ON A GENERALIZED FRAÎSSÉ LIMIT CONSTRUCTION

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ABSTRACT. In this paper, we present a slightly modified version of Fraïssé theory which is used in [Eag16] and [Mas16]. Using this version, we also show that every UHF algebra can be recognized as a Fraïssé limit of a class of C*-algebras of matrix-valued continuous functions on cubes with distinguished traces.

1. Introduction

Fraïssé theory was originally invented by Rolland Fraïssé in [Fra54]. The fundamental theorem of this theory claims that there is a bijective correspondence between the ultra-homogeneous structures and what we call Fraïssé classes. For an ultra-homogeneous structure, the corresponding Fraïssé class is its age, that is, the class of all finitely generated substructures; and the ultra-homogeneous structure is recovered as a generic inductive limit of members of the Fraïssé class, so that it is called the Fraïssé limit of the class.

This theory has been generalized to the setting of metric structures ([Sch07], [Ben15]). In [Ben15], a Fraïssé class of metric structures was defined as a class of finitely generated metric structures which satisfies the axioms called the hereditary property (HP), the joint embedding property (JEP), the near amalgamation property (NAP), the Polish property (PP), and the continuity property (CP). Then it was shown that there is a bijective correspondence between the approximately ultra-homogeneous structures and the Fraïssé classes as above.

The key idea of the proof of the fundamental theorem in [Ben15] was to use approximate isomorphisms. If \( \mathcal{X} \) is a Fraïssé class, then a structure \( M \) is called a \( \mathcal{X} \)-structure if its age is included in \( \mathcal{X} \); and an approximate \( \mathcal{X} \)-isomorphism from a \( \mathcal{X} \)-structure \( M \) into another \( \mathcal{X} \)-structure \( N \) is defined as a bi-Katětov map from \( |M| \times |N| \) to \([0, \infty]\) which approximately dominates a map of the form \((a, b) \mapsto d(\iota(a), \eta(b))\), where \( \iota \) and \( \eta \) are finite partial embeddings of \( M \) and \( N \) into some \( C \in \mathcal{X} \), respectively. Intuitively, an approximate \( \mathcal{X} \)-isomorphism from \( M \) into \( N \) should be thought of as a condition to be imposed on an embedding of a substructure of \( M \) into an extension of a substructure of \( N \). What is important is that we can consider compositions of approximate isomorphisms. Namely, if \( \varphi \) is an approximate isomorphism from \( M_1 \) into \( M_2 \) and \( \psi \) is an approximate isomorphism from \( M_2 \) into \( M_3 \), then one can consider a new approximate isomorphism \( \psi \varphi \) from \( M_1 \) into \( M_3 \). Thanks to this fact, we can prove that the limits of two generic inductive systems of members of a Fraïssé class \( \mathcal{X} \) are isomorphic to

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each other and are ultra-homogeneous, by carrying over a back-and-forth argument between them via approximate isomorphisms.

In [Eag16], a more relaxed version of Fraïssé theory was considered in order to recognize several well-known examples of operator algebras as Fraïssé limits. In their definition of Fraïssé classes, the axioms PP and CP were replaced by weaker conditions called the weak Polish property (WPP) and the Cauchy continuity property (CCP), and the axiom HP was omitted. Corresponding to this change, the definition of $\mathcal{H}$-structures for a Fraïssé class $\mathcal{K}$ was also modified: a structure $M$ is said to be a $\mathcal{K}$-structure if it is an inductive limit of members of $\mathcal{K}$.

Then it was claimed that every Fraïssé class has its limit, that is, for every Fraïssé class $\mathcal{K}$, there exists a unique $\mathcal{K}$-structure $M$ which is approximately $\mathcal{K}$-ultra-homogeneous in the sense that if $\iota_1, \iota_2$ are two embeddings of a member $A$ of $\mathcal{K}$, then for any finite subset $F \subseteq |A|$ and any $\varepsilon > 0$ there exists an automorphism $\alpha$ of $M$ with $d(\alpha \circ \iota_1(a), \iota_2(a)) < \varepsilon$ for all $a \in F$. The proof of this claim was not presented in [Eag16], because it was thought that the proof of [Ben15] would still work almost verbatim in this setting.

In order to adopt the proof of [Ben15], we first have to guess what is the appropriate definition of approximate isomorphisms in this relaxed setting. One of the candidates would be the same definition as the original one in [Ben15], but this does not seem to work, because within this setting one can no longer prove in the same way as [Ben15] that compositions of approximate isomorphisms between $\mathcal{K}$-structures are approximate isomorphisms (see Remark 4.8). This is a fatal gap, because this property is used to establish the existence and uniqueness of a limit of a Fraïssé class.

In this paper, we reconstruct the theory presented in [Eag16] and reveal the correct form of the fundamental theorem. Because of the gap explained in the previous paragraph, it turns out that the concept of $\mathcal{K}$-structures should have been more complicated, and the homogeneity property the generic limit satisfies is weaker than the original claim.

Simultaneously, we slightly generalize the theory so that we can deal with categories. The motivation of this generalization is the following. In [Eag16], the Jiang–Su algebra $\mathcal{Z}$ was recognized as a Fraïssé limit of the class of prime dimension drop algebras with distinguished faithful traces. An alternative proof of this fact was given in [Mas16], which was based on the fact that every unital embedding between dimension drop algebras is approximately diagonalizable. These results led to the prospect of giving a short proof of the fact that the Jiang–Su algebra is tensorially self-absorbing (i.e. $\mathcal{Z} \otimes \mathcal{Z} \approx \mathcal{Z}$), and the first step of such a short proof was expected to be showing that the class of tensor products of dimension drop algebras with distinguished faithful traces is also a Fraïssé class. For this, if we adopt the same strategy as [Mas16], then we should first show the statement that every unital embedding between tensor products of dimension drop algebras is approximately diagonalizable, which turns out to be false (cf. Remark 5.1). A natural solution to this problem would be simply restricting embeddings to approximately
diagonalizable ones, so that the object under consideration would not be a class but a category.

We should note that there is another research on Fraïssé theory for categories by Wiesław Kubis [Kub13]. His theory is developed within the theory of categories enriched over metric spaces, while our approach is based on the work by Itaï Ben Yaacov [Ben15].

This paper is organized as follows. In the second section, we recall the definition and properties of approximate isometries. The third section is devoted to metric structures and approximate isomorphisms. The existence and uniqueness of the Fraïssé limit is shown in the fourth section. The fifth section contains an application of this theory to UHF algebras.

2. Approximate isometries

In this section, we recall the definition and properties of approximate isometries. Our handling of them is based on [Ben15]. Proofs are reproduced for the convenience of the reader.

Let $X$ and $Y$ be metric spaces. We denote by $\text{JE}(X, Y)$ the set of all pairs $(\iota, \eta)$, where $\iota: X \to Z$ and $\eta: Y \to Z$ are isometries into some metric space $Z$. Each element of $\text{JE}(X, Y)$ is called a joint embedding of $X$ and $Y$.

**Definition 2.1.**

1. Let $X$ be a metric space. A map $\varphi: X \to [0, \infty]$ is said to be Katětov if it satisfies the inequalities
   $$\varphi(x) \leq d_X(x, x') + \varphi(x'), \quad d_X(x, x') \leq \varphi(x) + \varphi(x')$$
   for all $x, x' \in X$.

2. Suppose that $X$ and $Y$ are metric spaces. An approximate isometry from $X$ to $Y$ is a map $\varphi: X \times Y \to [0, \infty]$ which is separately Katětov. The class of all approximate isometries from $X$ to $Y$ is denoted by $\text{Apx}(X, Y)$. Note that, being a closed subset of $[0, \infty]^{X \times Y}$, the space $\text{Apx}(X, Y)$ is compact and Hausdorff with respect to the topology of pointwise convergence.

Intuitively, an approximate isometry is a condition to be imposed on joint embeddings. A joint embedding $(\iota, \eta) \in \text{JE}(X, Y)$ is said to satisfy an approximate isometry $\varphi$ from $X$ to $Y$ if the inequality
$$d(\iota(x), \eta(y)) \leq \varphi(x, y)$$
holds for all $x \in X$ and $y \in Y$. We shall denote by $\text{JE}^{\preceq\varphi}(X, Y)$ the class of all joint embeddings satisfying $\varphi$. Clearly, the condition $\varphi \equiv \infty$ is the weakest condition. Note that if an approximate isometry $\varphi$ from $X$ to $Y$ takes a finite value at some point, then it is real-valued, because if $\varphi(x_0, y_0) < \infty$, then
$$\varphi(x, y) \leq d_X(x, x_0) + \varphi(x_0, y_0) + d(y_0, y) < \infty.$$

**Example 2.2.**

1. For a joint embedding $(\iota, \eta)$ of $X$ and $Y$, the map
   $$(x, y) \mapsto d(\iota(x), \eta(y))$$
itself is an approximate isometry. We shall denote this approximate isometry by \( \varphi_{i, \eta} \). If \( \eta \) is equal to \( \text{id}_Y \), then \( \varphi_{i, \eta} \) is simply written as \( \varphi_i \).

We shall show that every approximate isometry is of this form unless it is equal to \( \text{id} \). To see this, let \( \varphi : X \times Y \to [0, \infty) \) be an approximate isometry and define a symmetric function \( \delta : (X \sqcup Y)^2 \to [0, \infty) \) by

\[
\delta(z, z') = \begin{cases} 
  d_X(z, z') & \text{if } z, z' \in X, \\
  \varphi(z, z') & \text{if } z \in X \text{ and } z' \in Y, \\
  d_Y(z, z') & \text{if } z, z' \in Y.
\end{cases}
\]

Then it is easy to see that \( \delta \) is a pseudo-metric. If \( \iota \) and \( \eta \) are canonical embeddings of \( X \) and \( Y \) into the quotient metric space \( X \sqcup \eta Y \), then \( d(\iota(x), \eta(y)) = \varphi(x, y) \), as desired. It follows that, for any approximate isometries \( \varphi \) and \( \psi \) from \( X \) to \( Y \), the inequality \( \varphi \leq \psi \) holds if and only if \( \text{JE}^{Z_\varphi}(X, Y) \) is included in \( \text{JE}^{Z_\psi}(X, Y) \), so the order \( \leq \) completely reflects the strength of conditions.

Note that a net \( \{\iota_\alpha\} \) of isometries from \( X \) into \( Y \) converges pointwise to an isometry \( \iota \) if and only if \( \{\varphi_{\iota_\alpha}\} \) converges to \( \varphi_\iota \). Indeed, if \( \{\iota_\alpha\} \) converges to \( \iota \), then

\[
\varphi_{\iota_\alpha}(x, y) = d(\iota_\alpha(x), y) \to d(\iota(x), y) = \varphi_\iota(x, y)
\]

for all \( x \in X \) and \( y \in Y \). Conversely, if \( \{\varphi_{\iota_\alpha}\} \) converges to \( \varphi_\iota \), then for any \( x \in X \) we have

\[
d(\iota_\alpha(x), \iota(x)) = \varphi_{\iota_\alpha}(x, \iota(x)) \to \varphi_\iota(x, \iota(x)) = d(\iota(x), \iota(x)) = 0.
\]

(2) If \( \varphi \) is an approximate isometry from \( X \) to \( Y \), then

\[
\varphi^*(y, x) := \varphi(x, y)
\]

defines an approximate isometry \( \varphi^* \) from \( Y \) to \( X \).

