Population-imbalance instability in a Bose-Hubbard ladder in the presence of a magnetic flux

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(Dated: April 24, 2015)

We consider a two-leg Bose-Hubbard ladder in the presence of a magnetic flux. We make use of mean-field, Bogoliubov, bosonization, renormalization group approaches to reveal a structure of ground-state phase diagrams in a weak-coupling regime relevant to cold atom experiments. It is found that except for a certain flux $\phi = \pi$, the system shows different properties as changing hoppings, which also leads to a quantum phase transition similar to the ferromagnetic XXZ model. This implies that population-imbalance instability occurs for certain parameter regime. On the other hand, for $\phi = \pi$, it is shown that an umklapp process caused by commensurability of a magnetic flux stabilizes a superfluid with chirality and the system does not experience such a phase transition.

PACS numbers: 67.85.-d, 05.30.Jp

I. INTRODUCTION

Quantum systems subject to high magnetic fields are known to acquire nontrivial characteristics such as the Hofstadter butterfly [1] and quantum Hall effect [2] in two dimensional systems. Recently, the so-called synthetic gauge fields [3–5] available in ultracold atomic systems pave the way to realizations of such systems. An advantage of cold atoms is that one can control geometry, dimension or quantum statistics of systems and parameters of microscopic Hamiltonians at unprecedented levels, which is used to explore non-trivial quantum states. Along these lines of researches, the Hofstadter Hamiltonian [6, 7] and Haldane topological model [8] have been realized in cold atoms.

In addition to such a non-trivial feature of non-interacting quantum matters, as well known, an interaction is a key ingredient for the diversity of nature. Indeed various phenomena such as superconductivity, superfluidity and Mott transition are understood as a consequence of the interactions. One can thus expect the interaction effects in such topological matters to cause further non-trivial nature, and effective approaches incorporating interactions are required in theory.

In one dimension one can successfully apply field theoretical approaches to incorporate interaction effects in a non-perturbative manner. Thus, reduction of dimensions would be a way to understand physics involving magnetic fields and correlations. The minimal model to show non-trivial effects in the presence of magnetic fields is a two-leg ladder, and the bosonic version has been first discussed in Ref. [9] in the context of Josephson junction arrays. In this study, it has been predicted that two different phases show up: Meissner and vortex phases. While in the former phase a chiral current analogous to a Meissner edge current is induced on the legs by a magnetic flux, in the latter it is significantly reduced due to penetration of vortices, which is analogous to field-induced vortices in type-II superconductors. Later on, the theoretical interest has been devoted to the strong correlated regime, and it has been demonstrated that commensurability of a particle filling poses a Mott insulator with chirality and interesting critical properties [10–14].

Remarkably, the two-leg bosonic ladder subject to a magnetic flux has been successfully realized in an experiment [15], where a weakly-interacting regime is concerned. In this experiment, it has been confirmed that behaviors of chiral currents are consistent with what has been predicted in Ref. [9]. Thus it seems that a basic consensus in a weak-coupling regime is obtained.

More recently, however, it has been argued in Ref. [16] that in a weak-coupling regime there should exist an additional phase where a spontaneous population imbalance between the legs occurs. This additional phase named a biased ladder phase has been shown with the theory of weakly-interacting Bose gases normally used in higher dimensions [17]. At the same time, in one dimension quantum fluctuations should be non-negligible in most cases. Thus it is worth considering whether the biased ladder phase is still robust against quantum fluctuations.

In this paper, we examine the two-leg Bose-Hubbard...
model in the presence of a magnetic flux in a weakly-interacting regime by means of a couple of effective-theory approaches. We show that a spontaneous population imbalance indeed occurs, and is stable against quantum fluctuation effects. In particular, we point out that the effective theory has the similarity to a ferromagnetic XXZ quantum spin model. This implies that the Heisenberg point exists in the phase diagram, where SU(2) symmetry shows up in the low-energy effective theory although the original Hamiltonian does not possess that symmetry. This situation is somewhat similar to a two-leg extended Bose-Hubbard system analyzed in Ref. [18], where the low-energy effective theory possesses an emergent symmetry.

We also state that unknap processes coming from commensurability of a flux should be seriously consid-
ered at the mean-field level, which has been overlooked in the previous studies. The unknap processes existing in $\phi = \pi$ unstabilize the biased ladder phase, and as a consequence, only the commensurate vortex phase is allowed.

The structure of the paper is as follows. In Sec. II we review structures of single-particle bands as a function of $\phi$, and low-energy effective Hamiltonian reflecting the band structure. The single-particle band bottom shows different topologies: A single minimum for a small flux, and double minima for a large flux. In Sec. III and IV we discuss physics separately for a single-minimum and for a double-minimum band structure, in which Meissner, vortex, and biased ladder phases are allowed depending on $\phi$ and $K/J$. Finally in Sec. V summary and perspective on a phase transition between the Meissner and biased ladder phases, and on a stronger interaction effect are provided.

II. FORMULATION OF THE PROBLEM

Following the setup in Ref. [15], we consider the following two-leg Bose-Hubbard ladder Hamiltonian:

$$H = -J \sum_{l=1}^{L} \sum_{p=1,2} (e^{iA_{l,p}^1} b_{l+1,p}^\dagger b_{l,p} + H.c.)$$

$$-K \sum_{l} (e^{iA_{l}^2} b_{l,1}^\dagger b_{l,2} + H.c.) + \frac{U}{2} \sum_{l,p} n_{l,p}(n_{l,p} - 1) |1|$$

where $J$ and $K$ are the hopping amplitudes along the leg and rung directions, respectively. The applied flux is introduced via the Peierls substitution, and the corresponding gauge fields along the chain and rung directions are denoted by $A_{l,p}^1$ and $A_{l}^2$, respectively. The flux $\phi$ is then given as $\phi = A_{l,1}^1 - A_{l+1,1}^1 - A_{l,2}^2 + A_{l+1,2}^2$. The technology of laser-assisted tunneling [6, 7, 19–21] generates spatially-dependent phase in rung hoppings, which leads to the $\phi$ flux per plaquette as described in Fig. II I. Namely, in this paper we choose the following gauge as $A_{l,p}^1 = 0$ and $A_{l}^2 = \phi l$. Taking into account the fact that an interatomic interaction is given by an s-wave scattering length, and the stability in bosonic systems, we restrict ourselves to a local repulsive interaction, $U > 0$.

Apparently, the Hamiltonian (1) is invariant under the simultaneous transformations, $b_{l,1(2)} \rightarrow b_{l,2(1)}$ and $\phi \rightarrow -\phi$. Thus, we can safely take the domain of definition in $\phi$ as $0 < \phi \leq \pi$.

By using the general relation between current and Hamiltonian, $J = -\frac{\partial H}{\partial A}$, where $A$ is the gauge field, we can define current operators along the legs and rungs as

$$j_{l,p} = iJ(b_{l+1,p}^\dagger b_{l,p}^1 - b_{l,p}^1 b_{l+1,p}^\dagger),$$

$$j_{l}^\perp = iK \left( e^{i\phi} b_{l,1}^\dagger b_{l,2} - e^{-i\phi} b_{l,2}^\dagger b_{l,1} \right).$$

For the sake of convenience, we also introduce chiral current along the legs:

$$j_c = j_{l,1}^\perp - j_{l,2}^\perp.$$  

We will see that $j_c$ and $j_{l,1}^\perp$ play important roles in characterizing each phase.

