Superstring Amplitudes as a Mellin Transform of Supergravity

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Abstract

At the tree level, the maximally helicity violating amplitudes of $N$ gauge bosons in open superstring theory and of $N$ gravitons in supergravity are known to have simple representations in terms of tree graphs. For superstrings, the graphs encode integral representations of certain generalized Gaussian hypergeometric functions of kinematic invariants while for supergravity, they represent specific kinematic expressions constructed from spinor–helicity variables. We establish a superstring/supergravity correspondence for this class of amplitudes, by constructing a mapping between the positions of gauge boson vertices at the disk boundary and the helicity spinors associated to gravitons. After replacing vertex positions by a larger set of $\frac{N(N-3)}{2}$ coordinates, the superstring amplitudes become (multiple) Mellin transforms of supergravity amplitudes, from the projective space into the dual Mellin space of $\frac{N(N-3)}{2}$ kinematic invariants. Similarly, inverse Mellin transforms transmute open superstrings into supergravity. We elaborate on the properties of multiple Mellin and inverse Mellin transforms in the framework of superstring/supergravity correspondence.
I. INTRODUCTION

Superstring scattering amplitudes are often considered far more involved than scattering amplitudes in quantum field theory. There are many efficient perturbative techniques available in quantum field theory, based on Feynman diagrams, recursion relations etc., and even some non-perturbative aspects of scattering amplitudes can be studied by using the AdS/CFT correspondence. Furthermore, a unified mathematical framework encompassing the complete perturbative S–matrix (all loops, arbitrary number of external particles) in terms of the Grassmannian description, has been proposed for maximally supersymmetric gauge theories \[1\].

In string theory, although there has been some steady progress over the last thirty years, two–dimensional world–sheet conformal field theory (CFT) still remains as the basic tool for computing scattering amplitudes. A rare newcomer to this research field has to digest several textbook chapters before even trying to reproduce the four–tachyon amplitude written by Veneziano in 1968 \[2\]. In order to compute the lowest order, semi–classical scattering amplitude of four gauge bosons in open superstring theory, one considers a disk world–sheet with four vertex operators inserted at the boundary. There is an integral to be performed, over the position of one of vertex operators, the other three being fixed by Möbius transformations. This integral yields a “special” function of kinematic invariants, the Euler beta function which summarizes virtual exchanges of gauge bosons and of the infinite tower of their string (Regge) excitations in all kinematic channels, but avoiding double–counting and implementing the world–sheet duality of the old “dual resonance model.”

More multiple integrals appear in the scattering amplitudes involving larger numbers of external gauge bosons (gluons) \[3, 4\]. In \(N\)–gluon amplitudes, kinematics are specified by \(\frac{N(N−3)}{2}\) Lorentz invariants, instead of just two Mandelstam’s variables (for \(N = 4\)), and there are \(N−3\) vertex positions to be integrated over the boundary. As a result, one obtains many generalized hypergeometric functions of many kinematic variables, instead of a single beta function of two variables. In spite of such complications, some significant progress has been accomplished over the last few years. Most notably, \(N\)–gluon superstring disk amplitudes have been expressed in terms of tree–level Yang–Mills amplitudes and \((N−3)!\) hypergeometric functions \[3, 6\]. More recently, a particularly simple formula has been derived for the maximally helicity violating (MHV) amplitudes \[7\], with the functions represented by tree
This paper begins with a simple observation of a similarity between semi-classical MHV amplitudes describing $N$ gauge bosons in open superstring theory and $N$–graviton MHV amplitudes in quantum field theory of supergravity. This similarity becomes most apparent when interpreting the amplitudes in terms of tree graphs. For superstrings, the vertices correspond to vertex positions at the disk boundary while for supergravity, they label the gravitons. Nevertheless, as shown in Section 2, the edge factors are identical, and both amplitudes can be expressed as certain minors of the same (generalized) Laplacian matrix. The only difference between supergravity and superstring is that in the latter case, there remain non–trivial integrations to be performed over the vertices.

In Section 3, we focus on the vertex integrations of superstring amplitudes. We uplift the vertex positions to a larger, $N(N-3)/2$–dimensional space, parameterized by Möbius invariant cross–ratios, in one-to-one correspondence with the kinematic invariants. The vertex integrations are lifted to the embedding projective space, to a surface localized by insertions of appropriate delta function constraints. We show that these integrals amount to a multi–dimensional Mellin transform of the supergravity amplitude, from Mellin position space of Möbius invariant cross–ratios to the dual space of kinematic variables.

In Section 4, we elaborate general aspects of multiple Mellin transforms in the string framework. We consider superstring amplitudes as Mellin amplitudes. An integral transform (multiple inverse Mellin transform) of the latter yields simple expressions in terms of products of delta–functions localizing on the two–dimensional world–sheet. Momentum dependence is reinstalled by applying Mellin transforms.

In Section 5, we point to future directions and discuss some broader implications of our results.

In Appendix A, we discuss polynomial reduction of a certain class of rational functions, which can be related to $(N–2)^{(N–4)}$ labelled trees on $N–2$ vertices (Cayley graphs). The latter serve as a basis for writing both the $N$–point graviton supergravity amplitude and the $N$–point superstring gluon amplitude. We prove that partial fraction decomposition reduces this set of rational functions to a basis of $(N–3)!$ elements, which are graphically related to Hamilton graphs. In Appendix B, we discuss some features of multiple Mellin transforms, detailing the $N = 5$ Cayley and $N = 6$ Hamilton bases. In Appendix C, we explicitly perform a quintuple inverse Mellin transform on a generic 5–point superstring form factor.
showing explicitly its result in terms of delta–functions in Mellin position space.

II. UNIFIED DESCRIPTION OF SUPERGRAVITY AND SUPERSTRING AMPLITUDES

A. Superstring amplitude

Our discussion builds on the previous studies of the tree-level (disk) superstring amplitudes for the scattering of $N$ gauge bosons, see Refs. \cite{4–10}. In Ref. \cite{7}, specific choices were made for three vertex positions and for the polarization vectors of $N−3$ gauge bosons with positive helicities. For the partial amplitude associated to the $\text{Tr}(T^{a_1} \cdots T^{a_N})$ Chan-Paton factor, $\text{PSL}(2, \mathbb{R})$ world-sheet invariance was used to fix $z_1 = −\infty$, $z_2 = 0$, $z_3 = 1$. Furthermore, the gauge freedom was exercised to select momentum $p_2$ as the reference vector, with the polarization vectors

$$\varepsilon^\mu_k = \sigma^\mu_{a\dot{a}} \frac{\lambda^a_k \lambda_{\dot{a}}}{(2k)} \, , \quad k \geq 4 \, ,$$

satisfying

$$\varepsilon_k p_2 = \varepsilon_k p_k = \varepsilon_k \varepsilon_i = 0 \, . \quad (2)$$

The starting point for the MHV superstring formula derived in Ref. \cite{7} is

$$A^S_N = \frac{1}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \int d\mu^S_N(z, s) \sum_{i \geq 4, j \geq 3} \prod_{i \neq j} \varepsilon_i p_j (z_{ij}) \, , \quad (3)$$

where the integral

$$\int d\mu^S_N(z, s) := \int_1^\infty dz_4 \ldots \int_1^\infty dz_{N−1} \prod_{2 \leq k < l \leq N} |z_{kl}|^{s_{kl}} \, , \quad s_{kl} = 2\alpha' p_k p_l \, , \quad (4)$$

and tilde refers to a specific $\text{PSL}(2, \mathbb{R})$ choice of the three vertex positions, with the integration domain over the remaining positions correlated with the Chan-Paton factor. Here, as usual, $z_{ij} = z_i − z_j$. In Eq. (3) the prime over the sum denotes exclusion of any index configuration involving a loop $(ij)(jk)\ldots(mi)$, thus eliminating all closed cycles $(z_{ij} z_{jk} \cdots z_{mi})^{−1}$ of single poles, in particular the double poles $(z_{ij})^{−2}$ which, upon integration, lead to tachyonic

1 Here and later, when discussing MHV graviton amplitudes, we follow the standard practice of omitting the “supersymmetric” $\langle 12 \rangle^4$ factors associated to two bosons with negative helicities in “mostly plus” amplitudes.
singularities. The remaining \((N - 2)^{(N - 4)}\) integrals are “transcendental” and are characterized by a special form of their small \(\alpha'\) (low-energy) expansions.

It is not accidental that, according to Cayley’s formula, \((N - 2)^{(N - 4)}\) is also the number of tree graphs with \(N - 2\) vertices. In fact, the amplitude (3) can be rewritten as

\[
A^S_N = \frac{1}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \left( \prod_{k=4}^{N} \langle 2k \rangle \right)^{-2} \int d\tilde{\mu}^S_N(z, s) \sum_{\text{trees}} \prod_{\text{edges}} \frac{s_{ij}}{z_{ij} z'_{ij}} ,
\]

where the sum extends over all tree graphs with vertices labelled by \(3, 4, \ldots, N\). Here, we introduced \(z'_{ij} \equiv z'_{i} - z'_{j} \) with

\[
z'_{ij} = \frac{\langle ij \rangle}{\langle 2i \rangle \langle 2j \rangle} ,
\]

where \(\lambda_1 \neq \lambda_2\) is an arbitrary spinor. Indeed, by using Schouten’s identity

\[
z'_{ij} = z'_{j} - z'_{i} = \frac{\langle ij \rangle}{\langle 2j \rangle \langle 2i \rangle} ,
\]

and the \(x\)-dependence cancels in \(z'_{ij}\). A similar identification between spinor brackets and free–fermion propagators on the complex plane has appeared before in [11]. Note, that the variables (7) satisfy the same partial fraction relations as the positions variables \(z_{ij}\)

\[
\frac{1}{z_{ij} z_{jk}} + \frac{1}{z_{ik} z_{kj}} + \frac{1}{z_{ij} z_{ki}} = 0 = \frac{1}{z'_{ij} z'_{jk}} + \frac{1}{z'_{ik} z'_{kj}} + \frac{1}{z'_{ij} z'_{ki}} ,
\]

reflecting

\[
\frac{\lambda_j}{\langle ij \rangle \langle jk \rangle} + \frac{\lambda_k}{\langle ik \rangle \langle jk \rangle} + \frac{\lambda_i}{\langle ij \rangle \langle ki \rangle} = 0 ,
\]

which is the relation underlying Schouten’s identity.

