This is the accepted manuscript made available via CHORUS. The article has been published as:

Algorithmic universality in F-theory compactifications
James Halverson, Cody Long, and Benjamin Sung
Phys. Rev. D 96, 126006 — Published 11 December 2017
DOI: 10.1103/PhysRevD.96.126006
On Algorithmic Universality in F-theory Compactifications

James Halverson, Cody Long, and Benjamin Sung

Department of Physics, Northeastern University
Boston, MA 02115-5000 USA

We study universality of geometric gauge sectors in the string landscape in the context of F-theory compactifications. A finite time construction algorithm is presented for $4d \times 2.96 \times 10^{753}$ F-theory geometries that are connected by a network of topological transitions in a connected moduli space. High probability geometric assumptions uncover universal structures in the ensemble without explicitly constructing it. For example, non-Higgsable clusters of seven-branes with intricate gauge sectors occur with probability above $1 - 1.01 \times 10^{-753}$, and the geometric gauge group rank is above 160 with probability 0.999995. In the latter case there are at least 10 $E_8$ factors, the structure of which fixes the gauge groups on certain nearby seven-branes. Visible sectors may arise from $E_6$ or $SU(3)$ seven-branes, which occur in certain random samples with probability $= 1/200$.

I. Introduction. String theory is a consistent theory of quantum gravity that naturally gives rise to interesting gauge and cosmological sectors. As such, it is a promising candidate for a unified theory. However, there is a vast landscape of four-dimensional metastable vacua that may realize different physics, making predictions difficult.

A possible way forward, as in many areas of physics, is to demonstrate universality in large ensembles. For string vacua, such as the oft-quoted $O(10^{500})$ type IIb flux vacua [1], studying universality via explicit construction is complex and impractical [2]. However, it may be possible to derive universality from a precise construction algorithm, rather than from the constructed ensemble. We refer to this as algorithmic universality, and find it a promising way forward in the string landscape.

We present such an algorithm in the context of $4d$ F-theory [3] compactifications. The ensemble is a collection of $4/3 \times 2.96 \times 10^{753}$ six-manifolds, perhaps the largest set of string geometries to date, that serve as the extra spatial dimensions. Their topological structure determines the $4d$ gauge group that arises geometrically from configurations of seven-branes that form a network of so-called non-Higgsable clusters (NHC) [4]. We establish that non-Higgsable clusters arise with probability above $1 - 1.01 \times 10^{-753}$ in this ensemble, and demonstrate that a rich minimal gauge structure arises with high probability. We also present results from random sampling that are potentially relevant for visible sectors.

A number of recent results suggest that NHC are important in the $4d$ F-theory landscape. They exist for generic vacuum expectation values of scalar fields (complex structure moduli), and therefore gauge symmetry does not require stabilization on subloci in moduli space [5], which can have high codimension [6]. Standard model structures may arise naturally [5], strong coupling is generic [7], and 4d NHC may exhibit features [8] (such as loops and branches) not present in 6d. NHC arise in the geometry with the largest number of flux vacua [9], and universally in known ensembles [10] closely related to ours. 6d NHC have been studied extensively [4, 11].

In Sec. II we review non-Higgsable clusters. In Sec. III we present our ensemble. In Sec. IV we exhibit universality. In Sec. V we discuss our results.

II. Seven-Branes and Non-Higgsable Clusters. A $4d$ F-theory geometry is a Calabi-Yau elliptic fibration $X$ over six extra spatial dimensions described by a complex threefold base space $B$ defined by the equation

$$y^2 = x^3 + fx + g$$

where $f$ and $g$ are polynomials in the coordinates of $B$; technically, $f \in \Gamma(O(-4K_B))$, $g \in \Gamma(O(-6K_B))$, with $K_B$ the canonical class. Seven-branes are localized on the discriminant locus $\Delta = 4f^3 + 27g^2 = 0 \subset B$.

