On the marginal deformations of general 
(0,2) non-linear sigma-models

Ido Adam

Abstract. In this note we explore the possible marginal deformations of general (0,2) non-linear sigma-models, which arise as descriptions of the weakly-coupled (large radius) limits of four-dimensional \( \mathcal{N} = 1 \) compactifications of the heterotic string, to lowest order in \( \alpha' \) and first order in conformal perturbation theory. The results shed light from the world-sheet perspective on the classical moduli space of such compactifications. This is a contribution to the proceedings of String-Math 2012.

1. Introduction

One possible way of obtaining gauge theories in dimensions lower than ten from superstring theory is to consider the heterotic string on \( \mathbb{R}^{d-1,1} \times M_{10-d} \), where \( \mathbb{R}^{d-1,1} \) is \( d \)-dimensional Minkowski space and \( M_{10-d} \) is a compact \((10-d)\)-dimensional manifold. For such compactifications one has also to specify the background gauge fields on \( M_{10-d} \) (a vector bundle \( V \)), which will break the large heterotic SO(32) or \( E_8 \times E_8 \) gauge group into smaller gauge groups more suitable for a realistic description of nature (for example getting an SU(5) GUT).

There is particular interest in compactifications with \( \mathcal{N} = 1 \) supersymmetry, since in that case there exist powerful non-renormalization theorems protecting the superpotential from perturbative \( \alpha' \) corrections and in many cases ensuring the existence of the vacuum in string perturbation theory. The non-renormalization theorem can be violated

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by instanton effects, but in some favorable cases these can be shown to be absent (see, for example, [6, 10, 4]).

It is a well known result that \( \mathcal{N} = 1 \) space-time supersymmetry in four dimensions requires that the local \((0,1)\) superconformal symmetry of the world-sheet description of the Ramond-Neveu-Schwarz superstring be enhanced to a global \((0,2)\) superconformal symmetry (in which the local one is embedded) and that extended space-time supersymmetry leads to even higher superconformal symmetry on the world-sheet [3, 2, 7]. When there is a geometric description of the compactification (i.e., as a manifold and vector bundle) the compact CFT on \( M \) can be described in the large radius limit as a \((0,2)\) superconformal non-linear sigma-model. The marginal deformations preserving the world-sheet supersymmetry (and hence the target-space one) would then correspond to massless modes parameterizing the moduli space of the compactification near the point represented by the world-sheet non-linear sigma-model. This led the authors of [9] to use an application of the methods of [8] to unitary two-dimensional \((0,2)\) SCFTs [1] in order to determine those moduli in the case of compactifications possessing \( G = G' \times E_8 \) space-time gauge symmetry where \( G' \) contains a non-anomalous \( U(1)_L \) symmetry.

In this publication we will examine to lowest order in \( \alpha' \) and to first order in conformal perturbation theory the general case of a gauge group \( G \) without requiring the existence of \( U(1)_L \) factors.

The paper is organized as follows. In section 2 we describe the general \((0,2)\) non-linear sigma-model, in section 3 we find the marginal deformations preserving the \((0,2)\) superconformal symmetry to lowest order in \( \alpha' \) and in conformal perturbation theory.

2. The general \((0,2)\) non-linear sigma-model

In this section the non-linear sigma-model describing the weakly-coupled limit of a compactification of the heterotic string on a complex manifold \( M \) with a complex vector bundle \( V \to M \) is constructed. For the background to be consistent, it has to satisfy the Bianchi identity 
\[
d\tilde{H} = \text{ch}_2(TM) - \text{ch}_2(V),
\]
where \( \tilde{H} = dB - \omega(TM) - \omega(V) \) is the torsion field shifted by the Chern-Simons three-forms constructed from the connection of the gauge vector bundle and the tangent bundle. We will further assume that the compactification has \( \mathcal{N} = 1 \) target space supersymmetry, so the sigma-model has \((0,2)\) global supersymmetry [3, 2, 7]. Target-space supersymmetry to one-loop order also implies the Hermitian Yang-Mills equations 
\[
\mathcal{F}_{ij} = \mathcal{F}_{\bar{i}\bar{j}} = 0, \quad g^{ij} \mathcal{F}_{ij} = 0,
\]
The curvature of the bundle, so in particular, \( V \) is a holomorphic vector bundle.

