Unitarily Equivalent Classes of First Order Differential Operators

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Abstract. The class of non-homogeneous operators which are based on the same vector field, when viewed as acting on appropriate Hilbert spaces, are shown to be isomorphic to each other. The method is based on expressing a first order non-homogeneous differential operator as a product of a scalar function, a differential operator, and the reciprocal scalar function.

1. Introduction

Let $M$ be an $n$-dimensional Riemannian manifold, $L$ be a continuous vector field on $M$, and denote by $C^k(M)$ the set of real valued functions which are differentiable of order $k$ on $M$. For each real valued continuous function $q : M \to \mathbb{R}$ there corresponds an operator $L + q : C^1(M) \to C^0(M)$ defined by $(L + q)\psi = L\psi + q\psi$, $\forall \psi \in C^1(M)$, where $L\psi$ is the Lie derivative of the function $\psi$ with respect to the field $L$, and $q\psi$ is the usual function multiplication. It is evident that the operator $L + q$ from the vector space $C^1(M)$ to the vector space $C^0(M)$ is linear. Let $\eta$ be a real valued
function, differentiable on $M$, and such that $(L + q)\eta = 0$. We aim in this work

(i) to show that the operators $L + q$ and $L$ are transformable to each other by

\[ L + q = \eta L \eta^{-1}. \]

(ii) to prove that, when defined on the appropriate Hilbert spaces of square integrable functions on $M$, the operators $L + q$ and $L$ are isomorphic.

An extension of this work in the domain of totally linear partial differential equation is given in [8], and an algebraic treatment of this extension is given in [9].

2. Factorization of a First Order Non-Homogeneous Operator

Let $\eta \in C^1(M)$ be a non-zero solution of the differential equation

\[ (L + q)\eta = 0, \]

or equivalently, $\eta$ is any element in the kernel of the linear operator $L + q$ that is different from zero. We assume that $\eta$ has no zeros in $M$, and hence $\eta^{-1}$ exists and of class $C^1(M)$. Now equation (1) is equivalent to

\[ q = -\eta^{-1}(L\eta). \]

We prove here a useful operator equality which will be used very often throughout this work.

Proposition 1. In $C^1(M)$ the following operator equality holds:

\[ \eta L \eta^{-1} = L + q. \]
Proof. \( \forall \psi \in C^1(M) \) we have

\[
(\eta L \eta^{-1})\psi = \eta(\eta^{-1} L - \eta^{-2}(L \eta))\psi = (L - \eta^{-1}(L \eta))\psi = (L + q)\psi.
\]

We have used equation (3) when making the last step.

Corollary 2. Equality (3) is equivalent to

\[
L = \eta^{-1}(L + q)\eta, \tag{4}
\]

which shows that all operators of the form \((L + q)\) which are constructed from the same field \(L\) may be transformed to \(L\) and accordingly to each other.

Corollary 3. From (4) we deduce that

\[
L^n = \eta^{-1}(L + q)^n \eta \quad \text{and} \quad (L + q)^n = \eta L^n \eta^{-1}, \tag{5}
\]

where \(n\) is a non-negative integer. If \(L\) is invertible then the latter relation holds for all integers. We assume in relation (4) that \(L\) is a \(C^n\) vector field, \(q\) and \(\eta\) are \(C^n\) and \(C^{n+1}\) functions respectively.

Corollary 4. If (4) holds, then it is easily checked that \((L + nq)\eta^n = 0\), and hence

\[
L = \eta^{-n}(L + nq)\eta^n. \tag{6}
\]

In general, and for any real number \(\alpha\), we have

\[
L = |\eta|^{-\alpha} (L + \alpha q) |\eta|^\alpha.
\]
Corollary 5. If $h$ is a real-valued continuous function on $M$ then $\eta^{-1}(L + q + h)\eta = L + h$. It follows that

$$(L + h)\psi = 0 \iff (L + q + h)(\eta\psi) = 0$$

Corollary 6. Take $h = -\lambda$ ($\lambda \in \mathbb{R}$) in corollary (5) to obtain

$$(L - \lambda)\psi_\lambda = 0 \iff (L + q - \lambda)(\eta\psi_\lambda) = 0.$$ 

Expressed in words, the last relation states that: if $\psi_\lambda$ is an eigenfunction of the operator $L$ belonging to the eigenvalue $\lambda$ then $\eta\psi_\lambda$ is an eigenfunction of the operator $L + q$ belonging to the same eigenvalue $\lambda$.

