POSITIVE TOPOLOGICAL ENTROPY FOR REEB FLOWS ON 3-DIMENSIONAL ANOSOV CONTACT MANIFOLDS

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Abstract. Let $(M, \xi)$ be a compact contact 3-manifold and assume that there exists a contact form $\alpha_0$ on $(M, \xi)$ whose Reeb flow is Anosov. We show this implies that every Reeb flow on $(M, \xi)$ has positive topological entropy, answering a question raised in [2]. Our argument builds on previous work of the author [2] and recent work of Barthelmé and Fenley [4]. This result combined with the work of Foulon and Hasselblatt [13] is then used to obtain the first examples of hyperbolic contact 3-manifolds on which every Reeb flow has positive topological entropy.

1. Introduction

In this work we show that the existence of an Anosov Reeb flow on a compact contact 3-manifold $(M, \xi)$ implies that all Reeb flows on $(M, \xi)$ have positive topological entropy. It then follows from the works of Katok [20, 21] and Lima and Sarig [23] that every Reeb flow on $(M, \xi)$ contains a “horseshoe” as a sub-system.

The combination of the works of Klingenberg [22] and Manning [26] implies that if a manifold $Q$ admits a Riemannian metric whose geodesic flow is Anosov, then for any Riemannian metric $g$ on $Q$ the associated geodesic flow has positive topological entropy. The results in the present paper can be seen as a generalization of this result to the world of contact 3-manifolds.

1.1. Basic notions.

1.1.1. Contact geometry and Reeb flows. We first recall some basic definitions from contact geometry. A 1-form $\alpha$ on a $(2n + 1)$-dimensional manifold $M$ is called a contact form if $\alpha \wedge (d\alpha)^n$ is a volume form on $M$. The hyperplane distribution $\xi = \ker \alpha$ is called the contact structure. For us a contact manifold will be a pair $(M, \xi)$ such that $\xi$ is the kernel of some contact form $\alpha$ on $M$. If $\alpha$ satisfies $\xi = \ker \alpha$, we will say that $\alpha$ is a contact form on $(M, \xi)$. Given a contact form $\alpha$, its Reeb vector field is the unique vector field $X_\alpha$ satisfying $\alpha(X_\alpha) = 1$.
and $i_{X_{a}}d\alpha = 0$, and the flow $\phi_{a}$ of $X_{a}$ is the Reeb flow of $\alpha$. The periodic orbits of $\phi_{a}$ are called the Reeb orbits of $\alpha$. The action $A(\gamma)$ of a Reeb orbit is defined by $A(\gamma) := \int_{\gamma} \alpha$. A contact form is hypertight if all its Reeb orbits are non-contractible. Using the Conley-Zehnder index one can define the parity of a non-degenerate Reeb orbit $\gamma$ of $\alpha$; see for example [6, 18] for precise definitions.

1.1.2. Dynamical systems. Let $Y$ be a smooth 3-manifold and $X$ be a smooth non-vanishing vector field on $Y$. We denote by $\phi_{X}$ the flow generated by $X$. We endow $Y$ with a Riemannian metric $g$, which defines a norm $|v|$ for every tangent vector $v$ in $M$. The flow $\phi_{X}$ is an Anosov flow if there exist continuous line fields $E_{S}$ and $E_{U}$ in $TY$, and numbers $\mu > 0$ and $A > 0$ such that

- at every point $y \in Y$ we have $T_{y}Y = \mathbb{R}X(y) \oplus E_{S}(y) \oplus E_{U}(y)$, where $\mathbb{R}X(y)$ is the 1-dimensional subspace of $T_{y}Y$ generated by $X(y)$,
- the line fields $E_{S}$ and $E_{U}$ are invariant under the flow $\phi_{X}$,
- $|D\phi_{X}^{t}(y)v| \geq Ae^{\mu t}|v|$ for every $y \in Y$, $v \in E_{U}(y)$ and $t \geq 0$,
- $|D\phi_{X}^{t}(y)v| \leq Ae^{-\mu t}|v|$ for every $y \in Y$, $v \in E_{S}(y)$ and $t \geq 0$.

Notice that it follows directly from these properties that all periodic orbits of an Anosov flow are hyperbolic.\footnote{Recall that a periodic orbit $\gamma$ of a flow $\phi$ is called hyperbolic if the differential of the first return map for of $\gamma$ is a hyperbolic matrix; for details see [27].} An Anosov flow on a 3-manifold is called transversely orientable if the line bundles $E_{S}$ and $E_{U}$ are trivial. Anosov flows occupy a very special place in the theory of dynamical systems. The first systematic study of this class of flows was done by Anosov [3] who showed, among other things, that they are structurally stable.

In this paper we establish the positivity of the topological entropy for all Reeb flows on certain contact 3-manifolds. The topological entropy is a non-negative number associated to a dynamical system which measures the complexity of the orbit structure of the system. We recall the definition of the topological entropy of a flow $\phi_{X}$ of a vector field $X$ on a compact manifold $M$. We endow $M$ with a distance function $d$. A set $S \subset M$ $(\epsilon, T)$-covers $(M, d)$ if for all points $x \in M$ we have $\max_{t \in [0, T]} d(\phi_{X}^{t}(x), \phi_{X}^{t}(S)) < \epsilon$. Letting $Cov_{\epsilon, d}^{T}(\phi_{X})$ be the minimal cardinality of an $(\epsilon, T)$-covering set in $(M, d)$ we have ([27, Chapter VIII])

$$h_{top}(\phi_{X}) = \lim_{\epsilon \to 0} \limsup_{T \to +\infty} \frac{\log Cov_{\epsilon, d}^{T}(\phi_{X})}{T}.$$ (1)

It is elementary to prove that $h_{top}(\phi_{X})$ does not depend on the distance $d$ but only on the topology of $M$. For flows on compact 3-manifolds generated by nowhere vanishing vector fields we have the following result:

**Theorem** (Katok [20, 21], Lima and Sarig [23], Sarig [28]). Let $X$ be a $C^{2}$ vector field without singularities on a compact 3-manifold $M$, and let $\phi_{X}$ denote its flow. If $h_{top}(\phi_{X}) > 0$ then there exists a hyperbolic periodic orbit of $\phi_{X}$ whose unstable and stable manifolds intersect transversely. As a consequence there exists a compact set $K$ invariant under $\phi_{X}$ such that the dynamics of the restriction of
φ_X to K is conjugate to the suspension of a subshift of finite type. It follows that the number \( P^T_{hyp}(\phi_X) \) of hyperbolic periodic orbits of \( \phi_X \) with period smaller than \( T \) grows exponentially with \( T \).

