Resistive-wall impedances of a thin non-evaporable getter coating on a conductive chamber

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The resistive-wall impedances of a thin non-evaporable getter (NEG) with constant conductivity coating on a copper chamber are studied in both longitudinal and transverse directions. The copper chamber has a finite thickness and is surrounded by air. As the frequency increases, wake fields see mostly the air first and then the copper chamber next, and then finally the NEG coating. Both longitudinal and transverse impedances slowly undergo transitions from the resistive-wall impedances of the copper-only chamber to those of the NEG-only chamber over a 0.1–100 GHz frequency range. They start to deviate from the conventional impedance lines for the NEG-only chamber at ∼100 GHz. They increase first to ∼1 THz and then decrease rapidly as a function of the frequency. Numerical examples of the resistive-wall impedances are presented up to ∼100 THz using similar parameters to those of the SLS-II (Swiss Light Source) upgrade. We briefly discuss the characteristic of a loss factor for the chamber with NEG coating.

1. Introduction

The resistive-wall impedances [1] (both longitudinal and transverse) of an infinitely long multi-layered chamber have been extensively studied by many researchers [2–15]. It is basically a two-dimensional problem, and it can be solved by the field-matching technique [16]. In synchrotron light sources [17,18], the bunch length varies from less than 1 mm to a few cm, and we need to investigate the behavior of impedances over a wide range of frequencies, even beyond 100 GHz.

In this paper we study the longitudinal and transverse impedances of a copper chamber coated with a non-evaporable getter (NEG) and the effects of the NEG thickness over a wide range of frequencies from very low frequency (∼Hz) to very high frequency (∼100 THz). Here, we assume for simplicity that all physical parameters such as the conductivity have no frequency dependence. For example, the relaxation time of the conductivity is ignored. For more accurate calculations, the frequency dependencies of the physical parameters would need to be measured by experiment. However, this is outside the scope of this paper.

In most parts of this paper we deal with the resistive wall impedance of a thin coating on a conductive chamber with finite conductivity and thickness. A thin NEG coating on vacuum chambers has been successfully used to achieve ultra-high vacuum in many accelerators, such as CERN LHC [19], ESRF [20], etc.
Some numerical examples are given in this paper using beam and machine parameters similar to those of the SLS-II (Swiss Light Source) upgrade [17,18]. We hope that the analysis will provide some clues on how to optimize the thickness of the NEG coating for similar electron machines.

In Sect. 2, we first try to simulate the resistive-wall impedance of a copper-only chamber, and discuss the limits and reliability of the method. In Sect. 3, theoretical approaches are applied to the case of a thin coating on the copper chamber up to THz frequencies, where the impedance has a peak in the THz range. In Sect. 4, we present simple formulae for the longitudinal and transverse impedances for the case of a thin coating on a conductive chamber, where the contribution from the induction term of the Maxwell–Ampère equation is taken into account. In Sect. 5, we briefly discuss the feature of a loss factor for the resistive-wall impedance with a NEG coating. The paper concludes in Sect. 6.

The formulae for the longitudinal and transverse impedances of a thin coating on a chamber with a finite conductivity and thickness are derived in Appendix A using the field-matching technique. The longitudinal impedance is approximated in Appendix B for an extremely thin NEG case. General formulae for longitudinal and transverse impedances for a perfectly conductive chamber (space charge impedances [21]) are derived in Appendix C.

2. Simulation approach to obtain resistive-wall impedances

When the conductivity of a chamber is much higher than $\epsilon_0\omega$, where $\epsilon_0$ is the dielectric constant of vacuum and $\omega$ is the angular frequency, the surface impedance technique [22], where the induction term of the Maxwell–Ampère equation is neglected, is applied to most simulations to estimate resistive-wall impedances. The three-dimensional simulation code CST STUDIO [23] has implemented a material with high conductivity by introducing “Lossy Metal.” The Wake Solver of the code may estimate the impedances by letting a beam pass through the chamber.

Figure 1 shows the simulation results for a chamber with inner radius ($d = 10$ mm and outer radius $a = 11$ mm whose conductivity is $\sigma_{3c} = 5.9 \times 10^7$ S m$^{-1}$. The red and the blue lines show the real and imaginary parts of the impedance, respectively. The results demonstrate that the method is not appropriate to estimate the resistive wall impedances due to the oscillation of the real part of the impedance.

![Fig. 1. Simulation results of the longitudinal impedance per unit length by Wake Solver for $\sigma_{3c} = 5.9 \times 10^7$ S m$^{-1}$, $d = 10$ mm, and $a = 11$ mm. The red and blue lines show the real and imaginary parts of the impedance, respectively.](image_url)
because it must not be negative from a physical point of view (a beam must lose its energy passing through the chamber).

From an experimental point of view, one way to measure the resistive-wall impedance is to stretch wires inside the chamber. Thus, it may be possible to obtain the resistive impedances by simulating this method [24]. Off-line measurements of the longitudinal and transverse impedances of a device under test (DUT) are typically done by stretching single or twin wires inside the DUT (see Fig. 2). By connecting both ends of the DUT to a network analyzer, the measured $S$-parameters are converted into impedances by using the standard log formulae [24]:

$$Z_L = -2Z_{cc} \log \frac{S_{21}^{DUT}}{S_{21}^{(ref)}},$$

$$Z_T = -\frac{2cZ_{dd}}{\omega \Delta^2} \log \frac{S_{dd21}^{DUT}}{S_{dd21}^{(ref)}},$$

where $\Delta$ is the interval between the twin wires, $Z_{cc}$ and $Z_{dd}$ are the characteristic impedances for the common and the differential modes, respectively, $S_{21}^{DUT}$, $S_{21}^{(ref)}$, $S_{dd21}^{DUT}$, and $S_{dd21}^{(ref)}$ are the transmission coefficients for the resistive chamber (DUT) and those for the perfectly conductive chamber, respectively. The subscripts cc and dd denote the common and differential modes, respectively.

Here, let us calculate the impedances by simulating the measurement setup. The Micro-Wave Solver in CST STUDIO [23] calculates $S$-parameters for the coaxial structure composed of the wires and the resistive chamber by adopting “Lossy Metal” as the ingredient of the chamber. The Micro-Wave Solver utilizes the technique of the surface impedance [22] to obtain the $S$-parameters, even if a resistive material with high conductivity is present in the simulation setup.

Figures 3 and 4 show the simulation results for a chamber with inner radius $d = 10\,\text{mm}$ and outer radius $a = 11\,\text{mm}$. In these simulations, we use a copper (whose conductivity is $\sigma_c = 5.9 \times 10^7\,\text{S}\,\text{m}^{-1}$) chamber as the resistive material. The black lines show the longitudinal and transverse impedances obtained by using Eqs. (1) and (2), respectively. The brown lines denote the theoretical impedances using the conventional formulae [1] given by

$$\frac{Z_L}{C} = Z_0 \sqrt{\frac{2\omega}{cZ_0\sigma_c}} \frac{(1 + f)}{4\pi d}.$$
Fig. 3. Simulation results (black) of the longitudinal impedance per unit length for the setup shown on the left of Fig. 2 with wire radius $\rho_w = 10\,\mu m$. The left and right figures show the real and imaginary parts of the impedance, respectively. The brown lines denote Eq. (3) with $\sigma_3c = 5.9 \times 10^7\,S\,m^{-1}$ for reference.

\[ Z_T = \frac{\rho}{C} \sqrt{\frac{Z_0 \sigma_3c}{2c}} \frac{(1 + j)}{\pi \sigma_3c \omega d^3}, \]  

where $j$ is the imaginary unit, $Z_0 = 120\pi\,\Omega$ is the impedance of free space, $C$ is the chamber length, $c$ is the velocity of light, and $\sigma_3c = 5.9 \times 10^7\,S\,m^{-1}$, for reference.

