ON THIN-COMPLETE IDEALS OF SUBSETS OF GROUPS

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Abstract. Let $\mathcal{F} \subset \mathcal{P}_G$ be a left-invariant lower family of subsets of a group $G$. A subset $A \subset G$ is called $\mathcal{F}$-thin if for any distinct elements $x, y \in G$, $xA \cap yA \in \mathcal{F}$. The family of all $\mathcal{F}$-thin subsets of $G$ is denoted by $\tau(\mathcal{F})$. If $\tau(\mathcal{F}) = \mathcal{F}$, then $\mathcal{F}$ is called thin-complete. The thin-completion $\tau^*(\mathcal{F})$ of $\mathcal{F}$ is the smallest thin-complete subfamily of $\mathcal{P}_G$ that contains $\mathcal{F}$.

Answering questions of Lutsenko and Protasov, we prove that a set $A \subset G$ belongs to $\tau^*(G)$ if and only if for any sequence $(g_n)_{n \in \omega}$ of non-zero elements of $G$ there is $n \in \omega$ such that

$$\bigcap_{i_0, \ldots, i_n \in (0, 1)} g_0^{i_0} \cdots g_n^{i_n} A \in \mathcal{F}.$$ 

Also we prove that for an additive family $\mathcal{F} \subset \mathcal{P}_G$ its thin-completion $\tau^*(\mathcal{F})$ is additive. If the group $G$ is countable and torsion-free, then the completion $\tau^*(\mathcal{F}_G)$ of the ideal $\mathcal{F}_G$ of finite subsets of $G$ is coanalytic and not Borel in the power-set $\mathcal{P}_G$ endowed with the natural compact metrizable topology.

1. Introduction

This paper was motivated by problems posed by Ie. Lutsenko and I.V. Protasov in a preliminary version of the paper [5] devoted to relatively thin sets in groups.

Following [4], we say that a subset $A$ of a group $G$ is thin if for any distinct points $x, y \in G$ the intersection $xA \cap yA$ is finite. In [5] (following the approach of [1]) Lutsenko and Protasov generalized the notion of a thin set to that of $\mathcal{F}$-thin set where $\mathcal{F}$ is a family of subsets of $G$. By $\mathcal{P}_G$ we shall denote the Boolean algebra of all subsets of the group $G$.

We shall say that a family $\mathcal{F} \subset \mathcal{P}_G$ is

- left-invariant if $xF \in \mathcal{F}$ for all $F \in \mathcal{F}$ and $x \in G$, and
- additive if $A \cup B \in \mathcal{F}$ for all $A, B \in \mathcal{F}$;
- lower if $A \in \mathcal{F}$ for any $A \subset B \in \mathcal{F}$;
- an ideal if $\mathcal{F}$ is lower and additive.

Let $\mathcal{F} \subset \mathcal{P}_G$ be a left-invariant lower family of subsets of a group $G$. A subset $A \subset G$ is defined to be $\mathcal{F}$-thin if for any distinct points $x, y \in G$ we get $xA \cap yA \in \mathcal{F}$. The family of all $\mathcal{F}$-thin subsets of $G$ will be denoted by $\tau(\mathcal{F})$. It is clear that $\tau(\mathcal{F})$ is a left-invariant lower family of subsets of $G$ and $\mathcal{F} \subset \tau(\mathcal{F})$. If $\tau(\mathcal{F}) = \mathcal{F}$, then the family $\mathcal{F}$ will be called thin-complete.

Let $\tau^*(\mathcal{F})$ be the intersection of all thin-complete families $\tilde{\mathcal{F}} \subset \mathcal{P}_G$ that contain $\mathcal{F}$. It is clear that $\tau^*(\mathcal{F})$ is the smallest thin-complete family containing $\mathcal{F}$. This family is called the thin-completion of $\mathcal{F}$.

The family $\tau^*(\mathcal{F})$ has an interesting hierarchic structure that can be described as follows. Let $\tau^0(\mathcal{F}) = \mathcal{F}$ and for each ordinal $\alpha$ put $\tau^\alpha(\mathcal{F})$ be the family of all sets $A \subset G$ such that for any distinct points $x, y \in G$ we get $xA \cap yA \in \bigcup_{\beta < \alpha} \tau^\beta(\mathcal{F})$. So,

$$\tau^\alpha(\mathcal{F}) = \tau(\tau^{<\alpha}(\mathcal{F})) \quad \text{where} \quad \tau^{<\alpha}(\mathcal{F}) = \bigcup_{\beta < \alpha} \tau^\beta(\mathcal{F}).$$

By Proposition 3 of [5], $\tau^*(\mathcal{F}) = \bigcup_{\alpha < |G|} \tau^\alpha(\mathcal{F})$.

The following theorem (that will be proved in Section 3) answers the problem of combinatorial characterization of the family $\tau^*(\mathcal{F})$ posed by Ie. Lutsenko and I.V. Protasov. Below by $e$ we denote the neutral element of the group $G$.

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Theorem 1.1. Let $F \subset P_G$ be a left-invariant lower family of subsets of a group $G$. A subset $A \subset G$ belongs to the family $\tau^*(F)$ if and only if for any sequence $(g_n)_{n \in \omega} \in (G \setminus \{e\})^\omega$ there is a number $n \in \omega$ such that

$$\bigcap_{k_0, \ldots, k_n \in \{0,1\}} g_0^{k_0} \cdots g_n^{k_n} A \in F.$$ 

We recall that a family $F \subset P_G$ is called additive if $\{A \cup B : A, B \in F\} \subset F$. It is clear that the family $F_G$ of finite subsets of a group $G$ is additive. If $G$ is an infinite Boolean group, then the family $\tau^*(F_G) = \tau(F_G)$ is not additive, see Remark 3 in [5]. For torsion-free groups the situation is totally different. Let us recall that a group $G$ is torsion-free if each non-zero element of $G$ has infinite order.

Theorem 1.2. For a torsion-free group $G$ and a left-invariant ideal $F \subset P_G$ the family $\tau^{<\alpha}(F)$ is additive for any limit ordinal $\alpha$. In particular, the thin-completion $\tau^*(F)$ of $F$ is an ideal in $P_G$.

We define a subset of a group $G$ to be $s$-thin if it belongs to the thin-completion $\tau^*(F_G)$ of the family $F_G$ of all finite subsets of the group $G$. By Proposition 3 of [5], for each countable group $G$ we get $\tau^*(F_G) = \tau^{<\omega_1}(F_G)$.

It is natural to ask if the equality $\tau^*(F_G) = \tau^{<\alpha}(F_G)$ can happen for some cardinal $\alpha < \omega_1$. If the group $G$ is Boolean, then the answer is affirmative: $\tau^*(F) = \tau^1(F)$ according to Theorem 1 of [5]. The situation is different for non-torsion groups:

Theorem 1.3. If an infinite group $G$ contains an abelian torsion-free subgroup $H$ of cardinality $|H| = |G|$, then $\tau^*(F_G) \neq \tau^\alpha(F_G) \neq \tau^{<\alpha}(F_G)$ for each ordinal $\alpha < |G|^+$.

Theorems 1.2 and 1.3 will be proved in Sections 4 and 6, respectively. In Section 7 we shall study the Borel complexity of the family $\tau^*(F_G)$ for a countable group $G$. In this case the power-set $P_G$ carries a natural compact metrizable topology, so we can talk about topological properties of subsets of $P_G$.

