Observable Perfect Equilibrium

Sam Ganzfried
Ganzfried Research
sam.ganzfried@gmail.com

Abstract

While Nash equilibrium has emerged as the central game-theoretic solution concept, many important games contain several Nash equilibria and we must determine how to select between them in order to create real strategic agents. Several Nash equilibrium refinement concepts have been proposed and studied for sequential imperfect-information games, the most prominent being trembling-hand perfect equilibrium, quasi-perfect equilibrium, and recently one-sided quasi-perfect equilibrium. These concepts are robust to certain arbitrarily small mistakes, and are guaranteed to always exist; however, we argue that neither of these is the correct concept for developing strong agents in sequential games of imperfect information. We define a new equilibrium refinement concept for extensive-form games called observable perfect equilibrium in which the solution is robust over trembles in publicly-observable action probabilities (not necessarily over all action probabilities that may not be observable by opposing players). Observable perfect equilibrium correctly captures the assumption that the opponent is playing as rationally as possible given mistakes that have been observed (while previous solution concepts do not). We prove that observable perfect equilibrium is always guaranteed to exist, and demonstrate that it leads to a different solution than the prior extensive-form refinements in no-limit poker. We expect observable perfect equilibrium to be a useful equilibrium refinement concept for modeling many important imperfect-information games of interest in artificial intelligence.

1 Introduction

When developing a strategy for a human or computer agent to play in a game, the starting point is typically a Nash equilibrium. Even if additional information is available about the opponents, e.g., from historical data or observations of play, we would often still opt to start playing a Nash equilibrium strategy until we are confident in our ability to successfully exploit opponents by deviating. It is well known that several conceptual and computational limitations exist for Nash equilibrium. For multiplayer and two-player non-zero-sum games, it is PPAD-hard to compute or approximate one Nash equilibrium, different Nash equilibria may give different values to the players, and following a Nash equilibrium strategy provides no performance guarantee. Even for two-player zero-sum games, in which these issues do not arise, there can still exist multiple Nash equilibria that we must select from. Therefore several solution concepts that refine Nash equilibrium in various ways have been proposed to help select one that is more preferable in some way. Most of the common equilibrium refinements are based on the idea of ensuring robustness against certain arbitrarily small “trembles” in players’ execution of a given strategy. Variants of these Nash equilibrium refinements have been devised for simultaneous strategic-form games as well as sequential games of perfect and imperfect information. In this paper we will be primarily interested in sequential games of imperfect information, which are more complex than the other games classes and have received significant interest recently in artificial intelligence due to their ability to model many important scenarios. To simplify analysis we will primarily be studying a subclass of these games in which there are two players, only one player has private information, and both players take a single action; however, our results apply broadly to extensive-form imperfect-information games. We will also be primarily focused on two-player zero-sum games, though some analysis also applies to two-player non-zero-sum and multiplayer games.
games. We will show that existing Nash equilibrium refinement concepts have limitations in sequential imperfect-information games, and propose the new concept of observable perfect equilibrium that addresses these limitations.

A strategic-form game (aka normal-form game) consists of a finite set of players $N = \{1, \ldots, n\}$, a finite set of pure strategies $S_i$ for each player $i \in N$, and a real-valued utility for each player for each strategy vector (aka strategy profile), $u_i : \times_i S_i \rightarrow \mathbb{R}$. A mixed strategy $\sigma_i$ for player $i$ is a probability distribution over pure strategies, where $\sigma_i(s_{i'})$ is the probability that player $i$ plays pure strategy $s_{i'} \in S_i$ under $\sigma_i$. Let $\Sigma_i$ denote the full set of mixed strategies for player $i$. A strategy profile $\sigma^* = (\sigma^*_1, \ldots, \sigma^*_n)$ is a Nash equilibrium if $u_i(\sigma^*_i, \sigma^*_{-i}) \geq u_i(\sigma_i, \sigma^*_{-i})$ for all $\sigma_i \in \Sigma_i$ for all $i \in N$, where $\sigma^*_{-i} \in \Sigma_{-i}$ denotes the vector of the components of strategy $\sigma^*$ for all players excluding player $i$. Here $u_i$ denotes the expected utility for player $i$, and $\Sigma_{-i}$ denotes the set of strategy profiles for all players excluding player $i$.

Nash equilibrium has emerged as the central solution concept in game theory, and is guaranteed to exist in all finite strategic-form games [23][24]. However, games may contain multiple Nash equilibria and it is not clear which one should be played. A popular refinement of Nash equilibrium is in all finite strategic-form games [23, 24]. However, games may contain multiple Nash equilibria and it is not clear which one should be played. A popular refinement of Nash equilibrium is perfect equilibrium if and only if $\sigma^*$ is a limit point of a sequence $\{\sigma(\epsilon)\}_{\epsilon \rightarrow 0}$ of totally mixed strategy profiles with the property that $\sigma^*$ is a best response for all players against every element $\sigma(\epsilon)$ in this sequence.

Theorem 1. Let $\sigma^*$ be a strategy profile of an $n$-player strategic-form game $G$. Then $\sigma^*$ is a trembling-hand perfect equilibrium if and only if $\sigma^*$ is a limit point of a sequence $\{\sigma(\epsilon)\}_{\epsilon \rightarrow 0}$ of totally mixed strategy profiles with the property that $\sigma^*$ is a best response for all players against every element $\sigma(\epsilon)$ in this sequence.

While the strategic form can be used to model simultaneous actions, settings with sequential moves are typically modelled using the extensive form representation. The extensive form can also model simultaneous actions, as well as chance events and imperfect information (i.e., situations where some information is available to only some of the agents and not to others). Extensive-form games consist primarily of a game tree; each non-terminal node has an associated player (possibly chance) that makes the decision at that node, and each terminal node has associated utilities for the players. Additionally, game states are partitioned into information sets, where the player whose turn it is to move cannot distinguish among the states in the same information set. Therefore, in any given information set, a player must choose actions with the same distribution at each state contained in the information set. If no player forgets information that he previously knew, we say that the game has perfect recall. A (behavioral) strategy for player $i$, $\sigma_i \in \Sigma_i$, is a function that assigns a probability distribution over all actions at each information set belonging to $i$. Similarly to strategic-form games, it can be shown that all extensive-form games with perfect recall contain at least one Nash equilibrium in mixed strategies. The concept of extensive-form trembling hand perfect equilibrium (EFTHEPE) is defined analogously to THPE by requiring that every action at every information set for each player is taken with non-zero probability in perturbed games. EFTHEPE are then limits of equilibria of such perturbed games as the tremble probabilities go to zero. It has been proven that an EFTHEPE exists in every extensive-form game [29].
2 Observable Perfect Equilibrium

In order to simplify our analysis we define a subset of extensive-form information games called two-player one-step extensive-form imperfect-information games (OSEFGs):

- There are two players, P1 and P2.
- Player 1 is dealt private information $\tau_1$ from a finite set $T_1$ uniformly at random.\footnote{Note that all of our analysis will still hold if we assume that $\tau_1$ is selected from an arbitrary probability distribution.}
- Player 1 can then choose action $a_1$ from finite set $A_1$.
- Player 2 observes the action $a_1$ but not $\tau_1$.
- Player 2 then chooses action $a_2$ from finite set $A_2$.
- Both players are then given payoff $u_i(\tau_1, a_1, a_2)$.

