CONVEX COMPACT SETS THAT ADMIT A LOWER SEMICONTINUOUS STRICTLY CONVEX FUNCTION

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Abstract. We study the class of compact convex subsets of a topological vector space which admits a strictly convex and lower semicontinuous function. We prove that such a compact set is embeddable in a strictly convex dual Banach space endowed with its weak* topology. In addition, we find exposed points where a strictly convex lower semicontinuous function is continuous.

1. Introduction

A well-known result of Hervé [6] says that a compact convex subset $K \subset X$ of a locally convex space is metrizable if and only if there exists $f: K \to \mathbb{R}$ which is both continuous and strictly convex. It happens that lower semicontinuity is a very natural hypothesis for a convex function, so it is natural to wonder if the existence of a strictly convex lower semicontinuous function on compact convex subset $K \subset X$ of a locally convex space enforces special topological properties on $K$. Ribarska proved [16, 17] that such a compact is fragmentable by a finer metric, and in particular it contains a completely metrizable dense subset. The third named author proved [14] that the same is true for the set of its extreme points $\text{ext}(K)$. On the other hand, Talagrand’s argument in [2, Theorem 5.2.(ii)] shows that $[0,\omega_1]$ is not embeddable in such a compact set. In addition, Godefroy and Li showed [5] that if the set of probabilities on a compact group $K$ admits a strictly convex lower semicontinuous function then $K$ is metrizable.

Our purpose here is to continue with the study of the class of compact convex subsets which admits a strictly convex lower semicontinuous function. We shall denote this class by $\mathcal{SC}$. The first remarkable fact that we have got is a Banach representation result.

Theorem 1.1. Let $X$ be a locally convex topological vector space and let $K \subset X$ be convex compact subset. Then there exists a function $f: K \to \mathbb{R}$ which is both lower semicontinuous and strictly convex and only if $K$ imbeds linearly into a strictly convex dual Banach space $Z$ endowed with its weak* topology.

Notice that the strictly convex norm of the dual Banach space in the statement is weak* lower semicontinuous, which is a stronger condition than just being a strictly convex Banach space isomorphic to a dual space.

If $f: K \to \mathbb{R}$ is a strictly convex function, then the symmetric defined by

$$
\rho(x,y) = \frac{f(x) + f(y)}{2} - f\left(\frac{x+y}{2}\right)
$$

is a strictly convex function on $K$.
provides a consistent way to measure diameters of subsets of $K$. This idea was successfully applied in renorming theory [9]. We will prove that every nonempty subset of $K$ has slices of arbitrarily small $\rho$-diameter, we can mimic some arguments of the geometric study of the Radon–Nikodým property which leads to results as the following one.

**Theorem 1.2.** Let $X$ be a locally convex topological vector space and let $f: X \to \mathbb{R}$ be lower semicontinuous, strictly convex and bounded on compact sets. Then for every $K \subset X$ compact and convex, the set of points in $K$ which are both exposed and continuity points of $f|_K$ is dense in $\text{ext}(K)$.

The organization of the paper is as follows. In the second section we present stability properties of the class $\mathcal{SC}$, which allow us to prove the embedding Theorem 1.1. The third and fourth sections are devoted to the search of faces and exposed points of continuity, respectively. Finally, a characterization of the class $\mathcal{SC}$ in terms of the existence of a symmetric with countable dentability index is given in Section 5.

## 2. Embedding into a dual space

Along this section $X$ will denote a locally convex topological vector space. Our first goal is to study the properties of following class of compact sets.

**Definition 2.1.** The class $\mathcal{SC}(X)$ consists of all the nonempty compact convex subsets $K$ of $X$ such that there exists a function $f: K \to \mathbb{R}$ which is lower semicontinuous and strictly convex. In addition, $\mathcal{SC}$ denotes the class composed of all the families $\mathcal{SC}(X)$ for any locally convex space $X$.

Since a lower semicontinuous function on a compact space attains its minimum, the function $f$ is bounded below. Later we shall show that we may always take $f$ to be bounded. Notice that metrizable convex compact admits continuous strictly convex functions, so they are in the class. In particular, if $X$ is metrizable then $\mathcal{SC}(X)$ contains all the convex compact subsets of $X$. If $X$ is a Banach space endowed with its weak topology, then $\mathcal{SC}(X)$ is made up of all convex weakly compact subsets as a consequence of the strictly convex renorming results for WCG spaces.

