Quantum phases of electric dipole ensembles in atom chips

Jiannis K. Pachos
Department of Applied Mathematics and Theoretical Physics,
University of Cambridge, Cambridge CB3 0WA, UK.
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We present how a phase factor is generated when an electric dipole moves along a closed trajectory inside a magnetic field gradient. The similarity of this situation with charged particles in a magnetic field can be employed to simulate condensed matter models, such as the quantum Hall effect and chiral spin Hamiltonians, with ultra cold atoms integrated on atom chips. To illustrate this we consider a triangular configuration of a two dimensional optical lattice, where the chiral spin Hamiltonian $\sigma_i \cdot \sigma_j \times \sigma_k$ can be generated between any three neighbours on a lattice yielding an experimentally implementable chiral ground state.

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The interaction of the electromagnetic field with charged particles has unveiled a variety of geometrical and topological effects, such as the Aharonov-Bohm effect [1], and plays a predominant role in the generation of collective phenomena, such as the quantum Hall effect and fractional statistics [2]. The Aharonov-Bohm effect associates a quantum phase with the state of a charged particle. This phase is proportional to the enclosed magnetic flux when the particle undergoes a looping trajectory and it is obtained even if the magnetic field is vanishing in the immediate neighbourhood of the trajectory. This intriguing fact inspired a series of extensions, e.g. the reciprocal or the dual of the original Aharonov-Bohm effect [3, 4, 5, 6]. Here, we shall consider another possible generalisation that can be applied to the neutral atom technology of Bose-Einstein condensates and provides a unique laboratory for the simulation of condensed matter models, e.g. chiral spin lattices or the quantum Hall effect, exhibiting advanced controllability and long coherence times. Such engineered systems can play a significant role in probing topological quantum simulations and error-free quantum information processing [7].

With advances in cold atom technology it has been possible to simulate many solid state effects [8, 9, 10] by employing neutral atoms and techniques from quantum optics. Initially, it seemed that the neutrality of the atoms significantly restricts the range of phenomena this technology can simulate. Nevertheless, great effort has been made to develop neutral atom techniques, generating suitable phase factors by laser radiation to simulate the behaviour of charged particles [8, 11, 12, 13]. These include proposals for the creation of the Hofstadter butterfly [12] and the lattice implementation of the quantum Hall effect [14]. These realisations are limited in terms of their possible applications and it still remains to develop a technique for the efficient simulation of the effect of electromagnetic fields on charged spin particles.

Here, an alternative approach is presented, which allows us to go well beyond these applications. In the following we consider the electric dipole moment of the atoms in the presence of an appropriate electromagnetic field. The resulting effect is equivalent to a charged particle moving in the presence of a magnetic field. This can simulate both the continuous case of electrons confined in a two dimensional plane, resulting in the quantum Hall effect [15, 16], as well as the case of discrete lattice systems, which can result in the realisation of chiral states [17]. The advantage of the present scheme is based on the simplicity of the required control procedure: utilising the atomic electric dipole as an additional degree of freedom provides adequate resources for probing even more complex structures, such as charged spin systems.

Let us first see the general setup for implementing interactions between a particle and an external electromagnetic field. Consider the case where a particle has a charge $e$, and an electric or a magnetic moment, given by $\vec{d}_e$ or $\vec{d}_m$ respectively. The minimal coupling of the particle with the electromagnetic field is given by substituting its momentum for

$$\vec{p} \to \vec{p} + e\vec{A} + \vec{d}_m \times \vec{E} + (\vec{d}_e \cdot \nabla)\vec{A},$$

where $\vec{E}$ is the electric field and $\vec{A}$ is the magnetic vector potential. The second term in (1) requires that the particle is charged and gives rise to the well known Aharonov-Bohm effect [1]. While its topological character with the employment of a magnetic solenoid is of great conceptual value, and has been verified experimentally [18], it has also given rise to a variety of applications in the solid state arena [19]. The third term in (1) is the origin of the Aharonov-Casher effect [8], which is reciprocal to the Aharonov-Bohm effect. It involves the circulation of a magnetic dipole around a charged straight line and it has been experimentally verified [20]. Recent experiments have been performed that generalise the Aharonov-Casher effect thereby partly overcoming original technological complications [21]. The fourth term involves the coupling of the electric dipole moment to a differential of the vector potential and its consequences in cold atom technology will be the focus of this Letter.