Upon compactification the gauge group structure of seven-branes gives rise to four-dimensional gauge sectors. It is controlled by $f$ and $g$, and for a typical $B$ the most general $f, g$ take the form $f = \tilde{f} \prod_i x_i^{l_i}$, $g = \tilde{g} \prod_i x_i^{m_i}$, so

$$\Delta = \tilde{\Delta} \prod_i x_i^{\min(l_i, 2m_i)} = \tilde{\Delta} \prod_i x_i^{n_i},$$

and therefore $f$, $g$, and $\Delta$ vanish along $x_i = 0$ to $mult_{x_i} = (f, g, \Delta) = (l_i, m_i, n_i)$. This seven-brane carries a gauge group $G_i$ given in Table I according to the Kodaira classification. In some cases further geometric data is necessary to uniquely specify $G_i$ [see e.g. [10] for conditions] but this data always exists for fixed $B$. For generic $f$ and $g$ a seven-brane on $x_i = 0$ requires $(l_i, m_i, n_i) \geq (1, 1)$.

Such a seven-brane is called a geometrically non-Higgsable seven-brane (NHC) because it carries a gauge group that cannot be removed by deforming $f$ or $g$. A NHC may have geometric gauge group

$$G \in \{E_8, E_7, E_6, F_4, SO(8), SO(7), G_2, SU(3), SU(2)\},$$

which could be broken by fluxes. We assume fluxes can be turned on in a large fraction of our geometries. A typical base $B$, as we will show in the strongest generality to date, has many non-Higgsable seven-branes that often intersect in pairs, giving rise to jointly charged matter. This is a geometrically non-Higgsable cluster (NHC). For brevity, we henceforth drop geometric and geometrically.

III. Large Landscapes of Geometries from Trees.
TABLE I. Kodaira fiber $F_i$, singularity, and gauge group $G_i$ on the seven-brane at $x_i = 0$ for given $l_i$, $m_i$, and $n_i$.

| $F_i$ | $I_0$ | $n_0$ | $m_0$ | $n_1$ | $m_1$ | $n_2$ | Sing | $G_i$ |
|-------|-------|-------|-------|-------|-------|-------|------|------|
| $I_n$ | $0$   | $0$   | $0$   | $0$   | $0$   | $0$   | none | none |
| $II$  | $1$   | $1$   | $2$   | $2$   | $3$   | $0$   | none | none |
| $III$ | $1$   | $2$   | $3$   | $4$   | $4$   | $4$   | $SU(2)$ or $SU(2)$ | $SU(3)$ |
| $IV$  | $2$   | $2$   | $3$   | $4$   | $5$   | $5$   | $SU(3)$ or $SO(7)$ or $G_2$ | $SO(8)$ |
| $IV^*$| $3$   | $5$   | $5$   | $5$   | $6$   | $6$   | $SO(2n - 4)$ or $SO(2n - 5)$ | $E_7$ |
| $III^*$| $3$  | $5$   | $5$   | $5$   | $6$   | $6$   | $E_6$ | $E_6$ |

We now introduce our construction, which utilizes building blocks in toric varieties that we call trees to systematically build up F-theory geometries. After describing the geometric setup and defining terms that simplify the discussion, we will present a criterion, classify all trees satisfying it, and build the F-theory geometries.

Our construction begins with a smooth weak-Fano toric threefold $B_i$, and then builds structure on top of it. Each geometry $B_i$ is determined by a fine regular star triangulation (FRST) of one of the $4319$ 3d reflexive polytopes [12]; there are an estimated $O(10^{15})$ such geometries [6]. The 2d faces of the 3d polytope are known as facets, and a triangulated polytope will have triangulated facets. Such $B_i$ do not support NHC; see [6].

Consider such a $B_i$ determined by an FRST of a 3d reflexive polytope $\Delta^e$, a triangulated facet $F$ in $\Delta^e$, and an edge between two points $v_1$ and $v_2$ in $F$ with associated homogeneous coordinates $x_1$ and $x_2$. Since $v_{1,2}$ are connected by an edge, $x_1 = x_2 = 0$ defines a Riemann surface (algebraic curve) in $B_1$, which can be “blown up” using a new ray $v_e = v_1 + v_2$ and subdividing cones using standard toric techniques. This is a topological transition that introduces a new (“exceptional”) divisor $e = 0$ in $B$, where $e$ is the coordinate associated to $v_e$. This process can be iterated, for example blowing up along $e = x_1 = 0$, which would add a new ray $v_e = v_1 + v_2$.

