{T^2} \sum_{q=0}^{T-1} \|\sum_{t=0}^{T-1} \Xi_t\|^2
\]

\[
\leq 0.1\eta \sum_{q=0}^{T-1} \|\nabla f(w_q)\|^2 + O\left(\eta C_2\right)
\]

where the last inequality follows from Lemma 3.2

For the second term \(\blacklozenge\), we have

\[
|\blacklozenge| \leq \frac{\eta}{T} \sum_{q=0}^{T-1} \sum_{t=0}^{T-1} \langle \nabla f(w_t) - \nabla f(w_q), \Xi_t \rangle \leq \frac{\eta}{T} \sum_{q=0}^{T-1} \sum_{t=0}^{T-1} \langle \nabla^2 f(w_0)(w_t - w_q), \Xi_t \rangle
\]

\[
+ \frac{\eta}{T} \sum_{q=0}^{T-1} \sum_{t=0}^{T-1} (\|w_t - w_0\| + \|w_q - w_0\|)\|w_t - w_q\| \Xi_t \|
\]

\[
\leq \frac{\eta}{T} \sum_{q=0}^{T-1} \sum_{t=0}^{T-1} \langle \nabla^2 f(w_0)(w_t - w_q), \Xi_t \rangle
\]

\[
+ \frac{\eta}{T} \sum_{q=0}^{T-1} \sum_{t=0}^{T-1} (\|w_t - w_0\| + \|w_q - w_0\|)\|w_t - w_q\| \Xi_t \|
\]

\[
\leq \frac{\eta}{T} \sum_{q=0}^{T-1} \sum_{t=0}^{T-1} \langle \nabla^2 f(w_0)(w_t - w_q), \Xi_t \rangle
\]

\[
+ \frac{\eta}{T} \sum_{q=0}^{T-1} \sum_{t=0}^{T-1} (\|w_t - w_0\| + \|w_q - w_0\|)\|w_t - w_q\| \Xi_t \|
\]
Using \( \|w_t - w_q\| \leq \|w_t - w_0\| + \|w_q - w_0\| \), one can derive

\[
\Diamond \leq \frac{\eta}{T} \sum_{t=0}^{T-1} \sum_{q=0}^{T-1} (\|w_t - w_0\| + \|w_q - w_0\|)^2 \cdot O(\sqrt{C_2})
\]

\[
\leq \eta \sum_{t=0}^{T-1} \|w_t - w_0\|^2 \cdot O(\sqrt{C_2})
\]

\[
\leq \eta^3 \sum_{t=0}^{T-1} \|\nabla f(w_0) + \cdots + \nabla f(w_{t-1}) + \Xi_0 + \cdots + \Xi_{t-1} + \xi_0 + \cdots + \xi_{t-1}\|^2 \cdot O(\sqrt{C_2})
\]

\[
\leq O(\sqrt{C_2} \eta^3 T^2) \sum_{t=0}^{T-1} \|\nabla f(w_t)\|^2 + O(\sqrt{C_2} \eta^3 T^2) + O(\eta^3 \nu^2 T^2 dC_1 \sqrt{C_2})
\]

As for \( \Diamond \),

\[
\left|\sum_{t=0}^{T-1} \langle \nabla^2 f(w_0)(w_t - w_q), \Xi_t \rangle\right| \leq \left|\sum_{t=q+1}^{T-1} \langle \nabla^2 f(w_0)(w_t - w_q), \Xi_t \rangle\right| + \left|\sum_{t=0}^{q-1} \langle \nabla^2 f(w_0)(w_t - w_q), \Xi_t \rangle\right|
\]

For the first term (and the second term is analogous), we have

\[
\left|\sum_{t=q+1}^{T-1} \langle \nabla^2 f(w_0)(w_t - w_q), \Xi_t \rangle\right|
\]

\[
= \eta \left|\sum_{t=q+1}^{T-1} \langle \nabla^2 f(w_0)(\nabla f(w_q) + \cdots \nabla f(w_{t-1}) + \Xi_q + \cdots + \xi_{t-1}), \Xi_t \rangle\right|
\]

\[
\leq \eta \left|\sum_{t=q+1}^{T-1} \langle \nabla^2 f(w_0)(\xi_q + \cdots + \xi_{t-1}), \Xi_t \rangle\right| + \eta \left|\sum_{t=q+1}^{T-1} \langle \nabla^2 f(w_0)(\Xi_q + \cdots + \Xi_{t-1}), \Xi_t \rangle\right|
\]

\[
\leq \eta \cdot O(\sqrt{dv^2 TC_1} \cdot \sqrt{TC_2}) + \eta \sum_{t=q}^{T-2} \|\nabla f(w_t)\| \|\Xi_{t+1} + \cdots + \Xi_{T-1}\| + \frac{\eta}{2} \|\Xi_q + \cdots + \Xi_{T-1}\|^2
\]

\[
\leq O(\eta \sqrt{TC_2}) \sum_{t=0}^{T-1} \|\nabla f(w_t)\| + O(T\eta C_2 + T\eta \nu^2 dC_1).
\]

Above, inequality \( \Box \) uses \( \|\Xi_0 + \cdots + \Xi_t\| \leq O(\sqrt{TC_2}) \) for \( C_2 = \alpha^2 \log \frac{mT}{p} + \frac{\log(T/p)}{m} \) (see Lemma 3.2) and a delicate application of Azuma’s inequality that we state at the end of this subsection (see Proposition 3.4); Inequality \( \bigcirc \) uses Young’s inequality and Lemma 3.2.

Putting this back to the formula of \( \Diamond \), we have

\[
\Diamond \leq O(\eta^2 \sqrt{TC_2}) \sum_{t=0}^{T-1} \|\nabla f(w_t)\| + O(T\eta^2 C_2 + T\eta^2 \nu^2 dC_1)
\]

\[
\leq 0.1 \eta \sum_{t=0}^{T-1} \|\nabla f(w_t)\|^2 + O(\eta^3 T^2 C_2 + T\eta^2 C_2 + T\eta^2 \nu^2 dC_1)
\]

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Finally, putting ◇ and ▽ back to ♠, and putting ♠ and ♠ back to (A.2) and (A.1), we have
\[
f(w_0) - f(w_T) \geq \eta \sqrt{d} \log(T) \sum_{t=0}^{T-1} \|\nabla f(w_t)\|^2
\]
\[\quad - C_2 \cdot O(\eta^2 d + T^2 \eta^2) - C_1 \cdot O(\eta T^2)\]
together with \(T = \frac{1}{100(1+\sqrt{C_2})}\) and \(\eta \leq 0.01\min\{1, \frac{1}{C_2}\}\), we have
\[
f(w_0) - f(w_T) \geq \eta \sqrt{d} \log(T) \sum_{t=0}^{T-1} \|\nabla f(w_t)\|^2 \geq C_2 \cdot O(\eta^2 d + T^2 \eta^2) - C_1 \cdot O(\eta T^2)\)
\[\quad - \eta \cdot O(C_2 + (C_2)^{1.5}) - O(\eta T^2)\]

\[\square\]

