On the condition of characteristic polynomials

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Abstract. We prove that the expectation of the logarithm of the condition number of each of the zeros of the characteristic polynomial of a complex standard Gaussian matrix is $\Omega(n)$. This may provide an explanation for the common wisdom in numerical linear algebra that advises against computing eigenvalues via root-finding for characteristic polynomials.

AMS subject classifications: 65F35, 65Y20, 68F15
Key words: eigenvalues, characteristic polynomial, condition, random matrices

1 Introduction

Common wisdom in numerical linear algebra advises against computing eigenvalues via root-finding for characteristic polynomials. Thus, the chapter on eigenvalues in Datta’s textbook [6, p. 372] begins with the following lines:

Because the eigenvalues of a matrix $A$ are the zeros of the characteristic polynomial $\det(A - \lambda I)$, one would naively think of computing the eigenvalues of $A$ by finding its characteristic polynomial and then computing its zeros by a standard root-finding method. Unfortunately, this is not a practical approach.

Unfortunately as well, there is no explanation on why this is not a practical approach. Complexity not being an issue (there is a vast literature on efficient algorithms for univariate polynomials root finding [15, 16]), it appears that numerical stability is.

According to Wilkinson [20, p. 13], “almost all of the algorithms developed before the 1950’s for dealing with the unsymmetric eigenvalue problem […] were based on some device for computing the explicit polynomial equation.” But in the early

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fifties this practice was put into question as some examples (notably the “perfidious polynomial”) showed an unexpectedly high sensitivity to root finding (i.e., were very poorly conditioned). Wilkinson writes [20, p. 3]: “Speaking for myself I regard it as the most traumatic experience in my career as a numerical analyst.”

Leaving aside the issue of why it was not expected that the perfidious polynomial would be ill-conditioned, it should suffice to us to observe that the phenomenon of ill-conditioned characteristic polynomials was so common that the practice of computing eigenvalues via finding their zeros was completely abandoned on the grounds of this ill-conditioning. This is explicit in the following passage by Trefethen and Bau [19, p. 190]:

Perhaps the first method one might think of would be to compute the coefficients of the characteristic polynomial and use a rootfinder to extract its roots. Unfortunately [...] this strategy is a bad one, because polynomial rootfinding is an ill-conditioned problem in general, even when the underlying eigenvalue problem is well-conditioned.

While this passage is right to point at numerical stability as the stumbling block for the computation of eigenvalues via root-finding for characteristic polynomials, its choice of words is somehow hapless as the statement “polynomial rootfinding is an ill-conditioned problem in general” may lead to the impression that there is a natural choice for all the involved ingredients, notably, for the basis we take for the space of degree \( n \) polynomials, for the probability distribution we draw the resulting coefficients from, and for the measure of condition used.

Maybe the least controversial of these ingredients is the probability distribution, as it is common practice to use the standard Gaussian (this goes back to the origins of modern numerical analysis [10] and is more recently found in the work by Borgwart [3], Demmel [7], and Smale [18] among others). The most common choice for condition measures is the normwise relative version of the condition number. Coupled with some choice of bases (notably, the standard monomial basis) this condition number is known to be small in general (see, e.g., [5, §17.8]). The fact that the last extends to multivariate polynomial systems is precisely what has allowed the recent advances in zero-finding for these systems [2, 4]. But there may as well be bases with respect to which this is no longer the case. These facts are for the normwise relative measure of condition. It is a must to observe, at this point, that for another common measure of condition, the so called componentwise, they don’t need to be true. We will return to this issue on §4.1.

Returning to the eigenvalue problem, it is known that Gaussian matrices are, on the average, well-conditioned with respect to the eigenpair problem [1, Thm. 2.14]. And, as we just mentioned, typical polynomials are well-conditioned for the computation of their zeros w.r.t. the monomial basis and normwise relative condition number.
Our goal in this paper is to show that in this setting characteristic polynomials of typical matrices are ill-conditioned for the computation of their zeros.

In order to precisely state the result, we recall that the condition of a complex polynomial $f$ at its zero $\zeta \in \mathbb{C}$ is defined as the worst possible magnification of the error in the returned zero $\tilde{\zeta}$ with respect to the size of a perturbation $\tilde{f}$ of $f$. In the relative normwise setting these errors are measured normwise for the polynomial $f$ and they are relative for both $f$ and $\zeta$ (see e.g., [8, §3]). That is,

$$\text{cond}(f, \zeta) := \lim_{\delta \to 0} \sup_{\|\tilde{f} - f\| \leq \delta} \frac{|\tilde{\zeta} - \zeta| \|f\|}{\|f - \tilde{f}\| |\zeta|}. \quad (1)$$

In what follows we assume that the norm above is with respect to the standard monomial basis. We also denote by $\chi_A$ the characteristic polynomial of a complex matrix $A \in \mathbb{C}^{n \times n}$, and we say that $A$ is standard Gaussian when the real and imaginary parts of its entries are independent standard Gaussian random variables. A result by Kostlan (Theorem 3 below) shows that for such a matrix we can individualize its $n$ different eigenvalues by the distribution of their moduli.

