Studying the scalar bound states of the $K\bar{K}$ system in the Bethe-Salpeter formalism

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Abstract

We study the possible bound states of the $K\bar{K}$ system in the Bethe-Salpeter formalism in the ladder and instantaneous approximations. We find that the bound states exist. However, these bound states have very small decay widths. Therefore, besides the possible $K\bar{K}$ component, there may be some other structures in the observed $f_0(980)$ and $a_0(980)$.

PACS Numbers: 11.10.St, 12.39.Mk, 12.39.Fe, 11.30.Rd

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1 Introduction

Although the dynamics of quarks and gluons at the low energy scale is expected to be relativistic and strongly coupled, the simple non-relativistic quark model successfully describes the properties of most light mesons ($q\bar{q}$) and baryons ($qqq$). However, exception occurs for some scalar particles. Just as stated in Ref. [1], “the features of QCD not (all) contained in the (simple) quark model”. To describe these overpopulated scalar particles, non-$q\bar{q}$ structures have been assigned to them for about three decades since the study in Ref. [2]. They have been regarded as four-quark states [2–5], or molecules composed of conventional particles [6–11] (e.g. $K\bar{K}$ for $f_0(980)$ and/or $a_0(980)$), etc.. Up to now, the puzzle about the nature of these scalar particles still remains unsolved. For example, to describe the recent experimental data [12] and the more accurate measurement by KLOE Collaboration [13], $f_0(980)$ and/or $a_0(980)$ were regarded as four-quark states [5] or molecular binding of $K\bar{K}$ [9] and both of them lead to results consistent with the experiments. Obviously, further investigation on the structure of these scalar particles is necessary.

On the other hand, more and more overpopulated states (especially those containing heavy flavors) have been discovered and confirmed by various experiments [14]. Due to the proximity of these particles’ masses to those of two lowest lying conventional particles (carrying certain heavy flavor(s)), one would naturally identify them as molecules of conventional particles (see, e.g. Refs. [11, 15]). Therefore, it is interesting to study whether this picture about these scalar systems is right or not.

In this paper we will focus on the scalar particles $f_0(980)$ and $a_0(980)$. One purpose of the present paper is to investigate whether the bound states of the $K\bar{K}$ system, interacting by exchanging various vector particles ($\rho$, $\omega$, $\phi$), exist. The other purpose is to discuss the extent to which the $K\bar{K}$ component contributes to the observed particles $f_0(980)$ and $a_0(980)$.

We choose the Bethe-Salpeter (BS) formalism (in the ladder approximation and the instantaneous approximation) as our starting point. The main reason for this is that, in comparison with the potential model (used in e.g. Refs. [10,11]), one can include some relativistic corrections automatically in the BS equation.

One may wonder whether the ladder approximation taken for our pseudo-scalar system in this paper is suitable. In fact, there have been some works in which the legitimacy of the application of the ladder approximation in the BS formalism has been
studied, see e.g. Refs. [16–21]. For example, in Ref. [16] it was shown that including only ladder graphs in the scalar-scalar system can not lead to the correct one-body limit. Furthermore, in the gauge theory, within the ladder approximation gauge invariance can not be maintained. To solve these problems, at least crossed-ladder graphs should be included [16, 17]. More recently, it was shown that the crossed-ladder graphs do contribute large corrections to the ladder approximation in some cases [18, 19]. For large enough coupling, the contribution from the crossed-ladder graphs becomes even more important than that from the ladder ones.

In our case, the square of the effective dimensionless coupling constant (see Eqs. (41)-(46) and (48) in Sect. 3) can be written as $g^2_{KKV}E^2/(4\pi M^2_K)$ which is greater than 3.* From the naïve point of view, for such a large coupling constant, the ladder approximation is not legitimate [19]. However, a closer examination shows that there is a significant difference between our case and the cases discussed in Refs. [18, 19] (see also Refs. [20, 21]), in which the mass of the exchanged particle is very small. On the contrary, the exchanged particles, $\rho$, $\omega$, $\phi$, in our case have large masses compared with the constituent particles $K$ and $\overline{K}$. We will show that the large masses of the exchanged particles suppress significantly the contribution of the crossed-ladder graphs since factors of the form $1/(p^2 - M^2_V)$ from the extra propagators in the crossed-ladder graphs lead to extra suppression (in powers of $1/M^2_V$). Therefore, in our case, the net contribution of the crossed-ladder graphs is in fact very small compared with that of the ladder graphs (more details are given in Sect. 3).

Since the contribution of the crossed-ladder graphs is small in our case the problems associated with the ladder approximation, if existing, will not be serious.

Another approximation we will take is the instantaneous approximation. In this approximation, the energy exchanged between the constituent particles of the binding system is neglected. This is appropriate if the relativistic effects in the system are small. Our calculations (in the ladder approximation and the instantaneous approximation) show that both the iso-scalar and iso-vector bound states of the $K\overline{K}$ system with small binding energy exist. This shows that the binding of the constituent particles is weak, hence the exchange of energy between them can be neglected.

However, regarding these bound states as the observed particles, $f_0(980)$ and $a_0(980)$, 

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* In our case, the binding energy is of order $\mathcal{O}(10^1)$ MeV, the total energy of the binding system is $E \approx 2M_K$, and the coupling $g_{KKV}$ is about 3 when $V = \rho, \omega$ and about $-3\sqrt{2}$ when $V = \phi$. 

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one can find that the decay widths of these bound states are too small to explain the experimental data. In other words, while the bound states of $K\bar{K}$ do contribute to the observed scalar particles, they themselves can not describe the full properties of these particles.

The remainder of this paper is organized as follows. In Sect. 2, we review the BS formalism for the system of two pseudo-scalar particles and discuss the normalization condition of the BS wave function. In Sect. 3, we discuss the bound state equations for the $K\bar{K}$ system in detail. The decays of the $K\bar{K}$ bound state to $\pi\pi$ and $\pi\eta$ final states are discussed in Sect. 4. The numerical results are presented in Sect. 5. The final section is reserved for some discussions and our conclusions.

2 The Bethe-Salpeter formalism

In this section we will review the general formalism of the BS equation and derive the BS equation for the system of two pseudo-scalar particles. We will also derive the normalization condition for the BS wave function. Let us start by defining the BS wave function for the bound state $|P\rangle$ of two pseudo-scalar particles as the following:

$$\chi_P(x_1, x_2) = \langle 0 | T \phi_1(x_1)\phi_2(x_2) | P \rangle = e^{-iP X} \chi_{\rho}(x),$$

(1)

where $\phi_1(x_1)$ and $\phi_2(x_2)$ are the field operators of two pseudo-scalar particles, respectively, $P$ denotes the total momentum of the bound state, and the relative coordinate $x$ and the center of mass coordinate $X$ are defined by

$$X = \eta_1 x_1 + \eta_2 x_2, \quad x = x_1 - x_2,$$

(2)

or inversely,

$$x_1 = X + \eta_2 x, \quad x_2 = X - \eta_1 x,$$

(3)

where $\eta_i = m_i/(m_1+m_2)$, $m_i$ ($i = 1, 2$) is the mass of the $i$-th constituent particle. The equation for the BS wave function can be derived from a four-point Green function,

$$S(x_1, x_2; y_2, y_1) = \langle 0 | T \phi_1(x_1)\phi_2(x_2)(\phi_1(y_1)\phi_2(y_2))^\dagger | 0 \rangle.$$

(4)

To obtain the BS equation, we express the above four-point Green function in terms of the four-point truncated irreducible kernel $\mathcal{K}$,

$$S(x_1, x_2; y_2, y_1) = S_{(0)}(x_1, x_2; y_2, y_1)$$

$$+ \int d^4u_1d^4u_2d^4v_1d^4v_2 S_{(0)}(x_1, x_2; u_2, u_1)\mathcal{K}(u_1, u_2; v_2, v_1)S(v_1, v_2; y_2, y_1),$$

