Infinite products of absolute zeta functions

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Abstract
We study some infinite products of absolute zeta functions. Especially, we consider the convergence and the rationality of them.

1 Introduction
It is well-known that some infinite products of zeta functions have interesting properties. For example the infinite product \( \prod_{n=1}^{\infty} \zeta(s+n) \) has an analytic continuation to all \( s \in \mathbb{C} \) as a meromorphic function, where \( \zeta(s) = \zeta_\mathbb{C}(s) \) is the Riemann zeta function: see Cohen–Lenstra [1] and Manin [7, §3.5]. We notice that
\[
\prod_{n=1}^{\infty} \zeta(s+n) = \sum_A |\text{Aut}(A)|^{-1} |A|^{-s},
\]
where \( A \) runs over all isomorphism classes of finite abelian groups.

In this paper we study infinite products of absolute zeta functions (zeta functions over \( \mathbb{F}_1 \)): see Soulé [8], Connes–Consani [2] and Kurokawa–Ochiai [4].

The simplest example is
\[
\prod_{n=1}^{\infty} \zeta_{\mathbb{F}_1}(s+n) = \prod_{n=1}^{\infty} \frac{1}{s+n},
\]
which diverges unfortunately.

We present some concrete examples of infinite products of absolute zeta functions such that
\[
\prod_{n=1}^{\infty} \zeta_{\text{GL}(3)/\mathbb{F}_1}(s+n) = \zeta_{\text{SL}(3)/\mathbb{F}_1}(s),
\]
where
\[
\zeta_{\text{GL}(3)/\mathbb{F}_1}(s) = \frac{(s-8)(s-7)(s-3)}{(s-9)(s-5)(s-4)}
\]
and
\[
\zeta_{SL(3)/F_1}(s) = \frac{(s-6)(s-5)}{(s-8)(s-3)}
\]

According to Kurokawa-Ochiai [4], for a suitable scheme \(X\) the absolute zeta function \(\zeta_{X/F_1}(s)\) is defined as
\[
\zeta_{X/F_1}(s) = \exp \left( \frac{\partial}{\partial w} Z_{X/F_1}(w, s) \bigg|_{w=0} \right)
\]
with
\[
Z_{X/F_1}(w, s) = \frac{1}{\Gamma(w)} \int_1^\infty |X(F_x)|x^{-s-1}(\log x)^{w-1} dx.
\]

**Theorem 1.** Let \(r \geq 2\) be an integer. Then we have
\[
\zeta_{SL(r)/F_1}(s) = \prod_{n=1}^{\infty} \zeta_{GL(r)/F_1}(s+n).
\]

**Example 1.** \((r = 2)\)
\[
\zeta_{SL(2)/F_1}(s) = \frac{s-1}{s-3},
\]
\[
\zeta_{GL(2)/F_1}(s) = \frac{(s-2)(s-3)}{(s-1)(s-4)}
\]
and
\[
\zeta_{SL(2)/F_1}(s) = \prod_{n=1}^{\infty} \zeta_{GL(2)/F_1}(s+n).
\]

**Remark 1.** \((r = 1)\)
\[
\zeta_{SL(1)/F_1}(s) = \frac{1}{s},
\]
\[
\zeta_{GL(1)/F_1}(s) = \frac{s}{s-1}
\]
and
\[
\zeta_{SL(1)/F_1}(s) \neq \prod_{n=1}^{\infty} \zeta_{GL(1)/F_1}(s+n).
\]

Let \(N\) be a positive integer. For a zeta function \(Z(s)\) we define a finite product \(Z^K_N(s)\) and an infinite product \(Z_\infty^N(s)\) as
\[
Z^K_N(s) = \prod_{k=0}^{K} Z(s+kN)
\]
and
\[
Z_\infty^N(s) = \prod_{k=0}^{\infty} Z(s+kN)
\]
respectively. Let
\[ Z(s) = \zeta_{\text{GL}(2)/\mathbb{F}_1}(s) = \frac{(s - 2)(s - 3)}{(s - 1)(s - 4)} \]
in Theorems 2–5.