A.3 Proof of Proposition 3.4

Proposition 3.4. Fix the dimension parameter \(d \geq 1\). Suppose \(\xi_0, \ldots, \xi_T \in \mathbb{R}^d\) are i.i.d. drawn from \(\mathcal{N}(0, I)\), and that \(\Delta_0, \ldots, \Delta_{T-1}\) are arbitrary vectors in \(\mathbb{R}^d\). Here, each vector \(\Delta_t\) with \(t = 1, \ldots, T-1\) can depend on \(\xi_0, \ldots, \xi_{t-1}\) but not on \(\xi_t, \ldots, \xi_{T-1}\). Suppose that these vectors satisfy \(\|\Delta_1 + \cdots + \Delta_t\|^2 \leq \mathcal{F}\) for every \(t = 1, \ldots, T-1\). Then, with probability at least \(1 - p\),
\[
\left|\sum_{t=1}^{T-1} \langle \xi_0 + \cdots + \xi_{t-1}, \Delta_t \rangle\right| \leq O(\sqrt{dT\mathcal{F} \log(T/p)})\,.
\]

Proof of Proposition 3.4. Using the identity formula\footnote{We thank an anonymous reviewer on openreview who pointed out this simpler proof to us.}
\[
\sum_{t=1}^{T-1} \langle \xi_0 + \cdots + \xi_{t-1}, \Delta_t \rangle = \left(\sum_{t=0}^{T-2} \xi_t^T \Delta_t\right) - \sum_{t=1}^{T-1} \langle \xi_t, \Delta_1 + \cdots + \Delta_t \rangle
\]
we have
\[
\left|\sum_{t=1}^{T-1} \langle \xi_0 + \cdots + \xi_{t-1}, \Delta_t \rangle\right| \leq \left|\sum_{t=0}^{T-2} \xi_t\right| + \left|\sum_{t=1}^{T-1} \langle \xi_t, \Delta_1 + \cdots + \Delta_t \rangle\right|.
\]
where the last inequality uses \(\|\xi_0 + \cdots + \xi_{t-2}\| \leq O(\sqrt{dT\mathcal{F} \log(T/p)})\) with probability at least \(1 - p/2\). Furthermore, we note that \(\xi_t\) is independent of \(\xi_0, \ldots, \xi_{t-1}, \Delta_1, \ldots, \Delta_t\) and \(\mathbb{E}[\xi_t] = 0\). Therefore, letting \(S_t = \langle \xi_t, \Delta_1 + \cdots + \Delta_t \rangle\), we have \(\mathbb{E}[S_t|\xi_0, \ldots, \xi_{t-1}] = 0\); furthermore, with probability at least \(1 - p/2\), it satisfies \(|S_t| \leq O(\sqrt{dT\mathcal{F} \log(T/p)})\) for every \(t\). Finally, by Azuma’s inequality, we have
\[
\left|\sum_{t=1}^{T-1} \langle \xi_t, \Delta_1 + \cdots + \Delta_t \rangle\right| \leq O(\sqrt{dT\mathcal{F} \log(T/p)})\,.
\]

