A quantum field theoretical approach to the $P$ vs. $NP$ problem via the sign of the fermionic Quantum Monte Carlo

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I present here a new method that allows the introduction of a discrete auxiliary symmetry in a theory in such a way that the eigenvalue spectrum of the fermion functional determinant is made up of complex conjugated pairs. The method implies a particular way of introducing and integrating over auxiliary fields related to a set of artificial shift symmetries. Gauge-fixing the artificial continuous shift symmetries in the direct and dual sectors leads to the implementation of direct and dual BRST-type global symmetries and of a symplectic structure over the field space (as prescribed by the Batalin-Vilkovisky method). A procedure similar to Kahler polarization in geometric quantization guarantees the possibility to choose a Kahler structure over the field space. This structure is generated by a special way of performing gauge fixing over the direct and dual sectors. The desired discrete symmetry appears to be induced by the Hodge-* operator. The particular extension of the field space presented here makes the operators of the de-Rham cohomology manifest. These become symmetries in the extended theory. This method implies the identification of the (anti)-BRST and dual-(anti)-BRST operators with the exterior derivative and cohomology manifest. These become symmetries in the extended theory. This method implies the identification of the (anti)-BRST and dual-(anti)-BRST operators with the exterior derivative and its dual in the context of the complex de-Rham cohomology. The novelty of this method relies on the fact that the field structure is doubled two times in order to make use of a supplemental symmetry prescribed by algebraic geometry. This leads to a generalization of Kramers theorem that avoids the Quantum Monte Carlo phase sign problem without any apparent increase in complexity.

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INTRODUCTION

The $P$ vs. $NP$ problem is known to have significant implications in many areas of science, not excluding physics, mathematics or computer science [1]. The main question is if some specific classes of problems can be efficiently treated via algorithmic methods. In this paper I present a constructive approach to this problem based essentially on some topological and geometrical arguments. While researchers speculate about this question [2] and give several interpretations of possible results there have been very few attempts to analyze the problem from perspectives other than purely algorithmic. As shown in ref. [3] the Quantum Monte Carlo phase sign problem can be mapped into a general $NP$-complete problem. I will follow this paper for a short introduction into the statistical aspects of the subject. The idea behind Monte Carlo simulations is to replace the direct calculation of sums of the form

$$< A > = \frac{1}{Z} \sum_{c \in \Omega} A(c) p(c); \quad Z = \sum_{c \in \Omega} p(c)$$

(1)

over a high dimensional space $\Omega$ of configurations $\{c\}$ with the sum over a set of $M$ configurations $\{c_i\}$ from $\Omega$ according to the distribution $p(c_i)$. The average is then calculated as

$$< A > \approx \bar{A} = \frac{1}{M} \sum_{i=1}^{M} A(c_i)$$

(2)

The statistical error of the above calculation is given by

$$\Delta A = \sqrt{\text{Var}_A (2\tau_A + 1)/M}$$

(3)

$\text{Var}_A$ being the variance of $A$ and $\tau_A$ measures the autocorrelations of the sequence $\{A(c_i)\}$.

The Monte Carlo approach permits the evaluation of the same average in polynomial time as long as $\tau_A$ does not increase faster than polynomial in the number of particles. For physical systems the sum one needs to calculate changes as follows

$$< A > = \frac{1}{Z} \text{Tr}[A \exp(-\beta H)]; \quad Z = \text{Tr}(\exp(-\beta H))$$

(4)

where $\beta$ is the inverse temperature and $Z$ is the partition function. A Monte Carlo technique can still be applied to reduce the exponential scaling of the problem, but, as specified in [3], only after the mapping of the quantum model on a classical one. The nature of this mapping is considered by [3], following [4], to be a Taylor expansion.

$$Z = \text{Tr}(\exp(-\beta H)) = \sum_{n=0}^{\infty} \frac{-\beta^n}{n!} \text{Tr}(H^n) =$$

(5)

$$= \sum_{n=0}^{\infty} \sum_{i_1,...,i_n} \frac{-\beta^n}{n!} < i_1|H|i_2 > ... < i_n|H|i_1 >$$

(6)

$$= \sum_{n=0}^{\infty} \sum_{i_1,...,i_n} p(i_1, ..., i_n) = \sum_c p(c)$$

(7)

For each order $n$ in the expansion, $n$ sums were inserted over a complete basis set of states $|i>$. The configurations are sequences of $n$ basis states and the weight $p(c)$ is
that although a change of the basis is possible. It is argued in [3] that a different quantization prescription and that a system can be mapped into an NP-complete problem into a problem that is NP-complete. This is of course correct if one follows the above steps. The main scope of this paper is to prove that some assumptions in [3] can be avoided when considering a different quantization prescription and that a system can be mapped into an NP-complete problem but still have a polynomial solution if analyzed from the perspective of the quantization of gauge theories. A gauge symmetry can be seen as a redundancy of the mathematical formulation. As shown by Batalin and Vilkovisky in [5] if one is willing to lose the explicit visibility of some properties one can reduce the gauge symmetry and transform a gauge theory into a non-gauge one. In this paper, I will follow the opposite path. I construct a theory that has artificial gauge symmetries introduced in such a way that a discrete symmetry to be associated with an artificial "time reversal" invariance appears. Following ref. [6] the presence of such a symmetry in a theory permits the avoidance of the sign problem. This construction is done by using the field-antifield quantization of gauge theories with general algebras. Moreover, I partially follow the description by Alfaro and Damgaard [8],[9] in order to show what is the effect of the quantization of a field theory with fermions, how the change in sign appears and how one can relate classical and quantum descriptions in a different way. I also make the connection between geometry (symmetry) and topology (cohomology) by introducing the BRST, anti-BRST and dual-(anti)BRST [10] operators associated to the de-Rham cohomology. I define and use the Hodge star operation [11] in this context in order to generate a discrete symmetry. I make use of the intrinsic symplectic structure of the general field-antifield functional space in order to generate a Kahler structure([12],[13]). I also use the fact that the extension of the field space towards an even dimensional space is always possible. The end result is a general quantum field theory free of the Monte Carlo sign problem and with no apparent exponential growth in complexity.

**QUANTIZATION PRESCRIPTIONS**

The idea of quantization has a vast history. Originally, physical variables have been promoted to operators with specific commutation rules. These encoded the first quantizations ever performed. They were followed by the second quantization prescriptions and the anti-commutators needed for the description of fermionic particles. Finally path integral quantization brought a completely new perspective on the procedure of quantization. While a classical theory is described by an action functional and a minimization prescription, a path integral quantization is constructed as a functional integral of the complex exponentiated action functional

\[ \exp(iS[\cdot]): C \rightarrow A \]  

where C is the configuration space and A is the resulting space. This definition is very formal. In practical situations the measure of the path integral is not always defined in the standard way. The configuration spaces are in general not even manifolds. Sometimes, in order to obtain pertinent results a so called "cohomological integration" is necessary. I start with a general field theory as described by the action S[φA]. The Batalin-Vilkovisky quantization prescription enlarges the field-space of the theory by introducing antifields (φ∗A) and gives a new canonical structure known as the antibracket [14],[20]. If one starts with the classical action (containing the usual number of fields) S[φA] the associated path integral is

\[ Z = \int [dφ^A] \exp \left[ i \frac{\hbar}{\beta} S[φ^A] \right] \]  

By performing the transformations ϕA → ϕA(x) − ϕ∗A(x) one constructs an action S[φA − ϕ∗A] invariant to a local shift symmetry

\[ \delta ϕ^A(x) = Θ(x) \]
\[ \delta ϕ^∗A(x) = Θ(x) \]  

where Θ(x) is arbitrary. In this way I constructed another field representation that contains a collective field ϕ∗A. One can in principle integrate over the collective field if one fixes the introduced gauge symmetry in the standard BRST manner: add an BRST-exact term in such a way that the local gauge symmetry is broken. This term must contain a ghost-antighost pair (cA(x), φ∗A(x)) and a Nakaniishi-Lautrup field BA(x). A global BRST symmetry should emerge. The transformation rules of the fields in this theory will be

\[ \delta ϕ^A(x) = c^A(x) \]
\[ \delta ϕ^∗A(x) = c^A(x) \]
\[ \delta c^A(x) = 0 \]
\[ \delta φ^A(x) = B_A(x) \]
\[ \delta B_A(x) = 0 \]  