In Ref. [7], we used partial fractioning (8) in \(z\)-variables to rewrite the amplitude (5) as a sum of chains (Hamiltonian paths) rooted at \(i = 3\), labeled by \((N - 3)!\) permutations \(P\) of \(4, 5, \ldots, N\):

\[
A^S_N = \frac{1}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \int d\tilde{\mu}^S_N(z, s) \sum_{P} \prod_{k=4}^{N} \frac{\langle 2|3 + \ldots + (k - 1)|k \rangle}{\langle 2k \rangle} \frac{1}{z_{(k-1)k}} .
\]

We refer the reader to Appendix A for a detailed exposition of Cayley graphs and their reduction to \((N - 3)!\) Hamilton graphs subject to partial fraction decomposition on the corresponding rational functions. The above result (10) can be also expressed as:

\[
A^S_N = \frac{1}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \left( \prod_{k=4}^{N} \langle 2k \rangle \right)^{-2} \int d\tilde{\mu}^S_N(z, s) \sum_{P} \prod_{i=4}^{N} \left( \frac{1}{z_{(i-1)j}} \sum_{m=3}^{i-1} s_{mi} \right) .
\]

Note that the integrand on the r.h.s. of Eq. (5) is symmetric under \(z \leftrightarrow z'\). Later, we will use partial fractioning in \(z'\) instead of \(z\), in order to make a direct connection with the general formula [5, 6] for superstring disk amplitudes.
B. Supergravity amplitude

The tree-level MHV formula for the scattering of $N$ gravitons can be written in many ways \cite{12–17}. In particular, in Ref. \cite{14}, Mason and Skinner recast the original formula of Berends, Giele and Kuijf \cite{12} into the following form \cite{126}:

$$A^G_N = \frac{1}{\langle 12 \rangle^2 \langle 23 \rangle^2 \langle 13 \rangle^2} \sum_{P} \frac{1}{\langle 1N \rangle} \prod_{k=4}^{N} \frac{(2|3+\ldots+(k-1)|k]}{(2k)} \frac{1}{\langle (k-1)k \rangle}. \quad (12)$$

Written in this way, the graviton amplitude bears a striking resemblance to the superstring amplitude of Eq. \cite{10}. Furthermore, we can also make a precise connection between the graphs representing the superstring amplitude \cite{5} and the graphs introduced in Refs. \cite{15, 17} to describe the graviton amplitude. To that end, it is most convenient to use the Feng–He’s version \cite{17} of Hodges’ determinant formula \cite{16}.

We begin by redefining

$$z_i \equiv \langle iy \rangle \langle xy \rangle \langle xi \rangle, \quad z'_i \equiv \langle ix \rangle \langle yx \rangle \langle yi \rangle, \quad (13)$$

where $\lambda_x$ and $\lambda_y$ are two arbitrary reference spinors. Furthermore, we have:

$$z_{ij} = \frac{\langle ij \rangle}{\langle xi \rangle \langle xj \rangle}, \quad z'_{ij} = \frac{\langle ij \rangle}{\langle yi \rangle \langle yj \rangle}, \quad (14)$$

which again obey the partial fraction relations \cite{8}. Although $z$ and $z'$ are related by $z_i z'_i = -\langle xy \rangle^{-2}$, they will be considered as independent variables\footnote{More precisely, the formula written below follows from \cite{14} after a trivial relabelling of graviton indices.}. Written in terms of these variables, up to an overall sign, the $N$-graviton MHV amplitude \cite{17} becomes

$$A^G_N = \left( \prod_{n=1}^{N} \langle xn \rangle \langle yn \rangle \right)^{-2} \frac{1}{z_{ij} z_{jk} z_{ki}} \frac{1}{z'_{ij} z'_{jk} z'_{ki}} \mid \Psi_{ijkl}^{rst} \mid, \quad (15)$$

where $\Psi$ is a $N \times N$ “weighted Laplacian” matrix with the elements

$$\psi_{ij} = \begin{cases} \frac{s_{ij}}{z_{ij} z'_{ij} \text{ if } i \neq j}, \\ -\sum_{j \neq i} \frac{s_{ij}}{z_{ij} z'_{ij}} \text{ if } i = j, \end{cases} \quad (16)$$

$$A^G_N = \frac{1}{\langle 12 \rangle^2 \langle 23 \rangle^2 \langle 31 \rangle^2} \sum_{P} \frac{1}{\langle 1N \rangle} \prod_{k=4}^{N} \frac{(2|3+\ldots+(k-1)|k]}{(2k)} \frac{1}{z_{(k-1)k}}, \quad (12)$$

which assumes the same form as the superstring amplitude \cite{10}, where $z$’s denote the vertex positions.
and $|\Psi|_{rst}^{ijk}$ denotes the minor determinant obtained after deleting three rows $i, j, k$ and three columns $r, s, t$.

For our purposes, it is sufficient to consider the case of $i = r, j = s, k = t$. Then, according to the matrix–tree theorem, the determinant is given by the sum of all forests consisting of three trees rooted at $i, j$ and $k$, with a combined number of $N - 3$ edges, each of them bringing a $\psi_{ij}$ factor. An even simpler expression can be obtained by choosing $x = i, y = j$ which sends $z_i \to \infty$ and $z'_j \to \infty$, thus leaving only single trees rooted at $k$. As a result, one obtains all trees with $N - 2$ vertices different from $i$ and $j$. For example, with the choice $x = i = r = 1, y = j = s = 2, k = t = 3$, one obtains (cf. \[13, 15\])

$$A^G_N = \frac{1}{\langle 12 \rangle^2 \langle 23 \rangle^2 \langle 31 \rangle^2} \left( \prod_{k=4}^N \langle 1k \rangle \langle 2k \rangle \right)^{-2} \sum_{\text{trees}} \prod_{\text{edges}} s_{ij} z_{ij} z'_{ij},$$

where the sum is over the same trees as in the superstring amplitude \[5\]. The edge factors become identical upon reverting to $z$'s defined as vertex positions.

C. Unified description

On the basis of preceding observations, we can set up a unified description of the amplitudes. First, we define the fundamental function:

$$\mathcal{M}_N(z, z', s)_{ijk}^{rst} \equiv \frac{1}{z_{ij} z_{jk} z_{ki}} \frac{1}{z'_{rs} z'_{st} z'_{tr}} |\Psi|_{ij}^{rst}.$$  

Then the graviton amplitude is:

$$A^G_N = \int d\mu^G_N(z, \lambda) \int d\mu^G_N(z', \lambda) \mathcal{M}_N(z, z', s)_{ijk}^{rst},$$

with the measures

$$\int d\mu^G_N(z, \lambda) = \int \prod_{i=1}^N dz_i \delta \left( \langle xi \rangle^2 z_i - \frac{\langle xi \rangle \langle yi \rangle}{\langle xy \rangle} \right),$$

$$\int d\mu^G_N(z', \lambda) = \int \prod_{i=1}^N dz'_i \delta \left( \langle yi \rangle^2 z'_i - \frac{\langle xi \rangle \langle yi \rangle}{\langle xy \rangle} \right).$$

On the other hand, the superstring amplitude is:

$$A^S_N = \int d\mu^S_N(z, s)_{ijk} \int d\mu^G_N(z', \lambda) \mathcal{M}_N(z, z', s)_{ijk}^{rst},$$
where:

$$\int d\mu^S_N(z, s)_{ijk} = z_{ij}z_{jk}z_{ki}\int_{D} \left( \prod_{l \leq N}^{' } dz_l \right) \prod_{m < n \leq N}^{' } |z_{mn}|^{s_{mn}}. \tag{22}$$

and the primes over products denote exclusion of the indices $i, j, k$. The domain of integration $D$ along the boundary of the disk is determined by the Chan–Paton factor. The factor $z_{ij}z_{jk}z_{ki}$ in the string measure can be identified as the standard reparametrization ghost correlator.

Finally, we wish to make a comment on the relation between Eq. (21) and the general formula [5, 6] for superstring disk amplitudes. Let us start from Eq. (21) with $i = r = N$ and set $z_N \to \infty$, as in Ref. [5, 6]. Next, we choose $y = j = s = N - 1$, which sets $z'_{N-1} = \infty$. Finally, we set $k = t = 1$. As a result, we obtain all tree graphs with the vertices labeled by $1, 2, \ldots, N - 2$. After partial fractioning in $z'$, in exactly the same way as it was done for $z$ in the derivation of Eq. (11), we obtain

$$A^S_N = \frac{1}{\langle(N-1)N\rangle \langle N \rangle \langle(N-1)1\rangle} \left( \prod_{i=2}^{N-2} \langle(N-1)i\rangle \right)^{-2} \int d\mu^S_N(z, s) \sum_{\mathcal{P}} \prod_{k=2}^{N-2} \left( \frac{1}{z'_{(k-1)k}} \sum_{m=1}^{k-1} s_{mk} \right), \tag{23}$$

where the permutations $\mathcal{P}$ are now acting on $2, \ldots, N - 2$. Here, $z'$ have already been fixed by the $d\mu^G_N(z', \lambda)$ integrations:

$$z'_i = \frac{\langle iN \rangle}{\langle(N-1)N\rangle \langle(N-1)i\rangle}, \quad z'_{ij} = \frac{\langle ij \rangle}{\langle(N-1)i\rangle \langle(N-1)j\rangle}. \tag{24}$$

After substituting these expressions, Eq. (23) becomes

$$A^S_N = \int d\mu^S_N(z, s) \sum_{\mathcal{P}} \left( \prod_{k=2}^{N-2} \sum_{m=1}^{k-1} s_{mk} z_{mk} \right) A^Y_N(1, 2, \ldots, N), \tag{25}$$

where $A^Y_N$ is the Yang–Mills MHV amplitude [18]:

$$A^Y_N(1, 2, \ldots, N) = \frac{1}{\langle 12 \rangle \langle 23 \rangle \cdots \langle N1 \rangle}. \tag{26}$$

In this way, we obtain the string amplitude in exactly the same form as in Ref. [5].
III. STRING THEORY IN MELLIN SPACE

A. Dual resonance models and multi–channel variables

In dual resonance models\(^4\), a generic \(N\)–point scattering process involves multiple resonance exchanges in various channels. A planar channel that includes external particles \(i, i+1, i+2, \ldots, j-1, j\) is labeled by \((i, j)\). For a given planar ordering \((1, 2, \ldots, N)\) of external particles, there exist \(\frac{N(N-3)}{2}\) planar channels. The basic property of dual models is that only \(N-3\) channels can simultaneously appear in a given \(N\)–point (planar) dual diagram. Actually, not all combinations of channels are allowed. For example, the adjacent channels like \((i, i+1)\) and \((i+1, i+2)\) cannot appear simultaneously (they are called dual or incompatible channels), while the so–called compatible (non–dual) channels can show up simultaneously\(^5\).

An elegant way to account for compatible and incompatible channels is to introduce the variables

\[
0 \leq u_{i,j} \leq 1 \quad \begin{cases} 
\text{if } i = 2, j = 3, \ldots, N-1, \\
\text{if } i = 3, \ldots, N-1 < j = 4, \ldots, N,
\end{cases}
\]

(27)

conjugate to the channels \((i, j)\). Such variables appeared first as the integration variables in the original constructions of the integral representation of the amplitude generalizing the Veneziano amplitude to an arbitrary number of external particles\(^6\). In the following, the above set \((27)\) of indices \((i, j)\) is denoted by \(P\). There are \(\frac{N(N-3)}{2}\) such coordinates \(u_{i,j}\), in one–to–one correspondence with the independent\(^6\) kinematic invariants

\[
s_{i,j} = \alpha'(k_i + k_{i+1} + \ldots + k_j)^2
\]

(28)

associated to the allowed planar channels \((i, j)\) of the \(N\)–point scattering amplitude. For a given channel \((i, j)\) with \(u_{i,j} = 0\), all incompatible channels \((k, l)\) are required to have \(u_{k,l} = 1\). These conditions can be summarized by the following \(\frac{N(N-3)}{2}\) (nonlinear) constraints

\[
u_{P} = 1 - \prod_{\tilde{P}} u_{\tilde{P}},
\]

(29)

with \(\tilde{P}\) the set of all channels incompatible to \(P\). This set of equations is sufficient for excluding simultaneous poles in incompatible channels. Only \(\frac{(N-2)(N-3)}{2}\) of the above con-
constraints \( \text{(29)} \) are independent, thus leaving \( N - 3 \) free variables which can be chosen as \( u_{2,j}, \ j = 3, \ldots, N - 1 \). In this way, a generic \( N \)-particle dual amplitude can be written as

\[
B_N(\{s_{k,l}\}, \{n_{k,l}\}) := \left( \prod_{i,j \in P} \int_0^\infty du_{i,j} u_{i,j}^{s_{i,j} - 1 + n_{i,j}} \theta(1 - u_{i,j}) \right) \delta(\{u_{p,q}\}),
\]

with a set \( n_{i,j} \) of integers and the product of \( \frac{(N-2)(N-3)}{2} \) delta functions

\[
\delta(\{u_{p,q}\}) = \prod_P \delta \left( u_P - 1 + \prod_{\tilde{P}} u_{\tilde{P}} \right),
\]

where the prime over product indicates exclusion of the channels \((2,j)\).