(3) Given \( \varphi \in \text{Apx}(X, Y) \) and \( \psi \in \text{Apx}(Y, Z) \), we define their composition by

\[
\psi \varphi(x, z) := \inf_{y \in Y} (\varphi(x, y) + \psi(y, z)).
\]

Here, we shall check that \( \psi \varphi \) is an approximate isometry from \( X \) to \( Z \). Indeed, if \( x \) and \( x' \) are points of \( X \), then

\[
\psi \varphi(x, z) = \inf_{y \in Y} (\varphi(x, y) + \psi(y, z)) \\
\leq \inf_{y \in Y} (d(x, x') + \varphi(x', y) + \psi(y, z)) \\
= d(x, x') + \psi \varphi(x' z)
\]

and

\[
d(x, x') \leq \inf_{x, y' \in X} (\varphi(x, y) + d(y, y') + \varphi(x', y')) \\
\leq \inf_{x, y' \in X} (\varphi(x, y) + \psi(y, z) + \varphi(x', y') + \psi(y', z)) \\
= \psi \varphi(x, z) + \psi \varphi(x' z),
\]

so \( \psi \varphi(x, \cdot, z) \) is Katětov for all \( z \in Z \). By symmetry \( \psi \varphi(x, \cdot) \) is also Katětov for all \( x \in X \), so \( \psi \varphi \) is an approximate isometry.

It is worth noting that if \( (t_1, t_2) \in \text{JE}^{Z_\varphi}(X, Y) \) and \( (t_2, t_3) \in \text{JE}^{Z_\psi}(Y, Z) \), then \( (t_1, t_3) \in \text{JE}^{Z_{\psi \varphi}}(X, Z) \), and \( \psi \varphi \) is the smallest approximate isometry satisfying this
property. Also, it can be easily seen that the equality $\varphi_{t,\eta} = \varphi^*_{\eta} \varphi_t$ holds for any joint embedding $(t, \eta)$.

(4) Let $X' \subseteq X$ and $Y' \subseteq Y$ be subspaces. If $\varphi$ is an approximate isometry from $X$ to $Y$, then its restriction $\varphi|_{X' \times Y'}$ is an approximate isometry from $X'$ to $Y'$. Note that, if $t: X' \to X$ and $\eta: Y' \to Y$ are the canonical embeddings, then $\varphi|_{X' \times Y'}$ is equal to $\varphi^*_{\eta} \varphi_t$. Now suppose that $\psi$ is an approximate isometry from $X'$ to $Y'$. The trivial extension of $\psi$ to $X \times Y$ is defined by $\psi|_{X \times Y} := \varphi^*_{\eta} \psi \varphi_t$. It is easy to show that $\psi|_{X \times Y}$ is the largest approximate isometry such that the restriction to $X' \times Y'$ is equal to $\psi$. More generally, an approximate isometry $\theta$ from $X$ to $Y$ satisfies $\theta \leq \psi|_{X \times Y}$ if and only if $\theta|_{X' \times Y'} \leq \psi$.

(5) If $\varphi$ is an approximate isometry from $X$ to $Y$ and $\varepsilon$ is a non-negative real number, then the relaxation of $\varphi$ by $\varepsilon$ is defined by $(x, y) \mapsto \varphi(x, y) + \varepsilon$. We simply denote this approximate isometry by $\varphi + \varepsilon$. Note that the operation of taking relaxations commutes with compositions.

**Definition 2.3.** An approximate isometry $\varphi$ from $X$ to $Y$ is said to be

- $\varepsilon$-total if $\varphi^* \varphi \leq \varphi_{\text{dists}} + 2\varepsilon$.
- $\varepsilon$-surjective if $\varphi^*$ is $\varepsilon$-total.
- $\varepsilon$-bijective if $\varphi$ is $\varepsilon$-total and $\varepsilon$-surjective.

If $\varphi$ and $\psi$ are approximate isometries from $X$ to $Y$ with $\psi \leq \varphi$, then clearly $\psi^* \psi \leq \varphi^* \varphi$. Therefore, if $\varphi$ is $\varepsilon$-total, then so is $\psi$. Similarly, if $\varphi$ is $\varepsilon$-surjective, then so is $\psi$.

**Proposition 2.4.** An approximate isometry $\varphi$ from $X$ to $Y$ is $\varepsilon$-total if and only if any $(t, \eta) \in \ JE_{\varepsilon}(X, Y)$ satisfies $d(t(x), \eta[Y]) \leq \varepsilon$ for each $x \in X$. In particular, if $Y$ is complete and $\varphi$ is $\varepsilon$-total for any $\varepsilon$, then it is of the form $\varphi_t$ for a unique isometry $t: X \to Y$.

**Proof.** Suppose that $\varphi$ is $\varepsilon$-total and let $(t, \eta)$ be in $\ JE_{\varepsilon}(X, Y)$. Then, for any $x \in X$, we have

$$
2 \inf_{y \in Y} d(t(x), \eta(y)) \leq \inf_{y \in Y} (\varphi(x, y) + \varphi^*(y, x)) = \varphi^* \varphi(x, x) \\
\leq \varphi_{\text{dists}}(x, x) + 2\varepsilon = 2\varepsilon,
$$
so $d(t(x), \eta[Y]) \leq \varepsilon$.

Conversely, suppose that $d(t(x), \eta[Y]) \leq \varepsilon$ holds for any $(t, \eta) \in \ JE_{\varepsilon}(X, Y)$ and any $x \in X$. Then $\varphi \neq \infty$, so it is of the form $\varphi_{t,\eta}$, and

$$
\varphi^* \varphi(x, x') = \inf_{y \in Y} (d(t(x), \eta(y)) + d(\eta(y), t(x'))) \\
\leq d(x, x') + 2 \inf_{y \in Y} d(t(x), \eta(y)) \\
\leq \varphi_{\text{dists}}(x, x') + 2\varepsilon.
$$

Let $\varphi$ be an approximate isometry from $X$ to $Y$. We set

$$\text{Apx}^{\varepsilon}_{\varphi}(X, Y) := \{\psi \in \text{Apx}(X, Y) \mid \psi \leq \varphi\}.$$
We also denote by \( \text{Apx}^{\leq}(X, Y) \) the interior of the closed subset \( \text{Apx}^{<}(X, Y) \) of the compact Hausdorff space \( \text{Apx}(X, Y) \), and write \( \psi \prec \varphi \) or \( \varphi \succ \psi \) if \( \psi \) belongs to \( \text{Apx}^{<}(X, Y) \). If \( \text{Apx}^{<}(X, Y) \) is nonempty, then \( \varphi \) is said to be strict. The class of all strict approximate isometries is denoted by \( \text{Stx}(X, Y) \).

It can be easily verified that the relation \( \prec \) is preserved under restrictions and trivial extensions. In particular, restrictions and trivial extensions of a strict approximate isometries are strict.

**Proposition 2.5.** For \( \varphi, \psi \in \text{Apx}(X, Y) \), the following are equivalent.

(i) The relation \( \psi \prec \varphi \) holds.

(ii) There exist finite subsets \( X_0 \subseteq X \) and \( Y_0 \subseteq Y \) and a positive real number \( \varepsilon \) such that the inequality \( \varphi \geq (\psi|_{X_0 \times Y_0})^{X \times Y} + \varepsilon \) holds.

(iii) Same as (ii), with \( \varepsilon \) replaced by \( \succ \).

Moreover, if these conditions are satisfied, then there exist then there exist finite subsets \( X_0 \subseteq X \) and \( Y_0 \subseteq Y \) and a rational-valued approximate isometry \( \rho \in \text{Apx}(X_0, Y_0) \) such that the relation \( \psi \prec \rho^{X \times Y} \prec \varphi \) holds.

**Proof.** First, suppose (i) holds. Then there exist finite subsets \( X_0 \subseteq X \) and \( Y_0 \subseteq Y \) and a positive real number \( \varepsilon \) such that the open neighborhood

\[
U := \{ \psi' \in \text{Apx}(X, Y) \mid |\psi'(x, y) - \psi(x, y)| < 2\varepsilon \text{ for any } x \in X_0, \ y \in Y_0 \}
\]

is included in \( \text{Apx}^{<}(X, Y) \). Clearly \( (\psi|_{X_0 \times Y_0})^{X \times Y} + \varepsilon \) belongs to \( U \), so (iii) follows.

It is trivial that (iii) implies (ii). Now assume (ii). Since \( \prec \) is preserved under trivial extensions, the relation \( \psi|_{X_0 \times Y_0} \prec \varphi|_{X_0 \times Y_0} + \varepsilon \) implies

\[
\psi \leq (\psi|_{X_0 \times Y_0})^{X \times Y} \prec (\varphi|_{X_0 \times Y_0})^{X \times Y} + \varepsilon \leq \varphi,
\]

so (i) holds.

Finally, in order to find \( \rho \) as in the statement, suppose \( \psi \prec \varphi \). Let \( X_0, Y_0 \) be as in the proof of (i) \( \Rightarrow \) (iii) above, and \( F_1, \ldots, F_n \) be the partition of \( X_0 \times Y_0 \) induced by \( \psi \). Without loss of generality, we may assume \( \psi|_{F_1} < \cdots < \psi|_{F_n} \). Take a function \( \delta: X_0 \times Y_0 \to (0, \varepsilon) \) so that

- \( \delta \) is constant on each \( F_i \);
- \( \delta|_{F_n} < \psi|_{F_n} - \psi|_{F_{n-1}} \);
- \( \delta|_{F_i} < \min\{\delta|_{F_{i+1}}, \psi|_{F_i} - \psi|_{F_{i-1}}\} \) for \( i = 2, \ldots, n-1 \);
- \( \delta|_{F_i} < \min\{\delta|_{F_2}, \psi|_{F_1}\} \) and
- \( \rho := \psi|_{X_0 \times Y_0} - \delta + \varepsilon \) is a rational valued function on \( X_0 \times Y_0 \).

We shall check that \( \rho \) is separately Katětov so that it is an approximate isometry. The inequality

\[
d(x, x') \leq \rho(x, y) + \rho(x', y)
\]
is obvious, because \( \rho \geq \psi|_{X_0 \times Y_0} \). On the other hand, for \( (x, y) \in F_i \) and \( (x', y) \in F_j \) with \( i < j \), we have
\[
\rho(x, y) = \psi|_{F_i} + \varepsilon - \delta|_{F_i} = \psi|_{F_j} + \varepsilon - \delta|_{F_j} - [(\psi|_{F_j} - \psi|_{F_i}) - \delta|_{F_j}] \\
\leq \psi|_{F_j} + \varepsilon - \delta|_{F_j} = \rho(x', y) \\
\leq d(x, x') + \rho(x', y)
\]
and
\[
\rho(x', y) = \psi(x', y) + \varepsilon - \delta|_{F_j} \\
\leq d(x', x) + \psi(x, y) + \varepsilon - \delta|_{F_j} \\
= d(x', x) + \rho(x, y),
\]
so \( \rho(\cdot, y) \) is Katětov for each \( y \in Y_0 \). By symmetry, \( \rho(x, \cdot) \) is also Katětov, whence \( \rho \) is an approximate isometry. Since clearly
\[
\psi|_{X_0 \times Y_0} \preceq \rho \preceq \psi|_{X_0 \times Y_0},
\]
the conclusion follows. \( \Box \)

**Lemma 2.6.** Let \( X, Y \) be metric spaces and \( X_0, X'_0 \subseteq X \) and \( Y_0, Y'_0 \subseteq Y \) be finite subsets. If \( X'_0 \) and \( Y'_0 \) are included in the \( \varepsilon/5 \)-neighborhoods of \( X_0 \) and \( Y_0 \) respectively, then for any \( \varphi \in \text{Ap}_X(X,Y) \), the inequality
\[
(\varphi|_{X_0 \times Y_0})^{X \times Y} + \varepsilon/5 \leq (\varphi|_{X'_0 \times Y'_0})^{X \times Y} + \varepsilon
\]
holds.