**Proposition 2.2.** The class $\mathcal{SC}$ satisfies the following stability properties:

a) $\mathcal{SC}(X)$ is stable by translations and homothetics;

b) $\mathcal{SC}$ is stable by Cartesian products;

c) $\mathcal{SC}$ is stable by linear continuous images;

d) If $A, B \in \mathcal{SC}(X)$, then $A + B \in \mathcal{SC}(X)$.

**Proof.** Statement a) is obvious. To prove b) suppose that $f_i$ witnesses $A_i \in \mathcal{SC}(X_i)$ for $i = 1, \ldots, n$. Then $\sum_{i=1}^n f_i \circ \pi_i$, where $\pi_i: \bigotimes_{i=1}^n X_i \to X_i$ is the coordinate projection, witnesses that $A_1 \times \cdots \times A_n \in \mathcal{SC}(\bigotimes_{i=1}^n X_i)$.

To prove c) assume that $A \in \mathcal{SC}(X)$ and $T: X \to Y$ is linear and continuous. Obviously $T(A)$ is convex and compact. Let $f: A \to \mathbb{R}$ be lower semicontinuous and strictly convex. It is straightforward to check that the function $g: T(A) \to \mathbb{R}$ defined by

$$g(y) = \inf \{ f(x) : x \in T^{-1}(y) \}$$

does the work. Finally, d) follows by a combination of b) and c).

We will need a kind of external convex sum of convex compact sets.

**Definition 2.3.** Given $A, B \subset X$ convex compact define a subset of $X \times X \times \mathbb{R}$ by

$$A \oplus B = \{(\lambda x, (1 - \lambda)y, \lambda) : x \in A, y \in B, \lambda \in [0, 1] \}.$$
Lemma 2.4. Let $A, B \subset X$ be convex compact subsets. Then
a) $A \oplus B$ is a convex compact subset of $X \times X \times \mathbb{R}$;
b) if $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ are convex, then $h : A \oplus B \to \mathbb{R}$ defined by
$$h((\lambda x, (1 - \lambda)y, \lambda)) = \lambda f(x) + (1 - \lambda)g(y)$$
is convex as well;
c) if $A, B \in \mathcal{SC}(X)$, then $A \oplus B \in \mathcal{SC}(X \times X \times \mathbb{R})$.

Proof. Compactness is clear in statement a). Given $(\lambda, x_i, (1 - \lambda_i)y_i, \lambda_i) \in A \oplus B$ for $i = 1, 2$, just observe that
$$\left(\frac{\lambda_1 x_1 + \lambda_2 x_2}{2}, \frac{(1 - \lambda_1)y_1 + (1 - \lambda_2)y_2}{2}, \frac{\lambda_1 + \lambda_2}{2}\right)$$
is convex. For the convexity of function $h$ notice that
$$h((\frac{\lambda_1 x_1 + \lambda_2 x_2}{2}, \frac{(1 - \lambda_1)y_1 + (1 - \lambda_2)y_2}{2}, \frac{\lambda_1 + \lambda_2}{2}))$$
$$= \frac{\lambda_1 + \lambda_2}{2} f(\frac{\lambda_1 x_1 + \lambda_2 x_2}{\lambda_1 + \lambda_2}) + \left(1 - \frac{\lambda_1 + \lambda_2}{2}\right) g(\frac{(1 - \lambda_1)y_1 + (1 - \lambda_2)y_2}{1 - \lambda_1 + (1 - \lambda_2)})$$
$$\leq \frac{\lambda_1 + \lambda_2}{2} \frac{\lambda_1 f(x_1) + \lambda_2 f(x_2)}{\lambda_1 + \lambda_2} + \left(1 - \frac{\lambda_1 + \lambda_2}{2}\right) \frac{(1 - \lambda_1)g(y_1) + (1 - \lambda_2)g(y_2)}{1 - \lambda_1 + (1 - \lambda_2)}$$
$$= \frac{1}{2} \left(h((\lambda_1 x_1, (1 - \lambda_1)y_1, \lambda_1)) + h((\lambda_2 x_2, (1 - \lambda_2)y_2, \lambda_2))\right).$$

If $f$ and $g$ were strictly convex, the above inequality for $h$ would become strict if $x_1 \neq x_2$ or $y_1 \neq y_2$. To overcome this difficulty consider the function
$$k((\lambda x, (1 - \lambda)y, \lambda)) = h((\lambda x, (1 - \lambda)y, \lambda)) + \lambda^2$$
and notice that $\lambda^2$ provides the strict inequality when $x_1 = x_2$ and $y_1 = y_2$. \hfill \Box

Proposition 2.5. Suppose that $A, B \in \mathcal{SC}(X)$. Then $\text{conv}(A \cup B) \in \mathcal{SC}(X)$ and $\text{aconv}(A) \in \mathcal{SC}(X)$.