It is clear that the argument we have presented to obtain equality (3) can be carried through without any change to the case in which the set of functions $C^k(M)$ is complex valued on $M$. The function $\eta$ is still real so that the expression $\eta L \eta^{-1}$ is real. From now on the set $C^k(M)$ will represent the set of complex valued functions which are differentiable of order $k$ on $M$. It is obvious that theorem 1 and the corollaries obtained from it are valid without modification.

Example 1. Take $L + q = \frac{d}{dx} + 2x : C^1(\mathbb{R}) \to C^0(\mathbb{R})$. Since $\eta = e^{-x^2}$ is a solution of (1), we have

$$\frac{d}{dx} + 2x = e^{-x^2} \frac{d}{dx} e^{x^2}.$$
It is obvious that every complex number $\lambda$ is an eigenvalue of the operator $\frac{d}{dx}$ to which an eigenfunction $\psi_\lambda = e^{\lambda x}$ belongs. In accordance with the last corollary, it is easily checked that $\lambda$ is an eigenvalue of the operator $\frac{d}{dx} + 2x$ to which the eigenfunction $\eta \psi_\lambda = e^{-x^2 + \lambda x}$ belongs.

3. Unitary Equivalence of $L + q$ and $L$.

Let $(L^2, d\tau) \equiv H$, where $d\tau$ is the volume element of $M$ defined by the Riemannian metric, be the natural Hilbert space of complex valued square integrable functions on $M$. The inner product in $H$ is denoted by $\langle, \rangle$, and the associated norm by $\|\|$, so that if $\psi \in H$ then $\|\psi\|^2 = \int_M \psi^* \psi \, d\tau < \infty$. Since the operator equality (3) is valid on $C^1(M)$, it is also valid when restricted to the set of differentiable functions in $H$ whose images under the action of either side of (3) is also in $H$. When restricted to $H$, the common domain of definition of either side of (3), is

$$D(L + q) = \{ \psi \in H : \psi \in C^1(M), (L + q)\psi \in H \}. \quad (7)$$

Denote by $(L^2, |\eta|^{2\alpha} d\tau) \equiv H_\eta$ the set of complex valued functions on $M$ such that

$$\psi \in H_\eta \Leftrightarrow \|\psi\|_{H_\eta}^2 \equiv \int_M \psi^* \psi |\eta|^{2\alpha} d\tau < \infty.$$ 

It is clear that $H_\eta$ is a Hilbert space, in which the inner product is defined by

$$\langle \psi, \phi \rangle_{H_\eta} = \int_M \psi^* \phi |\eta|^{2\alpha} d\tau.$$
It is also clear that
\[ \psi \in H_\eta \Leftrightarrow |\eta|^\alpha \psi \in H. \] (8)

Consider the operator
\[ |\eta|^\alpha : H_\eta \to H, \psi \to |\eta|^\alpha \psi. \]

Since
\[ <\psi | \phi>_{\eta} = \int_M \psi^* \phi |\eta|^{2\alpha} d\tau = \int_M (|\eta|^\alpha \psi)^* (|\eta|^\alpha \phi) d\tau = < |\eta|^\alpha \psi, |\eta|^\alpha \phi >, \]

\[ |\eta|^\alpha \] is an isometric operator. It is therefore linear and has an inverse
\[ |\eta|^{-\alpha} : H \to H_\eta, \psi \to |\eta|^{-\alpha} \psi, \]

which is also an isometry [1].

