Fundamental trade-off between the speed of light and photon current fluctuations in electromagnetically induced slow-light medium

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Electromagnetically induced slow-light medium is a promising system for quantum memory devices, but controlling its noise level remains a major challenge to overcome. This work provides a new fundamental trade-off relationship between the group velocity of light and photon current fluctuations in such media, modeled as coherently controlled three-level Λ-system interacting with bosonic bath. Considering the steady state limits of a newly derived Lindblad-type equation, we find that the Fano factor of the photon current maximizes to 3 at the minimal group velocity of light, which holds true universally regardless of detailed values of parameters characterizing the medium.

I. INTRODUCTION

Quantitative characterization of fluctuations in driven quantum dynamical processes has fundamental implications for quantum thermodynamics [1–6], and is a central issue to address for the development of efficient quantum information [7–9] and sensing devices [10–12]. To this end, significant theoretical advances have been made in recent years, for example, by identifying new relations and bounds for quantum fluctuations through quantum extensions [1] of thermodynamic uncertainty relations [19–21] and related quantum fluctuation theorems [5, 6, 22]. As yet, utilizing many of these relations for actual experimental measurements/developments requires further theoretical analyses that can provide concrete and experimentally testable relationships between physical observables. This work provides such an analysis for a well established process that utilizes coherent driving of laser pulses to slow down light propagation [23, 24] and establishes an important trade-off relationship for the first time.

There have been considerable efforts to develop optical quantum memory devices employing laser control [24,30] since Hau et al. [23] demonstrated extraordinary slowdown of the group velocity of light as low as 17 m/s in an ultracold gas medium of sodium atoms. The electronic states of a sodium atom constitute a Λ-type three-level system, which comprises two nearly degenerate ground states and a common excited state. Applying a control pulse to the Λ-system can eliminate the linear absorption of a resonant probe pulse. Depending on the intensity of the control pulse relative to the probe pulse, two distinct mechanisms, coherent population trapping (CPT) [31, 32] and electromagnetically induced transparency (EIT) [33] (see Supplemental Material (SM)), transform an otherwise absorbing medium into effectively transparent one and slow down the group velocity of the propagating probe pulse [25]. While conceptually clear, realization of an actual quantum memory device employing these phenomena remains challenging due to a substantial level of noise [34, 35], the precise origin of which is currently left unknown in many cases. Thus, elucidating the origin and intrinsic limit of such noise level is a crucial step in overcoming practical challenges.

The main objective of this work is to offer a quantitative understanding of how photon current fluctuations through a coherently controlled ensemble of Λ-systems change as the speed, i.e., the group velocity of light is reduced. In a recent work on a field-driven two-level system (TLS) weakly interacting with bosonic environment [17], we have shown that fluctuations in photon current are determined by the competition between the real and imaginary parts of the steady state coherence formed between the excited and ground states, such that the imaginary part of the coherence reduces the fluctuations, whereas the real part contributes to enhancing them [17]. Employing a similar formalism for the Λ-system and through careful theoretical analyses of a newly derived Lindblad-type equation, we discover a fundamental trade-off relationship between the speed of light and photon current fluctuations.

II. THEORETICAL MODEL

A three-level Λ-system comprised of the electronic states [1], [2], and [3] is coupled to a thermally-equilibrated bosonic bath at temperature $T$. The system is illuminated with control ($\alpha = c$) and probe
be described by the following Lindblad-type equation (RWA), we find that the dynamics of the $\Lambda$-system can be expressed as:

\[
\rho(t) = \frac{1}{i\hbar}\left[\rho, H_{\text{int}}\right] + D(\rho(t)),
\]

where $H_{\text{int}} = h(\omega_1 |1\rangle \langle 1| + \omega_2 |2\rangle \langle 2| + \omega_3 |3\rangle \langle 3|)$ with $\hbar \omega_i$ denoting the energy level of the $i$-th state, and $D(\rho(t))$ is a Lindblad-type dissipator.

Equation (1) can be transformed to $\dot{\rho}(t) = L\rho(t)$, where $\dot{\rho} \equiv \{\rho_{11}, \rho_{12}, \rho_{13}, \rho_{21}, \rho_{22}, \rho_{23}, \rho_{31}, \rho_{32}, \rho_{33}\}^T$ is vector representation of $\rho(t)$ in the rotating frame (see SM), and $L$ represents the Liouvillian super-operator expressed as $9 \times 9$ matrix in the Fock-Liouville space [33]. The steady-state value of each element $\rho_{ij}$ is calculated from $L\rho^{ss} = 0$ (see Eq. (26) in SM). Fig. 1 shows the population in each state ($\rho_{ii}^R$, which satisfies $\sum_{i=1,2,3} \rho_{ii}^{ss} = 1$) and coherences between the states $|i\rangle$ and $|j\rangle$ ($\rho_{ij}^{ss} = \rho_{ij}^R + i\rho_{ij}^I$, $i \neq j$, with $\rho_{ij}^R \equiv \text{Re}\{\rho_{ij}^{ss}\}$ and $\rho_{ij}^I \equiv \text{Im}\{\rho_{ij}^{ss}\}$) as a function of the detuning frequency of the probe pulse ($\delta \omega_p$).

At two-photon resonance ($\delta \omega_p = 2\omega_c = 0$) and $\Omega_c \approx \Omega_p$, the atom is locked in the states $|2\rangle$ and $|3\rangle$, without populating the state $|1\rangle$, i.e., $\rho_{22}, \rho_{33} \neq 0$ but $\rho_{11} = 0$ (panel (a) of Fig. 1B). In addition, except for the real part of coherence between $|2\rangle$ and $|3\rangle$ ($\rho_{23}^R \neq 0$), all the coherence terms vanish, such that $\rho_{12}^R = \rho_{12}^R = \rho_{13}^R = \rho_{13}^R = \rho_{23}^R = 0$. This situation corresponds to the CPT, where the effects of control and probe pulses are cancelled off via destructive interference, and the atomic state is delocalized between $|2\rangle$ and $|3\rangle$, forming a dark state. Since there is neither dispersion ($\rho_{13}^I = 0$) nor absorption of light ($\rho_{13}^R = 0$), the atomic medium look effectively transparent to the probe pulse.

### III. Transition Current, Fluctuations, and Fano Factor

Laser pulse applied to the system for a time interval sufficiently longer than the decay time ($\tau \equiv \gamma_1 t \gg 1$) establishes steady-state current of photon absorption and emission. With the net number of photon (or atomic) transitions in the $\Lambda$-system denoted as $n(\tau)$, where $n(\tau) > 0$ is for emissions and $n(\tau) < 0$ is for absorptions, the photon current at steady state, its fluctuations, and the corresponding Fano factor are

- **FIG. 1.** Optical properties of $\Lambda$-system as a function of detuning frequency ($\delta \omega_p$). (A) Schematic of the system consisting of 3 electronic states, $|1\rangle$, $|2\rangle$ and $|3\rangle$, interacting with the probe and control pulses of frequencies $\omega_p$ and $\omega_c$. Here, $\omega_{12}(\equiv \omega_1 - \omega_2)$ and $\omega_{13}(\equiv \omega_1 - \omega_3)$ are the resonant frequencies. Further, $\delta \omega_c = \omega_c - \omega_{12}$ and $\delta \omega_p = \omega_p - \omega_{13}$ denote the detuning frequencies. The condition $\delta \omega_p = \delta \omega_c = 0$ corresponds to the two-photon resonance. (B) Populations in $|1\rangle$, $|2\rangle$, and $|3\rangle$ are shown in the panel (a). Real and imaginary parts of the coherences $\rho_{12}, \rho_{13}$, and $\rho_{23}$ are depicted in (b), (c), and (d) as a function of $\delta \omega_p$, with the solid and dotted lines, respectively. Here, we have used $\gamma \equiv \gamma_1/\gamma_1 = 0.9$, $\delta \omega_c = 0$, $n_{ij} = 0$, $\Omega_c = 0.56$, and $\Omega_p = 0.50$. All the frequencies are scaled with $\gamma_{13}\approx 0.62 \times 10^{10}$ s$^{-1}$. 