**Theorem 2 (explicit formula).** For \( K = 0, 1, 2, \ldots \) we obtain the following results.

1. \[ Z^K_1(s) = \prod_{k=0}^{K} Z(s + k) = \frac{(s - 2)(s + K - 3)}{(s - 4)(s + K - 1)} \]
2. \[ Z^\infty_1(s) = \frac{s - 2}{s - 4} \] (1.1)
3. \[ Z^K_2(s) = \prod_{k=0}^{\infty} Z(s + 2k) = \frac{(s - 3)(s + 2K - 2)}{(s - 4)(s + 2K - 1)} \]
4. \[ Z^\infty_2(s) = \frac{s - 3}{s - 4} \] (1.2)

**Remark 2.** Let \( q \) be a prime power. The congruence zeta function \( \zeta_{X/\mathbb{F}_q}(s) \) associated with \( X \) over \( \mathbb{F}_q \) is defined by
\[ \zeta_{X/\mathbb{F}_q}(s) = \exp \left( \sum_{m=1}^{\infty} \frac{|X(\mathbb{F}_{q^m})|}{m} q^{-ms} \right). \]
The Hasse zeta function \( \zeta_{X/\mathbb{Z}}(s) \) associated with \( X \) over \( \mathbb{Z} \) is defined by
\[ \zeta_{X/\mathbb{Z}}(s) = \prod_{p \text{primes}} \zeta_{X/\mathbb{F}_p}(s). \]
In this case it is not difficult to show that \( \zeta_{X/\mathbb{Z}}(s) \) has a functional equation of the type
\[ \Gamma_{X/\mathbb{Z}}(s) \zeta_{X/\mathbb{Z}}(s) = \left( \Gamma_{X/\mathbb{Z}}(d - s) \zeta_{X/\mathbb{Z}}(d - s) \right) \left( -1 \right)^{n_1 + \cdots + n_r} \]
where \( d = \sum_{j=1}^{r} \frac{n_j(3n_j - 1)}{2} + 1 \) and \( \Gamma_{X/\mathbb{Z}}(s) \) is expressed in terms of a product/quotient of the gamma function. We call \( \Gamma_{X/\mathbb{Z}}(s) \) the gamma factor for \( \zeta_{X/\mathbb{Z}}(s) \). In [9], we study the rationality of gamma factors associated to certain Hasse zeta functions. We refer to Manin [7] and Connes-Consani [2] for gamma factors of Hasse zeta functions. Especially, we have
\[ \Gamma_{\text{GL}(2)/\mathbb{Z}}(s) = \frac{s - 3}{s - 4}. \]
The right side is the same formula as the right side of (1.2).
Theorem 3 (functional equation). For $K = 0, 1, 2, \ldots$ we obtain the following results.

1. $Z_1^K(5 - K - s) = Z_1^K(s)$.
2. $Z_1^\infty(6 - s) = Z_1^\infty(s)^{-1}$.
3. $Z_2^K(5 - 2K - s) = Z_2^K(s)$.
4. $Z_2^\infty(7 - s) = Z_2^\infty(s)^{-1}$.

Theorem 4 (convergence). $Z_N^\infty(s)$ converges and

$$Z_N^\infty(s) = \frac{\Gamma\left(\frac{s-1}{N}\right)\Gamma\left(\frac{s-4}{N}\right)}{\Gamma\left(\frac{s-2}{N}\right)\Gamma\left(\frac{s-3}{N}\right)}.$$  

Theorem 5 (rationality). $Z_N^\infty(s)$ is a rational function if and only if $N = 1, 2$.

Next, let

$$Z(s) = \frac{(s - c)(s - d)}{(s - a)(s - b)}.$$  

in Theorems 6–8.

Theorem 6. The following (1) and (2) are equivalent.

1. $Z_N^\infty(s)$ converges.
2. $a + b = c + d$.

Theorem 7. The following (1), (2) and (3) are equivalent.

1. $Z_N^\infty(s)$ converges.
2. [functional equation] $Z(a + b - s) = Z(s)$.
3. [absolute automorphy] $f\left(\frac{1}{x}\right) = x^{-(a+b)}f(x)$.

Theorem 8 (rationality). $Z_N^\infty(s)$ is a rational function if and only if $\zeta_N^a + \zeta_N^b = \zeta_N^c + \zeta_N^d$ with $\zeta_N = \exp\left(\frac{2\pi i}{N}\right)$.

