New Properties of Numbers of Plane Graphs

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Abstract

We explore various techniques for counting the number of straight-edge crossing-free graphs that can be embedded on a planar point set. In particular, we derive a lower bound on the ratio of the number of such graphs with \(m + 1\) edges to the number of graphs with \(m\) edges. We show how a relatively small improvement of this bound would improve existing bounds concerning numbers of plane graphs with a prescribed number of edges. Furthermore, we use a cross-graph charging scheme to derive lower bounds for the number of such graphs when the point set has few extreme points.

1 Introduction

A straight-edge crossing-free graph or a plane graph \(G\) is an embedding of a planar graph onto a planar point set \(S\) with edges that are straight-line segments that may only intersect at their endpoints. We say that a planar point set is in general position if no three points are collinear. In this paper all point sets will be assumed to be in general position. We are interested in questions of the form – what is the maximum/minimum number of straight-edge crossing-free graphs (with possible additional properties) that can be embedded on any set \(S\) of \(N\) points in the plane?\(^1\)

In 1982 Ajtai et al. \([2]\) showed that the number of plane graphs over any set of \(N\) points can never exceed a fixed exponential in \(N\), and gave the first upper bound of \(10^{13N}\). Over the past decade, progressively smaller upper bounds have been derived (e.g., see \([4, 8, 10, 11]\)). The number of plane graphs has also been studied from an algorithmic viewpoint. In 2011 Razen and Welzl \([9]\) showed that computing the number of plane graphs on a given planar point set can be done exponentially faster than enumerating them, and in 2016 Marx and Miltzow \([6]\) showed that the number of triangulations can be computed in sub-exponential time.

Work on the above class of problems has resulted in the development of several different combinatorial techniques. For example, the current best upper bound of \(O(10.05^N)\) on the number of perfect matchings was proved by Sharir and Welzl \([12]\) by considering a vertical decomposition of the plane and bounding the number of ways to add and remove an edge to an existing matching. Flajolet and Noy \([3]\) use techniques from analytic combinatorics to show that the number of plane graphs on \(N\) points is minimized when the points are in convex position, and that this number is \(\Theta(11.65^N)\). Sharir and Sheffer \([10]\) use a novel combinatorial technique called a cross-graph charging scheme to derive the best known upper bound bound of \(O^*(187.53^N)\) for the number of plane graphs.\(^2\) A charging scheme involves assigning charges to the vertices of a graph \(G\), and then moving

\(^1\)For a detailed list of up-to-date bounds on the number of different types of plane graphs that can be embedded on a set of \(N\) points, see a dedicated webpage by Adam Sheffer: https://adamsheffer.wordpress.com/numbers-of-plane-graphs/

\(^2\)In the notations \(O^*()\), \(\Theta^*()\), and \(\Omega^*()\), we neglect subexponential factors.
the charges between various vertices of $G$. The cross-graph charging scheme used in [10] involves moving charges between the vertices of different graphs over the same point set. Hoffmann et al. in [4] study flippable edges in triangulations to obtain the best known bounds of $O(141.07^N)$ and $O(160.55^N)$ for the maximum number of spanning trees and spanning forests that can be embedded on $N$ points respectively. They moreover derive upper bounds for the number of plane graphs with exactly $cN$ edges, fewer than $cN$ edges, and more than $cN$ edges, for any parameter $c$.

Before we state our main results, we introduce some useful notation and give formal definitions of some recurring concepts that we will use throughout this paper.

Given a set $S$ of $N$ points in the plane define $\mathcal{P}(S)$ as the set of all plane graphs of $S$. Let $\text{pg}(S) := |\mathcal{P}(S)|$, and let $\text{pg}_m(S)$ denote the number of plane graphs of $S$ with $m$ edges. Let $c$ be the number such that $m = cN$, where $0 \leq c < 3$ (a plane graph on $N$ vertices has at most $3N - 6$ edges). Let $\text{pg}(N) = \max_{|S| = N} \text{pg}(S)$ and $\text{pg}_m(N) = \max_{|S| = N} \text{pg}_m(S)$. For a given $c$, we define the increase rate $r_c(S)$ of a point set $S$ as the ratio $r_c = r_c(S) := \frac{\text{pg}_{cN+1}(S)}{\text{pg}_{cN}(S)}$. This measures the relative increase in the number of plane graphs when one more edge is in the graph. Moreover, let $h = h(S)$ denote the number of vertices that lie on the boundary of the convex hull of $S$, and let $\text{pg}_c^*(N)$ be the minimum value of $\text{pg}(S)$ taken over all sets of $N$ points with a triangular convex hull ($h = 3$).

Our first result is a lower bound on $r_c$.

**Theorem 1.1.** For every set $S$ of $N$ points in the plane with $h$ points on the boundary of its convex hull and $0 \leq c < 3$,

$$r_c(S) \geq \frac{(3-c)N - h - 3}{cN + 1}.$$

We show that a relatively small improvement of this bound is needed to improve the best upper bound on $\text{pg}_{cN}(N)$. We then compute $r_c(S)$ for a specific point set $S$ and use this to show that our bound on $r_c$ does not appear amenable for much improvement.

Our other result is a lower bound on the number of plane graphs with a triangular convex hull.

**Theorem 1.2.** $\text{pg}_{cN}^*(N) = \Omega^*(12.36^N)$.

Our method relies on charging schemes between objects from different plane graphs over the same point set. Our lower bound for $\text{pg}_{cN}^*(N)$ is not far off from the upper bound of $\text{pg}_{cN}^*(N) = O^*(23.3^N)$ which we derive via a simple construction. In [1] Aichholzer et al. show that the number of plane graphs that can be embedded on a set of $N$ points is minimized when the $N$ points are in convex position. This leads us to conjecture that our construction, which consists of points in convex chain configuration, minimizes $\text{pg}_{cN}(N)$.

## 2 Plane Graphs with a Prescribed Number of Edges

In this section we study the relation between the number of plane graphs with $cN$ edges and the number of plane graphs with $cN + 1$ edges. We first give a lower bound on the increase rate $r_c$ for every set $S$ of $N$ points, and outline how this can be used to derive bounds on $\text{pg}_{cN}(N)$. We then compute $r_c$ explicitly for a specific configuration of points. Comparing the two results indicates that our bound on $r_c$ cannot be improved significantly, though only a small improvement is needed to improve bounds on $\text{pg}_{cN}(N)$.

