Some estimates of Wang–Yau quasilocal energy

Pengzi Miao\textsuperscript{1}, Luen-Fai Tam\textsuperscript{2} and Naqing Xie\textsuperscript{3}

\textsuperscript{1} School of Mathematical Sciences, Monash University, Victoria, 3800, Australia
\textsuperscript{2} The Institute of Mathematical Sciences and Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong, People’s Republic of China
\textsuperscript{3} School of Mathematical Sciences, Fudan University, Shanghai 200433, People’s Republic of China

E-mail: Pengzi.Miao@sci.monash.edu.au, lftam@math.cuhk.edu.hk and nqxie@fudan.edu.cn

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Abstract
Given a spacelike 2-surface $\Sigma$ in a spacetime $N$ and a constant future timelike unit vector $T_0$ in $\mathbb{R}^{3,1}$, we derive upper and lower estimates of Wang–Yau quasilocal energy $E(\Sigma, X, T_0)$ for a given isometric embedding $X$ of $\Sigma$ into a flat 3-slice in $\mathbb{R}^{3,1}$. The quantity $E(\Sigma, X, T_0)$ itself depends on the choice of $X$; however, the infimum of $E(\Sigma, X, T_0)$ over $T_0$ does not. In particular, when $\Sigma$ bounds a compact domain $\Omega$ in a time symmetric 3-slice in $N$ and has nonnegative Brown–York quasilocal mass $m_{BY}(\Sigma, \Omega)$, our estimates show that $\inf_{T_0} E(\Sigma, X, T_0)$ equals $m_{BY}(\Sigma, \Omega)$. We also study the spatial limit of $\inf_{T_0} E(S_r, X_r, T_0)$, where $S_r$ is a large coordinate sphere in a fixed end of an asymptotically flat initial data set $(M, g, p)$ and $X_r$ is an isometric embedding of $S_r$ into $\mathbb{R}^3 \subset \mathbb{R}^{3,1}$. We show that if $(M, g, p)$ has future timelike ADM energy–momentum, then $\lim_{r \to \infty} \inf_{T_0} E(S_r, X_r, T_0)$ equals the ADM mass of $(M, g, p)$.

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1. Introduction
In [9, 10], Wang and Yau define a quasilocal energy $E(\Sigma, X, T_0)$ for a spacelike 2-surface $\Sigma$ in a spacetime $N$, where $X$ is an isometric embedding of $\Sigma$ into the Minkowski spacetime $\mathbb{R}^{3,1}$ and $T_0$ is a constant future timelike unit vector in $\mathbb{R}^{3,1}$. Under the assumptions that the mean curvature vector $H$ of $\Sigma$ in $N$ and the mean curvature vector $H_0$ of $\Sigma$ when embedded in $\mathbb{R}^{3,1}$ are both spacelike, $E(\Sigma, X, T_0)$ is defined and can be expressed as follows
They prove an important property on positivity of \( m_{\Sigma} \) embed 
\[ E(\Sigma, X, T_0) = \frac{1}{8\pi} \int_{\Sigma} \left\{ \sqrt{\langle H_0 \rangle^2 (1 + |\nabla \tau|^2)} + (\Delta \tau)^2 - \sqrt{|H|^2 (1 + |\nabla \tau|^2)} + (\Delta \tau)^2 \right\} 
+ \Delta \tau \left[ \sin^{-1} \left( \frac{\Delta \tau}{\sqrt{1 + |\nabla \tau|^2 |H_0|}} \right) - \sin^{-1} \left( \frac{\Delta \tau}{\sqrt{1 + |\nabla \tau|^2 |H|}} \right) \right] \right. \]
\[ - \left\langle \nabla_{\nabla \tau} J_0 \left| \frac{H_0}{|H_0|} \right| + \left( \nabla_{\nabla \tau} J \left| \frac{H}{|H|} \right| \right) \right\rangle \mathrm{d}V \Sigma. \]  
(1.1)

Here \( |H_0| = \sqrt{\langle H_0, H_0 \rangle}, |H| = \sqrt{\langle H, H \rangle}, \tau = -\langle X, T_0 \rangle \), \( \nabla \tau \) and \( \Delta \tau \) are the intrinsic gradient and the intrinsic Laplacian of \( \tau \) on \( \Sigma \); \( J \) is the future timelike unit normal vector field along \( \Sigma \) in \( N \) which is dual to \( H \) along the light cone in the normal bundle of \( \Sigma \). Namely, if \( e_1, e_2, e_3, e_4 \) are orthonormal tangent vectors of \( N \) such that \( e_1, e_2 \) are tangent to \( \Sigma, e_3 \) is spacelike with \( \langle e_3, H \rangle < 0 \) and \( e_4 \) is future timelike, then \( H = \langle H, e_3 \rangle e_3 - \langle H, e_4 \rangle e_4 \) and \( J = (\langle H, e_3 \rangle e_3 - \langle H, e_4 \rangle e_4) / \sqrt{1 + |\nabla \tau|^2 |H_0|} \) is defined similarly for \( \Sigma \) when embedded in \( \mathbb{R}^{3,1} \).

Wang and Yau [9, 10] define a new quasi-local mass of \( \Sigma \) in \( N \), which we denote by \( m_{\Sigma Y}(\Sigma) \), to be the infimum of \( E(\Sigma, X, T_0) \) over those \( X \) and \( T_0 \) such that the resulting \( \tau = -(T_0, X) \) is admissible (see [10, definition 5.1] for the definition of admissible data). They prove an important property on positivity of \( m_{\Sigma Y}(\Sigma) \) under the assumption that \( N \) satisfies the usual dominant energy condition. Note that if \( \Sigma \) has positive Gaussian curvature, then \( \tau = 0 \) is admissible [10, Remark 1.1]. Hence, \( m_{\Sigma Y}(\Sigma) \leq E(\Sigma, X, T_0) = m_{LY}(\Sigma) \), where \( X \) is an embedding of \( \Sigma \) into \( \mathbb{R}^{3} = \langle 0, x \rangle \in \mathbb{R}^{3,1} \), \( T_0 = (1, 0, 0, 0) \) and \( m_{LY}(\Sigma) \) is the Liu–Yau quasi-local mass of \( \Sigma \) [5, 6]. The Liu–Yau quasi-local mass \( m_{LY}(\Sigma) \) is the same as the Brown–York quasi-local mass \( m_{BY}(\Sigma, \Omega) \) [2, 3] if \( \Sigma \) bounds a compact time-symmetric hypersurface \( \Omega \).