The results demonstrate that the simulation technique by using the Micro-Wave Solver reproduces reasonably well the conventional formula of the resistive-wall impedance, except below around 1 GHz. The upper limit of the frequency of the impedance evaluated by the wire method is determined by the cut-off frequency of the waveguide structure, which is about 10 GHz in this setup. The lower limit of the frequency is given by $\delta \lesssim (a - d)$, which is roughly about $f \gtrsim 4\,kHz$, at which the skin depth ($\delta$) exceeds the thickness of the chamber ($a - d$). Theoretically, this is a limit of the application of the surface impedance theory. But, the simulation results shown in Figs. 3 and 4 seem to deviate from the conventional formulae (3) and (4) at less than around 1 GHz, which is much higher than the frequency ($\sim kHz$).
3. Overall behavior of the resistive wall impedance of a thin NEG coating on a conductive chamber

The formulae for the longitudinal and transverse impedances of a thin coating with a finite conductivity ($\sigma_{2c}$) and thickness ($d - b$) on a chamber with a finite conductivity ($\sigma_{3c}$) and thickness ($a - d$) are derived in Appendix A using the field-matching technique. In this section, we present some numerical examples using parameters similar to those of the vacuum chambers of the SLS-II upgrade plan [17,18]: $d = 10$ mm, $\sigma_{3c} = 5.952 \times 10^7$ S m$^{-1}$ for the copper chamber, and $\sigma_{2c} = 1.098 \times 10^6$ S m$^{-1}$ for the NEG coating [14]. From now on, we deal only with a relativistic beam, for simplicity. It is noticeable that all the physical parameters such as the conductivity are assumed to have no frequency dependence.

The black lines in Fig. 5 show the exact solutions of the real parts of the longitudinal (left) and transverse (right) impedances per unit length for a copper thickness of $a - d = 1$ mm and NEG thickness of $d - b = 2$ μm, respectively. The brown lines in both figures show the real part of the resistive-wall impedance of the copper-only chamber using the conventional formulae given by Eqs. (3) and (4). On the other hand, the purple lines show the real part of the resistive-wall impedance of the infinitely thick NEG coating by replacing $\sigma_{3c}$ with $\sigma_{2c}$ in the formulae.

We can see clearly that both the longitudinal and transverse impedances slowly undergo transitions from the resistive-wall impedances of the copper-only chamber to those of the NEG-only chamber over a $1$–$100$ GHz frequency range for a NEG thickness of $d - b = 2$ μm. Both longitudinal and transverse impedances start to deviate from the conventional impedance lines for the NEG-only chamber at $\sim 100$ GHz. They increase first to $\sim 1$ THz and then damp rapidly as a function of the frequency.

Now let us investigate the effects of the thickness of both the copper chamber and the NEG coating. We use the outer radius of the copper chamber $a$ (that gives the thickness of the copper chamber) and the inner radius of the NEG coating $b$ (that gives the thickness of the NEG coating) as free parameters.

Figure 6 shows the real parts of the longitudinal impedances for a copper thickness of $a - b = 1$ mm (left) and $a - b = 2$ mm (right). The green, black, blue, and red lines show the real parts of the

![Fig. 5. The real parts of the resistive-wall impedances per unit length for the 1 mm-thick copper chamber and the 2 μm-thick NEG coating (black). The right and left figures show the longitudinal and transverse impedances, respectively. The brown lines show the impedance of the copper-only chamber using the conventional formulae (3) and (4). The purple lines show the impedance of the NEG-only chamber using the conventional formulae (3) and (4).](https://academic.oup.com/ptep/article-abstract/2017/12/123G01/4746618)
Fig. 6. Dependence of the real part of the longitudinal impedance per unit length on the thickness of the NEG coating for two different copper chamber thicknesses (left: \(a - b = 1\) mm, right: \(a - b = 2\) mm). The green, black, blue, and red lines show the cases for 1 μm, 2 μm, 5 μm, and 10 μm, respectively.

longitudinal impedances when the thickness of the NEG coating is 1 μm, 2 μm, 5 μm, and 10 μm, respectively. The slow transitions from the resistive-wall impedances of the copper-only chamber to those of the NEG-only chamber can be found over a 0.1–100 GHz range in both left and right figures. The only difference between the two figures appears at very low frequencies (less than a few kHz) where the wake fields leak out of the copper chamber to the outside air. This difference may not be too important in ordinal light source rings where the revolution frequency is of the order of MHz.

In the high-frequency region where the skin depth is much smaller than the NEG thickness, all the lines converge to the same result. The frequencies where the skin depth becomes comparable to the thickness of the NEG coating are 230 GHz for 1 μm, 57.6 GHz for 2 μm, 9.2 GHz for 5 μm, and 2.3 GHz for 10 μm, respectively. We can see in the figures that the transitions of impedances from the copper-only case to the NEG-only case are almost completed at around those frequencies.

Figure 7 shows the real parts of the transverse impedances for a copper thickness of \(a - b = 1\) mm (left) and \(a - b = 2\) mm (right), respectively. The green, black, blue, and red lines show the real parts of the transverse impedances when the thickness of the NEG coating is 1 μm, 2 μm, 5 μm, and 10 μm, respectively. The brown and purple lines show the impedances of the copper-only chamber and the NEG-only chamber using the conventional formula (4), respectively. The slow transitions from the resistive-wall impedances of the copper-only chamber to those of the NEG-only chamber can be observed over a 0.1–100 GHz range in these transverse impedance cases as well.

Now, let us see more closely the impedances in the frequency range up to 100 GHz on a linear scale. Figures 8 and 9 show the real parts of the longitudinal and transverse impedances when the thickness of the copper chamber is 1 mm and the thickness of the NEG coating is 1 μm (green), 2 μm (black), 5 μm (blue), and 10 μm (red), respectively. The brown and purple lines show the real part of the impedances when the formulae (3) and (4) are used for the copper-only case and the NEG-only case, respectively. It can be seen that the real part of the longitudinal impedance increases almost linearly to the frequency when the thickness of the NEG coating is small (a few μm). Also, the impedance shape almost saturates at around 5 μm as a function of the NEG thickness. In other words, the real part of the longitudinal impedance is almost unchanged even if the thickness of the NEG coating is increased above 5 μm.
Fig. 7. Dependence of the real part of the transverse impedance per unit length on the thickness of the NEG coating for two different copper chamber thicknesses (left: $a - b = 1$ mm, right: $a - b = 2$ mm). The green, black, blue, and red lines show the cases for 1 μm, 2 μm, 5 μm, and 10 μm, respectively. The brown and purple lines show the impedances of the copper-only chamber and the NEG-only chamber using the conventional formula (4), respectively.

Fig. 8. Dependence of the real part of the longitudinal impedance per unit length on the thickness of the NEG coating for the 1 mm-thick copper chamber, in a different frequency range from Fig. 6 and depicted on a linear scale. The green, black, blue, and red lines show the real part of the longitudinal impedances for 1 μm, 2 μm, 5 μm, and 10 μm, respectively. The brown and purple lines show the impedances of the copper-only chamber and the NEG-only chamber using the conventional formula (3), respectively.

The transverse impedance behaves very differently from the longitudinal one. As the left panel of Figs. 7 and 9 show, it varies widely as a function of the thickness of the NEG coating. Above around 1 THz, the transverse impedance looks saturated as a function of the NEG thickness, but below that frequency, the transverse impedance increases significantly for a thicker NEG coating. When the NEG coating is very thin (such as 1 μm), the effect of the NEG coating may be unnoticeable and the impedance due to the NEG coating becomes important beyond 20 GHz. From a few kHz to 10 MHz, the wake fields see mostly the copper chamber so that the transverse impedance has a little dependence on the thickness of the NEG coating.

