Entanglement between distant qubits in cyclic XX chains

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We evaluate the exact concurrence between any two spins in a cyclic XX chain of \( n \) spins placed in a uniform transverse magnetic field, both at zero and finite temperature, by means of the Jordan-Wigner transformation plus a number parity projected statistics. It is shown that while at \( T = 0 \) there is always entanglement between any two spins in a narrow field interval before the transition to the aligned state, at low but non-zero temperatures the entanglement remains non-zero for arbitrarily high fields, for any pair separation \( L \), although its magnitude decreases exponentially with the field. It is also demonstrated that all associated limit temperatures approach a constant non-zero value in this limit, which decreases as \( L^{-2} \) for \( L \ll n \) but exhibit special finite size effects for distant qubits \( (L \approx n/2) \). Related aspects such as the different behavior of even and odd antiferromagnetic chains, the existence of \( n \) ground state transitions and the thermodynamic limit \( n \to \infty \) are also discussed.

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I. INTRODUCTION

Quantum entanglement denotes those correlations with no classical analogue that can be exhibited by composite quantum systems and that constitute one of the most fundamental features of quantum mechanics. It is considered an essential resource in the field of quantum information [1], where it plays a key role in various quantum information processing tasks such as quantum teleportation [2] and quantum cryptography [3]. It is also playing an increasingly important role in condensed matter physics, providing a new perspective for understanding quantum phase transitions and collective phenomena in strongly correlated systems [4, 5, 6, 7].

In particular, there has been considerable interest in investigating entanglement in quantum spin chains with Heisenberg interactions [8, 9], since they provide a scalable qubit representation apt for quantum processing tasks [10, 11] which can be realized in diverse physical systems. Studies of the pairwise entanglement in the Ising and XY models [4, 5, 12] and in the isotropic Heisenberg model [13, 14, 15, 16] at zero and finite temperature and in a transverse uniform field, as well as in diverse XX, XY and XYZ models for two or a small number of qubits [17, 18, 19, 20, 21], have been made. An important result is that the entanglement range may remain finite at a quantum phase transition, limited for instance to first and second neighbors in the Ising model [4, 5], in contrast with the behavior of the correlation length, which diverges at these points. Global thermal entanglement has also been studied [22], showing that limit temperatures for pairwise entanglement are lower bounds to those limiting entanglement between global partitions. A fundamental result for finite systems is that there is always a finite limit temperature for entanglement, since any mixed state becomes completely separable if it is sufficiently close to the full random state [23, 24].

In this work we analyze the entanglement between any two spins in a cyclic chain with nearest neighbor XX coupling in a transverse magnetic field (control parameter) by means of an exact analytic treatment valid for any spin number \( n \) and pair separation \( L \), based on the Jordan-Wigner mapping and the use of number parity projected statistics for \( T > 0 \). Recent studies in XX chains have focused either chains with a small number of spins [17, 21, 22], where results were obtained through direct diagonalization, or open chains at zero temperature and field [15]. We will show that the XX model offers very interesting properties such as entanglement between any pair (full range) in a finite field interval just before the critical point at \( T = 0 \), which subsists for large fields at low but non-zero temperatures \( T < T_L \). Moreover, limit temperatures \( T_L \) approach a non-zero limit for large fields, for all separations \( L \). It also displays \( n \) ground state transitions at analytic field values, entailing a stepwise variation of the entanglement range suitable for its use as an entanglement switch. Let us mention that XX chains have also been employed for entanglement teleporation [22].

Section II describes the formalism for evaluating the exact concurrence between arbitrary sites both at zero and finite temperature. Section III describes the main physical results, including the ground state transitions and concurrence both in ferro- and antiferromagnetic systems, and a detail study of the limit temperatures for entanglement. Conclusions are drawn in IV.

II. FORMALISM

We consider a cyclic chain of \( n \) spins with nearest neighbor XX coupling. The Hamiltonian reads

\[
H = b \mathbf{S}^2 - v \sum_{j=1}^{n} (s^x_j s^x_{j+1} + s^y_j s^y_{j+1}) \quad (1a)
\]

\[
= b \mathbf{S}^2 - \frac{1}{2} v \sum_{j=1}^{n} (s^+_j s^-_{j+1} + s^-_j s^+_{j+1}) , \quad (1b)
\]

