Abstract

In this paper we prove local in time well-posedness for the incompressible Euler equations in $\mathbb{R}^n$ for the initial data in $\mathcal{L}_{1(1)}^1(\mathbb{R}^n)$, which corresponds to a critical case of the generalized Campanato spaces $\mathcal{L}_{q(N)}^s(\mathbb{R}^n)$. The space is studied extensively in our companion paper[9], and in the critical case we have embeddings $B_{\infty,1}^1(\mathbb{R}^n) \hookrightarrow \mathcal{L}_{1(1)}^1(\mathbb{R}^n) \hookrightarrow C^{0,1}(\mathbb{R}^n)$, where $B_{\infty,1}^1(\mathbb{R}^n)$ and $C^{0,1}(\mathbb{R}^n)$ are the Besov space and the Lipschitz space respectively. In particular $\mathcal{L}_{1(1)}^1(\mathbb{R}^n)$ contains non-$C^1(\mathbb{R}^n)$ functions as well as linearly growing functions at spatial infinity. We can also construct a class of simple initial velocity belonging to $\mathcal{L}_{1(1)}^1(\mathbb{R}^n)$, for which the solution to the Euler equations blows up in finite time.

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1 Introduction

Let $0 < T < +\infty$ and $Q_T = \mathbb{R}^n \times (0,T)$ with $n \in \mathbb{N}, n \geq 2$. We consider the homogeneous incompressible Euler equations

\begin{equation}
\begin{aligned}
\partial_t v + (v \cdot \nabla)v &= -\nabla p \quad \text{in} \quad Q_T, \\
\nabla \cdot v &= 0 \quad \text{in} \quad Q_T,
\end{aligned}
\end{equation}

equipped with the initial condition

\begin{equation}
v = v_0 \quad \text{on} \quad \mathbb{R}^n \times \{0\},
\end{equation}

where $v = (v_1, \ldots, v_n) = v(x,t)$ represents the velocity of the fluid flows, and $p = p(x,t)$ denotes the scalar pressure. The system of Euler equations is of fundamental importance in the mathematical fluid mechanics (see e.g. books \cite{19, 18, 1} or survey paper \cite{12}). Therefore, many authors studied the local well-posedness/ill-posedness of the Cauchy problem (1.1)-(1.2) in various function spaces \cite{14, 15, 26, 24, 25, 13, 21, 17, 3, 6, 2, 17, 10, 11, 20}. In particular it is shown that the system (1.1)-(1.2) is locally well-posed in the critical Besov space $B^1_{\infty,1}(\mathbb{R}^n)$ \cite{21, 17}, but ill-posed in the Lipschitz space $C^{0,1}(\mathbb{R}^n)$ \cite{15}.

Our aim in this paper is to show the local well-posedness in a critical generalized Campanato space, which is embedded into $C^{0,1}(\mathbb{R}^n)$, but larger than the Besov space $B^1_{\infty,1}(\mathbb{R}^n)$. Furthermore, our function space include linearly growing functions at infinity as well as bounded functions. Furthermore it also contains non-$C^1(\mathbb{R}^n)$ functions as shown in our companion paper \cite{9}.

At a first glance one may think it is impossible to get result of local well-posedness in such function spaces due to the following example.

\begin{equation}
v(x,t) = \frac{T_s}{T_s - t}(x_1, -x_2)^\top, \quad (x,t) \in \mathbb{R}^2 \times (0, T_s),
\end{equation}

which solves (1.1) with $v_0(x) = (x_1, -x_2)^\top$ and $p(x,t) = \frac{1}{2(T_s - t)^2}(x_1^2 - x_2^2 + T_s(x_1^2 + x_2^2))$. Since $T_s > 0$ can be chosen arbitrarily small independent of the size of $v_0$, the Euler equations with linear growing initial data is in general ill-posed.

We observe that in the above solution one has freedom to choose the pressure with quadratic growth depending on both the time derivative of $v$ and the convection term.
In order to avoid such pathological case we shall restrict our class of solutions by imposing extra condition on choice of the pressure. More specifically will introduce a pressure operator \( \Pi = \Pi(v, v) \) such that possible linear growing solutions to (1.1) with \( \nabla p = \nabla \Pi \) are determined uniquely.

In this paper we call a pair \((v, p)\) a solution to the Euler equations if \((v, p) \in L^\infty(0, T; L^2_{\text{loc}}(\mathbb{R}^n)) \times L^\infty(0, T; L^2_{\text{loc}}(\mathbb{R}^n))\), both \(\nabla v\) and \(D^2 p\) are bounded in \(Q_T\), and (1.1) holds a.e. in \(Q_T\).

We start our discussion with the following notion of equivalent solutions.

**Definition 1.1.**

1. Two solutions \((v_1, p_1)\) and \((v_2, p_2)\) are called equivalent to each other \(v_1, p_1 \sim v_2, p_2\), if there exists \(\xi \in C^{1,1}([0, T]; \mathbb{R}^n)\) such that for almost every \((x, t) \in Q_T\)

\[
v_2(x, t) = v_1(x + \xi(t), t) - \dot{\xi}(t), \quad \nabla p_2(x, t) = \nabla p_1(x + \xi(t), t) - \ddot{\xi}(t).
\]

2. A solution \((v, p)\) is called centered, if

\[
(1.4) \quad v(0, t) = 0 \quad \forall t \in [0, T].
\]

3. We say a solution \((v, p)\) decays as \(|x| \to +\infty\), if there exists \((u, q) \sim (v, p)\) such that

\[
\frac{1}{r^n} \int_{B(r)} |u(x, t)| \, dx \to 0 \quad \text{as} \quad r \to +\infty \quad \text{for all} \quad t \in (0, T),
\]

where \(B(r)\) denotes the ball with radius \(r\), with its center at the origin.

**Remark 1.2.**

1. Clearly, the relation \(\sim\) between two solutions to the Euler equations defines an equivalence relation. Given a solution \((v, p)\) to (1.1) the set \([[(v, p)]\) containing all solutions to (1.1) which are equivalent to \((v, p)\) forms the unique equivalence class, which in particular contains \((v, p)\). Furthermore, each equivalence class \([[(v, p)]\) contains a centered solution. Indeed, we may find a solution \(\xi \in C^{1,1}([0, T]; \mathbb{R}^n)\) to the ordinary differential equations

\[
\dot{\xi}(t) = v(\xi(t), t), \quad t \in (0, T).
\]

Setting

\[
V(x, t) = v(x + \xi(t), t) - \dot{\xi}(t), \quad P(x, t) = p(x + \xi(t), t) - \ddot{\xi}(t) \cdot x, \quad t \in (0, T),
\]

it is obvious that \((V, P)\) is centered and \((V, P) \sim (v, p)\).

2. As an example of non-equivalent solutions in \(\mathbb{R}^2\) we consider \((v, p)\) and \((u, q)\) both satisfying the same initial condition (1.2), and defined by

\[
v(x, t) = (x_1 + x_2, x_1 - x_2), \quad -\nabla p(x, t) = (2x_1, 2x_2),
\]

\[
u(x, t) = (x_1 + e^t x_2, e^t x_1 - x_2), \quad -\nabla q(x, t) = ((e^{2t} + 1)x_1 + e^t x_2, (e^{2t} - 1)x_2 + e^t x_1).
\]

This example also shows that we cannot expect uniqueness in the class of solutions with linear growth at infinity without restriction of the pressure as mentioned above.

3. Let \((v, p)\) be a solution to (1.1) the fixed properties above. Suppose that \(v(t) \in L^2(\mathbb{R}^n)\) for all \(t \in [0, T]\), then it holds that \(\|v(t)\|_2 = \|v(0)\|_2\) for all \(t \in (0, T)\).
Indeed, by interpolating between $v \in L^\infty(0, T; L^2(\mathbb{R}^n))$ and $\nabla v \in L^\infty(Q_T)$ one has $v \in L^3(0, T; L^{\frac{3n}{n+3}}(\mathbb{R}^n))$, in which class we can perform integration by part in the convection term and the pressure term to make them vanish, and finally to get the desired energy conservation.

Let us introduce the spaces we will use throughout the paper. Let $N \in \mathbb{N} \cup \{0\} := \mathbb{N}_0$. By $\mathcal{P}_N$ ($\mathcal{P}_N^*$ respectively), denotes the space of all polynomial (all homogenous polynomials respectively) of degree less or equal $N$. We equip the space $\mathcal{P}_N$ with the norm $\|P\|_{(p)} = \|P\|_{L^p(B(1))}$. Note that since dim($\mathcal{P}_N$) $< +\infty$ all norms $\| \cdot \|_{(p)}$, $1 \leq p \leq +\infty$, are equivalent.

Let $f \in L^2_{loc}(\mathbb{R}^n), 1 \leq p \leq +\infty$. For $x_0 \in \mathbb{R}^n$ and $0 < r < \infty$ we define the oscillation

$$\text{osc}_{p,N}(f; x_0, r) := |B(r)|^{-\frac{1}{p}} \inf_{P \in \mathcal{P}_N} \|f - P\|_{L^p(B(x_0, r))}.$$  

Then, we define for $1 \leq q, p \leq +\infty$ and $s \in [0, N + 1)$ the spaces

$$\mathcal{L}_{q,p,N}(\mathbb{R}^n) = \left\{ f \in L_{loc}^p(\mathbb{R}^n) \mid |f|_{\mathcal{L}_{q,p,N}} := \sup_{x \in \mathbb{R}^n} \left( \sum_{j \in \mathbb{Z}} \left( 2^{-sj} \text{osc}_{p,N}(f; x, 2^j) \right)^s \right)^{\frac{1}{s}} < +\infty \right\},$$

Furthermore, by $\mathcal{L}^{k,s}_{q,p,N}(\mathbb{R}^n)$, $k \in \mathbb{N}$, we denote the space of all $f \in W_{loc}^{k,p}(\mathbb{R}^n)$ such that $D^k f \in \mathcal{L}^{s}_{q,p,N}(\mathbb{R}^n)$. The space $\mathcal{L}^{k,s}_{q,p,N}(\mathbb{R}^n)$ will be equiped with the norm

$$\|f\|_{\mathcal{L}^{k,s}_{q,p,N}} = \|D^k f\|_{\mathcal{L}^{s}_{q,p,N}} + \|f\|_{L^p(B(1))}, \quad f \in \mathcal{L}^{k,s}_{q,p,N}(\mathbb{R}^n).$$

Note that the oscillation introduced above is attained by a unique polynomial $P_\ast \in \mathcal{P}_N$. Below we recall basic properties on this space. According to the characterization theorem of the Triebel-Lizorkin spaces in terms of oscillation(cf. [23, Theorem, Chap. 1.7.3]), we have

$$\begin{aligned}
\left\{ \begin{array}{l}
f \in F^{s}_{r,q}(\mathbb{R}^n) \iff \|f\|_{L_{min}(r,q)} + \left( \sum_{j = -\infty}^0 \left( 2^{-sj} \text{osc}_{p,N}(f; \cdot, 2^j) \right)^q \right)^{\frac{1}{q}} \|L^r < +\infty.
0 < r < +\infty, 0 < q \leq \infty, \quad s > \left( \frac{1}{r} - \frac{1}{q} \right)_+, \quad s > \left( \frac{1}{q} - \frac{1}{p} \right)_+,
\end{array} \right.
\end{aligned}$$

and we could regard the spaces $\mathcal{L}^{s}_{q,p,N}(\mathbb{R}^n)$ as an extension of the limit case of $F^{s}_{r,q}(\mathbb{R}^n)$ as $r \to +\infty$. In case $q = +\infty$ and $s > 0$ we get the usual Campanato spaces with the isomorphism relation(cf. [2] [13])

$$\mathcal{L}_{N}^{0+p,q,N}(\mathbb{R}^n) \cong \mathcal{L}_{\infty,p,N}(\mathbb{R}^n).$$

Furthermore, in the case $N = 0$, $s = 0$ and $q = \infty$ we get the space of bounded mean oscillation, i.e.,

$$\mathcal{L}_{\infty,p,0}(\mathbb{R}^n) \cong BMO.$$
In case \( N = -1 \) and \( s \in (-\frac{n}{p}, 0) \) the above space coincides with the usual Morrey space \( \mathcal{M}^{n+ps}(\mathbb{R}^n) \). Our aim in this paper is to prove the local well-posedness of the Euler equations in the critical space \( \mathcal{L}^{1,(p,1)}(\mathbb{R}^n) \). We recall the following embedding relations (see \([9]\)).

\[
B^{1+\frac{n}{p}}_{r,1} \hookrightarrow B^{1}_{\infty,1}(\mathbb{R}^n) \hookrightarrow \mathcal{L}^{1,(p,1)}(\mathbb{R}^n) \hookrightarrow C^{0,1}(\mathbb{R}^n).
\]

Accordingly,

\[
\|\nabla u\|_\infty \leq c \|u\|_{\mathcal{L}^{1,(p,1)}}.
\]

Furthermore, for every \( f \in \mathcal{L}^{k,(p,k)}(\mathbb{R}^n), k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), there exists a unique \( \hat{P}_\infty^k(f) \in \hat{P}_k \), such that for all \( x_0 \in \mathbb{R}^n \)

\[
f \text{ converges asymptotically to } \hat{P}_\infty^k(f) \text{ as } |x| \to +\infty.
\]

The exact meaning of this asymptotic limit is stated in Theorem 2.6 (see also \([9, \text{ Section 2}]\)).

We also introduce the following critical homogenous space

\[
\hat{\mathcal{L}}^{1,(p,1)}(\mathbb{R}^n) = \left\{ u \in \mathcal{L}^{1,(p,1)}(\mathbb{R}^n) \left| u(0) = 0 \right. \right\}.
\]

The space \( \hat{\mathcal{L}}^{1,(p,1)}(\mathbb{R}^n) \) will be equipped with the homogenous norm

\[
\|u\|_{\hat{\mathcal{L}}^{1,(p,1)}} = |u|_{\mathcal{L}^{1,(p,1)}} + |P_\infty^0(\nabla u)|, \quad u \in \hat{\mathcal{L}}^{1,(p,1)}(\mathbb{R}^n).
\]

We recall that \( u \in \hat{\mathcal{L}}^{1,(p,1)}(\mathbb{R}^n) \) implies \( \nabla u \in \mathcal{L}^{0,(p,0)}(\mathbb{R}^n) \).

By \( \mathcal{L}^{1,\sigma}(\mathbb{R}^n) \) we denote the subspace of all \( u \in \mathcal{L}^{1,(p,1)}(\mathbb{R}^n) \) such that \( \nabla \cdot u = 0 \) almost everywhere in \( \mathbb{R}^n \). Next, we focus on the pressure \( p \), which satisfies the Poisson equation

\[
-\Delta p = \nabla v : (\nabla v)^T \quad \text{in } \mathbb{R}^n.
\]

In contrast to the decaying case this problem for \( \nabla p \) is not well posed in the space \( \mathcal{L}^{1,(p,1)}(\mathbb{R}^n) \), since if \( \nabla p \in \mathcal{L}^{1,(p,1)}(\mathbb{R}^n) \) solves (1.8), then the function \( p + Q \) for any \( Q \in \mathcal{P}_1 \) also solves it. The same problem occurs for the general Poisson equation. In order to have uniqueness of solution we add an asymptotic condition for \( \nabla p \) as \( |x| \to +\infty \) together with a condition at one point. We have the following

**Theorem 1.3.** For every matrix \( H = \{H_{\alpha \beta}\} \in \mathcal{L}^{1,(p,1)}(\mathbb{R}^n), a_0 \in \mathbb{R} \) and \( Q_\infty \in \hat{P}_1 \) there exists a unique solution \( f \in \mathcal{L}^{1,(p,1)}(\mathbb{R}^n) \) to the problem

\[
\begin{cases}
-\Delta f = \nabla \cdot \nabla \cdot H & \text{in } \mathbb{R}^n, \\
f(0) = a_0, \quad \hat{P}_\infty^1(f) = Q_\infty.
\end{cases}
\]

Furthermore, there exists a constant \( c = c(n, p) \) such that

\[
|f|_{\mathcal{L}^{1,(p,1)}} \leq c |H|_{\mathcal{L}^{1,(p,1)}}, \quad \|f\|_{\mathcal{L}^{1,(p,1)}} \leq c (|H|_{\mathcal{L}^{1,(p,1)}} + |a_0| + \|Q_\infty\|).
\]
The proof of Theorem 1.3 is based on Theorem 3.4 given in Section 3.

Next, we discuss the problem of defining the pressure. We first define an operator \( \nabla \Pi : \mathcal{L}^{1,1}_{1(p,1),\sigma}(\mathbb{R}^n) \times \mathcal{L}^{1,1}_{1(p,1)}(\mathbb{R}^n) \to \mathcal{L}^{1,1}_{1(p,1)}(\mathbb{R}^n) \) as follows. This definition is based on the following theorem, which is an immediate consequence of Theorem 3.5 (see also Remark 3.7)

**Theorem 1.4.** Let \( (u, v) \in \mathcal{L}^{1,1}_{1(p,1),\sigma}(\mathbb{R}^n) \times \mathcal{L}^{1,1}_{1(p,1)}(\mathbb{R}^n) \). There exists a function \( \pi \in \mathcal{L}^{1,1}_{1(p,1)}(\mathbb{R}^n) \) solving the Poisson equation

\[
\begin{cases}
-\Delta \pi = \nabla \cdot \nabla \cdot (u \otimes v) & \text{in } \mathbb{R}^n, \\
\nabla \pi(0) = 0, \quad \dot{P}_\infty(\nabla \pi) = -\frac{1}{n} P_\infty(\nabla u : (\nabla v)^\top) \mathbf{x},
\end{cases}
\]

which is unique up to a constant.

Now we are ready to introduce the following definition.

**Definition 1.5.** Let \( (u, v) \in \mathcal{L}^{1,1}_{1(p,1),\sigma}(\mathbb{R}^n) \times \mathcal{L}^{1,1}_{1(p,1)}(\mathbb{R}^n) \). Then by \( \nabla \Pi(u, v) \) we denote the unique function \( \nabla \pi \in \mathcal{L}^{1,1}_{1(p,1)}(\mathbb{R}^n) \), where \( \pi \in \mathcal{L}^{1,1}_{1(p,1)}(\mathbb{R}^n) \) stands for the solution of (1.11) according to Theorem 1.4.

In particular, in view of (3.49) and (3.50) (cf. Section 3) it holds

\[
\|\nabla \Pi(u, v)\|_{\mathcal{L}^{1,1}_{1(p,1)}} \leq c \left( \|u\|_{\mathcal{L}^{1,1}_{1(p,1)}} \|\nabla v\|_\infty + \|v\|_{\mathcal{L}^{1,1}_{1(p,1)}} \|\nabla u\|_\infty \right).
\]

We are now in a position to present our first main result.

**Theorem 1 (Local well posedness in \( \mathcal{L}^{1,1}_{1(p,1)}(\mathbb{R}^n) \)).** For every \( v_0 \in \mathcal{L}^{1,1}_{1(p,1),\sigma}(\mathbb{R}^n) \) there exists

\[
T_0 \geq \frac{1}{c\|v_0\|_{\mathcal{L}^{1,1}_{1(p,1)}}},
\]

and a unique solution \( v \in L^\infty(0, T_0; \mathcal{L}^{1,1}_{1(p,1),\sigma}(\mathbb{R}^n)) \) to (1.1), (1.2) in \( Q_{T_0} \) with pressure \( \pi \in L^\infty(0, T_0; L^2_{loc}(\mathbb{R}^n)) \) such that \( \nabla \pi \in L^\infty(0, T_0; \mathcal{L}^{1,1}_{1(p,1)}(\mathbb{R}^n)) \) and satisfying

\[
\nabla \pi = \nabla \Pi(v, v).
\]

**Remark 1.6.** In case of sublinear growing solutions the condition (1.14) is automatically satisfied for the function \( \nabla \pi(x, t) = \nabla p(x, t) - \nabla p(0, t) \), if \( (v, p) \) solves the Euler equations, using the arguments in Section 4.

Using the Galilean transform \( (x, t) = (y + ta, t), a \in \mathbb{R}^n, (x, t) \in Q_{T_0}, \) with \( a = -v_0(0) \), we obtain the following local well posedness in \( \mathcal{L}^{1,1}_{1(p,1)}(\mathbb{R}^n) \).
Theorem 2 (Local well posedness in $\mathcal{L}^1_{1(p,1)}(\mathbb{R}^n)$). For every $v_0 \in \mathcal{L}^1_{1(p,1)}(\mathbb{R}^n)$ with $\nabla \cdot v_0 = 0$ there exists

$$T_0 \geq \frac{1}{c\|v_0 - v_0(0)\|_{\mathcal{L}^1_{1(p,1)}}},$$

and a unique solution $v \in L^\infty(0,T_0; \mathcal{L}^1_{1(p,1)}(\mathbb{R}^n))$ to (1.1)-(1.2) in $Q_{T_0}$ with pressure $p \in L^\infty(0,T_0; L^2_{\text{loc}}(\mathbb{R}^n))$ such that $\nabla p \in L^\infty(0,T_0; \mathcal{L}^1_{1(p,1)}(\mathbb{R}^n))$ and for almost all $t \in (0,T)$

$$\nabla p(\cdot, t) - \nabla p(0, t) = \nabla \Pi(v(t), v(t)), \quad \nabla p(0(0), t, t) = 0.$$

Remark 1.7. In fact, our main result stated in Theorem 2 improves substantially previous result in [21] in both directions, in the sense of regularity and asymptotic behavior at infinity in space. Firstly, we recall that by (1.5)

$$B^1_{\infty,1}(\mathbb{R}^n) \hookrightarrow \mathcal{L}^1_{1(p,1)}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \hookrightarrow \mathcal{L}^1_{1(p,1)}(\mathbb{R}^n).$$

Secondly, according to [22, p. 85], (see also [1]) we have the embedding

$$B^1_{\infty,1}(\mathbb{R}^n) \hookrightarrow C^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n).$$

On the other hand, there exists a function $f \in \mathcal{L}^1_{1(p,1)}(\mathbb{R}^n)$, which is not in $C^1(\mathbb{R}^n)$ (see in [9] appendix). Consequently, $\mathcal{L}^1_{1(p,1)}(\mathbb{R}^n)$ contains less regular functions then $B^1_{\infty,1}(\mathbb{R}^n)$. In particular, this implies that we have local well posedness of the Euler equations for initial data not in $C^1(\mathbb{R}^n)$.

