Quantum Speedup in Adaptive Boosting of Binary Classification

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In classical machine learning, a set of weak classifiers can be adaptively combined to form a strong classifier for improving the overall performance, a technique called adaptive boosting (or AdaBoost). However, constructing the strong classifier for a large data set is typically resource consuming. Here we propose a quantum extension of AdaBoost, demonstrating a quantum algorithm that can output the optimal strong classifier with a quadratic speedup in the number of queries of the weak classifiers. Our results also include a generalization of the standard AdaBoost to the cases where the output of each classifier may be probabilistic even for the same input. We prove that the update rules and the query complexity of the non-deterministic classifiers are the same as those of deterministic classifiers, which may be of independent interest to the classical machine-learning community. Furthermore, the AdaBoost algorithm can also be applied to data encoded in the form of quantum states; we show how the training set can be simplified by using the tools of t-design. Our approach describes a model of quantum machine learning where quantum speedup is achieved in finding the optimal classifier, which can then be applied for classical machine-learning applications.

Introduction—One of the most fundamental topics in machine learning is how to train a machine to perform classification tasks given a set of labelled samples. Unlike human brains that can efficiently distinguish different objects with eyes, the classification capability of a learning machine is restricted by the amount of available information. In many practical circumstances, the performances of the classifiers could be quite weak, say, they are only slightly better than tossing a fair coin. A big question that attracted significant interests in the past is whether these weak classifiers can be efficiently combined to a strong classifier. Yoav Freund and Robert Schapire [1] provided a positive solution to this question in their Gódel Price winning work. Their pioneering work triggered a long line of follow-up work, and led to a well-known adaptive boosting algorithm that yields a strong classifier from an ensemble of weak classifiers, abbreviated as AdaBoost [2]. The original AdaBoost works well for binary classification problems, such as detecting human in images and videos [3, 4]. It can also be extended to multi-class classification tasks, e.g. face recognition [5, 6].

AdaBoost also works well with a large amount of widely used machine learning algorithms, such as support vector machine (SVM) [7], decision tree [8], rotation forest [9] and so on. With the help of these algorithms, it has been applied to a variety of scenarios since its appearance, including chemical and biological informatics, financial analysis, information extraction, computer vision and natural language processing, which are all important techniques that are spreading influences into modern industry. However, as most machine learning algorithms, AdaBoost requires a huge amount of data to train, and the complexity of the algorithm grows rapidly with the increase of sample size. This rapidly growing data unambiguously becomes a fundamental challenge for further development. Therefore, any improvement of the complexity of machine learning algorithms would be extremely valuable.

As a novel field lying at the intersection of quantum physics and computer science, quantum computing shows a potential of significantly improving classical algorithms. Quantum computation, taking advantages of the laws of quantum physics, has the ability to process data with much fewer registers than classical computing because different states of registers could exist at the same time as a superposition. Efficient quantum algorithms have been discovered since 1990s, such as Grover searching algorithm [10], quantum phase estimation algorithm [11], factoring integers, discrete logarithm [12], and solving linear system [13, 14]. Vast new algorithms are constructed based on them [15], and many of them are proved to be faster than any possible classical, and even probabilistic algorithms [16].

Considering the advantage of quantum computation, it is natural to investigate this new idea into the field of machine learning, seeking solutions to the “big data” challenge. Quantum machine learning, as an interdisciplinary field between machine learning and quantum computation, explores how to deal with big data with quantum computers. It has made great progress in recent years [17, 18]. It has become a matter of interest for the great potential of solving the challenge of “big data” [19-22]. Major breakthroughs include quantum
support vector machine [20, 23], quantum discriminant analysis [24] and quantum principal component analysis [25].

In this Letter, we revisit the original AdaBoost algorithm, and propose a quantum generalization with a quadratic speedup. Besides a quantum AdaBoost algorithm, we also show that the original AdaBoost algorithm has the similar performance even if the basis classifiers are relaxed to probabilistic ones. Our quantum AdaBoost has the potential to improve various existing quantum machine learning algorithms, such as quantum state discriminating algorithms [26, 27].

**Binary Classification**—Classification is the task of assigning same labels to a collection of inputs with the same structures. It is a crucial component in constructing supervised learning models. Specifically, in binary supervised learning, we are given a sample consisting of \( N \) labeled examples (data) \( \{(x_1, y_1), \ldots, (x_j, y_j), \ldots (x_N, y_N)\} \), where each \( x_j \) is a data point in a sample space \( \Omega \), and \( y_j \in \{1, -1\} \) is the label of \( x_j \). A binary classifier can be considered as a mapping \( H: \Omega \rightarrow \{-1, +1\} \).

However, in many cases, such an optimal classifier is hard to obtain, and only weak classifiers (or basis classifiers), whose performance may be slightly better then random guessing, are available. In [28], Kearns and Valiant proposed the following fundamental question: how to **boost** a set of weak classifiers \( \{H_1, H_2, \ldots, H_T\} \) to become a strong classifier? It turns out that such a task can be achieved with the boosting technique.

**Conventional AdaBoost**—The main idea of boosting is to construct a strong classifier \( H_{\text{strong}} \) given a collection of weak classifiers \( \{H_1, \ldots, H_T\} \):

\[
H_{\text{strong}}(x) = \text{sgn}(g_T(x)) ,
\]

where \( g_T(x) := \sum_{t=1}^{T} \alpha_t H_t(x) \) is a weighted sum of the \( T \) basis classifiers \( H_t(x) \), and the sign function \( \text{sgn}(v) \) gives +1 if \( v \) is positive and -1 otherwise.

Assume that each input \( x \sim D \) occurs with a probability denoted by \( p(x) \). The corresponding cost function \( C_T \) in boosting is the exponential error of \( g_T(x) \),

\[
C_T = \sum_{x} \prod_{t=1}^{T} p(x) e^{-\alpha_t H_t(x)y(x)} ,
\]

where the \( T \) coefficients \( \{\alpha_t\}_{t \in [T]} \) in \( g_T(x) \) have to be optimized. In the binary case, \( H_t(x), y(x) \in \{-1, +1\} \), we have \( H_t(x)y(x) = (-1)^{r_t(x)} \), where the indicator function \( r_t \) is 1 if the \( t \)th classifier \( H_t \) classifies \( x \) correctly, and 0 otherwise. With the indicator \( r_t \), \( C_T \) can be rewritten as

\[
C_T = \sum_{x} \prod_{t=1}^{T} p(x) e^{-\alpha_t (-1)^{r_t}r_t} .
\]

Note that, an optimal solution \( \{\alpha_t\}_{t \in [T]} \) exists because this is a convex optimization problem [2]. Moreover, this optimal solution would not be worse than any of the basis classifiers, as the choice of \( \alpha_t = 1 \) and \( \alpha_t \neq 0 \) reduces to a basis classifier, and this would not be better than the optimal solution. As a result, the strong classifier \( H_{\text{strong}} \) should be able to classify (i.e., \( g_T(x)y(x) > 0 \)) the inputs with a success rate better than any of the basis classifiers.