After a number of iterations the associated toric variety will have a collection of exceptional divisors with associated rays $v_{e_i} = a_i v_1 + b_i v_2$, which will appear to have formed a tree above the ground that connects $v_1$ and $v_2$ in $F$. Each $v_e$ is a leaf with height $h_e = a_i b_i$, and we will refer to trees built on edges within $F$ as edge trees. The height of a tree is the height of its highest leaf. As an example, $\{v_1 + v_2, v_1 + v_2, v_1 + v_2\}$ appears with a face on $F$, with vertices $v_1, v_2, v_3$ associated to $x_1, x_2, x_3$. Adding $v_e = v_1 + v_2 + v_3$ and subdividing appropriately blows up the point $x_1 = x_2 = x_3 = 0$ and produces a new toric variety. Again such blowups can be iterated. This process builds a collection of leaves $v_{e_i} = a_i v_1 + b_i v_2 + c_i v_3$ with $a_i, b_i, c_i \geq 0$ of height $h_{e_i} = a_i b_i + c_i$ that comprise a face tree. Face trees are built above the interior of the face due to the strict inequality in the definition. Note if one leaf coefficient was zero the associated leaf would be above an edge of the face, not above the face interior.

Geometries can be systematically constructed by adding a face tree to each face in each triangulated facet of $\Delta^e$, and then an edge tree to each edge. The associated smooth toric threefold $B$ has a collection of rays $v$, each of which can be written $v = a v_1 + b v_2 + c v_3$ with $v_i$ 3d cone vertices in $B_i$. If $(a, b, c) = (1, 0, 0)$ or some permutation thereof, $v \in \Delta^e$ and this height $h_v = 1$ “leaf” is more appropriately a root, since it is on the ground.

A natural question in systematically building up geometries is whether there is a maximal tree height. For a toric variety $B$ to be an allowed F-theory base it must not have any so-called $(4, 6)$ divisors (see Appendix), which we ensure by a simple height criterion proven in Prop. 1:

If $h_v \leq 6$ for all leaves $v \in B$, then there are no $(4, 6)$ divisors.

This condition is simple and sufficient, but not necessary, for the absence of $(4, 6)$ divisors. Nevertheless, it will allow us to build a large class of geometries.

The task is now clear: we must systematically build all topologically distinct edge trees and face trees of height $\leq 6$. Since the combinatorics are daunting, let us exemplify the problem for $h \leq 3$ trees. Viewing the facet head on, an edge in $F$ appears as

$$v_1 \quad v_2$$

$$1 \quad 1$$

with the vertices and their heights labeled. Adding $v_1 + v_2$ subdivides the edge, and further subdivision gives

$$1 \quad 2 \quad 3$$

$$\rightarrow$$

$$1 \quad 3 \quad 2 \quad 1$$ where we have dropped the vertex labels and kept the heights. The trees emerge out of the page, but visualization is made easier by projecting on to the edge; the right-most tree is the one previously presented vertically. There are five edge trees with height $\leq 3$. Similarly,
shows that there are 2 face trees of height ≤ 3. Here we have denoted the new edges by green lines since they do not sit in the facet. With our definitions, edge trees are built above an edge in the facet, whereas higher leaves in face trees may be built on new edges that do not sit in the facet. For example, a height 4 leaf could be added on any of the green lines above. A (tedious) straightforward calculation shows that the number of edge or face trees with \( h \leq N \) grows rapidly with \( N \), as in Table II.

| \( h \) | Probability |
|------|-------------|
| 3    | .99999998   |
| 4    | .999995     |
| 5    | .999997     |
| 6    | .999899     |

### TABLE III. The probability that a face tree with \( h \leq 6 \) has a leaf \( v \) with a given height \( h_v \).

Having classified the number of \( h \leq 6 \) face trees and edge trees, we now give a lower bound for the number of F-theory geometries that arise from building trees on an FRST of \( \Delta^6 \), denoted \( T(\Delta^6) \). We construct an ensemble \( S_{\Delta^6} \) of geometries by systematically putting \( h \leq 6 \) face trees on all faces \( F \) of \( T(\Delta^6) \) and then putting \( h \leq 6 \) edge trees on all edges \( E \) of \( T(\Delta^6) \). Using Table II, the size of \( S_{\Delta^6} \) is

\[
|S_{\Delta^6}| = 82 \# E_{\text{on } T(\Delta^6)} \times (4.19 \times 10^6) \# F_{\text{on } T(\Delta^6)}.
\]  

\( \# E \) and \( \# F \) are triangulation-independent and are entirely determined by \( \Delta^6 \) [13].

