Unified discrete mechanics III: the hyperincursive discrete Klein-Gordon equation bifurcates to the 4 incursive discrete Majorana and Dirac equations and to the 16 Proca equations

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Abstract. This paper presents the second order hyperincursive discrete Klein-Gordon equation in three spatial dimensions. This discrete Klein-Gordon equation bifurcates to 4 incursive discrete equations. We present the 4 incursive discrete Dubois-Ord-Mann real equations and the corresponding first order partial differential equations. In three spatial dimensions, the two oscillators given by these equations are now entangled with one spatial dimension, so the 4 equations form a whole. Then, we deduce the 4 incursive discrete Dubois-Majorana real equations and the corresponding first order partial differential equations. Next, these Dubois-Majorana real equations are presented in the generic form of the Dirac 4-spinors equation. In making a change in the indexes of the 4 functions in the Dubois-Majorana equations, the 4 first order partial differential equations become identical to the Majorana real 4-spinors equations. Then, we demonstrate that the Majorana equations bifurcate to the 8 real Dirac first order partial differential equations that are transformed to the original Dirac 4-spinors equations. Then, we give the 4 incursive discrete Dirac 4-spinors equations. Finally, we show that there are 16 discrete functions associated to the space and time symmetric discrete Klein-Gordon equation. This is in agreement with the Proca thesis on the 16 components of the Dirac wave function in 4 groups of 4 equations. In this paper, we restricted our derivation of the Majorana and Dirac equations to the first group of 4 equations depending on 4 functions.

1. Introduction
This paper deals with a survey of our recent research on a unified discrete mechanics with the formalism of the hyperincursive discrete equation separable into incursive discrete equations.

This paper is a continuation of my two papers [1,2] presented at the IXth and Xth Symposia on unified field mechanics, honoring noted French mathematical physicist Jean-Pierre Vigier.

The first paper concerns the hyperincursive algorithms of classical harmonic oscillator applied to quantum harmonic oscillator separable into incursive oscillators.

The second paper deals a unified discrete mechanics given by the bifurcation of the hyperincursive discrete harmonic oscillator, the hyperincursive discrete Schrödinger quantum equation, the hyperincursive discrete Klein-Gordon equation and the Dirac quantum relativist equations.

This paper is based on a series of classical papers given historically as follows.
In 1926, Oskar Klein [3] and Walter Gordon [4] presented independently what is called the Klein-Gordon equation. In 1928, Dirac [5] introduced the relativist quantum mechanics based on this Klein-Gordon equation. His fundamental equation is based on 4-spinors, and is given by 4 first order complex partial differential equations. All the work of Dirac is well explained in his book [6].

In 1930 and 1932, Al Proca [7, 8] proposed a generalization of the Dirac theory with the introduction of 4 groups of 4-spinors, and with 16 first order complex partial differential equations.

In 1937, Ettore Majorana [9] proposed a real 4-spinors Dirac equation, given by 4 first order real partial differential equations. Ettore Majorana disappears just after having written this fundamental paper. Eliano Pessa [10] presented a very interesting paper on the Majorana oscillator based of the 4 first order real partial differential equations.

In this paper, we will not introduce the theory of Klein-Gordon and Dirac, so an excellent introduction to quantum mechanics is given in the books of Albert Messiah [11].

In my second Vigier paper [2], I have demonstrated that the second order hyperincursive discrete Klein-Gordon equation bifurcates to the 4 Dirac first order equations, in one space dimension. I obtained 4 equations, similar to the equations that G N Ord and R B Mann [11] deduced differently from a stochastic model with difference equations. A generalization is given in this paper under the name of Dubois-Ord-Mann equation. The temporal Klein-Gordon equation bifurcates is similar in mathematical form to the hyperincursive discrete classical harmonic oscillator.

A good introduction to incursion and hyperincursion is given in the following series of papers on the total incursive control of linear, non-linear and chaotic Systems [13], on computing anticipatory systems with incursion and hyperincursion [14], on the computational derivation of quantum and relativist systems with forward-backward space-time shifts [15], a review of incursive, hyperincursive and anticipatory systems, with the foundation of anticipation in electromagnetism [16], then on the precision and stability analysis of Euler, Runge-Kutta and incursive algorithms for the harmonic oscillator [17], and finally, on the new concept of deterministic anticipation in natural and artificial systems [18].

I wrote a series of theoretical papers on the discrete physics with Adel Antippa on the harmonic oscillator via the discrete path approach [19], on anticipation, orbital stability, and energy conservation in discrete harmonic oscillators [20], on the dual incursive system of the discrete harmonic oscillator [21], on the superposed hyperincursive system of the discrete harmonic oscillator [22], on the incursive discretization, system bifurcation, and energy conservation [23], on the hyperincursive discrete harmonic oscillator [24], on the synchronous discrete harmonic oscillator [25], on the discrete harmonic oscillator, a short compendium of formulas [26], on the time-symmetric discretization of the harmonic oscillator [27], and finally, on the discrete harmonic oscillator, evolution of notation and cumulative erratum [28]. This discrete physics is based on the fundamental mathematical development of the hyperincursive and incursive discrete harmonic oscillator.

The main purpose of this paper is the bifurcation of our second order hyperincursive discrete Klein-Gordon equation firstly to the 4 incursive discrete Majorana real 4-spinors real equations, then to the 8 incursive discrete Dirac real 8-spinors real equations, that can be rewritten as the 4 incursive discrete Dirac complex 4-spinors equations.

This paper is essentially based on the recent papers that I presented on the hyperincursive and incursive discrete equations dealing with the following topics on reversible time, retardation and anticipation in the recursive, incursive and hyperincursive Klein-Gordon quantum systems [29], on the reversible hyperincursive discrete Schrödinger quantum harmonic oscillator separable into two non-reversible incursive discrete oscillators [30], on the deduction of the first order spinor equations of the Schrödinger quantum spatial harmonic oscillator with the emergence of the fundamental energy [31], on the generalisation to three spatial dimensions of the Dubois-Ord-Mann real 4-spinors equation from the hyperincursive discrete Klein-Gordon equation [32], on the deduction of the Majorana real 4-spinors generic Dirac equation from the computable hyperincursive discrete Klein-Gordon equation [33], on the hyperincursive discrete Klein-Gordon equation for computing the Majorana real 4-spinors equation and the real 8-spinors Dirac equation [34], and finally, on the bifurcation of the
hyperincursive discrete Klein-Gordon equation to real 4-spinors Dirac equation related to the Majorana equation.

The paper is organized as follows.

In section 2, we present the Klein-Gordon partial differential equation and the space and time symmetric second order hyperincursive discrete Klein-Gordon equation.