where \( s^x,y,z \) are the spin components (in units of \( \hbar \)) at site \( j \), \( s^\pm_j = s^x_j \pm is^y_j \), \( S^z = \sum_{j=1}^{n} s^z_j \) is the total spin along the
where \( \rho = \text{Tr} \exp[-\beta H] \) and \( T \) is the temperature (we set Boltzmann constant \( k = 1 \)). This entanglement is determined by the reduced pair density \( \rho_{ij} = \text{Tr}_n \rho_{ij} \rho(T) \) and can be measured through the concurrence

\[
C_{ij} = |2\lambda_M - \text{tr} R|_+, \quad R = \sqrt{\frac{1}{2} \rho_{ij} \rho_{ij}^\dagger \frac{1}{2}},
\]

where \( |u|_+ = (u + |u|)/2 \), \( \lambda_M \) denotes the largest eigenvalue of the hermitian matrix \( R \) and \( \rho_{ij} = 4s^y_i s^y_j \rho_{ij} s^y_i s^y_j \) is the spin flipped density (\( \text{tr} R \) is the fidelity between \( \rho_{ij} \) and \( \rho_{ij} \)). The entanglement of formation \( \frac{2}{\sqrt{n}} \) of the pair is \( E_{ij} = -\sum q_\pm \log q_\pm \), where \( q_\pm = (1 \pm \sqrt{1 - C_{ij}^2})/2 \) and is just an increasing function of \( C_{ij} \), with \( E_{ij} = C_{ij} = 1 \) (0) for a maximally entangled (separable) pair state.

Since \( H \) commutes with \( S^z \) and is invariant under translation and inversion, \( \rho_{ij} \) will commute with the pair spin component \( S^i_j = s^z_i + s^z_j \) and its elements will depend just on the separation \( |i-j| \). Hence, in the standard basis of \( S^i_j \) eigenstates, it must be of the form

\[
\rho_{ij} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & p_L & 0 & 0 \\
0 & 0 & p_L & 0 \\
0 & 0 & 0 & p_L
\end{pmatrix}, \quad L = |i-j|,
\]

where \( p_L + 2p_L + p_L = 1, p_L + p_L = 2 \) and

\[
p_L = \langle s^z_i + \frac{1}{2} (s^z_j + \frac{1}{2}) \rangle, \quad \alpha_L = \langle s^+_i s^-_j \rangle.
\]

Here \( \langle O \rangle \equiv \text{Tr} \rho(T) O \) denotes the thermal average of \( O \) and \( \langle s^i_j \rangle = (S^i_j)/n \) is the intensive magnetization. \( \rho_{ij} \) commutes as well with the total spin of the pair \( \langle S^i_j \rangle^2 = S^0 \cdot S^0 \), its eigenstates being the standard triplet states and singlet \( |\uparrow \downarrow \rangle, |\downarrow \uparrow \rangle \) and \( (|\uparrow \rangle \pm |\downarrow \rangle)/\sqrt{2} \), with eigenvalues \( p_L^2, p_L \pm \alpha_L \). The pair entanglement is obviously driven by the mixing coefficient \( \alpha_L \). The concurrence \( C_L \) becomes

\[
C_L = 2 \left[ |\alpha_L| - \sqrt{p_L^2 p_L^\dagger} \right] + ,
\]

so that \( \rho_{ij} \) is entangled if and only if \( |\alpha_L| > \sqrt{p_L^2 p_L^\dagger} \). This condition also follows from the PPT criterion.

### A. Exact energy levels

By means of the Jordan-Wigner transformation to fermion operators \( c^\dagger_j = s^+_j \exp[-i\pi \sum_{k=1}^{j-1} s^+_k s^-_k] \), we may rewrite \( H \) exactly as a bilinear form in \( c^\dagger_j, c_j \) for each value of the spin or fermion number parity

\[
P \equiv \exp[i\pi N], \quad N = \sum_{j=1}^N c^\dagger_j c_j = S^2 + n/2.
\]

The result for \( P = \sigma = \pm 1 \) is

\[
H_\sigma = \sum_{j=1}^n b(c^\dagger_j c_j - \frac{1}{2}) - \frac{1}{2} v(1 - \delta_{jn} \delta_{o1})(c^\dagger_j c_{j+1} + c^\dagger_{j+1} c_{j})
= \sum_{k \in K_o} \lambda_k (c^\dagger_k c_k - \frac{1}{2}), \quad \lambda_k = b - v \cos \omega_k,
\]

where the fermion operators \( c^\dagger_j \) are related to \( c^\dagger_j \) by a parity dependent discrete Fourier transform

\[
c^\dagger_j = \frac{1}{\sqrt{n}} \sum_{k \in K_o} e^{i \omega_k j} c^\dagger_k \quad \omega_k = 2\pi k/n ,
\]

For \( \sigma = 1 \) (integer) \( \lambda_k \) depend only on \( k \) and for even \( n \) the sign of \( v \) can be inverted by a local transformation \( s^i_j \rightarrow (-1)^j s^i_j \) (and that of \( b \) by \( s^y_j \rightarrow -s^y_j \)). The concurrence \( C_L \) will then exhibit the same properties, depending just on \( |b| \) and \( n \) even just on \( |v| \).

