PURE DISCRETE SPECTRUM IN SUBSTITUTION TILING SPACES

MARCY BARGE
Department of Mathematics
Montana State University
Bozeman, MT 59717, USA

SONJA ŠTIMAC
Department of Mathematics
University of Zagreb
Bijenička 30, 10 000 Zagreb, Croatia

R. F. WILLIAMS
Department of Mathematics
University of Texas
Austin, TX 78712, USA

(Communicated by J. M. Sanz-Serna)

Abstract. We introduce a technique for establishing pure discrete spectrum for substitution tiling systems of Pisot family type and illustrate with several examples.

1. Introduction

The study of Aperiodic Order received impetus in 1985 with the discovery of physical quasicrystals. These materials have enough long range order to produce a pure point X-ray diffraction spectrum yet they lack the strict periodicity of traditional crystals. Quasicrystals are modeled by self-affine tilings of Euclidean space and these tilings are effectively studied by means of an associated topological dynamical system, the tiling space. It has been established, through a beautiful circle of ideas and in considerable generality, that a pattern of points (atoms) produces pure point X-ray diffraction if and only if the associated tiling space has pure discrete dynamical spectrum ([15],[23]).

1991 Mathematics Subject Classification. Primary: 37B50, 52C22, 52C23; Secondary: 37B05, 11R06.

Key words and phrases. Pure discrete spectrum, tiling space, substitution, maximal equicontinuous factor, coincidence.

The second author was supported in part by the MZOS Grant 037-0372791-2802 of the Republic of Croatia.
We introduce in this article techniques for determining whether (or not) a tiling dynamical system has pure discrete spectrum. These techniques apply to tiling systems that arise from substitutions. In dimension one, a necessary condition for a substitution tiling system to have pure discrete spectrum is that the expansion factor of the substitution be a Pisot number ([32]). Recent results ([25]) show that if a substitution tiling system in any dimension has pure discrete spectrum, then the linear expansion associated with the substitution must be generally ‘Pisot’ in nature and our technique is, correspondingly, restricted to so-called Pisot family substitutions (see the next section for definitions).

Our approach has its origins in the balanced pair algorithm for symbolic substitutive systems initiated by Dekking ([14]). A geometric version of balanced pairs (overlap coincidences) was introduced by Solomyak ([32]) and considered for one-dimensional substitution tiling systems in [18], [3], [10], [29], [28], and [16], and for higher dimensions in [21], [22], [23], and [17]. Akiyama and Lee ([1]) have implemented overlap coincidence in computer language with an algorithm that will decide whether or not a particular substitution tiling system (with the Meyer property) has pure discrete spectrum.

Proposition 17.4 of [10] (with a generalization to the ‘reducible’ case following from Theorem 6.1 of [6]) states that it suffices to check the convergence of the balanced pair algorithm on a single balanced pair in order to determine pure discrete spectrum for a one-dimensional Pisot substitution tiling system. The main result of this article extends that result to Pisot family tiling spaces in all dimensions. (The proof of Theorem 16.3 of [10], from which the one-dimensional result follows, contains a gap that is not easily plugged by the techniques of that paper. We recover the one-dimensional case here as Corollary 2.)

In the next section we briefly review the relevant definitions and background for substitution tiling spaces. Two subsequent sections establish sufficient and necessary conditions for pure discrete spectrum and a final section provides examples in which pure discrete spectrum is established.

2. Definitions and background

The basic ingredients for an $n$-dimensional substitution are a collection $\{\rho_1, \ldots, \rho_l\}$ of prototiles, a linear expansion $\Lambda : \mathbb{R}^n \to \mathbb{R}^n$, and a substitution rule $\Phi$. Each prototile $\rho_i$ is an ordered pair $\rho_i = (C_i, i)$ in which $C_i$ is a compact and topologically regular ($C_i = \text{cl}(\text{int}(C_i))$) subset of $\mathbb{R}^n$. A translate $\rho_i + v := (C_i + v, i)$ of a prototile by a $v \in \mathbb{R}^n$ is called a tile. The support of a tile $\tau = \rho_i + v$ is the set $\text{spt}(\tau) := C_i + v$ and the interior of $\tau$ is the interior of its support: $\hat{\tau} := \text{int}(\text{spt}(\tau))$. A
collection of tiles with pairwise disjoint interiors is called a *patch*, the *support of a patch* $P$ is the union of the supports of its constituent tiles, $\text{spt}(P) := \cup_{\tau \in P} \text{spt}(\tau)$, and the *interior of a patch* $P$ is $\tilde{P} := \cup_{\tau \in P} (\tilde{\tau})$ (or sometimes $\text{int}(\tilde{P})$). A *tiling* is a patch whose support is all of $\mathbb{R}^n$.

Given a collection $\mathcal{A} = \{\rho_1, \ldots, \rho_l\}$ of prototiles, let $\mathcal{A}^*$ be the collection of all finite patches consisting of collections of translates of the prototiles. A *substitution* with linear expansion $\Lambda$ is a map $\Phi : \mathcal{A} \to \mathcal{A}^*$ with the property: $\text{spt}(\Phi(\rho_i)) = \Lambda\text{spt}(\rho_i)$ for each $i$. Such a $\Phi$ extends to a map from tiles to patches by $\Phi(\rho_i + v) := \Phi(\rho_i) + \Lambda v := \{\tau + \Lambda v : \tau \in \Phi(\rho_i)\}$ and to a map from patches to patches by $\Phi(P) := \cup_{\tau \in P} \Phi(\tau)$.

A substitution $\Phi$, as above, is *primitive* if there is $k \in \mathbb{N}$ so that $\Phi^k(\rho_i)$ contains a translate of $\rho_j$ for each $i, j \in \{1, \ldots, l\}$. That is equivalent to saying that $M^k$ has strictly positive entries, where $M = (m_{i,j})$ is the $l \times l$ transition matrix for $\Phi$: $m_{i,j}$ is the number of translates of $\rho_i$ that occur in $\Phi(\rho_j)$.

A patch $P$ is *allowed* for $\Phi$ if there is $k \in \mathbb{N}$ and $v \in \mathbb{R}^n$ so that $P \subset \Phi^k(\rho_i) + v$ for some $i$. The *tiling space* associated with a primitive substitution $\Phi$ is the set $\Omega_\Phi := \{T : T$ is a tiling and all finite patches in $T$ are allowed for $\Phi\}$ with a topology in which two tilings are close if a small translate of one agrees with the other in a large neighborhood of the origin. Let us be more precise about the topology. Given a tiling $T$, let $B_0[T] := \{\tau \in T : 0 \in \text{spt}(\tau)\}$ and, for $r > 0$, let $B_r[T] := \{\tau \in T : \text{spt}(\tau) \cap B_r(0) \neq \emptyset\}$. A neighborhood base at $T \in \Omega_\Phi$ is then $\{U_\varepsilon\}_{\varepsilon > 0}$ with $U_\varepsilon := \{T' \in \Omega_\Phi : B_{1/\varepsilon}[T'] = B_{1/\varepsilon}[T] - v$ for some $v \in B_{\varepsilon}(0)\}$. This is a metrizable topology (see, for example, [2]) and we denote by $d$ any metric that induces this topology. The map $T \mapsto \Phi(T)$ and the $\mathbb{R}^n$-action $(T,v) \mapsto T - v$ are continuous and these intertwine by $\Phi(T-v) = \Phi(T) - \Lambda v$.

A tiling $T$ has (translational) *finite local complexity* (FLC) if, for each $r \geq 0$, there are only finitely many patches, up to translation, of the form $B_r[T - v], v \in \mathbb{R}^n$. The substitution $\Phi$ is said to have finite local complexity if each of its admissible tilings has finite local complexity. Finally, $\Phi$ is *aperiodic* if, for $T \in \Omega_\Phi$ and $v \in \mathbb{R}^n, T - v = T \implies v = 0$.

Under the assumption that $\Phi$ is primitive, aperiodic, and has finite local complexity, the following facts are by now well known ([2],[30]):

- $\Omega_\Phi$ is a continuum (i.e., a compact and connected metrizable space),
- $\Phi : \Omega_\Phi \to \Omega_\Phi$ is a homeomorphism, and
- The $\mathbb{R}^n$-action on $\Omega_\Phi$ is strictly ergodic (i.e., minimal and uniquely ergodic).

All substitutions $\Phi$ in this article will be assumed to be primitive, aperiodic, and to have finite local complexity. We will denote by $\mu$ the unique translation invariant Borel probability measure on $\Omega_\Phi$. A
function $f \in L^2(\Omega_\Phi, \mu)$ is an eigenfunction of the $\mathbb{R}^n$-action with eigenvalue $b \in \mathbb{R}^n$ if $f(T - v) = e^{2\pi i (b, v)} f(T)$ for all $v \in \mathbb{R}^n$, and almost all $T \in \Omega_\Phi$. The $\mathbb{R}^n$-action on $\Omega_\Phi$ is said to have pure discrete spectrum if the linear span of the eigenfunctions is dense in $L^2(\Omega_\Phi, \mu)$. Solomyak proves in [31] that every eigenfunction is almost everywhere equal to a continuous eigenfunction. It is then a consequence of the Halmos-von Neumann theory that the $\mathbb{R}^n$-action on $\Omega_\Phi$ has pure discrete spectrum if and only if there is a compact abelian group $X$, an $\mathbb{R}^n$-action on $X$ by translation, and a continuous semi-conjugacy $g : \Omega_\Phi \to X$ of $\mathbb{R}^n$-actions that is a.e. one-to-one (with respect to Haar measure).

There are certain necessary conditions on the linear expansion $\Lambda$ for the substitution $\Phi$ in order for the $\mathbb{R}^n$-action on $\Omega_\Phi$ to have pure discrete spectrum (pds). For there to even be a substitution associated with $\Lambda$, it’s necessary that the eigenvalues of $\Lambda$ be algebraic integers ([19], [24]). For pds, the eigenvalues of $\Lambda$ must constitute a Pisot family ([25]).

**Definition 2.1.** The set $\text{spec}(\Lambda)$ of eigenvalues of $\Lambda$ is a Pisot family if, whenever $\lambda \in \text{spec}(\Lambda)$ and $\lambda'$ is an algebraic conjugate of $\lambda$ with $|\lambda'| \geq 1$, then $\lambda' \in \text{spec}(\Lambda)$ and is of the same multiplicity as $\lambda$.

Under the additional assumptions that $\Lambda$ is diagonalizable over $\mathbb{C}$ and all eigenvalues of $\Lambda$ are algebraic conjugates - say, of degree $d$ - and of the same multiplicity - say $m$ - Lee and Solomyak prove ([25]) that the $\mathbb{R}^n$-action on $\Omega_\Phi$ has a generous compliment of eigenfunctions.

**Definition 2.2.** A substitution is of $(m, d)$-Pisot family type (or, if $m$ and $d$ are unimportant, just of Pisot family type) if its linear expansion $\Lambda$ is diagonalizable over $\mathbb{C}$ and the eigenvalues of $\Lambda$ are algebraic conjugates of degree $d$ and multiplicity $m$ and constitute a Pisot family. $\Omega = \Omega_\Phi$ is then a Pisot family tiling space.

