MIRROR SYMMETRY FOR $\mathbb{P}^2$ AND TROPICAL GEOMETRY

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Introduction.

In [14, 15, 16, 17], Bernd Siebert and I have been working on a program designed to understand mirror symmetry via an algebro-geometric analogue of the Strominger-Yau-Zaslow program [32]. The basic idea is that the controlling objects in mirror symmetry are integral affine manifolds with singularities. One can view an integral affine manifold as producing a mirror pair of manifolds, one a symplectic manifold and one a complex manifold, each of twice the real dimension. These correspond to the $A$- and $B$-models of mirror symmetry. A great deal of the work carried out by myself and Siebert has been devoted to building up a dictionary between geometric notions on affine manifolds and objects in the $A$- and $B$-models. If mirror symmetry is to become self-evident from this process, one should be able to find a single geometric notion on an affine manifold which corresponds to both rational curves on the $A$-model side and corrections to period calculations on the $B$-model side. A conceptual proof of mirror symmetry would identify these objects in the world of integral affine geometry. (For a survey of this basic approach, the reader may consult [13]; however, while this paper is motivated by this program, it is largely self-contained.)

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In fact, much progress has been made in this direction. In the last five years, it has become apparent that tropical geometry is the relevant geometry on integral affine manifolds. In particular, thanks to work of Mikhalkin \cite{mikhalkin} in dimension two and work of Nishinou and Siebert \cite{nishinou_siebert} in higher dimensions, it is known that there is a correspondence between tropical rational curves in $\mathbb{R}^n$ and holomorphic rational curves in $n$-dimensional toric varieties. On the other hand, work of Siebert and myself \cite{siebert} has shown that from an integral affine manifold with singularities and some additional data, one can construct a maximally unipotent degeneration of varieties with effective anti-canonical class which is controlled by tropical data on the affine manifold. Thus one begins to see some connection between the two sides of mirror symmetry.

While this includes the Calabi-Yau case, there is still a great deal of work which needs to be done. In the $B$-model one needs to calculate periods and understand how these periods are related to the tropical data determining the degeneration. In the $A$-model, there is as yet no correspondence theorem between tropical curves and actual curves. So we are still some distance from a proof of mirror symmetry using these ideas, but the path is clear.

On the other hand, in the case of $\mathbb{P}^2$, curve-counting via tropical methods is already understood. So this raises the question: Is it possible, using the philosophy of my program with Siebert, to prove mirror symmetry for $\mathbb{P}^2$?

The mirror to $\mathbb{P}^2$ was introduced by Givental in \cite{givental}, in which he explicitly described small quantum cohomology of $\mathbb{P}^2$ via certain oscillatory integrals. Using work of Dubrovin \cite{dubrovin} on semi-simple Frobenius manifolds, one can see that mirror symmetry extends to big quantum cohomology. Work of Sabbah and Sabbah-Douai \cite{sabbah_douai, sabbah} and Barannikov \cite{barannikov} showed how to construct Frobenius manifold structures on the full moduli space of the mirror to $\mathbb{P}^2$. Here, we are interested in Barannikov's explicit description of this Frobenius manifold structure via oscillatory integrals to obtain a description of big quantum cohomology with gravitational descendents. The $B$-model description of small quantum cohomology for $\mathbb{P}^2$ is simpler, and its tropical interpretation has already been studied by Chan and Leung in \cite{chan_leung}.

It is clear that there should be a tropical argument for mirror symmetry; after all, Mikhalkin showed that big quantum cohomology for $\mathbb{P}^2$ can be computed tropically, while Barannikov showed that his $B$-model for the mirror to $\mathbb{P}^2$ calculates big quantum cohomology. The point of this paper is more philosophical. I claim that mirror symmetry for $\mathbb{P}^2$ is in fact equivalent to tropical genus zero curve counting in tropical $\mathbb{P}^2$, in a rather strong and transparent way. In fact, I believe the most conceptual way to prove mirror symmetry for $\mathbb{P}^2$ is via tropical geometry. The same will be true in any dimension, and the ideas in this paper should also generalize to other toric varieties, but for simplicity of exposition, and for maximum explicitness, for the most part we will stick to $\mathbb{P}^2$ in this paper.
The mirror to $X = \mathbb{P}^2$ is an algebraic torus $\hat{X} \cong (\mathbb{C}^*)^2$, best written as $V(x_0x_1x_2 - 1) \subseteq \mathbb{C}^3$. However, $\hat{X}$ must come along with a Landau-Ginzburg potential, a regular function $W: \hat{X} \to \mathbb{C}$. It is usually written as $W = x_0 + x_1 + x_2$, but the full complex moduli space of the mirror is the universal unfolding of this function $W$. The work of Barannikov [2] and Douai-Sabbah [6] generalised classic work of Kyoji Saito [31], showing how to construct a Frobenius manifold structure on this universal unfolding. Barannikov described this structure explicitly in terms of oscillatory integrals.

This procedure is quite subtle, and is explained in low-tech terms in §1 of this paper. The key point is identifying flat coordinates on the universal unfolding moduli space. The existence of these flat coordinates tells us that there is a canonical way of deforming the potential $W$. This raises the question of whether or not it is possible to write down such a canonical deformation directly. This paper answers this question positively.

The main idea is as follows. It has been understood since work of Cho and Oh [4] that the terms in the Landau-Ginzburg potential are in one-to-one correspondence with families of Maslov index two holomorphic disks in $\mathbb{P}^2$ with boundary on a given $T^2 \subseteq \mathbb{P}^2$, a fibre of the moment map on $\mathbb{P}^2$ induced by the toric structure on $\mathbb{P}^2$. Therefore, a natural idea for deforming this potential is to include some points in $\mathbb{P}^2$ and consider families of disks, again with boundary in a given $T^2$, which pass through a subset of the given points. We deform the potential by adding terms corresponding to families of such disks of the proper dimension.

Such a construction is in fact theoretically possible, and Fukaya, Oh, Ohta and Ono have shown in independent work [9] that this idea can be used. The approach here, in keeping with the general philosophy I have advocated with Bernd Siebert, is to replace holomorphic disks with tropical disks. Once this is done, there are no theoretical difficulties involved in defining the correct perturbation, and its calculation is a purely combinatorial problem. This construction is very simple, and is explained in detail in §2. These first two sections can be read as an extended introduction. We also note that work of Nishinou [27] shows that holomorphic disks can be counted tropically, so by working tropically from the beginning, we avoid a great deal of technical difficulties related to holomorphic disks.

The main theorem of the paper is then stated at the end of §3. Its main content is:

(1) The natural parameters appearing in the deformation of $W$ constructed in §2 using tropical disks are in fact flat coordinates.

(2) Mirror symmetry for $\mathbb{P}^2$ is equivalent to specific tropical formulas for descendent Gromov-Witten invariants for $\mathbb{P}^2$ of the form

$$\langle [pt], \cdots, [pt], \psi^k \alpha \rangle_d$$

where $[pt]$ denotes the cohomology class of a point and $\alpha \in H^*(\mathbb{P}^2, \mathbb{C})$. 
Expanding on this second point, we remark that it is standard to put the descendent Gromov-Witten invariants together into a generating function called the $J$-function. Similarly, once we have tropical definitions for these descendents, we can assemble these invariants in the same way to form the tropical $J$-function, which we write as $J_{\text{trop}}$. The philosophy of the paper is then exhibited by the following diagram:

![Diagram](image)

Theorem 3.8 identifies the relevant oscillatory integrals in the B-model with $J_{\text{trop}}$, while mirror symmetry identifies these same integrals with $J$. Hence the equality $J = J_{\text{trop}}$ suggested by the third side of the triangle is equivalent to mirror symmetry.

Note that for $k = 0$, we get ordinary Gromov-Witten invariants for $\mathbb{P}^2$, and since mirror symmetry for $\mathbb{P}^2$ is known, this gives a new proof of Mikhalkin’s tropical curve counting formula \[26\]. Furthermore, for $\alpha = [pt]$, these formulas agree with those recently discovered by Markwig and Rau in \[25\]. This latter work calculates the descendent invariants via tropical intersection theory. At this point in time, this method has not been extended to other choices of $\alpha$. The problem is that tropical intersection theory becomes more difficult when one has to worry about compactified tropical moduli spaces. By restricting to the case where $\alpha = [pt]$, this is not an issue. So there is yet no tropical derivation of the other descendent invariants.

§§4 and 5 are devoted to the proof of the main theorem and are focused on an explicit evaluation of the necessary oscillatory integrals. However, the crucial point, which is explored in §4, is that our deformation of the Landau-Ginzburg potential depends on the choice of a basepoint $Q \in \mathbb{R}^2$ (which we view as tropical $\mathbb{P}^2$), playing the tropical role of a choice of fibre of the moment map. There turns out to be a chamber structure in $\mathbb{R}^2$, so that as $Q$ moves around, the deformed Landau-Ginzburg potential changes via an explicit wall-crossing formula. These wall-crossings can be viewed as having to do with Maslov index zero disks, and this is the same wall-crossing phenomenon observed by Auroux in \[1\]. This chamber structure is formed by what Siebert and I call a scattering diagram,\[1\]It should not be viewed as a surprise that this gives a new proof of Mikhalkin’s formula. Indeed, the proof of Mikhalkin’s formula given by Gathmann and Markwig \[14\] shows that this formula gives a potential satisfying the WDVV equation; essentially the standard proofs of mirror symmetry for $\mathbb{P}^2$ show that the potential produced on the $B$-side also satisfies the WDVV equations.
which played a vital role in [17] and uses ideas originating in [23]. This is really the one point where this paper makes contact with the ideas in [17].

This point of contact should not be surprising. In [23] and [17], scattering diagrams are used to build, in the former article, non-Archimedean K3 surfaces, and in the latter article, degenerations of Calabi-Yau manifolds. It is expected that the scattering diagrams which appear in those papers should describe Maslov index zero disks on the mirror side. So again, wall crossing formulas are associated with the presence of Maslov index zero disks.

In §5, we complete the proof by evaluating the necessary period integrals. This is a rather involved and delicate process, but the calculation shows that we indeed have constructed flat coordinates and produced the tropical formulas for the descendent invariants mentioned above.

As mentioned above, there is some overlap between this work and work of Fukaya, Oh, Ohta and Ono [9]. Both Fukaya and I spoke on preliminary versions of our work at the final conference for the program in Real and Tropical Algebraic Geometry at EPFL in Lausanne in June 2008. At that time, I had not yet determined the formula for the descendent invariants \( \langle [pt], \ldots, [pt], \psi^k[\mathbb{P}^2] \rangle \). While our approaches are quite different, and Fukaya et al.’s work does not deal with descendents, his talk helped lead me to the correct formulation, for which I thank him.

I would also like to thank Paul Hacking, Claus Hertling, Ludmil Katzarkov, Sean Keel, Hannah Markwig, and Bernd Siebert for useful conversations, and thank Kwokwai Chan and Conan Leung for explaining their work to me during a very pleasant visit to CUHK, which motivated me to work on this problem. I would also like to thank Alexander Givental, who pointed out to me in 2004 that there was little hope of using tropical geometry to understand mirror symmetry unless one could find tropical descriptions of gravitational descendents. Finally, I would like to thank the referees for useful suggestions.

1. **Barannikov’s mirror symmetry for \( \mathbb{P}^n \)**

We begin by reviewing Barannikov’s description of the \( B \)-model for \( \mathbb{P}^n \) in terms of oscillatory integrals, giving a precise statement of mirror symmetry. For the purposes of this paper, we do not need the general formalism of semi-infinite variation of Hodge structures, but instead explain the approach as concretely as possible.

Let \( X = \mathbb{P}^n, \hat{X} = V(\prod_{i=0}^n x_i - 1) \subseteq \text{Spec} \mathbb{C}[x_0, \ldots, x_n] \). Here \( \hat{X} \) is isomorphic to \( (\mathbb{C}^\times)^n \), but we should consider \( \hat{X} \) as a Landau-Ginzburg model, with potential

\[
W_0 = \sum_{i=0}^n x_i.
\]
The pair \((\check{X}, W_0)\) is usually viewed as the mirror to \(X\). We then consider the universal unfolding of \(W_0\), with \(t = (t_0, \ldots, t_n)\),

\[ W_t = W_0 + \sum_{j=0}^{n} t_j W_j^0, \]

parameterized by the moduli space \(\mathcal{M} = \text{Spf} \mathbb{C}[t_0, \ldots, t_n]\), the completion at the origin of \(\mathbb{C}^{n+1}\) with coordinates \(t_0, \ldots, t_n\). Here \(\text{Spf}\) denotes the formal spectrum. One then considers the local system \(\mathcal{R}\) on \(\mathcal{M} \times \mathbb{C}^\times\) whose fibre at a point \((t, q)\) is the relative homology group \(H_n(\check{X}, \text{Re}(qW_t) \ll 0; \mathbb{C})\).

Then Barannikov shows first of all, that one can find a unique choice of the following data:

1. A (multi-valued) basis of sections of \(\mathcal{R}\), \(\Xi_0, \ldots, \Xi_n\), with \(\Xi_i\) uniquely determined modulo \(\Xi_0, \ldots, \Xi_{i-1}\).
2. A section \(s\) of \(\mathcal{R}^\vee \otimes \mathcal{O}_{\mathcal{M} \times \mathbb{C}^\times}\) defined by integration of a family of holomorphic forms on \(\check{X} \times \mathcal{M} \times \mathbb{C}^\times\) of the form

\[ e^{qW_t} f \, d\log x_1 \wedge \cdots \wedge d\log x_n, \]

where \(q\) is the coordinate on \(\mathbb{C}^\times\) and \(f\) is a regular function on \(\check{X} \times \mathcal{M} \times \mathbb{C}^\times\) with \(f|_{\check{X} \times \{0\} \times \mathbb{C}^\times} = 1\) and which extends to a regular function on \(\check{X} \times \mathcal{M} \times (\mathbb{C}^\times \cup \{\infty\})\).

This data must satisfy the following conditions:

1. The monodromy associated to \(q \mapsto qe^{2\pi i}\) in the local system \(\mathcal{R}\) is given, in the basis \(\Xi_0, \ldots, \Xi_n\), by \(\exp((n+1)2\pi i\text{N})\), where

\[ N = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \]

2\While this group is often written in this way, it should be defined more precisely as the space of rapid decay homology cycles \([19],[22]\) as follows. Choose a variety \(Y\) containing \(\check{X}\) with \(Y \setminus \check{X}\) a normal crossings divisor, such that the map \(W : \check{X} \to \mathbb{C}\) extends to a map \(W : Y \to \mathbb{P}^1\) with \(Y \setminus \check{X}\) mapping to \(\infty \in \mathbb{P}^1\). Let \(\check{Y}\) denote a real oriented blow-up of \(Y \setminus \check{X}\) in \(Y\). The exceptional locus contains a set \(Z\) consisting of all points \(b\) such that \(\text{Re}(qW(z)) \to -\infty\) as \(z \to b\). We then define the homology group as the relative homology \(H_n(\check{Y}, Z; \mathbb{C})\). The main point is that these are precisely the cycles over which it makes sense to integrate forms of the sort appearing in (M2).

3\The discussion in \([2]\) considers \(\mathcal{M}\) to be an analytic germ of \(0 \in \mathbb{C}^{n+1}\). This raises certain technical issues, because for deformations in the directions \(t_i, i \geq 2\), we obtain “non-tame” behaviour, and this relative homology group jumps, because the critical locus of the Landau-Ginzburg potential \(W_t\) jumps. We get around this problem by working formally around a neighbourhood where the relevant homology group is the correct one. It is not difficult to check that \([2]\) works in this context.
Note that this condition determines the basis $\Xi_0, \ldots, \Xi_n$ up to a change of basis matrix $S = (s_{ij})$ an upper triangular matrix with $s_{i-1,j-1} = s_{ij}$ for $1 \leq i, j \leq n$.

(M4) We identify a fibre of the dual local system $R^\vee$ with the ring $\mathbb{C}[\alpha]/(\alpha^{n+1})$, with $\alpha^i$ dual to $\Xi_i$. Note that $H^{2*}(\mathbb{P}^n, \mathbb{C}) \cong \mathbb{C}[\alpha]/(\alpha^{n+1})$, with the primitive positive generator of $H^2(\mathbb{P}^n, \mathbb{C})$ corresponding to $\alpha$. Under this isomorphism, the action of $(n + 1)N^t$ can be viewed as mirror to cupping with the anti-canonical class in $H^{2*}(\mathbb{P}^n, \mathbb{C})$.

The section $s$ of $R^\vee \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{M} \times \mathbb{C}^\times}$ yields an element of each fibre of $R^\vee$, which we can write as

$$s(t, q) = \sum_{i=0}^{n} \alpha^i \int_{\Xi_i} e^{qW^t} f \ d\log x_1 \wedge \cdots \wedge \ d\log x_n.$$ 

We then require that we can write

$$s(t, q) = q^{(n+1)\alpha} \sum_{i=0}^{n} \varphi_i(t, q)(\alpha/q)^i$$

for functions $\varphi_i$ satisfying

$$\varphi_i(t, q) = \delta_{0,i} + \sum_{j=1}^{\infty} \varphi_{i,j}(t) q^j.$$ 

for $0 \leq i \leq n$. (The shape of this formula determines the section $s$ uniquely; Barannikov refers to this as a normalization condition.) Here we use the expansion

$$q^{(n+1)\alpha} = \sum_{i=0}^{n} \frac{(n + 1)^i}{i!} (\log q)^i \alpha^i$$

to interpret $q^{(n+1)\alpha}$; this takes care of the multi-valuedness of the integrals. A consequence of these two conditions is that if we set

$$y_i(t) = \varphi_{i,1}(t), \quad 0 \leq i \leq n,$$

$y_0, \ldots, y_n$ form a system of coordinates on $\mathcal{M}$, which are called flat coordinates. Furthermore,

$$\lim_{q \to 0} \frac{\varphi_i(0, q)}{q^i} = \delta_{0,i}.$$ 

(This last condition, not explicitly mentioned in [2], fixes the basis $\Xi_0, \ldots, \Xi_n$ up to the action of monodromy.)

We will take the existence of such data satisfying these properties as given, and in some sense our goal will be to identify the flat coordinates $y_0, \ldots, y_n$ and the regular function $f$ tropically, and compare the results with the $A$-model data on $\mathbb{P}^n$. Before discussing the $A$-model data, it is worth noting that it is quite non-trivial to find $y_0, \ldots, y_n$ and $f$ in terms of $t$, and to the best of my knowledge, unlike in the Calabi-Yau case, this computation has
not been carried out in the literature even to low order in the \( t_i \)'s. Rather, Barannikov proves mirror symmetry for \( \mathbb{P}^n \) by showing that the formalism of semi-infinite variation of Hodge structures allows one to construct a Frobenius manifold structure on \( \mathcal{M} \) for which the \( y_i \)'s are flat coordinates. The \( A \)-model for \( \mathbb{P}^n \) also yields a Frobenius manifold, and the fact that these two Frobenius manifolds are isomorphic is shown first by identifying the two algebra structures at one point, and then using semi-simplicity of this Frobenius algebra structure and results of Dubrovin \([8]\).

We now consider the \( A \)-model for \( \mathbb{P}^n \), so we can state mirror symmetry for \( \mathbb{P}^n \). Let \( \overline{\mathcal{M}}_{0,m}(\mathbb{P}^n,d) \) denote the moduli space of stable maps of degree \( d \) from \( m \)-pointed curves of genus zero into \( \mathbb{P}^n \). Let \( \psi_1, \ldots, \psi_m \in H^2(\overline{\mathcal{M}}_{0,m}(\mathbb{P}^n,d), \mathbb{Q}) \) be the usual \( \psi \) classes, i.e., \( \psi_i \) is the first Chern class of the line bundle on \( \overline{\mathcal{M}}_{0,m}(\mathbb{P}^n,d) \) whose fibre at a point \([f,C,p_1,\ldots,p_m]\) is the cotangent line to \( C \) at the marked point \( p_i \). We have evaluation maps \( \text{ev}_i: \overline{\mathcal{M}}_{0,m}(\mathbb{P}^n,d) \rightarrow \mathbb{P}^n \), and define, for classes \( \beta_1, \ldots, \beta_m \in H^*(\mathbb{P}^n, \mathbb{Q}) \), the descendent Gromov-Witten invariant

\[
\langle \psi^{m_1} \beta_1, \ldots, \psi^{m_m} \beta_m \rangle_d = \int_{\overline{\mathcal{M}}_{0,m}(\mathbb{P}^n,d)} \bigwedge_{i=1}^{\sum} (\psi_i^{m_i} \wedge \text{ev}_i^* \beta_i).
\]

We can then write the precise statement of mirror symmetry in terms of the Givental \( J \)-function. This is a function \( J_{\mathbb{P}^n}(y_0, \ldots, y_n, q) \) with values in \( H^*(\mathbb{P}^n, \mathbb{C}) \). Let \( T_0, \ldots, T_n \) be generators of \( H^*(\mathbb{P}^n, \mathbb{Z}) \), with \( T_i \in H^{2i}(\mathbb{P}^n, \mathbb{Z}) \) positive. Then with \( \gamma = \sum_{i=2}^{n} y_i T_i \), the \( J \)-function is defined by

\[
J_{\mathbb{P}^n}(y_0, \ldots, y_n, q) = e^{q(y_0T_0+y_1T_1)} \left( T_0 + \sum_{i=0}^{n} \sum_{m \geq 0} \sum_{d \geq 0} \frac{1}{m!} \langle T_0, \gamma^m, T_{n-i} / (q^{-1} - \psi) \rangle_d e^{d y_1 T_i} \right).
\]

(See e.g., \([21]\), Definition 2.14 for this description of the \( J \)-function.) Here \( \gamma^m \) means we take \( \gamma \) \( m \) times, and \( 1/(q^{-1} - \psi) = q/(1 - q \psi) \) is expanded formally in \( q \psi \). We define \( J_i \) by writing

\[
J_{\mathbb{P}^n}(y_0, \ldots, y_n, q) = \sum_{i=0}^{n} J_i(y_0, \ldots, y_n, q) T_i.
\]

We then consider the following statement:

**Statement 1.1** (Mirror symmetry for \( \mathbb{P}^n \)). In the \( \mathbb{C} \)-vector space \( \mathbb{C}[y_0, \ldots, y_n, q] \),

\[
J_i = \varphi_i.
\]

This mirror symmetry statement was proved by Barannikov in \([2]\) for the part of the statement which does not involve gravitational descendents. In any event, genus zero descendent invariants can be reconstructed from the non-descendent invariants, but see \([20]\) for a more direct proof for the statement with gravitational descendents. However, the philosophy in this paper is not to prove it, but to prove its equivalence to a tropical statement which is stated precisely in Statement 3.10.
It is worthwhile expanding out the expression for the $J$-function. Recall by the Fundamental Class Axiom (see e.g. [5], page 305) that if $d > 0$, we have

$$\langle T_0, T_0^{m_0}, \ldots, T_n^{m_n}, \psi^\nu T_{n-i} \rangle_d = \langle T_0^{m_0}, \ldots, T_n^{m_n}, \psi^{-1} T_{n-i} \rangle_d$$

where a correlator involving $\psi^{-1}$ is interpreted as zero. Also note that $\langle T_0, \psi^\nu T_{n-i} \rangle_0 = 0$ since $\overline{\mathcal{M}}_{0,2}(\mathbb{P}^n, 0)$ is empty. Thus,

$$J_{\mathbb{P}^n} = e^{q(y_0 T_0 + y_1 T_1)} \left( T_0 + \sum_{i=0}^n \sum_{m_2+\cdots+m_n \geq 0} \sum_{\nu \geq 0} \langle T_0, T_2^{m_2}, \ldots, T_n^{m_n}, \psi^\nu T_{n-i} \rangle_d q^{\nu+1} e^{dy_1} \frac{y_2^{m_2} \cdots y_n^{m_n}}{m_2! \cdots m_n!} T_i \right)$$

$$\langle T_0, T_2^{m}, \psi^\nu T_{2-i} \rangle_0 = 0$$

unless $m + 1 = 2m + 2 - i + \nu$, for dimension reasons. Thus we need $0 \leq \nu = -m - 1 + i$, so we only get a contribution when $m = 1$, $i = 2$, as $i \leq 2$. But $\langle T_0, T_2, T_{2-i} \rangle_0 = \int_{\mathbb{P}^2} T_0 \cup T_2 \cup T_{2-i} = \delta_{2,i}$. Thus we get

$$J_{\mathbb{P}^2} = e^{q(y_0 T_0 + y_1 T_1)} \left( T_0 + \sum_{i=0}^2 \left( qy_2 \delta_{2,i} + \sum_{m_2 \geq 0} \sum_{d \geq 1, \nu \geq 0} \langle T_2^{m_2}, \psi^\nu T_{2-i} \rangle_d q^{\nu+2} e^{dy_1} \frac{y_2^{m_2}}{m_2!} T_i \right) \right).$$

2. Tropical geometry

We next review the definition of a tropical curve from [20], and then introduce the notion of a tropical disk.

We fix once and for all a lattice $M = \mathbb{Z}^n$, $N = \text{Hom}_\mathbb{Z}(M, \mathbb{Z})$ the dual lattice, $M_\mathbb{R} = M \otimes_{\mathbb{Z}} \mathbb{R}$, $N_\mathbb{R} = N \otimes_{\mathbb{Z}} \mathbb{R}$.

Let $\Gamma$ be a weighted, connected finite graph without bivalent vertices. Its set of vertices and edges are denoted $\Gamma^{[0]}$ and $\Gamma^{[1]}$, respectively, and $w_\Gamma : \Gamma^{[1]} \rightarrow \mathbb{N} = \{0, 1, \ldots\}$ is the weight function. An edge $E \in \Gamma^{[1]}$ has adjacent vertices $\partial E = \{V_1, V_2\}$. Let $\Gamma_{\infty}^{[0]} \subseteq \Gamma^{[0]}$ be the set of univalent vertices. We set

$$\Gamma := \Gamma \setminus \Gamma_{\infty}^{[0]}.$$

We write the set of vertices and edges of $\Gamma$ as $\Gamma^{[0]}$, $\Gamma^{[1]}$, and we have the weight function $w_\Gamma : \Gamma^{[1]} \rightarrow \mathbb{N}$. Some edges of $\Gamma$ are now non-compact, and these are called unbounded edges. We use the convention that the weights of unbounded edges are always zero or one, and the weights of all bounded edges are positive. In particular, we do not allow bounded
edges to be contracted in what follows (as is sometimes the case in the tropical geometry literature). Write $\Gamma^1 \subseteq \Gamma^1$ for the set of unbounded edges.

**Definition 2.1.** A parameterized $d$-pointed marked tropical curve in $M_\mathbb{R}$ with marked points $\{p_1, \ldots, p_d\}$ is a choice of inclusion $\{p_1, \ldots, p_d\} \hookrightarrow \Gamma^\infty_\infty$ written as $p_i \mapsto E_{p_i}$, and a continuous map $h : \Gamma \rightarrow M_\mathbb{R}$ satisfying the following conditions.

1. $w_\Gamma(E) = 0$ if and only if $E = E_{p_i}$ for some $i$.
2. $h|_{E_{p_i}}$ is constant, $1 \leq i \leq d$, while for every other edge $E \in \Gamma^1$, the restriction $h|_E$ is a proper embedding with image $h(E)$ contained in an affine line with rational slope.
3. For every vertex $V \in \Gamma^0$, the following balancing condition holds. Let $E_1, \ldots, E_m \in \Gamma^1$ be the edges adjacent to $V$, and let $m_i \in M$ be the primitive integral vector emanating from $h(V)$ in the direction of $h(E_i)$. Then
   \[ \sum_{j=1}^m w_\Gamma(E_j)m_j = 0. \]

We write a parameterized $d$-pointed tropical curve as
   \[ h : (\Gamma, p_1, \ldots, p_d) \rightarrow M_\mathbb{R}. \]

We write $h(p_i)$ for the point $h(E_{p_i})$.

An isomorphism of tropical curves $h_1 : (\Gamma_1, p_1, \ldots, p_d) \rightarrow M_\mathbb{R}$ and $h_2 : (\Gamma_2, p_1, \ldots, p_d) \rightarrow M_\mathbb{R}$ is a homeomorphism $\Phi : \Gamma_1 \rightarrow \Gamma_2$ respecting the marked edges and the weights with $h_1 = h_2 \circ \Phi$. A $d$-pointed tropical curve is an isomorphism class of parameterized $d$-pointed tropical curves. We never distinguish between a $d$-pointed tropical curve and a particular representative.

The genus of a tropical curve $h : \Gamma \rightarrow M_\mathbb{R}$ is the first Betti number of $\Gamma$. A rational tropical curve is a tropical curve of genus zero.

The combinatorial type of a marked tropical curve $h : (\Gamma, p_1, \ldots, p_d) \rightarrow M_\mathbb{R}$ is defined to be the homeomorphism class of $\Gamma$ with the marked points and weights, together with, for every vertex $V$ and edge $E$ containing $V$, the primitive tangent vector to $h(E)$ in $M$ pointing away from $V$.

We modify this definition slightly to define a tropical disk: (see [27], where these are called tropical curves with stops).

**Definition 2.2.** Let $\overline{\Gamma}$ be a weighted, connected finite graph without bivalent vertices as above, with the additional data of a choice of univalent vertex $V_{out}$, adjacent to a unique edge $E_{out}$. Let

\[ \Gamma' := (\overline{\Gamma} \setminus \overline{\Gamma}^0_{\infty}) \cup \{V_{out}\} \subseteq \overline{\Gamma}. \]
Suppose furthermore that $\Gamma'$ has first Betti number zero (i.e., $\Gamma'$ is a tree with one compact external edge and a number of non-compact external edges). Then a parameterized $d$-pointed tropical disk in $M_\mathbb{R}$ is a choice of inclusion $\{p_1, \ldots, p_d\} \hookrightarrow \Gamma[1]_\mathbb{R} \setminus \{E_{out}\}$ written as $p_i \mapsto E_{p_i}$ and a map $h : \Gamma' \to M_\mathbb{R}$ satisfying the same conditions as Definition 2.1 except there is no balancing condition at $V_{out}$.

An isomorphism of tropical disks $h_1 : (\Gamma'_1, p_1, \ldots, p_d) \to M_\mathbb{R}$ and $h_2 : (\Gamma'_2, p_1, \ldots, p_d) \to M_\mathbb{R}$ is a homeomorphism $\Phi : \Gamma'_1 \to \Gamma'_2$ respecting the marked edges and the weights with $h_1 = h_2 \circ \Phi$. A tropical disk is an isomorphism class of parameterized tropical disks.

The combinatorial type of a tropical disk $h : (\Gamma', p_1, \ldots, p_d) \to M_\mathbb{R}$ is defined to be the homeomorphism class of $\Gamma$ with the marked points, weights, and $V_{out}$, together with, for every vertex $V$ and edge $E$ containing $V$, the primitive tangent vector to $h(E)$ in $M$ pointing away from $V$.

We also recall Mikhalkin’s notion of multiplicity. For the remainder of the paper, we restrict to the case that $M$ is rank two, i.e., $M = \mathbb{Z}^2$. Much of what we say can be generalized to higher dimension, but for ease of exposition, we restrict to dimension two.

**Definition 2.3.** Let $h : \Gamma \to M_\mathbb{R}$ be a marked tropical curve or $h : \Gamma' \to M_\mathbb{R}$ be a marked tropical disk such that $\bar{\Gamma}$ only has vertices of valency one and three. The multiplicity of a vertex $V \in \Gamma[0]$ in $h$ is $\text{Mult}_V(h) = 1$ if one of the edges adjacent to $V$ has weight zero (i.e, is a marked unbounded edge), and otherwise

$$\text{Mult}_V(h) = w_1w_2|m_1 \wedge m_2| = w_1w_3|m_1 \wedge m_3| = w_2w_3|m_2 \wedge m_3|,$$

where $E_1, E_2, E_3 \in \Gamma[1]$ are the edges containing $V$, $w_i = w_\Gamma(E_i)$, and $m_i \in M$ is a primitive integral vector emanating from $h(V)$ in the direction of $h(E_i)$. The equality of the three expressions follows from the balancing condition.

The multiplicity of the curve or disk $h$ is then

$$\text{Mult}(h) := \prod_{V \in \Gamma[0]} \text{Mult}_V(h).$$

Note that in the case of the tropical disk, there is no contribution from $V_{out}$.

We now fix a complete rational polyhedral fan $\Sigma$ in $M_\mathbb{R}$, with $\Sigma[1]$ denoting the set of one-dimensional cones in $\Sigma$. We denote by $T_\Sigma$ the free abelian group generated by $\Sigma[1]$, and for $\rho \in \Sigma[1]$ we denote by $t_\rho$ the corresponding generator for $T_\Sigma$ and by $m_\rho$ the primitive generator of the ray $\rho$. Let $X_\Sigma$ denote the toric surface defined by $\Sigma$.

**Definition 2.4.** A tropical curve (or disk) $h$ is a tropical curve (or disk) in $X_\Sigma$ if every $E \in \Gamma[1]$ (or $E \in \Gamma[1] \setminus \{E_{out}\}$) has $h(E)$ either a point or a translate of some $\rho \in \Sigma[1]$. 

If the tropical curve or disk has $d$ unbounded edges which are translates of $\rho \in \Sigma[1]$, (remember these unbounded edges always have weight one), then the degree of $h$ is 
\[ \Delta(h) := \sum_{\rho \in \Sigma[1]} d_{\rho} \rho \in T_{\Sigma}. \]

We define 
\[ |\Delta(h)| := \sum_{\rho \in \Sigma[1]} d_{\rho}. \]

Fix points $P_1, \ldots, P_k \in M_{\mathbb{R}}$ general, and fix a general base-point $Q \in M_{\mathbb{R}}$. When we talk about general points in the sequel, we mean that there is an open dense subset (typically the complement of a finite union of polyhedra of codimension at least one) of $M_{\mathbb{R}}^k+1$ such that $(P_1, \ldots, P_k, Q) \in M_{\mathbb{R}}^k+1$ lies in this open subset. This choice of open subset will depend on particular needs.

