Entanglement classification via integer partitions

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Abstract
In Walter et al. (Science 340:1205, 2013), they gave a sufficient condition for genuinely entangled pure states and discussed SLOCC classification via polytopes and the eigenvalues of the single-particle states. In this paper, for $4n$ qubits, we show the invariance of algebraic multiplicities (AMs) and geometric multiplicities (GMs) of eigenvalues and the invariance of sizes of Jordan blocks (J Bs) of the coefficient matrices under SLOCC. We explore properties of spectra, eigenvectors, generalized eigenvectors, standard Jordan normal forms (SJNFs), and Jordan chains of the coefficient matrices. The properties and invariance permit a reduction in SLOCC classification of $4n$ qubits to integer partitions (in number theory) of the number $2^{2n} - k$ and the AMs.

Keywords SLOCC entanglement classification · Standard Jordan normal form · Jordan blocks · Algebraic multiplicity of eigenvalues · Geometric multiplicity of eigenvalues · Jordan chain · Generalized eigenvectors

1 Introduction
As the subtle properties of entangled states are applied in quantum information and computation, many efforts have contributed to understanding the different ways of entanglement [1]. Clearly, local quantum operations cannot change the non-local properties of a state. The entanglement for two and three qubits is well known. However, it is hard to classify multipartite entanglement for four or more qubits. To reach the purpose, SLOCC equivalence of two states of a multipartite system was proposed and formulated [2,3]. It is known that two states in the same SLOCC equivalence class can do the same tasks of quantum information theory, although with different success probabilities [3–5].

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Dür et al. classified three qubits into six SLOCC classes including the classes GHZ and W, and indicated that there are an infinite number of SLOCC classes for four or more qubits. In the pioneering work [4], Verstraete et al. classified the infinite number of SLOCC classes of four qubits into nine families under determinant one SLOCC by using a generalization of the singular value decomposition. After that, SLOCC classification of four qubits was studied deeply [6–16].

For SLOCC classification of n qubits, the previous articles proposed different SLOCC invariants: for example, the concurrence and 3-tangle [17]; local ranks for three qubits [3]; polynomial invariants [6,7,13,18–27] of which the invariant polynomials of degrees 2 for even n qubits [22], 4 for \( n \geq 4 \) (odd and even) qubits [22,27], and 6 for even \( n \geq 4 \) qubits [25]; the diversity degree and the degeneracy configuration of a symmetric state [28]; ranks of coefficient matrices [29–33]; and the entanglement polytopes [34]. Recently, spectra and SJNFs of 4 by 4 matrices have been used to investigate SLOCC classification of pure states of n qubits [35].

In this paper, we show the invariance of algebraic and geometric multiplicities of eigenvalues and sizes of JBs under SLOCC for 4\( n \) qubits. We investigate properties of spectra, eigenvectors, generalized eigenvectors, SJNFs, and Jordan chains of matrices \( \Phi_{22n+1} \). Via integer partitions, the properties, and the invariance, we classify pure states of 4\( n \) qubits, specially four qubits, under SLOCC.

This paper is organized as follows. In Sect. 2, we show the invariance of algebraic and geometric multiplicities of eigenvalues and sizes of JBs under SLOCC for 4\( n \) qubits. In Sect. 3, via integer partitions, we classify spectra of \( \Phi_{22n+1} \) and pure states of 4\( n \) qubits. In Sect. 4, via integer partitions, we classify SJNFs of \( \Phi_{22n+1} \) and pure states of 4\( n \) qubits.

### 2 Invariant AMs, GMs, and sizes of JBs under SLOCC

Let \( |\psi\rangle = \sum_{i=0}^{2^{4n}-1} a_i |i\rangle \) be any pure state of 4\( n \) qubits, where \( a_i \) are coefficients. It is well known that two 4\( n \)-qubit pure states |\( \psi \rangle \) and |\( \psi' \rangle \) are SLOCC equivalent if and only if there is an invertible local operator \( A_1 \otimes \cdots \otimes A_{4n} \) such that

\[
|\psi'\rangle = A_1 \otimes A_2 \otimes \cdots \otimes A_{4n} |\psi\rangle,
\]

where \( A_i \in CL(\mathbb{C}, 2) \) [3].

Let \( C_{q_1q_2\ldots q_{2n}} (|\psi\rangle) \) be the coefficient matrix of the state |\( \psi \rangle \) of 4\( n \) qubits, i.e. entries of the matrix are the coefficients of the state |\( \psi \rangle \), where \( q_1, q_2, \ldots, q_{2n} \) are chosen as row bits, while \( q_{2n+1}, q_{2n+2}, \ldots, q_{4n} \) are chosen as column bits. Clearly, \( C_{q_1q_2\ldots q_{2n}} \) is a 2\( 2n \) by 2\( 2n \) matrix.

It is known that for any two SLOCC equivalent pure states |\( \psi \rangle \) and |\( \psi' \rangle \) of 4\( n \) qubits, the matrices \( C_{q_1q_2\ldots q_{2n}} \) satisfy the following equation [29,30,33],

\[
C_{q_1q_2\ldots q_{2n}} (|\psi'\rangle) = \Delta_1 C_{q_1q_2\ldots q_{2n}} (|\psi\rangle) \Delta_2,
\]

where \( \Delta_1 = (A_{q_1} \otimes A_{q_2} \otimes \cdots \otimes A_{q_{2n}}) \) and \( \Delta_2 = (A_{q_{2n+1}} \otimes \cdots \otimes A_{q_{4n}})' \). Note that \( A' \) is the transpose of \( A \).
Let
\[
T = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & i & i & 0 \\
0 & -1 & 1 & 0 \\
i & 0 & 0 & -i
\end{pmatrix}
\]  
(3)
and
\[
U = T^\otimes n.
\]  
(4)

It is easy to see that \( T \) and \( U \) are unitary. We make a conjugation of \( C_{q_1q_2...q_{2n}}(|\psi\rangle) \) by the unitary matrix \( U \) in Eq. (4) as follows. Let
\[
\Gamma_{2n}(|\psi\rangle) = UC_{q_1q_2...q_{2n}}(|\psi\rangle)U^+
\]  
(5)
and
\[
\Gamma_{2n}(|\psi\rangle) = UC_{q_1q_2...q_{2n}}(|\psi\rangle)U^+.  
\]  
(6)

Let \( Q_1 = U \Delta_1 U^+ \) and \( Q_2 = U \Delta_2 U^+ \). From Eq. (2), we obtain
\[
UC_{q_1q_2...q_{2n}}(|\psi\rangle)U^+ = U \Delta_1 C_{q_1q_2...q_{2n}}(|\psi\rangle) \Delta_2 U^+
\]  
(7)
\[
= U \Delta_1 U^+ UC_{q_1q_2...q_{2n}}(|\psi\rangle)U^+ U \Delta_2 U^+,  
\]  
(8)
and then,
\[
\Gamma_{2n}(|\psi\rangle) = Q_1 \Gamma_{2n}(|\psi\rangle) Q_2,  
\]  
(9)

Clearly, \( \Gamma_{2n}(|\psi\rangle) \) is not similar to \( \Gamma_{2n}(|\psi\rangle) \).

Let us consider the matrix
\[
\Phi_{2n+1}(|\psi\rangle) = \begin{pmatrix}
\Gamma_{2n}(|\psi\rangle)
\end{pmatrix}^t
\]  
(10)

Via Eq. (9), a calculation derives
\[
\Phi_{2n+1}(|\psi\rangle) = O \Phi_{2n+1}(|\psi\rangle) O',
\]  
(11)
where
\[
O = \begin{pmatrix}
Q_1' \\
Q_2'
\end{pmatrix},
\]  
(12)

Clearly,
\[
O' O = \begin{pmatrix}
Q_1' Q_1 \\
Q_2' Q_2'
\end{pmatrix} = \begin{pmatrix} gI_{2n} \\
hI_{2n}
\end{pmatrix},
\]  
(13)

where
\[
g = \prod_{i=1}^{2n} \det A_{qi}
\]  
(14)
and
\[
h = \prod_{i=1}^{2n} \det A_{q_{2n+i}}
\]  
(15)
from Eqs. (A13, A15) in Appendix A.

Note that neither $Q_i$ nor $O$ is orthogonal except that $A_i \in SL(C, 2)$. Therefore, SLOCC cannot guarantee that $\Phi_{2n+1}(|\psi\rangle)$ and $\Phi_{2n+1}(|\psi\rangle)$ are similar. Anyway, from Eqs. (11, 13) we obtain

$$\Phi_{2n+1}(|\psi\rangle) = O \Phi_{2n+1}(|\psi\rangle) O^t O^{-1}$$

$$= O \Theta O^{-1},$$

(16)

where

$$\Theta = \Phi_{2n+1}(|\psi\rangle) O^t O$$

$$= \Phi_{2n+1}(|\psi\rangle) \left( gI_{2^{2n}} hI_{2^{2n}} \right)$$

$$= \left( g[\Gamma_{2^{2n}}(|\psi\rangle)]^t \right).$$

(17)

In general, a square complex matrix $M$ is similar to a block diagonal matrix

$$J = \begin{pmatrix} J_{i1}(\lambda_1) & & \\ & J_{i2}(\lambda_2) & \\ & & \ddots \\ & & & J_{im}(\lambda_m) \end{pmatrix},$$

(18)

where

$$J_{il}(\lambda_i) = \begin{pmatrix} \lambda_i & 1 \\ & \lambda_i \\ & & \ddots \\ & & & 1 \\ & & & \lambda_i \end{pmatrix}$$

(19)

is a standard Jordan block with the eigenvalue $\lambda_i$, where $i_l$ is the size of the block. Usually, $J$ is written as the direct sum

$$J = J_{i1}(\lambda_1) \oplus J_{i2}(\lambda_2) \oplus \cdots \oplus J_{im}(\lambda_m)$$

(20)

direct sum as

$$J = J_{i1}(\lambda_1) J_{i2}(\lambda_2) \cdots J_{im}(\lambda_m)$$

(21)

by omitting “$\oplus$”. We call Eq. (21) the SJNF of the matrix $M$.

In this paper, we define that two SJNFs

$$J_{i1}(\beta_1) J_{i2}(\beta_2) \cdots J_{ik}(\beta_k)$$

(22)

and

$$J_{i1}(\eta\beta_1) J_{i2}(\eta\beta_2) \cdots J_{ik}(\eta\beta_k),$$

(23)
where \( \eta \neq 0 \), are proportional. For example, the SJNFs \( J_1(1)J_2(1)J_3(2) \) and \( J_1(3)J_2(3)J_3(6) \) are proportional.

Though we cannot guarantee that \( \Phi_{2n+1}(|\psi\rangle) \) and \( \Phi_{2n+1}(|\psi\rangle) \) are similar, we can next show that their spectra and SJNFs are proportional.