**Theorem 2.3.** [8] If $\Phi$ is of $(m, d)$-Pisot family type, then there is a continuous finite-to-one semi-conjugacy $g : \Omega_\Phi \to X$ of the $\mathbb{R}^n$-action on $\Omega_\Phi$ with an action of $\mathbb{R}^n$ by translation on a group $X$. Moreover, there is a hyperbolic endomorphism $F : \mathbb{T}^{md} \to \mathbb{T}^{md}$ of the $md$-dimensional torus so that $X$ is the inverse limit $X = \hat{\mathbb{T}}^{md} := \varprojlim F$. The $\mathbb{R}^n$-action on $\hat{\mathbb{T}}^{md}$ is a Kronecker action by translation along the $n$-dimensional unstable leaves of the shift homeomorphism $\hat{F} : \hat{\mathbb{T}}^{md} \to \hat{\mathbb{T}}^{md}$ and, in addition, $g \circ \Phi = F \circ g$.

If the eigenvalues of $\Lambda$ are algebraic units then the $F$ above is an automorphism and $\mathbb{T}^{md}$ is simply the $md$-dimensional torus. Otherwise, $F$ is a $k - to - 1$ covering map for some $k > 1$ and $\hat{\mathbb{T}}^{md}$ is an $md$-dimensional solenoid. The group $\hat{\mathbb{T}}^{md}$, with the Kronecker action, is
the *maximal equicontinuous factor* of the $\mathbb{R}^n$-action on $\Omega$: If there is a semi-conjugacy $g' : \Omega \to Y$ with an equicontinuous $\mathbb{R}^n$-action on $Y$, then there is a semi-conjugacy $g'' : \hat{T}^{md} \to Y$ with $g' = g'' \circ g$.

Tilings $T, T' \in \Omega$ are *regionally proximal*, $T \sim_{rp} T'$, provided there are $T_k, T'_k \in \Omega$ and $v_k \in \mathbb{R}^n$ so that $d(T, T_k) \to 0$, $d(T', T'_k) \to 0$, and $d(T_k - v_k, T'_k - v_k) \to 0$ as $k \to \infty$. It is proved in [8] that if $\Omega$ is a Pisot family tiling space, then $T$ and $T'$ in $\Omega$ are regionally proximal if and only if $T$ and $T'$ are *strongly regionally proximal*, $T \sim_{srp} T'$: for each $k \in \mathbb{N}$ there are $T_k, T'_k \in \Omega$ and $v_k \in \mathbb{R}^n$ with $B_k[T_k] = B_k[T]$, $B_k[T'_k] = B_k[T']$, and $B_k[T_k - v_k] = B_k[T'_k - v_k]$. The notion of regional proximality extends in an obvious way to arbitrary group actions and Veech proves in [33] (see also [4]) that, in the case of minimal abelian actions on compact metric spaces, regional proximality is the equicontinuous structure relation.

**Theorem 2.4.** (Veech) If $G$ is an abelian group acting minimally on the compact metric space $X$, and $g : X \to X_{\text{max}}$ is the maximal equicontinuous factor, then $g(x) = g(x')$ if and only if $x \sim_{rp} x'$.

We thus have:

**Corollary 1.** If $\Omega$ is a Pisot family tiling space with maximal equicontinuous factor $g : \Omega \to \hat{T}^{md}$, then $g(T) = g(T')$ if and only if $T \sim_{srp} T'$.

3. **Sufficient conditions for pure discrete spectrum**

Given patches $Q, Q'$ (not necessarily allowed for $\Phi$) and $x \in \text{int}(\text{spt}(Q) \cap \text{spt}(Q'))$, we’ll say that $Q$ and $Q'$ are *coincident* at $x$ provided $B_0[Q - x] = B_0[Q' - x]$ and eventually coincident at $x$ if there is $k \in \mathbb{N}$ so that $B_0[\Phi^k(Q - x)] = B_0[\Phi^k(Q' - x)]$. We’ll say that $Q$ and $Q'$ are *densely eventually coincident* on overlap if $Q$ and $Q'$ are eventually coincident at a set of $x$ that is dense in $\text{int}(\text{spt}(Q) \cap \text{spt}(Q'))$, and, if this happens for tilings $Q, Q'$, we’ll say that $Q$ and $Q'$ are *densely eventually coincident*. If $Q$ is a finite patch and there is a basis $\{v_1, \ldots, v_n\}$ of $\mathbb{R}^n$ so that $\cup_{k_1, \ldots, k_n \in \mathbb{Z}}(Q + \sum k_i v_i)$ is a tiling of $\mathbb{R}^n$, we’ll denote this tiling by $\hat{Q}$. If $L$ is the lattice spanned by $\{v_1, \ldots, v_n\}$ over $\mathbb{Z}$, we’ll call the torus $\mathbb{R}^n/L$ the *torus spanned by $\{v_1, \ldots, v_n\}$*. Given $w_1, \ldots, w_k \in \mathbb{R}^n$, the $k$ translations $x + L \mapsto x + w_i + L$ generate a $\mathbb{Z}^k$-action on $\mathbb{R}^n/L$ which we’ll call the $\mathbb{Z}^k$-action generated by $\{w_1, \ldots, w_k\}$ on the torus spanned by $\{v_1, \ldots, v_n\}$. This action is *minimal* if its only closed invariant sets are $\emptyset$ and $\mathbb{R}^n/L$.

**Theorem 3.1.** Suppose that $\Phi$ is a substitution of Pisot family type and that $Q$ is a finite patch such that $\hat{Q} = \cup_{k_1, \ldots, k_n \in \mathbb{Z}}(Q + \sum k_i v_i)$ is a tiling of $\mathbb{R}^n$. Suppose also that the $\mathbb{Z}^k$-action generated by $\{w_1, \ldots, w_k\}$ on
the torus spanned by \{v_1, \ldots, v_n\} is minimal and that \(Q\) and \(Q - w_i\) are densely eventually coincident for each \(i = 1, \ldots, k\). Then the \(\mathbb{R}^n\)-action on \(\Omega_{\Phi}\) has pure discrete spectrum.

Proof. If \(Q\) and \(Q'\) are eventually coincident at \(x\), let us write \(Q - x \sim_c Q' - x\). Note that eventual coincidence at a point is an open condition, is translation equivariant, and only depends on local structure: if \(Q_1 - x \sim_c Q'_1 - x, B_0[Q_2 - y] = B_0[Q_1],\) and \(B_0[Q'_2 - y] = B_0[Q'_1]\), then \(Q_2 - (x + y) \sim_c Q'_2 - (x + y)\).

We first prove that if \(x_0, w \in \mathbb{R}^n\) are such that \(\bar{Q}\) and \(\bar{Q} - w\) are eventually coincident at \(x_0\), then \(\bar{Q}\) and \(\bar{Q} - w\) are densely eventually coincident. To this end, let \(U := \{x : \bar{Q} - x \sim_c \bar{Q} - w - x\}\), and \(U_i := \{x : \bar{Q} - x \sim_c \bar{Q} - w_i - x\}\) for each \(i = 1, \ldots, k\). Each of these sets is open, \(U\) is nonempty, and the \(U_i\) are dense. We aim to show that \(U\) is also dense. Let \(x \in \text{cl}(U)\), fix \(i \in \{1, \ldots, k\}\), and let \(\epsilon > 0\) be given. There is then \(x' \in U\) with \(|x' - x| < \epsilon/2\) and then \(x'' \in U \cap U_i \cap (U_i - w)\) with \(|x'' - x'| < \epsilon/2\). Then \(\bar{Q} - w_i - x'' \sim_c \bar{Q} - x'' \sim_c \bar{Q} - w_i - w \sim_c \bar{Q} - w'' - w - w_i\). That is, \(\bar{Q} - (x'' + w_i) \sim_c \bar{Q} - (x'' + w_i) - w\), with \(|(x'' + w_i) - (x + w_i)| < \epsilon\), and it follows that \(x + w_i \in \text{cl}(U)\). Thus if \((n_1, \ldots, n_k) \in \mathbb{N}^k\) and \(x \in \text{cl}(U)\), then \(x + n_1w_1 + \cdots + n_kw_k \in \text{cl}(U)\). Since the \(\mathbb{Z}^k\)-action generated by \(\{w_1, \ldots, w_k\}\) on the torus spanned by \(\{v_1, \ldots, v_n\}\) is minimal, \(U\) is dense; that is, \(\bar{Q}\) and \(\bar{Q} - w\) are densely eventually coincident.

If \(\bar{Q}\) and \(\bar{Q} - w_i\) are eventually coincident at \(x\), then \(\Phi^m(\bar{Q}) = \bar{Q}_m\) and \(\Phi^m(\bar{Q} - w_i) = \bar{Q}_m - \Lambda^m w_i\), where \(Q_m = \Phi^m(Q)\), are eventually coincident at \(\Lambda^m x\), for all \(m \in \mathbb{N}\). Also, the \(\mathbb{Z}^k\)-action generated by \(\{\Lambda^m v_1, \ldots, \Lambda^m v_n\}\) on the torus spanned by \(\{\Lambda^m v_1, \ldots, \Lambda^m v_n\}\) is minimal. Thus, given \(R\), we may assume (by replacing \(Q\) with \(Q_m\) for sufficiently large \(m\)) that \(Q\) contains every allowed patch of diameter less than \(R\). From this, we see that if \(S\) and \(S'\) are any two allowed finite patches that share a tile, then \(S\) and \(S'\) are densely eventually coincident on overlap. Indeed, we may assume that \(Q\) contains translated copies of both \(S\) and \(S'\). Let \(S - u \subset Q\) and let \(w\) be such that \(Q - w \supset S' - u\). There is then \(x \in \text{spt}(Q)\) so that \(Q\) and \(Q - w\) are coincident at \(x\) (since \(Q\) and \(Q - w\) agree on the overlap of \(S - u\) and \(S' - u\)). From the above, \(\bar{Q}\) and \(\bar{Q} - w\) are densely eventually coincident; in particular, \(S - u\) and \(S' - u\), hence \(S\) and \(S'\), are densely eventually coincident on overlap.

Now, if the \(\mathbb{R}^n\)-action on \(\Omega\) does not have pure discrete spectrum, then for each \(T \in \Omega\) there are finitely many, and at least two, \(T' \in \Omega\) so that \(T\) and \(T'\) are regionally proximal and \(T\) and \(T'\) don’t share a tile (see [8]). Pick \(T \in \Omega\) that is \(\Phi\)-periodic (such \(T\) always exist - in fact, for primitive \(\Phi\), they are dense in \(\Omega_{\Phi}\)). There is then \(T' \in \Omega\) that is
also \( \Phi \)-periodic, that is strongly regionally proximal with \( T \) (Corollary 1), but disjoint from \( T \). There are then \( T_0, T'_0 \in \Omega \) and \( v_0 \in \mathbb{R}^n \) so that
\[
B_0[T_0] = B_0[T], \ B_0[T'_0] = B_0[T'] \quad \text{and} \quad B_0[T_0 - v_0] = B_0[T'_0 - v_0].
\]
Let \( S \) and \( S' \) be the finite patches in \( T_0 \) and \( T'_0 \): \( S := B_0[T_0] \cup B_0[T_1 - v_0] + v_0 \) and \( S' := B_0[T'_0] \cup B_0[T'_1 - v_0] + v_0 \). Then \( S \) and \( S' \) are densely eventually coincident on overlap. In particular, there is \( x \) at which \( T \) and \( T' \) are eventually coincident. But \( T \) and \( T' \) are periodic, so eventual coincidence means that \( T \) and \( T' \) share a tile, contrary to assumption. Thus, the \( \mathbb{R}^n \)-action on \( \Omega \) must have pure discrete spectrum. \( \square \)

Suppose now that \( \phi : \mathcal{A} \to \mathcal{A}^* \), \( \mathcal{A} = \{1, \ldots, k\} \), is a substitution on \( k \) letters. The incidence matrix of \( \phi \) is the \( k \times k \) matrix \( M = M_\phi \) whose \( i, j \)-th entry is the number of \( i \)'s that appear in \( \phi(j) \). The substitution is primitive if some power of \( M \) has all positive entries; in this case there is a positive left eigenvector \( \omega = (\omega_1, \ldots, \omega_k) \) corresponding to the Perron-Frobenius eigenvalue \( \lambda \) of \( \phi \). We'll say that \( \phi \) is a Pisot substitution if \( \phi \) is primitive and \( \lambda \) is a Pisot number. Such a substitution on letters determines a geometrical substitution \( \Phi \) on prototiles \( \rho_i := [0, \omega_i], i = 1, \ldots, k \), that is of (1, \( d \))-Pisot family type, \( d \) being the algebraic degree of \( \lambda \).