**Proof.** For \( x' \in X'_0 \) and \( y' \in Y'_0 \), there exist \( x \in X_0 \) and \( y \in Y_0 \) with \( d(x', x) < \varepsilon/5 \) and \( d(y', y) < \varepsilon/5 \), so we have
\[
(\varphi|_{X_0 \times Y_0})^{X \times Y}(x', y') + \varepsilon/5 \leq d(x', x) + (\varphi|_{X_0 \times Y_0})^{X \times Y}(x, y) + d(y, y') + \varepsilon/5 \\
= d(x', x) + \varphi(x, y) + d(y, y') + \varepsilon/5 \\
\leq 2d(x', x) + \varphi(x', y') + 2d(y, y') + \varepsilon/5 \\
\leq \varphi(x', y') + \varepsilon.
\]
\( \Box \)

3. **Metric structures and approximate isomorphisms**

In this section, we recall the definition of metric structures and investigate fundamental properties of approximate isomorphisms. All the discussions are parallel to [Ben15].

By definition, a **language** is a set \( L \) such that each element of \( L \) is either a **function symbol** or a **relation symbol**. To each symbol \( S \) is associated a natural number \( n_S \), which is called the **arity of** \( S \), and a symbol with arity \( n \) is called an **n-ary symbol**. A 0-ary function symbol is often called a **constant symbol**.

An **\( L \)-structure** \( M \) is a complete metric structure \( M \), which is called the **domain** of \( M \), together with an **interpretation** of symbols of \( L \):
• to each \( n \)-ary relation symbol \( R \) is assigned a continuous map \( R^M \) from \( M^n \) to \( \mathbb{R} \); and

• to each \( n \)-ary function symbol \( f \) is assigned a continuous map \( f^M \) from \( M^n \) to \( M \).

For an \( L \)-structure \( M \), we shall denote its domain by \( |M| \).

An \( L \)-embedding of an \( L \)-structure \( N \) into another \( L \)-structure \( M \) is an isometry \( \iota \) from \( |N| \) into \( |M| \) such that

• for any \( n \)-ary relation symbol \( R \) and any elements \( a_1, \ldots, a_n \in |N| \), the equation
  \[ R^N(a_1, \ldots, a_n) = R^M(\iota(a_1), \ldots, \iota(a_n)) \]
  holds, and

• for any \( n \)-ary function symbol \( f \) and any elements \( a_1, \ldots, a_n \in |N| \), the equation
  \[ \iota(f^N(a_1, \ldots, a_n)) = f^M(\iota(a_1), \ldots, \iota(a_n)) \]
  holds.

For an \( L \)-embedding \( \iota: N \to M \) and a tuple \( \bar{a} = (a_1, \ldots, a_n) \in |N|^n \), we shall write the tuple \( (\iota(a_1), \ldots, \iota(a_n)) \in |M|^n \) as \( \iota(\bar{a}) \).

For a subset \( E \) of an \( L \)-structure \( M \), the \( L \)-substructure generated by \( E \) is denoted by \( \langle E \rangle \). If it coincides with \( M \), then \( E \) is said to be a generator of \( M \). If \( M \) is generated by a finite subset, then \( M \) is said to be finitely generated. A tuple \( \bar{a} = (a_1, \ldots, a_n) \in |M|^n \) is an ordered generator if \( \{a_i \mid i = 1, \ldots, n\} \) is a generator of \( M \).

In the sequel, we fix a language \( L \) and a category \( \mathcal{K} \) of finitely generated \( L \)-structures and \( L \)-embeddings. Embeddings and isomorphisms in \( \text{Mor}(\mathcal{K}) \) are often referred to as \( \mathcal{K} \)-embeddings and \( \mathcal{K} \)-isomorphisms respectively. A joint \( \mathcal{K} \)-embedding \( \langle \iota, \eta \rangle \) such that both \( \iota \) and \( \eta \) are \( \mathcal{K} \)-embeddings. We denote by \( \mathcal{J}E_{\mathcal{K}}(\mathcal{A}, \mathcal{B}) \) the class of all joint \( \mathcal{K} \)-embeddings of \( \mathcal{A} \) and \( \mathcal{B} \).

**Definition 3.1.** (1) Let \( \mathcal{A}, \mathcal{B} \) be objects of \( \mathcal{K} \) and \( \iota: \mathcal{A} \to \mathcal{B} \) be a finite partial isometry, that is, an isometry between finite subsets of \( |\mathcal{A}| \) and \( |\mathcal{B}| \). Then \( \iota \) is called a finite partial \( \mathcal{K} \)-isomorphism if

• the \( L \)-substructures \( \langle \text{dom} \iota \rangle \) and \( \langle \text{ran} \iota \rangle \) are objects of \( \mathcal{K} \);

• the canonical embeddings \( \langle \text{dom} \iota \rangle \to \mathcal{A} \) and \( \langle \text{ran} \iota \rangle \to \mathcal{B} \) are \( \mathcal{K} \)-embeddings; and

• \( \iota \) extends to a \( \mathcal{K} \)-isomorphism from \( \langle \text{dom} \iota \rangle \) onto \( \langle \text{ran} \iota \rangle \).

(2) Let \( \mathcal{A}, \mathcal{B} \) be objects of \( \mathcal{K} \). We denote by \( \text{Ap}_{2, \mathcal{K}}(\mathcal{A}, \mathcal{B}) \) the set of all approximate isometries from \( |\mathcal{A}| \) to \( |\mathcal{B}| \) which are of the form \( \varphi_{\iota, \eta}^{\mathcal{A} \times \mathcal{B}} \), where \( \iota: \mathcal{A} \to \mathcal{C} \) and \( \eta: \mathcal{B} \to \mathcal{C} \) are finite partial \( \mathcal{K} \)-isomorphisms into some object \( \mathcal{C} \) of \( \mathcal{K} \).

(3) A \( \mathcal{K} \)-structure is an \( L \)-structure \( M \) together with an inductive system of \( \mathcal{K} \)-embeddings

\[
\mathcal{A}_1 \xrightarrow{\iota_1} \mathcal{A}_2 \xrightarrow{\iota_2} \mathcal{A}_3 \xrightarrow{\iota_3} \cdots
\]

such that the inductive limit of the system as an \( L \)-structure is \( M \). We often write \( M = \bigcup_n \mathcal{A}_n \) identifying each \( \mathcal{A}_n \) as the corresponding \( L \)-substructure of \( M \). Note that \( M \) is not necessarily an object of \( \mathcal{K} \).
(4) For $\mathcal{K}$-structures $M = \bigcup_n A_n$ and $N = \bigcup_m B_m$, we define

$$\text{Apx}_{\mathcal{K}}(M, N) := \text{cl}\left( \bigcup_{n,m} \{ \psi : \exists \varphi \in \text{Apx}_{2,\mathcal{K}}(A_n, B_m), \psi \geq \varphi|_{M \times N} \} \right)$$

and call its elements approximate $\mathcal{K}$-isomorphisms. Also, we set

$$\text{Apx}_{\mathcal{K}}^e(M, N) := \text{Apx}_{\mathcal{K}}(M, N) \cap \text{Apx}_{\mathcal{K}}^e(|M|, |N|),$$

$$\text{Apx}_{\mathcal{K}}^{\ast}(M, N) := \text{Apx}_{\mathcal{K}}(M, N) \cap \text{Apx}_{\mathcal{K}}^{\ast}(|M|, |N|).$$

If $\text{Apx}_{\mathcal{K}}^e(M, N)$ is nonempty, then $\varphi$ is said to be strict. We denote the set of strict approximate $\mathcal{K}$-isomorphisms from $M$ to $N$ by $\text{Stx}_{\mathcal{K}}(M, N)$.

(5) An $L$-embedding $\iota$ of a $\mathcal{K}$-structure $M = \bigcup_n A_n$ into another $\mathcal{K}$-structure $N = \bigcup_m B_m$ is said to be $\mathcal{K}$-admissible if the corresponding approximate isometry $\varphi$, belongs to $\text{Apx}_{\mathcal{K}}(M, N)$. Two $\mathcal{K}$-structures are understood to be isomorphic if there exists a $\mathcal{K}$-admissible isomorphism between them.

An object $A$ of $\mathcal{K}$ can be canonically identified with a $\mathcal{K}$-structure obtained from the inductive system

$$A \xrightarrow{id} A \xrightarrow{id} \cdots,$$

so that we can consider $\text{Apx}_{\mathcal{K}}(A, B)$ for objects $A, B$ of $\mathcal{K}$. If $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are objects of $\mathcal{K}$ and $\iota: \mathcal{A} \rightarrow \mathcal{C}$ and $\eta: \mathcal{B} \rightarrow \mathcal{C}$ are $\mathcal{K}$-embeddings, then $\varphi, \eta$ belongs to $\text{Apx}_{\mathcal{K}}(\mathcal{A}, \mathcal{B})$, because it is the limit of

$$\{(\varphi, \eta)|_{A_0 \times B_0}\}^{\mathcal{A} \times \mathcal{B}}[A_0 \subseteq |A|, B_0 \subseteq |B| \text{ are finite generators}] \subseteq \text{Apx}_{\mathcal{K}}(\mathcal{A}, \mathcal{B}).$$

In particular, every $\mathcal{K}$-embedding is $\mathcal{K}$-admissible. On the other hand, note that there might be a $\mathcal{K}$-admissible isomorphism between objects of $\mathcal{K}$ which is not a morphism of $\mathcal{K}$. There can be even a $\mathcal{K}$-admissible $\iota: \mathcal{A} \rightarrow \mathcal{B}$ such that no net of $\mathcal{K}$-embeddings of $\mathcal{A}$ into $\mathcal{B}$ converges to $\iota$.

For any approximate $\mathcal{K}$-isomorphism $\varphi$ from $M$ to $N$, the set $\text{Apx}_{\mathcal{K}}^e(M, N)$ is obviously included in the relative interior of $\text{Apx}_{\mathcal{K}}^{\ast}(M, N)$ in $\text{Apx}_{\mathcal{K}}(M, N)$. The opposite inclusion also holds, because any relative interior point $\psi$ in $\text{Apx}_{\mathcal{K}}^{\ast}(M, N)$ satisfies (ii) in Proposition 2.5.

Given a subset $A$ of $\text{Apx}(X, Y)$, we shall define

$$A^{\uparrow} := \{ \psi \in \text{Apx}(X, Y) | \exists \varphi \in A, \psi \geq \varphi \}.$$
Proposition 3.3. The category $\mathcal{K}$ is said to satisfy

- the joint embedding property (JEP) if $JE_{\mathcal{K}}(\mathcal{A}, \mathcal{B})$ is nonempty for any objects $\mathcal{A}, \mathcal{B}$ of $\mathcal{K}$.
- the near amalgamation property (NAP) if for any objects $\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2$ in $\mathcal{K}$, any $\mathcal{K}$-embeddings $\iota_i : \mathcal{A} \to \mathcal{B}_i$, any finite subset $F \subseteq |\mathcal{A}|$ and any $\varepsilon > 0$, there exists a joint $\mathcal{K}$-embedding $(\eta_1, \eta_2)$ of $\mathcal{B}_1$ and $\mathcal{B}_2$ such that the inequality

$$d(\eta_1 \circ \iota_1(a), \eta_2 \circ \iota_2(a)) < \varepsilon$$

holds for all $a \in F$.

The following propositions are essential in proving the existence and uniqueness of Fraïssé limits in the next section. In fact, the gap of the theory presented in [Eag16] is also related to these propositions, as is explained in Remark 4.8.

Proposition 3.3. Suppose that $\mathcal{K}$ satisfies NAP. Then for any objects $\mathcal{A}, \mathcal{B}$ of $\mathcal{K}$ and any strict approximate $\mathcal{K}$-isomorphism $\varphi$ from $\mathcal{A}$ to $\mathcal{B}$, there exists a joint $\mathcal{K}$-embedding $(\iota, \eta)$ of $\mathcal{A}$ and $\mathcal{B}$ satisfying $\varphi_{\iota, \eta} < \varphi$.

