Pull-back of currents by holomorphic maps

Tien-Cuong Dinh and Nessim Sibony

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Abstract

We define the pull-back operator, associated to a meromorphic transform, on several types of currents. We also give a simple proof to a version of a classical theorem on the extension of currents.

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1 Introduction

Let $X$ and $X'$ be two (connected) complex manifolds of dimensions $k$ and $k'$ respectively. A holomorphic map $f : X \to X'$ induces a pull-back operator $f^*$ acting on smooth forms on $X'$ with values in the space of smooth forms on $X$. Our main purpose in this paper is to extend the previous operator to some classes of currents. This is a fundamental question in complex analysis because in many problems, one has to deal with singular objects like subvarieties or more generally positive closed currents.

Our first motivation comes from the theory of complex dynamical systems in higher dimension. Given a holomorphic self-map or more generally a multivalued meromorphic self-map $f : X \to X$, a main problem in dynamics is to construct interesting measures invariant by $f$. A general strategy is to construct invariant positive closed currents and then obtain invariant measures as intersection of such currents, see [21] for historical comments. So, a necessary step here is to define the pull-back operator on positive closed currents. Theorem 1 below allows to extend the construction of Green currents in [12] to arbitrary holomorphic correspondences with finite fibers between compact Kähler manifolds.

In order to avoid some trivial counter-examples, assume throughout the paper that $f$ is dominant, i.e. its image contains a non-empty open subset of $X'$. Otherwise, one might replace $X'$ by $f(X)$ which is a variety immersed in $X'$, possibly with singularities. In [19], Mêo proved that the operator $f^*$ can be continuously...
extended to positive closed currents of bidegree \((1, 1)\). He also constructed an example where one cannot extend \(f^*\) to positive closed currents of higher bidegree. Our main theorem is the following result.

**Theorem 1.1.** Let \(f : X \to X'\) be a holomorphic map. Assume that each fiber of \(f\) is either empty or is an analytic set of dimension \(\dim X - \dim X'\). Then the pull-back operator \(f^*\) can be extended to positive closed (resp. \(dd^c\)-closed) \((p, p)\)-currents on \(X'\). Moreover, if \(T\) is such a current then \(f^*(T)\) is a positive closed (resp. \(dd^c\)-closed) \((p, p)\)-current on \(X\) which depends continuously on \(T\) for the weak topology on currents. If \(T\) has no mass on a Borel set \(A \subset X'\), then \(f^*(T)\) has no mass on \(f^{-1}(A)\).

Observe that \(f\) is open hence we can consider that \(f\) is surjective by restricting \(X'\). Note that when \(f\) is a finite map between open subsets of \(C^k\), Méo gave in [19] a definition of \(f^*\) for positive closed \((p, p)\)-currents. He used potentials of currents and didn't consider the continuity of \(f^*\) and its independence of coordinates in \(C^k\) which are crucial here in order to extend \(f^*\) to the case of manifolds.

We deduce from the previous result the following corollary where \(\{\cdot\}\) denotes the class of a positive \(dd^c\)-closed \((p, p)\)-current in the cohomology group \(H^{p, p}(\cdot, \mathbb{C})\):

**Corollary 1.2.** Let \(f : X \to X'\) be a holomorphic map between compact Kähler manifolds. Assume that each fiber of \(f\) is an analytic set of dimension \(\dim X - \dim X'\). If \(T\) is a positive closed or \(dd^c\)-closed \((p, p)\)-current on \(X'\), then \(f^*\{T\} = \{f^*(T)\}\).

The linear operator \(f^* : H^{p, p}(X', \mathbb{C}) \to H^{p, p}(X, \mathbb{C})\) is induced by the action of \(f^*\) on smooth forms. Corollary 1.2 follows from the continuity in Theorem 1.1 and from our result in [14] which says that one can write \(T = T^+ - T^-\) where \(T^\pm\) can be approximated by smooth positive \(dd^c\)-closed forms.

Theorem 1.1 is also valid for another class of currents, useful in dynamics, that we call \textit{dsh currents}, see Theorem 3.4. It is still valid for some meromorphic maps or more generally for some meromorphic transforms, see Theorems 4.5 and 4.6. The case of compact Kähler manifolds will be discussed in Section 5.

The tools in order to prove the main results give also a simple proof of the following theorem on the extension of currents.

**Theorem 1.3.** Let \(F\) be a closed subset of a complex manifold \(X\). Let \(T\) be a positive \((p, p)\)-current on \(X \setminus F\). Assume that \(F\) is locally complete pluripolar and \(T\) has locally finite mass near \(F\). Assume also that there exists a positive \((p + 1, p + 1)\)-current \(S\) with locally finite mass near \(F\) such that \(dd^c T \leq S\) on \(X \setminus F\). Then \(dd^c T\) has locally finite mass near \(F\). Moreover, if \(\widetilde{T}\) and \(\widetilde{dd^c T}\) denote the extensions by zero of \(T\) and \(dd^c T\) on \(X\), then \(\widetilde{dd^c T} - \widetilde{dd^c T}\) is positive. If \(T\) is closed then \(\widetilde{T}\) is closed.
This result extends a classical Skoda’s theorem when $T$ is closed, see [22], [15] and [20]. For $S = 0$ it is proved by Dabbek-Elkhadhara-El Mir in [5], see also Remark 2.3, and is due to Alessandrini-Bassanelli [11] when $F$ is an analytic set and $dd^c T$ has bounded mass. Under the extra assumption that $dT$ is of order zero, the result was proved by the second author in [20]. In this case, we also have the formula $dT = dT$.

2 Extension of currents

In this section we give a simple proof of Theorem 1.3. We start with the following lemmas which are versions of the Chern-Levine-Nirenberg inequality [4, 6]. In what follows, $\omega$ denotes a hermitian $(1,1)$-form on a manifold $X$ of dimension $k$. If $T$ is a current of order zero, the mass of $T$ on a Borel set $K \subset X$ is denoted by $\|T\|_K$. The mass of $T$ on $X$ is denoted by $\|T\|$. If $T$ is a positive or a negative $(p,p)$-current, $\|T\|_K$ is equivalent to $|\int_K T \wedge \omega^{k-p}|$. We often identify these two quantities.