From Eq. (17), we have the following result.

**Lemma 1** Spectra and SJNFs of \( \Theta \) and \( \Phi_{2n+1}(|\psi\rangle) \) are proportional.

Property 1 in Appendix B means that their spectra are proportional.

We next show that SJNFs of \( \Theta \) and \( \Phi_{2n+1}(|\psi\rangle) \) are proportional. From the linear algebra, for any JB \( J_\tau(\lambda) \) of \( \Phi_{2n+1}(|\psi\rangle) \), \( \Phi_{2n+1}(|\psi\rangle) \) has a Jordan chain \( v_i, i = 1, 2, \ldots, r \), where \( v_1 \) is the eigenvector of \( \Phi_{2n+1}(|\psi\rangle) \) corresponding to the eigenvalue \( \lambda \). Here, let \( v_i \) be column vectors \( \begin{pmatrix} v_i' \\ v_i'' \end{pmatrix} \), where two blocks \( v_i' \) and \( v_i'' \) are of the same size. In light of Property 3 in Appendix B, we can construct a chain of \( s \) sizes. The following is our argument. Equation (16) implies that \( \sqrt{\sqrt{\lambda}} \) is the eigenvalue of \( \sqrt{\lambda} \).

Thus, Eq. (16) and Lemma 1 lead to the following theorem.

**Theorem 1** If the states \( |\psi\rangle \) and \( |\psi\rangle' \) of 4n qubits are SLOCC equivalent, then spectra and SJNFs of \( \Phi_{2n+1}(|\psi\rangle) \) and \( \Phi_{2n+1}(|\psi\rangle') \) are proportional, respectively.

The following is our argument. Equation (16) implies that \( \Phi_{2n+1}(|\psi\rangle) \) and \( \Theta \) are similar. Therefore, \( \Phi_{2n+1}(|\psi\rangle) \) and \( \Theta \) have the same spectra and SJNFs (ignoring the order of JBs). In light of Lemma 1, \( \Theta \) and \( \Phi_{2n+1}(|\psi\rangle) \) have the proportional spectra and SJNFs.

Restated in the contrapositive, the theorem reads: If spectra or SJNFs of two matrices \( \Phi_{2n+1}(|\psi\rangle) \) in Eq. (10) associated with two 4n-qubit pure states are not proportional, then the two states are SLOCC inequivalent.

For example, for four qubits, let \( |\gamma\rangle = \sum_{i,j,k,l \in \{0,1\}} |ijkl\rangle - |0000\rangle - |1111\rangle \). In light of Theorem 1, one can test that \( |\gamma\rangle \) is inequivalent to the states GHZ, W, Cluster, or the Dicke state \( |2,4\rangle \) under SLOCC.

From Theorem 1, we conclude the following corollary.

**Corollary 1** (Invariant AMs and GMs of eigenvalues and sizes of JBs) If two states \( |\psi\rangle \) and \( |\psi\rangle' \) of 4n qubits are SLOCC equivalent, then the matrices \( \Phi_{2n+1}(|\psi\rangle) \) and \( \Phi_{2n+1}(|\psi\rangle) \) have the same AMs and GMs, and the JBs of \( \Phi_{2n+1}(|\psi\rangle) \) with the eigenvalue \( \lambda \) and the JBs of \( \Phi_{2n+1}(|\psi\rangle') \) with the eigenvalue \( \sqrt{\lambda} \) have the same sizes.

It is easy to derive the corollary from Theorem 1. The following is our detailed argument. Equation (16) implies that \( \Phi_{2n+1}(|\psi\rangle) \) and \( \Theta \) are similar. Therefore, \( \Phi_{2n+1}(|\psi\rangle) \) and \( \Theta \) have the same spectra and SJNFs (ignoring the order of JBs).
(a). For the invariance of AMs

In light of Property 1 in Appendix B, if \( \Phi_{2n+1}(|\psi\rangle) \) has the characteristic polynomial

\[
\det(\lambda I_{2n+1} - \Phi_{2n+1}(|\psi\rangle)) = \lambda^{2k} (\lambda \pm \lambda_1)^{\ell_1} \ldots (\lambda \pm \lambda_s)^{\ell_s},
\]

(24)

where \( \lambda_i \neq 0 \) and \( \lambda_i \neq \lambda_j \) when \( i \neq j \), then \( \Theta \) has the characteristic polynomial

\[
\det(\lambda I_{2n+1} - \Theta) = \lambda^{2k} (\lambda \pm \sqrt{gh}\lambda_1)^{\ell_1} \ldots (\lambda \pm \sqrt{gh}\lambda_s)^{\ell_s}.
\]

(25)

Therefore, \( \Phi_{2n+1}(|\psi\rangle) \) and \( \Phi_{2n+1}(|\psi\rangle) \) have the same AMs.

(b). For the invariance of GMs

From Eqs. (24, 25), if \( \Phi_{2n+1}(|\psi\rangle) \) has the spectrum

\[\{0^{\otimes 2k}, (\pm \lambda_1)^{\otimes \ell_1}, \ldots, (\pm \lambda_s)^{\otimes \ell_s}\},\]

where \( \otimes \ell_i \) stands for the AM \( \ell_i \), then \( \Theta \) has the spectrum

\[\{0^{\otimes 2k}, (\pm \sqrt{gh}\lambda_1)^{\otimes \ell_1}, \ldots, (\pm \sqrt{gh}\lambda_s)^{\otimes \ell_s}\},\]

Let \( \mu_0 \) and \( \mu_0' \) be the GMs of the zero eigenvalue of \( \Phi_{2n+1}(|\psi\rangle) \) and \( \Theta \). In light of Property (2.2) in Appendix B, \( \mu_0 = \mu_0' \). In light of Property 7 in Appendix C, the eigenvalues \( \pm \lambda_i \) of \( \Phi_{2n+1}(|\psi\rangle) \) have the same GM, for example \( \mu_i \). Similarly, the eigenvalues \( \pm \sqrt{gh}\lambda_i \) of \( \Theta \) have the same GM, for example \( \mu_i' \). In light of Property (2.1) in Appendix B, \( \mu_i = \mu_i', i = 1, \ldots, s \). Thus, \( \Phi_{2n+1}(|\psi\rangle) \) has the set of GMs \{\( \mu_0, \mu_1, \ldots, \mu_s \}\}, and \( \Phi_{2n+1}(|\psi\rangle) \) has the set of GMs \{\( \mu_0', \mu_1', \ldots, \mu_s' \}\}. Therefore, \( \Phi_{2n+1}(|\psi\rangle) \) and \( \Phi_{2n+1}(|\psi\rangle) \) have the same GMs.

(c). For the invariance of sizes of JBs

In light of Property (3) in Appendix B, \( \Phi_{2n+1}(|\psi\rangle) \) has a JB with the size of \( r \) corresponding to the eigenvalue \( \lambda \) if and only if \( \Theta \) has a JB with the size of \( r \) corresponding to the eigenvalue \( \sqrt{gh}\lambda \). The conclusion is also true for the zero eigenvalue. Therefore, the corresponding JBs of \( \Phi_{2n+1}(|\psi\rangle) \) and \( \Phi_{2n+1}(|\psi\rangle) \) have the same sizes.

3 Classification of spectra of matrices \( \Phi_{2n+1}(|\psi\rangle) \) and pure states of \( 4n \) qubits via integer partitions of the number \( 2^{2n} - k \)

In this paper, \( \ell \) in \( \lambda^{\otimes \ell} \) indicates the AM of the eigenvalue \( \lambda \). If \( \ell = 1 \), then we write \( \lambda^{\otimes 1} \) as \( \lambda \). In this paper, let \( P(i) \) be the number of integer partitions of \( i \). Specially, \( P(0) = 1 \).
3.1 For 4n qubits via integer partitions

By means of Property 1 in Appendix C, spectra of $\Phi_{2^{2n}+1}(|\psi\rangle)$ in Eq. (10) are of the following form:

$$\{0^{\otimes 2k}, (\pm \lambda_1)^{\otimes \ell_1}, (\pm \lambda_2)^{\otimes \ell_2}, \ldots, (\pm \lambda_s)^{\otimes \ell_s}\},$$  \hspace{1cm} (26)

where $\lambda_i \neq 0, \lambda_i \neq \lambda_j$ when $i \neq j, 0 \leq k \leq 2^{2n}$, and $\ell_i$ is the AM of the eigenvalues $\pm \lambda_i$. Clearly, all the AMs satisfy the equation

$$2(\ell_1 + \ell_2 + \cdots + \ell_s) + 2k = 2^{2n+1}. \hspace{1cm} (27)$$

Because the eigenvalues $\pm \lambda_i$ have the same AM $\ell_i$, for the sake of simplicity and without loss of generality, we ignore $\pm$ in Eq. (26) when calculating AMs below.

3.1.1 A set of AMs is invariant under SLOCC and just an integer partition of the number $2^{2n} - k$

Let $\Xi$ be a set of AMs of eigenvalues in Eq. (26). Then,

$$\Xi = (2k; \ell_1, \ell_2, \ldots, \ell_s),$$  \hspace{1cm} (28)

where $2k$ is the AM of the zero eigenvalue, while $\ell_1, \ell_2, \ldots, \ell_s$ are the AMs of the different nonzero eigenvalues. From Eq. (27), it is clear that $(\ell_1, \ell_2, \ldots, \ell_s)$ is just an integer partition of the number $2^{2n} - k$, i.e.

$$\ell_1 + \ell_2 + \cdots + \ell_s = 2^{2n} - k. \hspace{1cm} (29)$$

In light of Corollary 1, $\Xi$ is invariant under SLOCC.

3.1.2 Classification via integer partitions of the number $2^{2n} - k$

Spectra are partitioned into different types  Next, we use $\Xi$ to label spectra ignoring values of eigenvalues. For example, we write $(0; 1, 1, 2)$ to label the spectrum $\{\lambda_1, \lambda_2, \lambda_3^{\otimes 2}\}$ of a matrix $\Phi_8$, where $(1, 1, 2)$ is an integer partition of 4.

We define that spectra of matrices $\Phi_{2^{2n}+1}(|\psi\rangle)$ in Eq. (10) belong to the same type if the spectra have the same AMs, i.e. the same $\Xi$ ignoring values of the eigenvalues. Thus, for spectra of the same type, the sets of AMs of nonzero eigenvalues are the same partition $(\ell_1, \ell_2, \ldots, \ell_s)$ of the number $2^{2n} - k$ for the same $k$.