In section 3, we present the Dubois-Ord-Mann real 4-spinors. It is important to know that the Dubois-Ord-Mann real 4-spinors belong to the group of the Majorana real 4-spinors. Indeed, with two successive permutations, the 4 first order partial differential equations of the Dubois-Ord-Mann 4-spinors transform to the 4 first order partial differential equations of the Majorana real 4-spinors equation given by Pessa [10]. The first permutation consists in permuting two space variables. The second permutation consists in permuting the indexes of the functions. This was demonstrated in my recent paper [35]. But it is interesting to analyze separately the transformation of the 4 Dubois-Ord-Mann equations, firstly, in permuting the two space variables, and secondly, in permuting the indexes of the functions. Indeed, in permuting only the two space variables, y and z, in the 4 Dubois-Ord-Mann equations, we obtain the 4 first order partial differential equations where the “Majorana Oscillator” presented in the paper of Pessa [10] is well put in evidence: the first two equations represent a first oscillator and the last two equations represents the second oscillator, that is an anti-oscillator. This is completely in agreement with my theory of the hyperincursive discrete harmonic oscillator that bifurcates to two incursive oscillators, each oscillator being the reverse of the other. This transformation of the Dubois-Ord-Mann equations, by permutation of the space variables, y and z, was presented in a recent paper [33], where the transformed equations were called the Dubois-Majorana equations. In this paper [33], we say explicitly what follows: “The Dubois Majorana real 4-spinors equation is in fact composed of a spatial oscillator entangled to a spatial harmonic oscillator, related to the particle electron and anti-particle positron in Majorana equation. …The Dubois-Majorana discrete real 4-spinors equation shows explicitly that the Majorana equation is in fact composed of a discrete hyperincursive spatial oscillator entangled to a spatial harmonic oscillator, related to the particle electron and anti-particle positron in Majorana equation. We have shown that the “Majorana Oscillator” in reference to the paper of Eliano Pessa [10] is explicitly explained by the hyperincursive discrete harmonic oscillator which is composed of 4 incursive discrete equations that are in fact the real 4-spinors time equation.” Moreover the 4 incursive discrete Dubois-Majorana equations can be used for the simulation of the “Majorana Oscillator”. The incursive discrete Dubois-Majorana real 4-spinors equations, is transformed to the Majorana equations [9, 10], by permutation of the indexes of the functions, as shown is our paper [33].

Section 4 gives 4 incursive discrete Dubois-Majorana real equations in three space dimensions (3D). The continuous Dubois-Majorana real equations are given by 4 first order partial differential equations. The two oscillators given by these equations are entangled with the spatial dimension z.

In section 5, then, we demonstrate that these Dubois-Majorana real equations represent a real 4-spinors in the generic form of the Dirac relativist quantum equation.

In section 6, we make a change in indexes of the 4 functions in the Dubois-Majorana equations that are then defined as the Majorana functions, so the transformed 4 first order partial differential equations are identical to the original Majorana equations.

In section 7, next, in defining the Majorana functions by 2-spinors real functions, after some mathematical manipulations, we demonstrate that the Majorana real 4-spinors equations bifurcate into the 8 real equations.

These 8 real first order partial differential equations represent the Dirac real 8-spinors equations that are transformed to the original Dirac complex 4-spinors equations.

Then the section 8 presents the 4 incursive discrete Dirac 4-spinors equations depending on 4 complex discrete Dirac wave functions.

In section 9, finally, we show that there are 16 complex functions associated to this second order hyperincursive discrete Klein-Gordon equation.

This is in agreement with the Proca thesis.
For Proca, the Dirac function has 16 components and not 4. But, in fact, these 16 components are divided in 4 groups of 4 functions with 4 equations. The first group is the fundamental group and the 3 other groups can be obtained from the first group. The first group is not separable except in particular cases.

In this paper, we restricted our derivation of the Majorana and Dirac equations to the first group of 4 equations depending to 4 functions.

2. The space and time symmetric second order hyperincursive discrete Klein-Gordon equation

In 1926, Oskar Klein [3] and Walter Gordon [4] published independently their famous equation, called the Klein-Gordon equation.

The Klein-Gordon equation with the function \( \varphi = \varphi(\mathbf{r}, t) \) in three spatial dimensions \( \mathbf{r} = (x, y, z) \) and time \( t \) is given by

\[
- \frac{\hbar^2}{2m} \varphi''(\mathbf{r}, t) = - \hbar^2 c^2 \nabla^2 \varphi(\mathbf{r}, t) + m^2 c^4 \varphi(\mathbf{r}, t)
\]

or, in the explicit form of the nabla operator \( \nabla \),

\[
- \frac{\hbar^2}{2m} \varphi''(\mathbf{r}, t) = - \hbar^2 c^2 \partial_x^2 \varphi + \hbar^2 c^2 \partial_y^2 \varphi + \hbar^2 c^2 \partial_z^2 \varphi + m^2 c^4 \varphi
\]

where \( \hbar \) is the constant of Plank, \( c \) is the speed of light, and \( m \) the mass.

As we will consider the discrete Klein-Gordon equation, we make the following usual change of variables

\[
q(\mathbf{r}, t) = \varphi(\mathbf{r}, t)
\]

\[
a = \omega = mc^2/\hbar
\]

where \( \omega \) is a frequency, so the Klein-Gordon equation (2) becomes

\[
\frac{\partial^2 q(\mathbf{r}, t)}{\partial t^2} = +c^2 \partial_x^2 q(\mathbf{r}, t)/\partial x^2 + c^2 \partial_y^2 q(\mathbf{r}, t)/\partial y^2 + c^2 \partial_z^2 q(\mathbf{r}, t)/\partial z^2 - a^2 q(\mathbf{r}, t)
\]

From the Klein-Gordon equation (5), the second order hyperincursive discrete Klein-Gordon equation [32,35] is given by

\[
q(x, y, z, t + 2\Delta t) - 2q(x, y, z, t) + q(x, y, z, t - 2\Delta t) = +B^2[q(x + 2\Delta x, y, z, t) - 2q(x, y, z, t) + q(x - 2\Delta x, y, z, t)]
\]

\[
+C^2[q(x, y + 2\Delta y, z, t) - 2q(x, y, z, t) + q(x, y - 2\Delta y, z, t)]
\]

\[
+D^2[q(x, y, z + 2\Delta z, t) - 2q(x, y, z, t) + q(x, y, z - 2\Delta z, t)] - A^2 q(x, y, z, t)
\]

where the following parameters \( A, B, C, \) and, \( D \),

\[
A = a (2\Delta t), \quad B = c (2\Delta t)/(2\Delta x), \quad C = c (2\Delta t)/(2\Delta y), \quad D = c (2\Delta t)/(2\Delta z)
\]

depend on the discrete interval of time \( \Delta t \), and the discrete intervals of space, \( \Delta x, \Delta y, \Delta z \), respectively.

