Functionals on the space of almost complex structures

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Abstract

We study functionals on the space of almost complex structures on a compact $\mathbb{C}$-manifold, whose variational properties could be used to tackle Yau’s Challenge.

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Introduction

This is supposed to be a step in the direction of understanding Yau’s Challenge, which is to determine if there are compact almost complex manifolds of dimension at least 3 that cannot be given an integrable almost complex structure [3], through the calculus of variations. S-T. Yau proposed devising a parabolic flow on the space of almost complex structures to study this question [2].

Let $X$ be a real $2n$-dimensional compact manifold, and $AC(X) = \{J \in C^\infty(X, \text{End}_{\mathbb{C}}(T_X)) \mid J^2 = -Id\}$ be the space of almost complex structures on $X$. This is an almost complex Fréchet manifold, and for any $J \in AC(X)$, $T_{AC(X), J} = \{h \in C^\infty(X, \text{End}_{\mathbb{C}}(T_X)) \mid J \circ h + h \circ J = 0\}$, which can be seen from the identity $0 = dJ^2 = d(J \circ J + J \circ dJ)$. An almost complex structure $\mathcal{J} : AC(X) \rightarrow \text{End}(T_{AC(X)})$ is given as $\mathcal{J}(J)(u) = J \circ u$, for any $J \in AC(X)$ and
Let \( u \in T_{AC(X),J} \). Let \( g \) be a fixed Riemannian metric on \( X \), and note that for any \( J \in AC(X) \), we get an almost hermitian metric \( g_J := \frac{1}{2}(g(\cdot,\cdot) + g(J,\cdot)) \).

We are looking for an energy functional \( F \) on \( AC(X) \) whose associated gradient flow is a parabolic PDE. Ideally, the critical points of \( F \) should be the integrable almost complex structures on \( X \), and the Euler-Lagrange equation of \( F \) should be elliptic so that the complex structures on \( X \) are energy minimizers. We would then expect any solution of the flow equation of \( F \) to converge to a genuine complex structure on \( X \). In some special cases, such as when \( AC(X) \) is connected (e.g. \( AC(S^6) \)), the non-existence of a flow solution might translate to the non-existence of complex structures. A more thorough development of these ideas will be the subject of future research. Here we only derive the Euler-Lagrange equations of the functionals \( N, \tilde{N} : AC(X) \rightarrow \mathbb{R}_{\geq 0} \),

\[
N(J) := \int_X \|N_J\|^2_{g_J} \text{vol}_g, \quad \text{and} \quad \tilde{N}(J) := \int_X \|N_J\|^2_{\tilde{g}_J} \text{vol}_{\tilde{g}_J},
\]

where \( \text{vol}_g \) is the Riemannian volume form, and \( \text{vol}_{g_J} \) is the volume form of \( \omega := \frac{i}{2}(g_J - \tilde{g}_J) \). Note that both \( N \), and \( \tilde{N} \) are identically zero on the integrable structures. A very similar, real version of \( N \) appears in [1].

We can think of first variations in terms of linear approximations. Let \( \text{Herm} \left( \Lambda^2 T^0_{1,1} \otimes T^1_{1,0} \right) \) denote the space of hermitian metrics on \( \Lambda^2 T^0_{X,1,1} \otimes T^1_{X,0} \). Let

\[
f : AC(X) \rightarrow C^\infty \left( X, \Lambda^2 T^0_{X,1,1} \otimes T^1_{X,0} \right) \times \text{Herm} \left( \Lambda^2 T^0_{X,1,1} \otimes T^1_{X,0} \right)
\]

be the function \( f(I) = (N, h) = (N(I), h(I)) = (N_I, \overline{g_I}^{-1} \wedge \overline{g_I}^{-1} \otimes g_I), \) and

\[
\phi : C^\infty \left( X, \Lambda^2 T^0_{X,1,1} \otimes T^1_{X,0} \right) \times \text{Herm} \left( \Lambda^2 T^0_{X,1,1} \otimes T^1_{X,0} \right) \rightarrow C^\infty (X, \mathbb{R}_{\geq 0})
\]

be the function \( \phi(N, h) = h^{-1} \wedge h^{-1} \otimes h(N, N) \), and define \( \psi := \phi \circ f \), where \( \psi(I) = \overline{g_I}^{-1} \wedge \overline{g_I}^{-1} \otimes g_I(N_I, N_I) =: \|N_I\|^2_{\tilde{g}_I} \). And now, let

\[
F : AC(X) \rightarrow C^\infty \left( X, \Lambda^2 T^0_{X,1,1} \otimes T^1_{X,0} \right) \times \text{Herm} \left( \Lambda^2 T^0_{X,1,1} \otimes T^1_{X,0} \right) \times \Omega^{n,n}(X),
\]

\[
F(I) = (N(I), h(I), \text{vol}(I)) = (N_I, \overline{g_I}^{-1} \wedge \overline{g_I}^{-1} \otimes g_I, \text{vol}_{g_I}), \) and
\]

\[
\Phi : C^\infty \left( X, \Lambda^2 T^0_{X,1,1} \otimes T^1_{X,0} \right) \times \text{Herm} \left( \Lambda^2 T^0_{X,1,1} \otimes T^1_{X,0} \right) \times \Omega^{n,n}(X) \rightarrow C^\infty (X, \mathbb{R}_{\geq 0}),
\]

where \( \text{vol} \) is the Riemannian volume form, and \( \text{vol}_{\tilde{g}_I} \) is the volume form of \( \omega := \frac{i}{2}(g_I - \tilde{g}_I) \).

And now, let \( \tilde{\Psi}(n, h) = h^{-1} \wedge h^{-1} \otimes h(n, n) \), and define \( \phi := \tilde{\Psi} \circ \Phi \), where \( \phi(I) = \tilde{g}_I^{-1} \wedge \tilde{g}_I^{-1} \otimes \tilde{g}_I(N_I, N_I) =: \|N_I\|^2_{\tilde{g}_I} \). And now, let

\[
\tilde{F} : AC(X) \rightarrow C^\infty \left( X, \Lambda^2 T^0_{X,1,1} \otimes T^1_{X,0} \right) \times \text{Herm} \left( \Lambda^2 T^0_{X,1,1} \otimes T^1_{X,0} \right) \times \Omega^{n,n}(X),
\]

\[
\tilde{F}(I) = (N(I), h(I), \tilde{\text{vol}}(I)) = (N_I, \tilde{\overline{g_I}}^{-1} \wedge \tilde{\overline{g_I}}^{-1} \otimes \tilde{g_I}, \tilde{\text{vol}}_{\tilde{g_I}}), \) and
\]

\[
\Phi : C^\infty \left( X, \Lambda^2 T^0_{X,1,1} \otimes T^1_{X,0} \right) \times \text{Herm} \left( \Lambda^2 T^0_{X,1,1} \otimes T^1_{X,0} \right) \times \Omega^{n,n}(X) \rightarrow C^\infty (X, \mathbb{R}_{\geq 0}),
\]
\[ \Phi(N,h,\text{vol}) = h^{-1} \wedge h^{-1} \otimes h(N,N)\text{vol}_H, \text{ and define } \Psi := \Phi \circ F \text{ so that } \Psi(I) = \bar{g}_I^{-1} \wedge \bar{g}_I^{-1} \otimes g_I(N_I,N_I)\text{vol}_{g_I}. \text{ Let } \gamma = \frac{1}{2}(g(u-,J) + g(J, u-)). \text{ Let } J \in AC(X), \text{ and } \delta J \text{ be a small perturbation of } J; \text{ i.e. if } u \text{ is a nearby structure in } AC(X), \text{ then } \delta J = (J + u) - J. \text{ Let } \delta N = N(J + \delta J) - N(J) = N_{J+u} - N_J = d_JN_J(u) + O(u^2), \delta h = h(J + \delta J) - h(J) = g_{J+u} - g_J = d_Jg_J(u) + O(u^2) = \gamma + O(u^2), \text{ and } d\text{vol} = vol(J + \delta J) - vol(J) = vol_{g_{J+u}} - vol_{g_J} = d_J(\text{vol}_{g_J}(u) + O(u^2)) = d_J(\text{vol}_{g_J})(u). \text{ Then,} \]

\[
d_J(\|N_J\|^2_{g_J})(u) \approx \|\Phi(N + \delta N,h + \delta h) - \Phi(N,h)\| + \|\Phi(N,h + \delta h) - \Phi(N,h)\|
\approx d_N\Phi(N,h) \cdot \delta N + d_h\Phi(N,h) \cdot \delta h
= d_N\|N_J\|^2_{g_{J+u}} \cdot d_JN_J(u) + d_{g_J}(\|N_J\|^2_{g_J}) \cdot \gamma,
\]