Associate to the points $P_1, \ldots, P_k$ the variables $u_1, \ldots, u_k$ in the ring 
\[ R_k := \mathbb{C}[u_1, \ldots, u_k] / (u_1^2, \ldots, u_k^2). \]

**Definition 2.5.** Let $h : (\Gamma', p_1, \ldots, p_d) \rightarrow M_{\mathbb{R}}$ be a tropical disk in $X_{\Sigma}$ with $h(V_{\text{out}}) = Q$, $h(p_j) = P_{i_j}$, $1 \leq i_1 < \cdots < i_d \leq k$. (This ordering removes a $d!$ ambiguity about the labelling of the marked points.) We say $h$ is a tropical disk in $(X_{\Sigma}, P_1, \ldots, P_k)$ with boundary $Q$.

The Maslov index of the disk $h$ is 
\[ MI(h) := 2(|\Delta(h)| - d). \]

**Lemma 2.6.** If $P_1, \ldots, P_k, Q$ are chosen in general position, then the set of Maslov index 2n tropical disks in $(X_{\Sigma}, P_1, \ldots, P_k)$ with boundary $Q$ is an $(n-1)$-dimensional polyhedral complex. The set of Maslov index 2n tropical disks with arbitrary boundary is an $(n+1)$-dimensional polyhedral complex.

**Proof.** This is a standard tropical general position argument. We sketch it here. Fix a combinatorial type of tropical disk with $d$ marked points, with degree $\Delta$. If the combinatorial type is general, then the domain $\Gamma'$ only has trivalent vertices apart from $V_{\text{out}}$. Such a tree has $|\Delta| + d - 1$ bounded edges (including $E_{\text{out}}$). A tropical disk $h : \Gamma' \rightarrow M_{\mathbb{R}}$ of this given combinatorial type is then completely determined by the position of $h(V_{\text{out}}) \in M_{\mathbb{R}}$ and the affine lengths of the bounded edges. This produces a cell in the moduli space $\mathcal{M}_{\Delta,d}^\text{disk}(X_{\Sigma})$ of all $d$-pointed tropical disks of degree $\Delta$. The closure of this cell is $(\mathbb{R}_{\geq 0})^{|\Delta(h)|+d-1} \times M_{\mathbb{R}}$. Also, there are only a finite number of combinatorial types of disks of a given degree. Thus $\mathcal{M}_{\Delta,d}^\text{disk}(X_{\Sigma})$ is a finite $(|\Delta|+d+1)$-dimensional polyhedral complex. Furthermore, we have a piecewise linear map $ev : \mathcal{M}_{\Delta,d}^\text{disk}(X_{\Sigma}) \rightarrow M_{\mathbb{R}}^d$, taking a disk $h$ to the tuple $(h(p_1), \ldots, h(p_d))$. 
Let $E \subseteq \mathcal{M}_{\Delta,d}^\mathrm{disk}(X_\Sigma)$ be the union of cells mapping under $\text{ev}$ to cells of codimension $\geq 1$ in $M_{\mathbb{R}}^d$; then $h(E)$ is a closed subset of $M_{\mathbb{R}}^d$. Thus, if $(P_{i_1}, \ldots, P_{i_d}) \in M_{\mathbb{R}}^d$ is not in this closed subset, for $1 \leq i_1 < \ldots < i_d \leq k$ distinct indices, then $\text{ev}^{-1}(P_{i_1}, \ldots, P_{i_d})$ is a codimension $2d$ subset of $\mathcal{M}_{\Delta,d}^\mathrm{disk}(X_\Sigma)$. Thus the dimension of the moduli space of tropical disks of a given degree $\Delta$ with arbitrary boundary in $(X_\Sigma, P_1, \ldots, P_k)$ is $|\Delta| + 1 - d = MI(h)/2 + 1$. Similarly, if we fix a general boundary point $Q$, the dimension is $MI(h)/2 - 1$, as claimed. □

**Definition 2.7.** Given the data $P_1, \ldots, P_k, Q \in M_{\mathbb{R}}$ general, let $h : (\Gamma', p_1, \ldots, p_d) \to M_{\mathbb{R}}$ be a Maslov index two marked tropical disk with boundary $Q$ in $(X_\Sigma, P_1, \ldots, P_k)$. Then we can associate to $h$ a monomial in $\mathbb{C}[T_\Sigma] \otimes_{\mathbb{C}} R_k[y_0]$, 

$$\text{Mono}(h) := \text{Mult}(h) z^{\Delta(h)} u_{I(h)},$$

where $z^{\Delta(h)} \in \mathbb{C}[T_\Sigma]$ is the monomial corresponding to $\Delta(h) \in T_\Sigma$, the subset $I(h) \subseteq \{1, \ldots, k\}$ is defined by

$$I(h) := \{ i \mid h(p_j) = P_i \text{ for some } j \},$$

and

$$u_{I(h)} = \prod_{i \in I(h)} u_i.$$

Define the $k$-pointed Landau-Ginzburg potential

$$W_k(Q) := y_0 + \sum_h \text{Mono}(h) \in \mathbb{C}[T_\Sigma] \otimes_{\mathbb{C}} R_k[y_0]$$

where the sum is over all Maslov index two disks $h$ in $(X_\Sigma, P_1, \ldots, P_k)$ with boundary $Q$. By Lemma 2.6, this is a finite sum for $P_1, \ldots, P_k, Q$ general.

Now restrict further to the case that $X_\Sigma$ is non-singular. There is an obvious map $r : T_\Sigma \to M$ given by $r(t_\rho) = m_\rho$, the primitive generator of $\rho$, and the assumption of non-singularity gives $r$ surjective. So there is a natural exact sequence

$$0 \to K_\Sigma \to T_\Sigma \xrightarrow{r} M \to 0$$

defining $K_\Sigma$. Dualizing this sequence gives

$$0 \to N \to \text{Hom}_\mathbb{Z}(T_\Sigma, \mathbb{Z}) \to \text{Pic} X_\Sigma \to 0.$$

See, e.g., [10], §3.4, for this description of the Picard group of $X_\Sigma$. After tensoring with $\mathbb{C}^\times$, we get an exact sequence

$$0 \to N \otimes \mathbb{C}^\times \to \text{Hom}(T_\Sigma, \mathbb{C}^\times) \xrightarrow{\kappa} \text{Pic} X_\Sigma \otimes \mathbb{C}^\times \to 0.$$

We set

$$\hat{X}_\Sigma := \text{Hom}(T_\Sigma, \mathbb{C}^\times) = \text{Spec} \mathbb{C}[T_\Sigma]$$
and we define the \textit{Kähler moduli space} of $X_\Sigma$ to be

$$\mathcal{M}_\Sigma := \text{Pic } X_\Sigma \otimes \mathbb{C}^\times = \text{Spec } \mathbb{C}[K_\Sigma].$$

We also have a morphism

$$\kappa : \tilde{X}_\Sigma \to \mathcal{M}_\Sigma.$$

We thicken $\mathcal{M}_\Sigma$ by setting

$$\mathcal{M}_{\Sigma,k} = \mathcal{M}_\Sigma \times \text{Spf } R_k[y_0],$$

and

$$\tilde{X}_{\Sigma,k} = \tilde{X}_\Sigma \times \text{Spf } R_k[y_0].$$

Then we have the family

$$\kappa : \tilde{X}_{\Sigma,k} \to \mathcal{M}_{\Sigma,k}.$$

$W_k(Q)$ is a regular function on $\tilde{X}_{\Sigma,k}$, so we can think of this as providing a family of Landau-Ginzburg potentials. Note that a fibre of $\kappa$ over a closed point of $\mathcal{M}_{\Sigma,k}$ is isomorphic to $N \otimes \mathbb{C}^\times$.

The sheaf of relative differentials $\Omega^1_{\tilde{X}_{\Sigma,k}/\mathcal{M}_{\Sigma,k}}$ is canonically isomorphic to $M \otimes \mathbb{Z} \mathcal{O}_{\tilde{X}_{\Sigma,k}}$, with $m \otimes 1$ corresponding to the differential

$$d\log m := \frac{d(z\overline{m})}{z\overline{m}};$$

here $\overline{m} \in T_\Sigma$ is any lift of $m \in M$, and $d\log m$ is well-defined as a relative differential independently of the lift. Thus a choice of generator of $\bigwedge^2 M \cong \mathbb{Z}$ determines a nowhere-vanishing relative holomorphic two-form $\Omega$, canonical up to sign. Explicitly, if $e_1, e_2 \in M$ is a positively oriented basis, then

$$\Omega = d\log e_1 \wedge d\log e_2.$$

\textit{Remark 2.8.} The function $W_k(Q)$ is intended to be the “correct” Landau-Ginzburg potential to describe the mirror to $X_\Sigma$, in the sense that $W_0(Q)$ is the expression usually taken to be the Landau-Ginzburg potential, and $W_k(Q)$ should be a canonical perturbation, in the sense that the parameters appearing in $W_k(Q)$ are closely related to flat coordinates. However, for general choice of $\Sigma$ this is not true, the chief problem being that there will be copies of $\mathbb{P}^1$ in the toric boundary of $X_\Sigma$ which do not deform to curves intersecting the big torus orbit of $X_\Sigma$. This is a standard problem in tropical geometry: so far, tropical geometry cannot “see” these curves. This is not a problem as long as $X_\Sigma$ is a product of projective spaces, so in particular, we will now restrict to the case of $X_\Sigma = \mathbb{P}^2$.

\footnote{This is almost, but not quite, the usual Kähler moduli space in the context of mirror symmetry (see, e.g., [5], §6.2). The Kähler moduli space of a Kähler manifold $X$ is generally the tube domain given by $(H^2(X, \mathbb{R}) + iK)/H^2(X, \mathbb{Z})$ for $K \subseteq H^2(X, \mathbb{R})$ the Kähler cone of $X$. In the case $X = X_\Sigma$, this is naturally an analytic open subset of $\mathcal{M}_\Sigma$.}
Example 2.9. Let $\Sigma$ be the fan depicted in Figure 2.1 so that $X_\Sigma = \mathbb{P}^2$. Here $T_\Sigma = \mathbb{Z}^3$ with basis $t_0, t_1, t_2$ corresponding to $\rho_0, \rho_1, \rho_2$, and we write $x_i$ for the monomial $z^{t_i} \in k[T_\Sigma]$. The map

$$\kappa : \text{Spec} \, k[T_\Sigma] \to M_\Sigma = \text{Spec} \, k[K_\Sigma]$$

is then a map

$$\kappa : (k^\times)^3 \to k^\times$$

given by $\kappa(x_0, x_1, x_2) = x_0 x_1 x_2$.

If we take $k = 0$, then there are precisely three Maslov index two tropical disks, as depicted in Figure 2.2. Thus we take

$$W_0(Q) = y_0 + x_0 + x_1 + x_2.$$

This is the standard Landau-Ginzburg potential for the mirror to $\mathbb{P}^2$ (except for the additional variable $y_0$). The formula in Definition 2.7 gives a deformation of this potential over the thickened moduli space $M_{\Sigma, k}$.

If we take $k = 1$, marking one point in $\mathbb{P}^2$, we obtain one additional disk, as depicted in Figure 2.3, and if we take $k = 2$ with $P_1$ and $P_2$ chosen as in Figure 2.4 we have three additional disks. Note the potential depends on the particular choices of the points $P_1, \ldots, P_k$ as well as $Q$. In the given examples, we have respectively

$$W_1(Q) = y_0 + x_0 + x_1 + x_2 + u_1 x_1 x_2$$
$$W_2(Q) = y_0 + x_0 + x_1 + x_2 + u_1 x_0 x_1 + u_2 x_0 x_1 + u_1 u_2 x_0 x_1^2.$$
Figure 2.2. Maslov index two tropical disks with no marked points.

Figure 2.3. The one additional Maslov index two tropical disk with $k = 1$.

Figure 2.4. The additional Maslov index two tropical disks with $k = 2$. 
3. Tropical descendental invariants and the main theorem

We will now define the tropical version of the descendental invariants and state the main theorem of the paper. These invariants will be defined as a count of certain tropical curves with vertices of high valency. These curves need to be counted with certain multiplicities. We begin with

**Definition 3.1.** Let $P_1, \ldots, P_k \in M_\mathbb{R}$ be general. Let $\Sigma$ be a complete fan in $M_\mathbb{R}$ defining a toric surface. Let $S \subseteq M_\mathbb{R}$ be a subset. Define

$$\mathcal{M}^{\text{trop}}_{\Delta, n}(X_\Sigma; P_1, \ldots, P_k, \psi^\nu S)$$

to be the moduli space of rational $(n + 1)$-pointed tropical curves in $X_\Sigma$

$$h : (\Gamma, p_1, \ldots, p_n, x) \to M_\mathbb{R}$$

degree $\Delta$ such that

1. $h(p_j) = P_{i_j}, 1 \leq i_1 < \cdots < i_n \leq k.$
2. The edge $E_x$ is attached to a vertex $V_x$ of $\Gamma$; the valency of this vertex is denoted $\text{Val}(x)$. Then $\text{Val}(x) = \nu + 3.$
3. $h(x) \in S.$

**Lemma 3.2.** For $P_1, \ldots, P_k \in M_\mathbb{R}$ general,

1. $\mathcal{M}^{\text{trop}}_{\Delta, n}(X_\Sigma; P_1, \ldots, P_k, \psi^\nu M_\mathbb{R})$ is a polyhedral complex of dimension $|\Delta| - n - \nu.$
2. $\mathcal{M}^{\text{trop}}_{\Delta, n}(X_\Sigma; P_1, \ldots, P_k, \psi^\nu C)$ is a polyhedral complex of dimension $|\Delta| - n - \nu - 1$ for $C$ a general translate of a tropical curve in $M_\mathbb{R}$.
3. $\mathcal{M}^{\text{trop}}_{\Delta, n}(X_\Sigma; P_1, \ldots, P_k, \psi^\nu Q)$ is a polyhedral complex of dimension $|\Delta| - n - \nu - 2$ for $Q \in M_\mathbb{R}$ a general point.

**Proof.** This is again standard, as in Lemma 2.6. The dimension count is as follows. Fix the combinatorial type of the curve to be generic, so that all vertices of $\Gamma$ are trivalent except for the vertex adjacent to $E_x$, which is $(\nu + 3)$-valent. Such a tree has $|\Delta| + n + 1$ unbounded edges, and then has $|\Delta| + n + 1 - (\nu + 3)$ bounded edges. The curves of this combinatorial type are then determined by the location of $h(x) \in M_\mathbb{R}$ and the lengths of the bounded edges, giving a cell of the form $(\mathbb{R} \geq 0)^{|\Delta|+n-\nu-2} \times M_\mathbb{R}$. Fixing $h(p_1), \ldots, h(p_n)$ then yields the desired dimension of the moduli space in (1) being

$$|\Delta| + n - \nu - 2n.$$

This gives (1). For (2) and (3), we consider the map

$$\text{ev}_x : \mathcal{M}^{\text{trop}}_{\Delta, n}(X_\Sigma; P_1, \ldots, P_k, \psi^\nu M_\mathbb{R}) \to M_\mathbb{R}$$
given by $\text{ev}_x(h) := h(x)$. Let $E_1$ be the union of cells in $\mathcal{M}^{\text{trop}}_{\Delta_n}(X_\Sigma, P_1, \ldots, P_k, \psi^\nu M_\mathbb{R})$ which map to codimension $\geq 1$ sets in $M_\mathbb{R}$, and let $E_2$ be the union of cells which map to points in $M_\mathbb{R}$. Then we need to choose the translate $C$ so that $C \cap h(E_1)$ is zero dimensional and $C \cap h(E_2) = \emptyset$. Similarly, we need to choose $Q \notin h(E_1)$. Then $\text{ev}_x^{-1}(C)$ or $\text{ev}_x^{-1}(Q)$ are the desired moduli spaces in cases (2) and (3) and are of the desired dimension. \qed

\textbf{Lemma 3.3.} Let $P_1, \ldots, P_k \in M_\mathbb{R}$ be general and $S \subseteq M_\mathbb{R}$ a subset. Let

$$h \in \mathcal{M}^{\text{trop}}_{\Delta_n}(X_\Sigma, P_1, \ldots, P_k, \psi^\nu S).$$

Let $\Gamma'_1, \ldots, \Gamma'_{n+2}$ denote the closures of the connected components of $\Gamma \setminus \{E_x\}$, with $h_i : \Gamma'_i \rightarrow M_\mathbb{R}$ the restrictions of $h$. Each disk $h_i$ is viewed as being marked by those points $p \in \{p_1, \ldots, p_n\}$ with $E_p \subseteq \Gamma'_i$. There is one special case to consider here: if $E_x$ and $E_p$, share a common vertex $V$, then we discard the edge $E_p$, from consideration as well, so we have disks $h_1, \ldots, h_{n+1}$. (Note that since $h(p_i) \neq h(p_j)$ for $i \neq j$, we never have $E_{p_i}$ and $E_{p_j}$ sharing a common vertex.)

(1) If $S = M_\mathbb{R}$ and $n = |\Delta| - \nu$, then either

(a) $E_x$ does not share a vertex with any of the edges $E_{p_i}$, and then $MI(h_i) = 2$ for all but two choices of $i$, for which $MI(h_i) = 0$.

(b) $E_x$ does share a vertex with one of the edges $E_{p_i}$, and then $MI(h_i) = 2$ for all $i$.

(2) If $S = C$ is a general translate of a tropical curve in $M_\mathbb{R}$ and $n = |\Delta| - \nu - 1$, then $MI(h_i) = 2$ for all but one $i$, and for this $i$, $MI(h_i) = 0$.

(3) If $S = \{Q\}$ for a general point $Q$ and $n = |\Delta| - \nu - 2$, then $MI(h_i) = 2$ for all $i$.

\textbf{Proof.} First note that the condition on $n$ and the generality of $P_1, \ldots, P_k, C$, and $Q$ guarantee by the previous lemma that the moduli space under consideration is zero dimensional. If any of the disks $h_i$ can be deformed while keeping its boundary $h_i(x)$ fixed, then this yields a non-trivial deformation of $h$, which does not exist. Thus by Lemma 2.6 we must have $MI(h_i) \leq 2$ in all cases. Let $n_i$ be the number of marked points on $h_i$. We note that

$$\sum_i \frac{MI(h_i)}{2} = \sum_i (|\Delta(h_i)| - n_i)$$

$$= \begin{cases} 
|\Delta(h)| - (n - 1) & \text{Case (1) (b)} \\
|\Delta(h)| - n & \text{otherwise}
\end{cases}$$

$$= \begin{cases} 
\nu & \text{Case (1) (a)} \\
\nu + 1 & \text{Case (1) (b)} \\
\nu + 1 & \text{Case (2)} \\
\nu + 2 & \text{Case (3)}
\end{cases}$$
Since there are $\nu + 2$ disks except in Case (1) (b), when there are $\nu + 1$ disks, the result follows. \qed

We can now define tropical analogues of the descendent Gromov-Witten invariants which appear in the Givental $J$-function. From now on in this section, we are only considering the case of $X_\Sigma = \mathbb{P}^2$, with $\Sigma$ the fan with rays generated by $m_0 = (-1, -1)$, $m_1 = (1, 0)$ and $m_2 = (0, 1)$, and $t_0, t_1, t_2$ the generators of $T_\Sigma$, with $r(t_i) = m_i$. Let

$$\Delta_d := d(t_0 + t_1 + t_2) \in T_\Sigma;$$

curves of degree $\Delta_d$ should be viewed as degree $d$ curves in $\mathbb{P}^2$.

**Definition 3.4.** Fix general points $Q, P_1, P_2, \cdots \in M_{\mathbb{R}}$. Let $L$ be the tropical line (a translate of the fan of $\mathbb{P}^2$) with vertex $Q$.

For a tropical curve $h$ in $\mathbb{P}^2$ with a marked point $x$, let $n_0(x), n_1(x)$ and $n_2(x)$ be the number of unbounded rays sharing a common vertex with $E_x$ in the directions $m_0, m_1$ and $m_2$ respectively. As in Lemma 3.3, we denote by $h_1, \ldots$ the tropical disks obtained by removing $E_x$ from $\Gamma$, with the outgoing edge of $h_i$ being $E_{i, \text{out}}$. Let $m(h_i) = w(E_{i, \text{out}})m_{\text{prim}}(h_i)$, where $m_{\text{prim}}(h_i) \in M$ is a primitive vector tangent to $h_i(E_{i, \text{out}})$ pointing away from $h(x)$.

Define

$$\text{Mult}_0^0(h) = \frac{1}{n_0(x)!n_1(x)!n_2(x)!};$$

$$\text{Mult}_1^1(h) = -\sum_{k=1}^{n_0(x)} \frac{1}{k} + \sum_{k=1}^{n_1(x)} \frac{1}{k} + \sum_{k=1}^{n_2(x)} \frac{1}{k} - \frac{n_0(x)!n_1(x)!n_2(x)!}{n_0(x)!n_1(x)!n_2(x)!};$$

$$\text{Mult}_2^2(h) = \frac{\left( \sum_{i=0}^2 \sum_{k=1}^{n_i(x)} \frac{1}{k} \right)^2 + \sum_{i=0}^2 \sum_{k=1}^{n_i(x)} \frac{1}{k^2}}{2n_0(x)!n_1(x)!n_2(x)!};$$

(1) We define

$$\langle P_1, \ldots, P_{3d-2-\nu}, \psi^\nu Q \rangle_{d}^{\text{trop}}$$

to be

$$\sum_h \text{Mult}(h)$$

where the sum is over all marked tropical rational curves

$$h \in \mathcal{M}_{\Delta_d, 3d-2-\nu}(P_1, \ldots, P_{3d-2-\nu}, \psi^\nu Q).$$

We define

$$\text{Mult}(h) := \text{Mult}_x^0(h) \prod_{V \in \Gamma^{[0]} \setminus V \in E_x} \text{Mult}_V(h).$$
(2) We define
\[ \langle P_1, \ldots, P_{3d-1-\nu}, \psi^\nu L \rangle_d^{\text{trop}} \]
as a sum
\[ \sum_h \text{Mult}(h) \]
where the sum is again over all marked tropical rational curves
\[ h : (\Gamma, p_1, \ldots, p_{3d-1-\nu}, x) \to \mathbb{M}_R \]
with \( h(p_i) = P_i \) and satisfying one of the following two conditions.
(a) \( h \in \mathcal{M}_{\Delta_d,3d-1-\nu}^{\text{trop}}(P_1, \ldots, P_{3d-1-\nu}, \psi^\nu L) \).
Furthermore, no unbounded edge of \( \Gamma \) having a common vertex with \( E_x \) other than \( E_x \) maps into the connected component of \( \mathbb{M}_R \setminus L \) containing \( h(x) \). By Lemma 3.3, there is precisely one \( j, 1 \leq j \leq \nu + 2 \), with \( \text{MI}(h_j) = 0 \). Suppose also the connected component of \( \mathbb{M}_R \setminus \{Q\} \) containing \( h(x) \) is \( Q + \mathbb{R}_{>0}m_i \). Then we define
\[ \text{Mult}(h) = |m(h_j) \wedge m_i| \text{Mult}_x^0(h) \prod_{V \in \Gamma \setminus V \in E_x} \text{Mult}_V(h). \]
(b) \( \nu \geq 1 \) and
\[ h \in \mathcal{M}_{\Delta_d,3d-1-\nu}^{\text{trop}}(P_1, \ldots, P_{3d-1-\nu}, \psi^{\nu-1}Q). \]
Then
\[ \text{Mult}(h) = \text{Mult}_x^1(h) \prod_{V \in \Gamma \setminus V \in E_x} \text{Mult}_V(h). \]
(3) We define
\[ \langle P_1, \ldots, P_{3d-\nu}, \psi^\nu M_\mathbb{R} \rangle_{d}^{\text{trop}} \]
as a sum
\[ \sum_h \text{Mult}(h) \]
where the sum is over all marked tropical rational curves
\[ h : (\Gamma, p_1, \ldots, p_{3d-\nu}, x) \to \mathbb{M}_R \]
such that \( h(p_i) = P_i \) and either
(a) \( h \in \mathcal{M}_{\Delta_d,3d-\nu}^{\text{trop}}(P_1, \ldots, P_{3d-\nu}, \psi^\nu M_\mathbb{R}) \)
and \( E_x \) does not share a vertex with any of the \( E_{p_i} \)'s. Furthermore, no unbounded edge of \( \Gamma \) having a common vertex with \( E_x \) other than \( E_x \) maps into the connected component of \( \mathbb{M}_R \setminus L \) containing \( h(x) \). By Lemma 3.3, there
are precisely two distinct \(j_1, j_2\) with \(1 \leq j_1, j_2 \leq \nu + 2\) such that \(MI(h_{j_1}) = 0\). Then we define

\[
\text{Mult}(h) = |m(h_{j_1}) \land m(h_{j_2})| \text{Mult}_x^0(h) \prod_{V \in \Gamma^{[0]} \atop V \not\in E_x} \text{Mult}_V(h).
\]

**(b)**

\[
h \in M_{\Delta,3d-\nu}^{\text{trop}}(P_1, \ldots, P_{3d-\nu}, \psi^\nu M_{\mathbb{R}})
\]

and \(E_x\) shares a vertex with \(E_{p_i}\). Furthermore, no unbounded edge of \(\Gamma\) having a common vertex with \(E_x\) other than \(E_x\) and \(E_{p_i}\) maps into the connected component of \(M_{\mathbb{R}} \setminus L\) containing \(h(x)\). Then we define

\[
\text{Mult}(h) = \text{Mult}_x^0(h) \prod_{V \in \Gamma^{[0]} \atop V \not\in E_x} \text{Mult}_V(h).
\]

**(c)** \(\nu \geq 1\) and

\[
h \in M_{\Delta,3d-\nu}^{\text{trop}}(P_1, \ldots, P_{3d-\nu}, \psi^{\nu-1} L).
\]

Furthermore, no unbounded edge of \(\Gamma\) having a common vertex with \(E_x\) other than \(E_x\) maps into the connected component of \(L \setminus \{Q\}\) containing \(h(x)\). By Lemma 3.3 there is precisely one \(j, 1 \leq j \leq \nu + 1\), with \(MI(h_{j}) = 0\). Suppose the connected component of \(L \setminus \{Q\}\) containing \(h(x)\) is \(Q + \mathbb{R}_{>0}m_i\). Then we define

\[
\text{Mult}(h) = |m(h_{j}) \land m_i| \text{Mult}_x^1(h) \prod_{V \in \Gamma^{[0]} \atop V \not\in E_x} \text{Mult}_V(h).
\]

**(d)** \(\nu \geq 2\) and

\[
h \in M_{\Delta,3d-\nu}^{\text{trop}}(P_1, \ldots, P_{3d-\nu}, \psi^{\nu-2} Q).
\]

Then

\[
\text{Mult}(h) = \text{Mult}_x^2(h) \prod_{V \in \Gamma^{[0]} \atop V \not\in E_x} \text{Mult}_V(h).
\]

In all cases \(S = \{Q\}, L\) or \(M_{\mathbb{R}}\), we define for \(\sigma \in \Sigma\),

\[
\langle P_1, \ldots, P_{3d-\nu-(2-\dim S)}, \psi^{\nu} S \rangle_{\Delta, \sigma}^{\text{trop}}
\]

to be the contribution to \(\langle P_1, \ldots, P_{3d-\nu-(2-\dim S)}, \psi^{\nu} S \rangle_{\Delta, \sigma}^{\text{trop}}\) coming from curves \(h\) with \(h(x)\) in the interior of \(\sigma + Q\). In (1), the only contribution comes from \(\sigma = \{0\}\), in (2), the contributions come from the zero and one-dimensional cones of \(\Sigma\), and in (3), the contributions come from all cones of \(\Sigma\).
Remarks 3.5. (1) Note that all moduli spaces involved are zero-dimensional for general choices of \( Q, P_1, \ldots \), so the sums make sense.

(2) The formula in Definition 3.4, (1), for \( \nu = 0 \), gives the standard tropical curve counting formula for the number of rational curves of degree \( d \) passing through \( 3d - 1 \) points. For \( \nu > 0 \), this coincides with the formula given by Markwig and Rau in [25]. In particular, by the results of that paper,

\[
\langle P_1, \ldots, P_{3d-2-\nu}, \psi^\nu Q \rangle_d^{\text{trop}} = \langle T_2^{3d-2-\nu}, \psi^\nu T_2 \rangle_d.
\]

(3) Clearly the formulas for the descendent invariants involving \( \psi^\nu L \) or \( \psi^\nu M_\mathbb{R} \) are rather more complicated and mysterious. Hannah Markwig informs me that if the tropical intersection procedure of [25] is carried out for the classes \( \psi^\nu L \) or \( \psi^\nu M_\mathbb{R} \), one obtains the contributions in Definition 3.4 only of the type (2) (a) and (3) (a) and (b). This does not give the correct formula for the descendent invariants. The likely explanation for this phenomenon is that [25] works with the non-compact moduli space of tropical rational curves in tropical \( \mathbb{P}^2 \). There should be a suitable compactification of this moduli space, and it is possible that tropical intersection theory applied to this compactified moduli space would yield the correct descendent invariants. If this is the case, then the contributions of the form (2) (b) or (3) (c)-(d) in Definition 3.4 could be viewed as boundary contributions.

It would be very interesting to learn if this is indeed the case. However, exploring this question would be considerably outside the scope of this paper. As a consequence, it is difficult to motivate the rather mysterious multiplicity formulas given in Definition 3.4. These are simply the formulas which emerge naturally from period integrals in §5.

(4) It is easy to see that \( \langle P_1, \ldots, P_{3d-1}, \psi^0 L \rangle_d^{\text{trop}} \) is \( d \) times the number of rational curves through \( 3d - 1 \) points. Indeed, the only contribution to this number comes from Definition 3.4 (2) (a). For each tropical rational curve \( h : \Gamma \to M_\mathbb{R} \) with \( 3d - 1 \) marked points passing through \( P_1, \ldots, P_{3d-1} \) we obtain a contribution for every point of \( h^{-1}(L) \) by marking that point with \( x \). The factor \( |m(h_j) \wedge m_i| \text{Mult}_x^0(h) \) for the multiplicity in this case gives the intersection multiplicity of \( h(\Gamma) \) with \( L \) at each point of \( h^{-1}(L) \), as defined in \( \text{[30]}, \text{§4}. \) By the tropical Bézout theorem (\text{[30]}, Theorem 4.2), the total contribution from \( h \) is then \( \langle h(\Gamma), L \rangle \text{Mult}(h) = d \text{Mult}(h) \).

Thus

\[
\langle P_1, \ldots, P_{3d-1}, \psi^0 L \rangle_d^{\text{trop}} = d\langle P_1, \ldots, P_{3d-1} \rangle_d^{\text{trop}} = d\langle T_2^{3d-1} \rangle_d = \langle T_2^{3d-1}, T_1 \rangle_d,
\]

by Mikhalkin’s formula and the Divisor Axiom.

(5) \( \langle P_1, \ldots, P_{3d}, \psi^0 M_\mathbb{R} \rangle_d^{\text{trop}} = 0 \). Indeed, the only possible contributions come from Definition 3.4 (3) (a), but there are no rational curves of degree \( d \) through \( 3d \) general points. Thus

\[
\langle P_1, \ldots, P_{3d}, \psi^0 M_\mathbb{R} \rangle_d^{\text{trop}} = \langle T_2^{3d}, T_0 \rangle_d,
\]
as both are zero.

One of our results will be the following:

**Theorem 3.6.** The invariants defined in Definition 3.4 are independent of the choice of the $P_i$'s and $Q$.

This will be proved in §5. However, this allows us to make the following definition.

**Definition 3.7.** We define

$$\langle T_{3d-\nu}, \psi_{\nu}^{T_2} \rangle_{d}^{\text{trop}} := \langle P_1, \ldots, P_{3d-\nu}, \psi_{\nu}^{T_2} \rangle_{d}^{\text{trop}}$$

where the $P_i$'s and $Q$ have been chosen generally.

We define

$$\langle T_m^m, \psi_{\nu}^{T_2} \rangle_{d}^{\text{trop}} := 0$$

if $m + i + \nu \neq 3d$.

