COLORINGS OF SIMPLICIAL COMPLEXES AND VECTOR BUNDLES OVER DAVIS-JANUSZKIEWICZ SPACES

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ABSTRACT. We show that coloring properties of a simplicial complex $K$ are reflected by splitting properties of a bundle over the associated Davis-Januszkiewicz space whose Chern classes are given by the elementary symmetric polynomials in the generators of the Stanley-Reisner algebra of $K$.

1. Introduction

For a simplicial complex $K$, Davis and Januszkiewicz constructed a family of spaces, all of which are homotopy equivalent, and whose integral cohomology is isomorphic to the associated Stanley-Reisner algebra $\mathbb{Z}[\langle K \rangle]$ [DJ, Section 4]. We denote a generic model for this homotopy type by $DJ(K)$. In the above mentioned influential paper, Davis and Januszkiewicz also constructed a particular complex vector bundle $\lambda$ over $DJ(K)$ whose Chern classes are given by the elementary symmetric polynomials in the generators of $\mathbb{Z}[\langle K \rangle]$ [DJ, Section 6]. This vector bundle is of particular interest. For example, if $K$ is the dual of the boundary of a simple polytope $P$, then the associated moment angle complex $Z_K$ is a manifold and the realification $\lambda_\mathbb{R}$ of $\lambda$ is stably isomorphic to the bundle given by applying the Borel construction to the tangent bundle of $Z_K$. And if $M^{2n}$ is a quasitoric manifold over $P$, then again the Borel construction applied to the tangent bundle of $M^{2n}$ produces a vector bundle stably isomorphic to $\lambda_\mathbb{R}$ [DJ, Theorem 6.6, Lemma 6.5].

Davis and Januszkiewicz also noticed that, if $K$ is the dual of the boundary of a polytope of dimension $n$ and admits a coloring with $n$ colors, the bundle $\lambda$ splits into a direct sum of $n$ complex line bundles and a trivial bundle [DJ, Section 6.2]. We are interested in generalizations of this observation. In fact, we will show that a simplicial complex admits a coloring with $r$ colors precisely when $\lambda$ splits stably into a direct sum of $r$ linear complex bundles and a trivial bundle. We will also show that a similar result holds for the realification of $\lambda$.

1991 Mathematics Subject Classification. 55R10, 57R22, 05C15.

Key words and phrases. Davis-Januszkiewicz spaces, vector bundle, characteristic classes, colorings, simplicial complexes.
To make our statements more precise we have to fix notation and recall some basic constructions. Let \( [m] := \{1, \ldots, m\} \) be the set of the first \( m \) natural numbers. A finite abstract simplicial complex \( K \) on \([m]\) is given by a set of faces \( \alpha \subseteq [m] \) which is closed under the formation of subsets. We consider the empty set \( \emptyset \) as a face of \( K \). The dimension \( \dim \alpha \) of a face \( \alpha \) is given in terms of its cardinality by \( |\alpha| - 1 \), and the dimension \( \dim K \) of \( K \) is the maximum of the dimensions of its faces.

The most basic examples are given by full simplices. For \( \alpha \subseteq [m] \) we denote by \( \Delta[\alpha] \) the simplicial complex which consists of all possible subsets of \( \alpha \). Then \( \Delta[\alpha] \) is an \((|\alpha| - 1)\)-dimensional simplex. The full simplex \( \Delta[m] \) contains \( K \) as a subcomplex, and if \( \sigma \subset K \) then \( \Delta(\sigma) \subset K \) is a subcomplex as well.

A regular \( r \)-paint coloring, an \( r \)-coloring for short, of a simplicial complex \( K \) is a non-degenerate simplicial map \( g: K \rightarrow \Delta[r] \), i.e. \( g \) maps each face of \( K \) isomorphically on a face of \( \Delta[r] \). The inclusion \( K \subset \Delta[m] \) always provides an \( m \)-coloring. If \( \dim(K) = n - 1 \), then \( K \) may only allow \( r \)-colorings for \( r \geq n \).