Two 3d reflexive polytopes give a far larger number \( |S_{\Delta^6}| \) than the others. They are the convex hulls \( \Delta_i^6 \) of pairs of the vertex sets

\[
S_1 = \{(-1,-1,-1),(-1,-1,1),(-1,1,-1),(-1,1,1),(1,-1,-1),(1,-1,1)\},
\]

\[
S_2 = \{(-1,-1,-1),(-1,-1,1),(-1,1,-1),(-1,1,1),(1,-1,-1),(1,-1,1)\}.
\]

\( T(\Delta_1^6) \) and \( T(\Delta_2^6) \) have the same number of edges and faces. Their largest facets are displayed in Fig. 1 and have \( \# E = 63 \) and \( \# F = 36 \). We compute

\[
|S_{\Delta_1^6}| = \frac{2.96}{3} \times 10^{755}, \quad |S_{\Delta_2^6}| = 2.96 \times 10^{755},
\]

where the factor of \( 1/3 \) is due to the symmetries discussed in the Appendix. All other polytopes \( \Delta^6 \) contribute negligibly:

\[
|S_{\Delta^6}| \leq 3.28 \times 10^{692} \text{ configurations.}
\]

This gives

\[
\# \text{4d F-theory Geometries} \geq \frac{4}{3} \times 2.96 \times 10^{755}.
\]

We see that NHC are universal in these ensembles.

We now wish to study physics in our ensemble. Consider a geometric assumption \( A_i \) and a physical property \( P_i \) such that \( A_i \implies P_i \). Our goal is to determine high probability assumptions that lead to interesting physical properties, computing \( P(A_i) \) since \( A_i \implies P_i \) ensures \( P(P_i) \geq P(A_i) \). We will focus on \( S_{\Delta_1^6} \) and \( S_{\Delta_2^6} \), since these are the dominant ensembles.

Consider first \( S_{\Delta_1^6} \) and let \( A_1 \) be the assumption that any simplex in an FRST of \( \Delta_1^6 \) containing a vertex of \( \Delta_1^6 \) has an \( h \geq 3 \) face tree on it. For the 3 symmetric facets of \( \Delta_1^6 \) there are 17 ways to choose simplices containing the vertices, and 1796 ways for its largest facet. The maximum number of simplices containing vertices is 24.

Using \( P(h \geq 3 \text{ tree on simplex}) \) from Table II,

\[
P(A_1 \text{ in } S_{\Delta_1^6}) \geq .9999998^{34} \approx .999995.
\]

There are \( 17^3 \times 1796 \) ways to choose simplices that contain the vertices, all of which yield \( G \geq F_4^{18} \times F_6^{10} \times U^9 \) which undercounts due to the facts that we choose to do face blowups followed by edge blowups to simplify the subdivision combinatorics, and that we have not taken into account the \( O(10^{15}) \) FRSTs of \( \Delta_2^6 \) and \( \Delta_2^6 \).

### IV. Universality and Non-Higgsable Clusters.

We now study universality in the dominant sets of F-theory geometries \( S_{\Delta_1^6} \) and \( S_{\Delta_2^6} \). We prove non-Higgsable cluster universality, minimal gauge group universality, and discuss results from random sampling.

#### Algorithmic Universality and Gauge Groups.

We wish to establish the likelihood that an F-theory base in \( S_{\Delta^6} \) or \( S_{\Delta^6} \) give rise to non-Higgsable seven-branes. The result arises from Prop. 2: if there is a tree anywhere on \( F \), even a single leaf, there is a non-Higgsable seven-brane on all divisors associated to interior points of \( F \). For any \( S_{\Delta^6} \) only one configuration has no trees, and therefore

\[
P(\text{NHC in } S_{\Delta^6}) \geq 1 - \frac{1}{|S_{\Delta^6}|}.
\]

This is always very close to one, and in particular

\[
P(\text{NHC in } S_{\Delta_1^6}) \geq 1 - 1.01 \times 10^{-755}.
\]

\[
P(\text{NHC in } S_{\Delta_2^6}) \geq 1 - .338 \times 10^{-755}.
\]
where $U \in \{G_2, F_4, E_6\}$, depending on details. All of these factors arise on the ground, and generally there will be many more factors from non-Higgsable seven-branes in the leaves. Here $E_6^{10}$ arises from an $E_6$ on every interior point of the large facet in $\Delta_5^3$, see Fig. 1. This set of statements defines physical property $P_1$, and since $A_1 \implies P_1$ we deduce $P(P_1 \in S_{\Delta_1^3}) \geq P(A_1 \in S_{\Delta_1^3}) \geq .999995$.