I make no assumptions about the Grassmann parity of the initial fields φA. The ghost numbers of the new fields will be

\[ gh(c^A) = 1; \ gh(φ^*_A) = -1; \ gh(B_A) = 0 \]
and $\delta$ is statistics changing. One can gauge fix the transformed action by adding

$$ - \delta[\phi_A^{*}\phi^A] = (-1)^{[(A)+1]}B_A\phi^A - \phi_A^{*}c^A $$

(14)

The collective field has been gauge fixed to zero. If one integrates out $B_A(x)$ one obtains

$$ Z = \int [d\phi_A][d\phi_A^*][dc_A][dB_A]\exp\left(\frac{i}{\hbar}S[\phi_A - \varphi_A]\right) - \int dx[(-1)^{(A)}B_A(x)\phi^A(x) + \phi_A^*(x)c^A(x)]] $$

(15)

(see ref. [20]), the above identity corresponds to what is called the master equation

$$ \frac{1}{2}(S_{ext}, S_{ext}) = - \int dx \frac{\delta r S_{ext}}{\delta \phi^A(x)}c^A(x) $$

(21)

The important aspect to be considered here is the right hand side term of the above formula. In this case the solution of the above expression implies an expansion in terms of the ghosts and the antighosts with a set of unknown coefficients

$$ S_{ext}[\phi^A, \phi_A^*, c^A] = S[\phi^A] + \sum_{n=1}^{\infty} a_n \phi_A^1...\phi_A^n c^1...c^n $$

(22)

Of course the choice of integrating over both the ghosts and the anti-ghosts is arbitrary. One can chose to integrate only over the ghost fields $c^A(x)$ but not over the corresponding anti-ghosts $\phi_A^*(x)$. The partition function becomes then

$$ Z = \int [d\phi_A][d\phi_A^*]\exp\left(\frac{i}{\hbar}S[\phi^A]\right) $$

(23)

On the side of the BRST algebra this change amounts in the way in which the non-propagating fields are replaced by their corresponding equations of motion. The direct method used above must be refined when dealing with fermionic type fields. The main question is how to replace $c^A$ inside the Green functions? The answer to this question will give the transformation rules for the fields that are not integrated out. Consider the identity

$$ \int [dc] F[c^B(y)]\exp[-\frac{i}{\hbar}\int dx\phi_A^*(x)c^A(x)] = F(i\hbar \frac{\delta l}{\delta \phi_B^*(y)})\exp[-\frac{i}{\hbar}\int dx\phi_A^*(x)c^A(x)] $$

(24)
It follows that the replacement of \( c^A \) with its equation of motion \((c^A(x) = 0)\) is not sufficient. One has to add what is called a "Quantum Correction" of the form \( \hbar \delta / \delta \phi^* \). What appears as "Quantum Correction" in the Green function results from our choice of integrating only over a ghost field and not over its associated anti-ghost. Essentially it is at this point in the quantization procedure where the difference between fermionic and bosonic fields appears. The BRST symmetry transformations have to change accordingly if the option of integrating only over the fermionic "half" of the field-antifield structure is chosen. Now, performing this replacement (meaningful only inside the path integral) the BRST transformation itself becomes

\[
\delta \phi^A(x) = i \hbar (-1)^c_A \frac{\delta^r}{\delta \phi^*_A(x)} \\
\delta \phi^*_A(x) = - \frac{\delta S}{\delta \phi^A(x)}
\]

(25)

where \( c_A \) is the Grassmann parity associated to the fields indexed by \( A \). It can be checked that this transformation leaves at least the combination of the measure and the action invariant. After integrating out the ghost the antibracket structure is modified. In order to see how, one can perform a variation of an arbitrary functional \( G[\phi^A, \phi^*_A] \). Inside the path integral we obtain the partition function as depending on the action functional described in terms of a set of fields \( \phi^A \)

\[
Z = Z_0 \int \exp(- \sum_i S_i[\phi^A]) D\phi^A
\]

(29)

More generally, the theory may have additional internal symmetries, generated by corresponding operators. In the supplement material \[20\] (section 2) I showed the procedure of doubling the field structure once. In this way a shift symmetry emerged. However, the action presents further flexibility. One can extend the field structure such that two BRST operators become manifest. In this way one implements the BRST-anti-BRST symmetry and the associated field structure \[7\]. This method allows Schwinger-Dyson equations as Ward identities as well.

The method is similar to what has been shown in ref. \[20\] and we obtain \( S_0[\phi] \rightarrow S_0[\phi - \varphi_{A1} - \varphi_{A2}] \). Two extra gauge symmetries arise for which a new structure of fields is introduced: two ghostfields \((c_{A1}, \phi^*_{A2})\) and two antighost fields \((\phi^*_{A1}, c_{A2})\). Of course this extends the symmetries allowed in the theory.

\[
\begin{align*}
\delta_1 \phi_A &= c_{A1} \\
\delta_1 \varphi_{A1} &= c_{A1} - \phi^*_{A2} \\
\delta_1 \varphi_{A2} &= \phi^*_{A2} \\
\delta_1 c_{A1} &= 0 \\
\delta_1 \phi^*_{A2} &= 0 \\
\delta_2 \phi_A &= c_{A2} \\
\delta_2 \varphi_{A1} &= - \phi^*_{A1} \\
\delta_2 \varphi_{A2} &= c_{A2} + \phi^*_{A1} \\
\delta_2 c_{A2} &= 0 \\
\delta_2 \phi^*_{A1} &= 0
\end{align*}
\]

(30)

Here \( \delta_1 \) and \( \delta_2 \) are respectively the BRST and anti-BRST transformations. The next step is to impose gauge fixing. This is done in the standard way by adding more bosonic fields, call them \( B_A \) and \( \lambda_A \). The BRST transformation

\[
\begin{align*}
\delta_1 B_A &= 0 \\
\delta_1 \lambda_A &= 0
\end{align*}
\]

(26)

The left derivative on the left side comes from the definition of \( c^A \) in \( S_{ext} \). In this case it can be considered simply as zero. I follow here closely the notation of reference \[8\] (see also ref. \[20\]). This equation describes the "quantum deformation" introduced in the antibracket structure

\[
\delta G[\phi^A, \phi^*_A] = \int dx [(G, S_{ext}) - i \hbar \Delta G]
\]

(27)

where

\[
\Delta = (-1)^{c_A+1} \frac{\delta^r \delta^r}{\delta \phi^*_A(x) \delta \phi^A(x)}
\]

(28)

Again, this term appears only as a consequence of the partial integration which allows us to expose the operator \( \delta / \delta \phi^*_A \) that otherwise acts only on a \( \delta \)-functional. Whenever one choses to keep only half of the field-antifield components in the theory and integrates over the ghosts obeying a fermi statistics the result will be a deformation of the antibracket structure that will lead to the Quantum Master Equation. One can already see that the mapping of the "quantum" problem to the "classical" problem as presented in reference \[3\] is correct but not unique. In the next chapter I show how one should extend the field structure of an arbitrary theory in order to obtain a sign-problem free theory. This prescription implies by no means any exponential increase in complexity if one considers what has been presented above.