As usually made in computer science, let us now introduce the discrete time \( t_k \), and the discrete spaces \( x_l, y_m, z_n \), as follows

\[
t_k = t_0 + k\Delta t, k = 0,1,2, \ldots
\]

where \( k \) is the integer time increment, and

\[
x_l = x_0 + l\Delta x, l = 0,1,2, \ldots
\]

\[
y_m = y_0 + m\Delta y, m = 0,1,2, \ldots
\]

\[
z_n = z_0 + n\Delta z, n = 0,1,2, \ldots
\]

where \( l, m, n \), are the integer space increments.
So, with these time and space increments, the second order hyperincursive discrete Klein-Gordon equation (6) becomes

\[
q(l, m, n, k + 2) - 2q(l, m, n, k) + q(l, m, n, k - 2) = \\
+ B^2 [q(l + 2, m, n, k) - 2q(l, m, n, k) + q(l - 2, m, n, k)] \\
+ C^2 [q(l, m + 2, n, k) - 2q(l, m, n, k) + q(l, m - 2, n, k)] \\
+ D^2 [q(l, m, n + 2, k) - 2q(l, m, n, k) + q(l, m, n - 2, k)] - A^2 q(l, m, n, k)
\]  

(10)

This equation without spatial components corresponding to a particle at rest is similar to the harmonic oscillator.

For a particle at rest, the Klein-Gordon equation (5), with the function \(q(t)\) depending only on the time variable, is given by

\[
\frac{\partial^2 q(t)}{\partial t^2} = -\frac{\hbar}{m} q(t)
\]

(11)

with the frequency \(\omega = \frac{mc^2}{\hbar}\) given by the equation (4).

This equation (11) is formally similar to the equation of the harmonic oscillator for which \(q(t)\) would represent the position \(x(t)\), and \(\frac{\partial q(t)}{\partial t}\) would represent the velocity \(v(t) = \frac{dx(t)}{dt}\).

So, with only the temporal component, the second order hyperincursive discrete Klein-Gordon equation (10) becomes

\[
q(k + 2) - 2q(k) + q(k - 2) = -A^2 q(k)
\]

(12)

that is similar to the second order hyperincursive equation of the harmonic oscillator.

This hyperincursive equation (12) is separable into a first discrete incursive oscillator depending on two functions defined by \(q_1(k), q_2(k)\), and a second incursive oscillator depending on two other functions defined by \(q_3(k), q_4(k)\), given by first order discrete equations.

So, the first incursive equations are given by:

\[
q_1(2k) = q_1(2k - 2) + A q_2(2k - 1) \\
q_2(2k + 1) = q_2(2k - 1) - A q_1(2k)
\]

(13a,b)

where \(q_1(2k)\) is defined of the even steps of the time, and \(q_2(2k + 1)\) is defined on the odd steps of the time.

And the second incursive equations are given by:

\[
q_3(2k) = q_3(2k - 2) - A q_4(2k - 1) \\
q_4(2k + 1) = q_4(2k - 1) + A q_3(2k)
\]

(14a,b)

where \(q_3(2k)\) is defined of the even steps of the time, and \(q_4(2k + 1)\) is defined on the odd steps of the time.

The second incursive system is the time reverse of the first incursive system in making the time inversion \(T\)

\[
T: \Delta t \rightarrow -\Delta t
\]

(15)

this gives an oscillator and its anti-oscillator.

In this paper we will present the bifurcation of this equation (10) to the 4 incursive discrete Majorana real equations which bifurcates to the 4 incursive discrete Dirac equations.

In the next section, we present the generalisation to three spatial dimensions (3D) of the one-dimension Dubois-Ord-Mann real 4-spinors equation [2] from the hyperincursive discrete Klein-Gordon equation [32].

3. The 4 incursive discrete Dubois-Ord-Mann equations in three spatial dimensions

In a recent paper, we presented the 4 incursive discrete Dubois-Ord-Mann real equations [32].
With the 4 discrete functions, \( q_i(l, m, n, k), i = 1, 2, 3, 4 \), the 4 incursive discrete Dubois-Ord-Mann equations [2, 29], in three spatial dimensions, are given by

\[
\begin{align*}
q_1(l, m, n, k + 1) &= q_1(l, m, n, k - 1) + B[q_2(l + 1, m, n, k) - q_2(l - 1, m, n, k)] \\
&\quad + C[q_3(l, m + 1, n, k) - q_3(l, m - 1, n, k)] + D[q_4(l, m, n + 1, k) - q_4(l, m, n - 1, k)] \\
&\quad - Aq_2(l, m, n, k)
\end{align*}
\] (16a)

\[
\begin{align*}
q_2(l, m, n, k + 1) &= q_2(l, m, n, k - 1) + B[q_1(l + 1, m, n, k) - q_1(l - 1, m, n, k)] \\
&\quad - C[q_3(l, m + 1, n, k) - q_3(l, m - 1, n, k)] - D[q_4(l, m, n + 1, k) - q_4(l, m, n - 1, k)] \\
&\quad + Aq_1(l, m, n, k)
\end{align*}
\] (16b)

\[
\begin{align*}
q_3(l, m, n, k + 1) &= q_3(l, m, n, k - 1) + B[q_4(l + 1, m, n, k) - q_4(l - 1, m, n, k)] \\
&\quad - C[q_2(l, m + 1, n, k) - q_2(l, m - 1, n, k)] + D[q_3(l, m, n + 1, k) - q_3(l, m, n - 1, k)] \\
&\quad + Aq_4(l, m, n, k)
\end{align*}
\] (16c)

\[
\begin{align*}
q_4(l, m, n, k + 1) &= q_4(l, m, n, k - 1) + B[q_3(l + 1, m, n, k) - q_3(l - 1, m, n, k)] \\
&\quad + C[q_1(l, m + 1, n, k) - q_1(l, m - 1, n, k)] - D[q_4(l, m, n + 1, k) - q_4(l, m, n - 1, k)] \\
&\quad - Aq_3(l, m, n, k)
\end{align*}
\] (16d)

where the following parameters \( A, B, C, \) and, \( D, \)

\[
A = a (2\Delta t), \quad B = c (2\Delta t)/(2\Delta x), \quad C = c (2\Delta t)/(2\Delta y), \quad D = c (2\Delta t)/(2\Delta z)
\] (17)

depend on the discrete interval of time \( \Delta t, \) and the discrete intervals of space, \( \Delta x, \Delta y, \Delta z, \) respectively.

Let us now transform the 4 discrete equations (20a,b,c,d) to partial differential equations.

The discrete functions \( q_j(x, y, z, t) = q_j(r, t), j = 1, 2, 3, 4 \) tend to the continuous functions \( \Psi_j(x, y, z, t) = \Psi_j(r, t) \), when the discrete space and time intervals tend to zero.