Since we are interested in the regime $J,K \gg U$, we start with diagonalizing the single-particle Hamiltonian. To this end, we perform gauge and Fourier transformations as $b_{l,1} = \frac{1}{\sqrt{2}} \sum_k e^{ikl} b_{k,1}$ and $b_{l,2} = \frac{1}{\sqrt{2}} \sum_k e^{ikl} b_{k,2}$. Then, by considering a unitary transformation for $b_{k,1}$ and $b_{k,2}$

$$\begin{pmatrix} b_{k,1} \\ b_{k,2} \end{pmatrix} = \begin{pmatrix} \cos \left( \frac{\xi_k}{2} \right) & -\sin \left( \frac{\xi_k}{2} \right) \\ \sin \left( \frac{\xi_k}{2} \right) & \cos \left( \frac{\xi_k}{2} \right) \end{pmatrix} \begin{pmatrix} \alpha_k \\ \beta_k \end{pmatrix},$$

where

$$\sin \left( \frac{\xi_k}{2} \right) = -\sqrt{1 - \frac{\sin \left( \frac{\phi}{2} \right) \sin k}{\sqrt{\left( \frac{K}{2J} \right)^2 + \sin^2 \left( \frac{\phi}{2} \right) \sin^2 k}}},$$

$$\cos \left( \frac{\xi_k}{2} \right) = \sqrt{1 + \frac{\sin \left( \frac{\phi}{2} \right) \sin k}{\sqrt{\left( \frac{K}{2J} \right)^2 + \sin^2 \left( \frac{\phi}{2} \right) \sin^2 k}}},$$

the single-particle Hamiltonian can be diagonalized [22, 23] as

$$H_0 = \sum_k (E_+(k) \alpha_k^\dagger \alpha_k + E_-(k) \beta_k^\dagger \beta_k).$$

Here, the single-particle spectrum is given by

$$E_\pm(k) = 2J \left[ -\cos \left( \frac{\phi}{2} \right) \cos k \pm \sqrt{\left( \frac{K}{2J} \right)^2 + \sin^2 \left( \frac{\phi}{2} \right) \sin^2 k} \right].$$  

(9)
which depicts the two-band structure as well as the $2\pi$ periodicity, reflecting the two-leg ladder geometry.

Unless a strong interaction is concerned, single-particle low-energy states, bottoms in the lowest band, play important roles in the low-energy many-body states. With this understanding, we neglect effects of the higher band $\alpha_k$ by keeping in mind the condition $J, K \gg U$.

### FIG. 2. Changes of topology in the lower band at a certain $\phi \neq \pi$. The direction of the arrows means the reduction of $K/J$. In a strong enough $K/J$, the band has a single minimum while in the opposite limit, the band of double well structure forms. In between, there exists a critical point in which the band bottom becomes quartic in $k$. The double-well structure is always maintained at $\phi = \pi$ regardless of values of $K/J$.

Let us look into a behavior of the lower band in more detail. We first obtain extrema via $\frac{\partial E}{\partial k} = 0$, which leads to

$$
\sin k \left[ \cos \left( \frac{\phi}{2} \right) - \frac{\sin^2 \left( \frac{\phi}{2} \right) \cos k}{\sqrt{\left( \frac{K}{2J} \right)^2 + \sin^2 \left( \frac{\phi}{2} \right) \sin^2 k}} \right] = 0.
$$

If $\phi \neq \pi$ with $K/J \gg 1$, $k = 0$ and $\pm \pi$ are the solution of Eq. (10), and $k = 0$ gives the minimum of the band. As in the case of the normal cosine band, the dispersion near $k = 0$ is approximated to be quadratic in $k$. It is straightforwardly shown that the condition for the single-minimum structure can be expressed as

$$
\left( \frac{K}{2J} \right)^2 > \frac{\sin^4 \left( \frac{\phi}{2} \right)}{1 - \sin^2 \left( \frac{\phi}{2} \right)}.
$$

As $K/J$ decreases, on the other hand, the double-well structure starts to show up. The critical point between the single- and double-minimum structure is given by

$$
\left( \frac{K}{2J} \right)^2 = \frac{\sin^4 \left( \frac{\phi}{2} \right)}{1 - \sin^2 \left( \frac{\phi}{2} \right)},
$$

which means the equality limit of the inequality in Eq. (11). Then the dispersion near the bottom becomes quartic in $k$. In the double-well structure case, two separated minima $\pm Q$ can be obtained from the factor of the square bracket in Eq. (10).

$$
Q = \sin^{-1} \left[ \sqrt{\sin^2 \left( \frac{\phi}{2} \right) - \left( \frac{K}{2J} \right)^2 \cot^2 \left( \frac{\phi}{2} \right)} \right].
$$

In addition, the dispersions around the minimum point are quadratic in $k$ as in the case of the single minimum. We emphasize that $\phi = \pi$ is special because the symmetric double-well structure is strongly protected and its minima are located at $Q = \pm \frac{\pi}{4}$ regardless of $K/J$.

The similar behavior is also found in the case of changes of $\phi$ with a fixed $K/J$. In that case it is shown that for the small enough $\phi$, the band forms single-minimum structure, and shows up the double-well structure when $\phi$ goes through a critical value $\phi_c$. A critical flux $\phi_c$ between these two structures is shown to be

$$
\sin^{-1} \left( \frac{\phi_c}{2} \right) = \sqrt{\left( \frac{K}{J} \right)^2 + 16 \left( \frac{K}{J} \right)^2 - \left( \frac{K}{J} \right)^2}.
$$

Based on the change of the band structure discussed above, let us incorporate interaction effects within a weakly-interacting regime, $J, K \gg U$. In this interaction regime, the bottoms of the band is also important for bosonic systems, which is an essentially different point from the fermion systems. Therefore, we first truncate all the effects involving the upper band ($\alpha_k$). The Hamiltonian (1) is reduced to

$$
H = \sum_k E_\gamma \beta^\dagger_{\gamma k} \beta_k + \frac{1}{2L} \sum_{k, k', k_1, k_2, k_3, k_4} \Gamma_{k_1, k_2, k_3, k_4} \beta^\dagger_{k_1} \beta^\dagger_{k_2} \beta_{k_3} \beta_{k_4},
$$

where

$$
\Gamma_{k_1, k_2, k_3, k_4} = U \sum_{m \in \mathbb{Z}} \delta_{k_1 + k_2 - k_3 - k_4, 2\pi m} \times \left[ \sin \left( \frac{\xi_{k_1}}{2} \right) \sin \left( \frac{\xi_{k_2}}{2} \right) \sin \left( \frac{\xi_{k_3}}{2} \right) \sin \left( \frac{\xi_{k_4}}{2} \right) \right.
$$

$$
+ \cos \left( \frac{\xi_{k_1}}{2} \right) \cos \left( \frac{\xi_{k_2}}{2} \right) \cos \left( \frac{\xi_{k_3}}{2} \right) \cos \left( \frac{\xi_{k_4}}{2} \right) \Big].
$$

Note that in contrast with a system in continuum space, we need to consider scattering processes involving a finite momentum transfer equal to the reciprocal lattice vector. In our model, the finite momentum transfer to $2\pi m$ with an integer $m$ is allowed, which is nothing but the umklapp process, and turns out to play a crucial role in the $\phi = \pi$ case.

In what follows, we separately look into many-body ground states in each topology of the single-particle band.

