Three-particle integrals with Bessel functions.

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Abstract

Analytical formulas for some useful three-particles integrals are derived. Many of these integrals include Bessel and/or trigonometric functions of one and two interparticle (relative) coordinates $r_{32}, r_{31}$ and $r_{21}$. The formulas obtained in such an analysis allow us to consider three-particle integrals of more complicated functions of relative/perimetric coordinates. In many actual problems such three-particle integrals can be found in matrix elements of the Hamiltonian and other operators.

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I. INTRODUCTION

The main goal of this study is to consider some special integrals which are closely related with various fundamental three- and few-body problems in physics. Possible application of such integrals include atomic, molecular and nuclear physics. In many of the problems in these areas one finds similar integrals which must be taken over three scalar distances and these three distances correspond to the sides (or ribs) of the triangle formed by the three ‘particles’. On the other hand, it is clear now that the general theory of three-particle integrals is a rapidly growing area of mathematical physics. The methods developed for numerical evaluations of different three-particle integrals can be used in various mathematical problems.

In general, the three-particle (or three-body) integral has the form

\[ I(\alpha, \beta, \gamma; F) = \int \int \int F(r_{32}, r_{31}, r_{21}) \exp(-\alpha r_{32} - \beta r_{31} - \gamma r_{21}) r_{32} r_{31} r_{21} dr_{32} dr_{31} dr_{21} \] (1)

where \( \alpha, \beta \) and \( \gamma \) are the three real values which are usually called and considered as the non-linear parameters. The function \( F(x, y, z) \) in Eq.(1) is an analytical function of each of the three real variables \( x, y, z \). The generalization of Eq.(1) to the case of complex variables is possible, but in this study we do not consider it. Moreover, to simplify the problem below we shall assume that the function \( F(r_{32}, r_{31}, r_{21}) \) does not grow ‘very rapidly’ when its arguments increase to the infinity. Such an assumption is needed to guarantee the convergence of all arising integrals (see below). The variables \( r_{ij} \) in Eq.(1) are the three relative coordinates \( r_{ij} = | \mathbf{r}_i - \mathbf{r}_j | \). These coordinates are scalars and they are symmetric upon permutation of their indexes \( r_{ij} = r_{ji} \). The relative coordinates are not truly independent of each other, since for these coordinates the six following conditions must always be obeyed: \( r_{ij} \geq | r_{ik} - r_{jk} | \) and \( r_{ij} \leq r_{ik} + r_{jk} \), where \( (i, j, k) = (1,2,3) \).

The integrals, Eq.(1), can be found in various few-body problems, but in Quantum three-body problems they play a central role, since the expressions for all matrix elements of the Hamiltonian and overlap matrices are reduced to Eq.(1). Furthermore, the explicit expressions for all expectation values (regular and singular) also reduce to the computation of formulas each of which coincides with Eq.(1). Different approaches to analytical and numerical computation of the three-body integrals Eq.(1) were developed in the middle of 1980’s [1], [2]. In particular, in those works it was shown that the most convenient and simple way to compute such integrals is based on the use of the three perimetric coordinates
u_1, u_2, u_3 [3], [4] instead of relative coordinates r_{32}, r_{31}, r_{21} mentioned above. The relation between these two set of coordinates is linear:

\[ u_1 = \frac{1}{2}(r_{31} + r_{21} - r_{32}) \quad u_2 = \frac{1}{2}(r_{21} + r_{32} - r_{31}) \quad u_3 = \frac{1}{2}(r_{32} + r_{31} - r_{21}) \quad (2) \]

The inverse relation takes the form \( r_{ij} = u_i + u_j \). The use of the three perimetric coordinates have a number of advantages in calculations of various three-particle integrals. For instance, the three perimetric coordinates \( u_1, u_2, u_3 \) are independent of each other and each of them varies between 0 and +\( \infty \). The substitution \((r_{32}, r_{31}, r_{21}) \rightarrow (u_1, u_2, u_3)\) drastically simplifies analytical and numerical computations of all three particle integrals. In perimetric coordinates \( u_1, u_2, u_3 \) the basic integral, Eq.(1), is written in the form

\[
I(\alpha, \beta, \gamma; F) = 2 \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} F(u_2 + u_3, u_1 + u_3, u_1 + u_2) \times 
\exp\left[-(\alpha + \beta)u_3 - (\alpha + \gamma)u_2 - (\beta + \gamma)u_1\right] (u_1 + u_2)(u_1 + u_3)(u_2 + u_3) \, du_1 du_2 du_3 \quad (3)
\]

where the factor 2 in the front of the integral is the Jacobian of the \((r_{32}, r_{31}, r_{21}) \rightarrow (u_1, u_2, u_3)\) transformation.

The integral, Eq.(3), thus takes the form of a Laplace transformation. It is also clear that the image function \( I \) will be written in the form \( I(\alpha + \beta, \alpha + \gamma, \beta + \gamma; F) \). In general, the calculation of the three-body integrals, Eq.(3), appears to be closely connected with the Laplace transform. Indeed, the tables of Laplace transormations are of great help in analytical and numerical computations of various three-body integrals. In our earlier studies we have derived the explicit formulas for a large number of three-body integrals. Such formulas include different regular and singular integrals, integrals with logarithmic terms, etc. However, the formulas for some important three particle integrals have never been derived in earlier studies. For instance, the integrals which contain one or two Bessel functions [5] and integrals in which one function of relative coordinate is represented in some ‘difficult’ forms, e.g., as the infinite power series, or as approximate expansions written in terms of other functions. In this work we consider some of such integrals.

II. EXPONENTIAL VARIATIONAL EXPANSION

As mentioned above the three-particle integrals, Eq.(1), arise in various three-body problems. In general, all problems related to the construction of highly accurate approximations
of the actual wave functions for bound states lead to such integrals. This means that the	hree-particle integrals, Eq. (1), can be found in all bound state problems, including trans-
sitions between bound states, time-evolution of bound states, etc. On the other hand,
analytical/numerical computation of the integrals, Eq. (1), is the central part of problems
related with the photodetachment and decays of bound states in three-body systems. Fortu-
nately, the three-particle integrals, Eq. (1), can be considered as the matrix elements between
two exponential basis functions written in relative coordinates. In general, the exponential
variational expansion of the three-body wave function for the bound $S(L = 0)$–states is
written in the form

$$\Psi = \frac{1}{2} (1 + \kappa \hat{P}_{21}) \sum_{i=1}^{N} C_i \exp (-\alpha_i r_{32} - \beta_i r_{31} - \gamma_i r_{21})$$