Finally, let

$$Z(s) = \zeta_{G_{m/F_1}}(s) = \prod_{\ell=0}^{r-1}(s - \ell)^{-\ell+1}\left(\frac{r}{\ell}\right).$$  

in Theorems 9–10.

Theorem 9. The following (1) and (2) are equivalent.

1. $Z_N^\infty(s)$ converges.
2. $r \geq 2$.

Theorem 10 (rationality). $Z_N^\infty(s)$ is a rational function if and only if $N = 1$. 

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2 Proof of Theorem 1.

Proof of Theorem 1. For \( r \geq 2 \) let

\[
f(x) = x^r - 1 \cdot (1 - x^{-2}) \cdots (1 - x^{-r}) = |\text{SL}(r, \mathbb{F}_x)|
\]

and

\[
g(x) = x^r - 1 \cdot (1 - x^{-1}) \cdots (1 - x^{-r}) = |\text{GL}(r, \mathbb{F}_x)|.
\]

Notice that \( g(x) = (x-1)f(x) \). Then using \( f(1) = g(1) = 0 \) we have (see [5])

\[
\zeta_{\text{SL}(r)/\mathbb{F}_1}(s) = \exp \left( \int_{1}^{\infty} \frac{f(x)x^{-s-1}}{\log x} \, dx \right)
\]

and

\[
\zeta_{\text{GL}(r)/\mathbb{F}_1}(s) = \exp \left( \int_{1}^{\infty} \frac{g(x)x^{-s-1}}{\log x} \, dx \right).
\]

From these expressions we have

\[
\frac{\zeta_{\text{SL}(r)/\mathbb{F}_1}(s+n-1)}{\zeta_{\text{SL}(r)/\mathbb{F}_1}(s+n)} = \exp \left( \int_{1}^{\infty} \frac{f(x)x^{-s-n} - x^{-s-n-1}}{\log x} \, dx \right)
\]

\[
= \exp \left( \int_{1}^{\infty} \frac{f(x)(x-1)x^{-s-n-1}}{\log x} \, dx \right)
\]

\[
= \exp \left( \int_{1}^{\infty} \frac{g(x)x^{-s-n-1}}{\log x} \, dx \right)
\]

\[
= \zeta_{\text{GL}(r)/\mathbb{F}_1}(s+n)
\]

for \( n \geq 1 \). Hence we get

\[
\prod_{n=1}^{N} \zeta_{\text{GL}(r)/\mathbb{F}_1}(s+n) = \prod_{n=1}^{N} \frac{\zeta_{\text{SL}(r)/\mathbb{F}_1}(s+n-1)}{\zeta_{\text{SL}(r)/\mathbb{F}_1}(s+n)}
\]

\[
= \frac{\zeta_{\text{SL}(r)/\mathbb{F}_1}(s)}{\zeta_{\text{SL}(r)/\mathbb{F}_1}(s+N)}.
\]

Thus it is sufficient to show that

\[
\lim_{N \to \infty} \zeta_{\text{SL}(r)/\mathbb{F}_1}(s+N) = 1
\]

for \( r \geq 2 \). Let \( f(x) = \sum_k a(k)x^k \in \mathbb{Z}[x] \) then

\[
\zeta_{\text{SL}(r)/\mathbb{F}_1}(s) = \prod_k (s-k)^{-a(k)}
\]

\[
= s^{-\sum_k a(k)} \prod_k (1 - \frac{k}{s})^{-a(k)}.
\]
By using $f(1) = 0 = \sum_k a(k)$ we obtain

$$\zeta_{\text{SL}(r)/F_1}(s) = \prod_k (1 - \frac{k}{s})^{-a(k)}.$$ 

Hence the expression

$$\zeta_{\text{SL}(r)/F_1}(s + N) = \prod_k (1 - \frac{k}{s + N})^{-a(k)}$$

implies

$$\lim_{N \to \infty} \zeta_{\text{SL}(r)/F_1}(s + N) = 1.$$ 

Hence, we obtain Theorem 1.

$$\square$$

### 3 Proof of Theorem 2–5.

**Proof of Theorem 2.** (1) We prove (1) by the induction on $K = 0, 1, 2, \ldots$. Put $K = 0$. Then we have

$$Z_1^0 = Z(s) = \frac{(s - 2)(s - 3)}{(s - 4)(s - 1)}.$$ 

Now we assume

$$Z_1^K(s) = \frac{(s - 2)(s + K - 3)}{(s - 4)(s + K - 1)}.$$ 

Then we have

$$Z_1^{K+1}(s) = Z_1^K(s)Z(s + K + 1) = \frac{(s - 2)(s + K - 3)}{(s - 4)(s + K - 1)} \frac{(s + K - 1)(s + K - 2)}{(s + K)(s + K - 3)} = \frac{(s - 2)(s + K - 2)}{(s - 4)(s + K)}.$$ 

Hence, we obtain (1).

