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Thierry Paul

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SYMBOLIC CALCULUS FOR SINGULAR CURVE OPERATORS

THIERRY PAUL

Abstract. We define a generalization of the Töplitz quantization, suitable for operators whose Töplitz symbols are singular. We then show that singular curve operators in Topological Quantum Fields Theory (TQFT) are precisely generalized Töplitz operators of this kind and we compute for some of them, and conjecture for the others, their main symbol, determined by the associated classical trace function.

in memory of Erik Balslev
from whom I learned so much
in mathematics and in physics

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1. INTRODUCTION

In 1925, Heisenberg invented quantum mechanics as a change of paradigm from (classical) functions to (quantum) matrices. He founded the new mechanics on the well known identity

\[ \frac{1}{\hbar}[Q, P] = 1 \]

that, a few months later, Dirac recognized as the quantization of the Poisson bracket

\[ \{q, p\} = 1. \]

Again a few years later, Weyl stated the first general quantization formula by associating to any function \( f(q, p) \) the operator

\[ F(Q, P) = \int \tilde{f}(\xi, x) e^{i\frac{qP - pQ}{\hbar}} d\xi dx \]

where \( \tilde{f} \) is the symplectic Fourier transform defined analogously by

\[ f(q, p) = \int \tilde{f}(\xi, x) e^{i\frac{qP - pQ}{\hbar}} d\xi dx. \]

Many years after was born the pseudodifferential calculus first establish by Calderon and Zygmund, and then formalized by Hörmander through the formula giving the integral kernel \( \rho_F \) for the quantization \( F \) of a symbol \( f \) in \( d \) dimensions as

\[ \rho_F(x, y) = \int f(q, p) e^{i\frac{p(x-y)}{\hbar}} \frac{dp}{(2\pi\hbar)^d} \]

A bit earlier had appeared, both in quantum field theory and in optics (Wick quantization) the (positive preserving) Töplitz quantization of a symbol \( f \)

\[ \text{Op}^T[f] = \int f(q, p)|q, p \times q, p| dq dq \]

where \( |q, p\) are the famous (suitably normalized) coherent states.

As we see, quantization is not unique. But all the different symbolic calculi presented above share, after inversion of the quantization formulæ written above, the same two first asymptotic features:

- the symbol of a product is, modulo \( \hbar \), the product of the symbols
- the symbol of the commutator divided by \( i\hbar \) is, modulo \( \hbar \) again, the Poison bracket of the symbols.
In other words, they all define a classical underlying space (an algebra of functions) endowed with a Poisson (of more generally symplectic) structure.

But it is very easy to show that this nice quantum/classical picture has its limits. And one can easily construct quantum operators whose classical limit will not follow the two items expressed above.

Consider for example the well known creation and annihilation operators $a^+ = Q + iP, a^- = Q - iP$. They act of the eigenvectors $h_j$ of the harmonic oscillator by

$$a^+ h_j = \sqrt{(j + \frac{1}{2})\hbar} h_{j+1}, \quad a^- h_j = \sqrt{(j - \frac{1}{2})\hbar} h_{j-1}.$$ 

Consider now the matrices

$$M_1^+ = \begin{pmatrix} 0 & 0 & 0 & 0 & \ldots & \ldots \\ 1 & 0 & 0 & 0 & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ldots & \ldots \\ 0 & \ldots & 0 & 1 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & 0 & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & 0 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix},$$

and its adjoint

$$M_1^- = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & \ldots \\ 0 & 0 & 1 & 0 & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ldots & \ldots \\ 0 & \ldots & 0 & 0 & 1 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & 1 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & 0 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}.$$

An elementary computation shows that

$$\mathcal{M}_1^+ = a^+(P^2 + Q^2)^{-1/2}, \quad \mathcal{M}_1^- = (P^2 + Q^2)^{-1/2} a^-.$$ 

Therefore, their (naively)expected leading symbols are $f^+(q, p) = \sqrt{\frac{q+ip}{q-ip}}$ and $f^-(q, p) = \sqrt{\frac{q-ip}{q+ip}}$ or, in polar coordinates $q + ip = \rho e^{i\theta}, f^\pm = e^{\pm i\theta}$.

If symbolic calculus would work the leading symbol of $\mathcal{M}_1^+ \mathcal{M}_1^-$ should be equal to 1 and $\mathcal{M}_1^+ \mathcal{M}_1^-$ should be therefore close to the identity $I$ as $\hbar \to 0$.

But

$$M_1^+ M_1^- = \begin{pmatrix} 0 & 0 & 0 & 0 & \ldots & \ldots \\ 0 & 1 & 0 & 0 & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ldots & \ldots \\ 0 & \ldots & 0 & 1 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & 0 & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix} = I - |h_0\rangle \langle h_0| \approx I,$$
The reason for this defect comes from the fact that the function $e^{i\theta} = \sqrt{z}$ is not a smooth function on the plane. In fact it is not even continuous at the origin: $F(z)$ can tend to any value in $\{e^{i\theta}, \theta \in \mathbb{R}\}$ when $z$ tends to zero.

Note finally that the commutator

$$[M_1^+, M_1^-] = \begin{pmatrix}
-1 & 0 & 0 & 0 & \ldots & \ldots \\
0 & 0 & 0 & 0 & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & 0 & 0 & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{pmatrix} = -|\bar{h}_0 \times h_0| \neq O(h), \quad (1.1)$$

so that its symbol doesn’t vanish at leading order, as expected by standard symbolic asymptotism.

One of the main goals of this paper is to define a quantization procedure which assigns a symbolic calculus to matrices presenting the pathologies analogous to the ones of $M_1^+, M_1^-$. We will state the results in the framework of quantum mechanics on the sphere $S^2$ as phase space. The reason of this is the fact that it is this quantum setting which corresponds to the asymptotism in Topological Quantum Fields Theory (TQFT) studied, among others, in [MP15].

This procedure will be a non-trivial extension to the Töplitz (anti-Wick) quantization already mentioned, and we will derive a suitable notion of symbol.

Indeed, another main goal of this article is to give a semiclassical settings to all curve-operators in TQFT in the case of the once punctured torus or the 4-times punctured sphere. In [MP15] was established that these curve-operators happen, for almost all colors associated to the marked points, to be Töplitz operators associated to the quantization of the two-sphere, in some asymptotics of large number of colors. It happens that this result applies for every curve whose classical trace function is a smooth function on the sphere. Since this trace function is shown to be the principal Töplitz-symbol of the curve operator, the lack of smoothness ruins the possibility of semiclassical properties for the curve operator in the paradigm of Töplitz quantization (see Section 4 below for a very short presentation of TQFT and the main results of [MP15]). In fact these singular trace functions are not even continuous at the two poles of the sphere, which suggests a kind of blow-up on the two singularities of the classical phase-space. This is not surprising that such a regularization should be done in a more simple way at the quantum level.
In the present paper we will show how to “enlarge” the formalism of Töplitz operators to “a-Töplitz operators”, in order to catch the asymptotics of the singular cases by semiclassical methods and compute the leading order symbols of some of them and conjecture them for the general singular curve operators. This principal symbol will be completely determined by the corresponding classical trace function, but will not be equal to it, for the reason that this enlarged a-Töplitz quantization procedure involves operator valued symbols. This construction will also be valid in the regular cases, where in this case the operator valued symbol is just a potential, hence it is defined by a function on the sphere whose leading behaviour is given by the trace function, as expected.

Therefore we are able in this paradigm to handle the large coloring asymptotism of all curve operators in the case of the once punctured torus or the 4-times punctured sphere (note that the method we use is able to give some partial results in higher genus cases).

We also study the natural underlying phase-space of our enlarged paradigm, the corresponding moduli space for TQFT, as a non-commutative space by identification with the non-commutative algebra of operator valued functions appearing at the classical limit for the symbol of the a-Töplitz operators, in the spirit of noncommutative geometry.

We will build the construction of the a-Töplitz quantization by showing its necessity on some toy matrices situations in Sections 3 after having defined in Section 2 the new Hilbert space on which these matrices will act, and before to show in Section 4 how general curve operators in TQFT enter this formalism.

The main results are Theorems 15 and 18, out of Definition 12 and Theorem 20 together with Section 4.4 below. The a-Töplitz operators are introduced in Definition 17.

Quantization of the sphere is briefly reviewed in Section 2.1, we won’t repeat it here. Let us just say that it consists in considering the sphere $S^2$ as the compactification of the plane $\mathbb{C}$. Hence one expect that the singular phenomenon which appeared above at the origin should now appear twice at the poles of the sphere. The quantum Hilbert space can be represented as the space of entire functions, square-integrable with respect to a measure $d\mu_N$ given in (3.6).