Given a word \( u \in \mathcal{A}^* \), the abelianization of \( u \) is the vector \( [u]_{ab} \in \mathbb{Z}^k \) with \( i \)-th entry equal to the number of \( i \)'s that appear in \( u \). A pair of words \( (u, v) \in \mathcal{A}^* \times \mathcal{A}^* \) is a balanced pair if \( [u]_{ab} = [v]_{ab} \). If \((u_1, v_1)\) and \((u_2, v_2)\) are balanced pairs, we will write \((u_1, v_1)(u_2, v_2) = (u_1u_2, v_1v_2)\) and call \((u_1, v_1)\) and \((u_2, v_2)\) factors of \((u_1u_2, v_1v_2)\). A balanced pair \((u, v)\) is irreducible if it has no (non-trivial) factors. Clearly, every balanced pair \((u, v)\) can be written (uniquely) as a product of irreducible balanced pairs: each of these irreducible balanced pairs is called an irreducible factor of \((u, v)\). A balanced pair of the form \((a, a)\), \( a \in \mathcal{A} \), is called a coincidence. Given a balanced pair \((u, v)\), we will say that the balanced pair algorithm for \((u, v)\) terminates with coincidence provided: (1) the collection \( \{(x, y) : (x, y) \text{ is an irreducible factor of } (\phi^m(u), \phi^m(v)) \text{ for some } m \in \mathbb{N} \} \) is finite; and (2) if \((x, y)\) is an irreducible factor of \((\phi^m(u), \phi^m(v))\) for some \( m \in \mathbb{N} \), then there is \( k \in \mathbb{N} \) so that \((\phi^k(x), \phi^k(y))\) has a coincidence as a factor. This terminology is due to [29]. It is a result of Livshits ([26]) that if \( w \) is right-infinite word fixed by \( \phi \), \( u \) is a prefix of \( w \), \( w = uv_1uv_2u \cdots \), and the balanced pair algorithm terminates with coincidence for all \((uv_i, v_iu)\), then the \( \mathbb{Z} \)-action on the substitutive system associated with \( \phi \) has pure discrete spectrum. The idea goes back to Michel ([27]) - see [29] for an actual algorithm based on the above, and a proof that termination with coincidence for all such \((uv_i, v_iu)\) implies that the \( \mathbb{R} \)-action on \( \Omega_\phi \) also has
pure discrete spectrum. The following theorem, which is an application of Theorem 3.1 to the usual Pisot substitutive case, asserts that it suffices to check only one (well chosen) \((uv, vu)\).

**Theorem 3.2.** Suppose that \(\phi : A \rightarrow A^*\) is a Pisot substitution with left Perron-Frobenius eigenvector \(\omega\) and suppose that \(u,v \in A^*\) are words with the properties: \(\langle [u]_{ab}, \omega \rangle\) and \(\langle [v]_{ab}, \omega \rangle\) are independent over \(Q\); and the balanced pair algorithm for \((uv, vu)\) terminates with coincidence. Then the \(\mathbb{R}\)-action on \(\Omega_\phi\) has pure discrete spectrum.

**Proof.** Let \(Q\) be a patch with underlying word \(uv\). The length of the support of \(Q\) is \(l := \langle [u]_{ab} + [v]_{ab}, \omega \rangle\) and \(Q = \cup_{n \in \mathbb{Z}} (Q + nl)\) is a periodic tiling of \(\mathbb{R}\). Let \(l_u := \langle [u]_{ab}, \omega \rangle\). The hypotheses imply that \(l_u\) is completely rationally independent of \(\{l\}\) and that \(Q\) and \(Q - l_u\) are densely eventually coincident. Indeed, given \(t \in \mathbb{R}\) and \(\epsilon > 0\), “termination” of the balanced pair algorithm guarantees that there is an \(n \in \mathbb{N}\) so that the patches of \(\Phi^n(Q)\) and \(\Phi^n((Q - l_u))\) lying over \(\lambda^n(t - \epsilon, t + \epsilon)\) contain patches \(P_{xy}\) and \(P_{yx}\), resp., with \(\text{support}(P_{xy}) = \text{support}(P_{yx})\), with \(P_{xy}\) and \(P_{yx}\) patches corresponding to words \(xy\) and \(yx\) and with \((xy, yx)\) an irreducible factor of \((\phi^n(uv), \phi^n(vu))\). The “coincidence” part of the balanced pair algorithm then gives an \(m \in \mathbb{N}\) so that \(\Phi^m(P_{xy})\) and \(\Phi^m(P_{yx})\) share a tile. Thus there is \(t' \in (t - \epsilon, t + \epsilon)\) so that \(B_0[\Phi^{m+n}(Q - t')] = B_0[\Phi^{m+n}(Q - l_u - t')]\). The result follows from Theorem 3.1. \(\square\)

The following corollary appears in [10]. The proof given there has a gap that is not easily fixed using the techniques of that paper.

**Corollary 2.** Suppose that \(\phi\) is a Pisot substitution whose incidence matrix has characteristic polynomial irreducible over \(Q\). If there are letters \(a \neq b\) in the alphabet for \(\phi\) so that the balanced pair algorithm for \((ab, ba)\) terminates with coincidence, then the \(\mathbb{R}\)-action on \(\Omega_\phi\) has pure discrete spectrum.

**Proof.** Irreducibility of the incidence matrix implies that the coordinates of the eigenvector \(\omega\) are independent over \(Q\), thus \(\langle [a], \omega \rangle\) and \(\langle [b], \omega \rangle\) are independent over \(Q\). \(\square\)

**Remark 1.** It may appear that the applicability of the methods presented above depends on the geometry of the tiles: as stated, Theorem 3.1 requires the existence of a patch that tiles \(\mathbb{R}^n\) periodically. But in fact if \(P\) is any patch and \(\{v_1, \ldots, v_n\}\) is any basis with \(\sigma := \{\sum_{i=1}^n t_i v_i : t_i \in [0,1]\} \subset \text{spt}(P)\) then one can consider the generalized patch \(Q\) consisting of the generalized tiles \((\text{spt}(\tau) \cap \sigma, \tau)\), for \(\tau \in P\) with \(\text{int}(\text{spt}(\tau)) \cap \sigma \neq \emptyset\). It makes perfectly good sense to speak of \(Q\) and
\(Q - v\) being densely eventually coincident and the proof of Theorem 3.1 goes through without change.

4. Necessary conditions for pure discrete spectrum

Throughout this section \(\Phi\) will be an \(n\)-dimensional substitution of \((m,d)\)-Pisot family type with tiling space \(\Omega = \Omega_\Phi\), inflation \(\Lambda\), and maximal equicontinuous factor \(g : \Omega \to \widehat{T}^{md}\). A vector \(v \in \mathbb{R}^n\) is a return vector for \(\Phi\) if for some (equivalently, every) \(T \in \Omega\), \(T - v \cap T \neq \emptyset\).

Let \(R = R_\Phi := \{v : v\) is a return vector for \(\Phi\}\) denote the set of return vectors for \(\Phi\).

Theorem 3.1 raises the question: If the \(\mathbb{R}^n\)-action on \(\Omega\) has pure discrete spectrum, for what \(v \in \mathbb{R}^n\) and what tilings \(S\) must it be the case that \(S\) and \(S - v\) are densely eventually coincident? It is clear (if \(S \in \Omega\)) that the collection of such \(v\) is countable: if \(S\) and \(S - v\) are densely eventually coincident, then \(\Phi^k(S)\) and \(\Phi^k(S - v) = \Phi^k(S) - \Lambda^k v\) share a tile for some \(k \in \mathbb{N}\) so that \(\Lambda^k v\) is a return vector, of which there are only countably many by FLC. We will see below (Proposition 1) that for the class of all tilings in \(\Omega\), the answer to the above question is precisely \(\bigcup_{k \in \mathbb{Z}} \Lambda^k R\). Under additional conditions this collection is a group and we will give it a homological interpretation.

For ease of exposition, we assume that the prototiles \(\rho_i\) of \(\Phi\) are polytopes and that the tiles in tilings in \(\Omega\) meet full face to full face in all dimensions (so each \(\rho_i\) has finitely many faces in each dimension, and if \(\tau, \sigma \in T \in \Omega\) meet in a point \(x\) that is in the relative interior of a face of \(\tau\) or \(\sigma\), then they meet in that entire face). We may assume that the prototiles ‘force the border’ (i.e., if \(T, T' \in \Omega\) share a tile \(\tau\) with \(0 \in \tau\), then there is \(k \in \mathbb{N}\) so that \(B_0[\Phi^k(T)] = B_0[\Phi^k(T')]\) - otherwise, replace the prototiles by the collared prototiles (see [2]). Let \(X\) be the Anderson-Putnam complex for \(\Phi\): \(X\) is a cell complex with one \(n\)-cell for each (collared) prototile and these \(n\)-cells are glued along faces by translation. If \(k\)-face \(f\) of \(\rho_i\) is glued to \(k\)-face \(g\) of \(\rho_j\), then there is \(v \in \mathbb{R}^n\) so that \(f = g - v\), and the gluing is defined by \(f \ni x \sim x + v \in g\).

The rule for deciding what faces are glued is as follows. The \(k\)-face \(f\) of \(\rho_i\) is adjacent to the \(k\)-face \(g\) of \(\rho_j\) if there are \(T \in \Omega\) and \(u, v \in \mathbb{R}^n\) so that \(\rho_i+u, \rho_j+v \in T\) and \(\rho_i+u \cap \rho_j+v = f+u = g+v\). Let gluable be the smallest equivalence relation containing the adjacency relation. Two \(k\)-faces of prototiles are glued in \(X\) if they are gluable. The substitution \(\Phi\) on the prototiles induces a continuous map \(f = f_\Phi : X \to X\) with the property that the inverse limit \(\lim f\) is homeomorphic with \(\Omega\) by a homeomorphism that conjugates the shift \(\hat{f}\) on \(\lim f\) with \(\Phi\) on \(\Omega\) ([2]).
We say that a patch $Q$ is \textit{admissible} if whenever tiles $\tau, \sigma \in Q$ meet along $k$-faces, those $k$-faces are gluable. Certainly every patch that is allowed for $\Phi$ is also admissible and if $Q$ is an admissible patch, then so is $\Phi(Q)$. For each admissible patch $Q$ there is then a continuous map $p^Q : spt(Q) \to X$ defined by $p^Q(x) = [x - v]$ provided $x \in spt(\rho_i) + v$ and $\rho_i + v \in Q$. Furthermore, if $f : X \to X$ is the continuous map induced by $\Phi$, then $f \circ p^Q = p^{\Phi(Q)} \circ \Lambda$ on $spt(Q)$ for all admissible patches $Q$.