Theorem 1.4. For a countable group $G$ and a countable ordinal $\alpha$ the subset $\tau^\alpha(F_G)$ of $P_G$ is Borel while $\tau^*(F_G) = \tau^{<\omega_1}(F_G)$ is coanalytic. If $G$ contains an element of infinite order, then the space $\tau^*(F_G)$ is coanalytic but not analytic.

2. Preliminaries on well-founded posets and trees

In this section we collect the neccessary information on well-founded posets and trees. A poset is an abbreviation from a partially ordered set. A poset $(X, \leq)$ is well-founded if each subset $A \subset X$ has a maximal element $a \in A$ (this means that each element $x \in A$ with $x \geq a$ is equal to $a$). In a well-founded poset $(X, \leq)$ to each point $x \in X$ we can assign the ordinal rank of the poset $X$ defined by the recursive formula:

$$\text{rank}_X(x) = \sup\{\text{rank}_X(y) + 1 : y > x\}$$

where $\sup \emptyset = 0$. Thus maximal elements of $X$ have rank 0, their immediate predecessors 1, and so on. If $X$ is not empty, then the ordinal rank $(X) = \sup\{\text{rank}_X(x) + 1 : x \in X\}$ is called the rank of the poset $X$. In particular, a one-element poset has rank 1. If $X$ is empty, then we put $\text{rank}(X) = 0$.

A tree is a poset $(T, \leq)$ with the smallest element $\theta_T$ such that for each $t \in T$ the lower set $\downarrow t = \{s \in T : s \leq t\}$ is well-ordered in the sense that each subset $A \subset \downarrow t$ has the smallest element. A branch of a tree $T$ is any maximal linearly ordered subset of $T$. If a tree is well-founded, then all its branches are finite.

A subset $S \subset T$ of a tree is called a subtree if it is a tree with respect to the induced partial order. A subtree $S \subset T$ is lower if $S = \downarrow S = \{t \in T : \exists s \in S \ t \leq s\}$.

All trees that appear in this paper are (lower) subtrees of the tree $X^{<\omega} = \bigcup_{n \in \omega} X^n$ of finite sequences of a set $X$. The tree $X^{<\omega}$ carries the following partial order:

$$(x_0, \ldots, x_n) \preceq (y_0, \ldots, y_m)$$

iff $n \leq m$ and $x_i = y_i$ for all $i \leq n$.

The empty sequence $\epsilon \in X^0$ is the smallest element of the tree $X^{<\omega}$. For a finite sequence $s = (x_0, \ldots, x_n) \in X^{<\omega}$ and an element $x \in X$ by $s \cdot x = (x_0, \ldots, x_n, x)$ we denote the concatenation of $s$ and $x$. So, $s \cdot x$ is one of $|X|$ many immediate successors of $s$. The set of all branches of $X^{<\omega}$ can be naturally identified with the countable power $X^\omega$. For each branch $s = (s_n)_{n \in \omega} \in X^\omega$ and $n \in \omega$ by $s | n = (s_0, \ldots, s_{n-1})$ we denote the initial interval of length $n$.

Let $\text{Tr} \subset P_X^{<\omega}$ denote the family of all lower subtrees of the tree $X^{<\omega}$ and $WF \subset \text{Tr}$ be the subset consisting of all well-founded lower subtrees of $X^{<\omega}$.

In Section 7 we shall exploit some deep facts about the descriptive properties of the sets $WF \subset \text{Tr} \subset P_X^{<\omega}$ for a countable set $X$. In this case the tree $X^{<\omega}$ is countable and the power-set $P_X^{<\omega}$ carries a natural compact
metrizable topology of the Tychonov power $2^{X^{<\omega}}$. So, we can speak about topological properties of the subsets $WF$ and $Tr$ of the compact metrizable space $P_{X^{<\omega}}$.

We recall that a topological space $X$ is **Polish** if $X$ is homeomorphic to a separable complete metric space. A subset $A$ of a Polish space $X$ is called

- **Borel** if $A$ belongs to the smallest $\sigma$-algebra that contains all open subsets of $X$;
- **analytic** if $A$ is the image of a Polish space $P$ under a continuous map $f : P \to A$;
- **coanalytic** if $X \setminus A$ is analytic.

By Souslin’s Theorem 14.11 [9], a subset of a Polish space is Borel if and only if it is both analytic and coanalytic. By $\Sigma^1_1$ and $\Pi^1_1$ we denote the classes of spaces homeomorphic to analytic and coanalytic subsets of Polish spaces, respectively.

A coanalytic subset $X$ of a compact metric space $K$ is called **$\Pi^1_1$-complete** if for each coanalytic set $C$ of the Cantor cube $2^\omega$ there is a continuous map $f : 2^\omega \to K$ such that $f^{-1}(X) = C$. It follows from the existence of a coanalytic non-Borel set in $2^\omega$ that each $\Pi^1_1$-complete set $X \subset K$ is non-Borel.

The following deep theorem is classical and belongs to Lusin, see [3] 32.B and 35.23.

**Theorem 2.1.** Let $X$ be a countable set.

1. The subspace $Tr$ is closed (and hence compact) in $P_{X^{<\omega}}$.
2. The set of well-founded trees $WF$ is $\Pi^1_1$-complete in $Tr$. In particular, $WF$ is coanalytic but not analytic (and hence not Borel).
3. For each ordinal $\alpha < \omega_1$ the subset $WF_\alpha = \{ T \in WF : \text{rank}(T) \leq \alpha \}$ is Borel in $Tr$.
4. Each analytic subspace of $WF$ lies in $WF_\alpha$ for some ordinal $\alpha < \omega_1$.

### 3. Combinatorial characterization of $\ast$-thin subsets

In this section we prove Theorem 1.1. Let $F \subset P_G$ be a left-invariant lower family of subsets of a group $G$. Theorem 1.1 trivially holds if $F = P_G$ (which happens if and only if $G \in F$). So, it remains to consider the case $G \notin F$. Let $e$ be the neutral element of $G$ and $G_o = G \setminus \{ e \}$. We shall work with the tree $G_o^{<\omega}$ discussed in the preceding section.

Let $A$ be any subset of $G$. To each finite sequence $s \in G_o^{<\omega}$ assign the set $A_s \subset G$, defined by induction: $A_\emptyset = A$ and $A_{s \cdot x} = A_s \cap x A_s$ and for $s \in G_o^{<\omega}$ and $x \in G_o$. Repeating the inductive argument of the proof of Proposition 2 [5], we can obtain the following direct description of the sets $A_s$:

**Claim 3.1.** For every sequence $s = (g_0, \ldots, g_n) \in G_o^{<\omega}$

$$A_s = \bigcap_{k_0, \ldots, k_n \in \{0,1\}} g_0^{k_0} \cdots g_n^{k_n} A.$$

The set

$$T_A = \{ s \in G_o^{<\omega} : A_s \notin F \}$$

is a subtree of $G_o^{<\omega}$ called the $\tau$-tree of the set $A$.

For a non-zero ordinal $\alpha$ let $-1 + \alpha$ be a unique ordinal $\beta$ such that $1 + \beta = \alpha$. For $\alpha = 0$ we put $-1 + \alpha = -1$. It follows that $-1 + \alpha = \alpha$ for each infinite ordinal $\alpha$.

**Theorem 3.2.** A set $A \subset G$ belongs to the family $\tau^\alpha(F)$ for some ordinal $\alpha$ if and only if its $\tau$-tree $T_A$ is well-founded and has rank($T_A$) $\leq -1 + \alpha + 1$.

