Partial existence result for Homogeneous Quasilinear parabolic problems beyond the duality pairing

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Abstract

In this paper, we study the existence of distributional solutions solving \((1.3)\) on a bounded domain \(\Omega\) satisfying a uniform capacity density condition where the nonlinear structure \(\mathcal{A}(x, t, \nabla u)\) is modelled after the standard parabolic \(p\)-Laplace operator. In this regard, we need to prove a priori estimates for the gradient of the solution below the natural exponent and a higher integrability result for very weak solutions at the initial boundary. The elliptic counterpart to these two estimates are fairly well developed over the past few decades, but no analogous theory exists in the quasilinear parabolic setting.

Two important features of the estimates proved here are that they are non-perturbative in nature and we are able to take non-zero boundary data. \textit{As a consequence, our estimates are new even for the heat equation on bounded domains.} This partial existence result is a nontrivial extension of the existence theory of very weak solutions from the elliptic setting to the quasilinear parabolic setting. Even though we only prove partial existence result, nevertheless we establish the necessary framework that when proved would lead to obtaining the full result for the homogeneous problem.

\textit{Keywords:} quasilinear parabolic equations, very weak solutions, initial boundary higher integrability, a priori estimates, existence.

\textit{2010 MSC:} 35D99, 35K59, 35K61, 35K92

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\textsuperscript{*}Supported by the National Research Foundation of Korea grant NRF-2015R1A2A1A15053024.
\textsuperscript{**}Supported by the National Research Foundation of Korea grant NRF-2015R1A4A1041675.

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In this paper, we are mainly interested in obtaining a priori estimates for (1.1) and (1.2) which will then be used to obtain the existence of very weak solutions to equations of the form (1.3). Here the nonlinearity \( A(x,t,\zeta) \) is modelled after the well known \( p \)-Laplace operator and \( \Omega \subset \mathbb{R}^n \) denotes a bounded domain with potentially nonsmooth boundary. The elliptic analogue of the estimates and existence theory studied in this paper are quite well understood. The first result for the elliptic homogeneous problem was proved in [20] with the sharp version of the a priori estimate and existence obtained recently in [5]. The parabolic counterpart to these questions have remained open for a long time and in this paper, we obtain some partial answers in this direction.

Weak solutions to (1.1) are in the space \( u \in L^2(0,T;L^2(\Omega)) \cap L^p(0,T;W^{1,p}_0(\Omega)) \) which allows one to use \( u \) as a test function. But from the definition of weak solution (see Definition 2.10), we see that the expression makes sense if we only assume \( u \in L^2(0,T;L^2(\Omega)) \cap L^s(0,T;W^{1,s}_0(\Omega)) \) for some \( s > \max\{p - 1,1\} \). But under this milder notion of solution called very weak solution, we lose the ability to use \( u \) as a test function. This difficulty was overcome in [22] where the method of Lipschitz truncation was developed to construct a suitable test function, which was then used to obtain interior higher integrability result below the natural exponent. This technique was subsequently extended in [2] to obtain analogous estimates up to the lateral boundary with zero boundary data. The extension of the higher integrability result for very weak solutions at the initial boundary seems to be nontrivial, mainly because of a lack of certain suitable cancellations and the lack of time derivative for the solutions.
In order to obtain the existence of very weak solution to (1.3), there are three main ingredients: first we need to obtain suitable a priori estimates that control the gradient of the solution in terms of the boundary data and secondly, we need to obtain higher integrability result for very weak solutions at the initial boundary and finally, we can combine the previous two estimates with standard compactness arguments to prove the existence result. In the subsequent three subsections, we shall discuss the main questions and the new tools that are going to be used in this paper.

In order to prove the main results of this paper, we need to develop new ideas which include bounds for a suitably modified maximal function on negative Sobolev spaces, careful construction of the Lipschitz truncation function at the initial boundary which preserves the necessary boundary values and the ability to handle non-zero boundary data which are complicated by the lack of a time derivative.

1.1. Discussion about the a priori estimate

In this subsection, we shall discuss the a priori estimates for very weak solutions

\[
\begin{aligned}
  \begin{cases}
    u_t - \text{div} A(x, t, \nabla u) = 0 & \text{on } \Omega \times (0, T), \\
    u = w & \text{on } \partial \Omega \times (0, T).
  \end{cases}
\end{aligned}
\] (1.1)

Since the notion of a solution does not a priori have any regularity in the time variable, the term $u_t$ (and hence $w_t$) is to be understood in the distributional sense. This complicates the construction of Lipschitz truncation due to [22] (see the boundary extension in [2]) which must be able to handle the boundary data as well as the distributional time derivative of $u$ and $w$. Since we are interested in controlling $|\nabla u|$ in terms of $|\nabla w|$ in suitable norms (whose exponents are below $p$), we are invariably forced to estimate $w_t$ in suitable negative Sobolev spaces (see Theorem 3.1).

We employ two main ideas to achieve this goal; the first is that the extension is very carefully obtained to preserve boundary data (see also [3] for more on this) and secondly, we define a new Maximal function (based on [7]) on appropriate negative Sobolev spaces. Combining these two ideas, we can develop the method Lipschitz truncation to handle non-zero boundary data and obtain a priori estimates below the natural exponent. These a priori estimates are new even for the heat equation on bounded domains.

In the process, we encounter a natural difficulty that arises due to the definition of the very weak solution. Since we make use of Steklov averages to define very weak solutions of (1.2), we are also forced to understand the relation between $\frac{d[w]_h}{dt}$ and $\frac{dw}{dt}$. It is well known that $\frac{d[w]_h}{dt}$ is a function whereas $\frac{dw}{dt}$ is a distribution and in general, the following does not hold:

\[ \frac{d[w]_h}{dt} \not\rightarrow \frac{dw}{dt} \quad \text{as } h \searrow 0. \]

To overcome this difficulty, we make an additional assumption regarding $\frac{dw}{dt}$ (see Remark 3.2) which enables us to obtain Lemma 4.8. While this assumption seems restrictive, in applications, the boundary data $w$ generally solves an analogous parabolic equation and thus, this difficulty goes away.

1.2. Discussion about the higher integrability estimate

In this subsection, we shall discuss the problem of obtaining higher integrability of very weak solutions to

\[
\begin{aligned}
  \begin{cases}
    u_t - \text{div} A(x, t, \nabla u) = 0 & \text{on } \Omega \times (0, T), \\
    u = w & \text{on } \partial \Omega \times \{t = 0\}.
  \end{cases}
\end{aligned}
\] (1.2)

In the interior, the result was proved in the seminal paper of [22] which laid the Foundation for the method of parabolic Lipschitz truncation. This was subsequently extended to the lateral boundary with zero boundary data in [2]. Following the idea of the proof in [22, 2], it becomes apparent that the method fails while handling non-zero boundary data at the initial boundary or at the initial boundary, the main obstruction being a lack of time derivative in general for the boundary data.

The second theorem we prove is the higher integrability for very weak solutions to (1.2) at the initial boundary with non-zero initial data. There are several comments to be made regarding this result and to understand the remarks, let us look at the higher integrability for weak solutions proved in [28, 29] at
the initial boundary. In those papers, they were able to take a suitable test function which exhibited a very crucial cancellation (see [27, Equation (5.6)]. In order to prove the higher integrability for very weak solutions, this cancellation needs to be preserved for the test function constructed through the method of Lipschitz truncation. In the interior case or at the lateral boundary with zero boundary data, such a cancellation trivially holds whereas it fails in general when handing non-zero boundary data at both the initial boundary and the lateral boundary.

In main theorem of this subsection Theorem 3.4, we obtain the higher integrability for very weak solutions at the initial boundary with non-zero initial data noting that the same estimate holds at the lateral boundary with non-zero boundary data and is optimal. In our construction, we are unable to recover the crucial cancellation of [27, Equation (5.6)] and hence at the initial boundary with non-zero data, our result is not optimal. Nevertheless, at least for zero initial data, the estimate is sharp. It would be interesting to obtain a modified construction of the Lipschitz test function that can preserve the necessary cancellations, which would then provide an optimal result at the initial boundary.

Let us now highlight some of the new ideas that are developed to obtain the result. Firstly, due to the presence of Steklov average in the Definition 2.10, the initial boundary value is not always preserved. Secondly, the problem with the lack of time derivative for the initial data is still present and to overcome this, we use the ideas developed in the proof of Theorem 3.1. It is interesting to note the unusual choice of the function (see (6.4)) used to perform the Lipschitz truncation upon. In particular, we handle the initial boundary problem as a problem at the lateral boundary, which leads to difficulties while applying the standard parabolic Poincaré’s inequality, and this is where we crucially exploit the fact that the extension constructed in (6.9) is zero on the bad part of the initial boundary. This is a very subtle technicality which originated from [2]. Once we have the modified construction, along with the bounds from Lemma 2.13, we can obtain a time localized version of the Caccioppoli type inequality followed by a reverse Hölder type inequality. Finally applying the parabolic Gehring’s lemma gives the desired higher integrability at the initial boundary. This result is new even for the heat equation on bounded domains.

1.3. Discussion about existence

In this subsection, we shall discuss the existence of very weak solutions to

\[
\begin{align*}
    u_t - \text{div } A(x, t, \nabla u) &= 0 \quad \text{on } \Omega \times (0, T), \\
    u &= u_0 \quad \text{on } \partial \Omega \times (0, T), \\
    u &= 0 \quad \text{on } \Omega \times \{t = 0\}.
\end{align*}
\] (1.3)

Before we explain the main result, let us discuss the main ideas behind the elliptic counterpart of the existence theory developed in [20, Theorem 2]. In that paper, the authors needed two main ingredients, first is an a priori estimate controlling the solution in terms of the boundary data and the second is an interior higher integrability result. Once both these estimates exist, then one can perform a standard approximation argument to get a sequence of weak solutions uniformly bounded in the right function spaces and then compactness methods can be used to deduce the converge of the approximate solution to the desired very weak solution.

In the parabolic setting, we also follow the same strategy. The desired a priori estimate is obtained in Theorem 3.1, but we now need higher integrability for very weak solutions in the interior (proved in [22] and at the initial boundary (see Corollary 3.5). Note that we can only use zero initial data to obtain the existence mainly because our higher integrability result at the initial boundary is not sharp for non-zero initial data.

We now need to first consider a suitable approximating sequence of solutions and show that this sequence converges to the desired very weak solution. To construct such a approximating sequence of solutions, the standard idea is to smoothen the given data (say by mollifying), but unfortunately, since \( \frac{du_0}{dt} \) is only a distribution, mollifying does not work. To overcome this difficulty, we assume either \( \frac{du_0}{dt} \) is an \( L^1 \) function, or more generally, we assume the existence of an approximating sequence satisfying (3.5) and (3.6) (see Remark 3.8 for more about the necessity of assuming the existence of an approximating sequence). Given such a sequence, we can then follow standard compactness arguments to obtain the desired very weak solutions to (1.3) which is Theorem 3.7.
1.4. Outline of the paper

In Section 2, we collect all the preliminary information along with structural assumptions regarding the nonlinearity and domain. In Section 3, we describe the main theorems and in Section 4, we shall recall and in some cases prove some well known lemmas that will be needed in the subsequent sections. In Section 5, we will obtain the proof of the a priori estimate from Theorem 3.1, in Section 6, we will obtain the proof of the higher integrability at the initial boundary as stated in Theorem 3.4 and finally in Section 7, we shall give the proof of the existence result from Theorem 3.7.

2. Preliminaries

2.1. Variational p-Capacity

Let \(1 < p < \infty\), then the variational \(p\)-capacity of a compact set \(K \subseteq \mathbb{R}^n\) is defined to be

\[
\text{cap}_{1,p}(K, \mathbb{R}^n) = \inf \left\{ \int_{\mathbb{R}^n} |\nabla \phi|^p \, dx : \phi \in C^\infty_c(\mathbb{R}^n), \ \chi_K(x) \leq \phi(x) \leq 1 \right\},
\]

where \(\chi_K(x) = 1\) for \(x \in K\) and \(\chi_K(x) = 0\) for \(x \notin K\). To define the variational \(p\)-capacity of an open set \(O \subseteq \mathbb{R}^n\), we take the supremum over the capacities of the compact sets contained in \(O\). The variational \(p\)-capacity of an arbitrary set \(E \subseteq \mathbb{R}^n\) is defined by taking the infimum over the capacities of the open sets containing \(E\). For further details, see [1, 19].

Let us now introduce the capacity density conditions which we later impose on the complement of the domain.

**Definition 2.1** (Uniform \(p\)-thickness). Let \(\hat{\Omega} \subseteq \mathbb{R}^n\) be a bounded domain and \(b_0, r_0\) be any two given positive constants. We say that the complement \(\hat{\Omega}^c := \mathbb{R}^n \setminus \hat{\Omega}\) is uniformly \(p\)-thick for some \(1 < p \leq n\) with constants \(b_0, r_0 > 0\), if the inequality

\[
\text{cap}_{1,p}(B_{r}(y_0) \cap \hat{\Omega}^c, B_{2r}(y_0)) \geq b_0 \cap_{1,p}(B_{r}(y_0), B_{2r}(y_0)),
\]

holds for any \(y_0 \in \partial \hat{\Omega}\) and \(r \in (0, r_0]\).

It is well-known that the class of domains with uniform \(p\)-thick complements is very large. They include all domains with Lipschitz boundaries or even those that satisfy a uniform exterior corkscrew condition, where the latter means that there exist constants \(c_0, r_0 > 0\) such that for all \(0 < t \leq r_0\) and all \(x \in \mathbb{R}^n \setminus \Omega\), there is \(y \in B_t(x)\) such that \(B_{t/c_0}(y) \subset \mathbb{R}^n \setminus \Omega\).

If we replace the capacity by the Lebesgue measure in Definition 2.1, then we obtain a measure density condition. A set \(E\) satisfying the measure density condition is uniformly \(p\)-thick for all \(p > 1\). If \(p > n\), then every non-empty set is uniformly \(p\)-thick. The following lemma from [27, Lemma 3.8] extends the capacity estimate in Definition 2.1 to make precise the notion of being uniformly \(p\)-thick:

**Lemma 2.2** ([27]). Let \(\hat{\Omega}\) be a bounded open set, and suppose that \(\mathbb{R}^n \setminus \hat{\Omega}\) is uniformly \(p\)-thick with constant \(b_0, r_0\). Choose any \(y \in \hat{\Omega}\) such that \(B_{2r}(y) \setminus \hat{\Omega} \neq \emptyset\), then there exists a constant \(b_1 = b_1(b_0, r_0, n, p) > 0\) such that

\[
\text{cap}_{1,p}(B_{2r}(y) \setminus \hat{\Omega}, B_{4r}(y)) \geq b_1 \text{cap}_{1,p}(B_{r}(y), B_{4r}(y)).
\]

Following the definition of \(p\)-thickness, a simple consequence of Hölder’s and Young’s inequality gives the following result (for example, see [27, Lemma 3.13] for the proof):

**Lemma 2.3.** Let \(1 < p \leq n\) be given and suppose a set \(E \subseteq \mathbb{R}^n\) is uniformly \(p\)-thick with constants \(b_0, r_0\). Then \(E\) is uniformly \(q\)-thick for all \(q \geq p\) with constants \(b_1, r_1\).

A very important result regarding the uniform \(p\)-thickness condition is that it has the self improving property (see [23] or [6, 26] for the details):

**Theorem 2.4** ([23]). Let \(1 < p \leq n\) be given and suppose a set \(E \subseteq \mathbb{R}^n\) is uniformly \(p\)-thick with constants \(b_0, r_0\). Then there exists an exponent \(q = q(n, p, b_0)\) with \(1 < q < p\) for which \(E\) is uniformly \(q\)-thick with constants \(b_1, r_1\).
We next state a generalized Sobolev-Poincaré’s inequality which was originally obtained by V. Maz’ya [25, Sec. 10.1.2] (see also [21, Sec. 3.1] and [1, Corollary 8.2.7]).

**Theorem 2.5.** Let $B$ be a ball and $\phi \in W^{1,p}(B)$ for some $p > 1$. Let $\kappa \in [1,n/(n-p)]$ if $1 < p < n$ and $\kappa \in [1,2]$ if $p = n$. Then there exists a constant $c = c(n,p) > 0$ such that

$$\left( \int_B |\phi|^p \, dx \right)^{\frac{1}{p}} \leq c \left( \frac{1}{\text{cap}_{1,p}(\Omega)} \int_B |\nabla \phi|^p \, dx \right)^{\frac{1}{p}},$$

where $N(\phi) = \{ x \in B : \phi(x) = 0 \}$.

2.2. Structural assumptions

In this subsection, we will mention all the assumptions we make on the operator $A(x,t,\zeta)$ as well as on the domain $\Omega$.

2.2.1. Assumptions on $A(x,t,\zeta)$

We shall now collect the assumptions on the nonlinear structure $A(\cdot,\cdot,\cdot)$. Let $T > 0$ be a fixed number, we then assume that the nonlinearity $A(\cdot,x,t,\zeta) : \Omega \times [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ is an Carathéodory function, i.e.,

$(x,t) \to A(x,t,\zeta)$ is measurable for every $\zeta \in \mathbb{R}^n$ and $\zeta \mapsto A(x,t,\zeta)$ is continuous for almost every $(x,t) \in \Omega \times [0,T]$.

We further assume that for a.e. $(x,t) \in \Omega \times [0,T]$ and for any $\zeta \in \mathbb{R}^n$, there exist two positive constants $\Lambda_0, \Lambda_1$ such that the following bounds are satisfied by the nonlinear structures:

$$\langle A(x,t,\zeta)\zeta,\zeta \rangle \geq \Lambda_0 |\zeta|^p - h_1 \quad \text{and} \quad |A(x,t,\zeta)| \leq \Lambda_1 |\zeta|^{p-1} + h_2,$$

where, the functions $h_1, h_2 : \Omega \times [0,T] \to \mathbb{R}$ are assumed to be measurable with bounded norm

$$h_0^\varphi := |h_1| + |h_2|^\frac{1}{\hat{q}-1} \quad \text{and} \quad \|h_0\|_{L^\hat{q}(\Omega \times [0,T])} < \infty \quad \text{for some} \quad \hat{q} \geq p.$$ (2.2)

An important aspect of the estimates obtained in this paper is that we do not make any assumptions regarding the smoothness of $A(x,t,\zeta)$ with respect to $x,t,\zeta$.

As the basic sets for our estimates, we will use parabolic cylinders where the radii in space and time are coupled. This is due to the fact that in the case that $p \neq 2$, the size of the cylinders intrinsically depends on the solution itself. This difficulty extends to the problems dealing with very weak solutions also.

In what follows, we will always assume the following restriction on the exponent $p$:

$$\frac{2n}{n+2} < p < \infty.$$ (2.3)

**Remark 2.6.** The restriction in (2.3) is necessary when dealing with parabolic problems because of the compact embedding $W^{1,p} \hookrightarrow L^2$. Since solutions to parabolic problems require us to deal with $L^2$-norm of the solution which comes from the time-derivative, this restriction is natural.

2.2.2. Assumptions on $\partial \Omega$

**Definition 2.7.** In this paper, we shall assume that the domain $\Omega$ is bounded and that its complement $\Omega^c$ is uniformly $p$-thick with constants $b_0, r_0$ in the sense of Definition 2.1.

Applying Theorem 2.4, we will henceforth fix the exponent $\varepsilon_0 = \varepsilon_0(n,p,b_0,r_0)$ to denote the self improvement property associated to $\partial \Omega$.

2.3. Function Spaces

Let $1 \leq \vartheta < \infty$, then $W_0^{1,\vartheta}(\Omega)$ denotes the standard Sobolev space which is the completion of $C_c^\infty(\Omega)$ under the $\| \cdot \|_{W_0^{1,\vartheta}}$ norm.

The parabolic space $L^\vartheta(0,T;W_0^{1,\vartheta}(\Omega))$ for any $\vartheta \in (1,\infty)$ is the collection of measurable functions $f(x,t)$ such that for almost every $t \in (0,T)$, the function $x \mapsto f(x,t)$ belongs to $W_0^{1,\vartheta}(\Omega)$ with the following norm being finite:

$$\|f\|_{L^\vartheta(0,T;W_0^{1,\vartheta}(\Omega))} := \left( \int_0^T \| f(\cdot,t) \|_{W_0^{1,\vartheta}(\Omega)}^\vartheta \, dt \right)^{\frac{1}{\vartheta}} < \infty.$$
Analogously, the parabolic space $L^{\vartheta}(0, T; W^{1,\vartheta}_0(\Omega))$ is the collection of measurable functions $f(x, t)$ such that for almost every $t \in (0, T)$, the function $x \mapsto f(x, t)$ belongs to $W^{1,\vartheta}_0(\Omega)$.

2.3.1. Negative Sobolev spaces

We denote $W^{-1,\vartheta}(\Omega) := \left(W^{1,\vartheta}_0(\Omega)\right)^*$ to be the usual dual space. Then we have the following well known lemma (see [12, Proposition 9.20] for the proof).

**Lemma 2.8.** Let $\Omega$ be any bounded domain, a function $\varphi \in W^{-1,\vartheta}(\Omega)$ if and only if there exists functions $\{\phi_0, \phi_1, \phi_2, \ldots, \phi_n\} \in L^{\vartheta'}(\Omega)$ such that

$$\langle \varphi, v \rangle = \int_{\Omega} \phi_0 v \, dx + \int_{\Omega} \sum_{i=1}^n \phi_i \frac{\partial v}{\partial x_i} \, dx \quad \forall \ v \in W^{1,\vartheta}_0(\Omega). \tag{2.4}$$

Moreover, there are $\{\phi_0, \phi_1, \phi_2, \ldots, \phi_n\} \in L^{\vartheta'}(\Omega)$ such that

$$\|\varphi\|_{W^{-1,\vartheta}(\Omega)} = \|\phi_0\|_{L^{\vartheta'}(\Omega)} + \sum_{i=1}^n \|\phi_i\|_{L^{\vartheta'}(\Omega)}. \tag{2.5}$$

Here we can formally integrate by parts (2.4) to get the representation

$$\varphi = \phi_0 - \sum_{i=1}^n \frac{\partial \phi_i}{\partial x_i} = \phi_0 - \text{div}(\phi_1, \phi_2, \ldots, \phi_n) = \phi_0 - \text{div} \vec{\phi}.$$

Since $\Omega$ is a bounded domain, we can take $\phi_0 = 0$.

An equivalent definition of the norm defined in (2.5) is given by

$$\|\phi\|_{W^{-1,\vartheta}(\Omega)} := \inf_{\phi = \varphi - \text{div} \psi} \|\varphi\|_{L^{\vartheta'}(\Omega)} + \|\psi\|_{L^{\vartheta'}(\Omega, \mathbb{R}^n)}, \tag{2.6}$$

where the infimum is taken over all representations of the form $\phi = \varphi - \text{div} \psi$ with $\varphi \in L^{\vartheta'}(\Omega)$ and $\psi \in L^{\vartheta'}(\Omega, \mathbb{R}^n)$.

2.4. Notion of Solution

There is a well known difficulty in defining the notion of solution for (1.1), (1.2) or (1.3) due to a lack of time derivative of $u$. To overcome this, one can either use Steklov average or convolution in time. In this paper, we shall use the former approach (see also [14, Page 20, Equation (2.5)] for further details).

We will use two equivalent notions of solutions depending on which equation we are handling.

2.4.1. Definition of Solution for (1.1) and (1.2)

Let us first define Steklov average as follows: let $h \in (0, 2T)$ be any positive number, then we define

$$u_h(\cdot, t) := \left\{ \begin{array}{ll}
\int_t^{t+h} u(\cdot, \tau) \, d\tau & t \in (0, T-h), \\
0 & \text{else}.
\end{array} \right. \tag{2.7}$$

We shall recall the following well known lemma regarding integral averages (for a proof in this setting, see for example [9, Chapter 8.2] for the details).

**Lemma 2.9.** Let $\psi : \mathbb{R}^{n+1} \to \mathbb{R}$ be an integrable function, $\lambda > 0$ be any fixed number and suppose $[\psi]_h(x, t) := \int_{t-h}^{t+h} \psi(x, \tau) \, d\tau$. Then we have the following properties:

(i) $[\psi]_h \to \psi$ a.e $(x, t) \in \mathbb{R}^{n+1}$ as $h \to 0$,

(ii) $[\psi]_h(x, \cdot)$ is continuous and bounded in time for a.e. $x \in \mathbb{R}^n$. 

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(iii) For any cylinder $Q_{r,\lambda^2} \subset \mathbb{R}^{n+1}$ with $r > 0$, there holds
\[
\iint_{Q_{r,\lambda^2}} [\psi]_h(x,t) \, dx \, dt \leq n \iint_{Q_{r,(\lambda+h)^2}} \psi(x,t) \, dx \, dt.
\]

(iv) The function $[\psi]_h(x,t)$ is differentiable with respect to $t \in \mathbb{R}$, moreover $[\psi]_h(x,\cdot) \in C^1(\mathbb{R})$ for a.e. $x \in \mathbb{R}^n$.

We shall now define the notion of very weak solution:

**Definition 2.10** (Very weak solution). Let $\beta \in (0,1)$ and $h \in (0,2T)$ be given and suppose $p - \beta > 1$. We then say $u \in L^2(0,T;L^2(\Omega)) \cap L^{p-\beta}(0,T;u_0 + W_0^{1,p-\beta}(\Omega))$ is a very weak solution of
\[
\begin{aligned}
&\left\{ \begin{array}{l}
\frac{\partial u}{\partial t} - \text{div} \, A(x,t,\nabla u) = 0 & \text{on } \Omega \times (0,T),
\end{array} \right.
\end{aligned}
\]
\[
\begin{aligned}
&u = u_0 & \text{on } \partial \Omega \times (0,T),
\end{aligned}
\]
\[
\begin{aligned}
&u = u_1 & \text{on } \Omega \times \{t = 0\}.
\end{aligned}
\]

if for any $\phi \in W_0^{1,p-\beta}(\Omega) \cap L^{\infty}(\Omega)$, the following holds:
\[
\int_{\Omega \times \{t\}} \frac{d[u]_h}{dt} \phi + ([A(x,t,\nabla u)]_h, \nabla \phi) \, dx = 0 \quad \text{for any } 0 < t < T - h.
\]

The initial condition is taken in the sense of $L^2(\Omega)$, i.e.,
\[
\int_B |u_h(x,0) - u_1(x)|^2 \, dx \xrightarrow{h \to 0} 0 \quad \text{for every } B \subset \Omega.
\]

**2.4.2. An equivalent Definition of very weak solution**

**Definition 2.11** (Very weak solution). Let $\beta \in (0,1)$ and $h \in (0,2T)$ be given and suppose $p - \beta > 1$. We then say $u \in L^2(0,T;L^2(\Omega)) \cap L^{p-\beta}(0,T;u_0 + W_0^{1,p-\beta}(\Omega))$ is a very weak solution of
\[
\begin{aligned}
&\left\{ \begin{array}{l}
\frac{\partial u}{\partial t} - \text{div} \, A(x,t,\nabla u) = 0 & \text{on } \Omega \times (0,T),
\end{array} \right.
\end{aligned}
\]
\[
\begin{aligned}
&u = u_0 & \text{on } \partial \Omega \times (0,T),
\end{aligned}
\]
\[
\begin{aligned}
&u = u_1 & \text{on } \Omega \times \{t = 0\}.
\end{aligned}
\]

if for any $\varphi \in C_0^\infty(\Omega_T)$, the following holds:
\[
\iint_{\Omega \times (0,t)} -u \varphi_t + (A(x,t,\nabla u), \nabla \varphi) \, dz = 0,
\]

and
\[
\int_B |u_h(x,0) - u_1(x)|^2 \, dx \xrightarrow{h \to 0} 0 \quad \text{for every } B \subset \Omega.
\]

**2.5. Some results about Maximal functions**

For any $f \in L^1(\mathbb{R}^{n+1})$, let us now define the strong maximal function in $\mathbb{R}^{n+1}$ as follows:
\[
\mathcal{M}(f)(x,t) := \sup_{Q_{a,b}(x,t)} \int_Q |f(y,s)| \, dy \, ds,
\]

where the supremum is taken over all parabolic cylinders $Q_{a,b}$ with $a,b \in \mathbb{R}^+$ such that $(x,t) \in Q_{a,b}$. An application of the Hardy-Littlewood maximal theorem in $x$- and $t$- directions shows that the Hardy-Littlewood maximal theorem still holds for this type of maximal function (see [24, Lemma 7.9] for details):
Lemma 2.12. If \( f \in L^1(\mathbb{R}^{n+1}) \), then for any \( \alpha > 0 \), there holds
\[
|\{ z \in \mathbb{R}^{n+1} : M(|f|)(z) > \alpha \}| \leq \frac{\alpha}{5^{n+2}}\|f\|_{L^1(\mathbb{R}^{n+1})},
\]
and if \( f \in L^\vartheta(\mathbb{R}^{n+1}) \) for some \( 1 < \vartheta \leq \infty \), then there holds
\[
\|M(|f|)\|_{L^{\vartheta}(\mathbb{R}^{n+1})} \leq C(\vartheta, \varrho)\|f\|_{L^\vartheta(\mathbb{R}^{n+1})}.
\]

Let us define the following new Maximal function defined in the dual Sobolev space: Let \( 1 < \vartheta < \infty \), then for any \( f \in W^{-1,\vartheta}(\mathbb{R}^n) \), we define
\[
\mathcal{M}^{-1,\vartheta}(f)(x) := \sup_{B \text{ ball}, B \ni x} \frac{1}{|B|^\vartheta}\|f\|_{W^{-1,\vartheta}(B)}.
\]  

We now have the following important boundedness result for \((2.9)\) obtained in [7, Proposition 2.5]. For the sake of completeness, we provide the proof.

