THE MODEL THEORY OF THE CURVE GRAPH

VALENTINA DISARLO, THOMAS KOBERDA, AND JAVIER DE LA NUEZ GONZÁLEZ

Abstract. In this paper we develop a bridge between model theory, geometric topology, and geometric group theory. We consider a surface $\Sigma$ of finite type and its curve graph $C(\Sigma)$, and we investigate the first-order theory of the curve graph in the language of graph theory. We prove that the theory of the curve graph is $\omega$–stable, give bounds on its Morley rank, and show that it has quantifier elimination with respect to the class of $\exists$–formulae. We also show that many of the complexes which are naturally associated to a surface are interpretable in the curve graph, which proves that these complexes are all $\omega$–stable and admit certain a priori bounds on their Morley ranks. We are able to use Morley rank to prove that several complexes are not bi–interpretable with the curve graph. As a consequence of quantifier elimination, we show that algebraic intersection number is not definable in the first order theory of the curve graph.

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1. **Introduction**

In this paper, we provide a systematic framework for studying objects in combinatorial topology from the point of view of model theory. Specifically, we are concerned with an orientable surface (that is, a real two-dimensional manifold) with compact interior and negative Euler characteristic, its mapping class group $\text{Mod}(\Sigma)$, consisting of homotopy classes of orientation preserving homeomorphisms of $\Sigma$, and the curve graph $C(\Sigma)$, which encodes the homotopy classes of simple closed curves on $\Sigma$. The reader is directed to Section 2 (and especially Subsection 2.1) for precise definitions. Our main results are about the model theory of the curve graph, thus investigating a geometrically/topologically defined object from a model-theoretic point of view. We draw inspiration and adapt many ideas from [BMPZ17], where a model-theoretic study of right-angled buildings was carried out.

1.1. **The model theory of the curve graph.** The curve graph was introduced by Harvey [Har81], who defined it as a surface-theoretic analogue of a building associated to a symmetric space of nonpositive curvature. The analogy with buildings is part of the inspiration for the present work. The curve graph of a surface is usually a locally infinite graph of infinite diameter, with complicated structure on both a local and global scale. Thus, one may suspect that reasonable notions classification (e.g. classifying the finite subgraphs of $C(\Sigma)$ up to isomorphism, for $S$ fixed) might be so chaotic as to be intractable. The primary purpose of this paper is to establish that, in spite of the apparent chaotic structure of the curve graph, its formal properties are tamer than one might initially imagine. We will view $C(\Sigma)$ as a graph, without any further structure. The language of (undirected) graphs has a single binary relation $E$ which is symmetric. In the context of the curve graph, the set $V$ of vertices of $C(\Sigma)$ is the universe, and $E$ is interpreted as adjacency in $C(\Sigma)$. We will write $\text{Th}(C(\Sigma))$ for the theory of $C(\Sigma)$. Precisely, this consists of the set of first order logical sentences which are satisfied by $C(\Sigma)$. The adjective **first order** refers to the fact that quantification is only allowed over individuals of the universe, as opposed to subsets of the universe.

The main results of this paper are listed in the remainder of this section.

**Theorem 1.1.** Let $\Sigma$ be a surface that is not a torus with two boundary components. Then $\text{Th}(C(\Sigma))$ has quantifier elimination with respect to the class of $\exists$–formulae.

Here, (absolute) quantifier elimination means that any first order predicate is equivalent in the theory to one which involves no quantifiers. A theory $T$ having relative quantifier elimination with respect to $\exists$–formulae means that every first order predicate is equivalent modulo $T$ to a Boolean combination of formulae which do not admit alternations between existential and universal quantifiers.
In the course of the proof of Theorem 1.1 we will construct an auxiliary structure which has absolute quantifier elimination, and then show that it is suitably bi–interpretable with the curve graph provided that the underlying surface is not a torus with two boundary components.

We note that one cannot hope for absolute quantifier elimination in Th(C(Σ)), since then every first order predicate about the graph C(Σ) would be determined by finitely many adjacency relations. For example, it is not difficult to write a first order formula which expresses that \( d_{C(\Sigma)}(a, b) = k \) and \( d_{C(\Sigma)}(a, c) = k + 1 \), where here we are measuring distance in the curve graph, and where \( k \geq 2 \) is an integer. In the language of graph theory, the quantifier–free types of the pair \((a, b)\) and the pair \((a, c)\) are the same. Therefore, the quantifier–free type of a pair cannot determine the type of a pair.

As a consequence of quantifier elimination, we have the following result:

**Corollary 1.2.** Let \( \Sigma \) be a surface of positive genus that is not a torus with fewer than two boundary components. The (absolute value of the) integral algebraic intersection number among curves on \( \Sigma \) is not a (parameter–free) definable relation in \( C(\Sigma) \).

We will in fact prove a significantly more general statement; cf. Corollary 4.14 below. The precise meaning of Corollary 1.2 is that the predicate \( \phi_n(x, y) \) stating “the algebraic (or geometric) intersection number between \( x \) and \( y \) is \( \pm n \)” is not definable. A straightforward modification of the argument we offer will even show that the \( (\mod 2) \) intersection number of curves is not a definable relation in \( C(\Sigma) \). That is, the relation \( R(x, y) \) given by \( (x, y) \in R \) if the algebraic intersection number of \( x \) and \( y \) is 0 \( \mod 2 \) is not definable by a first order predicate without parameters.

Corollary 1.2 seems counterintuitive at first, since algebraic intersection number is a homological invariant which is easy to compute once curves are identified with conjugacy classes in \( \pi_1(\Sigma) \). What the corollary is saying is that the first order theory of the graph structure of the curve graph is not well–suited to predicates encoding algebraic intersection number.

We are now in a position to state the second general result about the model theory of the curve graph.

**Theorem 1.3.** Let \( \Sigma \) be a surface. Then \( \text{Th}(C(\Sigma)) \) is \( \omega \)–stable. If \( S \) has genus \( g \) and has \( b \) boundary components and punctures, then the Morley rank of \( \text{Th}(C(\Sigma)) \) is bounded above \( \omega^{3g+b-2} \).

In precise terms, a theory is \( \omega \)–stable if there are only countably many types over a countable set of parameters. We refer the reader to Subsection 2.2 for an expanded discussion of this notion. In intuitive terms, it means that countable models for \( C(S) \) are “classifiable”. The Morley rank of a theory is a notion of dimension. See the discussion in Section 2. We will prove equalities of Morley ranks for certain
auxiliary structures that are bi–interpretable with the curve graph, and which we construct in the course of the paper. We note that the general strategies of our proofs of Theorem 1.1 and Theorem 1.3 are adapted from [BMPZ17].

1.2. Interpretability of other complexes. There are many graphs other than the curve graph which are naturally associated to a surface $\Sigma$ of finite type, and a general theme that can be observed in the literature on these complexes is that their automorphism groups tend to coincide with that of the curve graph, except in some sporadic low–complexity cases and a few notable exceptions. We would like to say, in precise terms, that most known graphs associated to $\Sigma$ are completely determined by the curve graph, and that in fact they can be reconstructed from a finite sequence of canonical first order operations.

For the remainder of this section, assume that $\Sigma$ is not a torus with two boundary components. The following result is a schematic in the sense that it is technical but has broad applicability to the first order theory of complexes associated to $\Sigma$. We state it here for easy reference.

**Corollary 1.4.** Let $X(\Sigma)$ be a graph with vertices $V(X(\Sigma))$ and edges $E(X(\Sigma))$. Let $G$ denote the mapping class group of $\Sigma$, and suppose that the natural action of $G$ on $C(\Sigma)$ induces an action on $X(\Sigma)$. Assume that the following conditions hold:

1. There exists a constant $N \geq 1$ such that each $v \in V(X(\Sigma))$ corresponds to a collection of at most $N$ curves or arcs;
2. There exists a constant $K \geq 0$ such that for every $v = \{\gamma_1, \ldots, \gamma_m\} \in V(X(\Sigma))$, we have that $i(\gamma_i, \gamma_j) \leq K$;
3. The quotient $E(X(\Sigma))/G$ is finite.

Then $X(\Sigma)$ is interpretable in $C(\Sigma)$. Consequently, $X(\Sigma)$ is $\omega$-stable and has finite Morley rank.

In item (2) of Corollary 1.4, the pairing $i$ denotes geometric intersection number. As a consequence of Corollary 1.4, we have the following conclusions about the first order theory of graphs associated to $\Sigma$.

**Corollary 1.5.** All of the following graphs are interpretable in the curve complex $C(\Sigma)$, and are therefore $\omega$-stable and have finite Morley rank:

1. the Hatcher-Thurston graph $\mathcal{HT}(\Sigma)$;
2. the pants graph $\mathcal{P}(\Sigma)$;
3. the marking graph $\mathcal{M}(\Sigma)$;
4. the non-separating curve graph $N(\Sigma)$;
5. the $k$-separating curve graph $C_k(\Sigma)$;
6. the Torelli graph $T(\Sigma)$;
(7) the k-Schmutz Schaller graph $S_k(\Sigma)$;
(8) the k-multicurve graph $MC_k(\Sigma)$;
(9) the arc graph $A(\Sigma)$;
(10) the k-multiarc graph $MA_k(\Sigma)$;
(11) the flip graph $F(\Sigma)$;
(12) the polygonalization graph $Pol(\Sigma)$;
(13) the arc-and-curve graph $AC(\Sigma)$;
(14) the graph of domains $D(\Sigma)$.

We briefly summarize these complexes for the convenience of the reader, and give some references for definitions and discussions of automorphism groups. The Hatcher-Thurston complex of a surface of positive genus consists of cut systems, i.e. systems of curves whose complement is a connected surface of genus zero, and whose edges correspond to elementary moves [HT80, IK07]. The pants graph consists of multicurves giving rise to a pants decomposition of the surface, with edges given by elementary moves [Mar04]. The marking graph consists of markings on the surface, which are pants decompositions together with transversal data, and whose edges are given by elementary moves [MM00]. The non-separating curve graph consists of simple closed curves on the surface whose complement is connected, and edges are given by disjointness [Irm06]. The separating curve graph has separating simple closed curves as its vertices and the edge relation is disjointness [Loo13, Kid11]. The Torelli graph consists of separating curves and bounding pairs, with the edge relation being disjointness [Kid11]. The Schmutz Schaller graph on a closed surface has nonseparating curves as its vertices, with geometric intersection number one being the edge relation [SS00]. The arc graph and multiarc graph consist of simple arcs or multiarcs with endpoints at distinguished marked points, with edge relation given by disjointness [IM10, EF17]. The flip graph and polygonalization graph consist of simple arc systems whose complements are a triangulation or a polygonal decomposition of the surface, respectively, with endpoints of the arcs lying at distinguished marked points. Edge relations are given by elementary moves [KP12a, AKP15, DPI18, BDT18]. The arc and curve graph consists of simple closed curves and simple arcs, with adjacency given by disjointness [KP12b]. The graph of domains consists of essential subsurfaces whose boundary components are essential, and adjacency is given by disjointness [MP12].

Strictly speaking, Corollary 1.4 does not apply to the complex of domains, though the proof of interpretability goes through without issue. See the proof of Corollary 4.12 below.

The decoration of a graph by a nonnegative integer $k \geq 0$ means that intersections between curves or arcs representing vertices are allowed in the edge relation, up to at most $k$ intersections.
In the standard literature on surface theory, many of the graphs discussed in Corollary 1.5 are referred to as “complexes” instead of as “graphs”. We stick to “graph” for consistency of terminology. In each case, the passage from the graph to the complex is simply by taking the flag complex, which is completely determined by its 1–skeleton. In particular, the relation expressing that $k + 1$ vertices bound a $k$–cell is a first order relation that is completely determined by the adjacency relation, and thus the first order theories of the graphs and the complexes are the same.

Corollary 1.5 suggests that many of the graphs labelled therein should be bi–interpretable with the curve graph, which would imply that the automorphism groups of these graphs coincide with that of the curve graph, i.e. the extended mapping class group. Ivanov conjectured that the automorphism group of a “natural” complex associated to a surface should be the extended mapping class group, and that the proof should factor through the automorphisms of the curve graph, whereby bi–interpretability provides a suitable framework for interpreting the meaning of “natural”. Significant progress on understanding Ivanov’s conjecture has been made by Brendle and Margalit [BM19], wherein they give conditions under which Ivanov’s conjecture is true and false, and adapt it to the study of normal subgroups of mapping class groups. There does not seem to be an omnibus result, even from the point of view of model theory. Though the graphs in Corollary 1.5 are interpretable in the curve graph, their theories are essentially different from that of the curve graph. This can be phrased in terms of Morley rank; for such a discussion, we direct the reader to Sections 4 and 11 below. Here, we opt for a description in terms of bi–interpretability. As an example of the failure of bi–interpretability, we have the following:

**Corollary 1.6.** Let $\Sigma$ be a surface of genus $g$ with $b$ boundary components. The curve graph of a surface $\Sigma$ is not interpretable in the pants graph of $\Sigma$, provided that $3g + b - 2 > 2$. The curve graph of a surface $\Sigma$ is not interpretable the separating curve graph of $\Sigma$, provided $g \geq 2$ and $b \leq 1$. The curve graph of a surface $\Sigma$ is not interpretable the arc graph of $\Sigma$, provided $g \geq 2$ and $b = 1$.

Finally, we emphasize the intellectual debt that this manuscript owes to the paper [BMPZ17]. Indeed, the ideas of building an auxiliary $L$–structure (Section 3), applications of geometry to characterizing types (Section 6), the notion of simple connectedness (Section 7), and the use of weak convexity in order to ultimately establish $\omega$–stability and versions of quantifier elimination (Sections 8, 9, and 10), are adapted from the arguments in [BMPZ17]. Suitable modifications to the context of mapping class groups are required, and the interpretability of other complexes (Section 4) is made possible through the framework in which we operate. Among the ingredients specific to the mapping class group situation is the use of the asymptotic geometry of the mapping class group and the Behrstock inequality.
ACKNOWLEDGEMENTS

The authors thank J. T. Moore, C. Perin, and K. Tent for interesting and helpful discussions. The authors thank Universität Heidelberg and University of the Basque Country (UPV/EHU) for hospitality while part of this research was carried out. The authors acknowledge support from U.S. National Science Foundation grants DMS 1107452, 1107263, 1107367 “RNMS: Geometric structures and Representation varieties” (the GEAR Network). The first author acknowledges support from the European Research Council under ERC-Consolidator grant 614733 (GEOMETRICSTRUCTURES) and the Olympia Morata Programme of Universität Heidelberg. The second author is partially supported by an Alfred P. Sloan Foundation Research Fellowship, by NSF Grant DMS-1711488, and by NSF Grant DMS-2002596. The third author has received funding from the European Research Council under the European Unions Seventh Framework Programme (FP7/2007-2013)/ERC Grant Agreements No. 291111 and No. 336983 and from the Basque Government Grant IT974-16.

2. BACKGROUND

In this section, we summarize relevant background from mapping class group theory and model theory, in order to make the present article as self–contained as possible.

2.1. Mapping class groups and curve graphs. A standard reference for the material in this section is the book by Farb and Margalit [FM11].

2.1.1. Generalities. Let \( \Sigma \) be an orientable surface of finite type. We will always assume that the Euler characteristic \( \chi(\Sigma) \) is negative, so that \( \Sigma \) admits a complete hyperbolic metric of finite area (possibly with cusps).

**Definition 2.1** (The mapping class group \( \text{Mod}(\Sigma) \)). The **(extended) mapping class group** \( \text{Mod}(\Sigma) \) is defined to be the group of the isotopy classes of the homeomorphisms of \( \Sigma \). We allow homeomorphisms to reverse the orientation of the surface and to permute its punctures.

Let \( \gamma: S^1 \to \Sigma \) be a homotopy class of maps of the circle into \( \Sigma \), which without loss of generality we assume to be smooth. We say that \( \gamma \) is an **simple closed curve** if in addition \( \gamma(S^1) \) represents a nontrivial conjugacy class in \( \pi_1(\Sigma) \), if some representative of \( \gamma \) is an embedding, and if \( \gamma(S^1) \) is not freely homotopic to a loop which encircles a puncture or boundary component of \( \Sigma \). We will often conflate \( \gamma \) with the image of a representative of \( \gamma \), which can always be chosen to be geodesic in a fixed hyperbolic metric on \( \Sigma \). The set of all simple closed curves on \( \Sigma \) is organized into the **curve graph** of \( \Sigma \).
Definition 2.2 (The curve graph $C(\Sigma)$). The curve graph $C(\Sigma)$ is a graph which consists of one vertex for each simple closed curve, and where $\gamma_1$ and $\gamma_2$ are adjacent if there are representatives of $\gamma_1(S^1)$ and $\gamma_2(S^1)$ which are disjoint in $\Sigma$. (This is equivalent to the statement that geodesic representatives of the corresponding loops are disjoint in $\Sigma$.)

We will often write $C_0(\Sigma)$ for the set of vertices of $C(\Sigma)$, and sometimes just $C_0$ if the underlying surface is clear. The curve graph has very complicated local and global structure. We record some of the facts which are useful for us.

Theorem 2.3 (See [FM11, MM99]). Let $\Sigma$ be a non-sporadic surface and let $C(\Sigma)$ be the curve graph of $\Sigma$. The following holds:

(1) The graph $C(\Sigma)$ is connected, locally infinite, and has infinite diameter.
(2) If $\Sigma' \subset \Sigma$ is an incompressible non-sporadic subsurface, then there is an inclusion of subgraphs $C(\Sigma') \subset C(\Sigma)$, and the image of this inclusion has diameter two in $C(\Sigma)$.
(3) The graph $C(\Sigma)$ is Gromov hyperbolic.

Here, by non-sporadic surfaces, we mean ones which admit two disjoint, non-isotopic simple closed curves. Sporadic surfaces are spheres with at most four punctures and tori with at most one puncture. For sporadic surfaces, there are suitable modifications of the definition of the curve graph which allow for analogues of Theorem 2.3.

The mapping class group acts simplicially on $C(\Sigma)$. Standard results from combinatorial topology of surfaces imply that, except for finitely many exceptions, the action of $\text{Mod}(\Sigma)$ on $C(\Sigma)$ is faithful.

Theorem 2.4 (Ivanov [Iva97], Luo [Luo00], Korkmaz [Kor99]). Let $\Sigma$ be an orientable surface of genus $g$ and $n$ punctures:

(1) If $\Sigma$ admits a pair of non-isotopic simple closed curves and if $(g, n) \neq (1, 2)$, then any automorphism of $C(\Sigma)$ is induced by a self-homeomorphism of the surface.
(2) Any automorphism of $C(\Sigma_{1,2})$ preserving the set of vertices represented by separating loops is induced by a self-homeomorphism of the surface.
(3) There is an automorphism of $C(\Sigma_{1,2})$ which is not induced by any homeomorphisms.

The action of $\text{Mod}(\Sigma)$ on $C(\Sigma)$ is cofinite, in the sense that the vertices and edges of $C(\Sigma)$ fall into finitely many orbits under the action of $\text{Mod}(\Sigma)$. Since $\text{Mod}(\Sigma)$ can invert edges of $C(\Sigma)$, we typically do not speak of a graph structure on the quotient $C(\Sigma)/\text{Mod}(\Sigma)$. 
2.1.2. Subsurface projections. Let $\Sigma'$ be an essential subsurface of $\Sigma$. We denote the curve graph of $\Sigma'$ simply by $C(\Sigma')$. Given $\alpha \in C_0$, we denote by $\pi_{\Sigma'}(\alpha)$ the (possibly empty) projection of $\alpha$ to $C(\Sigma')$. Following Masur–Minsky [MM99], the projection of a curve $\alpha$ to a subsurface $\Sigma' \subset \Sigma$ is obtained by taking a geodesic representative of $\alpha$ and of $\partial \Sigma'$, and then taking the boundary of a small tubular neighborhood of
\[ \partial \Sigma' \cup (\alpha \cap \Sigma'). \]
The result is a finite collection of simple closed curves of $\Sigma'$, some of which may be inessential or peripheral. Discarding those, we obtain a finite set of pairwise adjacent vertices in $C(\Sigma')$, which we take to be $\pi_{\Sigma'}(\alpha)$. Thus, projection gives a coarsely well-defined map from $C(\Sigma)$ to $C(\Sigma')$. We will write $d_{\Sigma'}(\alpha, \beta)$ as an abbreviation for
\[ d_{C(\Sigma')}(\pi_{\Sigma'}(\alpha), \pi_{\Sigma'}(\beta)). \]
We adopt the convention that the distance between the empty set and any set is infinite. Subsurface projections will play an essential role in establishing simple connectivity of certain relational structures in Section 7 below.

2.2. Model theory. For standard references for the material contained in this section see [TZ12, Mar06].

2.2.1. Languages, structures, theories, and models. We will work with several languages in this paper. The theory of the curve graph will be formulated in the first order language of graph theory, which consists of the single symmetric, binary relation $E$, denoting adjacency. For most of the discussion in this paper, we will work in an auxiliary structure adapted to a particular surface $S$, and which will encode relations for subsurfaces and mapping classes.

**Definition 2.5 ($L$–structure).** A language $L$ is a set of constants, function symbols and relation symbols. An $L$–structure $M$ is given by a set $M$ called the universe, together with an interpretation of the constants, relations, and functions of $L$. We say that a structure is relational if its underlying language has no functions and no constants.

One can use symbols in a language, together with logical connectives, variables, and quantifiers, in order to express conditions on tuples of elements in a structure. Such a condition among tuples is called a formula. A sentence is a formula without variables that are unbound by a quantifier.

For every sentence $\phi$ we write $M \models \phi$ if $\phi$ holds in $M$. We say that $M$ is a model of $\phi$ and $\phi$ holds in $M$. Similarly if $\Sigma$ is a set of sentences, then $M$ is a model of $\Sigma$ if all the sentences of $\Sigma$ if all the sentences of $\Sigma$ holds in $M$. 


Definition 2.6 (Consistent \( \mathcal{L} \)-theory). A theory \( T \) is given by a set of \( \mathcal{L} \)-sentences. A theory \( T \) is consistent if there is an \( \mathcal{L} \)-structure \( M \) where every sentence in \( T \) holds. We say that \( M \) is a model of \( T \).

The consequences of \( T \) are the sentences which hold in every model of \( T \). If \( \phi \) is a consequence of \( T \), we say that \( \phi \) follows from \( T \) (or \( T \) proves \( \phi \)) and write \( T \vdash \phi \).

Definition 2.7 (Complete theory). A consistent theory is called complete if for all \( \mathcal{L} \)-sentences \( \phi \) either \( T \vdash \phi \) or \( T \vdash \neg \phi \).

Theorem 2.8 (Compactness Theorem). A theory \( T \) is consistent if and only if every finite subset of \( T \) is consistent.

Let \( M \) be an \( \mathcal{L} \)-structure and \( A \subseteq M \). Then \( a \in M \) realizes a set of \( \mathcal{L}(A) \)-formulas \( p(x) \) (containing at most the free variable \( x \)), is \( a \) satisfies all formulas from \( p(x) \), in this case we write \( M \models p(a) \). We say that \( p(x) \) is finitely satisfiable in \( M \) if every finite subset of \( p \) is realizable in \( M \).

Definition 2.9 (Types). The set \( p(x) \) of \( \mathcal{L}(A) \)-formulae is a type over \( A \) if \( p(x) \) is finitely satisfiable in \( M \). If \( p(x) \) is maximal with respect to inclusion, we say that \( p(x) \) is a complete type. We say that \( A \) is the domain (or set of parameters) of \( p \).

A typical example of a type is the type determined by an element \( m \in M \):

\[
\text{tp}(m/A) = \{ \phi(x) \mid M \models \phi(m), \ \phi \text{ an } \mathcal{L}(A)-\text{formula} \}.
\]

An \( n \)-type is a finitely satisfiable set of formulae in \( n \) variables \( \{x_1, \ldots, x_n\} \). As for \( 1 \)-types, a maximal \( n \)-type is called complete. We denote by \( S_n(A) \) the set of \( n \)-types, and by

\[
S(A) := \bigcup_{n < \omega} S_n(A)
\]

the set of types. Let \( p \) be a type and let \( \phi \in p \). We say that \( \phi \) isolates \( p \) if for all \( \psi \in p \), we have

\[
\text{Th}(M) \models \phi(x) \rightarrow \psi(x).
\]

The set of complete \( n \)-types with parameters in \( A \) has a natural topology on it. A basis of open sets is given by formulae \( \phi \) with \( n \) free variables, and \( p \in U_\phi \) if and only if \( \phi \in p \). Completeness of types implies that these sets are in fact clopen. A type is isolated if and only if it is isolated in this topology. We call this topology the Stone topology.

