Systematic Analysis of Flow Distributions

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The information of the event-by-event fluctuations is extracted from flow harmonic distributions and cumulants, which can be done experimentally. In this work, we employ the standard method of Gram-Charlier series with the normal kernel to find such distribution, which is the generalization of recently introduced flow distributions for the studies of the event-by-event fluctuations. Also, we introduce a new set of cumulants $j_n\{2k\}$ which have more information about the fluctuations compared with other known cumulants. The experimental data imply that not only all of the information about the event-by-event fluctuations of collision zone properties and different stages of the heavy-ion process are not encoded in the radial flow distribution $p(v_n)$, but also the observables describing harmonic flows can generally be given by the joint distribution $P(v_1, v_2, \ldots)$. In such a way, we first introduce a set of joint cumulants $K_{nm}$, and then we find the flow joint distribution using these joint cumulants. Finally, we show that the Symmetric Cumulants $SC(2,3)$ and $SC(2,4)$ obtained from ALICE data are explained by the combinations $K_{22} + \frac{1}{2}K_{04} - K_{31}$ and $K_{22} + 4K_{11}^2$.

I. INTRODUCTION

The collective behavior of the initial fireball, which is created in heavy-ion collisions, can be experimentally measured by the anisotropic flow. Anisotropic flow is traditionally quantified with harmonics $v_n$, which have been measured by the several experimental groups at Relativistic Heavy Ion Collider (RHIC) and Large Hadron Collider (LHC) [1-8]. Due to the effect of reaction plane angle and low statistic at each event, anisotropic flow finding is experimentally challenging. There are several techniques to solve these problems [9-13]. One of them is a two-dimensional standardized cumulants, which can help us remove non-controllable effects of reaction plane angle and find flow harmonics by averaging over all events to increase statistic [13]. On the other hand, the experimental results show that the flow harmonics fluctuate event-by-event even if a specific centrality class is considered [14, 15]. The flow fluctuations contain the information of the collision geometry, quantum fluctuations at the initial state, and effects of different evolution stages in the heavy-ion process [16, 17]. The distribution of flow harmonic not only can solve the problems of the reaction plane angle effect and low statistic in a given event but also it can help us extract the information of observed event-by-event fluctuations. So, these issues motivate us to study the radial flow distributions $p(v_n)$.

Experimentally flow distributions for second, third, and fourth harmonics have been obtained using the unfolding method [18-19]. Also, it has been found that the Bessel Gaussian distribution describes the observed flow distributions in some centrality collisions [19].

It should be noted that the information of flow fluctuations not only are encoded in the flow harmonic distribution $p(v_n)$, but also this information can be extracted from radial cumulants $c_n\{2k\}$ [20, 21]. Consequently, finding the right set of cumulants and connecting them to flow harmonic distribution $p(v_n)$ can help us get closer to an exact interpretation of the event-by-event fluctuations. Thereby, different distributions and their cumulants have recently been introduced and investigated to explain the contributions of all evolution stages on the fluctuations. In Ref. [20], odd flow harmonic distributions have been obtained by employing two-dimensional standardized cumulants. In addition, using Gram-Charlier A series with orthogonal polynomials, $p_{odd}(v_n)$ has been found in Ref. [21]. The experimental data of the even flow harmonics cannot be explained by the Bessel Gaussian distribution in peripheral collisions. So, finding the corrections to the Bessel Gaussian distribution is crucial. Ref. [21] considered an ansatz series as the corrections to the Bessel Gaussian distribution. They employed moments to find the corresponding coefficients of this series. Their suggested flow distribution could decently explain both even and odd harmonics.

Experiments show that the event-plane correlations and symmetric cumulants are non-vanishing [22-24]. Thus, all of the information about the fluctuations can be extracted from a joint flow distribution $P(v_1, v_2, \ldots)$, which can explain the correlations between flow harmonics, event-by-event initial fluctuations, and correlations between different stages in heavy-ion collision processes. Now a question arises: Is there an unambiguous technique to find such radial flow distributions $p(v_n)$? Furthermore, can we find a joint flow distribution to interpret the most general form of the event-by-event fluctuations? The purpose of this paper is the answer to this question by introducing a systematic analysis of flow fluctuations so that we can find the cumulant coefficients and the consequently flow harmonic distributions.

In this work, we employ the standard method of the Gram-Charlier series with the normal kernel to introduce this analysis in Sec. II. Also, we show that using this technique, one can find the radial cumulants $c_n\{2k\}$ to
flow moments \( \langle v_{n}^{2k} \rangle \). In Sec. [11] expanding the relation between moment and cumulant characteristic functions to 2-dimension, we first rederive the relations between \( \langle v_{n}^{2k} \rangle \) and \( c_{n}(2k) \), and then we find the distribution of odd flow harmonics which has been found in Refs. [20] and [21]. After that, we find a general form of flow distribution which is true for both even and odd harmonics. In the path to find this distribution, we introduce a new set of cumulants \( j_{n}(2k) \) which have more information compared to other known cumulants such as \( c_{n}(2k) \) and \( q_{n}(2k) \) [21]. In the final step, we introduce a joint distribution of flow harmonics and its cumulants \( K_{nm} \) in Sec. [IV] We conclude Sec. [IV] by showing ALICE data.

Moreover, the simulation data can be explained by the obtained joint distribution of flow harmonics. We present a general method to find the flow distribution, because of the event-by-event fluctuations, finding the anisotropic flow is not straightforward. Hence, presenting a general method to find the flow distribution and its cumulants to explain the event-by-event flow fluctuations becomes important. In this section, we introduce such a method using the relation between moment and cumulant generating function. For simplicity we consider one dimensional generating functions. In statistics, the generating function of moments in one dimension is \( G(t) = \int dx \, e^{itx} p(x) \equiv \langle e^{itx} \rangle \). This is while the cumulant-generating function as the logarithm of the characteristic function is \( K(t) = \sum_{n=1}^{\infty} \frac{(it)^{n}}{n!} \kappa_{n} \), where \( G(t) = \exp \left[ \sum_{n=1}^{\infty} \frac{(it)^{n}}{n!} \kappa_{n} \right] \) [25][27]. Note that \( \kappa_{n} \) is the \( n \)th cumulants. Furthermore, the relation between cumulants and moments by using definitions of \( G(t) \) and \( K(t) \) is

\[
1 + \sum_{n=1}^{\infty} \frac{\mu_{n} t^{n}}{n!} = \exp \left( \sum_{n=1}^{\infty} \frac{\kappa_{n} t^{n}}{n!} \right),
\]

where \( \mu_{n} = \langle x^{n} \rangle \). The relation between n-th moment and cumulants can be obtained by differentiating both sides of Eq.(2) \( n \) times and evaluating the result at \( t = 0 \),

\[
K^{(n)}(t)|_{t=0} = (\log G(t))^{(n)}|_{t=0}.
\]