### 2.1. (0,2) superspace conventions.

The easiest way to write the most-general (0,2) non-linear sigma-model is in (0,2) superspace. We use mostly the conventions of \([13]\). In particular the right-moving supercharges and super-derivatives are given by

\[
Q = \frac{\partial}{\partial \theta} + i \bar{\theta} \partial, \quad \bar{Q} = -\frac{\partial}{\partial \bar{\theta}} - i \theta \bar{\partial}, \quad \mathcal{D} = \frac{\partial}{\partial \theta} - i \bar{\theta} \partial, \quad \bar{\mathcal{D}} = -\frac{\partial}{\partial \bar{\theta}} + i \theta \partial
\]

A generic (0,2) superfield is of the form

\[
\Phi = \phi + \sqrt{2} \theta \psi + i \bar{\theta} \bar{\psi} + i \theta \bar{\theta} F.
\]

A chiral superfield is constrained to satisfy \( \bar{\mathcal{D}} \Phi = 0 \), while an anti-chiral superfield satisfies \( \mathcal{D} \bar{\Phi} = 0 \). They are of the form

\[
\Phi = \phi + \sqrt{2} \theta \psi - i \bar{\theta} \bar{\partial} \phi, \quad \bar{\Phi} = \bar{\phi} - \sqrt{2} \bar{\theta} \bar{\psi} + i \theta \bar{\theta} \bar{\partial} \bar{\phi}.
\]

The model also includes chiral and anti-chiral Fermi superfields satisfying \( \bar{\mathcal{D}} \Gamma = 0 \) and \( \mathcal{D} \bar{\Gamma} = 0 \). Their form is

\[
\Gamma = \gamma - \sqrt{2} \theta G - i \bar{\theta} \bar{\partial} \gamma, \quad \bar{\Gamma} = \bar{\gamma} - \sqrt{2} \bar{\theta} \bar{G} + i \theta \bar{\theta} \bar{\partial} \bar{\gamma}.
\]

The Hermiticity conditions relating the two kinds of fields are the trivial ones \( \theta^\dagger = \bar{\theta}, \phi^\dagger = \bar{\phi}, \psi^\dagger = \bar{\psi} \) and \( \gamma^\dagger = \bar{\gamma} \).

### 2.2. The (0,2) non-linear sigma-model.

The general (0,2) non-linear sigma-model has been written by \([5]\). Its field content is comprised of the chiral and anti-chiral superfields \( \Phi^i, \bar{\Phi}^i \) (\( i, \bar{i} = 1, \ldots, n \)), which are coordinates of a complex manifold \( M \) of dimension \( n \) (for a Calabi-Yau three-fold, which is the case of most interest, \( n = 3 \)), and the chiral and anti-chiral Fermi superfields \( \Gamma^\alpha \) and \( \bar{\Gamma}^{\bar{\alpha}} \), which take values in the vector bundle \( V \) over \( M \). Their conformal weights and \( U(1)_R \) charges are given in Table 1.

The integrand of the superspace integral must be of conformal weight (1,0) and have no \( U(1)_R \) charge. The most general such action is of the form

\[
S = -\frac{i}{8\pi \alpha'} \int d^2 \! x d^2 \theta \left[ \mathcal{K}_i(\Phi, \bar{\Phi}) \partial \Phi^i - \bar{\mathcal{K}}_i(\Phi, \bar{\Phi}) \partial \bar{\Phi}^\dagger + i \left( \mathcal{H}_{\alpha \bar{\beta}}(\Phi, \bar{\Phi}) \Gamma^\alpha \Gamma^\bar{\beta} + 2 \mathcal{H}_{\alpha \bar{i}}(\Phi, \bar{\Phi}) \Gamma^\alpha \bar{\Gamma}^{\bar{i}} + \mathcal{H}_{\bar{i} \bar{\beta}}(\Phi, \bar{\Phi}) \bar{\Gamma}^{\bar{i}} \bar{\Gamma}^{\bar{\beta}} \right) \right],
\]

where \( \mathcal{K}_i(\Phi, \bar{\Phi}) \) and \( \bar{\mathcal{K}}_i(\Phi, \bar{\Phi}) \) can be regarded as the (1,0)- and (0,1)-forms \( \mathcal{K} = \mathcal{K}_i d \Phi^i \) and \( \bar{\mathcal{K}} = \bar{\mathcal{K}}_{\bar{i}} d \bar{\Phi}^{\bar{i}} \) on the target manifold and \( \mathcal{H}_{\alpha \bar{\beta}} \) is a Hermitian structure on the vector bundle.
Table 1. Conformal weights and right R-charges of the various superfields

|       | $h$ | $\bar{h}$ | $\bar{q}$ |
|-------|-----|----------|-----------|
| $\theta$ | 0   | $-\frac{1}{2}$ | $+1$     |
| $\bar{\theta}$ | 0   | $-\frac{1}{2}$ | $-1$     |
| $D_i, Q$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $-1$     |
| $\bar{D}_i, \bar{Q}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $+1$     |
| $\Phi^i, \bar{\Phi}^i$ | 0   | 0         | 0         |
| $\Gamma^a, \bar{\Gamma}^\bar{a}$ | $\frac{1}{2}$ | 0         | 0         |