Instead of trying to blow-up these two singularities at a “classical” (namely manifold) level, we will see that there is an easiest way of solving the problem by working directly at the “quantum”
of their symbols. Together with their products whose symbols are, at leading order, the (noncommutative) product $M$. Section 3.6 Definition 17, and Theorem 18 shows that matrices like $\gamma$, where now $\Sigma$ are operators, namely operators of the form where each $\psi^N_n$ is proportional to the elements of the canonical basis $\{\varphi^N_n, n = 1, \ldots, N - 1\}$, provides a decomposition of the identity on $\mathcal{H}_N$ endowed with a different Hilbert structure for which the $\psi^N_n$'s are normalized (see Section 2.3). The advantage of working with the left hand side of (1.2) instead of the usual decomposition of the identity using coherent states and leading to standard Töplitz quantization, is the fact that $\psi^a_z$ possess an extra parameter: the density $a$. Therefore one can “act” on $\psi^a_z$ not only by multiplication by a function $f(z)$ but by letting an operator valued function of $z$ acting on $a$. This leads to what is called in this paper $a$-Töplitz operators, namely operators of the form

$$\int_\mathbb{C} |\psi_z^{\Sigma(z)}|\langle \psi_z^a|d\mu_N(z),$$

where now $\Sigma(z)$ is, for each $z$, an operator acting on $\mathcal{S}(\mathbb{R})$. The precise definition is given in Section 3.6 Definition 17, and Theorem 18 shows that matrices like $M^\pm_1$ are $a$-Töplitz operators, together with their products whose symbols are, at leading order, the (noncommutative) product of their symbols.

Let us remark finally that, even at the limit $h = \frac{\pi}{N} = 0$, the symbol of $M^\pm_1$ is NOT $e^{\pm i\theta} = (z/\bar{z})^{\pm \frac{1}{2}}$. Traces of the noncommutative part of the symbol persist at the classical limit, as in [TP]. Therefore the “classical underlying phase-space” is not the 2-sphere anymore, but rather a noncommutative space identified with a non commutative algebra of such symbols playing the role of the commutative algebra of continuous functions on a standard manifold. A quick description of this space, inspired of course by noncommutative geometry [AC], is given in Section 3.7.

The construction dealing with $M^\pm_1$ can be in particular generalized to matrices of the form

$$M^N_\gamma = \begin{pmatrix}
\gamma_0(0) & \gamma_1(0) & \gamma_2(0) & \cdots & \cdots & \gamma_{N-1}(0) \\
\gamma_{-1}(0) & \gamma_0(1/N) & \gamma_1(1/N) & \cdots & \cdots & \gamma_{N-2}(1/N) \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\gamma_{-(N-2)}(0) & \cdots & \cdots & \gamma_{-1}((N-2)/N) & \gamma_0((N-2)/N) & \gamma_1((N-2)/N) \\
\gamma_{-(N-1)}(0) & \cdots & \cdots & \gamma_{-2}((N-3)/N) & \gamma_{-1}((N-2)/N) & \gamma_0((N-1)/N)
\end{pmatrix}. \quad (1.3)$$

Namely, instead of considering the quantization process related to the so-called coherent state family $\rho_z$ defined in (2.3) and which are (micro)localized at the points $z \in S^2$, we will consider families of states $\psi^a_z := \int_\mathbb{R} a(t)e^{\frac{i}{\hbar} e^{i(t)} t} \rho_x^a \frac{dz}{\sqrt{2\pi}}$, where $t(z) = \frac{\|z\|^2}{1+|z|^2}$ and $a \in \mathcal{S}(\mathbb{R})$ (see Section 2.2 for details). For $z$ not at the poles, $\psi^a_z$ is a Lagrangian (semiclassical) distribution (WKB state) localized on the parallel passing through $z$ [PU], but for $z$ close to the pole the states $\psi^a_z$ catches a different information. The equality (2.13):

$$\int_\mathbb{C} |\psi_z^a|^2 \langle \psi_z^a|d\mu_N(z) = \sum_{n=0}^{N-1} |\psi_n^N\rangle \langle \psi_n^N|,$$

(1.2)

where each $\psi^N_n$ is normalized (see Section 2.3). The advantage of working with the left hand side of (1.2) instead of the usual decomposition of the identity using coherent states and leading to standard Töplitz quantization, is the fact that $\psi^a_z$ possess an extra parameter: the density $a$. Therefore one can “act” on $\psi^a_z$ not only by multiplication by a function $f(z)$ but by letting an operator valued function of $z$ acting on $a$. This leads to what is called in this paper $a$-Töplitz operators, namely operators of the form

$$\int_\mathbb{C} |\psi_z^{\Sigma(z)}|\langle \psi_z^a|d\mu_N(z),$$

where now $\Sigma(z)$ is, for each $z$, an operator acting on $\mathcal{S}(\mathbb{R})$. The precise definition is given in Section 3.6 Definition 17, and Theorem 18 shows that matrices like $M^\pm_1$ are $a$-Töplitz operators, together with their products whose symbols are, at leading order, the (noncommutative) product of their symbols.

Let us remark finally that, even at the limit $h = \frac{\pi}{N} = 0$, the symbol of $M^\pm_1$ is NOT $e^{\pm i\theta} = (z/\bar{z})^{\pm \frac{1}{2}}$. Traces of the noncommutative part of the symbol persist at the classical limit, as in [TP]. Therefore the “classical underlying phase-space” is not the 2-sphere anymore, but rather a noncommutative space identified with a non commutative algebra of such symbols playing the role of the commutative algebra of continuous functions on a standard manifold. A quick description of this space, inspired of course by noncommutative geometry [AC], is given in Section 3.7.
It has been proven in [BGPU, AS] that such a family of matrices $M_{\gamma}^N$ is a Töplitz operator of symbol $\gamma(\tau, \theta) = \sum_{k=1}^{N-1} \gamma_k(\tau) e^{ik\theta}$ if and only if

$$\left(\tau(1-\tau)\right)^{\frac{\alpha}{2}} \gamma_k(\tau) \in C^\infty([0,1]), k = 1, \ldots, N - 1.$$  \hspace{1cm} (1.4)

Condition (1.4) expresses explicitly that $\gamma \in C^\infty(S_2)$.

In Section 3.6 theorem 18 we prove that (more general matrices than) the family $M_{\gamma}^N$ are $a$-Töplitz operators, and we compute their symbols, when (1.4) is replaced by the condition $^1$

$$\gamma_k(\tau) \in C^\infty([0,1]), k = 1 - N, \ldots, N - 1.$$  \hspace{1cm} (1.5)

Under (1.5) $\gamma \notin C^\infty(S_2)$ and one has to pass form the Töplitz to the $a$-Töplitz paradigm (note that $M_1^{\pm}$ indeed satisfy (1.5) and not (1.4)).

Let us finish this long introduction by giving the key ideas leading to the setting of our main result, Theorem 20. The reader can found in Section 4 a very short introduction to TQFT. Larger basics on TQFT can be found in [MP15] using the same vocabulary as the present paper together with a substantial bibliography.

Combinatorial curve operators are actions of the curves on a punctured surface $\Sigma$ on a finite dimensional vector space $V_r(\Sigma, c)$ indexed by a level $r$ and a coloring $c$ of the marked points taken in a set of $r$ colors. The dimension $N = N(r)$ of $V_r(\Sigma, c)$ will diverge as $r \to \infty$ and $\frac{1}{r}$ can be considered as a phenomenological Planck constant $\hbar$.

In [MP15] we provided the construction of an explicit orthogonal basis of $V_r(\Sigma, c)$ and we conjectured that any curve operator is expressed in this basis by a matrix essentially of the form $M_{\gamma}$. More precisely we showed that the conjecture is true in the case where $\Sigma$ is either the punctured 2-torus or the 4 times punctured sphere. Even more, we proved that (the matrix of) any curve operator belongs to the algebra generated by three matrices of the form $M_{\gamma}^N_{\Gamma_0}$, $M_{\gamma}^N_{\Gamma_1}$, $M_{\gamma}^N_{\Gamma_d}$

\footnote{The construction works certainly also for conditions of the type, e.g.,

$$\left(\tau(1-\tau)\right)^{\frac{\alpha}{2}} \gamma_k(\tau) \in C^\infty([0,1]), k = 1, \ldots, N - 1, \hspace{0.5cm} 0 \leq \alpha \leq 1$$

(or even more general ones), but since we don’t see any applications of these situations, we concentrate in this paper to the condition (1.5).}
defined in (1.3) where
\[
\begin{align*}
\Gamma_0^r(\tau, \theta) &= \gamma_0(\tau, r) \\
\Gamma_1^r(\tau, \theta) &= 2\gamma_1(\tau, r) \cos \theta \\
\Gamma_d^r(\tau, \theta) &= e^{\frac{2\pi}{\Gamma} \gamma_1(\tau, r) \cos (\theta + \tau)},
\end{align*}
\]
for two explicit families of functions $\gamma_0, \gamma_1$.

We showed in [MP15] that, for “most” values of the coloring of the marked points of $\Sigma$, the functions $\Gamma_0^r, \Gamma_1^r, \Gamma_d^r$ are smooth functions on the sphere, and that, indeed, the corresponding curve operators are standard Töplitz operators. This proves also that any curve operator is Töplitz, by the stability result by composition of the Töplitz class. Moreover the leading symbols of any curve operator happen to be the classical trace function associated to the corresponding curve (see [MP15] for details).

Theorem 20 of the present article express the same result for any coloring of the marked points, at the expense of replacing Töplitz quantization by $a$-Töplitz one. The only difference, unavoidably for the reason of the change of Töplitz paradigm, is the fact that the $a$-Töplitz leading symbol of the curve operator is not (and cannot) the classical trace function of the curve, but we are able to compute or conjecture it out of the trace function.

