A NOTE ON ISOPERIMETRIC INEQUALITIES OF GROMOV HYPERBOLIC MANIFOLDS AND GRAPHS

ÁLVARO MARTÍNEZ-PÉREZ AND JOSÉ M. RODRÍGUEZ

Abstract. In this paper we study the relationship of hyperbolicity and (Cheeger) isoperimetric inequality in the context of Riemannian manifolds and graphs. We characterize the hyperbolic manifolds and graphs (with bounded local geometry) verifying this isoperimetric inequality, in terms of their Gromov boundary improving similar results from a previous work. In particular, we prove that having a pole is a necessary condition and, therefore, it can be removed as hypothesis.

2010 AMS Subject Classification numbers: Primary 53C21, 53C23; Secondary 58C40.

Keywords: Cheeger isoperimetric constant; Gromov hyperbolicity; bounded local geometry; pole.

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Date: April 7, 2020.
1. Introduction

Isoperimetric inequalities are of interest in pure and applied mathematics (see, e.g., [19], [46]). There are close connections between isoperimetric inequality and some conformal invariants of Riemannian manifolds and graphs, namely Poincaré-Sobolev inequalities, the bottom of the spectrum of the Laplace-Beltrami operator, the exponent of convergence, and the Hausdorff dimensions of the sets of both bounded geodesics and escaping geodesics in a negatively curved surface (see [4], [10], [15] p.228], [20], [25], [26], [27], [28] p.333]). The Cheeger isoperimetric inequality is closely related to the project of Ancona on the space of positive harmonic functions of Gromov-hyperbolic manifolds and graphs ([5], [6] and [7]). In fact, in the study of the Laplace operator on a hyperbolic manifold or graph, Ancona obtained in those three last papers interesting results, under the additional assumption that the bottom of the spectrum of $X$ is positive. The well-known Cheeger inequality $\lambda_1(X) \geq \frac{1}{4} h(X)^2$, where $h(X)$ is the isoperimetric constant of $X$, guarantees that $\lambda_1(X) > 0$ when $h(X) > 0$ (see [14] for a converse inequality). Hence, the results of this paper are useful in order to apply these Ancona’s results.

There is a natural connection between hyperbolicity and Cheeger isoperimetric inequality. In fact, one of the alternative definitions of Gromov hyperbolicity uses some kind of isoperimetric inequality (see [3], [32]).

Cao proved in [18] that hyperbolicity with an extra hypothesis on the Gromov boundary implies (Cheeger) isoperimetric inequality (an extra hypothesis is necessary, since there exist hyperbolic graphs without isoperimetric inequality, as the Cayley graph of the group $\mathbb{Z}$).

In [39] we studied the relationship of hyperbolicity and Cheeger isoperimetric inequality in the context of Riemannian manifolds and graphs with bounded local geometry.

Given any Riemannian $n$-manifold $M$, the Cheeger isoperimetric constant of $M$ is defined as

$$h(M) = \inf_{A} \frac{\text{Vol}_{n-1}(\partial A)}{\text{Vol}_{n}(A)},$$

where $A$ ranges over all non-empty bounded open subsets of $M$, and $\text{Vol}_{k}(B)$ denotes the $k$-dimensional Riemannian volume of the set $B$.

Given any graph $\Gamma = (V, E) = (V(\Gamma), E(\Gamma))$, let us consider the natural length metric $d_{\Gamma}$ where every edge has length 1. For any graph $\Gamma$, any vertex $v \in V$ and any $k \in \mathbb{N}$, let $S(v, k) := \{w \in V | d_{\Gamma}(v, w) = k\}$. As usual, we denote by $B(v, k)$ and $\overline{B}(v, k)$ the open and closed balls, respectively.

The combinatorial Cheeger isoperimetric constant of $\Gamma$ is defined to be

$$h(\Gamma) = \inf_{A} \frac{|\partial A|}{|A|},$$

where $A$ ranges over all non-empty finite subsets of vertices in $\Gamma$, $\partial A = \{v \in \Gamma | d_{\Gamma}(v, A) = 1\}$ and $|A|$ denotes the cardinality of $A$.