(2) Let $K \to \infty$ in (1). Then we have

$$\lim_{K \to \infty} \frac{s + K - 3}{s + K - 1} = 1.$$ 

Thus we obtain

$$Z_1^\infty(s) = \frac{s - 2}{s - 4}.$$ 

(3) We prove (3) by the induction on $K = 0, 1, 2, \ldots$. Put $K = 0$. Then we have

$$Z_2^0(s) = Z(s) = \frac{(s - 3)(s - 2)}{(s - 4)(s - 1)}.$$
Now we assume
\[ Z^K_2(s) = \frac{(s-3)(s+2K-2)}{(s-4)(s+2K-1)} \]

Then we have
\[ Z^{K+1}_2(s) = Z^K_2(s)Z(s+2(K+1)) \]
\[ = \frac{(s-3)(s+2K-2)}{(s-4)(s+2K-1)} \cdot \frac{(s+2K)(s+2K-1)}{(s+2K+1)(s+2K-2)} \]
\[ = \frac{(s-3)(s+2K)}{(s-4)(s+2K+1)}. \]

Hence, we obtain (3).
(4) Let \( K \to \infty \) in (3). Then we have
\[ \lim_{K \to \infty} \frac{s+2K-2}{s+2K-1} = 1. \]

Thus we obtain
\[ Z^\infty_2(s) = \frac{s-3}{s-4}. \]  

**Proof of Theorem 3.** (1) For an integer \( N \geq 1 \) we prove
\[ Z^K_N(5 - KN - s) = Z^K_N(s) \]
more generally. Put \( K = 0 \). Then we have
\[ Z^0_N(s) = Z(s) \]
\[ = \frac{(s-2)(s-3)}{(s-1)(s-4)}. \]

This gives the functional equation
\[ Z(5 - s) = \frac{(3-s)(2-s)}{(4-s)(1-s)} = Z(s). \]  

(3.1)

For an integer \( K \geq 0 \) we have
\[ Z^K_N(s) = \prod_{k=0}^{K} Z(s + kN). \]

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This gives

\[ Z^K_N (5 - KN - s) = \prod_{k=0}^{K} Z(5 - KN - s + kN) \]
\[ = \prod_{k=0}^{K} Z(5 - (K - k)N - s) \]
\[ = \prod_{k=0}^{K} Z(5 - (kN + s)). \]

From (3.1) we obtain

\[ Z^K_N (5 - KN - s) = \prod_{k=0}^{K} Z(s + kN) = Z^K_N (s). \]

(2) By

\[ Z_1^\infty (s) = \frac{s - 2}{s - 4} \]

we have

\[ Z_1^\infty (6 - s) = \frac{4 - s}{2 - s} \]
\[ = \frac{s - 4}{s - 2} \]
\[ = Z_1^\infty (s)^{-1}. \]

(3) By the proof of (1) we obtain (3).

(4) By

\[ Z_2^\infty (s) = \frac{s - 3}{s - 4} \]

we have

\[ Z_2^\infty (7 - s) = \frac{4 - s}{3 - s} \]
\[ = \frac{s - 4}{s - 3} \]
\[ = Z_2^\infty (s)^{-1}. \]

\[ \square \]