At the limit,

\[
\Psi_j = \Psi_j(r, t) = \lim_{\Delta r \to 0, \Delta t \to 0} q_j(r, t), j = 1, 2, 3, 4
\] (18)

the 4 discrete equations (16a,b,c,d) tend to the following 4 first order partial differential equations representing the Dubois-Ord-Mann real 4-spinors equations

\[
\begin{align*}
+ \partial \Psi_1/\partial t &= + c \partial \Psi_2/\partial x + c \partial \Psi_4/\partial y + c \partial \Psi_1/\partial z - (mc^2/h)\Psi_2 \\
+ \partial \Psi_2/\partial t &= + c \partial \Psi_1/\partial x - c \partial \Psi_3/\partial y - c \partial \Psi_2/\partial z + (mc^2/h)\Psi_1 \\
+ \partial \Psi_3/\partial t &= + c \partial \Psi_4/\partial x - c \partial \Psi_2/\partial y + c \partial \Psi_3/\partial z + (mc^2/h)\Psi_4 \\
+ \partial \Psi_4/\partial t &= + c \partial \Psi_3/\partial x + c \partial \Psi_1/\partial y - c \partial \Psi_4/\partial z - (mc^2/h)\Psi_3
\end{align*}
\] (19a-c)

Let us notice that these two oscillators given by equations are entangled with the spatial dimension \( y. \)

The discrete Dubois-Ord-Mann real 4-spinors was obtained from the deterministic space and time symmetric hyperincursive Klein-Gordon equation. It could be interesting to see how to generalize the one-dimensional real stochastic difference equations of Ord and Mann to three spatial dimensions. It is important to know that the Dubois-Ord-Mann real 4-spinors belong to the group of the Majorana real 4-spinors. Indeed, with two successive permutations, the 4 first order partial differential equations of the Dubois-Ord-Mann 4-spinors transform to the 4 first order partial differential equations of the
Majorana real 4-spinors equation given by Pessa [10]. The first permutation consists in permuting two space variables. The second permutation consists in permuting the indexes of the functions. This was demonstrated in my recent paper [35]. But it is interesting to analyze separately the transformation of the 4 Dubois-Ord-Mann equations, firstly, in permuting the two space variables, and secondly, in permuting the indexes of the functions. Indeed, in permuting only the two space variables, y and z, in the 4 Dubois-Ord-Mann equations, we obtain the 4 first order partial differential equations where the “Majorana Oscillator” presented in the paper of Pessa [10] is well put in evidence: the first two equations represents a first oscillator and the last two equations represents the second oscillator, that is an anti-oscillator. This is completely in agreement with my theory of the hyperincursive discrete harmonic oscillator that bifurcates to two incursive oscillators, each oscillator being the reverse of the other. This transformation of the Dubois-Ord-Mann equations, by permutation of the space variables, y and z, was presented in a recent paper [33], where the transformed equations were called the Dubois-Majorana equations. In this paper [33], we say explicitly what follows: “The Dubois Majorana real 4-spinors equation is in fact composed of a spatial oscillator entangled to a spatial anti-oscillator, related to the particle electron and anti-particle positron in Majorana equation. … The Dubois-Majorana discrete real 4-spinors equation shows explicitly that the Majorana equation is in fact composed of a discrete hyperincursive spatial oscillator entangled to a spatial harmonic oscillator, related to the particle electron and anti-particle positron in Majorana equation. We have shown that the ‘Majorana Oscillator’ in reference to the paper of Eliano Pessa [10] is explicitly explained by the hyperincursive discrete harmonic oscillator which is composed of 4 incursive discrete equations that are in fact the real 4-spinors time equation.” Moreover the 4 incursive discrete Dubois-Majorana equations can be used for the simulation of the “Majorana Oscillator”.

4. The hyperincursive discrete and continuous Dubois-Majorana real equations
In inversing the space variables z and y, in the Dubois-Ord-Mann equations (16a,b,c,d), we obtain the Dubois-Majorana real 4-spinors equations [33,35].
So, the hyperincursive discrete Dubois-Majorana real equations are given by the following 4 discrete space time incursive equations of the 4 functions $q_i(l, m, n, k), i = 1,2,3,4$,

\[ q_1(l, m, n, k + 1) = q_1(l, m, n, k - 1) + B[q_2(l + 1, m, n, k) - q_2(l - 1, m, n, k)] + C[q_4(l, m + 1, n, k) - q_4(l, m - 1, n, k)] - Aq_2(l, m, n, k) \]  

\[ q_2(l, m, n, k + 1) = q_2(l, m, n, k - 1) + B[q_1(l + 1, m, n, k) - q_1(l - 1, m, n, k)] - C[q_2(l, m + 1, n, k) - q_2(l, m - 1, n, k)] - D[q_3(l, m, n + 1, k) - q_3(l, m, n - 1, k)] + Aq_1(l, m, n, k) \]  

\[ q_3(l, m, n, k + 1) = q_3(l, m, n, k - 1) + B[q_4(l + 1, m, n, k) - q_4(l - 1, m, n, k)] + C[q_3(l, m + 1, n, k) - q_3(l, m - 1, n, k)] - D[q_2(l, m, n + 1, k) - q_2(l, m, n - 1, k)] + Aq_4(l, m, n, k) \]  

\[ q_4(l, m, n, k + 1) = q_4(l, m, n, k - 1) + B[q_3(l + 1, m, n, k) - q_3(l - 1, m, n, k)] - C[q_4(l, m + 1, n, k) - q_4(l, m - 1, n, k)] + D[q_1(l, m, n + 1, k) - q_1(l, m, n - 1, k)] - Aq_3(l, m, n, k) \]
So, the continuous Dubois-Majorana real 4-spinors [33] are given by the following 4 first order partial differential equations, after inversion of space variables \(y\) and \(z\) in equations (19a,b,c,d), as follows

\[
\Psi_1 \frac{\partial \Psi_1}{\partial t} = c \frac{\partial \Psi_2}{\partial x} + c \frac{\partial \Psi_4}{\partial y} + \frac{mc^2}{\hbar} \Psi_2
\]

\[
\Psi_2 \frac{\partial \Psi_2}{\partial t} = c \frac{\partial \Psi_1}{\partial x} - c \frac{\partial \Psi_3}{\partial y} - \frac{mc^2}{\hbar} \Psi_1
\]

\[
\Psi_3 \frac{\partial \Psi_3}{\partial t} = c \frac{\partial \Psi_4}{\partial x} + c \frac{\partial \Psi_2}{\partial y} + \frac{mc^2}{\hbar} \Psi_4
\]

\[
\Psi_4 \frac{\partial \Psi_4}{\partial t} = c \frac{\partial \Psi_3}{\partial x} - c \frac{\partial \Psi_1}{\partial y} + \frac{mc^2}{\hbar} \Psi_3
\]

(21a)

(21b)

(21c)

(21d)

The two oscillators given by the equations (21a,b) and (21c,d) are now entangled with the spatial dimension \(z\).

The next section will give these Dubois-Majorana real 4-spinors in the generic form of the Dirac equation.

5. The Dubois-Majorana real 4-spinors equations in the generic form of the Dirac equation

In a recent paper [33,35], we have given these Dubois-Majorana real 4-spinors in the generic form of the Dirac equation.