Proof. Since $\varphi$ is strict, $\text{Apx}_{\mathcal{K}}(\mathcal{A}, \mathcal{B})$ is an open nonempty subset of $\text{Apx}_{\mathcal{K}}(\mathcal{A}, \mathcal{B})$. Therefore, it intersects with $\text{Apx}_{\mathcal{K}}(\mathcal{A}, \mathcal{B})^\dagger$ is a dense subset in other words, there exist an object $\mathcal{C}_0$ of $\mathcal{K}$ and finite partial $\mathcal{K}$-isomorphisms $\iota_0 : \mathcal{A} \to \mathcal{C}_0$ and $\eta_0 : \mathcal{B} \to \mathcal{C}_0$ such that the relation $\varphi_{\iota_0, \eta_0}^{\mathcal{A} \times \mathcal{B}} + \varepsilon < \varphi$ holds for some $\varepsilon > 0$. Put $\mathcal{A}_0 := \langle \text{dom} \iota_0 \rangle$ and $\mathcal{B}_0 := \langle \text{dom} \eta_0 \rangle$. By the definition of finite partial $\mathcal{K}$-isomorphisms, the canonical embeddings $\mathcal{A}_0 \to \mathcal{A}$ and $\mathcal{B}_0 \to \mathcal{B}$ are $\mathcal{K}$-embeddings.

Now, by NAP there exist $\mathcal{K}$-embeddings $\iota_1 : \mathcal{A} \to \mathcal{C}_1$ and $\iota'_1 : \mathcal{C}_0 \to \mathcal{C}_1$ such that the inequality $d(\iota_1(a), \iota'_1 \circ \iota_0(a)) < \varepsilon/3$ holds for all $a \in \text{dom} \iota_0$. Similarly, there are $\mathcal{K}$-embeddings $\eta_1 : \mathcal{B} \to \mathcal{C}_2$ and $\eta'_1 : \mathcal{C}_0 \to \mathcal{C}_2$ with $d(\eta_1(b), \eta'_1 \circ \eta_0(b)) < \varepsilon/3$ for all $b \in \text{dom} \eta_0$. Then, again by NAP, there exist $\mathcal{K}$-embeddings $\iota_2 : \mathcal{C}_1 \to \mathcal{C}$ and $\eta_2 : \mathcal{C}_2 \to \mathcal{C}$ with $d(\iota_2 \circ \iota'_1(c), \eta_2 \circ \eta'_1(c)) < \varepsilon/3$ for any $c \in \text{ran} \iota_0 \cup \text{ran} \iota_0$.

\[
\begin{array}{c}
\mathcal{A}_0 \xrightarrow{\iota_0} \mathcal{A} \\
\downarrow \iota_1 \downarrow \iota_1 \downarrow \iota_1 \\
\mathcal{B}_0 \xrightarrow{\eta_0} \mathcal{C}_0 \xrightarrow{\iota'_1} \mathcal{C}_1 \\
\downarrow \eta_1 \downarrow \eta_1 \downarrow \eta_1 \\
\mathcal{B} \xrightarrow{\eta_1} \mathcal{C}_0 \xrightarrow{\eta'_1} \mathcal{C}_2 \xrightarrow{\iota_2} \mathcal{C} \\
\end{array}
\]

Set $\iota := \iota_2 \circ \iota_1$ and $\eta := \eta_2 \circ \eta_1$. Then for $a \in \text{dom} \iota_0$ and $b \in \text{dom} \eta_0$, we have

\[
d(\iota(a), \eta(b)) = d(\iota_2 \circ \iota_1(a), \eta_2 \circ \eta_1(b)) \\
\leq d(\iota_2 \circ \iota'_1 \circ \iota_0(a), \eta_2 \circ \eta'_1 \circ \eta_0(b)) + 2\varepsilon/3 \\
\leq d(\iota_2 \circ \iota'_1 \circ \iota_0(a), \iota_2 \circ \iota'_1 \circ \eta_0(b)) + \varepsilon = d(\iota_0(a), \eta_0(b)) + \varepsilon,
\]

so $\varphi_{\iota_0, \eta_0}^{\mathcal{A} \times \mathcal{B}} + \varepsilon < \varphi$, as desired. \qed
Proposition 3.4. Let $M_1 = \bigcup_i A_i$, $M_2 = \bigcup_m B_m$ and $M_3 = \bigcup_n C_n$ be $\mathcal{K}$-structures. If $\varphi$ and $\psi$ belongs to $\text{Apx}_\mathcal{K}(M_1, M_2)$ and $\text{Apx}_\mathcal{K}(M_2, M_3)$ respectively, then the composition $\psi \varphi$ is in $\text{Apx}_\mathcal{K}(M_1, M_3)$.

Proof. First, assume that both $\varphi$ and $\psi$ are strict and $M_1$, $M_2$, $M_3$ are objects of $\mathcal{K}$. Then, by Proposition 3.3, there exist objects $D$ and $E$ of $\mathcal{K}$ and $\mathcal{K}$-embeddings $\iota_i : M_i \to D$ ($i = 1, 2$) and $\eta_j : M_j \to E$ ($j = 2, 3$) such that $\varphi_{1,2} < \varphi$ and $\varphi_{2,3} < \psi$. It follows from Proposition 2.5 that there exist a finite subset $F_0 \subseteq |M_2|$ and a positive real number $\varepsilon > 0$ with

\[
(\varphi_{1,2}|_{M_1 \times F_0})|_{M_1 \times M_2} + \varepsilon < \varphi, \quad (\varphi_{2,3}|_{F_0 \times M_3})|_{M_2 \times M_1} + \varepsilon < \psi.
\]

By NAP, we can find $\mathcal{K}$-embeddings $\theta_1 : D \to F$ and $\theta_2 : E \to F$ such that the inequality $d(\theta_1 \circ \iota_2(b), \theta_2 \circ \eta_2(b)) < 2\varepsilon$ holds for all $b \in F_0$.

\[
\begin{array}{cccc}
M_1 & \xrightarrow{\iota_1} & D & \xrightarrow{\iota_2} & E & \xrightarrow{\eta_2} & M_2 \\
\downarrow{\theta_1} & & \downarrow{\theta_1} & & \downarrow{\theta_2} & & \downarrow{\eta_1} & \downarrow{\eta_1} & \downarrow{\eta_1} & \downarrow{\eta_1} \\
M_3 & & & & & & & & & & M_3
\end{array}
\]

For $a \in |M_1|$ and $c \in |M_3|$, we have

\[
d(\theta_1 \circ \iota_1(a), \theta_2 \circ \eta_2(c)) \\
\leq \inf_{b \in F_0} \left[ d(\theta_1 \circ \iota_1(a), \theta_1 \circ \iota_2(b)) + d(\theta_1 \circ \iota_2(b), \theta_2 \circ \eta_2(c)) \right] \\
< \inf_{b \in F_0} \left[ d(\theta_1 \circ \iota_1(a), \theta_1 \circ \iota_2(b)) + d(\theta_2 \circ \eta_2(b), \theta_2 \circ \eta_3(c)) + 2\varepsilon \right] \\
=(\varphi_{2,3}|_{M_2 \times M_3} + \varepsilon)(\varphi_{1,2}|_{M_1 \times F_0} + \varepsilon)(a, c),
\]

so

\[
\varphi_{\theta_1 \circ \iota_1, \theta_2 \circ \eta_2 \circ \eta_1} \leq \left[ (\varphi_{2,3}|_{M_2 \times M_3} + \varepsilon)|_{M_2 \times M_3} \right] \left[ (\varphi_{1,2}|_{M_1 \times F_0} + \varepsilon)|_{M_1 \times M_2} \right] \leq \psi \varphi.
\]

Since $\varphi_{\theta_1 \circ \iota_1, \theta_2 \circ \eta_2 \circ \eta_1}$ is in $\text{Apx}_\mathcal{K}(M_1, M_3)$, so is $\psi \varphi$.

Next, assume that both $\varphi$ and $\psi$ are still strict, but $M_1$, $M_2$ and $M_3$ are general $\mathcal{K}$-structures. Then there exist sufficiently large $l, m, m', n \in \mathbb{N}$ and approximate $\mathcal{K}$-isomorphisms $\varphi'$ from $A_l$ to $B_m$ and $\psi'$ from $B_m$ to $C_n$ with $\varphi'|_{M_1 \times M_2} < \varphi$ and $\psi'|_{M_2 \times M_3} < \psi$. We may assume without loss of generality that $m$ is equal to $m'$, since in general, if $\iota : A \to A'$ and $\eta : B \to B'$ are $\mathcal{K}$-embeddings, then one can directly check from the definition that the trivial extension of an approximate $\mathcal{K}$-isomorphism in $\text{Apx}_{\mathcal{K}}(A, B)$ via these $\mathcal{K}$-embeddings belong to $\text{Apx}_{\mathcal{K}}(A', B')$. Also, we may assume that both $\varphi'$ and $\psi'$ are strict by Proposition 2.5. By what we proved in the preceding paragraph, $\psi' \varphi'$ is in $\text{Apx}_\mathcal{K}(A_l, C_n)$. By direct computation, one can check that

\[
(\psi'|_{M_2 \times M_3})(\varphi'|_{M_1 \times M_2}) = (\psi' \varphi)|_{M_1 \times M_0},
\]

so $\psi \varphi$ is in $\text{Apx}_\mathcal{K}(M_1, M_3)$. 
Finally, let \( \varphi \) and \( \psi \) be general approximate \( \mathcal{K} \)-isomorphisms between general \( \mathcal{K} \)-structures. Then there exist nets \( \{\varphi_n\} \) and \( \{\psi_n\} \) of strict approximate \( \mathcal{K} \)-isomorphisms which converge to \( \varphi \) and \( \psi \) respectively, and
\[
\psi \varphi = (\lim_n \psi_n)(\lim_n \varphi_n) \geq \lim_n (\psi_n \varphi_n) \in \text{Apx}_{\mathcal{K}}(M_1, M_3),
\]
so \( \psi \varphi \) belongs to \( \text{Apx}_{\mathcal{K}}(M_1, M_3) \).
\[\Box\]

**Corollary 3.5.** Extensions and restrictions of approximate \( \mathcal{K} \)-isomorphisms via \( \mathcal{K} \)-admissible embeddings are approximate \( \mathcal{K} \)-isomorphisms.

4. **Fraïssé categories and their limits**

Let \( L \) be a language and \( \mathcal{K} \) be a category of finitely generated \( L \)-structures and \( L \)-embeddings with JEP and NAP. For each \( n \in \mathbb{N} \), we denote by \( \mathcal{K}_n \) the class of all pairs \( \langle \bar{a}, \bar{b} \rangle \), where \( \bar{a} \) is an object of \( \mathcal{K} \) and \( \bar{a} \) is an ordered generator of \( \mathcal{A} \). We simply write \( \langle \bar{a} \rangle \) instead of \( \langle \mathcal{A}, \bar{a} \rangle \) when there is no danger of confusion.

For each \( n \), we consider a pseudo-metric on \( \mathcal{K}_n \) defined by
\[
d^\mathcal{K}_n(\langle \bar{a}, \bar{b} \rangle) := \inf \{ \max \varphi(a_i, b_i) \mid \varphi \in \text{Apx}_{\mathcal{K}}(\langle \bar{a}, \bar{b} \rangle) \}
\]
\[
= \inf \{ \max \varphi(a_i, b_i) \mid \varphi \in \text{Stx}_{\mathcal{K}}(\langle \bar{a}, \bar{b} \rangle) \}
\]
\[
= \inf \{ \max d(\iota(a_i), \eta(b_i)) \mid (\iota, \eta) \in \text{JE}_{\mathcal{K}}(\langle \bar{a}, \bar{b} \rangle) \},
\]
where \( a_i \) and \( b_i \) denotes the \( i \)-th component of \( \bar{a} \) and \( \bar{b} \) respectively. The fact that \( d^\mathcal{K}_n \) is indeed a pseudo-metric follows from JEP and NAP.