For four qubits, we obtain 12 different types of spectra of $\Phi_8$ without considering permutations of qubits in Table 1. In Table 1, $SP$ is short for a spectrum.
Pure states are partitioned into different groups  By letting pure states of $4n$ qubits with the same type of spectra of $\Phi_{22n+1}(\ket{\psi})$ in Eq. (10) belong to the same group, then each group can be characterized with a set $\Xi$ of AMs. Thus, SLOCC classification of $4n$ qubits is reduced to calculating integer partitions of the number $2^{2n} - k$ for each $k$, where $0 \leq k \leq 2^{2n}$.

One can know that for each partition $(\ell_1, \ell_2, \ldots, \ell_s)$ of $2^{2n} - k$, $(2k; \ell_1, \ell_2, \ldots, \ell_s)$ corresponds to a set of AMs of eigenvalues in Eq. (26). Different partitions of $2^{2n} - k$ correspond to different types of spectra and different groups of pure states. In light of Corollary 1, two pure states of $4n$ qubits belonging to different groups are SLOCC inequivalent.

For the fixed $k$, from Eq. (29) there are $P(2^{2n} - k)$ different partitions of $2^{2n} - k$. For all $k$, a calculation yields $\sum_{i=0}^{2_n} P(i)$ different partitions. From this, we can conclude the following theorem.

**Theorem 2**  Via partitions of $2^{2n} - k$, the matrices $\Phi_{22n+1}(\ket{\psi})$ in Eq. (10) have $\sum_{i=0}^{2_n} P(i)$ different types of spectra and pure states of $4n$ qubits are classified into $\sum_{i=0}^{2_n} P(i)$ different groups under SLOCC.

### 3.2 Classification of four qubits via integer partitions of $4 - k$

We first calculate partitions of $4 - k$, where $0 \leq k \leq 4$. For example, for $k = 1, 3$ ($= 4 - 1$) can be partitioned in the three distinct ways: $3, 1 + 2$, and $1 + 1 + 1$. Then, from the three partitions we obtain three sets of AMs: $(2;3), (2;1,2)$, and $(2;1,1,1)$. For all $k$, there are 12 integer partitions. So, there are 12 types of spectra of $\Phi_8$ and 12 groups of pure states without considering permutations of qubits. Refer to Table 1.
3.3 Detect genuinely entangled states of $4n$ qubits via the invariant $\Xi$

For four qubits, 3 of 12 groups in the first column of Table 3 include product states and we label the 3 groups with $\prec$. Thus, other 9 groups are genuinely entangled, i.e. each state of the 9 groups is genuinely entangled. For example, it is easy to check that $|\Upsilon\rangle$ is genuinely entangled. Note that when calculating the invariant $\Xi$ for product states, we use the coefficient matrix $C_{12}(|\psi\rangle)$.

For $4n$ qubits, if the spectrum of the matrix $\Phi_{22n+1}(|\psi\rangle)$ does not belong to the types which include spectra of the matrices $\Phi_{22n+1}(|\psi\rangle)$ in Eq. (10) for product states, then the state $|\psi\rangle$ is a genuinely entangled state.

4 Classification of SJNFs of matrices $\Phi_{22n+1}(|\psi\rangle)$ and pure states of $4n$ qubits via integer partitions of AMs

In this paper, we write the direct sum $J_m(\lambda) \oplus \cdots \oplus J_m(\lambda)$ as $J_m(\lambda) \oplus^j$ and the JB $J_1(a)$ as $a$.

4.1 The relation between the set of sizes of JBs with the zero eigenvalue and the integer partition of the AM of the zero eigenvalue

Let $P^*(2k)$ be the number of different SJNFs with the spectrum $0^{\otimes 2k}$ by Properties 1, 3, and 5 in Appendix C, where $P^*(0) = 1$. To calculate $P^*(2k)$, we give the following definition.

**Definition** If $m$ is partitioned into an even number of parts and in the partition if a part is an even number then the number of its occurrences is also even, then the partition is called a tri-even partition of $m$. For example, the partition $2 + 2 + 3 + 1 = 8$ is a tri-even partition of 8 because 8 is partitioned into four parts and “2” occurs twice.

One can check that in light of Properties 1, 3, and 5 in Appendix C, the set of sizes of JBs with the zero eigenvalue must be a tri-even partition of $2k$ for the spectrum $0^{\otimes 2k}$. Conversely, the JBs with the zero eigenvalue, of which the set of sizes is a tri-even partition of $2k$ for the spectrum $0^{\otimes 2k}$, must satisfy Properties 1, 3, and 5 in Appendix C.

For the spectrum $0^{\otimes 2n+1}$, we do not consider the integer partition $(1, \ldots, 1)$ of $2^{2n+1}$ which implies that the corresponding SJNF is the zero matrix, then $\Phi_{2^{2n+1}}(|\psi\rangle) = 0$, and then, all the coefficients of the corresponding state vanish.

Let $2k$ be a set of all the tri-even partition of $2k$, where $\emptyset = \phi$, which is the empty set. A simple calculation yields that $P^*(2) = 1$, $P^*(4) = 3$, $P^*(6) = 5$, and $P^*(8) = 10$. 

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4.2 Classification for $4n$ qubits via integer partitions of AMs

In light of Property 3 in Appendix C, the number of JBs corresponding to the zero eigenvalue of $\Phi_{2n+1}(|\psi\rangle)$ in Eq. (10) is even. In light of Property 7 in Appendix C, the numbers of JBs corresponding to the nonzero eigenvalues $\pm \lambda$ of $\Phi_{2n+1}(|\psi\rangle)$ in Eq. (10) are the same. In light of Property 8 in Appendix C, the corresponding JBs with the nonzero eigenvalues $\pm \lambda$ have the same size. Thus, SJNFs of $\Phi_{2n+1}(|\psi\rangle)$ in Eq. (10) with the spectrum in Eq. (26) are of the following form:

$$J_{\tau_1}(0) \ldots J_{\tau_{2m}}(0) J_{\alpha_1}(\pm \lambda_1) \ldots J_{\alpha_l}(\pm \lambda_1)$$
$$J_{\beta_1}(\pm \lambda_2) \ldots J_{\beta_l}(\pm \lambda_2) \ldots J_{\gamma_1}(\pm \lambda_s) \ldots J_{\gamma_{ls}}(\pm \lambda_s).$$

(30)

For four qubits, there are 43 different SJNFs of $\Phi_8$ in Table 2 without considering permutations of qubits. Note that in Table 2, $\lambda_i$ are the eigenvalues of $\Phi_8$, where $\lambda_i \neq 0$ and $\lambda_i \neq \lambda_j$ when $i \neq j$.

Note that Table 2 does not include the SJNFs: $\pm \lambda_1 J_2(0) J_4(0)$, $J_2(0) J_6(0)$ or $00 J_2(0) J_4(0)$. This is because these SJNFs do not satisfy Property 5.1 in Appendix C.

Note that each pair of JBs like $J_{\alpha_1}(\pm \lambda_1)$ in Eq. (30) have the same size. For the sake of simplicity and without loss of generality, we ignore $\pm$ in JBs in Eq. (30) when calculating sizes of JBs below. For example, for the SJNF $J_2(\pm \lambda_1) J_2(\pm \lambda_2)$ of $\Phi_8$, we only consider the sizes of the JBs $J_2(\lambda_1)$ and $J_2(\lambda_2)$, ignoring the sizes of the JBs $J_2(-\lambda_1)$ and $J_2(-\lambda_2)$.

4.2.1 A collection of sets of sizes of JBs with different eigenvalues is invariant under SLOCC and just a list of partitions of AMs

In Eq. (30), let $\tau$ be a set of sizes of JBs with the zero eigenvalue and $\pi_1$ (resp. $\pi_2$, ..., $\pi_s$) be a set of sizes of JBs with the eigenvalue $\lambda_1$ (resp. $\lambda_2$, ..., $\lambda_s$). From Eq. (30), we obtain

$$\tau = (\tau_1, \ldots, \tau_{2m}),$$
$$\pi_1 = (\alpha_1, \ldots, \alpha_l),$$
$$\pi_2 = (\beta_1, \ldots, \beta_l),$$
$$\vdots$$
$$\pi_s = (\gamma_1, \ldots, \gamma_{ls}).$$

Let

$$\vartheta = \{\tau; \pi_1, \pi_2, \ldots, \pi_s\}.$$  

(31)

In light of Corollary 1, $\vartheta$ is invariant under SLOCC.

Clearly, each SJNF can be described by the $\vartheta$. For example, for the SJNF $J_2(\lambda_1) J_2(\lambda_2)$, $\tau = \phi$ and $\vartheta = \{\phi; (2), (2)\}$. For the SJNF $J_2(\lambda_1) \lambda_2 \lambda_2$, $\vartheta = \{\phi; (2), (1, 1)\}$. We call $\vartheta$ the label of the SJNF.
From the above discussion, $\tau$ is just a tri-even partition of $2k$ (here, $2k$ is the AM of the zero eigenvalue), and $\pi_1$ (resp. $\pi_2, \ldots, \pi_s$) is just a partition of $\ell_1$ (resp. $\ell_2, \ldots, \ell_s$) which is the AM of the eigenvalue $\lambda_1$ (resp. $\lambda_2, \ldots, \lambda_s$). Refer to Eq. (26).

In this paper, let $\vec{l}$ stand for a set of all the integer partitions of $l$. For example, $\vec{2} = \{(2), (1, 1)\}$ and $\vec{3} = \{(3), (2, 1), (1, 1, 1)\}$. Thus, $\tau \in 2\vec{k}$, $\pi_i \in \vec{l_i}, i = 1, \ldots, s$. Clearly, $\vartheta$ is also a list of partitions of AMs (refer to Eq. (26)), and thus, each SJNF corresponds to a list of partitions of AMs ignoring values of eigenvalues.