2. Hilbert spaces associated to new quantizations of the sphere

2.1. The standard geometric quantization of the sphere. In this section we will consider the quantization of the sphere in a very down-to-earth way. See [GF, MP15] for more details.

Given an integer $N$, we define the space $H_N$ of polynomials in the complex variable $z$ of order strictly less than $N$ and set
\[
\langle P, Q \rangle = \frac{i}{2\pi} \int_{\mathbb{C}} \frac{P(z)\overline{Q(z)}}{(1 + |z|^2)^{N+1}} dz d\overline{z} \quad \text{and} \quad \varphi_n^N(z) = \sqrt{\frac{N!}{n!(N-1-n)!}} z^n.
\]
(2.1)

The vectors $(\varphi_n^N)_{n=0,\ldots,N-1}$ form an orthonormal basis of $H_N$.

By the stereographic projection
\[
S^2 \ni (\tau, \theta) \in [0, 1] \times S^1 \to z = \sqrt{\frac{\tau}{1-\tau}} e^{i\theta} \in \mathbb{C} \cup \{\infty\},
\]
The space $H_N$ can be seen as a space of functions on the sphere (with a specific behaviour at the north pole). Write
\[
d\mu_N = \frac{i}{2\pi} \frac{dz d\overline{z}}{(1 + |z|^2)^{N+1}}.
\]
(2.2)

As a space of analytic functions in $L^2(\mathbb{C}, d\mu_N)$, the space $H_N$ is closed.

For $z_0 \in \mathbb{C}$, we define the coherent state
\[
\rho_{z_0}(z) = N(1 + \overline{z_0}z)^{N-1}.
\]
(2.3)
These vectors satisfy $\langle f, \rho_{z_0} \rangle = f(z_0)$ for any $f \in \mathcal{H}_N$ and the orthogonal projector $\pi_N : L^2(\mathbb{C}, d\mu_N) \to \mathcal{H}_N$ satisfies $(\pi_N \psi)(z) = \langle \psi, \rho_z \rangle$.

For $f \in C^\infty(S^2, \mathbb{R})$ we define the (standard) Toeplitz quantization of $f$ as the operator
\[
T^N[f] : \mathcal{H}_N \to \mathcal{H}_N
\]
\[
T^N[f] := \int_{\mathbb{C}} f(z) |\rho_z\rangle \langle \rho_z| d\mu_N(z)
\]
i.e. $T^N[f] \psi := \int_{\mathbb{C}} f(z) \langle \rho_z, \psi \rangle_{\mathcal{H}_N} \rho_z d\mu_N(z) = \pi_N(f \psi)$ for $\psi \in \mathcal{H}_N$.

A Toeplitz operator on $S^2$ is a sequence of operators $(T_N) \in \text{End}(\mathcal{H}_N)$ such that there exists a sequence $f_k \in C^\infty(S^2, \mathbb{R})$ such that for any integer $M$ the operator $R^M_N$ defined by the equation
\[
T_N = \sum_{k=0}^M N^{-k} T_{f_k} + R^M_N
\]
is a bounded operator whose norm satisfies $\|R_M\| = O(N^{-M-1})$.

An easy use of the stationary phase Lemma shows that the (anti-)Wick symbol (also called Husimi function) of $T_f$, namely $\langle T_f | \rho_z, \rho_z \rangle_{\langle \rho_z, \rho_z \rangle}$ satisfies
\[
\frac{\langle T_f | \rho_z, \rho_z \rangle_{\langle \rho_z, \rho_z \rangle}}{\langle \rho_z, \rho_z \rangle} = f + \frac{1}{N} \Delta_S f + O(N^{-2}),
\]
where $\Delta_S = (1 + |z|^2)^2 \partial_z \partial_{\overline{z}}$ is the Laplacian on the sphere.

### 2.2. The building vectors

Let $a \in \mathcal{S}(\mathbb{R})$, $\|a\|_{L^2(\mathbb{R})} = 1$ and $z \in \mathbb{C}$. We define
\[
\psi^a_z = \int_{\mathbb{R}} a(t) e^{i \frac{\tau(z)}{\hbar} t} \rho e^{it} \frac{dt}{\sqrt{2\pi}}
\]
where $\tau(z) = \frac{|t|^2}{1+|t|^2}$ and $\hbar = \frac{\pi}{N}$.

Although we won’t need it in this paper, let us note that, when $z$ is far away from the origin and the point at infinity, $\psi^a_z$ is a lagrangian semiclassical distribution (WKB state) (by a similar construction as in [PU]).

Since, by (2.3), $\rho_z = \sum_{n=1}^{N-1} \frac{N!}{n!(N-1-n)!} \varphi^n_{n} \varphi^N_{n}$, we get that
\[
\psi^a_z = \sum_{n=0}^{N-1} \tilde{a} \left( \frac{\tau(z) - nh}{\hbar} \right) \sqrt{\frac{N!}{n!(N-1-n)!}} \varphi^n_{n} \varphi^N_{n} = \sum_{n=0}^{N-1} \tilde{a} \left( \frac{\tau(z) - nh}{\hbar} \right) \varphi^N_{n}(z) \varphi^N_{n}.
\]

where $\tilde{a}$ is the Fourier transform of $a$
\[
\tilde{a}(y) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ixy} a(x) dx.
\]

**Remark 1.** Note that, by (2.7), $\psi^a_z$ depends only on the values of $\tilde{a}$ on $[0, N]$. Therefore one can always restrict the choice of $a$ to the functions whose Fourier transform is supported on $[0, N]$. In the sequel of this article we will always do so.
Lemma 2.

\[ \int_{\mathbb{C}} |\psi_n^a\rangle \langle \psi_n^a| d\mu_N(z) = \sum_{n=0}^{N-1} C_n^N |\varphi_n^N\rangle \langle \varphi_n^N| \]  

(2.8)

with

\[ C_n^N = \frac{(N-1)!}{n!(N-1-n)!} \int_0^1 \left| \bar{a} \left( \frac{\tau - nh}{\hbar} \right) \right|^2 \left( \frac{\tau}{1-\tau} \right)^n \left( 1 - \tau \right)^{N-1} \frac{d\tau}{\hbar}. \]  

(2.9)

Moreover, as \( N - 1 = \frac{1}{\hbar} \to \infty \)

\[ C_n^N = 1 + O\left( \frac{1}{N} \right), \quad 0 < \hbar n < 1. \]  

(2.10)

\[ C_n^N \sim \frac{1}{n!} \int_0^\infty \left| \bar{a}(\lambda - n) \right|^2 \lambda^n e^{-\lambda} \sqrt{2\pi} \lambda d\lambda, \quad 0 \sim \hbar n. \]  

(2.11)

\[ C_n^N \sim C_{N-1-n}^N, \quad n \hbar \sim 1. \]  

(2.12)

**Proof.** Deriving (2.8) is a straightforward calculus after (2.7).

By the asymptotic formula for the binomial we get that, as \( N,n \to \infty \),

\[ \frac{(N-1)!}{n!(N-1-n)!} \sim \left( \frac{N-1}{N-1-n} \right)^n \left( 1 - \frac{n}{N-1} \right)^{(N-1)}. \]

Moreover since \( 0 < \hbar n < 1 \) we get since \( \bar{a} \) is fast decreasing at infinity,

\[ \int_0^1 \left| \bar{a} \left( \frac{\tau - nh}{\hbar} \right) \right|^2 \left( \frac{\tau}{1-\tau} \right)^n \left( 1 - \tau \right)^{N-1} \frac{d\tau}{\hbar} \sim \int_{-\infty}^{+\infty} \left| \bar{a} \left( \frac{\tau - nh}{\hbar} \right) \right|^2 \left( \frac{\tau}{1-\tau} \right)^n \left( 1 - \tau \right)^{N-1} \frac{d\tau}{\hbar} \]

and

\[ \frac{1}{\hbar} \left| \bar{a} \left( \frac{\tau - nh}{\hbar} \right) \right|^2 \to \|a\|_{L^2(\mathbb{R})}^2 \delta(\tau - nh) \text{ as } \hbar = \frac{1}{N-1} \to 0. \]

Therefore we get (2.10).

\[ \Box \]

Definition 3.

\[ \psi_n^N := \sqrt{C_n^N} \varphi_n^N. \]

This definition is motivated by (2.8) which actually reads

\[ \int_{\mathbb{C}} |\psi_n^a\rangle \langle \psi_n^a| d\mu_N(z) = \sum_{n=0}^{N-1} |\psi_n^N\rangle \langle \psi_n^N|. \]  

(2.13)

This leads to the following equality:

\[ \int_{\mathbb{C}} |\psi_n^a\rangle \langle \psi_n^a| d\mu_N(z) = 1_{H_N^a}, \]  

(2.14)

where \( H_N^a \) is the same space of polynomials as \( H_N \) but now endowed with the renormalized scalar product \( \langle \cdot , \cdot \rangle_a \) fixed by

\[ \langle \psi_m^N , \psi_n^N \rangle_a = \delta_{m,n}, \]  

(2.15)

and

\[ |\psi_n^a\rangle \langle \psi_n^a| \psi := \langle \psi_n^a , \psi \rangle_a |\psi_n^a\rangle, \quad \psi \in H_N^a. \]  