Now given a path $\gamma : [a, b] \to X$ and $x \in \mathbb{R}^n$, there is a unique (continuous) path $\bar{\gamma} : [a, b] \to \mathbb{R}^n$ with $\bar{\gamma}(a) = x$ and $p^Q(\bar{\gamma}(t)) = \gamma(t)$ whenever $\bar{\gamma}(t) \in spt(Q)$ for any admissible patch $Q$. We will call such a $\bar{\gamma}$ a \textit{lift} of $\gamma$ to $\mathbb{R}^n$. If $\gamma$ is a loop in $X$ then $\gamma$ is also a singular 1-cycle and we denote its homology class by $[\gamma] \in H := H_1(X; \mathbb{Z})$. If $T \in \Omega$ and $\tau, \tau + v \in T$ (thus $v$ is a return vector), there is a corresponding element $[\gamma] \in H$, where $\gamma(t) := p^T(x + tv)$ for fixed $x \in spt(\tau)$ and $t \in [0, 1]$. Note that $[\gamma]$ doesn’t depend on the choice of $x \in spt(\tau)$, but it does depend on $\tau$ (not just $v$). We will show that the reverse of this process is better behaved.

Given a path $\gamma : [a, b] \to X$, let $\bar{\gamma}$ be a lift of $\gamma$ to $\mathbb{R}^n$ and let $l(\gamma) := \bar{\gamma}(b) - \bar{\gamma}(a)$. This is well-defined, i.e., independent of lift, since all lifts of $\gamma$ are translates of one another (as is easily seen by uniqueness). It is also clear that if $\gamma = \gamma_1 \ast \gamma_2$ is a concatenation of two paths, then $l(\gamma) = l(\gamma_1) + l(\gamma_2)$ and if $\gamma^{-1}$ is the reverse of $\gamma$, then $l(\gamma^{-1}) = -l(\gamma)$. By lifting homotopies, one sees that $l$ induces a homomorphism from the fundamental group of $X$ to the additive group $\mathbb{R}^n$. We will see that $l$ also passes to homology.

**Lemma 4.1.** If $\gamma$ is a loop in $X$ and $0 = [\gamma] \in H$ then $l(\gamma) = 0$.

**Proof.** By the above remark on concatenations, we may assume that $\gamma(0)$ is a vertex in $X$. Suppose first that $n = 1$. We may then homotope $\gamma$ to a loop that doesn’t turn in the interior of any edge and this can be done without affecting $l$. We may assume that $\gamma$ itself has this property. Now $\gamma = \gamma_1 \ast \cdots \ast \gamma_k$ is a concatenation of paths, each running along a single edge exactly once. If $\gamma_i$ and $\gamma_j$ run along the same edge, but in different directions, then $l(\gamma_i) = -l(\gamma_j)$. Since $[\gamma] = 0$, each $\gamma_i$ can be paired with a $\gamma_j$ so that elements of a pair traverse the same edge, but in opposite directions. Thus $l(\gamma) = \sum l(\gamma_i) = 0$.

Suppose now that $n > 1$. After sufficient simplicial subdivision of $X$, we may homotope $\gamma$ to a loop that lies in the 1-skeleton of this subdivision, only self-intersects at vertices of this subdivision, and that has a continuous lift that also runs from $\bar{\gamma}(0)$ to $\bar{\gamma}(1)$. Let us replace $\gamma$ with such a curve. Let $\Gamma = \sum e_i$ be the simplicial 1-chain representing $\gamma$: the $e_i$ are the oriented 1-simplices in the image of $\gamma$ and we have
modified $\gamma$ so that $e_i \neq \pm e_j$ for $i \neq j$. Since $\gamma$ is homologous to zero in $X$, there is a simplicial 2-chain $\Sigma = \sum \sigma_j$ in the subdivided simplicial structure of $X$ with $\partial \Sigma = \Gamma$. After further subdivision, we may assume that each $\sigma_j$ has at most one edge on $\Gamma$.

Now there is a lift of $\Gamma$ to a 1-chain $\bar{\Gamma}$ in $\mathbb{R}^n$ with $\partial \bar{\Gamma} = \bar{\gamma}(1) - \bar{\gamma}(0)$. We may lift $\Sigma$ to a 2-chain $\bar{\Sigma} = \sum \bar{\sigma}_j$ in $\mathbb{R}^n$ in which each $\bar{\sigma}_j$ is a lift of $\sigma_j$ and $\bar{\sigma}_j$ has an edge on $\bar{\Gamma}$ if and only if $\sigma_j$ has an edge on $\Gamma$ (that is, $\sigma_j$ meets $\Gamma$ along $e_i$ if and only if $\bar{\sigma}_j$ meets $\bar{\Gamma}$ along $\bar{e}_i$).

Now $\partial(\Sigma) = \bar{\Gamma} + \bar{\Delta}$ with none of the elementary 1-simplices of $\bar{\Delta}$ in $\bar{\Gamma}$. Since $\partial^2(\Sigma) = 0$,

$$\partial(\bar{\Delta}) = \bar{\gamma}(0) - \bar{\gamma}(1).$$

Each edge of $\bar{\Delta}$ lies over some edge (of some simplex in $\Sigma$) that does not occur in $\partial \bar{\Sigma} = \Gamma$: write $\bar{\Delta} = \sum_i \sum_{k=1}^{j_i} (-1)^{m(i,k)} e'_{i,k}$ where, for each $i$, the $e'_{i,k}$ all lie over the same edge $e_i'$ of one of the $\sigma_j$ in $\Sigma$, and $e'_i \neq e'_s$ for $i \neq s$. Pushed forward into $X$, the equation $\partial(\Sigma) = \Gamma + \Delta$ reads $\partial \Sigma = \Gamma + \sum_i (\sum_{k=1}^{j_i} (-1)^{m(i,k)}) e'_i$. Since $\partial \Sigma = \Gamma$, we see that

$$\sum_{k=1}^{j_i} (-1)^{m(i,k)} = 0$$

for all $i$. Now, from the displayed equation above, we have the ‘formal’ statement

$$\bar{\gamma}(0) - \bar{\gamma}(1) = \sum_i \sum_{k=1}^{j_i} (-1)^{m(i,k)} \partial e'_{i,k}.$$

But this is also valid as a statement of equality between elements in the additive group $\mathbb{R}^n$ and, as group elements, $\partial e'_{i,k} = \partial e'_i$ for each $i, k$.

Thus the right hand side of the last displayed equation is, as a group element, $\sum_i (\sum_{k=1}^{j_i} (-1)^{m(i,k)}) \partial e'_i = 0$. Hence $l = 0$. □

Thus, since each element of $H$ is the homology class of a loop, $l : H \to \mathbb{R}^n$ given by $l([\gamma]) = l(\gamma)$ is a well-defined homomorphism. Note that if $\gamma$ is a loop defined by return vector $v$, then $l([\gamma]) = v$.

Lemma 4.2. $l \circ f_* = \Lambda \circ l$.

Proof. It follows from $f \circ p^Q = p^\Phi(Q) \circ \Lambda$ that if $\bar{\gamma}$ is a lift of $\gamma$ to $\mathbb{R}^n$, then $\Lambda \circ \bar{\gamma}$ is a lift of $f \circ \gamma$ to $\mathbb{R}^n$. Thus $l([f \circ \gamma]) = \Lambda \circ \bar{\gamma}(1) - \Lambda \circ \bar{\gamma}(0) = \Lambda(\bar{\gamma}(1) - \bar{\gamma}(0)) = \Lambda(l([\gamma]))$. □

Let us call a vector $v \in \mathbb{R}^n$ a generalized return vector for $\Phi$ if there are $v_i \in \mathbb{R}^n$ and $T_i \in \Omega$, $i = 1, \ldots, k$, with $v = v_1 + \cdots + v_k$, so that $B_0[T_i + v_i] = B_0[T_{i+1}]$ for $i = 1, \ldots, k - 1$, and $B_0[T_k + v_k] = B_0[T_1]$. Let $GR = GR(\Phi)$ be collection of generalized return vectors. It is a consequence of the next lemma that $GR(\Phi)$ is a subgroup of $\mathbb{R}^n$. 
Lemma 4.3. Let $X$ be the collared Anderson-Putnam complex for $\Phi$ and let $H = H_1(X; \mathbb{Z})$. Then $l(H) = \text{GR}(\Phi)$.

Proof. If $\gamma$ is a loop in $X$. Then $\gamma$ is homotopic to a loop that factors as a product of paths $\gamma_i$ with each $\gamma_i$ a two-piece piecewise linear path from a prototile $\rho_{\alpha(i)}$ to an adjacent prototile $\rho_{\alpha(i+1)}$. Each $\gamma_i$ is itself homotopic rel endpoints to a product of two-piece piecewise linear paths each making an allowed transition between prototiles (this from the definition of the Anderson-Putnam complex: the gluing of prototiles along faces is by the smallest equivalence relation that includes the relation defined by the allowed transitions). Thus the displacement of the lift of $\gamma$ is a generalized return vector.

Conversely, each generalized return vector $v$ determines a (at least one) loop $\gamma$ with $l([\gamma]) = v$. \hfill $\square$

The stable manifold of $T' \in \Omega$ is the set $W^s(T') := \{ T \in \Omega : d(\Phi^k(T), \Phi^k(T')) \to 0 \text{ as } k \to \infty \}$.

Lemma 4.4. Suppose that $\Omega = \Omega_\Phi$ is an $n$-dimensional Pisot family tiling space whose $\mathbb{R}^n$-action has pure discrete spectrum and suppose that $T \in W^s(T')$. Then $T$ and $T'$ are densely eventually coincident.

