Some Topology on Zero-Dimensional Subrings of Product of Rings

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Abstract. Let $R$ be a ring and $\{R_i\}_{i \in I}$ a family of zero-dimensional rings. We define the Zariski topology on $\mathcal{Z}(R, \prod R_i)$ and study their basic properties. Moreover, we define a topology on $\mathcal{Z}(R, \prod R_i)$ by using ultrafilters; it is called the ultrafilter topology and we demonstrate that this topology is finer than the Zariski topology. We show that the ultrafilter limit point of a collections of subrings of $\mathcal{Z}(R, \prod R_i)$ is a zero-dimensional ring. Its relationship with $F^{-}\lim$ and the direct limit of a family of rings are studied.

1. Introduction

All rings considered in this paper are assumed to be commutative, and have identity element. Let $S$ be a ring, we will denote by $\mathcal{Z}(S), \mathcal{V}(S), \mathcal{A}(S)$, respectively, the sets of zero-dimensional, von Neumann regular, artinian subrings of $S$. The study of zero-dimensionality in commutative rings has been widely treated in the literature with a purely algebraic approach, (see [6, 7]). Our purpose here is to give a new characterization of zero-dimensionality notion by using the ultrafilters topology and the $F^{-}\lim$.

Let $R$ be a subring of a ring $S$, we denote by $\mathcal{Z}(R, S)$ the set of intermediate zero-dimensional subrings of the pair $(R, S)$. We define Zarisky topology on $\mathcal{Z}(R, S)$, more precisely on $\mathcal{Z}(R, \prod R_i)$, where $\{R_i\}_{i \in I}$ is a family of zero dimensional rings, whose open sets are of the form $\mathcal{Z}(R[x], \prod R_i)$ such that $x \in \prod R_i$. Moreover, we define a topology on $\mathcal{Z}(R, \prod R_i)$ by using ultrafilters it’s called the ultrafilter topology and we demonstrate that this topology is finer than the Zariski topology. Based on the notion of the $F^{-}\lim$ one gives new rings can be expressible as direct union of artinian rings or zero dimensional rings with finite spectrum.

In the second section, we define the Zariski topology on $\mathcal{Z}(R, \prod R_i)$ and study their basic properties, The ultrafilter topology is studied in the third section . In the fourth section we define the $F^{-}\lim$-limit of a collection of zero-dimensional rings and we give his relationship with the direct union of subrings of $\prod R_i$.

2. Preliminaries

In this paper, we focus on intermediate zero-dimensional rings of a pair of rings. En particularly, we characterize these families by using special topologies. Now, let $I$ be a set and $\mathcal{F}$ be a collection of subsets of $I$, we define a ultrafilters topology on $I$ when $\mathcal{F}$ is a collection of clopen sets. We begin by giving some notations. Let $R$ be a subring of a ring $S$, we denote by $\mathcal{V}(S, R)$ and $\mathcal{A}(S, R)$, respectively, the set of von
Neumann regular and artinian subrings of $S$ that contain $R$. Thus, the following question arises naturally. Under what condition is a specified one of two sets nonempty?

Clearly $\mathcal{Z}(S, R)$ is nonempty if $\mathcal{V}(S, R)$ or $\mathcal{A}(S, R)$ is nonempty. On the other hand, suppose $R$ is a subring of the ring $S$, the following conditions are equivalent:

1. $\mathcal{Z}(R, S) \neq \emptyset$.
2. The power of the ideal $xS$ is idempotent for each $x$ in $R$.
3. For each finitely generated ideal $I$, the set $\{\text{Ann}_R(I)\}_{m=1}^{\infty}$ stabilizes for some $m \in \mathbb{N}$.

For proof see [3, Proposition 1] and [6, Theorem 1.6]

Now, assume that $\mathcal{Z}(R, S) \neq \emptyset$, is that $\mathcal{Z}(R, S)$ closed under arbitrary intersection?

**Theorem 2.1.** ([6, Theorem 2.1]) Suppose $\{R_a\}_{a \in A}$ is a nonempty family of zero-dimensional subrings of the ring $S$. Then $\bigcap_{a \in A} R_a$ is zero-dimensional subring of $S$.