- (α = p) laser pulses, $\mathbf{E}_\alpha = \mathcal{E}_\alpha \mathbf{\zeta}_\alpha (e^{i \omega_\alpha t} + e^{-i \omega_\alpha t})$, each with the amplitude $\mathcal{E}_\alpha$, the angular frequency $\omega_\alpha$, and the two polarization vectors satisfying $\mathbf{\zeta}_\alpha \cdot \mathbf{\zeta}_\alpha = 0$. The atoms in $|2\rangle$ and $|3\rangle$ states are excited to a common level $|1\rangle$ through interactions of transition dipoles, $d_2$ (between $|1\rangle$ and $|2\rangle$) and $d_3$ (between $|1\rangle$ and $|3\rangle$), with the pulses (see Fig. 1A). This is represented by an interaction Hamiltonian $H_{\text{int}} = -d_2^* \mathbf{E}_c - d_3^* \mathbf{E}_p$, for which two Rabi frequencies $\Omega_c$ and $\Omega_p$ characterizing the respective interaction strengths can be defined (see SM for details). The state $|1\rangle$ can either decay into $|2\rangle$ with a rate $\gamma_{12}$ or into $|3\rangle$ with $\gamma_{13}$. The transition between $|2\rangle$ and $|3\rangle$ is effectively spin-dissolved with $\gamma_{23} \ll \gamma_{12}, \gamma_{13}$. Employing the standard assumptions of the weak system-bath coupling, Born-Markov, and the rotating wave approximations (RWA), we find that the dynamics of the $\Lambda$-system can be described by the following Lindblad-type equation for the reduced density matrix $\rho(t)$ (see SM),

- $\partial_t \rho(t) = -(i/\hbar)[H_S + H_{\text{int}}, \rho(t)] + D(\rho(t))$, (1)

- where $H_S = h(\omega_1 |1\rangle \langle 1| + \omega_2 |2\rangle \langle 2| + \omega_3 |3\rangle \langle 3|)$ with $\hbar \omega_i$ denoting the energy level of the $i$-th state, and $D(\rho(t))$ is a Lindblad-type dissipator.

Equation (1) can be transformed to $\partial_t \rho(t) = L\rho(t)$ where $\dot{\rho} \equiv \{\rho_{11}, \rho_{12}, \rho_{13}, \rho_{21}, \rho_{22}, \rho_{23}, \rho_{31}, \rho_{32}, \rho_{33}\}^T$ is vector representation of $\rho(t)$ in the rotating frame (see SM), and $L$ represents the Liouvillian super-operator expressed as $9 \times 9$ matrix in the Fock-Liouville space [33].
\[ q(\gamma) = \frac{2(\gamma \xi^6 + 2\gamma \xi^4 + 2\xi^2 + 1)}{(\xi^2 + 1)(\xi^2 + \gamma^2)}, \]

where \( \xi = \Omega_p/\Omega_p \) is the experimentally controllable variable, and \( \gamma = \gamma_{12}/\gamma_{13} \) (see Eqs. (S29) and (S30) in SM). Note that the result of TLS, i.e., \( q(\gamma) = 0 \) is recovered under the limiting condition of \( \gamma > 1 \).

**Group velocity of probe field and Fano factor.** Since the group velocity of light is defined as \( v_g = (d\omega/d\omega)^{-1} \), where \( k(\omega) = \omega \eta(\omega)/c \) with \( \eta(\omega) \) denoting the real part of the refractive index and \( c \) speed of light in vacuum, a change in the refractive index gives rise to a change in the group velocity of probe field across the medium as follows (see SM)

\[
v_g = \frac{c}{1 + 2\pi N d_{13} \partial R / d\omega_p N_d \partial R / d\omega_p}.
\]

where \( N_d \equiv N |d_{13}|^2/N_{\Omega_p} \gamma_{13}^2 \) with \( N \) being the density of atoms comprising the medium of atomic vapor.

The condition of two-photon resonance (\( \delta\omega_p = \delta\omega_c \)) simplifies Eq. (1) with \( (\rho^{R}_{13})_{\delta\omega_p=0} = 0 \) (Fig. 1B, Fig. 3A inset, and see Eq. (S26)). Hence, \( v_g \) is greatly reduced by increasing the derivative term, \( (\partial R/\partial \omega_p)_{\delta\omega_p=0} \), namely, by increasing the variation of refractive index (or coherence) involving the states [1] and [3] with the probe frequency [23]. In fact, it is straightforward to show \( (\partial R/\partial \omega_p)_{\delta\omega_p=0} = \Omega_p^{-1}(\rho^{R}_{23})_{\delta\omega_p=0} = 0 \) (Eq. (S26)). Thus, \( v_g \) in Eq. (5) is determined by the strength of Raman coherence, i.e., the magnitude of the real part of coherence between the two ground states [2] and [3] at \( \delta\omega_p = 0 \) as follows.

\[
v_g = \frac{c}{1 + \mathcal{N} (\rho^{R}_{23})^2_{\delta\omega_p=0}},
\]

where \( \mathcal{N} \equiv 2\pi N d_{13} \omega_p/\Omega_p \) is a factor determined by the density of atoms comprising the medium, the magnitude of the transition dipole moment \( |d_{13}| \), the resonant and Rabi frequencies, \( \omega_p \) and \( \Omega_p \).

An important relationship between \( v_g \) and \( F \) for \( \Lambda \)-systems can be identified through \( \xi \) (see Fig. 2A for \( v_g = v_g(\xi) \)). Fig. 2B shows a curve of \( \mathcal{F} \) versus \( v_g \) parameterized with \( \xi = \delta\omega_p = \delta\omega_c = 0 \) for \( \gamma = 0.9 \), clarifying a trade-off relation between \( \mathcal{F} \) and \( v_g \) for experimentally relevant range of variable, \( \xi > 1 \). It is noteworthy that the Fano factor of photon transitions sharply increase to \( \mathcal{F} \approx 3 \) when \( v_g \) approaches its minimal value \( v_g \approx 7 \) m/s (Fig. 2B, magenta line), which is even smaller than the one experimentally reported [23].
FIG. 3. Effect of detuning. (A) Diagram of $F(δω_p, Ω_p)$ calculated for $δω_c = 0$ with $Ω_c = 0.56$, $γ = 0.90$, $A = 47$. (Inset) $F$, $R$, $I$, and $q$ as a function of $δω_p$ for $Ω_p = 0.5$. The blue vertical dashed line indicates the value of $δω_p (∼ 0.8)$ that gives rise to the minimal $F$. The range of transparency window ($Δ_p$) is indicated by the arrow. (B) Real ($R$) and (C) imaginary parts of coherence ($I$) and (D) the factor $q$ as a function of probe detuning $δω_p$ and driving frequency $Ω_p$.