Lemma 2.13. Let \( 1 < \vartheta < \infty \) be given and let \( q > \vartheta \) be fixed. Then for any \( f \in W^{-1,q}(\mathbb{R}^n) \), there holds
\[
\|\mathcal{M}^{-1,\vartheta}(f)\|_{L^q(\mathbb{R}^n)} \leq (\vartheta, q, n)\|f\|_{W^{-1,q}(\mathbb{R}^n)}.
\]

Proof. Applying Lemma 2.8, there exists \( \phi_0 \in L^\vartheta(\mathbb{R}^n) \) and \( \psi_0 \in L^q(\mathbb{R}^n, \mathbb{R}^n) \) such that \( f = \phi_0 - \text{div} \psi_0 \). Let \( B \subset \mathbb{R}^n \) be any given ball, then from (2.6), we see that
\[
\|f\|_{W^{-1,q}(B)} := \inf_{f = \phi - \text{div} \psi} \|\phi\|_{L^q(B)} + \|\psi\|_{L^q(B, \mathbb{R}^n)}.
\]

Using this, we get the following sequence of estimates:
\[
\frac{1}{|B|^\vartheta}\|f\|_{W^{-1,\vartheta}(B)}(x) = \inf_{f = \phi - \text{div} \psi} \frac{1}{|B|^\vartheta} \left( \|\phi\|_{L^\vartheta(B)} + \|\psi\|_{L^\vartheta(B, \mathbb{R}^n)} \right)
\leq \frac{1}{|B|^\vartheta} \left( \|\phi_0\|_{L^\vartheta(B)} + \|\psi_0\|_{L^\vartheta(B, \mathbb{R}^n)} \right)
= \left( \int_B |\phi_0|^\vartheta \, dx \right)^{\frac{1}{\vartheta}} + \left( \int_B |\psi_0|^\vartheta \, dx \right)^{\frac{1}{\vartheta}}
\leq M(|\phi_0|^\vartheta)^{\frac{1}{\vartheta}}(x) + M(|\psi_0|^\vartheta)^{\frac{1}{\vartheta}}(x).
\]

Here \( x \) is any point in the ball \( B \), since we have used uncentered Hardy-Littlewood maximal function as defined in (2.8). Now taking supremum over all balls \( B \ni x \) followed by taking the norm in \( L^\vartheta \), we get
\[
\left\| \sup_{B \ni x} \frac{1}{|B|^\vartheta}\|f\|_{W^{-1,\vartheta}(B)}(x) \right\|_{L^\vartheta(\mathbb{R}^n)} \leq M(|\phi_0|^\vartheta)^{\frac{1}{\vartheta}}_{L^\vartheta(\mathbb{R}^n)} + \|M(|\psi_0|^\vartheta)^{\frac{1}{\vartheta}}\|_{L^\vartheta(\mathbb{R}^n)}
\leq \|\phi_0\|_{L^\vartheta(\mathbb{R}^n)} + \|\psi_0\|_{L^\vartheta(\mathbb{R}^n)}
= C(\vartheta, q, n)\|f\|_{W^{-1,q}(\mathbb{R}^n)}.
\]

This completes the proof of the lemma.

2.6. Notations

We shall clarify the notation that will be used throughout the paper:

(i) We shall use \( \nabla \) or \( \text{div} \) to denote derivatives only with respect to the space variable \( x \).

(ii) We shall sometimes alternate between using \( \frac{df}{dt} \) \( \partial_t f \) and \( f' \) to denote the time derivative of a function \( f \).

(iii) We shall use \( D \) to denote the derivative with respect to both the space variable \( x \) and time variable \( t \) in \( \mathbb{R}^{n+1} \).
(iv) Let \( z_0 = (x_0, t_0) \in \mathbb{R}^{n+1} \) be a point and \( \rho, s > 0 \) be two given parameters and let \( \alpha \in (0, \infty) \). We shall use the following symbols to denote the following regions:

\[
I_s(t_0) := (t_0 - s, t_0 + s) \subset \mathbb{R}, \quad Q_{\rho,s}(z_0) := B_{\rho}(x_0) \times I_s(t_0) \subset \mathbb{R}^{n+1},
\]

\[
\alpha Q_{\rho,s}(z_0) := B_{\alpha \rho}(x_0) \times I_{\alpha s}(t_0) \subset \mathbb{R}^{n+1}, \quad \mathcal{H}_s(t_0) := \mathbb{R}^n \times I_s(t_0) \subset \mathbb{R}^{n+1},
\]

\[
\alpha \mathcal{H}_s(t_0) := \mathbb{R}^n \times I_{\alpha s}(t_0) \subset \mathbb{R}^{n+1}, \quad \Omega_{\rho,s}(z_0) := \Omega \cap B_{\rho}(x_0) \times I_s(t_0) \subset \mathbb{R}^{n+1},
\]

\[
\Omega_{\rho}(x_0) := \Omega \cap B_{\rho}(x_0) \subset \mathbb{R}^n.
\]

(v) We shall use the notation \( \{ t \leq 0 \} \) to denote the region \( \mathbb{R}^n \times (-\infty, 0] \). The region \( \{ t \geq 0 \} \) is analogously defined.

(vi) We shall use \( \int \) to denote the integral with respect to either space variable or time variable and use \( \iint \) to denote the integral with respect to both space and time variables simultaneously.

Analogously, we will use \( \int \) and \( \iiint \) to denote the average integrals as defined below: for any set \( A \times B \subset \mathbb{R}^n \times \mathbb{R} \), we define

\[
(f)_A := \frac{1}{|A|} \int_A f(x) \, dx, \quad (f)_{A \times B} := \frac{1}{|A \times B|} \int_{A \times B} f(x,t) \, dx \, dt.
\]

(vii) Given any positive function \( \mu \), we shall denote \( (f)_\mu := \int f \frac{\mu}{\| \mu \|_{L^1}} \, dm \) where the domain of integration is the domain of definition of \( \mu \) and \( dm \) denotes the associated measure.

(viii) Given any \( \lambda > 0 \), we can convert \( \mathbb{R}^{n+1} \) into a metric space where the parabolic cylinders correspond to balls under the parabolic metric given by:

\[
d_\lambda(z_1, z_2) := \max \left\{ |x_2 - x_1|, \sqrt{\frac{\lambda^{-2}}{\rho,s} |t_2 - t_1|} \right\}.
\]

(ix) In what follows, \( r_0 \) and \( b_0 \) will denote the constants arising from the assumption that \( \Omega^c \) is uniformly \( p \)-thick and denote \( \varepsilon_0 = \varepsilon_0(n, p, b_0, r_0) \) to be the self improvement exponent (see Definition 2.7).

3. Main Theorems

In this section, we will describe the main theorems that will be proved. Note that (2.3) is always in force. The first theorem is an a priori estimate that controls the gradient of the solution in terms of the boundary data.

**Theorem 3.1.** Let \( \Omega \) be a bounded domain whose complement is uniformly \( p \)-thick with constants \( (b_0, r_0) \) as in Definition 2.1. There exists \( \beta_1 = \beta_1(p, n, \Lambda_0, \Lambda_1, b_0, r_0) \) such that for any \( \beta \in (0, \beta_1) \), suppose

\[
w \in L^{p-\beta}(0, T; W^{1,p-\beta}(\Omega)), \quad \frac{dw}{dt} \in L^1(\Omega \times [0, T]) \quad \text{and} \quad \frac{dw}{dt} \in L^{\frac{p-\beta}{\beta}}(0, T; W^{-1,\frac{p-\beta}{\beta}}(\Omega)),
\]

be any given function. Then for any very weak solution \( u \in L^2(0, T; L^2(\Omega)) \cap L^{p-\beta}(0, T; W^{1,p-\beta}(\Omega)) \) solving (1.1), the following a priori estimate holds:

\[
\iint_{\Omega_T} |\nabla u|^{p-\beta} \, dz \leq (n,p, \beta, \Lambda_0, \Lambda_1) \iint_{\Omega_T} |\nabla w|^{p-\beta} \, dz + \iint_{\Omega_T} |h_0|^{p-\beta} \, dz + \left\| \frac{dw}{dt} \right\|_{L^{\frac{p-\beta}{\beta}}(0, T; W^{-1,\frac{p-\beta}{\beta}}(\Omega))}.
\]
Remark 3.2. The additional assumption $\frac{dw}{dt} \in L^1_{\text{loc}}(\Omega \times [0, T])$ in (3.1) can be replaced by the following weaker assumption: Let $t_1, t_2 \in [0, T], \phi \in C_c^\infty(\Omega)$ and $\varphi \in C^\infty[t_1, t_2]$, then assume the following holds

$$\int_{t_1}^{t_2} \left\langle \frac{d[w]}{dt}, \phi \right\rangle_{(W^{-1, \frac{p}{p-\beta}}(\Omega), W_p^0(\Omega)^\prime)}(t) \varphi(t) \, dt = \int_{t_1}^{t_2} \left[ \left\langle \frac{dw}{dt}, \phi \right\rangle_{(W^{-1, \frac{p}{p-\beta}}(\Omega), W_p^0(\Omega)^\prime)}(t) \right\rangle_h \right\rangle (t) \varphi(t) \, dt.$$  

(3.2)

This is necessary to obtain the estimate in Lemma 4.8. As a consequence, all the estimates in Section 4 are applicable provided (3.2) holds.

Heuristically speaking, the equality in (3.2) asks for a general form of the fundamental theorem of calculus to hold for distributions of the form $\frac{dw}{dt}$. In particular, we would need the following to hold for all $\phi \in C_c^\infty(\Omega)$ and $\varphi \in C_c^\infty(\mathbb{R})$:

$$\int_{\Omega} ((w\varphi)(x, b) - (w\varphi)(x, a)) \phi(x) \, dx = \int_{a}^{b} \int_{\Omega} \frac{d(w\varphi)}{dt}(x, t) \phi(x) \, dx \, dt.$$  

If the distribution $\frac{dw}{dt} \in L^1_{\text{loc}}(\Omega_T)$, then the above equality holds, which implies the equality in (3.2), see [18] for the details.

In the special case of the initial boundary value being zero, we get an analogous result stated below.

Corollary 3.3. Let $\Omega$ be a bounded domain whose complement is uniformly $p$-thick with constants $(b_0, r_0)$ as in Definition 2.1. Let

$$w \in L^{p-\beta}(0, T; W^{1,p-\beta}(\Omega)), \quad \frac{dw}{dt} \in L^1_{\text{loc}}(\Omega \times [0, T]) \quad \text{and} \quad \frac{dw}{dt} \in L^\frac{p}{p-\beta}(0, T; W^{-1, \frac{p}{p-\beta}}(\Omega)),$$

be any given function. Then there exists a $\beta_1 = \beta_1(p, n, \Lambda_0, \Lambda_1, b_0, r_0)$ such that for any $\beta \in (0, \beta_1)$ and any very weak solution $u \in L^2(0, T; L^q(\Omega)) \cap L^{p-\beta}(0, T; W^{1,p-\beta}(\Omega))$ solving

$$\begin{cases}
 u_t - \text{div} \mathcal{A}(x, t, \nabla u) = 0 & \text{on } \Omega \times (0, T), \\
 u = w & \text{on } \partial \Omega \times (0, T), \\
 u = 0 & \text{on } \Omega \times \{ t = 0 \},
\end{cases}$$

the following a priori estimate holds:

$$\int_{\Omega_T} |\nabla u|^{p-\beta} \, dz \leq (n, p, \beta, \Lambda_0, \Lambda_1) \int_{\Omega_T} (|\nabla w| + |b_0|)^{p-\beta} \, dz + \left\| \frac{dw}{dt} \right\|_{L^\frac{p}{p-\beta}(0, T; W^{-1, \frac{p}{p-\beta}}(\Omega))}.$$  

Here the initial condition is taken to hold in the sense

$$\int_{B} (|w_h(x, 0)|^2 + |u_h(x, 0)|^2) \, dx \xrightarrow{\text{h} \to 0} 0 \quad \text{for all } B \subset \Omega.$$  

The second theorem that we will prove is a higher integrability result for very weak solutions at the initial boundary.

Theorem 3.4. Let $w$ be such that

$$w \in L^p(0, T; W^{1,p}(\Omega)) \quad \text{and} \quad \frac{dw}{dt} \in L^\frac{p}{p-\beta}(0, T; W^{-1, \frac{p}{p-\beta}}(\Omega)) \cap L^1_{\text{loc}}(\Omega_T),$$

then there exists $\beta_2 = \beta_2(p, n, \Lambda_0, \Lambda_1) \in (0, 1)$ such that the following holds: for any $\beta \in (0, \beta_2)$ and any very weak solution $u \in L^2(0, T; L^q(\Omega)) \cap L^{p-\beta}(0, T; W^{1,p-\beta}(\Omega))$ solving (1.2) in $\Omega_T$, we have the following improved integrability $u \in L^2(0, T; L^q(\Omega)) \cap L^p(0, T; W^{1,p}(\Omega))$. In particular, for any fixed $\rho \in (0, \infty)$ and
where $h_0$ is from (2.2), $\bar{w}$ is as obtained in (6.5), $C = C(n, p, \Lambda_0, \Lambda_1)$ and

$$d := \begin{cases} 2 - \beta & \text{if } p \geq 2, \\ p - \beta - \frac{(2-p)\alpha}{2} & \text{if } p < 2. \end{cases} \quad (3.3)$$

Note that Theorem 3.4 holds under the weaker assumption from Remark 3.2 instead of the stronger assumption $\frac{d\bar{w}}{dt} \in L^1_{\text{loc}}(\Omega_T)$. In the special case of zero initial data, we have the following important corollary:

**Corollary 3.5.** There exists $\beta_2 = \beta_2(n, p, \Lambda_0, \Lambda_1) \in (0, 1)$ such that the following holds: for any $\beta \in (0, \beta_2)$ and any very weak solution $u \in L^2(0, T; L^2_{\text{loc}}(\Omega)) \cap L^p - (0, T; W^{1,p - \beta}_{\text{loc}}(\Omega))$ solving

$$\begin{cases} u_t - \text{div} \mathcal{A}(x, t, \nabla u) = 0 & \text{in } \Omega \times (0, T), \\ u = 0 & \text{on } \Omega \times \{t = 0\}, \end{cases}$$

we have the following improved integrability $u \in L^2(0, T; L^2_{\text{loc}}(\Omega)) \cap L^p(0, T; W^{1,p - \beta}_{\text{loc}}(\Omega))$. In particular, for any fixed $\rho \in (0, \infty)$ and $s \in (0, T/2)$, the following quantitative estimate holds: let $Q_{2\rho, 2s} \subseteq \Omega \times \mathbb{R}$ be any parabolic cylinder, then there holds

$$\iint_{Q_{\rho, s}} |\nabla u|^p \chi_{[0, T]} \, dz \leq C \left[ \iint_{Q_{2\rho, 2s}} (|\nabla u| + h_0)^{p - \beta} \chi_{[0, T]} \, dz \right]^{1 + \frac{\beta}{p}} + \iint_{Q_{2\rho, 2s}} (1 + h_0^p) \chi_{[0, T]} \, dz,$$

where $h_0$ is from (2.2), $C = C(n, p, \Lambda_0, \Lambda_1)$ and $d$ is from (3.3).

In the above theorem, it is easy to see that the choice of the parabolic cylinder $Q_{\rho, s}$ is made such that it crosses the initial boundary at $\{t = 0\}$. If the cylinder is completely contained in $\Omega \times (0, T)$, then there is nothing to prove as this is the main result obtained in [22]. Hence, the new contribution is only when the cylinder crosses the initial boundary.

**Remark 3.6.** In what follows, we will define $\beta_0 := \min\{\beta_1, \beta_2, \beta_{\text{int}}\}$ where $\beta_1$ is from Theorem 3.1, $\beta_2$ is from Corollary 3.5 and $\beta_{\text{int}}$ is the interior higher integrability exponent such that [22, Theorem 2.8] holds for all $\beta \in (0, \beta_{\text{int}}]$. All the subsequent results will hold for any $\beta \in (0, \beta_0)$.

Finally, we are ready to state the main existence theorem that will be proved in this paper.

**Theorem 3.7.** Let $\Omega$ be a bounded domain whose complement is uniformly $p$-thick with constants $(b_0, r_0)$ as in Definition 2.1, let $\beta \in (0, \beta_0)$ be given with $\beta_0 = \beta_0(n, p, \Lambda_0, \Lambda_1, b_0, r_0)$ as in Remark 3.6 and (2.2) holds. Moreover, let the nonlinearity $\mathcal{A}$ satisfy (2.1) and

$$\langle \mathcal{A}(x, t, \eta) - \mathcal{A}(x, t, \zeta), \eta - \zeta \rangle \geq \Lambda_0 (|\eta|^2 + |\zeta|^2)^{\frac{p}{p - \beta}} |\eta - \zeta|^2 \quad \forall (x, t) \in \Omega_T \quad \forall \eta, \zeta \in \mathbb{R}^n. \quad (3.4)$$

Suppose that the boundary condition satisfies

$$u_0 \in L^{p - \beta}(0, T; W^{1,p - \beta}_{\text{loc}}(\Omega)) \quad \text{with} \quad \frac{d u_0}{d t} \in L^{\frac{p}{p - \beta}}(0, T; W^{-1,\frac{p}{p - \beta}}_{\text{loc}}(\Omega)) \cap L^1_{\text{loc}}(\Omega_T),$$

$$u_0(\eta, t) = 0 \quad \text{on} \quad \partial \Omega \times (0, T).$$
Remark 3.8. In Theorem 3.7, the assumption $\frac{du_0}{dt} \in L^1(\Omega_T)$ can be used to construct an approximating sequence $u^k_0$ satisfying (3.5) and (3.6) through the process of mollification, i.e., if $\eta_k \in C^\infty_c(\mathbb{R}^{n+1})$ is the standard mollifier, then letting $u^k_0 := u_0 * \eta_k$ would be an admissible approximating sequence (see [12, Section 4.4] for more details).

On the other hand, the assumptions (3.5) and (3.6) are necessary if $\frac{du_0}{dt}$ is a distribution and not a function.

4. Some well known lemmas

4.1. Sobolev and Sobolev-Poincaré lemmas

Let us recall a time localised version of the parabolic Poincaré inequality proved in [3, Lemma 4.2].

Lemma 4.1. Let $\psi \in L^q(0,T;W^{1,q}(\Omega))$ with $q \in [1,\infty)$ and suppose that $B_{\rho} \subseteq \Omega$ be compactly contained ball of radius $\rho > 0$. Let $I \subseteq (0,T)$ be a time interval and $\xi(x,t) \in L^1(B_{\rho} \times I)$ be any positive function such that

$$\frac{\|\xi\|_{L^\infty(B_{\rho} \times I)}}{|B_{\rho} \times I|} \leq \frac{\|\xi\|_{L^1(B_{\rho} \times I)}}{|B_{\rho} \times I|}.$$  

and $\mu(x) \in C^\infty_c(B_{\rho})$ be such that $\int_{B_{\rho}} \mu(x) \, dx = 1$ with $|\mu| \leq \frac{1}{\rho^n}$ and $|\nabla \mu| \leq \frac{1}{\rho^{n+1}}$, then there holds:

$$\int_{B_{\rho} \times I} \left| \psi(z) \chi_j - \left( \frac{\psi \chi_j}{\rho} \right)_\xi \right| ^q \, dz \leq (n,q) \int_{B_{\rho} \times I} |\nabla \psi| ^q \chi_j \, dz + \sup_{t_1,t_2 \in I} \left| \left( \frac{\psi \chi_j}{\mu} \right)_{\mu} (t_2) - \left( \frac{\psi \chi_j}{\mu} \right)_{\mu} (t_1) \right| ^q,$$

where $(\psi)_\xi := \int_{B_{\rho} \times I} \psi(z) \xi \, dz$, $(\psi \chi_j)_\mu (t_i) := \int_{B_{\rho}} \psi(x,t_i) \mu(x) \chi_j \, dx$ and $J \subseteq (-\infty,\infty)$ be some fixed time-interval.

In the above Lemma 4.1, we can take any bounded region $\bar{\Omega}$ instead of $B_\rho$ such that $\bar{\Omega}^c$ is uniformly $p$-thick as in Definition 2.1. In that case, the constants will subsequently depend on the $p$-thickness constants of $\bar{\Omega}^c$.

We will need the following well known Gagliardo-Nirenberg’s inequality (see [10, Lemma 3.2] for the details):
Lemma 4.2. Let $B_\rho \subset \mathbb{R}^n$ with $\rho \in (0, 1]$ and $f \in W^{1, \vartheta}(B_\rho)$ and $1 \leq \sigma, \vartheta, r \leq \infty$ and $\delta \in (0, 1)$ be given satisfying
\[-\frac{n}{\sigma} \leq \delta \left(1 - \frac{n}{\vartheta}\right) - (1 - \delta)\frac{n}{r}.\] (4.1)
Then there exists a constant $C = C(n, \sigma, \vartheta)$ such that there holds
\[\int_{B_\rho} \frac{|f|^r}{\rho} \, dx \leq C \left( \int_{B_\rho} \frac{|f|^\vartheta}{\rho} + |\nabla f|^\sigma \, dx \right)^{\frac{r}{(1-\delta)\vartheta}} \left( \int_{B_\rho} \frac{|f|^r}{\rho} \, dx \right)^{\frac{1-\delta}{r}}.\]

4.2. Whitney type decomposition lemma
Let us first recall a well known Whitney type decomposition Lemma proved in [15, Lemma 3.1] or [11, Chapter 3]:

Lemma 4.3. Let $E$ be any closed set and $\lambda \in (0, \infty)$ be a fixed constant. Define $\gamma := \lambda^{2-p}$, then there exists a $\gamma$-parabolic Whitney covering $\{Q_i(z_i)\}$ of $E^c$ in the following sense:
(W1) $Q_j(z_j) = B_j(x_j) \times I_j(t_j)$ where $B_j(x_j) = B_{r_j}(x_j)$ and $I_j(t_j) = (t_j - \gamma r_j^2, t_j + \gamma r_j^2)$.
(W2) $d_\lambda(z_j, E) = 16r_j$.
(W3) $\bigcup_j \frac{1}{2}Q_j(z_j) = E^c$.
(W4) For all $j \in \mathbb{N}$, we have $8Q_j \subset E^c$ and $16Q_j \cap E \neq \emptyset$.
(W5) If $Q_j \cap Q_k \neq \emptyset$, then $\frac{1}{2}r_k \leq r_j \leq 2r_k$.
(W6) $\frac{1}{4}Q_j \cap \frac{1}{4}Q_k = \emptyset$ for all $j \neq k$.
(W7) $\sum_j \chi_{\frac{3}{4}Q_j}(z) \leq c(n)$ for all $z \in E^c$.

For a fixed $k \in \mathbb{N}$, let us define $A_k := \left\{ j \in \mathbb{N} : \frac{3}{4}Q_k \cap \frac{3}{4}Q_j \neq \emptyset \right\}$, then we have
(W8) For any $i \in \mathbb{N}$, we have $\#A_i \leq c(n)$.
(W9) Let $i \in \mathbb{N}$ be given and let $j \in A_i$, then $\max\{|Q_j|, |Q_i|\} \leq C(n)|Q_j \cap Q_i|$.
(W10) Let $i \in \mathbb{N}$ be given and let $j \in A_i$, then $\max\{|Q_j|, |Q_i|\} \leq \frac{3}{4}Q_j \cap \frac{3}{4}Q_i$.
(W11) Let $i \in \mathbb{N}$ be given, then for any $j \in A_i$, we have $\frac{3}{4}Q_j \subset 4Q_i$.

Subordinate to the above Whitney covering, we have an associated partition of unity which we recall in the following lemma.

Lemma 4.4. Associated to the covering given in Lemma 4.3, there exists functions $\{\Psi_j\}_{j \in \mathbb{N}} \in C^\infty_c \left( \frac{3}{4}Q_j \right)$ such that the following holds:
(W12) $\chi_{\frac{3}{4}Q_j} \leq \Psi_j \leq \chi_{\frac{3}{4}Q_j}$.
(W13) $\|\Psi_j\|_\infty + r_j\|\nabla\Psi_j\|_\infty + r_j^2\|\nabla^2\Psi_j\|_\infty + \lambda^{2-p}r_j^2\|\partial_t\Psi_j\|_\infty \leq C(n)$.
(W14) Let $i \in \mathbb{N}$ be given, then $\sum_{j \in A_i} \Psi_j(z) = 1$ for all $z \in \frac{3}{4}Q_i$. 

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4.3. A few other well known lemmas

Let us first recall the well known iteration lemma (see [10, Lemma 3.3] for the details):

**Lemma 4.5.** Let \( \delta \in (0,1) \), \( B \geq 0 \), \( A \geq 0 \), \( \alpha > 0 \) and \( 0 < r < \rho < \infty \) and let \( f \geq 0 \) be a bounded, measurable function satisfying

\[
 f(t_1) \leq \delta f(t_2) + A(t_2-t_1)^{-\alpha} + B \quad \text{for all } r \leq t_1 < t_2 \leq \rho,
\]

then there exists a constant \( C = C(\alpha, \delta) \) such that the following holds:

\[
 f(r) \leq C \left( A(\rho - r)^{-\alpha} + B \right).
\]

We will use the following result which can be found in [16, Theorem 3.1] (see also [13] where it was originally proved) for proving the Lipschitz regularity for the constructed test function. This very important simplification of the original technique from [22] first appeared in [11, Chapter 3].

**Lemma 4.6.** Let \( \gamma > 0 \) and \( D \subset \mathbb{R}^{n+1} \) be given. For any \( z \in D \) and \( r > 0 \), let \( Q_{r,\gamma,r}(z) \) be the parabolic cylinder centred at \( z \) with radius \( r \). Suppose there exists a constant \( C > 0 \) independent of \( z \) and \( r \) such that the following bound holds:

\[
 \frac{1}{|Q_{r,\gamma,r}(z) \cap D|} \int_{Q_{r,\gamma,r}(z) \cap D} \left| \frac{f(x,t) - (f)(Q_{r,\gamma,r}(z) \cap D)}{r} \right| \, dx \, dt \leq C \quad \forall \ z \in D \text{ and } r > 0,
\]

then \( f \in C^{0,1}(D) \) with respect to the metric \( d(z_1, z_2) := \max\{|x_1 - x_2|, \sqrt{\gamma^{-1}}|t_1 - t_2|\} \).

Finally, let us recall the parabolic version of the well known Gehring's lemma (for example, see [10, Lemma 6.4] for the details).

**Lemma 4.7.** Let \( \alpha_0 \geq 1 \), \( \kappa \geq 1 \), \( \varepsilon_0 > 0 \), \( p > 1 \) and \( \beta_{gh} > 0 \) be given. Let \( q \) be given such that \( 1 < p - \varepsilon_0 \leq q < p - 2\beta < p - \beta \) for some \( \beta \in (0, \beta_{gh}) \). Furthermore, for a cylinder \( Q_2 = Q_{2p,2p^2} \), let \( f \in L^{p-\beta}(Q_2) \) and \( g \in L^p(Q_2) \) for some \( \tilde{p} \geq p > 1 \) be given. Suppose for each \( \lambda \geq \alpha_0 \) and almost every \( z \in Q_2 \) with \( f(z) > \lambda \), there exists a parabolic cylinder \( Q = Q_{\rho,s}(z) \subset Q_2 \) such that

\[
 \frac{\lambda^{p-\beta}}{\kappa} \leq \int_Q f^{p-\beta}(\tilde{z}) \chi_{Q_{\rho,s}(z)} \, d\tilde{z} \leq \kappa \left( \int_Q f^q(\tilde{z}) \chi_{Q_{\rho,s}(z)} \, d\tilde{z} \right)^{\frac{\epsilon_0}{q}} + \kappa \int_Q g^{p-\beta}(\tilde{z}) \chi_{Q_{\rho,s}(z)} \, d\tilde{z} \leq \kappa^2 \lambda^{p-\beta},
\]

then there exists \( \delta_0 = \delta_0(\kappa, p, \beta, q, \varepsilon_0) \) and \( C = C(\kappa, p, \beta, q, \varepsilon_0) \), such that \( f \in L^{p-\beta+\delta_1}(Q_2) \) with \( \delta_1 = \min\{\delta_0, \tilde{p} - p + \beta\} \). This improved higher integrability comes with the following bound:

\[
 \int_{Q_2} f^{p-\beta+\delta}(\tilde{z}) \chi_{Q_{\rho,s}(z)} \, d\tilde{z} \leq \alpha_0^{\delta} \int_{Q_2} f^{p-\beta}(\tilde{z}) \chi_{Q_{\rho,s}(z)} \, d\tilde{z} + \int_{Q_2} g^{p-\beta+\delta}(\tilde{z}) \chi_{Q_{\rho,s}(z)} \, d\tilde{z} \quad \text{for all } \delta \in (0, \delta_1].
\]

4.4. Crucial lemma

The lemma concerns very weak solutions of (1.1) which will be used in Section 5 and Section 6.