A.4 Proof of Lemma 3.6

Proof of Lemma 3.6. Let us denote by \(r_t = \frac{\|\xi_t^T \Delta_t\|}{2} = -\frac{\|\xi_t^T \Delta_t\|^2}{4}\) and we know \(r_t \sim \mathcal{N}(0, \frac{\mathcal{F}^2}{4})\). We can write
\[
w_{t+1}^a - w_t^a = \eta r_t^\theta e_t + w_{t+1}^b - w_t^b - \eta (\nabla f(w_t^a) - \nabla f(w_t^b)) - \eta (\Xi_t^a - \Xi_t^b)
\]
Using the second-order smoothness, we have
\[
\nabla f(w_t^a) - \nabla f(w_t^b) = \int_0^1 \nabla^2 f(w_t^a + \tau (w_t^b - w_t^a))(w_t^a - w_t^b)d\tau
\]
\[\quad = \nabla^2 f(w_0) \cdot (w_t^a - w_t^b) + \theta_t
\]
for some vector $||\theta_t|| \leq \max(\|w^a_0 - w^b_0\|, \|w^a_0 - w^b_0\|) \cdot \|w^a_0 - w^b_0\|$. Therefore, we have
\[ w^a_{t+1} - w^b_{t+1} = \eta_t e_1 + (I - \eta \nabla^2 f(w_0))(w^a_t - w^b_t) - \eta(\Xi^a_t - \Xi^b_t + \theta_t) \]
Now, giving $\psi_0 = \bar{\psi}_0 = 0$, imagine two sequences
- $\psi_{t+1} = \eta_t e_1 + (I - \eta \nabla^2 f(w_0))\psi_t$ and
- $\bar{\psi}_{t+1} = \eta_t e_1 + (I - \eta \nabla^2 f(w_0))\bar{\psi}_t - \eta(\Xi^a_t - \Xi^b_t + \theta_t) = w^b_{t+1} - w^a_{t+1}
We will inductively prove $\|\psi_t - \bar{\psi}_t\| \leq \frac{1}{2}\|\psi_t\|$. On one hand, it is easy to see that $\psi_t$ is zero except in the first coordinate, in which it behaves as a Gaussian with zero mean and variance $\sum_{k=0}^{t-1}(1 + \eta \delta)^{2k} \cdot \frac{\Xi^2}{4^2} = \Theta\left(\frac{(1 + \eta \delta)^{2t}}{\eta \delta} \cdot \eta^2 \nu^2\right)$. By Gaussian tail bounds, we know that
- with probability at least 0.99, it satisfies $||\psi_t|| \leq O\left(\frac{\nu(1 + \eta \delta)^{t}}{\sqrt{\eta \delta}}\right)$ for every $t$
- with probability at least 0.99, it satisfies $\|\psi_T\| \geq \frac{1}{1000} (\frac{\nu(1 + \eta \delta)^{T}}{\sqrt{\eta \delta}})
In the rest of the proof, we condition on this event happens. We prove towards contradiction by assuming $\|w^a_0 - w^b_0\| \leq R$ and $\|w^b_0 - w^a_0\| \leq R$ for all $t \in [T]$.
We will inductively prove that $\|\psi_t - \bar{\psi}_t\| \leq \frac{1}{1000} (\frac{\nu(1 + \eta \delta)^{t}}{\sqrt{\eta \delta}})$. We calculate the difference
\[ \psi_t - \bar{\psi}_t = \eta \sum_{i=0}^{t-1} (I - \eta \nabla^2 f(w_0))^{t-1-i} (\Xi^a_t - \Xi^b_t + \theta_t) \]
Let $g = \frac{\psi_t - \bar{\psi}_t}{\|\psi_t - \bar{\psi}_t\|}$, then we can inner-product the above equation by vector $g$, which gives
\[ \|\psi_t - \bar{\psi}_t\| = \eta \sum_{i=0}^{t-1} \langle \Xi^a_t - \Xi^b_t + \theta_t, (I - \eta \nabla^2 f(w_0))^{t-1-i} g \rangle \]
\[ \leq \eta \sum_{i=0}^{t-1} \left( \langle \Xi^a_t - \Xi^b_t, (I - \eta \nabla^2 f(w_0))^{t-1-i} g \rangle + R \cdot O\left(\frac{\nu(1 + \eta \delta)^{i}}{\sqrt{\eta \delta}}\right) \cdot (1 + \eta \delta)^{t-1-i} \right) \]
\[ \leq \eta \sum_{i=0}^{t-1} \langle \Xi^a_t - \Xi^b_t, (I - \eta \nabla^2 f(w_0))^{t-1-i} g \rangle + O(R \eta T \frac{\nu C_1}{\sqrt{\eta \delta}} \nu(1 + \eta \delta)^{i}) \]
where the inequality uses $||\theta_t|| \leq R : \|\psi_t\| \leq R : (\|\psi_t\| + \|\psi_t - \bar{\psi}_t\|), \| (I - \eta \nabla^2 f(w_0))^{t-1-i} g \| \leq (1 + \eta \delta)^{t-1-i}$, and the inductive assumption. Let us call $M = (I - \eta \nabla^2 f(w_0))$, and focus on
\[ \left| \sum_{i=0}^{t-1} \langle \Xi^a_t, (I - \eta \nabla^2 f(w_0))^{t-1-i} g \rangle \right| \]
\[ = \|\Xi^a_t + \cdots + \Xi^a_t \cdot g\| + \sum_{i=0}^{t-2} \|\Xi^a_t + \cdots + \Xi^a_t \cdot M^{t-1-i} g - M^{t-2-i} g\| \]
\[ \leq \|\Xi^a_t + \cdots + \Xi^a_t \cdot g\| + \sum_{i=0}^{t-2} \|\Xi^a_t + \cdots + \Xi^a_t \cdot M^{t-1-i} g - M^{t-2-i} g\| \]
\[ \leq O(\sqrt{TC_2}) \left( \|g\| + \sum_{i=0}^{t-2} \|M^{t-1-i} g - M^{t-2-i} g\| \right) \] (using Lemma 3.2)
\[ \leq O(\sqrt{TC_2} \cdot (1 + \eta \delta)^{t-1}) \]
Together, we have
\[ \|\psi_t - \bar{\psi}_t\| \leq O(\eta \sqrt{TC_2}) \cdot (1 + \eta \delta)^t + O(R \eta T \frac{\nu C_1}{\sqrt{\eta \delta}} \nu(1 + \eta \delta)^{t}) \]
Under our assumption, we have \[ \| \psi_t - \bar{\psi}_t \| < \frac{1}{2000} \left( \frac{\sqrt{m}(1 + \eta \delta')}{\sqrt{3}} \right) \] and therefore \[ \| \bar{\psi}_T \| - \| \psi_T - \bar{\psi}_T \| \geq \frac{1}{2000} \left( \frac{\sqrt{m}(1 + \eta \delta')}{{\sqrt{3}}} \right). \] Thus, within \( T \) iterations, we have \( \| \bar{\psi}_T \| > R \) and this gives a contradiction. \( \square \)

**B** Final: Double Safe Guard

We now come to our final Algorithm 3 which is our perturbed SGD algorithm with two safeguards. The two safeguard algorithm naturally divides itself into epochs, each consisting of \( T_1 \) iterations. We will demonstrate that within most epochs, we make good progress. Thus, consider some iterate \( x_{mT_1} \), for some \( m < T/T_1 \).

Our goal will be to argue that we make good function value progress by iterate \( x_{(m+1)T_1} \), and that we do not settle into any saddle points. To slightly simplify notation, let \( w_0 = x_{mT_1} \), and let the sequence of iterates be \( w_0, \ldots, w_{T_1-1} \), so that \( w_{T_1-1} = x_{(m+1)T_1-1} \). For completeness’ sake we rewrite this as Algorithm 1.

**Algorithm 3** Perturbed SGD with double safe guard (for analysis purpose)

| Input: | \( w_0 \in \mathbb{R}^d \), set \( \text{good}_0 \supseteq \text{good} \), rate \( \eta > 0 \), lengths \( T_1 \geq T_0 \geq 1 \), threshold \( \Upsilon_1 > \Upsilon_0 > 0 \); |
|---|---|
| for | \( t \leftarrow 0 \) to \( T_1 - 1 \) do |
| for each | \( i \in \text{good} \) do |
| receive | \( \bar{\nabla}_{t,i} \in \mathbb{R}^d \) from machine \( i \); |
| \( A_i \leftarrow \sum_{k=0}^{t} \bar{\nabla}_{k,i} \) and \( B_i \leftarrow \sum_{k=\text{last}}^{t} \bar{\nabla}_{k,i} \); |
| end for |
| \( A_{\text{med}} \leftarrow A_i \) where \( i \in \text{good} \) is any machine s.t. \( \{ j \in \text{good} : \| A_j - A_i \| \leq \Upsilon_1 \} \geq m/2 \); |
| \( B_{\text{med}} \leftarrow B_j \) where \( i \in \text{good} \) is any machine s.t. \( \{ j \in \text{good} : \| B_j - B_i \| \leq \Upsilon_0 \} \geq m/2 \); |
| \( \text{good}_{t+1} \leftarrow \{ i \in \text{good} : \| A_i - A_{\text{med}} \| \leq 2\Upsilon_1 \wedge \| B_t - B_{\text{med}} \| \leq 2\Upsilon_0 \} \); |
| \( w_{t+1} = w_t - \eta \left( \bar{\xi}_t + \frac{1}{\| \text{good}_t \|} \sum_{i \in \text{good}_t} \bar{\nabla}_{t,i} \right) \); |
| end for |

Our main result is the following theorem.