To see the reliability of the result, we compare our results with those of the ImpedanceWake2D code [15] for the same set of parameters. The ImpedanceWake2D code generally calculates the
Fig. 9. Dependence of the real part of the transverse impedance per unit length on the thickness of the NEG coating for the 1 mm-thick copper chamber, in a different frequency range from Fig. 7 and depicted on a linear scale. The green, black, blue, and red lines show the real part of the transverse impedances for 1 μm, 2 μm, 5 μm, and 10 μm, respectively. The brown and purple lines show the impedances of the copper-only chamber and the NEG-only chamber using the conventional formula (4), respectively.

Fig. 10. Comparisons of the longitudinal impedances per unit length between the present theory (red) and the ImpedanceWake2D code [15] (blue). The left and right figures show the real and imaginary parts of the impedance, respectively. The purple line shows the impedance of the NEG-only chamber using the conventional formula (3).

longitudinal and transverse impedances by using sophisticated matrix formalism for multi-layered chambers [15]. Since the ImpedanceWake2D code includes the beam energy (Lorentz γ ) dependence of impedances, we restore it in our formulae for the comparisons (γ = 7460.52). Figures 10 and 11 show comparisons of the longitudinal and transverse impedances, respectively. The red lines show the present theoretical results. The blue lines show the results by ImpedanceWake2D [15]. Both results show excellent agreement.

Finally, let us see some simulation results for the same set of parameters. We applied the simulation technique explained in the previous section to the calculations. The “Lossy Metal” is chosen to be the copper, while the “Normal Metal” is the NEG. Otherwise, the wake fields cannot see the effect of the copper chamber. The results of the longitudinal and transverse impedances are shown in Figs. 12 and 13, respectively. The left and right figures show the real and imaginary parts of the impedances. The black and blue lines show the simulation and the theoretical results, respectively. The brown
Fig. 11. Comparisons of the transverse impedances per unit length between the present theory (red) and the ImpedanceWake2D code [15] (blue). The left and right figures show the real and imaginary parts of the impedance, respectively. The purple line shows the impedance of the NEG-only chamber using the conventional formula (4).

Fig. 12. Comparisons of the longitudinal impedances per unit length between the simulation results (black) and the present theory (blue). The left and right figures show the real and imaginary parts of the impedance, respectively. The brown line shows the impedance of the copper-only chamber by using the conventional formula (3).

lines show the impedances of the copper-only chamber by using the conventional formulae (3) or (4), for reference. As in the previous result for a single-layered resistive chamber shown in Figs. 3 and 4, the discrepancy between the simulation (black) and the theoretical (blue) results is also clearly identified below around 1 GHz in Figs. 12 and 13. However, from 1 GHz to 10 GHz, the agreement between them is rather good. Moreover, we can clearly see that the impedances rise apart from the impedance of the copper-only chamber (brown) in the frequency region. Notwithstanding these interesting findings about the simulation results, the theoretical approaches turn out to be more accurate from a practical point of view.

4. Simple formulae for the resistive-wall impedances

For the case of a chamber with an extremely thin NEG coating \((d - b \ll b\)\), the longitudinal impedance for a relativistic beam can be simplified by a single formula (see Appendix B). It is
expressed as

\[
\frac{Z_T}{C} \approx \frac{1}{2\pi b(\sigma_3 c + j\omega\epsilon_0)} \left[ \left( \frac{I_0(\tilde{v}_2 d)K_0(\tilde{v}_3 a) - I_0(\tilde{v}_3 a)K_0(\tilde{v}_2 d)}{\tilde{v}_2 d I_1(\tilde{v}_2 d)K_0(\tilde{v}_3 a) + I_0(\tilde{v}_3 a)K_1(\tilde{v}_2 d)} \right) + \frac{\tilde{v}_3 (\sigma_3 c + j\omega\epsilon_0)(I_0(\tilde{v}_2 d)K_0(\tilde{v}_3 a) - I_0(\tilde{v}_3 a)K_0(\tilde{v}_2 d))}{\tilde{v}_3 (\sigma_3 c + j\omega\epsilon_0) + I_0(\tilde{v}_2 d)K_0(\tilde{v}_3 a) + I_0(\tilde{v}_3 a)K_0(\tilde{v}_2 d)} \right]
\]

where \( I_n(z) \) and \( K_n(z) \) are the modified Bessel functions \([25]\),

\[
\tilde{v}_2 = \sqrt{jkZ_0\sigma_2 c},
\]

\[
\tilde{v}_3 = \sqrt{jkZ_0\sigma_3 c},
\]

and \( k = \omega/c \). Here, it is noticeable that we do not assume that the conductivities \( (\sigma_2 c, \sigma_3 c) \) are much larger than \( \omega\epsilon_0 \) in order to retain the effect due to the induction term, which is typically neglected in the surface impedance theory \([22]\).

For the transverse impedance, the formula is divided by the frequency region for simplification. For \( \delta = \sqrt{2/\mu_0\sigma_3 c \omega} > (a - b) \), it is approximated as

\[
\frac{Z_T}{C} \approx \text{numerator/denominator},
\]

where

\[
\text{numerator} = jZ_0[I_1(\tilde{v}_3 a)K_1(\tilde{v}_3 b) - I_1(\tilde{v}_3 b)K_1(\tilde{v}_3 a)] + a\tilde{v}_3[I'_1(\tilde{v}_3 a)K_1(\tilde{v}_3 b) - I'_1(\tilde{v}_3 b)K'_1(\tilde{v}_3 a)],
\]

\[
\text{denominator} = b^2 \pi [ab\tilde{v}_3^2(-I'_1(\tilde{v}_3 a)K'_1(\tilde{v}_3 b) + I'_1(\tilde{v}_3 b)K'_1(\tilde{v}_3 a)] + a\tilde{v}_3[I'_1(\tilde{v}_3 a)K_1(\tilde{v}_3 b) - I_1(\tilde{v}_3 b)K'_1(\tilde{v}_3 a)]
\]

Fig. 13. Comparisons of the transverse impedances per unit length between the simulation results (black) and the present theory (blue). The left and right figures show the real and imaginary parts of the impedance, respectively. The brown line shows the impedance of the copper-only chamber by using the conventional formula (4).
for a thin coating ($d - b \ll a - b$), and the conductivity $\sigma_3c$ is reasonably assumed to be much larger than $\omega e_0$ at the frequency, where the prime in the modified Bessel functions means the differential by their argument.

At high frequency ($\sqrt{jkZ_0\sigma_3}\omega \gg 1$ and $\sqrt{jkZ_0\sigma_2}\omega \gg 1$), we obtain

$$
\frac{Z_T}{C} \simeq -\frac{4Z_0b^2\nu_3^2\nu_2^4(k-jZ_0\sigma_2c)}{b^2\pi[A\cosh^2(\nu_2(d-b)) + B\sinh(2(\nu_2(d-b))) + C\sinh^2(\nu_2(d-b))]} \times \left[ \cosh(\nu_2(d-b)) + \frac{\nu_2(k-jZ_0\sigma_2c)}{\nu_3(k-jZ_0\sigma_2c)} \sinh(2(\nu_2(d-b))) + \frac{(k(\nu_2^2 - \nu_3^2)^2 + b^2\nu_4^2\nu_3^2(k-jZ_0\sigma_3c))\sinh^2(\nu_2(d-b))}{2b^2\nu_3^2\nu_4^2(k-jZ_0\sigma_2c)} \right],
$$

(11)

for $\sqrt{2/\mu_0\sigma_2c}\omega > (d - b)$, otherwise,

$$
\frac{Z_T}{C} \simeq -\frac{2jZ_0\nu_2^2\nu_1^2(\nu_2b)}{b^2\pi(-jZ_0\sigma_2c + 2k)(b\nu_2kK_0(\nu_2b) + \frac{(-2\nu_2^3 + (4b^2\nu_3^2)k^2)}{(-jZ_0\sigma_2c + 2k)}K_1(\nu_2b) + b\nu_2kK_2(\nu_2b))},
$$

