Shot Noise Induced Charge and Potential Fluctuations of Edge States in Proximity of a Gate

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Abstract. We evaluate the RC-time of edge states capacitively coupled to a gate located away from a QPC which allows for partial transmission of an edge channel. At long times or low frequencies the RC-time governs the relaxation of charge and current and governs the fluctuations of the equilibrium electrostatic potential. The RC-time in mesoscopic structures is determined by an electrochemical capacitance which depends on the density of states of the edge states and a charge relaxation resistance. In the non-equilibrium case, in the presence of transport, the shot noise leads to charge fluctuations in proximity of the gate which are again determined by the equilibrium electrochemical capacitance but with a novel resistance. The case of multiple edge states is discussed and the effect of a dephasing voltage probe on these resistances is investigated. The potential fluctuations characterized by these capacitances and resistances are of interest since they determine the dephasing rate in Coulomb coupled mesoscopic conductors.

1 Introduction

Dynamic fluctuations in mesoscopic conductors have attracted considerable attention. Most of the work has focused on the low frequency white noise limit of the current fluctuations that can be measured at the terminals of a conductor [1]. Much less is known, if we ask about fluctuations at higher frequencies. To be sure, there are a number of questions which can be asked in a frequency range for which the scattering matrix of the conductor can still be taken energy independent. All that matters in this regime is the frequency dependence of Fermi functions which govern the occupation of the states incident form a reservoir. Much more interesting problems arise if we ask questions which directly probe the energy dependence of the scattering matrix.

In this work we are concerned with charge and potential fluctuations in Coulomb coupled systems. Such systems are of increasing interest because one of the systems can serve as a measurement probe of the other system [2, 3]. Coulomb coupled mesoscopic systems are also of interest in the investigation of dephasing: through the long range Coulomb interactions the proximity of a mesoscopic conductor affects the dephasing rate in the other conductor [4, 5]. The dephasing rate is essentially determined by the fluctuations of the electrostatic potential which leads directly to the fluctuation of the phase of a carrier. Thus a theoretical description and experimental characterization of potential fluctuations is essential for an understanding of such problems. Perhaps the simplest
Coulomb coupled system consists of a mesoscopic capacitor: two small plates, separated by a barrier which is too high to permit carrier exchange, are each separately coupled to a reservoir. Such a system permits no dc-transport, but exhibits an ac-conductance and exhibits frequency dependent charge, potential and current fluctuations \[6, 7\]. From the point of view of the scattering theory of electrical transport, it is a simple example, in which the energy dependence of the scattering matrix is crucial. We are not merely testing the transmission probability of a conductor, nor the frequency dependence of the Fermi functions, but are now asking a question that is sensitive to the charge distribution and its dynamics. The questions we wish to address and illustrate with a simple example in this article are of this nature.

The dynamic behavior of a capacitor is determined by its RC-time. At long times, the relaxation of charge and current and the electrostatic potential is determined by this time. Thus it is interesting to ask: What is the RC-time of a phase-coherent conductor? Ref. [6, 7] considered two small conductors each of which is connected only via a single lead to an electron reservoir. The two conductors interact only via the long range Coulomb force. Assuming that the main effect of the Coulomb interaction is the energy cost to charge the system, Ref. [6] presents an answer in terms of the geometrical capacitance and the energy derivatives of the scattering matrix. The resulting capacitance is called an electrochemical capacitance \( C_{\mu} \), and the resistance of the structure is called a charge relaxation resistance \( R_q \), to distinguish them from their geometrical and classical counterparts. Note that such a system has an infinite dc-resistance and.

\[
\begin{align*}
\text{Fig. 1. Hall bar with a quantum point contact and a gate overlapping the edge of the conductor.}
\end{align*}
\]

thus the expression for the resistance \( R_q \) which governs the relaxation of charge looks very different from the scattering matrix expressions for dc-resistances of conductors with non-vanishing transmission probability.

The RC-time plays a central role also for mutually Coulomb coupled multiprobe conductors. In multiterminal structures, especially if they are ballistic,
additional inductive-like time-scales appear [8, 10]. However, as soon as we consider such a conductor not in isolation, but coupled to another gate or conductor, the RC-time remains a fundamental quantity: if we keep all external potentials of the conductor at the same value, we are again faced with a purely capacitive question: To what extent can we charge this conductor against the other nearby conductor or gate?