While Quantum Chromodynamics superseded the dual resonance model as the theory of strong interactions, all assumptions/axioms of duality have been later implemented in disk amplitudes of open superstring theory. Actually, the change of integration variables from the positions of vertex operators at the disk boundary to the multi–channel variables \( \text{(27)} \) is very useful for studying the singularity structure of the amplitudes \( \text{(6)} \) and allows rewriting a generic string “formfactor” in exactly the same form as the dual amplitude \( \text{(30)} \). This change of integration variables is described below.

**B. Pascal’s triangle of constraints**

Here, we give another representation of the constraints \( \text{(29)} \) which is more natural for the computations of open string disk amplitudes. Now the \( \frac{N(N-3)}{2} \) coordinates \( \text{(27)} \) are written as the Möbius–invariant cross–ratios of vertex positions at the boundary

\[
u_{i,j} = \frac{(z_i - z_j)(z_{i-1} - z_{j+1})}{(z_i - z_{j+1})(z_{i-1} - z_j)},
\]

with the indices \( i, j \in P \) specified in \( \text{(27)} \) and the cyclic identification \( k + N \equiv k \) (e.g. \( z_0 = z_N, \ z_{N+1} = z_1 \) etc.). By using elementary algebraic manipulations, it is easy to show that these coordinates do indeed satisfy Eq. \( \text{(29)} \).

The new set of (equivalent) constraints can be succinctly summarized by drawing a “Pascal’s triangle” of \( \frac{(N-2)(N-3)}{2} \) cells labeled by \( (k,l), \ k = 3, \ldots, N - 1 < l = 4, \ldots, N \) or equivalently, by the sequences \( (k,k+1,\ldots,l) \), see Figure 1.
Each cell has its ancestors at higher levels and descendants at lower levels, sharing a sequence of at least two indices. For example, in Figure 1, the cell $(4, N-1)$ has 3 ancestors: $(3, N-1), (4, N), (3, N)$, while the ancestors of $(3, 4)$ are: $(3, 5), (3, 6), \ldots, (3, N)$. Note that $(3, N)$, at the top of the triangle, is the primary ancestor to all cells. For each cell $(k, l)$, we define the homogenous functions

$$\rho_{kl}(u) = u_{k,l} \prod_a a(u_{k,l}) , \quad \sigma_{kl}(u) = \prod_{n=k}^{l-1} u_{2,n} ,$$

where $a(u_{k,l})$ are the ancestors of $u_{k,l}$. We also introduce the polynomials:

$$\alpha_{kl}(u) = \rho_{kl}(u) + \sigma_{kl}(u) - 1 .$$

To each cell of the triangle, we associate the constraint:

$$\alpha_{kl}(u) = 0 .$$
Here again, it is a matter of simple algebra to verify these constraints and to show that they are equivalent to Eq. (29), thus describing the same embedding of disk boundary in \( N(N-3) \) –dimensional projective space.

C. From string world–sheet to Mellin space

We wish to uplift the integrals (1) over string vertex positions \( z_1, z_2, \ldots, z_N \) from disk boundary to the projective space of conformal coordinates (32). The constraints (35) will be implemented by inserting the following product of delta functions

\[
\delta(\{u_{p,q}\}) = \prod_{l=4}^{N} \delta(\alpha_{l-1,l}) \prod_{k=3}^{l-2} \rho_{kl} \delta(\alpha_{kl}),
\]

which is equivalent to (31). In this way, the string integral measure (1) is replaced by the following integral:

\[
\int d\tilde{\mu}_N^S(z, s) = \int dM_N(u, s) := \left( \prod_{i,j \in P} \int_0^\infty du_{i,j} u_{i,j}^{s_{i,j}-1} \theta(1-u_{i,j}) \left( \prod_{l=4}^{N} \sigma_{3l}^{-1} \rho_{l-1,l} \right) \delta(\{u_{p,q}\}) \right).
\]

Here, the delta functions enforce the constraints (35), while the adjacent bracket comprises a Jacobian determinant, which follows from the differential:

\[
\left( \prod_{k=4}^{N} dz_k \right) \left( \prod_{i<j}^{N} |z_{ij}|^{s_{i,j}} \right) \frac{z_{12} z_{23} z_{31}}{\prod_{l=1}^{N} |z_l - z_{l+2}|} = \left( \prod_{i,j \in P} du_{i,j} u_{i,j}^{s_{i,j}} \right) \delta(\{u_{p,q}\}).
\]

Written explicitly in terms of the delta functions associated to the cells of Pascal’s triangle,

\[
\int dM_N(u, s) = \left( \prod_{i,j \in P} \int_0^\infty du_{i,j} u_{i,j}^{s_{i,j}-1} \theta(1-u_{i,j}) \left( \prod_{l=4}^{N} \sigma_{3l}^{-1} \prod_{l=3}^{l-1} \rho_{kl} \delta(\alpha_{kl}) \right) \right).
\]

The above integral represents a multi–dimensional Mellin transform \(^7\) directly from the string world–sheet boundary to the dual space of kinematic invariants \( s_{i,j} \), called Mellin space, thus side–stepping space–time.

\(^7\) Note the identity: \( \frac{z_{12} z_{23} z_{31}}{\prod_{l=1}^{N} |z_l - z_{l+2}|} = \det(\frac{\partial u_{p,q}}{\partial z_{mn}}) \prod_{2 \leq i < j \leq N-1}^{N} u_{N+2-j,N+2-i}^{1-j+i} = \left( \prod_{l=4}^{N} \sigma_{3l}^{-1} \rho_{l-1,l} \right) \left( \prod_{i,j \in P} u_{i,j} \right) \).

\(^8\) The (single variable) Mellin transformation is an operation \( M_f(s) \), which assigns a function \( M_f(s) := \int_0^\infty dx x^{s-1} f(x) \) of the complex variable \( s \) to each locally summable function \( f(x) \), which satisfies the following two conditions: (i) \( f(x) \) is defined for \( x > 0 \) and (ii) there exists a strip \( s_1 < \Re(s) < s_2 \) in the complex \( s \)–plane such that \( x^{s-1} f(x) \) is absolutely integrable w.r.t. \( x \in (0, \infty) \).
In the previous Section we concluded that both the (MHV) scattering amplitudes $A_{SN}^S$ of gauge bosons in superstring theory and the graviton amplitudes $A_{GN}^G$ of supergravity can be obtained from a single function $M_N$, cf. Eq. (18), by appropriate integrations as (19) and (21), respectively. Thus we can start from the supergravity amplitude (19) and replace the integral measure $\int d\mu_G^N(z,\lambda)$ by the Mellin transform (37)

$$\int d\mu_G^N(z,\lambda) \rightarrow \int dM_N(u,s),$$

which, in the specific gauge, takes us back to the tree–graph formula of Eq. (5) with

$$\int d\tilde{\mu}_S^N(z,s) \rightarrow \int dM_N(u,s)$$

and the edge factors expressed in terms of the position variables $u_{i,j}$ described in the following.

As written in Eq. (5), in addition to the position variables, the edge factors $s_{i,j}$ depend on the kinematic invariants $s_{i,j}$ belonging to the dual Mellin space. Through $z_{i,j}^i z_{i,j}^j = \partial_{z_{i,j}}^i z_{i,j}^j$, these can be represented by insertions of the respective differential operators acting in position space $\partial_{u_{i,j}}^i u_{i,j}^i$. After integrating by parts, their action can be redirected on the delta function constraint (36). One finds that each edge gives rise to a single derivative of the respective delta function

$$\frac{s_{i,j}}{z_{i,j}} \rightarrow -\sigma_{3j} \frac{\delta'(\alpha_{i,j})}{\delta(\alpha_{i,j})},$$

supplemented by the factor $\sigma_{3j}$. Eventually, by using the relation $x \delta'(x) = -\delta(x)$, the replacements (41) and (42) provide the final form of the amplitude (5):

$$A_N^S = \frac{1}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \left( \prod_{k=4}^{N} \langle 2k \rangle \right)^{-2} \int dM_N(u,s) \sum_{\text{trees}} \prod_{\text{edges}} \sigma_{3j} \alpha_{i,j} z_{i,j}^j.$$

Here, the sum extends over all tree graphs with vertices labeled by $i, j = 3, 4, \ldots, N$, as in Eq. (5), and we used the fact that graphs are unoriented to label edges by ordered pairs $i < j$. The above formula yields the superstring amplitude as a Mellin transform of the graviton supergravity amplitude (17).

**IV. SUPERSTRING AMPLITUDES AS MELLIN AMPLITUDES**

In quantum field theory, generic correlation functions do not assume a simple form in position space, therefore one performs a Fourier transform to momentum space, where the

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9 We refer the reader to Appendix B for further technical details and examples.
analytic properties of the amplitude like its pole structure become simple. For CFTs, the
Symanzik’s star operator converts position space integrals into (inverse) Mellin transforms
of the so–called Mellin amplitudes \[22\]. The latter depend on complex variables \( s_{i,j} \)
substituting for the kinematic invariants of the scattering amplitude. Hence for CFTs, a Mellin
transform is more appropriate and Mellin space serves as a natural momentum space. In this
space, the CFT amplitudes exhibit a universal behaviour. Thus Mellin transforms, although
less familiar to particle theorists than Fourier transforms
\[10\], are useful for representing cor-
relation functions in CFTs, so it is not too surprising that they appear in the context of the
two–dimensional string world–sheet. In this Section we want to investigate the connection
between generic dual (superstring) amplitudes \(30\) and Mellin amplitudes in more detail.