and

\[
d_J(\|N_J\|^2_{g_J} \text{vol}_{g_J})(u) \approx \|\Phi(N + \delta N,h + \delta h,\text{vol} + \delta \text{vol}) - \Phi(N,h + \delta h,\text{vol} + \delta \text{vol})\| + \\
\|\Phi(N,h + \delta h,\text{vol} + \delta \text{vol}) - \Phi(N,h,\text{vol} + \delta \text{vol})\| + \\
\|\Phi(N,h,\text{vol} + \delta \text{vol}) - \Phi(N,h,\text{vol})\|
\approx d_N\Phi(N,h) \cdot \delta N + d_h\Phi(N,h,\text{vol}) \cdot \delta h + d_{\text{vol}}\Phi(N,h,\text{vol}) \cdot \delta \text{vol}
= d_N\|N_J\|^2_{g_{J+u}} \cdot d_JN_J(u) + d_{g_J}(\|N_J\|^2_{g_J} \text{vol}_{g_{J+u}}) \cdot \gamma + \\
d_{\text{vol}}(\|N_J\|^2_{g_J} \text{vol}_{g_J}) \cdot d_J(\text{vol}_{g_J})(u).
\]

The Nijenhuis tensor \( N \) is \( g_J \)-orthogonal to \( dN_J(u)^2 \cdot 0 \) and \( dN_J(u)^1, 1 \). This is a consequence of the \( g_J \)-orthogonality of the holomorphic, and antiholomorphic tangent bundles of \( X \). From now on, all \( O(u^2) \)-terms will be omitted throughout with only a few exceptions in the last section. Then, we find that

\[
d_N(\|N_J\|^2_{g_{J+u}}) \cdot d_JN_J(u) = \left\langle d_JN_J(u), N_J \right\rangle_{g_{J+u}} + \left\langle d_JN_J(u), N_J \right\rangle_{g_{J+u}}
= 2\text{Re}\left[ \left\langle dN_J(0,2)(u), N_J \right\rangle_{g_{J+u}} \right].
\]

**Proposition 1.** The first variation of \( N \) is

\[
d_JN_J(u) = \int_X \left\{ 2\text{Re}\left[ \left\langle dN_J(0,2)(u), N_J \right\rangle_{g_{J+u}} \right] + d_{g_J}(\|N_J\|^2_{g_J}) \cdot \gamma \right\}\text{vol}_{g_J},
\]
and that of $\tilde{N}$ is

$$d_J \tilde{N}(f)(u) = \int_X 2\Re \left[ \langle dN_j^{0,2}(u), N_j \rangle_{g_J+u} \right] \text{vol}_{g_J+u} + \int_X d_{g_J} (\| N_j \|_{g_J+u}^2 \text{vol}_{g_J+u}) \cdot \gamma + \int_X d_{\text{vol}} (\| N_j \|_{g_J}^2 \text{vol}_{g_J}) \cdot d_J (\text{vol}_{g_J})(u).$$

In order to retrieve the Euler-Lagrange equations of interest, we need to integrate $2\Re \left[ \langle dN_j^{0,2}(u), N_j \rangle_{g_J+u} \right]$ by parts. We do this in the coordinates defined below.

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**Coordinates**

We try to develop intrinsic complex coordinates on the almost hermitian manifold $(X,J,g_J)$, centered at a given point $p \in X$, that are the next best alternative to both holomorphic coordinates, which exist only when $J$ is integrable, and geodesic coordinates at $p$, which exit iff the fundamental form $\omega$ of $g_J$ is Kähler.

Recall that if $(z_k)_{1 \leq k \leq n}$ are holomorphic coordinates on $U \subset X$, then $\partial_J z_k = 0$ on some neighborhood $U_x \subset X$ of every $x \in U$. We do not have that in the almost complex case. However, we can design complex coordinates, which will be denoted here by $w_k$, for which $\partial_J w_k$ is as close as possible to being zero on each $U_x$. First note that we can always find complex coordinates $z_k \in C^\infty(U_p, \mathbb{C})$, centered at $p$, such that $\partial_J z_k(p) = 0$. Then, $(d_{z_k}(p))_{1 \leq k \leq n} = (\partial_J z_k(p))_{1 \leq k \leq n}$ is a basis of $(T_{X,p}^{1,0})^\ast$, and so $(\overline{\partial_J z_k(p)})_{1 \leq k \leq n}$ is a basis of $(T_{X,p}^{0,1})^\ast$. Hence, we get a local frame $(\partial_J z_k)_{1 \leq k \leq n}$ of $(T_{X,p}^{0,1})^\ast$, and so

$$\partial_J z_k = \sum_{1 \leq l \leq n} f_{kl}(z) \overline{\partial_J z_l},$$

where $f_{kl} \in C^\infty(U_p, \mathbb{C})$, and $f_{kl}(p) = 0$. Note that if $z_k$ were holomorphic, all of the coefficient functions $f_{kl}$ would be identically zero. Here, for every $1 \leq l \leq n$, each of these functions has a Taylor expansion

$$f_{kl}(z) = \sum_{|\alpha|+|\beta| \leq N} c_{k\alpha l\beta} z^\alpha \overline{z}^\beta + O(|z|^{N+1}).$$
Given that we will only differentiate once, we may instead work with the truncation

\[ f_{kl}(z) = \sum_{1 \leq j \leq n} (a_{jkl}z_j + a'_{jkl}\bar{z}_j) + O(|z|^2). \]

Again, if the \( z_k \) were holomorphic, we would in particular have that \( a_{jkl} = a'_{jkl} = 0 \), for all \( 1 \leq j, k, l \leq n \). We wish to emulate this situation \( (a_{jkl} = a'_{jkl} = 0) \) in the almost complex case. Concretely, we are looking for new coordinates that annihilate as many of the coefficients \( a_{jkl} \), and \( a'_{jkl} \) as possible. To that end, let

\[ w_k = z_k + \sum_{1 \leq r, s \leq n} (\alpha_{krs}z_rz_s + \beta_{krs}z_r\bar{z}_s + \gamma_{krs}\bar{z}_r\bar{z}_s) + O(|z|^3). \]

We still have that \( w_k(p) = 0, \partial_j w_k(p) = 0, \) and \( dw_k(p) = dz_k(p) \), and we still get a local frame \( (\partial_j w_k)_{1 \leq k \leq n} \) of \( T^1_{X, p} \) so that

\[ \partial_j w_k = \sum_{1 \leq j, l \leq n} (b_{jkl}w_j + b'_{jkl}\bar{w}_j + O(|w|^2))\partial_j w_l. \]

We will see that the holomorphic condition prescribes \( \beta_{klm}, \) and \( \gamma_{klm}, \) while the geodesic condition can be used to solve for \( \alpha_{klm} \). The point is that we are reducing the problem of finding optimal complex coordinates on an almost hermitian manifold to finding \( \alpha_{krs}, \beta_{krs}, \gamma_{krs} \) that annihilate the maximum number of coefficients of the Taylor expansions of \( f_{kl}, \) and \( \omega_{\lambda\bar{\mu}}. \) We call \( w_k = z_k + \sum_{1 \leq r, s \leq n}(\alpha_{krs}z_rz_s + \beta_{krs}z_r\bar{z}_s + \gamma_{krs}\bar{z}_r\bar{z}_s) + O(|z|^3), \) \( 1 \leq k \leq n, \) with \( \alpha_{krs}, \beta_{krs}, \gamma_{krs} \) subject to these constraints almost holomorphic geodesic coordinates on \( X \) at \( p. \)

**Lemma 1.** Any complex coordinates \( z_k \in C^\infty(U_p, \mathbb{C}), 1 \leq k \leq n, \) on an almost hermitian manifold \( (X, J, g) \) that are centered at \( p, \) and for which \( \partial_j z_k(p) = 0, \) and \( \left( \frac{\partial}{\partial z_k}(p) \right)_{1 \leq k \leq n} \) is an orthonormal basis of \( T^1_{X, p} \), determine almost holomorphic geodesic coordinates at \( p. \) Specifically, if the Taylor expansion of \( \partial_j z_k \) on \( U_p \) is

\[ \partial_j z_k = \sum_{1 \leq j, l \leq n} (a_{kjl}z_j + a'_{kjl}\bar{z}_j + O(|z|^2))\partial_j z_l, \]
and if
\[ \omega_{m\bar{l}} = \delta_{ml} + \sum_{s=1}^{n} (\tau_{mls}z_s + \tau'_{mls}\bar{z}_s) + O(|z|^2), \]
then
\[ w_k = z_k - \sum_{1 \leq m, l \leq n} \left[ \frac{1}{4} (a_{klm} + a_{mkl} + \tau_{klm} + \tau_{mkl})z_lz_m + a_{klm}z_l\bar{z}_m + \frac{1}{4} (a'_{klm} + a'_{kml})\bar{z}_l\bar{z}_m \right] + O(|z|^3). \]