For a commutative ring \( R \) with unit we denote by \( R[m] := R[v_1, \ldots, v_m] \) the graded polynomial algebra generated by the algebraically independent elements \( v_1, \ldots, v_m \) of degree 2, one for each vertex of \( K \). For each subset \( \alpha \subseteq [m] \) we denote by \( v_{\alpha} := \prod_{j \in \alpha} v_j \) the square free monomial whose factors are in 1 to 1 correspondence with vertices contained in \( \alpha \).

The graded Stanley-Reisner algebra \( R[K] \) associated with \( K \) is defined as the quotient \( R[K] := R[m]/I_K \), where \( I_K \subset R[m] \) is the ideal generated by all elements \( v_{\alpha} \) such that \( \alpha \subseteq [m] \) is not a face of \( K \).

Since \( BT^m \) is an Eilenberg-MacLane space realizing the polynomial algebra \( \mathbb{Z}[m] \), the projection \( \mathbb{Z}[m] \rightarrow \mathbb{Z}[K] \) can be realized by a map \( f: DJ(K) \rightarrow BT^m \). We can think of \( T^m \) as the maximal torus of the unitary group \( U(m) \). The pull back along the composition \( DJ(K) \rightarrow BT^m \rightarrow BU(m) \) of the universal bundle over \( BU(m) \) gives a vector bundle \( \lambda \downarrow DJ(K) \). This is the vector bundle studied by Davis and Januszkiewicz and mentioned above. The total Chern class \( c(\lambda) = 1 + \sum_i c_i(\lambda) \) of \( \lambda \) is then given by \( c(\lambda) = \prod_{i=1}^m (1 + v_i) \in \mathbb{Z}[K] \).

The realification of a complex vector bundle \( \xi \) is denoted by \( \xi_{\mathbb{R}} \). Confusing notation we will denote by \( \mathbb{C} \) and \( \mathbb{R} \) a 1-dimensional trivial complex or real vector bundle over a space \( X \). Now we can state our main theorem.

**Theorem 1.1.** Let \( K \) be a finite simplicial complex over the vertex set \([m] \). Then the following conditions are equivalent.
(i) \( K \) admits an \( r \)-coloring \( K \rightarrow \Delta[r] \).
(ii) The vector bundle \( \lambda \) splits into a direct sum \((\bigoplus_{i=1}^r v_i) \oplus \mathbb{C}^{m-r} \) of \( r \) complex line bundles \( v_i \) and a trivial \((m-r)\)-dimensional complex bundle.
(iii) The realification \( \lambda_{\mathbb{R}} \) of \( \lambda \) splits into a sum \((\bigoplus_{i=1}^r \theta_i) \oplus \mathbb{R}^{2(m-r)} \) of \( r \) 2-dimensional real bundles \( \theta_i \) and a trivial \( 2(m-r) \)-dimensional
real bundle.

(iv) The vector bundle $\lambda$ is stably isomorphic to a direct sum $\bigoplus_{i=1}^{r} \nu_i$ of $r$ complex line bundles.

(v) The realification $\lambda_\mathbb{R}$ is stably isomorphic to a direct sum $\bigoplus_{i=1}^{r} \theta_i$ of $r$ 2-dimensional real bundles.

Several of our vector bundles will be constructed as homotopy orbit spaces. For a compact Lie group $G$ and a $G$-space $X$, the Borel construction or homotopy orbit space $EG \times_G X$ will be denoted by $X_{hG}$. If $\eta \downarrow X$ is an $n$-dimensional $G$-vector bundle over $X$ with total space $E(\eta)$, the Borel construction establishes a fibre bundle $E(\eta)_{hG} \rightarrow X_{hG}$. In fact, this is an $n$-dimensional vector bundle over $X_{hG}$, denoted by $\eta_{hG}$. For definitions and details see [S].

