Differential Calculus and Integration of Generalized Functions over Membranes

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Abstract

In this paper we continue the development of the differential calculus started in [2]. Guided by the topology introduced in [3] and [4] we introduce the notion of membranes and extend the definition of integrals, given in [2], to integral defined on membranes. We use this to prove a generalized version of the Cauchy formula and to obtain the Goursat Theorem for generalized holomorphic functions. We also show that the generalized transport equation can be solved giving an explicit solution.

1 Introduction

The theory of Colombeau generalized functions developed rapidly during the last years. It has useful applications and gives new inside where the classical theory does not (see [14]).

Having the algebraic theory ([5]) as a starting point, Aragona-Fernandez-Juriaans have developed a differential calculus which allows to introduce most notions of differential calculus and geometry into this context. Using the algebraic and differential theories, Aragona-Fernandez-Juriaans ([2]) were able to generalize a result of [7] on the existence of solutions for linear PDE’s. In [6] these algebraic and differential theories were also used to study algebraic properties of the algebra of Colombeau generalized functions.

In this paper we continue the development of the calculus started in [2]. After defining what we mean by an n-dimensional membrane we define the integral of a generalized function over a membrane. We then proceed to apply these notions and results. Among these applications are the Cauchy Formula in the context of generalized holomorphic functions and examples which show that for some linear operators the equation $L(u) = f$ can be solved explicitly.

Basic references for the theory of Colombeau generalized numbers, functions and their topologies are [1], [8], [10], [11], [12], [13], [15] and [18].

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2 Differential Calculus

Let $\Omega \subset \mathbb{R}^n$ be an open subset, $I = [0, 1]$ and $I_\eta := [0, \eta[\, for each $\eta \in I$. As usual, $\kappa$ denotes indistinctly $\mathbb{R}$ or $\mathbb{C}$. The definitions of the algebra of the simplified generalized functions, $G(\Omega)$, and the ring of the simplified generalized numbers, $\kappa$, are the ones given in [5].

In this section we continue the theory developed in [2] and we shall use results and notation from [2] and [5]. We remind that $\kappa : G(\Omega) \to C^\infty(\tilde{\Omega}_c, \kappa)$ is the embedding introduced in [2] and that the function

$$ (G(\Omega))^p \ni (f_1, \ldots, f_p) \longmapsto (\kappa f_1, \ldots, \kappa f_p) \in C^\infty(\tilde{\Omega}_c, \kappa)^p $$

will also be denoted by $\kappa$.

The results here presented can be proved using the same arguments of their classical analog observing that

$$ \lim_{x \to x_0} \frac{r(x)}{\alpha \log ||x-x_0||} = 0 \iff \lim_{x \to x_0} \frac{||r(x)||}{||x-x_0||} = 0, $$

where $r : A \to \mathbb{R}^n$ is a function, $A$ is an open subset of $\mathbb{R}^m$ and $x_0 \in A$.

**Theorem 2.1 (Chain Rule)** Let $U$ be an open subset of $\mathbb{R}^m$, $V$ an open subset of $\mathbb{R}^k$, $f : U \to V$ a function differentiable at $x_0 \in U$ and $g : V \to \mathbb{R}^s$ differentiable at $f(x_0)$. Then $g \circ f$ is differentiable at $x_0$ and $D(g \circ f)(x_0) = Dg(f(x_0)) \circ Df(x_0)$.

**Theorem 2.2** Let $U$ and $V$ be open subsets of $\mathbb{R}^n$, $f : U \to V$ a function with inverse $g : V \to U$. If $f$ is differentiable at $x_0 \in U$, $\det(Df(x_0)) \in \text{Inv}(\kappa)$ and $g$ is continuous in $y_0 := f(x_0)$, then $g$ is differentiable at $y_0$.

We now announce the two most classical theorems of differential calculus.

**Theorem 2.3 (Inverse Function Theorem)** Let $\Omega$ be an open and convex subset of $\mathbb{R}^n$, $f \in (G(\Omega))^n$ and $x_0 \in \tilde{\Omega}_c$ such that $\det(D\kappa(f))(x_0)) \in \text{Inv}(\kappa)$. Then there exist $U$ and $V$ open subsets of $\mathbb{R}^n$ such that $x_0 \in U$, $(\kappa(f))(x_0) \in V$ and $\kappa(f) : U \to V$ is a $C^\infty$-diffeomorphism.

**Theorem 2.4 (Implicit Function Theorem)** Let $\Omega$ be an open and convex subset of $\mathbb{R}^m \times \mathbb{R}^k$, $f \in (G(\Omega))^k$, $(x_0, y_0) \in \tilde{\Omega}_c$ such that $\kappa(f)(x_0, y_0) = 0$ and $\det(D\kappa(f))(x_0, y_0)) \in \text{Inv}(\kappa)$. Then there exist $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^k$ with $(x_0, y_0) \in U \times V \subset \tilde{\Omega}_c$ such that for all $x \in U$ there is a unique $g(x) \in V$ with $(\kappa(f))(x, g(x)) = 0$. Moreover the function $g : x \in U \mapsto g(x) \in V$ is $C^\infty$ and $Dg(x) = [D\kappa(f))(x, g(x))]^{-1}[−Dx(\kappa(f))(x, g(x))]$, for all $x \in U$.