(12)

where

\begin{align}
A &= -jb^2\nu_2^2\nu_3^2(k-jZ_0\sigma_2c)(8k^2 + 8bk^2\nu_3 - 4\nu_2^2 + b^2k^2\nu_3^2 - 4jbkZ_0\nu_2\nu_3\sigma_3c), \\
B &= \frac{b\nu_2}{2}[8jk^3(\nu_2^3 - \nu_3^2)^2 - 8jb^2k^3\nu_3^2(\nu_2^2 + \nu_2^3)] - 2jkb^2\nu_3^2(8k^2 - 4\nu_2^2 + b^2k^2\nu_3^2) + 4jk^2Z_0\nu_2^2\nu_3^2\nu_3^2(3\nu_3^2\sigma_3c + 2\nu_3^2\sigma_3c + \nu_2^2\sigma_2c) + 4jb^2kZ_0^2\nu_2^2\nu_3^2\nu_3^2(\nu_2^2\sigma_2c + \nu_2^2\sigma_3c) - 8jk^2Z_0\nu_2^2\nu_3^2(\sigma_2c + \sigma_3c) + 4bZ_0\nu_2^2\nu_3^4(\sigma_2c + \sigma_3c) - b^3k^2Z_0\nu_2^2\nu_3^4(\sigma_3c + \sigma_2c)], \\
C &= 8jk^3\nu_2^2 - 4jk\nu_2^2 + jb^2k^3\nu_3^2 - 16jk^3\nu_2^2\nu_3^2 + 8jk\nu_2^2\nu_3^2 - 10jbk^2\nu_2^2\nu_3^2 + 8jbk^2\nu_3^2(8k^2 - 4\nu_2^2 + b^2k^2\nu_3^2) - 8jb^2k^3\nu_3^2\nu_3^2 + 4jb^2kZ_0^2\nu_2^2\nu_3^4\sigma_3c - 8b^2k^2Z_0\nu_2^2\nu_3^2(\sigma_2c - \sigma_3c) - 8b^2k^2Z_0\nu_2^2\nu_3^4\sigma_3c - 4b^2Z_0\nu_2^2\nu_3^4\sigma_3c - b^3k^2Z_0\nu_2^2\nu_3^2(\sigma_2c + \sigma_3c).
\end{align}

(14)

(15)

Equation (8) is derived by assuming that the chamber is made only of copper, whose thickness is finite, and approximated for low frequency. Equation (11) is obtained by assuming the copper thickness is infinite and a thin coating ($b \simeq d$), where the Bessel functions are approximated by the hyperbolic functions by using the conditions ($\sqrt{jkZ_0\sigma_3c}\omega \gg 1$ and $\sqrt{jkZ_0\sigma_2c}\omega \gg 1$). Equation (12) is obtained by considering that the thickness of the NEG is infinite compared to the skin depth at high frequency, where the effect of the induction term is restored.

Figures 14 and 15 show the longitudinal and transverse impedances for a copper chamber thickness of 2 mm and a NEG coating thickness of 10 μm. The solid and dotted lines show the exact and approximated solutions given by Eqs. (5)–(15), respectively. They are in excellent agreement for
both the real and imaginary parts of the impedance. The formulae (5), (8), (11), and (12) have no convergence problem that some of the other methods may suffer [13].

5. Loss factor

Once the longitudinal impedance is calculated, a loss factor [16] is calculated by using

$$\kappa = \frac{1}{\pi} \int_0^\infty d\omega \text{Im}[Z_L(\omega)] \exp \left(-\frac{\omega^2}{c^2} \sigma_s^2 \right),$$

(16)

where we assume a Gaussian beam profile,

$$\rho_G = \exp \left(-\frac{z^2}{2\sigma_s^2} \right) \frac{1}{\sqrt{2\pi} \sigma_s},$$

(17)

in the longitudinal direction. When the RMS bunch length $\sigma_s$ is 3 mm, the loss factor per unit length is 0.0122 V (pCm)$^{-1}$ and 0.0741 V (pCm)$^{-1}$ for a NEG coating thickness of 1 μm and 10 μm, respectively. If we assume an SLS-II-like ring (circumference = 288 m) and use a total beam current

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**Fig. 14.** The longitudinal impedance per unit length for a 2 mm-thick copper chamber and a 10 μm-thick NEG coating. The solid and dotted lines show the exact and approximated solutions using the formula (5), respectively.

**Fig. 15.** The transverse impedance per unit length for a 2 mm-thick copper chamber and a 10 μm-thick NEG coating. The solid (red) and dotted lines show the exact and approximated solutions using the formula, respectively. The blue, green, and yellow lines are obtained by using Eqs. (8), (11), and (12), respectively.
of 400 mA and single bunch current of 1 mA, the power deposition on the chamber per unit length is about 30 W m\(^{-1}\) for a NEG thickness of 10 μm. It may not be significant power deposition that requires a special cooling system for the copper chamber.

The estimation significantly depends on the bunch length \(σ_s\). However, when we assume an infinitesimal beam, \(σ_s = 0\), we find that the loss factor does not depend on the conductivity of the material in most cases.

This can be proved as follows. First, let us modify Eq. (16) as

\[
κ = \Im \left[ \frac{1}{\pi} \int_0^\infty \frac{dω}{t} \int_{-\infty}^{\infty} d\omega' \frac{e^{-jωt}W_L(t)}{e^{j\omega't}W_L(0)} \right] = 2W_L(0), \tag{18}
\]

where \(W_L(t)\) is the longitudinal wake function. For an infinitely thick chamber with thin TiN coating, the impedance can be approximated as

\[
\frac{Z_L}{C} \approx \frac{1}{2\pi bσ_c} \left[ -\frac{1}{\sqrt{jkZ_0σ_2c}} + \frac{σ_c\sinh\sqrt{jkZ_0σ_2c}(b-d)}{\sqrt{bd}\sqrt{jkZ_0σ_3c}} \right] \tag{19}
\]

The wake function is calculated as

\[
W_L(t) = \int_{-\infty}^{\infty} \frac{dω}{2\pi} e^{j\zeta_3t} \frac{cZ_0e^{-j\frac{Z_0}{b^2}\sqrt{e^{j\frac{Z_0}{b^2}\omega}}} - e^{j\frac{Z_0}{b^2}(e^{j\frac{Z_0}{b^2}\omega})}}{2πb^2 \left(F_1(ω) - e^{j\frac{Z_0}{b^2}(e^{j\frac{Z_0}{b^2}\omega})} \right)} + \int_{-\infty}^{∞} \frac{dω}{2π} e^{j\zeta_3t} \frac{cZ_0e^{j\frac{Z_0}{b^2}\sqrt{e^{j\frac{Z_0}{b^2}\omega}}} - e^{j\frac{Z_0}{b^2}(e^{j\frac{Z_0}{b^2}\omega})}}{2πb^2 \left(F_2(ω) - e^{j\frac{Z_0}{b^2}(e^{j\frac{Z_0}{b^2}\omega})} \right)}, \tag{20}
\]

where \(ω\) is scaled to be a dimensionless variable and

\[
\zeta_3 = c \left(\frac{Z_0σ_3c}{b^2}\right)^{1/3}, \tag{21}
\]

\[
F_1(ω) = \frac{\cosh \left[ (d-b)e^{-j\frac{Z_0}{b^2}\sqrt{Z_0σ_2c}(Z_0σ_3c)\frac{1}{2}\sqrt{e^{j\frac{Z_0}{b^2}\omega}}} \right]}{1 + \sqrt{\frac{σ_2c}{σ_3c}} \sinh \left[ (d-b)e^{-j\frac{Z_0}{b^2}\sqrt{Z_0σ_2c}(Z_0σ_3c)\frac{1}{2}\sqrt{e^{j\frac{Z_0}{b^2}\omega}}} \right]}, \tag{22}
\]

\[
F_2(ω) = \frac{\cosh \left[ (d-b)e^{j\frac{Z_0}{b^2}\sqrt{Z_0σ_2c}(Z_0σ_3c)\frac{1}{2}\sqrt{e^{j\frac{Z_0}{b^2}\omega}}} \right]}{1 + \sqrt{\frac{σ_3c}{σ_2c}} \sinh \left[ (d-b)e^{j\frac{Z_0}{b^2}\sqrt{Z_0σ_2c}(Z_0σ_3c)\frac{1}{2}\sqrt{e^{j\frac{Z_0}{b^2}\omega}}} \right]}, \tag{23}
\]

