Perfect Prediction in Minkowski Spacetime: Perfectly Transparent Equilibrium for Dynamic Games with Imperfect Information

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Abstract  The assumptions of necessary rationality and necessary knowledge of strategies, also known as perfect prediction, lead to at most one surviving outcome, immune to the knowledge that the players have of them. Solutions concepts implementing this approach have been defined on both dynamic games with perfect information, the Perfect Prediction Equilibrium, and strategic games with no ties, the Perfectly Transparent Equilibrium.

In this paper, we generalize the Perfectly Transparent Equilibrium to games in extensive form with imperfect information and no ties. Both the Perfect Prediction Equilibrium and the Perfectly Transparent Equilibrium for strategic games become special cases of this generalized equilibrium concept. The generalized equilibrium, if there are no ties in the payoffs, is at most unique, and is Pareto-optimal.

We also contribute a special-relativistic interpretation of a subclass of the games in extensive form with imperfect information as a directed acyclic graph of decisions made by any number of agents, each decision being located at a specific position in Minkowski spacetime, and the information sets being derived from the causal structure. Strategic games correspond to a setup with only spacelike-separated decisions, and dynamic games to one with only timelike-separated decisions.

The generalized Perfectly Transparent Equilibrium thus characterizes the outcome and payoffs reached in a general setup where decisions can be located in any generic positions in Minkowski spacetime, under necessary rationality and necessary knowledge of strategies.

Keywords  Counterfactual dependency, Non-Cooperative Game Theory, Non-Nashian Game Theory, Spacetime, Superrationality, Preemption, Imperfect information

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1 Introduction

How often have I said to you that when you have eliminated the impossible, whatever remains, however improbable, must be the truth?

Sir Arthur Conan Doyle,
Sherlock Holmes,
The Sign of the Four (1890)

Most game-theoretical solution concepts, based on the work of John Nash (1951), assume that players make their decisions independently from each other.

This can be expressed in counterfactual terms (Lewis, 1973) by saying that an agent Mary picks some strategy S, the other agents anticipated that Mary was going to pick strategy S, and that if Mary had, counterfactually, picked a different strategy T, the other agents’ anticipations would still have been that Mary was going to pick strategy S. This assumption that a decision is independent, i.e., anything correlated to it could potentially have been caused by it1 (RENNER AND COLBECK, 2011), is often called free will or free choice in literature.

1.1 Non-Nashian game theory and Perfect Prediction

There is an emerging different line of research, non-Nashian game theory, which drops this assumption.

There is a specific class of non-Nashian game theoretical results based on the opposite assumption, namely, that agents are perfectly predictable, and that the prediction of a decision is perfectly correlated with this decision. This can be expressed in counterfactual terms by saying that an agent Mary picks some strategy S, the other agents anticipated that Mary was going to pick strategy S, and that if Mary had, counterfactually, picked a different strategy T, the other agents would have anticipated that Mary was going to pick strategy T. This idea was introduced by Dupuy (2000) and is called Perfect Prediction. This can be interpreted as a weaker form of free will in which the agents could have acted otherwise, but not precluding predictability.

Contrary to intuition, assuming that agents are perfectly predictable does not lead to a trivial theory, but instead translates to a fix-point problem: the apparent conflict between making a free decision and being predictable no matter what one does translates into finding those outcomes that are immune against their anticipation. Knowing which outcome will be reached in advance, the players play towards this outcome. In spite of knowing the outcome in advance, the players play towards it. Put more boldly, the very anticipation of a specific outcome by the players causes them to play towards this outcome. Dupuy (2000) conjectured that, under this assumption, interesting and desirable properties emerge, such as existence, uniqueness and Pareto-optimality of the outcome at hand.

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1 in special-relativistic terms, anything correlated to it must be in the future light cone of the decision
This conjecture was found to be correct for games in extensive form, with perfect information with no ties. The corresponding equilibrium is called the Perfect Prediction Equilibrium (Fourny et al., 2018). For games in normal form, the equilibrium is at most unique and Pareto-optimal, however, it does not exist for all games (Fourny, 2017). It was called Perfectly Transparent Equilibrium.

In this paper, we extend the Perfectly Transparent Equilibrium concept to any game in extensive form with imperfect information. The above counterpart equilibria are thus special cases of this new, more general equilibrium concept.