A closely related phenomenon occurs if we drive the conductor out of equilibrium by applying a dc voltage to it. Now at zero temperature the conductor exhibits shot noise [9, 1] which in addition to the usually investigated current fluctuations at the contacts of the conductor, generates charge fluctuations. These charge fluctuations depend again crucially on the capacitance of the mesoscopic conductor vis-a-vis other nearby conductors or gates. For small driving voltages, we find in fact that the capacitance is $C_{\mu}$ as in the equilibrium system. But a novel resistance appears [11], which we call $R_v$ to indicate that it is connected to a non-equilibrium state obtained by applying a voltage $V$ to one of the conductors.

The example which we treat in this work is shown in Fig. 1. A conductor subject to a high magnetic field with a quantum point contact (QPC) is capacitively coupled to a gate. The contacts of the conductor are labeled 1 and 2 and the gate contact is labeled 3. We assume that the magnetic field is in a range at which the only states at the Fermi energy which connect contacts 1 and 2 are edge states [12]. A similar geometry without the QPC was investigated by Chen et al. [13]. In this work it was shown that an oscillating voltage applied to the gate (contact 3) generates a current only at contact 2 but if the magnetic field polarity is reversed the induced current is found only at contact 1. Since coupling between the gate and the mesoscopic sample is purely capacitive, this experiment verifies a prediction [14] that capacitance coefficients are in general not even functions of magnetic field. The geometry with the QPC is inspired by a recent experiment of Sprinzak et al. [5] which investigates the dephasing of a double quantum dot due to the charge fluctuations generated by a current through the QPC. Here we will consider the geometry with the gate, instead of the double quantum dot. The conductor of Fig. 1 permits an investigation of the electrochemical capacitance $C_{\mu}$ and the resistances $R_q$ and $R_v$ of this structure. The relationship of these transport coefficients to the dephasing time is the subject of Ref. [15]. We will not review this part of Ref. [15] but only mention that related work [16] addresses this question invoking only the fluctuations of non-interacting electrons. Here we treat the fluctuations within a charge and current conserving self-consistent random-phase approximation (RPA) which represents a dynamical extension [7, 11, 15] of Ref. [17].

2 The scattering matrix

To be specific we consider the conductor shown in Fig. 1. Of interest is the current $dI_{\alpha}(\omega)$ at contact $\alpha$ of this conductor if an oscillating voltage $dV_\beta(\omega)$ is
applied at contact $\beta$. Here $\alpha$ and $\beta$ label the contacts of the conductor and the gate and take the values $1, 2, 3$. Furthermore, we are interested in the current noise spectrum $S_{I_\alpha I_\beta}(\omega)$ defined as $2\pi S_{I_\alpha I_\beta}(\omega)\delta(\omega + \omega') = 1/2\langle \hat{I}_\alpha(\omega)\hat{I}_\beta(\omega') + \hat{I}_\beta(\omega')\hat{I}_\alpha(\omega) \rangle$ and the fluctuation spectrum of the electrostatic potential. We assume that the charge dynamics is relevant only in the region underneath the gate. Everywhere else we assume the charge to be screened completely. This is a strong assumption: In reality the QPC is made with the help of gates (capacitors) and also exhibits its own capacitance [8]. Edge states might generate long range fields, etc. Thus the results presented below can only be expected to capture the main effects but can certainly be refined. We assume that the gate is a macroscopic conductor and screens perfectly.

The scattering matrix of the QPC alone can be described by $r \equiv s_{11} = s_{22} = -i R_{1/2}$ and $t \equiv s_{21} = s_{12} = T_{1/2}$ where $T = 1 - R$ is the transmission probability through the QPC. Here the indices 1 and 2 label the reservoirs (see Fig. 1). A carrier traversing the region underneath the gate acquires a phase $\phi(U)$ which depends on the electrostatic potential $U$ in this region. Since we consider only the charge pile up in this region all additional phases in the scattering problem are here without relevance. The total scattering matrix of the QPC and the traversal of the region $\Omega$ is then simply

$$s = \begin{pmatrix} r & t \\ te^{i\phi} & re^{i\phi} \end{pmatrix}. \quad (1)$$

If the polarity of the magnetic field is reversed the scattering matrix is given by $s_{\alpha\beta}(B) = s_{\beta\alpha}(-B)$, i.e. in the reversed magnetic field it is only the second column of the scattering matrix which contains the phase $\phi(U)$. In what follows, the dependence of the scattering matrix on the phase $\phi$ is crucial. We emphasize that the approach presented here can be generalized by considering all the phases of the problem and by considering these phases and the amplitudes to depend on the entire electrostatic potential landscape [7].