Let us expand \( G(t) \) in Eq.(2) to second order and set \( \kappa_{1} = \mu_{1} \equiv \mu \) and \( \kappa_{2} = \sigma^{2} \). The structure of the characteristic function, thus, becomes the following

\[
G(t) = \exp \left[ \sum_{n=3}^{\infty} \frac{\kappa_{n} (it)^{n}}{n!} + \kappa_{1}(it) + \kappa_{2} \frac{(it)^{2}}{2!} \right]
= \exp \left[ \sum_{n=3}^{\infty} \frac{\kappa_{n} (it)^{n}}{n!} e^{it\mu - \frac{t^{2} \sigma^{2}}{2}} \right],
\]

\[
= \exp \left[ \sum_{n=3}^{\infty} \frac{\kappa_{n} (it)^{n}}{n!} G_{N}(t) \right],
\]

with \( G_{N}(t) \equiv \exp \left[ it\mu - \frac{t^{2} \sigma^{2}}{2} \right] \). Note that integrating by parts gives \( (it)^{n} G_{N}(t) \) as the characteristic function of \( (-D)^{n} G_{N}(x) \), where \( D \) is the differential operator. On the other hand, we can find the probability density function \( p(x) \) by using the last line of defined moment-generating function in Eq.(3):

\[
p(x) = \frac{1}{2\pi} \int dt \, e^{-itx} G(t)
\approx \frac{e^{-\frac{(x-\mu)^{2}}{2\sigma^{2}}}}{\sqrt{2\pi}\sigma} \left( 1 + \sum_{n=3}^{\infty} \frac{\kappa_{n}}{n!} H_{c_{n}}(\frac{x-\mu}{\sigma}) \right).
\]

This technique is the standard method of finding Gram-Charlier series with the normal kernel [23]. In this method, one can find the probability density function (p.d.f) without any considered ansatz for the p.d.f.

To see how this method can help us to find the distribution of flow harmonics and cumulants, we first define the form of characteristic function using Eq.(2) as follows

\[
G(\lambda) = \langle e^{i\lambda \cdot v} \rangle = \langle e^{iv_{n} \lambda \cos(\Psi_{n} - \Psi_{\lambda})} \rangle,
\]

where we have used the notation \( \Psi_{n} = n\psi_{n} \). Since one-dimensional characteristic function is needed to find the relations between cumulants and moments in the case of flow harmonics, we can integrate over \( \Psi_{n} \) to have \( G(\lambda) \) [21],

\[
G(\lambda) = \langle J_{0}(\lambda v_{n}) \rangle.
\]

So, the relation between the generating functions of 2k-particle cumulants \( c_{n}(2k) \) [10][13] and flow magnitude moments \( \langle v_{n}^{2k} \rangle \) are

\[
\langle J_{0}(\lambda v_{n}) \rangle = 1 + \sum_{k=1}^{\infty} \frac{\lambda^{2k} \langle v_{n}^{2k} \rangle}{k!(4k^{2})^{2k}} = \exp \left( \sum_{k=1}^{\infty} \frac{c_{n}(2k) \lambda^{2k}}{4k!(4k^{2})^{2k}} \right),
\]

where \( I_{\nu}(x) \) is the modified Bessel function,

\[
I_{\nu}(x) = \sum_{k=0}^{\infty} \frac{1}{k!(k+\nu+1)} \left( \frac{x}{2} \right)^{2k+\nu}.
\]
In the results, 2k-particle cumulants $c_n \{2k\}$ can be given to the measured $v_n$ at each event by differentiating both sides of Eq. (5) at $\lambda = 0$:

\[
\begin{align*}
c_n \{2\} &= \langle v_n^2 \rangle, \\
c_n \{4\} &= \langle v_n^4 \rangle - 2 \langle v_n^2 \rangle, \\
c_n \{6\} &= 12 \langle v_n^2 \rangle^3 - 9 \langle v_n^4 \rangle \langle v_n^2 \rangle + \langle v_n^6 \rangle, \\
c_n \{8\} &= -144 \langle v_n^2 \rangle^4 + 144 \langle v_n^4 \rangle \langle v_n^2 \rangle^2 \\
&\quad - 16 \langle v_n^6 \rangle \langle v_n^2 \rangle - 18 \langle v_n^4 \rangle^2 + \langle v_n^8 \rangle, \\
&\quad \vdots
\end{align*}
\]

where the average $\langle v_n^m \rangle$ is performed with respect to the distribution $p(v_n)$ in Eq. (4). Finding the flow distribution is left to next section.

So far we have presented a well-known technique in statistic theory to find the probability distribution and its cumulants. As can be seen, using this technique we could find the 2k-particle cumulants. In the following, we first obtain the flow distribution of odd harmonics. Then we try to find a general probability distribution to explain the event-by-event fluctuation which is true for both odd and even flow harmonics.

### III. TWO-DIMENSIONAL CUMULANT AND MOMENT

As mentioned earlier, introducing a method to find flow harmonic distribution that extract the maximum amount of information is necessary. Here, we present a technique commonly used in statistics to achieve our goal. To find the relations between moments and cumulants of flow harmonics, we use the joint generating functions [20],

\[
\log \langle e^{\lambda_z + \lambda_z^*} \rangle = \sum_{k,l} \frac{\lambda_z^k \lambda_z^l}{k!l!} \kappa(k,l),
\]

where $\kappa(k,l)$ are joint cumulants. It is worth emphasizing that Eq. (7) is a general formula. Moreover, to find the desired flow distributions we need to modify Eq. (7) by choosing different definitions of $z$ and $\lambda$.