(5)
where $S_{(0)}$ is related to the forward scattering disconnected four-point amplitude,

$$S_{(0)}(x_1, x_2; y_2, y_1) = \Delta_1(x_1, y_1)\Delta_2(x_2, y_2),$$  

(6)

where $\Delta_i(x_i, y_i)$ is the complete propagator of the $i$-th particle,

$$\Delta_i(x, y) = \langle 0 | T \phi_i(x)\phi_i(y)^\dagger | 0 \rangle = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \Delta_i(k, m_i).$$  

(7)

From Eqs. (1) and (5) one can derive the following BS equation for the bound state of two pseudo-scalar particles:

$$\chi_p(x_1, x_2) = \int d^4u_1d^4u_2d^4v_1d^4v_2 S_{(0)}(x_1, x_2; u_2, u_1)K(u_1, u_2; v_2, v_1) \chi_p(v_1, v_2),$$  

(8)

or, by inverting $S_{(0)}$,

$$\int d^4y_1d^4y_2 S_{(0)}^{-1}(x_1, x_2; y_1, y_2) \chi_p(y_1, y_2) = \int d^4v_1d^4v_2 K(x_1, x_2; v_1, v_2) \chi_p(v_1, v_2).$$  

(9)

In this paper, we will investigate the BS equation in momentum space, in which the BS wave function is obtained as (using Eq. (1))

$$\chi_p(p_1, p_2) = \int d^4x_1d^4x_2 e^{ip_1x_1+ip_2x_2} \chi_p(x_1, x_2) = (2\pi)^4 \delta(p_1 + p_2 - P)\chi_p(p),$$  

(10)

where $p = \eta_2p_1 - \eta_1p_2$ is the relative momentum and $\chi_p(p) = \int d^4x e^{ipx} \chi_p(x)$. The Fourier transformation of the four-point Green function in Eq. (11) reads

$$S(x_1, x_2; y_2, y_1) = \int \frac{d^3P d^3P' d^4p d^4p'}{(2\pi)^{16}} e^{-iPX+iP'Y-ipv+ip'y} \tilde{S}(p, p', P, P'),$$  

(11)

with $\tilde{S}(p, p', P, P') = (2\pi)^4 \delta^4(P - P')\tilde{S}_P(p, p')$. Similarly, for the irreducible kernel we have

$$K(x_1, x_2; y_2, y_1) = \int \frac{d^3P d^3P' d^4p d^4p'}{(2\pi)^{16}} e^{-iPX+iP'Y-ipv+ip'y} \tilde{K}(p, p', P, P'),$$  

(12)

with $\tilde{K}(p, p', P, P') = (2\pi)^4 \delta^4(P - P')\tilde{K}_P(p, p')$. The relative momenta and the total momentum of the bound state in the equations are defined by

$$p = \eta_2p_1 - \eta_1p_2, \quad p' = \eta_2p_1' - \eta_1p_2', \quad P = p_1 + p_2 = p_1' + p_2',$$

(13)

or inversely,

$$p_1 = \eta_1P + p, \quad p_2 = \eta_2P - p, \quad p_1' = \eta_1P + p', \quad p_2' = \eta_2P - p'.$$  

(14)
We must note that the constituents of the bound state can not be on-shell, otherwise the bound state is not a really bound state. Consequently, $p_i^2 \neq m_i^2$ (and similar for $p'_i$).

Then, the inhomogeneous equation (15) in momentum space reads

$$
\int \frac{d^4k}{(2\pi)^4} \left[ I_P(p, k) + \overline{K}_P(p, k) \right] \tilde{S}_P(k, p') = (2\pi)^4 \delta(p - p'),
$$

where $I_P(p, k) = -(2\pi)^4 \delta^4(p - k) \Delta_1^{-1}(p_1, m_1) \Delta_2^{-1}(p_2, m_2)$. The BS equation (9) for the bound state in momentum space takes the following form:

$$
\int \frac{d^4k}{(2\pi)^4} \left[ I_P(p, k) + \overline{K}_P(p, k) \right] \chi_P(k) = 0.
$$

This is a homogeneous equation for the BS wave function.

From the BS bound state equation, Eq. (9) in coordinate space or Eq. (16) in momentum space, we can see that the BS wave function satisfies a homogeneous equation. Therefore, its normalization can not be determined from the bound state equation. To obtain the correct normalization of the BS wave function, following Ref. [22], we start by considering the contribution of the bound state. Let us first isolate the contributions from some possible bound states. Consider the case with $\min\{x_1^0, x_2^0\} > \max\{y_1^0, y_2^0\}$ and insert a complete set of states into the four-point Green function, we have

$$
S(x_1, x_2; y_1, y_2) = \sum_P \langle 0 | T \phi_1(x_1) \phi_2(x_2) | P \rangle \langle P | T \phi_2(y_2)^\dagger \phi_1(y_1)^\dagger | 0 \rangle \big|_{\min\{x_1^0, x_2^0\} > \max\{y_1^0, y_2^0\}}
$$

$$
= \int \frac{d^3P}{(2\pi)^3} e^{-iE_P(X^0 - Y^0) + i\mathbf{P} \cdot \mathbf{X} - \mathbf{Y}} \chi_P(x) \overline{\chi}_P(y) \big|_{\min\{x_1^0, x_2^0\} > \max\{y_1^0, y_2^0\}}.
$$

Furthermore, the requirement $\min\{x_1^0, x_2^0\} > \max\{y_1^0, y_2^0\}$ can be described by a theta-function,

$$
\theta \left( X^0 - Y^0 + \frac{\eta_2 - \eta_1}{2} (x^0 - y^0) - \frac{|x^0|}{2} - \frac{|y^0|}{2} \right).
$$

Using this representation and the contour-integral definition of the theta-function, Eq. (17) can be written as

$$
S(x_1, x_2; y_1, y_2) = i \int \frac{d^4P}{(2\pi)^4} e^{i\mathbf{P} \cdot (\mathbf{X} - \mathbf{Y}) - iP_0(X^0 - Y^0)} \chi_P(x) \overline{\chi}_P(y) \frac{1}{P_0 - E_P + i\epsilon}
$$

$$
\times e^{-i(P_0 - E_P) \left[ (\eta_2 - \eta_1)(x^0 - y^0) - |x^0| - |y^0| \right]/2}.
$$

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Hence, near the pole at $P^0 = E_\mathbf{P}$, we have
\[ \tilde{S}_\mathbf{P}(p, p') = \frac{i}{P^0 - E_\mathbf{P} + i\epsilon} \chi_\rho(p)\overline{\chi}_\rho(p') + \text{terms regular at } P^0 = E_\mathbf{P}. \tag{20} \]

Now, define an auxiliary quantity:
\[ Q_\mathbf{P}(p, p') = \int \frac{d^4k}{(2\pi)^4} (P^0 - E_\mathbf{P})\tilde{S}_\mathbf{P}(p, k) \frac{\partial}{\partial P^0} [I_\mathbf{P}(k, p') + \overline{K}_\mathbf{P}(k, p')] . \tag{21} \]

For convenience, we can imagine the arguments of $K_\mathbf{P}$, $I_\mathbf{P}$, $\tilde{S}_\mathbf{P}$, and $Q_\mathbf{P}$ as matrix indices and write the above quantity in a compact form:
\[ Q_\mathbf{P} = (P^0 - E_\mathbf{P})\tilde{S}_\mathbf{P} \frac{\partial}{\partial P^0} [I_\mathbf{P} + \overline{K}_\mathbf{P}] . \tag{22} \]

In terms of this notation we can also rewrite Eqs. (15), (16), and (20) as
\[ \tilde{S}_\mathbf{P}[I_\mathbf{P} + \overline{K}_\mathbf{P}] = 1, \tag{23} \]
\[ [I_\mathbf{P} + \overline{K}_\mathbf{P}] \chi_\rho = 0, \quad (P^0 = E_\mathbf{P}), \tag{24} \]
\[ \lim_{P^0 \to E_\mathbf{P}} (P^0 - E_\mathbf{P})\tilde{S}_\mathbf{P} = i \chi_\rho \overline{\chi}_\rho. \tag{25} \]