For $A ≫ 1$ (or $η ≃ 0$) with $δω_c = δω_p = 0$, the expressions of coherence terms (Eq. (S26)) are greatly simplified, enabling us to further clarify a relationship between $v_g$ and $F$. With $(ρ_{23}^R)^2_{δω_p=0} = ξ^2/(ξ^2 + 1)^2$, $ρ_{23}^R = ρ_{12}^R = ρ_{13}^F = ρ_{13}^F = 0$ (Eq. (S26)) and the expression of $q(\cdot)$ given in Eq. (4), the group velocity and the Fano factor read

$$v_g = \frac{c}{1 + N/(ξ + 1/ξ)^2}$$

and

$$F ≃ 1 + \frac{2(1 + γξ^2)}{(γ + ξ^2)}.$$  

From Eq. (7), it is clear that $v_g$ minimizes to $v_g^{\text{min}} = c/(1 + N/4)$ for $ξ = 1$, and saturates to $v_g = c$ for $ξ ≫ \sqrt{N}$ or $ξ \ll 1/\sqrt{N}$ (see Fig. 2A). Next, insertion of the two solutions of Eq. (7), $ξ(v_g) =$$1/2[(N/(c/v_g - 1) ± \sqrt{N/(c/v_g - 1)^2 - 4})$, to Eq. (8) yields $F = F_>(v_g)$ for $ξ(v_g) > 1$ (magenta line in Fig. 2B), and $F = F_<(v_g)$ for $ξ(v_g) < 1$ (blue line in Fig. 2B). We note that only the condition of $ξ(v_g) > 1$ is of practical relevance to the slow-light experiment because the current fluctuations are smaller and more controllable with $F_>(v_g) ≤ 3$. At $ξ(v_g) = 1$ or equivalently at $v_g = v_g^{\text{min}}$, one always obtains $F = 3$. The universality of this value is a key outcome of our analyses.

For more general case with $δω_p ≠ 0$ and $δω_c = 0$, the expression of $F$ is complicated; yet, $F$ is still an even function of $δω_p$ (Eq. (S26)). Confining ourselves to the condition $ξ > 1$, we resort to numerics to calculate $F(δω_p, Ω_p)$ (Fig. 3), finding that $F$ is maximized over the transparency window $Δ_p$, given by

$$Δ_p ≃ \left[\frac{∂ρ_{13}^F}{∂δω_p}|_{δω_p=0}\right]^{-1} = Ω_p(ξ^2 + 1)/ξ^2.$$  

Note that $Δ_p$ is narrow for the case of CPT ($ξ ≃ 1$) but is wide for EIT ($ξ ≫ 1$). Over the $Δ_p$, $ρ_{13}^F, ρ_{13}^F ≃ 0$ (Fig. 1B), and $R$ and $q$ display maximal contribution at two-photon resonance (Fig. 3 inset, B, and C), whereas $I ≃ 0$, i.e., the absorption is negligible (Fig. 3A inset and C).

It is interesting to note that the fluctuation in photon current is maximally suppressed under a detuning condition $δω_p ≠ 0$ where $I$ is maximized, $R ≃ 0$, and $q(\cdot) < 0$, resulting in $F < 1$ (Fig. 3A inset, B, and D); however, the value of $δω_p$ is beyond the transparency window, which is not suitable for generation of absorption-free slow light. This condition corresponds to the absorption doublet [39] involving the transitions from $|0\rangle$ to two eigenstates $|±\rangle$ comprised of the three electronic states $|1\rangle$, $|2\rangle$, and $|3\rangle$ (see Fig. 3B and Eq. (S5)).
IV. CONCLUDING REMARKS

This work has established a fundamental trade-off relationship between the group velocity and photon current fluctuations of light propagating through electromagnetically induced slow light medium as modeled by coherently controlled A-type three-level system interacting with bosonic bath. In particular, the Fano factor of net number of photon transitions \( n(\tau) \), which dictates the fluctuations of the laser power as \( \langle (\delta n(\tau))^2 \rangle \propto \langle (\delta I)^2 \rangle \), is maximized to \( F = 3 \coth(A/2) \) at the slowest group velocity, \( v_g \approx (4/N)c \). This indicates that slow light is attained at the expense of fluctuations of the irreversible photon current. This trade-off, which may be inevitable in the basic setup of CPT or EIT-based optical quantum memory device, is physically sensible in that as the light slows down, overall fluctuations in the photon current is enhanced over the prolonged travel time of the photon inside the medium. At two-photon resonance, the real part of coherence between the two ground states \( \rho_{23}^R \), which engenders slow light (Eq. [0]) and enhanced fluctuations of signal (Eq. (3)), is maximized at the regime corresponding to CPT, where the Rabi frequencies of control and probe pulses are identical (\( \xi = \Omega_p/\Omega_p = 1 \)).

Our results can also be applied to the medium consisting of \(^{133}\text{Cs} \) atoms, whose D1 line constitutes the three-level A-system. For Cs atoms, the frequency gap between the two ground states \( 6^2S_{1/2}(|F = 3\rangle) \) and \( 6^2S_{1/2}(|F = 4\rangle) \), where \( F \) stands for the total angular momentum quantum number, is \( \approx 9.2 \) GHz. The condition \( \rho_{23}^R \neq 0 \) and \( \rho_{32}^R = 0 \) signifies a Raman coherence between \( |F = 3\rangle \) and \( |F = 4\rangle \) effectively with no absorption. The slowest group velocity achievable for the case of CPT regime (\( \xi = 1 \)) of \(^{133}\text{Cs} \) vapor [40] is \( v_g \approx 38 \) m/s with \( N = 2\pi N_d(\omega_p/\Omega_p) \approx 3.2 \times 10^7 \), which is estimated from \( \Omega_p = 0.5 \), \( \omega_p = (2\pi c/\lambda_p)/\gamma_{13} \approx 2.1 \times 10^7 \) with \( \lambda_p \approx 894 \) nm [40] and \( \gamma_{13} \approx 10^8 \) s\(^{-1} \), and \( N_d = N|d_{13}|^2/(\hbar \Omega_p) \approx 0.12 \) with \( N \approx 10^{12} \) cm\(^{-3} \) and \( |d_{13}| = 2.7 \times 10^{-29} \) C m = 8.09 \times 10^{-18} \) statC-cm [11]. Our estimate for the slowest group velocity of light in the atomic vapor of cesium is amenable for an experimental verification.

Finally, we emphasize that the size of current fluctuations (or noise level) in three-level A-system at the slowest group velocity is universal (\( F = 3 \)) regardless of the atomic type, which warrants experimental confirmation. The formalism of this work can be extended to other types of systems, for example, with \( V \) and ladder structures, which are used for different applications [25, 42, 43] and also to Bose-Einstein condensates that can serve as media where the light can stop completely [25]. How the trade-off relationship becomes altered for the different systems and by additional effects due to non-Markovian or strongly coupled environments [44-46] remains an important theoretical issue that requires further investigation.

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V. SUPPLEMENTAL MATERIAL

A. Coherent population trapping (CPT) and electromagnetically induced transparency (EIT)

CPT. The absorption and dispersion profiles of probe pulse as a function of detuning ($\delta \omega_p$) are calculated in Fig. 1B in the main text. At the two-photon resonance ($\delta \omega_p = \delta \omega_c = 0$), both the coherences between the states $|1\rangle$ and $|3\rangle$ and between the states $|1\rangle$ and $|2\rangle$ vanish ($\rho_{13}^R = \rho_{13}^I = 0$ and $\rho_{12}^R = \rho_{12}^I = 0$ in Fig. 1B), which implies that the medium is effectively transparent to the probe and control pulses. The two light pulses interacting with the matter vanish via the destructive interference between two pathways between $|3\rangle \leftrightarrow |1\rangle \rightarrow |2\rangle$ and $|2\rangle \leftrightarrow |1\rangle \rightarrow |3\rangle$ (Fig. S1).

