Potential Integral Equations in Electromagnetics

Jie Li, Balasubramaniam Shanker
Department of Electrical and Computer Engineering
Michigan State University, East Lansing, MI 48824

Xin Fu
Department of Electronic and Electrical Engineering
The University of Hong Kong, Pokfulam, Hong Kong, China

Abstract—In this work, a new integral equation (IE) based formulation is proposed using vector and scalar potentials for electromagnetic scattering. The new integral equations feature decoupled vector and scalar potentials that satisfy Lorentz gauge. The decoupling of the two potentials allows low-frequency stability. The formulation presented also results in Fredholm integral equations of second kind. The spectral properties of second kind integral operators leads to a well-conditioned system.

I. INTRODUCTION

Recently two similar formulations [1, 2] have been proposed to solve the low frequency breakdown based on the auxiliary vector and scalar potentials instead of electromagnetic fields. In [1], two fully decoupled integral equations (IEs) are formulated for vector potential (A) and scalar potential (φ) respectively. The operators in those IEs are defined to result in IE formulations of the second kind, which is significantly useful in finite element analysis. Liu et al. [2] with A based integral equation also shows the low-frequency stability. In both papers, the vector potential and scalar potential are related through Lorentz gauge. However the second approach does not guarantee Fredholm integral equations of the second kind. What’s more, at low frequency (approaching to zero), the system equation is nearly a saddle point problem that needs special care. In this work, starting from the rigorously derived representation theorem for A [3], a new decoupled potential integral equation method is formulated for more generalized dielectric problems. The proposed formulation features stability and is comprised of Fredholm integral equations of the second kind. Detailed discussion including spectral analysis and verification will be presented at the conference. This paper lays out the basic formulation together with ideas and tools for spectral analysis.

II. DECOUPLED POTENTIAL FORMULATION

In this section, decoupled A − φ descriptions of scattering from the homogeneous body will be presented. The formulation will be consistent with Lorentz gauge. In this work, time harmonic factor e^{jωt} is assumed and suppressed.

A. Decoupled Potential Representations and Boundary Conditions

Vector potential A and scalar potential φ satisfy vector and scalar Helmholtz equations, respectively. The following representations for scattered potential can be derived;

\[ A^{scat} = S_k[n' \times \nabla' \times A(r')] + \nabla \times S_k[n' \times A(r')] - S_k[n' \nabla' \cdot A(r')] - \nabla S_k[n' \cdot A(r')] \equiv \mathcal{L}_k^A ([A]) \]

(1)

\[ \phi^{scat} = -S_k \left[ \frac{\partial \phi(r')}{\partial n'} \right] + D_k [\phi(r')] = \mathcal{L}_k^\phi ([\phi]) \]

(2)

where \( S_k \) and \( D_k \) are single and double layer potential operators respectively.

Electric and magnetic fields can be decomposed into A − φ forms (\( E = -j\omega A - \nabla \phi \) and \( H = \nabla \times A \)). From the following conditions of electric and magnetic fields,

\[ \hat{n} \times E_1 = \hat{n} \times E_2, \quad \hat{n} \times H_1 = \hat{n} \times H_2 \]

\[ \epsilon_1 \hat{n} \cdot E_1 = \epsilon_2 \hat{n} \cdot E_2 \quad \text{and} \quad \mu_1 \hat{n} \cdot H_1 = \mu_2 \hat{n} \cdot H_2 \]

(3)

one can set up two sets of stronger conditions for the two decoupled potentials. For scalar potential, one has

\[ \phi_1 = \phi_2 + V \quad \text{and} \quad \epsilon_1 \hat{n} \cdot \nabla \phi_1 = \epsilon_2 \hat{n} \cdot \nabla \phi_2, \]

(4)

where \( V \) is a jump term allowed in scalar potential, and each separated object corresponds to one constant value. For vector potential, one has

\[ \hat{n} \times A_1 = \hat{n} \times A_2, \quad \epsilon_1 \hat{n} \cdot A_1 = \epsilon_2 \hat{n} \cdot A_2 \quad \text{and} \quad \frac{1}{\mu_1} \hat{n} \times \nabla \times A_1 = \frac{1}{\mu_2} \hat{n} \times \nabla \times A_2. \]

(5)

Due to the two facts that (1) A and φ have to satisfy the Lorentz gauge and (2) the charge neutrality requirement should be imposed if not satisfied implicitly, one can get additional conditions for φ and A respectively: (1) \( \int \frac{\partial \phi}{\partial n} ds' = 0 \) for φ and (2) \( \nabla \cdot A_1 = \nabla \cdot A_2 + C \) and \( \int \hat{n} \cdot A_1 ds' = 0 \) for A. C is a jump term in divergence of A as in φ. It’s worth noting that two jump terms could be set to zero to derive even stronger boundary conditions. In each case, one can denote the application of boundary conditions on the interface as \( \{ \phi_2 \} = B(\{ \phi_1 \}, V) \) or \( \{ A_2 \} = B(\{ A_1 \}, C) \).

