Revan–degree indices on random graphs

R. Aguilar-Sánchez\textsuperscript{1}, I. F. Herrera-González\textsuperscript{2}, J. A. Méndez-Bermúdez\textsuperscript{3}, and José M. Sigarreta\textsuperscript{4}

\textsuperscript{1}Facultad de Ciencias Químicas, Benemérita Universidad Autónoma de Puebla, Puebla 72570, Mexico
\textsuperscript{2}Departamento de Ingeniería, Universidad Popular Autónoma del Estado de Puebla, Puebla, Pue., 72410, Mexico
\textsuperscript{3}Instituto de Física, Benemérita Universidad Autónoma de Puebla, Apartado Postal J-48, Puebla 72570, Mexico
\textsuperscript{4}Universidad Autónoma de Guerrero, Centro Acapulco CP 39610, Acapulco de Juárez, Guerrero, Mexico

ras747698@gmail.com, ivanfernando.herrera@upaep.mx, jmendezb@ifuap.buap.mx, josemariasigarretalmira@hotmail.com

(Received June 5, 2020)

Abstract

Given a simple connected non-directed graph $G = (V(G), E(G))$, we consider two families of graph invariants: $RX_{\Sigma}(G) = \sum_{uv \in E(G)} F(r_u, r_v)$ (which has gained interest recently) and $RX_{\Pi}(G) = \prod_{uv \in E(G)} F(r_u, r_v)$ (that we introduce in this work); where $uv$ denotes the edge of $G$ connecting the vertices $u$ and $v$, $r_u$ is the Revan degree of the vertex $u$, and $F$ is a function of the Revan vertex degrees. Here, $r_u = \Delta + \delta - d_u$ with $\Delta$ and $\delta$ the maximum and minimum degrees among the vertices of $G$ and $d_u$ is the degree of the vertex $u$. Particularly, we apply both $RX_{\Sigma}(G)$ and $RX_{\Pi}(G)$ on two models of random graphs: Erdős-Rényi graphs and random geometric graphs. By a thorough computational study we show that $\langle RX_{\Sigma}(G) \rangle$ and $\langle \ln RX_{\Pi}(G) \rangle$, normalized to the order of the graph, scale with the average Revan degree $\langle r \rangle$; here $\langle \cdot \rangle$ denotes the average over an ensemble of random graphs. Moreover, we provide analytical expressions for several graph invariants of both families in the dense graph limit.

*Corresponding author
1 Introduction

We can identify two families of graph invariants which have been extensively studied in chemical graph theory, namely

\[ X_\Sigma(G) = \sum_{uv \in E(G)} F(d_u, d_v) \] (1)

and

\[ X_\Pi(G) = \prod_{uv \in E(G)} F(d_u, d_v). \] (2)

Here \( uv \) denotes the edge of the graph \( G = (V(G), E(G)) \) connecting the vertices \( u \) and \( v \), \( d_u \) is the degree of the vertex \( u \), and \( F(x, y) \) is a given function of the vertex degrees, see e.g. [1]. While both \( X_\Sigma(G) \) and \( X_\Pi(G) \) are referred as topological indices in the literature, to make a distinction between them, here we name \( X_\Sigma(G) \) and \( X_\Pi(G) \) as topological indices (TIs) and multiplicative topological indices (MTIs), respectively.

In fact, within a statistical approach on random graphs, it has been recently shown that the average values of indices of the type \( X_\Sigma(G) \), normalized to the order of the graph \( n \), scale with the average degree \( \langle d \rangle \); see e.g. Refs. [2–6]. That is, \( \langle X_\Sigma(G) \rangle / n \) is a function of \( \langle d \rangle \) only:

\[ \langle X_\Sigma(G) \rangle / n \equiv f_\Sigma(\langle d \rangle). \] (3)

More recently, a number of new TIs with the form

\[ RX_\Sigma(G) = \sum_{uv \in E(G)} F(r_u, r_v) \] (4)

have been proposed and studied, see e.g. Refs. [7–10]. Above, \( r_u \) is the Revan vertex degree of the vertex \( u \) which is defined as

\[ r_u = \Delta + \delta - d_u, \] (5)

where \( \Delta \) and \( \delta \) are the maximum and minimum degrees among the vertices of the graph \( G \), respectively. Note that \( RX_\Sigma(G) \) is the Revan version of \( X_\Sigma(G) \).