A Riemannian manifold or graph $X$ satisfies the (Cheeger) isoperimetric inequality if $h(X) > 0$, since in this case

$$\text{Vol}_{n}(A) \leq h(X)^{-1} \text{Vol}_{n-1}(\partial A),$$

for every bounded open set $A \subseteq X$ if $X$ is a Riemannian $n$-manifold, and

$$|A| \leq h(X)^{-1}|\partial A|,$$

for every finite set $A \subseteq V(X)$ if $X$ is a graph.

Along the paper, we just consider manifolds and graphs $X$ which are connected. This is not a loss of generality, since if $X$ has connected components $\{X_i\}$, then $h(X) = \inf_{i} h(X_i)$.

Let $(X, d_X)$ and $(Y, d_Y)$ be two metric spaces. A map $f : X \rightarrow Y$ is said to be an $(\alpha, \beta)$-quasi-isometric embedding, with constants $\alpha \geq 1$, $\beta \geq 0$, if for every $x, y \in X$:

$$\alpha^{-1} d_X(x, y) - \beta \leq d_Y(f(x), f(y)) \leq \alpha d_X(x, y) + \beta.$$

The function $f$ is $\varepsilon$-full if for each $y \in Y$ there exists $x \in X$ with $d_Y(f(x), y) \leq \varepsilon$. 

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A map $f : X \to Y$ is said to be a quasi-isometry, if there exist constants $\alpha \geq 1$, $\beta, \varepsilon \geq 0$ such that $f$ is an $\varepsilon$-full $(\alpha, \beta)$-quasi-isometric embedding. Two metric spaces $X$ and $Y$ are quasi-isometric if there exists a quasi-isometry $f : X \to Y$. One can check that to be quasi-isometric is an equivalence relation.

A graph $\Gamma$ is said to be $\mu$-uniform if each vertex $p$ of $V$ has at most $\mu$ neighbors, i.e.,

$$\sup \{ ||N(p)|| \mid p \in V(\Gamma) \} \leq \mu.$$ 

If a graph $\Gamma$ is $\mu$-uniform for some constant $\mu$ we say that $\Gamma$ is uniform or that it has bounded local geometry.

A Riemannian $n$-manifold $M$ has bounded local geometry if there exist positive constants $r, c$, such that for every $x \in M$ there is a diffeomorphism $F : B(x, r) \to \mathbb{R}^n$ with

$$\frac{1}{c} d(x_1, x_2) \leq ||F(x_1) - F(x_2)|| \leq c d(x_1, x_2)$$

for every $x_1, x_2 \in B(x, r)$.

The injectivity radius $\text{inj}(x)$ of $x \in M$ is defined as the supremum of those $r > 0$ such that $B(x, r)$ is simply connected or, equivalently, as half the infimum of the lengths of the (homotopically non-trivial) loops based at $x$. The injectivity radius $\text{inj}(M)$ of $M$ is the infimum over $x \in M$ of $\text{inj}(x)$.

**Remark 1.1.** If $M$ has positive injectivity radius and a lower bound on its Ricci curvature, then $M$ has bounded local geometry [8].

A celebrated theorem of Kanai in [35] states that quasi-isometries preserve isoperimetric inequalities between Riemannian manifolds and graphs with bounded local geometry. This result also holds with weaker hypotheses in the context of Riemann surfaces [17, 31].

Let $X$ be a metric space. Fix a base point $o \in X$ and for $x, x' \in X$ let

$$(x|x')_o = \frac{1}{2} \left( d(x, o) + d(x', o) - d(x, x') \right).$$

The number $(x|x')_o$ is non-negative and it is called the Gromov product of $x, x'$ with respect to $o$.

**Definition 1.2.** A metric space $X$ is (Gromov) hyperbolic if it satisfies the $\delta$-inequality

$$(x|y)_o \geq \min \{(x|z)_o, (z|y)_o\} - \delta$$

for some $\delta \geq 0$, for every base point $o \in X$ and all $x, y, z \in X$.