To show the destructive quantum interference more explicitly, we consider an addition of two pulses with quantum coherence,

$$\hat{\rho}_{\text{sum}} = \hat{\rho}_{12} + \hat{\rho}_{13}. \quad (S1)$$

Note that $\hat{\rho}_{ij} = |\hat{\rho}_{ij}| \exp(i\theta_{ij})$ with $|\hat{\rho}_{ij}|^2 = (\rho_{ij}^R)^2 + (\rho_{ij}^I)^2$ and $\tan \theta_{ij} = (\rho_{ij}^I / \rho_{ij}^R)$. Numerical calculation using the results in Eq. (S26) gives rise to Fig. S2 and shows that the amplitude of $\hat{\rho}_{\text{sum}}$ vanishes at two-photon resonance ($\delta \omega_p = \delta \omega_c = 0$). Thus, the excitation transfer to the state $|1\rangle$ is negligible, and almost all the atomic population is trapped in the states $|2\rangle$ and $|3\rangle$ (Fig. 1B in the main text). The “coherent population trapping” (CPT) refers to such a trapping of population in the two ground states via a coherent superposition of the quantum states.

The destructive interference and hence population trapping in states $|2\rangle$ and $|3\rangle$ results in strong coupling between these states, which is reflected in the high value of $\rho_{23}^R$ (see Fig. 1B in the main text).

More complete physical interpretation of CPT can be given in terms of the basis representing the dressed (or eigen) states. Under the following unitary transformation, which is equivalent to describing the system in the rotating frame,

$$|\psi\rangle = \mathcal{U} |\phi\rangle, \quad (S2)$$

FIG. S1. (A) Bare state basis to show the paths involved in the destructive interference and (B) the corresponding dressed state picture for the weak probe field.

where $\mathcal{U} = e^{-i \omega_p t} |1\rangle \langle 1| - i (\omega_p - \omega_c) t |2\rangle \langle 2|$, the Schrödinger equation $\partial_t |\psi\rangle = -i \mathcal{H}/\hbar |\psi\rangle$ is written as $\partial_t |\phi\rangle = -i \mathcal{H}_{\text{eff}}/\hbar |\phi\rangle$ with

$$\mathcal{H}_{\text{eff}} = \mathcal{U}^\dagger \mathcal{H} \mathcal{U} - i \hbar \mathcal{U}^\dagger \frac{d\mathcal{U}}{dt} = -\hbar \delta \omega_p |1\rangle \langle 1| - \hbar (\delta \omega_p - \delta \omega_c) |2\rangle \langle 2| - \hbar (\Omega_p |3\rangle \langle 1| + \Omega_c |1\rangle \langle 2| + \text{h.c.}). \quad (S3)$$

When $\delta \omega_p = \delta \omega_c = \delta \omega$ is assumed for simplicity, the energy eigenvalues and eigenstates of $\mathcal{H}_{\text{eff}}$ are

$$\lambda_0 = 0 \quad \lambda_{\pm} = 0.5 \hbar \left( \delta \omega \pm \sqrt{\delta \omega^2 + 4(\Omega_p^2 + \Omega_c^2)} \right), \quad (S4)$$

and

$$|0\rangle = \cos \theta |3\rangle - \sin \theta |2\rangle, \quad |\pm\rangle = \sin \theta \cos \phi |3\rangle + \cos \theta \cos \phi |2\rangle - \sin \phi |1\rangle, \quad |\mp\rangle = \sin \theta \sin \phi |3\rangle + \cos \theta \sin \phi |2\rangle + \cos \phi |1\rangle, \quad (S5)$$

where the mixing angles $\theta$ and $\phi$ are defined as

$$\theta = \tan^{-1} (\Omega_p / \Omega_c), \quad \phi = 0.5 \tan^{-1} \left( 2 \sqrt{\Omega_p^2 + \Omega_c^2} / \delta \omega \right). \quad (S6)$$

Under the two-photon resonance condition ($\delta \omega = 0$), the eigenstate $|0\rangle$, a coherent superposition between the states $|2\rangle$ and $|3\rangle$, of the effective Hamiltonian (Eq. S3) has zero eigenvalue. Hence, the state $|0\rangle$ is a dark state that does not evolve with time, and is decoupled from the applied fields. Now the spontaneous emission from the state $|1\rangle$ always populates the quantum states $|2\rangle$ and $|3\rangle$. Therefore, irrespective of the initial condition, the atomic population is trapped in the dark state $|0\rangle$ for an extended period of time, $t \gg 1/\gamma$. This corresponds to the CPT.
**EIT.** For a strong control field (\(\xi = \Omega_c / \Omega_p \gg 1\)) and \(\delta \omega = 0\), a coherent superposition of states \(|1\rangle\) and \(|2\rangle\), produces the dressed states \(|\pm\rangle\), without affecting the state \(|3\rangle\) (\(|0\rangle\)) (Fig. S1). The three energy eigen-states and corresponding eigenvalues (inside parenthesis) are obtained as

\[
\begin{align*}
|0\rangle & = |3\rangle \quad (\lambda_0 = 0), \\
|\pm\rangle & = \frac{1}{\sqrt{2}}(|2\rangle \pm |1\rangle) \quad (\lambda_\pm = \pm \hbar \Omega_c). \quad (S7)
\end{align*}
\]

In this case, the transition amplitude at the resonant probe frequency (\(\delta \omega_p = 0\)) between the ground state \(|0\rangle = |3\rangle\) to the dressed states \(|\pm\rangle\) can be written as \(|3\rangle \vec{d} |+\rangle + |3\rangle \vec{d} |−\rangle \simeq \vec{d}_{32} + \vec{d}_{31} + \vec{d}_{32} - \vec{d}_{31} = 2 \vec{d}_{32} = 0\) because the electric dipole selection rule that disallows the transition from \(|2\rangle \rightarrow |3\rangle\) (\(\vec{d}_{32} = 0\)). Consequently, all the population is effectively confined in the dark state \(|0\rangle\). At \(\delta \omega_p = 0\), the media is transparent to the pulse, and does not absorb the probe pulse. This strong control field-induced (\(\Omega_c \gg \Omega_p\)) conversion of an absorptive medium to a transparent one is termed the electromagnetically induced transparency (EIT) \[39\]. The EIT creates the destructive interference between the transition pathways \(|3\rangle \rightarrow |1\rangle\) and \(|2\rangle \rightarrow |3\rangle\).

The energy gap between the dressed states is \(2\hbar \Omega_c\). Then, the conditions for the perfect resonance between \(|0\rangle\) and \(|\pm\rangle\) appears when \(\delta \omega_p = \pm \Omega_c\), resulting in the complete absorption of probe pulse, giving rise to the Aulter-Townes absorption doublet \[39\]. The off-resonant probe pulse (\(\delta \omega_p \approx 1\)) engenders the absorption doublet where again the dispersion becomes zero (\(\rho_{33} = 0\)), but this time the absorption (\(\rho_{13}\)) is maximized.