Proof. Since \( z_m\bar{\partial}_j z_l = O(|z|^2), \bar{z}_m\partial_j z_l = O(|z|^2), \) and \( \bar{\partial}_j z_l = \partial_j z_l, \) and since \( \gamma_{klm} \) is \((l,m)\)-symmetric,
\[ \bar{\partial}_j w_k = \partial_j z_k + \sum_{1 \leq l, m \leq n} \bar{\partial}_j (a_{klm}z_lz_m + \beta_{klm}z_l\bar{z}_m + \gamma_{klm}\bar{z}_lz_m) + O(|z|^3) \]
\[ = \partial_j z_k + \sum_{1 \leq l, m \leq n} (\beta_{klm}\bar{z}_l\partial_j z_m + (\gamma_{klm} + \gamma_{kml})z_l\bar{\partial}_j z_m) + O(|z|^2) \]
\[ = \sum_{1 \leq l, m \leq n} \left[ (a_{klm} + \beta_{klm})z_l + (a'_{klm} + 2\gamma_{klm})\bar{z}_l \right] \partial_j z_m + O(|z|^2). \]
Based on this calculation, \( \alpha_{klm} \) is free to be any complex number, while \( \beta_{klm} = -a_{klm}. \) And we may take, at best, the symmetric part of \( a'_{klm} + 2\gamma_{klm} \) to be zero, which is achieved by setting \( \gamma_{klm} = -\frac{1}{4}(a'_{klm} + a'_{kml}). \) So far, we gather that
\[ w_k = z_k + \sum_{1 \leq m, l \leq n} \left[ a_{klm}z_lz_m - a_{klm}z_l\bar{z}_m + \frac{1}{4} (a'_{klm} + a'_{kml})\bar{z}_l\bar{z}_m \right] + O(|z|^3). \]
Next, we optimize \( \alpha_{klm} \) subject to the constraint of \( w_k \) being geodesic coordinates at \( p. \) Since \( \alpha_{mlj} \) is \((l,j)\)-symmetric,
\[ \partial_j w_m = \partial_j z_m + \sum_{l, j=1}^{n} (2\alpha_{mlj}z_j + \beta_{mlj}\bar{z}_j)\partial_j z_l + O(|z|^2) \]
so that
\[ \bar{\partial}_j w_m = \bar{\partial}_j z_m + \sum_{l, j=1}^{n} (2\alpha_{mlj}z_j + \bar{\beta}_{mlj}\bar{z}_j)\bar{\partial}_j z_l + O(|z|^2). \]
Then, since \( O(|w|^2) = O(|z|^2) \),

\[
\frac{i}{2} \sum_{m,l=1}^{n} \left( \delta_{ml} + O(|w|^2) \right) \partial_j w_m \wedge \overline{\partial_j w_l} = \frac{i}{2} \sum_{m=1}^{n} \partial_j w_m \wedge \overline{\partial_j w_m} + O(|w|^2)
\]

\[
= \frac{i}{2} \sum_{m=1}^{n} \left( \partial_j z_m \wedge \overline{\partial_j z_m} + \sum_{l,j=1}^{n} \left( \frac{2\overline{\alpha_{mlj}} \bar{z}_j}{2} + \frac{2\overline{\beta_{mlj}} \bar{z}_j}{2} \right) \partial_j z_m \wedge \overline{\partial_j z_l} \right) + O(|z|^2) + O(|w|^2)
\]

\[
= \frac{i}{2} \sum_{l,m=1}^{n} \left( \delta_{ml} + \sum_{j=1}^{n} \left( \frac{2\overline{\alpha_{mlj}} + 2\overline{\alpha_{jm}} \bar{z}_j}{2} \right) \partial_j z_m \wedge \overline{\partial_j z_l} \right) + O(|z|^2) + O(|w|^2)
\]

\[
i.e. \quad \omega = \frac{i}{2} \sum_{m,l=1}^{n} \left( \delta_{ml} + O(|w|^2) \right) \partial_j w_m \wedge \overline{\partial_j w_l}, \text{ is a condition that can be attained, at best, by setting the } (m,j)-\text{symmetric part of } \alpha_{lmj} + \frac{1}{2}(\overline{\beta_{mlj}} - \tau_{mlj}) \text{ equal to zero. Thus, we may take}
\]

\[
\alpha_{lmj} = -\frac{1}{4} \left( \frac{\beta_{mlj} - \overline{\beta_{jm}} + \tau_{mlj} - \tau_{jm}}{2} \right)
\]

\[
= \frac{1}{4} \left( a_{mlj} + a_{jm} + \tau_{mlj} + \tau_{jm} \right),
\]

\[
\text{and therefore}
\]

\[
w_k = z_k + \sum_{1 \leq m,l \leq n} \left[ \frac{1}{4} \left( a_{lkm} + a_{mkl} + \tau_{lkm} + \tau_{mkl} \right) z_l z_m - a_{klm} \bar{z}_l \bar{z}_m - \frac{1}{4} \left( a'_{klm} + a'_{mkl} \right) \bar{z}_l \bar{z}_m \right] + O(|z|^3).
\]
Euler-Lagrange equations

Let \((w_k)_{k=1}^n\) be almost holomorphic geodesic coordinates at \(p\). We now have local coordinate frames \(\frac{\partial^{1,0}}{\partial w_k}\left|_{1 \leq k \leq n}\right.) of \(T^1_\ast X\), and \(\frac{\partial^{0,1}}{\partial \bar{w}_k}\left|_{1 \leq k \leq n}\right.) of \(T^0_\ast X\) with dual coframes \(\left( dw_k^{1,0}\right)_{1 \leq k \leq n}\) and \(\left( d\bar{w}_k^{0,1}\right)_{1 \leq k \leq n}\). We also have the local coordinate expressions

\[
(g_{\lambda \bar{\mu}})_{\lambda \bar{\mu}} = \left( \frac{\partial^{1,0}}{\partial w_\lambda}, \frac{\partial^{0,1}}{\partial \bar{w}_\mu} \right)_{g_j} = \delta_{\lambda \mu} + \sum_{m=1}^n (\tau_{\lambda \bar{\mu}} w_m + \tau'_{\lambda \bar{\mu}} \bar{w}_m) + O(|w|^2),
\]

\[
N_j = \sum_{i,j,k=1}^n N_{ij}^k d\bar{w}_i^{0,1} \wedge d\bar{w}_j^{0,1} \otimes \frac{\partial^{1,0}}{\partial \bar{w}_k},
\]

and likewise

\[
dN_j^{0,2}(u) = \sum_{i,j,k=1}^n (dN_j(u))_{ij}^k d\bar{w}_i^{0,1} \wedge d\bar{w}_j^{0,1} \otimes \frac{\partial^{1,0}}{\partial \bar{w}_k}.
\]

Here we write \(dV = \left( \frac{i}{2} \right)^n dw_1^{1,0} \wedge d\bar{w}_1^{0,1} \wedge \cdots \wedge dw_n^{1,0} \wedge d\bar{w}_n^{0,1}\), \(h := g_{j+u} = g_j + \gamma + O(u^2)\), and

\[
h_{ij} = \delta_{ij} + \sum_{m=1}^n (\tau_{ijm} w_m + \tau'_{ijm} \bar{w}_m) + O(|w|^2) + \gamma_{ij} + O(u^2),
\]

the components of \(h^{-1}\) then being

\[
h_{ij}^{-1} := \delta_{ij} - \sum_{m=1}^n (\tau_{ijm} \bar{w}_m + \tau'_{ijm} w_m) - \gamma_{ij} + \sum_{c,v=1}^n \left( \overline{\tau_{cv}} w_v + \overline{\tau'_{cv}} \bar{w}_v \right) \overline{\gamma}_{cj} + \sum_{c,v=1}^n \overline{\gamma}_{cv} \left( \overline{\tau}_{cjv} \bar{w}_v + \overline{\tau'_{cjv}} w_v \right) + O(|w|^2) + O(u^2).
\]

