The scaling dimension of low lying Dirac eigenmodes and of the topological charge density

1. INTRODUCTION

With modern computational power has come the ability to examine the low lying eigenvectors of the Dirac operator and hence their spatial correlation with instantons and other related objects thought to be involved in chiral symmetry breaking and confinement [1,2]. While these studies focused primarily on the local relationship between instantons and low-lying Dirac eigenmodes (LDEs), other models of confinement and chiral symmetry breaking involving objects of lower co-dimension are popular, based on monopoles, vortices, and hybrid objects [3]. Presumably these objects would have a rather different effect on the LDEs than 4-dimensional instantons, due to their different co-dimension. Furthermore, a recent study [4] has suggested a dense layered 3-dimensional structure to the LDEs.

One difficulty is the quantitative characterization of localization of the LDEs or related quantities such as the topological charge density. In [2] localization of the LDEs was studied using the inverse participation ratio (IPR) which yields a number characterizing the localization of an eigenmode.

By studying the scaling dimension of the IPR, we can find the co-dimension of the structures which localize the LDEs, thus giving some insight as to the possible confining objects and mecha-
2

2. INVERSE PARTICIPATION RATIO (IPR)

The IPR of a normalized field \( \rho_i(x) \) is defined as
\[
I = N \sum_x \rho_i^2(x)
\]
where \( N \) is the number of lattice sites \( x \). Here we use \( \rho_i(x) = \psi_i^\dagger(x) \psi_i(x) \) is the \( i \)-th, normalized (\( \sum_x \rho_i(x) = 1 \)), lowest eigenvector of the Dirac operator.

With this definition, \( I \) characterizes the inverse “fraction” of sites contributing significantly to the support of \( \rho(x) \) (we now drop the subscript \( i \)). A simple calculation shows that the IPR takes the following values for these simple situations:

- Unlocalized: \( \rho(x) = \text{const.} \quad I = 1 \)
- \( \delta \)-function: \( \rho(x) = \delta(x_o) \quad I = N \)
- Localized on fraction \( f \) of sites: \( I = 1/f \)

Suppose that the objects responsible for confinement, or indeed any physics governing the lowest Dirac eigenmodes, localize the LDEs. As the lattice spacing is reduced, the fraction of sites contributing significantly to the support of \( \rho(x) \) (we now drop the subscript \( i \)). A simple calculation shows that the IPR takes the following values for these simple situations:

\[
\begin{align*}
\text{Unlocalized:} & \quad \rho(x) = \text{const.} \quad I = 1 \\
\text{\( \delta \)-function:} & \quad \rho(x) = \delta(x_o) \quad I = N \\
\text{Localized on fraction} & \quad I = 1/f
\end{align*}
\]

We see clear evidence for lower dimensional scaling as \( a \to 0 \) with co-dimension between 2 and 3. Figure 1 shows both the distribution of the IPRs and the scaling of the averages. The data points are fit to \( a^{1-n} c_1 + a^n c_2 \), where \( c_1 \) and \( c_2 \) are constants and \( n = 1, 2, 3 \). The reduced chi squared values\(^2\) for the fits are 6.7, 3.2, and 45 for \( n = 1, 2, 3 \), respectively.

\[\text{Figure 1. Scaling of the average IPR as} \quad a \to 0\]

\[\text{The error bars shown here are corrected from those (much larger) shown at the conference.}\]
In figure 2 we show the behaviour of the IPR as we increase the volume at fixed lattice spacing; it is rather unaffected, as we expected above.

Figure 2. Scaling of the average IPR as $L \to \infty$ with $a = 0.12$ fm

4. TOPOLOGICAL CHARGE

We can also investigate the localization of the topological charge density by computing the IPR from $q(x) = F_{\mu\nu}F^{\mu\nu}$, where we have normalized $\sum_x |q(x)| = 1$. We have computed $q(x)$ by successive HYP smearing sweeps on the $a \to 0$ series, and show the results in Figure 3 (note that only 5 HYP smearing steps were performed on the $a = 0.12$ fm $L = 20$ lattice set). While this plot does not show us new information on the localization of the LDEs, it is nonetheless instructive.

First, we see that all lattices without smoothing have an IPR $= \pi/2$. This is the value expected if the field is a gaussian fluctuation at each site, regardless of its width. We further see the approach to a stable localization of topological charge versus HYP smearing as the lattice spacing is decreased (at fixed volume). We note that $<\text{IPR}>$ is not large, meaning that $q(x)$ is not strongly localized. Also, it increases as $a \to 0$ as for the LDEs.

5. CONCLUSIONS

The main result of this study is the indication of a localization of the low-lying Dirac eigenmodes on surfaces of co-dimension between 2 and 3, qualitatively supporting the center vortex or monopole pictures of confinement. Note, however, that the singularities of thin objects (vortices or monopoles) are expected to be smoothed out by the QCD interactions and become thick, with a size $\sim 1/\Lambda_{\text{QCD}}$. Thick objects fill a fixed fraction of space, not a divergent one. The indication we have, via the divergence of $<\text{IPR}>$ as $a \to 0$, of localization on singular manifolds, is remarkable, whatever these manifolds are.

REFERENCES

1. Ph. de Forcrand et. al., Nucl. Phys. Proc. Suppl. 73 (1999) 578; T. DeGrand and A. Hasenfratz, Phys. Rev. D64 (2001) 034512; T.L. Ivanenko and J.W. Negele, Nucl. Phys. Proc. Suppl. 63 (1998) 504
2. C. Gattringer et. al., Nucl. Phys. B617 (2001) 101; T. Kovacs, Phys. Rev. D67 (2003) 094501
3. M. Engelhardt, these proceedings.
4. I. Horvath et. al., Phys. Rev. D68 (2003) 114505; H.B. Thacker, these proceedings.
5. T. DeGrand, A. Hasenfratz, T.G. Kovacs Nucl. Phys. B505 (1997) 417