De Philippis, Guido; Palmieri, Luca; Rindler, Filip  
**On the two-state problem for general differential operators.** (English)  
Zbl 1403.35335  
Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods 177, Part B, 387-396 (2018).

In [Arch. Ration. Mech. Anal. 100, No. 1, 13–52 (1987; Zbl 0629.49020)], J. M. Ball and R. D. James proved the following:

Theorem. Let $\Omega \subseteq \mathbb{R}^d$ be an open, bounded and connected set and let $A, B$ be $m \times d$ matrices with rank $(A - B) \geq 2$. Then,

(i) If $u \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ satisfies the differential inclusion

$$Du(x) \in \{A, B\} \text{ for almost all } x \in \Omega,$$

then either $Du \equiv A$ or $Du \equiv B$.

(ii) Let $\{u_j\}_{j=1}^\infty \subseteq W^{1,\infty}(\Omega; \mathbb{R}^m)$ be a uniformly norm-bounded sequence such that

$$\text{dist}(Du_j, \{A, B\}) \to 0 \text{ in measure}.$$

Then, there is a subsequence such that either

$$\int_{\Omega} |Du_j(x) - A| \, dx \to 0 \text{ or } \int_{\Omega} |Du_j(x) - B| \, dx \to 0,$$

where $D$ denotes the gradient.

Property (i) is known as rigidity for exact solutions of (1), while (ii) is known as rigidity for approximate solutions of (1). The goal of this paper is to extend this result (and some related results) to more general operators $A$ where

$$Av \equiv \sum_{|\alpha|=k} A_\alpha \partial^\alpha v$$

where the sum is over all multi-indices $\alpha \in (\mathbb{N} \cup \{0\})^d$ and $A_\alpha \in \mathbb{R}^n \otimes \mathbb{R}^l$.

The authors note three reasons for this generalization: 1) it extends known results to general operators of any order, 2) the result allows the case of maps that are only bounded in $L^1$ and 3) the authors believe that this proof is more natural than ones currently in the literature. The proof makes use of ideas from [G. De Philippis and F. Rindler, Ann. Math. (2) 184, No. 3, 1017–1039 (2016; Zbl 1352.49050)]. The authors note that the proof can be modified to work for non-homogeneous operators and non-zero (but asymptotically vanishing) right-hand sides.

Reviewer: Daniel C. Biles (Nashville)

**MSC:**

35R70 PDEs with multivalued right-hand sides  
47F05 General theory of partial differential operators

**Keywords:**

differential inclusions; rigidity; two-state problem; gradient; approximate solutions; compensated compactness

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