The key feature of AdaBoost based on \( C_T \) is that one can adaptively compute \( \alpha_t \) using the recursive relation:

\[
C_t = C_{t-1} e^{\alpha_t H_t(x)y(x)} .
\]

We can determine the desired coefficient \( \alpha_t \) that minimizes \( C_t \) by differentiating \( \frac{\partial C_t}{\partial \alpha_t} \) and bring it to zero.

As a result, the optimal solution to the conventional AdaBoost is given by \( \alpha_t = \frac{1}{T} \ln \left( \frac{1}{R_t} \right) \) [2], where the weighted errors \( R_t := \mathbb{E}_{x \sim D}[W_t^x r_t^x] \). The weights \( W_t^x \) can be obtained adaptively with the information of \( R_1, \ldots, R_{t-1} \) (details can be found in section **Proof of Theorem 1** in appendix or [2]). In next section, we shall show that the conventional AdaBoost is simply a special case of probabilistic AdaBoost.

**Probabilistic AdaBoost**—Here we present a generalized AdaBoost method which can be applied to the cases where each basis classifiers are probabilistic. In conventional AdaBoost, each basis classifier \( H_t \) must produce the same label for each input \( x \), even if it is incorrect. We consider the setting where each basis classifier produces a label **probabilistically** for each input \( x \). Specifically, we define \( q_r(x) \) to be the conditional probability where \( H_t \) produces the label such that \( r_t \) belongs to \( \{0, 1\} \).

For further convenience here we define a binary string of length \( t \) as \( s_t := s_{t_1} s_{t_2} \cdots s_t \) to record the results for a sequence of basis classifiers for input \( x \). The joint probability to obtain the string \( s_t \) with input \( x \) is given by

\[
q(s_t, x) := p(x) \prod_{i=1}^{t} q(r_i^x) = s_t | x \).
\]

With a little abuse of notation, the cost function (exponential error (2)) is redefined for probabilistic case:

\[
C_T = \sum_{x, s_T} \prod_{t=1}^{T} q(s_t, x) e^{-\alpha_t (-1)^{r_t}r_t^x} .
\]

We shall show that (i) the solution of the probabilistic models of AdaBoost is a generalization of the conventional case, but (ii) their query complexities are the same, which are summarized by the following theorems:

**Theorem 1** (Optimal solution to probabilistic case).

The optimal solution to the probabilistic AdaBoost model, in terms of \( \{\alpha_t\} \) minimizing the cost function \( C_T \) (3), is given by

\[
\alpha_t = \frac{1}{2} \ln \frac{1 - \tilde{R}_t}{\tilde{R}_t} ,
\]

where \( \tilde{R}_t \) is the weighted error taking over the joint distribution \( q(s_t, x) \):

\[
\tilde{R}_t = \mathbb{E}_{x, s_t}[W_t^x r_t^x] = \sum_{x, s_t} q(s_t, x) W_t^x r_t^x .
\]
Here the weights \( W_{s_0}^x \) are obtained adaptively: starting from empty string \( s_0 \), \( W_{s_0}^x = 1 \), for \( k \in [0, t - 1] \)

\[
W_{s_{k+1}}^x = \frac{1}{2R_k} W_{s_k}^x \quad \text{if } s_{k+1} = 1
\]

\[
W_{s_{k+1}}^x = \frac{1}{2(1 - R_k)} W_{s_k}^x \quad \text{if } s_{k+1} = 0 .
\] (6)

The proof of theorem 1 is given in the appendix. The result of conventional AdaBoost is just a special case of theorem 1 while \( q_t(r_t^x|x) \) only take values in \( \{0, 1\} \).

Defining \( \hat{c} = \max_{x, s_t} \{W_{s_t}^x\} \) as the maximum value of \( W_{s_t}^x \) for all \( x \), the complexity to find the optimal solution is as follows.

**Theorem 2 (Query Complexity for probabilistic case).** There exists an algorithm that can approximate the optimal coefficients in theorem 1 with precision \( \epsilon \) with \( O \left( \frac{\epsilon^2}{\epsilon^2} T \right) \) queries of basis classifiers \( H_t \) and requires \( N = O \left( \frac{\epsilon^2}{\epsilon^2} \right) \) data points as training sample.

**Proof.** (Details see appendix.) The explicit algorithm is shown in algorithm 1. This is indeed a Monte Carlo method. Note that, although there are \( 2^T \) possible routines of \( s_T \) for each \( x \), according to a variation of Höeffding’s inequality (theorem 4), the precision of sampling \( \hat{R}_t = \mathbb{E}[W_{s_t}^x, r_t^x] \) only depends on the maximum and minimum value of the function to be averaged, but not the distribution. This can be seen as input \( x \) is only evaluated once with each classifier \( H_t \), for each branch \( s_T \), the evaluations behaves as if the classifiers are deterministic. Furthermore, each routine \( s_T \) obeys the same update rule; therefore the maximum value is bounded in the same way, i.e., only a sample \( S \) of size \( N = O \left( \frac{\epsilon^2}{\epsilon^2} \right) \) examples are required to estimate \( \hat{R}_t \) with \( \hat{R}_t := \frac{1}{N} \sum_{x \in S} W_{s_t}^x, r_t^x \) with precision \( \epsilon \). To complete the computation, this need to be repeated for each \( t \), and gives the total query complexity \( O(NT) = O \left( \frac{\epsilon^2}{\epsilon^2} T \right) \). □

**Algorithm 1 Classical AdaBoost**

1: Import \( H_1 \); \hspace{1cm} \( \triangleright \) The \( T \) basis classifiers
2: Input \( S \); \hspace{1cm} \( \triangleright \) The sample of size \( N \)
3: Initialize \( W_0^x = 1 \);
4: for \( t \) from 1 to \( T \) do
5: \hspace{1cm} for \( x \) in \( S \) do
6: \hspace{2cm} \( r_t^x \leftarrow H_t(x) \);
7: \hspace{1cm} end for
8: \hspace{1cm} end for
9: \hspace{1cm} \( \hat{R}_t \leftarrow \frac{1}{N} \sum_{x \in S} r_t^x W_{s_t}^x \); \hspace{1cm} \( \triangleright \) Take the average over \( x \)
10: for \( t \) from 1 to \( T \) do
11: \hspace{1cm} if \( r_t^x = 0 \) then
12: \hspace{2cm} \( W_{s_t}^x \leftarrow W_{s_t}^{x \cdot 1}/(2(1 - \hat{R}_t)) \);
13: \hspace{1cm} else
14: \hspace{2cm} \( W_{s_t}^x \leftarrow W_{s_t}^{x \cdot 1}/2 \hat{R}_t \);
15: \hspace{2cm} end if
16: \hspace{1cm} end for
17: Output \( \{\alpha_1, \ldots, \alpha_T\} \);

Quantum AdaBoost — Here we propose a quantum version of the AdaBoost algorithm, which provides a quadratic speed up of the query complexity in term of the sample size \( N \).

