A Note on the Possible Existence of an Instanton-like Self-Dual Solution to Lattice Euclidean Gravity

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The self-dual solution to lattice Euclidean gravity is constructed. In contrast to the well known Eguchi-Hanson solution to continuous Euclidean Gravity, the lattice solution is asymptotically globally Euclidean, i.e., the boundary of the space as \( r \to \infty \) is \( S^3 = SU(2) \).

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I. INTRODUCTION

Soon after the discovery of the self-dual (instanton) solution to the 4D Euclidean Yang-Mills theory [1], the self-dual solution to 4D Euclidean Gravity has been obtained [2, 3] (see also [4], [5], [6]). But the space-time topology of the gravitational instanton solution differs crucially from the topology of a real space-time. Therefore, though the action of the Eguchi-Hanson solution is zeroth, the physical meaning of the solution is not clear.

I construct here the analogue of the Eguchi-Hanson self-dual solution to the 4D Euclidean lattice Gravity with zeroth action. The solution transforms locally into the Eguchi-Hanson gravitational instanton, in contrast to the Yang-Mills one. Otherwise, a "cone-type" singularity (effective delta-function in the curvature at the instanton centre for \( r = a \)) would be necessary. This means that the space-time topology of the gravitational instanton solution differs crucially from the topology of a real space-time. Therefore, though the action of the Eguchi-Hanson solution is zeroth, the physical meaning of the solution is not clear.

This article does not contain any new result in comparison with [11], but the main result is obtained more rationally and transparently. The changes concerned only Section V.

A preliminary version of the work has been published in [7].

II. EGUCHI-HANSON SELF-DUAL SOLUTION

First of all it is necessary to describe shortly the Eguchi-Hanson self-dual solution to continuous Euclidean Gravity. Let \( \gamma^a \), \( \gamma^a \gamma^b + \gamma^b \gamma^a = 2\delta^{ab} \), \( a = 1, 2, 3, 4 \) be the Hermitian Dirac matrices \( 4 \times 4 \) in spinor representation \((\sigma^a, \alpha = 1, 2, 3, 4) \) are Pauli matrices:

\[
\gamma^0 = \begin{pmatrix} 0 & -i\sigma^a \\ i\sigma^a & 0 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

\[
\gamma^5 \equiv \gamma^1 \gamma^2 \gamma^3 \gamma^4 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^{ab} = \frac{1}{4} [\gamma^a, \gamma^b],
\]

\[
\sigma^{a4} = \frac{i}{2} \left( \sigma^a \begin{pmatrix} 0 & 0 \\ 0 & -\sigma^a \end{pmatrix} \right), \quad \sigma^{a\beta} = \frac{i\varepsilon_{a\beta\gamma}}{2} \begin{pmatrix} \sigma^\gamma & 0 \\ 0 & -\sigma^\gamma \end{pmatrix}.
\]

(2.1)

Let's consider the pure 4D Euclidean Gravity action in the Palatini form (the independent variables are tetrad and connection):

\[
\mathcal{A} = -\frac{1}{l^2} \int \text{tr} \gamma^5 R \wedge e \wedge e = \]

\[
= -\frac{1}{l^2} \int \left\{ R^\alpha_{\beta\mu} \wedge E^\alpha_{\beta\mu} - R^\alpha_{-\beta\mu} \wedge E^\alpha_{-\beta\mu} \right\},
\]

\[
R \equiv 2 (d\omega + \omega \wedge \omega) = \frac{2i}{2} \left( R^\alpha_{\beta\mu} \wedge \frac{2}{\varepsilon_{\alpha\beta\gamma}} \omega^\gamma_{\mu} \wedge \omega^\gamma_{\mu} \right) d x^\mu,
\]

\[
\omega^\alpha_{\beta\mu} = \frac{1}{2} \left\{ \omega^{\alpha}_{\beta\mu} + \frac{1}{2} \varepsilon_{\beta\gamma} \omega^\gamma_{\mu} \right\} d x^\mu,
\]

\[
R^\alpha_{\beta\mu} = 2 d\omega^\alpha_{\beta\mu} - \varepsilon_{\alpha\beta\gamma} \omega^\gamma_{\mu} \wedge \omega^\gamma_{\mu},
\]

\[
e \equiv \gamma^a e^a_{\beta\mu} d x^\mu,
\]

\[
E^\alpha_{\beta\mu} \equiv \left\{ \pm e^4_{\lambda\mu} - e^4_{\lambda\rho} + \varepsilon_{\lambda\rho\beta\gamma} e^\gamma_{\alpha\rho} \right\} d x^\lambda \wedge d x^\rho.
\]

(2.2)
One can take six 1-forms $\omega^a_{(\pm)}$ as independent variables instead of six 1-forms $\omega^{ab}$. Obviously, the representation \[2.2\] is consistent with the representation of the group $\text{Spin}(4) \approx \text{Spin}(4)_{(+)} \otimes \text{Spin}(4)_{(-)} \approx \text{SU}(2)_{(+)} \otimes \text{SU}(2)_{(-)}$.

The following equations are equivalent
\[
\omega = \pm \gamma^5 \omega \longleftrightarrow \omega^{ab} = \pm \frac{1}{2} \varepsilon_{abcd} \omega^{cd} \longleftrightarrow \omega^a_{(\pm)} = 0. \quad (2.3)
\]
Eqs. \[2.3\] imply the following one:
\[
R = \pm \gamma^5 R \longleftrightarrow R^{ab} = \pm \frac{1}{2} \varepsilon_{abcd} R^{cd} \longleftrightarrow R^a_{(\pm)} = 0. \quad (2.4)
\]
The action stationarity condition relative to the connection gives the equation
\[
\delta \mathfrak{A} / \delta \omega^{a}_{\mu} = 0 \longrightarrow d e^a + \omega^{ab} \wedge e^b = 0, \quad (2.5)
\]
which determines uniquely the connection forms for fixed forms $e^a$. Additionally, Eq. \[2.5\] imply the algebraic Bianchi identity for Riemannian tensor, the combination of which with Eq. \[2.4\] leads to the Einstein’s equation
\[
R_{ab} \equiv R^c_{ac} = 0. \quad (2.6)
\]
On the other hand, Einstein’s equation is equivalent to the action stationarity condition relative to the forms $e^a_{\mu}$:
\[
R_{ab} = 0 \longleftrightarrow \delta \mathfrak{A} / \delta e^a_{\mu} = 0. \quad (2.7)
\]
The question arises: why the additional Eq. \[2.3\] does not come into conflict with Eqs. \[2.5\] and \[2.7\]? To answer this question let’s consider the case with the lower sign in Eq. \[2.3\] when
\[
\omega^{a}_{(-)\mu} = 0. \quad (2.8)
\]
The combination of the part of Eqs. \[2.6\] $\delta \mathfrak{A} / \delta \omega^{a}_{(-)\mu} = 0$ and Eq. \[2.8\] gives the following 12 equations:
\[
\varepsilon^{\mu \nu \lambda \rho} \partial_{\nu} E^{a}_{(-)\lambda \rho} = 0. \quad (2.9)
\]
Now we must solve the system of equations \[2.7\], \[2.8\], \[2.9\] and
\[
\delta \mathfrak{A} / \delta \omega^{a}_{(+\mu}) = 0. \quad (2.10)
\]
Note that Eqs. \[2.3\] and \[2.7\] do not fix the variables $\omega^{ab}$, $e^a_{\mu}$ completely but up to the gauge (orthogonal) transformations. Here the gauge group leaves 6 unfixed functions.