We define the tropical $J$-function for $\mathbb{P}^2$ by

$$J_{\mathbb{P}^2}^{\text{trop}} := e^{q(y_0 T_0 + y_1 T_1)} \left( T_0^2 + \sum_{i=0}^{2} (qy_2 \delta_{i,2} + \sum_{d \geq 1} \sum_{\nu \geq 0} \langle T_{2d+i-2-\nu}, \psi_{\nu}^{T_{2-i}} \rangle_{d}^{\text{trop}} q^{\nu+i} e^{dy_1 \frac{y_2^{3d+i-2-\nu}}{(3d+i-2-\nu)!}} T_i \right)$$

$$= : \sum_{i=0}^{2} J_i^{\text{trop}} T_i.$$

The main theorem of the paper, to be proved in §5, is now

**Theorem 3.8.** A choice of general points $P_1, \ldots, P_k$ and $Q$ gives rise to a function $W_k(Q) \in \mathbb{C}[T_2] \otimes \mathbb{C} [y_0]$ by Definition 2.7, and hence a family of Landau-Ginzburg potentials on the family $\kappa : \tilde{X}_{\Sigma,k} \rightarrow \mathcal{M}_{\Sigma,k}$ with a relative nowhere-vanishing two-form $\Omega$ as defined in (2.2). This data gives rise to a local system $\mathcal{R}$ on $\mathcal{M}_{\Sigma,k} \otimes \text{Spec} \mathbb{C}[q, q^{-1}]$ whose fibre over $(\kappa, q)$ is $H_2((\tilde{X}_{\Sigma,k})_{\kappa}, \text{Re}(qW_0(Q)) \ll 0)$. There exists a multi-valued basis $\Xi_0, \Xi_1, \Xi_2$ of sections of $\mathcal{R}$ satisfying conditions (M1) and (M3) of §1 such that

$$\sum_{i=0}^{2} \alpha^i \int_{\Xi_i} e^{qW_k(Q)} \Omega = q^{3q} \sum_{i=0}^{2} \varphi_i \cdot (\alpha/q)^i$$

where

$$\varphi_i(y_0, \kappa, u_1, \ldots, u_k) = \delta_{0,i} + \sum_{j=1}^{\infty} \varphi_{i,j}(y_0, \kappa, u_1, \ldots, u_k) q^j,$$
for $0 \leq i \leq 2$, with

\[
\begin{align*}
\varphi_{0,1} &= y_0 \\
\varphi_{1,1} &= y_1 := \log(\kappa) \\
\varphi_{2,1} &= y_2 := \sum_{i=1}^{k} u_i.
\end{align*}
\]

Furthermore,

\[
\varphi_i = J^i_{\text{trop}}(y_0, y_1, y_2).
\]

Note one immediate corollary of this theorem:

**Corollary 3.9.** Let $\mathcal{M}_{\Sigma,k}$ be the formal spectrum of the completion of $\mathbb{C}[\mathcal{K}_\Sigma] \otimes_{\mathbb{C}} R_k[y_0]$ at the maximal ideal $(y_0, \kappa - 1, u_1, \ldots, u_k)$. This completion is isomorphic to $\mathbb{C}[y_0, y_1] \otimes_{\mathbb{C}} R_k$, with $y_1 := \log \kappa$, the latter expanded in a power series at $\kappa = 1$. Let

\[
\mathcal{X}_{\Sigma,k} = \mathcal{X}_{\Sigma,k} \times_{\mathcal{M}_{\Sigma,k}} \mathcal{M}_{\Sigma,k}.
\]

The regular function $W_k(Q)$ on $\mathcal{X}_{\Sigma,k}$ restricts to $W_0(Q) = x_0 + x_1 + x_2$ on the closed fibre of $\mathcal{X}_{\Sigma,k} \rightarrow \mathcal{M}_{\Sigma,k}$, and hence $W_k(Q)$ provides a deformation of this function over $\mathcal{M}_{\Sigma,k}$. Thus we have a morphism from $\mathcal{M}_{\Sigma,k}$ to the universal unfolding moduli space $\text{Spec} \mathbb{C}[y_0, y_1, y_2]$ with $y_0, y_1, y_2$ the flat coordinates of §1, (M4). This map is given by

\[
\begin{align*}
y_0 &\mapsto y_0 \\
y_1 &\mapsto \log \kappa \\
y_2 &\mapsto \sum_{i=1}^{k} u_i.
\end{align*}
\]

**Proof.** The universal unfolding of $x_0 + x_1 + x_2$ on $(\mathbb{C}^\times)^2 = V(x_0x_1x_2 - 1) \subseteq \mathbb{C}^3$ is a regular function $W$ on $(\mathbb{C}^\times)^2 \times \text{Spec} \mathbb{C}[y_0, y_1, y_2]$, and there is a map $\xi : \mathcal{M}_{\Sigma,k} \rightarrow \text{Spf} \mathbb{C}[y_0, y_1, y_2]$ and an isomorphism

\[
\eta : \mathcal{X}_{\Sigma,k} \rightarrow (\mathbb{C}^\times)^2 \times \mathcal{M}_{\Sigma,k}
\]

which is the identity on the closed fibre and such that, if $W' = W \circ (\text{id} \times \xi)$ is the pull-back of $W$ to $(\mathbb{C}^\times)^2 \times \mathcal{M}_{\Sigma,k}$, then $W_k(Q) = W' \circ \eta$. Then

\[
(\eta^{-1})^* (e^{qW_k(Q)} \Omega) = e^{qW'}(\eta^{-1})^*(\Omega).
\]

Of course $(\eta^{-1})^* \Omega = f \Omega$ for a regular function $f$ which is identically 1 on the closed fibre. Thus $(\eta^{-1})^* (e^{qW_k(Q)} \Omega)$ is a form of the sort allowed in (M2), and Theorem 3.8 shows this form satisfies the normalization conditions of (M4). Thus $\xi$ is given by $y_0 \mapsto y_0$, $y_1 \mapsto y_1$, $y_2 \mapsto \sum_{i=1}^{k} u_i$ because of the calculation of $\varphi_{i,1}$, $i = 0, 1, 2$ given in Theorem 3.8. 

Another corollary is the strong equivalence between mirror symmetry for $\mathbb{P}^2$ and tropical curve counting with descendents. We have the following statement:
Statement 3.10 (Tropical curve counting with descendents).

\[ J_{\mathbb{P}^2} = J_{\mathbb{P}^2}^{\text{trop}}. \]

Then

Corollary 3.11. Statement 3.10 and Statement 3.10 are equivalent. In particular, since Statement 3.10 is known to be true, Statement 3.10 is true.

This is the promised strong equivalence between mirror symmetry for \( \mathbb{P}^2 \) and tropical curve counting.

The proofs of Theorems 3.6 and 3.8 require explicit evaluation of the integrals \( \int_\Xi e^{qW_k(Q)} \Omega \) for various \( \Xi \). While it is not, in general, difficult to get explicit answers for these integrals, it is actually quite difficult to extract a useful combinatorial result from these answers. There is one particular case where this is not difficult, however, namely the case when \( \Xi = \Xi_0 \). Because of the simplicity of this case, in which one can see exactly how the oscillatory integral extracts Gromov-Witten invariants, I believe it is worth presenting this calculation before embarking on the main part of the proof of the main theorems.

We will need the following lemma, whose proof is given at the very end of §4.

Lemma 3.12. For \( \Xi \in H_2((\tilde{\mathcal{X}}_{\Sigma, k}), \text{Re}(qW_0(Q)) < 0) \), the integral

\[ \int_{\Xi} e^{qW_k(Q)} \Omega \]

is independent of the choice of \( Q \) and \( P_1, \ldots, P_k \).

Proposition 3.13. \( \langle P_1, \ldots, P_{3d-2-\nu}, \psi^\nu Q \rangle_{\mathbb{P}^2}^{\text{trop}} \) is independent of the choice of the points \( P_1, \ldots, P_{3d-2-\nu}, Q \) general. Furthermore, there exists a well-defined cycle \( \Xi_0 \) such that

\[ \int_{\Xi_0} e^{qW_k(Q)} \Omega = J_{\mathbb{P}^2}^{\text{trop}}. \]

Proof. Each fibre of \( \kappa \) is an algebraic torus isomorphic to \( N \otimes \mathbb{C}^\times \). We will take \( \Xi_0 \) to be the element of \( H_2(N \otimes \mathbb{C}^\times, \mathbb{C}) \) which is \( 1/(2\pi i)^2 \) times the homology class of the compact torus \( N \otimes U(1) \subseteq N \otimes \mathbb{C}^\times \). Since the choice of \( \Omega \) depended on a choice of orientation on \( M_\mathbb{R} \), hence an orientation on \( N_\mathbb{R} \), this defines an orientation on \( N \otimes U(1) \). Then it is very easy to compute the integral

\[ (3.3) \]

\[ \int_{\Xi_0} e^{qW_k(Q)} \Omega = e^{qy_0} \int_{\Xi_0} e^{q(W_k(Q)-y_0)} \Omega. \]

Indeed, this is just a residue calculation. We expand \( e^{q(W_k(Q)-y_0)} \Omega \) in a power series, obtaining a sum of terms of the form \( Cz^m \Omega \) for various \( m \in T_\Sigma \), \( C \in R_\kappa[q] \). Such a term contributes to the integral if and only if the image \( r(m) \) of \( m \) in \( M \) is zero, i.e., \( m = d(t_0 + t_1 + t_2) = \Delta_d \) for some \( d \geq 0 \), in which case the contribution is just \( C\kappa^d \).
These terms are easily interpreted. We write
\[
\exp(q(W_k(Q) - y_0)) = \sum_{n=0}^{\infty} \frac{q^n(W_k(Q) - y_0)^n}{n!}.
\]
The coefficient of \(q^0\) in this expansion is 1, contributing 1 to the integral. The coefficient of \(q\) is \(W_k(Q) - y_0\), and no term of this expression is of the form \(Cz^m\) with \(r(m) = 0\). So there is no contribution from these terms. Thus any other possible contribution comes from terms in \(q^{\nu+2}(W_k(Q) - y_0)^{\nu+2}/(\nu+2)!\) for \(\nu \geq 0\), and such a term can be written as
\[
q^{\nu+2} \prod_i \frac{1}{\nu_i!} \text{Mono}(h_i)^{\nu_i}
\]
with \(\sum \nu_i = \nu + 2\), with the \(h_i : \Gamma_i' \to M_\mathbb{R}\) being Maslov index two tropical disks. Note that \(\text{Mono}(h_i)^2 = 0\) unless \(h_i\) has no marked points, so \(\nu_i = 1\) unless \(h_i\) consists of just one unbounded edge.

The condition that this term contributes to the integral is saying precisely that these tropical disks can be glued to obtain a tropical curve. Indeed, we can take \(\nu_1\) copies of \(\Gamma_1', \nu_2\) copies of \(\Gamma_2', \) and so on, and glue these graphs together by identifying the outgoing vertices \(V_{\text{out}}\) on all of them to get a graph \(\Gamma\). We can add an additional marked unbounded edge to \(\Gamma\) at \(V_{\text{out}}\), labelled \(x\), so that now \(V_{\text{out}}\) is a \(\nu + 3\)-valent vertex of \(\Gamma\). The maps \(h_i\) then glue to give a map \(h : \Gamma \to M_\mathbb{R}\) with \(h(x) = Q\). This map satisfies the balancing condition at all vertices except perhaps at \(V_{\text{out}}\). To check balancing at \(V_{\text{out}}\), let \(E_{\text{out},i}\) denote the edge of the tropical disk \(h_i\) adjacent to the outgoing vertex \(V_{\text{out},i}\) of \(\Gamma_i'\). Suppose \(h_i\) has degree \(\Delta(h_i)\). Let \(v_i \in M\) be a primitive tangent vector based at \(Q\) pointing away from \(Q\) and tangent to \(h_i(E_{\text{out},i})\). Then by summing the balancing conditions at all vertices of \(\Gamma_i'\) other than \(V_{\text{out},i}\), one obtains
\[
(3.4) \quad w_{\Gamma_i}(E_{\text{out},i})v_i = r(\Delta(h_i)).
\]
So the condition that \(\prod_i \text{Mono}(h_i)^{\nu_i}\) gives a non-zero contribution to the integral \((5.3)\) is equivalent to
\[
(3.5) \quad 0 = \sum_i \nu_i r(\Delta(h_i)) = \sum_i \nu_i w_{\Gamma_i}(E_{\text{out},i})v_i,
\]
which is precisely the balancing condition for \(\Gamma\) at \(V_{\text{out}}\). Furthermore, the exponent of \(q\) determines the valency of \(\Gamma\) at \(V_{\text{out}}\). Thus we see that a term of \(\int_{\Xi_0} e^{q(W_k(q) - y_0)} \Omega\) of the form
\[
q^{\nu+2}K^d \prod_i \frac{1}{\nu_i!} \text{Mult}(h_i)^{\nu_i} u_I(h_i)
\]
gives a marked tropical curve \(h : (\Gamma, p_1, \ldots, p_{3d-2-\nu}, x) \to M_\mathbb{R}\) of degree \(d\) in \((\mathbb{P}^2, P_1, \ldots, P_k)\) with \(V_{\text{out}} \in \Gamma^{[0]}\) the vertex of \(E_x\) and \(\text{Val}(V_{\text{out}}) = \nu + 3\), \(h(x) = Q\).
Conversely, by Lemma 3.3 (3), if we have such a tropical curve, we can split this curve up into a collection of Maslov index two disks with endpoint $Q$. This in turn gives a term of $\int_{\Xi_0} e^{q(W_k(q)-y_0)\Omega}$ of the form \ref{eq:match}. This gives a one-to-one correspondence between such terms and curves.

Now given such a tropical curve $h$, let $n_0(x)$, $n_1(x)$ and $n_2(x)$ be the number of unbounded edges of $h$ with vertex $V_{\text{out}}$ in the directions $m_0$, $m_1$ and $m_2$ respectively. Then the term corresponding to such an $h$ is

$$\frac{1}{n_0(x)!n_1(x)!n_2(x)!} q^{\nu+2} \kappa^d u_I(h) \prod_i \text{Mult}(h_i) = \text{Mult}_0(h) q^{\nu+2} \kappa^d u_I(h) \prod_{V \in I} \text{Mult}_V(h).$$

Thus we see that for an index set $I = \{i_1, \ldots, i_{3d-2-\nu}\} \subseteq \{1, \ldots, k\}$, $i_1 < \cdots < i_{3d-2-\nu}$, the coefficient of $q^{\nu+2} \kappa^d u_I$ in $\int_{\Xi_0} e^{q(W_k(q)-y_0)\Omega}$ is

$$\langle P_{i_1}, \ldots, P_{i_{3d-2-\nu}}, \psi^\nu Q \rangle_{d}^{\text{trop}}.$$

By Lemma 3.12 the integral is independent of the position of the $P_i$’s and $Q$, and hence this number is independent of $I$, as can be seen simply by permuting the $P_i$’s. Thus this number can be interpreted as

$$\langle T_2^{3d-2-\nu}, \psi^\nu T_2 \rangle_{d}^{\text{trop}},$$

now shown to be completely independent of the choice of $P_1, \ldots, P_k$ and $Q$. In addition, we see that

$$\int_{\Xi_0} e^{qW_k(Q)\Omega} = e^{qy_0} \left( 1 + \sum_{d \geq 1} \sum_{\nu \geq 0} \langle T_2^{3d-2-\nu}, \psi^\nu T_2 \rangle_{d}^{\text{trop}} \kappa^d q^{\nu+2} \sum_{I \subseteq \{1, \ldots, k\}} \mu_I \right)$$

$$= e^{qy_0} \left( 1 + \sum_{d \geq 1} \sum_{\nu \geq 0} \langle T_2^{3d-2-\nu}, \psi^\nu T_2 \rangle_{d}^{\text{trop}} q^{\nu+2} \sum_{y_2 \in \mathbb{Z}} \frac{y_2}{(3d-2-\nu)!} \right),$$

where we take $y_2 = \sum_{i=1}^k u_i$ and formally take $y_1 = \log \kappa$. The latter is the expression for $J_0^{\text{trop}}$, hence the result. \hfill \square

**Example 3.14.** $\langle \psi^{3d-2} T_2 \rangle_{d}^{\text{trop}}$ is easily computed: there is only one tropical curve of degree $d$ with a vertex of valency $3d$ at $Q$, namely the curve which has $d$ legs of weight one in each of the three directions $(-1, -1)$, $(1, 0)$, and $(0, 1)$, and hence contributes a multiplicity of $1/(d!)^3$, so

$$\langle \psi^{3d-2} T_2 \rangle_{d}^{\text{trop}} = \frac{1}{(d!)^3}.$$

Using the formula for $W_1(Q) = y_0 + x_0 + x_1 + x_2 + u_1 x_1 x_2$ given in Example 2.9 we see that contributions to $\langle T_2, \psi^{3d-3} T_2 \rangle_{d}^{\text{trop}}$ comes from terms of the form $q^{3d-1} x_0^d x_1^{d-1} x_2^{d-1} (u_1 x_1 x_2)$, which shows that

$$\langle T_2, \psi^{3d-3} T_2 \rangle_{d}^{\text{trop}} = \frac{1}{(d!)(d-1)!(d-1)!}.$$
4. SCATTERING DIAGRAMS

It will be important for our task of computing the period integrals \( \int_{\Xi} e^{qW_k(Q)} \Omega \) to understand how the functions \( W_k(Q) \) depend on the choice of the base point \( Q \) as well as the \( P_i \)'s. It turns out that there is a chamber structure to \( M_\mathbb{R} \). For \( Q \) varying within a chamber, \( W_k(Q) \) is constant, and there are wall-crossing formulas for when \( Q \) moves between chambers. In fact, these wall-crossing formulas have already appeared in the context of scattering diagrams, which appeared in [17], following Kontsevich and Soibelman [23], to construct toric degenerations of Calabi-Yau manifolds. In addition, scattering diagrams have also been seen to play a role in enumerative and tropical geometry in [18] as well as in wall-crossing formulas for Donaldson-Thomas invariants in [24]. These types of wall-crossing formulas also appeared in [1], arising there, as here, from Maslov index zero disks. Our discussion here will make the enumerative relevance of these ideas clearer.

We shall repeat the definition of scattering diagram here, with slightly different definitions and conventions than was used in [17] or [18]. As in §2, we fix \( M \) a rank two lattice, and a complete fan \( \Sigma \) in \( M_\mathbb{R} \) defining a non-singular toric surface, giving an exact sequence

\[
0 \to K_\Sigma \to T_\Sigma \overset{r}{\to} M \to 0.
\]

**Definition 4.1.** Fix \( k \geq 0 \).

(1) A ray or line is a pair \((\mathfrak{d}, f_\mathfrak{d})\) such that

- \( \mathfrak{d} \subseteq M_\mathbb{R} \) is given by
  \[
  \mathfrak{d} = m'_0 - \mathbb{R}_{\geq 0}r(m_0)
  \]
  if \( \mathfrak{d} \) is a ray and
  \[
  \mathfrak{d} = m'_0 - \mathbb{R}r(m_0)
  \]
  if \( \mathfrak{d} \) is a line, for some \( m'_0 \in M_\mathbb{R} \) and \( m_0 \in T_\Sigma \) with \( r(m_0) \neq 0 \). The set \( \mathfrak{d} \) is called the support of the line or ray. If \( \mathfrak{d} \) is a ray, \( m'_0 \) is called the initial point of the ray, written as \( \text{Init}(\mathfrak{d}) \).
- \( f_\mathfrak{d} \in \mathbb{C}[z^{m_0}] \otimes \mathbb{C} R_k \subseteq \mathbb{C}[T_\Sigma] \otimes \mathbb{C} R_k[y_0] \).
- \( f_\mathfrak{d} \equiv 1 \mod (u_1, \ldots, u_k)z^{m_0} \).

(2) A scattering diagram \( \mathfrak{D} \) is a finite collection of lines and rays.

If \( \mathfrak{D} \) is a scattering diagram, we write

\[
\text{Supp}(\mathfrak{D}) := \bigcup_{\mathfrak{d} \in \mathfrak{D}} \mathfrak{d} \subseteq M_\mathbb{R}
\]

and

\[
\text{Sing}(\mathfrak{D}) := \bigcup_{\mathfrak{d} \in \mathfrak{D}} \partial \mathfrak{d} \cup \bigcup_{\dim \mathfrak{d}_1 \cap \mathfrak{d}_2 = 0} \mathfrak{d}_1 \cap \mathfrak{d}_2.
\]

Here \( \partial \mathfrak{d} = \{\text{Init}(\mathfrak{d})\} \) if \( \mathfrak{d} \) is a ray, and is empty if \( \mathfrak{d} \) is a line.
Construction 4.2. Given a smooth immersion $\gamma : [0, 1] \to M_R \setminus \text{Sing}(\mathcal{D})$ with endpoints not in any element of a scattering diagram $\mathcal{D}$, such that $\gamma$ intersects elements of $\mathcal{D}$ transversally, we can define a ring automorphism $\theta_{\gamma, \mathcal{D}} \in \text{Aut}(\mathbb{C}[T_\Sigma] \otimes_{\mathbb{C}} R_k[y_0])$, the $\gamma$-ordered product of $\mathcal{D}$. Explicitly, we can find numbers

$$0 < t_1 \leq t_2 \leq \cdots \leq t_s < 1$$

and elements $\mathfrak{d}_i \in \mathcal{D}$ such that $\gamma(t_i) \in \mathfrak{d}_i$ and $\mathfrak{d}_i \neq \mathfrak{d}_j$ if $t_i = t_j$, $i \neq j$, and $s$ taken as large as possible. Then for each $i$, define $\theta_{\gamma, \mathfrak{d}_i} \in \text{Aut}(\mathbb{C}[T_\Sigma] \otimes_{\mathbb{C}} R_k[y_0])$ to be

$$\theta_{\gamma, \mathfrak{d}_i}(z^m) = z^m f_{\mathfrak{d}_i}^{(n_0, r(m))}$$

$$\theta_{\gamma, \mathfrak{d}_i}(a) = a$$

for $m \in T_\Sigma$, $a \in R_k[y_0]$, where $n_0 \in \mathbb{N}$ is chosen to be primitive, annihilate the tangent space to $\mathfrak{d}_i$, and is finally completely determined by the sign convention that

$$\langle n_0, \gamma'(t_i) \rangle < 0.$$

We then define

$$\theta_{\gamma, \mathcal{D}} = \theta_{\gamma, \mathfrak{d}_s} \circ \cdots \circ \theta_{\gamma, \mathfrak{d}_1}.$$

There is still some ambiguity to the ordering if $\gamma$ crosses several overlapping rays. However, an easy check shows that automorphisms associated to parallel rays commute, so the order is irrelevant. Automorphisms associated with non-parallel rays do not necessarily commute, hence the need for $\gamma$ to avoid points of $\text{Sing}(\mathcal{D})$.

We will also allow the possibility that $\gamma$ is piecewise linear so that $\gamma'$ may not be defined at $t_i$, but still have $\gamma$ pass from one side of $\mathfrak{d}$ to the other, in which case we take $n_0$ so that $\gamma$ passes from the side of $\mathfrak{d}$ where $n_0$ is larger to the side it is smaller.

It is easy to check that $\theta_{\gamma, \mathcal{D}}$ only depends on the homotopy class of the path $\gamma$ inside $M_R \setminus \text{Sing}(\mathcal{D})$.

Example 4.3. Let $\mathcal{D} = \{(\mathfrak{d}_1, f_{\mathfrak{d}_1}), (\mathfrak{d}_2, f_{\mathfrak{d}_2}), (\mathfrak{d}_3, f_{\mathfrak{d}_3})\}$ with

$$\mathfrak{d}_1 = \mathbb{R}r(m_1), \quad f_{\mathfrak{d}_1} = 1 + c_1 w_1 z^{m_1},$$

$$\mathfrak{d}_2 = \mathbb{R}r(m_2), \quad f_{\mathfrak{d}_2} = 1 + c_2 w_2 z^{m_2},$$

$$\mathfrak{d}_3 = -\mathbb{R}_{\geq 0}(r(m_1 + m_2)), \quad f_{\mathfrak{d}_3} = 1 + c_1 c_2 w_{out} r(m_1) \wedge r(m_2) | z^{m_1 + m_2},$$

where $m_1, m_2 \in T_\Sigma \setminus K_\Sigma$, and $w_1, w_2$ and $w_{out}$ are the indices of $r(m_1), r(m_2)$ and $r(m_1 + m_2)$ respectively. The expression $|r(m_1) \wedge r(m_2)|$ denotes the absolute value of $r(m_1) \wedge r(m_2) \in \wedge^2 M \cong \mathbb{Z}$. Suppose $c_1, c_2 \in R_k$ satisfy $c_1^2 = c_2^2 = 0$. Then one can calculate that if $\gamma$ is a loop around the origin, then $\theta_{\gamma, \mathcal{D}}$ is the identity. See Figure 4.4, where $\theta_{\gamma, \mathcal{D}} = \theta_2^{-1} \theta_3 \theta_1^{-1} \theta_2 \theta_1$, with $\theta_i$ the automorphism coming from the first crossing of $\mathfrak{d}_i$.

\footnote{If $m \in M \setminus \{0\}$, the index of $m$ is the largest positive integer $w$ such that $m = wm'$ with $m' \in M$.}
We now relate this concept to the behaviour of the Landau-Ginzburg potential under change of the base-point $Q$. We fix general $P_1, \ldots, P_k \in M \mathbb{R}$, and will study the behaviour of $W_k(Q)$ as a function of the base-point $Q$. The discussion in this section will be for arbitrary complete fans $\Sigma$ in $M \mathbb{R}$ defining a non-singular toric surface.

First, we define a variant of tropical disk.

**Definition 4.4.** A tropical tree in $(X_\Sigma, P_1, \ldots, P_k)$ is a $d$-pointed tropical curve $h : (\Gamma, p_1, \ldots, p_d) \to M \mathbb{R}$ with $h(p_j) = P_{i_j}$, $1 \leq i_1 < \cdots < i_d \leq k$, along with the additional data of a choice of unmarked unbounded edge $E_{out} \in \Gamma^{[1]}$ such that for any $E \in \Gamma^{[1]} \setminus \{E_{out}\}$, $h(E)$ is a point or is a translate of some $\rho \in \Sigma^{[1]}$. The degree of $h$, $\Delta(h)$, is defined without counting the edge $E_{out}$, which need not be a translate of any $\rho \in \Sigma^{[1]}$.

The Maslov index of $h$ is

\[ \text{MI}(h) := 2(|\Delta(h)| - d). \]

Given $h$ and a point $V_{out}$ in the interior of $E_{out}$, we can remove the unbounded component of $E_{out} \setminus \{V_{out}\}$ from $\Gamma$ to obtain $\Gamma'$. Note $V_{out}$ is a univalent vertex of $\Gamma'$. Take $h' : \Gamma' \to M \mathbb{R}$ with $h' = h|_{\Gamma'}$. Then $h'$ is a tropical disk with boundary $h(V_{out})$ and Maslov index $\text{MI}(h') = \text{MI}(h)$, since $|\Delta(h')| = |\Delta(h)|$.

As in Lemma 2.6, standard tropical dimension counting arguments show that, for general choice of $P_1, \ldots, P_k$, a tropical tree $h$ moves in a family of dimension $\text{MI}(h)/2$. In particular, the set of Maslov index zero trees is a finite set, which we denote by Trees$(\Sigma, P_1, \ldots, P_k)$. As usual, with general choice of $P_1, \ldots, P_k$, we can assume all these trees are trivalent.

**Definition 4.5.** We define $\mathcal{D}(\Sigma, P_1, \ldots, P_k)$ to be the scattering diagram which contains one ray for each element $h$ of Trees$(\Sigma, P_1, \ldots, P_k)$. The ray corresponding to $h$ is of the form $(0, f_0)$, where

- $\mathfrak{d} = h(E_{out})$,
- $f_0 = 1 + w_{\Gamma}(E_{out}) \operatorname{Mult}(h) z^{\Delta(h)} u_I(h)$, where $u_I(h) = \prod_{i \in I(h)} u_i$ and $I(h) \subseteq \{1, \ldots, k\}$ is defined by $I(h) := \{i \mid h(p_j) = P_i \text{ for some } j\}$.
Example 4.6. $\mathcal{D}(\Sigma, P_1, P_2)$ is illustrated in Figure 4.2, where $\Sigma$ is the fan for $\mathbb{P}^2$, as given in Example 2.9.

Proposition 4.7. If $P \in \text{Sing}(\mathcal{D}(\Sigma, P_1, \ldots, P_k))$ is a singular point with $P \not\in \{P_1, \ldots, P_k\}$, and $\gamma_P$ is a small loop around $P$, then $\theta_{\gamma_P, \mathcal{D}(\Sigma, P_1, \ldots, P_k)} = \text{id}$.

Proof. Let $P$ be such a singular point. Suppose that $\mathfrak{d} \in \mathcal{D}(\Sigma, P_1, \ldots, P_k)$ has $\text{Init}(\mathfrak{d}) = P$, and let $h$ be the corresponding tree. Then the unique vertex $V$ of $\Gamma$ on $E_{\text{out}}$ has $h(V) = P$. By the generality assumption, $V$ is trivalent, so if we cut $\Gamma$ at $V$, we obtain two tropical disks $h_1' : \Gamma_1' \to M_\mathbb{R}$ and $h_2' : \Gamma_2 \to M_\mathbb{R}$ with boundary $P$ and $V_{\text{out}} = V$ in both cases. Now $\text{MI}(h) = \text{MI}(h_1') + \text{MI}(h_2')$, so $\text{MI}(h_1') = \text{MI}(h_2') = 0$ is the only possibility. Thus $h_1', h_2'$ extend to tropical trees $h_i : \Gamma_i \to M_\mathbb{R}$, with corresponding rays $\mathfrak{d}_1, \mathfrak{d}_2$. Note that $P \neq \text{Init}(\mathfrak{d}_1)$, $\text{Init}(\mathfrak{d}_2)$ and $I(h_1) \cap I(h_2) = \emptyset$. So every ray $\mathfrak{d}$ with $P = \text{Init}(\mathfrak{d})$ arises from the collision of two rays $\mathfrak{d}_1, \mathfrak{d}_2$ with $P \neq \text{Init}(\mathfrak{d}_i)$.

Conversely, if we are given two such rays $\mathfrak{d}_1, \mathfrak{d}_2$ passing through $P$ corresponding to trees $h_1$ and $h_2$ with $I(h_1) \cap I(h_2) = \emptyset$, we obtain a new tree by cutting $h_1$ and $h_2$ at $P$ to get Maslov index zero disks $h_i' : \Gamma_i' \to M_\mathbb{R}$ with boundary at $P$. Next glue $\Gamma_1'$ and $\Gamma_2'$ at the outgoing vertex $V$, and add an additional unbounded edge $E_{\text{out}}$ with endpoint $V$ to get a graph $\Gamma$. If $E_{\text{out}, 1}, E_{\text{out}, 2}$ are the two outgoing edges of $\Gamma_1'$ and $\Gamma_2'$ respectively, with primitive tangent vector to $h_i'(E_{\text{out}, i})$ pointing away from $P$ being $m_i'$, then we define $h : \Gamma \to M_\mathbb{R}$ to restrict to $h_i'$ on $\Gamma_i'$ and to take $E_{\text{out}}$ to the ray $P - \mathbb{R}_{\geq 0}(w_\Gamma(E_{\text{out}, 1})m_1' + w_\Gamma(E_{\text{out}, 2})m_2')$. By
taking $w_T(E_{out})$ to be the index of $w_T(E_{out,1})m_1' + w_T(E_{out,2})m_2'$, we find $h$ is balanced at $V$. Thus $h$ is a tropical tree, whose Maslov index is zero.

To prove the proposition, define a new scattering diagram $\mathcal{D}_P$, whose elements are in one-to-one correspondence with elements of $\mathcal{D}(\Sigma, P_1, \ldots, P_k)$ containing $P$. If $(\mathcal{d}, f_\mathcal{d}) \in \mathcal{D}(\Sigma, P_1, \ldots, P_k)$ is a ray containing $P$ then the corresponding element of $\mathcal{D}_P$ will be $(\mathcal{d}', f_\mathcal{d})$, where $\mathcal{d}'$ is the tangent line (through the origin) of $\mathcal{d}$ if $P \neq \text{Init}(\mathcal{d})$ and is the ray $\mathcal{d} - P$ with endpoint otherwise. If $\gamma_0$ is a loop around the origin with the same orientation as $\gamma_P$, then $\theta_{\gamma_0, \mathcal{D}_P} = \theta_{\gamma_P, \mathcal{D}(\Sigma, P_1, \ldots, P_k)}$.

First consider the simplest case, when $\mathcal{D}_P$ contains two lines and at most one ray. If the two lines correspond to trees $h_1$ and $h_2$, and $I(h_1) \cap I(h_2) \neq \emptyset$, then $h_1$ and $h_2$ cannot be glued as above since they pass through some common marked point $P_i$. Thus $\mathcal{D}_P$ contains no rays. In this case, the automorphisms associated to $\mathcal{d}_1$ and $\mathcal{d}_2$ commute by Example 4.3 since $u_{I(h_1)}u_{I(h_2)} = 0$, and so $\theta_{\gamma_0, \mathcal{D}_P}$ is the identity.

If, on the other hand, $I(h_1) \cap I(h_2) = \emptyset$, then $h_1$ and $h_2$ can be glued to obtain a new tree $h$, and $\mathcal{D}_P$ consists of three elements $\mathcal{d}_1$, $\mathcal{d}_2$ and $\mathcal{d}$, corresponding to $h_1$, $h_2$ and $h$ respectively. Now

$$f_\mathcal{d}_i = 1 + w_T(E_{out, i}) \text{Mult}(h_i)z^\Delta(h_i)u_{I(h_i)}$$

for $i = 1, 2$ and

$$f_\mathcal{d} = 1 + w_T(E_{out}) \text{Mult}(h)z^\Delta(h)u_{I(h)}$$

$$= 1 + w_T(E_{out, 1}) \text{Mult}(h_1) \text{Mult}(h_2) \text{Mult}_V(h)z^{\Delta(h_1) + \Delta(h_2)}u_{I(h_1)}u_{I(h_2)}$$

$$= 1 + w_T(E_{out, 1}) \text{Mult}(h_1) \text{Mult}(h_2)w_{1, 2}(E_{out, 2})m_1'm_2'z^{\Delta(h_1) + \Delta(h_2)}u_{I(h_1)}u_{I(h_2)}$$

$$= 1 + w_T(E_{out, 1}) \text{Mult}(h_1) \text{Mult}(h_2)|r(\Delta(h_1)) \cap r(\Delta(h_2))|z^{\Delta(h_1) + \Delta(h_2)}u_{I(h_1)}u_{I(h_2)}.$$ 

Thus from Example 4.3 $\theta_{\gamma_0, \mathcal{D}_P}$ is the identity.

For the general case, we have some finite set of lines in $\mathcal{D}_P$, along with some rays. Suppose that there are three lines in $\mathcal{D}_P$ corresponding to trees $h_1$, $h_2$ and $h_3$ with $I(h_1)$, $I(h_2)$ and $I(h_3)$ mutually disjoint. Then as in the case of two lines above, these trees can be glued at $P$, obtaining a Maslov index zero tree with a quadrivalent vertex. However, since $P_1, \ldots, P_k$ are in general position, no Maslov index zero tree has a vertex with valence > 3. Thus this case does not occur. On the other hand, given two lines corresponding to trees $h_1$, $h_2$ with $I(h_1) \cap I(h_2) = \emptyset$, these two trees can be glued as above at $P$ to obtain a new Maslov index zero tree. Thus the rays in $\mathcal{D}_P$ are in one-to-one correspondence with pairs of lines $\mathcal{d}_1, \mathcal{d}_2 \in \mathcal{D}_P$ corresponding to trees $h_1$ and $h_2$ with $I(h_1) \cap I(h_2) = \emptyset$. So we can write

$$\mathcal{D}_P = \{\mathcal{d}_1, \ldots, \mathcal{d}_n\} \cup \bigcup_{j=1}^m \mathcal{D}_i$$

where $\mathcal{d}_1, \ldots, \mathcal{d}_n$ are lines corresponding to trees $h$ such that $I(h) \cap I(h') \neq \emptyset$ for any Maslov index zero tree $h'$ with outgoing edge passing through $P$, and $\mathcal{D}_1, \ldots, \mathcal{D}_m$ are scattering
we are given the additional data of a monomial
and the ray corresponding to the tree obtained by gluing
and 2 at $P$.