**Remark 2.2.** Suppose $R$ is subring of the ring $S$. If $\mathcal{Z}(R, S) \neq \emptyset$, then Theorem 2.1 shows that $\mathcal{Z}(R, S) \neq \emptyset$ has a unique minimal element. We denote this element by $R_0$, and call it the minimal zero-dimensional extension of $R$ in $S$. Then for each $x$ in $R$, assume that $x^{m(x)}S$ is idempotent, and let $s_x$ be the pointwise inverse of $x^{m(x)}$ in $S$. By [6, Theorem 2.5] we have that $R^0 = R[s_x : x \in R]$.

We will work in at least ZFC, that is, Zermelo-Frankel set theory with the axiom of choice. We will in certain case use additional axioms. We recall that $\mathcal{F}$ is a filter on set $I$ if it is a subset of the power set of $I$ that satisfies the following conditions:

1. $\emptyset \not\in \mathcal{F}$ and $I \in \mathcal{F}$;
2. If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$;
3. If $A \in \mathcal{F}$ and $A \subset A' \subset I$, then $A' \in \mathcal{F}$.

A filter $\mathcal{F}$ on $I$ is called an ultrafilter if $\mathcal{F}$ is maximal with respect to being a filter, or equivalently, if whenever $A \subset I$, then either $A \in \mathcal{F}$ or $I \setminus A \in \mathcal{F}$. An ultrafilter $\mathcal{F}$ is called principal if there exists an element $i_0 \in I$ such that $\mathcal{F}$ consist of all subsets of $I$ that contain $i_0$. Other ultrafilters are called non-principal. We denote the collection of all ultrafilters on a set $I$ by $\hat{\mathcal{P}}(I)$.

**Definition 2.3.** Let $R$ be a subring of the ring $S$ and $S(R, S)$ be the set of all the rings between $R$ and $S$. Let $Y$ be a subset of $S(R, S)$ and $\mathcal{F}$ be an ultrafilter on $Y$. Set $Y_\mathcal{F} := \{a \in S : U_a \cap Y \in \mathcal{F}\}$. We call $Y_\mathcal{F}$ an ultrafilter limit point of $Y$ with $U_a := \{c \in S(R, S) : a \in c\}$.

**Lemma 2.4.** The set $Y_\mathcal{F}$ is a subring of $S$ contains $R$.

**Proof.** Let $x, y \in Y_\mathcal{F}$ then each of the sets $U_{xy}$ and $U_{x-y}$ contain $U_x \cap U_y \in \mathcal{F}$, from definition of an ultrafilter, we have $U_{xy}, U_{x-y} \in \mathcal{F}$, then $x - y, xy \in Y_\mathcal{F}$. Moreover, for each $a \in R$, $U_a = S(R, S) \in \mathcal{F}$, then $Y_\mathcal{U}$ contains $R$. Hence $Y_\mathcal{F}$ is a subring of $S$. □

A nonempty subset $Y$ of $S(R, S)$ is an ultrafilter closed if, for any ultrafilter $\mathcal{U}$ on $Y$, we have $Y_\mathcal{U} \in Y$. Then the ultrafilter closed sets of $S(R, S)$ are closed sets for a topology, called the ultrafilter topology (see [2]).

**Definition 2.5.** Let $X$ be a set and $\mathcal{F}$ be a nonempty collection of subsets of $X$. For each $Y \subseteq X$ and each ultrafilter $\mathcal{U}$ on $Y$, we define $Y_{(X, \mathcal{F})}(\mathcal{U}) := \{x \in X : \forall F \in \mathcal{F}, x \in F \iff F \cap Y \in \mathcal{U}\}$.