### B. Exact partition function and concurrence

The partition function \( Z \) of the system is to be evaluated in the full grand-canonical (GC) ensemble of the fermionic representation. However, due to the parity dependence of the latter, this requires a number parity projected statistics. \( Z \) can then be written as a sum of partition functions for each parity,

\[
Z = \text{Tr} \sum_{\sigma = \pm 1} \frac{1}{2}(1 + \sigma P) e^{-\beta H_{\sigma}} = \frac{1}{2} \sum_{\sigma = \pm 1} (Z^\sigma_0 + \sigma Z^\sigma_1),
\]

where \( \frac{1}{2}(1 + \sigma P) \) is the projector onto parity \( \sigma \) and

\[
Z^\sigma_0 = \text{Tr} P^\sigma e^{-\beta H_{\sigma}} = e^{\beta \lambda_k/2} \prod_{k \in K_o} (1 + (-1)^\nu e^{-\beta \lambda_k}),
\]

for \( \nu = 0, 1 \). The expectation value of an operator \( O \) can then be similarly expressed as

\[
\langle O \rangle = \frac{1}{2} Z^{-1} \sum_{\sigma = \pm 1} (Z^\sigma_0 (O)_0^\sigma + \sigma Z^\sigma_1 (O)_1^\sigma),
\]

\[
\langle O \rangle^\nu = (Z^\sigma_0)^{-1} \text{Tr} [P^\sigma e^{-\beta H_{\sigma}} O], \quad \nu = 0, 1.
\]
In the case of many-body fermion operators, the thermal version of Wick’s theorem cannot be applied in the final average (12), but it can be used for evaluating the partial averages (13) (as $P^\nu e^{-\beta H_\sigma} = e^{-\beta H_\sigma + i\nu\pi N}$ is still the exponential of a one-body operator), in terms of the contractions

$$g_L \equiv (c^\dagger L c)^\sigma \nu = \frac{1}{n} \sum_{k \in K_\sigma} \langle c^\dagger_k c^\dagger_k \rangle \nu \cos(L\omega_k),$$

(14)

where $\langle c^\dagger_k c^\dagger_k \rangle \nu = [1 + (-1)^\nu e^{\beta \lambda_k}]^{-1}$ (Eq. 13). As $s_i^x = c^\dagger_i c_i - \frac{1}{2}$, this leads to

$$\langle s_i^x \rangle \nu = -\frac{1}{2}, \langle (s_i^z + \frac{1}{2})(s_i^z + \frac{1}{2}) \rangle \nu = g_o^2 - g_L^2,$$

(15)

Using the identity $s_j^+ s_j^- = s_j^+ [\prod_{k=i+1}^{j-1} (s_k^+ s_k^- + s_k^- s_k^+)] s_j^-$ for $i < j$, with $s_j^+ s_{j+1}^- = c^\dagger_j c_{j+1}$, $s_j^+ s_{j+1}^- = c^\dagger_j c_{j+1}$, we also obtain

$$\langle s_j^+ s_j^- \rangle \nu = \frac{1}{2} \text{Det}(A_L),$$

(16)

where $A_L$ is the $L \times L$ matrix of elements

$$(A_L)_{ij} = 2g_{j+i-1} - \delta_{i,j-1},$$

(17)

i.e., $\text{Det}(A_1) = 2g_1$, $\text{Det}(A_2) = 4[g_0^2 - g_2(g_0 - \frac{1}{2})]$. All terms in (11) and (6) can then be exactly evaluated.

In the thermodynamic limit $n \to \infty$, and for finite $L \ll n$, we can ignore parity effects and replace sums over $k$ by integrals over $\omega \equiv \omega_k$. We can then directly employ Wick’s theorem in terms of the elements

$$g_L = \langle c^\dagger L c \rangle = \frac{1}{\pi} \int_0^\pi \frac{\cos(L\omega)}{1 + e^{-\beta b - \nu \cos(\omega)}} d\omega.$$

(18)

This leads to (Eq. 5)

$$p^L_+ = g_0^2 - g_L^2, \quad \alpha_L = \frac{1}{2} \text{Det}(A_L),$$

(19)

and $p^-_L = p^L_+ + 1 - 2g_0$, where $A_L$ is constructed with the elements (18). We then obtain the final expression

$$C_L = \left[\text{Det}(A_L) - 2 \sqrt{(g_0^2 - g_L^2)((1 - g_0)^2 - g_L^2)}\right].$$

(20)

Note that for $T \to 0$, Eq. (18) yields $g_L = 0$ for $b > |v|$ and $g_L = \sin(L\omega)/(L\pi)$ (with $g_0 = \omega/\pi$) for $|b| < |v|$, where $\cos(\omega) = b/|v|$.