**Lemma 2.1.** Let $U$ be an open subset of $X$. Let $K$ and $L$ be compact sets in $U$ with $L \subset K$. Assume that $T$ is positive and $dd^c T$ has order zero. Then there exists a constant $c_{K,L} > 0$ such that for every smooth bounded psh function $u$ on $U$ we have the following estimate

$$\int_L du \wedge d^c u \wedge T \wedge \omega^{k-p-1} \leq c_{K,L} \|u\|_{L^\infty(K)}^2 (\|T\|_K + \|dd^c T\|_K).$$

**Proof.** We can assume that $K$ is the unit ball in $\mathbb{C}^k$ and $L$ is the ball of center 0 and of radius $1 - 3\delta$, $0 < \delta < 1/4$, and $\omega$ is the canonical Kähler form on $\mathbb{C}^k$. Replacing $T$ by $T \wedge \omega^{k-p-1}$ and $u$ by $\frac{1}{4}\|u\|_{L^\infty(K)}^2 u + \frac{1}{4}$, we can assume that $p = k - 1$ and $0 \leq u \leq 1/2$. Let $\chi$ be a real-valued smooth function, $0 \leq \chi \leq 1$, supported on $K$ and such that $\chi = 1$ on $\{|z| < 1 - \delta\}$. Define

$$v(z) := \max_x (u(z), \delta^{-1}(|z|^2 - (1 - 2\delta)^2))$$

where $\max_x (x, y)$ is a smooth function on $\mathbb{R}^2$, convex increasing on each variable, $\max_x (x, y) \geq \max (x, y)$, and $\max_x (x, y)$ is equal to $\max (x, y)$ outside a small neighbourhood of $\{x = y\}$. Then $v$ is a positive smooth psh function equal to $\delta^{-1}(|z|^2 - (1 - 2\delta)^2)$ on $\{1 - \delta \leq |z| \leq 1\}$. In particular, we have $v(z) = \delta^{-1}(|z|^2 - (1 - 2\delta)^2)$ on the support supp$(d\chi)$ of $d\chi$ and $v = u$ on $L$. Then, since $v^2$ is psh,

$$\int_L du \wedge d^c u \wedge T = \int_L \left(\frac{1}{2} dd^c v^2 - v dd^c v\right) \wedge T$$

$$\leq \frac{1}{2} \int dd^c v^2 \wedge \chi T = \frac{1}{2} \int v^2 dd^c (\chi T)$$

$$= \frac{1}{2} \int v^2 (\chi dd^c T + T \wedge dd^c \chi) + \frac{1}{2} \int v^2 (-d^c \chi \wedge dT - d^c T \wedge d\chi).$$
Define $\tilde{v}(z) := \delta^{-1}(\|z\|^2 - (1 - 2 \delta)^2)$ on $\mathbb{C}^k$. Recall that $v = \tilde{v}$ on $\text{supp}(d\chi)$. Using an analogous computation as above, we obtain that the last integral is equal to

$$\int dd^c \tilde{v}^2 \wedge \chi T - \int \chi \tilde{v}^2 dd^c T - \int \tilde{v}^2 dd^c \chi \wedge T.$$ 

Hence

$$\int_L du \wedge d^c u \wedge T \leq \frac{1}{2} \int (v^2 - \tilde{v}^2)(\chi dd^c T + T \wedge dd^c \chi) + \frac{1}{2} \int dd^c \tilde{v}^2 \wedge \chi T.$$ 

The lemma follows.

**Lemma 2.2.** Under the assumptions of Lemma 2.1, we have

$$\|dd^c u \wedge T\|_L \leq c_{K,L} \|u\|_{L^\infty(K)}(\|T\|_K + \|dd^c T\|_K)$$

where $c_{K,L} > 0$ is a constant independent of $u$ and $T$.

**Proof.** Write

$$dd^c u \wedge T := udd^c T - dd^c (uT) + d(d^c u \wedge T) - d^c (du \wedge T).$$

Hence, Cauchy-Schwarz inequality and Lemma 2.1 applied to $L' := \text{supp}(\chi)$, imply

$$\|dd^c u \wedge T\|_L \leq \int \chi dd^c u \wedge T = \int \chi udd^c T - \int dd^c \chi \wedge uT - \int d\chi \wedge d^c u \wedge T + \int d^c \chi \wedge du \wedge T \lesssim \|u\|_{L^\infty(K)}(\|T\|_K + \|dd^c T\|_K).$$

End of the proof of Theorem 1.3— Recall that a complete locally pluripolar set is locally the pole set $\{\varphi = -\infty\}$ of a psh function $\varphi$. Since the problem is local, we can assume that $X$ is a ball in $\mathbb{C}^k$. Then, there is a sequence of smooth psh functions $u_n$, $0 \leq u_n \leq 1$, vanishing near $F$ and increasing to 1 on $X \setminus F$, see e.g. [21]. We have $u_n T \to \tilde{T}$ and as in (2.1)

$$u_n dd^c T - dd^c (u_n T) = dd^c u_n \wedge T - d(d^c u_n \wedge T) + d^c (du_n \wedge T).$$

We first show that $\partial u_n \wedge T \to 0$. This implies that $\bar{\partial} u_n \wedge T \to 0$ by conjugaison. We can assume that $p = k - 1$. Indeed, one only has to test forms of type $\varphi \Omega$ with $\varphi$ a smooth function and $\Omega$ a smooth positive closed form. Then we reduce the problem to the case $p = k - 1$ by replacing $T$ by $T \wedge \Omega$. 