3 Density of States Matrix Elements

To describe the charge distribution due to carriers in an energy interval $dE$ in our conductor, we consider the Fermi-field [1]

$$\hat{\psi}(r, t) = \sum_{\alpha m} \int dE \psi_{\alpha m}(r, E) \hat{a}_{\alpha m}(E) \exp(-iEt/\hbar) \quad (2)$$

which annihilates an electron at point $r$ and time $t$. The Fermi operator Eq. (2) is built up from all scattering states $\psi_{\alpha m}(r, E)$ which have unit incident amplitude in contact $\alpha$ in channel $m$. The operator $\hat{a}_{\alpha m}(E)$ annihilates an incident carrier in reservoir $\alpha$ in channel $m$. The local carrier density at at point $r$ and time $t$ is determined by $\hat{n}(r, t) = \hat{\psi}^\dagger(r, t)\hat{\psi}(r, t)$. We will investigate the density operator in the frequency domain, $\hat{n}(r, \omega)$. It is now very convenient and instructive to
consider an expression for the density operator not in terms of wave functions but more directly in terms of the scattering matrix. It can be shown [7], that the density operator \( \hat{n}(\mathbf{r}, \omega) \), in the zero frequency limit, can be written in the form

\[
\hat{n}(\mathbf{r}) = \sum_{\alpha\gamma\delta} \int dE \hat{a}^{\dagger}_{\gamma m}(E) n_{\gamma m\delta n}(\alpha, \mathbf{r}) \hat{a}_{\delta n}(E)
\]

where the elements \( n_{\gamma m\delta n} \) form a matrix of dimensions \( M_{\gamma} \times M_{\delta} \). Here \( M_{\gamma} \) is the number of channels at the Fermi energy in contact \( \gamma \). This matrix is given by [7]

\[
n_{\beta\gamma}(\alpha, \mathbf{r}) = -(1/4\pi i)[s^{\dagger}_{\alpha\beta}(\partial s_{\gamma\alpha}/\partial eU(\mathbf{r})) - (\partial s^{\dagger}_{\alpha\beta}/\partial eU(\mathbf{r}))s_{\gamma\alpha}].
\]

The low frequency charge dynamics can be found if these density of states matrix elements are known. Eq. (4) tells us that in order to find the carrier distribution and its fluctuations, we should introduce a small potential perturbation into the sample and find the scattering matrix which belongs to this perturbation. Clearly, such a detailed information requires a considerable effort and even more so, if we subsequently should solve the Poisson equation to find the electrostatic potential landscape which belongs to this density distribution. To proceed we introduce the simplifying assumption that it is only the charge pile-up near the gate which counts and moreover that the potential in this region \( \Omega \) can be described with a single potential parameter \( U \). All we need then is the density elements integrated over the region \( \Omega \). Instead of Eq. (3) we want to find

\[
\hat{N}(\mathbf{r}) = \sum_{\alpha\gamma\delta} \int_{\Omega} d^{3}r \int dE \hat{a}^{\dagger}_{\gamma m}(E) n_{\gamma m\delta n}(\alpha, \mathbf{r}) \hat{a}_{\delta n}(E)
\]

\[
\equiv \sum_{\alpha\gamma\delta} \int dE \hat{a}^{\dagger}_{\gamma m}(E) N_{\gamma m\delta n}(\alpha) \hat{a}_{\delta n}(E)
\]

with

\[
N_{\beta\gamma}(\alpha) = -(1/4\pi i)[s^{\dagger}_{\alpha\beta}(ds_{\gamma\alpha}/edU) - (ds^{\dagger}_{\alpha\beta}/edU)s_{\gamma\alpha}].
\]

Thus it is sufficient to find the variation in the scattering matrix for a potential that is uniform over the region of interest. In our example it is only the phase \( \phi \) in Eq. (1) which depends on \( U \). Thus we can evaluate the density of states elements if we know \( d\phi/edU \). But in the WKB-limit, which is sufficient for our purpose, \( d\phi/edU = -d\phi/dE \). However, \( d\phi/dE = 2\pi N \) where \( N \) is just the density of states of the edge state underneath the gate.

We are now ready to evaluate the density of states elements Eq. (4). For the specific example given by Eq. (1) we find that all elements with \( \alpha = 1 \) vanish: \( N_{11}(1) = N_{21}(1) = N_{12}(1) = N_{22}(1) = 0 \). There are no carriers incident from contact 1 or 2 which pass through region \( \Omega \) and leave the conductor through contact 1. The situation is different if we demand that the current leaves the sample through contact 2. Now we find

\[
N_{\beta\gamma}(2) = \begin{pmatrix} TN & t^{*}rN \\ r^{*}tN & RN \end{pmatrix},
\]
where, as already mentioned, $N$ is the density of states of carriers in the edge state underneath the gate. For the reverse magnetic field polarity all components of the matrix vanish except the elements $N_{22}(1) = T N$ and $N_{22}(2) = R N$.