The expression of \( E(\Sigma, X, T_0) \) is rather complicated. It is not clear if \( m_{LY}(\Sigma) \) can be achieved by some admissible data. In [11], restricting to an embedding \( X \) of \( \Sigma \) into \( \mathbb{R}^{3} \subset \mathbb{R}^{3,1} \), Wang and Yau study the spatial limit of \( E(\Sigma, X, T_0) \) on an asymptotically flat spacelike hypersurface. Motivated by their work, we want to get some lower and upper estimates of \( E(\Sigma, X, T_0) \) for a fixed 2-surface \( \Sigma \).

More precisely, let us assume that \( \Sigma \) has a positive Gaussian curvature. Isometrically embed \( \Sigma \) in \( \mathbb{R}^{3} = \langle 0, x \rangle \in \mathbb{R}^{3,1} \) and let \( X \) be the embedding. Let \( T_0 = (\sqrt{1 + |a|^2}, a) \), where \( a = (a_1, a_2, a_3) \in \mathbb{R}^{3} \), be a constant future timelike unit vector in \( \mathbb{R}^{3,1} \). In this work, we will prove the following estimates:
\[ \sqrt{1 + |a|^2} (m_{LY}(\Sigma) + C) - \sum_{i=1}^{3} a_i V_i \geq E(\Sigma, X, T_0) \]
\[ \geq \sqrt{1 + |a|^2} m_{LY}(\Sigma) - \sum_{i=1}^{3} a_i V_i \]  
(1.2)

where \( C \) is a constant depending only on \( |H_0|, |H| \) and their integrals on \( \Sigma \), and \( V = (V_1, V_2, V_3) \) is a constant vector in \( \mathbb{R}^{3} \) so that \(-\sum_{i=1}^{3} a_i V_i = \int_{\Sigma} \frac{1}{\sqrt{1 + |\nabla \tau|^2 |H_0|}} \mathrm{d}V \) in (1.1). If we let \( W = (m_{LY}(\Sigma), V) \), then \( \sqrt{1 + |a|^2} m_{LY}(\Sigma) - \sum_{i=1}^{3} a_i V_i \) is exactly \(-(T_0, W)\), and (1.2) can be written as
\[ -(T_0, W) + C \sqrt{1 + |a|^2} \geq E(\Sigma, X, T_0) \geq -(T_0, W). \]  
(1.3)

An immediate application of the estimates (1.2) is the following result relating the Wang–Yau quasi-local energy and the Brown–York quasi-local mass:

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With the above notation, suppose $\Sigma$ bounds a compact, time-symmetric hypersurface $\Omega$ in a spacetime $N$ satisfying the dominant energy condition. Suppose $\Sigma$ has positive Gaussian curvature and has positive mean curvature in $\Omega$ with respect to the outward unit normal, then

$$\inf_{T_0} E(\Sigma, X, T_0) = m_{\text{WY}}(\Sigma, \Omega).$$

Another application of (1.2) is in the study of the spatial limit of Wang–Yau quasilocal energy on an asymptotically flat spacelike hypersurface. Recall that a spacelike hypersurface $M$ with induced metric $g$ and second fundamental form $p$ in a spacetime $N$ is called asymptotically flat if there is a compact set $K$ such that $M \setminus K$ has finitely many ends, each of which is diffeomorphic to the complement of a Euclidean ball in $\mathbb{R}^3$ and such that the metric $g$ is of the form $g_{ij} = \delta_{ij} + a_{ij}$ so that

$$r |a_{ij}| + r^2 |\partial a_{ij}| + r^3 |\partial^2 a_{ij}| \leq C$$

and $p_{ij}$ satisfies

$$r^2 |p_{ij}| + r^3 |\partial p_{ij}| \leq C$$

for some constant $C$, where $r = |x|$ denotes the coordinate length in $\mathbb{R}^3$. In [11], Wang and Yau prove the following:

**Theorem 1.1** [11, theorem 3.1]. Suppose $S_r$ is the coordinate sphere of radius $r$ in an end of an asymptotically flat three-manifold $(M, g_{ij}, p_{ij})$ and $(E, P^1, P^2, P^3)$ is the ADM energy–momentum four-vector of this end; then

$$\lim_{r \to \infty} E(S_r, X_r, T_0) = \sqrt{1 + |a|^2} E + \sum_{i=1}^3 a^i P_i,$$

where $X_r$ is the (unique) isometric embedding of $S_r$ into $\mathbb{R}^3 \subset \mathbb{R}^{3,1}$ and $T_0 = (\sqrt{1 + |a|^2}, a_1, a_2, a_3)$ is an arbitrary constant timelike unit vector.

In fact, $X_r$ is only unique up to rigid motions in $\mathbb{R}^3$, and $E(S_r, X_r, T_0)$ may be different from $E(S_r, Y_r, T_0)$ if $Y_r = A \circ X_r$, where $A$ is a rigid motion of $\mathbb{R}^3$. The theorem is true and was proved for embeddings $X_r$ satisfying certain conditions using the results in [4]. See lemma 3.1 for details.

Since the Wang–Yau quasilocal mass $m_{\text{WY}}(\Sigma)$ is defined as the infimum of a class of $E(\Sigma, X, T_0)$, we would like to understand the asymptotical behavior of the infimum $\inf_{T_0} E(S_r, X_r, T_0)$ as $r \to \infty$. Note that in theorem 1.1, if $E > |P|$ and hence the ADM mass $m_{\text{ADM}}$ (see [1]) of $(M, g, p)$ is positive, then it is not hard to see that

$$\inf_{a \in \mathbb{R}^3} \left( \sqrt{1 + |a|^2} E + \sum_{i=1}^3 a^i P_i \right) = \sqrt{E^2 - |P|^2} = m_{\text{ADM}}. \quad (1.6)$$

Applying the estimates (1.2), we will prove the following:

With the notation given in theorem 1.1, suppose $N$ satisfies the dominant energy condition and suppose $N$ is not flat along $M$; then

$$\lim_{r \to \infty} \inf_{T_0} E(S_r, X_r, T_0) = m_{\text{ADM}}. \quad (1.7)$$

Again, since $m_{\text{WY}}(S_r)$ is defined to be the infimum of $E(S_r, X_r, T_0)$ over those isometric embeddings $X_r : \Sigma \hookrightarrow \mathbb{R}^{3,1}$ and constant future timelike unit vectors $T_0$ in $\mathbb{R}^{3,1}$ such that $\tau = -(X_r, T_0)$ are admissible, we remark that it is yet unclear whether $\lim_{r \to \infty} m_{\text{WY}}(S_r) = m_{\text{ADM}}$. 