### 2.1 Bounding the Increase Rate

We seek a bound of the form $r_c \geq g(c)$ where $g$ is a rational function of $c$. The following theorem by Hoffmann et al. [4] gives a formula to get upper bounds for $\text{pg}_{cN}(S)$.
Theorem 2.1. For every set $S$ of $N$ points in the plane and $0 \leq c < 3$,

$$\text{pg}_{cN}(S) = O^*(B(c)^N) \cdot \text{tr}(S),$$

where

$$B(c) := \frac{5^{5/2}}{8(c + t - 1/2)^{c+t-1/2}(3 - c - t)^{3-c-t}(2l)^t (1/2 - t)^{1/2-t}t},$$

and

$$t = \frac{1}{2} \left( \sqrt{(7/2)^2 + 3c + c^2 - 5/2 - c} \right).$$

Here, $\text{tr}(S)$ denotes the number of triangulations that can be embedded on $S$, for which the current best upper bound is $30^N$ due to Sharir and Sheffer [11]. The upper bound for $\text{pg}_{cN}(S)$ is maximized when $c = 19/12$. Suppose now for some fixed $c$ we get the bound $\text{pg}_{cN}(S) = O^*(\alpha^N)$ from the above formula. For any $g(c)$ with $g(c) \leq r_c$, or equivalently $\text{pg}_{cN+1}(S) \geq g(c)\text{pg}_{cN}(S)$, we have

$$\text{pg}_{(c-\delta)N}(S) \leq O^*(\alpha^N) \cdot \left( \min_{\lambda \in [c-\delta,c]} g(\lambda) \right)^{-\delta N} = O^*(\alpha^N) \cdot \max_{\lambda \in [c-\delta,c]} (g(\lambda)^{-\delta N}),$$

(1)

for some parameter $\delta$. Thus, improving the lower bound $g(c)$ improves the bound on $\text{pg}_{(c-\delta)N}(S)$ derived in this manner.

We now establish our lower bound on $r_c$ (Theorem 1.1 from the introduction). We restate the theorem for the convenience of the reader.

Theorem 1.1. For every set $S$ of $N$ points in the plane with $h$ points on the boundary of its convex hull and $0 \leq c < 3$,

$$r_c \geq \frac{(3-c)N - h - 3}{cN + 1}.$$ 

Proof. Let $\mathcal{X}$ and $\mathcal{Y}$ be the sets of all plane graphs with $cN$ edges and $cN+1$ edges on $S$, respectively. We construct a bipartite graph $G = (\mathcal{Y}, \mathcal{E})$ on $\mathcal{X} \cup \mathcal{Y}$ as follows: For every plane graph $G \in \mathcal{Y}$ we connect $G$ to every $G' \in \mathcal{X}$ that can be obtained by removing a single edge from $G$. Note that the degree of any $G \in \mathcal{Y}$ is the number of edges available for removal, which is $cN + 1$. Similarly, the degree of any $G \in \mathcal{X}$ is the number of ways we can add a single edge $e$ to $G$ such that $G \cup \{e\}$ is also a plane graph. We obtain a lower bound for the degree of any $G \in \mathcal{X}$ by completing $G$ to an arbitrary triangulation $T$, which by Euler’s formula contains $3N - 3 - h$ edges. Since $G$ has $cN$ edges, $T \setminus G$ contains $3N - 3 - h - cN$ edges, so at least $3N - 3 - h - cN$ edges can be added to $G$. Thus,

$$((3-c)N - 3 - h) \overset{\text{pg}_{cN}(S)}{\leq} |\mathcal{E}| = (cN + 1) \overset{\text{pg}_{cN+1}(S)}{\leq} |\mathcal{Y}|,$$

so we get

$$r_c \geq \frac{(3-c)N - h - 3}{cN + 1}.$$ 

To get a more accurate inequality, the factor of $(\min_{\lambda \in [c-\delta,c]} g(\lambda))^{-\delta N}$ in (1) can be replaced with the product $\prod_{(c-\delta)N \leq \lambda \leq cN} g(\lambda)$, but this has a negligible impact on the final bound.
Remark. In the case that \( c < \frac{1}{2} \), we may use the result that any triangulation over \( N \) vertices contains at least \( N/2 - 2 \) flippable edges \([5]\) to obtain a slightly improved bound on \( r_c \). If the number of edges in \( G \) is \( cN < N/2 \), then \( T \setminus G \) contains at least \( N/2 - 2 - cN = (1/2 - c)N - 2 \) flippable edges. Thus, there are \( (1/2 - c)N - 2 \) additional edges that can be added to \( G \) such that the resulting graph is a plane graph. This gives the bound

\[
  r_c \geq \frac{(7/2 - 2c)N - h - 5}{cN + 1}.
\]

If \( h \) is a small constant, we can use the inequality \( r_c \geq \frac{3}{c} \) in (1), since

\[
  3 - \frac{3}{c} - \frac{(3-c)N-h-3}{cN+1} = o(1),
\]

which is hidden by \( O^*(\cdot) \). Then, combining Theorem 1.1 with (1) allows us to derive bounds on \( \text{pg}_{cN}(N) \), but this method produces bounds that are slightly weaker than Theorem 2.1. For example, Theorem 2.1 gives

\[
  \text{pg}_{3N/4}(N) = O^*(127.5N) \quad \text{and} \quad \text{pg}_{2N/3}(N) = O^*(114.4N),
\]

but applying (1) with \( c = 3/4 \) and \( \delta = 1/12 \) gives \( \text{pg}_{2N/3}(N) = O^*(116.4N) \). When \( c < 1/2 \), using the inequality

\[
  r_c \geq \frac{3.5 - 2c}{c} \quad \text{(from the above remark) in (1)}
\]

also produces bounds that are slightly weaker than Theorem 2.1. For example, Theorem 2.1 gives

\[
  \text{pg}_{N/4}(N) = O^*(52.1N) \quad \text{and} \quad \text{pg}_{N/6}(N) = O^*(41.4N),
\]

but applying (1) with \( c = 1/4 \) and \( \delta = 1/12 \) gives \( \text{pg}_{N/6}(N) = O^*(42.3N) \).

However, only a small improvement is needed in our bound for \( r_c \) to give better bounds for \( \text{pg}_{cN}(N) \). For example, \( r_c \geq \frac{3.5 - c}{c} \) already gives stronger bounds for \( \text{pg}_{cN}(N) \) than Theorem 2.1.