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Let $\alpha$ be a test smooth $(0,1)$-form with support in a compact subset $K$ of $X$. We have to show that $\int_{\partial u_n} \alpha \wedge T$ converge to zero. Fix an $\varepsilon > 0$ and a neighbourhood $U_\varepsilon$ of $F$ such that

$$\int_{U_\varepsilon} i\alpha \wedge \bar{\alpha} \wedge \tilde{T} \leq \varepsilon^2.$$ 

Since $u_n$ converge locally uniformly to 1 on $X \setminus F$, Lemma 2.1 applied to $(u_n - 1)$, implies that locally on $X \setminus F$ the mass of $\partial u_n \wedge T$ tends to zero. Hence,

$$\lim_{n \to \infty} \int_{X \setminus U_\varepsilon} \partial u_n \wedge \alpha \wedge T = 0.$$ 

On the other hand, Cauchy-Schwarz inequality gives

$$\left| \int_{U_\varepsilon} i\partial u_n \wedge \alpha \wedge T \right| \leq \left( \int_{K} i\partial u_n \wedge \bar{\partial} u_n \wedge T \right)^{1/2} \left( \int_{U_\varepsilon} i\alpha \wedge \bar{\alpha} \wedge T \right)^{1/2}.$$ 

The second factor is bounded by $\varepsilon$. So it is enough to show that the integral on $K$ is uniformly bounded with respect to $n$. We cannot apply directly Lemma 2.1 since we don’t know that $dd^c T$ has finite mass. Let $\chi$ be a cutoff function, $0 \leq \chi \leq 1$ and $\chi = 1$ on $K$. We have, using a computation as in (2.1),

$$I_n := \int \chi^2 i\partial u_n \wedge \bar{\partial} u_n \wedge T$$

$$\leq \frac{1}{2} \int \chi^2 i\partial \bar{\partial} u_n^2 \wedge T = \frac{1}{2} \int u_n^2 i\partial \bar{\partial} (\chi^2 T)$$

$$\leq \frac{1}{2} \int u_n^2 [\chi^2 i\partial \bar{\partial} T + i\partial \bar{\partial} \chi^2 \wedge T] + 2 \left| \int u_n \bar{\partial} u_n \wedge \partial \chi^2 \wedge T \right|$$

The first term in the last line is bounded uniformly since $0 \leq u_n \leq 1$ and $dd^c T \leq S$. The identity $\partial \chi^2 = 2\chi \partial \chi$ and the Cauchy-Schwarz inequality imply that the last integral is bounded by

$$2 \left( \int \chi^2 i\partial u_n \wedge \bar{\partial} u_n \wedge T \right)^{1/2} \left( \int i\partial \chi \wedge \bar{\partial} \chi \wedge u_n^2 T \right)^{1/2}.$$ 

Since the last integral is uniformly bounded we have $I_n \leq \text{const}(1 + I_n^{1/2})$ and hence $I_n$ is bounded.

We have proved that $\partial u_n \wedge T \to 0$ and $\bar{\partial} u_n \wedge T \to 0$. Identity (2.2) implies that

$$u_n (dd^c T - S) - dd^c u_n \wedge T = -u_n S + dd^c (u_n T) - d(d^c u_n \wedge T) + d^c (du_n \wedge T).$$ 

The right hand side converges to $-\tilde{S} + dd^c \tilde{T}$ where $\tilde{S}$ is the trivial extension by zero of $S$ on $X$. Since both terms on the left hand side are negative currents their limit
values are negative. We then deduce that $-\tilde{S} + dd^{c}\tilde{T}$ is negative and hence $dd^{c}T$ has finite mass near $F$. Finally the left hand side of (2.2) converges to $dd^{c}T - dd^{c}\tilde{T}$ and hence the right hand side converge to $\lim dd^{c}u_{n} \wedge T$ which is positive. Hence $dd^{c}T - dd^{c}\tilde{T}$ is positive. When $T$ is closed, we have $d\tilde{T} = \lim du_{n} \wedge T = 0$, hence $\tilde{T}$ is closed. □

Remark 2.3. If $S$ is closed then $\tilde{S}$ is closed. It follows from Theorem 1.3 that $-dd^{c}\tilde{T} + S$ is positive and closed. Hence $\tilde{T}$ is dsh, see Section 3 for the definition. If $F$ is an analytic subset of codimension $\geq p + 2$ and if $T$ is $dd^{c}$-closed then $\tilde{T}$ is $dd^{c}$-closed. Indeed, in this case, $-dd^{c}\tilde{T}$ is a positive closed $(p + 1, p + 1)$-current; it should vanish on sets of codimension $\geq p + 2$.

3 Pull-back operator

We give here the proof of Theorem 1.4. We first consider a general local situation. Let $B_{k}$ denote the unit ball in $\mathbb{C}^{k}$ and $B_{k}(r)$ denote the ball of center 0 and of radius $r$ in $\mathbb{C}^{k}$. Let $\pi : \mathbb{C}^{k} \times \mathbb{C}^{k'} \to \mathbb{C}^{k'}$ be the canonical projection. Consider a subvariety $V$ of pure dimension $k' + l$ in $B_{k} \times B_{k'}$. We will denote by $[V]$ the current of integration on $V$. Assume that the fibers of $\pi|_{V}$ are either empty or of pure dimension $l$. Let $w = (z, z')$ denote the coordinates in $\mathbb{C}^{k} \times \mathbb{C}^{k'}$. Let $\mathcal{C}$ denote the critical set of $\pi|_{V}$ which contains the singularities of $V$. The set $\mathcal{C}$ is defined by the property that $\pi|_{V}$ is locally a submersion at every point $w \in V \setminus \mathcal{C}$.

Lemma 3.1. Let $L$ be a compact subset of $B_{k} \times B_{k'}$. Then, there exists $c_{L} > 0$ such that if $T$ is a positive smooth $(p, p)$-form on $B_{k'}$ then

$$\int_{V \cap L} \pi^{*}(T) \wedge (dd^{c}\|w\|^{2})^{k' + l - p} \leq c_{L}(\|T\| + \|dd^{c}T\|).$$

Proof. Since the problem is local we can assume that $0 \in V$ and prove the lemma for a small ball $L$ of center 0. Let $P$ be a complex plane of codimension $l$ in $\mathbb{C}^{k} \times \mathbb{C}^{k'}$ such that $\pi$ restricted to $V \cap P$ is discrete. Shrinking $B_{k}$ and $B_{k'}$ allows to assume that, for every small perturbation $P_{\epsilon}$ of $P$, $\pi$ restricted to $V \cap P_{\epsilon}$ defines a finite ramified covering of degree $d_{\epsilon}$ over $B_{k'}$. Slicing by $P_{\epsilon}$ reduces the problem to the case where $l = 0$ and $\pi|_{V}$ defines a ramified covering of the same degree $d_{\epsilon}$.