**B. Coherent control of dispersion of media**

The probe pulse-induced polarization of the \(\Lambda\)-system is quantified with the dipole moment between \(|1\rangle\) and \(|3\rangle\) per unit volume as \(P_{13} = N \langle \vec{d}_{3} \rangle = \chi_{13} \vec{E}_p\), where \(N\) is the number density of atoms. \(P_{13}\) is the linear susceptibility of the medium \[39\]. Since \(\langle \vec{d}_{3} \rangle = \text{Tr}(\hat{\rho} \vec{d}) = \rho_{31} \vec{d}_{31} + \rho_{31} \vec{d}_{31} = \rho_{13} e^{-i \omega_{13} t} \vec{d}_{31} + \rho_{31} e^{i \omega_{13} t} \vec{d}_{31} \simeq e^{i \omega_{13} t} \rho_{13} \vec{d}_{31}\), the linear susceptibility can be expressed as \(\chi_{13} = |P_{13}|/|\vec{E}_p| = N \rho_{13}\) with \(N = N \rho_{31}/\zeta_{p}\). For the medium with \(|\chi_{13}| \ll 1\), the refractive index, dielectric constant and linear susceptibility for the probe field are related with one another in Gaussian units as

\[
\eta_{13} = (\sqrt{\epsilon_{13}}) = \sqrt{1 + 4 \epsilon_{13} \chi_{13}} \simeq 1 + 2 \chi_{13} R + i 2 \chi_{13} I. \quad (S8)
\]

where \(\chi^R\) and \(\chi^I\) are the real and imaginary parts of the susceptibility. When the probe field, \(\vec{E}_p = e^{i k_p z} e^{-i \omega_p t}\), passes across the dielectric medium with a wave vector \(k_p\),

\[
k_p = \frac{\omega_p}{c} \eta_{13} = \frac{\omega_p}{c} \left(1 + 2 \pi \chi_{13} R + \frac{4 \pi \chi_{13} I}{c^2} \right). \quad (S9)
\]

it moves through the medium with a phase velocity \(c(1 + 2 \pi \chi_{13} R)/\omega_p\), and is also attenuated by the medium with an absorption coefficient \(\alpha\). Since \(\chi_{13} = N_d \rho_{13}\), the real and imaginary parts of the susceptibility is linked to the dispersion and absorption profiles of the medium, respectively, as \(\chi_{13}^R = N_d \rho_{13}\) and \(\chi_{13}^I = N_d \rho_{13}\).

**C. Evolution equation**

The total Hamiltonian in the presence of an external field is expressed as \[39\] \[47\]

\[
H = H_S + H_{int} + H_B + H_{SB}, \quad \text{(S10)}
\]

where

\[
H_{int} = -\hat{d}_2 \cdot \vec{E}_c - \hat{d}_3 \cdot \vec{E}_p
\]

\[
H_B = \sum_{\vec{k},\lambda} \hbar \omega_{\vec{k},\lambda} b_{\vec{k},\lambda}^\dagger b_{\vec{k},\lambda}
\]

\[
H_{SB} = \sum_{\vec{k},\lambda} \hbar \left[ (g_{\vec{k},\lambda})_{12} b_{\vec{k},\lambda}^\dagger |2\rangle + (g_{\vec{k},\lambda})_{13} b_{\vec{k},\lambda} |1\rangle \right] \langle 3 | + (g_{\vec{k},\lambda})_{13} b_{\vec{k},\lambda}^\dagger |3\rangle \langle 1 | \text{ (S11)}
\]

The control and probe fields, \(\vec{E}_\alpha(t) = \hat{e}_\alpha \zeta_\alpha (e^{i \omega_{\alpha} t} + e^{-i \omega_{\alpha} t})\) with \(\alpha = c, p\) where \(\hat{e}_\alpha\) is the unit vector representing the direction of polarization and \(\zeta_\alpha\) denotes the amplitude, interact with the \(\Lambda\)-system via the interaction energy Hamiltonian \(H_{int} = -\vec{d}_2 \cdot \vec{E}_c - \vec{d}_3 \cdot \vec{E}_p\), inducing the excitations of \(|2\rangle \rightarrow |1\rangle\) and \(|3\rangle \rightarrow |1\rangle\), respectively. The transition dipole operator is given by \(\hat{d} = \hat{d}_2 + \hat{d}_3 = \hat{d}_{12} |1\rangle \langle 2 | + \hat{d}_{21} |2\rangle \langle 1 | + \hat{d}_{13} |3\rangle \langle 1 | + \hat{d}_{31} |1\rangle \langle 3 | \text{ with the dipole matrix elements, } \hat{d}_{ij}\). Since the transition between \(|2\rangle\) and \(|3\rangle\) is effectively forbidden, \(\hat{d}_{31} = \hat{d}_{12} \approx 0\). The summation \(\sum_{\vec{k},\lambda}\) extends over the wavevector \(\vec{k}\) and polarization \(\lambda\). The symbols, \(b_{\vec{k},\lambda}^\dagger\) and \(b_{\vec{k},\lambda}\) denote the creation and annihilation operators of the harmonic
oscillators of angular frequency $\omega_k$ constituting the reservoir. The dipole coupling constant, $(g_{k,\lambda}^2)_{ij} = -i\sqrt{\omega_k/2\hbar\varepsilon_0 V} \hat{e}_{k,\lambda} \cdot \hat{d}_{ij}$ for $j \in 2, 3$, contains the information of polarization $\hat{e}_{k,\lambda}$, quantization volume $V$ and vacuum permittivity $\varepsilon_0$.

The density matrix for the total system, $\rho_{\text{tot}}(t)$, evolves with time, obeying the von Neumann equation, $d\rho_{\text{tot}}(t)/dt = -\frac{i}{\hbar}[H, \rho_{\text{tot}}]$. In the framework of Lindblad approach, the reduced density matrix after tracing out the bath degrees of freedom obeys the following evolution equation.

$$
\frac{d\rho(t)}{dt} = -\frac{i}{\hbar}[H_S + H_{\text{int}}, \rho] \\
+ \gamma_{12}(\bar{n}_{12} + 1) \left(\langle 2 | \langle 1 | \rho | 1 \rangle \langle 2 | - \frac{1}{2} \{\langle 1 \rangle | \rho \rangle \langle 1 | \rho \rangle \} \right) \\
+ \gamma_{12}\bar{n}_{12} \left(\langle 1 | \langle 2 | \rho | 2 \rangle \langle 1 | - \frac{1}{2} \{\langle 2 \rangle | \rho \rangle \langle 2 | \rho \rangle \} \right) \\
+ \gamma_{13}(\bar{n}_{13} + 1) \left(\langle 3 | \langle 1 | \rho | 1 \rangle \langle 3 | - \frac{1}{2} \{\langle 1 \rangle | \rho \rangle \langle 1 | \rho \rangle \} \right) \\
+ \gamma_{13}\bar{n}_{13} \left(\langle 1 | \langle 3 | \rho | 3 \rangle \langle 1 | - \frac{1}{2} \{\langle 3 \rangle | \rho \rangle \langle 3 | \rho \rangle \} \right), 
$$

where $\gamma_{1j} = 4\omega_1^2|d_{1j}|^2/(3\hbar c^3)$ is the spontaneous decay rate from the excited state $|1\rangle$ to the ground state $|j\rangle$ ($j = 2, 3$), $\bar{n}_{ij} = (e^{\beta\omega_{1j}} - 1)^{-1}$ is the mean number of thermal photons with $\beta = 1/k_BT$, and $\{A, B\}_+ \equiv AB + BA$ denotes the anti-commutator.

