HOMOTOPY 4-SPHERES ASSOCIATED TO AN INFINITE ORDER LOOSE CORK

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Abstract. We show the homotopy spheres $\Sigma_n = -W \sim_{f^n} W$, formed by doubling the infinite order loose-cork $(W, f)$ by iterates of the cork diffeomorphism $f : \partial W \rightarrow \partial W$, is obtained by Gluck twistings of $S^4$; then by this we show how to cancel 3-handles of $\Sigma_n$ and draw several handlebody pictures of $\Sigma_n$ without 3-handles.

0. Introduction

Let $(W, f)$ be the infinite order loose-cork of [A1], shown in Figure 1. As indicated in [A1], this $W$ can be identified with the one described in [G1]. Recall that the diffeomorphism $f : \partial W \rightarrow \partial W$ here is given by the $\delta$-move along the curve $\delta$ of Figure 2 as defined in [A1]. We will refer the iterates $f^n$ of $f$ as $\delta^n$-move, or just simply $\delta$-move.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.pdf}
\caption{W}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.pdf}
\caption{$\delta$ - move}
\end{figure}

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Consider the homotopy 4-spheres obtained by doubling of the contractible manifold $W$ by the iterates $f^n = f \circ f \circ ... \circ f$:

$$\Sigma_n = -W \prec f^n W$$

The obvious question is whether $\Sigma_n$ are all diffeomorphic to $S^4$, or if this family contains an exotic copy of $S^4$. Here we show that each $\Sigma_n$ is obtained by Gluck twisting $S^4$ along some knotted $S^2 \subset S^4$, and then present several 3-handle free handle pictures of $\Sigma_n$.

Along the way we will show that $\Sigma_n - B^4$ can be obtained by attaching a 2-handle to the ribbon complement $Q = B^4 - N(D^2)$ along the $n$-th iterate of a loop $\gamma \subset \partial Q$ by a diffeomorphism $f : \partial Q \to \partial Q$

$$\Sigma_n - B^4 = Q \prec f^n(\gamma) h^2_\gamma$$

where $D \subset B^4$ is the standard ribbon bounded by $\partial D = K \# K$, and $K$ is the figure-eight knot, and $N(D)$ denotes the tubular neighborhood of $D$. By using these we hope to study $\Sigma_n$ closely in a future paper.

1. CONSTRUCTION

Our first goal is to determine how the $\delta$-move diffeomorphism $f$ moves curves on the boundary $\partial W$ (see also [A3]). This is important, because by using this we will construct the handlebody picture of the manifolds $\Sigma_n$, by drawing the attaching circles of the dual handles of the upside down $-W$. This is a nontrivial task, because $\delta$-move is performed by first introducing and then canceling $2/3$-handle pairs. So the attaching $S^2$ of the 3-handle might puncture the dual 2-handle curves on $-\partial W$, forcing us to push them into the interior of $W$. To go around this problem, we will describe the $\delta$-move diffeomorphism in an alternative way, as a carving and uncarving operations, that is $1$- and $2$- handle exchanges in the interior (this is also referred as ”dot and zero exchanges” in short). This technique was exploited in [A3].

In Figure 3 we first replace dot with zero (turning 1-handle to 2-handle), then perform the 2-handle slide (indicated by the arrows), resulting the handlebody on the right. The reverse operation (i.e. going from right to left of the figure) can be obtained by first doing the 2-handle slide, indicated by the dotted arrows, and then by replacing “zero with dot”. Here we also traced the dual circles to the 2- handles during this operation (small red circles), where attaching 2-handles to these circles gives the double of $W$, which we denoted by $\Sigma_0$. To construct the handlebody of $\Sigma_n$, we need to modify $W \subset \Sigma_n$ along its boundary by a $\delta$-move.
Figure 4 indicates how the $\delta^n$-move $f : \partial W \to \partial W$ affects the dual handles (red) circles (figure is drawn for $n = 2$). Going back to the original Figure 1 via the reverse $\delta$-move (as indicated in Figure 3) shows that the effect of the $\delta$-move on dual circles, is as in Figure 5.