Thirdly, since $\mathcal{L}^1_{1(p,1)}(\mathbb{R}^n)$ contains linear growing functions, and therefore polynomials of degree less or equal one, $\mathcal{L}^1_{1(p,1)}(\mathbb{R}^n)$ is strictly larger then $B^1_{\infty,1}(\mathbb{R}^n)$ in the sense of asymptotic behavior at spatial infinity.

Next, we introduce the notion of equivalent solutions by using the change of coordinates $(x, t) = (x + \xi(t), t)$ for a given function $\xi \in C^{1,1}([0,T]; \mathbb{R}^n)$.

Definition 1.8. A solution $(v, p) \in L^\infty(0,T_0; \mathcal{L}^1_{1(p,1)}(\mathbb{R}^n) \times L^2_{\text{loc}}(\mathbb{R}^n))$ to the Euler equations (1.1) is called eligible if there exists a centered solution $(V, P) \sim (v, p)$ with

$$\nabla P(t) = \nabla \Pi(V(t), V(t)) \quad \forall t \in [0,T].$$

Remark 1.9. As an example of non-eligible solutions we have solutions in (1.3). In general, we may look for solutions $v(t) = A(t)x$, where $A \in \mathbb{R}^{n \times n}$ stands for a matrix $\text{trace}(A) = 0$. In (1.1) replacing $v$ by $Ax$ we obtain the equation

$$\partial_t Ax + A_2 = -\nabla \pi \quad \text{in} \quad Q_T.$$

The compatibility condition (1.14) yields $\nabla \pi = -\frac{1}{n} \text{trace}(A_2^2)x$. Inserting this identity into (1.19) and applying $\nabla$ to the resultant equations, we are led to the system of ODE's

$$\dot{A} + A_2 = \frac{1}{n} \text{trace}(A_2^2)I \quad \text{in} \quad [0,T].$$
This system, under the initial condition $A(0) = A_0$ has a unique local solution $A \in C^1([0,T)); \mathbb{R}^{n \times n})$. Applying trace to (1.20), we deduce that $\frac{d}{dt} \text{trace}(A) = 0$, which together with trace$(A_0) = 0$, yields trace$(A) = 0$. This shows that $v = Ax$ is a centered solution of the Euler equations, and by Theorem 1 this solution is unique. The following examples show the global existence and finite time blow up depending on the initial data in $\mathcal{L}^1_{1(p,1)}(\mathbb{R}^n)$.

1. **Global existence in $n = 2$**. Let $v_0(x) = A_0 x$, where $A_0 = \text{diag}(\lambda_{0,1}, \lambda_{0,2})$. The condition $\nabla \cdot v_0 = 0$ implies $\lambda_{0,2} = -\lambda_{0,1}$. Then the centered solution $v$ to (1.20) must be given by $v = Ax$, where $A$ solves (1.20). However, noting that $\lambda_{0,1}^2 = \frac{1}{2} \text{trace} A_0^2$, it readily seen that $A \equiv A_0$ is a global unique solution to (1.20) and $v = A_0 x$ is the global centered solution to (1.1).

2. **Global existence and finite time blow up in $n = 3$**. We begin our discussion with the global existence. Let $v_0 = A_0 x$ be given with $A_0 = \text{diag}(\lambda_{0,1}, \lambda_{0,3}, \lambda_{0,2})$ with $\sum_{i=1}^3 \lambda_{0,i} = 0$ such that all eigen values $\lambda_{0,i}$ are different. Then the solution $A$ to (1.20) has the form $A = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$ such that $\sum_{i=1}^3 \lambda_i = 0$. By $T_* > 0$ we denote the maximal time of existence of this solution, i.e. $\lambda_i$ solve the system of ODE

\begin{equation}
\dot{\lambda}_i + \lambda_i^2 = \frac{1}{3} |\lambda|^2 \quad \text{in} \quad (0, T_*), \quad i = 1, 2, 3.
\end{equation}

We claim $T_* = +\infty$. To see this, first we verify that all eigen values $\lambda_i(t)$ are different for all $t \in (0, T_*)$. In fact, in view of (1.21), the function $\mu = \lambda_i - \lambda_j$ for $i \neq j$ solves the ODE $\dot{\mu} + (\lambda_i + \lambda_j)\mu = 0$ in $(0, T_*)$. In case $\mu(t) = 0$ for some $t \in (0, T_*)$ it follows that $\mu \equiv 0$, which contradicts to $\mu(0) \neq 0$. We now define the differences and sum

$$
\mu_1 = \lambda_2 - \lambda_3, \quad \mu_2 = \lambda_3 - \lambda_1, \quad \mu_3 = \lambda_1 - \lambda_2,
$$

$$
\nu_1 = \lambda_2 + \lambda_3, \quad \nu_2 = \lambda_3 + \lambda_1, \quad \nu_3 = \lambda_1 + \lambda_2.
$$

Then (1.21) yields

\begin{equation}
\dot{\mu}_i + \nu_i \mu_i = 0 \quad \text{in} \quad (0, T_*), \quad i = 1, 2, 3.
\end{equation}

Solving this equations, we get

$$
\mu_i(t) = \mu_i(0) e^{-\nu_i(t)}, \quad t \in (0, T_*).
$$

Verifying that $\sum_{i=1}^3 \nu_i \equiv 0$, we obtain $\prod_{i=1}^3 \mu_i \equiv c_0 := \prod_{i=1}^3 \mu_{0,i}$. Accordingly,

\begin{equation}
\mu_3 = \frac{c_0}{\mu_1 \mu_2}.
\end{equation}

We now define $\alpha = \mu_1 + \mu_2 = -\mu_3$, and $\beta = \mu_1 - \mu_2$. We calculate,

$$
\nu_3 = \lambda_1 + \lambda_2 = \mu_1 - \mu_2 + 2\lambda_3 = \mu_1 - \mu_2 - 2\nu_3.
$$

Thus,

$$
\nu_3 = \frac{1}{3} (\mu_1 - \mu_2) = \frac{\beta}{3}.
$$
Inserting this identity into (1.22) for \(i = 3\), we get

\[
\dot{\alpha} + \frac{\beta}{3}\alpha = 0 \quad \Rightarrow \quad \alpha = \alpha(0)e^{-\frac{1}{3}\beta}.
\]

On the other hand, in view of (1.23) we infer

\[
\beta^2 = \mu_1^2 - 2\mu_1\mu_2 + \mu_2^2 = \alpha^2 - 4\mu_1\mu_2 = \alpha^2 - 4\epsilon_0\mu_0^{-1} = \alpha^2 + 4\epsilon_0\alpha^{-1}.
\]

Inserting \(\alpha = \alpha(0)e^{-\frac{1}{3}\beta}\), this yields

\[
\beta^2 = \alpha(0)e^{-\frac{2}{3}\beta} + 4\epsilon_0\alpha(0)^{-1}e^{\frac{1}{3}\beta} \quad \text{in} \quad (0, T_*).
\]

This shows that \(\beta\) is bounded, which also implies that \(\alpha\) is bounded. Hence, \(\lambda_i, i = 1, 2, 3\) are bounded. Whence, \(T_* = +\infty\).

Next, we give an example of finite time blow up. As we have seen above this is only possible if two eigenvalues are equal. Thus, we may assume that \(\lambda = \lambda_1 = \lambda_2 > 0\) and \(\lambda_2 = -2\lambda\). Then, in view of (1.21) \(\lambda\) solves the Riccati equation

\[
\dot{\lambda} = \frac{1}{3}\lambda^2 \quad \text{in} \quad (0, T_*),
\]

which has the unique solution

\[
\lambda(t) = \frac{3\lambda_0}{\lambda(0)t - 3}, \quad t \in (0, T_*), \quad T_* = \frac{3}{\lambda(0)}.
\]

For the case of initial data with sub linear growth we get the following third main result which can be directly compared with the known results in Besov spaces

**Theorem 3** (Local well posedness in \(L^1_{1(p,1)} \cap BMO\)). *For every \(v_0 \in L^1_{1(p,1)} \cap BMO\), there exists*

\[
T_0 \geq \frac{1}{c|v|_{L^1_{1(p,1)}}},
\]

*and a unique solution \((v, p) \in L^\infty(0, T_0; (L^1_{1(p,1)} \cap BMO) \times BMO) to (1.1)-(1.2) in \(Q_{T_0}\) such that \((p)_{0,1} = 0\). Such solution is also eligible.*

*In case \(v_0 \in L^\infty(\mathbb{R}^n)\) the above solution is bounded.*

We also are able to generalize the Baele-Kato-Majda condition\(^3\) to the non decaying case as follows.

**Theorem 4.** Let \(v_0 \in L^{1+\delta}_{q(p,1), \sigma}(\mathbb{R}^n) \cap L^{1}_{1(2,1)}(\mathbb{R}^n), 1 < p < +\infty, 1 \leq q \leq +\infty, \delta \in (0, 1),\) fulfilling

\[
\forall \varepsilon > 0 \quad \exists k \in \mathbb{N} \quad \text{such that} \quad \sup_{x_0 \in \mathbb{R}^n} \sum_{j=k}^{\infty} 2^{-j} \text{osc}(v_0; x_0, 2^j) \leq \varepsilon.
\]
Let \( v \in L^\infty_{\text{loc}}([0,T_*); \mathcal{L}^1_{1(2,1)}(\mathbb{R}^n)) \) be an eligible solution to (1.1) according to Theorem 2. Furthermore, assume that

\[
(1.28) \quad \int_0^{T_*} |\omega(\tau)|_{BMO} + |P^0_{\infty}(\nabla v(\tau))|d\tau < +\infty.
\]

Then, \( v \in L^\infty(0,T_\ast; (\mathcal{L}^{1+\delta}_{q(p,1)} \cap \mathcal{L}^1_{1(2,1)})(\mathbb{R}^n)) \), and the solution can be extended to \([0,T_*+\eta]\) for some \( \eta > 0 \).

Remark 1.10. 1. We wish to emphasize that in the case of sublinear growing initial data the condition (1.27) is obviously satisfied. Furthermore, in that case it holds \( P^0_{\infty}(\nabla v_0) = 0 \), and as shown in Section 7 this implies \( P^0_{\infty}(\nabla v(\tau)) = 0 \) for all \( \tau \in [0,T_\ast) \). Hence, (1.28) reduces to Kozono-Taniuchi’s condition in [16]

\[
(1.29) \quad \int_0^{T_*} |\omega(\tau)|_{BMO}d\tau < +\infty,
\]

which is a refined version of the Beale-Kato-Majda criterion[3].

Remark 1.11. The examples of solutions \((v,p) \in C([0,T_*); \mathcal{L}^1_{1(p,1)}(\mathbb{R}^n))\) in Remark 1.9 show that even if \( \omega(t) = 0 \) for all \( t \in [0,T_*) \), the solution can blow up at \( t = T_* \), and it holds \( \int_0^{T_*} |P^0_{\infty}(\nabla v(t))|dt = +\infty \), which implies the necessity of the second integrand of (1.28) in the case of solutions having linear growth at infinity.

2 Notations and preliminary lemmas

Let \( X = \{X_j\}_{j \in \mathbb{Z}} \) be a sequence of non-negative real numbers. We define \( S_{\alpha,q} : X = \{X_j\}_{j \in \mathbb{Z}} \mapsto Y = \{Y_j\}_{j \in \mathbb{Z}} \), where

\[
Y_j = S_{\alpha,q}(X)_j = 2^{j\alpha} \left( \sum_{i=j}^{\infty} (2^{-i\alpha} X_i)^{\frac{1}{q}} \right)^{\frac{1}{q}}, \quad j \in \mathbb{Z}.
\]

From the above definition it follows that

\[
(2.1) \quad \|S_{0,q}(X)\|_{\ell^\infty} \leq \|X\|_{\ell^q}, \quad \forall \ X \in \ell^q.
\]

Given \( X = \{X_j\}_{j \in \mathbb{Z}}, Y = \{Y_j\}_{j \in \mathbb{Z}} \), we denote \( X \leq Y \) if \( X_j \leq Y_j \) for all \( j \in \mathbb{Z} \). Throughout this paper, we frequently make use of the following lemma, which could be regarded as a generalization of the result in [4].

Lemma 2.1. For all \( \beta < \alpha \) and \( 0 < p \leq q \leq +\infty \) it holds

\[
(2.2) \quad S_{\beta,q}(S_{\alpha,p}(X)) \leq \frac{1}{1 - 2^{-(\alpha-\beta)}} S_{\beta,q}(X).
\]
For the proof see in [9, Section 2].

In what follows we provide important properties of the space $L^{k,s}_{\mathcal{B}(p,N)}(\mathbb{R}^n)$ such as embedding properties, equivalent norms. First, let us recall the definition of the generalized mean for distributions $f \in \mathcal{S}'$, where $\mathcal{S}$ denotes the usual Schwarz class of rapidly decaying functions. For $f \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$ we define the convolution

$$ f \ast \varphi(x) = \langle f, \varphi(x - \cdot) \rangle, \quad x \in \mathbb{R}^n, $$

where $\langle \cdot, \cdot \rangle$ denotes the dual pairing. Below we use the notation $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Then, $f \ast \varphi \in C^\infty(\mathbb{R}^n)$ and for every multi index $\alpha \in \mathbb{N}^n_0$ it holds

$$ D^\alpha(f \ast \varphi) = f \ast (D^\alpha \varphi) = (D^\alpha f) \ast \varphi. $$

Given $x_0 \in \mathbb{R}^n$, $0 < r < +\infty$ and $f \in \mathcal{S}'$ we define the mean

$$ [f]_{x,r}^\alpha = f \ast D^\alpha \varphi_r(x), $$

where $\varphi_r(y) = r^{-n} \varphi(r^{-1}(y))$, and $\varphi \in C^\infty_c(B(1))$ stands for the standard mollifier, such that $\int_{\mathbb{R}^n} \varphi dx = 1$. Note that in case $f \in L^1_{loc}(\mathbb{R}^n)$ we get

$$ [f]_{x,r}^0 = \int_{\mathbb{R}^n} f(x - y) \varphi_r(y) dy = \int_{B(x,r)} f(y) \varphi_{x,r}(-y) dy, $$

where $\varphi_{x,r} = \varphi_r(\cdot + x)$. Furthermore, from the above definition it follows that

$$ [f]_{x,r}^\alpha = (D^\alpha f) \ast \varphi_r(x) = [D^\alpha f]_{x,r}^0. $$

For $f \in L^1_{loc}(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}^n_0$ we immediately get

$$ [f]_{x,r}^\alpha \leq c r^{-|\alpha|-n} \|f\|_{L^1(B(x,r))} \quad \forall x \in \mathbb{R}^n, r > 0. $$

By the following lemma we introduce the mean polynomial $P^N_{x_0,r}(f)$ together with its properties. The proof of this and all other lemmas of this section can be found in [9, Section 3].

**Lemma 2.2.** Let $x_0 \in \mathbb{R}^n$, $0 < r < +\infty$ and $N \in \mathbb{N}_0$. For every $f \in \mathcal{S}'$ there exists a unique polynomial $P^N_{x_0,r}(f) \in \mathcal{P}_N$ such that

$$ [f - P^N_{x_0,r}(f)]_{x_0,r}^\alpha = 0 \quad \forall |\alpha| \leq N. $$

In addition, the mapping $P^N_{x_0,r} : L^p(B(x_0,r)) \rightarrow \mathcal{P}_N, 1 \leq p \leq +\infty$, defines a projection, i.e.

$$ P^N_{x_0,r}(Q) = Q \quad \forall Q \in \mathcal{P}_N, $$

$$ \|P^N_{x_0,r}(f)\|_{L^p(B(x_0,4r))} \leq c \|P^N_{x_0,r}(f)\|_{L^p(B(x_0,r))} \leq c \|P^N_{0,1}(f)\|_{L^p(B(x_0,r))}. $$
Let $f$ then

Furthermore, for all $f \in W^{p,j}(B(x_0, r))$, $1 \leq p < +\infty$, $1 \leq j \leq N + 1$, it holds

$$
\|f - P_{x_0,r}^N(f)\|_{L^p(B(x_0, r))} \leq c r^j \sum_{|\alpha| = j} \|D^\alpha f - D^\alpha P_{x_0,r}^N(f)\|_{L^p(B(x_0, r))}.
$$

Remark 2.3. From \(2.9\) with $j = N + 1$ we get the generalized Poincaré inequality

$$
\|f - P_{x_0,r}^N(f)\|_{L^p(B(x_0, r))} \leq c r^{N+1} \|D^{N+1}f\|_{L^p(B(x_0, r))}
$$

Corollary 2.4. For all $x_0 \in \mathbb{R}^n$, $0 < r < +\infty$, $N \in \mathbb{N}_0$, and $1 \leq p < +\infty$ it holds

$$
\|f - P_{x_0,r}^N(f)\|_{L^p(B(x_0, r))} \leq c \inf_{Q \in \mathcal{P}_N} \|f - Q\|_{L^p(B(x_0, r))} = c r^N \text{osc}(f; x_0, r).
$$

In our discussion below and in the sequel of the paper it will be convenient to work with smooth functions. Using the standard mollifier we get the following estimate in $\mathcal{L}^{k,s}_q(p,N)(\mathbb{R}^n)$ for the mollification.

Lemma 2.5. Let $\varepsilon > 0$. Given $f \in \mathcal{S}'$, we define the mollification

$$
f_\varepsilon(x) = \left[ f \right]_{\varepsilon}^0 = f * \varphi_\varepsilon(x), \quad x \in \mathbb{R}^n.
$$

1. For all $f \in \mathcal{L}^{k,s}_q(p,N)(\mathbb{R}^n)$, and all $\varepsilon > 0$ it holds

$$
|f_\varepsilon|_{\mathcal{L}^{k,s}_q(p,N)} \leq c |f|_{\mathcal{L}^{k,s}_q(p,N)}.
$$

2. Let $f \in L^p_{loc}(\mathbb{R}^n)$ such that for all $0 < \varepsilon < 1$,

$$
|f_\varepsilon|_{\mathcal{L}^{k,s}_q(p,N)} \leq c_0,
$$

then $f \in \mathcal{L}^{k,s}_q(p,N)(\mathbb{R}^n)$ and it holds $|f|_{\mathcal{L}^{k,s}_q(p,N)} \leq c_0$.

Next, we provide the following embedding properties. First, let us introduce the definition of the projection to the space of homogenous polynomial $\hat{P}_{x_0,r}^N : \mathcal{S}' \to \mathcal{P}_N$ defined by

$$
\hat{P}_{x_0,r}^N(f)(x) = \sum_{|\alpha| = N} \frac{1}{\alpha!} [f]_{x_0,r}^\alpha x^\alpha, \quad x \in \mathbb{R}^n.
$$

Clearly, for all $f \in \mathcal{S}'$ it holds

$$
D^\alpha \hat{P}_{x_0,r}^N(f) = \hat{P}_{x_0,r}^{N-|\alpha|}(D^\alpha f) \quad \forall |\alpha| \leq k.
$$
Theorem 2.6. 1. For every $N \in \mathbb{N}_0$ the following embedding holds true.

\begin{equation}
\begin{cases}
\mathcal{L}^N_{1(p,N)}(\mathbb{R}^n) \hookrightarrow C^{N-1,1}(\mathbb{R}^n) & \text{if } N \geq 1 \\
\mathcal{L}^0_{1(p,0)}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n) & \text{if } N = 0.
\end{cases}
\end{equation}

2. For every $f \in \mathcal{L}^N_{1(p,N)}(\mathbb{R}^n)$ there exists a unique $\hat{P}^N \in \hat{\mathcal{P}}_N$ such that for all $x_0 \in \mathbb{R}^n$

$$
\lim_{r \to \infty} \hat{P}_{x_0,r}(f) \to \hat{P}^N(f) \text{ in } \mathcal{P}_N.
$$

Furthermore, $\hat{P}^N : \mathcal{L}^N_{1(p,N)}(\mathbb{R}^n) \to \hat{\mathcal{P}}_N$ is a projection with the property

\begin{equation}
D^\alpha \hat{P}^N(f) = \hat{P}^{N-|\alpha|}(D^\alpha f) \quad \forall |\alpha| \leq N.
\end{equation}

3. For all $g, f \in \mathcal{L}^1_{1(p,1)}(\mathbb{R}^n)$ it holds

\begin{equation}
\hat{P}^1\big(\partial_k f\big) = \hat{P}^1\big(\partial_k g\big) \partial_k \hat{P}^1\big(\partial_k f\big) = \hat{P}^1\big(\partial_k g\big) \hat{P}^0\big(\partial_k f\big), \quad k = 1, \ldots, n.
\end{equation}

In addition, for $g \in C^{0,1}(\mathbb{R}^n; \mathbb{R}^n)$, and for all $f \in \mathcal{L}^0_{1(p,0)}(\mathbb{R}^n)$ it holds

\begin{equation}
\hat{P}^0\big(\partial_k f\big) := \lim_{r \to \infty} P^0_{1,r}(g \partial_k f) = 0, \quad k = 1, \ldots, n,
\end{equation}

where $g \partial_k f = \partial_k (g f)$.