Since the proofs do not differ much from the classical ones we omit them all.
Remark 2.5 Let $\Omega := [0\,\,1]\times \Omega \rightarrow \mathbb{R}$ the function defined by $f(x) = \alpha_1 x$. Then $f = \kappa F$, where $F = \{(x, y) \in [0, 1] \times \Omega \mapsto y\}$ has inverse $g = f^{-1}$, $g(x) = \alpha_{1} x$, but there do not exist $V$, an open subset of $\mathbb{R}$, and $h \in \mathcal{G}(V)$ such that $\kappa(h) = f^{-1}$.

The problem here is that there does not exists an open subset $V \subset \mathbb{R}$ such that $g(V_c) \subset \Omega_{c} \subset B_{1}(0)$. For if this were the case then, for $x_0 \in V \setminus \{0\}$ and $y_0 = (x_0)$, we would have that $\|g(y_0)\| = e\|y_0\| = e$ and hence $g(y_0) \notin B_{1}(0)$. Hence we may conclude that there does not exists an open subset $V \subset \mathbb{R}$ and $h \in \mathcal{G}(V)$ such that $g = \kappa(h)$. Indeed, if they did exist, then for the composition of $F$ and $h$ to make sense we must have that $h \in \mathcal{G}(V_c) = \mathcal{G}(V_c) = \mathcal{G}(V_c)$ and hence, $g$ would be defined on $V_c$ and so $\text{Im}(g) \subset \Omega_c$. This proves that $F$ does not have an inverse as a generalized function.

In the next section we introduce a way in which an inverse of the function defined above exists without appealing to the inverse defined in $\mathbb{R}$.

### 3 Integration on Membranes

Given $x_0 \in \mathbb{R}$ and $0 < r \in \mathbb{R}$ we define $V_r[x_0] := \{x \in \mathbb{R} | |x - x_0| \leq \alpha_r\}$, where the definition of absolute value was introduced in $[6]$ and $\alpha_r$ was defined in $[5]$. It is proved in $[3]$ and $[4]$ that $\{V_r[x_0] | 0 < r \in \mathbb{R}, x_0 \in \mathbb{R}\}$ is a basis of a topology in $\mathbb{R}$ which coincides with Scarpalezos’ sharp topology. It is easy to verify that $x \in V_r[x_0]$ if and only if there exist representatives $(x_0)$, $(x_0c)$ of $x$ and $x_0$ respectively such that $x_0 \in B_{r}(x_0c), \forall \in I$. So if $x_0 \in \Omega_{c}$, $(x_0c)$ is a representative of $x_0$ and $(M_r) := (B_{r}(x_0c) \cap \Omega_{c})$, then $V_r[x_0] = \{(x_0c) | x_0c \in M_r\}$.

Based on this fact we introduce the notion of membranes which will allow us to effectively integrate generalized functions.

We first start by defining what the subsets are over which we will be integrating generalized functions and then define how to integrate over these sets.

**Definition 3.1** We denote by $\mathcal{P}(\Omega)_M$ the family of subsets $(M_r)$ such that

1. $\exists K \subset \Omega$ a compact subset and $\eta \in I$ such that $M_r \subset K$ for $\in I_{\eta}$;

2. the characteristic function of $M_r$ is Riemann integrable for all $\in I_{\eta}$.

Any element of $\mathcal{P}(\Omega)_M$ is called a $n-$dimensional pre-membrane in $\Omega$.

Note that 1. implies that $\{(x_0c) | x_0c \in M_r, \forall \in I\} \subset \Omega_{c}$.

**Definition 3.2** Let $\gamma = (\gamma_{\epsilon})$ be a family of elements of $C^1([0, 1], \mathbb{R}^n)$. It is called a $n$-dimensional history or just history if $(\gamma_{\epsilon}([0, 1]))$ is a pre-membrane and there are $N \in \mathbb{N}, c > 0$ and $\eta \in I$ such that $|\gamma_{\epsilon}'(t)| \leq ce^{-N}, \forall \in I_{\eta}$ and $t \in [0, 1]$. The pre-membrane $(\gamma_{\epsilon}([0, 1]))$ is denoted by $\gamma^*.$

Two elements $(M_r)$ and $(M'_r)$ of $\mathcal{P}(\Omega)_M$ are said to be equivalent if there exists a null-function $(\Psi_{\epsilon}) \in \mathcal{N}(\Omega; \mathbb{R}^n)$ such that the function $\phi$ defined on $I \times \Omega$
So we can define the function \( \text{vol}(\cdot) \) satisfies \( \text{vol}(\cdot) = M'_{\varepsilon} \), \( \forall \varepsilon \in I \). This clearly defines an equivalence relation on \( P(\Omega)_M \). We denote the quotient space by \( P(\Omega)_M/\sim \) and call its elements \( n\)-dimensional membranes in \( \Omega \) or just membranes. It is easy to verify that if \( (M_{\varepsilon}) \) and \( (M'_{\varepsilon}) \) are equivalent and \( \gamma = (\gamma_{\varepsilon}) \) is a history such that \( \gamma^*(M_{\varepsilon}) \), then \( (\beta_{\varepsilon}) \) is a history and \( \beta^*(M'_{\varepsilon}) \), where \( \beta_{\varepsilon}(t) := \phi(\varepsilon, \gamma_{\varepsilon}(t)) \).