Definition 2.10 (\( \kappa \)-saturated). Let \( \kappa \) be an infinite cardinal. A \( \mathcal{L} \)-structure \( M \) is \( \kappa \)-saturated if for all \( A \subseteq M \) and \( p \in S(A) \) with \( |A| \leq \kappa \), the type \( p \) is realized in \( M \). We say that \( M \) is saturated if it is \( |M| \)-saturated.
To avoid having to switch models to realize types, we will often work in the **monster model**, a model where all types over subsets of the universe are realized. A complete theory with infinite models admits a model $\mathcal{M}$ such that all types over all subsets of $M$ are realized in $\mathcal{M}$. Up to some set-theoretic issues, $\mathcal{M}$ is unique up to isomorphism.

It is well–known that the monster model of a theory $T$ also enjoys all of the following properties:

- Any model of $T$ is elementarily embeddable in $\mathcal{M}$;
- Any elementary bijection between two subsets of $\mathcal{M}$ can be extended to an automorphism of $\mathcal{M}$.

2.2.2. **Definability and interpretability.** Let $X$ be an $\mathcal{L}$–structure. A subset $Y \subset X$ is **definable** if there is a formula $\phi$ and a finite tuple of parameters $a$ in $X$ such that

$$Y = \{y \mid X \models \phi(y, a)\}.$$ 

Thus, $Y$ is “cut out” by the formula $\phi$. We will say that $Y$ is $\emptyset$–**definable** if

$$Y = \{y \mid X \models \phi(y)\}.$$ 

In this case, we also say that $Y$ is **parameter–free definable**. We say a structure $Y$ is **interpretable** in a structure $X$ if, roughly, there is a definable subset $X_0$ of $X$ and a definable equivalence relation $\sim$ on $X_0$ such that $Y$ is isomorphic to $X_0/\sim$. More precisely, we will define interpretability as follows.

Let $f : A \to B$ be a function. We denote by $f^{\times r}$ the associated product function

$$f \times f \cdots \times f : A^r \to B^r.$$ 

Given an equivalence relation $E$ on a set $X$, we let $E'$ the natural equivalence relation it induces on $X'$.

**Definition 2.11** (Interpretation). Given two structures $\mathcal{M}$ and $\mathcal{N}$, a (parameter–free) **interpretation** of $\mathcal{M}$ in $\mathcal{N}$ is given by a tuple

$$\bar{\eta} = (\eta, k, X, E),$$

where the following holds:

- $k \in \mathbb{N}$;
- $X \subseteq N^k$ is a $\emptyset$–definable set;
- $E \subseteq X \times X$ a definable equivalence relation;
- $\eta : M \to X/E$ a bijection such that for every $r \geq 1$ any definable set $Y \subseteq M^r$ is the preimage of a unique, $E$-invariant, $\emptyset$–definable set $Y_\eta \subseteq X'$.

Notice that in order to verify a purported interpretation, it is suffices to check the relevant properties for the relations in the language of $\mathcal{M}$, and for the graphs of function symbols in the language of $\mathcal{M}$.
Given parameter-free interpretations
\[ \bar{\eta}_1 = (\eta_1, k_1, X_1, E_1) \]
of \( M_1 \) in \( M_2 \) and
\[ \bar{\eta}_2 = (\eta_2, k_2, X_2, E_2) \]
of \( M_2 \) in \( M_3 \), the composition
\[ \bar{\eta}_2 \circ \bar{\eta}_1 = (\eta_3, k_1, X_1, E_3) \]
is given as follows:

- \( X_3 := \eta_2[X_1] \subseteq X_2^{k_1} \subseteq M_1^{k_1 k_2} \),
- \( E_3 \) is the equivalence relation \( \eta_2[E_1] \subseteq X_3 \times X_3 \). Notice that this is automatically coarser than the restriction of \( E_2^{k_1} \) to \( X_3 \),
- \( \eta_3 \) is the composition of \( \eta_1 \) with the map \( \bar{\eta}_2 : X_1/E_1 \to X_3/E_3 \) through which
  \[ \eta_2^{x k_1} |_{X_1} : X_1 \to (X_2/E_2)^{k_1} \]
  factors in the natural way.

It can be shown that \( \bar{\eta}_3 \) is an interpretation of \( M_1 \) in \( M_3 \).

**Definition 2.12** (Definable interpretation). We say that an interpretation \( \bar{\eta} = (\eta, k, X, E) \) of a structure \( M \) in itself is **definable** if the relation
\[ \Gamma_{\bar{\eta}} = \{ (x, y) \in M^2 \mid \eta(x) = [y]_E \} \]
is \( \emptyset \)-definable.

**Definition 2.13** (Bi-interpretation). A bi-interpretation between structures \( M \) and \( N \) is a pair \( (\bar{\eta}, \bar{\zeta}) \), where \( \bar{\eta} \) is an interpretation of \( M \) in \( N \), where \( \bar{\zeta} \) is an interpretation of \( N \) in \( M \), and where both \( \bar{\zeta} \circ \bar{\eta} \) and \( \bar{\eta} \circ \bar{\zeta} \) are \( \emptyset \)-definable. Accordingly, we say that the two structures are bi-interpretable.

2.2.3. **Algebraic and definable closure.** Let \( X \) be an \( L \)-structure and let \( A \subset X \) be a set of parameters. An element \( a \in X \) is in the **definable closure** of \( X \) if \( a \) is the unique element satisfying a formula \( \phi \) with parameters in \( A \). The definable closure \( dcl(A) \) consists of all such elements \( a \). Similarly, the **algebraic closure** of \( A \) consists of elements \( a \in X \) for which there is a formula \( \phi \) with parameters in \( A \) which is satisfied by \( a \) and which has finitely many solutions. We write \( acl(A) \) for the algebraic closure of \( A \). Algebraic and definable closures are idempotents.

Algebraic and definable closures are invariant under elementary extensions. Moreover, if \( X \) is sufficiently saturated then algebraic and definable closures are characterized in terms of types. Namely, \( a \in dcl(A) \) if and only if \( tp(a/A) \) has a unique realization in \( X \), and \( a \in acl(A) \) if and only if \( tp(a/A) \) has finitely many realizations in \( X \).
If $X$ is the monster model of a theory, then $a \in \text{dcl}(A)$ if and only if $a$ is fixed by every automorphism of $X$ which fixes $A$. Similarly, $a \in \text{acl}(A)$ if and only if $a$ has finitely many orbits under the group of automorphisms of $X$ fixing $A$.

2.2.4. Imaginaries. Let $X$ be an $L$–structure and let $x$ and $y$ be $n$–tuples for some $n$. An equivalence formula $\phi(x, y)$ is a formula that is a symmetric and transitive relation. It is an equivalence relation on the set of tuples $a$ such that $\phi(a, a)$. An imaginary is an equivalence formula $\phi$ together with an equivalence class $[a]_\phi$. We say that $X$ has elimination of imaginaries if for every imaginary there is a formula $\psi$ such that $[a]_\phi$ consists of tuples satisfying $\psi(x, b)$.

A theory admits elimination of imaginaries if every model does. Not every theory $T$ admits elimination of imaginaries, though every theory $T$ can be embedded in a theory $T^{\text{eq}}$ which does. Models $X$ of a theory $T$ can be extended to structures $X^{\text{eq}}$, which consist of $M$ (called the home sort) together with all the definable equivalence relations on tuples in $M$ (called the imaginaries). The algebraic and definable closure in $M^{\text{eq}}$ are written $\text{acl}^{\text{eq}}$ and $\text{dcl}^{\text{eq}}$ respectively.

2.2.5. $\omega$–stability, Morley rank, and forking.

**Definition 2.14** ($\omega$–stability). Let $T$ be a complete theory with infinite models. We say that $T$ is $\omega$–stable if in each model of $T$ and for each countable set of parameters $A$, there are at most countably many $n$–types for each $n$.

**Example 2.15.** The following are classical examples of $\omega$–stable theories:

- the theory of algebraically closed fields $\text{ACF}$ (of characteristic 0 or $p$);
- every countable $\kappa$–categorical theory with $\kappa$ uncountable cardinal.

In the context of a complete theory $T$, the **Morley rank** is a notion of dimension for a formula with parameters in the monster model.

**Definition 2.16** (Morley rank). We define the **Morley rank of a formula** $RM(\phi)$ by transfinite induction as follows:

1. $RM(\phi(x)) \geq 0$ if $(\exists x)\phi(x)$.
2. For $\alpha = \beta + 1$, $RM(\phi(x)) \geq \alpha$ if and only if there exist an infinite family of formulas $\phi_i(x)$ which are pairwise inconsistent and $\forall x(\phi_i(x) \rightarrow \phi(x))$.
3. $RM(\phi(x)) \geq \delta$ for a limit ordinal $\delta$ if and only if for each $\alpha < \delta$, we have $RM(\phi(x)) \geq \alpha$.

We set $RM(\phi(x)) = \beta$ for the least $\beta$ such that $RM(\phi(x)) \geq \beta + 1$. If there is no such $\beta$, then $\phi$ is unranked. The **Morley rank of a type** $p \in S(A)$ is the smallest Morley rank of a formula $\phi \in p$ in that type. The **Morley rank of a countable theory** $T$ is defined as the Morley rank of the formula $x = x$, that is

$$RM(T) := RM(x = x).$$
In the Stone topology on types, Morley rank coincides with Cantor–Bendixson rank.

For a countable complete theory, the notion of \( \omega \)--stability is characterized by the existence of an ordinal--valued rank function \( r \), whose domain is the set of definable subsets of a (sufficiently saturated) model, and which has the following property: if \( X \) is a definable subset of the model, then \( r(X) > \alpha \) if there exists a countably infinite collection of pairwise disjoint definable sets \( \{ Y_i \subset X \mid i < \omega \} \) with \( r(Y_i) \geq \alpha \). The smallest such rank function is the Morley rank.

Let \( T \) be an \( \omega \)--stable theory, and let \( M \) and \( N \) be models of \( T \). Let \( p \) be a type of \( M \) and \( q \) be a type of \( N \) containing \( p \). We say that \( q \) is a forking extension if the Morley rank of \( q \) is smaller than that of \( p \), and non--forking if the Morley rank is the same. Terminologically and notationally, we say that a set \( A \) is independent from \( B \) over \( C \) if for every finite tuple \( a \) in \( A \), the type \( tp(a/BC) \) does not fork over \( C \), and we write \( A \downarrow \! \! C B \). For a particular tuple \( a \) in \( A \), we write \( a \downarrow \! \! C B \).

2.2.6. Quantifier elimination. Let \( T \) be an \( L \)--theory and let \( x \) be a multi--variable. We say that formulae \( \phi(x) \) and \( \psi(x) \) are equivalent modulo \( T \) if
\[
T \vdash (\forall x)(\phi(x) \leftrightarrow \psi(x)).
\]

**Definition 2.17** (Quantifier elimination). A theory \( T \) has quantifier elimination if every \( L \)--formula in the theory is equivalent modulo \( T \) to a quantifier--free formula. A theory \( T \) has relative quantifier elimination with respect to a class of formulae \( \mathcal{F} \) if every \( L \)--formula in the theory is equivalent modulo \( T \) to a quantifier free formula which allows elements of \( \mathcal{F} \) as predicates.

In our context, we will discuss quantifier elimination with respect to \( \exists \)--formulae, in which case we simply mean that a given first order formula will be equivalent modulo \( T \) to a Boolean combination of \( \exists \)--formulae. Here, \( \exists \)--formulae are predicates which allow quantifiers but do not admit nested alternations between existential and universal quantifiers.

One can always expand a given language \( L \) by adding predicates which were definable in \( L \), and this expansion does not affect the absolute model theory of \( L \)--structures since it only depends on the class of definable sets. Note that any theory can thus be embedded into one which has quantifier elimination at the expense of enlarging the language.

A useful criterion for proving that a theory has quantifier elimination relies on the following property.

**Definition 2.18** (Back-and-forth property). Let \( M \) and \( N \) be \( L \)--structures, and let \( I \) be a set of partial isomorphisms between \( M \) and \( N \). We say that \( I \) has the back--and--forth property if whenever \( \langle \overline{a}, \overline{b} \rangle \in I \) and \( c \in M \), there exists a \( d \in N \) such that \( \langle \overline{ac}, \overline{bd} \rangle \in I \). Dually, if \( d \in N \), there exists a \( c \in M \) such that \( \langle \overline{ac}, \overline{bd} \rangle \in I \).
In the definition of the back-and-forth property, we view substructures as tuples. If $\overline{a}$ denotes a tuple of elements in $M$ and $c \in M$, then $\overline{ac}$ is the concatenation of $\overline{a}$ and $c$.

The following result is a standard fact from model theory, and is essential for us.

**Theorem 2.19.** Let $T$ be a theory, and let $M$ and $N$ be models of $T$. Suppose that $I$ is a family of partial isomorphisms between $M$ and $N$. Then for each $f \in I$, the types of the domain and the range of $f$ are equal.

In particular, if $I$ is the family of partial isomorphisms between subsets of $\omega$-saturated models of $T$ and if $I$ has the back–and–forth property, then the quantifier–free type of a tuple in an arbitrary model of $T$ determines its type. Therefore, $T$ has quantifier elimination. The converse also holds.

In our context, we will be interested in enlarging the language of graph theory in order to embed the theory of the curve graph in a theory with quantifier elimination, and then interpreting the theory of the curve graph therein. This way, we cannot quite obtain absolute quantifier elimination for the theory of the curve graph, and instead we will obtain quantifier elimination relative to natural classes of formulae. As stated in the introduction, absolute quantifier elimination is not possible for the curve graph.

### 3. Framework

Let $\Sigma$ be a (large enough) bounded surface (with or without punctures) and $C$ its curve graph, whose set of vertices $C_0$ is the collection of homotopy classes of oriented essential non-peripheral simple closed curves in $\Sigma$. The pure (orientation preserving) mapping class group $G$ of $\Sigma$ acts faithfully on $C$.

**Definition 3.1 (Domains).** A domain $D$ is a subset of vertices $D \subseteq C_0(\Sigma)$ with the property that whenever

$$\{\alpha_1, \alpha_2, \ldots, \alpha_k\} \subseteq D,$$

then $\gamma \in D$ for a curve $\gamma \in C_0$ that is isotopic (perhaps peripherally) into the (possibly disconnected) subsurface

$$\text{Fill}(\alpha_1, \alpha_2, \ldots, \alpha_k) \subseteq \Sigma$$

filled by $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$. We will write $\text{Fill}(D)$ for the subsurface filled by the curves in $D$. We will sometimes call the resulting surface the **realization** $|D|$ of $D$.

We let $D_0 = D_0(\Sigma)$ be the collection of all domains $D \subseteq C_0(\Sigma)$. Note that a domain may be empty, in which case the realization of the domain is also empty.

The reason for the subscript in the definition of the set of all domains will become clear in the sequel.

**Example 3.2.** Here are some examples of domains:
(1) The full set of vertices of the curve graph $C_0(\Sigma)$ is a domain.
(2) For every $\alpha \in C_0(\Sigma)$, we have $\{\alpha\} \in D_0$. We will often write $\alpha \in D_0$ for short.
(3) For every pair of pants $P$ with essential boundary curves $\alpha, \beta, \gamma \in C_0(\Sigma)$ then $D := \{\alpha, \beta, \gamma\} \subset C_0(\Sigma)$ is a domain in $D$. Note that the realization $|D|$ is given by three pairwise disjoint annuli of core curves $\alpha, \beta, \gamma$, not by the pair of pants.

**Definition 3.3** (Domain associated to a subsurface). For an essential subsurface $\Sigma' \subset \Sigma$ the **associated domain** of $\Sigma'$ is defined to be

$$D_{\Sigma'} := C_0(\Sigma') \cup \partial \Sigma' \subset C_0(\Sigma).$$

**Definition 3.4** (Connected domains). We say that a domain $D \in D_0$ is **connected** if $D = D_{\Sigma'}$ for some connected subsurface $\Sigma' \subset \Sigma$ which is not a pair of pants.

Note that if $D = D_{\Sigma'}$ is connected then $|D| = \Sigma'$.

**Definition 3.5** (Complexity of a domain). For a connected $D \in D_0$, we define the **complexity** of $D$ as

$$k(D) := 3g' + b' - 2,$$

where $g'$ and $b'$ are respectively the genus and the number of boundary components of $\Sigma' = \text{Fill}(D)$.

**Lemma 3.6.** Let $\Sigma$ be an orientable surface of genus $g$ and with $b$ boundary components and punctures with $3g - 3 + b \geq 1$. Suppose that

$$\emptyset = D_0 \subsetneq D_1 \subsetneq D_2 \subsetneq \cdots \subsetneq D_k = C_0(\Sigma)$$

is a chain of connected domains. Then $k \leq 3g - 2 + b$, and any maximal chain has length exactly $3g - 2 + b$.

**Proof.** Consider a proper inclusion of domains $D_i \subsetneq D_{i+1}$ for $i > 0$, with underlying topological realizations $\Sigma_i \subsetneq \Sigma_{i+1}$ obtained via application of Fill. Suppose that there is an essential arc $\alpha \subset \Sigma_{i+1} \setminus \Sigma_i$ which meets boundary curves $B_1$ and $B_2$ of $\Sigma_i$. Note that it is possible that $B_1 = B_2$. We have that a tubular neighborhood of $B_1 \cup \alpha \cup B_2$ is homeomorphic to an essential pair of pants (and possibly to a torus with one boundary component if $\Sigma_i$ is itself an annulus. In this case, we may view $B_1 \cup \alpha \cup B_2$ as obtained from an annulus by attaching two cuffs of a pair of pants to the two boundary components). If no such arc $\alpha$ exists, then $\Sigma_i$ has two boundary components $B_1$ and $B_2$ which are isotopic to each other in $\Sigma_{i+1}$.

It follows that there exists an ascending chain of essential subsurfaces

$$\emptyset = \Sigma_0 \subsetneq \Sigma_1 \subsetneq \cdots \subsetneq \Sigma_m = \Sigma$$
and a strictly increasing function
\[ f : \{0, \ldots, n\} \rightarrow \{0, \ldots, m\} \]
satisfying the following conditions:

- We have \( \Sigma_{i+1} \setminus \Sigma_i \) is either an essential pair of pants or an annulus for all \( i > 0 \).
- The surface \( \Sigma_{f(i)} \) is the underlying surface of the domain \( D_i \).

Thus, we may assume that for \( i \geq 1 \), the surface \( \Sigma_i \) is obtained from \( \Sigma_{i-1} \) by attaching a pair of pants or by gluing together two boundary components of \( \Sigma_i \). It suffices to show that \( m = 3g - 2 + b \).

The Euler characteristic of \( \Sigma \) is \( 2 - 2g - b \). Gluing two surfaces \( S_1 \) and \( S_2 \) along a boundary curve results in a surface \( S_3 \) such that
\[ \chi(S_3) = \chi(S_1 \cup S_2). \]
The Euler characteristic of a pair of pants is \(-1\), so that \( \Sigma_m \) is built from \( 2g + b - 2 \) pairs of pants. We thus have that \( m \) can be estimated from the number of gluings that need to be made between boundary curves of these \( 2g + b - 2 \) pairs of pants in order to reassemble \( \Sigma_m \).

The total number of boundary curves among all the pairs of pants is \( 6g + 3b - 6 \). A total of \( b \) of these curves correspond to the boundary components of \( \Sigma \) and are not involved in any gluings. The remaining \( 6g + 2b - 6 \) are glued in pairs, which results in exactly \( 3g + b - 3 \) gluings.

Now, since \( m \) is assumed to be maximal, we must have that \( \Sigma_1 \) is an annulus. Applying the maximality of \( m \) again and the assumption that the corresponding domains are connected, we have that \( \Sigma_2 \) is either a torus with one boundary component or a sphere with four boundary components. In either case, \( \Sigma_2 \) is the surface obtained after the first gluing, so that
\[ m = 3g + b - 3 + 1 = 3g + b - 2, \]
as claimed. \( \square \)

**Definition 3.7** (Domain spanned by \( D_1 \) and \( D_2 \)). Given \( D_1, D_2 \in D_0 \), there exists a unique smallest \( D \in D_0 \) such that \( D_1, D_2 \subseteq D \). We will denote it by \( D_1 \vee D_2 \) and say that \( D_1 \vee D_2 \) is the **domain spanned by** \( D_1 \) and \( D_2 \).

**Example 3.8.** Let \( D_1 = D_{\Sigma_1} \) and \( D_2 = D_{\Sigma_2} \) be two connected domains with associated surfaces \( \Sigma_1, \Sigma_2 \subseteq \Sigma \). Then the following holds:

1. if \( \Sigma_1 \subseteq \Sigma_2 \) then \( D_1 \vee D_2 = D_2 \);
2. if \( \Sigma_1 \cap \Sigma_2 = \emptyset \) then \( D_1 \vee D_2 = D_1 \cup D_2 \);
3. if \( \Sigma_1 \cap \Sigma_2 \neq \emptyset \) then \( D_1 \vee D_2 = D_{\Sigma_1 \cup \Sigma_2} \).
In the previous example, if $\Sigma_1$ and $\Sigma_2$ intersect essentially (i.e. are not isotopic to disjoint surfaces), then $\Sigma_1 \cap \Sigma_2 \neq \emptyset$. If $\Sigma_1$ and $\Sigma_2$ are surfaces which share a boundary component and correspond to domains $D_1$ and $D_2$, then there is a unique smallest connected domain containing $D_1 \cup D_2$.

3.1. **Orthogonality.** Given $D \subseteq C_0$ and $\alpha \in C_0$, we write $\alpha \perp D$ if and only if $i(\alpha, \beta) = 0$ for all $\beta \in D$. Here, $i(\alpha, \beta)$ denotes the geometric intersection number of $\alpha$ and $\beta$, taken to be the minimal number of intersections over all representatives of $\alpha$ and $\beta$ in their respective isotopy classes. We write $D \perp D'$ if $\alpha \perp D'$ for all $\alpha \in D$. If $D = \emptyset$ the all $\alpha \in C_0$ are orthogonal to $D$.

**Example 3.9.** Let $D = D_\Sigma$ be a connected domain.

1. For every curve $\alpha \in \partial \Sigma'$ which is also essential in $\Sigma$, we have $\alpha \perp D$.
2. If $\Sigma \setminus \Sigma'$ is essential then every essential simple closed curve $\alpha$ on $\Sigma \setminus \Sigma'$ is orthogonal to the domain $D'$ associated to $\Sigma'$.

**Definition 3.10** (Orthogonal domains). For a domain $D \in \mathcal{D}_0$, we define its orthogonal complement as

$$D^\perp := \{ \alpha \in C_0 \mid \alpha \perp D \}.$$ 

Equivalently, $D^\perp \in \mathcal{D}_0$ consisting of curves which are (possibly peripherally) homotopic into $\Sigma \setminus \partial |D|$.

Note that if $D$ is connected then we have $\partial \Sigma' \subset D^\perp$ and $D^\perp = D_{\Sigma \setminus |D|}$. For $\gamma \in C_0$, we will often write simply $\gamma^\perp$, instead of $\{ \gamma \}^\perp$. The following proposition follows easily from our definitions.

**Proposition 3.11.** The following properties hold:

1. $D^\perp \in \mathcal{D}_0$;
2. $(D^\perp)^\perp = D$;
3. $(D_1 \cup D_2)^\perp = D_1^\perp \cap D_2^\perp$;
4. $(D_1 \cap D_2)^\perp = D_1^\perp \cup D_2^\perp$.

**Lemma 3.12.** For every $D \in \mathcal{D}_0$ there exist domains $D_1, \ldots, D_k \in \mathcal{D}_0$ which are connected and pairwise orthogonal such that $D = \bigvee_{i=1}^k D_i$.