**Definition 4.1.** The category \( \mathcal{K} \) is said to satisfy
- the **weak Polish property** (WPP) if \( \mathcal{K}_n \) is separable with respect to the pseudo-metric \( d^\mathcal{K}_n \) for each \( n \).
- the **Cauchy continuity property** (CCP) if
  (i) for any \( n \)-ary predicate symbol \( P \) in \( L \), the map
    \( \langle \mathcal{A}, (\bar{a}, \bar{b}) \rangle \mapsto P^\mathcal{A}(\bar{a}) \)
  from \( \mathcal{K}_{n+m} \) into \( \mathbb{R} \) sends Cauchy sequences into Cauchy sequences; and
  (ii) for any \( n \)-ary function symbol \( f \) in \( L \), the map
    \( \langle \mathcal{A}, (\bar{a}, \bar{b}) \rangle \mapsto \langle \mathcal{A}, (\bar{a}, \bar{b}, f^\mathcal{A}(\bar{a})) \rangle \)
  from \( \mathcal{K}_{n+m} \) into \( \mathcal{K}_{n+m+1} \) sends Cauchy sequences into Cauchy sequences.

**Remark 4.2.** If \( \mathcal{K} \) satisfies CCP, then \( d^\mathcal{K}_n(\langle \bar{a}, \bar{b} \rangle) \) is equal to zero if and only if there exists a \( \mathcal{K} \)-admissible isomorphism from \( \langle \bar{a} \rangle \) onto \( \langle \bar{b} \rangle \) which sends \( a_i \) to \( b_i \). To see this, first suppose that the map \( a_i \mapsto b_i \) extends to a \( \mathcal{K} \)-admissible isomorphism \( \iota \). Then \( (\varphi^\mathcal{A}_{\bar{a}\bar{b}})^{(\bar{a})\times(\bar{b})} + \varepsilon \) belongs to \( \text{Stx}_{\mathcal{K}}(\langle \bar{a}, \bar{b} \rangle) \), and
\[
d^\mathcal{K}_n(\langle \bar{a}, \bar{b} \rangle) \leq (\varphi^\mathcal{A}_{\bar{a}\bar{b}})^{(\bar{a})\times(\bar{b})}(a_i, b_i) + \varepsilon = \varepsilon
\]
for arbitrary \( \varepsilon > 0 \). Conversely, suppose \( d^\mathcal{K}_n(\langle \bar{a}, \bar{b} \rangle) = 0 \). Let \( D_\bar{a} \) be the set of all elements of \( \langle \bar{a} \rangle \) of the form \( g(\bar{a}) \), where \( g \) is a composition of functions equipped with \( \langle \bar{a} \rangle \), and \( D_\bar{b} \) be the set obtained from \( \langle \bar{b} \rangle \) by the same way. Then it
follows from CCP that the map \( a_i \mapsto b_i \) extends to an isometry from \( D_0 \) onto \( D_b \) and the interpretations of the symbols can be identified via this isometry, so that it extends to an \( L \)-isomorphism \( \iota \) from \( \langle \bar{a} \rangle \) onto \( \langle \bar{b} \rangle \). If \( \bar{c} = (g_1(\bar{a}), \ldots, g_n(\bar{a})) \) and \( \bar{d} = (g_1(\bar{b}), \ldots, g_n(\bar{b})) \), then \( d^K(\langle \bar{a}, \bar{c} \rangle, \langle \bar{b}, \bar{d} \rangle) = 0 \) by CCP, so there exists a joint \( \mathcal{K} \)-embedding \((\eta_1, \eta_2)\) of \( \langle \bar{a}, \bar{c} \rangle \) and \( \langle \bar{b}, \bar{d} \rangle \) such that the \( \eta_1(\bar{a}, \bar{c}) \) and \( \eta_2(\bar{b}, \bar{d}) \) are arbitrarily close to each other, whence \( \iota \) is \( \mathcal{K} \)-admissible.

**Definition 4.3.** (1) A category \( \mathcal{K} \) of finitely generated separable \( L \)-structures is called a \textit{Fraïssé category} if it satisfies JEP, NAP, WPP and CCP.

(2) Let \( \mathcal{K} \) be a Fraïssé class. A \( \mathcal{K} \)-structure \( M \) is called a \textit{Fraïssé limit} of \( \mathcal{K} \) if for any \( \mathcal{K} \)-structure \( N \) and any strict approximate \( \mathcal{K} \)-isomorphism \( \varphi \in \text{Stx}_\mathcal{K}(N, M) \), there exists a \( \mathcal{K} \)-admissible embedding \( \iota: N \rightarrow M \) with \( \varphi_i < \varphi \).

We shall begin with characterizing Fraïssé limits. Fix a Fraïssé class \( \mathcal{K} \).

**Definition 4.4.** A \( \mathcal{K} \)-structure \( M \) is said to be

- \( \mathcal{K} \)-universal if for any object \( A \) of \( \mathcal{K} \), there exists a \( \mathcal{K} \)-admissible embedding of \( A \) into \( M \).
- approximately \( \mathcal{K} \)-ultra-homogeneous if for any \( \langle \bar{a} \rangle \in \mathcal{K}_n \), any \( \varepsilon > 0 \) and any \( \mathcal{K} \)-admissible embeddings \( \iota, \eta: \langle \bar{a} \rangle \rightarrow M \), there exists a \( \mathcal{K} \)-admissible automorphism \( \alpha \) of \( M \) with max \( d(\alpha \circ \iota(a_i), \eta(a_i)) \leq \varepsilon \).

**Theorem 4.5.** For a \( \mathcal{K} \)-structure \( M \), the following are equivalent.

(i) The structure \( M \) is a Fraïssé limit of \( \mathcal{K} \).

(ii) For any object \( A \) of \( \mathcal{K} \) and any \( \varphi \in \text{Stx}_\mathcal{K}(A, M) \), there exists a \( \mathcal{K} \)-admissible embedding \( \iota: A \rightarrow M \) with \( \varphi_i < \varphi \).

(iii) If \( \langle \bar{a} \rangle \) is in \( \mathcal{K}_n \) and \( \varphi \) is a strict approximate \( \mathcal{K} \)-isomorphism from \( \langle \bar{a} \rangle \) to \( M \), then for any \( \varepsilon > 0 \) there is an approximate \( \mathcal{K} \)-isomorphism \( \psi \in \text{Stx}_\mathcal{K}(\langle \bar{a} \rangle, M) \) such that \( \psi|_{\text{Stx}M} \) is \( \varepsilon \)-total.

(iv) The structure \( M \) is \( \mathcal{K} \)-universal and approximately \( \mathcal{K} \)-ultra-homogeneous. Moreover, if a Fraïssé limit exists, then it is unique up to \( \mathcal{K} \)-admissible isomorphisms.

**Proof.** First, assume that \( M = \bigcup_n A_n \) and \( N = \bigcup_m B_m \) are \( \mathcal{K} \)-structures satisfying (iii). We shall show that if \( \varphi \) is a strict approximate \( \mathcal{K} \)-isomorphism from \( M \) to \( N \), then there exists a \( \mathcal{K} \)-admissible isomorphism \( \alpha \) from \( M \) onto \( N \) with \( \varphi_i < \varphi \). Since \( \varphi \) is strict, there exist an approximate \( \mathcal{K} \)-isomorphism \( \psi \) from \( M \) to \( N \), finite subsets \( E \subseteq \bigcup_n \{A_n\} \) and \( F \subseteq \bigcup_m \{B_m\} \), and a positive real number \( \varepsilon \leq 1 \) with

\[
(\psi|_{E\times F})^{\text{Max}N} + \varepsilon < \varphi.
\]

Take increasing sequences \( \{X_i\} \) and \( \{Y_j\} \) of finite sets such that

- \( X_0 = E \) and \( Y_0 = F \);
- \( X_i \subseteq \bigcup_n \{A_n\} \) and \( Y_j \subseteq \bigcup_m \{B_m\} \) for all \( i, j \); and
- \( \bigcup_i X_i \) and \( \bigcup_j Y_j \) are dense in \( |M| \) and \( |N| \) respectively.

We claim the existence of a sequence \( \{\psi_i\} \) of strict approximate \( \mathcal{K} \)-isomorphisms form \( M \) to \( N \) with the following properties.
(a) each $\psi_l$ is of the form $(\theta|_{X_0\times X_{j+1}})^{M\times N} + \varepsilon$ for some $\theta \in \text{Apx}_{\mathcal{X}}(M, N)$, where $\delta_l \leq 2^{-l}$ and $i(l), j(l) \uparrow \infty$;
(b) $\psi_{l+1} \prec \psi_l$; and
(c) $\psi_{l+1}|_{X_0\times N}$ is $\delta_l$-total if $l$ is even, while $\psi_{l+1}|_{M\times Y_{j+1}}$ is $\delta_l$-surjective if $l$ is odd.

The construction of such a sequence proceeds as follows. Set

$$\psi_0 := (\psi|_{X_0\times Y_2})^{[M\times |N|]} + \varepsilon.$$  

Assume $l$ is even and $\psi_l$ is given. Then, by assumption on $N$, one can find $\theta \prec \psi_l$ such that $\theta|_{X_0\times N}$ is $\delta_l/2$-total. Since $(\theta|_{X_i\times Y_j})^{M\times N} + \delta$ converges to $\theta$ as $i, j \to \infty$ and $\delta \to 0$, for sufficiently large $i(l + 1) > i(l)$ and $j(l + 1) > j(l)$ and sufficiently small $\delta_{l+1} < \delta_l/2$, we have

$$\psi_l \triangleright (\theta|_{X_{l+1}\times Y_{j+1}})^{M\times N} + \delta_{l+1}.$$  

We let $\psi_{l+1}$ be the right-hand side. Then it is clear that $\psi_{l+1}|_{X_0\times N}$ is $\delta_l$-total. The case $l$ is odd is similar, and the description of the construction of $\psi|_{X_1\times Y_2}$ is completed.

Now the sequence being decreasing, there exists the limit $\psi_\infty \in \text{Apx}_{\mathcal{X}}^\infty(M, N)$, which is clearly of the form $\varphi_\alpha$ for some isomorphism $\alpha : M \to N$ by Proposition 2.4, as desired.

(iii) $\Rightarrow$ (i) can be verified by the similar argument as above. Also, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (iv). It follows from (iii) $\Rightarrow$ (ii) that $M$ is $\mathcal{X}$-universal. Let $\iota, \eta : \langle \bar{a} \rangle \to M$ be $\mathcal{X}$-admissible embeddings and $\epsilon$ be a positive real number. Then

$$\varphi := ((\varphi_\eta \varphi_\iota^*)|_{\bar{a}\times \eta(\bar{a})})^{M\times M} + \epsilon$$

is in Stx$_{\mathcal{X}}(M, M)$, so by what we proved in the first paragraph, one can find a $\mathcal{X}$-admissible automorphism $\alpha$ of $M$ with $\varphi_\alpha \prec \varphi$. Since $\varphi(\iota(a_i), \eta(a_i)) = \epsilon$, the inequality $d(\alpha \circ \iota(a_i), \eta(a_i)) \leq \epsilon$ follows.

