February 24, 1998
physics/9802044
to appear in Lett. Math. Phys.

MIRROR SYMMETRY ON K3 SURFACES
AS A HYPERKÄHLER ROTATION

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Abstract. We show that under the hypotheses of [11], a mirror partner of a K3 surface $X$ with a fibration in special Lagrangian tori can be obtained by rotating the complex structure of $X$ within its hyperkähler family of complex structures. Furthermore, the same hypotheses force the $B$-field to vanish.

1. Introduction

According to the proposal of Strominger, Yau and Zaslow [11], the mirror partner of a K3 surface $X$ admitting a fibration in special Lagrangian tori should be identified with the moduli space of such fibrations (cf. also [7]). In more precise terms, the mirror partner $\check{X}$ should be identified with a suitable compactification of the relative Jacobian of $X'$, where $X'$ is an elliptic K3 surface obtained by rotating the complex structure of $X$ within its hyperkähler family of complex structures.

Morrison [10] suggested that such a compactification is provided by the moduli space of torsion sheaves of degree zero and pure dimension one supported by the fibers of $X'$. (It should be noted that whenever the fibration $X' \to \mathbb{P}^1$ admits a holomorphic section, as it is usually assumed in the physical literature, the complex manifolds $X'$ and $\check{X}$ turn out to be isomorphic). In [1] Morrison’s suggestion was implemented, and it was shown that the relative Fourier-Mukai
transform defined by the Poincaré sheaf on the fiber product $X' \times_{\mathbb{P}^1} \hat{X}$ enjoys some good properties related to mirror symmetry; e.g., it correctly maps D-branes in $X$ to D-branes in $\hat{X}$, preserves the masses of the BPS states, etc. (The fact that the Fourier-Mukai transform might describe some aspects of mirror symmetry was already suggested in [3].)

It remains to check that $\hat{X}$ is actually a mirror of $X$ in the sense of Dolgachev and Gross-Wilson, cf. [2, 3, 4]. In this note we show that this is indeed the case. Roughly speaking, we prove that whenever $X$ admits a fibration in special Lagrangian tori with a section, and also admits an elliptic mirror $\hat{X}$ with a section, then the complex structure of $\hat{X}$ is obtained by that of $X$ by redefining the B-field and then performing a hyperkähler rotation. A more precise statement is as follows. Let $M$ be a primitive sublattice of the standard 2-cohomology lattice of a K3 surface, and denote by $K_M$ the moduli space of pairs $(X, j)$, where $X$ is a K3 surface, and $j : M \to \text{Pic}(X)$ is a primitive lattice embedding. Let $T = M^\perp$. We assume that $T$ contains a $U(1)$ lattice $P$; this means that the generic K3 surface $X$ in $K_M$, possibly after a rotation of its complex structure within its hyperkähler family, admits a fibration in special Lagrangian tori with a section. After setting $\hat{M} = T/P$, we assume that the generic K3 surface in $K_{\hat{M}}$ is elliptic and has a section. These hypotheses force the B-field to be an integral class. Then, by setting to zero this class (as it seems to be suggested by the physics, since in string theory the B-field is a class in $H^2(X, \mathbb{R}/\mathbb{Z})$), and rotating the complex structure of $X$ within its hyperkähler family of complex structures, we associate to $X \in K_M$ a K3 surface $\hat{X}$ in $K_{\hat{M}}$ such that $\text{Pic}(\hat{X}) \simeq \hat{M}$.

2. Special Lagrangian fibrations and mirror K3 surfaces

We collect here, basically relying on [6, 9, 2, 5], some basic definitions and constructions about mirror families of K3 surfaces.

*Special Lagrangian submanifolds.* Let $X$ be an $n$-dimensional Kähler manifold with Kähler form $\omega$, and suppose that on $X$ there is a nowhere vanishing holomorphic $n$-form $\Omega$. One says that a real $n$-dimensional submanifold $\iota : Y \hookrightarrow X$ is *special Lagrangian* if $\iota^*\omega = 0$, and $\Omega$ can be chosen so that the form $\iota^*\text{Re }\Omega$ coincides with the volume form of $Y$. The moduli space of deformations of $Y$ through special Lagrangian submanifolds was described in [3].

Let $n = 2$, assume that $X$ is hyperkähler with Riemannian metric $g$, and choose basic complex structures $I, J,$ and $K$. These generate an $S^2$ of complex structures compatible with the Riemannian metric of $X$, which we shall call the *hyperkähler family* of complex structures of $X$.