The main result of this paper is the following.

**Theorem 1** Suppose that $A \in \mathbb{C}^{n \times n}$ is standard Gaussian and $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A$. For $1 \leq i \leq n$ we have

$$E \ln \text{cond}(\chi_A, \lambda_i) \geq \frac{1}{2} (n - 1) \ln n - 0.79 n - 0.5 i.$$

Moreover, there exists $K > 0$ such that for all $n$ we have

$$\min_{1 \leq i \leq n} E \ln \text{cond}(\chi_A, \lambda_i) \geq 0.05 n - K.$$

For the average logarithm of the condition number we obtain

$$E \left( \frac{1}{n} \sum_{i=1}^{n} \ln \text{cond}(\chi_A, \lambda_i) \right) \geq \frac{1}{2} (n - 1) \ln n - 1.54 n.$$

The third bound in Theorem 1 can be interpreted in terms of the so called standard distribution on the solution manifold. The latter is the set

$$V := \{(A, \lambda) \in \mathbb{C}^{n \times n} \times \mathbb{C} \mid \det(A - \lambda I) = 0\}$$

and the standard distribution on $V$ amounts to drawing $A$ from the standard Gaussian and then drawing one of its (almost surely) $n$ different eigenvalues from the uniform distribution. We denote this standard distribution on $V$ by $\rho_{\text{std}}$. The third bound can then be written as

$$E_{(A, \lambda) \sim \rho_{\text{std}}} \ln \text{cond}(A, \lambda) \geq \frac{1}{2} (n - 1) \ln n - 1.54 n.$$
Note that this implies, via Jensen’s inequality, that

\[ \mathbb{E}_{(A, \lambda) \sim \rho_{\text{std}}} \text{cond}^2(A, \lambda) \geq n^{n-1} e^{-3.08 n}. \]

The following result is a small improvement on this lower bound.

**Theorem 2**

\[ \mathbb{E}_{(A, \lambda) \sim \rho_{\text{std}}} (\text{cond}^2(\chi_A, \lambda)) \geq (n - 1)! 2^n. \]

After laying down some preliminaries, we prove these results in Section 3. Then, in Section 4, we discuss the robustness of Theorem 1 with respect to (some) changes in the way errors are measured for conditioning, and show some computer simulations. The latter are consistent with our findings.

**Acknowledgment.** We are grateful to Dennis Amelunxen for performing the computer simulations.

## 2 Preliminaries

### 2.1 Condition of univariate polynomials

The Euclidean norm of a complex polynomial \( f \in \mathbb{C}[X] \),

\[ f(X) = a_n X^n + a_{n-1} X^{n-1} + \cdots + a_1 X + a_0, \]  \hspace{1cm} (2)

is given by

\[ \|f\|^2 := \sum_{k=0}^{n} |a_k|^2. \]

For a simple zero \( \zeta \in \mathbb{C} \) of \( f \) the condition number \( \text{cond}(f, \zeta) \) in (1) then takes the following form (see, e.g., in [5, §14.1.1])

\[ \text{cond}(f, \zeta) = \frac{\|f\|}{|\zeta|} \frac{1}{|f'(\zeta)|} \| (1, |\zeta|, \ldots, |\zeta|^n) \|. \]  \hspace{1cm} (3)

We won’t use the following result in the proof of Theorems 1 and 2. But as we have not seen it anywhere and it has an interest per se, we offer here a proof.

**Proposition 1** If \( f \) is a complex polynomial of degree \( n \) and \( \zeta \in \mathbb{C} \) a zero of \( f \), then \( \text{cond}(f, \zeta) \geq \frac{1}{n} \).

**Proof.** Write \( r = |\zeta| \) so that we have

\[ \text{cond}(f, \zeta) = \frac{\|f\|}{|f'(\zeta)|} \cdot \frac{\|(1, r, r^2, \ldots, r^n)\|}{r}. \]
Assume $f$ is as in (2). Then
\[
|f'(\zeta)| = |na_n\zeta^{n-1} + \ldots + 2a_2\zeta + a_1| \\
\leq \|a_n, a_{n-1}, \ldots, a_1\| \cdot \|(n\zeta^{n-1}, (n-1)\zeta^{n-2}, \ldots, 2\zeta, 1)\| \\
\leq \|f\| \|(n\zeta^{n-1}, (n-1)\zeta^{n-2}, \ldots, 2r, 1)\| \leq \|f\| \| (r^{n-1}, r^{n-2}, \ldots, r, 1) \|,
\]
the first line by Cauchy-Schwartz. It follows that
\[
\text{cond}(f, \zeta) \geq \frac{\|(1, r, r^2, \ldots, r^n)\|}{nr \|(r^{n-1}, r^{n-2}, \ldots, r, 1)\|} \geq \frac{\|(1, r, r^2, \ldots, r^n)\|}{n \|(r^2, \ldots, r^n)\|} \geq \frac{1}{n}. \quad \square
\]