Now computing $\theta_{\gamma_0, D_p}$ is an exercise in commutators. Note that if $D_1, D_2 \in D_P$ correspond
to two trees $h_1, h_2$ with $I(h_1) \cap I(h_2) \neq \emptyset$, then as already observed, $\theta_{\gamma_0, D_1}$ and $\theta_{\gamma_0, D_2}$ commute. Thus after using this commutation, one can write

$$
\theta_{\gamma_0, D_p} = \left( \prod_{i=1}^{n} \theta_{\gamma_0, D_i} \circ \theta_{\gamma_0, D_i}^{-1} \right) \circ \prod_{j=1}^{m} \theta_{\gamma_0, D_j}.
$$

Of course $\theta_{\gamma_0, D_i} \circ \theta_{\gamma_0, D_i}^{-1} = \text{id}$ and $\theta_{\gamma_0, D_j} = \text{id}$ by the special case already carried out. Thus $\theta_{\gamma_0, D_p} = \text{id}$ in this general case.

**Remark 4.8.** Note that the rays in $D = D(\Sigma, P_1, \ldots, P_k)$ with endpoint $P_i$ are in one-to-one correspondence with Maslov index two disks in $(X_\Sigma, P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_k)$ with boundary $P_i$. Indeed, taking any such Maslov index two disk, extending the outgoing edge to get a tropical tree, we can mark the point on this outgoing edge which maps to $P_i$, thus getting a tropical tree with Maslov index zero in $(X_\Sigma, P_1, \ldots, P_k)$. The corresponding ray in $D$ has endpoint $P_i$. Conversely, given a ray in $D$ with endpoint $P_i$, this corresponds to a Maslov index zero tree such that the vertex $V$ adjacent to $V_{out}$ is the vertex of a marked edge $E_x$ mapping to $P_i$. By cutting this tree at $V$, removing the marked edge mapping to $P_i$, we get a Maslov index two disk with boundary $P_i$.

Furthermore, by the general position of the $P_j$, there are no rays in $D$ containing $P_i$ but which don’t have $P_i$ as an endpoint.

One benefit of this scattering diagram approach is that it is easy to describe the Maslov index two disks with boundary a general point $Q$, using what we call *broken lines*:

**Definition 4.9.** A broken line is a continuous proper piecewise linear map $\beta : (-\infty, 0] \to M_\mathbb{R}$ with endpoint $Q = \beta(0)$, along with some additional data described as follows. Let

$$
-\infty = t_0 < t_1 < \cdots < t_n = 0
$$

be the smallest set of real numbers such that $\beta|_{(t_{i-1}, t_i)}$ is linear. Then for each $1 \leq i \leq n$, we are given the additional data of a monomial $c_i z^{m_i^\beta} \in C[T_\Sigma] \otimes C R_k[y_0]$ with $m_i^\beta \in T_\Sigma$ and $0 \neq c_i \in R_k[y_0]$. Furthermore, this data should satisfy the following properties:

1. For each $i$, $r(m_i^\beta)$ points in the same direction as $-\beta'(t)$ for $t \in (t_{i-1}, t_i)$.
2. $m_i^\beta = t_\rho$ for some $\rho \in \Sigma[1]$ and $c_1 = 1$.
3. $\beta(t_i) \in \text{Supp}(D(\Sigma, P_1, \ldots, P_k))$ for $1 \leq i \leq n$. 


(4) If the image of $\beta$ is disjoint from $\text{Sing}(\mathfrak{D}(\Sigma, P_1, \ldots, P_k))$, and $\beta(t_i) \in \mathfrak{d}_1 \cap \cdots \cap \mathfrak{d}_s$ (necessarily this intersection is one-dimensional), then $c_{i+1}z^{m_{i+1}}$ is a term in
\[(\theta_{\beta, \mathfrak{d}_1} \circ \cdots \circ \theta_{\beta, \mathfrak{d}_s})(c_i z^{m_i}).\]

By this, we mean the following. Suppose $f_{\mathfrak{d}_j} = 1 + c_{\mathfrak{d}_j} z^{m_{\mathfrak{d}_j}}$, $1 \leq j \leq s$, with $c_{\mathfrak{d}_j}^2 = 0$, and $n \in \mathbb{N}$ is primitive, orthogonal to all the $\mathfrak{d}_j$’s, chosen so that
\[(\theta_{\beta, \mathfrak{d}_1} \circ \cdots \circ \theta_{\beta, \mathfrak{d}_s})(c_i z^{m_i}) = c_i z^{m_i} \prod_{j=1}^{s} (1 + c_{\mathfrak{d}_j} z^{m_{\mathfrak{d}_j}})^{\langle n, r(m_{\mathfrak{d}_j}) \rangle}.\]

Then we must have
\[c_{i+1}z^{m_{i+1}} = c_i z^{m_i} \prod_{j \in J} (\langle n, r(m_{\mathfrak{d}_j}) \rangle c_{\mathfrak{d}_j} z^{m_{\mathfrak{d}_j}})\]

for some index set $J \subseteq \{1, \ldots, s\}$. We think of this as $\beta$ being bent by time $t_i$ by the rays $\{\mathfrak{d}_j \mid j \in J\}$.

(5) If the image of $\beta$ is not disjoint from $\text{Sing}(\mathfrak{D}(\Sigma, P_1, \ldots, P_k))$, then $\beta$ is the limit of a family of broken lines which are disjoint from $\text{Sing}(\mathfrak{D}(\Sigma, P_1, \ldots, P_k))$. More precisely, there is:

- A continuous map $B : (-\infty, 0] \times [0, 1] \rightarrow M_{\mathbb{R}}$.
- Continuous functions $t_0, \ldots, t_n : [0, 1] \rightarrow [-\infty, 0]$ such that
  \[-\infty = t_0(s) \leq t_1(s) \leq \cdots \leq t_n(s) = 0\]
  for $s \in [0, 1]$, with strict inequality for $s < 1$.
- Monomials $c_i z^{m_i}$ for $1 \leq i \leq n$.

This data satisfies $B_s := B|_{(-\infty, 0] \times \{s\}}$ (with the data $c_i z^{m_i}$) is a broken line not passing through a point of $\text{Sing}(\mathfrak{D}(\Sigma, P_1, \ldots, P_k))$ for $s < 1$, and $\beta := B_1$.

Note that in taking such a limit, we might have $t_{i-1}$ and $t_i$ coming together for various $i$, so the limit might have fewer linear segments.

**Example 4.10.** Again, in the case of $\mathbb{P}^2$, $k = 2$, Figure 4.3 shows the broken lines with $\beta(0)$ the given point $Q$. The segments of each broken line are labelled with their corresponding monomial.

**Proposition 4.11.** If $Q \not\in \text{Supp}(\mathfrak{D}(\Sigma, P_1, \ldots, P_k))$ is general, then there is a one-to-one correspondence between broken lines with endpoint $Q$ and Maslov index two disks with boundary $Q$. In addition, if $\beta$ is a broken line corresponding to a disk $h$, and $cz^m$ is the monomial associated to the last segment of $\beta$ (the one whose endpoint is $t_n = 0$), then
\[cz^m = \text{Mono}(h).\]
Figure 4.3. The broken lines ending at $Q$.

**Proof.** We first prove the following claim:

**Claim:** Let $h : \Gamma' \to M_\mathbb{R}$ be a Maslov index two disk in $(X_\Sigma, P_1, \ldots, P_k)$ with boundary $Q' \in M_\mathbb{R}$. Suppose furthermore that all vertices of $\Gamma'$ except $V_{out}$ are trivalent and $h$ cannot be deformed continuously in a family of Maslov index two disks with boundary $Q'$. Then there is a uniquely determined subset $\Xi = \Xi(h) \subseteq \Gamma$ which is a union of edges of $\Gamma$ and is homeomorphic to $[-\infty, 0]$, connecting some point in $\Gamma_\infty \setminus \{V_{out}\}$ to $V_{out}$, satisfying:

1. $\Xi$ is disjoint from $\partial E_{p_i}$ for all $i$.
2. The restriction of $h$ to the closure of any connected component of $\Gamma' \setminus \Xi$ is a Maslov index zero disk.

**Proof.** We proceed inductively on the number of vertices of $\Gamma'$. If $\Gamma'$ has only one vertex, $V_{out}$, then $\Gamma'$ has only one edge and no marked edges. We simply take $\Xi(h)$ to consist of this edge.

For the induction step, let $\Gamma'$ have outgoing edge $E_{out}$ with vertices $V_{out}$ and $V$, and $h(V_{out}) = Q'$.

First, we will show that $V$ cannot be a vertex of some marked edge $E_{p_i}$. If it were, so that $h(V) = h(p_i) = P_j$ for some $j$, then we can cut $h$ at $V$ and remove the marked edge $E_{p_i}$. This gives a disk $h'$ with boundary $P_j$ but with one less marked point than $h$. Hence
$h'$ is Maslov index four, and thus by Lemma 2.6, $h'$ can be deformed in a one-parameter family while keeping the endpoint $P_j$ fixed. Note that for small deformations of $h'$, the edge $h'(E_{out}^i)$ does not change its tangent direction. Thus a deformation of $h'$ can be extended to a deformation of $h$. This contradicts the assumption that $h$ cannot be deformed. Thus $V$ cannot be a vertex of some $E_{pi}$.

Now split $h$ at $V$, letting $\Gamma'_1$ and $\Gamma'_2$ be the closures of the two connected components of $\Gamma' \setminus \{V\}$ not containing $V_{out}$. Let $h_i = h|_{\Gamma'_i}$. This gives two disks $h_1$, $h_2$ with boundary $h(V)$. We have $MI(h) = MI(h_1) + MI(h_2)$. Suppose $MI(h_1) \geq 4$. Then $h_1$ can be deformed leaving the endpoint $h(V)$ fixed, and by gluing such a deformation to $h_2$, we obtain a deformation of $h$, again a contradiction. Thus $MI(h_1), MI(h_2) \leq 2$, so we must have $MI(h_1) = 0$ and $MI(h_2) = 2$ or vice versa.

Without loss of generality, assume $MI(h_2) = 2$. Note that $h_2$ is now a Maslov index two disk with boundary $Q'' = h_2(V)$. If $h_2$ could be deformed in a family of disks with boundary $Q''$, then by gluing these deformations to $h_1$, we obtain a deformation of $h$, a contradiction. Thus $h_2$ satisfies the hypotheses of the Claim, but $\Gamma'_2$ has fewer vertices than $\Gamma'$. So $\Xi(h_2)$ exists by the induction hypothesis, and we can take $\Xi(h)$ to be

$$\Xi(h) = E_{out} \cup \Xi(h_2).$$

$\Xi(h)$ satisfies the two desired properties because $\Xi(h_2)$ does, $E_{out}$ is disjoint from $\partial E_{pi}$, for all $i$, and $h_1$ is a Maslov index zero disk. \hfill \Box

Now fix a Maslov index two disk $h : \Gamma' \to M_{\mathbb{R}}$ with boundary $Q$. By the generality of $Q, P_1, \ldots, P_k$, $h$ satisfies the hypotheses of the Claim. Taking $\beta = h|_{\Xi(h)}$, we see that $\beta$ is piecewise linear. Let $-\infty = t_0 < \cdots < t_n = 0$ be chosen as in the definition of broken line. Each $t_i$ corresponds to a vertex $V_i$ of $\Gamma'$. Of course $\Gamma' \setminus \{V_i\}$ for $i \neq n$ has two connected components not containing $V_{out}$, and the proof of the claim shows that restricting $h$ to the closure of one of these two connected components yields a Maslov index two disk with boundary $h(V_i)$ which we now call $h_i$. The other component similarly yields a Maslov index zero disk. Hence $\beta(t_i) \in \text{Supp}(\mathcal{D}(\Sigma, P_1, \ldots, P_k))$ for $1 \leq i \leq n$. We take the monomial $c_i z^{m_i}$ to be $\text{Mono}(h_i)$, and need to check that with this data $\beta$ is now a broken line.

We have just shown condition (3), and condition (2) is obvious. Condition (1) is immediate from the balancing condition for $h_i$. So for $Q$ sufficiently general, we only need to verify condition (4). We need to show that the monomial $\text{Mono}(h_{i+1})$ attached to the edge joining $V_i$ and $V_{i+1}$ arises from the monomial $\text{Mono}(h_i)$ attached to the edge joining $V_{i-1}$ and $V_i$, as in Condition (4). Suppose that the two subtrees in $\Gamma'$ rooted at $V_i$ are $g$ and $h_i$, with $MI(g) = 0$. Now

$$\text{Mono}(h_i) = \text{Mult}(h_i) z^{\Delta(h_i)} u_I(h_i),$$
and if $\mathfrak{d}$ is the ray corresponding to the tropical tree obtained from $g$, then

$$f_{\mathfrak{d}} = 1 + w_{\text{out}}(g) \text{Mult}(g) z^{\Delta(g)} u_{I(g)}.$$  

Here $w_{\text{out}}(g)$ denotes the weight of the outgoing edge of $g$. By the balancing condition, we can write $-w_{\text{out}}(g)m'_1 = r(\Delta(g))$, and $-w_{\text{out}}(h_i)m'_2 = r(\Delta(h_i))$ with $m'_1 \in M$ primitive, with $m'_1$ tangent to the outgoing edge of $g$ and $m'_2$ tangent to the outgoing edge of $h_i$. Choosing a basis for $M$ and writing $m'_i = (m'_{i1}, m'_{i2})$, we see that to define $\theta_{\beta, \mathfrak{d}}$, we take $n_0 = \pm (m'_{i12}, -m'_{i11})$, so

$$\theta_{\beta, \mathfrak{d}}(\text{Mono}(h_i)) = \text{Mult}(h_i) z^{\Delta(h_i)} u_{I(h_i)} \cdot
\cdot (1 + w_{\text{out}}(g) \text{Mult}(g) z^{\Delta(g)} u_{I(g)})(\pm (m'_{i12}, -m'_{i11}), -w_{\text{out}}(h_i)(m'_{21}, m'_{22}))
= \text{Mult}(h_i) z^{\Delta(h_i)} u_{I(h_i)}
+ \text{Mult}(h_i) \text{Mult}(g)|m'_1 \wedge m'_2| w_{\text{out}}(g) w_{\text{out}}(h_i) z^{\Delta(h_i) + \Delta(g)} u_{I(h_i)} u_{I(g)}.$$

Here the second term occurs with a plus sign since the exponent is always positive—the convention on $n_0$ says that $n_0$ should be negative on vectors pointing in the direction we cross $\mathfrak{d}$; but $m'_{2}$ is such a vector so $n_0$ is positive on $-m'_{2}$.

Now it is the second term we are interested in, and this is

$$\text{Mult}(h_i) \text{Mult}(g) \text{Mult}_{V_1(h_{i+1})} z^{\Delta(h_{i+1})} u_{I(h_{i+1})} = \text{Mult}(h_{i+1}) z^{\Delta(h_{i+1})} u_{I(h_{i+1})}
= \text{Mono}(h_{i+1})$$

as desired.

Conversely, given a broken line $\beta$, it is easy to construct the corresponding Maslov index two disk, by attaching Maslov index zero disks to the domain $(-\infty, 0]$ of $\beta$ at each bending point. In particular, if $\beta(t_i)$ lies in rays $\mathfrak{d}_1, \ldots, \mathfrak{d}_s \in \mathfrak{D}(\Sigma, P_1, \ldots, P_k)$, and $\beta$ is bent at time $t_i$ by a subset $\{\mathfrak{d}_j | j \in J\}$ of these rays, then for each $j \in J$, we attach the Maslov index zero disk with endpoint $\beta(t_i)$ corresponding to $\mathfrak{d}_j$ to $t_i \in (-\infty, 0]$. (Note that by general position of the $P_i$’s and $Q$, in fact we can assume that $\#J = 1$.) It is clear that this reverses the above process of passing from a Maslov index two disk to a broken line.

The first main theorem of this section explores how $W_k(Q)$ depends on $Q$.

**Theorem 4.12.** If $Q, Q' \in M_k \setminus \text{Supp}(\mathfrak{D}(\Sigma, P_1, \ldots, P_k))$ are general, and $\gamma$ is a path connecting $Q$ and $Q'$ for which $\theta_{\gamma, \mathfrak{D}(\Sigma, P_1, \ldots, P_k)}$ is defined, then

$$W_k(Q') = \theta_{\gamma, \mathfrak{D}(\Sigma, P_1, \ldots, P_k)}(W_k(Q)).$$

**Proof.** Let $\mathfrak{D} = \mathfrak{D}(\Sigma, P_1, \ldots, P_k)$. Let $\mathfrak{U}$ be the union of Supp($\mathfrak{D}$) and the union of images of all broken lines, with arbitrary endpoint, which pass through points of Sing($\mathfrak{D}$). Recall from Definition 4.9 (5), that such broken lines are limits of broken lines that don’t pass
through singular points of $\mathcal{D}$. It is clear that $\dim \mathcal{U} \leq 1$ (of course equal to 1 provided $k \geq 1$).

We will now define a continuous deformation of a broken line, much as we did in Definition 4.11 (5). This is a continuous map $B : (-\infty, 0] \times I \to M_\mathbb{R}$ with $I \subseteq \mathbb{R}$ an interval, continuous functions $t_0, \ldots, t_n : I \to [-\infty, 0]$ such that $-\infty = t_0(s) < t_1(s) < \cdots < t_n(s) = 0$ for $s \in I$, and monomials $c_iz^{m_i} \beta$, $1 \leq i \leq n$. This data satisfies the condition that $B_s := B|_{(-\infty, 0] \times \{s\}}$ is a broken line in the usual sense for all $s \in I$, with the data $t_0(s) < \cdots < t_n(s)$ and monomials $c_iz^{m_i} \beta$, $1 \leq i \leq n$.

We say $B_{s'}$ is a deformation of $B_s$ for $s, s' \in I$.

Note a broken line $\beta$ which does not pass through a point of $\text{Sing}(\mathcal{D})$ can always be deformed continuously. This can be done as follows. We translate the initial ray $\beta((-\infty, t_1])$ of $\beta$. Inductively, this deforms all the remaining segments of $\beta$. As long as one of the bending points does not reach a singular point of $\mathcal{D}$, each bending point remains inside exactly the same set of rays in $\mathcal{D}$, and therefore the deformed broken line can bend in exactly the same way as $\beta$. Thus we run into trouble building this deformation only when this deformation of $\beta$ converges to a broken line which passes through a point of $\text{Sing}(\mathcal{D})$, as then the set of rays containing a bending point may jump.

From this it is clear that as long as the endpoint of $\beta$ stays within one connected component of $M_\mathbb{R} \setminus \mathcal{U}$, $\beta$ can be deformed continuously. More precisely, if we consider a path $\gamma : [0, 1] \to \mathcal{U}$, for $\mathcal{U}$ a connected component of $M_\mathbb{R} \setminus \mathcal{U}$, and $\beta$ is a broken line with endpoint $\gamma(0)$, then there is a continuous deformation $B$ with $\beta = B_0$ and with $B_s(0) = \gamma(s)$, $0 \leq s \leq 1$.

By Proposition 4.11 the Maslov index two disks with boundary $Q$ are in one-to-one correspondence with the broken lines with endpoint $Q$ for $Q$ general. Thus by the above discussion, $W_\lambda(Q)$ is constant for $Q$ varying inside a connected component of $M_\mathbb{R} \setminus \mathcal{U}$.

We will now analyze carefully how broken lines change if their endpoint passes in between different connected components of $M_\mathbb{R} \setminus \mathcal{U}$. So now consider two connected components $\mathcal{U}_1$ and $\mathcal{U}_2$ of $M_\mathbb{R} \setminus \mathcal{U}$. Let $L = \overline{\mathcal{U}_1 \cap \mathcal{U}_2}$, and assume $\dim L = 1$. Let $Q_1$ and $Q_2$ be general points in $\mathcal{U}_1$ and $\mathcal{U}_2$, near $L$, positioned on opposite sides of $L$. Let $\gamma : [0, 1] \to M_\mathbb{R}$ be a short general path connecting $Q_1$ and $Q_2$ crossing $L$ precisely once. Let $s_0$ be the only time at which $\gamma(s_0) \in L$. By choosing $\gamma$ sufficiently generally, we can assume that $\gamma(s_0)$ is a point in a neighbourhood of which $\mathcal{U}$ is a manifold.

Let $\mathfrak{B}(Q_i)$ be the set of broken lines with endpoint $Q_i$. Let $n_0 \in N$ be a primitive vector annihilating the tangent space to $L$ and taking a smaller value on $Q_1$ than $Q_2$. We can decompose $\mathfrak{B}(Q_i)$ into three sets $\mathfrak{B}^+(Q_i)$, $\mathfrak{B}^-(Q_i)$, and $\mathfrak{B}^0(Q_i)$ as follows. For $\beta \in \mathfrak{B}(Q_i)$, let $m_\beta = \beta_*(-\partial/\partial t|_{t=0})$. Then $\beta \in \mathfrak{B}^+(Q_i)$, $\mathfrak{B}^-(Q_i)$, or $\mathfrak{B}^0(Q_i)$ depending on whether
\( \langle n_0, m_\beta \rangle > 0, \langle n_0, m_\beta \rangle < 0, \) or \( \langle n_0, m_\beta \rangle = 0. \) This gives decompositions

\[
\begin{align*}
W_k(Q_1) &= W_k^-(Q_1) + W_k^0(Q_1) + W_k^+(Q_1), \\
W_k(Q_2) &= W_k^-(Q_2) + W_k^0(Q_2) + W_k^+(Q_2).
\end{align*}
\]

We will show

\[
\begin{align*}
(4.1) \quad \theta_{\gamma,D}(W_k^-(Q_1)) &= W_k^-(Q_2), \\
(4.2) \quad \theta_{\gamma,D}^{-1}(W_k^+(Q_2)) &= W_k^+(Q_1), \\
(4.3) \quad W_k^0(Q_2) &= W_k^0(Q_1).
\end{align*}
\]

From this follows the desired identity

\[
\theta_{\gamma,D}(W_k(Q_1)) = W_k(Q_2),
\]

as \( \theta_{\gamma,D} \) is necessarily the identity on \( W_k^0(Q_1) \). One then uses this inductively to see that this holds for any path \( \gamma \) with endpoints in \( M_\mathbb{R} \setminus \U \) for which \( \theta_{\gamma,D} \) is defined.

**Proof of** (4.1) and (4.2). If \( \beta \) is a broken line with endpoint \( Q_1 \), then \( \beta([t_{n-1}, 0]) \cap L = \emptyset \) if \( \langle n_0, m_\beta \rangle \leq 0 \), while \( \beta([t_{n-1}, 0]) \cap L \neq \emptyset \) if \( \langle n_0, m_\beta \rangle > 0 \). (Here we are using \( Q \) very close to \( L \).) On the other hand, if \( \beta \) has endpoint \( Q_2 \), then \( \beta([t_{n-1}, 0]) \cap L = \emptyset \) if \( \langle n_0, m_\beta \rangle \geq 0 \) and \( \beta([t_{n-1}, 0]) \cap L \neq \emptyset \) if \( \langle n_0, m_\beta \rangle < 0 \).

To see, say, (4.1), we proceed as follows. Let \( \beta \in \mathcal{B}^-(Q_1) \). By the previous paragraph, \( \beta([t_{n-1}, 0]) \cap L = \emptyset \). Let \( c_n z^{m_\beta} \) be the monomial associated to the last segment of \( \beta \), and write \( \theta_{\gamma,D}(c_n z^{m_\beta}) \) as a sum of monomials \( \sum_{i=1}^{s_1} d_i z^{m_\gamma} \) as in Definition 1.9 (4). We can then deform \( \beta \) continuously along \( \gamma \) to time \( s_0 \). Indeed, by the definition of \( \U \), if \( \beta \) converged to a broken line through \( \text{Sing}(\mathcal{D}) \), the image of this broken line would be contained in \( \U \), and then \( \U \), already containing \( L \), would not be a manifold in a neighbourhood of \( \gamma(s_0) \).

Let \( \beta' \) be the deformation of \( \beta \) with endpoint \( \gamma(s_0) \). For \( 1 \leq i \leq s_1 \), we then get a broken line \( \beta_i' \) by adding a short line segment to \( \beta' \) in the direction \( -r(m_i) \), with attached monomial \( d_i z^{m_\gamma} \). This new broken line has endpoint in \( u_2 \), and hence can be deformed to a broken line \( \beta_i'' \in \mathcal{B}^-(Q_2) \). We note that the line may not actually bend at \( L \) if \( d_i z^{m_\gamma} \) is the term \( c_n z^{m_\beta} \) appearing in \( \theta_{\gamma,D}(c_n z^{m_\beta}) \). See Figure 4.4.

Conversely, any broken line \( \beta \in \mathcal{B}^-(Q_2) \) clearly arises in this way.

From this, (4.1) becomes clear. (4.2) is identical. \( \square \)

**Proof of** (4.3). We will show that there are partitions \( \mathcal{B}^0(Q_1) = \bigsqcup_{i=1}^s \mathcal{B}^1_i \) and \( \mathcal{B}^0(Q_2) = \bigsqcup_{i=1}^s \mathcal{B}^2_i \) such that for each \( i \), the contributions to \( W_k(Q_1) \) and \( W_k(Q_2) \) from \( \mathcal{B}^1_i \) and \( \mathcal{B}^2_i \) are the same.

For simplicity of exposition, we will describe this very explicitly in the case that any broken line with endpoint \( \gamma(s_0) \) passes through at most one point of \( \text{Sing}(\mathcal{D}) \); we leave it
Figure 4.4. Broken lines with endpoints $Q_1$ and $Q_2$. 

To the reader to deal with the general case: this is notationally, but not conceptually, more complicated.

Let $\beta_1 \in \mathcal{B}^0(Q_1)$. If $\beta_1$ deforms continuously to a broken line $\beta_2$ in $\mathcal{B}^0(Q_2)$, then $\beta_1$ and $\beta_2$ will each appear in one-element sets in the partition, say $\beta_1 \in \mathcal{B}_1^1$, $\beta_2 \in \mathcal{B}_2^2$, and clearly both these sets contribute the same term to $W_k(Q_1)$ and $W_k(Q_2)$.

Now suppose $\beta_1 \in \mathcal{B}^0(Q_1)$ cannot be deformed continuously to any $\beta_2 \in \mathcal{B}^0(Q_2)$. This means that there is a $B : (-\infty,0] \times [0,s_0) \to M_{\mathbb{R}}$ as in Definition 4.9, (5) such that $B|_{(-\infty,0] \times [0,s_0)}$ is a continuous deformation and $B_{s_0}$ is a broken line passing through a point of $\text{Sing}(\mathcal{D})$. Furthermore, there must be some $j$ such that $B(t_j(s_0),s_0) = P \in \text{Sing}(\mathcal{D})$. In other words, for $s$ near $s_0$, $B_s$ must bend near $P$, and this bending point approaches $P$. Otherwise, we could continue to deform our broken line continuously through $P$ as the line does not bend near $P$.

There are two cases we need to analyze: either $P \in \{P_1, \ldots, P_k\}$ or $P \notin \{P_1, \ldots, P_k\}$.

Case 1. $P = P_i$ for some $i$. Because all rays with endpoint $P_i$ involve the monomial $u_i$, a broken line can only bend along at most one ray with endpoint $P_i$, and as observed above, $\beta_1$ must bend along at least one such ray. So call the ray with endpoint $P_i$ along which $\beta_1$ bends $\mathcal{D}_1$, corresponding to a Maslov index zero tree $\bar{h}_1 : \bar{\Gamma}_1 \to M_{\mathbb{R}}$. This tree passes through $P_i$, and by cutting this tree at $P_i$ and removing the marked edge mapping to $P_i$, we obtain a Maslov index two disk $h_1 : \Gamma'_1 \to M_{\mathbb{R}}$ with boundary $P_i$. Let $\bar{\beta}_2$ be the broken line with endpoint $P_i$ corresponding to this Maslov index 2 disk. See Figure 4.5.

Next, recalling that $B(t_j(s_0),s_0) = P_i$, let $\bar{\beta}'_2 : [t_j(s_0),0] \to M_{\mathbb{R}}$ be the restriction of $B$ to $[t_j(s_0),0] \times \{s_0\}$: this is a piece of a broken line starting at $P_i$. We can then concatenate $\bar{\beta}_2$ with $\bar{\beta}'_2$ by identifying 0 in the domain of $\bar{\beta}_2$ with $t_j(s_0)$ in the domain of $\bar{\beta}'_2$, obtaining what we hope will be a broken line $\beta'_2$ passing through $P_i$.

Note that the broken line $B|_{(-\infty,t_j(s_0)] \times \{s_0\}}$ is a broken line with endpoint $P_i$, and hence corresponds to a Maslov index two disk $h_2 : \Gamma'_2 \to M_{\mathbb{R}}$ with endpoint $P_i$. By extending the
edge $E'_2$ of $\Gamma_2'$ to an unbounded edge, we get a tropical tree $\tilde{h}_2 : \tilde{\Gamma}_2 \to M_\mathbb{R}$, and once we mark the point on $\tilde{\Gamma}_2$ which maps to $P_i$, it becomes a Maslov index zero tree and hence corresponds to a ray $v_2 \in \mathcal{D}$ with endpoint $P_i$.

Note that the function attached to $v_2$ is $1 + w_{r_1}(E_{\text{out},i})u_i \text{Mono}(h_i)$. On the other hand, the monomial attached to the last segment of $B_{[(-\infty,t_j(s_0))\times\{s_0\}]_{\mathbb{R}}}$, i.e. $c_jz^{m_j^i}$, is $\text{Mono}(h_2)$, while the monomial attached to the last segment of $\tilde{\beta}_2$ is $\text{Mono}(h_1)$. Thus, in particular, the monomial $c_{j+1}z^{m_{j+1}^i}$ is obtained from the bend of $\beta_1$ at $v_1$, and hence is

$$w_{r_1}(E_{\text{out},1})\langle n_1, r(\Delta(h_2)) \rangle u_i \text{Mono}(h_1) \text{Mono}(h_2).$$

Here $n_1 \in N$ is primitive, orthogonal to $v_1$, and positive on $r(\Delta(h_2))$.

We can now deform $\beta_2'$ by moving the endpoint of $\tilde{\beta}_2$ along $v_2$ away from $P_i$, moving $\tilde{\beta}_2'$ along with it. However, we also need to keep track of monomials: we have to make sure that the monomial on the first segment of $\tilde{\beta}_2'$ is the one which would arise when $\tilde{\beta}_2'$ bends along $v_2$. However, this latter monomial is a term obtained by applying the automorphism associated to crossing $v_2$ to $\text{Mono}(h_1)$, and is thus precisely

$$w_{r_2}(E_{\text{out},2})\langle n_2, r(\Delta(h_1)) \rangle u_i \text{Mono}(h_1) \text{Mono}(h_2).$$

Again, $n_2 \in N$ is primitive, orthogonal to $v_2$, and positive on $r(\Delta(h_1))$. However, one sees easily that

$$w_{r_1}(E_{\text{out},1})\langle n_1, r(\Delta(h_2)) \rangle = w_{r_2}(E_{\text{out},2})\langle n_2, r(\Delta(h_1)) \rangle,$$

so (4.5) coincides with (4.4). As a result, $\beta_2'$ can now be deformed away from the singular point $P_i$, giving a broken line $\beta_2$ with endpoint $Q_2$. Note that in no way does this represent a continuous deformation: the broken line really jumps as it passes through $P_i$.

Note this process is reversible. If we start with $\beta_2$ and try to deform it through $P_i$ as above, we obtain $\beta_1$.

To conclude, in this case, we can take one-element sets in the partition of the form $\beta_1 \in \mathcal{B}^1_i$ and $\beta_2 \in \mathcal{B}^2_i$ for some $i$. They both give the same contribution to $W_k(Q)$.

Case 2. $P \notin \{P_1, \ldots, P_k\}$. Let $\mathcal{D}_P \subseteq \mathcal{D}(\Sigma, P_1, \ldots, P_k)$ be the subset of rays passing through $P$. Recall from the proof of Proposition 4.7 that we can write

$$\mathcal{D}_P = \{v_1, \ldots, v_n\} \cup \bigcup_{i=1}^m \mathcal{D}_i$$

where $v_1, \ldots, v_n$ are rays with $\text{Init}(v_i) \neq P$ corresponding to trees $h$ such that $I(h) \cap I(h') \neq \emptyset$ for any Maslov index zero tree $h'$ with outgoing edge passing through $P$, and $\mathcal{D}_1, \ldots, \mathcal{D}_m$ are scattering diagrams consisting of three rays, with two rays corresponding to trees $h_1$ and $h_2$ with $I(h_1) \cap I(h_2) = \emptyset$ and the third ray having initial point $P$, corresponding to the gluing of the two trees $h_1$ and $h_2$ at $P$. 
Now $\beta_1$ cannot bend at two different rays near $P$ corresponding to trees $h_1$ and $h_2$ with $I(h_1) \cap I(h_2) \neq \emptyset$, as $u_{I(h_1)}u_{I(h_2)} = 0$. This gives us two immediate possibilities. The first possibility is that $\beta_1$ only bends once near $P$ along a ray $d$ for which $P \neq \text{Init}(d)$. The second possibility is that there is some $i$ such that $\beta$ either bends once along a ray $d \in D_i$ with $P = \text{Init}(d)$ or $\beta$ bends along the ray $P \neq \text{Init}(d_1), \text{Init}(d_2)$.

