Abstract

This paper proves that for each positive integer $m$, there is a triangle-free planar graph $G$ which is not $(3m + [\frac{17}{11}] - 1, m)$-choosable.

Keywords: fractional choice number, multiple list colouring, triangle-free planar graphs, strong fractional choice number.

1 Introduction

Colouring of triangle free planar graphs has been studied extensively in the literature. It was proved by Grötzsch [3] that every triangle free planar graph is 3-colourable. On the other hand, Voigt [6] showed that there are triangle free planar graphs that are not 3-choosable. In this paper, we are interested in multiple list colouring of triangle free planar graphs.

A $b$-fold colouring of a graph is a graph $G$ is a mapping $\phi$ which assigns to each vertex $v$ of $G$ a set $\phi(v)$ of $b$ colours, so that adjacent vertices receive disjoint colour sets. An $(a, b)$-colouring of $G$ is a $b$-fold colouring $\phi$ of $G$ such that $\phi \subseteq \{1, 2, \ldots, a\}$ for each vertex $v$. The fractional chromatic number of $G$ is

$$\chi_f(G) = \inf \{ \frac{a}{b} : G \text{ is } (a,b)\text{-colourable} \}.$$ 

An $a$-list assignment of $G$ is a mapping $L$ which assigns to each vertex $v$ a set $L(v)$ of a permissible colours. A $b$-fold $L$-colouring of $G$ is a $b$-fold colouring $\phi$ of $G$ such that $\phi(v) \subseteq L(v)$ for each vertex $v$. We say $G$ is $(a, b)$-choosable if for any $a$-list assignment $L$ of $G$, there is a $b$-fold $L$-colouring of $G$. The fractional choice number of $G$ is

$$ch_f(G) = \inf \{ \frac{a}{b} : G \text{ is } (a,b)\text{-choosable} \}.$$ 

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It was proved by Alon, Tuza and Voigt [1] that for any finite graph $G$, $\chi(G) = ch_f(G)$ and moreover the infimum in the definition of $ch_f(G)$ is attained and hence can be replaced by minimum. This implies that if $G$ is $(a,b)$-colourable, then for some integer $m$, $G$ is $(am,bm)$-choosable. Recently, it was shown by Dvořák, Sereni and Volec [3] that an $n$-vertex triangle free planar graph $G$ has fractional chromatic number at most $9n^3n + 1$. Therefore, for each triangle free planar graph $G$, there is an integer $m$ such that $G$ is $(3m - 1, m)$-choosable.

A natural question is whether there is a constant $m$ such that every triangle free planar graph $G$ is $(3m, m)$-choosable. If not, what is the smallest real number $\epsilon$ such that every triangle free planar graph $G$ is $((3 + \epsilon)m, m)$-choosable.

For a positive real number $\alpha$, we say a graph $G$ is strongly fractional $\alpha$-choosable if for any positive integer $m$, $G$ is $([\alpha m], m)$-choosable. We define the strong fractional choice number $ch^*_f(G)$ of $G$ as

$$ch^*_f(G) = \inf\{\alpha : G \text{ is strongly fractional } \alpha\text{-choosable}\}.$$  

The strong fractional choice number of a family $\mathcal{G}$ of graphs is defined as

$$ch^*_f(\mathcal{G}) = \sup\{ch^*_f(G) : G \in \mathcal{G}\}.$$  

We are interested in the strong fractional choice number of the class of triangle free planar graphs.

The strong fractional choice number of planar graphs was studied in [10], where it was shown that for each positive integer $m$, there is a planar graph $G$ which is not $(4m + \lceil \frac{2m-1}{9} \rceil, m)$-choosable. Let $\mathcal{P}$ be the class of planar graphs. Then we have $ch^*_f(\mathcal{P}) \geq 4 + \frac{2}{9}$.

In this paper, we prove the following result. Let $\mathcal{Q}$ be the family of triangle free planar graphs.