Using the above equations and operating $Q_\mathbf{P}$ upon $\chi_\rho$ we have the normalization condition for the BS wave function,
\[ i \int \frac{d^4p \, d^4p'}{(2\pi)^8} \chi_\rho(p) \frac{\partial}{\partial P^0} [I_\mathbf{P}(p, p') + \overline{K}_\mathbf{P}(p, p')] \chi_\rho(p') = 1, \quad P^0 = E_\mathbf{P}. \tag{26} \]

The BS equation (16) is very complex. Without approximation we can not even write down the irreducible kernel and the propagators of particles explicitly. Since the binding of the $K\overline{K}$ system is weak we use the so-called instantaneous approximation:
\[ \overline{K}_\mathbf{P}(p, p') = \overline{K}_\mathbf{P}(\mathbf{p}, \mathbf{p}') . \]
Furthermore, the propagator is set to have the form of the free one. Then, the BS equation (16) becomes
\[ - (p_1^2 - m_1^2)(p_2^2 - m_2^2)\chi_\rho(p) = \int \frac{d^4p'}{(2\pi)^4}\overline{K}_\mathbf{P}(\mathbf{p}, \mathbf{p}')\chi_\rho(p') . \tag{27} \]

Now, we divide Eq. (27) by the two propagators on both sides and then perform the integration over $p^0$ and $p'^0$. Then we have
\[ \frac{E^2}{(E_1 + E_2)^2} - \frac{(E_1 + E_2)^2}{E_1 E_2} \chi_\rho(p) = \frac{i}{2} \int \frac{d^3p'}{(2\pi)^3} \overline{K}_\mathbf{P}(\mathbf{p}, \mathbf{p}')\chi_\rho(p'), \tag{28} \]
where $E_i = \sqrt{\mathbf{p}^2 + m_i^2}$, $E = P^0$, and the equal-time wave function is defined as
\[ \overline{\chi}_\rho(p) = \int dp^0 \chi_\rho(p). \tag{29} \]
While deriving Eq. (28) we have used the following result (in the rest frame of the bound state):

\[
\int_{-\infty}^{\infty} \frac{dp^0}{(p_1^2 - m_1^2 + i\epsilon)(p_2^2 - m_2^2 + i\epsilon)} = -i\pi \frac{(E_1 + E_2)/E_1E_2}{E^2 - (E_1 + E_2)^2},
\]

which can be obtained by choosing a proper contour. For convenience we define the following potential:

\[
V(p, p') = \frac{i}{E_1E_2(E_1 + E_2)} \overline{K}_P(p, p').
\]

Then, the BS bound state equation can be written as

\[
\left[ \frac{E^2}{(E_1 + E_2)^2} - 1 \right] \tilde{\chi}_\rho(p) = \frac{1}{2} \int \frac{d^3p'}{(2\pi)^3} V(p, p') \tilde{\chi}_\rho(p').
\]

Eq. (32) will be the starting point in our later numerical calculations.

For later convenience we also write out \(\chi_\rho(p)\) in terms of \(\tilde{\chi}_\rho(p)\). From Eqs. (27) and (28) we have

\[
\chi_\rho(p^0, p) = \frac{1}{(p_1^2 - m_1^2 + i\epsilon)(p_2^2 - m_2^2 + i\epsilon)} \int \frac{d^3p'}{(2\pi)^4} \overline{K}_P(p, p') \tilde{\chi}_\rho(p'),
\]

where \(p_1^2 - m_1^2 = (\eta_1 E + p^0)^2 - E_1^2, p_2^2 - m_2^2 = (\eta_2 E - p^0)^2 - E_2^2\).

### 3 The bound state(s) of the \(K\overline{K}\) system

In this section, we will study the possible bound state of the \(K\overline{K}\) system. The lowest lying particles with the strangeness numbers \(\pm 1\) form two isospin doublets: \((K^+, K^0)^T\) and \((-\overline{K}^0, K^-)^T\), where the superscript T denotes transpose. One can gather them into two fields, \(K_1\) and \(K_2\), which have the following expansion in momentum space

\[
K_1 = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_{K^+} e^{-ipx} + a_{K^-}^\dagger e^{ipx}),
\]

\[
K_2 = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_{K^0} e^{-ipx} + a_{K^0}^\dagger e^{ipx}).
\]

The conventions about the isospin multiplets used here and in the following are the same as those used in e.g. Ref. [23].
where \( E_p = \sqrt{p^2 + m_K^2} \) is the energy of the particles and we omit the effects of isospin violation so that the masses of the kaons are the same. These two fields can be grouped into an isospin doublet \( K = (K_1, K_2)^T \), which furnishes the fundamental representation of the isospin group \( SU(2)_f \).

The \( K\overline{K} \) system has isospin 1 or 0. The iso-scalar bound state can be written as

\[
|P\rangle_0 = \frac{1}{\sqrt{2}} |K^+ K^- + K^0 \overline{K}^0\rangle ,
\]

and the three components of the iso-vector states are

\[
|P\rangle_{1,0} = \frac{1}{\sqrt{2}} |K^+ K^- - K^0 \overline{K}^0\rangle , \quad |P\rangle_{1,1} = - |K^+ \overline{K}^0\rangle , \quad |P\rangle_{1,-1} = |K^- K^0\rangle .
\]

Let us now project the bound states on the field operators \( K_1 \) and \( K_2 \). From Eqs. (34) and (35) we have

\[
\langle 0 | T \{ K_i(x_1) K_j(x_2) \dagger \} |P\rangle_{I,I_3} = C_{ij}^{(I,I_3)} \chi_P(I)(x_1,x_2) ,
\]

where \( \chi_P(I) \) is the common BS wave function for the bound state with isospin \( I \) which depends only on the state \( |P\rangle_{I,I_3} \) (as will be shown later, the BS wave function only depends on \( I \) but not on \( I_3 \)) but not on the concrete field contents. The isospin coefficients \( C_{ij}^{(I,I_3)} \) for the iso-scalar state are

\[
C_{(0,0)}^{11} = C_{(0,0)}^{22} = 1/\sqrt{2} , \quad \text{else} = 0 ,
\]

and for the iso-vector state we have

\[
C_{(1,0)}^{11} = - C_{(1,0)}^{22} = 1/\sqrt{2} , \quad C_{(1,1)}^{12} = -1 , \quad C_{(1,-1)}^{21} = 1 , \quad \text{else} = 0 .
\]

Now consider the kernel. The BS equation (27) for the bound state can be written down schematically,

\[
\Delta_1^{-1} \Delta_2^{-1} C_{ij}^{(I)} \chi_P(I) = K_P^{ij,kl} C_{ik}^{(I)} \chi_P(I) ,
\]

where \( \Delta_1,2 \) are the propagators of the constituent particles. Then, from Eq. (39), for the iso-scalar case, we have (take \( ij = 11 \) as an example)

\[
\Delta_1^{-1} \Delta_2^{-1} \chi_P^{(0)} = (K_P^{11,11} + K_P^{11,22}) \chi_P^{(0)} .
\]

Similarly, for the iso-vector case, taking the \( I_3 = 0 \) component as an example, we have

\[
\Delta_1^{-1} \Delta_2^{-1} \chi_P^{(1,0)} = (K_P^{11,11} - K_P^{11,22}) \chi_P^{(1,0)} .
\]
From the above equations, we can see that if the iso-scalar bound state exists one cannot ensure the existence of the iso-vector bound state, and vice versa.