**Proposition 7.** The operator \( p = iL \) defined in \( H_\eta \) on the domain
\[ D(p) = \{ \psi \in H_\eta, \psi \in C^1(M), p\psi \in H_\eta \} \] (9)
is isomorphic to the operator \( P = i |\eta|^\alpha L |\eta|^{-\alpha} \) defined in \( H \) on the domain
\[ D(P) = \{ \psi \in H, |\eta|^{-\alpha} \psi \in C^1(M), P\psi \in H \}. \] (10)

**Proof.** According to the definition of unitarily equivalent operators [1] it is sufficient for the problem we consider to prove that \( D(P) = |\eta|^\alpha D(p) \). Indeed
\[ |\eta|^\alpha D(p) = \{ |\eta|^\alpha \psi \in H : \psi \in D(p) \}. \]
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\[= \{ |\eta|^\alpha \psi \in H : \psi \in H_\eta, \psi \in C^1(M), L\psi \in H_\eta \} \text{ by (9)}\]

\[= \{ |\eta|^\alpha \psi \in H : \psi \in C^1(M), |\eta|^\alpha L\psi \in H \} \text{ by (8)}\]

\[= \{ \phi \in H, |\eta|^{-\alpha} \phi \in C^1(M), |\eta|^\alpha L |\eta|^{-\alpha} \phi \in H \}\]

\[= D(P).\]

In the last step we set \(\phi = |\eta|^\alpha \psi\), and use the fact that \(\eta\) is in \(C^1(M)\) and has no zeros in \(M\), and hence \(\phi\) is in \(C^1(M)\) if and only if \(|\eta|^{-\alpha} \phi\) is in \(C^1(M)\).

Proposition 2, in essence, asserts that the replacement of the operator \(P\) by \(p\) amounts to replacing the volume element \(d\tau\) in \(M\) by \(|\eta|^{2\alpha} d\tau\).

4. The Case of Non-Homogeneous Symmetric Operators

It is well known [1, 2] that the operator \(\hat{P} = -i(L + q)\), with \(q = \frac{1}{2} \text{div} L\), defined on the domain

\[D_0(\hat{P}) = \{ \psi \in C^1_0(M) : \psi, \hat{P}\psi \in L^2(M) \}\]

is symmetric. The symbol \(C^1_0(M)\) refers to the set of continuously differentiable functions with compact support in \(M\). When the field \(L\) is complete the symmetric operator \(\hat{P}\) admits a self-adjoint extension, which we denote again by \(\hat{P}\), and is defined in \(L^2(M) \equiv (L^2, d\tau)\) on the domain

\[D(\hat{P}) = \{ \psi \in AC(M) : \psi, \hat{P}\psi \in L^2(M) \},\]

where \(AC(M)\) denotes the set of absolutely continuous functions on \(M\). By the spectral theory [3], there corresponds to the self-adjoint operator \(\hat{P}\) an one-parameter
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\[ U_t = \exp(i\hat{P}t) = \exp(i\eta\hat{p}\eta^{-1}t) \]  \hspace{1cm} (13)

of unitary transformation of \((L^2, d\tau)\). Now it is clear that the family of unitary operators \( \{V_t = \eta^{-1}U_t\eta, t \in \mathbb{R}\} \), acting in the Hilbert space \( H_\eta \equiv (L^2, \eta^2 d\tau) \), forms an one-parameter group of transformations of this space, which is isomorphic to the group \( U_t \) that acts in \( H \equiv (L^2, d\tau) \). The infinitesimal generator of the group \( V_t \), denoted by \( \hat{p}_0 \), is given by

\[ \hat{p}_0 = \frac{d}{dt}V_t \big|_{t=0} = \eta^{-1}\frac{d}{dt}U_t \big|_{t=0} \eta = \eta^{-1}\hat{P}\eta = \hat{p}. \]  \hspace{1cm} (14)

It follows that the isomorphic operators \( \hat{p} \) and \( \hat{P} \) generate one-parameter groups of \((L^2, \eta^2 d\tau)\) and \((L^2, d\tau)\) respectively, which are isomorphic to each other.