Observe that $u := (\pi|_{V})_{*}(\|w\|^{2})$ is a continuous psh function bounded by $2d_{\epsilon}$. Let $K$ be a compact subset of $B_{k'}$ such that $L' := \pi(L) \subseteq K$. Then

$$\int_{V \cap L} \pi^{*}(T) \wedge (dd^{c}\|w\|^{2})^{k' - p} \leq \int_{L'} T \wedge \pi_{*}[(dd^{c}\|w\|^{2})^{k' - p}] \leq \int_{L'} T \wedge (dd^{c}u)^{k' - p}.$$ 

Using Lemma 2.2 and the fact that $T$ is smooth, a simple induction on $r$ gives

$$\int_{L'} T \wedge (dd^{c}u)^{r} \wedge (dd^{c}\|w\|)^{k' - p - r} \leq c(\|T\| + \|dd^{c}T\|),$$
where the constant $c > 0$ depends only on $L$ and $d_t$ (this is important for the slicing). When $r = k' - p$, we obtain the lemma.

We define now the space of dsh currents.

**Definition 3.2.** A $(p,p)$-current $T$ on a complex manifold $X$ of dimension $k$, $0 \leq p \leq k - 1$, is dsh if there exist negative $(p,p)$-currents $T_i$ and positive closed $(p + 1, p + 1)$-currents $\Omega_i^\pm$ such that

$$T = T_1 - T_2 \quad \text{and} \quad dd^c T_i = \Omega_i^+ - \Omega_i^- \text{ for } i = 1, 2.$$  \hspace{1cm} (3.1)

Let $DSH^p(X)$ denote the space of dsh $(p,p)$-currents on $X$. We say that the dsh $(p,p)$-currents $T_{(n)}$ converge in $DSH^p(X)$ to $T$ if $T_{(n)} \to T$ in the sense of currents and if we can write as in (3.1)

$$T_{(n)} = T_{(1, n)} - T_{(2, n)} \quad \text{and} \quad dd^c T_{i,n} = \Omega_{i,n}^+ - \Omega_{i,n}^- \text{ for } i = 1, 2$$

so that the masses of $T_{i,n}$ and $\Omega_{i,n}^\pm$ are locally uniformly bounded.

Dsh currents have been introduced in [9, 12, 13]. They are stable under push-forward by holomorphic proper maps and under pull-back by holomorphic submersions. A good example of dsh currents to have in mind is the product $(\varphi_1 - \varphi_2)T$ where $\varphi_1, \varphi_2$ are bounded quasi-psh functions and where $T$ is a positive closed current. Recall that $\varphi$ is quasi-p.s.h. if locally it is a difference of a p.s.h. function with a smooth one.

**Lemma 3.3.** Let $(T_n)$ be a sequence of positive smooth $(p,p)$-forms on $B_{k'}$ which converge in $DSH^p(B_{k'})$ to a current $T$. Then $\pi^*(T_n)\wedge [V]$ converge in $DSH^{k'-l+p}(B_k \times B_{k'})$ to a current $S$. If $T$ has no mass on a Borel set $A$ then $S$ has no mass on $\pi^{-1}(A)$.

**Proof.** Since the problem is local it is sufficient to consider the case where 0 belongs to $V$ and where $L$ is a small ball of center 0 as in Lemma 3.1. Let $\Pi : \mathbb{C}^k \times \mathbb{C}^{k'} \to \mathbb{C}'$ a generic linear projection. Observe that test $(k' + l - p, k' + l - p)$-forms with support in $L$ are generated by forms of type $\varphi \wedge \Pi^*(\psi)$ where $\varphi$ is a form of small support in $B_k \times B_{k'}$ and $\psi$ is a form of maximal degree on $\mathbb{C}'$. It follows that slicing by fibers of $\Pi$ reduces the problem to the case where $l = 0$ and $\pi_{|V}$ defines a finite ramified covering of degree $d_t$ over $B_{k'}$. We use here the Lebesgue convergence theorem in order to show that if the slices converge pointwise for a fixed $\Pi$ and the mass is dominated then we have the convergence. The problem is to show that there is a unique cluster point for the sequence $\pi^*(T_n) \wedge [V]$. Let $\Sigma \subset V$ be the smallest analytic subset such that $\pi^*(T_n) \wedge [V]$ converge to a positive current $\tau_0$ on $B_k \times B_{k'} \setminus \Sigma$. Lemma 3.1 applied to $T_n$ and $dd^c T_n$ implies that $\tau_0$ and $dd^c \tau_0$ have finite masses near $\Sigma$. Consider the extensions by zero of $\tau_0$ through $\Sigma$ that we denote also by $\tau_0$. By Theorem 1.3 $dd^c \tau_0$ has order zero.
We want to prove that $\Sigma = \emptyset$. Assume that $\Sigma \neq \emptyset$. Replacing $B_k \times B_{k'}$ by a small neighbourhood of some generic (in the Zariski sense) point $a \in \Sigma$ allows to assume that $\Sigma$ is smooth and $\pi_{|\Sigma}$ is injective.

Now, consider a limit value $\tau$ of $\pi^*(T_n) \land [V]$ in $B_k \times B_{k'}$. Then $\tau - \tau_0$ is a positive current with support in $\Sigma$. Lemma 3.1 applied to $d\delta^c T_n$ implies that $d\delta^c (\tau - \tau_0)$ has order zero. The support theorem in [1] extends Federer’s theorem on flat currents [16] to currents of order zero with $d\delta^c$ of order zero. It implies that $\tau - \tau_0$ is a current of $\Sigma$. Let $\varphi$ be a test $(k' - p, k' - p)$-form with compact support in $B_{k'}$. We have

$$\lim_{n \to \infty} \langle \pi^*(T_n) \land [V], \pi^*(\varphi) \rangle = \lim_{n \to \infty} \langle T_n, \pi_* (\pi^*(\varphi) \land [V]) \rangle = \lim_{n \to \infty} \langle T_n, d_\tau \varphi \rangle = d_\tau \langle T, \varphi \rangle. \quad (3.2)$$

Hence, $\langle \tau - \tau_0, \pi^*(\varphi) \rangle$ is independent of the choice of $\tau$. Since $\tau - \tau_0$ is a current on $\Sigma$ and since $\pi_{|\Sigma}$ is injective, the last identity implies that $\tau - \tau_0$ is independent of the choice of $\tau$. In other words, $\pi^*(T_n) \land [V]$ converge. This contradicts the property that $\Sigma \neq \emptyset$.