The underlying Kripke semantics (Kripke, 1963) formalizing the reasoning with possible worlds, explicit accessibility relations and counterfactual functions was introduced in (Fourny, 2018). It also applies, with a few adaptations, to the generalized Perfectly Transparent Equilibrium.

1.2 Related work

Other non-Nashian results are found in literature, most of them with assumptions intermediate between Nashian Free Choice (independent decisions) and Perfect Prediction (decisions are always correctly anticipated). Joe Halpern dubbed this intermediate spectrum “translucency”. Solution concepts include Superrationality in symmetric games (Hofstadter, 1983), translucent games, minimax-rationalizability and individual rationality (Halpern and Pass, 2013), (Capraro and Halpern, 2015), Second-Order Nash Equilibria (Bilo and Flammini, 2011), Joint-selfish-rational equilibrium (Shiffrin et al., 2009), Program equilibria (?).

2 Games with imperfect information

We start with the core definitions of a game in extensive form with imperfect information, as typically encountered in literature.

What differentiates a game with imperfect information from a game with perfect information is that, when making a choice, there is some opacity regarding other agents’ choices, even though these choices are not in the future. For example, these other decisions are being made in separate rooms, with no communication. Nodes are thus grouped into information sets, and a choice of action has to be taken not knowing at which node, within this information set, one is playing.

Games with imperfect information thus also subsume games in normal form: a game in normal form can be expressed as a game in extensive form with imperfect information.

2.1 Formal definition

We take as the definition of a game with imperfect information the same as that of Jackson et al. (2019) at Stanford, making a few more properties explicit. An alternate definition based on sequences of actions is given by Osborne and Rubinstein (1994).
Definition 1 (Game with imperfect information) A game with imperfect information is a tuple \((N,A,H,Z,\chi,\rho,\sigma,u,I)\) where

- \(N\) is a set of players.
- \(A\) is a set of actions (common across all players).
- \(H\) is a set of choice (non-terminal) nodes.
- \(Z\) is a set of outcomes.
- \(\chi \in H \rightarrow \mathcal{P}(A)\) is the action function, assigning each choice node to the set of available actions at that node.
- \(\rho \in H \rightarrow N\) is the player function, assigning each choice node to a player.
- \(\sigma \in H \times A \rightarrow H \cup Z\) is the successor function, assigning each pair of choice node and action (available at that choice node) to a choice node or outcome.

There are a few constraints in \(\sigma\) to enforce that the game is a tree. First, it is injective. Second, \(\sigma(h,a)\) is only defined if \(a \in \chi(h)\). Lastly, \(\sigma\) must organize the choice nodes and outcomes in a single connected component: there can only be one root, as opposed to a forest.²

- \(u \in N \times Z \rightarrow \mathbb{R}\) is the utility function, assigning each player and outcome to a payoff. Since we are only interested in pure strategies, payoffs are ordinal, not cardinal, meaning that it only matters how they compare, but their absolute values do not. Literature thus also models utilities with an order relation, with no explicit payoff numbers. Using numbers improves readability and makes it easier to talk about examples.

- \(I\) is an equivalence relation on \(H\) that is compatible with the player function as well as with the action function. By convention the equivalence relation is expressed in terms of information sets, which are partitions \((I_{i,j})_{(i,j) \in N \times N}\), one for each player and integer index. Formally, it fulfills for any \(i\) and \(j\) that \(h \in I_{i,j} \implies \rho(h) = i\) as well as \(h, h' \in I_{i,j} \implies \chi(h) = \chi(h')\).

2.2 Notations

Since the definition of the solution concept in this paper involves large formulas, we use a few notations that make them easier to read.

For payoffs, we write \(u_i(z)\) rather than \(u(i,z)\) to denote the payoff of player \(i\) at outcome \(z\) for convenience.

For navigation in the tree, we write \(h \oplus a\) for \(\sigma(h,a)\).

We also write \(I_{i,j} \oplus a = \cup_{n \in I_{i,j}} \{n \oplus a\}\) for an information set \(I_{i,j}\) to denote all its successor nodes for a specific action. This is consistent with extending the application of a function to a set of inputs, as is common in algebra literature.

Finally, we use the letter \(\delta\) to denote the set of all the descendants of a node in the tree induced by \(\sigma\).