A key observation about the classical AdaBoost algorithm is that, the weights \( W_{s_t}^x \) are updated independently for each \( x \); also we are only interested in the average \( \hat{R}_t = \mathbb{E}[W_{s_t}^x, r_t^x] \). (5)

To translate algorithm 1 into a quantum algorithm, first we define \( |s_t \rangle \equiv |s_1 \rangle \otimes \cdots \otimes |s_t \rangle \) to be a register encoding the string \( s_t \). Provided that one can access each basis classifier as a query operator \( \hat{H}_i \) in quantum superposition, i.e.,

\[
\hat{H}_i |x \rangle_X |0 \rangle \rightarrow |x \rangle_X \left( \sqrt{q_i(0|x)} |0 \rangle + \sqrt{q_i(1|x)} |1 \rangle \right) ,
\] (7)

algorithm 1 can be sped up by applying phase estimation. The \( q_i(r_t^x|x) \) here works as the same role as the probabilistic case. Also, a register \( |W_{s_t}^x \rangle_M \) encoding the numerical value of weights \( W_{s_t}^x \) need to be introduced.

The algorithm 1 can be divided into three parts:

In lines 4-6, it queries each classifier with each example in sample \( S \). This can be done with \( t \) queries \( \bigotimes_{i=1}^t \hat{H}_i \) on each qubit in \( R_t \), one can obtain a state of the superposition of all branches \( s_t \): \( \bigotimes_{i=1}^t \hat{H}_i |x \rangle_X |0 \rangle_{R_t} \rightarrow \sum_{s_t} \sqrt{q(s_t, x)} |x \rangle_X |s_t \rangle_{R_t} \). (8)

Then in line 9, the algorithm evaluates \( \hat{R}_t \), which estimate \( \hat{R}_t \) well, by taking the average of \( W_{s_t}^x, r_t^x \). In quantum algorithm is can be done with phase estimation.

Finally, in lines 10-14, the algorithm updates the weights \( W_{s_t}^x \) with the information of \( \hat{R}_t \), which can be done in quantum computer with same gate complexity as classical as it is an arithmetic process.

Our main result for the quantum algorithm is as follows:

**Theorem 3 (Query Complexity for quantum case).** There exists a quantum algorithm that can approximate the optimal coefficients in theorem 1 with precision \( \epsilon \) with \( O \left( \frac{\epsilon}{\epsilon^2} T^2 \right) \) queries of quantum basis classifiers \( \hat{H}_t \) and requires \( N = O \left( \frac{\epsilon^2}{\epsilon^2} \right) \) data points as training sample.

**Proof.** At each iteration \( t \), the querying process can be simply done with \( \bigotimes_{i=1}^t \hat{H}_i \) as shown above.

Although the evaluating of \( \hat{R}_t \), which is taking the average of all branches, cannot be done in superposition as other steps, this can be achieved with phase estimation. With an ancillary qubit \( |W_{s_t}^x \rangle_M |s_t \rangle_{R_t} |0 \rangle \) it can be converted to

\[
\sqrt{1 - W_{s_t}^x/\hat{c}|W_{s_t}^x \rangle_M |s_t \rangle_{R_t} |0 \rangle} + \sqrt{W_{s_t}^x/\hat{c}|W_{s_t}^x \rangle_M |s_t \rangle_{R_t} |1 \rangle}
\]

if the last bit of \( R_t \) is in state \( |0 \rangle \) (Lemma 4 in [29]). The \( \hat{c} \) here
is divided to make sure $1 - W_{x_i}^*/\hat{c}$ is always real, such that this operation is valid.

If we start from the superposition of the whole sample, which is obtained from querying first $t$ classifiers $\mathcal{O}_{l=1}^t \mathcal{H}_l$: $\frac{1}{\sqrt{N}} \sum_{x, s_l} |x \rangle X \sqrt{q(s_l, x)} |W_{x_l}^z\rangle_M |s_l\rangle_{\mathcal{R}_l}$, this conditional operator gives

$$\left[ 1 - \frac{1}{\sqrt{N}} \sum_{x, s_l} q(s_l, x) r^2 \frac{W_{x_l}^z}{\epsilon} |\varphi_0\rangle + \sum_{x, s_l} q(s_l, x) r^2 \frac{W_{x_l}^z}{\epsilon} |\varphi_1\rangle \right] \frac{1}{\sqrt{1 - \hat{R}_t/\hat{c}}} \frac{1}{\sqrt{\hat{R}_t/\hat{c}}} |\varphi_1\rangle,$$

where

$$|\varphi_0\rangle = \frac{1}{\sqrt{N}} \sum_{x, s_l} \sqrt{q(s_l, x)} \sqrt{1 - r^2 \frac{W_{x_l}^z}{\epsilon}} |x \rangle |s_l\rangle |W_{x_l}^z\rangle |0\rangle$$

$$|\varphi_1\rangle = \frac{1}{\sqrt{N}} \sum_{x, s_l} \sqrt{q(s_l, x)} \frac{r^2 \frac{W_{x_l}^z}{\epsilon}}{\sqrt{\hat{R}_t/\hat{c}}} |x \rangle |s_l\rangle |W_{x_l}^z\rangle |1\rangle.$$  \hspace{1cm} (9)

It is not hard to check that they are normalized. Rewrite this as $\sin(\theta_l) |\varphi_0\rangle + \cos(\theta_l) |\varphi_1\rangle$. The whole procedure is then simply rotate the initial state by $\theta_l$. Such operation can be used for phase estimation [12], where $\theta_l$ can be approximated with a constant probability with $O(1/\delta)$ operations. The $\delta$ here is the precision of estimated $\theta_l$, which is at most linear to the precision of $\hat{R}_t/\hat{c} = \cos^2(\theta_l)$. $\hat{R}_t$ can be easily calculated from the value of $\theta_l$, with precision $\Delta \delta$. That is, choosing $\delta = \frac{\epsilon}{\hat{c}}$, which means perform $O(\frac{\epsilon}{\Delta \delta}) = \sqrt{N} \tau$ times of the operation, is enough to estimate $\hat{R}_t$ with precision $\epsilon$. Also, as discussed in the proof of theorem 2, $|\hat{R}_t - \hat{R}_t^{\text{true}}| \leq \epsilon$. Combine these precision together, it can be seen that our quantum algorithm achieves the same order of precision as the classical algorithm 1.

Finally, the weight updating part of the classical AdaBoost algorithm can be viewed as an operation on $|s_l\rangle_{\mathcal{R}_l}$ and $|W_{x_l}^z\rangle_M$ for each iteration $t$. With the updating rule (6), $W_{x_l}^z$ can be easily obtained with $t - 1$ iterations from $W_{x_l}^z|\mathcal{R}_l = 1$ with the information of $\hat{R}_1, \hat{R}_2, \cdots, \hat{R}_{t-1}$ and each bit in $\mathcal{R}_l$ as control bit. Since in iteration $t + 1$, the value $\hat{R}_t$ is fixed for all inputs $x$, the division of $\hat{R}_t$ can be applied simultaneously to all branches, regards the superposed nature of the weights. Since this is simply an arithmetic process, these operations can be implemented on quantum circuits with the same order of complexity as the classical circuits. Also there are no queries needed in this step. See appendix for the details of this implementation.