In the absence of world-sheet boundaries, the action is invariant under the transformations $K \rightarrow K + \partial f$, $\bar{K} \rightarrow \bar{K} + \bar{\partial} f$ for any (0,0)-form $f$, since these would shift the action by a total divergence. Similarly, shifting $K \rightarrow K + \omega$, $\bar{K} \rightarrow \bar{K} + \bar{\omega}$, where $\omega$ and $\bar{\omega}$ are a holomorphic (1,0)-form and an anti-holomorphic (0,1)-form, respectively, again only shifts the integral world-sheet integral by a total derivative, because $\omega$ depends only on the target-space-holomorphic fields $\Phi^i$ so

$$\int \Sigma d^2 \theta \omega(\Phi) = i \int \Sigma \frac{\partial}{\partial \phi^i} \omega(\phi) \bar{\partial} \phi^i = i \int \bar{\partial} \omega(\phi) = 0.$$ 

Requiring that the action $S$ be real leads to the Hermiticity conditions $K^i_\dagger = \bar{K}^i$, $\mathcal{H}^i_{\alpha \bar{\beta}} = \mathcal{H}^i_{\bar{\beta} \alpha}$ and $\mathcal{H}^i_{\alpha \bar{\beta}} = \mathcal{H}^i_{\bar{\beta} \alpha}$.

In order to get some intuition about the meaning of the various background superfields, we also write the action in component form (dropping total derivative terms which vanish in the absence of world-sheet boundaries):

$$S = -\frac{1}{2\pi \alpha'} \int d^2 z \left[ \frac{i}{2} g_{ij} (\partial \phi^i \bar{\partial} \phi^j + \bar{\partial} \phi^i \partial \phi^j) + \frac{1}{2} B_{ij} (\partial \phi^i \bar{\partial} \phi^j - \bar{\partial} \phi^j \partial \phi^i) 
+ i g_{ij} \bar{\psi}^j \Psi^i - i \bar{\psi}^j \left( \Omega_{jki} \bar{\partial} \phi^k + \Omega_{jki} \partial \phi^k \right) \psi^j - i \bar{\gamma}_a (\bar{\partial} \gamma^a + A^a_{ij} \partial \phi^j \gamma^i) 
- \frac{i}{2} \bar{A}^a_{ij} \bar{\partial} \phi^j \gamma^a \gamma^i 
- \frac{i}{2} A^a_{ij} \bar{\partial} \phi^j \gamma^a \bar{\gamma}^i 
- \frac{1}{2} \bar{F}^a_{ij} \psi^i \bar{\psi}^j \gamma^a \gamma^j 
+ \bar{F}^{a}_{ij} \psi^i \bar{\psi}^j \gamma^a \gamma^j \right],$$

(6)

where the metric and the $B$-fields are given by

$$g_{ij} = \frac{1}{2} (\partial_j K_i + \partial_i \bar{K}_j), \quad B_{ij} = \frac{1}{2} (\partial_j K_i - \partial_i \bar{K}_j),$$

(7)
and the torsion-twisted connection is

\[ \Omega^{-}_{jk\bar{i}} = \Gamma_{jk\bar{i}} - \frac{1}{2} H_{jk\bar{i}} , \quad \Omega^{\bar{j}}_{jk\bar{i}} = \Gamma_{jk\bar{i}} - \frac{1}{2} H_{jk\bar{i}} , \]

with \( \Gamma \) being the Christoffel symbol of the first kind associated with the metric \( g_{ij} \) and \( H = dB \) is the \( H \)-field. Furthermore, for brevity we define the holomorphic and anti-holomorphic connections on the vector bundle

\[ \mathcal{A}^{\alpha}_{i\bar{j}} \equiv \partial_{i} \mathcal{H}_{\beta \bar{\gamma}} \mathcal{H}^{\bar{\gamma} \alpha} , \quad \bar{\mathcal{A}}^{\bar{j}}_{i\bar{j}} \equiv \mathcal{H}^{\bar{\beta} \alpha} \partial_{i} \mathcal{H}_{\gamma \bar{\beta}} \]

(we use the notation that \( \mathcal{H}^{\alpha \beta} \) is the inverse Hermitian metric of \( \mathcal{H}_{\alpha \beta} \)) as well as the symbols

\[ \bar{\mathcal{F}}^{\bar{\alpha}}_{jk\bar{i}} \equiv \partial_{j} \mathcal{A}^{\alpha}_{k\bar{i}} - \mathcal{A}^{\alpha}_{ij\bar{k}} \mathcal{A}_{k\bar{i}}^{\bar{j}} , \quad \bar{\mathcal{F}}_{k\bar{j}i\bar{j}} \equiv \partial_{k} \mathcal{A}^{\alpha}_{j\bar{i}} - \mathcal{A}^{\alpha}_{k\bar{j}} \mathcal{A}_{j\bar{i}}^{\bar{k}} . \]

(10)

Note that these are not connections and curvatures of the vector bundle (the curvature is \( \mathcal{F}^{\alpha}_{ji\bar{j}} = \partial_{j} \mathcal{A}^{\alpha}_{i\bar{j}} \)). We hope that the use of the letters \( \mathcal{A} \) and \( \mathcal{F} \) will not cause any confusion.