**Lemma 4.8.** Let \( u \in L^2(0,T; L^2(\Omega)) \cap L^{p-\beta}(0,T; w + W_0^{1,p-\beta}(\Omega)) \) be a very weak solution of (1.1) for some \( 0 \leq \beta \leq \min\{1, p - 1\} \) and \( h \in (0,T) \). Let \( \phi(x) \in C_0^\infty(\Omega) \) and \( \varphi(t) \in C^\infty(\mathbb{R}) \) be non-negative functions and \( [u]_{h}, [w]_{h} \) be the Steklov average as defined in Lemma 2.9. Furthermore, let \( w \) satisfy the hypothesis in (3.1) (see Remark 3.2), then the following estimate holds for any time interval \( (t_1, t_2) \subset (0,T) \):

\[
 \left| \left( [u - w]_{h} \varphi \right)_{\phi} (t_2) - \left( [u - w]_{h} \varphi \right)_{\phi} (t_1) \right| \leq C(\Lambda_1, p) \left\| \nabla \phi \right\|_{L^\infty(\Omega)} \left\| \varphi \right\|_{L^\infty(t_1, t_2)} \int_{\Omega \times (t_1, t_2)} |(\nabla u)| + |h_0|^{p-1}h \, dz
\]

\[
 + \left\| \varphi \right\|_{L^\infty(t_1, t_2)} \int_{t_1}^{t_2} \left[ \int_{\Omega} \langle \nabla \phi, \nabla \phi \rangle \, dx \right]_h \, dt
\]

\[
 + \left\| \phi \right\|_{L^\infty(\Omega)} \left\| \varphi \right\|_{L^\infty(t_1, t_2)} \int_{\Omega \times (t_1, t_2)} |[u - w]_{h}| \, dz.
\]
\textbf{Proof.} Let us use \( \phi(x)\varphi(t) \) as a test function in Definition 2.10 solving (1.1) to get
\[
\int_{\Omega \times \{t\}} \frac{d[u]_h}{dt}(x,t)\phi(x)\varphi(t) \, dx + \langle [A(x,t,\nabla u)]_h, \nabla \phi \rangle (x,t)\varphi(t) \, dx = 0.
\]
Integrating over \((t_1, t_2)\), we get
\[
\int_{t_1}^{t_2} \int_{\Omega} \frac{d[u]_h}{dt}(x,t)\phi(x)\varphi(t) \, dx \, dt = -\int_{t_1}^{t_2} \int_{\Omega} \langle [A(x,t,\nabla u)]_h, \nabla \phi \rangle (x,t)\varphi(t) \, dx \, dt \\
- \int_{t_1}^{t_2} \int_{\Omega} \frac{d[w]_h}{dt}(x,t)\phi(x) \, dx \, dt \\
+ \int_{t_1}^{t_2} \int_{\Omega} [u - w]_h(x,t)\phi(x) \frac{d\varphi(t)}{dt} \, dx \, dt.
\]
We estimate each of the terms as follows:
\[
\left| ([u - w]_h\varphi) (t_2) - ([u - w]_h\varphi) (t_1) \right| \leq \begin{align*}
&\overset{(a)}{=} C(A_1, p)\|\nabla \phi\|_{L^\infty(\Omega)}\|\varphi\|_{L^\infty(t_1, t_2)} \int_{\Omega \times (t_1, t_2)} |(\nabla u) + [h_0])^{p-1}|_h \, dx \\
&+ \|\varphi\|_{L^\infty(t_1, t_2)} \int_{t_1}^{t_2} \left[ \frac{d[w]}{dt} \phi \right]_{(W^{-1,p-\beta}(\Omega), W^{1,p-\beta}(\Omega))} \, dt \\
&+ \|\phi\|_{L^\infty(\Omega)}\|\varphi\|_{L^\infty(t_1, t_2)} \int_{\Omega \times (t_1, t_2)} |[u - w]_h| \, dz \\
&\overset{(b)}{=} C(A_1, p)\|\nabla \phi\|_{L^\infty(\Omega)}\|\varphi\|_{L^\infty(t_1, t_2)} \int_{\Omega \times (t_1, t_2)} |(\nabla u) + [h_0])^{p-1}|_h \, dz \\
&+ \|\varphi\|_{L^\infty(t_1, t_2)} \int_{t_1}^{t_2} \left[ \int_{\Omega} \langle \bar{w}, \nabla \phi \rangle \, dx \right]_h \, dt \\
&+ \|\phi\|_{L^\infty(\Omega)}\|\varphi\|_{L^\infty(t_1, t_2)} \int_{\Omega \times (t_1, t_2)} |[u - w]_h| \, dz.
\end{align*}
\]
To obtain (a), we made use of assumptions (2.1) and (3.2) and to obtain (b), we made use of Lemma 2.8 to obtain the representation \( \frac{d\bar{w}}{dt} = \text{div} \bar{w} \) for some \( \bar{w} \in L^\frac{p}{p-1}(0, T; L^\frac{p}{p-1}(\Omega, \mathbb{R}^n)) \). This completes the proof of the lemma. \qed

\section{5. A priori estimates for the homogeneous problem}

In this section, let us fix an exponent \( q \) such that
\[
1 < p - \varepsilon_0 < q \leq p - 2\beta < p - \beta < p,
\]
where \( \varepsilon_0 = \varepsilon_0(n, p, b_0, r_0) \) is the exponent described in (ix) and \( \beta \) is a constant to be chosen sufficiently small later on.

Let us define the following new maximal functional for any \( \vartheta \in [1, \infty) \) and any \( Q \subset \mathbb{R}^{n+1} \):
\[
\mathcal{P}^{-1,\vartheta}(f\chi_Q)(x,t) := \sup_{(a,b) \ni \vartheta \ni B} \sup_{\Omega \ni B \ni x} \frac{1}{b - a} \int_a^b \frac{1}{|B|^{1\vartheta}} \|f\chi_q\|_{W^{-1,\vartheta}(B)\chi_Q} \, dt,
\]
where \( \| \cdot \|_{W^{-1,\vartheta}(B)} \) is defined in Lemma 2.8 and the supremum is taken over all parabolic cylinders of the form \( B \times (a, b) \) containing the point \( (x,t) \).

Given any \( f \in L^\frac{p}{p-1}(0, T; W^{-1,\frac{p}{p-1}}(\Omega)) \), from Lemma 2.8, we see that there exists a vector field \( \tilde{f} \in \mathbb{R}^n \) such that
\[
\text{div} \tilde{f} = f \
\]
$L^{p,q}_\Omega(\Omega,\mathbb{R}^n)$ satisfying $\|f\|_{L^{p,q}_\Omega(0,T;W^{-1},\mathbb{R}^n)} = \|\hat{f}\|_{L^{p,q}_\Omega(\Omega,\mathbb{R}^n)}$. Using this, we get

$$
\|P^{-\beta} \nabla(f_{\Omega_T})\|_{L^{p,q}_\Omega(\mathbb{R}^{n+1})} \leq \left\| \sup_{(a,b) \geq t} \sup_{B \ni x} \int_a^b \frac{1}{|B|} \|\nabla f\|_{L^{p,q}_\Omega(B,\mathbb{R}^n)} \chi_{[0,t]} \, ds \right\|_{L^{p,q}_\Omega(\mathbb{R}^{n+1})} \\
\leq \left\| \sup_{(a,b) \geq t} \sup_{B \ni x} \left( \int_a^b \frac{1}{|B|} \|\nabla f\|_{L^{p,q}_\Omega(B,\mathbb{R}^n)} \, ds \right) \right\|_{L^{p,q}_\Omega(\mathbb{R}^{n+1})} \\
= M(|\hat{f}|_{\nabla(f_{\Omega_T})}^{\frac{1}{\beta}}) \leq \|\hat{f}\|_{L^{p,q}_\Omega(\Omega,\mathbb{R}^n)}^{\frac{1}{\beta}} \|f\|_{L^{p,q}_\Omega(0,T;W^{-1},\mathbb{R}^n)} (5.3)
$$

To obtain (a), we made use of the standard strong maximal function bound from Lemma 2.12 along with the observation $\frac{p-\beta}{q} > 1$ and to obtain (b), we made use of Lemma 2.8.

Let $u$ and $w$ be as in Theorem 3.1, then define the following functions:

$$
v(z) = u(z) - w(z) \quad \text{and} \quad v_h(z) = [u - w]_h(z), \quad z \in \mathbb{R}^{n+1}, \quad (5.4)
$$

where $[\cdot]_h(z)$ denotes the usual Steklov average defined in (2.7). From (1.1) and Lemma 2.9, we see that

$$
v_h \rightharpoonup u \quad \text{and} \quad v(z) = 0 \quad \text{for} \quad z \in \partial_p(\Omega \times (0, T)).
$$

In subsequent calculations, we extend $v_h$ by zero outside $\Omega \times [0, \infty)$.

### 5.1. Construction of test function

Let us define the following function:

$$
g(z) := \max \left\{ M \left( |\nabla u|^q + (|\nabla u| + |h_0|)^q + |\nabla w|^q \right) \chi_{\Omega_T}^{1_q} (z), P^{-\beta} \nabla (w' \chi_{\Omega_T})^{1_q} (z) \right\}, \quad (5.5)
$$

where $v$ is defined in (5.4), $u$ and $w$ are as in the hypothesis of Theorem 3.1 and $q$ is from (5.1). We then have the following estimate for $g$:

$$
\|g\|_{L^{p,q}(\mathbb{R}^{n+1})} \overset{\text{Lemma 2.12}}{\leq} \left\| (|\nabla u| + |h_0|) \chi_{\Omega_T}\|_{L^{p,q}(\mathbb{R}^{n+1})}^{\frac{p-\beta}{p}} + \|\nabla w'\|_{L^{p,q}(\mathbb{R}^{n+1})} \right\|_{L^{p,q}(\mathbb{R}^{n+1})} \\
\overset{(5.3)}{\leq} \left\| (|\nabla u| + |h_0|) \chi_{\Omega_T}\|_{L^{p,q}(\mathbb{R}^{n+1})}^{\frac{p-\beta}{p}} + \|\nabla w'\|_{L^{p,q}(\mathbb{R}^{n+1})} \right\|_{L^{p,q}(\mathbb{R}^{n+1})} (5.6)
$$

For a fixed $\lambda > 0$, let us define the **good set** by

$$
E_\lambda := \{(x,t) \in \mathbb{R}^{n+1} : g(x,t) \leq \lambda\},
$$

and apply Lemma 4.3 and Lemma 4.4 with $E = E_\lambda^c$ to get a covering of $E_\lambda^c$. Recall that the intrinsic scaling is of the form

$$
\gamma := \lambda^{2-p} \quad \text{and} \quad Q_j(x,t) = B_{r_j}(x) \times (t_j - \gamma r_j^2, t_j + \gamma r_j^2). \quad (5.7)
$$

Now we define the following Lipschitz extension function as follows:

$$
v_{\lambda,h}(z) := v_h(z) - \sum_i \Psi_t(z)(v_h(z) - v_h^i), \quad (5.8)
$$
where
\[ v_h^i := \begin{cases} \\ 
\frac{1}{\|\Psi_i \|_{L^1(\Omega)}} \int_{\Omega} v_h(z) \Psi_i(z) \chi_{[0,T]} \, dz & \text{if } \frac{3}{4} Q_i \subset \Omega \times [0, \infty), \\ 0 & \text{else.} 
\end{cases} \] (5.9)

Note that \( u - w = 0 \) on \( \partial \Omega \times [0, T] \), which enables us to switch between \( \chi_{[0,T]} \) and \( \chi_{\Omega_T} \) without affecting the calculations.

Even though \( v_h(x, 0) \neq 0 \) in general, nevertheless the following observations regarding the initial boundary values hold:

- The initial condition \((u - w)(x, 0) = 0\) is to be understood in the sense \( |u - w|_{h}(\cdot, 0) \xrightarrow{h \to 0} 0 \) in \( L^2(\Omega) \).
- For \((x, 0) \in E_\lambda \), we have \( v_{\lambda,h}(x, 0) = v_h(x, 0) \).
- For \((x, 0) \notin E_\lambda \), we have \( v_{\lambda,h}(x, 0) = 0 \) by using (5.9).
- From Lemma 2.9, we see that \( v_{\lambda,h}(z) \xrightarrow{h \to 0} v_\lambda(z) \) almost everywhere.

For the rest of this section, let us denote
\[ 2\rho := \text{diam}(\Omega). \]

5.2. Bounds of \( v_{\lambda,h} \)

Let us now prove some estimates on the test function constructed in (5.8).

**Lemma 5.1.** For any \( z \in E_\lambda^c \), we have
\[ |v_{\lambda,h}(z)| \leq (n,p,q,\Lambda_1,b_0,r_0) \rho \lambda. \] (5.10)

**Proof.** By construction of the extension in (5.8), for \( z \in E_\lambda^c \), \( v_{\lambda,h}(z) = \sum_j \Psi_j(z)v_h^j \) with \( v_h^j = 0 \) whenever \( \frac{3}{4} Q_j \not\subset \Omega \times [0, \infty) \). Making use of (W12), that (5.10) follows if there holds
\[ |v_h^j| \leq (n,p,q,\Lambda_1,b_0,r_0) \rho \lambda. \] (5.11)

Note that we only have to consider the case \( \frac{3}{4} Q_j \subset \Omega \times [0, \infty) \), which automatically implies \( \frac{3}{4} r_j \leq \rho \). Thus we proceed as follows: let us define the following constant \( k_0 := \min\{\tilde{k}_1, \tilde{k}_2\} \) where \( \tilde{k}_1 \) and \( \tilde{k}_2 \) satisfy
\[ 2^{\tilde{k}_1 - 1} r_j < \rho \leq 2^{\tilde{k}_2} r_j, \]
\[ 2^{\tilde{k}_2 - 1} Q_j \subset \Omega \times [0, \infty) \quad \text{but} \quad 2^{k_0} Q_j \not\subset \Omega \times [0, \infty). \] (5.12)

The idea is to gradually enlarge \( \frac{3}{4} Q_j \) until it goes outside \( \Omega \times [0, \infty) \). As a consequence, we consider the following two subcases, first case where \( 2^{k_1} Q_j \) crosses the lateral boundary first, and second case when \( 2^{k_2} Q_j \) crosses the initial boundary first. Note that \( k_0 \) denotes the first scaling exponent under which either \( 2^{k_0} r_j \geq \rho \) occurs or \( 2^{k_0} Q_j \) goes outside \( \Omega \times [0, \infty) \).

Since we only consider the case \( \frac{3}{4} Q_i \subset \Omega \times [0, \infty) \), using triangle inequality, we get
\[ |v_h^j| \leq \sum_{m=0}^{k_0-2} \left[ \left( |u - w|_{h} \chi_{Q_{j,\rho}(\cdot)} \right)_{2^m Q_j} - \left( |u - w|_{h} \chi_{Q_{j,\rho}(\cdot)} \right)_{2^{m+1} Q_j} \right] + \left( |u - w|_{h} \chi_{Q_{j,\rho}(\cdot)} \right)_{2^{k_0-1} Q_j} \]
\[ =: \sum_{m=0}^{k_0-2} S_1^m + S_2. \] (5.13)

We shall now estimate \( S_1^m \) and \( S_2 \) as follows:
Estimate of $S^m_1$: Note that $2^{m+1}Q_j \subset \Omega \times [0, \infty)$. Thus applying Lemma 4.1 for any $\mu \in C^\infty_c(B_{2^{m+1}r_j}(x_j))$ with $|\mu(x)| \leq \frac{C(n)}{(2^{m+1}r_j)^n}$ and $|\nabla \mu(x)| \leq \frac{C(n)}{(2^{m+1}r_j)^{n+1}}$, we get

$$S^m_1 \leq (2^{m+1}r_j) \left( \int_{2^{m+1}Q_j} |\nabla [u - w]_h|^q \chi_{\Omega_T} \, dx \right)^{\frac{1}{q}} + (2^{m+1}r_j) \left( \sup_{t_1, t_2 \in 2^{m+1}Q_j} \left[ \left( \frac{|(u - w)_h(t_2) - [(u - w)_h(t_1)]}{2^{m+1}r_j} \right)^q \right] \right)^{\frac{1}{q}}. \tag{5.14}$$

By Lemma 2.8 and finally making use of (5.17) and (5.15) for any $J$ and finally to obtain (W4) and (5.7) as

$$\leq (2^{m+1}r_j)^\lambda + (2^{m+1}r_j) \left( \sup_{t_1, t_2 \in 2^{m+1}Q_j} \left[ \left( \frac{|(u - w)_h(t_2) - [(u - w)_h(t_1)]}{2^{m+1}r_j} \right)^q \right] \right)^{\frac{1}{q}}. \tag{W4}$$

Since $B_{2^{m+1}r_j}(x_j) \subset \Omega$, we can apply Lemma 4.8 with the test function $\phi(x) = \mu(x)$ and $\varphi(t) = 1$, which implies that for any $t_1, t_2 \in \frac{3}{4}I_j \cap [0, T]$

$$|(u - w)_h(t_2) - [(u - w)_h(t_1)]| \leq \|\nabla \mu\|_{L^\infty} \int_{2^{m+1}Q_j} \left[ (|\nabla u| + |h_0|)^{p-1} \right] \chi_{\Omega_T} \, dx \leq (2^{m+1}r_j)^\lambda. \tag{5.16}$$

The first term on the right hand side of (5.15) can be controlled using (W4) and (5.7) as

$$\|\nabla \mu\|_{L^\infty} \int_{2^{m+1}Q_j} \left[ (|\nabla u| + |h_0|)^{p-1} \right] \chi_{\Omega_T} \, dx \leq (2^{m+1}r_j)^\lambda. \tag{5.17}$$

We now estimate $J$ as follows:

$$J \leq \|\nabla \mu\|_{L^\infty} \|B_{2^{m+1}r_j}\| \int_{I_j} \left[ \int_{B_{2^{m+1}r_j}(x_j)} \left| \sum_{i=1}^n \left( \int_{B_{2^{m+1}r_j}(x_j)} \left| \nabla \phi_i(x) \right|^{\frac{q}{p-1}} \, dx \right)^{\frac{p-1}{q}} \right] \, dt \right] \tag{a}$$

$$\leq \|\nabla \mu\|_{L^\infty} \|B_{2^{m+1}r_j}\| \int_{I_j} \left[ \int_{B_{2^{m+1}r_j}(x_j)} \left| \int_{B_{2^{m+1}r_j}(x_j)} \left| \phi_i(x) \right|^{\frac{q}{p-1}} \, dx \right] \, dt \right] \tag{b}$$

$$\leq 2^{m+1}r_j \int_{B_{2^{m+1}r_j}(x_j)} \left| \int_{B_{2^{m+1}r_j}(x_j)} \left| \phi_i(x) \right|^{\frac{q}{p-1}} \, dx \right| \, dt \tag{c}$$

$$\leq 2^{m+1}r_j \gamma^{\lambda^{p-1}} = 2^{m+1}r_j \lambda. \tag{d}$$

To obtain (a), we made use of Hölder’s inequality, to obtain (b), we used Lemma 2.8, to obtain (c), we used (5.7) along with Lemma 2.9 and finally to obtain (d), we made use of (W4).

Thus combining (5.17) and (5.16) into (5.15) and finally making use of (5.14) along with (W4) and $\gamma := \lambda^{2-p}$, we get

$$S^m_1 \leq (2^{m+1}r_j) \left( \lambda^q + (\lambda^{p-1})^q \right)^{\frac{1}{q}} \leq (2^{m+1}r_j)^\lambda. \tag{5.18}$$

Estimate of $S_2$: For this term, we know that $2^{k_0-1}Q_j \not\subset \Omega \times [0, \infty)$, which implies $2^{k_0-1}Q_j$ crosses either the lateral boundary $\partial \Omega \times [0, \infty)$ or the initial boundary $\Omega \times \{t = 0\}$. We will consider both the cases separately and estimate $S_2$ as follows:

In the case $2^{k_0-1}Q_j$ crosses the lateral boundary $\partial \Omega \times [0, \infty)$ first, we can directly apply Poincaré’s inequality to obtain

$$\int_{2^{k_0-1}Q_j} |u - w| h \chi_{\Omega_T} \, dx \leq (2^{k_0}r_j) \left( \int_{2^{k_0}Q_j} |\nabla [u - w]_h|^q \chi_{\Omega_T} \, dx \right)^{\frac{1}{q}} \leq \rho \lambda. \tag{5.19}$$
To obtain (a), we made use of (W4) along with $2^{k_0 - 2}r_j \leq \rho$ given by (5.12).

In the case $2^{k_0}Q_j$ crosses the initial boundary $\Omega \times \{ t = 0 \}$ first, by enlarging the cylinder to $2^{k_1 + 1}Q_j$, we can find a cut-off function $\xi(x,t)$ such that

$$\text{spt } \xi(x,t) \subset 2^{k_1 + 1}Q_j \cap \mathbb{R}^n \times (-\infty, 0) \quad \text{and} \quad \| \xi \|_{L^\infty} \lesssim n \frac{\| \xi \|_{L^1}}{|2^{k_1 + 1}Q_j|},$$

by which and along with the fact that $v_h(z)\chi_{[0,T]} = 0$ on $\mathbb{R}^n \times (-\infty, 0)$, we get $\left( \left. v_h \chi_{[0,T]} \right| \right) = 0$. Thus applying Lemma 4.1, we get

$$\iint_{2^{k_0 + 1}Q_j} |v_h(z)| \chi_{[0,T]} \ dz = \iint_{2^{k_0 + 1}Q_j} \left| v_h(z) \chi_{[0,T]} - \left( \left. v_h \chi_{[0,T]} \right| \right) \right| \ dz$$

$$\lesssim (2^{k_0 + 1}r_j) \left( \iint_{2^{k_0 + 1}Q_j} |(u - w)| \chi_{[0,T]} \ dz \right)^{\frac{1}{q}}$$

$$+ (2^{k_0 + 1}r_j) \left( \sup_{t_1, t_2 \in 2^{k_0 + 1}Q_j \cap [0,T]} \left| \left( \left. \frac{1}{|\mu|} \left( [u - w] \chi_{[0,T]} \right) \right| (t_2) - \left( \left. \frac{1}{|\mu|} [u - w] \chi_{[0,T]} \right| (t_1) \right) \right| \right) \right)^{\frac{1}{q}}$$

$$\lesssim 2^{k_0 + 1}r_j \lambda \lesssim \rho \lambda. \quad (5.20)$$

To bound the first term in (a), we made use of (W1) along with (W4) and to bound the second term in (a), we proceed exactly as in (5.15) and finally to obtain (b), we made use of (5.12).

Combining (5.19) and (5.20), we get

$$S_2 \lesssim \rho \lambda. \quad (5.21)$$

Thus combining (5.18) and (5.21) into (5.13), we get

$$|v_h^j| \leq \sum_{m=0}^{k_0 - 2} S_1^m + S_2 \lesssim \left( \sum_{m=0}^{k_0 - 2} 2^{m+1}r_j + \rho \right) \lesssim \rho \lambda.$$

This completes the proof of the Lemma.

Now we prove a sharper estimate.

Lemma 5.2. For any $j \in A_i$, the following improved estimate holds:

$$|v_h^j - v_r^j| \lesssim \min \{ \rho, r_i \} \lambda.$$

Proof. We only have to consider the case $r_i \leq \rho$ because if $\rho \leq r_i$, we can directly use Lemma 5.1 to get the required conclusion.

We first consider the case that $\frac{3}{4}Q_i$ intersect the initial or lateral boundary. Initial Boundary Case

$\frac{3}{4}Q_i \subset \Omega \times \mathbb{R}$: Without loss of generality, we can assume $2Q_i \subset \Omega \times \mathbb{R}$. We now pick a

$$\xi(x,t) \in C_0^\infty(\mathbb{R}^{n+1}) \quad \text{with} \quad \text{spt } \xi \subset 2B_i \times (-\infty, 0).$$

We extend $u - w = 0$ on $2B_i \times (-\infty, 0)$, which implies $\left( \left. [u - w] \chi_{[0,T]} \right| \right) = 0$. Thus we get

$$|v_h^j| \lesssim \iint_{2Q_i} \left| [u - w] \chi_{[0,T]} - \left( \left. [u - w] \chi_{[0,T]} \right| \right) \right| \ dz$$

$$\lesssim r_i \left( \iint_{2Q_i} |\nabla v_h| \chi_{[0,T]} \ dz + \sup_{t_1, t_2 \in 2B_i \cap [0,T]} \left| \left( \left. \frac{1}{|\mu|} v_h \chi_{[0,T]} \right| (t_2) - \left( \left. \frac{1}{|\mu|} v_h \chi_{[0,T]} \right| (t_1) \right) \right| \right) \right)^{\frac{1}{q}}$$

$$\lesssim r_i \lambda. \quad (5.21)$$
To obtain (a), we made use of Lemma 4.1 and to obtain (b), we proceed similarly to how (5.15) was estimated.

**Lateral Boundary Case** \( \frac{3}{4} Q_i \cap (\Omega \times \mathbb{R})^c \neq \emptyset \): In this case, using Theorem 2.5 along with (W4), we get

\[
|v_h^i| \leq r_i \left( \iint_{2Q_i} \left| \frac{[u-w]_h}{r_i} \chi_{[0,T]} \right|^q \, dz \right)^{\frac{1}{q}} \leq r_i \left( \iint_{2Q_i} |\nabla [u-w]_h \chi_{[0,T]}| \, dz \right)^{\frac{1}{q}} \leq r_i \lambda. \tag{5.22}
\]

From (5.22) and (5.11), we see that the lemma is proved provided either \( v_h^i = 0 \) or \( v_h^i = 0 \).

**Now let us consider the case** \( \frac{3}{4} Q_i \subset \Omega \times (0, \infty) \). From the definition of \( v_h^i \) in (5.9), triangle inequality and (W10), we get

\[
|v_h^i - v_h^j| \leq \iint_{\frac{3}{4} Q_i} \left| v_h^i(z) \chi_{[0,T]} - v_h^j(z) \right| \, dz + \iint_{\frac{3}{4} Q_j} \left| v_h^i(z) \chi_{[0,T]} - v_h^j(z) \right| \, dz.
\]

We now apply Hölder’s inequality followed by Lemma 4.1 with \( \mu \in C_c^\infty \left( \frac{3}{4} B_i \right) \) satisfying \( |\mu(x)| \leq \frac{1}{r_i} \) and \( |\nabla \mu(x)| \leq \frac{1}{r_i^{n+1}} \) to estimate (5.23) as follows:

\[
\iint_{\frac{3}{4} Q_i} \left| v_h^i(z) \chi_{[0,T]} - v_h^j(z) \right| \, dz \leq r_i \left( \iint_{\frac{3}{4} Q_i} |\nabla v_h| \chi_{[0,T]} | \, dz \right)^{\frac{1}{q}} + r_i \left( \sup_{t_1, t_2 \in \frac{3}{4} Q_i \cap [0,T]} \left| \frac{([u-w]_h \mu)(t_2) - ([u-w]_h \mu)(t_1)}{r_i} \right| \right)^{\frac{1}{q}}.
\]

The first term on the right of (5.24) can be controlled using (W4) and the second term can be controlled similarly as (5.15). Thus we get

\[
\iint_{\frac{3}{4} Q_i} \left| v_h^i(z) \chi_{[0,T]} - v_h^j(z) \right| \, dz \leq r_i \lambda.
\]

This completes the proof of the Lemma. \( \square \)

**5.3. Bounds on derivatives of \( v_{i,h} \)**

Using Lemma 5.1 and Lemma 5.2, in the same spirit of [3], we can obtain the following estimates:

**Lemma 5.3.** Given any \( z \in E^\lambda_\rho \), we have \( z \in \frac{3}{4} Q_i \) for some \( i \in \mathbb{N} \). Then there holds

\[
|\nabla v_{i,h}(z)| \leq C(n,p,q,\Lambda_1,b_0,r_0) \lambda. \tag{5.25}
\]

**Lemma 5.4.** Let \( z \in E^\lambda_\rho \) and \( \varepsilon \in (0,1) \) be any number, then \( z \in \frac{3}{4} Q_i \) for some \( i \in \mathbb{N} \) from (W1). There exists a constant \( C = C(n) \) such that the following holds:

\[
|v_{i,h}(z)| \leq C \iint_{4Q_i} |v_h(\tilde{z})| \chi_{[0,T]} \, d\tilde{z} \leq \frac{Cr_i \lambda}{\varepsilon} + \frac{C\varepsilon}{\lambda r_i} \iint_{4Q_i} |v_h(\tilde{z})|^2 \chi_{[0,T]} \, d\tilde{z},
\]

\[
|\nabla v_{i,h}(z)| \leq C \frac{1}{r_i} \iint_{4Q_i} |v_h(\tilde{z})| \chi_{[0,T]} \, d\tilde{z} \leq \frac{CA \lambda}{\varepsilon} + \frac{C\varepsilon}{\lambda r_i} \iint_{4Q_i} |v_h(\tilde{z})|^2 \chi_{[0,T]} \, d\tilde{z},
\]

\[
|v_{i,h}(z)| \leq C \left( \min \{|\rho, r_i| \lambda + |v_h^i|\} \right) \leq C \left( \frac{r_i \lambda}{\varepsilon} + \frac{\varepsilon}{r_i \lambda} |v_h^i|^2 \right),
\]

\[
|\nabla v_{i,h}(z)| \leq C \frac{\lambda}{\varepsilon}. \tag{5.27}
\]
Lemma 5.5. Let $z \in E^c_\lambda$ and $\varepsilon \in (0, 1]$ be any number, then $z \in \frac{3}{4}Q_i$ for some $i \in \mathbb{N}$ from (W1). There exists a constant $C = C(n,p,q,A_1,b_0,r_0)$ such that the following estimates for the time derivative of $v_{\lambda,h}$ holds:

$$|\partial_t v_{\lambda,h}(z)| \leq C \frac{1}{\lambda^{2-p_r}} \int_{Q_i} |v_h(\tilde{z})| |\chi_{[0,T]}| d\tilde{z},$$

$$|\partial_t v_{\lambda,h}(z)| \leq C \frac{1}{\lambda^{2-p_r}} \min\{r_i, \rho\} \lambda.$$  \hspace{1cm} (5.28)

Lemma 5.6. For any $\vartheta \geq 1$, we have the following bound:

$$\int_{\Omega_T \setminus E} |v_{\lambda,h}(z)|^\vartheta \, dz \leq (n,p,q,A_1,b_0,\rho,\vartheta) \int_{\Omega_T \setminus E} |v_h(z)|^\vartheta \chi_{[0,T]} \, dz.$$  

Lemma 5.7. For any $1 \leq \vartheta \leq q$, there holds

$$\int_{\Omega_T \setminus E} |\partial_t v_{\lambda,h}(z)(v_{\lambda,h}(z) - v_h(z)))|^{\vartheta} \, dz \leq (n,p,q,A_1,b_0,\rho,\vartheta) \lambda^{\vartheta p} |\mathbb{R}^{n+1} \setminus E|.$$  

5.4. Lipschitz continuity of the test function

We shall now prove the Lipschitz continuity of $v_{\lambda,h}$ on $\mathcal{H} := \mathbb{R}^n \times [0,T]$.