We denote by $\delta(X)$ the sharp hyperbolicity constant of $X$:

$$\delta(X) = \sup \left\{ \min \{(x|z)_o, (z|y)_o\} - (x|y)_o \mid x, y, z, o \in X \right\}.$$ 

Hence, $X$ is hyperbolic if and only if $\delta(X) < \infty$.

The theory of Gromov hyperbolic spaces was introduced by M. Gromov for the study of finitely generated groups (see [32]). The concept of Gromov hyperbolicity grasps the essence of negatively curved spaces like the classical hyperbolic space, Riemannian manifolds of negative sectional curvature bounded away from 0, and of discrete spaces like trees and the Cayley graphs of many finitely generated groups. It is remarkable that a simple concept leads to such a rich general theory (see [3, 30, 32]). This theory has been developed from a geometric point of view to the extent of making hyperbolic spaces an important class of metric spaces to be studied on their own (see, e.g., [11, 13, 16, 30, 33]). In the last years, Gromov hyperbolicity has been intensely studied in graphs (see, e.g., [9, 12, 23, 34, 37, 38, 44, 48, 49, 50, 51, 54] and the references therein). Gromov hyperbolicity, specially in graphs, has found applications in different areas such as phylogenetics (see [23, 24]), real networks (see [1, 2, 22, 36, 43]) or the secure transmission of information and virus propagation on networks (see [33, 34]).

We want to remark that the main examples of hyperbolic graphs are the trees. In fact, the hyperbolicity constant of a metric space can be viewed as a measure of how “tree-like” the space is, since those spaces $X$ with $\delta(X) = 0$ are precisely the metric trees. This is an interesting subject since, in many applications, one
finds that the borderline between tractable and intractable cases may be the tree-like degree of the structure
to be dealt with (see, e.g., [21]).

In [39] we characterized, in terms of their Gromov boundary, the uniform hyperbolic graphs and a large
class of hyperbolic manifolds satisfying isoperimetric inequality (see, respectively, Theorems 3.4 and 4.6). In
Theorems 3.4, 4.6 and [18] Theorem 1.1] it is used the hypothesis of the existence of a pole (in fact, although
[18] Theorem 1.1] apparently uses a different hypothesis, actually, in hyperbolic spaces, it is equivalent to
the existence of a pole). The hypothesis on Gromov hyperbolicity is natural (we need it in order to deal with
the Gromov boundary). However, although the hypothesis on the pole is technically needed in the proofs,
it does not look natural. The goal of this paper is to remove this hypothesis in the statements of Theorems
the Gromov boundary). However, although the hypothesis on the pole is technically needed in the proofs,

2. SOME PREVIOUS RESULTS

Recall that a geodesic space is a metric space such that for every couple of points there exists a geodesic
joining them.

Definition 2.1. A geodesic space $X$ has a pole in a point $v$ if there exists $M > 0$ such that each point of $X$
lies in an $M$-neighborhood of some geodesic ray emanating from $v$.

If $X$ is a geodesic metric space and $x_1, x_2, x_3 \in X$, the union of three geodesics $[x_1 x_2]$, $[x_2 x_3]$ and $[x_3 x_1]$ is
a geodesic triangle that will be denoted by $T = \{x_1, x_2, x_3\}$ and we will say that $x_1, x_2$ and $x_3$ are the vertices
of $T$. We say that $T$ is $\delta$-thin if any side of $T$ is contained in the $\delta$-neighborhood of the union of the two other
sides. We denote by $\delta_{th}(T)$ the sharp thin constant of $T$, i.e., $\delta_{th}(T) := \inf\{\delta \geq 0 : T \text{ is } \delta\text{-thin}\}$. The space
$X$ is $\delta$-thin (or satisfies the Rips condition with constant $\delta$) if every geodesic triangle in $X$ is $\delta$-thin. We
 denote by $\delta_{th}(X)$ the sharp thin constant of $X$, i.e., $\delta_{th}(X) := \sup\{\delta_{th}(T) : T \text{ is a geodesic triangle in } X\}$.