### 2.2 Convex Chain Configuration

In this section we present a construction which indicates that our bound in Theorem 1.1 cannot be significantly improved. The point set \( S \) consists of \( N - 1 \) points arranged on a semicircle and an additional point \( v \) positioned sufficiently high above to form a triangular convex hull (see Figure 1). We call this convex chain configuration. Note that the condition on \( v \) implies that an edge from \( v \) to any point does not pass through the convex chain formed by the remaining points. We choose this configuration due to the fact that for \( N \) points the number of plane graphs is minimized when they are in convex position \([4]\).

![Figure 1: Points in convex chain configuration with all possible edges drawn. Type 1 edges are dotted, Type 2 edges are solid, and Type 3 edges are dashed.](image)

We now compute \( \text{pg}_{cN}(S) \), and use this to compute \( r_c(S) \).

**Lemma 2.2.** For every set \( S \) of \( N \) points in convex chain configuration,

\[
  \text{pg}_{cN}(S) = O^*(\tilde{f}(c, \frac{1}{4}(-1 + \sqrt{5c + 1}))^N),
\]

where

\[
  \tilde{f}(c, d) := \frac{4(1 + d)^{1+d}}{(1 - d)^{1-d}d^{2d}(c - d)^{c-d}(2 - c + d)^{2-c+d}}.
\]

Moreover, \( \text{pg}(S) = O^*(23.3N) \).
Proof. To compute $pg_{cN}(S)$ we partition edges into three types (see Figure [1]):

- Type 1: edges between two consecutive vertices on the convex chain (this includes the bottom edge).
- Type 2: edges between two non-consecutive vertices on the convex chain.
- Type 3: edges adjacent to the topmost vertex $v$.

In [7, p. 80] the following formula is given for the number of ways to choose $M$ non-intersecting diagonals from the interior of a convex $N$-gon:

$$\frac{1}{M+1} \binom{N-3}{M} \binom{N+M-1}{M}.$$ 

This counts exactly the number of ways to choose $M$ non-crossing Type 2 edges. Note that there are exactly $N-1$ Type 1 edges and $N-1$ Type 3 edges, none of which intersect. Let $f(cN,dN)$ be the number of plane graphs with $cN$ edges that can be drawn on $S$ such that $dN$ of the edges are of Type 2. We have

$$f(cN,dN) = \frac{1}{dN+1} \binom{N-3}{dN} \binom{(1+d)N-1}{dN} \binom{2N-2}{(c-d)N}.$$ 

We approximate $f(cN,dN)$ using Stirling’s approximation $n! \sim \sqrt{2\pi n(n/e)^n}$. Since we are not concerned with sub-exponential factors, we use $n! = \Theta^*(n/e)^n$. Thus we may approximate the binomial coefficient $\binom{n}{k}$ with

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \Theta^*\left(\frac{(n/e)^n}{(k/e)^k((n-k)/e)^{n-k}}\right) = \Theta^*\left(\frac{n^n}{k^k(n-k)^n(n-k)}\right).$$

Using this we get $f(cN,dN) = \Theta^*(\tilde{f}(c,d)^N)$, where

$$\tilde{f}(c,d) = \frac{4(1+d)^{1+d}}{(1-d)^{1-d}d^{2d}(c-d)^{c-d}(2-c+d)^{2-c+d}}.$$ 

Differentiating $\tilde{f}$ with respect to $d$ yields

$$\tilde{f}_d = \frac{4(1+d)^{1+d}}{(1-d)^{1-d}d^{2d}(c-d)^{c-d}(2-c+d)^{2-c+d}} \log \frac{(1-d)(1+d)(c-d)}{d^2(2-c+d)}.$$ 

The first term on the right hand side is always positive, and the natural logarithm is 0 exactly when $d = \frac{1}{4}(1 \pm \sqrt{8c+1})$. A tedious computation shows that the second derivative $\tilde{f}_dd$ evaluated at $d = \frac{1}{4}(1 \pm \sqrt{8c+1})$ is negative, so this is a maximum for $\tilde{f}$. A similar computation yields that $\tilde{f}(c,\frac{1}{4}(1 + \sqrt{8c+1}))$ attains a maximum of 23.3 at $c \approx 1.7$. Thus, $pg_{cN}(S) = O^*(\tilde{f}(c,\frac{1}{4}(1 + \sqrt{8c+1}))^N)$, and in particular $pg(S) = O^*(23.3^N)$. Note that this bound for $pg(S)$ also follows from the fact that there are $\Theta(11.65^N)$ plane graphs that can be embedded on the bottom $N-1$ points in convex position [3]. Indeed, there are $2^{N-1}$ combinations of Type 3 edges, so we get that in fact $pg(S) = \Theta((2 \cdot 11.65)^N) = \Theta(23.3^N)$. \[\square\]

We now compute the increase rate $r_c$ for this configuration. Suppose the additional edge is Type 2. Then,

$$r_c = \frac{f(dN+1,cN+1)}{f(dN,cN)} = \frac{(dN + N)(N - dN - 3)}{(dN + 1)(dN + 2)} \xrightarrow{N \to \infty} \frac{1 - d^2}{d^2}.$$
If the additional edge is Type 1 or Type 3 we have

\[ r_c = \frac{f(dN, cN + 1)}{f(dN, cN)} = \frac{(2 - c + d)N - 2}{(c - d)N + 1} \xrightarrow{N \to \infty} \frac{2 - c + d}{c - d}. \]

At \( d = \frac{1}{4}(-1 \pm \sqrt{8c + 1}) \), both ratios evaluate to \( r_c = \frac{-4c + \sqrt{1 + 8c + 7}}{4c - 1 + 8c + 1} \). However, this increase rate is not significantly greater than what our bound predicts, which indicates that our bound cannot be improved drastically (see Table 1).

| \( c \) | \( r_c(S) \), \( S \) in convex chain configuration | Lower bound predicted by Theorem 1.1 |
|---|---|---|
| 2/3 | 5.96 | 3.5 |
| 1 | 3 | 2 |
| 4/3 | 1.74 | 1.25 |
| 5/3 | 1.06 | 0.8 |

Table 1: Comparison of increase rate for convex chain configuration to the bound of Theorem 1.1

### 3 Plane Graphs with a Triangular Convex Hull

In this section we derive lower bounds on the minimum number of plane graphs with a triangular convex hull using a cross-graph charging scheme in the same spirit as [10]. In [1] Aichholzer et al. show that the number of plane graphs that can be embedded on a set of \( N \) points is minimized when the \( N \) points are in convex position. In [3] Flajolet and Noy show that this number is \( \Theta((6 + 4\sqrt{2})^N) \approx \Theta(11.65^N) \). Thus, every set of \( N \) points has \( \Omega(11.65^N) \) plane graphs. To the best of our knowledge, the setting of plane graphs with a triangular convex hull has not been previously considered, and thus \( \Omega(11.65^N) \) serves as a preliminary lower bound for the number of such graphs. Let \( \text{pg}_{\Delta}(N) \) be the minimum value of \( \text{pg}(S) \) taken over all sets \( S \) of \( N \) points with a triangular convex hull. Lemma 2.2 gives us the upper bound \( \text{pg}_{\Delta}(N) = O^*(23.3^N) \). For the remainder of this section all point sets will be assumed to have a triangular convex hull.