For mixed strategy $\sigma_1$ for player 1, $\sigma_1(\tau_1, a_1)$ denotes the probability that player 1 takes action $a_1 \in A_1$ with private information $\tau_1 \in T_1$. Similarly for mixed strategy $\sigma_2$ for player 2, $\sigma_2(a_1, a_2)$ denotes the probability that player 2 takes action $a_2 \in A_2$ following observed action $a_1 \in A_1$ of player 1.

Suppose we are in the position of player 2 responding to the observed action $a_1 \in A_1$. If both players are following a Nash equilibrium strategy, then we know that we are best responding to player 1’s strategy. However, suppose that we are following our component from Nash equilibrium strategy profile $\sigma^*$ in which $\sigma_1^*(\tau_1, a_1) = 0$ for all $\tau_1 \in T_1$. Our observation is clearly inconsistent with player 1 following $\sigma^*$, since they would never choose action $a_1$. Since our strategy is part of a Nash equilibrium, it ensures that player 1 cannot profitably deviate from $\sigma_1^*$ with any $\tau_1 \in T_1$ and take $a_1$; however, there may be many such strategies, and we would like to choose the best one given that we have actually observed player 1 irrationally playing $a_1$. In this situation, playing an EFTHPE strategy may ensure that we play a stronger strategy against this opponent, who has selected an action that they should not rationally play, since EFTHPE explicitly ensures robustness against the possibility of “trembling” and playing such an action with small probability.

Extensive-form trembling-hand perfect equilibrium assumes that all players take all actions at all information sets with non-zero probability. In the situation described above, we know that player 1 is taking $a_1$ at some information set with non-zero probability; however, we really have no further information beyond that. It is very possible that they are playing a strategy that takes $a_1' \in A_1$ with zero probability with all $\tau_1 \in T_1$. The core assumption of game theory is that, in the absence of any information the contrary, we assume that all players are behaving rationally. Now clearly that assumption is violated in this case when we observe player 1 irrationally playing $a_1$. However, it seems a bit extreme to now assume that all players are playing all actions with nonzero probability. If we assume that the opponent is playing as rationally as possible given our observations, then we would only consider trembles that are consistent with our observations of their play. Such trembles must satisfy $\sigma_1(\tau_1, a_1) > 0$ for at least one $\tau_1 \in T_1$, or alternatively, $\sum_{\tau_1 \in T_1} \sigma_1(\tau_1, a_1) > 0$. The concept of observable perfect equilibrium (OPE) captures this assumption that all players are playing as rationally as possible subject to the constraint that their play is consistent with our observations.

**Definition 1.** Let $G$ be a two-player one-step extensive-form imperfect-information game, and suppose that player 2 has observed public action $a_1$ from player 1. Then $\sigma^*$ is an observable perfect equilibrium if there is a sequence of perturbed games, in which player 1 is required to play $a_1$ with non-zero probability for at least one $\tau_1 \in T_1$, that converges to $G$, in which there is a sequence of Nash equilibria of the perturbed games that converges to $\sigma^*$. 
Proposition 1. Every observable perfect equilibrium is a Nash equilibrium.

Proof. Let $\sigma^*$ be an observable perfect equilibrium of $G$, where $a_1$ is the observed public action of player 1. Then there exists a sequence of games $\{G_\epsilon\}$ converging to $G$ in which player 1 is forced to play $a_1$ with non-zero probability for at least one $\tau_1 \in T_1$, and a sequence of Nash equilibria $\{(\sigma_1^\epsilon, \sigma_2^\epsilon)\}$ that converges to $\sigma^*$. Suppose that $\sigma^*$ is not a Nash equilibrium of the original game $G$. Suppose that player 2 can profitably deviate to $\sigma_2$. Then we have

$$u_2(\sigma_1^*, \sigma_2) > u_2(\sigma_1^*, \sigma_2^*)$$

By continuity of expected utility,

$$\lim_{\epsilon \to 0} u_2(\sigma_1^\epsilon, \sigma_2) > u_2(\lim_{\epsilon \to 0} \sigma_1^\epsilon, \lim_{\epsilon \to 0} \sigma_2^\epsilon)$$

So there exists some $\epsilon > 0$ for which

$$u_2(\sigma_1^\epsilon, \sigma_2) > u_2(\sigma_1^*, \sigma_2^*)$$

This contradicts the fact that $\sigma^\epsilon$ is a Nash equilibrium of $G_\epsilon$.

Now suppose player 1 can profitably deviate to $\sigma_1$. By similar reasoning as above, there exists some $\epsilon' > 0$ such that

$$u_1(\sigma_1, \sigma_2^*) > u_1(\sigma_1^*, \sigma_2^*)$$

for all $\epsilon \in (0, \epsilon')$. If $\sum_{\tau_1 \in T_1} \sigma_1(\tau_1, a_1) \geq \epsilon'$ then we are done, since $\sigma_1$ is a valid strategy in $G_{\epsilon'}$. Otherwise, let $\epsilon^* = \sum_{\tau_1 \in T_1} \sigma_1(\tau_1, a_1) < \epsilon'$. Then $\sigma_1$ is a valid strategy in $G_{\epsilon^*}$. So player 1 can profitably deviate from $\sigma_1^*$ in $G_{\epsilon^*}$. This contradicts the fact that $\sigma^* = \sigma^*$ is a Nash equilibrium of $G_{\epsilon^*}$.

So we have shown that neither player can profitably deviate from $\sigma^*$, and therefore $\sigma^*$ is a Nash equilibrium. \qed

We can extend Definition 1 to general $n$-player extensive-form imperfect-information games by adding analogous constraints for all observed actions (i.e., by requiring that the sum of the probabilities of strategy sequences taken consistent with our observations is nonzero). This is useful because we are no longer required to reason about trembles that are incompatible with the current path of play, which are irrelevant at this point. For general extensive-form games, there is a further consideration about what trembles should be considered for future moves beyond the current path of play. Extensive-form trembling hand perfect equilibrium assumes that all players may tremble in future actions, while an alternative concept called quasi-perfect equilibrium (QPE) assumes that only the opposing players tremble for future actions (even if we have trembled previously ourselves) [28]. The related concept of one-sided quasi-perfect equilibrium (OSQPE) assumes that only the opposing players can tremble at all and we cannot [9]. OSQPE is the most computationally efficient and is also the most similar to observable perfect equilibrium.

