DERIVED EQUIVALENCE OF SURFACE ALGEBRAS IN GENUS 0 VIA GRADED EQUIVALENCE

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ABSTRACT. We determine some of the derived equivalences of a class of gentle algebras called surface algebras. These algebras are constructed from an unpunctured Riemann surface of genus 0 with boundary and marked points by introducing cuts in internal triangles of an arbitrary triangulation of the surface. In particular, we fix a triangulation of a surface and determine when different cuts produce derived equivalent algebras.

1. INTRODUCTION

Let $T$ be a triangulation of a bordered unpunctured Riemann surface $S$ with a set of marked points $M$, and let $(Q_T, I_T)$ be the bound quiver associated to $T$ as in [4, 11]. The corresponding algebra $\Lambda_T = kQ_T/I_T$, over an algebraically closed field $k$, is a finite-dimensional gentle algebra [4] which is also the endomorphism algebra of the cluster-tilting object corresponding to $T$ in the generalized cluster category associated to $(S, M)$, see [1, 9, 10, 11]. Each internal triangle in the triangulation $T$ corresponds to an oriented 3-cycle in the quiver $Q_T$, and the relations for the algebra $B_T$ state precisely that the composition of any two arrows in an oriented 3-cycle is zero in $\Lambda_T$.

In [12], surface algebras were introduced as a new setting to describe the iterated tilted algebras of Dynkin type $A$ and $\tilde{A}$, corresponding to the case where $S$ is a disc and annulus respectively. In addition to the iterated tilted algebras of type $A$ or $\tilde{A}$ with global dimension 2, the authors obtained the larger class of surface algebras by realizing the concept of an admissible cut, as defined in [7], in the surface. This procedure increases the number of marked points in each boundary component while the number of edges in the triangulation remains fixed, so the resulting algebra comes from a partial triangulation of a surface. In terms of the quiver $Q_T$, we get a new quiver associated to this partial triangulation by removing one arrow from the oriented 3-cycles corresponding to internal triangles. The surface algebras that are not iterated tilted do not appear in any other known classification of algebras. In general, there are many different surface algebras that can arise even when we fix a triangulation. It is natural to ask how these new algebras are related to each other.

We focus on describing the derived categories of these algebras. This work is motivated by the fact that derived equivalence in the disc and annulus is relatively easy to check. For the surface algebras of the disc and annulus, derived equivalence is determined by the derived invariant of Avella-Alaminos and Geiss defined in [6]. This invariant is easy to calculate for surface algebras, see [12]. However, for surfaces with higher genus or with more than two boundary components, this invariant need not determine derived equivalence. However, using the $AG$-invariant, we can show that there may be several derived

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equivalence classes of algebras for a fixed triangulation of any surface other than the disc, see [12]. In fact, for surface algebras from the annulus, there must be at least two derived equivalence classes.

In this paper, we present a method for determining the derived equivalence of surface algebras coming from a fixed triangulation of $T$ of a surface with genus 0. That is, we restrict ourselves to considering those surface algebras that come from different cuts of the same triangulated surface. We do not attack this directly, rather, we take advantage of recent work by Amiot and Oppermann [2] in which they show that in certain cases derived equivalence is the same as considering graded equivalence with respect to a suitable grading of the arrows. In particular, this is true for surface algebras. This greatly simplifies the problem because we are able to describe the graded equivalences in terms of the cuts that define our algebras and automorphisms of the surface.

We denote cuts of a surface by $\chi$. A pair of two cuts $(\chi_1, \chi_2)$ is called equi-distributed if for each boundary component $B$ of $S$, the number of cuts in $\chi_1$ on $B$ is equal to the number of cuts in $\chi_2$ on $B$. When $\chi_1$ and $\chi_2$ are equi-distributed, we can view $\chi_1$ as being a permutation of $\chi_2$. Additionally, given a cut $\chi$ we get a grading on $\Lambda_T$ by assigning the weight 1 to each arrow removed from $Q_T$ by $\chi$ and 0 for all other arrows in $Q_T$. We denote the graded algebra obtained in this way by $\Lambda$. We have our first main theorem.

**Theorem.** Let $(S, M, T)$ be a triangulated bordered surface of genus 0 and $\Lambda_1$ and $\Lambda_2$ be surface algebras of type $(S, M, T)$ coming from admissible cuts $\chi_1$ and $\chi_2$. Then $\Lambda_1$ and $\Lambda_2$ are graded equivalent if there is an automorphism $f$ of the surface (up to isotopy) such that $f$ induces a quiver automorphism on $Q_T$ and $(\chi_1, f(\chi_2))$ or $(f(\chi_1), \chi_2)$ are equi-distributed.

Using the work by Amiot and Oppermann, this theorem becomes a statement about derived equivalence. The graded equivalence of $\Lambda_1$ and $\Lambda_2$ becomes a derived equivalence of $\Lambda_1$ and $\Lambda_2$.

**Corollary.** Let $\Lambda_i$ and $\chi_i$ be as in the theorem. Then $\Lambda_1$ and $\Lambda_2$ are derived equivalent if there is an automorphism $f$ of the surface (up to isotopy) such that $f$ induces a quiver automorphism on $Q_T$ and $(\chi_1, f(\chi_2))$ or $(f(\chi_1), \chi_2)$ are equi-distributed.

Related work has been done for unpunctured surfaces without cuts. Ladkani [17] uses quiver mutation to characterize the surfaces such that all the algebras arising from their triangulations are derived equivalent. Bobinski and Buan [8] classified the gentle algebras that are derived equivalent to cluster-tilted algebras of type $A$ and $\tilde{A}$, these arise from the triangulations of the disc and annulus. Their proof makes use Brenner-Butler tilting via reflections of gentle algebras. We realize a connection between these two methods of studying derived equivalence by characterizing the reflections of surface algebras as cut versions of mutation in the surface. Let $R_x$ denote the reflection of $Q$ at the vertex $x$ and $\mu_x$ the mutation at $x$. We have the following theorem.

**Theorem.** Let $(Q, I)$ be the quiver with relations of a surface algebra of type $(S, M, T)$. If $x$ is not the source of a relation in $(Q, I)$ and $R_x$ is defined, then there is an admissible cut of $\mu_x(Q_T)$ that gives $R_x(Q)$.

The use of reflections allows us to realize the derived equivalence of surface algebras coming from different triangulations. Additionally, this theorem gives us a way to realize derived equivalences of surface algebras in the module category. In the work of Amiot
and Opperman [2], they explicitly describe the tilting object associated to a graded equivalence. This tilting object is specifically described in the derived category; hence, the derived equivalences given by non-trivial automorphisms of the surface are necessarily given by tilting objects in the derived category that can not be viewed as sitting in the module category.

We would like to remark that results of Amiot and Oppermann in [3] give a complete description of the derived equivalence classes of surface algebras of type $\tilde{\mathbb{A}}$. They do this by considering graded mutations of quivers with potentials and introducing an invariant called the weight or algebra. Similar work for the surface algebras of type $\tilde{\mathbb{A}}$ was also done in [12] using the AG-invariant. Using reflections, we give an alternative realization of the derived equivalences of surface algebras of type $\tilde{\mathbb{A}}$.

In Sections 2 and 3 we introduce the necessary definitions and background about surface algebras and graded algebras. Section 3 ends with a partial description of the graded equivalences given by the identity map on $(S, M)$. Section 4 contains the main theorem of the paper, extending the description in Section 3 to other elements in the mapping class group of $(S, M)$. Note that the definition of the mapping class group is different from the usual definition. Section 5 reformulates the theorems about graded equivalences in terms of derived equivalences. The final section considers derived equivalences of surface algebras given by reflections of gentle algebras.

2. Preliminaries and Notation

In this section we give an alternative but equivalent definition of surface algebras from [12].

2.1. Triangulated surfaces. Let $S$ be a connected oriented unpunctured Riemann surface with boundary $\partial S$ and let $M$ be a non-empty finite subset of the boundary $\partial S$ with at least one point in each boundary component. The elements of $M$ are called marked points. We will refer to the pair $(S, M)$ simply as an unpunctured surface.

We say that two curves in $S$ do not cross if they do not intersect each other except that the endpoints may coincide.

Definition 1. An arc $\gamma$ in $(S, M)$ is a curve in $S$ such that
(a) the endpoints are in $M$,
(b) $\gamma$ does not cross itself,
(c) the relative interior of $\gamma$ is disjoint from $M$ and from the boundary of $S$,
(d) $\gamma$ does not cut out a monogon or a digon.

If $\gamma$ is called a generalized arc if it satisfies only conditions (a), (c) and (d).