Definition 2 (Descendant function) Given a game with imperfect information \((N,A,H,Z,\chi,\rho,\sigma,u,I)\), the descendant function maps any choice node or outcome to the
set of all its descendants, i.e., in its transitive and reflexive closure via $\sigma$. For $h \in H \cup Z$, 

$$\delta(h) = \{ h' \in H \cup Z | h = h' \lor \exists k \in \mathbb{N}^+, \exists (a_i)_{i=1..k} \in A^k, h' = h \oplus a_1 \ldots \oplus a_k \}$$

Whenever we have two nodes $h \neq h'$ such that $h' \in \delta(h)$, we introduce the notation $h'_h$ as the only action $a$ such that $h' \in \delta(h \oplus a)$. It always exists because if $h'$ is a descendant of $h$, then it is in one of its subtrees.

Following standard mathematical practice, we also use the notation $\delta(I_{i,j})$ where $I_{i,j}$ is an information set, to denote the set of all descendants of all nodes in this information set.

### 2.3 Canonical form of a game with imperfect information

Some games are allowed by the definition given in Section 2.1 but contain subtrees that cannot be reached by the definition of the game itself. This happens when a node is a descendant of another node that is in the same information set: if $h' \in \delta(h \oplus a)$ for some $a \in A$, then the nodes in $\delta(h' \oplus b)$ for any $b \neq a$ are never reached for any assignment of information sets to choices of actions.

Such subtrees can simply be pruned out with no changes to the semantics of the game. We call this the canonical form.

**Definition 3 (Canonical form of a game)** A game with imperfect information is a tuple $(N,A,H,Z,\chi,\rho,\sigma,u,I)$ is in canonical form if no information set contains two node that are different, and one is the descendant of the other:

$$\forall I_{i,j} \in I, \forall n \in I_{i,j}, \delta(n) \cap I_{i,j} = \{ n \}$$

A game can be put in canonical form by pruning all nodes $n$ that are the strict descendants of another node $m$ in their information set, replacing such a node $n$ by its subtree $n \oplus n_m$.

We assume in the remainder of this paper that all the considered games have been made canonical.

### 3 Games with imperfect information and Minkowski spacetime

#### 3.1 Minkowski spacetime

We now give an interpretation of a subcategory of games in extensive form with imperfect information in the context of special relativity. We consider a Lorentzian manifold in which the metric tensor is constant, i.e., it is the same at all positions across space and time. In this setting, the distance between two points is independent of the observer, assuming that inertial timeframes are obtained from each other by Lorentz transformations. This is known as Minkowski spacetime ([Minkowski, 1908](#)).
We are thus looking at a vector space \( \mathbb{R}^n \) endowed with a non-degenerate, symmetric bilinear form (a \( n \times n \) matrix). We assume a metric signature \((n - 1, 1)\), having in mind that, in practice, this is commonly \((3, 1)\). The first \((n - 1)\) coordinates are known as space, the last one as time.

Given two events (vectors) in spacetime (our vector space), we can calculate their distance in spacetime with the bilinear form. The sign of this distance allows us to classify pairs of vector as one of two cases:

- either they are timelike-separated, meaning that any observer (inertial timeframe) would see these two events occur after one another, in always the same order. We can thus say that one of the two events precedes the other, because its time coordinates are smaller than the other event’s time coordinates in any inertial timeframe.
- or they are spacelike-separated, meaning that the order in which these events occur depends on the observer. No signal can be sent between these two events, because it would involve faster-than-light travel, equivalent to travelling back in time for some other observer.

3.2 Decision points

Now that we have a Minkowski spacetime, we can place in it what we can call decision points at which agents in a set \( \hat{N} \) make decisions taken from a global set of actions \( \hat{A} \).

We denote a decision point \( \hat{I}_{i,j} \) in which agent \( i \in \hat{N} \) makes the decision, and \( j \in \mathbb{N}^* \) indexes all the decision points at which agent \( i \) makes the decision. We call \( \hat{I} \) the set of all decision points.

We denote \( \hat{\chi}(\hat{I}_{i,j}) \subseteq \hat{A} \) the set of possible actions that agent \( i \) can take at decision point \( \hat{I}_{i,j} \).