1.2. Main results. Let \((M, \xi)\) be a compact contact 3-manifold. We say that \((M, \xi)\) is an **Anosov contact manifold** if there exists a contact form \( \alpha_0 \) on \((M, \xi)\) whose Reeb flow is Anosov. Anosov contact structures were studied in [24, 29]. In [24] the authors establish the exponential growth of \( S^1 \)-equivariant symplectic homology for exactly fillable Anosov contact manifolds. In [29] the author establishes the exponential growth of cylindrical contact homology for Anosov contact 3-manifolds that admit a transversely orientable Anosov Reeb flow. The main dynamical consequence of these works is that for non-degenerate contact forms on such Anosov contact manifolds the number of Reeb orbits grows exponentially with the period.

The classical examples of Anosov contact 3-manifolds are the unit tangent bundles of higher genus surfaces endowed with the geodesic contact structure, and new examples were constructed by Foulon and Hasselblatt in [13]. Our main result is that if \((M, \xi)\) is an Anosov contact 3-manifold then every Reeb flow on \((M, \xi)\) has positive topological entropy. We establish this result using cylindrical contact homology via the following theorem, which is a modification of [2, Theorem 2].

**Theorem 1.** Let \( \alpha_0 \) be a non-degenerate hypertight contact form on a contact manifold \((M, \xi)\) and assume that the cylindrical contact homology of \( \alpha_0 \) has weak exponential homotopical growth with weight \( a > 0 \). Then for every \( C^2 \) contact form \( \alpha \) on \((M, \xi)\) the Reeb flow \( \phi_{\alpha} \) has positive topological entropy. More precisely, if \( f_{\alpha} \) is the function such that \( \alpha = f_{\alpha} \alpha_0 \), then

\[
\frac{\log h_{top}(\phi_{\alpha})}{\log f_{\alpha}} = \frac{a}{\max f_{\alpha}}. \tag{2}
\]

Our main theorem is the following:

**Theorem 2.** Let \((M, \xi)\) be a compact 3-dimensional contact manifold and \( \alpha_0 \) be a contact form on \((M, \xi)\) such that its Reeb flow is a transversely orientable Anosov flow. Then, for every \( 0 < a < h_{top}(\phi_{\alpha_0}) \), the cylindrical contact homology of \( \alpha_0 \) has weak exponential homotopical growth rate with weight \( a \). It follows that if \( \alpha \) is a contact form on \((M, \xi)\), and \( f_{\alpha} \) is the function such that \( f_{\alpha} \alpha_0 = \alpha \), we have

\[
\frac{\log h_{top}(\phi_{\alpha})}{\log f_{\alpha}} = \frac{h_{top}(\phi_{\alpha_0})}{\max f_{\alpha}}. \tag{3}
\]

This theorem establishes the positivity of topological entropy for all Reeb flows on an Anosov contact 3-manifold \((M, \xi)\) provided that \((M, \xi)\) admits a weak exponential homotopical growth rate with weight \( a \).
transversely orientable Anosov Reeb flow. The non-transversely orientable case is obtained as a corollary of our main theorem.

**Corollary 3.** Let \((M, \xi)\) be a compact contact 3-manifold and assume that there exists a contact form \(\alpha_0\) on \((M, \xi)\) whose Reeb flow is Anosov. Then every Reeb flow on \((M, \xi)\) has positive topological entropy. Moreover, if \(\alpha\) is a contact form on \((M, \xi)\), and \(f_\alpha\) is the function such that \(f_\alpha \alpha_0 = \alpha\), we have

\[
h_{\text{top}}(\phi_{\alpha}) \geq \frac{h_{\text{top}}(\phi_{\alpha_0})}{\max f_\alpha}.
\]

The lower bound for \(h_{\text{top}}(\phi_{\alpha})\) given in (2) is not necessarily optimal. Therefore, the lower bounds for \(h_{\text{top}}(\phi_{\alpha})\) in (3) and (4) may not be optimal. Finding such optimal bounds would be an interesting topic for future research.

One dynamical consequence we obtain combining Corollary 3 and the theorem of Katok, Lima and Sarig is that for any Reeb flow on an Anosov contact 3-manifold the number of hyperbolic Reeb orbits grows exponentially with the period. This is a strengthening of the results of [24, 29]. Another corollary of Theorem 2 is the first proof of existence of hyperbolic contact 3-manifolds\(^4\) on which every Reeb flow has positive topological entropy.

**Corollary 4.** There exists an infinite family \(\{(M_n, \xi_n)\}_{n \in \mathbb{N}}\) of compact hyperbolic 3-manifolds such that

- \(M_n\) and \(M_m\) are not diffeomorphic if \(n \neq m\), and
- every Reeb flow on \((M_n, \xi_n)\) has positive topological entropy.

*Proof of Corollary 4 assuming Theorem 2.* In [13] the authors showed that there exist infinitely many non-diffeomorphic hyperbolic 3-manifolds that admit contact forms whose Reeb flows are transversely orientable Anosov flows. This and Theorem 2 imply the corollary.