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2. Estimates of Wang–Yau quasilocal energy

In this section, we derive the main estimates of $E(\Sigma, X, T_0)$ for a given isometric embedding $X$ of $\Sigma$ into a flat 3-slice in $\mathbb{R}^{3,1}$. Precisely, let $\Sigma$ be a spacelike 2-surface in a spacetime $N$ such that $\Sigma$ has positive Gaussian curvature and the mean curvature vector $H$ of $\Sigma$ in $N$ is spacelike. Let $X$ be an isometric embedding of $\Sigma$ into $\mathbb{R}^{3} = \{(0, x) \in \mathbb{R}^{3,1}\}$ and let $T_0 \in \mathbb{R}^{3,1}$ be an arbitrary constant future timelike unit vector. Let $E(\Sigma, X, T_0)$ be given by (1.1) with $\tau = -\langle T_0, X \rangle$. Since $X$ is only unique up to a rigid motion of $\mathbb{R}^{3}$, we note that $E(\Sigma, X, T_0)$ depends on the choice of $X$, but the infimum

$$\inf_{T_0} E(\Sigma, X, T_0)$$  \hspace{1cm} (2.1)

does not. Hence, if it is finite, (2.1) gives a geometric invariant of $\Sigma$ in $N$.

First, we define a constant vector $V = V(\Sigma, X, T)$ associated with $\Sigma$ and $X$. Let $a_\Sigma(\cdot)$ be the connection 1-form on $\Sigma$ introduced in [10, (1.3)] which is defined as

$$a_\Sigma(Y) = \left( \nabla_Y N, \frac{H}{|H|} \right), \quad \forall Y \in T \Sigma. \hspace{1cm} (2.2)$$

Let $V = V(\Sigma)$ be the vector field on $\Sigma$ that is dual to $a_\Sigma(\cdot)$. Given an embedding $X : \Sigma \hookrightarrow \mathbb{R}^{3}$, we identify $V$ with $dX(V)$ through the tangent map $dX$ and hence view $V = V(\Sigma, X)$ as an $\mathbb{R}^{3}$-valued vector field along $\Sigma$. Now define

$$V(\Sigma, X) = \frac{1}{8\pi} \int_{\Sigma} V \, d\Sigma \in \mathbb{R}^{3} \hspace{1cm} (2.3)$$

and

$$V(\Sigma, X) = (m_{LY}(\Sigma), V(\Sigma, X)) \in \mathbb{R}^{3,1}. \hspace{1cm} (2.4)$$

Here $m_{LY}(\Sigma)$ is the Liu–Yau quasilocal mass defined in [5, 6], i.e.

$$m_{LY}(\Sigma) = \frac{1}{8\pi} \int_{\Sigma} (k_0 - |H|) \, d\Sigma, \hspace{1cm} (2.5)$$

where $k_0$ is the mean curvature of $\Sigma$ with respect to the outward unit normal when isometrically embedded in $\mathbb{R}^{3}$. Note that if $\Sigma$ bounds a compact, time-symmetric hypersurface $\Omega$ in $N$ and $\Sigma$ has a positive mean curvature in $\Omega$ with respect to the outward unit normal, then $m_{LY}(\Sigma)$ agrees with the Brown–York quasilocal mass $m_{BY}(\Sigma, \Omega)$ [2, 3].

If $X$ differs by a rigid motion in $\mathbb{R}^{3}$, then $V(\Sigma, X)$ differs by a corresponding rotation in $\mathbb{R}^{3}$ and $V(\Sigma, X)$ differs by a rotation in $\mathbb{R}^{3}$ considered as a Lorentzian transformation in $\mathbb{R}^{3,1}$.

**Theorem 2.1.** Let $\Sigma, X, T_0$ and $V = V(\Sigma, X)$ and $\mathcal{W} = \mathcal{W}(\Sigma, X)$ be given as above. The following are true:

(i) Suppose $V = (V_1, V_2, V_3)$; then

$$-\langle T_0, \mathcal{W} \rangle + C \sqrt{1 + |a|^2} = \sqrt{1 + |a|^2} (m_{LY}(\Sigma) + C) - \sum_{i=1}^{3} a^i V_i \geq E(\Sigma, X, T_0) \geq \sqrt{1 + |a|^2} m_{LY}(\Sigma) - \sum_{i=1}^{3} a^i V_i = -\langle T_0, \mathcal{W} \rangle,$$

where $T_0 = (\sqrt{1 + |a|^2}, a_1, a_2, a_3)$ and $C$ is the constant given by

$$C = \sup_{\Sigma} \left( \left| \frac{|H_0|^2}{|H|^2} + \frac{|H_0|}{|H|} - 2 \right| \frac{1}{8\pi} \int_{\Sigma} |(|H_0| - |H|)| \right).$$
Moreover, the equality
\[ E(\Sigma, X, T_0) = -\langle T_0, \mathcal{W} \rangle \] (2.7)
holds if and only if \( T_0 = (1, 0, 0, 0) \) or \( |H| = |H_0| \) everywhere on \( \Sigma \). In this second case, \( E(\Sigma, X, T_0) = -\sum_{i=1}^{3} a_i V_i, \forall T_0 \).