Each decision point has a location in spacetime. Given two decision points, we can thus say that they are either spacelike-separated, or timelike-separated based on their coordinates – or that they coincide if the distance is 0. The precise locations are irrelevant to us.

It is thus possible to list all decision points in a certain order that is compatible with their spacetime coordinates, i.e., whenever two decision points \( I \) and \( J \) are timelike-separated, then if \( I \) precedes \( J \) in spacetime, then \( I \) must precede \( J \) in the list. Decision points that are co-located must appear together in the list. Such a list is not unique, because spacelike-separation gives a few degrees of freedom, but it always exists. This is because, if it were impossible to build such a list because of a cycle in the order of events, we would be looking at a closed timelike curve, which does not exist in Minkowski spacetime\(^3\).

We denote this list of decision points \((\hat{I}_1, \hat{I}_2, ..., \hat{I}_n)\) where \( n \) is the total number of decision points. Thus, when an decision point has one index, we mean its absolute position in the ordered list selected in the former paragraph. When it has two indices,

\(^3\) but may exist in general relativity
we mean that the first index is the agent, and the second index its (arbitrary) index within the agent’s decision points.

It is crucial to distinguish between two orders: a partial order with timelike-separation semantics relative to spacetime, and a total order based on the selected ordered list of decision points, the latter being a superset of the former. We will denote the former \( \prec \) and the latter \(<\).

3.3 Contingency coordinates

Each decision point has, in addition to its spacetime coordinates, contingency coordinates. The contingency coordinates of a decision point are the actions that must be taken at previous, timelike-separated decision points for this decision point to actually be reached. In other words, the ability to make that decision must be caused by one specific set of events at decision points preceding it in spacetime.

For example, Peter at a first decision point \( I_1 \) may pick action \( a \) or \( b \), and John at a second decision point \( I_2 \not< I_1 \), may pick \( c \) or \( d \). If Peter has picked \( a \) and John picked \( c \), then Mary at a third decision point \( I_3 \) such that \( I_3 \succ I_1 \) and \( I_3 \succ I_2 \) can pick \( e \) or \( f \). We say that the contingency coordinates of Mary’s decision point are \((a,c)\), because Mary is only given this choice in case Peter picked \( a \) and John picked \( c \), and otherwise not.

The contingency coordinates of decision point \( \hat{I}_k \) are thus an assignment of an action \( a_{k,l} \), \( l \) to decision points \( \hat{I}_l \prec \hat{I}_k \) (so \( a_{k,l} \) is only defined for \( l < k \)), but with two constraints:

- Only those decision points \( \hat{I}_l \) such that \( \hat{I}_l \prec \hat{I}_k \) are assigned an action. The others are assigned a dummy value \( a_{k,l} = \bot \), which is only a notational convention.
- A decision point \( \hat{I}_l \) is only assigned an action \( a_{k,l} \) if \( (a_{l,1},...,a_{l,l-1}) = (a_{k,1},...,a_{k,l-1}) \). Otherwise, \( a_{k,l} = \bot \).
- Conversely, if a decision point \( \hat{I}_l \prec \hat{I}_k \), and if \( a_{k,l} \) if \( (a_{l,1},...,a_{l,l-1}) = (a_{k,1},...,a_{k,l-1}) \), then \( a_{k,l} \) must be defined, i.e., \( a_{k,l} \not= \bot \).

The contingency coordinates of all decision points can be represented on paper as a two-dimensional triangle of actions.

3.4 Possible histories

We now look at all possible worlds that can be instantiated from the decision points, i.e., considering all possible actions that can be taken.

A history is an assignment of actions to decision points, which we denote \((h_k)_{1 \leq k \leq n}\), allowing for some decision points to be unassigned (\( \bot \)). This happens when a decision is not happening in the considered possible world, because the past does not cause making that decision.

We model the agent’s preferences with a function \( \hat{u} \) mapping each history and agent to a number with ordinal semantics, which is equivalent to a total order relation over histories for each agent.

\footnote{The notation of causality, in this paper, coincides with timelike-separation. \( A \) causes \( B \) if \( A \prec B \). We carefully separate the notion of causality from that of counterfactuality.}
An incomplete history is an assignment of actions to decision points the first \( o \leq l < n \) decision points, but not to the remaining ones.