**Proof.** By induction on $\alpha$. Observe that $A \in \tau^0(F) = F$ if and only if $T_A = \emptyset$ if and only if rank($T_A$) $= 0 = 1 + 0 + 1$. So, Theorem holds for $\alpha = 0$.

Assume that for some ordinal $\alpha > 0$ and any ordinal $\beta < \alpha$ we know that a set $A \subset G$ belongs to $\tau^\beta(G)$ if and only if its $\tau$-tree $T_A$ is well-founded and has rank($T_A$) $\leq -1 + \beta + 1$. Given a subset $A \subset G$ we should check that $A \in \tau^\alpha(F)$ if and only if its $\tau$-tree $T_A$ is well-founded and has rank($T_A$) $\leq -1 + \alpha + 1$.

First assume that $A \in \tau^\alpha(F)$. Then for every $x \in G_o$ the set $A \cap x A$ belongs to $\tau^{\beta_x}(F) \subset \tau^{<\alpha}(F)$ for some ordinal $\beta_x < \alpha$. By the inductive assumption, the $\tau$-tree $T_{A \cap x A}$ is well-founded and has rank($T_{A \cap x A}$) $\leq -1 + \beta_x + 1$.

If $A \notin \tau(F)$, then $T_A \subset \{ s_0 \}$ and rank($T_A$) $\leq 1 \leq -1 + \alpha + 1$. So, we can assume that $A \notin \tau(F)$. In this case each point $x \in G_o = G_o^1$ considered as the sequence $(x) \in G^1$ of length 1 belongs to the $\tau$-tree $T_A$ of the set $A$. So we can consider the upper set $T_A(x) = \{ s \in T_A : s \geq x \}$ and observe that the subtree $T_A(x)$ of $T_A$...
is isomorphic to the $\tau$-tree $T_{A\cap xA}$ of the set $A \cap xA$ and hence $\operatorname{rank}(T_A(x)) = \operatorname{rank}(T_{A\cap xA}) \leq -1 + \beta_x + 1$. It follows that

$$\operatorname{rank}(T_A) = \operatorname{rank}T_A(s_0) + 1 = \left( \sup_{x \in G_0} \operatorname{rank}T_A(x) \right) + 1 = \left( \sup_{x \in G_0} \operatorname{rank}T_A(x) \right) + 1 \leq \left( \sup_{x \in G_0} (-1 + \beta_x + 1) \right) + 1 \leq -1 + \alpha + 1.$$ 

Now assume conversely that the $\tau$-tree $T_A$ of $A$ is well-founded and has $\operatorname{rank}(T_A) \leq -1 + \alpha + 1$. For each $x \in G_0$, find a unique ordinal $\beta_x$ such that $-1 + \beta_x = \operatorname{rank}T_A(x)$. It follows from

$$-1 + \beta_x + 2 = \operatorname{rank}T_A(x) + 2 \leq \operatorname{rank}T_A(s_0) + 1 = \operatorname{rank}(T_A) \leq -1 + \alpha + 1$$

that $\beta_x < \alpha$. Since the subtree $T_A(x) = T_A \cap x$ is isomorphic to the $\tau$-tree $T_{A \cap xA}$ of the set $A \cap xA$, we conclude that $T_{A \cap xA}$ is well-founded and has $\operatorname{rank}(T_{A \cap xA}) = \operatorname{rank}(T_A(x)) = \operatorname{rank}T_A(x) + 1 = -1 + \beta_x + 1$. Then the inductive assumption guarantees that $A \cap xA \in \tau^{\beta_x}(F) \subset \tau^{<\alpha}(F)$ and hence $A \in \tau^{\alpha}(F)$ by the definition of the family $\tau^{\alpha}(F)$. \qed

As a corollary of Theorem 3.2, we obtain the following characterization proved in [5]:

**Corollary 3.3.** A subset $A \subset G$ belongs to the family $\tau^n(F)$ for some $n \in \omega$ if and only if for each sequence $(g_i)_{i=0}^n \in G_0^{n+1}$ we get

$$\bigcap_{k_0, \ldots, k_n \in \{0, 1\}} g_k^{k_0} \cdots g_n^{k_n} A \subset F.$$

Theorem 3.2 also implies the following explicit description of the family $\tau^*(F)$, which was announced in Theorem 1.1.

**Corollary 3.4.** For a subset $A \subset G$ the following conditions are equivalent:

1. $A \in \tau^*(F)$;
2. the $\tau$-tree $T_A$ of $A$ is well-founded;
3. for each sequence $(g_i)_{i \in \omega} \in G_\omega$ there is $i \in \omega$ such that $(g_0, \ldots, g_i) \notin T_A$;
4. for each sequence $(g_i)_{i \in \omega} \in G_\omega$ there is $i \in \omega$ such that

$$\bigcap_{k_0, \ldots, k_n \in \{0, 1\}} g_k^{k_0} \cdots g_n^{k_n} A \subset F.$$

4. The additivity of the families $\tau^{<\alpha}(F)$

In this section we shall prove Theorem 1.2. Let $G$ be an infinite group and $e$ be the neutral element of $G$.

For a natural number $m$ let $2^m$ denote the finite cube $\{0, 1\}^m$. For vectors $g = (g_1, \ldots, g_m) \in (G \setminus \{e\})^m$ and $x = (x_1, \ldots, x_m) \in 2^m$ let

$$g^x = g_1^{x_1} \cdots g_m^{x_m} \in G.$$

A function $f : 2^m \to G$ to a group $G$ will be called cubic if there is a vector $g = (g_1, \ldots, g_m) \in (G \setminus \{e\})^m$ such that $f(x) = g^x$ for all $x \in 2^m$.

**Lemma 4.1.** If the group $G$ is torsion-free, then for every $n \in \mathbb{N}$, $m > (n-1)^2$, and a cubic function $f : 2^m \to G$ we get $|f(2^m)| > n$.

**Proof.** Assume conversely that $|f(2^m)| \leq n$. Consider the set $B = \{(k_1, \ldots, k_m) \in 2^m : \sum_{i=1}^m k_i = 1\}$ having cardinality $|B| = m > (n-1)^2$. Since $e \notin f(B)$, we conclude that $|f(B)| \leq |f(2^m)| - 1 \leq n - 1$ and hence $|f^{-1}(y)| \geq n$ for some $y \in f(B)$. Let $B_y = f^{-1}(y)$ and observe that $f(2^m) \supset \{e, y, y^2, \ldots, y^{B_y}\}$ and thus $|f(2^m)| \geq |B_y| + 1 \geq n + 1$, which contradicts our assumption. \qed

For every $n \in \mathbb{N}$ let $c(n)$ be the smallest number $m \in \mathbb{N}$ such that for each cubic function $f : 2^m \to G$ we get $|f(2^m)| > n$. It is easy to see that $c(n) \geq n$. On the other hand, Lemma 4.1 implies that $c(n) \leq (n-1)^2 + 1$ if $G$ is torsion-free.

For a family $F$ and a natural number $n \in \mathbb{N}$, let

$$\bigvee_n F = \{\cup A : A \subset F, |A| \leq n\}.$$
Lemma 4.2. Let $\mathcal{F} \subset \mathcal{P}_G$ be a left-invariant lower family of subsets in a torsion-free group $G$. For every $n \in \mathbb{N}$ we get
\[
\bigvee_{n} \tau(\mathcal{F}) \subset \tau^{c(n)-1}(\bigvee_{m} \mathcal{F})
\]
where $m = n^{2c(n)}$.