### III. BAND WITH A SINGLE MINIMUM

For weakly-interacting bosons in higher dimensions, the Gross-Pitaevskii (GP) approach as one of the mean-field theories is known to provide good results [17]. As a consequence, the system undergoes a Bose-Einstein condensate (BEC), which also implies spontaneous breaking of $U(1)$ symmetry. Thereby, a gapless excitation mode known as a Nambu-Goldstone (NG) mode is obtained.
However, it is also well known that for an interacting one-dimensional bosonic system, there exists neither BEC nor NG mode in the thermodynamic limit. Namely the mean-field analysis underestimates fluctuation effects, and cannot thus capture correct results in the one-dimensional cases. However, one-dimensional superfluids in the weakly-interacting regime show very slow power-law decay in such a way that the order effect works almost comparably to the quantum fluctuation. In addition, there is also an acoustic phonon mode similar to the NG mode, although it does not correspond to the spontaneous symmetry breaking. From these facts, it is found that the GP approach is not correct in a strict sense, but would provide a practically reasonable starting point to discuss the ground state and low-energy excitation structure. In addition the advantage of the GP approach is that both kinetic and interaction energies can be simultaneously taken into account at the mean-field level \cite{24}.

Based on the above observations, let us consider the system with the single band. Then, we first apply a continuum approximation to the Hamiltonian \cite{15}:

\[ H \approx \int dx \left[ -\beta \frac{\nabla^2}{2M}(x) + U \frac{\beta}{4}(x)\beta(x)\beta(x)\right], \]

\[ \beta(x) = \frac{1}{\sqrt{\beta}} \sum_k \beta_ke^{i{k}\cdot{x}}. \]

where \( \beta(x) \) is the mean density. We note that this is essential identical to the Lieb-Liniger model \cite{25}. As far as the weak-coupling limit is concerned, we may consider the following GP ground state:

\[ |GS\rangle = \frac{1}{\sqrt{N!}}(\beta_{k=0})^N|0\rangle, \]

where \( N \) is the number of bosons.

Let us next incorporate long-wavelength fluctuations which play crucial roles in low-energy properties. To this end, we adopt the hydrodynamic approach also known as the bosonization for bosons \cite{26,27}:

\[ \beta(x) \sim \left[n - \frac{\nabla \phi(x)}{\pi}\right] \frac{1}{2} \sum_{m=2} \sum_{k} \beta_0 e^{i{k}\cdot{x}}, \]

where \( n = N/L \) is the mean density. We introduced the density and phase fluctuations, \( \phi(x) \) and \( \theta(x) \), respectively, and the commutation relation between them is given by

\[ [\theta(x), \frac{1}{\pi} \nabla \phi(x')] = i\delta(x-x'). \]

Applying the above bosonization formula \cite{19} to Eq. (17), we obtain

\[ H_{\text{eff}} = \frac{\nu_0}{2\pi} \sqrt{\frac{\nu}{\beta}} \int dx \left[ \frac{1}{K_0}(\nabla \phi)^2 + K_0 (\nabla \theta)^2 \right], \]

where \( \nu_0 = \sqrt{\frac{\nu U}{\beta}} \) and \( K_0 = \pi \sqrt{\frac{\nu}{\beta \tilde{M}}} \). This is the Hamiltonian for the celebrated Tomonaga-Luttinger liquid (TLL), which corresponds to a \( c = 1 \) conformal field theory. It is remarkable that in this TLL Hamiltonian, the long-range order (LRO) of the single-particle density matrix, incorrectly predicted by the GP mean-field theory, is directly confirmed to be modified into a correct quasi-LRO of the algebraic decay \cite{27}; \( \langle \beta'(x)\beta(0) \rangle \sim (\frac{1}{x})^{1/2K_0} \).

Let us next look into the rung \cite{3} and chiral current \cite{4} by translating them in effective theory derived above. By using the bosonization formula \cite{19}, the current operators, Eqs. (3) and (4), are expressed as

\[ j^-(x) \sim 0, \]

\[ j_c(x) \sim 2nJ \sin \left( \frac{\phi}{2} \right) + O(\nabla^2 \theta), \]

where we point out that the rung current vanishes regardless of bosonization, while the chiral current has the nonzero constant term and the terms starting from \( \nabla^2 \theta \). This implies that a current fluctuation,

\[ \langle \delta j(x)\delta j(0) \rangle = \langle j(x)j(0) \rangle - \langle j(x) \rangle \langle j(0) \rangle, \]

is going to be zero in the rung current, and decays as \( 1/x^4 \) in the chiral current, which originates from \( \nabla^2 \theta \).

In addition, Eq. \( 23 \) shows that the chiral current increases with \( \phi \). These properties correspond to the Meissner phase introduced in Ref. \cite{9}, which has been derived in a condition \( J \gg K, U \) different from the present case.

### IV. Band with Double Minima

We next examine low-energy properties of the system with the double-well band structure where we can distinguish a commensurate wave number \( Q \), giving the lowest-energy single-particle states, from incommensurate one. In the commensurate cases, \( Q \) can be represented as \( Q = \pi p/q \), where \( p, q \) are coprime numbers.

The effect of commensurability is related to types of interactions. As pointed out in Ref. \cite{9}, a \( q \)-body interaction produces the serious effect for \( Q = \pi p/q \). Thus, if arbitrary multi-body interactions come into the low-energy effective theory, every commensurability should be taken care. Note that it does not mean that multi-body interactions are required at the microscopic level. Namely even if only two-body interactions are assumed in the microscopic Hamiltonian, multi-body interactions are generated as virtual multiple-scattering processes when we integrate out irrelevant high-energy degrees of freedom such as deriving a low-energy effective Hamiltonian and implementing a perturbative renormalization group theory.

However, such virtual multiple-scattering processes would be suppressed in the weakly-interacting case, and the relevant case would be only for \( Q = \pi/2 \) in which the two-body interaction yields the strong commensurability effect.

Here we first consider an incommensurate \( Q \) case in Sec. IV-A Next in Sec. IV-B we move on to the discus-
sion of the $Q = \frac{\pi}{2}$ ($\phi = \pi$) case as one of the commensurate cases. The other commensurability is also briefly discussed in Sec. [IV.C]

A. Incommensurate $Q$ case

In contrast with the single minimum case, the mean ground-state density with the double well structure depends on the couplings of the Hamiltonian. Following the analysis for a BEC on a double-well potential [17], we assume the following ansatz [10]:

$$|GS\rangle = \frac{1}{\sqrt{N!}}(e^{i\theta_+ x} \cos \gamma \beta_Q + e^{i\theta_- x} \sin \gamma \beta_Q)^N|0\rangle,$$

where $\gamma$ and $\theta_\pm$ are variational parameters.

By taking the expectation value of $H$ in Eq. (15) with the above ansatz, we obtain

$$\frac{E_0(\gamma, \theta_\pm)}{N} = E_\pm(Q) + \frac{U_n}{4} \left[ \left( \frac{3}{2} \sin^2 \xi_Q - 1 \right) \sin^2 2\gamma 
- \sin^2 \xi_Q + 2 \right],$$

$$\langle n_+ \rangle = \langle \beta_+^\dagger(x) \beta_+(x) \rangle = n \cos^2 \gamma,$$

$$\langle n_- \rangle = \langle \beta_-^\dagger(x) \beta_-(x) \rangle = n \sin^2 \gamma,$$

where $\beta_\pm(x) = \frac{1}{\sqrt{T}} \sum_k \xi_{\pm k} e^{ikx}$. We note that these mean-field values have no dependence in $\theta_\pm$, which implies that the problem is reduced to optimization of the single variational parameter, $\gamma$. Then, the optimized $\gamma$ is alternatively determined by whether

$$\frac{3}{2} \sin^2 \xi_Q < 1,$$

or

$$\frac{3}{2} \sin^2 \xi_Q > 1.$$ (29)

In the former case (29), the ground state is minimized by $\gamma = \pi/4$, where the populations at $k = \pm Q$ are the same: $\langle n_+ \rangle = \langle n_- \rangle$. Thus, at the mean-field level, we expect that there are two independent BECs in the ground state. We note that this is different from a BEC on a double-well potential, where the ground-state energy depends on the relative phase via a hopping term between the condensates [17]. By contrast, such a hopping does not exist in our system, and thus the mean-field ground state is free to the relative phase. We will come back to this point in an analysis in the $\phi = \pi$ case, where a relative phase dependence shows up via the umklapp process in a nontrivial manner.