However, the information of $\hat{R}_1, \cdots, \hat{R}_{t-1}$ are required for $t$th step, and hence above procedures have to be repeated for each $t$. Also, at $t$th step, the weights $W_{x_l}^z$ have to be adaptively updated from vary beginning as the measurements for phase estimation would disturb the quantum states. Therefore, at $t$th iteration, first $t$ basis classifiers need to be evaluated and hence the algorithm requires $\sum_{l=1}^t O(\hat{R}_t/\hat{c}) = O(\hat{R}_t^2 T^2)$ queries in total. As we choose $N = O\left(\frac{1}{\epsilon^2}\right)$, the query complexity can be rewritten as $O(\sqrt{NT^2})$.

Quantum Learning—Our quantum AdaBoost algorithm is also valid even if the inputs or basis classifiers are quantum, e.g., in the task of quantum template matching [26] that aims to classify quantum states. In such an application, without loss of generality, we can consider the $t$-th binary classifier to be a two-outcome POVM $\{M_{t-1}', M_{t+1}'\}$, where $M_{t-1} + M_{t+1}' = \mathbb{I}$ are positive semidefinite operators. Denote the true label of $\rho_x$ by $y(\rho_x) \in \{-1, +1\}$. It follows that $r^2 = 0$ if the measurement outcome $h$ is equal to $y(\rho_x)$, and $r^2 = 1$ otherwise. Then the error probability of the $t$-th classifier on input state $\rho_x$ is given by $q_t(r^2 = 1 | \rho_x) = tr[M_{t-1}^* \rho_x]$, where $M_{t+1}^* := M_{t+1}$. Now consider the task whose goal is to classify all pure states $|\psi\rangle \langle \psi|$ into two groups, where each state is sampled from the Haar measure on the space of density operators. In general, a single POVM may not work well. However, if $T$ copies of the same state are available, then our quantum AdaBoost algorithm could be used to generate a strong POVM (aka. binary classifier) from individual weak POVM to improve the performance.

Finally, we remark that the theory of $t$-design [30] could help in the implementation of our algorithm. In AdaBoost algorithm, the update rule requires taking expectation over the Haar measure on the space of density operators. In general, a single POVM may not work well. However, if $T$ copies of the same state are available, then our quantum AdaBoost algorithm could be used to generate a strong POVM (aka. binary classifier) from individual weak POVM to improve the performance.

Discussion and Conclusion — In this article, we considered the conventional AdaBoost algorithm for binary classification tasks and extended it to probabilistic and quantum classifiers. The probabilistic extension is a straightforward analogy to quantum algorithms due to the probabilistic nature of quantum physics. As a result, the conventional AdaBoost algorithm can be perfectly recovered under the probabilistic extension. Furthermore, we showed that there exists a quantum algorithm that can optimize the AdaBoost model to the same precision as the classical algorithm with a quadratic speedup in terms of query complexity. Table I illustrates the complexities of AdaBoost algorithms in different scenarios. In realistic circumstances, $N \gg T$ holds in the application of AdaBoost, hence, the query complexity in the
TABLE I. Query Complexity of AdaBoost Models

| AdaBoost Model | Type of Basis Classifier | Query Complexity |
|----------------|--------------------------|------------------|
| Conventional   | D                        | $O(NT)$          |
| Probabilistic  | D/P                      | $O(NT)$          |
| Quantum        | D/P/Q                    | $O(\sqrt{NT})$  |

$^a$ D for deterministic classifier, P for probabilistic classifier, and Q for quantum classifier.

quantum case performs better than that in the classical cases.

Boosting a collection of weak classifiers into a strong classifier is, in particular, suitable for building quantum learning machines because weak classifiers are easier to implement under current quantum hardware technology [31]. Similar boosting ideas will most likely appear in designing noisy intermediate-scale quantum (NISQ) devices, and our quantum boosting algorithm can be employed to further improve the efficiency including those in [31].

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Proof of Theorem 1

Here we perform our analysis for the probabilistic case, which can degenerate to the conventional AdaBoost if the outputs of classifiers are certain. Moreover, we assume a certain target label \(y(x)\) for all \(x \in \Omega\), where \(\Omega\) is the sample space of all possible inputs. Let \(p(x)\) be the probability mass function defined on \(\Omega\).

The goal of AdaBoost is to find the optimal coefficients \(\{\alpha_t\}\) of the linear model \(g_t := \sum_{t=1}^{T} \alpha_t H_t\) based on the basis classifiers \(H_t : \Omega \to \{+1, -1\}\) with the minimum exponential error, which is the average of \(e^{-g_t(x)y(x)}\) over the joint distribution of inputs and classifiers. Here \(\{H_t\}\) are random variables which yield the conditional probabilities \(\mathbb{P}[H_t(x) = y(x)|x]\). Let \(r_t^x := \frac{1}{2}(1 - H_t(x) \cdot y(x))\), that is \(r_t^x = 0\) if \(H_t(x) = y(x)\), and \(r_t^x = 1\) otherwise. Then \(\mathbb{P}[H_t(x) = y(x)|x]\) is fully determined by the conditional probability mass function \(q_t(r_t^x|x)\)

The exponential error as the cost function \(C_T := \mathbb{E}[e^{g_T(x)y(x)}]\) yields

\[
C_T = \sum_x p(x) \prod_{t=1}^{T} \left( \sum_{r_t^x = 0,1} q_t(r_t^x|x) e^{-\alpha_t(-1)^{r_t^x}} \right) \tag{11}
\]

because \(H_t(x)y(x) = (-1)^{r_t^x}\). In AdaBoost, the optimization problem is done by adding each term into \(g_T(x)\) one by one with the optimal weight \(\alpha_t\) at \(t^{th}\) iteration. Let \(C_t\) be the exponential error of the first \(t\) terms of \(g_T\)

Let \(s_t \in \mathbb{B}^t\) be a binary string \(s_1s_2\cdots s_t\). Let \(w_{s_t}^x := \prod_{i=1}^{t} e^{-\alpha_i(-1)^{y_i}|x}\), and let \(q(s_t, x) := p(x) \prod_{i=1}^{t} q_i(r_i^x = s_i|x)\). Then equation (11) gives

\[
C_t = \sum_{x,s \in \Omega \times s_{t-1}} q(s_{t-1}, x) w_{s_{t-1}}^x \left( \sum_{r_t^x = 0,1} q_t(r_t^x|x) e^{-\alpha_t(-1)^{r_t^x}} \right). \tag{12}
\]