Note that if \( (M_{\varepsilon}) \) and \( (M'_{\varepsilon}) \) are equivalent pre-membranes then
\[
\{(x_{\varepsilon}) \mid x_{\varepsilon} \in M_{\varepsilon}, \forall \varepsilon \in I\} = \{(y_{\varepsilon}) \mid y_{\varepsilon} \in M'_{\varepsilon}, \forall \varepsilon \in I\},
\]
so we can define the function
\[
j : P(\Omega)_M/\sim \ni [(M_{\varepsilon})] \mapsto \{(x_{\varepsilon}) \mid x_{\varepsilon} \in M_{\varepsilon}, \forall \varepsilon \in I\} \subset \tilde{\Omega}_{\varepsilon}.
\]

Since \( \text{vol}(M_{\varepsilon}) \) is uniformly bounded for small \( \varepsilon \), we can define \( \text{vol}(X) \) by \( \text{vol}(X) := [\varepsilon \to \text{vol}(M_{\varepsilon})] \).

From here on we shall write \( X = [(M_{\varepsilon})] \) instead of \( X = j\left((M_{\varepsilon})\right) \). If we drop condition 2. of definition 3.1 we will speak of a pseudo-membrane, i.e., a family of subsets satisfying only the first condition of definition 3.1 will be called a pseudo-membrane. The image by \( j \) of a pseudo-membrane shall still be called a pseudo-membrane.

**Lemma 3.3** Let \( (M_{\varepsilon}) \) be a pseudo-membrane. Then \( j([(M_{\varepsilon})]) = j([(M_{\varepsilon})]) \), where \( \overline{M_{\varepsilon}} \) is the topological closure of \( M_{\varepsilon} \).

**Proof.** It is enough to prove that \( j([(M_{\varepsilon})]) \subset j([(M_{\varepsilon})]) \). Choose \( x \in j([(M_{\varepsilon})]) \); then \( x = [(x_{\varepsilon})] \), with \( x_{\varepsilon} \in \overline{M_{\varepsilon}} \). For each \( x_{\varepsilon} \) we may choose \( y_{\varepsilon} \in M_{\varepsilon} \) such that \( |x_{\varepsilon} - y_{\varepsilon}| < \exp(\frac{-1}{\varepsilon}) \). Since \( \exp(\frac{-1}{\varepsilon}) \) is a null-element we are done.

If \( x, y \) are points in \( \mathbb{K}^n \) we define the *generalized distance* between them as
\[
d(x, y) = \text{dist}(x, y) := [\varepsilon \to \text{dist}(x_{\varepsilon}, y_{\varepsilon})].
\]
This is a well defined element of \( \mathbb{R} \). We recall also that in [3] it is proved that if \( w \in \mathbb{K} \) is a non-zero element then there exists an idempotent \( e \in \mathbb{K} \) and \( r \in \mathbb{R} \) such that \( e \cdot |w| > e \cdot \alpha r \). If \( x_0 \in \mathbb{K}^n \) then \( V_r[x_0] = \{x \in \mathbb{K}^n \mid d(x, x_0) < \alpha r\} \). In [3] it is proved that these sets are a basis of neighborhood of the sharp topology of \( \mathbb{K}^n \). We will use these observations in what follows.

**Proposition 3.4** Pseudo-membranes are closed in the sharp topology. Moreover, if \( M = j([(M_{\varepsilon})]) \) and \( M_{\varepsilon} \) is convex, for small \( \varepsilon \), then \( M \) is not open.

**Proof.** Let \( M \) be a pseudo-membrane and choose \([(x_{\varepsilon})] = x \notin M \). We may suppose, by Lemma 3.3 that \( M = j([(M_{\varepsilon})]) \), with all \( M_{\varepsilon} \) closed. It follows that \( \text{dist}(x, M) = d := [\varepsilon \to \text{dist}(x_{\varepsilon}, M_{\varepsilon})] \) is a non-zero element of \( \mathbb{R} \) and hence there exists an idempotent \( e \in \mathbb{K} \) and \( r \in \mathbb{R} \) such that \( e \cdot |w| > e \cdot \alpha r \). Now let \( r < s \) and \( y \in V_s[x] \cap M \). Then, \( d \leq \text{dist}(x, y) < \alpha s < \alpha r \) and thus \( e \cdot \alpha e < e \cdot d \leq e \cdot \text{dist}(x, y) < e \cdot \alpha s < e \cdot \alpha r \), a contradiction.

Let \( x \in M \) be an interior point whose representative \((x_{\varepsilon})\) satisfies \( x_{\varepsilon} \in \partial M_{\varepsilon} \). Since \( x \) is an interior point, there exists \( r \in \mathbb{R} \) such that \( V_r[x] \subset M \). Since the
$M_{\varepsilon}$’s are convex, we may choose points $y_{\varepsilon}$ of norm 1 such that $z_{\varepsilon} := x_{\varepsilon} + \varepsilon^ny_{\varepsilon}$ satisfies $d(z_{\varepsilon}, M_{\varepsilon}) \geq \varepsilon^r$. Set $z = [(z_{\varepsilon})]$; then $\text{dist}(z, M) \geq \alpha_\varepsilon$ and hence $z \not\in M$. On the other hand $z \in V_r[x] \subset M$, a contradiction.