**Structure of the paper.** In Section 2 we review the basics of pseudoholomorphic theory in exact symplectic cobordisms and recall the definition of cylindrical contact homology. In Section 3 we prove Theorem 1. Section 4 is devoted to the proofs of our main results, Theorem 2 and Corollary 3.

2. PSEUDOHOLONOMIC CURVES AND CYLINDRICAL CONTACT HOMOLOGY ON FREE HOMOTOPY CLASSES

We review the basics of pseudoholomorphic curves in exact symplectic cobordisms. For more details we refer the reader to [2, 10, 19].

\(^4\)Another approach to construct hyperbolic contact 3-manifolds on which every Reeb flow has positive topological entropy is the theme of a joint project of the author, Vincent Colin and Ko Honda, and is based on combining the methods of [1] and [8].
2.1. Cylindrical almost complex structures. Let \((M, \xi)\) be a contact manifold and \(\alpha\) a contact form on \((M, \xi)\). The symplectization of \((M, \alpha)\) is the product \(\mathbb{R} \times M\) with the symplectic form \(d(e^s \alpha)\).\(^5\) The 2-form \(d\alpha\) restricts to a symplectic form on the vector bundle \(\xi\) and it is well-known that the set \(j(\alpha)\) of \(d\alpha\)-compatible almost complex structures on \(\xi\) is non-empty and contractible.

For \(j \in j(\alpha)\) we can define an \(\mathbb{R}\)-invariant almost complex structure \(J\) on \(\mathbb{R} \times M\) by demanding that
\[
J \partial_s = X_\alpha, \quad J|_\xi = j.
\]
We will denote by \(\mathcal{J}(\alpha)\) the set of almost complex structures in \(\mathbb{R} \times M\) that are \(\mathbb{R}\)-invariant and satisfy (5) for some \(j \in j(\alpha)\).

2.2. Exact symplectic cobordisms with cylindrical ends. Let \((W, \omega = d\kappa)\) be an exact symplectic manifold without boundary, and let \((M^+, \xi^+)\) and \((M^-, \xi^-)\) be contact manifolds with contact forms \(\alpha^+\) and \(\alpha^-\). We say that \((W, \omega = d\kappa)\) is an exact symplectic cobordism from \(\alpha^+\) to \(\alpha^-\) if there exist subsets \(W^+, W^-\) and \(\tilde{W}\) of \(W\) and diffeomorphisms \(\Psi^+: W^+ \rightarrow [0, +\infty) \times M^+\) and \(\Psi^-: W^- \rightarrow (-\infty, 0] \times M^-\), such that
\[
\tilde{W} \text{ is compact, } W = W^+ \cup \tilde{W} \cup W^-, \text{ and } W^+ \cap W^- = \emptyset,
\]
\[
(\Psi^+)^*(e^s \alpha^+) = \kappa \text{ and } (\Psi^-)^*(e^s \alpha^-) = \kappa.
\]

In a cobordism, we say that an almost complex structure \(\mathcal{J}\) is asymptotically cylindrical if \(\mathcal{J}\) coincides with \(J^\pm \in \mathcal{J}(C^\pm \alpha^\pm)\) in the region \(W^\pm\), and \(\mathcal{J}\) is compatible with \(\omega\) in \(\tilde{W}\), where \(C^\pm > 0\) are constants.

For fixed \(J^\pm \in \mathcal{J}(C^\pm \alpha^\pm)\), we let \(\mathcal{J}(J^-, J^+)\) be the set of asymptotically cylindrical almost complex structures in \((W, \omega)\) coinciding with \(J^\pm\) on \(W^\pm\). The space \(\mathcal{J}(J^-, J^+)\) is non-empty and contractible.

2.3. Splitting symplectic cobordisms. Let \(\alpha^+\), \(\alpha\) and \(\alpha^-\) be contact forms on \((M, \xi)\). We assume that there exist exact symplectic cobordisms from \(\alpha^+\) to \(\alpha\) and from \(\alpha\) to \(\alpha^-\) which are diffeomorphic to \(\mathbb{R} \times M\). Then, for \(\epsilon > 0\) sufficiently small, we also have exact symplectic cobordisms from \(\alpha^+\) to \((1 + \epsilon)\alpha\) and from \((1 - \epsilon)\alpha\) to \(\alpha^-\) which are diffeomorphic to \(\mathbb{R} \times M\). Then, following the recipe for gluing exact symplectic cobordisms in [7], one constructs for each \(R > 0\) an exact symplectic form \(\omega_R = d\kappa_R\) on \(W = \mathbb{R} \times M\), where

- \(\kappa_R = e^{s-R-2} \alpha^+\) in \([R + 2, +\infty) \times M\),
- \(\kappa_R = f(s) \alpha\) in \([-R, R] \times M\),
- \(\kappa_R = e^{s+R} \alpha^-\) in \((-\infty, -R - 2] \times M\), and
- \(f: [-R, R] \rightarrow [1 - \epsilon, 1 + \epsilon]\) satisfies \(f(-R) = 1 - \epsilon, f(R) = 1 + \epsilon\), and \(f' > 0\).

The cobordism \((\mathbb{R} \times M, \omega_R)\) is called an exact splitting symplectic cobordism from \(\alpha^+\) to \(\alpha^-\) along \(\alpha\) with length \(R\). We consider a compatible asymptotically cylindrical almost complex structure \(\mathcal{J}_R\) on \((\mathbb{R} \times M, \omega_R)\); but we require one extra condition:
\[
\mathcal{J}_R \text{ coincides with } J \in \mathcal{J}(\alpha) \text{ in } [-R, R] \times M.
\]
\(^5\) \(s\) denotes the \(\mathbb{R}\) coordinate in \(\mathbb{R} \times M\)
Again we divide $W$ into regions:

\[ W^+ = [R + 2, +\infty) \times M \]
\[ W(\alpha^+, \alpha) = [R, R + 2] \times M \]
\[ W(\alpha) = [-R, R] \times M \]
\[ W(\alpha, \alpha^-) = [-R - 2, -R] \times M \]
\[ W^- = (-\infty, -R - 2] \times M. \]

The family $\{(\mathbb{R} \times M, \partial R, \tilde{f}_R)|R > 0\}$ is called a splitting family from $\alpha^+$ to $\alpha^-$ along $\alpha$.