CONSTRUCTION OF THE THEORY

My method relies at a first level on the field-antifield formalism as introduced by Batalin and Vilkovisky \[5\] and at a second level on an innovative use of some algebraic geometry and topology theorems \[11\]. I will regard

\[
\delta G[\phi^A, \phi^*_A] = \int dx \frac{\delta^r G}{\delta \phi^A(x)} \frac{\delta^l S_{ext}}{\delta \phi^*_A(x)} + (i \hbar (-1)^{c_A} \frac{\delta^r}{\delta \phi^*_A(x)} - \int dx \frac{\delta^r G}{\delta \phi^*_A(x)} \frac{\delta^l S_{ext}}{\delta \phi^A(x)}
\]

(26)
rules extend according to

\[
\begin{align*}
\delta_1 c_{A2} &= B_A \\
\delta_2 c_{A1} &= -B_A \\
\delta_1 B_A &= 0 \\
\delta_2 B_A &= 0 \\
\delta_1 \phi_1^2 &= \lambda_A - \frac{B_A}{2} \\
\delta_2 \phi_2^2 &= -\lambda_A - \frac{B_A}{2} \\
\delta_1 \lambda_A &= 0 \\
\delta_2 \lambda_A &= 0
\end{align*}
\]

These rules imply the nilpotency conditions \(\delta_1^2 = \delta_2^2 = \delta_1 \delta_2 + \delta_2 \delta_1 = 0\). The action invariant under this BRST symmetry contains the terms of \(S_0[\phi_A - \varphi_{A1} - \varphi_{A2}]\) plus some gauge fixing terms

\[
S_{\text{col}} = \frac{1}{2} \delta_1 \delta_2 [\varphi_{A1} - \varphi_{A2}] = -(\varphi_{A1} + \varphi_{A2}) \lambda_A + \frac{B_A}{2} (\varphi_{A1} - \varphi_{A2}) + (-1)^a \phi_A^* c_{Aa}
\]

Here summation over \(a = 1, 2\) is implied. Using the transformation

\[
\varphi_{A+} = \varphi_{A1} \pm \varphi_{A2}
\]

we obtain the gauge fixed action

\[
S_{gf} = S_0[\phi_A - \varphi_{A+}] - \varphi_{A+} \lambda_A + \frac{B_A}{2} \varphi_{A-} + (-1)^a \phi_A^* c_{Aa}
\]

For the sake of generality I follow the notation in reference [7] and define

\[
\delta_a \phi_A = R_{Aa}(\phi_A)
\]

where \(R_{Aa}\) is the BRST-anti-BRST symmetry transformation associated to an initial intrinsic gauge symmetry. Apart from this, the transformations associated to the additional artificial shift symmetries will be added. In the case \(a = 1\) we have the BRST transformation rules whereas in the case \(a = 2\) we have the anti-BRST transformation rules. The two collective fields are denoted by \(\varphi_{A1}\) and \(\varphi_{A2}\) or generally \(\varphi_{Aa}\). The transformation will be \(\phi_A - \varphi_{A1} - \varphi_{A2}\). The field multiplets are used as the ghosts \((c_{A1}, \phi_{A1}^*)\) and the antighosts \((\phi_{A2}^*, c_{A2})\). For \(a = 1, c_{Aa}\) is a ghost while for \(a = 2, c_{Aa}\) is an antighost. The BRST-anti-BRST transformations are

\[
\begin{align*}
\delta_a \phi_A &= c_{Aa} \\
\delta_a \phi_{A2} &= \delta_a [c_{Aa} - \epsilon_{ac} \phi_A^* c_{Aa}] - R_{Aa}(\phi_A - \varphi_{A1} - \varphi_{A2}) + (1 - \delta_{ab}) \epsilon_{ac} \phi_A^* c_{Aa}
\end{align*}
\]

Here I imply no summation over \(a\). Also, here \(\epsilon_{ac}\) is the antisymmetric tensor. The extra fields \(B_A\) and \(\lambda_A\) are introduced and we have extra transformation rules

\[
\begin{align*}
\delta_a c_{Ab} &= \epsilon_{ab} B_A \\
\delta_a B_A &= 0 \\
\delta_a \phi_{A}^b &= -\delta_a^b (-1)^a \lambda_A + \frac{1}{2} (B_A + \frac{\delta R_{A1}(\phi_A - \varphi_{A1} - \varphi_{A2})}{\phi_B} R_{B2}(\phi_B - \varphi_{B1} - \varphi_{B2})) \\
\delta_a \lambda_A &= 0
\end{align*}
\]

The gauge fixing procedure must occur in a BRST-anti-BRST invariant way. The inclusion of the terms involving \(\delta \Sigma_{B1} R_B\) as well as the additional terms in eq. (36) as a modification of the traditional BRST transformation rules is done in order to encode the nilpotency of the BRST-anti-BRST transformation in a way that is independent of the gauge algebra. The additional fields \(B_A\) and \(\lambda_A\) have the role of imposing the nilpotency at the level of the transformation rules of the original fields (see section 2 of ref. [20]). Any off-shell non-nilpotency is thus shifted to the transformation rules of the collective fields. There are various ways in which more than one gauge symmetry can be encoded in the BRST transformation rules. Also parts of some transformation rules can be transferred to transformation rules of additional fields. These properties have many possible applications. Here I make use of them in order to avoid the sign problem. One can introduce a matrix \(M^{AB}\) which is invertible and has the property

\[
M^{AB} = (-1)^{\epsilon_A \epsilon_B} M_{BA}
\]

It also makes all the entries between the Grassmann odd and Grassmann even sectors vanish. This means that the term \(\phi_A M^{AB} \phi_B\) has ghostnumber zero and even Grassmann parity. One can gauge fix to zero the collective terms

\[
\begin{align*}
S_{\text{col}} &= -\varphi_{A+} M^{AB} \lambda_B + \frac{1}{2} \varphi_{A-} M^{AB} B_B + (-1)^a (-1)^{\epsilon_B \epsilon_A} \phi_A^* M^{AB} B_{Ba} + \\
&+ \frac{1}{2} \varphi_{A-} M^{AB} \delta B_1 (\phi_A - \varphi_{B1}) R_{B2}(\phi_B - \varphi_{C+}) + (-1)^{a+1} (-1)^{\epsilon_B \epsilon_A} \phi_A^* M^{AB} R_{Ba}(\phi_B - \varphi_{B+})
\end{align*}
\]

Here the summation over \(a\) is implied. The sum of the two collective fields is fixed to zero, \(\phi_A^a\) are the source terms for the BRST-anti-BRST transformations and the difference between the two collective fields \(\varphi_{A-}\) is the source of the mixed transformations. The original gauge symmetry can be fixed in an extended BRST-invariant way by adding the variation of a gauge boson \(\Psi(\phi)\) of ghostnumber zero.

\[
S_\Psi = \frac{1}{2} \epsilon^{ab} \delta_a \delta_b \Psi(\phi)
\]

The gauge fixed action can be written as:

\[
S_{gf} = S_0[\phi_A - \varphi_{A+}] + S_{\text{col}} + S_\Psi
\]

where

\[
S_{\text{col}} = -\frac{1}{4} \epsilon^{ab} \delta_a \delta_b (\varphi_{A1} M^{AB} \varphi_{B1} - \varphi_{A2} M^{AB} \varphi_{B2})
\]

At this moment we have a gauge fixed action with a BRST-anti-BRST symmetry. Although the matrix \(M\) can be eliminated in the end, it has the potential to introduce a metric on the space spanned by the fields of the trivial system and the ghosts [29]. The same idea
can be used to introduce a Kahler structure on the field space. The introduction of the internal space is required due to the definition of the Hodge dual operation. As can be seen in [20] (section 5) and in [21] the internal space allows the definition of the Hodge operator in any dimension. This will extend the applicability of the procedure described in [24] for arbitrary dimensions of the original theory (considering the suitable generalization of the indices of the operators). The discrete symmetry is obtained by introducing new collective fields and imposing a dual-space gauge fixing that generates a Kahler structure. The Hodge star operator that relates the direct and dual emerging global continuous symmetry transformations will induce a discrete symmetry in the final theory. This symmetry can be associated to a form of artificial discrete invariance (see example at the end of section 6 in ref [20] or ref [22]-[23]). Now I am focusing on the method that generates a time-reversal type symmetry. One spans an internal space by the introduction of a new set of fields and an equivalent of the $M$ matrix. Take the action $S_{\text{col}}$ used above. Now introduce an internal space index for the collective fields $\varphi_{\alpha A}^{\Omega}$, define their dual with respect to the internal space:

$$\varphi_{\alpha A}^{\Omega} = 1/2 \epsilon^{\Omega\Gamma} \varphi_{\alpha A}^{\Gamma} \tag{43}$$

and rewrite the fields as

$$\varphi_{\alpha A}^{\pm \Omega} = 1/2 (\varphi_{\alpha A}^{\Omega} \pm i \varphi_{\alpha A}^{\Omega}) \tag{44}$$