In fact, electric dipoles can give rise to a variety of dif-
different phenomena that generate quantum phases when they undergo a cyclic trajectory \( C \). Starting from (1) and by employing Stokes’s theorem the phase factor contribution to the final state of the dipole is eventually given by

\[
\phi = \int_S \left[ \nabla \times (\vec{B} \times \vec{d}_e) + \nabla \times \nabla (\vec{d}_e \cdot \vec{A}) \right] \cdot d\vec{s},
\]

(2)

where \( S \) is a surface bordered by the cyclic path \( C \) and \( d\vec{s} \) is its elementary area. The second term on the right hand side is the curl of the gradient of \( \vec{d}_e \cdot \vec{A} \) and can be taken to be zero, assuming continuity. The case where \( \vec{d}_e \cdot \vec{A} \) is a multi-variable function, as might be produced when the dipole passes through a magnetic sheet, results in topological effects that have been studied in [6]. Hence, from (2), we obtain

\[
\phi = \int_S (\vec{d}_e \cdot \nabla)\vec{B} = \vec{d}_e \cdot \nabla \vec{B} \cdot d\vec{s}.
\]

(3)

The second term on the right hand side is zero as can be seen from the Maxwell equations. Alternatively, a configuration with an infinite chain of magnetic monopoles, that alter the usual Maxwell equations, can lead to the generation of a topological phase dual to the Aharonov-Casher one [4]. For \( \nabla \cdot \vec{B} = 0 \) we finally obtain

\[
\phi = \int_S (\vec{d}_e \cdot \nabla)\vec{B} \cdot d\vec{s}.
\]

(4)

This equation can be viewed as the differential Aharonov-Bohm effect. By inspection of relation (1) we see that a nontrivial phase can be produced if we generate an inhomogeneous magnetic field in the neighbourhood of the dipole. In particular, a non-zero gradient of the magnetic field component perpendicular to the surface \( S \), varying in the direction of the dipole, ensures a non-zero phase factor. The dual effect with a magnetic dipole and a gradient of an electric field can also give a similar phase factor that originates from the third term in (1), while a homogeneous electric field will just orient the electric dipole appropriately. As a realisation of (1), we can take \( S \) to lie on the \( x-y \) plane and \( \vec{d}_e \) to be perpendicular to the surface \( S \). A non-zero phase, \( \phi \), is produced if there is a non-vanishing gradient of the magnetic field along the \( z \) direction (see Figure 1(a)). Alternatively, if \( \vec{d}_e \) is along the surface plane, then a non-zero phase is produced if the \( z \) component of the magnetic field has a non-vanishing gradient along the direction of \( \vec{d}_e \) as shown in Fig. 1(b). Furthermore let us consider the case where the gradient of the magnetic field is spatially restricted, e.g. in a finite interval along the \( x \) direction. Then a trajectory of an electric dipole outside this area, \( S \), gives rise to a topological phase independent of the local details of the trajectory.

FIG. 1: The path circulation of the electric dipole in the inhomogeneous magnetic field. Fig. (a) depicts magnetic field lines sufficient to produce the appropriate non-vanishing gradient of \( B_z \) along the \( z \) axis. Fig. (b) depicts only \( B_z \) and how it varies along the direction of the dipole, \( \vec{d}_e \).

These configurations of magnetic field gradients have to satisfy the Maxwell equations, \( \nabla \cdot \vec{B} = 0 \) and \( \nabla \times \vec{B} = 0 \). Indeed, for \( \partial B_z / \partial x \neq 0 \) one should allow for an additional variation of the \( x \) component of the magnetic field along the \( z \) direction with equal gradient. It can be verified easily that this satisfies Maxwell equations and provides the nontrivial phase factor given in (4). The same holds in the presence of a uniform electric field used to orient the electric dipole appropriately.