Let $M^{2n}$ be a quasitoric manifold over the simple polytope $P$. That is that $M^{2n}$ carries a $T^n$-action, which is locally standard and that the orbit space $M^{2n}/T^n = P$ is a simple polytope. The Borel construction produces a space $(M^{2n})_{hT^n}$, which is homotopy equivalent to $DJ(K_P)$, where $K_P$ is the simplicial complex dual to the boundary of $P$. For details see [DJ] Section 4.2. Let $\tau_M$ denote the tangent bundle of $M^{2n}$. Davis and Januczkiewicz showed that the vector bundle $((\tau_M)_{hT^n} \downarrow DJ(K))$ and $\lambda_\mathbb{R}$ are stably isomorphic as real vector bundles over $DJ(K_P)$ [DJ Section 6]. We can draw the following corollary of Theorem 1.1.

**Corollary 1.2.** Let $M^{2n}$ be a quasitoric manifold over a simple polytope $P$. Let $K_P$ be the simplicial complex dual to the boundary of $P$. If the tangent bundle $\tau_M$ of $M^{2n}$ is stably equivariantly isomorphic to a direct sum of $r$ 2-dimensional equivariant $T^n$-bundles over $M^{2n}$, then $K_P$ admits an $r$-coloring.

The paper is organized as follows. For the proof of our main theorem we will need two different models for $DJ(K)$. They are discussed in the next section. In Section 3 we will use some geometric constructions to produce a splitting of $\lambda$ from a given coloring. The final section contains the proof of Theorem 1.1.

If not specified otherwise, $K$ will always denote an $(n-1)$-dimensional finite simplicial complex with $m$-vertices.

We would like to thank Nigel Ray and Natalia Dobrinskaya for many helpful discussions.

2. Models for $DJ(K)$

Let $\text{cat}(K)$ denote the category whose objects are the faces of $K$ and whose arrows are given by the subset relations between the faces. $\text{cat}(K)$ has an initial object given by the empty face. Given a pair $(X,Y)$ of pointed topological space we can define covariant functors

$$X^K, (X,Y)^K : \text{cat}(K) \rightarrow \text{Top}.$$
The functor $X^K$ assigns to each face $α$ the cartesian product $X^α$ and to each morphism $i_{α,β}$ the inclusion $X^α ⊂ X^β$ where missing coordinates are set to the base point $∗$. If $α = ∅$, then $X^α$ is a point. And $(X, Y)^K$ assigns to $α$ the product $X^α × Y^{[m]\setminus α}$ and to $i_{α,β}$ the coordinate wise inclusion $X^α × Y^{[m]\setminus α} ⊂ X^β × Y^{[m]\setminus β}$. The inclusions $X^α ⊂ X^m = X^n$ and $X^α × Y^{[m]\setminus α} ⊂ X^m$ establish inclusions
\[
\text{colim}_{\text{CAT}(K)} X^K \rightarrow X^m, \quad \text{colim}_{\text{CAT}(K)} (X, Y)^K \rightarrow X^m.
\]

We are interested in two particular cases, namely the functor $X^K$ for the classifying space $BT = CP^\infty$ of the 1-dimensional circle $T$ and the functor $(X, Y)^K$ for the pair $(D^2, S^1)$. The colimit
\[
Z_K := \text{colim}_{\text{CAT}(K)} (D^2, S^1)^K
\]
is called the moment angle complex associated to $K$. The inclusions $Z_K ⊂ (D^2)^m ⊂ C^m$ allow to restrict the standard $T^m$-action on $C^m$ to $Z_K$. The Borel construction produces a fibration
\[
q_K : (Z_K)_{hT^m} \rightarrow BT^m
\]
with fiber $Z_K$. Moreover, $B_T K := (Z_K)_{hT^m}$ is a realization of the Stanley-Reisner algebra $Z[K]$ and a model for $DJ(K)$. That is there exists an isomorphism $H^*(B_T K; Z) ≅ Z[K]$ such that the map $H^*(q_K; Z)$ can be identified with the map $Z[m] \rightarrow Z[K]$ [DJ, Theorem 4.8]. We will use this model for geometric construction with our vector bundles.