### 2.2 Distribution of eigenvalues of Gaussian matrices

Let $A = (z_{ij}) \in \mathbb{C}^{n \times n}$ be a random matrix such that for all $i, j$, the real part $\Re(z_{ij})$ and the imaginary part $\Im(z_{ij})$ of $z_{ij}$ are independent standard Gaussian random variables. Note that $E |z_{ij}|^2 = 2$. The resulting distribution of matrices is sometimes called the complex Ginibre ensemble. We will also say that $A$ is standard Gaussian and write $A \sim N(0, \text{Id})$ for this. Ginibre [9] showed that the density of the joint probability distribution of the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $A$ is given by
\[
\rho(\lambda_1, \ldots, \lambda_n) = C_n e^{-\frac{1}{2} \sum_{i=1}^{n} |\lambda_i|^2} \prod_{i<j} |\lambda_i - \lambda_j|^2, \quad (4)
\]
where $C_n^{-1} = 2^{-\frac{n(n+1)}{2}} \pi^n \prod_{k=1}^{n} k!$.

Based on Ginibre’s formula, Eric Kostlan [14] observed that, surprisingly, the squared absolute values $|\lambda_i|^2$ of the eigenvalues $\lambda_i$ of a standard Gaussian matrix $A$ are distributed like independent $\chi^2$ random variables. This insight will be crucial for our analysis.

**Theorem 3** For a standard Gaussian $A \in \mathbb{C}^{n \times n}$ with eigenvalues $\lambda_1, \ldots, \lambda_n$, the set $\{|\lambda_1|^2, \ldots, |\lambda_n|^2\}$ is distributed like the set $\{\chi_2^2, \ldots, \chi_{2n}^2\}$, where $\chi_2, \ldots, \chi_{2n}$ are independent $\chi^2$ random variables with $2, \ldots, 2n$ degrees of freedom, respectively. \(\square\)

### 2.3 Some useful bounds

We collect here some known facts related to $\chi^2$-distributions needed for the proof of the main result.

The psi function, also called the logarithmic digamma function, is defined as the logarithmic derivative of the gamma function:
\[
\psi(x) := \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}.
\]
It satisfies the recursion $\psi(x + 1) = \psi(x) + \frac{1}{x}$ for $x > 0$, and $\psi(1) = -\gamma \approx -0.577$, where $\gamma$ denotes the Euler-Mascheroni constant. Therefore, for positive $m, n \in \mathbb{N}$,

$$\psi(n) = -\gamma + \sum_{k=1}^{n-1} \frac{1}{k} \geq \ln n - \gamma \quad \text{and} \quad \psi(m + n) - \psi(m) = \sum_{k=m}^{m+n-1} \frac{1}{k}. \quad (5)$$

We say that a nonnegative random variable $X$ is $\chi$-distributed with $n$ degrees of freedom, written $X \sim \chi^2_n$, if $X^2 \sim \chi^2_n$.

**Lemma 1** Suppose that $r_1, r_2, \ldots$ is a sequence of independent $\chi$-distributed random variables, where $r_i \sim \chi^2_i$. Then we have for $i, j \geq 1$

$$\mathbb{E} \ln r_i^2 = \psi(i) + \ln 2 \geq \mathbb{E} \ln r_i^2 = \ln 2 - \gamma > 0.1159,$$

and hence

$$\mathbb{E} \ln \frac{r_j^2}{r_i^2 + r_j^2} = \psi(j) - \psi(i + j).$$

Moreover, if $j \geq 2$, we have

$$\mathbb{E} \ln \frac{r_i + r_j}{r_j} \leq \sqrt{\frac{i}{j-1}}.$$

**Proof.** The density of $\chi^2_n$ is given by $\rho(q) = 2^{-\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)^{-1} q^{\frac{n}{2}-1} e^{-\frac{q}{2}}$. Therefore, substituting $v = q/2$,

$$\mathbb{E} \ln \chi_n^2 = \frac{1}{2\pi \Gamma\left(\frac{n}{2}\right)} \int_0^\infty q^{\frac{n}{2}-1} e^{-\frac{q}{2}} \ln(q) \, dq = \frac{1}{\Gamma\left(\frac{n}{2}\right)} \int_0^\infty v^{\frac{n}{2}-1} e^{-v} (\ln(v) + \ln(2)) \, dq = \psi(n/2) + \ln(2),$$

where we used [11, 4.352-1] for the last equality. This shows the first assertion.