In the same way introduce another matrix $N$ that can be factorized as:

$$N^{\Omega \Gamma} = 1/2 (h^{\Omega \Gamma} - if^{\Omega \Gamma})$$

$$N^{\Omega \Gamma} = 1/2 (h^{\Omega \Gamma} + if^{\Omega \Gamma}) \tag{45}$$

The matrices $f$ and $h$ are completely arbitrary as long as the matrix $N$ can be decomposed in the above way. Replacing this into the action together with a corresponding change in the fields we obtain a Kahler structure imposed over the manifold of the field-antifield formalism. This procedure may be related to the idea of Kahler polarization in the geometric quantization. The additional terms obtained in the matrix are now of the form:

$$S_{\text{col}} = -1/4 \epsilon^{ab} \delta_b \delta_0 \delta_e \left( \varphi_{\alpha A}^{\Omega} N^{\Omega \Gamma} \varphi_{B}^{\Gamma} - \varphi_{\alpha A}^{\Omega} \overline{N}^{\Omega \Gamma} \varphi_{B}^{\Gamma} \right) \tag{46}$$

where now, the $\delta$ and $\tilde{\delta}$ operators correspond to the dual BRST transformations. Their form depends on the practical calculation. The most general expression that can be written here is

$$\delta_{DA} \Phi_A = \delta_{DA} \varphi_{\alpha A} = \varphi_{\alpha A}^{*}$$

$$\delta_{DA} \varphi_{Ab} = \delta_{Ab} \varphi_{\alpha A} - \epsilon_{ac} c_A^c - R_{Ac}$$

$$\delta_{DA} c_{Ab} = -\delta_a ((-1)^{a} \lambda_A + 1/2 \left( B_A + \delta_{Ab} R_{B1} \right) \right)$$

$$\delta_{DA} B_A = 0$$

$$\delta_{DA} \lambda_A = 0 \tag{47}$$

Where the same convention remains valid as for the BRST and anti-BRST transformations. However, several expressions may be altered according to the particularities of each theory. The expressions for the case of 2D QED are given in section 6 of [20]. $S_{\text{col}}$ is the equivalent of the collective term in the action for the new degrees of freedom constructed to introduce the Kahler structure. The matrices $N_{\Omega \Gamma}$ and $\overline{N}_{\Omega \Gamma}$ allow me to write the gauge fixing term in such a way that a Kahler structure becomes visible. They may be compared to the choice of a polarization set over the field space although here the scope is another. $M^{AB}$ is considered implicitly. The same method that allows the $M$ matrix to vanish eliminates the $N$ matrix as well if this is our intention. This would lead to losing the “polarization” that makes the Kahler structure visible (please note that I use the term “polarization” in a non-rigorous sense referring only to the way in which the fields can be partitioned). We now have a Kahler structure imposed on our original action. The dual-BRST symmetry is the BRST symmetry created by the new collective fields together with their trivial system. It is the analogue of the co-derivative from algebraic geometry. In this way we obtained the so-called de-Rham cohomology operators that are now identified with the (anti)-BRST and dual-(anti)-BRST operators.

The connection between them is given by the Hodge star operator which can be constructed independent of the dimension of the original field space if one follows the prescription of constructing the internal spaces as described above. In this case the Hodge duality plays the role of a discrete symmetry transformation (section 6, ref. [20]).

As noted in reference [13] and [15] the field-anti-field setup is amenable to the construction of a Kahlerian structure imposed on the system of fields. Here, the Hodge star induces a symmetry that can be identified with time-reversal in the case of Kahlerian structures. If one thinks at the antipode in a Hopf algebra one can see that there are not few similarities between the Hodge star operator and the antipode. All one has to do is to suitably introduce fields and antifields via appropriate trivial symmetries such that the antipodal structure (associated to the Hodge dual) becomes visible. In the context of the field-antifield approach the structure of the emerging fermionic determinant will be

$$\text{det}(D) = \begin{pmatrix} iT_1 & 0 \\ 0 & -iT_1 \end{pmatrix} \tag{48}$$

where $T_1$ results from the construction of the Kahler structure. This assures that for the extended field space the sign of the fermionic determinant is always positive and allows us to avoid the sign problem.

In order to assess the complexity of the final problem, considering the fact that the fermionic determinant does not change sign (this can be interpreted in the formalism of the first chapter as the weights $p(c)$ being positive)
the introduction of a single shift symmetry using one set of fields. In the above constructions the number of fields has been doubled two times so I went from a theory containing $N$ fields to a theory containing $4N$ fields. Also, additional fields have been added each time in order to ensure the desired gauge fixing. The additional fields on the BRST and anti-BRST branches are related so the construction of the BRST-anti-BRST structure required $4N$ fields and the dual counterpart required another $4N$ fields. This amounts to a theory containing $8N$ fields globally. Considering that half of the fields live in the internal space and have a controlled behavior and also that the increase in the field number is polynomial, the method should not add exponential complexity.

CONCLUSION

In conclusion, I showed that it is possible to introduce an auxiliary discrete symmetry that mimics time-reversal and that this symmetry can be used in order to avoid the sign problem.

The method used here is fairly general and I foresee various applications in computer science, condensed matter, exotic states of matter, etc. In general it opens completely new perspectives on notions like symmetry and it identifies a new and interesting connection between geometry (symmetry) and topology.

SUPPLEMENTAL NOTES

1. Leibniz rules for derivation

I am following here and in the main paper the rules of reference [8] for the left and right derivatives. Accordingly

$$\frac{\delta^i (FG)}{\delta A} = \frac{\delta^i F}{\delta A} G + (-1)^{\epsilon_F A} F \frac{\delta^i G}{\delta A}$$

$$\frac{\delta^r (FG)}{\delta A} = F \frac{\delta^r G}{\delta A} + (-1)^{\epsilon_A G} \frac{\delta^r F}{\delta A} G$$

which amount to the following relation between left and right derivatives in general

$$\frac{\delta^i F}{\delta A} = (-1)^{\epsilon_A (\epsilon_F + 1)} \frac{\delta^r F}{\delta A}$$

2. Doubling the fields

In the main paper, a discussion about the BRST-anti-BRST formalism as induced by two shifts in the field space is presented. Here, I show the main idea related to the introduction of a single shift symmetry using one set of collective fields. I will regard the partition function as depending on the action functional described in terms of a set of fields $\phi^A$

$$Z = Z_0 \int \exp \left( - \sum_i S_i [\phi^A] \right) D\phi^A$$

More generally, the theory may have additional internal symmetries, generated by corresponding operators. Let me now double the fields by introducing a collective field $\phi^A$ that induces a shift symmetry in the theory:

$$\phi^A \rightarrow \phi^A - \varphi^A$$

No assumption regarding the statistics of the $\phi^A$ fields is required. They can be fermionic or bosonic. The new shift symmetry must be gauge fixed and for this I have to introduce a ghost and a trivial system in the form of a multiplet consisting of an antifield $\phi^A_*$ and an auxiliary field $B_A$. After gauge fixing a global BRST symmetry emerges in general. In the present case however, in order to maintain the triviality of the extended BRST symmetry that encompasses also the shift symmetry the BRST transformation rules change in the following way:

$$\delta \phi^A = c^A$$

$$\delta \varphi^A = c^A - R^A [\phi^A - \varphi^A]$$

$$\delta c^A = 0$$

$$\delta \phi^A_* = B_A$$

$$\delta B_A = 0$$

where $c^A$ is the ghost field and $\delta$ is the BRST transformation to be associated with the total BRST-type symmetries. $R^A [\phi^A - \varphi^A]$ represents a formal definition of the BRST symmetry associated to a possible intrinsic initial gauge symmetry. There exist a freedom to shift the original gauge symmetry $R^A [\phi^A - \varphi^A]$ between the BRST transformation of the fields and the BRST transformation of the collective fields. In this way a possible off-shell non-nilpotency in the transformation rules is transferred to the transformation rules of the collective field. The $\frac{\delta^r R_{A_i}}{\delta \phi^A_*} R_{B_2}$ term in equation (37) of the main article has the special role of restoring the nilpotency of the BRST transformations independent of the closure of the algebra. This was one of the first applications of the collective field formalism to the quantization of theories involving algebras that do not close (i.e. algebras of the gauge symmetry generators depending on the form of the field equations of motion). As an example of such theories one may quote supergravity. The ghost numbers of the new fields are

$$gh(c^A) = 1; \quad gh(\phi^A_*) = -1; \quad gh(B_A) = 0$$

Although this artificial and certainly trivial continuous symmetry is easy to eliminate at this level, it is of major importance as a tool for generating new discrete symmetries. Considering the new continuous shift symmetry,
one has to gauge fix it. This can be done adding the following terms in form of BRST transformations:
\[ S_{gf} = S_0[\phi^A - \varphi^A] - \delta[\phi^*_A \phi^A] + \delta \Psi[\phi^A] = \]
\[ S_0[\phi^A - \varphi^A] + \phi^*_A R^A[\phi^A - \varphi^A] - \phi^*_A c^A + \frac{\delta \Psi}{\delta \phi^A} c^A - \varphi^A B_A = \]
\[ S_{BV}[\phi^A - \varphi^A] - \phi^*_A c^A + \frac{\delta \Psi}{\delta \phi^A} c^A - \varphi^A B_A \]
(56)
where \( S_{BV} \) is called the Batalin-Vilkovisky action. It incorporates the original action and the terms arising from other possible internal gauge symmetries. \( \Psi \) is a gauge fixing bosonic functional depending only on the original fields. By this I define a new gauge fixed action. Note that the nilpotency \( \delta^2 = 0 \) of the BRST transformation assures the overall invariance. The partition function is obtained:
\[ Z = \int [d\phi^A][d\phi^*_A] e^{-S_{BV}[\phi^A, \phi^*_A]} \] (57)
Where \( \Psi[\phi^A] \) is defined considering the condition imposed in the resulting delta-function. One must underline that the gauge fixing procedure must keep the gauge independence of the full partition function including the integration measure.