#### 3.1 Setup

Most of this subsection is taken from [10], as we require a very similar list of definitions and tools.

Given two vertices \( p, q \) of a plane graph \( G \), we say that \( p \) sees \( q \) in \( G \) if the (straight) edge \( pq \) does not cross any edge of \( G \). The degree of a ving \( v = (p, G) \) is the degree of \( p \) in \( G \); a ving of degree \( i \) is called an \( i \)-ving. We denote by \( N_G(v) \) (or simply \( N(v) \) when the underlying graph is understood) the set of vertices that are connected to \( p \) by an edge in \( G \). We say that points \( s, t \in N(v) \) are consecutive if \( t \) is immediately after \( s \) in the enumeration of \( N(v) \) in which we shoot a ray down from \( v \) and rotate it clockwise about \( v \), labeling the points of \( N(v) \) as they are hit by the ray. We say that a ving \( v = (p, G) \) is an \( x \)-ving if it satisfies the following properties:

1. We cannot increase the degree of \( p \) by inserting additional (straight) edges to \( G \) (that is, \( p \) cannot see any vertex that is not connected to it in \( G \)).

2. \( p \) does not lie on the boundary of the convex hull of \( G \).

We say that an \( x \)-ving \( v = (p, G) \) is an \( x_i \)-ving if \( v \) is also an \( i \)-ving. Note that every \( x \)-ving has degree at least 3. We say that a ving \( u = (p, G') \) corresponds to the \( x \)-ving \( v = (p, G) \) if \( G \) is
obtained by inserting into $G'$ all the edges that connect $p$ to the points that it sees in $G'$ (and are not already in the graph). Notice that every ving corresponds to a unique $x$-ving. Given a plane graph $G \in \mathcal{P}(S)$, we denote by $v_i(G)$ the number of $i$-vings with $G$, for $i \geq 0$, and by $v_x(G)$ the number of $x$-vings in $G$. Finally, the expected value of $v_x(G)$, for a graph chosen uniformly at random from $\mathcal{P}(S)$, is denoted as $\hat{v}_x(S)$. More formally, $\hat{v}_x = \hat{v}_x(S) := \mathbb{E}\{v_x(G)\} = \frac{\sum_{G \in \mathcal{P}(S)} v_x(G)}{\text{pg}(S)}$. A similar notation, $\hat{v}_i(S)$, applies to the expected value of $v_i(G)$.

The following lemma, whose proof is similar to that of Lemma 2.1 in [10], gives us a relation between $\hat{v}_x$ and $\text{pg}_\Delta(N)$.

**Lemma 3.1.** For $N \geq 3$, let $C > 0$ be a real number such that $\hat{v}_x(S) \leq C(N - 3)$ holds for every set $S$ of $N$ points in the plane with a triangular convex hull. Then $\text{pg}_\Delta(N) \geq \frac{1}{C} \text{pg}_\Delta(N - 1)$.

**Proof.** Let $S$ be a set that minimizes $\text{pg}(S)$ among all sets of $N$ points in the plane with triangular convex hull. Then, we can obtain certain plane graphs of $S$ by choosing a point $q \in S$ that does not lie on the boundary of the convex hull of $S$ and a plane graph $G$ of $S \setminus \{q\}$, inserting $q$ into $G$, and then connecting $q$ to all the vertices that it can see in $G$. It is then clear that a plane graph $G$ of $S$ can be obtained in exactly $v_x(G)$ ways in this manner. Thus,

$$\hat{v}_x \cdot \text{pg}(S) = \sum_{G \in \mathcal{P}(S)} v_x(G) = \sum_{q \in S} \text{pg}(S \setminus \{q\}).$$

The leftmost expression equals $\hat{v}_x \cdot \text{pg}_\Delta(N)$, and the rightmost expression is at least $(N - 3) \cdot \text{pg}_\Delta(N - 1)$. Hence, with $\hat{v}_x \leq C(N - 3)$, we have $\text{pg}_\Delta(N) = \text{pg}(S) \geq \frac{N - 3}{\hat{v}_x} \cdot \text{pg}_\Delta(N - 1) \geq \frac{1}{C} \cdot \text{pg}_\Delta(N - 1)$.

### 3.2 Charging Scheme

To get upper bounds for $\hat{v}_x$ (in order to apply Lemma 3.1), we use a charging scheme that gives a unit of charge to every $x$-ving, and then moves charge from $x$-vings to higher degree $x$-vings. We say an $x_i$-ving $v = (p, G)$ reduces to an $x_j$-ving $v' = (p, G')$ for $j > i$ if we can obtain $v'$ by removing an edge from $G$ of the form $ab$, where $a, b \in N_G(v)$, and then connecting $p$ to all newly visible vertices (see Figure 2).

![Figure 2: $x_4$-ving with vertex $p$ that reduces to the $x_5$-ving with vertex $p$ obtained by deleting edge $ac$, and then connecting $p$ to the visible vertex $b$.](image)

The following lemma shows that although not every $x$-ving necessarily reduces to a higher degree $x$-ving, $x$-vings with a neighborhood in convex position always reduce to at least one higher degree $x$-ving.