Lemma 5.8. The function $v_{\lambda,h}$ from (5.8) is $C^{0,1}(\mathcal{H})$ with respect to the parabolic metric given by

$$d_{\lambda}(z_1, z_2) := \max\left\{|x_2 - x_1|, \sqrt{\lambda^p - 2}|t_2 - t_1|\right\}.$$  

Proof. Let us consider a parabolic cylinder $Q_r(z) := Q_{r,\gamma^2}(z) := Q = B \times I$ for some $z \in \mathcal{H}$ and $r > 0$ (recall $\gamma = \lambda^{2-p}$ which is the intrinsic scaling from Lemma 4.3). To prove the lemma, we make use of Lemma 4.6 and obtain the following bound:

$$I_r(z) := \int_{Q \cap \mathcal{H}} \left| \frac{v_{\lambda,h}(\tilde{z}) - (v_{\lambda,h})_{Q \cap \mathcal{H}}}{r} \right| d\tilde{z} \leq o(1),$$

where $o(1)$ denotes a constant independent of $z \in \mathcal{H}$ and $r > 0$ only. We will split the proof into several subcases and proceed as follows:

Case 2$Q \subset E^c_\lambda$: In this case, from (W3), we see that $z \in \frac{3}{4}Q_i$ for some $i \in \mathbb{N}$. From the construction in (5.8), we see that $v_{\lambda,h} \in C^{\infty}(E^c_\lambda)$ which combined with the mean value theorem gives

$$I_r(z) \leq \frac{1}{r} \int_{Q \cap \mathcal{H}} \int_{Q \cap \mathcal{H}} |v_{\lambda,h}(\tilde{z}_1) - v_{\lambda,h}(\tilde{z}_2)| \, d\tilde{z}_1 \, d\tilde{z}_2 \leq \sup_{\tilde{z} \in Q \cap \mathcal{H}} \left( |\nabla v_{\lambda,h}(\tilde{z})| + \lambda^{2-p_r} |\partial_t v_{\lambda,h}(\tilde{z})| \right).$$

Let us pick some $\tilde{z}_0 \in Q \subset E^c_\lambda$, then $\tilde{z}_0 \in Q_j$ for some $j \in \mathbb{N}$. Thus we can make use of (5.25) and (5.28) to get

$$|\nabla v_{\lambda,h}(\tilde{z}_0)| + \lambda^{2-p_r} |\partial_t v_{\lambda,h}(\tilde{z}_0)| \leq \lambda + \lambda^{2-p_r} \frac{1}{\lambda^{2-p_r}} r_j \lambda.$$  \hspace{1cm} (5.29)

In (5.29), we need to understand the relation between $r_j$ and $r$. To this end, from $2Q \subset E^c_\lambda$, we see that

$$r \leq d_{\lambda}(\tilde{z}_0, E_\lambda) \leq d_{\lambda}(\tilde{z}_0, z_j) + d_{\lambda}(z_j, E_\lambda) \leq r_j + 16r_j = 17r_j.$$  \hspace{1cm} (5.30)

Combining (5.29) and (5.30), we get

$$|\nabla v_{\lambda,h}(\tilde{z}_0)| + \lambda^{2-p_r} |\partial_t v_{\lambda,h}(\tilde{z}_0)| \leq \lambda.$$  

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Case \(2Q \notin E_\lambda^c\): In this case, we shall split the proof into three subcases:

**Subcase** \(2Q \subset \mathbb{R}^n \times (-\infty, T]\) or \(2Q \subset \mathbb{R}^n \times [0, \infty)\): In this situation, it is easy to see that the following holds:

\[|Q \cap \mathcal{H}| \geq |Q|.\]  

(5.31)

We apply triangle inequality and estimate \(I_r(z)\) by

\[
I_r(z) \leq \iint_{Q \cap \mathcal{H}} \frac{|v_{\lambda,h}^r(\hat{z}) - v_h^r(\hat{z})|}{r} + \frac{|v_h(\hat{z}) - (v_h)_{Q \cap \mathcal{H}}|}{r} + \frac{(v_h)_{Q \cap \mathcal{H}} - (v_{\lambda,h})_{Q \cap \mathcal{H}}}{r} \, d\hat{z} \leq 2J_1 + J_2,
\]

(5.32)

where we have set

\[J_1 := \iint_{Q \cap \mathcal{H}} \frac{|v_{\lambda,h}^r(\hat{z}) - v_h^r(\hat{z})|}{r} \, d\hat{z} \quad \text{and} \quad J_2 := \iint_{Q \cap \mathcal{H}} \frac{|v_h(\hat{z}) - (v_h)_{Q \cap \mathcal{H}}|}{r} \, d\hat{z}.
\]

(5.33)

We now estimate each of the terms of (5.33) as follows:

**Estimate for \(J_1\):** From (5.8), we get

\[
J_1 \leq \sum_{i \in \mathbb{N}} \frac{1}{|Q \cap \mathcal{H}|} \iint_{Q \cap \mathcal{H} \cap \frac{3}{4}Q_i} \frac{|v_h(\hat{z})\chi_{[0,T]} - v_h^i|}{r} \, d\hat{z}.
\]

(5.34)

Let us fix an \(i \in \mathbb{N}\) and take two points \(\hat{z}_1 \in Q \cap \frac{3}{4}Q_i\) and \(\hat{z}_2 \in E_\lambda \cap 2Q_i\). Let \(z_i\) denote the center of \(\frac{3}{4}Q_i\), making use of (W2) along with the trivial bound \(d_\lambda(\hat{z}_1, \hat{z}_2) \leq 4r\) and \(d_\lambda(z_i, \hat{z}_1) \leq 2r_i\), we get

\[
16r_i = d_\lambda(z_i, E_\lambda) \leq d_\lambda(z_i, \hat{z}_1) + d_\lambda(\hat{z}_1, \hat{z}_2) \leq 2r_i + 4r \quad \Rightarrow \quad 2r_i \leq r.
\]

(5.35)

Note that (5.31) holds and thus summing over all \(i \in \mathbb{N}\) such that \(Q \cap \mathcal{H} \cap \frac{3}{4}Q_i \neq \emptyset\) in (5.34) and making use of (5.35), we get

\[
J_1 \leq \sum_{i \in \mathbb{N}} \frac{3^4 |Q_i|}{|Q \cap \mathcal{H}|} \iint_{\frac{3}{4}Q_i} \frac{|v_h(\hat{z})\chi_{[0,T]} - v_h^i|}{r} \, d\hat{z} \overset{(a)}{\leq} \sum_{i \in \mathbb{N}} \iint_{\frac{3}{4}Q_i} \frac{|v_h(\hat{z})\chi_{[0,T]} - v_h^i|}{r_i} \, d\hat{z} \overset{(b)}{\leq} \lambda.
\]

To obtain (a), we made use of (5.31) and (5.35), to obtain (b), we follow the calculation from bounding (5.24).

**Estimate for \(J_2\):** Note that \(Q \cap \mathcal{H}\) is another cylinder. *In the case \(Q \subset \Omega \times \mathbb{R}\), choose a cut-off function \(\mu \in C_c^\infty(\Omega)\) and apply Lemma 4.1 to get*

\[
J_2 \leq \left( \iint_{Q \cap \mathcal{H}} |\nabla v_h|^q \chi_{\Omega_T} + \sup_{t_1, t_2 \in I} \left| \frac{(v_h\chi_{[0,T]})(t_2) - (v_h\chi_{[0,T]})(t_1)}{r} \right|^q \right)^{\frac{1}{q}}.
\]

(5.36)

Recall that we are in the case \(2Q \cap E_\lambda \neq \emptyset\) and \(2Q \cap E_\lambda^c \neq \emptyset\). We can now proceed as in (5.15) to get

\[
J_2 \leq \lambda.
\]

(5.36)

*On the other hand, if \(Q \notin \Omega \times \mathbb{R}\), then we can apply Theorem 2.5 directly and make use of the fact that \(2Q \cap E_\lambda \neq \emptyset\) to get*

\[
J_2 \leq \left( \iint_{Q \cap \mathcal{H}} |\nabla v_h(\hat{z})\chi_{[0,T]}|^q \, d\hat{z} \right)^{\frac{1}{q}} \leq \lambda.
\]
Subcase 2$Q \cap \mathbb{R}^n \times (-\infty, 0] \neq \emptyset$ and $2Q \cap \mathbb{R}^n \times [T, \infty) \neq \emptyset$ AND $\gamma r^2 \leq T$: In this case, we see that

\[ |Q \cap \mathcal{H}| \geq |B_1| r^n \times \frac{T}{2} \]

We apply triangle inequality and estimate $I_r(z)$ as we did in (5.32) to get

\[ I_r(z) \leq 2J_1 + J_2, \]

where we have set

\[ J_1 := \iint_{Q \cap \mathcal{H}} \left| \frac{v_{\lambda,h}(\tilde{z}) - v_h(\tilde{z})}{r} \right| d\tilde{z} \quad \text{and} \quad J_2 := \iint_{Q \cap \mathcal{H}} \left| \frac{v_h(\tilde{z}) - (v_h)_{Q \cap \mathcal{H}}}{r} \right| d\tilde{z}. \]

We estimate $J_1$ as follows

\[ J_1 \leq \sum_{i \in \mathbb{N}} \frac{|Q_i|}{|Q \cap \mathcal{H}|} \iint_{\frac{3}{2}Q_i} \left| \frac{v_h(\tilde{z})\chi_{[0,T]} - v_h^i}{r} \right| d\tilde{z} \]
\[ \leq \frac{r^{n+2}}{r^n T} \sum_{i \in \mathbb{N}} \iint_{\frac{3}{2}Q_i} \left| \frac{v_h(\tilde{z})\chi_{[0,T]} - v_h^i}{r} \right| d\tilde{z} \]
\[ \leq \frac{r^{n+2} \gamma}{r^n T} \sum_{i \in \mathbb{N}} \iint_{\frac{3}{2}Q_i} \left| \frac{v_h(\tilde{z})\chi_{[0,T]} - v_h^i}{r} \right| d\tilde{z} \]
\[ \leq \lambda \frac{r^2 \gamma}{T}. \]

To obtain (a), we proceeded similarly to (5.24) and to obtain (b), we made use of $\gamma r^2 \leq T$. The estimate of $J_2$ is already obtained in (5.36) which shows

\[ J_2 \leq \lambda. \]

Subcase 2$Q \cap \mathbb{R}^n \times (-\infty, 0] \neq \emptyset$ and $2Q \cap \mathbb{R}^n \times [T, \infty) \neq \emptyset$ AND $\gamma r^2 > T$: Using triangle inequality, we get

\[ \iint_{Q \cap \mathcal{H}} \left| \frac{v_{\lambda,h}(\tilde{z}) - (v_{\lambda,h})_{Q \cap \mathcal{H}}}{r} \right| d\tilde{z} \leq \frac{1}{|Q \cap \mathcal{H}|} \iint_{Q \cap \mathcal{H}} |v_{\lambda,h}(\tilde{z})| d\tilde{z} \]
\[ \leq \frac{1}{|Q \cap \mathcal{H}|} \iint_{Q \cap \mathcal{H} \cap E_\lambda} |v_{\lambda,h}(\tilde{z})| d\tilde{z} + \frac{1}{|Q \cap \mathcal{H}|} \iint_{Q \cap \mathcal{H} \setminus E_\lambda} |v_{\lambda,h}(\tilde{z})| d\tilde{z}. \]

We have $v_{\lambda,h} = v_h$ on $E_\lambda$. On $\Omega_T \setminus E_\lambda$, we can apply Lemma 5.1 to obtain the following bound:

\[ \iint_{Q \cap \mathcal{H}} \left| \frac{v_{\lambda,h}(\tilde{z}) - (v_{\lambda,h})_{Q \cap \mathcal{H}}}{r} \right| d\tilde{z} \leq \frac{1}{r^n T} \iint_{\Omega_T} |v_h(\tilde{z})| d\tilde{z} + \frac{1}{|Q \cap \mathcal{H}|} \iint_{Q \cap \mathcal{H} \setminus E_\lambda} \rho \lambda d\tilde{z} \]
\[ \leq \left( \gamma \frac{\lambda}{T} \right)^{\frac{1}{2}} \frac{1}{T} \|v_h\|_{L^1(\Omega_T)} + \rho \lambda. \]

This completes the proof of the Lipschitz regularity of $v_{\lambda,h}$.

\[ \square \]

5.5. Two crucial estimates for the test function

We shall now prove the first crucial estimate which holds on each time slice.
Lemma 5.9. For any \( i \in \mathbb{N} \) and any \( 0 < \varepsilon \leq 1 \), there exists a positive constant \( C(n, p, q, \Lambda_1, b_0, r_0) \) such that for almost every \( t \in [0, T] \), there holds
\[
\left| \int_{\Omega} (v(x, t) - v^i) v\lambda(x, t) \Psi_i(x, t) \, dx \right| \leq C \left( \frac{\lambda^p}{\varepsilon} |4Q_i| + \varepsilon |4B_i| |v^i|^2 \right). \tag{5.37}
\]

Proof. Let us fix any \( t \in [0, T] \), \( i \in \mathbb{N} \) and take \( \Psi_i(y, \tau)v_{\lambda,h}(y, \tau) \) as a test function in (1.1). Further integrating the resulting expression over \( \left( t_i - \gamma \left( \frac{3}{4} r_i \right)^2, t \right) \) or \( (0, t) \) depending on the location of \( \frac{3}{4} Q_i \), along with making use of the fact that \( \Psi_i(y, t_i - \gamma (3r_i/4)^2) = 0 \) or \( v_{\lambda,h}(y, 0) = 0 \) for \( (y, 0) \in E_y \), we get for any \( a \in \mathbb{R} \), the equality
\[
\int_{\Omega} \left( (v_h - a)\Psi_i v\lambda,h \right)(y, t) \, dy = \int_{\Omega} \partial_t \left( [u - w]_h \Psi_i v\lambda,h - a \Psi_i v\lambda,h \right)(y, \tau) \, dy \, d\tau
\]
\[
- \int_{\Omega} \partial_t \left( [u - w]_h \Psi_i v\lambda,h \right) \, dy \, d\tau + \int_{\Omega} \partial_t \left( a \Psi_i v\lambda,h \right) \, dy \, d\tau.
\]

We can estimate \( |\nabla(\Psi_i v\lambda)| \) using the chain rule and (W13), to get
\[
|\nabla(\Psi_i v\lambda)| \leq \frac{1}{r_i} |v\lambda| + |\nabla v\lambda|. \tag{5.39}
\]

Similarly, we can estimate \( |\partial_t (\Psi_i v\lambda)| \) using the chain rule and (W13), to get
\[
|\partial_t (\Psi_i v\lambda)| \leq \frac{1}{r_i} |v\lambda| + |\partial_t v\lambda|. \tag{5.40}
\]

Let us take \( a = v_h^i \) in the (5.38) followed by letting \( h \searrow 0 \) and making use of (5.39), (2.1) and crucially the assumption from Remark 3.2 (more specifically (3.2)), we get
\[
\left| \int_{\Omega} \left( (v - v^i) \Psi_i v\lambda \right)(y, t) \, dy \right| \leq J_1 + J_2 + J_3, \tag{5.41}
\]
where we have set
\[
J_1 := \int_{\Omega} \int_{\Omega_x} (|\nabla u| + |h_0|)^{p-1} \left( \frac{1}{r_i} |v\lambda| + |\nabla v\lambda| \right) \chi_{4Q_i \cap \Omega_T} \, dy \, d\tau,
J_2 := \int_{0}^{T} \left| \partial_t \left( \Psi_i v\lambda \right) \right|_{(W^{-1, \frac{p}{p-\alpha}} (\Omega), W^{\frac{p}{p-\alpha}} (\Omega))} \, d\tau = \int_{0}^{T} \int_{\Omega} \left| \partial_t (\nabla(\Psi_i v\lambda)) \chi_{4Q_i \cap \Omega_T} \right| \, dy \, d\tau,
J_3 := \int_{\Omega} \int_{\Omega_x} |v - v^i| |\partial_t (\Psi_i v\lambda)| \chi_{4Q_i \cap \Omega_T} \, dy \, d\tau.
\tag{5.42}
\]

Let us now estimate each of the terms as follows:

**Bound for** \( J_1 \): If \( p \leq r_i \), we can directly use Hölder’s inequality, Lemma 5.1, Lemma 5.3 and (W4), to find that for any \( \varepsilon \in (0, 1] \), there holds
\[
J_1 \leq \left( \frac{1}{r_i} \rho \lambda + \lambda \right) |Q_i| \left( \int_{16Q_i} (|\nabla u| + |h_0|)^q \chi_{\Omega_T} \, dy \, d\tau \right)^{\frac{p-1}{q}} \leq \frac{\lambda^p}{\varepsilon} |4Q_i|. \tag{5.43}
\]
In the case \( r_i \leq \rho \), we make use of (5.26), (W4) and Lemma 5.3 along with the fact \( |Q_i| = |B_i| \times 2\lambda^2 r_i^2 \), to get

\[
J_1 \leq \frac{1}{r_i} \left( \frac{r_i \lambda}{\varepsilon} + \frac{\varepsilon}{\lambda r_i} |v_i|^2 \right) + \lambda \left( 4Q_i \left( \iint_{4Q_i} (|\nabla u| + |h_0|)^\gamma \chi_{3r_i} \, dy \, dt \right) \right)^\frac{p-1}{\gamma}.
\]

(5.44)

Thus combining (5.44) and (5.43), we get

\[
J_1 \leq \frac{\lambda^p}{\varepsilon} |4Q_i| + \chi_{r_i \leq \rho} \varepsilon |4B_i||v_i|^2,
\]

where we have set \( \chi_{r_i \leq \rho} = 1 \) if \( r_i \leq \rho \) and \( \chi_{r_i \leq \rho} = 0 \) else.

**Bound for \( J_2 \):** In this case, we can directly use Lemma 5.3 and (W4) to get for any \( \varepsilon \in (0, 1] \), the bound

\[
J_2 \leq \frac{\lambda^p}{\varepsilon} |4Q_i| + \chi_{r_i \leq \rho} \varepsilon |4B_i||v_i|^2.
\]

**Bound for \( J_3 \):** Substituting (5.27), (5.28) and (W13) into (5.40), for any \( \varepsilon \in (0, 1] \), there holds

\[
|\partial_t (\Psi v_\lambda)(z)| \leq \frac{1}{\gamma r_i^2} \left( \frac{r_i \lambda}{\varepsilon} + \frac{\varepsilon}{r_i \lambda} |v_i|^2 \right) + \frac{1}{\gamma r_i^2} \min\{r_i, \rho\} \lambda \approx \frac{1}{\gamma r_i^2} \left( \frac{r_i \lambda}{\varepsilon} + \frac{\varepsilon}{r_i \lambda} |v_i|^2 \right).
\]

(5.47)

Making use of (5.47) in the expression for \( J_3 \) in (5.42), we get

\[
J_3 \leq \frac{1}{\gamma r_i^2} \left( \frac{r_i \lambda}{\varepsilon} + \frac{\varepsilon}{r_i \lambda} |v_i|^2 \right) \iint_{4Q_i} |v - v_\lambda| \chi_{3r_i} \, dy \, dt.
\]

We can now proceed similarly to (5.24) to get

\[
J_3 \leq \frac{1}{\gamma r_i^2} \left( \frac{r_i \lambda}{\varepsilon} + \frac{\varepsilon}{r_i \lambda} |v_i|^2 \right) r_i \lambda |Q_i| \leq \frac{\lambda^p}{\varepsilon} |4Q_i| + \varepsilon |4B_i||v_i|^2.
\]

(5.48)

Substituting the estimates (5.45), (5.46) and (5.48) into (5.41) completes the proof of the lemma.

We now come to essentially the most important estimate which will be needed to prove the difference estimate:

**Lemma 5.10.** There exists a positive constant \( C = C_{\nu, p, q, \lambda_1, \nu_0, r_0} \) such that the following estimate holds for every \( t \in [0, T] \):

\[
\int_{Q \cap E_\lambda} (|v|^2 - |v - v_\lambda|^2)(x, t) \, dx \geq -CX^p |R^{n+1} \setminus E_\lambda|.
\]

(5.49)
Proof. Let us fix any \( t \in [0, T] \) and any point \( x \in \Omega \setminus E_\lambda^t \). Now define

\[ \Upsilon := \{ i \in \mathbb{N} : \text{spt}(\Psi_i) \cap \Omega \times \{ t \} \neq \emptyset \text{ and } |v_i| + |v^i_\lambda| \neq 0 \text{ on } \text{spt}(\Psi_i) \cap (\Omega \times \{ t \}) \}. \]

Hence we only need to consider \( i \in \Upsilon \). Noting that \( \sum_{i \in \Upsilon} \Psi_i(\cdot, t) \equiv 1 \) on \( \mathbb{R}^n \cap E_\lambda^t \), we can rewrite the left-hand side of (5.49) as

\[
\int_{\Omega \setminus E_\lambda^t} (|v|^2 - |v - v^i_\lambda|^2)(x, t) \, dx = \sum_{i \in \Upsilon} \int_{\Omega} \Psi_i(|v|^2 - |v - v^i_\lambda|^2) \, dx \\
= \sum_{i \in \Upsilon} \int_{\Omega} \Psi_i(z) \left(|v|^2 + 2v^i_\lambda (v - v^i)\right) \, dx - \sum_{i \in \Upsilon} \int_{\Omega} \Psi_i(z)|v - v^i|^2 \, dx \\
:= J_1 - J_2.
\]

**Estimate of J_1:** Using (5.37), we get

\[
J_1 \geq \sum_{i \in \Upsilon} \int_{\Omega} \Psi_i(z)|v|^2 \, dz - \varepsilon \sum_{i \in \Upsilon} |4B_i||v|^2 - \sum_{i \in \Upsilon} \frac{\lambda^p}{\varepsilon} |4Q_i|. \tag{5.50}
\]

From (5.9), we have \( v^i = 0 \) whenever \( \text{spt}(\Psi_i) \cap \Omega^c \neq \emptyset \). Hence we only have to sum over all those \( i \in \Upsilon \) for which \( \text{spt}(\Psi_i) \subset \Omega \times (0, \infty) \). In this case, we make use of a suitable choice for \( \varepsilon \in (0, 1) \), and use (W7) along with (W12) and estimate (5.50) from below to get

\[
J_1 \geq -\lambda^p |\mathbb{R}^{n+1} \setminus E_\lambda|.
\]  

**Estimate of J_2:** For any \( x \in \Omega \setminus E_\lambda^t \), we have from (W14) that \( \sum_j \Psi_j(x, t) = 1 \), which gives

\[
\Psi_i(z)|v^i_\lambda(z) - v^i|^2 \leq \Psi_i(z) \sum_{j \in A_i} (v^i - v^j)^2 \leq \min\{\rho, r_i\}^2 \lambda^2. \tag{5.52}
\]

To obtain (a) above, we made use of Lemma 5.2 along with (W8). Substituting (5.52) into the expression for \( J_2 \) and using \( |Q_i| = |B_i| \times 2\gamma r_i^2 \), we get

\[
J_2 \leq \sum_{i \in \Upsilon} |B_i| \frac{\gamma r_i^2 \lambda^2}{\gamma} \leq \lambda^p |\mathbb{R}^{n+1} \setminus E_\lambda|. \tag{5.53}
\]

Substituting (5.51) and (5.53) into Subsection 5.5, we get the desired estimate. This completes the proof of the lemma. \( \square \)

5.6. **A priori estimate - Proof of Theorem 3.1**

Consider the following cut-off function \( \zeta_c \in C^\infty(0, \infty) \) such that \( 0 \leq \zeta_c(t) \leq 1 \) and

\[
\zeta_c(t) = \begin{cases} 
1 & \text{for } t \in (0 + \varepsilon, T - \varepsilon) \\
0 & \text{for } t \in (-\infty, 0) \cup (T, \infty). 
\end{cases}
\]

It is easy to see that

\[
\zeta'_c(t) = 0 \quad \text{for } t \in (-\infty, 0) \cup (0 + \varepsilon, T - \varepsilon) \cup (T, \infty), \\
|\zeta'_c(t)| \leq \frac{c}{\varepsilon} \quad \text{for } t \in (0, 0 + \varepsilon) \cup (T - \varepsilon, 0).
\]

Let \( h \in (0, T) \) be the Steklov averaging exponent. Without loss of generality, we shall always take \( h \geq 2\varepsilon \) since we will take limits in the following order \( \lim_{h \to 0} \varepsilon \to 0 \).

Let us use \( v_{\lambda, h} \zeta_c \) constructed in (5.8) as a test function in (1.1) to get

\[
\iint_{\Omega_T} \frac{d|u|}{dt} v_{\lambda, h} \zeta_c \, dx \, dt + \iint_{\Omega_T} \left\langle [A(x, t, \nabla u)]_h, \nabla v_{\lambda, h} \zeta_c \right\rangle dx \, dt = \iint_{\Omega_T} \frac{d|w|}{dt} v_{\lambda, h} \zeta_c \, dx \, dt, \tag{5.54}
\]

which we express as

\[
L_1 + L_2 = L_3.
\]
**Estimate of $L_1$:** Recalling (5.4), we get

$$L_1 = \int_0^T \int_\Omega \frac{dv_h(y,s)}{ds} \partial_{h \lambda}(y,s) \zeta(s) \, dy \, ds$$

$$= \frac{1}{2} \int_0^T \int_\Omega \left( \left( (v_h)^2 - (v_{\lambda h} - v_h)^2 \right) \partial_{h \lambda}(s) \right) \, dy \, ds + \int_0^T \int_{\Omega \setminus E_\lambda} \frac{dv_{\lambda h}}{ds} (v_{\lambda h} - v_h) \zeta(s) \, dy \, ds$$

$$- \frac{1}{2} \int_0^T \int_\Omega \frac{d\zeta}{ds} \left( (v_h)^2 - (v_{\lambda h} - v_h)^2 \right) \, dy \, ds$$

$$:= J_1(T) - J_1(0) + J_2 - J_3, \quad (5.55)$$

where we have set

$$J_1(s) := \frac{1}{2} \int_\Omega \left( (v_h)^2 - (v_{\lambda h} - v_h)^2 \right) \zeta(s) \, dy.$$

Since $\zeta(s) = 0$ at $s = T$, we have

$$J_1(0) = J_1(T) = 0. \quad (5.56)$$

Form Lemma 5.7 applied with $\vartheta = 1$, we have the bound

$$|J_2| \leq \int_{\Omega \setminus E_\lambda} \left| \frac{dv_{\lambda h}}{ds} (v_{\lambda h} - v_h) \right| \, dy \, ds \leq \lambda |\partial^{n+1} \setminus E_\lambda|. \quad (5.57)$$

In order to estimate $- \int_0^T \int_\Omega \frac{d\zeta}{ds} (v_h^2 - (v_{\lambda h} - v_h)^2) \, dy \, ds$, we take limits first in $\varepsilon \downarrow 0$ followed by $h \downarrow 0$ to get

$$- \int_0^T \int_\Omega \frac{d\zeta}{ds} (v_h^2 - (v_{\lambda h} - v_h)^2) \, dy \, ds \lim_{\lambda \to \lambda_0} \lim_{h \to 0} \int_\Omega (v^2 - (v_{\lambda} - v)^2)(x,T) \, dx$$

$$- \int_\Omega (v^2 - (v_{\lambda} - v)^2)(x,0) \, dx. \quad (5.58)$$

For the second term on the right of (5.58), we observe that $v_{\lambda} = v$ on $E_\lambda$. Note that $v_{\lambda}(\cdot, 0) = 0 = v(\cdot, 0)$ by the initial condition. Thus, the second term on the right of (5.58) vanishes and so

$$- \int_0^T \int_\Omega \frac{d\zeta}{ds} (v_h^2 - (v_{\lambda h} - v_h)^2) \, dy \, ds \lim_{\lambda \to \lambda_0} \lim_{h \to 0} \int_\Omega (v^2 - (v_{\lambda} - v)^2)(x,T) \, dx. \quad (5.59)$$

**Estimate of $L_2$:** We split $L_2$ and make use of the fact that $v_{\lambda h}(z) = v_h(z) \overset{(5.4)}{=} [u-w]_h(z)$ for all $z \in E_\lambda \cap \Omega_T$ to get

$$L_2 = \int_{\Omega \setminus E_\lambda} \langle [A(x,t,\nabla u)]_h, \nabla[u-w]_h \rangle \zeta \, dz + \int_{\Omega \setminus E_\lambda} \langle [A(x,t,\nabla u)]_h, \nabla v_{\lambda h} \rangle \zeta \, dz$$

$$=: L_2^1 + L_2^2. \quad (5.60)$$

**Estimate of $L_2^1$:** Using ellipticity from (2.1), we get

$$L_2^1 \geq \int_{\Omega \setminus E_\lambda} \left| [\nabla u]^p - [h_0]_h \right| \zeta \, dz - \int_{\Omega \setminus E_\lambda} \left| [\nabla u]^{p-1} + [h_0]^{p-1} \right| [\nabla w] |\zeta| \, dx \, dt$$

$$\lim_{\lambda \to \lambda_0} \lim_{h \to 0} \int_{\Omega \setminus E_\lambda} \left| \nabla u \right|^p - \left| h_0 \right|^p \, dz - \int_{\Omega \setminus E_\lambda} \left( [\nabla u]^{p-1} + [h_0]^{p-1} \right) |\nabla w| \, dx \, dt. \quad (5.61)$$

**Estimate of $L_2^2$:** Using the bound from Lemma 5.3, we get

$$|L_2^2| \leq \lambda \int_{\Omega \setminus E_\lambda} \left| [\nabla u]^{p-1} + [h_0]^{p-1} \right| \zeta \, dz \lim_{\lambda \to \lambda_0} \lim_{h \to 0} \int_{\Omega \setminus E_\lambda} \left( [\nabla u]^{p-1} + [h_0]^{p-1} \right) \, dz. \quad (5.62)$$
Combining (5.61) and (5.62) with (5.60), we get
\[
L_2 \geq \iint_{\Omega_T \cap E_\lambda} |\nabla u|^p - |h_0|^p \, dz - \iint_{\Omega_T \cap E_\lambda} (|\nabla u|^{p-1} + |h_0|^{p-1}) |\nabla w| \, dz - \lambda \int_{\Omega_T \cap E_\lambda} (|\nabla u|^{p-1} + |h_0|^{p-1}) \, dz.
\]  
(5.63)

Estimate of \( L_3 \): Analogous to estimate for \( L_2 \), we split \( L_3 \) into integrals over \( E_\lambda \) and \( E_\lambda^c \), to find
\[
L_3 \overset{(a)}{=} \int_0^T \left[ \int_\Omega \left\langle \frac{dw}{dt}, v_{\lambda,h} \right\rangle_{(W^{-1}, \frac{\partial}{\partial t}, E_\lambda)} (x,t) \right]_{h} \zeta_\varepsilon(t) \, dt
\]
\[
\overset{\text{Lemma 2.8}}{=} \int_0^T \left[ \int_\Omega \left\langle \omega, \nabla v_{\lambda,h} \right\rangle (x,t) \right]_{h} \zeta_\varepsilon(t) \, dt
\]
\[
\overset{(b)}{\leq} \int_0^T \left[ \int_{\Omega \cap E_\lambda^c} \langle \omega, \nabla (u-w) \rangle (x,t) \right]_{h} \zeta_\varepsilon(t) \, dt
\]
\[
\overset{\text{Lemma 5.3}}{=} \lim_{\varepsilon \to 0^+} \int_{\Omega \cap E_\lambda} |\omega| |\nabla (u-w)| \, dz + \lambda \int_{\Omega_T \cap E_\lambda} |\omega| \, dz.
\]  
(5.64)

To obtain (a), we made use of the weaker assumption (3.2) (see Remark 3.2 for the details) and to obtain (b), we made use of (5.8) and Lemma 5.3.