Let us now consider the implementation of the phase factor [11] with neutral atoms confined on a two dimensional plane. The electric dipole moment of an atom, which can be induced with the application of an external homogeneous electric field, is of the order \( |d_e| \approx n^2 e a_0 \), where \( n \) is the principal quantum number of the electron orbit, \( e \) is its charge and \( a_0 \) is the Bohr radius. It is even plausible to consider the employment of Rydberg atoms for which \( n \) is quite large [22]. By applying an external macroscopic magnetic field we achieve gradients of the order \( \partial B_z / \partial x \approx 10^2 - 10^3 \) Gauss/m. Much larger gradients of the magnetic field are feasible with atom chip technology due to the miniaturisation of their structure. For example, gradients of the order of \( 10^8 \) Gauss/m have already been reported at Caltech for the manipulation of ultra cold neutral atoms [22]. This technology should prove very fruitful for the future of Bose-Einstein experiments where whole atomic ensembles can be mounted on atom chips [24, 25, 26]. With these at hand we can calculate that for atomic number \( n = 1 \) and for the circulation of an area of the order of the optical wavelength (\( \sim 1 \mu \text{m} \)) squared one can achieve phases of the order of \( \phi \approx 8 > 2\pi \). Hence, by varying the gradient of the magnetic field or even the orientation of the electric dipole via an additional electric field it is possible to obtain an arbitrary value for the phase factor \( e^{i\phi} \). This provides a great degree of controllability when simulating charged particles in the presence of magnetic fields of arbitrarily large intensity. Next we shall see applications of this property.

An important tool in the manipulation of atomic ensembles is the employment of optical lattices. They can generate one, two or three dimensional structures of potential minima for the atoms. In particular, a one dimen-
sional optical lattice is sufficient to confine the atomic ensemble in parallel planes above an atom chip, providing two dimensional confinement, a necessary condition for the implementation of the proposed phase factor with atomic ensembles. For a sufficiently high flux of the effective magnetic field through an area $S$, given by $e^*B^*S/\hbar$, and low two-dimensional densities of atoms, $n_{2D}$, one can obtain $\nu = n_{2D}/(e^*B^*/\hbar) = 1/2$. This corresponds to the fractional quantum Hall effect [16, 17] which can be described by the $m = 2$ Laughlin wavefunction [28].

Moreover, additional optical lattices can be applied to create a regular structure on this plane in order to simulate two dimensional topological spin effects. This is achieved by generating complex tunneling interactions along the planar lattice sites. In particular, consider two species of atoms, namely $\sigma = \{\uparrow, \downarrow\}$, that are superimposed with particular configurations of optical lattices. The evolution of the system, for atoms restricted in the lowest Bloch-band, is described by the Bose-Hubbard Hamiltonian that is comprised of tunneling transitions of atoms between neighbouring sites of the lattice, $V = -\sum_{\sigma} (J_\uparrow a_\sigma^\dagger a_{\sigma+1\uparrow} + H.c.)$, and collisional interactions between atoms in the same site, $H' = \frac{1}{2} \sum_{\sigma} U_{\sigma\sigma} a_\sigma^\dagger a_\sigma^\dagger a_{\sigma} a_{\sigma}$. For the two dimensional confinement, a necessary condition to create a regular structure on this plane in order to manifest the breaking of the symmetry under time reversal, $T$, in our model, eventually producing an effective Hamiltonian that is not invariant under complex conjugation of the tunneling couplings. Moreover, as the second order perturbation theory is manifestly $T$ symmetric, we need to consider the third order. In that case the effective Hamiltonian becomes

$$H_{\text{eff}} = \sum_{\langle i,j \rangle} \left[ \tau^{(1)}(\sigma_i^\uparrow \sigma_j^\uparrow + \tau^{(2)}(\sigma_i^\uparrow \sigma_j^\downarrow + \sigma_i^\downarrow \sigma_j^\uparrow) + \tau^{(3)}(\sigma_i^\uparrow \sigma_j^\downarrow - \sigma_i^\downarrow \sigma_j^\uparrow) \right] + \tau^{(4)} \sum_{\langle i,j,k \rangle} \vec{\sigma}_i \cdot \vec{\sigma}_j \times \vec{\sigma}_k,$$

with $\vec{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$ and $\langle ... \rangle$ denoting nearest neighbour sites. The couplings appearing in (5) are given by

$$\tau^{(1)} = J_{\text{eff}}^{(2)}(\frac{1}{U_{\uparrow\uparrow}} - \frac{1}{2U_{\uparrow\downarrow}}) + \langle \uparrow\uparrow \downarrow \rangle,$$

$$\tau^{(2)} = J_{\text{eff}}^{(4)} U_{\uparrow\uparrow} + \langle \uparrow\downarrow \downarrow \rangle,$$

$$\tau^{(3)} = J_{\text{eff}}^{(4)} U_{\uparrow\downarrow} + \langle \uparrow\downarrow \downarrow \rangle,$$

$$\tau^{(4)} = \frac{1}{U_{\uparrow\downarrow}^2} + \langle \uparrow\downarrow \downarrow \rangle - \langle \uparrow\downarrow \downarrow \rangle.$$