Proof. Consider the map $T : X \times X \times \mathbb{R} \to X$ defined by $T((x, y, t)) = x + y$ and observe that $T(A \cup B) = \text{conv}(A \cup B)$. Since $T$ is linear and continuous, the combination of the previous results gives us that $\text{conv}(A \cup B) \in \mathcal{SC}(X)$. The application to the symmetric convex hull follows by applying it with $B = -A$. \hfill \Box

Lemma 2.6. Let $B \subset X$ be a symmetric compact convex set and let $Z = \text{span}(B)$. Then the following hold:
a) $Z$, with the norm given by the Minkowski functional of $B$, is isometric to a dual Banach space;
b) $B$ imbeds linearly into $(Z, w^*)$;
c) if $f : X \to \mathbb{R}$ is convex and lower semicontinuous, then $f|_Z$ is weak* lower semicontinuous.

Proof. Notice that $Z = \bigcup_{n=1}^{\infty} nB$, and thus the Minkowski functional of $B$ is a norm on $Z$. Of course, $B$ is the unit ball of $Z$ endowed with this norm. By a result of Dixmier-Ng, see for instance [10], the space $Z$ is isometric to the dual of the Banach space $W$ of all linear functionals $f$ on $Z$ such that $f|_B$ is $\tau$-continuous. If $f : X \to \mathbb{R}$ is convex and lower semicontinuous, then the sets $\{f \leq a\}$ are convex and closed for any $a \in \mathbb{R}$. We have $\{f|_Z \leq a\} = \{f \leq a\} \cap Z$, and thus $\{f|_Z \leq a\} \cap nB = \{f \leq a\} \cap nB$ is compact, and so it is weak* compact as subset.
of $Z$ for every $n \in \mathbb{N}$. By the Banach-Dieudonné theorem, $\{ f | z \leq a \}$ is a weak* closed subset of $Z$. \qed

Proof of Theorem 2.7. Let $B = \operatorname{aconv}(K)$ which is in $\mathcal{SC}(X)$. The function $f$ witnessing that $B \in \mathcal{SC}(X)$ is weak* lower semicontinuous and strictly convex. By Lemma 2.6, we only need to renorm the dual space $Z$. Notice that the function $f$ can be taken symmetric and bounded. Indeed, for the symmetry just take $g(x) = f(x) + f(-x)$. Now apply the Baire theorem to the $B = \bigcup_{n=1}^{\infty} g^{-1}((-\infty, n])$ to obtain a set of the form $\lambda B$ with $\lambda > 0$ where $g$ is bounded. Then redefine $f$ as $f(x) = g(\lambda x)$.

Without loss of generality we may assume that $f$ takes values in $[0, 1]$. Consider the function defined on $B_Z$ by

$$h(x) = \frac{1}{2}(3|x| + f(x))$$

and consider the set $C = \{ x \in B_Z : h(x) \leq 1 \}$. Clearly $\frac{1}{2}B_Z \subset C \subset 2B_Z$, and $C$ is convex, symmetric and weak* closed. Moreover, if $h(x) = h(y) = 1$, then $h(\frac{x+y}{2}) < 1$. Therefore, $C$ is the unit ball of an equivalent strictly convex dual norm on $Z$. \qed

**Corollary 2.7.** If $K \in \mathcal{SC}(X)$, then it is witnessed by the square of a lower semicontinuous strictly convex norm defined on $\operatorname{span}(K)$.

We shall finish this section by showing the connection between the class $\mathcal{SC}$ and $(\ast)$ property. The following notion was introduced in [11] in order to characterize dual Banach spaces that admit a dual strictly convex norm:

**Definition 2.8.** A compact space $K$ is said to have $(\ast)$ if there exists a sequence $(U_n)_{n=1}^{\infty}$ of families of open subsets of $K$ such that, given any $x, y \in K$, there exists $n \in \mathbb{N}$ such that:

a) $\{ x, y \} \cap U_n$ is non-empty;

b) $\{ x, y \} \cap U$ is at most a singleton for every $U \in U_n$.