**Theorem B.1.** Let \( C_3 = \alpha^2 + \frac{1}{m} \). Suppose we pick parameters \( p, \delta \in (0,1) \), \( \eta \leq \tilde{O}(\frac{\delta}{C_3}) \), \( \nu^2 = \tilde{O}(C_3) \), \( T_0 = \tilde{\Theta}(\frac{1}{\eta}) \), \( T_1 = \tilde{\Theta}(\frac{1}{\eta^p}) \geq T_0 \), \( \Upsilon_1 = \tilde{\Theta}(\sqrt{T_1}) \), and \( \Upsilon_0 = \tilde{\Theta}(\sqrt{T_0}) \). Then, starting from \( w_0 \),

(a) with probability at least 1 - \( p \) we have

\[
  f(w_0) - f(w_{T_1}) \geq 0.7\eta \sum_{t=0}^{T_1-1} \left( \| \nabla f(w_t) \|^2 - \tilde{O}(\eta C_3d) \right).
\]

(b) As long as \( \| w_t - w_0 \| \geq R \) for some \( t \in \{1, 2, \ldots, T_1\} \) and for \( R = \tilde{\Theta}(\delta) \leq \frac{\delta}{2} \), then with probability at least 1 - \( p \) we have then

\[
  f(w_0) - f(w_{T_1}) \geq 0.5\eta \sum_{t=0}^{T_1-1} \left( -\tilde{O}(\eta C_3d) \right) + \tilde{\Omega}(\delta^3)
\]

(c) if \( \lambda_{\min}(\nabla^2 f(w_0)) \leq -\delta \), we also have with probability at least 0.45,

\[
  f(w_0) - f(w_{T_1}) \geq 0.5\eta \sum_{t=0}^{T_1-1} \left( -\tilde{O}(\eta C_3d) \right) + \tilde{\Omega}(\delta^3)
\]

**B.1** Why Theorem B.1 Implies Theorem 2.3

Using the parameter choice \( \eta = \tilde{\Theta}(\frac{\delta^2}{C_3d}) \) from Theorem 2.3 we know \( \tilde{O}(\eta C_3d) \leq 0.1\epsilon^2 \). We claim two things:
• For at least 90% of the epochs, they must satisfy (denoting by \(w_0\) and \(w_{T_1}\) the beginning and ending points of this epoch)

\[
    f(w_0) - f(w_{T_1}) \leq 20 \frac{f(x_0) - \min f(x)}{T/T_1} \leq \varepsilon^{1.5}
\]

The last inequality uses our choice of \(T\) and \(\delta = \Theta(\sqrt{\varepsilon})\).

The reason for this is by way of contradiction. Suppose for at least 10% of the epochs it satisfies \(f(w_0) - f(w_{T_1}) > 20 \frac{f(x_0) - \min f(x)}{T/T_1}\), then, for the remainder of the epochs, they must at least satisfy \(f(w_0) - f(w_{T_1}) \geq -0.7\eta T_1 \cdot 0.1\varepsilon^2\). Summing over all the epochs, we shall obtain \(f(x_0) - f(x_T) > f(x_0) - \min f(x)\) but this gives a contradiction.

• For at least 40% of the epochs, they must satisfy the three properties from Theorem B.1.

In particular, for at least 30% of the epochs, they must satisfy both. Since \(\varepsilon^{1.5}\) is so small that

\[
    \varepsilon^{1.5} \geq f(w_0) - f(w_{T_1}) \geq 0.5\eta \sum_{t=0}^{T_1-1} \left(-\tilde{O}(\eta C_3d)\right) + \Omega(\delta^3) \geq \tilde{O}(\delta^3) - 0.05\eta T_1\varepsilon^2
\]

would give a contradiction (for instance, one can choose \(\delta\) to be slightly larger than \(\sqrt{\varepsilon}\) by some log factors), this means, for those 30% of the epochs, they must satisfy:

- \(\varepsilon^{1.5} \geq 0.7\eta \sum_{t=0}^{T_1-1} (\|\nabla f(w_t)\|^2 - 0.1\varepsilon^2)\)
- \(\|w_t - w_0\| \leq \frac{\delta}{\varepsilon}\) for every \(t = 1, 2, \ldots, T_1\), and
- \(\nabla^2 f(w_0) \succeq -\delta I\)

The latter two properties together implies \(\nabla^2 f(w_t) \succeq -\frac{\delta}{\varepsilon} I\) for every \(t = 1, 2, \ldots, T_1\) (by the second-order smoothness). The first property implies for at least 90% of the iterations \(t\) in this epoch, they must satisfy \(\|\nabla f(x)\| \leq \varepsilon\). This finishes the proof of Theorem 2.3.

### B.2 Proof of Theorem B.1

We first have the following lemma

**Lemma B.2** (double safe guard). In Algorithm 3, suppose \(\Sigma_1 = 8\sqrt{T_1 \log(16mT_1/p)}\) and \(\Sigma_0 = 8\sqrt{T_0 \log(16mT_1/p)}\). Then, with probability at least \(1 - p/2\), for every \(t = 0, \ldots, T_1 - 1\),

- \(\sigma_t g_t \succeq \text{good}\).
- \(\|\sigma_t\|^2 \leq O\left(\frac{\log(T_1/p)}{m}\right), \|\Delta_t\|^2 \leq \alpha^2, \|\xi_t\|^2 \leq O(\nu^2 d \log(T_1/p))\),
- \(\|\sigma_0 + \cdots + \sigma_{t-1}\|^2 \leq O\left(\frac{T_1 \log(T_1/p)}{m}\right), \|\sigma_{last} + \cdots + \sigma_{t-1}\|^2 \leq O\left(\frac{T_1 \log(T_1/p)}{m}\right)\)
- \(\|\Delta_0 + \cdots + \Delta_{t-1}\|^2 \leq O(\alpha^2 T_1 \log(mT_1/p))\) and \(\|\Delta_{last} + \cdots + \Delta_{t-1}\|^2 \leq O(\alpha^2 T_0 \log(mT_1/p))\)
- \(\|\xi_0 + \cdots + \xi_{t-1}\|^2 \leq O(\nu^2 d T_1 \log(T_1/p))\) and \(\|\xi_{last} + \cdots + \xi_{t-1}\|^2 \leq O(\nu^2 d T_0 \log(T_1/p))\).

We call this probabilistic event \(\text{Event}_{w_0}^{\text{double}}(w_0)\) and \(\Pr[\text{Event}_{w_0}^{\text{double}}(w_0)] \geq 1 - p/2\).