Let $A_2$ be the assumption that there exists a $h = 5$ face tree somewhere on the large facet $F$ in $\Delta_5^3$. Knowing $F = 36$ on $F$ and using Table II, we compute $P(A_2 \in S_{\Delta_1^3}) = (1 - (1 - .999997)^{36}) \approx 1 - 10^{-199}$. Let $A_3$ be that $A_2$ and $A_3$ hold, so $P(A_3) = P(A_1)P(A_2) = P(A_1)$. Then given $A_3$ a short calculation shows that the $h = 5$ tree on $F$ enhances $E_6$ in $P_1$ to $E_8$, giving 10 $E_8$'s on the ground. $P_1$ with this enhancement defines $P_3$.

Similar results hold for $S_{\Delta_2^3}$. Let $A_1$ be the assumption that any simplex in an FRST of $\Delta_2^3$ containing a vertex of $\Delta_2^3$ has an $h \geq 3$ face tree on it. This ensures that $G \geq F_4^{15} \times E_7^9 \times U^{12}$. However, this is quickly enhanced to $G \geq F_4^{18} \times E_8^{10} \times U^9$, via a $h = 5$ face tree on each face, and a $1 - 6.55 \times 10^{-8}$ probability blow-up along an edge connecting the point $\{-1, 2, -1\}$ to one of the points $\{-1, 1, n\}$, where $n = -1 \ldots 3$. The existence of these edges is independent of triangulation. Summarizing, the probability that a geometry in our set has $G \geq F_4^{18} \times E_8^{10} \times U^9$ on the ground is $\geq .999995$. This minimal group for $P_3$ on $S_{\Delta_2^3}$ matches that of $P_3$ on $S_{\Delta_1^3}$.

It is natural to ask whether this structure on the ground constrains the gauge structure in the trees. In Prop. 3 it is shown that the gauge group on a leaf $v$ in a tree built above $E_8$'s on the ground is determined by the leaf height $h_v$. The result is that a $h_v = 1, 2, 3, 4, 5, 6$ leaf above $E_8$ roots has Kodaira fiber $F_v = II^*, IV_{ns}^*, I_{h,n,s}^*, IV_{ns}, H, -$ with gauge group $G_v = E_8, F_4, G_2, SU(2), - -, -$ respectively.

This leads to a high probability result about the structure of the gauge group. Since $A_3 \implies P_3$, which has at least 10 $E_8$ factors nearby another, $P_3$ also has

$$G \geq E_8^{10} \times F_4^{18} \times U^9 \times P^H_4 \times G^H_2 \times A^H_4,$$

where $H_i$ is the number of height $i$ leaves in trees built on $E_8$ roots, and $rk(G) \geq 160 + 4H_2 + 2H_3 + H_4$. There are $H_6$ Kodaira type $II$ seven-branes that do not carry a gauge group but realize Argyres-Douglas theories on D3 probes. The first $F_4$ and also the $U$ factors may be enhanced, but the other factors are fixed. The probability of this physical property is $P(P_3) \geq P(A_3) \approx .999995$. This non-trivial minimal gauge structure is universal in our large ensembles given by $S_{\Delta_1^3}$ and $S_{\Delta_2^3}$.

**Random Samples and Geometric Visible Sectors.** It may be possible to accommodate visible sectors from flux breaking these gauge sectors, but it is also interesting to study whether gauge factors $E_6$ and/or $SU(3)$ arise with high probability. We have not yet discovered a high probability simple geometric assumption that leads to $E_6$ or $SU(3)$. However, it is possible that they arise regularly, but due to a complex geometric assumption.

This idea can be tested by random sampling. Let $B$ be an F-theory base obtained by adding face trees then edge trees at random, followed by edge trees at random, to the “pushing” triangulation [13] of $\Delta_3^3$. We studied an ensemble $S_r$ of $10^9$ such random samples and found $P(SU(3) or E_6 in S_r) \approx 1/200$, and that at least 36 of the points in $\Delta_3^3$ carried $E_8$, a significant enhancement beyond $P_3$. Furthermore, in our sample we found that $E_6$ only arose on the point $(1, -1, -1)$, which is the only vertex of $\Delta_3^3$ that is not in the largest facet. Similar results and probabilities also hold using these techniques on $\Delta_2^3$. It would be interesting to study random samples of other triangulations, or to see if other geometric assumptions imply these enhancements. We leave the systematic study of geometric visible sectors to future work.