Now we prove that $S$ has no mass on $\pi^{-1}(A)$. By slicing, one can assume that $\pi_{|V}$ is finite. Let $\Sigma \subset V$ be an analytic set such that $S$ has no mass on $\pi^{-1}(A) \setminus \Sigma$. One can choose $\Sigma$ minimal in the sense that no proper analytic set in $\Sigma$ satisfies the same property. As above, if $\Sigma$ is not empty, we can assume that it is a smooth submanifold of $B_k \times B_{k'}$ and $\pi_{|\Sigma}$ is injective. Let $\tau_0$ be the restriction of $S$ to $B_k \times B_{k'} \setminus \Sigma$ and let $\tau_0$ its extension by zero. Then Remark 2.3 implies that $\tau_0$ is dsh and hence $S_{|\Sigma} := S - \tau_0$ is a positive dsh current with support in $\Sigma$. By the support theorem [1], this is a current on $\Sigma$. One deduces from identity (3.2) that $\langle S_{|\Sigma}, \pi^*(\varphi|A) \rangle = d_\tau \langle T, \varphi|_{A \cap \pi(\Sigma)} \rangle$. Hence, since $T$ has no mass on $A$, the previous integrals vanish. The property that $\pi_{|\Sigma}$ is injective implies that $S_{|\Sigma}$ has no mass on $\pi^{-1}(A)$ which contradicts the definition of $\Sigma$. 

End of the proof of Theorem 1.1 — Let $\pi_1 : X \times X' \to X$ and $\pi_2 : X \times X' \to X'$ denote the canonical projections. Let $\Gamma$ denote the graph of $f$ in $X \times X'$. For $T$ smooth, we have $f^*(T) = (\pi_1)_* (\pi_2^*(T) \land [\Gamma])$. We have to show for the general case that $\pi_2^*(T) \land [\Gamma]$ is well defined and then define $f^*(T) := (\pi_1)_* (\pi_2^*(T) \land [\Gamma])$. Note that since $\pi_1$ is proper on $\Gamma$, the operator $(\pi_1)_*$ is well defined on currents supported on $\Gamma$.

On a small open subset $U$ of $X'$, we approximate $T$ by smooth positive closed (resp. $d\delta^c$-closed) forms $T_n$ and define

$$\pi_2^*(T) \land [\Gamma] := \lim_{n \to \infty} \pi_2^*(T_n) \land [\Gamma] \quad \text{in } \pi_2^{-1}(U).$$

Lemma 3.3 implies that the limit exists and does not depend on the choice of $T_n$. This implies also that if $U$ and $U'$ are small open subsets of $X'$, our construction gives two currents which coincide in $\pi_2^{-1}(U \cap U')$. Hence, $\pi_2^*(T) \land [\Gamma]$ and $f^*(T)$...
are globally well defined. Since $\pi_2^*(T_n) \wedge [\Gamma]$ are positive and closed (resp. $dd^c$-closed), $\pi_2^*(T) \wedge [\Gamma]$ and $f^*(T)$ are positive and closed (resp. $dd^c$-closed). The continuity of $T \mapsto \pi_2^*(T) \wedge [\Gamma]$, and then the continuity of $T \mapsto f^*(T)$, follow from Lemma 3.3. If $T$ has no mass on a set $A \subset X'$, Lemma 3.3 shows also that $\pi_2^*(T) \wedge [\Gamma]$ has no mass on $\pi_2^{-1}(A) \cap \Gamma$. Hence $f^*(T)$ has no mass on $f^{-1}(A)$.

Using Lemma 3.3, we prove in the same way the following result.

**Theorem 3.4.** Let $f$ be as in Theorem 1.1. Then the pull-back operator

$$f^* : \text{DSH}^p(X') \to \text{DSH}^p(X)$$

is well defined, continuous and commutes with $dd^c$. Moreover, if a dsh current $T$ on $X'$ has no mass on a Borel set $A \subset X'$ then $f^*(T)$ has no mass on $f^{-1}(A)$.

**Remark 3.5.** If $g : X' \to X''$ is another holomorphic map with fibers of pure dimension $\dim X' - \dim X''$ then the continuity of the pull-back operator implies that

$$(g \circ f)^*(T) = f^*(g^*(T))$$

for the classes of current under consideration. Indeed, this identity holds for smooth forms.

### 4 Meromorphic transforms

Meromorphic transforms (MT for short) were considered in [14] in order to treat with the same method different problems in dynamics and in the study of distribution of varieties (see also [7, 8]). In this Section we recall the definition of MT and introduce the pull-back operator on currents associated to a MT.

**Definition 4.1.** 1 A *meromorphic transform* $F$ of codimension $l$, $0 \leq l \leq k - 1$, from $X$ onto $X'$ is a finite holomorphic chain $\Gamma = \sum \Gamma_i$ such that

- $\Gamma_i$ is an irreducible analytic subset of dimension $k' + l$ of $X \times X'$.
- $\pi_1$ restricted to each $\Gamma_i$ is proper.
- $\pi_2$ restricted to each $\Gamma_i$ is dominant.

The second item is always verified when $X'$ is compact. We do not assume that the $\Gamma_i$ are smooth or distinct. Of course we can write $\Gamma = \sum n_j \Gamma'_j$ where $n_j$ are positive integers and $\Gamma'_j$ are distinct irreducible analytic sets. Then a generic point in the support $\bigcup \Gamma'_j$ of $\Gamma$ belongs to a unique $\Gamma'_j$ and $n_j$ is called the *multiplicity* of $\Gamma$ at $x$. In what follows we always write $\Gamma$ as $\sum \Gamma_i$. The indices $i$ allow to count the multiplicities.