In Ref. [20], an expansion of flow distribution for odd harmonics has been found (also see Eq.(24) in Ref. [21]). To reproduce this expansion, we have to set $z \equiv V_n$ and $\lambda \equiv (\lambda_x - i \lambda_y)/2$ in Eq. (7). By replacing these considerations in Eq. (7), we have

\[
\begin{align*}
\langle e^{v_n x \lambda_x + v_n y \lambda_y} \rangle
&= \exp \left[ \sum_{kl} \frac{(\lambda_x + i \lambda_y)^k (\lambda_x - i \lambda_y)^l}{2^{(k+l)} k! l!} c_n \{k,l\} \right].
\end{align*}
\]

Note that here we use the common notation of $c_n$ for 2k-particle cumulants. The cumulant $c_n \{k,l\}$ can be obtained by differentiating both sides of Eq. (7):

\[
\begin{align*}
\frac{\partial^{k+l}}{\partial \lambda_x^k \partial \lambda_y^l} \left( e^{v_n x \lambda_x + v_n y \lambda_y} \right)
&= \exp \left[ \sum_{kl} \frac{(\lambda_x + i \lambda_y)^k (\lambda_x - i \lambda_y)^l}{2^{(k+l)} k! l!} c_n \{k,l\} \right],
\end{align*}
\]

and evaluating the results at $\lambda_x = 0$ and $\lambda_y = 0$. Note that in Eq.(8) only terms with $k = l$ are non-zero. Setting $k = l$ the relations in Eq. (6) are reproduced [1]. To find the odd flow distributions [2], we use the first line of Eq.(4) [7] and the Fourier transformation of characteristic function, $\lambda_x^2 + \lambda_y^2 \rightarrow \partial_x^2 + \partial_y^2$. The probability distribution for odd harmonics, thus, becomes

\[
\begin{align*}
p_{\text{odd}}(v_n x, v_n y)
&= \exp \left[ \sum_{k=2} c_n \{2k\} \frac{(\partial_x^2 + \partial_y^2)^k}{4^k (k!)^2} \right] \frac{1}{\pi c_n \{2\}^2} e^{-\frac{v_n^2}{c_n \{2\}^2}}.
\end{align*}
\]

If we rewrite this distribution in polar coordinates, $v_n^2 = v_n x^2 + v_n y^2$, we can obtain the radial odd flow distribution,

\[
\begin{align*}
\int dv_n x dv_n y p_{\text{odd}}(v_n x, v_n y)
&= \int \frac{v_n dv_n d\Psi_n}{\pi c_n \{2\}^2} \exp \left[ \sum_{k=2} c_n \{2k\} \frac{D^k_{v_n, \Psi_n}}{4^k (k!)^2} \right] e^{-\frac{v_n^2}{c_n \{2\}^2}} \\
&\approx \int \frac{v_n dv_n d\Psi_n}{\pi c_n \{2\}^2} \left[ 1 + \sum_{k=2} c_n \{2k\} \frac{D^k_{v_n}}{4^k (k!)^2} \right] e^{-\frac{v_n^2}{c_n \{2\}^2}} \\
&= \int dv_n p_{\text{odd}}(v_n),
\end{align*}
\]

where $D_{v,\Psi}$ represent $D_v + (1/v^2) \partial^2_{\Psi}$ and $D_v$ is $\partial^2_v + (1/v) \partial_v$. Therefore, the radial distribution of odd flow harmonics $p_{\text{odd}}(v_n)$ is

\[
\begin{align*}
p_{\text{odd}}(v_n) &= \frac{2v_n}{c_n \{2\}^2} \left[ 1 + \sum_{k=2} c_n \{2k\} \frac{D^k_{v_n}}{4^k (k!)^2} \right] \exp \left[ -\frac{v_n^2}{c_n \{2\}^2} \right].
\end{align*}
\]

The form of $p_{\text{odd}}(v_n)$ can be found in terms of cumulants by evaluating the $k$th derivative of $-(\frac{v_n^2}{c_n \{2\}^2})$ (see Appendix A) and letting $2\sigma^2 = c_n \{2\}$ for odd harmonics [20, 21],

\[
\begin{align*}
p_{\text{odd}}(v_n) &= \left( \frac{v_n}{\sigma^2} \right) e^{-\frac{v_n^2}{2\sigma^2}} \left[ 1 + \sum_{k=2} \frac{(-1)^k \Gamma_{\text{odd}}(2k-2)}{k! \Gamma_k} L_k(v_n^2/(2\sigma^2)) \right].
\end{align*}
\]

1 Note that to find the averaged flow magnitude we have to integrate the generating moments over $\phi_\lambda$ in polar coordinates

\[
G(\lambda) = \frac{1}{2\pi} \int_0^{2\pi} d\phi_\lambda < e^{\lambda \cdot v} >,
\]

where $\lambda_x = \lambda \cos \phi_\lambda$ and $\lambda_y = \lambda \sin \phi_\lambda$.

2 Here we use “odd” because $\langle v_{n,x} \rangle = 0$ similar zero net triangularity for $n = 3$.

3 In the integration we use $\lambda_x \rightarrow -i \lambda_x$ and $\lambda_y \rightarrow -i \lambda_y$. 
where $\Gamma_{2k} = c_n \{2k\}j_n \{2k\}$. The expansion (11) is exactly the flow distribution found in Ref. [20] which can explain any event-by-event flow fluctuations of odd harmonics.

Because $p_{\text{odd}}(v_{n,x},v_{n,y})$ is rotationally symmetric $(\bar{v}_{2n+1} = \langle v_{2n+1,x} \rangle = 0)$ and consequently the main features of 2D and radial odd flow distribution are the same, obtaining distribution (11) is simple. But this case is not true for even flow harmonics, since $v_{2n} \neq 0$. This is because even flow distributions are not rotationally symmetric, and reshuffling $(v_{n,x},v_{n,y})$ leads to a partial loss of information of $p_{\text{even}}(v_{n,x},v_{n,y})$. Hence, the main challenge is to find a radial flow distribution which can give a good approximation of flow fluctuations for even $n$ so that the least amount of information is lost.

In the following, we begin to find the flow harmonic distribution and its cumulants by assuming non-zero $\bar{v}_n$. Modifying the relation (7) for even flow harmonics, the relation of moment and cumulant generating functions in 2D with $k = l$ can be rewritten as

$$\langle \epsilon(v_{n,x}-\bar{v}_n)\lambda_x+v_{n,y}\lambda_y \rangle = \exp \left[ \sum_k \frac{(\lambda_x^2 + \lambda_y^2)^k}{2k!(k)!^2} j_n \{2k\} \right].$$

where we consider $z = V_n - \bar{v}_n$ and $\lambda = \frac{\lambda_x - \lambda_y}{2}$. We simply use the notation $W_n = V_n - \bar{v}_n$ as a shifted flow vector so that $\langle W_n \rangle = 0$. The reason for choosing $k = l$ is to avoid obtaining complex $j_n$ cumulants. By differentiating both sides of Eq. (12) at $\lambda_x = 0$ and $\lambda_y = 0$, one can find the relations between $j_n \{2k\}$ and moments,