Combining (5.55), (5.56), (5.57), (5.59), (5.63) and (5.64) with (5.54), we get
\[
\int_\Omega (v^2 - (v_\lambda - v)^2)(x,T) \, dx + \iint_{\Omega_T \cap E_\lambda} |\nabla u|^p - |h_0|^p \, dz \leq \iint_{\Omega_T \cap E_\lambda} (|\nabla u|^{p-1} + |h_0|^{p-1}) |\nabla w| \, dz
\]
\[
+ \lambda \int_{\Omega \cap E_\lambda} (|\nabla u|^{p-1} + |h_0|^{p-1}) \, dz
\]
\[
+ \int_{\Omega_T \cap E_\lambda} |\omega| \, dz
\]
\[
+ \lambda \int_{\Omega_T \cap E_\lambda^c} |\omega| \, dz
\]
\[
+ \lambda^p |\mathbb{R}^{n+1} \setminus E_\lambda|.
\]  
(5.65)

In fact, if we consider a cut-off function \( \zeta_{\varepsilon}^{t_0}(\cdot) \) for some \( t_0 \in (0, T) \), where
\[
\zeta_{\varepsilon}^{t_0}(t) = \begin{cases} 
1 & \text{for } t \in (0 + \varepsilon, t_0 - \varepsilon) \\
0 & \text{for } t \in (-\infty, 0) \cup (t_0, \infty),
\end{cases}
\]
we get the following analogue of (5.65):
\[
\int_\Omega (v^2 - (v_\lambda - v)^2)(x,t_0) \, dx + \int_0^{t_0} \int_{\Omega \cap E_\lambda} |\nabla u|^p - |h_0|^p \, dx \, dt \leq \iint_{\Omega_T \cap E_\lambda} (|\nabla u|^{p-1} + |h_0|^{p-1}) |\nabla w| \, dz
\]
\[
+ \lambda \int_{\Omega \cap E_\lambda} (|\nabla u|^{p-1} + |h_0|^{p-1}) \, dz
\]
\[
+ \int_{\Omega_T \cap E_\lambda} |\omega| \, dz
\]
\[
+ \lambda \int_{\Omega_T \cap E_\lambda^c} |\omega| \, dz
\]
\[
+ \lambda^p |\mathbb{R}^{n+1} \setminus E_\lambda|.
\]  
(5.66)
Using Lemma 5.10, we get for almost every \( t \in (0, T) \),
\[
\int_{\Omega} |(v)^2 - (v_\lambda - v)^2|(y, t) \, dy \geq \int_{E^\lambda_\Omega} |v(x, t)|^2 \, dx - \lambda^p |\mathbb{R}^{n+1} \setminus E^\lambda_\Omega|.
\] (5.67)

Thus combining (5.67) with (5.66), we get
\[
\sup_{t \in (0, T)} \int_{E^\lambda_\Omega} |v(x, t)|^2 \, dx + \int_0^t \int_{\Omega \cap E^\lambda_\Omega} |\nabla u|^p - |h_0|^p \, dx \, dt \leq \int_{\Omega_\lambda \cap E^\lambda_\Omega} (|\nabla u|^{p-1} + |h_0|^{p-1}) |\nabla w| \, dz \\
+ \lambda \int_{\Omega_\lambda \setminus E^\lambda_\Omega} (|\nabla u|^{p-1} + |h_0|^{p-1}) \, dz \\
+ \lambda \int_{\Omega_\lambda \cap E^\lambda_\Omega} |\omega| |\nabla(u - w)| \, dz \\
+ \lambda \int_{\Omega_\lambda \setminus E^\lambda_\Omega} |\omega| \, dz \\
+ \lambda^p |\mathbb{R}^{n+1} \setminus E^\lambda_\Omega|.
\] (5.68)

Since \( \int_{E^\lambda_\Omega} |v(x, t)|^2 \, dx \) occurs on the left hand side of (5.68) and is positive, we can ignore this term. Let us now multiply (5.68) with \( \lambda^{-1-\beta} \) and integrating over \( (0, \infty) \) with respect to \( \lambda \), we get
\[
K_1 \leq K_2 + K_3 + K_4 + K_5 + K_6,
\] (5.69)

where we have set
\[
K_1 := \int_0^\infty \lambda^{-1-\beta} \int_{\Omega_\lambda \cap E^\lambda_\Omega} (|\nabla u|^p - |h_0|^p) \, dz \, d\lambda,
K_2 := \int_0^\infty \lambda^{-1-\beta} \int_{\Omega_\lambda \cap E^\lambda_\Omega} (|\nabla u| + |h_0|)^{p-1} |\nabla w| \, dz \, d\lambda,
K_3 := \int_0^\infty \lambda^{-\beta} \int_{\Omega_\lambda \cap E^\lambda_\Omega} (|\nabla u| + |h_0|)^{p-1} \, dz \, d\lambda,
K_4 := \int_0^\infty \lambda^{-1-\beta} \int_{\Omega_\lambda \cap E^\lambda_\Omega} |\omega| |\nabla(u - w)| \, dz \, d\lambda,
K_5 := \int_0^\infty \lambda^{-\beta} \int_{\Omega_\lambda \cap E^\lambda_\Omega} |\omega| \, dz \, d\lambda,
K_6 := \int_0^\infty \lambda^{-1-\beta} \lambda^p |\mathbb{R}^{n+1} \setminus E^\lambda_\Omega| \, d\lambda.
\]

Let us now estimate each of the \( \{K_i\}_{i=1}^6 \) as follows:

**Estimate of \( K_1 \):** Applying Fubini, we get
\[
K_1 \geq \frac{1}{\beta} \int_{\Omega_\lambda} g(z)^{-\beta} (|\nabla u|^p - |h_0|^p) \, dz.
\]

Using Young’s inequality along with (5.6), we get for any \( \epsilon_1 > 0 \),
\[
\int_{\Omega_\lambda} |\nabla u|^{p-\beta} \, dz \leq C(\epsilon_1) \beta K_1 + (\epsilon_1 + \beta) \int_{\Omega_\lambda} (|\nabla u| + |h_0|)^{p-\beta} \, dz + \epsilon_1 \int_{\Omega_\lambda} |\nabla w|^{p-\beta} \, dz \\
+ \epsilon_1 \|w\|_{L^{p-\beta}(0; T; W^{-1}(0, t; \mathbb{R}^{n+1}))}.
\] (5.70)

**Estimate of \( K_2 \):** Again by Fubini, we get
\[
K_2 = \frac{1}{\beta} \int_{\Omega_\lambda} g(z)^{-\beta} (|\nabla u| + |h_0|)^{p-1} |\nabla w| \, dz.
\]
From the definition of \( g(z) \) in (5.5), we see that for \( z \in \Omega_T \), we have \( g(z) \geq (|\nabla u| + |h_0|)(z) \) which implies \( g(z)^{-\beta} \leq (|\nabla u| + |h_0|)^{-\beta}(z) \). Applying Young’s inequality, for any \( \epsilon_2 > 0 \), we get
\[
K_2 \leq \frac{C(\epsilon_2)}{\beta} \int_{\Omega_T} |\nabla w|^{p-\beta} \, dz + \frac{\epsilon_2}{\beta} \int_{\Omega_T} (|\nabla u| + |h_0|)^{p-\beta} \, dz. \tag{5.71}
\]

**Estimate of \( K_3 \):** Again applying Fubini, we get
\[
K_3 = \frac{1}{1-\beta} \int_{\Omega_T} g(z)^{1-\beta} (|\nabla u| + |h_0|)^{p-1} \, dz
\leq \int_{\Omega_T} g(z)^{p-\beta} \, dz + \int_{\Omega_T} (|\nabla u| + |h_0|)^{p-\beta} \, dz \tag{5.72}
\leq \int_{\Omega_T} (|\nabla u| + |h_0|)^{p-\beta} \, dz + \int_{\Omega_T} |\nabla u|^{p-\beta} \, dz + \|w'\|_{L^{p-\beta} \Omega_T}.
\]

**Estimate of \( K_4 \):** Again by Fubini, we get
\[
K_4 = \frac{1}{\beta} \int_{\Omega_T} g(z)^{-\beta} |\nabla u - \nabla w| \, dz.
\]

From the definition of \( g(z) \) in (5.5), we see that for \( z \in \Omega_T \), we have \( g(z) \geq |\nabla u - \nabla w|(z) \) which implies \( g(z)^{-\beta} \leq |\nabla u - \nabla w|^{-\beta}(z) \). Applying Young’s inequality, for any \( \epsilon_3 > 0 \), we get
\[
K_4 \leq \frac{1}{\beta} \int_{\Omega_T} |\nabla u - \nabla w|^{1-\beta} \, dz
\leq \frac{C(\epsilon_3)}{\beta} \int_0^T \int_{\Omega} |\tilde{a}|^{p-\beta} \, dx \, dt + \frac{\epsilon_3}{\beta} \int_{\Omega_T} |\nabla u - \nabla w|^{p-\beta} \, dx \, dt
\leq \frac{C(\epsilon_3)}{\beta} \|w'\|_{L^{p-\beta} (0,T;\Omega_T)} + \frac{\epsilon_3}{\beta} \int_{\Omega_T} |\nabla w|^{p-\beta} \, dz.
\tag{5.73}
\]

**Estimate of \( K_5 \):** Again applying Fubini, we get
\[
K_5 = \frac{1}{1-\beta} \int_{\Omega_T} g(z)^{1-\beta} |\tilde{w}| \, dz
\leq \int_{\Omega_T} g(z)^{p-\beta} \, dz + \int_{\Omega_T} |\tilde{w}|^{p-\beta} \, dz
\leq \int_{\Omega_T} (|\nabla u| + |h_0|)^{p-\beta} \, dz + \int_{\Omega_T} |\nabla u|^{p-\beta} \, dz + \|w'\|_{L^{p-\beta} \Omega_T}. \tag{5.74}
\]

To obtain (a), we made use of (5.6) and Lemma 2.8.

**Estimate of \( K_6 \):** Applying the layer cake representation followed by (5.6), we get
\[
K_6 = \frac{1}{p-\beta} \int_{\mathbb{R}^{n+1}} g(z)^{p-\beta} \, dz
\leq \int_{\Omega_T} (|\nabla u| + |h_0|)^{p-\beta} \, dz + \int_{\Omega_T} |\nabla u|^{p-\beta} \, dz + \|w'\|_{L^{p-\beta} \Omega_T}. \tag{5.75}
\]

Combining (5.70), (5.71), (5.72), (5.73), (5.74) and (5.75) with (5.69), we get
\[
\int_{\Omega_T} |\nabla u|^{p-\beta} \, dz \leq [C(\epsilon_1)(\epsilon_2 + \beta + \epsilon_3) + \epsilon_1] \int_{\Omega_T} |\nabla u|^{p-\beta} \, dz
+ [C(\epsilon_1)(\epsilon_2 + \beta + 1) + \epsilon_1] \int_{\Omega_T} |h_0|^{p-\beta} \, dz
+ [C(\epsilon_1)(C(\epsilon_2) + \beta + \epsilon_3) + \epsilon_1] \int_{\Omega_T} |\nabla w|^{p-\beta} \, dz
+ C(\epsilon_1)(C(\epsilon_3) + \beta) \|w'\|_{L^{p-\beta} (0,T;\Omega_T)}. \tag{5.76}
\]
Choosing $\epsilon_1$ small followed by choosing $\epsilon_2, \epsilon_3, \beta$ small, we get the desired estimate. This completes the proof of the theorem.

6. Higher integrability at the initial boundary

Let us consider a cylinder $Q = Q_{\rho,s}(z_0)$ centered at a point $z_0 = (x_0, t_0) \in \Omega \times \mathbb{R}$ such that

$$Q \cap \Omega \times [0, \infty) \neq \emptyset \quad \text{and} \quad Q \cap \Omega \times (-\infty, 0) \neq \emptyset.$$ 

In particular, we take a cylinder that crosses the initial time slice $\{ t = 0 \}$. Furthermore, assume that the cylinder $Q$ satisfies

$$Q = B_{\rho} \times I_s \quad \text{and} \quad 16Q \subset \Omega \times \mathbb{R}. \quad (6.1)$$

We shall suppress writing the center of the cylinder $z_0 = (x_0, t_0)$ henceforth unless necessary. Corresponding to this cylinder, let us take the following test functions:

$$\eta(x) \in C_c^\infty(B_{8\rho}) : \quad \eta(x) \equiv 1 \quad \text{on } B_{4\rho}, \quad |\nabla \eta| \leq \frac{C}{\rho}, \quad (6.2)$$

$$\zeta(t) \in C_c^\infty(I_{8s}) : \quad \zeta(t) \equiv 1 \quad \text{on } I_{4s}, \quad |\zeta'| \leq \frac{C}{s}. \quad (6.3)$$

For any given $(x, t) \in \Omega \times [0, T)$, we define the following function and its corresponding Steklov average to be

$$v(x, t) := (u(x, t) - w(x, t))\eta(x)\zeta(t) \quad \text{and} \quad v_h(x, t) := [u - w]_h(x, t)\eta(x)\zeta(t). \quad (6.4)$$

From (1.2) and Lemma 2.9, we see that

$$v_h \xrightarrow{h \to 0} v \quad \text{and} \quad v(z) = 0 \quad \text{for } z \in \Omega \times \{ t = 0 \}.$$ 

In subsequent calculations, we extend $v$ by zero to $\Omega \times (-\infty, 0]$.

6.1. Construction of test function

In what follows, we shall use Lemma 2.8 to obtain a representation

$$\frac{d\tilde{w}}{dt} = \text{div} \tilde{w} \quad \text{in } \Omega, \quad (6.5)$$

for some $\tilde{w} \in L^{p,\infty}(0, T; L^{p,\infty}(\Omega))$.

Again, let us fix the following choice of exponents:

$$1 < q \leq p - 2\beta < p - \beta < p, \quad (6.6)$$

where $\beta$ is a constant to be chosen sufficiently small later on. Define the following function

$$g(z) := \max \{ G_1(z), G_2(z), G_3(z), G_4(z), G_5(z) \}. \quad (6.7)$$

with

$$G_1(z) := \mathcal{M}(|\nabla u| + |h_0|)^q \chi_{16Q \cap \Omega_T} \chi(z),$$

$$G_2(z) := \mathcal{M}(|\nabla u|^q \chi_{16Q \cap \Omega_T}) \chi(z),$$

$$G_3(z) := \mathcal{M}(|u - w|^q \rho^{a} \chi_{16Q \cap \Omega_T}) \chi(z),$$

$$G_4(z) := \mathcal{M}(|\nabla w|^q \chi_{16Q \cap \Omega_T}) \chi(z),$$

$$G_5(z) := \mathcal{P}^{-1,1}(w', \chi_{16Q \cap \Omega_T})(z). \quad (6.8)$$

where $v$ is defined in (6.4) and $u$ is as defined in the hypothesis of Theorem 3.4 and $G_5$ is defined using (6.5) as follows (see also (5.2)):

$$\mathcal{P}^{-1,1}(w', \chi_{16Q \cap \Omega_T})(x, t) := \sup_{(a, b) \ni t} \sup_{B \ni x} \int_a^b \int_B |\tilde{w}| \chi_{16Q \cap \Omega_T} \, dx \, dt.$$
Let us now define the good set to be

\[ E_\lambda = \{ z \in \mathbb{R}^{n+1} : g(z) \leq \lambda \}, \]

and apply Lemma 4.3 and Lemma 4.4 with \( E = E_\lambda \) to get a covering of \( E_\lambda \). Recall that the intrinsic scaling is of the form

\[ \gamma := \lambda^{2-p} \quad \text{and} \quad Q_j(x,t) = B_{r_j}(x) \times (t_j - \gamma r_j^2, t_j + \gamma r_j^2). \]

Now we define the following Lipschitz extension function as follows:

\[ v_{\lambda,h}(z) := v_h(z) - \sum_i \Psi_i(z)(v_h(z) - v_i^h). \tag{6.9} \]

where

\[
v_i^h := \begin{cases} \frac{1}{\| \Psi_i \|_{L^1(\frac{3}{4}Q_i)}} \int_{\frac{3}{4}Q_i} v_h(z)\Psi_i(z)\chi_{[0,T]} \, dz & \text{if } \frac{3}{4}Q_i \subset 8B \times [0, \infty), \\ 0 & \text{else}. \end{cases}
\]

Note that \( u = 0 \) on \( \Omega \times \{ t = 0 \} \), which along with (6.1) enables us to switch between \( \chi_{[0,T]} \) and \( \chi_{\Omega_T} \) without affecting the calculations.

**Assumption 6.1.** Let \( \alpha_0 \in \mathbb{R}^+ \) be such that the following is satisfied:

\[
\alpha_0^{p-\beta} \leq \int_Q (|\nabla u| + |h_0|)^{p-\beta} \chi_{16Q\cap \Omega_T} \, dz + \int_Q |\nabla w|^{p-\beta} \chi_{16Q\cap \Omega_T} \, dz + \int \frac{d}{dt} \left( \chi_{16Q\cap \Omega_T} \right) \left| \frac{\alpha}{\rho} \right| L^{n-1} \left( 0, T; W^{-1} \frac{\alpha}{\rho} \right) \Omega, 
\]

where \( Q = Q_{p,s} = Q_{p, \alpha_0^{p-\beta}} \) and constant depends on universal constants. Noting (6.7), it is easy to see that there exists a universal positive constant \( c_e = c_e(n, p, \Lambda_0, \Lambda_1) \) such that for all \( \lambda \geq c_e \alpha_0 \), we have \( E_\lambda \neq \emptyset \).

Note here that we made use of (6.5) and denoted

\[
\frac{1}{|16Q|} \left| \frac{d}{dt} \chi_{16Q\cap \Omega_T} \right| L^{n-1} \left( 0, T; W^{-1} \frac{\alpha}{\rho} \right) \Omega := \int_{16Q} |\alpha| \left| \frac{\alpha}{\rho} \right| L^{n-1} \left( 0, T; W^{-1} \frac{\alpha}{\rho} \right) \Omega. 
\]

Let us first prove an important bound for \( g \) as defined in (6.7):

**Lemma 6.2.** Let Assumption 6.1 be in force and let \( \alpha_0 \in \mathbb{R}^+ \) and \( c_e \) be as in Assumption 6.1 and \( g(z) \) be as in (6.7), then the following holds for any \( q < \theta \leq p - \beta \):

\[
\int_{\mathbb{R}^{n+1}} |g(z)|^\theta \, dz \leq |Q| \alpha_0^\theta. 
\]

**Proof.** We proceed as follows:

\[
\int_{\mathbb{R}^{n+1}} |g(z)|^\theta \, dz \overset{(a)}{\leq} \int_{16Q\cap \Omega_T} (|\nabla u| + |h_0|)^\theta + |\nabla w|^\theta + |\nabla v|^\theta + \left( \frac{|v|}{\rho} \right)^\theta \, dz
\]

\[ \overset{(b)}{=} \int_{16Q\cap \Omega_T} (|\nabla u| + |h_0| + |\nabla w|)^\theta \, dz + \int_{16Q\cap \Omega_T} \left( \frac{|u - w|}{\rho} \right)^\theta \, dz \tag{6.10} \]

\[ \overset{(c)}{=} |Q| \alpha_0^\theta + \int_{16Q\cap \Omega_T} \left( \frac{|u - w|}{\rho} \right)^\theta \, dz. \]
To obtain (a), we applied the standard maximal function estimate along with the bound from (5.3), to obtain (b), we made use of (6.4) which gives $\nabla v = (u - w)\zeta \nabla \eta + \eta \nabla (u - w)$ and finally to obtain (c), we made use of the hypothesis from Assumption 6.1.

To control the last term of (6.10), let us use the cut-off function $\xi \in C^\infty_c (16Q \cap \Omega_T \cap \{ t \leq 0 \})$ from Lemma 4.1 and note that $(u - w)_\xi = 0$. Thus we can apply Lemma 4.1 to get

$$
\iint_{16Q \cap \Omega_T} \left( \frac{|u - w|}{\rho} \right)^\vartheta \, dz = \iint_{16Q} \left( \frac{|(u - w) - (u - w)_\xi|}{\rho} \right)^\vartheta \chi_{[0,T]} \, dz
$$

$$
\leq \iint_{16Q} |\nabla u - \nabla w|^\vartheta \chi_{[0,T]} \, dz
$$

$$
+ \sup_{t_1, t_2 \in 16I \cap [0,T]} \left( \frac{|(u - w)_{\mu} (t_2) - (u - w)_{\mu} (t_1)|}{\rho} \right)^\vartheta.
$$

Assumption 6.1

To control the last term of (6.11), we apply Lemma 4.8 with $\varphi \equiv 1$ and $\phi(x) = \mu(x)$ to get for any $t_1, t_2 \in 16I \cap [0,T]$,

$$
| (u - w)_{\mu} (t_2) - (u - w)_{\mu} (t_1) | \leq \| \nabla \mu \|_{L^\infty(16B_\rho)} \iint_{8Q} (|\nabla u| + |h_0|)^{p-1} \chi_{[0,T]} \, dz
$$

$$
+ \| \nabla \mu \|_{L^\infty(16B_\rho)} |Q| \iint_{16Q \cap \Omega_T} |w| \chi_{16Q \cap \Omega_T} \, dz
$$

(6.12)

To obtain (a), we made use of Lemma 4.8 and to obtain (b), we made use of Assumption 6.1.

Combining (6.10), (6.11) and (6.12), we get the desired estimate.

By the choice of the cylinder, we see that $t_0 - s < 0 < t_0 + s$. As a consequence, let us define the followings:

$$
\mathcal{H} := \mathbb{R}^n \times [0, t_0 + s] \cap \Omega_T, k\mathcal{H} := \mathbb{R}^n \times [0, t_0 + k^2 s] \cap \Omega_T,
$$

$$
\Theta := \{ i \in \mathbb{N} : \frac{3}{4} Q_i \cap 2\mathcal{H} \neq \emptyset \},
$$

$$
\Theta_1 := \{ i \in \mathbb{N} : 8Q_i \subset \mathbb{R}^n \times (-\infty, t_0 + 16s) \},
$$

$$
\Theta_2 := \Theta \setminus \Theta_1.
$$

Recall from (6.2) that on $|t_0 - 16s, t_0 + 16s|$, we have $\zeta(t) = 1$ and in particular on $Q_i$ for any $i \in \Theta_1$.

6.2. Bounds of $v_{\lambda,h}$

The first lemma is a rough bound of $v_{\lambda,h}$:

**Lemma 6.3.** Let $z \in E_\lambda^c$, then

$$
|v_{\lambda,h}(z)| \lesssim_{(n)} \rho \lambda.
$$

**Proof.** From (6.9), we see that $v_{\lambda,h}(z) = \sum_{i} \Psi_i(z) \psi^i_{h}$, which along with Lemma 2.9 and (W4) gives the following bound:

$$
|v_{\lambda,h}^i| \leq \rho \iint_{4Q_i} \frac{|\psi^i_{h}(\tilde{z})|}{\rho} \chi_{[0,T]} \, d\tilde{z} \leq \rho \iint_{16Q_i} \frac{|v(\tilde{z})|}{\rho} \chi_{[0,T]} \, d\tilde{z} \leq \rho \lambda.
$$

Let us now prove an improved bound of $v_{\lambda,h}$.
Lemma 6.4. For any $i \in \Theta_1$ and any $j \in A_i$ where $\Theta_1$ is from (6.14) and $A_i$ is from Lemma 4.3, there holds

$$|v^i_h - v^j_h| \leq (n, p, \Lambda_1) \min \{\rho, r_i\} \lambda.$$  

Proof. If $\rho \leq r_i$, then the result follows from Lemma 6.3. Hence, we only have to consider the case $r_i \leq \rho$.

Suppose either $\frac{3}{4} Q_i \cap \{t \leq 0\} \neq \emptyset$ or $\frac{3}{4} Q_i \cap (8B)^c \times [0, \infty) \neq \emptyset$, i.e., $\frac{3}{4} Q_i$ either crosses the initial boundary or the lateral boundary.

- In the case $\frac{3}{4} Q_i \cap \{t \leq 0\} \neq \emptyset$, then from the fact $j \in A_i$, we can assume $\frac{3}{4} Q_i \subset 8B \times [0, \infty)$, otherwise, $v^j_h = 0$ and there is nothing to prove. Thus, we can apply Poincaré's inequality after enlarging to $4Q_j$ which gives $\frac{3}{4} Q_i \subset 4Q_j$ from (W11)

$$|v^j_h| \leq \iint_{4Q_j} |v^j_h| \chi_{[0, T]} \, dz = r_j \iint_{4Q_j} \left| \frac{v^j_h(z)}{r_j} \right| \chi_{[0, T]} \, dz \leq r_j \iint_{4Q_j} |\nabla v^j_h| \chi_{[0, T]} \, dz \leq r_j \lambda.$$  

- In the case $\frac{3}{4} Q_i \cap \{t \leq 0\} \neq \emptyset$, then again we can assume $\frac{3}{4} Q_i \subset 8B \times [0, \infty)$, otherwise $v^j_h = 0$ and there is nothing to prove. From (W11), we see that $\frac{3}{4} Q_i \subset 4Q_j$, which along with (6.9) and a cut-off function $\xi \in C_c^\infty(4Q_j \cap \{t \leq 0\})$ implies $\left( v^j_h \chi_{[0, T]} \right)_{\xi} = 0$. Thus, for any $\mu \in C_c^\infty(4B_j)$ with $|\mu| \leq \frac{C}{r_j^n}$ and $|\nabla \mu| \leq \frac{C}{r_j^{n+1}}$, we get

$$|v^j_h| \leq r_j \iint_{4Q_j} \left| \frac{v^j_h(z)}{r_j} \chi_{[0, T]} \right| \, dz \leq r_j \iint_{4Q_j} |\nabla v^j_h| \chi_{[0, T]} \, dz + \sup_{t_1, t_2 \in 4I_j, \cap [0, T]} \left| \left( v^j_h \chi_{[0, T]} \right)_{\mu} (t_2) - \left( v^j_h \chi_{[0, T]} \right)_{\mu} (t_1) \right| \overset{(a)}{\leq} r_j \lambda.$$  

To obtain (a), we controlled the first term using (W4) and the second term is controlled similar to (6.18).

Let us come to the case where $\frac{3}{4} Q_i \subset 8B \times [0, \infty)$. We see that (see [2, Lemma 3.7] for the details)

$$|v^i_h - v^j_h| \leq \iint_{Q_i} |v^i_h(z) \chi_{[0, T]} - v^j_h(z)| \, dz + \iint_{Q_j} |v^i_h(z) \chi_{[0, T]} - v^j_h(z)| \, dz.$$  

(6.15)

Since $i \in \Theta_1$, we must have $\zeta \equiv 1$ on $Q_i$, thus applying Lemma 4.1, for any $\mu \in C_c^\infty(B_i)$ with $|\mu| \leq \frac{C}{r_i^n}$ and $|\nabla \mu| \leq \frac{C}{r_i^{n+1}}$, we get

$$\iint_{Q_i} |v^i_h(z) \chi_{[0, T]} - v^j_h(z)| \, dz \leq r_i \iint_{Q_i} |\nabla v^i_h| \chi_{[0, T]} \, dz + \sup_{t_1, t_2 \in I_i} \left| \left( v^i_h \chi_{[0, T]} \right)_{\mu} (t_2) - \left( v^i_h \chi_{[0, T]} \right)_{\mu} (t_1) \right|.$$  

(6.16)

The first term on the right hand side of (6.16) can be controlled easily using (W4) to get

$$\iint_{Q_i} |\nabla v^i_h| \chi_{[0, T]} \, dz \leq \lambda.$$  

(6.17)
To control the second term on the right hand side of (6.16), let us apply Lemma 4.8 with \( \phi(x) = \eta(x)\mu(x) \) and \( \varphi(t) \equiv \zeta(t) \equiv 1 \) along with to get

\[
\sup_{t_1, t_2 \in [0, T]} \left| (v_h)_\mu (t_2) - (v_h)_\mu (t_1) \right| \lesssim \left( \frac{1}{r_i \lambda^{N}} + \frac{1}{r_i^{n+1}} \right) \int_{Q_i} |\nabla u| + |h_0|^{p-1} \chi_{[0,T]} \, dz
\] 

\[
+ \left( \frac{1}{r_i \lambda^{N}} + \frac{1}{r_i^{n+1}} \right) \int_{I_i} \left[ \int_{B_i} |\bar{w}| \chi_{16Q \cap \Omega_T} \, dx \right] \, dt.
\]

Since \( |B_i| = c(n)r_i^n \), \( |I_i| = \lambda^2 r_i^2 \), after using the fact that \( r_i \leq \rho \), we make use of Remark 3.2 to get

\[
\sup_{t_1, t_2 \in I_i} \left| (v_h)_\mu (t_2) - (v_h)_\mu (t_1) \right| \leq r_i^{2-p} \left( \iint_{16Q_i} (|\nabla u| + |h_0|^{p-1}) \chi_{[0,T]} \, dz + \iint_{16Q_i} |\bar{w}| \chi_{16Q \cap \Omega_T} \, dz \right)
\]  

(W4) \( \lesssim r_i \lambda. \)  

Thus combining (6.18) and (6.17) into (6.16) gives the desired estimate which proves the lemma.  

