A NEW CHARACTERIZATION OF THE CLIFFORD TORUS VIA SCALAR CURVATURE PINCHING

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Abstract. Let $M^n$ be a compact hypersurface with constant mean curvature $H$ in $S^{n+1}$. Denote by $S$ the squared norm of the second fundamental form of $M$. We prove that there exists a positive constant $\gamma(n)$ depending only on $n$ such that if $|H| \leq \gamma(n)$ and $\beta(n, H) \leq S \leq \beta(n, H) + \frac{\gamma(n)^2}{n^2}$, then $S \equiv \beta(n, H)$ and $M$ is one of the following cases: (i) $S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}})$, $1 \leq k \leq n-1$; (ii) $S^1(\sqrt{\frac{1}{1+\mu^2}}) \times S^{n-1}(\sqrt{\frac{n-1}{\mu^2}})$. Here $\beta(n, H) = n + \frac{n^3}{2(n-1)}H^2 + \frac{n(n-2)}{4(n-1)^2}n^2H^4 + 4(n-1)H^2$ and $\mu = \frac{n|H|+\sqrt{n^2H^2+4(n-1)}^2}{2(n-1)^2}$. This provides a new characterization of the Clifford torus.

1. Introduction

Let $M^n$ be an $n$-dimensional compact hypersurface with constant mean curvature in an $(n+1)$-dimensional unit sphere $S^{n+1}$. Denote by $R$, $H$ and $S$ the scalar curvature, the mean curvature and the squared norm of the second fundamental form of $M$, respectively. It follows from the Gauss equation that $R = n(n-1) + n^2H^2 - S$. A famous rigidity theorem due to Simons, Lawson, and Chern, do Carmo and Kobayashi ([11], [14], [20]) says that if $M$ is a closed minimal hypersurface in $S^{n+1}$ satisfying $S \leq n$, then $S \equiv 0$ and $M$ is the great sphere $S^n$, or $S \equiv n$ and $M$ is the Clifford torus $S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}})$, $1 \leq k \leq n-1$. Afterward, Li-Li [16] improved Simons’ pinching constant for $n$-dimensional closed minimal submanifolds in $S^{n+p}$ to $\max\{\frac{n-2}{2(1/p) + 3n}\}$. Further developments on this rigidity theorem have been made by many other authors (see [10], [12], [17], [24], [25], [31], etc.). In 1970’s, Chern proposed the following conjecture.

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Chern Conjecture. (See [11], [32]) Let $M$ be a compact minimal hypersurface in the unit sphere $\mathbb{S}^{n+1}$.

(A) If $S$ is constant, then the possible values of $S$ form a discrete set. In particular, if $n \leq S \leq 2n$, then $S = n$, or $S = 2n$. 
(B) If $n \leq S \leq 2n$, then $S \equiv n$, or $S \equiv 2n$.

In 1983, Peng and Terng ([18], [19]) made a breakthrough on the Chern conjecture, and proved the following

**Theorem A.** Let $M$ be a compact minimal hypersurface in the unit sphere $\mathbb{S}^{n+1}$.

(i) If $S$ is constant, and if $n \leq S \leq n + \frac{1}{12n}$, then $S = n$.
(ii) If $n \leq 5$, and if $n \leq S \leq n + \tau_1(n)$, where $\tau_1(n)$ is a positive constant depending only on $n$, then $S \equiv n$.

During the past three decades, there have been some important progress on these aspects (see [4], [8], [9], [13], [21], [22], [29], [30], [33], etc.). In 1993, Chang [4] proved Chern Conjecture (A) in dimension three. Yang-Cheng [30] improved the pinching constant $\frac{1}{12n}$ in Theorem A(i) to $\frac{2}{3}$. Later, Suh-Yang [21] improved this pinching constant to $\frac{2}{11}n$.

In 2007, Wei and Xu [22] proved that if $M$ is a compact minimal hypersurface in $\mathbb{S}^{n+1}$, $n = 6, 7$, and if $n \leq S \leq n + \tau_2(n)$, where $\tau_2(n)$ is a positive constant depending only on $n$, then $S \equiv n$. Later, Zhang [33] extended the second pinching theorem due to Peng-Terng [19] and Wei-Xu [22] to the case of $n = 8$. Recently, Ding and Xin [13] obtained the striking result, as stated

**Theorem B.** Let $M$ be an $n$-dimensional compact minimal hypersurface in the unit sphere $\mathbb{S}^{n+1}$, and $S$ the squared length of the second fundamental form of $M$. If $n \geq 6$, and if $n \leq S \leq \frac{24}{23}n$, then $S \equiv n$, i.e., $M$ is a Clifford torus.

The rigidity problem for hypersurfaces of constant mean curvature is much more complicated than the minimal hypersurface case (see [2], [3], [5], [23], [24], [26], [27]). For example, the famous Lawson conjecture [15], verified by Brendle [3], states that the Clifford torus is the only compact embedded minimal surface with genus 1 in $\mathbb{S}^3$. Recently, Andrews and Li [2] proved a beautiful classification theorem for constant mean curvature tori in $\mathbb{S}^3$, which implies that constant mean curvature tori in $\mathbb{S}^3$ include the Clifford torus as well as many other CMC surfaces.

Let $M$ be an $n$-dimensional compact submanifold with parallel mean curvature in the unit sphere $\mathbb{S}^{n+p}$. We put

$$\alpha(n, H) = n + \frac{n^3}{2(n-1)} H^2 - \frac{n(n-2)}{2(n-1)} \sqrt{n^2 H^4 + 4(n-1)H^2},$$
\[ \beta(n, H) = n + \frac{n^3}{2(n-1)} H^2 + \frac{n(n-2)}{2(n-1)} \sqrt{n^2 H^4 + 4(n-1)H^2}, \]
\[ \lambda = \frac{n|H| + \sqrt{n^2 H^2 + 4(n-1)}}{2(n-1)}, \quad \mu = \frac{n|H| + \sqrt{n^2 H^2 + 4(n-1)}}{2}. \]

In [23, 24], the first author proved the generalized Simons-Lawson-Chern-dO-Carmo-Kobayashi theorem for submanifolds with parallel mean curvature in a sphere.

**Theorem C.** Let \( M \) be an \( n \)-dimensional oriented compact submanifold with parallel mean curvature in an \((n+p)\)-dimensional unit sphere \( \mathbb{S}^{n+p} \). If \( S \leq C(n, p, H) \), then \( M \) is either a totally umbilic sphere, a Clifford torus in an \((n+1)\)-sphere, or the Veronese surface in \( \mathbb{S}^4(\frac{1}{\sqrt{1+H^2}}) \). In particular, if \( M \) is a compact hypersurface with constant mean curvature in \( \mathbb{S}^{n+1} \), and if \( S \leq \alpha(n, H) \), then \( M \) is either a totally umbilic sphere, or a Clifford torus. Here the constant \( C(n, p, H) \) is defined by

\[ C(n, p, H) = \begin{cases} 
\alpha(n, H), & \text{for } p = 1, \text{ or } p = 2 \text{ and } H \neq 0, \\
\min \left\{ \alpha(n, H), \frac{1}{3}(2n + 5nH^2) \right\}, & \text{otherwise.}
\end{cases} \]

The second pinching theorem for \( n(\leq 8) \)-dimensional hypersurfaces with small constant mean curvature was proved for \( n \leq 7 \) by Cheng-He-Li [7] and Xu-Zhao [28], and for \( n = 8 \) by Chen-Li [6] and Xu [29], respectively. In [27], the authors obtained the following second pinching theorem for hypersurfaces with small constant mean curvature in spheres.

**Theorem D.** Let \( M \) be an \( n \)-dimensional compact hypersurface with constant mean curvature in the unit sphere \( \mathbb{S}^{n+1} \). There exist two positive constants \( \gamma_0(n) \) and \( \delta_0(n) \) depending only on \( n \) such that if \( |H| \leq \gamma_0(n) \), and \( \beta(n, H) \leq S \leq \beta(n, H) + \delta_0(n) \), then \( S = \beta(n, H) \) and \( M \) is one of the following cases: (i) \( \mathbb{S}^k(\sqrt{\frac{k}{n}}) \times \mathbb{S}^{n-k}(\sqrt{\frac{n-k}{n}}) \), \( 1 \leq k \leq n-1 \); (ii) \( \mathbb{S}^1(\sqrt{\frac{1}{1+\mu^2}}) \times \mathbb{S}^{n-1}(\sqrt{\frac{\mu}{1+\mu^2}}) \).