$$j_n \{2\} = \langle w_n^2 \rangle,$$

$$j_n \{4\} = \langle w_n^4 \rangle - 2\langle w_n^2 \rangle^2,$$

$$j_n \{6\} = \langle w_n^6 \rangle + 12\langle w_n^2 \rangle^3 - 9\langle w_n^2 \rangle \langle w_n^4 \rangle,$$

$$j_n \{8\} = \langle w_n^8 \rangle - 144\langle w_n^2 \rangle^4 + 144\langle w_n^4 \rangle^2 \langle w_n^4 \rangle^2 - 16\langle w_n^6 \rangle^2 \langle w_n^2 \rangle^2 - 18\langle w_n^4 \rangle^4,$$

$$\vdots$$

where $w_n^2 = |W_n|^2 = (v_{n,x} - \bar{v}_n)^2 + v_{n,y}^2$. As can be seen, by choosing $\bar{v}_n = 0$, 2k-particle correlation functions $c_n \{2k\}$ can be recovered. In the result, the shifted cumulants $j_n \{2k\}$ are the generalized forms of 2k-particle cumulants where the collision geometry effects were extracted. Fig. 1 presents the cumulants $j_n \{2k\}$ for $n = 2$, obtained from the iEBE-VISHNU output, and the comparisons between them. Note that we have $j_n \{2k\} = c_n \{2k\}$ for odd harmonics. As demonstrated in this figure, the differences between $j_n \{4\}$, $j_n \{6\}$, and $j_n \{8\}$ are sensible in peripheral central collisions such that the relation between $j_n \{2k\}$ is

$$j_n \{2\} \gg j_n \{4\} \gg j_n \{6\} \gg j_n \{8\} \gg \cdots .$$

We expected this relation, because the experimental results [2] show that $p_{\text{even}}(v_n)$ have a deviation from Bessel-Gaussian. This deviation is more pronounced in peripheral collisions where the Bessel-Gaussian distribution can not explain experimental data. Furthermore, we expect that the cumulants $j_n \{2k\}$ can quantify the main

![Fig. 1](image-url)
features of a distribution near Bessel-Gaussian.

In Ref. [21], a new set of cumulants \( q_n\{2k\} \) has been defined to study the distributions near Bessel-Gaussian. These cumulants have also been obtained from 2k-particle correlation functions \( c_n\{2k\} \). Replacing the definitions of \( q_n\{2k\} \) (see Eq.(36) in Ref.[21]) in Eq.(13), one can find

\[
j_n\{2\} = q_n\{2\}, \text{ and } j_n\{2k\} = q_n\{2k\} + ..., \text{ for } k \geq 2.
\]

A comparison between \( j_n\{2k\} \) and \( q_n\{2k\} \) are presented in Fig. 2. As can be seen, the difference between these sets of cumulants for \( k \geq 2 \) is significant, especially for mid-centralities and peripheral collisions. This means that the amount of encoded information in these two sets are different. It should be noted the cumulant set \( q_n\{2k\} \) has been defined by using the moments of the radial flow distribution \( p_y(v_n; \bar{v}_n) \) in Ref. [21], but here we only used the relation between the joint cumulant and moment generating functions to find \( j_n\{2k\} \). This means that our technique does not require any knowledge about the flow distributions.

The main challenge is finding the form of flow harmonic distributions by considering \( \bar{v}_n \neq 0 \). If we obtain the Fourier transformation of joint characteristic function of moments in Eq.\((12)\), \( \lambda_x \rightarrow -i \partial_x \) and \( \lambda_y \rightarrow -i \partial_y \), the 2D distribution \( p(v_n,x,v_n,y) \) is obtained as

\[
p(v_n,x,v_n,y) = \exp \left[ \sum_{k=2}^{n} \frac{j_n\{2k\} D^k}{4^k(k!)^2} F(v_n,x,v_n,y) \right], \tag{15}
\]

where \( D \) is the differential operator with respect to \( \lambda_x \) and \( \lambda_y \). Also, the distribution \( \sqrt{2\pi}\sigma F(v_n,x,v_n,y) \) is a 2D Gaussian distribution with mean \( \bar{v}_n \) and standard deviation \( \sqrt{f_n\{2\}}/2 \). After some calculations in Cartesian coordinates, we have

\[
D^k F(v_n,x,v_n,y) = \left( -1 \right)^k 4^k k! \frac{j_n\{2k\}}{j_n\{2\}}^k F(v_n,x,v_n,y) L_k \left( \frac{\bar{v}_n^2}{j_n\{2\}} \right). \tag{16}
\]

Since we follow the radial flow distribution, Eq.\((16)\) can be written in polar coordinates as follows

\[
D^k_{v_n,\Psi_n} F(v_n; \bar{v}_n, \Psi_n) = \left( -1 \right)^k 4^k k! \frac{j_n\{2k\}}{j_n\{2\}}^k F(v_n; \bar{v}_n, \Psi_n) \times \left( L_k \left( \frac{\bar{v}_n^2}{j_n\{2\}} \right) + A_k + B_k \right). \tag{17}
\]

where the terms of \( A_k \) and \( B_k \) are

\[
A_k = \alpha_k, \quad B_k = \sum_{i=1}^{k} \beta_{kl} \cos l \Psi_n.
\]

The derivation of the \( n \)th derivative of \( F(v_n,x,v_n,y) \) in Eq.\((17)\) and definitions of the coefficients \( \alpha \) and \( \beta \) are in

\[\footnote{A flow distribution that works for even harmonics is a general distribution that is true for all harmonics. Furthermore, we use the notation \( p(v_n,x,v_n,y) \) instead of \( p_{even}(v_n,x,v_n,y) \) for non-rotational symmetric flow distribution.}
the Appendix [B]. If we integrate “\(F(v_n; \bar{v}_n, \Psi_n)\) cos \(l \Psi_n\)” over \(\Psi_n\), we find that

\[
\int_0^\infty v_n dv_n \int_0^{2\pi} d\Psi F(v_n; \bar{v}_n, \Psi_n) \cos l \Psi_n
\]

\[
= \int_0^\infty dv_n \left( \frac{2v_n}{j_n(2)} \right) e^{-\frac{v_n^2 + \bar{v}_n^2}{4mv_n^2}} I_0 \left( \frac{2v_n \bar{v}_n}{j_n(2)} \right)
\]

\[
= \int_0^{2\pi} d\theta \left( \frac{2v_n}{j_n(2)} \right) I_0 \left( \frac{2v_n \bar{v}_n}{j_n(2)} \right).
\]