The proof is a direct corollary of Lemma 3.2, by combining events \(\text{Event}_{w_0}^{\text{single}}(w_0), \text{Event}_{w_0}^{\text{single}}(w_0), \text{Event}_{w_0}^{\text{single}}(w_0), \text{Event}_{w_0}^{\text{single}}(w_2T_0)\) and so on. The next lemma is a simple corollary by repeatedly applying Lemma 3.3. It proves Theorem B.1a.

**Lemma B.3** (modified from Lemma 3.3). Denote by \(C_1 = \log(T_1/p)\) and \(C_2 = \alpha^2 \log mT_1/p + \log(T_1/p)/m\). Suppose \(\eta \leq 0.01 \min\{1, \frac{1}{C_2}\}, T_0 = \frac{1}{1000\eta(1 + \sqrt{C_2})}\) and \(T_1 \geq T_0\). We start from \(w_0\) and apply Algorithm 3. Under event \(\text{Event}_{w_0}^{\text{double}}(w_0)\), it satisfies

\[
    f(w_0) - f(w_{T_1}) \geq 0.7\eta \sum_{t=0}^{T_1-1} \left(\|\nabla f(w_t)\|^2 - \eta \cdot O(C_2 + (C_2)^1.5) - O(C_1 \nu^2 \eta (d + \sqrt{C_2}))\right)
\]

The next lemma can be easily derived from Lemma 3.6.
Lemma B.4 (modified from Lemma 3.6). Suppose
\[ R = \Theta\left(\frac{\delta}{\sqrt{C_1 \log \left(\frac{\delta^3}{\eta C_2}\right)}}\right) \quad \text{and} \quad \nu^2 = \Theta\left(C_2 \log \frac{\delta^3}{\eta C_2}\right) \]
Suppose \( \eta \leq 0.01 \min\{1, \frac{\delta^3}{C_2}\} \), \( T_0 = \frac{1}{1000(1+\sqrt{C_2})} \) and \( T_1 = \Theta\left(\frac{1}{\eta\nu^2} \log \frac{\delta^3}{\eta C_2}\right) \geq T_0 \). Let \( w_0 \in \mathbb{R}^d \) be any point in the space and suppose \( \lambda_{\text{min}}(\nabla^2 f(w_0)) \leq -\delta \) for some \( \delta \geq 0 \). Given two coupled sequences defined as before, under events \( \text{Event}^{\text{double}}_{T_1,T_0}(w^a_0) \) and \( \text{Event}^{\text{double}}_{T_1,T_0}(w^b_0) \), we have with probability at least 0.98
\[
\max \{ f(w^a_0) - f(w^a_{T_1}), f(w^b_0) - f(w^b_{T_1}) \} \\
\geq 0.5\eta \sum_{t=0}^{T_1-1} \left( -\eta \cdot O(C_2 + (C_2)^{1.5}) - O(C_1 \nu^2 \eta(d + \sqrt{C_2})) \right) + \Omega\left(\frac{\delta^3}{C_1 \log \frac{\delta^3}{\eta C_2}}\right)
\]
Lemma B.4 directly proves the second half of Theorem B.1c, because given two coupled sequences with the same marginal distribution, we have
\[
\Pr[f(w^a_0) - f(w^b_{T_1}) \geq X] \geq \frac{1}{2} \Pr[\max \{ f(w^a_0) - f(w^a_{T_1}), f(w^b_0) - f(w^b_{T_1}) \} \geq X]
\]
Proof of Lemma B.4. Our choice on \( r \) and \( R \) satisfy the requirement of Lemma 3.6. Suppose without loss of generality that the \( w^a_t \) sequence leaves \( w_0 \) by more than \( R \). Let \( T^a_t \) be the first iteration \( t \leq T_1 \) in which \( \|w^a_t - w^a_0\| \geq R \).
\[
\|w^a_{T^a_t} - w^a_0\|^2 = \eta^2 \|\nabla f(w^a_0) + \cdots + \nabla f(w^a_{T^a_t-1}) + \xi^a_0 + \cdots + \xi^a_{T^a_t-1}\|^2 \\
\leq O(\eta^2 T_1) \sum_{t=0}^{T^a_t-1} \|\nabla f(w^a_t)\|^2 + O(C_2 \eta^2 T_1) + O(C_1 \eta^2 \nu^2 T_1 d)
\]
Combining this with Lemma B.3, we have
\[
f(w^a_0) - f(w^a_{T^a_t}) \geq 0.5\eta \sum_{t=0}^{T^a_t-1} \left( \|\nabla f(w^a_t)\|^2 - \eta \cdot O(C_2 + (C_2)^{1.5}) - O(C_1 \nu^2 \eta(d + \sqrt{C_2})) \right) + \frac{\|w^a_{T^a_t} - w^a_0\|^2}{100\eta T_1} \\
\geq 0.5\eta \sum_{t=0}^{T^a_t-1} \left( \|\nabla f(w^a_t)\|^2 - \eta \cdot O(C_2 + (C_2)^{1.5}) - O(C_1 \nu^2 \eta(d + \sqrt{C_2})) \right) + \frac{R^2}{100\eta T_1}
\]
Combining this with Lemma B.3 again but for the remainder iterations, we have
\[
f(w^a_0) - f(w^a_{T_1}) \geq 0.5\eta \sum_{t=0}^{T_1-1} \left( \|\nabla f(w^a_t)\|^2 - \eta \cdot O(C_2 + (C_2)^{1.5}) - O(C_1 \nu^2 \eta(d + \sqrt{C_2})) \right) + \frac{R^2}{100\eta T_1}
\]
In fact, the above same proof of Lemma B.4 also implies Theorem B.1b. These together finish the proof of Theorem B.1.

C More on Experiments

We conduct experiments on training a deep residual network (ResNet-20) [14] on the CIFAR-10 image classification task [18].

C.1 Setting and Implemented Methods

In all of our experiments, we use 10 workers and the mini-batch size (per worker) is 8. We run all algorithms for 200 epochs, with initial learning rate \( \eta = 0.1 \), but the learning rate decreases by a factor of 10 on epochs 80, 120, and 160.
The ideal baseline is gradient “mean without attacks”, and we compare against (Naive) Mean, Geometric Median \[10\], Coordinate-wise Median \[33,34\], Krum \[7\], and Zeno \[30\] with attacks. We set \(\alpha = 0.4\) so there are 4 Byzantine workers. (This exceeds the fault-tolerance of Krum, and so we also tested Krum with only 3 Byzantine workers.) We formally define those prior works as follows.