Note that the second pinching constant \( \delta_0(n) \) in Theorem D is equal to \( O(\frac{1}{\sqrt{n}}) \). The purpose of this paper is to give a new characterization of the Clifford torus. We prove the following theorem.

**Main Theorem.** Let \( M \) be an \( n \)-dimensional compact hypersurface with constant mean curvature in the unit sphere \( \mathbb{S}^{n+1} \). There exists a positive constant \( \gamma(n) \) depending only on \( n \) such that if \( |H| \leq \gamma(n) \), and \( \beta(n, H) \leq S \leq \beta(n, H) + \frac{n}{24} \), then \( S = \beta(n, H) \) and \( M \) is one of
the following cases: (i) $S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}}), 1 \leq k \leq n - 1$; (ii) $S^1(\frac{1}{\sqrt{1+\mu^2}}) \times S^{n-1}(\frac{1}{\sqrt{1+\mu^2}})$.

Our main theorem generalizes Theorem B and improves Theorem D as well. The key ingredient of our proof is to estimate $A - 2B$. By using the parameter method, we obtain an upper bound for $A - 2B$ in the form of $\frac{1}{4}|S + 4 + C(n)G^{1/3} + q(n, H)|\nabla h|^2$. Then we derive a new integral estimate with two parameters. By choosing suitable values of the parameters, we show that $\nabla h = 0$ which implies that $M$ is a Clifford torus.

2. Preliminaries

Let $M^n$ be an $n(\geq 2)$-dimensional compact hypersurface with constant mean curvature in the unit sphere $S^{n+1}$. We shall make use of the following convention on the range of indices:

$1 \leq i, j, k, \ldots, \leq n$.

We choose a local orthonormal frame $\{e_1, e_2, \ldots, e_{n+1}\}$ near a fixed point $x \in M$ over $S^{n+1}$ such that $\{e_1, e_2, \ldots, e_n\}$ are tangent to $M$.

Let $\{\omega_1, \omega_2, \ldots, \omega_{n+1}\}$ be the dual frame fields of $\{e_1, e_2, \ldots, e_{n+1}\}$. Denote by $R$, $H$, $h$ and $S$ the scalar curvature, the mean curvature, the second fundamental form and the squared length of the second fundamental form of $M$, respectively. Then we have

\[ h = \sum_{i,j} h_{ij} \omega_i \otimes \omega_j, \quad h_{ij} = h_{ji}, \]

\[ S = \sum_{i,j} h_{ij}^2, \quad H = \frac{1}{n} \sum_i h_{ii}, \]

\[ R = n(n - 1) + n^2H^2 - S. \]

Choose $e_{n+1}$ such that $H \geq 0$. Denote by $h_{ijk}$, $h_{ijkl}$ and $h_{ijklm}$ the first, second and third covariant derivatives of the second fundamental tensor $h_{ij}$, respectively. Then we have

\[ \nabla h = \sum_{i,j,k} h_{ijk} \omega_i \otimes \omega_j \otimes \omega_k, \quad h_{ijk} = h_{ikj}, \]

\[ h_{ijkl} = h_{ijlk} + \sum_m h_{mj}R_{mikl} + \sum_m h_{im}R_{mjkl}, \]

\[ h_{ijklm} = h_{ijkm} + \sum_r h_{jr}R_{rlm} + \sum_r h_{irk}R_{rjlm} + \sum_r h_{jir}R_{rlm}. \]
Take a suitable orthonormal frame \( \{ e_1, e_2, \ldots, e_n \} \) at \( x \) such that \( h_{ij} = \lambda_i \delta_{ij} \) for all \( i, j \). Then we have

\[
\frac{1}{2} \Delta S = S(n - S) - n^2 H^2 + nH f_3 + |\nabla h|^2,
\]

\[
\frac{1}{2} \Delta |\nabla h|^2 = (2n + 3 - S)|\nabla h|^2 - \frac{3}{2} |\nabla S|^2 + |\nabla^2 h|^2 + 3(2B - A) + 3nHC,
\]

where

\[
f_k = \sum_i \lambda_i^k, \quad A = \sum_{i,j,k} h_{ijk}^2 \lambda_i^2, \quad B = \sum_{i,j,k} h_{ijk}^2 \lambda_i \lambda_j, \quad C = \sum_{i,j,k} h_{ijk}^2 \lambda_i.
\]

Following [27], we have

\[
|\nabla^2 h|^2 \geq \frac{3}{4} \sum_{i \neq j} t_{ij}^2 = \frac{3}{4} \sum_{i,j} t_{ij}^2,
\]

\[
3(A - 2B) \leq \sigma S|\nabla h|^2,
\]

where \( t_{ij} = h_{ijij} - h_{jiij} \) and \( \sigma = \frac{\sqrt{17} + 1}{2} \). Moreover, we have

\[
\int_M (A - 2B) dM = \int_M (nH f_3 - S^2 - f_3^2 + S f_4 - \frac{|\nabla S|^2}{4}) dM.
\]

Denote by \( \phi := \sum_{i,j} \phi_{ij} \omega_i \otimes \omega_j \) the trace free second fundamental form of \( M \). By diagonalizing \( h_{ij} \), we have \( \phi_{ij} = \mu_i \delta_{ij} \), where \( \mu_i = \lambda_i - H \). Putting \( \Phi = |\phi|^2 \) and \( \bar{f}_k = \sum_i \mu_i^k \), we get \( \Phi = S - nH^2 \) and \( f_3 = \bar{f}_3 + 3H \Phi + nH^3 \). From (2.7), we obtain

\[
\frac{1}{2} \Delta \Phi = -F(\Phi) + |\nabla \phi|^2,
\]

where \( F(\Phi) = \Phi^2 - n\Phi - nH^2 \Phi - nH \bar{f}_3 \). Therefore, we have

\[
|\nabla \Phi|^2 = \frac{1}{2} \Delta (\Phi)^2 - \Phi \Delta \Phi = \frac{1}{2} \Delta (\Phi)^2 + 2F(\Phi) - 2\Phi |\nabla \phi|^2,
\]

\[
\int_M F(\Phi) dM = \int_M |\nabla \phi|^2 dM.
\]

Put \( \beta_0(n, H) = \beta(n, H) - nH^2 \). When \( \Phi \geq \beta_0(n, H) \), it’s seen from Proposition [I (i) that

\[
F(\Phi) \geq 0.
\]

Moreover, if \( F(\Phi) = 0 \), then \( \Phi = \beta_0(n, H) \).
Set $G = \sum_{i,j} (\lambda_i - \lambda_j)^2 (1 + \lambda_i \lambda_j)^2$. Then we have

\begin{equation}
G = 2[S f_4 - f_3^2 - S^2 - S(S-n) + 2nH f_3 - n^2 H^2].
\end{equation}

This together with (2.7) and (2.11) implies

\begin{equation}
\frac{1}{2} \int_M GdM = \int_M [(A - 2B) - |\nabla h|^2 + \frac{1}{4} |\nabla S|^2]dM.
\end{equation}

The following propositions will be used in the proof of Main Theorem.

**Proposition 1.** (See [24], [33])

(i) Let $a_1, a_2, \ldots, a_n$ be real numbers satisfying $\sum_i a_i = 0$ and $\sum_i a_i^2 = \mathcal{A}$. Then

$$|\sum_i a_i^3| \leq \frac{n-2}{\sqrt{n(n-1)}} \mathcal{A}^{\frac{3}{2}},$$

and the equality holds if and only if at least $n-1$ numbers of $a_i$'s are same with each other.