**Proof.** Let $D \in \mathcal{D}_0$, we have that

$$|D| = \Sigma_1 \cup \ldots \cup \Sigma_k,$$

where each $\Sigma_i$ is a suitable essential, subsurface of $\Sigma$, and where for $i \neq j$, we have that $\Sigma_i \cap \Sigma_j = \emptyset$ (up to isotopy). We now have $D = D_1 \cup \ldots \cup D_k$ with $D_i = D_{\Sigma_i}$.
As the subsurfaces share at most a boundary component, it follows that $D_i \perp D_j$ for $i \neq j$. □

**Definition 3.13** (Transverse domains). If $D$ and $D'$ are both non-orthogonal and incomparable (with respect to the inclusion relation), we say that they are **transverse**, and we write $D \cap D'$. □

**Definition 3.14** (Boundary of a domain). Given $D \in \mathcal{D}_0$, we define $\partial D := D \cap D^\perp$ the **boundary** of $D$. This coincides with the collection of curves $\alpha \in C$ parallel to a boundary of $|D|$. If $\alpha \in \partial D$, then we say that $\alpha$ is **peripheral** in $D$.

We will require one further notion of orthogonality, which we will call **strong orthogonality**, and which we will denote by $\perp^*$. It is important to note that strong orthogonality is not a symmetric relation. We introduce this definition purely for technical purposes, and it will only come into play in Section 8.

**Definition 3.15** (Strong orthogonality). Given $D_1, D_2 \in \mathcal{D}_0$, we will say that $D_1$ is **strongly orthogonal** to $D_2$ and write $D_1 \perp^* D_2$ if both the following hold:

1. $D_1 \perp D_2$;
2. $D_2 \cap \partial D_1$.

Notice that if neither $D_1$ nor $D_2$ are annular then the notions of orthogonality and strong orthogonality coincide.

### 3.2. The auxiliary $\mathcal{L}$-structure $\mathcal{M}(\Sigma)$

In this section we will define the auxiliary $\mathcal{L}$-structure $\mathcal{M}_0(\Sigma)$ and related structures $\mathcal{M}(\Sigma)$ that we will work with in for most of the rest of the paper.

**Definition 3.16** ($D$-related). Choose arbitrary orientations on each simple closed curve in $C_0(\Sigma)$. Given mapping classes $\sigma, \tau \in G$ and $D \in \mathcal{D}_0$, we say that $\sigma$ and $\tau$ are **$D$-related** if

$$\sigma^{-1} \circ \tau \in \text{Stab}^+_D.$$ 

That is, $\sigma^{-1} \circ \tau$ preserves all the curves in $D^\perp$ with their orientation.

Note that the requirement is superfluous when $D^\perp$ is connected and not an annulus, since then the fact that the curves are stabilized with their orientation is automatic.

We let $\mathcal{L}_0(\mathcal{D}_0)$ be a first order language containing a binary relation symbol $R_D$ for each $D \in \mathcal{D}_0$ and a binary relation $R_g$ for every $g \in G$, that is

$$\mathcal{L}_0(\mathcal{D}_0) := (\{R_D\}_{D \in \mathcal{D}_0}, \{R_g\}_{g \in G}).$$

We build a $\mathcal{L}_0(\mathcal{D}_0)$-structure whose universe is $G$, where

- $R_g(x, y)$ holds if and only if $y = xg$;
- $R_D(x, y)$ holds for $D \in \mathcal{D}_0$ if and only if $x$ and $y$ are $D$-related.
When no confusion can arise, we will write $L_0$ instead of $L_0(D_0)$. We emphasize that $C_0(\Sigma)$ is considered to be an element of $D_0$, so that for all $g \in G$ we have $R_C(1, g)$.

Let $W$ the collection of finite words in the alphabet $A_0 = D_0 \cup (G \setminus 1)$.

**Definition 3.17** (The model $M_0(\Sigma)$). Consider the language $L(D_0)$ obtained from $L_0(D_0)$ by adding a relation symbol $R_w$ for each tuple $w \in W$. Writing $w = (\delta_1, \delta_2, \ldots, \delta_k)$, we obtain a structure $M_0$ by interpreting each $R_w$ as the relation $R_{\delta_1} \circ R_{\delta_2} \cdots \circ R_{\delta_k}$.

In other words,

$$R_w(x, y) \iff \exists z_0 \exists z_1 \ldots \exists z_k \left( z_0 = x \land (z_1 = y) \land \bigwedge_{i=0}^{k-1} R_{\delta_i}(z_i, z_{i+1}) \right).$$

This structure will be called the **mapping class group model**, and its universe will be written $M$, in order to distinguish group elements viewed as elements of the universe from group elements parametrizing relation symbols. When no confusion can arise, we will write $L$ instead of $L(D_0)$.

**Definition 3.18** (Downward closed). A subset $D \subset D_0$ is **downward closed** if for every proper subdomain $D \subset D$ and every $E \subset D$, we have $E \subset D$ as well.

**Definition 3.19** (The models $M(\Sigma)$ for $D \subset D_0$). Let $D \subset D_0$ be a $G$–invariant and downward closed subset of $D_0$ that contains $C$. We define the model $M(\Sigma)$ to have the same universe as $M_0(\Sigma)$, but we restrict to the alphabet $A = D \cup (G \setminus 1)$. These models will collectively be called **restricted mapping class group models**.

**Definition 3.20** (Language $L(D)$). Let $D \subset D_0$ be $G$–invariant and downward closed. The **language adapted to $D$** is the sublanguage $L(D) \subset L_0$ defined by including only symbols of the form $R_w$ where no $E \subset D_0 \setminus D$ appears in $w$. In other words, $w$ is a word in the restricted alphabet $A$.

**Remark 3.21.** It is crucial to note that, whereas the group $G$ is a group of automorphisms of $C(\Sigma)$ (viewed as a structure in the language of graph theory), it is generally not a group of automorphisms of the structures $M(D)$. This is because the relations are not $G$ invariant. The group $G$ is a group of permutations of the universe $M$.

**Remark 3.22.** In light of Remark 3.21, the group $G$ can play several roles. It is identified with the universe of $M$, and it appears as a relational symbol. In the latter of these roles, it is natural for the group $G$ to act on words on the right. In the sequel,
we will also need to consider $G$ as a group of automorphisms of $C(\Sigma)$. In this case, it is convenient to view curves as left cosets of their stabilizer, and then the group $G$ acts naturally on the left.

Let $\text{Th}(\mathcal{M})$ be the theory of $\mathcal{M}$, and let $\mathcal{N}$ be a model of $\text{Th}(\mathcal{M})$ with universe $\mathcal{N}$. Given $a, b \in \mathcal{N}$ and $w = (\delta_1, \ldots, \delta_k) \in \mathcal{W}$ such that $\mathcal{N} \models R_w(a, b)$, we define a $w$-path from $a$ to $b$ to be a sequence

$$s : a = a_0, a_1, \ldots, a_k = b$$

such that $a_i \in \mathcal{N}$ for $0 \leq i \leq k$, and such that

$$\mathcal{N} \models \bigwedge_{i=0}^{k-1} R_{\delta_i}(z_i, z_{i+1})$$

for all $i = 0, \ldots, k$.

![Figure 1. A $w$-path witnessing $R_w(a, b)$](image)

4. Interpretations

In this section, we will prove some general results about interpretations and bi-interpretations of various structures with $\mathcal{M}_0$ and $\mathcal{M}$. We will note consequences concerning quantifier elimination and $\omega$–stability, though we will relegate proofs of those properties to later sections.

The following lemma is a general result which will allow us to prove that many complexes associated to surfaces are interpretable in $\mathcal{M}_0$ and in the curve graph. Recall that a structure is relational if its underlying language has no functions and no constants.

**Lemma 4.1.** Suppose that the mapping class group $G$ acts by isomorphisms on a relational structure $\mathcal{B}$ over a language $\mathcal{L}_B$. Let $\mathcal{D} \subseteq \mathcal{D}_0$ be $G$–invariant and downward closed, and suppose furthermore that:

1. The universe $B$ of $\mathcal{B}$ is a union of finitely many $G$–orbits.
2. For each symbol $R^{(k)} \in \mathcal{L}_B$, the image of $R_B \subseteq B^k$ in $G \setminus B^k$ is either finite or cofinite.
(3) For any \( b \in B^{<\omega} \) there is a (necessarily unique) \( D_b \in \mathcal{D} \) such that \( \text{Stab}_G(b) \) contains \( \text{Stab}^+(D_b) \) with finite index.

Then there is an interpretation \( \tilde{\zeta} \) of \( \mathcal{B} \) in \( M \) by which an element \( b \in B \) is sent to the corresponding imaginary element in \( M/\text{Stab}(b) \).

**Proof.** For \( b \in B \), we define the equivalence relation \( S_b \) on \( G \) by \( (g, h) \in S_b \) if and only if \( g(b) = h(b) \). Notice that this equivalence relation is definable in \( M \) without parameters, since we can get express it as a finite union of relations of the form \( R_{\sigma, D_b} \), with \( \sigma \in G \). If \( \{b_1, b_2, \ldots, b_m\} \) are representatives of the orbits of \( B \) under \( G \), then there is a natural bijection between \( B \) and

\[
\prod_{j=1}^m M/S_{b_j}.
\]

Observe that this latter union can be encoded as the quotient of a definable subset of \( M \) by a definable equivalence relation.

Now, let \( R^{(k)} \in \mathcal{L}_B \) be a relation. Without loss of generality, we assume that

\[
R_{\mathcal{B}}/G \subseteq B^k/G
\]

is finite. It suffices to show that for each fixed

\[
\tilde{i} \in \{1, 2, \ldots m\}^k,
\]

the intersection

\[
R_{\mathcal{B}} \cap (O(b_{i_1}) \times O(b_{i_2}) \times \cdots \times O(b_{i_k}))
\]

is the pullback of a formula \( \psi^R_i(x_1, \ldots x_k) \) which is equivariant under

\[
S_{b_{i_1}} \times \cdots \times S_{b_{i_k}},
\]

upon restriction to

\[
O(b_{i_1}) \times O(b_{i_2}) \times \cdots \times O(b_{i_k}).
\]

Let \( \mathcal{T}_\tilde{i} \) be a finite collection of \((k-1)\)-tuples \((g_{1}, \ldots, g_{k-1})\) of mapping classes such that every tuple

\[
(b_1, b_2, \ldots, b_k) \in O(b_{i_1}) \times O(b_{i_2}) \times \cdots \times O(b_{i_k})
\]

is the translate of a tuple of the form

\[
(b_{i_1}, g_1(b_{i_2}), g_2(b_{i_3}), \ldots, g_{k-1}(b_{i_k})).
\]

Then, we take the (abbreviated) quantifier free formula

\[
\psi^R_i(x) = \bigvee_{\tilde{i} \in \{1, \ldots, m\}^k} \bigvee_{\tilde{\tau} \in \mathcal{T}_\tilde{i}} \bigwedge_{j=2}^k (S_{b_{i_1}} \circ R_{\tau_{j-1}} \circ S_{b_{i_j}})(x_1, x_j),
\]

which has the desired property. \( \square \)
Remark 4.2. The encoding of 
\[ \prod_{j=1}^{k} M/S_{b_j} \]
as a quotient by a single equivalence relation can be done in such a way that the resulting interpretation \( \psi^R \) of \( R \) is quantifier free.

The following results from Theorem 9.5 below.

**Corollary 4.3.** Let \( \mathcal{B} \) be interpretable in \( M \) as in Lemma 4.1. Then \( \mathcal{B} \) is \( \omega \)-stable.

**Corollary 4.4.** The theory of the curve graph is \( \omega \)-stable.

Now, let \( \mathcal{D} \) and \( \mathcal{B} \) be as in Lemma 4.1. Assume furthermore that \( \mathcal{D} \) is the collection of all \( D \in \mathcal{D}_0 \) such that \( \text{Stab}^+(D) \) is the stabilizer of a suitable \( b \in B^{<\omega} \). For a tuple \( b \in B^{<\omega} \), we let \( \phi_b(x) \) be a formula isolating \( \text{tp}(b) \). Suppose that 
\[ \delta = (\delta_1, \delta_2, \ldots, \delta_k) \]
is a tuple of elements of \( B \) with trivial stabilizer, and let \( X \subseteq C^{[W_1]} \) be the set of realizations of \( \text{tp}(\delta) \equiv \phi_\delta(x) \). The map \( \eta \) defined by
\[ \eta: g \mapsto g\delta \]
is a bijection. We call such a \( \delta \) a rigid tuple.

**Lemma 4.5** (\( \mathcal{B} \) and \( M \) are bi-interpretable). Suppose that \( \mathcal{B} \) admits a rigid tuple. We have that \( \bar{\eta} = (\eta, X, =) \) is an interpretation of \( M \) in \( \mathcal{B} \). If \( \bar{\zeta} \) is the interpretation of \( \mathcal{B} \) in \( M \) provided by Lemma 4.1 then \( (\bar{\zeta}, \bar{\eta}) \) is a bi-interpretation between \( \mathcal{B} \) and \( M \).

**Proof.** For \( g \in G \) and \( D \in \mathcal{D} \), it suffices to check that the relations \( R_g \) and \( R_D \) are the preimages of the set of solutions of suitable definable relations \( \psi_g \) and \( \psi_D \) on \( X \) under \( \eta \times \eta \).

For \( g \in G \), we write 
\[ \psi_g(x, y) \equiv \bigwedge_{j=1}^{k} \phi_{g,\delta_j,\delta}(y_j, x). \]

It follows that:
\[ \mathcal{B} \models \psi_g(\eta(h), \eta(h')) \iff \mathcal{B} \models \bigwedge_{j=1}^{k} \phi_{g,\delta_j,\delta}(h' \cdot \delta_j, h \cdot \delta) \iff \]
\[ \mathcal{B} \models \bigwedge_{j=1}^{k} \phi_{g,\delta_j,\delta}(h^{-1}h' \cdot \delta_j, \delta) \iff h^{-1}h' = g \iff \mathcal{M} \models R_g(h, h'). \]
For $D \in D_0$, choose a finite tuple $(c_1, c_2, \ldots, c_l) \in B^{<\omega}$ such that $\text{Stab}^+(D^\perp)$ coincides with the stabilizer of $(c_1, c_2, \ldots, c_l)$, and let

$$
\psi_D(x, y) \equiv \exists z_1, z_2, \ldots, z_l \bigwedge_{j=1}^l \phi_{c_j,\delta}(z_j, x) \land \phi_{c_j,\delta}(z_j, y)
$$

It is straightforward to verify that $\psi_D$ interprets $R_D$.

Now, let $\tilde{\xi}_1 = \bar{\eta} \circ \bar{\zeta}$ and $\tilde{\xi}_2 = \bar{\zeta} \circ \bar{\eta}$. We need to show that the underlying maps $\xi_1$ and $\xi_2$ are definable without parameters in $B$ and in $M$, respectively.

To this end, let $\{b_1, b_2, \ldots, b_m\}$ the representatives of the orbits of $B$ under $G$ as used in the construction of $\bar{\zeta}$. Given $b \in B$, we denote by $\hat{b}$ the unique representative $b_i$ satisfying $O_G(b_i) = O_G(b)$.

On the one hand, we have that $\bar{\eta}$ sends $g \in M$ to $g\delta \in B^k$, which in turn gets sent to a $k$-tuple of equivalence classes $(gS\delta_j)_{j=1}^k$ by $\bar{\zeta}$. Note that $gS\delta_j = \{h \in M \mid S_{\delta_j}(g, h)\}$.

Since $S_{\delta_j}$ is definable in $M$ without parameters, definability of a predicate $\Gamma_{\tilde{\xi}_2}$ cutting out the graph of $\xi_2$ follows.

On the other hand, $\bar{\zeta}$ sends $b \in B$ to the unique coset $g_bS_{\hat{b}} \subseteq M = G$ consisting of those elements sending $\hat{b}$ to $b$, which is in turn sent by $\eta$ to the collection $X_b := \{h\delta \mid h \in g_bS_{\hat{b}}\}$.

Note, however, that

$$
X_b = \{g_bx \mid \text{tp}(x, \hat{b}) = \text{tp}(\delta, \hat{b})\} = \{g_bx \mid \text{tp}(g_bx, g_b\hat{b}) = \text{tp}(\delta, \hat{b})\} = \{y \mid \text{tp}(y, b) = \text{tp}(\delta, \hat{b})\}.
$$

By assumption, $\phi_{b}(u, x)$ is chosen such that $\phi(\beta, x)$ defines the set $X_\beta$ for any $\beta \in B$ satisfying $\hat{\beta} = \hat{\alpha}$. It follows that we may set

$$
\Gamma_{\tilde{\xi}_1}(x, y) \equiv \bigvee_{j=1}^k \phi_{b_j}(x) \land \phi_{b_j,\delta}(x, y),
$$

whence the desired conclusion follows. See Definition 2.12 for the definition of $\Gamma_{\tilde{\xi}_1}(x, y)$.

**Remark 4.6.** Suppose that for each tuple $\alpha$, we have that $\phi_{\alpha}$ is an existential formula. Then for all $w \in \mathcal{W}$, the relation

$$
\eta[R_w] \subseteq B^k \times B^k
$$

is definable in $B^k$.
and the predicate $\Gamma_{\xi_1}(x,y)$ are definable by an existential formula.

The following lemma relies on Theorem 10.18 which says that $M$ has absolute quantifier elimination.

**Lemma 4.7 (B has relative QE).** Assume the hypotheses of Lemma 4.5. Suppose furthermore that the type of each finite tuple in $M$ is isolated by an existential formula. Then $B$ has quantifier elimination relative to the collection of $\exists$–formulae.

**Proof.** Let $\phi(x)$ be a (possibly multivariable) parameter–free formula in $L_B$. The interpretation $\zeta[\phi(x)]$ is equivalent to a Boolean combination of quantifier–free formulae in $\text{Th}(M)$, by Theorem 10.18. In view of Remark 4.6, we wish to show that $\psi \equiv \eta[\zeta[\phi(x)]]$ is equivalent to a Boolean combination of formulae which are existential or universal.

Note that under $\eta$, relations in $M$ are interpreted in $B$ via existential formulae. Thus, an atomic quantifier–free formula in $M$ can be interpreted as an existential or universal formula in $B$.

It suffices to show that the pullback $\psi'(x)$ by $\xi_1$ of an existential or universal formula $\psi(y)$ contained in the image of $\xi_1$ is existential or universal, respectively. To this end, we note two different ways of expressing $\psi'(y)$:

$$B \models \forall x (\psi'(x) \leftrightarrow (\exists y \psi(y) \land \Gamma_{\eta\zeta}(x,y)))$$

$$B \models \forall x (\psi'(x) \leftrightarrow (\forall y \neg \Gamma_{\eta\zeta}(x,y) \lor \psi(y))).$$

The first of these expressions can be used for existential formulae, and the second for universal formulae. This establishes the lemma. □

**Definition 4.8 (Strongly rigid tuple).** We say that a tuple $W$ of vertices a graph $(V,E)$ is **strongly rigid** if any for any partial isomorphism of graphs $f : W \rightarrow W' \subseteq V$, there exists a unique automorphism $g$ of $(V,E)$ extending $f$.

The following observation is straightforward.

**Observation 4.9.** Let $B$ be a graph. The type of a strongly rigid tuple of elements is isolated by a quantifier-free formula. If a graph $B$ admits an exhaustion by strongly rigid tuples with trivial point-wise stabilizer, then every orbit of a tuple $\alpha$ of elements is the set of solutions of a suitable existential formula $\phi_{\alpha}(x)$ which isolates $tp(\alpha)$.

In [AL16], Aramayona and Leininger prove the following:

**Theorem 4.10 (Aramayona-Leininger [AL16]).** Let $C(\Sigma)$ be the curve graph of a closed orientable surface which is not a torus with two punctures. Then there is a sequence

$$W_1 \subset W_2 \subset \cdots \subset W_n \subset \cdots$$
of finite strongly rigid tuples of vertices of $\mathcal{C}(\Sigma)$ with trivial fix-point stabilizer such that
\[
\bigcup_{n \geq 1} W_n = V(\mathcal{C}(\Sigma)).
\]

The following is a consequence of Theorem 4.10, Theorem 9.1, and Lemma 4.7.

**Corollary 4.11** ($\mathcal{C}(\Sigma)$ has relative QE). *Suppose $\Sigma$ is not a torus with two boundary components. Then the structures $\mathcal{C}(\Sigma)$ and $\mathcal{M}(\Sigma)$ are bi-interpretable, and $\mathcal{C}(\Sigma)$ has quantifier elimination relative to the class of $3$-formulae.*

Many other geometric graphs associated to a surface $\Sigma$ can be interpreted in the curve graph, and $\omega$–stability and Morley rank bounds can be obtained quite easily.

Let $X(\Sigma)$ be a graph with sets of vertices $V(X(\Sigma))$ and sets of edges $E(X(\Sigma))$. We say that $X(\Sigma)$ is a **geometric graph** if the following conditions are satisfied:

1. The mapping class group $G$ acts on $X(\Sigma)$ via its action on curves and arcs, and $V(X(\Sigma))$ consists of finitely many $G$–orbits;
2. There exists a constant $N \geq 1$ such that each $v \in V(X(\Sigma))$ corresponds to a collection of $N$ curves and/or arcs;
3. There exists a constant $K \geq 0$ such that for all $v = \{\gamma_1, \ldots, \gamma_m\} \in V(X(\Sigma))$,
   the geometric intersection number satisfies $i(\gamma_i, \gamma_j) \leq K$ for all $i, j \in \{1, \ldots, m\}$;
4. The set $E(X(\Sigma))/G$ is finite.

For a geometric graph $X(\Sigma)$, let $v = \{\gamma_1, \ldots, \gamma_m\}$ be a vertex $X(\Sigma)$ and $D_v \in \mathcal{D}$ be the domain associated to the subsurface $\text{Fill}(v)$ filled by the curves and/or arcs defining $v$; that is, $D_v := D_{\text{Fill}(v)}$. Let $k(X(\Sigma))$ be the maximal length of an ascending chain
\[
\emptyset \subsetneq D_1 \subsetneq \ldots \subsetneq D_n = \mathcal{C}(\Sigma),
\]
where for $1 \leq i < n$ we have that $D_i$ is a connected domain contained in $D_i^\perp$.

**Corollary 4.12** ($\omega$-stability of other geometric graphs). *Suppose that $X(\Sigma)$ is a geometric graph. We have that $X(\Sigma)$ is interpretable in $\mathcal{C}(\Sigma)$ and its Morley rank is bounded above as follows:
\[
RM(X(\Sigma)) \leq \omega^{k(X(\Sigma))}.
\]
In particular, $X(\Sigma)$ is $\omega$-stable.*

**Proof.** We apply Lemma 4.1 and Corollary 4.3. The bound of the Morley rank follows from Theorem 11.1 and Corollary 11.4 below. \qed
Corollary 4.12 implies Corollary 1.5 through standard combinatorial topology arguments.

**Corollary 4.13.** Let \( \Sigma \) be a surface of genus \( g \) and \( b \) punctures, let \( P = P(\Sigma) \) denote the pants graph of \( \Sigma \), let \( S = S(\Sigma) \) denote the separating curve graph, and let \( A = A(\Sigma) \) denote the arc graph. For \( S \), assume that \( g \geq 2 \) and \( b \leq 1 \), and for \( A \), assume that \( g \geq 2 \) and \( b = 1 \). Then we have

\[
RM(A) \leq \omega^{3g-4}, \quad RM(S) \leq \omega^{3g-4}, \quad RM(P) \leq \omega^2,
\]

and \( C(\Sigma) \) is not interpretable in \( S \) nor in \( A \), nor in \( P \) provided \( 3g + b - 2 > 2 \).

**Proof.** A vertex \( v \) of \( A \) is given by a properly embedded essential arc \( \alpha \), with both endpoints on the boundary component. If \( \alpha \) is nonseparating then the result of cutting \( \Sigma \) open along \( \alpha \) is a surface of genus \( g - 1 \) and one boundary component. If \( \alpha \) is separating then each component of the result of cutting \( \Sigma \) open along \( \alpha \) has genus at most \( g - 1 \) and has one boundary component. The longest chain of domains in \( D_v^- \) is \( 3g - 4 \), so that \( k(A) \leq 3g - 4 \).

A vertex \( v \) of \( S \) is given by a separating curve \( \gamma_v \) on \( \Sigma \). We have that \( \Sigma \setminus \gamma_v \) is the disjoint union of surfaces \( \Sigma_1 \) and \( \Sigma_2 \), and that both \( \Sigma_1 \) and \( \Sigma_2 \) must have positive genus. If \( \Sigma_1 \) is exactly a torus with at most two boundary components, then \( \Sigma_2 \) is homeomorphic to a surface with one boundary component and genus lower by one. In any case, the longest chain of domains in \( D_v^- \) is \( 3g - 4 \). Thus, \( k(S) \leq 3g - 4 \).