2.4. **Pseudoholomorphic curves.** Let $(S, i)$ be a closed Riemann surface without boundary and $\Gamma \subset S$ a finite set. Let $\alpha$ be a contact form on $(M, \xi)$ and $J \in \mathcal{J}(\alpha)$. A finite energy pseudoholomorphic curve in the $(\mathbb{R} \times M, J)$ is a map $\tilde{w} = (s, \upsilon) : S \sim \Gamma \rightarrow \mathbb{R} \times M$ that satisfies

\[ \tilde{d}_J(\tilde{w}) = d\tilde{w} \circ i - J \circ d\tilde{w} = 0, \]
\[ E(\tilde{w}) = \sup_{q \in \mathcal{E}} \int_{S \sim \Gamma} \tilde{w}^* d(q\alpha) < +\infty, \]

where $\mathcal{E} = \{ q : \mathbb{R} \rightarrow [0, 1]; q' \geq 0 \}$. The quantity $E(\tilde{w})$ is called the Hofer energy and was introduced in [16]. The operator $\tilde{d}_J$ is called the Cauchy-Riemann operator for the almost complex structure $J$.

For an exact symplectic cobordism $(W, \omega)$ from $\alpha^+$ to $\alpha^-$ as considered above, and $\tilde{f} \in \mathcal{J}(J^-, J^+)$, a finite energy pseudoholomorphic curve is again a map $\tilde{w} : S \sim \Gamma \rightarrow W$ satisfying:

\[ d\tilde{w} \circ i = \tilde{f} \circ d\tilde{w}, \]
\[ 0 < E_{\alpha^-}(\tilde{w}) + E_{\epsilon}(\tilde{w}) + E_{\alpha^+}(\tilde{w}) < +\infty, \]

for $E_{\alpha^+}(\tilde{w}) = \sup_{q \in \mathcal{E}} \int_{\tilde{w}^{-1}(W(\epsilon) \times J^+)} \tilde{w}^* d(q\alpha^\pm)$ and $E_{\epsilon}(\tilde{w}) = \int_{\tilde{w}^{-1}(W(\alpha, \alpha^-))} \tilde{w}^* \omega$. These energies were also introduced in [16].

In splitting symplectic cobordisms we use a slightly modified version of energy. We require

\[ 0 < E_{\alpha^-}(\tilde{w}) + E_{\alpha^-, \alpha}(\tilde{w}) + E_{\alpha}(\tilde{w}) + E_{\alpha, \alpha^+}(\tilde{w}) + E_{\alpha^+}(\tilde{w}) < +\infty, \]

where

\[ E_{\alpha}(\tilde{w}) = \sup_{q \in \mathcal{E}} \int_{\tilde{w}^{-1}(W(\alpha))} \tilde{w}^* d(q\alpha), \]
\[ E_{\alpha^-}(\tilde{w}) = \int_{\tilde{w}^{-1}(W(\alpha, \alpha^-))} \tilde{w}^* \omega, \]
\[ E_{\alpha^-, \alpha}(\tilde{w}) = \int_{\tilde{w}^{-1}(W(\alpha^-, \alpha))} \tilde{w}^* \omega, \]

and $E_{\alpha^-}(\tilde{w})$ and $E_{\alpha^+}(\tilde{w})$ are as above.

For the remainder of this section we assume that the contact forms $\alpha, \alpha^+$ and $\alpha^-$ are non-degenerate. The elements of the set $\Gamma \subset S$ are called punctures of
the pseudoholomorphic curve \( \tilde{w} \). According to \([16, 17]\) punctures fall into two classes, positive and negative ones. For \( z \in \Gamma \) we take a neighborhood \( U \subset S \) of \( z \) that admits a holomorphic chart \( \psi_U : (U, z) \to (\mathbb{D}, 0) \). Using polar coordinates \((r, t) \in (0, +\infty) \times S^1 \) we can write \( x \in (\mathbb{D} \sim 0) \) as \( x = e^{-r} t \). It is shown in \([16, 17]\) that if \( z \) is a positive/negative puncture then there exists a Reeb orbit \( \gamma^\pm \) of \( X_{\alpha^\pm} \) such that for \( \tilde{w} \circ \psi_U^{-1}(r, t) = (s(r, t), w(r, t)) \) we have that \( s(r, t) \to \pm \infty \) as \( r \to +\infty \) and

- \( w^r(t) = w(r, t) \) converges in \( C^\infty \) to a Reeb orbit \( \gamma^\pm \) of \( \pm X_{\alpha^\pm} \), exponentially in \( r \) and uniformly in \( t \).

In this case we say that \( \tilde{w} \) is asymptotic to \( \gamma^\pm \) at the positive/negative puncture \( z \). The discussion above can be summarized by saying that at their punctures finite energy pseudoholomorphic curves detect Reeb orbits.

We denote by \( \mathcal{M}(\gamma, \gamma'; J) \) the moduli space of pseudoholomorphic cylinders, modulo biholomorphic reparametrisation of the domain and \( \mathbb{R} \) translations in the symplectization, with one positive puncture asymptotic to a non-degenerate Reeb orbit \( \gamma \) and one negative puncture asymptotic Reeb orbits \( \gamma' \). It is well-known that the linearization \( \mathcal{D}\psi_J \) of \( \psi_J \) at any element of \( \mathcal{M}(\gamma, \gamma'; J) \) is a Fredholm map. We denote by \( \mathcal{M}^k(\gamma, \gamma'; J) \) the subspace of \( \mathcal{M}(\gamma, \gamma'; J) \) whose elements have Fredholm index \( k \). Along the same lines one defines moduli spaces \( \mathcal{M}(\gamma^+, \gamma^-; J) \) and \( \mathcal{M}^k(\gamma^+, \gamma^-; J) \) in exact symplectic cobordisms. For more details we refer the reader to the original sources \([10, 7]\) or \([2, \text{Section 2}]\).

