Conserved Quantities in $f(R)$ Gravity via Noether Symmetry *

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(Received 1 March 2012)

We investigate $f(R)$ gravity using the Noether symmetry approach. For this purpose, we consider Friedmann Robertson–Walker (FRW) universe and spherically symmetric spacetimes. The Noether symmetry generators are evaluated for some specific choice of $f(R)$ models in the presence of the gauge term. Further, we calculate the corresponding conserved quantities in each case. Moreover, the importance and stability criteria of these models are discussed.

PACS: 04.50.Kd, 98.80.–k, 02.20.Sv DOI: 10.1088/0256-307X/29/8/080402

Astrophysical data from different sources such as cosmic microwave background fluctuations,¹¹ Supernovae Ia (SNIa)¹², x-ray experiments,⁴¹ and large scale structure ᵴ¹⁴ indicate that our universe is currently expanding at an accelerated rate. Higher-dimensional theories¹⁹ like M-theory or string theory may explain this accelerated expansion. Another explanation comes from modification of Einstein’s theory with some inverse curvature terms which cause increase in gravity.⁶ However, modified gravity with inverse curvature terms is known to be unstable and does not pass some solar system tests.⁷ This discrepancy can be removed by including higher derivative terms. In particular, the viability⁶ can be achieved with squared curvature terms. It is now thought that the current cosmic expansion can be justified if some suitable powers of curvature are added to the usual Einstein–Hilbert action.⁹ Thus it would be interesting to investigate the universe in the context of modified or alternative theories of gravity. The $f(R)$ theory of gravity, which involves a generic function of Ricci scalar in standard Einstein–Hilbert Lagrangian, is an attractive choice.

In recent years, many authors investigated $f(R)$ gravity in different contexts. Felice and Tsujikawa¹¹ gave a detailed review about $f(R)$ theories of gravity. A similar work has been reported by Sotiriou and Faraoni.¹² Hendi and Momeni¹³ explored black hole solutions in $f(R)$ gravity. Moon and Myung¹⁴ gave the stability analysis of the Schwarzschild black hole in this theory. The geodesic deviation equation in metric $f(R)$ gravity was obtained by Guarnizo et al.¹⁴ Upadhye and Hu²¹ investigated the existence of relativistic stars in $f(R)$ gravity. The metric $f(R)$ theories of gravity are generalized to five-dimensional spacetimes by Huang et al.¹⁶ They showed that expansion and contraction of the extra dimension prescribed a smooth transition from deceleration phase to acceleration phase. The stability conditions for $f(R)$ models have been discussed by Starobinsky.¹⁷ Multamäki and Vilja¹⁸,¹⁹ explored spherically symmetric vacuum and non-vacuum solutions in $f(R)$ theory of gravity. Shojai and Shojai²⁰ calculated exact spherically symmetric interior solutions in the metric version of $f(R)$ gravitational theory. Azadi et al.²¹ investigated cylindrically symmetric vacuum solutions in this theory. Plane symmetric solutions were studied by Sharif and Shamir.²² The same authors²¹ investigated the solutions for Bianchi types I and V models for both vacuum and non-vacuum case.

The field equations in $f(R)$ gravity are fourth order partial differential equations (PDEs) when the function is assumed to have the terms like $R^2$. However, the order could be higher if the terms like $R^3$ and $R^4$ etc. are included. On the other hand, Lie’s theory gives a systematic and mathematical way to investigate the solutions of differential equations. The application of Lie group theory for the solution of a nonlinear ordinary differential equation is one of the most fascinating and significant areas of research. It is mentioned here that from Lie’s theory one can not only construct a class of exact solutions but also find new solutions using different invariant transformations. It also gives the most widely applicable technique to find the closed form solution of differential equations. Investigation of these solutions plays a vital role in the understanding of the physical aspects of these differential equations.