(iv) $\Rightarrow$ (ii). Suppose that $A$ is an object of $\mathcal{X}$ and $\varphi$ is a strict approximate $\mathcal{X}$-isomorphism from $A$ to $M$. By assumption, there exists a $\mathcal{X}$-admissible embedding $\iota : A \to M = \bigcup_n A_n$, so it suffices to show that there is a $\mathcal{X}$-admissible automorphism $\alpha$ of $M$ with $\varphi_\alpha \prec \varphi$, or equivalently, $\varphi_\alpha \prec \varphi_\alpha^*$. To see this, find sufficiently large $n$ and finite partial $\mathcal{X}$-isomorphisms $\iota_1, \iota_2$ from $A_n$ into some object $C$ of $\mathcal{X}$ with

$$(\varphi_{12})^{M\times M} + \epsilon \prec \varphi_\alpha^*.$$  

Since there exists a $\mathcal{X}$-admissible embedding of $C$ into $M$, and since $M$ is approximately $\mathcal{X}$-ultra-homogeneous, there exists a $\mathcal{X}$-admissible embedding $\eta : C \to M$ with $d(b, \eta \circ \iota_2(b)) < \epsilon/2$ for $b \in \text{dom} \iota_2$. Again by the $\mathcal{X}$-ultra-homogeneity of $M$, one can find a $\mathcal{X}$-admissible automorphism $\alpha$ of $M$ such that the inequality $d(\alpha(a), \eta \circ \iota_1(a)) < \epsilon/2$ holds for all $a \in \text{dom} \iota_1$. Then

$$d(\alpha(a), b) \leq d(\alpha(a), \eta \circ \iota_1(a)) + d(\eta \circ \iota_2(a), \eta \circ \iota_2(b)) + d(\eta \circ \iota_2(b), b) \leq d(\iota_1(a), \iota_2(b)) + \epsilon = \varphi_{12}(a, b) + \epsilon,$$

which completes the proof. □
Next, we shall prove the existence of the Fraïssé limit. For this, we need the following lemma which claims that, in order to see (iii) in Theorem 4.5, we only have to check a countable dense part.

**Lemma 4.6.** Let $\mathcal{M}$ be a $\mathcal{K}$-structure and $M_0$ be a countable dense subset of $|\mathcal{M}|$. Suppose that, for each $n \in \mathbb{N}$, a countable dense subset $\mathcal{K}_n,0$ of $\mathcal{K}_n$ is given. Then, in order for $\mathcal{M}$ to be the Fraïssé limit of $\mathcal{K}$, it is sufficient that, for any $n \in \mathbb{N}$, any $\langle \bar{a} \rangle \in \mathcal{K}_n,0$, any finite subset $F \subseteq M_0$ and any $\varphi \in \text{Stx}_{\mathcal{K}}(\langle \bar{a} \rangle, \mathcal{M})$ which is rational-valued on $\bar{a} \times F$, there exists $\psi \in \text{Ap}_\bar{a}^G(\langle \bar{a} \rangle, \mathcal{M})$ such that $\psi|_{\bar{a} \times M}$ is $\epsilon$-total.

**Proof.** Let $\mathcal{B}$ be an object of $\mathcal{K}$ and $\varphi$ be a strict approximate $\mathcal{K}$-isomorphism from $\mathcal{B}$ to $\mathcal{M}$, and take $\varphi' \in \text{Stx}_{\mathcal{K}}(\mathcal{B}, \mathcal{M})$. Then there exists an arbitrarily large finite subsets $F \in |\mathcal{B}|^n$ and $G \subseteq M_0$ and arbitrarily small $\epsilon > 0$ with

$$\varphi'' := (\varphi'|_{F \times G})|_{\bar{a} \times M} + \epsilon < \varphi.$$

Without loss of generality, we may assume that $F$ is a generator of $\mathcal{B}$. Let $\bar{b} = (b_1, \ldots, b_n)$ be an enumeration of $F$. Take $\langle \bar{a} \rangle \in \mathcal{K}_n,0$ with $d(\bar{a}, \langle \bar{b} \rangle) < \epsilon/4$ and find a joint $\mathcal{K}$-embedding $(\iota, \eta)$ of $\langle \bar{a} \rangle$ and $\langle \bar{b} \rangle$ satisfying $\max_i \varphi_{i,n}(a_i, b_i) < \epsilon/4$. Then, being a restriction of an extension of a strict approximate $\mathcal{K}$-isomorphism, $\varphi''_{i,n}$ is strict, so there exists $\psi' \in \text{Stx}_{\mathcal{K}}(\langle \bar{a} \rangle, \mathcal{M})$ which is rational-valued on $\bar{a} \times F$ and satisfies $\psi' < \varphi''_{i,n}$, by Proposition 2.5. By assumption, we can take $\psi'' \in \text{Ap}_\bar{a}^G(\langle \bar{a} \rangle, \mathcal{M})$ such that $\psi''|_{\bar{a} \times M}$ is $\epsilon/4$-total. Then, $(\psi''_{i,n})|_{\bar{a} \times M}$ is $\epsilon/2$-total, and

$$(\psi''_{i,n})|_{\bar{a} \times M} \leq (\psi''_{i,n})|_{\bar{a} \times M} \leq \varphi''|_{\bar{a} \times M} + \epsilon/2,$$

since $\varphi''_{i,n}|_{\bar{a} \times (\bar{a})}$ is $\epsilon/4$-total. Now take a finite subset $H \subseteq |\mathcal{M}|$ such that $G$ is included in $H$ and $(\psi''_{i,n})|_{\bar{a} \times H}$ is $3\epsilon/4$-total, and put

$$\psi := ((\psi''_{i,n})|_{\bar{a} \times H})|_{\bar{a} \times M} + \epsilon/4.$$ 

Then $\psi|_{\bar{a} \times M}$ is $\epsilon$-total, and

$$\psi \leq (\psi''|_{\bar{a} \times H})|_{\bar{a} \times M} + 3\epsilon/4 \leq (\varphi'|_{F \times G})|_{\bar{a} \times M} + \epsilon < \varphi.$$ 

Since this shows that (iii) in Theorem 4.5 holds, it follows that $\mathcal{M}$ is the Fraïssé limit of $\mathcal{K}$.

**Theorem 4.7.** Every Fraïssé category has its limit.

**Proof.** Take a countable dense subset $\mathcal{K}_n,0 \subseteq \mathcal{K}_n$ for each $n$. In view of Proposition 3.3, we can inductively find a $\mathcal{K}$-structure $\mathcal{A}_k$, a $\mathcal{K}$-embedding $\iota_{k-1} : \mathcal{A}_{k-1} \rightarrow \mathcal{A}_{k}$ and a countable dense subset $A_{k,0} \subseteq |\mathcal{A}_{k}|$ so that, if $\langle \bar{a} \rangle$ is in $\mathcal{K}_n,0$, if $F$ is a finite subset of $A_{k,0}$ and if $\varphi$ is a strict approximate $\mathcal{K}$-isomorphism from $\langle \bar{a} \rangle$ to $\mathcal{A}_{k}$ which is rational-valued on $\bar{a} \times F$, then there exists a $\mathcal{K}$-embedding $t : \langle \bar{a} \rangle \rightarrow \mathcal{A}_l$ for some $l > k$ with $\varphi_i < \varphi_{i,l} \varphi$, where $\iota_{i,k}$ denotes the composition of $\iota_{i} \ldots \iota_{i-1}$. Then the $\mathcal{K}$-structure obtained from the inductive system satisfies the assumption in the previous lemma, so we are done.

**Remark 4.8.** Let $\mathcal{K}$ be a Fraïssé category. If every $L$-embedding between objects of $\mathcal{K}$ is a morphism of $\mathcal{K}$, then $\mathcal{K}$ is a Fraïssé class in the sense of [Eag16].
notice that there is a subtle difference between our result (Theorems 4.5 and 4.7) and [Eag16, Theorem 2.8]. We proved that if \( \mathcal{M} \) is the limit of a Fraïssé class \( \mathcal{K} \), if \( \iota, \eta : A \to M \) are \( \mathcal{K} \)-admissible embedding of an object \( A \) of \( \mathcal{K} \), and if \( F \) is a finite subset of \([A]\), then for any \( \varepsilon > 0 \) there exists a (\( \mathcal{K} \)-admissible) automorphism \( \alpha \) of \( M \) with \( d(\alpha \circ \iota(a), \eta(a)) < \varepsilon \) for all \( a \in F \). On the other hand, it is claimed in [Eag16, Definition 2.6 and Theorem 2.8] that even if \( \iota \) and \( \eta \) are not \( \mathcal{K} \)-admissible, one can still find an automorphism with the same property.

In order to obtain the result claimed in [Eag16, Theorem 2.8], one might modify the definition of approximate \( \mathcal{K} \)-isomorphisms as following. First, for \( \mathcal{K} \)-structures \( \mathcal{M} \) and \( \mathcal{N} \), define \( \text{Apx}_{\mathcal{K}}(\mathcal{M}, \mathcal{N}) \) as the set of all approximate isomorphisms of the form \( \varphi_{\iota, \eta} \), where \( \iota \) and \( \eta \) are finite partial \( \mathcal{L} \)-isomorphisms from \( \mathcal{M} \subseteq \mathcal{N} \) into some \( A \in \mathcal{K} \) such that the structures \( \langle \text{dom} \ i \rangle, \langle \text{ran} \ i \rangle, \langle \text{dom} \ \eta \rangle \) and \( \langle \text{ran} \ \eta \rangle \) are in \( \mathcal{K} \). Then define \( \text{Apx}_{\mathcal{K}}(\mathcal{M}, \mathcal{N}) \) as the closure of the set of all approximate isomorphisms which dominate some element of \( \text{Apx}_{\mathcal{K}}(\mathcal{M}, \mathcal{N}) \). If one could prove Propositions 3.3 and 3.4 with this modified definition, then one would be able to obtain the desired result by simply copying the proofs in this paper.

However, with this modified definition, the proof of Proposition 3.4 no longer works. The problem lies in the second paragraph, where the proof is reduced to the case that all the relevant structures are objects of \( \mathcal{K} \). To see the difficulty, let \( \mathcal{M}_1, \mathcal{M}_2 \) and \( \mathcal{M}_3 \) be \( \mathcal{K} \)-structures, and \( \varphi \in \text{Apx}_{\mathcal{K}}(\mathcal{M}_1, \mathcal{M}_2) \) and \( \psi \in \text{Apx}_{\mathcal{K}}(\mathcal{M}_2, \mathcal{M}_3) \) be strict. Then there exist finite partial \( \mathcal{L} \)-embeddings \( \iota_i : \mathcal{M}_i \to \mathcal{A} \) (\( i = 1, 2 \)) and \( \eta_j : \mathcal{M}_j \to \mathcal{B} \) (\( j = 2, 3 \)), where \( \mathcal{A} \) and \( \mathcal{B} \) are members of \( \mathcal{K} \), and a positive real number \( \varepsilon \) such that

\[
\varphi' := \varphi_{\iota_1, \iota_2}|_{\mathcal{M}_1 \times \mathcal{M}_2} + \varepsilon \circ \varphi, \quad \psi' := \varphi_{\eta_2, \eta_3}|_{\mathcal{M}_2 \times \mathcal{M}_3} + \varepsilon \circ \psi.
\]

It is true that there exists an \( \mathcal{L} \)-embedding \( \iota \) of a member \( C \) of \( \mathcal{K} \) into \( \mathcal{M}_2 \) such that the image of \( \iota \) ALmost includes both \( \text{dom} \ \iota_2 \) and \( \text{dom} \ \eta_1 \), and in order to reduce the proof, it is expected to show that the restrictions \( \varphi'^*_i \varphi' \) and \( \psi'_i \varphi' \) are approximate isomorphisms in the sense of the modified definition; but how?

Note that with the original definition (i.e., Definition 3.1), this difficulty could be avoided. This is because we could take \( \iota \) above so that the image genuinely includes the domains of both \( \iota_2 \) and \( \eta_1 \), whence the restrictions are trivially approximate isomorphisms.

5. UHF algebras

In this section, we give an application of our theory to C*-algebras. This is a generalization of the results in Section 3 of [Mas16].