**Proof.** Fix any $A \in \bigvee \tau(\mathcal{F})$ and write it as the union $A = A_1 \cup \cdots \cup A_n$ of sets $A_1, \ldots, A_n \in \tau(\mathcal{F})$. The inclusion $A \in \tau^{c(n)-1}(\bigvee_{m} \mathcal{F})$ will follow from Corollary 3.3 as soon as we check that
\[
\bigcap_{x \in 2^{c(n)}} g^x A \in \bigvee_{m} \mathcal{F}
\]
for each vector $g \in (G \setminus \{e\})^{c(n)}$. De Morgan’s law guarantees that
\[
\bigcap_{x \in 2^{c(n)}} g^x \left( \bigcup_{i=1}^{n} A_i \right) = \bigcup_{f \in n^{2c(n)}} g^x A_{f(x)}.
\]
So, the proof will be complete as soon as we check that for every function $f : 2^{c(n)} \rightarrow n$ the set $\bigcap_{x \in 2^{c(n)}} g^x A_{f(x)}$ belongs to $\mathcal{F}$. The vector $g \in (G \setminus \{e\})^{c(n)}$ induces the cube function $g : 2^{c(n)} \rightarrow G$, $g : x \mapsto g^x$. The definition of the function $c(n)$ guarantees that $|g(2^{c(n)})| > n$. The function $f : 2^{c(n)} \rightarrow n$ can be thought as a coloring of the cube $2^{c(n)}$ into $n$ colors. Since $|g(2^{c(n)})| > n$, there are two points $y, z \in 2^{c(n)}$ colored by the same color such that $g(y) \neq g(z)$. Then $g^y = g(y) \neq g(z) = g^z$ but $f(y) = f(z) = k$ for some $k \leq n$. Consequently,
\[
\bigcap_{x \in 2^{c(n)}} g^x A_{f(x)} \subset g^y A_k \cap g^z A_k \in \mathcal{F}
\]
because the set $A_k \in \tau(\mathcal{F})$.

Now consider the function $c : \mathbb{N} \times \omega \rightarrow \omega$ defined recursively as $c(n, 0) = 0$ for all $n \in \mathbb{N}$ and $c(n, k + 1) = c(n, k) + c(n^{2c(n)}, k)$ for $(n, k) \in \mathbb{N} \times \omega$. Observe that $c(n, 1) = c(n) - 1$ for all $n \in \mathbb{N}$.

**Lemma 4.3.** If the group $G$ is torsion-free and $\mathcal{F} \subset \mathcal{P}_G$ is a left-invariant ideal, then
\[
\bigvee_{n} \tau^{k}(\mathcal{F}) \subset \tau^{c(n,k)}(\mathcal{F})
\]
for all pairs $(n, k) \in \mathbb{N} \times \omega$.

**Proof.** By induction on $k$. For $k = 0$ the equality $\bigvee_{n} \tau^{0}(\mathcal{F}) = \mathcal{F} = \tau^{c(n,0)}(\mathcal{F})$ holds because $\mathcal{F}$ is additive. Assume that Lemma is true for some $k \in \omega$. By Lemma 4.2 and by the inductive assumption, for every $n \in \mathbb{N}$ we get
\[
\bigvee_{n} \tau^{k+1}(\mathcal{F}) \subset \bigvee_{n} \tau(\tau^{k}(\mathcal{F})) \subset \tau^{c(n)-1}(\bigvee_{n^{2c(n)}} \tau^{k}(\mathcal{F})) \subset \tau^{c(n)-1}(\tau^{c(n^{2c(n)}, k)}(\mathcal{F})) = \tau^{c(n)-1+c(n^{2c(n)}, k)}(\mathcal{F}) = \tau^{c(n,k+1)}(\mathcal{F}).
\]

Now we are able to present:

**Proof of Theorem 4.4.** Assume that $G$ is a torsion-free group $G$ and $\mathcal{F} \subset \mathcal{P}_G$ is a left-invariant ideal. By transfinite induction we shall prove that for each limit ordinal $\alpha$ the family $\tau^{<\alpha}(\mathcal{F})$ is additive. For the smallest limit ordinal $\alpha = 0$ the additivity of the family $\tau^{0}(\mathcal{F}) = \mathcal{F}$ is included into the hypothesis. Assume that for some non-zero limit ordinal $\alpha$ we have proved that the families $\tau^{<\beta}(\mathcal{F})$ are additive for all limit ordinals $\beta < \alpha$. Two cases are possible:

1) $\alpha = \beta + \omega$ for some limit ordinal $\beta$. By the inductive assumption, the family $\tau^{<\beta}(\mathcal{F})$ is additive. Then Lemma 4.3 implies that the family $\tau^{<\alpha}(\mathcal{F}) = \tau^{<\beta}(\tau^{<\beta}(\mathcal{F}))$ is additive.

2) $\alpha = \sup B$ for some family $B \not\subseteq \alpha$ of limit ordinals. By the inductive assumption for each limit ordinal $\beta \in B$ the family $\tau^{<\beta}(\mathcal{F})$ is additive and then the union $\tau^{<\alpha}(\mathcal{F}) = \bigcup_{\beta \in B} \tau^{<\beta}(\mathcal{F})$ is additive.

\[
\tau^{<\alpha}(\mathcal{F}) = \bigcup_{\beta \in B} \tau^{<\beta}(\mathcal{F})
\]
is additive too.

This completes the proof of the additivity of the families $\tau^{<\alpha}(F)$ for all limit ordinals $\alpha$. Since the torsion-free group $G$ is infinite, the ordinal $\alpha = |G|^+$ is limit and hence the family $\tau^*(F) = \tau^{<\alpha}(F)$ is additive. Being left-invariant and lower, the family $\tau^*(F)$ is a left-invariant ideal in $P_G$. \hfill \qed

Remark 4.4. Theorem 1(2) of [5] is not true for an infinite Boolean group $G$. In this case Theorem 1(2) of [5] implies that $\tau^*(F_G) = \tau(F_G)$. Then for any infinite thin subset $A \subset G$ and any $x \in G \setminus \{e\}$ the union $A \cup xA$ is not thin as $(A \cup xA) \cap x(A \cup xA) = A \cup xA$ is infinite. Consequently, the family $\tau^*(F_G) = \tau(F_G)$ is not additive.

5. $h$-Invariant families of subsets in groups

Let $G$ be a group and $h : H \to K$ be an isomorphism between subgroups of $G$. A family $F$ of subsets of $G$ is called $h$-invariant if a subset $A \subset H$ belongs to $F$ if and only if $h(A) \in F$.

Example 5.1. The ideal $F_{\mathbb{Z}}$ of finite subsets of the group $\mathbb{Z}$ is $h$-invariant for each isomorphism $h_k : \mathbb{Z} \to k\mathbb{Z}$, $h : x \mapsto kx$, where $k \in \mathbb{N}$.

Proposition 5.2. Let $h : H \to K$ be an isomorphism between subgroups of a group $G$. For any $h$-invariant family $F \subset P_G$ and any ordinal $\alpha$ the family $\tau^\alpha(F)$ is $h$-invariant.