A vertex \( v \) of \( P \) is given by a pants decomposition of \( \Sigma \). If \( D \) is a nontrivial, connected, proper domain contained in \( D_v^- \) then \( D \) is a component of the pants decomposition. In particular, \( k(P) = 2 \).

Suppose \( C(\Sigma) \) were interpretable in \( P \). Write \( \mathcal{D} \) for the \( G \)-invariant downward closed collection of domains generated by pants decompositions of \( \Sigma \). Then \( \mathcal{M} = \mathcal{M}(\mathcal{D}) \) has Morley rank \( \omega^2 \) by Corollary 11.4 and \( X \) is interpretable in \( \mathcal{M}(\mathcal{D}) \). We have a chain of interpretations as follows: \( \mathcal{M}(\mathcal{D}_0) \) is interpretable in \( C(\Sigma) \) is interpretable in \( P \) is interpretable in \( \mathcal{M}(\mathcal{D}) \), and \( RM(\mathcal{M}(\mathcal{D}_0)) = \omega^{3g+b-2} \). Theorem 11.1 gives the desired contradiction, since it follows that the rank of a definable set of imaginaries is bounded from above by that of a definable set of real tuples. The arguments for \( A \) and \( S \) are analogous. \( \square \)

A philosophical reason for these bounds on Morley rank can be formulated thus: in all three graphs, one is highly constrained in the edge relation: one does not have a maximal subsurface curve graph’s worth of choices for a neighbor in these graphs. This is especially apparent in the pants graph. The bounds on the Morley rank can thus be thought of as a rigorous expression of this restriction.

To close this section, we give a proof of Corollary 1.2 as announced in the introduction. We will in fact prove a somewhat more general statement.
Corollary 4.14. Let $\Sigma$ be a surface that is not a torus with two boundary components. Let $Q \subseteq C(\Sigma)^2$ be a binary predicate with the property that for any $n \in \mathbb{N}$, there exist pairs $(\alpha, \beta) \in Q$ and $(\alpha', \beta') \in C(\Sigma)^2 \setminus Q$ with
$$d(\alpha, \beta), d(\alpha', \beta') \geq n.$$ Then $Q$ is not $\emptyset$-definable in $C(\Sigma)$.

The following is immediate from the existence of pseudo-Anosov elements in the Torelli group (which requires us to exclude a torus with at most one boundary component), which have positive translation length in $C(\Sigma)$ and which preserve the algebraic intersection pairing.

Corollary 4.15. Suppose $\Sigma$ has positive genus and is not a torus with fewer than three boundary components. For each $n > 0$, the predicates that state that two simple closed curves have integral algebraic intersection number $\pm n$ is not $\emptyset$-definable. Similarly, the predicate that states that two curves have algebraic intersection number $0 \mod 2$ is not $\emptyset$-definable.

Proof of Corollary 4.14. Suppose the contrary. Lemma 4.1 implies that $C(\Sigma)$ is interpretable in $M$. A predicate $Q$ is defined by a parameter free formula $\theta$ in $L(D_0)$, in the sense that $Q(x, y)$ is given by a conjunction of formulae interpreting $C$ in $M$, together with $\theta$ itself. Since $M$ has quantifier elimination by Theorem 10.18, we may assume that $\theta$ is quantifier free, so that $\theta$ is a Boolean combination of relations in the language $L(D_0)$, some positive and some negated.

If $D$ is a proper domain, then the relation $R_D$ gives rise to a priori bounds on distances between curves that are $D$-related, as is proved in Lemma 6.2 below. In terms of mapping classes and curves, this means that if $g$ is a mapping class preserving $D$ and if $\gamma \in C_0(\Sigma)$, then there is a $K = K(D, \gamma)$ such that $d_{C(\Sigma)}(\gamma, g(\gamma)) \leq K$.

Expressing $\theta$ as a Boolean combination of relational formulae $R_{w_i}$, suppose
$$\{w_1, \ldots, w_k\}$$
are the words which appear in relations, where each domain appearing in each $w_i$ is proper. Let $\gamma \in C_0(\Sigma)$ be a simple closed curve. From Lemma 6.2, we have that there exists a bound $K = K(\gamma, w_1, \ldots, w_k)$ such that
$$\bigvee_{i=1}^k R_{w_i}(\xi(\gamma), \zeta(\gamma')) \rightarrow d_{C(\Sigma)}(\gamma, \gamma') \leq K.$$ We remark that here we are slightly abusing notation since $\zeta(\gamma)$ is an equivalence class, though it is not difficult to see that a choice of representative within an equivalence class does not affect any substantive change.

By assumption, for any bound $K$, there exist curves $\{\alpha, \beta, \alpha', \beta'\}$, with
$$d_{C(\Sigma)}(\alpha, \beta) > K, \quad d_{C(\Sigma)}(\alpha', \beta') > K,$$
and such that $C(\Sigma) \models Q(\alpha, \beta) \land \neg Q(\alpha', \beta')$. Thus, given a finite list of words \( \{w_1, \ldots, w_k\} \) wherein all the domains occurring in these words are proper, there exist curves \( \{\alpha, \beta, \alpha', \beta'\} \) such that

\[
\mathcal{M} \models \left( \bigwedge_{i=1}^{k} -R_{w_i}(\zeta(\alpha), \zeta(\beta)) \right) \land \left( \bigwedge_{i=1}^{k} -R_{w_i}(\zeta(\alpha'), \zeta(\beta')) \right),
\]

and such that $C(\Sigma) \models Q(\alpha, \beta) \land \neg Q(\alpha', \beta')$, and one can assume that the distances between these curves in $C(\Sigma)$ exceed any prescribed bound.

We are therefore reduced to two cases. If there exists a finite list of words \( \{w_1, \ldots, w_k\} \) such that

\[
\mathcal{M} \models \left( \forall x \forall y \left( \bigwedge_{i=1}^{k} -R_{w_i}(x, y) \rightarrow \theta(x, y) \right) \right),
\]

then there exist curves \( \{\alpha, \beta, \alpha', \beta'\} \) such that

\[
\mathcal{M} \models \theta(\zeta(\alpha), \zeta(\beta)) \land \theta(\zeta(\alpha'), \zeta(\beta')) \land C(\Sigma) \models Q(\alpha, \beta) \land \neg Q(\alpha', \beta').
\]

Otherwise, we have

\[
\mathcal{M} \models \left( \forall x \forall y \left( \theta(x, y) \rightarrow R_w(x, y) \right) \right),
\]

for some word $w$, wherein all domains are proper. It follows in this case that there exist curves \( \{\alpha, \beta, \alpha', \beta'\} \) such that

\[
\mathcal{M} \models -\theta(\zeta(\alpha), \zeta(\beta)) \land -\theta(\zeta(\alpha'), \zeta(\beta')) \land C(\Sigma) \models Q(\alpha, \beta) \land \neg Q(\alpha', \beta').
\]

In both cases, we obtain a contradiction. \( \square \)

5. The relational theory of $\mathcal{M}(\Sigma)$

In this section, we record a calculus for manipulating the relations $R_w$ occurring in the language $\mathcal{L} = \mathcal{L}(D)$.

5.1. Preliminaries. We note some properties enjoyed by the relations $R_g$ in $\mathcal{M}$. All of them are elementary conditions, which is to say they can be captured by a sentence of $\mathcal{L}$ and thus will hold in any model of Th($\mathcal{M}$).

(a) $R_{1_g}$ is the identity relation.
(b) $R_{gh} = R_{hg}$ for any $g, h \in G$.
(c) $R_D \cap R_{D'} = R_{D \cap D'}$.
(d) Suppose that $R_D(1, g)$ for $g \in G$. Then $R_{D,g} = R_{g,D} = R_D$.
(e) $R_D \subseteq R_{D'}$ if and only if $D \subseteq D'$. In particular, in this situation $R_{D',1_D} = R_{D',D} = R_{D',D'}$.
(f) Suppose $D_1, D_2 \in D$ satisfy $D_1 \perp D_2$. Then, for any $x, y$ such that $R_{D_1,D_2}(x, y)$ we also have $R_{D_2,D_1}(x, y)$.
(g) Suppose $D \in D$ and $g \in G$. Then $R_{g,D}(x, y)$ is equivalent with $R_{g^{-1}D,g}$. 

Let $\mathcal{N}$ be an $\aleph_0$-saturated model of $T$. Notice that in the original structure $\mathcal{M}$, the transitive closure of $\{R_{D_1}, R_{D_2}\}$ coincides with $R_{D_1 \cup D_2}$. This equality cannot be expressed by a first order $\mathcal{L}$–sentence and will not hold in $\mathcal{N}$ as soon as $\mathcal{N}$ is saturated.

**Definition 5.1** (Inclusion relation $\subseteq$ in $\mathcal{A}$). We extend the inclusion relation $\subseteq$ to the whole of $\mathcal{A}$ by declaring the following:
- $g \subseteq D$ if and only if $R_D(1, g)$;
- $g \subseteq h$ if and only if $g = h$.

**Definition 5.2** (Support of $g$). Let $g \in G$. The *support of* $g$, which we denote by $\text{supp} g$, is the isotopy class of the smallest essential subsurface $\Sigma_0 \subseteq \Sigma$ for which there exists a homeomorphism representative $\tilde{g}$ which restricts to the identity on $\Sigma \setminus \Sigma_0$.

**Definition 5.3** (Orthogonal words). We say that $g \in G \setminus \{1\}$ is *orthogonal* to $D \in \mathcal{D}$, and write

$$g \perp D,$$

if $gD = D$. Similarly, given $u \in W$ and $D \in \mathcal{D}$ we say that $u$ is *orthogonal* to $D$, and write

$$u \perp D,$$

if each letter of $u$ is orthogonal to $D$.

Strongly orthogonal sequences are defined analogously to orthogonal ones.

**Definition 5.4** (Strongly orthogonal words). Given $u \in W$ and $D \in \mathcal{D}$ we say that $u$ is *strongly orthogonal* to $D$, and write

$$u \perp^* D,$$

if each domain occurring in $u$ is strongly orthogonal to $D$, and each group element occurring in $u$ is orthogonal to $D$.

Notice that in any model of $\text{Th}(\mathcal{M})$, for a given $g \in G$ and $x \in M$, there is a unique $y$ such that $xR_g y$. In this situation we might write $y = xg$ and $xG = \{xg\}_{g \in G}$.

**Definition 5.5** ($R^*_D$-relation). Let $D \in \mathcal{D}$ be given. Let

$$\mathcal{A}(D) = \{X \in \mathcal{A} \mid X \subseteq D\},$$

and denote by $\mathcal{W}(D)$ the collection of all $w \in \mathcal{W}$ containing only instances of letters in $\mathcal{A}(D)$. We consider the following type-definable relation:

$$R^*_D(x, y) \equiv R_D(x, y) \land \bigwedge_{w \in \mathcal{W}(D)} \neg R_w(x, y)$$
The idea is that $R^*_p(x,y)$ holds whenever $x,y$ are $R_D$-related in a generic fashion. In other words they are not linked by any relation stronger than $R_D$; we will later show that in this situation $R_D$ is in some sense not only a minimal but the minimal relation between $x$ and $y$.

Using well-established ideas from mapping class group theory (which we will make explicit in the sequel), it is not difficult to see that the $R^*_p$ is finitely satisfiable in the usual mapping class group, which is to say there are mapping classes which are $D$–related but not $w$–related for elements $w \in \mathcal{W}_0 \subset \mathcal{W}$, where $\mathcal{W}_0$ is taken to be finite. However, it is generally not possible to realize $R^*_p$ in the usual mapping class group, since mapping classes supported on a given sufficiently complicated subsurface $\Sigma_0 \subset \Sigma$ are often expressible as products of mapping classes supported on proper subsurfaces of $\Sigma_0$. This observation will be later codified by the statement that $R^*_p$ is consistent and non-algebraic; see Corollary [6.3] and Lemma [7.2].

**Definition 5.6** (Strict $w$-sequence). Given $w \in \mathcal{W}$, write

$$w = \delta_1 \delta_2 \cdots \delta_k,$$

and let $R^*_w(x,y)$ stand for the composition of relations

$$R^*_{\delta_1} \circ R^*_{\delta_2} \circ \cdots \circ R^*_{\delta_k}$$

under the convention $R^*_g = R_g$ for $g \in G$. In other words, $R^*_w(x,y)$ is equivalent to the existence of a sequence

$$x_0 = x, x_1, \ldots, x_k = y$$

such that $R^*_w(x_{i-1}, x_i)$ for all $1 \leq i \leq k$. We will refer to any sequence as above as a strict $w$-sequence from $x$ to $y$.

Though it is not entirely obvious from the definition, one can use compactness to show that $R^*_w$ is expressible as a parameter–free type-definable condition as well.

**Remark 5.7.** We write $\phi(x,y) \in R^*_p(x,y)$ and $\phi(y) \in R^*_p(a,y)$ to indicate that $\phi$ is a finite subconjunction of terms, where the terms appear in the infinite conjunction defining the corresponding relations in $\mathcal{P}$. Thus, the inclusion relations have as their target the collection of all finite conjunctsions of relations $R^*_p(x,y)$ and $R^*_p(a,y)$.

**Definition 5.8** (The inclusion relation $\subseteq$ on $\mathcal{W}$). Given $u, v \in \mathcal{W}$ we write:

$$u \subseteq v$$

if we can write $v = v_1 v_2 \cdots v_k$ with $v_j \in \mathcal{A}$ and $u = u_1 u_2 \cdots u_k$ where for all $1 \leq j \leq k$ either:

- $v_j \in G$ and $u_j = v_j$
- $v_j \in D$ and either $u_j = D$ or $u_j \in \mathcal{W}(D)$

We write $u \subset v$ if $u_j \in \mathcal{W}(Z_j)$ for at least one index $j$ above.
This leads to the following observation:

**Observation 5.9.** Let \( w \in \mathcal{W} \) and suppose that 
\[
\{ z_i \}_{i=0}^{k}
\]
a \( w \)-sequence from \( x \) to \( y \). Then either \( \{ z_i \} \) is strict and thus lies in \( R_w^k(x,y) \), or there exists a \( w' \subset w \) and a refinement \( \{ z'_i \} \) of \( \{ z_i \} \) which is a \( w' \)-sequence from \( x \) to \( y \).

**Lemma 5.10** (Refining a \( w \)-sequence to a strict sequence). Given a \( w \)-sequence 
\[
\{ z_i \}_{i=0}^{k}
\]
between points \( x \) and \( y \) in a model \( \mathcal{N} \) of \( \text{Th}(\mathcal{M}) \), there exists a finite refinement \( \{ z'_i \} \) of \( \{ z_i \} \) and a \( w' \subset w \) such that \( \{ z'_i \} \) is a strict \( w' \)-sequence from \( x \) to \( y \).

Unfortunately, the existence of a strict sequence of a given type \( w \) between \( x \) and \( y \) does not determine in general the type \( \text{tp}(x,y) \). In order to fix this issue, one must require \( w \) to contain no redundancies, i.e. that \( w \) be reduced in a very specific sense. This is the focus of the next subsubsection.

5.2. **Word equivalence and cancellation.** Throughout this subsection and for the rest of the manuscript, we will implicitly assume (unless otherwise noted) that each word \( w \) has **connected letters**, that is if \( D \subseteq \mathcal{D} \) occurs in \( w \) then \( D \) is a connected domain.

Consider a word
\[
w = (\delta_1, \delta_2, \delta_3, \ldots, \delta_k) \in \mathcal{W}.
\]
We can obtain a new \( w' \in \mathcal{W} \) by applying one of the following fundamental moves to \( w \).

- **(Rm)** Deleting an instance of \( \text{Id}_G \).
- **(Cmp)** Replacing a subword of the form \( g_1, g_2 \) by \( g_1g_2 \).
- **(Swp)** Replacing a subsequence of the form \( (D_1, D_2) \) where \( D_1 \perp D_2 \) by \( (D_2, D_1) \).

We sometimes call this a **transposition**.

- **(Jmp)** Replacing a subsequence of the form \( (D, g) \) by \( (g, g^{-1}(D)) \) or vice versa.
- **(Abs\( G \))** Replacing a subword of the form \( (g, D) \) or \( (D, g) \) with \( D \), where \( g \subset D \).
- **(Abs\( \cap \))** Replacing a subsequence \( (D, D') \) or \( (D', D) \) by \( D \) in case \( D \subsetneq D' \).
- **(Abs\( = \))** Replacing a subsequence of the form \( (D, D) \) by \( D \).

We will occasionally say that two letters in a word \( w \in \mathcal{W} \) which can be transposed **commute** with each other. We remark that this is slightly misleading terminology, since a domain is not orthogonal to itself and hence does not commute with itself in the sense of transpositions. The move \( \text{Abs}_\cap \) provides a suitable framework for dealing with instances of the same domain within a word. We remark that if \( D \) is a non-annular domain and \( E \) is an annular domain that is peripheral to \( D \), then \( D \) and \( E \) commute with each other.
Definition 5.11 (Permutation of w). We say that w is a permutation of w' if it can be obtained from w by repeated applications of \{(Swp), (Jmp), (Cmp), (Rm)\} and their inverses.

Finally, consider the following class of moves:
(C) Replacing a subsequence of the form (D, D) by a word in \(W'(D)\).

We refer to all the operations listed above as elementary moves.

Definition 5.12 (Reduced word). We say that w is reduced if it cannot be brought by a permutation to a word to which one of
\{(Abs\(\subseteq\)), (Abs\(\subseteq\)), (C)\}
can be applied.

Definition 5.13 (Reduction of words, concatenation). We say that w' is a reduct of w (equivalently, w reduces to w') if w' can be obtained from w by a successive application of elementary moves. We say that w reduces to w' without cancellation if it reduces to w' without application of (C). We write \([w]\) for the equivalence class of a reduced word up to permutations, and \(w_1 \simeq w_2\) with \(w_2\) reduced if \(w_1 \in [w_2]\) without cancellation. Words \(w_1\) and \(w_2\) can be combined to a word \(w_1 w_2\) by concatenation.

The following observation follows from embedding words in a partially commutative monoid, and applying the solution to the word problem therein. The reader will find a discussion which implies this lemma in the proof of Lemma 5.21 below.

Lemma 5.14 (Existence of reduced words). Up to permutation, there is a unique reduced word \(w'\) that can be obtained from any given word \(w \in W\), provided the reduction is performed without cancellation.

Definition 5.15 (Partial order on words). We say that \(w_1 \preceq w_2\) if for some (possibly empty) collection of domains \(\{D_1, \ldots, D_k\}\) appearing in \(w_2\), we can replace each \(D_i\) by a word in \(W'(D_i)\) and apply a permutation to obtain \(w_1\).

The partial order on words, together with the complexity \(k(D)\) of a domain, allow us to define the associated ordinal of a word.

Definition 5.16 (Associated ordinal). Let \(w \in W\). We define the associated ordinal \(\text{Or}(w)\) of w inductively.
(1) For \(D = \emptyset\), we define \(\text{Or}(D) = 0\).
(2) For \(g \in M\), we define \(\text{Or}(g) = 0\).
(3) For \(\emptyset \neq D \subseteq C\) a domain of complexity \(k\), we define \(\text{Or}(D) = \omega^k\).
(4) Let \(w_1, w_2 \in W\) be words for which Or is defined. We define
\[\text{Or}(w_1 w_2) = \text{Or}(w_1) \oplus \text{Or}(w_2)\;\]
that is, we take the symmetric sum of the two ordinals.

The last part of the definition serves to distinguish $M$ from $M_0$. We remark that $\omega^k_{\text{max}}$ is the foundation rank of the partial order on words. Notice that the moves

$$\{(\text{Rm}), (\text{Cmp}), (\text{Swp}), (\text{Jmp})\}$$

preserve $\text{Or}(w)$, while the moves

$$\{(\text{Abs}_-), (\text{Abs}_-), (\text{C})\}$$

strictly decrease it.

A pertinent (if rather trivial) observation is the following:

**Observation 5.17.** Either $\text{Or}(w_1) = \text{Or}(w_2)$, in which case $w_1$ and $w_2$ are permutations of each other, or else $\text{Or}(w_1) < \text{Or}(w_2)$ or $\text{Or}(w_2) < \text{Or}(w_1)$.

5.3. **Straightening paths.** Let $u \in \mathcal{W}$, suppose that

$$a = \{a_i\}_{i=0}^k$$

is a $u$-sequence from $a_0$ to $a_k$. We assume that $u$ contains a subword which admits the application of some elementary move as above.

If (Rm) can be applied to $u_i = \text{Id}_G$ then there is a repetition $a_i = a_{i+1}$, and the result of deleting $a_i$ yields a $u'$-sequence from $a_0$ to $a_k$.

Suppose now that $\{a_i, a_{i+1}, a_{i+2}\}$ is a subsequence corresponding to the subword word $u_i u_{i+1}$, where $u_i u_{i+1}$ now admits the application of an elementary move. Let $a'$ be the result of deleting $a_{i+1}$ from the sequence $a$. We distinguish several cases, according to the types of moves that can be applied:

1. In case of

$$\{(\text{Cmp}), (\text{Abs}_G), (\text{Abs}_-)\},$$

the sequence $a'$ is an $u'$-sequence from $x$ to $y$, where $u'$ is the result of the application of the particular move to $u$, along the subword $u_i u_{i+1}$. It can be easily checked that in this case $a'$ is a strict $u'$-sequence if and only if $a$ is a strict $u$-sequence.

2. In case of (Swp) or (Cmp) being applied to the subword $u_i u_{i+1}$, then by property (f) or (g) respectively there exists $a'_{i+1}$ such that the result of replacing $a_{i+1}$ by $a'_{i+1}$ is an $u'$-sequence, where $u$ is the result of applying (Swp) or (Cmp) to $u$ respectively. If $a$ is strict then so is $a'$.

3. If $(u_i, u_{i+1}) = (D, D)$ for some $D \in \mathcal{D}$, then clearly $a'$ is an $u_0$-sequence from $a_0$ to $a_k$, where $u_0$ is the result of applying move $(\text{Abs}_-)$ to $u_i u_{i+1}$. However, if $a$ is strict there are two mutually incompatible situations:

   - If $R^+_D(a_i, a_{i+2})$ then $a'$ is a strict $u_0$-sequence from $a_0$ to $a_k$. 

\begin{itemize}
\item $R_w(a_i, a_{i+2})$ for some $w \in W(D)$, in which case $a'$ is a (possibly non-strict) $u_1$-sequence from $a_0$ to $a_k$, where $u_1$ is the result of replacing $DD$ by $w$ via a move of type $(C)$.
\end{itemize}

\textit{Remark 5.18.} Notice that given $u$ the reduction process sketched above works starting from any (strict) $u$-sequence from $a_0$ to $a_k$. This freedom of choice will be of importance for Lemma \textbf{5.4}.

We say that $w$ and $w'$ are equivalent if one can get from one to the other applying moves from $(Rm)$ to $(Abs_\leq)$ and their inverses. The following lemma is straightforward, and we omit a proof.

\textbf{Lemma 5.19.} If $w$ can be obtained from $w'$ by repeated application of the moves from $(Rm)$ to $(Abs_\leq)$ and their inverses, then

$$\text{Th}(\mathcal{M}) \vdash (\forall x \forall y)(R_w(x, y) \leftrightarrow R_{w'}(x, y)).$$

If $w \simeq w'$, then

$$\text{Th}(\mathcal{M}) \vdash (\forall x \forall y)(R^*_w(x, y) \leftrightarrow R^*_{w'}(x, y)).$$

For all $w, w' \in W$, if $w$ reduces to $w'$ then

$$\text{Th}(\mathcal{M}) \vdash (\forall x \forall y)(R_{w'}(x, y) \rightarrow R_w(x, y)).$$

Combining Lemma \textbf{5.10} with Observation \textbf{5.17} one gets the following consequence:

\textbf{Corollary 5.20.} Let $N$ be a model of Th($\mathcal{M}$), and let $a, b \in N$ be such that $R_u(a, b)$ for a suitable $u \in W$. Then there exists a reduct $w \in W$ of $u$ such that $R^*_w(a, b)$.