We will denote the set $Y_{(X, \mathcal{F})}(\mathcal{U})$ simply by $Y(\mathcal{U})$, when no confusion can arise.
Lemma 2.6. ([1, Lemma 2.5]) Let $X$ be a set, $\mathcal{F}$ be a given nonempty collection of subsets of $X$ and $Y \subseteq Z \subseteq X$. Let $\mathcal{U}$ be an ultrafilter on $Y$, $T \in \mathcal{U}$ and let $\mathcal{U}_f$ and $\mathcal{U}^\ast$, respectively, the ultrafilter defined by

$$\mathcal{U}_f := \{U \cap T : U \in \mathcal{U}\}, \quad \mathcal{U}^\ast := \{Z' \subseteq Z : Z' \cap Y \in \mathcal{U}\}.$$

Then we have

$$Y(\mathcal{U}) = T(\mathcal{U}_f) = Z(\mathcal{U}^\ast).$$

Remark 2.7. Let $X$ be a set and $\mathcal{F}$ be a nonempty collection of subsets of $X$ that is closed under complements. Then, for any subset $Y$ of $X$ and any ultrafilter $\mathcal{U}$ on $Y$, we have

$$Y(\mathcal{U}) = \bigcap \{F \in \mathcal{F} : F \cap Y \in \mathcal{U}\}.$$

Definition 2.8. Let $X$ be a set and $\mathcal{F}$ be a nonempty collection of subsets of $X$. Then, we say that a subset $Y$ of $X$ is $\mathcal{F}$–stable under ultrafilter if $Y(\mathcal{U}) \subseteq Y$, for each ultrafilter $\mathcal{U}$ on $Y$.

Let $X$ be a set and $\mathcal{F}$ be a nonempty collection of subsets of $X$. Then the family of all the subsets of $\mathcal{F}$–stable under ultrafilter is the collection of the closed sets form a topology on $X$. We will call it the $\mathcal{F}$–ultrafilter topology on $X$, and denote by $X^{\mathcal{F}–ultra}$ the set $X$ endowed with the $\mathcal{F}$–ultrafilter topology.

Proposition 2.9. ([1, Proposition 2.13]) Let $X$ be a set, $\mathcal{F}$ be a nonempty collection of subsets of $X$. Then, for each subspace $Y$ of $X^{\mathcal{F}–ultra}$, we have

$$Ad(Y) = \bigcup \{Y(\mathcal{U}) : \mathcal{U} \in \beta(Y)\}.$$

Remark 2.10. If $\mathcal{F} \subseteq C$ are collections of subsets of $X$, then the $C$–ultrafilter topology is finer than or equal to the $\mathcal{F}$–ultrafilter topology. In fact, for each subset $Y$ of $X$ and each ultrafilter $\mathcal{U}$ on $Y$, we have $Y(\mathcal{U}) \subseteq Y_{\mathcal{F}}(\mathcal{U}) \subseteq Y_C(\mathcal{U})$.

In the following example we give some relation between the $\mathcal{F}$–ultrafilter topology and ultrafilter topology for particular cases.

Example 2.11. 1. Let $A$ be a ring and $\mathcal{P}$ be the collection of all the principal open subsets of $X := Spec(A)$. Then, the $\mathcal{P}$–ultrafilter topology of $X$ is equal to the ultrafilter topology.

2. Let $K$ be a field, $A$ be a subring of $K$ and $C$ be the natural basis of open sets for the Zariski topology on the spectral space $Zar(K|A)$ of all the valuation domains of a field $K$ containing a fixed subring $A$ of $K$. Then, the $C$–ultrafilter topology is equal to the ultrafilter topology on $Zar(K|A)$.

Now, we are interested in the topological structure on $Z(R, \prod R_i)$. Let $R$ be a ring and $\{R_i\}_{i \in \mathcal{I}}$ be a family of zero-dimensional rings such that $R$ is imbeddable into $\prod R_i$. The set $Z(R, \prod R_i)$ endowed with a topological structure defined by taking, as a basis for the open sets, the subsets

$$B_S := \{T \in Z(R, \prod R_i) \setminus S \subseteq T\}.$$

For $S$ varying in $B_{fin}(\prod R_i)$. This topology is called the Zariski topology on $Z(R, \prod R_i)$.