Using Eq. (17), we find that

\[
\int_0^\infty v_n dv_n \int_0^{2\pi} d\Psi_n D_{v_n, \Psi_n} F(v_n; \bar{v}_n, \Psi_n)
\]

\[
= \int_0^\infty dv_n \left( -1 \right)^k \frac{2^k k!}{j_n(2)^k} \int_0^{2\pi} d\theta \left( \frac{2v_n}{j_n(2)} \right) I_0 \left( \frac{2v_n \bar{v}_n}{j_n(2)} \right)
\]

\[
= \frac{j_n(2k)}{4^k(k!)^2} \int_0^{2\pi} d\theta \left( \frac{2v_n}{j_n(2)} \right) I_0 \left( \frac{2v_n \bar{v}_n}{j_n(2)} \right)
\]

\[
= \left( L_k \frac{v_n^2 + \bar{v}_n^2}{j_n(2)^2} \right) + \alpha_k I_0 \left( \frac{2v_n \bar{v}_n}{j_n(2)} \right) + \beta_k I_1 \left( \frac{2v_n \bar{v}_n}{j_n(2)} \right).
\]

The radial flow distribution \(p(v_n; \bar{v}_n)\) using Eq. (19) can be obtained

\[
p_q(v_n; \bar{v}_n)
\]

\[
= \int_0^{2\pi} d\Psi_n v_n p(v_n; \bar{v}_n, \Psi_n)
\]

\[
= \int_0^{2\pi} d\Psi_n v_n \left[ 1 + \sum_{k=2}^{\infty} \frac{j_n(2k) D_{v_n, \Psi_n}^k}{4^k(k!)^2} \right] F(v_n; \bar{v}_n, \Psi_n)
\]

\[
= \left[ \frac{\alpha_k'}{k!} \right] \left[ \alpha_k I_0 \left( \frac{2v_n \bar{v}_n}{j_n(2)} \right) + \beta_k I_1 \left( \frac{2v_n \bar{v}_n}{j_n(2)} \right) \right],
\]

where \(\alpha_k' = \frac{L_k v_n^2 + \bar{v}_n^2}{j_n(2)^2} \) and \(\gamma_k = j_n(2k)/j_n(2)^k = q_n(2k)/q_n(2)^k + \cdots\). Note that we have assumed \(\gamma_0 = 1\) and \(\gamma_1 = \alpha_0 = \beta_0 = 0\) in Eq. (20). The first term \((q = 0)\) of \(p_q(v_n; \bar{v}_n)\) is a Bessel-Gaussian distribution. Other terms are the corrections to the Bessel-Gaussian distribution.

Fig. 3 compares the obtained distribution from iEBEVISHNU with estimated distribution in Eq. (20). In this figure, we investigate different truncations of \(p_q(v_n; \bar{v}_n)\) for \(q = 0, 2, 3, 4\) plotted by black, red, blue, green lines, respectively. Since the main shortcoming of the Bessel-Gaussian distribution compared with the simulation data are in peripheral collisions, we only show the results in \(65 - 70\%\), \(70 - 75\%\), and \(75 - 80\%\) centrality classes. As demonstrated in this figure, the generated data cannot be described by the black curve, which corresponds to the Bessel-Gaussian distribution.

Also, studying \(\chi^2/\mathrm{NDF}\) for the Bessel-Gaussian distribution and \(p_q(v_n; \bar{v}_n)\) for \(q = 2, 3, 4\) plotted in Fig. 4 for the investigated centralities in Fig. 3. As can be seen, the values of \(\chi^2/\mathrm{NDF}\) associated \(p_q(v_n; \bar{v}_n)\) are more closer to 1 comparing with the Bessel Gaussian distribution.

The results of Fig. 3 and 4 show that the distribution of elliptic flow is different from the Bessel Gaussian distribution. So, the corrections to the Bessel Gaussian distribution becomes important which is described by \(p_q(v_n; \bar{v}_n)\).
IV. JOINT DISTRIBUTION OF FLOW HARMONICS

The information of the event-by-event flow fluctuations is encoded in the joint flow harmonic distribution \( p(v_1, v_2, \ldots) \), as mentioned in Sec. I. Therefore, using joint cumulant and moment generating functions, we obtain the joint distribution of flow harmonics in this section. To do this, we consider the relation between joint generating function of moments and cumulants as follows

\[
\langle e^{W_n \lambda_n + W_m \lambda_m} \rangle = \exp \left[ \sum_{k,l=0} \frac{(\lambda_n)^k (\lambda_m)^l}{k!! l!!} K_{kl} \right], \tag{21}
\]

where \( W_n \) and \( K_{nm} \) are shifted flow vectors and joint flow cumulants, respectively. The relations between \( K_{kl} \) and moments are

\[
K_{0,0} = K_{1,0} = K_{0,1} = 0,
K_{1,1} = \langle W_n W_m^* \rangle = \langle v_n v_m \cos(\Psi_1 - \Psi_2) \rangle - \bar{v}_n \bar{v}_m,
K_{2,0} = \langle |W_n|^2 \rangle = \langle v_n^2 \rangle - \bar{v}_n^2,
K_{0,2} = \langle |W_m|^2 \rangle = \langle v_m^2 \rangle - \bar{v}_m^2,
\]

\[
\vdots
\]

Note that because the average of sifted flow vector \( \langle W_n \rangle \) is zero the cumulants \( K_{1,0} \) and \( K_{0,1} \) are zero for all harmonics. Using Eq. (21), we can rewrite the cumulants of flow joint distribution in terms of \( j_n \{2k\} \) and flow corre-

One way to investigate the event-by-event flow fluctuations is by measuring the correlation between the magnitudes of different flow harmonics using a cumulant analysis. These new observables are commonly known as Symmetric Cumulants (SC). Recently, ALICE has measured \( SC(2,3) \) and \( SC(2,4) \) as a function of centrality at center-of-mass energy per nucleon pair \( \sqrt{s} = 2.76 \) TeV, with transverse momentum in the range of 0.2 < \( p_T \) < 5 GeV. In this paper, we show that these experimental data can be explained by a combination of joint cumulants \( \mathcal{K} \). Fig. 5 present a comparison between simulation and experimental data. It is worth mentioning that using VISHNU output \( p_T \) is in the range 0.28 < \( p_T \) < 4 GeV. As can be seen, there is a mismatch between \( SC(2,3) \) obtained from simulation and experimental data. But the experimental data can be described by combination \( \mathcal{K}_{22} + \frac{1}{2} \mathcal{K}_{04} - \mathcal{K}_{31} \). Also, one can find that \( SC(2,4) = \mathcal{K}_{22} + 4 \mathcal{K}_{11}^2 \) can explain the ALICE data. Given these agreements, we encourage the experimentalists to consider the cumulants \( \mathcal{K} \) as new observables.