4. For all $v \in \mathcal{L}^1_{1(p,1)}(\mathbb{R}^n; \mathbb{R}^n)$ with $\nabla \cdot v = 0$ almost everywhere in $\mathbb{R}^n$ and $f \in \mathcal{L}^1_{1(p,1)}(\mathbb{R}^n)$ it holds

\begin{equation}
\hat{P}^0\big(\nabla v \cdot \nabla f\big) = \hat{P}^0\big(\nabla v\big) \cdot \hat{P}^0\big(\nabla f\big).
\end{equation}

Next, we have the following norm equivalence, which is similar to the properties of the usual Campanato spaces.

Lemma 2.7. Let $1 \leq p < +\infty$, $1 \leq q \leq +\infty$, and $N, N' \in \mathbb{N}_0$, $N < N'$, $s \in [-\frac{n}{p}, N+1)$. If $f \in \mathcal{L}^{k,s}_{q(p,N')}(\mathbb{R}^n)$, and satisfies

\begin{equation}
\lim_{m \to \infty} \hat{P}_{0,2m}^L(D^k f) = 0 \quad \forall L = N + 1, \ldots, N'.
\end{equation}

then $f \in \mathcal{L}^{k,s}_{q(p,N)}(\mathbb{R}^n)$ and it holds,

\begin{equation}
|f|_{\mathcal{L}^{k,s}_{q(p,N')}} \leq |f|_{\mathcal{L}^{k,s}_{q(p,N)}} \leq c |f|_{\mathcal{L}^{k,s}_{q(p,N')}}.
\end{equation}

Remark 2.8. For all $f \in \mathcal{L}^s_{q(p,N)}(\mathbb{R}^n)$, $1 \leq p < +\infty$, $1 \leq q \leq +\infty$, $s \in [-\frac{n}{p}, N+1)$, the condition (2.20) is fulfilled, and therefore (2.21) holds for all $f \in \mathcal{L}^s_{q(p,N)}(\mathbb{R}^n)$ under the assumptions on $p, q, s, N$ and $N'$ of Lemma 2.7. To verify this fact we observe that for $f \in \mathcal{L}^0_{q(p,N)}(\mathbb{R}^n)$

\begin{equation}
\sup_{m \in \mathbb{Z}} 2^{-Nm} \text{osc}_{p,N} \big(f, 0, 2^m\big) \leq |f|_{\mathcal{L}^s_{q(p,N)}}.
\end{equation}
Then for $L \in \mathbb{N}$, $L > N$, we estimate for all multi index $\alpha$ with $|\alpha| = L$

$$|D^\alpha \tilde{P}_{0,2^m}^L(f)| = |D^\alpha \tilde{P}_{0,2^m}^L((f - P_{0,2^m}^N))| \leq c2^{-Lm} \text{osc}_{p,N}(f,0,2^m)$$

$$\leq c2^{m(N-L)}|f|_{L^q_{[p,N]}} \to 0 \quad \text{as} \quad m \to +\infty.$$ 

Hence, (2.20) is fulfilled.

**Remark 2.9.** In case $q = \infty$, since $L^s_{[p,N]}(\mathbb{R}^n)$ coincides with the usual Campanato space, and Lemma 2.7 reduces to the well known result (cf. [13, p. 75]).

We also have the following growth properties of functions in $L^q_{[p,N]}(\mathbb{R}^n)$ as $|x| \to +\infty$ (see [9]).

**Lemma 2.10.** Let $N \in \mathbb{N}_0$. Let $f \in L^q_{[p,N]}(\mathbb{R}^n), 1 \leq q \leq +\infty, 1 \leq p < +\infty, s \in [N,N+1)$.

1. In case $s \in (N,N+1)$ it holds

$$|f(x)| \leq c(1 + |x|^s)\|f\|_{L^q_{[p,N]}} \quad \forall x \in \mathbb{R}^n.$$  

2. In case $s = N$ it holds

$$|f(x)| \leq c(1 + \log(1 + |x|)\|f\|_{L^q_{[p,N]}} \quad \forall x \in \mathbb{R}^n.$$  

Here $q' = \frac{q}{q-1}$, and the constant $c = \text{const} > 0$, depends on $q, p, s, N$ and $n$.

### 3 Calderón-Zygmund estimate involving $L^q_{p,N}$ norm

In this section we establish the Calderón-Zygmund type estimate for our spaces. For this purpose let us introduce the partition of unity, which will be used in what follows. We set $U_j = B(2^{j+1}) \setminus \overline{B(2^j)}$, $j \in \mathbb{Z}$. Clearly, $\{U_j\}$ is a local finite covering of $\mathbb{R}^n \setminus \{0\}$. By $\{\psi_j\}$ we denote a corresponding partition of unity of radial symmetric functions $\psi_j \in C^\infty(U_j)$, such that $0 \leq \psi_j \leq 1$ in $U_j$, $|D^k\psi_j| \leq c_k 2^{-kj}$ in $U_j$ and $\sum_{j \in \mathbb{Z}} \psi_j = 1$ on $\mathbb{R}^n \setminus \{0\}$. We have the following.

**Lemma 3.1.** Let $K \in C^2(\mathbb{R}^n \setminus \{0\})$ be a Calderón-Zygmund kernel, i.e.

(i) $K(x) \sim t^n K(tx)$ for all $x \in \mathbb{R}^n \setminus \{0\}, t > 0.$

(ii) $\int_{\partial B(1)} K(x)dS = 0.$

By $\{\psi_j\}$ we denote a partition of unity introduced above. Let $m, k \in \mathbb{Z}, m < k$. Define

$$T_m^k(h)(x) = \sum_{i=m}^{k} \int_{\mathbb{R}^n} h(x-y)K(y)\psi_i(y)dy, \quad x \in \mathbb{R}^n, \quad h \in L^1_{\text{loc}}(\mathbb{R}^n),$$

then for all $1 < p < +\infty, 1 \leq q \leq +\infty, s \in [0,N+1), N \in \mathbb{N}_0$ the operator $T_m^k$ is uniformly bounded in $L^q_{[p,N]}(\mathbb{R}^n)$, i.e. it holds

$$|T_m^k(h)|_{L^q_{[p,N]}} \leq c_{N,s,q,n}|h|_{L^q_{[p,N]}} \quad \forall h \in L^q_{[p,N]}(\mathbb{R}^n).$$

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Proof: Let $K : \mathbb{R}^n \setminus \{0\}$ be a Calderón-Zygmund kernel. Let $h \in L_{q(p,N)}^s(\mathbb{R}^n)$. Given $m, M \in \mathbb{Z}, m < M$ we set

$$f^k_m(x) = T_m^k(h)(x) = \sum_{i = m}^{k} \int h(x-y)K(y)\psi_i(y)dy, \quad x \in \mathbb{R}^n.$$  

Let $x_0 \in \mathbb{R}^n$ be arbitrarily chosen. Let $j \in \mathbb{Z}$ be fixed. Our aim will be to evaluate the $\text{osc}_{p,N}(f^k_m; x_0, 2^j)$. We decompose $f^k_m$ into the sum $g^k_m + G^k_m$ by means of

$$g^k_m(x) = \sum_{i = m}^{k} \int h(x-y)K(y)\psi_i(y)dy,$$

$$G^k_m(x) = \sum_{i > m}^{k} \int h(x-y)K(y)\psi_i(y)dy, \quad x \in \mathbb{R}^n.$$  

Let $x \in B(x_0, 2^j)$ arbitrarily chosen, but fixed. Defining $Q \in \mathcal{P}_N$ by means of

$$Q(x) = \sum_{i = m}^{k} \int P_{x_0, 2^j+2}(h)(\cdot - y)K(y)\psi_i(y)dy, \quad x \in \mathbb{R}^n,$$

it can be checked easily that

$$g^k_m(x) - Q(x) = \sum_{i = m}^{k} \int_{B(2^{j+1})} (h(x-y) - P_{x_0, 2^j+2}(h)(x-y))K(y)\psi_i(y)dy$$

$$= \sum_{i = m}^{k} \int_{B(x, 2^{j+1})} (h(y) - P_{x_0, 2^j+2}(h)(y))K(x-y)\psi_i(x-y)dy.$$  

Clearly, $B(x, 2^{j+1}) \subset B(x_0, 2^{j+2})$ and $\text{supp}(\psi_i(x - \cdot)) \subset B(x, 2^{j+1})$ for all $i \leq j$. This shows that

$$g^k_m(x) - Q(x) = \sum_{i = m}^{k} \int_{\mathbb{R}^n} \chi_{B(x_0, 2^{j+2})}(y)(h(y) - P_{x_0, 2^j+2}(h)(y))K(x-y)\psi_i(x-y)dy$$

$$= \sum_{i = m}^{k} \int_{\mathbb{R}^n} \chi_{B(x_0, 2^{j+2})}(h - P_{x_0, 2^{j+2}}(h)) \psi_i(y)dy.$$  

By virtue of the well known Calderón-Zygmund inequality in $L^p$ we find

$$\|g^k_m(x) - Q\|_{L^p(B(x_0, 2^j))} \leq c\|h - P_{x_0, 2^j+2}(h)\|_{L^p(B(x_0, 2^{j+2}))} \leq c2^{jN\frac{p}{p,N}} \text{osc}_p(h; x_0, 2^{j+2}).$$
Next, we estimate $G_m^k$. Let $\alpha \in \mathbb{N}_0^n$ be any multi index with $|\alpha| = N + 1$. Clearly,

$$D^\alpha G_m^k(x) = \sum_{\substack{i = m\in \mathbb{N}_0^n \setminus j \geq j}} \int_{\mathbb{R}^n} (h(x - y) - P_{x_0,2^{i+2}}^N (h)(x - y)) D^\alpha (K(y) \psi_1(y)) dy.$$ 

Let $i \in \{j + 1, \ldots, k\}$. Noting that $B(x, 2^{i+1}) \subset B(x_0, 2^{i+1} + 2^i) \subset B(x_0, 2^{i+2})$, and employing Jensen’s inequality, we estimate

$$\left| \int_{\mathbb{R}^n} (h(x - y) - P_{x_0,2^{i+2}}^N (h)(x - y)) D^\alpha (K(y) \psi_1(y)) dy \right|$$

$$\leq c \int_{B(2^{i+1})} |h(x - y) - P_{x_0,2^{i+2}}^N (h)(x - y)| 2^{-i(n + N + 1)} dy$$

$$= c \int_{B(x, 2^{i+1})} |h(y) - P_{x_0,2^{i+2}}^N (h)(y)| 2^{-i(n + N + 1)} dy$$

$$\leq c \int_{B(x, 2^{i+2})} |h(y) - P_{x_0,2^{i+2}}^N (h)(y)| 2^{-i(n + N + 1)} dy$$

$$\leq c 2^{-i(2)(\frac{\alpha}{p} + N + 1)} \left( \int_{B(x, 2^{i+2})} |h(y) - P_{x_0,2^{i+2}}^N (h)(y)|^p dx \right)^{\frac{1}{p}}$$

$$\leq c 2^{-i(2)(N + 1)} \text{osc}_{p,N}^g(h; x_0, 2^{i+2}).$$

Summing over $i = j + 1$ to $k$ to both sides of the above inequality and multiplying the result by $2^{i(N + 1)}$, we get

$$2^{i(N + 1)} \| D^{N+1} G_m^k(x) \|_{L^\infty(B(x_0, 2^{i}))} \leq c S_{N+1,1}(\text{osc}_{p,N}^g(h; x_0))_j. \quad (3.3)$$

Thanks to Poincaré’s inequality (3.3) implies

$$\| G_m^k - P_{x_0,2^{i+2}}^N (G_m^k) \|_{L^p(B(x_0, 2^{i}))} \leq c 2^{i(\frac{\alpha}{p} + N + 1)} \| D^{N+1} G_m^k(x) \|_{L^\infty(B(x_0, 2^{i}))}$$

$$\leq c 2^{i(\frac{\alpha}{p})} S_{N+1,1}(\text{osc}_{p,N}^g(h; x_0))_j. \quad (3.4)$$

Furthermore, noting that $P_{x_0,2^{i+2}}^N (f_m) = P_{x_0,2^{i+2}}^N (g_m^k + P_{x_0,2^{i+2}}^N (G_m^k)$, we infer

$$\| f_m^k - P_{x_0,2^{i+2}}^N (f_m^k) \|_{L^p(B(x_0, 2^{i}))}$$

$$\leq \| g_m^k - P_{x_0,2^{i+2}}^N (g_m^k) \|_{L^p(B(x_0, 2^{i}))} + \| G_m^k - P_{x_0,2^{i+2}}^N (G_m^k) \|_{L^p(B(x_0, 2^{i}))}$$

$$\leq c \| g_m^k - \mathcal{Q} \|_{L^p(B(x_0, 2^{i}))} + \| G_m^k - P_{x_0,2^{i+2}}^N (G_m^k) \|_{L^p(B(x_0, 2^{i}))}.$$

Combining this inequality with (3.2) and (3.4), we obtain

$$\text{osc}_{p,N}^g(f_m^k; x_0, 2^{i})$$

$$\leq c \text{osc}_{p,N}^g(h; x_0, 2^{i+2}) + c S_{N+1,1}(\text{osc}_{p,N}^g(h; x_0))_j$$

$$\leq c S_{N+1,1}(\text{osc}_{p,N}^g(h; x_0))_j. \quad (3.5)$$
We now perform \( S_{s,q} \) to the both sides of (3.5), and use Lemma 3.1 with \( X = \text{osc}(h; x_0) \), with \( p = 1, \alpha = N + 1, \beta = s \). Then taking the supremum over \( x_0 \in \mathbb{R}^n \) on both sides, we obtain

\[
(3.6) \quad |f^k_m|_{\mathcal{L}^q_{p,N}} \leq c|h|_{\mathcal{L}^q_{p,N}}.
\]

Whence, (3.1).

**Lemma 3.2.** Let \( N \in \mathbb{N}_0 \). Let \( \{\varphi_k\} \) be a sequence of functions in \( C^l_c(\mathbb{R}^n), l \in \mathbb{N}, l \geq N + 1 \), with \( \text{supp}(\varphi_k) \subset \mathbb{R}^n \setminus B(2^k) \), and

\[
(3.7) \quad |D^\alpha \varphi_k| \leq c|y|^{-n-|\alpha|} \quad \text{in} \quad \mathbb{R}^n \quad \forall |\alpha| \leq l.
\]

Let \( 1 \leq q \leq +\infty, s \in [0, N + 1) \). Given \( f \in L^p_{loc}(\mathbb{R}^n) \) such that

\[
(3.8) \quad \sup_{x_0 \in \mathbb{R}^n} \sum_{j=1}^{\infty} 2^{-jq} \text{osc}_{p,N}(f; x_0, 2^j)^q < +\infty.
\]

Define,

\[
w^k(x) = \int_{\mathbb{R}^n} f(x - y)\varphi_k(y)dy, \quad x \in \mathbb{R}^n, \quad k \in \mathbb{N}.
\]

Then for every multi index \( \alpha \) with \( |\alpha| \in \{N + 1, \ldots, l\} \),

\[
(3.9) \quad D^\alpha w^k \to 0 \quad \text{uniformly in} \quad \mathbb{R}^n \quad \text{as} \quad k \to +\infty.
\]

**Proof:** Let \( \{\psi_j\} \) denote the partition of unity introduced in the beginning of this section. Since \( \text{supp}(\varphi_k) \subset \mathbb{R}^n \setminus B(2^k) \) we get

\[
w^k(x) = \sum_{j=k-1}^{\infty} \int_{\mathbb{R}^n} f(x - y)\varphi_k(y)\psi_j(y)dy, \quad x \in \mathbb{R}^n, \quad k \in \mathbb{N}.
\]

Let \( \alpha \in \mathbb{N}_0^n \) be any multi index with \( |\alpha| \in \{N + 1, \ldots, l\} \). We calculate

\[
D^\alpha w^k(x) = \sum_{j=k-1}^{\infty} \int_{\mathbb{R}^n} (f(x - y) - P_{x_0, 2^{j+2}}^N(x - y))D^\alpha(\varphi_k(y)\psi_j(y))dy, \quad x \in \mathbb{R}^n.
\]

Let \( j \in \mathbb{Z}, j \geq k - 1 \). Fix \( x_0 \in \mathbb{R}^n \). Noting that \( B(x, 2^{j+1}) \subset B(x_0, 2^{j+1} + 2^k) \subset B(x_0, 2^{j+2}) \) for all \( x \in B(x_0, 2^k) \), observing (3.7), and arguing as in the proof of Lemma 3.1 we find for all \( x \in B(x_0, 2^k) \),

\[
\left| \int_{\mathbb{R}^n} (f(x - y) - P_{x_0, 2^{j+2}}^N(x - y))D^\alpha(\varphi_k(y)\psi_j(y))dy \right|
\]

\[
\leq c \int_{B(2^{j+1})} |f(x - y) - P_{x_0, 2^{j+2}}^N(x - y)||2^{-j|\alpha|}|dy
\]

\[
\leq c2^{-j|\alpha|} \text{osc}_{p,N}(f; x_0, 2^{j+2}).
\]
This together with Hölder’s inequality yields
\[
|D^\alpha w^k(x_0)| \leq \|D^\alpha w^k\|_{L^\infty(B(x_0,2^k))} \\
\leq c \sum_{j=k}^{\infty} 2^{-j|\alpha|} \text{osc}_{p,N}(f;x_0,2^{j+2}) \leq c \sum_{j=k}^{\infty} 2^{-j(|\alpha|-s)} 2^{-js} \text{osc}_{p,N}(f;x_0,2^j) \\
\leq c\left( \sum_{j=k}^{\infty} 2^{-j(|\alpha|-s)s'} \right)^{\frac{1}{s'}} \left( \sup_{x \in \mathbb{R}^n} \sum_{j=1}^{\infty} 2^{-jsq} \text{osc}_{p,N}(f;x,2^j)^q \right)^{\frac{1}{q'}}. 
\]
(3.10)

Observing (3.8), the right-hand side tends to zero as \( k \to +\infty \) uniformly in \( x_0 \in \mathbb{R}^n \) we get the claim.

Next, we apply Lemma 3.1 to the Laplace equation

\[
(3.11) \quad -\Delta f = \sum_{\alpha,\beta=1}^{n} \partial_\alpha \partial_\beta H_{\alpha\beta} \text{ in } \mathbb{R}^n.
\]

Let \( H \in L^2_{\text{loc}}(\mathbb{R}^n;\mathbb{R}^{n^2}) \). A function \( f \in L^2_{\text{loc}}(\mathbb{R}^n) \) is called a very weak solution to (3.11) if the following identity holds for all \( \phi \in C^\infty_c(\mathbb{R}^n) \)

\[
(3.12) \quad -\int_{\mathbb{R}^n} f \Delta \phi dx = \int_{\mathbb{R}^n} H : \nabla^2 \phi dx.
\]

Below we shall also make use of the following.

**Lemma 3.3.** Let \( \{f_k\} \) be a bounded sequence in \( \mathcal{L}^s_{q(p,N)}(\mathbb{R}^n) \). Suppose there exists \( f \in L^p_{\text{loc}}(\mathbb{R}^n) \) such that

\[
(3.13) \quad f_k \to f \text{ in } L^p_{\text{loc}}(\mathbb{R}^n) \text{ as } k \to +\infty.
\]

Then \( f \in \mathcal{L}^s_{q(p,N)}(\mathbb{R}^n) \) and it holds

\[
(3.14) \quad |f|_{\mathcal{L}^s_{q(p,N)}} \leq \sup_{k \in \mathbb{N}} |f_k|_{\mathcal{L}^s_{q(p,N)}}.
\]

**Proof:** Let \( m, l \in \mathbb{Z}, m < l \) be arbitrarily chosen. By means of (3.13) we get for all \( x_0 \in \mathbb{R}^n \)

\[
\sum_{j=m}^{l} \text{osc}_{p,N}(f;x_0,2^j) = \lim_{k \to \infty} \sum_{j=m}^{l} \text{osc}_{p,N}(f_k;x_0,2^j) \leq \sup_{k \in \mathbb{N}} |f_k|_{\mathcal{L}^s_{q(p,N)}}.
\]

Passing \( m \to -\infty \) and \( l \to +\infty \) and taking the supremum over all \( x_0 \in \mathbb{R}^n \), we obtain the claim (3.14).

We have following
Theorem 3.4. Let $N \in \{0,1\}$, $s \in [0,N+1)$. For every $H \in L^s_{q(p,N)}(\mathbb{R}^n;\mathbb{R}^n)$ there exists a unique very weak solution $f \in L^s_{q(p,N)}(\mathbb{R}^n)$ to (3.11) such that

$$P_{0,1}^N(f) = 0.$$  

In particular, the following estimate holds true

$$|f|_{L^s_{q(p,N)}} \leq c |H|_{L^s_{q(p,N)}},$$

where $c = \text{const} > 0$, depending only on $n,q,N$ and $s$.

Proof: By $K_{\alpha \beta}$ we denote the kernel $\partial_\alpha \partial_\beta \Gamma$, where $\Gamma$ stands for the Newtonian potential in $\mathbb{R}^n$, i.e.

$$\Gamma(x) = \begin{cases} \frac{1}{n|B(1)||x|^{n-2}} & \text{if } n \geq 3 \\ \frac{1}{2\pi \log |x|} & \text{if } n = 2. \end{cases}$$

It is readily seen that $K_{\alpha \beta}$ is a Calderón-Zygmund kernel. Let $H \in L^s_{q(p,N)}(\mathbb{R}^n;\mathbb{R}^n)$. We define, for $m,k \in \mathbb{Z}$, $m < k$,

$$f^k_m(x) = \sum_{i=m}^{k} \int_{\mathbb{R}^n} H(x-y) : K(y)\psi_i(y)dy, \quad x \in \mathbb{R}^n.$$ 

According to Lemma 3.1 it holds

$$|f^k_m|_{L^s_{q(p,N)}} \leq c |H|_{L^s_{q(p,N)}},$$

where the constant $c > 0$ does not depend on $m,k \in \mathbb{Z}$.