For the charge and its fluctuations underneath the gate it is not relevant through which contact carriers leave. The charge pile up and its fluctuations are thus governed by a matrix
\[
N_{\beta\gamma} = \sum_{\alpha} N_{\beta\gamma}(\alpha)
\]
which is obtained by summing over the contact index $\alpha$ from the elements given by Eq. (4). For our example the density matrix elements for the charge are thus evidently given by $N_{\beta\gamma} = N_{\beta\gamma}(2)$ whereas for the reversed magnetic field polarity we have $N_{11} = T N$, $N_{22} = R N$ and $N_{21} = N_{21} = 0$.

Furthermore, we will make use of the *injectivity* of a contact into the region $\Omega$ and will make use of the emissivity of the region $\Omega$ into a contact. The injectivity of contact $\alpha$ is the charge injected into a region in response to a voltage variation at this contact, independently through which contact the carriers leave the sample [14]. The injectivities of contact 1 and 2 are
\begin{align*}
N_1 &= N_{11}(1) + N_{11}(2) = T N \\
N_2 &= N_{22}(1) + N_{22}(2) = R N
\end{align*}
Note that the sum of the injectivities of both contacts is just the density of states $N$ underneath the gate. The *emissivity* of region $\Omega$ is the portion of the density of states of carriers which leave the conductor through contact $\alpha$ irrespective from which contact they entered the conductor [14]. We find emissivities
\begin{align*}
N^{(1)} &= N_{11}(1) + N_{22}(1) = 0 \\
N^{(2)} &= N_{11}(2) + N_{22}(2) = N
\end{align*}
Any charge accumulation or depletion is only felt in contact 2. The injectivities and emissivities in the magnetic field $B$ are related by reciprocity to the emissivities and injectivities in the reversed magnetic field, $N_{\alpha}(B) = N^{(\alpha)}(-B)$ and $N^{(\alpha)}(B) = N_{\alpha}(-B)$. In contrast, the density of states $N$ is an even function of magnetic field.

4 The Poisson Equation: The effective interaction

Thus far we have only considered bare charges. The true charge, however, is determined by the long range Coulomb interaction. First we consider the screening of the average charges and in a second step we consider the screening of charge fluctuations. We describe the long range Coulomb interaction between the charge on the edge state and on the gate with the help of a geometrical capacitance $C$. The charge on the edge state beneath the gate is determined by the voltage difference between the edge state and the gate $dQ = C(dU - dV_g)$, where $dU$ and
\( dV_g \) are deviations from an equilibrium reference state. On the other hand the charge beneath the gate can also be expressed in terms of the injected charges \( e^2N_1dV_1 \) in response to a voltage variation at contact 1 and \( e^2N_2dV_2 \) in response to a voltage variation at contact 2. Furthermore, the injected charge leads to a response in the internal potential \( dU \) which in turn generates a screening charge \(-e^2NdU\) proportional to the density of states. Thus the Poisson equation for the charge underneath the gate is

\[
dQ = C(dU - dV_g) = e^2N_1dV_1 + e^2N_2dV_2 - e^2NdU
\]

and the charge on the gate is given by \(-dQ = C(dV_g - dU)\). Solving Eq. (13) for \( dU \) gives

\[
dU = G_{\text{eff}}(CdV_g + e^2N_1dV_1 + e^2N_2dV_2),
\]

where \( G_{\text{eff}} = (C + e^2N)^{-1} \) is an effective (RPA) interaction which gives the potential underneath the gate in response to an increment in the charge.

## 5 Admittance

Consider now the low-frequency conductance: To leading order in the frequency \( \omega \) we write

\[
G_{\alpha\beta}(\omega) = G_{\alpha\beta}(0) - i\omega E_{\alpha\beta} + \omega^2 K_{\alpha\beta} + O(\omega^3).
\]

Here \( G_{\alpha\beta}(0) \) is the dc-conductance matrix, \( E_{\alpha\beta} \) is the emittance matrix, and \( K_{\alpha\beta} \) is a second order dissipative contribution to the frequency dependent admittance. The zero-frequency dc-conductance matrix has only four non-vanishing elements which are given by \( G \equiv G_{11} = G_{22} = -G_{12} = -G_{21} = (e^2/h)T \). Ref. [14] showed that the emittance matrix \( E \) is given by

\[
E_{\alpha\beta} = e^2N_{\beta\beta}(\alpha) - e^2N^{(\alpha)}G_{\text{eff}}N_{\beta}
\]