For the second assertion we use $\ln \frac{r_i + r_j}{r_j} = \ln \left(1 + \frac{r_i}{r_j}\right) \leq \frac{r_i}{r_j}$. Note that $\frac{r_i^2}{r_j^2}$ has the law of an $F$-distribution with parameters $2i$ and $2j$, whose expectation is known [13, p. 326] to be equal to $2j/(2j - 2)$. Therefore

$$\mathbb{E} \frac{r_i}{r_j} \leq \sqrt{\mathbb{E} \frac{r_i^2}{r_j^2}} = \sqrt{\frac{i}{j-1}}. \quad \square$$

**Lemma 2** Suppose that $r_1, r_2, \ldots$ is a sequence of independent $\chi$-distributed random variables, where $r_i \sim \chi^2_i$. Then we have for $i \geq 2$ and $j \geq 1$

$$\mathbb{E} \ln \frac{r_i r_j}{r_i + r_j} \geq \mathbb{E} \ln \frac{r_i^2 r_j}{r_i^2 + r_j^2} - \frac{1}{2} \ln 2.$$
Proof. We first note that for $i \geq 2$ and $j \geq 1$:
\[
\psi(i) + \psi(j) - \psi(i + j) \geq \psi(2) + \psi(j) - \psi(2 + j).
\] (6)
Indeed, this means $\psi(i + j) - \psi(i) \leq \psi(2 + j) - \psi(2)$, which by (5) is equivalent to
\[
\sum_{k=i}^{i+j-1} \frac{1}{k} = \sum_{\ell=2}^{j+1} \frac{1}{\ell + (i-2)} \leq \sum_{\ell=2}^{j+1} \frac{1}{\ell},
\]
which is obviously true.

Using Lemma 1 and inequality (6) we deduce that
\[
\mathbb{E} \ln \frac{\gamma_{i}^{2} \gamma_{j}^{2}}{\gamma_{i}^{2} + \gamma_{j}^{2}} \geq \mathbb{E} \ln \frac{\gamma_{i}^{2} \gamma_{j}^{2}}{\gamma_{i}^{2} + \gamma_{j}^{2}}.
\] (7)
Now we use the fact $\gamma_{i}^{2} + \gamma_{j}^{2} \leq (\gamma_{i} + \gamma_{j})^{2} \leq 2 (\gamma_{i}^{2} + \gamma_{j}^{2})$ to obtain, for $i \geq 2$,
\[
\mathbb{E} \ln \frac{\gamma_{i} \gamma_{j}}{\gamma_{i} + \gamma_{j}} = \frac{1}{2} \mathbb{E} \ln \frac{\gamma_{i}^{2} \gamma_{j}^{2}}{\gamma_{i}^{2} + \gamma_{j}^{2}} \geq \frac{1}{2} \mathbb{E} \ln \frac{\gamma_{i}^{2} \gamma_{j}^{2}}{\gamma_{i}^{2} + \gamma_{j}^{2}} - \frac{1}{2} \ln 2
\]
\[
\geq \frac{1}{2} \mathbb{E} \ln \frac{\gamma_{i} \gamma_{j}}{\gamma_{i} + \gamma_{j}} - \frac{1}{2} \ln 2 \geq \frac{1}{2} \mathbb{E} \ln \frac{\gamma_{i}^{2} \gamma_{j}^{2}}{\gamma_{i}^{2} + \gamma_{j}^{2}} - \frac{1}{2} \ln 2
\]
\[
= \mathbb{E} \ln \frac{\gamma_{i} \gamma_{j}}{\gamma_{i} + \gamma_{j}} - \frac{1}{2} \ln 2. \quad \square
\]

3 Proofs of the Main Results

Proof of Theorem 1. We denote by $\chi_{A}$ the characteristic polynomial of a complex matrix $A \in \mathbb{C}^{n \times n}$ and by $\{\lambda_{1}, \ldots, \lambda_{n}\}$ the multiset of its eigenvalues, so that $\chi_{A} = (X - \lambda_{1}) \cdots (X - \lambda_{n})$. We rely on Theorem 3 to associate the index $i$ to the eigenvalue satisfying $|\lambda_{i}|^{2} \sim \chi_{2i}$.

In a first step we provide a lower bound for the condition of the pair $(\chi_{A}, \lambda_{i})$ as characterized in (3). Since $(-1)^{n} \det(A)$ is the constant term of $\chi_{A}$, we have
\[
\|\chi_{A}\| \geq |\det A| = |\lambda_{1}| \cdots |\lambda_{n}|.
\]
We also use the facts that, for each $1 \leq i \leq n$, we have $\|(1, |\lambda_{i}|, \ldots, |\lambda_{i}|^{n})\| \geq |\lambda_{i}|^{n-1}$ and
\[
|\chi'_{A}(\lambda_{i})| = \prod_{j \neq i} |\lambda_{j} - \lambda_{i}|.
\]
Replacing the last three relations in (3) we obtain, for each $1 \leq i \leq n$,
\[
\text{cond}(\chi_{A}, \lambda_{i}) = \frac{\|\chi_{A}\|}{|\lambda_{i}|} \frac{1}{|\chi'_{A}(\lambda_{i})|} \|(1, |\lambda_{i}|, \ldots, |\lambda_{i}|^{n})\| \geq \prod_{j \neq i} \frac{|\lambda_{i}||\lambda_{j}|}{|\lambda_{i} - \lambda_{j}|} \geq \prod_{j \neq i} \frac{|\lambda_{i}||\lambda_{j}|}{|\lambda_{i}| + |\lambda_{j}|},
\] (8)
the last by the triangle inequality.

