Full Electroweak $\mathcal{O}(\alpha)$ corrections to Higgs boson production processes with the beam polarization at the International Linear Collider†

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We present the full $\mathcal{O}(\alpha)$ electroweak radiative corrections to 9 Higgs boson production with the beam polarization at the International Linear Collider (ILC). The computation is performed with the help of GRACE-Loop. We compared the full $\mathcal{O}(\alpha)$ electroweak radiative corrections with the factorized initial state radiation (ISR) effects to estimate the pure weak corrections. This reveals the pure weak corrections are not negligible and should be taken into account for the precision measurements of the Higgs boson property at the ILC experiments.

Subject Index

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1 Introduction

The ILC is a future linear electron-positron collider, operating at 250 to 500 GeV center-of-mass energies with high luminosity which can be extended to 1 TeV in the upgrade stage. It is, based on 1.3 GHz superconducting radio-frequency (SCRF) accelerating technology [2],[3]. The ILC is proposed to be constructed in the Kitakami Mountains in Tohoku area, Japan. It is an international project running for more than 20 years collaborated by more than 300 institutes, universities, and laboratories.

The major physics aim of the ILC is to determine the future direction of particle physics via the precise measurements of the couplings of the Higgs boson with other elementary particles. A wide variety of elementary particles and nuclear physics can be studied.

On July 4 in 2012, the Higgs boson discovery was announced by physicists from two experiments (the CMS group and the ATLAS group) of the LHC at CERN. A new particle with a mass of approximately 125 GeV and several other properties of the Higgs boson, as predicted by the SM, were observed in Refs.[4] and [5]. The Higgs boson discovery made the SM of particle physics complete.

The prediction of additional Higgs bosons is one of the prominent features of possible physics of beyond the standard model, BSM, which leads to an extended Higgs-boson sector. Searching for higher-mass Higgs bosons is also very interesting. However, the hypothesis that the Higgs boson is heavy and approaches the theoretical upper bound has been validated by the expected exclusion regions with 300 fb$^{-1}$ and 3 ab$^{-1}$ LHC data. Therefore, the ILC is an exemplary collider to study the 125 GeV Higgs boson. With this Higgs particle, the first course at the ILC would be at threshold around $\sqrt{s} = 250$ GeV, which displays the clear peak cross section for the process $e^+e^- \rightarrow ZH$. At this energy, the precise measurements of Higgs recoil mass for the Higgs-strahlung process $e^+e^- \rightarrow ZH$ with subsequent $Z \rightarrow l^+l^- (l = e, \mu)$ decay is one of the most important measurements. This measurement allows a model-independent absolute measurement of the $g_{HZZ}$ coupling. In this setting, it is possible to measure the proportion for all of Higgs boson decays including common final states and invisible decays with great accuracy.

It is not necessary to observe the Higgs decay because corresponding hidden decay is observable. Nevertheless, the $e^+e^- \rightarrow ZH$ process can be used to measure diverse branching ratios for various Higgs decay processes. the $Z \rightarrow q\bar{q}$ and $Z \rightarrow \nu\bar{\nu}$ processes were included in this analysis to increase the statistics. The Higgs boson can also decay into a pair of W-bosons.
However, the measurement of the WW-fusion process at $\sqrt{s} = 250$ GeV is quite difficult. Conversely, the W-pair production process becomes active at $\sqrt{s} = 500$ GeV, and hence the energy setting of the ILC is expected to be changed, accordingly [3].

In this paper, we have studied 9 processes of $e^+e^- \rightarrow f\bar{f}H$ with the polarized electron and positron beams of center-of-mass energy at 250 GeV, i.e. the production of (1) muon pair and Higgs boson, (2) electron-positron pair and Higgs boson, (3) tau-pair and Higgs boson, (4) muon-neutrino pair and Higgs boson, (5) electron-neutrino pair and Higgs boson, (6) up-quark pair and Higgs boson, (7) down-quark pair and the Higgs boson, (8) charm-quark pair and the Higgs boson, and (9) bottom-quark pair and the Higgs boson (note that we did not consider the production of top quark pair, strange quark pair, and tau-neutrino pair). We calculated the $\mathcal{O}(\alpha)$ corrections using the on-shell renormalization scheme to each of these 9 reactions as well as the one-photon emission process.

In this work, we have used the GRACE-Loop system to calculate the amplitudes automatically. We have developed two sets of Fortran codes.

The first set was used to calculate the $O(\alpha)$ corrections corresponding to the one-loop diagrams and the one-photon emission processes; in this case, all the fermion masses were kept non-zero, except the neutrino masses, and the arbitrary longitudinal polarization of the input electron and positron beams were available. The second set was used to treat the effects of initial-state radiation (ISR). The effects of one-loop weak corrections were determined by comparing the $O(\alpha)$ corrections with the ISR effects.

We introduce “NOLFS (NO Light-Fermion Scalar coupling) approximation”, where the couplings between the light-fermions (except for the bottom- and top-quarks) and scalar particles are neglected, but the masses of all fermions are retained. We confirmed the one-loop amplitudes generated using the NOLFS approximation to be consistent with those obtained from the Monte-Carlo integration of the full Lagrangian. Therefore, in this work, we present and discuss the results obtained using the NOLFS approximation.

The structure of this paper is as follows; after the introduction, we will explain the calculation framework of GRACE-Loop in Section II. In Section III, we will define the beam polarization of the ILC beams. The recoil mass analysis is introduced in Section IV. In Section V, we will explain the radiator method of the initial state radiation. Then we will present the numerical results of the leptonic processes in Section VI and those of quark processes in Section VII. Then we will make a discussion in Section VIII. Section IX is devoted to the conclusion of this paper.
2 GRACE-Loop and the calculation of $e^+e^- \rightarrow f \bar{f}H$

The GRACE-Loop system is a programming system for calculating full one-loop electroweak cross sections and the tree-level automatically, with beam polarizations based on the SM and Minimal Supersymmetric Standard Model (MSSM) of high energy physics. This program was created and developed by the Minami-Tateya group at the High Energy Accelerator Research Organization, KEK [6]. It uses the symbolic manipulation system FORM [7] to calculate all Dirac and tensor algebra in $n$ dimensions.

In the GRACE system [6], at the tree level, the beam polarization effects are taken into account from the beginning to produce the helicity amplitudes, which are evaluated by a numerical way. Nevertheless, at the full one-loop electroweak corrections with the GRACE-Loop system, it adopts the trace techniques to get the interference between the tree amplitudes and one-loop ones using the symbolical manipulation with the FORM system. Then, the polarization effects of the beams are included with the insertion of the projection operators, $\frac{1 \pm \gamma_5}{2}$ into amplitudes. It makes the program size become much larger.

It has been successfully tested for a variety of one-loop $2 \rightarrow 2$ electroweak processes [8]. It also provided the first results on the full one-loop radiative corrections to $e^+e^- \rightarrow \nu \bar{\nu}H$ [9] which has been confirmed by an independent calculation [10].

The GRACE-Loop system primarily focuses on evaluating one-loop corrections to the SM processes during electron-positron collisions. In addition, it can calculate one-loop corrections to the MSSM [11].

It should be noted the FULL model contains all of the couplings of the scalar particles such as the Higgs boson or pseudo-Goldstone scalar bosons and fermions in the SM are included.

In this paper, the bottom- and top-quark are considered to be heavy fermions and other fermions are recognized to be light ones.

When the two cases of the FULL model and NOLFS approximation are compared, the number of tree diagrams and one-loop Feynman diagrams in the former is much larger than in the latter; thus, the integration over phase space is not practical in the former, as shown in Table 1.

From Table 1 we observe that the number of one-loop Feynman diagrams of the $\mu^+\mu^-H$ and $\tau^+\tau^-H$ processes have the same, with 208 and 2235 diagrams with the NOLFS approximation and the FULL model, respectively. Similarly, the number of one-loop Feynman diagrams of $u\bar{u}H$, $d\bar{d}H$ and $c\bar{c}H$ processes have the same, with 209 and 2327 diagrams
Table 1 The numbers of Feynman diagrams for the NOLFS approximation and the FULL model of 9 processes

| Graph information | one-loop | Tree | 5-point | one-loop | Tree | 5-point |
|-------------------|----------|------|---------|----------|------|---------|
| $\mu^+ \mu^- H$   | 208      | 1    | 10      | 2235     | 21   | 170     |
| $e^+ e^- H$       | 416      | 2    | 20      | 4470     | 42   | 340     |
| $\tau^+ \tau^- H$| 208      | 1    | 28      | 2235     | 21   | 188     |
| $\nu_\mu \bar{\nu}_\mu H$ | 122    | 1    | 6       | 604      | 4    | 36      |
| $\nu_e \bar{\nu}_e H$ | 219   | 2    | 15      | 1350     | 12   | 113     |
| $u\bar{u} H$      | 209      | 1    | 10      | 2327     | 21   | 174     |
| $d\bar{d} H$      | 209      | 1    | 10      | 2327     | 21   | 174     |
| $c\bar{c} H$      | 209      | 1    | 10      | 2327     | 21   | 174     |
| $b\bar{b} H$      | 651      | 6    | 29      | 2327     | 21   | 193     |

For all electroweak processes we adopt the on-shell renormalisation scheme according to [12] and [8]. The one-loop scalar integrals with the width are evaluated using the LoopTools packages [13], on the other hand, the detail of the reduction on the tensor integrals and the 5-point functions based on the Feynman parameters are described in [8].