**Definition C.1** (GeoMed \[10\]). The geometric median of \(\{y_1,\ldots,y_m\}\), denoted by \(\text{geo\_med}\{y_1,\ldots,y_m\}\), is

\[
\text{geo\_med}\{y_1,\ldots,y_m\} \overset{\text{def}}{=} \arg\min_{y \in \mathbb{R}^d} \sum_{i=1}^{m} \|y - y_i\|
\]

In our experiments, we choose the geometric median from set \(\{y_1,\ldots,y_m\}\).

**Definition C.2** (coordinate-wise median \[33,34\]). Coordinate-wise median \(g = \text{med}\{y_1,\ldots,y_m\}\) is defined as a vector with its \(k\)-th coordinate being \(g[k] = \text{med}\{y_1[k],\ldots,y_m[k]\}\) for each \(k \in [d]\), where \(\text{med}\) is the usual (one-dimensional) median.

**Definition C.3** (Krum \[7\]).

\[
KR\{y_1,\ldots,y_m\} \overset{\text{def}}{=} y_k \quad \text{where} \quad k = \arg\min_{i \in [m]} \sum_{i \to j} \|y_i - y_j\|^2
\]

and \(i \to j\) is the indices of the \(m - b - 2\) nearest neighbours of \(y_i\) in \(\{y_1,\ldots,y_m\}\) by Euclidean distances.

Note that Krum requires \(2b + 2 < m\). So, we have also repeated the experiments for Krum with 3 Byzantine workers (out of 10 workers) to be more fair.

**Definition C.4** (Zeno \[30\]).

\[
\text{Zeno}_b\{y_1,\ldots,y_m\} = \frac{1}{m - b} \sum_{i = 1}^{m-b} \tilde{y}(i)
\]

where \(\{\tilde{y}(i): i \in [m]\}\) are the gradient estimators with the \(m - b\) highest “scores”, and the so-called stochastic descendant score for any gradient estimator \(u\), based on the current parameter \(x\), learning rate \(\eta\), and a constant weight \(\rho > 0\), is defined as:

\[
\text{Score}_{\eta,\rho}(u, x) = f_r(x) - f_r(x - \eta u) - \rho\|u\|^2
\]

\(f_r(x) - f_r(x - \eta u)\) is the estimated descendant of the loss function and \(\rho\|u\|^2\) is the magnitude of the update.

**Safeguard SGD.** To provide the strongest comparison against prior works, we have made universal choices for our safeguard window sizes (across all attackers), and provide an automatic way to select the safeguard thresholds.

Specifically, for our algorithm with a single safeguard we have used a universal window size \(T = 3\) epochs, and for our algorithm with double safeguards we have used window sizes \(T_0 = 1\) epoch and \(T_1 = 6\) epochs.

As for the procedure to tune thresholds, for each attacker, we pre-run the experiment for 20 epochs with some sufficiently large pre-chosen threshold value (e.g. \(\Sigma = 100\) or \(\Sigma_0 = 100\) and \(\Sigma_1 = 100\)), and then automatically select a good final threshold. Let us describe this process in the case of a single safeguard (and double safeguard is similar). In each iteration, the master examines \(\{B_i | i \in \text{good}\}\) to determine some minimal \(\Sigma_{\text{temp}}\): it satisfies that at least half of the workers satisfy \(\|B_i - B_{\text{med}}\| \leq \Sigma_{\text{temp}}\) and there exists at least one worker \(j \in \text{good}\) such that \(\|B_j - B_{\text{med}}\| > 2\Sigma_{\text{temp}}\). Then, \(\Sigma_{\text{temp}}\) is added to the temporary threshold list (after rounding to the first decimal point). In the end, the automated process will recommend the most frequently repeated items in the temporary threshold list, as the final choice of threshold. We emphasize that this procedure only requires knowing the \(\alpha < 0.5\) upper bound on the number/fraction of possibly faulty nodes.

**C.2 Experiment Results by Attacks**

**C.2.1 Sign-Flipping Attack**

Recall that, in a sign-flipping attack, each Byzantine worker sends the negative gradient to the master. The results for this attack are shown in Figure 3. From the plots, one can see that our single and double
safe-guard algorithms *both* outperform prior works, even though some of these algorithms make additional assumptions, and closely track the performance of the ideal baseline (without failures). Under this attacker, single and double safeguard algorithms perform similarly.

(Parameters: \( m = 10, \alpha = 0.4 \). Single safe-guard threshold \( \mathcal{T} = 2 \), window size \( T = 3 \) epochs, variance \( \nu^2 = 10^{-8} \); double safe-guard thresholds \( \mathcal{T}_0 = 1, \mathcal{T}_1 = 2 \) and window sizes \( T_0 = 1 \) epoch, \( T_1 = 6 \) epochs, variance \( \nu^2 = 10^{-8} \). Zeno Batch size \( n_r = 4 \) and constant weight \( \rho = 0.0005 \).)

C.2.2 Delayed-Gradient Attack

Recall that, in a delayed-gradient attack, each Byzantine worker sends an old gradient to the master. In our experiments, the delay is of \( D = 1000 \) iterations. The results are shown in Figure 4. From the plots, one can see that our single and double safe-guard algorithms *both* outperform prior works, and closely track the performance of the ideal baseline (without failures). Under this attacker, single and double safeguard algorithms perform similarly.

(Parameters: \( m = 10, \alpha = 0.4 \). Single safe-guard threshold \( \mathcal{T} = 2 \), window size \( T = 3 \) epochs, variance \( \nu^2 = 10^{-8} \); double safe-guard thresholds \( \mathcal{T}_0 = 2, \mathcal{T}_1 = 7 \) and window sizes \( T_0 = 1 \) epoch, \( T_1 = 6 \) epochs, variance \( \nu^2 = 10^{-8} \). Zeno Batch size \( n_r = 4 \) and constant weight \( \rho = 0.0005 \).)
C.2.3 Label-Flipping Attack

Recall that, in the label-flipping attack, each Byzantine worker computes its gradient based on the cross-entropy loss with flipped labels: for CIFAR-10, label $\ell \in \{0, ..., 9\}$ is flipped to $9 - \ell$.

The results are shown in Figure 5. From the plots, one can see that our double safe-guard algorithm outperforms prior works, and closely tracks the performance of the ideal baseline (without failures). Under this attacker, the double safeguard algorithm performs slightly better than single safeguard.