Here we are using the agreement that $\bigcup U_n = \bigcup\{ U : U \in U_n \}$. Recall that if $K$ is a subset of a locally convex topological vector space then a slice of $K$ is an intersection of $K$ with an open halfspace. If the elements of $\bigcup_{n=1}^{\infty} U_n$ can be taken to be slices of $K$, then $K$ is said to have $(\ast)$ with slices. It is shown in [11] Theorem 2.7 that if $Z$ is a dual Banach space then $(B_Z, w^*)$ has $(\ast)$ with slices if and only if $Z$ admits a dual strictly convex norm.

**Corollary 2.9.** Let $(X, \tau)$ be locally convex topological vector space and $K \subset X$ be compact and convex. Then $K \in \mathcal{SC}(X)$ if and only if $K$ has $(\ast)$ with slices.

**Proof.** By Lemma 2.6, we may assume that $K \subset Z = \operatorname{span}(K)$ has $(\ast)$ with weak* slices. It follows from [11] Proposition 2.2 that then there is a lower semicontinuous strictly convex function defined on $K$. On the other hand, assume that $\phi$ witnesses $K \in \mathcal{SC}(X)$. For $f \in (X, \tau)^*$ and $r \in \mathbb{R}$, denote $S(f, r) = \{ x \in K : f(x) > r \}$. Consider the families $\{ U_{qr} \}_{q, r \in \mathbb{Q}}$ of open subsets given by

$$U_{qr} = \{ S(f, r) : f \in (X, \tau)^*, S(f, r) \cap \{ x : \phi(x) \leq q \} = \emptyset \} .$$

Let $x \neq y$ be in $K$. We may assume that $\phi(x) \leq \phi(y)$. Since $\phi$ is strictly convex, there exists $q \in \mathbb{Q}$ such that $\phi(\frac{x+y}{2}) < q < \phi(y)$. By the Hahn–Banach theorem, there is $f \in (X, \tau)^*$ and $r \in \mathbb{Q}$ such that $\sup \{ f(z) : \phi(z) \leq q \} < r < f(y)$. Therefore, $S(f, r) \cap \{ z : \phi(z) \leq q \} = \emptyset$ and $\{ x, y \} \cap \bigcup U_{qr} \neq \emptyset$.

Suppose that $x, y \in S(g, r) \subset U_{qr}$. Then $g(x), g(y) > r$ implies $g(\frac{x+y}{2}) > r$. Hence $x + y \notin \{ z : \phi(z) \leq q \}$, a contradiction. So $\{ x, y \} \cap S(g, q)$ is at most a singleton for each $S(g, q) \in U_{qr}$.
3. Faces of continuity

We will assume along the section that $Z = W^*$ is a dual Banach space endowed with the weak* topology. Therefore any unspecified topological concept (compact, open, . . . ) is always referred to the weak* topology. The elements of $W$ will be considered as functionals on $Z$. Other topological ingredient that we will use is a symmetric $\rho: Z \times Z \to [0, +\infty)$. Recall that a symmetric satisfies $\rho(x, y) = \rho(y, x)$ and $\rho(x, y) = 0$ if and only if $x = y$. Since a symmetric does not satisfy the triangle inequality, its associated topology is complicated to handle. Nevertheless we have a natural notion of diameter associated to $\rho$ defined by

$$\rho\text{-diam}(A) = \sup\{\rho(x, y) : x, y \in A\}$$

Let us recall the definition of face of a convex set.

**Definition 3.1.** Let $C \subset Z$ be closed and convex. We say that a closed subset $F \subset C$ is a face if there is a continuous affine function $w: C \to \RR$ such that

$$F = \{x \in C : w(x) = \sup\{w, C\}\}.$$ 

In that case we say that the face is produced by $w$. In addition, we say that a point $x \in C$ is a exposed point of $C$ if $\{x\}$ is a face of $C$.

Sometimes the face is produced by an element of the dual. Nevertheless, there may exist continuous affine functions on $C$ that are not the restriction of an element of the dual.

We shall need the following lemma.

**Lemma 3.2** (Lemma 3.3.3 of [1]). Suppose that $w \in W$ and $\|w\| = 1$. For $r > 0$ denote by $V_r$ the set $rB_Z \cap w^{-1}(0)$. Assume that $x_0$ and $y$ are points of $Z$ such that $w(x_0) > w(y)$ and $\|x_0 - y\| \leq r/2$. If $u \in W$ satisfies that $\|u\| = 1$ and $w(x_0) > \sup\{u, y + V_r\}$, then $\|w - u\| \leq \frac{2}{r}\|x_0 - y\|$.