Like contingency coordinates, a history must respect constraints, i.e., when the past history of a decision point (list of actions taken at previous decision points in the total order) matches its contingency coordinates, this decision point must be assigned an action, and conversely.

Formally, an incomplete history \((h_1,...,h_m)\) matches contingency coordinates \((a_{m,1},...,a_{m,m})\) if \(\forall 1 \leq k \leq l, (a_{m,k} \neq \bot \implies h_k = a_{m,k})\).

A history is consistent if, for any \(m\) such that \(1 \leq m \leq n\), all its prefix incomplete histories \((h_1,...,h_{m-1})\) match the contingency coordinates of \(I_m\) when \(h_m \neq \bot\), and conversely. This definition of consistency is extended to incomplete histories. If a history or an incomplete history is consistent, then all its prefixes are consistent as well.

We denote the set of all consistent histories \(\hat{Z}\), and the set of all consistent incomplete histories \(\hat{H}\), excluding those that end with \(\bot\), and excluding those that, if padded with \(\bot\) assignments for remaining decision points, match a history in \(\hat{Z}\).

3.5 Construction of the game with imperfect information

The decision points can then be seen as a partition of \(\hat{H}\), in which an incomplete history of size \(m < n\) is mapped to information set \(I_{m+1}\).

We define the function \(\hat{\rho}\) mapping an incomplete history \((h_1,...,h_m)\) to the agent in \(\hat{N}\) who decides at decision point \(I_{m+1}\).

We define the function \(\hat{\sigma}\) mapping an incomplete history (possibly empty if \(m = 0\)) \((h_1,...,h_m)\in \hat{H}\) and an action \(a \in \hat{\chi}(I_{m+1})\) to the (incomplete or complete) history \((h_1,...,h_m,\bot,...,\bot,a)\in \hat{H}\cup \hat{Z}\), where the position at which \(a\) appears is taken as the first possible position \(m\) such that \((h_1,...,h_m,\bot,...,\bot)\) matches the contingency coordinates of \(I_{m+1}\). The resulting (incomplete or complete) history is thus consistent by construction. If such a position does not exist, then we pad the history with \(\bot\) until the last position, in which case the result is a complete history. \(\hat{\sigma}\) is injective by construction, and the root of the so obtained tree is the empty history.

\((\hat{N},\hat{A},\hat{H},\hat{Z},\hat{\rho},\hat{\sigma},\hat{u},\hat{I})\) is then a game in extensive form with imperfect information, on which we can compute equilibria (Nash, PTE, etc). This game has a natural interpretation in which the players are agents located in Minkowski spacetime making decisions. The information sets are interpreted as the situations in which decisions are spacelike-separated and thus no signal can be sent between two agents.

Not all games with imperfect information can be obtained in this way. Further work includes characterizing this subclass of games for which an underlying semantics involving agents making decisions in Minkowski spacetime exists.

In the rest of this paper, we do have in mind that subclass of games as our prominent use cases, however, the generalized Perfectly Transparent Equilibrium is defined

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\[5\] This is an easier way to express that we build an equivalence relation on consistent (complete or incomplete) histories by trimming \(\bot\) assignments and comparing the remaining non-dummy actions, and that we quotient the set of consistent histories by this equivalent relation.
on all canonical games, even if they cannot be interpreted with agents located in Minkowski spacetime.

3.6 Link with quantum theory

This construct with decision points in spacetime is to put in direct perspective with Renner and Colbeck (2011)'s spacetime random variables (SV) in thought experiments modelling (i) physicists picking a measurement axis and (ii) the choice (by the universe) of the measurement outcome. Renner and Colbeck (2011) showed that the predictive power of quantum theory cannot be improved under a strong assumption of free choice.

Games with imperfect information compatible with an underlying spacetime semantics provide an alternate framework to discuss quantum measurements involving a game between agents and the universe. Such a framework implies that the universe has some utility or preference function, an idea that is not unusual in the history of science. For example, as pointed by Dupuy in e-mail discussions, the path followed by light can be expressed as a minimization problem amongst possible worlds. Many problems in mechanics can be expressed as the minimization or maximization of some quantity, e.g., the principle of least action. These formulations can be recast as a utility maximization problem.