Proof of Theorem 4. Since

\[ Z(s + kN) = (s - 1 + kN)^{-1}(s - 2 + kN)(s - 3 + kN)(s - 4 + kN)^{-1} \]
\[ = (k + \frac{s - 1}{N})^{-1}(k + \frac{s - 2}{N})(k + \frac{s - 3}{N})(k + \frac{s - 4}{N})^{-1} \]
Thus we obtain
\[ t \in \mathbb{N} \]
and
\[ \Gamma(x + 1) = x\Gamma(x) , \]
we have
\[ Z(s + kN) = \frac{\Gamma\left(\frac{s-1}{N} + k\right) \Gamma\left(\frac{s-2}{N} + k + 1\right) \Gamma\left(\frac{s-3}{N} + k + 1\right)}{\Gamma\left(\frac{s}{N} + k\right) \Gamma\left(\frac{s}{N} + k + 1\right)} . \]
Hence, we obtain
\[ Z_N^K(s) = \prod_{k=0}^{K} Z(s + kN) \]
\[ = \frac{\Gamma\left(\frac{s-1}{N}\right) \Gamma\left(\frac{s-2}{N} + K + 1\right) \Gamma\left(\frac{s-3}{N} + K + 1\right)}{\Gamma\left(\frac{s}{N} + K + 1\right)} \]
\[ = \frac{\Gamma\left(\frac{s-1}{N}\right) \Gamma\left(\frac{s}{N} + K + 1\right)}{\Gamma\left(\frac{s}{N} + K + 1\right)} . \]
As \( K \to \infty \) for \( \alpha \in \mathbb{C} \) by the Stirling’s formula we have
\[ \Gamma(\alpha + K + 1) \sim \sqrt{2\pi K} (\alpha + K)^{\alpha + K + \frac{1}{2}} e^{-(\alpha + K)} \]
\[ = \sqrt{2\pi K} (\alpha + K + \frac{1}{2})(1 + \frac{\alpha}{K})^{\alpha + K + \frac{1}{2}} e^{-(\alpha + K)} \]
\[ \sim \sqrt{2\pi K} (\alpha + K)^{\frac{1}{2}} e^{-\alpha} e^{-K} \]
\[ = \sqrt{2\pi K} (\alpha + K)^{\frac{1}{2}} e^{-K} . \]
Using this formula to \( \alpha = \frac{s-1}{N}, \frac{s-2}{N}, \frac{s-3}{N}, \frac{s-4}{N} \), we have
\[ \lim_{K \to \infty} \frac{\Gamma\left(\frac{s-1}{N} + K + 1\right) \Gamma\left(\frac{s-3}{N} + K + 1\right)}{\Gamma\left(\frac{s}{N} + K + 1\right)} = 1 . \]
Thus we obtain
\[ Z_N^\infty(s) = \lim_{K \to \infty} Z_N^K(s) \]
\[ = \frac{\Gamma\left(\frac{s-1}{N}\right) \Gamma\left(\frac{s-4}{N}\right)}{\Gamma\left(\frac{s}{N} + \frac{1}{2}\right)} . \]

**Proof of Theorem 5.** By (1.1) and (1.2) \( Z_1^\infty(s) \) and \( Z_2^\infty(s) \) are rational functions. Now we assume \( N \geq 3 \). Then \( Z_N^\infty(s) \) has poles at
\[ s = 1 - nN \quad (n = 0, 1, 2, \ldots) \]
\( \Gamma\left(\frac{s-1}{N}\right) \) has poles of order 1.
\[ \Gamma\left(\frac{s-2}{N}\right) \Gamma\left(\frac{s-3}{N}\right) \]
\[ = \Gamma(-(n + \frac{1}{N})) \Gamma(-(n + \frac{2}{N})) \]
\( (\neq 0) \) is a finite value.
\[ \Gamma\left(\frac{s-4}{N}\right) \]
\( (\neq 0) \) is a finite value. \( (N = 3) \), \( (N \geq 4) \). Hence, \( Z_N^\infty(s) \) is not a rational function when \( N \geq 3 \).
4 Proof of Theorem 6–8.