Remark 2.12. If $S := \{x_1, x_2, ..., x_n\}$ with $x_j \in \prod R_i$ for each $j \in \{1, ..., n\}$, then

$$B_S := Z(R[x_1, x_2, ..., x_n], \prod R_i).$$

Therefore the collection of subsets $\mathcal{B} := \{Z(R[x], \prod R_i) : x \in \prod R_i\}$ is a base for the Zariski topology on $Z(R, \prod R_i)$.

As a simple consequence of the previous remark, if $Z(R, \prod R_i) \neq \emptyset$, then for each $x \in R$, there exists $m(x) \in \mathbb{N}$ such that $x^{m(x)} \prod R_i$ is idempotent, let $s_x$ be the pointwise inverse of $x^{m(x)}$ in $\prod R_i$, then $R^0 = R[s_x : x \in R]$, with $R^0$ is a unique minimal zero-dimensional in $Z(R, \prod R_i)$.

Moreover, the collection of subsets $\mathcal{B} := \{Z(R[t], \prod R_i) : x \in \prod R_i\}$ is a base for the Zariski topology, as $R[x_1 : x \in R] \subseteq Z(R, \prod R_i)$, then there is an element $t \in \prod R_i$ such that $R[s_x : x \in R] \subseteq Z(R[t], \prod R_i)$, then $R[t] \subseteq R[s_x : x \in R]$, and as $R[s_x : x \in R]$ is an unique minimal element, then $R[t] = R[s_x : x \in R]$ or $\dim(R[t]) \neq 0$. 

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3. The Ultrafilter Topology on \( \mathcal{Z}(R, \prod R_i) \)

It is worth reminding that not all rings admit a zero-dimensional subring. Particularly, infinite product of rings. Now, let \( R \) be a ring and \( \{R_i\}_{i \in I} \) a family of zero-dimensional rings such that \( R \) is imbeddable into \( \prod R_i \). The main goal is to study the behavior of \( \mathcal{Z}(R, \prod R_i) \) with respect to the ultrafilter topology and will compare it with the Zariski topology.

We start with following properties.

**Theorem 3.1.** Let \( R \) be a ring and \( \{R_i\}_{i \in I} \) a family of zero-dimensional rings such that \( R \) is imbeddable into \( \prod R_i \) with \( \mathcal{Z}(R, \prod R_i) \neq \emptyset \), if \( Y \) is a nonempty subset of \( \mathcal{Z}(R, \prod R_i) \) and \( \mathcal{U} \) is an ultrafilter on \( Y \), then:

1. \( R_{\mathcal{U}} := \{x \in \prod R_i / B_{[x]} \cap Y \in \mathcal{U}\} \in \mathcal{Z}(R, \prod R_i) \).
2. The collection of all subsets \( Y \) of \( \mathcal{Z}(R, \prod R_i) \) stable for ultrafilters (i.e.
   for each \( \mathcal{U} \in \beta(Y) \), \( R_{\mathcal{U}} \in Y \)) is the family of closed sets for a topology on \( \mathcal{Z}(R, \prod R_i) \) called the ultrafilter topology of \( \mathcal{Z}(R, \prod R_i) \).

**Proof.** 1. Let \( C := \{B_S : S \in B_{fin}(\prod R_i)\} \) be the natural basis of open sets of the Zariski topology of \( \mathcal{Z}(R, \prod R_i) \). If \( Y \) is a subset of \( \mathcal{Z}(R, \prod R_i) \) and \( \mathcal{U} \) is an ultrafilter on \( Y \), we have:

\[
x \in Y_C(\mathcal{U}) \iff \forall B_S \in C, x \in B_S \Rightarrow B_S \cap Y \in \mathcal{U}
\]

\[
\Leftrightarrow B_{[x]} \cap Y \in \mathcal{U}
\]

Then \( Y_C(\mathcal{U}) = \{R_{\mathcal{U}}\} \), as \( C \) is closed under complements. According to Remark 2.7, we have that \( R_{\mathcal{U}} = \bigcap \{F \in C : F \cap Y \in \mathcal{U}\} \), then \( R_{\mathcal{U}} \) is a zero-dimensional ring by Theorem 2.1. On the other hand, \( R \subseteq R_{\mathcal{U}} \) because \( B_{[x]} = \mathcal{Z}(R, \prod R_i) \) for each \( x \in R \).