The boundary segments of $S$ are those curves that connect two marked points and lie entirely on the boundary of $S$ without passing through a third marked point.

We consider generalized arcs up to isotopy inside the class of such curves. Moreover, each generalized arc is considered up to orientation, so if a generalized arc has endpoints $a, b \in M$ then it can be represented by a curve that runs from $a$ to $b$, as well as by a curve that runs from $b$ to $a$.

For any two arcs $\gamma, \gamma'$ in $S$, let $e(\gamma, \gamma')$ be the minimal number of crossings of $\gamma$ and $\gamma'$, that is, $e(\gamma, \gamma')$ is the minimum of the numbers of crossings of curves $\alpha$ and $\alpha'$, where $\alpha$ is isotopic to $\gamma$ and $\alpha'$ is isotopic to $\gamma'$. Two arcs $\gamma, \gamma'$ are called non-crossing if $e(\gamma, \gamma') = 0$.

A triangulation is a maximal collection of non-crossing arcs. The arcs of a triangulation cut the surface into triangles. Since $(S, M)$ is an unpunctured surface, the three sides of
each triangle are distinct (in contrast to the case of surfaces with punctures). A triangle in $T$ is called an **internal triangle** if none of its sides are a boundary segment. We often refer to the triple $(S, M, T)$ as a **triangulated surface**.

### 2.2. Jacobian algebras from surfaces

Let $Q = (Q_0, Q_1, s, t)$ be a quiver with vertex set $Q_0$, $Q_1$ the arrow set, and $s, t: Q_1 \to Q_0$ are maps that assign to each arrow $\alpha$ its source $s(\alpha)$ and target $t(\alpha)$. For $v, v' \in Q_0$, we let $Q_1(v, v')$ denote the set of arrows from $v$ to $v'$.

If $T = \{ \tau_1, \tau_2, \ldots, \tau_n \}$ is a triangulation of an unpunctured surface $(S, M)$, we define a quiver $Q_T$ as follows. Each arc in $T$ corresponds to a vertex of $Q_T$. We will denote the vertex corresponding to $\tau_i$ simply by $i$. The number of arrows from $i$ to $j$ is the number of triangles $\triangle$ in $T$ such that the arcs $\tau_i, \tau_j$ form two sides of $\triangle$, with $\tau_j$ following $\tau_i$ when going around the triangle $\triangle$ in the counter-clockwise orientation, see Figure 1 for an example. For clarity we suppress the $\tau$ notation when there is no possibility of confusion.

Note that the interior triangles in $T$ correspond to certain oriented 3-cycles in $Q_T$.

Following [4, 16], let $W$ be the sum of all oriented 3-cycles in $Q_T$ coming from internal triangles. Then $W$ is a potential, in the sense of [13], which gives rise to a Jacobian algebra $\Lambda_T = \text{Jac}(Q_T, W)$, which is defined as the quotient of the path algebra of the quiver $Q_T$ by the two-sided ideal $I_T$ generated by the subpaths of length two of each oriented 3-cycle in $Q_T$.

### 2.3. Cutting a surface

Let $(S, M)$ be a surface without punctures, $T$ a triangulation, $Q_T$ the corresponding quiver, and $\Lambda_T$ the Jacobian algebra. Throughout this section, we assume that, if $S$ is a disc, then $M$ has at least 5 marked points, thus we exclude the disc with 4 marked points.

**Definition 2.** Recall that the interior triangles of $T$ distinguish certain oriented 3-cycles in the quiver $Q_T$. Let $\mathcal{I}$ denote the set of internal triangles of $(S, M, T)$. We define an **admissible cut** of $T$ to be a function $\chi: \mathcal{I} \to M$ that selects a vertex in each internal triangle of $T$.

In addition to selecting a marked point $\chi(\triangle) = v$ on the surface, this map also distinguishes the two edges $\tau_i$ and $\tau_j$ in $T$ incident to $v$ in $\triangle$. We call the image of $\chi$ in $\triangle$ a **local cut** of $(S, M, T)$, denoted $\chi_{v, i, j}$ or $\chi_{i, j}$ when there is no cause for confusion. We will always write $\chi_{i, j}$ when the corresponding arrow is $i \to j$ in $Q_T$. Graphically, we will denote a local cut in $(S, M, T)$ by bisecting the marked point $\chi(\triangle)$ between the corresponding edges $\tau_i$ and $\tau_j$, see Figure 2. The decorated surface corresponding to $\chi$ is denoted $(S, M^\dagger, T^\dagger)$.
Definition 3. Note that a local cut of \((S, M, T)\) distinguishes an arrow in the quiver \(Q_T\) associated to the triangulation. Let \(\chi\) be an admissible cut of \((S, M, .)\). By an abuse of notation, let \(\chi\) also denote this collection of arrows, then we define the surface algebra \(\Lambda_{T^+}\) of type \((S, M)\) associated to \(\chi\) to be the quotient \(kQ_T/I_T \cup \chi\), we let \(I_T\) denote the corresponding ideal of relations on \(Q_{T^+}\).

Example 4. Here we present an admissible cut of the surface \((S, M, T)\) in Figure 1 and the associated quiver of \(\Lambda_{T^+}\).

See [12] for a complete description of surface algebras arising from admissible cuts in terms of partially triangulated surfaces and partial cluster-tilting objects.

3. Graded Equivalence

Ultimately, we are interested in describing the derived equivalence classes of surface algebras. To this end, we are led to investigate graded equivalences of graded algebras because of a theorem of Amiot and Oppermann in [2, Theorem 5.6] showing a strong connection between the two types of equivalences.

In this section we introduce the concept of graded equivalence and seek to give our first criteria for graded equivalence of surface algebras.

3.1. Graded algebras. We will only consider \(\mathbb{Z}\)-graded algebras, however, the following definitions can be re-stated for any group \(G\), as in [15]. We will simply refer to \(\mathbb{Z}\)-gradings as gradings.

A weight function on \(Q\) is a function \(w: Q_1 \to \mathbb{Z}\), that is, a function that assigns an integer to each arrow of \(Q\). We can naturally extend the weight function to paths in \(Q\), by setting \(w(e_1) = 0\) for each stationary path in \(Q\) and \(w(\alpha_1 \cdots \alpha_r) = w(\alpha_1) + \cdots + w(\alpha_r)\) for each path in \(Q\) with length \(r \geq 1\). This induces a grading on \(kQ\) with \(kQ = \bigoplus_{p \in \mathbb{Z}} kQ^p\), where \(kQ^p\) is generated by the set of paths with weight \(p\). A relation \(r\) is homogeneous of degree \(p\) if \(r \in kQ^p\) for some \(p\). The grading induced by \(w\) gives a grading on \(kQ/I\) if and only if \(I\) is generated by homogeneous relations, not necessarily all of the same degree.
Let \( \Lambda = \bigoplus_{p \in \mathbb{Z}} \Lambda^p \) be a graded algebra. As in [15], we denote by \( \text{gr} \Lambda \) the category of finitely generated graded modules over \( \Lambda \). For a graded module \( M = \bigoplus_{p \in \mathbb{Z}} M^p \), we define \( M(q) := \bigoplus_{p \in \mathbb{Z}} M^{p+q} \). That is, the \( p \) graded part of \( M(q) \) is the \( p + q \) graded part of \( M \).

We use this grading shift to define a new category that will, in some ways, take on the role of the derived category. Of course, this new category is relatively simpler.

**Definition 5.** Given a graded algebra \( \Lambda = kQ/I \) induced by a weight \( w \), we define the covering of \( \Lambda \)

\[
\text{Cov}(\Lambda) := \text{add} \{ \Lambda(p) : p \in \mathbb{Z} \} \subseteq \text{gr} \Lambda.
\]

Let \( F: \text{gr} \Lambda \to \text{mod} \Lambda \) be the functor that forgets the grading. We associate to \( \text{Cov}(\Lambda) \) the quiver with relations \((Q^*, I^*)\) defined by

\[
Q_0^* = Q_0 \times \mathbb{Z},
Q_1^*((v, i), (v', j)) = \{ \alpha \in Q_1(v, v') : w(\alpha) = j - i \}.
\]

Note that \( Q^* \) is infinite. The map \( F \) induces a projection \( Q^* \to Q \), we will also refer to this as \( F \). We define the relations on \( Q \) by \( p \in I^* \) if \( F(p) \in I \). We partition the vertices of \( Q^* \) into levels where \( (v, i) \) is of level \( i \). If \( w(\alpha) > 0 \), we refer to the copies of \( \alpha \) in \( Q^* \) as bridge arrows, these arrows connect different levels of \( Q^* \).