Proof of Theorem 6. (1) We prove Theorem 6 by the same way of the proof of Theorem 4. Since

\[
Z(s + kN) = (s - a + kN)^{-1}(s - b + kN)^{-1}(s - c + kN)(s - d + kN)
= (k + s/N - a/N)^{-1}(k + s/N - b/N)^{-1}(k + s/N - c/N)(k + s/N - d/N)
\]

we have

\[
Z(s + kN) = \frac{\Gamma(\frac{s-a}{N})}{\Gamma(\frac{s-a}{N} + k + 1)} \frac{\Gamma(\frac{s-b}{N})}{\Gamma(\frac{s-b}{N} + k + 1)} \frac{\Gamma(\frac{s-c}{N})}{\Gamma(\frac{s-c}{N} + k + 1)} \frac{\Gamma(\frac{s-d}{N})}{\Gamma(\frac{s-d}{N} + k)}.
\]

Hence, we obtain

\[
Z_N^S(s) = \frac{\Gamma(\frac{s-a}{N})}{\Gamma(\frac{s-a}{N} + K + 1)} \frac{\Gamma(\frac{s-b}{N})}{\Gamma(\frac{s-b}{N} + K + 1)} \frac{\Gamma(\frac{s-c}{N})}{\Gamma(\frac{s-c}{N} + K + 1)} \frac{\Gamma(\frac{s-d}{N})}{\Gamma(\frac{s-d}{N} + K + 1)}.
\]

Here as \( K \to \infty \) by the Stirling’s formula we have

\[
\Gamma(\frac{s-a}{N} + K + 1) \sim \sqrt{2\pi K^{\frac{s-a}{N} + K + \frac{1}{2}}} e^{-K},
\]
\[
\Gamma(\frac{s-b}{N} + K + 1) \sim \sqrt{2\pi K^{\frac{s-b}{N} + K + \frac{1}{2}}} e^{-K},
\]
\[
\Gamma(\frac{s-c}{N} + K + 1) \sim \sqrt{2\pi K^{\frac{s-c}{N} + K + \frac{1}{2}}} e^{-K},
\]
\[
\Gamma(\frac{s-d}{N} + K + 1) \sim \sqrt{2\pi K^{\frac{s-d}{N} + K + \frac{1}{2}}} e^{-K}.
\]

These formula give

\[
\frac{\Gamma(\frac{s-a}{N} + K + 1)\Gamma(\frac{s-d}{N} + K + 1)}{\Gamma(\frac{s-a}{N} + K + 1)\Gamma(\frac{s-b}{N} + K + 1)} \sim K^{\frac{s-b-c-d}{N}}.
\]

Hence, the convergence of \( Z_N^S(s) \) is equivalent to \( a + b = c + d \). Then we have

\[
Z_N^S(s) = \frac{\Gamma(\frac{s-a}{N})\Gamma(\frac{s-b}{N})}{\Gamma(\frac{s-c}{N})\Gamma(\frac{s-d}{N})}.
\]

Proof of Theorem 7. From Theorem 5 the convergence of \( Z_N^S(s) \) is equivalent to \( a + b = c + d \). So we show (2) and (3) are equivalent to \( a + b = c + d \) respectively.
(2)

\[ Z(a + b - s) = Z(s) \]

\[ \iff \frac{(a + b - c - s)(a + b - d - s)}{(a - s)(b - s)} = \frac{(s - c)(s - d)}{(s - a)(s - b)} \]

\[ \iff \frac{(s - (a + b - c))(s - (a + b - d))}{(s - a)(s - b)} = \frac{(s - c)(s - d)}{(s - a)(s - b)} \]

\[ \iff (s - (a + b - c))(s - (a + b - d)) = (s - c)(s - d) \]

\[ \iff s^2 - (2a + 2b - c - d)s + (a + b - c)(a + b - d) = s^2 - (c + d)s + cd \]

\[ \iff a + b = c + d. \]

(3)

\[ f\left(\frac{1}{x}\right) = x^{-(a + b)} f(x) \]

\[ \iff x^{a+b}(x^a - x^b - x^c - x^d) = x^a + x^b - x^c - x^d \]

\[ \iff x^{a+b-c} + x^{a+b-d} = x^c + x^d \]

\[ \iff a + b = c + d. \]

\[ \square \]

Proof of Theorem 8. For \( f(x) = x^a + x^b - x^c - x^d \) we put

\[ f_N^\infty(x) = \frac{f(x)}{1 - x^{-N}}. \]

Then \( f_N^\infty(x) \) satisfies the absolute automorphy:

\[ f_N^\infty\left(\frac{1}{x}\right) = -x^{-(a+b+N)} f_N^\infty(x). \]