1. Assume $H \in C^\infty(\mathbb{R}^n)$. Using integration by parts, and noting that

$$\sum_{i=m}^{m+2} \int_{B(2^{m+3})} \partial_\beta \Gamma(y) \partial_\alpha \psi_i(y)dy = 0,$$

we get

$$f^k_m(x) = \sum_{i=m}^{k} \int_{\mathbb{R}^n} \partial_\alpha H_{\alpha \beta}(x-y) \partial_\beta \Gamma(y)\psi_i(y)dy$$

$$- \sum_{i=k-2}^{k} \int_{B(2^{k+1})} H_{\alpha \beta}(x-y) \partial_\beta \Gamma(y)\partial_\alpha \psi_i(y)dy$$

$$- \sum_{i=m}^{m+2} \int_{B(2^{m+3})} (H_{\alpha \beta}(x-y) - H_{\alpha \beta}(x)) \partial_\beta \Gamma(y)\partial_\alpha \psi_i(y)dy.$$
Clearly, from the above identity we deduce that $f^k_m(x) \to f^k(x)$ as $m \to -\infty$ for all $x \in \mathbb{R}^n$, where

$$f^k(x) = \sum_{i=-\infty}^{k} \int_{\mathbb{R}^n} \partial_\alpha H_{\alpha\beta}(x - y) \partial_\beta \Gamma(y) \psi_i(y) dy - \sum_{i=k-2}^{k} \int_{B(2k+1)} H_{\alpha\beta}(x - y) \partial_\beta \Gamma(y) \partial_\alpha \psi_i(y) dy. $$

By Lebesgue’s theorem of dominated convergence we see that $f^k_m \to f^k$ in $L^p_{\text{loc}}(\mathbb{R}^n)$ as $m \to -\infty$. Applying integration by parts, we find that

$$f^k(x) = \int_{\mathbb{R}^n} \partial_\alpha \partial_\beta H_{\alpha\beta}(x - y) \Gamma(y) \chi_k(y) dy - \sum_{j=k-2}^{k} \int_{B(2k+1)} H_{\alpha\beta}(x - y)(\partial_\beta \Gamma(y) \partial_\alpha \psi_j(y) + \partial_\alpha \Gamma(y) \partial_\beta \psi_j(y)) dy,$$

where

$$\chi_k(y) = \begin{cases} \sum_{j=-\infty}^{k} \psi_j(y) & \text{if } y \in \mathbb{R}^n \setminus \{0\}, \\ 1 & \text{if } y = 0. \end{cases}$$

In addition, (3.14) together with (3.17) implies

$$|f^k|_{\mathcal{L}^p_q(p,N)} \leq c|H|_{\mathcal{L}^p_q(p,N)}. $$

Set $g^k := f^k - P_{0,1}^N(f^k)$. We easily get for $j \in \mathbb{N}$

$$g^k = f^k - P_{0,2i}^N(f^k) + \sum_{i=1}^{j} (P_{0,2i}^N(f^k) - P_{0,2i-1}^N(f^k)) = f^k - P_{0,2}^N(f^k) + \sum_{i=1}^{j} P_{0,2i}^N(f^k - P_{0,2i-1}^N(f^k)). $$

Thus, by the triangle inequality along with (2.7) we find

$$\|g^k\|_{L^p(B(2i))} \leq c2^{\frac{3j}{p}} \text{osc}_{p,N}(f^k, 0, 2^j) + \sum_{i=1}^{j} \|P_{0,2i}^N(f^k - P_{0,2i-1}^N(f^k))\|_{L^p(B(2i))} \leq c2^{\frac{3j}{p}} \text{osc}_{p,N}(f^k, 0, 2^j) + c \sum_{i=1}^{j} 2^{\frac{3j}{p}} \text{osc}_{p,N}(f^k, 0, 2^j). $$

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Applying Hölder’s inequality, and using (3.18), we find
\[ \left\| g^k \right\|_{L^p(B(2^j \mathbb{R}^n))} \leq c 2^{j^s + j N} |f^k|_{L^q_{q(p,N)}} \leq c 2^{j^s + j N} |H|_{L^q_{q(p,N)}}. \]

Thus, \( \{g^k\} \) is bounded in \( L^p_{\text{loc}}(\mathbb{R}^n) \). Noting that for all \( x_0 \in \mathbb{R}^n \) it holds \( \text{osc}(g^k, x_0) = \text{osc}(f^k, x_0) \), owing to (3.18), we infer
\[ |g^k|_{L^q_{q(p,N)}} \leq c |H|_{L^q_{q(p,N)}}. \]

Thus, by the compact embedding \( L^q_{q(p,N)}(\mathbb{R}^n) \hookrightarrow L^p_{\text{loc}}(\mathbb{R}^n) \), eventually passing to a subsequence, we get a function \( f \in L^p_{\text{loc}}(\mathbb{R}^n) \) such that
\[ g^k \rightarrow f \quad \text{in} \quad L^p_{\text{loc}}(\mathbb{R}^n) \quad \text{as} \quad m \rightarrow +\infty. \]

This together with (3.14) and (3.19) shows that \( g \in L^q_{q(p,N)}(\mathbb{R}^n) \), and satisfies the inequality
\[ |g|_{L^q_{q(p,N)}} \leq c |H|_{L^q_{q(p,N)}}. \]

Setting
\[ \varphi_k = - \sum_{j=k-2}^{k} \left( \Gamma(y) \partial_{\alpha} \partial_{\beta} \psi_j(y) + \partial_{\alpha} \Gamma(y) \partial_{\beta} \psi_j(y) + \partial_{\beta} \Gamma(y) \partial_{\alpha} \psi_j(y) \right), \]
we may write \( f^k = f^k_1 + f^k_2 \), where
\[ f^k_1(x) = \int_{\mathbb{R}^n} \partial_{\alpha} \partial_{\beta} H_{\alpha \beta}(x - y) \Gamma(y) \chi_k(y) dy \]
\[ f^k_2(x) = \int_{\mathbb{R}^n} H_{\alpha \beta}(x - y) \varphi_k(y) dy, \quad x \in \mathbb{R}^n. \]

Clearly, \( \varphi_k \in C_c^{N+1}(\mathbb{R}^n) \) with \( \text{supp}(\varphi_k) \subset \mathbb{R}^n \setminus B(2^{k-3}) \) and satisfying condition (3.7) of Lemma 3.2. Thus, thanks to Lemma 3.2
\[ D^{N+1} f^k_2 \rightarrow 0 \quad \text{uniformly in} \quad \mathbb{R}^n \quad \text{as} \quad k \rightarrow +\infty. \]

Let \( \phi \in C_c^\infty(\mathbb{R}^n) \) be arbitrarily chosen. Employing (3.22), recalling that \( N \leq 1 \), we immediately verify that
\[ \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} f^k_2(x) \Delta \phi(x) dx = 0. \]
Using Fubini’s theorem, and applying integration by parts, we calculate

\[
\begin{align*}
\int_{\mathbb{R}^n} f_1^k(x)\Delta\phi(x)dx \\
= & \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \partial_\alpha \partial_\beta H(y)\Gamma(x-y)\chi_k(x-y)\Delta\phi(x)dydx \\
= & \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \partial_\alpha \partial_\beta H(y)\Gamma(x-y)\chi_k(x-y)\Delta\phi(x)dydx \\
= & -\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \partial_\alpha \partial_\beta H(y)\phi(y)dy \\
& + 2\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \partial_\alpha \partial_\beta H(y)\nabla\Gamma(x-y) \cdot \nabla\chi_k(x-y)\phi(x)dydx \\
& + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \partial_\alpha \partial_\beta H(y)\Gamma(x-y)\Delta\chi_k(x-y)\phi(x)dydx \\
= & \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \partial_\alpha \partial_\beta H(y)\phi(y)dy + I_k + II_k.
\end{align*}
\]

Again applying integration by parts, we infer

\[
I_k = 2\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} H(y)\partial_\alpha \partial_\beta (\nabla\Gamma(x-y) \cdot \nabla\chi_k(x-y))\phi(x)dydx
= 2(-1)^{N-1}\int_{\mathbb{R}^n} \partial_\alpha \partial_\beta \int_{\mathbb{R}^n} H(x-y)\nabla\Gamma(y) \cdot \nabla\chi_k(y)dy\phi(x)dx.
\]

Using Lemma 3.2, we get \(I_k = o(1)\). By a similar reasoning we see that \(II_k = o(1)\). By means of these properties together with \(\chi_k \to 1\) uniformly on each ball as \(k \to +\infty\), along with (3.23), we deduce that

\[
(3.24) \quad \lim_{k \to \infty} \int_{\mathbb{R}^n} f^k(x)\Delta\phi(x)dx = -\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \partial_\alpha \partial_\beta H(y)\phi(y)dy.
\]

On the other hand, recalling the definition of \(g^k\), we see that

\[
-\int_{\mathbb{R}^n} f^k(x)\Delta\phi(x)dx = -\int_{\mathbb{R}^n} g^k(x)\Delta\phi(x)dx.
\]

By the aid of (3.20) letting \(k \to +\infty\) on the right-hand side, and using (3.24), we obtain the identity

\[
(3.25) \quad -\int_{\mathbb{R}^n} f(x)\Delta\phi(x)dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \partial_\alpha \partial_\beta H(y)\phi(y)dy.
\]
Accordingly, \( f \) is a very weak solution to (3.12). Recalling that \( P^N_{0,1}(g_k) = 0 \) for all \( k \in \mathbb{N} \) thanks to (3.20) it holds \( P^N_{0,1}(f) = 0 \).

Now, let \( H \in \mathcal{L}^s_{q(p,N)}(\mathbb{R}^n) \) be arbitrarily chosen. By \( H \), for \( \varepsilon > 0 \) we denote the standard mollification of \( H \). According to Lemma 2.6 it satisfies

\[
|H_\varepsilon|_{\mathcal{L}^s_{q(p,N)}} \leq c|H|_{\mathcal{L}^s_{q(p,N)}}.
\]

From the previous step we get a solution \( f_\varepsilon \in \mathcal{L}^s_{q(p,N)}(\mathbb{R}^n) \) to (3.12) such that \( P^N_{0,1}(f_\varepsilon) = 0 \). According to (3.21), the following a priori estimate holds

\[
|f_\varepsilon|_{\mathcal{L}^s_{q(p,N)}} \leq c|H|_{\mathcal{L}^s_{q(p,N)}}.
\]

Set \( g_\varepsilon = f_\varepsilon - P^N_{0,1}(f_\varepsilon) \). As above we verify that \( \{g_\varepsilon\} \) is bounded in \( L^p_{\text{loc}}(\mathbb{R}^n) \) and in \( \mathcal{L}^s_{q(p,N)}(\mathbb{R}^n) \). By similar argument to the above we get a function \( f \in \mathcal{L}^s_{q(p,N)}(\mathbb{R}^n) \) together with a sequence \( \varepsilon_k \downarrow 0 \) as \( k \to +\infty \) such that

\[
f_\varepsilon \to f \quad \text{in} \quad L^p_{\text{loc}}(\mathbb{R}^n) \quad \text{as} \quad k \to +\infty.
\]

In addition, it holds

\[
|f|_{\mathcal{L}^s_{q(p,N)}} \leq c|H|_{\mathcal{L}^s_{q(p,N)}}.
\]

Let \( \phi \in C_c^\infty(\mathbb{R}^n) \). Since \( f_\varepsilon \) solves (3.12) with \( H_\varepsilon \) in place of \( H \), we infer that the following identity holds true

\[
- \int_{\mathbb{R}^n} f_\varepsilon \Delta \phi \, dx = \int_{\mathbb{R}^n} H_\varepsilon : D^2 \phi \, dx.
\]

Letting \( k \to +\infty \) on both sides of the above identity, and making use of (3.28), we are led to

\[
- \int_{\mathbb{R}^n} f \Delta \phi \, dx = \int_{\mathbb{R}^n} H : D^2 \phi \, dx.
\]

This shows that \( f \) is a very weak solution to (3.12) satisfying (3.15).

**Uniqueness.** Let \( \overline{f} \) be another very weak solution to (3.12) satisfying (3.15). Then \( f - \overline{f} \in \mathcal{L}^s_{q(p,N)}(\mathbb{R}^n) \) and \( f - \overline{f} \) is harmonic. By the virtue of the Caccioppoli inequality for harmonic functions we get

\[
\left( \int_{B(2^j)} |D^{N+1}(f - \overline{f})|^p \, dx \right)^{\frac{1}{p}} \leq c2^{-j(N+1)} \left( \int_{B(2^{j+1})} |(f - \overline{f}) - P^N_{0,2^{j+1}}(f - \overline{f})|^p \, dx \right)^{\frac{1}{p}}
\]

\[
\leq c2^{-j(N+1)} \max_{p,N}(f - \overline{f}, 0, 2^{j+1}) \leq c2^{-j(N+1 - s)}|f - \overline{f}|_{\mathcal{L}^s_{q(p,N)}}.
\]
Since the right-hand side tends to zero as $j \to +\infty$, we deduce that $D^{N+1}(f - \bar{f}) = 0$. Hence, $f - \bar{f} \in P_N$. Observing (3.15), it follows that $f - \bar{f} = 0$. This completes the proof of the uniqueness.

We are now in a position to prove Theorem 1.3.

**Proof of Theorem 1.3** Let $H \in L^1_{1,p,(1)}(\mathbb{R}^n)$. Thanks to Theorem 3.4 there exists a unique very weak solution $g \in L^1_{1,p,(1)}(\mathbb{R}^n)$ to (3.11) such that $P^1_{0,1}(g) = 0$. Let $a \in \mathbb{R}^n$ and $Q_\infty \in \hat{P}_1$. We define

$$f(x) = g(x) - g(0) + a - \hat{P}^1_{\infty}(g)(x) + Q_\infty(x), \quad x \in \mathbb{R}^n.$$ 

Clearly, $f \in L^1_{1,p,(1)}(\mathbb{R}^n)$ is a very weak solution to (3.11) satisfying $f(0) = a$ and $P^1_{\infty}(f) = Q_\infty$. Assume $\bar{f}$ is another very weak solution to (3.11) satisfying $\bar{f}(0) = a$ and $P^1_{\infty}(\bar{f}) = Q_\infty$. Then by Weyl’s lemma $f - \bar{f}$ is harmonic. Using Liouville’s theorem we see that $f - \bar{f} \in P_1$. Owing to $f(0) - \bar{f}(0) = 0$ and $\nabla(f - \bar{f}) = \nabla \hat{P}^1_{\infty}(f - \bar{f}) = 0$ we get $f = \bar{f}$.

**Definition of the Helmholtz-Leray projection.** Let $u \in L^1_{1,p,(1)}(\mathbb{R}^n)$. Applying Theorem 1.3 with $a = 0$ and $Q_\infty = -\frac{1}{n} P_\infty^0(\nabla \cdot u) x_\alpha$, $\alpha = 1, \ldots, n$, we get a unique very weak solution $w = Q^\sharp(1)(u) \in L^1_{1,p,(1)}(\mathbb{R}^n)$ to the equation

$$
\begin{cases}
-\Delta w_\alpha = -\partial_\alpha \nabla \cdot u & \text{in } \mathbb{R}^n, \quad \alpha = 1, \ldots, n. \\
w_\alpha(0) = 0, \quad P^0_\infty(w_\alpha) = -\frac{1}{n} P^0_\infty(\nabla \cdot u) x_\alpha.
\end{cases}
$$

Then we define $\mathbb{P} : L^1_{1,p,(1)}(\mathbb{R}^n) \to L^1_{1,p,(1)}(\mathbb{R}^n)$ by means of

$$\mathbb{P}u = u - Q^\sharp(1)(u), \quad u \in L^1_{1,p,(1)}(\mathbb{R}^n).$$

Thanks to Theorem 1.3 both $Q^\sharp$ and $\mathbb{P}$ are bounded operators. It is readily seen that

$$\int_{\mathbb{R}^n} \mathbb{P}u \cdot \nabla \phi dx = 0 \quad \text{for all } \phi \in C_0^\infty(\mathbb{R}^n),$$

which shows that $\nabla \cdot \mathbb{P}u = 0$. On the other hand, if $u \in L^1_{1,p,(1)}(\mathbb{R}^n)$ with $\nabla \cdot u = 0$ then $\Delta Q^\sharp(u) = 0$ and $P^1_\infty(Q^\sharp(u)) = 0$, which implies that $Q^\sharp(u)$ is constant. Observing (3.30), we infer $Q^\sharp(u) = 0$ and

$$Q^\sharp(u) = \hat{P}^1_{\infty}(Q^\sharp(u)) = 0.$$ 

Accordingly, $\mathbb{P}(u) = u$, which shows that $\mathbb{P} : L^1_{1,p,(1)}(\mathbb{R}^n) \to L^1_{1,p,(1)}(\mathbb{R}^n)$ defines a projection onto divergence free fields.

In what follows we consider the equation (3.12) for matrices $H = u \otimes v$. We first prove the following lemma, which covers the situation $\nabla \cdot u = 0$ and $v = (h, 0, \ldots, 0)$.

**Theorem 3.5.** Let $N \in \{0, 1\}$, $1 < p < +\infty$, $1 \leq q \leq +\infty$, $s \in [0, N + 1)$. Let $u, v \in \mathcal{L}^{s}_{q(p,N)}(\mathbb{R}^n) \cap \mathcal{L}^1_{1,p,(1)}(\mathbb{R}^n)$, such that

$$\nabla \cdot u = 0 \quad \text{a.e. in } \mathbb{R}^n.$$
Then for every \( l \in \{1, \ldots, n\} \) there exists a unique solution \( f \in L^q_{q(p,N)}(\mathbb{R}^n) \cap L^1_{1(p,1)}(\mathbb{R}^n) \) to the equation

\[
(3.32) \quad -\Delta f = \partial_t \nabla \cdot ((v \cdot \nabla)u) \quad \text{in} \quad \mathbb{R}^n,
\]

such that

\[
(3.33) \quad \dot{P}_\infty(f) = -\frac{1}{n} P_{\infty}^0(\nabla u : (\nabla v)^\top) x_l, \quad f(0) = 0.
\]

In addition, it holds

\[
(3.34) \quad \begin{cases}
|f|_{\mathcal{L}^1_{q,1}(p,1)} \leq c \left( \|\nabla u\|_{\infty} |v|_{\mathcal{L}^1_{q}(p,N)} + \|\nabla v\|_{\infty} |u|_{\mathcal{L}^1_{q}(p,N)} \right), \\
|f|_{\mathcal{L}^q_{q,N}(N,N)} \leq c \left( \|\nabla u\|_{\infty} |v|_{\mathcal{L}^q_{q,N}} + \|\nabla v\|_{\infty} |u|_{\mathcal{L}^q_{q,N}} \right).
\end{cases}
\]

In particular, \( \nabla f \in L^\infty(\mathbb{R}^n) \), and it holds

\[
(3.35) \quad \|\nabla f\|_\infty \leq c \left\{ \|\nabla u\|_{\infty} |v|_{\mathcal{L}^1_{1}(p,N)} + \|\nabla v\|_{\infty} |u|_{\mathcal{L}^1_{1}(p,N)} \right\}.
\]

In addition, given \( 1 < r < p \), the following inequality holds for all \( j \in \mathbb{Z} \)

\[
(3.36) \quad (\text{osc}_{r,N}(f;x_0))_j \leq c \left( |\nabla u|_{BMO} + \sup_{i \geq j} |P_{x_0,2^r}(\nabla u)| \right) S_{N+1,1}(\text{osc}(v;x_0))_j + c \left( |\nabla v|_{BMO} + \sup_{i \geq j} |P_{x_0,2^r}(\nabla v)| \right) S_{N+1,1}(\text{osc}(u;x_0))_j.
\]

**Proof:** Let \( l \in \{1, \ldots, n\} \). We define, for \( m, k \in \mathbb{Z}, m < k \),

\[
f^k_m(x) = \sum_{j=m}^{k} \int_{\mathbb{R}^n} \partial_\beta u_\alpha(x - y) v_\beta(x - y) \partial_\delta \partial_\alpha(\Gamma(y)\psi_j(y))dy, \quad x \in \mathbb{R}^n.
\]

Let \( j \in \mathbb{Z} \) be fixed. We decompose \( f^k_m \) into the sum \( g^k_m + C^k_m \), where

\[
g^k_m(x) = \sum_{\substack{i=m \leq j \leq k \in \mathbb{Z}}} \int \partial_\beta u_\alpha(x - y) v_\beta(x - y) \partial_\delta \partial_\alpha(\Gamma(y)\psi_j(y))dy,
\]

\[
C^k_m(x) = \sum_{\substack{i=m \leq j \leq k \in \mathbb{Z}}} \int \partial_\beta u_\alpha(x - y) v_\beta(x - y) \partial_\delta \partial_\alpha(\Gamma(y)\psi_j(y))dy, \quad y \in \mathbb{R}^n.
\]

Setting

\[
Q(x) = \sum_{\substack{i=m \leq j \leq k \in \mathbb{Z}}} P_{x_0,2^{i+j+2}}^0(\partial_\beta u_\alpha) P_{x_0,2^{i+j+2}}^N(\partial_\beta)(x - y) \partial_\delta \partial_\alpha(\Gamma(y)\psi_i(y))dy \in \mathcal{P}_N,
\]