(4)

where $\kappa = \pm 1$ for symmetric systems and $\kappa = 0$ otherwise (see below). The generalization
of Eq. (1) to the case of bound states with arbitrary angular momentum $L$ is written in the
form [6]

$$\Psi = \frac{1}{2} (1 + \kappa \hat{P}_{21}) \sum_{i=1}^{N} \sum_{\ell_1} C_i \mathcal{Y}_{LM}^{\ell_1, \ell_2}(r_{31}, r_{32}) \exp (-\alpha_i r_{32} - \beta_i r_{31} - \gamma_i r_{21})$$

(5)

A slightly more complicated (but much more flexible!) generalization of Eq. (1) to the case
of arbitrary $L$ takes the form [7]

$$\Psi_{LM} = \frac{1}{2} (1 + \kappa \hat{P}_{21}) \sum_{i=1}^{N} \sum_{\ell_1} C_i \mathcal{Y}_{LM}^{\ell_1, \ell_2}(r_{31}, r_{32}) \phi_i(r_{32}, r_{31}, r_{21}) \exp (-\alpha_i u_1 - \beta_i u_2 - \gamma_i u_3) \times$$

$$\times \exp (i\delta_i u_1 + w_i u_2 + f_i u_3)$$

(6)

where $C_i$ are the linear (or variational) parameters, $\alpha_i, \beta_i, \gamma_i, \delta_i, e_i$ and $f_i$ are the real non-
linear parameters and $i$ is the imaginary unit. In the last equations all exponents contain
the three perimetric coordinates $u_1, u_2, u_3$ instead of relative coordinates $r_{32}, r_{31}, r_{21}$ used in
Eq. (1). The functions $\mathcal{Y}_{LM}^{\ell_1, \ell_2}(r_{31}, r_{32})$ in Eqs. (5) and (6) are the bipolar harmonics [8] of the
two vectors $r_{31} = r_{31} \cdot n_{31}$ and $r_{32} = r_{32} \cdot n_{32}$. The bipolar harmonics are defined as follows

$$\mathcal{Y}_{LM}^{\ell_1, \ell_2}(x, y) = x^{\ell_1} y^{\ell_2} \sum_{\ell_1, \ell_2} C_{\ell_1 \ell_2 \ell_1 m_1; \ell_2 m_2} Y_{\ell_1 m_1}(n_x) Y_{\ell_2 m_2}(n_y)$$

(7)

where $C_{\ell_1 m_1; \ell_2 m_2}^{LM}$ are the Clebsch-Gordan coefficients (see, e.g., [8]) and the vectors $n_x = \frac{x}{x}$
and $n_y = \frac{y}{y}$ are the corresponding unit vectors constructed for arbitrary non-zero vectors $x$
and $y$. Also, in this equation $L$ is the total angular momentum of the three-body system, i.e.
\[ \hat{L}^2 \Psi_{LM} = L(L+1) \Psi_{LM}, \text{ while } M \text{ is the eigenvalue of the } \hat{L}_z \text{ operator, i.e. } \hat{L}_z \Psi_{LM} = M \Psi_{LM}. \]

In actual calculations it is possible to use only those bipolar harmonics for which \( \ell_1 + \ell_2 = L + \epsilon \), where \( \epsilon = 0 \) or 1. The first choice of \( \epsilon \) (i.e. \( \epsilon = 0 \)) corresponds to the natural spatial parity \( \chi_P = (-1)^L \) of the wave functions \[9\]. The second choice (i.e. \( \epsilon = 1 \)) represents states with the unnatural spatial parity \( \chi_P = (-1)^{L+1} \). In almost all works on highly accurate bound state computations only the bound states of natural parity are considered. In real physical systems only such states are stable.

The polynomial-type functions \( \phi_i(r_{32}, r_{31}, r_{21}) \) are used in Eq.(1) to represent the interparticle correlations at short distances. In general, such simple polynomial functions allow one to increase the overall flexibility of the variational expansion Eq.(11). In many studies, however, these additional functions are chosen in the form \( \phi_i(r_{32}, r_{31}, r_{21}) = 1 \) for \( i = 1, \ldots, N \), since the overall convergence rate of the variational expansion, Eq.(6), is already very high. The operator \( \hat{P}_{21} \) in Eq.(11) is the permutation of the identical particles in symmetric three-body systems, where \( \kappa = \pm 1 \), otherwise \( \kappa = 0 \).

Note that the basis functions, Eq.(4), are \((2L + 1)\)-dimensional vectors, while all matrix elements of the Coulomb three-body Hamiltonian matrix and overlap matrix are scalars. In reality, one finds no contradiction here, since the angular integral of the products of the two and three bipolar harmonics \( \mathcal{Y}^{\ell_1, \ell_2}_{LM}(r_{31}, r_{32}) \) equals the products of some linear functions of the \( r_{31}^2, r_{32}^2, r_{31} \cdot r_{32} \) variables and \( 6j \)- and \( 9j \)-symbols, respectively (see, e.g., \[10\] and \[11\]). The scalar variables \( r_{31}^2( = u_3^2 + u_1^2 + 2u_1u_3), r_{32}^2( = u_3^2 + u_2^2 + 2u_2u_3), r_{31} \cdot r_{32} = \frac{1}{2}(r_{32}^2 + r_{31}^2 - r_{21}^2) \) are easily expressed as quadratic functions of the perimetric coordinates. Indeed, by using the relations between the relative and perimetric coordinates one finds

\[
\begin{align*}
    r_{31}^2 &= u_3^2 + u_1^2 + 2u_1u_3, \\
    r_{32}^2 &= u_3^2 + u_2^2 + 2u_2u_3, \\
    r_{31} \cdot r_{32} &= \frac{1}{2}(r_{32}^2 + r_{31}^2 - r_{21}^2) = u_3^2 + u_1u_3 + u_2u_3 - u_1u_2
\end{align*}
\]

Numerical computations of the corresponding radial integrals are slightly more complicated than in the case of \( L = 0 \). However, all integrals needed in actual bound state computations based on the exponential variational expansion, Eq.(6), are written in the form of Eq.(11), or can be reduced to such a form. This fact explains our permanent interest in developing of the new analytical/numerical approaches to calculations of the three-body integrals, Eq.(11).
III. INTEGRALS OF POLYNOMIAL FUNCTIONS