6.3. Bounds on derivatives of \( v_{\lambda,h} \)  
Using bounds from Lemma 6.3 and Lemma 6.4, following the calculations in [2], we can obtain the following lemmas which estimate the derivatives of \( v_{\lambda,h} \):

Lemma 6.5. Given any \( z \in E^c_\lambda \), from Lemma 4.3, we have \( z \in \frac{3}{4} Q_i \) for some \( i \in \mathbb{N} \). If either \( i \in \Theta_1 \) or \( i \in \Theta_2 \) with \( \rho \leq r_i \), then

\[
|\nabla v_{\lambda,h}| \lesssim (n,p,\Lambda_1) \lambda.
\]

Lemma 6.6. Let \( z \in E^c_\lambda \) and \( \varepsilon > 0 \) be any number, then there exists a constant \( C(n) \) such that the following holds:

\[
|v_{\lambda,h}(z)| \leq C \iint_{4Q_i} |v_{h}(\tilde{z})| \chi_{[0,T]} \, d\tilde{z} \leq \frac{C r_i \lambda}{\varepsilon} + \frac{C \varepsilon}{\lambda r_i} \iint_{4Q_i} |v_{h}(\tilde{z})|^2 \chi_{[0,T]} \, d\tilde{z},
\]

\[
|\nabla v_{\lambda,h}(z)| \leq C \frac{1}{r_i} \iint_{4Q_i} |v_{h}(\tilde{z})| \chi_{[0,T]} \, d\tilde{z} \leq \frac{C \lambda}{\varepsilon} + \frac{C \varepsilon}{\lambda r_i} \iint_{4Q_i} |v_{h}(\tilde{z})|^2 \chi_{[0,T]} \, d\tilde{z}.
\]

Lemma 6.7. Let \( z \in E^c_\lambda \) and \( \varepsilon \in (0,1) \) be given, from Lemma 4.3, we have \( z \in \frac{3}{4} Q_i \) for some \( i \in \mathbb{N} \). Suppose \( i \in \Theta_1 \), then there holds:

\[
|v_{\lambda,h}(z)| \leq C(n,p,\Lambda_1) \left( \min\{\rho, r_i\} \lambda + |v_{h}(z)| \right) \leq C(n,p,b_0,.r_0,\Lambda_1) \left( \frac{r_i \lambda}{\varepsilon} + \frac{\varepsilon}{r_i \lambda} |v_{h}(z)| \right)
\]

\[
|\nabla v_{\lambda,h}(z)| \leq C(n) \frac{\lambda}{\varepsilon}.
\]

Lemma 6.8. Let \( z \in E^c_\lambda \), then from Lemma 4.3, we have \( z \in \frac{3}{4} Q_i \) for some \( i \in \mathbb{N} \). Suppose \( i \in \Theta_2 \), then there holds:

\[
|v_{\lambda,h}(z)| \leq C(n) \left( r_i \lambda + \frac{\lambda^{1-p} r_i}{s} \iint_{4Q_i} |v_{h}(\tilde{z})|^2 \chi_{[0,T]} \, d\tilde{z} \right),
\]

\[
|\nabla v_{\lambda,h}(z)| \leq C(n) \left( \lambda + \frac{\lambda^{1-p}}{s} \iint_{4Q_i} |v_{h}(\tilde{z})|^2 \chi_{[0,T]} \, d\tilde{z} \right).
\]

Lemma 6.9. Let \( z \in E^c_\lambda \), then from Lemma 4.3, there exists an \( i \in \mathbb{N} \) such that \( z \in \frac{3}{4} Q_i \). Then the following estimates for the time derivative of \( v_{\lambda,h} \) holds:

\[
|\partial_t v_{\lambda,h}(z)| \leq C(n) \frac{1}{\gamma T} \iint_{4Q_i} |v_{h}(\tilde{z})| \chi_{[0,T]} \, d\tilde{z}.
\]
If \( i \in \Theta_1 \), then we have
\[
|\partial_i v_{\lambda,h}(z)| \leq C_{(n,p,\Lambda_1)} \frac{1}{\gamma r_i} \min\{r_i,\rho\} \lambda.
\]

If \( i \in \Theta_2 \), then there holds
\[
|\partial_i v_{\lambda,h}(z)| \leq C_n \frac{\rho \lambda}{s}.
\]

As a consequence of the above lemmas, we have the following lemma which controls the integral of \( v_{\lambda,h} \).

**Lemma 6.10.** From (6.9), we extend \( v_{\lambda,h} \) by zero in the region \( 8Q \setminus E_\lambda \cap \{t \leq 0\} \). From this, for any \( \vartheta \in [1,p-\beta] \), we have the following bound:
\[
\iint_{8Q \setminus E_\lambda} |v_{\lambda,h}(z)|^\vartheta dz \leq C_{(n,p,\Lambda_1)} \iint_{8Q \setminus E_\lambda} |v_h(z)|^\vartheta \chi_{[0,T]} dz.
\]

### 6.4. Two important intermediate estimates

In order to prove the Lipschitz continuity of \( v_{\lambda,h} \), we need to obtain a suitable control of integrals over \( Q_1 \), which will be done in the following two lemmas. The first one is an estimate for cylinders in \( \Theta_1 \).

**Lemma 6.11.** For any \( i \in \Theta_1 \), we have the following estimate:
\[
\iint_{\frac{3}{4}Q_i} \left| \frac{v_h(z) - v_{\lambda,h}(z)}{r_i} \right|^q \chi_{[0,T]} dz \leq C_{(n,p,\Lambda_0,\Lambda_1)} \lambda^q.
\]

**Proof.** For any \( z \in \left( \frac{3}{4}Q_i \right) \), using (6.9) along with triangle inequality and (W13), we get
\[
\iint_{\frac{3}{4}Q_i} |v_h(z) - v_{\lambda,h}(z)|^q \chi_{[0,T]} dz \leq \iint_{\frac{3}{4}Q_i} |v_h(z) - v_i|^q dz + \sum_{j \in \mathcal{A}_i} \iint_{\frac{3}{4}Q_i} \left| v_j^i - v_{\lambda,h}^{i,j} \right|^q dz := J_1 + J_2. \quad (6.19)
\]

We shall estimate each of the terms of (6.19) as follows (note that \( i \in \Theta_1 \)). Note that \( J_1 \) is exactly as in (6.15), which implies
\[
J_1 \leq (r_i \lambda)^q. \quad (6.20)
\]

In order to estimate \( J_2 \), we can directly use Lemma 6.4 to get
\[
J_2 \leq (r_i \lambda)^q. \quad (6.21)
\]

Substituting (6.20) and (6.21) into (6.19) and making use of (W8), the lemma follows.

The second lemma is more involved to prove, as it concerns cylinders in \( \Theta_2 \).

**Lemma 6.12.** Given any \( i \in \Theta_2 \) and \( 8Q_i \subset \mathbb{R}^n \times \mathbb{R}^+ \), let \( \alpha_0 \) and \( c_0 \) be as in Assumption 6.1, then for any \( \lambda \geq c_0 \alpha_0 \), there holds
\[
\iint_{\frac{3}{4}Q_i} \left| \frac{v_h(z) \chi_{[0,T]} - v_i^h}{r_i} \right|^q dz \leq (n,p,q,\Lambda_0,\Lambda_1,c_0) \lambda^q \left( 1 + \left( \frac{\lambda^{2-p}}{\alpha_0^{2-p}} \right) + \left( \frac{\lambda^{2-p}}{\alpha_0^{2-p}} \right)^{\frac{p}{p-1}} + \left( \frac{\lambda^{2-p}}{\alpha_0^{2-p}} \right)^{\frac{2p}{p-1}} \right)^q.
\]

**Proof.** Let us first note that without loss of generality, we can take \( r_i \leq \rho \), otherwise we can directly apply Lemma 6.4 in the case of \( r_i \leq \rho \) to get
\[
\iint_{\frac{3}{4}Q_i} \left| \frac{v_h(z) \chi_{[0,T]} - v_i^h}{r_i} \right|^q dz = \iint_{\frac{3}{4}Q_i} \left| \frac{v_h(z) \chi_{[0,T]} - v^h_{\lambda,h}}{r_i} \right|^q dz \leq \frac{\rho}{r_i} \lambda^q \leq \lambda^q. \quad (6.22)
\]

- If \( v_i^h = 0 \), then either \( \frac{3}{4}Q_i \cap (8B)^c \times [0,\infty) \neq \emptyset \) or \( \frac{3}{4}Q_i \cap \{t \leq 0\} \neq \emptyset \). In each of those cases, we can estimate as follows:
In the case \( \frac{3}{4} Q_i \) crosses the lateral edge, we can directly apply Theorem 2.5 with \( \kappa = 1 \) to get

\[
\iint_{\frac{3}{4} Q_i} \left| \frac{v_h(z) \chi_{[0,T]}}{r_i} \right|^q \, dz \leq \iint_{\frac{3}{4} Q_i} |\nabla v_h(z)|^q \chi_{[0,T]} \, dz \overset{(W4)}{\lesssim} \lambda^q.
\]

In the case \( \frac{3}{4} Q_i \) crosses the initial edge, we take a cut-off function \( \xi \in C^\infty_c(Q_j \cap \{t \leq 0\}) \) from which we see that \( (v_h)_\xi = 0 \). Making use of Lemma 4.1, for any \( \mu \in C^\infty_c(B_i) \) with \( |\mu| \leq \frac{C}{r_i^n} \) and \( |\nabla \mu| \leq \frac{C}{r_i^{n+1}} \), we get

\[
\iint_{\frac{3}{4} Q_i} \left| \frac{v_h(z) \chi_{[0,T]}}{r_i} \right|^q \, dz \leq \iint_{\frac{3}{4} Q_i} \left| \frac{v_h(z) \chi_{[0,T]} - (v_h)_\xi}{r_i} \right|^q \, dz \\
\leq \iint_{\frac{3}{4} Q_i} |\nabla v_h|^q \chi_{[0,T]} \, dz + \sup_{t_1, t_2 \in t_i} \left| \left( v_h \chi_{[0,T]} \right)_\mu (t_1) - \left( v_h \chi_{[0,T]} \right)_\mu (t_2) \right|^q. \tag{6.23}
\]

The first term on the right hand side of (6.23) can be estimated using (W4), to estimate the second term on the right hand side of (6.23), let us apply Lemma 4.8 with \( \phi(x) = \mu(x) \eta(x) \) and \( \varphi(t) = \zeta(t) \), which gives for any \( t_1, t_2 \in I_i \cap [0, T] \), the following sequence of estimates:

\[
| (v_h)_\mu (t_2) - (v_h)_\mu (t_1) | \overset{(a)}{\leq} \left( \frac{1}{pr_i^p} + \frac{1}{r_i^{n+1}} \right) \int_{t_1}^{t_2} \int_{\frac{1}{3} B_i} [ |\nabla u| + |h_0| ]_{h}^{p-1} \, dz \, dt \\
+ \left( \frac{1}{pr_i^p} + \frac{1}{r_i^{n+1}} \right) \int_{t_1}^{t_2} \int_{\frac{1}{3} B_i} |\tilde{u}|_{16Q} \, dx \, dz \\
+ \frac{1}{r_i^3} \int_{t_1}^{t_2} \int_{\frac{1}{2} B_i} |u-w|_h \chi_{[0,T]} \, dz \overset{(b)}{\leq} r_i \lambda^{2-p} \iint_{\frac{3}{4} Q_i} [ |\nabla u| + |h_0| ]_{h}^{p-1} \chi_{[0,T]} \, dz \\
+ r_i \lambda^{2-p} \int_{t_1}^{t_2} \left[ \int_{\frac{1}{3} B_i} |\tilde{u}|_{16Q} \, dx \right]_h \, dt \\
+ \frac{r_i^3 \lambda^{2-p}}{s} \iint_{\frac{3}{4} Q_i} [ |u-w|_h ]_{h} \, dz. \tag{6.24}
\]

To obtain (a) and (b), we made use of (6.2), (6.3) and the bounds for \( \mu \) along with (W1) and Remark 3.2. The first term and second term on the right hand side of (6.24) can be controlled using (W4) to get

\[
r_i \lambda^{2-p} \iint_{\frac{3}{4} Q_i} [ |\nabla u| + |h_0| ]_{h}^{p-1} \chi_{[0,T]} \, dz + r_i \lambda^{2-p} \int_{t_1}^{t_2} \left[ \int_{\frac{1}{3} B_i} |\tilde{u}|_{16Q} \, dx \right]_h \, dt \leq r_i \lambda. \tag{6.25}
\]

To estimate the third term on the right hand side of (6.24), let us take a cut-off function \( \xi \in C^\infty_c(4Q_i \cap \{t \leq 0\}) \), from which we note that for \( h \) sufficiently small, using Lemma 4.1 and the hypothesis that
$Q_i$ crosses the initial boundary, we observe $([u - w]_{h})_{\xi} = 0$. Thus we get

$$
\iint_{4Q_i} \frac{|[u - w]_{h}|}{r_i} \chi_{[0, T]} \, dz = \iint_{4Q_i} \frac{|[u - w]_{h} \chi_{[0, T]} - ([u - w]_{h})_{\xi}}{r_i} \, dz
$$

\[ \leq (a) \iint_{4Q_i} \frac{||\nabla u - \nabla w||_{h} \chi_{[0, T]} \, dz}{r_i} + \sup_{t_1, t_2 \in 4L \cap [0, T]} \frac{|([u - w]_{h})_{\mu}(t_1) - ([u - w]_{h})_{\mu}(t_2)|}{r_i} \, dz \tag{6.26} \]

\[ \leq (b) \lambda + \frac{||\nabla \mu||_{L^\infty(4Q_i)}}{r_i} \iint_{4Q_i} \frac{||\nabla u || + |h_0||^{p-1}_{h} \chi_{[0, T]} \, dz}{r_i} + \frac{||\nabla \mu||_{L^\infty(4Q_i)}}{r_i} \iint_{4L \cap [0, T]} \left[ \int_{B_i} |\bar{w}| \chi_{16Q \cap \Omega_T} \, dx \right]_h \, dt \]

\[ \leq (c) \lambda. \]

To obtain (a), we made use of Lemma 4.1, to obtain (b), we made use of Lemma 4.8 along with (W4) and finally to obtain (c), we used (W1) and (W4) along with Remark 3.2.

Combining (6.26), (6.25) and (6.24) into (6.23) followed by the restriction $r_i \leq \rho$ and Assumption 6.1, we get

$$
\iint_{4Q_i} \left| \frac{v_h(z) \chi_{[0, T]}}{r_i} \right|^q \, dz \leq \chi_q + \chi_q \left( \frac{\lambda_{2-p}}{\alpha_{0, 3-p}} \right) \cdot
$$

\[ \cdot \text{Now we consider the case when } v_i^{(i)} \neq 0. \text{ Again without loss of generality, we can assume } r_i \leq \rho \text{ because if } \rho \leq r_i, \text{ we can proofed as in (6.22). Since } i \in \Theta_2, \text{ we have } \chi_{r_i}^{2} \geq s. \text{ Now applying Lemma 4.1 with } \mu(x) \in C_{c}^{\infty}(B_i) \text{ satisfying } ||\mu|| \leq \frac{C(n)}{r_i} \text{ and } ||\nabla \mu|| \leq \frac{C(n)}{r_i}, \text{ we get}
$$

$$
\iint_{4Q_i} \left| \frac{v_h(z) \chi_{[0, T]}}{r_i} - v_i^{(i)} \right|^q \, dz \leq \iint_{Q_i} \left| \nabla v_h \right|^q \chi_{[0, T]} \, dz + \sup_{t_1, t_2 \in L_i} \frac{|(v_h \chi_{[0, T]})_{\mu}(t_1) - (v_h \chi_{[0, T]})_{\mu}(t_2)|}{r_i} \tag{6.27} \]

The first term on the right hand side of (6.27) can be estimated using (W4). To estimate the second term on the right hand side of (6.27), let us apply Lemma 4.8 with $\phi(x) = \mu(x) \eta(x)$ and $\varphi(t) = \zeta(t)$, which gives for any $t_1, t_2 \in L_i \cap [0, T]$, the following sequence of estimates:

\[ |(v_h)_{\mu}(t_2) - (v_h)_{\mu}(t_1)| \leq ||(\eta \mu)||_{L^\infty(4B_i)} \chi_{[0, T]} \iint_{t_1}^{t_2} \int_{B_i} \left| \nabla u \right| + |h_0|^{p-1}_{h} \chi_{[0, T]} \, dz + \int_{t_1}^{t_2} \int_{B_i} |\bar{w}| \, dx \right]_h \, dt + \int_{t_1}^{t_2} \int_{B_i} |\bar{w}| \, dx \right]_h \, dt

\[ \leq (a) \left( \frac{1}{\rho r_i} + \frac{1}{r_i^{n+1}} \right) \iint_{t_1}^{t_2} \int_{B_i} \left| \nabla u \right| + |h_0|^{p-1}_{h} \, dz + \frac{1}{\rho r_i} \cdot \frac{1}{r_i^{n+1}} \iint_{t_1}^{t_2} \int_{B_i} |\bar{w}| \, dx \right]_h \, dt + \frac{1}{\rho r_i} \cdot \frac{1}{r_i^{n+1}} \iint_{t_1}^{t_2} |u - w|_h \chi_{[0, T]} \, dz

\[ \leq (b) r_i \chi_{2-p} \iint_{4Q_i} \left| \nabla u \right| + |h_0|^{p-1}_{h} \chi_{[0, T]} \, dz + \frac{1}{\rho r_i} \cdot \frac{1}{r_i^{n+1}} \iint_{B_i} |\bar{w}| \, dx \right]_h \, dt + \frac{1}{\rho r_i} \cdot \frac{1}{r_i^{n+1}} \iint_{B_i} \int_{t_1}^{t_2} \int_{B_i} |\bar{w}| \, dx \right]_h \, dt + \rho |Q| \cdot \frac{1}{r_i^{n+1}} \iint_{4Q_i} \left| \nabla u \right| + |h_0|^{p-1}_{h} \chi_{[0, T]} \, dz.

\[ (6.28) \]
To obtain (a), we made use of Lemma 4.1, to obtain (b), we made use of Lemma 4.8 along with (W4) and finally to obtain (c), we made use of (W4) along with (6.2).

Combining (6.29), (6.28) and (6.27) along with the bound $\lambda^{2-p}r_{i}^{2} \geq s$ and Assumption 6.1, we get

$$\int_{\frac{1}{2}Q_{i}} \frac{|v_{h}(z)\chi_{[0,T]} - v_{h}|^{q}}{r_{i}} \, dz \leq \lambda^{q} \left( 1 + \left( \frac{\lambda^{2-p}}{\alpha_{0}^{2-p}} \right) \frac{n+1}{2} + \left( \frac{\lambda^{2-p}}{\alpha_{0}^{2-p}} \right) \frac{n-1}{2} \right).$$

This completes the proof of the Lemma.

**Remark 6.13.** In the case $p \geq 2$, the estimate in Lemma 6.12 takes the form

$$\int_{\frac{1}{2}Q_{i}} \frac{|v_{h}(z)\chi_{[0,T]} - v_{h}|^{q}}{r_{i}} \, dz \leq (n,p,q) \lambda^{q}.$$

To obtain analogous cleaner estimate in the case $p < 2$, we can use the unified intrinsic scaling approach developed in [4]. This cleaner estimate will not be needed in this paper and hence we leave the details to the interested reader.

### 6.5. Lipschitz continuity of the test function

**Lemma 6.14.** The extension $v_{\lambda,h}$ from (6.9) is $C^{0,1}(\mathcal{H})$ with respect to the parabolic metric (2.10). Here $\mathcal{H}$ is as defined in (6.13).

### 6.6. Two crucial estimates for the test function

Before we state the two crucial lemmas, let us collect a few consequences of the estimates proved in the previous subsections. The first estimate is very similar to [2, Lemma 3.16].

**Lemma 6.15.** For any $1 \leq \vartheta \leq q$, there exists a positive constant $C(n,p,\Lambda_{1}, \vartheta) > 0$ such that the following holds:

$$\int_{2\mathcal{H}\setminus E_{\lambda}} |\partial_{x}v_{\lambda,h}(z)(v_{\lambda,h}(z) - v_{h}(z))|^{\vartheta} \, dz \leq C \lambda^{p} |\mathbb{R}^{n+1} \setminus E_{\lambda}| + \frac{1}{s} \int_{\frac{1}{2}Q_{i}} |v_{h}(z)|^{2q} \chi_{[0,T]} \, dz.$$

Analogous to [2, Lemma 3.19], we have the following lemma:
Lemma 6.16. For any $i \in \Theta$ and $k \in \{0, 1\}$, there exists a positive constant $C(n, p, q, \Lambda_0, \Lambda_1)$ such that there holds:
$$\int_{\frac{1}{2}Q, \frac{1}{8}Q} \frac{1}{|\nabla u| + |h_0| |v_{\lambda, h}| |\chi_{[0, T]}|} \ dx \leq C \rho^{1-k} \left( \lambda^p |4Q_i| + \frac{\chi_{i \in \Theta_2}}{s} \int_{4Q_i} |v_h|^2 \chi_{[0, T]} \ dx \right).$$
Here we have used the notation $\chi_{i \in \Theta_2} = 1$ if $i \in \Theta_2$ and $\chi_{i \in \Theta_1} = 0$ if $i \in \Theta_1$ and $\nabla^0 v_{\lambda, h} := v_{\lambda, h}$.

We also have the estimate
$$\int_{\frac{1}{2}Q, \frac{1}{8}Q} |\tilde{w}| |\nabla u| v_{\lambda, h} |\chi_{[0, T]}| \ dx \leq C \rho^{1-(k)} \left( \lambda^p |4Q_i| + \frac{\chi_{i \in \Theta_2}}{s} \int_{4Q_i} |v_h|^2 \chi_{[0, T]} \ dx \right).$$

Corollary 6.17. There exists a positive constant $C(n, p, q, \Lambda_0, \Lambda_1)$ such that the following estimate holds for any $i \in \{0, 1\}$:
$$\int_{sB \times 2I \setminus E_x} |\nabla u| + |h_0|^2 |v_{\lambda, h}| \chi_{[0, T]} \ dx \leq C \rho^{1-k} \left( \lambda^p |R^{n+1} \setminus E_x| + \frac{1}{s} \int_{4Q_i} |v_h|^2 \chi_{[0, T]} \ dx \right).$$

We also have the estimate
$$\int_{sB \times 2I \setminus E_x} |\tilde{w}| |\nabla v_{\lambda, h}| \chi_{[0, T]} \ dx \leq C \rho^{1-(k)} \left( \lambda^p |R^{n+1} \setminus E_x| + \frac{1}{s} \int_{4Q_i} |v_h|^2 \chi_{[0, T]} \ dx \right).$$

The first crucial estimate on each time slice follows analogous to [2, Lemma 3.21] and takes the form.

Lemma 6.18. For any $i \in \Theta_1$ and any $0 < \varepsilon \leq 1$, for almost every $t \in (0, t_0 + 4s) := 2I \cap [0, T]$, there holds
$$\left| \int_{16B} (v(x, t) - v_i\chi_{[0, T]}(x, t) \Psi_i(x, t) \ dx \right| \leq (\frac{\lambda^p}{8} |4Q_i| + \varepsilon |4B_i| |v_i|^2).$$

In the case $i \in \Theta_2$, for almost every $t \in (t_0 + 4s)$, there holds
$$\left| \int_{16B} v(x, t) v_i\chi_{[0, T]}(x, t) \Psi_i(x, t) \ dx \right| \leq (\frac{\lambda^p}{8} |4Q_i| + \varepsilon |u - w|^2 \chi_{[0, T]} \ dx \right).$$

We now come to essentially the most important estimate which will be used to obtain the Caccioppoli inequality. The proof is very similar to [2, Lemma 3.22] and will be omitted.

Lemma 6.19. There exists a positive constant $C(n, p, q, \Lambda_0, \Lambda_1)$ such that the following estimate holds for every $t \in [0, t_0 + 4s] := 2I \cap [0, T]$:
$$\int_{sB \setminus E_x} (|v_h|^2 - |v_h - v_i\chi|^2)(x, t) \ dx \geq C \left( -\lambda^p |R^{n+1} \setminus E_x| - \frac{1}{s} \int_{4Q_i} |u - w|^2 \chi_{[0, T]} \ dx \right).$$

6.7. Caccioppoli type inequality

We shall prove the Caccioppoli inequality in this subsection.

Lemma 6.20. Let $\alpha_0$ and $c_\rho$ be as in Assumption 6.1, then there exists constants $C = C(n, p, q, \Lambda_0, \Lambda_1)$ and $\beta_0 = \beta_0(n, p, \Lambda_0, \Lambda_1) \in (0, 1)$ small such that the following holds. For some $\beta \in (0, \beta_0)$, suppose that $u \in L^2(0, T; L^2(\Omega) \cap L^{p, \hat{p}}(0, T; W_0^{1, p, \hat{p}}(\Omega)))$ is any weak solution of (1.2) in the sense of Definition 2.10, then there holds
$$\alpha_0^{p, \hat{p}} + \sup_{t \in I \cap (t_0, T)} \int_B \mathcal{M}(x, t)^{-\beta} \left| \frac{u - w}{\rho} \right|^2 (x, t) \ dx$$
$$\leq \int_{sQ} \left[ \alpha_0^{p, \hat{p}} \left( \frac{|u - w|}{\rho} \right)^2 + \left( \frac{|u - w|}{\rho} \right)^{p, \hat{p}} \right] \chi_{[0, T]} \ dz$$
$$\quad + \int_{sQ} |h_0|^p \chi_{[0, T]} \ dz + \int_{sQ} |\nabla u|^p \chi_{[0, T]} \ dz + \int_{sQ} |\nabla u|^p \chi_{[0, T]} \ dz,$$
where we have set $\mathcal{M}(x, t) := \max \{g(x, t), \alpha_0\}$. 41
Proof. Pick any \( t_1 \in (0, t_0 + s) \) and consider the cut-off function \( \chi_{0,t_1}^\varepsilon \in C_c^\infty(0, t_1) \) such that

\[
\chi_{0,t_1}^\varepsilon (t) = \begin{cases} 1 & \text{for } t \in (0 + \varepsilon, t_1 - \varepsilon) \\ 0 & \text{for } t \in (\infty, 0) \cup (t_1, \infty). \end{cases}
\]  

(6.30)

Let us use \( v_{\lambda,h}(x,t) \eta(x) \chi_{0,t_1}^\varepsilon \) as a test function in (1.2) where \( v_{\lambda,h} \) is from (6.9) and \( \eta \) is from (6.2). Integrating over \((0, t_1)\), we get

\[
L_1 + L_2 := \int_0^{t_1} \left[ \int_{16B} \frac{d[u]_{\lambda}}{dt} \eta(x)v_{\lambda,h}(x,t) + \left\langle [A(x,t,\nabla u)]_{\lambda}, \nabla (\eta v_{\lambda,h}^\varepsilon) \right\rangle \, dx \right] \chi_{0,t_1}^\varepsilon (t) \, dt = 0.
\]  

(6.31)

Estimate of \( L_1 \): Note \( \zeta = 1 \) on \((0, t_1)\), from which we get

\[
\int_0^{t_1} \int_{16B} \frac{d[u]_{\lambda}}{dt} v_{\lambda,h} \eta \chi_{0,t_1}^\varepsilon (t) \, dz = \int_0^{t_1} \int_{16B} \frac{dv_{\lambda,h}}{dt} (v_{\lambda,h} - v_h) \chi_{0,t_1}^\varepsilon (t) \, dz + \frac{1}{2} \int_0^{t_1} \int_{16B} \frac{d}{dt} \left( (v_h^2) - (v_{\lambda,h} - v_h)^2 \right) \chi_{0,t_1}^\varepsilon (t) \, dz - \frac{1}{2} \int_0^{t_1} \int_{16B} \frac{d}{dt} \left( (v_h^2) - (v_{\lambda,h} - v_h)^2 \right) \chi_{0,t_1}^\varepsilon (t) \, dz 
+ \int_0^{t_1} \int_{16B} \frac{d[w]_{\lambda}}{dt} \eta v_{\lambda,h} \chi_{0,t_1}^\varepsilon (t) \, dz.
\]  

(6.32)

From (6.30), we see that \( \int_0^{t_1} \int_{16B} \frac{d}{dt} \left( (v_h^2) - (v_{\lambda,h} - v_h)^2 \right) \chi_{0,t_1}^\varepsilon (t) \, dx \, dt = 0 \) since \( \chi_{0,t_1}^\varepsilon (0) = 0 \) and \( \chi_{0,t_1}^\varepsilon (t_1) = 0 \).