Now comes a crucial point: A reader gazing at the first picture of Figure 4 might conclude that $\delta$-move doesn’t move the dual circles, because $n$ and $-n$ twists cancel each other. Here are two explanations: First of all, here we are dealing with circles-with-dots not framed circles, transferring twist across them has the affect of changing the carvings (i.e. changing the interiors). Secondly, the original $\delta$ move takes place on $W$, not on the homotopy ball $\Sigma_n - B^4 = W \subset [\text{dual 2-handles}]$, that is $\delta$ may not be an on unknot on $\partial (\Sigma_n - B^4)$. Surprisingly, we can obtain Figure 5 by performing $\delta$-move to $\Sigma_n - B^4$ (the first picture of Figure 3 with dual handles) by using the curve $d$ of Figure 6. This $d$ is in fact an unknot on $\partial (\Sigma_n - B^4)$, which can be checked by the boundary correspondence of Figure 3. Also $d$ happens to be an unknot on $\partial W$, so we could use $d$ for the place of $\delta$ to serve for the dual purpose.

To sum up, the first picture of Figure 7 represents a handlebody of $\Sigma_n$. Now it is easy to check that the middle dotted curve in the second picture of Figure 7 is an unknot (to see this, do the reverse $\delta$-move go back to the first picture of Figure 5, and then observe that in the presence of the dual 2-handles, the dotted circle becomes an unknot there). From this we see that $\Sigma_n$ is obtained from $S^4 = \Sigma_0$ by Gluck twisting (this requires a simple check here, namely remove the dotted circle, and the $-1$ twist on the curves it links from the middle of Figure 7 then see that you get $S^4$). Now by using this unknot, we can attach a 2/3-handle pair (the new 2-handle is the 0-framed dotted curve in the figure). Next we employ a trick , which was used solving the “Cappell-Shaneson homotopy sphere problem” (Figure 14.11 of [A2]): After the obvious handle slide over the middle 0-framed 2-handle in Figure 7 we obtain the pictures of Figure 8 where we can see two cancelling 1/2 -handle pairs! The two 0-framed middle 2-handles cancel the two 1-handles (represented with large dotted circles)! So this picture can be thought of a handlebody without 1-handles, and hence turning it upside down we will get a handlebofy without 3-handles! Having noted this, we can turn this handlebody upside down (as the process described in [A2]). That is, we ignore the cancelled 1/2 handle pairs, and carry the duals of the remaining 2-handles to the boundary of $\#3(S^1 \times B^3)$ by a diffeomorphism.
Now our task is to find a diffeomorphism from the boundary of the pictures of Figure 8 to \( \#3(S^1 \times B^3) \) and carry the dual 2-handles. By applying the Figure 3 boundary identification, we see that the boundary of the last picture of Figure 8 can be identified with the boundary of the first picture of Figure 9, then the obvious isotopy gives the second picture of Figure 9. Note that we don’t draw 3- and 4-handles here, the handlebodies of \( \Sigma_n \) and \( \Sigma_n - B^4 \) will be drawn the same.

Again by applying the reverse boundary identification of Figure 3 to Figure 9 we get Figure 10 which is a handlebody picture of \( \Sigma_n \) without 3-handles! Finally the indicated simple handler slide gives Figure 11 (the picture is drawn for \( n = 2 \)). To indicate how the pattern changes as we increase \( n \to n + 1 \), in Figure 12 we drew \( \Sigma_n \) for \( n = 1 \).

Now let us check the claim 2 of Section 0. We will demonstrate a proof for \( \Sigma_1 \) (from this the reader can see the proof for the general case). For this we first isotope Figure 12 to Figure 13, then do the handle slides and cancellations of the figures Figures 13 \( \sim .. \sim 18 \) as indicated in the pictures. During these operations we trace the ribbon which the unknot \( T \) of Figure 13 bounds. In this figure this ribbon is the trivial ribbon bounding the unknot, where its ribbon move indicated with an unknotted arc in Figure 15. But during the handle slide Figure 15 \( \to \) Figure 16 this trivial ribbon turns into the nontrivial ribbon \( D \), mentioned above. By performing the ribbon move in Figure 16 along the indicated dotted arc, we get Figure 17 (the dotted blue line of his figure is the dual of dotted red line of Figure 16). Then the 1/2 handle cancellation by using the \( -1 \) framed 2-handle, gives Figure 18 which is \( \Sigma_1 \). To see the general pattern, we can apply the same steps to Figure 11 rather than Figure 12 then we see that we get Figure 19 picture of \( \Sigma_2 \). Now the handlebody patterns of \( \Sigma_n \) is as required in 2.