In what follows we write \( r_j := |\lambda_j| \), for \( j = 1, \ldots, n \). Recall, these are independent random variables with \( r_j \sim \chi^2_j \). From (8) we get for fixed \( i \geq 1 \),
\[
\ln \text{cond}(\chi_A, \lambda_i) \geq \sum_{j \neq i} \ln \frac{r_i r_j}{r_i + r_j},
\]
and hence
\[
E \ln \text{cond}(\chi_A, \lambda_i) \geq \sum_{j \neq i} \left( E \ln r_i + E \ln \frac{r_j}{r_i + r_j} \right). 
\tag{9}
\]
To bound the first term in the right-hand side, we combine Lemma 1 with (5) to obtain
\[
E \ln r_i = \frac{1}{2} \psi(i) + \frac{1}{2} \ln 2 \geq \frac{1}{2} \ln i - \frac{1}{2} \gamma + \frac{1}{2} \ln 2. \tag{10}
\]
We bound the second term in the right-hand side of (9) using that \((r_i + r_j)^2 \leq 2(r_i^2 + r_j^2)\),
\[
\ln \frac{r_j}{r_i + r_j} = \frac{1}{2} \ln \frac{r_j^2}{(r_i + r_j)^2} \geq \frac{1}{2} \ln \frac{r_j^2}{2(r_i^2 + r_j^2)} = \frac{1}{2} \ln \frac{r_j^2}{r_i^2 + r_j^2} - \frac{1}{2} \ln 2.
\]
Using Lemma 1 and (5) one more time, we deduce
\[
E \ln \frac{r_j}{r_i + r_j} \geq \frac{1}{2} (\psi(j) - \psi(i + j)) - \frac{1}{2} \ln 2 = -\frac{1}{2} \sum_{k=j}^{i+j-1} \frac{1}{k} - \frac{1}{2} \ln 2.
\]
Replacing this bound and (10) in (9), we obtain
\[
E \ln \text{cond}(\chi_A, \lambda_i) \geq \frac{1}{2} (n - 1) \ln i - \frac{1}{2} \gamma (n - 1) - \frac{1}{2} \sum_{j=1}^{n} \sum_{k=j}^{i+j-1} \frac{1}{k}. \tag{11}
\]
We can further bound the last term in the right-hand term by exchanging the order of summation. For fixed \( 1 \leq k \leq i + n - 1 \) we have at most \( k \) choices of \( j \), since \( 1 \leq j \leq k \). Therefore,
\[
\frac{1}{2} \sum_{j=1}^{n} \sum_{k=j}^{i+j-1} \frac{1}{k} \leq \frac{1}{2} \sum_{k=1}^{n+i-1} \frac{k}{k} = \frac{1}{2} (n + i - 1).
\]
We thus obtain from (11)
\[
E \ln \text{cond}(\chi_A, \lambda_i) \geq \frac{1}{2} (n - 1) \ln i - \frac{1}{2} \gamma n + \frac{1}{2} - \frac{1}{2} \sum_{j=1}^{n} \sum_{k=j}^{i+j-1} \frac{1}{k} \geq \frac{1}{2} (n - 1) \ln i - 0.79 n - 0.5 i, \tag{12}
\]
which proves the first assertion. Averaging this over $i = 1, \ldots, n$, we obtain

$$\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \ln \text{cond}(\chi_{A}, \lambda_{i}) \geq \frac{1}{2} (n-1) \frac{1}{n} \sum_{i=1}^{n} \ln i - 0.79 n - 0.5 \frac{1}{n} \sum_{i=1}^{n} i \geq \frac{1}{2} (n-1) (\ln n - 1) - 0.79 n - \frac{n + 1}{4},$$

where we have used that

$$\sum_{i=1}^{n} \ln i \geq \int_{1}^{n} \ln x \, dx \geq n (\ln n - 1).$$

Therefore,

$$\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \ln \text{cond}(\chi_{A}, \lambda_{i}) \geq \frac{1}{2} (n-1) \ln n - 1.54 n,$$

which proves the third assertion.

Note that the general lower bound (12) is not useful for small values of $i$. This is why the proof of the second assertion on the minimum of the expectations over $1 \leq i \leq n$ needs an extra argument.