In the sequel we will require the equations of motion for the various superfields. Since the fields are either chiral or anti-chiral, we need their variations to obey the chirality/anti-chirality constraints. This is easily done by writing the variations as

\[ \delta \Phi^{i} = \bar{D} \delta X^{i} , \quad \delta \bar{\Phi}^{\bar{i}} = D \delta \bar{X}^{\bar{i}} , \quad \delta \Gamma^{\alpha} = \bar{D} \delta \Lambda^{\alpha} , \quad \delta \bar{\Gamma}^{\bar{\alpha}} = D \delta \bar{\Lambda}^{\bar{\alpha}} . \]

The equations of motion thus obtained are

\[ E_{4}^{\Phi} = 2H_{jk\bar{i}} \bar{D} \Phi^{k} \bar{D} \bar{\Phi}^{j} - 2 \bar{D}(g_{ij} \partial \bar{\Phi}^{j}) + i \bar{D}(\partial_{i} \mathcal{H}_{\alpha \bar{\beta}} \Gamma^{\alpha} \bar{\Gamma}^{\beta}) - 
\]

\[ -2i \bar{D}(\partial_{i} \mathcal{H}_{\alpha \bar{\beta}} \bar{\Gamma}^{\beta}) \Gamma^{\alpha} + i \bar{D}(\partial_{i} \mathcal{H}_{\alpha \bar{\beta}} \bar{\Gamma}^{\beta}) \Gamma^{\alpha} = 0 , \]

\[ E_{4}^{\bar{\Phi}} = 2D(g_{ji} \partial \Phi^{j}) + 2H_{k\bar{j}i} D \Phi^{k} \partial \bar{\Phi}^{j} + i D(\partial_{i} \mathcal{H}_{\alpha \bar{\beta}} \bar{\Gamma}^{\alpha} \bar{\Gamma}^{\beta}) + 
\]

\[ +2i D(\partial_{i} \mathcal{H}_{\alpha \bar{\beta}} \bar{\Gamma}^{\beta}) \Gamma^{\alpha} + i D(\partial_{i} \mathcal{H}_{\alpha \bar{\beta}} \bar{\Gamma}^{\beta}) \Gamma^{\alpha} = 0 , \]

\[ E_{4}^{\Gamma} = \bar{D} \mathcal{H}_{\alpha \bar{\beta}} \Gamma^{\beta} + \bar{D}(\mathcal{H}_{\alpha \bar{\beta}} \bar{\Gamma}^{\beta}) = 0 , \]

\[ E_{4}^{\bar{\Gamma}} = D(\mathcal{H}_{\beta \bar{\alpha}} \bar{\Gamma}^{\alpha}) - D \mathcal{H}_{\beta \bar{\alpha}} \bar{\Gamma}^{\alpha} = 0 . \]

(11)

(12)

(13)

(14)

(15)

3. Marginal deformations

In this section we consider the marginal deformations to the lowest order in \( \alpha' \) and first order in conformal perturbation theory.
It can be seen that much like in the four-dimensional case \[8\] there are no Kähler deformations of the form
\[\int d^2 z \overline{D} \overline{X} .\]
The only type of marginal deformations are of the form
\[(16)\quad S_{\mathcal{W}} = -\frac{i}{8\pi \alpha'} \int d^2 z \mathcal{D} \mathcal{W} + \text{h.c.} ,\]
where \(\mathcal{W}\) must be a chiral primary of weights \((1, \frac{1}{2})\) and \(U(1)_R\) charge +1. (For \(S_{\mathcal{W}}\) to be truly marginal, these conditions should hold to any order in conformal perturbation theory and in \(\alpha'\).)