This is a convex function respect to \(\alpha_t\) and an unique solution to the problem exists at the extreme. Taking its derivative to 0 gives

\[
\sum_{x,s \in \Omega \times s_{t-1}} q(s_{t-1}, x) w_{s_{t-1}}^x (-q_t(0|x)e^{-\alpha_t} + q_t(1|x)e^{\alpha_t}) = 0 \tag{13}
\]

and hence

\[
e^{2\alpha_t} \sum_{x,s \in \Omega \times s_{t-1}} q(s_{t-1}, x) w_{s_{t-1}}^x q_t(1|x) = \sum_{x,s \in \Omega \times s_{t-1}} q(s_{t-1}, x) w_{s_{t-1}}^x q_t(0|x). \tag{14}
\]

That is

\[
\alpha_t = \frac{1}{2} \ln \frac{\sum_{x,s \in \Omega \times s_{t-1}} q(s_{t-1}, x) w_{s_{t-1}}^x q_t(0|x)}{\sum_{x,s \in \Omega \times s_{t-1}} q(s_{t-1}, x) w_{s_{t-1}}^x q_t(1|x)}. \tag{15}
\]

Let

\[
\tilde{R}_t := \frac{\sum_{x,s \in \Omega \times s_{t-1}} q(s_{t-1}, x) w_{s_{t-1}}^x q_t(1|x)}{\sum_{x,s \in \Omega \times s_{t-1}} q(s_{t-1}, x) w_{s_{t-1}}^x}. \tag{16}
\]

Then the optimal weight of each iteration is

\[
\alpha_t = \frac{1}{2} \ln \left( \frac{1 - \tilde{R}_t}{\tilde{R}_t} \right). \tag{17}
\]

In the following, we demonstrate that the optimal weight \(\{\alpha_t\}\) can be adaptively obtained. When \(t = 1\), initialize \(w_{s_0}^x = 1\). Thus

\[
\tilde{R}_1 = \frac{\sum_{x} p(x)q_1(1|x)}{\sum_{x} p(x)} = \frac{\sum_{x} p(x)q_1(1|x)}{\mathbb{E}_x q_1(r_1^x)} = \mathbb{E}_x q_1[r_1^x] \tag{18}
\]

which is exactly the generalization error of \(H_1\).

Let \(Z_t = \sum_{x,s_{t-1}} q(s_{t-1}, x) w_{s_{t-1}}^x\) be the \(t^{th}\) normalization factor. Then

\[
\tilde{R}_t = \frac{\sum_{x,s_{t-1}} q(s_{t-1}, x) w_{s_{t-1}}^x q_t(1|x)}{Z_t} = \frac{\sum_{x,s_{t-1}} q(s_{t-1}, x) q_t(r_t^x|x) w_{s_{t-1}}^x}{Z_t} \tag{19}
\]

By definition \(w_{s_t}^x = w_{s_{t-1}r_t}^x = w_{s_{t-1}}^x e^{-\alpha_t(-1)^{r_t}}\). Therefore

\[
Z_{t+1} = \sum_{x,s_t} q(s_t, x) w_{s_t}^x = \sum_{x,s_{t-1}} q(s_{t-1}, x) w_{s_{t-1}}^x q_t(0|x)e^{-\alpha_t} + q_t(1|x)e^{\alpha_t}) = e^{-\alpha_t}(Z_t \tilde{R}_t e^{2\alpha_t} + Z_t(1 - \tilde{R}_t)) = e^{-\alpha_t}(Z_t \tilde{R}_t \frac{1 - \tilde{R}_t}{\tilde{R}_t} + Z_t(1 - \tilde{R}_t)) = 2e^{-\alpha_t}(1 - \tilde{R}_t)Z_t. \tag{20}
\]

Let

\[
W_{s_{t+1}}^x := \frac{w_{s_t}^x}{Z_{t+1}} \tag{21}
\]

Therefore, all the values of \(W_{s_t}^x\) can be obtained by iterating with the information of \(\tilde{R}_t\). It is not hard to check
that (21) is equivalent to the updating rule (6). These values again yield \( R_{t+1} = E[W^x_{s_{t+1}}, r^x_{t+1}] \) for next iteration, and therefore every \( \alpha_t \) could be determined analytically in this manner.

Proof of Theorem 2

In the section Proof of Theorem 1, a theoretical optimal solution to the AdaBoost Model is derived. However, in practice, the underlying distribution of inputs is unknown, and therefore the values of \( q(s_t | x) \) are impossible to be evaluated. Also, usually the training algorithm cannot cover the whole sample space (otherwise the explicit relationship between inputs and output are known, and machine learning is unnecessary).

Similar to other machine learning tasks, this problem is solved by sampling. Clearly, with a underlying distribution \( D \) on the sample space \( \Omega \), each \( r^x \) can be viewed as a random variable on the sample space. This can be done with an interesting result derived from Hoeffding’s inequality.

**Theorem 4 (Hoeffding’s inequality).** If a sample \( S \) of size \( N \) is drawn from a distribution \( D \) on a sample space \( \Omega \), then given a random variable \( X \) on \( \Omega \) and any positive number \( \epsilon > 0 \)

\[
P \left( \left| \frac{1}{N} \sum_{x \in S} X(x) - E[D][X(x)] \right| \geq \epsilon \right) \leq 2 \exp \left( -\frac{2N\epsilon^2}{\epsilon^2} \right) \tag{22}
\]

where \( c := \sup X - \inf X \).

The key point here is that, though \( E[D][X(x)] \) cannot be evaluated in practice, \( \frac{1}{N} \sum_{x \in S} X(x) \) is computable, and it approximates \( E[D][X(x)] \) well when \( N = |S| \) is large.

According to equation (19), \( \hat{R}_t = E[q(s_t | x) W^x_{s_t} r^x_{t}] \). For a sample \( S \) of pairs \((x, s_t)\) drawn from the distribution \( q(s_t | x) \), let

\[
\hat{R}_t := \frac{1}{N} \sum_{x \in S} [W^x_{s_t} r^x_{t}] \tag{23}
\]

Then theorem 4 shows that

\[
P \left( \left| \hat{R}_t - \hat{R}_t \right| \geq \epsilon \right) \leq 2 \exp \left( -\frac{2N\epsilon^2}{\epsilon^2} \right) \tag{24}
\]

where \( N \) is the size of \( S \) and \( c_t := \max(W^x_{s_t} r^x_{t}) - \min(W^x_{s_t} r^x_{t}) \).

To be noticed, the value of \( W^x_{s_t} \) is derived with iteration according to equation (21). Since \( W^x_{s_t} \equiv 1 \), \( W^x_{s_t} \) is always positive, which means \( \min(W^x_{s_t} r^x_{t}) \) is always non-negative as well. Further, max \( W^x_{s_t} r^x_{t} \) \( \leq \max W^x_{s_t} := c_t \).

Therefore, for a target precision \( \epsilon \) of \( R_t \), a sample with size \( N = \mathcal{O} \left( \frac{c_t^2}{\epsilon^2} \right) \) is good enough to achieve the goal with a constant probability. Nevertheless, the size of sample have to be determined before hand; and hence we should choose

\[
N = \mathcal{O} \left( \frac{c_t^2}{\epsilon^2} \right) \tag{25}
\]

where \( \hat{c} = \max \{c_t\} \).