Eqs. \[2.3\] do not fix the quantities $E^{a}_{(-)\lambda \rho}$ completely but up to summands of the form $\left( \partial_{\rho} \Psi^{a}_{(-)\lambda} - \partial_{\lambda} \Psi^{a}_{(-)\rho} \right)$ where $\Psi^{a}_{(-)\lambda}$ are 3 arbitrary vector fields (12 functions altogether). But each of three vector fields $\Psi^{a}_{(-)\lambda}$ contains only 3 independent functions due to invariance of the expression $\left( \partial_{\lambda} \Psi^{a}_{(-)\rho} - \partial_{\rho} \Psi^{a}_{(-)\lambda} \right)$ relative to the changes $\Psi^{a}_{(-)\lambda} \longrightarrow \Psi^{a}_{(-)\lambda} + \partial_{\lambda} \phi^{a}_{(-)}$. As a result, Eqs. \[2.3\] fix no more than additional $12 - 3 \times 3 = 3$ functions. This means that the gauge subgroup Spin(4)$_{(-)}$ is broken by Eq. \[2.5\]. So we see that the system of equations \[2.7\]–\[2.10\] is consistent, though it fixes the gauge subgroup Spin(4)$_{(-)}$. The Eguchi-Hanson solution is the simplest solution of the system. Let’s write out this solution \[2\], \[3\].

Let $x^i = (r, \theta, \varphi, \psi)$, where $(\theta, \varphi, \psi)$ be the Euler angles. The cartesian coordinates $x^a$ in $\mathbb{R}^4$ are connected with the coordinates $x^a$ as follows:
\[
z_1 \equiv x^1 + ix^2 = r \cos \frac{\theta}{2} \exp \left[ i \frac{1}{2} (\psi + \varphi) \right], \quad (2.11)
\]
\[
z_2 \equiv x^3 + ix^4 = r \sin \frac{\theta}{2} \exp \left[ i \frac{1}{2} (\psi - \varphi) \right]. \quad (2.11)
\]
There is a one-to-one correspondence between these two coordinate systems if the Euler angles vary in the ranges
\[
0 \leq \theta \leq \pi, \quad 0 \leq \varphi \leq 2\pi, \quad 0 \leq \psi \leq 4\pi. \quad (2.12)
\]
The Eguchi-Hanson solution for the metrics $d s^2 = (e^a_{\mu} d x^\mu)^2$ has the form
\[
e^a_{\mu} d x^\mu = \begin{pmatrix}
\frac{r}{2} (\sin \psi d \theta - \sin \theta \cos \psi d \varphi) \\
\frac{r}{2} (\cos \psi d \theta + \sin \theta \sin \psi d \varphi) \\
- \frac{g(r)}{r} (\cos \theta d \varphi + d \psi) \\
g(r)^{-1} d r
\end{pmatrix},
\]
\[
g(r) = \sqrt{1 - \frac{a^4}{r^4}}, \quad r \geq a. \quad (2.13)
\]
For $a = 0$, the metrics \[2.13\] transforms to the 4D Euclidean metrics in Euler angles on the $S^3$ with ranges \[2.12\]. But $0 \leq \psi \leq 2\pi$ for the Eguchi-Hanson solution (when $a \neq 0$) since the points with coordinates $\psi$ and $(\psi + 2\pi)$ and the same $(r, \theta, \varphi)$ are identified. Otherwise, in order for the Chern-Gauss-Bonnet theorem to be satisfied in the case \[2.12\], a ”cone-type” singularity (effective delta-function in the curvature at the instanton centre for $r = a$) would be necessary (see \[3\]).

The connection 1-forms are given by Eqs. \[2.8\] and:
\[
\omega^{1}_{(+)} = \frac{2g}{r} e^1 = g (\sin \psi d \theta - \sin \theta \cos \psi d \varphi), \quad \omega^{2}_{(+)} = \frac{2g}{r} e^2 = g (\cos \psi d \theta + \sin \theta \sin \psi d \varphi), \quad \omega^{3}_{(+)} = 2 \left( \frac{2}{rg} - \frac{g}{r} \right) e^3 = - \left( 1 + \frac{a^4}{r^4} \right) (\cos \theta d \varphi + d \psi). \quad (2.14)
\]
We have
\[
\int_{r=\text{Const}} \rightarrow \infty \left( \frac{1}{2} \omega^1_{(+)} \right) \wedge \left( \frac{1}{2} \omega^2_{(+)} \right) \wedge \left( \frac{1}{2} \omega^3_{(+)} \right) = -\pi^2 \quad (2.15)
\]
and
\[
\int_{r=\text{Const}} \rightarrow 0 \left( \frac{1}{2} \omega^1_{(+)} \right) \wedge \left( \frac{1}{2} \omega^2_{(+)} \right) \wedge \left( \frac{1}{2} \omega^3_{(+)} \right) = 0 \quad (2.16)
\]
for the range $0 \leq \psi \leq 2\pi$ and orientation $\theta, \varphi, \psi, r$. The integral (2.13) would be equal to $(-\pi^2)$ for any $0 < r = \text{Const} < \infty$ in the case $a = 0$ (the Euclidean metrics in Euler angles). Thus, the boundary conditions (2.13)-(2.16) determine the instanton Eguchi-Hanson solution with the same integration constant $a$ as in the relation (2.16). This means that the system of equations (2.11)-(2.14) together with the boundary conditions (2.13)-(2.16) possess unique solution (2.16) for the centrally symmetrical metrics anzatzs with the same integration constant $a$ as in the relation (2.16).