---

1This definition differs slightly from the definitions given in the previous references.
Define formally $F := π_2 \circ (π_{1|Γ})^{-1}$ and for $A ⊂ X$ and $B ⊂ X'$

$$F(A) := π_2(π_1^{-1}(A) ∩ Γ) \text{ and } F^{-1}(B) = π_1(π_2^{-1}(B) ∩ Γ).$$

So a generic fiber $F^{-1}(x')$ is either empty or an analytic subset of pure dimension $l$ of $X$. The sets

$$I_1 := \{ x ∈ X, \dim π_1^{-1}(x) ∩ Γ > k' + l - k \}$$

and

$$I_2 := \{ x' ∈ X', \dim π_2^{-1}(x') ∩ Γ > l \}$$

are the first and second indeterminacy sets of $F$, they are of codimension $≥ 2$.

When $I_2 = ∅$ we say that $F$ is pure. When $π_1$ restricted to each $Γ$ is surjective, we say that $F$ is complete. A complete MT $F$ of codimension 0 between manifolds of same dimension is called a meromorphic correspondence. If moreover $π_1$ restricted to $Γ$ is a finite map, then $F$ is called holomorphic correspondence. If generic fibers of $π_{1|Γ}$ contain only one point we obtain a dominant meromorphic map from $X$ onto $X'$.

The following proposition extends known results, see e.g. [19].

**Proposition 4.2.** Let $φ$ be a quasi-psh function on $X'$ and $Φ$ be a $(p,q)$-form, $k' + l - k ≤ p,q ≤ k'$, whose coefficients are dominated by $|φ|$. Let $F$ be a MT as above. Then the $(k-k'-l+p,k-k'-l+q)$-current

$$F^*(Φ) := (π_1)_*(π_2^*(Φ) ∧ [Γ])$$

is well defined and has locally finite mass. Let $Φ_n$ be a sequence of $(p,q)$-forms whose coefficients are dominated by $|φ|$. If $Φ_n → Φ$ in the sense of currents, then $F^*(Φ_n) → F^*(Φ)$. If $F$ is complete then $F^*(Φ)$ has $L^1_{loc}$ coefficients.

**Proof.** Observe that $(π_1)_*$ acts continuously on currents with support in $Γ$ since $π_1$ restricted to $Γ$ is proper. So, we only have to define $π_2^*(Φ) ∧ [Γ]$.

The function $φ \circ π_2$ is quasi-psh on $X × X'$. Its restriction to any component of $Γ$ is not identically $-∞$. Hence, $φ \circ π_2$ is a quasi-psh function on $Γ$. In particular, this function is $[Γ]$-integrable. For quasi-psh functions on singular varieties, see [17]. We can also use a desingularization $τ : ˆΓ → Γ$ and replace $π_i$ by $π_i \circ τ$ in order to reduce the integrability of $φ \circ π_2$ to the smooth case.

We deduce that $π_2^*(Φ)$ restricted to $Γ$ is a form with coefficients bounded by a quasi-psh function. It follows that $π_2^*(Φ) ∧ [Γ]$ is well defined and has locally finite mass. Hence, $F^*(T)$ is well defined and has locally finite mass. Moreover, we have $π_2^*(Φ_n) ∧ [Γ] → π_2^*(Φ) ∧ [Γ]$ which implies the convergence in the proposition.

When $F$ is complete, $F^*(Φ)$ has no mass on sets of Lebesgue measure zero. Hence, it has coefficients in $L^1_{loc}$. Note that when $F$ is not complete we can have “vertical” components of $Γ$. 

\[\square\]
Corollary 4.3. Let \( F, p \) and \( q \) be as above. Assume that \( X \) and \( X' \) are compact Kähler manifolds. Then the operator

\[
F^* : H^{p,q}(X' \cup \mathbb{C}) \to H^{k-k'-l+p,k-k'-l+q}(X \cup \mathbb{C})
\]

is well defined and is linear.

Proof. When \( \Phi \) is a smooth closed \((p,q)\)-form then \( F^*(\Phi) \) is well defined and is a closed current of bidegree \((k-k'-l+p,k-k'-l+q)\) since \( F^* \) commutes with the operators \( \partial \) and \( \overline{\partial} \). It follows that \( F^* \) is well defined on the cohomology groups.

Assume that \( k - k' - l + 1 \geq 0 \). This condition is necessary so that the pull-back operator on \((1,1)\)-currents is meaningful. The definition of this operator on positive closed \((1,1)\)-currents under a holomorphic map is classical [19] and it can be extended easily to the case of a MT.

Proposition 4.4. Let \( T \) be a positive closed current of bidegree \((1,1)\) on \( X' \). Then \( F^*(T) \) is well defined and is positive closed current of bidegree \((k-k'-l+1,k-k'-l+1)\) which depends continuously on \( T \). If \( X \) and \( X' \) are compact Kähler manifolds, we have \( \{F^*(T)\} = F^*\{T\} \).

Proof. We can write locally \( T = dd^c u \) with \( u \) psh and define

\[
\pi_2^*(T) \wedge [\Gamma] := dd^c((u \circ \pi_2)[\Gamma]) \quad \text{and} \quad F^*(T) := (\pi_1)_*(\pi_2^*(T) \wedge [\Gamma]).
\]

Since \( u \circ \pi_2 \) is psh, the current \((u \circ \pi_2)[\Gamma]\) is well defined. Hence, \( F^*(T) \) is well defined. Since \( dd^c \) and \((\pi_1)_* \) commutes, the definition is independent of the choice of \( u \). The current \( \pi_2^*(T) \wedge [\Gamma] \) is positive and closed. Hence, so is \( F^*(T) \). If \( T_n \to T \), we can write locally \( T_n = dd^c u_n \) with and \( u_n \to u \) in \( \mathcal{L}^1 \). So the continuity is clear. The assertion on \( \{F^*(T)\} \) follows from this continuity and a regularization of \( T \), see [6, 11].

Using Lemma 3.3 we easily extends Theorems 1.1 and 3.4 and Corollary 1.2 to the case of a pure MT. In particular, the following results hold for dominant meromorphic maps with finite fibers.