The interactions among the kaons and vector particles, $\rho$, $\omega$, $\phi$, at the level of hadrons are described by the $SU(3)_V \times SU(3)_A$ chiral dynamics. The relevant interaction vertices are (see e.g. Ref. [11])

\[
\mathcal{L}_{KK\rho} = ig_{KK\rho} K^+(\vec{\tau} \cdot \vec{\rho}) K + c.c.,
\]

\[
\mathcal{L}_{KK\omega} = ig_{KK\omega} K^+(\partial_\mu K) \omega^\mu + c.c.,
\]

\[
\mathcal{L}_{KK\phi} = ig_{KK\phi} K^+(\partial_\mu K) \phi^\mu + c.c.,
\]

where c.c. is the complex conjugate of the first term and $g_{KKV}$ ($V$ can be $\rho$, $\omega$, and $\phi$) are the coupling constants which can be related to $g_{\rho\pi\pi}$ in the $SU(3)_f$ limit,

\[
g_{KK\rho} = g_{\rho\pi\pi}/2, \quad g_{KK\omega} = g_{\rho\pi\pi}/2, \quad g_{KK\phi} = -g_{\rho\pi\pi}/\sqrt{2}.
\]

The $\rho\pi\pi$ coupling is determined by $g_{\rho\pi\pi} = M_\rho/\sqrt{2}f_\pi \approx 6$ [24], where $M_\rho$ is the mass of $\rho$ and $f_\pi$ is the decay constant of the pion.

From the above observations, at the tree level, in $t$-channel we have the following kernel for the BS equation in the so-called ladder approximation:

\[
\mathcal{K}(p_1, p_2; p'_2, p'_1; M_V) = -i(2\pi)^4 \delta^4(p'_1 - p_1 + p'_2 - p_2)
\times c_f g_{KKV}^2 \frac{(p_1 + p'_1) \cdot (p_2 + p'_2) + (p_1^2 - p'_1^2)(p_2^2 - p'_2^2)/M_V^2}{(p_1 - p'_1)^2 - M_V^2},
\]

where $c_f$ is the isospin coefficient: $c_0 = 3, 1, 1$ and $c_1 = -1, 1, 1$ for $\rho, \omega, \phi$, respectively. These results are consistent with those in Ref. [7]. In Eq. (48) we have used the following propagator for a massive vector meson:

\[
\Delta_{\mu\nu}(p, M_V) = \frac{-i}{p^2 - M_V^2} (g_{\mu\nu} - p_\mu p_\nu/M_V^2).
\]

From the above analysis, we can see that the BS wave function depends only on the isospin $I$ but not on its component $I_3$. This is because we have only considered strong interactions which preserve the isospin symmetry. Therefore, we will omit the $I_3$ label and write $\chi_P^{(I,I_3)}$ simply as $\chi_P^{(I)}$ from now on.

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\footnote{When the exchanging meson is $\rho$, we have \( \mathcal{K}_{\rho}^{11,22}(\rho) = 2\mathcal{K}_{\rho}^{11,11}(\rho) \) and \( \mathcal{K}_{\rho}^{12,12}(\rho) = -\mathcal{K}_{\rho}^{11,11}(\rho) \); when the exchanging meson is $\omega$, we have \( \mathcal{K}_{\rho}^{11,22}(\omega) = 0 \) and \( \mathcal{K}_{\rho}^{12,12}(\omega) = \mathcal{K}_{\rho}^{11,11}(\omega) \); the case for $\phi$ is the same as that for $\omega$.}
In the following discussion and calculation in this paper, we will take the kernel (48) as our starting point (this is the so-called ladder approximation) and will not consider those non-ladder (e.g. crossed) graphs. Before proceeding, let us discuss briefly the contribution of the crossed-ladder graphs. Our aim is to estimate the ratio of their contribution to that of the ladder graph. To make the calculation of the (4-th order) crossed graph tractable, the following simplification are taken: terms carrying $p^2, p'^2, p \cdot p'$, etc. are omitted in the calculation. Since we will stay in the rest frame of the binding system, we also set $p \cdot P$ and $p' \cdot P$ to zero (the instantaneous approximation). These approximations will be appropriate if the half width of the BS wave function is small enough compared with the masses $M_V$ and $M_K$. From the numerical results (in Sect. 5), we can see that the half width of the BS wave function is about 0.1 GeV, which is indeed very small compared with the masses $M_V$ and $M_K$. The physical picture for this approximation is that the configurations with small momenta are dominant in the model.

Taking these approximations, we calculate explicitly the contribution from the (4-th order) crossed graph. We take the case where the exchanged particle is $\rho$ as an example, other two cases with $\omega$ and $\phi$ as exchanged particles give similar results. We work with the dimensional regularization method and the minimal subtraction scheme while calculating the crossed graph. The numerical results show that the ratio of the contribution from the crossed graph to that from the ladder (2nd order) graph has the following form:

$$0.12 - 0.01 \ln \frac{\mu^2}{1\text{GeV}^2},$$

where $\mu$ is the renormalization scale. Since only one (but not all) higher order graph is calculated, the result depends on this renormalization scale. It is natural to take $\mu$ to be around 1 GeV, which is the scale of chiral symmetry breaking. From this result, we can see that the ratio of the contribution from the crossed graph to that from the ladder one is less than 15% (in the case where $\omega$ is the exchanged particle the result is almost the same while in the case where $\phi$ is the exchanged particle, we have the ratio $\approx 25\%$).

For comparison, let us reduce manually the masses of the exchanged particles to $0.15M_K$, which is the case discussed in Refs. [18, 19]. We find that this ratio will rise to 300% or even more: $3.84 + 3.38 \ln \frac{\mu^2}{1\text{GeV}^2}$. This result is consistent with those shown in Refs [18, 19].
Now, let us consider only the ladder approximation and proceed by taking the instantaneous approximation, \( p^0 = 0 \) and \( p'^0 = 0 \) in the kernel \([48]\), and stay in the center-of-mass frame of the bound state, \( P = 0 \). Then, the potential in Eq. \((31)\) due to the exchange of a vector meson \( V \) becomes (using Eq. \((14)\))

\[
V^{(I)}(p, p'; M_V) = c_I U(p, p'; M_V)
= c_I \frac{-g_{KVV}^2}{E_1 E_2 (E_1 + E_2)} \frac{(p + p')^2 + 4\eta_1 \eta_2 E_1 + (p^2 - p'^2) / M_V^2}{(p - p')^2 + M_V^2}. \tag{51}
\]

In order to describe the phenomena in the real world, we should include a form factor at each interacting vertex of hadrons to include the finite-size effects of these hadrons. For the meson \((q \bar{q})\) case, the form factor is assumed to take the following form \([7]\):

\[
F(k) = \frac{2\Lambda^2 - M_V^2}{2\Lambda^2 + k^2}, \quad k = p - p', \tag{52}
\]

where \( \Lambda \) is a cutoff parameter which will be adjusted to give the solution of the BS equation. At the lowest order, the BS equation includes \( F^2 \) in its kernel, i.e. \( V \rightarrow V \cdot F^2 \).

The most important term in the numerator of Eq. \((51)\) is \( 4\eta_1 \eta_2 E_2 \). Other terms are small since the momenta of the constituent particles of the binding system are small. After transforming into the form in coordinate space, similar to the case in Ref. \([11]\), one can see that the potential is in fact a Yukawa-like potential (the sum of a Yukawa potential and several derivatives of the Yukawa potential).