First, let us present here our formula for the three-body integral, Eq.(11), which includes the polynomial function $F(r_{32}, r_{31}, r_{21}) = r_{32}^k r_{31}^l r_{21}^n$. In this case the integral is designated as $\Gamma_{k,l,n}(\alpha, \beta, \gamma)$ and it is written in the form

$$\Gamma_{k,l,n}(\alpha, \beta, \gamma) = \int \int \int \frac{\Gamma(\alpha + \beta + \gamma)}{\Gamma(\alpha)} dr_{32} dr_{31} dr_{21} \tag{9}$$

where all indexes $k, l, n$ are assumed to be non-negative. In perimetric coordinates the integral, Eq.(10), is written in the form

$$\Gamma_{k,l,n}(\alpha, \beta, \gamma) = 2 \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} (u_2 + u_3)^k (u_1 + u_3)^l (u_1 + u_2)^n \times \exp[-(\alpha + \beta)u_3 - (\alpha + \gamma)u_2 - (\beta + \gamma)u_1] du_1 du_2 du_3 \tag{10}$$

Analytical evaluation of the integral, Eq.(10), is straightforward. Finally, one finds the following formula

$$\Gamma_{k,l,n}(\alpha, \beta, \gamma) = 2 \sum_{k_1=0}^{k} \sum_{l_1=0}^{l} \sum_{n_1=0}^{n} C^{k}_{k_1} C^{l}_{l_1} C^{n}_{n_1} \frac{(l - l_1 + k_1)!}{\alpha + \beta} \frac{(k - k_1 + n_1)!}{\alpha + \gamma} \frac{(n - n_1 + l_1)!}{\beta + \gamma} \times \frac{\Gamma(\alpha + \beta + \gamma)}{\Gamma(\alpha)} \tag{11}$$

where $C^{m}_{M}$ is the number of combinations from $M$ by $m$ ($m$ and $M$ are the non-negative integers). The formula, Eq.(11), can also be written in a few other equivalent forms. The function $\frac{n!}{x^{n+1}}$, in Eq.(11) is the $A_n(X)$ function introduced by Larson [12]. For the first time, one of the authors produced the formula, Eq.(11), in the middle of 1980’s (see [1], [2] and references therein) for the first time.

Note that in some of our earlier works the following integral in perimetric coordinates was considered as the basic three-body integral:

$$B(a, b, c; p_1, p_2, p_3; q_0, q_1, q_2, q_3; s) = \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \frac{u_1^{p_1} u_2^{p_2} u_3^{p_3} \exp(-a u_1 - b u_2 - c u_3)}{(q_0 + q_1 u_1 + q_2 u_2 + q_3 u_3)^s} du_1 \times \exp(-q_0 x) x^{s-1} dx \tag{12}$$

where $\Gamma(x)$ is the Euler’s gamma-function (see, e.g., [13], [14]). This integral depends upon eleven parameters $a, b, c, p_1, p_2, p_3; q_0, q_1, q_2, q_3$ and $s$. Formally, all these parameters must be real and positive (or non-negative). In particular, for $s = 1, q_0 = 1$ and $q_1 = q_2 = q_3 = 0$ one
finds from the last formula

\[ B(a, b, c; p_1, p_2, p_3; 1, 0, 0, 0, 1) = \frac{\Gamma(p_1 + 1)\Gamma(p_2 + 1)\Gamma(p_3 + 1)}{a^{p_1 + 1}b^{p_2 + 1}c^{p_3 + 1}} \]  

(13)

This formula leads to the generalization of Eq.(11) to the case of non-integer values of \(P_1, p_2, p_3\). It is often used to operate with the modified basis sets, e.g., with the basis set which includes semi-integer powers of perimetric coordinates.

In some related three-body problems, e.g., scattering, one finds a number of advantages of using some non-exponential basis sets, e.g., power-type wave functions of the relative and/or perimetric coordinates. In such cases we need to determine different basic integrals. In this study we chose not to discuss this interesting problem. Note only the following formula which arises in the case of power-type basis functions

\[ G(p_1, p_2, p_3; q_0, q_1, q_2, q_3; s) = \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \frac{u_1^{p_1} u_2^{p_2} u_3^{p_3}}{(q_0 + q_1 u_1 + q_2 u_2 + q_3 u_3)^s} du_1 du_2 du_3 \]

\[ = \frac{\Gamma(p_1 + 1)\Gamma(p_2 + 1)\Gamma(p_3 + 1)}{\Gamma(s)q_1^{p_1+1}q_2^{p_2+1}q_3^{p_3+1}} \cdot \frac{\Gamma(s - p_1 - p_2 - p_3 - 3)}{q_0^{s-p_1-p_2-p_3-3}} \]

(14)

where it is assumed that \(s > p_1 + p_2 + p_3 + 3\) and all values \(p_i, q_i (i = (1,2,3))\) and \(q_0\) must be positive.

IV. DERIVATION OF THE RELATED INTEGRALS

By using the expression for the \(\Gamma_k;\ell;n\) integral, Eq.(10), we can obtain analytical formulas for various three-particle integrals. First, consider the matrix elements of the real (analytical) functions \(f(r_{21})\) and \(F(r_{21})\) which are represented by the following series:

\[ f(r_{32}) = \sum_n A_n r_{32}^n \quad \text{and} \quad F(r_{32}) = \sum_n A_n r_{32}^n \exp(-B_n r_{32}) \]  

(15)

where the number of terms can be finite, or infinite. The computation of the matrix elements of these functions is reduced to the calculation of the two following sums

\[ M_f = \sum_n A_n \Gamma_{n+1:1:1}(\alpha, \beta, \gamma) \]  

(16)

and

\[ M_F = \sum_n A_n \Gamma_{n+1:1:1}(\alpha + B_n, \beta, \gamma) \]  

(17)
where the integrals $\Gamma_{n;k;l}$ are defined in Eq.(1). If the coefficients $A_n$ in these expansions rapidly decrease with $n$, then one needs to compute only a finite number of terms in such sums. For instance, if the coefficients in Eq.(15) are the power-type functions of some small parameter, then the series in Eq.(16) and Eq.(17) converge rapidly. This is the case in various atomic problems related to QED applications, when $A_n \sim \alpha^n$, where $\alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}$ is the dimensionless fine-structure constant.