We first consider the case where $i = 1$. Lemma 1 tells us that $\mathbb{E} \ln r_1 > 0.057$ and

$$0 \leq \mathbb{E} \ln \frac{r_1 + r_j}{r_j} \leq \sqrt{\frac{1}{j-1}} < 0.007,$$

provided $j \geq j_1 := 20410$, which implies in that case

$$\mathbb{E} \ln \frac{r_1 r_j}{r_1 + r_j} = \mathbb{E} \ln r_1 + \mathbb{E} \ln \frac{r_j}{r_1 + r_j} \geq 0.057 - 0.007 = 0.05.$$

If we denote by $K_1$ the sum of $\mathbb{E} \ln \frac{r_1 + r_j}{r_1 r_j}$ over $1 \leq j \leq j_1 - 1$, then we obtain with (9) that, for all $n \geq j_1$,

$$\mathbb{E} \ln \text{cond}(\chi_{A}, \lambda_1) \geq -K_1 + \sum_{j=j_1}^{n} 0.05 = 0.05 \cdot n - (K_1 + 0.05 j_1). \quad (13)$$

We study now the case where $2 \leq i \leq n$. Combining (9) with Lemma 2 we get

$$\mathbb{E} \ln \text{cond}(\chi_{A}, \lambda_i) \geq \sum_{j \neq i} \mathbb{E} \ln \frac{r_i r_j}{r_i + r_j} \geq \sum_{j \neq i} \left( \mathbb{E} \ln \frac{r_2 r_j}{r_2 + r_j} - \frac{1}{2} \ln 2 \right).$$

We have $\mathbb{E} \ln r_2 - \frac{1}{2} \ln 2 = \frac{1 - \gamma}{2} > 0.2$ by Lemma 1. Choose $j_2 = 180$ such that $\sqrt{\frac{3}{j_2 - 1}} \leq \frac{1 - \gamma}{4}$. Using again Lemma 1 we obtain, for $j \geq j_2$,

$$\mathbb{E} \ln \frac{r_2 r_j}{r_2 + r_j} - \frac{1}{2} \ln 2 = \mathbb{E} \ln r_2 - \frac{1}{2} \ln 2 + \mathbb{E} \ln \frac{r_j}{r_2 + r_j} \geq \frac{1 - \gamma}{2} - \sqrt{\frac{2}{j-1}} \geq \frac{1 - \gamma}{4}.$$
Writing
\[ K_2 := -\min_{2 \leq i < j_2} \sum_{\substack{1 \leq j < j_2 \atop j \neq i}} \left( \mathbb{E} \ln \frac{r_{2r_j}}{r_2 + r_j} - \frac{1}{2} \ln 2 \right) \]
we conclude that for all \( 2 \leq i \leq n \) and all \( n \geq j_2 \),
\[ \mathbb{E} \ln \text{cond}(\chi_A, \lambda_i) \geq -K_2 + \sum_{\substack{j_2 \leq j \leq n \atop j \neq i}} \frac{1 - \gamma}{4} \geq \frac{1 - \gamma}{4} (n - j_2) - K_2 \geq 0.1 n - (0.11 j_2 + K_2). \]
The second assertion of the theorem follows from this bound and (13) taking
\[ K := \max\{0.05 j_1 + K_1, 0.11 j_2 + K_2\}. \]

**Proof of Theorem 2.** The joint density of the eigenvalues of a Gaussian matrix \( A \) given in (4) is invariant under permutations of the \( \lambda_i \)'s. Hence, for all \( i \leq n \),
\[ \mathbb{E}(\text{cond}^2(\chi_A, \lambda_i)) = \mathbb{E}(\text{cond}^2(\chi_A, \lambda_n)). \]

We will therefore compute the expectation for \( i = n \) (note that this is just for notational convenience: \( \lambda_n \) now is not the eigenvalue whose modulus squared is \( \chi_{2n}^2 \)-distributed).

We know that
\[ \text{cond}^2(\chi_A, \lambda_n) \geq \prod_{j=1}^n \frac{\lambda_j^2}{|\lambda_n|^2} \prod_{j<n} \frac{|\lambda_n|^{2n}}{|\lambda_n - \lambda_j|^2} = \prod_{j<n} |\lambda_j|^2 \prod_{j<n} |\lambda_n|^2 \prod_{j<n} |\lambda_n - \lambda_j|^2. \]

Then, because the density of \((\lambda_1, \ldots, \lambda_n)\) is (4),
\[ \mathbb{E}(\text{cond}^2(\chi_A, \lambda_n)) \geq \int \prod_{j<n} |\lambda_j|^2 \prod_{j<n} \frac{|\lambda_n|^{2n}}{|\lambda_n - \lambda_j|^2} C_n e^{-\frac{1}{2} \sum_{i=1}^{n-1} |\lambda_i|^2} \prod_{j<i} |\lambda_i - \lambda_j|^2 d\lambda_1 \ldots d\lambda_n \]
\[ = \int C_n e^{-\frac{1}{2} \sum_{i=1}^{n-1} |\lambda_i|^2} \prod_{j<n} |\lambda_j|^2 \prod_{j<i} |\lambda_i - \lambda_j|^2 d\lambda_1 \ldots d\lambda_{n-1} \cdot \int |\lambda_n|^{2n} e^{-\frac{1}{2} |\lambda_n|^2} d\lambda_n. \]
The first integral yields
\[ \frac{C_n}{C_{n-1}} \mathbb{E}_{A \sim N(0, I_{n-1})} (|\lambda_1|^2 \ldots |\lambda_{n-1}|^2) = \frac{C_n}{C_{n-1}} 2^{n-1} (n - 1)! = \frac{1}{2\pi n} \]
because of Theorem 3 and the fact that \( \frac{C_n}{C_{n-1}} = \frac{1}{\pi n^{2n}} \).