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we see that
\[ g_m^k(x) - Q(x) \]
\[ = \sum_{i=m_i}^{k} \int \left( \partial_{\beta} u_{\alpha} v_{\beta} - P_{x_0,2j+2}^0(\partial_{\beta} u_{\alpha}) P_{x_0,2j+2}^N(v_{\beta}) \right) (x - y) \partial_{\alpha} (\Gamma(y) \psi(y)) dy \]
\[ = \sum_{i=m_i}^{k} \int \partial_{\beta} u_{\alpha} (v_{\beta} - P_{x_0,2j+2}^N(v_{\beta})) (x - y) \partial_{\alpha} (\Gamma(y) \psi(y)) dy \]
\[ + \sum_{i=m_i}^{k} \int (\partial_{\beta} u_{\alpha} - P_{x_0,2j+2}^0(\partial_{\beta} u_{\alpha})) P_{x_0,2j+2}^N(v_{\beta}) (x - y) \partial_{\alpha} (\Gamma(y) \psi(y)) dy \]
\[ = J_1(x) + J_2(x). \]

(i) **Estimation of J** 1: Arguing as in the proof of Lemma 3.1, using Calderon-Zygmund inequality and Hölder’s inequality, we find for all 1 < r ≤ p
\[ \| J_1 \|_{L^r(B(x_0,2))} \leq c \left\| \nabla u \cdot (v - P_{x_0,2j+2}^N(v)) \right\|_{L^r(B(x_0,2j+2))} \]
\[ \leq c \left\| \nabla u \right\|_{L^{\frac{pr}{p-r}}(B(x_0,2j+2))} \left\| v - P_{x_0,2j+2}^N(v) \right\|_{L^p(B(x_0,2j+2))}. \]

(ii) **Estimation of J** 2: Applying integration by parts and recalling that \( \nabla \cdot u = 0 \), we infer
\[ J_2(x) = - \sum_{i=m_i}^{k} \int \left( u_{\alpha} - P_{x_0,2j+2}^N(u_{\alpha}) \right) P_{x_0,2j+2}^0(\nabla \cdot v) (x - y) \partial_{\alpha} (\Gamma(y) \psi(y)) dy \]
\[ + \sum_{i=m_i}^{k} \int (u_{\alpha} - P_{x_0,2j+2}^N(u_{\alpha})) P_{x_0,2j+2}^0(\partial_{\alpha} v_{\beta}) (x - y) \partial_{\beta} (\Gamma(y) \psi(y)) dy. \]

Once more applying Calderon-Zygmund’s inequality using Poincaré’s inequality, arguing as above, we obtain for 1 < r ≤ p
\[ \| J_2 \|_{L^r(B(x_0,2))} \leq c \left\| u - P_{x_0,2j+2}^N(u) \right\|_{L^p(B(x_0,2j+2))} \left\| \nabla v \right\|_{L^{\frac{pr}{p-r}}(B(x_0,2j+2))}. \]

Employing the two estimates for J 1 and J 2, we get
\[ \| g_m^k - Q \|_{L^r(B(x_0,2))} \leq c \left\| \nabla u \right\|_{L^{\frac{pr}{p-r}}(B(x_0,2j+2))} \left\| v - P_{x_0,2j+2}^N(v) \right\|_{L^p(B(x_0,2j+2))} \]
\[ + c \left\| u - P_{x_0,2j+2}^N(u) \right\|_{L^p(B(x_0,2j+2))} \left\| \nabla v \right\|_{L^{\frac{pr}{p-r}}(B(x_0,2j+2))}. \]

(3.37)

By the aid of (3.37), we deduce the following estimate for the oscillation of \( g_m^k \).
\[ \text{osc}_{r,N}(g_m^k, x_0, 2j) \]
\[ \leq c \left\| \nabla u \right\|_{L^{\frac{pr}{p-r}}(B(x_0,2j+2))} \text{osc}(v, x_0, 2j+2) + c \text{osc}(u, x_0, 2j+2) \left\| \nabla v \right\|_{L^{\frac{pr}{p-r}}(B(x_0,2j+2))}. \]

(3.38)

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Next, we estimate $G^k_m$. Let $\sigma \in \mathbb{N}_0^d$ be any multi index with $|\sigma| = N + 1$. Clearly,

$$D^\sigma G^k_m(x)$$

$$= \sum_{i=m}^{k} \int_{i>j} \left( \partial_\beta u_\alpha v_\beta - P^0_{x_0,2^{i+2}}(\partial_\beta u_\alpha) P^N_{x_0,2^{i+2}}(v_\beta) \right) (x - y) D^\sigma \partial_\alpha \Gamma(y) \psi(y) dy$$

$$= \sum_{i=m}^{k} \int_{i>j} \partial_\beta u_\alpha(x - y)(v_\beta(x - y) - P^N_{x_0,2^{i+2}}(v_\beta)(x - y)) D^\sigma \partial_\beta \Gamma(y) \psi(y) dy$$

$$+ \sum_{i=m}^{k} \int_{i>j} (\partial_\beta u_\alpha(x - y) - P^0_{x_0,2^{i+2}}(\partial_\beta u_\alpha)) P^N_{x_0,2^{i+2}}(v_\beta)(x - y) D^\sigma \partial_\alpha \Gamma(y) \psi(y) dy.$$

Let $j \leq i \leq k$. Noting that $B(x, 2^{i+1}) \subset B(x_0, 2^{i+1} + 2^j) \subset B(x_0, 2^{i+2})$, and employing Jensen’s inequality, we infer

$$\left| \int_{\mathbb{R}^n} \partial_\beta u_\alpha(x - y)(v_\beta(x - y) - P^N_{x_0,2^{i+2}}(v_\beta)(x - y)) D^\sigma \partial_\beta \Gamma(y) \psi(y) dy \right|$$

$$\leq c \int_{B(2^{i+1})} |\nabla u(x - y)||v(x - y) - P^N_{x_0,2^{i+2}}(v)(x - y)| 2^{-i(n+N+1)} dy$$

$$= c \int_{B(x, 2^{i+1})} |\nabla u(y)||v(y) - P^N_{x_0,2^{i+2}}(v)(y)| 2^{-i(n+N+1)} dy$$

$$\leq c \|\nabla u\|_{L^p'(B(x_0, 2^{i+2}))} 2^{-i(n+N+1)} \left( \int_{B(x_0, 2^{i+2})} |v - P^N_{x_0,2^{i+2}}(v)|^p dy \right)^{\frac{1}{p}}$$

$$\leq c 2^{-i\frac{n}{p'}} \|\nabla u\|_{L^p'(B(x_0, 2^{i+2}))} 2^{-i(N+1)} \text{osc}(v; x_0, 2^{i+2}).$$

Similarly, using integration by parts along with (3.31), we get

$$\left| \int_{\mathbb{R}^n} (\partial_\beta u_\alpha(x - y) - P^0_{x_0,2^{i+2}}(\partial_\beta u_\alpha)) P^N_{x_0,2^{i+2}}(v_\beta)(x - y) D^\sigma \partial_\alpha \Gamma(y) \psi(y) dy \right|$$

$$\leq c 2^{-i\frac{n}{p'}} \|\nabla v\|_{L^p'(B(x_0, 2^{i+2}))} 2^{-i(N+1)} \text{osc}(u; x_0, 2^{i+2}).$$

This yields

$$\|D^N G^k_m\|_{L^\infty(B(x_0, 2^i))}$$

$$\leq c \sum_{i=m}^{k} 2^{-i\frac{n}{p'}} \|\nabla u\|_{L^p'(B(x_0, 2^{i+2}))} 2^{-i(N+1)} \text{osc}(v; x_0, 2^{i+2})$$

$$+ c \sum_{i=m}^{k} 2^{-i\frac{n}{p'}} \|\nabla v\|_{L^p'(B(x_0, 2^{i+2}))} 2^{-i(N+1)} \text{osc}(u; x_0, 2^{i+2}).$$

(3.39)
Noting that for all \( i \geq j \)

\[
2^{-i \frac{p'}{p}} \| \nabla u \|_{L^{p'}(B(x_0,2^{i+2}))} \leq c |\nabla u|_{BMO} + c \sup_{i\geq j} |P_{x_0,2^i}(\nabla u)|
\]

\[
2^{-i \frac{p'}{p}} \| \nabla v \|_{L^{p'}(B(x_0,2^{i+2}))} \leq c |\nabla v|_{BMO} + c \sup_{i\geq j} |P_{x_0,2^i}(\nabla v)|
\]

we infer from (3.39)

\[
2^{j(N+1)} \| D^N G_m \|_{L^\infty(B(x_0,2^j))} \leq c \left( |\nabla u|_{BMO} + \sup_{i\geq j} |P_{x_0,2^i}(\nabla u)| \right) S_{N+1,1}(\text{osc}(v; x_0))_j
\]

\[
+ c \left( |\nabla v|_{BMO} + \sup_{i\geq j} |P_{x_0,2^i}(\nabla v)| \right) S_{N+1,1}(\text{osc}(u; x_0))_j.
\]

(3.40)

With the help of Poincaré's inequality (2.10) along with (3.40) we find

\[
\text{osc}_{r,N}(G^k_m; x_0)_j
\]

\[
\leq 2^{-j \frac{p'}{p}} \| G^k_m - P_{x_0,2^j} (G^k_m) \|_{L^{p'}(B(x_0,2^j))}
\]

\[
\leq c 2^{j(N+1)} \| D^{N+1} G^k_m \|_{L^\infty(B(x_0,2^j))}
\]

\[
\leq c \left( |\nabla u|_{BMO} + \sup_{i\geq j} |P_{x_0,2^i}(\nabla u)| \right) S_{N+1,1}(\text{osc}(v; x_0))_j
\]

\[
+ c \left( |\nabla v|_{BMO} + \sup_{i\geq j} |P_{x_0,2^i}(\nabla v)| \right) S_{N+1,1}(\text{osc}(u; x_0))_j.
\]

(3.41)

Combining this inequality with (3.38) and (3.41), and noting that for all \( j \in \mathbb{Z} \)

\[
\text{osc}_{p,N}(u; x_0, 2^{i+j+2}) \leq c S_{N+1,1}(\text{osc}_{p,N}(u; x_0))_j,
\]

\[
\text{osc}_{p,N}(v; x_0, 2^{i+j+2}) \leq c S_{N+1,1}(\text{osc}_{p,N}(v; x_0))_j,
\]

we obtain in case \( r = p \)

\[
\text{osc}_{p,N}(f^k_m; x_0)
\]

\[
\leq c \| \nabla u \|_{L^\infty} S_{N+1,1}(\text{osc}_{p,N}(v; x_0)) + c \| \nabla v \|_{L^\infty} S_{N+1,1}(\text{osc}_{p,N}(u; x_0)).
\]

(3.42)

and in case \( 1 < r < p \)

\[
(\text{osc}_{r,N}(f^k_m; x_0))_j
\]

\[
\leq c \left( |\nabla u|_{BMO} + \sup_{i\geq j} |P_{x_0,2^i}(\nabla u)| \right) S_{N+1,1}(\text{osc}_{p,N}(v; x_0))_j
\]

\[
+ c \left( |\nabla v|_{BMO} + \sup_{i\geq j} |P_{x_0,2^i}(\nabla v)| \right) S_{N+1,1}(\text{osc}_{p,N}(u; x_0))_j.
\]

(3.43)

We are now in a position to apply Lemma 2.1 with \( \alpha = N+1, \beta = s, p = 1 \). Performing \( S_{s,q} \) to both sides of (3.42), we get

\[
|f^k_m|_{L^s_{q(p,N)}} \leq c \left( \| \nabla u \|_{L^\infty} \| \nabla v \|_{L^s_{q(p,N)}} + \| \nabla v \|_{L^\infty} \| \nabla u \|_{L^s_{q(p,N)}} \right).
\]

(3.44)
In particular, for $N = 1, s = 1$ and $q = 1$ it follows that

\begin{equation}
|f_m|_{L^1_{p,N}} \leq c \left( \|\nabla u\|_\infty \|\nabla v\|_{L^1_{p,N}} + \|\nabla v\|_\infty \|\nabla u\|_{L^1_{p,N}} \right).
\end{equation}

Set $g_m = f_m - P_{0,1}(f_m)$. Clearly, $\{g_m\}$ is bounded in $L^s_{p,N}(\mathbb{R}^n) \cap L^1_{p,1}(\mathbb{R}^n) \rightarrow C^{0,1}(\mathbb{R}^n)$. Arguing as in the proof of Theorem 3.4, we get a very weak solution $g \in L^s_{q(p,N)} \cap L^1_{p,1}(\mathbb{R}^n)$ with $\nabla g = L^\infty(\mathbb{R}^n)$ to

\[-\Delta g = \partial_k \nabla \cdot ((v \cdot \nabla)u) \quad \text{in} \quad \mathbb{R}^n.
\]

We make the ansatz: $f = g + Ax + b$. Clearly, for all $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$, $f \in L^s_{q(p,N)} \cap L^1_{p,1}(\mathbb{R}^n)$ is a very weak solution to (3.32). The condition $f(0) = 0$ implies $b = -g(0)$, while the first condition in (3.33) implies

\[P^0_\infty(\nabla u : (\nabla v)^\top_{x_l} = P^1_\infty(f) = P^1_\infty(g + Ax + b) = P^1_\infty(g) + Ax.
\]

Setting $Ax = P^0_\infty(\nabla u : (\nabla v)^\top_{x_l} = P^1_\infty(g)$, the function $f$ fulfills (3.33). Arguing as in the proof of Theorem 3.4, the estimate (3.34) follows from (3.41) and (3.45). Furthermore, the estimate (3.36) follows from (3.43) after passing $k \rightarrow +\infty$ and $m \rightarrow -\infty$.

**Uniqueness.** Let $\tilde{f} \in L^s_{q(p,N)} \cap L^1_{p,1}(\mathbb{R}^n)$ be a second solution which satisfies (3.33). Clearly, $f - \tilde{f}$ is harmonic. Arguing as in the proof of Theorem 1.3, we conclude $f - \tilde{f} \in \mathcal{P}_1$, and by (3.33) $f = \tilde{f}$.}

**Remark 3.6.** Noting that $\operatorname{osc}(f^k_{m,\Pi(x_0,2^l)} \rightarrow \operatorname{osc}(f; x_0, 2^l)$ as $k \rightarrow +\infty$ and $m \rightarrow -\infty$, we infer from (3.42) the estimate

\begin{equation}
\operatorname{osc}(f; x_0) \leq c \|\nabla u\|_\infty S_{N+1,1}(\operatorname{osc}(v; x_0)) + c \|\nabla v\|_\infty S_{N+1,1}(\operatorname{osc}(u; x_0)).
\end{equation}

**Remark 3.7.** By $L^1_{p,1,\sigma}(\mathbb{R}^n)$ we define the space of all $u \in L^1_{p,1}(\mathbb{R}^n)$ such that $\nabla \cdot u = 0$ in $\mathbb{R}^n$. Then by Theorem 3.5, we are able to construct the pressure $\pi \in L^1_{p,1,\sigma}(\mathbb{R}^n)$, by the relation $\nabla \pi = f$, where $f \in L^1_{p,1}(\mathbb{R}^n)$ is the unique very weak solution to (3.32), (3.33). In fact, from (3.32) it follows that $\nabla \times f$ is harmonic and bounded. Thus, by Liouville’s theorem for harmonic functions we see that $\nabla \times f$ is constant. On the other hand, by means of (3.33) we find

\[\nabla \times f = P^0_\infty(\nabla \times f) = \nabla \times P^1_\infty(f) = \nabla \times (P^0_\infty(\nabla u : (\nabla v)^\top_{x_l} = 0.
\]

Thus, $f = \nabla \pi$ for a unique $\pi \in L^1_{p,1}(\mathbb{R}^n)$ fulfilling $[\pi]^0_{0,1} = 0$. Setting $\nabla \Pi(u, v) = \nabla \pi$, defines a linear mapping $\nabla \Pi : L^1_{p,1,\sigma}(\mathbb{R}^n) \times L^1_{p,1}(\mathbb{R}^n) \rightarrow L^1_{p,1}(\mathbb{R}^n)$. Accordingly, it holds

\begin{align*}
\Delta \pi &= \nabla u : (\nabla v)^\top \quad \text{in} \quad \mathbb{R}^n, \\
\hat{P}^1_\infty(\nabla \pi) &= -\frac{1}{n} P^0_\infty(\nabla u : (\nabla v)^\top_{x_l}.
\end{align*}
In addition, from (3.34) we deduce that $\nabla \Pi$ is bounded. More precisely,
\begin{equation}
|\nabla \Pi(u, v)|_{L^1(p, 1)} \leq c \left( \|u\|_{L^\infty} |v|_{L^1(p, 1)} + \|v\|_{L^\infty} |u|_{L^1(p, 1)} \right),
\end{equation}
and in view of (3.33)
\begin{equation}
|\hat{P}_\infty^1(\nabla \Pi(u, v))| \leq c |P^0_\infty(\nabla u)| \|P^0_\infty(\nabla v)|.
\end{equation}
In case $u, v \in \mathcal{L}^q_{q(p, N)}(\mathbb{R}^n)$ we get from (3.34)
\begin{equation}
|\nabla \Pi(u, v)|_{L^q_q(p, N)} \leq c \left\{ \|u\|_{L^\infty} |v|_{L^q_q(p, N)} + \|v\|_{L^\infty} |u|_{L^q_q(p, N)} \right\}.
\end{equation}
In addition, given $1 < r < p$, in view of (3.42) for all $j \in \mathbb{Z}$ it holds
\begin{align}
(osc(\nabla \Pi(u, v); x_0))_j & \leq c \left( |\nabla u|_{BMO} + \sup_{i \geq j} |P^0_{x_0, 2^i}(\nabla u)| \right) S_{N+1, 1}(osc(v; x_0))_j \\
& + c \left( |\nabla v|_{BMO} + \sup_{i \geq j} |P^0_{x_0, 2^i}(\nabla v)| \right) S_{N+1, 1}(osc(u; x_0))_j.
\end{align}
We also get the following pressure estimate for the case of sublinear growth less then $\frac{1}{2}$. The following theorem will be used in the proof of Theorem 3.8.

**Theorem 3.8.** Let $1 \leq q \leq +\infty$ and $s \in [0, \frac{1}{2})$. Let $v \in \mathcal{L}^q_{q(p, 0), \sigma}(\mathbb{R}^n) \cap C^{0, 1}(\mathbb{R}^n)$. Let $1 < r < (1 - s)p$ with $r \geq \frac{2}{3}$. There exists a unique very weak solution $\pi \in \mathcal{L}^{2 - (1 - s)p}_{q(r, 0), \sigma}(\mathbb{R}^n)$ to the equation
\begin{equation}
-\Delta \pi = \nabla \cdot \nabla \cdot (v \otimes v) \quad \text{in} \quad \mathbb{R}^n
\end{equation}
with $P^0_{0, 1}(\pi) = 0$. Furthermore, for all $x_0 \in \mathbb{R}^n$ and $j \in \mathbb{Z}$ it holds
\begin{equation}
(osc(\pi; x_0, 2^j)) \leq c \|\nabla v\|_{L^\infty}^{2 - \frac{2}{r}} S_{1, 1}(\left( 2^{(2 - \frac{2}{r})i} \right)_{p, 0}(osc(v; x_0, 2^j))_j)
\end{equation}

**Proof:** Let $1 < r < (1 - s)p$ with $r \geq \frac{2}{3}$. Let $m, k \in \mathbb{Z}$ with $m < k$. Define,
\begin{equation}
f^m_k(x) = \sum_{i=m}^{k} \int_{\mathbb{R}^n} v_\alpha(x - y)v_\beta(x - y)\partial_\alpha \partial_\beta(\Gamma(y)\psi_i(y))dy, \quad x \in \mathbb{R}^n.
\end{equation}
Let $j \in \{m, \ldots, k-1\}$ be fixed. Our aim is to estimate $osc_{r, 0}(f^m_k; x_0, 2^j)$. We decompose $f^m_k$ into the sum $g^m_k + G^m_k$, where
\begin{align}
g^m_k(x) &= \sum_{i=m}^{j} \int_{\mathbb{R}^n} v_\alpha(x - y)v_\beta(x - y)\partial_\alpha \partial_\beta(\Gamma(y)\psi_i(y))dy, \\
G^m_k(x) &= \sum_{i=j+1}^{k} \int_{\mathbb{R}^n} v_\alpha(x - y)v_\beta(x - y)\partial_\alpha \partial_\beta(\Gamma(y)\psi_i(y))dy, \quad x \in \mathbb{R}^n.
\end{align}
Recalling that $\nabla \cdot v = 0$, we obtain
\[
g^k_m(x) = \sum_{l=m_{2^n}}^j \int (v_\alpha(x-y) - P_{x_0,2^{j+2}}^0)(v_\beta(x-y) - P_{x_0,2^{j+2}}^0) \partial_\alpha \partial_\beta (\Gamma(y) \psi_\gamma(y)) dy.
\]

Arguing as in the proof of Theorem 3.1, by the aid of Calderón-Zygmund’s estimate along with Hölder’s inequality we find
\[
\|g^k_m\|_{L^r(B(x_0,2^j))} \leq c \|v - P_{x_0,2^{j+2}}^0(v)\|_{L^{2r}(B(x_0,2^{j+2}))}^2 \leq c 2^{j(2-\frac{p}{r})}\|\nabla v\|_{L^\infty}^{2-\frac{p}{r}} \|v - P_{x_0,2^{j+2}}^0(v)\|_{L^p(B(x_0,2^{j+2}))}^{\frac{p}{r}}.
\]

This shows that
\[
(3.55) \quad \text{osc}(g^k_m; x_0, 2^j) \leq c 2^{j(2-\frac{p}{r})}\|\nabla v\|_{L^\infty}^{2-\frac{p}{r}} \text{osc}(v; x_0, 2^{j+2})^{\frac{p}{r}}.
\]