The following lemma establishes the predictability of the elementary moves as operations on words in $\mathcal{A}$. The proof makes a detour through combinatorial group theory.

\textbf{Lemma 5.21.} Suppose that a word $w$ can be written as $w = uDv$, where $uD$ and $Dv$ are both reduced. Then all the instances of $D$ in $w$ survive in any reduct of $w$. In particular $w$ cannot reduce to the identity.

\textit{Proof.} Let $u$ and $v$ be expressed as

$$uD = g_1D_1g_2D_2 \cdots g_kD_kD$$

and

$$Dv = DE_1h_1E_2h_2 \cdots E_jh_j,$$

where here $g_i, h_i \in G$ and where $D_i, E_i \in D$, and where both of these words are reduced up to applications of the move $(Rm)$. This way, we allow elements of $G$ to
be the identity, thus serving as dummy “placeholders” without otherwise affecting the irreducibility of the words. For nontrivial group elements \( g_i \) and \( h_i \) occurring in the expressions for \( u \) and \( v \) respectively, we assume that no permutation can be applied so that the move \( (\text{Abs}_G) \) is applicable. In order to apply the move \( (\text{Swp}) \), it is necessary for there to be two letters of the form \( D_i D_j \) which are adjacent and not separated by a nontrivial group element, and such that \( D_i \perp D_j \). Thus, in the expressions for \( u \) and \( v \), any nontrivial group element \( g_i \) or \( h_i \) forms a \( (\text{Swp}) \)--wall, which is to say that it is not possible to apply \( \text{Swp} \) to move \( D_i \) to the right of \( g_j \neq 1 \) for some \( j > i \), nor is it possible to apply \( \text{Swp} \) to move \( E_i \) to the left for \( h_j \neq 1 \) for some \( j \leq i \). Thus, we may assume that

\[
u D = D_1 \cdots D_k D
\]

and that

\[
D v = D E_1 E_2 \cdots E_j.
\]

Here, the reader may imagine that we have moved all nontrivial group elements to the far left of \( u \) and to the far right of \( v \) by applications of the move \( (\text{Imp}) \). This sort of construction will recur later in this paper, in the guise of left and right normal forms (cf. Definition 8.7 below).

Now, we can consider the formal strings of domains

\[
\{D_1, \ldots, D_k, E_1, \ldots, E_j, D\},
\]

and embed them in a partially commutative monoid \( Y \). Precisely, we proceed as follows. Each domain on this list corresponds to a connected subsurface. We take one positive generator for each distinct connected subsurface, and impose the relation that two generators \( X_i \) and \( X_j \) commute if the corresponding subsurfaces are orthogonal.

This partially commutative monoid embeds in the corresponding partially commutative group, i.e. a right-angled Artin group; this fact is true for all Artin–Tits monoids and their corresponding Artin–Tits groups \([\text{Par}02]\). The solution to the word problem in right-angled Artin groups says that a word \( w_1 \) and \( w_2 \) represent the same element in the group if and only if \( w_1 \) can be transformed into \( w_2 \) by applying a sequence of free reductions, insertions of words of the form \( gg^{-1} \), and by switching adjacent letters which commute with each other. If \( w_1 \) and \( w_2 \) are words which cannot be shortened by commutations and free reductions, then \( w_1 \) and \( w_2 \) are equal as elements in the right-angled Artin group if and only if \( w_1 \) can be transformed into \( w_2 \) by a sequence of commutations of adjacent letters (see \([\text{LWZ}90, \text{CF}69]\)).

In \( Y \), we are only considering positive generators and because the move \( (C) \) is ruled out in the words \( uD \) and \( Dv \), we have that no free reductions occur. Thus, two words in the generators of \( Y \) represent the same element of \( Y \) if and only if one can be transformed into the other by a sequence of commutations of adjacent letters.
which correspond to domains that can be transposed. That is, we only allow local moves of the form $X_i X_j$ replaced by $X_j X_i$. In particular, two elements of $Y$ which are equal in the right-angled Artin group generated by $Y$ were already equal in $Y$.

Now, suppose that $D$ cancels with the occurrence of the letter $D$ in $u$ or in $v$. Here, we are of course referring to reduction not in the group theoretic sense, but in the sense of move (C). It follows then that there is a reduction diagram for $w$ (see [Kim08, KK15], for example). That is, either $u = u_1 D u_2$ with each surface in $u_2$ orthogonal to $D$, or $v = v_1 D v_2$, with each surface in $v_1$ orthogonal to $D$. It follows that either $u D$ or $D v$ is reducible, a contradiction. \[\square\]

Embedding formal strings of domains in a partially commutative monoid as in Lemma 5.21 gives us the following:

**Corollary 5.22.** The operation of concatenation followed by reduction without cancellation, viewed as an operation on equivalence classes $[w]$ of reduced words $w \in W$, is well-defined and associative.

We denote the operation above by $\ast$.

5.3.1. **Transitivity.** Let $M$ be a structure in a given first order language $\mathcal{L}$ such that $\text{Aut}(M)$ acts transitively on its universe $M$. It is easy to check that for an arbitrary $\mathcal{L}$–formula in one free variable $\phi(x)$, the theory $\text{Th}(M)$ contains the sentence

$$(\exists x)\phi(x) \rightarrow (\forall x)\phi(x).$$

One can use this to prove that if $N$ is a model of $\text{Th}(M)$ and if $p(x, y)$ a consistent type in two variables, the types $p(x, a)$ and $p(a, x)$ are consistent for arbitrary $a \in N$.

**Lemma 5.23.** Let $\kappa$ be an infinite cardinal. Suppose that we are given a tuple $a = (a_i)_{i \in I}$ of parameters contained in fewer than $\kappa$–many orbits of a $\kappa$–saturated model $N$ of $\text{Th}(M)$, a basepoint $a_0$ indexed by $i_0 \in I$, and $D \in D$ such that for any $a \in A$, we have $R_u(a_0, a)$ for some $u \perp D$. Let $a'_i \in N$ be a point such that $R_D(a_0, a'_i)$. Then $a'_i$ extends to a tuple $(a'_i)_{i \in I}$ satisfying both $(a'_i)_{i \in I} \equiv (a)_{i \in I}$ and $R_D(a, a'_i)$ for all $i \in I$.

Here, the symbol $\equiv$ is used to denote elementary equivalence, so that the types of these tuples coincide.

**Proof of Lemma 5.23**. Let $x = (x_i)_{i \in I}$ be tuple of variables. Fix an arbitrary formula $\psi(x)$ in the quantifier–free type $\text{qftp}_1(A)$. Let $I_0$ be the finite subset of $I$ consisting of indices appearing in $\psi$. We may assume by hypothesis that $i_0 \in I_0$, and that for all $i \in I_0$, the formula $\psi$ implies $R_u(x_{i_0}, x_i)$ for a suitable $u \perp D$. Let $(g_i)_{i \in I_0}$ be a tuple of points of $M$, viewed as elements of $G$, which witness $\psi(x_{i_0})$. Pick an arbitrary $h \in G$ such that $M \models R_D(1, h)$. For $i \in I_0$, we set $g'_i := h^{x_i_{i_0}^{-1}} g_i$. 
As the tuple \((g_i)_{i \in I_0}\) is in the orbit of \((g_i)_{i \in h}\) under the action of \(\text{Aut}(\mathcal{M})\), it clearly satisfies \(\psi\) as well. On the other hand a simple calculation yields
\[
\mathcal{M} \models R_h(g_i, g'_i),
\]
where here
\[
h_i = h_i^{(s_i^{-1} \cdot s_i)}.
\]
Now \(\alpha = g_i^{-1} \cdot g_i \in \text{Stab}^+(D)\) and \(h \in \text{Stab}^+(D^\perp)\), so that \(h \alpha \in \text{Stab}^+(D^\perp)\) as well. Hence \(R_D(g_i, g'_i)\) for all \(i \in I\). The desired result follows by compactness. □

5.4. Symmetric decomposition. In the sequel, we will require a minor variation on Proposition 5.9 from [BMPZ17], which is called the symmetric decomposition lemma. Even though the proof is nearly identical to what is given in that paper, we will record the details because the words that we consider may have group elements in them, in contrast to [BMPZ17].

**Lemma 5.24.** Let \(u\) and \(v\) be reduced words. Then up to permutations, there are unique decompositions
\[
\begin{align*}
u &\simeq gu_1u'w, \quad v \simeq vwv_1h,\\
\end{align*}
\]
which satisfy the following conditions:

1. The letters \(g\) and \(h\) are group elements, and no group elements occur in \(u_1u'w\) and \(vwv_1\).
2. The word \(w\) is a commuting word, i.e. all domains occurring in \(w\) are pairwise orthogonal;
3. The word \(u'\) is properly left-absorbed by \(v_1\);
4. The word \(v'\) is properly right-absorbed by \(u_1\);
5. The words \(\{u', w, v'\}\) pairwise commute;
6. The word \(u_1wv_1\) is reduced.

Thus, \(uv \simeq gu_1wv_1h\).

**Proof.** Applying the moves \((\text{Jmp})\) and \((\text{Cmp})\), we may assume that \(u = gu_0\) and \(v = v_0h\), where \(u_0\) and \(v_0\) are reduced words with no occurrences of group elements.

We may now proceed by induction on the sum of the lengths of \(u_0\) and \(v_0\). If the word \(u_0v_0\) is already reduced then we set \(u_1 = u_0\) and \(v_1 = v_0\), and \(v', u', w\) to be trivial.

Since otherwise \(u_0v_0\) is not reduced, without loss of generality, we may write \(u_0 = u'_0D\), with \(D\) left–absorbed by \(v_0\), so that \(Dv_0 \simeq v_0\). By induction, find words \(\{y_1, y', z, x', x_1\}\) which satisfy the conclusions of the lemma, and such that
\[
u_0' = y_1y'z, \quad v_0 = zx'x_1.
\]

Since \(u_0\) is reduced, we have that \(D\) cannot be left–absorbed by \(z\) nor \(x'\). It follows that \(D\) is orthogonal to \(zx'\), and must be absorbed by \(x_1\). If \(D\) is properly absorbed
by $x_1$, then we set
\[ u' = y'D, \quad w = z, \quad v_1 = y_1. \]
If $D$ is absorbed via move $(\mathrm{Abs}_w)$ then we may write (up to permutation) $x_1 = Dx'_1$, whereby we set
\[ u'_1 = y'_1, \quad w = zD, \quad v_1 = x'_1. \]

Now, clearly $gu_0v_0h \simeq gu_1wv_1h$. The word $w$ is the longest common terminal segment of $u_0$ and $v_0^{-1}$, and $u_1w$ is the longest common initial subword of $u_0$ and $[u_0] \ast [v_0]$, since otherwise $v_1 = Dv'_1$ and $u' = Du''$, which contradicts the fact that $u'$ is properly left–absorbed by $v_1$. Similarly, $wv_1$ is the longest common terminal subword of $v_0$ and $[u_0] \ast [v_0]$. This establishes uniqueness of the decomposition. \[ \Box \]

6. Displacement and types

In this and the following section, the action of $G$ will generally appear on the left (cf. Remark 3.22). We will generally reserve the letters \{g, h, k\} (possibly with subscripts and superscripts) for group elements, and all other letters will denote elements of the universe on which group elements act.

Let $D \subseteq D_0$ be a $G$–invariant and downward closed family of domains. Consider a connected domain $D \in D$. By definition, $D$ is identified with the curve graph of the underlying realized topological surface $|D|$, together with some boundary curves. We recall the notation $C(D)$, which denotes the curve graph of $|D|$.

If $g \in \text{Stab}^+_G(D^\perp)$, which is to say $R_D(1, g)$, then $g$ acts on $C(D)$. The kernel
\[ K \trianglelefteq \text{Stab}^+_G(D^\perp) \]
of this action consists of all the $g \in G$ such that $R_{\partial D}(1, g)$, which is to say the group generated by the Dehn twists over connected components of the inner boundary of $|D|$. The following is clear:

**Observation 6.1.** Fix a connected non–annular $D \in D$ and let $u \in \mathcal{W}(D)$. If $g \in \text{Stab}^+(D^\perp)$, then whether the relation $R_{u, \partial D}(1, g)$ holds in $M$ or not depends only on the action of $g$ on $C(D)$.

In fact, more can be said:

**Lemma 6.2.** Let $D \in D$ be connected. Given $w \in \mathcal{W}(D)$ and any $\alpha \in C(D)$, there is a constant $K = K(w, \alpha) > 0$ such that for each pair $g, g' \in M$ satisfying $R_w(g, g')$, we have $d_D(g \cdot \alpha, g' \cdot \alpha) \leq K$.

**Proof.** We may assume that $D$ is not annular so that $|D|$ contains more than one curve, since otherwise $w$ can only contain letters from $G$. Fix $\alpha \in C(D)$. The proof is by induction on $|w|$. Suppose that $w$ is of the form $(v, D')$, where $D' \subseteq D$. Choose $\beta$ in $(D \cap \partial D') \setminus \partial D$, so that $\beta$ is a boundary curve of $D'$ which is non-peripheral in
Let $N = d_D(\alpha, \beta)$. If $M \models R_w(g, g')$, then there exists $h \in M$ such that both $M \models R_v(g, h)$ and $M \models R_D(h, g')$ hold. Since $h \cdot \beta = g' \cdot \beta$, we have
\[ d_D(h \cdot \alpha, g' \cdot \alpha) \leq 2N. \]

Using the triangle inequality and the induction hypothesis, it follows that
\[ d_D(g \cdot \alpha, g' \cdot \alpha) \leq K_0 + 2N =: K \]
satisfies the required properties, where $K_0 = K(v, \alpha)$ is the constant provided by the induction hypothesis. The case in which $w$ is of the form $v g$ follows by an identical argument, mutatis mutandis. □

Since there are elements $g \in \text{Stab}^+(D^\perp)$ that act on $C(D)$ with arbitrarily large translation length, we obtain:

**Corollary 6.3.** Let $\mathcal{D} \subseteq \mathcal{D}_0$. The type $R_D^+(x, y)$ is consistent for any $D \in \mathcal{D}$.

**Proof.** Suppose
\[ \{w_1, \ldots, w_k\} \subseteq \mathcal{W}(D). \]

Since for each $i$ the word $w_i$ is a finite word consisting of instances of $\mathcal{A}(D)$, we have that for each $\alpha \in C(D)$ there is an absolute bound $C_i$ such that if $R_{w_i}(1, g)$, then $d_D(\alpha, g \cdot \alpha) \leq C_i$. However, there exists an $h \in G$ such that $R_\alpha(1, h)$ and
\[ d_D(\alpha, h \cdot \alpha) \geq C = \max_{1 \leq i \leq k} C_i + 1. \]

Thus, the type $R_D^+$ is finitely satisfiable and hence consistent. □

A converse result can be obtained using the fact that the edge relation in a curve graph is the union of finitely many distinct topological configurations.

**Lemma 6.4.** Given a connected proper domain $D \in \mathcal{D}$, a finite collection $F \subseteq C(D)$, and a constant $K > 0$, there exists a $\phi \in R_D^+$ such that $M \models \phi(1, g)$ implies
\[ d_D(F, g \cdot F) > K. \]

We do not consider the case where $D = C$, and indeed if $D = C$ then the statement of the lemma is not true.

**Proof of Lemma 6.4.** We argue the contrapositive, so that given $K$ we want to prove the existence of a collection $\mathcal{W}_0$ of finitely many words in $\mathcal{W}(D)$ such that
\[ d_D(F, g \cdot F) \leq K \]
implies $M \models R_u(1, g)$ for some $u \in \mathcal{W}_0$.

Since the mapping class group acts by isometries on the curve graph and thus preserves diameters of subsets of the curve graph, we may assume without loss of generality that $F$ consists of a single curve $\gamma$. 
If $D$ is annular, then the Dehn twist $\tau$ about the core curve has the property that $R_D(x,y)$ holds if and only if $y = x t^n$ for some $n$. We have that $C(|D|)$ is quasi-isometric to a line on which $\tau$ acts as a loxodromic element, whence the result follows easily.

Otherwise, $|D|$ is a surface with boundary of genus $g$ and with $b$ boundary components, verifying the inequality $3g + b > 3$. Here, we remind the reader that a pair of pants is treated as three disjoint annuli and is therefore not a connected domain. The graph $C' = C(|D|)$ is a locally infinite graph of infinite diameter, with vertices curves in $D \setminus \partial D$ and edges between pairs of curves with minimal intersection. The vertices and edges of $C'$ fall into finitely many orbits under the mapping class group $H$ of $|D|$. Choose $A \subset D$ a finite set of representatives from every orbit of vertices. Writing $S = |D|$, if $\gamma_1$ and $\gamma_2$ are disjoint simple closed curves on $S$, then there are only finitely many topological types of surfaces of the form $S \setminus \{\gamma_1 \cup \gamma_2\}$. It follows that there exists a finite collection

$$\{h_1, h_2, \ldots, h_r\} \subset \text{Stab}^+(D^\perp)$$

such that if $\{a, b\}$ is an edge in $C'$ with $a \in A$, then there is an $a' \in A$ and

$$g \in \text{Stab}^+(D^\perp) \cap \text{Stab}(a)$$

such that

$$\{a, b\} = g \cdot \{a, h_j \cdot a'\}.$$  

for a suitable index $j$. We write $E$ for the finite set of unordered pairs

$$\{\{a, h_j \cdot a'\} \mid a \in A, 1 \leq j \leq r\}.$$  

Thus, if $\gamma' \in H \cdot \{\gamma\}$ lies at distance at most $K$ from $\gamma$ in $C'$ then there exists a path

$$\gamma_0 = \gamma, \gamma_1, \ldots, \gamma_t = \gamma',$$

where $t \leq K$ and where $\{\gamma_i, \gamma_{i+1}\}$ is a translate of an element of $E$. We now claim that there is a finite collection $\mathcal{W}_0$ such that if $d_D(\gamma, g \cdot \gamma) \leq K$ then $R_w(1, g)$ for some $w \in \mathcal{W}_0$. By induction, suppose we have built a finite collection of such words for some value of $K$. Thus, for each $i$ there is an element $h_i \in H$ and $a_i \in A$ such that $\gamma_i = h_i \cdot a_i$, and by induction we may suppose that $R_w(1, h_i)$ for some $w \in \mathcal{W}_0$. The edge $\{\gamma_i, \gamma_{i+1}\}$ is a translate of $e_i \in E$ by an element $h'_i \in H$. Note that $h_i$ and $h'_i$ differ by an element

$$k_i \in \text{Stab}(\gamma_i) = h_i \text{Stab}(a_i)h_i^{-1},$$

so that $h'_i = k_i \cdot h_i$. But then $k_i = h_i k'_i h_i^{-1}$ for some $k'_i \in \text{Stab}(a_i)$, so that $h'_i = h_i k'_i$. Since $R_D(1, k'_i)$ for $D' = D \cap \{a\}^\perp$, we may enlarge $\mathcal{W}_0$ by finitely many words to make $R_w(1, h'_i)$ for some $w \in \mathcal{W}_0$. By the same argument, we see that $R_w(1, h_{i+1})$
for some $w \in \mathcal{W}_0$, again at the cost of increasing the size of $\mathcal{W}_0$ by finitely many words. This establishes the lemma.

7. Simple connectedness

The goal of this section is to show that $\text{Th}(\mathcal{M})$ enjoys a model-theoretic property called \textbf{simple connectedness} as introduced by [BMPZ17], which is made precise below in Lemma 7.2 and the preceding discussion, together with Lemma 7.3. In order to establish simple connectedness, we will require some nontrivial results from surface theory.

7.1. Certifying non-relatedness. The following result is a rephrasing of Theorem 4.3 in [Beh06], and is commonly known as the Behrstock inequality. We note that in the original, the inequality is given in terms of complete markings:

**Theorem 7.1.** Let $\Sigma$ be a surface such that $3g - 3 + b \geq 1$. There exists a constant $C \geq 0$ such that given any pair $X_1$ and $X_2$ of essential connected subsurfaces of $\Sigma$ which are not pairs of pants and satisfy $X_1 \cap X_2$, and a curve $\alpha \in C$ with non trivial projection to both $X_1$ and $X_2$, we have:

$$\min\{d_{x_1}(\partial X_2, \alpha), d_{x_2}(\partial X_1, \alpha)\} \leq C$$

The following lemma is the key ingredient allowing one to describe the structure of a general model of $\text{Th}(\mathcal{M})$. We first give the reader an intuitive idea of its function.

Suppose we are given a reduced word $w$ and a curve $\alpha \in C$ such that $\alpha$ meets at least one surface appearing in $w$ in an essential way, and let $h$ satisfy $R_w(1, h)$. Then usually we will have that $\alpha \neq h \cdot \alpha$, in which case we say that $\alpha$ is perturbed by $h$. The content of the lemma is that in fact $\alpha$ will be perturbed by any $h$ satisfying the relation $R_w(1, h)$ in a sufficiently generic way. More precisely, given an arbitrary $\beta \in C$, there is an explicit, nonempty subset $\psi_{\alpha, \beta}(x, y) \subset R_w^*(x, y)$ such that $\psi_{\alpha, \beta}(1, h)$ implies $h \cdot \alpha \neq \beta$. The same conclusion will hold equivariantly, with $(g, gh, g \cdot \alpha)$ in place of $(1, h, \alpha)$, where here $g \in G$ is arbitrary.

**Lemma 7.2.** Suppose that

$$w = D_1D_2 \cdots D_kh$$

is a given reduced word, that $g \in G$, and that $\alpha \in C$ with $h(\alpha) \leq D_j$ for some $1 \leq j \leq k$. Then there exist formulae $\phi_i(x, y) \in R_w^{D_j}$ for $1 \leq i \leq k$ such that

$$M \models \forall x_0 \forall x_1 \cdots \forall x_k \left( \bigwedge_{i=1}^k \phi_i(x_{i-1}, x_i) \rightarrow \neg R_{g, x_0}^{\alpha}(x_0, x_k) \right).$$

In Lemma 7.2 we implicitly assume that the domains occurring in $w$ lie in $\mathcal{D}$. 
Proof of Lemma 7.2. Clearly, we can assume that each of the $D_j$ is connected. We need to show that for any $\alpha \in C$ with $\alpha \not\perp \bigvee_{i=1}^k D_i$ and $\beta \in C$ arbitrary, there exist formulae $\phi_i(x, y)$ as above such that for any sequence

$$h_1, h_2, \ldots, h_k$$

of elements of $G$ satisfying $\phi_i(1, h_i)$ for $1 \leq i \leq k$, the element

$$g = h_1 h_2 \cdots h_k$$

cannot send $\alpha$ to $\beta$.

To begin, let $C$ be the constant provided by Theorem 7.1. From Lemma 6.4, for each $1 \leq j \leq k$, there exists a formula $\phi_j(x, y) \in R_{D_j}^+(x, y)$ with the property that for arbitrary $g, h \in M$, the condition $M \models \phi_i(g, h)$ implies

$$d_{D_j}(g \cdot A_j, h \cdot A'_j) > 3C,$$

where here

$$A_j = \pi_{D_j} \left( \{ \beta \} \cup \bigcup_{t < j} \partial D_t \right)$$

$$A'_j = \pi_{D_j} \left( \{ \alpha \} \cup \bigcup_{t > j} \partial D_t \right).$$

Let $h_1, h_2, \ldots, h_k$ be chosen so that $\phi_i(1, h_i)$ for all $1 \leq i \leq k$. Write

$$h_{i, j} = h_i h_{i+1} \cdots h_j$$

for $0 \leq i < j \leq k$.

Let $j_0$ be the maximum index $1 \leq j \leq k$ for which $\alpha$ is not orthogonal to $D_j$. Note that this implies $\alpha = h_{j_0+1,k} \cdot \alpha$.

If for all $j < j_0$ we have $D_j \perp D_{j_0}$, then we obtain the conclusion of the lemma. Indeed, in this case $h_{1,j_0-1}$ fixes $D_{j_0}$, and thus

$$\pi_{D_{j_0}}(h_{1,j_0-1} \cdot \beta) = \pi_{D_{j_0}}(h_{j_0-1} \cdot \cdot \cdot D_{j_0}(\beta) = h_{1,j_0-1} \cdot \pi_{D_{j_0}}(\beta) = \pi_{D_{j_0}}(\beta).$$

It follows easily then that

$$d_{D_{j_0}}(g \cdot \alpha, \beta) = d_{D_{j_0}}(h_{1,j_0-1}h_{j_0,k} \cdot \alpha, h_{1,j_0-1} \cdot \beta) = d_{D_{j_0}}(h_{j_0}h_{j_0+1,k} \cdot \alpha, \beta) = d_{D_{j_0}}(h_{j_0} \cdot \alpha, \beta) > C.$$

Here, we implicitly allow the possibility that $\pi_{D_{j_0}}(\beta) = \emptyset$, since then whereas this estimate is no longer valid, it is obvious that because $w$ is not orthogonal to $\alpha$, we cannot have $g \cdot \alpha = \beta$. 