Until now, a continuous shift-symmetry has been introduced and gauge fixed.

3. antibracket

The antibracket is a new structure introduced in the process of BV-BRST quantization. It is defined considering two Grassmann functionals \( F \) and \( G \) as
\[ (F, G) = \frac{\delta F}{\delta \phi^A(x)} \frac{\delta G}{\delta \phi^*_A(x)} - \frac{\delta F}{\delta \phi^*_A(x)} \frac{\delta G}{\delta \phi^A(x)} \] (58)
involving alternate functional differentiation with respect to the fields and antifields (\( r \) and \( l \) superscripts meaning right and left derivatives respectively). The antibracket has some important properties: it changes the statistics as
\[ \epsilon[(F, G)] = \epsilon(F) + \epsilon(G) + 1 \] (59)
and satisfies the following relation
\[ (F, G) = -(-1)^{(r(F)+1)(r(G)+1)}(G, F) \] (60)
where \( \epsilon \) is the Grassmann parity operator. Using this structure the Batalin-Vilkovisky prescription can be written as
\[ \frac{1}{2}(W, W) = ih\Delta W \] (61)
where
\[ \Delta = (-1)^{c_A+1} \frac{\delta r}{\delta \phi^A} \frac{\delta r}{\delta \phi^*_A} \] (62)
\( W \) is called the "quantum action" and is a solution of the above equation. If it can be expanded in powers of \( \hbar \) one obtains:
\[ W = S + \sum_{n=0}^{\infty} \hbar^n M_n \] (63)
The boundary conditions should make this coincide with the classical action when all antifields are removed (\( \phi^*_A = 0 \)). To the lowest order one recovers the classical master equation \((S, S) = 0 \).

4. Hodge star and Hodge duality

Let \((M, g)\) be a \( N = 2d \)-dimensional manifold for which we can define the * operator in the following way [16]:
\[ \alpha \wedge * \beta = g_p(\alpha, \beta)dv_g \]; \( \alpha, \beta \in \wedge^N \) (64)
We have also that \((**)=1 \) on \( \wedge^N \) which means that \( \wedge^N \) splits into eigenspaces as
\[ \wedge^N = \wedge_+ + \wedge_- \] (65)
where the two eigenspaces correspond to eigenvalues +1 and -1 respectively. A \( d \)-form which belongs to \( \wedge_+ \) is called self-dual whereas if it belongs to the other eigenspace it is called anti-self-dual. An important remark to be done here is that given a \( p \)-vector \( \lambda \in \wedge^pV \) then \( \forall \theta \in \wedge^{d-p}V \) there exists the wedge product such that \( \lambda \wedge \theta \in \wedge^d \). The (anti)BRST and dual-(anti)BRST operators are then equivalent to the operators:
\[ \delta, \delta^* \] (66)
\[ \delta, \delta^* = *\delta_{a,b}* : \wedge^k \rightarrow \wedge^{k-1} \] (67)
\[ \Delta = \delta_{a,b}(*\delta_{a,b}*) + (*\delta_{a,b}*)\delta_{a,b} : \wedge^k \rightarrow \wedge^k \] (68)
In the context of algebraic geometry these are in order: the exterior differential, the coexterior(dual) differential and the Laplace operator. The exact and co-exact forms are orthogonal. The Hodge theorem allows the identification of a unique representative for each cohomology class as belonging to the Kernel of the Laplacian defined for the specific complex manifold. If this is put together with the definition of the Kahler manifold we obtain extra (discrete) symmetries in the Hodge structure of the manifold. In the main paper the dual operators acting on the field space have been introduced in a general context. For a practical description in the context of field-spaces see ref. [22]. There the author starts from a field theory with physical terms and identifies the discrete symmetry as the one induced by the Hodge-* operator in the
physical context. In the current approach, the BV formalism generates the usual even dimensional symplectic space. Dualization of the BRST-anti-BRST operators in this work is done using the extended symplectic field structure. One is not supposed to assume physicality of the terms involved.

5. Internal spaces and duality

The use of internal spaces in order to naturally define duality operations is not new. In fact I follow here reference [21] to show that the construction of an internal space is useful in this context and that a discrete $Z_2$ symmetry can appear.

I start by following reference [21] with an example of even dimensional $(2n)$ electrodynamics. Let $A$ be a general $(n-1)$ form and $F_{k_1,...,k_n}$ its associated field strength:

$$F_{k_1...k_n} = \partial[k_n] A_{k_1...k_{n-1}}$$

$$\star F_{k_1...k_n} = \frac{1}{n!} \epsilon^{k_1...k_{2n}} F_{k_{n+1}...k_{2n}}$$

Given the action, the equation of motion and the Bianchi identity as

$$S = - e_n \int d^{2n} x F_{k_1...k_n} F^{k_1...k_n}$$

$$\partial_{k_1} F_{k_1...k_n} = 0$$

$$\partial_{k_1} \star F_{k_1...k_n} = 0$$

($e_n$ is a constant, $k_j$ is the tensorial index) we can see that at the level of the Bianchi identity and the equation of motion the dual operation is a symmetry. Nevertheless, in general the second power of the dual operation has a different structure depending on the dimension of the space:

$$\star \star F = \begin{cases} F & \text{if } D = 4k - 2 \\ -F & \text{if } D = 4k \end{cases}$$

As one can see the dual $\star$ is not well defined for the 2-dimensional $(2D)$ scalar or for the 4k-2 dimensional extensions. Its definition has been enlarged [21] by making an internal structure of the potentials in the theory manifest. One should note that this has been achieved by using a canonical transformation and that the same can be achieved via BRST. I will enlarge the set of fields (alternatively the Hilbert space) by giving them an internal structure of the form $(\alpha, \beta)$. The dual operation is now defined as

$$\tilde{F}^\alpha = \epsilon^{\alpha\beta} \star F^\beta, \ D = 4k$$

$$\tilde{F} = F$$

In this case self and anti self dualities are well defined in any $D = 2k$ dimensional space. One can start with the first order form of the theory:

$$S = \int d^D x [\Pi \cdot \dot{A} - \frac{1}{2} \Pi \cdot \Pi - \frac{1}{2} B \cdot B + A_0 (\partial \cdot \Pi)]$$