Thus, inspired by the scaling law of Eq. (3), in this paper we explore the statistical properties of \( \langle RX_\Sigma(G) \rangle \) on random graphs and look for the scaling parameter and the corresponding scaling law. Moreover, to complete the panorama of Revan-degree–based indices, we introduce Revan versions of MTIs:

\[ RX_\Pi(G) = \prod_{uv \in E(G)} F(r_u, r_v), \] (6)

and also study their statistical and scaling properties on random graphs.
2 Statistical analysis of Revan-degree–based TIs on random graphs

Among the recently introduced Revan-degree–based indices, $RX_Σ(G)$, we can mention [7–9]

$$R_1(G) = \sum_{uv \in E(G)} r_u + r_v, \quad R_2(G) = \sum_{uv \in E(G)} r_u r_v,$$

and

$$FR(G) = \sum_{uv \in E(G)} r_u^2 + r_v^2,$$

and

$$RSO(G) = \sum_{uv \in E(G)} \sqrt{r_u^2 + r_v^2}.$$

Evidently, these TIs are the Revan versions of the first and second Zagreb indices [11],

$$M_1(G) = \sum_{uv \in E(G)} d_u + d_v, \quad M_2(G) = \sum_{uv \in E(G)} d_u d_v,$$

the forgotten index [12]

$$F(G) = \sum_{uv \in E(G)} d_u^2 + d_v^2,$$

and the Sombor index [13]

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2},$$

respectively.

In what follows we compute $R_1(G)$, $R_2(G)$, $FR(G)$ and $RSO(G)$ on two models of random graphs: Erdős-Rényi (ER) graphs and random geometric (RG) graphs. ER graphs [14-15] $G_{ER}(n, p)$ are formed by $n$ vertices connected independently with probability $p \in [0, 1]$. While RG graphs [16-17] $G_{RG}(n, r)$ consist of $n$ vertices uniformly and independently distributed on the unit square, where two vertices are connected by an edge if their Euclidean distance is less or equal than the connection radius $\ell \in [0, \sqrt{2}]$.

Moreover, since a given parameter pair [(n, p) or (n, \ell)] represents an infinite-size ensemble of random [ER or RG] graphs, the computation of a graph invariant on a single graph may be irrelevant. In contrast, the computation of the average value of a graph invariant over a large ensemble of random graphs, all characterized by the same parameter pair, may provide useful average information about the full ensemble. This statistical approach, well known in random matrix theory studies, has been recently applied to random graphs and networks by means of degree–based TIs, see e.g. Refs. [2-6, 10].
In Fig. 1 we present the average values of the Revan-degree–based TIs $R_1(G_{ER})$, $R_2(G_{ER})$, $FR(G_{ER})$ and $RSO(G_{ER})$ as a function of the probability $p$ of Erdős-Rényi graphs $G_{ER}(n, p)$ of sizes $n \in [125, 1000]$. Dashed lines are (a) $\langle M_1(G_{ER}) \rangle$, (b) $\langle M_2(G_{ER}) \rangle$, (c) $\langle F(G_{ER}) \rangle$, and (d) $\langle SO(G_{ER}) \rangle$. Each data value was computed by averaging over $10^6$ random graphs $G_{ER}(n, p)$.

2.1 Revan-degree–based TIs on Erdős-Rényi graphs

In Fig. 1 we present the average values of the Revan-degree–based TIs $R_1(G_{ER})$, $R_2(G_{ER})$, $FR(G_{ER})$ and $RSO(G_{ER})$ as a function of the probability $p$ of ER graphs of four different sizes $n$ (full lines). For comparison purposes in each panel of Fig. 1 we include the corresponding average degree–based TIs; that is, we plot the average values of $M_1(G_{ER})$, $M_2(G_{ER})$, $F(G_{ER})$ and $SO(G_{ER})$, respectively (dashed lines).

It is interesting to note, from Fig. 1, that $\langle RX_{\Sigma}(G_{ER}) \rangle \approx \langle X_{\Sigma}(G_{ER}) \rangle$ once $p > 0.01$. Moreover, given that $RX_{\Sigma}(G_{ER})$ and $X_{\Sigma}(G_{ER})$ have the same functional form on $r$ and $d$, respectively, $\langle RX_{\Sigma}(G_{ER}) \rangle \approx \langle X_{\Sigma}(G_{ER}) \rangle$ must be the consequence of

$$\langle r(G_{ER}) \rangle = \langle \Delta(G_{ER}) \rangle + \langle \delta(G_{ER}) \rangle - \langle d(G_{ER}) \rangle \approx \langle d(G_{ER}) \rangle,$$