The order of magnitude of the functions \(F_1(ω)\) and \(F_2(ω)\) is less than or equal to one. If we neglect their frequency dependencies and express them as \(F_3\), the \(ω\)-integration is performed as

\[
W_L(t) = \frac{cZ_0}{2πb^2} \int_{-∞}^{∞} \frac{dω}{2π} e^{j\zeta_3t} \left[ e^{j\frac{Z_0}{b^2}\sqrt{e^{j\frac{Z_0}{b^2}\omega}}} \right] + \frac{cZ_0}{2πb^2} \int_{-∞}^{∞} \frac{dω}{2π} e^{j\zeta_3t} \left[ e^{j\frac{Z_0}{b^2}(e^{j\frac{Z_0}{b^2}\omega})} \right] = \frac{cZ_0}{πb^2} \sum_{n=0}^{∞} \frac{2^n(-F_3)^n(\zeta_3t)^{3n}}{Γ\left(\frac{3n}{2} + 1\right)}. \tag{24}
\]

Equation (24) reproduces Eq. (A.87) in Ref. [7] when \(F_3\) is identical to one. The conductivity appears in the parameter \(ζ_3\), which means that the wake function at the origin \(t = 0\) (the loss factor) does not depend on the conductivity.
6. Conclusions

We have found that both the longitudinal and transverse impedances of a thin NEG coating on a copper chamber are in the middle of the transit states between a copper-only chamber and NEG-only chamber in the frequency range of 0.1–100 GHz in terms of the conventional resistive-wall impedance formulae. In this frequency range, the longitudinal impedance seems to be saturated at around 5 μm as a function of the thickness of the NEG coating, but the transverse impedance still has a large dependence on the thickness of the NEG coating.

At the SuperKEKB [26], they have considered using a NEG coating on chambers in the interaction region to achieve a high vacuum there. They think that the NEG thickness needs to be at least several μm for effective pumping and for a long lifetime.\(^1\) In this regard, the choice of the NEG thickness may have a great impact on the impedance budget of a machine.

At very high frequencies over \(\sim 100\) GHz, the impedances deviate from the conventional impedance lines for the NEG-only chamber due to the effect of the induction term. We need precise calculations using the elaborate formulae such as the present theory, or the more sophisticated ImpedanceWake2D code [15]. Only for an impedance between a few kHz and 10 MHz, where the resistive-wall impedance may excite a Robinson instability [1], one may use the conventional formula for the copper-only chamber.

The present theory and findings can provide guidance for the design of vacuum chambers with a thin coating and a tool to estimate their longitudinal and transverse resistive-wall impedances, though constant conductivities are assumed over a wide frequency range to 100 THz.

Once the longitudinal impedance is found, the loss factor can be calculated. In most cases, the loss factor for an infinitesimal beam does not depend on the conductivity of the chamber material as in the case of a single-layered resistive chamber. The characteristic can be a benchmark to evaluate the reliability for impedance calculations.

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Appendix A. Resistive-wall impedances of a thin coating on a conductive chamber

Let us consider a conductive chamber with inner and outer radii \(d\) and \(a\), respectively, and with conductivity \(\sigma_{3c}\). The inner surface is coated by a thin film with conductivity \(\sigma_{2c}\), and the thickness is given by \(d - b\), i.e. the total thickness of the chamber is \(a - b\).

A.1. Particular solutions of Maxwell equations with beam density and currents

Let us assume that a particle travels along a beam pipe with velocity \(\beta c\) in the longitudinal direction, where \(\beta\) is the Lorentz \(\beta\) and \(c\) is the velocity of light. The Maxwell equations can be written as wave equations. Assuming that electromagnetic fields have a time dependency of \(e^{i\omega t}\), they become Helmholtz equations:

\[
(\Delta + k^2\beta^2)\vec{E} = jk\beta Z_0\vec{j} + \vec{\nabla}(cZ_0\vec{\rho}),
\]

\[
(\Delta + k^2\beta^2)\vec{H} = -\vec{\nabla} \times \vec{j},
\]

\(^1\)Y. Suetsugu, private communications.
where \( \tilde{\rho} \) and \( \tilde{j} \) are the charge density and the current density of the beam, respectively, \( k = \omega / \beta c \), and \( Z_0 = 120\pi \, \Omega \) is the impedance of free space. In the cylindrical coordinates \((\rho, \theta, z)\) for an axially symmetric structure, the wave equation for the longitudinal components of the electric and magnetic fields contain no transverse field component. They are decoupled. For the longitudinal field, there is a source term \( cZ_0 \partial \tilde{\rho} / \partial z + jk\beta Z_0 j_z \), while the \( z \)-component of \( \vec{V} \times \vec{j} \) vanishes for particles with a longitudinal velocity only.

When the beam with its charge \( q \) travels along the pipe at a constant radial offset position \( \rho = r_b \), \( \theta = \theta_b \), the charge density is expressed as

\[
\tilde{\rho} = \frac{i_m}{r_{b}^{1+m}} \delta(\rho - r_b) \delta(\theta - \theta_b) \delta(z - \beta ct) = \sum_{m=0}^{\infty} \frac{dk}{2\pi} i_m \rho_m, \tag{A.3}
\]

\[
\rho_m = \frac{1}{\pi r_{b}^{1+m}} (1 + \delta_{m0}) \delta(\rho - r_b) \cos m(\theta - \theta_b) e^{-jk(z - \beta ct)}, \tag{A.4}
\]

\[
i_m = qr_{b}^{m}, \tag{A.5}
\]

where \( \delta(x) \) is the \( \delta \)-function, \( \delta_{\rho}(\theta) \) is the periodic \( \delta \)-function, and \( \delta_{m,n} \) is the Kronecker \( \delta \). Since the general solutions of Maxwell equations are obtained by the superposition of those for \( i_m \rho_m \), we choose \( i_m \rho_m \) as the source term. Let us define the source field specified with superscript \( S \) as the solution which satisfies the Maxwell equations with \( \rho_m, \tilde{j}_m \) and vanishes at \( \rho \rightarrow \infty \). It is given by

\[
H^S_P = E^S_\theta = H^S_z = 0, \tag{A.6}
\]

\[
E^S_z = \begin{cases} 
\frac{j k c Z_0 h(k \rho)}{2\pi \gamma} K_0(\bar{k} \rho) e^{-jkz} & \text{for } \rho > r_b, \\
\frac{j k c Z_0 k(k \rho)}{2\pi \gamma} I_0(\bar{k} \rho) e^{-jkz} & \text{for } r_b > \rho,
\end{cases} \tag{A.7}
\]

\[
\frac{\beta}{Z_0} E^S_\rho = H^S_\theta = \begin{cases} 
\frac{\beta k c Z_0 l(k \rho)}{2\pi \gamma} K_1(\bar{k} \rho) e^{-jkz} & \text{for } \rho > r_b, \\
-\frac{\beta k c Z_0 k(k \rho)}{2\pi \gamma} I_1(\bar{k} \rho) e^{-jkz} & \text{for } r_b > \rho,
\end{cases} \tag{A.8}
\]