### Table 2

| $SP$ | SJNF | SJNF |
|------|------|------|
| $SP_1$ | $J_4(\pm \lambda_1)$ | $J_2(\pm \lambda_1) J_2(\pm \lambda_1)$ |
|       | $J_3(\pm \lambda_1) \pm \lambda_1$ | $(\pm \lambda_1)^{\oplus 2} J_2(\pm \lambda_1)$ |
|       | $(\pm \lambda_1)^{\oplus 4} \frac{\pi_1}{2}$ | $J_2(\pm \lambda_1) J_2(\pm \lambda_2)$ |
| $SP_2$ | $\pm \lambda_1 J_3(\pm \lambda_2)$ | $\pm \lambda_1 \pm \lambda_2 J_2(\pm \lambda_2)$ |
|       | $\pm \lambda_1(\pm \lambda_2)^{\oplus 3}$ | |
| $SP_3$ | $\pm \lambda_1 \pm \lambda_2 J_2(\pm \lambda_3)$ | $\pm \lambda_1 \pm \lambda_2(\pm \lambda_3)^{\oplus 2}$ |
| $SP_4$ | $(\pm \lambda_1)^{\oplus 2}(\pm \lambda_2)^{\oplus 2}$ | $(\pm \lambda_1)^{\oplus 2} J_2(\pm \lambda_2)$ |
|       | $J_2(\pm \lambda_1) J_2(\pm \lambda_2)$ | |
| $SP_5$ | $\pm \lambda_1 \pm \lambda_2 \pm \lambda_3 \pm \lambda_4$ | $00 J_2(\pm \lambda_1) \pm \lambda_1$ |
|       | $00(\pm \lambda_1)^{\oplus 3}$ | |
| $SP_6$ | $00 J_3(\pm \lambda_1)$ | $00 J_2(\pm \lambda_1) \pm \lambda_1$ |
|       | $00(\pm \lambda_1)^{\oplus 3}$ | |
| $SP_7$ | $00 \pm \lambda_1 J_2(\pm \lambda_2)$ | $00 \pm \lambda_1 J_1(\pm \lambda_2)^{\oplus 2}$ |
| $SP_8$ | $00 \pm \lambda_1 \pm \lambda_2 \pm \lambda_3$ | $00 \pm \lambda_1 \pm \lambda_2 \pm \lambda_3$ |
| $SP_9$ | $J_2(0)^{\oplus 2} J_2(\pm \lambda_1)$ | $J_2(0)^{\oplus 2}(\pm \lambda_1)^{\oplus 2}$ |
|       | $J_3(0) 0 J_2(\pm \lambda_1)$ | $J_3(0) 0(\pm \lambda_1)^{\oplus 2}$ |
|       | $0^{\oplus 4} J_2(\pm \lambda_1)$ | $0^{\oplus 4} \pm \lambda_1 \pm \lambda_1$ |
| $SP_{10}$ | $J_2(0)^{\oplus 2} \pm \lambda_1 \pm \lambda_2$ | $J_3(0) 0 \pm \lambda_1 \pm \lambda_2$ |
|       | $0^{\oplus 4} \pm \lambda_1 \pm \lambda_2$ | |
| $SP_{11}$ | $0 J_3(0) \pm \lambda_1$ | $J_3(0)^{\oplus 2} \pm \lambda_1$ |
|       | $J_3(0)^{\oplus 3} \pm \lambda_1$ | $J_2(0)^{\oplus 2} 0 \pm \lambda_1$ |
|       | $0^{\oplus 6} \pm \lambda_1 \frac{\pi_2}{2}$ | |
| $SP_{12}$ | $J_7(0) 0$ | $J_5(0) J_3(0)$ |
|       | $J_4(0)^{\oplus 2}$ | $J_5(0)^{\oplus 2} J_3(0)^{\oplus 2} \frac{\pi_4}{2}$ |
|       | $J_2(0)^{\oplus 4} \frac{\pi_3}{2}$ | $J_3(0)^{\oplus 2} 0^{\oplus 2} \frac{\pi_4}{2}$ |
|       | $J_2(0)^{\oplus 2} J_3(0) 0 \frac{\pi_5}{2}$ | $J_3(0)^{\oplus 2} \frac{\pi_5}{2}$ |
|       | $J_2(0)^{\oplus 2} 0^{\oplus 4} \frac{\pi_6}{2}$ | $J_2(0)^{\oplus 2} 0^{\oplus 4} \frac{\pi_7}{2}$ |

$\frac{\pi_1}{2}$ includes the product states $|\text{EPR}_{13}| |\text{EPR}_{24}$ and $|\text{EPR}_{14}| |\text{EPR}_{23}$.

$\frac{\pi_2}{2}$ includes the product state $|\text{EPR}_{12}| |\text{EPR}_{34}$.

$\frac{\pi_3}{2}$ includes the product states $|00| |\text{EPR}_{24}$, $|00| |\text{EPR}_{23}$, $|00| |\text{EPR}_{14}$, and $|00| |\text{EPR}_{13}$.

$\frac{\pi_4}{2}$ includes the product states $|i\rangle |\text{GHZ}_{jkl}$.

$\frac{\pi_5}{2}$ includes the product states $|i\rangle |\text{W}_{jkl}$.

$\frac{\pi_6}{2}$ includes the full separate state $|0000\rangle$.

$\frac{\pi_7}{2}$ includes the product states $|00| |\text{EPR}_{12}| |\text{EPR}_{34}$ and $|00| |\text{EPR}_{12}$.
4.2.2 Classification of SJNFs and pure states via integer partitions of AMs

SJNFs are partitioned into different types For example, for the SJNFs $J^2(\lambda_1)\lambda_2\lambda_2$ and $\lambda_1\lambda_1 J^2(\lambda_2)$, one can see that one of the two SJNFs can be obtained from the other one by renaming $\lambda_1$ as $\lambda_2$ and $\lambda_2$ as $\lambda_1$ simultaneously. Here, we consider that these two SJNFs possess the same type. Note that for the SJNF $J^2(\lambda_1)\lambda_2\lambda_2$, the label is $\vartheta = \{\phi; (2), (1, 1)\}$, and for the SJNF $\lambda_1\lambda_1 J^2(\lambda_2)$, the label is $\vartheta' = \{\phi; (1, 1), (2)\}$. Here, we also consider that $\vartheta = \vartheta'$ ignoring the order of $(2)$ and $(1, 1)$.

Generally, for two labels $\vartheta = \{\tau; \pi_1, \pi_2, \ldots, \pi_s\}$ and $\vartheta' = \{\tau'; \pi'_1, \pi'_2, \ldots, \pi'_s\}$, we define that $\vartheta = \vartheta'$ if and only if

$$\tau = \tau'$$

and

$$\{\pi_1, \pi_2, \ldots, \pi_s\} = \{\pi'_1, \pi'_2, \ldots, \pi'_s\}$$

ignoring the order of $\pi_1, \pi_2, \ldots, \pi_s$ and the order of $\pi'_1, \pi'_2, \ldots, \pi'_s$.

We can next define that two SJNFs possess the same type if and only if their labels are equal.

For four qubits, we obtain 43 types of SJNFs. Refer to Table 2 and the second and third columns of Table 3.

Pure states are partitioned into different families By letting states with the same type of SJNFs of $\Phi_{22n+1}(|\psi\rangle)$ in Eq. (10) belong to the same family, then each family can be described with an invariant $\vartheta = \{\tau; \pi_1, \pi_2, \ldots, \pi_s\}$. Thus, SLOCC classification of $4n$ qubits is reduced to calculating integer partitions of AMs.

We next explain how to calculate all the integer partitions of AMs. We first calculate partitions of $2^{2n} - k$ for each $k$, where $0 \leq k \leq 2^{2n}$. Then, for each partition $(\ell_1, \ell_2, \ldots, \ell_s)$ of $2^{2n} - k$, we calculate partitions of $\ell_i$ ($i = 1, \ldots, s$) and tri-even partitions of $2k$. Conversely, let $\tau \in 2k, \pi_i \in \ell_i, i = 1, \ldots, s$. Then, the list of partitions $\vartheta = \{\tau; \pi_1, \pi_2, \ldots, \pi_s\}$ corresponds to a collection of sets of sizes of JBs of $\Phi_{22n+1}(|\psi\rangle)$ in Eq. (10) for $4n$ qubits.

One can see that different $\vartheta$ correspond to different types of SJNFs of $\Phi_{22n+1}(|\psi\rangle)$ in Eq. (10) and different families of pure states. In light of Corollary 1, two states belonging to different families are SLOCC inequivalent.

In Appendix D, a calculation shows that there are $\eta - 1$ different lists of partitions of AMs, where $\eta$ is defined in Eq. (D3). Then, we can conclude the following theorem.

**Theorem 3** Via partitions of AMs, i.e. via partitions of $\ell_i$ ($i = 1, \ldots, s$) and tri-even partitions of $2k$ in each partition $(\ell_1, \ell_2, \ldots, \ell_s)$ of $2^{2n} - k$, where $0 \leq k \leq 2^{2n}$,
Table 3  SLOCC classification of four qubits

| Ξ   | φ       | φ       |
|-----|---------|---------|
| (0;4) <₁ | {φ:(4)} | {φ:(2,2)} |
|      | {φ:(1,1,2)} | {φ:(1,1,1,1)} ₁₁ |
|      | {φ:(3,1)} | |
| (0;1,3) | {φ:(1),(3)} | {φ:(1),(1,2)} |
|      | {φ:(1),(1,1,1)} | |
| (0;1,1,2) | {φ:(1),(1),(2)} | {φ:(1),(1),(1,1)} |
|      | {φ:(1),(1,1)} | {φ:(1),(2)} |
|      | {φ:(2),(2)} | |
| (0;1,1,1,1) | {φ:(1),(1),(1),(1)} | |
| (2;3) | {(1,1):(3)} | {(1,1):(2,1)} |
|      | {(1,1):(1,1,1)} | |
| (2;1,2) | {(1,1),(1),(2)} | {(1,1),(1,1,1)} |
|      | {(1,1),(1),(1)} | |
| (2;1,1,1) | {(1,1),(1),(1),(1)} | |
| (4;2) | {(2,2),(2)} | {(2,2):(1,1)} |
|      | {(3,1),(2)} | {(3,1):(1,1)} |
|      | {(1,1,1,1),(2)} | {(1,1,1,1),(1,1)} |
| (4;1,1) | {(2,2),(1),(1)} | {(3,1),(1),(1)} |
|      | {(1,1,1,1),(1),(1)} | |
| (6;1) <₂ | {(1,5),(1)} | {(3,3),(1)} |
|      | {(2,2,1,1),(1)} | {(1,1,1,1,1),(1)} ₂₂ |
|      | {(3,1,1,1),(1)} | |
| (8;₃) <₃ | {(7,1);} | {(5,3);} |
|      | {(4,4);} | {(2,2,2,2);} ₃₃ |
|      | {(3,3,1,1);} ₄₄ | {(2,2,3,1);} ₅₅ |
|      | {(5,1,1,1);} ₆₆ | {(2,2,1,1,1);} ₇₇ |
|      | {(3,1,1,1,1,1);} ₈₈ |

φ is the empty set. (· · · ) is the set of sizes of JBs with the zero eigenvalue
₁₁ includes \(|EPR\)_13\(_13|EPR\)_24\(_24 and |EPR\)_14\(_14|EPR\)_23\(_23
₂₂ includes \(|EPR\)_12\(_12|EPR\)_34\(_34
₃₃ includes \(|00\)_13\(_13|EPR\)_24\(_24, |00\)_14\(_14|EPR\)_23\(_23, |00\)_23\(_23|EPR\)_14\(_14, and \(|00\)_24\(_24|EPR\)_13\(_13
₄₄ includes \(|0\)_1j\(_1j|GHZ\)_jkl\(_jkl
₅₅ includes \(|0\)_j\(_j|W\)_jkl\(_jkl
₆₆ includes \(|0000\)_jkl\(_jkl
₇₇ includes \(|00\)_12\(_12|EPR\)_34\(_34 and |00\)_34\(_34|EPR\)_12\(_12
₁₁ includes \(|EPR\)_13\(_13|EPR\)_24\(_24 and |EPR\)_14\(_14|EPR\)_23\(_23
₅₅ includes \(|EPR\)_12\(_12|EPR\)_34\(_34
₃₃ includes \(|0\)_j\(_j|W\)_jkl\(_jkl, \(0\)_j\(_j|GHZ\)_jkl\(_jkl, \(0\)_j\(_j|EPR\)_jkl\(_jkl, and \(|0000\)_jkl\(_jkl, where \(|GHZ\)_jkl\(_jkl is a 3-qubit GHZ state, \(|W\)_jkl\(_jkl is a 3-qubit W state, and \(|EPR\)_jkl\(_jkl is a 2-qubit EPR state.
\(\Phi_{2n+1}(|\psi\rangle)\) in Eq. (10) has \(\eta - 1\) different types of SJNFs, and then, pure states of \(4n\) qubits are classified into \(\eta - 1\) different families.