Buchstaber and Panov gave a different construction for $DJ(K)$. They showed that $c(K) := \text{colim}_{\text{CAT}(K)} BT^K$ is homotopy equivalent to $B_T K$ and that the inclusion
\[
c(K) \rightarrow BT^m
\]
is homotopic to $q_K$ [BP, Theorem 6.29]. In particular, each face $α ∈ K$ defines a map $h_α : BT^α \rightarrow c(K)$. The model $c(K)$ will be used to produce a coloring from a stable splitting of $λ$.

**Remark 2.1.** If $K$ is the triangulation of an $(n-1)$-dimensional sphere, the moment angle complex $Z_K$ is a manifold. In this case, the tangent bundle $τ_Z$ is a $(m+n)$-dimensional $T^m$-equivariant vector bundle, which satisfies the analogue of Corollary [LZ]. If $τ_Z$ is stably equivariantly isomorphic to a direct sum of $r$ 2-dimensional equivariant $T^m$-bundles over $Z_K$, then $K$ admits an $r$ coloring. Again this follows from the fact that $(τ_Z)_{hT^m}$ and $λ_λ$ are stably isomorphic [DJ, Section 6].

### 3. Geometric constructions

The $m$-dimensional torus $T^m$ acts coordinate wise on $C^m$. And the diagonal action of $T^m$ on $C^m × Z_K$ makes the projection $C^m × Z_K \rightarrow Z_K$ onto the second factor into a $T^m$-equivariant complex vector bundle over $Z_K$, denoted by $X$. An application of the Borel construction produces the bundle $λ := λ_{hT^m} \downarrow B_T K$ over $B_T K$ whose total Chern
class is given by \( c(\lambda) = \prod_{i}(1 + v_i) \in \mathbb{Z}[K] \) and whose classifying map is the composition \( B_TK \overset{\phi_K}{\longrightarrow} BT^m \longrightarrow BU(m) \). Since \( T^m \) acts coordinatewise on \( \mathbb{C}^m \), both bundles, \( \lambda' \) and \( \lambda \) split into a direct sum of (equivariant) line bundles. Let \( \mathbb{C}_j \) denote the \( j \)-th component of \( \mathbb{C}^m \). In particular, \( T^m \) acts on \( \mathbb{C}_j \) via the projection \( T^m \longrightarrow T^j \) onto the \( j \)-th component of \( T^m \). The vector bundle \( \lambda'_j := \mathbb{C}_j \times Z_K \) is \( T^m \)-equivariant, and \( \lambda_j := (\lambda'_j)_h T^m \) is a 1-dimensional complex vector bundle over \( B_TK \). We have \( \lambda' \cong \bigoplus_j \lambda'_j \) and \( \lambda \cong \bigoplus_j \lambda_j \). All this can be found in [DJ, Section 6].

If \( g: K \longrightarrow \Delta[r] \) is an \( r \)-coloring we want to construct an equivariant splitting of \( \lambda' \downarrow Z_K \) into a direct sum of \( T^m \)-equivariant complex line bundles and a trivial bundle \( \mathbb{C}^{m-r} \). We will use ideas of Davis and Januczkiewicz discussed in [DJ, Section 6.2]. For each \( i \in [r] \) we denote by \( S_i := g^{-1}(i) \subset [m] \) the preimage of \( i \) and by \( s_i := |S_i| \) the order of \( S_i \). There are two vector bundles associated with \( S_i \), namely the tensor product \( \nu_i := \bigotimes_{j \in S_i} \lambda'_j \) of all complex line bundles associated to the vertices contained in \( S_i \) and the direct sum \( \nu_i := \bigoplus_{j \in S_i} \lambda'_j \) of all these line bundles. Both are \( T^m \)-equivariant vector bundles over \( Z_K \).

Lemma 3.1. For all \( i \in [r] \), there exists an \( T^m \)-equivariant vector bundle isomorphism \( \nu_i \oplus \mathbb{C}^{s_i-1} \longrightarrow \eta_i \).

For simplicial complexes dual to the boundary of simple polytopes, the claim is already stated in [DJ, Section 6.2]. We will give here a different proof.