(ii) If \( \mathcal{W} \) is future timelike, then
\[
\sqrt{-\langle \mathcal{W}, \mathcal{W} \rangle} + C \frac{m_{LY}(\Sigma)}{\sqrt{-\langle \mathcal{W}, \mathcal{W} \rangle}} \geq E(\Sigma, X, T_0^*) \\
\geq \inf_{T_0^*} E(\Sigma, X, T_0) \geq \sqrt{-\langle \mathcal{W}, \mathcal{W} \rangle},
\]
where \( T_0^* = \frac{\mathcal{W}}{\sqrt{-\langle \mathcal{W}, \mathcal{W} \rangle}}. \)

(iii) The vector \( \dot{\mathcal{V}} \) is the gradient of the function \( E(a) = E(\Sigma, X, T_0) \) at \( a = (0, 0, 0) \). If in addition \( m_{LY}(\Sigma) \geq 0 \), then \( \mathcal{V} = 0 \) if and only if
\[
\inf_{T_0^*} E(\Sigma, X, T_0) = m_{LY}(\Sigma).
\] (2.9)

**Proof.** (i) Let \( \{e_\alpha | \alpha = 1, 2\} \) be an orthonormal frame of \( T_p \Sigma \) for any \( p \in \Sigma \). Identifying \( e_\alpha \) with \( dX(e_\alpha) \) through the tangent map \( dX \), we have
\[
\nabla \tau = -\sum_{\alpha=1}^{2} (e_\alpha(a, X)) e_\alpha = -\sum_{\alpha=1}^{2} \langle a, e_\alpha \rangle e_\alpha.
\]
Hence,
\[
\left< \nabla^N_{\tau} \frac{J}{|H|}, \frac{H}{|H|} \right> = -\sum_{\alpha=1}^{2} \langle a, e_\alpha \rangle \left< \nabla^N_{e_\alpha} \frac{J}{|H|}, \frac{H}{|H|} \right> = -\langle a, \mathcal{V} \rangle,
\] (2.10)
where \( \mathcal{V} \) is the vector field on \( \Sigma \) dual to the connection 1-form \( \alpha_\tau(\cdot) \) defined in (2.2). On the other hand,
\[
-\left< \nabla^{\rho, \rho}_{\tau} \frac{J_0}{|H_0|}, \frac{H_0}{|H_0|} \right> = 0
\] (2.11)
since \( X(\Sigma) \) lies in \( \mathbb{R}^3 \). Therefore, by (1.1), (2.10) and (2.11), we have
\[
E(\Sigma, X, T_0) = \hat{E}(\Sigma, X, T_0) - \langle a, \mathcal{V} \rangle,
\] (2.12)
where
\[
\hat{E}(\Sigma, X, T_0) = \frac{1}{8\pi} \int_\Sigma \left\{ \sqrt{|H_0|^2(1 + |\nabla \tau|^2) + (\Delta \tau)^2} - \sqrt{|H|^2(1 + |\nabla \tau|^2) + (\Delta \tau)^2} \\
- \Delta \tau \left[ \sinh^{-1} \left( \frac{\Delta \tau}{\sqrt{1 + |\nabla \tau|^2}|H_0|} \right) - \sinh^{-1} \left( \frac{\Delta \tau}{\sqrt{1 + |\nabla \tau|^2}|H|} \right) \right] \right\} dV_\Sigma.
\] (2.13)
We first prove that
\[
\hat{E}(\Sigma, X, T_0) \geq \sqrt{1 + |a|^2} m_{LY}(\Sigma).
\] (2.14)
To prove (2.14), we use spherical coordinates on \( \mathbb{R}^3 \). Let \( \rho \geq 0 \) be a scalar and \( \omega \in S^2 \) be a unit vector in \( \mathbb{R}^3 \). Writing \( a = \rho \omega \), we have
\[
\tau = -\langle a, X \rangle = -\rho \langle \omega, X \rangle,
\]
\[
\Delta \tau = -\langle a, D X \rangle = -\langle \rho \omega, H_0 \rangle = \rho k_0 \langle \omega, e^{\hat{h}_0} \rangle,
\]
\[
1 + |\nabla \tau|^2 = 1 + \rho^2(1 - \langle \omega, e^{\hat{h}_0} \rangle^2),
\]
and
\[
$$
\begin{align*}
\Delta \tau &= -\langle a, D X \rangle = -\rho \omega \cdot H_0 \\
&= \rho k_0 \langle \omega, e^{\hat{h}_0} \rangle,
\end{align*}
$$

where \( k_0 \) is the scalar curvature of the manifold.

\[
\sqrt{|H_0|^2(1 + |\nabla \tau|^2) + (\Delta \tau)^2} - \sqrt{|H|^2(1 + |\nabla \tau|^2) + (\Delta \tau)^2} \\
- \Delta \tau \left[ \sinh^{-1} \left( \frac{\Delta \tau}{\sqrt{1 + |\nabla \tau|^2}|H_0|} \right) - \sinh^{-1} \left( \frac{\Delta \tau}{\sqrt{1 + |\nabla \tau|^2}|H|} \right) \right] \\
&\geq \sqrt{1 + |a|^2} m_{LY}(\Sigma).
\]
where $H_0$ is the mean curvature vector of $X(\Sigma)$ in $\mathbb{R}^3 \subset \mathbb{R}^{3,1}$, $e^{H_0} = -\frac{H_0}{|H_0|}$ is the outward unit normal to $X(\Sigma)$ in $\mathbb{R}^3$ and $k_0 = |H_0|$ is the mean curvature of $X(\Sigma)$ with respect to $e^{H_0}$ in $\mathbb{R}^3$.

In what follows, we let

$$p = \langle \omega, e^{H_0} \rangle, \quad q = \sqrt{1 - \langle \omega, e^{H_0} \rangle^2}$$

be two functions on $\Sigma$. In terms of $p, q$, we have

$$\Delta_1 \tau = k_0 p, \quad 1 + |\nabla \tau|^2 = 1 + \rho^2 q^2$$

and

$$\frac{\Delta \tau}{\sqrt{1 + |\nabla \tau|^2}} = k_0 \frac{\rho p}{\sqrt{1 + \rho^2 q^2}}.$$  

The quantity $\tilde{E}(\Sigma, X, T_0)$ now becomes a function of $\rho$ and $\omega$, say $\tilde{E}(\rho, \omega)$. By (2.13), we have

$$\tilde{E}(\rho, \omega) = \tilde{E}(\Sigma, X, T_0)$$

$$= \frac{1}{8\pi} \int_{\Sigma} k_0 \left[ \sqrt{1 + \rho^2 - \sqrt{\rho^2 q^2 + t^2(1 + \rho^2 q^2)}} \right]$$

$$+ k_0(\rho p) \left[ \sinh^{-1} \left( \frac{\rho p}{\sqrt{1 + \rho^2 q^2}} \frac{k_0}{k} \right) - \sinh^{-1} \left( \frac{\rho p}{\sqrt{1 + \rho^2 q^2}} \right) \right]$$

where $k = |H| > 0$ and $H$ is the mean curvature vector of $\Sigma$ in $N$. Henceforth, we omit writing the volume form $dv_{\Sigma}$ for simplicity.