The produced matrix elements to the electroweak corrections by the GRACE-Loop, are checked by performing three kinds of tests at a random point in phase space. For these tests to be passed one works in quadruple precision.

We first have to check the ultraviolet finiteness of the results. This test applies to the whole set of the virtual one-loop diagrams. In order to conduct this test we regularise any infrared divergence by giving the photon a fictitious mass (we set this at $\lambda = 10^{-15}\text{GeV}$).

In the intermediate step of the symbolic calculation dealing with loop integrals (in $n$-dimension), we extract the regulator constant $C_{\text{UV}} = 1/\varepsilon - \gamma_E + \log 4\pi$, $n = 4 - 2\varepsilon$ and treat this as a parameter. The ultraviolet finiteness test is performed by varying the dimensional regularisation parameter $C_{\text{UV}}$. This parameter could then be set to 0 in further computation.
By varying $C_{UV}$ from 0 to 100 for the $e^+e^- \rightarrow \mu^+\mu^- H$ process of the FULL model with $k_c = 10^{-1}$ GeV, $\lambda = 10^{-17}$ GeV and $\Gamma_Z = 0$ GeV at $\sqrt{s} = 250$ GeV, we obtained $C_{UV}$ independence with an agreement up to 33 digits as shown in Table 2.

| $C_{UV}$ | Evaluation of the 1 loop amplitude |
|----------|------------------------------------|
| 0        | 0.127132019079521763060824923741166 |
| 100      | 0.127132019079521763060824923741166 |

Table 2  Changing $C_{UV}$ from 0 to 100 of $e^+e^- \rightarrow \mu^+\mu^- H$ of the FULL model at a phase point in the arbitrary unit with $k_c = 10^{-1}$ GeV, $\lambda = 10^{-17}$ GeV and $\Gamma_Z = 0$ GeV at $\sqrt{s} = 250$ GeV.

As the second test, the infrared finiteness was performed by including both loop and bremsstrahlung contributions and checking that there is no dependence on the fictitious photon mass $\lambda$.

Fictitious photon mass($\lambda$) independence was achieved by varying $\lambda$ from $10^{-17}$ GeV to $10^{-19}$ GeV with $C_{UV} = 100$ and $\Gamma_Z = 0$ GeV has up to 14-digit agreement as shown in Table 3.

| $\lambda$ [GeV] | Evaluation of the 1 loop amplitude |
|----------------|------------------------------------|
| $10^{-17}$     | 0.127132019079521763060824923741166 |
| $10^{-19}$     | 0.127132019079522845528273933268792 |

Table 3  $\lambda$ independence of $e^+e^- \rightarrow \mu^+\mu^- H$ of the FULL model at a phase space point in the arbitrary unit by changing from $10^{-17}$ GeV to $10^{-19}$ GeV with $C_{UV} = 0$ at $\sqrt{s} = 250$ GeV and $\Gamma_Z = 0$ GeV.

A crucial test concerns the gauge parameter independence of the results. Gauge parameter independence of the result is performed through a set of five gauge fixing parameters. For the latter a generalised non-linear gauge fixing condition [11], [8] has been chosen.

$$\mathcal{L}_{GF} = -\frac{1}{\xi_W} \left| (\partial_\mu - ie\tilde{\alpha}_A_\mu - igc_W\tilde{\beta}Z_{\mu})W_{\mu}^+ + \xi_W \frac{g}{2}(v + \tilde{\delta}H + i\tilde{\kappa}\chi_3)\chi^+ \right|^2$$

$$-\frac{1}{2\xi_Z}(\partial.Z + \xi_Z \frac{g}{2c_W}(v + \tilde{\epsilon}\mathcal{H})\chi_3)^2 - \frac{1}{2\xi_A}(\partial.A)^2 .$$

(1)

The $\chi$ represent the Goldstone. We take the ’t Hooft-Feynman gauge with $\xi_W = \xi_Z = \xi_A = 1$ so that no “longitudinal” term in the gauge propagators contributes. Not only this makes
the expressions much simpler and avoids unnecessary large cancelations, but it also avoids the need for high tensor structures in the loop integrals. The use of the five parameters, $\tilde{\alpha}, \tilde{\beta}, \tilde{\delta}, \tilde{\kappa}, \tilde{\varepsilon}$ is not redundant as often these parameters check complementary sets of diagrams.

Non-linear gauge parameter-independence was confirmed by varying $\tilde{\alpha}, \tilde{\beta}, \tilde{\delta}, \tilde{\kappa}, \tilde{\varepsilon}$ from $(0,0,0,0,0)$ to $(10,20,30,40,50)$ with $C_{UV} = 100$ at $\sqrt{s} = 250$ GeV and $\Gamma_{Z} = 0$ GeV in order to avoid the violation of the gauge invariance, which resulted in an agreement up to 15 digits as shown in Table 4.

| $\tilde{\alpha}, \tilde{\beta}, \tilde{\delta}, \tilde{\kappa}, \tilde{\varepsilon}$ | Evaluation of the 1 loop amplitude |
|-----------------------------|----------------------------------|
| 0,0,0,0,0                  | 0.127132019079521763060824923741166 |
| 10,20,30,40,50             | 0.127132019079522096127732311288128 |

Table 4 Non-linear gauge parameters independence of the FULL model by changing $\tilde{\alpha}, \tilde{\beta}, \tilde{\delta}, \tilde{\kappa}, \tilde{\varepsilon}$ at a phase point in the arbitrary unit from $(0,0,0,0,0)$ to $(10,20,30,40,50)$ of $e^{+}e^{-} \rightarrow \mu^{+}\mu^{-}H$ with $k_c = 10^{-1}$ GeV, $\lambda = 10^{-17}$ GeV at $\sqrt{s} = 250$ GeV and $\Gamma_{Z} = 0$ GeV.

We also will show the comparison between the FULL model and the NOLFS approximation at a phase space point in Table 5.

| FULL model v.s.NOLFS approximation | Evaluation of the 1 loop amplitude |
|-----------------------------------|----------------------------------|
| FULL model                       | 0.127132019079521763060824923741166 |
| NOLFS approximation               | 0.127180115426583029147877823561430 |

Table 5 Comparison of the 1 loop amplitudes at a phase point between the FULL model and the NOLFS approximation in the arbitrary unit for the $e^{+}e^{-} \rightarrow \mu^{+}\mu^{-}H$ with $k_c = 10^{-1}$ GeV at $\sqrt{s} = 250$ GeV and $\Gamma_{Z} = 0$ GeV in order to avoid the violation of the gauge invariance.

An additional stability test on the NOLFS approximation concerns the bremsstrahlung part. It relates to the independence in the parameter $k_c$ which is a soft photon cut parameter that separates soft photon radiation and the hard photon performed by the Monte-Carlo integration package of the BASES [15]. Here, we have implemented the finite fixed width in all Z-boson propagators as $P_Z(q) = [q^2 - m_Z^2 + i m_Z \Gamma_Z]^{-1}$ with $\Gamma_Z = 2.49$ GeV [16] in order to avoid hitting the pole of the Z-boson propagators in the amplitudes.