For the general case, we define $j_0$ as before. We inductively construct a descending sequence

$$j_0 > j_1 > \cdots > j_t$$

of indices by setting $j_{k+1}$ to be the largest index $j$ which is smaller than $j_k$ and such that $D_j \subsetneq D_{j_k}$. Eventually, we obtain and index $j_t$ such that $D_{j_t}$ is orthogonal to $D_j$ for all $j < j_t$. Necessarily, $D_{j_t} \cap D_{j_{t+1}}$ for $0 \leq \ell \leq t-1$. Indeed, $D_{j_t}$ and $D_{j_{t+1}}$ are not orthogonal by construction. Moreover, they are incomparable since otherwise $w$ would not be reduced, as every letter occurring in $w$ between these surfaces is orthogonal to $D_{j_t}$.

**Claim 1.** For all $1 \leq \ell \leq t$, we have

$$h_{j_{t+1,k}} \cdot \alpha \subsetneq D_{j_{\ell}}$$

and

$$d_{D_{j_{\ell}}}(h_{j_{t+1,k}} \cdot \alpha, \partial D_{j_{\ell-1}}) \leq C.$$

The conclusion of the lemma follows from the case $\ell = t$ of Claim 1 above for the same choices of $\phi_j$ as in the case $t = 0$ considered previously. Indeed, since $\phi_j(1, h_{j_t})$ holds, we have

$$d_{D_{j_t}}(\beta, h_{j_t} \cdot \partial D_{j_{t-1}}) > 2C.$$

On the other hand, Claim 1 asserts that

$$d_{D_{j_t}}(h_{j_{t+1,k}} \cdot \alpha, \partial D_{j_{t-1}}) \leq C,$$

which in turn implies

$$d_{D_{j_t}}(h_{j_{t,k}} \cdot \alpha, h_{j_t} \cdot \partial D_{j_{t-1}}) \leq C.$$

This allows us to conclude that

$$d_{h_{j_{t+1}, \partial D_{j_t}}}(\beta, g \cdot \alpha) = d_{D_{j_t}}(\beta, h_{j_{t,k}} \cdot \alpha) > C,$$

since $D_i \perp D_{j_t}$ for $i < j_t$. Thus, we obtain $\beta \neq g \cdot \alpha$.

It remains only to prove the claim. We proceed by induction on $\ell$. Suppose that for some $1 \leq \ell < t$ we have already successfully shown that

$$d_{D_{j_{\ell}}}(h_{j_{\ell+1,k}} \cdot \alpha, \partial D_{j_{\ell-1}}) \leq C.$$

Then, we have

$$d_{D_{j_{\ell}}}(h_{j_{\ell,k}} \cdot \alpha, h_{j_{\ell}} \cdot \partial D_{j_{\ell-1}}) \leq C.$$

The choice of $h_{j_{\ell}}$ implies that

$$d_{D_{j_{\ell}}}(\partial D_{j_{\ell+1}}, h_{j_{\ell}} \cdot \partial D_{j_{\ell-1}}) > 2C.$$

The triangle inequality then implies that

$$d_{D_{j_{\ell}}}(\partial D_{j_{\ell+1}}, h_{j_{\ell,k}} \cdot \alpha) > C.$$
Theorem \ref{thm:local} then shows

\[ d_{D_{j_{l+1}}} (h_{j_{l}, k} \cdot \alpha, \partial D_{j_{l}}) \leq C. \]

Since \( D_s \perp D_{j_{l}} \) for all \( j_{l+1} < s < j_{l} \), the left hand side of this last inequality is equal to

\[ d_{D_{j_{l+1}}} (h_{j_{l+1}, 1} \cdot \alpha, \partial D_{j_{l}}), \]

which establishes the claim. \hfill \( \Box \)

7.2. Parametrizing the quantifier–free type of a pair of elements. For \( w \in \mathcal{W} \), we let \( w^{-1} \) be the result of writing the letter occurring in the expression for \( w \) in reverse order, and by replacing each occurrence of \( g \in G \) with \( g^{-1} \). It follows by definition that \( R_w(x, y) \) if and only if \( R_{w^{-1}}(y, x) \).

\textbf{Lemma 7.3.} Let \( N \) be a model of \( M \). Suppose we are given two elements \( a, b \in N \), and let \( u \) and \( v \) be reduced words such that \( R_w^* (a, b) \) and \( R_w^* (a, b) \). Then \( u \simeq v \).

\textit{Proof.} We will write \( D \subseteq D_0 \) as before. We proceed by induction on \( \text{Or}(u) \cup \text{Or}(v) \). If the concatenation \( uv^{-1} \) is not reducible, then either both \( u \) and \( v \) contain only elements from \( G \), or else Lemma \ref{lem:rede} leads to a contradiction.

We may thus assume assume that \( uv^{-1} \) is reducible and that both \( u \) and \( v \) have syllables coming from \( D \). This means that there exist comparable elements \( D, E \in D \) such that \( u \simeq u_0 D \) and \( v \simeq v_0 E \). In view of Lemma \ref{lem:compression} take \( c \) such that \( R_{u_0}^* (a, c) \) and \( R_{c_0}^* (c, b) \) and \( d \) such that \( R_{v_0}^* (a, d) \) and \( R_{d_0}^* (d, b) \).

Without loss of generality we can assume that \( E \subseteq D \). Consider first the case in which \( R_{u_0}^* (c, d) \). Then \( R_{u_0}^* (a, a) \). By virtue of Corollary \ref{cor:compression} there exists a word

\[ w \subseteq u_0 D v_0^{-1} \]

such that \( R_{w}^* (a, a) \). By Lemma \ref{lem:rede}, we necessarily have that \( [w] \neq [1] \), which contradicts Lemma \ref{lem:rede}. The remaining possibility is that \( E = D \) and \( R_{u_0}^* (c, d) \) for some \( u_1 \in \mathcal{W}(D) \). In this case, applying the induction hypothesis to the pairs \( (a, d) \in N^2 \) and \( (u_0 u_1, v) \in \mathcal{W}^2 \) instead of \( (a, c) \) and \( (u, v) \) yields \( u_0 u_1 \simeq v_0 \), which implies

\[ u = u_0 D \simeq u_0 u_1 D \simeq v_0 D = v, \]

the desired conclusion. \hfill \( \Box \)

\textbf{Definition 7.4 \( (\delta(a, b)) \).} We denote the unique reduced class \( [u] \) such that \( R_u^* (a, b) \) by \( \delta(a, b) \).

\textbf{Observation 7.5.} Given \( a, a', a'' \in N \) words in \( \delta(a, a'') \) are reducts of concatenations of representatives of \( \delta(a, a') \) and \( \delta(a', a'') \).
Corollary 7.6. Given tuples $a = (a_i)_{i \in I}$ and $a' = (a'_i)_{i \in I}$ of elements from $N$, we have $\text{qftp}(a) = \text{qftp}(a')$ if and only if $\delta(a_i, a_j) = \delta(a'_i, a'_j)$ for all $i, j \in I$. In particular, the set $\delta(a, b)$ determines the quantifier free type $\text{qftp}^{a,b}(A)$.

Proof. Combining Lemma 7.3 and Corollary 5.20, we have that for any $u \in \mathcal{W}$, the validity in $N$ of $R_u(a, b)$ is equivalent to any representative of $\delta(a, b)$ being a reduct of $u$. Since the language under consideration contains only binary relations, the result follows. \qed

8. Weakly convex sets and their extensions

The goal of this section is to establish a certain technical result, Lemma 8.16, which will allow us to establish a suitable version of quantifier elimination and stability for $\text{Th}(\mathcal{M})$. The essential point is to apply Theorem 2.19.

We will work in the universe $N$ of a fixed, sufficiently saturated model $N$ of the theory $\text{Th}(\mathcal{M})$.

An expression such as $x_A$ will denote the (possibly infinite) tuple of variables $(x_a)_{a \in A}$. The expression $\text{qftp}^a(A)$ will denote the quantifier-free type of $A$ with $x_a$ in place of $a \in A$. That is, this is a type that for all pairs $a, a' \in A$ contains a formula $R_u(x_a, x_{a'})$ in case $R_u(a, a')$ holds, and the formula $\neg R_u(x_a, x_{a'})$ otherwise.

Definition 8.1. We say that a subset $A \subseteq N$ is weakly convex if

- The set $A$ is a union of orbits, i.e. $aG \subseteq A$ for all $a \in A$.
- For all pairs $a, a' \in A$ lying in the same connected component, and an arbitrary representative $w \in \delta(a, a') \subseteq \mathcal{W}$, there exists a strict sequence from $a$ to $a'$ which is entirely contained in $A$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{weak_convexity}
\caption{Definition 8.1. Weak convexity asserts the existence of the dotted path.}
\end{figure}
Definition 8.2. Given $A \subset N$, an element $a_0 \in A$, and a domain $D \in \mathcal{D}$, we let $p_{a_0,A}^D(x,y_A)$ be the type stating that $R_D(x,y_A)$ but $\neg R_u(x,y_A)$ for all $a \in A$ and $u \in \mathcal{W}(D)$. We say that $b$ is step $D$ away from $A$ if there is $a_0 \in A$ such that $\mathcal{N} \models p_{a_0,A}^D(b,A)$. In such a setup, we will refer to $a_0$ as a basepoint for $b$ in $A$.

Lemma 8.3. For all choices of sets $A$ of parameters, basepoints $a_0 \in A$, and domains $D \in \mathcal{D}$, we have that the type $p_{a_0,D}^A$ is consistent.

Proof. This can be shown using the same argument used to prove that $R_{D}^a(x,y)$ is consistent in Corollary 6.3. Specifically, we need 

$$ \mathcal{N} \models (\forall y_1, \ldots, y_k)(\exists x)R_D(y_1, x) \land \bigwedge_{j=1}^{k} \bigwedge_{u \in \mathcal{W}(D)} \neg R_u(y_j, x). $$

Given arbitrary mapping classes $\{g_1, \ldots, g_k\}$ on a surface $S$ and $B \in \mathbb{N}$, there exists a pseudo-Anosov mapping class $h$ such that the curve graph translation length of $g_i^{-1}h$ is at least $B$ for all $i$, whence the lemma follows easily. □

We will refer to the following lemma as the gate property.

Lemma 8.4. Suppose that $A \subset N$ is weakly convex and that $b \in N$ is one step $D$ away from $A$, with basepoint $a_0$. Then for all $a \in A$, the class $\delta(b,a)$ is the unique equivalence class of reduced words generated by $(D, \delta(a_0,a))$ without using move (C). In particular, if $R_u(b,a)$ for some $u \in \mathcal{W}$ and if $a \in A$, then there is a domain $E \in \mathcal{D}$ occurring in $u$ such that $k(E) \geq k(D)$.

We remind the reader that $k(E)$ is the maximal length of a strictly ascending chain of proper, nontrivial subdomains of $E$. In the case that the language $\mathcal{D} \neq \mathcal{D}_0$ is restricted, we of course insist that these subdomains lie in $\mathcal{D}$.

Proof of Lemma 8.4. Let $a \in A$ and let $u \in \delta(a_0,a)$. Suppose that in the process of reduction of $Du$ to $\delta(b,a)$, some cancellation takes place. Then one can write $u \simeq Du_2 = u'$ (cf. Remark 5.18). Since $A$ is weakly convex, there is a strict $u'$--path

$$ a_0, a_1, \ldots, a_k = a $$

that is strictly contained in $A$. Observe that $R_D(b,a_1)$ holds by transitivity of $R_D$. If we have $R_{D}^a(b,a_1)$, then we would have $\delta(b,a) = [Du']$, which contradicts our assumption that no cancellation occurs. Otherwise, the point $a_1 \in A$ witnesses $\mathcal{N} \models p_{a_0,A}^D(b,A)$. □

Given a set $A \subset N$ and $b \in N$, we write

$$ \text{Or}(b,A) = \min\{\text{Or}(\delta(b,a))\}_{a \in A}. $$
If $\text{Or}(w) = \text{Or}(b, A)$ and $\{b = b_0, b_1, \ldots, b_k\}$ a strict $w$–sequence from $a$ to some $b_k = a \in A$, then we say that $\bar{b}$ is a minimizing sequence from $b$ to $A$ and $w$ a minimizing word from $b$ to $A$.

Given two sequences $s, s'$ of points in $N$, we say that they fellow travel with each other if the $G$-orbits of points in the sequences coincide. Given $w, w' \in \mathcal{W}$ we say that $w \equiv w'$ if it is possible to obtain one from the other by applying the moves

$$\{(\text{Cmp}), (\text{Swp}), (\text{Rm}), (\text{Abs}_G)\}$$

and their inverses.

**Observation 8.5.** If $w \equiv w'$, then for any strict $w$–path $p$, there is a unique strict $w'$–path $p'$ between the same endpoints that fellow-travels with $p$.

The proof of the following lemma is straightforward, and we omit the proof.

**Lemma 8.6.** Let $w$ be a reduct without cancellation of a word of the form $Dv$, where $v$ is reduced. Then one of the following holds:

- There exists some $w' \in [[w]]$ of the form $uDu'$, where $u \perp^* D$ and where $uu' \in [[v]]$. In particular, $uu'$ is reduced.
- There exists some $w' \in [[w]]$ of the form $uEu'$, where $u \perp^* D$, where $D \subseteq E$, and $uEu' \in [[v]]$.

We write $[w] \perp D$ if $w' \perp D$ for some $w' \in [w]$. Similarly, we write $[w] \perp^* D$ if $w' \perp^* D$ for some $w' \in [w]$.

**Definition 8.7.** We say that a reduced word $w \in \mathcal{W}$ is a left (right) normal form if $w$ has at most one occurrence of a letter from $G$, and it is the leftmost (rightmost) letter, which we refer to as its $G$–term.

**Lemma 8.8.** Given a reduced $w \in \mathcal{W}$, there exits a unique left (right) normal form $w' \in [[w]]$. Given two left (right) normal forms $w \equiv w'$, we can pass from one to the other by an iterated application of $(\text{Swp})$. If $w \perp D$, then $w' \perp^* D$, where here $w'$ is the corresponding left (right) normal form.

The proof of Lemma 8.8 is straightforward.

**Lemma 8.9.** Let $[w]$ be an equivalence class of reduced words and let $D \in \mathcal{D}$. The following are equivalent:

- $(a)$ $w' \perp D$ for all $w' \in [w]$ in left (right) normal form;
- $(b)$ $[w] \perp D$;
- $(c)$ For all $w' \in [w]$, we have $\text{Th}(\mathcal{M}) \vdash (\forall x \forall y) R_{w'}(x, y) \rightarrow R_{D^+}(x, y)$.

**Proof.** The implication $(a) \Rightarrow (b)$ is clear. For $(b) \Rightarrow (c)$, notice that the conclusion is clearly true for $w' = w$. Since

$$\text{Th}(\mathcal{M}) \vdash (\forall x \forall y) R_{w}(x, y) \leftrightarrow R_{D}(x, y),$$
for all pairs of equivalent (reduced) words \( u, u' \), the conclusion follows.

Let us now show implication \( \neg(a) \Rightarrow \neg(c) \). Assume there exists \( w' \in [w] \) in normal form such that \( w \perp D \). One of the following two cases occurs.

1. \( D \perp E \) for any \( E \in \mathcal{D} \) occurring in \( w \), but the \( G \)-term of \( w \) is not in \( \text{Stab}^+ (D) \).
2. There exists a \( E \in \mathcal{D} \) occurring in \( w \) such that \( E \perp D \).

The first case is in clear contradiction with \( (c) \), since under this assumption we have

\[
\text{Th}(\mathcal{M}) \models \forall x \forall y \ R_{w'}(x, y) \rightarrow \neg R_{D^\perp}(x, y).
\]

Consider the second case. Write \( w = gu \) where \( u \in \mathcal{D}^* \). There is some \( \alpha \in D \) which is not orthogonal to some \( E \) appearing in \( u \). By Lemma 7.2 there exists an \( h \in G \) such that \( R_u(1, h) \), such that \( h(\alpha) \neq g^{-1}(\alpha) \). The element \( h' = gh \) satisfies \( R_u(1, h') \). But \( \mathcal{M} \models \neg R_{D^\perp}(1, h') \) since \( h' \) does not fix \( \alpha \), contrary to \( (c) \).

Observation 8.10. If \( w \) is the reduct of a concatenation of words \( v_1, v_2 \) where \( [v_i] \perp D \), then \( [w] \perp D \).

The following is a consequence of Lemma 8.8 and Lemma 8.9, which is a natural refinement of Observation 8.10.

Lemma 8.11. Given a reduced \( w \in \mathcal{W} \) and \( D \in \mathcal{D} \), we have \([w] \perp^* D\) if and only if for any \( w' \in [w] \) there exists some \( w'' \in ([w']) \) such that \( w'' \perp^* D \). The same conclusion holds with \( \perp^* \) replaced by \( \perp \).

Proof. Suppose that \([w] \perp^* D \). Lemma 8.9 implies that \( w' \perp D \) for the left (right) normal form of \( w \). None of the letters in \( w' \) can absorb \( D \), since by Lemma 8.8, they all appear in some \( u \equiv w \) with \( u \perp^* D \).

Lemma 8.12. Let \( D \in \mathcal{D} \) and let \( v, w \in \mathcal{W} \) be reduced words such that \( v, w \perp^* D \). Then for any reduct \( u \) of \( vw \), there is \( u' \in [[u]] \) such that \( u' \perp^* D \). That it, \( g \perp D \) for all \( g \in G \) appearing in \( D \), and \( E \perp^* D \) for all \( E \) appearing in \( u' \).

Proof. We may assume that the reduction of \( vw \) does not involve the inverse of a composition move (Cmp). Notice that the application of any other move to a word \( u \) preserves the property \( u \perp D \). It is also clear that moves of type

\[
\{(\text{Swp}), (\text{Cmp}), (\text{Abs}_+), (\text{Abs}_-), (\text{Abs}_G)\}
\]

applied to \( u \) preserve the property \( u \perp^* D \). Given \( g \perp D \) and \( E \perp^* D \), we necessarily have \( g^{-1}(E) \perp^* D \), so that orthogonality is preserved by moves of type (Jmp) as well. Finally, notice that if \( E \perp^* D \), then \( v \perp^* D \) for all \( v \in \mathcal{W}(E) \). Indeed, \( v \perp D \) and \( D \perp E' \) for any \( E' \in \mathcal{D} \) with \( E' \subseteq E \).
Definition 8.13. Given \( D \in \mathcal{D} \), a set of parameters \( A \), and a basepoint \( a_0 \in A \), we define:

\[
\text{con}^1_A(a_0, D) = \{ a \in A \mid \delta(a_0, a) \perp^* D \}
\]
\[
\text{tcon}^1_A(a_0, D) = \{ ag \mid a \in \text{con}^1_A(a_0, D), g \in G \}.
\]

Lemma 8.14. If \( A \) is weakly convex, then so is \( \text{tcon}^1_A(a_0, D) \).

Proof. Let \( C_0 = \text{con}^1_A(a_0, D) \) and \( C = \text{tcon}^1_A(a_0, D) \). We have the following sublemma:

Lemma 8.15. For all choices of \( c, c' \in C_0 \), for all reduced \( u \in \delta(c, c') \), and for all strict \( u \)-paths \( p \) from \( c \) to \( c' \) in \( A \), there exists a \( u' \in [[u]] \) and a strict \( u' \)-path \( p' \) from \( c \) to \( c' \) that fellow travels with \( p \), and that is entirely contained in \( C_0 \).

Proof. Let \( w_1 \in \delta(c, a_0) \) and \( w_2 \in \delta(a_0, c) \) both be strongly orthogonal to \( D \). Notice that \( \delta(c, c') \) is a reduct of \( w_1w_2 \) and that \( w_i \perp^* D \) by definition, so that Lemma 8.12 implies the existence of a \( u' \in [[u]] \) such that \( u \perp^* D \). By Observation 8.5 there exists a path \( p' \) from \( c \) to \( c' \) of type \( u' \), which fellow travels with \( u \). On the one hand, \( p' \) is clearly contained in \( A \), since \( A \) is a union of \( G \)-orbits. On the other hand, for an arbitrary point \( b \) appearing in \( p' \), we have that \( \delta = \delta(a_0, b) \) is a reduct of the concatenation of \( w_1^{-1} \) with an initial segment of \( u' \), both of which are strongly orthogonal to \( D \). A further application of Lemma 8.12 yields \( \delta \perp^* D \), and thus \( b \in C_0 \). \( \Box \)

Now, let \( c, c' \in C \), let \( u \in \delta(c, c') \), and let \( g, g' \in G \) be arbitrary. Choose \( d, d' \in C_0 \) such that \( R_g(c, d) \) and such that \( R_{g'}(c', d') \). The reduced word

\[
v = g^{-1}ug' \in \delta(c, d)
\]

lies in \( \delta(d, d') \). By Lemma 8.11 there exists a \( v' \in [[v]] \) together with a \( v' \)-path \( p_1 \) from \( d \) to \( d' \). On the other hand, \( p_1 \) fellow travels with a unique strict \( v \)-path \( p_2 \) from \( d \) to \( d' \), which is entirely contained in \( C = C_0G \). Dropping the first and last points in \( p_2 \) yields a strict \( u \)-path from \( c \) to \( c' \) which is entirely contained in \( C \). \( \Box \)

The following is the key technical result of the entire paper, and establishes the back–and–forth property needed to prove quantifier elimination in \( \mathcal{M} \).

Lemma 8.16. Let \( A \subseteq N \) be a weakly convex set of cardinality less than \( \kappa \), let \( a_0 \in A \) be a basepoint, and let \( D \in \mathcal{D} \). Let \( C_0 \subseteq C \) be the set of parameters given by \( C_0 = \text{con}^1_A(a_0, D) \), and write \( C = \text{tcon}^1_A(a_0, D) \). Consider the type:

\[
q^D_{a_0, A}(x_c, y_A) := p^D_{a_0, A}(x_{a_0}, y_A) \cup \text{qftp}^{x_c}(C) \cup \text{qftp}^{x_A}(A) \cup \{ R_D(x_c, y_c) \mid c \in C_0 \}.
\]

The following conclusions hold:
It follows that $D_Pu$ by Lemma 8.4, and of $\delta$, that whenever $c$ and Lemma 8.4, it is enough to prove that $R$.

Clearly, $\delta(c', d')$. Then we have that $\delta(a', a'')$ is a reduct without cancellation of both $\delta(a', a'') = \{ R_D(x_c, y_c) \mid c \in C_0 \}$

says that each point in the new copy of $C_0$ (in the $x$–variables) is $D$–related to the corresponding point in the original copy of $C_0$.

Proof of Lemma 8.16 Item (i) is a particular instance of Lemma 5.23 and follows from the consistency of the relevant types.

For (ii), let $(c, a) \in C \times A$. By Corollary 7.6 it suffices to show that $q_{a_0, A}^D$ completely determines the value of $\delta(x_c, y_a)$. Now, for arbitrary choices of $g \in G$ and $c, d \in N$, the type $\delta(cg, d)$ is completely determined by $\delta(c, d)$. By virtue of this and Lemma 8.4 it is enough to prove that $q_{a_0, A}^D(x_c, y_a) \models p_{A,c}^D(x_c, y_a)$

whenever $c \in C_0$.

So, suppose that we are given $C'$ and $A''$ such that $C'A'' \models q_{a_0, A}^D(x_c, y_a).$

Clearly, $R_D(c', c'')$ for any $c \in C_0$. Now, assume for a contradiction that there exists a $c \in C_0$ and $a \in A$ such that $\delta(c', a'') = \{ u \mid u \in W(D) \}$. Then we have $[D\delta(a', a'')] = [D\delta(a_0, a)]$

by Lemma 8.4 and of $[\delta(a_0, c')u] = [\delta(a_0, c)u]$. Now, the definition of $C_0$ implies that $\delta(a_0, a)$ is strongly orthogonal to $D$. Since $u \in W(D)$, we have that if $D$ is annular, then any domain occurring in $u$ is empty. It follows that $D$ cannot occur in $[\delta(a_0, c)u]$. In the case where $D$ is non-annular, and
$D$ cannot be absorbed by any domain occurring in $[\delta(a_0,c)u]$. This contradicts the assumption that $\delta(a',d'')$ is a reduct without cancellation of $D\delta(a_0,a)$.