In the latter case (30), the ground state is characterized by $\gamma = 0$ or $\pi/2$, where the mean density becomes $\langle n_+ \rangle$, $\langle n_- \rangle = (n, 0)$ or $(0, n)$. This is the solution such that all the bosons occupy either at $k = Q$ or at $k = -Q$. Thus, the mean-field theory shows that $Z_2$ symmetry is spontaneously broken in the ground state, and a single BEC occurs simultaneously.

The transition between these mean-field ground states occurs at $\frac{\pi}{2} \sin^2 \xi_Q = 1$, which turns out to be rewritten as

$$\left( \frac{K}{2J} \right)^2 = \frac{\sin^4(\phi/2)}{\frac{\pi}{2} - \sin^2(\phi/2)}.$$ (31)

What is important is that the above critical $K/J$ is smaller than another critical $K/J$, given in Eq. (12), between the single- and double-minimum band topology. Namely it means that the solution (31) always exists in the regime of $K/J$ and $\phi$ in which the double-minimum band structure comes out. Therefore, we see that the transition between the mean-field ground states always occurs at a certain $K/J$ given by $\phi > 0$.

Let us next look into fluctuation effects based on the above mean-field analyses. As in the case of the single-minimum band, we approximate the Hamiltonian as [13]

$$H \approx \int dx \left[ -\sum_{j=\pm} \beta_j^\dagger(x) \beta_j(x) \right] + \frac{U}{8} \left( \frac{2 + \sin^2 \xi_Q}{8} (n_+ - n_-)^2 + \frac{U(2 - 3 \sin^2 \xi_Q)}{8} (n_+ - n_-)^2 \right],$$ (32)

where $\frac{1}{M^*} = \frac{d^2 E_{\pm}(\pm Q)}{dx^2}$ is the effective mass.

We first incorporate fluctuation effects in the case of the mean density $\langle n_+ \rangle = \langle n_- \rangle = n/2$ which is stable when Eq. (29) is obeyed. By using bosonization formula (15) in $\theta_\pm$, we obtain

$$H_{\text{eff}} = \sum_{\nu = s, a} \frac{v_\nu}{2\pi} \int dx \left[ K_\nu (\nabla \theta_\nu)^2 + \frac{(\nabla \phi_\nu)^2}{K_\nu} \right] + g \int dx \cos(\sqrt{S} \phi_a),$$ (33)

where we have introduced the symmetric (anti-symmetric) fields, $\phi_{\nu(a)} = \frac{1}{\sqrt{2}} (\phi_+ + (-) \phi_-)$ and $\theta_{\nu(a)} = \frac{1}{\sqrt{2}} (\theta_+ + (-) \theta_-)$. The velocities, $v_s$ and $v_a$, and TLL parameters, $K_s$ and $K_a$, appearing in the quadratic parts of the Hamiltonian are specified by

$$v_s = \sqrt{\frac{nU(2 + \sin^2 \xi_Q)}{2M^*}},$$ (34)

$$v_a = \sqrt{\frac{nU(2 - 3 \sin^2 \xi_Q)}{2M^*}},$$ (35)

$$K_s = \sqrt{n \frac{2M^* U(2 + \sin^2 \xi_Q)}{2M}},$$ (36)

$$K_a = \sqrt{n \frac{2M^* U(2 - 3 \sin^2 \xi_Q)}{2M}},$$ (37)

and the coupling of the cosine term is given by

$$g = n^2 U \sin^2 \xi_Q/2.$$ (38)

In the effective Hamiltonian (33), the symmetric and anti-symmetric fields are decoupled. Especially, the symmetric part is the conventional TLL Hamiltonian. On
the other hand, the anti-symmetric part seems not to be the TLL Hamiltonian due to the presence of the cosine term. Thus, the low-energy properties of the system are determined by the relevancy of this cosine term in the sense of the renormalization group. To this end, we implement the perturbative renormalization group analysis. By treating the coupling constant \( g \) as a perturbative parameter, we obtain \[ \frac{dg/v_a}{dl} = 2(1 - K_a)g/v_a, \] (39)

\[
\begin{align*}
  j_c(x) &\sim nJ[4 \sin^2 \left( \frac{\xi_Q}{2} \right) \sin \left( \frac{\phi}{2} + Q \right) - \sqrt{2} \sin^2 \left( \frac{\xi_Q}{2} \right) \cos \left( \frac{\phi}{2} + Q \right) \nabla \theta_a \\
  &+ \sqrt{2} \cos^2 \left( \frac{\xi_Q}{2} \right) \cos \left( \frac{\phi}{2} - Q \right) \nabla \theta_a - 4 \sin \left( \frac{\xi_Q}{2} \right) \cos \left( \frac{\phi}{2} \right) \cos \left( Q(2x - \sqrt{2} \theta_a) \right) \bigg] ,
\end{align*}
\] (40)

\[
\begin{align*}
  j^\perp(x) &\sim nK \left( \sin^2 \left( \frac{\xi_Q}{2} \right) - \cos^2 \left( \frac{\xi_Q}{2} \right) \right) \sin(2Qx - \sqrt{2} \theta_a). 
\end{align*}
\] (41)

By taking the averages of the these quantities, we obtain

\[
\langle j_c(x) \rangle \sim 4nJ \sin^2 \left( \frac{\xi_Q}{2} \right) \sin \left( \frac{\phi}{2} + Q \right) \\
\langle j^\perp(x) \rangle \sim 0.
\] (42)

Seemingly, the above properties coincide with those in the Meissner phase discussed in Sec. III since the radial current is vanished \[29\] and there is a non-zero chiral current. However, the chiral current decreases with \( \phi \), which is the opposite tendency to the Meissner phase. This implies that there exists a correlation between the chains. This phase corresponds to the (incommensurate) vortex phase introduced in Ref. \[19\]. Indeed, a signature of the vortices is found in the correlations of the current fluctuations, which are calculated as

\[
\langle \delta j_c(x) \delta j_c(0) \rangle \sim \frac{n^2J^2}{K_a} \left[ \sin^2 \left( \frac{\xi_Q}{2} \right) \cos \left( \frac{\phi}{2} + Q \right) + \cos^2 \left( \frac{\xi_Q}{2} \right) \cos \left( \frac{\phi}{2} - Q \right) \right] \frac{1}{x^2}
\] (44)

\[
\langle \delta j^\perp(x) \delta j^\perp(0) \rangle \sim n^2K^2 \left[ \sin^2 \left( \frac{\xi_Q}{2} \right) - \cos^2 \left( \frac{\xi_Q}{2} \right) \right] \frac{1}{x^1/K_a}.
\] (45)

They have now oscillation components decaying with a power law. Recalling \( K_a \gg 1 \), these power-law decays are extremely slow.