Letting \( \varepsilon \to 0 \) in (6.32), we get

\[
\int_0^{t_1} \int_{16B} \frac{d[u]_{\lambda}}{dt} v_{\lambda,h} \eta \, dz = \int_0^{t_1} \int_{16B} \frac{dv_{\lambda,h}}{dt} (v_{\lambda,h} - v_h) \, dz - \frac{1}{2} \int_0^{t_1} \int_{16B} \left( v_h^2 - (v_{\lambda,h} - v_h)^2 \right) (x,0) \, dz + \frac{1}{2} \int_0^{t_1} \int_{16B} \left( v_h^2 - (v_{\lambda,h} - v_h)^2 \right) (x,t_1) \, dz + \int_0^{t_1} \int_{16B} \frac{d[w]_{\lambda}}{dt} \eta (v_{\lambda,h} (z)) \, dz = J_2 - J_1(0) + J_3(t_1) + J_3.
\]  

(6.33)

Let us now estimate each of the terms as follows:

Estimate of \( J_1 \): Taking absolute values and making use of Lemma 6.15, we get

\[
|J_1| \leq \int_{2\mathcal{N}\setminus E_\lambda} \left| \frac{dv_{\lambda,h}}{dt} (v_{\lambda,h} - v_h) \right| \chi_{[0,T]} \, dz \leq \lambda^p |\mathbb{R}^{n+1} \setminus E_\lambda| + \frac{1}{s} \int_{sQ} |u-w|_0^2 \chi_{[0,T]} \, dz 
\lim_{h \to 0} \lambda^p |\mathbb{R}^{n+1} \setminus E_\lambda| + \frac{1}{s} \int_{sQ} |u-w|_0^2 \chi_{[0,T]} \, dz.
\]  

(6.34)

Estimate of \( J_1(0) \): Since we have \( v = v_0 = 0 \) on \( \{ t = 0 \} \), we see that \( v = v_\lambda = 0 \) on \( E_\lambda \cap \{ t = 0 \} \) and on \( E_\lambda^c \cap \{ t = 0 \} \), we have \( v_\lambda = 0 \) from (6.9) and hence

\[
\lim_{h \to 0} J_1(0) = 0.
\]  

(6.35)

Estimate for \( J_1(t_1) \): We can take \( h \searrow 0 \) followed by making use of Lemma 6.19, we get

\[
\lim_{h \to 0} J_1(t_0) = \frac{1}{2} \int_{16B} |(v)^2 - (v_\lambda - v)^2| (y,t) \, dy 
\geq \int_{E_\lambda^c} |v(x,t)|^2 \, dx - \lambda^p |\mathbb{R}^{n+1} \setminus E_\lambda| - \frac{1}{s} \int_{sQ} |u-w|_0^2 \chi_{[0,T]} \, dz.
\]  

(6.36)
Remark 3.2 (6.36)

and (6.37)

Corollary 6.17 (6.34)

almost every

Multiplying

Noting that \( \lim \)

Estimate of

\( \bar{\lambda} \)

Estimate of

\( t \)

\( \lim_{h \to 0} \)

Almost every

Estimate of \( L_2 \): We decompose the expression as

\[
L_2 = \left[ \int_0^{t_1} \int_{E^1_x} + \int_0^{t_1} \int_{16B \setminus E^1_x} \right] \left( [A(x, t, \nabla v)]_h, \nabla (\eta v_{\lambda, h}) \right) \chi_{[0, T]} \, dz
\]

\[
= L^1_2 + L^2_2. \tag{6.37}
\]

Estimate of \( L^2_2 \): Using the chain rule, (2.1), (6.2) along with Corollary 6.17, we get

\[
L^2_2 \leq \int_0^{t_1} \int_{E^1_x} \left| \nabla u \right|^{p-1} \left| \nabla (\eta v_{\lambda, h}) \right| \, dz
\]

\[
\leq \sum_{k=0}^{1} \rho^{k-1} \int_{(8B \times 2t) \setminus E^1_x} \left| \nabla u \right|^{p-1} \left| \nabla (\eta v_{\lambda, h}) \right| \, dz
\]

\[
\leq \lambda^{p} |\mathbb{R}^{n+1} \setminus E_{\lambda}| + \frac{1}{s} \int_{8Q} \left| u - w \right|^{2} \chi_{[0, T]} \, dz \tag{6.38}
\]

In the above estimate, we made use of the bound \( \left| v_h \right| \leq \left| u - w \right|_h \) which follows from (6.4).

Noting that \( \lim_{h \to 0} L^1_2 = \int_0^{t_1} \int_{E_{\lambda}(\lambda)} \langle A(y, \tau, \nabla u), \nabla (\eta v_{\lambda}) \rangle \chi_{[0, T]} \, dz \), we combine (6.38), (6.37), (6.36), (6.35), (6.34) and (6.33), followed by making use of (6.31), we get

\[
\int_{E_{\lambda}} |v(x, t)|^2 \, dx + \int_0^{t_1} \int_{E^1_x} \langle A(y, \tau, \nabla u), \nabla (\eta v_{\lambda}) \rangle \chi_{[0, T]} \, dz
\]

\[
\leq \lambda^{p} |\mathbb{R}^{n+1} \setminus E_{\lambda}| + \frac{1}{s} \int_{8Q} \left| u - w \right|^{2} \chi_{[0, T]} \, dz + \int_0^{t_1} \int_{16B \setminus E^1_x} \langle \bar{w}, \nabla (\eta^2 (u - w)) \rangle \chi_{[0, T]} \, dz. \tag{6.39}
\]

Multiplying (6.39) by \( \lambda^{-1-\beta} \) and integrating from \( (c, \alpha_0, \infty) \) (recall that \( c \) is as in Assumption 6.1), for almost every \( t \in (t_0 + s, t_0 + 4s) \) (actually holds for any \( t \in (0, t_0 + 4s) \)), we get

\[
K_1 + K_2 \leq K_3 + K_4 + K_5, \tag{6.40}
\]

where we have set

\[
K_1 := \frac{1}{2} \int_{c, \alpha_0}^{\infty} \lambda^{-1-\beta} \int_{E^1_x} |v(y, t)|^2 \, dy \, d\lambda,
\]

\[
K_2 := \int_{c, \alpha_0}^{\infty} \lambda^{-1-\beta} \int_0^t \int_{E^1_x} \langle A(y, \tau, \nabla u), \nabla (\eta^2 (u - w)) \rangle \chi_{[0, T]} \, dy \, d\tau \, d\lambda,
\]

\[
K_3 := \int_{c, \alpha_0}^{\infty} \lambda^{-1-\beta} \lambda^{p} |\mathbb{R}^{n+1} \setminus E_{\lambda}| \, d\lambda,
\]

\[
K_4 := \frac{1}{s} \int_{c, \alpha_0}^{\infty} \lambda^{-1-\beta} \int_{8Q} \left| u - w \right|^{2} \chi_{[0, T]} \, dy \, d\tau \, d\lambda,
\]

\[
K_5 := \int_{c, \alpha_0}^{\infty} \lambda^{-1-\beta} \int_0^t \int_{E^1_x} \langle \bar{w}, \nabla (\eta^2 (u - w)) \rangle \chi_{[0, T]} \, dy \, d\tau \, d\lambda.
\]
We now define the truncated Maximal function $\mathcal{M}(z) := \max\{g(z), \alpha_0\}$ and then estimate each of the $K_i$ for $i \in \{1, 2, 3, 4\}$ as follows:

**Estimate of $K_1$:** By applying Fubini, we get

$$K_1 \geq \frac{1}{\beta c_1} \int_{SB} \mathcal{M}(y, t)^{-\beta} |v(y, t)|^2 \, dy. \quad (6.41)$$

**Estimate of $K_2$:** Again applying Fubini, we get

$$K_2 = \frac{1}{\beta c_1} \int_0^t \int_{SB} \mathcal{M}(y, \tau)^{-\beta} \langle A(y, \tau, \nabla u), \nabla (\eta^2(u - w)) \rangle \, dy \, d\tau.$$

Applying chain rule along with $A$ gives

$$\beta C_2 K_2 = \int_0^t \int_{SB} \mathcal{M}(y, \tau)^{-\beta} \langle A(y, \tau, \nabla u), \nabla \eta^2 \rangle (u - w) \, dy \, d\tau$$

$$\geq \int_{Q} \mathcal{M}(y, \tau)^{-\beta} |\nabla u|^2 \chi_{[0, T]} \, dy \, d\tau - \int_{Q} \mathcal{M}(y, \tau)^{-\beta} |h_0|^p \chi_{[0, T]} \, dy \, d\tau$$

$$- \int_{Q} \mathcal{M}(y, \tau)^{-\beta} (|\nabla u|^{p-1} + |h_0|^{p-1}) \frac{|u - w|}{\rho} \chi_{[0, T]} \, dy \, d\tau$$

$$:= A_1 + A_2 + A_3 + A_4.$$

**Estimate of $A_1$:** Note that $\eta \equiv 1$ on $B$. Let $S := \{z \in Q \cap \{t \geq 0\} : |\nabla u(z)| \geq \beta g(z)\}$, then we get

$$\int_{Q} |\nabla u|^{p-\beta} \chi_{[0, T]} \, dz = \int_{S} |\nabla u|^{p-\beta} \, dz + \int_{Q \setminus S} |\nabla u|^{p-\beta} \chi_{[0, T]} \, dz$$

$$\leq \beta^{-\beta} \int_{Q} \mathcal{M}(z)^{-\beta} |\nabla u|^{p} \chi_{[0, T]} \, dz + \beta^{-p-\beta} \int_{Q \setminus S} \mathcal{M}(z)^{p-\beta} \chi_{[0, T]} \, dz$$

$$\leq \frac{1}{\rho} \int_{Q \setminus S} \mathcal{M}(z)^{p-\beta} \chi_{[0, T]} \, dz + \beta^{-p-\beta} |Q| \alpha_0^{p-\beta}.$$

**Estimate of $A_2$:** From $(6.7)$, we see that $\chi_{Q \cap \{t \geq 0\}}(|\nabla u(z)| + |h_0(z)|) \leq \mathcal{M}(z)$ for a.e. $z \in \mathbb{R}^n$, which gives

$$A_2 = \int_{Q \cap \{t \geq 0\}} \mathcal{M}(z)^{-\beta} |h_0|^p \chi_{[0, T]} \, dz \leq \int_{Q} |h_0|^{p-\beta} \chi_{[0, T]} \, dz. \quad (6.44)$$

**Estimate of $A_3$:** We use the bound $\chi_{Q \cap \{t \geq 0\}}(|\nabla u(z)| + |h_0(z)|) \leq \mathcal{M}(z)$ for a.e. $z \in \mathbb{R}^n$, along with Young’s inequality and Assumption 6.1, to get

$$A_3 \leq \int_{Q} (|\nabla u| + |h_0|)^{p-1-\beta} \frac{|u - w|}{\rho} \chi_{[0, T]} \, dz \leq \varepsilon |Q| \alpha_0^{p-\beta} + C(\varepsilon) \int_{Q \setminus S} \frac{|u - w|}{\rho} \chi_{[0, T]} \, dz. \quad (6.45)$$

**Estimate of $A_4$:** Similar to the calculations for $A_3$, we get

$$A_4 \leq \varepsilon |Q| \alpha_0^{p-\beta} + C(\varepsilon) \int_{Q \setminus S} |\nabla w|^{p-\beta} \chi_{[0, T]} \, dz.$$
Estimate of $K_3$: Applying the layer-cake representation (see for example [17, Chapter 1]), we get

$$K_3 \leq \frac{1}{p - \beta} \int_{\mathbb{R}^{n+1}} \mathcal{M}(z)^{p - \beta} \, dz \overset{\text{Lemma 6.2}}{\lesssim} |Q| \alpha_0^{p - \beta}. \quad (6.46)$$

Estimate of $K_4$: Again applying Fubini, we get

$$K_4 = \frac{1}{s} \int_{c0_0}^\infty \lambda^{-1 - \beta} \int_{\mathcal{S}_Q} |u - w|^2 \chi_{[0, T]} \, dz \, d\lambda = \frac{1}{\beta} \int_{\mathcal{S}_Q} \left( a_0^{-\beta} \frac{|u - w|^2}{s} \right) \chi_{[0, T]} \, dz. \quad (6.47)$$

Estimate of $K_5$: Applying Fubini, we get

$$K_5 = \frac{1}{\beta c_1^\beta} \int_{0}^{t} \int_{\mathcal{S}_B} \mathcal{M}(y, \tau)^{-\beta} \langle \bar{w}, \nabla (\eta^2 (u - w)) \rangle \chi_{[0, T]} \, dz \leq \frac{1}{\beta c_1^\beta} \int_{\mathcal{S}_Q} \mathcal{M}(y, \tau)^{-\beta} \left( \frac{|u - w|}{\rho} + |\nabla (u - w)| \right) \chi_{[0, T]} \, dz. \quad (6.48)$$

From (6.8), we have $\mathcal{M}(y, \tau) \geq \frac{|u - w|}{\rho}$ as well as $\mathcal{M}(y, \tau) \geq |\nabla u|$ and $\mathcal{M}(y, \tau) \geq |\nabla w|$ which combined with Hölder’s inequality and Assumption 6.1 gives

$$K_5 \leq \frac{1}{\beta c_1^\beta} \int_{\mathcal{S}_Q} \left( \frac{|u - w|^2}{\rho} \right)^{1-\beta} + \left( \frac{|\nabla u|^2}{\rho} \right)^{1-\beta} + \left( \frac{|\nabla w|^2}{\rho} \right)^{1-\beta} \chi_{[0, T]} \, dz \leq \frac{c(\varepsilon)}{\beta c_1^\beta} \int_{\mathcal{S}_Q} \left| \bar{w} \right|^{p-\beta} \chi_{[0, T]} \, dz + \frac{\varepsilon}{\beta c_1^\beta} \int_{\mathcal{S}_Q} \left| u - w \right|^{p-\beta} \chi_{[0, T]} \, dz + \frac{\varepsilon}{\beta c_1^\beta} \alpha_0^{p-\beta} |Q|. \quad (6.49)$$

Substituting (6.43), (6.44) and (6.45) into (6.42) followed by combining (6.41), (6.46), (6.47) and (6.48) into (6.40), we get

$$\frac{1}{2\beta} \int_B \mathcal{M}(y, t)^{-\beta} \left| u - w \right|^2 (y, t) \, dy + \frac{1}{\beta} \int_Q |\nabla u|^{p-\beta} \chi_{[0, T]} \, dz \leq \frac{1}{\beta} \int_{\mathcal{S}_Q} |\nabla u|^{p-\beta} \chi_{[0, T]} \, dz + \frac{1}{\beta} \int_{\mathcal{S}_Q} |h_0|^{p-\beta} \chi_{[0, T]} \, dz + \frac{1}{\beta} \int_{\mathcal{S}_Q} \alpha_0^{-\beta} \frac{|u - w|^2}{s} \chi_{[0, T]} \, dz + \frac{\varepsilon}{\beta c_1^\beta} \int_{\mathcal{S}_Q} |\nabla u|^{p-\beta} \chi_{[0, T]} \, dz + \frac{\varepsilon}{\beta c_1^\beta} \int_{\mathcal{S}_Q} |\nabla w|^{p-\beta} \chi_{[0, T]} \, dz + \frac{\varepsilon}{\beta c_1^\beta} \alpha_0^{p-\beta} |Q|. \quad (6.50)$$

Multiplying the above expression by $\beta$ followed by choosing $\beta \in (0, \beta_0)$ and $\varepsilon \in (0, 1)$ small and then using the intrinsic scaling $s = \rho^2 \alpha_0^{p-2}$ along with Assumption 6.1, we get

$$\int_B \mathcal{M}(y, t)^{-\beta} \left| u - w \right|^2 (y, t) \, dy + |Q| \alpha_0^{p-\beta} \leq \int_{\mathcal{S}_Q} |h_0|^{p-\beta} \chi_{[0, T]} \, dz + \int_{\mathcal{S}_Q} |\nabla u|^{p-\beta} \left( \frac{|u - w|}{\rho} \right)^2 \chi_{[0, T]} \, dz + \int_{\mathcal{S}_Q} |\nabla w|^{p-\beta} \chi_{[0, T]} \, dz + \int_{\mathcal{S}_Q} |\bar{w}|^{\frac{p-\beta}{p}} \chi_{16Q \cap \Omega_T} \, dz. \quad (6.51)$$

Rearranging the above expression and dividing throughout by $|Q|$, we get

$$\sup_{t \in [0, T]} \alpha_0^{p-2} \int_B \mathcal{M}(y, t)^{-\beta} \left| \frac{u - w}{\rho} \right|^2 (y, t) \, dy + \alpha_0^{p-\beta} \leq \int_{\mathcal{S}_Q} |h_0|^{p-\beta} \chi_{[0, T]} \, dz + \int_{\mathcal{S}_Q} |\nabla u|^{p-\beta} \left( \frac{|u - w|}{\rho} \right)^2 \chi_{[0, T]} \, dz + \int_{\mathcal{S}_Q} |\nabla w|^{p-\beta} \chi_{[0, T]} \, dz + \int_{\mathcal{S}_Q} |\bar{w}|^{\frac{p-\beta}{p}} \chi_{16Q \cap \Omega_T} \, dz. \quad (6.52)$$

This completes the proof of the Lemma.
6.8. Some consequences of Caccioppoli inequality

**Lemma 6.21.** Let $\kappa \geq 1$, then there exists $\beta_0(n,p,q,\Lambda_0,\Lambda_1,\kappa)$ such that for any $\beta \in (0,\beta_0)$ and any very weak solution $u \in L^2(0,T;L^2(\Omega)) \cap L^{p-\beta}(0,T;W^{1,p-\beta}(\Omega))$ of (1.2), the following holds: Let $Q_{\rho,s}(x_0,t_0) = B_{\rho}(x_0) \times [t_0-s,t_0+s]$ be the parabolic cylinder with $t_0-s \leq 0 < t_0+s$ and $s = \rho^2 \alpha_0^2 p$ for some $\alpha_0 > 0$ as in Assumption 6.1. Let $\alpha Q$ be a rescaled parabolic cylinder for some $\alpha \in (1,8)$ and also suppose that

$$\iiint_{\alpha Q} (|\nabla u| + |h_0|)^{p-\beta} \chi_{\alpha Q \cap \Omega_T} \, dz + \int_{\alpha Q} |\nabla w|^{p-\beta} \chi_{\alpha Q \cap \Omega_T} \, dz + \frac{1}{|\alpha Q|} \int_{\partial (\alpha Q \cap \Omega_T)} |\nabla w| \chi_{\alpha Q \cap \Omega_T} \, d\Gamma \leq \kappa \alpha_0^{p-\beta}.$$  

(6.49)

Let us define

$$J := \sup_{t \in I \cap [0,T]} \int_B \left( \frac{|u-w|}{\rho} \right)^{2} M(x,t)^{-\beta} \, dx,$$

where $M(z) := \max\{g(z),\alpha_0\}$ is the same as in the proof of Lemma 6.20 but with $16Q \cap \Omega_T$ replaced by $\alpha Q \cap \Omega_T$ in (6.8).

For any $1 \leq \sigma \leq \max\{2,p-\beta\}$, with $r = \frac{2(p-\beta)}{p}$ and $\vartheta = \max\left\{1, \frac{n \sigma}{n+r}\right\}$, there exists a universal positive constant $C = C(n,p,\sigma,\kappa)$ such that the following holds:

$$\iint_{Q} \left( \frac{|u-w|}{\rho} \right)^{\sigma} \chi_{[0,T]} \, dz \leq C \left( \alpha_0^{p-\beta}(\sigma-\vartheta) \right) \left( \iint_{Q} \left( \frac{|u-w|}{\rho} \right)^{r} \, dz \right)^{\frac{\sigma-\vartheta}{r}} \chi_{[0,T]} \, dz.$$  

(6.50)

Proof. In order to prove the lemma, we want to make use of Lemma 6.2. First, we note that the choice of $\sigma, \vartheta, r$ with $\delta = \frac{\vartheta}{\sigma}$ satisfies (4.1). Applying Lemma 6.2, we get

$$\iint_{Q} \left( \frac{|u-w|}{\rho} \right)^{\sigma} \chi_{[0,T]} \, dz \leq C \iint_{I} \left( \int_{B_{\rho}} \left( \frac{|u-w|}{\rho} \right)^{\vartheta} + |\nabla u - \nabla w|^{\vartheta} \, dx \right) \left( \int_{B_{\rho}} \left( \frac{|u-w|}{\rho} \right)^{r} \, dx \right)^{\frac{\sigma-\vartheta}{r}} \chi_{[0,T]} \, dt. \quad (6.51)$$

With $g(z)$ as in the hypothesis, we can apply Hölder’s inequality following by taking the supremum over $t \in I \cap [0,T]$, to get

$$\int_{B} \left( \frac{|u-w|}{\rho} \right)^{r} (x,t) \, dx = \int_{B} \left( \frac{|u-w|}{\rho} \right)^{2} (x,t) M(x,t)^{-\beta} \chi_{[0,T]} \, dx \leq J^{\frac{\sigma}{\sigma-\vartheta}} \left( \int_{B} \left( \frac{|u-w|}{\rho} \right)^{2} \chi_{[0,T]} \, dx \right)^{\frac{2-r}{2}}.$$  

(6.52)

If $\beta_0$ is chosen sufficiently small, then for any $\beta \in (0,\beta_0]$, we can get $\frac{r p}{\beta(p-\beta)} > 0$, which along with (6.50) and (6.51) gives

$$\iint_{Q} \left( \frac{|u-w|}{\rho} \right)^{\sigma} \chi_{[0,T]} \, dz \leq J^{\frac{\sigma-\vartheta}{r}} \iint_{I} \left( \int_{B_{\rho}} \left( \frac{|u-w|}{\rho} \right)^{\vartheta} + |\nabla u - \nabla w|^{\vartheta} \, dx \right) \left( \int_{B} (M(z))^{p-\beta} \, dx \right)^{\frac{\beta(p-\beta)}{rp}} \chi_{[0,T]} \, dt \leq J^{\frac{\sigma-\vartheta}{r}} \left( \int_{Q} M(z)^{p-\beta} \, dz \right)^{\frac{\beta(p-\beta)}{rp}} \chi_{[0,T]} \, dz \times \left( \int_{Q} M(z)^{p-\beta} \, dz \right)^{\frac{\beta(p-\beta)}{rp}}.$$  

Making use of (6.49), we can follow the proof of Lemma 6.2 to obtain the bound

$$\iint_{Q} M(z)^{p-\beta} \, dz \leq \alpha_0^{p-\beta}.$$  

Using the identity $\frac{\beta(p-\beta)(p-\beta)}{rp} = \frac{\beta(p-\beta)}{2}$, we get the desired estimate. 

\end{proof}
Lemma 6.22. Let $\frac{2n}{n+2} < p < 2 + \beta$, then under the assumptions of Lemma 6.20, there holds

$$\iint_{Q} |u-w|^2 \chi_{[0,T]} \, dz \lesssim_{(n,p,\Lambda,\alpha,\beta,\rho_0)} \rho^2 \alpha_0^2.$$ 

Proof. Let us choose $1 \leq \alpha_1 < \alpha_2 \leq 16$. Making use of Lemma 6.21 with $\sigma = 2$ gives

$$\iint_{\alpha_1 Q} \left| \frac{u-w}{\rho} \right|^2 \chi_{[0,T]} \, dz \lesssim (\alpha_0^\beta J)^{\frac{2-p}{p}} \left( \iint_{\alpha_1 Q} \left| \frac{u-w}{\rho} \right|^{\frac{p-\beta}{p-\rho \beta (2-\sigma)}} + |\nabla u - \nabla w|^{\frac{p-\beta}{p-\rho \beta (2-\sigma)}} \chi_{[0,T]} \, dz \right)^{\frac{p-\beta (2-\sigma)}{p}},$$

where $r = \frac{2(p-\beta)}{p}$, $\vartheta = \max \left\{ 1, \frac{2n}{n+2} \right\}$ and

$$J = \sup_{t \in (0,1]} \iint_{\alpha_1 B} \left| \frac{u-w}{\rho} \right|^2 \mu(\cdot, t)^{-\beta} \, dx.$$ 

From a calculation similar to Lemma 6.20 applied over $\alpha_1 Q$ and $\alpha_2 Q$ for $1 \leq \alpha_1 < \alpha_2 \leq 16$ and corresponding cut-off functions

$$\eta \in C^\infty_c(\alpha_2 B) \quad \text{with} \quad \eta \equiv 1 \text{ on } \alpha_1 B \quad \text{and} \quad \zeta \in C^\infty_c(\alpha_1 I) \quad \text{with} \quad \zeta \equiv 1 \text{ on } \alpha_1 I,$$

along with an application of Young’s inequality and Assumption 6.1 (note we have $p-\beta < 2$), we get

$$\alpha_0^\beta J \lesssim \iint_{\alpha_2 Q} \left| \frac{u-w}{\alpha_2 \rho - \alpha_1 \rho} \right|^2 \chi_{[0,T]} \, dz + \alpha_0^{2-p+\beta} \iint_{\alpha_2 Q} \left| \frac{u-w}{\alpha_2 \rho - \alpha_1 \rho} \right|^{p-\beta} \chi_{[0,T]} \, dz + \alpha_0^2 \iint_{\alpha_2 Q} \left| h \right|^{\frac{p-\beta}{p-\rho \beta (2-\sigma)}} \chi_{[0,T]} \, dz + \iint_{\alpha_2 Q} \left| \nabla \left| \nabla w \right|^{\frac{p-\beta}{p-\rho \beta (2-\sigma)}} \chi_{[0,T]} \, dz \right)^{\frac{p-\beta (2-\sigma)}{p}} \chi_{[0,T]} \, dz \right)^{\frac{p-\beta (2-\sigma)}{p}},$$

Combining (6.52) and (6.53), along with applying Young’s inequality, we get

$$\iint_{\alpha_1 Q} |u-w|^2 \chi_{[0,T]} \, dz \leq C \rho^\beta \alpha_0^2 \left( \iint_{\alpha_2 Q} \left| \frac{u-w}{\alpha_2 \rho - \alpha_1 \rho} \right|^2 \chi_{[0,T]} \, dz + \alpha_0^2 \right)^{\frac{2-p}{p-\rho \beta (2-\sigma)}} 
\leq \frac{1}{2} \iint_{\alpha_2 Q} |u-w|^2 \chi_{[0,T]} \, dz + C(\alpha_2 - \alpha_1)^{-2(\frac{p}{q}-1)} \rho^2 \alpha_0^2.$$ 

We can now use Lemma 4.5 to absorb the first term on the right of (6.54) which proves the lemma.

6.9. Reverse Hölder inequality

Lemma 6.23. Suppose that Assumption 6.1 holds over $\{Q, 16^2 Q\}$ instead of $\{Q, 16Q\}$ where $Q = Q_{p,\alpha_0^2-p^2}(x_0,t_0)$, then there holds

$$\alpha_0^{p-\beta} \lesssim \left( \iint_{16Q} |\nabla u|^{q_0} \chi_{[0,T]} \, dz \right)^{\frac{1}{q_0}} \iint_{16Q} |\Xi|^{p-\beta} \chi_{[0,T]} \, dz,$$

where

$$q_0 := \begin{cases} \max\{q, \bar{q}\}, & \bar{q} = \frac{np(p-\beta)}{p(n+2) - \beta(2 + p - \beta)} \quad \text{if} \quad p - \beta \geq 2 \\ \max\{q, \bar{q}\}, & \bar{q} = \frac{2np}{p(n+2) - 4\beta} \quad \text{if} \quad \frac{2n}{n+2} < p - \beta < 2, \end{cases}$$

and

$$\Xi := |h_0| + |\nabla w| + |\overline{\omega}|^{\frac{1}{2}}.$$
Proof. Since Assumption 6.1 is satisfied over \( \{Q, 16^2Q\} \), but with a different universal constant, we can apply the Caccioppoli inequality from Lemma 6.20 to get

\[
\alpha_0^{p-\beta} \leq \alpha_0^{p-2-\beta} \iint_{16Q} \frac{|u-w|}{\rho}^2 \chi_{[0,T]} \, dz + \iint_{16Q} \frac{|u-w|}{|x|^p} \chi_{[0,T]} \, dz + \iint_{16Q} |\Xi|^{p-\beta} \chi_{16Q \cap \Omega} \, dz
\]

\[
= C_{c_{\nu}} (I_2 + I_{p-\beta} + \iint_{16Q} |\Xi|^{p-\beta} \chi_{16Q \cap \Omega} \, dz),
\]

where we have set \( I_\sigma := \alpha_0^{p-\beta-\sigma} \iint_{16Q} \frac{|u-w|^\sigma}{\rho} \chi_{[0,T]} \, dz \) for \( \sigma = 2 \) or \( \sigma = p - \beta \) and \( \Xi \) is from (6.56). Thus, we can apply Lemma 6.21 to get

\[
I_\sigma = \alpha_0^{p-\beta-\sigma} \iint_{16Q} \frac{|u-w|^\sigma}{\rho} \chi_{[0,T]} \, dz \leq \alpha_0^{p-\beta-\sigma} \left( \alpha_0^{\frac{\sigma}{p}} J \right)^{\frac{p}{p-\sigma}} \iint_{16Q} \left( |\nabla u - \nabla w|^{\beta(\sigma-\beta)} + \frac{|u-w|^\sigma}{\rho} \right) \chi_{[0,T]} \, dz,
\]

where \( r = \frac{2(p-\beta)}{p} \), \( \beta = \max \{1, \frac{n\sigma}{n+r} \} \) and

\[
J := \sup_{t \in \mathbb{R} \cap [0,T]} \left\{ \iint_{16B} \frac{|u-w|}{\rho}^2 \mathcal{M}(\cdot, t) \, dz \right\}.
\]

Again, we apply Lemma 6.20 to estimate \( J \) to get

\[
\alpha_0^{\beta} J \leq C_{c_{\nu}} \left( \iint_{16^2Q} \frac{|u-w|}{\rho}^2 \chi_{[0,T]} \, dz + \alpha_0^{2-p+\beta} \iint_{16^2Q} \frac{|u-w|}{\rho}^{p-\beta} \chi_{[0,T]} \, dz + \alpha_0^{2-p+\beta} \iint_{16^2Q} |\Xi|^{p-\beta} \chi_{[0,T]} \, dz \right).
\]

To estimate the first term on the right of (6.59), we split into two cases:

**In the case** \( p - \beta \leq 2 \), we directly apply Lemma 6.22 to get

\[
\iint_{16^2Q} \frac{|u-w|}{\rho}^2 \chi_{[0,T]} \, dz \leq \alpha_0^2.
\]

**In the case** \( p - \beta > 2 \), we get the following sequence of estimates. Firstly, since \( u-w = 0 \) for \( \{t \leq 0\} \), which gives \( (u-w)\xi = 0 \) where \( \xi \) is defined analogous to Lemma 4.1 but on \( 16^2Q \cap \{t \leq 0\} \). Thus applying Hölder’s inequality, we get

\[
\iint_{16^2Q} \frac{|u-w|}{\rho}^2 \chi_{[0,T]} \, dz \leq \left( \iint_{16^2Q} \frac{|(u-w) - (u-w)\xi|}{\rho}^{p-\beta} \chi_{[0,T]} \, dz \right)^{\frac{2}{p-\beta}} =: H^{p-2}. \]

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To estimate the term $H$ occurring on the right hand side of (6.61), we proceed as follows:

\[
H \overset{(a)}{\leq} \iint_{16^2Q} |\nabla u - \nabla w|^p \chi_{[0,T]} \, dz + \sup_{t_1,t_2 \in 16^2Q \cap \{t \geq 0\}} \frac{|(u-w)_{t_1}(t_2) - (u-w)_{t_1}(t)|}{\rho}^{p-1} \chi_{[0,T]} \, dz
\]

\[
\overset{(b)}{\leq} \iint_{16^2Q} |\nabla u - \nabla w|^p \chi_{[0,T]} \, dz + \left( \frac{\|\nabla u\|_{L^\infty(16^2B)}}{\rho} \iint_{16^2Q} |\nabla u| + |h_0|^{p-1} \chi_{[0,T]} \, dz \right)
\]

\[
\overset{(c)}{\leq} \iint_{16^2Q} |\nabla u - \nabla w|^p \chi_{[0,T]} \, dz + \left( \alpha_0^{2-p} \iint_{16^2Q} |\nabla u| + |h_0|^{p-1} \chi_{[0,T]} \, dz \right)
\]

\[
\overset{(d)}{\leq} \alpha_0^{p-\beta} + \left( \alpha_0^{2-p} \alpha_0^{p-1} \right)^{p-\beta} = \alpha_0^{p-\beta}.
\]

To obtain (a), we apply Lemma 4.1 with $\mu \in C^\infty(16^2B)$ (which is just a rescaled version of (6.2)). To obtain (b), we apply Lemma 4.8 with $\phi = \mu$ and $\varphi = 1$. To obtain (c), we note that $|16^2Q| \equiv \rho^{n+2} \alpha_0^{-p}$ and finally to obtain (d), we make use of the hypothesis of the lemma (more specifically Assumption 6.1).