Now consider part (iii). We need to show that for all pairs of elements $e'_1 \in C'$ and $e''_2 \in A''$ corresponding to points $e_1 \in C$ and $e_2 \in A$ respectively, and for all $w \in \delta(e_1,e_2)$, there exist a strict $w$–sequence from $e'_1$ to $e''_2$ which is strictly contained in $C'A''$. By replacing $e_1$ by an element in its $G$–orbit, we may clearly assume that $e_1 \in C_0$. We know that $\delta(e'_1,e''_2)$ is the result of reducing $D\delta(e_1,e_2)$ without cancellation.

Suppose first that $D$ is not absorbed by any domain occurring in $\delta(e_1,e_2)$. By Observation 8.5 and Lemma 8.6, we may assume that $w$ is of the form $uDu'$, where $u \perp^* D$ and where

$$v = uu''u' \in \delta(e_1,e_2).$$

Here, the word $u''$ is absorbed by $D$. Let $\{d_1, \ldots, d_k\}$ be a $uu''u'$–path in $A$, with $d_1 = e_1$ and $d_k = e_2$, and let $d_i$ and $d_j$ denote the points in this sequence corresponding to the final letters of $u$ and $u''$ respectively. Notice that

$$\{d_1, \ldots, d_i\} \subset C,$$

since $u \perp^* D$. Thus, we may consider the paths

$$\{d'_1, \ldots, d'_l\} \subset A', \quad \{d''_j, d''_{j+1}, \ldots, d''_k\} \subset A''.$$

Since $u'' \in \mathcal{W}(D)$, we have that $d'_i$ and $d''_j$ are strictly $D$–related, so weak convexity holds for these points and the word $w$.

Finally, we suppose that $D$ is absorbed by a domain in $\delta(e_1,e_2)$, so that we may assume that prior to application of the absorption move $(\text{Abs}_\subset)$, the word $w$ is of the form $uDEu'$, so that we may write

$$w = uEu' \in \delta(e_1,e_2) = \delta(e'_1,e''_2),$$

with $u \perp^* D$. Note that $u$ may not be strongly orthogonal to $E$. Retaining the notation that $d'$ is the terminal point in a $u$–sequence in $C'$ starting at $e'_1$ and that $d''$ is the initial point of a $u'$–sequence terminating at $e''_2$, we have that $R_D(d',f)$ and $R_E(f,d'')$ for a suitable point $f \in C'A''$. We have that $R_E(d',d'')$ already, so that the resulting path in $C'A''$ is in fact a $w$–path, as desired. \hfill \Box

9. Relative quantifier elimination and $\omega$-stability

We are now ready to prove some of the main results of this paper. We include this section for completeness, as the arguments are nearly identical to those in [BMPZ17].
Relative quantifier elimination. Our preliminary result on quantifier elimination is as follows.

**Theorem 9.1.** The quantifier-free type of a weakly convex set of a model of \( \text{Th}(\mathcal{M}) \) determines its type.

**Proof.** We will show that given two \( \omega \)-saturated models \( \mathcal{N}_1 \) and \( \mathcal{N}_2 \), the collection of partial isomorphisms between weakly convex subsets of \( \mathcal{N}_1 \) and \( \mathcal{N}_2 \) has the back-and-forth property.

Let \( \mathcal{N} \) and \( \mathcal{N}' \) be two given \( \omega \)-saturated models of \( \text{Th}(\mathcal{M}) \). Let \( \phi : A \cong \tilde{A} \) be a given isomorphism between two weakly convex sets, each of which is the union of finitely many \( G \)-orbits, and let and \( c \in N\setminus A \). We will show that the map \( \phi \) extends to a partial isomorphism \( \tilde{\phi} : B \to \tilde{B} \), where here \( B \) and \( \tilde{B} \) are both weakly convex and \( B \) contains \( a \). Notationally, we will write that \( \phi \) sends \( a \) to \( \tilde{a} \). By a straightforward induction on the length of a minimal path from \( c \) to \( A \), we may assume that \( c \) is a step \( D \)-away from \( A \) over a basepoint \( a_0 \), for a suitable \( D \) and \( a_0 \).

Part (i) of Lemma 8.16 yields realizations \( B \) and \( \tilde{B} \) of of \( q_{a_0,A}^D \) in \( \mathcal{N} \) and \( \mathcal{N}' \) extending \( cA \) and \( \tilde{c}\tilde{A} \) respectively.

Both \( B \) and \( \tilde{B} \) are weakly convex by item (iii) of Lemma 8.16. Finally, item (ii) furnishes a unique extension of \( \phi \) to \( \tilde{\phi} : B \to \tilde{B} \).

9.2. \( \omega \)-stability. Let \( \mathcal{N} \) be an \( \omega \)-saturated model of \( \text{Th}(\mathcal{M}) \) as before. Given \( a, b \in N \), we define \( \text{Or}(a, b) = \text{Or}(\delta(a, b)) \). We say that \( a_0 \in A \) is a basepoint for \( b \) in \( A \) if \( \text{Or}(b, a_0) = \text{Or}(b, A) \).

The following lemmas are straightforward.

**Lemma 9.2.** Let \( A \subset N \) be a weakly convex set, and let \( a_0 \in A \) be a basepoint for a point \( b \in N\setminus A \). Let \( w \in \delta(b, a_0) \), and suppose that \( b' \) is the penultimate point in a strict \( w \)-sequence from \( b \) to \( a_0 \). Suppose furthermore that \( R_D^*(b', a_0) \) for \( D \in \mathcal{D} \). Then \( b' \) is \( D \)-step away from \( A \) with basepoint \( a_0 \). Moreover, for any weakly convex set \( B \) containing \( \{a_0\} \cup A \) as constructed by Lemma 8.16, the element \( b' \) is a basepoint for \( b \) in \( B \).

**Lemma 9.3.** If \( A \subset N \) is weakly convex and if \( a_0 \in A \) is a basepoint for \( b \in N\setminus A \), then for all \( a \in A \), the class \( \delta(b, a) \) is the unique equivalence class of reduced words that can be obtained by reduction from \( \delta(b, a_0)\delta(a_0, a) \) without using the move \( (C) \).

**Proof.** This follows from repeated application of Lemma 9.2.

We obtain the following corollaries.
Corollary 9.4. Let $\mathcal{N}$ be a model of $\text{Th}(\mathcal{M})$, let $A \subseteq \mathcal{N}$ be weakly convex, and let $b, b' \in \mathcal{N}$. If there is an $a_0 \in A$ that is a basepoint for both $b$ and $b'$ and if $\delta(b, a_0) = \delta(b', a_0)$, then $\text{tp}(b/A) = \text{tp}(b'/A)$.

Let $w = \delta(b, a_0)$. We write $p^w_{a_0, A}$ for the type furnished in the corollary.

Proof of Corollary 9.4. First we get $\text{qftp}(b/A) = \text{qftp}(b'/A)$ by Corollary 7.6 and Lemma 8.4. By Theorem 9.1 we have $\text{tp}(b/A) = \text{tp}(b'/A)$. □

Theorem 9.5. $\text{Th}(\mathcal{M})$ is $\omega$-stable.

Proof. Let $\mathcal{N}$ be a sufficiently saturated model of $\text{Th}(\mathcal{M})$. We need to show that given a countable set of parameters $A \subseteq \mathcal{N}$, there are at most countably many distinct types in one variable over $A$. If $A$ is weakly convex, this follows immediately from Corollary 9.4 since there are only countably many choices for $a_0$ and countably many choices for $\delta(b, a_0)$. For the general case, note that an arbitrary countable subset of $\mathcal{N}$ is contained in a countable weakly convex set, by repeated applications of the construction in Lemma 8.16. □

10. Quantifier elimination in $\mathcal{M}$

In this section, we improve Theorem 9.1 to absolute quantifier elimination in $\text{Th}(\mathcal{M})$. We retain notation from the previous section, so that $\mathcal{N}$ denotes a countable $\omega$–saturated model of $\text{Th}(\mathcal{M})$. Denote by $\hat{\mathcal{N}}$ the collection of all imaginaries of the form $[a]_D$, where $a \in \mathcal{N}$ and $D \in \mathcal{D}$. This includes elements of $\mathcal{N}$ under the identification of the element $a$ with $[a]_\emptyset$. Given some set $A \subseteq \mathcal{N}$ we denote by $\hat{A}$ the collection of all the classes of the form $[a]_D$ with $a \in A$. Denote by $\text{acl}(A)$ the intersection of $\text{acl}^\mathcal{N}(A)$ with $\hat{\mathcal{N}}$.

Given $D \in \mathcal{D}$ and $a \in \mathcal{N}$ we let $\langle [a]_D \rangle_G$ be the collection of all classes of the form $[a']_E$, where $a' = ag$ for some $g$ such that $D \subseteq g(E)$. Notice that

$$\langle [a]_D \rangle_G \subseteq \text{dcl}^\mathcal{N}(a, b).$$

Lemma 10.1. If $[a']_E$ with $a' = ag$ for some $g \in G$ such that $D \not\subseteq g(E)$, then the orbit of $[a']_E$ under the action of the group of automorphisms of $aG$ which preserve $[a]_D$ is infinite.

Proof. Indeed, let $h \in G$. There is a unique automorphism $\phi_h$ of the orbit $aG$ sending $a$ to $ah$, which sends $ak$ to $akh$ for any $k \in G$. In particular, if we let $Q = \text{Stab}^+(E)_{\perp}$ then $\phi_h(a') = \phi_h(ag) = ahg$ and $\phi_h([a']_E) = \phi_h([a']_E)$ if and only if $hgQ = h'gQ$, i.e. if and only if $(h^{-1}h')^g \in Q$, i.e., if and only if $h^{-1}h' \in Q^g \triangleq \text{Stab}^+(g(E)_{\perp})$. □

Suppose we are given a weakly convex set $B$ and a minimizing path $p$ from $a$ to $B$, with basepoint $b_0 \in B$. We denote by $\mathcal{H}(p, B)$ the collection of all weakly
convex sets that are furnished by concatenating one-step extensions along $p$ as in Lemma 8.16. The following is an easy observation which is left to the reader:

**Observation 10.2.** Let $P$ and $B$ be as above and let $P'$ be a path (of the same length) which fellow travels with $P$. Then $\mathcal{H}(P, B) = \mathcal{H}(P', B)$.

Given a strict path $P$ from $a \in N$ to $b \in N$ we shall write $\mathcal{H}_P$ in place of $\mathcal{H}(P, bG)$ and given two points $a, b$ we will write $\mathcal{H}_{a,b}$ for the union of $\mathcal{H}_P$ where $P$ ranges among all strict paths between $a$ and $b$.

**Observation 10.3.** Given two $w$-paths with endpoints $ab$ and $a'b'$ respectively, an iterated application of Lemma 8.16 yields the existence of an isomorphism between any $H \in \mathcal{H}_P$ and any $H' \in \mathcal{H}_P$ sending $ab$ to $a'b'$. Thus, $P \equiv P'$ by relative quantifier elimination.

**Lemma 10.4.** Suppose that we are given strict minimizing path $P$ of type $w$ between a point $a \in N$ and a weakly convex $B$ with basepoint $b_0$. Then:

1. For an arbitrary $H \in \mathcal{H}(P, B)$ and $c \in H$ there exist
   \[\{c_1 \in cG, c_0 \in H, b_1 \in B\}\]
   where $c_0$ lies in strict paths from $a$ to $b_0$ and from $a$ to $d_1$, where
   \[\delta(b_0, b_1) = \delta(c_0, c_1) \perp \delta(c_0, b_0),\]
   and where both $\delta(c_0, b_0)$ and $\delta(c_0, c_1)$ have representatives in $\mathcal{D}^*$.

2. Let $c, c' \in H$ be such that $\delta(a, c) = \delta(a, c')$ and such that the corresponding points $b_1$ and $b'_1$ can be taken to be equal. Then we have $c' = cg$ for some $g \perp \delta(b, b_0)$ and $g \perp w_2$.

**Proof.** We may assume that $\delta(a, b) \cap \mathcal{D}^* \neq \emptyset$. Assume that $P$ is given by
   \[a_0 = a, a_1, a_2, \ldots, a_k = b_0,\]
   where $R_D(a_i, a_{i+1})$.

   By induction, we may assume the result holds for given data in which the path is of length strictly smaller than $k$. Set $H_k := B$ and for $0 \leq i \leq k - 1$, let $H_i$ be a one-step extension of $H_{i-1}$ of type $D_i$ through $a_i$ with basepoint $a_{i+1}$ so that $H_0 = H$.

   Both statements are clearly true for $c \in H_k$. It remains to show that the validity of the statement for all pairs $c, c' \in H_{j-1}$ implies its validity for $c, c' \in H_j \setminus H_{j-1}$ as well.

   Pick $c \in H_j \setminus H_{j-1}$. There exists $\tilde{c} \in cG$ and $d \in H_{j-1}$ such that $\delta(a_j, \tilde{c}) = \delta(a_{j+1}, d) \perp^* D_j$ and has representatives in $\mathcal{D}^*$ and $\delta(\tilde{c}, d) \approx D_j$.

   By induction, we know that there are $d_0, d_1 \in H_{j+1}$ such that $d_1 \in dG$ and such that $d_0$ lies in strict paths from $a_j$ to $b_0$ and $d_1$ respectively. Moreover,
   \[\delta(d_1, b_1) = \delta(d_1, b_0) \perp \delta(d_0, d_1) = \delta(b_0, b_1),\]
and both $\delta(d_0, d_1)$ and $\delta(d_0, b_0)$ have representatives in $\mathcal{D}^*$.

Now, since $\delta(a_{j+1}, d) \perp^* D_j$ there must be some $d'_0 \in d_0G$ such that

$$\delta(a_{j+1}, d'_0) \perp D_j, \quad \delta(d'_0, d) \perp D_j.$$  

Since $\delta(d_0, d_1) \cap \mathcal{D}^* \neq \emptyset$, we may assume there is $h \in G$ such that $d = d_1h$ and $d'_0 = d_0h$.

By construction there is $e'_0 \in H_j \setminus H_{j-1}$ such that $\delta(a_j, e'_0) = \delta(a_{j+1}, d'_0)$ and $\delta(e'_0, c) = \delta(d'_0, d)$. Consider $c_0 = e'_0h^{-1}$ and $c_1 = \delta h^{-1}$, and let $D'_j = h(D_j)$. Notice that $R^*_D(c_0, d_0)$ and $R^*_D(c_1, d_1)$; similarly,

$$\delta(c_0, c_1) = \delta(d_0, d_1) \perp^* D'_j,$$

since $\delta(d_0, d_1) \perp D'_j$. The latter orthogonality relation implies that

$$\delta(c_0, c_1) = \delta(d_0, d_1) = \delta(b_0, b_1) \perp \delta(c_0, b_0).$$

On the other hand, $c_0$ is in a strict path from $a_j$ to $c_1$. On the other hand, for $w \in \delta(d_0, b_0)$, the word $D'_j w$ must be reduced, by virtue of the minimality of $P$. Therefore, it must lie in some strict path from $a_j$ to $b_0$ as well.

Now let $c' \in H_j \setminus H_{j-1}$ be such that $\delta(a_0, c') = \delta(a_0, c)$ and $\delta(c', a_k) = \delta(c, a_k)$. The previous paragraph shows that we may assume there is $c'_0 \in H_{j-1}$ such that

$$\delta(a_j, c_0) = \delta(a_j, c'_0), \quad \delta(c'_0, a_k) = \delta(c_0, a_k),$$

and $R^*_D(c', c'_0)$. The induction hypothesis implies that $c'_0 \in c_0G$. But then the construction of $H_j$ from $H_{j+1}$ implies that $c' \in cG$. \hfill $\square$

**Definition 10.5.** Given $B, B' \subseteq N$ we say that a map $f : B \to B'$ is a homomorphism if

$$\delta(f(b_1), f(b_2)) \leq \delta(b_1, b_2)$$

for any $b_1, b_2 \in B$. If $A \subseteq \hat{B}$, we say that $f$ is an $A$–**homomorphism** if it preserves each class in $A$.

Given $A \subseteq \hat{B}$ we say that $B$ is strongly incompressible over $A$ if all $A$–homomorphisms $f : B \to B'$ are isomorphic embeddings.

**Lemma 10.6.** Suppose $A \not\subseteq B \subseteq N$, where $A$ is weakly convex. Then there is is a homomorphic retraction $f : B \to A$. In particular, if $B$ is weakly convex and tight over some $A \subseteq \hat{B}$, then any $A$–homomorphism from $B$ to itself is an isomorphism.

**Proof.** We follow the proof of Lemma 7.11 in [BMPZ17]. We first note that there is a homomorphic retraction from $A$ to itself. Let $A \subseteq H \subseteq N$ be a maximal weakly convex subset that admits a retraction to $A$, and let $C = H \cap B$. We claim that $C = B$. If not, let $b \in B \setminus C$. The Lemma 8.16 furnishes a weakly convex extension
$H'$ of $H$ containing $b$ which retracts to $H$, and which by composition retracts to $A$. This violates the maximality of $H$.

\begin{lemma}
Given a strict path $P$ from $a \in N$ to $b \in N$, any $H \in \mathcal{H}_P$ is strongly incompressible over \{a, b\}.
\end{lemma}

\begin{proof}
Let $P$ denote the path $a = a_0, a_1, \ldots, a_r = b$. We may assume that the type of $P$ is a word $\in \mathcal{D}^*$. Consider an element $H \in \mathcal{H}_P$, and let $\phi : H \to N$ be an $\{a, b\}$–homomorphism. Let

$$H_0 = H, H_1, H_2, \ldots, H_r$$

be as in the proof of part 2 of Lemma 10.4, where here $B = bG$. We will show that $\phi_{1|H_k}$ is an isomorphism, by reverse induction. The base case $k = r$ is obvious. Now, assume that $\phi_{1|H_k}$ is an isomorphism for $k > 0$, and let $\delta(a_{k-1}, a_k) = D$.

Now,

$$H_{k-1} = \tilde{C}_0 G \cup H_k,$$

where $a_{k-1} \in \tilde{C}_0$, and where there is an isomorphism $\lambda$ between $\tilde{C}_0$ and

$$C_0 := \{ c \in H_k \mid \delta(a_k, c) \perp^* D \} \subseteq H_k$$

such that $\hat{c}$ is a step $D$ away from $H_k$, with basepoint $\lambda(c_0)$, and with $\lambda(a_{k-1}) = a_k$.

By the proof of Lemma 8.16, it suffices to show that $\phi(c)$ is one step away from $\phi(H_k)$ for any $c \in \tilde{C}_0$. Take an arbitrary $c \in \tilde{C}_0$, and assume for a contradiction that there exists $d \in \phi(H_k)$ such that

$$\delta(\phi(\lambda(c)), d) = [u] \in \mathcal{W}(D).$$

Let

$$[w] = \delta(a, a_{k-1}), \quad [w'] = \delta(a_k, b).$$

We know that $\delta(a_k, c) = [v]$ for some reduced $v \perp^* D$. Let $[v'] = \delta(c, b)$. By part 1 of Lemma 10.4, we know that $v v'$ is reduced and equivalent to $w'$. Let $[u] = \delta(c, d)$. We may assume $u \in \mathcal{D}^*$. We then have that the words $\delta(a_k, d)$ and $\delta(d, b)$ are reducts of $vu$ and $uv'$ respectively. Lemma 10.4 also tells us that $\delta(a_k, d)\delta(a_k, b)$ is reduced. Necessarily, $\delta(d, b) = [v']$, since otherwise

$$[w'] = \delta(a_k, d)\delta(d, b) = [Ew']$$

for some $E \subseteq D$, and the word $w Dw'$ could not have been reduced.

Thus, we conclude that

$$Or(\delta(a_{k-1}, b)) \leq Or(v) \perp Or(\delta(c, \phi(\lambda(c)))) \perp Or(v') < Or(\delta(a_{k-1}, b)),$$

which since $\phi$ is a homomorphism, implies the contradictory statement

$$Or(\delta(a, \phi(a_{k-1}))) \perp Or(\delta(a_{k-1}, b)) < Or(\delta(a, b)).$$

This concludes the proof.
\end{proof}
Corollary 10.8. Let $P$ and $P'$ be strict $w$-paths between points $a, b$ and $a', b'$ respectively, $H \in \mathcal{H}_{a,b}$ and $H' \in \mathcal{H}_{a',b'}$. Then there is an automorphism of $N$ sending $P$ to $P'$ and $H$ to $H'$.

Proof. By saturation and Observation 10.3, there is an isomorphic embedding $\phi : H \to N$ sending $P$ to $P'$. Now take a retraction $\pi : N \to H'$. Since $\phi(H)$ is clearly also strongly incompressible over $a', b'$ the restriction of $\pi$ to $\phi(H)$ is an isomorphic embedding. We claim that it is in fact surjective. Indeed, otherwise the retraction of $N$ onto $\pi(\phi(H))$ would yield an isomorphism between $H'$ and some proper subset of itself, which is impossible since by construction $H'$ comprises only finitely many $G$-orbits. Therefore the map $\pi \circ \phi$ is an isomorphism between $H$ and $H'$ sending $P$ to $P'$, which by relative quantifier elimination must extend to an automorphism of $N$. \qed

We now adapt the concept of the wobbling path from [BMPZ17].

Observation 10.9. Given a reduced $w \in \mathcal{W} \cap D^*$, there is a (possibly disconnected) $D \in D$ such that $[u] * [w] = [w]$ if and only if $u \in \mathcal{W}(D) \cup D$. The domain $D$ only depends on the equivalence class of $w$.

We denote the domain $D$ furnished by Observation 10.9 by $LA(w)$. Given reduced words $w$ and $w'$, we let $w \bowtie w' := LA(w^{-1}) \cap LA(w')$.

Lemma 10.10. Let

$$w = D_1D_2 \cdots D_k \in \mathcal{W} \cap D^*$$

be a reduced word, and let

$$a_0, a_1, \ldots, a_k \quad \text{and} \quad a'_0, a'_1, \ldots, a'_k$$

be two strict paths of type $w$ between two points $a_0 = a'_0$ and $a_k = a'_k$. For each $1 \leq i \leq k$, let

$$u_i = D_1D_2 \cdots D_{i-1}, \quad v_i = D_iD_{i+1} \cdots D_k.$$

Then $R_{E_i}(a_i, a'_i)$ for any such $i$, where $E_i = u_i \bowtie v_i$.

Proof. Following the proof of Lemma 6.19 in [BMPZ17], we proceed by induction on $i < k$. Suppose first that $i = 1$. Then we have that $a_1$ and $a'_1$ are related to $a_0$ by $D_1$, so that $R_{D_1D_1}(a_1, a'_1)$. It follows that any reduced word $w_1$ such that $R_{w_1}(a_1, a'_1)$ is absorbed by $D_1$. Similarly, we have that

$$R_{D_2D_1 \cdots D_k}(a_1, a'_1),$$

since $a_1$ and $a'_1$ are related to $a_k$ by $D_2D_3 \cdots D_k$. Since the word $D_2 \cdots D_k$ is reduced and

$$w_1D_2 \cdots D_k \simeq D_2 \cdots D_k,$$

we must have that $w_1$ is fully absorbed by $D_2 \cdots D_k$. It follows that $a_1$ and $a'_1$ are related by an $E_1$ which is absorbed by both $D_1$ and $D_2D_3 \cdots D_k$. 

Now suppose that $w_i$ and $w_{i+1}$ are reduced words such that
\[ R_{w_i}(a_i, a'_i) \quad \text{and} \quad R_{w_{i+1}}(a_{i+1}, a'_{i+1}). \]
By induction, $w_i$ is fully absorbed by $D_1 \cdots D_i$ and by $D_{i+1} \cdots D_k$. We may (up to equivalence) write $w_i$ as a product $w'_i w''_i$, where $w'_i$ is left absorbed by $D_{i+1}$ and where $w''_i$ is orthogonal to $D_{i+1}$ and is left absorbed by $D_{i+2} \cdots D_k$. We therefore have that $D_{i+1} w_i D_{i+1}$ reduces to $w''_i D_{i+1} D_{i+1}$, which then further reduces to $w_{i+1}$. It follows that $w_{i+1}$ is a reduct of $w''_i y_i$, where $y_i$ is absorbed by $D_{i+1}$.