The perfectly transparent equilibrium shows, in this framework, that an alternate theory based on weakened free choice can be constructed (Fourmy, 2019), which eliminates impossible worlds (Rantala, 1982; Kripke, 1965), and under some conditions singles out at most one specific actual world.

4 Computation of the Perfectly Transparent Equilibrium

We now consider any canonical game in extensive form with perfect information, and give the algorithm for computing the Perfectly Transparent Equilibrium. This is done by iterative elimination of those outcomes that cannot be the result of the game, assuming that the players perfectly predict them and are rational in all possible worlds.

We start with an initial set of outcomes containing all the outcomes of the game, $\mathcal{S}_0 = \mathbb{Z}$. For each $k$, we eliminate more outcomes from $\mathcal{S}_k$, building $\mathcal{S}_{k+1}$.

4.1 Reached information sets

Once it has been established that any outcome outside $\mathcal{S}_k$ is impossible, it is possible to derive the information that some information sets must be reached by the game: no matter what outcome would be reached, the information set intersects with the path leading to it, meaning that reaching this outcome involves making a decision at this information set no matter what.
Definition 4 (Reached information set) Given a game with imperfect information 
\((N, A, H, Z, \chi, \rho, \sigma, u, I)\) in canonical form, and given the set \(\mathcal{S}_k\) of surviving outcomes at step \(k \in \mathbb{N}\), an information set \(I_{i,j}\) is reached at step \(k\) if:

\[ \mathcal{S}_k \subseteq \delta(I_{i,j}) \]

We denote \(\mathcal{R}_k\) the set of all such information sets at step \(k\).

Reached information sets grow with each step, i.e., the algorithm is nothing else than a forward induction going from the root all the way to at most one surviving outcome.

4.2 Preemption

Knowing that some information sets are guaranteed to be reached, we can eliminate outcomes that cannot possibly be known as the equilibrium if the agents are rational in all possible worlds.

Such an outcome \(o \in \mathcal{S}_k\) is characterized by the fact that for some agent making a decision at some reached information set, a deviation from \(o\) by picking a different action that the one leading to \(o\) guarantees to this agent a minimum payoff that is greater than the one she would obtain at \(o\).

Preempted outcomes are characterized as those that do not Pareto-dominate the maximins of the agents, i.e., the highest payoff they can guarantee themselves by a smart choice of actions. If an outcome, for some agent, is worse than its maximin, and the agent knows that this is the result of the game, then this is incompatible with their rationality: being rational, they would have picked a different action. This is a proof by reductio ad absurdum that this outcome cannot be reached by the game. This is very similar to the normal-form version of the Perfectly Transparent Equilibrium.

Preemption can only be done by deviating at information sets that are reached by the game. This is because otherwise, a preemptive action could itself be preempted, as originally pointed out by [Dupuy (2000)] in his seeding work. If a preemption is carried out at an information set that is known to be reached, then no subsequent preemption can invalidate this preemption reasoning later on. This dependency resolution is addressed with the forward induction mechanism. This is the exact same idea as in the Perfect Prediction Equilibrium: the reasoning only involves nodes that are on the equilibrium path, which avoids backward induction paradoxes [Pettit and Sugden (1989)], because the justification of each choice is never motivated by what would happen at an information set that is actually not reached.

Definition 5 (Preempted outcomes) Given a game with imperfect information \((N, A, H, Z, \chi, \rho, \sigma, u, I)\) in canonical form, and given the set \(\mathcal{S}_k\) of surviving outcomes at step \(k \in \mathbb{N}\) as well as the set of reached information sets \(\mathcal{R}_k\) at step \(k\), we say that an outcome \(z \in \mathcal{S}_k\) is preempted by player \(i \in N\) at a reached information set \(I_{i,j} \in \mathcal{R}_k\) if:
\[ u_i(z) < \max_{a \in \chi(I_{i,j})} \min_{z' \in \mathcal{J}_k \cap \delta(I_{i,j} \oplus a)} u_i(z') \]
\[
\text{s.t.} \quad \mathcal{J}_k \cap \delta(I_{i,j} \oplus a) \neq \emptyset
\]

We call \( \mathcal{P}_k \) the set of all outcomes that are preempted for some reached information set \( I_{i,j} \) by some player \( i \in N \).