As it is written, Eq. (16) applies only to the elements where \( \alpha \) and \( \beta \) take the values 1 or 2. The remaining elements can be obtained from current conservation (which demands that the elements of each row and column of this matrix add up to zero) or can be obtained directly by using a more general formula [8]. For our example we find an emittance matrix,

\[
E = C_\mu \begin{pmatrix} 0 & 0 & 0 \\ T & R & -1 \\ -T & -R & 1 \end{pmatrix},
\]

with an electrochemical capacitance of the conductor vis-a-vis the gate given by \( C_\mu = C e^2N/(C + e^2N) \). Eq. (17) determines the displacement currents in response to an oscillating voltage at one of the contacts. There is no displacement current at contact 1 (the elements of the first row vanish) which is consequence of our assumption that charge pile up occurs only underneath the gate. The emittance matrix in the magnetic field \( B \) and in the magnetic field \(-B\) are related by reciprocity, \( E_{\alpha\beta}(B) = E_{\beta\alpha}(-B) \). For the reverse polarity, a voltage
oscillation at contact 1 generates no displacement currents (the elements of the first column vanish).

The emittance matrix element $E_{21}$ is positive and thus has the sign not of a capacitive but of an inductive response. The elements of row 3 and column 3 are a consequence of purely capacitive coupling and have the sign associated with the elements of a capacitance matrix. Thus these elements represent the capacitance matrix elements which can be measured in an ac-experiment. Note that the capacitances $E_{31} ≡ C_{31}$ and $E_{32} ≡ C_{32}$ depend not only on the density of states and geometrical capacitances but also on transmission and reflection probabilities. Measurement of these capacitances provides thus a direct confirmation of the concept of partial density of states [14, 8]. Furthermore, we see that for instance $C_{31}(B) ≡ E_{31} = T C_{\mu}$ but $C_{31}(-B) = 0$. A similarly striking variation of the capacitance coefficients was observed in the experiment of Chen et al. [13] in the integer quantum Hall effect and in Refs. [19] in the fractional quantum Hall effect.

6 Bare charge fluctuations

Let us now turn to the charge fluctuations. With the help of the charge density matrix the low frequency limit of the bare charge fluctuations can be obtained [6, 11, 15]. It is given by

$$S_{NN}(\omega) = \hbar \sum_{\gamma} \int dE F_{\gamma\delta}(E, \omega) Tr[N_{\gamma\delta}(E, E + \hbar\omega) N_{\gamma\delta}^\dagger(E, E + \hbar\omega)]$$

(18)

where the elements of $N_{\gamma\delta}$ are in the zero-frequency limit of interest here given by Eq. (8) and $F_{\gamma\delta} = f_{\gamma}(E)(1 - f_{\delta}(E + \hbar\omega)) + f_{\delta}(E + \hbar\omega)(1 - f_{\gamma}(E))$ is a combination of Fermi functions. Using only the zero-frequency limit of the elements of the charge operator determined above gives,

$$S_{NN}(\omega) = \hbar N^2 \left[ T^2 \int dE F_{11}(E, \omega) + TR \int dE F_{12}(E, \omega) + TR \int dE F_{21}(E, \omega) + R^2 \int dE F_{22}(E, \omega) \right].$$

(19)

At equilibrium all the Fermi functions are identical and we obtain $S_{NN}(\omega) = \hbar N^2 \int dE F(E, \omega)$ which in the zero-frequency limit is

$$S_{NN}(\omega) = \hbar N^2 kT$$

(20)

and at zero-temperature to leading order in frequency is,

$$S_{NN}(\omega) = \hbar N^2 \hbar \omega.$$  

(21)

In the zero-temperature, zero-frequency limit, in the presence of a current through the sample, we find for the charge fluctuations associated with shot noise

$$S_{NN}(\omega) = \hbar N^2 TR e |V|.$$  

(22)

However, the bare charge fluctuations are not by themselves physically relevant.
7 Fluctuations of the true charge

To find the fluctuations of the true charge we now write the Poisson equation for the fluctuating charges. All contact potentials are at their equilibrium value, \( dV_1 = dV_2 = dV_g = 0 \). The fluctuations of the bare charge now generate fluctuations in the electrostatic potential. Thus the electrostatic potential has also to be represented by an operator \( \hat{U} \). Furthermore, the potential fluctuations are also screened. As in the case of the average charges we take the screening to be proportional to the density of states \( N \) but replace the c-number \( U \) by its operator expression \( \hat{U} \). The equation for the fluctuations of the true charge is thus

\[
\dot{\hat{Q}} = C \dot{\hat{U}} = e\hat{N} - e^2 \hat{N} \hat{U}
\]

whereas the fluctuation of the charge on the gate is simply \( -\dot{\hat{Q}} = -C \dot{U} \).