A. Mellin amplitudes

In Refs. \[24, 25\] an exact correspondence between conformal field theories in \(D\) dimensions
and dual resonance models in \(D’\) dimensions has been established. Correlation functions
in the \(D\)–dimensional CFT are related to Mellin amplitudes of the dual resonance model
\[2\] through an inverse Mellin transformation\[11\]. The latter relates a conformal \(N\)–point
function \(G_N(\{x_r\}) := G(x_1, \ldots, x_N)\) with positions \(x_r\) in \(D\) dimensions and depending on
\(m = \frac{1}{2}N(N – 3)\) anharmonic cross–ratios \(\omega_{i,j}\) to a scattering amplitude \(M(\{s_{k,l}\})\) in the
dual resonance model in \(D’\) dimensions as

\[ G_N(\{x_r\}) = (2\pi i)^{-m} \left( \prod_{i,j \in P} \int_{-\infty+c}^{+\infty+c} ds_{i,j} \omega_{i,j}^{-s_{i,j}} \right) \left( \prod_{i<j} \Gamma(s_{ij}) \right) M(\{s_{k,l}\}), \tag{44} \]

with \(m\) complex variables \(s_{i,j}\) to be related to the \(m\) kinematic invariants \[28\] and the
cross–ratios \(\omega_{i,j} = \frac{(x_i-x_j)(x_{i-1}-x_{j+1})}{(x_i-x_{j+1})(x_{i-1}-x_j)}\) for \(D = 2\) and \(\omega_{i,j} = \frac{|x_i-x_j|^2|x_{i-1}-x_{j+1}|^2}{|x_i-x_{j+1}|^2|x_{i-1}-x_j|^2}\) for \(D > 2\). In
\[44\] the integration is over a suitable choice of contour in the complex variables \(s_{i,j}\) and the
set of indices \(i, j\) is defined in \(27\). The relation \[44\] has recently been applied to conformal
theories in \(D = 4\) dimensions to derive conformal correlators in AdS/CFT backgrounds

\[10\] In fact, the theory of Mellin transform is equivalent to that of Fourier transforms in the complex
plane \[23\]: For the Mellin transform \(M_f(s)\) of a function \(f\) with \(s \in \mathbf{C}\) we have \((2\pi)^{-1/2} M_f(is) =
(2\pi)^{-1/2} \int_{-\infty}^{\infty} dy \ e^{-isy} g(y)\), with the latter being the Fourier transform of \(g(y) := f(e^y)\).

\[11\] The (single variable) inverse Mellin transform \(f\) of a function \(M_f(s)\) is given by the complex integral
\(f(x) = 2\pi \int_{-\infty+c}^{+\infty+c} ds \ x^{-s} M_f(s)\), provided that the latter converges absolutely along the line \(s = c\) for any
real value \(c\) with \(s_1 < c < s_2\), the function \(M_f(s)\) is analytic in the strip \(s_1 < \Re(s) < s_2\) and goes to zero
uniformly for increasing \(\Im(s)\).
from Mellin space \[26\], to find Feynman rules for Mellin amplitudes \[27\], and to rewrite dual conformal integrals of perturbative scattering amplitudes in \(\mathcal{N} = 4\) SYM \[28\].

Furthermore, it has been speculated in \[24\], that the Mellin amplitude \(\mathcal{M}(\{s_{k,l}\})\), which shares exact duality, \textit{i.e.} meromorphy in \(s_{i,j}\) with simple poles in single variables, crossing symmetry and factorization, may actually be derived from correlators in string theory. In fact, in the following\(^{12}\) we shall start from the specific \(N\)-point Mellin amplitudes

\[
\hat{\mathcal{M}}(\{s_{k,l}\}) = B_N(\{s_{k,l}\}, \{n_{k,l}\})
\]

(45)

describing dual (superstring) \(N\)-point amplitudes \[30\] and consider their inverse Mellin transforms \[44\]. At a practical level in this case the objects \(\hat{\mathcal{M}}(\{s_{k,l}\})\) represent complicated multiple Gaussian hypergeometric functions encoding the infinite heavy string states and their inverse Mellin transforms should give some simple function \(G(\{x_r\})\) describing correlators of a conformal field theory.

Finally, integral transformations on string form factors \[30\] have already been considered in the past for computing high–energy limits, dispersion relations and discontinuities of dual amplitudes. \textit{E.g.} in Ref. \[29\] a dual amplitude \(B_N\), as a function of its own set of (planar) kinematic invariants \(s_{i,j}\), is written as a multiple beta–transform on some conjugate set of the variables \(s_{k,l}\). For \(N = 4\) this transformation gives rise to \[30\]

\[
B(s, u) = \int_{-i\infty+c}^{+i\infty+c} d\sigma \int_{-i\infty+c}^{+i\infty+c} d\tau \, B(s, \sigma) \, B(u, \tau) \, B(1 - \sigma, 1 - \tau),
\]

with \(s = s_{1,2}\), \(u = s_{2,3}\) and the Euler beta function \(B(s, u) = \frac{\Gamma(s)\Gamma(u)}{\Gamma(s+u)}\). However, as we will see in the next paragraph multiple Mellin transforms and its application to distributional delta–functions provide a novel direction.

\[B.\] \textbf{String form factors, inverse Mellin transforms and space–time correlators}

According to Section 3 for a set of integers \(n_{i,j}\) the form factors \[30\] of the \(N\)-point string amplitude can be written in terms of integrals over the \(m := \frac{1}{2}N(N - 3)\) coordinates \[27\] with the product of delta functions \(\delta(\{u_{k,l}\})\) given in \[31\] or \[36\]. We refer the reader to

\[12\] In the sequel we shall work with the reduced Mellin amplitude \(\hat{\mathcal{M}}(\{s_{k,l}\}) = \mathcal{M}(\{s_{k,l}\}) \left( \prod_{i<j} \Gamma(s_{ij}) \right) \[25\].
Ref. [6] for a detailed exposition and application of (30) in view of the hypergeometric function structure of superstring amplitudes. E.g. for \( N = 4 \) the expression (30) gives

\[
B_4(\{s_{k,l}\}, \{n_{k,l}\}) = \int_0^1 du_{1,2} \int_0^1 du_{2,3} u_{1,2}^{s_{12} - 1 + n_{1,2}} u_{2,3}^{s_{23} - 1 + n_{2,3}} \delta(u_{1,2} + u_{2,3} - 1)
\]

\[
= \frac{\Gamma(s + n_{1,2}) \Gamma(u + n_{2,3})}{\Gamma(s + u + n_{1,2} + n_{2,3})},
\]

while for \( N = 5 \) we have

\[
B_5(\{s_{k,l}\}, \{n_{k,l}\}) = \int_0^1 du_{1,2} \int_0^1 du_{2,3} \int_0^1 du_{3,4} \int_0^1 du_{4,5} \int_0^1 du_{1,5} u_{1,2}^{s_{12} - 1 + n_{1,2}} u_{2,3}^{s_{23} - 1 + n_{2,3}} u_{3,4}^{s_{34} - 1 + n_{3,4}}
\]

\[
\times \delta(u_{1,2} + u_{2,3}u_{1,5} - 1) \delta(u_{3,4} + u_{2,3}u_{4,5} - 1)
\]

\[
\times \delta(u_{4,5} + u_{3,4}u_{1,5} - 1) = \frac{\Gamma(s_{2} + n_{2,3}) \Gamma(s_{3} + n_{3,4}) \Gamma(s_{4} + n_{4,5}) \Gamma(s_{5} + n_{1,5})}{\Gamma(s_{2} + s_{3} + n_{2,3} + n_{3,4}) \Gamma(s_{4} + s_{5} + n_{4,5} + n_{1,5})}
\]

\[
\times {}_3F_2 \left[ \begin{array}{c} s_{2} + n_{2,3}, s_{5} + n_{1,5}, s_{3} + s_{4} - s_{1} + n_{3,4} + n_{4,5} - n_{1,2}, 1 \\ s_{2} + s_{3} + n_{2,3} + n_{3,4}, s_{4} + s_{5} + n_{4,5} + n_{1,5} \end{array} \right],
\]

with\(^{14} s_{1} = s_{1,2}, s_{2} = s_{2,3}, s_{3} = s_{3,4}, s_{4} = s_{4,5} \) and \( s_{5} = s_{5,1} \).

Let us now compute the integral (44) (with the reduced Mellin amplitude (15)) for the two amplitudes (46) and (47), respectively. For \( N = 4 \) we perform the double inverse Mellin transformation on (46):

\[
\frac{1}{(2\pi i)^2} \int_{-i\infty + \epsilon}^{+i\infty + \epsilon} ds \int_{-i\infty + \epsilon}^{+i\infty + \epsilon} du \ u_{1,2}^{-s} u_{2,3}^{-u} \frac{\Gamma(s + n_{1,2}) \Gamma(u + n_{2,3})}{\Gamma(s + u + n_{1,2} + n_{2,3})}
\]

\[
= u_{1,2}^{n_{1,2}} u_{2,3}^{n_{2,3}} \delta(1 - u_{1,2} - u_{2,3}) \theta(1 - u_{1,2}) \theta(1 - u_{2,3}).
\]

To show (48) we have used [31]

\[
\frac{1}{2\pi i} \int_{-i\infty + \epsilon}^{+i\infty + \epsilon} ds \ x^{-s} \frac{\Gamma(s)}{\Gamma(s + a + 1)} = \begin{cases} 
\frac{(1-x)^a}{\Gamma(a+1)}, & 0 < x \leq 1, \\
0, & x > 1,
\end{cases}
\]

for \( \Re(a) > -1 \) and the inverse Mellin transformation

\[
\delta(x - y) = \delta(y - x) = \frac{1}{2\pi i} \int_{-i\infty + \epsilon}^{+i\infty + \epsilon} ds \ x^{-s} y^{s-1}, \quad x, y > 0
\]

\(^{13} \) Note, that the coordinates (27) are subject to the identifications \( u_{i,j} = u_{j+1,i-1} \) and \( u_{k,N} = u_{k,k-1}, \ k \geq 3. \)

\(^{14} \) Note, that the integers \( n_{i,j} \) are subject to the identifications \( n_{i,j} = n_{j+1,i-1} \) and \( n_{k,N} = n_{i,k-1}, \ k \geq 3, \)

i.e. \( n_{1,3} = n_{4,5}, n_{1,5} = n_{2,4} \) and \( n_{3,5} = n_{1,2}. \)

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following from the Mellin transformation of the \( \delta \)-function \( y^{s-1} \, \theta(y) = \int_0^\infty dx \, x^{s-1} \, \delta(x-y) \) for \( y > 0 \) [32]. The extension of the Mellin transformation to a larger framework, in which Dirac delta and other singular functions can be treated, has mainly been established by Kang [32]. For \( N = 5 \) we must consider the quintuple inverse Mellin transformation on [47]:

\[
\frac{1}{(2\pi i)^5} \int_{-i\infty+c}^{+i\infty+c} ds_1 \int_{-i\infty+c}^{+i\infty+c} ds_2 \int_{-i\infty+c}^{+i\infty+c} ds_3 \int_{-i\infty+c}^{+i\infty+c} ds_4 \int_{-i\infty+c}^{+i\infty+c} ds_5 \, u_{1,2}^{-s_1} u_{2,3}^{-s_2} u_{3,4}^{-s_3} u_{4,5}^{-s_4} u_{5,1}^{-s_5}
\]

\[
\times \frac{\Gamma(s_2 + n_{2,3}) \Gamma(s_3 + n_{3,4})}{\Gamma(s_2 + s_3 + n_{2,3} + n_{3,4})} \frac{\Gamma(s_4 + n_{4,5}) \Gamma(s_5 + n_{1,5})}{\Gamma(s_4 + s_5 + n_{4,5} + n_{1,5})} \times F_2 \left[ \begin{array}{l} s_2 + n_{2,3}, s_5 + n_{4,5}, s_3 + s_4 - s_1 + n_{3,4} + n_{4,5} - n_{3,5}, s_2 + s_3 + n_{2,3} + n_{3,4}, s_4 + s_5 + n_{4,5} + n_{1,5} \\ \end{array} ; 1 \right]
\]

\[
= u_{1,2}^{n_{1,2}} u_{2,3}^{n_{2,3}} u_{3,4}^{n_{3,4}} u_{4,5}^{n_{4,5}} u_{5,1}^{n_{5,1}} \delta(u_{4,5} + u_{3,4}u_{1,5} - 1) \delta(u_{1,2} + u_{2,3}u_{1,5} - 1)
\]

\[
\times \delta(u_{3,4} + u_{2,3}u_{4,5} - 1) \theta(1 - u_{1,2}) \theta(1 - u_{2,3}) \theta(1 - u_{3,4}) \theta(1 - u_{4,5}) \theta(1 - u_{1,5}) .
\]

The details of these integrations are displayed in Appendix C.