Proof. For simplification we drop the subindex \( i \) in the notation and assume that \( S = [s] \). We will think of \( \mathbb{C}^{s-1} \subset \mathbb{C}^s \) as the subspace given by \( \{(x_1, \ldots, x_s) \in \mathbb{C}^s | \sum_{k} x_k = 0 \} \). We define a map

\[
  f: \mathbb{C} \times \mathbb{C}^{s-1} \times Z_K \longrightarrow \mathbb{C}^s \times Z_K
\]

by \( f(y, x, z) := (u, z) \) where the \( j \)-th coordinate \( u_j \) of \( u \) is given by \( u_j := y \prod_{k \neq j, k \in [s]} \overline{z}_k + x_j z_j \). Here, \( \overline{z}_k \) denotes the complex conjugate of \( z_k \). If \( T^m \) acts on \( \mathbb{C} \) via the map \( t \mapsto \prod_{j \in [s]} t_j \), trivially on \( \mathbb{C}^{s-1} \) and on \( \mathbb{C}^s \) via the projection \( T^m \longrightarrow T^s \) onto the first \( s \) coordinates, one can easily show that this map is \( T^m \)-equivariant. Moreover, with these actions, the source is the total space of the bundle \( \nu \oplus \mathbb{C}^{s-1} \downarrow Z_K \) and the target the total space of \( \eta \downarrow Z_K \). Since both sides have the same dimension, it is only left to show that \( f \) is fiber wise a monomorphism.

By construction, any subset \( \{j, k \} \subset [s] \) is a missing face in \( K \). Since \( Z_K = \bigcup_{h \in K} (D^2)^{\alpha} \times (S^1)^{[m] \setminus \alpha} \), the space \( (D^2)^{j \in K} \times (S^1)^{[m] \setminus \{i, j \}} \) is not contained in \( Z_K \) and for \( z = (z_1, \ldots, z_m) \in Z_K \) there is at most one coordinate among \( z_1, \ldots, z_s \) which is trivial.

Now we assume that \( f(y, x, z) = (0, z) \). In particular, we have \( x_j z_j = -y \prod_{k \neq j} \overline{z}_k \). If one of the coordinates \( z_j \) vanishes, say \( z_1 = 0 \), then
\[ z_j \neq 0 \text{ for } j \neq 1 \text{ and hence } y = 0 \text{ as well as } x_j = 0 \text{ for } j \neq 1. \]

Since \( \sum_j x_j = 0 \), we also have \( x_1 = 0 \).

If \( z_j \neq 0 \) for all \( j \), then \( x_j \) is defined by \( y \prod_{k \neq j} \bar{x}_k/z_j = y \sum_j \prod_{k \neq j} \bar{x}_k/z_j \). Multiplying with \( \prod_j z_j \) shows
\[
\sum_j x_j \prod_{k \neq j} \bar{x}_k/z_j = 0 \text{ and hence that } y = 0 \text{ as well as } x_j = 0 \text{ for all } j.
\]

This shows that \( f \) is a fiber wise monomorphism and finishes the proof.

**Corollary 3.2.** Let \( K \longrightarrow \Delta[r] \) be an \( r \)-coloring of a finite simplicial complex. Then the following holds:

(i) The bundle \( \lambda' \downarrow Z_K \) splits equivariantly into a direct sum of \( r \) equivariant complex line bundles and a trivial bundle.

(ii) The bundle \( \lambda \downarrow DJ(K) \) splits into a direct sum of \( r \) complex line bundles and a trivial bundle.

**Proof.** By Proposition 3.1 we have
\[
\lambda' \cong \bigoplus_{j=1}^r \bigoplus_{i \in S_j} \lambda'_j \cong \bigoplus_{j=1}^r (\nu'_j \oplus \mathbb{C}^{s_i-1}) \cong \bigoplus_{j=1}^r \nu'_j \oplus \mathbb{C}^{m-r}.
\]

This proves the first part, the second follows from the first by applying the Borel construction.

4. **Proof of Theorem 1.1**

The proof needs some preparation. For topological spaces \( X \) and \( Y \) we denote by \([X, Y]\) the set of homotopy classes of maps from \( X \) to \( Y \) and for two compact Lie groups \( G \) and \( H \) by \( \text{hom}(H, G) \) the set of Lie group homomorphism \( H \longrightarrow G \).