In the BASES operation, to integrate one-loop amplitude, we set 10 iteration steps for the grid optimization and 200 iteration steps for the accumulation with 40,000 sampling
points. it takes 16 hours for $\mu^+\mu^-H$ and other processes; almost 103 hours for $b\bar{b}H$, with 16 Xeon 3.20 GHz CPU with the size of memory 128 GB. The typical integration errors are 0.06% for the each step of the loop calculations.

| Processes | $k_c = 10^{-1}$ GeV | $k_c = 10^{-3}$ GeV | $k_c = 10^{-5}$ GeV |
|-----------|----------------------|----------------------|----------------------|
| $\mu^+\mu^-H$ | -4.14(08) | -4.14(08) | -4.23(08) |
| $e^+e^-H$ | -4.45(09) | -4.53(09) | -4.44(09) |
| $\tau^+\tau^-H$ | -4.54(08) | -4.62(08) | -4.59(08) |
| $\nu_\mu\bar{\nu}_\mu H$ | -4.37(13) | -4.40(14) | -4.45(16) |
| $\nu_e\bar{\nu}_e H$ | -3.56(12) | -3.55(12) | -3.55(12) |
| $u\bar{u}H$ | -6.75(10) | -6.75(10) | -6.70(10) |
| $d\bar{d}H$ | -4.97(10) | -5.06(10) | -5.12(10) |
| $c\bar{c}H$ | -6.41(10) | -6.39(10) | -6.46(10) |
| $b\bar{b}H$ | -6.50(11) | -6.49(11) | -6.44(11) |

Table 6 $\delta_{Total}$ in % with various $k_c$ values , $k_c=10^{-1}$ GeV, $k_c=10^{-3}$ GeV, and $k_c=10^{-5}$ GeV of all 9 processes without beam polarizations at $\sqrt{s} = 250$ GeV.

From this table, we claim that the accuracy of our $\mathcal{O}(\alpha)$ corrections is approximately 0.1% for all 9 processes.

As final checks, we will show the comparison between the unpolarized cross sections of $e^+e^- \rightarrow \nu\bar{\nu}H$ between the current results and those of German group [10] at center-of-mass energy of $\sqrt{s} = 500$ GeV because in those days, the Higgs boson was not yet confirmed.
Table 7 Comparison between the unpolarized cross sections of $e^+e^- \rightarrow \nu \bar{\nu}H$ between the current results and those of German group [10] at $\sqrt{s} = 500$ GeV.

| $m_H$ (GeV) | $m_W$ (GeV) | $\sigma_{\text{Tree}}$ (pb) | $\sigma_{\text{O}}$ (pb) | $\delta_{\text{Total}}\%$ |
|------------|-------------|-----------------------------|---------------------------|---------------------------|
| 150        | 80.3767     | 6.1072(9)$\times 10^{-2}$   | 6.075(6)$\times 10^{-2}$  | -0.5                      |
|            |             | 6.1076(5)$\times 10^{-2}$   | 6.080(2)$\times 10^{-2}$  | -0.4 Ref.[10]             |
| 200        | 80.3571     | 3.7302(5)$\times 10^{-2}$   | 3.705(4)$\times 10^{-2}$  | -0.7                      |
|            |             | 3.7293(3)$\times 10^{-2}$   | 3.709(2)$\times 10^{-2}$  | -0.6 Ref.[10]             |
| 250        | 80.3411     | 2.110(2)$\times 10^{-2}$    | 2.059(2)$\times 10^{-2}$  | -2.4                      |
|            |             | 2.1134(1)$\times 10^{-2}$   | 2.060(1)$\times 10^{-2}$  | -2.5 Ref.[10]             |
| 300        | 80.3275     | 1.0744(7)$\times 10^{-2}$   | 1.0258(7)$\times 10^{-2}$ | -4.5                      |
|            |             | 1.07552(7)$\times 10^{-2}$  | 1.0282(4)$\times 10^{-2}$ | -4.4 Ref.[10]             |
| 350        | 80.3158     | 4.6077(4)$\times 10^{-3}$   | 4.172(2)$\times 10^{-3}$  | -9.5                      |
|            |             | 4.6077(2)$\times 10^{-3}$   | 4.181(1)$\times 10^{-3}$  | -9.3 Ref.[10]             |

3 Beam polarization

Here we would like to define the beam polarization and the realistic polarization status of the proposed ILC. The left-handed polarization degree of the electron beam is defined as,

$$p_e = (N_{e_R} - N_{e_L})/(N_{e_L} + N_{e_R}),$$  \hspace{1cm} (2)

where $N_{e_R}$ and $N_{e_L}$ are the number of the right-handed and left-handed electrons in the beam, respectively , and

$$p_p = (N_{p_R} - N_{p_L})/(N_{p_R} + N_{p_L}),$$  \hspace{1cm} (3)

$N_{p_R}$ and $N_{p_L}$ are the number of the right-handed and left-handed positrons in the beam, respectively.

When a normalization $N_L + N_R = 1$ is used, the normalized number of left-handed and right-handed electrons can be obtained as $N_L = \frac{1-p_e}{2}$ and $N_R = \frac{1+p_e}{2}$, respectively.

Therefore, the cross sections for a given combination of the electron and positron beam polarization can be written as:

$$\sigma(p_e, p_p) = \frac{1}{4} \{ (1-p_e)(1+p_p)\sigma_{LR} + (1+p_e)(1-p_p)\sigma_{RL}$$

$$+ (1-p_e)(1-p_p)\sigma_{LL} + (1+p_e)(1+p_p)\sigma_{RR} \}$$  \hspace{1cm} (4)
where \( \sigma_{LR} \) stands for the cross section with the 100% left-handed polarized electron \((p_e = -1)\) and the 100% right-handed polarized positron \((p_p = +1)\) beams. The cross sections \( \sigma_{RL}, \sigma_{LL} \) and \( \sigma_{RR} \) are defined analogously.

The ILC proposed polarizations of the design values are

\[
p_e = \frac{0.1 - 0.9}{1} = -0.8, \tag{5}
\]

and

\[
p_p = \frac{0.65 - 0.35}{1} = 0.3. \tag{6}
\]

In the following discussions, we use the following short notations for the various polarization conditions:

- \( p_e = 0, p_p = 0 \) UP: unpolarized,
- \( p_e = +1, p_p = +1 \) RR: right \( e^- \) right \( e^+ \),
- \( p_e = -1, p_p = -1 \) LL: left \( e^- \) left \( e^+ \),
- \( p_e = +1, p_p = -1 \) RL: right \( e^- \) left \( e^+ \),
- \( p_e = -1, p_p = +1 \) LR: left \( e^- \) right \( e^+ \),
- \( p_e = -0.8, p_p = 0.3 \) ILC: the ILC proposed polarization.

4 Recoil mass analysis

We introduce the recoil mass analysis that we conducted to make a model-independent measurement of the coupling between the Higgs and Z bosons, (i.e., \( g_{HZZ} \) coupling), using the recoil mass distribution in \( e^+e^- \toZH \) with \( Z \to \mu^+\mu^- \)[17].

In electron-positron collisions at \( \sqrt{s} = 250 \) GeV, the main Higgs production mechanism is the Higgs-strahlung process \( e^+e^- \to HZ \). For \( m_H = 125 \) GeV, the cross section for the s-channel process is maximal close to \( \sqrt{s} = 250 \) GeV.

The total \( HZ \) cross section is proportional to the square of the coupling between Higgs and Z bosons, that is, \( \sigma(e^+e^- \to HZ) \propto g_{HZZ}^2 \), and the cross sections of the decays to the final state in \( H \to XX \) can be expressed as

\[
\sigma(e^+e^- \to HZ) \times BR(H \to XX) \propto \frac{g_{HZZ}^2 \times g_{HXX}^2}{\Gamma_H}.
\]

In this study, the cross section of \( e^+e^- \to HZ \) was measured using the recoil mass technique. We considered \( \sigma(e^+e^- \to HZ) \), and the recoil mass can be expressed as

\[
m_{\text{rec}}^2 = s - 2\sqrt{s}(E_{\mu^+} + E_{\mu^-}) + m_{\mu^+\mu^-}^2, \tag{7}
\]
where $\sqrt{s}$ is the center-of-mass energy, and $E_{\mu^+}$ and $E_{\mu^-}$ are the energies of the two muons and $m_{\mu^+\mu^-}^2$ is the invariant mass of muon and anti-muon from $Z$ decay.

At $\sqrt{s} = 250$ GeV, where the energy of the muons from $Z$ decay approximately scales as $\sqrt{s}$, the width of recoil mass distribution increases significantly with increasing center-of-mass energy. Therefore, the leptonic (in particular, muonic) recoil mass analysis leads to a higher precision on $g_{HZZ}$ for $\sqrt{s} = 250$ GeV, where $\sigma(\mathcal{H}Z)$ is the largest and the recoil mass peak is relatively narrow.

Using this technique, one can determine the absolute branching ratios of Higgs boson decays, including those of the invisible decays.

It should be noted that in this recoil mass distribution, the simple ISR and the detector resolution effects are included, which leads to the appearance of the tail structure in Fig. 1.