Note that

$$\frac{i\sigma^\alpha}{2} \omega^{a}_{(+)} = U^{-1} dU \quad \text{as} \quad r \to \infty,$$

where

$$U = \exp \left( -\frac{i\sigma^3}{2} \varphi \right) \exp \left( i\sigma^2 \frac{t}{2} \right) \exp \left( -i\sigma^3 \frac{\theta}{2} \right).$$

According to the Eq. (2.17)

$$\frac{1}{12} \left( \text{tr} \left( U^{-1} dU \right) \wedge \left( U^{-1} dU \right) \wedge \left( U^{-1} dU \right) \right) =$$

$$= \left( \frac{1}{2} \omega^i_{(+)} \right) \wedge \left( \frac{1}{2} \omega^j_{(+)} \right) \wedge \left( \frac{1}{2} \omega^k_{(+)} \right) \quad \text{as} \quad r \to \infty.$$

(2.19)

Taking into account Eqs. (2.15) and (2.17), we obtain for the angle ranges (2.12):

$$\frac{1}{12} \left( \text{tr} \left( U^{-1} dU \right) \wedge \left( U^{-1} dU \right) \wedge \left( U^{-1} dU \right) \right) = -2\pi^2.$$

(2.20)

This equality means that Eq. (2.18) gives the smooth mapping of the space-time hypersurface $S^3$ prescribed by the Euler angles (see Eqs. (2.11) with a fixed parameter $r$) into the SU(2) group space, and the degree of the mapping is equal to $(-1)$.

Write out also the Riemannian curvature 2-form:

$$R^0_{(-)} = 0,$$

$$R^1_{(+)} = \frac{8a^4}{r^6} \left( \epsilon^1 \wedge \epsilon^4 + \epsilon^2 \wedge \epsilon^3 \right),$$

$$R^2_{(+)} = \frac{8a^4}{r^6} \left( \epsilon^1 \wedge \epsilon^3 - \epsilon^2 \wedge \epsilon^4 \right),$$

$$R^3_{(+)} = \frac{16a^4}{r^6} \left( \epsilon^1 \wedge \epsilon^2 + \epsilon^3 \wedge \epsilon^4 \right).$$

(2.21)

III. THE LATTICE GRAVITY MODEL

The next step is to adumbrate the model of lattice gravity which is used here. A detailed description of the model is given in [8]-[10].

The orientable 4-dimensional simplicial complex and its vertices are designated as $\mathcal{R}$ and $a^\nu$, the indices $\nu = 1, 2, \ldots$, $9^\mu \to \infty$ and $\mathcal{W}$ enumerate the vertices and 4-simplices, correspondingly. It is necessary to use the local enumeration of the vertices $a^\nu$ attached to a given 4-simplex: the all five vertices of a 4-simplex with index $\mathcal{W}$ are enumerated as $a_{\mathcal{W}ij}, i = 1, 2, 3, 4, 5$. The later notations with extra index $\mathcal{W}$ indicate that the corresponding quantities belong to the 4-simplex with index $\mathcal{W}$. The Levi-Civita symbol with in pairs different indexes $\varepsilon_{\mathcal{W}ijklm} = \pm 1$ depending on whether the order of vertices $\delta_{\mathcal{W}ijklm} = a_{\mathcal{W}ij}, a_{\mathcal{W}jk}, a_{\mathcal{W}km}, a_{\mathcal{W}lm}$ defines the positive or negative orientation of 4-simplex $\delta_{\mathcal{W}}$. An element of the group Spin(4) and an element of the Clifford algebra

$$\Omega_{\mathcal{W}ij} = \Omega_{\mathcal{W}ij}^{(-1)} \equiv \exp (\omega_{\mathcal{W}ij}), \quad \omega_{\mathcal{W}ij} \equiv \frac{1}{2} \sigma^{ab} \omega^{ab}_{\mathcal{W}ij};$$

$$\hat{e}_{\mathcal{W}ij} = \hat{e}_{\mathcal{W}ij} \equiv e_{\mathcal{W}ij} \gamma^a \equiv -\Omega_{\mathcal{W}ij} \hat{e}_{\mathcal{W}ij} \Omega_{\mathcal{W}ij}^{(-1)}.$$ (3.1)

are assigned for each oriented 1-simplex $a_{\mathcal{W}ij}a_{\mathcal{W}ji}$. The lattice analog of the action (2.2) has the form

$$\Phi = \frac{1}{5 \times 24} \sum_{\mathcal{W}, i,j,k,l,m} \varepsilon_{\mathcal{W}ijklm} \text{tr} \gamma^5 \times$$

$$\times \left\{ -\frac{2}{\ell_P^2} \Omega_{\mathcal{W}mi} \Omega_{\mathcal{W}ij} \Omega_{\mathcal{W}jm} \tilde{e}_{\mathcal{W}mk} \omega_{\mathcal{W}ml} \right\}.$$ (3.2)

This action is invariant relative to the gauge transformations

$$\hat{\Omega}_{\mathcal{W}ij} = S_{\mathcal{W}i} \Omega_{\mathcal{W}ij} S_{\mathcal{W}j}^{-1}, \quad \hat{e}_{\mathcal{W}ij} = S_{\mathcal{W}i} e_{\mathcal{W}ij} S_{\mathcal{W}j}^{-1}, \quad S_{\mathcal{W}i} \in \text{Spin}(4).$$ (3.3)

It is natural to interpret the quantity

$$l^2_{\mathcal{W}ij} = \frac{1}{4} \text{tr} \left( \hat{e}_{\mathcal{W}ij} \right)^2 = \sum_{a=1}^{4} (\epsilon_{\mathcal{W}ij}^a)^2 \sim l_P^2$$ (3.4)

as the square of the length of the edge $a_{\mathcal{W}ij}a_{\mathcal{W}ji}$, and the parameter $l_P$ is of the order of the lattice spacing. Thus, the geometric properties of a simplicial complex prove to be completely defined.

Now, let us show in the limit of slowly varying fields, that the action (3.2) reduces to the continuous gravity action (2.2).