Maxwell’s Gauss constraint can be generalized to be precisely the extended curl $(\epsilon \partial) = \epsilon_{k_1k_2...k_{D-1}} \partial_{k_{D-1}}$. Then

$$\Pi = (\epsilon \partial) \cdot \phi$$

$$B = (\epsilon \partial) \cdot A$$

where $\phi$ is a $(\frac{D}{2} - 1)$-form potential, $A$ is a generalization of the vector potential, $A_0$ is the general multiplier that enforces the Gauss constraint, the antisymmetrization of $\partial$ is defined as

$$(\epsilon \partial) = \epsilon_{k_1k_2...k_{D-1}} \partial_{k_{D-1}}$$

and in general the notation

$$\Phi \cdot \Psi = \Phi[k_1...k_{D-1}] \Psi[k_1...k_{D-1}]$$

is used to imply antisymmetrization via the brackets. Now I construct an internal space of potentials where duality symmetry is manifest ($\Phi^+$ and $\Phi^-$ represent the new field structure). The dual projection can be defined now as a canonical transformation of the fields in the following way:

$$A = (\Phi^+ + \Phi^-)$$

$$\Pi = \eta (\epsilon \partial) (\Phi^+ - \Phi^-)$$

$$\eta = \pm 1$$

The action can be rewritten in terms of these fields as

$$S = \int d^D x [\eta \dot{\Phi}^{(\alpha)} \sigma_3^{\alpha\beta} B^{(\beta)} + \dot{\Phi}^{(\alpha)} \epsilon^{\alpha\beta} B^{(\beta)}] - B^{(\beta)} \cdot B^{(\beta)}$$

where $B^{(\beta)} = (\epsilon \partial \cdot \Phi^{(\beta)})$ and $\sigma_3^{(\alpha\beta)}$ and $\sigma_2^{(\alpha\beta)} = i \epsilon^{(\alpha\beta)}$ are the Pauli matrices. We see that the symplectic part factorizes in two parts: one involving the third Pauli matrix and the other one the second Pauli matrix. For a dimension $D = 4k$ the first term is the generalization of the 2D chiral bosons. The $Z_2$ symmetry manifests itself in the transformation $\Phi^{(\pm)} \leftrightarrow \Phi^{(\mp)}$. The second term becomes a total derivative. For $D = 2K$ the first term becomes a total derivative and the second term explicitly shows the symmetry of $SO(2)$. Although the
complete diagonalization of the action in 3D cannot be done in coordinate space a dual projection is possible in the momentum space [21]. Let me introduce a two-basis \( \{ \hat{e}_a(k, x), a = 1, 2 \} \) with \( (k, x) \) being conjugate variables and the orthonormalization condition given as

\[
\int dx \hat{e}_a(k, x) \hat{e}_b(k', x) = \delta_{ab}(k, k') \tag{86}
\]

The vectors in the basis can be chosen to be eigenvectors of the Laplacian, \( \nabla^2 = \partial \theta \) and

\[
\nabla^2 \hat{e}_a(k, x) = -\omega^2(k) \hat{e}_a(k, x) \tag{87}
\]

The action of \( \partial \) over the \( \hat{e}_a(k, x) \) basis is

\[
\partial \hat{e}_a(k, x) = \omega(k) M_{ab} \hat{e}_b(k, x) \tag{88}
\]

The two previous equations give

\[
\tilde{M} M = -I \tag{89}
\]

where \( \tilde{M}_{ab} = M_{ba} \). The canonical scalar and its conjugate momentum have the following expansion

\[
\Phi(x) = \int dk q_a(k) \hat{e}_a(k, x) \tag{90}
\]

\[
\Pi(x) = \int dk p_a(k) \hat{e}_a(k, x) \tag{91}
\]

where \( q_a \) and \( p_a \) are the expansion coefficients. The action appears in this representation as a two dimensional oscillator. The phase space is now four dimensional, representing two degrees of freedom per mode,

\[
S = \int dk \{ p_a \dot{q}_a - \frac{1}{2} p_a p_a - \frac{\omega^2}{2} q_a q_a \} \tag{92}
\]

now we can introduce the following canonical transformation

\[
p_a(k) = \omega(k) \epsilon_{ab} (\varphi_a^{(+)} - \varphi_b^{(-)}) \tag{93}
\]

\[
q_a(k) = (\varphi_a^{(+)} + \varphi_a^{(-)}) \tag{94}
\]

The action becomes \( S = S_+ + S_- \) where

\[
S_{\pm} = \int dk \omega(k)(\pm \dot{q}_a \epsilon_{ab} q_b - \omega(k) q_a q_a) \tag{95}
\]

As expected, this action presents the \( Z_2 \) symmetry under the transformation \( \varphi_a^a \rightarrow \sigma^\alpha \varphi_a^\beta \).

This is particular example. However, the field-antifield prescription used in the main paper has practically a similar role and is defined in general. It generates a symplectic even dimensional field space suitable for quantization. It also defines an analogues for the Hodge-* operators.

6. Hodge star as discrete symmetry

For an example of how the Hodge star induces a discrete symmetry I follow ref. [22]-[24]. The main idea there was to represent the Hodge decomposition operators \( (d, \delta, \Delta) \) as some symmetries of a given BRST invariant Lagrangean of a gauge theory. In general, the Hodge decomposition theorem states that on a compact manifold any \( n \)-form \( f_n(n = 0, 1, 2, \ldots) \) can be uniquely represented as the sum of a harmonic form \( h_n(\Delta h_n = 0, dh_n = 0, \delta h_n = 0) \), an exact form \( d\epsilon_{n-1} \) and a co-exact form \( \delta c_{n+1} \) as

\[
f_n = h_n + d\epsilon_{n-1} + \delta c_{n+1} \tag{96}
\]

where here \( d \) is the exterior derivative, \( \delta \) is its dual and \( \Delta \) is the Laplacian operator \( \Delta = d\delta + \delta d \). In order to identify the dual BRST transformation, one has to observe that while the direct BRST transformations leave the two form \( F = dA \) in the construction of a gauge theory invariant and transform the Dirac fields like a local gauge transformation, the dual-BRST transformations leave the previous gauge fixing term invariant and transform the Dirac fields like a chiral transformation. So, as a practical example, I can start like the authors of [24] from a BRST invariant lagrangean for QED noting that generalizations for non-abelian gauge theories with interactions exist in the literature as well.

\[
L_B = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(i \gamma^\mu \partial_\mu - m) \psi - e \bar{\psi} \gamma^\mu A_\mu \psi + B(\partial A) + \frac{1}{2} B^2 - i \partial_\mu \bar{C} \partial^\mu C \tag{97}
\]

\( F^{\mu\nu} \) being the field strength tensor, \( B \) is the Nakanishi-Lautrup auxiliary field and \( C, \bar{C} \) are the anticommuting ghosts. The BRST transformations that leave this Lagrangian invariant are

\[
\begin{align*}
\delta_B A_\mu &= \eta \partial_\mu C & \delta_B \psi &= -ie C \psi \\
\delta_B C &= 0 & \delta_B \bar{C} &= i \eta B \\
\delta_B \bar{\psi} &= i e C \bar{\psi} & \delta_B F^{\mu\nu} &= 0 \\
\delta_B (\partial A) &= \eta \Box C & \delta_B B &= 0 \tag{98}
\end{align*}
\]
where $\eta$ is an anticommuting space-time independent transformation parameter. Particularizing for the 2 dimensional case the Lagrangian becomes

$$L_B = -\frac{1}{2}E^2 + \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - e\bar{\psi}\gamma^\mu A_\mu \psi + B(\partial A) + \frac{1}{2}B^2 - i\partial_\mu \bar{C}\partial^\mu C$$

(99)

and this can be rewritten after introducing another auxiliary field $B$ as

$$L_B = BE - \frac{1}{2}E^2 + \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - e\bar{\psi}\gamma^\mu A_\mu \psi + B(\partial A) + \frac{1}{2}B^2 - i\partial_\mu \bar{C}\partial^\mu C$$

(100)

The dual BRST symmetry operators to be associated to the theory above in the 2 dimensional case are [24]

$$\begin{align*}
\delta_DA_\mu &= -\eta\epsilon_\mu\partial_\mu \bar{C} \\
\delta_DC &= -i\eta\bar{B} \\
\delta_D\bar{C} &= -i\eta \bar{C} \gamma_5 \psi \\
\delta_D\psi &= i\eta \bar{C} \gamma_5 \bar{\psi} \\
\delta_D(\partial A) &= 0 \\
\delta_DB &= 0 \\
\delta_DB &= 0
\end{align*}$$

Moreover, as noted in reference [24] the interacting Lagrangian in 2 dimensions is invariant under the following transformations

$$\begin{align*}
C &\rightarrow \pm i\gamma_5 \bar{C} \\
\bar{B} &\rightarrow \mp i\gamma_5 B \\
A_0 &\rightarrow \pm i\gamma_5 A_1 \\
A_1 &\rightarrow \pm i\gamma_5 A_0 \\
E &\rightarrow \pm i\gamma_5 (\partial A) \\
(\partial A) &\rightarrow \pm i\gamma_5 E \\
\bar{\psi} &\rightarrow \mp \bar{\psi} \\
\bar{\psi} &\rightarrow \pm \bar{\psi}
\end{align*}$$

(102)

Reference [24] shows that these are the analogues of the Hodge duality ($\ast$) for this particular example and that they induce a discrete symmetry. One can also verify that

$$\ast (\ast \Phi) = \pm \Phi$$

(103)

where for ($+$) the generic field $\Phi$ is $\psi$, $\bar{\psi}$ and for ($-$) $\Phi$ represents the rest of the fields. One can also observe that for the direct and dual BRST symmetries

$$\delta_D\Phi = \pm \ast \delta_B \ast \Phi$$

(104)

is valid. It has been known before that the above statements are valid for any even dimensional theory [22] and applications for $D = 4$, (3,1) and $D = 6$ dimensional theories have been given. However, combining the ideas presented in the main paper with the observations in ref. [21] and some theorems of algebraic topology and geometry one can generalize the applicability of this method to any dimension. While it is true that in some cases non-local transformations emerge ([25]-[27]) the method described in this paper is simply a mathematical trick that allows the construction of dual theories with no sign problems so physical meaning of the artificial transformations is irrelevant.