### 4.3 Classification of four qubits

We first calculate partitions of \(4 - k\) for each \(k\), where \(0 \leq k \leq 4\). For all \(k\), there are 12 partitions. Then, for each partition \((\ell_1, \ell_2, \ldots, \ell_s)\) of \(4 - k\), we calculate partitions of \(\ell_i\) (\(i = 1, \ldots, s\)) and tri-even partitions of \(2k\). For example, let \(k = 0\), for the partition \((2, 2)\) of 4, we calculate partitions of 2; then, we obtain three different lists of partitions: \(\{\phi; (1, 1), (1, 1)\}, \{\phi; (1, 1), (2)\}, \text{and} \{\phi; (2), (2)\}\).

For four qubits, in total there are 43 different lists of partitions. Thus, we obtain 43 different types of SJNFs and 43 SLOCC inequivalent families of pure states without considering permutations of qubits. Refer to the second and third columns of Table 3.

Furthermore, for each type of SJNFs, we can give a state of four qubits for which \(\Phi_8\) has the corresponding type.

Note that Table 3 does not include the following \(\vartheta\): \(\{(2, 4); (1)\}, \{(2, 6); \}\), \(\{(1, 1, 2, 4); \}\). This is because the corresponding SJNFs do not satisfy Property 5.1 in Appendix C.

### 4.4 Detect genuinely entangled states of \(4n\) qubits via the invariant \(\vartheta\)

For four qubits, 7 of 43 families (refer to the second and third columns of Table 3) include product states and we label the 7 families with \(\ddagger\) in Table 3. Thus, other 36 families are genuinely entangled, i.e. each state of the 36 families is genuinely entangled. For example, it is easy to check that \(|\Upsilon\rangle\) is genuinely entangled. Note that when calculating the invariant \(\vartheta\) for product states we use the coefficient matrix \(C_{12}(|\psi\rangle)\).

One can see that only four families \(L_{\ast}^{\ast}ab_3, L_{a_4}, L_{05\oplus 3}, \text{and} L_{07\oplus 1}\) of Verstraete et al.’s nine families are genuinely entangled, where \(L_{ab}^{\ast}\) is obtained by replacing the last two + signs of \(L_{ab}\) with – signs [9].

For \(4n\) qubits, if the SJNF of the matrix \(\Phi_{2n+1}(|\psi\rangle)\) does not belong to the types which include SJNFs of matrices \(\Phi_{2n+1}(|\psi\rangle)\) in Eq. (10) for product states, then the state \(|\psi\rangle\) is a genuinely entangled state.

### 5 Comparison with Verstraete et al.’s nine families

Via the complex SVD, Verstraete et al. partitioned pure states of four qubits into nine families: \(G_{abcd}, L_{abc2}, L_{a2b2}, L_{ab_3}, L_{a_4}, L_{a_20_3\oplus 1}, L_{0_5\oplus 3}, L_{07\oplus 1}, \text{and} L_{03_10_3\oplus 1}\) up to permutations of the qubits under determinant 1 SLOCC [4].

In this paper, we show that if two pure states of \(4n\) qubits are SLOCC equivalent, then the spectra and SJNFs of their matrices \(\Phi_{2n+1}(|\psi\rangle)\) in Eq. (10) are proportional. It means the invariance of AMs and GMs and the sizes of JBs. Via integer partitions of \(2^{2n} - k\), we can partition pure states of \(4n\) qubits into \(\sum_{i=0}^{2^{2n}} P(i)\) different groups under SLOCC without considering permutations of qubits. Specially, pure states of
four qubits are partitioned into 12 types. Via integer partitions of AMs, we can partition pure states of $4n$ qubits into $\eta - 1$ ($\eta$ is defined in Eq. (D3)) different families under SLOCC without considering permutations of qubits. Specially, pure states of four qubits into are partitioned into 43 families.

Chterental and Djoković pointed out an error in Verstraete et al.’s nine families by indicating that the family $L_{abj}$ is SLOCC equivalent to the subfamily $L_{abc2}(a = c)$ of the family $L_{abc2}$ [9]. The statement was corrected in [33], where it was deduced that when $a \neq 0$, the family $L_{abj}$ is SLOCC equivalent to the subfamily $L_{abc2}(a = c)$ of the family $L_{abc2}$, while $a = 0$, $L_{abj}$ and $L_{abc2}(a = c)$ are SLOCC inequivalent. In light of Theorem 1, we can also show that $L_{abj}(a = 0)$ and $L_{abc2}(a = c = 0)$ are SLOCC inequivalent because the matrices $\Phi_8$ have SJNFs $J_3(0) J_3(0) \pm b$ and $J_2(0) J_2(0) \pm b00$ for $L_{abj}(a = 0)$ and $L_{abc2}(a = c = 0)$, respectively.

For the completeness of Verstraete et al.’s nine families, Chterental and Djoković changed the family $L_{abj}$ as the family $L_{abj}^*$ defined above. A calculation yields that the states $L_{abj}(a = b = 0)$, $L_{abj}^*(a = b = 0)$, and $|0\rangle(|000\rangle + |111\rangle)$, which is the representative state of the family $L_{03\oplus03\oplus1}$, have the same Jordan block structure $J_3(0) J_3(0) 00$ though the states $L_{abj}(a = b = 0)$ and $|0\rangle(|000\rangle + |111\rangle)$, and the states $L_{abj}^*(a = b = 0)$ and $|0\rangle(|000\rangle + |111\rangle)$ are SLOCC inequivalent, respectively. Note that $L_{abj}(a = b = 0)$ is SLOCC equivalent to $L_{abj}^*(a = b = 0)$ [30].

Recall that a family is defined as having Jordan and degenerated Jordan blocks of specific dimension (see the proof of Theorem 2 on page 3 of [4]). So, via the definition for the families, the states $L_{abj}(a = b = 0)$ and $|0\rangle(|000\rangle + |111\rangle)$ should belong to the same family, and the states $L_{abj}^*(a = b = 0)$ and $|0\rangle(|000\rangle + |111\rangle)$ should belong to the same family. Unfortunately, they are partitioned into different families. Clearly, the definition for the families and the representative states are not consistent and pure states of four qubits are partitioned into the nine families incompletely. These errors are avoided in this paper. In this paper, the three states $L_{abj}^*(a = b = 0)$, $L_{abj}(a = b = 0)$, and $|0\rangle(|000\rangle + |111\rangle)$ are included in one family.

6 Summary

In this paper, we show that algebraic and geometric multiplicities of eigenvalues and sizes of JBs of $\Phi_{22n+1}(|\psi\rangle)$ in Eq. (10) are invariant under SLOCC. Thus, we have invariants $\Xi = (2k; \ell_1, \ell_2, \ldots, \ell_s)$ and $\vartheta = \{\tau; \pi_1, \pi_2, \ldots, \pi_s\}$, where $\tau \in 2k$, $\pi_i \in \overline{\ell_i}$ ($i = 1, \ldots, s$), $2k$ is the AM of the zero eigenvalue, $\ell_1, \ell_2, \ldots, \ell_s$ are the AMs of the nonzero eigenvalues, $\overline{\ell_i}$ is a set of all the integer partitions of $\ell_i$, and $(\ell_1, \ell_2, \ldots, \ell_s)$ is just a partition of $2^{2n} - k$. Note that $\vartheta$ is also a collection of sets of sizes of JBs.

For $4n$ qubits, for all $k$ there are $\sum_{i=0}^{2^{2n}} P(i)$ different partitions of $2^{2n} - k$. For four qubits, for all $k$ there are 12 partitions of $4 - k$. Refer to the first column of Table 3. Thus, for $4n$ qubits, we obtain $\sum_{i=0}^{2^{2n}} P(i)$ different types of spectra and then classify pure states of $4n$ qubits into $\sum_{i=0}^{2^{2n}} P(i)$ different groups. Specially, pure states of four (eight) qubits are partitioned into 12 (915) groups.
Furthermore, for each partition \((\ell_1, \ell_2, \ldots, \ell_s)\) of \(2^2n - k\), by calculating partitions of \(\ell_i\) and tri-even partitions of \(2k\) we can obtain \(\eta - 1\) different lists of partitions, then \(\eta - 1\) different types of SJNFs of \(\Phi_{22n+1}(|\psi\rangle)\) in Eq. (10) and \(\eta - 1\) different families of pure states of \(4n\) qubits. Specially, for four qubits, we obtain 43 families. Refer to the second and third columns of Table 3. We show that 9 of 12 groups and 36 of 43 families are genuinely entangled.

We also show that if spectra or SJNFs of two matrices \(\Phi_{22n+1}(|\psi\rangle)\) in Eq. (10) associated with two \(4n\)-qubit pure states are not proportional, then the two states are SLOCC inequivalent.