**Remark.** However, \( \hat{c} \) might not be small when \( T \) is large, which indicates that AdaBoost may not be good if the model does not converge fast with the number of classifiers used. These might be improved by other boosting algorithms, e.g. LogitBoost, Gradient Boosting, XGBoosting.

As long as we obtain a sample \( S \) of size \( N \), according to theorem 4, the algorithm 1 approximates \( \hat{R}_t \) well. This algorithm evaluates each data \( x \in S \) for each classifier \( H_t \), and therefore requires \( \mathcal{O}(NT) \) queries.

Quantum Simulation of Classical Process

This section reviews some results from Kitaev’s paper [11] that simulate classical Boolean circuits with quantum circuits. For convenient, without loose of generality the classical registers are denoted with Dirac notations here.

According to lemma 1 and 7 in [11], if a function \( F : \mathbb{B}^n \rightarrow \mathbb{B}^m \) can be computed with \( L \) Boolean operations \( g \in \mathcal{B} \), which is a small set of Boolean operations, then it can be computed with \( 2L + m \) operations in the basis \( \mathcal{B}_r \). The basis \( \mathcal{B}_r \) is defined in a way that, for each \( g \in \mathcal{B} : |x⟩_X \rightarrow |g(x)⟩_B \), there is a \( g_r \in \mathcal{B}_r : |x⟩_X |v⟩_B \rightarrow |x⟩_X |v \oplus g(x)⟩_B \). Also the operation to copy a state

\[
τ_{A,B} : |x⟩_A |v⟩_B \rightarrow |x⟩_A |v \oplus x⟩_B \tag{26}
\]

(which is indeed a CNOT gate) have to be included into \( \mathcal{B}_r \).

Furthermore, we say a circuit computes a Boolean function \( F \), if it converts \( |x⟩_X \rightarrow |F(x)⟩_B \). With basis \( \mathcal{B}_r \), this computation is performed as \( |x⟩_X |0⟩_B \rightarrow |x⟩_X |F(x)⟩_B \).

However, one may only need partial information about the output \( F(x) \). Classically, it is free to readout part of the the output bits and drop the rest. Nevertheless, in quantum computation, dropping those “garbage” bits \((|\text{gar}(x)⟩)\) would destroy the quantum state if they are in superposition. But as shown above, \( |x⟩_X |0⟩_B \rightarrow |x⟩_X |F(x)⟩_B \) can be constructed with \( 2L + m \) reversible gates. Divide register \( B \) into two parts \((B_1, B_2)\), and then \( |F(x)⟩_B \) is \( |f(x)⟩_B_1 \otimes |\text{gar}(x)⟩_B_2 \) (it is always separable as the initial states are all tensor product states), above process is then

\[
|x⟩_X |0⟩_B \rightarrow |x⟩_X |f(x)⟩_B_1 \otimes |\text{gar}(x)⟩_B_2 .
\]
By repeating this process on an extra register $B' = (B'_1, B'_2)$, the process
\[ |x\rangle_X |0\rangle_{B_1} |0\rangle_{B_2} |0\rangle_{B'_1} |0\rangle_{B'_2} \]
\[ \rightarrow |x\rangle_X |H(x)\rangle_{B_1} |\text{gar}(x)\rangle_{B_2} |f(x)\rangle_{B'_1} |\text{gar}(x)\rangle_{B'_2} \]
can be constructed.

If the input state is on quantum registers and it is in superposition
\[ \left( \frac{1}{\sqrt{N}} \sum_x |x\rangle_X \right) |0\rangle_{B_1} |0\rangle_{B_2} |0\rangle_{B'_1} |0\rangle_{B'_2}, \]
this process will give
\[ \left( \frac{1}{\sqrt{N}} \sum_x |x\rangle_X |f(x)\rangle_{B_1} |\text{gar}(x)\rangle_{B_2} |f(x)\rangle_{B'_1} |\text{gar}(x)\rangle_{B'_2} \right). \]

Then the pairwise operation $\tau_{B'_2, B'_2}$ (26) is performed between the “garbage” states on $B_2$ and $B'_2$, which gives
\[ \left( \frac{1}{\sqrt{N}} \sum_x |x\rangle_X |f(x)\rangle_{B_1} |\text{gar}(x)\rangle_{B_2} |f(x)\rangle_{B'_1} |\text{gar}(x)\rangle_{B'_2} \right) \]
\[ = \left( \frac{1}{\sqrt{N}} \sum_x |x\rangle_X |f(x)\rangle_{B_1} |0\rangle_{B_2} |f(x)\rangle_{B'_1} |\text{gar}(x)\rangle_{B'_2} \right). \]

Finally, the original process is performed again on $B'$ which ends up at
\[ \left( \frac{1}{\sqrt{N}} \sum_x |x\rangle_X |f(x)\rangle_{B_1} \right) |0\rangle_{B_2} |0\rangle_{B'_1} |0\rangle_{B'_2}. \]

Since the appending registers $B_2, B'_1, B'_2$ are all end up at $|0\rangle$, it is free to drop them after computation.

In summary, the process $|x\rangle_X |0\rangle \rightarrow |x\rangle_X |f(x)\rangle$ can be achieved with $O(L)$ quantum gates even for computation in superposition. This fact indicates that each arithmetic part in our quantum algorithm can be performed with the same complexity of the classical algorithm. Since all ancillary registers always start and end at $|0\rangle$, they are neglected in our notation for simplicity.

Note that, although above result is only valid for Boolean functions, as how modern computers work, these Boolean operations are indeed universal. In case people want to deal with real numbers on computers, those values have to be encoded into binary strings up to some precision.

**Example 1.** The updating rule (6) is purely arithmetic. This can be viewed as repeating controlled operation $U$ on a register $M$, encoding a numerical value $\xi$ in terms of binary strings $|\xi\rangle$ up to some precision. Each application of $U$ is controlled by each qubit of the string $|s_i\rangle_{R_i} := |s_1\rangle \otimes \cdots \otimes |s_i\rangle$. More precisely,
\[ U_i = U_0 \otimes |s_i = 0\rangle \langle s_i = 0| + U_1 \otimes |s_i = 1\rangle \langle s_i = 1|, \]
where $U_0 |\xi\rangle_M = |\frac{\xi}{2^{1 + \beta M}}\rangle_M$; $U_1 |\xi\rangle_M = |\frac{\xi}{2^{1 + \beta M}}\rangle_M$. As a result, lines 10-14 in algorithm 1 can be performed in quantum circuits with the same order of gates as classical circuit. Additionally, this can be done in superposition for all $x$, and hence the “for” loop in classical algorithm can be done in one shot.

Similarly, another step for phase estimation in our algorithm can be done with this method.