7. Kahler manifolds

Kahler manifolds are particularly interesting for the current problem. In general having a differential manifold $\mathcal{M}$ and a tensor of type $(1, 1)$ $J$ such that $\forall p \in \mathcal{M}$, $J_p^2 = -1$, the tensor $J$ will give a structure to $\mathcal{M}$ with the property that the eigenvalues of it will be of the form $\pm i$. This means that $J_p$ is an even dimensional matrix and $\mathcal{M}$ is an even manifold. From the same definition it follows that $J_p$ can divide the complexified tangent space at $p$ in two disjoint vector subspaces

$$T_p\mathcal{M}^C = T_p\mathcal{M}^+ \oplus T_p\mathcal{M}^-$$

(105)

$$T_p\mathcal{M}^\pm = \{Z \in T_p\mathcal{M}^C | J_pZ = \pm iZ\}$$

(106)

One can introduce two projection operators of the form

$$P^\pm : T_p\mathcal{M}^C \rightarrow T_p\mathcal{M}^\pm$$

(107)

$$P^\pm = \frac{1}{2}(1 \pm iJ_p)$$

(108)

which will decompose $Z$ as $Z = Z^+ + Z^-$. This construction will generate a holomorphic and an antiholomorphic sector: $Z^\pm = P^\pm Z \in T_p\mathcal{M}^\pm$, $T_p\mathcal{M}^+$ being the
holomorphic sector. A complex manifold appears when demanding that given two intersecting charts \((U_i, \gamma_i)\) and \((U_j, \gamma_j)\), the map \(\psi_{ij} = \gamma_j \phi_i^{-1}\) from \(\gamma_i(U_i \cap U_j)\) to \(\gamma_j(U_i \cap U_j)\) is holomorphic. Here \(\gamma_i\) and \(\gamma_j\) are chart homeomorphisms and \(\psi_{ij}\) is the transition map. In this case the complex structure is given independently from the chart by

\[
J_p = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \forall p \in \mathcal{M}
\]  

(109)

In the complex case there is a unique chart-independent decomposition in holomorphic and antiholomorphic parts. This means we can now choose as a local basis for those subspaces the vector \((\frac{\delta}{\delta z^\mu}, \frac{\delta}{\delta \bar{z}^\nu})\) where \((z^\mu, \bar{z}^\nu)\) are the complex coordinates such that the complex structure becomes

\[
J_p = \begin{pmatrix} i1 & 0 \\ 0 & -i1 \end{pmatrix} \quad \forall p \in \mathcal{M}
\]  

(110)

If we add a Riemannian metric \(g\) to the complex manifold and demand that the metric satisfies \(g_p(J_pX, J_pY) = g_p(X, Y)\), \(\forall p \in \mathcal{M}\) and \(X, Y \in T_p\mathcal{M}\) then the metric is called hermitian and \(\mathcal{M}\) is called a hermitian manifold. A complex manifold always admits a hermitian metric. Using the base vectors of the complexified \(T_p\mathcal{M}^C\) we can always write the metric locally as

\[
g = g_{\mu\bar{\nu}} dz^\mu \otimes d\bar{z}^\nu + g_{\mu\bar{\nu}} d\bar{z}^\mu \otimes dz^\nu
\]  

(111)

If we have a hermitian manifold \((\mathcal{M}, g)\) with \(g\) hermitian metric and a fundamental 2-tensor \(\Omega\) whose action on vectors \(X\) and \(Y\) \(\in T_p\mathcal{M}\) is

\[
\Omega_p(X, Y) = g_p(J_pX, Y)
\]  

(112)

then we call \(\Omega_p(X, Y)\) a Kahler form. With this definition the Kahler form has some very useful properties. Firstly it is antisymmetric

\[
\Omega(X, Y) = g(J^2X, JY) = -g(X, JY) = -\Omega(Y, X)
\]  

(113)

Then it is invariant under the action of the complex structure

\[
\Omega(JX, JY) = \Omega(X, Y)
\]  

(114)

and under complexification

\[
\Omega_{\mu\bar{\nu}} = ig_{\mu\bar{\nu}} = 0
\]  

(115)

\[
\Omega_{\bar{\mu}\nu} = ig_{\mu\bar{\nu}} = 0
\]  

(116)

\[
\Omega_{\mu\bar{\nu}} = -\Omega_{\bar{\nu}\mu} = ig_{\mu\bar{\nu}}
\]  

(117)

thus leading to

\[
\Omega = ig_{\mu\bar{\nu}} dz^\mu \wedge d\bar{z}^\nu
\]  

(118)

A Kahler manifold is a hermitian manifold \((\mathcal{M}, g)\) whose Kahler form \(\Omega\) is closed \((d\Omega = 0)\). \(g\) is called a Kahler metric. The closing condition defines a differential equation for the metric.

\[
d\Omega = (\delta + \bar{\delta})g_{\mu\bar{\nu}} dz^\mu \wedge d\bar{z}^\nu = 0
\]  

(119)

This leads to the relations

\[
\frac{\delta g_{\mu\bar{\nu}}}{\delta z^\lambda} = \frac{\delta g_{\lambda\bar{\nu}}}{\delta \bar{z}^\mu}
\]  

(121)

\[
\frac{\delta g_{\mu\bar{\nu}}}{\delta \bar{z}^\lambda} = \frac{\delta g_{\bar{\mu}\lambda}}{\delta z^\nu}
\]  

(122)

The solution of the above equation takes the form

\[
g_{\mu\bar{\nu}} = \delta_{\mu\bar{\nu}} K_i
\]  

(123)

on a chart \(U_i\) included in the manifold \(\mathcal{M}\). \(K_i\) is called Kahler potential.

\[
K_i : U_i \rightarrow \mathbb{R}
\]  

(124)

\[
K_i = K_i^i
\]  

(125)

The Kahler form can be locally expressed in terms of the Kahler potential as

\[
\Omega = i\delta \bar{\delta} K_i
\]  

(126)

The definition given above is the most general one. In the main paper this method will be used for the specific case of the Quantum Monte Carlo phase sign problem.

8. BRST-anti-BRST, Kahler partitioning and dual gauge fixing

One important aspect discussed in the main paper is the simultaneous direct and dual gauge fixing of artificial shift symmetries on a complexified space. This is done using some special properties of the matrices \(M\) and \(N\). Following reference [7] the matrix \(M\) insures the simultaneous gauge fixing of the collective fields in a BRST-anti-BRST invariant way. This matrix must be invertible and may have complex numbers as entries. While acting on the field space it must have the symmetry property \(M_{AB}^{\alpha \beta} = (-1)^{\epsilon_{\alpha \beta}^r} M_{BA}^{\alpha \beta}\). It must also insure that \(\phi_A M^{AB} \phi_B\) has global ghostnumber zero, where here, \(\phi_A\) and \(\phi_B\) are arbitrary fields from the theory. In the discussion of reference [7] no other requirements on the \(M\) matrix are needed. Geometric quantization follows several important steps. The first would be the
construction of a symplectic manifold $\mathcal{M}$ of even dimension ($\text{dim}(\mathcal{M}) = 2n$) using the BV, BRST or field-antifield prescriptions. The next step is called “polarization” and involves the selection of $n$ directions over this manifold on which the resulting quantum states should depend. The probably best known polarizations produce the Schrodinger or momentum representations in basic quantum mechanics. These are however not the only ones. While the Batalin-Vilkovisky procedure generates

$$T_{(1,0)} = \{ v \in T_x \mathcal{M}^C | J_x(v) = iv \};$$

$$T_{(0,1)} = \{ v \in T_x \mathcal{M}^C | J_x(v) = -iv \}$$

One may observe that $M_{AB}$ has the potential to induce a specific metric over the field space constructed from the original fields and the additional ghosts, antighosts, ghost-for-ghosts, etc. In order to use this potential for the current problem I introduce two other matrices

$$N^\Omega \Gamma = \frac{1}{2}(h^{\Omega \Gamma} - if^{\Omega \Gamma}),$$

$$\overline{N}^\Omega \Gamma = \frac{1}{2}(h^{\Omega \Gamma} + if^{\Omega \Gamma}).$$