These are the same assumptions made in [11] on physical grounds.
Denote by \( \omega_I, \omega_J \) and \( \omega_K \) the Kähler forms corresponding to the complex structures \( I, J \) and \( K \). The 2-form \( \Omega_I = \omega_I + i \omega_K \) never vanishes, and is holomorphic with respect to \( I \). Thus, submanifolds of \( X \) that are special Lagrangian with respect to \( I \), are holomorphic with respect to \( J \) (this is a consequence of Wirtinger’s theorem, cf. \[6\]). If \( X \) is a complex K3 surface that admits a foliation by special Lagrangian 2-tori (in the complex structure \( I \)), then in the complex structure \( J \) it is an elliptic surface, \( p : X' \to \mathbb{P}^1 \). If one wants \( X \) to be compact then one must allow the fibration \( p : X' \to \mathbb{P}^1 \) to have some singular fibers, cf. \[8\].

**Mirror families of K3 surfaces** \[2\]. Let \( L \) denote the lattice over \( \mathbb{Z} \)
\[
L = U(1) \perp U(1) \perp U(1) \perp E_8 \perp E_8
\]
(by “lattice over \( \mathbb{Z} \)” we mean as usual a free finitely generated \( \mathbb{Z} \)-module equipped with a symmetric \( \mathbb{Z} \)-valued quadratic form). If \( X \) is a K3 surface, the group \( H^2(X, \mathbb{Z}) \) equipped with the cohomology intersection pairing is a lattice isomorphic to \( L \).

If \( M \) is an even nondegenerate lattice of signature \((1, t)\), a \( M \)-polarized K3 surface is a pair \((X, j)\), where \( X \) is a K3 surface and \( j : M \to \text{Pic}(X) \) is a primitive lattice embedding. One can define a coarse moduli space \( \mathbf{K}_M \) of \( M \)-polarized K3 surfaces; this is a quasi-projective algebraic variety of dimension \( 19 - t \), and may be obtained by taking a quotient of the space
\[
D_M = \{ \mathcal{C} \Omega \in \mathbb{P}(M^\perp \otimes \mathbb{C}) | \Omega \cdot \Omega = 0, \Omega \cdot \bar{\Omega} > 0 \}
\]
by a discrete group \( \Gamma_M \) (which is basically the group of isometries of \( L \) that fix all elements of \( M \)) \[2\].

A basic notion to introduce the mirror moduli space to \( \mathbf{K}_M \) is that of admissible \( m \)-vector. We shall consider here only the case \( m = 1 \). Let us pick a primitive sublattice \( M \) of \( L \) of signature \((1, t)\).

**Definition 2.1.** A 1-admissible vector \( E \in M^\perp \) is an isotropic vector in \( M^\perp \) such that there exists another isotropic vector \( E' \in M^\perp \) with \( E \cdot E' = 1 \).

After setting
\[
\tilde{M} = E^\perp / \mathbb{Z}E
\]
one easily shows that there is an orthogonal decomposition \( M^\perp = P \oplus \tilde{M} \), where \( P \) is the hyperbolic lattice generated by \( E \) and \( E' \). The orthogonal of \( E \) is taken here in \( M^\perp \). The mirror moduli space to \( \mathbf{K}_M \) is the space \( \tilde{\mathbf{K}}_M \). Of course one has
\[
\dim \mathbf{K}_M + \dim \tilde{\mathbf{K}}_M = 20.
\]
The operation of taking the “mirror moduli space” is a duality, i.e. \( \tilde{\mathbf{K}}_M \simeq M \) (this works so because we consider the case of a 1-admissible vector, and is no longer true for \( m > 1 \)).
The interplay between special Lagrangian fibrations and mirror K3 surfaces.

Let again $M$ be an even nondegenerate lattice of signature $(1, t)$, and suppose that $X$ is K3 surface such that $\text{Pic}(X) \cong M$. The transcendental lattice $T$ (the orthogonal complement of $\text{Pic}(X)$ in $H^2(X, \mathbb{Z})$) is an even lattice of signature $(2, 19 - t)$. Let $\Omega = x + iy$ be a nowhere vanishing, global holomorphic two-form on $X$. Being orthogonal to all algebraic classes, the cohomology class of $\Omega$ spans a space-like 2-plane in $T \otimes \mathbb{R}$. The moduli space of K3 such that $\text{Pic}(X) \cong M$ is parametrized by the periods, whose real and imaginary parts are given by intersection with $x$ and $y$, respectively. Indeed, one should recall that if we fix a basis of the cohomology lattice $H^2(X, \mathbb{Z})$ given by integral cycles $\alpha_i$, $i = 1, \ldots, 22$, every complex structure on $X$ is uniquely determined, via Torelli’s theorem, by the complex valued matrix whose entries $\varpi_i$ are given by the intersections of the cycles $\alpha_i$ with the class of the holomorphic two-form $\Omega$, i.e. $\varpi_i = \alpha_i \cdot \Omega$. This shows that generically neither $x$ nor $y$ are integral classes in the cohomology ring. However, if we make the further request that there is a 1-admissible vector in $T$, and make some choices, one of the two classes is forced to be integral.