Indeed, in defining the 2-spinors real functions

\[
\varphi_a = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \varphi_b = \begin{pmatrix} \Psi_3 \\ \Psi_4 \end{pmatrix},
\]

(22)

the two equations (21a,b) and (21c,d) are transformed to the two 2-spinors real equations:

\[
\frac{\partial \varphi_a}{\partial t} = +c \sigma_1 \varphi_a/\partial x + c \sigma_3 \varphi_a/\partial y - c \sigma_2 \varphi_a/\partial z + \frac{mc^2}{\hbar} \sigma_2 \varphi_a
\]

\[
\frac{\partial \varphi_b}{\partial t} = +c \sigma_1 \varphi_b/\partial x + c \sigma_3 \varphi_b/\partial y + c \sigma_2 \varphi_b/\partial z - \frac{mc^2}{\hbar} \sigma_2 \varphi_b
\]

(23a)

(23b)

where the real 2-spinors matrices \(\sigma_1, \sigma_2, \sigma_3\), are defined by

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

(24a,b,c)

and 2-Identity

\[
\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2,
\]

(24d)

with the properties

\[
\sigma_1^2 = \sigma_0, \sigma_2^2 = -\sigma_0, \sigma_3^2 = \sigma_0,
\]

(25)

\[
\sigma_1 \sigma_2 = \sigma_3, \sigma_2 \sigma_3 = -\sigma_1, \sigma_3 \sigma_2 = -\sigma_1, \sigma_1 \sigma_3 = -\sigma_2, \sigma_1 \sigma_1 = \sigma_2
\]

(26)

These real spinor matrices are related to the Pauli spin matrices,

\[
\sigma_x = \sigma_1, \sigma_y = i \sigma_2, \sigma_z = \sigma_3
\]

(27a,b,c)

In defining the 4-spinors real function

\[
\Psi = \begin{pmatrix} \varphi_a \\ \varphi_b \end{pmatrix},
\]

(28)

the two equations (23a,b) can be regrouped to the following 4-spinors real equation

\[
\frac{\partial \Psi}{\partial t} = -c \alpha_1 \varphi_a/\partial x - c \alpha_2 \varphi_a/\partial y - c \alpha_3 \varphi_a/\partial z + \frac{mc^2}{\hbar} \alpha_0 \Psi
\]

(29)

where the real 4-spinors matrices \(\alpha_1, \alpha_2, \alpha_3, \alpha_0\), are given by

\[
\alpha_1 = -\begin{pmatrix} \sigma_1 \\ 0 \\ 0 \end{pmatrix}, \alpha_2 = -\begin{pmatrix} 0 \\ \sigma_3 \\ 0 \end{pmatrix}, \alpha_3 = -\begin{pmatrix} 0 \\ \sigma_2 \\ 0 \end{pmatrix}, \alpha_0 = \begin{pmatrix} \sigma_2 \\ 0 \\ -\sigma_2 \end{pmatrix},
\]

(30a,b,c,d)

with the properties
\[ \alpha_1^2 = I_4, \alpha_2^2 = I_4, \alpha_3^2 = I_4, \alpha_0^2 = -I_4 \] (31)

and 4-Identity
\[ I_4 = \begin{pmatrix} 1_2 & 0 \\ 0 & 1_2 \end{pmatrix} \] (32)

\[ \alpha_0 \alpha_1 = \begin{pmatrix} -\sigma_2 \sigma_1 \\ 0 \\ 0 \\ \sigma_2 \sigma_1 \end{pmatrix}, \quad \alpha_0 \alpha_2 = \begin{pmatrix} 0 \\ \sigma_2^2 \\ 0 \\ 0 \end{pmatrix}, \quad \alpha_0 \alpha_3 = \begin{pmatrix} -\sigma_2 \sigma_3 \\ 0 \\ 0 \\ \sigma_2 \sigma_3 \end{pmatrix}, \quad \alpha_0 \alpha_4 = 0, \]
\[ \alpha_1 \alpha_2 = \begin{pmatrix} 0 \\ -\sigma_1 \sigma_2 \\ 0 \\ 0 \end{pmatrix}, \quad \alpha_1 \alpha_3 = \begin{pmatrix} \sigma_1 \sigma_3 \\ 0 \\ \sigma_1 \sigma_3 \\ 0 \end{pmatrix}, \quad \alpha_1 \alpha_4 = 0, \]
\[ \alpha_2 \alpha_3 = \begin{pmatrix} 0 \\ 0 \\ -\sigma_2 \sigma_3 \\ 0 \end{pmatrix}, \quad \alpha_2 \alpha_4 = 0, \quad \alpha_3 \alpha_4 = 0 \] (33)

that give the following properties
\[ \alpha_0 \alpha_1 + \alpha_1 \alpha_0 = 0, \quad \alpha_0 \alpha_2 + \alpha_2 \alpha_0 = 0, \quad \alpha_0 \alpha_3 + \alpha_3 \alpha_0 = 0, \]
\[ \alpha_1 \alpha_2 + \alpha_2 \alpha_1 = 0, \quad \alpha_1 \alpha_3 + \alpha_3 \alpha_1 = 0, \quad \alpha_2 \alpha_3 + \alpha_3 \alpha_2 = 0 \] (34)

In multiplying the 4-spinors real equation (29) by \( i \hbar \), one obtains the 4-spinors equation
\[ + i \hbar \partial \Psi / \partial t = - c \alpha_x \hbar \partial \Psi / \partial x - c \alpha_y \hbar \partial \Psi / \partial y - c \alpha_z \hbar \partial \Psi / \partial z + (mc^2 / \hbar) i \hbar \omega_0 \Psi \] (35)

and in introducing the quantum 3-momentum-vector
\[ p = (p_1, p_2, p_3) \] given by
\[ p_1 = -i \hbar \partial / \partial x, p_2 = -i \hbar \partial / \partial y, p_3 = -i \hbar \partial / \partial z, \] (36a,b,c)

with the energy \( E \)
\[ E = i \hbar \partial / \partial t \] (36d)

the 4-spinors equation (35) is represented in the generic form of the Dirac equation as
\[ + i \hbar \partial \Psi / \partial t = \alpha_x p_1 \Psi + \alpha_y p_2 \Psi + \alpha_z p_3 \Psi + mc^2 \beta \Psi \] (37)

where
\[ \alpha_x = \alpha_1, \quad \alpha_y = \alpha_2, \quad \alpha_z = \alpha_3, \] (38a,b,c)

and
\[ \beta = i \alpha_0 \] (38d)

Now let us show that this Dubois-Majorana real 4-spinors can be simply transformed to the original Majorana equation [33,35].

6. The continuous and hyperincursive discrete Majorana real 4-spinors equations

Recently, we have shown that this Dubois-Majorana real 4-spinors can be simply transformed to the original Majorana equation [33,35].

In the Dubois-Majorana equations 4.2-a-b-c-d, let us make the change in indexes \( j \) of the 4 functions \( \Psi_j = \Psi_j(x, y, z, t), \ j = 1,2,3,4, \) as follows
\[ \Psi_1 = \Psi_4, \quad \Psi_2 = \Psi_2, \quad \Psi_3 = \Psi_1, \quad \Psi_4 = \Psi_3, \] (39)

where the functions
\[ \Psi_j = \Psi_j(x, y, z, t), \ j = 1,2,3,4, \] (40)

define the Majorana functions.