Remark 3. We call the absolute automorphy (see [4, 5, 6, 9] for detail) of the condition

\[ f\left(\frac{1}{x}\right) = Cx^{-D} f(x) \]

with \( C = \pm 1 \) and \( D \in \mathbb{R} \). Moreover we define the absolute zeta function \( \zeta_f(s) \) of \( f \) by

\[ \zeta_f(s) := \exp\left( \frac{\partial}{\partial w} Z_f(w, s) \bigg|_{w=0} \right) \]

with

\[ Z_f(w, s) := \frac{1}{\Gamma(w)} \int_1^\infty f(x)x^{-s-1}(\log x)^{w-1}dx. \]

Absolute zeta functions were studied by Soulé [9] and Connes and Consani [2].
Now we have
\[ Z_N^\infty(s) = \zeta_f^\infty(s) \]
\[ = \frac{\Gamma\left(\frac{s-a}{N}\right)\Gamma\left(\frac{s-b}{N}\right)}{\Gamma\left(\frac{s-c}{N}\right)\Gamma\left(\frac{s-d}{N}\right)}. \]

Hence, \( Z_N^\infty(s) \) is a rational function \( \iff f_N^\infty(x) \in \mathbb{Z}[x, x^{-1}] \iff f(\zeta_N) = 0 \) with \( \zeta_N = \exp\left(\frac{2\pi i}{N}\right) \). Thus \( Z_N^\infty(s) \) is a rational function if and only if \( \zeta_N^a + \zeta_N^b = \zeta_N^c + \zeta_N^d \).

Remark 4. Since
\[ \zeta_N^a + \zeta_N^b = \zeta_N^\frac{a+b}{2} (\zeta_N^{-\frac{a+b}{2}} + \zeta_N^{\frac{a+b}{2}}) \]
\[ = 2\zeta_N^\frac{a+b}{2} \cos\left(\frac{|a-b|}{N} \pi\right), \]
\[ \zeta_N^c + \zeta_N^d = 2\zeta_N^\frac{c+d}{2} \cos\left(\frac{|c-d|}{N} \pi\right), \]
the condition of \( N \) is given by
\[ \cos\left(\frac{|a-b|}{N} \pi\right) = \cos\left(\frac{|c-d|}{N} \pi\right) \]
\[ \iff \sin\left(\frac{|a-b|}{N} \pi + \frac{|c-d|}{N} \pi\right) \sin\left(\frac{|a-b|}{N} \pi - \frac{|c-d|}{N} \pi\right) = 0 \]
\[ \iff N \left|\frac{|a-b|}{2} + \frac{|c-d|}{2}\right| \text{ or } N \left|\frac{|a-b|}{2} - \frac{|c-d|}{2}\right| = 0. \]

5 Proof of Theorem 9–10.

Proof of Theorem 9. Let \( r = 1 \). Since
\[ Z(s) = \zeta_{G_m/F_1}(s) \]
\[ = \frac{s}{s-1}, \]
we have
\[ Z_N^K(s) = \prod_{k=0}^{K} Z(s + kN) \]
\[ = \prod_{k=0}^{K} \frac{s + kN}{s - 1 + kN}. \]
Hence, when \( N = 1 \) we obtain

\[
Z^K_1(s) = \prod_{k=0}^{K} \frac{s + k}{s - 1 + k} = \frac{s + K}{s - 1}
\]

and

\[
Z^\infty_1(s) = \lim_{K \to \infty} Z^K_1(s) = \infty.
\]

Now we assume \( N \geq 2 \). We notice that

\[
Z^K_N(s) = \zeta_{f_{1,N}}(s)
\]

with

\[
f^K_{1,N}(x) = (x - 1)(1 + x^{-N} + \cdots + x^{-KN})
\]

\[
= (x - 1)\frac{1 - x^{-(K+1)N}}{1 - x^{-N}}
\]

\[
= \frac{x + x^{-(K+1)N} - 1 - x^{1-(K+1)N}}{1 - x^{-N}}.
\]

Using the Stirling’s formula and

\[
Z^K_N(s) = \frac{\Gamma_1(s - 1, (N))
\Gamma_1(s + (K + 1)N, (N))}{\Gamma_1(s, (N))
\Gamma_1(s + (K + 1)N - 1, (N))}
\]

\[
= \frac{\Gamma\left(\frac{x-1}{N}\right)
\Gamma\left(\frac{1}{N} + K + 1\right)}{\Gamma\left(\frac{x}{N}\right)
\Gamma\left(\frac{1}{N} + K + 1\right)},
\]

we have

\[
Z^\infty_N(s) = \lim_{K \to \infty} Z^K_N(s) = \infty.
\]