Proof. Let $g : \Omega \to \hat{T}^{md}$ be the map onto the maximal equicontinuous factor, let $\hat{F} : \hat{T}^{md} \to \hat{T}^{md}$ be the hyperbolic automorphism with $g \circ \Phi = \hat{F} \circ g$, and let $z \mapsto z - v$ denote the Kronecker action on $\hat{T}^{md}$, so that $g(T-v) = g(T) - v$ for all $T \in \Omega$ and $v \in \mathbb{R}^n$. Suppose that $T \in W^s(T')$ but that $0 \neq U \subset \mathbb{R}^n$ is such that $T-x \notin W^s(T'-x)$ for all $x \in U$. For fixed $x_0 \in U$, let $T_k := \Phi^k(T-x_0)$, $T'_k := \Phi^k(T'-x_0)$, $z_k := g(T_k)$, and $z'_k := g(T'_k)$. Then $d(z_k, z'_k) \to 0$ as $k \to \infty$. Choose $k_i$ with $T_{k_i} \to S$, $T'_{k_i} \to S'$; then also $z_{k_i}, z'_{k_i} \to z := g(S) = g(S')$. Since the $\mathbb{R}^n$-action on $\Omega$ has pure discrete spectrum, $S$ and $S'$ are proximal. Thus, given $r > 0$, there is $y = y(r) \in \mathbb{R}^n$ so that $B_r[S-y] = B_r[S'-y]$ (see [8]). From the local product structure on $\hat{T}^{md}$, there is $\epsilon > 0$ so that if $0 < |w| < \epsilon$ and $w \in \mathbb{R}^n$, then $\bar{z}-w \notin W^s_\epsilon(\bar{z}) := \{ \bar{z}' : d(\hat{F}^k(\bar{z}'), \hat{F}^k(\bar{z})) \leq \epsilon \text{ for all } k \geq 0 \}$, for all $\bar{z} \in \hat{T}^{md}$. Let $r$ be large enough so that if $T, T' \in \Omega$ and $B_r[T] = B_r[T']$, then $g(T) \in W^s_\epsilon(g(T'))$. Let $y = y(r)$, as above, and let $i$ be large enough so that $B_i[T_{k_i}-y-w] = B_i[T'_{k_i}-y]$ with $|w| < \epsilon$. We may also take $i$ large enough so that $x_0 + \Lambda^{-ki}y \in U$. In $\hat{T}^{md}$ it is the case that if $\bar{z} \in W^s_\epsilon(\bar{z}')$, then $\bar{z} - v \in W^s_\epsilon(\bar{z}' - v)$ for all $v \in \mathbb{R}^n$. Thus, we may further increase $i$ so that $z_{k_i}-y \in W^s_\epsilon(z'_{k_i}-y)$. But $z_{k_i}-y-w \in W^s_\epsilon(z'_{k_i}-y)$ and $|w| < \epsilon$. Thus $w = 0$ so that $T_{k_i}-y \in W^s(T'_{k_i}-y)$, and then $T-x \in W^s(T'-x)$ for $x := x_0 + \Lambda^{-ki}y \in U$, contradicting the choice of $U$. Thus $T-x \in W^s(T'-x)$ for a dense set
of \( x \in \mathbb{R}^n \) and it only remains to observe that \( T \) is eventually coincident with \( T' \) at \( x \) if and only if \( T \in W^s(T') \) (see, for example [11]).

**Proposition 1.** Suppose that \( \Omega_\Phi \) is an \( n \)-dimensional Pisot family tiling space whose \( \mathbb{R}^n \)-action has pure discrete spectrum and that \( T \in \Omega \). Then

\[
\{ v \in \mathbb{R}^n : T - v \text{ is densely eventually coincident with } T \} = \bigcup_{k \in \mathbb{Z}} A^k \mathcal{R}.
\]

Proof. We have already observed that

\[
\{ v \in \mathbb{R}^n : T - v \text{ is densely eventually coincident with } T \} \subset \bigcup_{k \in \mathbb{Z}} A^k \mathcal{R}.
\]

Suppose that \( v \in \mathcal{R} \) and \( k \in \mathbb{Z} \). There is then \( \tau \in \Phi^{-k}(T) - v \cap \Phi^{-k}(T) \). Pick \( x \) in the interior of the support of \( \tau \). Then \( \Phi^{-k}(T) - v - x \in W^s(\Phi^{-k}(T) - x) \). It follows that \( T - \Lambda^k v - \Lambda^k x \in W^s(T - \Lambda^k x) \) so that, by Lemma 4.4, \( T - \Lambda^k v - \Lambda^k x \) is densely eventually coincident with \( T - \Lambda^k x \). Thus \( T - \Lambda^k v \) is densely eventually coincident with \( T \). \( \Box \)

For merely admissible tilings \( S \) it may (as far as we know) happen that \( S \) and \( S - v \) are densely eventually coincident for some \( v \notin \bigcup_{k \in \mathbb{Z}} A^k \mathcal{R} \). But something like the other direction of containment in Proposition 1 remains true (the following proposition is, in fact, exactly the other direction of containment for ‘unimodular’ \( \Phi \) - see Proposition 3).

**Proposition 2.** Suppose that \( \Omega_\Phi \) is an \( n \)-dimensional Pisot family tiling space whose \( \mathbb{R}^n \)-action has pure discrete spectrum. Then for any admissible tiling \( S \) of \( \mathbb{R}^n \) for \( \Phi \), and any \( v \in GR(\Phi) \), \( S \) and \( S - v \) are densely eventually coincident.

Proof. Let \( \sim_{dc} \) be the densely eventually coincident relation (\( T \sim_{dc} T' \) means \( T - x \) is eventually coincident with \( T' - x \) for a dense set of \( x \in \mathbb{R}^n \)). Since eventual coincidence at a point is an open condition, \( \sim_{dc} \) is an equivalence relation. Suppose that \( T_i \in \Omega \) and \( v_i \in \mathbb{R}^n \), \( i = 1, \ldots, k \), are such that \( B_0[T_i + v_i] = B_0[T_{i+1}] \), for \( i = 1, \ldots, k - 1 \), and \( B_0[T_k + v_k] = T_1 \). From Lemma 4.4, \( T_i \sim_{dc} T_{i+1} \sim_{dc} T_i + v_i \), for \( i = 1, \ldots, k \), with \( T_{k+1} := T_k + v_k \), and \( T_1 \sim_{dc} T_{k+1} \). Clearly, \( T \sim_{dc} T' \Rightarrow T - w \sim_{dc} T' - w \), so we have \( T_1 + v \sim_{dc} T_{k+1} \), with \( v := v_1 + \cdots + v_k \). Thus \( T_1 + v \sim_{dc} T_1 \). Now let \( T \) be any element of \( \Omega \) and let \( T_1 \) and \( v \) be as above. Let \( w \in \mathbb{R}^n \) be so that \( B_0[T - w] = B_0[T_1] \). Then \( T - w \sim_{dc} T_1 \sim_{dc} T_{1} + v \). But also, \( T - w - v \sim_{dc} T_1 - v \), so \( T - v \sim_{dc} T \). That is, \( T - v \sim_{dc} T \) for all \( T \in \Omega \) and all \( v \in GR(\Phi) \).

Now suppose that \( S \) is admissible for \( \Phi \) and let \( v \in GR(\Phi) \). Let \( \sigma, \tau \in S \) be such that \( U = U(\sigma, \tau) := \text{int}(\text{spt}(\sigma)) \cap (\text{int}(\text{spt}(\tau)) - v) \neq \emptyset \). Since \( S \) is admissible, there are \( T_1, \ldots, T_k \in \Omega_\Phi \) with \( \sigma \in T_1 \), \( \tau \in T_2 \), and \( T_{i+1} \cap T_i \neq \emptyset \) for \( i = 1, \ldots, k - 1 \). Restricting \( \sim_{dc} \) to \( U \) (that is,
by $T \sim_{U} T'$ we mean $T - x$ is eventually coincident with $T' - x$ for a dense set of $x \in U$) we have: $S \sim_{U} T_1 \sim_{U} T_1 + v \sim_{U} T_2 + v \sim_{U} T_3 + v \sim_{U} \cdots \sim T_k + v \sim_{U} S + v$. As $\cup_{\sigma, \tau} U(\sigma, \tau)$ is dense in $\mathbb{R}^n$, we have $S \sim_{dc} S + v$. \hfill \square

**Example 4.5. (Table substitution)** The Table substitution $\Psi$ has inflation $\Lambda = 2I$ and prototiles $\rho_1 = [0, 2] \times [0, 1]$, $\rho_2 = [0, 1] \times [0, 2]$ that substitute as pictured in Figure 1. Let $Q = \{x_1\}$ and $v = (1, 0)$.

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics[scale=0.5]{rectangle_1.png}
\end{array}
\sim_{\Psi} \begin{array}{c}
\includegraphics[scale=0.5]{rectangle_2.png}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\includegraphics[scale=0.5]{rectangle_3.png}
\end{array}
\sim_{\Psi} \begin{array}{c}
\includegraphics[scale=0.5]{rectangle_4.png}
\end{array}
\end{array}
\]

**Figure 1.** Subdivision for the table.

Then $Q$ is admissible and $v$ is a return vector, but $Q$ and $Q - v$ are easily seen to be nowhere eventually coincident. Thus the $\mathbb{R}^2$-action on $\Omega_{\Psi}$ does not have pure discrete spectrum.

**Proposition 3.** Suppose that $\Omega_{\Psi}$ is an $n$-dimensional substitution tiling of $(m, d)$-Pisot family type whose $\mathbb{R}^n$ action has pure discrete spectrum. If the product of the nonzero eigenvalues of $f_* : H_1(X; Q) \to H_1(X; Q)$ is $\pm 1$, then $GR(\Phi) = \cup_{k \in \mathbb{Z}} \Lambda^k \mathcal{R}$.

**Proof.** Suppose that $u \in GR(\Phi)$. If $T \in \Omega$ then $T$ and $T - u$ are densely eventually coincident by Proposition 2. By Proposition 1, $u \in \cup_{k \in \mathbb{Z}} \Lambda^k v$.

Conversely, if the product of the nonzero eigenvalues of $f_* : H_1(X; Q) \to H_1(X; Q)$ is $\pm 1$, there is $j \in \mathbb{N}$ and a subgroup $H_{ER}$ of the free part $H_{\text{free}}$ of $H_1(X; \mathbb{Z})$ so that $f_\gamma^j(H_{\text{free}}) = H_{ER}$ and $f_\gamma | H_{ER} : H_{ER} \to H_{ER}$ is invertible. Let $v \in \mathcal{R}$ and let $[\gamma] \in H_1(X, \mathbb{Z})$ be so that $l([\gamma]) = v$. There are then $[\alpha] \in H_{ER}$ and $[\beta] \in H_1(X; \mathbb{Z})$ so that $[\gamma] = [\alpha] + [\beta]$ and $f_\gamma^j([\beta])$ is a torsion element. It follows from Lemma 4.2 that $l([\beta]) = 0$ and hence $l([\alpha]) = v$. We have $\Lambda^k v = \Lambda^k l([\alpha]) = l((f_* | H_{ER})^k([\alpha])) \in GR(\Phi)$, by Lemmas 4.2 and 4.3. \hfill \square

Even when the $\mathbb{R}^n$ action on an $n$-dimensional Pisot family tiling space has pure discrete spectrum, it is not necessarily the case that $GR(\Phi) = \cup_{k \in \mathbb{Z}} \Lambda^k \mathcal{R}$. For example, the Period Doubling substitution with tiles of unit length and inflation factor 2 (presented symbolically by $a \mapsto ab$, $b \mapsto aa$) has $GR(\Phi) = \mathbb{Z}$ and $\cup_{k \in \mathbb{Z}} \Lambda^k \mathcal{R} = \mathbb{Z}[1/2]$. To obtain equality, it is necessary that the eigenvalues of $\Lambda$ be algebraic units.

Substitutions satisfying the first set of hypotheses in the following corollary are called *unimodular, homological Pisot* and it is conjectured
(a version of the ‘Pisot Conjecture’) that their associated $\mathbb{R}^n$-actions always have pure discrete spectrum (see [7]).

**Corollary 3.** Suppose that $\Phi$ is an $n$-dimensional substitution of $(m, d)$-Pisot family type with $\dim (\tilde{H}^1(\Omega_\Phi; \mathbb{Q})) = md$ and also that the eigenvalues of the expansion $\Lambda$ are algebraic units. If the $\mathbb{R}^n$-action on $\Omega_\Phi$ has pure discrete spectrum then $GR(\Phi) = \bigcup_{k \in \mathbb{Z}} \Lambda_k$.