**Lemma 3.2.** If $v$ is an $x$-ving such that $N(v)$ is in convex position, then $v$ reduces to at least one higher degree $x$-ving.
**Proof.** Note that in order for an $x_i$-ving $v$ not to reduce to any higher degree $x$-vings, all vertices that are not adjacent to $v$ must be prevented from seeing $v$ by at least two edges between neighbors of $v$. In other words, for $v$ to see a vertex it is not adjacent to, at least two edges $ab, cd$, for $a, b, c, d \in N(v)$ not necessarily distinct, must be removed from the graph (for example, in Figure 3 any vertices that lie in wedge $\angle avd$ or $\angle cvd$ that are not contained triangle $avc$ are blocked from $v$ by two edges). When $N(v)$ is in convex position, no two edges between neighbors of $v$ can prevent $v$ from seeing the same set of vertices. Hence if $N(v)$ is in convex position, $v$ reduces to at least one higher degree $x$-ving.

![Figure 3: An $x_4$-ving that potentially does not reduce to any higher degree $x$-vings.](image)

We now derive our first bound for $\hat{v}_x$.

**Lemma 3.3.** For every set $S$ of $N$ points in the plane, $\hat{v}_x(S) \leq \frac{N-3}{12}$.

**Proof.** Consider a set $S$ of $N$ points. We start by giving a unit charge to every $x$-ving of every plane graph embedded on $S$. Fix a positive integer $M$ and nonnegative reals $c_3, c_4, \ldots, c_{M-1}$. For each $x_3$-ving, we move $c_3$ units of charge to one of the higher degree $x$-vings it reduces to. For each $x_4$-ving, we then move $c_4$ units of charge to one of the higher degree $x$-vings it reduces to. We continue this process for $x_i$-vings for each $i < M$, moving $c_i$ units of charge to one of the higher degree $x$-vings it reduces to (and doing nothing if the $x$-ving does not reduce to any higher degree $x$-vings). If an $x_i$-ving reduces to multiple higher degree $x$-vings, we arbitrarily choose one. Finally, each $x_i$-ving (for each $i < N$) redistributes its charge evenly to itself and the $2^i - 1$ lower degree vings that correspond to it. Note that none of these lower degree vings are $x$-vings.

If we show that every $x_i$-ving receives a charge of at most $t$ units after redistribution, then we would have

$$\hat{v}_x \cdot \text{pg}(S) = \sum_{G \in \mathcal{P}(S)} v_x(G) = \text{total charge} \leq (N - 3)t \cdot \text{pg}(S),$$

since none of the vings which are hull vertices receive any charge. This would then lead to the bound $\hat{v}_x \leq (N - 3)t$.

Denote by $\text{ch}(v)$ the charge on the $x_i$-ving $v$ before the redistribution step.

If $v$ is an $x_3$-ving then $N(v)$ is convex. Lemma 3.2 then implies that $v$ reduces to a higher degree $x$-ving, so $\text{ch}(v) \leq 1 - c_3$. At most two $x_3$-vings can reduce to a given $x_4$-ving, so if $v$ is an $x_4$-ving such that $N(v)$ is in convex position, we have by Lemma 3.2 $\text{ch}(v) \leq 1 + 2c_3 - c_4$. If $v$ is an $x_4$-ving that does not reduce to any higher degree $x$-ving, $N(v)$ is not in convex position, and thus only one $x_3$-ving can reduce to $v$. So in this case, $\text{ch}(v) \leq 1 + c_3$.

Suppose that $v$ is an $x_i$-ving for $5 \leq i < M$. Note that the maximum number of $x$-vings that could potentially reduce to $v$ is $\binom{i}{2} - i = \frac{i(i-3)}{2}$ (the number of possible edges between any two non-consecutive vertices of $N(v)$), so we get the bound $\text{ch}(v) \leq 1 + \frac{i(i-3)}{2} \cdot \max\{c_3, \ldots, c_{i-1}\} - c_i$ if
\(v\) reduces to a higher degree \(x\)-ving. If either \(i \geq M\) or \(v\) does not reduce to a higher degree \(x\)-ving, then the bound is simply 
\[
\text{ch}(v) \leq 1 + \frac{i(i-3)}{2} \cdot \max\{c_{3}, \ldots, c_{M-1}\}.
\]

![Figure 4: Example of an \(x_7\)-ving that is charged by the \(x_{5}\)-ving obtained by drawing the edge \(ab\) with \(d(a, b) = 2\).](image)

We can improve the bounds on \(\text{ch}(v)\) when the points of \(N(v)\) are in convex position. For any two vertices \(a, b \in N(v)\) define their distance \(d(a, b)\) to be the number of edges adjacent to \(v\) that intersect the edge \(ab\). For example, in Figure 4, \(d(a, b) = 2\). Note that drawing the edge \(ab\) and removing the edges it intersects gives an \(x_{i-d(a,b)}\)-ving that reduces to \(v\). For example, in Figure 4, the \(x_7\)-ving \(v\) is charged by the \(x_5\)-ving obtained by drawing \(ab\) and removing the edges it intersects.

For \(j < i\), the number of \(x_j\)-vings that reduce to \(v\) is at most \(i\), so we immediately get the bound 
\[
\text{ch}(i) \leq 1 + ic_{i-1} + ic_{i-2} + \cdots + ic_3 - c_i\text{ (note that all our bounds for }\text{ch}(v)\text{ when }N(v)\text{ is in convex position will include a }-c_i\text{ term due to Lemma 3.2).}
\]
However, in this bound for \(\text{ch}(i)\) we are overcounting the number of \(x\)-vings that can reduce to \(v\). Note that if there are \(k\) \(x_j\)-vings that reduce to \(v\), where \(j = i - \ell\) for some distance \(\ell \neq i/2 - 1\), then there are \(k\) pairs of vertices \((a, b) \in N(v) \times N(v)\) such that \(d(a, b) = \ell\). This means that there can be at most \(i - k\) pairs of vertices in \(N(v) \times N(v)\) with distance \(i - \ell - 2\), so at most \(i - k\) \(x_{\ell+2}\)-vings reduce to \(v\). If \(i\) is even, then there are at most \(i/2\) pairs of neighbors of \(v\) that are at a distance of \(i/2 - 1\). Thus, the number of \(x_{i/2+1}\)-vings that reduce to \(v\) is at most \(i/2\). Note that these observations are independent of the relative position of \(v\) in the interior of the convex hull of \(N(v)\).