2. Suppose that \( A, B \) are ultrafilter closed of \( \mathcal{Z}(R, \prod R_i) \) (i.e.
   closed set for an ultrafilter topology of \( \mathcal{Z}(R, \prod R_i) \)), and \( \mathcal{U} \) be an ultrafilter on \( Y = A \cup B \). Take into account the properties of ultrafilters, we can assume that \( A \in \mathcal{U} \). According to proof of (1) and Definition 2.8 , \( A \) is \( C \)-stable with \( C := \{B_S : S \in B_{fin}(\prod R_i)\} \), by Lemma 2.6, we have \( \{R_{\mathcal{U}}\} = Y_C(\mathcal{U}) = A_C(\mathcal{U}_A) \subseteq A \subseteq Y \), then \( Y \) is ultrafilter closed.

Now, let \( C \) be a collection of ultrafilter closed on \( \mathcal{Z}(R, \prod R_i) \). Let \( \mathcal{U} \) be an ultrafilter on \( X = \bigcap C := \{C / C \in C\} \). By Lemma 2.6, for each \( C \in C \), we have \( C(\mathcal{U}^C) = X(\mathcal{U}) \), and thus \( X(\mathcal{U}) \subseteq X \). Then \( X \) is ultrafilter closed. \( \square \)

**Theorem 3.2.** Let \( R \) be a ring and \( \{R_i\}_{i \in I} \) a family of zero-dimensional rings such that \( R \) is imbeddable into \( \prod R_i \) with \( \mathcal{Z}(R, \prod R_i) \neq \emptyset \).

1. The ultrafilter topology is finer than the Zariski topology on \( \mathcal{Z}(R, \prod R_i) \).
2. The basic open sets of the Zariski topology on \( \mathcal{Z}(R, \prod R_i) \) are both open and closed by the ultrafilter topology.

**Proof.** 1. Since \( C := \{B_S : S \in B_{fin}(\prod R_i)\} \) is a natural basis of open sets on the Zariski topology of \( \mathcal{Z}(R, \prod R_i) \), it is enough to prove that \( O := \mathcal{Z}(R, \prod R_i) \backslash B_S \) is stable for ultrafilter. Assume, by contradiction, that there exists an ultrafilter \( \mathcal{U} \) on \( O \) such that \( R_{\mathcal{U}} \neq O \). It follows that \( S \subseteq R_{\mathcal{U}} \), and then \( B_S \cap O \in \mathcal{U} \), for every \( x \in S \). Then \( B_S \cap O \in \mathcal{U} \), because \( S \) is finite. This is a contradiction by the definition of \( O \).

2. Direct consequence of the Theorem 3.1 and the Remark 2.12. \( \square \)

**Remark 3.3.** Let \( R \) be a ring and \( \{R_i\}_{i \in I} \) a family of zero-dimensional rings such that \( R \) is imbeddable into \( \prod R_i \). According to Remark 2.7 and [6] the set \( \mathcal{A}(R, \prod R_i) \) is not \( C \)-stable, where \( C := \{B_S : S \in B_{fin}(\prod R_i)\} \) is the basis of open sets of the Zariski topology on \( \mathcal{Z}(R, \prod R_i) \).
4. The \( F^- \) – lim of a Collection of Zero-Dimensional Rings

Let \( R \) be a subring of a ring \( S \). The first goal of this section is to define the \( F^- \) – lim of the set \( \mathcal{Z}(R, S) \). Then we give a characterization of \( \mathcal{Z}(R, S) \) by using the \( F^- \) – lim. Thereby, its relationship with ultrafilter limit and the direct limit of a family of rings.