for \( m = 0 \), and

\[
E^S_z = 0, \tag{A.9}
\]

\[
E^S_z = \begin{cases} 
\frac{j k c Z_0 k(k \rho)}{\rho \pi r_b^{2}} K_0(\bar{k} \rho) \cos m(\theta - \theta_b) e^{-jkz} & \text{for } \rho > r_b, \\
\frac{j k c Z_0 k(k \rho)}{\rho \pi r_b^{2}} I_0(\bar{k} \rho) \cos m(\theta - \theta_b) e^{-jkz} & \text{for } r_b > \rho,
\end{cases} \tag{A.10}
\]

\[
-\frac{Z_0}{\beta} H^S_\rho = E^S_\theta = \begin{cases} 
\frac{m c Z_0 k(k \rho)}{\rho \pi r_b^{2}} K_0(\bar{k} \rho) \sin m(\theta - \theta_b) e^{-jkz} & \text{for } \rho > r_b, \\
\frac{m c Z_0 k(k \rho)}{\rho \pi r_b^{2}} I_0(\bar{k} \rho) \sin m(\theta - \theta_b) e^{-jkz} & \text{for } r_b > \rho,
\end{cases} \tag{A.11}
\]

\[
\frac{\beta}{Z_0} E^S_\rho = H^S_\theta = \begin{cases} 
\frac{\beta k c Z_0 l(k \rho)}{2\pi \gamma} (K_{m-1}(\bar{k} \rho) + K_{m+1}(\bar{k} \rho)) \cos m(\theta - \theta_b) e^{-jkz} & \text{for } \rho > r_b, \\
-\frac{\beta k c Z_0 k(k \rho)}{2\pi \gamma} (I_{m-1}(\bar{k} \rho) + I_{m+1}(\bar{k} \rho)) \cos m(\theta - \theta_b) e^{-jkz} & \text{for } r_b > \rho,
\end{cases} \tag{A.12}
\]

for \( m > 0 \), where \( j \) is the imaginary unit, \( \gamma \) is the Lorentz \( \gamma \), \( k = 2\pi f / \beta c \), \( \bar{k} = k / \gamma \), and \( K_m(z) \) and \( I_m(z) \) are the modified Bessel functions, respectively [21,25].
Next, let us solve the Maxwell equation so that it satisfies the boundary conditions due to the conductive chamber with the thin coating, and obtain the longitudinal impedance of the beam.

A.2. Longitudinal impedance

General solutions (in particular \(E_z, H_\theta\) for \(m = 0\) are expressed as

\[
E_z = E_z^S + A(k)e^{-jkz}I_0(\tilde{k}\rho),
\]
\[
H_\theta = H_\theta^S + \frac{j\beta\gamma}{Z_0}A(k)e^{-jkz}I_1(\tilde{k}\rho),
\]

in the vacuum chamber \((\rho < b)\):

\[
E_z = e^{-jkz}(C_1(k)I_0(v_2\rho) + C_2(k)K_0(v_2\rho)),
\]
\[
H_\theta = \frac{(\sigma_{2c} + j\omega\varepsilon_0)}{v_2}e^{-jkz}(C_1(k)I_1(v_2\rho) - C_2(k)K_1(v_2\rho)),
\]

in the coating film with conductivity \(\sigma_{2c}\) \((b < \rho < d)\):

\[
E_z = e^{-jkz}(D_1(k)I_0(v_3\rho) + D_2(k)K_0(v_3\rho)),
\]
\[
H_\theta = \frac{(\sigma_{3c} + j\omega\varepsilon_0)}{v_3}e^{-jkz}(D_1(k)I_1(v_3\rho) - D_2(k)K_1(v_3\rho)),
\]

in the conductive material with conductivity \(\sigma_{3c}\) \((d < \rho < a)\); and

\[
E_z = E(k)e^{-jkz}K_0(\tilde{k}\rho),
\]
\[
H_\theta = -\frac{j\beta\gamma}{Z_0}E(k)e^{-jkz}K_1(\tilde{k}\rho),
\]

outside the chamber \((\rho > a)\), where \(v_2 = \sqrt{k^2/\gamma^2 + jk\beta Z_0\sigma_{2c}}, v_3 = \sqrt{k^2/\gamma^2 + jk\beta Z_0\sigma_{3c}}, \) and \(A(k), C_1(k), C_2(k), D_1(k), D_2(k), \) and \(E(k)\) are arbitrary coefficients. The matching conditions on each surface are specified by \(\rho = b, \rho = d, \) and \(\rho = a\):

\[
\frac{jkcZ_0I_0(\tilde{k}r_b)}{2\pi\gamma^2}K_0(\tilde{k}b) + A(k)I_0(\tilde{k}b) = C_1(k)I_0(v_2b) + C_2(k)K_0(v_2b),
\]
\[
\frac{\beta kcI_0(\tilde{k}r_b)}{2\pi\gamma}K_1(\tilde{k}b) + \frac{j\beta\gamma}{Z_0}A(k)I_1(\tilde{k}b) = \frac{(\sigma_{2c} + j\omega\varepsilon_0)}{v_2}(C_1(k)I_1(v_2b) - C_2(k)K_1(v_2b)),
\]
\[
C_1(k)I_0(v_2d) + C_2(k)K_0(v_2d) = D_1(k)I_0(v_3d) + D_2(k)K_0(v_3d),
\]
\[
\frac{(\sigma_{2c} + j\omega\varepsilon_0)}{v_2}(C_1(k)I_1(v_2d) - C_2(k)K_1(v_2d)) = \frac{(\sigma_{3c} + j\omega\varepsilon_0)}{v_3}(D_1(k)I_1(v_3d) - D_2(k)K_1(v_3d)),
\]
\[
D_1(k)I_0(v_3a) + D_2(k)K_0(v_3a) = E(k)K_0(\tilde{k}a),
\]
\[
D_1(k)I_1(v_3a) - D_2(k)K_1(v_3a) = -\frac{j\beta\gamma v_3}{(\sigma_{3c} + j\omega\varepsilon_0)Z_0}E(k)K_1(\tilde{k}a).
\]

These equations are composed of six linear equations with six unknown coefficients \((A, C_1, C_2, D_1, D_2, \) and \(E)\). Consequently, they give a unique solution for the coefficient \(A(k)\) straightforwardly by using Mathematica [27].
A.3. Transverse impedance

General solutions (in particular $E_z, E_\theta, H_z, H_\theta$) for $m = 1$ are expressed as

$$E_z = i_1 (E_z^S + A(k) I_1 (\tilde{k} \rho) \cos(\theta - \theta_b) e^{-jkz}),$$

$$H_\theta = i_1 \left( H_\theta^S + \frac{j \nu}{\bar{k}} \left( \frac{B(k) I_1 (\tilde{k} \rho)}{\rho} + \frac{\beta \bar{k} A(k)}{Z_0} I_1^I (\tilde{k} \rho) \right) \cos(\theta - \theta_b) e^{-jkz} \right),$$

$$H_z = i_1 B(k) I_1 (\tilde{k} \rho) \sin(\theta - \theta_b) e^{-jkz},$$

$$E_\theta = i_1 \left( E_\theta^S - \frac{j \beta \gamma Z_0}{k} \left( \tilde{k} B(k) I_1^I (\tilde{k} \rho) + \frac{A(k)}{Z_0 \beta} I_1 (\tilde{k} \rho) \right) \sin(\theta - \theta_b) e^{-jkz} \right),$$

inside the vacuum chamber ($\rho < b$);

$$E_z = (C_3 (k) I_1 (v_2 \rho) + C_4 (k) K_1 (v_2 \rho)) \cos(\theta - \theta_b) e^{-jkz},$$