**FACT.** Applying Stokes’ Theorem and using the exactness of the symplectic cobordisms considered above we obtain that the energy \( E(\tilde{w}) \) of \( \tilde{w} \) satisfies \( E(\tilde{w}) \leq 5A(\tilde{w}) \) where \( A(\tilde{w}) \) is the sum of the action of the Reeb orbits detected by positive punctures of \( \tilde{w} \) counted with multiplicity; see \([7, 18]\) or \([19, \text{Section 2}]\).

### 2.5. Cylindrical contact homology in special homotopy classes.

We denote by \((M, \xi)\) a contact manifold endowed with a non-degenerate hypertight contact form \( \alpha_0 \).

Let \( \Lambda(M) \) be the set free homotopy classes of loops in \( M \). Let \( \rho \in \Lambda(M) \) be a free homotopy class which contains only simple Reeb orbits of \( \alpha_0 \), and let \( \mathcal{P}_\rho(\alpha_0) \) be the set of Reeb orbits of \( \alpha_0 \) that belong to \( \rho \). By the works \([9, 5]\), we know that there exists a generic subset \( \mathcal{J}^0_{\text{reg}}(\alpha_0) \) of \( \mathcal{J}(\alpha_0) \) such that for all \( J \in \mathcal{J}^0_{\text{reg}}(\alpha_0) \) we have the following:

- for all Reeb orbits \( \gamma_1, \gamma_2 \in \rho \), the moduli space of pseudoholomorphic cylinders \( \mathcal{M}(\gamma_1, \gamma_2; J) \) is transversely cut out.

For \( \rho \) as above, we let \( \mathcal{C}^0_{\text{cyl}}(\alpha_0) \) be the \( \mathbb{Q} \) vector-space generated by \( \mathcal{P}_\rho(\alpha_0) \).

We then define, for \( J \in \mathcal{J}^0_{\text{reg}}(\alpha_0) \):

\[
\partial^0_J(\gamma) = \sum_{\gamma' \in \mathcal{P}_\rho(\alpha_0)} C^0(\gamma, \gamma'; J)\gamma'
\]

\(\text{\footnotesize{\textsuperscript{6}This is the case, for example, if \( \rho \) is a primitive free homotopy class in \( \Lambda(M) \).}}\)
where $C^0(\gamma, \gamma'; J)$ is the number of points of the moduli space $\mathcal{M}(\gamma, \gamma'; J)$.

For $\alpha_0$ and $\rho$ as above and $J \in \mathcal{J}_{\text{reg}}(\alpha_0)$, Eliashberg, Givental and Hofer [10] showed:

**Proposition 5** (Eliashberg-Givental-Hofer, [10]). Let $(M, \xi)$ be a contact manifold with a non-degenerate hypertight contact form $\alpha_0$. Let $\rho \in \Lambda(M)$ a free homotopy class which contains only simple Reeb orbits of $\alpha_0$, and pick $J \in \mathcal{J}_{\text{reg}}(\alpha_0)$. Then $d^0_J$ is well-defined and $(d^0_J)^2 = 0$. Under these conditions we let $\text{Cyl}_\rho^0(\alpha_0)$ denote the homology of the pair $(\text{Cyl}_\rho^0(\alpha_0), d^0_J)$.

**Remark.** Let $\text{Cyl}_{\text{odd}}^0(\alpha_0)$ and $\text{Cyl}_{\text{even}}^0(\alpha_0)$ be the subspaces of $\text{Cyl}_\rho^0(\alpha_0)$ generated only by odd and even Reeb orbits, respectively. Then it is clear that $\text{Cyl}_{\text{odd}}^0(\alpha_0) = \text{Cyl}_{\text{even}}^0(\alpha_0) \oplus \text{Cyl}_{\text{even}}^0(\alpha_0)$, and it is well-known (see [6, 10]) that $d^0_J(\text{Cyl}_{\text{odd}}^0(\alpha_0)) \subset \text{Cyl}_{\text{even}}^0(\alpha_0)$ and $d^0_J(\text{Cyl}_{\text{even}}^0(\alpha_0)) \subset \text{Cyl}_{\text{odd}}^0(\alpha_0)$.

Let $(M, \xi)$ be a contact manifold with a non-degenerate hypertight contact form $\alpha_0$. Let $\alpha^+ = C\alpha_0$ and $\alpha^- = c\alpha_0$ where $C > c > 0$ are constants, and $\rho \in \Lambda(M)$ be a free homotopy class that contains only simple Reeb orbits of $\alpha_0$. Pick an almost complex structure $J \in \mathcal{J}_{\text{reg}}(\alpha_0)$, and set $J^\pm = J$. Let $(\mathbb{R} \times M, \omega)$ be an exact symplectic cobordism from $C\alpha_0$ to $c\alpha_0$. The techniques of [9] show that there exists a generic subset $\mathcal{J}_{\text{reg}}(J^-, J^+) \subset \mathcal{J}(J^-, J^+)$ such that if $\tilde{J} \in \mathcal{J}_{\text{reg}}(J^-, J^+)$, then

- for all Reeb orbits $\gamma_1, \gamma_2 \in \rho$, the moduli space of pseudoholomorphic cylinders $\mathcal{M}(\gamma_1, \gamma_2; \tilde{J})$ is transversely cut out.