As a corollary we have the following result.

**Corollary 3.5** Let $(M_n)$ be a decreasing sequence of pseudo-membranes with diameters tending to zero. Then $\bigcap_{n \in \mathbb{N}} M_n$ consists of a single point.

**Proof.** This is clear since $\mathbb{R}^n$ is a complete metric space and pseudo-membranes are closed. 

Observe that the element $x = [\{(1/\varepsilon)\}] \approx 0$ and has norm 1. It follows that $B_1(0)$ is a proper subset of $\overline{\mathcal{M}}_0 = \{ x \in \mathbb{R}^n \mid x \approx 0 \}$. Observe also that $M = B_1(0)$ is not a membrane: in fact suppose that $M = \{ M_{\varepsilon} \}$ and let $N_{\varepsilon}$ be the convex hull of $M_{\varepsilon}$. As proved above, we may suppose that $M_{\varepsilon}$ is closed for all $\varepsilon$. As is easily seen, $N = \{ N_{\varepsilon} \}$ is contained in the convex hull of $M$. Since $M$ is a subring of $\mathbb{R}$, it follows that it equals its convex hull. It follows that $M = N$ and hence is not open because all the $N_{\varepsilon}$’s are convex, a contradiction.

**Example 3.6** Let $x \in \mathbb{R}^n$, $r \in \text{Inv}(\mathbb{R})$ and let $(x_{\varepsilon})$, $(x'_{\varepsilon})$ be representatives of $x$ and $y_{\varepsilon}$, $(r_{\varepsilon})$, $(r'_{\varepsilon})$ be representatives of $r$. Consider the pre-membranes $M_{\varepsilon} := B_{r_{\varepsilon}}(x_{\varepsilon})$ and $M'_{\varepsilon} := B_{r'_{\varepsilon}}(x'_{\varepsilon})$. Define $(\Psi_{\varepsilon})$ by

$$
\Psi_{\varepsilon}(y) := \frac{r'_{\varepsilon} - r_{\varepsilon}}{r_{\varepsilon}}(y - x_{\varepsilon}) + \frac{r_{\varepsilon} - r'_{\varepsilon}}{r_{\varepsilon}}(x'_{\varepsilon} - x_{\varepsilon}).
$$

Then $(\Psi_{\varepsilon}) \in \mathcal{N}(\mathbb{R}^n)$ and $\phi(\varepsilon, w) := w + \Psi_{\varepsilon}(w)$, $\forall (\varepsilon, w) \in I \times \mathbb{R}^n$, satisfies $\phi(\varepsilon, M_{\varepsilon}) = M'_{\varepsilon}$, $\forall \varepsilon \in I$. Hence $(M_{\varepsilon})$ and $(M'_{\varepsilon})$ are equivalent. When $r = \alpha_s$, then $j(\{M_{\varepsilon}\})$ is just $V_s[x]$. Its volume is $\text{vol}(V_s[x]) = \pi\alpha_s^2 = \pi\alpha_{2s}$.

For this reason we call $V_s[x]$ a *generalized ball* whose center is $x$ and whose radius is $\alpha_s$. By a *generalized sphere* we shall mean a set of the form $\{ x \in \Omega_{\varepsilon} \mid ||x - x_0|| = \alpha_s \}$, for some $s \in \mathbb{R}$ and $x_0 \in \Omega_{\varepsilon}$.

We are now in position to define a notion of integration of generalized functions that is consistent with the differential calculus we have developed so far. For this we need the following result.

**Proposition 3.7** Let $f \in \mathcal{G}(\Omega)$, $(f_{\varepsilon})$ a representative of $f$ and $\lambda$ the Lebesgue measure on $\mathbb{R}^n$.

1. If $(M_{\varepsilon})$ is a pre-membrane, then the function $\varepsilon \mapsto \int_{M_{\varepsilon}} f_{\varepsilon} \, d\lambda$ is moderate (is null if $(f_{\varepsilon})$ is null).

2. If $(M_{\varepsilon})$ and $(M'_{\varepsilon})$ are equivalent pre-membranes, then

$$
[\varepsilon \mapsto \int_{M_{\varepsilon}} f_{\varepsilon} \, d\lambda] = [\varepsilon \mapsto \int_{M'_{\varepsilon}} f_{\varepsilon} \, d\lambda].
$$
3. If \((M_e)\) and \((M'_e)\) are equivalent pre-membranes and \((g_e)\) is a representative of \(f\), then \(\varepsilon \mapsto \int_{M_e} f_e \, d\lambda = \varepsilon \mapsto \int_{M'_e} g_e \, d\lambda\).