$$H_\theta = \frac{jk}{v_2^2} \left[ \frac{C_1 (k) I_1 (v_2 \rho) + C_2 (k) K_1 (v_2 \rho)}{\rho} + \frac{(\sigma_2 c + j \omega \epsilon_0) v_2 (C_3 (k) I_1^I (v_2 \rho) + C_4 (k) K_1^I (v_2 \rho))}{jk} \right]$$

$$\times \cos(\theta - \theta_b) e^{-jkz},$$

$$H_z = (C_1 (k) I_1 (v_2 \rho) + C_2 (k) K_1 (v_2 \rho)) \sin(\theta - \theta_b) e^{-jkz},$$

$$E_\theta = -\frac{jk}{v_2^2} \left( \frac{C_3 (k) I_1 (v_2 \rho) + C_4 (k) K_1 (v_2 \rho)}{\rho} + \beta Z_0 v_2 \left( C_1 (k) I_1^I (v_2 \rho) + C_2 (k) K_1^I (v_2 \rho) \right) \right)$$

$$\times \sin(\theta - \theta_b) e^{-jkz},$$

in the coating film ($b < \rho < d$);

$$E_z = (D_3 (k) I_1 (v_3 \rho) + D_4 (k) K_1 (v_3 \rho)) \cos(\theta - \theta_b) e^{-jkz},$$

$$H_\theta = \frac{jk}{v_3^2} \left( \frac{D_1 (k) I_1 (v_3 \rho) + D_2 (k) K_1 (v_3 \rho)}{\rho} + \frac{(\sigma_3 c + j \omega \epsilon_0) v_3 (D_3 (k) I_1^I (v_3 \rho) + D_4 (k) K_1^I (v_3 \rho))}{jk} \right)$$

$$\times \cos(\theta - \theta_b) e^{-jkz},$$

$$H_z = (D_1 (k) I_1 (v_3 \rho) + D_2 (k) K_1 (v_3 \rho)) \sin(\theta - \theta_b) e^{-jkz},$$

$$E_\theta = -\frac{jk}{v_3^2} \left( \frac{D_3 (k) I_1 (v_3 \rho) + D_4 (k) K_1 (v_3 \rho)}{\rho} + \beta Z_0 v_3 \left( D_1 (k) I_1^I (v_3 \rho) + D_2 (k) K_1^I (v_3 \rho) \right) \right)$$

$$\times \sin(\theta - \theta_b) e^{-jkz}.$$
in the conductive material ($d < \rho < a$); and

\[
E_z = i_1 E_4(k) K_1(\tilde{k}\rho) \cos(\theta - \theta_b)e^{-j\kappa z}, \tag{A.40}
\]

\[
H_\theta = \frac{j\gamma}{k} \left( \frac{i_1 E_2(k) K_1(\tilde{k}\rho)}{\rho} + \frac{\beta^* i_1 E_4(k)}{Z_0} K'_1(\tilde{k}\rho) \right) \cos(\theta - \theta_b)e^{-j\kappa z}, \tag{A.41}
\]

\[
H_z = i_1 E_2(k) K_1(\tilde{k}\rho) \sin(\theta - \theta_b)e^{-j\kappa z}, \tag{A.42}
\]

\[
E_\theta = -\frac{j\beta \gamma Z_0}{k} \left( \tilde{k} i_1 E_2(k) K'_1(\tilde{k}\rho) + \frac{i_1 E_4(k)}{Z_0 \beta \rho} K_1(\tilde{k}\rho) \right) \sin(\theta - \theta_b)e^{-j\kappa z}, \tag{A.43}
\]

outside the chamber ($a < \rho$), where $A(k), B(k), C_1(k), C_2(k), C_3(k), C_4(k), D_1(k), D_2(k), D_3(k), D_4(k), E_2(k)$, and $E_4(k)$ are arbitrary coefficients determined by the boundary conditions on $\rho = b$, $\rho = d$, and $\rho = a$:

\[
\frac{jkc Z_0 I_1(\tilde{k}r_b) K_1(\tilde{k}b)}{\pi r_b \gamma^2} = -I_1(\tilde{k}b) A(k) + I_1(v_2b) C_3(k) + K_1(v_2b) C_4(k), \tag{A.44}
\]

\[
\frac{\beta k c I_1(\tilde{k}r_b)(K_0(\tilde{k}b) + K_2(\tilde{k}b))}{2\pi r_b \gamma} = \frac{j\gamma \beta I'_1(\tilde{k}b)}{k_b} A(k) - \frac{j\gamma I_1(\tilde{k}b)}{Z_0} B(k) + \frac{jk I_1(v_2b)}{v_2^2} C_1(k)\tag{A.45}
\]

\[
+ \frac{jk K_1(v_2b)}{v_2^2} C_2(k) + \left( \frac{\sigma_{2c} + j\omega \epsilon_0 I'_1(v_2b)}{v_2} \right) C_3(k) + \left( \frac{\sigma_{2c} + j\omega \epsilon_0 K'_1(v_2b)}{v_2} \right) C_4(k),
\]

\[
0 = -I_1(\tilde{k}b) B(k) + I_1(v_2b) C_1(k) + K_1(v_2b) C_2(k), \tag{A.46}
\]

\[
\frac{c Z_0 I_1(\tilde{k}r_b) K_1(\tilde{k}b)}{b \pi r_b} = \frac{j\gamma I_1(\tilde{k}b)}{k_b} A(k) + j\beta \gamma Z_0 I'_1(\tilde{k}b) B(k) - \frac{jk Z_0 I'_1(v_2b)}{v_2} C_1(k)\tag{A.47}
\]

\[
- \frac{jk Z_0 K'_1(v_2b)}{v_2} C_2(k) - \frac{jk I_1(v_2b)}{v_2^2} C_3(k) - \frac{jk K_1(v_2b)}{v_2^2} C_4(k),
\]

at ($\rho = b$);

\[
C_3(k) I_1(v_2d) + C_4(k) K_1(v_2d) - D_3(k) I_1(v_3d) - D_4(k) K_1(v_3d) = 0, \tag{A.48}
\]

\[
C_1(k) \frac{I_1(v_2d)}{v_2^2} + C_2(k) \frac{K_1(v_2d)}{v_2^2} + C_3(k) \frac{(\sigma_{2c} + j\omega \epsilon_0 I'_1(v_2d))}{jk v_2} + C_4(k) \frac{(\sigma_{2c} + j\omega \epsilon_0 K'_1(v_2d))}{jk v_2} = 0,
\]

\[
- D_1(k) \frac{I_1(v_3d)}{v_3^2} - D_2(k) \frac{K_1(v_3d)}{v_3^2} - D_3(k) \frac{(\sigma_{2c} + j\omega \epsilon_0 I'_1(v_3d))}{jk v_3} - D_4(k) \frac{(\sigma_{2c} + j\omega \epsilon_0 K'_1(v_3d))}{jk v_3} = 0, \tag{A.49}
\]

\[
C_1(k) I_1(v_2d) + C_2(k) K_1(v_2d) - D_1(k) I_1(v_3d) - D_2(k) K_1(v_3d) = 0, \tag{A.50}
\]

\[
C_1(k) \frac{\beta Z_0 I'_1(v_2d)}{v_2} + C_2(k) \frac{\beta Z_0 K'_1(v_2d)}{v_2} + C_3(k) \frac{I_1(v_2d)}{v_2^2} + C_4(k) \frac{K_1(v_2d)}{v_2^2} = 0,
\]

\[
- D_1(k) \frac{\beta Z_0 I'_1(v_3d)}{v_3} - D_2(k) \frac{\beta Z_0 K'_1(v_3d)}{v_3} - D_3(k) \frac{I_1(v_3d)}{v_3^2} - D_4(k) \frac{K_1(v_3d)}{v_3^2} = 0. \tag{A.51}
\]
at \((\rho = d)\); and

\[
D_3(k)I_1(v_3a) + D_4(k)K_1(v_3a) - E_4(k)K_1(\bar{k}a) = 0,
\]