We next consider fluctuation effects in the case of the biased mean density \( \langle n_+ \rangle, \langle n_- \rangle = (n, 0) \) or \( (0, n) \). For the sake of simplicity, let us take the case of \( \langle n_+ \rangle, \langle n_- \rangle = (n, 0) \) \[30\]. Namely all the bosons only populate around \( k = Q \). The degrees of freedom around \( k = -Q \) are completely suppressed as far as the interaction is weak enough, and we may consider only the degrees of freedom around \( k = Q \). Then the Hamiltonian simplified by the long-wave-length approximation is given as

\[
H = \int dx \left[ -\beta^\dagger(x) \frac{\nabla^2}{2M^*} \beta(x) + \frac{U(2 - \sin^2 \xi_Q)}{4} n^2 \right].
\] (46)

Furthermore, by the bosonization formula \[19\], we obtain

\[
H_{\text{eff}} = \frac{\vec{v}}{2\pi} \int dx \left[ \vec{K}(\nabla \theta)^2 + \frac{(\nabla \phi)^2}{K} \right],
\] (47)

where the velocity and TLL parameter are, respectively,

\[
\vec{v} = \sqrt{\frac{nU(2 - \sin^2 \xi_Q)}{M^*}}, \quad \vec{K} = \frac{n}{M^*U(2 - \sin^2 \xi_Q)}. \] (48) (49)
Unlike the equal density case, the effective theory is described by a single TLL.

Let us next look at the currents. They are bosonized as given by

\[ j_c(x) \sim 4nJ \left[ \sin^2 \left( \frac{\xi Q}{2} \right) \sin \left( \frac{\phi + Q}{2} \right) \right] - 2nJ \left[ \sin^2 \left( \frac{\xi Q}{2} \right) \cos \left( \frac{\phi}{2} + Q \right) - \cos^2 \left( \frac{\xi Q}{2} \right) \cos \left( \frac{\phi}{2} - Q \right) \right] \nabla \theta , \quad (50) \]

\[ j^\perp(x) \sim 0 , \quad (51) \]

where the rung current is shown to be zero regardless of the bosonization as in the case of the Meissner phase. The averages of them are calculated as

\[ \langle j_c(x) \rangle \sim 4nJ \sin^2 \left( \frac{\xi Q}{2} \right) \sin \left( \frac{\phi + Q}{2} \right) , \quad (52) \]

\[ \langle j^\perp(x) \rangle \sim 0 , \quad (53) \]

which look the same as those of the \( \langle n_+ \rangle = \langle n_- \rangle \) case, i.e., Eqs. (42) and (43). However, this does not mean that all the low-energy properties coincide with the equal-density case. Indeed, we find that the difference occurs in the current fluctuations as

\[ \langle \delta j_c(x) \delta j_c(0) \rangle \sim \frac{2n^2J^2}{K} \left[ \sin^2 \left( \frac{\xi Q}{2} \right) \cos \left( \frac{\phi + Q}{2} \right) - \cos^2 \left( \frac{\xi Q}{2} \right) \cos \left( \frac{\phi}{2} - Q \right) \right]^2 \frac{1}{x^2} , \quad (54) \]

\[ \langle \delta j^\perp(x) \delta j^\perp(0) \rangle \sim 0 . \quad (55) \]

Thus, in contrast with the vortex phase in the \( \langle n_+ \rangle = \langle n_- \rangle \) case, the current fluctuations in this phase do not have an oscillating component.

A peculiarity of this phase is seen in the density in each leg. To see this, we define a magnetization, i.e., population imbalance between the legs, as \( m \equiv \langle n_1 \rangle - \langle n_2 \rangle \). According to the mean-field theory \[ \text{MF} \], the magnetization can be calculated as

\[ m = -n \cos \xi Q . \quad (56) \]

We note that the magnetization value \( m \) is robust even when the quantum fluctuation up to the bosonization level is incorporated. As shown in Fig. 3, \( m \) takes a nonzero value as far as the stability condition of the phase is met, which means the spontaneous imbalance of the populations between the legs occurs. This phase corresponds to the biased ladder phase called in Ref. [110], which has been first demonstrated within the GP mean-field theory. What is addressed here is that even in the presence of the quantum fluctuation the \( Z_2 \) symmetry breaking between the populations in the doubly-fold lowest energy in the double-well band is maintained, and the biased ladder phase is thus stable at the full quantum level. This would be reasonable once one recalls the fact that even though a continuous \( U(1) \) symmetry cannot be broken in quantum one-dimensional systems, a spontaneous breaking of a discrete symmetry is possible.

1. Analogy with ferromagnetic XXZ model

Let us now examine the nature in the phases and transitions between them, from the viewpoint of symmetry. To this end, we focus on the low-energy Hamiltonian \[ H_{\text{MF}} \] which is invariant under the continuous transformations, \( \beta_\pm \rightarrow e^{i\theta} \beta_\pm \), and discrete transformation, \( \beta_+ \rightarrow \beta_- \). This implies that symmetry of the low-energy Hamiltonian \[ H_{\text{MF}} \] is \( U(1)_+ \times U(1)_- \times Z_2 \), where the subscript
\[ \pm \text{of } U(1) \text{ represents the corresponding symmetry in } \beta_\pm. \]

We also note that the above symmetry can also be represented as \( U(1)_V \times U(1)_A \times \mathbb{Z}_2 \) where \( U(1)_V \) and \( U(1)_A \) represent the vector \( U(1) \) symmetry, \( \beta_\pm \rightarrow e^{i\theta} \beta_\pm \), and the axial \( U(1) \) symmetry, \( \beta_\pm \rightarrow e^{\pm i\theta} \beta_\pm \), respectively \( [31] \).

In the vortex phase, the low-energy effective properties are captured by the two independent TLLs, in which the two independent \( U(1) \) symmetries are hold; namely no symmetry breaking occurs in this phase. In the biased ladder phase, on the other hand, the low-energy effective properties are described by the single TLL reflecting the acoustic phonon excitation around one of the two minimum-energy state in the double-well band. The \( \mathbb{Z}_2 \) symmetry turns out to be broken, since the minima of the band are degenerate and one of the minima is spontaneously chosen.

It is important to clarify the transition point between the vortex and biased ladder phases, where we need a special attention. First, we address that at this transition point, symmetry of the low-energy Hamiltonian is enlarged. To see this, it is convenient to introduce a two-component spinor,

\[ \vec{\beta} = \left( \beta_+ \beta_- \right). \]  

(57)

In this spinor representation, we can define \( U(1)_V \times SU(2) \) transformations as \( \vec{\beta} \rightarrow e^{i\alpha_1}e^{i\alpha_2}e^{i\alpha_3}e^{i\alpha_4} \vec{\beta} \), where \( \sigma_i \) (\( i = x, y, z \)) is the Pauli matrix, and \( \alpha_i \) (\( i = 1, \cdots, 4 \)) represents an angle of \( U(1)_V \times SU(2) \). Then, the low-energy Hamiltonian \( [32] \) is shown to be invariant under the above transformations at the transition point where the term proportional to \( (n_+ - n_-)^2 \) disappears, i.e., the Hamiltonian has \( U(1)_V \times SU(2) \) symmetry. This implies that we can define the following conserved charges:

\[ S_+ = \int dx \beta_+^\dagger \beta_- , \]  

(58)

\[ S_- = \int dx \beta_-^\dagger \beta_+ , \]  

(59)

\[ S_z = \int dx [n_+ - n_-] , \]  

(60)

\[ N_V = \int dx [n_+ + n_-] . \]  

(61)

Here, Eqs. (58)-(60) constitute \( SU(2) \) charges and Eq. (61) originates from \( U(1)_V \) symmetry. In addition, the Hamiltonian \( [32] \) at the transition point is identical to the two-component bosonic Yang-Gaudin model where the Bethe ansatz solution is available \( [32, 33] \). So far the followings on two-component bosonic Yang-Gaudin model are known: As usual, there is a TLL associated with \( U(1)_V \) symmetry. In addition, the \( SU(2) \) symmetry corresponding to Eqs. (58)-(60) is spontaneously broken in the ground state, which is essentially corresponds to the physics of the Heisenberg ferromagnet \( [32] [34] \). Due to the spontaneous breaking of the \( SU(2) \) symmetry, we expect that there is a NG mode whose dispersion is quadratic in \( k \) and there exist an infinite number of the degenerate ground states and one of them is selected spontaneously. This is significant difference from the biased ladder phase where there only exists double degeneracy.