Theorem 4.5. Let \( F : X \to X' \) be a pure MT of codimension \( l \) between complex manifolds \( X \), \( X' \) of dimensions \( k \) and \( k' \) respectively. Let \( T \) be a positive closed (resp. \( dd^c \)-closed) \((p,p)\)-current on \( X' \). Then \( F^*(T) \) is a positive closed (resp. \( dd^c \)-closed) current of bidegree \((k-k'+p-l,k-k'+p-l)\) which depends continuously on \( T \). Moreover, if \( T \) has no mass on \( F(A) \), with \( A \subset X' \) a Borel set, then \( F^*(T) \) has no mass on \( A \). If \( X \) and \( X' \) are compact Kähler manifolds, we have \( \{F^*(T)\} = F^*\{T\} \).
Theorem 4.6. Let $F$ and $A$ be as in Theorem 4.5. Then the pull-back operator

$$F^* : \text{DSH}^p(X') \to \text{DSH}^{k-k'+p-l}(X)$$

is well defined, continuous and commutes with $dd^c$. Moreover, if a dsh current $T$ on $X'$ has no mass on $F(A)$, then $F^*(T)$ has no mass on $A$.

5 The case of compact Kähler manifolds

In general it is not possible to define the pull-back of a current under a holomorphic map, in a consistent way. A simple example in [19] shows that the mass of the pull-back could be infinite around exceptional fibers. However the situation in the compact case is different.

From now on $(X, \omega)$ and $(X', \omega')$ denote compact Kähler manifolds of dimension $k$ and $k'$ respectively. Consider a MT $F : X \to X'$ of codimension $l$ of graph $\Gamma$. Let $\mathcal{C}$ be the analytic subset of $\Gamma$ defined by the property that $\pi_2$ restricted to $\Gamma \setminus \mathcal{C}$ has locally only empty fibers or fibers of dimension pure $l$. Let $\pi$ denote the restriction of $\pi_2$ to $\Gamma \setminus \mathcal{C}$. If $T$ is a positive closed (resp. $dd^c$-closed) current of bidegree $(p, p)$, $k'+l-k \leq p \leq k'$, on $X'$, then by Theorem 4.5, the current $\pi^*(T)$ is well defined on $\Gamma \setminus \mathcal{C}$. We will show that $\pi^*(T)$ has finite mass; this allows us to extend $\pi^*(T)$ through $\mathcal{C}$. Denote by $(\pi_2|_{\Gamma})^*(T)$ the trivial extension of $\pi^*(T)$. By Theorem 4.5, $dd^c(\pi_2|_{\Gamma})^*(T) \leq 0$. In our situation, since $X'$ is compact Stokes formula implies that for any closed form $\Omega$ of right bidegree

$$\langle dd^c(\pi_2|_{\Gamma})^*(T), \Omega \rangle = \langle (\pi_2|_{\Gamma})^*(T), dd^c\Omega \rangle = 0.$$

In particular, the mass of $dd^c(\pi_2|_{\Gamma})^*(T)$ is zero, then $(\pi_2|_{\Gamma})^*(T)$ is $dd^c$-closed. We call the strict transform of $T$ by $F$ the current

$$F^*(T) := (\pi_1)_*(\pi_2|_{\Gamma})^*(T).$$

Proposition 5.1 (see also [10, 11]). Let $F$ and $T$ be as above. Then $F^*(T)$ is positive closed (resp. $dd^c$-closed). Moreover, there exists a constant $c > 0$ independent of $T$ such that $\|F^*(T)\| \leq c \|T\|$. The operator $T \mapsto F^*(T)$ is lower semi-continuous in the sense that if $T_n \to T$ then every cluster value $\tau$ of $F^*(T_n)$ satisfies $\tau \geq F^*(T)$. If $F$ is complete and if $T$ has no mass on analytic sets then $F^*(T)$ has no mass on analytic sets.

Proof. We first prove that $(\pi_2|_{\Gamma})^*(T)$ has finite mass. By [11], there are positive closed (resp. $dd^c$-closed) smooth forms $T_n^\pm$ with cohomology classes bounded by $c'\|T\|$, $c' > 0$, such that $T_n^\pm \to T^\pm$ and $T^+ - T^- = T$. We have

$$\| (\pi_2|_{\Gamma})^*(T) \| \leq \| (\pi_2|_{\Gamma})^*(T^+) \| = \lim_{n \to \infty} \| (\pi_2|_{\Gamma})^*(T_n^+) \| \leq \lim_{n \to \infty} \| (\pi_2)^*(T_n^+) \wedge [\Gamma] \|.$$
Since the cohomological classes \( \{T_n^+\} \) of \( T_n^+ \) are uniformly bounded with respect to \( n \), \( \|(\pi_2)^*(T_n^+) \wedge [\Gamma]\| \), which can be computed cohomologically, are uniformly bounded. More precisely, we have \( \|(\pi_2)^*(T_n^+) \wedge [\Gamma]\| \leq c'\|T\|, \) \( c' > 0 \), for \( n \) large enough. Hence \( F^*(T) \) is well defined and \( \|F^*(T)\| \leq c\|T\| \) with \( c > 0 \) independent of \( T \).

By definition, \( (\pi_2|\Gamma)^*(T) = \lim(\pi_2|\Gamma)^*(T_n) \) on \( \Gamma \setminus \mathcal{C} \). Since \( (\pi_2|\Gamma)^*(T) \) has no mass on \( \mathcal{C} \), it follows that \( (\pi_2|\Gamma)^*(T) \) is smaller than any cluster value of \( (\pi_2|\Gamma)^*(T_n) \). Hence \( \tau \geq F^*(T) \). The last statement follows from Theorem 5.4. \( \square \)

**Definition 5.2.** If \( \{F^*(T)\} = F^*\{T\} \) we say that \( F^*(T) \) is well defined and we write \( F^*(T) := F^*(T) \). We call \( F^*(T) \) the total transform of \( T \).

For currents in a projective space, this definition has been used in order to study the dynamics of birational maps [13] (see also [2]). The following result justifies the previous definition.

**Proposition 5.3.** The operator \( F^* \) is continuous in the following sense. Let \( T_n \) and \( T \) be positive closed \((dd^c\text{-closed})\) currents such that \( T_n \to T \). Assume that \( F^*(T_n) \) and \( F^*(T) \) are well defined in the above sense. Then \( F^*(T_n) \to F^*(T) \).