The two examples \( N = 4 \) and \( N = 5 \) demonstrate, that the inverse Mellin transformation of the string form factors [30] essentially picks up the delta–functions describing the duality constraint equations [35]. Hence, for any \( N \) the (multiple) inverse Mellin transform of [30] yields:

\[
G_N(\{u_{i,j}\}) := (2\pi i)^{-m} \left( \prod_{(i,j) \in P} \int_{-i\infty+c}^{+i\infty+c} ds_{i,j} \, u_{i,j}^{-s_{i,j}} \right) B_N(\{s_{k,l}\}, \{n_{k,l}\})
\]

\[
= \left( \prod_{(i,j) \in P} u_{i,j}^{n_{i,j}} \theta(1 - u_{i,j}) \right) \delta(\{u_{k,l}\}) .
\]

It is interesting to look at the analogue of the \( \alpha' \)-expansion in Mellin position space. E.g. for \( N = 4 \) Eq. [5] becomes:

\[
A_4^S = \frac{1}{\langle 12 \rangle \langle 31 \rangle \langle 34 \rangle \langle 24 \rangle} \int \tilde{\mu}_4^S(z, s) \frac{s_{34}}{z_{34}}
\]

\[
= \frac{1}{\langle 12 \rangle \langle 31 \rangle \langle 34 \rangle \langle 24 \rangle} G(s) \Gamma(u) \Gamma(s + u).
\]

A double inverse Mellin transform on

\[
\frac{s \, \Gamma(s) \, \Gamma(u)}{\Gamma(s + u)} = 1 + \frac{s}{u} - \zeta(2) \left( s^2 + su \right) + O(\alpha'^3)
\]
yields the corresponding relation in Mellin position space:

\[ u_{1,2} \delta'(1 - u_{1,2} - u_{2,3}) \theta(1 - u_{1,2}) \theta(1 - u_{2,3}) \tag{55} \]
\[ = \theta(1 - u_{1,2}) \theta(1 - u_{2,3}) \left\{ \delta(1 - u_{1,2}) \delta(1 - u_{2,3}) + u_{1,2} \delta'(1 - u_{1,2}) \right. \]
\[ - \zeta(2) \left[ u_{1,2}^2 \delta''(1 - u_{1,2}) \delta(1 - u_{2,3}) + u_{1,2} \delta'(1 - u_{1,2}) \delta'(1 - u_{2,3}) \right] \} + \ldots . \]

It is straightforward to derive similar expansions for \( N \geq 5 \). It would be interesting to relate the amplitudes (52) in position space with the results in [33], i.e. relating aspects of motivic multiple zeta values to the space of Dirac delta and other singular functions.

C. Mellin position space and conformal cross–ratios

According to (32) the \( m \) channel variables \( u_{i,j} \) can be identified with the anharmonic ratios, i.e. \( \omega_{i,j} = u_{i,j} = \frac{(z_i - z_j)(z_i - z_{j+1})}{(z_i - z_{j+1})(z_i - z_j)} \). With this choice all duality constraint equations (35) are satisfied reducing the \( m \) anharmonic ratios to \( N - 3 \) independent fundamental cross–ratios. Rewriting (30) in terms of (32) reveals the invariance group \( PSL(2, \mathbb{R}) \). E.g. for \( N = 4 \) the two variables \( u_{1,2} \) and \( u_{2,3} \) are identified as

\[ u_{1,2} = \frac{z_{12} z_{34}}{z_{13} z_{24}} , \quad u_{2,3} = \frac{z_{23} z_{14}}{z_{13} z_{24}} , \tag{56} \]

with:

\[ u_{2,3} = 1 - u_{1,2} . \tag{57} \]

Furthermore, for \( N = 5 \) we have the following relations

\[ u_{1,2} = \frac{z_{12} z_{35}}{z_{13} z_{25}} , \quad u_{2,3} = \frac{z_{23} z_{14}}{z_{13} z_{24}} , \quad u_{3,4} = \frac{z_{34} z_{25}}{z_{24} z_{35}} \]
\[ u_{4,5} = \frac{z_{13} z_{45}}{z_{35} z_{14}} , \quad u_{5,1} = \frac{z_{24} z_{15}}{z_{14} z_{25}} , \tag{58} \]

with:

\[ u_{1,2} = 1 - u_{2,3} u_{1,5} , \quad u_{3,4} = \frac{1 - u_{2,3}}{1 - u_{2,3} u_{1,5}} , \quad u_{4,5} = \frac{1 - u_{1,5}}{1 - u_{2,3} u_{1,5}} . \tag{59} \]

The \( N \) real variables \( z_i \) are associated to each external leg \( i \). It has been known since the early days of superstring theory that the integrals over vertex positions on a disk boundary can be replaced by a different set of variables. From the set \( z_1, \ldots z_N \) one picks three positions, say \( z_a, z_b, z_c \) and and employs \( PSL(2, \mathbb{R}) \) invariance setting them to specific values.
For instance, we can choose $z_1 = -\infty$, $z_2 = 0$, $z_3 = 1$ as in Section 2. With \ref{eq:Bz} and \ref{eq:delta1}, the amplitude \ref{eq:amp} takes the form \ref{eq:finalamp}:

$$B_N(\{s_{k,l}\}, \{n_{k,l}\}) = \int_{-\infty}^{\infty} \prod_{i=1}^{N} \frac{\theta(z_i - z_{i+1})}{d\theta d\theta d\theta} \frac{z_{ab} z_{bc} z_{ca}}{\prod_{i=1}^{N} |z_i - z_{i+2}|} \left( \prod_{i<j}^{N} |z_{ij}| s_{ij} \tilde{n}_{ij} \right), \quad (60)$$

with:

$$\tilde{n}_{ij} = n_{i,j} + n_{i+1,j-1} - n_{i+1,j} - n_{i,j-1}. \quad (61)$$

V. CONCLUDING REMARKS

In this work, we argued that in the case of MHV helicity configurations, $N$–gluon superstring amplitudes are given by Mellin transforms of $N$–graviton supergravity amplitudes, as written in Eq.\ref{eq:MHV}. The most pressing question is whether this result can be extended to other helicity configurations, at least at the semi–classical level, that is promoted to a general relation between all string disk amplitudes and tree–level graviton amplitudes of supergravity. In principle, all necessary ingredients are available for answering this question: superstring amplitudes are written in Refs. \cite{10} and \cite{5, 6} while (very plausible conjectures for) supergravity amplitudes can be found in Refs. \cite{35, 36}. Furthermore, the recursive techniques developed for supergravity in Refs. \cite{12, 13}, may be helpful. Nevertheless, it may take quite a tour de force to make a connection between the two sides.

The description \ref{eq:MHV} of superstring scattering amplitudes as Mellin transforms of supergravity amplitudes or generically the inverse Mellin transform \ref{eq:invmellin} of string form factors \ref{eq:formfactors} into products of delta–functions localizing in Mellin position space might point towards a dual description of perturbative string theory. Basic building blocks of the latter are graphs and delta–functions assembled by rules coming from the Pascal’s triangle in Figure 1. Moving to Mellin position space, which bypasses Koba–Nielsen factors, might allow to directly compute on–shell superstring scattering amplitudes without resorting to the conventional evolution through space–time. From a mathematical point of view the distributional setting of Mellin transformation converting string form factors \ref{eq:formfactors} to Dirac delta–functions \ref{eq:delta1} in Mellin position space provides a new ground for studying superstring amplitudes, \textit{e.g.} for recursion relations, by working directly in position space.
So what if all superstring amplitudes are given by some Mellin transforms of supergravity amplitudes? Can we “trivialize” string theory? The key to the answer lies in Figure 1. In order to compute the transform, supergravity amplitudes are uplifted to the Mellin position space and then integrated over a surface constrained by the Pascal’s triangle of nonlinear equations. Understanding the nature of this embedding should help unraveling a deeper superstring/supergravity correspondence. Superstring theory may well be supergravity in a brilliant disguise.

Acknowledgements

St.St. would like to thank the Institute for Advanced Study at Princeton for its hospitality during initiating this work. This material is based in part upon work supported by the National Science Foundation under Grant No. PHY-0757959. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

Appendix A: From Cayley graphs to Hamiltonian graphs

In this Appendix we discuss polynomial reduction of the set of rational functions in $z_{ij}$, which appears in (5). The latter can be related to $(N - 2)^{(N-4)}$ labelled trees. We shall prove, that partial fraction decomposition reduces this set of rational functions to a basis of $(N - 3)!$ elements, which appears in Eq. (10).

Let us first introduce some common notion in graph theory. A graph $G$ constitutes a set of vertices $V$ and a set of edges $E$, with each edge $e$ being a pair of two different vertices $v_1, v_2$ and no more than one edge between two vertices. A tree graph is a connected graph without cycles. The graph $P_n$ is simply a path on $n$ vertices. A spanning tree of a connected graph is a tree comprising all vertices. In a complete graph every two of its vertices are adjacent. The degree $deg(V)$ of the vertex $V$ is the number of edges attached to it. A rooted tree is a tree with one vertex designed as a root. Finally, a Hamiltonian path is a path in an undirected graph, that visits each vertex exactly once.

The tree diagrams of interest are Cayley graphs $C_n$. The latter describe labelled trees on $n$ vertices. According to Cayley there are $n^{n-2}$ of them \[37\]. For $n = 3$ we have the three
diagrams with vertices $i, j$ and $k$, depicted in Figure 2:

![Cayley graphs for $n = 3$](image)

FIG. 2. Cayley graphs for $n = 3$.

On the other hand, for $n = 4$ we have the following sixteen diagrams with vertices $i, j, k$ and $l$:

![Cayley graphs for $n = 4$](image)

FIG. 3. Cayley graphs for $n = 4$.

As advocated in Section 2 we assign to each product of rational functions a tree graph, e.g. the following rational function are associated with the corresponding diagrams

\[
\frac{1}{z_{ij}z_{jk}z_{kl}} = \frac{1}{z_{ij}z_{jk}z_{jl}} = \frac{1}{z_{ij}z_{jk}z_{jl}},
\]

respectively. Partial fraction decomposition on rational functions (A1) gives rise to relations between trees. In Section 2 we have attributed to each Cayley graph $C_n$ the corresponding rational function $C_{\sigma}^{N}$ in the coordinates $z_{i}$, with $n = N - 2$. Many of them can be related by
partial fractioning. In this Appendix we want to find those functions, which can no longer be related subject to partial fraction relations, \( i.e. \) they form a basis. As we shall see their corresponding trees represent a special subset of the full set of labelled trees \( C_n \).