**V. Discussion.** We have presented a construction algorithm for $\frac{1}{4} \times 2.96 \times 10^{735}$ geometries for 4d F-theory compactifications. This number is only a lower bound and may be enlarged in at least three ways: by relaxing the requirement of edge blowups after face blowups, by taking into account the $O(10^{15})$ FRS triangulations of 3d reflexive polytopes, and by considering blow-ups of non-toric intersections of seven-branes.

We have initiated the study of this ensemble by focusing on the geometric gauge group. Using knowledge of the construction algorithm, we derived the existence of universal properties for the minimal geometric gauge group on non-Higgsable clusters. High rank groups are generic, as is the existence of at least 10 $E_8$ factors on the ground. The gauge group on leaves above these $E_8$ factors on the ground is fixed entirely by their height. Such large gauge sectors motivate dark glueballs; see [14].

There are many directions for future work. For example, it would be interesting to study the number of consistent fluxes per geometry and how they alter the gauge group, to perform a statistical analysis of the gauge group in the leaves, or to analyze physics that arises from blow-ups of non-toric intersections of seven-branes. Perhaps most pressing is that, though we have demonstrated that the gauge group is generically high rank and have reviewed some possible realizations of the standard model discussed in [5], it is not yet clear whether the standard model is realized with high probability in our ensemble.

We believe that this is the first time that such a large ensemble has been systematically studied in string theory. In our view, the crucial ingredient that made the results possible are what we call algorithmic universality: derivation of universality from a construction algorithm, rather than an explicitly constructed ensemble or random sampling. Given the plethora of large ensembles in string theory and the infeasibility of constructing all of them, universality of this sort may play a critical role in making the string landscape tractable.

**A. Appendix: Technical Subtleties.** We now address technical subtleties that are important for establishing, but not understanding, results in the main text.

**Polytope Symmetries and Toric Morphisms.** In
Suppose each leaf \( m_F \in \Delta \). \((m_F, v) \geq -1 \forall v \in \Delta^o \). Now suppose \( h_v \leq 6/n, n \in \mathbb{N} \) for all rays \( v = av_1 + bv_2 + cv_3 \in B \), with \( v_i \) 3d cone vertices in \( B \). Then \( (nm_F, v) \geq -n(a + b + c) = -nh_v \geq -6 \) for all rays \( v \) and therefore \( nm_F \in \Delta_g \). Here we denote \( \Delta_g, \Delta_g^s \) as the polytopes corresponding to \( (\mathcal{O}(−4K_B)), (\mathcal{O}(−6K_B)) \), respectively. If \( h_v \leq 6 \forall v \), then \( m_F \in \Delta_g \). This monomial has multiplicity of vanishing \((v, m_F)\) of 5 for any \( v \) in or above \( F \). which protects \( v \) from being a \((4,6)\) divisor. If \( h_v \leq 6 \forall v \) then \( m_F \in \Delta_g \forall F \) and there is a monomial that prevents each divisor from being \((4,6)\).

It is also simple to see that in our ensemble, \( f \) and \( g \) can only vanish to multiplicities less than \((8,12)\) along curves and orders less than \((12,18)\) at points, respectively. Consider any toric curve \( C = D_s \cdot D_t \subset B \). Take \( v_s = \sum_i a_i,v_i \) and \( v_t = \sum_i a_i,v_i \) and define \( a = \sum_i a_i,s \) and \( b = \sum_i a_i,t \). Let \( F \) be a facet on which or above which \( v_s \) and \( v_t \) sit, with \( m_F \) the dual facet. As an element of \( \Delta_g \) the associated monomial may be written \( s^{(m,v)} + 6f(m,v) + 6 \times \ldots \), and the monomial vanishes to multiplicity \((m,v) + (n,v)\) of 12 = \( a + b + 12 \) along \( C \). For \( g \) to vanish to multiplicity 12 along a curve, this requires \( a + b < 0 \), which cannot happen. A similar argument shows that \( g \) cannot vanish to order 18 or higher at points. On the other hand, our ensemble is generated by a series of repeated blowups along curves and points, and one can pass to a Calabi-Yau minimal Weierstrass model only if the MOV is \( (4,6) \) for a curve, and \( (8,12) \) for a point. One can achieve the required MOV by tuning in complex structure moduli space, but one has to ensure that no infinite distance singularities (as in the above) are introduced in the process. However, it is simple to see that the desired MOV can be achieved, without introducing any disallowed singularities, by simple tuning without turning off the non-monomial corresponding to \( m_F \), for all \( F \).