So, with these changes of indexes, the 4 equations (21a,b,c,d) are transformed to the following 4 first order partial differential equations...
which are identical to the original Majorana equations [9], e.g. equations (4a,b,c,d) in Pessa [10].

Let us remark that in 1937, Ettore Majorana published his last paper on his Majorana Equation [9], before his mysterious disappearance.

The “Majorana Oscillator”, in reference to the paper of Eliano Pessa [10], is explicitly written by the real 4-spinors time differential equations, for particles at rest, as follows:

\[ + \frac{\partial \Psi_1}{\partial t} = +c \frac{\partial \Psi_4}{\partial x} - c \frac{\partial \Psi_2}{\partial y} + c \frac{\partial \Psi_3}{\partial z} - (mc^2/h) \Psi_4 \] \hspace{1cm} (41a)

\[ + \frac{\partial \Psi_2}{\partial t} = +c \frac{\partial \Psi_4}{\partial x} - c \frac{\partial \Psi_3}{\partial y} - c \frac{\partial \Psi_1}{\partial z} + (mc^2/h) \Psi_3 \] \hspace{1cm} (41b)

\[ + \frac{\partial \Psi_3}{\partial t} = +c \frac{\partial \Psi_2}{\partial x} + c \frac{\partial \Psi_1}{\partial y} + c \frac{\partial \Psi_4}{\partial z} - (mc^2/h) \Psi_2 \] \hspace{1cm} (41c)

\[ + \frac{\partial \Psi_4}{\partial t} = +c \frac{\partial \Psi_1}{\partial x} + c \frac{\partial \Psi_3}{\partial y} - c \frac{\partial \Psi_2}{\partial z} + (mc^2/h) \Psi_1 \] \hspace{1cm} (41d)

which are similar to the Dubois-Majorana temporal equations given by equations (21a,b,c,d).

In view of simulating this “Majorana Oscillator”, let us explicit the hyperincursive discrete Majorana equations.

As presented in my recent paper [33,35], in making the change in indexes of the functions

\[ q_j = q_j(x,y,z,t), j = 1,2,3 \]

of the real equations in the Dubois-Majorana discrete 4-spinors equations:

\[ \bar{q}_1 = q_4, \quad \bar{q}_2 = q_2, \quad \bar{q}_3 = q_1, \quad \bar{q}_4 = q_3, \] \hspace{1cm} (43)

where the functions \( \bar{q}_j = \bar{q}_j(x,y,z,t) = \bar{q}_j(l,m,n,k), j = 1,2,3,4 \), define discrete Majorana functions, we obtain the 4 discrete incursive equations of the functions \( \bar{q}_j, j = 1,2,3,4 \):

\[ \bar{q}_1(l,m,n,k+1) = \bar{q}_1(l,m,n,k-1) + B[\bar{q}_4(l+1,m,n,k) - \bar{q}_4(l-1,m,n,k)] - C[\bar{q}_1(l+1,m,n,k) - \bar{q}_1(l-1,m,n,k)] + D[\bar{q}_3(l,m,n+1,k) - \bar{q}_3(l,m,n-1,k)] \] \hspace{1cm} (44a)

\[ -A[\bar{q}_2(l,m,n,k)] \]

\[ \bar{q}_2(l,m,n,k+1) = \bar{q}_2(l,m,n,k-1) + B[\bar{q}_3(l+1,m,n,k) - \bar{q}_3(l-1,m,n,k)] - C[\bar{q}_2(l+1,m,n,k) - \bar{q}_2(l-1,m,n,k)] - D[\bar{q}_1(l,m,n+1,k) - \bar{q}_1(l,m,n-1,k)] \] \hspace{1cm} (44b)

\[ +A[\bar{q}_4(l,m,n,k)] \]

\[ \bar{q}_3(l,m,n,k+1) = \bar{q}_3(l,m,n,k-1) + B[\bar{q}_2(l+1,m,n,k) - \bar{q}_2(l-1,m,n,k)] + C[\bar{q}_3(l+1,m,n,k) - \bar{q}_3(l-1,m,n,k)] + D[\bar{q}_1(l,m,n+1,k) - \bar{q}_1(l,m,n-1,k)] \] \hspace{1cm} (44c)

\[ -A[\bar{q}_4(l,m,n,k)] \]

\[ \bar{q}_4(l,m,n,k+1) = \bar{q}_4(l,m,n,k-1) + B[\bar{q}_3(l+1,m,n,k) - \bar{q}_3(l-1,m,n,k)] + C[\bar{q}_4(l+1,m,n,k) - \bar{q}_4(l-1,m,n,k)] - D[\bar{q}_2(l,m,n+1,k) - \bar{q}_2(l,m,n-1,k)] \] \hspace{1cm} (44d)

\[ +A[\bar{q}_1(l,m,n,k)] \]
with
\[ \bar{A} = A = a(2\Delta t), \quad \bar{B} = B = c \Delta t/\Delta x, \quad \bar{C} = C = c \Delta t/\Delta y, \quad \bar{D} = D = c \Delta t/\Delta z \]
where \( \Delta t \) and \( \Delta x, \Delta y, \Delta z \) are the discrete intervals of time and space respectively.

7. The bifurcation of the Majorana real 4-spinors equation to the Dirac real 8-spinors equations

Recently, we demonstrated that Majorana 4-spinors equations bifurcate simply to the Dirac real 8-spinors equations [34,35].

First, let us consider the inverse parity space, in inverting the sign of the space variables in the Majorana equations (41a,b,c,d),

\[ \frac{\partial \Psi}{\partial t} = -c \frac{\partial \Psi}{\partial x} - c \frac{\partial \Psi}{\partial y} - c \frac{\partial \Psi}{\partial z} - (mc^2/h)\Psi \]

In defining the 2-spinors real functions,
\[ \varphi_a = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \varphi_b = \begin{pmatrix} \Psi_3 \\ \Psi_4 \end{pmatrix}, \]

the two equations (46a,b) and (46c,d) are transformed to the two 2-spinors real equations

\[ \frac{\partial \varphi_a}{\partial t} = -c \sigma_1 \frac{\partial \varphi_b}{\partial x} + c \sigma_0 \frac{\partial \varphi_a}{\partial y} - c \sigma_2 \frac{\partial \varphi_b}{\partial z} + (mc^2/h)\sigma_2 \varphi_a \]
\[ \frac{\partial \varphi_b}{\partial t} = -c \sigma_1 \frac{\partial \varphi_a}{\partial x} - c \sigma_0 \frac{\partial \varphi_b}{\partial y} - c \sigma_2 \frac{\partial \varphi_a}{\partial z} + (mc^2/h)\sigma_2 \varphi_a \]

where the real 2-spinors matrices \( \sigma_1, \sigma_2, \sigma_3 \), are defined by
\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

and 2-Identity
\[ \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2 \]

