Front Propagation in Reaction-Superdiffusion Dynamics - Taming Lévy Flights with Fluctuations

D. Brockmann and L. Hufnagel

Max-Planck-Institut für Strömungsforschung, Bunsenstr. 10, 37073 Göttingen, Germany and Kavli Institute for Theoretical Physics, University of California Santa Barbara, CA 93106, USA

We investigate front propagation in a reacting particle system in which particles perform scale-free random walks known as Lévy flights. The system is described by a fractional generalization of a reaction-diffusion equation. We focus on the effects of fluctuations caused by a finite number of particles. We show that, in spite of superdiffusive particle dispersion and contrary to mean field theoretical predictions, wave fronts propagate at constant velocities, even for very large particle numbers. We show that the asymptotic velocity scales with the particle number and obtain the scaling exponent.

PACS numbers: 05.70.Ln, 02.50.Ey, 05.40.Fb, 87.23.Cc

One of the fundamental processes involved in non-equilibrium pattern formation is the spatial propagation of interfaces or fronts. Front propagation usually emerges when a local reaction dynamics interplays with diffusion in space of the reacting agents and has been observed in a wide range of physical, chemical and biological systems [1]. Prominent examples are patterns of bacterial colonies [2], spreading phenomena in population genetics [3, 4] and interface dynamics in superconductors [5]. One of the most prominent models which displays propagating fronts is the Fisher-Kolmogorov-Petrovsky-Piscounov (FKPP) equation for the spatial concentration $u(x,t)$ of a reacting agent,

$$\partial_t u = \gamma u (1 - u) + \Delta u, \quad (1)$$

where diffusive motion of the reacting agents is assumed. However, this assumption cannot be justified for a number of systems. In fact, superdiffusive dispersion in space has been observed in a wide range of physical and biological systems, e.g. intermittent chaotic systems [6], bacterial motion [7], and foraging patterns of albatrosses [8].

Superdiffusive stochastic motion is usually characterized by a lack of scale in the microscopic step distribution. One of the most successful theoretical concepts devised for the understanding of superdiffusion is a class of random walks known as Lévy flights [9]. A Lévy flight consists of random single steps $\Delta x$ which are drawn from an inverse power-law pdf (probability density function) $p(\Delta x) \sim |\Delta x|^{-(1+\mu)}$ characterized by a Lévy exponent $0 < \mu < 2$. Due to the heavy tail, the variance in step size is divergent, the process lacks a spatial scale and the position $X(t)$ of a Lévy flight scales heuristically with time $t$ as $X(t) \sim t^{1/\mu}$. The associated diffusion equations contain fractional generalizations of ordinary derivatives [10, 11]. These fractional Fokker-Planck equations can exhibit behaviors strikingly different from ordinary ones [12] and have found wide application in physics, e.g. protein motion on folded hetero-polymers [13] and the dynamics of modern epidemics [14, 15].

In two recent studies wave front dynamics was shown to be drastically different from ordinary reaction-diffusion dynamics when the reacting agents move superdiffusively [16, 17]. The authors considered fractional generalizations of Eq. (1) and showed that the spatio-temporal shape $u(x,t)$ of the leading edge of a propagating front has a power-law tail along the spatial coordinate and accelerates exponentially in time as opposed to the constant velocity and exponential decay in space exhibited by ordinarily diffusive system. The predictions made by mean field theory thus indicate that scale-free, superdiffusive dispersion of the reacting agents excludes constant velocity wave fronts and induces an entirely different spatio-temporal behavior.

However, as has been shown in a number of recent studies, the effect of fluctuations can be rather profound in these systems [18, 19, 20]. For instance, fluctuations can destabilize homogeneous wave fronts in two-dimensional systems [21]. A finite albeit large number $N$ of particles or reacting agents leads to a multiplicative noise term in the reaction-diffusion equations and although the variance of the noise is of order $1/N$, corrections to macroscopic quantities such as the front speed $v$ scale as $(\log N)^{-2}$, a significant correction to the mean field approximation even for very large values of $N$. Brunet and Derrida [22, 23] extended mean field dynamics by an effective cutoff parameter $\varepsilon$ for the concentration of particles below which no reaction and hence no exponential growth of the leading edge of a front is possible. Despite the fact that a rigorous equivalence with multiplicative noise is still lacking, the effective cutoff approach is very intuitive and in remarkable agreement with simulations of the full probabilistic dynamics.