In two-player one-step extensive-form imperfect-information games QPE and EFTHPE are identical, since both players only take a single action along the path of play. They both require that both player 1 and player 2 put non-zero probability trembles on all actions. Both of these potentially differ from OSQPE, which from the perspective of player 2 requires that player 1 puts non-zero probability trembles on all possible actions, while player 2 does not have this requirement. Note that all three of these concepts potentially differ from observable perfect equilibrium (from the perspective of player 2), which requires only that player 1 makes a non-zero probability tremble at some information set consistent with taking the observed action $a_1$ and no requirement on player 2.

Observable perfect equilibrium is fundamentally different from other equilibrium refinements in that it is dependent on the action taken by the other player. For each observed action $a_1$ we compute a potentially
different strategy for ourselves, conditionally on having observed $a_1$. In contrast, all other equilibrium refinement concepts compute a full strategy profile for all players in advance of game play. If play contains longer sequences of actions than just a single move for each player, we can recompute our OPE strategy after each new observation (again by assuming positive probability trembles for all actions consistent with the path of play). Note that in aggregate over all information sets an OPE still defines a full strategy; we just do not need to compute it in entirety to implement it.

We can view the relation between OPE and other solution concepts analogously to the relation between endgame solving [13] and standard offline game solving in large imperfect-information games. Previously the standard approach for approximating Nash equilibrium strategies in large imperfect-information games was to first apply an abstraction algorithm to create a significantly smaller game that is strategically similar to the full game [27] [3] [15] [16] [17] [30] [20], then solve the abstract game using an equilibrium-finding algorithm such as counterfactual regret minimization (CFR) [31] or a generalization of Nesterov’s excessive gap technique [13]. With endgame solving [13], the portion of the game tree that we have reached is solved in real time to a finer degree of granularity than in the offline abstract equilibrium computation. This focused computation led to superhuman play in two-player [4] [22] and six-player no-limit Texas hold ’em [5]. OPE similarly achieves computational advantages over other equilibrium refinement concepts such as TFPE, QPE, and OSQPE, by focusing computation in real time only on the portion of the game tree we have reached (and observed an opponent’s “tremble”). However, unlike endgame solving, OPE still guarantees that the computed strategy remains a Nash equilibrium, while it has been shown that endgame solving may produce strategies that are not Nash equilibria of the full game (even if the trunk strategy were an exact Nash equilibrium) [13].

In order to show existence of observable perfect equilibrium and an algorithm for its computation, we first review results for the related concept of one-sided quasi-perfect equilibrium. One-sided quasi-perfect equilibrium assumes that the game is two-player zero-sum and that we play the role of the “machine player” while the opponent is the “human player.” A key subroutine in the computation of one-sided quasi-perfect equilibrium in two-player zero-sum games is the solution to an optimization formulation for an $\epsilon$-quasi-perfect equilibrium strategy profile (where the trembles for the human player are lower bounded by $\ell_h(\epsilon)$). The vector $\ell_h(\epsilon)$ has entries $\ell_h(\epsilon)[\sigma] = \epsilon|\sigma|$, where $|\sigma|$ denotes the number of actions for player $h$ in the sequence $\sigma$. In the OSEFG setting there is just a single action per player per sequence, so we have $l_h(\epsilon) = \epsilon$. It has been shown that the optimization formulation (1) corresponds to a one-sided quasi-perfect equilibrium [9]. In (1), $x_m$ is the vector of the machine player’s strategy sequences, $x_h$ is the vector of the human player’s strategy sequences, and $A_m$ is the payoff matrix from the machine player’s perspective. The vector $f_h$ and matrix $F_h$ are constants that encode the sequence-form representation of the information set structure for the human player, and $f_m$, $F_m$ are analogous for the machine player [21].

$$\max_{x_m} \min_{x_h} x_m^T A_m x_h \quad \text{s.t.} \quad F_m x_m = f_m$$
$$x_m \geq 0$$
$$F_h x_h = f_h$$
$$x_h \geq \ell_h(\epsilon) \quad (1)$$

The following result has been proven from this formulation [9].

**Lemma 1.** Consider the bilinear saddle point problem (1). Then, for any $\epsilon > 0$ for which the domain of the minimization problem is nonempty, any solution to (1) is a one-sided $\epsilon$-quasi-perfect strategy profile.

From Lemma [1] they are able to prove the existence of one-sided quasi-perfect equilibrium as a corollary [9].

**Corollary 1.** Every two-player zero-sum extensive-form game with perfect recall has at least one one-sided quasi-perfect equilibrium.
We can obtain analogous results for observable perfect equilibrium using similar reasoning. We refer to the players as we did originally, where player 2 corresponds to ourselves (i.e., the “machine player”), and player 1 corresponds to the opponent (i.e., the “human player”). Note that the optimization formulation (2) differs from (1) only in the constraints for player 1’s strategy vector. The inequality \(c^T x_1 \geq \epsilon\) encodes the constraint that the sum of the probabilities that player 1 takes strategy sequences that are consistent with our observations of play so far is at least \(\epsilon\), where \(c\) is a constant vector.

\[
\begin{align*}
\max_{x_2} \min_{x_1} & \quad x_2^T A_2 x_1 \\
\text{s.t.} & \quad F_2 x_2 = f_2 \\
& \quad x_2 \geq 0 \\
& \quad F_1 x_1 = f_1 \\
& \quad x_1 \geq 0 \\
& \quad c^T x_1 \geq \epsilon 
\end{align*}
\]

Lemma 2. Consider the bilinear saddle point problem (2). Then, for any \(\epsilon > 0\) for which the domain of the minimization problem is nonempty, any solution to (2) is an \(\epsilon\)-observable-perfect strategy profile.

Corollary 2. Every two-player zero-sum extensive-form game with perfect recall has at least one observable perfect equilibrium.

For one-sided quasi-perfect equilibrium, it has been shown that formulation (1) implies the following linear programming formulation (3) for an \(\epsilon\)-quasi-perfect equilibrium strategy for the machine player [9]. In (3), \(v\) is a new vector of free variables, while all other quantities are the same as in (1).

\[
\begin{align*}
\arg \max_{x_m, v} & \quad (A_m \ell_h(\epsilon))^T x_m + (f_h - F_h \ell_h(\epsilon))^T v \\
\text{s.t.} & \quad A_m^T x_m - F_h v \geq 0 \\
& \quad F_m x_m = f_m \\
& \quad x_m \geq 0 \\
& \quad v \text{ free}
\end{align*}
\]

Proposition 2. Any limit point of solutions to the trembling linear program (3) as the trembling magnitude \(\epsilon \to 0^+\) is a one-sided quasi-perfect equilibrium strategy for the machine player.