(2.16)
2.3. **The Hilbert structure.** The Hilbert scalar product on $H^a_N$ is obtained out of (2.15) by bi-linearity. Since any polynomial $f$ satisfies

$$f = \sum_{n=0}^{N-1} \langle \varphi^N_n, f \rangle \varphi^N_n = \frac{1}{C_N} \langle \psi^N_n, f \rangle \psi^N_n,$$

we get

$$\langle f, g \rangle_a := \sum_{n=0}^{N-1} \frac{1}{(C_N)^2} \langle f, \psi^N_n \rangle \langle \psi^N_n, g \rangle = \sum_{n=0}^{N-1} \frac{1}{C_N} \langle f, \varphi^N_n \rangle \langle \psi^N_n, g \rangle.$$

Note that $\langle \cdot, \cdot \rangle_a$ is not given by an integral kernel. But if we “change” of representation and define $F(z) := \langle \psi^a_z, f \rangle$, $G(z) := \langle \psi^a_z, g \rangle$ then, by (2.13) we have

$$\langle f, g \rangle_a = \langle F, G \rangle = \int_{\mathbb{C}} F(z) G(z) d\mu_N(z).$$

Let us remark finally that

$$F(z) = \int_{\mathbb{R}} a(t) e^{i(t-z)t} f(e^{it} z) \frac{dt}{\sqrt{2\pi}}$$

and

$$f = \int_{\mathbb{C}} F(z) |\psi^a_z\rangle dz,$$

namely

$$f(z') = \int_{\mathbb{C}} F(z) \psi^a_z(z') dz.$$

3. **Singular quantization**

This section is the heart for the present paper. We will first show how operators on $H^a_N$ defined as matrices on the basis $\{\psi^M_n\}$ act on the building operators $\psi^a_z$ by action on $a$ (section 3.3). This will allow us, in section 3.4, to assign to each of these matrices symbols whose symbolic calculus is studied in section 3.5. This will lead us finally to section 3.6 where we define the $a$-Töplitz quantization.

3.1. **A toy model case.** Let us consider the $N \times N$ matrix

$$M_1 = \begin{pmatrix}
0 & 1 & 0 & 0 & \ldots & 0 \\
1 & 0 & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & 1 & 0 & 1 \\
0 & \ldots & 0 & 0 & 1 & 0 \\
0 & \ldots & 0 & 0 & 0 & 1 \\
\end{pmatrix} =: M_1 + M_1^+ + M_1^-$$

where

$$M_1^+ = \begin{pmatrix}
0 & 0 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & 1 & 0 & 0 \\
0 & \ldots & 0 & 0 & 1 & 0 \\
0 & \ldots & 0 & 0 & 0 & 0 \\
\end{pmatrix}$$

and

$$M_1^- = \begin{pmatrix}
0 & 1 & 0 & 0 & \ldots & 0 \\
1 & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & 0 & 1 & 0 \\
0 & \ldots & 0 & 0 & 0 & 1 \\
0 & \ldots & 0 & 0 & 0 & 0 \\
\end{pmatrix}.$$
and let us consider the operator $\mathcal{M}_1 = \mathcal{M}_1^+ + \mathcal{M}_1^-$ on $\mathcal{H}_N^a$ whose matrix on the orthonormal basis 
\{\psi^N_n, n = 0, \ldots, N - 1\} is $M$. That is
\[
\begin{aligned}
\mathcal{M}_1^+ \psi^N_0 &= \frac{1+1}{2} \psi^N_1 \\
\mathcal{M}_1^+ \psi^N_i &= \frac{1+1}{2} \psi^N_{i+1}, \quad 1 \leq i \leq N - 2 \\
\mathcal{M}_1^+ \psi^N_{N-1} &= \frac{1+1}{2} \psi^N_{N-2}
\end{aligned}
\]

Proposition 4.

\[M_1 \psi^a_z = \psi^\Sigma_1(z)^a\]

where the operator $\Sigma_1(z)$ is given by (3.5) below.

Proof. By (2.7) we get that, calling $D^N_n = (C^N_n)^{2 \over h^2}$ (once again $h = \frac{1}{N-1}$),

\[
\psi^a_z = \sum_{n=0}^{N-2} \tilde{a} \left( \frac{\tau(z) - n h}{h} \right) \sqrt{N} \sqrt{\binom{N-1}{n}} \bar{z}^n D^N_n \psi^n_n.
\]

Therefore

\[
\begin{aligned}
\mathcal{M}_1 \psi^a_z
&= \tilde{a} \left( \frac{\tau(z)}{h} \right) \sqrt{N} D^N_0 \psi^N_1 \\
&\quad + \sum_{n=1}^{N-2} \tilde{a} \left( \frac{\tau(z) - n h}{h} \right) \sqrt{N} \sqrt{\binom{N-1}{n}} \bar{z}^n D^N_n (\psi^n_{n-1} + \psi^n_{n+1}) \\
&\quad + \tilde{a} \left( \frac{\tau(z) - 1}{h} \right) \sqrt{N} \bar{z}^{N-1} D^N_{N-1} \psi^N_{N-2}
\end{aligned}
\]

\[
= \tilde{a} \left( \frac{\tau(z)}{h} \right) \sqrt{N} D^N_0 \psi^N_1 + \tilde{a} \left( \frac{\tau(z) - 1}{h} \right) \sqrt{N} \bar{z}^{N-1} D^N_{N-1} \psi^N_{N-2}
\]

\[
+ \sum_{n=0}^{N-3} \tilde{a} \left( \frac{\tau(z) - (n + 1) h}{h} \right) \sqrt{N} \sqrt{\binom{N-1}{n+1}} \bar{z}^{n+1} D^N_{n+1} \psi^n_n
\]

\[
+ \sum_{n=2}^{N-1} \tilde{a} \left( \frac{\tau(z) - (n - 1) h}{h} \right) \sqrt{N} \sqrt{\binom{N-1}{n-1}} \bar{z}^{n-1} D^N_{n-1} \psi^n_n
\]

\[
= \sum_{n=0}^{N-2} \tilde{a} \left( \frac{\tau(z) - (n + 1) h}{h} \right) \sqrt{N} \sqrt{\binom{N-1}{n+1}} \bar{z}^{n+1} D^N_{n+1} \psi^n_n
\]

\[
+ \sum_{n=1}^{N-1} \tilde{a} \left( \frac{\tau(z) - (n - 1) h}{h} \right) \sqrt{N} \sqrt{\binom{N-1}{n-1}} \bar{z}^{n-1} D^N_{n-1} \psi^n_n
\]

(3.1)

Let us consider the sum in (3.2). On can write it as

\[
\psi_{sud} = \sum_{n=0}^{N-1} \frac{1}{2} \bar{\mu}(n) \tilde{a} \left( \frac{\tau(z) - (n - 1) h}{h} \right) \sqrt{N} \sqrt{\binom{N-1}{n}} \bar{z}^n D^N_n \psi^n_n
\]

\[
= \sum_{n=0}^{N-1} \frac{1}{2} \bar{\mu}(n) \tilde{a} \left( \frac{\tau(z) - (n - 1) h}{h} \right) \psi_n(z) \varphi_n
\]
with
\[ \mu(n) = \begin{cases} \sqrt{\frac{n}{N-n}}, & n > 0 \\ 0, & n = 0 \end{cases} \] (3.3)

We get that
\[ \psi_{sud} = \psi_z b_{sud} \]
where
\[ b_{sud} = \sqrt{\frac{C_N}{C_{N-1}}} \mu \left( \frac{\tau(z)}{\hbar} - i \partial_x \right) \frac{e^{ix}}{z} a = \Sigma_1 a. \]

Here we have denote by \(\frac{C_N}{C_{N-1}}\) the function defined out of (2.9) by
\[ \frac{C_N}{C_{N-1}} : \xi \in [0, N] \rightarrow \sqrt{\frac{\xi}{N-\xi}} \int_0^1 \left| \frac{\xi}{h} \left( \frac{\tau - \xi h}{1-\tau} \right) \right|^2 \left( \frac{\tau}{1-\tau} \right)^\xi (1-\tau)^{N-1} \frac{d\tau}{h}, \] (3.4)
and \(\sqrt{\frac{C_N}{C_{N-1}}} \mu\) is meant as the product of the two functions, i.e. \(\sqrt{\frac{C_N}{C_{N-1}}} \mu(\xi) = \sqrt{\frac{C_N}{C_{N-1}}} (\xi) \mu(\xi)\) using (3.4). Note finally that, by the band limited hypothesis on \(a\) in Remark 1, \(b_{sud}\) is well defined.