Under these conditions one has (see [10]) a map $\Phi^0: \text{Cyl}_{\text{reg}}^0(\alpha_0) \rightarrow \text{Cyl}_{\text{reg}}^0(\alpha_0)$ by counting rigid pseudoholomorphic cylinders. Precisely, for each $\gamma \in \mathcal{P}_\rho(\alpha_0)$ we set $\Phi^0(\gamma) = \sum_{\gamma' \in \mathcal{P}(\alpha_0)} \#(\mathcal{M}^0(\gamma_1, \gamma_2; \tilde{J})) \gamma'$ and extend it to all $\text{Cyl}_{\text{reg}}^0(\alpha_0)$ by linearity. It is part of the philosophy introduced in [10] that homotopic cobordisms should induce chain homotopic maps for contact homologies. In our situation this is expressed by the following result:

**Proposition 6** (Eliashberg-Givental-Hofer). Let $(M, \xi)$ be a contact manifold with a non-degenerate hypertight contact form $\alpha_0$. Let $\alpha^+ = C\alpha_0$ and $\alpha^- = c\alpha_0$ where $C > c > 0$ are constants, and $\rho \in \Lambda(M)$ be a free homotopy class containing only simple Reeb orbits of $\alpha_0$. Choose an almost complex structure $J \in \mathcal{J}_{\text{reg}}(\alpha_0)$, and set $J^\pm = J$. Let $(W = \mathbb{R} \times M, \omega)$ be an exact symplectic cobordism from $C\alpha_0$ to $c\alpha_0$, and choose a regular almost complex structure $\tilde{J} \in \mathcal{J}_{\text{reg}}(J^-, J^+)$. Then, the map $\Phi^0: \text{Cyl}_{\text{reg}}^0(\alpha_0) \rightarrow \text{Cyl}_{\text{reg}}^0(\alpha_0)$ as constructed above is well-defined. Moreover, if there is a homotopy $(\mathbb{R} \times M, \omega_t)$ of exact symplectic cobordisms from $C\alpha_0$ to $c\alpha_0$, with $\omega_0 = \omega$ and $\omega_1 = d(e^t\alpha_0)$, it follows that the map $\Phi^0: \text{Cyl}_{\text{reg}}^0(\alpha_0) \rightarrow \text{Cyl}_{\text{reg}}^0(\alpha_0)$ is chain homotopic to the identity.
3. Growth of cylindrical contact homology and lower bounds for $h_{\text{top}}$

We start introducing the notion of weak exponential homotopical growth of cylindrical contact homology.

**Definition 7.** Let $(M, \xi)$ be a contact manifold and $\alpha_0$ be a non-degenerate hypertight contact form on $(M, \xi)$. For $T > 0$ let $\tilde{\Lambda}^T(\alpha_0) \subset \Lambda(M)$ be the set that contains free homotopy classes $\rho$ that satisfy

- all Reeb orbits of $X_{\alpha_0}$ in $\rho$ are simply covered, and $CH^\rho_{\text{cyl}}(\alpha_0) \neq 0$,
- there is a finite subset $\{\gamma_1^\rho, ..., \gamma_{k_\rho}^\rho\} \subseteq P^\rho(\alpha_0)$ and rational numbers $q_1^\rho, ..., q_{k_\rho}^\rho$,
- such that every element of $\{\gamma_1^\rho, ..., \gamma_{k_\rho}^\rho\}$ has action $\leq T$, $d^\rho_j(\sum_{i=1}^{k_\rho} q_i^\rho \gamma_i^\rho) = 0$ and $[\sum_{i=1}^{k_\rho} q_i^\rho \gamma_i^\rho] \in C^\rho_{\text{cyl}}(\alpha_0)$ is non-zero.

We define $N^T_{\text{cyl}}(\alpha_0) := \#\tilde{\Lambda}^T(\alpha_0)$.

**Definition 8.** We say that the cylindrical contact homology of $(M, \alpha_0)$ has weak exponential homotopical growth with weight $a > 0$ if there exist a number $b$ and a sequence $T_n \to +\infty$, such that $N^T_{\text{cyl}}(\alpha_0) \geq e^{aT_n+b}$ for all $T_n$.

We now prove Theorem 1, that is a strengthening of [2, Theorem 2].

*Proof of Theorem 1.* The proof is almost identical to the one of [2, Theorem 2]. We write $E = \max f_a$. We assume first that $\alpha$ is non-degenerate and $C^\infty$. For every $\epsilon > 0$ we can construct an exact symplectic cobordism from $(E+\epsilon)\alpha_0$ to $\alpha$. Analogously, for $c > 0$ small enough, it is possible to construct an exact symplectic cobordism from $\alpha$ to $c\alpha_0$.

Let $\rho \in \tilde{\Lambda}^T(\alpha_0)$. Using this cobordism, we construct a splitting family $(\mathbb{R} \times M, \omega_R, J_R)$ from $(E+\epsilon)\alpha_0$ to $c\alpha_0$, along $\alpha$, such that for every $R > 0$, $(\mathbb{R} \times M, \omega_R, J_R)$ is homotopic to the symplectization of $\alpha_0$ and $J_R$ is regular. By the methods of [5] we can choose a regular $J_0$ that coincides with a fixed $J_\rho \in J_\rho(a_0)$ on the positive and negative ends of the cobordism and with a fixed $J \in J(\alpha)$ on $[-R,R] \times M$. Since $J_R \in J_\rho(J_0, J_0)$, the exact symplectic cobordism $(\mathbb{R} \times M, \omega_R, J_R)$ induces a map $\Psi_{J_R} : CH^\rho_{\text{cyl}}(\alpha_0) \to CH^\rho_{\text{cyl}}(\alpha_0)$. As $(\mathbb{R} \times M, \omega_R, J_R)$ is homotopic to the symplectization of $\alpha_0$, we know from Proposition 6 that $\Psi_{J_R}$ is the identity.