Consider a certain 4D sub-complex of complex $\mathcal{R}$ with the trivial topology of four-dimensional disk. Realize geometrically this sub-complex in $\mathbb{R}^4$. Suppose that the geometric realization is an almost smooth four-dimensional surface [12]. Thus each vertex of the sub-complex acquires the coordinates $x^\mu$ which are the coordinates of the vertex image in $\mathbb{R}^4$:

$$x^\mu_{\mathcal{W}i} = x^\mu_0 (a_{\mathcal{W}i}) \equiv x^\mu (a^\nu), \quad \mu = 1, 2, 3, 4$$ (3.5)

We stress that these coordinates are defined only by their vertices rather than by the higher dimension simplices to which these vertices belong; moreover, the correspondence between the vertices from the considered subset and the coordinates (3.5) is one-to-one.
We have
\[ |x^\mu_{W_1} - x^\mu_{W_2}| \sim l_p. \tag{3.6} \]

It is evident that the four vectors
\[ d x^\mu_{W_{ij}} \equiv x^\mu_{W_1} - x^\mu_{W_2}, \quad i = 1, 2, 3, 4 \tag{3.7} \]
are linearly independent and
\[ \begin{vmatrix} dx^1_{W_1} & dx^2_{W_1} & \cdots & dx^4_{W_1} \\ \vdots & \vdots & \cdots & \vdots \\ dx^1_{W_4} & dx^2_{W_4} & \cdots & dx^4_{W_4} \end{vmatrix} \geq 0, \tag{3.8} \]
depending on whether the frame \((X^W_{m_1}, \ldots, X^W_{m_4})\) is positively or negatively oriented. Here, the differentials of coordinates \((3.7)\) correspond to one-dimensional simplices \(a_{W_j}a_{W_i}\), so that, if the vertex \(a_{W_i}\) has coordinates \(x^\mu_{W_i}\), then the vertex \(a_{W_j}\) has the coordinates \(x^\mu_{W_j} + d x^\mu_{W_{ij}}\).

In the continuous limit, the holonomy group elements \((3.1)\) are close to the identity element, so that the quantities \(\omega_{i}^b\) tend to zero being of the order of \(O(d x^\mu)\). Thus one can consider the following system of equation for \(\omega_{W_{ij}}\)
\[ \omega_{W_{ij}} d x^\mu_{W_i} = \omega_{W_i}, \quad i = 1, 2, 3, 4. \tag{3.9} \]

In this system of linear equation, the indices \(W\) and \(m\) are fixed, the summation is carried out over the index \(\mu\), and index runs over all its values. Since the determinant \((3.8)\) is nonzero, the quantities \(\omega_{W_{ij}}\) are defined uniquely. Suppose that a one-dimensional simplex \(X^W_{m_i}\) belongs to four-dimensional simplices with indices \(W_1, W_2, \ldots, W_r\). Introduce the quantity
\[ \omega_{\mu} \left( \frac{1}{2} (x^\mu_{W} + x^\mu_{W_i}) \right) \equiv \frac{1}{r} \left\{ \omega_{W_1 \mu} + \ldots + \omega_{W_r \mu} \right\}, \tag{3.10} \]
which is assumed to be related to the midpoint of the segment \([x^\mu_{W_1}, x^\mu_{W_i}]\). Recall that the coordinates \(x^\mu_{W_i}\) as well as the differentials \((3.7)\) depend only on vertices but not on the higher dimensional simplices to which these vertices belong. According to the definition, we have the following chain of equalities
\[ \omega_{W_1 m_i} = \omega_{W_2 m_i} = \ldots = \omega_{W_r m_i}. \tag{3.11} \]

It follows from \((3.7)\) and \((3.9)\) - \((3.11)\) that
\[ \omega_{\mu} \left( x^\mu_{W} + \frac{1}{2} d x^\mu_{W_i} \right) d x^\mu_{W_i} = \omega_{W_i}. \tag{3.12} \]
The value of the field element \(\omega_{\mu}\) in \((3.12)\) is uniquely defined by the corresponding one-dimensional simplex.

Next, we assume that the fields \(\omega_{\mu}\), smoothly depend on the points belonging to the geometric realization of each four-dimensional simplex. In this case, the following formula is valid up to \(O((d x)^2)\) inclusive
\[ \Omega_{W_{mi}} \Omega_{W_{ij}} \Omega_{W_{jm}} = \exp \left[ \frac{1}{2} \mathcal{R}_{\mu \nu} (x^\mu_{W_i}) d x^\mu_{W_{mi}} d x^\nu_{W_{mj}} \right], \tag{3.13} \]
where
\[ \mathcal{R}_{\mu \nu} = \partial_{\mu} \omega_{\nu} - \partial_{\nu} \omega_{\mu} + [\omega_{\mu}, \omega_{\nu}]. \tag{3.14} \]

When deriving formula \((3.13)\), we used the Hausdorff formula.

In exact analogy with \((3.9)\), let us write out the following relations for a tetrad field without explanations
\[ \hat{e}_{W_{ij}} d x^\mu_{W_i} = \hat{e}_{W_{mi}}. \tag{3.15} \]

Applying formulas \((3.13)\) - \((3.15)\) to the discrete action \((3.2)\) and changing the summation to integration we find that in the long-wavelength limit the lattice action \((3.2)\) transforms into continuous action \((2.2)\) and any information about lattice is forgotten in the main approximation.

**IV. THE SELF-DUAL SOLUTIONS TO LATTICE GRAVITY**

Now let us consider the self-dual solution to lattice gravity. We have the lattice analogue of Eqs. \((2.5)\) and \((2.7)\);
\[ \delta \mathcal{A} / \delta \omega^{\alpha}_{W(-)W_{mi}} = 0, \tag{4.1} \]
\[ \delta \mathcal{A} / \delta \hat{e}^{\alpha}_{W_{mi}} = 0. \tag{4.2} \]

Because of the action \((3.2)\) is a homogeneous quadratic function of the variables \([e]\), so
\[ 2 \mathcal{A} = \sum_{(e)} e^{\alpha}_{W_{mi}} (\delta \mathcal{A} / \delta \hat{e}^{\alpha}_{W_{mi}}) = 0 \tag{4.3} \]
on the mass shell according to Euler theorem. Let’s impose the additional conditions (compare with Eqs. \((3.5)\))
\[ \omega^{\alpha}_{W(-)W_{mi}} = 0. \tag{4.4} \]

Combining Eqs. \((4.1)\) with index \((-)\) and \((4.4)\) we obtain:
\[ \frac{\delta \mathcal{A}}{\delta \omega^{\alpha}_{(-)W_{mi}}} \omega^{\alpha}_{(-)W_{mi}} = 0 = \frac{2}{l_p^2} \sum_{W} \sum_{j,k,l} e^{\alpha}_{W'mijkl} \left\{ E^{\alpha}_{(-)W'ijkl} + E^{\alpha}_{(-)W'mijkl} + E^{\alpha}_{(-)W'mijkl} + E^{\alpha}_{(-)W'mijkl} + (m \leftrightarrow i) \right\} = 0, \]
\[ E^{\alpha}_{m'kl} \equiv \frac{1}{3} \left( E^{\alpha}_{sm'} + E^{\alpha}_{sk'lm} + E^{\alpha}_{s[k'l]} \right), \]
\[ E^{\alpha}_{s[m'k]} \equiv \frac{1}{2} \left( e^{\alpha}_{smk} e^{\alpha}_{sm} - e^{\alpha}_{smk} e^{\alpha}_{sm} \right) + \varepsilon_{\alpha \beta \gamma} e^{\alpha}_{smk} e^{\alpha}_{sm}. \tag{4.5} \]
The index $\mathcal{W}'$ in (4.5) enumerates all 4-simplices which contain a marked 1-simplex $a\omega\alpha\omega\omega$. As in continuous case, the system of equations (4.1), (4.2) and (4.3) is equivalent to the system of equations (4.4) with index $(+), (4.2), (4.3)$ and (4.4). It will be shown that the square brackets give null equation under the sum (4.5). Therefore only the first term in the parentheses in Eq. (4.6) is significant.