It is well-known that a geodesic metric space is hyperbolic if and only if it satisfies the Rips condition for
some constant (see, e.g., [3 30]). In the classical references on this subject (see, e.g., [3 30]) appear many
different definitions of Gromov hyperbolicity, which are equivalent in the sense that if $X$ is $\delta$-hyperbolic with
respect to one definition, then it is $\delta'$-hyperbolic with respect to another definition (for some $\delta'$ related to $\delta$).

In order to consider a graph $G$ as a geodesic metric space, identify (by an isometry) any edge $uv \in E(G)$
with the interval $[0,1]$ in the real line; then the edge $uv$ (considered as a graph with just one edge) is isometric
to the interval $[0,1]$. Thus, the points in $G$ are the vertices and, also, the points in the interior of any edge
of $G$. In this way, any connected graph $G$ has a natural distance defined on its points, induced by taking
shortest paths in $G$, and we can see $G$ as a metric graph.

Let us adapt the following definition from [16] where we introduce the constant $\varepsilon_0$ for convenience. Notice
that for bounded metric spaces both definitions coincide. Since herein this property will be always applied
to compact spaces all the results work as well with the original definition.

Definition 2.2. Given a metric space $(X,d)$ and a constant $S > 1$, we say that $(X,d)$ is $S$-uniformly perfect
if there exists some $\varepsilon_0 > 0$ such that for every $x \in X$ and every $0 < \varepsilon \leq \varepsilon_0$ there exist a point $y \in X$ such
that $\frac{\varepsilon}{S} < d(x,y) \leq \varepsilon$. We say that $(X,d)$ is uniformly perfect if there exists some $S$ such that $(X,d)$ is
$S$-uniformly perfect.

3. HYPERBOLIC GRAPHS

Let us recall the concepts of geodesic and sequential boundary of a hyperbolic space and some basic
properties. For further information and proofs we refer the reader to [11 16 30 32].

Let $X$ be a hyperbolic space and $o \in X$ a base point.

The relative geodesic boundary of $X$ with respect to the base-point $o$ is

$$\partial^0_oX := \{[\gamma] : [0, \infty) \rightarrow X \text{ is a geodesic ray with } \gamma(0) = o\},$$

where $\gamma_1 \sim \gamma_2$ if there exists some $K > 0$ such that $d(\gamma_1(t), \gamma_2(t)) < K$, for every $t \geq 0$. 

In fact, the definition above is independent from the base point. Therefore, the set of classes of geodesic rays is called geodesic boundary of $X$, $\partial^\infty X$. Herein, we do not distinguish between the geodesic ray and its image.

A sequence of points $\{x_i\} \subset X$ converges to infinity if

$$\lim_{i,j \to \infty} (x_i|x_j)_o = \infty.$$ 

This property is independent of the choice of $o$ since

$$|(x|x')_o - (x|x')_{o'}| \leq d(o,o')$$

for any $x,x',o,o' \in X$.

Two sequences $\{x_i\}, \{x'_i\}$ that converge to infinity are equivalent if

$$\lim_{i \to \infty} (x_i|x'_i)_o = \infty.$$ 

Using the $\delta$-inequality, we easily see that this defines an equivalence relation for sequences in $X$ converging to infinity. The sequential boundary at infinity $\partial_\infty X$ of $X$ is defined to be the set of equivalence classes of sequences converging to infinity.

Note that given a geodesic ray $\gamma$, the sequence $\{\gamma(n)\}$ converges to infinity and two equivalent rays induce equivalent sequences. Thus, in general, $\partial^\infty X \subseteq \partial_\infty X$.

We say that a metric space is proper if every closed ball is compact. Every uniform graph and every complete Riemannian manifold are proper geodesic spaces.

**Proposition 3.1.** [[11] Chapter III.H, Proposition 3.1] If $X$ is a proper hyperbolic geodesic space, then the natural map from $\partial^\infty X$ to $\partial_\infty X$ is a bijection.