(128)

Their role is to induce a special gauge fixing that generates a Kahler structure over the field space. That gauge fixing can be done by choosing a metric over the symplectic BV (or BRST) field space has been shown in reference [29]. Apart from the standard BRST-anti-BRST operators, algebraic geometry defines also the dual-BRST-anti-BRST operators. These are related to the direct operators via a Hodge star transformation. Moreover, the Hodge star operation induces an extra discrete symmetry. The Kahler structure imposed by the $N^\Omega \Gamma$ matrices assures that this symmetry is of the form of an anti-unitary time reversal operation, as required to solve the sign problem (see ref. [6]). Polarization has two main parts. First it induces a form of partitioning of the field space in “momentum” and “position” types variables. Second, it imposes a condition that eliminates half of these variables from the definition of the wavefunction. In this case the last part is not of interest. For the first part however one can consider the manifold $T^* \mathcal{M}$ and define a complex basis $\{ z_j, \bar{z}_j \}$. The symplectic form becomes $\omega = \frac{i}{2} d\bar{z}_j \wedge z_j$ and the complex structure is defined by the

$$\varphi^\Omega \Gamma = \frac{1}{2}(\varphi^\Omega \pm i \varphi^\Omega), \quad N^\Omega \Gamma = \frac{1}{2}(h^{\Omega \Gamma} - if^{\Omega \Gamma}),$$

$$\overline{N}^\Omega \Gamma = \frac{1}{2}(h^{\Omega \Gamma} + if^{\Omega \Gamma}).$$

(130)

This will extend the $\varphi^\Omega$ potential while the matrices $N$ and $\overline{N}$ will mix the extension with the original terms. In the 2n dimensional manifold the procedure of generating the $n$ dimensional quantum space has additional freedom. This leads to a different form in which the variables (fields) can be partitioned, called the Kahler polarization. A procedure very similar to the construction of a polarization is used here in order to introduce a complex structure over the symplectic manifold. This generates a split of the field structure into two distinct components.

9. Kahler duality transformation and symmetry

I show here that via a suitable shift in the field space a theory can be constructed that has the precise form as the one given in the main article for the Kahler-extended formulation. Let the Lagrangean be

$$L = L_0 + f_{\Omega \Gamma} \varphi^\Omega \varphi^\Gamma$$

(129)

The Lagrangean can be extended by shifting terms and fields

$$L = L_0 + f_{\Omega \Gamma} \varphi^\Omega \varphi^\Gamma + h_{\Omega \Gamma} \varphi^\Omega \varphi^\Gamma$$

(131)

this way at a first instance one obtains
and in the end
\[ L = L_0[\Phi] + L_{\text{col}} \] (132)
where \( \Phi \) is the general notation for any field occurring in the theory. Now one has to gauge fix this by the equation in the main paper
\[ L_{\text{col}} = -\frac{1}{4} e^{ab} \delta_a \delta_b \delta(\varphi^+ \Omega N \varphi^+ \Gamma - \varphi^- \Omega \bar{N} \varphi^- \Gamma) \] (133)
But again, here one can make use of the freedom in the definition of the matrices \( N \) and \( \bar{N} \). Combined with the metric induced by the matrix \( M \), the dualization and the Hodge star operator inducing a discrete symmetry one can generate a splitting of the field space in blocks such that the final field structure is of the form similar to the Kahler structure. One observes that it is of no importance what kind of fields one considers (Grassmann or bosonic) because the whole set of original fields is in the end split into two blocks after the introduction of the Kahler “partitioning”. As a consequence this method works for theories combining bosons and fermions with no additional problems. In fact due to the specific way in which the symplectic and Kahler structures are constructed one can also identify an artificially induced symmetry between fermions and bosons.

10. The Jacobian

The two main ideas of this paper (symmetry out of cohomology and dual gauge fixing) define a new way in which symmetry can be regarded. Instead of considering it as given by nature, here, some discrete symmetries are used as artificial tools that can be added or removed from the theory. In order to make this clear I use the field-antifield formalism. What one usually considers when studying theoretical problems are actions that have some importance what kind of fields one considers (Grassmann or bosonic) because the whole set of original fields is in the end split into two blocks after the introduction of the Kahler “partitioning”. As a consequence this method works for theories combining bosons and fermions with no additional problems. In fact due to the specific way in which the symplectic and Kahler structures are constructed one can also identify an artificially induced symmetry between fermions and bosons.

\[ \omega = dy^i \wedge dy^j \omega_{ij} \] (135)
\[ d\omega = 0 \] (136)
Finally an antibracket structure emerges
\[ \{ A, B \} = A \delta^i_j \omega^{ij} \partial_j B \] (137)
By introducing the internal space in the way explained in chapter 3 of the main article (equations 43-45) one extends the space again. Now $D = 2d$ and I define the hodge star operation and its associated duality. Having the Kahler structure defined by

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$  \hspace{1cm} (138)$$

and going to a complex coordinate basis

$$z^a = (z^\alpha, \bar{z}^\alpha) \quad \bar{z}^a = (\bar{z}^\alpha, \bar{\bar{z}}^\alpha), \quad \alpha = 1, 2, ..., d$$  \hspace{1cm} (139)$$

$$z^\alpha = x^\alpha + i x^{d+\alpha} \quad \bar{z}^\alpha = \bar{x}^\alpha + i \bar{x}^{d+\alpha}$$  \hspace{1cm} (140)$$

we obtain a supermanifold with a Kahlerian geometry and an equivalent change in the representation of the antibracket. Following reference [17] (for the sake of brevity I will not perform the calculations here again) the change in the metric which amounts to the redefinition of the poisson bracket (generalized to the antibracket in our situation)

$$\{f, g\} = \sum_{\alpha\beta} \Omega^{\alpha,\beta} \frac{\partial f}{\partial \eta^\alpha} \frac{\partial g}{\partial \eta^\beta}$$  \hspace{1cm} (141)$$

modifies the expression of the integration measure taking the change of the metric in the definition of the antibracket and mapping it onto the structure of the resulting global block-determinant. (see eq. (11)-(15) and (17)-(18) of ref. [17]). This ensures that the discrete symmetry affects the resulting determinant in the desired way.

Another way of looking at this discrete symmetry is to consider it as induced by the antipode of a hopf-algebra (the vector space analogue of the Hodge star). Only after one constructs the global BRST-anti-BRST and dual-BRST-anti-BRST symmetries will the discrete symmetry emerge. The method of constructing the first two symmetries already implies the inclusion of the Jacobian of the considered transformations. This will correctly modify action as well as the measure of integration (see [8],[9]).

One may also notice that here, I used the de-Rham cohomology and Hodge duality in order to generate a discrete symmetry. Further symmetries could be obtained considering other topological properties like cobordism or Morse-surgery.

11. Practical calculation

I present here preliminary results obtained by applying a path integral Monte Carlo method to a simple oscillator-quartic anharmonic potential. While this is not a numerical proof of validity it may be considered as a test for a known case. The results presented in the left figure have been obtained using the corrective series expansion in the form of a effective potential [28]. The dual gauge fixing method used to produce the right figure had no need for such corrective expansions and converged to approximately the same values. The number of iterations appear larger in the gauge fixing method but one has to consider that in the left figure the cost of constructing the effective corrective potential is not considered. That effective potential calculation introduced several other terms and became intractable for higher orders. I am aware that this example is not specifically related to the fermionic sign problem. The figures presented here aim just to show that my method is valid and consistent with known results. Further investigation of the effects of this method in more relevant physical situations (especially fermionic problems) is obviously desirable.

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