**Theorem 1** For each positive integer $m$, there is a triangle-free planar graph $G$ which is not $(3m + \lceil \frac{m}{17} \rceil - 1, m)$-choosable. Consequently, $ch^*_f(\mathcal{Q}) \geq 3 + \frac{1}{17}$.

The $m = 1$ case of Theorem 1 is equivalent to say that there are non-3-choosable triangle-free planar graphs, which was proved by Voigt [6]. A smaller triangle-free planar graph was constructed by Glebov, Kostochka and Tashkinov [1].

## 2 The proof of Theorem 1

In this section, $m$ is a fixed positive integer. We shall construct a triangle-free planar graph $G$ which is not $(3m + \lceil \frac{m}{17} \rceil - 1, m)$-choosable.

**Lemma 2** Let $H$ be the graph as shown in Figure 1. Let $\epsilon$ be the real number such that $\epsilon m = \lceil \frac{m}{17} \rceil - 1$. Let $A, B$ be disjoint sets of $m$ colours. Then there is a list assignment $L$ of $H$ for which the following hold:
1. $L(u) = A$ and $L(v) = B$.

2. $|L(s)| = 3m + \epsilon m$ for each vertex $s \neq u, v$.

3. There is no $m$-fold $L$-colouring of $H$.

**Proof.** Let $A, B, C, D, E$ be disjoint colour sets, where $|A| = |B| = |C| = m$, $|D| = 2m$ and $|E| = \epsilon m$.

Let $L$ be the list assignment of $H$ defined as follows:

- $L(u) = A$ and $L(v) = B$.
- $L(x_1) = L(x_2) = L(x_3) = L(w) = A \cup B \cup C \cup D$.
- $L(y_1) = L(y_3) = L(z_5) = A \cup D \cup E$.
- $L(y_4) = L(z_1) = L(z_3) = B \cup D \cup E$.
- $L(y_2) = L(y_5) = L(z_2) = L(z_4) = C \cup D \cup E$.

We shall show that $H$ is not $L$-colourable. Assume to the contrary that $\phi$ is an $L$-colouring of $H$. 

![Figure 1: The graph $H$](image)
The vertices \( y_1, y_2, y_3, y_4, y_5 \) induce a 5-cycle. So each colour in \( D \cup E \) can be used at most twice on these five vertices. At least \((1 - \epsilon)m\) colours of \( C \) are used on vertex \( x_3 \). Hence at most \( em \) of colours from \( C \) can be used at vertex \( y_5 \). Similarly, at most \( \epsilon m \) of colours from \( C \) can be used at vertex \( y_2 \). Assume \( \tau m \) colours from \( A \) are used at vertex \( y_3 \). Since altogether we use \( 5m \) colours to colour these five vertices, we conclude \( \tau + 4\epsilon + 4 \geq 5 \). Hence

\[
\tau \geq 1 - 4\epsilon.
\]

As \( m \) colours from \( C \cup E \) is used by \( x_1 \), we know that at most \( \epsilon m \) colours of \( C \cup E \) are used at vertex \( x_2 \). As at least \((1 - 4\epsilon)m\) of \( A \) are used at \( y_3 \), so at most \( 4\epsilon m \) colours of \( A \) are used at vertex \( x_2 \).

So at least \((1 - 5\epsilon)m\) colours from \( B \) are used at vertex \( x_2 \).

At most \( \epsilon m \) colours from \( C \) are used at vertex \( z_2 \). At most \( 5\epsilon m \) colours from \( B \) are used at vertex \( z_2 \). At most \( 4\epsilon m \) colours from \( A \) are used at vertex \( z_5 \). Each colour from \( D \cup E \) is used at most twice among vertices \( z_1, z_2, z_3, z_4, z_5 \). Assume \( \sigma m \) colours from \( C \) are used at vertex \( z_4 \). Then \( \sigma + 12\epsilon + 4 \geq 5 \). Hence

\[
\sigma \geq 1 - 12\epsilon.
\]

Therefore, at most \( 4\epsilon m \) colours from \( A \) can be used at vertex \( w \), at most \( 12\epsilon m \) colours from \( C \) can be used at vertex \( w \) and at most \( \epsilon m \) colours from \( E \) can be used at vertex \( w \). So the total number of colour available to \( w \) is at most \( 17\epsilon m \). Since \( \epsilon < \frac{1}{17} \), we arrived at a contradiction. ■

It can be verified that for the list assignment \( L \) defined above, if \( E \) is a set of \( \left[ \frac{m}{17} \right] \) colours, then there is an \( L \)-colouring of \( H \).