Now, having the joint cumulants enables us to obtain the joint distribution of flow harmonics. To do this, one should find a form of cumulative characteristic function...
$G(\lambda_n, \lambda_m)$ by expanding it to $k + l = 2$,
$$
\exp \left( G(\lambda_n, \lambda_m) \right) \\
= \exp \left[ \sum_{k+l \geq 3} \tilde{K}_{kl}(\lambda_n)^k(\lambda_m)^l \right] \\
\times \exp \left[ \lambda_n^2\tilde{K}_{20} + \lambda_m^2\tilde{K}_{02} + \lambda_n\lambda_m\tilde{K}_{11} \right] \\
= \exp \left[ \sum_{k+l \geq 3} \tilde{K}_{kl}(\lambda_n)^k(\lambda_m)^l \right] \mathcal{N}(\lambda_n, \lambda_m).
$$

where the standard joint cumulants $\tilde{K}_{mn}$ are $K_{mn}/(m!n!)$.

Applying Fourier transforming to both sides of Eq. (21), we get
$$
\int dW_n dW_m \mathcal{P}(W_n, W_m) e^{W_n\lambda_n + W_m\lambda_m} \\
= \exp \left[ \sum_{k+l \geq 3} \tilde{K}_{kl}(\partial_n)^k(\partial_m)^l \right] \mathcal{N}(\lambda_n, \lambda_m),
$$

where $dW_n = dw_{n,x} dw_{n,y}$. Eventually, we find the joint distribution $\mathcal{P}(W_n, W_m)$ as
$$
\mathcal{P}(W_n, W_m) \\
= \frac{1}{2\pi\Delta} \exp \left[ \sum_{k+l \geq 3} \tilde{K}_{kl}(\partial_n)^k(\partial_m)^l \right] \\
\times \exp \left[ -\tilde{K}_{02}w_m^2 + \tilde{K}_{20}w_n^2 - \tilde{K}_{11}(w_{n,x}w_{m,x} + w_{n,y}w_{m,y}) \right] \\
\approx \left[ 1 + \sum_{k+l \geq 3} \tilde{K}_{kl}(\partial_n)^k(\partial_m)^l \right] \mathcal{N}(W_n, W_m)
$$

where $\Delta$ defined $(4\tilde{K}_{20}\tilde{K}_{02} - \tilde{K}_{22}^2)^{1/2}$ or in the simplified case $(\lambda_n J_0(\tilde{K}_{j2})^2 - \text{Re}[(\tilde{K}_{j2}^2)]^{1/2})$. Note that if we only consider $\mathcal{N}(W_n, W_m)$ as the first term of $\mathcal{P}(W_n, W_m)$ and compare it with bivariate normal distribution, we find that

$$
\sigma_n^2 = 2\tilde{K}_{20} = \langle |W_n|^2 \rangle, \\
\sigma_m^2 = 2\tilde{K}_{20} = \langle |W_m|^2 \rangle, \\
\rho_{nm} = \tilde{K}_{11} \frac{2}{\sqrt{\tilde{K}_{20}\tilde{K}_{02}}} = \frac{\text{Re}[\langle W_n W_m^* \rangle]}{\sqrt{\langle |W_n|^2 \rangle \langle |W_m|^2 \rangle}}.
$$

These results show that the general joint distribution of flow vectors can be obtained,
$$
\mathcal{P}(W_1, W_2, ..., W_n) \\
\approx \left[ 1 + \sum_{k_1 + \ldots + k_n \geq 3} \tilde{K}_{k_1...k_n}(\partial_1)^{k_1} \ldots (\partial_n)^{k_n} \right] \times \mathcal{N}(W_1, W_2, ..., W_n),
$$

by defining the joint cumulant and moment generating function relation,
$$
\langle e^{W_1\lambda_1 + \ldots + W_n\lambda_n} \rangle \\
= \exp \left[ \sum_{k_1 + \ldots + k_n \geq 0} \frac{(\lambda_1)^{k_1} \ldots (\lambda_n)^{k_n}}{k_1! \ldots k_n!} \tilde{K}_{k_1...k_n} \right],
$$

where $K_{k_1...k_n}$ are the generalized joint cumulants. Note that the distribution $\mathcal{N}(W_1, W_2, ..., W_n)$ in Eq. (26) is a generalization of the one-dimensional normal distribution to higher dimensions which is dubbed as the joint normal distribution. But it should be noted that since the souls of the multivariate normal distribution and the distribution $\mathcal{N}(W_1, W_2, ..., W_n)$ in Eq. (27) are different, we have $\int dW_1 \cdots dW_n\mathcal{N}(W_1, W_2, ..., W_n) \neq 1$. So, in the following, we use normalized kernel $\mathcal{N}(W_1, W_2, ..., W_n)$ to find the joint distribution of flow magnitudes.

Let us return to the computation of the joint radial distribution of two flow harmonics using Eq. (24). In Ref. [30], a technique has been introduced that enables us to study the correlations between any rapidity windows and any harmonics. In this technique denoting the relative angle $\Phi = \Psi_m - \Psi_n^*$ and averaging over reaction plane angle, the joint radial flow distribution is obtained as

$$
\int dv_n dv_m \mathcal{P}(v_n; \bar{v}_n, v_m; \bar{v}_m) \\
= \int v_n dv_n v_m dv_m \int d\mathcal{P}(W_n, W_m) \frac{d\mathcal{P}(W_n, W_m)}{d\Psi_n d\Psi_m} d\Phi d\Psi_n d\Phi d\Psi_m \delta(\Phi - \Psi_m + \Psi_n).
$$

To study the joint radial flow distribution, we consider the first term in Eq. (24) for simplicity. Inserting it in Eq. (28) we obtain the joint distribution
$$
\int dv_n dv_m \mathcal{P}_1(v_n; \bar{v}_n, v_m; \bar{v}_m) \\
= \int dv_n dv_m \mathcal{P}_1(v_n; \bar{v}_n, v_m; \bar{v}_m) \mathcal{P}_1(v_n; \bar{v}_n, v_m; \bar{v}_m)
$$

In 2D, the probability density function of a vector $[x', y']$ is
$$
f(x', y') = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1 - \rho^2}} \times \exp \left( -\frac{1}{2(1-\rho^2)} \frac{x'^2}{\sigma_x^2} + \frac{y'^2}{\sigma_y^2} - 2\rho x'y' / \sigma_x\sigma_y \right),
$$

where $\sigma$ and $\rho$ are the standard deviation and the Pearson correlation, respectively.