Thus combining (6.60), (6.61), (6.62) with (6.59), we get

\[
\alpha_0^\beta J \leq \alpha_0^2.
\]

From the choice $\vartheta = \max \left\{ 1, \frac{n\sigma}{n+r} \right\}$ along with $r = \frac{2(p-\beta)}{p}$, (6.55) and $\sigma = \max\{2,p-\beta\}$, we see that

\[
\vartheta \frac{rp}{rp - \beta} \leq \vartheta \frac{rp}{rp - \beta} \leq \vartheta \leq \vartheta \leq q_0,
\]

provided $\beta \ll 1$ is sufficiently small (see [10] 192p for the details.)

Making use of (6.63) in (6.58) and the observation (6.64) along with Young’s inequality, for any $\varepsilon > 0$, we get

\[
I_\sigma \leq \varepsilon \alpha_0^{p-\beta} + C_\varepsilon \left[ \iint_{16Q} \left( \frac{|u-w|}{\rho} \right)^{q_0} + |\nabla u - \nabla w|^{q_0} \chi_{[0,T]} \, dz \right]^{\frac{1}{q_0}}.
\]

To estimate the second term on the right of (6.65), we can proceed similarly to (6.61) and (6.62) to get

\[
\iint_{16Q} \left( \frac{|u-w|}{\rho} \right)^{q_0} + |\nabla u - \nabla w|^{q_0} \chi_{[0,T]} \, dz \leq \iint_{16Q} |\nabla u - \nabla w|^{q_0} \chi_{[0,T]} \, dz
\]

\[
+ \left( \alpha_0^{2-p} \iint_{16Q} |\nabla u| + |h_0|^{p-1} \chi_{[0,T]} \, dz \right)^{q_0}.
\]

To estimate second term of right hand side, we use Hölder’s inequality and Assumption 6.1 to discover

\[
\iint_{16Q} |\nabla u|^{p-1} \chi_{[0,T]} \, dz \leq \left( \iint_{16Q} |\nabla u|^{p} \chi_{[0,T]} \, dz \right)^{\frac{p}{p-1}} \left( \iint_{16Q} |\nabla u|^{q_0} \chi_{[0,T]} \, dz \right)^{\frac{p-1}{q_0}}
\]

\[
\leq \alpha_0^{-p-2} \left( \iint_{16Q} |\nabla u|^{q_0} \chi_{[0,T]} \, dz \right)^{\frac{1}{q_0}}.
\]

Combining (6.67) into (6.66) and making use of (6.65), we get

\[
I_\sigma \leq CC_\varepsilon \alpha_0^{p-\beta} + C_\varepsilon \left( \iint_{16Q} |\nabla u|^{q_0} \chi_{[0,T]} \, dz \right)^{\frac{p-\beta}{q_0}} + C_\varepsilon \iint_{16Q} |\nabla |^{p-\beta} \chi_{16Q \cap \Omega_T} \, dz.
\]
Combining (6.68) and (6.57), we get

$$\alpha_0^{p-\beta} \leq CC_{\text{case}} \varepsilon \alpha_0^{p-\beta} + C \varepsilon \left( \iint_{16Q} |\nabla u|^{q_0} \chi_{[0,T]} \, dz \right)^{\frac{p-\beta}{q_0}} + \iint_{16Q} |\Xi|^{p-\beta} \chi_{16Q\cap \Omega_T} \, dz. $$

Choosing $\varepsilon$ small, the lemma follows. \qed

6.10. Higher integrability at initial boundary - Proof of Theorem 3.4

We are now ready to prove Theorem 3.4. The calculations follows very similar to [10, Theorem 2.1] with a few modifications. For the sake of completeness, we provide the rough sketch below.

Without loss of generality, we can assume $\rho = 1$ and $z_0 = (0,0) \in \partial \Omega \times [0,T]$. We take $Q_1 := Q_{(\rho,\delta)}(0,0)$ and $Q_2 := Q_{(2\rho,2\delta^2)}(0,0)$ and for any $z \in Q_2$, define the parabolic distance of $z$ to $\partial Q_2$ by

$$d_p(z) := \inf_{\xi \in \mathbb{R}^{n+1}\setminus Q_2} \min \{|x - \xi|, \sqrt{|t-\eta|}\}. $$

Furthermore, let $\beta_0$ be the constant such that Lemma 6.23 holds for any $\beta \in (0,\beta_0)$ and $p - \beta > \frac{2n}{n+2}$. For $z \in Q_2$, let us define the following function:

$$\psi(z) := (|\nabla u(z)| + |\Xi(z)|) \chi_{[0,T]} \quad \text{and} \quad f(z) := d_p^\alpha(z) \psi(z) \quad \text{with} \quad \alpha := \frac{n+2}{d}, \quad (6.69)$$

where $d$ is as defined in (3.3) and $\Xi$ is defined in (6.56). Finally we define $\alpha_0$ to be

$$\alpha_0^d := \iint_{Q_2} \psi(z)^{p-\beta} \, dz + 1. \quad (6.70)$$

Let $\lambda_0$ be any number such that

$$\lambda_0 \geq b \frac{1}{2} \alpha_0 \quad \text{where} \quad b := 2^{10(n+2)}. \quad (6.71)$$

Now suppose that $z \in Q_2$ with $f(z) > \lambda_0$, then let us denote the parabolic distance of $z$ to $\partial Q_2$ by $r_z := d_p(z)$ and define the intrinsic scaling factor as

$$\gamma = \gamma(z) := (r_z^{-\alpha} \lambda_0)^{2-p} = (d_p(z)^{-\alpha} \lambda_0)^{2-p}. \quad (6.72)$$

In order to prove higher integrability, we want to apply Lemma 4.7. So the rest of the proof is devoted to ensuring that all the hypotheses of Lemma 4.7 are satisfied.

Case $p \geq 2$: Let us note that $r_z^n \leq 2^n \leq b^4 \alpha_0 \leq \lambda_0$, which implies $\gamma = (r_z^{-\alpha} \lambda_0)^{2-p} \leq 1$. Hence we shall consider intrinsic cylinders of the type $Q_2(R, \gamma R^2)$ with $0 < R \leq r_z$.

In order to apply Lemma 4.7, we need to find an appropriate intrinsic parabolic cylinder around $z$ on which all the hypotheses of Lemma 6.23 are satisfied. In order to do this, let us first take $R$ such that $r_z \leq 2^0 R < 2^0 r_z$. In this case, there holds:

$$\iint_{Q_2(R, \gamma R^2)} \psi(z)^{p-\beta} \, dz \leq \frac{|Q_2|}{|Q_2(R, \gamma R^2)|} \iint_{Q_2} \psi(z)^{p-\beta} \, dz \leq \frac{2^n+2}{R^{n+2}\gamma} \alpha_0^d \leq \frac{2^{10(n+2)} \lambda_0}{r_z^{n+2}} \frac{\lambda_0^d}{b} = (r_z^{-\alpha} \lambda_0)^{p-\beta}. \quad (6.73)$$

Furthermore, by Lebesgue differentiation theorem, for every $z \in Q_2$ with $f(z) > \lambda_0$, there holds

$$\lim_{r \searrow 0} \iint_{Q_3(r, \gamma r^2)} \psi(z)^{p-\beta} \, dz = \psi(z)^{p-\beta} (6.69)^{6.72} = (r_z^{-\alpha} f(z))^{p-\beta} > (r_z^{-\alpha} \lambda_0)^{p-\beta}. \quad (6.74)$$
Thus from (6.73) and (6.74), we observe that there should exist \( \rho \in \left(0, \frac{r_1}{2^q}\right) \), such that

\[
\iint_{Q_1(r, \rho R^2)} \psi(z)^{p-\beta} \, dz = (r_3^{-\alpha} \lambda_0)^{p-\beta},
\]

\[
\iint_{Q_2(R, \rho R^2)} \psi(z)^{p-\beta} \, dz \leq (r_3^{-\alpha} \lambda_0)^{p-\beta}, \quad \forall \ R \in [\rho, r_3].
\]

We now set \( Q := Q_3(r, \rho R^2) \), then \( 2^q Q \subset Q_2 \), thus all the hypotheses of Lemma 6.23 are satisfied with \((r_3^{-\alpha} \lambda_0, 1)\) instead of \((\alpha_0, \kappa)\), i.e., the following holds:

\[
(r_3^{-\alpha} \lambda_0)^{p-\beta} = \iint_{Q} \psi(z)^{p-\beta} \, dz \quad \text{and} \quad \iint_{2^q Q} \psi(z)^{p-\beta} \, dz \leq (r_3^{-\alpha} \lambda_0)^{p-\beta}. \quad (6.75)
\]

In the case \( Q \cap \{ t \leq 0 \} \neq \emptyset \), we can apply Lemma 6.23 and in the case \( 2^q Q \subset \Omega \times [0, T] \), we are in the interior case and can apply [10, Lemma 6.3] with \( \theta \) replaced by \( \Xi \) since \( |\theta| \leq |\Xi| \) (which was first proved in [22]) to get

\[
(r_3^{-\alpha} \lambda_0)^{p-\beta} \leq \left( \iint_{2^q Q} |\nabla u|^q \chi_{[0, T]} \, dz \right)^{\frac{p-\beta}{q}} + \iint_{2^q Q} |\Xi|^{p-\beta} \chi_{[0, T]} \, dz. \quad (6.76)
\]

Since \( 2^q \rho \leq r_3 \) and \( \gamma \leq 1 \), we also have for all \( z \in 2^q Q \) that

\[
d_p(z) \leq \min\{r_3 - 2^q \rho, \sqrt{r_3^2 - \gamma (2^q \rho)^2}\} \leq \frac{3}{2} r_3, \quad (6.77)
\]

\[
d_p(z) \geq \min\{r_3 - 2^q \rho, \sqrt{r_3^2 - \gamma (2^q \rho)^2}\} \geq \frac{1}{2} r_3. \quad (6.78)
\]

Now substituting (6.77) and (6.78) into (6.69), we find

\[
c^{-1} f(z) \leq r_3^{\alpha} \psi(z) \leq c f(z), \quad \forall \ z \in 2^q Q \quad \text{with} \quad c = c(n) > 1. \quad (6.79)
\]

We now claim the following estimate holds:

\[
\lambda_0^{p-\beta} \overset{(a)}{\lesssim} \iint_{2^q Q} f(z)^{p-\beta} \, dz \overset{(b)}{\lesssim} \left( \iint_{2^q Q} f(z)^q \, dz \right)^{\frac{p-\beta}{q}} + \iint_{2^q Q} (r_3^\alpha |\Xi(z)|)^{p-\beta} \, dz \overset{(c)}{\lesssim} \chi_0^{p-\beta}. \quad (6.80)
\]

**Estimate (a):** This follows easily by making use of (6.75) and (6.79) and subsequently enlarging the parabolic cylinder \( Q \).

**Estimate (b):** This is obtained by the following chain of estimates:

\[
\iint_{2^q Q} f(z)^{p-\beta} \, dz \overset{(6.79)}{\leq} r_3^{\alpha(p-\beta)} \iint_{2^q Q} \psi(z)^{p-\beta} \, dz \overset{(6.75)}{\leq} \lambda_0^{p-\beta}
\]

\[
\overset{(6.76)}{\leq} r_3^{\alpha(p-\beta)} \left( \iint_{2^q Q} |\nabla u|^q \chi_{[0, T]} \, dz \right)^{\frac{p-\beta}{q}} + r_3^{\alpha(p-\beta)} \iint_{2^q Q} |\Xi(z)|^{p-\beta} \chi_{[0, T]} \, dz
\]

\[
\overset{(6.79)}{\leq} \left( \iint_{2^q Q} f(z)^q \, dz \right)^{\frac{p-\beta}{q}} + \iint_{2^q Q} (r_3^\alpha |\Xi(z)|)^{p-\beta} \chi_{[0, T]} \, dz.
\]

**Estimate (c):** This follows by applying Jensen’s inequality (since \( q < p - \beta \)) along with the bound (6.75) to get:

\[
\left( \iint_{2^q Q} f(z)^q \, dz \right)^{\frac{p-\beta}{q}} + \iint_{2^q Q} (r_3^\alpha |\Xi(z)|)^{p-\beta} \chi_{[0, T]} \, dz \overset{(6.79)}{\leq} \iint_{2^q Q} f(z)^{p-\beta} \, dz \overset{(6.75)}{\leq} \lambda_0^{p-\beta}.
\]
Thus (6.80) holds and as a consequence, we can apply Lemma 4.7 over $2^8Q$ to see that for any $\beta \in (0, \beta_0]$, there exists $\delta_0 = \delta_0(n, p, \beta_0, r_0, \varepsilon_0, \Delta_0, \Lambda_1, \beta) > 0$ such that $f \in L^{\beta_0+\delta_1}_{\text{loc}}(Q_2)$ with $\delta_1 = \min\{\delta_0, \tilde{p} - p + \beta\}$. This is quantified by the estimate:

$$
\iint_{Q_2} f(z)^{\beta_0+\delta_1} dz \leq \alpha_0 \iint_{Q_2} f(z)^{\beta_0} dz + \iint_{Q_2} (r_3^\alpha|\Xi(z)|)^{\beta_0+\delta_1} \chi_{[0, T]} dz \quad \forall \delta \in (0, \delta_1].
$$

By iterating the previous arguments, for any $\beta \in (0, \varepsilon_{\text{geh}}]$ where $\varepsilon_{\text{geh}} > 0$ is the gain in higher integrability coming from Lemma 4.7, we obtain the bound

$$
\iint_{Q_2} f(z)^p dz \leq \alpha_0 \iint_{Q_2} f(z)^{p-\beta} dz + \iint_{Q_2} |\Xi(z)|^p \chi_{[0, T]} dz.
$$

(6.81)

For any $z \in Q_1$, we have $d_p(z) \geq \min\{1, \sqrt{3}\} \geq 1$, $|Q_2|/|Q_1| = C(n)$, which implies the following bounds hold:

$$
|\nabla u(z)| \leq \psi(z) \quad \forall z \in Q_1 \cap \mathbb{R}^n \times [0, T],
$$

(6.82)

$$
\iint_{Q_1} |\nabla u|^{p+\beta} \chi_{[0, T]} dz \leq \iint_{Q_2} f(z)^p dz,
$$

(6.83)

$$
\psi(z) \leq 2^\alpha \psi(z) \quad \forall z \in Q_2, \text{ since } d_p(z) \leq 2.
$$

(6.84)

Using (6.82), (6.83), (6.84) along with (6.81) and making use of (6.69) and (6.70), we get

$$
\iint_{Q_1} |\nabla u|^p \chi_{[0, T]} dz \leq \alpha_0 \iint_{Q_2} \psi(z)^{p-\beta} dz + \iint_{Q_2} \Xi^p \chi_{[0, T]} dz \\
\leq \left( \iint_{Q_2} (|\nabla u| + |\Xi|)^{p-\beta} \chi_{[0, T]} dz \right)^{1+\frac{\beta}{p}} + \iint_{Q_2} (1 + \Xi^p) \chi_{[0, T]} dz.
$$

Substituting the expression for $\Xi$ from (6.56), we get

$$
\iint_{Q_1} |\nabla u|^p \chi_{[0, T]} dz \leq \left( \iint_{Q_2} (|\nabla u| + |h_0|)^{p-\beta} \chi_{[0, T]} dz \right)^{1+\frac{\beta}{p}} + \iint_{Q_2} (1 + h_0^p) \chi_{[0, T]} dz \\
+ \iint_{Q_2} |\nabla w|^{p-\beta} \chi_{[0, T]} dz + \iint_{Q_2} |\nabla w|^p \chi_{[0, T]} dz + \iint_{Q_2} |\vec{w}|^p \chi_{[0, T]} dz.
$$

This proves the asserted estimate.

**Case $\frac{2n}{n+2} < p < 2$:** The basic change with respect to the case $p \geq 2$ is that, we now switch to the subquadratic scaling, i.e., we consider intrinsic cylinders of the type $Q_3(\gamma^{-\frac{1}{2}}R, R^2)$.

The parameter $\alpha_0$ is still given by (6.70) and $\lambda_0$ is chosen as in (6.71) and $\gamma$ is again given as in (6.72), where $z \in Q_2$ with $f(z) > \lambda_0$. But in contrast to $p \geq 2$ case, we have $\gamma = (r_3^{-\alpha}\lambda_0)^{2-p} \geq 1$. Hence for $R \in (0, r_1)$, we have $Q_3(\gamma^{-\frac{1}{2}}R, R^2) \subset Q_2$. Now once again, in order to apply Lemma 4.7, we need to find a suitable intrinsic parabolic cylinder around $z$, which enables us to apply Lemma 6.23 or [10, Lemma 6.3]. We observe, from the definition (6.69) that $n+2 = ad$ (recall $d$ is defined as in (3.3)) and $(2-p)n/2 + d = p - \beta$, which gives

$$
\iint_{Q_3(\gamma^{-\frac{1}{2}}R, R^2)} \psi(z)^{p-\beta} dz \leq \frac{|Q_2|}{|Q_3(\gamma^{-\frac{1}{2}}R, R^2)|} \iint_{Q_2} \psi(z)^{p-\beta} dz = \frac{2n+2}{R^{\alpha+2\gamma-\frac{2}{2}}} \alpha_0 \leq (r_3^{-\alpha}\lambda_0)^{p-\beta}.
$$

Now we can continue as in the $p \geq 2$ case to obtain the desired conclusion.

This completes the proof of the theorem. \qed

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7. Proof of Theorem 3.7

Let us first use the approximation \( u_0^k \in C^2(0,T;L^2(\Omega)) \cap L^p(0,T;W^{1,p}(\Omega)) \) given in the hypotheses of Theorem 3.7 satisfying (3.5) and (3.6). Subordinate to this sequence, there exists a unique weak solution \( u^k \in C^0(0,T;L^2(\Omega)) \cap L^p(0,T;u_0^k + W^{1,p}_0(\Omega)) \) solving

\[
\begin{cases}
    u_t^k - \text{div} A(x,t,\nabla u^k) = 0 & \text{on } \Omega \times (0,T), \\
    u^k = u_0^k & \text{on } \partial \Omega \times (0,T), \\
    u^k = 0 & \text{on } \Omega \times \{ t = 0 \}.
\end{cases}
\]  

(7.1)

in the sense of Definition 2.10, i.e., the following holds for any \( \varphi \in W^{1,2}(0,T;L^2(\Omega)) \cap L^p(0,T;W^{1,p}(\Omega)) \),

\[
\int_{\Omega} u^k \varphi(x,t) \, dx + \int_{\Omega \times (0,T)} \left\{ -u^k \varphi_t + (A(x,t,\nabla u^k),\nabla \varphi) \right\} \, dz = \int_{\Omega} u_0^k(x,0) \varphi(x,0) \, dx = 0.
\]  

(7.2)

In order to pass through the limit in (7.2), it would suffice to show the following convergence results:

- \( u^k \to u \) in \( C^0(0,T;L^2_{\text{loc}}(\Omega)) \).
- \( \{ u^k \} \) is precompact in \( L^p(0,T;W^{1,p}_{\text{loc}}(\Omega)) \).

Note that we only require \( \{ u^k \} \) is precompact in \( L^p(0,T;W^{1,p}_{\text{loc}}(\Omega)) \), which is away from the lateral boundary.

Let \( i, j \in \mathbb{N} \) be any two exponents, since \( u^i \) and \( u^j \) are regular weak solutions, then from standard energy estimates obtained locally in space, for any \( 4B \subset \Omega \) and any non-negative function \( \varphi \in C^\infty_c(2B) \) with \( \varphi \equiv 1 \) on \( B \), using \( (u^i - u^j)\varphi^p \) as a test function in (7.2), we get the estimate

\[
\sup_{t \in [0,s]} \int_{B} (u^i(t,x) - x) \, dx + \int_{B \times [0,s]} \left\{ A(x,t,\nabla u^i) - A(x,t,\nabla u^j), \nabla u^i - \nabla u^j \right\} \, dz \leq \int_{B \times [0,s]} \left( (|\nabla u^i|^2 + |h_0|)^{p-1} + (|\nabla u^j|^2 + |h_0|)^{p-1} \right) |u^i - u^j| |\nabla \varphi| \, dz.
\]  

(7.3)

In the case \( p \geq 2 \), making use of (3.4), we get

\[
\int_{B \times [0,s]} |\nabla u^i(x,t) - \nabla u^j(x,t)|^p \, dx \leq \int_{B \times [0,s]} \left( |\nabla u^i(x,t)|^2 + |\nabla u^j(x,t)|^2 \right)^{\frac{p}{2}} |\nabla u^i(x,t) - \nabla u^j(x,t)| \, dx \leq \int_{B \times [0,s]} \langle A(x,t,\nabla u^i) - A(x,t,\nabla u^j), \nabla u^i - \nabla u^j \rangle \, dx,
\]

and for \( p \leq 2 \), we have

\[
\int_{B \times [0,s]} |\nabla u^i(x,t) - \nabla u^j(x,t)|^p \, dx = \int_{B \times [0,s]} \left( |\nabla u^i(x,t)|^2 + |\nabla u^j(x,t)|^2 \right)^{\frac{p}{2}} |\nabla u^i(x,t) - \nabla u^j(x,t)|^p \, dx \leq \epsilon \int_{B \times [0,s]} |\nabla u^i|^p + |\nabla u^j|^p \, dx + C(\epsilon) \int_{B \times [0,s]} \left( |\nabla u^i|^2 + |\nabla u^j|^2 \right)^{\frac{p}{2}} |\nabla u^i(x,t) - \nabla u^j(x,t)|^2 \, dx \leq \epsilon \int_{B \times [0,s]} |\nabla u^i|^p + |\nabla u^j|^p \, dx + C(\epsilon) \int_{B \times [0,s]} \langle A(x,t,\nabla u^i) - A(x,t,\nabla u^j), \nabla u^i - \nabla u^j \rangle \, dx.
\]

Using (2.1) and Hölder’s inequality in (7.3), we get

\[
\begin{align*}
\sup_{t \in [0,s]} \int_{B} (u^i(t,x) - x) \, dx + \int_{B \times [0,s]} |\nabla u^i(x,t) - \nabla u^j(x,t)|^p \, dx & \leq \epsilon \int_{B \times [0,s]} |\nabla u^i(x,t)|^p + |\nabla u^j(x,t)|^p \, dx \\
+ C(\epsilon)\|\nabla \varphi\|_{L^{\infty}(2B)} \left( \int_{2B \times [0,s]} |\nabla u^i|^p + |\nabla u^j|^p + |h_0|^p \, dx \right)^{\frac{1}{p}} \left( \int_{2B \times [0,s]} |u^i - u^j|^p \, dx \right)^{\frac{1}{p}}.
\end{align*}
\]  

(7.4)
In particular, the standard energy estimate takes the form
\[
\sup_{t \in [0, s]} \int_B (u^i)^2 \, dx + \iint_{B \times [0, s]} |\nabla u^i|^p \, dz \leq \left( \iint_{2B \times [0, s]} |\nabla \varphi|^p \, dz \right)^{1/p} \left( \left( \iint_{2B \times [0, s]} |h_0|^p \, dz \right)^{1/p} + \left( \iint_{2B \times [0, s]} |u|^p \, dz \right)^{1/p} \right) \cdot \left( \iint_{2B \times [0, s]} |u|^p \, dz \right)^{1/p}.
\]

Using the interior higher integrability from [22] and the initial boundary higher integrability from Corollary 3.5, we get
\[
\iint_{2B \times [0, s]} |\nabla u^i|^p \, dz \lesssim \left( \frac{1}{s|B|} \right)^{\frac{\beta}{p}} \left( \iint_{\Omega} (|\nabla u^i| + |h_0|)^{p-\beta} \, dz \right)^{1+\frac{\beta}{p}} + \iint_{4B \times [0, s]} (1 + |h_0|^p) \, dz.
\]

Now we make use of Corollary 3.3 along with the hypotheses (3.5) and (3.6), we can bound the right hand side of (7.5) uniformly by a term depending only on \( u \) and \( h_0 \), in particular, there holds
\[
\iint_{2B \times [0, s]} |\nabla u^i|^p \, dz \lesssim \left( \frac{1}{s|B|} \right)^{\frac{\beta}{p}} \left( \iint_{\Omega} (|\nabla u^i| + |h_0|)^{p-\beta} \, dz \right)^{1+\frac{\beta}{p}} + \iint_{4B \times [0, s]} (1 + |h_0|^p) \, dz
\]
\[\quad =: R^i.
\]

Denoting the term appearing on the right hand side of (7.6) by \( R \), we use the above estimate in (7.4) to get
\[
\sup_{t \in [0, s]} \int_B (u^i - u^j)^2 \, dx + \iint_{B \times [0, s]} |\nabla u^i - \nabla u^j|^p \, dz
\]
\[\quad \leq \epsilon R + C(\epsilon) \left( \frac{s}{|B|} \right)^{\frac{\beta}{p} - 1} \left( \int_{2B \times [0, s]} |u^i - u^j|^p \, dz \right)^{\frac{1}{p}}.
\]

We shall now show how to obtain the necessary convergence results that allows us to pass through the limit in (7.2). To do this, we follow the structure from [31, Proof of Theorem 2] (see also [8] for the details).

**Step 1:** First, we want to obtain a limit for \( u^i \). In order to do this, we use the higher integrability for very weak solutions from (7.6) to see that \( \nabla u^i \) is uniformly bounded in \( L^p(0, T; L^p_{\text{loc}}(\Omega)) \). Also from Theorem 3.1, we see that \( u^i - u_0 \) is uniformly bounded in \( L^p(0, T; W_{0, \text{loc}}^{-1, p-\beta}(\Omega)) \) so that \( u^i \) is bounded in \( L^p(0, T; W_{\text{loc}}^{-1, p-\beta}(\Omega)) \). Thus there exists a subsequence (still denoted by \( \{u^i\} \) such that
\[ u^i \to u \quad \text{weakly in} \ L^p(0, T; W_{\text{loc}}^{-1, p-\beta}(\Omega)). \]

**Step 2:** From (7.1) and **Step 1**, we see that \( u^i \) is uniformly bounded in \( L^\frac{p}{p-\gamma}(0, T; W_{\text{loc}}^{-1, p-\gamma}(\Omega)) \). Applying Lions-Aubin Lemma (see [30, Proposition 1.3 on page 106]), there exists a subsequence such that
\[ u^i \to u^j \quad \text{weakly in} \ L^p(0, T; W_{\text{loc}}^{-1, p-\beta}(\Omega)), \]
\[ u^i \to u \quad \text{strongly in} \ L^p(0, T; L^p_{\text{loc}}(\Omega)). \]

**Step 3:** As a direct consequence of **Step 2** and (7.7), we take \( \epsilon \) sufficiently small followed by taking \( i \) and \( j \) large enough to obtain
\[ \nabla u^i \to \nabla u \quad \text{a.e in} \ L^p(0, T; W_{\text{loc}}^{-1, p}(B)) \quad \text{for any} \ B \subset \Omega. \]

**Step 4** From (2.1), we see that \( A(x, t, \nabla u^i) \) is bounded in \( L^\frac{p}{p-\gamma}(0, T; W_{\text{loc}}^{-1, p-\gamma}(\Omega)) \). Thus, as a consequence of **Step 3**, we find that
\[ A(x, t, \nabla u^i) \to A(x, t, \nabla u) \quad \text{weakly in} \ L^\frac{p}{p-\gamma}(0, T; W_{\text{loc}}^{-1, p-\gamma}(\Omega)). \]

From the above convergence estimates, we can now take \( \lim_{k \to \infty} \) in (7.2) to obtain the existence of a very weak solution \( u \in C^0(0, T; L^2_{\text{loc}}(\Omega)) \cap L^{p-\beta}(0, T; u_0 + W_{0}^{-1, p-\beta}(\Omega)) \) of (1.2). This completes the proof of the theorem. \( \Box \)
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