Since $R_{D_{i+2} \cdots D_k}(a_{i+1}, a_k)$ and since $D_{i+2} \cdots D_k$ is reduced, we have that $y_i$ must be left absorbed by $D_{i+2} \cdots D_k$. It follows then that $w_{i+1}$ is left absorbed by $D_{i+2} \cdots D_k$. Since $w''_i$ is orthogonal to $D_{i+1}$ and is right absorbed by $D_1 \cdots D_i$, we may assume that $w''_i y_i$ is absorbed by $D_{i+1} \cdots D_k$. Since $y_i$ is absorbed by $D_{i+1}$, we see that $w_{i+1}$ is right absorbed by $D_1 \cdots D_{i+1}$. The lemma follows.

We define Osc to be the collection of all imaginary classes whose orbit to equivalence is automatically strict, the latter can be expressed by a first order formula. For general $a, b \in N$ we define $\text{Osc}(a, b)$ by $\text{Osc}(a', b)$ where $a' \in aG$ is such that $\delta(a, b) \cap D^* = \emptyset$.

**Observation 10.11.** The definition of $\text{Osc}(a, b)$ is independent of the choice of $a'$ as above.

**Observation 10.12.** We have $\text{Osc}(a, b) \subseteq \text{dcl}^\text{eq}(a, b)$

**Proof.** We may assume $\delta(a, b) \cap D^* = \emptyset$. Indeed, any $e \in \text{Osc}_w(a, b)$ is in the definable closure of some $[a_i]_{D_i}$, where $a_i$ is the $i^{th}$ step of some strict path of type $w$ from $a$ to $b$, while in turn $[a_i]_{D_i}$ can be uniquely characterized as the $D_i$ class of the $i^{th}$ step of some strict path of type $w$ from $a$ to $b$. Since any path of type $w$ from $a$ to $b$ is automatically strict, the latter can be expressed by a first order formula.

Recall that $\text{acl}^\text{eq}(A)$ stands for the collection of all imaginary classes whose orbit under the point-wise stabilizer of $A$ is finite. From the Lemma above we recover the following corollary:

**Corollary 10.13.** Let $a, b \in N$ then $\hat{N} \cap \text{acl}^\text{eq}(a, b) = \text{Osc}(a, b)$. In particular,
\[ \hat{N} \cap \text{acl}^\text{eq}(a, b) \subseteq \text{dcl}^\text{eq}(a, b). \]

**Proof.** It suffices to prove that $\hat{\text{acl}}(a, b) \subseteq \text{Osc}(a, b)$. We may assume that $\delta(a, b) \cap D^* \neq \emptyset$. Let $e \in \hat{N} \setminus \text{Osc}(a, b)$. Let $P$ be a strict path from $a$ to $b$ and pick some $H \in \mathcal{H}_p$. If $e \notin \hat{H}$, then we are done by Lemma [10.1]. Assume now that $e = [c]_E$ with $c \in H$. We know that there is some $c'$ of the form $ch$ that occurs in some strict path of type $w_1 \ast w_2 \in D^*$ from $a$ to $b$, where $\delta(a, c) = [w_1]$ and $\delta(c, w_2) = [w_2]$. Let $D = w_1 \ast w_2$. Since $[c]_D \notin \text{acl}^\text{eq}(a, b)$, it follows from Lemma [10.1] that the orbit of $e$ under the action of $\text{Stab}^+(D^\perp)$ on the right on $(cG)/R_E$ is infinite. By
Lemma 10.8, this implies that the orbit of $e$ under $\text{Aut}_{a,b}(N)$ is infinite and hence $e \notin \text{acl}^{\text{eq}}(a,b)$. □

Given $n_0 \in N$ and reduced $w \in \mathcal{W}$ let $\nu_{n_0,N}$ be the type furnished by Corollary 9.4

**Lemma 10.14.** Given two reduced words $w, w' \in \mathcal{W}$ and $n_0, n'_0 \in N$ the equality $\nu_{n_0,N} = \nu_{n'_0,N}$ holds if and only if there is $g \in G$ such that $w' \simeq wg$ and $R_{Dg}(n_0, n'_0)$, where where $D = \text{LA}(w^{-1})$. In particular, in the situation above, we have

$$\text{Or}(w) = \text{Or}(w * \delta(n_0, n'_0)).$$

**Proof.** We will limit ourselves to proving the direction from left to right, since the converse implication follows easily from the associativity of reduction without cancellation (see Corollary 5.22). Clearly, $\sigma(\nu_{n_0,N}) = \nu_{\sigma(n_0),N}$, whence the last claim follows.

The type $\nu_{n_0,N}$ implies that $\delta(x, n'_0) \simeq w * \delta(n_0, n'_0)$ and the same holds after exchanging the role of $(w, n_0)$ and $(w', n_0)$. It follows that each of $w, w'$ is an initial sub-word of the other up to equivalence and thus $\text{Or}(w) = \text{Or}(w')$. As $\nu_{n_0,g,n_0} = \nu_{n_0,g}$, we may assume $w = w'$. In this case the result follows immediately from the definition of $\text{LA}(w^{-1})$. □

This, together with Lemma 10.13 implies that given $a, b \in N$ any $\text{acl}^{\text{eq}}(a, b)$--invariant 1-type (i.e. one invariant under all automorphisms fixing $\text{acl}^{\text{eq}}(a, b)$) is actually $ab$ invariant. In other words, we have:

**Corollary 10.15.** Given $a, b \in N$ all types in $S^1(ab)$ are stationary; that is, all such types admit a unique non-forking extension to any set of parameters $B \supset \{a, b\}$.

The following is an adaptation of Proposition 7.21 in [BMPZ17]. Recall that the notation $w^{-1}$ for a word denotes the word obtained by writing the letters occurring in it in reverse order, and inverting the group elements which appear.

**Lemma 10.16.**

(1) Suppose that we are given $u, v, w \in \mathcal{W}$ such that $uv$ reduces to $w$. Then there are decompositions

$$u \simeq u_1 \alpha^{-1} s^{-1},$$
$$v \simeq s \beta v_1,$$
$$w \simeq u_1 x v_1,$$

where $\{\alpha, \beta, x\}$ pairwise commute, where $x$ is properly right-absorbed by $s$, where $\alpha$ is properly left-absorbed by $v_1$, and where $\beta$ is right-absorbed by $u_1$. These decompositions are unique up to permutations.
(2) If we are given \( a, b, c \in N \) such that
\[
\delta(a, b) = u, \quad \delta(b, c) = v, \quad \delta(a, c) = w,
\]
then there is a strict path \( P \) of type \( w \) from \( a \) to \( c \), a weakly convex set \( H \in \mathcal{H}_p \) and a basepoint \( b_0 \) of \( b \) in \( H \) such that:
\[
\delta(a, b_0) = [u_1], \\
\delta(b_0, c) = [xv_1], \\
\delta(b, b_0) = [sa\beta].
\]

We recall many of the details of the proof for the convenience of the reader.

**Proof of Lemma 10.16** We first prove the existence of such a decomposition. Suppose \( R_w(a, c) \), that \( R_u(a, b) \) and \( R_v(b, c) \). We consider the collection of weakly convex sets \( \mathcal{H}_p \) containing both \( a \) and \( c \). For a strict path \( P \) from \( a \) to \( c \) and let \( b_0 \) be a basepoint on \( P \). Let
\[
[w_1] = \delta(a, b_0), \quad [w_2] = \delta(b_0, c), \quad [y] = \delta(b, b_0).
\]
We choose \( H \in \mathcal{H}_p \) and \( y \) in order to minimize \( Or(y) \). This implies that no initial and no terminal segment of \( y \) is contained in \( w_1 \) \( \wr \) \( w_2 \).

We now have that \( yw_1^{-1} \cong u^{-1} \) and that \( yw_2 \cong v \). By Lemma 5.24 we have unique decompositions
\[
w_1 \cong u_1x'\theta, \quad y \cong s_1^{-1}\beta, \quad u \cong u_1\theta s_1,
\]
where \( x' \) and \( \beta \) commute, \( x' \) is properly left-absorbed by \( s_1 \), where \( \theta \) is word made up of pairwise commuting letters which commutes with both \( x' \) and \( \beta \), and where \( \beta \) is right-absorbed by \( u_1 \). Moreover, the only group elements occurring in these decompositions are the initial letter of \( u_1 \) and the terminal letter of \( s_1 \). By expanding \( u_1 \) to include \( \theta \), we may assume \( \theta \) is trivial. Expanding \( u_1 \) further, we may also assume that \( x' \) is trivial. Analogously and by performing similar expansions if necessary, we write
\[
w_2 \cong xv_1, \quad y \cong s_2\alpha, \quad v \cong s_2v_1,
\]
where \( x \) commutes with \( \alpha \), where \( x \) is properly right-absorbed by \( s_2 \) and where \( \alpha \) is left-absorbed by \( v_1 \).

Observe that \( y \cong s_2\alpha \cong s_1^{-1}\beta \). The minimality assumption on \( y \) (that is, no initial or terminal segment of \( y \) is contained in \( w_1 \) \( \wr \) \( w_2 \)) now implies that no terminal segment of \( \alpha \) can coincide with a terminal segment of \( \beta \). It follows that all terminal segments of \( \alpha \) commute with \( \beta \), whence \( \alpha \) commutes with \( \beta \) and is a terminal segment of \( s_1^{-1} \) (cf. Lemma 5.3 of [BMPZ17]). Repeating this line of reasoning for \( \beta \) and \( s_2 \), we are able to write
\[
s_1^{-1} \cong s\alpha, \quad s_2 \cong s\beta.
\]
We now claim that $\beta$ and $x$ are strongly orthogonal (in the sense that $\beta$ and $x$ are orthogonal and no letter in $x$ is absorbed by $\beta$). Were this not the case, we would be able to write $x \simeq x_1sx_2$, where $x_1$ and $\beta$ commute but where $\beta$ does not commute with $s$. We have that $x$ is right-absorbed by $s_2$, which implies that $s$ must be absorbed by $\beta$. It follows that $s$ is right-absorbed by $u_1$. Since $x_1$ commutes with $\beta$, we obtain that $s$ commutes with $x_1$. We thus see that $w = u_1xv_1$ is not reduced, which is a contradiction. It follows that $x$ is properly right-absorbed by $s$.

We may now write
\[ u \simeq u_1a^{-1}s^{-1}, \quad v = s\beta v_1, \quad w = w_1w_2 \simeq u_1xv_1, \quad y \simeq sa\beta. \]

We may assume that $\alpha$ is properly left-absorbed by $v_1$. Were this not the case, we obtain
\[ \alpha \simeq \alpha'\eta, \quad \eta v_2 \simeq v_1, \]
where $\eta$ is made up of pairwise commuting letters, where $\alpha'$ is left absorbed by $v_2$, and where $\alpha'$ commutes with $\eta$.

Since $w_2 \simeq \eta xv_2$, we apply Lemma 10.14 to assume that there is a basepoint $b'_0$ in the path $P$ such that $R_\eta(b_0, b'_0)$. The word $\eta$ is right-absorbed by $y \simeq s\beta\alpha'\eta$, we may substitute
\[ u_1 \mapsto u_1\eta, \quad v_1 \mapsto v_2, \quad \alpha \mapsto \alpha', \quad \beta \mapsto \beta\eta, \]
in order to obtain new words $\{u_1, v_1, s, x, \alpha, \beta\}$ with the desired properties.

The uniqueness part of the lemma is mostly formal and is a reprise of the proof of Proposition 7.21 in [BMPZ17]. The main point is that the classes of $\{u_1, v_1, x, \alpha, \beta, s\}$ are canonically defined from the classes $\{[u], [v], [u] * [v] = [w]\}$, and hence are unique up to permutations. We omit further details. $\square$

The following is essentially the same as the proof of Corollary 7.22 in [BMPZ17]. The roles of Lemma 6.4 and Corollary 6.5 are played by Lemmas 8.16 and Theorem 9.1.

**Corollary 10.17.** Given $a, b, c \in N$, the type $tp(abc)$ is uniquely determined by $(\delta(a, b), \delta(b, c), \delta(a, c))$.

**Proof.** Let $abc$ and $a'b'c'$ two triples of points such that
\[ (\delta(a, b), \delta(b, c), \delta(a, c)) = (\delta(a', b'), \delta(b', c'), \delta(a', c')). \]
Lemma 10.16 yields weakly convex sets $H \in \mathcal{H}_{a',c'}$ and $H' \in \mathcal{H}_{a',c'}$, together with basepoints $b_0$ for $b$ in $H$ and $b'_0$ for $b'$ in $H'$, with the property that
\[ \delta(b_0, a) = \delta(b'_0, a'), \quad \delta(b_0, c) = \delta(b'_0, c'), \quad \delta(b_0, b) = \delta(b'_0, b'). \]
By Lemma 10.8 there is an isomorphism between $H$ and $H'$ which sends $b_0$ to $b'_0$. An iterated application of 8.16 yields an extension to an isomorphism between
weakly convex sets sending $b$ to $b'$ so that $tp(abc) = tp(a'b'c')$ by virtue of Theorem 9.11.

**Theorem 10.18.** $\text{Th}(M)$ has absolute quantifier elimination.

**Proof.** By virtue of Proposition 7.6, it suffices to show that for all pairs of $k$–tuples $\{a, a'\} \subseteq N^k$ which satisfy $\delta(a_i, a_j) = \delta(a'_i, a'_j)$ for all $1 \leq i < j \leq k$, it is possible to construct weakly convex sets $A$ and $A'$ extending $a$ and $a'$ respectively and an isomorphism from $A$ to $B$ sending $a_i \mapsto b_i$ for all $1 \leq i \leq k$. By a simple limit argument, it suffices to show that given an isomorphism $\phi$ between subsets $C$ and $C'$ of $N$ sending a pair of points $(a, b)$ to $(a', b')$ and given any $w \in \delta(a, b)$, there are strict paths $p$ and $p'$ of type $w$ from $a$ to $b$ and from $a'$ to $b'$ respectively such that $\phi$ extends to an isomorphism

$$C \cup p \to C' \cup p'.$$

Given a strict path $p_0$ of type $w$ from $a$ to $b$, let $q(x, y, z) = tp(p_0, a, b)$. Note that this type is uniquely determined by $w$.

We let $p$ be a realization of $q(x, a, b)$ independent from $C$, and $p'$ a realization of $q(x, a', b')$ independent from $C'$. Denote by $\psi$ the unique isomorphism between $p$ and $p'$ taking $a$ to $a'$ and $b$ to $b'$. For $c \in C$, we have that $tp(cab) = tp(\phi(c)a'b')$, by Corollary 10.17. By Lemma 10.15 both $tp(c/ab)$ and $tp(\phi(c)/a'b')$ are stationary, which together with the fact that $tp(abp) = tp(a'b'p')$ implies that $tp(cp) = tp(\phi(c)p')$. Since our language consists only of binary relations, this suffices to show that $\phi \cup \psi$ is an isomorphism. $\square$

11. Morley Rank

We now turn our attention to the problem of finding upper and lower bounds for the Morley rank of types in $\text{Th}(M(D))$ for a given $D \subseteq D_0$ as in Definition 3.19.

**Theorem 11.1.** For any $r > 0$ the Morley Rank of $M'$ is at most $\binom{r+1}{2} k(\Sigma)$, where $k(\Sigma)$ is the length $k$ of a chain connected domains $\emptyset = D_0 \subseteq D_1 \subseteq D_2 \cdots \subseteq D_k = C$ in $\mathcal{D}$.

**Proof.** Let $\mathcal{N}$ be a $\omega$-saturated model of $\text{Th}(M)$. Consider the collection $\mathcal{S}$ of triples of the form $\langle \bar{w}, \bar{b}, \bar{v} \rangle$, where

$$\bar{w} = (w_j)_{j=1}^r, \ \bar{v} = (v_{i,j})_{1 \leq i < j \leq r}$$

are tuples of of reduced words, and $\bar{b} \in N'$. To any $\tau = (\bar{w}, \bar{b}, \bar{v}) \in \mathcal{S}$, we associate the $r$-variable formula

$$\psi_\tau(x) \equiv \bigwedge_{j=1}^r R_{w_j}(x_j, b) \land \bigwedge_{1 \leq i < j \leq r+1} R_{v_{i,j}}(x_i, x_j),$$
together with the ordinal
\[ \text{Or}(\tau) := \left( \bigoplus_{1 \leq j \leq r} \text{Or}(w_j) \right) \oplus \left( \bigoplus_{1 \leq i < j \leq r} \text{Or}(v_{i,j}) \right) \].

It suffices to prove by induction that:
- \( \text{RM}(\psi_\tau) \leq \text{Or}(v) \) for any \( \tau \in X \)

By quantifier elimination, any type \( q \in S'(N) \) containing \( \psi_\tau \) except at most one must contain a formula of one of the following forms:
- \( R_{w'_j}(x, b) \) where \( b \in N \) and \( \text{Or}(w'_j) < \text{Or}(w_j) \) for some \( 1 \leq j \leq r \);
- \( R_{v'_{i,j}}(x_i, x_j) \) for some proper reduct \( v'_{i,j} \) of \( v_{i,j} \) and \( 1 \leq i < j \leq r \).

In both cases, \( q \) contains a formula of the form \( \psi_{\tau'} \) for some \( \tau' \in X \) with \( \text{Or}(\tau') < \text{Or}(\tau) \). By induction, \( \text{RM}(\psi_{\tau'}) \leq \text{Or}(\tau') < \text{Or}(\tau) \) and thus \( \text{RM}(q) < \text{Or}(v) \). It follows from the characterization of the Morley rank of a type over \( N \) as its Cantor–Bendixon rank as a point in the space \( S'(N) \) and that of a formula as the Cantor–Bendixon rank of the corresponding clopen subset of \( S'(N) \) that \( \text{RM}(\psi_\tau) \leq \text{Or}(\tau) \), as desired. \( \square \)

The Morley Rank can be also bounded from below using the same strategy as in [BMPZ17]. Throughout the discussion \( N \) will be the monster model of \( \text{Th}(M) \).

**Lemma 11.2.** Suppose \( a \in N \) and let \( B \subseteq C \subseteq N \) be weakly convex sets. Then we have \( a \downarrow_B C \) if and only if there is a basepoint for \( a \) in \( C \) that lies in \( B \).

**Proof.** On the one hand, if there is a basepoint \( b_0 \in B \), then by Observation [10.14], we see that \( \text{tp}(a/C) \) extends to a global \( B \)-invariant type, and therefore \( a \downarrow_B C \).

Assume now that no basepoint for \( a \) in \( C \) lies in \( B \). By virtue of Lemma [10.14], this means that given a basepoint \( c_0 \) for \( a \) in \( C \), we have that
\[ \delta(a, b) = \delta(a, c_0) \ast \delta(c_0, b) \neq \delta(a, c_0) \]
for all \( b \in B \). Let \( b_0 \) be a basepoint for \( c_0 \) in \( B \), let \( [v] = \delta(c_0, b_0) \), and let \( [w] = \delta(a, c_0) \).

Suppose \( q = d_{b_0,N}^{\delta(c_0,b_0)}(x) \in S^1(N) \) is a \( B \)-invariant type extending \( \text{tp}(c_0, B) \), as given in the previous paragraph. Let \( (c_i)_{i \in \omega} \) be a Morley sequence of \( q \) over \( B \) starting with \( c_0 \); that is, \( c_j \in N \) is chosen to witnesses the type \( q(x)\restrict_{B \cup \{c_i\}\setminus c_j} \);

For \( \ell \in \omega \), let \( p_{\ell} = d_{c_0,N}^w \). For \( \ell < j \) we have \( p_{\ell} \neq p_j \), since \( p_{\ell} \) states that
\[ \delta(x, c_j) = [w] \ast \delta(c_j, c_\ell) = [w] \ast ([v] \ast [v^{-1}]) \neq ([w] \ast [v]) \ast [v^{-1}] \neq [w] \]
by the associativity of \( \ast \) (cf. Corollary [5.22]). It is a standard fact that the sequence \( \ell \) is indiscernible over \( B \) (i.e. any two tuples with increasing indices have the same type over \( B \)) and that this implies \( p_0 \) is a forking extension of \( p_0 \restrict_B = \text{tp}(a/B) \).
Since $p_0$ is a non-forking extension of $p_0|_{C} = tp(a/C)$ by the previous paragraph, necessarily $tp(a/C)$ is a forking extension of $tp(a/B)$.

Given $v, w \in \mathcal{W}$, we write $v \prec'_0 w$ if $v \simeq v_0v_1$ and $w \simeq v_0D$, where $v_1 \in \mathcal{W}(D) \cap \mathcal{D}^*$, and where none of the letters in $v_1$ are contained in $\partial D$. We let $\prec'$ be the transitive closure of $\prec'_0$. Clearly $\prec' \subset \prec$, so it is well-founded. Let $\text{Or}'$ be the corresponding foundation rank.

Any finite $\prec'$ descending chain starting at a word $w$ gives rise to a chain of extensions of any type $p^w_{b_0,B}$, where $B$ is weakly convex, which follows from an iterated application of the following Lemma.

**Lemma 11.3.** Let $a \in N$, and let $B \subseteq N$ by a weakly convex set such that there is a minimizing sequence of type $wD$ from $a$ to $B$. Let $v \in \mathcal{W}(D)$ be a (possibly empty) word such that $vw$ is reduced. Then there is a weakly convex set $B'$ containing $B$ and a minimizing sequence of type $wv$ from $a$ to $B$.

**Proof.** We may assume that $w \in \mathcal{W} \cap \mathcal{D}^*$. Suppose that

$$a = a_0, a_1, a_2, \ldots, a_r = b_0 \in B$$

is a minimizing sequence of type $w$ from $a$ to $B$ and pick $a' \in N$ such that $\delta(a, a') = [v]$. Notice that $a'$ must be one step $D$ away from $B$ with basepoint $b_0$, just like $a_{r-1}$. Let $B'$ be one-step extension of $B$ through $a'$. We claim that for any strict $w$-sequence

$$a_{r-1}', a_r', \ldots, a_s' = a'$$

from $a_{r-1}$ to $a'$, the resulting sequence

$$a_0, a_2, \ldots, a_{r-1}, a_r', \ldots, a_s'$$

is a minimizing sequence from $a$ to $B'$.

Suppose not. Since $vw$ is reduced, we have that there exists a $b_1 \in B'$ such that $\delta(a, b_1) = [w']$, with $\text{Or}(w') < \text{Or}(vw)$. Since

$$[w'] = [w] * [v] * \delta(a', b_1),$$

this can only take place if at least one cancellation move is used in the reduction process. The fact that $v \perp D$, together with the assumption that none of the letters in $v_1$ are contained in $\partial D$, implies that such a cancellation involves a letter in $\delta(a', b_1)$, together with a letter in $w$. It follows easily that this contradicts the minimality of the original sequence $a_0, a_1, \ldots, a_r$. □

Recall that in the $\omega$-stable context, a type extension $p \subset q$ is non-forking if and only if $RM(p) = RM(q)$. It follows that the foundation rank (on the class of complete types over varying sets of parameters) of the relation $\prec'$, given by $q \prec' p$ if and only if $q$ is a forking extension of $p$ (known as the Lascar U-rank) bounds the
Morley rank of a type from below. Putting this information together with Lemmas 11.2 and 11.3 yields some lower bounds on the Morley Rank.

**Corollary 11.4.** Let \( b \in N \) and let \( B \subseteq N \) be weakly convex. Then we have \( \text{RM}(p^w_{b,B}) \geq \text{Or}(w) \). In particular,

\[
\text{Or}(R^*_D(x,a)) = \text{Or}(R_D(x,a)) = \text{Or}(D)
\]

for any \( D \in \mathcal{D}\setminus\{C\} \), and

\[\text{RM}(x = x) = \omega^{k(\mathcal{D})} .\]

If \( \mathcal{D} = \mathcal{D}_0 \), then

\[\text{RM}(x = x) = \omega^{3g+b-2},\]

where \( g \) is the genus of the underlying surface \( \Sigma \), and \( b \) the number of punctures.

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