**Proof.** The assertion 1. is obvious. For 2. and 3., let \(\Psi = [(\Psi_e)] \in \mathcal{N}(\Omega; \mathbb{R}^n)\) and \(\phi(\varepsilon, x) := x + \Psi_e(x), \forall (\varepsilon, x) \in I \times \Omega, \text{ such that } \phi(\varepsilon, M_e) = M'_e, \forall \varepsilon \in I.\)

Denote by \(D\phi\) the Jacobian matrix of \(\phi\) and let \(J\) be its determinant. Then we have that \(J = 1 + \tau\) with \(\tau \in \mathcal{N}(\Omega)\) and

\[
\left| \int_{M_e} f_e \, d\lambda - \int_{M'_e} f_e \, d\lambda \right| \leq \int_{M_e} |f_e(\phi(\varepsilon, x)) - f_e(\phi(\varepsilon, x))| \, dx + \int_{M'_e} |f_e(\phi(\varepsilon, x))\tau(\varepsilon, x)| \, dx.
\]

Choosing a compact subset \(K\) containing \(\cup_{\varepsilon \in I}(M_e \cup M'_e)\), we can find \(x_1, \ldots, x_s \in K\) and \(r_1, \ldots, r_s > 0\) such that

\[
K \subset L_1 := \bigcup_{1 \leq j \leq s} B_{r_j}(x_j) \subset L := \bigcup_{1 \leq j \leq s} B'_{r_j}(x_j) \subset \Omega.
\]

As \(\Psi \in \mathcal{N}(\Omega; \mathbb{R}^n)\) there is \(\eta_1 \in I\) such that \(\phi_e(B'_{r_j}(x_j)) \subset B'_{2r_j}(x_j), \forall \varepsilon \in I_{\eta_1}.\) Since \(f\) is moderate there are \(N \in \mathbb{N}, c > 0\) and \(\eta \in I_{\eta_1}\) such that \(\max \{||\nabla f(x)||, |f_e(x)|\} \leq c\varepsilon^{-N}, \forall x \in L, \forall \varepsilon \in I_{\eta_1}.\) Hence, by the Mean Value Theorem, noting that \(B'_{r_j}(x_j) \subset B'_{2r_j}(x_j)\) and \(B'_{r_j}(x_j)\) are convex for all \(1 \leq j \leq s,\) we conclude that

\[
\int_{M_e} |f_e(x) - f_e(\phi(\varepsilon, x))| \, dx \leq \sum_{1 \leq j \leq s} \int_{B'_{r_j}(x_j)} |f_e(x) - f_e(\phi(\varepsilon, x))| \, dx
\]

\[
\leq \sum_{1 \leq j \leq s} \int_{B'_{r_j}(x_j)} c\varepsilon^{-N} |\Psi(\varepsilon, x)| \, dx,
\]

\(\forall \varepsilon \in I_{\eta}.\) Using that \(\Psi \in \mathcal{N}(\Omega; \mathbb{R}^n), \tau \in \mathcal{N}(\Omega)\) and \(|f_e(\phi(\varepsilon, x))\tau(\varepsilon, x)| \leq \varepsilon^{-N}|\tau(\varepsilon, x)|, \forall \varepsilon \in I_{\eta_1}, \forall x \in L_1,\) we conclude that 2. holds. For 3. note that

\[
\left| \int_{M_e} f_e \, d\lambda - \int_{M'_e} g_e \, d\lambda \right| \leq \left| \int_{M_e} f_e \, d\lambda - \int_{M'_e} f_e \, d\lambda \right| + \left| \int_{M'_e} f_e - g_e \, d\lambda \right|.
\]

The result now readily follows from the others assertions.  \(\blacksquare\)

The proposition above guarantees that the following definition makes sense.

**Definition 3.8** Let \(f = [(f_e)] \in \mathcal{G}(\Omega), M = [(M_e)]\) a membrane of \(\mathbb{R}^n\) and \(\lambda\) the Lebesgue measure on \(\mathbb{R}^n\). The generalized number

\[
\int_M f := [\varepsilon \mapsto \int_{M_e} f_e \, d\lambda]
\]

is called the integral of \(f\) over the membrane \(M\).

**Definition 3.9** Let \(\Omega \subset \mathbb{R}^n, f = [(f_e)] \in (\mathcal{G}(\Omega))^n\) and \(\gamma = (\gamma_e)\) a history.

1. If \(K = \mathbb{R}\), then the generalized number

\[
\int_\gamma f \, d\gamma := [\varepsilon \mapsto \int_0^1 < f_e(\gamma(t)) \gamma'_e(t) > \, dt]
\]

is called the (line) integral of \(f\) along \(\gamma\), where \(< \cdot , \cdot >\) denotes the standard inner product of \(\mathbb{R}^n\).
2. If \( n = 2, \Omega \subset \mathbb{C}, \mathbb{K} = \mathbb{C} \) and \( f \in \mathcal{G}(\Omega) \), then the generalized number

\[
\int_{\gamma} f \, dz := (e \mapsto \int_{0}^{1} f_{e}(\gamma_{e}(t)) \gamma'_{e}(t) \, dt)
\]

is called the integral of \( f \) along \( \gamma \).

It is easy to verify that these definitions make sense.