for large $p$. Indeed, in Fig. 2(a) we plot $\langle r(G_{ER}) \rangle$ (full lines) and $\langle d(G_{ER}) \rangle$ (dashed lines) and clearly verify that $\langle r(G_{ER}) \rangle \approx \langle d(G_{ER}) \rangle$ for large $p$. Thus, the approximation in Eq. (7) implies that $\langle d(G_{ER}) \rangle \approx [\langle \Delta(G_{ER}) \rangle + \langle \delta(G_{ER}) \rangle]/2$. This rough estimate of the mean from the max and min values is validated in Fig. 2(b) where we contrast $[\langle \Delta(G_{ER}) \rangle + \langle \delta(G_{ER}) \rangle]/2$ with $\langle d(G_{ER}) \rangle$ and show that they certainly coincide for large enough $p$. 
Therefore, in the dense limit, i.e. when \( \langle d(G_{ER}) \rangle \gg 1 \), we can estimate the Revan-degree–based TIs by the use of the approximations \( r_u \approx r_v \approx \langle r(G_{ER}) \rangle \) and \( \langle d(G_{ER}) \rangle \approx \langle r(G_{ER}) \rangle \). For example, for \( R_1(G_{ER}) \) we can write

\[
R_1(G_{ER}) = \sum_{uv \in E(G_{ER})} r_u + r_v \approx \sum_{uv \in E(G_{ER})} 2 \langle r(G_{ER}) \rangle \approx n \langle d(G_{ER}) \rangle \langle r(G_{ER}) \rangle \approx n \langle r(G_{ER}) \rangle^2
\]

or

\[
\frac{R_1(G_{ER})}{n} \approx \langle r(G_{ER}) \rangle^2.
\] (8)

Above we have used \( |E(G_{ER})| = n \langle d(G_{ER}) \rangle / 2 \). Similar approximations give

\[
\frac{R_2(G_{ER})}{n} \approx \frac{1}{2} \langle r(G_{ER}) \rangle^3,
\] (9)

\[
\frac{FR(G_{ER})}{n} \approx \langle r(G_{ER}) \rangle^3,
\] (10)

and

\[
\frac{RSO(G_{ER})}{n} \approx \frac{1}{\sqrt{2}} \langle r(G_{ER}) \rangle^2.
\] (11)

From Eqs. (8)-(11) we can see that the ratio \( RX_\Sigma(G_{ER})/n \) should depend on \( \langle r(G_{ER}) \rangle \) only in the dense limit.

Then, in Fig. 3 we plot \( \langle RX_\Sigma(G_{ER}) \rangle / n \) vs. \( \langle r(G_{ER}) \rangle \) (full lines) and observe a good correspondence with Eqs. (8)-(11) (orange dashed lines) in the dense limit, i.e. when \( \langle r(G_{ER}) \rangle \geq 10 \). Furthermore, except for a small-size effect evident at small \( \langle r(G_{ER}) \rangle \), we notice that the curves \( \langle RX_\Sigma(G_{ER}) \rangle / n \) vs. \( \langle r(G_{ER}) \rangle \) do not depend on \( n \) (that is, the curves for different graph sizes fall one on top of the other) even for \( \langle r(G_{ER}) \rangle < 10 \).
Figure 3. (a) $\langle R_1(\mathit{G_{ER}}) \rangle / n$, (b) $\langle R_2(\mathit{G_{ER}}) \rangle / n$, (c) $\langle \mathit{FR}(\mathit{G_{ER}}) \rangle / n$, and (d) $\langle \mathit{RSO}(\mathit{G_{ER}}) \rangle / n$ as a function of the average Revan vertex degree $\langle r(\mathit{G_{ER}}) \rangle$ of Erdős-Rényi graphs $\mathit{G_{ER}}(n,p)$ of sizes $n \in [125,1000]$. Dashed lines are (a) $\langle M_1(\mathit{G_{ER}}) \rangle / n$, (b) $\langle M_2(\mathit{G_{ER}}) \rangle / n$, (c) $\langle \mathit{F}(\mathit{G_{ER}}) \rangle / n$, and (d) $\langle \mathit{SO}(\mathit{G_{ER}}) \rangle / n$ as a function of the average degree $\langle d(\mathit{G_{ER}}) \rangle$. Same data of Fig. 1. Orange dashed lines are (a) Eq. (8), (b) Eq. (9), (c) Eq. (10), and (d) Eq. (11). The vertical magenta dashed lines indicate $\langle r(\mathit{G_{ER}}) \rangle = 10$.