**Proof.** As in Proposition 5.1 \( \|F^*(T_n)\| \) is bounded uniformly on \( n \). We can assume that \( F^*(T_n) \) converge to a current \( \tau \). Proposition 5.1 implies that \( \tau \geq F^*(T) \). On the other hand, since \( F^* \) acts continuously on cohomology groups, we have

\[
\{\tau\} = \lim\{F^*(T_n)\} = \lim F^*\{T_n\} = F^*\{T\} = \{F^*(T)\}.
\]

It follows that \( \|\tau\| = \|F^*(T)\| \) and hence \( \tau = F^*(T) \). \( \square \)

Now we study the case of \( dd^c\text{-closed} \) \((1,1)\)-currents.

**Theorem 5.4.** Let \( F : X \to X' \) be a MT between compact Kähler manifolds. If \( T \) is a positive \( dd^c\text{-closed} \) \((1,1)\)-current on \( X' \). Then there is a unique \( dd^c\text{-closed} \) extension \( F^*(T) \) of \( F^*(T) \) such that \( T \mapsto F^*(T) \) is continuous for the weak topology on currents. Moreover, we have \( \{F^*(T)\} = F^*\{T\} \).

For the proof we need the following fact.

**Proposition 5.5.** Let \( f : X \to X' \) be a holomorphic surjective map between compact Kähler manifolds of dimension \( k \) and \( k' \). Let \( I_2 \) be the set of \( x' \in X' \) such that \( \dim f^{-1}(x') > k - k' \). Then the components of codimension 1 of \( f^{-1}(I_2) \) are cohomologically independent.

**Proof.** By Stein’s factorization theorem [18], E.G.A III 4.3.3, there exist a normal space \( Y \), a holomorphic map \( h : X \to Y \) and a finite morphism \( g : Y \to X' \) such that \( f = g \circ h \). Moreover generic fibers of \( h \) are connected. We can replace
X' by Y and assume that generic fibers of f are connected, but X' may have singularities.

Let \( E_i \) be components of codimension 1 of \( f^{-1}(I_2) \). We have to show that the classes \( \{E_i\} \) are linearly independent. Let \( c_i \in \mathbb{R} \) such that the class of \( \sum c_i E_i \) vanishes. Then there is an integrable function \( u \) such that \( dd^c u = \sum c_i [E_i] \). This function is pluriharmonic out of \( f^{-1}(I_2) \).

Let \( \Omega \) be a closed smooth \((k-k',k-k')\)-form on \( X \). Then \( f_*(u\Omega) \) is \( dd^c \)-closed. It follows that the current \( f_*(u\Omega) \) is equal to a function which is pluriharmonic out of \( I_2 \). Since \( I_2 \) has codimension \( \geq 2 \) this function can be extended to a pluriharmonic function on the compact space \( X' \) and hence should be constant.

We show that \( u \) is constant on each generic fiber of \( f \). The case \( k = k' \) is clear. One consider the case \( k > k' \). In a neighborhood of a generic point \( x \in X \) one can find a coordinate system \((z,z')\) so that \( x = 0 \) and \( f(z,z') = z' \). If \( \varphi \) is a smooth function with support in this neighborhood, then \( \Omega := dd^c \varphi \wedge (dd^c \|z\|^2)^{k-k'-1} \) is a closed form. Since \( f_*(u\Omega) \) is a constant function vanishing near \( I_2 \), we have \( f_*(u\Omega) = 0 \). It follows that

\[
\int u(z,0)dd^c \varphi(z,0) \wedge (dd^c \|z\|^2)^{k-k'-1} = 0
\]

every \( \varphi \). Hence \( u(z,0) \) is constant. Since generic fibers of \( f \) are connected, \( u \) is constant on each generic fiber of \( f \). Then there is a function \( u' \) on \( X' \) such that \( u = u' \circ f \) almost everywhere.

Now consider a strictly positive closed form \( \Omega \). We have \( f_*(u\Omega) = u'f_*(\Omega) \). On the other hand, since \( f_*(\Omega) \) is a closed \((0,0)\)-current, it is given by a constant function. This function is not zero because \( \Omega \) is strictly positive. Finally, the fact that \( f_*(u\Omega) \) is given by a constant function implies that \( u' \) is constant and then \( u \) is constant. Consequently, \( c_i = 0 \) for every \( i \).

**Proof of Theorem 5.4.** We can assume that the graph \( \Gamma \) of \( F \) is irreducible and that \( \Gamma \) is smooth. Otherwise, we consider a blow-up \( \tau : \tilde{\Gamma} \to \Gamma \) and use \( \pi_i \circ \tau \) instead of \( \pi_{i|\Gamma} \).

By [14], \( T \) is a difference of currents which can be approximated by smooth positive \( dd^c \)-closed \((1,1)\)-forms. We can assume that \( T \) is the limit of smooth positive \( dd^c \)-closed \((1,1)\)-forms \( T_n \). Recall that \( F^* \) is well defined on smooth forms. Then the uniqueness in the theorem is clear.

The masses of \((\pi_{2|\Gamma})^*(T_n)\) are computed cohomologically; they are bounded uniformly on \( n \). We only have to show that \((\pi_{2|\Gamma})^*(T_n)\) converge. If \( \tau \) is a limit value then \( \{\tau\} = (\pi_{2|\Gamma})^*\{T\} \) and \( \tau = (\pi_{2|\Gamma})^*(T) \) is a pluriharmonic \((1,1)\)-current with support in \( \pi_2^{-1}(I_2) \cap \Gamma \). If the \( E_i \) are components of codimension 1 of \( \pi_2^{-1}(I_2) \cap \Gamma \) then there are real numbers \( c_i \) such that \( \tau = (\pi_{2|\Gamma})^*(T) + \sum c_i [E_i] \). Proposition 5.3 implies that \( c_i \) are uniquely determined by the identity \( \{\tau\} = \{(\pi_{2|\Gamma})^*T\} + \sum c_i \{[E_i]\} \). The theorem follows. □

\[ \text{we thank F. Campana who told us this argument} \]
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Tien-Cuong Dinh
Institut de Mathématique de Jussieu
Plateau 7D, Analyse Complex
175 rue du Chevaleret
75013 Paris, France
dinh@math.jussieu.fr

Nessim Sibony
Mathématique - Bâtiment 425
UMR 8628
Université Paris-Sud
91405 Orsay, France
nessim.sibony@math.u-psud.fr