This approach can also be used to produce analytical formulas for more complicated integrals, e.g., the general three-body integrals with the Bessel function $j_L(Vr_{32})$. First, let us obtain the computational formula for the following integral

$$B^{(0)}_{k;l;n}(\alpha, \beta, \gamma; V) = \int \int \int r_{32}^k r_{31}^l r_{21}^n j_0(Vr_{32}) \exp(-\alpha r_{32} - \beta r_{31} - \gamma r_{21}) dr_{32} dr_{31} dr_{21} \quad (18)$$

By using the formula $j_0(x) = \frac{\sin x}{x}$ and Eq.(16) one finds

$$B^{(0)}_{k;l;n}(\alpha, \beta, \gamma; V) = \sum_{q=0}^{\infty} \frac{(-1)^q V^{2q}}{(2q+1)!} \Gamma_{k+2q;l;n}(\alpha, \beta, \gamma) \approx \sum_{q=0}^{q_{\text{max}}} \frac{(-1)^q V^{2q}}{(2q+1)!} \Gamma_{k+2q;l;n}(\alpha, \beta, \gamma) \quad (19)$$

The integral $B^{(0)}_{k;l;n}(\alpha, \beta, \gamma; V)$ in the last equation converges for all $V$, but for $V \leq 1$ it converges very rapidly. In reality, the maximal value of the index $q$ (or $q_{\text{max}}$) in Eq.(19) is finite. Numerical investigations indicate that to stabilize 15 decimal digits for $V \leq 1$ one needs to use in Eq.(19) $q_{\text{max}} = 20$ to 40. For $V \geq 2$ the value of $q_{\text{max}}$ rapidly increases up to 50 - 70 and even 100. The same conclusion is true about the convergence of the three-body integrals with the Bessel function $j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$. This integral takes the form

$$B^{(1)}_{k;l;n}(\alpha, \beta, \gamma; V) = \sum_{q=0}^{\infty} \frac{(-1)^q (2q + 2)V^{2q+1}}{(2q+3)!} \Gamma_{k+2q+1;l;n}(\alpha, \beta, \gamma) \quad (20)$$

The three-body integrals with the lowest order Bessel functions $j_0(x)$ and $j_1(x)$ are of great interest in applications involving the decays and photodetachment of atoms/ions in their ground states. The results of numerical computations of some integrals $\Gamma_{k;l;n}(\alpha, \beta, \gamma)$, $B^{(0)}_{k;l;n}(\alpha, \beta, \gamma; V)$ and $B^{(1)}_{k;l;n}(\alpha, \beta, \gamma; V)$ can be found in Tables I and II. For all integrals with Bessel functions shown in Table II we restricted to the accuracy $\approx 1 \cdot 10^{-15}$. To determine the integrals $B^{(0)}_{k;l;n}(\alpha, \beta, \gamma; V)$ and $B^{(1)}_{k;l;n}(\alpha, \beta, \gamma; V)$ to such an accuracy for $V \leq 1$ it was sufficient to use 30 terms in Eqs.(19) and (20). For $V = 2$ we used up to 75 terms in these formulas.
The formulas for the three-body integrals $B_{k,l,m}^{(L)}(\alpha, \beta, \gamma; V)$ with other spherical Bessel functions $j_L(Vr_{32})$ can be obtained by using the same procedure. The result is

$$B_{k,l,m}^{(L)}(\alpha, \beta, \gamma; V) = V^L \sum_{\kappa=0}^{\infty} \frac{(-1)^{\kappa} V^{2\kappa}}{2^{\kappa} \kappa! \Gamma_{k+L+2\kappa,l,m}(\alpha, \beta, \gamma)}$$

(21)

where $\kappa$ is integer number. In Eq.(21) we have used the following formula for the $j_L(x)$ Bessel function

$$j_L(z) = z^L \sum_{k=0}^{\infty} \frac{(-1)^{k} z^{2k}}{2^k k!(2L + 2k + 1)!!}$$

(22)

Another approach for derivation of these formulas is based on the use of the well-known recursion formulas for the spherical Bessel functions \[13\] and analytical formulas for the three-body integrals containing the $j_0(Vr_{32})$ and $j_1(Vr_{32})$ Bessel functions, Eqs.(19) and (20). The formulas for the integrals containing the spherical Bessel functions of different arguments, e.g., $j_L(Vr_{31})$ and/or $j_L(Vr_{21})$, are easily obtained from the expressions for the integrals with the $j_L(Vr_{32})$ functions by applying a set of different $\alpha \rightarrow \beta \rightarrow \gamma$ substitutions. Here we do not want to produce these formulas, since it is rather a technical problem which step-by-step repeats the procedure described above for deriving the formulas for the integrals with the $j_0(Vr_{32})$ and $j_1(Vr_{32})$ Bessel functions. Instead we consider in the next Section a more interesting and actual problem which is closely related with analytical and numerical computation of other three-body integrals with modified Bessel functions $K_0(r)$, $K_1(r)$ and $K_2(r)$ functions.

V. MATRIX ELEMENTS OF THE UEHLING POTENTIAL

As is well known (see, e.g., \[15\], \[16\]) in the lowest order approximation the effect of vacuum polarisation between two interacting electric charges is described by the Uehling potential $U(r)$ \[17\]. In \[18\] we have derived the closed analytical formula for the Uehling potential. For atomic systems this formula is written in the following three-term form (in atomic units $\hbar = 1$, $m_e = 1$, $e = 1$)

$$U(2br) = \frac{2\alpha Q}{3\pi} \cdot \frac{1}{r} \left[ \int_1^{+\infty} \exp(-2\alpha^{-1} \xi r)(1 + \frac{1}{2\xi^2}) \frac{\sqrt{\xi^2 - 1}}{\xi^2} d\xi \right]$$

$$= \frac{2\alpha Q}{3\pi r} \left[ \left( 1 + \frac{b^2 r^2}{3} \right) K_0(2br) - \frac{br}{6} K_1(2br) - \left( \frac{b^2 r^2}{3} + \frac{5}{6} \right) K_2(2br) \right] ,$$

(23)
where the notation $Q$ stands for the electric charge of the nucleus, $b = \alpha^{-1}$ and $\alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}$ is the dimensionless fine-structure constant. Here and below $\hbar = \frac{h}{2\pi}$ is the reduced Planck constant (also called the Dirac constant), $e$ is the electric charge of the positron and $m_e$ is the mass of the electron (= mass of the positron). In Eq. (23) $K_0(a)$ is the modified Bessel function of zero order (see, e.g, [13]), i.e.

$$K_0(z) = \int_0^\infty \exp(-z \cosh t)dt = \sum_{k=0}^{\infty} (\psi(k+1) + \ln 2 - \ln z) \frac{z^{2k}}{2^{2k}(k!)^2},$$

where $\psi(k)$ is the Euler psi-function defined by Eq.(8.362) from [13]. The functions $Ki_1(z)$ and $Ki_2(z)$ in Eq.(23) are the recursive integrals of the $K_0(z)$ function, i.e.

$$Ki_1(z) = \int_z^\infty Ki_0(z)dz, \quad Ki_n(z) = \int_z^\infty Ki_{n-1}(z)dz, \quad (24)$$

where $n \geq 1$.