For every $\xi, \xi' \in \partial_\infty X$, its Gromov product with respect to the base point $o \in X$ is defined as

$$(\xi|\xi')_o = \inf \liminf_{i \to \infty} (x_i|x'_i)_o,$$

where the infimum is taken over all sequences $\{x_i\} \in \xi$, $\{x'_i\} \in \xi'$.

A metric $d$ on the sequential boundary at infinity $\partial_\infty X$ of $X$ is said to be visual, if there are $o \in X$, $a > 1$ and positive constants $c_1, c_2$, such that

$$c_1 a^{-d(\xi,\xi')} \leq d(\xi,\xi') \leq c_2 a^{-d(\xi,\xi')}$$

for all $\xi,\xi' \in \partial_\infty X$. In this case, we say that $d$ is a visual metric with respect to the base point $o$ and the parameter $a$.

**Theorem 3.2.** [[16] Theorem 2.2.7] Let $X$ be a hyperbolic space. Then for any $o \in X$, there is $a_o > 1$ such that for every $a \in (1, a_o]$ there exists a metric $d$ on $\partial_\infty X$, which is visual with respect to $o$ and $a$.

**Remark 3.3.** Notice that for any visual metric, $\partial_\infty X$ is bounded and complete.

**Theorem 3.4.** Given a hyperbolic uniform infinite graph $\Gamma$ with a pole, then $h(\Gamma) > 0$ if and only if $\partial_\infty \Gamma$ is uniformly perfect for some visual metric.

### 4. Hyperbolic Manifolds

Let us recall the following definitions from [[35]].

A subset $A$ in a metric space $(X,d)$ is called $r$-separated, $r > 0$, if $d(a,a') \geq r$ for any distinct $a,a' \in A$. Note that if $A$ is maximal with this property, then the union $\bigcup_{a \in A} B_r(a)$ covers $X$. A maximal $r$-separated set $A$ in a metric space $X$ is called an $r$-approximation of $X$.

Let $X$ be a complete Riemannian manifold and denote by $d$ the induced metric. Given any $\varepsilon$-approximation $A_\varepsilon$ of $X$, the graph $\Gamma_{A_\varepsilon} = (V,E)$ with $V = A_\varepsilon$ and $E : = \{xy \mid x,y \in A_\varepsilon \text{ with } 0 < d(x,y) \leq 2\varepsilon\}$ is called an $\varepsilon$-net.
Proposition 4.1. [35, Lemma 4.5] Suppose that $X$ is a complete Riemannian manifold with bounded local geometry and let $\Gamma$ be an $\varepsilon$-net in $X$. Then, $h(X) > 0$ if and only if $h(\Gamma) > 0$.

Note that the results in [35] require $M$ to have positive injectivity radius and a lower bound on its Ricci curvature instead of bounded local geometry, but the proofs in [35] just use that there are uniform lower and upper bounds for the volume of the balls $B(x, r)$ which do not depend on $x \in M$ for $0 < r < r_0$ (and we have these uniform bounds with bounded local geometry). Hence, the results in [35] also hold with the weaker hypothesis of bounded local geometry.

Proposition 4.2. [35, Lemma 2.5] Suppose that $X$ is a complete Riemannian manifold with bounded local geometry and let $\Gamma$ be an $\varepsilon$-net in $X$. Then, $X$ and $\Gamma$ are quasi-isometric.

Theorem 4.3. [30, p.88] If $f : X \to Y$ is a quasi-isometry between geodesic spaces, then $X$ is hyperbolic if and only if $Y$ is hyperbolic.

Proposition 4.4. [39, Proposition 5.6] Suppose $X, Y$ are proper hyperbolic geodesic spaces and $f : X \to Y$ is a quasi-isometry. If $X$ has a pole in $v$, then $Y$ has a pole in $f(v)$.

Lemma 4.5. [35, Lemma 2.3] Every $\varepsilon$-net in a complete Riemannian manifold with bounded local geometry is uniform.