Extraneous Zeeman terms in the $z$ direction can be dealt with by Raman transitions that effectively create a compensating field of the form $B_z = 2(J_{\text{eff}}^{(2)} / U_{\uparrow\uparrow} - J_{\text{eff}}^{(4)} / U_{\uparrow\downarrow})$. The coupling constants, $\tau^{(i)}$, can take various values. For example, one can choose $U_{\uparrow\downarrow} \rightarrow \infty$ (e.g. by employing fermionic atoms) so that $\tau^{(2)}$ vanishes. For certain values of the collisional couplings one can also vary the tunneling couplings such that $\tau^{(1)} = 0$. Moreover, by appropriately tuning the phase of the tunneling couplings one can choose either $\tau^{(3)}$ or $\tau^{(4)}$ to vanish.

Remarkably, with this physical proposal, a chiral three-spin interaction appears in (5), which can be isolated, especially from the Zeeman terms that are predominant in equivalent solid state systems. This interaction term is also known in the literature as the chirality operator [17]. It breaks the time reversal symmetry of the system as a consequence of the externally applied field. A chiral spin state is then a state for which the expectation value of the chirality operator has a nonzero value independent of the position of the plaquette $\langle i, j, k \rangle$.

As a particular example, for studying the behaviour of the three spin interaction term, we take a hexagonal configuration of 19 spin (see Fig. 2a). On this lattice we simulate Hamiltonian [5] for $\tau^{(1)} = 0$, $\tau^{(2)} = \tau^{(4)}$ ($\tau^{(4)} > 0$) and $\tau^{(3)} = 0$ with the additional condition that the spins on the boundary experience a strong magnetic field oriented in the $z$-direction. A numerical simulation
has been performed to obtain the energy gap, $\Delta$, above the ground state (see Fig. 2(b)) as well as the chirality of the ground state, $\chi \equiv \langle \vec{\sigma}_i \cdot \vec{\sigma}_j \times \vec{\sigma}_k \rangle$, for each triangular plaquette of neighbouring sites $i$, $j$ and $k$ (see Fig. 2(c) and (d)). Indeed, Fig. 2(b) shows that there is criticality behaviour for $\tau_c \approx -0.7$ and $\tau_{c_2} \approx -0.1$, where the energy gap becomes zero. In Fig. 2(d) the chirality for different values of $\tau$ and for different plaquettes on the plane numbered by $M$ is displayed where a reconstruction of the chirality on the plane is given in $\text{Fig. 2(c,1)}$, (c,2) and (c,3) for the three distinctive areas.

For $\tau < \tau_{c_1}$ the exchange interaction, $H_e = \sum_{(ij)} (\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y)$, dominates the chiral interaction, $H_c = \sum_{(ijk)} \vec{\sigma}_i \cdot \vec{\sigma}_j \times \vec{\sigma}_k$, forcing the spins to be aligned, pointing upwards in agreement to the boundary conditions. In this case $\chi$ is almost zero (see Fig. 2(c,3)). This holds also when we consider the Ising or a Zeeman interaction dominating $H_c$. For $\tau_{c_1} < \tau$, the interaction $H_c$ dominates giving a non-zero value to $\chi$. At $\tau_{c_1}$ and $\tau_{c_2}$ the chiral order parameter jumps indicating a quantum phase transition towards a chiral phase of the spin system. In particular, the plaquettes that have two sites on the boundary have small $\chi$ as both of the boundary spins tend to have parallel orientation. There is a small jump in $\chi$ around $\tau_{c_2}$ as $H_e$ induces a transition between energetically favourable states from ferromagnetic ($\tau < 0$) to antiferromagnetic ones ($\tau > 0$), for which case $\chi$ increases due to frustration. As can be easily calculated for a single triangle, $H_e$ has doubly degenerate ground states. This degeneracy is actually lifted by the chiral interaction giving a non-zero $\chi$ for $\tau > 0$. Indeed, one finds that on a triangle the common ground state of the two interactions is $|\Psi\rangle = \frac{1}{\sqrt{2}}(|↑↑↓⟩ + \omega |↓↓↑⟩ + \omega^2 |↑↓↑⟩)$ which has non-vanishing chirality. This fact is in agreement with the persistence of chirality for large $\tau$ as shown in Fig. 2(d), which is the regime of the adopted perturbation theory. This experimentally feasible domain of couplings exhibits as a ground state a chiral spin state that may be possible to detect with the state of the art technology.

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