The Noether symmetry approach is an important aspect of Lie theory. This is the most elegant and systematic approach to compute conserved vectors, given by Noether in 1918. The conservation laws play a vital role in the study of physical phenomenon. The integrability for PDEs depends on a number of conservation laws. Another important aspect of conservation laws is that they are helpful in the numerical...
integration of PDEs. There are a number of methods developed for the construction of conservation laws such as the Noether theorem\cite{25,26} for variational problems, the partial Noether theorem for variational and non variational structures\cite{27} and the multiplier approach.\cite{28} Computer packages for the construction of conserved quantities are reported by many authors, e.g. Wolf,\cite{29} Wolf et al.\cite{30} Göktaş et al.\cite{31–34} The Maple code to compute conservation laws based on the multiplier approach was introduced by Cheviakov.\cite{35}

The Noether theorem states that any differentiable symmetry of the action of a physical system has a corresponding conservation law. The main feature of this theorem is that it may provide information regarding the conservation laws in the theory of relativity. Conservation laws of linear and angular momentum can be well explained by the translational and rotational symmetries using the Noether theorem.\cite{36} It has many applications in theoretical physics. In recent years, many authors have used this theorem in different cosmological contexts. Capozziello et al.\cite{37} discussed \( f(R) \) gravity for spherically symmetric spacetime using Noether symmetry. Flat FRW universe has been discussed in Palatini \( f(R) \) gravity by Kucukakca and Cemeli\cite{38} Jamil et al.\cite{39} investigated the \( f(R) \) Tachyon model via the Noether symmetry approach. Hussain et al.\cite{40} found Noether symmetries for the flat FRW model using the gauge term in metric \( f(R) \) gravity. Energy distribution of the Bardeen model is given by Sharif and Waheed\cite{41} using approximate symmetry method. The same authors\cite{42} re-scaled the energy in the stringy charged black hole solutions using approximate symmetries. In a recent paper, we \cite{43} have found a new class of plane symmetric solutions in metric \( f(R) \) gravity using Lie point symmetries.

In this Letter, we focus our attention to investigate the Noether symmetries of FRW and spherically symmetric spacetimes in the context of metric \( f(R) \) gravity. First, we present some basics of \( f(R) \) theory of gravity. Then we calculate the Noether symmetries of FRW and the spherically symmetric spacetimes, respectively.

The action for four-dimensional \( f(R) \) theory of gravity in gravitational units \((8\pi G = 1)\) is given by\cite{37}

\[
S_{f(R)} = \int \sqrt{-g} f(R) + L_m d^4x,
\]

where \( L_m \) is the matter Lagrangian and \( f(R) \) is a general function of the Ricci scalar. The standard Einstein–Hilbert action can be obtained by taking \( f(R) = R \). Variation of this action with respect to the metric tensor yields the field equations

\[
f'(R)R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} - \nabla_\mu \nabla_\nu f'(R) + g_{\mu\nu} \square f'(R) = \kappa T^m_{\mu\nu},
\]

where \( \square \) denotes the derivatives with respect to \( R, \kappa \) is a constant coupling in gravitational units, \( T^m_{\mu\nu} \) is the standard matter energy-momentum tensor and \( \nabla_\mu \) defined as the covariant derivative. The field equations can be expressed in an alternative form familiar with general relativity (GR) field equations as

\[
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T^c_{\mu\nu} + \tilde{T}^c_{\mu\nu},
\]

where \( \tilde{T}^m_{\mu\nu} = T^m_{\mu\nu}/f'(R) \) and the energy-momentum tensor for gravitational fluid is given by

\[
T^c_{\mu\nu} = \frac{1}{f'(R)} \left[ \frac{1}{2} g_{\mu\nu} \left( f(R) - Rf'(R) \right) + f'(R)\alpha\beta \left( g_{\alpha\mu} g_{\beta\nu} - g_{\mu\nu} g_{\alpha\beta} \right) \right].
\]

It is clear from Eq. (4) that the energy-momentum tensor for gravitational fluid \( T^c_{\mu\nu} \) contributes matter part from geometric origin. This approach seems interesting as it may provide all the matter components which are required to investigate the dark part of our universe. Thus it is expected that \( f(R) \) theory of gravity may give fruitful results to understand the phenomenon of the expansion of the universe.