We combine the above observations to arrive at the bounds for \(\text{ch}(v)\), where \(v\) is an \(x\)-ving with a convex neighborhood. If \(v\) is an \(x_i\)-ving for some \(5 \leq i < M\), we have
\[
\text{ch}(v) \leq \begin{cases} 
1 + i \sum_{\ell=1}^{(i-3)/2} \max(c_{i-\ell}, c_{i+2}) - c_i & \text{if } i \text{ is odd} \\
1 + i \sum_{\ell=1}^{(i-4)/2} \max(c_{i-\ell}, c_{i+2}) + \frac{i}{2}c_{i/2+1} - c_i & \text{if } i \text{ is even}
\end{cases}
\]

If \(i \geq M\) the bound for \(\text{ch}(v)\) is identical barring the transferring of charge prior to redistribution. Table 2 summarizes the upper bounds on \(\text{ch}(v)\) based on what type of \(x\)-ving \(v\) is.

After we run the charging scheme and the redistribution step, each ving will end up with a charge that is at most the maximum value of \(\frac{\text{ch}(v)}{2\deg v}\), where \(\text{ch}(v)\) is at most the maximum of the upper bounds given in Table 2.

An important observation is that high degree \(x\)-vings do not play a significant role in the charging scheme. The charge on any given \(x_i\)-ving \(v\) and the \(2^i - 1\) lower degree vings that correspond to it is at most \(\frac{\text{ch}(v)}{2^i}\). The denominator \(2^i\) grows significantly faster than \(\text{ch}(v) = O(i^2)\). In fact, if \(v\) is an \(x_i\)-ving for \(i > 12\), we get that \(\frac{\text{ch}(v)}{2^i}\) is small enough such that its reciprocal value is greater than 23.3, which is our upper bound for \(\text{pg}_{\Delta}(N)\) (Lemma 2.2). Thus, such large degree \(x\)-vings do not affect the maximum possible charge that can end up on a ving after the redistribution step.
Condition on \( x_i \)-ving \( v \) & Upper bound on \( \text{ch}(v) \) \\
\( i = 3 \) & \( 1 - c_3 \) \\
\( i = 4 \), and \( v \) does not reduce to any higher degree \( x \)-vings & \( 1 + c_3 \) \\
\( i = 4 \), and \( N(v) \) is convex & \( 1 + 2c_3 - c_4 \) \\
\( 5 \leq i < M \) and \( N(v) \) is convex & \[
1 + i \sum_{\ell=1}^{(i-3)/2} \max\{c_{i-\ell}, c_{\ell+2}\} - c_i \quad \text{if } i \text{ is odd}
\]
& \[
1 + i \sum_{\ell=1}^{(i-4)/2} \max\{c_{i-\ell}, c_{\ell+2}\} + \frac{i}{2} c_{i/2+1} - c_i \quad \text{if } i \text{ is even}
\] \\
\( i \geq M \) and \( N(v) \) is convex & \[
1 + i \sum_{\ell=1}^{(i-3)/2} \max\{c_{i-\ell}, c_{\ell+2}\} \quad \text{if } i \text{ is odd}
\]
& \[
1 + i \sum_{\ell=1}^{(i-4)/2} \max\{c_{i-\ell}, c_{\ell+2}\} + \frac{i}{2} c_{i/2+1} \quad \text{if } i \text{ is even}
\] \\
\( i \geq 5 \), \( N(v) \) is non-convex, and \( v \) reduces to at least one higher degree \( x \)-ving & \[
1 + i \sum_{\ell=1}^{(i-3)/2} \max\{c_{i-\ell}, c_{\ell+2}\} \quad \text{if } i \text{ is odd}
\]
& \[
1 + i \sum_{\ell=1}^{(i-4)/2} \max\{c_{i-\ell}, c_{\ell+2}\} + \frac{i}{2} c_{i/2+1} \quad \text{if } i \text{ is even}
\] \\
\( i \geq 5 \), \( N(v) \) is non-convex, and \( v \) does not reduce to any higher degree \( x \)-vings & \[
1 + \frac{i(i-3)}{2} \cdot \max\{c_3, \ldots, c_{i-1}\} - c_i
\]

Table 2: Upper bounds on \( \text{ch}(v) \).

We ran Mathematica computations to choose values for \( c_3, \ldots, c_{M-1} \) to minimize the maximum value of the set of possible charges, for several values of \( M \), and found that choosing \( M = 11 \) and \( c_3 = 0.333333, c_4 = 0.333333, c_5 = 0.387846, c_6 = 0.544067, c_7 = 0.83942, c_8 = 1.15292, c_9 = 1.04764, \) and \( c_{10} = 1.34586 \) gives a maximum possible charge of 0.083333 (the only constraints on \( c_3, \ldots, c_{M-1} \) were that every \( x \)-ving gives away a non-negative amount of charge, and that no \( x \)-ving gives away more charge than it holds). Choosing larger values of \( M \) had a negligible affect on the maximum charge. This maximum charge has a reciprocal value of 12, so we may conclude that the average value of \( v_x \) is at most \( \frac{N-3}{12} \).

Combining Lemmas 3.3 and 3.1 and using an obvious induction on \( N \) (starting with the constant value \( p_{\Delta}(11) \)), we obtain

**Theorem 3.4.** \( p_{\Delta}^{-}(N) = \Omega^*(12^N) \).

### 3.3 Improvements

In coming up with a bound for \( \text{ch}(v) \) that works for every possible degree of an \( x \)-ving \( v \) we were unable to take into account some more subtle properties of low degree \( x \)-vings. We now derive an improved bound, by studying \( \text{ch}(v) \) more carefully for \( x_5 \)-vings \( v \).

**Lemma 3.5.** For every set \( S \) of \( N \) points in the plane, \( \hat{v}_x(S) \leq \frac{N-3}{12N} \).

**Proof.** The main structure of the previous charging scheme still applies to this lemma, but we will treat \( x_5 \)-vings separately. We first count the number of lower degree \( x \)-vings that can possibly...
contribute charge (directly or indirectly) to a given $x_5$-ving $v$. We then transfer all the charge from these lower degree $x$-ving to $v$, and then redistribute to all vings that either correspond to $v$ or to one of the lower degree $x$-ving that contributed charge to $v$.

For an $x_5$-ving $v$, let $H(v)$ be the set of vertices that lie on the boundary of the convex hull of $N(v)$. We split up our analysis into several cases based on the size of $H(v)$ and the position of $v$ in the interior of $H(v)$ with respect to the diagonals between points in $N(v)$.

\[ \text{Case 1:} \quad |H(v)| = 5. \]

Suppose $v$ lies in region $X$. In this case, there are at most five $x_4$-ving that reduce to $v$, and no $x_3$-ving reduce to $v$. The number of $x_3$-ving that can possibly contribute charge to $v$ is at most the Catalan number $C_3 = 5$, which counts the number of triangulations of a regular pentagon. This is due to the one-to-one correspondence between triangulations of $N(v)$ and $x_3$-ving that contribute charge to $v$ resulting from sending a triangulation $T$ to the triangle containing $v$.