Here, we focus on the effect of fluctuations on reaction-superdiffusion kinetics. We show that for arbitrarily small fluctuations (i.e. arbitrarily large particle numbers $N$), wave fronts propagate asymptotically at constant velocities. Furthermore, we show that as soon as fluctuations enter the description the algebraic tail along the spatial coordinate of the leading edge disappears and is replaced by an exponential decay. Thus, despite the fact that reacting agents move superdiffusively in space, the wave front patterns are qualitatively the same as in the ordinary diffusion case. We show that a front speed $v$ is selected after a transient time and that $v$ scales with particle number as $v \sim N^{1/\mu}$ for Lévy exponents $\mu < 2$. The results reported here are rather counterintuitive, deviate strongly from the predictions cast by mean field theory, and indicate that fluctuations affect reactions-superdiffusion sys-
respectively. Furthermore, particles of both types may jump
from position $x$ to position $y$ with a probability density rate
$f(|x - y|)$ which we assume to be a decreasing function of
distance $|x - y|$. The dynamic stochastic variables are the
particle numbers $n_A(x, t)$ and $n_B(x, t)$ of particles in an volume
of size $\Omega$ around $x$ of type $A$ and $B$, respectively. The volume
$\Omega$ is assumed to be large enough to contain a large number
of particles but small compared to the spatial extend of the
system. The total number of particles in $\Omega$ around $x$ is given
by $N(x, t) = n_A(x, t) + n_B(x, t)$. Without spatial dispersion
of particles the local dynamics is governed by the master equation

$$\partial_t p(n, t) = \sum_m w(n|m)p(m, t) - w(m|n)p(n, t)$$

for the probability $p(n, t)$ of finding a number $n = n_A$
of particles of type $A$ at a location $x$ with initial condition
$p(n, 0) = \delta(n - n_0)$ and the rate

$$w(n|m) = \frac{k_1}{\Omega} n(N-m)\delta_{n,m+1} + \frac{k_2}{\Omega} m(N-m)\delta_{n,m-1}. \quad (5)$$

The dynamics of the expectation value of $\langle n(t) \rangle$ is governed by

$$\partial_t \langle n(t) \rangle = \frac{\gamma}{\Omega} \langle n(t) (N - n(t)) \rangle,$$  

where $\gamma = k_1 - k_2$. Note that $\langle n(t) \rangle$ is continuous in $[0, N]$. In
a spatially extended system the number of particles is a function
of position, i.e. $n = n(x, t)$. Apart from normalization, $\langle n(x, t) \rangle$ may be interpreted as the probability of finding an $A$
particle in the volume $\Omega$. Dispersion contributes to the change
of particles $\partial_t \langle n(x) \rangle$ according to

$$\partial_t \langle n(x, t) \rangle = \int dy \left[ f(|x - y|) \left[ \langle n(y, t) \rangle - \langle n(x, t) \rangle \right] \right], \quad (7)$$

incorporating the probability density rate (Eq. (3)) of jumping
from $y$ to $x$. Eq. (7) defines the operator $L$ acting on the field
$\langle n(x, t) \rangle$. Denoting the spatial density of particles by
$u(x, t) = n(x, t)/\Omega$ and combining Eq. (6) with (7) yields

$$\partial_t \langle u \rangle = \gamma \langle u (1-u) \rangle + L \langle u \rangle,$$   \quad (8)

where we have set without loss of generally the maximum density $N/\Omega$ to unity.