First we shall discuss the dual Banach case.

**Proposition 3.3.** Let $f: Z \to \RR$ be a convex lower semicontinuous function which is bounded on compact subsets. If $K \subset Z$ is compact convex, then there exists a $G_δ$ dense set of elements of $W$ producing faces where $f|K$ is constant and continuous.

**Proof.** Define the pseudo-symmetric $\rho$ by the formula

$$\rho(x, y) = \frac{f(x)^2 + f(y)^2}{2} - \left(f\left(\frac{x + y}{2}\right)\right)^2.$$ 

We claim that $\rho(x, y) = 0$ implies $f(x) = f(y) = f\left(\frac{x + y}{2}\right)$ (in particular, if $f$ were strictly convex, $\rho$ would be a symmetric). Indeed, it follows easily from this observation

$$\rho(x, y) \geq \frac{f(x)^2 + f(y)^2}{2} - \left(f\left(\frac{x + y}{2}\right)\right)^2 = \left(f(x) - f\left(\frac{x + y}{2}\right)\right)^2 \geq 0.$$ 

Now we claim that the set $G(K, \varepsilon)$ is open and dense in $W$ for $K \subset Z$ compact convex and $\varepsilon > 0$, where

$$G(K, \varepsilon) = \{w \in W : \exists a < \sup\{w, K\}, \rho\text{-diam}(K \cap \{w > a\}) < \varepsilon\}.$$ 

Suppose that $w \in G(K, \varepsilon)$. If $w' \in W$ is close enough to $w$ to fulfill that

$$\sup\{w', K\} > \sup\{w', K \cap \{w \leq a\}\}$$

then $w' \in G(K, \varepsilon)$ as well. Thus $G(K, \varepsilon)$ is open. In order to see that it is also dense, fix $w \in W$ and $\delta \leq 1/4$. Take $x \in K$ and $y \in Z$ with $w(x) > a > w(y)$ for some $a \in \RR$. Take $r = \sup\{\|x' - y\|, x' \in K\}/2\delta$, consider the set $V_r$ given by Lemma 3.2 and define the set $C = \overline{\text{conv}}(K \cup (y + V_r))$. By [13 Theorem 1.1], the
halfspace \{ w > a \} contains a point \( x_0 \in \text{ext}(C) \) where \( f|_C \) is continuous. Notice that \( x_0 \in \text{ext}(K) \) and \( \|x_0 - y\| \leq r/2 \). There exists \( u \in W \) and \( b \in \mathbb{R} \) such that \( u(x_0) > b \), \( C \cap \{ u > b \} \subset C \cap \{ w > a \} \) and \( \rho\text{-diam}(C \cap \{ u > b \}) < \varepsilon \). In particular \( \rho\text{-diam}(K \cap \{ u > b \}) < \varepsilon \). Since \( C \cap \{ u > b \} \) does not meet \( y + V_r \), we have \( u(x_0) > \sup\{ u, y + V_r \} \). Thus, \( \|w - u\| \leq \frac{\delta}{2} \|x_0 - y\| \leq \delta \). That completes the proof of the density of \( G(K, \varepsilon) \) in \( W \).

By the Baire theorem, the set \( G(K) = \cap_{n=1}^\infty G(K, 1/n) \) is dense. If \( w \in G(K) \) and \( s = \sup\{ w, K \} \) then

\[
\lim_{t \to s^-} \rho\text{-diam}(K \cap \{ w > t \}) = 0.
\]

In particular, the face \( F = K \cap \{ w = s \} \) satisfies that \( \rho\text{-diam}(F) = 0 \). That implies that \( f \) is constant on \( F \). Moreover, we claim that any point \( x \in F \) is a point of continuity of \( f|_K \). If \( (x_\alpha) \subset K \) is a net with limit \( x \), then \( \lim_\alpha w(x_\alpha) = w(x) \). Therefore \( \lim_\alpha \rho(x_\alpha, x) = 0 \). It follows that \( \lim_\alpha f(x_\alpha) = f(x) \), so \( f|_K \) is continuous at \( x \).

Now the above result can be translated into a more general setting.

**Proposition 3.4.** Let \( f : X \to \mathbb{R} \) be a convex lower semicontinuous function which is bounded on compact subsets. Then for every compact convex subset \( K \subset X \) and every open slice \( S \subset K \), there is a face \( F \subset S \) of \( K \) such that \( f|_K \) is constant and continuous on \( F \).