(Parameters: $m = 10$, $\alpha = 0.4$. Single safe-guard threshold $T = 0.4$, window size $T = 3$ epochs, variance $\nu^2 = 10^{-8}$; double safe-guard thresholds $T_0 = 2$, $T_1 = 7$ and window sizes $T_0 = 1$ epoch, $T_1 = 6$ epochs, variance $\nu^2 = 10^{-8}$. Zeno Batch size $n_r = 4$ and constant weight $\rho = 0.0005$.)
C.2.4 Zero-Gradient Attack

Recall that, in the zero-gradient attack, each Byzantine worker sends zero gradient to the master. The result is shown in Figure 6. From the plots one can see that our single and double safe-guard algorithms, together with Zeno and Mean outperform other methods, and they closely track the performance of the ideal baseline (without failures). It should not be surprising that the (Naive) Mean method performs well under zero-gradient attack: taking the naive mean now behaves equivalently as the perfect baseline with decreased learning rate.

(Parameters: $m = 10$, $\alpha = 0.4$. Single safe-guard threshold $\Sigma = 2$, window size $T = 3$ epochs, variance $\nu^2 = 10^{-8}$; double safe-guard thresholds $\Sigma_0 = 1$, $\Sigma_1 = 2$ and window sizes $T_0 = 1$ epoch, $T_1 = 6$ epochs, variance $\nu^2 = 10^{-8}$. Zeno Batch size $n_r = 4$ and constant weight $\rho = 0.0005$.)
C.2.5 Safeguard Attack

Recall that, in the safeguard attack, Byzantine workers send negative but re-scaled gradient to the master. We choose the re-scale factor so that it hardly triggers the safe-guard conditions at the master. From our experiment, choosing the re-scale factor as 0.4 in most cases do not trigger the safe-guard conditions, while re-scale factor 0.5 enables the algorithm to catch some Byzantine workers.

Our results are shown in Figure 8 (for re-scale factor 0.4) and Figure 9 (for re-scale factor 0.5). In Figure 8, the performance of our (single and double) safeguard algorithms indeed get impacted, but our algorithm still outperforms or matches all prior works. The performance of our (single and double) safeguard algorithms hardly get impacted, and closely tracks the performance of the ideal baseline (and outperforms other prior works significantly).

| Defense               | Best Test Accuracy | Last Test Accuracy | Best Loss | Last Loss |
|-----------------------|--------------------|--------------------|-----------|-----------|
| Mean without Failures | 90.8               | 88.1               | 0.30      | 0.36      |
| Double Safe-guard     | 91.4               | 88.5               | 0.26      | 0.36      |
| Single Safe-guard     | 90.8               | 88.8               | 0.27      | 0.36      |
| Zeno                  | 86.5               | 84.8               | 0.45      | 0.50      |
| Krum / Krum (α=0.3)   | -24.1 (α=0.3)      | -20.4 (α=0.3)      | - > 3 (α=0.3) | - > 3 (α=0.3) |
| GeoMed                | 16.4               | 11.8               | > 3       | > 3       |
| Coordinate-wise Median| 23.8               | 18.5               | 2.16      | 2.35      |

Figure 7: Full results under safe-guard attack with re-scale factor = 0.5.

(Figure 8 parameters: m = 10, α = 0.4. Single safe-guard threshold η = 2, window size T = 3 epochs, variance υ^2 = 10^{-8}; double safe-guard thresholds η_0 = 1, η_1 = 2 and window sizes T_0 = 1 epoch, T_1 = 6 epochs, variance υ^2 = 10^{-8}. Zeno Batch size n_r = 4 and constant weight ρ = 0.0005.)

(Figure 9 parameters: m = 10, α = 0.4. Single safe-guard threshold η = 0.8, window size T = 3 epochs, variance υ^2 = 10^{-8}; double safe-guard thresholds η_0 = 0.8, η_1 = 2 and window sizes T_0 = 1 epoch, T_1 = 6
epochs, variance $\nu^2 = 10^{-8}$. Zeno Batch size $n_r = 4$ and constant weight $\rho = 0.0005$.)

**C.3 Variance Attack**

The Variance attack follows the strategy prescribed by [6], by which Byzantine workers collude in order to shift the mean among all gradients by a factor $\beta$ times the standard deviation of the gradient, while staying within population variance. More precisely, the maximal change to the mean that can be applied by an attacker without the fear of being detected, by using the properties of the Normal distribution, specifically the cumulative standard normal function, and compute the maximal possible shift so that the attackers’s values stay within population variance. (Please see [6, Algorithm 3] for a precise description. Our $\beta$ is called $z_{\text{max}}$ in their notation.) We implement this strategy coordinate-wise, as in the original implementation of [6]. Their work observes that the shift $\beta$ can be non-trivial in practice, since stochastic gradients tend to have large variance in neural network training (which we also observed in our setup). Critically, the attack cannot be defended against by historyless algorithms, as the attacker’s values are statistically indistinguishable from a regular execution in a single iteration. In our setting, for 10 total nodes and $\alpha = 0.4$, $\beta$ is upper bounded by 0.3 (following the classic tables for the cumulative normal). We ran the same attack in the setup from their paper (50 nodes total, of which 24 are Byzantine, which allows $\beta \sim 1.5$) and observed exactly the same outcome. Results for this experiment are given in Table 2, while the convergence of different schemes under this attack is presented in Figure 10.
Figure 9: Convergence under safe-guard attack with re-scale factor = 0.5.

Figure 10: Convergence under variance attack with std dev factor = 0.3.

As shown by the results, our algorithm provably circumvents this attack, and recovers full accuracy. This is because both variants of the algorithm successfully tag the bad nodes after a few hundred iterations. This is explained by the fact that the algorithm has memory: in particular, Byzantine nodes following this strategy will progressively diverge from the (honest) “median” $B_{\text{med}}$ (at a “linear” rate), and therefore will eventually exceed the threshold and be marked as malicious by the algorithm. (In practice, the norm of their $B_i$ vector increases roughly monotonically until it surpasses the threshold.)

(Figure 10 parameters: $m = 10$, $\alpha = 0.4$. Single safe-guard threshold $\mathcal{T} = 0.8$, window size $T = 3$ epochs,
variance $\nu^2 = 10^{-8}$; double safe-guard thresholds $T_0 = 0.8, T_1 = 2$ and window sizes $T_0 = 1$ epoch, $T_1 = 6$ epochs, variance $\nu^2 = 10^{-8}$. Zeno Batch size $n_r = 4$ and constant weight $\rho = 0.0005$. Weight decay was enabled.)

C.4  Full Comparison Table

Please see Table 2.

C.5  Attack Against the Convex Algorithm of [1]

We now briefly describe an attack against this algorithm. The attack specifically leverages the fact that the algorithm does not use sliding windows.

One can first run the vanilla SGD to compute “the maximum deviation per good worker” for the accumulation vector used by the algorithm $\sum_{t=0}^T \nabla_t$. This maximum deviation is therefore a lower bound for the threshold used in their algorithm. Next, we design an attacker who evenly distributes this total allowed deviation to e.g. 5 consecutive epochs, and behaves honestly for the remaining epochs. Such an attacker cannot be identified by this algorithm, because its total deviation across all the iterations is identical to that of a good worker. However, this leads the algorithm to divergence.