Then, for the bound state of the \( K\bar{K} \) system, the BS equation \((32)\) becomes

\[
\left[ \frac{E^2}{(E_1 + E_2)^2} - 1 \right] \tilde{\chi}_\rho^{(I)}(|p|) = \frac{1}{2} \int \frac{d^3p'}{(2\pi)^3} V^{(I)}_{\text{eff}}(p, p') F(k)^2 \tilde{\chi}_\rho^{(I)}(|p'|), \tag{53}
\]

where the effective potential is (depending on isospin \( I \))

\[
V^{(0)}_{\text{eff}}(p, p') = 3U(p, p'; M_\rho) + U(p, p'; M_\omega) + U(p, p'; M_\phi),
\]

\[
V^{(1)}_{\text{eff}}(p, p') = -U(p, p'; M_\rho) + U(p, p'; M_\omega) + U(p, p'; M_\phi). \tag{54}
\]

If we are interested in the ground state of the BS equation, the corresponding BS wave function is in fact rotational invariant, i.e. \( \tilde{\chi}_\rho(p) \) depends only on the norm of the three momentum, \( |p| \). Therefore, after completing the azimuthal integration, the above BS equation becomes a one-dimensional-integral equation, which reads

\[
\tilde{\chi}_\rho^{(I)}(|p|) = \int d|p'| V^{(I)}_{\text{ld}}(|p|, |p'|) \tilde{\chi}_\rho^{(I)}(|p'|), \tag{56}
\]

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where \( V^{(I)}_{1d}(|p|, |p'|) \) are one-dimensional effective potentials: \( V^{(0)}_{1d}(|p|, |p'|) = 3U_{1d}(M_\rho) + U_{1d}(M_\omega) + U_{1d}(M_\phi) \), and \( V^{(I)}_{1d}(|p|, |p'|) = -U_{1d}(M_\rho) + U_{1d}(M_\omega) + U_{1d}(M_\phi) \) with

\[
U_{1d}(M_\nu) = -\frac{g_{KK}^2}{4(2\pi)^2} \frac{E_1 + E_2}{E_1E_2[E^2 - (E_1 + E_2)^2]} \left| \frac{p'}{p} \right| (V_1 + V_2 + V_3),
\]

where

\[
V_1 = -4|p||p'| (2\Lambda^2 - M_\nu^2) \frac{2\Lambda^2 + 2(2\eta_1\eta_2 E^2 + |p|^2 + |p'|^2) + (|p|^2 - |p'|^2)/M_\nu^2}{2\Lambda^2 + (|p| + |p'|)^2} - \frac{2\Lambda^2 + (|p| + |p'|)^2}{2\Lambda^2 + (|p| - |p'|)^2},
\]

\[
V_2 = \left[ M_\nu^2 + 2(2\eta_1\eta_2 E^2 + |p|^2 + |p'|^2) + (|p|^2 - |p'|^2)/M_\nu^2 \right] \ln \frac{M_\nu^2 + (|p| + |p'|)^2}{M_\nu^2 + (|p| - |p'|)^2},
\]

\[
V_3 = -\left[ M_\nu^2 + 2(2\eta_1\eta_2 E^2 + |p|^2 + |p'|^2) + (|p|^2 - |p'|^2)/M_\nu^2 \right] \ln \frac{2\Lambda^2 + (|p| + |p'|)^2}{2\Lambda^2 + (|p| - |p'|)^2}.
\]

### 4 The decay width of the \( K\bar{K} \) system

To find out the bound states of the \( K\bar{K} \) system, one only needs to solve the homogeneous BS equation. However, when we want to calculate physical quantities such as the decay width we have to face the problem of the normalization of the BS wave function. In the following we will discuss the normalization of the BS wave function \( \tilde{\chi}_{p}(|p|) \).

Substituting the relation between \( \chi_{p}(p) \) and \( \tilde{\chi}_{p}(|p|) \), Eq. (33), and Eqs. (31) (51) into the normalization equation (26) one arrives at the following normalization equation for \( \tilde{\chi}_{p}(|p|) \) (after carrying out some \( p_0 \)-integrations with proper contours):

\[
-\frac{1}{4\pi^4} \int \frac{d^3p}{(2\pi)^3} \tilde{\chi}^{(I)}_{p}(|p|)^2 R - \frac{2\eta_1\eta_2}{\pi^2} \int \frac{d^3p d^3p'}{(2\pi)^6} \tilde{\chi}^{(I)}_{p}(|p|) \tilde{\chi}^{(I)}_{p'}(|p'|) F^2 H^{(I)} = 1, \tag{57}
\]

where \( F(= \frac{2\Lambda^2 - M_\nu^2}{2\Lambda^2 + (p - p')^2}) \) is the form factor, \( H^{(0)} = 3H(M_\rho) + H(M_\omega) + H(M_\phi) \) and

\[
H^{(I)} = -H(M_\rho) + H(M_\omega) + H(M_\phi) \text{ with } H(M_\nu) = \frac{g_{KK}^2}{(p - p')^2 + M_\nu^2},
\]

and

\[
R = -E \left[ -2E^2(E_1^2 - E_2^2)(E_1\eta_1 - E_2\eta_2) + E^4(E_1\eta_1 + E_2\eta_2)
+ (E_1^2 - E_2^2)(E_1^3\eta_1 + 3E_1E_2^2\eta_1 - 3E_1^2E_2\eta_2 - E_2^3\eta_2) \right]
\times \left\{ 2E_1E_2 \left[ E^4 + (E_1^2 - E_2^2)^2 - 2E^2(E_1^2 + E_2^2) \right] \right\}^{-1}
\]

After completing the azimuthal integration in Eq. (57) we have

\[
-\frac{1}{2\pi^4} \int d|p||p|^2 \tilde{\chi}^{(I)}_{p}(|p|)^2 R
- \frac{\eta_1\eta_2 E}{8\pi^6} \int d|p| d|p'| |p||p'| \tilde{\chi}^{(I)}_{p}(|p|) \tilde{\chi}^{(I)}_{p'}(|p'|) T^{(I)} = 1, \tag{58}
\]

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where \( T^{(0)} = 3T(M_\rho) + T(M_\omega) + T(M_\phi) \) and \( T^{(1)} = -T(M_\rho) + T(M_\omega) + T(M_\phi) \) with

\[
T(M_V) = g_{KKV}^2 \left[ \frac{2\Lambda^2 - M_V^2}{2\Lambda^2 + (|p| + |p'|)^2} - \frac{2\Lambda^2 - M_V^2}{2\Lambda^2 + (|p| - |p'|)^2} \right. \\
\left. + \ln \frac{2\Lambda^2 + (|p| - |p'|)^2}{2\Lambda^2 + (|p| + |p'|)^2} - \ln \frac{M_V^2 + (|p| - |p'|)^2}{M_V^2 + (|p| + |p'|)^2} \right]
\]

for each vector meson with mass \( M_V \) and coupling \( g_{KKV} \).

If the wave function obtained in the previous section (which will be calculated numerically in the following section) does not satisfy this normalization equation but gives some constant \( c^2 \neq 1 \) for the expression on the left hand side of Eq. (58), one needs only make the replacement \( \tilde{\chi}_\rho(|p|) \to \tilde{\chi}_\rho(|p|)/|c| \) to ensure the correct normalization of the BS wave functions.

If the molecular binding is dominant in the \( K\bar{K} \) system, then the possible bound states of the \( K\bar{K} \) system are most likely related to the two particles which are denoted by \( f_0(980) \) and \( a_0(980) \) in the review of PDG [26], since they are just below the threshold of the free \( K\bar{K} \) system and, up to now, can not be assigned with the common \( q\bar{q} \) structure. One possibility is that they are (mainly) molecular states of other conventional particles, e.g. \( K \) and \( \bar{K} \). However, to identify the possible molecules of \( K\bar{K} \) with these scalar particles, we should also identify other properties (other than the binding energy) of the molecules with those of the scalar particles measured by experiments. Among these properties, an important one is the decay width of the bound state.