Suppose $v$ lies in region $Y$. In this case, there are at most four $x_4$-ving that reduce to $v$. There are five $x_3$-ving that contribute charge to $v$ by reducing to an $x_4$-ving that reduces to $v$; these arise from the triangulations of $N(v)$. There is also one additional $x_3$-ving that directly reduces to $v$. This $x_3$-ving is obtained by drawing one diagonal in the interior of $H(v)$, rather than by triangulating it.

Suppose $v$ lies in region $Z$. In this case, there are at most three $x_4$-ving that reduce to $v$. There are five $x_3$-ving that contribute charge to $v$ by reducing to an $x_4$-ving that reduces to $v$; these arise from the triangulations of $N(v)$. There are also two additional $x_3$-ving that directly reduce to $v$. These $x_3$-ving are obtained by drawing one diagonal in the interior of $H(v)$, rather than by triangulating it.

\[ \text{Case 2:} \quad |H(v)| = 4. \]

Let $T_1$, $T_2$, and $T_3$ be the three triangulations of $N(v)$, according to Figure 6.

Suppose $v$ lies in region $X$. In this case, there are at most four $x_4$-ving that reduce to $v$, and no $x_3$-ving reduce to $v$. There are three $x_3$-ving that contribute charge to $v$ by reducing to an

![Figure 5: The number of $x_3$-ving and $x_4$-ving that can contribute charge to a given $x_5$-ving $v$ is determined by the region (enclosed by solid lines) in the interior of $H(v)$ that contains $v$.](image)
$x_4$-ving that reduces to $v$; these arise from the triangulations of $N(v)$ (and as before, this is due to the one-to-one correspondence between triangulations of $N(v)$ and these $x_3$-vings arising from sending a triangulation $T$ to the triangle containing $v$).

Suppose $v$ lies in region $Y$ (without loss of generality, suppose it lies in the left portion of $Y$). In this case, there are at most three $x_4$-vings that reduce to $v$. There are three $x_3$-vings that contribute charge to $v$ by reducing to an $x_4$-ving that reduces to $v$; these arise from the triangulations of $N(v)$. Moreover, triangulation $T_1$ gives an $x_3$-ving that directly reduces to $v$ by only drawing the first diagonal that prevents $v$ from seeing the remaining two vertices, rather than fully triangulating $N(v)$.

Suppose $v$ lies in region $Y'$ (without loss of generality, suppose it lies in the left portion of $Y'$). In this case, there are at most three $x_4$-vings that reduce to $v$. $T_1$ gives an $x_3$-ving that reduces to an $x_4$-ving that reduces to $v$. If $v$ lies below the dashed line that passes through $Y'$, there are two additional $x_3$-vings that may contribute charge to $v$. In triangulations $T_2$ and $T_3$, $v$ is contained in the same triangle, and these together yield two $x_3$-vings that contribute charge to $v$; one of which directly reduces to $v$ by taking $T_3$ and only drawing the first diagonal that prevents $v$ from seeing the remaining two vertices. If $v$ lies above the dashed line that passes through $Y'$, there is one additional $x_3$-ving that may contribute charge to $v$. In triangulations $T_2$ and $T_3$, $v$ is contained in the same triangle, and these together yield one $x_3$-ving that contributes charge to $v$ by reducing to an $x_4$-ving that reduces to $v$.

Suppose $v$ lies in region $Z$ (without loss of generality, suppose it lies in the left portion of $Z$). In this case, there are at most two $x_4$-vings that reduce to $v$. Triangulation $T_1$ gives two $x_3$-vings that contribute charge to $v$: one of which directly reduces to $v$ by only drawing the first diagonal that prevents $v$ from seeing the remaining two vertices. If $v$ lies below the dashed line that passes through $Z$, there are two additional $x_3$-vings that may contribute charge to $v$. In triangulations $T_2$ and $T_3$, $v$ is contained in the same triangle, and these together yield two $x_3$-vings that contribute charge to $v$: one of which directly reduces to $v$ by only drawing the first diagonal that prevents $v$ from seeing the remaining two vertices. If $v$ lies above the dashed line that passes through $Z$, there is only one additional $x_3$-ving that may contribute charge to $v$. In triangulations $T_2$ and $T_3$, $v$ is contained in the same triangle, and these together yield one $x_3$-ving that contributes charge to $v$ by reducing to an $x_4$-ving that reduces to $v$.

Suppose $v$ lies in region $Z'$. In this case, there are at most two $x_4$-vings that reduce to $v$. If $v$ lies in the leftmost or the rightmost region of $Z'$ determined by the two dashed lines that pass through $Z'$, there are two $x_3$-vings that contribute charge to $v$. In triangulations $T_1$, $T_2$, and $T_3$, $v$ is contained in the same triangle, and these together yield at most two $x_3$-vings that contribute charge to $v$: one of which directly reduces to $v$ by taking one of $T_1$, $T_2$, or $T_3$ and only drawing the first diagonal that prevents $v$ from seeing the remaining two vertices. If $v$ lies in the center region of $Z'$ determined by the two dashed lines that pass through $Z'$, there is one $x_3$-ving that contributes charge to $v$.
charge to \( v \). In triangulations \( T_1, T_2, \) and \( T_3, v \) is contained in the same triangle, and these together yield at most one \( x_3 \)-ving that contributes charge to \( v \) by reducing to an \( x_4 \)-ving that reduces to \( v \).

Suppose \( v \) lies in region \( Z'' \). In this case, there are at most two \( x_4 \)-vings that reduce to \( v \). There are five \( x_3 \)-vings that may contribute charge to \( v \). Triangulations \( T_1 \) and \( T_2 \) each give two \( x_3 \)-vings that contribute charge to \( v \): one of which directly reduces to \( v \) obtained by only drawing the first diagonal that prevents \( v \) from seeing the remaining two vertices. Triangulation \( T_3 \) gives one \( x_3 \)-ving that reduces to an \( x_4 \)-ving that reduces to \( v \).