From [15, Theorem 0.1] we have,

**Proposition 6.** Let \( \Lambda \) be a finite dimensional graded algebra and \((Q, I)\) a quiver with relations and weight \( w \) such that \( \Lambda \cong kQ/I \) and the grading on \( \Lambda \) is induced by \( w \), then \( \text{mod} kQ^*/I^* \cong \text{mod Cov}(\Lambda) \cong \text{gr} \Lambda \).

Additionally, we recall from [2, Theorem 2.11],

**Proposition 7.** Let \( \Lambda \) be an algebra with two different gradings. We denote by \( \text{Cov}(\Lambda_1) \) the covering corresponding to the first grading, and \( \text{Cov}(\Lambda_2) \) the covering corresponding to the second grading. Then the following are equivalent:

1) There is an equivalence \( U: \text{mod Cov}(\Lambda_1) \cong \text{mod Cov}(\Lambda_2) \) such that the following diagram commutes.

\[
\begin{array}{ccc}
\text{mod Cov}(\Lambda_1) & \to & \text{mod Cov}(\Lambda_2) \\
\downarrow U & & \downarrow U \\
\text{mod} \Lambda & \to & \text{mod} \Lambda
\end{array}
\]

2) There exist a map \( r: Q_0 \to \mathbb{Z} \) with \( r(i) = r_i \) and an isomorphism of graded algebras

\[
f: \Lambda_2 \cong \bigoplus_{p \in \mathbb{Z}} \text{Hom}_{\text{Cov}(\Lambda_1)}(\bigoplus_{i=1}^n P_i(r_i), \bigoplus_{i=1}^n P_i(r_i + p))
\]

where \( \Lambda_1 \cong \bigoplus_{i=1}^n P_i \) in \( \text{gr} \Lambda_1 \).

In this case we say that the gradings are equivalent.

**Remark 8.** The isomorphism \( f: \Lambda_2 \cong \bigoplus_{p \in \mathbb{Z}} \text{Hom}_{\text{Cov}(\Lambda_1)}(\bigoplus_{i=1}^n P_i(r_i), \bigoplus_{i=1}^n P_i(r_i + p)) \) may arise by first applying an automorphism to the quiver \( Q^* \) of \( \text{Cov}(\Lambda_1) \). The simplest case to consider is when this automorphism is the identity on \( Q^* \), when this happens, the graded equivalence can be checked via purely combinatorial methods involving the quiver \( Q^* \) associated to \( \text{Cov}(\Lambda_1) \). In particular, let \( w_2 \) be the weight determined by the grading of \( \Lambda_2 \), then verifying (2) reduces to finding vertices \((v, i)\) and \((v', j)\) of \( Q^* \) such that if \( \alpha: v \to v' \)
and \( w_2(\alpha) = k \), there is an arrow \( (v, i) \rightarrow (v', j + k) \) in \( Q^* \). Then we can define the map \( r \) such that \( r(v) = i \) and \( r(v') = j \). We will use this fact in the proof of the main theorem. The algebras \( \Lambda_1 \) and \( \Lambda_2 \) are graded equivalent if such a choice can be made simultaneously for each vertex. For brevity we will later refer to this as being graded equivalent via the identity. Notice that we must have \( j > i \), because \( \alpha \) is a bridge arrow, which by definition must always point in an increasing direction. See example 12.

We will not consider the surfaces algebras as graded algebras. However, the cut defining a surface algebra does induce a grading on the algebra coming from the original triangulation.

**Definition 9.** Let \( \Lambda \) be a surface algebra coming from an admissible cut of \((S, M, T)\). Let \( \tilde{\Lambda} \) denote the Jacobian algebra coming from \((Q_T, W)\) with a grading given by the weight

\[
w(\alpha) = \begin{cases} 0 & \text{if } \alpha \in Q_T \cap Q_T^1, \\ 1 & \text{if } \alpha \in Q_T \setminus Q_T^1. \end{cases}
\]

This weight is homogeneous for all relations in \((Q_T, W)\), hence it induces a grading on \( \tilde{\Lambda} \).

### 3.2. Graded equivalence and surface algebras

In this section we describe when two surface algebras are graded equivalent via the identity. To that end we begin by finding the required integers \( r_i \), as in Proposition 7 and Remark 11, for those vertices corresponding to edges in \((S, M, T)\) incident to a cut. Throughout we fix two different admissible cuts \( \chi_1 \) and \( \chi_2 \) of \((S, M, T)\) with \( Q \) the corresponding cut quivers, \( \Lambda \) the corresponding surface algebras, \( \tilde{\Lambda} \) the corresponding graded Jacobian algebras, and \( Q^* \) the quiver of Cov(\( \Lambda_1 \)).

**Definition 10.** Given a pair of cuts \((\chi_1, \chi_2)\) let \( \{ \tau_i, \ldots, \tau_k \} \) be the set of edges in \((S, M, T)\) such that \( \tau_i \) is the edge of a triangle in which \( \chi_1 \) and \( \chi_2 \) differ and \( \tau_i \) is incident to both cuts. We call the edges in \( \{ \tau_i, \ldots, \tau_k \} \) sliding edges. Notice that there is at most one sliding edge for each internal triangle of \((S, M, T)\). Additionally, each sliding edge is associated with at least one internal triangle; however, there may be sliding edges \( \tau_i \) associated with two different triangles. When necessary we may distinguish between the different types of sliding edges as one-sliding and two-sliding edges, respectively.

**Remark 11.** Recall that the local cut \( \chi_{i,j} \) denotes the cut which removes the arrow \( i \rightarrow j \). Let \( \tilde{\Lambda}_1 \sim \tilde{\Lambda}_2 \) be graded equivalent via the identity. By considering the orientation of the arrows which are cut and definition 9 of the weight given by a cut, we give an explicit formula for determining the function \( r \) from Proposition 7 (2) on triangles containing sliding edges. Since the weight of an arrow is at most 1, the value of \( r \) can only differ by one near sliding edges. We first consider triangles where \( \tau_i \) is a two-sliding edge, so there are internal triangles \( \triangle = \tau_i \tau_j \tau_k \) and \( \triangle' = \tau_i \tau'_j \tau'_k \). For \( \tau_i \) to be a two sliding edge, when we restrict to \( \triangle \) and \( \triangle' \), we must have

\[(a) \ \chi_1 = \chi_{ki} \chi_{kli} \text{ and } \chi_2 = \chi_{ij} \chi_{kj} \] or \[(b) \ \chi_1 = \chi_{ij} \chi_{ki} \text{ and } \chi_2 = \chi_{kl} \chi_{kli}, \]

see Figure 3. If we let \( r(i) \) be any integer, then a graded equivalence via the identity implies that we must have \( r(\ell) = r(i) + 1 \) for \( \ell = j', k' \) and \( r(\ell) = r(i) - 1 \) for \( \ell = j, k \) in the first case, in the second case we must have \( r(\ell) = r(i) + 1 \) for \( \ell = j, j', k, k' \). In both cases, the full subquiver on the \( P_i(r_i) \) in \( Q^* \) contains the bridge arrows associated to \( \chi_1 \).

Now we consider the triangles \( \tau_i \tau_j \tau_k \) where \( \tau_i \) is a one-sliding edge. Then we must have

\[(a) \ \chi_1 = \chi_{ij} \text{ and } \chi_2 = \chi_{ki}, \] or \[(b) \ \chi_1 = \chi_{ki} \text{ and } \chi_2 = \chi_{ij}, \]
see Figure 4. If we let \( r(i) \) be any integer, then in the first case we must choose \( r(\ell) = r(i) + 1 \) for \( \ell = j, k \). In the second case, \( r(\ell) = r(i) - 1 \) for \( \ell = j, k \). Again, in both cases the full subquiver on the \( P_1(r) \) contains the bridge arrow associated to \( \chi_1 \).
and $\Lambda_i$ given by $\chi_i$. The quiver of $\text{Cov}(\tilde{\Lambda}_1)$ is given in Figure 6. Letting $P_i(r_i)$ be given by the circled vertices. Then

$$\tilde{\Lambda}_2 \cong \bigoplus_{p \in \mathbb{Z}} \text{Hom}_{\text{Cov}(\Lambda_1)} \left( \bigoplus_{i=1}^n P_i(r_i), \bigoplus_{i=1}^n P_i(r_i+p) \right)$$

is graded equivalent via the identity. There are three level partitions; the component of level $-1$ consists of the vertices 12, 13, 14, and 15 along with the arrow $13 \to 14, 14 \to 15$ and $15 \to 12$.