The first possibility is easy to analyze: the singular point in fact plays no role, and the broken line $\beta$ can be deformed continuously through $P$. We were assuming this was not possible.

The second possibility is quite delicate. Since the broken line only interacts with rays in some $D_i$, we can assume $D_i = D_P$, so we are in the situation depicted in Figure 4.6 where we have precisely three rays through $P$. We will write $D_P = \{d_1, d_2, d_3\}$ as depicted in that figure. We will find in this case that the broken lines we are interested in cannot be deformed through $P$. Rather, one broken line will split up into two broken lines, or two broken lines will turn into one.

We note $D_P$ splits $M_R$ up into five cones, labelled $\sigma_1, \ldots, \sigma_5$ in Figure 4.6. Now a broken line $\beta$ which bends twice near $P$ or bends along the ray with initial point $P$ will have an attached monomial $cz^m$ as $\beta$ approaches a neighbourhood of $P$ and an attached monomial $c'z^{m+m_{d_1}+m_{d_2}}$ once $\beta$ leaves the neighbourhood of $P$. Without loss of generality, assume $P = 0$. The behaviour of $\beta$ as it moves through $P$ will depend on which of the cones $\sigma_1, \ldots, \sigma_5$ contain $r(m)$ and $-r(m+m_{d_1}+m_{d_2})$.

Figure 4.7 now gives a case by case description in the case that $r(m)$ and $r(m+m_{d_1}+m_{d_2})$ are not tangent to any of the rays in $D_P$, so $r(m)$ and $-r(m+m_{d_1}+m_{d_2})$ each lie in the interior of one of $\sigma_1, \ldots, \sigma_5$. In Case I, $r(m) \in \sigma_1$; in Case II, $r(m) \in \sigma_2$; and in Case III, $r(m) \in \sigma_4$. We can ignore the possibility of $r(m)$ being in $\sigma_3$ or $\sigma_5$ as by symmetry, these behave in the same way as $\sigma_1$ or $\sigma_4$ respectively.
The finer classification comes from considering in which cone \(-r(m + m_{\sigma_1} + m_{\sigma_2})\) lies. We will go through this classification for each of the three cases.

Consider first Case I. As \(-r(m)\) lies in the cone \(\sigma_3\) spanned by \(r(m_{\sigma_2})\) and \(-r(m_{\sigma_1})\), \(-r(m + m_{\sigma_1} + m_{\sigma_2})\) lies in the cone spanned by \(r(m_{\sigma_2})\) and \(-r(m_{\sigma_1} + m_{\sigma_2})\), i.e. the union of \(\sigma_3\) and \(\sigma_4\). In Case I (a), \(-r(m + m_{\sigma_1} + m_{\sigma_2})\) lies in \(\sigma_3\), and in Case I (b), \(-r(m + m_{\sigma_1} + m_{\sigma_2})\) lies in \(\sigma_4\).

Consider next Case II. As \(r(m)\) is in the interior of the cone generated by \(r(m_{\sigma_1})\) and \(r(m_{\sigma_2})\), \(-r(m + m_{\sigma_1} + m_{\sigma_2})\) lies in the interior of the cone generated by \(-r(m_{\sigma_1})\) and \(-r(m_{\sigma_2})\). Hence \(-r(m + m_{\sigma_1} + m_{\sigma_2})\) lies in either \(\sigma_4\) or \(\sigma_5\). The situation is symmetric with respect to \(\sigma_4\) and \(\sigma_5\), so we only consider the case of \(\sigma_4\). This gives Case II.

Finally, consider Case III. As \(r(m)\) lies in the interior of the cone generated by \(-r(m_{\sigma_1} + m_{\sigma_2})\) and \(-r(m_{\sigma_1})\), \(-r(m + m_{\sigma_1} + m_{\sigma_2})\) lies in the cone generated by \(\pm r(m_{\sigma_1})\) and \(r(m_{\sigma_2})\). Thus \(-r(m + m_{\sigma_1} + m_{\sigma_2})\) lies in \(\sigma_1\), \(\sigma_5\) or \(\sigma_2\), giving cases III (a), III (b), and III (c) respectively.

Figure 4.6 illustrates all six of these possibilities. In each case we draw all possible types, up to continuous deformation, of broken lines near \(P\) of the sort we are considering. There is always one that bends precisely once near \(P\), along \(\mathfrak{d}_3\). There are then always two other types of lines which bend twice, once on each of \(\mathfrak{d}_1\) and \(\mathfrak{d}_2\). For example, consider Case I (a). A broken line coming in from cone \(\sigma_1\) hits \(\mathfrak{d}_1\) or \(\mathfrak{d}_2\) first, depending on its position. It would then bend as depicted, and next hits \(\mathfrak{d}_2\) or \(\mathfrak{d}_1\), bending again. Note that if a line hits \(\mathfrak{d}_1\) first but does not bend there, but then bends at \(\mathfrak{d}_2\), the next line segment remains in the cone \(\sigma_3\) and thus cannot hit \(\mathfrak{d}_1\) again. Similarly a broken line which hits \(\mathfrak{d}_2\) first but fails to bend there never hits \(\mathfrak{d}_2\) again. Thus the possibilities depicted for Case I (a) are the only possible broken lines which bend along both \(\mathfrak{d}_1\) and \(\mathfrak{d}_2\) or along \(\mathfrak{d}_3\).

The remaining cases are similar; we leave it to the reader to check in each case that there are precisely two types of broken lines which bend both at \(\mathfrak{d}_1\) and \(\mathfrak{d}_2\), as depicted.

Consider case I (a). The broken line \(\beta_i\) is either the line on the left or one of the two on the right. In the first case, take \(\mathfrak{B}_i^1 = \{\beta_1\}\) for some choice of \(i\), and take \(\mathfrak{B}_i^2\) to consist
of the two broken lines on the right. In the second case, we interchange this choice of $\mathcal{B}_i^1$ and $\mathcal{B}_i^2$. We need only check that these two sets contribute the same amount to $W_k$.

This in fact follows from Proposition 4.7. Indeed, consider two paths $\gamma_1$ and $\gamma_2$ with the same endpoints, starting in $\sigma_1$ and ending in $\sigma_3$, but with $\gamma_1$ going around the left of $P$ and $\gamma_2$ going around the right. Then by Proposition 4.7, $\theta_{\gamma_1, \mathcal{D}} = \theta_{\gamma_2, \mathcal{D}}$. Call the three broken lines in the figure for Case I (a), from left to right, $\beta_1$, $\beta_2$ and $\beta_3$, so that $\mathcal{B}_i^1 = \{\beta_1\}$ and $\mathcal{B}_i^2 = \{\beta_2, \beta_3\}$ or vice versa.

Recalling that the monomial attached to the $j$-th segment of each of these broken lines is $cz^m$, then the monomial attached to the $j + 2$-nd segment of $\beta_1$ is, by definition, the term of $\theta_{\gamma_1, \mathcal{D}}(cz^m)$ of the form $c'z^{m+m_2}$. Similarly, the sum of the two monomials attached to the $j + 2$-nd segment of $\beta_2$ and the $j + 1$-st segment of $\beta_3$ is the term of $\theta_{\gamma_2, \mathcal{D}}(cz^m)$ of the form $c'z^{m+m_2 + m_2}$, precisely because such terms appear either when we cross $\mathcal{D}_1$ and $\mathcal{D}_2$, or when we cross $\mathcal{D}_3$. But since

$$\theta_{\gamma_1, \mathcal{D}}(cz^m) = \theta_{\gamma_2, \mathcal{D}}(cz^m)$$

by Proposition 4.7, we see that the contribution to $W_k$ from $\mathcal{B}_i^1$ and $\mathcal{B}_i^2$ are the same.

The other cases are essentially the same: In Case I (b), we take $\mathcal{B}_i^1$ to be the two left-most lines and $\mathcal{B}_i^2$ the right-most line, or vice-versa. In Case II, we take $\mathcal{B}_i^1$ to be the two upper lines and $\mathcal{B}_i^2$ to be the lower line, or vice versa. In Case III (a), we partition by taking the two lower lines and the upper line, and in Case III (c) we take the two upper lines and the lower line. In each of these cases, the argument is then identical to the argument given in Case I (a).

Case III (b) is slightly more delicate. Here we take $\mathcal{B}_i^1$ be the left-most line and $\mathcal{B}_i^2$ to be the other two lines, or vice versa. In this case, it is simplest to calculate the contributions of the two sets of broken lines directly. Choose $n_1, n_2 \in N$ primitive, annihilating the tangent spaces to $\mathcal{D}_1$, $\mathcal{D}_2$ respectively, and so that $\langle n_1, r(m_{\mathcal{D}_1}) \rangle > 0$ and $\langle n_2, r(m_{\mathcal{D}_1}) \rangle < 0$. Then $n_3 = (w_1 n_1 + w_2 n_2)/w_{\text{out}} \in N$, annihilates the tangent space to $\mathcal{D}_3$, and $\langle n_3, r(m_{\mathcal{D}_1}) \rangle < 0$, as is easily checked.

Label the broken lines from left to right $\beta_1, \beta_2, \beta_3$. Now the monomials attached to the $j + 2$-nd segment of $\beta_1$ and $\beta_2$ and the $j + 1$-st segment of $\beta_3$ are

$$\beta_1 : \langle n_1, r(m) + r(m_{\mathcal{D}_1}) \rangle \langle n_2, r(m) \rangle c_1 c_2 w_1 w_2 z^{m + m_2}$$
$$\beta_2 : \langle n_2, r(m) + r(m_{\mathcal{D}_1}) \rangle \langle n_1, r(m) \rangle c_1 c_2 w_1 w_2 z^{m + m_2}$$
$$\beta_3 : \langle n_3, r(m) \rangle r(m_{\mathcal{D}_1}) \wedge r(m_{\mathcal{D}_2}) c_1 c_2 w_{\text{out}} z^{m + m_2}$$
But $|r(m_{\beta_1}) \wedge r(m_{\beta_2})| = w_1 \langle n_1, r(m_{\beta_1}) \rangle = -w_2 \langle n_2, r(m_{\beta_2}) \rangle$. So the total contribution from $\beta_2$ and $\beta_3$ is, leaving off the common term $cc_1cc_2z^{m_1+m_2}$,

$$\langle n_2, r(m) \rangle + r(m_{\beta_1}) \rangle |n_1, r(m_1)r(m)w_1w_2 + \langle n_3, r(m) \rangle |r(m_{\beta_2}) \rangle w_2 = \left( \langle n_2, r(m) \rangle |n_1, r(m_1)r(m)w_1w_2 + \langle n_2, r(m_{\beta_1}) \rangle |n_1, r(m_{\beta_2}) \rangle w_1w_2 \right)$$

Thus the contribution from $\beta_1$ agrees with the sum of the contributions from $\beta_2$ and $\beta_3$.

There is one degenerate situation we still have to deal with, namely that one of $r(m)$ or $-r(m + m_{\sigma_1} + m_{\sigma_2})$ are parallel to one of the rays in $D_p$. These can be viewed as a degenerate version of one of the cases considered above, and requires another case-by-case analysis, which we shall leave to the reader. In all cases, one of the broken lines disappears. Let us illustrate what happens in one case, in which, in Case I (a), the incoming segment of the broken lines is parallel to $d_2$. In this case, we only have two possible broken lines, as depicted as in Figure 4.8. There is no broken line which starts in $\sigma_5$ and bends first at $d_2$ and then at $d_1$.

This does not present any difficulty for our analysis, however. As before, pick two paths $\gamma_1$ and $\gamma_2$ going from $\sigma_1$ to $\sigma_3$, with $\gamma_1$ on the left. Note that from the explicit analysis given in Case I (a), the broken line on the left, $\beta_1$, contributes the $z^{m_1+m_2}$ term to $\theta_{\gamma_1,D}(cz^m)$, while the broken line on the right provides the only contribution to the $z^{m_1+m_2}$ term in $\theta_{\gamma_2,D}(cz^m)$, precisely because $\theta_{\gamma_2,D}(cz^m) = 0$.

The same happens in all parallel cases, and we omit the details. $\square$

We now turn our attention to studying the dependence of $W_k(Q)$ on the points $P_1, \ldots, P_k$. For this discussion, it will be useful to explain that the automorphisms arising from scattering diagrams are elements of an interesting group, variants of which first appeared in [23] and then was used in [17] and [18]. We summarize this point of view.

We denote the module of log derivations of $C[T_{\Sigma}] \otimes_R R_k[\gamma_0]$ to be the module

$$\Theta(C[T_{\Sigma}] \otimes_R R_k[\gamma_0]) := \text{Hom}_Z(M, C[T_{\Sigma}] \otimes_R R_k[\gamma_0]) = (C[T_{\Sigma}] \otimes_R R_k[\gamma_0])) \otimes Z N.$$

An element $f \otimes n$ is written as $f \partial_n$, and acts as a derivation on $C[T_{\Sigma}] \otimes_R R_k[\gamma_0]$ over $C[K_{\Sigma}] \otimes_R R_k[\gamma_0]$ via

$$f \partial_n(z^m) = f(n, r(m))z^m.$$

Given $\xi \in m_{R_k} \Theta(C[T_{\Sigma}] \otimes_R R_k[\gamma_0])$, where $m_{R_k} = (u_1, \ldots, u_k)$ is the maximal ideal of $R_k$, we define

$$\exp(\xi) \in \text{Aut}(C[T_{\Sigma}] \otimes_R R_k[\gamma_0])$$
Figure 4.7. Various possibilities for broken lines passing through singular points. The dots indicate bending points.
by

$$\exp(\xi)(a) = a + \sum_{i=1}^{\infty} \frac{\xi^i(a)}{i!}. $$

This is a finite sum given the assumption on $\xi$.

Now let

$$\mathfrak{v}_{\Sigma,k} = \bigoplus_{m \in T_{\Sigma}} z^m(m_{R_k} \otimes r(m)^\perp) \subseteq \Theta(\mathbb{C}[T_{\Sigma}] \otimes_{\mathbb{C}} R_k[y_0]).$$

Set

$$\mathbb{V}_{\Sigma,k} = \{\exp(\xi) | \xi \in \mathfrak{v}_{\Sigma,k}\}.$$  

Note the Lie bracket on $\mathfrak{v}_{\Sigma,k}$ is given by

$$[z^m \partial_n, z^{m'} \partial_{n'}] = z^{m+m'}\left(\langle n, r(m')\rangle \partial_{n'} - \langle n', r(m)\rangle \partial_n\right).$$

Then $\mathfrak{v}_{\Sigma,k}$ is closed under Lie bracket, and hence $\mathbb{V}_{\Sigma,k}$ is a group, with multiplication given by the Baker-Campbell-Hausdorff formula. We note that for $m \in T_{\Sigma}$ with $r(m) \neq 0$, $n \in \mathbb{N}$ with $\langle n, r(m) \rangle = 0$, $I \subseteq \{1, \ldots, k\}$ non-empty, $c \in \mathbb{C}$,

$$\exp(c u_I z^m \partial_n)(z^{m'}) = z^{m'}(1 + cu_I \langle n, r(m') \rangle z^m),$$

so for any scattering diagram $\mathfrak{D}$, we have $\theta_{\gamma, \mathfrak{D}} \in \mathbb{V}_{\Sigma,k}$. Furthermore, $\mathbb{V}_{\Sigma,k}$ is generated by automorphisms of the form $\exp(c u_I z^m \partial_n)$.

The original version of this group introduced in [23] was defined as a group of Hamiltonian symplectomorphisms, and it is convenient to use this identification here. In particular, the holomorphic symplectic form $\Omega$ was given by a choice of generator of $\wedge^2 M$, i.e., an identification $\wedge^2 M \cong \mathbb{Z}$. This gives an isomorphism $M \cong N$ well-defined up to sign; we view $m \in M$ as an element $X_m$ of $N$ via the linear map $m' \mapsto m \wedge m' \in \wedge^2 M \cong \mathbb{Z}$.

Suppose $\mathfrak{d}$ is a ray or line with $f_{\mathfrak{d}} = 1 + w(m)c u_I z^m$ where $w(m)$ is the index of $r(m)$. Then the automorphism $\theta$ obtained by crossing $\mathfrak{d}$ is $\exp(\pm c u_I z^m X_{r(m)})$. In fact $z^m X_{r(m)}$ is
the Hamiltonian vector field\(^6\) associated to the function \(f = -z^m\). So for \(f \in m_{R_k}(\mathbb{C}[T_x] \otimes \mathbb{C} R_k[y_0])\), we write
\[
X_f \in m_{R_k} \Theta(\mathbb{C}[T_x] \otimes \mathbb{C} R_k[y_0])
\]
for the Hamiltonian vector field induced by \(f\). This is convenient for writing the following easily checked standard lemma:

**Lemma 4.13.** If \(f \in m_{R_k}(\mathbb{C}[T_x] \otimes \mathbb{C} R_k[y_0])\) and \(\theta \in \mathbb{V}_{\Sigma,k}\), then
\[
\theta \circ X_f \circ \theta^{-1} = X_{\theta(f)}.
\]

We will need a three-dimensional version of scattering diagrams.

**Definition 4.14.** Let \(L \subseteq \mathbb{R}\) be a closed interval. Let \(\pi_1\) and \(\pi_2\) be the projections of \(M_\mathbb{R} \times L\) onto \(M_\mathbb{R}\) and \(L\) respectively. A *scattering diagram* in \(M_\mathbb{R} \times L\) is a set \(\mathcal{D}\) consisting of pairs \((\mathcal{d}, f_\mathcal{d})\) such that

- \(\mathcal{d} \subseteq M_\mathbb{R} \times L\) is a convex polyhedral subset of dimension two such that \(\pi_2(\mathcal{d})\) is one-dimensional. Furthermore there is a one-dimensional subset \(\mathcal{b} \subseteq M_\mathbb{R} \times L\) and an element \(m_0 \in T_x\) with \(r(m_0) \neq 0\) such that
  \[
  \mathcal{d} = \mathcal{b} - \mathbb{R}_{\geq 0}(r(m_0), 0).
  \]
- \(f_\mathcal{d} \in \mathbb{C}[z^{m_0}] \otimes \mathbb{C} R_k \subseteq \mathbb{C}[T_x] \otimes \mathbb{C} R_k[y_0]\).
- \(f_\mathcal{d} \equiv 1 \mod (u_1, \ldots, u_k) z^{m_0}\).

We define
\[
\text{Sing}(\mathcal{D}) = \bigcup_{\mathcal{d} \in \mathcal{D}} \partial \mathcal{d} \cup \bigcup_{\substack{\mathcal{d}_1, \mathcal{d}_2 \in \mathcal{D} \\dim \mathcal{d}_1 \cap \mathcal{d}_2 = 1}} \mathcal{d}_1 \cap \mathcal{d}_2.
\]
This is a one-dimensional subset of \(M_\mathbb{R} \times L\). Let \(\text{Interstices}(\mathcal{D})\) be the finite set of points where \(\text{Sing}(\mathcal{D})\) is not a manifold. In keeping with the language of [17], we will denote by \(\text{Joints}(\mathcal{D})\) the set of closures of the connected components of \(\text{Sing}(\mathcal{D}) \setminus \text{Interstices}(\mathcal{D})\), calling elements of \(\text{Joints}(\mathcal{D})\) and \(\text{Interstices}(\mathcal{D})\) *joints* and *interstices* respectively. We call a joint *horizontal* if its image under \(\pi_2\) is a point; otherwise we call a joint *vertical*.

For a path \(\gamma\) in \((M_\mathbb{R} \times L) \setminus \text{Sing}(\mathcal{D})\), one can define an element
\[
\theta_{\gamma,\mathcal{D}} \in \mathbb{V}_{\Sigma,k}
\]
exactly as in the case of a scattering diagram in \(M_\mathbb{R}\). Indeed, we just need to define the automorphism \(\theta_{\gamma,\mathcal{d}}\) when \(\gamma\) crosses \((\mathcal{d}, f_\mathcal{d})\) at time \(t_i\). Assuming \(\gamma\) crosses \(\mathcal{d}\) transversally, then \(\pi_1(\gamma'(t_i))\) is not parallel to \(r(m_0)\). So we can choose \(n_0 \in \mathbb{N}\) primitive with \(\langle n_0, r(m_0) \rangle = 0\) and \(\langle n_0, \pi_1(\gamma'(t_i)) \rangle < 0\). We define as usual
\[
\theta_{\gamma,\mathcal{d}}(z^m) = z^m f_\mathcal{d}^{\langle n_0, r(m) \rangle}.
\]
\(^6\)Given a function \(f\) on \(M \otimes \mathbb{C}^x\), the Hamiltonian vector field associated to it is the vector field \(X_f\) such that \(\iota(X_f)\Omega = df\).
Again, it is easy to check that $\theta_{\gamma,\mathcal{D}}$ only depends on the homotopy type of the path $\gamma$ inside $(M_\mathbb{R} \times L) \setminus \text{Sing}(\mathcal{D})$.

A \textit{broken line} in $M_\mathbb{R} \times L$ is a map $\beta : (-\infty, 0] \to M_\mathbb{R} \times L$, along with data $t_0 < \cdots < t_n$ and monomials $c_i z_i^{m_i}$, such that

1. $\pi_2 \circ \beta$ is constant, say with image $P \in L$.
2. $\pi_1 \circ \beta$ is a broken line in the sense of Definition 4.9 with respect to the scattering diagram $\mathcal{D}_P$ in $M_\mathbb{R}$ given, after identifying $M_\mathbb{R} \times \{P\}$ with $M_\mathbb{R}$, by

$$
\mathcal{D}_P := \{(\delta \cap (M_\mathbb{R} \times \{P\}), f_\delta) \mid (\delta, f_\delta) \in \mathcal{D} \text{ such that } \delta \cap (M_\mathbb{R} \times \{P\}) \neq \emptyset\}.
$$

We now reach the last goal of our section: we wish to understand how $W_k(Q)$ varies as the points $P_1, \ldots, P_k$ are varied.

**Theorem 4.15.** Let $W$ and $W'$ be $W_k(Q)$ for two different choices of general points $P_1, \ldots, P_k$ and $P'_1, \ldots, P'_k$. Then

$$W' = \theta(W)$$

for some $\theta \in \mathcal{V}_{\Sigma, k}$.

**Proof.** We shall show this result by induction on $k$, noting that the base case $k = 1$ is obvious, as moving $P_1$ and keeping $Q$ fixed is the same thing as moving $Q$ and keeping $P_1$ fixed.

It is clearly enough to show this result in the case that only $P_1$ changes. So consider a choice of general points $P_1, \ldots, P_k$ and $P'_1$. Consider the line segment $L$ joining $P_1$ and $P'_1$. For all but a finite number of points $P \in L$, we can assume $P, P_2, \ldots, P_k$ will be sufficiently general so that $\text{Trees}(\Sigma, P, P_2, \ldots, P_k)$ is finite, and all elements of this set are trivalent. This gives rise to a family of scattering diagrams $\mathcal{D}(\Sigma, P, P_2, \ldots, P_k)$, $P \in L$. We can put these scattering diagrams together into a scattering diagram $\tilde{\mathcal{D}} = \tilde{\mathcal{D}}(\Sigma, L, P_2, \ldots, P_k)$ in $M_\mathbb{R} \times L$. $\tilde{\mathcal{D}}$ is determined by the requirement that for $P \in L$ general,

$$\mathcal{D}(\Sigma, P_2, \ldots, P_k) = \{(\tilde{\delta} \cap (M_\mathbb{R} \times \{P\}), f_\tilde{\delta}) \mid (\tilde{\delta}, f_\tilde{\delta}) \in \tilde{\mathcal{D}} \text{ such that } \tilde{\delta} \cap (M_\mathbb{R} \times \{P\}) \neq \emptyset\}.$$

To keep track of the dependence of $W_k(Q)$ on the point $P \in L$, we write $W_k(Q; P)$. We wish now to show that if $\gamma$ is a general path in $M_\mathbb{R} \times L$ joining $(Q, P_1)$ to $(Q, P'_1)$, then

$$W_k(Q; P'_1) = \theta_{\gamma, \tilde{\mathcal{D}}}(W_k(Q; P_1)).$$

It is enough to show

1. $W_k(Q; P)$ is constant for $(Q, P) \in M_\mathbb{R} \times L$ varying within a connected component of $(M_\mathbb{R} \times L) \setminus \text{Supp}(\tilde{\mathcal{D}})$.
2. For two such connected components separated by a wall $(\tilde{\delta}, f_\tilde{\delta})$ and points $(Q, P)$, $(Q', P')$ on either side of the wall, we have (4.6) for $\gamma$ a short path joining $(Q, P)$ with $(Q', P')$. 


Once we show (1), Theorem 4.12 already shows (2): as there are no walls in \( \tilde{\mathcal{D}} \) projecting to points in \( L \), we can always choose points \((Q, P), (Q', P')\) on opposite sides of a wall with \( P = P' \), and then we are in the case already shown in Theorem 4.12. So we only need to show (1).

To show (1), we use the same technique we used for the variation of \( Q \), deforming broken lines. Take \((Q, P), (Q', P')\) general within a connected component of \((M_R \times L) \setminus \text{Supp}(\tilde{\mathcal{D}})\) and move from \((Q, P)\) to \((Q', P')\) via a general path \( \gamma \). Consider broken lines in \( M_R \times L \) with endpoint \( \gamma(t) \). As \( t \) varies, we can continuously deform a broken line with endpoint \( \gamma(t) \) unless the broken line converges to one passing through a singular point of \( \tilde{\mathcal{D}} \). However, since such a family of broken lines traces out a two-dimensional subset of \( M_R \times L \), by choosing \( \gamma \) sufficiently general, we can be sure that none of these broken lines converge to broken lines passing through interstices, as interstices are codimension three. However, they can pass through joints, and this requires some care.

The first observation is that we have already analyzed in the proof of Theorem 4.12 what happens if a broken line passes through a vertical joint. Indeed, we can just as well assume that \( \gamma \) has been chosen so that at a time \( t_0 \) when a broken line passes through a vertical joint, \( \pi_2(\gamma(t)) \) remains constant for \( t \) in a neighbourhood of \( t_0 \). Then we are in precisely the situation analyzed in Theorem 4.12.

So we only need to see what happens if a broken line passes through a horizontal joint. Note that horizontal joints occur when two or more parallel rays in a scattering diagram come together as the point \( P \) varies; this can typically lead to values of \( P \) with families of Maslov index zero disks or the existence of Maslov index \(-2\) disks.

In fact, it is enough to show that if \( j \) is a horizontal joint and \( \gamma_j \) is a small loop in \( M_R \times L \) around the joint, then \( \theta_{\gamma_j, \tilde{\mathcal{D}}} = \text{id} \). Indeed, if \( j \) projects to \( P \in L \), \( j \) is contained in some polygons \( \tilde{\mathcal{D}}_1, \ldots, \tilde{\mathcal{D}}_n \in \tilde{\mathcal{D}} \), and necessarily for \( P' \in L \) near \( P \), \( \tilde{\mathcal{D}}_i \cap (M_R \times \{P'\}) \) is either a ray parallel to \( j \) or is empty. Thus as \( P' \in L \) moves from one side of \( P \) to the other, some parallel rays \( \mathcal{D}_1, \ldots, \mathcal{D}_p \) in \( \mathcal{D}(\Sigma, P', P_2, \ldots, P_k) \) come together to yield the joint and then turn into parallel rays \( \mathcal{D}'_1, \ldots, \mathcal{D}'_p \) on the other side of \( P \). Let \( \mathcal{D}_1, \mathcal{D}_2 \) be the scattering diagrams in \( M_R \) given by \( \mathcal{D}(\Sigma, P', P_2, \ldots, P_k) \) for \( P' \) very close to \( P \), but on opposite sides of \( P \). Let \( \gamma \) be a path which is a short line segment crossing \( j \), so that we can write

\[
\theta_{\gamma, \mathcal{D}_1} = \theta_{\gamma, \mathcal{D}_1} \circ \ldots \theta_{\gamma, \mathcal{D}_p},
\theta_{\gamma, \mathcal{D}_2} = \theta_{\gamma, \mathcal{D}_1'} \circ \ldots \theta_{\gamma, \mathcal{D}_p'}.
\]

(Note the ordering is immaterial as all these automorphisms commute). But \( \theta_{\gamma_j, \tilde{\mathcal{D}}} = \theta_{\gamma_j, \mathcal{D}_1} \circ \theta_{\gamma_j, \mathcal{D}_2} \), so if \( \theta_{\gamma_j, \tilde{\mathcal{D}}} = \text{id} \), we have \( \theta_{\gamma, \mathcal{D}_1} = \theta_{\gamma, \mathcal{D}_2} \). This means that, by Definition 4.9 (4), broken lines will behave in the same way on either side of \( P \) near the joint \( j \). Note that the actual set of broken lines on either side may be different, because we are not claiming that
the set \( \{ \theta_{\gamma, a_1}, \ldots, \theta_{\gamma, a_p} \} \) coincides with \( \{ \theta_{\gamma, a'_1}, \ldots, \theta_{\gamma, a'_p} \} \), but rather the total contribution from bends along the two sets of broken lines remains the same.

To show \( \theta_{\gamma, \tilde{D}} = \text{id} \) for each horizontal joint, we use a method introduced in [17], Proposition 3.17. For \( I \subseteq \{1, \ldots, k\} \), define

\[
\text{Ideal}(I) := \langle u_i | i \not\in I \rangle \subseteq \mathbb{C}[T_\Sigma] \otimes_\mathbb{C} R_k[[y_0]].
\]

We will proceed by induction, showing

**Claim.** For \( k' \geq 0 \) and \( \#I = k' \), we have

\[
\theta_{\gamma, \tilde{D}} \equiv \text{id} \mod \text{Ideal}(I)
\]

for every horizontal joint \( j \).

The base case with \( k' = 0 \) is trivial, because all automorphisms are trivial modulo the ideal \( (u_1, \ldots, u_k) \). So assume the claim for all \( k'' < k' \). Fix a set \( I \) with \( \#I = k' \). Fix an orientation on \( M_\mathbb{R} \times L \), so that if any joint \( j \) is given an orientation, this determines the orientation of a loop \( \gamma_j \) around \( j \), using, say, the right-hand rule. We wish to study \( \theta_{\gamma, \tilde{D}} \) for \( j \) horizontal.

Note that as \( \theta_{\gamma, \tilde{D}} \) for \( j \) horizontal only involves a composition of automorphisms associated to parallel rays, we can in fact write

\[
\theta_{\gamma, \tilde{D}}(z^{m'}) = f_j(z^{n_j}) z^{m'}
\]

for some \( n_j \in \mathbb{N} \) primitive and zero on the tangent space to \( j \). Also,

\[
f_j \in \mathbb{C}[\{ m \in T_\Sigma | r(m) \text{ is tangent to } j \}] \otimes_\mathbb{C} R_k[[y_0]].
\]

Note that \( f_j \) depends on the choice of sign of \( n_j \). Assume we have chosen these consistently, in the sense that if any two joints \( j \) and \( j' \) have the same tangent space, then \( n_j = n_{j'} \).

We need to show \( f_j \equiv 1 \mod \text{Ideal}(I) \). Fix some \( m \in T_\Sigma \). For each horizontal joint \( j \), let the term in \( f_j \mod \text{Ideal}(I) \) involving \( z^m \) be \( c_{m,j} z^m \). Thus \( c_{m,j} = \tilde{c}_{m,j} \prod_{i \in I} u_i \) for some \( \tilde{c}_{m,j} \in \mathbb{C} \) since \( f_j \equiv 1 \mod \text{Ideal}(I') \) for any \( I' \not\subseteq I \). Note that \( f_j \) is a product of polynomials of the form \( 1 + cz^m \) with \( r(m') \neq 0 \). It then follows that if \( \tilde{c}_{m,j} \neq 0 \), then \( r(m) \neq 0 \). So we will assume \( r(m) \neq 0 \). We will also include here the case that \( j \) is a vertical joint, by setting \( \tilde{c}_{m,j} = 0 \) for vertical joints. Note that \( \tilde{c}_{m,j} \) depends on the orientation on \( j \). A change of orientation of \( j \) changes the direction of \( \gamma_j \), replacing \( f_j \) with \( f_j^{-1} \). This changes the sign of \( \tilde{c}_{m,j} \). As a result, we can view

\[
j \mapsto \tilde{c}_{m,j}
\]

as a 1-chain for the one-dimensional simplicial complex \( \text{Sing}(\tilde{D}) \). Here the choice of orientation on \( j \) implicit.

**Subclaim.** \( j \mapsto \tilde{c}_{m,j} \) is a 1-cycle.
Proof. We need to check the 1-cycle condition at each interstice of $\tilde{\mathcal{D}}$, so let $(Q, P) \in \text{Interstices}(\tilde{\mathcal{D}})$. Consider a small two-sphere $S$ in $M_R \times L$ with center $(Q, P)$. Then suppose that $x_1, \ldots, x_s \in S$ are distinct points such that

$$\{x_1, \ldots, x_s\} = \bigcup_{j \in \text{Joints}(\tilde{\mathcal{D}})} j \cap S.$$ 

Choose a base-point $y \in S$, $y \notin \text{Supp}(\tilde{\mathcal{D}})$. We can choose small counterclockwise loops $\gamma_1, \ldots, \gamma_s$ in $S$ around $x_1, \ldots, x_s$ and paths $\beta_i$ joining $y$ with the base-point of $\gamma_i$ in such a way that

$$\beta_1 \gamma_1 \beta_1^{-1} \ldots \beta_s \gamma_s \beta_s^{-1} = 1$$

in $\pi_1(S \setminus \{x_1, \ldots, x_s\}, y)$. Because $\theta_{\gamma, \tilde{\mathcal{D}}}$ only depends on the homotopy type of $\gamma$ in $(M_R \times L) \setminus \text{Sing}(\tilde{\mathcal{D}})$, we obtain the equality

$$\theta_{\gamma}^{-1} \circ \theta_{\gamma} \circ \theta_{\beta} \circ \cdots \circ \theta_{\beta}^{-1} \circ \theta_{\gamma} \circ \theta_{\beta} = \text{id}. \quad (4.7)$$

Here, we have dropped the $\tilde{\mathcal{D}}$’s in the subscripts.