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Appendix A: A calculation of \(Q_i Q^t_i\)

We calculate \(Q_i Q^t_i, i = 1, 2\), as follows. First we show that

\[
U^+ U^* = \nu \otimes 2^n,
\]  

(A1)

where \(U^*\) is a complex conjugate of \(U\). Equation (A1) holds from \(T^+ T^* = \nu \otimes \nu\) and \(U^+ U^* = T^+ T^* \otimes \cdots \otimes T^+ T^*\), where \(\nu = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\). Then, a calculation yields

\[
Q_1 Q^t_1 = U \Delta_1 U^+ U^* \Delta_1^t U^t.
\]  

(A2)

Via Eq. (A1),

\[
Q_1 Q^t_1 = U \Delta_1 \nu \otimes 2^n \Delta_1^t U^t.
\]  

(A3)

Using the definitions for \(U\) and \(\Delta_1\), a straightforward calculation derives

\[
Q_1 Q^t_1 = T \otimes (\otimes_{i=1}^{2n} A_{q_i}) \nu \otimes 2^n (\otimes_{i=1}^{2n} A^t_{q_i}) (T^t) \otimes 2^n
\]

\[
= [T (A_{q_1} \otimes A_{q_2}) \nu \otimes 2^n (A^t_{q_1} \otimes A^t_{q_2}) T^t] \otimes \cdots \otimes
\]

\[
[T (A_{q_{2n-1}} \otimes A_{q_{2n}}) \nu \otimes 2^n (A^t_{q_{2n-1}} \otimes A^t_{q_{2n}}) T^t].
\]  

(A4)

Next, we reduce Eq. (A4). It is easy to test

\[
A_i \nu A^t_i = (\det A_i) \nu
\]  

(A5)

and

\[
(A_i \otimes A_j) \nu \otimes 2^n (A^t_i \otimes A^t_j)
\]

\[
= A_i \nu A^t_i \otimes A^t_j \nu A^t_j
\]

\[
= (\det A_i) \nu \otimes (\det A_j) \nu
\]

\[
= (\det A_i)(\det A_j) \nu \otimes 2^n.
\]  

(A6)

(A7)

(A8)
Thus, via Eqs. (A8), (A4) reduces to

\[
Q_1 Q_1' = [T(\det A_{q_1} \det A_{q_2})u^\otimes 2 T'] \otimes \cdots \otimes [T(\det A_{q_{2n-1}} \det A_{q_{2n}})u^\otimes 2 T']
\]

\[
= [(\det A_{q_1}) T u^\otimes 2 T'] \otimes \cdots \otimes [(\det A_{q_{2n-1}}) T u^\otimes 2 T']
\]

\[
= (\Pi_{i=1}^{2n} \det A_{q_i})(T u^\otimes 2 T' \otimes \cdots \otimes T u^\otimes 2 T').
\]  

(A9)

(A10)

(A11)

One can check that

\[
T u^\otimes 2 T' = I_4.
\]  

(A12)

Thus, from Eqs. (A11, A12), we obtain

\[
Q_1 Q_1' = (\Pi_{i=1}^{2n} \det A_{q_i})I_{2^{2n}}.
\]  

(A13)

A calculation also yields

\[
Q_1 Q_1' = Q_1' Q_1.
\]  

(A14)

Similarly,

\[
Q_2 Q_2' = Q_2' Q_2 = (\Pi_{i=1}^{2n} \det A_{q_{2n+i}})I_{2^{2n}}.
\]  

(A15)

**Appendix B: Proportional relations**

Let

\[
M_{2n} = \begin{pmatrix} 0 & m \\ m' & 0 \end{pmatrix},
\]

\[
D_{2n} = \begin{pmatrix} 0 & hm' \\ gm' & 0 \end{pmatrix},
\]

(B1)

(B2)

where \(m\) is an \(n\) by \(n\) matrix and \(g\) and \(h\) are nonzero complex numbers.

Property (1). \(\det(\lambda I - D_{2n}) = \det(\lambda^2 I - ghmm')\). Let \(\lambda^2 = \sigma\). Then, we obtain \(\det(\sigma I - ghmm')\). Let \(a^2\) be an eigenvalue of \(mm'\). Then, \(gha^2\) is an eigenvalue of \(ghmm'\), \(\pm a\) are eigenvalues of \(M_{2n}\), and \(\pm \sqrt{gh} a\) are eigenvalues of \(D_{2n}\). Therefore, spectra of \(D_{2n}\) and \(M_{2n}\) are proportional.

Property (2.1). Let

\[
V = \begin{pmatrix} v' \\ v'' \end{pmatrix},
\]

(B3)

where \(v'\) and \(v''\) are \(n \times 1\) vectors, be an eigenvector of \(M_{2n}\) corresponding to the eigenvalue \(\lambda \neq 0\). Then,

\[
M_{2n} V = \lambda V.
\]  

(B4)
Let
\[ W = \left( \frac{\sqrt{h/g}v'}{v''} \right), \tag{B5} \]

Then, via Eq. (B4) one can check that
\[ D_{2n}W = \sqrt{gh\lambda}W. \tag{B6} \]

It means that \( W \) is an eigenvector of \( D_{2n} \) corresponding to the eigenvalue \( \sqrt{gh\lambda} \).

One can see that if there are \( s \) linearly independent eigenvectors \( \left( \frac{v'_i}{v''_i} \right), \ i = 1, 2, \ldots, s \), corresponding to the eigenvalue \( \lambda \) of \( M_{2n} \), then there are \( s \) linearly independent eigenvectors \( \left( \frac{\sqrt{h/g}v'_i}{v''_i} \right), \ i = 1, 2, \ldots, s \), corresponding to the eigenvalue \( \sqrt{gh\lambda} \) of \( D_{2n} \). Therefore, the eigenvalue \( \lambda \) of \( M_{2n} \) and the eigenvalue \( \sqrt{gh\lambda} \) of \( D_{2n} \) possess the same geometry multiplicity. It implies that \( M_{2n} \) and \( D_{2n} \) have the same number of JBs corresponding to the eigenvalues \( \lambda \) and \( \sqrt{gh\lambda} \).

Property (2.2). Let \( V \) be an eigenvector of \( M_{2n} \) corresponding to the zero eigenvalue. Then, one can check that \( V \) is also an eigenvector of \( D_{2n} \) corresponding to the zero eigenvalue. It means that the zero eigenvalue of \( M_{2n} \) and the zero eigenvalue of \( D_{2n} \) possess the same eigenspace and of course the same geometry multiplicity. Thus, \( M_{2n} \) and \( D_{2n} \) have the same number of JBs corresponding to the zero eigenvalue.

Property (3). \( M_{2n} \) has a JB with the size of \( r \) corresponding to the eigenvalue \( \lambda \) if and only if \( D_{2n} \) has a JB with the size of \( r \) corresponding to the eigenvalue \( \sqrt{gh\lambda} \). The property is also true when \( \lambda = 0 \).

Suppose that \( M_{2n} \) has a JB with the size of \( r \) corresponding to the eigenvalue \( \lambda \) (\( \lambda \) may be zero). Then, there exists a Jordan chain with the size of \( r \) corresponding to the eigenvalue \( \lambda \) \cite{37}. Let the Jordan chain be
\[ v_i = \left( \frac{v'_i}{v''_i} \right), \ i = 1, 2, \ldots, r, \tag{B7} \]

where \( v'_i \) and \( v''_i \) are \( n \times 1 \) vectors, \( v_1 \) is the eigenvector, and \( v_i \) satisfy
\[ (M_{2n} - \lambda I_{2n}) v_i = v_{i-1}, \ i = 2, \ldots, r. \tag{B8} \]

From the Jordan chain, we construct the following chain:
\[ z_1 = \left( \frac{\sqrt{h/g}v'_1}{v''_1} \right), \tag{B9} \]
\[ z_i = \left( \frac{\frac{1}{g(\sqrt{h/g})^{i-1}}v'_i}{(\sqrt{g h})^{i-1}v''_i} \right), \ i = 2, \ldots, r. \tag{B10} \]
One can test that $z_1$ is an eigenvector of $D_{2n}$ corresponding to the eigenvalue $\sqrt{gh}\lambda$. A calculation yields that

$$D_{2n}z_i = \begin{pmatrix} 0 & hm \\ gm^t & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{g(\sqrt{gh})^{i-2}}v'_i \\ \frac{1}{(\sqrt{gh})^{i-1}}v''_i \end{pmatrix}$$

$$= \begin{pmatrix} \frac{h}{g(\sqrt{gh})^{i-2}}mv''_i \\ \frac{g}{g(\sqrt{gh})^{i-1}}v'\end{pmatrix}$$

$$= \begin{pmatrix} \frac{g(\sqrt{gh})^{i-2}mv'_i}{g(\sqrt{gh})^{i-1}} \\ \frac{1}{(\sqrt{gh})^{i-2}}m^tv'_i \end{pmatrix}$$

$$= \begin{pmatrix} \frac{g(\sqrt{gh})^{i-3}mv''_i}{g(\sqrt{gh})^{i-2}} \\ \frac{1}{g(\sqrt{gh})^{i-1}}m^tv''_i \end{pmatrix},$$

$$\sqrt{gh}\lambda I_{2n}z_i = \sqrt{gh}\lambda \begin{pmatrix} \frac{1}{g(\sqrt{gh})^{i-2}}v'_i \\ \frac{1}{(\sqrt{gh})^{i-1}}v''_i \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{g(\sqrt{gh})^{i-3}}\lambda v'_i \\ \frac{1}{(\sqrt{gh})^{i-2}}\lambda v''_i \end{pmatrix},$$

and

$$z_{i-1} = \begin{pmatrix} \frac{1}{g(\sqrt{gh})^{i-2}}v'_{i-1} \\ \frac{1}{(\sqrt{gh})^{i-1}}v''_{i-1} \end{pmatrix}.$$  

From Eqs. (B8, B13, B14, B15), we can show that

$$\left(D_{2n} - \sqrt{gh}\lambda I_{2n}\right)z_i = z_{i-1}, i = 2, \ldots, r.$$  

Thus, we obtain a Jordan chain $z_1, \ldots, z_r$ corresponding to the eigenvalue $\sqrt{gh}\lambda$ of $D_{2n}$. It means that the two Jordan chains have the same size. Note that the Jordan chain $z_1, \ldots, z_r$ corresponds to the JB of size $r$ corresponding to the eigenvalue $\sqrt{gh}\lambda$ of $D_{2n}$. Conversely, it is also true.

**Appendix C: Properties of the matrix $M_{2n}$**

Let

$$M_{2n} = \begin{pmatrix} 0 & m \\ m^t & 0 \end{pmatrix},$$

where $m$ is an $n$ by $n$ matrix. We calculate the characteristic polynomial of $M_{2n}$ below.

$$\det(\lambda I_{2n} - M_{2n}) = \det\left(\lambda^2 I_n - mm^t\right) = \det\left(\lambda^2 I_n - m^tm\right).$$
Equation (C2) leads to Property 1.

**Property 1**

1.1. \( \lambda \) is an eigenvalue of \( M_{2n} \) if and only if \( \lambda^2 \) is an eigenvalue of \( m'm \) and \( mm't \), respectively. Thus, the nonzero eigenvalues of \( M_{2n} \) are \( \pm \lambda_i, \ i = 1, 2, \ldots \).