After eliminating the terms violating the energy conservation [32], which amounts to taking the rotating wave approximation (RWA), the energy hamiltonian for the light-matter interaction is simplified to

$$
H_{\text{int}} \simeq -\hbar\Omega_c \left(e^{-i\omega_c t}|1\rangle \langle 2| + e^{i\omega_c t}|2\rangle \langle 1|\right) \\
- \hbar\Omega_p \left(e^{-i\omega_p t}|1\rangle \langle 3| + e^{i\omega_p t}|3\rangle \langle 1|\right)
$$

where $\Omega_c = \zeta_c |\hat{e}_c \cdot \hat{d}_{12}|/\hbar$ and $\Omega_p = \zeta_p |\hat{e}_p \cdot \hat{d}_{13}|/\hbar$ corresponds to the driving (Rabi) frequencies. With $H_S$ in Eq. (S11), $H_{\text{int}}$ in Eq. (S13), and transformations into rotating frame which lead to $\rho_{ii} \rightarrow \tilde{\rho}_{ii}$, $\rho_{12} \rightarrow \rho_{12}e^{-i\omega_c t}$, $\rho_{13} \rightarrow \rho_{13}e^{-i\omega_p t}$, and $\rho_{23} \rightarrow \rho_{23}e^{-i(\omega_p - \omega_c)t}$ (see the next section Transformation to the rotating frame), the transformed matrix elements $\tilde{\rho}_{ij}$'s evolve with time as follows.

$$
\frac{d\tilde{\rho}_{22}}{dt} = \gamma(\bar{n}_{12} + 1)\tilde{\rho}_{11} + i\Omega_c\tilde{\rho}_{12} - i\Omega_c\tilde{\rho}_{21} - \gamma\bar{n}_{12}\tilde{\rho}_{22} \\
\frac{d\tilde{\rho}_{33}}{dt} = (\bar{n}_{13} + 1)\tilde{\rho}_{11} + i\Omega_p\tilde{\rho}_{13} - i\Omega_p\tilde{\rho}_{31} - \bar{n}_{13}\tilde{\rho}_{33} \\
\frac{d\tilde{\rho}_{12}}{dt} = -i\Omega_c\tilde{\rho}_{11} + \left[i\delta\omega_c - \frac{\gamma}{2}(\bar{n}_{12} + 1) - (\bar{n}_{13} + 1)\right]\tilde{\rho}_{12} \\
+ i\Omega_p\tilde{\rho}_{22} - i\Omega_p\tilde{\rho}_{32} \\
\frac{d\tilde{\rho}_{13}}{dt} = -i\Omega_p\tilde{\rho}_{11} + \left[i\delta\omega_p - \frac{\gamma}{2}(\bar{n}_{12} + 1) - (\bar{n}_{13} + 1)\right]\tilde{\rho}_{13} \\
+ i\Omega_p\tilde{\rho}_{23} - i\Omega_p\tilde{\rho}_{33} \\
\frac{d\tilde{\rho}_{23}}{dt} = i\Omega_p\tilde{\rho}_{11} - i\Omega_p\tilde{\rho}_{21} \\
+ \left[i(\delta\omega_p - \delta\omega_c) - \frac{(\gamma\bar{n}_{12} + \bar{n}_{13})}{2}\right]\tilde{\rho}_{23}, \quad (S14)
$$

where the equations are rescaled with $\gamma_{13}$, redefining the parameters and variables, such that $\tau = \gamma_{13}t$, $\gamma = \gamma_{12}/\gamma_{13}$. Hereafter, we implicitly assume that all the rates including $\Omega_c$, $\Omega_p$, $\delta\omega_c$, and $\delta\omega_p$ are those scaled with $\gamma_{13}$, e.g., $\Omega_c/\gamma_{13} \rightarrow \Omega_c$, $(\omega_c - \omega_{12})/\gamma_{13} \rightarrow \delta\omega_c$ and so forth. The equations for the remaining elements are obtained from the constraints $\sum_i \rho_{ii} = 1$ and $\rho_{ij} = \rho_{ij}^*$ for $i \neq j$.

D. Transformation to the rotating frame

The following operation transforms the state vector $|\psi\rangle$ in the rotating frame into the one in the stationary frame $|\psi\rangle$.

$$
|\psi\rangle = U(t) |\phi\rangle, \quad (S15)
$$

with $U(t) = e^{-i\omega_c t}|1\rangle \langle 1|-i(\omega_c - \omega_p) t/2)|2\rangle \langle 2|$. Then, the density matrix $\tilde{\rho} = |\phi\rangle \langle \phi| \rho U^\dagger (= U \rho U^\dagger)$.

The Baker-Campbell-Hausdorff formula,

$$
e^{sA}B e^{-sA} = B + \frac{s}{1!}[A, B] + \frac{s^2}{2!}[A, [A, B]] \cdots
$$

enables one to rewrite the diagonal elements as $\tilde{\rho}_{jj} = \rho_{jj}$, and the off-diagonal elements as $\tilde{\rho}_{12} = \rho_{12} e^{i\omega_c t}$, $\tilde{\rho}_{13} = \rho_{13} e^{i\omega_p t}$, and $\tilde{\rho}_{23} = \rho_{23} e^{i(\omega_p - \omega_c)t}$.

E. The method of cumulant generating function

The elements of the reduced density matrix in Fock-Liouville space, $\tilde{\rho} = (\tilde{\rho}_{11}, \tilde{\rho}_{12}, \tilde{\rho}_{21}, \tilde{\rho}_{13}, \tilde{\rho}_{23}, \tilde{\rho}_{31}, \tilde{\rho}_{32}, \tilde{\rho}_{33})^T$ obeys the Liouville equation

$$
\partial_\tau \tilde{\rho}(\tau) = L \tilde{\rho}(\tau), \quad (S16)
$$
where $\mathcal{L}$ is the Liouvillian super-operator expressed as $9 \times 9$ matrix, and formally evolves with time as $\dot{\hat{g}}(\tau) = e^{\epsilon \mathcal{L}} \hat{g}(0)$. The vector $\dot{\hat{g}}(\tau)$ is decomposed into $\hat{g}(n, \tau)$ which denotes the probability that $n$ net photons have been emitted to the environment, such that $\dot{\hat{g}}(\tau) = \sum_n \hat{g}(n, \tau)$. The $\hat{g}(n, \tau)$ satisfies the $n$-resolved master equation

$$\begin{align*}
\partial_\tau \hat{g}(n, \tau) &= \mathcal{L}_0 \hat{g}(n, \tau) \\
&+ \mathcal{L}_+ \hat{g}(n-1, \tau) + \mathcal{L}_- \hat{g}(n+1, \tau), \quad (S17)
\end{align*}$$

where the generators $\mathcal{L}_+$ and $\mathcal{L}_-$ are the off-diagonal element of the $\mathcal{L}$ corresponding to the emissions ($\mathcal{L}_{22,11}, \mathcal{L}_{33,11}$) and absorption ($\mathcal{L}_{11,22}, \mathcal{L}_{11,33}$), respectively, and $\mathcal{L}_0$ is for the rest of the elements. Discrete Laplace transform $\hat{\hat{g}}_z(\tau) = \sum_n \hat{g}(n, \tau) e^{inz}$, which satisfies $\lim_{z \to 0} \hat{\hat{g}}_z(\tau) = \hat{g}(\tau)$, casts Eq. (S17) into

$$\begin{align*}
\partial_\tau \hat{\hat{g}}_z(\tau) &= \mathcal{L}(z) \hat{\hat{g}}_z(\tau) \quad (S18)
\end{align*}$$

with $\mathcal{L}(z) \equiv \mathcal{L}_0 + iz \mathcal{L}_+ + ez \mathcal{L}_-$. Proceeding with $\delta \omega_{pc} = \delta \omega_p - \delta \omega_c$, $A_1 = \gamma (\bar{n}_{12} + 1) + (\bar{n}_{13} + 1)$, $A_2 = \gamma (2\bar{n}_{12} + 1)/2 - (\bar{n}_{13} + 1)/2$, $A_3 = \gamma (\bar{n}_{12} + 1)/2 - (2\bar{n}_{13} + 1)/2$, and $A_6 = (\gamma \bar{n}_{12} + \bar{n}_{13})/2$. Note that at $\tau \gg 1$, $\rho_z(\tau)$ is related with the largest eigenvalue $\lambda_0(z)$ of the modified super-operator $\mathcal{L}(z)$ as follows

$$\hat{\hat{g}}_z(\tau) = e^{\mathcal{L}(z)\tau} \hat{\hat{g}}_z(0) \sim e^{\lambda_0(z)\tau}. \quad (S20)$$