We now distinguish between two cases.

Case 1. The interstice $(Q, P)$ does not satisfy $Q \in \{P, P_2, \ldots, P_k\}$. Then by Proposition 4.7, $\theta_{\gamma_i} = \text{id}$ for each $\gamma_i$ which is a loop around a vertical joint containing $(Q, P)$. On the other hand, modulo $\text{Ideal}(I)$, for $\gamma_i$ around a horizontal joint $j_i$, by the induction hypothesis, $f_{j_i}$ is of the form $1 + (\cdots) \prod_{i \in I} u_i$. One then checks $\theta_{\gamma_i}$ necessarily commutes, modulo $\text{Ideal}(I)$, with any element of $\mathbb{V}_{\Sigma, k}$. This can easily be seen as in Example 4.3 using the fact that $u_j \prod_{i \in I} u_i \equiv 0 \mod I$ for any $j$. Thus in particular, $\theta_{\gamma_i}$ commutes with $\theta_{\beta_i}$. Thus (4.7) becomes

$$\prod_{i} \theta_{\gamma_i} \equiv \text{id} \mod \text{Ideal}(I)$$

where the product is over all $\gamma_i$ around horizontal joints. Applying this identity to a monomial $z^{m'}$, we obtain

$$\prod f_{j_i}^{(m_i, r(m'))} z^{m'} = z^{m'} \mod \text{Ideal}(I),$$

which, after expansion, gives the identity

$$\sum \langle n_i, r(m') \rangle c_{m, i} = 0 \mod \text{Ideal}(I) \quad (4.8)$$

for any $m' \in T_{\Sigma}$. Now a monomial $z^m$ can only appear in $f_{j_i}$ if $r(m)$ is in fact tangent to $j_i$, so the only horizontal joints containing $(Q, P)$ with $c_{m, j} \neq 0$ are the joints contained in the affine line $(Q, P) + \mathbb{R}(r(m), 0)$. Let $s$ be the number of joints contained in this line containing $(Q, P)$. Then either $s = 0, 1$ or 2. If $s = 0$, there is nothing to prove. If $s = 1$, with $j_i$ the only such joint, it follows from (4.8) that $\bar{c}_{m, j_i} = 0$. If $s = 2$, let $j_{i_1}, j_{i_2}$ be the two such joints. Then (4.8) implies that $\bar{c}_{m, j_{i_1}} = \bar{c}_{m, j_{i_2}}$, assuming $j_{i_1}$ and $j_{i_2}$ are oriented in the same direction. This shows that the 1-cycle condition holds at $(Q, P)$.
Case 2. The interstice \((Q, P)\) satisfies \(Q \in \{P, P_2, \ldots, P_k\}\), say \(Q = P_i\). We’ll write \(i = 1\) if \(Q = P\). The argument is almost the same, but now there are two vertical joints, say \(j_1\) and \(j_2\) with endpoint \((Q, P)\), with \(j_1, j_2 \subseteq \{P_i\} \times L\) if \(i > 1\) and \(j_1, j_2 \subseteq \{(P', P')| P' \in L\} \subseteq M_R \times L\) if \(i = 1\). Without loss of generality we can take the base-point \(y\) near \(x_1\) and assume \(\beta_1\) is a constant path so that \(\theta_{\beta_1} = id\). The argument will be the same as in Case 1 once we show

\[
(4.9) \quad \theta_{\beta_2}^{-1} \circ \theta_{\gamma_2} \circ \theta_{\beta_2} \circ \theta_{\gamma_1} = id.
\]

To do so, consider the scattering diagram \(\mathcal{D}(\Sigma, P', P_2, \ldots, P_k)\) for \(P' \in L\), \(P'\) near \(P\) but \(P' \neq P\). By Remark 4.8, the rays emanating from \(P_i\) \((P' \text{ if } i = 1)\) in \(\mathcal{D}(\Sigma, P', P_2, \ldots, P_k)\) are in one-to-one correspondence with the terms in \(W_{k-1}(P; P') - y_0\), where \(W_{k-1}(P; P')\) denotes \(W_{k-1}(P_i)\) computed using the marked points \(P', P_2, \ldots, P_{i-1}, P_{i+1}, \ldots, P_k\) (or \(P_2, \ldots, P_k\) if \(i = 1\)). In particular, given a term \(cz^m\) in \(W_{k-1}(P; P') - y_0\), the corresponding ray carries the function \(1 + u_i c w(m) z^m\), where \(w(m)\) is the index of \(r(m)\). Note that if \(\gamma\) is a simple loop around \(P_i\), then the contribution to \(\theta_{\gamma, \mathcal{D}(\Sigma, P', P_2, \ldots, P_k)}\) from such a ray is \(\exp(\pm X_{u_i} z^m)\). Here the sign only depends on the orientation of \(\gamma\) and the chosen identification of \(\Lambda^2 M\) with \(\mathbb{Z}\). All automorphisms attached to the rays emanating from \(P_i\) commute by Example 3.3 because \(u_i^2 = 0\), so

\[
\theta_{\gamma, \mathcal{D}(\Sigma, P', P_2, \ldots, P_k)} = \prod \exp(\pm X_{u_i} z^m) = \exp(\pm X_{u_i}(W_{k-1}(P; P') - y_0)).
\]

Here the product is over all terms \(cz^m\) appearing in \(W_{k-1}(P; P') - y_0\). Furthermore, if \(P' \in \pi(j_1) \setminus \{P\}\) and \(P'' \in \pi(j_2) \setminus \{P\}\), then by 4.10 applied inductively to \(k - 1\) points if \(i \neq 1\), and by Theorem 4.12 if \(i = 1\),

\[
u_i W_{k-1}(P; P') = u_i \theta_{j_2} (W_{k-1}(P; P')).
\]

It then follows from Lemma 4.13 that

\[
\theta_{\gamma_2} = (\theta_{\beta_2} \circ \theta_{\gamma_1} \circ \theta_{\beta_2}^{-1})^{-1},
\]

the last inverse on the right since \(\gamma_1\) and \(\gamma_2\) are homotopic to loops in \(M_R \times \{P'\}\) and \(M_R \times \{P''\}\) respectively with opposite orientations. This shows (4.9). We can then finish as in Case 1.

This completes the proof of the subclaim. \(\square\)

To complete the proof of the claim, hence the theorem, we now note that the cycle \(\sigma\) given by \(j \mapsto \bar{c}_{m,j}\) is in fact zero. Indeed, picking a given joint with \(\bar{c}_{m,j} \neq 0\), the fact that \(\sigma\) is a cycle implies that the line containing \(j\) can be written as a union of joints \(j'\) with orientation compatible with that on \(j\), with \(\bar{c}_{m,j'} = \bar{c}_{m,j}\). However, there must be one joint \(j'\) contained in this line which is unbounded in the direction \(r(m)\). But none of the polyhedra of \(\bar{\mathcal{D}}\) containing \(j'\) can involve a monomial of the form \(z^m\), since a ray carrying a monomial \(z^m\) is unbounded only in the direction \(-r(m)\). Thus \(0 = \bar{c}_{m,j'} = \bar{c}_{m,j}\) as desired. \(\square\)
Lemma 4.16. Let \( \theta \in \mathbb{V}_{\Sigma,k} \), and suppose \( f \) is in the ideal generated by \( u_1, \ldots, u_k \) in \( \mathbb{C}[T_{\Sigma}] \otimes_{\mathbb{C}} R_k[y_0] \). Then for any cycle \( \Xi \in H_2((\hat{\mathcal{X}}_{\Sigma,k}), \Re(qW_0(Q)) \ll 0; \mathbb{C}) \),

\[
\int_{\Xi} e^{q(W_0(Q)+f)} \Omega = \int_{\Xi} e^{q\theta(W_0(Q)+f)} \Omega.
\]

Proof. We continue to use the fixed identification \( \wedge^2 M \cong \mathbb{Z} \) given by \( \Omega \). It is enough to show the lemma for \( \theta = \exp(cz^{m_0}X_{r(m_0)}) \) with \( m_0 \in T_{\Sigma}, r(m_0) \neq 0 \) and \( c^2 = 0 \), as such elements generate \( \mathbb{V}_{\Sigma,k} \). Note that if \( W_0(Q) + f = \sum_m c_m z^m \), then

\[
\theta(W_0(Q) + f) = \sum_m c_m (z^m + \langle X_{r(m_0)}, r(m) \rangle cz^{m_0+m})
\]

and

\[
e^{q\theta(W_0(Q)+f)} = e^{q(W_0(Q)+f)}(1 + \sum_m qcc_m \langle X_{r(m_0)}, r(m) \rangle z^{m_0+m}).
\]

Furthermore, \( d(z^m \text{ dlog}(z^{m_0})) = -\langle X_{r(m_0)}, r(m) \rangle z^m \Omega \). Thus

\[
(e^{q\theta(W_0(Q)+f)} - e^{q(W_0(Q)+f)}) \Omega = e^{q(W_0(Q)+f)} \left( q \sum_m ccc_m \langle X_{r(m_0)}, r(m) \rangle z^{m_0+m} \right) \Omega
\]

\[
= -d(cz^{m_0}e^{q(W_0(Q)+f)} \text{ dlog}(z^{m_0})).
\]

The result then follows from Stoke’s theorem and the fact that \( e^{q(W_0(Q)+f)} \) goes to zero rapidly on the unbounded part of \( \Xi \).

\[\square\]

Proof of Lemma 3.12. This now follows immediately from Theorems 4.12, 4.15, and Lemma 4.16.

\[\square\]

5. Evaluation of the period integrals

Our main goal in this section is the computation of the integrals

\[
\int_{\Xi} e^{qW_0(Q)} \Omega
\]

in the case of \( \mathbb{P}^2 \). In doing so, we will prove Theorems 3.6 and 3.8. We continue with the notation \( \Sigma, T_{\Sigma}, t_i, \rho_i \) of Example 2.9 as well as \( m_i = r(t_i) \).
Lemma 5.1. There is a (multi-valued) basis $\Xi_0, \Xi_1, \Xi_2$ of sections of $\mathcal{R}$ such that, with $x_0 x_1 x_2 = 1$,

$$
\sum_{i=0}^{2} \alpha^i \int_{\Xi_i} e^{q(x_0 + x_1 + x_2)} \Omega = q^{3\alpha} \left( \sum_{d=0}^{\infty} \frac{q^d}{(dl)^3} - 3\alpha \sum_{d=1}^{\infty} \frac{q^d}{(dl)^3} \sum_{k=1}^{d} \frac{1}{k} + \frac{9}{2} \alpha^2 \sum_{d=1}^{\infty} \frac{q^d}{(dl)^3} \left( \sum_{k=1}^{d} \frac{1}{k} \right)^2 + \frac{1}{3} \sum_{k=1}^{d} \frac{1}{k^2} \right)
$$

$$
= q^{3\alpha} \sum_{d=0}^{\infty} q^{3d} (B_0(d) + \alpha B_1(d) + \alpha^2 B_2(d)),
$$

where the last equality defines the numbers $B_0(d), B_1(d), B_2(d)$.

Proof. This was shown in [2], Proposition 3.1. In particular, each integral $f_i = \int_{\Xi_i} e^{q(x_0 + x_1 + x_2)} \Omega$ can be shown to satisfy the differential equation

$$(qd/dq)^3 f_i = 27 q^3 f_i,$$

which can then be solved using a recurrence relation to obtain the above solutions. \qed

We can use this to compute the integrals we are interested in by writing

$$
\int_{\Xi_i} e^{qW_k(Q)} \Omega = e^{qW_0} \int_{\Xi_i} e^{q(x_0 + x_1 + x_2)} e^{q(W_k(Q) - W_0(Q))} \Omega.
$$

The factor $e^{q(W_k(Q) - W_0(Q))}$ can then be expanded in a Taylor series, noting that in any term, each monomial in $W_k(Q) - W_0(Q)$ can appear at most once, because it has a coefficient of square zero; thus this expansion is quite easy and is finite. Thus we only need to calculate, with $x_0 x_1 x_2 = \kappa$,

$$
\sum_{i=0}^{2} \alpha^i \int_{\Xi_i} e^{q(x_0 + x_1 + x_2)} x_0^{n_0} x_1^{n_1} x_2^{n_2} \Omega.
$$

Lemma 5.2. With $\Xi_0, \Xi_1, \Xi_2$ as in Lemma 5.1 but with $x_0 x_1 x_2 = \kappa$,

$$
\sum_{i=0}^{2} \alpha^i \int_{\Xi_i} e^{q(x_0 + x_1 + x_2)} x_0^{n_0} x_1^{n_1} x_2^{n_2} \Omega = q^{3\alpha} \kappa \sum_{i=0}^{2} \psi_i(n_0, n_1, n_2) \alpha^i,
$$

where

$$
\psi_i(n_0, n_1, n_2) = \sum_{d=0}^{\infty} D_i(d, n_0, n_1, n_2) q^{3d - n_0 - n_1 - n_2} \kappa^d
$$

with $D_i$ given as follows. First,

$$
D_0(d, n_0, n_1, n_2) = \begin{cases} 
\frac{1}{(d-n_0)!(d-n_1)!(d-n_2)!} & \text{if } d \geq n_0, n_1, n_2 \\
0 & \text{otherwise}
\end{cases}
$$
Second, if \( d \geq n_0, n_1, n_2 \), then
\[
D_1(d, n_0, n_1, n_2) = -\sum_{k=1}^{d-n_0} \frac{1}{k} + \sum_{k=1}^{d-n_1} \frac{1}{k} + \sum_{k=1}^{d-n_2} \frac{1}{k}
\]
while if \( n_0, n_1 \leq d < n_2 \), then
\[
D_1(d, n_0, n_1, n_2) = \frac{(-1)^{n_2-d-1}(n_2 - d - 1)!}{(d - n_0)!(d - n_1)!}
\]
with similar expressions if instead \( d < n_0 \) or \( d < n_1 \). If \( d \) is smaller than two of \( n_0, n_1, n_2 \), then
\[
D_1(d, n_0, n_1, n_2) = 0.
\]

Third, if \( d \geq n_0, n_1, n_2 \), then
\[
D_2(d, n_0, n_1, n_2) = \left( \sum_{i=0}^{2} \sum_{k=1}^{d-n_i} \frac{1}{k} \right)^2 + \sum_{i=0}^{2} \sum_{k=1}^{d-n_i} \frac{1}{k^2}
\]
while if \( n_0, n_1 \leq d < n_2 \),
\[
D_2(d, n_0, n_1, n_2) = \frac{(-1)^{d-n_2}(n_2 - d - 1)!}{(d - n_0)!(d - n_1)!} \left( \sum_{k=1}^{d-n_0} \frac{1}{k} + \sum_{k=1}^{d-n_1} \frac{1}{k} + \sum_{k=1}^{d-n_2} \frac{1}{k} \right),
\]
with similar expressions if instead \( d < n_0 \) or \( d < n_1 \). If \( n_0 \leq d < n_1, n_2 \), then
\[
D_2(d, n_0, n_1, n_2) = \frac{(-1)^{n_1+n_2}(n_1 - d - 1)!}{(d - n_0)!}(n_2 - d - 1)!
\]
with similar expressions if instead \( n_1 \leq d < n_0, n_2 \) or \( n_2 \leq d < n_0, n_1 \). Finally, if \( d < n_0, n_1, n_2 \), then
\[
D_2(d, n_0, n_1, n_2) = 0.
\]

Proof. Consider the integral
\[
I_i(a_0, a_1, a_2) = \int_{\mathbb{Z}_i} e^{a_0 x_0 + a_1 x_1 + a_2 x_2 \Omega},
\]
with \( a_0, a_1, a_2 \in \mathbb{C}^\times \) and \( x_0 x_1 x_2 = 1 \). Then
\[
\frac{\partial^{n_0+n_1+n_2}}{\partial a_0^{n_0} \partial a_1^{n_1} \partial a_2^{n_2}} I_i = \int_{\mathbb{Z}_i} e^{a_0 x_0 + a_1 x_1 + a_2 x_2} x_0^{-n_0} x_1^{-n_1} x_2^{-n_2} \Omega.
\]
Evaluate this at \( a_0 = a_1 = a_2 = q^{1/3} \kappa \) and make the change of variables \( x_i \mapsto x_i/\kappa^{1/3} \) in the integral. Note that as \( \Omega = \frac{dx_0 \wedge dx_1 \wedge dx_2}{x_1 x_2} \), such a change of variables does not affect \( \Omega \). Then using \( x_0 x_1 x_2 = \kappa \) we obtain
\[
\frac{\partial^{n_0+n_1+n_2}}{\partial a_0^{n_0} \partial a_1^{n_1} \partial a_2^{n_2}} I_i \bigg|_{a_i=q^{1/3}} = \int_{\mathbb{Z}_i} e^{g(x_0 + x_1 + x_2)} \kappa^{-(n_0+n_1+n_2)/3} x_0^{-n_0} x_1^{-n_1} x_2^{-n_2} \Omega.
\]
On the other hand, $I_i$ can be calculated by making the substitution

$$
\begin{align*}
    x_0 &\mapsto (a_1a_2/a_0^2)^{1/3}x_0 \\
    x_1 &\mapsto (a_0a_2/a_1^2)^{1/3}x_1 \\
    x_2 &\mapsto (a_0a_1/a_2^2)^{1/3}x_2
\end{align*}
$$

in $I_i$ which gives

$$
I_i(a_0, a_1, a_2) = \int_{\mathbb{R}} e^{(a_0a_1a_2)^{1/3}x_0+x_1+x_2} \Omega.
$$

Thus we can compute $\sum_{i=0}^2 \alpha^i I_i(a_0, a_1, a_2)$ by substituting in $q = (a_0a_1a_2)^{1/3}$ in the formula of Lemma 5.1. To differentiate the resulting expression, note that under this substitution, $q^{3\alpha+3d}$ becomes $(a_0a_1a_2)^{\alpha+d}$ and

$$
\frac{\partial^{n_0+n_1+n_2}}{\partial a_0^{n_0} \partial a_1^{n_1} \partial a_2^{n_2}} (a_0a_1a_2)^{\alpha+d} \bigg|_{a_i=q^{1/3}} = q^{3\alpha+3d-n_0-n_1-n_2} \kappa^{\alpha+d-(n_0+n_1+n_2)/3} \prod_{k=1}^{n_0} (\alpha + d - k + 1) \prod_{k=1}^{n_1} (\alpha + d - k + 1) \prod_{k=1}^{n_2} (\alpha + d - k + 1)
$$

$$
= q^{3\alpha+3d-n_0-n_1-n_2} \kappa^{\alpha+d-(n_0+n_1+n_2)/3} \left( C_0(d, n_0, n_1, n_2) + \alpha C_1(d, n_0, n_1, n_2) + \alpha^2 C_2(d, n_0, n_1, n_2) \right),
$$

where the last equality defines $C_0, C_1$ and $C_2$. One then sees that

$$
\psi_i(n_0, n_1, n_2) = \sum_{d=0}^{\infty} \sum_{k=0}^{\infty} B_k(d) C_{i-k}(d, n_0, n_1, n_2) q^{3d-n_0-n_1-n_2} \kappa^d
$$

with the $B_i$’s defined in Lemma 5.1. Furthermore, computing the $C_i$’s, we see

$$
C_0(d, n_0, n_1, n_2) = \begin{cases} 
(d!)^3 & \text{if } d \geq n_0, n_1, n_2 \\
0 & \text{otherwise.} 
\end{cases}
$$

If $d \geq n_0, n_1, n_2$, then

$$
C_1(d, n_0, n_1, n_2) = \frac{(d!)^3}{(d-n_0)!(d-n_1)!(d-n_2)!} \left( \sum_{k=d-n_0+1}^{d} \frac{1}{k} + \sum_{k=d-n_1+1}^{d} \frac{1}{k} + \sum_{k=d-n_2+1}^{d} \frac{1}{k} \right),
$$

while if one of $n_0, n_1, n_2$ is larger than $d$, we have

$$
C_1(d, n_0, n_1, n_2) = \prod_{k=d-n_0+1}^{d} k \prod_{k=d-n_1+1}^{d} k \prod_{k=d-n_2+1}^{d} k.
$$

Otherwise

$$
C_1(d, n_0, n_1, n_2) = 0.
$$
If \( d \geq n_0, n_1, n_2 \) then

\[
C_2(d, n_0, n_1, n_2) = \frac{(dl)^3}{2(d - n_0)!(d - n_1)!(d - n_2)!} \left( \left( \sum_{k=d-n_0+1}^{d} \frac{1}{k} + \sum_{k=d-n_1+1}^{d} \frac{1}{k} + \sum_{k=d-n_2+1}^{d} \frac{1}{k} \right)^2 - \left( \sum_{k=d-n_0+1}^{d} \frac{1}{k^2} + \sum_{k=d-n_1+1}^{d} \frac{1}{k^2} + \sum_{k=d-n_2+1}^{d} \frac{1}{k^2} \right) \right).
\]

If \( n_1, n_2 \leq d < n_0 \), then

\[
C_2(d, n_0, n_1, n_2) = \left( \prod_{k=d-n_0+1}^{d} k \right) \frac{(dl)^2}{(d - n_1)!(d - n_2)!} \left( \sum_{k=d-n_0+1}^{d} \frac{1}{k} + \sum_{k=d-n_1+1}^{d} \frac{1}{k} + \sum_{k=d-n_2+1}^{d} \frac{1}{k} \right).
\]

We have similar expressions if \( d < n_1 \) or \( d < n_2 \). If two of \( n_0, n_1 \) and \( n_2 \) are larger than \( d \), then

\[
C_2(d, n_0, n_1, n_2) = \prod_{k=d-n_0+1}^{d} k \prod_{k=d-n_1+1}^{d} k \prod_{k=d-n_2+1}^{d} k.
\]

Finally, if \( n_0, n_1, n_2 > d \), then

\[
C_2(d, n_0, n_1, n_2) = 0.
\]

A laborious calculation now gives the forms given in the Lemma for the coefficients \( D_i \).

**Remark 5.3.** The coefficient \( D_i(d, n_0, n_1, n_2) \) can be written in a more uniform way, which is convenient for computation on a computer algebra system. Obviously

\[
D_0(d, n_0, n_1, n_2) = \frac{1}{\Gamma(d - n_0 + 1)\Gamma(d - n_1 + 1)\Gamma(d - n_2 + 1)}.
\]

For \( i = 1 \), we obtain

\[
D_1(d, n_0, n_1, n_2) = -\frac{3\gamma + \Psi_0(d - n_0 + 1) + \Psi_0(d - n_1 + 1) + \Psi_0(d - n_2 + 1)}{\Gamma(d - n_0 + 1)\Gamma(d - n_1 + 1)\Gamma(d - n_2 + 1)}.
\]

Here \( \gamma \) is Euler’s constant and \( \Psi_0 \) is the digamma function, \( \Psi_0 = \Gamma'/\Gamma \), with \( \Psi_0(d) = -\gamma + \sum_{k=1}^{d-1} 1/k \) for \( d \geq 1 \). If \( \max(n_0, n_1, n_2) > d \), this is interpreted as a limit, using \( \lim_{x \to d} \Psi_0(x)/\Gamma(x) = (-1)^{d+1}d! \), for \( d > 0 \) an integer.

Finally, \( D_2(d, n_0, n_1, n_2) \) is

\[
\frac{\pi^2 + 18\gamma^2 + 12\gamma \left( \sum_{k=0}^{d} \Psi_0(d - n_k + 1) \right) + 2 \left( \sum_{k=0}^{d} \Psi_0(d - n_k + 1) \right)^2 - 2 \sum_{k=0}^{d} \Psi_1(d - n_k + 1)}{4\Gamma(d - n_0 + 1)\Gamma(d - n_1 + 1)\Gamma(d - n_2 + 1)}.
\]

Here \( \Psi_1 = \Psi_0' \), and \( \Psi_1(d) = \pi^2/6 - \sum_{k=1}^{d-1} 1/k^2 \) for \( d > 0 \) an integer. Again, with the appropriate limit interpretation, this covers all cases.
Definition 5.4. For $m \in T$, $m = \sum_{i=0}^{2} n_i t_i$ with $n_i \geq 0$ for all $i$, define

$$
\psi_i(m) := \psi_i(n_0, n_1, n_2)
$$

$$
D_i(d, m) := D_i(d, n_0, n_1, n_2)
$$

and

$$
|m| := n_0 + n_1 + n_2.
$$

We will now start on our proof of Theorems 3.6 and 3.8, beginning with the following definition.

Definition 5.5. Fix $P_1, \ldots, P_k$ general. For $Q$ general, let $S_k$ (or $S_k(Q)$ if the dependence on $Q$ needs to be emphasized) be a finite set of triples $(c, \nu, m)$ with $c \in R_k$ a monomial such that

$$
e^{q(W_k(Q) - W_0(Q))} = \sum_{(c, \nu, m) \in S_k} c \psi^\nu z^m,
$$

with each term $c \psi^\nu z^m$ of the form $\prod_{i=1}^{\nu} \text{Mono}(h_i)$ for $h_1, \ldots, h_\nu$ distinct Maslov index two tropical disks with boundary $Q$.

Let

$$
L_i^d = L_i^d(Q) := \sum_{(c, \nu, m) \in S_k} c \psi^{3d+\nu-|m|} D_i(d, m).
$$

We can now clarify what needs to be proved. The following lemma reduces the two theorems to three equalities.

Lemma 5.6. Let $Q$ be chosen generally, and let $L$ be the tropical line with vertex $Q$. The three equalities

$$
L_0^d = \delta_{0,d} + \sum_{\nu \geq 0} \sum_{I:\{1, \ldots, k\} \atop I_1 < \cdots < I_{3d-\nu-2}} \langle P_1, \ldots, P_{3d-\nu-2}, \psi^{\nu} L \rangle_d \psi^{\nu+2}
$$

$$
L_1^d = \sum_{\nu \geq 0} \sum_{I:\{1, \ldots, k\} \atop I_1 < \cdots < I_{3d-\nu-2}} \langle P_1, \ldots, P_{3d-\nu-1}, \psi^{\nu} L \rangle_d \psi^{\nu+1}
$$

$$
L_2^d = y_2 q^{-1} \delta_{0,d} + \sum_{\nu \geq 0} \sum_{I:\{1, \ldots, k\} \atop I_1 < \cdots < I_{3d-\nu}} \langle P_{1}, \ldots, P_{3d-\nu}, \psi^{\nu} M_{2} \rangle_d \psi^{\nu+1}
$$

imply Theorems 3.6 and 3.8.
Proof. Let us be precise about what needs to be shown to prove Theorems 3.6 and 3.8. If we write, for $0 \leq i \leq 2$, 

$$K_i^{\text{trop}} = \sum_{d \geq 1} \sum_{\nu \geq 0} \langle T_2^{3d+i-2-\nu}, \psi^\nu T_2^{-i} \rangle_d^{\text{trop}} q^{\nu+2} \kappa_d \frac{y_2^{3d+i-2-\nu}}{(3d+i-2-\nu)!},$$

then

$$J_0^{\text{trop}} = e^{q y_0} (1 + K_0^{\text{trop}})$$

$$J_1^{\text{trop}} = e^{q y_0} (q y_1 (1 + K_0^{\text{trop}}) + K_1^{\text{trop}})$$

$$J_2^{\text{trop}} = e^{q y_0} \left( q \frac{y_2^2}{2} (1 + K_0^{\text{trop}}) + q y_1 K_1^{\text{trop}} + q y_2 + K_2^{\text{trop}} \right)$$

We wish to compare these expressions with the expressions obtained via period integrals over $\Xi_0$, $\Xi_1$ and $\Xi_2$. Take for the $\Xi_i$ the cycles given by Lemma 5.1. Consider the $\varphi_i$'s defined using these cycles in (3.2). Expanding the integral in (3.2) by using Lemma 5.2 and $\kappa^\alpha = e^{y_1 \alpha} = 1 + y_1 \alpha + y_1^2 \alpha^2 / 2$, the left-hand side of (3.2) is

$$\sum_{(c,\nu,m) \in S_k} c e^{q y_0} q^{3\alpha+\nu} \kappa^\alpha \sum_{i=0}^2 \psi_i(m) \alpha^i = q^3 e^{q y_0} \sum_{(c,\nu,m) \in S_k} c q^{\nu} \sum_{i=0}^2 \sum_{k=0}^i \frac{y_1^k}{k!} \psi_{i-k}(m) \alpha^i.$$ 

Comparing this with the right-hand side of (3.2), we get

$$\varphi_0 = e^{q y_0} \sum_{(c,\nu,m) \in S_k} c q^{\nu} \psi_0(m)$$

$$\varphi_1 = e^{q y_0} \sum_{(c,\nu,m) \in S_k} c q^{\nu+1} (y_1 \psi_0(m) + \psi_1(m))$$

$$\varphi_2 = e^{q y_0} \sum_{(c,\nu,m) \in S_k} c q^{\nu+2} \left( \frac{y_1^2}{2} \psi_0(m) + y_1 \psi_1(m) + \psi_2(m) \right).$$

Thus to show $\varphi_i = J_i^{\text{trop}}$, we need to show the following three equalities:

(5.5) \hspace{1cm} \sum_{(c,\nu,m) \in S_k} c q^{\nu} \psi_0(m) = 1 + K_0^{\text{trop}}

(5.6) \hspace{1cm} \sum_{(c,\nu,m) \in S_k} c q^{\nu} \psi_1(m) = q^{-1} K_1^{\text{trop}}

(5.7) \hspace{1cm} \sum_{(c,\nu,m) \in S_k} c q^{\nu} \psi_2(m) = q^{-2} (q y_2 + K_2^{\text{trop}}).
Then using the expansion for $\psi_i$ in Lemma 5.2, (5.5), (5.6) and (5.7) are equivalent, if we compare the coefficients of $\kappa^d$ on both sides, to:

\begin{equation}
L_0^d = \delta_0,d + \sum_{\nu \geq 0} \langle T_2^{3d-\nu-2}, \psi^\nu T_2 \rangle_d^{\text{trop}} \frac{y_2^{3d-\nu-2}}{(3d-\nu-2)!} q^\nu+2,
\end{equation}

\begin{equation}
L_1^d = \sum_{\nu \geq 0} \langle T_2^{3d-\nu-1}, \psi^\nu T_1 \rangle_d^{\text{trop}} \frac{y_2^{3d-\nu-1}}{(3d-\nu-1)!} q^\nu+1,
\end{equation}

\begin{equation}
L_2^d = y_2 q^{-1} \delta_0,d + \sum_{\nu \geq 0} \langle T_2^{3d-\nu}, \psi^\nu T_0 \rangle_d^{\text{trop}} \frac{y_2^{3d-\nu}}{(3d-\nu)!} q^\nu.
\end{equation}

Now suppose we have shown (5.2), (5.3) and (5.4). The left-hand sides of these equations come from the period integrals, and hence are independent of the locations of $Q$ and $P_1, \ldots, P_k$. So the right-hand side is also independent of the locations of $Q$ and $P_1, \ldots, P_k$. So in particular, once we show (5.2), (5.3) and (5.4), we find that the invariants $\langle T_2^{3d-\nu-2}, \psi^\nu T_2 \rangle_d^{\text{trop}}$ are well-defined, showing Theorem 3.6, and also showing (5.8), (5.9) and (5.10), hence $\phi_i = J_i^{\text{trop}}$. In particular, $\phi_{i,1} = y_i$ for $0 \leq i \leq 2$. This gives Theorem 3.8. □

We have in fact already taken care of (5.2):

Lemma 5.7. (5.2) holds.

Proof. In fact Proposition 3.13 shows the equivalent statement that $\phi_0 = J_0^{\text{trop}}$. Note however that the proof of Proposition 3.13 was carried out using a specific choice of $\Xi_0$, which a priori may not be the same $\Xi_0$ given by Lemma 5.1. However, one checks easily that the values for the integrals $\int_{\Xi_0} e^{q(x_0+x_1+x_2)} x_0^{n_0} x_1^{n_1} x_2^{n_2} \Omega$ given in Lemma 5.2 agree with the integrals over the $\Xi_0$ used in the proof of Proposition 3.13. Thus the argument still works. □

We will now refine the expressions $L_i^d$ which we need to compute.

Definition 5.8. For each cone $\sigma \in \Sigma$, $\sigma$ is the image under $r$ of a proper face $\tilde{\sigma}$ of the cone $K \subseteq T_\Sigma \otimes \mathbb{R}$ generated by $t_0, t_1, t_2$ (i.e., the first octant). For $d \geq 0$, denote by $K_d \subseteq K$ the cube

\[ K_d = \left\{ \sum_{i=0}^{2} n_i t_i \mid 0 \leq n_i \leq d \right\} \]

and for $\sigma \in \Sigma$, define

\[ \tilde{\sigma}_d := (\tilde{\sigma} + K_d) \setminus \bigcup_{\tau \in \sigma} (\tilde{\tau} + K_d). \]

Here $+$ denotes Minkowski sum.

Example 5.9. We have the following examples of $\tilde{\sigma}_d$. Let $m = \sum_i n_i t_i \in K$. 