Similarly we get that the sum in (3.1) is
\[ \psi_{nord} = \sum_{n=0}^{N-1} \bar{\psi}_{n} \left( \frac{\tau(z) - (n+1)h}{h} \right) \sqrt{\left( \frac{N-1}{n} \right)^2 c_n} D_n^N \psi_n^N \]
with
\[ \nu(n) = \begin{cases} \sqrt{\frac{n-1}{n+1}}, & n < N-1 \\ 0, & n = N-1 \end{cases} \]

So
\[ \psi_{nord} = \psi_z b_{nord} \]
where
\[ b_{nord} = \sqrt{\frac{C_N}{C_{N+1}}} \nu^N \left( \frac{\tau(z)}{h} - i \partial_x \right) \bar{a} e^{-ix} a = \Sigma_1^+ a. \]

We define
\[ \Sigma_1(z) = \sqrt{\frac{C_N}{C_{N+1}}} \mu^N \left( \frac{\tau(z)}{h} - i \partial_x \right) \frac{e^{ix}}{z} + \sqrt{\frac{C_N}{C_{N-1}}} \nu^N \left( \frac{\tau(z)}{h} - i \partial_x \right) \bar{a} e^{-ix}. \] (3.5)

\[ = \sqrt{\frac{C_N}{C_{N+1}}} \Sigma_1^+(z) + \sqrt{\frac{C_N}{C_{N-1}}} \Sigma_1^-(z) \]

where
\[ \mu^N = \chi_{\left[ -\frac{1}{2}, N-\frac{1}{2}\right]} \mu, \quad \nu^N = \chi_{\left[ -\frac{1}{2}, N-\frac{1}{2}\right]} \nu, \]
(3.6)
\( \chi \in C^\infty(\mathbb{R}) \) satisfies

\[
\chi_{[a,b]}(\xi) = \begin{cases} 
0 & \text{if } \xi \leq a \\
\chi'(\xi) > 0 & \text{if } a < \xi < a + \frac{1}{2} \\
1 & \text{if } a + \frac{1}{2} \leq \xi \leq b = \frac{1}{2} \\
\chi'(\xi) < 0 & \text{if } b - \frac{1}{2} < \xi < b \\
0 & \text{if } b \leq \xi 
\end{cases} \tag{3.7}
\]

and \( \mu^N \left( \frac{\tau(z)}{h} - i\partial_x \right) \) and \( \nu^N \left( \frac{\tau(z)}{h} - i\partial_x \right) \) are defined by the spectral theorem applied to the operator \(-i\partial_x\) acting on \(L^2(\mathbb{R})\). Moreover \( \psi_{\text{boud}}, \psi_{\text{bord}} \) depend only on \( \mu^N(\frac{z}{h} - n), \nu^N(\frac{z}{h} - n) \), so that they depend only on the properties (3.7) of \( \chi \).

In order to make the notations a bit lighter, we will skip the over-script \( N \) in \( \mu^N \) and \( \nu^N \) in the sequel of the paper.

3.2. (General) trigonometric matrices. Let

\[
\Sigma_1^+(z) = \Sigma_1^- = \mu^N \left( \frac{\tau(z)}{h} - i\partial_x \right) \frac{e^{ix}}{z}, \quad \Sigma_1^-(z) = \Sigma_1^+ = \nu^N \left( \frac{\tau(z)}{h} - i\partial_x \right) ze^{-ix} \tag{3.8}
\]

as defined by (3.5).

We get easily the following result.

Lemma 5.

\[
\Sigma_1^+ \Sigma_1^- = \chi \left[ \frac{1}{2}, N - \frac{1}{2} \right] \left( \frac{\tau(z)}{h} - i\partial_x \right) \tag{3.9}
\]

\[
\Sigma_1^- \Sigma_1^+ = \chi \left[ - \frac{1}{2}, N - \frac{3}{2} \right] \left( \frac{\tau(z)}{h} - i\partial_x \right) \tag{3.10}
\]

\[
[\Sigma_1^-, \Sigma_1^+] = \chi \left( \frac{\tau(z)}{h} - i\partial_x \right) \tag{3.11}
\]

is

\[
\chi(\xi) = \begin{cases} 
0 & \text{if } \xi \leq -\frac{1}{2} \\
0 & \text{if } -\frac{1}{2} < \xi < 0 \\
1 & \text{if } 0 \leq \xi \leq \frac{1}{2} \\
\chi' < 0 & \text{if } \frac{1}{2} < \xi < 1 \\
0 & \text{if } 1 \leq \xi \leq N - 2 \\
\chi' < 0 & \text{if } N - 2 < \xi < N - \frac{3}{2} \\
-1 & \text{if } N - \frac{3}{2} \leq \xi \leq N - 1 \\
0 & \text{if } N - 1 \leq \xi \leq N - \frac{1}{2} \\
0 & \text{if } N - \frac{1}{2} \leq \xi \leq 1
\end{cases}
\]

Remark 6. When \( z \) is far away from the origin or the infinity, the “symbol” \( \Sigma \) at \( z \) is just an operator of multiplication, therefore “commutative”. And it is as expected equal to, basically, \( 2 \cos \theta \). But \( 2 \cos \theta \) is not regular at the two poles, and the trace of this singularity is the fact that \( \Sigma(z) \) becomes a non-local operator when \( z \) close to the poles, coming from the fact that the vector field expressed by the transport equation becomes infinite.
\textbf{Remark 7.} By (2.8) we have that
\[
\int_{\mathbb{C}} |\psi_{\alpha}^a(x, y)|^2 d\mu_N(z) = C_L^N
\]
where $L\varphi_n = n\varphi_n$. Therefore we could also look at matrices acting on $\mathcal{H}^N$ instead of $\mathcal{H}_a^N$ by conjugation by $C_L^N$. But this doesn’t give anything interesting for symbols.

Let us generalize this to the situation where $M$ has the form, for $\alpha \in C^\infty([0, 1]) \cap L^\infty([0, 1])^2$,
\[
M_{1, \alpha} = \begin{pmatrix}
0 & \alpha(\hbar) & 0 & 0 & \cdots & 0 \\
\alpha(\hbar) & 0 & \alpha(2\hbar) & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & \alpha((N-3)\hbar) & 0 & \alpha((N-2)\hbar) \\
0 & \cdots & 0 & 0 & \alpha((N-2)\hbar) & 0
\end{pmatrix}
\]

The operator $M_{1, \alpha}$ on $\mathcal{H}_N^a$ whose matrix on the orthonormal basis $\{\psi_n, n = 0, \ldots, N - 1\}$ is $M_{1, \alpha}$ becomes
\[
\begin{align*}
M_{1, \alpha}\psi_0^N &= \alpha(\hbar)\psi_1^N \\
M_{1, \alpha}\psi_1^N &= \alpha((i-1)\hbar)\psi_{i-1}^N + \alpha((i+1)\hbar)\psi_{i+1}^N, \quad 1 \leq i \leq N - 2 \\
M_{1, \alpha}\psi_{N-1}^N &= \alpha((N-2)\hbar)\psi_{N-2}^N
\end{align*}
\]
The same type of computations contained in the proof of Proposition 4 provides, thanks to Lemma 5, the proofs of the next Propositions 8, 9 and 10 below.

\textbf{Proposition 8.}
\[
M_{1, \alpha}\psi_x^a = \psi_z^{\Sigma_{1, \alpha}(z)}
\]
where
\[
\Sigma_{1, \alpha}(z) = \frac{e^{ix}}{z}(\alpha(\kappa)\mu) \left( \frac{\tau(z)}{\hbar} + \frac{1}{2} - i\partial_x \right) + \bar{z}e^{-ix}(\alpha(\kappa)\nu) \left( \frac{\tau(z)}{\hbar} - \frac{1}{2} - i\partial_x \right).
\]  
(3.12)

In particular if $\sqrt{\tau(1 - \tau)}\alpha(\tau) \in C^\infty([0, 1])$, so that $\alpha(\tau)e^{i\theta} \in C^\infty(S^2)$, then, for all $z \in S^2$, $\Sigma_{1, \alpha}(z) \sim 2\alpha(\tau(z))\cos 2(x + \theta(z))$ as $N \to \infty$.

Otherwise, this last asymptotic equality is valid only for $z$ away from the two poles.

\footnote{by this we mean that $\alpha$ is bounded on $[0, 1]$ and $C^\infty$ on any open subset of $[0, 1]$.}
Let now, again for \( \beta \in C^\infty([0,1]) \cap L^\infty([0,1]) \),

\[
M_{2,\beta} = \begin{pmatrix}
0 & 0 & \beta(h) & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & \beta(2h) & 0 & \ldots & 0 \\
\beta(2h) & 0 & 0 & \beta(3h) & \ldots & 0 \\
\ldots & 0 & \beta((N-4)h) & 0 & 0 & \beta((N-3)h) \\
0 & \ldots & 0 & \beta((N-3)h) & 0 & 0 \\
0 & \ldots & 0 & 0 & \alpha((N-2)h) & 0 & 0
\end{pmatrix}
\]

The operator \( M_{2,\beta} \) on \( \mathcal{H}_N^\beta \) whose matrix on the orthonormal basis \( \{\psi_n^N, n = 0, \ldots, N-1\} \) is \( M_{2,\beta} \) becomes

\[
\begin{align*}
M_{2,\beta}\psi_0^N &= \alpha(h)\psi_0^N \\
M_{2,\beta}\psi_1^N &= \beta(2h)\psi_0^N \\
M_{2,\beta}\psi_i^N &= \beta((i-2)h)\psi_{i-2}^N + \alpha((i+2)h)\psi_{i+2}^N, \quad 2 \leq i \leq N-3 \\
M_{2,\beta}\psi_{N-2}^N &= \alpha((N-4)h)\psi_{N-4}^N \\
M_{2,\beta}\psi_{N-1}^N &= \alpha((N-3)h)\psi_{N-3}^N
\end{align*}
\]

**Proposition 9.**

\[
M_{2,\beta}\psi^a_z = \psi^a_{z^{\Sigma_{2,\beta}(z)\alpha}}
\]

where

\[
\Sigma_{2,\beta}(z) = \frac{e^{2iz}}{2\pi}(\beta(h)\mu_2) \left( \frac{\tau(z)}{h} + \frac{3}{2} - i\varphi_x \right) + \frac{3}{2} e^{-i2z}(\beta(h)\nu_2) \left( \frac{\tau(z)}{h} - \frac{3}{2} - i\varphi_x \right).
\] (3.13)

with

\[
\mu_2(n) = \sqrt{\frac{(n)(n-1)}{(N+1-n)(N-n)} D_{n+2}^N} \quad \text{and} \quad \nu_2(n) = \sqrt{\frac{(N-1-n)(N-2-n)}{(n+2)(n+1)} D_{n-2}^N}
\]

And again if \( \tau(1-\tau)\beta(\tau) \in C^\infty([0,1]) \), so that \( \beta(\tau)e^{i\theta} \in C^\infty(S^2) \), then, for all \( z \in S^2 \), \( \Sigma_{2,\beta}(z) \sim 2\beta(\tau(z)) \cos 2(x + \theta(z)) \) as \( N \to \infty \).