With the inversion between \( \sigma_0 \) and \( \sigma_2 \), in introducing the tensor product by \( -\sigma_2 \), the functions \( \bar{\Psi}_j \)

\[ \bar{\Psi}_j = \begin{pmatrix} \Psi_{j,1} \\ \Psi_{j,2} \end{pmatrix}, j = 1,2,3,4, \]

bifurcate to two functions
\[ -\sigma_2 \Psi_j = -\sigma_2 \begin{pmatrix} \Psi_{j,1} \\ \Psi_{j,2} \end{pmatrix} = -\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Psi_{j,1} \\ \Psi_{j,2} \end{pmatrix} = \begin{pmatrix} \Psi_{j,1} \\ \Psi_{j,2} \end{pmatrix}, j = 1,2,3,4 \]

So the Majorana real 4-spinors equation bifurcates into the Dirac real 8-spinors equations

\[ \frac{\partial \Psi_{1,1}}{\partial t} = -c \frac{\partial \Psi_{1,1}}{\partial x} - c \frac{\partial \Psi_{1,2}}{\partial y} - c \frac{\partial \Psi_{3,1}}{\partial z} + (mc^2/h)\Psi_{1,2} \]
\[ \frac{\partial \Psi_{1,2}}{\partial t} = -c \frac{\partial \Psi_{1,2}}{\partial x} + c \frac{\partial \Psi_{1,1}}{\partial y} + c \frac{\partial \Psi_{3,2}}{\partial z} + (mc^2/h)\Psi_{2,2} \]
\[ \frac{\partial \Psi_{3,1}}{\partial t} = -c \frac{\partial \Psi_{3,1}}{\partial x} - c \frac{\partial \Psi_{3,2}}{\partial y} - c \frac{\partial \Psi_{1,1}}{\partial z} - (mc^2/h)\Psi_{3,2} \]
\[ \frac{\partial \Psi_{4,1}}{\partial t} = -c \frac{\partial \Psi_{4,1}}{\partial x} + c \frac{\partial \Psi_{1,1}}{\partial y} + c \frac{\partial \Psi_{2,1}}{\partial z} - (mc^2/h)\Psi_{4,2} \]
These 8 real first order partial differential equations represent real 8-spinors equations that are similar to the original Dirac [5,6] complex 4-spinors equations.

In defining the wave function
\[ \Psi(x, y, z, t) = \Psi_{1,1} + i \Psi_{1,2}, \]
with the imaginary number \( i \), we obtain the original Dirac equation as a complex 4-spinors equation
\[ + \partial_{1,2} t = -c \partial \Psi_{4,2}/\partial x + c \partial \Psi_{4,1}/\partial y - c \partial \Psi_{3,2}/\partial z - (mc^2/h)\Psi_{1,1} \quad (53a) \]
\[ + \partial_{2,2} t = -c \partial \Psi_{3,2}/\partial x - c \partial \Psi_{3,1}/\partial y + c \partial \Psi_{4,2}/\partial z - (mc^2/h)\Psi_{2,1} \quad (53b) \]
\[ + \partial_{3,2} t = -c \partial \Psi_{2,2}/\partial x + c \partial \Psi_{2,1}/\partial y - c \partial \Psi_{1,2}/\partial z + (mc^2/h)\Psi_{3,1} \quad (53c) \]
\[ + \partial_{4,2} t = -c \partial \Psi_{1,2}/\partial x - c \partial \Psi_{1,1}/\partial y + c \partial \Psi_{2,2}/\partial z + (mc^2/h)\Psi_{4,1} \quad (53d) \]

8. The 4 incursive discrete Dirac 4-spinors equations
Recently, we have presented the 4 incursive discrete Dirac complex equations [34]. Let us define the discrete Dirac wave functions
\[ Q_j(l, m, n, k) = Q_{j,1} + i Q_{j,2}, j = 1, 2, 3, 4, \]
corresponding to the Dirac wave functions equation (54), where \( i \) is the imaginary number. The 4 incursive discrete Dirac equations of the discrete wave functions are then given by
\[ Q_1(l, m, n, k + 1) = Q_1(l, m, n, k - 1) - B[Q_4(l + 1, m, n, k) - Q_4(l - 1, m, n, k)] + i C[Q_4(l, m + 1, n, k) - Q_4(l, m - 1, n, k)] - D[Q_3(l, m, n + 1, k) - Q_3(l, m, n - 1, k)] - i A Q_1(l, m, n, k) \quad (57a) \]
\[ Q_2(l, m, n, k + 1) = Q_2(l, m, n, k - 1) - B[Q_3(l + 1, m, n, k) - Q_3(l - 1, m, n, k)] - i C[Q_3(l, m + 1, n, k) - Q_3(l, m - 1, n, k)] + D[Q_4(l, m, n + 1, k) - Q_4(l, m, n - 1, k)] + i A Q_2(l, m, n, k) \quad (57b) \]
\[ Q_3(l, m, n, k + 1) = Q_3(l, m, n, k - 1) - B[Q_2(l + 1, m, n, k) - Q_2(l - 1, m, n, k)] + i C[Q_2(l, m + 1, n, k) - Q_2(l, m - 1, n, k)] - D[Q_4(l, m, n + 1, k) - Q_4(l, m, n - 1, k)] + i A Q_3(l, m, n, k) \quad (57c) \]
\[ Q_4(l, m, n, k + 1) = Q_4(l, m, n, k - 1) - B[Q_1(l + 1, m, n, k) - Q_1(l - 1, m, n, k)] - i C[Q_1(l, m + 1, n, k) - Q_1(l, m - 1, n, k)] + D[Q_2(l, m, n + 1, k) - Q_2(l, m, n - 1, k)] + i A Q_4(l, m, n, k) \quad (57d) \]
with

\[ A = 2\omega \Delta t, \quad B = c \Delta t/\Delta x, \quad C = c \Delta t/\Delta y, \quad D = c \Delta t/\Delta z \]  

(58)

where \( \Delta t, \Delta x, \Delta y, \Delta z \) are the discrete intervals of time and space respectively.

9. The hyperincursive discrete Klein-Gordon equation bifurcates to the 16 Proca equations

Let us show that there are 16 complex functions associated to this second order hyperincursive discrete Klein-Gordon equation.

This equation without spatial components, corresponding to a particle at rest, is similar to the harmonic oscillator.

For a particle at rest, the Klein-Gordon equation (5), with the function \( q(t) \) depending only on the time variable, is given by

\[ \frac{\partial^2 q(t)}{\partial t^2} = -\alpha^2 q(t) \]  

(59)

with the frequency, \( \alpha = \omega = mc^2/\hbar \), given by the equation (4).

This equation (11) is formally similar to the equation of the harmonic oscillator for which \( q(t) \) would represent the position \( x(t) \), and \( \frac{\partial q(t)}{\partial t} \) would represent the velocity \( v(t) = \frac{dx(t)}{dt} \).