**Proof.** By Lemma 2.6, \( Z = \bigcup_{n=1}^\infty n \text{acov}(K) \) is a dual Banach space and \( f|_Z \) is weak* lower semicontinuous. Then we can apply the previous proposition.

It is clear that the last two results are true for countably many functions simultaneously.

**Remark 3.5.** We do not know if the function \( f \) in Proposition 3.3 and 3.4 can be assumed to be defined only on \( K \). Notice that if \( \| \| \) is a strictly convex norm on \( Z \) then \( f(x) = -\sqrt{1 - \|x\|^2} \) is a strictly convex weak* lower semicontinuous function on \( (B_Z, w^*) \) that cannot be extended to a convex function on \( Z \).

4. Exposed points

Notice that if a strictly convex function is constant on a face of a compact \( K \), then necessarily that face should be an exposed point of \( K \). Having this in mind, Propositions 3.3 and 3.4 can be rewritten. As in the previous section \( Z = W^* \) is a dual Banach space endowed with the weak* topology and we understood all the topological notions referred to that topology.

**Proposition 4.1.** Let \( f : Z \to \mathbb{R} \) be a strictly convex lower semicontinuous function which is bounded on compact subsets. If \( K \subset Z \) is compact convex, then there exists a \( G_\varepsilon \) dense set of elements of \( W \) exposing points of \( K \) at which \( f|_K \) is continuous.

**Proof.** It follows straightforward from Proposition 3.3.

In particular, we retrieve the following result, which is usually proved in the frame of Gâteaux Differentiability Spaces [13, Corollary 2.39 and Theorem 6.2].

**Corollary 4.2 (Asplund, Larman–Phelps).** Let \( Z \) be a strictly convex dual Banach space. Then every convex compact is the closed convex hull of its exposed points.

**Proof of Theorem 1.2.** It follows straightforward from Proposition 3.4.

**Corollary 4.3.** Assume that \( K \in \mathcal{SC}(X) \). Then \( K \) is the closed convex hull of its exposed points.
Proof. Thanks to Theorem 1.1, it can be reduced to the previous corollary. □

Notice that the previous result is far from being a characterization. For instance, consider \( X = C([0,\omega_1])^* \) and \( K = (B_X, w^*) \). Then \( X \) has the Radon–Nikodým Property and thus there exist strongly exposed points of \( K \) [1 Theorem 3.5.4]. Nevertheless, Talagrand’s argument in [2, Theorem 5.2.(ii)] shows that \( K \not\in SC(X, w^*) \).

Indeed, the result of Larman and Phelps mentioned above states that Banach spaces for which each weak* compact convex subset has an exposed point are exactly dual spaces of a Gâteaux Differentiability Space.

Remark 4.4. A point \( x \) in a subset \( C \) of a normed space \( (Z, \| \cdot \|) \) is said to be a farthest point in \( C \) if there exists \( y \in Z \) such that \( \| y - x \| \geq \sup \{ \| y - c \| : c \in C \} \).

If \( \| \cdot \| \) is strictly convex then every farthest point of \( C \) is exposed by a functional in \( Z^* \). In addition, it was shown in [3] that there exists a weak* compact subset of \( \ell_1 \) with oscillation less than \( K \), which is beyond the ordinals. The existence of exposed points does not imply the existence of farthest points. On the other hand, suppose that \( Z \) is a strictly convex dual Banach space, \( C \) is a compact subset of \( Z \) and \( x \) is a farthest point in \( C \) with respect to \( y \in Z \). Consider the symmetric \( \rho(u,v) = \frac{\| u - v \|^2 + \| u + v \|^2}{2} - \| u + v - y \|^2 \).

Then \( x \) is a \( \rho \)-denting point of \( C \), that is, admits slices with arbitrarily small \( \rho \)-diameter. Indeed, if \( \delta = \frac{1}{1+2\| x - y \| + 2\| y \|} \) then every slice of \( C \) that does not meet \( B(y, \| y - x \| - \delta) \) has \( \rho \)-diameter less than \( \varepsilon \).

Typically a variational principle provides strong minimum for certain functions after a small perturbation. But in the compact setting, a lower semicontinuous function already attains its minimum. Nevertheless, inspired by Stegall’s variational principle [4 Theorem 11.6], we have obtained the following result.

Proposition 4.5. Suppose that \( K \in SC(X) \) and let \( f: K \to \mathbb{R} \) be a lower semicontinuous function. Given \( \varepsilon > 0 \), there exists an affine continuous function \( w \) on \( K \) with oscillation less than \( \varepsilon \) such that \( f + w \) attains its minimum exactly at one point. Moreover, if \( X \) is a dual Banach space then \( w \) can be taken from the predual with norm less than \( \varepsilon \).