Thus \( \dot{\hat{Q}} \) is the charge operator which determines the dipole which forms between the charge on the edge state and the charge on the gate. Solving Eq. (23) for the potential operator \( \hat{U} \) and using this result to find the fluctuations of the charge \( \dot{\hat{Q}} \) gives

\[
S_{QQ}(\omega) = e^2 C^2 G_{eff}^2 S_{NN}(\omega) = 2C_\mu^2 (1/2e^2)(S_{NN}(\omega)/N^2). \tag{24}
\]

We now discuss three limits of this result.

8 Equilibrium and non-equilibrium charge relaxation resistance

At equilibrium, in the zero-frequency limit, the charge fluctuation spectrum can be written with the help of the equilibrium charge relaxation resistance \([6, 7, 11]\) \( R_q \),

\[
S_{QQ}(\omega) = 2C_\mu^2 R_q kT. \tag{25}
\]

For our specific example\([15]\), we find using Eqs. (20) and (24),

\[
R_q = h/2e^2. \tag{26}
\]

The charge relaxation resistance is universal and equal to half a resistance quantum as expected for a single edge state \([8]\). At equilibrium the fluctuation spectrum is via the fluctuation dissipation theorem directly related to the dissipative part of the admittance. We could also have directly evaluated the element \( K_{33} \) of Eq. (15) to find \( K_{33} = C_\mu^2 R_q \). Second at equilibrium, but for frequencies which are large compared to the thermal energy, but small compared to any intrinsic excitation frequencies, we find that zero-point fluctuations give rise to a noise power spectral density

\[
S_{QQ}(\omega) = 2C_\mu^2 R_q \hbar \omega \tag{27}
\]
which is determined by the charge relaxation resistance Eq. (26). Third, in the
presence of transport, we find in the zero-frequency, zero-temperature limit, a
charge fluctuation spectrum [11],
\[ S_{QQ}(\omega) = 2C_p^2 R_v e|V|, \]  
(28)
where |V| is the voltage applied between the two contacts of the sample and a
non-equilibrium charge relaxation resistance which for our example is given by
[15]
\[ R_v = (h/e^2)T\mathcal{R}. \]  
(29)
It is maximal for a semi-transparent QPC, \( T = \mathcal{R} = 1/2 \).

The current at the gate due to the charge fluctuations is \( dI_g = -i\omega dQ(\omega) \)
and thus its fluctuation spectrum is given by \( S_{t_x t_x}(\omega) = \omega^2 S_{QQ} \). The potential
fluctuations are related to the charge fluctuations by \( d\hat{U} = d\hat{Q}/C \) and thus the
spectral density of the potential fluctuations is \( S_{UU}(\omega) = C^{-2} S_{QQ} \). Thus the
charge relaxation resistance determines, together with the electrochemical and
geometrical capacitance, the fluctuations of the charge, the potential and the
current induced into the gate. Since dephasing rates can be linked to the low
frequency limit of the potential fluctuations [18] the resistances \( R_q \) and \( R_v \) also
determine the dephasing rate in Coulomb coupled mesoscopic conductors [15].

9 Several edge states

Let us next consider the case, where there are several edge states. A QPC in
a high magnetic field permits perfect transmission of the outer edge states (be-
longing to the lower Landau levels) and it is only the innermost edge state which
is partially transmitted or reflected at the QPC. Let us just consider two edge
states: the outer edge state labeled 1 is perfectly transmitted \( T_1 = 1 \), whereas
the inner edge state labeled 2 has a transmission probability \( T_2 \equiv T \) which might
take any value between zero and one. The outer edge state, with transmission
probability 1 is entirely noiseless as far as the shot noise in the total current is
concerned [1]. One might thus be tempted to think that such a perfectly trans-
mitted edge state plays no role at all. That however is not the case. Our result
involves screening in an essential manner and the charge fluctuations in one of
the edge states can now be screened by charge accumulation or depletion in the
other edge state. The screening properties depend on the electrostatic interac-
tion between the two edge states. Thus the answer we obtain depends on the
detailed electrostatic assumptions which we invoke to treat this problem. Here,
to provide a simple discussion, we assume that the two edge states are so close,
that they can be described with a common electrostatic potential \( U \). If we denote
the density of states of the edge states 1 and 2 in the region \( \Omega \) of interest by \( N_1 \)
and $N_2$ a detailed consideration, repeating the procedure given above for one edge state only, leads to an equilibrium charge relaxation resistance \[15\]

\[
R_q = \frac{h}{2e^2} \frac{N_1^2 + N_2^2}{(N_1 + N_2)^2} \tag{30}
\]