Since $k_0 > 0$, we can rewrite $\tilde{E}(\rho, \omega)$ as

$$\tilde{E}(\rho, \omega) = \frac{1}{8\pi} \int_{\Sigma} k_0 \left[ \sqrt{1 + \rho^2 - \sqrt{\rho^2 p^2 + t^2(1 + \rho^2 q^2)}} \right]$$

$$+ k_0(\rho p) \left[ \sinh^{-1} \left( \frac{\rho p}{\sqrt{1 + \rho^2 q^2}} \frac{t}{f} \right) - \sinh^{-1} \left( \frac{\rho p}{\sqrt{1 + \rho^2 q^2}} \right) \right]$$

where $t = \frac{k_0}{\rho} > 0$ is a function on $\Sigma$. Let

$$f(\rho, \omega) = \frac{\rho p}{\sqrt{1 + \rho^2 q^2}}.$$  

Then

$$1 + \rho^2 q^2 = \frac{1 + \rho^2}{1 + f^2}, \quad p = \frac{f}{\rho (1 + f^2)^{1/2}}.$$  

Now define

$$B(\rho, \omega) = \sqrt{1 + \rho^2} \cdot \sqrt{\rho^2 p^2 + t^2(1 + \rho^2 q^2)}.$$  

$$= \sqrt{1 + \rho^2} \left( 1 - \frac{(t^2 + f^2)^{1/2}}{(1 + f^2)^{1/2}} \right)$$

and

$$F(\rho, \omega) = \rho p \left[ \sinh^{-1} \left( \frac{f}{t} \right) - \sinh^{-1}(f) \right].$$

By (2.18), we have

$$\tilde{E}(\rho, \omega) = \frac{1}{8\pi} \int_{\Sigma} k_0[B(\rho, \omega) + F(\rho, \omega)].$$

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Direct calculation gives
\[
\frac{\partial f}{\partial \rho} = \frac{\rho(1 + \rho^2 q^2)^{-\frac{3}{2}}}{f(1 + f^2)}, \quad (2.24)
\]
\[
\frac{\partial B}{\partial \rho} = \frac{\rho}{(1 + \rho^2)^{\frac{3}{2}}} \frac{(1 - t^2) f^2 + \rho^2 (t^2 + f^2)}{\rho(t^2 + f^2)^{\frac{3}{2}}(1 + \rho^2)^{\frac{3}{2}}}, \quad (2.25)
\]
\[
\frac{\partial F}{\partial \rho} = \frac{\rho}{(1 + f^2)^{\frac{3}{2}}} \left[ \sinh^{-1} \left( f \left( \frac{t}{f} \right) \right) + \frac{f(1 + f^2)}{(1 + \rho^2)(t^2 + f^2)^{\frac{3}{2}}} \right], \quad (2.26)
\]
where we assume \( \rho > 0 \) whenever \( \rho \) appears in a denominator. By (2.25) and (2.26), we have
\[
\frac{\partial \tilde{E}}{\partial \rho} = \frac{\rho}{(1 + \rho^2)^{\frac{3}{2}}} \int_{\Sigma} k_0 \left[ \frac{\partial B}{\partial \rho} + \frac{\partial F}{\partial \rho} - \frac{\rho}{(1 + \rho^2)^{\frac{3}{2}}}(1 - t) \right] = \frac{1}{8\pi} \int_{\Sigma} k_0 (\Phi(t) - \Phi(1)), \quad (2.27)
\]
where
\[
\Phi(t) = \frac{1}{\rho(1 + f^2)^{\frac{3}{2}}(1 + \rho^2)^{\frac{3}{2}}} \left[ -f(1 + \rho^2) \sinh^{-1} \left( \frac{f}{t} \right) + (\rho^2 - f^2)(t^2 + f^2)^{\frac{3}{2}} \right] - \frac{\rho}{(1 + \rho^2)^{\frac{3}{2}}}. \quad (2.28)
\]
Now
\[
\frac{\partial \Phi}{\partial t} = \left[ (1 - t^2) f^2 + \rho^2 (t^2 + f^2) \right] \frac{1}{\rho(1 + f^2)^{\frac{3}{2}}(1 + \rho^2)^{\frac{3}{2}}} - 1 \frac{\rho}{(1 + \rho^2)^{\frac{3}{2}}}. \quad (2.29)
\]
Hence if \( 0 < t \leq 1 \),
\[
\frac{\partial \Phi}{\partial t} \geq \frac{\rho}{(1 + \rho^2)^{\frac{3}{2}}} \left[ \frac{(1 + \left( \frac{t}{f} \right)^2)^{\frac{3}{2}}}{(1 + f^2)^{\frac{3}{2}}} - 1 \right] \geq 0. \quad (2.30)
\]
Similarly, if \( t > 1 \), then
\[
\frac{\partial \Phi}{\partial t} \leq \frac{\rho}{(1 + \rho^2)^{\frac{3}{2}}} \left[ \frac{(1 + \left( \frac{t}{f} \right)^2)^{\frac{3}{2}}}{(1 + f^2)^{\frac{3}{2}}} - 1 \right] \leq 0. \quad (2.31)
\]
Therefore, \( \Phi(1) = \max_{t>0} \Phi(t) \). By (2.27), we have
\[
\frac{\partial \tilde{E}}{\partial \rho} \geq \frac{\rho}{(1 + \rho^2)^{\frac{3}{2}}} \text{mLY}(\Sigma) \quad (2.32)
\]
for all $\rho > 0$. Integrating (2.32) and noting that $\dot{E}(0, \omega) = m_{LY}(\Sigma)$, we conclude

$$E(\rho, \omega) \geq \sqrt{1 + \rho^2 m_{LY}(\Sigma)},$$

(2.33)

which proves (2.14). Now the lower bound of $E(\Sigma, X, T_0)$ in (2.6) follows directly from (2.12) and (2.14).