Proof. By the embedding it is enough to consider the Banach case. Let \( m \) be the minimum of \( f \) and take \( M > 0 \) such that \( K \subset MB_X \). Consider the compact set

\[
H = \{ (x,t) : f(x) \leq t \leq m + \varepsilon M \}
\]

and take its convex closed envelop \( A \). By Proposition 2.5, \( A \in SC(X \times \mathbb{R}) \). The functional on \( X \times \mathbb{R} \) given by \((0,1)\) attains its minimum on \( A \). Proposition 4.3 provides a small perturbation of the form \((w,1)\), with \( \|w\| < \varepsilon \), attaining its minimum on \( A \) at one single point \((x_0,t_0)\). Notice that \( t_0 = f(x_0) \) and \( f(x_0) + w(x_0) \leq m + \varepsilon M \). If \( y \in K \), then either \( f(y) \leq m + \varepsilon M \) and \( y, f(y) \in A \), or \( f(y) > m + \varepsilon M \geq f(x_0) + w(x_0) \). □

5. Ordinal indices

Let \( K \) be a convex and compact subset of a locally convex topological vector space and \( \rho \) a symmetric on \( K \). We consider the following set derivations:

\[
[K]_\varepsilon = \{ x \in K : x \in S \text{ slice of } K \Rightarrow \rho\text{-diam}(S) \geq \varepsilon \};
\]

\[
(K)_\varepsilon = \{ x \in K : x \in U \text{ open } \Rightarrow \rho\text{-diam}(S) \geq \varepsilon \}.
\]

The iterated derived sets are defined as \([K]_{\varepsilon}^{\alpha+1} = [[K]_{\varepsilon}^{\alpha}], (K)_{\varepsilon}^{\alpha+1} = \langle (K)_{\varepsilon}^{\alpha} \rangle \rangle \) and intersection in case of limit ordinals. If there exists some ordinal such that \([K]_{\varepsilon}^{\alpha} = \emptyset \), then we set \( Dz_\rho(K, \varepsilon) = \min \{ \alpha : [K]_{\varepsilon}^{\alpha} = \emptyset \} \). Otherwise, we take \( Dz_\rho(K, \varepsilon) = \infty \), which is beyond the ordinals. The \( \rho \)-dentability index of \( K \) is...
defined by \( D_z(K) = \sup_{\epsilon > 0} D_z(K, \epsilon) \). The \( \rho \)-Szlenk index of \( K \), \( S_{z\rho}(K) \), is defined the same way. Obviously \( S_{z\rho}(K) \leq D_{z\rho}(K) \). Set derivations with respect to a symmetric were introduced in \([7]\) in order to characterize dual Banach spaces admitting a dual strictly convex norm.

**Proposition 5.1.** Let \( K \) be a convex compact subset of a locally convex space. Then the following assertions are equivalent:

a) \( K \in SC \);

b) there exists a symmetric \( \rho \) on \( K \) such that \( D_{z\rho}(K) = \omega \);

c) there exists a symmetric \( \rho \) on \( K \) such that \( D_{z\rho}(K) = \omega_1 \).

**Proof.** Let \( f \) be a bounded function witnessing that \( K \in SC \) and assume that \( f \) takes values in \([0,1]\). For a fixed \( \epsilon > 0 \), take \( N > 1/\epsilon \) and define the closed convex subsets \( F_n = \{ x \in K : f(x) \leq 1 - n/N \} \) for \( n = 0, \ldots, N \). Take

\[
\rho(x,y) = \frac{f(x) + f(y)}{2} - f\left(\frac{x + y}{2}\right).
\]

We claim that \([K]^\epsilon_n \subseteq F_1\). Let \( x_0 \in K \setminus F_1 \). By the Hahn–Banach theorem, there exists a slice \( S \) of \( K \) such that \( x_0 \in S \) and \( S \cap F_1 = \emptyset \). If \( x, y \in S \), then \( \frac{x+y}{2} \in S \) and \( \rho(x,y) \leq 1 - (1 - 1/N) = 1/N \). Thus, \( \rho(S_1) < \epsilon \) and \( x_0 \notin [K]^\epsilon \). By iteration, we get that \([K]^\epsilon_n \subseteq F_X \) and hence \([K]^\epsilon_{n+1} = \emptyset \). Therefore, \( D_{z\rho}(K, \epsilon) < \omega \) for each \( \epsilon > 0 \).