Equations (4.3) do not fix the quantity $E_{(-)}^\alpha \mathcal{W}'[jkkl]$ completely but up to summands of the kind

$$E_{(-)}^\alpha \mathcal{W}'[jkkl] \rightarrow E_{(-)}^\alpha \mathcal{W}'[jkkl] + \left( \Psi_{(-)}^\alpha \mathcal{W}'[jkl] + \Psi_{(-)}^\alpha \mathcal{W}'[ijkl] \right),$$

and the lattice 1-form $\Psi_{(-)}^\alpha \mathcal{W}'[km] = -\Psi_{(-)}^\alpha \mathcal{W}'[mk]$ can be varied according to

$$\Psi_{(-)}^\alpha \mathcal{W}'[mk] \rightarrow \Psi_{(-)}^\alpha \mathcal{W}'[mk] + \left( \phi_{(-)}^\alpha \mathcal{W}'[mk] - \phi_{(-)}^\alpha \mathcal{W}'[mk] \right).$$

It follows from here that Eqs. (4.5) fix no more than 3 additional real-valued parameters at each vertex of complex, leading to the fixation of gauge subgroup Spin(4).-. Let’s prove the given statements. For this end one must check that the equation

$$\sum_{\mathcal{W}'} \left\{ \sum_{j,k,l} \varepsilon_{\mathcal{W}'[ijkl]} \left( \Psi_{(-)}^\alpha \mathcal{W}'[jkl] + \Psi_{(-)}^\alpha \mathcal{W}'[ijkl] \right) \right\} = 0$$

is satisfied identically. It is evident that the braces in Eq. (4.8) vanishes identically at each fixed value of index $\mathcal{W}'$ if $\Psi_{(-)}^\alpha \mathcal{W}'[mk] = \left( \phi_{(-)}^\alpha \mathcal{W}'[mk] - \phi_{(-)}^\alpha \mathcal{W}'[mk] \right)$.

Consider two adjacent positively oriented 4-simplices

$$s_{4}^\mathcal{W} = a\omega\alpha\omega\omega\alpha\omega\omega\omega, \quad a_{4}^\mathcal{W} = a\omega\alpha\omega\omega\alpha\omega\omega\omega,$$

so that

$$\varepsilon_{\mathcal{W}[ijkl]} = \varepsilon_{\mathcal{W}[mikjl]} = 1. \quad (4.10)$$

Select from the sum (4.8) two summands corresponding to the 4-simplices (4.9):

$$\sum_{j,k,l} \varepsilon_{\mathcal{W}[ijkl]} \left( \Psi_{(-)}^\alpha \mathcal{W}'[jkl] + \Psi_{(-)}^\alpha \mathcal{W}'[ijkl] \right) + \sum_{j,k,l'} \varepsilon_{\mathcal{W}[ijkl']} \left( \Psi_{(-)}^\alpha \mathcal{W}'[jkl'] + \Psi_{(-)}^\alpha \mathcal{W}'[ijkl'] \right) + \sum_{j,k,l} \varepsilon_{\mathcal{W}[ijkl]} \left( \Psi_{(-)}^\alpha \mathcal{W}'[jkl] + \Psi_{(-)}^\alpha \mathcal{W}'[ijkl] \right),$$

We see that the quantities $\Psi_{(-)}^\alpha \mathcal{W}'[mk]$ belonging to the common 3-simplices of the adjacent 4-simplices cancel on in (4.11) as a consequence of Eqs. (4.10) and

$$(\Psi_{(-)}^\alpha \mathcal{W}'[jkl] + \Psi_{(-)}^\alpha \mathcal{W}'[jkl']) = 0. \quad (5.2)$$

V. LATTICE ANALOGUE OF THE EGUCHI-HANSON SOLUTION

Now proceed to study the lattice analog of the Eguchi-Hanson solution. But it is impossible to give the irregular lattice solution in an explicit form in contrast to the continuous case. Thus, the problem reduces to the solution existence proof and finding of its asymptotics.

Suppose that the complex $\mathcal{R}$ is finite but it contains gigillion simplices, so that the number of vertices $\infty \gg 1$. Suppose also that $\mathcal{R}$ can be considered as a triangulation of a part of $\mathcal{R}^+$ and $\partial \mathcal{R} \approx S^3$. It is assumed in a wide vicinity of $\partial \mathcal{R}$ the long-wavelength limit is valid and the continuous solution (5.1), (5.2) approximates correctly the exact lattice solution and the hyper-surface $\partial \mathcal{R}$ is given by the Eq. $r = R \sim \text{Const} \rightarrow \infty$. Denote by $\mathcal{E} \subset \mathcal{R}$ a finite sub-complex containing the centre of instanton with the boundary $\partial \mathcal{E} \neq S^3$. Evidently, the Euler characteristics $\chi(\mathcal{E}) = \chi(\mathcal{R}) = 1$. We have the exact lattice equivalents of the instanton boundary conditions (2.13) and (2.14):

$$\frac{1}{12 \cdot 4!} \sum_{\mathcal{S}(\partial \mathcal{E})} \sum_{ijklm} \varepsilon_{ijklm} \left( \Omega_{(+)}^{-1} \mathcal{S}_{ij} \Omega_{(+)} \mathcal{S}_{jk} \right) \left( \Omega_{(+)}^{-1} \mathcal{S}_{km} \Omega_{(+)} \mathcal{S}_{lm} \Omega_{(+)} \mathcal{S}_{kl} \right) = -2\pi^2, \quad (5.1)$$

$$\sum_{\mathcal{S}(\partial \mathcal{E})} \sum_{ijklm} \varepsilon_{ijklm} \left( \Omega_{(+)}^{-1} \mathcal{S}_{ij} \Omega_{(+)} \mathcal{S}_{jk} \right) \left( \Omega_{(+)}^{-1} \mathcal{S}_{km} \Omega_{(+)} \mathcal{S}_{lm} \Omega_{(+)} \mathcal{S}_{kl} \right) = 0. \quad (5.2)$$

Here the indices $\mathcal{S}(\partial \mathcal{E})$ and $\mathcal{S}(\partial \mathcal{E})$ enumerate 3-simplices on the boundaries $\partial \mathcal{E}$ and $\partial \mathcal{E}$, correspondingly, and the Levi-Civita symbol $\varepsilon_{ijklm} = \pm 1$ depending on whether the order of vertices $s_{3}^\mathcal{S} = a\omega\alpha\omega\omega\omega\alpha\omega\omega\omega$ defines the positive or negative orientation of this 3-simplex.