**Proposition 13.** Let $\chi_1$ and $\chi_2$ be two admissible cuts of a surface $(S, M, T)$ such that $\tilde{\Lambda}_1$ and $\tilde{\Lambda}_2$ are graded equivalent via the identity. The connected components of each level partition of $\text{Cov}(\tilde{\Lambda}_1)$ determine a connected region in $S$ bounded by the sliding edges of $(\chi_1, \chi_2)$ and $\partial S$.

**Proof.** Let $Q'$ be the quiver of $\text{Cov}(\tilde{\Lambda}_1)$ and $\mathcal{C}$ be a level connected component of $Q'$. By definition, $\mathcal{C}$ can touch other connected components only by bridge arrows, which are associated to sliding edges of $(\chi_1, \chi_2)$. Recall that arrows correspond to triangles in $(S, M, T)$. Because $\mathcal{C}$ is a connected subgraph of $Q'$, $\mathcal{C}$ must correspond to some contiguous collection of triangles in $(S, M, T)$, denote this collect by $\mathcal{E}$. Further, because we can only leave $\mathcal{E}$ via bridge arrows, we must also have that $\mathcal{E}$ is bounded by $\partial S$ and sliding edges.

Note that if $\mathcal{E}$ consists of a single vertex, then $\mathcal{E}$ will consist of a single sliding edge, this edge must be two-sliding. In all other cases $\mathcal{E}$ will have positive area.

**Remark 14.** Proposition 13 implies that any two edges $\tau_j$ and $\tau_k$ contained in the interior of the same bounded region must have the same value $r_j = r_k$, because these regions are determine by the connected components of that level.

**Definition 15.** The pair $(\chi_1, \chi_2)$ is called equi-distributed if for each boundary component $B$, we have $|\text{Im} \chi_1 \cap B| = |\text{Im} \chi_2 \cap B|$, meaning the the number of cuts in $\chi_1$ on $B$ is equal to the number of cuts in $\chi_2$ on $B$.

**Theorem 16.** Let $(S, M, T)$ be a triangulated bordered surface of genus 0 and $\Lambda_1$ and $\Lambda_2$ surface algebras of type $(S, M, T)$. Then $\tilde{\Lambda}_1$ and $\tilde{\Lambda}_2$ are graded equivalent via the identity if and only if $(\chi_1, \chi_2)$ is equi-distributed.
Proof. First we assume that \((\chi_1, \chi_2)\) is equi-distributed. Set \(Q_1\) to be the quiver of \(\Lambda_1\), \(Q\) the quiver of \((S,M,T)\), and \(Q'\) the quiver associated to \(\text{Cov}(\Lambda_1)\). By determining the associated level partitions in \(Q'\) we will explicitly describe the function \(r: Q_0 \to \mathbb{Z}\) so that we have

\[
\Lambda_2 \sim \bigoplus_{p \in \mathbb{Z}} \text{Hom}_{\text{Cov}(\Lambda_1)} \left( \bigoplus_{i=1}^{n} P_i(\langle r_i \rangle), \bigoplus_{i=1}^{n} P_i(\langle r_i + p \rangle) \right)
\]

Because of Proposition 13 and Remark 14 it is sufficient to only determine the values for \(r\) near sliding edges. The value for other edges will be induced by the choices at the sliding edges.

The process is to choose (at random) a bounded region and assign \(r = 0\) to each internal arc of that region. Applying Remark 11, we then proceed to assign values of \(r\) to each sliding edge bounding the chosen region as well as the neighboring regions. We then reiterate this process with each neighboring region and so on. The primary work of the proof is to show that such a choice is well defined for all of \(S\). Assume first that \(S\) has at least two boundary components.

Let \(\{\mathcal{C}_i\}\) \(i=0, \ldots, r\) be the bounded regions given by \((\chi_1, \chi_2)\) and let \(r_i\) be the corresponding value of \(r\) for \(\mathcal{C}_i\). We now consider \(r\) as the function \(r: S \to \mathbb{Z}\) by setting \(r(x) = r_i\) for \(x \in \mathcal{C}_i\). Fix \(i\) and a point \(x_0 \in \mathcal{C}_i\) let \(\rho\) be a non-contractible loop based at \(x_0\), without loss of generality we may let \(i = 0\). We may assume that \(r(x_0) = 0\). Because \(S\) is genus zero, the loop divides \(S\) into two parts, the inside (to the right) of \(\rho\) and the outside (to the left)
of $\rho$. We want to show that as we travel along $\rho$, in either direction, and apply Remark 11 to determine the value of $r$ as we change bounded regions, we recover that $r(x_0) = 0$ as we cross back into $C_0$. Let $r'_0$ be value of $r$ as we cross back into $C_0$.

For each sliding edge $\tau$ intersecting $\rho$ we associate two integers $\Delta r_\tau$ and $\Delta \chi_\tau$. Let $C$ and $D$ be the components that are bound by $\tau$ and $D$ follows $C$ with respect to $\rho$, then $\Delta r_\tau := r(D) - r(C)$. Further, let $a_\tau$ be the number of local cuts from $\chi_1$ incident to $\tau$ on boundary components inside of $\rho$ and $b_\tau$ the number of local cuts from $\chi_2$ on boundary components inside of $\rho$ and incident to $\tau$, we define $\Delta \chi_\tau = b_\tau - a_\tau$. For each sliding edge and choice of $\rho$, since $r$ is chosen as in Remark 11, then $\Delta \chi_\tau = -\Delta r_\tau$. This can be shown by considering cases. See Figures 3 and 4.

The number $\Delta \chi_\rho = \sum_i \Delta \chi_i$ measures the total change in the number of cuts on the boundary components inside of $\rho$. Similarly, $\Delta r_\rho = \sum_i \Delta r_i$ measures the total change in $r$ after one iteration of $\rho$. Hence, if $(\chi_1, \chi_2)$ is equi-distributed, then $\Delta \chi_\rho = 0$. Therefore, $\Delta r_\rho = -\Delta \chi_\rho = 0$. It follows that if $r_0 = 0$, as desired. Because $\rho$ is arbitrary, we see that the choice of $r$ given by Remark 11 is well-defined.

Conversely, assume that $(\chi_1, \chi_2)$ is not equi-distributed. Then in the above analysis we must have $\Delta r_\rho = -\Delta \chi_\rho \neq 0$ for some loop $\rho$. It follows that there is no consistent way to define the function $r: Q_0 \to \mathbb{Z}$. It follows that $\Lambda_1$ and $\tilde{\Lambda}_2$ are not graded equivalent. □

**Remark 17.** We remark that the above theorem does not hold for higher genus. Let $S$ be the torus with one boundary component. Let $M$ be a single point on the boundary and consider the triangulation $T$ in Figure 8. Because there is only one boundary component, Proposition 7 would imply that any two admissible cuts should be graded equivalent via the identity. However, the cuts $\chi_{1,2,3,4}$ and $\chi_{1,2,3,4}$ are easily shown to not be graded equivalent via the identity. Let $\Lambda_1$ be given by $\chi_{1,2,3,4}$ and $\Lambda_2$ be given by $\chi_{1,2,3,4}$. Because the induced weight on the arrows $1 \to 2$, $2 \to 3$ and $1 \to 3$ does not change between the two cuts, we must have $r_1 = r_2 = r_3$, where $r_i$ is as in Proposition 7. Additionally, because the weight on the arrow $4 \to 1$ changes we must have $r_1 \neq r_4$, but the weight on $2 \to 4$ does not change so $r_2 = r_4$, hence we must also have $r_1 = r_4$, a contradiction.
4. Boundary Permutations

At this point we have determined that we get graded equivalent algebras when we permute local cuts along a fixed boundary component. In this section we will show that we can also permute cuts among different boundary components.

We define the mapping class group of \((S, M)\) as in [5]. Set \(\text{Homeo}^{+}(S, M)\) to be the set of orientation preserving homeomorphism from \(S\) to \(S\) that send \(M\) to \(M\). Note that if a boundary component \(C_1\) is mapped to a component \(C_2\), then the two components must have the same number of marked points. We say that a homeomorphism \(f\) is isotopic to the identity relative to \(M\), if \(f\) is isotopic to the identity via a homotopy that fixes \(M\) point-wise. Then we set \(\text{Homeo}_0(S, M)\) to be the homeomorphisms isotopic to the identity relative to \(M\). The mapping class group of \((S, M)\) is

\[
\mathcal{M}(S, M) = \text{Homeo}^{+}(S, M) / \text{Homeo}_0(S, M)
\]

For \(f \in \mathcal{M}(S, M)\) we define \(f\) at an admissible cut \((S, M^1, T^1)\) by setting \(f(\chi_{i,j}) = \chi_{f(i), f(j)}\) for each local cut. By construction this induces a graded isomorphism of \(\Lambda_{T^1}\) and \(\Lambda_{f(T^1)}\) because it explicitly sends arrows of weight one to arrows of weight one.