(ii) Assume $f_n(t) = 17[t - 2 - \eta(n)][3(n-2)t + (n+2)\eta(n) + 10 - 4n]$ and $g_n(t) = [8 + 16\eta(n)](4t - 2 - 3\sqrt{-2t^2 + 2t + 8})$, for $2 \leq t \leq 1 + \sqrt{\frac{17}{2}}$ and $4 \leq n \leq 5$. Here $\eta(4) = 0.16$ and $\eta(5) = 0.23$. Then

$$h_n(t) = f_n(t) - g_n(t) \leq 0.$$

**Proposition 2.** Let $a_{ij}, b_i$ $(i, j = 1, \ldots, n)$ be real numbers satisfying

\begin{equation}
\sum_i b_i = 0, \quad \sum_i b_i^2 = \mathcal{B} > 0, \quad \sum_{i,j} (b_i + b_j) a_{ij} = \mathcal{C}.
\end{equation}

Then we have

\begin{equation}
\sum_i a_{ii}^2 + 3 \sum_{i \neq j} a_{ij}^2 \geq \frac{3\mathcal{C}^2}{2(n+4)\mathcal{B}}.
\end{equation}

**Proof.** Applying the Lagrange multiplier method, we compute the minimum value $L_{\min}$ of $L = \sum_i a_{ii}^2 + 3 \sum_{i \neq j} a_{ij}^2$ with constraints $\sum_i b_i = 0, \sum_i b_i^2 = \mathcal{B}, \sum_{i,j} (b_i + b_j) a_{ij} = \mathcal{C}$. We consider the function $L^* = \sum_i a_{ii}^2 + 3 \sum_{i \neq j} a_{ij}^2 + \lambda \sum_i b_i + \mu(\sum_i b_i^2 - \mathcal{B}) + \nu[\sum_{i,j} (b_i + b_j) a_{ij} - \mathcal{C}]$. At the extreme points of $L$, we have

\begin{equation}
\frac{\partial L^*}{\partial a_{ii}} = 2a_{ii} + 2\nu b_i = 0, \quad \text{for } 1 \leq i \leq n,
\end{equation}

\begin{equation}
\frac{\partial L^*}{\partial a_{ij}} = 6a_{ij} + \nu(b_i + b_j) = 0, \quad \text{for } 1 \leq i, j \leq n, \ i \neq j.
\end{equation}
It follows from (2.18) and (2.19) that

\[(2.21) \quad 2 \sum_i a_{ii}^2 + 2\nu \sum_i a_{ii}b_i = 0,\]

\[(2.22) \quad 6 \sum_{i\neq j} a_{ij}^2 + \nu \sum_{i\neq j} a_{ij}(b_i + b_j) = 0.\]

Hence

\[(2.23) \quad 2\left(\sum_i a_{ii}^2 + 3 \sum_{i\neq j} a_{ij}^2 \right) = -\nu \mathcal{E}.\]

On the other hand, we have

\[(2.24) \quad 2 \sum_i b_i a_{ii} + 2\nu \sum_i b_i^2 = 0,\]

\[(2.25) \quad 6 \sum_{i\neq j} (b_i + b_j)a_{ij} + \nu \sum_{i\neq j} (b_i + b_j)^2 = 0.\]

Combining (2.20), (2.24) and (2.25), we have

\[(2.26) \quad \nu = \frac{3\mathcal{E}}{(n + 4)\mathcal{B}}.\]

This together with (2.23) and the fact that the Hessian matrix of \(L\) is identically positive definite implies that

\[(2.27) \quad L_{\min} = \frac{3\mathcal{E}^2}{2(n + 4)\mathcal{B}}.\]

q.e.d.

### 3. Estimate for \(A - 2B\)

It plays a crucial role to estimate for \(A - 2B\) in our work. Using an analogous argument as in the proof of Lemma 3.2 in [33], we get the following

**Lemma 1.** Let \(M\) be an \(n\)-dimensional closed hypersurface with constant mean curvature in the unit sphere \(\mathbb{S}^{n+1}\). If \(\lambda_1^2 - 4\lambda_1\lambda_2 \geq tS\), for some \(t \in \left[2, \frac{1+\sqrt{17}}{2}\right]\), then \((\lambda_1^2 - 4\lambda_1\lambda_2) - (\lambda_i^2 - 4\lambda_1\lambda_i) \geq rS\), for \(i \neq 1, 2\), where \(r = \frac{16t-8-12\sqrt{17}-2t^2+2t+8}{17}\).

For the low dimensional cases, we give an up bound for \(A - 2B\).
Lemma 2. If \(4 \leq n \leq 5\), then
\[
3(A - 2B) \leq [2 + \eta(n)]S|\nabla h|^2,
\]
where \(\eta(4) = 0.16\) and \(\eta(5) = 0.23\).

Proof. We put
\[
w_i = h_{ii}, \quad w = \sum_{i \neq 1} w_i^2 + \frac{1}{3} w_1^2,
\]
\[
f = \sum_{i \neq 1} (\lambda_i^2 - 4\lambda_1\lambda_i)w_i^2 - \lambda_1^2 w_1^2.
\]
If \(\lambda_i^2 - 4\lambda_1\lambda_i \leq [2 + \eta(n)]S\), for any \(i \neq 1\), we have \(f \leq [2 + \eta(n)]Sw\).
Otherwise, without loss of generality, we assume that \(\lambda_i^2 - 4\lambda_1\lambda_i = tS\) for some \(t \in [2, \frac{1 + \sqrt{17}}{2}]\), and \(w_1 = zw_2\). Since \(H\) is a constant, we have
\[
(w_1 + w_2)^2 = \left(\sum_{i \neq 1, 2} w_i\right)^2.
\]
Hence
\[
\sum_{i \neq 1, 2} w_i^2 \geq \frac{(1 + z)^2}{n - 2} w_2^2,
\]
\[
\lambda_1^2 \geq (t - 2)S.
\]
From (3.1), (3.2) and Lemma 1, we have
\[
f - [2 + \eta(n)]Sw \leq [t - 2 - \eta(n)]Sw_2^2 + [t - r - 2 - \eta(n)]S \sum_{i \neq 1, 2} w_i^2
\]
\[
- (t - 2 + \frac{2 + \eta(n)}{3})Sw_1^2
\]
\[
\leq [t - 2 - \eta(n)]Sw_2^2 + \frac{t - r - 2 - \eta(n)}{n - 2}(1 + z)^2 Sw_2^2
\]
\[
- (t - 2 + \frac{2 + \eta(n)}{3}) z^2 Sw_2^2,
\]
where \(r = \frac{16t^2 - 8 - 12\sqrt{2} + 2t + 8}{2t} \). Set \(K(n, t, z) = t - 2 - \eta(n) + \frac{t - r - 2 - \eta(n)}{n - 2}(1 + z)^2 - (t - 2 + \frac{2 + \eta(n)}{3}) z^2\). (3.3) becomes
\[
f - [2 + \eta(n)]Sw \leq K(n, t, z)Sw_2^2.
\]
From (3.1), (3.2) and Lemma 1, we have
\[
f - [2 + \eta(n)]Sw \leq [t - 2 - \eta(n)]Sw_2^2 + [t - r - 2 - \eta(n)]S \sum_{i \neq 1, 2} w_i^2
\]
\[
- (t - 2 + \frac{2 + \eta(n)}{3})Sw_1^2 - t - 2 + \frac{2 + \eta(n)}{3} z^2 Sw_2^2,
\]
where \(r = \frac{16t^2 - 8 - 12\sqrt{2} + 2t + 8}{2t} \). Set \(K(n, t, z) = t - 2 - \eta(n) + \frac{t - r - 2 - \eta(n)}{n - 2}(1 + z)^2 - (t - 2 + \frac{2 + \eta(n)}{3}) z^2\). (3.3) becomes
\[
f - [2 + \eta(n)]Sw \leq K(n, t, z)Sw_2^2.
\]
Noting that \(\frac{\partial K(n, t, z)}{\partial z}|_{z = z_0} = 0\), we have
\[
K(n, t, z) \leq K(n, t, z_0) = \frac{h_n(t)}{51L(n, t)}.
\]
where \( L(n, t) = r + 2 + \eta(n) - t + (n - 2)(t - 2 + \frac{2+\eta(n)}{2}) \), and \( h_n(t) \) is defined as in Proposition 1 (ii). This together with Proposition 1 (ii) implies

\[
(3.6) \quad f \leq [2 + \eta(n)]Sw.
\]

Similarly, for any fixed \( j \), we have

\[
f_j = \sum_{i \neq j} (\lambda^2_j - 4\lambda_j \lambda_i)h^2_{ij} - \lambda^2_j h^2_{jjj}
\]

\[
(3.7) \quad \leq (2 + \eta(n))S(\sum_{i \neq j} h^2_{ij} + \frac{1}{3}h^2_{jjj}).
\]

Hence

\[
3(A - 2B) = \sum_{i,j,k \text{ distinct}} [2(\lambda^2_i + \lambda^2_j + \lambda^2_k) - (\lambda_i + \lambda_j + \lambda_k)^2]h^2_{ijk}
\]

\[
-3 \sum_i \lambda^2_i h_{ii}^2 + 3 \sum_i \sum_j (\lambda^2_j - 4\lambda_i \lambda_j)h^2_{ij}
\]

\[
\leq 2S \sum_{i,j,k \text{ distinct}} h^2_{ijk} + 3 \sum_j \sum_{i \neq j} [(\lambda^2_j - 4\lambda_i \lambda_j)h^2_{ij} - \lambda^2_j h^2_{jjj}]
\]

\[
\leq (2 + \eta(n))S(\sum_{i,j,k \text{ distinct}} h^2_{ijk} + 3 \sum_j \sum_{i \neq j} h^2_{ij} + \sum_j h^2_{jjj})
\]

\[
(3.8) \quad = (2 + \eta(n))S \sum_{i,j,k \text{ distinct}} h^2_{ijk}.
\]

This proves Lemma 2.

q.e.d.