1.2. The AM of the zero eigenvalue of \( M_{2n} \) is even.

**Property 2**

If \( V \) in Eq. (B3) is an eigenvector of \( M_{2n} \) corresponding to the zero eigenvalue, then \( V_1 = \begin{pmatrix} v' \\ 0 \end{pmatrix} \) (if \( v' \neq 0 \)) and \( V_2 = \begin{pmatrix} 0 \\ v'' \end{pmatrix} \) (if \( v'' \neq 0 \)) are also eigenvectors of \( M_{2n} \) corresponding to the zero eigenvalue. Clearly, \( V \) is a linear combination of \( V_1 \) and \( V_2 \), i.e. \( V = V_1 + V_2 \).

**Proof**

From that \( M_{2n} V = 0 \), we obtain

\[
\begin{align*}
mv'' &= 0, \\
m'v' &= 0.
\end{align*}
\]

It is easy to verify that \( V_1 = \begin{pmatrix} v' \\ 0 \end{pmatrix} \) (if \( v' \neq 0 \)) and \( V_2 = \begin{pmatrix} 0 \\ v'' \end{pmatrix} \) (if \( v'' \neq 0 \)) are also eigenvectors of \( M_{2n} \) corresponding to the zero eigenvalue. □

**Property 3**

The GM of the zero eigenvalue of \( M_{2n} \) is \( 2(n - rk(m)) \), where \( rk \) stands for “rank”. Thus, there are \( 2(n - rk(m)) \) JBs corresponding to the zero eigenvalue of \( M_{2n} \).

**Proof**

From the linear algebra, it is easy to see that Property 3 holds. We want to prove it differently next. From [37], we know that the generalized eigenvector of rank 1 is just an eigenvector. For \( M_{2n} \), let \( \chi_1 \) be the number of linear independent generalized eigenvectors of rank 1 corresponding to the zero eigenvalue. Then, from [37]

\[
\chi_1 = 2n - rk(M_{2n}).
\]

It is easy to see that \( rk(M_{2n}) = 2 \ast rk(m) \). □

**Property 4**

A basis of the zero eigenspace of \( M_{2n} \) can be obtained via the bases of the zero eigenspaces of \( m \) and \( m' \) as follows. Let \( v'_1, v'_2, \ldots , v'_{n-rk(m)} \) be all the linearly independent eigenvectors of \( m' \) corresponding to the zero eigenvalue and \( v''_1, v''_2, \ldots , v''_{n-rk(R)} \) be all the linearly independent eigenvectors of \( m \) corresponding to the zero eigenvalue. Then,

\[
\left\{ \begin{pmatrix} v'_1 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} v'_{n-rk(m)} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ v''_1 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ v''_{n-rk(m)} \end{pmatrix} \right\}
\]

is a basis of the zero eigenspace of \( M_{2n} \).
Proof Let $V$ in Eq. (B3) be an eigenvector of $M_{2n}$ corresponding to the zero eigenvalue. Then, by Eqs. (C3, C4), $v''$ is an eigenvector of $m$ corresponding to the zero eigenvalue if $v'' \neq 0$ and $v'$ is an eigenvector of $m'$ corresponding to the zero eigenvalue if $v' \neq 0$. Conversely, if $v'$ (resp. $v''$) is an eigenvector of $m'$ (resp. $m$) corresponding to the zero eigenvalue, then $\begin{pmatrix} v' \\ 0 \end{pmatrix}$ (resp. $\begin{pmatrix} 0 \\ v'' \end{pmatrix}$) is an eigenvector of $M_{2n}$ corresponding to the zero eigenvalue. From Eqs. (C3, C4), we know that $m$ and $m'$ have $n - rk(m)$ linearly independent eigenvectors corresponding to the zero eigenvalue, respectively. Thus, Property 4 holds and we have Property 3 again. \[ \square \]

Property 5.1 For $M_{2n}$, let $\chi_\ell$ be the number of linear independent generalized eigenvectors of rank $\ell$ corresponding to the zero eigenvalue [37]. Then, $\chi_{2k} + \chi_{2k+1}$, where $k \geq 1$, must be even.

Proof From [37],
\[
\chi_{2k} = rk(M_{2n}^{2k-1}) - rk(M_{2n}^{2k})
\] (C7)
and
\[
\chi_{2k+1} = rk(M_{2n}^{2k}) - rk(M_{2n}^{2k+1}),
\] (C8)
where $k \geq 1$. Then,
\[
\chi_{2k} + \chi_{2k+1} = rk(M_{2n}^{2k-1}) - rk(M_{2n}^{2k+1}).
\] (C9)

Let us compute $M_{2n}^{2k+1}$ next.
\[
M_{2n}^{3} = \begin{pmatrix} m'mm' \end{pmatrix},
\] (C10)
\[
M_{2n}^{2k+1} = \begin{pmatrix} \left(M'\right)' \end{pmatrix},
\] (C11)
where $M' = mm'mm' \cdots mm'm$.

It is easy to check that $rk(M_{2n}^{2k+1}) = 2 \times rk(M')$. Similarly, the number $rk(M_{2n}^{2k-1})$ is even. Therefore, $\chi_{2k} + \chi_{2k+1}$ is even. Specially, $\chi_2 + \chi_3$ is even. \[ \square \]

Property 5.2 The number of the occurrences of JBs with the same odd size corresponding to the zero eigenvalue of $M_{2n}$ may be even or odd.

Proof For the JB $J_{2k+1}(0)$ corresponding to the eigenvector $x_1$, there is a Jordan chain $x_1, x_2, \ldots, x_{2k+1}$ corresponding to the zero eigenvalue, where $x_i$ is the generalized eigenvector of rank $i$ of $M_{2n}$. Clearly, $x_2j$ is the generalized eigenvector of rank $2j$ and $x_{2j+1}$ is the one with rank $2j + 1$, where $j = 1, \ldots, k$. Thus, the chain adds 1 to $\chi_{2j}$ and 1 to $\chi_{2j+1}$, respectively, $j = 1, \ldots, k$. That is, the chain adds 2 to the number $\chi_{2j} + \chi_{2j+1}$, $j = 1, \ldots, k$. One can know that any number of occurrences of JBs with the same odd size corresponding to the zero eigenvalue will not change the parity of $\chi_{2k} + \chi_{2k+1}$. Therefore, Property 5.2 holds. \[ \square \]
Property 5.3 The number of the occurrences of the JBs with the same even size corresponding to the zero eigenvalue must be even.

Proof For the JB $J_{2k}(0)$ with $k \geq 1$ corresponding to the eigenvector $y_1$, there is a Jordan chain $y_1, y_2, \ldots, y_{2k}$ corresponding to the zero eigenvalue, where $y_i$ is the generalized eigenvector of rank $i$ of $M_{2n}$. Thus, $y_{2j}$ is the generalized eigenvector of rank $2j$, while $y_{2j+1}$ is the one with rank $2j+1$, where $j = 1, \ldots, k-1$. Thus, the chain adds 1 to $\chi_{2j}$ and 1 to $\chi_{2j+1}$, respectively, $j = 1, \ldots, k-1$. □

Clearly, $y_{2k}$ is the generalized eigenvector of rank $2k$. Thus, it adds 1 to $\chi_{2k}$. But the chain does not include the generalized eigenvector of rank $2k+1$. Thus, it adds 0 to $\chi_{2k+1}$. It means that the chain will change the parity of $\chi_{2k} + \chi_{2k+1}$.

Accordingly, for the $2l + 1$ occurrences of the JB $J_{2k}(0)$ with $k \geq 1$, the corresponding $2l + 1$ Jordan chains include $2l + 1$ generalized eigenvectors of the same rank $2k$, but the chains do not have any generalized eigenvector of rank $2k + 1$. Thus, in light of Property 5.2, the number $\chi_{2k} + \chi_{2k+1}$ will be an odd number. It does not satisfy Property 5.1.

For the $2l$ occurrences of the JB $J_{2k}(0)$ with $k \geq 1$, the corresponding $2l$ Jordan chains include $2l$ generalized eigenvectors of the same rank $2k$, but the chains do not have any generalized eigenvector of rank $2k + 1$. Thus, in light of Property 5.2, the number $\chi_{2k} + \chi_{2k+1}$ will be an even number.

One can see that $\chi_{2k} + \chi_{2k+1}$ is even permits that the size of a JB with the zero eigenvalue is odd or even.

For example, a calculation shows that for four qubits, $\Phi_8$ has the SJNFs $J_4(0) \oplus 2$, $J_2(0) \oplus 4 \oplus J_1(0) \oplus 4$, and $J_2(0) \oplus 4 \oplus J_3(0) \oplus J_1(0)$ and $J_2(0) \oplus 4$ for the states $L_{ab}(a = 0)$, $L_{ab}20(2c)$, $L_{ab}20(0)$, and $L_{ab}20(0)$, respectively. In detail, $J_4(0)$ occurs twice, and $J_2(0)$ occurs twice, twice, and for four times in the above SJNFs. For these SJNFs, $\chi_{2k} + \chi_{2k+1}$ is even.

One can know that $\Phi_8$ does have SJNFs $\pm \lambda J_2(0)J_4(0)$, $J_2(0)J_6(0)$ or $0J_2(0)J_4(0)$ because for these SJNFs $\chi_2 + \chi_3$ is odd. Note that $J_2(0)$, $J_4(0)$, and $J_6(0)$ occur once in the above different SJNFs.

Property 6 Let $V$ in Eq. (B3) be an eigenvector of $M_{2n}$ corresponding to the nonzero eigenvalue $\lambda$. Then, $v' \neq 0$ and $v'' \neq 0$.

Proof From the equation $(M_{2n} - \lambda I_{2n}) V = 0$, we obtain

$$m v'' = \lambda v', \quad (C12)$$

$$m' v' = \lambda v''. \quad (C13)$$

Then from Eqs. (C12, C13), it is easy to show that $v' \neq 0$ and $v'' \neq 0$. In other words, the vectors of the forms $\begin{pmatrix} v' \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ v'' \end{pmatrix}$ are not eigenvectors of $M_{2n}$ corresponding to nonzero eigenvalues.