We recall now a result from [5] (although in a slightly weaker form).

**Proposition 2.2.** There exists in $T$ a 1-admissible vector if and only if there is a complex structure on $X$ such that $X$ has a special Lagrangian fibration with a section.

So we consider on $X$ a complex structure satisfying this property (it follows from [5] that, if we fix a hyperkähler metric on $X$, this complex structure belongs to the same hyperkähler family as the one we started from). As a direct consequence we have

**Proposition 2.3.** If there exists a 1-admissible vector in $T$ one can perform a hyperkähler rotation of the complex structure and choose a nowhere vanishing two-form $\Omega$, holomorphic in the new complex structure, whose real part $\Re \Omega$ is integral.

**Proof.** By Proposition 1.3 of [5] the existence of a 1-admissible vector implies the existence on $X$ of a special Lagrangian fibration with a section. On the other hand by [5] what is special Lagrangian in a complex structure is holomorphic in the complex structure in which the Kähler form is given by $\Re \Omega$. Thus in this complex structure the Picard group is nontrivial, which implies that the surface is algebraic, i.e. $\Re \Omega$ is integral.

3. The construction

We introduce now a moduli space $\tilde{K}_M$ parametrizing $M$-polarized K3 surfaces together with of a 1-admissible vector in $T = M^\perp$. The generic K3
surface $X$ in $\tilde{K}_M$ admits a fibration in special Lagrangian tori with a section; the primitive $U(1)$ sublattice $P$ of the transcendental lattice $T$ associated with the 1-admissible vector is generated by the class of the fiber and the class of the section. We fix a marking of $X$, i.e., a lattice isomorphism $\psi: H^2(X, \mathbb{Z}) \to L$. We have an isomorphism

\[ L \simeq M \oplus P \oplus \tilde{M}, \]

where $\tilde{M} = T/P$. The fact that $\tilde{M} \simeq M$ implies that the moduli spaces $\tilde{K}_M$ and $\tilde{K}_\tilde{M}$ are isomorphic. Generically, we may assume that $M \simeq \psi(\text{Pic}(X))$.

One easily shows that the following assumptions are generically equivalent to each other (where “generically” means that this holds true for $X$ in a dense open subset of $\tilde{K}_M$):

(i) The lattice $\tilde{M}$ contains a primitive $U(1)$ sublattice $P'$.

(ii) The generic K3 surface in the mirror moduli space $K_\tilde{M}$ is an elliptic fibration with a section.

(iii) $X$ carries two fibrations in special Lagrangian tori admitting a section, in such a way that the corresponding $U(1)$ lattices $P$, $P'$ are orthogonal.

The two $U(1)$ lattices $P$ and $P'$ are interchanged by an isometry of $L$. Thus, the operation of exchanging them has no effect on the moduli space $K_M$ (although it does on $D_M$).

We shall assume one of these equivalent conditions. The form (ii) of the second condition shows that we are working exactly under the same assumptions that in [11] are advocated on physical grounds.

In the complex structure of $X$ we have fixed at the outset we have the Kähler form $\omega$ and the holomorphic two-form $\Omega = x + iy$, with $x$ an integral class. Condition (iii) means that $P'$ is calibrated by $x$. If we perform a rotation around the $y$ axis, mapping the pair $(\omega, x)$ to $(x, -\omega)$, we still obtain an algebraic K3 surface $X'$ whose Picard group contains $P'$.\footnote{Since we are fixing a marking of $X$ in the following we shall often confuse the lattices $H^2(X, \mathbb{Z})$ and $L$.}

Now we want to show that the Kähler class of $X'$ is a space-like vector contained in the hyperbolic lattice $P'$. We remind here that the explicit mirror map in [2] and [3] is given in terms of a choice of a hyperbolic sublattice of the transcendental lattice. Let $D_M$ be defined as in Section 2, and let

\[ T_M = \{ B + i\omega \in M \otimes \mathbb{C} \mid \omega \cdot \omega > 0 \} = M \times V(M)^+. \]