The International Linear Collider (ILC) is a proposed international $e^+e^-$ linear collider with beam energies ranging from 250 GeV to 1 TeV. ILC is supposed to start at the center-of-mass energy of 250 GeV in its initial stage to obtain high statistics measurements of the Higgs boson. Note that this center-of-mass energy was used for the calculations in this paper.

![Fig. 1](image.png)  
Recoil mass distribution of the Higgs-strahlung process $e^+e^- \rightarrow \mu^+\mu^-H$ ($e^+e^- \rightarrow ZH$ followed by $Z \rightarrow \mu^+\mu^-$) with 250 fb$^{-1}$ for $m_h = 120$ GeV at $\sqrt{s} = 250$ GeV [17].
Owing to parity violation in weak interactions, beam polarization effects are essential to resolve new phenomena beyond the standard model (BSM), and hence, studying these effects is necessary. Beam polarization and its importance in studying physics at the $e^+e^-$ collider have been discussed in detail over the past decades. A precise measurement to study the properties of the Higgs boson is one of the key targets of the ILC experiments. However, to achieve this, one needs to know the $\mathcal{O}(\alpha)$ corrections to the underlying processes.

5 Radiator method

The effect of the initial photon emission can be factorized when the total energy of the emitted photons is sufficiently small compared to the beam energy or for small angle (co-linear) emissions. This approximation is referred to as the ”soft-collinear photon approximation (SPA)”. Under SPA, the corrected cross sections with ISR, that is, $\sigma_{ISR}$, can be obtained from the tree cross sections $\sigma_{Tree}$ using a radiator function $H(x,s)$ as follows:

$$\sigma_{ISR} = \int_0^1 dx \, H(x,s)\sigma_{Tree}(s(1-x)), \quad (8)$$

where $s$ is the square of the CM energy and $x$ is the energy fraction of an emitted photon.

The total cross section with higher-order QED corrections to ISR can be calculated using the following function [18]:

$$\sigma_{ISR} = \int_0^{2/\sqrt{s}} dx_1 \int_0^{1-x_1} dx_2 \, D(x_1,s)D(x_2,s)\sigma_{Tree}(sx_1x_2). \quad (9)$$

where the structure function $D(x,s)$, which is corresponding to square root of the radiator function $H(x,s)$, gives a probability to emit a photon with energy fraction of $x$ at the CM energy square $s$.

In this method, electron and positron can emit different energies, and thus finite boost of the CM system can be treated. The structure function can be obtained as

$$D(1 - x, s)^2 = H(x, s) = \Delta_2 \beta x^{\beta -1} - \Delta_1 \beta \left(1 - \frac{x}{2}\right)$$

$$+ \frac{\beta^2}{8} \left[-4(2 - x) \log x - \frac{1 + 3(1 - x)^2}{x} \log (1 - x) - 2x\right], \quad (10)$$
where

\[ \beta = \frac{2\alpha}{\pi} \left( \log \frac{s}{m_e^2} - 1 \right), \]

\[ \Delta_2 = 1 + \delta_1, \quad \Delta_1 = 1 + \delta_1 \]

\[ \delta_1 = \frac{\alpha}{\pi} \left( \frac{3}{2} \log \frac{s}{m_e^2} + \frac{\pi^2}{3} - 2 \right). \]

This result is obtained based on perturbative calculations of initial-state photon emission diagrams up to one-loop order. The Ref. [18] shows the formula for the \( \mathcal{O}(\alpha^2) \) ISR effects. On the other hand, we are interested in separating the \( \mathcal{O}(\alpha) \) photonic corrections from the full \( \mathcal{O}(\alpha) \) electroweak corrections. It is known that such a separation cannot be realized based on the diagrammatic calculations due to the \( SU(2)_L \times U(1) \) gauge invariance. On the other hand, the exponentiation method has no such problems estimating the initial state photonic effects. However, it is exponentiated because we would like to extract \( \mathcal{O}(\alpha) \) photonic effects as precisely as possible. We neglect the term \( \delta_2 \) which is just the term of the \( \mathcal{O}(\alpha^2) \) ISR formula in the Ref. [18].

6 Results of the lepton processes with the polarization

A canonical set of the input for the electroweak parameters for the numerical evaluations are chosen as follows;

| Parameter | Value |
|-----------|-------|
| \( m_e \) | 0.51099906 x 10^{-3} GeV [16] |
| \( m_\mu \) | 105.6583389 x 10^{-3} GeV [16] |
| \( m_\tau \) | 1.771 GeV [16] |
| \( m_{\nu_e} \) | 0 GeV |
| \( m_{\nu_\mu} \) | 0 GeV |
| \( m_{\nu_\tau} \) | 0 GeV |
| \( m_u \) | 0.0063 GeV [19] |
| \( m_d \) | 0.0063 GeV [19] |
| \( m_s \) | 0.0094 GeV [19] |
| \( m_c \) | 1.5 GeV [19] |
| \( m_t \) | 172.9 GeV [16] |
| \( m_Z \) | 91.1876 GeV [16] |
| \( m_W \) | 80.366 GeV [19] |
| \( m_H \) | 125.1 GeV [16] |
| \( m_b \) | 4.7 GeV [19] |
| \( \alpha \) | 1/137.035999084 [16] |
| \( \Gamma_Z \) | 2.49 GeV [16] |

Table 8 Canonical input parameters

Here the light quark masses are taken from the numerical program \( \text{MW}.f \) written by Z.Hioki [19] to reproduce the total hadronic cross sections of \( e^+e^- \) collisions in the low energy regions by means of the dispersion relations, and which also calculates the electroweak one-loop corrected \( m_W \) from \( \alpha, m_Z, m_{Higgs}, m_e, m_\mu, m_\tau, m_u, m_d, m_s, m_c, m_b \) and \( m_t \) as inputs parameters to be \( m_W = 80.366 \) GeV.
6.1 $e^+e^- \rightarrow \mu^+\mu^- H$

Now we are going to show five types of typical one-loop Feynman diagrams with typical three-point functions in Fig. 2, a typical four-point one, a typical five-point one in Fig. 3 and a typical fish type diagram in Fig. 4 respectively. The treatment of these kinds of n-point Feynman diagrams in GRACE-Loop is discussed in detail in [8, 20].

Fig. 2  A typical final and initial vertex correction Feynman diagrams of $e^+e^- \rightarrow \mu^+\mu^- H$

Fig. 3  Typical 4-point and 5-point functions of Feynman diagrams of $e^+e^- \rightarrow \mu^+\mu^- H$

Table 9 shows the total cross sections of the tree level, ones of the $\mathcal{O}_\alpha$ corrected with the NOLFS approximation and ones including the ISR effects. It shows also the total ratios,
\( \delta_{O_\alpha} \) with the NOLFS approximation and \( \delta_{ISR} \) with the ISR effects for various polarization conditions (i.e., UP, RR, LL, RL, LR, and ILC) defined in Section 3. The convergence of phase space integration both of the one-loop effects and also of the ISR effects are very bad for the case of LL and RR because of the tiny cross section of the tree in \( 10^{-11} \) pb due to the very small electron mass effects. Even though, for the case of the ILC proposed polarization with \( p_e = -0.8 \), \( p_p = 0.3 \), the convergence of the phase space integration is very good and we get the reliable numbers in 0.1% accuracy.

![Feynman Diagram](image)

**Fig. 4** A typical Fish-type Feynman diagram of \( e^+e^- \rightarrow \mu^+\mu^-H \)

|     | \( \sigma_{Tree} \) (pb) | \( \sigma_{O_\alpha} \) (pb) | \( \delta_{Total} \)% | \( \sigma_{ISR} \) (pb) | \( \delta_{ISR} \)% |
|-----|-----------------|-----------------|-----------------|-----------------|-----------------|
| UP  | \( 7.021(2) \times 10^{-3} \) | \( 6.724(1) \times 10^{-3} \) | -4.2            | \( 6.312(4) \times 10^{-3} \) | -9.9            |
| RR  | \( 2.16(1) \times 10^{-11} \) | \( 8.4(2) \times 10^{-7} \)  | 3.91 \times 10^6 | \( 9.62(3) \times 10^{-13} \) | -95.6           |
| LL  | \( 2.16(1) \times 10^{-11} \) | \( 8.45(2) \times 10^{-7} \)  | 3.91 \times 10^6 | \( 7(2) \times 10^{-11} \)   | 224.1           |
| RL  | \( 1.108(2) \times 10^{-2} \) | \( 1.194(1) \times 10^{-2} \) | 7.7             | \( 1.534(3) \times 10^{-3} \) | -19.3           |
| LR  | \( 1.709(2) \times 10^{-2} \) | \( 1.497(1) \times 10^{-2} \) | -12.1           | \( 9.944(2) \times 10^{-2} \)  | -9.9            |
| ILC | \( 1.035(2) \times 10^{-2} \) | \( 9.182(1) \times 10^{-3} \) | -11.3           | \( 9.321(4) \times 10^{-3} \)  | -9.9            |