**7-Branes and Gauge Enhancement.** We now prove some useful results that allow us to determine a universal minimal gauge sector in our ensemble, as well as show that NH 7-branes are ubiquitous.

**Proposition 2.** Suppose \( \exists v \) in or above a facet \( F \), i.e. \( v = av_1 + bv_2 + cv_3 \) with \( v_i \) simplex vertices in \( F \), such that \( h_v \geq 2 \). Then there is a non-Higgsable seven-brane on the divisor associated to each interior point of \( F \).

Proof. Then \((6m_F, v) = −6h_v \leq −12 \) implies \( 6m_F \notin \Delta_g \). Similarly, \( 4m_F \notin \Delta_f \). Since any point \( p \) interior to \( F \) has \((m,p) = −1 \iff m = m_F \) and reflexive polytopes of dimension three are normal, i.e. any \( m_f \in \Delta_f \) \((m_f \in \Delta_g) \) has \( m_f = \sum_i m_i \in \Delta \) \((m_g = \sum_i m_i \in \Delta) \). from which \( (m_f, p) = −4 \iff m_f = 4m_F \) and \((m,g,p) = −6 \iff m_g = 6m_F \). Therefore, if there is any tree on \( F \) then \( 4m_F \notin \Delta_f \) and \( 6m_F \notin \Delta_g \). By normality, for any \( p \) interior to \( F \) this gives \( 2m_f \in \Delta_f \) \((m_f, p) = −4 \) and \( 2m_g \in \Delta_g \) \((m_g, p) = −6 \), and therefore \( ord_p(f,g) > (0,0) \), which implies there is a non-Higgsable seven-brane on the divisor associated to \( p \).

---

1 We thank D. Morrison for discussions on this and related points.
Proposition 3. Let \( v \) be a leaf \( v = av_1 + bv_2 + cv_3 \) with \( v_i \) simplex vertices in \( F \). If the associated divisors \( D_{1,2,3} \) carry a non-Higgsable \( E_6 \) seven-brane, and if \( v \) has height \( h_v = 1, 2, 3, 4, 5, 6 \) it also has Kodaira fiber \( F_v = I^{*}, IV^{*}_{ns}, IV^{*}_{ns}I, IV^{*}_{ns}, II^{*}, \) and gauge group \( G_v = E_8, F_4, G_2, SU(2), - , - , \) respectively.

\textbf{Proof.} The height criterion gives \(\text{mult}_v(g) \leq 6 - h_v\). If \( v = av_1 + bv_2 + cv_3 \) with \( v_i \) each carrying \( E_8 \), then \( (m_g, v) \geq 0, (m_g, v) \geq -1, \forall m_f \in \Delta_f \) and \( \forall m_g \in \Delta_g \). This gives \( (m_g, v) \geq 0, (m_g, v) \geq -(a+b+c) = -h_v \). Together, we see \(\text{mult}_v(f) \geq 4, \text{mult}_v(g) \geq 6 - h_v\). For \( h_v = 1, 5, 6 \) this fixes \( G_v \), but to determine \( G_v \) for \( h_v = 2, 3, 4 \) we must study \( v \) to arrive at additional consistent tree configurations via blowing down at intermediate steps. We did not consider such possibilities, for combinatorial reasons.

However, such questions about mixing blow-ups and blow-downs are the subject of Oda’s Weak and Strong Factorization conjectures. The former states that any proper birational morphism \( X \to Y \) of complete, nonsingular varieties in characteristic zero factors into a sequence of smooth blow-ups and blow-downs. The latter conjectures that the morphism factors into a sequence of successive blow-ups followed by a sequence of successive blow-downs; it is open in dimension 3 and higher.

An interesting physical question arises in this context. By weak Oda, two trees are related by a sequence of blow-ups and blow-downs. However, if each sequence between fixed \( X \) and \( Y \) gives rise to an intermediate variety \( X_i \) with a \((4, 6)\) divisor, then the moduli space of four-dimensional F-theory compactifications is disconnected.