**Case 3:** \( |H(v)| = 3 \).

![Figure 7: Triangulations of \( N(v) \) when \( |H(v)| = 3 \).](image)

Let \( T_1 \) and \( T_2 \) be the two triangulations of \( N(v) \), according to Figure 7.

Suppose \( v \) lies in region \( X \). In this case, there are at most three \( x_4 \)-vings that reduce to \( v \), and no \( x_3 \)-vings that directly reduce to \( v \). There are two \( x_3 \)-vings that contribute charge to \( v \) by reducing to an \( x_4 \)-ving that reduces to \( v \); these arise from the triangulations of \( N(v) \).

Suppose \( v \) lies in region \( Y \) (without loss of generality, suppose it lies in the left portion of \( Y \)). In this case, there are at most two \( x_4 \)-vings that reduce to \( v \). There are two \( x_3 \)-vings that contribute charge to \( v \) by reducing to an \( x_4 \)-ving that reduces to \( v \); these arise from the triangulations of \( N(v) \). If \( v \) lies to the right of the dashed line that passes through \( Y \), triangulation \( T_1 \) gives an additional \( x_3 \)-ving that directly reduces to \( v \), obtained by only drawing the first diagonal that prevents \( v \) from seeing the remaining two vertices.

Suppose \( v \) lies in region \( Y' \). In this case, there are at most two \( x_4 \)-vings that reduce to \( v \). In triangulations \( T_1 \) and \( T_2 \), \( v \) is contained in the same triangle, and these together yield only one \( x_3 \)-ving that reduces to an \( x_4 \)-ving that reduces to \( v \). There are no \( x_3 \)-vings obtained in this manner than directly reduce to \( v \), since there is no edge that can simultaneously prevent \( v \) from seeing both remaining vertices.

Suppose \( v \) lies in region \( Z \) (without loss of generality, suppose it lies in the left portion of \( Z \)). In this case, there is at most one \( x_4 \)-ving that reduces to \( v \). If \( v \) lies in the upper or bottom region of \( Z \) determined by the two dashed lines that pass through \( Z \), there are two \( x_3 \)-vings that contribute charge to \( v \). In triangulations \( T_1 \) and \( T_2 \), \( v \) is contained in the same triangle, and these together yield two \( x_3 \)-vings that contribute charge to \( v \): one of which directly reduces to \( v \) by only drawing the first diagonal that prevents \( v \) from seeing the remaining two vertices. If \( v \) lies in the center region of \( Z \) determined by the two dashed lines that pass through \( Z \), there is one \( x_3 \)-ving that contributes charge to \( v \). In triangulations \( T_1 \) and \( T_2 \), \( v \) is contained in the same triangle, and these together yield one \( x_3 \)-ving that contributes charge to \( v \) by reducing to an \( x_4 \)-ving that reduces to \( v \).

Suppose \( v \) lies in region \( Z' \). In this case, there is at most one \( x_4 \)-ving that reduces to \( v \). If \( v \) lies in the leftmost (rightmost) region of \( Z' \) determined by the two dashed lines that pass through \( Z' \), there are three \( x_3 \)-vings that contribute charge to \( v \). Triangulation \( T_1 \) (\( T_2 \)) gives one \( x_3 \)-ving that contributes charge to \( v \) by reducing to an \( x_4 \)-ving that reduces to \( v \), and triangulation \( T_2 \) (\( T_1 \))
gives two $x_3$-vings that contribute charge to $v$; one of which directly reduces to $v$, obtained by only drawing the first diagonal that prevents $v$ from seeing the remaining two vertices. If $v$ lies in the center region of $Z'$ determined by the two dashed lines that pass through $Z'$, there are four $x_3$-vings that contribute charge to $v$. Triangulations $T_1$ and $T_2$ each give two $x_3$-vings that contribute charge to $v$: one that reduces to an $x_4$-ving that reduces to $v$, and one that directly reduces to $v$ obtained by only drawing the first diagonal that prevents $v$ from seeing the remaining two vertices.

**Completing the analysis**

Table 3 summarizes the charge contributions we have accounted for:

| $|H(v)| = 5$ | $|H(v)| = 4$ | $|H(v)| = 3$ |
|---|---|---|
| $v \in X Y Z X Y Y'$ | $Z Z'$ | $Z'' X Y Y' Z Z'$ |
| $x_4$-vings | 5 | 4 | 3 |
| 2 | 3 |
| 2 | 3 |
| 5 | 2 |
| $x_3$-vings | 5 | 6 | 7 |
| 3 | 4 | 2 or 3 |
| 3 or 4 | 1 or 2 |
| 2 |

Table 3: Number of $x_3$-vings and $x_4$-vings that can contribute charge to any given $x_5$-ving.

We run the same charging scheme as in Lemma 3.3 with the following modification. For every $x_5$-ving $v$, the $x_3$-vings and $x_4$-vings that would have contributed charge to $v$ in the previous charging scheme now transfer all their charge to $v$. $v$ then transfers $c_5$ units of charge to a higher degree $x$-ving it reduces to (if it reduces to any). Suppose $v$ was charged by $n_4 x_4$-vings and $n_3 x_3$-vings. In the final redistribution step, $v$ evenly redistributes its charge to the $2^5 + n_4 \cdot 2^4 + n_3 \cdot 2^3$ vings that correspond to $v$ or to one of the lower degree $x$-vings that transferred charge to $v$. Thus, the maximum charge that could end up on any of these vings is max \( \left( \frac{1+n_4+n_3}{2^5+n_4\cdot 2^4+n_3\cdot 2^3} \right) \), taken over all entries \((n_4,n_3)\) in Table 3.

Replacing the upper bounds for terms of the form $\frac{\text{ch}(v)}{2^{n x}}$ for vings corresponding to an $x_5$-ving $v$ in the computation from Lemma 3.3 with the newly derived bound yields that when $M = 11$ and $c_3 = 0.369874$, $c_4 = 0.446263$, $c_5 = 0.195332$, $c_6 = 0.508932$, $c_7 = 0.665472$, $c_8 = 1.29213$, $c_9 = 1.24496$, and $c_{10} = 0.935081$, any ving ends up with a charge of at most 0.0808824. This maximum charge has a reciprocal value of 12.36, so we may conclude that the average value of $v_x$ is at most $\frac{N-3}{12.36}$.

Combining Lemmas 3.5 and 3.1 yields

**Theorem 1.2** $\operatorname{pg}_\Delta(N) = \Omega^*(12.36^N)$.

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