Therefore, a scaling relation for $\langle RX_{\Sigma}(\mathit{G_{ER}}) \rangle$ can be stated as

$$\frac{\langle RX_{\Sigma}(\mathit{G_{ER}}) \rangle}{n} \approx g_{\Sigma}(\langle r(\mathit{G_{ER}}) \rangle).$$ (12)

Note that scaling (12) is the Revan version of scaling (3). Also note that in those expressions we deliberately named the functions on the rhs as $g_{\Sigma}$ and $f_{\Sigma}$, respectively, to stress that they are different. Nevertheless, as can be clearly seen in Fig. 3 where we also include the curves $\langle X_{\Sigma}(\mathit{G_{ER}}) \rangle / n$ vs. $\langle d(\mathit{G_{ER}}) \rangle$ (dashed lines), once $\langle r(\mathit{G_{ER}}) \rangle \geq 10$ the curves $\langle X_{\Sigma}(\mathit{G_{ER}}) \rangle / n$ vs. $\langle d(\mathit{G_{ER}}) \rangle$ and $\langle RX_{\Sigma}(\mathit{G_{ER}}) \rangle / n$ vs. $\langle r(\mathit{G_{ER}}) \rangle$ coincide. This means that Eqs. (8,11) with $RX \rightarrow X$ and $r \rightarrow d$ also describe the corresponding degree–based indices $X_{\Sigma}(\mathit{G_{ER}})$ when $\langle d(\mathit{G_{ER}}) \rangle \geq 10$; or equivalently, the functions $f_{\Sigma}$ and $g_{\Sigma}$ in the scalings (3) and (12), respectively, must be equal in the dense limit.

### 2.2 Revan-degree–based TIs on random geometric graphs

Now, in Fig. 4 we present the average values of the Revan-degree–based TIs $R_1(\mathit{G_{RG}})$, $R_2(\mathit{G_{RG}})$, $\mathit{FR}(\mathit{G_{RG}})$ and $\mathit{RSO}(\mathit{G_{RG}})$ as a function of the connection radius $\ell$ of RG graphs...
Figure 4. (a) \(\langle R_1(G_{RG}) \rangle\), (b) \(\langle R_2(G_{RG}) \rangle\), (c) \(\langle FR(G_{RG}) \rangle\), and (d) \(\langle RSO(G_{RG}) \rangle\) as a function of the connection radius \(\ell\) of random geometric graphs \(G_{RG}(n, \ell)\) of sizes \(n \in [125, 1000]\). Dashed lines are (a) \(\langle M_1(G_{RG}) \rangle\), (b) \(\langle M_2(G_{RG}) \rangle\), (c) \(\langle F(G_{RG}) \rangle\), and (d) \(\langle SO(G_{RG}) \rangle\). Each data value was computed by averaging over \(10^6\) random graphs \(G_{RG}(n, \ell)\).

of four different sizes \(n\) (full lines). In this figure we also include the corresponding average degree–based TIs as dashed lines. In addition, in Figs. 5(a) and 5(b) we plot \(\langle r(G_{RG}) \rangle\) (full lines) and \([\langle \Delta(G_{RG}) \rangle + \langle \delta(G_{RG}) \rangle]/2\) (full lines) as a function of \(\ell\), respectively.

For comparison purposes, Figs. 4 and 2 for ER graphs are equivalent to Figs. 4 and 5 for RG graphs. In fact, all observations and conclusions made in the previous Subsection for ER graphs are also valid for RG graphs, namely:

(i) \(\langle RX_\Sigma(G_{RG}) \rangle \approx \langle X_\Sigma(G_{RG}) \rangle\) for large \(\ell\), see Fig. 4

(ii) \(\langle d(G_{RG}) \rangle \approx [\langle \Delta(G_{RG}) \rangle + \langle \delta(G_{RG}) \rangle]/2\) for large \(\ell\), see Fig. 5(b), thus

(iii) \(\langle r(G_{RG}) \rangle \approx \langle d(G_{RG}) \rangle\) for large \(\ell\), see Fig. 5(a). Therefore,

(iv) Eqs. 11 with \(G_{ER} \rightarrow G_{RG}\) should also be valid for RG graphs in the dense limit.

This is indeed verified in Fig. 6 where we have plotted \(\langle RX_\Sigma(G_{RG}) \rangle/n\) vs. \(\langle r(G_{RG}) \rangle\) (full lines) together with Eqs. 11 (orange dashed lines).