For $\rho \in \tilde{\Lambda}^T(\alpha_0)$, let $\{\gamma_1^\rho, ..., \gamma_{k_\rho}^\rho\} \subseteq P^\rho(\alpha_0)$ and $q_1^\rho, ..., q_{k_\rho}^\rho$ be, respectively, the Reeb orbits with action $\leq T$ and rational numbers given in Definition 7, such that $d^\rho_j(\sum_{i=1}^{k_\rho} q_i^\rho \gamma_i^\rho) = 0$ and $[\sum_{i=1}^{k_\rho} q_i^\rho \gamma_i^\rho] \in CH^\rho_{\text{cyl}}(\alpha_0)$ is non-zero. Since $\Psi_{J_R} : CH^\rho_{\text{cyl}}(\alpha_0) \to CH^\rho_{\text{cyl}}(\alpha_0)$ is the identity, it follows that $\Psi_{J_R}([\sum_{i=1}^{k_\rho} q_i^\rho \gamma_i^\rho]) \neq 0$. We then conclude that for every $R$ there exists some $i_R \in \{1, ..., k_\rho\}$ and a finite energy pseudoholomorphic cylinder $\tilde{w}_R$ in $(\mathbb{R} \times M, J_R)$ positively asymptotic to $\gamma_i^\rho$ and negatively asymptotic to an orbit in $P^\rho(\alpha_0)$. Once these pseudoholomorphic cylinders are obtained one proceeds as in the proof of [2, Theorem 2].
Specifically, one takes a sequence \( R_n \to +\infty \) and studies the behavior of the sequence \( \tilde{w}_{R_n} \). A subsequence of \( \tilde{w}_{R_n} \) converges to a pseudoholomorphic building \( \tilde{w} \) in the sense of [7]. Reasoning exactly as in Step 2 of the proof of [2, Theorem 2] one concludes that the topological properties of the building \( \tilde{w} \) imply the existence of a Reeb orbit \( \gamma_\rho \) of \( \alpha \) in the free homotopy class \( \rho \) and with action \( \leq ET \).

Let \( N_{X_\alpha}(T) \) be the number of free homotopy classes in \( M \) that contain a Reeb orbit of \( \alpha \) with action \( \leq T \). Reasoning as in Step 3 of the proof of [2, Theorem 2] we conclude that \( N_{X_\alpha}(T_n) \geq \#\Lambda^T_0(\alpha_0) \geq e^{at_{\alpha_0} + b} \) for all elements of the sequence \( T_n \). Applying [2, Theorem 1] we obtain \( h_{\text{top}}(\phi_\alpha) \geq \frac{b}{\rho} \). This proves the theorem in the case that \( \alpha \) is \( C^\infty \) and non-degenerate. The case of a \( C^2 \) possibly degenerate contact form \( \alpha \) is done exactly as in Step 4 of the proof of [2, Theorem 2].

\[ \square \]

4. Weak exponential homotopical growth of cylindrical contact homology for Anosov Reeb flows

We first recall some facts about Anosov flows on compact 3-dimensional manifolds.

**Proposition 9** (Anosov [3], Fenley [12]). Let \((M, \xi)\) be a compact contact 3-manifold and \( \alpha_0 \) a contact form on \((M, \xi)\) whose Reeb flow is a transversely orientable Anosov flow. Then \( \alpha_0 \) is hypertight and all simple Reeb orbits of \( \alpha_0 \) are contained in primitive free homotopy classes.

Crucial for the proof of Theorem 2 is the work of Barthelmé and Fenley [4], where it is shown that the number of free homotopy classes in a compact 3-manifold containing periodic orbits with action \( \leq T \) of an Anosov flow grows exponentially with \( T \). We need a small improvement of their result.

**Proposition 10.** Let \((M, \xi)\) be a compact contact 3-manifold and \( \alpha_0 \) a contact form on \((M, \xi)\) such that its Reeb flow is a transversely orientable Anosov flow. Let \( \Lambda^T_0(\alpha_0) \) be the set of primitive free homotopy classes of \( M \) that contain a Reeb orbit of \( \alpha_0 \) with action \( \leq T \). Then, for every \( 0 < a < h_{\text{top}}(\phi_{\alpha_0}) \), there exists a monotone sequence \( T_n \to +\infty \) and a number \( b \) such that \( \#\Lambda^T_0(\alpha_0) \geq e^{at_n + b} \).

**Proof.** We define \( \Lambda^T(\alpha_0) \) as the set of free homotopy classes in \( M \) that contain some Reeb orbit of \( \alpha_0 \) with action \( \leq T \) and let \( 0 < a < h_{\text{top}}(\phi_{\alpha_0}) \). We then invoke [4, Theorem A]. It implies that, for \( 0 < c < h_{\text{top}}(\phi_{\alpha_0}) - a \) there exist a monotone sequence \( T_n \to +\infty \) and a number \( \hat{b} \) such that

\[
\#\Lambda^T(\alpha_0) \geq e^{(a+c)T_n + \hat{b}}.
\]

We choose \( c > 0 \) such that all Reeb orbits of \( \alpha_0 \) have action \( \geq c \). Let \( \mathcal{N}_{\text{simp}}^T(\alpha_0) \) be the number of free homotopy classes in \( M \) which contain a simple Reeb orbit of \( \alpha_0 \) with period \( \leq T \). It is elementary to show that

\[
\left[ \frac{T}{c} \right] \mathcal{N}_{\text{simp}}^T(\alpha_0) \geq \#\Lambda^T(\alpha_0).
\]
It follows from Proposition 9 that \( N^T_{\text{simp}}(a_0) = \# \tilde{\Lambda}^T_0(a_0) \), because all simple Reeb orbits of \( a_0 \) are in primitive free homotopy classes. We thus obtain

\[
\left\lceil \frac{T}{c} \right\rceil \# \tilde{\Lambda}^T_0(a_0) \geq \# \Lambda^T(0).
\]

Inequalities (15) and (13) imply that \( \# \tilde{\Lambda}^T_0(a_0) \geq \frac{e^{(a+1)T} + b}{c \# T_0(\alpha_0)}. \) It is then immediate that there exist a monotone sequence \( T_n \to +\infty \) and a number \( b \) such that

\[
\# \tilde{\Lambda}^T_0(a_0) \geq e^{aT_n+b}.
\]

**Proof of Theorem 2.** Let \( 0 < a < h_{\text{top}}(\phi_{\alpha_0}) \). The main idea of the proof is to use the fact that all Reeb orbits of a transversely orientable Anosov Reeb flow are even to show that for every \( \rho \in \tilde{\Lambda}^T_0(a_0) \) the differential \( d^p_f(\gamma) \) vanishes. Once this is established we obtain the exponential growth rate with weight \( a \) from Proposition 10.