**Table 9** The cross section of \( e^+e^- \rightarrow \mu^+\mu^-H \) with various conditions of the beam polarization and without experimental cuts.
6.2 \( e^+e^- \rightarrow e^+e^-H \)

Note that the notations and symbols used for the \( e^+e^- \rightarrow e^+e^-H \) and other processes are the same as those for the \( e^+e^- \rightarrow \mu^+\mu^-H \) process, for the UP, RR, LL, RL, LR, and ILC. In this process, because of sizable t-channel amplitude contribution, \( \sigma_{\alpha(ILC)}e^+e^-H = 9.841(2) \times 10^{-3} \text{ pb} \) is larger than \( \sigma_{\alpha(ILC)}\mu^+\mu^-H = 9.182(1) \times 10^{-3} \text{ pb} \) in Section 6.1 approximately 7\% at the proposed ILC with \( p_e = -0.8, p_p = 0.3 \). Because it is not needed to form spin-1 state at t-channel, LL and RR cross sections are sizable even at the tree level. Table 10 shows detailed values.

|        | \( \sigma_{Tree} \) (pb) | \( \sigma_{\alpha} \) (pb) | \( \delta_{Total}\% \) | \( \sigma_{ISR} \) (pb) | \( \delta_{ISR}\% \) |
|--------|---------------------------|-----------------------------|-------------------------|---------------------------|-------------------------|
| UP     | 7.714(2) \times 10^{-3}   | 7.348(1) \times 10^{-3}     | -4.7                    | 6.314(4) \times 10^{-3}   | -6.7                    |
| RR     | 6.874(4) \times 10^{-4}   | 6.221(1) \times 10^{-4}     | -9.4                    | 3.080(2) \times 10^{-4}   | -88.8                   |
| LL     | 6.874(4) \times 10^{-4}   | 6.221(1) \times 10^{-4}     | -9.4                    | 3.080(2) \times 10^{-4}   | -55.4                   |
| RL     | 1.149(2) \times 10^{-2}   | 1.236(1) \times 10^{-2}     | 7.6                     | 9.769(2) \times 10^{-3}   | -15.0                   |
| LR     | 1.799(3) \times 10^{-2}   | 1.576(1) \times 10^{-2}     | -12.4                   | 1.487(3) \times 10^{-2}   | -17.3                   |
| ILC    | 1.119(1) \times 10^{-2}   | 9.841(2) \times 10^{-3}     | -12.0                   | 9.159(4) \times 10^{-3}   | -18.1                   |

**Table 10** The cross section of \( e^+e^- \rightarrow e^+e^-H \) with various conditions of the beam polarization and without experimental cuts.

6.3 \( e^+e^- \rightarrow \tau^+\tau^-H \)

There is a question about tau particle which we are interested in: whether the tau mass can be neglected or not.

\[
\sigma_{\alpha(ILC)\tau^+\tau^-H} = 9.145(2) \times 10^{-3} \text{ pb} \quad \text{and} \quad \sigma_{\alpha(ILC)\mu^+\mu^-H} = 9.2100(1) \times 10^{-3} \text{ pb}
\]

in Section 6.1 are quite similar as we expected because all of final radiation effects like \( \ln \left( \frac{m_{\tau}^2}{s} \right) \) are canceled according to the Kinoshita-Lee-Nauenberg theorem [21],[22]. The terms from the initial state radiation of \( \ln \left( \frac{m_e^2}{s} \right) \) of \( e^+e^- \) still remain, which have the same effects for both processes of \( \tau^+\tau^-H \) and \( \mu^+\mu^-H \). More results for the case of other polarization conditions will be shown in Table 11.

6.4 \( e^+e^- \rightarrow \nu_\mu\bar{\nu}_\mu H \)

The difference between \( \delta_{Total} \) and \( \delta_{ISR} \) is approximately 5\%. We skip \( e^+e^- \rightarrow \nu_\tau\bar{\nu}_\tau H \) process because it is quite similar to this one. More results are provided in Table 12.
Table 11  The cross section of $e^+e^- \rightarrow \tau^+\tau^-H$ with various conditions of the beam polarization and without experimental cuts.

|     | $\sigma_{Tree}$ (pb) | $\sigma_{\alpha}$ (pb) | $\delta_{Total}$% | $\sigma_{ISR}$ (pb) | $\delta_{ISR}$% |
|-----|----------------------|-----------------------|--------------------|---------------------|-----------------|
| UP  | 7.018(2)×10^{-3}     | 6.708(5)×10^{-3}      | -4.4               | 6.294(4)×10^{-3}    | -10.3           |
| RR  | 2.10(2)×10^{-9}      | 8.44(3)×10^{-7}       | 4.0×10^4          | 8.20(7)×10^{-12}    | -99.6           |
| LL  | 2.10(2)×10^{-9}      | 8.44(3)×10^{-7}       | 4.0×10^4          | 8.3(7)×10^{-12}     | -99.6           |
| RL  | 1.1078(2)×10^{-2}    | 1.189(6)×10^{-2}      | 7.3                | 9.17(2)×10^{-3}     | -10.5           |
| LR  | 1.7021(4)×10^{-2}    | 1.491(1)×10^{-2}      | -12.4              | 1.524(2)×10^{-2}    | -10.5           |
| ILC | 1.0343(2)×10^{-2}    | 9.145(2)×10^{-3}      | -11.6              | 9.261(4)×10^{-3}    | -10.5           |

Table 12  The cross section of $e^+e^- \rightarrow \nu_\mu\bar{\nu}_\mu H$ with various conditions of the beam polarization and without experimental cuts.

|     | $\sigma_{Tree}$ (pb) | $\sigma_{\alpha}$ (pb) | $\delta_{Total}$% | $\sigma_{ISR}$ (pb) | $\delta_{ISR}$% |
|-----|----------------------|-----------------------|--------------------|---------------------|-----------------|
| UP  | 1.3891(6)×10^{-2}    | 1.328(3)×10^{-2}      | -4.4               | 1.249(1)×10^{-2}    | -10.1           |
| RR  | 2.71(1)×10^{-12}     | 1.669(7)×10^{-6}      | 6.16×10^7         | 1.971(6)×10^{-12}   | -27.3           |
| LL  | 2.71(1)×10^{-12}     | 1.669(7)×10^{-6}      | 6.15×10^7         | 2.307(7)×10^{-12}   | -14.9           |
| RL  | 2.191(1)×10^{-2}     | 2.409(7)×10^{-2}      | 10.0               | 1.965E(1)×10^{-2}   | -10.3           |
| LR  | 3.367(1)×10^{-2}     | 2.905(1)×10^{-2}      | 13.7               | 3.022(1)×10^{-2}    | -10.2           |
| ILC | 2.045(1)×10^{-2}     | 1.782(1)×10^{-2}      | -12.9              | 1.837(1)×10^{-2}    | -10.2           |

6.5  $e^+e^- \rightarrow \nu_e\bar{\nu}_e H$

$e^+e^- \rightarrow \nu_e\bar{\nu}_e H$ has t-channel. Table 13 shows the results of $e^+e^- \rightarrow \nu_e\bar{\nu}_e H$ process.

|     | $\sigma_{Tree}$ (pb) | $\sigma_{\alpha}$ (pb) | $\delta_{Total}$% | $\sigma_{ISR}$ (pb) | $\delta_{ISR}$% |
|-----|----------------------|-----------------------|--------------------|---------------------|-----------------|
| UP  | 2.067(1)×10^{-2}     | 2.002(9)×10^{-2}      | -3.1               | 1.764(2)×10^{-2}    | -14.6           |
| RR  | 3.05(2)×10^{-12}     | 3.53(2)×10^{-6}       | 1.16×10^8         | 2.043(8)×10^{-12}   | -33.0           |
| LL  | 3.05(2)×10^{-12}     | 3.53(2)×10^{-6}       | 1.16×10^8         | 2.385(9)×10^{-12}   | -21.8           |
| RL  | 2.191(1)×10^{-2}     | 2.414(7)×10^{-2}      | 10.2               | 1.905(1)×10^{-2}    | -10.3           |
| LR  | 6.076(1)×10^{-2}     | 5.612×10^{-2}         | -7.6               | 5.085(2)×10^{-2}    | -16.3           |
| ILC | 3.632(1)×10^{-2}     | 3.357(3)×10^{-2}      | -7.6               | 3.044(2)×10^{-2}    | -16.2           |