Now, we will proceed to study the decay widths of the \( K\bar{K} \) bound states and compare them with those of \( f_0(980) \) and \( a_0(980) \) and see whether the assignment of the molecular states with them is suitable. Since the dominant decay channels of \( f_0(980) \) and \( a_0(980) \) are \( \pi\pi \) and \( \eta\pi \), respectively, we will study the decay widths of the above bound states into \( \pi\pi \) and \( \eta\pi \). The relevant interaction vertices are (see e.g. Ref. [11])

\[
\mathcal{L}_{\pi K K^*} = ig_{sKK^*} \left[ \partial_\mu K^\dagger (\vec{\tau} \cdot \vec{\pi}) K^{*\mu} - K^\dagger (\vec{\tau} \cdot \partial_\mu \vec{\pi}) K^{*\mu} \right] + \text{c.c. ,} \quad (59)
\]

\[
\mathcal{L}_{\eta K K^*} = ig_{\eta KK^*} \left[ \partial_\mu K^\dagger K^{*\mu} \eta - K^\dagger K^{*\mu} \partial_\mu \eta \right] + \text{c.c. ,} \quad (60)
\]

where c.c. denotes the complex conjugate of the previous terms. The coupling constants are related to \( g_{\rho\pi\pi} \) in the following way:

\[
g_{sKK^*} = g_{\rho\pi\pi}/2 \, , \quad g_{\eta KK^*} = -\sqrt{3}g_{\rho\pi\pi}/2 \, . \quad (61)
\]
The differential decay width of the bound state can be written as [26]

\[ d\Gamma = \frac{1}{32\pi^2} |\mathcal{M}|^2 \frac{|q|}{E^2} d\Omega, \]

(62)

where \( |q| \) is the norm of the three-momentum of the particles in the final state in the rest frame of the bound state. \( \mathcal{M} \) is the Lorentz invariant decay amplitude of the process. The lowest order decay amplitude can be written as (the decay to \( \pi\eta \) will be considered later on)

\[
\langle \pi^a(q_1)\pi^b(q_2)|\frac{i^2}{2i} \int d^4x d^4y T \{ \mathcal{L}_{\pi KK^*}(x)\mathcal{L}_{\pi KK^*}(y) \}|P\rangle = \frac{g_{\pi KK^*}^2}{\sqrt{2E_{\pi}^a 2E_{\pi}^b}} \int \frac{d^4p}{(2\pi)^4} \Delta_{\mu\nu}(k, M_{K^*})i^2(k + 2q_2)^\mu(k - 2q_1)^\nu F(|k|)^2 \\
\times \left\{ (\tau^a \tau^b)_{ij} \bigg|_{k = q + p} + (\tau^b \tau^a)_{ij} \bigg|_{k = q - p} \right\} C_{ij}^{(I)} \chi_p^{(I)}(p) \\
\times (2\pi)^4 \delta^4(P - q_1 - q_2),
\]

(63)

where \( q_i \ (i = 1, 2) \) is the momentum of the \( i \)-th particle in the final state and \( E_{\pi}^a = \sqrt{q_1^2 + m_{\pi}^2} \), \( E_{\pi}^b = \sqrt{q_2^2 + m_{\pi}^2} \). The coefficients \( C_{ij}^{(I)} \) in Eq. (63) are the isospin factors which have been given in Eqs. (39) and (40), \( q \equiv \eta_2 q_1 - \eta_1 q_2 \) which is not the relative momentum of particles in the final state (note that \( \eta_1 \) and \( \eta_2 \) are defined as \( \eta_i = m_i/(m_1 + m_2) \), and \( m_1 \) and \( m_2 \) are the masses of the component particles of the bound states but not of the final states). In deriving the above equation the following propagator for the vector kaons has been used:

\[
\langle 0|T \{ K_{i}^{* \mu}(x) K_{j}^{\nu}(y) \}|0\rangle = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \Delta_{\mu\nu}(k, M_{K^*}) \delta_{ij},
\]

(64)

where \( i \) and \( j \) are isospin indices.

The Lorentz-invariant decay amplitude of the \( K\bar{K} \) bound state to \( \pi\pi \) is then

\[
\mathcal{M}_{(I)}^{(\pi\pi)} = iC_{(I)}^{(\pi\pi)} g_{\pi KK^*}^2 \sqrt{2E} \int \frac{d^4p}{(2\pi)^4} \left\{ \pm F(|k|)^2 \Delta_{\mu\nu}(k, M_{K^*})i^2(k + 2q_2)^\mu(k - 2q_1)^\nu \right|_{k = q - p} \\
+ (k \rightarrow q + p + (\eta_1 - \eta_2)P) \right\} \chi_p^{(I)}(p),
\]

(65)

where “+” and “−” in “±” are for the iso-scalar and iso-vector channels, respectively. For the iso-scalar channel, from Eq. (39) we have \( (\tau^a \tau^b)_{ij} C_{(0)}^{ij} = \frac{1}{\sqrt{2}} \text{Tr}(\tau^a \tau^b) = \sqrt{2}\delta^{ab} \). Since the iso-scalar \( \pi\pi \) final states reads \footnote{For the states in isospin multiplets, we use the same conventions as those in Ref. [23]. That is to say, we have \( |\pi, \pm 1\rangle = \mp|\pi^\pm\rangle = \mp\frac{1}{\sqrt{2}}|\pi^1 \pm i\pi^2\rangle \) and \( |\pi, 0\rangle = |\pi^3\rangle \).}

\[
|\pi\pi\rangle_{(0, 0)} = -\frac{1}{\sqrt{3}} \left| \pi^+ \pi^- + \pi^- \pi^+ + \pi^0 \pi^0 \right\rangle = -\frac{1}{\sqrt{3}} \left| \pi^1 \pi^1 + \pi^2 \pi^2 + \pi^3 \pi^3 \right\rangle,
\]

(66)
we have \( c^{(\pi \pi)}_{(0)} = -\sqrt{6} \). For the iso-vector channel, using Eq. (40) we have \( c^{(\pi \pi)}_{(1)} = 2 \) since the iso-vector \( \pi \pi \) final state is

\[
|\pi \pi\rangle_{(1,\pm 1)} = -\frac{1}{\sqrt{2}} \left| \pi^\pm \pi^0 - \pi^0 \pi^\pm \right\rangle = \frac{1}{2} \left| \pi^0 \pi^1 - \pi^1 \pi^0 \mp i\pi^0 \pi^2 \mp i\pi^2 \pi^0 \right\rangle ,
\]

\[
|\pi \pi\rangle_{(1,0)} = -\frac{1}{\sqrt{2}} \left| \pi^+ \pi^- - \pi^- \pi^+ \right\rangle = \frac{i}{\sqrt{2}} \left| \pi^1 \pi^2 - \pi^2 \pi^1 \right\rangle .
\]

Note that this isospin coefficient is independent of the component \( I_3 \) (notice that our convention for the kaon state is different from that in Refs. \([7, 10]\)). This iso-vector final state is anti-symmetric, so there is in fact no S-wave \( \pi \pi \) final state with \( I = 1 \).

Now, let us turn to \( \pi \eta \) final state. The lowest order matrix element for the decay of the \( K\overline{K} \) system into \( \pi \eta \) is

\[
\langle \pi^a(q_1) \eta(q_2) | \mathcal{L}_{\pi KK^*}(x) \mathcal{L}_{\eta KK^*}(y) \rangle | P \rangle
\]

\[
= \frac{g_{sKK^*} g_{sKK^*}}{\sqrt{2E_{\pi^a} E_{\eta}}^2} \langle \pi^a_i q_j | \mathcal{L}_{\eta KK^*} \rangle \frac{d^4 p}{(2\pi)^4} \left\{ F(|k|^2) \Delta_{\mu \nu}(k, M_{K^*}) i^2(k + 2q_2) \mu^2(k - 2q_1) \nu \right\}_{k=q-p} + (k \rightarrow q + p + (\eta_1 - \eta_2) P) \chi_{\rho}(p) (2\pi)^4 \delta^4(P - q_1 - q_2) .
\]

Then, the Lorentz-invariant decay amplitude is (only the iso-vector channel contributes)

\[
\mathcal{M}^{(\pi \eta)}_{(1)} = i c^{(\pi \eta)}_{(1)} g_{sKK^*} g_{sKK^*} \sqrt{2E} \frac{d^4 p}{(2\pi)^4} \left\{ F(|k|^2) \Delta_{\mu \nu}(k, M_{K^*}) i^2(k + 2q_2) \mu^2(k - 2q_1) \nu \right\}_{k=q-p} + (k \rightarrow q + p + (\eta_1 - \eta_2) P) \chi_{\rho}(p) .
\]

The components of \( \pi \eta \) are

\[
|\pi \eta\rangle_{(1,\pm 1)} = \mp |\pi^\pm \eta\rangle = \mp \frac{1}{\sqrt{2}} \left| \pi^1 \pm i\pi^2 \eta \right\rangle ,
\]

\[
|\pi \eta\rangle_{(1,0)} = |\pi^0 \eta\rangle = |\pi^3 \eta\rangle .
\]

Then, from Eq. (40), we have \( c^{(\pi \eta)}_{(1)} = \sqrt{2} \) which is again independent of \( I_3 \).