Property 7 The GMs of the nonzero eigenvalues $\pm \lambda$ of $M_{2n}$ both are $n - rk(m' m - \lambda^2 I_n)$. Thus, there are $n - rk(m' m - \lambda^2 I_n)$ JBs corresponding to the nonzero eigenvalues $\pm \lambda$ of $M_{2n}$, respectively.
Proof Let $\chi_1(\lambda)$ (resp. $\chi_1(-\lambda)$) be the number of linear independent generalized eigenvectors of rank 1 corresponding to the nonzero eigenvalue $\lambda$ (resp. $-\lambda$). One can know that $\chi_1(\lambda)$ (resp. $\chi_1(-\lambda)$) is just the GMs of the nonzero eigenvalues $\lambda$ (resp. $-\lambda$) of $M_{2n}$. Then, from [37]

\[
\chi_1(\lambda) = 2n - rk(M_{2n} - \lambda I_{2n}) \\
= 2n - rk\left( \begin{array}{cc} -\lambda I_n & m \\ m^t & -\lambda I_n \end{array} \right). \tag{C14}
\]

\[
\chi_1(-\lambda) = 2n - rk(M_{2n} + \lambda I_{2n}) \\
= 2n - rk\left( \begin{array}{cc} \lambda I_n & m \\ m^t & \lambda I_n \end{array} \right). \tag{C15}
\]

To calculate the ranks of the matrices $\left( \begin{array}{cc} -\lambda I_n & m \\ m^t & -\lambda I_n \end{array} \right)$ and $\left( \begin{array}{cc} \lambda I_n & m \\ m^t & \lambda I_n \end{array} \right)$, we do the following operations:

\[
\left( \begin{array}{cc} I_n & 0 \\ \frac{1}{\lambda} m^t & I_n \end{array} \right) \left( \begin{array}{cc} -\lambda I_n & m \\ m^t & -\lambda I_n \end{array} \right) = \left( \begin{array}{cc} -\lambda I_n & m \\ 0 & \frac{1}{\lambda} m^t m - \lambda I_n \end{array} \right) \tag{C16}
\]

and

\[
\left( -I_n \begin{array}{cc} 0 & 1 \\ \frac{1}{\lambda} m^t & -I_n \end{array} \right) \left( \begin{array}{cc} \lambda I_n & m \\ m^t & \lambda I_n \end{array} \right) = \left( \begin{array}{cc} -\lambda I_n & -m \\ 0 & \frac{1}{\lambda} m^t m - \lambda I_n \end{array} \right). \tag{C17}
\]

From the linear algebra, since $\left( \begin{array}{cc} I_n & 0 \\ \frac{1}{\lambda} m^t & I_n \end{array} \right)$ is full rank, via Eq. (C16) we obtain

\[
rk\left( \begin{array}{cc} -\lambda I_n & m \\ m^t & -\lambda I_n \end{array} \right) = \rk\left( \begin{array}{cc} -\lambda I_n & m \\ 0 & \frac{1}{\lambda} m^t m - \lambda I_n \end{array} \right) = n + \rk\left( \begin{array}{cc} 1 \\ \frac{1}{\lambda} m^t m - \lambda I_n \end{array} \right) = n + \rk\left( \begin{array}{cc} 1 \\ (m^t m - \lambda^2 I_n) \end{array} \right) = n + \rk(m^t m - \lambda^2 I_n). \tag{C18}
\]

From the linear algebra, since $\left( -I_n \begin{array}{cc} 0 & 1 \\ \frac{1}{\lambda} m^t & -I_n \end{array} \right)$ is full rank, via Eq. (C17) we obtain

\[
rk\left( \begin{array}{cc} \lambda I_n & m \\ m^t & \lambda I_n \end{array} \right) = \rk\left( \begin{array}{cc} -\lambda I_n & -m \\ 0 & \frac{1}{\lambda} m^t m - \lambda I_n \end{array} \right)
\]
\[ n + rk \left( \frac{1}{\lambda} m^t m - \lambda I_n \right) = n + rk(m^t m - \lambda^2 I_n). \] (C19)

From Eqs. (C14, C18), \( \chi_1(\lambda) = 2n - [n + rk(m^t m - \lambda^2 I_n)] = n - rk(m^t m - \lambda^2 I_n). \) Clearly, \( m^t m - \lambda^2 I_n \) is a characteristic matrix of \( m^t m \) in \( \lambda^2. \) From Eqs. (C15, C19), \( \chi_1(-\lambda) = n - rk(m^t m - \lambda^2 I_n) \). Therefore, \( \chi_1(\lambda) = \chi_1(-\lambda) \) and then Property 7 holds.

By Property 1.1, when \( \lambda \) is an eigenvalue of \( M_{2n} \), then \( \lambda^2 \) is an eigenvalue of \( m^t m. \) It is well known that roots of the equation \( \det(m^t m - \lambda^2 I_n) = 0 \) are eigenvalues of \( m^t m. \) Thus, \( 0 \leq rk(m^t m - \lambda^2 I_n) < n, \) and then, \( 0 < \chi_1(\lambda) \leq n. \) When \( \lambda^2 \) is not an eigenvalue of \( m^t m, \) i.e. \( \lambda \) is not an eigenvalue of \( M_{2n}, \) then \( \det(m^t m - \lambda^2 I_n) \neq 0, \) i.e. \( rk(m^t m - \lambda^2 I_n) = n. \) Thus, \( \chi_1(\lambda) = 0. \) \( \square \)

**Property 8** The Jordan chain with the nonzero eigenvalue \( \lambda \) corresponding to the eigenvector \( \left( \begin{array}{c} v'_1 \\ v''_1 \end{array} \right) \) and the Jordan chain with the nonzero eigenvalue \( -\lambda \) corresponding to the eigenvector \( \left( \begin{array}{c} -v'_1 \\ v''_1 \end{array} \right) \) have the same size. Thus, their corresponding JBs have the same size.

**Proof** Let

\[ v_i = \left( \begin{array}{c} v'_i \\ v''_i \end{array} \right), i = 1, 2, \ldots, r, \] (C20)

where \( v'_i \) and \( v''_i \) are \( n \times 1 \) vectors, be a Jordan chain with the nonzero eigenvalue \( \lambda \) corresponding to the eigenvector \( v_1 = \left( \begin{array}{c} v'_1 \\ v''_1 \end{array} \right). \) By Property 6, \( v'_1 \neq 0 \) and \( v''_1 \neq 0. \) Then, by the definition of Jordan chain [37],

\[ (M_{2n} - \lambda I_{2n})v_1 = 0 \] (C21)

and

\[ (M_{2n} - \lambda I_{2n})v_k = v_{k-1}, k \geq 2. \] (C22)

Let

\[ \omega_1 = \left( \begin{array}{c} -v'_1 \\ v''_1 \end{array} \right), \] (C23)

\[ \omega_2 = \left( \begin{array}{c} v'_2 \\ -v''_2 \end{array} \right), \] (C24)

\[ \vdots \]

\[ \omega_l = (-1)^{l+1} \left( \begin{array}{c} -v'_l \\ v''_l \end{array} \right), l \geq 2. \] (C25)

It is easy to check that

\[ (M_{2n} + \lambda I_{2n})\omega_1 = 0 \] (C26)

\( \square \) Springer
and
\[(M_{2n} + \lambda I_{2n}) \omega_k = \omega_{k-1}, \quad k \geq 2. \tag{C27}\]

Here, $\omega_1$ is an eigenvector of $M_{2n}$ corresponding to $-\lambda$. Let $s$ be the size of the Jordan chain with the nonzero eigenvalue $-\lambda$ corresponding to the eigenvector $\omega_1$. Clearly, $s \geq r$. Conversely, similarly, we can show that $r \geq s$. Thus, $s = r$. \qed

**Appendix D: The number of different lists of partitions of AMs**

We define a product of sets $L$ and $M$ as $L \times M = \{ \{l, m\} | l \in L \text{ and } m \in M \}$, and we define that $\{l, m\}$ is an unordered list of partitions. Thus, $\{l, m\} = \{m, l\}$. By the definition, $2 \times 2 = \{(2), (2), (1, 1), (1, 1)\}$. Note that $\{(2), (1, 1)\} = \{(1, 1), (2)\}$.

From Eq. (30), let $\Gamma_1 = \tilde{\omega}_2^k \times \tilde{\ell}_1 \times \tilde{\ell}_2 \times \cdots \times \tilde{\ell}_s$. (D1)

From the above discussion, we consider that $\tilde{\ell}_1 \times \tilde{\ell}_2 \times \cdots \times \tilde{\ell}_s$ is an unordered list of partitions. Note that some $\tilde{\ell}_i$ in a set of AMs $\{2k; \ell_1, \ell_2, \ldots, \ell_s\}$ from Eq. (26) may occur twice or more. For example, $\Phi_8$ has the spectrum $\{\pm \lambda_1^{\otimes 2}, \pm \lambda_2^{\otimes 2}\}$ and the set of the AM is $\{0; 2, 2\}$.

First, let us compute how many different lists of partitions there are from the product set $\tilde{I} \times \cdots \times \tilde{I}$. We consider distributing $j$ indistinguishable balls into $P(l)$ distinguishable boxes. Let $\rho(l, j) = \left( \frac{j + P(l) - 1}{j} \right)$. Thus, there are $\rho(l, j)$ distributing ways without exclusion [36]. Via the probability model, $\tilde{I} \times \cdots \times \tilde{I}$ has $\rho(l, j)$ different lists of partitions. Specially, $\tilde{2} \times \tilde{2}$ has $\rho(2, 2) = 3$ different lists of partitions.

It is easy to check that $\tilde{2} \times \tilde{3}$ has $P(2)P(3) = 6$ different lists of partitions. When $l$, $k$, and $m$ are distinct from each other, $\tilde{l} \times \tilde{k} \times \tilde{m}$ has $P(l)P(k)P(m)$ different lists of partitions.

Let us compute how many different lists of partitions there are from the product set $\Gamma$ in Eq. (D1) for all $k$. For the sake of clarity, we rewrite $\Gamma$ in Eq. (D1) as follows:

$$\Gamma = \tilde{\omega}_2^k \times \tilde{\omega}_1 \times \cdots \times \tilde{\omega}_1 \times \tilde{\theta} \times \cdots \times \tilde{\theta} \times \cdots \times \tilde{\zeta} \times \cdots \times \tilde{\zeta}, \tag{D2}$$

where $\omega_1, \ldots, \omega_i, \theta, \ldots,$ and $\zeta$ are different from each other. From Eq. (D2), for all $k$ we obtain

$$\eta = \sum_{k=0}^{2n} P^*(2k) \sum_{\omega} P(\omega_1) \cdots P(\omega_i) \rho(\theta, j) \ldots \rho(\zeta, m) \tag{D3}$$
different lists of partitions, where $\pi = \{\pi_1, \ldots, \pi_i, \theta, \ldots, \theta, \ldots, \xi, \ldots, \xi\}$, which is a partition of $2^{2n} - k$, and the second sum is evaluated over all the partitions of $2^{2n} - k$.

To compute $P^*(2^{2n+1})$, from $2^{2n+1}$ we should remove the partition $(1, \ldots, 1)^{2n+1}$, which means that the SJNF of the $\Phi_{2^{2n+1}}(|\psi\rangle)$ in Eq. (10) is the zero matrix, and then, $\Phi_{2^{2n+1}}(|\psi\rangle)$ in Eq. (10) is the zero matrix. Therefore, in total we obtain $\eta - 1$ different lists of partitions of AMs in $(2k; \ell_1, \ell_2, \ldots, \ell_s)$.

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