The scenario discussed above reminds us of the similarity to the ferromagnetic \( XXZ \) model. When we bosonize the \( XXZ \) model, the low-energy effective theory is described as the sine-Gordon model. When the \( XY \) in-plane anisotropy is strong, the theory is renormalized to the TLL which corresponds to the so-called \( XY \) phase, and upon approaching the isotropic point the velocity and Luttinger parameter of the \( XY \) phase, respectively, go to zero and infinity, which implies the quadratic dispersion of an excitation \( [27] \). Furthermore, beyond the isotropic point, i.e., Ising anisotropy, the \( XXZ \) model undergoes the so-called ferromagnetic Ising phase, where the excitations are massive, and the \( \mathbb{Z}_2 \) spin-inverse symmetry is spontaneously broken.

In our case, the anti-symmetric sector in the vortex phase \( [33] \) exhibits the same effective theory as the \( XXZ \) model, and the velocity \( [35] \) and Luttinger parameter \( [37] \) show the same behavior as those in the isotropic limit of the ferromagnetic \( XXZ \) model. Going beyond the Heisenberg point, we encounter the biased ladder phase described by a single TLL, which corresponds to the ferromagnetic Ising phase in the ferromagnetic \( XXZ \) model. It is interpreted that for the biased ladder phase the anti-symmetric sector goes away into the high energy regime, which would corresponds to the massive excitation in the ferromagnetic Ising phase. Namely such a massive excitation should describe the change of the populations on the band bottoms is regarded as the high energy one in our approach, which is excluded in Eq. (47). In Sec. IV A 2, it is shown that such a massive excitations may be incorporated with the Bogoliubov theory.

2. Bogoliubov spectrum in biased ladder phase

To see some insight into the biased ladder phase, let us here apply the Bogoliubov theory \( [10] \). Since the Bogoliubov theory is based on the expansion from the GP solution, it must underestimate fluctuation effects. However, this does not mean that all the results by means of the Bogoliubov theory are incorrect as pointed out in Sec. III. At least, an excitation spectral feature in the low-energy limit for weakly interacting one-dimensional bosons is expected to reproduce the correct behavior. Indeed, it is known that linear excitation feature occurring in the Lieb-Liniger model \( [25] \) and quadratic excitation feature occurring in the two-component Yang-Gaudin model \( [33] \) can be captured by the Bogoliubov theory.

As usual, by applying \( \beta_k = \sqrt{N_0} \delta_{k,\xi} \pm \bar{\beta}_k \), where \( \bar{\beta}_k \) is the fluctuation field, to Eq. (15), we obtain the Bogoliubov Hamiltonian to be correct up to second order in the
fluctuation field
\[ H_{\text{Bog}} \sim \sum_{k>0} \beta_{Q+k} \bar{\beta}_{Q-k} \]
\[ = (\beta_{Q+k}, \bar{\beta}_{Q-k}) \begin{pmatrix} \zeta(k) & \eta(k) \\ \eta(k) & \zeta(-k) \end{pmatrix} \begin{pmatrix} \bar{\beta}_{Q+k} \\ \bar{\beta}_{Q-k} \end{pmatrix}, \]  
(62)

where the matrix elements are defined as
\[ \zeta(k) = E_-(Q + k) - E_-(Q) + 2Un \left[ \sin^2 \frac{\xi_Q}{2} \sin^2 \frac{\xi_{Q+k}}{2} + \cos^2 \frac{\xi_Q}{2} \cos^2 \frac{\xi_{Q+k}}{2} \right] - Un \left[ \sin^4 \frac{\xi_Q}{2} + \cos^4 \frac{\xi_Q}{2} \right], \]  
(63)

\[ \eta(k) = Un \left[ \sin^2 \frac{\xi_Q}{2} \sin \frac{\xi_{Q+k}}{2} \sin \frac{\xi_{Q-k}}{2} + \cos^2 \frac{\xi_Q}{2} \cos \frac{\xi_{Q+k}}{2} \cos \frac{\xi_{Q-k}}{2} \right]. \]  
(64)

To diagonalize the above Hamiltonian, a simple scheme is to consider the following eigenvalue problem [35, 36]:
\[ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \zeta(k) \\ \eta(k) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \epsilon \begin{pmatrix} u \\ v \end{pmatrix}. \]  
(65)

Then, the Bogoliubov spectra \( \pm \epsilon \) are obtained as eigenvalues of the above matrix equation. The Bogoliubov spectrum is shown to be \[ 16 \] over the linear spectrum.

\[ \epsilon(k) = \pm \left( \frac{\zeta(k) - \zeta(-k)}{2} \right) + \sqrt{\left( \zeta(k) + \zeta(-k) \right)^2 - \eta^2(k)}. \]  
(66)

A behavior of the Bogoliubov spectrum is shown in Fig. 4. When looking at \( k \to 0 \), we have the linear spectrum, which can be regarded as the TLL. In addition, there is a local minimum around \( k = -Q \), which is interpreted as the massive excitation originating from the \( Z_2 \) symmetry breaking. We point out that the similar situation also occurs in a BEC in a shaken optical lattice, which has been recently realized in Ref. [37]. The energy gap around \( k \approx -Q \) \( (k \approx -Q) \), is found to go closed as \( Z_2 \) symmetry, which is restored. This scenario may remind one of the ferromagnetic transition in the \( XXZ \) model when approaching the Heisenberg point from the Ising anisotropic side, the Ising gap will collapse and the dispersion turns to quadratic in \( k \).

B. Commensurate Q case \( Q = \pi/2 \)

Let us next consider a commensurability effect, i.e., \( Q = \pi/2 \) \( (\phi = \pi) \) case where we require considerable attention. In this case, the double-well band structure is always maintained and its minima are located at \( k = \pm \pi/2 \) regardless of the ratio \( K/J \). A peculiarity is an emergence of the umklapp scattering processes between two energy minima in the band, which has been overlooked so far. To consider this, we turn back to the mean-field ansatz [25]. The form of this ansatz includes the couplings of the far-separated states \( k = \pm Q \), and thus automatically allows us to take into account the umklapp process involving the large momentum transfer. Based on the mean-field ansatz, the ground-state energy is straightforwardly calculated as
\[ \frac{E_0(\gamma, \theta_{\pm})}{N} = E_-(Q) + \frac{Un}{4} \left[ -\sin^2 \xi_Q + 2 \right. \]  
\[ + \left. \left\{ \frac{3}{2} + \cos(2\theta_{\pm} - 2\phi_{\pm}) \right\} \sin^2 \xi_Q - 1 \right] \sin^2 2\gamma \]  
(67)

Note that it differs from Eq. (26) due to the presence of the umklapp scattering. As can be clear from the above expression, the energy must be minimized when the relative phase satisfies \( \theta_{+} - \theta_{-} = \pm \frac{\pi}{2} \). We notice that this is different from the case of a BEC in a double-well potential, where the relative phase is zero in the ground state [17]. This difference originates from the fact that the relative phase dependence is caused by a hopping (kinetic) term such as \( -J \cos(\theta_{-} - \theta_{+}) \) in a BEC on a double-well potential while that in our model originates from the interaction term in our model.