Note that the angles $\Phi$ and $\Psi$ are in the range of $[0, \pi]$ and $[0, 2\pi]$, respectively.

Here to normalize distribution $\mathcal{N}(W_n, W_m)$, we assume $\Delta \to \Delta/\sqrt{2}$.
where
\[
\chi_{mn} \equiv \frac{4v_n v_m}{\pi \Delta^2} \exp \left[ -\frac{\Delta^2}{2\Delta_{12}} - \frac{\Delta^2}{2\Delta_{20}} - \frac{\Delta^2}{2\Delta_{11}} \right],
\]
\[
\zeta_1 \equiv v_n \left( \frac{2\bar{v}_n}{\Delta^2/\Delta_{12}} - \frac{\bar{v}_m}{\Delta^2/\Delta_{11}} \right),
\]
\[
\zeta_2 \equiv v_m \left( \frac{2\bar{v}_m}{\Delta^2/\Delta_{20}} - \frac{\bar{v}_n}{\Delta^2/\Delta_{11}} \right),
\]
\[
\zeta_3 \equiv \frac{\bar{v}_n \bar{v}_m}{\Delta^2/\Delta_{11}}.
\]

Note that Eq. (29) is the first approximation of the radial joint distribution of any two flow harmonics. To study the distribution \( \mathcal{P}(v_n; \bar{v}_n, v_m; \bar{v}_m) \), we investigate it for \( v_2 \) and \( v_3 \). In this case, since the correlation \( \rho(V_2, V_3) \) approximately vanishes, the joint cumulant \( \bar{K}_{11} \) can be ignored and thus \( \Delta^2 \) is equal to \( j_2 \{ 2 \} j_3 \{ 2 \} \). Also, since the triangular flow distribution is rotationally symmetric, the triangularity is zero. Concerning these variables, the joint distribution of second and third harmonics can be rewritten
\[
\mathcal{P}_1(v_2; \bar{v}_2, v_3; 0) = \frac{4v_2 v_3}{j_2 \{ 2 \} j_3 \{ 2 \}} \exp \left[ -\frac{v_2^2 + \bar{v}_2^2}{j_2 \{ 2 \}} - \frac{v_3^2}{j_3 \{ 2 \}} \right] I_0 \left( \frac{2v_2 \bar{v}_2}{j_2 \{ 2 \}} \right).
\]

Eq. (30) show that because there is a negligible correlation between \( V_2 \) and \( V_3 \), the first approximation of joint distribution \( \mathcal{P}(v_2; \bar{v}_2, v_3; 0) \) is \( p_0(v_3; 0)p_0(\bar{v}_2) \). Note that \( p_1(v_n; \bar{v}_n) \) is the first truncation of the distribution Eq. (30) which is called the Bessel-Gaussian distribution. Fig. (6) present the smooth density histogram of \( v_2 \) and \( v_3 \), which is obtained by using the results of the event-by-event \( 3 + 1 \)D viscous hydrodynamics at center-of-mass energy per nucleon pair \( \sqrt{s} = 5.02 \) TeV. These data are obtained with the wounded-quark initial conditions, and the contour plot of the first term of distribution \( \mathcal{P}(v_2; \bar{v}_2, v_3; 0) \) in \( 30 - 40\% \) centrality. As can be seen in this figure, the results show that there is a decent agreement between theory and simulation data. To find the best estimation, one have to insert the complete form of Eq. (24) in Eq. (28).

V. CONCLUSION

In this paper, we employed the relation between joint cumulant and moment generating function of \( v_{n,x} \) and \( v_{n,y} \) to relate the radial flow distribution to cumulants by using the standard method of finding Gram-Charlier series. We have found a general flow distribution in Eq. (20) by using Fourier transformation both sides of Eq. (12). It is an expansion around Bessel-Gaussian distribution where the coefficients of the expansion have been written in terms of a general set of cumulants \( j_n \{ 2k \} \). We have shown that \( p(v_n; \bar{v}_n) \) can explain the generated data in the peripheral collisions, by assuming \( \bar{v}_n \neq 0 \) for even harmonics. Our results indicate a significant improvement over the Bessel-Gaussian distribution. Also, we have obtained the odd flow distribution which has been found in Refs. 20 and 21 by setting \( \bar{v}_n = 0 \). The general cumulants \( j_n \{ 2k \} \) were written in terms of moments \( \langle w_n^k \rangle \) where \( w_n \) is the magnitude of the shifted flow vector. If one assumes \( \bar{v}_n = 0 \), the cumulants \( j_n \{ 2k \} \) would be \( 2k \)-particle correlation functions \( c_n \{ 2k \} \) which can be observed experimentally. Also, we have shown that the cumulants \( j_n \{ 2k \} \), which is obtained from the relation between joint cumulant and moment generating functions, are more general than the cumulants \( q_n \) found in previous works. In the final step, we have studied the joint
distribution of flow harmonics and presented a general form for $P(W_1, W_2, ..., W_n)$. To do this, we introduced new observables $K_{mn}$ and showed that the experimental data for symmetric cumulants $SC(2, 3)$ and $SC(2, 4)$ can be explained by combinations of these observables. So, we think that the cumulants $K_{mn}$ can be interesting observables for experimentalists. We also obtained the joint radial distribution of the two flow harmonics and showed that the first terms of this distribution for $v_2$ and $v_3$ can justify the simulation data. Investigating joint distributions of other flow harmonics is left to future studies.