Next, we shall find the Noether symmetries of FRW spacetime. The FRW metric is given by

\[
d\tau^2 = dt^2 - a^2 \left[ \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right],
\]

where \( a \) is a function of cosmic time \( t \) and known as the scale factor of the universe and \( d\Omega^2 = r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \). The curvature parameter \( k \) is 0, 1 or \(-1\), which represents a flat, open or closed universe, respectively. The corresponding Lagrangian is given by\cite{44}

\[
L = 6a\dot{a}^2 \dot{f} + 6a^2 \dot{\dot{R}} \cdot \dot{R} + a^3 (f - R \dot{f}) - 6ka f' + a^3 P.
\]

Here dot denotes the derivative with respect to \( t \) and the fluid pressure \( P \) is given by

\[
P = m\omega a^{-3(1+\omega)},
\]

where \( m \) is an arbitrary real constant while \( \omega \) is the equation of state parameter. The Noether symmetry generator of Eq. (7) is given by

\[
X = \tau(t, a, R) \frac{\partial}{\partial t} + \psi(t, a, R) \frac{\partial}{\partial a} + \phi(t, a, R) \frac{\partial}{\partial R}.
\]

We can find Noether symmetries by using the following equation:

\[
X^{[1]} L + (D\tau)L = DB(t, a, R).
\]

where \( X^{[1]} \) is the first prolongation\cite{45} given by

\[
X^{[1]} = X + \dot{\psi}(t, a, R) \frac{\partial}{\partial a} + \dot{\phi}(t, a, R) \frac{\partial}{\partial R}.
\]
we have explored a more general solution of the de-
terning equations by taking

\begin{align}
\dot{\psi} = \frac{\partial \psi}{\partial t} + \frac{a}{\partial a} \frac{\partial \psi}{\partial a} + \frac{R}{\partial R} \frac{\partial \psi}{\partial R} - 2 \frac{\partial \tau}{\partial a} - \dot{a} \dot{R} \frac{\partial \tau}{\partial R}, \\
\dot{\phi} = \frac{\partial \phi}{\partial t} + \frac{a}{\partial a} \frac{\partial \phi}{\partial a} + \frac{R}{\partial R} \frac{\partial \phi}{\partial R} - 2 \frac{\partial \tau}{\partial a} - \dot{a} \dot{R} \frac{\partial \tau}{\partial a},
\end{align}

(12, 13)

where

\begin{align}
\dot{\psi} &= \frac{\partial \psi}{\partial t} + \frac{a}{\partial a} \frac{\partial \psi}{\partial a} + \frac{R}{\partial R} \frac{\partial \psi}{\partial R} - 2 \frac{\partial \tau}{\partial a} - \dot{a} \dot{R} \frac{\partial \tau}{\partial R}, \\
\dot{\phi} &= \frac{\partial \phi}{\partial t} + \frac{a}{\partial a} \frac{\partial \phi}{\partial a} + \frac{R}{\partial R} \frac{\partial \phi}{\partial R} - 2 \frac{\partial \tau}{\partial a} - \dot{a} \dot{R} \frac{\partial \tau}{\partial a},
\end{align}

(12, 13)

The Noether symmetry generators turn out to be

\begin{align}
X_1 &= \frac{\partial}{\partial t} + \frac{2}{3} \frac{\partial}{\partial a} - 2 R \frac{\partial}{\partial R}, \\
X_2 &= \frac{\partial}{\partial t}, \\
X_3 &= t a^{-1} \frac{\partial}{\partial a} - 2 R t a^{-2} \frac{\partial}{\partial R}, \\
X_4 &= a^{-1} \frac{\partial}{\partial a} - 2 R a^{-2} \frac{\partial}{\partial R}.
\end{align}

(29, 30, 31, 32)

These generators form a four-dimensional algebra with the commutators listed in Table 1.