(v) Finally, from Fig. 6 the scaling law

\[
\frac{\langle RX_\Sigma(G_{RG}) \rangle}{n} \approx g_\Sigma(\langle r(G_{RG}) \rangle)
\]
Figure 5. (a) Average Revan vertex degree $\langle r(G_{RG}) \rangle$ and (b) $[\langle \Delta(G_{RG}) \rangle + \langle \delta(G_{RG}) \rangle]/2$ as a function of the connection radius $\ell$ of random geometric graphs $G_{RG}(n, \ell)$ of sizes $n \in [125, 1000]$. Dashed lines are the corresponding average degrees $\langle d(G_{RG}) \rangle$. Each data value was computed by averaging over $10^6$ random graphs $G_{RG}(n, \ell)$.

Figure 6. (a) $\langle R_1(G_{RG}) \rangle/n$, (b) $\langle R_2(G_{RG}) \rangle/n$, (c) $\langle FR(G_{RG}) \rangle/n$, and (d) $\langle RSO(G_{RG}) \rangle/n$ as a function of the average Revan vertex degree $\langle r(G_{RG}) \rangle$ of random geometric graphs $G_{RG}(n, \ell)$ of sizes $n \in [125, 1000]$. Dashed lines are (a) $\langle M_1(G_{RG}) \rangle/n$, (b) $\langle M_2(G_{RG}) \rangle/n$, (c) $\langle F(G_{RG}) \rangle/n$, and (d) $\langle SO(G_{RG}) \rangle/n$ as a function of the average degree $\langle d(G_{RG}) \rangle$. Same data of Fig. 5. Orange dashed lines are (a) Eq. (8), (b) Eq. (9), (c) Eq. (10), and (d) Eq. (11) with $G_{ER} \rightarrow G_{RG}$. The vertical magenta dashed lines indicate $\langle r(G_{RG}) \rangle = 10$. 
can be stated.

3 Statistical analysis of Revan-degree–based MTIs on random graphs

We now introduce the multiplicative versions of the Revan-degree–based TIs, $R_{\Pi}(G)$, studied in the previous Section:

$$R_{1\Pi}(G) = \prod_{uv \in E(G)} (r_u + r_v), \quad R_{2\Pi}(G) = \prod_{uv \in E(G)} r_u r_v,$$

$$FR_{\Pi}(G) = \prod_{uv \in E(G)} \sqrt{r_u^2 + r_v^2},$$

and

$$RSO_{\Pi}(G) = \prod_{uv \in E(G)} \sqrt{r_u^2 + r_v^2}.$$

These MTIs are the Revan versions of the multiplicative Zagreb indices [18] [19], the multiplicative forgotten index

$$\Pi^*_1(G) = \prod_{uv \in E(G)} d_u + d_v, \quad \Pi^*_2(G) = \prod_{uv \in E(G)} d_u d_v,$$

the multiplicative Sombor index

$$F_{\Pi}(G) = \prod_{uv \in E(G)} d_u^2 + d_v^2,$$

and the multiplicative Sombor index

$$SO_{\Pi}(G) = \prod_{uv \in E(G)} \sqrt{d_u^2 + d_v^2},$$

respectively. Note that (as far as we know) neither $F_{\Pi}(G)$ nor $SO_{\Pi}(G)$ have been considered before.

It is fair to recall that a statistical study of degree–based MTIs, $X_{\Pi}(G)$, on random graphs has been already reported in Ref. [20]. There, the multiplicative Zagreb indices, the multiplicative Randić connectivity index, the multiplicative harmonic index, the multiplicative sum-connectivity index, the multiplicative inverse degree index, as well as the Narumi-Katayama index were applied to ER graphs, RG graphs and bipartite random graphs. There it was demonstrated that $\langle \ln X_{\Pi}(G) \rangle$ normalized to the order of the graph scale with the corresponding average degree:

$$\frac{\langle \ln X_{\Pi}(G) \rangle}{n} = f_{\Pi}(\langle d \rangle).$$ (16)
Figure 7. (a) $\langle \ln R_{1\Pi}(G_{ER}) \rangle$, (b) $\langle \ln R_{2\Pi}(G_{ER}) \rangle$, (c) $\langle \ln FR_{\Pi}(G_{ER}) \rangle$, and (d) $\langle \ln RSO_{\Pi}(G_{ER}) \rangle$ as a function of the probability $p$ of Erdős-Rényi graphs $G_{ER}(n, p)$ of sizes $n \in [125, 1000]$. Dashed lines are (a) $\langle \ln \Pi_{1}(G_{ER}) \rangle$, (b) $\langle \ln \Pi_{2}(G_{ER}) \rangle$, (c) $\langle \ln F_{\Pi}(G_{ER}) \rangle$, and (d) $\langle \ln SO_{\Pi}(G_{ER}) \rangle$. Each data value was computed by averaging over $10^6$ random graphs $G_{ER}(n, p)$.