In the calculation we stay in the rest frame of the bound state and hence \( P = (E, \mathbf{0}) \). In this frame the momenta of the two particles in the final state can be taken as: \( q_1 = (E'_1, \mathbf{q}) \), \( q_2 = (E'_2, -\mathbf{q}) \). Therefore, \( q = \eta_2 q_1 - \eta_1 q_2 = (\eta_2 E'_1 - \eta_1 E'_2, \mathbf{q}) \). When the final state is \( \pi \pi \), \( E'_1 = E_{\pi a} \) and \( E'_2 = E_{\pi b} \) while when the final state is \( \pi \eta \), \( E'_1 = E_{\pi a} \) and \( E'_2 = E_{\eta} \). To calculate the amplitude, we first carry out the azimuthal integration of the spatial part of \( p \), the result having the following structure:

\[
\int \frac{d^4 p}{(2\pi)^4} \left\{ F(|k|^2) \Delta_{\mu \nu}(k, M_{K^*}) i^2(k + 2q_2) \mu^2(k - 2q_1) \nu \chi_{\rho}(p) \right\} (p)
\]

\[
= -i^2 \int_{-\infty}^{\infty} dp^0 \int_{0}^{\infty} d|p| |p|^2 f(p^0) \chi_{\rho}(\pm p^0, |p|) ,
\]

\[15\]
where

\[
f(p^0) = \frac{(M_{K^*}^2 - 2\Lambda^2)^2}{(-2p^0q^0 + |p|^2 + |q|^2 + s_4 + 2\Lambda^2 + i\epsilon)^2}
\]

\[
\times \left\{ 2\frac{(-2p^0q^0 + |p|^2 + |q|^2 + s_4 + 2\Lambda^2)(-2p^0q^0 + |p|^2 + |q|^2 - s_1 + s_2s_3/M_{K^*}^2 + 2\Lambda^2)}{[(|p| - |q|)^2 + 2\Lambda^2] [(|p| + |q|)^2 + 2\Lambda^2]}
\right.
\]

\[
\left. + \frac{s_1 + s_4 - s_2s_3/M_{K^*}^2}{2|p||q|} \ln \left[ \frac{2p^0q^0 + 2|p||q| - s_4}{2p^0q^0 - 2|p||q| - s_4} \cdot \frac{(|p| - |q|)^2 + 2\Lambda^2}{(|p| + |q|)^2 + 2\Lambda^2} \right] \right\}.
\]

Now we will give some explanations about Eq. (71). The results for \( k = q - p \) and \( k = q + p + (\eta_1 - \eta_2)P \) have the same structures. We have changed the sign of the imaginary part of the pole for \( k = q + p + (\eta_1 - \eta_2)P \) by taking the variable transformation \( p^0 \rightarrow -p^0 \). When \( k = q - p \) we will take \( \chi_p(t)(+p^0, |p|) \) in Eq. (71) and \( s_i \) \( i = 1, \ldots, 4 \) are defined by

\[
s_1 = p^2 + q^2 + 4(\eta_1^2 - \eta_1)P^2 + 2(2\eta_1 - 1)(p \cdot P + q \cdot P),
\]

\[
s_2 = p^2 - q^2 + 2\eta_1(p \cdot P - q \cdot P),
\]

\[
s_3 = p^2 - q^2 + 2(\eta_1 - 1)(p \cdot P - q \cdot P),
\]

\[
s_4 = p^2 + q^2 - M_{K^*}^2,
\]

while when \( k = q + p + (\eta_1 - \eta_2)P \) we will take \( \chi_p(t)(-p^0, |p|) \) in Eq. (71) and \( s_i \) \( i = 1, \ldots, 4 \) are defined by

\[
s_1 = p^2 + q^2 - P^2,
\]

\[
s_2 = p^2 - q^2 - (2\eta_1 - 1)P^2 - 2(\eta_1 - 1)p \cdot P - 2\eta_1q \cdot P,
\]

\[
s_3 = p^2 - q^2 + (2\eta_1 - 1)P^2 - 2\eta_1p \cdot P - 2(\eta_1 - 1)q \cdot P,
\]

\[
s_4 = p^2 + q^2 - M_{K^*}^2 + (2\eta_1 - 1)^2P^2 - 2(2\eta_1 - 1)p \cdot P + 2(2\eta_1 - 1)q \cdot P.
\]

Now, we can substitute Eq. (33) into Eq. (71) and complete the \( p^0 \)-integration by choosing proper contours. From the expression of \( f(p^0) \) above we can see that the all the poles come from the BS wave function. This is because the denominator in \( f(p^0) \) (neglecting the isospin violation, then \( \eta_1 = \eta_2 \))

\[-2p^0q^0 + |p|^2 + |q|^2 + s_4 + 2\Lambda^2 = (p^0 - q^0)^2 + 2\Lambda^2 - M_{K^*}^2.
\]

is positive when \( \Lambda > M_{K^*}/\sqrt{2} \), which is satisfied in our case (see the discussion in the next section). The remaining contour integration over \( p^0 \) is straightforward and the
result reads

\[ \int \frac{d^4p}{(2\pi)^4} \left\{ F(|k|)^2 \Delta_{\mu\nu}(k, M_{K^*}) i^2(k + 2q_2)^\mu(k - 2q_1)^\nu \left( \left| \begin{array}{c} \xi = 0 + q_1 \nu \\ q = q + p \end{array} \right| \right) \chi_p^{(I)}(p) \right\} \]

\[ = \frac{i^4}{(2\pi)^3} \int_0^{\infty} d|p||p|^2 \left[ \left| \xi_1 f(p^0) \right|_{p^0 = -\eta_1 E - E_1} + \xi_2 f(p^0) \right|_{p^0 = -\eta_2 E - E_2} \]

\[ \pm \xi_3 f(p^0) \right|_{p^0 = \eta_1 E - E_1} \pm \xi_4 f(p^0) \right|_{p^0 = -\eta_2 E - E_2} \chi_p^{(I)}(|p|) , \]  

where \( \xi_1 = E_2(E - E_1 - E_2)/[(E_1 + E_2)(E + E_1 - E_2)] \), \( \xi_2 = E_1(E + E_1 + E_2)/[(E_1 + E_2)(E + E_1 - E_2)] \), \( \xi_3 = E_2(E + E_1 + E_2)/[(E_1 + E_2)(E - E_1 + E_2)] \), and \( \xi_4 = E_1(E - E_1 - E_2)/[(E_1 + E_2)(E - E_1 + E_2)] \). If \( \eta_1 = \eta_2 \) we have \( \xi_1 = \xi_4 \) and \( \xi_2 = \xi_3 \), then for the iso-vector \( \pi\pi \) final state, the decay width is zero.

Once we have obtained the BS wave function of the ground state \( \chi_p^{(I)}(p) \) (the numerical calculation will be carried out in the next section), we will take the wave function as input to calculate the decay amplitudes in Eqs. (65) and (69).

### 5 Numerical analysis and results

The cutoff \( \Lambda \) in our model is not a free parameter in principle. It contains the information about the non-point interaction due to the structures of hadrons. In Ref. [7], the cutoff for the interaction of \( K\bar{K}\rho \) is taken to be rather large (about 3.18 GeV in our notation). On the other hand, in the study of baryons in the quark-diquark picture, the cutoff in the form factors associated with the diquark-gluon-diquark interaction is taken to be about 1.27 GeV [25]. In this work, we shall treat the cutoff \( \Lambda \) in the form factors as a parameter varying in a much wider range \((0.8, 4.8)\) GeV, in which we will try to search for possible solutions of the \( K\bar{K} \) bound states.