Since \(\mathcal{D}^2 = 0\), \(S_{\mathcal{W}}\) clearly remains unmodified under \(\mathcal{W} \rightarrow \mathcal{W} + \partial \mathcal{Z}\), where \(Y\) has conformal weight \((1, 0)\) and R-charge +2. In the absence of world-sheet boundaries, it is also invariant under \(\mathcal{W} \rightarrow \mathcal{W} + \partial \mathcal{Z}\) with \(\mathcal{Z}\) being a superfield of weight \((0, \frac{1}{2})\) and R-charge +1, and \(\partial \mathcal{Z}\) is required to be chiral. Finally, if we deform using \(\mathcal{W}' = \mathcal{W} + \overline{D} X\) one obtains an equivalent deformation \(S_{\mathcal{W}'}\) because
\[(17)\quad S_{\mathcal{W}'} - S_{\mathcal{W}} = \int d^2 z \mathcal{D} \overline{D} X,\]
which is a trivial deformation.

A short note about the condition of chirality is in order. Working in conformal perturbation theory, we should expand the deformed action around the undeformed conformal theory. Therefore, we should treat the deformation \(S_{\mathcal{W}}\) as a series of operator insertions in the undeformed correlation function evaluated at the conformal point. Insertions in a path integral satisfy the equations of motion of the undeformed action (up to possible contact terms with other insertions). Another point of view is that terms in the action that are proportional to the equations of motion can be removed by a field redefinition. Henceforth, on-shell will always mean on-shell with respect to the undeformed equations of motion.

Since our analysis is done at the first order in conformal perturbation theory and at tree-level in \(\alpha'\), all the fields have their classical dimensions and we can treat the deformation as a classical object. The most general deformation with the required \((1,1)\) conformal weight and R-charge +1 is
\[(18)\quad \mathcal{W} = (\Lambda^\alpha_{ij}(\Phi, \overline{\Phi}) \Gamma^\alpha_a \Gamma^b + \Lambda^\alpha_{ij} \Gamma^\alpha_a \Gamma^b + \Lambda^\alpha_{i\overline{j}}(\Phi, \overline{\Phi}) \Gamma^\alpha_a \Gamma^\beta \overline{\Gamma}^\beta \overline{D} \Phi^i +
+ (Y^i_{ij}(\Phi, \overline{\Phi}) \partial \Phi^i + g_{ik} Z^i_{ij}(\Phi, \overline{\Phi}) \partial \Phi^k) \overline{D} \Phi^j) ,\]
where \( \Lambda_{ij}^{\alpha} = H^{\alpha\gamma}\Lambda_{ij\gamma} \) (\( \Lambda_{ij\gamma} = -\Lambda_{ij\gamma} \)) and \( \Lambda_{ij}^{\alpha} = H^{\gamma\alpha}\Lambda_{ij\gamma\delta} \) (\( \Lambda_{ij\gamma\delta} = -\Lambda_{ij\gamma\delta} \)). A term of the form

\[
\tilde{\mathcal{W}} = (\Omega_{\alpha\beta}(\Phi, \bar{\Phi})\Omega_{\alpha} + \Omega_{\alpha\beta}(\Phi, \bar{\Phi})\bar{\Omega}_{\alpha})(\mathcal{D}\bar{\Omega}^{\beta} + \bar{\Lambda}_{\gamma}^\alpha \mathcal{D}\bar{\Phi}^\gamma \bar{\Gamma}^\alpha)
\]

(where the derivative has been replaced by a gauge-covariant derivative to maintain gauge-invariance) does not appear because it can be absorbed in the deformation \([48] \) by using the undeformed equations of motion.

For the theory to be well defined on the entire compact space, the deformation parameters must be sections of the appropriate bundles:

\[
\Lambda \in \Gamma(\Omega^{0,1} \otimes \text{End} \, \mathcal{V}) \quad , \quad Y \in \Gamma(\Omega^{1,1}) \quad , \quad Z \in \Gamma(\Omega^{0,1} \otimes T^{1,0}M) \text{ .}
\]

The deformation \([48] \) is not manifestly chiral as it depends on anti-chiral fields as well as chiral ones. However, as discussed above, it needs only be chiral on-shell in order to preserve (0,2) supersymmetry in conformal perturbation theory. On-shell