Theorem 4.6. [39, Theorem 5.12] Let $X$ be a non-compact complete Riemannian manifold with bounded local geometry. Assume that $X$ is hyperbolic and has a pole. Then, $h(X) > 0$ if and only if $\partial_\infty X$ is uniformly perfect.

In [18] appear several sufficient conditions in order to guarantee that a Riemannian manifold is hyperbolic. See also [52] for the case of Riemannian surfaces.

5. Main results

In this section we prove that having a pole is a necessary condition for a hyperbolic uniform graph or a complete Riemannian manifold with bounded local geometry satisfying isoperimetric inequality. Therefore, it can be removed as hypothesis in the statements of Theorems 3.4 and 4.6.

In fact, we are going to prove the following results.

Theorem 5.1. Given a hyperbolic uniform graph $\Gamma$, then $h(\Gamma) > 0$ if and only if $\partial_\infty \Gamma$ is uniformly perfect for some visual metric and $\Gamma$ is an infinite graph with a pole.

Theorem 5.2. Let $X$ be a hyperbolic complete Riemannian manifold with bounded local geometry. Then, $h(X) > 0$ if and only if $\partial_\infty X$ is uniformly perfect and $X$ is non-compact and has a pole.

These results are a consequence of Theorems 3.4, 4.6 and the two following results.

Theorem 5.3. Let $\Gamma$ be a uniform graph. If $\Gamma$ is hyperbolic and $h(\Gamma) > 0$, then $\Gamma$ is an infinite graph with a pole.

Theorem 5.4. Let $X$ be a complete Riemannian manifold with bounded local geometry. If $X$ is hyperbolic and $h(X) > 0$, then $X$ is non-compact and has a pole.

Let us now proceed with the proof of Theorem 5.3.

Proof. Since $h(\Gamma) > 0$, we have that $\Gamma$ is an infinite graph.

Seeking for a contradiction assume that $\Gamma$ does not have a pole. Let $\mu$ be a constant such that $\Gamma$ is $\mu$-uniform.

Fix $v \in V(\Gamma)$ and denote by $K$ the union of the geodesic rays starting from $v$. Since $\Gamma$ does not have a pole, for each $n$ there exists $v_n \in V(\Gamma)$ with $d_\Gamma(v_n, K) \geq 4n$. Let $\eta_n : [0, \ell_n] \to \Gamma$ be a geodesic joining $v_n$ with $K$ and such that $\eta_n(0) = v_n$ and $d_\Gamma(\eta_n, K) = L(\eta_n) = \ell_n \geq 4n$. 
Given \( s \in \mathbb{R} \), denote by \( \lfloor s \rfloor \) the lower integer part of \( s \), i.e., the largest integer not greater than \( s \). Assume that the ball \( B_t(\eta_n(t), [\delta_{th}(\Gamma)] + 1) \) intersects every geodesic joining \( v_n \) with some point of \( K \), for some \( t \in \mathbb{Z}^+ \) with \( [\delta_{th}(\Gamma)] < t \leq 4n \leq \ell_n \). Thus, the connected component \( A_n \) of \( \Gamma \setminus B_t(\eta_n(t), [\delta_{th}(\Gamma)] + 1) \) containing \( v_n \) satisfies that its boundary \( \partial A_n \) is contained in the sphere \( S_t(\eta_n(t), [\delta_{th}(\Gamma)]) \).

Since \( \Gamma \) is \( \mu \)-uniform,

\[
|\partial A_n| \leq |S_t(\eta_n(t), [\delta_{th}(\Gamma)])| \leq \mu^{|\delta_{th}(\Gamma)|}.
\]

Thus

\[
t - [\delta_{th}(\Gamma)] = |V(\Gamma) \cap \eta_n([0, t - [\delta_{th}(\Gamma)] - 1])| \leq |A_n| \leq h(\Gamma)^{-1}|\partial A_n| \leq h(\Gamma)^{-1} \mu^{|\delta_{th}(\Gamma)|},
\]

and we conclude that

\[
t \leq [\delta_{th}(\Gamma)] + h(\Gamma)^{-1} \mu^{|\delta_{th}(\Gamma)|} =: M.
\]