**Table 1. Commutator relations.**

|       | $X_1$ | $X_2$ | $X_3$ | $X_4$ |
|-------|-------|-------|-------|-------|
| $X_1$ | 0     | $X_2$ | $X_3/3$ | $4X_4/3$ |
| $X_2$ | $-X_1$ | 0     | $-X_4$ | 0     |
| $X_3$ | $-X_2/3$ | $X_4$ | 0     | 0     |
| $X_4$ | $-4X_3/3$ | 0     | 0     | 0     |

Here the Lie bracket $[X_i, X_j]$ is defined by the following unique relation

$$
[X_i, X_j] = X_i \left( X_j \right) - X_j \left( X_i \right),
$$

where $i, j = 1, 2, 3, 4$. Moreover, the first integrals in this case are

\begin{align}
I_1 &= 9 a t a^2 R^{1/2} + \frac{1}{2} a^3 t R^{3/2} + \frac{9}{2} a^2 t a R R^{-1/2} \\
&- 3 t a^2 R R^{-1/2} - 3 a^2 t a R R^{-1/2}, \\
I_2 &= - 9 a t a^2 R^{1/2} + \frac{1}{2} a^3 t R^{3/2} - \frac{9}{2} a^2 t a R R^{-1/2}, \\
I_3 &= 9 a R^{1/2} - 9 t a R^{1/2} - \frac{9}{2} a R R^{-1/2}, \\
I_4 &= - 9 a R^{1/2} - \frac{9}{2} a R R^{-1/2}.
\end{align}

**Case 2: Non-Flat Non-Vacuum Universe ($k \neq 0$, $\omega \neq 0$).** For this case, the determining equations yield a solution

\begin{align}
\tau &= c_1 t + c_2, \\
\psi &= \frac{2 c_1 a^2 + 3 c_3 t + 3 c_4}{3 a}, \\
\phi &= - 2 R c_1 a^2 + c_3 t + c_4, \\
B &= 9 c_3 a \sqrt{R} + c_5.
\end{align}

(25, 26, 27, 28)

Here the gauge term turns out to be constant which can be taken as zero. It is mentioned here that this solution is for an arbitrary $f(R)$. The Noether symmetry generator turns out to be

$$
X = \frac{\partial}{\partial t}.
$$

(33, 34, 35)
The corresponding first integral becomes
\[ I = 6a\alpha^2 f' + 6a^2 \dot{a} R f'' - a^3 (f - R f') + 6ka f' - m\omega a^{-3\zeta}. \] (36)

We search the Noether symmetries of static spherically symmetric spacetime\cite{46}
\[ ds^2 = Adt^2 - \left( \frac{dr^2}{A} + r^2 d\Omega^2 \right), \] (37)
where \( d\Omega^2 = r^2 (d\theta^2 + \sin^2 \theta d\phi) \) and \( A \) is the function of \( r \). The corresponding Lagrangian is given by
\[ L = r^2 (f' - R f') + 2f' \left( 1 - a \frac{dA}{dr} \right) + f'' r^2 \left( \frac{dR}{dr} \right) \left( \frac{dA}{r} \right), \] (38)
where primes denote the derivatives with respect to \( R \).

The corresponding Noether symmetry generator is given by
\[ X = \tau(r, R, A) \frac{\partial}{\partial r} + \psi(r, R, A) \frac{\partial}{\partial R} + \phi(r, R, A) \frac{\partial}{\partial A}, \] (39)
The Noether symmetries can be computed by using the equation
\[ X^{[1]} L + (D_r)L = DB(t, a, R), \] (40)
where \( X^{[1]} \) is the first prolongation\cite{45} given by
\[ X^{[1]} = X + \psi(r, R, A) \frac{\partial}{\partial R} + \phi(r, R, A) \frac{\partial}{\partial A}, \] (41)
in which the top primes represent the derivatives with respect to \( r \),
\[ \dot{\psi} = \frac{\partial \psi}{\partial r} + \dot{R} \frac{\partial \psi}{\partial R} - \dot{R} \frac{\partial \psi}{\partial A} - \dot{A} \frac{\partial \psi}{\partial A} - \dot{R} \frac{\partial \psi}{\partial A}, \] (42)
\[ \dot{\phi} = \frac{\partial \phi}{\partial r} + \dot{R} \frac{\partial \phi}{\partial R} + \dot{A} \frac{\partial \phi}{\partial A} - \dot{A} \frac{\partial \phi}{\partial A} - \dot{R} \frac{\partial \phi}{\partial A}. \] (43)