Note that in contrast to the single edge state, now $R_q$ depends explicitly on the densities of states. We can expect that the density of states $N_2$ of the inner edge state is typically larger than the density of states of the outer edge state since the potential for the inner edge state is much shallower. In this case $N_2 >> N_1$ and $R_q$ for the two edge states will in fact be the same as for one edge state only.

In contrast, for samples with a sharp edge, we can expect that both density of states are comparable, and thus $R_q$ for two edge states will be nearly a factor 2 smaller than the $R_q$ of a single edge state only.

Similarly, if we investigate $R_v$ for two edge states, we find \[15\]

\[
R_v = \frac{h}{2e^2} \frac{N_2^2}{(N_1 + N_2)^2} TR. \tag{31}
\]

Again the density of states of the two edge states appear now explicitly. The density of states of the outer edge state appears only in the denominator since it plays a role only in screening but it is not a primary source of charge fluctuations. In the limit $N_2 >> N_1$ of a shallow edge the outermost edge state is unimportant, whereas for a steep edge if both density of states are comparable, $R_v$ is reduced by a factor 4 compared to the case of a single edge state only.

Form the above results it is obvious how the formulas must be written if there is one edge state which is partially reflected or transmitted and many edge states which are perfectly transmitted.

10 Phase Randomization

Is the result given above sensitive to phase? Experimentally this question is investigated by Sprinzak et al. [5]. Our result for one channel, Eq. (29), contains only transmission probabilities. To investigate this question, we consider, like the experiment, an additional contact between the QPC and the region $\Omega$ as shown in Fig. 2. The contact will be considered as a voltage probe. An ideal voltage probe exhibits infinite impedance at all frequencies. Consequently, the net current at the voltage probe vanishes at every instant of time. Thus the voltage of the probe becomes a fluctuating quantity. Despite the fact that the total current vanishes, carriers leave the sample through this contact, and are replaced by carriers which enter from the reservoir. Carriers leaving into the reservoir and carriers reentering the conductor from the reservoir have no phase relationship and consequently a voltage probe acts as a dephaser [12].

The voltage probe changes the conductor: if we include the gate we now deal with a four probe conductor. We keep for the gate the label 3 and designate the voltage probe as contact 4. Since the potential is a function of time, we must also
know the dynamic conductance of the system. To begin we consider the general relation between currents and voltages of our four-terminal conductor. This relation takes the form of a Langevin equation which includes the fluctuating currents at the terminals as noise sources \[ dI_\alpha(\omega) = \sum_\beta G_{\alpha\beta}(\omega) dV_\beta(\omega) + \delta I_\alpha(\omega). \] (32)

Here \( G_{\alpha\beta}(\omega) \) is the self-consistent dynamic conductance and \( \delta I_\alpha(\omega) \) are the (self-consistent) frequency-dependent current fluctuations at the contacts of the conductor. Since the current spectrum at the gate is second order in frequency, it is sufficient to calculate the current amplitudes to first order in frequency. We thus need \( G_{\alpha\beta}(\omega) \) only to first order in frequency and write \( G_{\alpha\beta}(\omega) = G_{\alpha\beta}(0) - i\omega E_{\alpha\beta} + O(\omega^2) \). Here \( G_{\alpha\beta}(0) \) is the dc-conductance which for \( \nu - 1 \) perfectly transmitted channels and one partially transmitted channel at the QPC is given by \( G_{11} = -G_{12} = -G_{41} = (e^2/h)(\nu - 1 + T) \), \( G_{22} = G_{44} = -G_{24} = -(e^2/h)\nu \) and \( G_{12} = -(e^2/h)R \). All other elements vanish. Repeating the calculation which led to Eq. (16) for the conductor of Fig. (2), we find \( E_{23} = -E_{24} = E_{33} = -E_{34} = -i\omega C_\mu \) with \( C_\mu \) as given in Eq. (16). Inserting these results into Eq. (32) and holding all potentials, except \( dV_4 \) at their equilibrium value gives for \( I_3 \) and \( I_4 \),

\[
\begin{align*}
I_3 &= -i\omega C_\mu dV_4 \\
I_4 &= \frac{e^2}{h} \nu dV_4 + \delta I_4
\end{align*}
\] (33)