In order to obtain an upper bound for $E(\Sigma, X, T_0)$, by (2.29) and using the fact $f^2 \leq \rho^2$, we have, if $0 < t_0 \leq t \leq 1$,

$$\frac{\partial \Phi}{\partial t} \leq \left(\frac{1}{t^2} + \frac{1}{t} - 2\right) \frac{\rho}{(1 + \rho^2)^{\frac{3}{2}}}$$

$$\leq \left(\frac{1}{t_0^2} + \frac{1}{t_0} - 2\right) \frac{\rho}{(1 + \rho^2)^{\frac{3}{2}}}.$$  

(2.34)

Hence by the mean value theorem, if $0 < t_0 \leq 1$,

$$\Phi(1) - \Phi(t_0) \leq (1 - t_0) \left(\frac{1}{t_0^2} + \frac{1}{t_0} - 2\right) \frac{\rho}{(1 + \rho^2)^{\frac{3}{2}}}. $$

(2.35)

Similarly, if $t_0 \geq t \geq 1$, we have

$$\frac{\partial \Phi}{\partial t} \geq \left(\frac{1}{t^2} + \frac{1}{t} - 2\right) \frac{\rho}{(1 + \rho^2)^{\frac{3}{2}}}$$

$$\geq \left(\frac{1}{t_0^2} + \frac{1}{t_0} - 2\right) \frac{\rho}{(1 + \rho^2)^{\frac{3}{2}}}.$$  

(2.36)

Thus, (2.35) is also true for $t_0 \geq 1$. By (2.27) and (2.35),

$$\frac{\partial \dot{E}}{\partial \rho} - \frac{\rho}{(1 + \rho^2)^{\frac{3}{2}}} m_{LY}(\Sigma) \leq \frac{\rho}{(1 + \rho^2)^{\frac{3}{2}}} \sup_{\Sigma} \left(\frac{|H_o|^2}{|H|^2} + \frac{|H_o|}{|H|} - 2\right) \left(\frac{1}{8\pi} \int \sum |(\|H_o\| - |H|)|. $$

(2.37)

From this, it is easy to see that the upper bound for $E(\Sigma, X, T_0)$ in (2.6) holds.

To prove the rest of (i), we note that if $T_0 = (1, 0, 0, 0)$, then $E(\Sigma, X, T_0) = m_{LY}(\Sigma)$; if $|H| = |H_o|$, then $E(\Sigma, X, T_0) = E(\Sigma, X, T_0) = 0$. Hence, (2.7) holds automatically in either case. Now suppose (2.7) is true for some $T_0$ with $a = \rho_0 \omega$ where $\rho_0 > 0$, then by the proof of (2.33), we have $\Phi(t) = \Phi(1)$ everywhere on $\Sigma$ for any $0 < \rho \leq \rho_0$. A detailed examination of (2.29)–(2.31) shows that, at points $x \in \Sigma$ where $f = 0$, we have $\frac{\partial \Phi}{\partial t_0} = 0$, $\forall t$, while at points $x \in \Sigma$ where $f \neq 0$, we have $\frac{\partial \Phi}{\partial t_0} > 0$, $\forall t = 0 < \rho < 1$, and $\frac{\partial \Phi}{\partial t_0} < 0$ for all $\rho > 0$. Therefore, the fact $\Phi(t) = \Phi(1)$ on $\Sigma$ implies that $t = 1$ on the subset $\{f \neq 0\} \subset \Sigma$. On the other hand, since $\rho_0 > 0$, we know $f = 0$ if and only if $p = 0$ or equivalently $\langle \omega, e^{\omega t_0} \rangle = 0$. Since $X(\Sigma)$ is a strictly convex closed surface in $\mathbb{R}^3$, the Gauss map that sends a point on $\Sigma$ to its outward unit normal $e^{\omega t_0}$ is a diffeomorphism from $\Sigma$ to $\mathbb{S}^2$. Therefore, the set $\{\langle \omega, e^{\omega t_0} \rangle = 0\}$ is a closed embedded curve in $\Sigma$. Consequently, its complement $\{f \neq 0\}$ is dense in $\Sigma$. Therefore, $t = 1$ and hence $|H| = |H_o|$ everywhere on $\Sigma$. In this case, by definition, $\dot{E}(\Sigma, X, T_0) = 0$, and by (2.12), $E(\Sigma, X, T_0) = -\sum_{i=1}^{3} a_i \nu_i$.

(ii) Suppose $\nu$ is future timelike, then $-(T_0, \nu)$ attains its minimum over all future timelike unit vectors $T_0$ at $T_0 = \nu/\sqrt{-\langle \nu, \nu \rangle}$. Now (2.8) follows directly from (2.6).

(iii) At $\rho = 0$, we observe that $f, \frac{\partial \Phi}{\partial \rho}, \frac{\partial T}{\partial \rho}$ all equal zero by (2.19), (2.25) and (2.26); hence $\frac{\partial \nu}{\partial \rho} = 0$ by (2.23). Therefore, the gradient of $E(a)$ at $a = (0, 0, 0)$ is $-\nu$ by (2.12). Now if (2.9) holds, then $a = (0, 0, 0)$ is a critical point of $E(a)$ by the fact that $E(0) = m_{LY}(\Sigma)$; therefore $-\nu = 0$. On the other hand, suppose $-\nu = 0$, by (2.6) we have $E(\Sigma, X, T_0) \approx 1 + |a|^2 m_{LY}(\Sigma)$, which implies $E(\Sigma, X, T_0) \approx m_{LY}(\Sigma)$ by the assumption that $m_{LY}(\Sigma) \approx 0$. Hence, (2.9) holds.
This completes the proof of theorem 2.1.

Suppose $\Sigma$ bounds a compact domain $\Omega$ in a time-symmetric slice in $N$, then one would like to compare the Brown–York mass of $\Sigma$ and the Wang–Yau energy of $\Sigma$. As an immediate corollary of theorem 2.1, we have

**Corollary 2.1.** Suppose $\Sigma$ bounds a compact, time-symmetric hypersurface $\Omega$ in a spacetime $N$ satisfying the dominant energy condition. Suppose $\Sigma$ has positive Gaussian curvature and positive mean curvature $k$ in $\Omega$ with respect to the outward unit normal $\nu$, then

$$\inf_{T_0} E(\Sigma, X, T_0) = m_{BY}(\Sigma, \Omega),$$

and the infimum is achieved at $T_0 = (1, 0, 0, 0)$. Moreover, if $m_{BY}(\Sigma, \Omega) > 0$, then $(1, 0, 0, 0)$ is the unique absolute minimum point of $E(\Sigma, X, T_0)$ when viewed as a function of $T_0$.