When particles perform ordinary random walks, i.e. if the
jump rate $f(|x - y|)$ is equipped with a length scale $\sigma$, the
operator $L$ can be approximated by the ordinary Laplacian $\Delta$
on scales larger than $\sigma$. However, this description is no longer
valid if $f(x) \sim |x|^{-(1+\mu)}$ with $0 < \mu < 2$. In this case
individual jumps lack a scale, particle perform Lévy flights, and
$L$ is proportional to a non-local singular integral operator,

$$\Delta^{\mu/2}u(x, t) = C_\mu \int dy \left[ \frac{u(y, t) - u(x, t)}{|x - y|^{1+\mu}} \right]. \quad (9)$$

where $C_\mu$ is a constant [24]. In Fourier space the operator
$\Delta^{\mu/2}$ is equivalent to a multiplication by $|k|^{\mu}$, generalizing the
well known $k^2$ factor corresponding to the ordinary Laplacian
which is why $\Delta^{\mu/2}$ is frequently referred to as a fractional
Laplacian.
the wavefronts on a double-logarithmic scale. After a transient phase, the shape of the front approaches a steady state with a sharply decreasing boundary at intermediate values for the concentration and an algebraic tail for large $x$ with concentrations $u(x,t)$ below the cutoff $\varepsilon$. Fig. 4 displays the steady state front shape on a semilogarithmic scale. As opposed to the algebraic tail predicted by mean field theory, the boundary is an exponential function of relative position $z = x - vt$, i.e. $u(z) \sim \exp(-\lambda z)$, characterized by a spatial scale $\lambda$. In summary, the characteristic spatio-temporal wave front solution of Eq. (11) for large times is given by $u(x,t) \sim \exp[-\lambda(x-vt)]$, for $1/N < u < 1$ followed by a power-law tail $u(x,t) \sim (x-vt)^{-(1+\mu)}$ for $u < 1/N$. The decay parameter $\lambda$ and the velocity $v$ depend on the particle number $N$. Qualitatively, the this dependence can be determined in the moving reference frame under the assumption that $u(x,t) = u(x-vt) = u(\zeta)$. Figs. 1(b) and 2 suggest that $u(z) = u_1(z) = \exp(-\lambda z)$ for $z > z^*$ and $\varepsilon < u \ll 1$ and $u(z) = u_2(z) = A/z^{1+\mu}$ for $z > z^* (u < \varepsilon)$ where $z^*$ marks the crossover between exponential and algebraic decay. Inserted into Eq. (11) yields

$$v \lambda u_1 \approx u_1 + \Delta \mu/2 u \quad z < z^*$$  \hspace{1cm} (12)

$$-v u_2' = \Delta \mu/2 u \quad z > z^*.$$  \hspace{1cm} (13)

Near the crossover the values of the dispersion in both equations are approximately the same, i.e. $(\Delta \mu/2 u)(z^* - \Delta z) \approx (\Delta \mu/2 u)(z^* + \Delta z)$ and both equations can be combined to

$$v \lambda u_1 \approx v u_1 - v u_2.$$  \hspace{1cm} (14)

The dependence of $\lambda$ on $N$ can be obtained by the short time dynamics. Consider an initial condition $u(x,0) = \exp(-\lambda x)$ on the half-line $x \geq 0$. After a short time $\Delta t \ll 1$ the wave front is approximately given by $u(x,\Delta t) \approx \exp(-\lambda x + \Delta t/\lambda x^{1+\mu})$. The crossover $\varpi$ can be defined as the point at which both terms are of the same order of magnitude, implicitly provided by $\lambda \exp(-\varpi) = \Delta t/\varpi^{1+\mu}$. Since $\lambda \ll 1$ one can approximately solve for the crossover, $\varpi \approx \ln(\lambda \Delta t)/\lambda$. In order for the exponential to remain invariant under the dynamics, $\lambda$ must be chosen such that the crossover coincides with the effective cutoff, i.e. $u(\varpi,\Delta t) = \varepsilon$ which implies the scaling relation $\lambda \propto \varepsilon^{1/\mu}$. With Eq. (14) one obtains the scaling law

$$v \propto N^{1/\mu}$$  \hspace{1cm} (15)

for the velocity $v$.