Arguing as in Lemma 3.1, we obtain
\[
2^j\|\nabla G^k_m\|_{L^\infty(B(x_0,2^j))} \leq c \|\nabla v\|_{L^\infty}^{2-\frac{p}{r}} S_{1,1}(\{2^{(2-\frac{p}{r})i}\text{osc}(v; x_0, 2^i)^{\frac{p}{r}}\}_i)_j.
\]

Using Poincaré’s inequality, this yields
\[
(3.56) \quad \text{osc}(G^k_m; x_0, 2^j) \leq c \|\nabla v\|_{L^\infty}^{2-\frac{p}{r}} S_{1,1}(\{2^{(2-\frac{p}{r})i}\text{osc}(v; x_0, 2^i)^{\frac{p}{r}}\}_i)_j.
\]

Combining (3.55) and (3.56), and letting $m \to -\infty$, we arrive at
\[
(3.57) \quad \text{osc}(f^k; x_0, 2^j) \leq c \|\nabla v\|_{L^\infty}^{2-\frac{p}{r}} S_{1,1}(\{2^{(2-\frac{p}{r})i}\text{osc}(v; x_0, 2^i)^{\frac{p}{r}}\}_i)_j.
\]

Recalling that $r < p(1-s)$, we have $2 - (1 - s)^\frac{p}{r} < 1$. Applying $S_{2-(1-s)^\frac{p}{r},q}$ to both sides of (3.57), thanks to Lemma 2.1 with $\alpha = 1$ and $\beta = 2 - (1 - s)^\frac{p}{r}$ we get
\[
S_{2-(1-s)^\frac{p}{r},q}(\text{osc}(f^k; x_0))_j \leq c \|\nabla v\|_{L^\infty}^{2-\frac{p}{r}} |v|_{H^{2-\frac{p}{2r}}(p,0)}^{\frac{p}{2} - 1} S_{2-(1-s)^\frac{p}{r},q}(\{2^{(2-\frac{p}{r})i}\text{osc}(v; x_0, 2^i)^{\frac{p}{r}}\}_i)_j.
\]

Multiplying both sides by $2^{j(2-(1-s)^\frac{p}{r})}$, and taking the supremum over all $x_0 \in \mathbb{R}^n$, we arrive at
\[
(3.58) \quad |f^k|_{H^{2-(1-s)^\frac{p}{r}}(p,0)} \leq c \|\nabla v\|_{L^\infty}^{2-\frac{p}{r}} |v|_{H^{2-\frac{p}{2r}}(p,0)}^{\frac{p}{2} - 1}.
\]
Set \( g^k = f^k - P_{0,1}^0(f^k) \). In view of (3.58) \( \{g^k\} \) is bounded in \( \mathcal{L}_{q(r,0)}^{2-(1-s)r}(\mathbb{R}^n) \). Thus, by the compact embedding \( \mathcal{L}_{q(r,0)}^{2-(1-s)r}(\mathbb{R}^n) \hookrightarrow L_{loc}^r(\mathbb{R}^n) \), eventually passing to a subsequence, we get \( \pi \in \mathcal{L}_{q(r,0)}^{2-(1-s)r}(\mathbb{R}^n) \) such that

\[
(3.59) \quad g^k \to \pi \quad \text{in} \quad L_{loc}^r(\mathbb{R}^n) \quad \text{as} \quad k \to +\infty.
\]

Arguing as in the proof of Theorem 3.5, it can be checked that \( \Delta \pi = -\nabla \cdot \nabla \cdot (v \otimes v) \) in the sense of distributions. Using (3.59), we immediately get (3.54) from the first inequality in (3.57) and \( P_{0,1}^0(\pi) = 0 \) from \( P_{0,1}^0(g^k) = 0 \).

**Uniqueness.** Assume there is a second very weak solution \( \overline{\pi} \in \mathcal{L}_{q(r,0)}^{2-(1-s)r}(\mathbb{R}^n) \) to (3.53). Then by Weyl’s Lemma \( \pi - \overline{\pi} \) is harmonic. Thus, by Liouville’s theorem of harmonic functions it follows that \( \pi - \overline{\pi} \) is constant. Taking into account the condition \( P_{0,1}^0(\pi - \overline{\pi}) = 0 \) we obtain \( \pi = \overline{\pi} \).

A careful inspection of the proof of Theorem 3.8 shows that we may remove the condition \( v \in C^{0,1}(\mathbb{R}^n) \) in case \( 2 < p < +\infty \) and \( r = \frac{p}{q} \). Thus, we have the following

**Corollary 3.9.** Let \( 1 \leq q \leq +\infty, 2 < p < +\infty \) and \( s \in [0, \frac{1}{2}] \). Let \( v \in \mathcal{L}_{q(p,0),\sigma}(\mathbb{R}^n) \). There exists a unique very weak solution \( \pi \in \mathcal{L}_{q(p,0),\sigma}(\mathbb{R}^n) \) to (3.53) with \( P_{0,1}^0(\pi) = 0 \). Furthermore, it holds

\[
(3.60) \quad |\pi|_{\mathcal{L}_{q(p,0),\sigma}^{2s}} \leq c|v|_{\mathcal{L}_{q(p,0),\sigma}^{-s}}^2.
\]

In particular, if \( v \in \text{BMO} \), then \( \pi \in \text{BMO} \) and it holds

\[
(3.61) \quad |\pi|_{\text{BMO}} \leq c|v|_{\text{BMO}}^2.
\]

## 4 Proof of Theorem 1

Before turning to the proof of Theorem 1 we state the following local energy inequality for the transport equation, which is proved in [9].

**Lemma 4.1.** Given \( v \in L^1(0, T; C^{0,1}(\mathbb{R}^n)) \) and \( g \in L^1(0, T; L_{loc}^p(\mathbb{R}^n)) \), let \( f \in C([0, T]; W_{loc}^{1,p}(\mathbb{R}^n)) \) be a weak solution to the transport equation

\[
(4.1) \quad \partial_t f + v \cdot \nabla f = g \quad \text{in} \quad Q_T.
\]

Let \( N \in \mathbb{N}_0 \). Then the following inequality holds for all \( t \in [0, T] \)

\[
\text{osc}_{p,\max \{2N-1,N\}} \left( f(t); x_0; \frac{r}{2} \right) \leq c \text{osc}_{p,N}(f(0); x_0, r) + c r^{-1} \int_0^t \|v(\tau)||_{L^\infty(B(x_0,r))} \text{osc}_{p,N}(f(\tau); x_0, 2r) d\tau
\]

\[
+ c \int_0^t \|\nabla \cdot v(\tau)||_{L^\infty(B(x_0,r))} \text{osc}_{p,N}(f(\tau); x_0, 2r) d\tau
\]

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\[ + \delta_{N\epsilon} c \int_0^t \frac{\text{osc}(v(\tau); x_0, r)}{p_N} \| \nabla P_{x_0,r}^N(f(\tau)) \|_{L^\infty(B(x_0,r))} d\tau \]
\[ + c \int_0^t \frac{\text{osc}(g(\tau); x_0, r)}{p_N} d\tau, \]

(4.2)

where \( \delta_{N\epsilon} = 0 \) if \( N = 0 \) and 1 otherwise.

Remark 4.2. Given \( \nu \in L^1(0,T; C^{0,1}(\mathbb{R}^n, \mathbb{R}^n)) \), and \( \pi \in L^1(0,T; W^{1,2}_{loc}(\mathbb{R}^n, \mathbb{R}^n)) \), let \( f \in L^\infty(0,T; C^{0,1}(\mathbb{R}^n, \mathbb{R}^n)) \) with \( \nabla \cdot f = 0 \) be a weak solution to the system

(4.3) \[ \partial_t f + (v \cdot \nabla) f = -\nabla \pi \quad \text{in} \quad Q_T. \]

Then, (4.2) can be replaced by

\[ \frac{\text{osc}(f(t); x_0, \frac{r}{2})}{2} \leq c \frac{\text{osc}(f(0); x_0, r)}{2} + cr^{-1} \int_0^t \| v(\tau) \|_{L^\infty(B(x_0,r))} \frac{\text{osc}(f(\tau); x_0, 2r)}{2} d\tau \]
\[ + c \int_0^t \| \nabla \cdot v(\tau) \|_{L^\infty(B(x_0,r))} \frac{\text{osc}(f(\tau); x_0, 2r)}{2} d\tau \]
\[ + c \int_0^t \frac{\text{osc}(v(\tau); x_0, r)}{2} \| \nabla P_{x_0,r}^1(f(\tau)) \| d\tau \]
\[ + c \int_0^t \frac{\text{osc}(\nabla \pi(\tau); x_0, r)}{2} d\tau. \]

(4.4)

Proof of Theorem

The proof of Theorem is based on a fixed point argument using Banach’s fixed point theorem. Let \( v_0 \in \mathcal{L}_{1(p,1)}^0(\mathbb{R}^n) \) be arbitrarily chosen. Let \( T_0 = \frac{1}{c \| v_0 \|_{\mathcal{L}_{1(p,1)}^0}} \), with a constant \( c > 0 \) which will be specified below. We construct an operator \( T : L^\infty(0,T_0; \mathcal{L}_{1(p,1)}^0(\mathbb{R}^n)) \to L^\infty(0,T_0; \mathcal{L}_{1(p,1)}^0(\mathbb{R}^n)) \) as follows. Given \( u \in L^\infty(0,T_0; \mathcal{L}_{1(p,1)}^0(\mathbb{R}^n)) \) by \( T(u) := v \in L^\infty(0,T_0; \mathcal{L}_{1(p,1)}^0(\mathbb{R}^n)) \) we denote the unique solution to the model problem

(5.4) \[
\begin{cases}
\partial_t v + \langle P(u) \cdot \nabla \rangle v = -\nabla \Pi(P(u), u) & \text{in} \quad Q_{T_0}\\
v = v_0 & \text{on} \quad \mathbb{R}^n \times \{0\}.
\end{cases}
\]

Here \( \nabla \Pi(P(u), u) \in L^\infty(0,T_0; \mathcal{L}_{1(p,1)}^0(\mathbb{R}^n)) \), and \( \pi := \Pi(P(u(\tau)), u(\tau)) \in \mathcal{L}_{1(1)}^0(\mathbb{R}^n) \) stands for the unique solution to the Poisson equation

(5.5) \[
\begin{cases}
-\Delta \pi(\tau) = \langle \nabla P(u(\tau)) \cdot (\nabla u(\tau)) \rangle^T & \text{in} \quad \mathbb{R}^n\\
\nabla \pi(0) = 0, \quad \hat{P}_\infty^1(\nabla \pi(\tau)) = -\frac{1}{\sigma} \hat{P}_\infty^0(\nabla P(u(\tau)) \cdot (\nabla (u(\tau))^T)x
\end{cases}
\]
with \((\pi(\tau))_{B(1)} = 0\). According to \([3.49], [3.50]\) (cf. Remark \[3.7\]) the following estimate holds true for all \(\tau \in [0, T_0]\)
\[
\|\nabla \Pi(P(u(\tau)), u(\tau))\|_{L^1_{t,p}(\mathbb{R}^1)} \leq c\|u(\tau)\|^2_{L^1_{t,p}(\mathbb{R}^1)}.
\]
Furthermore, we wish to remark that \([9, \text{Theorem 1.2}]\) ensures the existence and uniqueness of the solution \(v = T(u)\).

Let \(x_0 \in \mathbb{R}^n\). Let \(0 < t \leq T_0\) be arbitrarily chosen, but fixed. By \(\xi \in C^{1,1}([0, T])\) we denote the unique solution to the ODE
\[
(4.8) \quad \dot{\xi}(\tau) = u(x_0 + \xi(\tau), \tau), \quad \tau \in [0, t], \quad \xi(t) = 0.
\]
We set
\[
V(x, \tau) = v(x + \xi(\tau), \tau),
\]
\[
U(x, \tau) = u(x + \xi(\tau), \tau) - \dot{\xi}(\tau),
\]
\[
g(x, \tau) = -\nabla \Pi(P(u(\tau)), u(\tau))(x + \xi(\tau), \tau), \quad (x, \tau) \in Q_t.
\]
It is readily seen that \(V\) solves the transport equation
\[
(4.9) \quad \partial_t V + (U \cdot \nabla)V = g \quad \text{in} \quad Q_t.
\]
Furthermore, it holds
\[
(4.10) \quad U(x_0, \tau) = 0 \quad \forall \tau \in (0, t], \quad V(t) = v(t).
\]
Let \(j \in \mathbb{Z}\) be arbitrarily chosen. Observing \([4.12]\) with \(V\) (\(U\) respectively) in place of \(v\) (\(u\) respectively), \(r = 2^{j+1}\), and \(N = 1\), using \([4.10]\), we obtain
\[
osc_{p,1}(v(t); x_0, 2^j) \leq c\osc_{p,1}(V(0); x_0, 2^{j+1})
\]
\[
+ c \int_0^t \|\nabla u(\tau)\|_\infty osc_{p,1}(V(\tau); x_0, 2^{j+2})d\tau
\]
\[
(4.11) \quad + c \int_0^t osc_{p,1}(U(\tau); x_0, 2^{j+1})\|\nabla v(\tau)\|_\infty d\tau + c \int_0^t osc_{p,1}(g(\tau); x_0, 2^{j+1})d\tau.
\]
Noting that all functions \(U, V\) and \(g\) belong to \(L^\infty(0, T; \mathcal{L}^1_{t,p}(\mathbb{R}^n))\), we may multiply both sides by \(2^{-j}\), apply the sum over \(j \in \mathbb{Z}\) on both sides and take the supremum over \(x_0 \in \mathbb{R}^n\). This yields
\[
|v(t)|_{\mathcal{L}^1_{t,p}(\mathbb{R}^1)} \leq |V(0)|_{\mathcal{L}^1_{t,p}(\mathbb{R}^1)}
\]
\[
+ c \int_0^t \|\nabla u(\tau)\|_\infty |V(\tau)|_{\mathcal{L}^1_{t,p}(\mathbb{R}^1)} d\tau
\]
\[
(4.12) \quad + c \int_0^t |U(\tau)|_{\mathcal{L}^1_{t,p}(\mathbb{R}^1)} \|\nabla v(\tau)\|_\infty d\tau + c \int_0^t |g(\tau)|_{\mathcal{L}^1_{t,p}(\mathbb{R}^1)} d\tau.
\]
Obviously, for all \( \tau \in (0, t) \),
\[
|V(\tau)|_{\mathcal{X}_1^{1}(p, 1)} = |v(\tau)|_{\mathcal{X}_1^{1}(p, 1)},
\]
\[
|U(\tau)|_{\mathcal{X}_1^{1}(p, 1)} = |u(\tau)|_{\mathcal{X}_1^{1}(p, 1)}.
\]

Furthermore, thanks to (4.7), we see that for all \( \tau \in (0, t) \),
\[
|g(\tau)|_{\mathcal{X}_1^{1}(p, 1)} = |\nabla P(u(\tau)), u(\tau))|_{\mathcal{X}_1^{1}(p, 1)} \leq c\|u(\tau)\|^2_{\mathcal{X}_1^{1}(p, 1)}.
\]

Inserting the estimates above into (4.12) we are led to
\[
|v(t)|_{\mathcal{X}_1^{1}(p, 1)} \leq c|v_0|_{\mathcal{X}_1^{1}(p, 1)} + c\int_0^t (|u(\tau)|_{\mathcal{X}_1^{1}(p, 1)}|v(\tau)|_{\mathcal{X}_1^{1}(p, 1)} + |u(\tau)|_{\mathcal{X}_1^{1}(p, 1)}^2) \, d\tau.
\]

In order to estimate \( |\mathcal{P}^1_{\infty}(v)| \), we argue as follows. Applying \( \mathcal{P}^1_{\infty} \) to both sides of the equation (4.5), and using (4.6), along with (2.17) and (2.19), we see that \( P_v = \mathcal{P}^1_{\infty}(v) \), \( P_u = \mathcal{P}^1_{\infty}(u) \) and \( \mathcal{P}_u = \mathcal{P}^1_{\infty}(\mathbb{P}(u)) \) solve the following transport equation
\[
\frac{d}{dt} P_v + (\mathcal{P}_u \cdot \nabla) P_v = \frac{1}{n} \nabla \mathcal{P}_u : (\nabla P_v) \top x \quad \text{in} \quad Q_{T_0}.
\]

Note that from the definition of \( \mathbb{P} \) we get
\[
\mathcal{P}_u = \mathcal{P}^1_{\infty}(u) - \frac{1}{n} P^0_{\infty}(\nabla \cdot u)x = P_u - \frac{1}{n} (\nabla \cdot P_u)x,
\]

Applying \( \nabla \) to both sides of (4.14), we see that \( A := \nabla P_v \in C^1([0, T_0]; \mathbb{R}^{n^2}) \) solves the ODE
\[
\frac{d}{dt} A + \nabla \mathcal{P}_u \cdot A = \frac{1}{n} \nabla \mathcal{P}_u : (\nabla P_u) \top I \quad \text{in} \quad (0, T_0).
\]

Multiplying both sides by \( \frac{\partial A(t)}{\partial t} A(t) \), integrating the result over \( (0, t) \), \( t \in (0, T_0] \) and applying integration by parts, we obtain
\[
|A(t)| \leq |A(0)| + c\int_0^t |A(\tau)| \left( |\nabla \mathcal{P}_u(\tau)| + |\nabla \mathcal{P}_u(\tau) : (\nabla P_u(\tau)) \top \right) \, d\tau
\]
\[
\leq |A(0)| + c\int_0^t |A(\tau)| \left( |P^0_{\infty}(\nabla u(\tau))| + |P^0_{\infty}(\nabla u(\tau))|^2 \right) \, d\tau.
\]

Combining (4.13) and (4.16), we obtain for all \( t \in (0, T_0] \)
\[
\|v(t)\|_{\mathcal{X}_1^{1}(p, 1)} \leq c\|v_0\|_{\mathcal{X}_1^{1}(p, 1)} + c\int_0^t (|u(\tau)|_{\mathcal{X}_1^{1}(p, 1)}|v(\tau)|_{\mathcal{X}_1^{1}(p, 1)} + |u(\tau)|_{\mathcal{X}_1^{1}(p, 1)}^2) \, d\tau.
\]
Using Gronwall’s Lemma, we infer from (4.17) for all $t \in [0, T_0]$

$$
\|v(t)\|_{\mathcal{H}_{1,p,1}}^2 \leq c_0 \left\{ \|v_0\|_{\mathcal{H}_{1,p,1}}^2 + \int_0^{T_0} \|u(\tau)\|^2_{\mathcal{H}_{1,p,1}} d\tau \right\} \exp \left( c_0 \int_0^{T_0} \|u(\tau)\|_{\mathcal{H}_{1,p,1}} d\tau \right).
$$

(4.18)

Assume that $\|u\|^2_{L^\infty(0,T_0;\mathcal{H}_{1,p,1})} \leq 2c_0e\|v_0\|_{\mathcal{H}_{1,p,1}}$, and

$$
T_0 = \frac{1}{8c_0^2e^2\|v_0\|_{\mathcal{H}_{1,p,1}}}.
$$

Then, (4.18) gives

$$
\|v\|^2_{L^\infty(0,T_0;\mathcal{H}_{1,p,1})} \leq c_0 \left\{ \|v_0\|_{\mathcal{H}_{1,p,1}} + T_04c_0^2e^2\|v_0\|^2_{\mathcal{H}_{1,p,1}} \right\} \exp \left( 2c_0^2eT_0\|v_0\|_{\mathcal{H}_{1,p,1}} \right)
$$

$$
\leq 2c_0e\|v_0\|_{\mathcal{H}_{1,p,1}}.
$$

This shows that $T|_M : M \to M$, where

$$
M = \left\{ u \in L^\infty(0,T_0;\mathcal{H}_{1,p,1})(\mathbb{R}^n) \mid \|u\|_{L^\infty(0,T_0;\mathcal{H}_{1,p,1})} \leq 2c_0e\|v_0\|_{\mathcal{H}_{1,p,1}} \right\}.
$$

Proof that $T|_M$ is a contractive. Let $u_1, u_2 \in M$ be given. Set $v_i = T(u_i), i = 1, 2$, and define $w = v_1 - v_2$. Then $w$ solves the transport equation

$$
\begin{cases}
\partial_t w + (\mathbb{P}(u_1) \cdot \nabla)w = -\nabla \Pi(\mathbb{P}(u_1), u_1 - u_2) - \nabla \Pi(\mathbb{P}(u_1 - u_2), u_2) & \text{in } Q_{T_0} \\
w = 0 & \text{on } \mathbb{R}^n \times \{0\}.
\end{cases}
$$

(4.19)

Arguing as above, we get the estimate

$$
\|w(t)\|_{\mathcal{H}_{1,p,1}} \leq c \int_0^{t} \|u_1(\tau)\|_{\mathcal{H}_{1,p,1}} \|w(\tau)\|_{\mathcal{H}_{1,p,1}} d\tau
$$

$$
+ \int_0^{t} \|u_1(\tau) - u_2(\tau)\|_{\mathcal{H}_{1,p,1}} \|v_1(\tau)\|_{\mathcal{H}_{1,p,1}} d\tau
$$

$$
+ \int_0^{t} (\|u_1(\tau)\|_{\mathcal{H}_{1,p,1}} + \|u_2(\tau)\|_{\mathcal{H}_{1,p,1}}) \|u_1(\tau) - u_2(\tau)\|_{\mathcal{H}_{1,p,1}} d\tau.
$$
Applying Gronwall’s lemma, we arrive at
\[ \|w(t)\|_{\dot{X}^1_{1,p}(\mathbb{R}^n)} \]
\[ \leq c_0 \int_0^{T_0} (\|u_1(\tau)\|_{\dot{X}^1_{1,p}(\mathbb{R}^n)} + \|u_2(\tau)\|_{\dot{X}^1_{1,p}(\mathbb{R}^n)}) \|u_1(\tau) - u_2(\tau)\|_{\dot{X}^1_{1,p}(\mathbb{R}^n)} d\tau \times \]
\[ \times \exp \left(c_0 \int_0^{T_0} \|u_1(\tau)\|_{\dot{X}^1_{1,p}(\mathbb{R}^n)} \right) d\tau \]
\[ \leq 6c_0^2 e^{2T_0}\|v_0\|_{\dot{X}^1_{1,p}(\mathbb{R}^n)} \|u_1 - u_2\|_{L^2(0,T_0;\dot{X}^1_{1,p}(\mathbb{R}^n))} \leq \frac{2}{3}\|u_1 - u_2\|_{L^2(0,T_0;\dot{X}^1_{1,p}(\mathbb{R}^n))}. \]

By virtue of Banach’s fixed point theorem there exists a unique fixed point \( v \in M \) such that \( \mathcal{T}(v) = v \).