**Theorem 18.** Let \((S, M, T)\) be a triangulated bordered surface of genus 0 and \(\Lambda_1\) and \(\Lambda_2\) be surface algebras of type \((S, M, T)\) coming from admissible cuts \(\chi_1\) and \(\chi_2\). Then \(\Lambda_1\) and \(\Lambda_2\) are graded equivalent if there is an element \(f \in \mathcal{M}(S, M)\) such that \(f\) induces a quiver automorphism on \(Q_T\) and \((\chi_1, f(\chi_2))\) or \((f(\chi_1), \chi_2)\) are equi-distributed.

**Proof.** Assume \((\chi_1, f(\chi_2))\) is equi-distributed, hence \(\tilde{\Lambda}_1\) and \(f(\tilde{\Lambda}_2)\) are graded equivalent by Theorem 16. By construction the extension of \(f\) to the cut surface induces a graded isomorphism of \(\tilde{\Lambda}_2\) and \(f(\tilde{\Lambda}_2)\). It follows that \(\tilde{\Lambda}_1\) and \(\tilde{\Lambda}_2\) are graded equivalent. \(\square\)

**Remark 19.** Note that the theorem excludes the use of the Dehn twists in \(\mathcal{M}(S, M)\).

In particular, this is because the Dehn twists can never change the configuration of the cuts in \((S, M, T)\). In general, a mapping class \(f \in \mathcal{M}(S, M)\) satisfying Theorem 18 will have to leave \(T\) invariant, the set of all such \(f\) will be a small subset of \(\mathcal{M}(S, M)\). On the other hand, let \(\Lambda_1\) and \(\Lambda_2\) be graded equivalent algebras coming from admissible cuts \(\chi_1\) of a surface \((S, M, T)\) such that the map \(f : \tilde{\Lambda}_1 \rightarrow \tilde{\Lambda}_2\) induces an isomorphism of quivers \(f : \tilde{Q}_1 \rightarrow \tilde{Q}_2\). Because arrows are associated to triangles, \(f\) induces a map \(f_S : (S, M, T) \rightarrow (S, M, T)\). To understand what this map is, we consider \((S, M, T)\) as a CW-complex where the 0-skeleton is \(M\), the 1-skeleton is given by \(T\) and the boundary segments, and the 2-skeleton is given by the ideal triangles. For convenience we use the following definition.

**Definition 20.** Let \((S, M, T)\) be a triangulated surface without punctures. There are three triangle types, we call those triangles with two edges in the boundary corner triangles, triangles with one edge in the boundary basic triangles, and triangles with no edge in the boundary internal triangles. Notice that there is a unique edge in \(T\) associated to each corner triangle.

Before analyzing the map \(f_S\), we note the following fact. Given a two dimensional finite CW-complex \(S\) and \(f_1\) a continuous self mapping on the one-skeleton of \(S\), there is a continuous map \(f : S \rightarrow S\) that restricts to \(f_1\). This map is given by considering barycentric coordinates on the homeomorphic image of each face into a convex open subset of \(\mathbb{R}^2\). Hence, it is enough to understand \(f_S\) on \(T\) and boundary segments.
The induced map on $(S, M, T)$ is given by first fixing and defining the map on a representative for each isotopy class in $T$. Notice that $Q_T = \tilde{Q}_1 = \tilde{Q}_2$ by assumption. Hence, we can view $f$ as an automorphism on $Q_T = (Q_0, Q_1)$. Considering $f$ as a map on vertices $Q_0$, we define $f_1$ on the edges of the triangulation by $f_1(\tau_i) = \tau_j$ when $f(i) = j$ in $Q_0$. Note that $f$ preserves orientation in $Q$ because it is a quiver automorphism. So, in the surface, if $\tau_i$ is incident to $\tau_j$ with $\tau_j$ following $\tau_i$ in the counter-clockwise direction, then $f_1(\tau_j)$ is incident to and follows $f_1(\tau_i)$ in the counter-clockwise direction. It follows that $f_1$ preserves triangle types and orientation, that is, the edges defining a basic, internal or corner triangle respectively and further those edges will be the same relative orientation. We can extend the definition of $f_1$ to boundary segments, because of this preservation of triangle type, as follows. Let $\triangle$ be a basic triangle with edges $\tau_i$, $\tau_j$ and boundary segment $b$, we define $f_1(b)$ to be the boundary segment incident to $f_1(\tau_i)$ and $f_1(\tau_j)$. Similarly, $f_1$ maps the corner triangle with edge $\tau_i$ and boundary segments $b$ and $b'$ with $b$ following $b'$ in the counter-clockwise direction to the corner triangle with edge $f_1(\tau_i)$ and boundary segments $f_1(b)$ and $f_1(b')$ with $f_1(b)$ following $f_1(b')$ in the counter-clockwise direction. By construction this map will preserve the on the 1-skeleton, hence the induced map $f_S$ will preserve the orientation of $S$. As a result we have the following partial converse to Theorem 18.

**Theorem 21.** Let $\tilde{\Lambda}_1$ and $\tilde{\Lambda}_2$ be graded equivalent algebras coming from admissible cuts $\chi_1$ and $\chi_2$ of a surface $(S, M, T)$ such that the map $f: \Lambda_1 \to \Lambda_2$ induces an automorphism on $Q_T$. Then, there is mapping class $f_S \in \mathcal{M}(S, M)$ that induces the graded isomorphism $f$ and $(f(\chi_1), \chi_2)$ are equi-distributed.

**Proof.** Let $f_S$ be given as in the above discussion and let $\tilde{\Lambda}'$ be the graded algebra given by $f_S(\chi_1)$. Note that $\tilde{\Lambda}'$ need not be $\Lambda_2$, but, by construction it will be graded equivalent to $\Lambda_2$, call the corresponding equivalence $g$. Indeed, $\Lambda_2$ and $\tilde{\Lambda}'$ are graded equivalent via the identity. That is, $g$ induces the identity map on the level of quivers. This follows immediately by carefully unwinding the definitions. By assumption $\Lambda_1$, $\tilde{\Lambda}_2$, and $\tilde{\Lambda}'$ have quiver $Q_T$. Let $\phi: Q_T \to Q_T$ denote the automorphism induced by $f$. By construction, $f_S$ induces the same a quiver automorphism $\phi$. Hence, we have the following commutative diagrams

$$
\begin{array}{ccc}
\tilde{\Lambda}_1 & \xrightarrow{f} & \tilde{\Lambda}_2 \\
\tilde{\Lambda}' & \xrightarrow{g} & Q_T \\
\end{array}
$$

Therefore, the map induced by $g$ must be the identity map. It follows from Theorem 16, that $(f(\chi_1), \chi_2)$ are equi-distributed. \square

**Example 22.** We give an example of a graded equivalence given by a non-trivial mapping class. Let $(S, M, T)$ be given as in Figure 9. The quiver of the triangulation is $Q_T$

$$
\begin{array}{cccc}
2 & 3 & 4 \\
\downarrow & \downarrow & \downarrow \\
1 & 5 \\
\uparrow & \uparrow & \uparrow \\
8 & 7 & 6 \\
\end{array}
$$
Let $\chi_1 = \chi_{1.2} \chi_{7.1}$ and $\chi_2 = \chi_{8.7} \chi_{3.1}$, given by the red and blue lines respectively. The corresponding surface algebras $\Lambda_1$ and $\Lambda_2$ are derived equivalent by Theorem 24. The required automorphism of the surface $f$ can be realized by a rotation of the universal cover of $S$ that fixes a lift of the $\tau_1$, see Figure 9. This map will induce the quiver automorphism given by the map on vertices

\[
\begin{align*}
1 & \mapsto 1 \\
2 & \mapsto 8 \\
3 & \mapsto 7 \\
4 & \mapsto 6 \\
5 & \mapsto 5 \\
6 & \mapsto 2 \\
7 & \mapsto 3 \\
8 & \mapsto 2.
\end{align*}
\]

Note that the image of $\chi_1$ under this map is not $\chi_2$, but $(f(\chi_1), \chi_2)$ is equi-distributed.

Theorem 18 does not tell us how to identify the homeomorphism of the surface that gives rise to the graded equivalence. Naturally, we want to determine which automorphisms of the surface determine a graded equivalence. A minimal combinatorial description can be given if we ignore some of the surface structure and consider the automorphism in combinatorial terms of the marked points, boundary components and triangles. In these terms, finding automorphisms that induce a graded equivalence is equivalent to finding permutations of the boundary components and of the marked points such that the corresponding map on the set of triangles sends neighboring triangles to neighboring triangles and boundary components to boundary components. Under the permutation of boundary components, a component can only be sent to another component with the same local configuration of triangles incident to the component. Similarly, a marked point must be sent to a marked point with the same number and type of incident triangles, these triangles must occur in the same order in the counter-clockwise direction.