By substituting this relative phase \( \theta_{+} - \theta_{-} = \pi/2 \) or \(-\pi/2 \) into Eq. (67), the ground-state energy is going to be
\[ \frac{E_0(\gamma)}{N} = E_-(Q) + \frac{Un}{4} \left[ \left( \sin^2 \xi_Q - 1 \right) \sin^2 2\gamma \right. \]  
\[ - \left. \sin^2 \xi_Q + 2 \right] \]  
(68)

Since \( \sin^2 \xi_Q \leq 1 \), the ground state can be uniquely determined by \( \gamma = \pi/4 \), which is independent of \( K/J \). This ground state leads to the balanced density \( \langle n_{+} \rangle = \langle n_{-} \rangle \). Thus, the biased ladder phase is washed out in the presence of the umklapp process.
Let us next consider quantum fluctuations from the mean-field solution. In this case, we need to retain the following process in the effective Hamiltonian (69):

\[ H_{\text{umklapp}} = U \sin^2 \xi Q \int dx [\beta_+^\dagger \beta_+ \beta_- + h.c.]. \] (69)

Note that due to this term (69), symmetry of the Hamiltonian is lowered from \( U(1) \times U(1) \times Z_2 \) to \( U(1) \nu \times Z_2 \). Thus, the axial \( U(1) \) symmetry \((\beta_+ \rightarrow e^{i \phi} \beta_+)\) disappears from the low-energy Hamiltonian, and the continuous symmetry remaining turns out to be only the vector \( U(1) \nu \) symmetry \((\beta_+ \rightarrow e^{i \phi} \beta_\pm)\). On the other hand, the \( Z_2 \) symmetry \((\beta_+ \rightarrow \beta_\pm)\) remains in the presence of the umklapp term (69).

Let us next perform bosonization as follows:

\[
H = \sum_{\mu=s,a} \frac{v_\mu}{2\pi} \int dx \left[ K_\mu (\nabla \phi_\mu)^2 + \frac{\langle \nabla \phi_\mu \rangle^2}{K_\mu} \right] \\
-g_1 \int dx \cos(\sqrt{8} \phi_a) - g_2 \int dx \cos(\sqrt{8} \phi_a),
\] (70)

where \( g_1 = n^2 \sin^2 \xi Q / 4 \), \( g_2 = \frac{n^2}{4} \sin^2 \xi Q \), and

\[
v_s = \sqrt{\frac{nU}{M^*}},
\] (71)

\[
v_a = \sqrt{\frac{nU (1 - \sin^2 \xi Q)}{M^*}},
\] (72)

\[
K_s = \sqrt{\frac{n}{4M^* U}},
\] (73)

\[
K_a = \sqrt{\frac{n}{4M^* U (1 - \sin^2 \xi Q)}}.
\] (74)

An essential difference from the incommensurate \( \phi \) case is the presence of \( \cos(\sqrt{8} \theta_a) \) which comes from \( H_{\text{umklapp}} \). Furthermore we move on to the renormalization group analysis to see the low-energy properties of the system. The parameters in Eq. (70) are found to obey the following renormalization group equations as

\[
\frac{dg_1/v_a}{dt} = 2(1 - K_a)g_1/v_a,
\] (75)

\[
\frac{dg_2/v_a}{dt} = 2 \left( 1 - \frac{1}{K_a} \right) g_2/v_a.
\] (76)

Since \( K_a \gg 1 \) in the weakly interacting case assumed here, it is found from Eqs. (75) and (76) that \( g_1/v_a \) and \( g_2/v_a \) is rapidly renormalized, respectively, to being zero and divergent as \( l \) increases. Namely, \( \cos(\sqrt{8} \theta_a) \) in Eq. (70) is highly relevantly retained in the effective Hamiltonian in the low-energy limit while \( \cos(\sqrt{8} \phi_a) \) irrelevantly goes away as in the case of \( Q \neq \pi/2 \). The renormalized effective theory is

\[
H = \sum_{\mu=s,a} \frac{v_\mu}{2\pi} \int dx \left[ K_\mu (\nabla \phi_\mu)^2 + \frac{\langle \nabla \phi_\mu \rangle^2}{K_\mu} \right] \\
-g_2 \int dx \cos(\sqrt{8} \phi_a).
\] (77)

Thus, the anti-symmetric sector becomes gapful due to the fixed relative phase \( \langle \cos(\sqrt{8} \phi_a) \rangle = 1 \), which is consistent with the mean-field scenario, deduced by Eq. (67), with the umklapp scattering.

Let us also look at the currents in this phase. By means of the bosonization formula, the averaged currents are calculated as

\[
\langle j_c(x) \rangle \sim 2nJ \sin \xi Q (-1)^x,
\] (78)

\[
\langle j^\dagger(x) \rangle \sim nK \cos \xi Q (-1)^x.
\] (79)

Thus, the current pattern shows up due to the umklapp effects. In a similar manner, we can evaluate the current correlations, which turn out to be zero (or exponentially decay in \( x \)). This phase is called (commensurate) vortex or chiral superfluid phase.

The possible phase diagram in the weak coupling regime \( J, K \gg U \) is summarized in Fig. 5.

**FIG. 5.** Schematic phase diagram (a) for \( Q \neq \pi/2 \) and (b) for \( Q = \pi/2 \). An emergent \( SU(2) \) symmetry shows up at the boundary between incommensurate vortex and biased ladder phases, which is marked with the red circle in (a).

### 1. Transition between commensurate and incommensurate fluxes

Now we discuss properties in the vicinity of the commensurate flux \( \phi = \pi \). To this end, we first estimate the gap coming from the coupling \( \cos(\sqrt{8} \theta_a) \) in Eq. (71) based on the renormalization group equation [27]. As mentioned above, this coupling \( g_2 \) is highly relevant in \( K_a \gg 1 \) and is therefore expected to flow to the strong coupling regime \( g_2 \rightarrow \infty \). Thus, we introduce a typical length scale \( l^* \) estimated from Eq. (76) as

\[
\rho^* \sim \left( \frac{v_2}{g_2} \right)^{-\frac{1}{2}} \pi^2
\] (80)

and stop the renormalization group flow at the length scale \( l^* \). On the other hand, if we are in the strong coupling, we may approximate the cosine term as \( g_2 \cos(\sqrt{8} \theta_a) \approx g_2[1 - 4\theta_2^2(x)] \). Therefore, applying the expansion to the renormalized effective theory (77), the Hamiltonian is easily diagonalized, and the consequently estimated gap \( \Delta(l^*) \) for the renormalized coupling constant \( g_2(l^*) \) at a termination of the renormalization group
flow is found to be $\Delta(l^*) \sim \sqrt{2/(K_a)} \sim \sqrt{\frac{v_0}{K_a}}$. Taking into account that the gap is renormalized as $\Delta(l) = e^l \Delta$, we obtain

$$\Delta \sim g_2 \frac{1}{\sqrt{K_a}} \frac{v_0}{K_a}$$

(81)

Based on this estimation, we next consider the situation in which $Q$ is slightly deviated from a commensurate point $Q = \pi/2$, and write $Q$ as $Q = \frac{\pi}{2} + \delta$, where $\delta$ is assumed to be small. Then, the umklapp term (69) is bosonized as

$$H_{\text{umklapp}} \sim g_2 \cos(\sqrt{8}Q_a + \delta x),$$

(82)

which is namely the oscillating term as a function of $x$. We may safely make the replacement of $\delta \to 0$ if $\delta$ is irrelevant. However, if $\delta$ is relevant, the umklapp term experiences a strong oscillation, and cancels out in the renormalized Hamiltonian. In general, the transition from the irrelevant and relevant $\delta$ or vice versa is known to occur around $\Delta \sim \delta$, and called a commensurate-incommensurate transition \[27\]. In our cases, the transition between the chiral superfluid and vortex phases corresponds to such a commensurate-incommensurate transition \[27\], because once $\delta$ becomes relevant, the effective theory reduces to that of the two independent TLLs, which is identical to that of the vortex phase.