\[
\mathcal{D}\mathcal{W} = \left( \partial_j \Lambda_{ij}^{\alpha} + \mathcal{A}_{ij}^\alpha \Lambda_{ij}^{\gamma} + \frac{i}{2} Z_{ik}^j \bar{\mathcal{F}}_{\alpha}^{\bar{\gamma}} \right) \bar{\Gamma}_\alpha \Gamma^\beta \mathcal{D}\bar{\Phi}^{j} \mathcal{D}\bar{\Phi}^{i} + \\
+ \left( \partial \Lambda_{ij}^{\alpha} - \Lambda_{ij}^{\alpha} \Lambda_{ij}^{\gamma} + \frac{i}{2} Z_{ik}^j \bar{\mathcal{F}}_{\alpha}^{\bar{\gamma}} \right) \bar{\Gamma}_\alpha \Gamma^\beta \mathcal{D}\bar{\Phi}^{j} \mathcal{D}\bar{\Phi}^{i} + \\
+ \left( \partial \Lambda_{ij}^{\alpha} - \Lambda_{ij}^{\alpha} \Lambda_{ij}^{\gamma} + \frac{i}{2} Z_{ik}^j \bar{\mathcal{F}}_{\alpha}^{\bar{\gamma}} \right) \bar{\Gamma}_\alpha \Gamma^\beta \mathcal{D}\bar{\Phi}^{j} \mathcal{D}\bar{\Phi}^{i} + \\
+ (\partial_k Y_{ij}^k + Z_j^l H_{ik}) \partial \bar{\Phi}^i \mathcal{D}\bar{\Phi}^j \mathcal{D}\bar{\Phi}^k + g_{ik} \partial_i Z_j^k \partial \bar{\Phi}^k \mathcal{D}\bar{\Phi}^j \mathcal{D}\bar{\Phi}^i .
\]

Requiring that \( \mathcal{D}\mathcal{W} = 0 \) yields the following constraints of the deformation parameters

\[
\mathcal{H}_{\alpha\gamma} \left( \partial_j \Lambda_{ij}^{\gamma} + \mathcal{A}_{ij}^\gamma \Lambda_{ij}^{\gamma} + \frac{i}{2} Z_{ik}^j \bar{\mathcal{F}}_{\alpha}^{\bar{\gamma}} \right) - (\alpha \leftrightarrow \beta) = 0 ,
\]

\[
\partial_j \Lambda_{ij}^{\alpha} - \Lambda_{ij}^{\alpha} \Lambda_{ij}^{\gamma} - i Z_{ik}^j \bar{\mathcal{F}}_{\alpha}^{\bar{\gamma}} = 0 ,
\]

\[
\mathcal{H}_{\gamma\alpha} \left( \partial_j \Lambda_{ij}^{\gamma} - \mathcal{A}_{ij}^\gamma \Lambda_{ij}^{\gamma} + \frac{i}{2} Z_{ik}^j \bar{\mathcal{F}}_{\alpha}^{\bar{\gamma}} \right) - (\bar{\alpha} \leftrightarrow \bar{\beta}) = 0 ,
\]

\[
\partial_k Y_{ij}^k - H_{ik} \partial_i Z_j^k = 0 ,
\]

\[
\partial_i Z_j^i = 0 ,
\]

(19)

where \([\ldots]\) denotes anti-symmetrization with respect to space indices only and indices between bars are excluded from the anti-symmetrization.

As discussed earlier, deformations are subject to the equivalence relation \( \mathcal{W} \simeq \mathcal{W} + \mathcal{D}X + \partial Z \). The most general \( X \) of weight (1,0) and
R-charge 0 is (again at the classical level)
\[
X = \lambda_{\alpha\beta}(\Phi, \tilde{\Phi}) \Gamma^\alpha \Gamma^\beta + \lambda^{\alpha\beta}(\Phi, \tilde{\Phi}) \tilde{\Gamma}_\alpha \tilde{\Gamma}_\beta + \lambda_{\alpha\beta}(\Phi, \tilde{\Phi}) \tilde{\Gamma}_\alpha \Gamma^\beta + \mu_i(\Phi, \tilde{\Phi}) \partial \Phi^i + g_{ij} \zeta^i(\Phi, \tilde{\Phi}) \partial \Phi^j ,
\]
where
\[
\lambda^\alpha_{\beta}, \lambda^{\alpha\beta}, \lambda^\alpha_{\beta} \in \Gamma(\text{End } V) , \quad \mu \in \Gamma(\Omega^{1,0}(M)) , \quad \zeta \in \Gamma(\mathbf{T}^{1,0}M) .
\]

The most general $\mathcal{Z}$ of weight $(0, \frac{1}{2})$ and R-charge $+1$ is
\[
\mathcal{Z} = \xi_j(\Phi, \tilde{\Phi}) \mathcal{D} \Phi^j .
\]

$\partial \mathcal{Z}$ is chiral on-shell provided
\[
\nabla^- \partial_j \xi_i = 0 , \quad \nabla^- \partial_j \xi_i = 0 ,
\]
\[
g^{ij}(\partial_i \xi_{[k} \tilde{F}^\alpha_{\beta]} - \partial_j \xi_{[k} \tilde{F}^\alpha_{\beta]}) = 0 ,
\]
\[
g^{ij}(\partial_i \xi_{[k} \tilde{F}^\alpha_{\beta]} - \partial_j \xi_{[k} \tilde{F}^\alpha_{\beta]}) = 0 ,
\]
\[
g^{ij}(\partial_i \xi_{[k} \tilde{F}^\alpha_{\beta]} - \partial_j \xi_{[k} \tilde{F}^\alpha_{\beta]}) = 0 .
\]