We shall consider the language of unital tracial C*-algebra. The language \( L_{TC} \) consists of the following symbols:

- two constant symbols 0 and 1;
- an unary function symbol \( \lambda \) for each \( \lambda \in \mathbb{C} \), which are to be interpreted as multiplication by \( \lambda \);
- an unary function symbol \( * \) for involution;
- a binary function symbol \( + \) and \( \cdot \);
**Remark 5.1.** Here, we shall give an example of $L_{TC}$-morphism between objects of $\mathcal{K}_v$ which cannot be approximated by diagonalizable ones with respect to point-norm topology. We assume that $2$ is in $\mathbb{N}_v$ and use $\mathbb{D} := \{ z \in \mathbb{C} \mid |z| \leq 1 \}$ instead of
Lemma 5.2. \( \beta \) from Proof.

Lemma 5.3. map such that any interval of the form \([a, b] \cap \mathbb{R} \) is diagonalizable if and only if \( \alpha \) is impossible. We note that this is also a counterexample of [Lin09, Theorem 6.3].

For any \( p \), there exists a \( \mathcal{X} \)-homomorphism \( \alpha_{p} \cdot \tau_{p} \) onto \( \langle \mathcal{A}_{p,n}, \tau_{p} \rangle \).

Proof. (i) Let \( \alpha \) be the homeomorphism of \([0, 1]^{p} \) with \( \alpha_{p} \cdot \tau_{p} \). Then the induced \( \star \)-homomorphism \( \alpha^{*} : f \mapsto f \circ \alpha \) is the desired one.

(ii) We may assume \( \tau = \tau_{k} \) by (i). It suffices to show the case \( p = 2 \). Let \( \beta : [0, 1] \rightarrow [0, 1]^{2} \) be the Hilbert curve [Hil91], which is a surjective continuous map such that any interval of the form \([k/4^{i}, k + 1/4^{i}] \) is sent to a square of the form \([k_{1}/2^{j}, k_{1} + 1/2^{j}] \times [k_{2}/2^{j}, k_{2} + 1/2^{j}] \), so that \( \beta_{k} \cdot \tau_{k} \). Then the \( \star \)-homomorphism \( \beta^{*} : f \mapsto f \circ \beta \) is the desired one. \( \square \)

Lemma 5.3. Suppose that \( t_{1}, t_{2} : \langle \mathcal{A}_{p,n}, \tau \rangle \rightarrow \langle \mathcal{A}_{p',n'}, \tau \rangle \) are \( \mathcal{X} \)-embeddings of the form

\[ t_{i}(f) = \text{diag}(f \circ t_{1,i}, \ldots, f \circ t_{k,i}) \]
If the diameter of the range of $t_{ij}$ is less than $\delta$ for all $i$ and $l$, then there exists a permutation $\sigma \in S_k$ such that the inequality $||t_{l,1} - t_{\sigma(l),2}|| < 2\delta$ holds for all $l$.

**Proof.** For each $l$, let $S_l$ be the set of all $l'$ with $\text{Im} \ t_{l,1} \cap \text{Im} \ t_{l',2} \neq \emptyset$. Then, for any $F \subseteq \{1, \ldots, k\}$, we have

$$\bigcup_{l \in F} S_l = \sum_{l \in F} \tau(\text{Im} \ t_{l,2}) \geq k\tau\left(\bigcup_{l \in F} \text{Im} \ t_{l,1}\right) \geq |F|,$$

since $t_1$ and $t_2$ are trace-preserving. By Hall’s marriage theorem there exists a permutation $\sigma \in S_k$ with $t_{\sigma(l),2} \in S_l$ for all $l$. Now the inequality $||t_{l,1} - t_{\sigma(l),2}|| < 2\delta$ is clear.

**Theorem 5.4.** The category $\mathcal{K}_c$ is a Fraïssé category.

**Proof.** JEP is a direct consequence of Lemma 5.2 and the fact that if $n$ divides $n'$, then there exists a $\mathcal{K}_c$-embedding from $\langle A_{1,n}, \tau_1 \rangle$ to $\langle A_{n',n}, \tau_1 \rangle$ defined by $f \mapsto \text{diag}[f, \ldots, f]$. For NAP, let $\iota_i$ be $\mathcal{K}_c$-embeddings from $\langle A_{p_0,n_0}, \tau_0 \rangle$ into $\langle A_{p_n,n}, \tau_i \rangle$ for $i = 1, 2$, and suppose that a finite subset $F$ of $A_{p_0,n_0}$ and a positive real number $\epsilon > 0$ are given. Our goal is to find $\mathcal{K}_c$-embeddings $\eta_i$ from $\langle A_{p_n,n}, \tau_i \rangle$ into some object $\langle A_{p_n,n}, \tau_3 \rangle$ such that the inequality $d(\eta_1 \circ t_1(f), \eta_2 \circ t_2(f)) < \epsilon$ holds for all $f \in F$. To see this, take $\delta > 0$ so that $|t - t'| < \delta$ implies $||f(t) - f(t')|| < \epsilon$. Apply JEP to find $\mathcal{K}_c$-embeddings $\eta_i$ from $\langle A_{p_n,n}, \tau_i \rangle$ into some object $\langle A_{p_n,n'}, \tau' \rangle$. By Proposition 5.2, we may assume without loss of generality that $\tau' = \tau_3$ and $p' = 1$. Now, since $\eta_1' \circ \iota_i$ is a $\mathcal{K}_c$-isomorphism, it is of the form

$$\eta_1' \circ \iota_i(f) = \text{Ad}(v_i')(\text{diag}[f \circ t_{i,1}', \ldots, f \circ t_{i,n}', 1]).$$

Take sufficiently large natural number $m$ such that $n'm$ is in $\mathbb{N}_c$ and $|s - s'| < 1/m$ implies $|t_{i,1}(s) - t_{i,1}(s')| < \delta/2$ for all $l$ and $i$. Define $r_c : [0, 1] \to [0, 1]$ by $r_c(x) := (x + c - 1)/m$ for $c = 1, \ldots, m$, and let $\rho$ be a $\mathcal{K}_c$-embedding from $\langle A_{1,n}, \tau_1 \rangle$ into $\langle A_{1,n'm}, \tau_1 \rangle$ of the form

$$\rho(f) = \text{diag}[f \circ r_1, \ldots, f \circ r_m].$$

Then $\rho \circ \eta_1' \circ \iota_i$ is of the form

$$\rho \circ \eta_1' \circ \iota_i(f) = \text{Ad}(v_i')(\text{diag}[f \circ t_{1,i}, \ldots, f \circ t_{k,i}]),$$

where the diameter of the image of $t_{it}$ is less than $\delta/2$ for all $l$ and $i$. By Lemma 5.3, we may assume without loss of generality that the inequality $||t_{l,1} - t_{l,2}|| < \delta$ holds for all $l$. It can be easily verified that $\eta_1 := \rho \circ \eta_1'$ and $\eta_2 : \text{Ad}(v_1v_2') \circ \rho \circ \eta_2'$ are the desired $\mathcal{K}_c$-embeddings.

WPP is clear, because up to $\mathcal{K}_c$-isomorphisms, there are only countable many objects in $\mathcal{K}_c$. Also, CCP automatically follows from the fact that all the relevant functions are $1$-Lipschitz on the unit ball.

We shall find a concrete description of the limit of $\mathcal{K}_c$. For this, the following proposition is useful.

**Proposition 5.5.** Let $\mathcal{K}$ be a Fraïssé class and $\mathcal{M} = \bigcup_n A_n$ be a $\mathcal{K}$-structure. Denote by $\iota_{k,j}$ the canonical $\mathcal{K}$-embedding from $A_j$ into $A_k$. Suppose that the following two conditions hold:
(a) Any object $C$ of $\mathcal{K}$ is $\mathcal{K}$-embeddable into $\mathcal{A}_n$ for some $n$.
(b) Given a finite subset $F \subseteq |\mathcal{A}_n|$, a positive real number $\varepsilon$ and a $\mathcal{K}$-embedding $\eta: \mathcal{A}_i \to \mathcal{A}_j$ for some $j > i$, one can find $k > j$ and a $\mathcal{K}$-automorphism $\alpha \in \mathcal{A}_k$ such that the inequality $d(\alpha \circ t_{k,j} \circ \eta(a), t_{k,j}(a)) < \varepsilon$ holds for all $a \in F$.

Then $\mathcal{M} = \bigcup_n \mathcal{A}_n$ is the Fraïssé limit of $\mathcal{K}$.

**Proof.** We shall check (iii) in Theorem 4.5. Let $\varepsilon$ be a positive real number, $\mathcal{B}$ be an object of $\mathcal{K}$ and $\varphi$ be in $\text{Stx}_{\mathcal{K}}(\mathcal{B}, \mathcal{M})$. Then one can find finite subsets $F_1 \subseteq |\mathcal{B}|$ and $F_2 \subseteq |\mathcal{A}_n|$, an object $C$ of $\mathcal{K}$, and $\mathcal{K}$-embeddings $\iota: \mathcal{B} \to C$ and $\eta: \mathcal{A}_i \to C$ such that the relation

$$(\varphi_{|\mathcal{B}|^{F_1 \times F_2}})^{\mathcal{B} \times \mathcal{M}} < \varphi$$

holds. By assumption (a), there exists a $\mathcal{K}$-embedding $\theta$ of $C$ into some $\mathcal{A}_j$ with $j > i$. Then one can find a $\mathcal{K}$-automorphism $\alpha \in \text{Aut}(\mathcal{A}_k)$ for some $k > j$ such that the inequality

$$d(\alpha \circ t_{k,j} \circ \theta \circ \eta(a), t_{k,j}(a)) < \varepsilon$$

holds for all $a \in F_2$, by assumption (b). Now, for $b \in F_1$ and $a \in F_2$, we have

$$d(\alpha \circ t_{k,j} \circ \theta \circ \iota(b), t_{k,j}(a))$$

$$< d(\alpha \circ t_{k,j} \circ \theta \circ \iota(b), \alpha \circ t_{k,j} \circ \theta \circ \eta(a)) + \varepsilon$$

$$= d(\iota(b), \eta(a)) + \varepsilon,$$

whence

$$\varphi_{|\mathcal{B}|^{F_1 \times F_2}} \leq (\varphi_{|\mathcal{B}|^{F_1 \times F_2}})^{\mathcal{B} \times \mathcal{M}} < \varphi,$$

which completes the proof. $\square$

**Corollary 5.6.** Let $\mathcal{M} = \bigcup_j (\mathcal{A}_{p_j n_j}, \tau_j)$ be a $\mathcal{K}_v$-structure and $\iota_{k,j}$ denote the canonical $\mathcal{K}$-embedding from $\langle \mathcal{A}_{p_j n_j}, \tau_j \rangle$ into $\langle \mathcal{A}_{p_k n_k}, \tau_k \rangle$. Suppose the following conditions hold:

(a) $p_j \geq 1$.
(b) For any $n \in \mathbb{N}_v$, there exists $j \in \mathbb{N}$ such that $n$ divides $n_j$.
(c) For any $j \in \mathbb{N}$ and $\varepsilon > 0$ there exists $k > j$ such that $t_{k,j}$ is of the form

$$t_{k,j}(f) = \text{Ad}(\nu)(\text{diag}[f \circ t_1, \ldots, f \circ t_m]),$$

where the diameter of the image of $t_l$ is less than $\varepsilon$ for all $l$.

Then $\mathcal{M} = \bigcup_j (\mathcal{A}_{p_j n_j}, \tau_j)$ is the Fraïssé limit of $\mathcal{K}_v$.