In fact, partial fractioning allows to reduce any tree with vertices comprising several branchings to a tree diagram with vertices to which at most two edges are attached, \( i.e. \) trees with vertices \( v \) of degree \( \text{deg}(v) \geq 3 \) can be always be brought to a sum of trees with vertices \( v \) of degree \( \text{deg}(v) \leq 2 \). \( E.g. \) for a rational function corresponding to a tree with one vertex \( j \) of degree \( \text{deg}(j) = 3 \) we have the following decomposition

\[
\frac{1}{z_{ij} z_{jk} z_{jl}} = \left\{ \frac{1}{z_{ij}} \left( \frac{1}{z_{jl} z_{lk}} + \frac{1}{z_{jk} z_{kl}} \right) ,
\right. \\
\frac{1}{z_{ij} z_{ij}} + \frac{1}{z_{ij} z_{ij}} \frac{1}{z_{jk}} \right. ,
\]

which graphically can be depicted as shown in Figure 4:

\[
\text{FIG. 4. Partional fractioning converts a vertex } j \text{ of } \text{deg}(j) = 3 \text{ to a vertex of } \text{deg}(j) = 2.
\]

Hence, we may convert the vertex \( j \) of degree \( \text{deg}(j) = 3 \) to a vertex of degree \( \text{deg}(j) = 2 \) by moving the vertex \( l \) either to the right hand side of \( j \) or to its left hand side. More generally, for a tree diagram with \( n \) vertices \( i_l \), \( l = 1, \ldots, n \) and one vertex \( i_2 \) having \( \text{deg}(i_2) = 3 \) we obtain \( n-2 \) tree diagrams with \( \text{deg}(i_2) = 2 \) and with the vertex \( i_n \) moved to the right hand side of \( i_2 \). Alternatively, we obtain two trees by moving the vertex \( i_n \) to the left hand side of the vertex \( i_2 \):

\[
\sum_{l=2}^{n-1} \left\{ \frac{1}{z_{i_l} z_{i_l}} \right. \\
\left. \frac{1}{z_{i_l} z_{i_l}} \frac{1}{z_{i_l} z_{i_l}} \right. ,
\]

In terms of partional fraction relations the tree diagrams \( A3 \) describe the two decomposi-
respectively. Three comments need to be made in the following. The above reasoning has been established for the vertex $i_n$ having degree $\text{deg}(i_n) = 1$, i.e. no further edges are attached to it. If the degree of the vertex $i_n$ was $\text{deg}(i_n) = d > 1$, in most of the diagrams on the right hand side of (A3) the degree of this vertex would become $\text{deg}(i_n) = d + 1 > 2$. However, in this case by repeated use of (A3) the multiple branching at $i_n$ can eventually be removed ending up at diagrams with $\text{deg}(i_n) \leq 2$ on the right hand side of (A3). Furthermore, if for the vertex $i_2$ the degree was $\text{deg}(i_2) = d > 3$ by applying (A3) once we would get $\text{deg}(i_2) = d - 1$ in the diagrams on the right hand side. Again, repeated use of (A3) eventually provides diagrams for which $\text{deg}(i_2) = 2$. Finally, if there were branching not only at the vertex $i_2$ but also at other vertices, i.e. $\text{deg}(i_l) > 2$, $i_l$, $l = 3, \ldots, n - 2$, the above reasoning applies as well.

To conclude, by repeated application of (A3) any rational function associated to one of the $n^{n-2}$ Cayley trees $C_n$ can be reduced to a sum of rational functions corresponding to tree diagrams $P_n$ with vertices of degree at most two. There are $\frac{1}{2}n!$ of the latter. Hence, in the following we only consider those $\frac{1}{2}(N - 2)!$ rational functions, which are associated to tree diagrams $P_n$. From those diagrams we can single out rooted trees with the vertex $a$ designated as a root, i.e. $\text{deg}(a) = 1$. Let us consider the following rational function

$$
\frac{1}{z_{ij} z_{ja} z_{ak}} = \frac{1}{z_{ik} z_{kj} z_{ja}} + \frac{1}{z_{ki} z_{ij} z_{ja}} = \frac{1}{z_{ij} z_{jk} z_{ak}} + \frac{1}{z_{ik} z_{kj} z_{ja}} = \frac{1}{z_{ij} z_{jk} z_{ak}} - \frac{1}{z_{ik} z_{kj} z_{ja}} = \frac{1}{z_{ik} z_{jk} z_{ja}} + \frac{1}{z_{ki} z_{ij} z_{ja}}.
$$

(Eqs. (A5)–(A7) demonstrate, that the rational function corresponding to the diagram (A5) can always be written as a sum of rational functions (A7) referring to rooted trees with
vertex \( a \) as their root. Hence, in the tree diagram (A5) the vertex \( a \) can always be moved to the boundary of the tree diagram. The above reasoning has been established for a tree \( P_n \) with \( n = 4 \). However, the same arguments apply for any tree diagram \( P_n \) with vertex \( a \) not at the boundary of the tree: if in (A5) there was a path on some vertices attached to the vertex \( k \) by (A3) in (A7) those vertices can be moved to the left hand side of the three diagrams. The same argument can be used in the case of a path on some vertices attached to vertex \( i \). Note, that during the step from (A6) to (A7) in the second diagram of (A6) according to (A3) we only could move the vertex \( k \) to the left in order to leave the vertex \( a \) at the boundary. This is way we cannot repeat the steps (A5)–(A7) to single out a second vertex and move it to the boundary.

To summarize, in the tree diagrams \( P_n \) we can single out one vertex \( a \) allowing to focus on rooted tree graphs with \( a \) designated as a root connected by a path on the remaining \( n - 1 \) vertices. The latter can be permuted, hence in total there are \((n - 1)!\) of them, shown in Figure 5. Their corresponding rational functions comprise a minimal basis subject to partial fraction decomposition. Hence, all rational functions described by Cayley graphs can be written in terms of a \((n - 1)!\)-dimensional basis corresponding to rooted trees of \( P_n \).

\[ \left(z_{12\sigma} z_{2\sigma 3\sigma} \cdots z_{(n-1)\sigma n\sigma}\right)^{-1} \]

FIG. 5. Hamilton graphs corresponding to a minimal basis subject to partial fractionning.

**Appendix B: Multiple Mellin transforms for superstrings**

In the superstring amplitude (5) the sum over \((N - 2)^{(N-4)}\) Cayley graphs amounts to considering for each graph a certain integral over the \( N \) world-sheet coordinates \( z_i \)

\[ \int d\mu_N^{s}(z, s) \ R(\{z_{kl}\}) , \quad R(\{z_{kl}\}) = \prod_{\text{edges}} \frac{s_{ij}}{z_{ij}} , \]  

(B1)

whose rational function \( R(\{z_{kl}\}) \) is graphically described by edges and vertices labelled by 3, 4, \ldots, \( N \), cf. Appendix A. To derive (43) we wrote the integral (B1) as Mellin trans-
form (37)
\[ \int dM_N(u, s) \ M(\{u_{p,q}\}) = \prod_{\text{edges}} \sigma_{3j} \partial \ln \delta(\alpha_{ij}) \] (B2)
of a function $M(\{u_{p,q}\})$ in Mellin position space $u_{i,j}$, which in turn according to (52) gives
rise to the correlation function
\[ G(\{u_{p,q}\}) = \left( \prod_{(i,j) \in P} \theta(1 - u_{i,j}) \right) \left( \prod_{l=4}^{N} \sigma_{3-1}^{-1} \rho_{l-1, l} \right) \delta(\{u_{p,q}\}) M(\{u_{p,q}\}) \] (B3)
in inverse Mellin space. Alternatively, according to (45) the form factors (B1) can be con-
sidered as Mellin amplitudes (30), whose inverse Mellin transform (52) must agree with
(B3).

For a given form factor $B_N(\{s_{k,l}\}, \{n_{k,l}\})$ a momentum insertions $s_{p,q}$ (with $(p, q) \in P$)
can easily be incorporated in its inverse Mellin transform (52) by acting on the latter through
the respective derivatives in position space $u_{p,q}$ as:
\[ (2\pi i)^{-m} \left( \prod_{(i,j) \in P} \int_{-i\infty+c}^{-i\infty+c} ds_{i,j} u_{i,j}^{-s_{i,j}} \right) s_{p,q} B_N(\{s_{k,l}\}, \{n_{k,l}\}) \]
\[ = -(2\pi i)^{-m} u_{p,q} \left\{ \frac{\partial}{\partial u_{p,q}} \left( \prod_{(i,j) \in P} \int_{-i\infty+c}^{-i\infty+c} ds_{i,j} u_{i,j}^{-s_{i,j}} \right) B_N(\{s_{k,l}\}, \{n_{k,l}\}) \right\} \]
\[ = -u_{p,q} \frac{\partial}{\partial u_{p,q}} \left\{ \left( \prod_{(i,j) \in P} u_{i,j}^{n_{i,j}} \theta(1 - u_{i,j}) \right) \delta(\{u_{k,l}\}) \right\} . \] (B4)
This way partial integration relations between different form factors become partial integra-
tion relations in Mellin position space.

1. Mellin transforms for $N = 5$

Let us consider the Cayley basis for $N = 5$. In the following we use: $x_1 := u_{1,2}$, $x_2 := u_{2,3}$, $x_3 := u_{3,4}$, $x_4 := u_{4,5}$, $x_5 := u_{5,6}$. The Mellin measure (37) is:
\[ \int dM_5(u, s) = \left( \prod_{i=1}^{5} \int_{0}^{\infty} x_i^{s_i+1} \theta(1 - x_i) \right) \frac{x_1}{x_2 x_3 x_5} \delta(\{x_j\}) , \] (B5)
with the corresponding product of delta–functions (36):
\[ \delta(\{x_j\}) = x_1 \delta(\alpha_{34}) \delta(\alpha_{45}) \delta(\alpha_{35}) \]
\[ = x_1 \delta(1 - x_2 - x_1 x_3) \delta(1 - x_5 - x_1 x_4) \delta(1 - x_1 - x_2 x_5) . \] (B6)

In the following Table I for each Cayley tree–graph and its corresponding integral (B1), we display its Mellin representation (B2) and the underlying correlation function (B3) in inverse Mellin space.

| tree–graph | tree–graph | \( R \) \( \prod_{\text{edges}} \sigma_{3j} \) | \( M \) | inverse Mellin space \( G \) |
|------------|------------|-----------------|---------|-----------------|
| 3 4 5 1 \( \sigma_{34} \sigma_{35} \) | \( \frac{s_{34} s_{45}}{z_{34} z_{45}} \) | \( x_2^2 x_5 \frac{\delta'(\alpha_{34}) \delta'(\alpha_{45})}{\delta(\alpha_{34}) \delta(\alpha_{45})} \) | \( x_1^3 x_3 x_4 \delta'(\alpha_{34}) \delta(\alpha_{45}) \) |
| 4 5 3 1 \( \sigma_{35}^2 \) | \( \frac{s_{45} s_{35}}{z_{45} z_{35}} \) | \( x_2^2 x_5 \frac{\delta'(\alpha_{45}) \delta'(\alpha_{35})}{\delta(\alpha_{45}) \delta(\alpha_{35})} \) | \( x_1^3 x_3 x_4 x_5 \delta'(\alpha_{45}) \delta'(\alpha_{35}) \) |
| 5 3 4 1 \( \sigma_{34} \sigma_{35} \) | \( \frac{s_{34} s_{35}}{z_{34} z_{35}} \) | \( x_2^2 x_5 \frac{\delta'(\alpha_{34}) \delta'(\alpha_{35})}{\delta(\alpha_{34}) \delta(\alpha_{35})} \) | \( x_1^3 x_3 x_4 \delta'(\alpha_{34}) \delta(\alpha_{35}) \) |

TABLE I. Cayley basis and inverse Mellin transform for \( N = 5 \).