Precisely, \( \phi_{\alpha_0} \) is non-degenerate since the fact that it is Anosov implies that all its Reeb orbits are hyperbolic. By Proposition 5, for each \( \rho \in \tilde{\Lambda}^T_0(a_0) \) the homology \( CH^p_{\text{cyl}}(\alpha_0) \) is defined. Moreover, since the Reeb flow of \( \alpha_0 \) is a transversely orientable Anosov flow implies that all Reeb orbits of \( \alpha_0 \) are even; see [29].

By the Remark after Proposition 5, the differential \( d^p_f(\gamma) \) vanishes for all \( \rho \in \tilde{\Lambda}^T_0(a_0) \). This implies that each Reeb orbit \( \gamma \in \rho \) is a closed and non-exact element of the chain complex \( (CH^p_{\text{cyl}}(\alpha_0), d^p) \). Therefore \( C^p_{\text{cyl}}(\alpha_0) \) is defined. Moreover, since the Reeb flow of \( \alpha_0 \) is a transversely orientable Anosov flow implies that all Reeb orbits of \( \alpha_0 \) are even; see [29].

By Proposition 10 it follows that there exist a monotone sequence \( T_n \to +\infty \) and a number \( b \) such that

\[
N^T_{\text{cyl}}(a_0) = \# \tilde{\Lambda}^T_0(a_0) \geq \# \tilde{\Lambda}^T_0(a_0) \geq e^{aT_n+b},
\]

which finishes the proof of the first statement of the theorem. The second statement of Theorem 2 follows from combining the first with Theorem 1.

**Proof of Corollary 3.** Because of Theorem 2 we must only treat the case where the Reeb flow of \( \alpha_0 \) is not transversely orientable. As observed in [11], there exists a double covering \( \pi_2 : \tilde{M} \to M \) such that the flow of the pullback vector field \( \pi_2^*X_{\alpha_0} \) is a transversely orientable Anosov flow. The vector field \( \pi_2^*X_{\alpha_0} \) is also the Reeb vector field of the contact form \( \tilde{\alpha}_0 := \pi_2^*\alpha_0 \) on the contact manifold \( (\tilde{M}, \tilde{\xi}) = \pi_2^*\xi \).

Taking \( \alpha \) to be a contact form on \( (M, \xi) \), we define \( \tilde{\alpha} := \pi_2^*\alpha \). If \( f_{\tilde{\alpha}} \) is the function such that \( \alpha = f_{\tilde{\alpha}}\alpha_0 \) and \( f_{\alpha} \) is the function such that \( \tilde{\alpha} = f_{\alpha}\tilde{\alpha}_0 \), it is clear that \( \max(f_{\tilde{\alpha}}) = f_{\alpha} \). We then have

\[
h_{\text{top}}(\phi_{\alpha_0}) = h_{\text{top}}(\phi_{\tilde{\alpha}}) \geq \frac{h_{\text{top}}(\phi_{\tilde{\alpha}_0})}{\max(f_{\tilde{\alpha}})} = \frac{h_{\text{top}}(\phi_{\alpha_0})}{\max(f_{\alpha})}.
\]
where the inequality in (18) follows from Theorem 1, and the equalities follow from the fact that the topological entropy of flows is preserved under double coverings; see Lemma 11 below.

The following lemma is certainly classical, but as we found no proof of it in the literature we present one for completeness.

**Lemma 11.** Let $M$ be a compact manifold, $X$ be a smooth vector field in $M$ with flow $\phi_X$, and $\pi_2 : \tilde{M} \to M$ be a double covering. Define $\tilde{X} := \pi_2^* X$ and denote by $\phi_{\tilde{X}}$ the flow of $\tilde{X}$. Then $h_{\text{top}}(\phi_X) = h_{\text{top}}(\phi_{\tilde{X}})$.

**Proof.** We choose an auxiliary Riemannian metric $g$ on $M$ that generates a distance function $d_g$ in $M$, let $\tilde{g} = \pi_2^* g$, and let $d_{\tilde{g}}$ be the distance function generated by $\tilde{g}$ on $\tilde{M}$. Let $\epsilon_g$ be the injective radius of both $g$ and $\tilde{g}$. It is clear that, for $0 < \epsilon < \epsilon_g$, if $S(\epsilon, T)$-covers $(M, d_g)$ then $\pi_2^{-1}(S(\epsilon, T))$-covers $(\tilde{M}, d_{\tilde{g}})$, which implies that $\text{Cov}^T_{\epsilon, d_g}(\phi_X) \leq 2 \text{Cov}^T_{\epsilon, d_{\tilde{g}}}(\phi_{\tilde{X}})$. Conversely, for $0 < \epsilon < \epsilon_g$, if $\tilde{S}(\epsilon, T)$-covers $(\tilde{M}, d_{\tilde{g}})$, then $\pi_2(\tilde{S}(\epsilon, T))$-covers $(M, d_g)$, which implies that $\text{Cov}^T_{\epsilon, d_{\tilde{g}}}(\phi_{\tilde{X}}) \leq \text{Cov}^T_{\epsilon, d_g}(\phi_X)$. It then follows from the definition of topological entropy (1) that $h_{\text{top}}(\phi_X) = h_{\text{top}}(\phi_{\tilde{X}})$. □

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