**Definition 4.1.** Let \( A \) be a set, \( S(A) \) be the set of all subsets of \( A \) and let \( I \) be an infinite set. Let \( \{S_i\}_{i \in I} \) be a family of \( S(A) \), and let \( F^- \) be an ultrafilter on \( I \), then we define the \( F^- \) – \( \lim \) of \( \{S_i\}_{i \in I} \) by:

\[
F^- \lim_{i \in I} S_i := \{a \in A : \{i \in I : a \in S_i\} \in F^-\}.
\]

We note that the set \( F^- \lim_{i \in I} S_i \) is a subset of \( A \) and we have that:

\[
F^- \lim_{i \in I} S_i = \bigcup_{x \in F^-} \bigcap_{i \in X} S_i.
\]

**Proposition 4.2.** Let \( R \) is subring of the ring \( S \) such that \( \mathcal{Z}(R, S) \neq \emptyset \). Let \( \{R_i\}_{i \in I} \in \mathcal{Z}(R, S) \), and \( F^- \) is an ultrafilter on \( I \). Then the ring \( F^- \lim_{i \in I} R_i \) is a direct union of zero-dimensional subrings of \( S \).

**Proof.** By Definition 4.1, we have that \( F^- \lim_{i \in I} R_i = \bigcup_{i \in F^-} (\cap_{i \in A} R_i) \), and according to Theorem 2.1 \( \cap_{i \in A} R_i \) is a zero-dimensional ring for each \( A \) in \( F^- \). On the other hand, if \( A \in F^- \) and \( A \subset A' \subset I \), then \( A' \in F^- \). Then the union is direct.

**Proposition 4.3.** Let \( \{R_i\}_{i \in I} \) be a nonempty family of zero-dimensional subrings of a ring \( S \) and \( F^- \) be an ultrafilter on \( I \). Let \( S_X = \cap_{i \in X} R_i \) for each \( X \in F^- \), if for each \( X \) some \( R_i \), \( F^- \lim_{i \in I} R_i \) is an artinian reduced ring, \( F^- \lim_{i \in I} R_i \) is a direct union of artinian rings.

**Proof.** Let \( I \) be set and let \( F^- \) an ultrafilter on \( I \). For each nonempty family \( \{R_i\}_{i \in I} \) of zero-dimensional subrings of a ring \( S \), let \( S_X = \cap_{i \in X} R_i \) for each \( X \in F^- \). Assuming that some \( R_i \) in \( \{R_i\}_{i \in I} \) is an artinian reduced ring then \( S_X \) is a zero-dimensional sub-ring of artinian reduced ring (because \( S_X = \cap_{i \in X} R_i \subseteq R_i \)). That means, \( S_X \) is a zero-dimensional reduced ring with only finitely many idempotents, therefore is an artinian ring. Then similar proof of the Lemma 4.2 may show that \( F^- \lim_{i \in I} R_i \) is a direct union of artinian rings.

**Lemma 4.4.** Let \( X \subseteq \mathcal{Z}(R, \Pi R) \) and \( U \) be an ultrafilter on \( X \), then for each subset \( \{S_j : j \in J\} \subseteq X \) and each ultrafilter \( F^- \) on \( J \). We have \( F^- \lim_{j \in J} S_j \in \mathcal{Z}(R, \Pi R) \), and \( S_U = F^- \lim_{j \in J} S_j \) with \( S_U \) is the ultrafilter limit of \( X \).

**Proof.** Let \( \sigma : J \rightarrow X \) be a bijection, and let \( F^- = \{\sigma^{-1}(F) : F \in U\} \). Then \( F^- \) is an ultrafilter on \( J \). For each \( j \in J \), we put \( \sigma(j) = S_j \). Then,

\[
a \in S_U \iff B_{|\sigma(j)} \cap X \in U \iff \{j \in J : a \in S_j\} \in F^- \iff a \in F^- \lim_{j \in J} S_j.
\]

Therefore, \( S_U = F^- \lim_{j \in J} S_j \). On other hand, according to Theorem 3.1 \( F^- \lim_{j \in J} S_j \in \mathcal{Z}(R, \Pi R) \).