Alternatively, considering the integrals (B1) as Mellin amplitudes (B3) gives the following dictionary, depicted in Table II.

| formfactor | \( z \)-space: | \( u \)-space: | inverse Mellin space: |
|------------|-----------------|-----------------|-----------------|
| \( \frac{1}{z_{34} z_{45}} \) \( \prod_{i<j} \frac{\delta(1-x_i)}{1-x_i} \) \( \prod_{(i,j) \in P} u_{i,j}^{n_{i,j}-1} \) in Eq. (50) | \( \prod_{(i,j) \in P} u_{i,j}^{n_{i,j}} \) in Eq. (52) |
| \( \frac{1}{z_{45} z_{53}} \) | \( \frac{1}{z_{13} z_{23} z_{34} z_{45} z_{51}} \) | \( x_1^{-1} x_2^{-1} x_3^{-1} x_4^{-1} x_5^{-1} \) | \( 1 \) |
| \( \frac{1}{z_{34} z_{53}} \) | \( \frac{1}{z_{12} z_{23} z_{34} z_{45} z_{51}} \) | \( x_1^{-1} x_2^{-1} x_3^{-1} x_4^{-1} \) | \( x_3 x_5 \) |
| \( \frac{1}{z_{34} z_{53}} \) | \( \frac{1}{z_{13} z_{23} z_{34} z_{45} z_{51}} \) | \( x_1^{-1} x_2^{-1} x_3^{-1} x_5^{-1} \) | \( x_4 \) |

TABLE II. Cayley basis and corresponding Mellin amplitudes for \( N = 5 \).

With the information displayed in Table II we can compute the inverse Mellin transform (B4) of the formfactors supplemented by the momentum insertions. Following the rule (B4) we obtain from the last column of Table II

\[
\frac{s_{34} s_{45}}{z_{34} z_{45}} \simeq x_3 \frac{\partial}{\partial x_3} x_4 \frac{\partial}{\partial x_4} \left\{ \left( \prod_{i=1}^5 \theta(1-x_i) \right) \delta(\{j\}) \right\}
\]

\[ = x_1^3 x_3 x_4 \delta'(1-x_2-x_1 x_3) \delta'(1-x_5-x_1 x_4) \delta(1-x_1-x_2 x_5) , \]
\[
\frac{s_{35}s_{45}}{z_{35}z_{45}} \simeq x_4 \frac{\partial}{\partial x_4} \left( x_1 \frac{\partial}{\partial x_1} - x_3 \frac{\partial}{\partial x_3} - x_4 \frac{\partial}{\partial x_4} \right) \left\{ \left( \prod_{i=1}^{5} \theta(1-x_i) \right) x_3 x_5 \delta(\{x_j\}) \right\} \\
= x_1^3 x_3 x_4 x_5 \delta(1-x_2-x_1 x_3) \delta'(1-x_5-x_1 x_4) \delta'(1-x_1-x_2 x_5),
\]

\[
\frac{s_{34}s_{35}}{z_{34}z_{35}} \simeq x_3 \frac{\partial}{\partial x_3} \left( x_1 \frac{\partial}{\partial x_1} - x_3 \frac{\partial}{\partial x_3} - x_4 \frac{\partial}{\partial x_4} \right) \left\{ \left( \prod_{i=1}^{5} \theta(1-x_i) \right) x_4 \delta(\{x_j\}) \right\} \\
= x_1^3 x_3 x_4 \delta'(1-x_2-x_1 x_3) \delta(1-x_5-x_1 x_4) \delta'(1-x_1-x_2 x_5),
\]
in agreement with the last column of Table I.

2. Mellin transforms for \( N = 6 \)

Let us consider the \( N = 6 \) Cayley basis and define: \( x_1 := u_{1,2}, \ x_2 := u_{2,3}, \ x_3 = u_{3,4}, \ x_4 = u_{4,5}, \ x_5 = u_{5,6}, \ x_6 = u_{1,6} \) and \( y_1 := u_{1,3}, \ y_2 := u_{2,4}, \ y_3 := u_{3,5}. \) The Mellin measure (37) is:

\[
\int dM_6(u, s) = \left( \prod_{i=1}^{6} \int_0^\infty x_i^{s_{i+1}} \theta(1-x_i) \right) \left( \prod_{i=1}^{3} \int_0^\infty y_i^{s_{i+2}} \theta(1-y_i) \right) \\
\times \frac{x_1^2 y_1 y_3}{x_2^4 x_6^2 y_2} \delta(\{x_i\}, \{y_j\}), \tag{B7}
\]

with the corresponding product of delta–functions (36)

\[
\delta(\{x_i\}, \{y_j\}) = x_1^3 y_1 y_3 \delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6, \tag{B8}
\]

with:

\[
\delta_1 := \delta(\alpha_{35}) = \delta(x_1 y_3 + x_2 y_2 - 1), \quad \delta_2 := \delta(\alpha_{46}) = \delta(x_1 y_1 + x_6 y_2 - 1),
\]
\[
\delta_3 := \delta(\alpha_{36}) = \delta(x_2 x_6 y_2 + x_1 - 1), \quad \delta_4 := \delta(\alpha_{56}) = \delta(x_1 x_5 y_1 + x_6 - 1),
\]
\[
\delta_5 := \delta(\alpha_{34}) = \delta(x_1 x_3 y_3 + x_2 - 1), \quad \delta_6 := \delta(\alpha_{45}) = \delta(x_1 x_4 y_1 y_3 + y_2 - 1). \tag{B9}
\]

In the following Table III for each Hamilton tree–graph and its corresponding integral (B1), we display its Mellin representation (B2) and the underlying correlation function (B3) in inverse Mellin space.
Alternatively, considering the integrals (B.1) as Mellin amplitudes (30) gives the following dictionary, depicted in Table IV.

| formfactor | $z$–space: $\prod_{i<j}^6 \frac{1}{|z_i-z_{i+2}|} \prod_{i<j} z_{i,j}^{-\hat{r}_{i,j}}$ in Eq. (60) | $u$–space: $\prod_{(i,j)\in P} u_{i,j}^{n_{i,j}-1}$ in Eq. (50) | inverse Mellin space: $\prod_{(i,j)\in P} u_{i,j}^{n_{i,j}}$ in Eq. (52) |
|-----------|-------------------------------------------------|-------------------------------------------------|-------------------------------------------------|
| $1$ in $\frac{34}{35} \frac{45}{56}$ | $z_{12} z_{23} z_{34} z_{45} z_{56} z_{16}$ | $x_1 x_2 x_3 x_4 x_5 x_6 y_1 y_2 y_3$ | $1$ |
| $1$ in $\frac{34}{36} \frac{45}{56}$ | $z_{12} z_{23} z_{34} z_{45} z_{56} z_{16}$ | $x_1 x_2 x_3 x_4 x_5 x_6 y_1 y_2 y_3$ | $x_4 x_6 y_3$ |
| $1$ in $\frac{34}{36} \frac{45}{56}$ | $z_{12} z_{23} z_{34} z_{45} z_{56} z_{16}$ | $x_1 x_2 x_3 x_4 x_5 x_6 y_1 y_2 y_3$ | $x_3 x_6 y_2 y_3$ |
| $1$ in $\frac{35}{46} \frac{45}{56}$ | $z_{12} z_{23} z_{34} z_{45} z_{56} z_{16}$ | $x_1 x_2 x_3 x_4 x_5 x_6 y_1 y_2 y_3$ | $x_3 x_5 y_2$ |
| $1$ in $\frac{35}{46} \frac{45}{56}$ | $z_{12} z_{23} z_{34} z_{45} z_{56} z_{16}$ | $x_1 x_2 x_3 x_4 x_5 x_6 y_1 y_2 y_3$ | $x_3 x_5 x_6 y_2 y_3$ |
| $1$ in $\frac{35}{46} \frac{45}{56}$ | $z_{12} z_{23} z_{34} z_{45} z_{56} z_{16}$ | $x_1 x_2 x_3 x_4 x_5 x_6 y_1 y_2 y_3$ | $x_3 x_4 x_6 y_2 y_3$ |

TABLE III. Hamilton basis and inverse Mellin transform for $N = 6$.

TABLE IV. Hamilton basis and corresponding Mellin amplitudes for $N = 6$.

With the information displayed in Table IV we can compute the inverse Mellin transform (52) of the formfactors supplemented by the momentum insertions. Following the rule (B.1)
we obtain from the last column of Table IV

\[
\frac{s_{34} s_{45} s_{56}}{z_{34} z_{45} z_{56}} \simeq x_3 x_4 x_5 \frac{\partial}{\partial x_3} \frac{\partial}{\partial x_4} \frac{\partial}{\partial x_5} \left\{ \left( \prod_{i=1}^{6} \theta(1 - x_i) \right) \left( \prod_{i=1}^{3} \theta(1 - y_i) \right) \delta(\{x_i\}) \delta(\{y_j\}) \right\}
\]

\[= x_1^6 x_3 x_4 x_5 y_3^3 \delta_1 \delta_2 \delta_3 \delta_4' \delta_5 \delta_6',
\]

\[
\frac{s_{34} s_{46} s_{56}}{z_{34} z_{46} z_{56}} \simeq x_4 x_5 \frac{\partial}{\partial x_4} \frac{\partial}{\partial x_5} \left( y_1 \frac{\partial}{\partial y_1} - x_4 \frac{\partial}{\partial x_4} - x_5 \frac{\partial}{\partial x_5} \right)
\]

\[\left\{ \left( \prod_{i=1}^{6} \theta(1 - x_i) \right) \left( \prod_{i=1}^{3} \theta(1 - y_i) \right) x_4 x_5 y_3 \delta(\{x_i\}) \delta(\{y_j\}) \right\}
\]

\[= x_1^6 x_3 x_4 x_5 y_3^3 \delta_2 \delta_3 \delta_4' \delta_5 \delta_6',
\]

\[
\frac{s_{36} s_{45} s_{56}}{z_{36} z_{45} z_{56}} \simeq x_4 \frac{\partial}{\partial x_4} \left( y_4 \frac{\partial}{\partial y_3} - x_3 \frac{\partial}{\partial x_3} - x_4 \frac{\partial}{\partial x_4} \right)
\]

\[\left\{ \left( \prod_{i=1}^{6} \theta(1 - x_i) \right) \left( \prod_{i=1}^{3} \theta(1 - y_i) \right) x_3 x_5 y_2 \delta(\{x_i\}) \delta(\{y_j\}) \right\}
\]

\[= x_1^6 x_3 x_4 x_5 y_3^3 \delta_2' \delta_3 \delta_4 \delta_5 \delta_6',
\]

\[
\frac{s_{36} s_{46} s_{56}}{z_{36} z_{46} z_{56}} \simeq x_4 \frac{\partial}{\partial x_4} \left( y_1 \frac{\partial}{\partial y_1} - x_4 \frac{\partial}{\partial x_4} - x_5 \frac{\partial}{\partial x_5} \right)
\]

\[\left\{ \left( \prod_{i=1}^{6} \theta(1 - x_i) \right) \left( \prod_{i=1}^{3} \theta(1 - y_i) \right) x_3 x_5 y_2 \delta(\{x_i\}) \delta(\{y_j\}) \right\}
\]

\[= x_1^6 x_3 x_4 x_5 y_3^3 \delta_2' \delta_3 \delta_4 \delta_5 \delta_6',
\]

\[
\frac{s_{35} s_{46} s_{56}}{z_{35} z_{46} z_{56}} \simeq x_5 \frac{\partial}{\partial x_5} \left( y_2 \frac{\partial}{\partial y_3} - x_3 \frac{\partial}{\partial x_3} - x_4 \frac{\partial}{\partial x_4} \right)
\]

\[\left\{ \left( \prod_{i=1}^{6} \theta(1 - x_i) \right) \left( \prod_{i=1}^{3} \theta(1 - y_i) \right) x_3 x_5 y_2 \delta(\{x_i\}) \delta(\{y_j\}) \right\}
\]

\[= x_1^6 x_3 x_4 x_5 y_3^3 \delta_2' \delta_3 \delta_4 \delta_5 \delta_6',
\]

30
in agreement with the last column of Table III.