Note that if \( n = 1 \) and \( ([a_{\varepsilon}, b_{\varepsilon}]) \) is a membrane, then the history \( \gamma = (\gamma_{\varepsilon}) \) where \( \gamma_{\varepsilon}(t) := a_{\varepsilon} + t(b_{\varepsilon} - a_{\varepsilon}), \forall t \in [0, 1] \), can be identified with the element \((a, b) := (\langle [a_{\varepsilon}], [b_{\varepsilon}] \rangle \rangle) \in \mathbb{R}^{2\varepsilon} \) and \( \int_{\langle [a_{\varepsilon}, b_{\varepsilon}] \rangle} f = \int_{\gamma} f \, d\gamma = \int_{0}^{1} f, \) where the last integral is the one given in [2, section 4]. Conversely an element \((c, d) := (\langle c_{\varepsilon}, d_{\varepsilon} \rangle) \in \mathbb{R}^{2\varepsilon} \) defines a history \( \beta = (\beta_{\varepsilon}) \) such that \( \int_{\beta} f \, d\beta = \int_{c}^{d} f \) (it is enough to define \( \beta_{\varepsilon} = c_{\varepsilon} + t(d_{\varepsilon} - c_{\varepsilon}) \), if \( c_{\varepsilon} \leq d_{\varepsilon} \) and \( \beta_{\varepsilon} = d_{\varepsilon} + t(d_{\varepsilon} - c_{\varepsilon}) \), if \( c_{\varepsilon} > d_{\varepsilon} \), \( \forall t \in [0, 1] \). In this case, the definition given here agrees with the one given in [2, section 4].

4 Calculus on Membranes

In this section we give some applications of the theory developed in the previous section.

**Proposition 4.1** Let \( f \in \mathcal{G}(\Omega), M \) a membrane of \( \mathbb{R}^{n} \) and \( \lambda \) the Lebesgue measure on \( \mathbb{R}^{n} \). Then we have:

1. There exists \( x_{0} \in \Omega_{\varepsilon} \) such that \( \int_{M} f = \text{vol}(M) f(x_{0}). \)

2. There exists \( r \in \mathbb{R} \) such that \( |\int_{M} f \, d\lambda| \leq \text{vol}(M)\alpha_{r}. \)

**Proof.** The first item follows readily from its classical analog and the second one follows from the first one and the definition of \( \mathbb{K}. \)

In what follows \( e_{i} \) will stand for \((0, \cdots, 1, \cdots) \in \mathbb{K}^{n} \) and \( \langle \cdot, \cdot \rangle \) will denote the standard bilinear form induced by the standard inner product of \( \mathbb{K}^{n}. \)

It is easily seen that if \( f : \Omega_{\varepsilon} \to \mathbb{K} \) is differentiable at \( x_{0} \in \Omega_{\varepsilon}, \) then there exist a continuous function \( \phi \) with \( \phi(x_{0}) = 0 \) and such that \( f(x) - f(x_{0}) = \langle \nabla f(x_{0})((x-x_{0}) + \phi(x)\alpha_{-\log ||x-x_{0}||}) \rangle. \) From this and the fact that if \( \gamma : \mathbb{K}_{\varepsilon} \to \Omega_{\varepsilon} \) is differentiable at \( t_{0} \) then

\[
\frac{||\gamma(t) - \gamma(t_{0})||}{||t - t_{0}||} = \frac{||\gamma(t) - \gamma(t_{0}) - \gamma'(t_{0})(t - t_{0}) + \gamma'(t_{0})(t - t_{0})||}{||t - t_{0}||}
\]

\[
\leq \max \left\{ \frac{||\gamma(t) - \gamma(t_{0}) - \gamma'(t_{0})(t - t_{0})||}{\alpha_{-\log ||t-t_{0}||}}, \frac{||\gamma'(t_{0})(t - t_{0})||}{\alpha_{-\log ||t-t_{0}||}} \right\}
\]
\[
= \max \left\{ \left\| \frac{\gamma(t) - \gamma(t_0) - \gamma'(t_0)(t - t_0)}{\alpha - \log \|t - t_0\|} \right\|, \|\gamma'(t_0)\| \right\}
\]

We deduce, using standard techniques of differential calculus, the chain rule for curves:

**Theorem 4.2** Let \( f : \tilde{\Omega} \to \mathbb{K} \) and \( \gamma : \tilde{\mathbb{K}} \to \tilde{\Omega} \). If \( \gamma \) is differentiable at \( t_0 \) and \( f \) is differentiable at \( x_0 = \gamma(t_0) \) then \( F := f \circ \gamma \) is differentiable at \( t_0 \) and 
\[
F'(t_0) = (\nabla f(\gamma(t_0)))' \gamma'(t_0).
\]

A history \( \gamma = (\gamma_\varepsilon) \) is closed if \( \gamma_\varepsilon \) is a closed curve for small \( \varepsilon \) and \( \gamma \) is simple if for small \( \varepsilon \) we have that \( \gamma_\varepsilon \) is a simple curve. In an obvious way we define positively and negatively oriented histories. We say that \( \gamma \) is contractible if \( \gamma_\varepsilon \) is homotopic to \( 0 \) for small \( \varepsilon \).

If \( \Omega \subset \mathbb{R}^2 \) and \( f = [(f_\varepsilon)] \in (\mathcal{G}(\Omega))^2 \) then define the generalized function \( \text{rot}(f) := [(x \in \Omega \mapsto \text{rot} f_\varepsilon(x))] \). This is obviously a well defined element of \((\mathcal{G}(\Omega))^3\). We can now state the **Generalized Green Theorem**.

**Theorem 4.3 (Green’s theorem)** Let \( \Omega \subset \mathbb{R}^2 \), \( \lambda \) the Lebesgue measure on \( \mathbb{R}^2 \), \( f \in (\mathcal{G}(\Omega))^2 \) and \( \gamma = (\gamma_\varepsilon) \) a closed, simple, contractible and positively oriented history. If \( M = [(\gamma_\varepsilon([0,1]))] \), then 
\[
\int_{\gamma} f d\gamma = \int_M \langle \text{rot}(f) | e_3 \rangle d\lambda.
\]

Most other theorems of classical differential calculus can now, in a very natural way, be translated to this context. Since we have also the notion of a generalized manifold these results should also be extended to generalized manifolds.