Proof. If $\alpha = 0$ the $h$-invariance of $\tau^0(F) = F$ follows from our assumption. Assume that for some ordinal $\alpha$ we have established that the families $\tau^\beta(F)$ are $h$-invariant for all ordinals $\beta < \alpha$. Then the union $\tau^{<\alpha}(F) = \bigcup_{\beta < \alpha} \tau^\beta(F)$ is also $h$-invariant.

We shall prove that the family $\tau^\alpha(F)$ is $h$-invariant. Given a set $A \subset H$ we need to prove that $A \in \tau^\alpha(F)$ if and only if $h(A) \in \tau^\alpha(F)$.

Assume first that $A \in \tau^\alpha(F)$. To show that $h(A) \in \tau^\alpha(F)$, take any element $y \in G \setminus \{e\}$. If $y \notin K$, then $h(A) \cap yh(A) = \emptyset \in \tau^{<\alpha}(F)$. If $y \in K$, then $y = h(x)$ for some $x \in H$ and then $h(A) \cap yh(A) = h(A \cap xA) \in \tau^{<\alpha}(F)$ since $A \cap xA \in \tau^{<\alpha}(F)$ and the family $\tau^{<\alpha}(F)$ is $h$-invariant.

Now assume that $A \notin \tau^\alpha(F)$. Then there is an element $x \in G \setminus \{e\}$ such that $A \cap xA \notin \tau^{<\alpha}(F)$. Since $A \subset H$, the element $x$ must belong to $H$ (otherwise $A \cap xA = \emptyset \in \tau^{<\alpha}(F)$). Then for the element $y = h(x)$ we get $h(A) \cap yh(A) \notin \tau^{<\alpha}(F)$ by the $h$-invariance of the family $\tau^{<\alpha}(F)$. Consequently, $h(A) \notin \tau^\alpha(F)$.

Corollary 5.3. Let $h : H \to K$ be an isomorphism between subgroups of a group $G$. For any $h$-invariant family $F \subset P_G$ the family $\tau^*(F)$ is $h$-invariant.

Definition 5.4. A left-invariant family $F \subset P_G$ of subsets of a group $G$ is called

- auto-invariant if $F$ is $h$-invariant for each injective homomorphism $h : G \to G$;
- sub-invariant if $F$ is $h$-invariant for each isomorphism $h : H \to K$ between subgroups $K \subset H$ of $G$;
- strongly invariant if $F$ is $h$-invariant for each isomorphism $h : H \to K$ between subgroups of $G$.

It is clear that

- strongly invariant $\Rightarrow$ sub-invariant $\Rightarrow$ auto-invariant

Remark 5.5. Each auto-invariant family $F \subset P_G$, being left-invariant is also right-invariant.

Proposition 6.2 implies:

Corollary 5.6. If $F \subset P_G$ is an auto-invariant (sub-invariant, strongly invariant) family of subsets of a group $G$, then so are the families $\tau^*(F)$ and $\tau^\alpha(F)$ for all ordinals $\alpha$.

It is clear that the family $F_G$ of finite subsets of a group $G$ is strongly invariant. Now we present some natural examples of families, which are not strongly invariant. Following [2], we call a subset $A$ of a group $G$

- large if there is a finite subset $F \subset G$ with $G = FA$;
- small if for any large set $L \subset G$ the set $L \setminus A$ remains large.

It follows that the family $S_G$ of small subsets of $G$ is a left-invariant ideal in $P_G$. According to [2], a subset $A \subset G$ is small if and only if for every finite subset $F \subset G$ the complement $G \setminus FA$ is large. We shall need the following (probably known) fact.

Lemma 5.7. Let $H$ be a subgroup of finite index in a group $G$. A subset $A \subset H$ is small in $H$ if and only if $A$ is small in $G$. 

**Proof.** First assume that $A$ is small in $G$. To show that $A$ is small in $H$, take any large subset $L \subset H$. Since $H$ has finite index in $G$, the set $L$ is large in $G$. Since $A$ is small in $G$, the complement $L \setminus A$ is large in $G$. Consequently, there is a finite subset $F \subset G$ such that $F \setminus (L \setminus A) = G$. Then for the finite set $F_H = F \cap H$, we get $F_H(L \setminus A) = H$, which means that $L \setminus A$ is large in $H$.

Now assume that $A$ is small in $H$. To show that $A$ is small in $G$, it suffices to show that for every finite subset $F \subset G$ the complement $G \setminus FA$ is large in $G$. Observe that $(G \setminus FA) \cap H = H \setminus F_HA$ where $F_H = F \cap H$. Since $A$ is small in $H$, the set $H \setminus F_HA$ is large in $H$ and hence large in $G$ (as $H$ has finite index in $G$). Then the set $G \setminus FA \supset H \setminus F_HA$ is large in $G$ too. □

**Proposition 5.8.** Let $G$ be an infinite abelian group.

1. If $G$ is finitely generated, then the ideal $SG$ is strongly invariant.
2. If $G$ is infinitely generated free abelian group, then the ideal $SG$ is not auto-invariant.

**Proof.** 1. Assume that $G$ is a finitely generated abelian group. To show that $SG$ is strongly invariant, fix any isomorphism $h : H \to K$ between subgroups of $G$ and let $A \subset H$ be any subset. The groups $H, K$ are isomorphic and hence have the same free rank $r_0(H) = r_0(K)$. If $r_0(H) = r_0(K) < r_0(G)$, then the subgroups $H, K$ have infinite index in $G$ and hence are small. In this case the inclusions $A \in SG$ and $h(A) \in SG$ hold and so are equivalent.

If the free ranks $r_0(H) = r_0(K)$ and $r_0(G)$ coincide, then $H$ and $K$ are subgroups of finite index in the finitely generated group $G$. By Lemma 5.7, a subset $A \subset H$ is small in $G$ if and only if $A$ is small in $H$ if and only if $h(A)$ is small in the group $h(H) = K$ if and only if $h(A)$ is small in $G$.

2. Now assume that $G$ is an infinitely generated free abelian group. Then $G$ is isomorphic to the direct sum $\oplus \mathbb{Z}$ of $\kappa = |G| \geq \aleph_0$ many copies of the infinite cyclic group $\mathbb{Z}$. Take any subset $\lambda \subset \kappa$ with infinite complement $\kappa \setminus \lambda$ and cardinality $|\lambda| = |\kappa|$ and fix an isomorphism $h : H \to G$ of the group $G = \oplus \mathbb{Z}$ onto its subgroup $H = \oplus \mathbb{Z}$. The subgroup $H$ has infinite index in $G$ and hence is small in $G$. Yet $h^{-1}(H) = G$ is not small in $G$, witnessing that the ideal $SG$ of small subsets of $G$ is not auto-invariant. □

### 6. Thin-completeness of the families $\tau^\alpha(F)$

In this section we shall prove that in general the families $\tau^\alpha(F)$ are not thin-complete. Our principal result is the following theorem that implies Theorem 1.3 announced in the Introduction.

**Theorem 6.1.** Let $G$ be a group containing a free abelian subgroup $H$ of cardinality $|H| = |G|$. If $F$ is a sub-invariant ideal of subsets of $G$ such that $\tau(F) \cap \mathcal{P}_H \not\in F$, then $\tau^*(F) \neq \tau^\alpha(F) \neq \tau^{<\alpha}(F)$ for all ordinals $\alpha < |G|^+$. We divide the proof of this theorem in a series of lemmas.