Appendix C: Inverse Mellin–Barnes transformation for $N = 5$

In this Appendix we prove the equality (51):

\[
\frac{1}{(2\pi i)^5} \int_{-i\infty+c}^{+i\infty+c} ds_1 \int_{-i\infty+c}^{+i\infty+c} ds_2 \int_{-i\infty+c}^{+i\infty+c} ds_3 \int_{-i\infty+c}^{+i\infty+c} ds_4 \int_{-i\infty+c}^{+i\infty+c} ds_5 \; u_{1,2}^{-s_1} u_{2,3}^{-s_2} u_{3,4}^{-s_3} u_{4,5}^{-s_4} u_{1,5}^{-s_5} \times \frac{\Gamma(s_2 + n_{23}) \Gamma(s_3 + n_{34}) \Gamma(s_4 + n_{45}) \Gamma(s_5 + n_{15})}{\Gamma(s_2 + s_3 + n_{23} + n_{34}) \Gamma(s_4 + s_5 + n_{15} + n_{45})} \times \sum_{c=0}^{3} F_2 \left[ \begin{array}{c} s_2 + n_{23}, s_5 + n_{15}, s_3 + s_4 - s_1 + n_{34} + n_{45} - n_{12} \\ s_2 + s_3 + n_{23} + n_{34}, s_4 + s_5 + n_{15} + n_{45} \end{array} ; 1 \right] \delta(u_{4,5} + u_{3,4} u_{1,5} - 1) \delta(u_{1,2} + u_{2,3} u_{1,5} - 1) \delta(u_{3,4} + u_{2,3} u_{4,5} - 1) \delta(1 - u_{1,2}) \delta(1 - u_{1,2}) \delta(1 - u_{3,4}) \delta(1 - u_{4,5}) \delta(1 - u_{1,5}) \cdot
\]

After writing the hypergeometric function $3F_2$ as power series:

\[
\frac{\Gamma(s_2 + n_{23}) \Gamma(s_3 + n_{34}) \Gamma(s_4 + n_{45}) \Gamma(s_5 + n_{15})}{\Gamma(s_2 + s_3 + n_{23} + n_{34}) \Gamma(s_4 + s_5 + n_{15} + n_{45})} \times \sum_{c=0}^{3} F_2 \left[ \begin{array}{c} s_2 + n_{23}, s_5 + n_{15}, s_3 + s_4 - s_1 + n_{34} + n_{45} - n_{12} \\ s_2 + s_3 + n_{23} + n_{34}, s_4 + s_5 + n_{15} + n_{45} \end{array} ; 1 \right] \frac{1}{n!} \frac{\Gamma(s_2 + n_{23} + n)}{\Gamma(s_2 + s_3 + n_{23} + n_{34} + n)} \times \frac{\Gamma(s_3 + n_{34}) \Gamma(s_4 + n_{45})}{\Gamma(s_4 + s_5 + n_{15} + n)} \frac{1}{\Gamma(s_3 + s_4 - s_1 + n_{34} + n_{45} - n_{12} + n)} \frac{1}{\Gamma(s_3 + s_4 - s_1 + n_{34} + n_{45} - n_{12} + n)} ,
\]

in (C1) we perform each of the five integrations, separately. The integrations over $s_5, s_2$ and $s_1$ give

\[
\frac{1}{2\pi i} \int_{-i\infty+c}^{+i\infty+c} ds_5 \; u_{1,5}^{-s_5} \frac{\Gamma(s_5 + n_{15} + n)}{\Gamma(s_4 + s_5 + n_{45} + n_{15} + n)} = u_{1,5}^{n_{15} + n} \Gamma(s_4 + n_{45})^{-1} (1 - u_{1,5})^{s_4 + n_{45} - 1} ,
\]

\[
0 < u_{1,5} \leq 1 ,
\]

\[
31
\]
\[
\frac{1}{2\pi i} \int_{-i\infty + c}^{+i\infty + c} ds_2 \ u_{2,3}^{-s_2} \ \frac{\Gamma(s_2 + n_{23} + n)}{\Gamma(s_2 + s_3 + n_{23} + n_{34} + n)}
= u_{2,3}^{n_{23} + n} \ \Gamma(s_3 + n_{34} - 1) \ (1 - u_{2,3})^{s_3 + n_{34} - 1}, \quad 0 < u_{2,3} \leq 1,
\]

\[
\frac{1}{2\pi i} \int_{-i\infty + c}^{+i\infty + c} ds_1 \ u_{1,2}^{-s_1} \ \frac{\Gamma(\alpha - s_1 + n)}{\Gamma(\alpha - s_1)} = u_{1,2}^{1-\alpha} \ \delta^{(n)}(u_{1,2} - 1),
\]

\[\alpha = s_3 + s_4 + n_{34} + n_{45} - n_{12}, \quad 0 < u_{1,2} \leq 1,\]

respectively. For the first two integrals we have applied the relation (49), while for the last integral we have used

\[
\frac{1}{2\pi i} \int_{-i\infty + c}^{+i\infty + c} ds \ x^{-s} \ a^{s-1-n} \ \frac{\Gamma(s)}{\Gamma(s - n)} = (-1)^n \ \delta^{(n)}(x - a), \quad a > 0,
\]

which follows from the Mellin transformation \((n \geq 0)\):

\[
\int_0^\infty \! dx \ x^{s-1} \ \delta^{(n)}(x - a) = \begin{cases} 
(-1)^n \ \frac{\Gamma(s)}{\Gamma(s - n)} \ a^{s-1-n}, \quad a > 0, \\
0, \quad a \leq 0.
\end{cases}
\]

The above relation (C3) can be proven by applying the fundamental equation, which defines derivatives of the delta–function \(\delta\)

\[
\int f(x) \ \delta^{(n)}(x - a) \ dx = - \int \frac{\partial f}{\partial x} \ \delta^{(n-1)}(x - a) \ dx \equiv (-1)^n \ f^{(n)}(a)
\]

for any function \(f\) which has continuous derivatives at least up to the \(n\)-th order in some neighbourhood of the point \(x = a\) [38].

After collecting all \(s_3\)-- and \(s_4\)--dependent terms we are left with the following two integrations

\[
\frac{1}{2\pi i} \int_{-i\infty + c}^{+i\infty + c} ds_3 \ (u_{1,2} u_{3,4})^{-s_3} \ (1 - u_{2,3})^{s_3-1} = \delta(1 - u_{2,3} - u_{1,2} u_{3,4}), \quad u_{1,2} u_{3,4} > 0,
\]

\[
\frac{1}{2\pi i} \int_{-i\infty + c}^{+i\infty + c} ds_4 \ (u_{1,2} u_{4,5})^{-s_4} \ (1 - u_{1,5})^{s_4-1} = \delta(1 - u_{1,5} - u_{1,2} u_{4,5}), \quad u_{1,2} u_{4,5} > 0,
\]

respectively. The latter are evaluated by using (50).

Putting together all \(n\)--independent terms gives:

\[
u_{2,3}^{n_{23}} \ u_{1,5}^{n_{15}} \ (1 - u_{2,3})^{n_{34}} \ (1 - u_{1,5})^{n_{45}} \ u_{1,2}^{1+n_{12}-n_{34}-n_{45}} \ \delta(1 - u_{1,5} - u_{1,2} u_{4,5}) \ \delta(1 - u_{2,3} - u_{1,2} u_{3,4})
= u_{1,2}^{1+n_{12}} \ u_{2,3}^{n_{23}} \ u_{3,4}^{n_{34}} \ u_{4,5}^{n_{45}} \ u_{1,5}^{n_{15}} \ \delta(1 - u_{1,5} - u_{1,2} u_{4,5}) \ \delta(1 - u_{2,3} - u_{1,2} u_{3,4}).
\]

(C4)
The remaining \( n \)–dependent terms conspire into the sum:

\[
\sum_{n=0}^{\infty} \frac{1}{n!} \delta^{(n)}(u_{1,2} - 1) \ (u_{2,3} \ u_{1,5})^n = \delta(u_{1,2} - 1 + u_{2,3}u_{1,5}) .
\] (C5)

For the above sum we have used the relation

\[
\sum_{n=0}^{\infty} \frac{x^n}{n!} \delta^{(n)}(y) = \delta(x + y) ,
\] (C6)

which can be derived by first writing \( \delta^{(n)}(y) = \frac{\delta^n}{\delta y^n} \int_{-\infty}^{+\infty} dk \ e^{2\pi iky} = \int_{-\infty}^{+\infty} dk \ (2\pi ik)^n \ e^{2\pi iky} \), then evaluate the sum \( \sum_{n=0}^{\infty} \frac{(2\pi ikx)^n}{n!} = e^{2\pi i k x} \) and eventually perform the integration \( \int_{-\infty}^{+\infty} dk \ e^{2\pi i ky} e^{2\pi i k x} = \delta(x + y) \).

Finally, putting (C4) and (C5) together gives the final result (C1):

\[
u_{1,2}^{1+n_{12}} \ u_{2,3}^{n_{23}} \ u_{3,4}^{n_{34}} \ u_{4,5}^{n_{45}} \ u_{1,5}^{n_{15}}
\times \ \delta(1-u_{1.5}-u_{1,2}u_{4,5}) \ \delta(1-u_{2,3}-u_{1,2}u_{3,4}) \ \delta(u_{1,2} - 1 + u_{2,3}u_{1,5})
\]

\[
= u_{1,2}^{n_{12}} \ u_{2,3}^{n_{23}} \ u_{3,4}^{n_{34}} \ u_{4,5}^{n_{45}} \ u_{1,5}^{n_{15}}
\times \ \delta(u_{1,2} + u_{2,3}u_{1,5} - 1) \ \delta(u_{3,4} + u_{2,3}u_{4,5} - 1) \ \delta(u_{4,5} + u_{3,4}u_{1,5} - 1) ,
\] (C7)

with the constraints \( 0 < u_{1,2}, u_{2,3}, u_{3,4}, u_{4,5}, u_{1,5} \leq 1 \). In (C7) the last equality follows from the following \( \delta \)–function identity:

\[
\delta(\{u_{p,q}\}) = u_{1,2}^{-1} \ \delta\left(u_{4,5} - \frac{1-u_{1,5}}{1-u_{2,3}u_{1,5}}\right) \ \delta\left(u_{3,4} - \frac{1-u_{2,3}}{1-u_{2,3}u_{1,5}}\right) \ \delta(u_{1,2} - 1 + u_{2,3}u_{1,5})
\]

\[
= u_{1,2} \ \delta(1-u_{1,5}-u_{1,2}u_{4,5}) \ \delta(1-u_{2,3}-u_{1,2}u_{3,4}) \ \delta(1-u_{1,2} - u_{2,3}u_{1,5})
\]

\[
= \delta(u_{1,2} + u_{2,3}u_{1,5} - 1) \ \delta(u_{3,4} + u_{2,3}u_{4,5} - 1) \ \delta(u_{4,5} + u_{3,4}u_{1,5} - 1) .
\] (C8)

Note, that the last line corresponds to (31) for \( N = 5 \), while the second last originates from (36).
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