(The relations [12]

\[
\Gamma_{kij} = \frac{1}{2} H_{kij} = -\frac{1}{2}(\partial_k g_{ij} - \partial_i g_{kj}) ,
\]
\[
\Gamma_{iij} = \frac{1}{2}(\partial_i g_{ij} + \partial_j g_{ij}) , \quad H_{iij} = \partial_i g_{ij} - \partial_j g_{ii}
\]
were used to rewrite the result in terms of the $H$-twisted connection $\nabla^- .)$

Putting all these together, the equivalence relation $\mathcal{W} \simeq \mathcal{W} + \mathcal{D} X + \partial \mathcal{Z}$ in component form are
\[
\Lambda_{i\beta}^\alpha \simeq \Lambda_{i\beta}^\alpha + \mathcal{H}^{\alpha\gamma} \partial_i \lambda_{\gamma\beta} + \tilde{\mathcal{A}}_{i\gamma}^\alpha \lambda_{\beta}^\gamma + \frac{i}{2}(\zeta^j + g^j_k \xi_k) \tilde{\mathcal{F}}_{ji\beta}^\alpha ,
\]
\[
\Lambda_{i\beta}^\alpha \simeq \Lambda_{i\beta}^\alpha + \partial_i \lambda_{\alpha\beta} - 2\lambda^{\alpha\gamma} \tilde{\mathcal{A}}_{i\gamma}^\beta - i(\zeta^j + g^j_k \xi_k) \tilde{\mathcal{F}}_{ij\beta}^\alpha ,
\]
\[
\Lambda_{i\beta}^\alpha \simeq \Lambda_{i\beta}^\alpha + \partial_i \lambda_{\alpha\beta} - \tilde{\mathcal{A}}_{i\beta}^\alpha \lambda^\alpha_{\gamma} - \frac{i}{2}(\zeta^j + g^j_k \xi_k) \tilde{\mathcal{F}}_{ij\beta}^\alpha ,
\]
\[
Y_{ij} \simeq Y_{ij} + \partial_j \mu_i + \partial_i \zeta_j + H_{ijk}(\zeta^k + g^k_l \xi_l) ,
\]
\[
Z_{ji} \simeq Z_{ji} + \partial_j(\zeta^i + g^i_k \xi_k) + g^i_k (\partial_k \xi_j - \partial_j \xi_k) .
\]

### 4. An example: $G = G' \times E_8$, $U(1)_L \subset G'$

In this section we reconsider the case in which the bundle’s surviving structure group is $G = G' \times E_8$ with $G'$ containing a $U(1)_L$ factor [9]. The analysis here differs from that in [9] by the inclusion of bundle deformations which break the $U(1)_L$ symmetry.
In this case

\begin{equation}
\mathcal{H}_{\alpha\beta} = \mathcal{H}_{\bar{\alpha}\bar{\beta}} = 0 ,
\end{equation}

from which it follows that

\begin{equation}
\mathcal{A}_{i\bar{j}}^\alpha = \bar{\mathcal{A}}_{i\bar{j}}^\bar{\alpha} = 0 , \quad \tilde{\mathcal{F}}_{j\bar{k}\bar{\beta}}^\bar{\alpha} = \bar{\mathcal{F}}_{j\bar{k}\bar{\beta}}^\bar{\alpha} = 0 , \quad \tilde{\mathcal{F}}_{\bar{j}\bar{k}\beta}^\alpha = \mathcal{F}_{\bar{j}\bar{k}\beta}^\alpha .
\end{equation}

Thus, the constraints on the deformations (19) become

\begin{align}
\mathcal{H}_{\alpha \gamma} (\partial_{ij} \Lambda^\gamma_{j\bar{i}} + \bar{A}^\gamma_{j\bar{i}} \Lambda^\gamma_{\bar{i}j}) - (\alpha \leftrightarrow \beta) &= 0 , \\
\partial_{ij} \Lambda^\alpha_{j\bar{i}} - iZ_{\bar{i}k}^j \mathcal{F}_{j\bar{k}\beta}^\alpha &= 0 , \\
\mathcal{H}_{\gamma \bar{\alpha}} (\partial_{ij} \Lambda^\gamma_{j\bar{i}} - \bar{A}^\gamma_{j\bar{i}} \Lambda^\gamma_{\bar{i}j}) - (\alpha \leftrightarrow \beta) &= 0 , \\
\partial_k Y_{ij} - H_{ikj} Z_j^l &= \partial_k Y_{ik} - H_{ikj} Z_j^l , \\
\partial_k Z_j^i &= 0 .
\end{align}