This linear programming formulation leads to a polynomial-time algorithm for computation of one-sided quasi-perfect equilibrium by solving the problem for consecutively smaller values of \(\epsilon\) until a termination criterion is met [9]. It is argued that this computation is more efficient than analogous algorithms for extensive-form trembling-hand perfect equilibrium and quasi-perfect equilibrium, because the OSQPE formulation only depends on \(\epsilon\) through the objective, while EFTHPE depends on \(\epsilon\) through the left-hand-side of constraints, and QPE depends on \(\epsilon\) in both the right-hand-side of constraints and the objective. We now provide a linear program formulation for \(\epsilon\)-observable perfect equilibrium that is analogous to (3), which also implies a polynomial-time algorithm for computation of observable-perfect equilibrium with the same computational advantages as for OSQPE. Note that in our new LP formulation (4) \(\epsilon\) occurs only in the objective, as it did in (3). Note also that our objective has only about half as many terms as the corresponding OSQPE objective, suggesting that OPE can be computed faster than OSQPE. This agrees with the intuition described above, since OPE only considers trembles for the opponent consistent with our observations of the path of play, while OSQPE considers all possible (past and future) trembles for the opponent.
arg max_{x_2,w,v} \ f_1^T v + \epsilon w \\
subject to \ F_1^T v + wc = A_2^T x_2 \\
F_2 x_2 = f_2 \\
x_2 \geq 0 \\
w \geq 0 \\
v \text{ free} \quad (4)

Proposition 3. Any limit point of solutions to the trembling linear program (4) as the trembling magnitude $\epsilon \to 0^+$ is an observable perfect equilibrium strategy for player 2.

Proof. First we create the dual of the minimization problem inside of (2):

$$
\begin{align*}
\max_{w,v} & \quad f_1^T v + \epsilon w \\
\text{s.t.} & \quad F_1^T v + wc = A_2^T x_2 \\
& \quad w \geq 0 \\
& \quad v \text{ free}
\end{align*}
$$

Next we substitute this new dual formulation back into the outer maximization problem to create a single maximization problem for player 2’s strategy:

$$
\begin{align*}
\arg \max_{x_2,w,v} & \quad f_1^T v + \epsilon w \\
\text{s.t.} & \quad F_1^T v + wc = A_2^T x_2 \\
& \quad F_2 x_2 = f_2 \\
& \quad x_2 \geq 0 \\
& \quad w \geq 0 \\
& \quad v \text{ free}
\end{align*}
$$

3 No-limit poker

In this section we illustrate how observable perfect equilibrium leads to a different strategy profile than the other equilibrium refinement concepts in no-limit poker. Poker has been a major AI challenge problem in recent years, with no-limit Texas hold ‘em in particular being the most popular variant for humans. No-limit Texas hold ‘em is a large sequential game of imperfect information, and just recently computers have achieved superhuman performance, first in the two-player variant [22, 4] and subsequently for six players [5]. These agents attempt to compute approximations of Nash equilibrium strategies by first running an abstraction algorithm to create a smaller strategically-similar game, and then solving the abstract game using an equilibrium-finding algorithm such as counterfactual regret minimization. Counterfactual regret minimization (CFR) is an iterative self-play procedure that has been proven to converge to Nash equilibrium in two-player zero-sum [31], though it has been demonstrated to not converge to Nash equilibrium in a simplified three-player poker game [11]. The key insight that led to superhuman play was to combine these abstraction and equilibrium-finding approaches with endgame solving [13], in which the portion of the game we have reached during a hand is resolved in a finer granularity in real time. It is somewhat remarkable that these approaches have achieved such strong performance despite numerous theoretical limitations: the abstraction algorithms have no performance guarantee, endgame solving has no performance guarantee, CFR does not guarantee convergence to Nash equilibrium for more than two players, and furthermore even
if CFR did converge for more than two players there can be multiple Nash equilibria and following one has no performance guarantee. It turns out that even ignoring all of these theoretical limitations, there is an additional challenge present. Even if we are in the two-player zero-sum setting and are able to compute an exact Nash equilibrium, the game may contain many Nash equilibria, and we would like to choose the “best” one. As we will see, even the simplest two-player no-limit poker game contains infinitely many Nash equilibria.

In the no-limit clairvoyance game [2], player 1 is dealt a winning hand (W) and a losing hand (L) each with probability $\frac{1}{2}$. (While player 2 is not explicitly dealt a “hand,” we can view player 2 as always being dealt a medium-strength hand that wins against a losing hand and loses to a winning hand.) Both players have initial chip stacks of size $n$, and they both ante $0.50$ (creating an initial pot of $1$). P1 is allowed to bet any integral amount $x \in [0, n]$ (a bet of 0 is called a check) [2]. Then P2 is allowed to call or fold (but not raise). This game clearly falls into the class of one-step extensive-form imperfect-information games. The game is small enough that its solution can be computed analytically (even for the continuous version) [2].

- P1 bets $n$ with prob. 1 with a winning hand.
- P1 bets $n$ with prob. $\frac{n}{1+n}$ with a losing hand (and checks otherwise).
- For all $x \in (0, n]$, P2 calls a bet of size $x$ with prob. $\frac{1}{1+x}$.

It was shown by Ankenman and Chen [2] that this strategy profile constitutes a Nash equilibrium. (They also show that these frequencies are optimal in many other poker variants.) Here is a sketch of that argument.

**Proposition 4.** The strategy profile presented above is a Nash equilibrium of the no-limit clairvoyance game.

**Proof.** Player 2 must call a bet of size $x$ with probability $\frac{1}{1+x}$ in order to make player 1 indifferent between betting $x$ and checking with a losing hand. For a given $x$, player 1 must bluff $\frac{x}{1+x}$ as often as he value bets for player 2 to be indifferent between calling and folding. Given these values, the expected payoff to player 1 of betting size $x$ is $v(x) = \frac{x}{2(1+x)}$. This function is monotonically increasing, and therefore player 1 will maximize his payoff with $x = n$. □

Despite the simplicity of this game, the solution has been used in order to interpret bet sizes for the opponent that fall outside an abstracted game model by many strong agents for full no-limit Texas hold ’em [12, 10, 19]. Thus, its solution still captures important aspects of realistic forms of poker played competitively.

It turns out that player 2 does not need to call a bet of size $x \neq n$ with exact probability $\frac{1}{1+x}$; he need only not call with such an extreme probability that player 1 has an incentive to change his bet size to $x$ (with either a winning or losing hand). In particular, it can be shown that player 2 need only call a bet of size $x$ with any probability (which can be different for different values of $x$) in the interval $\left[\frac{1}{1+x}, \min \left\{ \frac{n}{x(1+n)}, 1 \right\} \right]$ in order to remain in equilibrium.

**Proposition 5.** A strategy profile $\sigma^*$ in the no-limit clairvoyance game is a Nash equilibrium if and only if under $\sigma^*$ player 1 bets $n$ with probability 1 with a winning hand and bets $n$ with probability $\frac{n}{1+n}$ with a losing hand, and for all $x \in (0, n]$ player 2 calls vs. a bet of size $x$ with probability in the interval $\left[\frac{1}{1+x}, \min \left\{ \frac{n}{x(1+n)}, 1 \right\} \right]$.