The second integral yields
\[ \int |\lambda|^{2n} e^{-\frac{1}{2} |\lambda|^2} d\lambda = 2\pi \int_0^\infty r^{2n+1} e^{-\frac{1}{2} r^2} dr \]
\[ = 2\pi \int_0^\infty (2u)^n e^{-u} du = \pi 2^n \Gamma(n + 1) = \pi 2^{n+1} n!, \]
where we changed variables \( r = |\lambda| \) and \( u = r^2/2 \) in the first and second equalities, respectively. The result follows. \( \square \)
4 Numerical simulations and additional remarks

4.1 Numerical simulations

We have performed some computer experiments to gauge the actual behavior of conditioning for the characteristic polynomials of typical matrices. Specifically, for each \( n = 2, \ldots, 100 \), we have generated 10,000 Gaussian matrices in \( \mathbb{C}^{n \times n} \) and computed, in each case, the average of the logarithms of the following two quantities:

\[
\text{cond}_{\text{min}}(\chi_A) := \min_{i \leq n} \text{cond}(\chi_A, \lambda_i)
\]

and

\[
\text{cond}_{\text{max}}(\chi_A) := \max_{i \leq n} \text{cond}(\chi_A, \lambda_i).
\]

Then, to visualize the growth of these averages with \( n \) we have plotted (smoothed curves corresponding to) the graphs of the following functions of \( n \),

\[
\frac{\text{Avg}(\ln \text{cond}_{\text{min}}(\chi_A))}{n} \quad \text{and} \quad \frac{\text{Avg}(\ln \text{cond}_{\text{max}}(\chi_A))}{n}.
\]

The resulting figure looks as follows.

![Graph showing the growth of averages](image)

We observe that, on the average, \( \frac{\ln \text{cond}_{\text{min}}(\chi_A)}{n} \) stabilizes on a value around 0.05 whereas \( \frac{\ln \text{cond}_{\text{max}}(\chi_A)}{n} \) grows in what appears to be a logarithmic manner. We therefore plotted the curve for \( \frac{\text{Avg}(\ln \text{cond}_{\text{max}}(\chi_A))}{n \ln(n)} \) and still observed a very gentle growth. So we finally did so for \( \frac{\text{Avg}(\ln \text{cond}_{\text{max}}(\chi_A))}{n \ln(n) \ln(\ln n)} \) (and \( n \geq 4 \)) and obtained the following figure.
The value of \( \frac{\text{Avg}(\ln \text{cond}_{\max}(A))}{n \ln(n) \ln(\ln n)} \) appears to stabilize at around 0.25 but we note that the numerical results here are not enough to determine whether \( E \log \text{cond}_{\max}(\chi_A) \) grows as \( n \ln(n) \) or as \( n \ln(n) \ln(\ln n) \).

The condition number we have considered in all the previous development is defined in a relative normwise manner. It measures errors in the approximation \( \tilde{f} \) of a polynomial \( f \) by the quotient \( \| \tilde{f} - f \| / \| f \| \). It should come as no surprise that in the case of \( \chi_A \) with \( A \) Gaussian, the condition number \( \text{cond}(\chi_A, \lambda) \) will be large for each \( \lambda \). After all, we are allowing errors proportional to \( | \det(\chi_A) \| \) in coefficients which we expect to be much smaller than this determinant. A different way to measure the error in \( f \) is the componentwise. If \( f = a_n X^n + \cdots + a_1X + a_0 \) then we measure the error of an approximation \( \tilde{f} \) by \( \max_{0 \leq i \leq n} | \tilde{a}_i - a_i | / | a_i | \). This leads to the componentwise condition number

\[
\text{Cw}(f, \zeta) := \lim_{\delta \to 0} \sup_{| \tilde{a}_i - a_i | \leq \delta | a_i |} \max_{0 \leq i \leq n} | \tilde{a}_i - a_i | / | \zeta |.
\]

It turns out that this condition number has a simple characterization (take \( m = 1 \) in [21, Thm. 2.1])

\[
\text{Cw}(f, \zeta) = \frac{1}{| \zeta | / | f'(\zeta) |} \sum_{i=0}^{n} | a_i | | \zeta |^i.
\]

We have used this expression to perform some computations, similar to the preceding ones (10,000 Gaussian matrices in \( \mathbb{C}^{n \times n} \) for each of \( n = 2, \ldots, 100 \), to gauge the average behavior of

\[
\text{Cw}_{\max}(\chi_A) := \max_{i \leq n} \text{Cw}(\chi_A, \lambda_i).
\]