**Example 2.** There exists an operation $Q_t$ such that for $\xi \in [0, \tilde{c}]$,
\[ Q_t |\xi\rangle_M |0\rangle = |\xi\rangle_M (\sqrt{1 - \frac{\xi}{\tilde{c}}} |0\rangle + \sqrt{\frac{\xi}{\tilde{c}}} |1\rangle). \quad (27) \]

The requirement of $\xi \in [0, \tilde{c}]$ is presented to make sure $\cos^{-1} \sqrt{\frac{\xi}{\tilde{c}}}$ is a real number, and therefore, the state $|\xi\rangle_M = \frac{\xi}{\tilde{c}}$ can be constructed on an ancillary register “anc”. Here $|A\rangle_{\text{anc}} = |a_1\rangle |a_2\rangle \cdots |a_m\rangle$, where $0, a_1 a_2 \cdots a_m$ is the binary representation of the real number $A$ up to some precision. The process to compute
\[ |\xi\rangle_M |0\rangle \rightarrow |\xi\rangle_M |\cos^{-1} \frac{\xi}{\tilde{c}}\rangle_{\text{anc}} \]

is arithmetic. By further appending an additional qubit $|0\rangle$ to the system, an operation can be constructed as lemma 4 in [20], such that it converts $|\xi\rangle_M |A = \cos^{-1} \frac{\xi}{\tilde{c}}\rangle_{\text{anc}} |0\rangle$ to
\[ |\xi\rangle_M |A\rangle_{\text{anc}} (\sin(A) |0\rangle + \cos(A) |1\rangle) \]
\[ = |\xi\rangle_M |A\rangle_{\text{anc}} (\sqrt{1 - \frac{\xi}{\tilde{c}}} |0\rangle + \sqrt{\frac{\xi}{\tilde{c}}} |1\rangle). \]

Finally the register $|A\rangle_{\text{anc}}$ can be cleared and dropped with the garbage dropping technique above. This whole process is exactly the operation $Q_t$.

The operations in these examples would be useful in next section.

**Proof of Theorem 3**

In the Quantum AdaBoost Algorithm, the computation other than the average of $r^T_W s_i$ can be performed in parallel on the whole sample. That is, for every initial state $|x\rangle_X$, where $x$ is the data points of a sample $S$ of size $N$ drawn from the sample space $\Omega$, the classical algorithm outputs $|r^T_W s_i\rangle_M$ to the register $M$, which encoding the numerical value of $r^T_W s_i$. Note that $|r^T_W s_i\rangle_M$ there is the state corresponding to the binary value of $r^T_W s_i$, as how modern computer saves numerical values. With this property, the AdaBoost algorithm can be performed by following adaptive procedure:
At \( t \)-th iteration, given the classical information of \( \hat{R}_1, \ldots, \hat{R}_{t-1} \) (where \( \hat{R}_t \) can be obtained in \( t \)-th iteration), initialize the state of three registers \( X, M, R_t \) as
\[
\frac{1}{\sqrt{N}} \sum_{x \in S} |x \rangle_X \otimes |W_{s_{t-1}^x}^x \rangle_M \otimes |0 \rangle^t_{R_t},
\]
with access to the quantum oracle \( \hat{H}_1 \oslash \ldots \oslash \hat{H}_t \) defined in (7), one can obtain
\[
\frac{1}{\sqrt{N}} \sum_{x \in S} \sum_{s_t} |x \rangle_X \otimes |W_{s_t}^{x_{t-1}} \rangle_M \otimes \sqrt{q(s_t|x)} |s_t \rangle_{R_t}.
\]
With the classical information of \( \hat{R}_1, \ldots, \hat{R}_{t-1} \), one can update the register \( M \) with the updating rule (6) (which is a classical arithmetic process shown in example 1) to the state
\[
\frac{1}{\sqrt{N}} \sum_{x \in S} \sum_{s_t} |x \rangle_X \otimes |W_{s_{t-1}^x}^{x_{t-1}} \rangle_M \otimes \sqrt{q(s_t|x)} |s_t \rangle_{R_t}.
\]
Compose the whole arithmetic process that converts (28) to (30) and rewrite it as \( \mathcal{A}_t \):
\[
\mathcal{A}_t \frac{1}{\sqrt{N}} \sum_{x \in S} |x \rangle_X \otimes |W_{s_t}^{x_{t-1}} \rangle_M \otimes |0 \rangle_{R_t} = \frac{1}{\sqrt{N}} \sum_{x \in S} \sqrt{q(s_t|x)} |x \rangle_X \otimes |W_{s_t}^{x_{t-1}} \rangle_M \otimes |s_t \rangle_{R_t}.
\]
With an extra working register, apply the operation in example 2 to the final state in (31)
\[
\mathcal{Q}_t \mathcal{A}_t \frac{1}{\sqrt{N}} \sum_{x \in S} |x \rangle_X \otimes |W_{s_t}^{x_{t-1}} \rangle_M \otimes |0 \rangle_{R_t} = \frac{1}{\sqrt{N}} \sum_{x \in S} \sqrt{q(s_t|x)} \sqrt{1 - \hat{R}_t/\hat{c}} |x \rangle_X \otimes |W_{s_t}^{x_{t-1}} \rangle_M \otimes |s_t \rangle_{R_t},
\]
where
\[
|\phi_0 \rangle = \frac{1}{\sqrt{N}} \sum_{x, s_t} \sqrt{q(s_t|x)} \frac{\sqrt{1 - \hat{R}_t/\hat{c}}}{\sqrt{1 - \hat{R}_t}} |x \rangle_X |W_{s_t}^{x_{t-1}} \rangle_M |s_t \rangle_{R_t},
\]
\[
|\phi_1 \rangle = \frac{1}{\sqrt{N}} \sum_{x, s_t} \sqrt{q(s_t|x)} \frac{\sqrt{\hat{R}_t/\hat{c}}}{\sqrt{\hat{R}_t}} |x \rangle_X |W_{s_t}^{x_{t-1}} \rangle_M |s_t \rangle_{R_t}.
\]
Note that for each \( x \), \( \sum_{s_t} q(s_t|x) = 1 \).
According to the definition in equation (23), the result of (32) is indeed
\[
\sqrt{1 - \hat{R}_t/\hat{c}} |\phi_0 \rangle + \sqrt{\hat{R}_t/\hat{c}} |\phi_1 \rangle.
\]
This can be rewrite as
\[
|\psi_0 \rangle := \sin(\theta_t) |\phi_0 \rangle |0 \rangle + \cos(\theta_t) |\phi_1 \rangle |1 \rangle,
\]
which performs a rotation of angle \( \theta_t \).