In the original formulation of the no-limit clairvoyance game [2] player 1 is allowed to bet any real value in $[0, n]$, making the game a continuous game, since player 1’s pure strategy space is infinite. For simplicity we consider the discrete game where player 1 is restricted to only betting integer values, though much of our equilibrium analysis will still apply for the continuous version as well.

Note that if $x$ is required to be integral then we always have $x \geq \frac{n}{1+n}$, and $\min \left\{ \frac{n}{x(1+n)}, 1 \right\} = \frac{n}{x(1+n)}$. However, our solution also holds for the continuous game.
Proof. We have already argued that the Nash equilibrium strategy for player 1 is unique. Note that for \( x \in (0, n) \), player 1 will never actually bet \( x \) in equilibrium, and player 2 must call with probability that ensures that player 1 will not want to deviate and bet \( x \). Suppose player 2 calls with probability \( p' < \frac{1}{1+x} \), for some \( x \in (0, n) \). If player 1 bets \( x \) with a losing hand, expected payoff is

\[
p'(\frac{x}{1+x} + 0.5) + (1 - p')(0.5)
\]

\[
= p'(1 - x) + 0.5
\]

\[
> \frac{-1 - x}{1 + x} + 0.5
\]

\[
= -0.5
\]

If instead player 1 checks with a losing hand, the expected payoff is -0.5. So player 1 will strictly prefer to bet \( x \) than to check, and will have incentive to deviate from his equilibrium strategy.

Now suppose player 2 calls with probability \( p' > \min \left\{ \frac{n}{x(1+n)}, 1 \right\} \), for some \( x \in (0, n) \). First suppose that \( \frac{n}{x(1+n)} \leq 1 \). If player 2 bets \( x \) with a winning hand, his expected payoff is

\[
p'(0.5 + x) + (1 - p')(0.5)
\]

\[
= 0.5p' + p'x + 0.5 - 0.5p'
\]

\[
> \frac{x}{x(1+n)} + 0.5
\]

\[
= \frac{1.5n + 0.5}{1 + n}
\]

If instead player 1 bets \( n \), the expected payoff is

\[
\frac{1}{1+n}(0.5 + n) + \left(1 - \frac{1}{1+n}\right)(0.5)
\]

\[
= \frac{1.5n + 0.5}{1 + n}
\]

So player 1 will strictly prefer to bet \( x \) than to bet \( n \), and will have incentive to deviate from his equilibrium strategy.

Our analysis so far has shown that for \( \frac{n}{x(1+n)} \leq 1 \), the strategy profile is a Nash equilibrium if and only if player 2 calls a bet of size \( x \) with probability in the interval \( \left[ \frac{1}{1+x}, \frac{n}{x(1+n)} \right] \) for all \( x \in (0, n] \).

Now suppose that \( \frac{n}{x(1+n)} > 1 \). Suppose that player 2 calls a bet of \( x \) with probability 1. If player 1 bets \( x \), his expected payoff is

\[
1(0.5 + x) + 0(0.5)
\]

\[
= x + 0.5
\]

\[
< \frac{n}{1+n} + 0.5
\]

\[
= \frac{1.5n + 0.5}{1 + n}
\]

If instead player 1 bets \( n \), the expected payoff as shown above is \( \frac{1.5n + 0.5}{1 + n} \). So player 1 will not have incentive to deviate and bet \( x \) instead of \( n \).

\( \square \)
So we have shown that player 2 has infinitely many Nash equilibrium strategies that differ in their frequencies of calling vs. “suboptimal” bet sizes of player 1. Which of these strategies should we play when we encounter an opponent who bets a suboptimal size? One argument for calling with probability at the lower bound of the interval—\(\frac{1}{1+\epsilon}\)—is as follows (note that the previously-computed equilibrium strategy uses this value [2]). If the opponent bets \(x\) as opposed to the optimal size of \(n\) that he should bet in equilibrium, then a reasonable deduction is that he isn’t even aware that \(n\) would have been the optimal size, and believes that \(x\) is optimal. Therefore, it would make sense to play a strategy that is an equilibrium in the game where the opponent is restricted to only betting \(x\) (or to betting 0, i.e., checking). Doing so would correspond to calling a bet of \(x\) with probability \(\frac{1}{1+\epsilon}\). The other equilibria pay more heed to the concern that the opponent could exploit us by deviating to bet \(x\) instead of \(n\); but we need not be as concerned about this possibility, since a rational opponent who knew to bet \(n\) would not bet \(x\).

One could use a similar argument to defend calling with probability at the upper bound of the interval—\(\min \left\{ \frac{n}{x(1+n)}, 1 \right\}\). If the opponent somehow knew that betting \(n\) was part of an optimal strategy but did not know that checking was, then perhaps we should follow an equilibrium of the game where the opponent is restricted to only betting \(x\) or \(n\), in which case our calling frequency should focus on dissuading the opponent from betting \(x\) with a winning hand instead of \(n\).

The first argument seems much more natural than the second, as it seems much more reasonable that a human is aware they should check sometimes with weak hands, but may have trouble computing that \(n\) is the optimal size and guess that it is \(x\). However, both arguments could be appropriate depending on assumptions about the reasoning process of the opponent. The entire point of Nash equilibrium as a prescriptive solution concept is that we do not have any additional information about the players’ reasoning process, so will opt to assume that all players are fully rational. If any additional information is available—such as historical data (either from our specific opponents’ play or from a larger population of players), observations of play from the current match, a prior distribution, or any other model of the reasoning mechanism of the opponents—then we should clearly utilize this information and not simply follow a Nash equilibrium. Without any such additional information, it does not seem clear whether we should call with the lower bound probability, upper bound probability, or a value in the middle of the interval. The point of the equilibrium refinements we have considered is exactly to help us select between equilibria in a theoretically principled way in the absence of any additional information that could be used to model the specific opponents.

For the remainder of our analysis we will restrict our attention to the no-limit clairvoyance game with \(n = 2\), with extensive-form game tree given by Figure [1].

According to our above analysis, the unique Nash equilibrium strategy for player 1 is to bet 2 with probability 1 with a winning hand, to bet 2 with probability \(\frac{2}{3}\) with a losing hand, and to check with probability \(\frac{1}{3}\). The Nash equilibrium strategies for player 2 are to call a bet of 2 with probability \(\frac{1}{2}\), and to call a bet of 1 with probability in the interval \([\frac{1}{2}, \frac{2}{3}]\). As it turns out, the unique trembling-hand perfect equilibrium strategy for player 2 is to call vs. a bet of 1 with probability \(\frac{2}{3}\). Since this is a one-step extensive-form imperfect-information game, this is also the unique quasi-perfect equilibrium. And since player 2's strategy is fully mixed, this is also the unique one-sided quasi-perfect equilibrium. However, the unique observable perfect equilibrium strategy for player 2 is to call with probability \(\frac{5}{9}\). Interestingly, the OPE corresponds to a different strategy for this game than all the other refinements we have considered, and none of them correspond to the “natural” argument for calling with probability \(\frac{1}{2}\) based on an assumption.