Since the long-wavelength limit is valid as $r \rightarrow \infty$, one can use the instanton solution for the dynamic variables (2.13)-(2.14) in this region:

$$\Omega_{(+)} \mathcal{W} \mathcal{M} \approx 1 + \frac{\varepsilon_{\alpha}^\mathcal{E}}{2} \omega_{(+)}^\alpha \mu d x_{\mu}^\mathcal{W}.$$
Therefore the sum in (5.1) transforms into integral (2.15) with the only difference that now the angle $\psi$ varies in the interval (2.12), and so the boundary condition (5.1) is just.

To implement the boundary condition (5.2) we suggest the following solutions on the sub-complex $\mathfrak{t}$:

Consider the following solutions of Eqs. (4.1), (4.2) and (4.3) on $\mathfrak{t}$:

$$\Omega^{(+)}_{Wij} = -1, \quad \Omega^{(-)}_{Wij} = 1, \quad s^i_{Wij} \in \mathfrak{t}. \quad (5.3)$$

From here it follows that

$$\Omega^{(+)}_{Wmi} \Omega^{(+)}_{Wj} \Omega^{(+)}_{Wjm} = -1 \quad \text{on} \quad \mathfrak{t}. \quad (5.4)$$

The equalities (5.4) hold true if we perform a gauge transformation (see (5.3)) with $S^{(+)}_{Wi} = \pm 1$, $S^{(-)}_{Wi} = 1$. The boundary condition (5.2) is true for each configuration obtained in such a way.

Obviously, 1-form $\varepsilon_{Wij}$ can be considered as a 1-cochain on complex, and the quantity

$$E^{ab}_{\mathfrak{W}[kl]} = (E^{a}_{\mathfrak{W}[kl]}), \quad (E^{a}_{\mathfrak{W}[kl]}) = \varepsilon_{abcd} e^c_{\mathfrak{W}mk} e^d_{\mathfrak{W}nl}, \quad (5.5)$$

is a 2-cochain which is the superposition of the exterior products of 1-cochains $e^a_{Wij}$.

It may be verified (compare with (4.5)) that the left hand side of Eq. (4.1) is nothing but the exterior lattice derivative of a 2-cochain (5.5), and Eq. (4.1) implies that the derivative is equal to zero in the case of (5.4), i.e. the quantity (5.5) is a cocycle [14]:

$$\sum_{W'} \varepsilon_{W'mijkl} E^{ab}_{W'[jkl]} = 0, \quad a_{Wm} a_{Wij} \in \mathfrak{t}, \quad a_{Wm} a_{Wij} \notin \partial \mathfrak{t}. \quad (5.6)$$

Evidently, Eqs. (5.6) are satisfied for

$$e^a_{V_1V_2} = \phi^a_{V_2} - \phi^a_{V_1}, \quad a_{V_1V_2} \in \mathfrak{t}, \quad a_{V_1V_2} \notin \partial \mathfrak{t}. \quad (5.7)$$

where $\phi^a$ is any scalar field on $\mathfrak{t}$. This statement is checked easily by a direct calculation and it follows from the fact that the cochain (5.6) is the superposition of the exterior products of exact 1-forms in the case (5.7).

Since the second cohomology group $H^2(\mathfrak{t})$ is a lattice cocontunity:

$$E^{ab}_{\mathfrak{W}[kl]} = (\Phi^{ab}_{\mathfrak{W}mk} - \Phi^{ab}_{\mathfrak{W}nl}), \quad \Phi^{ab}_{\mathfrak{W}mk} = - \Phi^{ab}_{\mathfrak{W}nl}, \quad (5.8)$$

Eqs. (5.6) fix no more than 6 real numbers at each vertex of the sub-complex $\mathfrak{t}$ for the reason that the lattice 1-form $\Phi^{ab}_{\mathfrak{W}mk}$ is determined up to the lattice gradient $(\phi^a_{Wk} - \phi^a_{Wm})$. So it follows from Eqs. (5.3) (just as like in the case of (4.1)) that now not only the gauge sub-group Spin(4) is broken to on $\mathfrak{t} \setminus \partial \mathfrak{t}$ almost wholly except for the center of the sub-group SU(2) (+) (see Eqs. (4.10)-(4.11)).

From here and throughout the following discussions the pairs of indices $(\mathcal{V}_1 \mathcal{V}_2)$ enumerate 1-simplices $a_{\mathcal{V}_1 \mathcal{V}_2} \in \mathfrak{t}$.

The action (3.2) is equal to zero identically for the configurations (4.4) and any values of $e^a_{\mathcal{V}_1 \mathcal{V}_2}$, where 1-simplex $a_{\mathcal{V}_1 \mathcal{V}_2} \in \mathfrak{t}$, $a_{\mathcal{V}_1 \mathcal{V}_2} \notin \partial \mathfrak{t}$. Therefore, Eqs. (4.2) are satisfied automatically in this case, they do not give any constraint onto the corresponding 1-forms $e^a_{\mathcal{V}_1 \mathcal{V}_2}$ in addition to Eqs. (5.8):

$$\delta \mathcal{A} / \delta e^a_{\mathcal{V}_1 \mathcal{V}_2} = 0, \quad a_{\mathcal{V}_1 \mathcal{V}_2} \in \mathfrak{t}, \quad a_{\mathcal{V}_1 \mathcal{V}_2} \notin \partial \mathfrak{t}. \quad (5.9)$$

Now let us prove that the lattice configuration of the variables $\Omega^{(+)}_{V_1V_2}$ on the sub-complex $\mathfrak{t}$ as in (5.3) is broken to on $\mathfrak{t}$ almost wholly except for the center of the sub-group SU(2) (+) (see Eqs. (4.10)-(4.11)).