Table 13  The cross section of $e^+e^- \rightarrow \nu_e\bar{\nu}_e H$ with various conditions of the beam polarization and without experimental cuts.
7 Results of the quark processes

7.1 $e^+e^- \rightarrow u \bar{u} H$

QCD corrections up to 4-loop level is well known, and with just in one-loop corrections is popular as the simple factor of $1 + \frac{\alpha_s}{\pi} \simeq 1.03\%$, where $\alpha_s$ is the coupling constant of the strong interaction ($\alpha_s \simeq 1/10$) thus we will not discuss about it in this paper. Table 14 shows calculation results of the first quark process $e^+e^- \rightarrow u \bar{u} H$.

|          | $\sigma_{\text{Tree}}$ (pb) | $\sigma_{\alpha}$ (pb) | $\delta_{\text{Total}}\%$ | $\sigma_{\text{ISR}}$ (pb) | $\delta_{\text{ISR}}\%$ |
|----------|-------------------------------|-------------------------|-----------------------------|----------------------------|---------------------------|
| UP       | $2.424(1) \times 10^{-2}$     | $2.258(2) \times 10^{-2}$ | -6.8                        | $2.178(2) \times 10^{-2}$  | -10.1                     |
| RR       | $3.65(2) \times 10^{-11}$     | $2.915(1) \times 10^{-6}$ | $7.98 \times 10^6$          | $8(4) \times 10^{-11}$     | 11.9                      |
| LL       | $3.65(2) \times 10^{-11}$     | $2.92(1) \times 10^{-6}$ | $7.98 \times 10^6$          | $8(4) \times 10^{-11}$     | 11.9                      |
| RL       | $3.824(1) \times 10^{-2}$     | $4.32(1) \times 10^{-2}$ | 13.1                        | $3.431(1) \times 10^{-2}$  | -10.0                     |
| LR       | $5.875(1) \times 10^{-2}$     | $4.809(1) \times 10^{-2}$ | -18.2                       | $5.274(1) \times 10^{-2}$  | -10.0                     |
| ILC      | $3.570(1) \times 10^{-2}$     | $2.934(1) \times 10^{-2}$ | -17.8                       | $3.206(2) \times 10^{-2}$  | -10.0                     |

Table 14 The cross section of $e^+e^- \rightarrow u \bar{u} H$ with various conditions of the beam polarization and without experimental cuts.

7.2 $e^+e^- \rightarrow d \bar{d} H$

The iso-spin of $u$ quark is up, on the other hand, the iso-spin of $d$ quark is down, which explains the difference between $\sigma_{\alpha}^{(\text{ILC})}d \bar{d}H = 4.070(6) \times 10^{-2}$ pb and $\sigma_{\alpha}^{(\text{ILC})}u \bar{u}H = 2.948(4) \times 10^{-2}$ pb as in the section 7.1. Table 15 shows the results of $e^+e^- \rightarrow d \bar{d} H$ process.

|          | $\sigma_{\text{Tree}}$ (pb) | $\sigma_{\alpha}$ (pb) | $\delta_{\text{Total}}\%$ | $\sigma_{\text{ISR}}$ (pb) | $\delta_{\text{ISR}}\%$ |
|----------|-------------------------------|-------------------------|-----------------------------|----------------------------|---------------------------|
| UP       | $3.112(1) \times 10^{-2}$     | $2.961(1) \times 10^{-2}$ | -4.8                        | $2.796 \times 10^{-2}$     | -10.2                     |
| RR       | $1.40(1) \times 10^{-11}$     | $3.74(2) \times 10^{-6}$ | $2.7 \times 10^{-7}$        | $1.9(8) \times 10^{-10}$   | 1257                      |
| LL       | $1.40(1) \times 10^{-11}$     | $3.74(2) \times 10^{-6}$ | $2.7 \times 10^{-7}$        | $1.8(8) \times 10^{-10}$   | 1186                      |
| RL       | $4.907(1) \times 10^{-2}$     | $5.205(1) \times 10^{-2}$ | 6.1                         | $6.769(2) \times 10^{-2}$  | -10.6                     |
| LR       | $7.567(1) \times 10^{-2}$     | $6.650(1) \times 10^{-2}$ | -12.1                       | $4.402(2) \times 10^{-2}$  | -10.3                     |
| ILC      | $4.581(1) \times 10^{-2}$     | $4.070(6)E \times 10^{-2}$ | -11.5                       | $4.115(2) \times 10^{-2}$  | -10.2                     |

Table 15 The cross section of $e^+e^- \rightarrow d \bar{d} H$ with various conditions of the beam polarization and without experimental cuts.
7.3 $e^+e^- \rightarrow c\bar{c}H$

Table 16 shows the cross sections of $e^+e^- \rightarrow c\bar{c}H$. The mass of charm quark is 1.5 GeV and the mass of up quark is 58 MeV. Because there is no large mass effect from the final state radiation, the results are quite similar as expected. Because of this observation, we skip $e^+e^- \rightarrow s\bar{s}H$ process because this process has the same characteristic of $e^+e^- \rightarrow d\bar{d}H$ process.

|   | $\sigma_{\text{Tree}}$ (pb) | $\sigma_{\mathcal{O}_\alpha}$ (pb) | $\delta_{\text{Total}}$ | $\sigma_{\text{ISR}}$ (pb) | $\delta_{\text{ISR}}$ |
|---|-----------------|-----------------|------------------|-----------------|------------------|
| UP | $2.421(1)\times10^{-2}$ | $2.259(1)\times10^{-2}$ | $-6.8$ | $2.175(2)\times10^{-2}$ | $-10.2$ |
| RR | $1.482(7)\times10^{-11}$ | $2.91(1)\times10^{-6}$ | $1.96\times10^{7}$ | $1.4155(8)\times10^{-11}$ | $-4.9$ |
| LL | $1.482(7)\times10^{-11}$ | $2.91(1)\times10^{-6}$ | $1.96\times10^{7}$ | $1.4600(8)\times10^{-11}$ | $-1.5$ |
| RL | $3.819(1)\times10^{-2}$ | $4.300(1)\times10^{-2}$ | $12.6$ | $3.425(2)\times10^{-2}$ | $-10.3$ |
| LR | $5.890(1)\times10^{-2}$ | $4.771(1)\times10^{-2}$ | $-19.0$ | $5.266(1)\times10^{-2}$ | $-10.6$ |
| ILC | $3.566(1)\times10^{-2}$ | $2.926(4)\times10^{-2}$ | $-18.0$ | $3.202(2)\times10^{-2}$ | $-10.2$ |

This table 16 gives the cross sections of $e^+e^- \rightarrow c\bar{c}H$ with various conditions of the beam polarization and without experimental cuts.

7.4 $e^+e^- \rightarrow b\bar{b}H$

Table 17 summaries cross sections of $e^+e^- \rightarrow b\bar{b}H$. We keep the bottom-Yukawa coupling with NOLFS approximation for $e^+e^- \rightarrow b\bar{b}H$ but not in other processes. Let’s compare the results of $e^+e^- \rightarrow d\bar{d}H$ and $e^+e^- \rightarrow b\bar{b}H$ of the proposed ILC, $p_e = -0.8, p_p = 0.3$ at the $\delta_{\text{Total}(ILC)}$, which are $-11.5\%$ and $12.5\%$, respectively. This $1\%$ difference is caused by the bottom-Yukawa coupling. With ISR, the results for $e^+e^- \rightarrow d\bar{d}H$ and $e^+e^- \rightarrow b\bar{b}H$ are quite similar as expected. This difference between $\delta_{\text{Total}(ILC)}$ and $\delta_{\text{ISR}(ILC)}$ is very interesting. Because this $1\%$ difference originated from the bottom-Yukawa coupling of the bottom quark, though it maybe hard to observe at the experiments.