On the other hand, the transition between the commensurate vortex and biased ladder phases may not be captured by this scenario of the incommensurate-commensurate transition because the effective theory in the biased ladder phase is not the two independent TLLs.

It is also interesting to consider the transition between the commensurate vortex and Meissner phases. However, it is difficult to describe such a transition by means of our approach. Thus, it would be worthwhile examining the nature of these transitions in a numerical simulation.

**C. Other commensurability effect**

So far, we have discussed only the $\phi = \pi$ case as a commensurate case. The point on the $\phi = \pi$ case is that the biased ladder phase is suppressed by the presence of the umklapp scattering. Here we consider the other commensurate case in order to see roles of general commensurability, and we will see that the $\phi = \pi$ commensurability is special.

As heretofore, let us first consider the mean-field approximation for the Hamiltonian \[15\]. Then, it turns out that as far as such a Hamiltonian is concerned, we always have the energy expression \[26\] regardless of commensurability of $Q$. Thus, the mean-field phase diagram is identical to the incommensurate $Q$ case.

Let us next consider the bosonization to see quantum fluctuation effects. For the incommensurate vortex phase at the mean-field level, it is expected that the higher order perturbation theory generates cosine terms in $\theta_a$ at a commensurate $Q$ as shown in Ref. \[9\]. Since many of such cosine terms are relevant for $K_a \gg 1$, we find that the incommensurate vortex phase is replaced by the commensurate vortex phase by quantum fluctuation effects if the coupling of the cosine term is larger than the temperature \[27\].

On the other hand, for the biased ladder phase at the mean-field level, we find that a cosine term to fix $\theta_a$ does not show up by the bosonization since the mean density in one of the wells is equal to zero. Namely, the biased ladder phase at such a commensurate $Q$ is robust.

**V. SUMMARY AND PERSPECTIVE**

We have examined the two-leg Bose-Hubbard ladder model subject to a magnetic field flux. We have particularly revealed the structure of the phase diagram in a weak-coupling regime by using a couple of the effective theory methods. What we stress is that we have also found the so-called biased-ladder phase, first predicted by the GP mean-field approach, to be robust against quantum fluctuations. It has also been shown that the transition between the biased-ladder and vortex phase has a similarity as that of the ferromagnetic $XXZ$ model where the emergent $SU(2)$ symmetry comes out at the transition between the biased ladder and vortex phases. In the case of the ladder system subject to the magnetic flux, an incommensurate effect works to phase degrees of freedom, which produces a kind of umklapp processes. By incorporating such an umklapp process at the mean-field level, we have shown that the biased-ladder state tends to be unstabilized by the umklapp process, and turns out to be forbidden for the case of $\phi = \pi$.

**A. Transition between Meissner and biased ladder phases**

As seen in Sec. \[11\] the dispersion becomes quartic at the critical point between the single- and double-minimum band structures. In the absence of an interaction, one can naively expect that all the bosons condense at the lowest energy, and just forms a BEC which is the same as the case of the quadratic dispersion.

A question is what happens in the presence of an interaction. Here we briefly discuss a possible scenario.

We first point out that the similar situation can be also considered for one-dimensional two-component bosons with spin-orbit couplings, where the bare single-particle dispersion becomes quartic at a certain value of the spin-orbit coupling and biased chemical potential between the two species. By employing the hydrodynamic approach and Gaussian approximation, it is shown that the low-energy effective theory undergoes non-TLL \[28\]. Then the excitation is still gapless, but is no longer identical to an acoustic phonon: the quadratic-dispersion mode. It would rather be that of the Heisenberg ferromagnet and
two-component bosonic Yang-Gaudin model where the spontaneous symmetry breaking and NG mode show up. Interestingly, however, the off-diagonal density matrix is shown to decay exponentially as

\[ \langle b^\dagger_{x,p} b_{x',p'} \rangle \sim ne^{-|x'|/\xi_c}, \]

where \( \xi_c \) is the correlation length given by \( \xi_c = \sqrt{2\rho_0/(mg\lambda^2)} \) with a mean-density \( \rho_0 \), atomic mass \( m \), density-density interaction \( g \) and spin-orbit coupling \( \lambda \). Thus, it means that even one-dimensional superfluidity is destroyed.

From the above example, we can expect the same physics in our model. Namely the system might form such a non-TLL when the system transits from the Meissner to the biased-ladder state. However, then the Meissner current would be predicted to be still protected because its presence is guaranteed by the two band structure from the ladder geometry and flux \[9, 13\].

In addition to the physical properties, this problem on interacting bosons for quartic bare dispersion would have another interesting aspect. In general, it is expected that an existence of gapless modes supports LRO or quasi-LRO while our model is an exceptional case on this statement. Thus, the profound understanding on the role of gapless modes in one dimension remains an open question.

**B. Stronger \( U \) effect**

In the paper, we have analysed the system under the condition \( K, J \gg U \). A natural question to come up then is what happens when the system with a stronger interaction \( U \) is concerned. In many of one-dimensional systems, both the weak- and strong-coupling analyses are continuously connected to each other consequently \[27\], but we would think that the strong coupling regime includes different physics in our model. This is because while in a weak coupling, the low-energy properties can be well captured by assuming the quasi-condensates at well-separated lowest energy single particle states in the double-well band structure, such a picture is no longer applicable in a strong coupling due to a hard-core feature analogous to Fermi statistics. Namely the double-well band feature can be no longer important in low-energy physics if we naively assume fermion-like occupation of the particles in the band picture as a strong coupling limit, and the different properties from those for the weak coupling should be then found. Indeed, the recent numerical analysis \[39\] in a strong coupling shows that the biased-ladder phase is not found while the presence of the Meissner and vortex phases is confirmed.

Let us make a further consideration on the physics in the intermediate interaction. Then it is convenient to take the generalized mean-field ansatz \[16\],

\[ |GS'\rangle = \frac{1}{\sqrt{N!}} \left( e^{i\theta_+\cos\gamma\beta_k^+ + i\theta_-\sin\gamma\beta_k^+} \right)^N |0\rangle, \]

where now the wave-vector \( k \) pointing the bottoms of the band is also treated as a variational parameter. This generalization means that a modification of the band structure by an interaction is taken into account. As shown in Ref. \[16\], the variational approach shows that the optimized value of \( k \) decreases with \( U \), and approaches zero at a certain \( U_c \). It is not clear whether this mean-field ansatz correctly capture the physics in the regime \( U \sim U_c \), but we can naively guess, at least, from this discussion, that the interaction works so as to collapse the double-well band structure.

Combining the mean-field and numerical result, the following scenario can be deduced: The biased-ladder state for the weak-interaction goes unstable as \( U \) increases; it eventually transits at \( U = U_c \) to Meissner state, and such a Meissner state continues to that of the strong coupling regime which is found in Ref. \[39\]. To test this scenario, or to precisely estimate the critical \( U_c \), an unbiased numerical simulation would be necessary.

**ACKNOWLEDGEMENTS**

S.U. is supported by the Swiss National Science Foundation under Division II.

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