**Proof.** Let $n$ be the future timelike unit normal to $\Omega$ in $N$. Since $\Omega$ is time-symmetric and $k = -\langle H, \nu \rangle > 0$, we have $H = -kn$ and $\nabla^N_Y J = \nabla^N_Y n = 0$ for any vector $Y$ tangent to $\Omega$. Hence, the vector field $V$ vanishes pointwise on $\Sigma$. As a result, $V = 1/16\pi \int_{\Sigma} V^i d\nu = 0$. By theorem 2.1 and the fact $m_{LY}(\Sigma) = m_{BY}(\Sigma, \Omega)$ under the assumptions, we have

$$E(\Sigma, X, T_0) \geq \sqrt{1 + |a|^2} m_{BY}(\Sigma, \Omega).$$  \tag{2.38}

On the other hand, by the positivity results on the Brown–York mass [8], we have

$$m_{BY}(\Sigma, \Omega) \geq 0.$$  \tag{2.39}

Therefore, it follows from (2.38) and (2.39) that

$$E(\Sigma, X, T_0) \geq m_{BY}(\Sigma, \Omega),$$  \tag{2.40}

where $m_{BY}(\Sigma, \Omega)$ is the value of $E(\Sigma, X, T_0)$ when $T_0 = (1, 0, 0, 0)$. If $m_{BY}(\Sigma, \Omega) > 0$, (2.38) further implies

$$E(\Sigma, X, T_0) > m_{BY}(\Sigma, \Omega)$$  \tag{2.41}

for any $T_0 \neq (1, 0, 0, 0)$. The corollary is thus proved. \hfill \Box

### 3. Large sphere limit of Wang–Yau quasilocal energy at spatial infinity

In this section, we study the asymptotical behavior of $E(S_r, X_r, T_0)$ in an asymptotically flat spacelike hypersurface $(M, g, p)$ in a spacetime $N$, where $S_r$ is the coordinate sphere in a fixed end of $M$ and $X_r$ is a suitably chosen embedding of $S_r$ into $\mathbb{R}^3 = \{(0, x) \in \mathbb{R}^{3,1}\}$.

First, we recall the definition of the ADM energy–momentum of $(M, g, p)$. Let $\{y^i | i = 1, 2, 3\}$ be an asymptotic flat coordinate chart on $M$, the ADM energy–momentum [1] of $(M, g, p)$ is a four-covector

$$(E, P_1, P_2, P_3),$$

where

$$E = \lim_{r \to \infty} \frac{1}{16\pi} \int_{S_r} (\partial_j g_{ij} - \partial_i g_{jj}) v^j d\nu,$$

is the ADM energy of $(M, g, p)$ and

$$P_k = \lim_{r \to \infty} \frac{1}{16\pi} \int_{S_r} 2(p_{ik} - \delta_{ik} p_{jj}) v^j d\nu$$

for $k = 1, 2, 3$. The ADM energy–momentum satisfies the Einstein’s field equations with $\Lambda = 0$.

Similarly, we shall construct the quasilocal mass and energy–momentum of $(M, g, p)$.
is the ADM linear momentum of $(M, g, p)$ in the $y^k$-direction. Here $S_r$ is the coordinate sphere $\{|y| = r\}$ and $\nu^i = \frac{\partial}{\partial y^i}$ is the outward unit normal to $S_r$.

Under the assumptions (1.4) and (1.5) on $(M, g, p)$, we have the following fact from [4, lemma 2.3].

**Lemma 3.1.** Let $(M, g, p)$ be an asymptotically flat spacelike hypersurface in a spacetime $N$. Let $Y = (y^1, y^2, y^3)$ be the asymptotically flat coordinates on a fixed end of $M$. There exist an $r_0$ and a constant $C$ independent of $r$ such that for $r \geq r_0$, there is an isometric embedding $X_r = (x^1, x^2, x^3)$ of $S_r = \{|y| = r\}$ into the Euclidean space $\mathbb{R}^3$ such that

$$|X_r - Y| + r\|\nabla X_r - \nabla Y\|_h + r^2\|H_0\| + r|n^0_r - n_r| \leq C,$$

where $X_r$ and $Y$ are considered as $\mathbb{R}^3$-valued functions on the sphere $S_r$, $H_0$ is the mean curvature vector of $X_r(S_r)$, the gradient $\nabla$ and the norm $\|\cdot\|_h$ are taken with respect to the induced metric $h_r$ on $S_r$, $n^0_r$ is the unit outward normal of $X_r(S_r)$ and $n_r = Y/|Y|$.

We will always work with the embedding $X_r$ of $S_r$ provided by lemma 3.1. Given such an $X_r$, we let $V_r = V(S_r, X_r), W_r = W(S_r, X_r)$ which are defined by (2.3) and (2.4).

**Theorem 3.1.** Let $(M, g, p), S_r, Y$ and $X_r$ be as in lemma 3.1. Let $V_r, W_r$ be given in (3.1).

Let $E(S_r, X_r, T_0)$ be the Wang–Yau quasilocal energy of $S_r$, where $T_0 = (\sqrt{1 + |a|^2}, a)$ is any constant future timelike unit vector in $\mathbb{R}^3$ with $a = (a_1, a_2, a_3)$.

Then the following are true:

(i) $\lim_{r \to \infty} W_r = (E, -P_1, -P_2, -P_3)$, \hspace{2cm} (3.2)

where $(E, -P_1, -P_2, -P_3)$ is the ADM energy–momentum four vector of $(M, g, p)$.

(ii) $E(S_r, X_r, T_0) = -\langle T_0, W_r \rangle + e(r)(1 + |a|^2)^{\frac{1}{2}}$ \hspace{2cm} (3.3)

where $e(r)$ is a quantity such that $e(r) \to 0$ as $r \to \infty$ uniformly in $a$.