Fig. 3 shows the front velocity $v$ as a function of particle number $N$ obtained by numerical integration of the dynamics (Eq. (11)). The numerics agree well with the scaling law (15) over several orders of magnitude and several choices of the Lévy exponent $\mu$. 

$$\varrho = \varepsilon \propto \lambda^{-1/\mu}.$$  \hspace{1cm} (16)
Figure 3: Asymptotic front velocity $v$ as a function of the particle number $N$ for different Lévy exponents $\mu = 1.2, 1.5$ and 1.8 (circles, diamonds and triangles, respectively). The dashed line indicates the scaling $v(N) \sim N^{1/\mu}$.

The front propagation characteristics of the macroscopic description in terms of the reaction-superdiffusion equation with effective cutoff (Eq. (11)) coincides with our simulations of the microscopic dynamics defined by Eqs. (2) and (3). The simulation results are summarized in Fig. 4. As in Fig. 1(a) the front velocity $v$ is constant after a transient phase, the velocity obeys the scaling law (15) and the wave front decays exponentially in space.

We are convinced that our results are of major importance for the understanding of front propagation in pattern forming systems in which the reactive agents defy the rules of ordinary diffusion. We have shown that constant velocity fronts are typical for pulled front dynamics, contrary to what is expected from mean field approximations and we believe that our results will contribute to the understanding of more complex pattern forming systems such as the geographic spread of human epidemics (15).

This research was supported in part by the National Science Foundation under grant No. PHY99-07949. Discussions with T. Geisel, L. Sander, W. Noyes, V. Hardapple and M. Boone Jr. are gratefully acknowledged.

[1] M. Cross and P. Hohenberg, Rev. Mod. Phys. 65, 851 (1993).
[2] I. Golding, Y. Kozlovsky, I. Cohen, and E. Ben-Jacob, Physica A 260, 510 (1998).
[3] R. A. Fisher, Ann. Eugen. 7, 355 (1937).
[4] D. G. Aronson and H. F. Weinberger, Adv. Math. 30, 33 (1978).
[5] S. J. D. Bartolo and A. T. Dorsey, Phys. Rev. Lett. 77, 4442 (1996).
[6] T. Geisel, J. Nierwetberg, and A. Zacherl, Phys. Rev. Lett. 54, 616 (1985).
[7] M. Levandowsky, B. S. White, and F. L. Schuster, Acta Protozool. 36, 237 (1997).
[8] G. M. Viswanathan, V. Afanasyev, S. V. Buldyrev, E. J. Murphy, P. A. Prince, and H. E. Stanley, Nature 381, 413 (1996).
[9] M. F. Shlesinger, G. M. Zaslavsky, and U. Frisch, eds., Lévy Flights and Related Topics in Physics, Lecture Notes in Physics (Springer Verlag, Berlin, 1995).
[10] R. Metzler and J. Klafter, Phys. Rep. 339, 1 (2000).
[11] I. M. Sokolov, J. Klafter, and A. Blumen, Physics Today 55, 48 (2002).
[12] D. Brockmann and T. Geisel, Phys. Rev. Lett. 90, 170601 (2003).
[13] D. Brockmann and T. Geisel, Phys. Rev. Lett. 91, 048303 (2003).
[14] L. Hufnagel, D. Brockmann, and T. Geisel (2004), submitted.
[15] D. Brockmann and L. Hufnagel, in preparation.
[16] R. Mancinelli, D. Vergni, and A. Vulpiani, Europhys. Lett. 60, 532 (2002).
[17] D. del Castillo-Negrete, B. A. Carreras, and V. E. Lynch, Phys. Rev. Lett. 91, 018302 (2003).
[18] L. Pechenik and H. Levine, Phys. Rev. E 59, 3893 (1999).
[19] E. Brunet and B. Derrida, J. Stat. Phys. 103, 269 (2001).
[20] C. R. Doering, C. Mueller, and P. Smereka, Physica A 325, 243 (2003).
[21] D. A. Kessler and H. Levine, Nature 394, 556 (1998).
[22] E. Brunet and B. Derrida, Phys. Rev. E 56, 2597 (1997).
[23] D. A. Kessler, Z. Ner, and L. M. Sander, Phys. Rev. E 58, 107 (1998).
[24] D. Brockmann and I. Sokolov, Chem. Phys. 284, 409 (2002).