For the higher dimensional cases, we obtain the following estimate of \( A - 2B \).

**Lemma 3.** If \( n \geq 6 \) and \( n \leq \frac{16}{19}n \), then

\[
3(A - 2B) \leq [S + 4 + C_3(n)G^{1/3} + q_5(n, H)]\|\nabla h\|^2.
\]

Here \( C_3(n) = \left( \frac{3}{\sqrt{6} - 4} \right)^{1/3} (6 - \sqrt{6} - 13p)^2 \), \( p = \frac{1}{13(n-2)} \), \( q_5(n, H) = \left\{ 2\left( \frac{16n}{13} \right)^{2/3} \left( \frac{15n}{3n-16} \right)^{1/3} p_1^{1/3} + (64n^2 - 4)^{2/3} \left( \frac{15n}{3n-16} + \frac{n^2H}{n-2} + \frac{2nH}{n-2} \sqrt{\frac{32n}{15}} \right) \right\} H^{1/3} \)

and \( p_1 = 5 + \frac{3}{2} \left( \frac{15n}{3n-16} - 1 \right) + \frac{3}{2} \left( \frac{15n}{3n-16} - 1 \right)^{-1} \).

**Proof.** For fixed distinct \( i, j, k \), we put

\[
\varphi = \lambda^2_i + \lambda^2_j + \lambda^2_k - 2\lambda_i \lambda_j - 2\lambda_j \lambda_k - 2\lambda_i \lambda_k,
\]

\[
\psi = \lambda^2_j - 4\lambda_i \lambda_j.
\]
If \( \lambda_i, \lambda_j, \lambda_k \) have the same sign, it is easy to see that \( \varphi \leq S \). Without loss of generality, we suppose

\[
(3.10) \quad \lambda_i \lambda_j \leq \lambda_j \lambda_k \leq 0, \lambda_i \lambda_k \geq 0.
\]

Putting \( \lambda_i = -x \lambda_j, \lambda_k = -y \lambda_j, x \geq y \geq 0 \), we have

\[
\varphi = \lambda_i^2 + \lambda_j^2 + \lambda_k^2 + 2(x + y - xy)\lambda_j^2
\]

\[
\leq S + 4 + 2[x\lambda_j^2 - 1 + (1 - x)y\lambda_j^2 - 1].
\]

Setting \( a = x\lambda_j^2 - 1, b = y\lambda_j^2 - 1, c = (1 - x)y\lambda_j^2 - 1 \), we rewrite (3.11) as follow

\[
(3.12) \quad \varphi \leq S + 4 + 2(a + c).
\]

Since

\[
(3.13) \quad \lambda_j^2 + \frac{1}{n - 1}(nH - \lambda_j)^2 \leq S \leq \frac{16n}{15},
\]

we have

\[
(3.14) \quad \lambda_j^2 \leq \frac{16(n - 1)}{15} + 2H\lambda_j - nH^2
\]

\[
\leq \frac{16(n - 1)}{15} + 2H\sqrt{\frac{16n}{15}}.
\]

Hence

\[
(3.15) \quad \frac{16(n - 1)}{15} \lambda_j^2 \leq 1 + q_1,
\]

where \( q_1 = q_1(n, H) := \frac{\sqrt{15n}}{2(n - 1)}H \).

When \( c \geq 0 \), we get \( x \leq 1 \) and \( a \geq b \geq c \geq 0 \). Then we have

\[
(3.16) \quad c \leq x(1 - x)\lambda_j^2 + q_1 - \frac{15}{16(n - 1)}\lambda_j^2
\]

\[
\leq \frac{4n - 19}{32n - 17}(x + 1)^2\lambda_j^2 + q_1
\]

\[
\leq (1 - \frac{16}{5n})\frac{2n - 2}{16n - 1}(x + 1)^2\lambda_j^2 + q_1.
\]

Similarly, we have

\[
(3.17) \quad a \leq x\lambda_j^2 + q_1 - \frac{15}{16(n - 1)}\lambda_j^2
\]

\[
\leq \frac{4n - 4}{16n - 1}(x + 1)^2\lambda_j^2 + q_1.
\]
and

\[
b \leq y\lambda_j^2 + q_1 - \frac{15}{16(n-1)}\lambda_j^2
\]

(3.18)

\[
b \leq \frac{4n - 4}{16n - 1}(y + 1)^2\lambda_j^2 + q_1.
\]

So,

\[
(a + c)^3 = a^3 + c^3 + 3(a^2c + ac^2)
\]

\[
\leq a^3 + c^3 + 3\left(1 + \frac{\epsilon}{2}\right)a^2 \left(1 - \frac{16}{5n}\right)\frac{2n - 2}{16n - 1}(x + 1)^2\lambda_j^2 + q_1
\]

\[
+ \frac{1}{2\epsilon}b^2 \left(\frac{4n - 4}{16n - 1}(y + 1)^2\lambda_j^2 + q_1\right)
\]

\[
= a^3 + b^3 + 3\left(\frac{2n - 2}{16n - 1}\right)\left[1 + \frac{\epsilon}{2}\right]a^2 \left(1 - \frac{16}{5n}\right)(x + 1)^2
\]

\[
+ \frac{1}{2\epsilon}b^2(y + 1)^2\lambda_j^2 + \left[3(1 + \frac{\epsilon}{2})a^2 + \frac{3}{2\epsilon}b^2\right]q_1,
\]

(3.19)

for \(\epsilon > 0\). Letting \((1 + \frac{\epsilon}{2})(1 - \frac{16}{5n}) = \frac{1}{\epsilon}\), we have \(\epsilon = \sqrt{\frac{15n - 16}{5n - 16}} - 1\). Since

\[
G \geq 2(\lambda_i - \lambda_j)^2(\lambda_i\lambda_j + 1)^2 + 2(\lambda_j - \lambda_k)^2(\lambda_j\lambda_k + 1)^2
\]

(3.20)

\[= 2[(x + 1)^2a^2 + (y + 1)^2b^2]\lambda_j^2,
\]

we have

\[
(a + c)^3 \leq a^3 + b^3 + 3\frac{2n - 2}{(16n - 1)\epsilon}[a^2(x + 1)^2 + b^2(y + 1)^2]\lambda_j^2
\]

\[
+ \left[3(1 + \frac{\epsilon}{2})a^2 + \frac{3}{2\epsilon}b^2\right]q_1
\]

\[
\leq a^2 \left[\frac{4n - 4}{16n - 1}(x + 1)^2\lambda_j^2 + q_1\right] + b^2 \left[\frac{4n - 4}{16n - 1}(y + 1)^2\lambda_j^2 + q_1\right]
\]

\[
+ \frac{3}{(16n - 1)\epsilon}[a^2(x + 1)^2 + b^2(y + 1)^2]\lambda_j^2
\]

\[
+ \left[3(1 + \frac{\epsilon}{2})a^2 + \frac{3}{2\epsilon}b^2\right]q_1
\]

\[
= \frac{2n - 2}{16n - 1}(2 + \frac{3}{\epsilon})[a^2(x + 1)^2 + b^2(y + 1)^2]\lambda_j^2
\]

\[
+ \left[(4 + \frac{3\epsilon}{2})a^2 + (1 + \frac{3}{2\epsilon})b^2\right]q_1
\]

(3.21)

\[
\leq \frac{n - 1}{16n - 1}(2 + \frac{3}{\epsilon})G + q_2.
\]
where \( q_2 = q_2(n, H, a, b) = \left[(4 + \frac{3\epsilon}{2})a^2 + (1 + \frac{3\epsilon}{2})b^2\right]q_1 \).