Lemma 2.

\[
\frac{\partial^{1,0} h_{r \bar{s}}}{\partial w_i}(p) = \tau_{r \bar{s} i} + O(u), \quad \frac{\partial^{0,1} h_{r \bar{s}}}{\partial \bar{w}_i}(p) = \tau'_{r \bar{s} i} + O(u),
\]

\[
\frac{\partial^{1,0} h_{r \bar{s}}}{\partial w_i}(p) = -\tau_{r \bar{s} i} + O(u), \quad \frac{\partial^{0,1} h_{r \bar{s}}}{\partial \bar{w}_i}(p) = -\tau'_{r \bar{s} i} + O(u).
\]
Proof. These equalities are a consequence of \( \gamma_{ij} \), and hence its derivatives with respect to \( w_m \) and \( \bar{w}_m \), being of order \( O(u) \). \qed

Lemma 3.

\[
\left\langle dN_{J}^{0,2}(u), N_{J} \right\rangle_{g_{\mathcal{L}u}} = 2\left( h^{s} h_{i} h_{k} h_{q} f_{j} \frac{\partial^{1,0} u_{s}^{k}}{\partial w_{i}} + h^{s} h_{i} h_{k} h_{q} \left( \frac{\partial^{0,1} u_{s}^{k}}{\partial w_{i}} + \frac{\partial^{0,1} u_{s}^{k}}{\partial \bar{w}_{i}} \right) + \right.
\]

\[
h^{s} h_{j} h_{k} \frac{\partial^{1,0} u_{s}^{k}}{\partial \bar{w}_{i}} f_{j} N_{\bar{m} \bar{\rho}}^{q} - 2h^{s} h_{i} h_{k} h_{q} \left( \frac{\partial^{0,1} u_{s}^{k}}{\partial w_{i}} + \frac{\partial^{0,1} u_{s}^{k}}{\partial \bar{w}_{i}} \right) N_{\bar{m} \bar{\rho}}^{q} + \right.
\]

\[
2h^{s} h_{i} h_{k} h_{q} \left( u \left[ \frac{\partial^{0,1} u_{s}^{k}}{\partial w_{i}} f_{j} \right], d w_{k}^{1,0} \right) N_{\bar{m} \bar{\rho}}^{q}.
\]

Proof. First, note that

\[
dN_{J}(u(\zeta, \eta)) = u(\zeta, \eta) + J(\zeta, u(\eta)) + u(J(\zeta, \eta)) - u(\zeta, J(\eta)) - J(\zeta, u(\eta)).
\]

Since \( u = u^{s} d w_{r}^{1,0} \otimes \frac{\partial^{1,0}}{\partial w_{s}} + u^{s} d w_{r}^{1,0} \otimes \frac{\partial^{0,1}}{\partial \bar{w}_{s}} + u^{s} d \bar{w}_{r}^{1,0} \otimes \frac{\partial^{1,0}}{\partial \bar{w}_{s}} + u^{s} d \bar{w}_{r}^{1,0} \otimes \frac{\partial^{0,1}}{\partial \bar{w}_{s}} \)

and \( J = f_{j}^{r} d w_{i}^{1,0} \otimes \frac{\partial^{1,0}}{\partial w_{r}} + u_{i}^{s} d w_{i}^{1,0} \otimes \frac{\partial^{0,1}}{\partial w_{s}} + u_{i}^{s} d \bar{w}_{i}^{1,0} \otimes \frac{\partial^{1,0}}{\partial \bar{w}_{s}} + u_{i}^{s} d \bar{w}_{i}^{1,0} \otimes \frac{\partial^{0,1}}{\partial \bar{w}_{s}} \)

it follows that

\[
J \left[ \frac{\partial^{0,1}}{\partial w_{i}}, \frac{\partial^{0,1}}{\partial w_{j}} \right] = \left( \frac{\partial^{0,1} u_{s}^{k}}{\partial w_{i}} f_{j}^{r} + \frac{\partial^{0,1} u_{s}^{k}}{\partial w_{j}} f_{j}^{r} \right) \frac{\partial^{1,0}}{\partial w_{s}} + \left( \frac{\partial^{0,1} u_{s}^{k}}{\partial \bar{w}_{i}} f_{j}^{r} + \frac{\partial^{0,1} u_{s}^{k}}{\partial \bar{w}_{j}} f_{j}^{r} \right) \frac{\partial^{0,1}}{\partial \bar{w}_{s}}.
\]

and

\[
u \left[ \frac{\partial^{0,1}}{\partial w_{i}}, \frac{\partial^{0,1}}{\partial w_{j}} \right] = \left( \frac{\partial^{0,1} f_{j}^{r}}{\partial w_{i}} u_{j}^{v} + \frac{\partial^{0,1} f_{j}^{r}}{\partial w_{j}} u_{j}^{v} \right) \frac{\partial^{1,0}}{\partial w_{s}} + \left( \frac{\partial^{0,1} f_{j}^{r}}{\partial \bar{w}_{i}} u_{j}^{v} + \frac{\partial^{0,1} f_{j}^{r}}{\partial \bar{w}_{j}} u_{j}^{v} \right) \frac{\partial^{0,1}}{\partial \bar{w}_{s}}.
\]

A similar computation shows that

\[
J \left[ \frac{\partial^{0,1}}{\partial w_{i}}, u \frac{\partial^{0,1}}{\partial w_{j}} \right] = \left( \frac{\partial^{0,1} f_{j}^{r}}{\partial w_{i}} u_{j}^{v} + \frac{\partial^{0,1} f_{j}^{r}}{\partial \bar{w}_{i}} u_{j}^{v} \right) \frac{\partial^{1,0}}{\partial w_{s}} + \left( \frac{\partial^{0,1} f_{j}^{r}}{\partial \bar{w}_{i}} u_{j}^{v} + \frac{\partial^{0,1} f_{j}^{r}}{\partial \bar{w}_{i}} u_{j}^{v} \right) \frac{\partial^{0,1}}{\partial \bar{w}_{s}}.
\]

Therefore,

\[
J \left[ \frac{\partial^{0,1}}{\partial w_{i}}, u \frac{\partial^{0,1}}{\partial w_{j}} \right] + J \left[ \frac{\partial^{0,1}}{\partial w_{j}}, u \frac{\partial^{0,1}}{\partial w_{i}} \right] = \left( \frac{\partial^{0,1} f_{j}^{r}}{\partial w_{i}} u_{j}^{v} + \frac{\partial^{0,1} f_{j}^{r}}{\partial \bar{w}_{i}} u_{j}^{v} \right) \frac{\partial^{1,0}}{\partial w_{s}} + \left( \frac{\partial^{0,1} f_{j}^{r}}{\partial \bar{w}_{i}} u_{j}^{v} + \frac{\partial^{0,1} f_{j}^{r}}{\partial \bar{w}_{i}} u_{j}^{v} \right) \frac{\partial^{0,1}}{\partial \bar{w}_{s}}.
\]
Lemma 4. At $p$, we have that

\[
\frac{\partial^{1,0}}{\partial w_i} \left[ h_s^m h_i^p h_{k\bar{q}} f_j^i N_{\bar{m}p} \det(h) \right] u_k^s = \left( -\tau_{s\bar{m}i} f_j^i N_{s m}^{k} - \tau_{i\bar{p}j} f_j^i N_{s p}^{k} + \tau_{k\bar{q}j} f_j^i N_{s q}^{k} + \partial^{1,0} \frac{\partial^{1,0} N_{s q}^{k}}{\partial w_i} + \tau_{c\bar{e}j} f_j^i N_{s q}^{k} \right) u_k^s,
\]