Let \( G \) be a compact connected Lie group with maximal torus \( j : T_G \hookrightarrow G \) and Weyl group \( W_G \). Since the action of \( W_G \) on \( T_G \) is induced by conjugation with elements of \( G \), the composition of \( w \in W_G \) and \( j \) induces a map between the classifying spaces homotopic to \( Bj \). And passing to classifying spaces followed by composing with \( Bj \) induces a map \( \text{hom}(H, T_G) \longrightarrow [BH, BG] \) which factors through the orbit space of the \( W_G \)-action on \( \text{hom}(H, T_G) \) and provides a map \( \text{hom}(H, T_G)/W_G \longrightarrow [BH, BG] \).

The following two facts may be found in [N] and are needed for the proof of our main theorem.

**Theorem 4.1.** [N] Let \( G \) be a connected compact Lie group and \( S \) a torus.

(i) The map \( \text{hom}(S, T_G)/W_G \longrightarrow [BS, BG] \) is a bijection.

(ii) The map \( [BS, BG] \longrightarrow \text{Hom}(H^*(BG; \mathbb{Q}), H^*(BS; \mathbb{Q})) \) is an injection.

The rational cohomology \( H^*(BG; \mathbb{Q}) \cong H^*(BT_G; \mathbb{Q})^{W_G} \) is the ring of polynomial invariants of the induced \( W_G \)-action on the polynomial algebra \( H^*(BT_G; \mathbb{Q}) \). For \( G = SO(2k+1) \) the maximal torus \( T_{SO(2k+1)} = \)
$T^k$ is an $k$-dimensional torus and we can identify $H^*(BTSO(2k+1)); \mathbb{Z}$ with $\mathbb{Z}[k] = \mathbb{Z}[v_1, \ldots, v_k]$. The Weyl group $W_{SO(2k+1)}$ is the wreath product $\mathbb{Z}/2! \Sigma_i$ where $(\mathbb{Z}/2)^k$ acts on $T^k$ via coordinate wise complex conjugation and $\Sigma_k$ via permutations of the coordinates. The rational cohomology of $BSO(2k+1)$ is then given by

$$H^*(BSO(2k+1); \mathbb{Q}) \cong \mathbb{Q}[k]^{\mathbb{Z}/2\Sigma_k} \cong \mathbb{Q}[p_1, \ldots, p_k].$$

The classes $p_i$ are already defined over $\mathbb{Z}$. On the one hand $p_i \in H^i(BSO(2k+1); \mathbb{Z})$ is the universal $i$-th Pontrjagin class for oriented bundles and on the other hand $p_i = (-1)^i \sigma_i(v_1^2, \ldots, v_k^2) \in \mathbb{Z}[k]^{\mathbb{Z}/2\Sigma_k}$ is up to a sign the $i$-th elementary symmetric polynomial in the squares of the generators of $\mathbb{Z}[k]$. In particular, for an oriented $(2k+1)$-dimensional real vector bundle $\rho$ over a space $X$, the total Pontrjagin class $p(\rho) = 1 + \sum_{i=1}^k p_i(\rho)$ determines completely the map $H^*(BSO(2k+1); \mathbb{Q}) \longrightarrow H^*(X; \mathbb{Q})$ induced by the classifying map $\rho: X \longrightarrow BSO(2k+1)$.

**Example 4.2.** Let $\rho: BT^* \longrightarrow BSO(2k+1)$ be the composition of a coordinate wise inclusion $\hat{\rho}: BT^* \longrightarrow BT^k$ followed by the maximal torus inclusion $BT^k \longrightarrow BSO(2k+1)$. Then the total Pontrjagin class of $\rho$ is given by $p(\rho) = \prod_i (1 - v_i^2)$, where we identify $H^*(BT^*; \mathbb{Z})$ with $\mathbb{Z}[v_1, \ldots, v_s]$.