**Lemma 6.2.** Let $h : H \to K$ be an isomorphism between subgroups of a group $G$, $F$ be an $h$-invariant left-invariant lower family of subsets of $G$. If a subset $A \subset H$ does not belong to $\tau^\alpha(F)$ for some ordinal $\alpha$, then for every point $z \in G \setminus \{e\}$ the set $h(A) \cup zh(A) \notin \tau^{\alpha+1}(F)$.

**Proof.** Proposition 5.2 implies that $h(A) \notin \tau^\alpha(F)$. Since

$$(h(A) \cup zh(A)) \cap z^{-1}(h(A) \cup zh(A)) \supset h(A) \notin \tau^\alpha(F),$$

the set $h(A) \cup zh(A) \notin \tau^{\alpha+1}(F)$ by the definition of $\tau^{\alpha+1}(F)$. □

In the following lemma for a subgroup $K$ of a group $H$ by

$$Z_H(K) = \{ z \in H : \forall x \in K \ z x z = xz \}$$

we denote the centralizer of $K$ in $H$.

**Lemma 6.3.** Let $h : H \to K$ be an isomorphism between subgroups $K \subset H$ of a group $G$ such that there is a point $z \in Z_H(K)$ with $z^2 \notin K$. Let $F \subset \mathcal{P}_H$ be an $h$-invariant left-invariant ideal. If a subset $A \subset H$ belongs to the family $\tau^\alpha(F)$ for some ordinal $\alpha$, then $h(A) \cup zh(A) \in \tau^{\alpha+1}(F)$.

**Proof.** By induction on $\alpha$. For $\alpha = 0$ and $A \in F$ the inclusion $h(A) \cup zh(A) \in F \subset \tau^0(F)$ follows from the $h$-invariance and the additivity of $F$.

Now assume that for some ordinal $\alpha$ we have proved that for every $\beta < \alpha$ and $A \in \mathcal{P}_H \cap \tau^\beta(F)$ the set $h(A) \cup zh(A)$ belongs to $\tau^{\beta+1}(F)$. Given any set $A \in \mathcal{P}_H \cap \tau^\alpha(F)$, we need to prove that $h(A) \cup zh(A) \in \tau^{\alpha+1}(F)$. This will follow as soon as we check that $(h(A) \cup zh(A)) \cap y(h(A) \cup zh(A)) \in \tau^\alpha(F)$ for every $y \in G \setminus \{e\}$. 

If $y \notin K \cup zK \cup z^{-1}K$, then  
\[(h(A) \cup zh(A)) \cap y(h(A) \cup zh(A)) \subseteq (K \cup zK) \cap y(K \cup zK) = \emptyset \in \tau^{\alpha+1}(F).\]

So, it remains to consider the case $y \in K \cup zK \cup z^{-1}K \subseteq H$. If $y \in K$, then  
\[(h(A) \cup zh(A)) \cap y(h(A) \cup zh(A)) = (h(A) \cap yh(A)) \cup z(h(A) \cap yh(A)).\]

Since $y \in K$, there is an element $x \in H$ with $y = h(x)$. Since $A \in \tau^{\alpha}(F)$, $A \cap xA \in \tau^{\beta}(F)$ for some $\beta < \alpha$ and then  
\[(h(A) \cup zh(A)) \cap y(h(A) \cup zh(A)) = h(A \cap xA) \cup zh(A \cap xA) \in \tau^{\beta+1}(F) \subseteq \tau^{\alpha}(F)\]

by the inductive assumption. If $y \in zK$, then $z^{2} \notin K$ implies that  
\[(h(A) \cup zh(A)) \cap y(h(A) \cup zh(A)) = zh(A) \cap yh(A) \subseteq h(A) \in \tau^{\alpha}(F)\]

by the $h$-invariance and the left-invariance of the family $\tau^{\alpha}(F)$, see Proposition [5.2].

If $y \in z^{-1}K$, then by the same reason,  
\[(h(A) \cup zh(A)) \cap y(h(A) \cup zh(A)) = h(A) \cap yzh(A) \subseteq h(A) \in \tau^{\alpha}(F).\]

\[\Box\]

Given an isomorphism $h : H \to K$ between subgroups $K \subseteq H$ of a group $G$, for every $n \in \mathbb{N}$ define the iteration $h^{n} : H \to K$ of the isomorphism $h$ letting $h^{1} = h : H \to K$ and $h^{n+1} = h \circ h^{n}$ for $n \geq 1$.

The isomorphism $h : H \to K$ will be called expanding if $\bigcap_{n \in \mathbb{N}} h^{n}(H) = \{e\}$.

**Example 6.4.** For every integer $k \geq 2$ the isomorphism  
\[h_{k} : \mathbb{Z} \to k\mathbb{Z}, \quad h_{k} : x \mapsto kx,\]

is expanding.

**Lemma 6.5.** Let $h : H \to K$ be an expanding isomorphism between torsion-free subgroups $K \subseteq H$ of a group $G$ and $F \subseteq P_{G}$ be an $h$-invariant left-invariant ideal of subsets of $G$. For any limit ordinal $\alpha$ and family $\{A_{\alpha}\}_{\alpha < \omega} \subseteq \tau^{<\alpha}(F)$ of subsets of the group $H$, the union $A = \bigcup_{\alpha \in \omega} h^{\alpha}(A_{\alpha})$ belongs to the family $\tau^{\alpha}(F)$.

**Proof.** First observe that $\{h^{\alpha}(A_{\alpha})\}_{\alpha \in \omega} \subseteq \tau^{<\alpha}(F)$ by Proposition [5.2]. To show that $A = \bigcup_{\alpha \in \omega} h^{\alpha}(A_{\alpha}) \in \tau^{\alpha}(F)$ we need to check that $A \subseteq \tau^{<\alpha}(F)$ for all $x \in G \setminus \{e\}$. This is trivially true if $x \notin H$ as $A \subseteq H$. So, we assume that $x \in H$. By the expanding property of the isomorphism $h$, there is a number $\ell \in \omega$ such that $x \notin h^{\ell}(H)$. Put $B = \bigcup_{\ell = \ell}^{\infty} h^{\ell}(A_{\alpha})$ and observe that $A \subseteq B \cup xB \subseteq \tau^{<\alpha}(F)$ as $\tau^{<\alpha}(F)$ is additive according to Theorem [1.2].

\[\Box\]

**Lemma 6.6.** Assume that a left-invariant ideal $F$ on a group $G$ is $h$-invariant for some expanding isomorphism $h : H \to K$ between torsion-free subgroups $K \subseteq H$ of $G$ such that $Z_{H}(K) \not\subseteq K$. If $\tau(F) \cap P_{H} \subseteq F$, then $\tau^{\alpha}(F) \neq \tau^{<\alpha}(F)$ for all ordinals $\alpha < \omega_{1}$.

**Proof.** Fix any point $z \in Z_{K}(H) \setminus K$. Since $H$ is torsion-free, $z^{2} \neq e$. Since the isomorphism $h$ is expanding, $z^{2} \notin h^{m}(H)$ for some $m \in \mathbb{N}$. Replacing the isomorphism $h$ by its iterate $h^{m}$, we lose no generality assuming that $z^{2} \notin h(H) = K$.

By induction on $\alpha < \omega_{1}$ we shall prove that $\tau^{\alpha}(F) \cap P_{H} \neq \tau^{<\alpha}(F) \cap P_{H}$.