(A.52)

\[
D_1(k)\frac{ki_1(v_3a)}{v_3^2a} + D_2(k)\frac{kK_1(v_3a)}{v_3^2a} + D_3(k)\frac{(\sigma_3e + j\omega\epsilon_0)I_1'(v_3a)}{jv_3} + D_4(k)\frac{(\sigma_3e + j\omega\epsilon_0)K_1'(v_3a)}{jv_3}
\]

\[
- E_2(k)\frac{\gamma K_1(\bar{k}a)}{ka} - E_4(k)\frac{\gamma\beta K_1'(\bar{k}a)}{Z_0} = 0,
\]

(A.53)

\[
D_1(k)I_1(v_3a) + D_2(k)K_1(v_3a) - E_2(k)K_1(\bar{k}a) = 0,
\]

(A.54)

\[
D_1(k)\frac{k\beta Z_0 I_1'(v_3a)}{v_3} + D_2(k)\frac{k\beta Z_0 K_1'(v_3a)}{v_3} + D_3(k)\frac{ki_1(v_3a)}{v_3^2a} + D_4(k)\frac{kK_1(v_3a)}{v_3^2a}
\]

\[
- E_2(k)\beta\gamma Z_0 K_1(\bar{k}a) - E_4(k)\frac{\gamma K_1(\bar{k}a)}{ka} = 0,
\]

(A.55)

at \((\rho = a)\). In the above formulae, the prime in the modified Bessel functions means the differential by their argument.

Using the Panofsky–Wenzel theorem \([1,28]\) and extracting the transverse impedance of the perfectly conducting chamber—Eq. (C.8)—we obtain the expression for the transverse impedance of the resistive-wall chamber as

\[
\frac{Z_T}{C} = -\frac{A(k)}{2\gamma c\beta} + \frac{kZ_0 I_1(\bar{k}r_b)K_1(\bar{k}a)}{j2\pi\beta r_b\gamma^3 I_1(\bar{k}a)}.
\]

(A.56)

Since the formalism is developed in the frequency domain, it is easy to incorporate frequency dependence of the conductivity into the present theory.

### Appendix B. Derivation of approximate formula for the longitudinal impedance of a thin coating on a conductive chamber

For an infinitesimal coating, the formula for the longitudinal impedance for a relativistic beam does not depend on whether or not the outermost layer is directly covered by the perfectly conductive wall. Thus, we expect that the longitudinal impedance for a thin coating \((d - b \ll a - b)\) on a conductive chamber is approximately derived by attaching a perfectly conductive wall on the outermost layer. It is expressed as

\[
Z_L = -\frac{A(k)}{c},
\]

(B.1)

by solving

\[
A(k) + v_2dD_1\left[-(I_0(v_2b)K_1(v_2d) + K_0(v_2b)I_1(v_2d))\left(I_0(v_3d) - \frac{I_0(v_3a)}{K_0(v_3a)}K_0(v_3d)\right)
\right.

\[
-\left(I_0(v_2b)K_0(v_2d) - K_0(v_2b)I_0(v_2d)\right)\frac{v_2(\sigma_3e + j\omega\epsilon_0)}{v_3(\sigma_2e + j\omega\epsilon_0)}\left(I_1(v_3d) + \frac{I_0(v_3a)}{K_0(v_3a)}K_1(v_3d)\right)
\]

\[
= 0,
\]

(B.2)
When a beam with the current density \( j_z = \beta c (1 - \Theta(\rho - \sigma)) e^{-jkz}/(\pi \sigma^2) \), where \( \Theta(x) \) is the step function, passes through the chamber, the source field, in particular \( E_z \), is expressed as

\[
E_z^S = \frac{jcZ_0 e^{-jkz}}{\pi \sigma^2} \left( \int_0^\rho d\rho r_b I_0(\tilde{k}\rho) K_0(\tilde{k}\rho) + \int_\rho^\sigma d\rho r_b K_0(\tilde{k}\rho) I_0(\tilde{k}\rho) \right)
\]

\[
= \frac{jcZ_0}{\pi \sigma^2} \left( \frac{1}{\tilde{k}} - \sigma I_0(\tilde{k}\rho) K_1(\tilde{k}\sigma) \right) e^{-jkz} \quad \text{for } \rho \leq \sigma, \tag{C.1}
\]

\[
E_z^S = \frac{jcZ_0 e^{-jkz}}{\pi \sigma^2} \int_0^\sigma d\rho r_b I_0(\tilde{k}\rho) K_0(\tilde{k}\rho) = \frac{jcZ_0}{\pi \sigma^2} I_1(\tilde{k}\sigma) K_0(\tilde{k}\rho) e^{-jkz} \quad \text{for } \rho \geq \sigma. \tag{C.2}
\]

When the chamber is made of a perfectly conductive material, the longitudinal electric field inside the chamber is given by

\[
E_z = \frac{jcZ_0}{\pi \sigma^2} \left( \frac{1}{\tilde{k}} - \sigma I_0(\tilde{k}\rho) K_1(\tilde{k}\sigma) \right) e^{-jkz} - \frac{jcZ_0}{\pi \sigma^2} \frac{I_1(\tilde{k}\sigma) K_0(\tilde{k}\rho)}{I_0(\tilde{k}\rho)} \frac{I_0(\tilde{k}\rho)}{I_0(\tilde{k}\rho)} e^{-jkz} \quad \text{for } \rho \leq \sigma, \tag{C.3}
\]

\[
E_z = \frac{jcZ_0}{\pi \sigma^2} I_1(\tilde{k}\sigma) K_0(\tilde{k}\rho) e^{-jkz} - \frac{jcZ_0}{\pi \sigma^2} \frac{I_1(\tilde{k}\sigma) K_0(\tilde{k}\rho)}{I_0(\tilde{k}\rho)} \frac{I_0(\tilde{k}\rho)}{I_0(\tilde{k}\rho)} e^{-jkz} \quad \text{for } \rho \geq \sigma. \tag{C.4}
\]
The coupling impedance $Z_L$ is defined as the average of the longitudinal electric field (normalized by the beam current) over the beam cross-section. For a pencil beam, we obtain

$$Z_L = \frac{j k Z_0 K_0(\bar{k} b)}{2 \pi \beta \gamma^2 I_0(\bar{k} b)}.$$  (C.5)

When a beam with an azimuthal dependency of $j_z = q \beta c \delta(\rho - r_b) \cos \theta e^{-jkz}/\pi r_b$ is running inside the chamber, the longitudinal electric field is given by

$$E_z = i \frac{j k c Z_0 I_1(\bar{k} r_b)}{\pi r_b \gamma^2} \left( K_1(\bar{k} \rho) - K_1(\bar{k} b) \right) \frac{I_1(\bar{k} \rho)}{I_1(\bar{k} b)} \cos(\theta - \theta_b)e^{-jkz} \quad \text{for } \rho > r_b,$$  (C.6)

$$E_z = i \frac{j k c Z_0 I_1(\bar{k} \rho)}{\pi r_b \gamma^2} \left( K_1(\bar{k} r_b) - K_1(\bar{k} b) \right) \frac{I_1(\bar{k} b)}{I_1(\bar{k} b)} \cos(\theta - \theta_b)e^{-jkz} \quad \text{for } \rho < r_b,$$  (C.7)

where $i_1 = q r_b$. Using the Panofsky–Wenzel theorem [1,28], we can obtain the transverse wake forces from the longitudinal electric fields. The transverse impedance is thus given by

$$Z_T = \frac{k Z_0}{2 \pi \beta r_b \gamma^2} \left( K_1(\bar{k} r_b) - \frac{I_1(\bar{k} r_b)}{I_1(\bar{k} b)} K_1(\bar{k} b) \right).$$  (C.8)

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