Recall that we may associate a cluster algebra to a triangulated surface, see [14]. The mapping classes of $(S, M, T)$ that correspond to graded equivalences will correspond to cluster automorphisms, defined in [5], which fix (up to a permutation) the cluster corresponding to the triangulation.

5. Derived Equivalence

All of this work to describe the graded equivalences of surface algebras has been done with the goal of determining derived equivalences. We restate a theorem of [2] in terms of surface algebras.

**Theorem 23** ([2, Theorem 5.6]). *Let $\Lambda_1$ and $\Lambda_2$ be surface algebras coming from admissible cuts $\chi_1$ and $\chi_2$. Then $\Lambda_1$ and $\Lambda_2$ are derived equivalent if and only if $\tilde{\Lambda}_1$ and $\tilde{\Lambda}_2$ are graded equivalent.*

We can now reformulate the theorems of section 2 and 3.
Theorem 24. Let $\Lambda_1$ and $\Lambda_2$ be surface algebras of type $(S, M, T)$ coming from admissible cuts $\chi_1$ and $\chi_2$ respectively. Then $\Lambda_1$ and $\Lambda_2$ are derived equivalent if there is an element $f \in \mathcal{A}(S, M)$ such that $f$ induces a quiver automorphism on $Q_T$ and $(\chi_1, f(\chi_2))$ or $(f(\chi_1), \chi_2)$ are equi-distributed.

The proof of Theorem 23 explicitly describes the tilting object associated to a given graded equivalence. If we have

$$\tilde{\Lambda}_2 \sim_{\mathbb{Z}} \bigoplus_{p \in \mathbb{Z}} \text{Hom} \text{Cov}(\Lambda_1) \left( \bigoplus_{i=1}^{n} P_i(r_i), \bigoplus_{i=1}^{n} P_i(r_i + p) \right)$$

Then $\bigoplus_{i=1}^{n} F^{-\gamma} P_{1}$ is the tilting object in $\mathcal{D}^b(\text{mod} \Lambda_1)$ that gives the derived equivalence of $\Lambda_1$ and $\Lambda_2$. Where $F := \mathbb{S}[-2]$ with $\mathbb{S}$ the Serre functor of $\mathcal{D}^b(\text{mod} \Lambda_1)$.

6. Reflections of Gentle Algebras

In the theory of cluster algebras, quiver mutation plays an important role. For cluster-algebras from surfaces this mutation can be realized in the surface as a flip of an edge in the triangulation. In this section we will show that a similar idea exists for surface algebras via tilting modules, instead of tilting objects in the derived category. These reflections induced derived equivalences via an explicit tilting object in module category. In contrast to the derived equivalences obtained via mapping classes, the derived equivalences obtained via reflections need not be between surface algebras of the same triangulation, in fact most are not. Additionally, reflections allow us to describe some derived equivalences of surface algebras in terms of tilting modules, instead of tilting objects in the derived category.

6.1. Definitions. We begin by recalling definitions.

Definition 25. The mutation of $Q$ at vertex $j$, denoted $\mu_j(Q)$, is the quiver obtained from $Q$ by the following procedure:

1. Reverse each arrow incident to $j$.
2. For all paths $i \to j \to k$ in $Q$, we introduce an arrow $i \to k$ in $\mu_j(Q)$.
3. Delete all 2-cycles that may have been generated.

In a triangulated surface without punctures $(S, M, T)$, each edge $\tau$ of the triangulation is contained in exactly two distinct triangles that form a quadrilateral in which $\tau$ is a diagonal. The mutation of the triangulation, $\mu_{\tau}(T)$, is given by $T \setminus \{\tau\} \cup \{\tau'\}$ where $\tau'$ is the other diagonal of the quadrilateral containing $\tau$. If $j \in Q_0$ corresponds to $\tau_j$, then $\mu_j(Q)$ is the quiver of $\mu_{\tau_j}(T)$.

Definition 26. A finite dimensional $k$-algebra $\Lambda$ is called gentle if the bound quiver $(Q,I)$ associated to $\Lambda$ satisfies:

1. For each $i \in Q_0$, $\# \{\alpha \in Q_1 : s(\alpha) = i \} \leq 2$ and $\# \{\alpha \in Q_1 : t(\alpha) = i \} \leq 2$.
2. For each $\beta \in Q_1$, $\# \{\alpha \in Q_1 : s(\beta) = t(\alpha) \text{ and } \alpha \beta \notin I \} \leq 1$ and $\# \{\gamma \in Q_1 : s(\gamma) = t(\beta) \text{ and } \beta \gamma \notin I \} \leq 1$.
3. The ideal $I$ is generated by paths of length 2.
4. For each $\beta \in Q_1$, $\# \{\alpha \in Q_1 : s(\beta) = t(\alpha) \text{ and } \alpha \beta \in I \} \leq 1$ and $\# \{\gamma \in Q_1 : s(\gamma) = t(\beta) \text{ and } \beta \gamma \in I \} \leq 1$.

Surface algebras are gentle [6, 12]. For the remainder of the section we assume that $Q$ is a gentle quiver with relations $I$. 
Definition 27. Let $i$ be a vertex of $Q$ such that for each arrow $\alpha \in Q_1$ with $s(\alpha) = i$ there exists $\beta_\alpha \in Q_1$ with $r(\beta_\alpha) = i$ and $\beta_\alpha \alpha \notin I$. The reflection of $Q$ at vertex $i$, denoted $R_i(Q) := (Q'_0, Q'_1, s', t')$, is the quiver with relations $I'$ obtained from $Q$ as follows:

- The vertices and arrows of $Q'$ are the vertices and arrows of $Q$, that is $Q'_0 = Q_0$ and $Q'_1 = Q_1$, only the maps $s$ and $t$ change.
- We define

$$s' \alpha := \begin{cases} i & \text{if } t(\alpha) = i, \\ s(\beta_\alpha) & \text{if } s(\alpha) = i, \\ s(\alpha) & \text{otherwise}, \end{cases}$$

$$t' \alpha := \begin{cases} i & \text{if } \exists \beta \in Q_1 \text{ such that } r(\beta) = i \text{ and } s(\beta) = t(\alpha) \text{ and } \alpha \beta \in I, \\ i & \text{if } \exists \beta \in Q_1 \text{ such that } r(\beta) = i \text{ and } s(\beta) = t(\alpha) \text{ and } \alpha \beta \in I, \\ t(\alpha) & \text{otherwise}. \end{cases}$$

- We define $I' := I_1 \cup I_2 \cup I_3$ where

$$I_1 = \{ \beta \alpha : t(\alpha) = i \text{ and } \exists \gamma \in Q_1 \text{ such that } \gamma \neq \alpha, t(\gamma) = i, \text{ and } \beta \gamma \in I \},$$

$$I_2 = \{ \alpha \beta : r(\beta) \neq i \text{ and } s(\beta) \neq i \},$$

$$I_3 = \{ \beta_\alpha \alpha : s(\alpha) = i \}.$$

Notice that the arrow $\alpha \in Q$ is also denoted $\alpha \in R_i(Q)$, the only difference is the definition of the source and target function. When we define the relations in $R_i(Q)$, we use the composition of arrows in $R_i(Q)$ but use the original functions $s$ and $t$ from $Q$ when selecting which arrows are in a relation. Many examples will be given below.