Specifically, suppose 4 Byzantine workers all maliciously report their stochastic gradients multiplied by the scalar $-5$, and the remaining 6 good workers report their true stochastic gradients. One can verify numerically that this attacker can run for 5 consecutive epochs (say, epochs $a, a+1, a+2, a+3, a+4$) without being caught by the algorithm. Now,

- if $a \leq 75$, within just 1 epoch of attack, the neural net weights diverge (value NaN).
- if $80 \leq a \leq 115$, this attack is applied after the first learning rate decay. Within just 1 epoch of the attack, the objective explodes and accuracy becomes 10% (random), and within 3 epochs the algorithm diverges completely.
- if $120 \leq a \leq 155$, this attack is after the second learning rate decay. Within just 2 epochs of attack, the accuracy drops to 11%. Later, the accuracy never recovers above 40%.

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| Attack             | Algorithm         | Best Test Accuracy | Test Accuracy (Epoch 200) | Best Cross Entropy on Testing Set | Cross Entropy on Testing Set (Epoch 200) |
|--------------------|-------------------|-------------------|--------------------------|----------------------------------|------------------------------------------|
| Delayed gradient   | Mean without Failures | 90.8              | 88.1                     | 0.30                             | 0.36                                     |
|                    | Double Safe-guard  | 91.7              | 88.2                     | **0.26**                         | **0.33**                                 |
|                    | Single Safe-guard  | 91.1              | **88.9**                 | 0.26                             | 0.35                                     |
|                    | Zeno               | 79.6              | 76.3                     | 0.64                             | 0.69                                     |
|                    | Krum / Krum (α=0.3) | 23.4/ 31.7 (α=0.3) | 19.7/ 28.6 (α=0.3)      | 2.04/ 1.84 (α=0.3)               | 2.06/ 1.90 (α=0.3)                       |
|                    | GeoMed             | 23.5              | 17.7                     | 2.03                             | 2.05                                     |
|                    | Coordinate-wise Median | 28.1            | 23.5                     | 1.96                             | 2.04                                     |
| Label-flipping     | Mean without Failures | 90.8              | 88.1                     | 0.30                             | 0.36                                     |
|                    | Double Safe-guard  | 90.6              | **87.8**                 | **0.29**                         | **0.36**                                 |
|                    | Single Safe-guard  | 85.5              | 83.6                     | 0.83                             | 0.88                                     |
|                    | Zeno               | 85.3              | 82.0                     | 0.85                             | 0.91                                     |
|                    | Krum / Krum (α=0.3) | 74.1/ 85.3 (α=0.3) | 71.6/ 81.0 (α=0.3)      | 1.07/ 0.50 (α=0.3)               | 1.14/ 0.62 (α=0.3)                       |
|                    | GeoMed             | 74.9              | 72.7                     | 1.08                             | 1.11                                     |
|                    | Coordinate-wise Median | 30.1            | 25.9                     | 1.94                             | 2.00                                     |
| Sign-flipping      | Mean without Failures | 90.8              | 88.1                     | 0.30                             | 0.36                                     |
|                    | Double Safe-guard  | 90.5              | 88.0                     | 0.30                             | 0.37                                     |
|                    | Single Safe-guard  | 90.9              | **88.3**                 | **0.27**                         | **0.34**                                 |
|                    | Zeno               | 74.4              | 71.2                     | 0.82                             | 0.86                                     |
|                    | Krum / Krum (α=0.3) | 54.5/ 72.3 (α=0.3) | 53.2/ 67.2 (α=0.3)      | 1.28/ 0.83 (α=0.3)               | 1.33/ 0.96 (α=0.3)                       |
|                    | GeoMed             | 55.5              | 53.8                     | 1.25                             | 1.30                                     |
|                    | Coordinate-wise Median | 22.7            | 19.4                     | 2.20                             | 2.36                                     |
| Zero gradient      | Mean without Failures | 90.8              | 88.1                     | 0.30                             | 0.36                                     |
|                    | Double Safe-guard  | 90.3              | 87.0                     | 0.31                             | 0.41                                     |
|                    | Single Safe-guard  | 90.6              | **87.5**                 | **0.29**                         | **0.38**                                 |
|                    | Zeno               | 88.6              | 87.2                     | 0.36                             | 0.41                                     |
|                    | Krum / Krum (α=0.3) | 11.8/ 12.7 (α=0.3) | 8.0/ 8.0 (α=0.3)        | 2.30/ 2.30 (α=0.3)               | 2.30/ 2.31 (α=0.3)                       |
|                    | GeoMed             | 11.8              | 8.0                      | 2.30                             | 2.30                                     |
|                    | Coordinate-wise Median | 29.4            | 22.8                     | 1.95                             | 2.01                                     |
| Safe-guard attack  | Mean without Failures | 90.8              | 88.1                     | 0.30                             | 0.36                                     |
| (rescaling-factor = 0.4) | Double Safe-guard  | 88.3              | **86.1**                 | **0.39**                         | **0.44**                                 |
|                    | Single Safe-guard  | 87.5              | 84.7                     | 0.39                             | 0.45                                     |
|                    | Zeno               | 87.7              | 84.9                     | 0.44                             | 0.49                                     |
|                    | Krum / Krum (α=0.3) | 12.8 (α=0.3)      | 10.9 (α=0.3)             | > 3 (α=0.3)                     | > 3 (α=0.3)                              |
|                    | GeoMed             | 21.3              | **12.2**                 | > 3                              | > 3                                      |
|                    | Coordinate-wise Median | 24.9            | 19.5                     | 2.13                             | 2.27                                     |
| Variance attack    | Mean without Failures | 90.8              | 87.4                     | 0.21                             | 0.4                                      |
| (rescale-factor = 0.3) | Double Safe-guard  | 90.8              | **88.2**                 | **0.26**                         | **0.38**                                 |
|                    | Single Safe-guard  | 91.0              | 87.4                     | 0.29                             | 0.38                                     |
|                    | Zeno               | 29.0              | 22.0                     | 1.95                             | 2.06                                     |
|                    | Krum (α=0.3)        | 24.2 (α=0.3)      | 14.2 (α=0.3)             | 2.53 (α=0.3)                    | 2612.3 (α=0.3)                           |
|                    | Coordinate-wise Median | 26.4            | 14.2                     | 2.51                             | 12106.37                                |

Table 2: Table of results for CIFAR10 dataset and ResNet20 model.