Otherwise, this last asymptotic equality is valid only for \( z \) away from the two poles.

Let us finally remark that when \( M_{0,\gamma} \) is diagonal with diagonal matrix elements \( \gamma(ih) \), then \( \Sigma_{0,\gamma} = \gamma(\tau(z))\text{Id} \), where \( \text{Id} \) is the identity on \( L^2(\mathbb{R}) \).

**3.3. Action of a general matrix.** For \( k = -(N-1), \ldots, N-1 \), let us call \( N_{k; \gamma_k} \) the matrix with non zero coefficients lying only on the \( k \)th diagonal and being equal to \( \gamma_k(j) = \gamma_k(j), k \leq j \leq N - i - k \). That is to say:

\[
N_{k; \gamma_k} = (M_1^+)^k M_{0; \gamma_k}.
\]
Let moreover
\[ \mu_k(n) = \prod_{j=0}^{k-1} \mu(n - j) = \sqrt{\frac{n}{k}} \left( \frac{N - n + k - 1}{k} \right)^{-1} \prod_{j=0}^{k-1} \chi_{\frac{1}{2}N - \frac{j}{2}}(n - j) \quad k > 0 \]
\[ \mu_0(n) = 1 \]
\[ \mu_k(n) = \prod_{j=k-1}^{0} \nu(n + j) = \sqrt{\frac{N - 1 - n}{k}} \left( \frac{n + k}{k} \right)^{-1} \prod_{j=k-1}^{0} \chi_{\frac{1}{2}N - \frac{j}{2}}(n + j) \quad k < 0 \]
The same arguments as in the proofs of Propositions 4, 8 and 9 leads easily to the following more general result.

**Proposition 10.**
\[ \mathcal{M}_{k; \gamma_k} \psi_x = \psi_z^{\Sigma_{k; \gamma_k}(z)a} \]
where
\[ \Sigma_{k; \gamma_k}(z) = \frac{e^{ikx}}{z^k} (\gamma_k(h) \mu_k) \left( \frac{\tau(z)}{h} - i\partial_x \right) \sqrt{\frac{C_{-i\partial_x + k}}{C_{i\partial_x}}} + \frac{z^k e^{-ikx} (\gamma_k(h) \nu_k) \left( \frac{\tau(z)}{h} - i\partial_x \right)}{\sqrt{\frac{C_{-i\partial_x + k}}{C_{i\partial_x}}}} \]
\[ (3.14) \]
And again if \((\tau(1 - \tau))_{\gamma_k} \in C^\infty([0,1]), so that \gamma_k(\tau)e^{i2\vartheta} \in C^\infty(S^2), then, for all \(z \in S^2, \Sigma_{k; \gamma_k}(z) \sim 2\gamma_k(\tau(z)) \cos(k(x + \theta(z)) as N \to \infty. \)
Otherwise, this last asymptotic equality is valid only for \(z\) away from the two poles.

### 3.4. Symbol.

Let us first remark the following co-cycle property.

**Lemma 11.**
\[ \sqrt{\frac{C_{-i\partial_x + k'}}{C_{i\partial_x}}} e^{ikx} \sqrt{\frac{C_{-i\partial_x + k}}{C_{i\partial_x}}} = e^{ikx} \sqrt{\frac{C_{-i\partial_x + k + k'}}{C_{i\partial_x}}} \]
so that
\[ \frac{e^{ikx}}{z^{k'}} (\gamma_{k'}(h) \mu_{k'}) \left( \frac{\tau(z)}{h} - i\partial_x \right) \sqrt{\frac{C_{-i\partial_x + k}}{C_{i\partial_x}}} e^{ikx} \frac{e^{ikx}}{z^{k'}} (\gamma_k(h) \mu_k) \left( \frac{\tau(z)}{h} - i\partial_x \right) \sqrt{\frac{C_{-i\partial_x + k}}{C_{i\partial_x}}} \]
\[ = \frac{e^{ikx}}{z^{k'}} (\gamma_{k'}(h) \mu_{k'}) \left( \frac{\tau(z)}{h} - i\partial_x \right) e^{ikx} (\gamma_k(h) \mu_k) \left( \frac{\tau(z)}{h} - i\partial_x \right) \sqrt{\frac{C_{-i\partial_x + k + k'}}{C_{i\partial_x}}} \]
Let us denote by \(N_{k; \gamma_k}\) the operator whose matrix on the basis \(\{\psi_n^N, n = 0 \ldots N - 1\}\) is \(N_{k; \gamma_k}.\)
We define the symbol of $\mathcal{N}_{k;\tau}$ at the point $z$ as the operator
\[
\tilde{\sigma}_{k;\gamma}(z) := e^{ikx} (\gamma_k(h)\mu_k) \left( \frac{\tau(z)}{\hbar} - i\partial_x \right) = e^{ikx} \gamma_k(\tau(z) - i\hbar\partial_x)\mu_k \left( \frac{\tau(z)}{\hbar} - i\partial_x \right) \tag{3.15}
\]
acting on $L^2(\mathbb{R})$.

**Definition 12.** Let $\gamma(\tau, \theta) = \sum_{k=-K}^{K} \gamma_k(\tau)e^{ik\theta}$ be a trigonometric function on the sphere with each $\gamma_k \in C^\infty([0, 1]) \cap L^\infty([0, 1])$.

Let
\[
\mathcal{N}_\gamma = \sum_{-(N-1)}^{N-1} \mathcal{N}_{k;\gamma} \quad \text{where} \quad (\mathcal{N}_{k;\gamma})_{ij} = \delta_{j,i+k}\gamma_k((k - \frac{(-1)^k - 1}{2})\hbar) \tag{3.16}
\]
and $\mathcal{N}_\gamma$ the operator whose matrix on the basis $\{\psi_n^N\}$ is $\mathcal{N}_\gamma$.

We call symbol of $\mathcal{N}_\gamma$ at the point $z \in S^2$ the operator
\[
\sigma[\mathcal{N}_\gamma](z) = \sum_{k=-(N-1)}^{N-1} \tilde{\sigma}_{k;\gamma}(z) \tag{3.17}
\]
where $\tilde{\sigma}_{k;\gamma}$ is given by (3.15).

Let us finish this section by giving a more global “quantization” type definition of the symbol. This end of Section 3.4 is not necessary for the understanding of the rest of the paper.

Note that
\[
\tilde{\sigma}_{k;\gamma} = \left( \mu_N \left( \frac{\tau(z)}{\hbar} - i\partial_x \right) e^{ix} \right)^k \gamma_k(\tau(z) - i\hbar\partial_x)
\]
\[
= \left( \mu_N \left( \frac{\tau(z)}{\hbar} - i\partial_x \right) e^{i(|z|+\theta(z))} \gamma_k(\tau(z) - i\hbar\partial_x) \right)^k
\]
\[
= e^{ik\theta(z)} \left( \mu_N \left( \frac{\tau(z)}{\hbar} - i\partial_x \right) e^{ix} \right)^k \gamma_k(\tau(z) - i\hbar\partial_x)
\]
\[
= e^{ik\theta(z)} \left( \frac{\tau(z) - i\hbar\partial_x}{|z|^2} \right)^k \gamma_k(\tau(z) - i\hbar\partial_x)
\]
\[
= \left( \frac{Z(z)}{|z|} \right)^k \gamma_k(\tau(z) - i\hbar\partial_x).
\]
Here the operator $Z(z)$ is the canonical (anti) pseudodifferential quantization of the canonical function $Z(x, \tau) := \sqrt{1-\tau} e^{ix}$, “shifted by $(\tau(z), \theta(z))$ where $z = \sqrt{\tau(z) - \tau \theta(z)}$, that is $Z_z(\tau, x) = f(\tau + \tau(z), x + \theta(z))$.

More precisely, the (anti) pseudodifferential quantization of a function $g$ is the pseudodifferential quantization of $G$ where one put all the differential part on the left (rather than on the right for the standard pseudodifferential calculus introduced at the beginning of Section 1.