Note that, when a preemption of an outcome \( z \) occurs, the \( \arg\max a \in \chi(I_{i,j}) \) is never \( z_{I_{i,j}} \) because \( u_i(z) \) is never strictly smaller than the minimum in the subtree to which \( z \) belongs, \( z_{I_{i,j}} \).

4.3 Recursion

Knowing the preempted outcomes at step \( k \), one deduces the new set of surviving outcomes, \( \mathcal{J}_{k+1} \).

**Definition 6 (Surviving outcomes)** Given a game with imperfect information \( (N, A, H, Z, \chi, \rho, \sigma, u, I) \) in canonical form, and given the set \( \mathcal{J}_k \) of surviving outcomes at step \( k \in \mathbb{N} \) as well as the set of reached information sets \( \mathcal{R}_k \) at step \( k \) and the preempted outcomes \( \mathcal{P}_k \) at set \( k \), the surviving outcomes at step \( k + 1 \) are:

\[ \mathcal{J}_{k+1} = \mathcal{J}_k \setminus \mathcal{P}_k \]

The recursion is initialized with \( \mathcal{J}_0 = Z \).

4.4 The equilibrium

The equilibrium then consists of the surviving outcomes.

**Definition 7 (Perfectly transparent equilibrium)** Given a game with imperfect information \( (N, A, H, Z, \chi, \rho, \sigma, u, I) \) in canonical form, an outcome \( z \in Z \) is a perfectly transparent equilibrium if it survives all rounds of elimination:

\[ z \in \bigcap_{k \in \mathbb{N}} \mathcal{J}_k \]

This surviving outcomes is at most unique for games with no ties in the payoffs.

**Theorem 1 (Uniqueness)** Given a game with imperfect information \( (N, A, H, Z, \chi, \rho, \sigma, u, I) \) in canonical form and with no ties, the perfectly transparent equilibrium is unique.
Proof (Uniqueness) The sequence \((S_k)\) is strictly decreasing until it reaches a singleton or the empty set. At every step \(k\), if we consider the lowest common ancestor of all surviving outcomes \(n\), the information set \(I_{i,j}\) it belongs to is reached. Let us consider the player \(i\) playing at this information set, and the outcome \(z\) with the lowest payoff for this player (which is unique because there are no ties).

Let us now consider an action \(a\) different from \(z_n\), so that \(\delta(n \oplus a)\) intersects with \(S_k\). Such an action \(a\) exists, because otherwise \(n\) would not be the lowest common ancestor: \(n \oplus z_n\) would be as well. Thus we have \(\delta(I_{i,j} \oplus a) \cap S_k \neq \emptyset\), meaning that the maximum at least involves another subtree.

Since the maximum also considers a different subtree at \(n\), this maximum is greater than \(u_i(z)\) because we picked this value to be minimal and there are no ties between payoffs for any agent.

Thus, the limit of the sequence \((S_k)\) can only be a singleton or an empty set.

Finally, this surviving outcome, if it exists, is always Pareto optimal.

Theorem 2 (Pareto optimality) Given a game with imperfect information \((N,A,H,Z,\chi,\rho,\sigma,u,I)\) in canonical form and with no ties, the perfectly transparent equilibrium, if it exists, is Pareto optimal in \(Z\).

Proof (Uniqueness) Assuming the PTE \(z\) is not Pareto optimal. Let \(z'\) be a Pareto improvement of \(z\). Let \(k\) be the step at which \(z'\) was eliminated, \(i\) the player who preempted it and \(I_{i,j}\) the reached information set that preempted it. Thus \(u_i(z')\) is smaller than the agent’s maximin, considering all players and all reached information sets. Thus \(u_i(z) < u_i(z')\) and \(z\) is preempted at this step too, which contradicts the fact that it is the PTE.

5 Conclusion

We have generalized the perfect prediction framework to any games with imperfect information. The underlying interpretation of the Perfectly Transparent Equilibrium is that, given agents endowed with weak free choice (“they could have acted otherwise”) and making decisions at any locations in Minkowski spacetime, it is the only possible world compatible with the fact that they are necessarily rational, and that perfectly predict each other’s choices.

This interpretation shows that perfect prediction cohabits well with imperfect information: information is not only deducted from actual signals, constrained by the speed of light; under a weaker form of free choice, information can also be deducted by logical reasoning, eliminating logically impossible worlds.

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