B. Well-conditioned Integral Equations

Given the representation theorems and the boundary conditions, one can construct integral equations for dielectrics. Without loss of generality, one can set the jump terms to zero and represent the total φ and \( \partial_n \phi \) just outside the interface as

\[ \phi_i = \phi_{iinc} \delta_{i11} + \mathcal{L}_k^{\phi} \{ \phi_i \} \]

(6a)

\[ \partial_n \phi_1 = \partial_n \phi_{iinc} \delta_{i11} + \partial_n \mathcal{L}_k^{\phi} \{ \phi_i \} \]

(6b)

where \( \delta_{i11} = 1 \) and \( \delta_{12} = 0 \), and the operand \( \{ \phi_i \} \) denotes all the possible components required as in Eq[2] to represent the scattered scalar potential. Linear combination of \( \phi_1 \) and \( \phi_2 \) will give us one IE with four unknowns associated with fields just outside and inside of the interface, as does the linear combination of their normal derivative terms. Applying boundary
conditions given earlier would reduce those unknowns into two unknowns (associated with outside of the interface) and the jump (associated with each separated body). Therefore, on the interface, one has

\[
\begin{align*}
(1 + \alpha)\phi_1 - X[\phi_1, \alpha] &= \phi_{\text{inc}} \quad (7a) \\
\beta_1 + \epsilon_2 \partial \phi_1 \over \partial n - \frac{\partial}{\partial n} X[\phi_1, \beta] &= \{\phi_1\} = \partial \phi_{\text{inc}} \over \partial n \quad (7b)
\end{align*}
\]

where \(X[\phi_1, \sigma] = L^\sigma_{k_1}[(\{\phi_1\}) + \sigma L^\sigma_{k_2}[B(\{\phi_1\})] \) with \(\sigma\) being \(\alpha\) or \(\beta\).

After rewriting the IEs with explicit single and double layer operators, one can get a system with its diagonal being identity operator plus compact operators. To obtain a second kind linear system, especially when the iterative method is used, one needs to handle the hypersingular term carefully. Therefore, on the interface, one has \(\phi_{\text{inc}}\) that needs to be imposed if the discretization scheme cannot guarantee that.

Following a similar procedure, one can get the vector potential integral equations. By denoting necessary components of \(A\) with \(\{A\} = [\hat{n} \times \nabla \times A, \hat{n} \times A, \nabla \cdot A, \hat{n} \cdot A]\) \(= [A^{v1}, A^{v2}, A^s, A^\delta]^T\), one can get the following vector potential integral equations.

\[
\begin{align*}
(1 + \alpha)A^{v1}_{\text{inc}} - \hat{n} \times \nabla \times X[A_1, \alpha] &= A^{v1}_{\text{inc}} \quad (8a) \\
(1 + \beta)A^{v2}_{\text{inc}} - \hat{n} \times X[A_1, \beta] &= A^{v2}_{\text{inc}} \quad (8b) \\
(1 + \gamma)A^s_{\text{inc}} - \nabla \cdot X[A_1, \gamma] &= A^s_{\text{inc}} \quad (8c) \\
(1 + \delta)A^\delta_{\text{inc}} - \hat{n} \cdot X[A_1, \delta] &= A^\delta_{\text{inc}} \quad (8d)
\end{align*}
\]

where \(X[A_1, \sigma] = L^A_{k_1}[(\{A_1\}) + \sigma L^A_{k_2}[B(\{A_1\})] \) with \(\sigma\) being \(\alpha, \beta, \gamma\) or \(\delta\). Similarly as in scalar potential case, specific linear factors \(\sigma\) should be chosen to cancel the singularity beyond \(1/2\) to produce compact off-diagonal operators. This is very similar to the situation in Müller formulation. Then one can also show that this IE set is also of the second kind, different from the extension of the \([5]\) in \([4]\).