Now suppose that \( D_{z\rho}(K) = \omega_1 \). Notice that indeed \( D_{z\rho}(K) < \omega_1 \). By Corollary \([2,4]\) it suffices to show that \( K \) has (*) with slices. For each \( n \in \mathbb{N} \) and \( \alpha < D_{z\rho}(K, 1/n) \) consider the family

\[
U_{\alpha,n} = \left\{ S : \text{slice of } K, [K]^{\alpha+1}_{1/n} \cap S = \emptyset, \rho\text{-diam}([K]^{\alpha}_{1/n} \cap S) < 1/n \right\}.
\]

Given distinct \( x, y \in K \), take \( n \) so that \( \rho(x,y) > 1/n \) and let \( \alpha \) be the least ordinal such that \( \{ x, y \} \cap [K]^{\alpha+1}_{1/n} \) is at most a singleton. Then it is clear that there is a slice in \( U_{\alpha,n} \) containing either \( x \) or \( y \), and no slice in \( U_{\alpha,n} \) contains both points. \( \square \)

**Remark 5.2.** By using deep results of descriptive set theory, Lancien proved in \([8]\) that there exists a universal function \( \psi : \{0, \omega_1\} \to \{0, \omega_1\} \) such that \( D_{\psi}(\|B_X\|) \leq \psi(S_{\|B_X\|}(\|B_X\|)) \) whenever \( X \) is a Banach space such that \( S_{\|B_X\|}(\|B_X\|) < \omega_1 \). We do not know if a similar statement holds when the norm is replaced by a symmetric.

We shall show that we cannot change symmetric by metric in Proposition \([5,3]\). That would imply that \( K \) is a Gruenhage compact, which is a strictly stronger condition that being in \( SC \) \([18, Theorem 2.4]\). By \([19, Lemma 7.1 \) and Proposition 7.4], a compact space \( K \) is Gruenhage if and only if there exists a countable set \( D \), a family of closed sets \( \{ A_d : d \in D \} \) and families \( \{ U_{d} \}_{d \in D} \) of open sets such that the family \( \{ A_d \cap U : U \in U_d \} \) is pairwise disjoint for each \( d \in D \) and the family \( \{ A_d \cap U : U \in U_d \} \) separates the points of \( K \).

**Proposition 5.3.** Let \( K \) be a compact space. Then the following assertions are equivalent:

a) \( K \) is Gruenhage;

b) there exists a metric \( d \) on \( K \) such that \( Sz_d(K) \leq \omega \);

c) there exists a metric \( d \) on \( K \) such that \( Sz_d(K) \leq \omega_1 \).

**Proof.** If \( K \) is a Gruenhage compact space, then the same construction used in the proof of \([15, Theorem 2.8]\) provides a metric on \( K \) such that \( Sz_d(K) \leq \omega \).

Now assume that \( d \) is a metric on \( K \) with countable Szlenk index. Let \( B = \bigcup_{m \in \mathbb{N}} B_m \) be a basis of the metric topology such that every \( B_m \) is discrete. Consider the open sets \( U^\alpha_{1/\alpha} = \bigcup \{ U : U \text{ open}, [K]^{\alpha}_{1/\alpha} \cap U \subset V \} \) and the families \( U^\alpha_{1/\alpha} = \)
\[ \{ U_V^{n,\alpha} : V \in B_m \}. \] Then \( \{(K)^{2-n}_\alpha \cap U : U \in U_m^{n,\alpha}\} \) is pairwise disjoint for each \( n, m \in \mathbb{N} \) and \( \alpha < Dz(K,2^{-n}). \) Given distinct \( x, y, z \in K \) take \( V \in B_m \) such that \( x \in V \) and \( y \notin V. \) Fix \( n \) such that \( B_2(x,2^{-n+1}) \subset V. \) Let \( \alpha \) be the least ordinal so that \( x \notin (K)^{2-n}_\alpha \). Then there is an open subset \( U \) of \( K \) such that \( x \in (K)^{2-n}_\alpha \cap U \) and \( \text{diam}((K)^{2-n}_\alpha \cap U) \leq 2^{-n}. \) Thus \( x \in (K)^{2-n}_\alpha \cap U_V^{n,\alpha} \subset V, \) so \( y \notin (K)^{2-n}_\alpha \cap U_V^{n,\alpha}. \)

\[ \square \]

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