---

4Observe that this game explicitly shows that Theorem [1] does not hold in general for extensive-form games, since all the Nash equilibria in this game satisfy the alternative formulation of trembling-hand perfect equilibrium. To see this, consider the sequence of strategies for player 1 that bet 1 with probability \(\epsilon\) with a winning hand and with probability \(\frac{\epsilon}{2}\) with a losing hand. This sequence will converge to the unique Nash equilibrium strategy for player 1 as \(\epsilon \to 0\), and furthermore player 2 is indifferent between calling and folding vs. a bet of 1 against all of these strategies, so all of player 2’s Nash equilibrium strategies are best responses. So the equivalent formulation of trembling-hand perfect equilibrium is only valid for simultaneous strategic-form games and does not apply to extensive-form games.
about the typical reasoning of human opponents. The OPE value of \( \frac{5}{2} \) corresponds to the solution assuming only that player 1 has bet 1 but that otherwise all players are playing as rationally as possible. Note also that the OPE does not simply correspond to the average of the two interval boundaries, which would be \( \frac{7}{12} \).

**Proposition 6.** In the no-limit clairvoyance game with \( n = 2 \), the unique extensive-form trembling-hand perfect equilibrium strategy for player 2 is to call vs. a bet of 1 with probability \( \frac{2}{3} \).

**Proof.** Let \( \mu \) denote the Nash equilibrium strategy profile where player 2 calls vs. a bet of 1 with probability \( \frac{2}{3} \). Consider the game \( G_\epsilon \) where each action probability must be at least \( \epsilon \). Consider the following strategy \( \sigma^*_{\epsilon,1} \) for player 1. With a winning hand player 1 bets 2 with probability \( 1 - 2\epsilon \), and bets 1 with probability \( 2\epsilon \). With a losing hand player 1 bets 2 with probability \( \frac{2(1-2\epsilon)}{3} \), bets 1 with probability \( \epsilon \) and bets 0 with remaining probability \( 1 - \frac{2(1-2\epsilon)}{3} - \epsilon = \frac{1+\epsilon}{3} \). Consider whether player 1 can profitably deviate to strategy \( \phi \) from this strategy when player 2 follows \( \mu \).

First, we calculate expected payoff for player 1 of playing \( \sigma^*_{\epsilon,1} \) against \( \mu \):

\[
\begin{align*}
\frac{1}{2} & \left( (1-2\epsilon) \left( \frac{1}{3}(2.5) + \frac{2}{3}(0.5) \right) + (2\epsilon) \left( \frac{2}{3}(1.5) + \frac{1}{3}(0.5) \right) \right) \\
& + \frac{1}{2} \left( \frac{2(1-2\epsilon)}{3} \left( \frac{1}{3}(-2.5) + \frac{2}{3}(0.5) \right) + \epsilon \left( \frac{2}{3}(-1.5) + \frac{1}{3}(0.5) \right) + \left( \frac{1+\epsilon}{3}(-0.5) \right) \right) \\
= & \frac{1}{2} \left( (1-2\epsilon) \left( \frac{7}{6} \right) + (2\epsilon) \left( \frac{7}{6} \right) \right) + \frac{1}{2} \left( \frac{2(1-2\epsilon)}{3} \left( -0.5 \right) + \epsilon \left( -\frac{5}{6} \right) + \left( \frac{1+\epsilon}{3}(-0.5) \right) \right) \\
= & \frac{7}{12} + \frac{1}{2} \left( -\frac{1}{3} + \frac{2\epsilon}{3} - \frac{5\epsilon}{6} - \frac{1}{6} - \frac{\epsilon}{6} \right) \\
= & \frac{1}{3} - \frac{\epsilon}{6}
\end{align*}
\]

If instead player 1 plays \( \phi \), expected payoff is

\[
\frac{1}{2} \left( (1 - \phi(W,1)) \left( \frac{7}{6} \right) + \phi(W,1) \left( \frac{7}{6} \right) \right)
\]
\[ + \frac{1}{2} \left( \phi(L, 2)(-0.5) + \phi(L, 1) \left( -\frac{5}{6} \right) + (1 - \phi(L, 2) - \phi(L, 1))(-0.5) \right) \]
\[ = \frac{7}{12} + \frac{1}{2} \left( -\frac{\phi(L, 1)}{3} - 0.5 \right) \]
\[ = \frac{1}{3} - \frac{\phi(L, 1)}{6} \]

Since \( \phi(L, 1) \geq \epsilon \), this quantity is maximized with \( \phi(L, 1) = \epsilon \), producing expected payoff \( \frac{1}{3} - \frac{\epsilon}{6} \). Since this is the same expected payoff as playing \( \sigma^*_{\epsilon, 1} \), player 1 cannot profitably deviate from \( \sigma^*_{\epsilon, 1} \). So the strategy profile where player 1 follows \( \sigma^*_{\epsilon, 1} \) and player 2 follows \( \mu \) is a Nash equilibrium of \( G_\epsilon \). By taking the limit as \( \epsilon \to 0 \) it follows that \( \mu \) is a trembling-hand perfect equilibrium of the original game \( G \).

Now let \( \mu_\alpha \) denote the Nash equilibrium strategy profile where player 2 calls vs. a bet of 1 with probability \( \alpha \), where \( \frac{1}{2} \leq \alpha < \frac{2}{3} \). Suppose that player 1 plays strategy \( \phi \) against this strategy in \( G_\epsilon \). Then player 1’s expected payoff is:

\[ \frac{1}{2} \left( (1 - \phi(W, 1)) \left( \frac{1}{3}(2.5) + \frac{2}{3}(0.5) \right) + \phi(W, 1) (\alpha(1.5) + (1 - \alpha)(0.5)) \right) \]
\[ + \frac{1}{2} \left( \phi(L, 2) \left( \frac{1}{3}(-2.5) + \frac{2}{3}(0.5) \right) + \phi(L, 1) (\alpha(-1.5) + (1 - \alpha)(0.5)) \right) \]
\[ + \frac{1}{2} ((1 - \phi(L, 2) - \phi(L, 1))(-0.5)) \]
\[ = \frac{1}{2} \left( (1 - \phi(W, 1)) \left( \frac{7}{6} \right) + \phi(W, 1) (\alpha + 0.5) \right) \]
\[ + \frac{1}{2} \left( \phi(L, 2) (-0.5) + \phi(L, 1) (-2\alpha + 0.5) + (1 - \phi(L, 2) - \phi(L, 1))(-0.5) \right) \]
\[ = \frac{1}{2} \left( \phi(W, 1) \left( \alpha - \frac{2}{3} \right) + \frac{7}{6} + \phi(L, 1)(-2\alpha + 1) - 0.5 \right) \]
\[ = \frac{1}{2} \left( \phi(W, 1) \left( \alpha - \frac{2}{3} \right) + \phi(L, 1)(-2\alpha + 1) + \frac{2}{3} \right) \]

For \( \alpha \in \left( \frac{1}{2}, \frac{2}{3} \right) \), this is maximized in \( G_\epsilon \) with \( \phi(W, 1) = \epsilon \), \( \phi(L, 1) = \epsilon \). However, if \( \phi(W, 1) = \phi(L, 1) = \epsilon \), then player 2’s expected payoff of calling vs. a bet of 1 will be \( \epsilon(-1.5) + \epsilon(1.5) = 0 \), while the expected payoff of folding vs. a bet of 1 is -0.5. So player 2 will strictly prefer to call vs. a bet of 1, and can profitably deviate from \( \mu_\alpha \). So there is no Nash equilibrium of \( G_\epsilon \) in which player 2 plays \( \mu_\alpha \), and therefore there cannot be a sequence of equilibria converging to \( \mu_\alpha \) in \( G \) as \( \epsilon \to 0 \). So \( \mu_\alpha \) is not trembling-hand perfect.