Namely, for any function $g(e^{i\theta}, \tau)$ on the sphere, we define $\text{Op}_z^{\text{APD}}[g]$ and $\text{Op}_z^{\text{APD}}[g]$ by their integral kernels

$$\text{Op}_z^{\text{APD}}[g](x, y) = \int g(y, \tau) e^{i\tau(x-y)/\hbar} d\tau/(2\pi\hbar)$$

$$\text{Op}_z^{\text{APD}}[g](x, y) = \int g(y + \theta(z), \tau + \tau(z)) e^{i\tau(x-y)/\hbar} d\tau/(2\pi\hbar),$$

one has

$$Z(z) = \text{Op}_z^{\text{APD}}[Z_z] = \text{Op}_z^{\text{APD}}[Z].$$ (3.18)

and

$$\tilde{\sigma}_{k;\gamma_k}(z) = \left(\text{Op}_z^{\text{APD}} \left( \frac{Z_z}{|z|} \right) \right)^k \text{Op}_z^{\text{APD}}(\gamma_k).$$

**Definition 13.** For any trigonometric polynomial on the sphere $s = s(e^{i\theta}, \tau) = \sum e^{ikx} s_k(\tau)$ we define

$$\text{Op}_z[s] = \sum_k \left( \text{Op}_z^{\text{APD}} \left( \frac{Z_z}{|z|} \right) \right)^k \text{Op}_z^{\text{APD}}(s_k).$$

Let us now define the “naive” symbol of $\mathcal{N}$ as the function

$$s_{\mathcal{N}}(\tau, \theta) = \sum_{k=-(N-1)}^{N-1} e^{ik\theta} \gamma_k(\tau) = \gamma(\theta, \tau).$$ (3.19)

**Proposition 14.**

$$\sigma[\mathcal{N}](z) = \text{Op}_z[s_{\mathcal{N}}].$$

3.5. **Symbolic calculus.**

As a direct corollary of (the second part of) Lemma 11 we get the following result.

**Theorem 15.**

$$\sigma[\mathcal{N}^d \mathcal{N}] = \sigma[\mathcal{N}^d] \sigma[\mathcal{N}].$$

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one can also say that $\text{Op}_z[s] = s^{\text{PS}} \left( \text{Op}_z^{\text{APD}} \left( \frac{Z_z}{|z|} \right), \text{Op}_z^{\text{APD}}(\tau) \right)$, where $s^{\text{PS}}$ is the pseudodifferential ordering of the trigonometric polynomial $s$, that is the one with all the $\text{Op}_z^{\text{APD}}(Z_z)$ on the left.
Let us define
\[ C(z) := \sum_{k=-N+1}^{N-1} \frac{C_{\gamma, k}^{N}(z) e^{-ik\theta(z)}}{C_{\gamma, k}^{N}(z)} e^{ik\theta(z)} \]  
and the convolution
\[ \sigma \ast C(z) := \int_{S^1} \sigma(z e^{-i\theta}) C(|z e^{i\theta}|) d\theta. \]  

Proposition 16.
\[ \mathcal{N}_z \psi = \psi \sigma(N) \ast C(z). \]

Proof. By decomposition on \( k \)-diagonal parts of \( \mathcal{N} \), Proposition 16 is a direct consequence of Proposition 10 and the fact that
\[ \Sigma_{k, \gamma_k}(z) = \tilde{\sigma}_{k, \gamma_k} \ast C(z). \]

3.6. \( a \)-Töplitz quantization.

Definition 17. To a (trigonometric) family \( z \mapsto \Sigma(z) \) of (bounded) operators on \( L^2(\mathbb{R}) \) we associate the operator \( \text{Op}_a^T(\Sigma) \) on \( H_a^N \) defined by
\[ \text{Op}_a^T(\Sigma) = \int_{S^2} |\psi_z \Sigma C(z)| a \langle \psi_z^a | d\mu_N(z) \]

The following result is one of the main of this paper: it express that any trigonometric matrix, as defined by (3.16), is \( a \)-Töplitz operator, and that its \( a \)-Töplitz symbol is exactly the symbol, as defined by (12).

Theorem 18. Let \( \gamma, \gamma' \) and \( \mathcal{N}_\gamma, \mathcal{N}_{\gamma'} \) as in Definition 12. Then
\[ \mathcal{N}_\gamma = \text{Op}_a^T(\sigma[\mathcal{N}_\gamma]) \]
\[ \mathcal{N}_{\gamma'} = \text{Op}_a^T(\sigma[\mathcal{N}_{\gamma'}]) \]
\[ \mathcal{N}_\gamma \mathcal{N}_{\gamma'} = \text{Op}_a^T(\sigma[\mathcal{N}_\gamma] \sigma[\mathcal{N}_{\gamma'}]) \]

Proof. Theorem 18 is verbatim a straightforward consequence of Theorem 15.

Remark 19. Although we don’t want to prove it here in order not to introduce too much semi-classical technicalities, let us mention that, in the case where the symbol of an \( a \)-Töplitz operator is just a regular potential (multiplication operator by a function of \( x \)), then one can show that the \( a \)-Töplitz operator is actually a standard Töplitz operator. Conversely, a standard Töplitz operator is an \( a \)-Töplitz operator with a symbol which is a potential.
3.7. Classical limit and underlying “phase-space”. We can rewrite the general structure of the symbol of an $a$-Töplitz operator $T$ has the form (near the south pole where $\tau \sim |z| \sim 0$)

$$\sigma(z) = S(1 - i \frac{\hbar}{\tau(z)} \partial_x, x + \theta, \tau(z) - i\hbar \partial_x, \hbar)$$

where the function $S$ is $2\pi$ periodic in the second variable and the quantization present in the two first variables is the one of antipseudodifferential calculus.

The function $S$ satisfies

$$S(1 + \xi, x + \theta, \tau(z) + \xi, \hbar) \to S(1, x + \theta, \tau(z), \hbar) = \gamma(\tau(z), \theta + x) \quad \text{as } \xi \to 0,$$

where $\gamma(\tau, \theta, h)$ is the so-called naive symbol of $T$.

As $h \to 0$, $z \neq 0$,

$$\sigma(z) \to \gamma(\tau, \theta + x)$$

but the limit $h, z \to 0$ is multivalued. Indeed as

$$\begin{align*}
\begin{cases}
\hbar &\to 0 \\
z &\to 0 \\
\frac{\hbar}{\tau(z)} &= \hbar_0
\end{cases}
\end{align*}$$

we have

$$\sigma(z) \to S(1 - i\hbar_0 \partial_x, e^{i(x+\theta)}, 0, 0).$$

And the “classical” noncommutative multiplication for the function $S$ is given by:

$$S \# S'(1 - h_0 \xi, \theta + x, \tau, 0) = S(1 - h_0 \xi, \theta + x + i\partial_{\xi'}, \tau, 0)S'(1 - h_0 \xi', \theta + x, \tau, 0)|_{\xi' = \xi}$$

$$:= S(1 - h_0 \xi, \theta + x + i\partial_{\xi}, \tau, 0)S'(1 - h_0 \xi, \theta + x, \tau, 0)$$

This define the classical phase-space, as a noncommutative algebra of functions i.e.a noncommutative blow up of the singularity.

4. Application to TQFT

In this section we apply the results of the preceding one and show that any curve operator in TQFT of the case of the once punctured torus or the 4-times punctured sphere. We first introduce in a very fast way curve operators. For more details, the reader can consult [MP15] which is precisely referred in the next sections, and [A06, A10, A11, BHMV, BP00, FWW, G86, H90, MV94, RT91, TW05, TU91, W89].
4.1. The curve operators in the case of the once punctured torus or the 4-times punctured sphere. To any closed oriented surface $\Sigma$ with marked points $p_1, \ldots, p_n$, any integer $r > 0$ and any coloring $c = (c_1, \ldots, c_n), c_i \in \{1, \ldots, r - 1\}$ of the marked points, TQFT provides, by the construction of [BHMV], a finite dimensional hermitian vector space $V_r(\Sigma, c)$ together with a basis $\{\varphi_n, n = 1, \ldots, \dim(V_r(\Sigma, c))\}$ of this space (see Sections 2.1 and 2.5 in [MP15]).

On the other (classical) side, to each $t \in \pi_1(\Sigma, \{p_1, \ldots, p_n\}) \rightarrow SU_2$ s.t. $\forall i, \text{tr} \rho(\gamma_i) = 2 \cos(t_i)$ we can associate the moduli space:

$$M(\Sigma, t) = \{\rho : \pi_1(\Sigma \setminus \{p_1, \ldots, p_n\}) \rightarrow SU_2 \text{ s.t. } \forall i, \text{tr} \rho(\gamma_i) = 2 \cos(t_i)\} / \sim$$

where one has $\rho \sim \rho'$ if there is $g \in SU_2$ such that $\rho' = g\rho g^{-1}$ and $\gamma_i$ is any curve going around $p_i$.

When $\Sigma$ is either a once punctured torus or a 4-times punctured sphere, $M(\Sigma, t)$ is symplectomorphic to the standard sphere $S^2 = \mathbb{CP}^1$.

To any curve $\gamma$ on the surface $\Sigma$ (that is, avoiding the marked points) we can associate two objects: a quantum one, the curve operator $T_\gamma$ acting on $V_r(\Sigma, c)$, and a classical one, the function $f_\gamma$ on the symplectic manifold $M(\Sigma, t)$.