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Appendix A: General Form of Radial Derivatives

If we differentiate 1D normal distribution, $\mathcal{N}(r) = e^{-\frac{r^2}{2\alpha^2}}/\sqrt{\pi \alpha}$ with $a = 2\sigma^2$, $n$ times, the result of each time is approximately a Laguerre polynomial,

$$
k = 1: D_1^r N(r) = D_r(N(r) L_0(\frac{r^2}{\alpha})),
$$

$$
k = 2: D_2^r N(r) \approx D_r(N(r) L_1(\frac{r^2}{\alpha})),
$$

$$
k = 3: D_3^r N(r) \approx D_r(N(r) L_2(\frac{r^2}{\alpha})),
$$

$$
\vdots
$$

$$
k = n: D_n^r (e^{-\frac{r^2}{2\alpha^2}}) \approx D_r(N(r) L_n(\frac{r^2}{\alpha})).
$$

Note that the radial derivative is $D_r = \partial_r + (1/r)\partial_r$. Using above derivative in Eq. (10), we can rewrite the $p(v_n)$ as follows

$$
p_{\text{odd}}(v_n) = \frac{2v_n}{c_n \{2\}} \left[ e^{-\frac{v_n^2}{\sqrt{\pi} \alpha}} + \sum_{k=2}^{\infty} \frac{c_n \{2k\}}{4^k k!} \right]
$$

$$
\times \left[ (-4)^{k-1} (k-1)! D_{v_n}(e^{-\frac{v_n^2}{\sqrt{\pi} \alpha}} L_{k-1}(\frac{v_n^2}{c_n \{2\}})) \right].
$$

If we differentiate $N(r) L_{k-1}(\frac{r^2}{\alpha})$ in the radial direction,

$$
D_r(N(r) L_{k-1}(\frac{r^2}{\alpha})) = -4n \frac{a}{a} N(r) L_k(\frac{r^2}{a}),
$$

and then replace it in Eq. (A2), in the result we have the distribution of odd flow harmonics as

$$
p_{\text{odd}}(v_n) = \left( \frac{2v_n}{c_n \{2\}} \right) e^{-\frac{v_n^2}{\sqrt{\pi} \alpha}}
$$

$$
\times \left[ 1 + \sum_{k=2}^{\infty} \frac{(-1)^k c_n \{2k\}}{k! c_n \{2\}^k} L_k(v_n^2/c_n \{2\}) \right].
$$

Appendix B: Two dimensional derivatives

As mentioned in Sec. III to find a general flow distribution we can have

$$
p(v_{n,x}, v_{n,y}) = \exp \left[ \sum_{k=2}^{\infty} \frac{j_n(2k)}{4^k (k!)^2} \mathcal{F}(v_{n,x}, v_{n,y}) \right],
$$

by using Eq. (12) and considering the relation

$$
\int D\lambda (\lambda_x^2 + \lambda_y^2)^k e^{-i(v_{n,x} - \bar{v}_n)\lambda_x - i v_{n,y} \lambda_y} \times e^{-(\lambda_x^2 + \lambda_y^2) / 2} / (2\pi)^{k/2},
$$

where $D\lambda = d\lambda_x d\lambda_y$. If we evaluate the derivative $D$ for $k = 1, 2, ..., k$ in Eq. (B1), we have

$$
k = 1: D^1 \mathcal{F}(v_{n,x}, v_{n,y}) =
$$

$$
-\frac{4}{j_n \{2\}} \mathcal{F}(v_{n,x}, v_{n,y}) L_1(\frac{(v_{n,x} - \bar{v}_n)^2 + v_{n,y}^2}{j_n \{2\}}),
$$

$$
k = 2: D^2 \mathcal{F}(v_{n,x}, v_{n,y}) =
$$

$$
\frac{32}{j_n \{2\}^2} \mathcal{F}(v_{n,x}, v_{n,y}) L_2(\frac{(v_{n,x} - \bar{v}_n)^2 + v_{n,y}^2}{j_n \{2\}}),
$$

$$
k = 3: D^3 \mathcal{F}(v_{n,x}, v_{n,y}) =
$$

$$
-\frac{384}{j_n \{2\}^3} \mathcal{F}(v_{n,x}, v_{n,y}) L_3(\frac{(v_{n,x} - \bar{v}_n)^2 + v_{n,y}^2}{j_n \{2\}}),
$$

\vdots

$$
k = n: D^n \mathcal{F}(v_{n,x}, v_{n,y}) =
$$

$$
\left( (-1)^n 4^n n! \right) \mathcal{F}(v_{n,x}, v_{n,y}) L_n(\frac{(v_{n,x} - \bar{v}_n)^2 + v_{n,y}^2}{j_n \{2\}}).
$$

Note that the calculations of Eq. (B3) are obtained by using Cartesian partial derivatives. To find radial flow distribution we have to integrate over azimuthal angle. Therefore, it is better to write down in polar coordinates,

$$
k = 1: D^1_{v_n, \Psi_n} \mathcal{F}(v_n; \bar{v}_n, \Psi_n) =
$$

$$
-\frac{4}{a} \mathcal{F}(v_n; \bar{v}_n, \Psi_n) \left( L_1(\frac{v_n^2 + \bar{v}_n^2}{j_n \{2\}}) + A_1 + B_1 \right),
$$

$$
k = 2: D^2_{v_n, \Psi_n} \mathcal{F}(v_n; \bar{v}_n, \Psi_n) =
$$

$$
\frac{32}{a^2} \mathcal{F}(v_n; \bar{v}_n, \Psi_n) \left( L_2(\frac{v_n^2 + \bar{v}_n^2}{j_n \{2\}}) + A_2 + B_2 \right),
$$

$$
k = 3: D^3_{v_n, \Psi_n} \mathcal{F}(v_n; \bar{v}_n, \Psi_n) =
$$

$$
-\frac{384}{a^3} \mathcal{F}(v_n; \bar{v}_n, \Psi_n) \left( L_3(\frac{v_n^2 + \bar{v}_n^2}{j_n \{2\}}) + A_3 + B_3 \right),
$$

\vdots

$$
k = n: D^n_{v_n, \Psi_n} \mathcal{F}(v_n; \bar{v}_n, \Psi_n) =
$$

$$
\left( (-1)^n 4^n n! \right) \mathcal{F}(v_n; \bar{v}_n, \Psi_n) \left( L_n(\frac{v_n^2 + \bar{v}_n^2}{j_n \{2\}}) + A_n + B_n \right),
$$

(B4)
where $A_k$ and $B_k$ are

\begin{align}
A_1 &= 0, \\
B_1 &= \frac{2v_n\bar{v}_n}{j_n(2)} \cos \Psi_n, \\
A_2 &= \frac{v_n^2\bar{v}_n^2}{j_n(2)}, \\
B_2 &= \frac{2v_n\bar{v}_n}{j_n(2)}\left(2L_1\left(\frac{v_n^2 + \bar{v}_n^2}{2j_n(2)}\right)\cos \Psi_n + \frac{v_n^2\bar{v}_n^2}{j_n(2)} \cos 2\Psi_n, \right) \\
&\vdots \\
A_k &= \alpha_k, \\
B_k &= \sum_{l=1}^{k} \beta_{kl} \cos l\Psi_n.
\end{align}

(B5)

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