\[
\frac{\partial^{0,1}}{\partial \bar{w}_i} \left[ h_s^m h_i^p h_{k\bar{q}} f_j^i N_{\bar{m}p} \det(h) \right] u_j^s = \left( -\tau_{s\bar{m}i} f_j^i N_{s m}^{k} - \tau_{i\bar{p}j} f_j^i N_{s p}^{k} + \tau_{k\bar{q}j} f_j^i N_{s q}^{k} + \partial^{1,0} \frac{\partial^{1,0} N_{s q}^{k}}{\partial \bar{w}_i} + \tau_{c\bar{e}j} f_j^i N_{s q}^{k} \right) u_j^s,
\]
\[
\frac{\partial^{0,1}}{\partial \bar{w}_i} \left[ h^{i\bar{m}} h^{i\bar{p}} h_{k\bar{q}} j^k_j N^q_{\bar{m}\bar{p}} \det(h) \right] u^s_j= \left( - \tau_{\bar{m}i\bar{m}} j^k_j N^k_{\bar{m}j} - \tau_{\bar{p}j\bar{p}} j^k_j N^k_{\bar{p}j} + \tau'_{k\bar{q}j} j^k_j N^q_{\bar{q}j} + \right. \\
\left. \frac{\partial^{0,1} j^k_j}{\partial \bar{w}_i} \right) N^k_{\bar{m}j} + j^k_j \frac{\partial^{0,1} N^k_{\bar{m}j}}{\partial \bar{w}_i} + j^k_j N^k_{\bar{m}j} \left( \sum_{c=1}^{n} \tau'_{c\bar{c}j} \right) u^s_j,
\]

and

\[
\frac{\partial^{0,1}}{\partial \bar{w}_i} \left[ h^{\bar{s}n} h^{\bar{i}\bar{p}} h_{\bar{k}\bar{q}} j^i_j N^q_{\bar{k}m} \det(h) \right] u^k_i= \left( - \tau_{\bar{s}m\bar{n}i} j^i_j N^i_{\bar{n}m} - \tau_{\bar{p}\bar{j}i} j^i_j N^i_{\bar{p}j} + \tau'_{\bar{k}\bar{q}j} j^i_j N^q_{\bar{q}j} + \right. \\
\left. \frac{\partial^{0,1} j^i_j}{\partial \bar{w}_i} \right) N^i_{\bar{n}m} + j^i_j \frac{\partial^{0,1} N^i_{\bar{n}m}}{\partial \bar{w}_i} + j^i_j N^i_{\bar{n}m} \left( \sum_{c=1}^{n} \tau'_{c\bar{c}j} \right) u^k_i.
\]

Proof. Note that \( h_{ij}(p) = \delta_{ij} + \gamma_{ij}(p) = \delta_{ij} + O(u) \), and \( h^{ij}(p) = \delta_{ij} - \bar{\gamma}_{ij} = \delta_{ij} + O(u) \). Then using Lemma [2],

\[
\frac{\partial^{1,0}}{\partial \bar{w}_i} h^{\bar{s}n} h^{\bar{i}\bar{p}} h_{\bar{k}\bar{q}} = - \tau_{\bar{s}m\bar{n}i} \delta_{jp} \delta_{kq} + O(u), \quad h^{\bar{s}n} \frac{\partial^{1,0} h^{\bar{i}\bar{p}}}{\partial \bar{w}_i} h_{\bar{k}\bar{q}} = - \tau_{\bar{i}p\bar{j}} \delta_{sm} \delta_{kq} + O(u),
\]

and

\[
h^{\bar{s}n} h^{\bar{i}\bar{p}} \frac{\partial^{1,0} h_{\bar{k}\bar{q}}}{\partial \bar{w}_i} = \tau_{\bar{k}qj} \delta_{sm} \delta_{jp} + O(u).
\]

Now since \( \det(h(p)) = 1 + \sum_{c=1}^{n} \gamma_{c\bar{c}c} = 1 + O(u) \),

\[
\frac{\partial^{1,0}}{\partial \bar{w}_i} h^{\bar{s}n} h^{\bar{i}\bar{p}} h_{\bar{k}\bar{q}} j^i_j N^q_{\bar{k}m} \det(h) u^k_i= \left( \frac{\partial^{1,0} h^{\bar{s}n} h^{\bar{i}\bar{p}} h_{\bar{k}\bar{q}}}{\partial \bar{w}_i} \right) + h^{\bar{s}n} \frac{\partial^{1,0} h^{\bar{i}\bar{p}}}{\partial \bar{w}_i} h_{\bar{k}\bar{q}} + \\
h^{\bar{s}n} h^{\bar{i}\bar{p}} \frac{\partial^{1,0} h_{\bar{k}\bar{q}}}{\partial \bar{w}_i} j^i_j N^q_{\bar{k}m} \det(h) u^k_i= \left( - \tau_{\bar{s}m\bar{n}i} \delta_{jp} \delta_{kq} + \tau_{\bar{i}p\bar{j}} \delta_{sm} \delta_{kq} + \tau_{\bar{k}qj} \delta_{sm} \delta_{jp} \right),
\]

Moreover, \( h^{\bar{s}n} h^{\bar{i}\bar{p}} h_{\bar{k}\bar{q}}(p) = \delta_{sm} \delta_{jp} \delta_{kq} + O(u) \), and since

\[
\det(h) = 1 + \sum_{c,m=1}^{n} \tau_{c\bar{c}m} w_m + \tau_{c\bar{c}m} \bar{w}_m + \sum_{c=1}^{n} \gamma_{c\bar{c}} + O(|w|^2),
\]

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\[ \frac{\partial^{1,0} \det(h)}{\partial w_i} (p) = \sum_{c=1}^{n} \tau_{c\bar{c}} + \sum_{c=1}^{n} \frac{\partial^{1,0} \gamma_{c\bar{c}}}{w_i} (p) = \sum_{c=1}^{n} \tau_{c\bar{c}} + O(u), \]

implying that

\[ h^{s\bar{m}} h^{j\bar{p}} h_{k\bar{q}} \frac{\partial^{1,0} J^i_j}{\partial w_i} \left( J^i_j N^{\bar{q}}_{\bar{m}\bar{\rho}} \det(h) \right) u_s^k = h^{s\bar{m}} h^{j\bar{p}} h_{k\bar{q}} \left( \frac{\partial^{1,0} J^i_j}{\partial w_i} N^{\bar{q}}_{\bar{m}\bar{\rho}} \det(h) + J^i_j \frac{\partial^{1,0} N^{\bar{q}}_{\bar{m}\bar{\rho}}}{\partial w_i} \right) u_s^k \]

\[ = \delta_{sm} \delta_{jp} \delta_{kq} \left( \frac{\partial^{1,0} J^i_j}{\partial w_i} N^{\bar{q}}_{\bar{m}\bar{\rho}} + J^i_j \frac{\partial^{1,0} N^{\bar{q}}_{\bar{m}\bar{\rho}}}{\partial w_i} + J^i_j N^{\bar{q}}_{\bar{m}\bar{\rho}} \sum_{c=1}^{n} \tau_{c\bar{c}} \right) u_s^k. \]

(2)

Now, using equations [1] and [2] we see that

\[ \frac{\partial^{1,0}}{\partial w_i} \left[ h^{s\bar{m}} h^{j\bar{p}} h_{k\bar{q}} J^i_j N^{\bar{q}}_{\bar{m}\bar{\rho}} \det(h) \right] u_s^k = \left[ \left( - \tau_{smi} \delta_{jp} \delta_{kq} - \tau_{j\bar{p}i} \delta_{sm} \delta_{kq} + \tau_{k\bar{q}i} \delta_{sm} \delta_{jp} \right) J^i_j N^{\bar{q}}_{\bar{m}\bar{\rho}} + \delta_{sm} \delta_{jp} \delta_{kq} \left( \frac{\partial^{1,0} J^i_j}{\partial w_i} N^{\bar{q}}_{\bar{m}\bar{\rho}} + J^i_j \frac{\partial^{1,0} N^{\bar{q}}_{\bar{m}\bar{\rho}}}{\partial w_i} + J^i_j N^{\bar{q}}_{\bar{m}\bar{\rho}} \sum_{c=1}^{n} \tau_{c\bar{c}} \right) \right] u_s^k \]

\[ = \left( - \tau_{smi} J^i_j N^{\bar{k}}_{\bar{m}\bar{j}} - \tau_{j\bar{p}i} J^i_j N^{\bar{k}}_{\bar{m}\bar{\rho}} + \tau_{k\bar{q}i} J^i_j N^{\bar{q}}_{\bar{m}\bar{\rho}} + \frac{\partial^{1,0} J^i_j}{\partial w_i} N^{\bar{k}}_{\bar{m}\bar{j}} + J^i_j \frac{\partial^{1,0} N^{\bar{k}}_{\bar{m}\bar{j}}}{\partial w_i} + J^i_j N^{\bar{k}}_{\bar{m}\bar{j}} \sum_{c=1}^{n} \tau_{c\bar{c}} \right) u_s^k. \]