Note that scaling (16) can be considered as the multiplicative version of scaling (3).

Then, in what follows we apply the Revan-degree–based MTIs defined above on ER and RG graphs.

In Figs. 7 and 8 we present the average values of the Revan-degree–based MTIs $R_{1\Pi}(G)$, $R_{2\Pi}(G)$, $FR_{\Pi}(G)$ and $RSO_{\Pi}(G)$ for ER (Fig. 7) and RG (Fig. 8) graphs of four different sizes $n$ (full lines). In these figures we also plot the corresponding average degree–based MTIs as dashed lines. For both random graph models we observe that $\langle RX_{\Pi}(G) \rangle \approx \langle X_{\Pi}(G) \rangle$ for large enough $p$ and large enough $\ell$, respectively.

Indeed, as for the Revan-degree–based TIs, here we can also estimate the Revan-degree–based MTIs in the dense limit by the use of the approximations $r_u \approx r_v \approx \langle r(G) \rangle$ and $\langle d(G) \rangle \approx \langle r(G) \rangle$. Thus, for $R_{1\Pi}(G)$ we write

$$R_{1\Pi}(G) = \prod_{uv \in E(G)} r_u + r_v \approx \prod_{uv \in E(G)} 2 \langle r(G) \rangle \approx [2 \langle r(G) \rangle]^{\langle d(G) \rangle / 2}$$

which leads to

$$\ln R_{1\Pi}(G) \approx \frac{1}{2} n \langle d(G) \rangle \ln[2 \langle r(G) \rangle] \approx \frac{1}{2} n \langle r(G) \rangle \ln[2 \langle r(G) \rangle]$$
or
\[
\ln \frac{R_{1\Pi}(G)}{n} \approx \frac{1}{2} \langle r(G) \rangle \ln[2 \langle r(G) \rangle].
\] (17)

Similar approximations give
\[
\ln \frac{R_{2\Pi}(G)}{n} \approx \langle r(G) \rangle \ln \langle r(G) \rangle ,
\] (18)
\[
\ln \frac{FR_{\Pi}(G)}{n} \approx \langle r(G) \rangle \ln[\sqrt{2} \langle r(G) \rangle] ,
\] (19)
and
\[
\ln \frac{RSO_{\Pi}(G)}{n} \approx \frac{1}{2} \langle r(G) \rangle \ln[\sqrt{2} \langle r(G) \rangle] .
\] (20)

Note that Eqs. (17-20) should work for both ER and RG graphs.