The calculation of the matrix elements of the Uehling potential with the use of our three term formula, Eq.(23), leads to the following three-body integrals:

$$K_{p;ln}^{(k,l,n)}(\alpha, \beta, \gamma) = \int \int \int r_{32}^k r_{31}^l r_{21}^n Ki_0(r_{32}) \exp(-\alpha r_{32} - \beta r_{31} - \gamma r_{21})dr_{32}dr_{31}dr_{21} \quad (25)$$

for $p = 0, 1, 2$ and two other similar integrals which contain $Ki_0(r_{31})$ and $Ki_0(r_{21})$. This problem can be solved approximately by using the known analytical formulas for the modified Bessel function $K_0(z)$ and for the two lowest recursive integrals of this function. However, the overall accuracy of the final solution is not very high. There are some advanced methods which can be used to solve this problem, but at this moment we also trying to apply a few alternative approaches. One of these methods is based on the original integral representation for the Uehling potential, Eq.(23). Below, we consider this procedure in detail.

First, consider the matrix elements of the regular Yukawa-type interparticle potential which is written in the form $V_Y (r) = V_0 \frac{\exp(-\mu r)}{r}$. The matrix element between the two exponential basis functions takes the following form

$$V_0 \int \int \int r_{31} r_{21} \exp[-(\alpha + \mu)r_{32} - \beta r_{31} - \gamma r_{21}]dr_{32}dr_{31}dr_{21} = V_0 \Gamma^{(0,1,1)}(\alpha + \mu, \beta, \gamma) \quad (26)$$

in the case of the $V_Y (r_{32})$ interaction. Note also that each of the integrals $\Gamma^{(0,1,1)}$ contains only four terms. This allows one to obtain the following formula in the case of the $U_{21}$ potential, which is the $(21)$-component of the total Uehling potential, Eq.(23):

$$\overline{U}_{21}(2\xi) = \int \int \frac{\exp(-2\xi r_{21})}{r_{21}} \exp(-\alpha r_{32} - \beta r_{31} - \gamma r_{21})r_{32}r_{31}r_{21}dr_{32}dr_{31}dr_{21} =$$
\[
\frac{2}{(\alpha + \beta)(\alpha + \gamma + 2b\xi)(\beta + \gamma + 2b\xi)} \left[ \frac{2}{(\alpha + \beta)^2} + \frac{1}{(\beta + \gamma + 2b\xi)(\alpha + \beta)} + \frac{1}{(\alpha + \gamma + 2b\xi)(\alpha + \beta)} \right] + \frac{1}{(\beta + \gamma + 2b\xi)(\alpha + \gamma + 2b\xi)} ,
\]

where \( \alpha + \beta > 0, \alpha + \gamma > 0, \beta + \gamma > 0 \) and \( \xi > 0 \). The first factor 2 in the numerator is the Jacobian of the linear transformation from relative to perimetric coordinates. Analogous expressions can be obtained for the \( U(r_{32}) \) and \( U(r_{31}) \) Yukawa-type potentials. In fact, such formulas can be derived from Eq.(27) simply by performing cyclic permutations of three parameters \( \alpha, \beta \) and \( \gamma \).

The final formula for the matrix elements of the Uehling potential in an arbitrary Coulomb three-body system is written in the form

\[
\frac{2\alpha}{3\pi} \int_1^\infty \left[ q_1 q_2 U_{21}(2b\xi) + q_1 q_3 U_{31}(2b\xi) + q_2 q_3 U_{32}(2b\xi) \right] \left( 1 + \frac{1}{2\xi^2} \right) \frac{\sqrt{\xi^2 - 1}}{\xi^2} d\xi ,
\]

where \( q_i \ (i = 1, 2, 3) \) are the particle charges expressed in atomic units. The expressions for the \( U_{ij}(2b\xi) \) terms are obtained from Eq.(27). By using the formulas, Eqs.(27) and (28), we have developed a number of effective numerical methods for accurate evaluation of the three-particle integrals arising in the expansion of the Uehling potential.

VI. THREE-PARTICLE INTEGRALS WITH TWO BESSEL FUNCTIONS

A number of actual problems in modern atomic physics lead to three-particle integrals with two Bessel functions. For instance, the probability of formation of the negatively charged tritium ion (T\(^-\)) during the \((n,^{3}\text{He}; \ ^1\text{H}, \ ^3\text{H})\) nuclear reaction in the two-electron \(^3\text{He}-\)atom is reduced to the calculation of the following integral (‘probability amplitude’ for more details, see, e.g., [19])

\[
A_{if} = \int \int \int \Phi_{T^-}(r_{32}, r_{31}, r_{21}) j_0(V_i \cdot r_{32}) j_0(V_i \cdot r_{31}) \Psi_{\text{He}}(r_{32}, r_{31}, r_{21}) r_{32} r_{31} dr_{32} dr_{31} dr_{21}
= \frac{1}{V_t^2} \int \int \int \Phi_{T^-}(r_{32}, r_{31}, r_{21}) \sin(V_i \cdot r_{32}) \sin(V_i \cdot r_{31}) \Psi_{\text{He}}(r_{32}, r_{31}, r_{21}) r_{32} dr_{32} dr_{31} dr_{21} \ (29)
\]

where \( V_i \) is the speed of the tritium nucleus after the nuclear reaction in the \(^3\text{He}\) atom. To obtain the formula, Eq.(29), we have used the known fact from atomic physics that the negatively charged hydrogen ion has only one bound \( 1^1S(L = 0)\)–state. Also, to derive Eq.(29) we restrict ourselves to the case when the incident \(^3\text{He}\) atom is in its ground \( 1^1S(L = 0)\)–state.
As mentioned in the second Section the wave functions of the ground $1^1S(L = 0)$—states in the two-electron H$^-$ ion and He atom are usually approximated with the use of highly accurate variational expansion written in the relative/perimetric coordinates. The most advanced of such expansions is the exponential variational expansion in relative/perimetric coordinates which takes the following form (for the bound $S(L = 0)$—states in the three-body systems):

$$\psi(r_{32}, r_{31}, r_{21}) = \sum_{i=1}^{N} C_i \exp(-\alpha_i r_{32} - \beta_i r_{31} - \gamma_i r_{21})$$

(30)

where $C_i$ are the variational coefficients and $N$ is the total number of terms in the trial function $\psi(r_{32}, r_{31}, r_{21})$. The probability amplitude $A_{if}$ is written as the double sum of the following three-particle integrals

$$B_{k:l:n}^{(00)}(\alpha, \beta, \gamma) = \int \int \int \exp(-\alpha r_{32} - \beta r_{31} - \gamma r_{21}) \sin(V_t \cdot r_{32}) \sin(V_t \cdot r_{31}) r_{12} dr_{32} dr_{31} dr_{21}$$