The action (3.2) is equal to zero identically for the configurations (4.4), (4.3) and any values of $e^a_{\mathcal{V}_1 \mathcal{V}_2}$, where 1-simplex $a_{\mathcal{V}_1 \mathcal{V}_2} \in \mathfrak{t}$, $a_{\mathcal{V}_1 \mathcal{V}_2} \notin \partial \mathfrak{t}$. Therefore, Eqs. (4.2) are satisfied automatically in this case, they do not give any constraint onto the corresponding 1-forms $e^a_{\mathcal{V}_1 \mathcal{V}_2}$ in addition to Eqs. (5.8):

$$\delta \mathcal{A} / \delta e^a_{\mathcal{V}_1 \mathcal{V}_2} = 0, \quad a_{\mathcal{V}_1 \mathcal{V}_2} \in \mathfrak{t}, \quad a_{\mathcal{V}_1 \mathcal{V}_2} \notin \partial \mathfrak{t}. \quad (5.9)$$

Now let’s consider $\Omega^{(+)}_{V_1V_2}$ as a constrained-free functional (3.2) on sub-complex $(\text{int} \mathfrak{t}) \setminus \mathfrak{t}$. Each of the elements $\Omega^{(+)}_{V_1V_2}$ is a smooth matrix function on the sum of two unit 3D spheres $S^3_{(+)}(V_1V_2) \cup S^3_{(-)}(V_1V_2)$. Therefore the action $\mathcal{A}_{\Omega}$ is a smooth real function on the compact manifold without boundary

$$\mathcal{C} = \bigcup_{(\mathcal{V}_1 \mathcal{V}_2)} \left( (S^3_{(+)}(V_1V_2) \cup S^3_{(-)}(V_1V_2) \right) \quad (5.11)$$

for any fixed set of values $\{e\}$. So there exist the points $p_{\mathfrak{t}} \in \mathcal{C}$ at which the action $\mathcal{A}_{\Omega}$ is extremal and therefore

$$\partial \mathcal{A}_{\Omega} / \partial e^{a}_{\mathcal{V}_1 \mathcal{V}_2} \bigg|_{p_{\mathfrak{t}} \in \mathcal{C}} = 0. \quad (5.12)$$

Further Eqs. (5.12) are assumed to be true.

Recall that the sets of equations (4.1) with the sign (−) and (4.4) are equivalent to the set of equations (see Eqs. (4.15))

$$\Phi^{a}_{\mathcal{V}_1 \mathcal{V}_2} \equiv - \frac{\ell_p}{2} \frac{\delta \mathcal{A}}{\delta e^{a}_{\mathcal{V}_1 \mathcal{V}_2}} |_{\omega(-)=0} = \sum_{W^j k, l} \varepsilon_{\mathcal{V}_1 \mathcal{V}_2 jkl} E^{a}_{\mathcal{V}_1 \mathcal{V}_2 jkl} = 0 \quad (5.13)$$
and Eqs. (4.3). Here the quantities $\omega_{m}a_{W_{i}}$ and $\varepsilon_{W_{m}i_{1}i_{2}}$ are renamed as $a_{V_{1}}a_{V_{2}}$ and $\varepsilon_{V_{1}V_{2}V_{3}V_{4}}$, correspondingly. Further the set of equations (5.13) on $\mathcal{R}\setminus\mathfrak{r}$ is considered as the set of constraints.

Let’s resolve the constraints (5.13) on the sub-complex $(\mathcal{R}\setminus\mathfrak{r})$ relative to a subset of variables $\{\mathfrak{g}_{q}\}$, $q = 1, \ldots, Q \sim 3\mathcal{N}$, describing the gauge transformations $\text{Spin}(4)|_{-}$. We denote by $\{e^{'j}\}$ the rest of variables $\{e\}$. So we have on the sub-complex $(\mathcal{R}\setminus\mathfrak{r})$:

(i) the set of variables $\{e\}$ is equivalent to the set of variables $\{e', \mathfrak{g}\}$,

(ii) the hypersurface (5.13) is represented as the set of analytical functions $\mathfrak{g}(e')$ and it is designated as $\mathcal{E}$.

The hypersurface $\mathcal{E}$ exhibits the following properties:

a) The manifold $\mathcal{E}$ is boundaryless. This property follows from the implicit function theorem.

b) The hypersurface $\mathcal{E}$ is a closed set. This property follows from the fact that any accumulation point of $\mathcal{E}$ in the space $\{e\}$ belong to the hypersurface due to analyticity of the constraints (5.13).

Denote by $S^H$ the hypersphere

$$\sum (e_{V_{1}V_{2}}^{a})^{2} = l_{F}^{2}\mathcal{N},$$

(5.14)

and

$$\mathfrak{c} = S^H \cap \mathcal{E}, \quad \partial \mathfrak{c} = 0.$$  

(5.15)

The manifold $\mathfrak{c}$ is compact and boundaryless.

It follows from here that the points $q_{c} \in \mathfrak{c}$ exist at which the action $\mathfrak{A}(e', \mathfrak{g})$ is extremal and so

$$\left. \partial \mathfrak{A}(e', \mathfrak{g}) / \partial e' \right|_{q_{c} \in \mathfrak{c}} = 0.$$  

(5.16)

We have also

$$\partial \mathfrak{A}(e', \mathfrak{g}) / \partial \mathfrak{g} = 0.$$  

(5.17)

The set of Eqs. (5.17) follow from the fact that variations of variables $\{\mathfrak{g}\}$ lead to variations of the constraints (5.13) only. But herein the corresponding variations of the action $\mathfrak{A}$ are equal to zero due to Eqs. (4.3) and since the contribution of the constraints (5.13) into the action looks like as

$$\Delta \mathfrak{A} \sim \sum \omega_{(-)} \Phi_{(-)}.$$  

The totality of equations (5.16) and (5.17) is equivalent to the system of equations

$$\partial \mathfrak{A}(e') / \partial e'|_{S^H} = 0.$$  

(5.18)

Resolve the constraint (5.14) relative to $e^{1}_{V_{m-1}V_{m}}$:

$$e^{1}_{V_{m-1}V_{m}} = \pm f(\ldots), \quad a_{V_{m-1}}a_{V_{m}} \in \partial \mathfrak{N},$$  

$$f = \sqrt{l_{F}^{2}\mathcal{N} - \sum (e_{V_{1}V_{2}}^{a})^{2}}.$$  

(5.19)

Here the symbol $\sum'$ is the same as in (5.14) only that the variable $e^{1}_{V_{m-1}V_{m}}$ is excluded. It follows from (5.18), (5.19):

$$\frac{\partial \mathfrak{A}}{\partial e^{a}_{V_{1}V_{2}}} \pm \left( \frac{\partial \mathfrak{A}}{\partial e^{1}_{V_{m-1}V_{m}}} \cdot \frac{\partial f}{\partial e^{a}_{V_{1}V_{2}}} \right) = 0.$$  

(5.20)