## 5 The Generalized Cauchy Formula

In this section \( \Omega \subset \mathbb{C} \) and \( \mathbb{K} = \mathbb{C} \). Let \( \gamma = (\gamma_\varepsilon) \) be a closed, simple, contractible history and \( z_0 = [(z_{0\varepsilon})] \in \overline{\mathbb{C}} \) such that \( z_{0\varepsilon} \) belong to the bounded connected component of \( \Omega \setminus \gamma_\varepsilon([0,1]), \forall \varepsilon \in \mathbb{I} \). As \( \gamma^* \) is a pre-membrane we can define the generalized number 
\[
d(z_0, \gamma^*) := [\varepsilon \mapsto d(z_{0\varepsilon}, \gamma_\varepsilon([0,1]))]
\]

where \( d(z_{0\varepsilon}, \gamma_\varepsilon([0,1])) \) is the distance of \( z_{0\varepsilon} \) to the set \( \gamma_\varepsilon([0,1]) \). Note that, if \( d(z_0, \gamma) \in \text{Inv}(\overline{\mathbb{C}}) \), then \( z - z_0 \in \text{Inv}(\overline{\mathbb{C}}) \) for all \( z \in [\gamma^*] \).

Let \( \Omega \) be a simply connected domain and \( f \in \mathcal{H}\mathcal{G}(\Omega) \). In [2] we proved that any such \( f \) has a convergent Taylor series. In [16] is proved that \( f \) has a representative \( (f_\varepsilon) \) such that \( f_\varepsilon \) is holomorphic for all \( \varepsilon \in \mathbb{I} \). Using this and the classical Cauchy Theorem we get the **G-Cauchy Formula**.
Theorem 5.1 (G-Cauchy Formula) Let $\Omega$ be a simply connected set, $\gamma$ be a closed, simple, contractible, positively oriented history and $z_0 = [(z_{0e})] \in \overline{\mathbb{C}}$ such that $z_{0e}$ belong to the bounded connected component of $\Omega \setminus \gamma_e([0,1])$, $\forall \varepsilon \in I$. If $d(z_0, \gamma^*) \in \text{Inv}(\mathbb{C})$ and $f = [(f_\varepsilon)] \in \mathcal{H}\mathcal{G}(\Omega)$, then

$$(\kappa(f))(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - z_0} dz := [\varepsilon \mapsto \frac{1}{2\pi i} \int_{\gamma_\varepsilon} \frac{f_\varepsilon(z)}{z - z_0} dz].$$

Now we shall prove the Goursat Theorem in this context. In [2] this theorem was proved with the condition that $f$ is sub-linear.

Theorem 5.2 (Goursat Theorem) Let $f \in \mathcal{H}\mathcal{G}(\Omega)$. Then $\kappa(f)$ is analytic in $\Omega_c$.

**Proof.** Let $z_0 = [(z_{0e})] \in \overline{\mathbb{C}}$ and $f = [(f_\varepsilon)]$. Then there are $\eta_1 \in I$ and $K \subset \Omega$ compact sets such that $z_{0e} \in K$, $\forall \varepsilon \in I_{\eta_1}$. From this we can choose $R > 0$ such that $B_R(z_{0e}) \subset \Omega$, $\forall \varepsilon \in I_{\eta_1}$. Let $\rho < r/4 < R/2$ and $\eta_2 \in I_{\eta_1}$ such that $\varepsilon^\eta < \rho$, $\forall \varepsilon \in I_{\eta_2}$. Define $\gamma_e(t) := z_{0e} + r e^{2\pi i t}$, $\forall t \in [0,1], \forall \varepsilon \in I_{\eta_2}$ and $\gamma^e := \gamma_{\eta_2/2}$, $\forall \eta_2 \leq \varepsilon \leq 1$. So $\gamma = (\gamma_e)$ is a closed, simple, contractible and positively oriented history.

Let $z = [(z_e)] \in V_{\eta_1}$. Then there is $\eta \in I_{\eta_2}$ such that $\forall t \in [0,1]$ one has $\varepsilon^\eta - |z_e - z_{0e}| \geq -\varepsilon^\eta$ and $|z_e - \gamma_e(t)| = |z_e - z_{0e} - r e^{2\pi i t}| \geq r - |z_e - z_{0e}| \geq r - 2\varepsilon^\eta \geq r - 2\rho > 2\rho$. Thus $d(z, \gamma^*) \in \text{Inv}(\mathbb{C})$. Fix $\varepsilon \in I_{\eta}$, $w \in \gamma_e([0,1])$ and note that

$$f_\varepsilon(w) = \frac{f_\varepsilon(w)}{w - z_{0e}} \sum_{n=0}^{\infty} \left(\frac{z_e - z_{0e}}{w - z_{0e}}\right)^n = \sum_{n=0}^{\infty} f_\varepsilon(w) \left(\frac{z_e - z_{0e}}{w - z_{0e}}\right)^n,$$

and so

$$2\pi i f_\varepsilon(z_e) = \int_{\gamma_e} \frac{f_\varepsilon(w)}{w - z_e} dw = \sum_{n=0}^{\infty} \int_{\gamma_e} f_\varepsilon(w) \left(\frac{z_e - z_{0e}}{w - z_e}\right)^n dw.$$