When the ground state is non-degenerate, Eqs. (19) - (20) are also exactly valid for finite $n$ in the $T \to 0$ limit, using the exact contractions

$$g_L = \langle c^\dagger O \rangle_L \equiv \frac{1}{n} \sum_{k \in \text{occ.}} \cos(L\omega_k),$$

(21)

where $\langle O \rangle_L$ denotes ground state average and the sum runs over the occupied levels (see next section).

III. RESULTS

A. Ground state transitions and concurrence

Let us first describe the behavior in the $T \to 0$ limit. As $[H, N] = 0$, the ground state of $H$ can be characterized by the fermion number $N$, i.e., the total spin component $M = N - n/2$ in the spin representation. Since $\lambda_k$ in (7) becomes negative for $b < v \cos(\omega_k)$, the ground state will exhibit $n$ transitions $N \to N + 1$ as $b$ decreases from $|v|$ to $-|v|$, starting from $N = 0$ (the aligned state $M = -n/2$ for $b > |v|$ ($\lambda_k > 0 \forall k$) and ending with $N = n$ ($M = n/2$) for $b < -|v|$ ($\lambda_k < 0 \forall k$).

For $v > 0$, the first transition $0 \to 1$ occurs at

$$b_1 = v,$$

i.e., when the lowest negative parity level $\lambda_0 = b - v$ becomes negative. It represents for $b > v$ the entangled-separable border at $T = 0$. For $b$ just below $b_1$, the ground state is the one-fermion state $c_0^\dagger |0\rangle = \frac{1}{\sqrt{n}} \sum_j c_j^\dagger |0\rangle$, i.e., the $W$-state ($|[\downarrow \downarrow \ldots \downarrow] + |[\downarrow \downarrow \ldots \downarrow] + \ldots)/\sqrt{n}$, which exhibits a constant concurrence

$$C_L = 2/n, \quad (N = 1)$$

(22)

for any separation $L$ (Fig. 1). Hence, the transition at $b = b_1$ is from a fully separable state for $b > v$ (aligned state) to a state where any pair is equally entangled.

Due to the parity dependence of the energy levels, the next transition $1 \to 2$ (existing for $n > 4$) does not take place when the next $\lambda_k$ becomes negative ($b = v \cos(\omega_k)$ but rather when the lowest $\sigma = 1$ level crosses the previous $\sigma = -1$ level, i.e., when $2\lambda_{k+1/2} = \lambda_0$, which leads to $b_2 = v(2\cos(\pi/n) - 1)$. In general, for $v > 0$ the transitions $N - 1 \to N$ occur at $b_N = 2v\cos(\omega_k) - b_{N-1}$, with $k = (N-1)/2$, which leads to the critical fields

$$b_N = \frac{v(\cos(\omega_k) - \sin(\omega_k) \tan(\pi/2n))}{\cos(\pi/(2n))}, \quad 1 \leq N \leq n,$$

(23)

i.e., $b_N = v(\cos(\omega_k) - \sin(\omega_k) \tan(\pi/2n))$. Thus, $b_N < b_{N-1}$, with $b_{n-N+1} = -b_N$ and $b_N \approx v \cos(\omega_k)$ for large $n$.

Eq. (22) is valid for $b_2 < b < b_1$. The exact expression for $C_L$ at the other $N$-fermion ground states is given by Eq. (20) with the elements (21), which become

$$g_L = \frac{\sin(NL\pi/n)}{n\sin(L\pi/n)},$$

(24)

with $g_0 = N/n = \lim_{L \to 0} g_L$. For $N = 1, g_L = 1/n \forall L$ and Eq. (20) leads to Eq. (22).