So, with only the temporal component, the second order hyperincursive discrete Klein-Gordon equation (10) becomes

\[ q(k + 2) - 2q(k) + q(k - 2) = -Aq(k) \]  

(60)

that is similar to the second order hyperincursive discrete equation of the harmonic oscillator [1].

This hyperincursive equation (60) is separable into a first discrete incursive oscillator depending on two functions defined by \( q_1(k), q_2(k) \), and a second incursive oscillator depending on two other functions defined by \( q_3(k), q_4(k) \), given by first order discrete equations.

So the first incursive equations are given by:

\[ q_1(2k) = q_1(2k - 2) + Aq_2(2k - 1) \]
\[ q_2(2k + 1) = q_2(2k - 1) - Aq_1(2k) \]  

(61a,b)

where \( q_1(2k) \) is defined of the even steps of the time, and \( q_2(2k + 1) \) is defined on the odd steps of the time.

And the second incursive equations are given by:

\[ q_3(2k) = q_3(2k - 2) - Aq_4(2k - 1) \]
\[ q_4(2k + 1) = q_4(2k - 1) + Aq_3(2k) \]  

(62a,b)

where \( q_3(2k) \) is defined of the even steps of the time, and \( q_4(2k + 1) \) is defined on the odd steps of the time.

The second incursive system is the time reverse of the first incursive system in making the discrete time inversion \( T \)

\[ T: \Delta t \rightarrow -\Delta t \]  

(63)

which gives an oscillator and its anti-oscillator.

In defining the following 2 complex functions, where \( i \) is the imaginary number,

\[ q_{13}(2k) = q_1(2k) + i q_3(2k) \]
\[ q_{24}(2k + 1) = q_2(2k + 1) - i q_4(2k + 1) \]  

(64a,b)

the 4 real incursive equations (13a,b) and (14a,b) are transformed to 2 complex incursive equations

\[ q_{13}(2k) = q_{13}(2k - 2) + Aq_{24}(2k - 1) \]
\[ q_{24}(2k + 1) = q_{24}(2k - 1) - Aq_{13}(2k) \]  

(65a,b)
So the hyperincursive equation for a particle at rest shows a temporal bifurcation into an oscillatory equation and an anti-oscillatory equation.

For a moving particle, the 3 discrete space-symmetric terms in equation (10)

\[ q(l+2, m, n, k) - 2q(l, m, n, k) + q(l-2, m, n, k) \]
\[ q(l, m+2, n, k) - 2q(l, m, n, k) + q(l, m-2, n, k) \]
\[ q(l, m, n+2, k) - 2q(l, m, n, k) + q(l, m, n-2, k) \]

are similar to the discrete time-symmetric term \[ q(l, m, n, k + 2) - 2q(l, m, n, k) + q(l, m, n, k - 2) \].

The two complex functions bifurcate for even and odd steps of space \( x \), giving 4 complex functions depending on 4 discrete incursive equations.

These 4 complex functions bifurcate for even and odd steps of space \( y \), giving 8 complex functions depending on 8 discrete incursive equations.

Finally, these 8 complex functions bifurcate for even and odd steps of space \( z \), giving 16 complex functions depending on 16 incursive discrete equations.

But if we consider the space variable as a set of the 3 space variables

\[ r = (x, y, z) \] (66)

the two complex functions bifurcate for even and odd steps of the space variable \( r = (x, y, z) \), giving 4 complex functions depending on 4 discrete incursive equations, which correspond to a discrete parity inversion \( P \)

\[ P: \Delta r \rightarrow -\Delta r \] (67)

In conclusion, with the discrete time inversion and the parity, we define a group of 4 incursive discrete equations with 4 functions.

This is in agreement with the thesis of Proca [7,8].

Indeed, as demonstrated by Al Proca [7,8] in 1930 and 1932, the Klein-Gordon equation admits in the general case a total of 16 functions.

Classically, for the well-known Dirac equation, there are 4 complex wave functions.

Proca demonstrated that there are 4 fundamental equations of 4 wave functions for the Dirac equation

\[ \varphi_{r,s} \quad \text{for} \quad r = 1,2,3,A, \text{and} \quad s = 1 \] (68)

and the other 3 x 4 equations are similar to these 4 equations.

Proca classified the 16 equations in 4 groups of 4 functions:

I. 4 equations of the 4 functions \( \varphi_{r,s} \quad \text{for} \quad r = 1,2,3,A, \text{and} \quad s = 1 \)
II. 4 equations of the 4 functions \( \varphi_{r,s} \quad \text{for} \quad r = 1,2,3,A, \text{and} \quad s = 2 \)
III. 4 equations of the 4 functions \( \varphi_{r,s} \quad \text{for} \quad r = 1,2,3,A, \text{and} \quad s = 3 \)
IV. 4 equations of the 4 functions \( \varphi_{r,s} \quad \text{for} \quad r = 1,2,3,A, \text{and} \quad s = 4 \)

In each group, the 4 equations depend on 4 functions which are not separable except in particular cases.

In this paper we restricted our analysis to the first group of 4 functions in studying the case of the Majorana and Dirac equations.

10. Conclusion
The main purpose of this paper is the bifurcation of our second order hyperincursive discrete Klein-Gordon equation firstly to the 4 incursive discrete Majorana real 4-spinors real equations, then to the 8
incursive discrete Dirac real 8-spinors real equations, that can be rewritten as the 4 incursive discrete Dirac complex 4-spinors equations.

From the Klein-Gordon equation, we presented the second order hyperincursive discrete Klein-Gordon equation.

We present the 4 incursive discrete Dubois-Ord-Mann real equations and the corresponding first order partial differential equations. In three spatial dimensions, the two oscillators given by these equations are now entangled with one spatial dimension, so the 4 equations form a whole.

Then, we deduce the 4 incursive discrete Dubois-Majorana real equations and the corresponding first order partial differential equations. Next, these Dubois-Majorana real equations are presented in the generic form of the Dirac 4-spinors equation.

In making a change in the indexes of the 4 functions in the Dubois-Majorana equations, the 4 first order partial differential equations become identical to the Majorana real 4-spinors equations.

Then, we demonstrate that the Majorana equations bifurcate to the 8 real Dirac first order partial differential equations that are transformed to the original Dirac 4-spinors equations. Then, we give the 4 incursive discrete Dirac 4-spinors equations.

Finally, we show that there are 16 complex functions associated to this second order hyperincursive discrete Klein-Gordon equation. This is in agreement with the Proca thesis. For Proca, the Dirac function has 16 components and not 4. But, in fact, these 16 components are divided in 4 groups of 4 functions with 4 equations. The first group is the fundamental group and the 3 other groups can be obtained from the first group. The first group is not separable except in particular cases.

In this paper, we restricted our derivation of the Majorana and Dirac equations to the first group of 4 equations depending to 4 functions.

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