Thus

$$2\pi i (\kappa(f))(z) = \left[ \int_{\gamma} \frac{f(w)}{w - z} dw \right] = \left[ \left( \sum_{n=0}^{\infty} \int_{\gamma_e} f_\varepsilon(w) \left(\frac{z_e - z_{0e}}{w - z_{0e}}\right)^n dw \right) \right].$$

Using that $||z - z_0|| < 1$, $\gamma^*$ is a pre-membrane and $(f_\varepsilon)$ is moderate it is not difficult to prove that $\sum_{n=0}^{\infty} \int_{\gamma_e} f_\varepsilon(w) \left(\frac{z_e - z_{0e}}{w - z_{0e}}\right)^n dw$ converges (since that $\lim_{n \to \infty} \int_{\gamma_e} f_\varepsilon(w) \left(\frac{z_e - z_{0e}}{w - z_0}\right)^n dw = 0$) and that

$$\sum_{n=0}^{\infty} \int_{\gamma_e} f_\varepsilon(w) \left(\frac{z_e - z_{0e}}{w - z_{0e}}\right)^n dw = \left[ \sum_{n=0}^{\infty} \int_{\gamma_e} f_\varepsilon(w) \left(\frac{z_e - z_{0e}}{w - z_{0e}}\right)^n dw \right].$$
Hence $2\pi i(\kappa(f))(z) = \sum_{n=0}^{\infty} \int_{\gamma} \frac{f(w)}{(w-z_0)^{n+1}} dw (z-z_0)^n$, \(\forall z = [(z_\varepsilon)] \in V_{\rho}[z_0] \).

6 The Transport Equation

In this section we shall consider the transport equation with generalized coefficients. We prove that we have now all the tools to give a classical solution of this problem.

Let \(\Omega = \mathbb{R}^n \times [0, \infty[\) and \(u : \tilde{\Omega}_c \to \mathbb{R}\) be a differentiable function. Denote by \(u_t\) the partial derivative of \(u\) in the last variable and let \(\nabla u = (\nabla_x u, u_t)\), where \(\nabla_x u\) is the gradient vector of \(u\) with respect to the first \(n\) variables. For \(b \in \mathbb{R}^n\) and \(g \in C^1(\tilde{\mathbb{R}}^n_c, \mathbb{R})\) we consider the transport equation

\[
 u_t + \left< \nabla_x u | b \right> = 0 \quad \text{in} \quad \tilde{\Omega}_c, \quad u(x, 0) = g(x), \quad \forall x \in \mathbb{R}^n_c.
\]

(1)

Let \(w(x, t) := g(x - tb)\), \(\forall (x, t) \in \tilde{\Omega}_c\). By Theorem 4.2, we have that \(w\) is a solution of (1). Moreover, if \(g \in \kappa(G(\mathbb{R}^n))\) and \(v \in G(\Omega)\) is a solution of (1) in \(G(\Omega)\), then \(\kappa v = w\).

Just like in the classical case, we can get an explicit solution of the boundary value problem

\[
 u_t + \left< \nabla_x u | b \right> = \kappa(f) \quad \text{in} \quad \tilde{\Omega}_c, \quad u(x, 0) = g(x), \quad \forall x \in \mathbb{R}^n_c,
\]

(2)

where \(f \in G(\mathbb{R}^n \times ]-a, \infty[)\) for some \(a > 0\). In this case, let

\[
 w(x, t) := g(x - tb) + \int_{M_t} f(x + sb, t + s) d\lambda(s), \quad \forall (x, t) \in \tilde{\Omega}_c,
\]

where \(\lambda\) is the Lebesgue measure on \(\mathbb{R}\) and if \(t = [(t_\varepsilon)]\), then \(M_t = [([-t_\varepsilon, 0])]\). So \(w\) is a solution of (2). Besides, if \(g \in \kappa(G(\mathbb{R}^n))\) and \(v \in G(\Omega)\) is a solution of (2) in \(G(\Omega)\), then \(\kappa v = w\).

This proves that the differential calculus we developed allows us to solve the generalized transport equation just like in the classical case. Using the solution of this equation, we can solve, for \(n = 1\) and \(g, h \in G(\mathbb{R})\), the boundary value problem

\[
 u_{tt} - u_{xx} = 0 \quad \text{in} \quad \tilde{\Omega}_c, \quad u(x, 0) = (\kappa g)(x) \quad \text{and} \quad u_t(x, 0) = (\kappa h)(x), \quad \forall x \in \mathbb{R}^n_c,
\]

giving a formula for its solution just as is done in the classical case. In this case a solution is the function

\[
 w(x, t) = \frac{1}{2} [g(x + t) + g(x - t)] + \frac{1}{2} \int_{M_{xt}} h d\lambda, \quad \forall (x, t) \in \tilde{\Omega}_c,
\]

where \(\lambda\) is the Lebesgue measure on \(\mathbb{R}\) and if \(x = [(x_\varepsilon)]\) and \(t = [(t_\varepsilon)]\), then \(M_{xt} = [(x_\varepsilon - t_\varepsilon, x_\varepsilon + t_\varepsilon)]\).

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