**Proof.** Since $\lim f : X \to X$ is homeomorphic with $\Omega_\Phi$, $\tilde{H}^1(\Omega_\Phi; \mathbb{Q})$ is isomorphic with $\lim (\tilde{f}^\ast) : H^1(X; \mathbb{Q}) \to H^1(X; \mathbb{Q})$. This direct limit is isomorphic with $g^\ast : \tilde{H}^1(\mathbb{T}^{md}; \mathbb{Q}) \to \tilde{H}^1(\Omega_\Phi; \mathbb{Q})$ is an injection and that the eigenvalues of $F^\ast : \tilde{H}^1(\mathbb{T}^{md}; \mathbb{Q}) \to \tilde{H}^1(\mathbb{T}^{md}; \mathbb{Q})$ are precisely the eigenvalues of $\Lambda$, together with all their algebraic conjugates, all of multiplicity $m$ ($\mathbb{T}^{md} = \mathbb{T}^{md}$ and $\tilde{F} = F$ since the eigenvalues of $\Lambda$ are units). Thus $g^\ast$ is surjective as well and $\hat{f}^\ast : \tilde{H}^1(\lim f; \mathbb{Q}) \to \tilde{H}^1(\lim f; \mathbb{Q})$, being conjugate with $\Phi^\ast$ and hence with $F^\ast$, has those same eigenvalues. Since $\hat{f}^\ast$ is conjugate with $f^\ast$ restricted to its eventual range, and the latter is (by the Universal Coefficient Theorem) just the transpose (dual) of $f_\ast : H_1(X; \mathbb{Q}) \to H_1(X; \mathbb{Q})$, restricted to its eventual range, we conclude that the eigenvalues of $f_\ast$ are precisely those of $\Lambda$, together with their algebraic conjugates, all of multiplicity $m$. As these eigenvalues are assumed to be units, the product of the nonzero eigenvalues of $f_\ast$ is $\pm 1$ and Proposition 3 applies. □

**Remark 2.** The counterexamples in [7] to a nonunimodular homological Pisot conjecture are also counterexamples to a straightforward extension of Corollary 3 to include $\Lambda$ with non unit eigenvalues. This seems to be a subtle issue. Also, Theorem 3.1 is sometimes not effective for $\Phi$ with such $\Lambda$. Consider again, for example, the Period Doubling substitution and let $Q$ be the patch consisting of a single $a$ tile. Then $\bar{Q}$ and $\bar{Q} - v$ are densely eventually coincident if and only if $v \in \mathbb{Z}[1/2]$. But for no such $v = a1$ are $a$ and $1$ independent over $\mathbb{Q}$. That is, the hypotheses of the theorem are never satisfied.

5. Applications

Given a substitution $\Phi$ and tilings $S$ and $S'$ by prototiles for $\Phi$, an overlap for $S$ and $S'$ is a pair of tiles $\{\tau, \tau'\}$ with $\tau \in S$, $\tau' \in S'$, and $\tau \cap \tau' \neq \emptyset$. The overlap $\{\tau, \tau'\}$ leads to coincidence if there is $x \in \tau \cap \tau'$ so that $S$ and $S'$ are eventually coincident at $x$. It is clear that if all pairs $\{\tau, \tau'\}$ that occur as overlaps for $\Phi^k(S)$ and $\Phi^k(S')$, $k \in \mathbb{N}$, lead to coincidence, then $S$ and $S'$ are densely eventually coincident. Pisot family substitution tilings have the Meyer property
(the set $\mathcal{R}$ of return vectors is relatively dense and uniformly discrete, and $\mathcal{R} - \mathcal{R}$ is uniformly discrete; see [25]) from which it follows that if $Q$ is an admissible patch for $\Phi$ that tiles periodically and $v \in GR(\Phi) \cup \cup_{k \in \mathbb{Z}} \Lambda^k \mathcal{R}$, then the collection of all translation equivalence classes of overlaps for $\Phi^k(\bar{Q}), \Phi^k(\bar{Q} - v), k \in \mathbb{N}$, is finite. In the examples of this section we implement Theorem 3.1 by finding appropriate $Q$ and $v$ so that each of the finitely many types of overlap that occur for $\Phi^k(\bar{Q}), \Phi^k(\bar{Q} - v), k \in \mathbb{N}$, leads to coincidence.

**Remark 3.** Our meaning of the term overlap is slightly nonstandard. More typically in the literature, an overlap is (a translation equivalence class of) a pair of tiles $(\tau, \tau' + v)$ with $\tau, \tau' \in T \in \Omega, v \in \mathcal{R}$, so that $\bar{\tau} \cap (\bar{\tau'} + v) \neq \emptyset$. In the algorithm of [1], it must be checked that all overlaps lead to coincidence. In particular, all overlaps must first be located. The (potentially) significant advantage offered by Theorem 3.1 is that it allows one to avoid altogether the task of finding all overlaps.

**Example 5.1. (Octagonal substitution)**

The (undecorated) octagonal substitution is a 2-dimensional Pisot family substitution with inflation $\Lambda = (1 + \sqrt{2})I$. There are 20 prototiles: four unmarked rhombi, $\rho_i = r^i(\rho_0), i = 0, \ldots, 3$, with $r^4(\rho_0) = \rho_0$, $r$ being rotation through $\pi/4$, and sixteen marked isosceles right triangles $\tau_i = r^i(\tau_0), \tau'_i = r^i(\tau'_0), i = 0, \ldots, 7$, with $\tau_i$ and $\tau'_i$ having the same support but bearing different marks. Let $\rho_0$ have vertices $(0, 0), (1, 0), (\sqrt{2}/2, \sqrt{2}/2)$, and $(1 + \sqrt{2}/2, \sqrt{2}/2)$, and let $\tau_0$ and $\tau'_0$ have vertices $(0, 0), (1, 0), (1, 1)$. The octagonal substitution, $\Phi_O$, is described in Figure 2.

![Octagon subdivision rule. The large shaded and unshaded triangles are $\tau'_3$ and $\tau_3$, resp.](image)

Let $Q$ be the unit square patch $Q = \{\tau_0, \tau'_4 + (1, 1)\}$ and let $w_1 = (1 - \sqrt{2}/2, \sqrt{2}/2)$. The following sequence of pictures (Figures 3 - 5) proves that $\bar{Q}$ and $\bar{Q} - w_1$ are densely eventually coincident.

Indeed, we see that each overlap that occurs for $\Phi_O^3(\bar{Q}), \Phi_O^3(\bar{Q} - w_1)$ already occurs (up to translation) for $\Phi_O^2(\bar{Q}), \Phi_O^2(\bar{Q} - w_1)$ and leads to coincidence in $\Phi_O^4(\bar{Q}), \Phi_O^4(\bar{Q} - w_1)$.

A similar argument shows that $\bar{Q}$ and $\bar{Q} - w_2$, with $w_2 = (\sqrt{2}/2, \sqrt{2}/2)$, are also densely eventually coincident. Since the $\mathbb{Z}^2$-action generated
Figure 3. Red overlay of the patch $\Phi^2_{\mathcal{O}}(Q)$ with background $\Phi^2_{\mathcal{O}}(\bar{Q} - w_1)$.

by \{w_1, w_2\} on the torus spanned by \{(1,0),(0,1)\} is minimal, the $\mathbb{R}^2$-action on $\Omega_{\mathcal{O}}$ has pure discrete spectrum. (This conclusion is well known: octagonal tilings can be obtained by a cut-and-project method with canonical window and such a provenance always guarantees pure discrete spectrum - see, for example, [5].)
Figure 4. $\Phi^3_Q (Q)$ and $\Phi^3_Q (\bar{Q} - w_1)$.

Figure 5. $\Phi^4_Q (Q)$ and $\Phi^4_Q (\bar{Q} - w_1)$.

Example 5.2. (Arnoux-Rauzy substitutions)

V. Berthé, T. Jolivet, and A. Siegel have obtained the result of this subsection by different methods ([12]).
We will say that words \( w \) and \( w' \) in the alphabet \( \mathcal{A} = \{1, \ldots, d\} \) are cyclically equivalent if there are non-degenerate words \( u, v \) with \( w = uv \) and \( w' = vu \). Such a cyclic equivalence is non-parallel if the abelianizations \([u]_{ab}\) and \([v]_{ab}\) are non-parallel vectors. For each \( i \in \mathcal{A} \), let \( \sigma_i : \mathcal{A} \to \mathcal{A}^* \) be the substitution defined by \( \sigma_i(j) = ji \), for \( j \neq i \), and \( \sigma_i(i) = i \).

**Lemma 5.3.** Suppose that \( u^i \in \mathcal{A}^* \), \( i = 1, \ldots, d \), are cyclically equivalent, and \( k \in \{2, \ldots, d\} \) and \( j \in \{1, \ldots, k\} \). Then \( \sigma_j(u^i) = v^1, \ldots, \sigma_j(u^k) = v^k \), \( \sigma_j((k + 1)u^{k+1}) = (k + 1)v^{k+1}, \ldots, \sigma_j(du^d) = dv^d \), with \( v^i, i = 1, \ldots, d \), cyclically equivalent.

**Proof.** Let \( k \in \{2, \ldots, d\} \), \( j \in \{1, \ldots, k\} \) and \( l \in \{k + 1, \ldots, d\} \). Then \( \sigma_j(lu^l) = lj \sigma_j(u^l) = lj(w^l)j \). Therefore, \( v^i, i \in \{1, \ldots, k\} \), and \( w^l j, l = k + 1, \ldots, d \), are cyclically equivalent, and hence \( v^i, i \in \{1, \ldots, k\} \), and \( v^l = jw^l, l = k + 1, \ldots, d \), are cyclically equivalent. \( \square \)

**Lemma 5.4.** Suppose that \( u^i, i = 1, \ldots, d \), are cyclically equivalent and \( k \in \{3, \ldots, d\} \). Then \( \sigma_k(u^1) = v^1, \ldots, \sigma_k(u^{k-1}) = v^{k-1} \), \( \sigma_k(ku^k) = v^k \), \( \sigma_k((k + 1)u^{k+1}) = (k + 1)v^{k+1}, \ldots, \sigma_k(du^d) = dv^d \), with \( v^i, i = 1, \ldots, d \), cyclically equivalent.

**Proof.** Let \( k \in \{3, \ldots, d\} \) and \( l \in \{k + 1, \ldots, d\} \). Then \( \sigma_k(ku^k) = k \sigma_k(u^k) = k(w^k)k \) and \( \sigma_k(lu^l) = lk \sigma_k(u^l) = lk(w^k)l \). Therefore, \( v^i, i \in \{1, \ldots, k - 1\} \), \( w^k k \) and \( w^l k, l = k + 1, \ldots, d \), are cyclically equivalent, and hence \( v^i, i \in \{1, \ldots, k-1\} \), \( v^k = kw^k \) and \( v^l = kw^l, l = k + 1, \ldots, d \), are cyclically equivalent. \( \square \)

Given a word \( w = w_1 w_2 \cdots w_k \in \mathcal{A}^* \), let \( \sigma_w := \sigma_{w_1} \circ \cdots \circ \sigma_{w_k} \). A substitution \( \phi : \mathcal{A} \to \mathcal{A}^* \) is Arnoux-Rauzy if \( \phi = \sigma_w \) for some \( w \) with \([w]_{ab}\) strictly positive (strict positivity guarantees that \( \phi \) is primitive and that the fixed words \( \phi^\infty(i) \) of Proposition 4 are infinite).