(iii) Suppose $(E, -P_1, -P_2, -P_3)$ is future timelike. Then

$$\lim_{r \to \infty} \inf_{T_0} E(S_r, X_r, T_0) = m_{\text{ADM}} = \lim_{r \to \infty} E(S_r, X_r, T_0^*)$$ \hspace{2cm} (3.4)

where $m_{\text{ADM}} = \sqrt{E^2 - |P|^2}$ is the ADM mass of $(M, g, p)$ and $T_0^* = W_r/\sqrt{-\langle W_r, W_r \rangle}$ for sufficiently large $r$.

**Proof.**

(i) By the proof of [11, theorem 3.1], we know

$$\lim_{r \to \infty} \frac{1}{8\pi} \int_{S_r} \left( \frac{\nabla^N}{|H|} \frac{J}{|H|} H \right) dv_r = \sum_{i=1}^3 a^i P_r.$$ \hspace{2cm} (3.5)

By (2.10), we have

$$\left( \frac{\nabla^N}{|H|} \frac{J}{|H|} H \right) = -\langle a, V \rangle,$$ \hspace{2cm} (3.6)
where $V = V_r$ is the vector field on $S_r$ dual to the 1-form $\alpha_e(\cdot)$ defined in (2.2). Therefore, by the definition of $V_r$ and (3.5) and (3.6),

$$\lim_{r \to \infty} \langle a, V_r \rangle = \lim_{r \to \infty} \frac{1}{8\pi} \int_{S_r} \langle a, V \rangle = -\sum_{i=1}^{3} a^i P_i.$$  

(3.7)

Since $a = (a^1, a^2, a^3)$ can be chosen arbitrarily, (3.7) implies

$$\lim_{r \to \infty} V_r = -(P_1, P_2, P_3).$$  

(3.8)

Next, we show

$$\lim_{r \to \infty} m_{LY}(S_r) = E.$$  

(3.9)

By [4], we have $\lim_{r \to \infty} m_{BY}(S_r, \Omega_r) = E$. Hence, it suffices to prove

$$\lim_{r \to \infty} (m_{LY}(S_r) - m_{BY}(S_r, \Omega_r)) = 0.$$  

(3.10)

It follows from the definitions of $m_{LY}(S_r)$ and $m_{BY}(S_r, \Omega_r)$ that

$$m_{LY}(S_r) - m_{BY}(S_r, \Omega_r) = \int_{S_r} (k - |H|) \, dv_r$$

$$= \int_{S_r} \left( k - \sqrt{k^2 - (\text{tr}_{S_r} p)^2} \right) \, dv_r,$$  

(3.11)

where $k$ is the mean curvature of $S_r$ in $(M, g)$ with respect to the outward unit normal $\nu_r$ and $\text{tr}_{S_r} p$ denotes the trace of $p$ restricted to $S_r$. Write $\nu_r = \nu_i \frac{\partial}{\partial y_i}$, by [4, Lemma 2.1], we have

$$\nu_i = y_i r + O(r^{-1}) \quad \text{and} \quad k = \frac{2}{r} + O(r^{-2}).$$  

(3.12)

Hence,

$$\text{tr}_{S_r} p = g^{ij} p_{ij} - p(\nu_r, \nu_r) = O(r^{-2}).$$  

(3.13)

Therefore,

$$k - \sqrt{k^2 - (\text{tr}_{S_r} p)^2} = \frac{(\text{tr}_{S_r} p)^2}{k + \sqrt{k^2 - (\text{tr}_{S_r} p)^2}} = O(r^{-3}),$$  

(3.14)

which together with (3.11) implies (3.10).

(ii) By (2.6) in theorem 2.1, we have

$$|E(S_r, X_r, T_0) - (-\langle T_0, W_r \rangle)| \leq C_r (1 + |a|^2)^{\frac{3}{2}},$$  

(3.15)

where

$$C_r = \sup_{S_r} \left( \left| \frac{|H_0|^2}{|H|^2} + \frac{|H_0|}{|H|} - 2 \right| \right) \frac{1}{8\pi} \int_{S_r} |(|H_0| - |H|)|.$$  

By lemma 3.1 and the fact $|H| = 2r^{-1} + O(r^{-2})$ in (3.12), we have

$$\lim_{r \to \infty} C_r = 0.$$  

(3.16)

Therefore, (3.3) follows from (3.15) and (3.16).
(iii) Since \((E, -P_1, -P_2, -P_3)\) is future timelike, by (i) \(W_r\) is future timelike if \(r\) is sufficiently large. For such an \(r\), by (2.8) in theorem 2.1,
\[
\sqrt{-\langle W_r, W_r \rangle} + C_r \frac{m_{\text{LY}}(S_r)}{\sqrt{-\langle W_r, W_r \rangle}} \geq E(S_r, X_r, T_{0r}^+) \\
\geq \inf_{T_0} E(S_r, X_r, T_0) \geq \sqrt{-\langle W_r, W_r \rangle}.
\]
\[(3.17)\]
Now (3.4) follows from (3.17), (3.16), (3.9) and the fact that
\[
\lim_{r \to \infty} \sqrt{-\langle W_r, W_r \rangle} = m_{\text{ADM}}.
\]
This completes the proof of theorem 3.1.

**Corollary 3.1.** Let \((M, g, p)\) be an asymptotically flat spacelike hypersurface in a spacetime \(N\) satisfying the dominant energy condition. With the same notation as in theorem 3.1, if \(N\) is not flat along \(M\), then
\[
\lim_{r \to \infty} \inf_{T_0} E(S_r, X_r, T_0) = m_{\text{ADM}}.
\]

**Proof.** By the positive mass theorem of Schoen–Yau [7] and Witten [12], the ADM energy–momentum of \((M, g, p)\) is future timelike unless \(N\) is flat along \(M\). The result now follows from theorem 3.1.

Suppose there is an end of \((M, g, p)\) such that \(m_{\text{ADM}} = 0\), by [7, 12] we know \(N\) is flat along \(M\), \((M, g, p)\) has only one end and can be isometrically embedded in the Minkowski space \(\mathbb{R}^{3,1}\); moreover, \(E = 0, P_i = 0, i = 1, 2, 3\). By [10, 11], we then have
\[
\lim_{r \to \infty} \inf_{\{\text{admissible } T_0\}} E(S_r, X_r, T_0) = 0.
\]
It is still unclear whether the following is true:
\[
\lim_{r \to \infty} \inf_{T_0} E(S_r, X_r, T_0) = 0.
\]

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