Again, using Lemma [2] we find that

\[ \frac{\partial^{0,1} h^{i\bar{m}} h^{j\bar{p}} h_{k\bar{q}}}{\partial \bar{w}_i} J^k_s N^{\bar{q}}_{\bar{m}\bar{\rho}} \det(h) u_j^s = \left( \frac{\partial^{0,1} h^{i\bar{m}}}{\partial \bar{w}_i} h^{j\bar{p}} h_{k\bar{q}} + h^{i\bar{m}} \frac{\partial^{0,1} h^{j\bar{p}}}{\partial \bar{w}_i} h_{k\bar{q}} + h^{i\bar{m}} h^{j\bar{p}} \frac{\partial^{0,1} h_{k\bar{q}}}{\partial \bar{w}_i} \right) J^k_s N^{\bar{q}}_{\bar{m}\bar{\rho}} \det(h) u_j^s \]

\[ = \left( - \tau_{im\bar{i}} \delta_{jp} \delta_{kq} - \tau_{j\bar{p}i} \delta_{im} \delta_{kq} + \tau_{k\bar{q}i} \delta_{im} \delta_{jp} \right) J^k_s N^{\bar{q}}_{\bar{m}\bar{\rho}} u_j^s. \]
Since
\[
\frac{\partial^{0.1} \det (h)}{\partial w_i}(p) = \sum_{c=1}^{n} \tau'_{c\ell i} + O(u),
\]
then
\[
h^{i\ell} h^{j\ell} h_{kq} \frac{\partial^{0.1}}{\partial w_i} (J^k_s N^q_{mp} \det (h)) u^s_j = h^{i\ell} h^{j\ell} h_{kq} \left( \frac{\partial^{0.1}}{\partial w_i} N^q_{mp} \det (h) + J^k_s \frac{\partial^{0.1} N^q_{mp}}{\partial w_i} \right) u^s_j + J^k_s \frac{\partial^{0.1} N^q_{mp}}{\partial w_i} u^s_j,
\]
confirming that
\[
\frac{\partial^{0.1}}{\partial w_i} \left[ h^{i\ell} h^{j\ell} h_{kq} J^k_s N^q_{mp} \det (h) \right] u^s_j = \left[ \left( -\tau_{i\ell m} \delta_{jq} \delta_{lk} - \tau_{ijp} \delta_{im} \delta_{lk} + \tau'_{kqj} \delta_{im} \delta_{jq} \right) J^k_s N^q_{mp} + \delta_{im} \delta_{jq} \delta_{lk} \right] u^s_j.
\]

\[
\text{Lemma 5. Let } g'_{ij} = \frac{1}{2} g \left( \frac{\partial^{1.0}}{\partial w_i}, J \frac{\partial^{1.0}}{\partial w_j} \right), g_{ij} = \frac{1}{2} g \left( \frac{\partial^{0.1}}{\partial w_i}, J \frac{\partial^{1.0}}{\partial w_j} \right), \text{ and so forth. Then,}
\]
\[
d_{g_l} (\|N\|_{g_{l+1}}^2 \text{vol}_{g_{l+1}}) \cdot \gamma (p) = \left( u_s^k \left( 2 g'_{kj} N^q_{ij} - g'_{kj} N^q_{ij} \right) + u_s^k \left( 2 g'_{kj} N^q_{ij} - g'_{kj} N^q_{ij} \right) \right) \text{d}V,
\]
and
\[
d_{\text{vol}} (\|N\|_{g_l}^2 \text{vol}_{g_l}) \cdot d_f (\text{vol}_{g_l}) (u) (p) = \left( u_s^k g'_{kj} + u_s^k g'_{kj} + u_s^k g'_{kj} + u_s^k g'_{kj} \right) \text{N}^p_{ij} \text{d}V.
\]
Proof. First note that

\[d_{g_l}(\|N_j\|^2_{g_{l+\gamma}} \cdot \gamma(p) = (\|N_j\|^2_{l+\gamma} - \|N_j\|^2_{l})(1 + tr(\gamma)) dV = -\left(\gamma_{im} N_{ij}^k N_{im}^k + \gamma_{im} N_{ij}^k N_{mj}^k - \gamma_{km} N_{ij}^m N_{ij}^m\right)dV = -2\gamma_{im} N_{ij}^k N_{im}^k - \gamma_{km} N_{ij}^m N_{ij}^m\] dV,

and that

\[d_{\text{vol}}(\|N_j\|^2_{g_l}) \cdot d_{f}(\text{vol}_{g_l})(u)(p) = \|N_j\|^2_{l+\gamma} (\text{vol}_{l+\gamma} - \text{vol}_{l}) = |N_{ij}^k|^2 2 tr(\gamma) dV.

Also, \(\gamma_{km} = u^v_k g^\nu_{\nu m} + u^v_k g^\nu_{\nu \bar{m}} + u^\nu_m g^\nu_{\nu k} + u^\nu_m g^\nu_{\nu \bar{k}}\). Equation 4 is obtained by a relabeling of indices in \(\gamma_{mj}, \gamma_{mi}, \) and \(\gamma_{km}\) so that the upper index of \(u\) is \(k\) (or \(\bar{k}\)) and the lower index is \(s\) (or \(\bar{s}\)), and collecting terms with the same \(u\)-coefficients. Equation 5 follows after writing \(tr(\gamma) = \sum_{s=1}^{n} \gamma_{s\bar{s}} = u^v_s g_{s\bar{s}} + u^\nu_s g_{s\bar{s}} + u^v_s g_{s\bar{s}} + u^\nu_s g_{s\bar{s}}\), and a similar relabelling of indices. □

**Proposition 2.** Suppose that \(u\) is compactly supported in \(U_p\). Let \(1 \leq p, q \leq n\).

The Euler-Lagrange system of equations of \(\tilde{N}\) at \(p\) is

\[\tilde{T}_p^q := 4 \frac{\partial_1 f}{\partial \tilde{w}_i} N_{ij}^q - (2 g_{dq} N_{ij}^q N_{ip}^q - g_{q\bar{m}} N_{ij}^p N_{ij}^m) + g_{\bar{q}p} |N_{ij}^q|^2 = 0,
\]

\[\tilde{T}_p^\bar{q} := -(2 g_{dq} N_{ij}^q N_{ip}^q - g_{q\bar{m}} N_{ij}^p N_{ij}^m) + g_{\bar{q}p} |N_{ij}^q|^2 = 0,
\]

\[\tilde{T}_p^q := 4 \left[ (J(\frac{\partial_0}{\partial \tilde{w}_j}) \omega_{mp} N_{ij}^q + (J(\frac{\partial_0}{\partial \tilde{w}_j}) \omega_{mj}) N_{ip}^q - (J(\frac{\partial_0}{\partial \tilde{w}_j}) \omega_{q\bar{m}} N_{ij}^p N_{ij}^m) - \frac{\partial_1}{\partial \tilde{w}_i} (j^q N_{ij}^q) \right]
\]

\[-\frac{\partial_0}{\partial \tilde{w}_i} (j^q N_{ij}^q) - (J(\frac{\partial_0}{\partial \tilde{w}_j}) \omega_{c\bar{c}}) N_{ij}^q + J_q \left( \frac{\partial_0}{\partial \tilde{w}_i} \omega_{c\bar{c}} N_{ij}^q + \frac{\partial_0}{\partial \tilde{w}_i} \omega_{c\bar{c}} N_{ij}^q \right)
\]

\[-\frac{\partial_0}{\partial \tilde{w}_i} \omega_{jm} N_{ij}^q + \sum_{c=1}^{n} \frac{\partial_0}{\partial \tilde{w}_i} \omega_{c\bar{c}} N_{ij}^q - \frac{\partial_0}{\partial \tilde{w}_i} (j^q N_{ij}^q) \right] - \frac{\partial_1}{\partial \tilde{w}_i} (j^q N_{ij}^q) + \frac{\partial_1}{\partial \tilde{w}_i} (j^q N_{ij}^q)
\]

\[-(2g_{q\bar{m}} N_{ij}^q N_{ip}^q - g_{q\bar{m}} N_{ij}^p N_{ij}^m) + g_{\bar{q}p} |N_{ij}^q|^2 = 0,
\]
Proof. The procedure is to use Lemma 4 to integrate by parts the terms involving derivatives of \( u \) in the first variation of \( \mathcal{N} \) (Proposition 4), and then isolate \( u \) in the resulting formula by writing it as the \( g_{ij} \)-inner product of a tensor, the Euler-Lagrange equation at \( p \), and \( u \). Lemmas 3, 5 lead to