In Eq. (40), \( B \) is called the gauge function with \( D \) defined as
\[ D \equiv \frac{\partial}{\partial t} + \dot{\phi} \frac{\partial}{\partial a} + \dot{R} \frac{\partial}{\partial R}. \] (44)
Substituting Eq. (38) into Eq. (40) and after some manipulations, we obtain an over determined system of linear PDEs, i.e.
\[ \tau_A = 0, \] (45)
\[ \tau_R = 0, \] (46)
\[ \psi_A = 0, \] (47)
\[ \phi_R = 0, \] (48)
\[ \phi_r = B_R, \] (49)
\[ 2r (\tau f'' + f' \tau_r) + r^2 (\psi f'' + \phi A f' + \psi R f' - \tau_r f'') = 0, \] (50)
\[ -2r f' - 2r (\psi f'' + \phi A f') + r^2 f' \psi_r = B_A, \] (51)
\[ 2r \tau (f - R f') - 2f' \phi - 2R f'' \psi + 2f'' \phi - 2f' \psi A \]
\[ - 2r f \phi_r + r^2 \tau R f' + 2r \tau f' - 2\tau_r A f' = B_r, \] (52)

When \( f(R) \) is arbitrary, we obtain the trivial results, i.e.
\[ \tau = 0, \ \psi = 0, \ \phi = 0. \] (53)
However we use a well-known form of \( f(R) \), i.e.
\[ f(R) = f_0 R^n, \quad n \neq 0, 1. \] (54)
This function has been widely used in different cosmological contexts. The determining equations, using Eq. (54) yields
\[ \tau = c_1 r, \ \psi = -3c_1 R_n, \ \phi = c_1 (2An - 3A - 2n + 3), \] (55)
where the gauge term is zero in this case. Thus the Noether symmetry generator in this case is given by
\[ X = r \frac{\partial}{\partial r} - 3 \frac{R}{n} \frac{\partial}{\partial R} + \frac{1}{n} \frac{R}{2An - 3A - 2n + 3} \frac{\partial}{\partial A}, \] (56)
and the conserved quantity turns out to be
\[ I = r(n - 1) \left[ r^2 R^n + 6(A - 1)R^{n-1} + 5r\dot{A}R^{n-1} \right. \]
\[ . \left. - (2n - 3)(A - 1)r\dot{R}R^{n-2} + r^2 \dot{A}\dot{R}R^{n-2} \right]. \] (57)

In summary, we have investigated the Noether symmetries in metric \( f(R) \) gravity. FRW and spherically symmetric spacetimes are considered for this purpose. We present a general solution of determining equations for FRW universe with the gauge term. In fact, we obtain four Noether symmetry generators. A non-zero gauge term is obtained, which depends on Ricci scalar \( R \) and scale factor \( a \). It would be worthwhile to mention here that the solution already obtained by Jamil et al.\cite{40} is a subcase for the flat universe with zero gauge term. Moreover, in palatini \( f(R) \) gravity, a non-zero time dependent gauge term is obtained\cite{38}. For the non-flat universe, we obtain one symmetry generator which is a translation of time coordinate. The first integrals are obtained in each case. The interesting feature of the cosmological model for \( f(R) \) is that it gives a negative deceleration parameter which is consistent with the recent experimental results to justify the accelerated expansion of universe. It has been shown\cite{41} that for \( a = a_0 t^2 \) and \( f(R) = f_0 R^{3/2} \), the deceleration parameter is \(-\frac{1}{2}\) for the flat universe.

The spherically symmetric spacetimes yield a set of eight linear PDEs. These equations are solved for two cases of \( f(R) \): The first case involves an arbitrary function of the Ricci scalar which gives a trivial solution. However, we obtain a non-trivial solution in the second case when \( f(R) = f_0 R^n \). The corresponding conserved quantity is also obtained in this case. This cosmological model has been used extensively in the
recent literature. In particular, the well known $f(R)$ model with inverse curvature term, corresponding to $n = -1$, predicts later time accelerated expansion of the universe.\cite{47}

Moreover, the stability conditions for $f(R)$ models are $f''(R) > 0$ and $f'''(R) > 0$.\cite{17} It would be worth mentioning here that the model $f(R) = f_0 R^{3/2}$ satisfies these conditions for $f_0 > 0$ and $R > 0$. These conditions are also satisfied by the model $f(R) = f_0 R^n$ when $f_0 > 0$, $n - 1 > 0$ and $R > 0$.

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