Here $V(M)^+$ is the component of the positive cone in $M \otimes \mathbb{R}$ that contains the Kähler form of $X$. The space $T_M$ can be regarded as a (covering of the) moduli space of “complexified Kähler structures” on $X$. Let $M' = T/P' \simeq \tilde{M}$. By [4] Proposition 1.1, the mirror map is an isomorphism

\[ \phi: T_{M'} \to D_M, \]

\footnote{Then one shows that the direct sum $P \oplus P'$ is an orthogonal summand of $T$.}
\[ \phi(\hat{B} + i\hat{\omega}) = \hat{B} + E' + \frac{1}{2}(\hat{\omega} \cdot \hat{\omega} - \hat{B} \cdot \hat{B})E + i(\hat{\omega} - (\hat{\omega} \cdot \hat{B})E). \]

Here \(E\) and \(E'\) are the two isotropic generators of the \(U(1)\) lattice \(P'\), while \(\hat{B}\) is what the physicists call the B-field. Our holomorphic two-form \(\Omega\) is of course of the form \(\phi(\hat{B} + i\hat{\omega})\) for suitable \(\hat{B}\) and \(\hat{\omega}\), since \(\phi\) is an isomorphism. The Kähler class of \(X'\) is given by

\[ x = \Re \phi \Omega = \hat{B} + E' + \frac{1}{2}(\hat{\omega} \cdot \hat{\omega} - \hat{B} \cdot \hat{B})E \]

and the new global holomorphic two-form is \(-\omega + iy\). Since \(\hat{B}\) is orthogonal to \(E\) and \(E'\), it is an integral class.

However, the Picard lattice of the K3 surface \(X'\) is generically not isomorphic to \(\hat{M}\). A better choice is suggested by the physics. Indeed in most string theory models the B-field is regarded as a Chern-Simons term, namely, as a class in \(H^2(X, \mathbb{R}/\mathbb{Z})\); so, if we consider the projection \(\lambda: H^2(X, \mathbb{R}) \to H^2(X, \mathbb{R}/\mathbb{Z})\), the relevant moduli space should be

\[ \tilde{T}_{M'} = \lambda(M' \otimes \mathbb{R}) \times V(M')^+ \]

instead of \(T_{M'}\). To take this suggestion into account we set \(\hat{B} = 0\). Since \(y = \hat{\omega} - (\hat{\omega} \cdot \hat{B})E\), this changes the complex structure in \(X'\). Moreover, \(x\) lies now in \(P'\).

So, let us now consider the intersection of \(P \otimes \mathbb{R}\) with the spacelike two-plane \(\langle \Omega \rangle\) spanned by \(\Omega\). This cannot be trivial, since \(P\) is hyperbolic and \(T \otimes \mathbb{R}\) is of signature \((2, 19 - t)\). So we have a real space-like class in \(P \otimes \mathbb{R} \cap \langle \Omega \rangle\) that is orthogonal to \(x\) by construction and thus must be equal (up to a scalar factor) to \(y\). But then, in the complex structure in which the Kähler form is given by \(x\), all the cycles of \(\hat{M}\) are orthogonal to the new holomorphic two-form, given by \(\omega + iy\), and therefore are algebraic. (Notice that the class \(y\) is not integral.)

### 4. Conclusions

A first conclusion we may draw is that the hypotheses of [11] force the B-field to be integral, namely, to be zero as a class \(\hat{B} \in H^2(X, \mathbb{R}/\mathbb{Z})\). Moreover, starting from a K3 surface \(X\) in \(\bar{K}_M\), the construction in the previous section singles out a point in the variety \(\bar{K}_{\hat{M}}\); so we have established a map

\[ \mu: \bar{K}_M \to \bar{K}_{\hat{M}} \]

which is bijective by construction, and deserves to be the called the mirror map. This map consists in setting \(\hat{B}\) to zero (as a class in \(H^2(X, \mathbb{Z})\)) and then performing a hyperkähler rotation.

If we do not set \(\hat{B}\) to zero, we obtain a family of K3 surfaces, labelled by the possible values of \(\hat{B} \in M' \simeq \hat{M}\). Its counterpart under mirror symmetry is a family of K3 surfaces labelled by \(M\). The two families are related by a hyperkähler rotation.
Acknowledgements. We thank C. Bartocci, I. Dolgachev and D. Hernández Ruipérez for useful comments and discussions. This work was partly supported by Ministero dell’Università e della Ricerca Scientifica e Tecnologica through the research project “Geometria reale e complessa.”

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