**Definition 4.5.** Let \( I \) be an arbitrary set and let \( F^- \) be any ultrafilter on \( I \). Let \( x_i \in X \) for \( i \in I \), we say that \( x \) is an ultralimit of \( x_i \) with respect to \( F^- \) if only if for every open set \( O \) of \( X \) with \( x \in X \cap \{i \in I : x_i \in O\} \in F^- \) when denoted by \( \lim_{F^-} x_i = x \).

We say that \( X \) is \( F^- \) – complete if and only if for all choices of \( x_i \in X \) there is a \( x \in X \) such that \( \lim_{F^-} x_i = x \).

We recall that \( X \) is ultracomplete if for every set \( I \), all sequences \( x_i \in X \) for \( i \in I \) and every ultrafilter \( F^- \) on \( I \), there is an \( x \in X \) with \( \lim_{F^-} x_i = x \), this is the equivalent of saying that \( X \) is \( F^- \) – complete for every \( I \) and every ultrafilter \( F^- \) on \( I \).

**Proposition 4.6.** Let \( \{S_i\}_{i \in I} \in S(A) \) for each \( i \in I \) and \( F^- \) be an ultrafilter on \( I \). We have \( \lim_{F^-} S_i = F^- \lim_{i \in I} S_i \).
Proof. Let $S_i \subseteq A$ for some $i \in I$, we claim that $\lim_{\mathcal{F}} S_i = S$ where $S = \mathcal{F} - \lim_{\mathcal{G}} S_i$. Suppose that $O$ is an open set in $S(A)$ such that $S \in O$. On the other hand let $B_{FG} = \{S \subseteq A : F \subseteq S \text{ and } S \cap G = \emptyset\}$ where $F,G$ are finite subsets of $A$. Each $B_{FG}$ is a basic open set of $S(A)$ such that $S \in B_{FG} \subseteq O$. But this asserts that $F \subseteq S$. Let $g \in G$, $g \notin S$. So $\{i : F \subseteq S_i\} \subseteq \mathcal{F}$ and also for each $g \in G$, $\{i : g \in S_i\} \notin \mathcal{F}$. It follows that for each $g \in G$, $\{i : g \notin S_i\} \in \mathcal{F}$. Thus $\{i : S_i \in B_{FG}\} \subseteq \mathcal{F}$. Thus the claim. \hfill $\square$

**Proposition 4.7.** ([8, Proposition 1.9]) $X$ is Hausdorff if and only if for all $I$, and all ultrafilters $\mathcal{F}$ on $I$ with all sequences $x_i \in X$ for $i \in I$ and if $\lim_{\mathcal{F}} x_i$ exists, then this limit is unique.

Before studying the relations between the notion of $\mathcal{F} - \lim$ and the direct union of the rings, give us a topological property of $\mathcal{Z}(R, \prod R_i)$ using the definition and the previous property.

**Proposition 4.8.** Let $R$ be a ring and $\{R_i\}_{i \in I}$ a family of zero-dimensional rings with $\mathcal{Z}(R, \prod R_i) \neq \emptyset$ then:

1. $\mathcal{Z}(R, \prod R_i)$ is Hausdorff.
2. $\mathcal{Z}(R, \prod R_i)$ is compact if and only if $R_{U} \neq \emptyset$ for each ultrafilter $U$ on $\mathcal{Z}(R, \prod R_i)$.

Proof. 1. Let $X \subseteq \mathcal{Z}(R, \prod R_i)$ and $U$ is an ultrafilter on $X$. Suppose that $J$ is an indexed set, $\mathcal{F}$ is an ultrafilter on $J$, and $S_i \in X$ by Lemma 4.4, $S_{\mathcal{F}} = \mathcal{F} - \lim_{\mathcal{G}} S_i$ with $S_{\mathcal{F}}$ is the ultrafilter limit of $X$, according to proof of Theorem 3.1 $\mathcal{F} - \lim_{\mathcal{G}} S_i$ is unique. On the other hand, by Proposition 4.6, Definition 4.5 and Proposition 4.7 we have that $\mathcal{Z}(R, \prod R_i)$ is a Hausdorff space.