It’s noted that the charge neutrality mentioned earlier should be imposed if the discretization scheme cannot guarantee that.

III. ANALYSIS ON THE SPHERE

In this section, the integral system is solved on a PEC sphere of radius \(r = a\) for a special case where the scalar potential vanishes \([2], [3]\). Those two unknown components of \(A\) for PEC case are \(A^{v1}\) and \(A^{v2}\) and hence only two equations out of Eqs. \([8]\) There are several possible choices that will give very different behaviors in terms of eigen properties and conditioning at low frequency. It can be shown that the first one and the last one should be chosen to lead to second kind IEs. Other reduced forms include the scheme used in \([2]\), which is not optimal.

The unknown quantities can be represented by vector and scalar spherical harmonics as in \([5]\). Following \([5]\), one can easily show that the mode orthogonality still holds in the discrete system. Hence the system is block-diagonal, with each of the block (corresponding to mode \((n, m)\)) being \(3 \times 3\) in size. Analytic derivations lead to only four nonzero matrix entry in each block \([Z_{ij}^{nm}]\) for mode \((n, m)\). The nonzero entries can be evaluated analytically (given as follows), which allows analytic inversion of the system.

\[
\begin{align*}
Z_{11}^{nm} &= a^2 + j a^2 z^{(1)}_n(ka)Z_n^{(1)}(ka) \\
Z_{22}^{nm} &= a^2 - j a^2 z^{(1)}_n(ka)Z_n^{(4)}(ka) \\
Z_{31}^{nm} &= \frac{a^2 \sqrt{n(n + 1)}}{j k} \left[ z^{(4)}_n(ka)Z_n^{(1)}(ka) + ka z^{(1)}_n(ka)Z_n^{(4)}(ka) \right] \\
Z_{33}^{nm} &= a^2 - j k^2 a^4 z^{(4)}_n(ka)z_1^{(1)}(ka)
\end{align*}
\]

where \(z^{(1)}(\cdot)\) and \(z^{(4)}(\cdot)\) are spherical Bessel function and spherical Hankel function of the second kind respectively, and \(Z^{(1)}(\cdot)\) and \(Z^{(4)}(\cdot)\) are spherical Riccati Bessel function and spherical Riccati Hankel function of the second kind respectively.

Together with the canonical incidence \(A_{\text{inc}}\), one can get the analytic solution of the scattering problem. Given plane wave incidence, one can construct the corresponding \(A - \phi\) representation, which are again represented by using incoming spherical vector and scalar wave functions. Results have been obtained for the above IEs, matching exactly those from Mie series approach.

More importantly, given the analytic matrix entries, the explicit “impedance” matrix can be used to show the spectral property of the integral equations derived in the previous section. Detailed analysis will be presented at the conference.

IV. CONCLUSION

Starting from the representation theorems for both scalar and vector potentials, one can derive a decoupled and the second kind Fredholm integral equation based formulation for electromagnetic scattering. The new IE features frequency stability and spectral properties that are essential to solve the linear system, especially when the iterative method is used. Note that this work is not immune from interior resonance at higher frequency for the reduced PEC case. Similar techniques as in combined field integral equation and decoupled potential IE in \([1]\) are expected to remove the resonance in existing operators.

REFERENCES

[1] F. Vico, M. Ferrando, L. Greengard, and Z. Gimbutas, “The Decoupled Potential Integral Equation for Time-Harmonic Electromagnetic Scattering,” Communications on Pure and Applied Mathematics, vol. 69, no. 4, pp. 771–812, Apr. 2016.
[2] Q. S. Liu, S. Sun, and W. C. Chew, “An Integral Equation Method based on Vector And Scalar Potential Formulations,” in 2015 IEEE International Symposium on Antennas and Propagation & USNC/URSI National Radio Science Meeting, IEEE, Jul. 2015, pp. 744–745.
[3] W. C. Chew, “Vector Potential Electromagnetics with Generalized Gauge for Inhomogeneous Media: Formulation,” Progress In Electromagnetics Research, vol. 149, pp. 69–84, 2014.
[4] Q. S. Liu, S. Sun, W. C. Chew, and L. J. Jiang, “Potential Based Integral Equation Method for Dielectric Problems,” in 2016 USNC/URSI National Radio Science Meeting, 2016.
[5] J. Li and B. Shanker, “Time-Dependent Debye-Mie Series Solutions for Electromagnetic Scattering,” IEEE Transactions on Antennas and Propagation, vol. 63, no. 8, 2015.