For $N \geq 2$, $C_L$ will depend on the separation $L$, decreasing almost linearly with $L$ for not too small $n$, as seen in Fig. 1. A series expansion of (20) yields the initial trend $C_L \approx (2N/n)[1 - \pi L \sqrt{(N^2 - 1)/3}/n]$ for $NL \ll n$. The extent of pairwise entanglement decreases then rapidly as $N$ increases (inset in Fig. 1), the separation between most distant entangled qubits
The lowest negative parity level is now \( n = 0 \) but remains maximum if \( n > T \). For \( L \ll n \), \( C_L \approx 2/n \), in agreement with (22), whereas for most distant qubits \( (L = \lfloor n/2 \rfloor) \), \( C_L = 2 \sin(\pi/(2n))/n \approx \pi/n^2 \). Hence, for large \( L \) a significant odd-even difference in \( C_L \) arises if \( v < 0 \), even for large qubit number \( n \), due to the ground state degeneracy of the odd system.

The next transition for \( v < 0 \) and \( n \) odd occurs when \( \lambda_n/2 + \lambda_{n/2-1} = \lambda_{[n/2]} \), i.e., at \( b_{\nu}^* = |v|[1 + \cos(2\pi/n)] - b_1 \), and in general at \( b_{\nu}^- = |v| \sum_{k=N/2-1}^{N/2} \cos \omega_k - b_{N-1}^- \), which leads to the smaller critical fields

\[
b_{\nu}^- = b_N \cos(\pi/n), \quad 1 \leq N \leq n, \quad (26)
\]

being \( L_m = ([n + 1.79]/3.57) \) for \( n = 2 \) and, roughly, \( L_m = ([n + 4]/(2N)) \) for \( 2 < n \ll n/2 \).

Just first and second neighbors \((L = 1, 2)\) remain entangled for \(|b| < 0.65e \forall n \) and \(|b| < 0.82e \) \((N \gtrsim n/5)\) for \( n \rightarrow \infty \) whereas only adjacent pairs \((L = 1)\) remain entangled for \(|b| < 0.26e \forall n \neq 5 \) and \(|b| < 0.5e \) \((N \gtrsim n/3)\) for \( n \rightarrow \infty \) \((n = 5\) second neighbors are entangled \( \forall b > 0 \)).

The concurrence of adjacent pairs increases first linearly with \( N \) \((C_1 \approx 2N/n \) for \( n \ll n \)) and becomes maximum for \( N = n/2 \) \((n \geq 4)\), where \( g_1 = 1/(n \sin(\pi/n)) \approx 1/\pi \) for large \( n \) and \( C_1 = 2g_1(1 + g_1) - 1/2 \approx 0.339 \).

For odd \( n \), results for \( v < 0 \) must be separately examined. The lowest negative parity level is now \( \lambda_{\pm[n/2]} = b - |v| \cos(\pi/n) \), so that the first transition occurs at

\[
b_1^* = |v| \cos(\pi/n), \quad (v < 0, n \text{ odd}),
\]

with the ground state two-fold degenerate after the transition \((k = \pm [n/2])\). The concurrence of the mixture \[ \frac{1}{2} \sum_{k=\pm[n/2]} |k\rangle \langle k| \] of the two degenerate ground states

\[
|k\rangle = \frac{1}{\sqrt{\rho}} \sum_j e^{-i\omega_j k} c_j^0 \] (the \( T \to 0 \) limit of \( \rho(T) \)) is

\[
C_L^- = 2 \cos(L\pi/n), \quad (N = 1) \quad (25)
\]

which is again non-zero \( \forall L \) \((n \text{ is odd})\) although it now decays as \( L \) increases \((\text{bottom panel in Fig. 1})\).

For \( L \ll n \), \( C_L \approx 2/n \), in agreement with (22), whereas for most distant qubits \((L = \lfloor n/2 \rfloor) \), \( C_L = 2 \sin(\pi/(2n))/n \approx \pi/n^2 \). Hence, for large \( L \) a significant odd-even difference in \( C_L \) arises if \( v < 0 \), even for large qubit number \( n \), due to the ground state degeneracy of the odd system.

The next transition for \( v < 0 \) and \( n \) odd occurs when \( \lambda_n/2 + \lambda_{n/2-1} = \lambda_{[n/2]} \), i.e., at \( b_{\nu}^* = |v|[1 + \cos(2\pi/n)] - b_1 \), and in general at \( b_{\nu}^- = |v| \sum_{k=N/2-1}^{N/2} \cos \omega_k - b_{N-1}^- \), which leads to the smaller critical fields

\[
b_{\nu}^- = b_N \cos(\pi/n), \quad 1 \leq N \leq n, \quad (26)
\]

where \( b_N \) are the fields (23). Ground states remain two-fold degenerate \( \forall N \neq 0, n \), since there is just one fermion in the highest occupied level \((k = \pm (n - N)/2))\).