Now suppose that \( \alpha = \frac{1}{2} \). Then the expected payoff is maximized in \( G_\epsilon \) with \( \phi(W, 1) = \epsilon \), while \( \phi(L, 2) \) can be any value at least \( \epsilon \), since the coefficient is \(-2\alpha + 1 = 0 \). Against this strategy, player 2’s expected payoff of calling vs. a bet of 1 is

\[ \epsilon(-1.5) + \phi(L, 2)(1.5) \geq \epsilon(-1.5) + \epsilon(1.5) = 0. \]

And as before the expected payoff of folding vs a bet of 1 is -0.5. So again player 2 will strictly prefer to call vs. a bet of 1, and \( \mu_\alpha \) is not trembling-hand perfect.

\begin{flushright}
\( \square \)
\end{flushright}

**Corollary 3.** In the no-limit clairvoyance game with \( n = 2 \), the unique quasi-perfect equilibrium strategy for player 2 is to call vs. a bet of 1 with probability \( \frac{2}{3} \). This is also the unique one-sided quasi-perfect equilibrium strategy for player 2.
Proposition 7. In the no-limit clairvoyance game with \( n = 2 \), the unique observable perfect equilibrium strategy for player 2 is to call vs. a bet of 1 with probability \( \frac{5}{9} \).

Proof. Let \( \mu \) denote the Nash equilibrium strategy profile where player 2 calls vs. a bet of 1 with probability \( \frac{5}{9} \). Consider the game \( G_\epsilon \) where the sum of the probability that player 1 bets 1 with a winning hand and with a losing hand is at least \( \epsilon \). Consider the following strategy \( \sigma^*_\epsilon,1 \) for player 1. With a winning hand player 1 bets 2 with probability \( 1 - \frac{2\epsilon}{3} \) and bets 1 with probability \( \frac{2\epsilon}{3} \). With a losing hand player 1 bets 2 with probability \( \frac{2(3 - 2\epsilon)}{9} \), bets 1 with probability \( \frac{\epsilon}{3} \) and bets 0 with remaining probability \( 1 - \frac{2(3 - 2\epsilon)}{9} - \frac{\epsilon}{3} = \frac{3 + \epsilon}{9} \).

Consider whether player 1 can profitably deviate to strategy \( \phi \) from this strategy when player 2 follows \( \mu \). First, we calculate expected payoff for player 1 of playing \( \sigma^*_\epsilon,1 \) against \( \mu \):

\[
\frac{1}{2} \left( \frac{1}{3} (1 - \frac{2\epsilon}{9}) \left( \frac{1}{3} (2.5) + \frac{2}{3} (0.5) \right) + \frac{2\epsilon}{3} \left( \frac{5}{9} (1.5) + \frac{4}{9} (0.5) \right) \right) \\
+ \frac{1}{2} \left( \frac{2(3 - 2\epsilon)}{9} \left( \frac{1}{3} (-2.5) + \frac{2}{3} (0.5) \right) + \frac{\epsilon}{3} \left( \frac{5}{9} (-1.5) + \frac{4}{9} (0.5) \right) + \frac{3 + \epsilon}{9} (-0.5) \right) \\
= \frac{1}{2} \left( \frac{2\epsilon}{27} + \frac{7}{6} \right) + \frac{1}{2} \left( -\frac{1}{3} + \frac{2\epsilon}{9} - \frac{1}{3} + \frac{11\epsilon}{18} - \frac{1}{6} - \frac{\epsilon}{18} \right) \\
= \frac{1}{3} - \frac{\epsilon}{18}
\]

If instead player 1 plays \( \phi \), expected payoff is

\[
\frac{1}{2} \left( (1 - \phi(W, 1)) \left( \frac{7}{6} \right) + \phi(W, 1) \left( \frac{19}{18} \right) \right) \\
+ \frac{1}{2} \left( \phi(L, 2) (-0.5) + \phi(L, 1) \left( -\frac{11}{18} \right) + (1 - \phi(L, 2) - \phi(L, 1))(-0.5) \right) \\
= \frac{1}{2} \left( \frac{7}{6} - \frac{\phi(W, 1)}{9} \right) + \frac{1}{2} \left( -0.5 - \frac{\phi(L, 1)}{9} \right) \\
= \frac{1}{3} - \frac{\phi(W, 1)}{18} - \frac{\phi(L, 1)}{18}
\]

Given the constraint that \( \phi(W, 1) + \phi(L, 1) \geq \epsilon \), this is maximized at \( \phi(W, 1) + \phi(L, 1) = \epsilon \), producing expected payoff \( \frac{1}{3} - \frac{\epsilon}{18} \). Since this is the same expected payoff as playing \( \sigma^*_\epsilon,1 \), player 1 cannot profitably deviate from \( \sigma^*_\epsilon,1 \). So the strategy profile where player 1 follows \( \sigma^*_\epsilon,1 \) and player 2 follows \( \mu \) is a Nash equilibrium of \( G_\epsilon \). By taking the limit as \( \epsilon \to 0 \) it follows that \( \mu \) is an observable perfect equilibrium of the original game \( G \).