Note that $\lambda_0(z)$ is obtained from the characteristic polynomial of $\mathcal{L}(z)$

$$\det |\lambda(z)\mathcal{I} - \mathcal{L}(z)| = \sum_{n=0}^{9} a_n(z)\lambda^n(z) = 0. \quad (S21)$$

The cumulant generating function $\mathcal{G}(z, \tau)$, defined by

$$\mathcal{G}(z, \tau) = \ln \langle e^{inz} \rangle = \ln \sum_n P(n, \tau) e^{inz}, \quad (S22)$$

with $P(n, \tau) = \sum_{a=1,2,3} \hat{\rho}_{aa}(n, \tau)$ and $\sum_n P(n, \tau) = 1$, allows one to calculate the $k$-th cumulant

$$\langle \langle n^k \rangle \rangle(\tau) = \frac{\partial^k \mathcal{G}(z, \tau)}{\partial z^k} \bigg|_{z=0}. \quad (S23)$$

For $\tau \gg 1$, $\mathcal{G}(z, \tau) \sim \lambda_0(z)\tau$. Therefore, we obtain the steady-state relation

$$\lim_{\tau \to \infty} \frac{\langle \langle n^k \rangle \rangle(\tau)}{\tau} = \frac{\partial^k \lambda_0(z)}{\partial z^k} \bigg|_{z=0} \quad (S24)$$

The general expressions for the Fano factor and coherences at steady states are too lengthy to display; however, for the case of resonant control pulse ($\delta \omega_c = 0$) with $A \gg 1$ (or $\bar{n} \sim 0$), the coherences and Fano factor at steady state are simplified and written in a manageable form.

F. Coherences and Fano factor

The general expressions for the Fano factor and coherences at steady states are too lengthy to display; however, for the case of resonant control pulse ($\delta \omega_c = 0$) with $A \gg 1$ (or $\bar{n} \sim 0$), the coherences and Fano factor at steady state are simplified and written in a manageable form.
\[ \begin{align*}
\hat{p}_{11} &= \frac{4(\gamma + 1)\Omega_p^2 \delta \omega_p^2}{D} \\
\hat{p}_{22} &= \frac{\Omega_p^2 \{ (\gamma + 1)^2 + 4\Omega_p^2 \} \delta \omega_p^2 + 4(\Omega_p^2 + \Omega_p^2)(\Omega_p^2 + \gamma \Omega_p^2)\}}{D} \\
\hat{p}_{33} &= \frac{\Omega_p^2 \{ 4\delta \omega_p^2 + \{ (\gamma + 1)^2 - 8\Omega_p^2 + 4\Omega_p^2 \} \delta \omega_p^2 + 4(\Omega_p^2 + \Omega_p^2)(\Omega_p^2 + \gamma \Omega_p^2)\}}{D} \\
\hat{p}_{12}^R &= -\frac{4\Omega_p^2 \Omega_p^2 (\Omega_p^2 + \gamma \Omega_p^2) \delta \omega_p}{D} \\
\hat{p}_{12}^I &= \frac{2\gamma(\gamma + 1)\Omega_p^2 \delta \omega_p}{D} \\
\hat{p}_{13}^R &= \frac{4\Omega_p^2 \Omega_p^2 (\Omega_p^2 + \gamma \Omega_p^2 - \delta \omega_p^2) \delta \omega_p}{D} \\
\hat{p}_{13}^I &= \frac{2(\gamma + 1)\Omega_p^2 \Omega_p \delta \omega_p}{D} \\
\hat{p}_{23}^R &= \frac{4\Omega_p^2 \Omega_p^2 \{ \Omega_p^2 \delta \omega_p^2 - (\Omega_p^2 + \Omega_p^2)(\Omega_p^2 + \gamma \Omega_p^2)\} }{D} \\
\hat{p}_{23}^I &= -\frac{2(\gamma + 1)(\Omega_p^2 + \gamma \Omega_p^2) \Omega_p \delta \omega_p}{D}
\end{align*} \]

with \( D = 4\Omega_p^2 \delta \omega_p^4 + \{ \gamma(\gamma + 1)^2 \Omega_p^2 + (\gamma + 1)(\gamma + 1 + 8\Omega_p^2) \Omega_p^2 - 8\Omega_p^4 \} \delta \omega_p^2 + 4(\Omega_p^2 + \Omega_p^2)^2(\Omega_p^2 + \gamma \Omega_p^2). \)

The coefficients of the characteristic polynomial of \( \mathcal{L}(z) \) (Eq. (S21)) at \( z = 0 \), which are required for evaluating the quantities in Eq. (S26), are obtained as follows.

\[ \begin{align*}
a_0'(0) &= a_0''(0) = (\gamma + 1)^3 \Omega_p^2 \delta \omega_p^4, \\
a_1'(0) &= -(\gamma + 1) \left[ \Omega_p^2 \delta \omega_p^4 + \{ (\gamma + 1) \Omega_p^2 (\gamma + 8\Omega_p^2 + 1) - 8\Omega_p^4 + \gamma(\gamma + 1)^2 \Omega_p^2 \} (\delta \omega_p^2/4) + (\Omega_p^2 + \Omega_p^2)^2(\Omega_p^2 + \gamma \Omega_p^2) \right], \\
a_0'(0) &= (\gamma + 1) \left[ \gamma \Omega_p^2 \delta \omega_p^4 + \gamma \Omega_p^2 \{ (\gamma + 1)^2 - 8\Omega_p^2 \} (\gamma + 1) \Omega_p^2 (20\Omega_p^2 + \gamma + 1) \} (\delta \omega_p^2/4) + (\Omega_p^2 + \Omega_p^2)^2(\gamma \Omega_p^2 + \Omega_p^2) \right], \\
a_2(0) &= \frac{1}{16} \left[ -4 \{ 8(\gamma + 2)\Omega_p^2 + (\gamma + 1)^3 \} \delta \omega_p^4 + \{ 64(\gamma + 2)\Omega_p^4 - 8\Omega_p^2 (3(\gamma + 1)^2 + 4(6\gamma + 7)\Omega_p^2) - (\gamma + 1) \{ (\gamma + 1)^4 + 8(\gamma + 1)(\gamma + 1)\Omega_p^2 + 16\Omega_p^4 \} \} \delta \omega_p^2 - 4(\Omega_p^2 + \Omega_p^2) \{ 8(\gamma + 2)\Omega_p^4 + (\gamma + 1)\Omega_p^2 (5 + 1) + 24\Omega_p^4 \} + 8(\gamma + 1)\Omega_p^4 + (\gamma + 1)^2 (5\gamma + 1)\Omega_p^4 \right].
\end{align*} \]

It can be shown that

\[ \frac{2(a_0'(0))^2 a_2(0) - 2a_0''(0)a_1(0)a_1'(0)}{a_0''(0)(a_1(0))^2} = 2 \sum_{i<j} (\hat{p}_{ij}^R)^2 - 6 \sum_{i<j} (\hat{p}_{ij}^I)^2 + q(\Omega_p, \Omega_p, \delta \omega_p, \gamma) \]

where

\[ q(\Omega_p, \Omega_p, \delta \omega_p, \gamma) = \frac{2q_n}{q_d} \]