2. Assume that $\mathcal{Z}(R, \prod R_i) \neq \emptyset$ then by [8, Theorem 1.6] we have that: $\mathcal{Z}(R, \prod R_i)$ is compact if and only if $\mathcal{Z}(R, \prod R_i)$ is ultracomplete. Moreover, by Definition 4.5, for every set $J$, all sequences $\{T_i\}_{i \in J} \in \mathcal{Z}(R, \prod R_i)$ for $j \in J$ there exists a $R_{T_i} = \mathcal{F} - \lim_{\mathcal{G}} T_j$ for every ultrafilter $\mathcal{F}$ on $J$, and it is equivalent to $R_{U} \neq \emptyset$ for each ultrafilter $U$ in $\mathcal{Z}(R, \prod R_i)$. \hfill $\square$

**Corollary 4.9.** Let $A$ be a nonempty subset of $\mathcal{A}(R, \prod R_i)$ and $U$ an ultrafilter in $A$, then $A_{U}$ the ultrafilter limit point of $A$ is a zero-dimensional ring.

Proof. We know that every artinian ring is a zero-dimensional ring, then $A$ is a nonempty subsets of $\mathcal{Z}(R, \prod R_i)$, by Theorem 3.1, we have that $A_{U}$ is a zero-dimensional ring. \hfill $\square$

By Remark 3.3 the ring $A_{U}$ is not necessary artinian.

**Definition 4.10.** Let $R$ be a ring and $S$ a ring containing $R$. An element $x \in S$ is said to be integral over $R$ if there exists an integer $n$ and elements $r_1, \ldots, r_n$ in $R$ such that

$$x^n + r_1x^{n-1} + \ldots + r_{n-1}x + r_n = 0.$$  

This equation is called an equation of integral dependence of $x$ over $R$ (of degree $n$). The set of all elements of $S$ that are integral over $R$ is called the integral closure of $R$ in $S$. If every element of $S$ is integral over $R$, we say that $S$ is integral over $R$.

**Lemma 4.11.** Let $R$ be a noetherian ring and $R_{a}$ be a family of zero-dimensional rings, and let $I \mathcal{Z}(R, \prod R_a)$ be the set of all the rings $C \subseteq \mathcal{Z}(R, \prod R_a)$ such that $C$ is integrally closed of $R$ in $\prod R_a$, then:

1. $\forall S_i \in I \mathcal{Z}(R, \prod R_a) \exists T_i \in \mathcal{A}(R, \prod R_a)$ such that $S_i = \lim_{\mathcal{G}} T_i$.

2. For each family $\{S_j : j \in I\} \subseteq I \mathcal{Z}(R, \prod R_a)$ we have that $\{K_i : i \in I\} \subseteq \mathcal{A}(R, \prod R_a)$ such that $\mathcal{F} - \lim_{\mathcal{G}} S_j = \lim_{\mathcal{G}} K_i$.  

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Proof. 1. Let $I\mathcal{Z}(R, \prod R_{\alpha})$ be the set of all the subrings $C \in \mathcal{Z}(R, \prod R_{\alpha})$ such that $C$ is integrally closed of $A$ in $\prod R_{\alpha}$ and let $S_j \in I\mathcal{Z}(A, \prod R_{\alpha})$, then by proof of [7, corollary 5.5] $S_j$ is a direct union of artinian subrings, from where $\exists T^j_i \in \mathcal{A}(R, \prod R_{\alpha})$ such that
\[ S_j = \lim_{i \in I} T^j_i. \]
with $I$ is a direct set.

2. According to Lemma 4.4 and Theorem 3.1 and [1, Proposition 3.6], we have that $\mathcal{F} - \lim_{j \in J} S_j$ is a zero-dimensional integral closed of $R$ in $\prod R_{\alpha}$, then $\mathcal{F} - \lim_{j \in J} S_j \in I\mathcal{Z}(R, \prod R_{\alpha})$ and by (1) we have that
\[ \mathcal{F} - \lim_{j \in J} S_j = \lim_{i \in I} K_i \]
with $\{K_i : i \in I\} \subseteq \mathcal{A}(A, \prod R_{\alpha})$. \qed

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