Let \( |\psi_0 \rangle := \cos(\theta_t) |\phi_0 \rangle |0 \rangle - \sin(\theta_t) |\phi_1 \rangle |1 \rangle \). After a Pauli-\( Z \) operation is performed on the last register of \( |\psi_0 \rangle \), it is transformed to
\[
\sin(\theta_t) \sin(\theta_t) |\psi_0 \rangle + \cos(\theta_t) |\psi_1 \rangle
\]
\[
- \cos(\theta_t) \cos(\theta_t) |\psi_0 \rangle - \sin(\theta_t) \sin(\theta_t) |\psi_1 \rangle
\]
\[
= \cos(2\theta_t) |\psi_0 \rangle + \sin(2\theta_t) |\psi_1 \rangle.
\]

Let \( \mathcal{G}_t := Q_t A_t \). Apply the inverse operation \( \mathcal{G}_t \) to (36), so that \( |\psi_0 \rangle \) is mapped back to the initial state (28). Note that, \( |\psi_1 \rangle \) is orthogonal to \( |\psi_0 \rangle \) and our operation is unitary. Therefore, if an operation \( U_{\perp} \) only inverse the amplitude of the every state perpendicular to the initial state (analogy to the diffusion operator in Grover’s algorithm [10]) is applied and the operation \( \mathcal{G}_t \) is performed again, \( |\psi_0 \rangle \) would be left unchanged. This procedure gives
\[
\cos(2\theta_t) |\psi_0 \rangle - \sin(2\theta_t) |\psi_1 \rangle
\]
\[
= \cos(2\theta_t) |\sin(\theta_t) |\phi_0 \rangle |0 \rangle + \cos(\theta_t) |\phi_1 \rangle |1 \rangle
\]
\[
- \sin(2\theta_t) |\cos(\theta_t) |\phi_0 \rangle |0 \rangle - \sin(\theta_t) |\phi_1 \rangle |1 \rangle
\]
\[
= \cos(3\theta_t) |\phi_0 \rangle |0 \rangle + \sin(3\theta_t) |\phi_1 \rangle |1 \rangle.
\]

In conclusion, \((\mathcal{G}_t U_{\perp} \mathcal{G}_t^\dagger)^k \mathcal{G}_t \) converts the initial state to
\[
\cos((2k + 1) \theta_t) |\phi_0 \rangle |0 \rangle + \sin((2k + 1) \theta_t) |\phi_1 \rangle |1 \rangle.
\]
Such operation provides the possibility to estimate \( \theta_t \) with phase estimation algorithm.
To fairly compare the query complexities, we want to constrain the results from both classical and quantum algorithm to the same precision. In order to approximate \( \hat{R}_t \) with the target precision \( O(\epsilon) \), the phase estimation algorithm have to estimate \( \hat{R}_t = |\cos^2(\theta_t)\hat{c}| \) with precision \( \epsilon \), and as shown in (25), a sample of size \( N = O\left(\frac{1}{\epsilon^2}\right) \) is enough to estimate each \( \hat{R}_t \) with \( \hat{R}_t \) with precision \( \epsilon \).
In the \( t \)-th step of our quantum algorithm, by choosing number of iterations in (38) as \( k = O\left(\frac{1}{\epsilon^2}\right) \), the phase estimation process could read out the value of \( \hat{\theta}_t \), such that \( |\hat{\theta}_t - \theta_t| \leq \delta \).
In order to estimates \( \hat{R}_t \) with the same precision as the classical algorithm, we need to bound \( \hat{\theta}_t \) to make sure
\[
|\hat{\theta}_t \cos^2(\hat{\theta}_t) - \hat{R}_t| \leq \epsilon.
\]
This can be done by choose a proper \( \delta \). Then, the task of our analysis is to bound the value of \( \delta \) in terms of \( \epsilon \) and \( c \) as in the classical case.
Let \( \hat{c} := \hat{\cos}^2(\hat{\theta}_t) - \hat{R}_t \), then \( |\hat{\theta}_t - \theta_t| \leq \delta \) gives
\[
|\cos^{-1}(\sqrt{\frac{\hat{R}_t + \hat{c}}{\hat{c}}}) - \cos^{-1}(\sqrt{\frac{\hat{R}_t}{\hat{c}}})| \leq \delta.
\]
Since \( |\hat{c}| \leq \epsilon \), \( \hat{c} \) is
a small number, and hence

\[ \cos^{-1}\left(\frac{\hat{R}_t + \hat{\epsilon}}{\hat{c}}\right) - \cos^{-1}\left(\frac{\hat{R}_t}{\hat{c}}\right) \approx \hat{\epsilon} \frac{d}{dx} \cos^{-1}(\sqrt{x}) \bigg|_{x = \hat{R}_t / \hat{c}}. \]  

(39)

When \( \hat{R}_t / \hat{c} \sim 0 \), \( \frac{d}{dx} \cos^{-1}(\sqrt{x}) \bigg|_{x = \hat{R}_t / \hat{c}} \) is almost a constant, and \( \hat{\theta}_t - \theta_t = O\left(\frac{\hat{\epsilon}}{\hat{c}}\right) \leq \delta \). This is usually true since \( 0 \leq \hat{R}_t \leq 1 \) and \( \hat{c} \gg 1 \). Note that \( \hat{c} = \max\{W^t_x\} = \prod_{t=1}^{T} \max\left\{ \frac{1}{2\hat{R}_t}, \frac{1}{2(1-\hat{R}_t)} \right\} \), and \( \max\left\{ \frac{1}{2\hat{R}_t}, \frac{1}{2(1-\hat{R}_t)} \right\} \geq 1 \).

To make sure \( \hat{c}\cos^2(\hat{\theta}_t) - \hat{R}_t \leq \epsilon \), or equivalently \( |\hat{\epsilon}| \leq \epsilon \), the optimal \( \delta \) can be chosen is \( O\left(\frac{\hat{\epsilon}}{\hat{c}}\right) \). This gives \( k = O\left(\frac{\hat{\epsilon}}{\hat{c}}\right) \).

Moreover, for \( t^{th} \) iteration, the step (29) requires \( t \) queries. So the query complexity for each iteration is \( O\left(\frac{\hat{\epsilon}}{\hat{c}}t\right) \).

Nevertheless, in order to obtain the value of \( \hat{R}_t \), each quantum iteration \( t \) is followed with a measurement. The information of \( W^t_x \) saved in superposition would be disrupted and thus it have to be evaluated from every beginning every time. Therefore the overall complexity is \( O\left(\frac{\hat{\epsilon}}{\hat{c}} \sum_{t=1}^{T} t\right) = O\left(\frac{\hat{\epsilon}}{\hat{c}} T^2\right) \), comparing to the classical case, which is \( O\left(\frac{\hat{\epsilon}}{\hat{c}} T\right) \). As discussed in remark at the end of the section Proof of Theorem 1, AdaBoost algorithm may not work well if it does not converge within a small number of iterations. Therefore, the \( T \) here may be considered as a small constant.

Also, for both quantum and classical algorithms, we use \( N = O\left(\frac{\hat{\epsilon}}{\hat{c}}^2\right) \), the query complexity of classical algorithm can be rewritten as \( O(NT) \) and the quantum query complexity is then \( O\left(\sqrt{NT^2}\right) \).

This quantum algorithm could give the same result of the classical algorithm with the same order of precision \( \epsilon \) with same success probability.