\[
\tilde{T}_p := 4 \left[ j_q \left( \frac{\partial^{0,1} \omega_{m_i} N_{j_m}}{\partial \bar{w}_i} + \frac{\partial^{0,1} \omega_{m_p}}{\partial \bar{w}_i} N_{j_p}^{m} - \frac{\partial^{0,1} \omega_{j_m}}{\partial \bar{w}_i} N_{i_m}^{m} - \sum_{c=1}^{n} \frac{\partial^{0,1} \omega_{c c}}{\partial \bar{w}_i} N_{i_p}^{c} \right) \right] - \frac{\partial^{0,1}}{\partial \bar{w}_i} \left( j_q \left( N_{j_i}^{m} \right) \right) - (2 g_{i_m} N_{i_p}^{m} N_{j_i}^{p} - g_{j_m} N_{i_p}^{m} N_{i_j}^{p}) + g_{j_m} |N_{i_j}^{p}|^2 = 0,
\]

and that of \( \mathcal{N} \) is

\[
\tilde{T}_p - g_{i_m} |N_{i_j}^{p}|^2 = 0,
\]

\[
\tilde{T}_g + 4 j_q \sum_{c=1}^{n} \frac{\partial^{0,1} \omega_{c c}}{\partial \bar{w}_i} N_{i_p}^{m} - g_{i_m} |N_{i_j}^{p}|^2 = 0.
\]

Proof. The procedure is to use Lemma 4 to integrate by parts the terms involving derivatives of \( u \) in the first variation of \( \mathcal{N} \) (Proposition 4), and then isolate \( u \) in the resulting formula by writing it as the \( g_{ij} \)-inner product of a tensor, the Euler-Lagrange equation at \( p \), and \( u \). Lemmas 3, 5 lead to

\[
d_j \tilde{\mathcal{N}}(f)(u) = 4 \Re \left[ \int_X \left( h^{s \bar{m} i} h^{j \bar{p}} h_{k \bar{q}} f_j i \frac{\partial^{1,0} u^k}{\partial \bar{w}_i} + h^{s \bar{m} i} h^{j \bar{p}} h_{k \bar{q}} \left( \frac{\partial^{0,1} u^k}{\partial \bar{w}_i} f_j i + \frac{\partial^{0,1} u^k}{\partial \bar{w}_i} f_j i \right) \right) \right] (p) + 4 \Re \left[ \int_X \left( h^{s \bar{m} i} h^{j \bar{p}} h_{k \bar{q}} \left( u^k \frac{\partial^{0,1} f_j i}{\partial \bar{w}_i} + u^k \frac{\partial^{0,1} f_j i}{\partial \bar{w}_i} \right) N_{i_m}^{m} \text{vol}_{g_{j+u}} \right) \right] (p)
\]

\[
- 4 \Re \left[ \int_X \left( h^{s \bar{m} i} h^{j \bar{p}} h_{k \bar{q}} \left( u^k \frac{\partial^{0,1} f_j i}{\partial \bar{w}_i} + u^k \frac{\partial^{0,1} f_j i}{\partial \bar{w}_i} \right) N_{i_m}^{m} \text{vol}_{g_{j+u}} \right) \right] (p)
\]

\[
+ \int_X \left( u^k \left( 2 g_{j k} N_{i_j}^{i} N_{i_j}^{m} - g_{k m} N_{i_j}^{i} N_{i_j}^{m} \right) + u^k \left( 2 g_{j k} N_{i_j}^{i} N_{i_j}^{m} - g_{k m} N_{i_j}^{i} N_{i_j}^{m} \right) + u^k \left( 2 g_{j k} N_{i_j}^{i} N_{i_j}^{m} - g_{k m} N_{i_j}^{i} N_{i_j}^{m} \right) dV +
\]
\[
\int_X \left( u_s^{j} g_{ks}^{j} + u_s^{j} g_{ks}^{j} + u_s^{k} g_{ks}^{j} + u_s^{k} g_{ks}^{j} \right) |N_{ij}^{2}|^2 dV
\]

\[
= 4 \text{Re} \left[ \int_X \left( \tau_{smi}^j j_{N_{mj}}^{k} + \tau_{ji}^j j_{N_{pj}}^{k} - \tau_{k\bar{q}i}^j j_{N_{sj}}^{q} - \frac{\partial^{1,0} j_i^j}{\partial \bar{w}_i} - j_i^j \frac{\partial^{1,0} N_{ij}^{k}}{\partial \bar{w}_i} \right) u_s^{j} dV + \int_X \left( \tau_{imi}^j j_{N_{mj}}^{k} + \tau_{ji}^j j_{N_{pj}}^{k} - \tau_{k\bar{q}i}^j j_{N_{sj}}^{q} \right) u_s^{k} dV + \int_X \left( \tau_{imi}^j j_{N_{mj}}^{k} + \tau_{ji}^j j_{N_{pj}}^{k} - \tau_{k\bar{q}i}^j j_{N_{sj}}^{q} \right) u_s^{k} dV \right]
\]

\[
+ 4 \text{Re} \left[ \int_X \left( \frac{\partial^{0,1} j_i^j}{\partial \bar{w}_i} u_s^{k} + \frac{\partial^{0,1} j_i^j}{\partial \bar{w}_i} u_s^{k} \right) |N_{ij}^{2}|^2 dV \right]
\]

\[
- \int_X \left( u_s^{j} \left( 2 g'_{ij} N_{ij}^{q} - g_{km} N_{ij}^{q} - g'_{ij} N_{ij}^{m} N_{ij}^{m} \right) + u_s^{j} \left( 2 g'_{ij} N_{ij}^{q} - g_{km} N_{ij}^{q} - g'_{ij} N_{ij}^{m} N_{ij}^{m} \right) + u_s^{k} \left( 2 g'_{km} N_{ij}^{q} - g_{kv} N_{ij}^{q} - g'_{km} N_{ij}^{m} N_{ij}^{m} \right) - \int_X \left( u_s^{j} g_{ks}^{j} + u_s^{j} g_{ks}^{j} + u_s^{k} g_{ks}^{j} + u_s^{k} g_{ks}^{j} \right) |N_{ij}^{2}|^2 dV
\]

\[
= \text{Re} \left[ \int_X \left( \left( \tau_{smi}^j j_{N_{mj}}^{k} + \tau_{smi}^j j_{N_{pj}}^{k} \right) N_{sj}^{k} + \left( \tau_{ji}^j j_{N_{pj}}^{k} + \tau_{ji}^j j_{N_{pj}}^{k} \right) N_{sj}^{k} \right) \right]
\]

\[
- \left( \tau_{kmi}^j j_{N_{mj}}^{m} - \left( \frac{\partial^{1,0} j_i^j}{\partial \bar{w}_i} + \frac{\partial^{0,1} j_i^j}{\partial \bar{w}_i} - j_i^j \frac{\partial^{1,0} N_{ij}^{k}}{\partial \bar{w}_i} \right) \right)
\]

\[
- j_i^j \frac{\partial^{0,1} N_{ij}^{k}}{\partial \bar{w}_i} - N_{sj}^{k} \left( j_j^i \left( \sum_{c=1}^{n} \tau_{cci}^k + \sum_{c=1}^{n} \tau_{cci}^2 \right) \right) + \tau_{imi}^j N_{ij}^{m} \right]
\]
\[
\tilde{T}_{sm} l_k l_{im} - \tau_{jmi} k n_{is} - \frac{\partial^0.1 j^l_k}{\partial \bar{w}_i} N_{ijs} - J^l_k \frac{\partial^0.1 N_{ijs}}{\partial \bar{w}_i} - J^l_k \left( \sum_{c=1}^n \tau^*_{c\bar{c}i} \right) \\
- \frac{\partial^1.0 j^l_k}{\partial \bar{w}_k} N_{sij} + \frac{\partial^0.1 j^l_i}{\partial \bar{w}_i} (N_{ij}^m) - \left( 2g'_{km} N^v_i N^m_{is} - g'_{kv} N^v_i N^m_{ij} + g'_{ks} N^v_{ij} \right) u^k + \left( 4\left( \tilde{T}_{imi} l_i k N^1_{ms} + \tilde{T}_{sm} i l_n N^1_{is} - \tau_{jmi} k n_{is} - \frac{\partial^0.1 j^l_k}{\partial \bar{w}_i} N_{ijs} - J^l_k \frac{\partial^0.1 N_{ijs}}{\partial \bar{w}_i} - J^l_k \left( \sum_{c=1}^n \tau^*_{c\bar{c}i} \right) \right) - \left( 2g'_{km} N^v_i N^m_{is} - g'_{kv} N^v_i N^m_{ij} + g'_{ks} N^v_{ij} \right) u^k + \left( 4\frac{\partial^0.1 j^l_k}{\partial \bar{w}_i} N^l_{ij} \right) u^k + \left( 4\frac{\partial^0.1 j^l_k}{\partial \bar{w}_i} N^l_{ij} \right) u^k \right) dV \\
\]