Let \( p = \left( \frac{3m + \epsilon m}{m} \right) \), and let \( G \) be obtained from the disjoint union of \( p^2 \) copies of \( H \) by identifying all the copies of \( u \) into a single vertex (also named as \( u \)) and all the copies of \( v \) into a single vertex (also named as \( v \)). It is obvious that \( G \) is a triangle-free planar graph.

Now we show that \( G \) is not \((3m + \epsilon m, m)\)-choosable. Let \( X \) and \( Y \) be two disjoint sets of \( 3m + \epsilon m \) colours. Let \( L(u) = X \) and \( L(v) = Y \). There are \( p^2 \) possible \( m \)-fold \( L \)-colourings of \( u \) and \( v \). Each such a colouring \( f \) corresponds to one copy of \( H \). In that copy of \( H \), define the list assignment as in the proof of Lemma \([2]\) by replacing \( A \) with \( \phi(u) \) and \( B \) with \( \phi(v) \). Now Lemma 2 implies that no \( m \)-fold colouring of \( u \) and \( v \) can be extended to an \( m \)-fold \( L \)-colouring of \( G \). This completes the proof of Theorem \([2]\).

3 Some open questions

We may define the choice number of a class \( \mathcal{G} \) of graphs as \( ch(\mathcal{G}) = \max\{ch(G) : G \in \mathcal{G} \} \). Then \( ch(\mathcal{Q}) = 4 \). The strong fractional choice number of graphs is intended to provide a finer scale for measuring the choosability of graphs. Indeed, it was conjectured by Erdős, Rubin and Taylor \([2]\) that if a graph \( G \) is \((a, b)\)-choosable, then for any positive
integer \( m \), \( G \) is \((am, bm)\)-choosable. If this conjecture is true, then \( ch^*_f(G) \leq ch(G) \) for any graph \( G \). However, the conjecture of Erdős, Rubin and Taylor is widely open. So the question whether \( ch^*_f(G) \leq ch(G) \) remains an open question. Nevertheless, since triangle free planar graphs are 3-degenerate, we know that for any positive integer \( m \), every triangle free planar graph \( G \) is \((4m, m)\)-choosable. So \( ch^*_f(\mathcal{Q}) \leq 4 \). A challenging problem is to determine the value of \( ch^*_f(\mathcal{Q}) \).

**Question 3** What is the value of \( ch^*_f(\mathcal{Q}) \)?

Question 3 maybe very difficult. The following question is easier, but also remains open.

**Question 4** Is there a positive integer \( m \) such that every triangle free planar graph is \((4m-1, m)\)-choosable?

If such an \( m \) exists, then the smallest possible value of \( m \) is 2.

**Question 5** Is it true that every triangle free planar graph is \((7, 2)\)-choosable?

If the conjecture of Erdős, Rubin and Taylor is true, then a positive answer to Question 4 would imply that \( ch^*_f(\mathcal{Q}) \leq 4 - \epsilon \) for some \( \epsilon > 0 \) and a positive answer to Question 5 would imply that \( ch^*_f(\mathcal{Q}) \leq 3.5 \).

It is known \([9]\) that for any finite graph \( G \), \( ch^*_f(G) \) is a rational number. It remains an open question as which rational numbers are the strong fractional choice number of graphs. The question restricted to planar graphs and triangle free planar graphs are also interesting.

**Question 6** For which rational numbers \( r \), there is a planar graph \( G \) with \( ch^*_f(G) = r \)? For which rational numbers \( r \), there is a triangle free planar graph \( G \) with \( ch^*_f(G) = r \)?

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