The noise spectrum at the voltage probe at low frequencies is just the spectrum of the noise of a QPC \( S_{I_4}^0(\omega) = 2e^2/Tr e|V| \) where we have added an upper index 0 to indicate that it is the spectrum for zero external impedance. Note that
there is no noise source to order $\omega$ in the total current for $I_3$. (The lowest order in frequency which is dissipative is proportional to $\omega^2$). For an ideal (infinite impedance) voltage probe we have $I_4 = 0$ and consequently

$$dV_4(\omega) = -\frac{\hbar}{e^2 \nu} \delta I_4(\omega)$$

(34)

Inserting this result in the equation for $I_3$ we find $S_{I_3I_3}(\omega) = \omega^2 C_\nu^2 S_{I_4I_4}(\omega)/\nu^2$. Using the shot noise power spectrum for $S_{I_4I_4}(\omega)$ gives for the spectrum at the gate $S_{I_3I_3}(\omega) = 2\omega^2 C_\nu^2 R_v e |V|$ with [15]

$$R_v = \frac{\hbar}{e^2 \nu^2} TR$$

(35)

Eq. (35) makes now an interesting prediction. For one edge state only, the dephasing voltage probe has no effect. The fluctuations observed at the gate remain unchanged. If there are several edge states, the voltage probe does have an effect since the voltage probe re-injects an equal current into all edge states. The difference between Eq. (35) and Eq.(30) is, however quite subtle. $R_v$ as given above is simply inversely proportional to the square of the number of edge states. Without the voltage probe we have seen that $R_v$ varies between $\frac{1}{e^2} TR$ for a steep edge and $\frac{1}{4e^2} TR$ for a shallow edge. Thus for a steep edge introducing a voltage probe has a considerable effect, whereas for a shallow edge introducing a voltage probe has no effect at all.

Apparently, in the experiment [5] the voltage probe is not ideal. Instead of an infinite impedance it might, at the relevant frequency, exhibit a finite impedance $Z_{ext}(\omega)$. We assume that the external impedance arises from a macroscopic circuit and its noise is voltage independent. In the presence of a finite impedance the current Eq. (33) can also be expressed as $I_4 = -Z_{ext}(\omega) V_4$. Consequently, instead of Eq. (34) we find

$$\delta V_4 = -\frac{Z_{ext}}{1 + G_0 Z_{ext}} \delta I_4$$

(36)

where we have introduced the abbreviation $G_0 = \nu e^2/\hbar$. Repeating the considerations given above, we find for the resistance $R_v$

$$R_v = \frac{e^2}{\hbar} \frac{|Z_{ext}|^2}{1 + G_0 |Z_{ext}|^2} TR$$

(37)

This consideration shows that a finite external impedance reduces the current fluctuations induced into the gate. Clearly this is simply a consequence of the fact that for a finite external impedance part of the current is "lost" at the voltage probe. This effect becomes significant when $Z_{ext}(\omega)$ at the frequency of interest becomes smaller than $G_0^{-1}$. 


11 Discussion

In this work, we have illustrated the calculation of charge and potential fluctuations for a simple problem: A Hall conductor with a QPC has on its side a gate which couples capacitively to the edge states. We have asked: What is the current induced into this gate due to the shot noise generated at the QPC. The simplifying assumption we have made is that the conductor remains charge neutral everywhere except near the gate where a charge pile-up limited by the Coulomb interaction between gate and edge is permitted. This allows a solution in terms of one fluctuating potential only.

Independent of the detailed discussion it is clear that the non-equilibrium resistance $R_v$ reflects the shot noise. The theoretical question concerns only the factor of proportionality. If we measure $R_v$ in units of $R_0 = h/2e^2TR$, we find that for one edge state $R_v/R_0$ is universal, whereas in the presence of a number of edge states it is not-universal, except if an ideal voltage probe completely equilibrates different channels, in which case we find $R_v/R_0 = 1/\nu^2$, where $\nu$ is the number of edge states. In Ref. [5] it is argued that the dephasing rate (which is proportional to $R_v$) should be periodic in a phase with period $\pi$ even for a single edge state. In contrast, in our our result [15], Eq. (29), such a periodic factor does not appear. We conclude by mentioning that the approach outlined here can be generalized to hybrid normal-superconducting systems [20].

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