When \( c < 0 \) and \( a \geq 0 \), (3.21) becomes
\[
(a + c)^3 \leq a^3 \leq \frac{n-1}{16n-1}(2 + \frac{3\epsilon}{\epsilon})G + q_2.
\]

This yields
\[
a + c \leq \left(\frac{n-1}{16n-1}(2 + \frac{3\epsilon}{\epsilon})G + q_2\right)^{1/3}
\]
\[
\leq \left[\frac{n-1}{16n-1}(2 + \frac{3\epsilon}{\epsilon})G\right]^{1/3} + q_2^{1/3}.
\]

So,
\[
\varphi \leq S + 4 + 2\left[\frac{n-1}{16n-1}(2 + \frac{3\epsilon}{\epsilon})G\right]^{1/3} + 2q_2^{1/3}
\]
\[
= S + 4 + C'_3(n)G^{1/3} + 2q_2^{1/3}.
\]

Here \( C'_3(n) = \frac{8(n-1)}{16n-1}(2 + \frac{3\epsilon}{\epsilon}) \).

When \( c < 0 \) and \( a < 0 \), it’s obvious that (3.24) holds.

To derive an upper bound for \( \psi \), it’s sufficient to estimate \( \psi \) on \( T = \{x \in M; \psi > S + 4\} \). At a fixed point \( x \in T \), we have \( \psi > S + 4 \). This implies \( \lambda_i\lambda_j < 0 \). Thus, there exists \( t > 0 \), such that \( \lambda_i = -t\lambda_j \). Then we get
\[
S \geq \lambda_i^2 + \lambda_j^2 + \frac{1}{n-2}(nH - \lambda_i - \lambda_j)^2
\]
\[
= \frac{n-1}{n-2}(\lambda_i^2 + \lambda_j^2) + \frac{2}{n-2}\lambda_i\lambda_j + \frac{n^2H^2}{n-2} - \frac{2nH}{n-2}(\lambda_i + \lambda_j).
\]

Hence
\[
\psi \leq S - \frac{n-1}{n-2}\lambda_i^2 - \frac{1}{n-2}\lambda_j^2 - \frac{2}{n-2}\lambda_i\lambda_j - r_1 - 4\lambda_i\lambda_j
\]
\[
= S + \left(-\frac{n-1}{n-2}t^2 + \frac{4n-6}{n-2}t - \frac{1}{n-2}\right)\lambda_j^2 - r_1,
\]
where \( r_1 = r_1(n, H, \lambda_i, \lambda_j) = \frac{n^2H^2}{n-2} - \frac{2nH}{n-2}(\lambda_i + \lambda_j) \). From (3.15), we have
\[
\psi \leq S + 4 + \left(-\frac{n-1}{n-2}t^2 + \frac{4n-6}{n-2}t - \frac{1}{n-2}\right)\lambda_j^2 - r_1
\]
\[
\leq S + 4 + \left(-\frac{n-1}{n-2}t^2 + \frac{4n-6}{n-2}t\right)
\]
\[
- \frac{1}{n-2}\lambda_j^2 - \frac{15}{4n-4}\lambda_j^2 + 4q_1 - r_1
\]
\[
\leq S + 4 + \left(-\frac{n-1}{n-2}t^2 + \frac{4n-6}{n-2}t\right)
\]
\[
- \frac{4}{n-2}\lambda_j^2 + r_2.
\]
Here \( r_2 = 4q_1 - r_1 \). Since

\[
(3.28) \quad \psi \leq S - \lambda_i^2 - 4\lambda_i \lambda_j = S + (4t - t^2)\lambda_j^2
\]

and

\[
(3.29) \quad -\frac{n-1}{n-2}t^2 + \frac{4n-6}{n-2}t - \frac{4}{n-2} \leq \left( t - \frac{12}{13(n-2)} \right)(4-t),
\]

we obtain

\[
(3.30) \quad (\psi - S - 4)^3 \leq \frac{((4t - t^2)\lambda_j^2 - 4)^2}{(1 + t)^2} \left( t - \frac{12}{13(n-2)} \right)(4-t)^2 + r_2
\]

Here \( r_3 = [(4t - t^2)\lambda_j^2 - 4]r_2 \). From (3.28), we get

\[
(3.31) \quad (4t - t^2)\lambda_j^2 > 4.
\]

Hence \( 4t - t^2 > 0 \), i.e., \( 0 < t < 4 \). We define an auxiliary function

\[
\xi(t) := \frac{(t-12p)(4-t)^3}{(1+t)^2}, \quad 0 < t < 4,
\]

where \( p = \frac{1}{13(n-2)} \). Letting \( \frac{d\xi(t)}{dt} \bigg|_{t=t_0} = 0 \), we get

\[
t_0 = \sqrt{9p^2 + 54p + 6 + 3p - 2},
\]

and

\[
\xi_0 := \xi(t_0) = \frac{(2 - 2t_0 + 18p)(4-t_0)^2}{1 + t_0}.
\]

It is easy to see that \( \frac{d^2\xi(t)}{dt^2} \bigg|_{t=t_0} < 0 \). Hence \( t_0 \) is the maximum point of \( \xi(t) \). Since \( t_0 \leq \sqrt{6} + 10p - 2 + 3p \) and \( \zeta(t) = \frac{(2-2t+18p)(4-t)^2}{1+t} \) is decreasing in \( t \), we have

\[
(3.32) \quad \xi(t) \leq \xi_0 \leq \frac{(6 - 2\sqrt{6} - 8p)(6 - \sqrt{6} - 13p)^2}{\sqrt{6} - 1 + 13p} = 2C_3(n)^3.
\]

Here \( C_3(n) = \left[ \frac{(6 - 2\sqrt{6} - 8p)(6 - \sqrt{6} - 13p)^2}{\sqrt{6} - 1 + 13p} \right]^{1/3} \). By the definition of \( G \), we have

\[
(3.33) \quad G \geq 2(\lambda_i - \lambda_j)^2(1 + \lambda_i \lambda_j)^2 = 2(t + 1)^2\lambda_i^2(t\lambda_j^2 - 1)^2.
\]
From (3.30) and (3.32), we obtain
\[
(\psi - S - 4)^3 \leq \xi(t)(1 + t^2)(t\lambda_j^2 - 1)^2\lambda_j^2 + r_3 \\
\leq C_3(n)^3G + r_3 \\
(3.34) \\
\leq (C_3(n)G^{1/3} + r_3^{1/3})^3,
\]
i.e.,
\[
\psi - S - 4 \leq C_3(n)G^{1/3} + r_3^{1/3}.
\]
Therefore, at any point \(x \in M\), (3.34) holds.

By a direct computation, we have
\[
C_3(7) \geq \lim_{n \to \infty} C_3'(n) \quad \text{and} \quad C_3'(6) \leq C_3(6).
\]
Noting that \(C_3'(n)\) and \(C_3(n)\) are both increasing in \(n\), we get
\[
C_3(n) \geq C_3'(n), \quad \text{for} \quad n \geq 6.
\]
Now we give proper upper bounds for \(r_3\) and \(q_2\). Since \(4t - t^2 \leq 4t - t^2|_{t=2} = 4\), we have
\[
(3.35) \quad (4t - t^2)\lambda_j^2 - 4 \leq 4 \times \frac{16n}{15} - 4,
\]
and
\[
(3.36) \quad |\lambda_i + \lambda_j| \leq \sqrt{2(\lambda_i^2 + \lambda_j^2)} \leq \sqrt{\frac{32n}{15}}.
\]
Then we obtain
\[
r_3 = [(4t - t^2)\lambda_j^2 - 4]^2r_2 \\
\leq [(4t - t^2)\lambda_j^2 - 4]^2[4q_1 + \frac{n^2H^2}{n-2} + \frac{2nH}{n-2}|\lambda_i + \lambda_j|] \\
(3.37) \leq q_3,
\]
where \(q_3 = q_3(n, H) = (\frac{64n}{15} - 4)^2 \left(4q_1 + \frac{n^2H^2}{n-2} \right)^2 + \frac{2nH}{n-2}\sqrt{\frac{32n}{15}}H\). Since \(x \geq y \geq 0\), we have
\[
(3.38) \quad a^2 = (x\lambda_j^2 - 1)^2 \leq x^2\lambda_j^2 = \lambda_i^2\lambda_j^2 \leq \left(\frac{16}{15}n\right)^2,
\]
and
\[
(3.39) \quad b^2 = (y\lambda_j^2 - 1)^2 \leq y^2\lambda_j^2 = \lambda_i^2\lambda_j^2 \leq \left(\frac{16}{15}n\right)^2.
\]
Hence
\[
(3.40) \quad q_2 = \left[(4 + \frac{3\epsilon}{2})a^2 + (1 + \frac{3\epsilon}{2})b^2\right]q_1 \leq q_4,
\]
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where \( q_4 = q_4(n, H) = \left( \frac{16}{15} n \right)^2 p_1 q_1, p_1 = 5 + \frac{3\sigma}{2} + \frac{3}{2\pi} \). Thus, we have