For $\alpha = 1$ the non-equality $\tau^{\alpha}(F) \cap P_{H} \neq \tau^{1}(F) \cap P_{H}$ is included into the hypothesis. Assume that for some ordinal $\alpha < \omega_{1}$ we proved that $\tau^{\beta}(F) \cap P_{H} \neq \tau^{<\beta}(F) \cap P_{H}$ for all ordinals $\beta < \alpha$.

If $\alpha = \beta + 1$ is a successor ordinal, then by the inductive assumption we can find a set $A \in \tau^{\beta}(F) \setminus \tau^{<\beta}(F)$ in the subgroup $H$. By Lemmas [5.2] and [3.3], $A \cup zA \in \tau^{\beta+1}(F) \setminus \tau^{<\beta}(F)$ and $\tau^{\alpha}(F) \setminus \tau^{<\alpha}(F)$ and we are done.

If $\alpha$ is a limit ordinal, then we can find an increasing sequence of ordinals $\{\alpha_{n}\}_{n \in \omega}$ with $\alpha = \sup_{n \in \omega} \alpha_{n}$. By the inductive assumption, for every $n \in \omega$ there is a subset $A_{n} \subseteq H$ with $A_{n} \in \tau^{\alpha_{n}+1}(F) \setminus \tau^{\alpha_{n}}(F)$. Then we can put $A = \bigcup_{n \in \omega} h^{\alpha_{n}}(A_{n})$. By Proposition [5.2] for every $n \in \omega$, we get  
\[h^{\alpha_{n}}(A_{n}) \subseteq \tau^{\alpha_{n}+1}(F) \setminus \tau^{\alpha_{n}}(F)\]

and thus $A \notin \tau^{<\alpha}(F)$ for all $n \in \omega$, which implies that $A \notin \tau^{\alpha}(F)$. On the other hand, Lemma [6.5] guarantees that $A \in \tau^{\alpha}(F)$.

\[\Box\]
Lemma 6.7. Assume that a left-invariant ideal $F$ on a group $G$ is $h$-invariant for some isomorphism $h : H \to K$ between torsion-free subgroups $K \subset H$ of $G$ such that $z^2 \notin K$ for some $z \in Z_K(H)$. Assume that for an infinite cardinal $\kappa$ there are isomorphisms $h_n : H \to H_n, n \in \kappa$, onto subgroups $H_n \subset H$ such that $F$ is $h_n$-invariant and $H_n \cdot H_m \cap H_n \cdot H_l = \{e\}$ for all indices $n, m, k, l \in \kappa$ with $\{n, m\} \cap \{k, l\} = \emptyset$.

If $\tau(F) \cap P_H \notin F$, then $\tau^\alpha(F) \neq \tau^\alpha(F) \cap P_H$ for all ordinals $\alpha < \kappa^+$.

Proof. By induction on $\alpha < \kappa^+$ we shall prove that $\tau^\alpha(F) \cap P_H \neq \tau^\alpha(F) \cap P_H$.

For $\alpha = 1$ the non-equivalency $\tau^1(F) \cap P_H \neq \tau^0(F) \cap P_H$ is included into the hypothesis. Assume that for some ordinal $\alpha < \kappa^+$ we proved that $\tau^\beta(F) \cap P_H \neq \tau^\beta(F) \cap P_H$ for all ordinals $\beta < \alpha$.

If $\alpha = \beta + 1$ is a successor ordinal, then by the inductive assumption we can find a set $A \in \tau^\beta(F) \setminus \tau^\beta(F)$ in the subgroup $H$. By Lemmas 6.2 and 6.3 $h(A) \cup zh(A) \in \tau^{\beta+1}(F) \setminus \tau^\beta(F)$ and we are done.

If $\alpha$ is a limit ordinal, then we can fix a family of ordinals $(\alpha_n)_{n \in \kappa}$ with $\alpha = \sup_{n \in \kappa}(\alpha_n + 1)$. By the inductive assumption, for every $n \in \kappa$ there is a subset $A_n \subset H$ such that $A_n \in \tau^{\alpha_n+1}(F) \setminus \tau^\alpha(F)$.

After a suitable shift, we can assume that $e \notin A_n$. Since the ideal $F$ is $h_n$-invariant, $h_n(A_n) \in \tau^{\alpha_n+1}(F) \setminus \tau^\alpha(F)$ according to Lemma 5.2.

Then the set $A = \bigcup_{n \in \omega} h_n(A_n)$ does not belong to $\tau^\alpha(F)$. The inclusion $A \in \tau^\alpha(F)$ will follow as soon as we check that $A \cap xA \in \tau^{<\alpha}(F)$ for all $x \in G \setminus \{e\}$. This is clear if $A \cap xA$ is empty. If $A \cap xA$ is not empty, then $x \in h_n(A_n) h_m(A_m)^{-1} \subset H_n H_m$ for some $n, m \in \kappa$. Taking into account that $H_n H_m \cap H_k H_l = \{e\}$ for all $k, l \in \kappa \setminus \{n, m\}$ and $e \notin A$, we conclude that $A \cap xA \subset h_n(A_n) \cup h_m(A_m) \cup xh_n(A_n) \cup xh_m(A_m) \in \tau^{<\alpha}(F)$ as $\tau^{<\alpha}(F)$ is additive according to Theorem 1.2.

Let us recall that a family $F$ of subsets of a group $G$ is called auto-invariant if for any injective homomorphism $h : G \to G$ a subset $A \subset G$ belongs to $F$ if and only if $h(A) \in F$.

Lemma 6.8. Let $G$ be a free abelian group $G$ and $F$ be an auto-invariant ideal of subsets of $G$. If $F$ is not thin-complete, then for each ordinal $\alpha < |G|^+$ the family $\tau^\alpha(F)$ is not thin-complete.

Proof. Being free abelian, the group $G$ is generated by some linearly independent subset $B \subset G$. Consider the isomorphism $h : G \to 3G$ of $G$ onto the subgroup $3G = \{g^3 : g \in G\}$ and observe that $h$ is expanding and for each $z \in B$ we get $z^2 \notin 3G$. The ideal $F$ being auto-invariant, is $h$-invariant. Applying Lemma 6.6, we conclude that $\tau^\alpha(F) \neq \tau^{<\alpha}(F)$ for all ordinals $\alpha < \omega_1$. If the group $G$ is countable, then this is exactly what we need.

Now consider the case of uncountable $G = |G|$. Being free abelian, the group $G$ is isomorphic to the direct sum $\oplus^\kappa \mathbb{Z}$ of $\kappa$-many copies of the infinite cyclic group $\mathbb{Z}$. Write the cardinal $\kappa$ as the disjoint union $\kappa = \bigcup_{\alpha \in \kappa} \kappa_\alpha$ of $\kappa$ many subsets $\kappa_\alpha \subset \kappa$ of cardinality $|\kappa_\alpha| = \kappa$. For every $\alpha \in \kappa$ consider the free abelian subgroup $G_\alpha = \oplus^{\kappa_\alpha} \mathbb{Z}$ of $G$ and fix any isomorphism $h_\alpha : G \to G_\alpha$. It is clear that $G_\alpha \oplus G_\beta \cap G_\gamma \oplus G_\delta = \{0\}$ for all ordinals $\alpha, \beta, \gamma, \delta \in \kappa$ with $\{\alpha, \beta\} \cap \{\gamma, \delta\} = \emptyset$.

Being auto-invariant, the ideal $F$ is $h_\alpha$-invariant for every $\alpha \in \kappa$. Now it is legal to apply Lemma 6.7 to conclude that $\tau^\alpha(F) \neq \tau^{<\alpha}(F)$ for all ordinals $\alpha < \kappa^+$.