The following picture, plotting \( \frac{\text{Avg}(\ln \text{Cw}_{\max}(\chi_A))}{\ln n} \), suggests that \( \text{Cw}_{\max}(\chi_A) \) has a sub-linear (or maybe linear) growth.
The good behavior of $C_{w_{\text{max}}}(\chi_A)$ stands in contrast with the numerical instability observed when computing the zeros of characteristic polynomials. The most likely explanation is that the algorithms that compute $\chi_A$ from $A$ produce forward-errors that are not componentwise small. We have browsed the literature in search of some bound for this forward-error (for some algorithm) and have found none. The closest result we found is in a paper by Ipsen and Rehman that gives upper bounds (Theorem 3.3 and Remark 3.4 in [12]) for componentwise errors in $\chi_A$ due to normwise measured perturbations on $A$. These upper bounds are absolute (i.e., not relative to the moduli of the coefficients of $\chi_A$). The corresponding relative bounds are small for the “extreme” coefficients (corresponding to large and small degree) but may be large for the “middle” coefficients. But it is unclear whether this is so because the relative condition number for these coefficients are large or because the upper bounds in [12] are not sharp.

4.2 Additional remarks

Condition numbers depend on the way errors (for both input data and output) are measured. When norms are used to do so, the choice of a particular norm plays a role as well. In all our previous development we have used the Euclidean norm in the space of polynomials. For reasons to be explained soon, another common choice in this space is the Weyl norm, which, for $f$ as in (2), is given by

$$
\|f\|_W^2 := \sum_{k=0}^{n} \binom{n}{k}^{-1} |a_k|^2. \tag{14}
$$

Again, for a simple zero $\zeta \in \mathbb{C}$ of $f$, the condition number $\text{cond}_W(f, \zeta)$ induced by the Weyl norm is shown (see, e.g., [5, §14.1.1]) to have the form

$$
\text{cond}_W(f, \zeta) = \frac{\|f\|_W}{|\zeta| |f'(\zeta)|} \frac{1}{(1 + |\zeta|^2)^{\frac{n}{2}}}. \tag{15}
$$
Since $\|\chi_A\|_W \geq |\det(A)|$ and $(1 + |\lambda_i|^2)^{\frac{n}{2}} \geq |\lambda_i|^{n-1}$ we have

$$\text{cond}_W(\chi_A, \lambda_i) \geq \prod_{j \neq i} \frac{|\lambda_i||\lambda_j|}{|\lambda_i| + |\lambda_j|}.$$  

That is, (8) holds for $\text{cond}_W(\chi_A, \lambda_i)$ and, hence, Theorem 1 holds as well (as the proof of this theorem is just a bound for the quantity on the right-hand side above).

Yet another change of setting consists of homogenizing the polynomial $f$ so that its zeros $\xi$ now live in $\mathbb{P}^1(\mathbb{C})$. For the homogenization $f^h(X, Z) = a_nX^n + a_{n-1}X^{n-1}Z + \cdots + a_1XZ^{n-1} + a_0Z^n$ of $f$ in (2) it is common to consider the Weyl norm above because it is invariant under the action of the unitary group. That is, for all unitary matrices $U \in \mathbb{C}^{2 \times 2}$,

$$\|f^h(X, Z)\| = \|f^h(U(X, Z))\|_W.$$  

The major difference in the notion of condition, however, now comes from the fact that errors in the zero $\xi$ are measured with the Riemannian distance in $\mathbb{P}^1(\mathbb{C})$, i.e., with angles between lines in $\mathbb{C}^2$. The condition thus obtained is therefore not affected by the distortions produced by the zeros of $f$ in $\mathbb{C}$ becoming large (in modulus). This condition number was introduced by Mike Shub and Steve Smale (they named it $\mu(f^h, \xi)$) who, in addition, showed the following characterization [17],

$$\mu(f^h, [\xi: 1]) = \text{cond}_W(f, \xi) \frac{|\xi|}{\sqrt{1 + |\xi|^2}}.$$  

Here $\xi \in \mathbb{C}$ and $[\xi: 1]$ is its image under the standard inclusion $\mathbb{C} \hookrightarrow \mathbb{P}^1(\mathbb{C})$. Then,

$$\mu(\chi_A^h, [\lambda_i: 1]) = \frac{\|\chi_A^h\|_W}{|\lambda_i|} \frac{1}{|\chi_A'(\lambda_i)|} \frac{(1 + |\lambda_i|^2)^{\frac{n}{2}} |\lambda_i|}{\sqrt{1 + |\lambda_i|^2}} \geq \frac{1}{|\chi_A'(\lambda_i)|} \frac{(|\det(A)|)^{\frac{1}{d}}}{(1 + |\lambda_i|^2)^{\frac{d}{2}}} \geq \prod_{j \neq i} \frac{|\lambda_i||\lambda_j|}{|\lambda_i| + |\lambda_j|}.$$  

That is, (8) also holds for $\mu(\chi_A^h, [\lambda_i: 1])$ and with it, Theorem 1.

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1The writing in terms of $\text{cond}_W(f, \xi)$ is ours. Furthermore, we note that the formula for $\mu$ in [17, p. 6] has a typo — $\|u\|^{d-1}$ should be $\|u\|^d$ — and that we have disregarded the “normalization factor” $d^{1/2}$.  

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