(31)

The theory of these integrals has not been developed in earlier studies. It was shown in [19] that the integral, Eq. (31), is reduced to the following double sum (here we apply the Cauchy formula)

$$B_{0:0:1}^{(00)}(\alpha, \beta, \gamma; V) = \sum_{n=0}^{\infty} (-1)^n \frac{V^{2n}}{(2n + 2)!} \sum_{\mu=0}^{\kappa} \frac{2\mu+1}{(2\mu+1)!} \sum_{\nu=0}^{\kappa} \frac{2\nu+1}{(2\nu+1)!} \Gamma_{k+2\mu+1;l+2\nu+1;n+1}(\alpha, \beta, \gamma)$$

(32)

where $C_{n}^{k}$ is the number of combinations from $n$ by $k$ ($n \geq k$) and $\Gamma_{k;l:n}(a, b, c)$ is the basic three-particle integral defined above. Note that the integral, Eq. (31), can easily be computed with the use of complex arithmetic. However, such methods are difficult to use in the general case. In our study to check the formula, Eq. (32), we have used both approaches.

The integral $B_{0:0:1}^{(00)}(\alpha, \beta, \gamma; V)$ belongs to the new class of three-particle integrals $B_{k:l:n}^{(L_1L_2)}(\alpha, \beta, \gamma; V)$. Such integrals contain the two Bessel functions $j_{L_1}(V r_{32})$ and $j_{L_2}(V r_{31})$. In this Section we consider one approach developed recently for computations of such integrals. First, by using Eq. (22) one finds the following formula for the product of the two Bessel functions $j_{L_1}(ax)$ and $j_{L_2}(by)$:

$$j_{L_1}(ax)j_{L_2}(by) = a^{L_1}x^{L_1}b^{L_2}y^{L_2} \sum_{p=0}^{\infty} \sum_{q=0}^{p} \frac{(-1)^p}{2^p p!} C_p^q \frac{a^{2q}b^{2q}y^{2p-2q}}{(2L_1 + 2q + 1)!!(2L_2 + 2p - 2q + 1)!!}$$

(33)

where $p$ and $q$ are both integer non-negative numbers and $C_p^q$ is the binomial coefficient. In our case we have $a = b = V, x = r_{32}$ and $y = r_{31}$. Therefore, the formula for the
\( B_{k,l,n}^{(L_1L_2)}(\alpha, \beta, \gamma; V) \) integrals take the form:

\[
B_{k,l,n}^{(L_1L_2)}(\alpha, \beta, \gamma; V) = \sum_{p=0}^{\infty} \frac{(-1)^p V^p}{2^p p!} \sum_{q=0}^{p} \frac{C_q^p}{(2L_1 + 2q + 1)!!(2L_2 + 2p - 2q + 1)!!} \times \\
\Gamma_{k+L_1+2q,l+L_2+2p-2q,n}(\alpha, \beta, \gamma) \tag{34}
\]

This formula is appropriate in all applications where \( V \leq 10 \). For \( V \leq 1 \) the overall convergence rate of Eq. (34) for the \( B_{k,l,n}^{(L_1L_2)}(\alpha, \beta, \gamma; V) \) integrals is fast, while for \( 1 \leq V \leq 2 \) it is relatively fast and for \( 5 \leq V \leq 10 \) the convergence rate can be considered as moderate. In the cases when \( V \geq 15 - 20 \) one needs to develop some other methods of computation of the \( B_{k,l,n}^{(L_1L_2)}(\alpha, \beta, \gamma; V) \) integrals, but we do not pursue this here.

VII. THREE-PARTICLE INTEGRALS OF MORE COMPLICATED FUNCTIONS

By using the formulas derived above for three-particle integrals with one and two Bessel functions we can obtain some useful formulas for more complicated three-particle integrals. In reality, one finds a large number of three-particle integrals which can be approximated by the integrals with Bessel function(s). In this study we restrict ourselves to the consideration of the following integral:

\[
J(\alpha, \beta, \gamma; t) = \int \int \int \cos \sqrt{r_{32}^2 - 2tr_{32}} \cdot \exp(-\alpha r_{32} - \beta r_{31} - \gamma r_{21}) dr_{32} dr_{31} dr_{21} \tag{35}
\]

Such integrals and their \( t \)--derivatives arise, e.g., in the problem of electron scattering on the electric dipole formed by the two heavy, positively charge particles. The formula for numerical evaluation of the \( J(\alpha, \beta, \gamma; t) \) integral is written in the form

\[
J(\alpha, \beta, \gamma; t) = \sum_{\kappa=0}^{\infty} \frac{t^\kappa}{\kappa!} B_{k+1:l:n}^{(\kappa-1)}(\alpha, \beta, \gamma; 1) \tag{36}
\]

All terms with \( \kappa \geq 1 \) in this formula are determined directly with the use of formulas given above. Calculations of the term with \( \kappa = 0 \) contains an additional complication, since in this case \( \kappa - 1 = -1 \) and we need to define the integral \( B_{k+1:l:n}^{(-1)}(\alpha, \beta, \gamma; 1) \). By using the formulas (10.1.11) and (10.1.12) from [14] one finds \( xj_{-1}(x) = j_0(x) - xj_1(x) \). Therefore, for the \( B_{k+1:l:n}^{(-1)}(\alpha, \beta, \gamma; 1) \) integral we have:

\[
B_{k+1:l:n}^{(-1)}(\alpha, \beta, \gamma; 1) = B_{k:l:n}^{(0)}(\alpha, \beta, \gamma; 1) - B_{k+1:l:n}^{(1)}(\alpha, \beta, \gamma; 1) \tag{37}
\]
Now, we can use the formula, Eq. (36), to approximate the integral $J(\alpha, \beta, \gamma; t)$ to arbitrarily high, in principle, numerical accuracy. There are many other uses for the integrals containing one and two Bessel functions and some functions of relative coordinates $r_{32}, r_{31}$ and $r_{21}$. A number of such formulas will be considered in our next study.

VIII. CONCLUSION

We have considered the problems of analytical and numerical computation of the three-body (exponential) integrals which contain different Bessel functions. For a number of such integrals we have derived closed analytical formulas and/or developed effective numerical methods. Our main interest is related to the integrals which contain the $j_0(kr_{ij})$ and $j_1(kr_{ij})$ Bessel functions, since such integrals play a central role in various problems on photodetachment of the ground $S(L = 0)$--states of different atomic and molecular systems. The formulas for the integrals with the spherical Bessel functions $j_L(kr_{ij})$ for $L \geq 2$ can be derived by applying the same procedure. The case of integrals with the modified Bessel functions, e.g., with the $K_n(br_{ij})$ functions, is more difficult for analytical consideration. However, such integrals are needed in various problems, including derivation of the closed formulas for the matrix elements of the Uehling potential.