\[
3(A - 2B) \leq \sum_{i,j,k \text{ distinct}} h_{ijk}^2 (\lambda_i^2 + \lambda_j^2 + \lambda_k^2 - 2\lambda_i \lambda_j - 2\lambda_j \lambda_k - 2\lambda_i \lambda_k) \\
+ \sum_{i \neq j} h_{ij}^2 (\lambda_j^2 - 4\lambda_i \lambda_j) \\
\leq \sum_{i,j,k \text{ distinct}} h_{ijk}^2 (S + 4 + C_3(n)G^{1/3} + \frac{2q_1^{1/3}}{3}) \\
+ \sum_{i \neq j} h_{ij}^2 (S + 4 + C_3(n)G^{1/3} + r_3^{1/3}) \\
(3.41) \leq (S + 4 + C_3(n)G^{1/3} + q_5)\|\nabla h\|^2,
\]

where \( q_5 = q_5(n, H) = 2q_4^{1/3} + q_3^{1/3} \). This completes the proof of Lemma 3. q.e.d.

4. PROOF OF MAIN THEOREM

Let \( x_n \) be the unique positive solution of the following equation

\[
\frac{n^3}{2(n - 1)} x + \frac{n(n - 2)}{2(n - 1)} \sqrt{n^2 x^2 + 4(n - 1)x} = \frac{n}{15} - \frac{n}{23}.
\]

Then we have

\[
\frac{n^3}{2(n - 1)} H^2 + \frac{n(n - 2)}{2(n - 1)} \sqrt{n^2 H^4 + 4(n - 1)H^2} \leq \frac{n}{15} - \frac{n}{23},
\]

for \( H \leq \gamma_1(n) := \sqrt{x_n} \).

Now we are in a position to prove our main theorem.

**Proof of Main Theorem.** For the low dimensional cases \((2 \leq n \leq 5)\), it follows from (2.8), (2.9) and (2.17) that

\[
(4.1) \int_M [(S - 2n - \frac{3}{2})\|\nabla h\|^2 + \frac{3}{2}(A - 2B) + \frac{9}{8}|\nabla S|^2 - 3nHC]dM \geq 0,
\]
for $H \leq \gamma_1(n)$. By the definition of $F(\Phi)$, we get

$$
\frac{F(\Phi)}{\Phi} = \frac{\Phi^2 - n\Phi - nH^2\Phi - nH\bar{f}_3}{\Phi}
= \Phi - n - nH^2 - \frac{nH\bar{f}_3}{\Phi}
\geq \Phi - \beta_0(n, H) - nH^2 - \frac{nH}{\Phi} \frac{n-2}{\sqrt{n(n-1)}} \Phi^{3/2}
\geq \Phi - \beta_0(n, H) - nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} \sqrt{\frac{16}{15} nH}
(4.2)
= \Phi - \beta_0 - q_6,
$$

where $q_6 = q_6(n, H) = nH^2 + \frac{4n(n-2)}{\sqrt{15(n-1)}} H$. Similarly, we have

$$(4.3) \quad \frac{F(\Phi)}{\Phi} \leq \Phi - \beta_0 + D_0 + q_6,$$

where $D_0 = D_0(n, H) = \beta_0(n, H) - n$. On the other hand, we have

$$
\frac{\bar{f}_3}{\Phi} \leq \frac{n-2}{\sqrt{n(n-1)}} \sqrt{\Phi} \leq \frac{n-2}{\sqrt{n(n-1)}} \sqrt{\frac{16n}{15}}.
(4.4)
$$

It follows from (4.3) and (4.4) that

$$
\Phi F(\Phi) = (F(\Phi) + n\Phi + nH^2\Phi + nH\bar{f}_3) \frac{F(\Phi)}{\Phi}
\leq \left( \Phi - \beta_0 + D_0 + q_6 + n + nH^2 + \frac{4n(n-2)H}{\sqrt{15(n-1)}} \right) F(\Phi).
(4.5)
$$

Since

$$(4.6) \quad |C| \leq \sqrt{S} |\nabla h|^2, \ n \leq S,$$

we have

$$
-3nH \int_M C dM \leq 3nH \int_M \sqrt{S} |\nabla h|^2 dM
\leq \frac{3\sqrt{15} H}{\sqrt{n(n-1)}} S |\nabla h|^2 dM.
(4.7)
$$
Here \( C = \sum_{i,j,k} h_{ijk}^2 \lambda_i \). Combining (2.13), (4.1), (4.3), (4.7) and the condition \( \beta(n, H) \leq S \leq \beta(n, H) + \frac{n}{27} \), we obtain

\[
0 \leq \int_M \left\{ \frac{9}{4}(\Phi - \beta_0)F(\Phi) + \left[ \frac{9}{4}(D_0 + q_6 + n + nH^2 + \frac{4n(n-2)H}{\sqrt{15(n-1)}}) \right. \right.
\]
\[
+ S - 2n - \frac{3}{2} - \frac{9}{4}\Phi + 3\sqrt{nH}S \left| \nabla h \right|^2 + \frac{3}{2}(A - 2B) \left( A - 2B \right) \} \} dM
\]
\[
\leq \int_M \left\{ \left[ \frac{9}{4}\left( \frac{n}{2} + n + D_0 + q_6 + nH^2 + \frac{4n(n-2)H}{\sqrt{15(n-1)}} \right) \right. \right.
\]
\[
+ S - 2n - \frac{3}{2} - \frac{9}{4}\Phi + \frac{16n\sqrt{nH}}{5} \left| \nabla h \right|^2 + \frac{3}{2}(A - 2B) \} \} dM
\]
\[
= \int_M \left[ \left( \frac{n}{4} + \frac{9n}{92} - \frac{5S}{4} - \frac{3}{2} + q_7 \right) \right.
\]
\[
x\left| \nabla h \right|^2 + \frac{3}{2}(A - 2B) \right\} dM,
\]

(4.8)

where \( q_7 = q_7(n, H) = \frac{9}{4} \left( D_0 + q_6 + 2nH^2 + \frac{4n(n-2)H}{\sqrt{15(n-1)}} \right) + \frac{16n\sqrt{nH}}{5} \).

When \( 2 \leq n \leq 3 \), substituting (2.10) into (4.8), we get

\[
0 \leq \int_M \left[ \frac{n}{4} + \frac{9n}{92} + \left( \frac{\sigma}{2} - \frac{5}{4} \right) \left( n + D_0 + nH^2 + \frac{n}{23} \right) \right.
\]
\[
- \frac{3}{2} + q_7 \left] \left| \nabla h \right|^2 dM \right.
\]
\[
\leq \int_M \left[ \frac{3}{46}(24\sigma - 67) + q_7 \right.
\]
\[
+ \left( \frac{\sigma}{2} - \frac{5}{4} \right) \left( D_0 + nH^2 \right) \right\} \left| \nabla h \right|^2 dM.
\]

(4.9)

There exists a positive constant \( \gamma_2(n) \) depending only on \( n \) such that \( q_7 + \left( \frac{\sigma}{2} - \frac{5}{4} \right) \left( D_0 + nH^2 \right) \leq 0.1 \) for \( H \leq \gamma_2(n) \), which implies that the coefficient of the integral in (4.9) is negative. This together with (4.9) implies \( \nabla h = 0 \).