In order to verify that \( v \) is a solution to (1.1) it only remains to show that \( \nabla \cdot v = 0 \) or what is equivalent to \( \mathbb{P}(v) = v \). First note that due to the definition of \( \mathcal{T} \), the function \( v \in L^\infty(0,T_0;\dot{X}^1_{1,p}(\mathbb{R}^n)) \) solves the transport equation
\[ \begin{cases} \partial_t v + (\mathbb{P}(v) \cdot \nabla)v = -\nabla \Pi(\mathbb{P}(v),v) & \text{in } \mathbb{R}^n, \\ v = v_0 & \text{on } \mathbb{R}^n \times \{0\}. \end{cases} \]
(4.20)

Applying \( \nabla \cdot \) to both sides of (4.20), we see that
\[ \begin{cases} \partial_t \nabla \cdot v + (\mathbb{P}(v) \cdot \nabla)\nabla \cdot v + \nabla \mathbb{P}(v) \cdot (\nabla v) = (\nabla v) \nabla \Pi(\mathbb{P}(v),v) & \text{in } \mathbb{R}^n, \\ \nabla \cdot v = 0 & \text{on } \mathbb{R}^n \times \{0\}. \end{cases} \]

Accordingly, \( \nabla \cdot v \in L^\infty(Q_{T_0}) \) solves the transport equation with zero data. The strong-weak uniqueness [9, pp. 46] implies \( \nabla \cdot v = 0 \). This completes the proof of Theorem 1.

\( \blacksquare \)

5 Proof of Theorem 2

Let \( v_0 \in \mathcal{L}^1_{1,p}(\mathbb{R}^n) \). Then \( v_0 - v_0(0) \in \dot{\mathcal{L}}^1_{1,p}(\mathbb{R}^n) \). According to Theorem 1 there exists \( T_0 \geq \frac{1}{c|v_0-v_0(0)||\dot{X}^1_{1,p}(\mathbb{R}^n)|} \) and a unique solution \( \tilde{v} \in L^\infty(0,T_0;\dot{\mathcal{L}}^1_{1,p}(\mathbb{R}^n)) \) to the Euler equations (1.1), (1.2) with \( v_0 - v_0(0) \) in place of \( v_0 \) and pressure \( \tilde{p} \in L^\infty(0,T_0;L^2_{loc}(\mathbb{R}^n)) \) such that \( \nabla \tilde{p} \in L^\infty(0,T_0;\dot{\mathcal{L}}^1_{1,p}(\mathbb{R}^n)) \)
\[ \nabla \tilde{p} = \nabla \Pi(v,\tilde{v}). \]

Setting \( v(x,t) = \tilde{v}(x - tv_0(0),t) + v_0(0) \), we see that \( v \in L^\infty(0,T_0;\mathcal{L}^1_{1,p}(\mathbb{R}^n)) \) and solves the Euler equations (1.1), (1.2) with pressure \( p(x,t) = \tilde{p}(x - v_0(0)t,t) \). We now verify that for almost all \( t \in (0,T_0) \)
\[ \nabla p(t) - \nabla p(0,t) = \nabla \Pi(v(t),v(t)). \]
Clearly, as $\nabla \cdot v = 0$ in $Q_{T_0}$ we find that $\pi = p(t) - \nabla p(0, t) \cdot x$ solves the Poisson equation

$$\Delta \pi = \nabla v(t) : (\nabla v)^\top = \nabla \cdot ((v(t) \cdot \nabla) v(t)).$$

Obviously, it holds $\nabla \pi(0) = 0$. It only remains to verify the asymptotics as $|x| \to +\infty$. By the definition of $p$ along with (5.1), recalling the definition of $\nabla \Pi$, it follows that

$$\hat{P}_N^1(\nabla \pi) = \hat{P}_N^1(\nabla p(t)) = \frac{1}{n} P_N^0(\nabla \tilde{v}(t) : (\nabla \tilde{v}(t))^\top)$$

$$= \frac{1}{n} P_N^0(\nabla v(t) : (\nabla v(t))^\top).$$

6 Proof of Theorem 3

Let $0 < T < +\infty$. We begin our discussion with the following oscillation estimate. Let $v \in L^1(0, T; C^{0,1}(\mathbb{R}^n))$. Let $x_0 \in \mathbb{R}^n$. We call $\xi \in C^1([0, T])$ an canceling shift in $x_0$ if $\xi$ satisfies the following ODE

$$\dot{\xi}(t) = v(x_0 + \xi(t), t) \quad \forall t \in (0, T).$$

Theorem 6.1. Let $1 \leq q \leq +\infty, N \in \mathbb{N}_0$, $s \in [0, N + 1)$, and let $P \in L^\infty(0, T; \mathcal{P}_N)$. Let $v \in L^\infty(0, T; C^{0,1}(\mathbb{R}^n))$ be a solution to the Euler equations

$$\begin{cases}
    \nabla \cdot v = 0 & \text{in } Q_T, \\
    \partial_t v + (v \cdot \nabla) v = -\nabla \Pi(v, v) + P & \text{in } Q_T, \\
    v = v_0 & \text{on } \mathbb{R}^n \times \{0\},
\end{cases}$$

with initial value $v_0 \in \mathcal{L}_{q[p, N]}^s(\mathbb{R}^n)$. Then $v \in L^\infty(0, T; \mathcal{L}_{q[p, N]}^s(\mathbb{R}^n))$.

Furthermore, the following estimate holds true for all $x_0 \in \mathbb{R}^n$ and $t \in [0, T]$, and all characteristics $\xi \in C^{1,1}([0, T])$ satisfying (6.1)

$$osc_{p, max\{2N-1, N\}}^{osc}(v(t), x_0 + \xi(t))$$

$$\leq c S_{N+1, 1}^{p, N}(osc(v_0, x_0 + \xi(0))) + c \int_0^t \|\nabla v(\tau)\|_\infty S_{N+1, 1}^{p, N}(osc(v(\tau), x_0 + \xi(\tau)))d\tau.$$

Furthermore, it holds

$$|v(t)|_{\mathcal{V}_{q[p, N]}^s} \leq c |v_0|_{\mathcal{V}_{q[p, N]}^s} \exp\left(c \int_0^t \|\nabla v(\tau)\|_\infty d\tau\right).$$

Proof: 1. First, let us consider the case $N \geq 1$. Let $x_0 \in \mathbb{R}^n$ be fixed. Let $\xi \in C^{1,1}([0, T])$ satisfying (6.1). We set

$$V(x, \tau) = v(x + \xi(\tau), \tau) - \dot{\xi}(\tau), \quad \Pi(x, \tau) = \pi(x + \xi(\tau), \tau),$$

$$\tilde{P}(x, \tau) = P(x + \xi(\tau), \tau), \quad (x, \tau) \in Q_T,$$
where $\nabla \pi = \nabla \Pi(v, v)$. By the definition of $\Pi$ it follows that

\begin{equation}
\nabla \Pi = \nabla \Pi(V; V).
\end{equation}

(6.5)

Clearly, $(V, P)$ solves the transformed Euler equations

\begin{equation}
\partial_t V + (V \cdot \nabla)V = -\nabla \Pi + \tilde{P} - \tilde{\xi} \quad \text{in} \quad Q_T.
\end{equation}

(6.6)

According to (4.2) in Lemma 4.1 with $V, P$

\begin{equation}
\nabla \Pi \text{ solves the transformed Euler equations}
\end{equation}

Clearly, $(6.6)$ in Lemma 4.1 with $v = u = V, g = -\nabla \Pi + \tilde{P} - \tilde{\xi}$ and $r = 2^{j+1}$ we get

\begin{equation}
\text{osc}_{p,2N-1}(V(t); x_0, 2^j) \leq c \text{osc}_{p,N}(V(0); x_0, 2^{j+1})
\end{equation}

(6.7)

\begin{equation}
+ c \int_0^t \|\nabla v(\tau)\|_{\infty} \text{osc}_{p,N}(V(\tau); x_0, 2^{j+2})d\tau + c \int_0^t \text{osc}_{p,N}(\nabla \Pi(\tau); x_0, 2^{j+1})d\tau.
\end{equation}

(6.8)

Thanks to (3.46) (cf. Remark 3.6) with $u = v = V$ it holds

\begin{equation}
\text{osc}_{p,N}(\nabla \Pi(\tau); x_0, 2^{j+1}) \leq c \|\nabla v(\tau)\|_{\infty} S_{N+1,1}(\text{osc}_{p,N}(V(\tau); x_0))_j, \quad j \in \mathbb{Z}.
\end{equation}

(6.9)

By the aid of Gronwall’s lemma we obtain from (6.9) for all $t \in [0, T]$

\begin{equation}
\text{osc}_{p,N}(V(t); x_0) \leq S_{N,1}(\text{osc}_{p,N}(V(t); x_0))
\end{equation}

(6.11)

By the aid of Gronwall’s lemma we obtain from (6.11) for all $t \in [0, T]$

\begin{equation}
\text{osc}_{p,N}(V(t); x_0) \leq S_{N,1}(\text{osc}_{p,N}(V(t); x_0))
\end{equation}

(6.12)

\begin{equation}
\leq c S_{s'',N}(\text{osc}_{p,N}(V(0); x_0)) \exp \left( c \int_0^t \|\nabla v(\tau)\|_{\infty} d\tau \right).
\end{equation}

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Let \( t \in (0, T] \) be arbitrarily chosen, but fixed. Since the constant in the above estimate does not depend on the choice of the characteristic, we may choose \( \xi \in C^{1,1}([0, T]) \) such that \( \xi(t) = 0 \). Then \( V(t) = v(t) \). Thus, replacing in (6.12) \( V(t) \) by \( v(t) \), operating \( S_{s,q} \) to both sides of (6.12), multiplying the result by \( 2^{-js} \), and using (2.2) of Lemma 2.1 we arrive at

\[
\|2^{-sj} \text{osc}(v(t); x_0, 2^j)\|_{L^q_p} \leq c |v_0|_{L^q_{p,N}} \exp \left( c \int_0^t \|\nabla v(\tau)\|_\infty \, d\tau \right).
\]

In (6.13) taking the supremum over all \( x_0 \in \mathbb{R}^n \), we obtain

\[
|v(t)|_{L^q_{p,N}} \leq c |v_0|_{L^q_{p,N}} \exp \left( c \int_0^t \|\nabla v(\tau)\|_\infty \, d\tau \right).
\]

Whence, \( v \in L^\infty(0, T; L^q_{p,N}) \).

2. The case \( N = 0, s \in (0, 1) \). Noting that \( L^q_{s,p,N}(\mathbb{R}^n) \hookrightarrow L^q_{s,p,1}(\mathbb{R}^n) \), we get from the case \( N = 1 \) that \( v \in L^\infty(0, T; L^q_{s,p,1}(\mathbb{R}^n)) \) together with the estimate (6.14). On the other hand, in view of Lemma 2.7, from \( v_0 \in L^q_{s,p,0}(\mathbb{R}^n) \) we deduce that

\[
\hat{P}_1^\infty(v_0) = 0.
\]

Applying \( \hat{P}_1^\infty \) to (6.2), and \( P \in L^\infty(0, T; \mathcal{P}_0) \), using (2.17), (2.19) and (3.48), we see that \( Q(t) = \hat{P}_1^\infty(v(t)) \), solves the equations in \( Q_T \)

\[
\partial_t Q + (Q \cdot \nabla)Q = \frac{1}{n} \nabla Q \cdot \nabla Q^T x, \quad Q(0) = 0.
\]

Applying \( \nabla \) to the above equation, we see that \( A(t) = \nabla Q(t) \) solves the ODE

\[
\partial_t A + A^2 = \frac{1}{n} \text{tr}(A^2) \quad \text{in} \quad (0, T), \quad A(0) = 0.
\]

With the help of Gronwall’s lemma, we easily get \( A(t) = 0 \) for all \( t \in [0, T] \). Once more appealing to Lemma 2.7 it follows that \( v(t) \in L^q_{s,p,0}(\mathbb{R}^n) \), and (6.14) together with (2.21) implies

\[
|v(t)|_{L^q_{s,p,0}} \leq c |v_0|_{L^q_{s,p,0}} \exp \left( c \int_0^t \|\nabla v(\tau)\|_\infty \, d\tau \right).
\]

Whence, \( v \in L^\infty(0, T; L^q_{s,p,0}(\mathbb{R}^n)) \) and (6.3) holds. 

### Remark 6.2.
1. Firstly, we wish to remark that Theorem 6.1 still holds under weaker assumption \( v \in L^1(0, T; C^{0,1}(\mathbb{R}^n)) \) together with the assumption \( v_0 \in L^q_{s,p,N}, \sigma(\mathbb{R}^n) \cap L^1_{s,p,N,\sigma}(\mathbb{R}^n) \). In fact, from Theorem 2 we get \( v \in L^\infty_{\text{loc}}(0, T_*; L^1_{s,p,N,\sigma}(\mathbb{R}^n)) \) for a maximal time \( T_* > 0 \). Assume \( T_* \leq T \). Then thanks to (6.14) we get \( v \in L^\infty(0, T_*; L^1_{s,p,N,\sigma}(\mathbb{R}^n)) \).
In case $T = T_*$ we get the claim. In the other case since $v(T_*) \in \mathcal{L}^1_{1(p,1),\sigma}(\mathbb{R}^n)$ we are in a position to apply again Theorem 2 which shows that $L^\infty(0, T_*$ + $\delta; \mathcal{L}^1_{1(p,1),\sigma}(\mathbb{R}^n))$ for some $\delta > 0$, which clearly contradicts to the definition of $T_*$. Whence, the claim.

2. As Corollary of the first remark we get the local well-posedness of the Euler equations in $\mathcal{L}^s_{p(N),\sigma}(\mathbb{R}^n) \cap \mathcal{L}^1_{1(p,1),\sigma}(\mathbb{R}^n)$ for $N \in \mathbb{N}_0, 1 < p < +\infty, 1 \leq q \leq +\infty, s \in [0, N + 1)$.

Next, we provide the following uniqueness result

**Lemma 6.3.** Let $(v, p), (u, q) \in L^\infty(0, T; \mathcal{L}^1_{1(p,1),\sigma}(\mathbb{R}^n))$ are two solutions to (1.1), (1.2). Assume that $\hat{P}_\infty^0(\nabla v_0) = 0$, and

\begin{equation}
\tag{6.16}
P_\infty^0(D^2p) = P_\infty^0(D^2q) = 0 \quad \text{in} \quad (0, T).
\end{equation}

Then $(v, p) = (u, q)$.

**Proof:** Set $A(t) = P_\infty^0(\nabla u(t)), t \in (0, T)$. As in the proof of Theorem 1 we get

\[ \hat{A} + A^2 = P_\infty^0(D^2p) = 0 \quad \text{in} \quad (0, T). \]

Owing to $A(0) = \hat{P}_\infty^0(\nabla v_0) = 0$ it follows that $A(t) = 0$ for all $t \in (0, T)$. Whence, $\hat{P}_\infty^0(\nabla v) = 0$ in $(0, T)$. Analogously, we see that $\hat{P}_\infty^0(\nabla u) = 0$ in $(0, T)$.

Next, let $\xi \in C^{1,1}([0, T])$ be the unique solution to the ODE

\[ \dot{\xi}(t) = v(\xi(t), t) \quad t \in (0, T), \quad \xi(0) = 0. \]

Set

\[ V(x, t) = v(\xi(t), t) - \dot{\xi}(t), \quad P(x, t) = p(\xi(t), t), \quad (x, t) \in QT. \]

Clearly, $(V, P) \sim (v, p)$ and $(V, P)$ is a centered solution to (1.1), (1.2). In addition, $P_\infty^0(D^2p) = 0$ implies $P_\infty^0(D^2P) = 0$, and $P_\infty^0(\nabla v) = 0$ implies $P_\infty^0(\nabla V) = 0$. Hence,

\[ \hat{P}_\infty^1(\nabla P) = 0 = -\frac{1}{n} P_\infty^0(\nabla v : (\nabla v)^\top)x. \]

Noting that (1.1) and $V(0, t) = 0$ implies $\nabla P(0, t) = 0$ we infer $P = \Pi(V, V)$. This shows that $(V, P)$ is eligible. Similar, there exists a unique centered solution $(U, Q) \sim (u, q)$. Since $P_\infty^0(D^2q) = 0$ and $P_\infty^0(\nabla u) = 0$ this solution is eligible too. According to Theorem 1 these solutions are unique, which gives $(V, P) = (U, Q)$. Accordingly, $(v, p) = (u, q)$. \(\blacksquare\)

**Proof of Theorem 3.** Let $v_0 \in \mathcal{L}^1_{1(p,1),\sigma}(\mathbb{R}^n) \cap BMO$. Set $u_0 = v_0 - v_0(0) \in \mathcal{L}^1_{1(p,1),\sigma}(\mathbb{R}^n) \cap BMO$. According to Theorem 1 there exists a unique centered eligible solution $(u, \pi) \in L^\infty(0, T_0; \mathcal{L}^1_{1(p,1),\sigma}(\mathbb{R}^n) \times L^2_{B^\infty}(\mathbb{R}^n))$ to (1.1) with $u_0$ in place of $v_0$, where $T_0 > 0$ satisfies $T_0 \geq \frac{1}{c\|u_0\|_{\mathcal{L}^1_{1(p,1)}}} = \frac{1}{c\|v_0\|_{\mathcal{L}^1_{1(p,1)}}}$. By the definition of eligible centered solutions to (1.1) it holds

\begin{equation}
\tag{6.17}
\nabla \pi(t) = \nabla \Pi(u(t), u(t)) \quad \forall t \in [0, T].
\end{equation}
As it has been proved in Theorem 6.1, \( \dot{P}_\infty^1(u_0) = 0 \) implies \( \dot{P}_\infty^1(u(\tau)) = 0 \) for all \( \tau \in (0,T_0) \). Thus, it holds

\[
(6.18) \quad -\Delta \pi = \nabla u : (\nabla u)^\top \quad \text{in} \quad \mathbb{R}^n, \quad \dot{P}_\infty^1(\nabla \pi) = 0.
\]

Let \( x_0 \in \mathbb{R}^n \). In view of (6.3) of Theorem 6.1 with \( N = 0, s = 0 \) and \( q = \infty \) we have \( u \in L^\infty(0,T_0;BMO) \) and it holds for all \( t \in (0,T_0) \)

\[
(6.19) \quad |u(t)|_{BMO} \leq c|v_0|_{BMO} \exp \left( c \int_0^t \|\nabla u(\tau)\|_{\infty} d\tau \right).
\]

Let \( \tau \in (0,T) \). Thanks to Corollary 3.9 with \( s = 0 \) and \( q = \infty \) there exists a unique very weak solution \( \pi_0(\tau) \in \mathcal{L}_{0(p,0)}^0(\mathbb{R}^n) \cong BMO \) to

\[
-\Delta \pi_0(\tau) = \nabla \cdot \nabla \cdot (v(\tau) \otimes v(\tau)) \quad \text{in} \quad \mathbb{R}^n
\]

with \( P_{0,1}^0(\pi_0(\tau)) = 0 \). This implies that \( P(\tau) := \pi(\tau) - \pi_0(\tau) \) is harmonic. By virtue of the Liouville theorem for harmonic functions it follows that \( P(\tau) \in \mathcal{P}_2 \). Noting that \( \dot{P}_\infty^1(\nabla \pi_0(\tau)) = 0 \), and observing (6.18), we see that \( \dot{P}_\infty^1(\nabla P(\tau)) = 0 \). Accordingly, \( P(\tau) \in \mathcal{P}_1 \), and \( \nabla P(\tau) \) is constant for all \( \tau \in [0,T_0] \). Set \( \eta(\tau) = \nabla P(\tau) \). Define,

\[
\xi(t) = -\int_0^t \int_0^\tau \eta(s)d\sigma d\tau - tv_0(0), \quad t \in [0,T_0].
\]

Set

\[
v(x,t) = u(x + \xi(t),t) - \dot{\xi}(t),
\]

\[
p(x,t) = \pi(x + \xi(t),t) - P(x + \xi(t),t) = \pi_0(x + \xi(t),t), \quad (x,t) \in Q_{T_0}.
\]

Clearly, \((v,p) \in L^\infty(0,T;\mathcal{L}_{1(p,1),\sigma}^1(\mathbb{R}^n) \cap BMO \times BMO)\) solves the Euler equations (1.1), (1.2). Eventually, replacing \( p \) by \( p - P_{0,1}^0(p) \) we may assume that \( P_{0,1}^0(p) = 0 \) is satisfied. Furthermore, thanks to (3.61) it holds

\[
(6.20) \quad \|p(\tau)\|_{BMO} \leq c\|v(\tau)\|_{BMO}^2 \quad \forall \tau \in (0,T_0).
\]

**Uniqueness.** Let \((\tilde{v},\tilde{p}) \in L^\infty(0,T;\mathcal{L}_{1(p,1),\sigma}^1(\mathbb{R}^n) \cap BMO \times BMO)\) be another solution to (1.1), (1.2), with \( P_{0,1}^0(\tilde{p}) = 0 \). From Theorem 3.4 we get \( \nabla \tilde{p}(t) \in \mathcal{L}_{1(p,1)}^1(\mathbb{R}^n) \). Clearly, \( \lim_{m \to \infty} P_{0,2m}^2(\tilde{p}(t)) = 0 \). This shows that \( P_{\infty}^0(D^2\tilde{p}(t)) = 0 \) for all \( t \in [0,T] \). Analogously, \( P_{\infty}^0(D^2p(t)) = 0 \) for all \( t \in [0,T] \). Applying Lemma 2.10 with \( s < 1 \) we see that \( P_{\infty}^0(\nabla \tilde{v}_0) = 0 \). We are now in a position to apply Lemma 6.3 which yields \((v,p) = (\tilde{v},\tilde{p})\).