Definition 28. Dually, we define the co-reflection at $i$. Let $i$ be a vertex of $Q$ such that for each arrow $\alpha \in Q_1$ with $r(\alpha) = i$ there exists $\beta_\alpha \in Q_1$ with $s(\beta_\alpha) = i$ and $\alpha \beta_\alpha \notin I$. The coreflection of $Q$ at vertex $i$, denoted $R^*_i(Q) := (Q'_0, Q'_1, s', t')$, is the quiver with relations $I'$ obtained from $Q$ as follows:

- The vertices and arrows of $Q'$ are the vertices and arrows of $Q$, that is $Q'_0 = Q_0$ and $Q'_1 = Q_1$, only the maps $s$ and $t$ change.
- We define

$$s' \alpha := \begin{cases} t(\alpha) & \text{if } s(\alpha) = i, \\ i & \text{if } \exists \beta \in Q_1 \text{ such that } s(\beta) = i \text{ and } \beta \alpha \in I, \\ s(\alpha) & \text{otherwise}. \end{cases}$$

$$t' \alpha := \begin{cases} i & \text{if } s(\alpha) = i, \\ t(\beta_\alpha) & \text{if } t(\alpha) = i, \\ t(\alpha) & \text{otherwise}. \end{cases}$$

- We define $I' := I_1 \cup I_2 \cup I_3$ where

$$I_1 = \{ \alpha \beta : s(\alpha) = i \text{ and } \exists \gamma \in Q_1 \text{ such that } \gamma \neq \alpha, s(\gamma) = i, \text{ and } \gamma \beta \in I \},$$

$$I_2 = \{ \alpha \beta : s(\beta) \neq i \text{ and } t(\beta) \neq i \},$$

$$I_3 = \{ \alpha \beta_\alpha : t(\alpha) = i \}.$$
The reflection of a quiver gives a Brenner-Butler tilt of the corresponding algebra. Let \((Q, I)\) be the quiver of a gentle algebra \(\Lambda\) and \((Q', I') = R_i(Q)\), then \(kQ'/I' \cong \text{End}(T)\) where

\[
T = \tau^{-1}S_i \oplus \bigoplus_{j \in Q_0 \setminus \{i\}} \Lambda e_j,
\]

with \(S_i\) the simple representation at \(i\).

6.2. Mutations and Reflections. We will show that most reflections can be described in terms of mutations and admissible cuts.

**Theorem 29.** Let \(Q\) be a quiver of a surface algebra given by an admissible cut of an algebra from a triangulated surface with quiver 
\(\tilde{Q}\). If \(i\) is not the source in \(Q\) of a relation and \(R_i\) is defined, then there is an admissible cut of \(\mu_i(\tilde{Q})\) that gives \(R_i(Q)\). Dually, if \(i\) is not the target of a relation and \(R_{-i}\) is defined, then there is an admissible cut of \(\mu_i(\tilde{Q})\) that gives \(R_{-i}(Q)\).

**Remark 30.** The definition of \(\mu_i\) and \(R_i\) are local to the vertex \(i\). Specifically, the construction of \(Q\) from the triangulation of a surface is sufficiently restrictive that the only possible changes between \(Q\) and either \(\mu_i(Q)\) or \(R_i(Q)\) can occur in arrows that start or end within a two vertex neighborhood of \(i\). Hence, in the proof of the proposition it suffices to only consider the local configurations of \(Q\) near \(i\).

**Proof.** We only present those configurations without double arrows. In each configuration, we can retrieve those configurations with double edges by identifying the white vertices. In the very first configuration we may also identify the black vertices, but we may not identify the white and black vertices at the same time.

Because we only consider surface algebras of admissible cuts, there are no overlapping relations in \(Q\). This follows from that the fact that there are no overlapping relations in \(\tilde{Q}\) outside of the 3-cycles which are cut in \(Q\). Hence, there are 10 possible local configurations near \(i\) at which we can reflect and satisfy the assumptions of the theorem. We will provide a dictionary for these 10 configurations. First note that if \(i\) is a sink that is not the end of any relations, then mutation and reflection have the exact same effect on \(Q\). We will not include this case below. Throughout the proof, relations will be indicated by dashed lines.

First, assume that \(i\) is the source of at least one arrow and is not the target of any relation. Because of the restrictions on where we may reflect, we get the following three possibilities.

The corresponding reflections are
The quivers $\tilde{Q}$ corresponding to $Q$ before the reflection are

and mutations at $i$

In each of these cases it is clear that if we cut $\mu_i(\tilde{Q})$ at the arrows $\alpha$ such that $t(\alpha) = i$, we will recover $R_i(Q)$.

We now consider those configurations in which $i$ is the target of a relation. There are five such configurations in which we may reflect at $i$. In these cases we must consider a local picture that is a two vertex neighborhood of $i$. First consider those configurations when $i$ is a sink.

The reflection at $i$ for each configuration is

The quivers $\tilde{Q}$ corresponding to $Q$ before the reflection are

The mutation at $i$ gives
In each of these local configurations, if we cut the arrow(s) $\alpha$ with $s(\alpha) \neq i \neq t(\alpha)$, then we recover $R_i(Q)$.

If $i$ is neither a source nor a sink, and we may reflect at $i$, then we have one of the following local configurations

The reflections at $i$ are

The quivers $\widetilde{Q}$ corresponding to $Q$ before the reflection are

The mutations at $i$ are

In the first case we recover $R_i(Q)$ by cutting the arrow $\alpha$ with $t(\alpha) = i$. Note that this is well-defined because there is only one cycle. In the second case we must cut the two arrows marked $\alpha$ and $\beta$ in the diagram. 

Unfortunately, this type of proof does not really explain what is happening. The connection with mutation becomes more when we translate the above dictionary into the cut surface. Like cluster mutations, we can express reflections as an operation on the edges in the triangulation of $(S, M, T)$. We list a local configuration at a vertex $i$ and corresponding local picture in $(S, M, T)$. The corresponding reflection at $i$ is given to the right. The red lines represent which vertices are cut, the line passing between edges $i$ and $j$ represents either $\chi_{i,j}$ or $\chi_{j,i}$ depending on the orientation of triangle. As in the proof of Theorem 29 we do not include pictures for those configurations with double arrows, those are 'degenerate' cases of the pictures given.
Definition 31. Let \((S, M, T)\) be a triangulated surface and \(\tau \in T\) the diagonal of a rectangle with vertices \(abcd\) such that the endpoints of \(\tau\) are at \(b\) and \(d\). We define \(\tau^\perp\) to be the arc that is the other diagonal of \(abcd\). A clockwise twist of \(\tau\) is an free isotopy \(\Phi: S \times [0, 1] \rightarrow S\) with \(\tau^\perp\) such that the endpoints of \(\Phi(\tau, t)\) are contained in the edges \(bc\) and \(ad\) for each \(t\). Similarly, a counterclockwise twist is given by a free isotopy \(\Phi\) such that the endpoints of \(\Phi(\tau, t)\) are contained in the edges \(ab\) and \(cd\).

![Figure 10](image-url)  
**Figure 10.** The clockwise twist of \(\tau\). The dashed line represents \(\Phi\) at time \(t\).

We can view the twist operation as an operation on the triangulation \((S, M, T)\). The twist at \(\tau\) produces a new triangulation \((S, M, T')\) which differs from \(T\) at only \(\tau\). Depending on the types of edges bounding the rectangle containing \(\tau\), the types of triangles defined by \(T'\) may be different than the types of triangles defined by \(T\). For example, if the vertex \(i\), corresponding to \(\tau_i\) is a sink and the end of a relation, then the rectangle containing \(\tau_i\) has exactly one internal triangle while the rectangle containing the twist of \(\tau_i\), \(\tau_i^\perp\), does not contain an internal triangle, see the dictionary table above.

Using the above dictionary we have the following proposition.

**Proposition 32.** Let \((S, M, T)\) be a triangulated surface and \(\chi\) an admissible cut of \((S, M, T)\). Let \(\tau\) be an arc of \(T\) contained in a rectangle \(abcd\) such that \(\tau\) is not the source of a relation in \(Q_T\). Then the (co-) reflection at \(\tau\) is given by a (counter-) clockwise twist \(\Phi\) which does not pass through any local cut of \(\chi\). Further, if the twist results in at least one internal triangle and

1. if the original cut vertices of \(abcd\) are still contained in internal triangles, the local cuts in the rectangle containing \(\tau\) does not change vertices; or,
2. if the original cut vertex of \(abcd\) is no longer contained in an internal triangle, the new cut is incident to \(\Phi(\tau, 1)\) at the same endpoint of \(\tau\) as the cut incident to \(\Phi(\tau, 0)\).

Otherwise the (co-) refection does not result in any internal triangles, so \(\chi\) has one less local cut. \(\square\)

In most cases the local cuts do not change vertices. A change in the location of a local cut only occurs when the internal triangle it associated with is destroyed by the reflection. The reflection need not create a new internal triangle, but when it does this new internal triangle will have a local cut.

**Corollary 33.** Let \((S, M, T)\) and \((S, M, T')\) be two triangulations of the same unpunctured surface and \(\Lambda\) and \(\Lambda'\) surface algebras corresponding to admissible cuts \(\chi\) and \(\chi'\) respectively. If there is a sequence of reflections of the type described in the above dictionary such that \((\chi, \chi')\) are equi-distributed, then \(\Lambda\) and \(\Lambda'\) are derived equivalent.
Proof. This follows immediately from Theorem 24 and Proposition 32. \qed

6.3. Reflections in a strip. Throughout the remainder of this section we fix $S$ to be an annulus. We use the above dictionary to provide an explicit method to construct a sequence of derived equivalences between surface algebras of $S$. In particular, we re-prove Theorem 16 for the annulus. This proof gives a more explicit construction of the derived equivalence in terms of module categories and tilting than is obtained via the direct application of Theorem 16.