By Theorem 4.1 up to homotopy every vector bundle $\omega: BT^* \longrightarrow BSO(2k+1)$ is the composition of a lift $\hat{\omega}: BT^* \longrightarrow BT^k$ and $B_j$. If $p(\omega) = p(\rho)$ then both maps $\rho$ and $\omega$ are homotopic and the underlying homomorphisms $j_\omega, j_\rho: T^* \longrightarrow T^k$ of the lifts $\hat{\omega}$ and $\hat{\rho}$ differ only by an element of the Weyl group (Theorem 4.1). In particular, since $j_\rho$ is given by a coordinatewise inclusion, the homomorphism $j_\omega$ also is a coordinate wise inclusion possibly followed by complex conjugation on some coordinates.

**Proof of Theorem 4.1:** If $K \longrightarrow \Delta[r]$ is an $r$-coloring, Corollary 3.2 provides the appropriate splitting for $\lambda$. The splitting conditions on $\lambda$ can be put into a hierarchy, a splitting of $\lambda$ establishes a splitting of $\lambda_{\mathbb{R}}$ and a stable isomorphism between $\lambda$ and a direct sum of $r$ complex line bundles, and the two latter conditions a stable isomorphism between $\lambda_{\mathbb{R}}$ and a direct sum of $r$ 2-dimensional real bundles. It is only left to show that this last stable isomorphism allows to construct a coloring.

We will work with the model $c(K)$ for $DJ(K)$ and again describe vector bundles over $c(K)$ by their classifying maps. In particular, for $t \geq 2m$ the real vector bundle $\rho_t := \lambda_{\mathbb{R}} \oplus R^{t-2m}$ is a map $\rho_t: c(K) \longrightarrow BO(t)$. Since we are considering stable splittings, we can pass from $\rho_t$ to $\rho_{t+1}$, if necessary, and assume that $t = 2s + 1$ is odd. This will simplify the discussion. For example, if $t$ is odd, we have $BO(t) \simeq BSO(t) \times B\mathbb{Z}/2$. And since $c(K)$ is simply connected, the bundle $\rho_t$ has a unique orientation given by the first coordinate of the map $\rho_t: c(K) \longrightarrow BSO(t) \times B\mathbb{Z}/2$. We also denote this map by $\rho_t$. The
total Pontrjagin class of $\rho_t$ is given by $p(\rho_t) = \prod_{i=1}^{m}(1-v_i^2)$ [DJ, Section 6].

Let $\phi: BT^r \to BSO(t)$ be the map induced by the composition of the coordinate wise inclusion $T^r \subset T^s$ into the first $r$ coordinates followed by the maximal torus inclusion $T^s = T_{SO(t)} \subset SO(t)$. A splitting $\rho_t \cong (\bigoplus_{j=1}^r \theta_j) \oplus \mathbb{R}^{t-2r}$ establishes a map $\hat{\rho}_t: c(K) \to BT^r$ such that $\phi \hat{\rho}_t \simeq \rho_t$.

Now let $\alpha \in K$ be a face and $h_\alpha: BT^\alpha \to c(K)$ the associated map. The composition $\hat{\rho}_t h_\alpha: BT^\alpha \to BT^r$ determines a unique homomorphism $j_\alpha: T^\alpha \to T^r$. The total Pontrjagin class of $\phi \hat{\rho}_t h_\alpha$ is given by $p(\phi \hat{\rho}_t h_\alpha) = \prod_{i \in \alpha}(1-v_i^2)$. Example 4.2 shows that $j_\alpha$ is a coordinate wise inclusion $T^\alpha \to T^k$ possibly followed by complex conjugation on some coordinates. The coordinate wise inclusion defines an injection $\alpha \to [r]$. Since for any inclusion of faces $\beta \subset \alpha$ the restriction $(\hat{\rho}_t h_\alpha)|_{BT^\beta}$ equals the composition $\hat{\rho}_t h_\beta$, the underlying homomorphisms satisfies the formula $j_\alpha|_\beta = j_\beta$. We can conclude that the collection of all these maps defines a map $[m] \to [r]$, whose restriction to any face of $K$ is an injection. This establishes a non degenerate simplicial map $K \to \Delta[r]$ which is an $r$-coloring for $K$. \qed

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