Eq. (25) holds for \( b_2^* < b < b_1^* \). The expression of \( C_L^- \) for general \( N \) in the \( T \to 0 \) limit can be similarly obtained by using Eqs. (20)-(21) for each of the degenerate ground states and taking then the average. The final result is

\[
C_L^- = \left[ \text{Re}[\text{Det}(A_L^-)] \right] - 2 \sqrt{(g_{b2}^2 - g_b^2)(1 - g_b^2)^2} \quad (27)
\]

where \( A_L^- \) is constructed with the elements

\[
g_b^- = g_b e^{iL\pi/n}, \quad (28)
\]

with \( g_b \) given again by (21). For \( N = 1 \) (27) leads to Eq. (25). The behavior of \( C_L^- \) for \( N \geq 2 \) is similar to that of \( C_L \) (Eq. (20)), although it is smaller than \( C_L \) (due to the ground state degeneracy) and its decay with \( L \) is less linear (see bottom panel). For instance, for \( L = 1 \) \( \text{Re}(\text{Det}(A_L^-)) = \text{Det}(A_L) \cos(\pi/n) \), whence \( C_L^- \approx C_1 \), with \( C_L^- \to C_1 \) for large \( n \).

Let us finally mention that for \( n \to \infty \) and \( \pi N/n \to \omega \), with \( L \) finite, Eqs. (24)-(28) coincide both exactly with the limit of (18) for \( T \to 0 \), where \( b_N \to v \cos \omega \).

**B. Results for finite temperatures**

Illustrative results for \( n = 14 - 15 \) and the thermodynamic limit \( n \to \infty \) are depicted in Figs. 23-24. For \( T \) close to 0, the concurrence exhibits a stepwise behavior in finite chains, in agreement with the \( T = 0 \) transitions previously described, presenting dips at the critical fields \((23-26)\) due to the ground state degeneracy at these points (level crossing). It is also verified that \( C_L \) is smaller in odd antiferromagnetic chains, particularly for large \( L \) close to \( n/2 \), in agreement with Eqs. (25)-(27).

While at \( T = 0 \) there is no entanglement in the ground state for \( b > b_1 \), a fundamental result for \( T > 0 \) is that...
\( \rho(T) \) remains entangled for all fields \( b > b_1 \) if \( T \) is sufficiently low, leading to a small but non-zero concurrence \( C_L \) for any separation \( L \) if \( 0 < T < T_L(b) \). Moreover, the limit temperature \( T_L(b) \) approaches a non-zero limit \( T_L \) for \( b \to \infty \) \( \forall L \), being practically constant for \( b > |v| \) (and \( b \) if \( L = 1 \)). This behavior applies for any \( n \), including \( n \to \infty \) as well as the special case \( v < 0 \) and \( n \) odd, as seen in the lower panels of Figs. 2-3.

In order to rigorously prove the previous behavior, we note that for \( b > |v| \gg kT (e^{-\beta(b-|v|)} \ll 1) \), we may keep just zero, one and two fermion states in \( \exp[-\beta H] \), i.e.,

\[
Z \approx e^{\beta bn/2} \left[ 1 + \sum_{k \in K_-} e^{-\beta \lambda_k} + \sum_{k < k' \in K_+} e^{-\beta(\lambda_k + \lambda_{k'})} \right]
\]

and similarly for \( \rho(T) \). This leads to \( \alpha_L = e^{-\beta b} I_L^+ (\beta v) \) and \( \rho_{hh}^+ \approx e^{-2\beta b} I_0^{+2} (\beta v) - I_L^{+2} (\beta v) \) up to lowest order in \( e^{-\beta b} \), where

\[
I_L^+ (\beta v) = \frac{1}{n} \sum_{k \in K_+} e^{\beta v \cos \omega_k \cos (L \omega_k)}. \tag{29}
\]

Hence, up to first order in \( e^{-\beta b} \) we obtain

\[
C_L \approx 2 e^{-\beta b} \left[ I_L (\beta v) - \sqrt{I_0^{+2} (\beta v) - I_L^{+2} (\beta v)} \right]. \tag{30}
\]

Thus, as \( b \) increases the concurrence decreases exponentially with the field when it is positive, but the limit temperature \( T_L(b) \) becomes constant, as the entanglement condition \( C_L > 0 \) becomes \( b \)-independent, i.e.,

\[
I_L (\beta v) > \sqrt{I_0^{+2} (\beta v) - I_L^{+2} (\beta v)}. \tag{31}
\]