Definition 34. Let $\chi$ be a cut of the triangulation $(S, M, T)$, $B$ be a boundary component of $S$ and $\triangle$ a triangle in $T$. We set $\chi(B)$ to be the number of local cuts in $\chi$ on $B$ and $\chi_\triangle(B)$ the number of local cuts in $\chi$ on $B$ contained in $\triangle$.

Note that while $0 \leq \chi(B) \leq n$, where $n$ is the number of internal triangles, we always have $\chi_B(\triangle)$ is either zero or one.

Lemma 35. Fix a boundary component $B$ and cuts $\chi_1$ and $\chi_2$ such that $\chi_1(B) = \chi_2(B)$. Define $D = \{ \triangle : \chi_1(\triangle) \neq \chi_2(\triangle) \}$. Then $\#D = 2m$ for some $m \in \mathbb{N}$. Further for each triangle $\chi \in D$ there is a corresponding triangle $\chi'$ with $\chi_1(\chi') = \chi_2(\chi)$.

Remark 36. Because of the restriction that $\chi_1(B) = \chi_2(B)$ and that $S$ is the annulus, the set $D$ does not depend on $B$.

Proof. We claim that we can pair up all of the triangles in $D$, that is there is some bijection $D \rightarrow D$ with no fixed points such that $\chi_1(\chi) = \chi_2(\chi)$. Notice that we can write $\chi_1(B) = \sum_\triangle \chi_1(\triangle)$ where we sum over those triangles $\triangle$ incident to $B$, similarly for $\chi_2(B)$. Then we have

$$0 = \chi_1(B) - \chi_2(B) = \sum_\triangle \chi_1(\triangle) - \chi_2(\triangle).$$

Further, we can restrict the sum to only those triangles in $D$ because we clearly get cancellation for those triangles not in $D$.

$$0 = \chi_1(B) - \chi_2(B) = \sum_{\triangle \in D} \chi_1(\triangle) - \chi_2(\triangle).$$

It follows that for each triangle $\chi \in D$ there is a distinct corresponding $\chi'$ with $\chi_1(\chi) = \chi_2(\chi)$ and hence $\#D = 2m$ for some $m \in \mathbb{N}$. \qed

In the subsequent lemmas we assume that the algebras $\Lambda_1$ and $\Lambda_2$ come from admissible cuts $\chi_1$ and $\chi_2$ respectively. We further assume that $\#D = 2$ and $\chi_1(B) = \chi_2(B)$. Set $D = \{ \triangle_1, \triangle_2 \}$. These lemmas will form the base step in the induction argument of Corollary 40. Note that a triangle $\triangle$ is in $D$ if the local cut of $\chi_1$ in $\triangle$ changes boundary components when we consider $\chi_2$. The goal in each lemma is to focus on one triangle $\triangle$ in $D$ and find a sequence of reflections that allows us to swap the the local cut in $\triangle$ from one boundary component to the other.

Lemma 37. If $\triangle_1$ shares an edge with $\triangle_2$, then $\Lambda_1$ is derived equivalent to $\Lambda_2$.

Proof. Let $\tau$ be the edge shared between $\triangle_1$ and $\triangle_2$. The fourth and fifth reflections in the dictionary show us that there are always suitable reflections (or co-reflections) such that both local cuts are incident to $\tau$. Then $R_1R_2$ is a sequence of reflections that send either $(S, M, T)$ to $(S', M', T')$ or vice versa. \qed

Lemma 38. If there is exactly one triangle separating $\triangle_1$ and $\triangle_2$, then $\Lambda_1$ and $\Lambda_2$ are derived equivalent.
Figure 11. The possible arrangements of $\triangle_1$ and $\triangle_2$ in Lemma 38. The red lines represent $\chi_1$, the blue dashed lines $\chi_2$.

Figure 12. The possible arrangements of $\triangle_1$ and $\triangle_2$ in Lemma 39. The red lines represent $\chi_1$, the blue dashed lines $\chi_2$.

**Proof.** In Figure 11 we see the four possible arrangements of $\triangle_1$ and $\triangle_2$. In each case we reduce to Lemma 37 by a reflection at $i$. Specifically, the desired sequence of reflections is $R_i R_j R_i R_j$. Note that in each of these pictures we have assumed that the left most triangle was always cut along the upper boundary component. By flipping each picture along the horizontal axis, we can see each situation with the left most triangle cut in the lower boundary component. In these cases the desired reduction come from the co-reflection at $j$. □

**Lemma 39.** If there are exactly two triangles separating $\triangle_1$ and $\triangle_2$, then $\Lambda_1$ and $\Lambda_2$ are derived equivalent.

**Proof.** We proceed as in Lemma 38. The possible configurations for $\triangle_1$ and $\triangle_2$ are shown in Figure 12. As in Lemma 38, we focus on those cases were the left most triangle is cut in the upper boundary. First note that case (d) reduces to (c) by a reflection at $j$. A reflection at $i$ then reduces (c) to Lemma 37. Let $R^*$ denote the corresponding sequence of reflections from Lemma 37. Then the desired sequence of reflections in case (c) is $R_i R^* R_j$. 


Similarly, (a) and (b) reduce to Lemma 37 by a reflection at \( j \). By reflecting at \( j \) we introduce a new cut triangle connecting \( \Delta_1 \) and \( \Delta_2 \). The cut will be in the lower boundary and upper boundary for case (a) and (b) respectively. We will explicitly discuss the sequence of reflections in case (a), the reflections for case (b) can be found in a similar manner. We may then apply Lemma 37 to this new triangle and \( \Delta_1 \), so as to move the cut in the upper boundary to the lower. We then apply Lemma 37 to the new middle triangle and \( \Delta_2 \), to move the cut in \( \Delta_2 \) to the upper boundary.

Using the above lemmas we get the following special case of Theorem 24.

**Corollary 40.** Let \( S \) be an annulus and \( \Lambda_1 \) and \( \Lambda_2 \) be algebras coming from \( \chi_1, \chi_2 \) respectively. If \( \chi_1(B) = \chi_2(B) \) for both boundary components \( B \) in \( S \), that is \( (\chi_1, \chi_2) \) are equi-distributed, then \( \Lambda_1 \) is derived equivalent to \( \Lambda_2 \).

**Proof.** Let \( D = \{ \triangle : \chi_1(\triangle) \neq \chi_2(\triangle) \} \), we begin by assuming that \( \#D = 2 \), say \( D = \{ \triangle_1, \triangle_2 \} \). In this setup we may even assume that there are no internal triangles separating \( \Delta_1 \) and \( \Delta_2 \), the process we will describe is transitive between internal triangles. We will show, by induction, that there is a sequence of reflections that allow us to swap the cuts in \( \Delta_1 \) and \( \Delta_2 \).

Throughout we will denote cuts as in the dictionary, by red lines bisecting the cut vertex between the endpoints of the resulting relation. Let \( \Delta_1 \) be the triangle containing \( m \) and \( i \). We focus on the different configurations for \( \Delta_1 \), the different cases corresponding to different configurations of \( \Delta_2 \) are hidden and dealt with in the induction step.

The initial case. By Lemmas 37, 38, 39 we can resolve \( \Delta_1 \) and \( \Delta_2 \) when there are 0, 1 or 2 triangles separating them. Now assume that we can resolve \( \Delta_1 \) and \( \Delta_2 \) with up to \( t \) triangles separating them. Let \( R^* \) denote the composition of reflections necessary for the induction hypothesis. Then we have one of the following picture for \( (S,M,T) \):

![Diagram](attachment:image.png)

Note that case (d) reduces to (c) by a reflection at \( j \), hence we only focus on (a), (b), and (c). The desired sequence of reflections is \( R, R^* R, R, R_f, R, R, R^* R_f, R, R^* R \) for (a), (b) and (c) respectively. For example, using the dictionary we get the following sequence of pictures in Figure 13. Note that if the cut incident to \( m \) (resp. \( \ell \) ) had been at the other vertex, a double (co)-reflection at \( m \) (resp. \( \ell \) ), would give us the above pictures.

The proof for \( \#D = 2 \) generalizes for arbitrary \( \#D = 2m \) by applying this proof to pairs \( \Delta_1 \) and \( \Delta_2 \) in \( D \) with a minimal number of triangles separating them, doing so until all pairs have been resolved. □
Figure 13. The sequence of reflections for case (a) of Corollary 40.

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