There is the estimation

$$l_{F}^{2}\mathcal{N} \sim R^4.$$  

(5.21)

According to (3.22) and the Eguchi-Hanson solution (2.21)

$$\frac{\partial \mathfrak{A}}{\partial e^{1}_{V_{m-1}V_{m}}} \sim \mathcal{R}^{ab} |_{\text{not far from } \partial \mathcal{R}} \sim R^{-6} \sim \mathcal{R}^{-3/2},$$

(5.22)

and therefore

$$\frac{\partial \mathfrak{A}}{\partial e^{1}_{V_{1}V_{2}}} \sim R^{-6}.$$  

(5.23)

Since the number of 1-simplexes $a_{V_{1}V_{2}} \subset \mathfrak{r}$ is of the order of $R^4$, so the estimation

$$\sum_{(V_{1}V_{2})} e^{a}_{V_{1}V_{2}} \left( \frac{\partial \mathfrak{A}}{\partial e^{a}_{V_{1}V_{2}}} \cdot \frac{\partial f}{\partial e^{1}_{V_{m-1}V_{m}}} \right) \sim R^{-2}.$$  

(5.24)

holds. Now due to (5.24) and (4.3) we have

$$2 \mathfrak{A} = \sum_{(V_{1}V_{2})} e^{a}_{V_{1}V_{2}} \frac{\partial \mathfrak{A}}{\partial e^{a}_{V_{1}V_{2}}} =$$

$$= \sum_{(V_{1}V_{2})} e^{a}_{V_{1}V_{2}} \left( \frac{\partial \mathfrak{A}}{\partial e^{1}_{V_{m-1}V_{m}}} \cdot \frac{\partial f}{\partial e^{1}_{V_{1}V_{2}}} \right) \sim R^{-2}.$$  

(5.25)

With the help of Eqs. (5.23) and (5.25) we obtain the final result:

$$\frac{\partial \mathfrak{A}}{\partial e^{1}_{V_{1}V_{2}}} \rightarrow 0, \quad \mathfrak{A} \rightarrow 0$$

as $\mathcal{R} \rightarrow \infty$, $R \rightarrow \infty$. (5.26)

So, it is proved the existence of simultaneous solution of equations of Eqs. (4.4), (4.2), (4.3) with boundary conditions (5.3), (5.10) on infinite complex $\mathfrak{r} \approx \mathbb{R}^4$.

VI. CONCLUSION

Aforesaid means that the lattice analogue of the Eguchi-Hanson self-dual solution to continuous Euclidean Gravity does exist for the complex $\mathfrak{r} \approx \mathbb{R}^4$.

It is crucially important that the problem of possible singularities for the curvature tensor does not exist on the lattice gravity. In continuous gravity for the $\psi$ range (2.12), the manifold would have "cone-tip" singularities at $r = a$; this implies the necessity of delta-functions in
the curvature at \( r = a \) (see [4, 5]). But delta-functions transforms into Kronecker symbol which is of the order of unity in discrete mathematics. The same is true with respect to the lattice analogue of the curvature tensor \( \Omega_{\alpha \beta \gamma} \approx \pm 1 \). This is the reason why one can take the range of angles \( \phi = \frac{\pi}{2} \) in a lattice gravity. Moreover, all lattice equations are satisfied and the action for the instanton solution is equal to zero.

Thus, the setting of the problem for finding a self-dual solution to lattice Euclidean gravity is as follows: one must solve (in anti-instanton case) the difference lattice system of equations (4.1) with indices (+), (4.2), (4.4), the contraints (5.13), and the boundary conditions (5.3), (5.10).

Here some questions are not enough clear or remain unclear.

1) Is the offered solution with \( \chi(\hat{F}) = 1 \) stable or it can be contracted smoothly into the trivial one?

The answer to this question seems to be as follows. Let’s consider a smooth path in the configuration space

\[
\Omega_{\nu_1 \nu_2}(t), \quad e^{(\alpha)}_{\nu_1 \nu_2}(t), \quad 0 \leq t \leq 1,
\]

such that

\[
\Omega_{\nu_1 \nu_2}(0) = \Omega_{\nu_1 \nu_2(\text{inst})}, \quad e^{(\alpha)}_{\nu_1 \nu_2}(0) = e^{(\alpha)}_{\nu_1 \nu_2(\text{inst})}, \quad \Omega_{\nu_1 \nu_2}(1) = 1, \quad e^{(\alpha)}_{\nu_1 \nu_2}(1) = \phi^{(\alpha)}_{\nu_2} - \phi^{(\alpha)}_{\nu_1}, \quad (6.1)
\]

Here the lower index (inst) designates the discussed self-dual solution. The field configuration \( \{\Omega_{\nu_1 \nu_2}(1), \quad e^{(\alpha)}_{\nu_1 \nu_2}(1)\} \) is a trivial solution of Eqs. \( (4.1), (4.2) \). Evidently, the global continuous description of this trivial solution is possible. Nevertheless, the considered instanton contraction scenario seems to be unsatisfactory since to do this, one must contract topologically non-trivial connection elements (see (5.1) and (5.10)) into unit in the infinite space-time.

Another instanton contraction scenario which is described in continual limit by limit process \( a \to 0 \) and in the lattice case by reducing the sub-complex \( \hat{\mathcal{A}} \) up to its disappearance also seems to be unsatisfactory. Indeed, in this case we would have resulted in the failure of the Chern-Gauss-Bonnet theorem in \( \mathbb{R}^4 \). So, this scenario is also impossible, it can be considered as a decreasing of the instanton scale \( a \) up to a lattice scale.

Therefore, the considered instanton solution for the case \( \chi(\hat{F}) = 1 \) seems to be stable.

2) The case \( \chi(\hat{F}) = (2k + 1) \), \( k = 1, 2, \ldots \) and \( \partial \Omega = \mathcal{S} \approx S^3 \) is interesting, but it is not considered here. So, the question remains unanswered: for what values of \( \chi(\hat{F}) \) the lattice self-dual solution does exist and it would be stable?

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[12] Here, by an almost smooth surface, we mean a piecewise smooth surface consisting of flat four-dimensional simplices, such that the angles between adjacent 4-simplices tend to zero and the sizes of these simplices are commensurable.
[13] Hence the statement that the square brackets give null equation under the sum (4.5) is proved.
[14] It was proved to be the case for the quantity \( E^{(\alpha)}_{\nu_1 \nu_2}(\hat{\mathcal{A}}[mkl]) \) (see (4.5)). The corresponding provement for the quantity \( E^{(\alpha)}_{\nu_1 \nu_2}(\hat{\mathcal{A}}[mkl]) \) is the same.