Now let \( \mu_\alpha \) denote the Nash equilibrium strategy profile where player 2 calls vs. a bet of 1 with probability \( \alpha \), where \( \frac{1}{2} \leq \alpha \leq \frac{2}{3} \). Suppose that player 1 plays strategy \( \phi \) against this strategy in \( G_\epsilon \). Then player 1’s expected payoff is:

\[
\frac{1}{2} \left( (1 - \phi(W, 1)) \left( \frac{1}{3} (2.5) + \frac{2}{3} (0.5) \right) + \phi(W, 1) (\alpha (1.5) + (1 - \alpha)) (0.5) \right)
\]
\[
+ \frac{1}{2} \left( \phi(L, 2) \left( \frac{1}{3}(-2.5) + \frac{2}{3}(0.5) \right) + \phi(L, 1) \left( \alpha(-1.5) + (1 - \alpha)(0.5) \right) \right) \\
+ \frac{1}{2} \left( (1 - \phi(L, 2) - \phi(L, 1))(-0.5) \right) \\
= \frac{1}{2} \left( (1 - \phi(W, 1)) \left( \frac{7}{6} \right) + \phi(W, 1) (\alpha + 0.5) \right) \\
+ \frac{1}{2} \left( \phi(L, 2) \left(-0.5\right) + \phi(L, 1) \left(-2\alpha + 0.5\right) + (1 - \phi(L, 2) - \phi(L, 1))(-0.5) \right) \\
= \frac{1}{2} \left( \phi(W, 1) \left( \alpha - \frac{2}{3} \right) + \frac{7}{6} + \phi(L, 1)(-2\alpha + 1) - 0.5 \right) \\
= \frac{1}{2} \left( \phi(W, 1) \left( \alpha - \frac{2}{3} \right) + \phi(L, 1)(-2\alpha + 1) + \frac{2}{3} \right)
\]

We want to maximize this subject to \(\phi(W, 1) + \phi(L, 1) \geq \epsilon\). For \(\frac{1}{2} < \alpha < \frac{5}{9}\), it is maximized by \(\phi(W, 1) = 0, \phi(L, q) = \epsilon\). Against this strategy, player 2’s expected payoff of calling vs. a bet of 1 is strictly larger than folding, so player 2 can profitably deviate from \(\mu_\alpha\). So there is no Nash equilibrium of \(G_\epsilon\) in which player 2 plays \(\mu_\alpha\), and therefore there cannot be a sequence of equilibria converging to \(\mu_\alpha\) in \(G\) as \(\epsilon \to 0\). So \(\mu_\alpha\) is not observable perfect.

For \(\frac{5}{9} < \alpha < \frac{2}{3}\), the expression is maximized by \(\phi(W, 1) = \epsilon, \phi(L, q) = 0\). Against this strategy, player 2 strictly prefers to fold vs. a bet 1 of than to call. So \(\mu_\alpha\) is not observable perfect.

For \(\alpha = \frac{1}{2}\), the expression is maximized by \(\phi(W, 1) = 0\) and \(\phi(L, q) \geq \epsilon\). Against this strategy, player 2 strictly prefers to call vs. a bet 1 of than to fold. So \(\mu_\alpha\) is not observable perfect.

For \(\alpha = \frac{2}{3}\), the expression is maximized by \(\phi(W, 1) \geq \epsilon\) and \(\phi(L, q) = 0\). Against this strategy, player 2 strictly prefers to fold vs. a bet 1 of than to call. So \(\mu_\alpha\) is not observable perfect.

The OPE solution can be motivated intuitively by the following reasoning. First, note that if player 1 bets the optimal size of 2 with a winning hand against an equilibrium strategy (which calls a bet of 2 with probability \(\frac{1}{3}\)), expected payoff is

\[
\frac{1}{3}(3) + \frac{2}{3}(1) = \frac{5}{3}.
\]

If player 1 bets 1 with a winning hand against a strategy that calls with probability \(\alpha\), expected payoff is

\[
\alpha(2) + (1 - \alpha)(1) = \alpha + 1.
\]

The difference between these two values is

\[
\frac{5}{3} - (\alpha + 1) = \frac{2}{3} - \alpha.
\]

If player 1 bets 2 with a losing hand against an equilibrium strategy, expected payoff is

\[
\frac{1}{3}(-2) + \frac{2}{3}(1) = 0.
\]

Similarly, if player 1 bets 0 with a losing hand against an equilibrium strategy, expected payoff is 0. If player 1 bets 1 with a losing hand against a strategy that calls with probability \(\alpha\), expected payoff is

\[
\alpha(-1) + (1 - \alpha)(1) = 1 - 2\alpha.
\]

The difference between these two values is

\[
0 - (1 - 2\alpha) = 2\alpha - 1.
\]
These two differences are equal iff
\[
\frac{2}{3} - \alpha = 2\alpha - 1
\]
\[
3\alpha = \frac{5}{3}
\]
\[
\alpha = \frac{5}{9}.
\]

So the OPE is the unique equilibrium where player 1 loses the same amount of expected payoff with both types of mistakes (betting 1 with a winning hand and betting 1 with a losing hand). In all other equilibria, player 1 loses more by making one type of mistake over the other. So in the setting where we have no information regarding the type of the mistake the opponent is making (i.e., we have a uniform prior distribution over the opponent’s type of mistake), the OPE solution is the most sensible.

4 Conclusion

We presented a new solution concept for sequential imperfect-information games called observable perfect equilibrium that captures the assumption that all players are playing as rationally as possible given the fact that some players have taken observable suboptimal actions. We believe that this is more compelling than other solution concepts that assume that one or all players make certain types of mistakes for all other actions including those that have not been observed. We showed that every observable perfect equilibrium is a Nash equilibrium, which implies that observable perfect equilibrium is a refinement of Nash equilibrium. We also showed that observable perfect equilibrium is always guaranteed to exist. We showed that an OPE can be computed in polynomial time in two-player zero-sum games based on repeatedly solving a linear program formulation. We also argued that computation of OPE is more efficient than computation of the related concept of one-sided quasi-perfect equilibrium, which in turn has been shown to be more efficient than computation of quasi-perfect equilibrium and extensive-form trembling-hand perfect equilibrium.

We demonstrated that observable perfect equilibrium leads to a different solution in no-limit poker than EFTHPE, QPE, and OSQPE. While we only considered a simplified game called the no-limit clairvoyance game, this game encodes several elements of the complexity of full no-limit Texas hold ‘em, and in fact conclusions from this game have been incorporated into some of the strongest agents for no-limit Texas hold ‘em. So we expect our analysis to extend to significantly more complex settings than the example considered. We think that observable perfect equilibrium captures the theoretically-correct solution concept for sequential imperfect-information games, and furthermore is more practical to compute than other solutions.

Future work should explore the scalability of the algorithm we have presented. While algorithms based on solving linear programs run in polynomial time, they often run into memory and speed issues that prevent them from being competitive with algorithms such as counterfactual regret minimization and fictitious play on extremely large games. Perhaps these algorithms can be modified in such a way that they are guaranteed to converge to an observable perfect equilibrium in two-player zero-sum games (currently they are just guaranteed to converge to Nash equilibrium and not to any specific refinement).

While the focus of this paper has been on two-player games, and two-player zero-sum games specifically, the concept of observable perfect equilibrium is generally applicable to multiplayer games and non-zero-sum games as well. It has been argued that one-sided quasi-perfect equilibrium is inappropriate for non-zero-sum (and multi-player) games and should not be used [9]. By contrast, we see no reason why observable perfect equilibrium cannot be applied to these games with the same theoretical and computational advantages as for the two-player zero-sum setting. In the future we would like to study applicability of observable perfect equilibrium to these settings and develop new algorithms for its computation.
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