(S29)
with
\[ q_n = 16 \gamma \Omega_c^4 \delta \omega_p^8 - 8 \gamma \Omega_c^2 \left( 8 \Omega_c^4 - \{ (\gamma + 1)^2 + 2 \Omega_p^2 \} \Omega_c^2 + (\gamma + 1)^2 \Omega_p^2 \right) \delta \omega_p^6 + \left[ 96 \gamma \Omega_c^8 - 16 \gamma \Omega_c^6 \{ (\gamma + 1)^2 - (\gamma + 2) \Omega_p^2 \} + (\gamma + 1) \Omega_c^2 \{ (\gamma + 1)^3 + 4 (\gamma + 1) \Omega_p^2 - 32 \Omega_p^4 \} - 2 \gamma \Omega_c^2 \Omega_p^2 \{ (\gamma + 1)^2 + 6 (\gamma + 1)^2 \Omega_p^2 + 16 \Omega_p^4 \} + \gamma (\gamma + 1) 4 \Omega_p^4 \right] \delta \omega_p^4 - 4 \{ 16 \gamma \Omega_c^{16} - 2 \gamma \Omega_c^8 \{ (\gamma + 1)^2 - 2 (2 \gamma + 7) \Omega_p^2 \} + \Omega_c^2 \Omega_p^2 \{ - (\gamma - 3 + 3 \gamma + 2) + 4 (3 \gamma^2 + \gamma + 1) \Omega_p^2 \} + 2 \gamma \Omega_c^2 \Omega_p^2 \{ (\gamma + 1)^2 - 2 (\gamma + 1)^2 \Omega_p^2 - 1) - 2 \gamma (\gamma + 1) \Omega_p^4 \} \right] \delta \omega_p^2 + 16 (\Omega_c^2 + \Omega_p^2)^2 (\Omega_c^6 + \Omega_p^6) \delta \omega_p^6 + 2 \gamma (\gamma + 1)^2 \Omega_p^2 \Omega_p^2 + 2 \gamma \Omega_c^2 \Omega_p^4 + \Omega_p^6) \]
\[ q_d = \left[ 4 \Omega_c^2 \delta \omega_p^4 - \{ 8 \Omega_c^4 - (\gamma + 1)^2 \Omega_c^2 - 2 \gamma \Omega_c^2 \Omega_p^2 \right] \delta \omega_p^2 + 4 (\Omega_c^2 + \Omega_p^2)^2 (\Omega_c^6 + \Omega_p^6) \right] \] 

For \( \delta \omega_p = 0 \),
\[ q(\cdot) \big|_{\delta \omega_p = 0} = \frac{32 (\Omega_c^2 + \Omega_p^2)^2 (\Omega_c^6 + \Omega_p^6) \left( \gamma \Omega_c^6 + 2 \gamma \Omega_c^4 \Omega_p^2 + 2 \gamma \Omega_c^2 \Omega_p^4 + \Omega_p^6 \right)}{4 (\Omega_c^2 + \Omega_p^2)^2 (\Omega_c^6 + \Omega_p^6)} \]
\[ = \frac{2(\gamma \xi^6 + 2 \gamma \xi^4 + 2 \xi^2 + 1)}{\left( (\xi^2 + 1)(\xi^2 + \gamma)^2 \right)} \] 

(S30)

Whereas \( q = 0 \) in a coherently driven TLS [17], \( q(\Omega_c, \Omega_p, \delta \omega_p, \gamma) \neq 0 \) in the \( \Lambda \)-system contributes to the Fano factor of the transition current.

Although the expressions for \( a_0(z) \) and \( a_1(z) \) are lengthy and complicated, the total transition current \( (j) \) is straightforwardly decomposed into the two parts, \( (j) = (j)_{12} + (j)_{13} \) with
\[ (j)_{12} = \gamma (\tilde{n}_{12} + 1) \rho_{11}^{ss} - \gamma \tilde{n}_{12} \rho_{22}^{ss} = 2 \Omega_c \rho_{11} \] (S31)

and
\[ (j)_{13} = (\tilde{n}_{13} + 1) \rho_{11}^{ss} - \tilde{n}_{13} \rho_{22}^{ss} = 2 \Omega_p \rho_{13} \] (S32)

The first equalities of Eqs. [S31] and [S32] are consistent with the definition of reaction current between two discrete states in classical Markov jump system, and this can also be related with the imaginary part of coherence between the two quantum states, which is called \textit{current-coherence relation} [19].

G. Relation between \( v_g \) and \( \mathcal{F} \)

For the case of resonant control pulse \( (\delta \omega_c = 0) \) with \( A \gg 1 \) (or \( \tilde{n} \sim 0 \)), when \( (\partial \rho_{13}^{R} / \partial \omega_p)_{\delta \omega_p = 0} = \Omega_p^{-1} \xi^2 / (\xi^2 + 1)^2 \) is inserted to Eq. (6), we get an expression of the group velocity in terms of \( \xi \).
\[ v_g = \frac{c}{1 + \frac{N \xi^2}{(\xi^2 + 1)^2}} \] (S33)

with \( N \equiv 2 \pi N_i \delta \omega_p / \Omega_p \).

For the two-photon resonance \( (\delta \omega_c = \delta \omega_p = 0) \), the Fano factor is contributed only by the real part of coherence between \( |2 \rangle \) and \( |3 \rangle \) \( (p_{23}^{R} \neq 0) \) while others vanish \( (p_{12}^{R} = p_{12}^{I} = p_{13}^{R} = p_{13}^{I} = p_{23}^{R} = 0) \), which simplifies \( \mathcal{F} \) into
\[ \mathcal{F} = 1 + 2 \left( p_{23}^{R} \right)^2 \bigg|_{\delta \omega_p = 0} + q(\xi, \gamma) \] (S34)

with
\[ \left( p_{23}^{R} \right)^2 \bigg|_{\delta \omega_p = 0} = \frac{\xi^2}{(\xi^2 + 1)^2} \]

\[ q(\xi, \gamma) \bigg|_{\delta \omega_p = 0} = \frac{2 \xi^6 \gamma + 2 \xi^4 \gamma + 2 \xi^2 + 1}{(\xi^2 + 1)^2 (\xi^2 + \gamma)} \]. (S35)

Insertion of Eq. (S35) into Eq. (S34) yields Eq. (8).

H. Laser power and Rabi frequency

For a plane wave the average intensity can be expressed as
\[ \left\langle I_\alpha \right\rangle = \frac{c}{8 \pi} \xi^2 \alpha \in c, p. \] (S36)

Now by considering the polarization of incident light parallel to the dipole, we can write \( \zeta_\alpha = \hbar \Omega_\alpha / |d_{ij}| \) which yields
\[ \left\langle I_\alpha \right\rangle = \frac{c h^2 \Omega_\alpha^2}{8 \pi |d_{ij}|^2}, \] (S37)
and from the spontaneous decay we know \( \left( \frac{\hbar}{|d_{ij}|} \right)^2 = 16\pi^2 \hbar/3\gamma_{ij}\lambda_\alpha^3 \). Thus, we obtain the relationship between the average intensity of the laser pulse \( \langle I_\alpha \rangle \), reported in the literature, and other quantities,

\[
\langle I_\alpha \rangle = \frac{2\pi\hbar c \Omega_\alpha^2}{3\gamma_{ij}\lambda_\alpha^3}.
\] (S38)