Proof of Theorem 6.7. Let $F$ be a sub-invariant ideal of subsets of a group $G$ and let $H \subset G$ be a free abelian subgroup of cardinality $|H| = |G|$. Assume that $\tau(F) \cap P_H \notin F$.

Consider the ideal $F' = P_H \cap F$ of subsets of the group $H$. By transfinite induction it can be shown that $\tau^\alpha(F') = P_H \cap \tau^\alpha(F)$ for all ordinals $\alpha$.

The sub-invariance of $F'$ implies the sub-invariance (and hence auto-invariance) of $F'$. By Lemma 6.8, we get $\tau^\alpha(F') \neq \tau^{<\alpha}(F')$ for each $\alpha < |H|^+ = |G|^+$. Then also $\tau^\alpha(F) \neq \tau^\alpha(F) \neq \tau^{<\alpha}(F)$ for all $\alpha < |G|^+$.

7. The descriptive complexity of the family $\tau^\alpha(F)$

In this section given a countable group $G$ and a left-invariant monotone subfamily $F \subset P_G$, we study the descriptive complexity of the family $\tau^\alpha(F)$, considered as a subspace of the power-set $P_G$ endowed with the compact metrizable topology of the Tychonov product $2^G$ (we identify $P_G$ with $2^G$ by identifying each subset $A \subset G$ with its characteristic function $\chi_A : G \to \{0, 1\}$).

Theorem 7.1. Let $G$ be a countable group and $F \subset P_G$ be a Borel left-invariant lower family of subsets of $G$.

1. For every ordinal $\alpha < \omega_1$ the family $\tau^\alpha(F)$ is Borel in $P_G$.
2. The family $\tau^\omega(F) = \tau^{<\omega_1}(F)$ is coanalytic.
3. If $\tau^\alpha(F) \neq \tau^\alpha(F)$ for all $\alpha < \omega_1$, then $\tau^\omega(F)$ is not Borel in $P_G$. 


Proof. Let us recall that \( G_0 = G \setminus \{ e \} \).

In Section 3 to each subset \( A \subset G \) we assigned the \( \tau \)-tree
\[
T_A = \{ s \in G_0^{\omega} : A_s \notin F \},
\]
where for a finite sequence \( s = (g_0, \ldots, g_{n-1}) \in G_0^n \subset G_0^{\omega} \) we put
\[
A_s = \bigcap_{x_0, \ldots, x_{n-1} \in 2^n} g_0^{x_0} \cdots g_{n-1}^{x_{n-1}} A.
\]

Consider the subspaces \( WF \subset Tr \) of \( P_{G_0^{\omega}} \), consisting of all (well-founded) lower subtrees of the tree \( G_0^{\omega} \).

**Claim 7.2.** The function
\[
T_s : P_G \to Tr, \ T_s : A \mapsto T_A
\]
is Borel measurable.

**Proof.** The Borel measurability of \( T_s \) means that for each open subset \( U \subset Tr \) the preimage \( T_{s}^{-1}(U) \) is a Borel subset of \( P_G \). Let us observe that the topology of the space \( Tr \) is generated by the sub-base consisting of the sets
\[
\langle s \rangle^+ = \{ T \in Tr : s \notin T \} \text{ and } \langle s \rangle^- = \{ T \in Tr : s \notin T \} \text{ where } s \in G_0^{\omega}.
\]
Since \( \langle s \rangle^- = Tr \setminus \langle s \rangle^+ \), the Borel measurability of \( T_s \) will follow as soon as we check that for every \( s \in G_0^{\omega} \) the preimage \( T_{s}^{-1}(\langle s \rangle^+) = \{ A \in P_G : s \in T_A \} \) is Borel.

For this observe that the function
\[
f : P_G \times G_0^{\omega} \to P_G, \ f : (A, s) \mapsto A_s,
\]
is continuous. Here the tree \( G_0^{\omega} \) is endowed with the discrete topology.

Since \( F \) is Borel in \( P_G \), the preimage \( \mathcal{E} = f^{-1}(P_G \setminus F) \) is Borel in \( P_G \times G_0^{\omega} \). Now observe that for every \( s \in G_0^{\omega} \) the set
\[
T_{s}^{-1}(\langle s \rangle^+) = \{ A \in P_G : s \in T_A \} = \{ A \in P_G : (A, s) \in \mathcal{E} \}
\]
is Borel. \( \square \)

By Theorem 3.2 \( \tau^*(F) = T_{\alpha}^{-1}(WF) \) and \( \tau^a(F) = T_{\alpha}^{-1}(WF_{-1+\omega+1}) \) for \( \alpha < \omega_1 \). Now Theorem 2.1 and the Borel measurability of the function \( T_s \) imply that the preimage \( T_{s}^{-1}(WF) \) is coanalytic while \( \tau^a(F) = T_{\alpha}^{-1}(WF_{-1+\omega+1}) \) is Borel for every \( \alpha < \omega_1 \), see [3] 14.4.

Now assuming that \( \tau^{\alpha+1}(F) \neq \tau^a(F) \) for all \( \alpha < \omega_1 \), we shall show that \( \tau^*(F) \) is not Borel. In the opposite case, \( \tau^*(F) \) is analytic and then its image \( T_s(\tau^*(F)) \subset WF \) under the Borel function \( T_s \) is an analytic subspace of \( WF \), see [3] 14.4. By Theorem 2.1, \( T_s(\tau^*(F)) \subset WF_{\alpha+1} \) for some infinite ordinal \( \alpha < \omega_1 \) and thus \( \tau^*(F) = T_{\alpha}^{-1}(WF_{\alpha+1}) = \tau^a(F) \), which is a contradiction. \( \square \)

Theorems 6.1 and 7.4 imply:

**Corollary 7.3.** For any countable non-torsion group \( G \) the ideal \( \tau^*(F_G) \subset P_G \) is coanalytic but not analytic.

By [3] 26.4, the \( \Sigma_1^1 \)-Determinacy (i.e., the assumption of the determinacy of all analytic games) implies that each coanalytic non-analytic space is \( \Pi_1^1 \)-complete. By [6], the \( \Sigma_1^1 \)-Determinacy follows from the existence of a measurable cardinal. So, the existence of a measurable cardinal implies that for each countable non-torsion group \( G \) the subspace \( \tau^*(F_G) \subset P_G \), being coanalytic and non-analytic, is \( \Pi_1^1 \)-complete.

**Question 7.4.** Is the space \( \tau^*(F_Z) \) \( \Pi_1^1 \)-complete in ZFC?

**References**

[1] T. Banakh, N. Lyaskovska, Completeness of translation-invariant ideals in groups, Ukr. Mat. Zh. 62:8 (2010), 1022–1031.
[2] A. Bella, V.I. Malykhin, On certain subsets of a group, Questions Answers Gen. Topology 17:2 (1999), 183–197.
[3] A. Kechris, Classical Descriptive Set Theory, Springer, 1995.
[4] Ie. Lutsenko, I.V. Protasov, Sparse, thin and other subsets of groups, Internat. J. Algebra Comput. 19 (2009), no. 4, 491–510.
[5] Ie. Lutsenko, I.V. Protasov, Relatively thin and sparse subsets of groups, Ukr. Math. Zh. (submitted).
[6] D.A. Martin, Measurable cardinals and analytic games, Fund. Math. 66 (1970), 287–291.

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