**Proposition 4.** If \( \phi = \sigma_w \) is Arnoux-Rauzy on \( d \) letters, then there are prefixes \( P_i \) of the fixed words \( \phi^\infty(i) = P_i, i = 1, \ldots, d \) so that there is a non-parallel cyclic equivalence between \( P_i \) and \( P_j \) for \( i \neq j \in \{1, \ldots, d\} \).

**Proof.** By symmetry of \( \sigma_i \)'s we may assume, after renumbering, that \( w = Y d X_{d-2} d - 1 \ldots 3 X_j 21^n \), with \( X_j \) having only 1's, \( \ldots, (j + 1)'s \).
Note that
\[
\sigma_{21^n}(1) = 12 \\
\sigma_{21^n}(2) = 21212 \ldots \\
\sigma_{21^n}(3) = 3212 \ldots \\
\vdots \\
\sigma_{21^n}(d) = d212 \ldots
\]

By Lemma 5.3 (for \( k = 2 \)) we have that
\[
\sigma_X, \sigma_{21^n}(1) = v_1^1 \\
\sigma_X, \sigma_{21^n}(2) = v_1^2 \\
\sigma_X, \sigma_{21^n}(3) = 3v_1^3 \ldots \\
\vdots \\
\sigma_X, \sigma_{21^n}(d) = dv_1^d \ldots
\]
with \( v_1^i, i = 1, \ldots, d, \) cyclically equivalent.

By Lemma 5.4 (for \( k = 3 \)) we have that
\[
\sigma_X, \sigma_{21^n}(1) = v_2^1 \\
\sigma_X, \sigma_{21^n}(2) = v_2^2 \\
\sigma_X, \sigma_{21^n}(3) = v_2^3 \ldots \\
\sigma_X, \sigma_{21^n}(4) = 4v_2^4 \ldots \\
\vdots \\
\sigma_X, \sigma_{21^n}(d) = dv_2^d \ldots
\]
with \( v_2^i, i = 1, \ldots, d, \) cyclically equivalent, etc.

Let \( P_i := \sigma_Y(v_{2d}^i) \). Then \( P_i \) and \( P_j \) are cyclically equivalent for \( i \neq j \).
If \( P_1 = uv \) with \( u, v \) non-degenerate, then \( M_\phi[1]_{ab} = [\phi(1)]_{ab} = [P_1]_{ab} = [P_i]_{ab} = [u]_{ab} + [v]_{ab} \). So if \( [v]_{ab} = t[u]_{ab} \), then \( \frac{1}{1-t1} [1]_{ab} = M_\phi^{-1}[u]_{ab} \). But \( M_\phi^{-1}[u]_{ab} \) is in \( \mathbb{Z}^d \) and \( \frac{1}{1-t1} [1]_{ab} \) is not. Thus the cyclic equivalences are non-parallel. \( \square \)

If \( \phi \) is an Arnoux-Rauzy substitution on \( d \) letters with Pisot inflation factor (that is, the Perron-Frobenius eigenvalue of \( \lambda \) of \( M_\phi \) is Pisot) we’ll say that \( \phi \) is Pisot Arnoux-Rauzy and irreducible Pisot Arnoux-Rauzy if \( M_\phi \) is irreducible. It is shown in [3] that all Arnoux-Rauzy substitutions on three letters are irreducible Pisot. If \( \phi \) is Pisot Arnoux-Rauzy, the maximal equicontinuous factor for the \( \mathbb{R} \)-action on the associated 1-dimensional tiling space \( \Omega_\phi \) is an \( \mathbb{R} \)-action on the \( m \)-torus \( \mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m, m = \text{deg}(\lambda) \) (this is because \( \text{det}(M_\phi) = 1 \), so \( \lambda \) is a unit).
The maximal equicontinuous factor map $g : \Omega_\Phi \to \mathbb{T}^m$ semi-conjugates $\Phi$ with a hyperbolic toral automorphism $F : \mathbb{T}^m \to \mathbb{T}^m$.

**Lemma 5.5.** If $\phi$ is a Pisot Arnoux-Rauzy substitution, $T$ and $T'$ in $\Omega_\Phi$ are tilings that have a common vertex, and $n_k \to \infty$ is such that $\Phi^{n_k}(T) \to S \in \Omega_\Phi$ and $\Phi^{n_k}(T') \to S' \in \Omega_\Phi$, then $g(S) = g(S')$.

**Proof.** Suppose that $T$ and $T'$ have a common vertex at $t_0$. Then $\Phi(T)$ and $\Phi(T')$ share a finite (of type $w_1$) with right vertex at $\lambda t_0$. Then, for $t$ slightly bigger than $t_0$, $T - t$ is in the $\Phi$-stable manifold of $T' - t$ and $g(T - t) = g(T) - t$ is in the F-stable manifold of $g(T' - t) = g(T') - t$. But then $g(T)$ is in the same $F$-stable manifold as $g(T')$ so that $g(S) = g(\lim_{k \to \infty} \Phi^{n_k}(T)) = \lim_{k \to \infty} F^{n_k}(g(T)) = \lim_{k \to \infty} F^{n_k}(g(T')) = g(\Phi^{n_k}(T')) = g(S')$. \hfill \Box

Let $cr = cr(\phi)$ be the coincidence rank of the Pisot substitution $\phi$: $cr := \max \{ T_1, \ldots , T_m : g(T_i) = g(T_j) \text{ and } T_i \cap T_j = \emptyset \text{ for } i \neq j \}$. The coincidence rank is finite and $g$ is a.e. $cr$-to-1 (see [10] or [8]). Furthermore, if $g(T_i) = g(T_j)$ and $T_i \cap T_j = \emptyset$, then $T_i$ and $T_j$ are nowhere eventually coincident ([8]). Let $T_1, \ldots , T_{cr} \in \Omega_\Phi$ be tilings with $g(T_i) = g(T_j)$ and $T_i$ nowhere eventually coincident with $T_j$ for $i \neq j$, let $V_i$ be the collection of vertices of $T_i$, let $V := \bigcup_{i=1}^{cr} V_i$, and write $V = \{ v_n : v_n < v_{n+1}, n \in \mathbb{Z} \}$. Note that if $\phi$ is Arnoux-Rauzy, then $V_i \cap V_j = \emptyset$ for $i \neq j$ (if $T$ and $T'$ share a vertex, then $\Phi(T)$ and $\Phi(T')$ share a tile of type $w_1$). For each $n \in \mathbb{Z}$ and $i \in \{ 1, \ldots , cr \}$, let $\tau_{n(i)}$ be the tile of $T_i$ whose support meets $(v_n, v_{n+1})$. The pair $C_n := (\{ v_n, v_{n+1} \}, \{ \tau_{n(1)}, \ldots , \tau_{n(cr)} \})$ is called a configuration. Configurations $C_n$ and $C_m$ have the same type if there is $t \in \mathbb{R}$ so that $[v_n, v_{n+1}] - t = [v_m, v_{m+1}]$ and $\{ \tau_{n(1)} - t, \ldots , \tau_{n(cr)} - t \} = \{ \tau_{m(1)}, \ldots , \tau_{m(cr)} \}$. By a result of [8], there are only finitely many types of configurations.

**Lemma 5.6.** If $\phi$ is Arnoux-Rauzy, there are $m < n$ so that the configurations $C_n$ and $C_m$ have the same type, but $C_{n+1}$ and $C_{m+1}$ have different types.

**Proof.** Otherwise, there is $L > 0$ so that $C_n$ and $C_{n+L}$ have the same type for all $n \in \mathbb{Z}$. For $n \in \mathbb{Z}$ let $i(n) \in \{ 1, \ldots , cr \}$ be (uniquely) defined by: $v_n$ is a vertex of a tile in $T_{i(n)}$. There are then $n \in \mathbb{Z}$ and $k \in \mathbb{N}$ so that $i(n) = i(n + kL)$. But then $i(n + jkL) = i(n)$ for all $j \in \mathbb{Z}$ and the tiling $T_{i(n)}$ is periodic. \hfill \Box

For an Arnoux-Rauzy substitution $\phi$ on $d$ letters, let $S_i \in \Omega_\Phi$ denote the tiling fixed by $\Phi$ containing a tile of type $i$ with the origin at its left endpoint. (The underlying word of $S_i$ is $\phi^\infty(w_1,i)$.)
Lemma 5.7. If $\phi$ is a Pisot Arnoux-Rauzy substitution, then there are $i \neq j$ so that the fixed tilings $S_i, S_j \in \Omega_\phi$ are densely eventually coincident.

Proof. By Lemma 5.6, there are $m < n$ so that the configurations $C_n$ and $C_m$ have the same type, but $C_{n+1}$ and $C_{m+1}$ have different types. Let $i$ be such that vertex $v_{n+1}$ is the initial vertex of a tile $\tau$ of type $i$ in $T_{i(n+1)}$ and let $j$ be such that vertex $v_{m+1}$ is the initial vertex of a tile $\tau'$ of type $j$ in $T_{i(m+1)}$. Note that $\{\tau_{n+1}(1) - v, \ldots, \tau_{n+1}(cr) - v\} \setminus \{\tau - v\} = \{\tau_{m+1}(1), \ldots, \tau_{m+1}(cr)\} \setminus \{\tau'\}$, with $v := v_{n+1} - v_{m+1}$. Let $b := \min\{v_{m+2}, v_{n+2} - v\}$. Suppose that there are $t_0$ and $\epsilon > 0$ so that $(t_0 - \epsilon, t_0 + \epsilon) \subset (v_{m+1}, b)$ and $\tau_{m+1}(1), \ldots, \tau_{m+1}(cr), \tau - v$ are pairwise nowhere eventually coincident on $(t_0 - \epsilon, t_0 + \epsilon)$. There is then $n_k \to \infty$ so that $\Phi^{n_k}(T_j - t_0) \to T'_j \in \Omega_\phi$ for $j = 1, \ldots, cr$ and $\Phi^{n_k}(T_{i(n+1)} - v - t_0) \to T'_{cr+1} \in \Omega_\phi$. The tilings $T'_j, j = 1, \ldots, cr + 1$, are then pairwise nowhere eventually coincident and they all have the same image under $\phi$, by Lemma 5.5. This violates the definition of $cr$. As the elements of $\{\tau_{m+1}(1), \ldots, \tau_{m+1}(cr)\}$, and of $\{\tau_{m+1}(1), \ldots, \tau_{m+1}(cr)\} \setminus \{\tau - v\}$ are pairwise nowhere eventually coincident, it must be the case that there is $t \in (t_0 - \epsilon, t_0 + \epsilon)$ with $\tau'$ and $\tau - v$ eventually coincident at $t$. That is, $\tau'$ and $\tau - v$ are densely eventually coincident on $(v_{m+1}, b)$. This means that there is $\delta > 0$ so that $S_i$ and $S_j$ are densely eventually coincident on $(0, \delta)$. As $\Phi^k(S_i) = S_i$ and $\Phi^k(S_j) = S_j$ for all $k$, we have that $S_i$ and $S_j$ are densely eventually coincident on $(0, \lambda^k\delta)$ for all $k$, so $S_i$ and $S_j$ are densely eventually coincident.