2. Rolling versus carving

Notice that the loop \( c \subset \partial M_n = S^3 \), which links the ribbon in Figure 18 (and in Figure 19) is the unknot in \( S^3 \). This is because doing \( -1 \) surgery to \( c \) (which corresponds to putting 0-framing on \( c \) on the figure) gives \( S^3 \), hence by Property \( P \) the loop \( c \) must be the unknot. Now we can attach a cancelling 2/3 handle pair to \( \Sigma_n \) along \( c \) (this corresponds to adding \( +1 \) framed 2-handle to \( c \)). This gives an alternative description of \( \Sigma_n \) which contains a copy of \( W \):

\[
(3) \quad \Sigma_n - B^4 = W \leftarrow f_n(\gamma) \ h_{\gamma}^2
\]
This is because $W$ is in the form $W = Q \prec h_c^2$, where $Q = B^4 - N(D)$ and $h_c$ is a $+1$-framed $2$ handle attached along $c$ ([A1] Remark 1, and [G1]), i.e. $\Sigma_n - B^4$ is obtained by attaching $2$-handle to $W$ along the $n$-th iterate of a loop $\gamma \subset \partial Q$ by some diffeomorphism $f : \partial Q \to \partial Q$.

**Remark 1.** The handlebody picture of $\Sigma_n$ (Figures 18 and 19) shows that, by changing the carving, which $K\#K$ bounds in $B^4$ by a diffeomorphism will move the position of the $2$-handle $\gamma$ to the $2$-handle of Figure 21, which can easily be identified with $B^4$. This diffeomorphism is obtained by first moving the knot $K\#K$ by an isotopy $g_t : S^3 \to S^3$ back to itself as indicated in Figure 20 along the dotted arrow (i.e. rolling one of the factors of the connected sum over $K\#K$ back to itself), then letting $g_1(K\#K)$ bound the standard slice disk $D$ in $B^4$, which $K\#K$ bounds. Call this new ribbon disk $D'$. Recall from [A4] that we could have relatively exotic but diffeomorphic ribbon complements in $B^4$, the complements of $D$ and $D'$ exhibit the similar curious property here. Reader should compare this to the infinitely many absolutely exotic manifolds of [A3], which also decompose as $A$. So to study $\Sigma_n$ we have two options: (1) Either attach the rolled $2$-handle to the standard ribbon complement $B^4 - D$ as in Figure 19 or (2) Attach the standard $2$ handle of Figure 21 to the nonstandard ribbon complement $B^4 - D'$, carved by rolling. This will be studied in another paper.

Reader should compare this paper to [G1] and [RR]. Note that “Dehn twist” referred there corresponds to the $\delta$-move to the curve $\delta \subset \partial W$. Patient reader can check that by tracing the steps outlined by Remark 1 of [A1], one gets the identification of Figure 24. Then by doubling and connect summing the circle $\delta_1$ and the arc $\delta_2$ one can recover the position of $\delta$ on the right picture of $W$ Figure 24 (cf. [A2]).

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Figure 3. Changing the carvings

Figure 4. Affect of δ-move on the boundary, n=2

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Figure 5. $\Sigma_2$

Figure 6. $d \subset \Sigma_0 = B^4$
Figure 7. $\Sigma_n$

Figure 8. Turning $\Sigma_n$ upside down
Figure 9. $\Sigma_n$, for $n=2$

Figure 10. $\Sigma_n$, for $n = 2$
Figure 11. $\Sigma_n$, for $n = 2$

Figure 12. $\Sigma_1$
Figure 13. $\Sigma_1$

Figure 14. $\Sigma_1$
Figure 15. $\Sigma_1$

Figure 16. $\Sigma_1$
Figure 17. $\Sigma_1$

Figure 18. $\Sigma_1$
Figure 19. $\Sigma_n, n = 2$

Figure 20. Rolling $f^n$

Figure 21. $\Sigma_0 = B^4$
Figure 22. Rolling 2-handle $\gamma$ by $f^n$

Figure 23. Carving ribbon 1-handle by $f^n$

Figure 24. $\delta$-move $\leadsto$ Dehn surgery