When \( 4 \leq n \leq 5 \), by Lemma 2, we have

\[
3(A - 2B) \leq [2 + \eta(n)]S\left| \nabla h \right|^2,
\]
where $\eta(4) = 0.16$ and $\eta(5) = 0.23$. Similarly, we get
\[
0 \leq \int_M \left[ \frac{n}{4} + \frac{9n}{92} + \left( \frac{2 + \eta}{2} - \frac{5}{4} \right)n - \frac{3}{2} + q_7 \right] |\nabla h|^2 dM
= \int_M \left( \frac{9n}{92} + \frac{9\eta}{2} - \frac{3}{2} + q_7 \right) |\nabla h|^2 dM
\leq \int_M \left( \frac{45}{92} + 5 \times 0.23 \right) \frac{2}{2} + q_7 \right) |\nabla h|^2 dM.
\]
(4.10)

We choose a positive constant $\gamma_2(n)$ depending only on $n$ such that $q_7 \leq 0.1$ for $H \leq \gamma_2(n)$. Therefore, the coefficient of the integral in (4.10) is negative. This together with (4.10) implies $\nabla h = 0$.

For higher dimensional cases ($n \geq 6$), we define $u_{ijkl} = \frac{1}{4}(h_{ijkl} + h_{lijk} + h_{klij} + h_{jkli})$. Notice that $u_{ijkl}$ is symmetric in $i, j, k, l$. Then we have
\[
\sum_{i,j,k,l} (h_{ijkl}^2 - u_{ijkl}^2) = \frac{1}{16} \sum_{i,j,k,l} [(h_{ijkl} - h_{lijk})^2 + (h_{ijkl} - h_{klij})^2 + (h_{lijk} - h_{klij})^2 + (h_{lijk} - h_{klij})^2]
\geq \frac{6}{16} \sum_{i \neq j} [(h_{ijij} - h_{jiji})^2 + (h_{ijji} - h_{jiji})^2]
\geq \frac{3}{4} G,
\]
(4.11)

\[
\sum_{i,j,k,l} u_{ijkl}^2 \geq \sum_i u_{iii}^2 + 3 \sum_{i \neq j} u_{iijj}^2.
\]
(4.12)

Since $\sum_i \mu_i = 0$, $\sum_i \mu_i^2 = \Phi$ and $\sum_{i,j} (\mu_i + \mu_j) u_{ijij} = -F(\Phi)$, using Proposition 2 we obtain
\[
|\nabla^2 h|^2 \geq \frac{3}{4} G + \frac{3F^2(\Phi)}{2(n+4)\Phi}.
\]
(4.13)

For $0 < \theta < 1$, $H \leq \gamma_1(n)$ and $\beta(n, H) \leq S \leq \beta(n, H) + \delta(n)$, we have
\[
\int_M |\nabla^2 h|^2 dM \geq \left( \frac{3(1-\theta)}{4} + \frac{3\theta}{4} \right) \int_M GdM + \int_M \frac{3F^2(\Phi)}{2(n+4)\Phi} dM.
\]
(4.14)
Combining (2.8), (2.17) and (4.14), using Lemma 3 and Young’s inequality, we drive the following inequality.

\[
\frac{3(1-\theta)}{4} \int_M G dM + \int_M \frac{3F^2(\Phi)}{2(n+4)\Phi} dM \\
\leq \int_M \left[ (S - 2n - 3) |\nabla h|^2 + \frac{3}{2} |\nabla S|^2 \\
+ 3(A - 2B) - 3nHC - \frac{3\theta}{4} G \right] dM \\
= \int_M (S - 2n - 3 + \frac{3\theta}{2}) |\nabla h|^2 dM + (3 - \frac{3\theta}{2}) \int_M (A - 2B) dM \\
+ \left( \frac{3}{2} - \frac{3\theta}{8} \right) \int_M |\nabla S|^2 dM - 3nH \int_M C dM \\
\leq \int_M (S - 2n - 3 + \frac{3\theta}{2}) |\nabla h|^2 dM \\
+ \left( 1 - \frac{\theta}{2} \right) \int_M \left( S + 4 + C_3(n)G^{1/3} + q_5 \right) |\nabla h|^2 dM \\
+ \left( \frac{3}{2} - \frac{3\theta}{8} \right) \int_M |\nabla S|^2 dM - 3nH \int_M C dM \\
\leq \int_M \left[ (2 - \frac{\theta}{2}) S - 2n + 1 - \frac{\theta}{2} + (1 - \frac{\theta}{2})q_5 \right] |\nabla h|^2 dM \\
+ \frac{3(1-\theta)}{4} \int_M G dM + C_4 \int_M |\nabla h|^3 dM \\
+ \left( \frac{3}{2} - \frac{3\theta}{8} \right) \int_M |\nabla S|^2 dM - 3nH \int_M C dM,
\]

(4.15) where \( C_4 = C_4(n, \theta) = \frac{4}{3} C_3(n)^{3/2} (1 - \frac{\theta}{2})^{3/2} (1 - \theta)^{-1/2}. \)

From (2.12), we get

\[
\int_M |\nabla \phi| \Delta \Phi = -F(\Phi)|\nabla \phi| + |\nabla \phi|^3.
\]

(4.16) This together with the divergence theorem and Cauchy-Schwarz’s inequality implies

\[
\int_M |\nabla h|^3 dM \leq \int_M F(\Phi)|\nabla \phi| dM + \epsilon \int_M |\nabla^2 \phi|^2 dM \\
+ \frac{1}{16\epsilon} \int_M |\nabla \Phi|^2 dM,
\]

(4.17) for \( \epsilon > 0. \) From (4.12), we have

\[
\int_M \frac{3F^2(\Phi)}{2(n+4)\Phi} dM \geq \int_M \frac{3}{2(n+4)}(\Phi - \beta_0 - q_0) F(\Phi) dM.
\]

(4.18)
Combining (4.7), (4.15), (4.17) and (4.18), we obtain

\[
0 \leq \int_M \left[ (2 - \frac{\theta}{2} + 3\sqrt{nH})S - 2n + 1 - \frac{\theta}{2} + (1 - \frac{\theta}{2})q_5 \right] |\nabla h|^2 dM \\
+ C_4 \left( \int_M F(\Phi)|\nabla \phi|dM + \epsilon \int_M |\nabla^2 \phi|^2 dM + \frac{1}{16\epsilon} \int_M |\nabla \Phi|^2 dM \right) + \frac{3}{2} - \frac{3\theta}{8} \int_M |\nabla \Phi|^2 dM - \int_M \frac{3}{2(n + 4)}(\Phi - \beta_0 - q_6)F(\Phi)dM.
\]

(4.19)

It follows from (2.8) and (2.10) that

\[
\int_M |\nabla \Phi|^2 dM = \int_M (\Phi - \beta_0)F(\Phi)dM + \int_M (\beta_0 - \Phi)|\nabla \phi|^2 dM.
\]

(4.21)

Thus, (4.19) becomes

\[
0 \leq \int_M \left\{ \left[ 2 - \frac{\theta}{2} + 3\sqrt{nH} + \epsilon C_4(\sigma + 1 + 3\sqrt{nH}) \right] (\Phi - \beta_0 + \beta) \\
- 2n + 1 - \frac{\theta}{2} + (1 - \frac{\theta}{2})q_5 - \epsilon C_4(2n + 3) \right\} |\nabla \phi|^2 dM \\
+ \left( \frac{3}{2} - \frac{3\theta}{8} + \frac{C_4}{16\epsilon} + \frac{3\epsilon C_4}{2} \right) \int_M |\nabla S|^2 dM \\
+ C_4 \int_M F(\Phi)|\nabla \phi|dM - \int_M \frac{3}{2(n + 4)}(\Phi - \beta_0 - q_6)F(\Phi)dM \\
= \int_M \left\{ D \left[ 2 - \frac{\theta}{2} + 3\sqrt{nH} + \epsilon C_4(\sigma + 1 + 3\sqrt{nH}) \right] + 1 - \frac{\theta}{2}(n + 1) \\
+ 3n\sqrt{nH} + (1 - \frac{\theta}{2})q_5 + \frac{3}{2(n + 4)}q_6 + \epsilon C_4(\sigma n + 3n^\frac{3}{2}H - n - 3) \right\} \times |\nabla \phi|^2 dM \\
+ C_4 \int_M F(\Phi)|\nabla \phi|dM - \left[ 1 - \frac{\theta}{4} + \frac{C_4}{8\epsilon} - 3\sqrt{nH} \right. \\
+ \epsilon C_4(2 - \sigma - 3\sqrt{nH}) \right] \int_M (\Phi - \beta_0)|\nabla \phi|^2 dM \\
+ \left( \frac{3}{4} - \frac{3\theta}{4} + \frac{C_4}{8\epsilon} + 3\epsilon C_4 - \frac{3}{2(n + 4)} \right) \int_M (\Phi - \beta_0)F(\Phi)dM,
\]