Still, another way of reaching the latter result, concerning the operator \( \hat{p} \), is based on the following observation. Let \((x^1, \ldots, x^n)\) be a chart on \( M \) in which the metric form is \( ds^2 = g_{ij}dx^i dx^j \), and set \( g = \det(g_{ij}) \). The volume element of \( M \) generated by the metric is \( d\tau = \sqrt{g}dx^1 \ldots dx^n \). The divergence of the vector field \( L = \xi^i \partial/\partial x^i \) is given by \[ \text{div}L = g^{-1/2}(g^{1/2}\xi^i)_i, \] where comma denotes differentiation with respect to a coordinate. It is apparent that defining the operator \( \hat{p} \) as acting in the Hilbert space \( H_\eta \equiv (L^2, \eta^2 d\tau) \) is equivalent to define in \( M \) a weighted volume element by \( \eta^2 d\tau \). We shall denote \( M \) when endowed with this new volume element by \( M_\eta \).
The divergence of the vector field $L$ in $M_\eta$ will be denoted by $\text{Div} L$. Now

$$\text{Div} L = (\eta^{-2} g^{-1/2})(\eta^2 g^{1/2} \xi_i)_i = \text{div} L + 2\eta^{-1} \xi_i \eta_i$$

$$= \text{div} L + 2\eta^{-1}(L\eta) = \text{div} L + 2(\frac{1}{2}\text{div} L) = 0.$$ 

Thus the vector field $L$ in $M_\eta$ is incompressible \[6\]. Now the operator $\hat{p} = -iL$ defined on the domain

$$D_0(\hat{p}) = \{ \psi \in C_0^1(M) : \psi, \hat{p}\psi \in L^2(M_\eta) \},$$

where $L^2(M_\eta) \equiv (L^2, \eta^2 d\tau)$, is symmetric. If $L$ is complete then the operator $\hat{p}$ defined on $D(\hat{p}) = \{ \psi \in AC(M) : \psi, \hat{p}\psi \in L^2(M_\eta) \}$ is self-adjoint. The one-parameter group $V_t$ of $L^2(M_\eta)$ generated by the self-adjoint operator $\hat{p}$ is given by $V_t = \exp(i\hat{p}t)$.

Expanding $V_t$ and $U_t$ and comparing the resulting expressions we find that $U_t = \eta V_t \eta^{-1}$, which shows that these two groups are isomorphic.

Related works on incompressible fields and on generalized quantum momentum observables are found in \[10\] and \[11\] respectively.

**Example 2.** Let our manifold $M$ be the semi-axis $(0, \infty)$, and consider the one-parameter group of transformation of $M$, $x \in M \rightarrow X = xe^t \in M$, where $t \in \mathbb{R}$.

The infinitesimal generator of this group is the complete vector field $L = x \frac{d}{dx}$. Since $\text{div} L = 1$, the operator $\hat{P} = i(x \frac{d}{dx} + \frac{1}{2})$ defined on the domain \[12\] is self-adjoint.

Noting that $\eta = x^{-1/2}$ is a solution of \[1\] we have $\hat{P} = i x^{1/2} \frac{d}{dx} x^{1/2}$. There corresponds to $\hat{P}$ an one-parameter group of transformation $U_t = \exp(x^{1/2} \frac{d}{dx} x^{1/2} t)$ of $L^2(0, \infty)$. 
The vector field in $M_\eta$ is incompressible, for $\text{Div}(x \frac{d}{dx}) = \eta^{-2}(\eta^2 x)_x = 0$. Therefore

the operator $\hat{p}$ defined in

$$\{ \psi \in AC(0, \infty) : \psi, x \frac{d\psi}{dx} \in L^2((0, \infty)_\eta) \}$$

is self-adjoint. Comparing the expansions of the one-parameter groups $U_t$ and $V_t = \exp(ixd/dx)$, we verify easily that $U_t = \eta V_t \eta^{-1}$.

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