We may use \( g_j(p) = 1 \) to induce the inner product \( \langle T, V \rangle = \sum_{s,k=1}^n (T^k_s V^k_s + T^k_s V^k_s + T^k_s V^k_s) \) of any \( T, V \in C^\infty(X, \text{End}(T_X^C)) \). Consider now the tensor \( \tilde{T}_j = \tilde{T}_p d w^p \otimes \frac{\partial^1.0}{\partial w^p} + \tilde{T}_p d w^p \otimes \frac{\partial^1.0}{\partial w^p} + \tilde{T}_p d w^p \otimes \frac{\partial^1.0}{\partial w^p} + \tilde{T}_p d w^p \otimes \frac{\partial^1.0}{\partial w^p} \), where

\[
\tilde{T}_p = 4 \frac{\partial^0.1 j^l_k}{\partial \bar{w}_i} N^v_i N^m_{ij} - \left( 2g'_{dq} N^v_i N^m_{ij} - g'_{dq} N^v_i N^m_{ij} + g'_{dp} N^v_{ij} \right) u^k + \left( 4\frac{\partial^0.1 j^l_k}{\partial \bar{w}_i} N^l_{ij} \right) u^k + \left( 4\frac{\partial^0.1 j^l_k}{\partial \bar{w}_i} N^l_{ij} \right) u^k, \\
\]
\[ \tilde{T}_p^q = 4 \left( \tau_{pmi} J_i^j + \tau_{pmi} J_i^j \right) N_{ij}^m + \left( \tau_{jmi} J_i^j + \tau_{jmi} J_i^j \right) N_{ij}^m \\
- \left( \tau_{qmj} J_i^j + \tau_{qmj} J_i^j \right) N_{ij}^m \\
\frac{\partial^1,0 J_i^j}{\partial w_i} + \frac{\partial^0,1 J_i^j}{\partial \tilde{w}_i} \right) N_{ij}^m - J_i^j \frac{\partial^1,0 N_{ij}^m}{\partial \tilde{w}_i} \\
- \frac{\partial^1,0 J_i^j}{\partial \tilde{w}_i} - \frac{\partial^0,1 J_i^j}{\partial w_i} \right) + \left( \sum_{c=1}^n \tau_{c\bar{c}} \right) \\
- \left( \frac{\partial^1,0 J_i^j}{\partial \tilde{w}_i} \right) - \left( 2 \bar{g}_{qm} N_{ip}^v N_{im}^v - g_{qij} N_{ij}^{v^2} \right) + g_{qij} |N_{ij}^v|^2. \]

and where

\[ \tilde{T}_p^q = 4 \left( \tau_{imj} J_i^j N_{ip}^m + \tau_{pmi} J_i^j N_{ip}^m \right) - \tau_{jmi} J_i^j N_{ip}^m - \frac{\partial^1,0 J_i^j}{\partial w_i} N_{ip}^m - \frac{\partial^0,1 J_i^j}{\partial \tilde{w}_i} N_{ip}^m \right) - J_i^j \frac{\partial^1,0 N_{ip}^m}{\partial \tilde{w}_i} \left( \sum_{c=1}^n \tau_{c\bar{c}} \right) \\
- \frac{\partial^1,0 J_i^j}{\partial \tilde{w}_i} N_{ip}^m \right) - \left( 2 \bar{g}_{qm} N_{ip}^v N_{im}^v - g_{qij} N_{ij}^{v^2} \right) + g_{qij} |N_{ij}^v|^2. \]

Then,

\[ d_J \tilde{N}(J)(u) = \Re \left\{ \int_X \langle \tilde{T}_j, \bar{w} \rangle dV \right\}, \]

and so \( \tilde{T}_j = 0 \) is the Euler-Lagrange equation at \( p \).

We can rewrite \( T_p^q \), and \( \tilde{T}_p^q \) somewhat more meaningfully, using that

\[ J \left( \frac{\partial^0,1}{\partial w_i} \right) = J_i^j \frac{\partial^0,1}{\partial w_i} + J_i^j \frac{\partial^0,1}{\partial \tilde{w}_i}, \]

and the hermitian nature of the fundamental form \( \omega \):

\[ T_p^q = 4 \left[ J_i^j \left( \frac{\partial^0,1 \omega_{mi}}{\partial \tilde{w}_i} N_{ip}^m + \frac{\partial^0,1 \omega_{mp}}{\partial \tilde{w}_i} N_{im}^m - \frac{\partial^0,1 \omega_{jm}}{\partial \tilde{w}_i} N_{ij}^m - \sum_{c=1}^n \frac{\partial^0,1 \omega_{c\bar{c}}}{\partial \tilde{w}_i} N_{ij}^m \right) \\
- \frac{\partial^0,1 \omega_{i\bar{p}}}{\partial \tilde{w}_i} \right] - \left( 2 \bar{g}_{qm} N_{ip}^v N_{im}^v - g_{qij} N_{ij}^{v^2} \right) + g_{qij} |N_{ij}^v|^2, \]

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\[ T^q_p = 4 \left[ J \left( \frac{\partial^{0,1}}{\partial \bar{w}_j} \right) \omega_{mp} \right] N^q_{mj} + \left( J \left( \frac{\partial^{0,1}}{\partial \bar{w}_j} \right) \omega_{mj} \right) \bar{N}^q_{pj} - \left( J \left( \frac{\partial^{0,1}}{\partial \bar{w}_j} \right) \omega_{qm} \right) \bar{N}^m_{pj} - \frac{\partial^{1,0}}{\partial \bar{w}_i} \left( J^i \bar{N}^q_{pj} \right) \\
- \frac{\partial^{0,1}}{\partial \bar{w}_i} (J^i \bar{N}^q_{pj}) - \left( J \left( \frac{\partial^{0,1}}{\partial \bar{w}_j} \right) \left( \sum_{c=1}^{n} \omega_{cc} \right) \right) \bar{N}^q_{pj} + J^i \left( \frac{\partial^{0,1}}{\partial \bar{w}_i} \omega_{mj} \right) \bar{N}^j_{mj} + \frac{\partial^{0,1}}{\partial \bar{w}_i} \omega_{mp} \bar{N}^j_{pj} \\
- \frac{\partial^{0,1}}{\partial \bar{w}_i} \omega_{jm} \bar{N}^m_{ip} \sum_{c=1}^{n} \frac{\partial^{0,1}}{\partial \bar{w}_i} \omega_{cc} \bar{N}^j_{ip} \right) - \frac{\partial^{0,1}}{\partial \bar{w}_i} \left( J^i \bar{N}^j_{ip} \right) - \frac{\partial^{1,0}}{\partial \bar{w}_i} \left( J^i \bar{N}^j_{ip} \right) + \frac{\partial^{0,1}}{\partial \bar{w}_i} \left( J^i \bar{N}^q_{ij} \right) \right) \\
- \left( 2g^q_{im} \bar{N}^m_{ip} \bar{N}^p_{ij} - g^q_{ip} \bar{N}^p_{ij} \right) + g^q_{ip} |\bar{N}^p_{ij}|^2. \]

In the case of \( \mathcal{N} \), there is no \( d_{\text{Vol}} \|N_j\|_{g_j}^2 \cdot d_{\text{Vol}} g_j(\text{id}) \) term in the first variation, neither are there derivatives of the Riemannian volume form. We can recover the Euler-Lagrange equation of \( \mathcal{N} \) at \( p \) from that of \( \mathcal{N} \) to find that it is a tensor equation \( T_{\mathcal{N}} = 0 \), where the components of \( T_{\mathcal{N}} \) are of the claimed form. \( \square \)

All integrable almost complex structures on \( X \) are critical points of both \( \mathcal{N} \), and \( \mathcal{N} \), but they are likely not the only ones. It is also unclear if \( \mathcal{N} \) has any advantages over \( \mathcal{N} \). It might make sense to try eliminating the derivatives of \( \mathcal{N} \) that appear in the Euler-Lagrange equation of \( \mathcal{N} \) since they do not directly contain information about integrability.

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