8 Recoil mass distribution of $e^+e^- \rightarrow \mu^+\mu^-H$ with beam polarization effects

In this section, we discuss the ZH recoil mass distribution $ZH \rightarrow \mu^+\mu^-H$ with $\mathcal{O}(\alpha)$ corrections and beam polarization effects. To conduct a more realistic analysis, we apply the following three experimental cuts:
Table 17  The cross section of $e^+e^- \rightarrow b\bar{b}H$ with various conditions of the beam polarization and without experimental cuts.

|     | $\sigma_{Tree}$ (pb) | $\sigma_{\mathcal{O}_a}$ (pb) | $\delta_{Total}\%$ | $\sigma_{ISR}$ (pb) | $\delta_{ISR}\%$ |
|-----|----------------------|---------------------|------------------|-------------------|-----------------|
| UP  | 3.078(1)$\times 10^{-2}$ | 2.88(1)$\times 10^{-2}$ | -6.3            | 2.765(2)$\times 10^{-2}$ | -10.2          |
| RR  | 7.40(4)$\times 10^{-12}$ | 3.69(2)$\times 10^{-6}$ | 5.0$\times 10^7$ | 5.70(3)$\times 10^{-12}$ | -22.9          |
| LL  | 7.40(4)$\times 10^{-12}$ | 3.69(2)$\times 10^{-6}$ | 5.0$\times 10^7$ | 6.31(3)$\times 10^{-12}$ | -14.6          |
| RL  | 4.855(1)$\times 10^{-2}$ | 5.06(1)$\times 10^{-2}$ | 4.2             | 4.035(1)$\times 10^{-2}$ | -10.2          |
| LR  | 7.461(2)$\times 10^{-2}$ | 6.46(1)$\times 10^{-2}$ | -13.5           | 6.691(2)$\times 10^{-2}$ | -10.2          |
| ILC | 4.533(1)$\times 10^{-2}$ | 3.969(4)$\times 10^{-2}$ | -12.5           | 4.068(2)$E\times 10^{-2}$ | -10.3          |

1. The angular cuts on $\theta_{\mu^+}, \theta_{\mu^-}$

$$10^\circ < \theta_{\mu^+}, \theta_{\mu^-} < 170^\circ,$$  \hspace{1cm} (11)

where $\theta_{\mu^+}$ and $\theta_{\mu^-}$ are the scattering angles of anti-muon and muon, respectively.

2. The energy cuts on $E_{\mu^+}, E_{\mu^-}$

$$E_{\mu^+}, E_{\mu^-} > 10 \text{ GeV},$$ \hspace{1cm} (12)

where $E_{\mu^+}$ and $E_{\mu^-}$ are the energies of anti-muon and muon, respectively.

3. The invariant mass cut $M_{\mu^+\mu^-}$

$$m_Z - 3\Gamma_Z < M_{\mu^+\mu^-} < m_Z + 3\Gamma_Z.$$ \hspace{1cm} (13)

where $M_{\mu^+\mu^-}$ is the invariant mass of anti-muon and muon particles; $\Gamma_Z = 2.49 \text{ GeV}$ \cite{16} is the width of the Z-boson.

Next, we present the results of the ZH recoil mass distribution with the $\mathcal{O}(\alpha)$ corrections of $e^+e^- \rightarrow \mu^+\mu^-H$ at $\sqrt{s} = 250 \text{ GeV}$, after applying the three types of experimental cuts as defined above. The recoil mass distribution is shown in Fig. 5.

The black line shows the tree level distribution, the blue line shows the ISR corrected distribution and the red line shows the $\mathcal{O}(\alpha)$ corrected distribution. The black line only shows the one bin peak and of which position indicates the mass of Higgs boson. On including the $\mathcal{O}(\alpha)$ corrections, a tail structure appears owing to the ISR effects, which is called the radiative tail. The height of peak is substantially reduced owing to the one-photon emission effect. However, the ISR effects are included, the height of the peak increases again because...
Fig. 5 Recoil mass distribution of $e^+e^- \rightarrow \mu^+\mu^-H$ after applying the experimental cuts as in Eqs (4.3), (4.4),(4.5) at $\sqrt{s} = 250$ GeV. The bin width is 0.3 GeV.

of the higher-order radiation effect. This swing back effect can be seen, occasionally when comparing the higher-order and $O(\alpha)$ corrections.

After applying the three experimental cuts, we obtained the total cross sections at $O(\alpha)$ and those including ISR effects and the total ratios at $O(\alpha)$ and those including ISR effects as shown in Table 18

| $\sigma_{Tree}$ (pb) | $\sigma_{O(\alpha)}$ (pb) | $\sigma_{ISR}$ (pb) | $\delta_{Total\%}$ | $\delta_{ISR\%}$ |
|----------------------|---------------------------|---------------------|--------------------|------------------|
| $9.143(1) \times 10^{-3}$ | $7.910(1) \times 10^{-3}$ | $8.236(4) \times 10^{-3}$ | $-13.2$            | $-9.9$           |

Table 18 Summary table of cross sections and total ratios of $e^+e^- \rightarrow \mu^+\mu^-H$ with experimental cuts and with ILC proposed polarization with $p_e = -0.8$, $p_p = 0.3$.

9 Discussion

First, we discuss the behavior of high-order corrections for the $e^+e^- \rightarrow \mu^+\mu^-H$ process in detail. For this process, various conditions of beam polarization without experimental cuts are shown in Table 9. At the tree level, the cross sections corresponding to the RR and LL are negligibly small compared with those of RL and LR since RR and LL polarizations cannot produce spin-1 boson. Small cross sections of RR and LL polarizations are resulted
from spin-flip effect, which are proportional to $|\frac{m_e^2}{s}| \propto \left(\frac{10^{-3}}{10^2}\right)^2 \sim 10^{-10}$. These results are consistent with $\frac{LL+RR}{LR+RL} \sim 10^{-11}$. Moreover, $\sigma_{O_\alpha}$ gives sizeable cross sections on RR and LL polarizations, because $O_\alpha$ amplitude includes spin-flip diagrams of the initial state radiation. For LL and RR, although the cross sections of LL and RR at the tree level are very small, the very large total ratios $\delta_{Total} \sim 10^6$ are observed.

$$\sigma_{O_\alpha(ILC)}(\mu^+\mu^-H) = 9.182(1) \times 10^{-2} \text{ pb is larger than } \sigma_{O_\alpha(UP)}(\mu^+\mu^-H) = 6.724(1) \times 10^{-3} \text{ pb}$$

thanks to the advantages of the beam polarization of the linear collider. It has significantly better statistics of Higgs with the same luminosity.

Next, we discuss the effect of higher-order corrections on weak interactions and photonic interactions. Note that $\sigma_{O_\alpha}$ includes both QED and weak corrections. However, it is impossible in general to clearly separate these two effects. We calculated $\delta_{ISR}$ as an approximation for QED corrections because ISR corrections dominate over all QED corrections. In this approximation, the weak corrections can be estimated as $\sigma_{weak} = \sigma_{O_\alpha} - \sigma_{ISR}$. $\delta_{ISR(RL)}$ and $\delta_{ISR(LR)}$ were obtained to be $-19.3\%$ and $-9.9\%$, respectively; however, $\delta_{Total(RL)} = 7.7\%$, and $\delta_{Total(LR)} = -12.1\%$. Thus the pure weak corrections corresponding to (RL) is 27% but that corresponding to (LR) is only -2%. From these results, it is clear that the experiments with beam polarization needs the precise calculations of the electroweak $\sigma_{O_\alpha}$ corrections. $\delta_{Total(ILC)} = -11.3\%$, $\delta_{ISR(ILC)} = -9.9\%$, and the weak correction for the proposed ILC polarization, $p_e = -0.8, p_p = 0.3$ is approximately $-1\%$. This result is sufficient to motivate the ILC to consider the contribution of the pure weak correction. Additionally $\delta_{weak}$ are necessary to compare with the naive estimation from the running alpha effects of $+14\% = (+7)\% + (+7)\%$.

| Processes     | $\sigma_{Tree}$ (pb) | $\sigma_{O_\alpha}$ (pb) | $\delta_{Total}%$ | $\sigma_{ISR}$ (pb) | $\delta_{ISR}%$ |
|---------------|-----------------------|---------------------------|-------------------|---------------------|-----------------|
| $\mu^+\mu^-H$ | $1.035(2)\times10^{-2}$ | $9.182(1)\times10^{-3}$   | -11.3             | $9.321(4)\times10^{-3}$ | -9.9            |
| $e^+e^-H$     | $1.119(1)\times10^{-2}$ | $9.841(2)\times10^{-3}$   | -12.0             | $9.159(4)\times10^{-3}$ | -18.1           |
| $\tau^+\tau^-H$ | $1.034(1)\times10^{-2}$ | $9.145(2)\times10^{-3}$   | -11.6             | $9.261(4)\times10^{-3}$ | -10.5           |
| $\nu_\mu\bar{\nu}_\mu H$ | $2.045(1)\times10^{-2}$ | $1.782(1)\times10^{-2}$   | -12.9             | $1.837(1)\times10^{-2}$ | -10.2           |
| $\nu_e\bar{\nu}_e H$ | $3.632(1)\times10^{-2}$ | $3.357(3)\times10^{-2}$   | -7.6              | $3.044(2)\times10^{-2}$ | -16.2           |

Table 19  Lepton processes with the ILC proposed beam polarization with $p_e = -0.8, p_p = 0.3$ at $\sqrt{s} = 250$ GeV and without experimental cuts.
We have summarized lepton processes $e^+e^- \rightarrow f \bar{f} H$ (i.e., $\mu^+\mu^- H$, $e^+e^- H$, $\tau^+\tau^- H$, $\nu_\mu\bar{\nu}_\mu H$, and $\nu_e\bar{\nu}_e H$) in Table 19.