Eq. (31) is always satisfied for sufficiently small but positive \( T \), for any distance \( L \), ensuring a non-zero concurrence and limit temperature \( T_L(b) \) for any \( b > |v| \). This is easy to prove for \( v > 0 \), where for \( T \to 0^+ \), \( I_L (\beta v) \approx e^{\beta v}/n \geq I_0^{+2} (\beta v) \approx 2e^{2\beta v \cos (\pi/n)}/n \). It also holds for \( v < 0 \) (not odd), since in this case, for \( T \to 0^+ \), \( I_L (\beta v) \approx e^{\beta v \cos (\pi/n) \cos (L \pi/n)}/n \), whereas the r.h.s of (31) becomes \( \approx \sqrt{2e^{2\beta v \cos (\pi/n) \sin (L \pi/n)}} \leq \sqrt{2} I_L (\beta v) \).

In the thermodynamic limit \( n \to \infty \), and for finite \( L \ll n \), we may neglect parity effects and just replace

\[
I_L (\beta v) \to 1 \pi \int_0^\pi e^{\beta v \cos \omega \cos (L \omega)} d\omega = I_L (\beta v), \tag{32}
\]

where \( I_L(x) \) is the modified Bessel function of the first kind \( (I_L(x) \approx e^x [1 + (1 - 4L^2)/8x]/\sqrt{2\pi} x \) for \( x \to \infty \), with \( I_L(-x) = (-)^L I_L(x) \)). Eq. (30) becomes then identical with the result obtained from Eqs. (18)-(20). Eq. (31) becomes then

\[
\sqrt{2} I_L (\beta v) \geq I_0 (\beta v), \tag{33}
\]

which is again always satisfied for sufficiently low \( T \) \( \forall L \). The limit temperatures \( T_L = T_L(\infty) \) are then determined for \( n \to \infty \) by the equation \( \sqrt{2} I_L (\beta v) = I_0 (\beta v) \), which leads to \( T_L \approx 0.486|v| \), \( T_2 \approx 0.16|v| \)

\[
T_L \approx |v| \ln 2/\sqrt{\lambda}, \tag{34}
\]

FIG. 2: (Color online) Concurrence (top) and limit temperatures for entanglement \( T_L(b) \) (bottom) for pairs \( i, i + L \) as a function of the magnetic field, for \( n = 14 \) qubits and in the thermodynamic limit \( n \to \infty \). The concurrence is plotted close to the \( T \to 0 \) limit (\( T = 0.005 \pi \)). All limit temperatures remain constant for \( b/v \to \infty \) (see text). Results for even \( n \) lie mostly above those for \( n \to \infty \), particularly for large \( L \) (where they saturate) and are independent of the sign of \( v \).
for large $L$ (as $I_L(x)/I_0(x) \approx e^{-L^2/(2x^2)}$ for $x \gtrsim L$). Thus, $T_L(b)$ decreases as the inverse square of the pair distance $L$ for large $b$. The maximum value attained by $C_L$ for $b > |v|$ becomes nevertheless small and decays exponentially with both $b$ and $L^2$ ($C_L \approx e^{-b/|v|}L^2/|v|}$ for $T = |v|t/L^2 < T_L$, with $f(t) = \sqrt{2t/\pi}e^{-t^2/(2t) - \sqrt{1 - 1/e}}$). Eq. (31) also indicates roughly the value of $T_L(b)$ at the critical region $b \approx |v|$, since it is almost constant for $b \gtrsim |v|$ (Figs. 2,3).

On the other hand, for large $L \approx n/2$ the projected expression (30) is required even for large $n$. For instance, for even $n$ and $L = n/2$, $\cos(L\omega_k) = 0 ((-1)^k$ for $k$ half-integer (integer). Hence, in this case $I_{n/2}^{+}(3v) = 0$, while for $v > 0$ and large $n$,

$$I_{n/2}^{+}(\beta v) \approx e^{\beta v} \theta_2(4)(e^{-2\beta v n^2/n^2})/n ,$$

after replacing $\omega_k \approx 1 - \omega_k^2/2$ ($\theta_2(u) \equiv 2\sum_{k=1}^{\infty} u^{k^2}$, $\theta_4(u) \equiv 1 + 2\sum_{k=1}^{\infty} (-1)^k u^{k^2}$, denote the Elliptic Theta functions). These results also hold approximately for large odd $n$ and $L = [n/2]$. Eq. (31) becomes then $I_{n/2}^{+}(\beta v) > I_{0}^{+}(\beta v)$, and since $\theta_2(u) = \theta_4(u)$ for $u = e^{-\pi}$, it leads to the limit temperature

$$T_{[n/2]} \approx 2\pi v/n^2 ,$$

for the most distant pairs and $v > 0$. It is greater than Eq. (34) for $L = n/2$ by a factor $\pi/(2\ln 2) \approx 2.27$.