Theorem 5.8. If $\phi$ is an irreducible Pisot Arnoux-Rauzy substitution, then the $\mathbb{R}$-action on $\Omega_\phi$ has pure discrete spectrum.

Proof. Let $S_i$ and $S_j$, $i \neq j$, be the fixed tilings in $\Omega_\phi$ that are densely eventually coincident (Lemma 5.7), and let $Q_i$ and $Q_j$ be their initial patches, corresponding to the prefixes $P_i$ and $P_j$ that are non-parallel cyclically equivalent (Proposition 4). Let $u$ and $v$ be non-degenerate words with non-parallel abelianizations so that $P_i = uv$ and $P_j = vu$, let $l_u$ be the length of the patch corresponding to $u$, and let $l_v$ be the length of $Q_i$. Suppose that $rl_u - sl = 0$ for integers $r, s$. The length of a tile of type $i$ is given by $\langle [i]_{ab}, \omega \rangle$ where $\omega = (\omega_1, \ldots, \omega_d)$ is a positive left Perron eigenvector of $M_\phi$. We have $0 = rl_u - sl = r\langle [u]_{ab}, \omega \rangle - s\langle [u + v]_{ab}, \omega \rangle = \langle (r - s)[u]_{ab} - s[v]_{ab}, \omega \rangle$. But the $\omega_i$ are independent over $\mathbb{Q}$ (by irreducibility), so the only integer vector orthogonal to $\omega$ is $0$. Thus $(r - s)[u]_{ab} = s[v]_{ab}$, and since $[u]_{ab}$ and $[v]_{ab}$ are not parallel, $r = 0 = s$. Thus, $l_u$ is irrationally related to $l$. We have that $Q_i$ and $Q_j = Q_i - l$ are densely eventually coincident. By Theorem 3.1, the $\mathbb{R}$-action on $\Omega_\phi$ has pure discrete spectrum. \qed
Example 5.9. (Arbitrary compositions of the Rauzy and modified Rauzy substitutions)

Let us consider the Rauzy substitution $\tau_1$: $1 \to 12$, $2 \to 13$ and $3 \to 1$, and modified Rauzy substitutions defined as follows, $\tau_2$: $1 \to 12$, $2 \to 31$ and $3 \to 1$; $\tau_3$: $1 \to 21$, $2 \to 13$ and $3 \to 1$; and $\tau_4$: $1 \to 21$, $2 \to 31$ and $3 \to 1$. It is easy to check that applying $\tau_1$, $\tau_2$, $\tau_3$ or $\tau_4$ to any pair in the list below results in a pair that can be factored as a product of pairs in the list below (or of the duals $(v, u)$ of pairs $(u, v)$ in the list) and coincidences.

List of irreducible pairs: $(12, 21)$, $(13, 31)$, $(123, 231)$, $(321, 132)$, $(213, 312)$, $(1123, 3112)$, $(3211, 2113)$, $(1213, 3112)$, $(1231, 3112)$, $(1231, 2113)$, $(1321, 2113)$, $(1213, 1321)$, $(1213, 1231)$, $(1231, 3112)$, $(3212, 11312)$, $(1231, 1321)$, $(1213, 1321)$, $(1231, 2131)$, $(1231, 2131)$, $(1321, 2131)$, $(1213, 3121)$, $(1213, 3121)$, $(1231, 1132)$, $(1231, 2131)$, $(1213, 1321)$, $(1213, 1231)$, $(1231, 3112)$, $(1231, 2113)$, $(1213, 3121)$, $(1213, 3121)$, $(1321, 2113)$, $(1213, 1321)$, $(1213, 1231)$, $(1231, 3112)$, $(1231, 2113)$, $(1321, 2113)$, $(1213, 3121)$, $(1213, 3121)$, $(1321, 2113)$, $(1213, 1321)$, $(1213, 1231)$, $(1321, 2113)$, $(1213, 3121)$, $(1213, 3121)$.

Note that for any $i, j \in \{1, 2, 3, 4\}$ and any $a, b \in \{1, 2, 3\}$, $(\tau_i \circ \tau_j(a), \tau_i \circ \tau_j(b))$ is either of the form $(uk\ldots, vk\ldots)$ or $(\ldots ku, \ldots kv)$ for some $k \in \{1, 2, 3\}$ and (possibly empty) words $u, v$ with $[u]_{ab} = [v]_{ab}$. Thus if $\tau$ is any finite composition of the $\tau_i$’s, the balanced pair algorithm for $\tau$ applied to $(12, 21)$ terminates with coincidence. Since such $\tau$ are Pisot with irreducible incidence matrix, the following is a consequence of Corollary 2.

Theorem 5.10. If $\tau$ is any finite composition of Rauzy and modified Rauzy substitutions, the $\mathbb{R}$-action on $\Omega_\tau$ has pure discrete spectrum.

References

[1] (MR2764877)[10.1016/j.aim.2010.07.019] S. Akiyama and J. Y. Lee, Algorithm for determining pure pointedness of self-affine tilings, Adv. Math., 226 (2011), 2855–2883.
[2] (MR1631708)[10.1017/S0143385798100457] J. E. Anderson and I. F. Putnam, Topological invariants for substitution tilings and their associated $C^*$-algebras, Ergodic Theory & Dynamical Systems, 18 (1998), 509–537.
[3] (MR1838930) P. Arnoux and S. Ito, Pisot substitutions and Rauzy fractals, Bull. Belg. Math Soc., 8 (2001), 18–2007.
[4] (MR0956049) J. Auslander,“Minimal Flows and Their Extensions,” North-Holland Mathematical Studies, 153, North-Holland, Amsterdam, New York, Oxford, and Tokyo, 1988.
[5] (MR2084582)[10.1515/crll.2004.064] M. Baake and R. V. Moody, Weighted Dirac combs with pure point diffraction, J. Reine Angew. Math., 573 (2004), 61–94. .
[6] (MR2290779)[10.5802/aif.2238] V. Baker, M. Barge and J. Kwapisz, Geometric realization and coincidence for reducible non-unimodular Pisot tiling spaces with an application to $\beta$-shifts, J. Instit. Fourier, 56 (2006), 2213–2248.

[7] (MR2897733) M. Barge, H. Bruin, L. Jones and L. Sadun, Homological Pisot substitutions and exact regularity, To appear in Israel J. Math., preprint, arXiv1001.2027.

[8] M. Barge and J. Kellendonk, Proximality and pure point spectrum for tiling dynamical systems, preprint, arXiv1108.4065.

[9] M. Barge, J. Kellendonk and S. Schmeiding, Maximal equicontinuous factors and cohomology of tiling spaces, To appear in Fund. Math., preprint, arXiv1204.1432.

[10] (MR2262174)[10.1353/ajm.2006.0037] M. Barge and J. Kwapisz, Geometric theory of unimodular Pisot substitutions, Amer J. Math., 128 (2006), 1219–1282.

[11] M. Barge and C. Olimb, Asymptotic structure in substitution tiling spaces, To appear in Ergodic Theory & Dynamical Systems, preprint, arXiv1101.4902.

[12] V. Berthé, T. Jolivet and A. Siegel, Substitutive Arnoux-Rauzy substitutions have pure discrete spectrum, preprint, arXiv1108.5574.

[13] (MR2191748) V. Berthé and A. Siegel, Tilings associated with beta-numeration and substitutions, Integers: electronic journal of combinatorial number theory, 5 (2005), A02.

[14] (MR0461470) F. M. Dekking, The spectrum of dynamical systems arising from substitutions of constant length, Z. Wahrscheinlichkeitstheorie verw. Gebiete, 41 (1978), 221–239.

[15] (MR1224190)[10.1063/1.530108] S. Dworkin, Spectral theory and X-ray diffraction, J. Math. Phys., 34 (1993), 2965–2967.

[16] (MR1970385) N. P. Fogg, “Substitutions in Dynamics, Arithmetics and Combinatorics,” Lecture notes in mathematics, (eds. V. Berthé, S. Ferenczi, C. Mauduit and A. Siegel), Springer-Verlag, 2002.

[17] (MR2301525)[10.1007/s00454-006-1280-9] D. Fretlöh and B. Sing, Computing modular coincidences for substitution tilings and point sets, Discrete Comput. Geom., 37 (2007), 381–407.

[18] (MR2254640)[10.1007/BF02771781] S. Ito and H. Rao, Atomic surfaces, tiling and coincidence I. Irreducible case, Israel J. Math., 153 (2006), 129–156.

[19] R. Kenyon, Ph. D. Thesis, Princeton University, 1990.

[20] (MR2587839) R. Kenyon and B. Solomyak, On the characterization of expansion maps for self-affine tilings, Discrete Comput. Geom., 43 (2010), 577–593.

[21] (MR2360242)[10.1016/j.geomphys.2007.07.003] J. Y. Lee, Substitution Delone multisets with pure point spectrum are inter-model sets, Journal of Geometry and Physics, 57 (2007), 2263–2285.

[22] (MR1811757)[10.1007/s004540010083] J. Y. Lee and R. Moody, Lattice substitution systems and model sets, Discrete Comput. Geom., 25 (2001), 173–201.

[23] (MR1976605)[10.1007/s00454-003-0781-z] J. Y. Lee, R. Moody and B. Solomyak, Consequences of Pure Point Diffraction Spectra for Multiset Substitution Systems, Discrete Comp. Geom., 29 (2003), 525–560.

[24] (MR2383764)[10.1007/s00454-008-9054-1] J. Y. Lee and B. Solomyak, Pure point diffractive substitution Delone sets have the Meyer property, Discrete Comp. Geom., 34 (2008), 319–338.
25

[25] (MR2851885) J. Y. Lee and B. Solomyak, *Pisot family self-affine tilings, discrete spectrum, and the Meyer property*, preprint, arXiv1002.0039.

[26] (MR1155902) A. N. Livshits, *Some examples of adic transformations and substitutions*, Selecta Math. Sovietica, 11 (1992), 83–104.

[27] (MR0486421)[10.1007/BF00641410] P. Michel, *Coincidence values and spectra of substitutions*, Zeit. Wahr., 42 (1978), 205–227.

[28] (MR2721985) A. Siegel and J. Thuswaldner, *Topological properties of Rauzy fractals*, preprint.

[29] (MR1941235)[10.4153/CMB-2002-062-3] V. F. Sirivent and B. Solomyak, *Pure discrete spectrum for one-dimensional substitution systems of Pisot type*, Canad. Math. Bull., 45 (2002), 697–710. Dedicated to Robert V. Moody.

[30] (MR1637896)[10.1007/PL00009386] B. Solomyak, *Nonperiodicity implies unique composition for self-similar translationally finite tilings*, Discrete Comput. Geometry, 20 (1998), 265–279.

[31] (MR2405614) B. Solomyak, *Eigenfunctions for substitution tiling systems*, Advanced Studies in Pure Mathematics, 49 (2007), 433–454.

[32] (MR1452190)[10.1017/S0143385797008498] B. Solomyak, *Dynamics of self-similar tilings*, Ergodic Theory & Dynamical Systems, 17 (1997), 695–738.

[33] W. A. Veech, *The equicontinuous structure relation for minimal Abelian transformation groups*, Amer. J. of Math. 90 (1968), 723-732.

Received July 2011; revised April 2012.

E-mail address: barge@math.montana.edu
E-mail address: sonja@math.hr
E-mail address: bob@math.utexas.edu