Next let \( r \geq 2 \). Then

\[
Z(s) = \zeta_{G_{in}/\bar{G}_1}(s)
\]

\[
= \prod_{\ell=0}^{r} \frac{1}{s - \ell} (-1)^{r-\ell+1} \left( \begin{array}{c} r \\ \ell \end{array} \right),
\]

\[
Z^K_N(s) = \prod_{k=0}^{K} Z(s + kN).
\]
Since

\[ Z(s + kN) = \prod_{\ell=0}^{r} (s - \ell + kN)^{-1}\left(\frac{r}{\ell}\right) \]

\[ = \prod_{\ell=0}^{r} \frac{(k + s - \ell)}{N}^{-1}\left(\frac{r}{\ell}\right) \]

\[ = \prod_{\ell=0}^{r} \frac{\Gamma\left(\frac{s - \ell + k}{N}\right)}{\Gamma\left(\frac{s - \ell}{N} + k\right)}^{-1}\left(\frac{r}{\ell}\right), \]

\[ Z^K_N(s) = \prod_{\ell=0}^{r} \left( \prod_{k=0}^{K} \frac{\Gamma\left(\frac{s - \ell + k + 1}{N}\right)}{\Gamma\left(\frac{s - \ell}{N} + k\right)} \right)^{-1}\left(\frac{r}{\ell}\right) \]

\[ = \prod_{\ell=0}^{r} \left( \frac{\Gamma\left(\frac{s - \ell + K + 1}{N}\right)}{\Gamma\left(\frac{s - \ell}{N}\right)} \right)^{-1}\left(\frac{r}{\ell}\right) \]

\[ = \prod_{\ell=0}^{r} \Gamma\left(\frac{s - \ell}{N}\right)^{-1}\left(\frac{r}{\ell}\right) \]

\[ \times \prod_{\ell=0}^{r} \Gamma\left(\frac{s - \ell}{N} + K + 1\right)^{-1}\left(\frac{r}{\ell}\right) \]

As \( K \to \infty \) by the Stirling’s formula we have

\[ \Gamma\left(\frac{s - \ell}{N} + K + 1\right) \sim \sqrt{2\pi K} e^{-K} \left( \frac{s - \ell}{N} + \frac{1}{2} \right). \]

Hence, we have

\[ \lim_{K \to \infty} \prod_{\ell=0}^{r} \Gamma\left(\frac{s - \ell}{N} + K + 1\right)^{-1}\left(\frac{r}{\ell}\right) = 1, \]

where we used

\[ \sum_{\ell=0}^{r} (-1)^{r-\ell} \left(\frac{r}{\ell}\right) = 0, \]

\[ \sum_{\ell=0}^{r} (-1)^{r-\ell} \ell \left(\frac{r}{\ell}\right) = 0. \]

Then we have

\[ Z_N^\infty(s) = \prod_{k=0}^{\infty} \zeta_{G_m,G_1}(s + kN) \]

\[ = \prod_{\ell=0}^{r} \Gamma\left(\frac{s - \ell}{N}\right)^{-1}\left(\frac{r}{\ell}\right) \]

\[ \square \]
Proof of Theorem 10. Let

$$f_{r,N}^\infty(x) = \frac{(x - 1)^r}{1 - x^{-N}}.$$  

Since

$$Z_N^\infty(s) = \prod_{k=0}^{\infty} \zeta_{G_m/F_1}(s + kN) = \zeta_{f_{r,N}^\infty}(s),$$

$Z_N^\infty(s)$ is a rational function $\Leftrightarrow \frac{(x - 1)^r}{1 - x^{-N}} \in \mathbb{Z}[x, x^{-1}] \Leftrightarrow N = 1$. Especially, when $N = 1$ using

$$f_{r,1}(x) = \frac{(x - 1)^r}{1 - x^{-1}} = x(x - 1)^{r-1},$$

$$\prod_{k=0}^{\infty} \zeta_{G_m/F_1}(s + k) = \zeta_{G_m^{-1}/F_1}(s - 1).$$

\[\square\]

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