The components of \( Z \) should satisfy

\begin{align}
\nabla_k \partial_\beta \xi_j &= 0 , \\
\nabla_k \partial_\beta Z_j^i &= 0 , \\
g^{ij} (\partial_\beta \xi_m) \mathcal{F}_{k\bar{j}\beta}^\alpha - \partial_\beta \xi_m \mathcal{F}_{m\bar{j}\beta}^\alpha &= 0 .
\end{align}

The equivalence relations (29) are then reduced to

\begin{align}
\Lambda^\delta_{i\bar{j}} &\simeq \Lambda^\delta_{j\bar{i}} + \mathcal{H}_{\delta \gamma} \partial_\gamma \lambda_{j\bar{i}} , \\
\Lambda^\alpha_{i\bar{j}} &\simeq \Lambda^\alpha_{j\bar{i}} + \partial_\gamma \lambda_{j\bar{i}} - i (\zeta^j + g^{\bar{k}j} \xi) \mathcal{F}_{j\bar{k}\beta}^\alpha , \\
\Lambda^\alpha_{i\bar{j}} &\simeq \Lambda^\alpha_{j\bar{i}} + \partial_\gamma \lambda_{j\bar{i}} - \bar{A}^\gamma_{i\bar{j}} \Lambda^\gamma_{j\bar{i}} , \\
Y_{ij} &\simeq Y_{ji} + \partial_j \mu_i + \partial_i \xi_j + H_{ijk} (\zeta^k + g^{\bar{k}i} \xi) , \\
Z_j^i &\simeq Z_j^i + \partial_j (\zeta^i + g^{\bar{k}i} \xi^k) + g^{\bar{k}i} (\partial_k \xi_j - \partial_j \xi_k) .
\end{align}

These are the same as the results obtained in [9] with the addition of deformations which break the U(1)_L symmetry.

We can bring the extra deformation parameterized by \( \Lambda^\delta_{i\bar{j}} \) and \( \Lambda^\alpha_{i\bar{j}} \) to a nicer form. Doing a little algebra yields

\begin{align}
\mathcal{H}_{\alpha \gamma} \partial_j \Lambda^\gamma_{i\bar{j}} &= \partial_j \Lambda_{i\alpha\beta} - \partial_j \mathcal{H}_{\alpha \gamma} \Lambda^\gamma_{i\bar{j}} , \\
\mathcal{H}_{\alpha \gamma} \mathcal{A}^\gamma_{i\bar{j}} &\Gamma^\delta_{i\bar{j}} = \partial_j \mathcal{H}_{\alpha \gamma} \Lambda^\gamma_{i\bar{j}} ,
\end{align}

so we can rewrite (32) and its associated equivalence relation as

\begin{equation}
\partial_{ij} \Lambda^\delta_{i\alpha\beta} = 0 , \quad \Lambda_{i\alpha\beta} \simeq \Lambda_{i\alpha\beta} + \partial_\gamma \lambda_{i\alpha\beta} .
\end{equation}
Hence, these extra deformations are elements of $H^1(M, V^* \wedge V^*)$. A similar manipulation of (36) gives

$$\partial_i \Lambda^\alpha \wedge \Lambda^\beta = 0,$$

$$\Lambda^\alpha \wedge \Lambda^\beta \simeq \Lambda^\alpha \wedge \partial_i \Lambda^\beta,$$

which are elements of $H^1(M, V \wedge V)$.

In particular, for the heterotic string compactified on a Calabi-Yau three-fold and the standard embedding of the tangent bundle in the vector bundle, which breaks the first $E_8$ into $E_6$, these new deformations will be in $H^1(M, T^*M \wedge T^*M) \simeq H^{2,1}(M)$ and $H^1(M, TM \wedge TM) \simeq H^{1,1}(M)$.

These are the same as the results obtained in [9] with the addition of deformations that break the $U(1)_L$ symmetry. To our knowledge, these deformations are new. A possible application of these new deformations is breaking the $E_6$ gauge group of the standard embedding to $SO(10)$. In this case the $78$ adjoint representation of $E_6$ is decomposed under its $SO(10) \times U(1)_L$ into $45_0 \oplus 16_{-3} \oplus \overline{16}_3 \oplus 1_0$. A deformation breaking the $U(1)_L$ should Higgs all but the $45$ of $SO(10)$. The $1$ is clearly lifted and the two spinor representations must become massive as well for the consistency of the low-energy effective theory.

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Instituto de Física Teórica, Universidade Estadual Paulista, R. Dr. Bento T. Ferraz 271, Bloco II, 01140-070, São Paulo, SP, Brasil,
E-mail address: idoadam.physics@yahoo.com