Therefore, in Figs. 8 and 9 we plot \( \langle RX_{\Pi}(G) \rangle /n \) vs. \( \langle r(G) \rangle \) (full lines) for ER and RG graphs, respectively, together with Eqs. (17-20) (orange dashed lines). Indeed, we observe a very good correspondence between predictions (17-20) and the numerical data for both random graph models in dense limit, i.e. when \( \langle r(G) \rangle \geq 10 \). From these figures, except for a small-size effect at small \( \langle r(G_{\text{ER}}) \rangle \), we can state the scaling of the Revan-degree–based
Figure 9. (a) $\langle \ln R_{1\Pi}(G_{ER}) \rangle / n$, (b) $\langle \ln R_{2\Pi}(G_{ER}) \rangle / n$, (c) $\langle \ln FR_{\Pi}(G_{ER}) \rangle / n$, and (d) $\langle \ln RSO_{\Pi}(G_{ER}) \rangle / n$ as a function of the average Revan vertex degree $\langle r(G_{ER}) \rangle$ of Erdős-Rényi graphs $G_{ER}(n, p)$ of sizes $n \in [125, 1000]$. Dashed lines are (a) $\langle \ln \Pi_{1\Pi}(G_{ER}) \rangle / n$, (b) $\langle \ln \Pi_{2\Pi}(G_{ER}) \rangle / n$, (c) $\langle \ln F_{\Pi}(G_{ER}) \rangle / n$, and (d) $\langle \ln S_{\Pi}(G_{ER}) \rangle / n$ as a function of the average degree $\langle d(G_{ER}) \rangle$. Same data of Fig. 7. Orange dashed lines are (a) Eq. (17), (b) Eq. (18), (c) Eq. (19), and (d) Eq. (20). The vertical magenta dashed lines indicate $\langle r(G_{ER}) \rangle = 10$. 
Figure 10. (a) $\langle \ln R_1(\Pi(G_{RG})) \rangle /n$, (b) $\langle \ln R_2(\Pi(G_{RG})) \rangle /n$, (c) $\langle \ln F_{\Pi}(G_{RG}) \rangle /n$, and (d) $\langle \ln S_{\Pi}(G_{RG}) \rangle /n$ as a function of the average Revan vertex degree $\langle r(G_{RG}) \rangle$ of random geometric graphs $G_{RG}(n, \ell)$ of sizes $n \in [125, 1000]$. Dashed lines are (a) $\langle \ln \Pi_1^*(G_{RG}) \rangle /n$, (b) $\langle \ln \Pi_2^*(G_{RG}) \rangle /n$, (c) $\langle \ln F_1^*(G_{RG}) \rangle /n$, and (d) $\langle \ln S_{\Pi}^*(G_{RG}) \rangle /n$ as a function of the average degree $\langle d(G_{RG}) \rangle$. Same data of Fig. 8. Orange dashed lines are (a) Eq. (17), (b) Eq. (18), (c) Eq. (19), and (d) Eq. (20). The vertical magenta dashed lines indicate $\langle r(G_{RG}) \rangle = 10$. 
MTIs as
\[
\frac{\langle \ln RX_{\Pi}(G) \rangle}{n} \approx g_{\Pi}(\langle r(G) \rangle) .
\] (21)
Note that scalings (16) and (21) indeed coincide for \( \langle r(G) \rangle \geq 10 \) as can be clearly seen in Figs. 9 and 10 where we also include the curves \( \langle X_{\Pi}(G_{ER}) \rangle / n \) vs. \( \langle d(G_{ER}) \rangle \) (dashed lines). This means that Eqs. (17-20) with \( RX \rightarrow X \) and \( r \rightarrow d \) also describe the corresponding degree–based indices \( X_{\Pi}(G) \) when \( \langle d(G) \rangle \geq 10 \); or equivalently, the functions \( f_{\Pi} \) and \( g_{\Pi} \) in the scalings (16) and (21), respectively, must be equal in the dense limit.

4 Summary
Motivated by potential theoretical–practical applications of topological indices, in this work we perform a thorough numerical study of two families of Revan-degree–based graph invariants: \( RX_{\Sigma}(G) = \sum_{uv \in E(G)} F(r_u, r_v) \) and \( RX_{\Pi}(G) = \prod_{uv \in E(G)} F(r_u, r_v) \). In particular while \( RX_{\Sigma}(G) \) has gained interest recently, see e.g. Refs. [7–10], we are introducing \( RX_{\Pi}(G) \) here. Specifically, we have considered the Revan-degree–based versions of the first and second Zagrev indices, the forgotten index, and the Sombor index. We have applied both \( X_{\Sigma}(G) \) and \( X_{\Pi}(G) \) on ensembles of Erdős–Rényi graphs and random geometric graphs, see Figs. 1, 4, 7 and 8.

We would like to add that we have also introduced here the multiplicative forgotten index \( F_{\Pi}(G) \) and the multiplicative Sombor index \( SO_{\Pi}(G) \), see Eqs. (14) and (15), respectively.

On the one hand we have shown that \( \langle RX_{\Sigma}(G) \rangle \) and \( \langle \ln RX_{\Pi}(G) \rangle \), normalized to the order of the graph \( n \), scale with the average Revan degree \( \langle r \rangle \); that is,
\[
\frac{\langle RX_{\Sigma}(G) \rangle}{n} \approx g_{\Sigma}(\langle r(G) \rangle) \quad \text{and} \quad \frac{\langle \ln RX_{\Pi}(G) \rangle}{n} \approx g_{\Pi}(\langle r(G) \rangle),
\] (22)
see Figs. 3 [4, 9 and 10]. On the other hand we have provided expressions for both \( \langle RX_{\Sigma}(G) \rangle \) and \( \langle \ln RX_{\Pi}(G) \rangle \) in the dense graph limit, see Eqs. (8-11) and Eqs. (17-20), respectively.