\textbf{Acknowledgments.} We thank W. Cunningham, T. Eliassi-Rad, J. Goodrich, D. Krioukov, B. Nelson, W. Taylor, J. Tian, and especially D.R. Morrison for discussions. J.H. is supported by National Science Foundation Grant PHY-1620526.

[1] R. Bousso and J. Polchinski, JHEP 06, 006 (2000), arXiv:hep-th/0004134 [hep-th]; S. Ashok and M. R. Douglas, \textit{ibid}. \textbf{01}, 060 (2004), arXiv:hep-th/0307049 [hep-th]; F. Denef and M. R. Douglas, \textit{ibid}. \textbf{05}, 072 (2004), arXiv:hep-th/0404116 [hep-th].

[2] F. Denef and M. R. Douglas, Annals Phys. \textbf{322}, 1096 (2007), arXiv:hep-th/0602072 [hep-th]; M. Cvetic, I. Garcia-Etxebarria, and J. Halverson, Fortsch. Phys. \textbf{59}, 243 (2011), arXiv:1009.5386 [hep-th].

[3] C. Vafa, Nucl. Phys. \textbf{B469}, 403 (1996), arXiv:hep-th/9602022 [hep-th]; D. R. Morrison and C. Vafa, \textit{ibid}. \textbf{B476}, 437 (1996), arXiv:hep-th/9603161 [hep-th].

[4] D. R. Morrison and W. Taylor, Central Eur. J. Phys. \textbf{10}, 1072 (2012), arXiv:1201.1943 [hep-th].

[5] A. Grassi, J. Halverson, J. Shaneson, and W. Taylor, JHEP \textbf{01}, 086 (2015), arXiv:1409.8925 [hep-th].

[6] A. P. Braun and T. Watari, JHEP \textbf{01}, 047 (2015), arXiv:1408.6167 [hep-th]; T. Watari, \textit{ibid}. \textbf{11}, 065 (2015), arXiv:1506.08433 [hep-th]; J. Halverson and J. Tian, Phys. Rev. \textbf{D95}, 026005 (2017), arXiv:1610.08864 [hep-th].

[7] J. Halverson, Nucl. Phys. \textbf{B919}, 267 (2017), arXiv:1603.01639 [hep-th].

[8] D. R. Morrison and W. Taylor, JHEP \textbf{05}, 080 (2015), arXiv:1412.6112 [hep-th].

[9] W. Taylor and Y.-N. Wang, JHEP \textbf{12}, 164 (2015), arXiv:1511.03209 [hep-th].

[10] J. Halverson and W. Taylor, JHEP \textbf{09}, 086 (2015), arXiv:1506.03204 [hep-th]; W. Taylor and Y.-N. Wang, \textit{ibid}. \textbf{01}, 137 (2016), arXiv:1510.04978 [hep-th].

[11] D. R. Morrison and W. Taylor, Fortsch. Phys. \textbf{60}, 1187 (2012), arXiv:1204.0283 [hep-th]; W. Taylor, JHEP \textbf{08}, 032 (2012), arXiv:1205.0952 [hep-th]; D. R. Morrison and W. Taylor, (2014), arXiv:1404.1572 [hep-th]; G. Martini and W. Taylor, \textit{ibid}. \textbf{06}, 061 (2015), arXiv:1404.6300 [hep-th]; S. B. Johnson and W. Taylor, \textit{ibid}. \textbf{10}, 23 (2014), arXiv:1406.0514 [hep-th]; W. Taylor and Y.-N. Wang, (2015), arXiv:1504.07689 [hep-th].

[12] M. Kreuzer and H. Skarke, Adv. Theor. Math. Phys. \textbf{2}, 847 (1998), arXiv:hep-th/9805190 [hep-th].

[13] J. A. De Loera, J. Rambau, and F. Santos, \textit{Triangulations: Structures for Algorithms and Applications}, 1st ed. (Springer Publishing Company, Incorporated, 2010).

[14] J. Halverson, B. D. Nelson, and F. Ruehle, Phys. Rev. \textbf{D95}, 043527 (2017), arXiv:1609.02151 [hep-ph]; A. Soni and Y. Zhang, (2016), arXiv:1610.06931 [hep-ph]; R. da Rocha, (2017), arXiv:1701.00761 [hep-ph]; B. S. Acharya, M. Fairbairn, and E. Hardy, (2017), arXiv:1704.01804 [hep-ph]; A. Soni, H. Xiao, and Y. Zhang, (2017), arXiv:1704.02347 [hep-ph].