**Proof.** This is immediate from Lemmas 5.2 and 5.3 and Proposition 5.5. $\square$

Take an increasing sequence $\{n_j\} \subseteq \mathbb{N}_v$ so that (b) in 5.6 is satisfied. Define $t_l: \mathcal{A}_{1, n_l} \to \mathcal{A}_{1, n_{l+1}}$ as the $*$-homomorphism of the form

$$t_l(f) = \text{diag}[f \circ r_1, \ldots, f \circ r_m],$$

where $r_1, \ldots, r_m$ are as in the proof of Theorem 5.4. Then the diagram

$$\langle \mathcal{A}_{1, n_1}, \tau_1 \rangle \xrightarrow{t_1} \langle \mathcal{A}_{1, n_2}, \tau_2 \rangle \xrightarrow{t_2} \langle \mathcal{A}_{1, n_3}, \tau_3 \rangle \xrightarrow{t_3} \cdots$$

$$\langle \mathcal{B}_{n_1}, \text{tr} \rangle \xrightarrow{t_{\mathcal{B}}^{n_1}} \langle \mathcal{B}_{n_2}, \text{tr} \rangle \xrightarrow{t_{\mathcal{B}}^{n_2}} \langle \mathcal{B}_{n_3}, \text{tr} \rangle \xrightarrow{t_{\mathcal{B}}^{n_3}} \cdots$$
commutes, where $\mathbb{M}_n$ is canonically identified with the C*-subalgebra of constant functions on the interval $[0, 1]$. Since the upper inductive system satisfies the assumption of 5.6 and the limit of the lower inductive system is clearly dense in that of the upper one, it follows that the Fraïssé limit of $\mathcal{K}_v$ is isomorphic to the UHF algebra of type $v$ as C*-algebras (See [Dav96, Example III.5.1] for the definition).

We conclude this section by showing that all $L_{TC^*}$-embeddings into the Fraïssé limit of $\mathcal{K}_v$ is indeed $\mathcal{K}_v$-admissible, so that the gap explained in Remark 4.8 disappears in this case. To see this, we use the following lemmas [Dav96, Exercise II.8 and Lemma III.3.2].

**Lemma 5.7.** Let $f$ be a continuous function on a compact subset $X$ of $\mathbb{C}$. Then, for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $a$ and $b$ are normal elements of a C*-algebra $\mathcal{A}$ with $\|a - b\| < \delta$, then $\|f(a) - f(b)\| < \varepsilon$.

**Lemma 5.8.** For any $\varepsilon > 0$ and $n \in \mathbb{N}$, there exists $\delta > 0$ such that if $\mathcal{A}$ and $\mathcal{B}$ are C*-subalgebras of a unital C*-algebra $\mathcal{D}$, if $\dim \mathcal{A}$ is less than $n$, and if $\{e_{ij}^{(k)}\}$ is a system of matrix units which spans $\mathcal{A}$ and satisfies $d(e_{ij}^{(k)}; \mathcal{B}) < \delta$, then there exists a unitary $u$ in $\mathcal{D}$ with $\|u - 1\| < \varepsilon$ and $\text{Ad}(u)[\mathcal{A}] \subseteq \mathcal{B}$.

**Lemma 5.9.** Let $\{e_{ij}\}$ be the system of standard matrix units of $\mathbb{M}_n$ and $a$ be an element of $\mathbb{M}_m \otimes \mathbb{M}_n$ satisfying $\|a(1 \otimes e_{ij}) - (1 \otimes e_{ij})a\| < \varepsilon$. Then the inequality $\|a - (1 \otimes \text{tr})(a)\| < n^2 \varepsilon$ holds.

**Proof.** If $a$ is represented as $\sum a_{ij} \otimes e_{ij}$, then one can easily verify the inequality

$$\|a_{ij} \otimes e_{ij} - \delta_{ij} \sum_k a_{kk} \otimes e_{kk}/n\| < \varepsilon,$$

from which the conclusion follows. \qed

**Theorem 5.10.** Every $L$-embedding from an object of $\mathcal{K}_v$ into the Fraïssé limit of $\mathcal{K}_v$ is $\mathcal{K}_v$-admissible.

**Proof.** Let $\mathcal{M}$ be the Fraïssé limit of $\mathcal{K}_v$ and $\iota: \langle \mathcal{A}_{p,n}, \tau \rangle \to \mathcal{M}$ be an $L$-embedding. Our goal is to show that $\iota$ can be approximated by $\mathcal{K}_v$-embeddings with respect to the topology of pointwise convergence. For simplicity, we only show the case $p = 1$ and $\tau = \tau_1$. Set

$$G := \{1 \otimes e_{ij} \mid i,j = 1, \ldots, n\} \cup \{\text{id}_{[0,1]} \otimes 1\} \subseteq C[0, 1] \otimes \mathbb{M}_n \simeq \mathcal{A}_{1,n},$$

where $\{e_{ij}\}$ is the system of standard matrix units of $\mathbb{M}_n$, and note that $G$ is a generator of $\mathcal{A}_{1,n}$. Given $\varepsilon > 0$, it suffices to find a $\mathcal{K}_v$-embedding $\eta$ of $\langle \mathcal{A}_{1,n}, \tau_1 \rangle$ into $\mathcal{M}$ satisfying $\|\iota(g) - \eta(g)\| < \varepsilon$ for all $g \in G$. For this, take $N \in \mathbb{N}$ with $1/N < \varepsilon/6$ and $nN \in \mathbb{N}_v$. For $c, d \in \mathbb{N}$ with $0 \leq c < d \leq N$, define a continuous function $f_{c,d}$ on $[0, 1]$ by

$$f_{c,d}(t) := \begin{cases} 0 & (t \notin [(c - 1)/N, (d + 1)/N]) \\ 1 & (t \in [(c - 1)/N, (d + 1)/N]) \\ Nt - c + 1 & (t \in [(c - 1)/N, c/N]) \\ -Nt + d - 1 & (t \in [d/N, (d + 1)/N]). \end{cases}$$
Then by Lemma 5.7, there exists positive $\delta < \varepsilon/2$ such that if $a$ is a normal element of $\mathcal{M}$ with $\|a - \iota(id_{[0,1]} \otimes 1)\| < \delta$, then the inequality $\|f(c,d)(a) - \iota(f(c,d) \otimes 1)\| < 1/\sqrt{N}$ holds for all $c, d \in \mathbb{N}$ with $0 \leq c < d \leq N$. Take such $\delta$ and set $\delta' := \delta/(6n^2 + 1)$.

Let 

\[
\begin{align*}
\langle \mathcal{A}_{1,n_1}, \tau_1 \rangle &\xrightarrow{\iota_1} \langle \mathcal{A}_{1,n_2}, \tau_1 \rangle \xrightarrow{\iota_2} \langle \mathcal{A}_{1,n_3}, \tau_1 \rangle \xrightarrow{\iota_3} \cdots \\
\langle \mathcal{M}_{n_1}, \text{tr} \rangle &\xrightarrow{\iota_1} \langle \mathcal{M}_{n_2}, \text{tr} \rangle \xrightarrow{\iota_2} \langle \mathcal{M}_{n_3}, \text{tr} \rangle \xrightarrow{\iota_3} \cdots
\end{align*}
\]

be the inductive system we saw before Lemma 5.7. Then, by Lemma 5.8, there exists a unitary $u$ in $\mathcal{M}$ with $\|u - 1\| < \delta'$ and $e'_{ij} := u[\iota(1 \otimes e_{ij})]u^* \in \bigcup_k \mathcal{M}_{n_k}$. We shall denote by $\mathcal{B}$ the finite dimensional simple C*-subalgebra generated by $\{e'_{ij}\}$. Note that the inequality $\|\iota(1 \otimes e_{ij}) - e'_{ij}\| < 2\delta' \leq \varepsilon$ holds for all $i, j$. Also, if $\mathcal{B}$ is included in $\mathcal{M}_{n_k}$, then $\mathcal{M}_{n_k}$ is canonically isomorphic to $\mathcal{B} \otimes \mathcal{M}_{n_k/n}$. Now, take $a \in \bigcup_k \mathcal{M}_{n_k}$ with $\|a - \iota(id_{[0,1]} \otimes 1)\| < \delta'$. By Lemma 5.7, we may assume without loss of generality that $a$ is a positive element with $\|a\| \leq 1$. Then $\|ae'_{ij} - e'_{ij}a\| < 6\delta$, so by Lemma 5.9, there exists a positive element $a' \in \bigcup_k \mathcal{M}_{n_k}$ which commutes with every element of $\mathcal{B}$ and satisfies the inequalities $\|a' - \iota(id_{[0,1]} \otimes 1)\| < (6n^2 + 1)\delta' \leq \delta$ and $\|a'\| \leq 1$. By definition of $\delta$, we have $\|f(c,d)(a') - \iota(f(c,d) \otimes 1)\| < 1/\sqrt{\sqrt{N}}$ for $0 \leq c < d \leq N$.

Let $k_0$ be sufficiently large so that both $\mathcal{B}$ and $a'$ is included in $\mathcal{M}_{n_{k_0}}$ and $m := n_{k_0}/n$ is a multiple of $N$. Since the commutant $\mathcal{B}' \cap \mathcal{M}_{m_{k_0}}$ is canonically isomorphic to $\mathcal{M}_{m}$, the positive element $a'$ can be identified with a diagonal matrix of $\mathcal{M}_{m}$, say $\text{diag}(t_1, \ldots, t_m)$. Without loss of generality, we may assume $t_1 \leq \cdots \leq t_m$. Then we have

\[
\text{tr}(\text{diag}(f(c,d)(t_1), \ldots, f(c,d)(t_m))) = \text{tr}^\mathcal{M}(f(c,d)(a')) \geq \tau_1(f(c,d) \otimes 1) - 1/\sqrt{\sqrt{N}} = (d - c)/\sqrt{N}.
\]

This inequality together with Hall’s marriage theorem implies that the real numbers $t_{mc/N+1}, \ldots, t_{m(c+1)/N}$ are included in $[(c - 1)/N, (c + 2)/N]$. Consequently, the element

\[
a'' := \text{diag}(r_1, \ldots, r_m) \otimes 1 \in C([0,1], \mathcal{M}_{m}) \otimes \mathcal{B} \cong \mathcal{A}_{1,n_{k_0}}
\]

satisfies $\|a'' - a'\| < 3/\sqrt{N} < \varepsilon/2$, where $r_1, \ldots, r_m$ are as in the proof of Theorem 5.4. One can easily check that the $\mathcal{K}$-embedding $\eta: \mathcal{A}_{1,n} \to \mathcal{A}_{1,n_{k_0}}$ defined by

\[
\eta(1 \otimes e_{ij}) := e'_{ij}, \quad \eta(id_{[0,1]} \otimes 1) = a''
\]

has the desired property. □

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References

[Ben15] Itaï BenYaacov, Fraïssé Limits of Metric Structures. *Journal of Symbolic Logic* 80 (2015), no. 1, 100–115.

[Dav96] Kunneth R. Davidson, C*-Algebras by Example. Fields Institute Monographs, 6. American Mathematical Society, 1996.

[Eag16] C. J. Eagle, I. Farah, B. Hart, B. Kadets, V. Kalashnyk and M. Lupini, Fraïssé Limits of C*-Algebras. *Journal of Symbolic Logic* 81 (2016), no. 2, 755–773.

[Fra54] Roland Fraïssé, Sur l’extension aux relations de quelques propriétés des ordres. *Annales Scientifiques de l’École Normale Supérieure*, Sér. 3, 71 no. 4 (1954), 363–388.

[Hil91] David Hilbert, Ueber die stetige Abbildung einer Line auf ein Flächenstück. *Mathematische Annalen* 38 (1891), no. 3, 459–460.

[Kub13] Wiesław Kubis, Metric-enriched categories and approximate Fraïssé limits. arXiv: 1210.6506v3 (2013, preprint).

[Lin09] Huaxin Lin, Approximately diagonalizing matrices over C(Y). *Proceedings of the National Academy of Sciences of the United States of America* 109 (2012), no. 8, 2842–2847.

[Mas16] Shuhei Masumoto, The Jiang–Su algebra as a Fraïssé limit. To appear in *Journal of Symbolic Logic*.

[Sch07] Konstantinos Schoretsanitis, Fraïssé Theory for Metric Structures. PhD thesis, the Graduate College of the University of Illinois at Urbana-Champaign, 2007.

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