Eq. (30) does not hold for $v < 0$ if $n$ is odd. In this case we may directly employ the asymptotic expression of Eq. (31) for $T \rightarrow 0^+$, which for large $n$ yields

$$T_{[n/2]} \approx \frac{|v|\pi^2}{2n^2 \ln[2\sqrt{2n/\pi}]} , \quad (v < 0, \; n \text{ odd}) .$$

Thus, in this case there is an additional logarithmic factor in the denominator, which makes $T_{[n/2]}$ lower than Eq. (34) and also Eq. (34) for $L = n/2$, originating an odd-even staggering of $T_{n/2}$ if $v < 0$.

The behavior of $T_L$ is depicted in Fig. 1. It is seen that for $L \gtrsim n/4$, it deviates from the $1/L^2$ law given by Eq. (31), approaching the values given by Eqs. (36) or (37) for $L \approx n/2$. Fig. 2 depicts the typical thermal behavior of $C_L$ for $v > 0$ near the transition at $b = b_1$. For $b < b_1$ there is entanglement between all pairs if $T$ is lower than a certain temperature, given approximately by Eq. (34). It also shows the reentry of $C_L$ for $T > 0$ for $b > v$, which is quite prominent for low $L$.

Finally Fig. 3 depicts the typical behavior with the qubit number $n$ of the concurrence and limit temperature. We have chosen a separation $L = 2$. Although the thermodynamic limit is on the average rapidly approached, the stepwise behavior of $C_L$ at low $T \approx 0.01v$ remains visible even for $n = 40$, and deviations in the limit temperature can be significant at the onset. They are as well significant for small $n \lesssim 10$.

An interesting feature is that the slope of $T_L(b)$ can be negative in this region, a fact already seen in Fig. 2 for $n = 14$, and visible here for $n = 8$ and $n = 20$. This occurs when the value of the onset field $b_c$ for finite $n$ (which for $L = 2$ corresponds to $b_2, b_4, b_r$ and $b_{14}$ for $n = 6, 8, 20$ and 40) lies above the value for $n \rightarrow \infty$, as occurs.
for \( n = 8, 20 \). In these cases there is a small field interval below \( b_c \), where entanglement between second neighbors exists only above a threshold temperature \( T_l(b) > 0 \), up to the higher limit temperature \( T_u(b) \).

A final comment is that we have checked all expressions by comparison with calculations for low \( n \leq 10 \) based on the direct diagonalization of \( H \). In particular, for \( n = 2 \), the entanglement condition \([51]\) becomes exact \( \forall b \), as in this case there are just one and two-fermion states, reducing to \( \sinh(\beta v) > 1 \) and leading to the known limit temperature \( T_1 = v/\ln(1 + \sqrt{2}) \) \([10]\).

**IV. CONCLUSIONS**

We have provided an exact analytic treatment of the entanglement between arbitrary pairs in cyclic \( XX \) chains in the presence of a transverse magnetic field, valid both at zero and finite temperatures and for *any* qubit number \( n \). We have shown that in spite of its simplicity, this system exhibits very interesting features such as a discrete set of \( [n/2] \) different entangled ground states at \( T = 0 \) (and \( b > 0 \)), which can be easily selected by adjusting the magnetic field across the critical values \([20] \) or \([26] \), and which develop increasing entanglement ranges, reaching always full range (all pairs entangled) in an interval \( b_2 < b < b_1 \), even for odd antiferromagnetic chains.

Moreover, while at \( T = 0 \) the ground state is fully separable for \( b > b_1 \), we have rigorously proved that for \( T > 0 \) there is a small but non-zero entanglement between *any* pair for all fields \( b > b_1 \) if \( T \) is sufficiently low, which decays exponentially with the field and with the square of the separation \( L \). Limit temperatures \( T_l \) are roughly independent of \( b \) for \( b > v \) and decay as \( L^{-2} \) for \( L \approx n/4 \), but tend to saturate at \( T_l[n/2] \) (Eq. [30]) for most distant pairs \( L \approx n/2 \) if \( n \) is even or \( v > 0 \). We have also shown that due to degeneracy of the ground state, pairwise entanglement in odd antiferromagnetic chains is weaker, particularly for distant pairs, where odd-even effects in the concurrence and \( T_l \) persist for all \( n \).

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