Derivation of simple analytical formulas for the three-particle integrals with one and two Bessel functions has a great value in numerous applications to atomic and nuclear physics. On the other hand, it is a very interesting mathematical problem, since the three relative coordinates $r_{32} = r_2, r_{31} = r_1$ and $r_{12}$ always form a triangle. In fact, we are dealing with a new class of multiple integrals which have special form. Such integrals play a central role in various problems of three- and few-body physics. Note here that three-particle integrals with one and two Bessel functions are of paramount importance for atomic analysis of the species arising during the nuclear $(n; t)$--, $(n; \alpha)$--, and $(n; p)$--reactions in few electron atoms. Another interesting problem is analytical/numerical calculations of the exponential three-body integrals which include functions approximated by series explicitly written in terms of the spherical Bessel functions (and other Bessel functions). In earlier studies numerical evaluation of such integrals was a very difficult problem which had no effective and reliable solution.

Appendix
Addition theorem for the spherical Bessel functions

Let us discuss here the statement known as the addition theorem for the spherical Bessel functions. This theorem plays a great role in the physics of three-body systems. On the other hand, it is of interest for the general theory of Bessel functions. First, consider the familiar Rayleigh expansion of a plane wave

$$\exp(ik \cdot r_{21}) = \sum_{L=0}^{\infty} l^L (2L+1) j_L(kr) P_L(\cos\Theta_{21})$$ (38)

where $j_L(x)$ is the spherical Bessel function of scalar argument $x$, $\mathbf{a} \cdot \mathbf{b}$ designates the scalar product of the two vectors $\mathbf{a}$ and $\mathbf{b}$, while $P_L(y)$ is the Legendre polynomial. In Eq.(38) the notation $\Theta_{21}$ stands for the angle between the $\mathbf{k}$ and $r_{21}$ vectors.

Now, suppose we have the three-body system (123) and we need to re-write the Rayleigh expansion, Eq.(38), in a different form which contains the plane waves of the two ‘new’ scalar products $\mathbf{k} \cdot r_{31}$ and $\mathbf{k} \cdot r_{32}$. Such problems always arise in three-body physics. It is clear that these two plane waves must uniformly be related to each other. The goal of this Appendix is to investigate such a relation in detail. Since the three particles in any three-body system always form a triangle, then we can write $r_{21} = r_{31} - r_{32}$. Now, one finds from Eq.(38)

$$\exp(ik \cdot r_{21}) = \exp(ik \cdot r_{31}) \exp(-ik \cdot r_{32})$$ (39)

By using the Rayleigh expansions twice for the right-hand side of the last equation we obtain

$$\sum_{L=0}^{\infty} l^L (2L+1) j_L(kr_{21}) P_L(\cos\Theta_{21}) = \sum_{L_1=0}^{\infty} l^{L_1} (2L_1+1) j_{L_1}(kr_{31}) P_{L_1}(\cos\Theta_{31}) \cdot \sum_{L_2=0}^{\infty} l^{L_2} \times$$

$$(2L_2+1) j_{L_2}(kr_{32}) P_{L_2}(\cos\Theta_{32})$$ (40)

where the notations $\Theta_{31}$ and $\Theta_{32}$ designate the angles between the $\mathbf{k}$ vector and $r_{31}$ and $r_{32}$ vectors, respectively.

The right-hand side of the last equation can be transformed with the use of the Cauchy product defined by a discrete convolution as follows:

$$\sum_{L=0}^{\infty} \left[ \sum_{\ell=0}^{L} l^\ell (-1)^{L-\ell} (2\ell+1)(2L-2\ell+1) j_\ell(kr_{31}) j_{L-\ell}(kr_{32}) P_\ell(\cos\Theta_{31}) P_{L-\ell}(\cos\Theta_{32}) \right]$$ (41)

It is crucially important here that each of the series in the Rayleigh expansion of a plane wave see, Eq.(38) and Eq.(39) converge absolutely [20]. From here and Eq.(40) we can
write

\[(2L + 1) j_L(kr_{21}) P_L(\cos \Theta_{21}) = (-1)^L \sum_{\ell=0}^{L} (-1)^\ell (2\ell + 1) (2L - 2\ell + 1) j_\ell(kr_{31}) j_{L-\ell}(kr_{32}) \times P_\ell(\cos \Theta_{31}) P_{L-\ell}(\cos \Theta_{32}) \quad (42)\]

By multiplying both sides of the last equation by \( P_L(\cos \Theta_{21}) \) and performing the integration over the \( \Theta_{21}, \Theta_{32} \) and \( \phi_{32} \) angles (or over all possible orientations of the \( \mathbf{r}_{32} \) vector) we find the following expression

\[j_L(kr_{21}) = \frac{(-1)^L}{4} \sum_{\ell=0}^{L} (-1)^\ell (2\ell + 1) (2L - 2\ell + 1) j_\ell(kr_{31}) j_{L-\ell}(kr_{32}) \times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P_\ell(\cos \Theta_{31}) P_{L-\ell}(\cos \Theta_{32}) P_L(\cos \Theta_{21}) \sin \Theta_{21} d\Theta_{21} \sin \Theta_{32} d\Theta_{32} \quad . \quad (43)\]

This formula contains the spherical Bessel functions \( j_L(kr_{21}) \) written in terms of the spherical Bessel functions \( j_n(kr_{32}) \) and \( j_n(kr_{31}) \) of two other arguments, where the three vectors \( \mathbf{r}_{32}, \mathbf{r}_{31}, \mathbf{r}_{21} \) form a triangle, i.e. \( \mathbf{r}_{32} + \mathbf{r}_{21} = \mathbf{r}_{31} \). As it follows from the definition of the \( \cos \Theta_{ij} \) angles we always have \( \Theta_{21} + \Theta_{31} + \Theta_{32} = \pi \). The formula, Eq.(43), is known as the addition theorem for the spherical Bessel functions.

[1] A.M. Frolov, *Highly accurate variational solutions for the Coulomb three-body problem.* Preprint IAE-4274/12, 21 p. (1986) (in Russian) (1986), unpublished.

[2] A.M. Frolov, Sov. Phys. JETP 92, 1100 (1987) [Zh. Eksp. Teor. Fiz. 92, 1959 (1987)].

[3] C.L. Pekeris, Phys. Rev. 112, 1649 (1958).