Then using the expansion for $\psi_i$ in Lemma 5.2, (5.5), (5.6) and (5.7) are equivalent, if we compare the coefficients of $\kappa^d$ on both sides, to:

\begin{equation}
L_0^d = \delta_0,d + \sum_{\nu \geq 0} \langle T_2^{3d-\nu-2}, \psi^\nu T_2 \rangle_d^{\text{trop}} \frac{y_2^{3d-\nu-2}}{(3d-\nu-2)!} q^\nu+2,
\end{equation}

\begin{equation}
L_1^d = \sum_{\nu \geq 0} \langle T_2^{3d-\nu-1}, \psi^\nu T_1 \rangle_d^{\text{trop}} \frac{y_2^{3d-\nu-1}}{(3d-\nu-1)!} q^\nu+1,
\end{equation}

\begin{equation}
L_2^d = y_2 q^{-1} \delta_0,d + \sum_{\nu \geq 0} \langle T_2^{3d-\nu}, \psi^\nu T_0 \rangle_d^{\text{trop}} \frac{y_2^{3d-\nu}}{(3d-\nu)!} q^\nu.
\end{equation}

Now suppose we have shown (5.2), (5.3) and (5.4). The left-hand sides of these equations come from the period integrals, and hence are independent of the locations of $Q$ and $P_1, \ldots, P_k$ by Lemma 3.12. So the right-hand side is also independent of the locations of $Q$ and $P_1, \ldots, P_k$. So in particular, once we show (5.2), (5.3) and (5.4), we find that the invariants $\langle T_2^{3d-\nu-2}, \psi^\nu T_2 \rangle_d^{\text{trop}}$ are well-defined, showing Theorem 3.6, and also showing (5.8), (5.9) and (5.10), hence $\phi_i = J_i^{\text{trop}}$. In particular, $\phi_{i,1} = y_i$ for $0 \leq i \leq 2$. This gives Theorem 3.8. □

We have in fact already taken care of (5.2):

Lemma 5.7. (5.2) holds.

Proof. In fact Proposition 3.13 shows the equivalent statement that $\phi_0 = J_0^{\text{trop}}$. Note however that the proof of Proposition 3.13 was carried out using a specific choice of $\Xi_0$, which a priori may not be the same $\Xi_0$ given by Lemma 5.1. However, one checks easily that the values for the integrals $\int_{\Xi_0} e^{q(x_0+x_1+x_2)} x_0^{n_0} x_1^{n_1} x_2^{n_2} \Omega$ given in Lemma 5.2 agree with the integrals over the $\Xi_0$ used in the proof of Proposition 3.13. Thus the argument still works. □

We will now refine the expressions $L_i^d$ which we need to compute.

Definition 5.8. For each cone $\sigma \in \Sigma$, $\sigma$ is the image under $r$ of a proper face $\tilde{\sigma}$ of the cone $K \subseteq T_\Sigma \otimes \mathbb{R}$ generated by $t_0, t_1, t_2$ (i.e., the first octant). For $d \geq 0$, denote by $K_d \subseteq K$ the cube

\[ K_d = \left\{ \sum_{i=0}^{2} n_i t_i \mid 0 \leq n_i \leq d \right\} \]

and for $\sigma \in \Sigma$, define

\[ \tilde{\sigma}_d := (\tilde{\sigma} + K_d) \setminus \bigcup_{\tau \in \sigma} (\tilde{\tau} + K_d). \]

Here $+$ denotes Minkowski sum.

Example 5.9. We have the following examples of $\tilde{\sigma}_d$. Let $m = \sum_i n_i t_i \in K$. 

• If $\sigma = \emptyset$, then $m \in \tilde{\sigma}_d$ if and only if $d \geq \max\{n_0, n_1, n_2\}$.
• If $\sigma = \rho_0$, then $m \in \tilde{\sigma}_d$ if and only if $n_1, n_2 \leq d < n_0$.
• If $\sigma = \rho_1 + \rho_2$, then $m \in \tilde{\sigma}_d$ if and only if $n_0 \leq d < n_1, n_2$.

**Definition 5.10.** For $\sigma \in \Sigma$, define

$$L^d_{i,\sigma} = L^d_{i,\sigma}(Q) := \sum_{(c,\nu, m) \in S_k} cq^{3d+\nu-|m|} D_i(d, m).$$

**Lemma 5.11.**

1. $L^d_{i} = \sum_{\sigma \in \Sigma} L^d_{i,\sigma}$.
2. $L^d_{i,\sigma}(Q) = \sum_{\nu \geq 1} \sum_{I \subseteq \{1, \ldots, k\}} \langle P_{i_1}, \ldots, P_{i_{3d-2+i-\nu'}}, \psi^\nu S \rangle_{d,\{0\}}^{\text{trop}} u_I q^{\nu+2-i}$

where $S = Q, L$ the tropical line with vertex $Q$, or $M_R$ in the cases $i = 0, 1$ and 2. Here, the meaning of the notation on the right-hand side with subscript $\{0\} \in \Sigma$ is defined in (3.1).

**Proof.** (1) just follows from Lemma 5.2 which tells us that $D_i(d, m) = 0$ if $m \notin \bigcup_{\sigma \in \Sigma} \tilde{\sigma}_d$.

(2) This is essentially the same argument as made in the proof of Proposition 3.13. Let $(c, \nu, m) \in S_k$ with $m = \sum_{i=0}^{2} n_i t_i$. Then $(c, \nu, m)$ contributes to $L^d_{i,\{0\}}$ only if $n_0, n_1, n_2 \leq d$. Write

$$cq^\nu z^m = q^\nu \prod_{i=1}^{\nu} \text{Mult}(h_i) z^{\Delta(h_i)} u_{I(h_i)}$$

for $h_i : \Gamma'_i \to M_R$, $1 \leq i \leq \nu$, Maslov index two disks with boundary $Q$. Let $\Gamma$ be the graph obtained by identifying the outgoing vertices $V_{\text{out}, i}$ of $\Gamma'_1, \ldots, \Gamma'_\nu$ to get a single vertex $V_{\text{out}}$ and then adding $(d - n_0) + (d - n_1) + (d - n_2) + 1$ additional unbounded edges with vertex $V_{\text{out}}$. We define $h : \Gamma \to M_R$ to be $h_i$ on each subgraph $\Gamma'_i \subseteq \Gamma$. Furthermore, for $0 \leq i \leq 2$, $h$ maps $d - n_i$ of the new unbounded edges to the ray $Q + R_{\geq 0} m_i$. Finally, the last unbounded ray is labelled with an $x$ and is contracted by $h$. Just as in the argument of Proposition 3.13, $h$ is now a balanced tropical curve.

The contribution of this term $cq^\nu z^m$ to $L^d_{i,\{0\}}$ is then

$$q^{3d+\nu-n_0-n_1-n_2} D_i(d, n_0, n_1, n_2) \prod_{i=1}^{\nu} \text{Mult}(h_i) u_{I(h_i)}$$

$$= q^{3d+\nu-n_0-n_1-n_2} \text{Mult}_i^l(h) \prod_{V \notin \{p|}} \text{Mult}_V(h),$$

comparing the definitions of $\text{Mult}_i^l(h)$ and $D_i(d, n_0, n_1, n_2)$. Note that the valency $\text{Val}(V_{\text{out}})$ of the vertex $V_{\text{out}}$ in $h$ is $\nu+3d-(n_0+n_1+n_2)+1$. Suppose that $I(h) = \{i_1, \ldots, i_{3d-2+i-\nu'}\}$
for some \( \nu' \). Noting that \( h \) is obtained by gluing \( \text{Val}(V_{\text{out}}) - 1 \) Maslov index two disks, we see that

\[
\text{Val}(V_{\text{out}}) - 1 = \sum_{i=1}^{\nu} (|\Delta(h_i)| - \#I(h_i)) + (d - n_1) + (d - n_2) + (d - n_3)
\]

\[
= 3d - (3d - 2 + i - \nu') = \nu' + 2 - i.
\]

Then the curve \( h \) contributes precisely the correct contribution, as given by Definition \ref{def:3.4} (1) (a), (2) (b), or (3) (d), to

\[
\langle P_1, \ldots, P_{3d - 2 + i - \nu'}, \psi_{\nu'} S \rangle_{\text{trop}}^{\text{prop}} u_I(h) q^{\nu' + 2 - i}.
\]

Conversely, given any curve \( h \) contributing to the above quantity, it follows from Lemma \ref{lem:3.3} (3), that \( h \) will arise in the above manner from some term \((c, \nu, m) \in S_k\).

Next, we need to understand the asymptotic behaviour of \( L^{d}_{i, \sigma}(Q) \).

**Lemma 5.12.** Let \( \omega \in \Sigma \), and let \( v \in \omega \) be non-zero (hence ruling out \( \omega = \{0\} \)). Then

\[
(5.11) \quad \lim_{s \to \infty} L^{d}_{i, \omega}(Q + sv) = 0.
\]

**Proof.** We first note that with \( \omega \neq \{0\} \),

\[
(5.12) \quad \text{if } m \in \bar{\omega}_d, \text{ then } r(m) \in \bigcup_{\sigma \in i \Sigma} \text{Int}(\sigma).
\]

Next, for sufficiently large \( s \), \( Q + sv \) lies in an unbounded connected component \( \mathcal{C} \) of \( M_R \setminus \text{Supp}(\mathcal{D}) \), where \( \mathcal{D} = \mathcal{D}(\Sigma, P_1, \ldots, P_k) \). By taking \( s \) sufficiently large, we can assume \( \mathcal{C} \) is the last component entered as \( s \to \infty \). To show \((5.11)\), it will be enough to show that if \( Q + sv \in \mathcal{C} \), there exists a convex cone \( K' \subseteq M_R \) with \( K' \cap \bigcup_{\omega \in \Sigma} \text{Int}(\sigma) = \emptyset \) such that \( W_k(Q + sv) - W_0(Q + sv) \) only contains monomials \( z^m \) with \( r(m) \in K' \). It then follows that all monomials \( z^m \) in \( \exp(q(W_k(Q + sv) - W_0(Q + sv))) \) satisfy \( r(m) \in K' \), and hence by \((5.12)\), \( m \not\in \bar{\omega}_d \). This implies \((5.11)\).

So we study monomials \( z^m \) appearing in \( W_k(Q + sv) - W_0(Q + sv) \) and construct a cone \( K' \) with the desired properties. We will make use of the asymptotic cone to the closure \( \overline{\mathcal{C}} \) of \( \mathcal{C} \), \( \text{Asym}(\overline{\mathcal{C}}) \), which is defined to be the Hausdorff limit \( \lim_{\epsilon \to 0} \epsilon \overline{\mathcal{C}} \). Note that the connected components of \( M_R \setminus \mathcal{D}(\Sigma, P_1) \) are \( P_1 - \text{Int}(\sigma) \) where \( \sigma \) runs over the maximal cones of \( \Sigma \). Since \( \text{Supp}(\mathcal{D}(\Sigma, P_1)) \subseteq \text{Supp}(\mathcal{D}) \), one sees that \( \text{Asym}(\overline{\mathcal{C}}) \) is contained in some cone \( -\sigma \) with \( \sigma \in \Sigma \) maximal and \( \sigma \cap \omega \neq \{0\} \). Note also that \( \text{Asym}(\overline{\mathcal{C}}) \) can be a ray if the unbounded edges of \( \overline{\mathcal{C}} \) are parallel. Let \( \partial_1, \partial_2 \) denote the two unbounded edges of \( \overline{\mathcal{C}} \).

Now for general \( s \), a term \( cz^m \) in \( W_k(Q + sv) \) corresponds to a broken line \( \beta \) with given data \( -\infty = t_0 < \cdots < t_\rho = 0 \), \( m_\beta \in T_\Sigma \) as in Definition \ref{def:4.9} and \( m = m_\beta \). If \( -r(m) \not\in \mathbb{R}_{>0}v \), then for \( s \) sufficiently large, with \( + \) denoting Minkowski sum,

\[
Q + sv \not\in \mathbb{R}_{>0}(-r(m)) + (\partial \overline{\mathcal{C}} \setminus (\partial_1 \cup \partial_2)).
\]
Indeed, $\partial \mathcal{C} \setminus (\mathcal{D}_1 \cup \mathcal{D}_2)$ is bounded, so the asymptotic cone of the right-hand side is $\mathbb{R}_{\geq 0}(-r(m))$, which does not contain $v$ by assumption. Thus, taking a sufficiently large $s$, we note $\beta$ cannot last enter $\mathcal{C}$ via $\partial \mathcal{C} \setminus (\mathcal{D}_1 \cup \mathcal{D}_2)$ since the last line segment of $\beta$ is in the direction $-r(m)$. So for sufficiently large $s$, $\beta$ must enter $\mathcal{C}$ by crossing one of $\mathcal{D}_1$ or $\mathcal{D}_2$. In what follows, we will not need to study the case $-r(m) \in \mathbb{R}_{>0}v$ as the cone $K'$ we construct will always contain $-v$.

We can now assume that for large $s$, $\beta$ enters $\mathcal{C} = \mathcal{C}_n$ from another unbounded connected component $\mathcal{C}_{n-1}$ of $M_\mathbb{R} \setminus \text{Supp}(\mathcal{D})$. Necessarily, the $m^\beta_i$ attached to $\beta$ while $\beta$ passes through $\mathcal{C}_{n-1}$ satisfies $-r(m^\beta_i) \notin \text{Asym}(\mathcal{C}_{n-1})$. Indeed, otherwise $\beta$ could not hit an unbounded edge of $\mathcal{C}_{n-1}$. Again, for large enough $s$, one sees similarly that $\beta$ must enter $\mathcal{C}_{n-1}$ through the other unbounded edge of $\mathcal{C}_{n-1}$, and we can then continue this process inductively, with $\beta$ passing only through unbounded edges via a sequence of unbounded components $\mathcal{C}_0, \ldots, \mathcal{C}_n$.

When $\beta$ bends, it then always bends outward, as depicted in Figure 5.1. From this we make the following two observations:

(C1) If the edges corresponding to $\mathcal{D}_1$ and $\mathcal{D}_2$ of $\text{Asym}(\mathcal{C})$ are generated by $v_1$, $v_2$ respectively (possibly $v_1 = v_2$) and $\beta$ enters $\mathcal{C}$ by crossing $\mathcal{D}_i$, then $-r(m)$ lies in a half-plane with boundary $\mathbb{R}v_i$ containing $\text{Asym}(\mathcal{C})$; otherwise, $\beta$ cannot reach the interior of $\mathcal{C}$.

(C2) For any $j$, $1 \leq j \leq p$, $-r(m)$ lies in the half-plane with boundary $\mathbb{R}r(m^\beta_j)$ containing $v_i$ corresponding to the edge $\mathcal{D}_i$ that $\beta$ crosses to enter $\mathcal{C}$. This follows from the behaviour described above about how $\beta$ bends.

Without loss of generality, let us assume for the ease of drawing pictures that $\omega = \rho_2$ or $\rho_1 + \rho_2$ and $\text{Asym}(\mathcal{C}) \subseteq -(\rho_0 + \rho_1)$. See Figure 5.2. Note that as depicted there, we must have $v_2 \in \rho_1 + \rho_2$. 

![Figure 5.1](image_url)
We analyze the possibilities for $\beta$: we have three cases, based on whether the initial direction of $\beta$ is $-m_1$, $-m_2$, or $-m_0$.

Case 1. $r(m_0^\beta) = m_1$. Then $\beta$ must enter $C$ via $d_2$. By (C1), $-r(m)$ lies in the half-plane with boundary $Rv_2$ containing $\text{Asym}(\overline{C})$, and by (C2), $-r(m)$ lies in the half-plane with boundary $Rm_1$ containing $\text{Asym}(\overline{C})$. Thus $-r(m) \in (-R_{\geq 0}m_1 + R_{\geq 0}v_2)$.

Case 2. $r(m_0^\beta) = m_2$. Then either $R_{\geq 0}m_2 \subseteq \text{Asym}(\overline{C})$ or $\text{Asym}(\overline{C}) \subseteq \rho_1 + \rho_2$ since $v \in \text{Asym}(\overline{C})$. In the first case, $\beta$ has no opportunity to bend, so corresponds to the monomial $x_2$, which doesn’t appear in $W_k(Q + sv) - W_0(Q + sv)$. In the second case, $\beta$ bends at time $t_1$ as it crosses a ray $d \in D$ with $f_0 = 1 + c_d z^{m_d}$ with $-r(m_0) \in \text{Int}(\rho_1 + \rho_2)$. Now $r(m_0^\beta) = m_2 + r(m_0)$, so it follows that $-r(m_0^\beta) \in \rho_1 + \rho_2$. (Here we use integrality of $m_0$ and $m_2 = (0, 1)$.) Thus by (C1) and (C2), $-r(m) \in (R_{\geq 0}m_1 + R_{\geq 0}v_1)$.

Case 3. $r(m_0^\beta) = m_0$. In this case $\beta$ must enter $C$ through the edge $d_1$ since $\text{Asym}(\overline{C}) \subseteq -(\rho_0 + \rho_1)$. Then one sees from (C1) and (C2) that $-r(m) \in (R_{\geq 0}(-m_0) + R_{\geq 0}v_1)$.

We now see that if $R_{\geq 0}m_2 \subseteq \text{Asym}(C)$, (which always happens if $v$ is proportional to $m_2$, in particular when $\omega = \rho_2$), then of these three cases, only cases 1 and 3 can occur, and in fact $r(m), -v \in \rho_0 + \rho_1$. Thus $K' = \rho_0 + \rho_1$ is the desired cone, proving the claim in this case.

If $R_{\geq 0}m_2 \not\subseteq \text{Asym}(C)$, then $v$ is not proportional to $m_2$ and $\omega = \rho_1 + \rho_2$. In this case, the above three cases show that $-r(m)$ is always contained in the upper half-plane. Thus $K'$ the lower half-plane is the desired cone, proving the claim in this case. □
The next step is to explain how $L_{i,\sigma}^d(Q)$ depends on $Q$ via a wall-crossing formula. While of course $L_i^d$ is independent of $Q$, the way the terms in $L_i^d$ are redistributed among the expressions $L_{i,\sigma}^d(Q)$ is key to the calculations.

**Definition 5.13.** Let $\mathcal{D} = \mathcal{D}(\Sigma, P_1, \ldots, P_k)$. Let $\mathcal{C}_1, \mathcal{C}_2$ be two connected components of $M_\mathcal{D} \setminus \text{Supp}(\mathcal{D})$ with $\dim \mathcal{C}_1 \cap \mathcal{C}_2 = 1$. Let $Q_i \in \mathcal{C}_i$ be general points, and let $\gamma$ be a path from $Q_1$ to $Q_2$, passing through $\text{Supp}(\mathcal{D})$ only at one time $t_0$, with $\gamma(t_0) \notin \text{Sing}(\mathcal{D})$. Let $\delta \in \mathcal{D}$ be a ray with $\gamma(t_0) \in \delta$, and let $n_\delta \in N$ be a primitive vector which is orthogonal to $\delta$ and satisfies $\langle n_\delta, \gamma'(t_0) \rangle < 0$. Writing $f_0 = 1 + c_\delta z^{m_0}$, note that

$$\theta_{\gamma,0}(z^m) = z^m + c_\delta \langle n_\delta, r(m) \rangle z^{m+m_0}.$$ 

Now take a pair $\omega \subsetneq \tau$ with $\omega, \tau \in \Sigma$ and $\dim \tau = \dim \omega + 1$. Note there is a unique index $j \in \{0, 1, 2\}$ such that $m_j \notin \omega$ but $m_j \in \tau$; call this index $j(\omega, \tau)$. Then define

$$L_{i,\omega,\tau}^d(\gamma) := \sum_{(c, \nu, m) \in S_k(Q_1)} \langle n_\delta, m_j(\omega, \tau) \rangle c_\delta c D_i(d, m + m_\delta + t_j(\omega, \tau)) q^{\nu + 3d - |m + m_\delta|},$$

where the sum is over all $(c, \nu, m) \in S_k(Q_1)$ such that $m + m_\delta \in \hat{\omega}_d$ but $m + m_\delta + t_j(\omega, \tau) \notin \hat{\tau}_d$. If $(c, \nu, m) \in S_k(Q_1)$ satisfies this condition, then we say the term $c q^\nu z^m$ contributes to $L_{i,\omega,\tau}^d(\gamma)$.

Define

$$L_{i,\omega,\tau}^d := \sum_{\delta} L_{i,\delta,\omega,\tau}^d,$$

where the sum is over all $\delta \in \mathcal{D}$ with $\gamma(t_0) \in \delta$.

For an arbitrary path $\gamma$ in $M_\mathcal{D} \setminus \text{Sing}(\mathcal{D})$ with $\gamma(0) = Q$, $\gamma(1) = Q'$, choose a partition of $[0, 1]$, $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $\gamma|_{[t_{j-1}, t_j]}$ is a path of the sort considered above, connecting endpoints in adjacent connected components. Then define

$$L_{i,\omega,\tau}^d := \sum_{j=1}^n L_{i,\gamma|_{[t_{j-1}, t_j]}},$$

Lemma 5.14. Let $P_1, \ldots, P_k$ be general. Let $\gamma$ be a path in $M_\mathcal{D} \setminus \text{Sing}(\mathcal{D})$ with $\gamma(0) = Q$, $\gamma(1) = Q'$. Then for $\dim \rho = 1$, $\rho \in \Sigma$,

$$L_{i,\rho}^d(Q') - L_{i,\rho}^d(Q) = L_{i,\rho,0}^d - \sum_{\sigma \in \Sigma, \rho \subseteq \sigma} L_{i,\sigma,\rho}^d.$$

while for $\dim \sigma = 2$, $\sigma \in \Sigma$,

$$L_{i,\sigma}^d(Q') - L_{i,\sigma}^d(Q) = \sum_{\rho \in \Sigma, \dim \rho = 1, \rho \subseteq \sigma} L_{i,\gamma,\rho}^d.$$
Proof. It is enough to show this for $\gamma$ a short path connecting $Q$ and $Q'$ in two adjacent components $C_1$ and $C_2$ of $M_\mathbb{R} \setminus \text{Supp}(\mathcal{D})$ as in Definition 5.13. Suppose that at time $t_0$, $\gamma(t_0) \in \mathfrak{d}_1 \cap \cdots \cap \mathfrak{d}_s$ for rays $\mathfrak{d}_1, \ldots, \mathfrak{d}_s \in \mathcal{D}$. Of course, $\dim \mathfrak{d}_i \cap \mathfrak{d}_j = 1$. We can then write, for $n_{\mathfrak{d}} = n_{\mathfrak{d}_i}$ for any $i$,

$$
\theta_{\gamma, \mathcal{D}}(z^m) = z^m \prod_{i=1}^s f_{n_{\mathfrak{d}_i}}^{(n, r(m))} \\
= z^m \prod_{i=1}^s (1 + c_{n, r(m)}^{(n, r(m))}) \\
= z^m + \sum_{i=1}^s c_{n, r(m)}^{(n, r(m))} z^{m+i}.
$$

Here the last equality follows from $c_{n, r(m)} = 0$ for $i \neq j$. This is the case by the assumption that $P_1, \ldots, P_k$ are general. Indeed, if $c_{n, r(m)} \neq 0$, then the Maslov index zero trees $h_i$ and $h_j$ corresponding to $\mathfrak{d}_i$ and $\mathfrak{d}_j$ would have $I(h_i) \cap I(h_j) = \emptyset$. However, a generic perturbation of the marked points with indices in $I(h_i)$ would deform $\mathfrak{d}_i$ without deforming $\mathfrak{d}_j$, so that $\mathfrak{d}_i \cap \mathfrak{d}_j = \emptyset$.

Now

$$
W_k(Q') = \theta_{\gamma, \mathcal{D}}(W_k(Q))
$$

by Theorem 4.12. Using the expansion (5.1) and $W_0(Q) = y_0 + \sum_{j=0}^2 z^{t_j}$,

$$
\exp \left( q(W_k(Q') - W_0(Q')) \right) = \exp \left( q(\theta_{\gamma, \mathcal{D}}(W_k(Q)) - W_0(Q)) \right) = \theta_{\gamma, \mathcal{D}}(\exp(q(W_k(Q) - W_0(Q)))) \cdot \exp \left( q(\theta_{\gamma, \mathcal{D}}(W_0(Q)) - W_0(Q)) \right) \\
= \theta_{\gamma, \mathcal{D}} \left( \sum_{(c,m) \in S_k(Q)} c^\nu z^m \right) \left( 1 + q \sum_{\ell=1}^s \sum_{j=0}^2 c_{n, m}^{(n, r(m))} z^{m+i+j} \right) \\
= \exp \left( q(W_k(Q) - W_0(Q)) \right) + \sum_{(c,\nu, m) \in S_k(Q)} \sum_{\ell=1}^s \left( c^\nu c_{n, m}^{\nu}(n, r(m)) z^{m+i+j} + q \sum_{j=0}^2 (n, m_j) z^{m+i+j} \right).
$$

We interpret this as follows. For each $(c, \nu, m) \in S_k(Q)$ and each $\ell$, look at the four terms

$$
c^\nu c_{n, m}^{\nu}(n, r(m)) z^{m+i+j} + q \sum_{j=0}^2 (n, m_j) z^{m+i+j}.
$$

These four terms contribute the expression

$$
c^\nu c_{n, m}^{\nu}(n, r(m)) D_1(d, m + m_\ell) + \sum_{j=0}^2 (n, m_j) D_1(d, m + m_\ell + t_j).
$$
to $L^d_i(Q')$. One can check that in fact this total contribution is zero, either by direct but tedious checking from the formulas for $D_i$, or by applying Lemma $4.16$ with $f = cq^{\nu-1} z^m$ and $\theta = \theta_{\gamma, \Sigma}$.

Now if $m + m_{\theta_i}$ and $m + m_{\theta_i} + t_j$, $0 \leq j \leq 2$, all lie in the same $\bar{\omega}_d$, then these terms produce no total contribution to $L^d_{i,\tau}(Q')$ for any $\tau \in \Sigma$, including $\tau = \omega$. On the other hand, these four terms can contribute to different $L^d_{i,\omega}(Q')$’s if $m + m_{\theta_i}$ and $m + m_{\theta_i} + t_j$, $j = 0, 1, 2$, don’t all lie in $\bar{\omega}_d$ for the same $\omega \in \Sigma$. This can happen only if $m + m_{\theta} \in \bar{\omega}_d$ but $m + m_{\theta} + t_j \in \bar{\tau}_d$ for some $j$ with $\omega \subseteq \tau \in \Sigma$ with $\dim \tau = \dim \omega + 1$ and $m_j \in \tau$, $m_j \notin \omega$. In this case, $L^d_{i,\tau}(Q') - L^d_{i,\omega}(Q)$ has a contribution of the form $cc_{\theta_i}(n_{\theta}, m_j)q^{3d+\nu-|m+m_{\theta}|} D_i(d, m + m_{\theta} + t_j)$. Thus $L^d_{i,\omega}(Q') - L^d_{i,\omega}(Q)$ must have a contribution coming from the same term, but with opposite sign. This gives the Lemma.

We can now use the asymptotic behaviour of the expressions $L^d_{i,\omega}(Q)$ and the above wall-crossing formula to rewrite the needed expressions:

**Lemma 5.15.** Let $\gamma_j$ be the straight line path joining $Q$ with $Q + sm_j$ for $s \gg 0$. Let $\gamma_{j,j+1}$ be the loop based at $Q$ which passes linearly from $Q$ to $Q + sm_j$, then takes a large circular arc to $Q + sm_{j+1}$, and then proceeds linearly from $Q + sm_{j+1}$ to $Q$. Here we take $j$ modulo $3$, and $\gamma_{j,j+1}$ is always a counterclockwise loop. Let $\sigma_{j,j+1} = \rho_j + \rho_{j+1}$, a two-dimensional cone in $\Sigma$. Then

$$L^d_{i}(Q) - L^d_{i,\{0\}}(Q) = - \sum_{j=0}^{2} L^d_{i,\gamma_j,\{0\} \rightarrow \rho_j} - \sum_{j=0}^{2} L^d_{i,\gamma_{j,j+1} \rightarrow \sigma_{j,j+1} \rightarrow \rho_{j+1} \rightarrow \sigma_{j,j+1}}.$$ 

**Proof.** By Lemma $5.12$, $L^d_{i,\sigma}(Q + sm_j) = 0$ for any $\sigma \in \Sigma$ with $\rho_j \subseteq \sigma$. Thus by $(5.13)$ and $(5.14)$, we have

$$L^d_{i,\rho_j}(Q) = -L^d_{i,\gamma_j,\{0\} \rightarrow \rho_j} + \sum_{\substack{\sigma \in \Sigma, \
 \rho_j \subseteq \sigma}} L^d_{i,\gamma_j,\rho_j \rightarrow \sigma},$$

$$L^d_{i,\sigma_{j,j+1}}(Q) = - \sum_{\substack{\rho \in \Sigma, \
 \dim \rho = 1, \
 \rho \not\subseteq \sigma_{j,j+1}}} L^d_{i,\gamma_j,\rho \rightarrow \sigma_{j,j+1}}.$$ 

Note we have broken symmetry for the second equation.
Adding together contributions from the $\rho_j$’s and $\sigma_{j_1,j_2}$’s, we see from Lemma 5.11 (1), that

$$L_i^d(Q) - L_i^d(0)(Q) = -\sum_{j=0}^{2} L_i^d_{i,\gamma_j}(0) - \rho_j - (L_i^d_{i,\gamma_0,\rho_1 - \sigma_0,1} - L_i^d_{i,\gamma_1,\rho_1 - \sigma_0,1}) - (L_i^d_{i,\gamma_1,\rho_2 - \sigma_1,2} - L_i^d_{i,\gamma_2,\rho_2 - \sigma_1,2}) - (L_i^d_{i,\gamma_2,\rho_0 - \sigma_2,0} - L_i^d_{i,\gamma_0,\rho_0 - \sigma_2,0}).$$

Again by Lemma 5.12, it follows that the contribution to $L_i^d_{i,\gamma,j+1,\rho,j+1 - \sigma,j,j+1}$ from the large circular arc is zero. Hence

$$L_i^d(Q) - L_i^d(0)(Q) = -\sum_{j=0}^{2} L_i^d_{i,\gamma_j}(0) - \rho_j - \sum_{j=0}^{2} L_i^d_{i,\gamma,j+1,\rho,j+1 - \sigma,j,j+1},$$

the desired result. \qed

We have already interpreted $L_i^d(0)(Q)$ in Lemma 5.11 (2), so it remains to interpret the remaining terms on the right-hand side of the above lemma.

Lemma 5.16.

$$-L_i^d_{i,\gamma,j}(0) - \rho_j = \sum_{\nu \geq i-1} \sum_{\nu \in \{1,\ldots,k\} \setminus \{i_1,\ldots,i_{\ell d - 2 + i - \nu}\}} (P_{i_1,\ldots,i_{\ell d - 2 + i - \nu}} S)_{trop}^u \varphi^{\nu + 2 - i}$$

for $S = Q, L$ or $M_R$ for $i = 0, 1$ and $i = 2$ respectively. Here, as usual, $L$ is a tropical line with vertex $Q$.

Proof. This is vacuous for $i = 0$, so we assume $i \geq 1$. Without loss of generality, consider $L_i^d_{i,\gamma,0}(0) - \rho_0$. This quantity is a sum of contributions from each point $P \in Q + (\rho_0 \setminus \{0\})$ which is the intersection of $Q + \rho_0$ with a ray $d \in \mathcal{D} = \mathcal{D}(\Sigma, P_1, \ldots, P_k)$. Let us consider the contribution to $L_i^d_{i,\gamma,0}(0) - \rho_0$ from a small segment $\gamma$ of $\gamma_0$ which only crosses $d$. Let $\gamma$ run from $Q_1$ to $Q_2$. Now $d$ corresponds to a Maslov index zero tree passing through $P$, and by cutting it at $P$, we obtain a Maslov index zero disk $h_1 : \Gamma_1 \rightarrow M_R$ with boundary $P$. Then

$$f_0 = 1 + w_{\Gamma_1} (E_{out,1}) \text{Mult}(h_1) z^{\Delta(h_1)} u_{I(h_1)}.$$  

Furthermore a term $cz^{m}q^\nu$ in $\exp(q(W_k(Q_i) - W_0(Q_i)))$ arises from $\nu$ distinct Maslov index two disks with boundary $Q_1$, say $h_2, \ldots, h_{\nu+1}$ (each with at least one marked point), and the term contributed is

$$q^\nu \prod_{i=2}^{\nu+1} \text{Mult}(h_i) z^{\Delta(h_i)} u_{I(h_i)}.$$
In order for this term to contribute to \( L_{i,\gamma,\{0\}}^d \rightarrow \rho_0 \), \( m + m_0 = \sum_{i=1}^{\nu+1} \Delta(h_i) \) must be of the form \( dt_0 + n_1t_1 + n_2t_2 \) with \( n_1, n_2 \leq d \). The disks \( h_2, \ldots, h_{\nu+1} \) deform to disks with boundary at \( P \), which we also call \( h_2, \ldots, h_{\nu+1} \). Write these disks as \( h_i : \Gamma_i' \rightarrow M_{\mathbb{R}} \). Each \( \Gamma_i', 1 \leq i \leq \nu + 1 \), has a vertex \( V_{\text{out},i} \).

Using this data, we can construct an actual tropical curve as follows. Let \( \Gamma \) be the graph obtained by identifying all the outgoing vertices \( V_{\text{out},i} \) in \( \Gamma_1', \ldots, \Gamma_{\nu+1}' \), to obtain a graph with a distinguished vertex \( V_{\text{out}} \), and then attaching \( (d - n_1) + (d - n_2) + 1 \) additional unbounded edges with vertex \( V_{\text{out}} \). We then define \( h : \Gamma \rightarrow M_{\mathbb{R}} \) to agree with \( h_i \) on \( \Gamma_i' \subseteq \Gamma \).

We have \( h \) taking the first \( d - n_1 \) new unbounded edges to \( P + \mathbb{R}_{\geq 0}m_1 \); the second \( d - n_2 \) new unbounded edges to \( P + \mathbb{R}_{\geq 0}m_2 \); and the last unbounded edge is contracted, and marked with the label \( x \). Note \( \Gamma \) has valency at \( V_{\text{out}} \) given by \( \text{Val}(V_{\text{out}}) = \nu + 1 + (d - n_1) + (d - n_2) + 1 \). Thus we obtain a parameterized curve \( h : \Gamma \rightarrow M_{\mathbb{R}} \) with \( h(x) = P \). The balancing condition needs to be checked at \( V_{\text{out}} \), but as in (3.4) and (3.5), the fact that\( -\langle n_2, m_0 \rangle w_{t(h)}w_{R_{\text{out},1}} \left( \prod_{i=1}^{\nu+1} \text{Mult}(h_i) \right) D_{i}(d, d + 1, n_1, n_2)q^{\nu+3d-(d+n_1+n_2)} \).