In addition, we have found that \( \langle r(G) \rangle \approx \langle d(G) \rangle \) and \( \langle RX_{\Sigma}(G) \rangle \approx \langle X_{\Sigma}(G) \rangle \) in the dense limit, i.e. when \( \langle r(G) \rangle \geq 10 \). This makes the scalings in (22) to reproduce the scalings [3] and [16] reported in Refs. [2 [6] and [20], respectively. Therefore, Eqs. (8-11) and Eqs. (17-20) also describe the corresponding degree-based topological indices in the dense limit. Furthermore, it is relevant to stress that the clear difference
between Revan-degree–based indices and degree–based indices for $\langle r(G) \rangle < 10$, makes Revan-degree–based indices particularly useful in that regime where they could provide additional information to standard degree–based indices.

We hope that our study may motivate further computational as well as theoretical studies of Revan-degree–based topological indices.

ACKNOWLEDGEMENTS

J.A.M.-B. thanks support from CONACyT (Grant No. 286633) and VIEP-BUAP (Grant No. 100405811-VIEP2022), Mexico. J.M.S. acknowledges financial support from Agencia Estatal de Investigación (PID2019-106433GB-I00 / AEI / 10.13039/501100011033), Spain.

References

[1] I. Gutman, Degree–based topological indices, *Croat. Chem. Acta* **86** (2013) 351–361.

[2] C. T. Martínez-Martínez, J. A. Méndez-Bermúdez, J. M. Rodríguez, J. M. Sigarreta Almira, Computational and analytical studies of the Randic index in Erdős-Rényi models, *Appl. Math. Comput.* **377** (2020) 125137.

[3] R. Aguilar-Sánchez, I. F. Herrera-González, J. A. Méndez-Bermúdez, J. M. Sigarreta, Computational properties of general indices on random networks, *Symmetry* **12** (2020) 1341.

[4] C. T. Martínez-Martínez, J. A. Méndez-Bermúdez, J. M. Rodríguez, J. M. Sigarreta, Computational and analytical studies of the harmonic index in Erdős-Rényi models, *MATCH Commun. Math. Comput. Chem.* **85** (2021) 395.

[5] R. Aguilar-Sánchez, J. A. Méndez-Bermúdez, J. M. Rodríguez, J. M. Sigarreta, Normalized Sombor indices as complexity measures of random networks, *Entropy* **23** (2021) 976.

[6] J. A. Mendez-Bermudez, R. Aguilar-Sanchez, R. Abreu-Blaya, J. M. Sigarreta, Stolarsky–Puebla index, *Discrete Math. Lett.* **9** (2022) 10–17.
[7] V. R. Kulli, Revan indices of oxide and honeycomb networks, *Inter. J. Math. Appl.* **5** (2017) 663–667.

[8] V. R. Kulli, F-Revan index and F-Revan polynomial of some families of benzenoid systems, *J. Global Res. Math. Archives* **5** (2018) 1–6.

[9] V. R. Kulli, I. Gutman, Revan Sombor index, *J. Math. Inform.* **22** (2022) 23–27.

[10] V. R. Kulli, J. A. Méndez-Bermúdez, J. M. Rodríguez, J. M. Sigarreta, Topological and statistical study of Revan Sombor indices, submitted (2022).

[11] I. Gutman, N. Trinajstić, Graph theory and molecular orbitals. Total ?-electron energy of alternant hydrocarbons, *Chem. Phys. Lett.* **17** (1972) 535–538.

[12] B. Furtula, I. Gutman, A forgotten topological index, *J. Math. Chem.* **53** (2015) 1184–1190.

[13] I. Gutman, Geometric approach to degree–based topological indices: Sombor indices, *MATCH Commun. Math. Comput. Chem.* **86** (2021) 11–16.

[14] R. Solomonoff, A. Rapoport, Connectivity of random nets, *Bull. Math. Biophys.* **13** (1951) 107–117.

[15] P. Erdös, A. Rényi, On random graphs, *Publ. Math. (Debrecen)* **6** (1959) 290–297.

[16] J. Dall, M. Christensen, Random geometric graphs, *Phys. Rev. E* **66** (2002) 016121.

[17] M. Penrose, Random Geometric Graphs (Oxford University Press, Oxford, 2003).

[18] M. Eliaši, A. Iranmanesh, I. Gutman, Multiplicative versions of first Zagreb index, *MATCH Commun. Math. Comput. Chem.* **68** (2012) 217–230.

[19] R. Kazemi, Note on the multiplicative Zagreb indices, *Discrete Appl. Math.* **198** (2016) 147–154.

[20] R. Aguilar-Sanchez, J. A. Méndez-Bermúdez, J. A. Mendez, J. M. Sigarreta, Computational properties of multiplicative topological indices on random networks, submitted (2022).