- $T_\gamma$ is obtained by a combinatorial topological construction recalled in Sections 2.3 and 2.4 in [MP15]. By the identification of the finite dimensional space $V_r(\Sigma, c)$ with the Hilbert space of the quantization of the sphere $\mathcal{H}_N$ defined in Section 2.1 with $N := \dim(V_r(\Sigma, c))$, through $\{\varphi_n, n = 1, \ldots, \dim(V_r(\Sigma, c))\} \leftrightarrow \{\psi_n^N, n = 1, \ldots, N\}$, $T_\gamma$ can be seen as a matrix on $\mathcal{H}_N$. One of the main results of [MP15] was to prove that this matrix is a trigonometric one in the sense of Section 3.2.

- $f_\gamma : M(\Sigma, t) \rightarrow [0, \pi]$ is defined by

$$\rho \mapsto f_\gamma(\rho) := -\text{tr} \rho(\gamma). \quad (4.1)$$

The asymptotism considered in [MP15] consists in letting $r \rightarrow \infty$ and considering a sequence of colorings $c_r$ such that $\pi_{c_r}^\infty$ converges to $t$ and the dimension of $V_r(\Sigma, c_r) := N$, grows linearly with $r$. One sees immediately that, by the identification $V_r(\Sigma, c) \leftrightarrow \mathcal{H}_N$, this corresponds to the semiclassical asymptotism $N \rightarrow \infty$.

The main result of [MP15] states that, for generic values of $t$,

$T_\gamma$ is a (standard) Töplitz operator of leading symbol $f_\gamma$.

The generic values of $t$ are the one for which $f_\gamma$ considered as a function on $S^2$ by the symplectic isomorphism mentioned earlier, belongs to $C^\infty(S^2)$.

4.2. Main result. It is easy to see that, for the remaining non generic values of $t$, $T_\gamma$ is not a standard Töplitz operator. Nevertheless, it happens that it is an $a$-Töplitz one.
Theorem 20. Let again $\Sigma$ be either the once punctured torus or the 4-times punctured sphere. For all values of $t$, the sequence of matrices $(T^T_{\gamma})$ are the matrices in the basis $\{\psi^N_n\}_{n=0,\ldots,N-1}$ of a family of a-Töplitz operators on $H^N_\Sigma$ with symbol $\sigma^T_{\gamma}$ satisfying, away of the two poles,

$$\sigma^T_{T^T_{\gamma}}(z) = f_\gamma(e^{-ix}z) + O(\sqrt{h})$$

(4.2)

where $f_\gamma$ is the trace function defined by (4.1).

4.3. Proof of Theorem 20. We give the proof in the case where $\Sigma$ is the once punctured torus.

In [MP15] we proved that any curve on $\Sigma$ is generated by the curves $\gamma, \delta, \zeta$ described in Section 3 of [MP15]. Therefore any curve operator belongs to the algebra generated by three matrices $T^\gamma_r, T^\delta_r, T^\zeta_r$ explicitly given by Proposition 3.1 and the end of Section 3.2 in [MP15].

The explicit expressions of the matrix elements of $T^\gamma_r, T^\delta_r, T^\zeta_r$ in [MP15], recalled in Section 4.4, shows clearly that these three triangular matrices are of the form (3.16). Therefore we know from Theorem 18 that they define three a-Töplitz operators, and therefore, again by Theorem 18, any curve operator is an a-Töplitz operator.

The first assertion of Theorem 20 is proven.

Moreover, we proved in [MP15] that the naive symbol of $T^\gamma_r$ as defined by (3.19), is equal, out of the poles and modulo $h$, to the trace function. Using now (3.3), (3.6) and (3.15) we find easily that as $h \sim 0$,

$$\gamma_k(\tau(z) - ih\hat{c}_x)\mu_k \left( \frac{\tau(z)}{h} - i\hat{c}_x \right) e^{ikx} \frac{n}{2} \sim \gamma_k(\tau(z)) \left( \frac{\sqrt{\tau(z)}}{1 - \tau(z)} \right)^k e^{ikx} \frac{n}{2} = \gamma_k(\tau(z)) e^{-ik(\theta(z)-x)},$$

which gives (4.2) after summation on $k$.

4.4. Examples. The symbol of a general curve operator is given by (3.15) out of is matrix elements. But we were not able in this paper to rely it, near the poles, to the trace functions in general. But in the cases of the three operators $T^\gamma_r, T^\delta_r, T^\zeta_r$ we can do it.

We find, by Proposition 3.1 and the relabelling of indices in the item 1. in Section 4.2 in [MP15], that

$$T^\gamma_r \psi_n = -2\cos \left( \frac{\pi}{N}(n + \frac{a+1}{2}) \right) \psi_n$$
$$T^\delta_r \psi_n = u_{n+1} \psi_{n+1} + u_n \psi_{n-1}$$
$$T^\zeta_r \psi_n = u_{n+1} e^{i\frac{\pi}{N}(n-a)} \psi_{n+1} + u_n e^{-i\frac{\pi}{N}(n+a)} \psi_{n-1},$$

Note that in [MP15] we proven this type of result by another method since we wanted also to define a symbol inside the interior of $\Sigma$ in the singular cases. Since we proved that in any coloring the three matrices are a-Töplitz operators, we don’t need such a direct result here.
where \(a \in \mathbb{N}, a \text{ odd},\) is the color assigned to the marked point and

\[
\sigma_{\nu}\tau = \left(\frac{\sin\left(\frac{n}{N}(n + a)\cos\left(\frac{n}{N}n\right)\right)}{\sin\left(\frac{n}{N}(n + \frac{a+1}{2})\cos\left(\frac{n}{N}(n + \frac{a-1}{2})\right)\right)}\right)^{\frac{1}{2}}.
\]

Defining as before \(h = \frac{\pi}{N},\) and \(\alpha = ha,\) we obtain that the naive symbols of \(T^\alpha, T^\beta, T^c,\) as defined in (3.19), are

\[
\sigma_{\nu}(\tau, \theta, h) = -2\cos\left(\frac{\alpha + h}{2}\right)
\]

\[
\sigma_{\nu}(\tau, \theta, h) = -\left(\frac{\sin(\tau + \alpha + h)\sin(\tau + h)}{\sin(\tau + \alpha + h/2)^{1/2}}\right)^{1/2} - \left(\frac{\sin(\tau + \alpha)\sin(\tau)}{\sin(\tau + \alpha + h/2)^{1/2}}\right)^{1/2} e^{-i\theta}
\]

\[
\sigma_{\nu}(\tau, \theta, h) = e^{i\frac{h}{2}}\sigma_{\nu}(\tau, \theta + \tau, h)
\]

and, as \(h \to 0,\)

\[
\sigma_{\nu}(\tau, \theta, 0) = -2\cos\left(\frac{\alpha}{2}\right)
\]

\[
\sigma_{\nu}(\tau, \theta, 0) = -\left(\frac{\sin(\tau + \alpha)\sin(\tau)}{\sin(\tau + \alpha + h/2)^{1/2}}\right)^{1/2} - \left(\frac{\sin(\tau + \alpha)\sin(\tau)}{\sin(\tau + \alpha + h/2)^{1/2}}\right)^{1/2} e^{-i\theta}
\]

\[
\sigma_{\nu}(\tau, \theta, 0) = \sigma_{\nu}(\tau, \theta + \tau, 0).
\]

We see that, for all values of \(a, \sigma_{\nu}(\tau, \theta, 0) \in C^\infty(S_2).\) For \(a > 0,\)

\[
\sigma_{\nu}(\tau, \theta, 0) \sim -\sqrt{\tau} \cos \theta, \quad \text{as } \tau \to 0,
\]

so that, for \(a > 0, \sigma_{\nu}(\tau, \theta, 0), \sigma_{\nu}(\tau, \theta, 0) \in C^\infty(S_2).\)

But when \(a = 0\) then

\[
\sigma_{\nu}(\tau, \theta, 0) = -2\cos \theta
\]

\[
\sigma_{\nu}(\tau, \theta, 0) = -2\cos(\theta + \tau)
\]

so that \(\sigma_{\nu}(\tau, \theta, 0), \sigma_{\nu}(\tau, \theta, 0)\) are singular on the sphere. Note that, for \(a = 0,\) the corresponding moduli space \(\mathcal{M}\) is also singular (see Remark 4.13 in [MP15]).

In fact \(T^\delta = -M_1\) where \(M_1\) is precisely the toy matrix defined in Section 3.1 above. The same way, \(T^c = -M_1,\) as defined in Section 3.2 with \(\kappa(\tau) = e^{i\tau}.\) Therefore its symbol, together with the one of \(T^c,\) is given out of the trace functions of \(\delta\) and \(\zeta\) by Definition 12. In other words,

\[
T^\eta = Op_n^{\tau}\sigma[N_{\eta}], \quad \eta = \gamma, \delta, \zeta.
\]

where \(f_\eta = -tr\rho(\eta)\) is the trace function defined by (4.1).

Straightforward but tedious computations show that the symbols of all the curve operators on the 4th-punctured sphere are also given out of the corresponding trace functions by Definition 12. This suggest the conjecture in the following section.
5. A conjecture

**Conjecture:** Any curve operator \( T_\gamma \) is an \( a \)-Töplitz operator whose symbol satisfies

\[
\sigma_{T_\gamma} = \sigma[N_{-tr\rho(\gamma)}] + O(h).
\]

in the sense of Definition 12.

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