(4.22)
where $D = D(n, H) = \beta(n, H) - n$. On the other hand, it follows from (4.3) that

$$C_4 \int_M F(\Phi)|\nabla \phi|dM \leq 2C_4(\beta_0 + \delta)\epsilon \int_M F(\Phi)dM$$

$$+ \frac{C_4}{8(\beta_0 + \delta)\epsilon} \int_M F(\Phi)|\nabla \phi|^2dM$$

$$\leq 2C_4(\beta_0 + \delta)\epsilon \int_M F(\Phi)dM$$

$$+ \int_M \left( \frac{C_4 \Phi}{8(\beta_0 + \delta)\epsilon} (\Phi - \beta_0 + D_0 + q_6)|\nabla \phi|^2dM \right)$$

$$\leq 2C_4(n + D_0 + \delta)\epsilon \int_M F(\Phi)dM$$

$$(4.23)$$

$$+ \int_M \frac{C_4}{8\epsilon}(\Phi - \beta_0 + D_0 + q_6)|\nabla \phi|^2dM.$$
Let \( \epsilon = \sqrt{\frac{\delta}{8(\sigma n + n - 3 + 5\delta)}} \) and \( \theta = 0.86 \). Then

\[
C_4(n) = \frac{4}{9} \times 0.573/2 \times 0.14^{-1/2} \times \sqrt{\frac{3 - \sqrt{6} - 4p}{\sqrt{6} - 1 + 13p}} (6 - \sqrt{6} - 13p),
\]

where \( p = \frac{13}{13(n-2)} \leq \frac{1}{13} \). Notice that \( C_4(n) \) is increasing in \( n \). Thus, we have

\[
C_4(n) \leq \lim_{l \to \infty} C_4(l) \approx 1.1185 < 1.1186, 0 < \epsilon \leq \frac{1}{6} \text{ and } \sigma = \frac{\sqrt{17} + 1}{2}.
\]

So, \( 0.785 + C_4\epsilon(2 - \sigma) > 0 \). Let \( \delta(n) = \frac{n}{27} \). There exists a positive constant \( \gamma_2(n) \) depending only on \( n \) such that

\[
q_8 \leq 0.001, \text{ for } H \leq \gamma_2(n).
\]

Here \( q_8 = q_8(n, H) = D \left[ 2 - \frac{\theta}{2} + 3\sqrt{n}H + \epsilon C_4(\sigma + 1 + 3\sqrt{n}H) \right] + 3n^3\epsilon C_4H + 3n\sqrt{n}H + \left( 1 + \frac{\theta}{2} \right) q_5 + \frac{3}{2(n+4)} q_6 + 2C_4D_0\epsilon + C_8 \left( D_0 + q_6 \right) + 3\sqrt{n}H + 3\epsilon C_4\sqrt{n}H. \]

Then we have

\[
0 \leq \left[ 1 - \frac{\theta}{2} (n+1) + \epsilon C_4(\sigma n + n - 3 + 5\delta) + \frac{C_4\delta}{8\epsilon} + \left( 3 - \frac{3\theta}{4} - \frac{3}{2(n+4)} \right) \delta + q_8 \right] \int_M |\nabla \phi|^2 dM
\]

\[
\leq \left( -0.43n + 0.57 \right) + C_4\sqrt{\frac{\delta}{2}(\sigma n + n - 3 + 5\delta)}
\]

(4.25)

\[
+ (2.355 - \frac{3}{2(n+4)})\delta + 0.001 \right) \int_M |\nabla \phi|^2 dM.
\]

When \( n \geq 75 \), we have

\[
C_4\sqrt{\frac{\delta}{2}(\sigma n + n - 3 + 5\delta)} \leq 1.1186 \times \sqrt{\frac{n}{46}(\sigma n + n + \frac{5n}{23})} \leq 0.3207n.
\]

(4.26)

Hence

\[
0 \leq \left[ -0.43n + 0.571 + 0.3207n \right.
\]

(4.27)

\[
+ \left( 2.355 - \frac{3}{2(n+4)} \right) \frac{n}{23} \int_M |\nabla \phi|^2 dM.
\]

Since \( \frac{3}{2(n+4)} \frac{n}{23} \geq 0.061 \), we have

(4.28) \( 0 \leq \left( -0.43n + 0.571 + 0.3207n + \frac{2.355n}{23} - 0.061 \right) \int_M |\nabla \phi|^2 dM. \)

Notice that the coefficient of the integral in (4.28) is negative. This shows that \( \nabla \phi = 0 \).
When $6 \leq n \leq 74$, we note that $C_4(n) = \frac{4}{3} \times 0.57^{3/2} \times 0.14^{-1/2} \times \sqrt{\frac{3-\sqrt{6-4p}}{\sqrt{6-1+13p}}} (6-\sqrt{6-13p})$, $p = \frac{1}{13(n-2)}$ and $\delta = \frac{\pi}{25}$. This implies that the coefficient of the integral in $(4.25)$ is negative. Thus, we have $\nabla \phi = 0$.

So, we conclude that

$$\nabla \phi = \nabla h = 0,$$

for $n \geq 2$ and $\gamma(n) \leq H \leq \gamma(n)$, where $\gamma(n) = \min\{\gamma_1(n), \gamma_2(n)\}$. This together with $(2.14)$ and $(2.15)$ implies that $F(\Phi) = 0$ and $\Phi = \beta_0(n, H)$.

When $H = 0$, $\Phi = \beta_0(n, H)$ becomes $\Phi = n$, i.e., $M$ is one of the Clifford torus

$$S^k\left(\sqrt{\frac{k}{n}}\right) \times S^{n-k}\left(\sqrt{\frac{n-k}{n}}\right), 1 \leq k \leq n - 1.$$

When $H \neq 0$, using Proposition 1 (i), we get

$$\lambda_1 = \cdots = \lambda_{n-1} = H - \frac{\beta(n, H) - nH^2}{n(n-1)},$$

$$\lambda_n = H + \frac{(n-1)(\beta(n, H) - nH^2)}{n}.$$

Therefore, $M$ is the Clifford torus

$$S^1\left(\frac{1}{\sqrt{1+\mu^2}}\right) \times S^{n-1}\left(\frac{\mu}{\sqrt{1+\mu^2}}\right)$$

in $S^{n+1}$, where $\mu = \frac{nH+\sqrt{n^2H^2+4(n-1)}}{2}$.

This completes the proof of Main Theorem. q.e.d.

Finally we would like to propose the following conjectures.

**Conjecture A.** Let $M$ be a closed hypersurface with constant mean curvature $H$ in $S^4$. We have

(i) If $\beta(3, H) \leq S \leq 6 + 9H^2$, then $S \equiv \beta(3, H)$ or $S \equiv 6 + 9H^2$, i.e., $M$ is a Clifford torus or a tube of the Veronese surface.

(ii) If $S \geq 6 + 9H^2$, then $S \equiv 6 + 9H^2$, i.e., $M$ is a tube of the Veronese surface.

In particular, if $H = 0$, the problem above is still open. In the case where $M$ is a closed hypersurface with constant mean curvature and constant scalar curvature in $S^4$, Conjecture A was solved by Almeida-Brito [1] and Chang [5]. It is well known that the possible values of the squared length of the second fundamental forms of all closed isoparametric hypersurfaces with constant mean curvature $H$ in the unit sphere...
form a discrete set $I(\subset \mathbb{R})$. The following conjecture can be viewed as a general version of the Chern conjecture.

**Conjecture B.** Let $M$ be an $n$-dimensional closed hypersurface with constant mean curvature $H$ in the unit sphere $S^{n+1}$.

(i) Assume that $a < b$ and $[a, b] \cap I = \{a, b\}$. If $a \leq S \leq b$, then $S \equiv a$ or $S \equiv b$, and $M$ is an isoparametric hypersurface in $S^{n+1}$.

(ii) Set $c = \sup_{t \in I} t$. If $S \geq c$, then $S \equiv c$, and $M$ is an isoparametric hypersurface in $S^{n+1}$.

In particular, if $a = nH^2$ and $b = \alpha(n, H)$, Theorem C provides an affirmative answer to Conjecture B (i).

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