It only remains to prove that \( v_0 \in L^\infty(\mathbb{R}^n) \) implies that \( v \in L^\infty(Q_{T_0}) \). Since \( v \in L^\infty(0,T_0;C^{0,1}(\mathbb{R}^n)) \), to verify the claim it will be sufficient to prove

\[
(6.21) \quad |P_{x_0,1}^0(v(t))| \leq c \left\{ |v_0|_{\infty} + \int_0^t |v(\tau)|_{BMO} d\tau \right\} \exp \int_0^t \|\nabla v(\tau)\|_{\infty} d\tau \quad \forall x_0 \in \mathbb{R}^n.
\]
In fact, applying $P^0_{x_0,1}$ to both sides of (1.1), we get the identity

$$
\frac{d}{dt}P^0_{x_0,1}(v) = -P^0_{x_0,1}(v \cdot \nabla)v - P^0_{x_0,1}(\nabla p).
$$

(6.22)

In particular, for all $m,k$.

The first term can be estimated by $|P^0_{x_0,1}(v(\tau))||\nabla v(\tau)||_\infty$, while the remaining two terms are bounded by $|v(\tau)|^2_{BMO} + |p(\tau)|_{BMO} \leq c|v(\tau)|^2_{BMO}$, where we have used (6.20). Thus,

$$
\frac{d}{dt}P^0_{x_0,1}(v) \leq P^0_{x_0,1}(v)||\nabla v||_\infty + c|v|^2_{BMO}.
$$

Using Gronwall’s lemma, we get (6.21). We now easily estimate

$$
|v(x_0, \tau)| \leq |v(x_0, \tau) - P^0_{x_0,1}(v(\tau))| + |P^0_{x_0,1}(v(\tau))| \leq c||\nabla v(\tau)||_\infty + |P^0_{x_0,1}(v(\tau))|.
$$

Together with (6.21) we see that $v \in L^\infty(Q_{T_0})$. This completes the proof of Theorem 3.

7 Proof of Theorem 4

The proof of Theorem 4 will be carried out, using logarithmic Sobolev type inequality similarly to the decaying case. We provide such inequality for the space $L^{1+\delta}_{1(p,1)}(\mathbb{R}^n)$.

**Lemma 7.1** (Logarithmic inequality). Let $u \in L^{1+\delta}_{1(p,1)}(\mathbb{R}^n) \cap BMO_1, \delta \in (0, 1)$. Then for all $x_0 \in \mathbb{R}^n$ and all $k \in \mathbb{Z}$ it holds

$$
\sum_{j=-\infty}^{k} 2^{-j} \text{osc}_{p,1}(u; x_0, 2^j) \leq c2^{k \delta} + c|\nabla u|_{BMO} \log(1 + |u|_{L^{1+\delta}_{1(p,1)}}).
$$

(7.1)

In particular, for all $m, k \in \mathbb{N}$ with $m < k$ it holds

$$
|P^0_{x_0,2^m}(\nabla u)| \leq c2^{k \delta} + |P^0_{x_0,2^k}(\nabla u)| + c|\nabla u|_{BMO} \log(1 + |u|_{L^{1+\delta}_{1(p,1)}}).
$$

(7.2)

**Proof:** 1. Let $k \in \mathbb{Z}$, and let $l \in \mathbb{N}$, specified below. Using Hölder’s inequality and Poincaré’s inequality, we easily get

$$
\sum_{j=-\infty}^{k} 2^{-j} \text{osc}_{p,1}(u; x_0, 2^j)
$$

$$
= \sum_{j=-\infty}^{k-l} 2^{-j} \text{osc}_{p,1}(u; x_0, 2^j) + \sum_{j=k-l+1}^{k} 2^{-j} \text{osc}_{p,1}(u; x_0, 2^j)
$$

$$
\leq \frac{2^{\delta(k-l)}}{1-2^\delta} |u|_{L^{1+\delta}_{1(p,1)}} + cl|\nabla u|_{BMO}.
$$
Choosing $$l = \left\lfloor \frac{1}{\log 2} \log(1 + |u|_{L^{1+\delta}_{2(1,p,1)}}) \right\rfloor + 1$$, we infer from the above estimate

$$\sum_{j=-\infty}^{k} 2^{-j} \text{osc}(u; x_0, 2^j) \leq c 2^{dk} + c |\nabla u|_{\text{BMO}} \log(1 + |u|_{L^{1+\delta}_{2(1,p,1)}}).$$

Whence, (7.1).

2. Let $$m, k \in \mathbb{Z}, m < k$$. Arguing as in the proof of Theorem 2.6 and using (7.1), we estimate

$$|P^0_{x_0,2^m}(\nabla u)| \leq |P^0_{x_0,2^k}(\nabla u)| + \sum_{j=m}^{k} 2^{-j} \text{osc}(u; x_0, 2^j)$$

(7.3)

$$\leq c 2^{dk} + |P^0_{x_0,2^k}(\nabla u)| + c |\nabla u|_{\text{BMO}} \log(1 + |u|_{L^{1+\delta}_{2(1,p,1)}}).$$

This completes the proof of (7.2).

**Proof of Theorem 4**

1. First applying the known Calderon-Zygmund estimate in BMO to the Biot-Savart formula, we get the estimate

$$|\nabla v(\tau)|_{\text{BMO}} \leq c |\omega(\tau)|_{\text{BMO}} \quad \forall \tau \in [0, T_*).$$

(7.4)

2. Let $$x_0 \in \mathbb{R}^n$$ be fixed. Let $$k \in \mathbb{Z}$$ be appropriately chosen, which will be specified below. Our aim is to provide an uniform bound for $$\sup_{j \geq k} |P^0_{x_0,2^j}(\nabla u)|$$. Let $$t \in (0, T_*)$$ be fixed. Let $$\xi \in C^{1,1}([0, T_*])$$ be a characteristic such that

$$\dot{\xi}(t) = P^0_{x_0,2^k}(v(\cdot + \xi(t), t)) \quad \forall t \in (0, T_*), \quad \xi(t) = 0.$$

We set

$$V(x, t) = v(x + \xi(t), t) - \dot{\xi}(t), \quad \Pi(x, t) = \pi(x + \xi(t), t) + \ddot{\xi}(t)x, \quad (x, t) \in Q_{T_*}.$$ 

Clearly,

$$P^0_{x_0,2^k}(V(t)) = 0 \quad \forall t \in (0, T_*),$$

and $$(V, \Pi)$$ solves the Euler equations.

(7.5)

$$\begin{cases}
\nabla \cdot V = 0 & \text{in } Q_{T_*}, \\
\partial_t V + (V \cdot \nabla)V = -\nabla \Pi & \text{in } Q_{T_*}.
\end{cases}$$

In view (4.4), putting $$f = V, \pi = \Pi$$ and $$r = 2^{j+2}$$ therein, we find

$$\text{osc}(v(t); x_0, 2^j) \leq c \text{osc}(v_0; x_0, 2^{j+1}) + c 2^{-j} \int_0^t \|V(\tau)\|_{L^\infty(B(x_0,2^{j+1}))} \text{osc}(V(\tau); x_0, 2^{j+2}) d\tau$$

$$+ c \int_0^t \text{osc}(V(\tau); x_0, 2^{j+1}) |P^0_{x_0,2^{j+1}}(\nabla V(\tau))| d\tau$$

(7.6)

$$+ c \int_0^t \text{osc}(\nabla \Pi(\tau); x_0, r) d\tau.$$
Using Poincare’s inequality along with (7.4), we find for all $i \geq j$

$$|P_{x_0, 2^i}^0 (V(\tau)) - P_{x_0, 2^{i+1}}^0 (V(\tau))| \leq c 2^i |\omega(\tau)|_{BMO} + c 2^i \sup_{i \geq j} |P^0_{x_0, 2^i} (\nabla V(\tau))|.$$  

(7.7)

Hence, recalling that $P_{x_0, 2^k}^0 (V(\tau)) = 0$, using triangle inequality, we estimate for all $j \geq k$

$$|P_{x_0, 2^i}^0 (V(\tau))| \leq |P_{x_0, 2^k}^0 (V(\tau))| + c \sum_{i=k}^j 2^i \left( |\omega(\tau)|_{BMO} + c \sup_{i \geq k} |P^0_{x_0, 2^i} (\nabla V(\tau))| \right)$$

(7.8) \[ \leq c 2^j \left( |\omega(\tau)|_{BMO} + c \sup_{i \geq k} |P^0_{x_0, 2^i} (\nabla V(\tau))| \right). \]

By virtue of Sobolev-Poincaré’s inequality together with (7.8) and (7.4) we estimate for all $\tau \in (0, T)$

$$2^{-j} \|V(\tau)\|_{L^\infty(B(x_0, 2^{i+1}))}$$

$$\leq c 2^{-j} \|\nabla V(\tau) - \nabla P_{x_0, 2^{i+1}}^1 (V(\tau))\|_{L^2(B(x_0, 2^{i+1}))} + c 2^{-j} \|P_{x_0, 2^{i+1}}^1 (V(\tau))\|_{L^2(B(x_0, 2^{i+1}))}$$

$$\leq c \|\nabla v(\tau)|_{BMO} + c 2^{-j} \|P_{x_0, 2^{i+1}}^0 (V(\tau))\| + c \|P_{x_0, 2^{i+1}}^0 (\nabla V(\tau))\|$$

$$\leq c |\omega(\tau)|_{BMO} + c \|P_{x_0, 2^{i+1}}^0 (\nabla V(\tau))\|.$$  

We also need to estimate the pressure. Noting that

$$\text{osc} \left( \nabla \pi(\tau); x_0, 2^j \right) = \text{osc} \left( \nabla \Pi(V(\tau), V(\tau)); x_0, 2^j \right)$$

consulting (3.52) with $r = \frac{2n}{n+2}$ and $p = 2$, and applying (7.4), we get, for all $\tau \in [0, T_*), x_0 \in \mathbb{R}^n$ and $j \in \mathbb{Z}$,

$$\text{osc} \left( \nabla \Pi(\tau); x_0, 2^j \right) \leq c \left( |\omega(\tau)|_{BMO} + \sup_{i \geq j} |P_{x_0, 2^i}^0 (\nabla V(\tau))| \right) S_{2,1}(\text{osc}(V(\tau); x_0))_j.$$  

(7.9)

Inserting the above estimates into the right-hand side of (7.6) along with (7.4) and (7.3), we get for all $j \in \mathbb{Z}$,

$$\text{osc}_{2,1} (v(t); x_0, 2^j) \leq c \text{osc}_{2,1} (V(0); x_0, 2^j)$$

$$+ c \int_0^t \left( |\omega(\tau)|_{BMO} + \sup_{i \geq j} |P_{x_0, 2^i}^0 (\nabla V(\tau))| \right) S_{2,1}(\text{osc}(V(\tau); x_0))_j d\tau.$$  

(7.10)

Applying $S_{1,1}$ to both sides of the above inequality, and using Lemma 2.1, we get,

$$S_{1,1}(\text{osc}_{2,1} (v(t); x_0))_k$$

$$\leq c S_{1,1}(\text{osc}_{2,1} (V(0); x_0))_k + c \int_0^t \left( |\omega(\tau)|_{BMO} + \sup_{i \geq k} |P_{x_0, 2^i}^0 (\nabla V(\tau))| \right) S_{1,1}(\text{osc}(V(\tau); x_0))_k d\tau.$$  

(7.11)
On the other hand, estimating for all $i \in \mathbb{Z}$
\[
|P_{x_0,2^i}(\nabla V(\tau)) - P_{x_0,2^{i+1}}(\nabla V(\tau))| \leq c2^{-i}\text{osc}(V(\tau); x_0, 2^{i+1}),
\]
using triangle inequality, we find for $l \in \mathbb{Z}$
\[
|P_{x_0,2^l}(\nabla V(\tau))| \leq c \sum_{i=l}^{\infty} 2^{-i}\text{osc}(V(\tau); x_0, 2^i) + c|P_\infty(\nabla v(\tau))|.
\]
In particular,
\[
\sup_{i \geq k} |P_{x_0,2^i}(\nabla V(\tau))| \leq c \sum_{i=k}^{\infty} 2^{-i}\text{osc}(V(\tau); x_0, 2^i) + c|P_\infty(\nabla v(\tau))|.
\]
Combining this estimate with (7.11), multiplying the resultant inequality by $2^k$, and taking the supremum over all $x_0 \in \mathbb{R}^n$, we arrive at
\[
\beta_k(t) \leq c_0\beta_k(0) + c_0 \int_0^t \alpha(\tau)\beta_k(\tau) + \beta_k(\tau)^2d\tau,
\]
where
\[
\alpha(\tau) = |\omega(\tau)|_{BMO} + |P_\infty(\nabla v(\tau))|,
\]
\[
\beta_k(\tau) = \sup_{x_0 \in \mathbb{R}^n} \sum_{i=k}^{\infty} 2^{-i}\text{osc}(v(\tau); x_0, 2^i), \quad \tau \in [0, T].
\]
According to our assumption (1.28) we have $\alpha \in L^1(0, T*)$. We define
\[
\varepsilon = \frac{1}{2c_0c_1eT_*}, \quad \text{where} \quad c_1 := c_0e^\int_0^{T*} \alpha(\tau)d\tau.
\]
Observing (1.27), we may choose $k \in \mathbb{Z}$ such that $\beta_k(0) \leq \varepsilon$. Applying Gronwall’s lemma, we deduce from (7.14) for all $t \in (0, T*)$
\[
\beta_k(t) \leq c_1\varepsilon e^{c_1\sup_{\tau \leq t} \beta_k(\tau)}.
\]
Without loss of generality we may assume that $c_1 \geq 1$. Clearly, $\beta(0) < \frac{1}{c_0T_*}$. Assume there exists $t \in [0, T_*]$ such that $\beta_k(t) = \frac{1}{c_0T_*}$ and $\sup_{\tau \leq t} \beta(\tau) \leq \frac{1}{c_0T_*}$. Then (7.15) would imply that
\[
\beta_k(t) = \frac{1}{c_0T_*} \leq c_1\varepsilon e = \frac{1}{2c_0T_*},
\]
which is a contradiction. Consequently,
\[
\beta_k(t) \leq \frac{1}{c_0T_*} \quad \forall \ t \in [0, T_*].
\]
3. We verify that \( v_0 \in L_{1(2,1)}^1(\mathbb{R}^n) \). In fact, since \( \nabla v_0 \in L^\infty(\mathbb{R}^n) \) by the help of H"older’s inequality and Poincaré’s inequality we easily get

\[
\text{osc}(v_0; x_0, 2^j) \leq c 2^{j \frac{q}{1-q}} \text{osc}(v_0; x_0, 2^j) \| \nabla v_0 \|_\infty^{\frac{q}{1-q}}
\]

By means of Hölder’s inequality we find

\[
\sum_{j \in \mathbb{Z}} 2^{-j(1+\frac{q}{1-q})} \text{osc}(v_0; x_0, 2^j) \leq \sum_{j = -\infty}^0 2^{-j(1+\frac{q}{1-q})} \text{osc}(v_0; x_0, 2^j) + c \| \nabla v_0 \|_\infty
\]

\[
\leq 2 \sum_{j = -\infty}^0 2^{j \frac{q}{1-q} - j \frac{q}{1-q}} \text{osc}(v_0; x_0, 2^j) \| \nabla v_0 \|_\infty^{\frac{q}{1-q}} + c \| \nabla v_0 \|_\infty
\]

\[
\leq c \| v_0 \|_\infty \| \nabla v_0 \|_\infty^{\frac{q}{1-q}} + c \| \nabla v_0 \|_\infty.
\]

4. Let \( k \in \mathbb{Z} \) chosen such that \( \beta_k(0) \leq \varepsilon \). Let \( x_0 \in \mathbb{R}^n \). Let \( j \in \mathbb{Z} \). By \((V, P)\) we denote a centered solution in \( x_0 \) to \((\text{II})\), which is equivalent to \((v, p)\). Since \( V(x_0, \tau) = 0 \) for all \( \tau \in (0, T) \), this yields

\[
\lim_{i \to -\infty} P^0_{x_0, 2^i}(V(\tau)) = 0.
\]

Using \((7.17)\) and triangle inequality, we get

\[
2^{-j} |P^0_{x_0, 2^j}(V(\tau))| \leq 2^{-j} \sum_{i = -\infty}^j \left( |\omega|_{\text{BMO}} + \sup_{i \in \mathbb{Z}} |P^0_{x_0, 2^i}(\nabla V(\tau))| \right)
\]

\[
\leq c \left( |\omega|_{\text{BMO}} + \sup_{i \in \mathbb{Z}} |P^0_{x_0, 2^i}(\nabla V(\tau))| \right).
\]

Inserting \((7.17)\) into the right-hand side of \((7.10)\), we obtain

\[
\text{osc}(v(t); x_0, 2^j) \leq c \text{osc}(V(0); x_0, 2^j)
\]

\[
+ \int_0^t \left( |\omega(\tau)|_{\text{BMO}} + \sup_{i \in \mathbb{Z}} |P^0_{x_0, 2^i}(\nabla V(\tau))| \right) S_{2,1}(\text{osc}(V(\tau); x_0)) d\tau.
\]

We proceed with the estimation of \( \sup_{i \in \mathbb{Z}} |P^0_{x_0, 2^i}(\nabla V(\tau))| \). Clearly, by \((7.12)\) we see that for all \( i \in \mathbb{Z} \)

\[
|P^0_{x_0, 2^i}(\nabla V(\tau))| \leq c \sum_{m \in \mathbb{Z}} 2^{-m} \text{osc}(V(\tau); x_0, 2^m) + c |P^0_{\infty}(\nabla v(\tau))| + c \beta_k(\tau) + c |P^0_{\infty}(\nabla v(\tau))|.
\]
By the aid of (7.1) with \( p = 2 \) and \( \frac{\delta}{4} \) in place of \( \delta \) (cf. Lemma 7.1) together with (7.4) we find

\[
\sup_{i \in \mathbb{Z}} |P_{x_0,2i}^0(\nabla V(\tau))| \leq c \left( 2^{\frac{\delta}{4}} + \beta(\tau) + |\omega(\tau)|_{BMO \log(1 + |v(\tau)|_{\mathcal{L}^{1+\frac{\delta}{4}}})} + |P_0^0(\nabla V(\tau))| \right).
\]

Inserting this estimate into the right-hand side of (7.18), and applying \( S_{1+\frac{\delta}{4},1} \) to both sides, using Lemma 2.1, we are led to

\[
S_{1+\frac{\delta}{4},1}(\text{osc}_{2,1}(v(t); x_0))_j \leq c S_{1+\frac{\delta}{4},1}(\text{osc}_{2,1}(V(0); x_0))_j
\]

\[
(7.19)
\]

\[
+ c \int_0^t \left( \alpha(\tau) \log(1 + |v(\tau)|_{\mathcal{L}^{1+\frac{\delta}{4},1}}) + 2^{\delta k(\tau)} \right) S_{1+\frac{\delta}{4},1}(\text{osc}_{2,1}(V(\tau); x_0))_j d\tau.
\]

Multiplying both sides of (7.19) by \( 2^{-j(1+\frac{\delta}{4})} \), taking the supremum over all \( x_0 \in \mathbb{R}^n \) after summing over \( j \in \mathbb{Z} \), and observing (7.16), we deduce that

\[
|v(t)|_{\mathcal{L}^{1+\frac{\delta}{4},1}_1} \leq c|v_0|_{\mathcal{L}^{1+\frac{\delta}{4},1}_1} + c \int_0^t (1 + \alpha(\tau)) \log(e + |v(\tau)|_{\mathcal{L}^{1+\frac{\delta}{4},1}_1}) |v(\tau)|_{\mathcal{L}^{1+\frac{\delta}{4},1}_1} d\tau,
\]

for a constant \( c > 0 \) independent of \( t \). Applying Gronwall’s lemma, we obtain from (7.20) that

\[
|v(t)|_{\mathcal{L}^{1+\frac{\delta}{4},1}_1} \leq \exp \left[ c|v_0|_{\mathcal{L}^{1+\frac{\delta}{4},1}_1} \exp \left( \int_0^t (1 + \alpha(\tau)) d\tau \right) \right].
\]

Accordingly, \( v \in L^\infty(0, T^*; \mathcal{L}^{1+\frac{\delta}{4}}_{1,1}(\mathbb{R}^n)) \). Taking into account (7.16), we see that \( v \in L^\infty(0, T^*; \mathcal{L}^{1}_{1,1}(\mathbb{R}^n)) \). In particular, \( \nabla v \) is bounded. Repeating the above argument and recalling \( v_0 \in \mathcal{L}^{1+\delta}_{q(p,1)}(\mathbb{R}^n) \), we obtain \( v \in L^\infty(0, T^*; \mathcal{L}^{1+\delta}_{q(p,1)}(\mathbb{R}^n)) \), which completes the proof of the theorem.

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