Hence, if \( 2n > M \), there exists a geodesic \( \sigma_n \) from \( v_n \) to \( K \) such that \( \sigma_n \cap B_t(\eta_n(2n), [\delta_{th}(\Gamma)] + 1) = \emptyset \). Note that \( d_\Gamma(\eta_n(2n), K) = \ell_n - 2n \geq 2n \). Let \( x_n \) and \( y_n \) be the endpoints of \( \eta_n \) and \( \sigma_n \) in \( K \), respectively, and consider the geodesic triangle \( T_n = \{v_n, x_n, y_n\} \) in \( \Gamma \). Since \( \delta_{th}(\Gamma) < [\delta_{th}(\Gamma)] + 1 \), we have

\[
d_\Gamma(\eta_n(2n), \sigma_n) \geq [\delta_{th}(\Gamma)] + 1 > \delta_{th}(\Gamma),
\]

and so, there exists \( z_n \in [x_n, y_n] \) with \( d_\Gamma(\eta_n(2n), z_n) \leq \delta_{th}(\Gamma) \) (see Figure 1).

Consider now the geodesic triangle \( T_n = \{v, x_n, y_n\} \) in \( \Gamma \), and \( z_n \in [x_n, y_n] \). Since \( [v x_n] \cup [v y_n] \subset K \), we have

\[
\delta_{th}(\Gamma) \geq d_\Gamma(z_n, [v x_n] \cup [v y_n]) \geq d_\Gamma(z_n, K) \geq d_\Gamma(\eta_n(2n), K) - d_\Gamma(\eta_n(2n), z_n) \geq 2n - \delta_{th}(\Gamma),
\]

and so, \( \delta_{th}(\Gamma) \geq n \) for every \( n \). This is the contradiction we were looking for, since \( \Gamma \) is hyperbolic, and we conclude that \( \Gamma \) has a pole. \( \square \)

Finally, let us now proceed with the proof of Theorem 5.4.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{If there is a geodesic \( \sigma_n = [v_n y_n] \) from \( v_n \) to \( K \) which does not intersect the ball \( B_t(\eta_n(2n), \delta') \), where \( \delta' := [\delta_{th}(\Gamma)] + 1 \), then there is a point \( z_n \in [x_n, y_n] \) which is far from \( K \).}
\end{figure}
Proof. Since $h(X) > 0$, we have that $X$ is non-compact.

Let $\Gamma$ be an $\varepsilon$-net in $X$. Since $h(X) > 0$, Proposition 4.1 gives that $h(\Gamma) > 0$.

Note that $X$ is a proper geodesic space since it is a complete Riemannian manifold. Also, $\Gamma$ is a proper geodesic space since it is a uniform graph by Lemma 4.5.

By Proposition 4.2, $X$ and $\Gamma$ are quasi-isometric and so, $\Gamma$ is hyperbolic by Theorem 4.3. Theorem 5.3 gives that $\Gamma$ has a pole, and by Proposition 4.4, $X$ has a pole.

□

Remark 5.5. Theorem 5.1 also allows to improve Theorem 4.12 in [40] which is based in the same results.

In the same way, having a pole is a necessary condition and can be removed as hypothesis.

Remark 5.6. Theorem 5.11 in [42] states that for geodesic, visual, Gromov hyperbolic spaces, having uniformly perfect boundary at infinity is equivalent to being uniformly equilateral. Thus, theorems 5.1 and 5.2 can also be written using this property and without using the boundary at infinity.

Acknowledgments: We would like to thank to Paloma Martín for her support during this research. This work was supported in part by two grants from Ministerio de Economía y Competitividad, Agencia Estatal de Investigación (AEI) and Fondo Europeo de Desarrollo Regional (FEDER) (MTM2015-63612P, MTM2016-78227-C2-1-P and MTM2017-90584-REDT), Spain.

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