Let us first solve the BS bound state equation (56) numerically. We discretize the integral equation (56) into a matrix eigenvalue equation by the Gaussian quadrature method. For each pair of trial values of the cutoff \( \Lambda \) and the binding energy \( E_b \) of the \( K\bar{K} \) system (which is defined as \( E_b = E - m_1 - m_2 \)), we will obtain all the eigenvalues of this eigenvalue equation. The eigenvalue closest to 1.0 for a pair of \( \Lambda \) and \( E_b \) will be selected out and called “the-trial-eigenvalue”. Fixing a value of the cutoff \( \Lambda \) and varying the binding energy \( E_b \) (from 0 to \(-100\) MeV) we will obtain a series of “the-trial-eigenvalue”s. For some (not all) values of the cutoff, we will find that the
Table 1: For the iso-scalar $K\overline{K}$ system, there are five regions of the cutoff $\Lambda$. In each region, for any value of the cutoff, the series of “the-trial-eigenvalue”’s cross over the exact eigenvalue 1.0 (at certain binding energy $E_b \in [-1, -99]$ MeV).

| $E_b$ (MeV) | $\Lambda$ (GeV) |
|------------|-----------------|
| -1         | 1.1360 2.0793 2.7352 3.5453 4.7633 |
| -99        | 1.2162 2.0979 2.7444 3.5524 4.7697 |

Table 2: The decay widths ($\Gamma_{\pi\pi}^{l=0}$) corresponding to the five cutoff-regions when the binding energy $E_b \approx -20$ MeV.

| $\Lambda$ (GeV) | 1.1700 2.0862 2.7385 3.5479 4.7656 |
| $\Gamma$ (MeV)  | 0.671 4.758 7.516 11.102 11.949 |

corresponding series cross over 1.0 in the range of $E_b \in (0, -100)$ MeV. The task is then to find out all these cutoff values (which are, in fact, some continuous regions).

In searching for the possible solutions in the iso-scalar channel of the $K\overline{K}$ system and its contribution to $f_0(980)$ ($I^G(J^P_C) = 0^+(0^{++})$), we find several regions of the cutoff. The results are listed in Table 1. If we want to identify the iso-scalar bound state of the $K\overline{K}$ system with the observed $f_0(980)$, we should let the binding energy approximately be $-20$ MeV. For the above five regions, this is equivalent to set the cutoff to be

$$\Lambda \approx 1.1700, \ 2.0862, \ 2.7385, \ 3.5479, \ 4.7656 \ \text{GeV},$$

(73)

respectively. From Eqs. (65) and (62), taking the corresponding BS wave function as input, the decay width of the iso-scalar $K\overline{K}$ to $\pi\pi$ can be obtained. The results are listed in Table 2. From the PDG’s review [26] the full width of $f_0(980)$ is $\Gamma = 40$ to 100 MeV ($\pi\pi$ dominant). Therefore, the results in Table 2 show that $f_0(980)$ can not be completely the $K\overline{K}$ iso-scalar bound state.

Now, let us turn to the iso-vector $K\overline{K}$ bound state. We find the following two regions of the cutoff in this case:

$$\Lambda \in (2.1160, 2.2213) \ \text{and} \ (4.4998, 4.5147) \ \text{GeV}.$$  

(74)

That is to say, e.g., from 0.99 to 1.01.
The corresponding decay widths of the $K\bar{K}$ system into $\pi\eta$ with the binding energy $E_b = -20$ MeV are given by

\begin{align}
\Gamma^{(I=1)}_{\pi\eta} &= 1.329 \text{ MeV}, \quad \Lambda = 2.1590 \text{ GeV}, \quad (75) \\
\Gamma^{(J=1)}_{\pi\eta} &= 0.031 \text{ MeV}, \quad \Lambda = 4.5056 \text{ GeV}. \quad (76)
\end{align}

The full width of $a_0(980)$ is $\Gamma = 50$ to 100 MeV ($\pi\eta$ dominant) [26]. Although the $K\bar{K}$ iso-vector bound state does contribute to $a_0(980)$ ($I^G(J^{PC}) = 1^-(0^{++})$), just as in the case of $f_0(980)$, $a_0(980)$ can not be completely the $K\bar{K}$ iso-vector bound state.

6 Conclusions and discussions

In this paper we derive the BS equation for the $K\bar{K}$ system, study the possible bound states of this system, and calculate their decay widths in the BS formalism. In our model, we have used the ladder approximation. This approximation has been questioned and is found not to be a good one in some models where higher order graphs give even more important contribution than the ladder graph [18–21]. However, in our case, we have shown explicitly that crossed-ladder graphs are suppressed greatly comparing with the ladder graphs due to the large masses of the exchanged particles. This makes the ladder approximation be legitimate in our model. In addition, based on the fact that the $K\bar{K}$ system is weakly bound, we have used the instantaneous approximation in the BS equation, in which the energy exchange between the constituent particles is neglected. Since the constituent particles and the exchanged particles in the $K\bar{K}$ system are not point-like, we introduce a form factor including a cutoff $\Lambda$ which reflects the effects of the structure of these particles. Since $\Lambda$ is controlled by non-perturbative QCD and can not be determined at present, we let it vary in a reasonable range within which we try to find possible bound states of the $K\bar{K}$ system.

From the calculating results we find that there exist bound states of the $K\bar{K}$ system. Unfortunately, we can not determine the binding energy uniquely. The binding energy depends on the value of the cutoff $\Lambda$. For the iso-scalar $K\bar{K}$ system, we find five cutoff regions in which the solutions (with the binding energy $E_b \in (0,-100)$ MeV) to the ground state of the BS equation can be found (in unit of GeV):

\[ \Lambda \sim (1.136,1.216), \ (2.079,2.098), \ (2.735,2.744), \ (3.545,3.552), \ (4.763,4.770). \]
From these results, we can see that, except for the first interval, these regions are very narrow. For the iso-vector case, we find two regions (in unit of GeV),

\[ \Lambda \sim (2.1160, 2.2213), \quad (4.4998, 4.5147). \]

How to fix the cutoff (then the binding energy can be predicted), which is equivalent to how to determine the finite size effects of hadrons in the calculation, is beyond the scope of this paper. If we treat the binding energy as an input \( E_b = -20 \text{ MeV} \), we find that the corresponding BS wave function gives too small decay widths, i.e.

\[ \Gamma^{(I=0)}_{\pi\pi} = 0.671, \ 4.758, \ 7.516, \ 11.102, \ 11.949 \ \text{MeV}, \]
corresponding to the five cutoff regions, respectively. For the iso-vector case we have

\[ \Gamma^{(I=1)}_{\pi\eta} = 1.329, \ 0.0305 \ \text{MeV}, \]
corresponding to the two cutoff regions, respectively.

The authors in Ref. [10] concluded that the model with the one-meson-exchange potential from chiral dynamics, which is also used in this work, is sufficient to bind the \( KK \) system into a molecule which has the same mass and decay width as those of the iso-scalar \( f_0(980) \). From our calculation, however, we find that even these \( (KK) \) bound states could contribute to the observed scalar particles, the portion should be small \[ \| \]

We prefer to draw the conclusion that there may be some more important structures besides the \( KK \) molecule in the observed overpopulated scalar particles (e.g. \( f_0(980) \) and \( a_0(980) \)). Obviously, to resolve this problem further investigations are required.

Acknowledgments. One of us (XHW) is grateful to Dr. Wei Zhang for the help on Fortran programing. This work was supported in part by National Natural Science Foundation of China (Project Number 10675022), the Key Project of Chinese Ministry of Education (Project Number 106024) and the Special Grants from Beijing Normal University.

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