Note \( n_2 \) is primitive, annihilates \( r(m_2) \), and must be positive on \( -m_0 \). Furthermore, after choosing an isomorphism \( \bigwedge^2 M \cong \mathbb{Z} \), \( w(E_{\text{out},1})n_2 \) can be identified, up to sign, with \( X_{r(m_2)} \). Thus setting \( m(h_1) = r(m_2) \) as in Definition 3.4, we see that\( -\langle n_2, m_0 \rangle w_{R_{\text{out},1}} = |m(h_1) \wedge m_0| \).

Thus (5.15) coincides with\( |m(h_1) \wedge m_0| u_{t(h)} D_i(d, d + 1, n_1, n_2) \left( \prod_{\nu \in \Gamma^{[0]} \setminus \nu \in E_x} \text{Mult}_V(h) \right) q^{\text{Val}(V_{\text{out}}) - 2} \).

Now \( D_i(d, d + 1, n_1, n_2) = \text{Mult}_i^{-1}(h) \) as defined in Definition 3.4 via direct comparison with the definitions of the \( D_i \)'s. Furthermore, if \( I(h) = \{i_1, \ldots, i_{3d - 2 + i - \nu'} \} \) for some \( \nu' \), we see that, as \( h \) is obtained by gluing one Maslov index zero disk to \( \text{Val}(V_{\text{out}}) - 2 \) Maslov index two disks, we have\( \text{Val}(V_{\text{out}}) - 2 = \sum_{i=1}^{\nu+1} (|\Delta(h_i)| - \#(h_i)) + (d - n_1) + (d - n_2) = 3d - (3d - 2 + i - \nu') = \nu' + 2 - i. \)
Thus, by Definition 3.4 the term under consideration contributes to $-L^{d}_{i,\gamma,\rho_{j+1} \rightarrow \sigma_{j},j+1}$ by exactly the same amount that the curve $h$ contributes to

$$\langle P_{i}, \ldots, P_{i_{3d-2+i}'}, \psi^{\nu'} S \rangle^{\text{trop}}_{d,\sigma_{j},j+1} u_{I(h)} q^{\nu'+2-i},$$

as desired.

Conversely, given any curve $h$ contributing to $\langle P_{i}, \ldots, P_{i_{3d-2+i}'}, \psi^{\nu'} S \rangle^{\text{trop}}_{d,\sigma_{j},j+1}$ with $h(E_{x}) = P \in \rho \setminus \{0\}$, the procedure of Lemma 3.3 shows $h$ must arise in precisely the way described above.

Lemma 5.17.

$$-L^{d}_{i_{s},j,j+1,\rho_{j+1} \rightarrow \sigma_{j},j+1} = \sum_{i_{s},t,} \sum_{P_{i} \in \sigma_{j},j+1} u_{i_{s}} q^{-1} + \sum_{\nu \geq 0} \sum_{I \subseteq \{1, \ldots, k\}} \sum_{l = (i_{1}, \ldots, i_{3d-2+i}' \nu)} 1_{i_{1} \ldots < i_{3d-2+i}' \nu} \langle P_{i}, \ldots, P_{i_{3d-2+i}'}, \psi^{\nu} S \rangle^{\text{trop}}_{l,\sigma_{j},j+1} u_{I(h)} q^{\nu' + 2-i}$$

for $S = Q$, $L$ or $M_{\mathbb{R}}$ for $i = 0, 1$ and $i = 2$ respectively.

Proof. First note this is vacuous for $i = 0$ or 1 as both sides are zero, so we can assume $i = 2$. Second, if $C_{1}$ and $C_{2}$ are closures of two connected components of $M_{\mathbb{R}} \setminus \text{Supp} \mathfrak{D}$, where $\mathfrak{D} = \mathfrak{D}(\Sigma, P_{1}, \ldots, P_{k})$, $\dim C_{1} \cap C_{2} = 1$, and $\gamma$ is a short path from $\text{Int}(C_{1})$ into $\text{Int}(C_{2})$ just crossing $\text{Int}(C_{1} \cap C_{2})$ once, then $L^{d}_{i_{s},\gamma,\rho_{j+1} \rightarrow \sigma_{j},j+1}$ is independent of $\gamma$ and its endpoints. Furthermore, reversing the direction of $\gamma$ changes the sign of $L^{d}_{i_{s},\gamma,\rho_{j+1} \rightarrow \sigma_{j},j+1}$. So a simple homological argument shows that

$$L^{d}_{i_{s},\gamma,\rho_{j+1} \rightarrow \sigma_{j},j+1} = \sum_{P_{i} \in \text{Sing}(\mathfrak{D}) \cap \sigma_{j},j+1} L^{d}_{i_{s},\gamma_{P},\rho_{j+1} \rightarrow \sigma_{j},j+1}$$

where $\gamma_{P}$ is a small counterclockwise loop around the singular point $P$. This localizes the calculation to the singular points of $\mathfrak{D}$ in $\sigma_{j},j+1$. Now such a singular point $P$ is either in $\{P_{1}, \ldots, P_{k}\}$ or not; this will give us cases (3) (b) and (3) (a) of Definition 3.4 respectively. To save on typing, we set

$$L_{P,j} := L^{d}_{2,\gamma_{P},\rho_{j+1} \rightarrow \sigma_{j},j+1}.$$

Case 1. $P \not\in \{P_{1}, \ldots, P_{k}\}$. Fix a base-point $Q'$ near $P$. Consider a term $cq^{\nu} z^{m}$ in $\exp(q(W_{k}(Q') - W_{0}(Q'))) (5.16)$ of the form

$$cq^{\nu} z^{m} = q^{\nu} \prod_{i=3}^{\nu+2} (\text{Mult}(h_{i}) z^{\Delta(h_{i})} u_{I(h_{i})})$$

where the $h_{i}$'s are Maslov index two disks with boundary $Q'$, but none of the $h_{i}$'s come from broken lines which bend near $P$. As a result, this term appears in $\exp(q(W_{k}(Q'') - W_{0}(Q''))) (5.16)$ for all $Q''$ general in a small open neighbourhood of $P$.

Suppose that such a term $cq^{\nu} z^{m}$ contributes to $L_{P,j}$ when $\gamma_{P}$ crosses a ray $\mathfrak{d} \in \mathfrak{D}$ with $P \in \mathfrak{d}$, $P \neq \text{Init}(\mathfrak{d})$. But $\gamma_{P}$ crosses this ray $\mathfrak{d}$ twice, in opposite directions, so $cq^{\nu} z^{m}$ will
contribute to $L_{P,j}$ twice, but with opposite signs. Thus these contributions cancel, and don’t contribute to the total in $L_{P,j}$.

Thus we only need analyze contributions arising when $\gamma_P$ crosses a ray $\varnothing$ with $\text{Init}(\varnothing) = P$ or contributions from monomials as in (5.16) where some of the $h_i$’s come from broken lines which do bend near $P$. As in the proof of Theorem 4.12 we can in fact assume that there are precisely three rays, $\varnothing_1$, $\varnothing_2$, $\varnothing_3$ passing through $P$, with $\text{Init}(\varnothing_1), \text{Init}(\varnothing_2) \neq P$ and $\text{Init}(\varnothing_3) = P$. Now $\varnothing_1, \varnothing_2$ correspond to Maslov index zero trees passing through $P$, and by cutting them, we obtain Maslov index zero disks $h_i : \Gamma_i^* \to M_R$, $i = 1, 2$ with boundary $P$, and for $i = 1, 2$,

$$f_{\varnothing_i} = 1 + w_{\Gamma_i}(E_{\text{out},i}) \text{ Mult}(h_i)z^{\Delta(h_i)}u^{I(h_i)}.$$  

We now analyze how additional terms $cq^\nu z^m$ which can contribute to $L_{P,j}$ may arise. In what follows, assume that $cq^\nu z^m$ is as in (5.16) in which none of the broken lines corresponding to $h_3, \ldots, h_{\nu+2}$ bend at $\varnothing_1, \varnothing_2$ or $\varnothing_3$. Write

$$m + \Delta(h_1) + \Delta(h_2) = \sum_{j=0}^{2} n_j t_j.$$  

We have the following possibilities of additional contributions:

(I) $cq^\nu z^m$ may contribute to $L_{P,j}$ when $\gamma_P$ crosses $\varnothing_3$. This contribution can only occur if $n_{j+2} \leq n_j = d < n_{j+1}$.

(II) After crossing $\varnothing_1$, new terms of the form (leaving off the coefficients) $z^{m+\Delta(h_1)}$ and $z^{m+\Delta(h_1)+t_j}$, $k = 0, 1, 2$ may appear in $\exp(q(W_k - W_0))$. Thus when we cross $\varnothing_2$, these new terms may contribute to $L_{P,j}$. Note that $z^{m+\Delta(h_1)}$ only contributes when crossing $\varnothing_2$ if $n_{j+2} \leq n_j = d < n_{j+1}$. The term $z^{m+\Delta(h_1)+t_j}$ only contributes if $n_{j+2} \leq d$, $n_j = d - 1$ and $d < n_{j+1}$. The term $z^{m+\Delta(h_1)+t_j+1}$ only contributes if $n_{j+2} \leq n_j = d < n_{j+1}$. The term $z^{m+\Delta(h_1)+t_j+2}$ only contributes if $n_{j+2} < n_j = d < n_{j+1}$.

(III) After crossing $\varnothing_2$, new terms of the form (leaving off the coefficients) $z^{m+\Delta(h_2)}$ and $z^{m+\Delta(h_2)+t_j}$, $k = 0, 1, 2$ may appear in $\exp(q(W_k - W_0))$. Thus when we cross $\varnothing_1$, these new terms may contribute to $L_{P,j}$. Note that $z^{m+\Delta(h_2)}$ only contributes when crossing $\varnothing_2$ if $n_{j+2} \leq n_j = d < n_{j+1}$. The term $z^{m+\Delta(h_2)+t_j}$ only contributes if $n_{j+2} \leq d$, $n_j = d - 1$ and $d < n_{j+1}$. The term $z^{m+\Delta(h_2)+t_j+1}$ only contributes if $n_{j+2} \leq n_j = d < n_{j+1}$. The term $z^{m+\Delta(h_2)+t_j+2}$ only contributes if $n_{j+2} < n_j = d < n_{j+1}$.

There are now three cases when these additional contributions to $L_{P,j}$ occur.

Case 1(a). $n_{j+2} \leq n_j = d < n_{j+1}$. In this case, (leaving off the coefficients), $z^m$ gives a contribution to $L_{P,j}$ of type (I) when $\gamma_P$ crosses $\varnothing_3$, and $z^{m+\Delta(h_1)}$, $z^{m+\Delta(h_1)+t_j+1}$, or $z^{m+\Delta(h_1)+t_j+2}$ (if $n_{j+2} < d$) may give contributions of type (II) or (III) when $\gamma_P$ crosses
$d_1$ and $d_2$. Now note that the total change to $L^d_{i,\sigma_{j+1}}$ due to these monomials as we traverse the loop $\gamma_P$ is the sum of the contributions of these monomials to $L^d_{i,\gamma_P,\rho_j-\sigma_{j+1}}$ and $L^d_{i,\gamma_P,\rho_{j+1}-\sigma_{j+1}}$. However, the total contribution to the change of $L^d_{i,\sigma_{j+1}}$ is necessarily zero, as $\gamma_P$ is a loop, and because $d < n_{j+1}$, none of these monomials contribute to any change of $L^d_{i,\gamma_P,\rho_j-\sigma_{j+1}}$. Hence the total contribution of these monomials to $L^d_{i,\gamma_P,\rho_{j+1}-\sigma_{j+1}} = L_{P,j}$ is also zero.

Case 1(b). $n_{j+2} \leq d$, $n_j = d - 1$, $d < n_{j+1}$. In this case only the terms $z^{m+\Delta(h_1)+t_j}$, $i = 1, 2$, may contribute. However, the same argument as in Case 1(a) shows that the total contribution from these terms is zero.

Case 1(c). $n_{j+2} \leq n_j = n_{j+1} = d$. In this case, contributions to $L_{P,j}$ only arise from terms of the form $z^{m+\Delta(h_1)+t_j}$. Choose $n_0$ so that at the first time $t_i$ when $\gamma_P$ passes through $d_i$, $\langle n_0, \gamma_P(t_i) \rangle < 0$. By interchanging the labelling of $d_1$ and $d_2$ and choosing the base-point $Q'$ appropriately, we can assume firstly that $\gamma_P$ passes initially through $d_1$ and then $d_2$, and secondly that $\langle n_0, m_{j+1} \rangle \geq 0$ for $i = 1, 2$. Write $f_{n_0} = 1 + c_{n_0} z^{m_0}$, for $i = 1, 2$.

Then the term

$$\langle n_{0_1}, m_{j+1} \rangle c_{n_0} q^{\nu+1} z^{m+\Delta(h_1)+t_j+1}$$

appears in $\exp \left( q(W_k(\gamma_P(t)) - W_0(\gamma_P(t))) \right)$ right after $\gamma_P$ crosses $d_1$ the first time (and disappears when we cross $d_1$ for the second time), and hence when $\gamma_P$ crosses $d_2$ for the first time, we obtain a contribution to $L_{P,j}$ of

$$\langle n_{0_2}, m_j \rangle \langle n_{0_1}, m_{j+1} \rangle c_{n_0} c_{n_2} D_2(d, m + \Delta(h_1) + \Delta(h_2) + t_j + t_{j+1}) q^{\nu+3d-m+\Delta(h_1)+\Delta(h_2)}.$$

On the other hand, the term $\langle n_{0_2}, m_{j+1} \rangle c_{n_0} q^{\nu+1} z^{m+\Delta(h_2)+t_j+1}$ appears after $\gamma_P$ crosses $d_2$ for the first time (and disappears when we cross $d_2$ for the second time), and hence when $\gamma_P$ crosses $d_1$ for the second time, we obtain a contribution to $L_{P,j}$ of

$$\langle -n_{0_1}, m_j \rangle \langle n_{0_2}, m_{j+1} \rangle c_{n_0} c_{n_2} D_2(d, m + \Delta(h_1) + \Delta(h_2) + t_j + t_{j+1}) q^{\nu+3d-m+\Delta(h_1)+\Delta(h_2)}.$$

Note

$$\langle n_{0_2}, m_j \rangle \langle n_{0_1}, m_{j+1} \rangle - \langle n_{0_1}, m_j \rangle \langle n_{0_2}, m_{j+1} \rangle = -|n_{0_1} \wedge n_{0_2}| = -|m_{\text{prim}}(h_1) \wedge m_{\text{prim}}(h_2)|$$

as $n_{0_1}, n_{0_2}$ form a positively oriented basis of $N_R$, and $m_j, m_{j+1}$ form a positively oriented basis of $M_R$.

Now the Maslov index two disks $h_3, \ldots, h_{\nu+2}$ deform to disks with boundary $P$, which we also call $h_3, \ldots, h_{\nu+2}$. We can then glue together the disks $h_1, \ldots, h_{\nu+2}$ along with $d - n_{j+2}$ copies of the Maslov index two disks with no marked points in the direction $m_{j+2}$. These are glued at their respective outgoing vertices, yielding a vertex $V_{\text{out}}$, and we add one additional unbounded edge $E_x$ with the label $x$, also attached to the vertex $V_{\text{out}}$. This yields a graph $\Gamma$, whose valency at $V_{\text{out}}$ is $\text{Val}(V_{\text{out}}) = \nu + 3 + d - n_{j+2}$. Thus we obtain
a parameterized curve $h : \Gamma \to M_\mathbb{R}$ with $h(x) = P$. Again one easily checks the balancing condition at $V_{out}$.

Thus the total contribution arising in the ways analyzed from $cq^\nu z^m$ to $-L_{P,j}$ is

$$
|m^{\text{prim}}(h_1) \wedge m^{\text{prim}}(h_2)|w_1(E_{out,1})w_2(E_{out,2})\text{Mult}_x(h)\left(\prod_{V \in \Gamma \setminus [0]} \text{Mult}_V(h)\right)q^{v+3d-(2d+n_j+2)}u_{I(h)}
$$

$$
= |m(h_1) \wedge m(h_2)|\text{Mult}_x(h)\left(\prod_{V \in \Gamma \setminus [0]} \text{Mult}_V(h)\right)q^{\text{Val}(V_{out})-3}u_{I(h)}.
$$

One sees that if $I(h) = \{i_1, \ldots, i_{3d-\nu}\}$ for some $\nu'$, then since $h$ is obtained by gluing two Maslov index zero disks with $\text{Val}(V_{out}) - 3$ Maslov index two disks, we have

$$
\text{Val}(V_{out}) - 3 = \sum_{i=1}^{\nu+2}(|\Delta(h_i)| - #I(h_i)) + d - n_j + 2
$$

$$
= 3d - (3d - \nu') = \nu'.
$$

Thus we see that the coefficient of the contributions analyzed above from $cq^\nu z^m$ to $-L_{P,j}$ is precisely the contribution of $h$ to

$$
(5.17) \quad (P_1, \ldots, P_{3d-\nu'}, \psi^{\nu'} M_\mathbb{R})^{\trop}_{d,\sigma_{j+1}} u_{I(h)}q^{\nu'}
$$
as desired.

Conversely, given an $h$ contributing to (5.17) with $h(x) = P$, one can cut it at $P$, using Lemma 3.3, decomposing it into tropical disks. Then we see $h$ arises precisely as above. Thus we see that $-L_{P,j}$ is the contribution to (5.17) from maps with $h(E_x) = P$.

Case 2. $P = P_i$ for some $i$. Again, choose a basepoint $Q'$ near $P_i$. By Remark 4.8, there is a one-to-one correspondence between rays in $\mathcal{D}$ containing $P_i$ and Maslov index two disks with boundary $P_i$ not having $P_i$ as a marked point. With $Q'$ sufficiently near $P_i$, these Maslov index two disks deform to ones with boundary at $Q'$, so the Maslov index two disks with boundary $P_i$ not having $P_i$ as a marked point are in one-to-one correspondence with the Maslov index two disks with boundary $Q'$ not having $P_i$ as a marked point.

If we are interested in terms in $\exp(q(W_k(Q') - W_0(Q')))$, which may contribute to $L_{P,i,j}$, we only need to look at those terms in $\exp(q(W_k(Q') - W_0(Q')))$. We do not have $u_i$ as a factor, as any term that does will not produce any new terms as we cross a ray through $P_i$. So consider a term $cq^\nu z^m$ of the form

$$
(5.18) \quad cq^\nu z^m = q^{\nu} \prod_{p=1}^{\nu} \text{Mult}_p(h_p)z^{\Delta(h_p)}u_{I(h_p)},
$$

where each of these disks $h_p$ with boundary $Q'$ does not pass through $P_i$, and hence corresponds to a disk with boundary $P_i$, which we also write as $h_p : \Gamma'_p \to M_\mathbb{R}$. By extending these disks to trees and marking $P_i$, we obtain Maslov index zero trees, corresponding to
rays $\mathfrak{d}_p$ in $\mathfrak{D}$ with initial point $P_i$. In addition, we have rays $c_p \in \mathfrak{D}$, $p = 0, 1, 2$, with initial point $P_i$, corresponding to the three Maslov index two disks with boundary $Q'$ with no marked points. These do not appear in $W_k(Q') - W_0(Q')$, so are distinct from the $\mathfrak{d}_p$'s.

In what follows, we write $m = \sum_{p=1}^\nu \Delta(h_p) = \sum_{j=0}^2 n_j t_j$, and take $n_{\mathfrak{d}_p}$ and $n_{c_p}$ to have their sign chosen so that they are negative on $\gamma$ when $\gamma$ crosses the corresponding ray. Note that as $\gamma_p$ is counterclockwise, if we use the identification $\wedge^2 M \cong \mathbb{Z}$ given by the standard orientation, i.e., $m_1 \wedge m_2 \mapsto 1$, then $w_{\gamma_p}(E_{\text{out},j}) n_{\mathfrak{d}_p} = X_r(\Delta(h_p))$. So

$$
(5.19) \quad \sum_{p=1}^\nu w_{\gamma_p}(E_{\text{out},j}) n_{\mathfrak{d}_p} = X_r(m).
$$

On the other hand, $\langle n_{\mathfrak{d}_{1,1}}, m_j \rangle = -1$ and $\langle n_{\mathfrak{d}_{1,2}}, m_j \rangle = 1$.

We can now view this term $cq^\nu z^m$ as giving rise to contributions to $L_{P,j}$ in the following four ways:

(I) $\gamma_p$ crosses $\mathfrak{d}_l$ for some $1 \leq l \leq \nu$. Then the term

$$
q^{\nu-1} \prod_{p=1, p \neq j}^\nu \text{Mult}(h_p) z^{\Delta(h_p)} u_I(h_p)
$$

contributes to $L_{P,j}$ if $n_{j+2} \leq n_j = d < n_{j+1}$, in which case the contribution is

$$
\langle n_q, m_j \rangle \left( \prod_{p=1}^\nu \text{Mult}(h_p) u_I(h_p) \right) u_i w_{\gamma_1}(E_{\text{out},j}) D_2(d, m + t_j) q^{\nu+3d-|m|-1}.
$$

Note such a contribution requires $\nu > 0$.

(II) $\gamma_p$ crosses $c_j$. If $c q^\nu z^m$ contributes to $L_{P,j}$ when $\gamma_p$ crosses $c_j$, its contribution would involve a factor of $\langle n_{c_j}, m_j \rangle = 0$, hence there is no contribution.

(III) $\gamma_p$ crosses $c_{j+1}$. We get a contribution from $c q^\nu z^m$ if $n_{j+2} \leq n_j = d \leq n_{j+1}$, in which case the contribution is

$$
\langle n_{c_{j+1}}, m_j \rangle \left( \prod_{p=1}^\nu \text{Mult}(h_p) u_I(h_p) \right) u_i D_2(d, m + t_j + t_{j+1}) q^{\nu+3d-|m|-1}.
$$

(IV) $\gamma_p$ crosses $c_{j+2}$. We get a contribution from $c q^\nu z^m$ if $n_{j+2} < n_j = d < n_{j+1}$, in which case we get

$$
\langle n_{c_{j+2}}, m_j \rangle \left( \prod_{p=1}^\nu \text{Mult}(h_p) u_I(h_p) \right) u_i D_2(d, m + t_j + t_{j+2}) q^{\nu+3d-|m|-1}.
$$

We now consider three cases.

Case 2(a). $n_{j+2} \leq n_j = d < n_{j+1}$, $\nu > 0$. In this case ignoring the common factors

$$
q^{\nu+3d-|m|-1} u_i \prod_{p=1}^\nu \text{Mult}(h_p) u_I(h_p),
$$

...
the total contribution is, using (5.19) and Lemma 5.2

\[
\left\langle \sum_{l=1}^{\nu} w_{r_{l}}(E_{out,l})n_{q_{l}}, m_{j} \right\rangle D_{2}(d, m + t_{j}) - D_{2}(d, m + t_{j} + t_{j+1}) + \begin{cases} D_{2}(d, m + t_{j} + t_{j+2}) & n_{j+2} < d \\ 0 & n_{j+2} = d \end{cases}
\]

\[
= r(m) \wedge m_{j}(-1)^{n_{j}+n_{j+1}+1} \frac{(n_{j} - d)! (n_{j+1} - d - 1)!}{(d - n_{j+2})!} \]

\[-(-1)^{n_{j}+n_{j+1}+2} \frac{(n_{j} - d)! (n_{j+1} - d)!}{(d - n_{j+2})!} \]

\[+ (-1)^{n_{j}+n_{j+1}+1} \frac{(n_{j} - d)! (n_{j+1} - d - 1)!}{(d - n_{j+2})!} (d - n_{j+2}) \]

\[= \left( (n_{j+2} - n_{j+1}) + (n_{j+1} - d) + (d - n_{j+2}) \right) \left( -1 \right)^{n_{j}+n_{j+1}+1} \frac{(n_{j} - d)! (n_{j+1} - d - 1)!}{(d - n_{j+2})!} \]

\[= 0. \]

So there is no contribution to \( L_{P,j} \) from this case.

Case 2(b). \( n_{j+2} \leq n_{j} = d = n_{j+1}, \nu > 0 \). In this case we only get a contribution from (III). In this case, we can glue together the disks \( h_{1}, \ldots, h_{\nu} \) along with \( d - n_{j+2} \) copies of the Maslov index two disk with no marked points in the direction \( m_{j+2} \). These are glued at their respective outgoing vertices, yielding a vertex \( V_{out} \), and we add two additional marked unbounded edges \( E_{x} \) and \( E_{p_{j}} \) for some \( l \) attached to \( V_{out} \). This yields a graph \( \Gamma \), whose valency at \( V_{out} \) is \( \text{Val}(V_{out}) = \nu + (d - n_{j+2}) + 2 = \nu + 3d - |m| + 2 \). Thus we obtain a parameterized curve \( h: \Gamma \rightarrow M_{R} \) with \( h(V_{out}) = h(x) = h(p_{l}) = P_{l} \). The contribution to \(-L_{P,j}\) is then easily seen by inspection to be

\[
(5.20) \quad \text{Mult}_{x_{1}}(h) \left( \prod_{l \in [0]} \text{Mult}_{x_{l}}(h) \right) u_{I(h)}q^{\text{Val}(V_{out})-3}. 
\]

Suppose that \( I(h) = \{i_{1}, \ldots, i_{3d-\nu'}\} \) for some \( \nu' \), recalling \( i \in I(h) \) since we added the marked edge \( E_{p_{l}} \) mapping to \( P_{l} \). Since \( h \) is obtained by gluing \( \text{Val}(V_{out}) - 2 \) Maslov index two disks, we have

\[
\text{Val}(V_{out}) - 2 = \sum_{i=1}^{\nu} (|\Delta(h_{i})| - \#I(h_{i})) + d - n_{j+2}
\]

\[= 3d - (3d - \nu' - 1) = \nu' + 1. \]

Thus we see that (5.20) is precisely the contribution of \( h \) to

\[
\langle P_{i_{1}}, \ldots, P_{i_{3d-\nu'}}, \psi^{\nu'} M_{R}^{\text{trop}} \rangle_{d, \sigma_{j+1}}^{\text{trop}} u_{I(h)}q^{\nu'}
\]
from Definition 3.4 (3) (b). As in the other cases we have considered, conversely any such curve \( h \) will give rise to the correct monomial \( cq^\nu z^m \) by cutting the curve at \( P \).

Case 2(c). \( \nu = 0 \). There is only one element \((c, \nu, m) \in S_k \) with \( \nu = 0 \), namely \((1, 0, 0) \) corresponding to the constant monomial 1. So \( n_0 = n_1 = n_2 = 0 \) and we have no contribution unless \( d = 0 \). Again, this contribution to \( L_{P, j} \) only arises from (III), and is

\[
\langle n_{\ell+1}, m_j \rangle u_i D_2(0, t_j + t_{j+1}) q^{-1} = -u_i q^{-1}.
\]

This gives the remaining claimed terms in \(-L_{P, j} \).

We finally have:

\[\text{Proof of Theorems 3.6 and 3.8.} \] By Lemma 5.6, it is enough to prove (5.2), (5.3) and (5.4). We have (5.2) by Lemma 5.7. The remaining two equations follow from Lemmas 5.11 (2), 5.15, 5.16 and 5.17. \(\square\)

6. Generalizations

I would like to comment briefly on possible generalizations of the results of this paper. One expects the version of mirror symmetry considered in this paper to hold more broadly for any non-singular projective toric variety. It should be no particular problem generalizing these results to \( \mathbb{P}^n \); one simply includes collections of linear subspaces of all codimensions \( \geq 2 \) to obtain the necessary perturbations. This produces higher dimensional scattering diagrams, and the arguments of \( \S 5 \) should go through, although the formulas will get progressively more complicated.

More of an issue is generalizing to other toric varieties. As mentioned in Remark 2.8, a basic problem with tropical geometry as currently understood is that it can’t account for curves with irreducible components mapping into the toric boundary. Even for surfaces, the only cases in which tropical geometry correctly computes quantum cohomology is \( \mathbb{P}^2 \) and \( \mathbb{P}^1 \times \mathbb{P}^1 \); in particular, the arguments of this paper should work equally well for \( \mathbb{P}^1 \times \mathbb{P}^1 \). To go further, one needs to improve the tropical understanding of these issues, which we leave for others. But the expectation is that once one correctly counts possible disks, then the methods of this paper should still work.

It is also possible to combine the ideas in this paper of using tropical geometry to count Maslov index two disks with the techniques of [17]. If one starts with an integral affine manifold with singularities which is non-compact, then one obtains something like a scattering diagram governing a degeneration of non-compact varieties. This data is called a structure in [17]. One expects the structure to encode Maslov index zero disks in the mirror. Then one can again describe a Landau-Ginzburg potential in terms of broken lines. This will be taken up elsewhere.
References

[1] D. Auroux: Mirror symmetry and T-duality in the complement of an anticanonical divisor, J. Gökova Geom. Topol. GGT 1 (2007), 51–91.
[2] S. Barannikov: Semi-infinite Hodge structures and mirror symmetry for projective spaces, preprint, 2000.
[3] K. Chan, N.-C. Leung: Mirror symmetry for toric Fano manifolds via SYZ transformations, preprint, 2008.
[4] C.-H. Cho, Y.-G. Oh: Floer cohomology and disc instantons of Lagrangian torus fibers in Fano toric manifolds, Asian J. Math. 10 (2006), 773–814.
[5] D. Cox, S. Katz: Mirror symmetry and algebraic geometry, Mathematical Surveys and Monographs, 68. American Mathematical Society, Providence, RI, 1999.
[6] A. Doan, C. Sabbah: Gauss-Manin systems, Brieskorn lattices and Frobenius structures. I, Proceedings of the International Conference in Honor of Frédéric Pham (Nice, 2002). Ann. Inst. Fourier (Grenoble) 53 (2003), 1055–1116.
[7] A. Doan, C. Sabbah: Gauss-Manin systems, Brieskorn lattices and Frobenius structures. II, Frobenius manifolds, 1–18, Aspects Math., E36, Vieweg, Wiesbaden, 2004.
[8] B. Dubrovin: Geometry of 2D topological field theories, Integrable systems and quantum groups (Montecatini Terme, 1993), 120–348, Lecture Notes in Math., 1620, Springer, Berlin, 1996.
[9] K. Fukaya, Y.-G. Oh, H. Ohta, K. Ono: Lagrangian Floer theory on compact toric manifolds II: Bulk deformations, preprint, 2008.
[10] W. Fulton, Introduction to toric varieties, Annals of Mathematics Studies, 131, The William H. Roever Lectures in Geometry. Princeton University Press, Princeton, NJ, 1993. xii+157 pp.
[11] A. Gathmann, H. Markwig: Kontsevich’s formula and the WDVV equations in tropical geometry, Adv. Math. 217 (2008) 537–560.
[12] A. Givental: Homological geometry and mirror symmetry, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), 472–480, Birkhäuser, Basel, 1995.
[13] M. Gross: The Strominger-Yau-Zaslow conjecture: From torus fibrations to degenerations, to appear in Proceedings of Symposia of Pure Mathematics, Seattle 2005.
[14] M. Gross, B. Siebert: Affine manifolds, log structures, and mirror symmetry, Turkish J. Math. 27 (2003), 33–60.
[15] M. Gross, B. Siebert: Mirror symmetry via logarithmic degeneration data I, J. Differential Geom. 72 (2006), 169–338.
[16] M. Gross, B. Siebert: Mirror symmetry via logarithmic degeneration data II, preprint 2007.
[17] M. Gross, B. Siebert: From real affine geometry to complex geometry, preprint, 2007.
[18] M. Gross, R. Pandharipande, B. Siebert: The tropical vertex, preprint, 2009.
[19] M. Hien: Periods for flat algebraic connections, preprint, 2008.
[20] H. Iritani: Quantum D-modules and generalized mirror transformations, Topology 47 (2008), 225–276.
[21] H. Iritani: An integral structure in quantum cohomology and mirror symmetry for toric orbifolds, preprint, 2009.
[22] L. Katzarkov, M. Kontsevich, T. Pantev: Hodge theoretic aspects of mirror symmetry, preprint, 2008.
[23] M. Kontsevich, Y. Soibelman: *Affine structures and non-Archimedean analytic spaces*, in: *The unity of mathematics* (P. Etingof, V. Retakh, I.M. Singer, eds.), 321–385, Progr. Math. 244, Birkhäuser 2006.

[24] M. Kontsevich, Y. Soibelman: *Stability structures, motivic Donaldson-Thomas invariants and cluster transformations*, preprint, 2008.

[25] H. Markwig, J. Rau: *Tropical descendent Gromov-Witten invariants*, preprint, 2008.

[26] G. Mikhalkin: *Enumerative tropical algebraic geometry in \( \mathbb{R}^2 \)*, J. Amer. Math. Soc. 18 (2005), 313–377.

[27] T. Nishinou: *Disc counting on toric varieties via tropical curves*, preprint (2006).

[28] T. Nishinou, B. Siebert: *Toric degenerations of toric varieties and tropical curves*, Duke Math. J. 135 (2006), 1–51.

[29] C. Sabbah: *Déformations isomonodromiques et variétés de Frobenius*, Savoirs Actuels (Les Ulis). Mathématiques (Les Ulis), EDP Sciences, Les Ulis; CNRS Éditions, Paris, 2002. xvi+289 pp.

[30] J. Richter-Gebert, B. Sturmfels, T. Theobald: *First steps in tropical geometry*, in Idempotent mathematics and mathematical physics, 289–317, Contemp. Math., 377, Amer. Math. Soc., Providence, RI, 2005.

[31] K. Saito: *Period mapping associated to a primitive form*, Publ. RIMS, Kyoto Univ. 19, (1983), 1231-1264.

[32] A. Strominger, S.-T. Yau, and E. Zaslow: *Mirror Symmetry is T-duality*, Nucl. Phys. B479 (1996), 243–259.

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