[4] R. McWeeny and B.T. Sutcliffe, *Methods of Molecular Quantum Mechanics* (Academic Press, New York, (1969)), Chp. 7.

[5] A.M. Frolov and D.M. Wardlaw, Phys. Rev. A, 79, 032703 (2009).

[6] A.M. Frolov and V.D. Efros, JETP Letters, 39, 544 (1984) [Pis’ma Zh. Eksp. Teor. Fiz. 39, 449 (1984)].

[7] A.M. Frolov, Phys. Rev. E 64, 036704 (2001).

[8] D.A. Varshalovich, A.N. Moskalev and V.K. Khersonskii, *Angular Momentum in Quantum Mechanics. Non-Relativistic Theory*, 3rd. edn. (Oxford, England, Pergamonn Press (1977)).

[9] L.D. Landau and E.M. Lifshitz, *Quantum Mechanics. Non-Relativistic Theory*, 3rd. edn. (Oxford, England, Pergamonn Press (1977)).
[10] V.D. Efros, Sov. Phys. JETP, 63, 5 (1986) [Zh. Eksp. Teor. Fiz. 90, 10 (1986)].
[11] A.M. Frolov and V.H. Smith, Jr., Phys. Rev. A, 53, 3853 (1996).
[12] S. Larsson, Phys. Rev. 169, 49 (1968).
[13] I.S. Gradstein and I.M. Ryzhik, Tables of Integrals, Series and Products, (6th revised ed., Academic Press, New York, (2000)).
[14] Handbook of Mathematical Functions, edited by M. Abramowitz and I.A. Stegun, (Dover, New York, (1972)).
[15] A.I. Akhiezer and V.B. Beresteskii, Quantum Electrodynamics, (4th Ed., Nauka (Science), Moscow (1981)), Chps. 4 and 5 (in Russian).
[16] W. Greiner and J. Reinhardt, Quantum Electrodynamics. (4th. Ed., Springer Verlag, Berlin, (2010)).
[17] E.A. Uehling, Phys. Rev. 48, 55 (1935).
[18] A.M. Frolov and D.M. Wardlaw, Eur. Phys. Jour. B 63, 339 (2012).
[19] A.M. Frolov, Eur. Phys. J. D 67, 126 (2013).
[20] W. Rudin, Principles of the Mathematical Analysis (McGraw-Hill Book Company, New York, (1964)), Chp. 3.
TABLE I: Numerical values of some three-body integrals $\Gamma_{k;l;n}(\alpha, \beta, \gamma)$. 

| $k$ | $l$ | $m$ | $\alpha$ | $\beta$ | $\gamma$ | $\Gamma_{k;l;n}(\alpha, \beta, \gamma)$ | $\Gamma_{k;l;n}(\alpha, \beta, \gamma)$ |
|-----|-----|-----|---------|---------|----------|-----------------------------------|-----------------------------------|
| 0   | 2   | 1   | 2.35    | 1.41    | 0.567    | 0.13248880489827E+00 -0.567        | 0.484535355001714E+01            |
| 1   | 2   | 1   | 2.35    | 1.41    | 0.567    | 0.105479781157007E+00 -0.567       | 0.462617958665629E+01            |
| 2   | 2   | 1   | 2.35    | 1.41    | 0.567    | 0.123759737118974E+00 -0.567       | 0.683620356276100E+01            |
| 3   | 2   | 1   | 2.35    | 1.41    | 0.567    | 0.190628938378487E+00 -0.567       | 0.138242778966704E+02            |
| 4   | 2   | 1   | 2.35    | 1.41    | 0.567    | 0.36205286177389E+00 -0.567        | 0.356816617385975E+02            |
| 5   | 2   | 1   | 2.35    | 1.41    | 0.567    | 0.81567409095427E+00 -0.567        | 0.112342033402992E+03            |
| 6   | 2   | 1   | 2.35    | 1.41    | 0.567    | 0.212162348108085E+01 -0.567       | 0.41792699357783E+03             |
| 7   | 2   | 1   | 2.35    | 1.41    | 0.567    | 0.6255939350668E+01 -0.567         | 0.179435469496013E+04            |
| 8   | 2   | 1   | 2.35    | 1.41    | 0.567    | 0.20551903374530E+02 -0.567        | 0.873301210942717E+04            |
| 9   | 2   | 1   | 2.35    | 1.41    | 0.567    | 0.745934650018583E+02 -0.567       | 0.475056243580342E+05            |

TABLE II: Numerical values of some three-body integrals $B_{k;l;n}^{(0)}(\alpha, \beta, \gamma; V)$ and $B_{k;l;n}^{(1)}(\alpha, \beta, \gamma; V)$. 

| $k$ | $l$ | $m$ | $\alpha$ | $\beta$ | $\gamma$ | $V$ | $B_{k;l;n}^{(0)}(\alpha, \beta, \gamma; V)$ | $B_{k;l;n}^{(1)}(\alpha, \beta, \gamma; V)$ |
|-----|-----|-----|---------|---------|----------|-----|-----------------------------------|-----------------------------------|
| 3   | 2   | 1   | 2.35    | 1.41    | 0.567    | 0.25 | 0.18233241643012516E+00           | 0.290930992106451E-01            |
| 5   | 2   | 1   | 2.35    | 1.41    | 0.567    | 0.25 | 0.75291471135429875E+00           | 0.166432412830887E+00            |
| 3   | 2   | 1   | 2.35    | 1.41    | 0.567    | 0.50 | 0.15968050735256670E+00           | 0.522255954684081E-01            |
| 5   | 2   | 1   | 2.35    | 1.41    | 0.567    | 0.50 | 0.59041249572520414E+00           | 0.278021233893212E+00            |
| 3   | 2   | 1   | 2.35    | 1.41    | 0.567    | 1.00 | 0.94868174980045456E-01           | 0.691516883096556E-01            |
| 5   | 2   | 1   | 2.35    | 1.41    | 0.567    | 1.00 | 0.2065506256710767E+00            | 0.27492833359198E+00             |
| 3   | 2   | 1   | 2.35    | 1.41    | 0.567    | 1.50 | 0.40374337963233781E-01           | 0.554457473644749E-01            |
| 5   | 2   | 1   | 2.35    | 1.41    | 0.567    | 1.50 | 0.20605506256710767E+00           | 0.27492833359198E+00             |
| 3   | 2   | 1   | 2.35    | 1.41    | 0.567    | 2.00 | 0.11173049407361310E-01           | 0.34084106321226E-01             |
| 5   | 2   | 1   | 2.35    | 1.41    | 0.567    | 2.00 | -0.35522376544132919E-01          | 0.316198754574614E-01            |