We present two separate figures for the cross sections and total ratios, for convenience. Fig. 6 shows the cross sections at the tree level, as well as $\sigma_{\cal O}$, and ISR cross sections. Fig. 7 shows $\delta_{Total}$ and $\delta_{ISR}$ of the four leptonic processes.

![Cross sections of the leptonic processes with the ILC proposed beam polarization](image)

**Fig. 6** Cross sections of the leptonic processes with the ILC proposed beam polarization, $p_e = -0.8, p_p = 0.3$ at $\sqrt{s} = 250$ GeV and without experimental cuts.

Owing to the sizable contribution of the t-channel amplitude, $\sigma_{\cal O}(ILC)\nu_e\bar{\nu}_e H = 3.632(1) \times 10^{-2}$ pb is larger than $\sigma_{\cal O}(ILC)\nu_\mu\bar{\nu}_\mu H = 1.782(1) \times 10^{-2}$ pb. $\delta_{Total}(ILC)\nu_\mu\bar{\nu}_\mu H = -12.9\%$ and $\delta_{ISR}(ILC)\nu_\mu\bar{\nu}_\mu H = -10.2\%$, and hence, the pure weak correction for $\nu_\mu\bar{\nu}_\mu H$ is approximately $-3\%$ whereas, $\delta_{Total}(ILC)\nu_e\bar{\nu}_e H = -7.6\%$ and $\delta_{ISR}(ILC)\nu_e\bar{\nu}_e H = -16.2\%$, hence the pure weak corrections for $\nu_e\bar{\nu}_e H$ is approximately $+9\%$. Moreover, $\mu^+\mu^- H$ and $e^+e^- H$ have the same tendency as $\nu_\mu\bar{\nu}_\mu H$ and $\nu_e\bar{\nu}_e H$. From this discussion, it is clear that although t-channel cross sections at the tree level are small, they are important for $\delta_{Total}(ILC)\nu_\mu\bar{\nu}_\mu H = -12.9\%$ and $\delta_{Total}(ILC)\nu_e\bar{\nu}_e H = -7.6\%$; this 5.3\% difference maybe owing to the charge/neutral channel or iso-spin up/down type.

Next, we discuss the quark processes. $e^+e^- \rightarrow f \bar{f} H$ (i.e., $u\bar{u} H$, $d\bar{d} H$, $c\bar{c} H$, $b\bar{b} H$), which are summarized in the Table 20.
Fig. 7  Ratios of the leptonic processes with the ILC proposed beam polarization, $p_e = -0.8$, $p_p = 0.3$ at $\sqrt{s} = 250$ GeV and without experimental cuts.

| Processes  | $\sigma_{Tree}$ (pb) | $\sigma_{O\alpha}$ (pb) | $\delta_{Total}$ | $\sigma_{ISR}$ (pb) | $\delta_{ISR}$ |
|------------|----------------------|--------------------------|------------------|---------------------|----------------|
| $u\bar{u}H$ | $3.570(1) \times 10^{-2}$ | $2.934(1) \times 10^{-2}$ | -17.8           | $3.206(2) \times 10^{-2}$ | -10.0          |
| $d\bar{d}H$ | $4.581(1) \times 10^{-2}$ | $4.070(1) \times 10^{-2}$ | -11.5           | $4.115(2) \times 10^{-2}$ | -10.2          |
| $c\bar{c}H$ | $3.566(1) \times 10^{-2}$ | $2.926(1) \times 10^{-2}$ | -18.0           | $3.215(2) \times 10^{-2}$ | -10.2          |
| $b\bar{b}H$ | $4.533(1) \times 10^{-2}$ | $3.969(4) \times 10^{-2}$ | -12.5           | $4.068(2) \times 10^{-2}$ | -10.3          |

Table 20  Quark processes with the ILC proposed beam polarization, $p_e = -0.8$, $p_p = 0.3$ at $\sqrt{s} = 250$ GeV and without experimental cuts.

Again, we present two separate figures for the cross sections and total ratios. Fig. 8 shows the cross sections at the tree level, as well as the $\sigma_{O\alpha}$, and ISR cross sections. Fig. 9 shows $\delta_{Total}$ and $\delta_{ISR}$ of the four quark processes.

$\delta_{ISR}$ is approximately $-10\%$ for all processes.

$\delta_{Total}$ of the up-type quarks is $-18\%$, while $\delta_{Total}$ of the down-type quarks is $-12\%$ because of the charge difference or up/down type with different iso-spins.

$\sigma_{O\alpha(ILC)u\bar{u}H} = 2.933(4) \times 10^{-2}$ pb is larger than $\sigma_{O\alpha(ILC)c\bar{c}H} = 2.926(4) \times 10^{-2}$ pb as expected. These exhibit the opposite tendency to leptons.
Fig. 8  Cross sections of the quark processes with the ILC proposed beam polarization, $p_e = -0.8, p_p = 0.3$ at $\sqrt{s} = 250$ GeV and without experimental cuts.

$$\delta_{Total(ILC)}d\bar{d}H = -11.5\%$$ and $$\delta_{Total(ILC)}b\bar{b}H = -12.5\%$$, with the 1% difference arising because of the Yukawa coupling. Weak corrections for the up-type and down-type quarks are approximately $-8\%$ and $-2\%$, respectively.

Next, we compare the lepton and quark channels. For leptons, $\delta_{ISR}$ for s-channel and t-channel processes proximate $-10\%$ and $-18\%$, respectively; for quarks, $\delta_{ISR}$ is approximately $-10\%$ for all the cases with the ILC polarized beams and without experimental cuts.

Comparing the lepton and quark channels, the difference between the up-type and the down-type are clearly observed in quarks; however, this is not observed for lepton.

We also calculated the recoil mass distribution after applying three experimental cuts for the $e^+e^- \rightarrow \mu^+\mu^-H$ process at the $O(\alpha)$ corrections and including the ISR effects. The following results were obtained: $\delta_{Total} = -13.2\%$ and $\delta_{ISR} = -9.9\%$, such that the weak correction was estimated to be $-3\%$. Thus, to measure the $g_{HZZ}$ coupling within 1% accuracy, the weak correction of $-3\%$ cannot be neglected.
Fig. 9  Ratios of the quark processes with the ILC proposed beam polarization $p_e = -0.8, p_p = 0.3$ at $\sqrt{s} = 250$ GeV and without experimental cuts.

10  Conclusion

We presented the full $\mathcal{O}(\alpha)$ electroweak radiative corrections to associated 9 Higgs fermion pair production in $e^+e^-$ collisions with the beam polarization at the International Linear Collider(ILC). The computation was performed with the help of GRACE-loop. We compared the full $\mathcal{O}(\alpha)$ electroweak radiative corrections with the factorized initial state radiation(ISR) effects to estimate the pure weak corrections. The matrix element to be produced by GRACE-loop was tested by (1) the renormalization,(2)infrared divergence-free, (3)the hard photon cut parameters ($k_c$) independence in 0.1%, (4) independence of the non-linear gauge parameters.

Our calculations have the following features: (1) the beam polarization effect was considered, (2) the mass effect of all the fermions except the neutrinos was retained, and (3) Yukawa coupling of the bottom quark was included. To the best of our knowledge, this is the first detailed discussion of the $e^+e^- \rightarrow f\bar{f}H$ processes. Furthermore, we compared our results with those of multiple photon emission from electrons and positrons in the ISR process and highlighted that the $\mathcal{O}(\alpha)$ corrections presented in this work are important for the analysis of experimental data. In the case of the $e^+e^- \rightarrow \mu^+\mu^-H$ process with the experimental cuts and the proposed ILC beam polarization condition, the pure weak correction was estimated
to be \(-3\%\). Thus, in order to measure the $g_{HZZ}$ coupling within 1% accuracy, the pure weak $\mathcal{O}(\alpha)$ corrections cannot be neglected.

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