BIGEOMETRIC CESÀRO DIFFERENCE SEQUENCE SPACES
AND HERMITE INTERPOLATION

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Abstract. In this paper, we introduce some difference sequence spaces in bi-
geometric calculus. We determine the α-duals of these sequence spaces and
study their matrix transformations. We also develop an interpolating poly-
nomial in bigeometric calculus which is analogous to the classical Hermite
interpolating polynomial.

1. Introduction

Bigeometric calculus is one of the non-Newtonian calculi developed by Grossman
et al. [11, 10] during the years 1967-1983. In bigeometric calculus, changes and
accumulations in arguments and values of a function are measured by ratios and
products respectively, whereas in classical calculus, changes are measured by dif-
fferences and accumulations are measured by sums. The bigeometric calculus may
be considered as a byproduct of ambiguity among scholars in choosing either dif-
fferences or ratios for the estimation of deviations. Galileo observed that the ratios
are more convenient in measuring deviations.

The important applications of bigeometric calculus are seen in fractal dynamics
of materials [22], fractal dynamics of biological systems [23], etc. Moreover, Multi-
plicative calculus is used to establish non-Newtonian Runge-Kutta methods [1],
Lorenz systems [2], and some finite difference methods [21]. Some non-Newtonian
Hilbert spaces [15] are constructed by Kadak et al. Some non-Newtonian met-
ric spaces and its applications can be seen in [3, 7]. Çakmak and Başar [5] have
constructed the non-Newtonian real field \( \mathbb{R}(N) \) and defined the sequence spaces
\( w(N), l_\infty(N), c(N), c_0(N) \) and \( l_p(N) \) in \( \mathbb{R}(N) \). The matrix transfor-
mations between these spaces are also studied by them [6]. While Kadak [12, 16, 13, 14], in a
series of papers, has made a significant contribution in constructing non-Newtonian
sequence spaces and in studying their Köthe-Toeplitz duals and matrix transfor-
mations.

The classical difference sequence spaces are first introduced by Kizmaz [18]. He
has studied the spaces \( Z(\Delta) \) for \( Z = l_\infty, c \) or \( c_0 \) which are defined as
\begin{equation}
Z(\Delta) = \{ x = (x_k) : (\Delta x_k) \in Z \},
\end{equation}
where \( \Delta x_k = x_k - x_{k+1} \). Later on, Et and Çolak [9] have generalized these spaces
by replacing the first order difference operator with \( m \)-th order difference operator
and hence defined the spaces \( Z(\Delta^m) \) for \( Z = l_\infty, c \) or \( c_0 \) as follows:
\begin{equation}
Z(\Delta^m) = \{ x = (x_k) : (\Delta^m x_k) \in Z \},
\end{equation}

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sequence space; matrix transformation; Hermite interpolation.
where $\Delta^m x_k = \Delta^{m-1}(x_k - x_{k+1}) = \sum_{v=0}^{m} \binom{m}{v} (-1)^v x_{k+v}$. Following these spaces, Orhan [20] has studied the Cesàro difference sequence spaces $C_p(\Delta)$ and $C_{\infty}(\Delta)$ for $1 \leq p < \infty$. Subsequently, Et [8] has used the $m$-th order difference operator instead of first order difference operator in the spaces of Orhan and constructed the spaces

\begin{equation}
C_p(\Delta^m) = \left\{ x = (x_k) : \sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} \Delta^m x_k \right|^p < \infty \right\}
\end{equation}

and

\begin{equation}
C_{\infty}(\Delta^m) = \left\{ x = (x_k) : \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} \Delta^m x_k \right| < \infty \right\}.
\end{equation}

Recently, Boruah and Hazarika [4] have defined the geometric difference sequence spaces $X(\Delta_G)$ for $X = l_G^1, c_G^1$ or $c_G^0$ which are analogous to the spaces of Kizmaz (1.1). They have studied duals of these spaces and constructed geometric Newton’s forward and backward interpolation formulae.

In this paper, we have introduced some Cesàro difference sequence spaces in bigeometric calculus analogous to the classical sequence spaces as defined in (1.3) and (1.4). We have also studied bigeometric $\alpha$-duals and matrix transformations of these spaces and formulated the Hermite interpolation formula in bigeometric calculus.

2. Preliminaries

**Definition 2.1.** (Arithmetic system)[11, 10]: An arithmetic system consists of a set $R$ with four operations namely addition, subtraction, multiplication, division and an ordering relation that satisfy the axioms of a complete ordered field. The set $R$ is called a realm, and the members of the set $R$ are called numbers of the system. The set of all real numbers $\mathbb{R}$ with the usual $+,-,\times,/\text{ and } <$ is known as the classical arithmetic system.

**Definition 2.2.** (Generator)[11, 10]: A one-to-one function $\phi : \mathbb{R} \to \mathbb{R}$ is called a generator whose range $A$ is a subset of $\mathbb{R}$.

**Definition 2.3.** ($\phi$-arithmetic)[11, 10]: The $\phi$-arithmetic with generator $\phi$ is an arithmetic system whose realm is the range $A$ of $\phi$ and the operations $\phi$-addition $'+_\phi$', $\phi$-subtraction $'-'_\phi$, $\phi$-multiplication '$\times_\phi$', $\phi$-division $'/_\phi$ and $\phi$-order $'<_\phi'$ are defined as follows:

- $u +_\phi v = \phi^{-1}(u + \phi^{-1}(v))$,
- $u -_\phi v = \phi^{-1}(u) - \phi^{-1}(v))$,
- $u \times_\phi v = \phi^{-1}(u) \times \phi^{-1}(v))$,
- $u /_\phi v = \phi^{-1}(u) / \phi^{-1}(v))$,
- and $u <_\phi v$ if and only if $\phi^{-1}(u) < \phi^{-1}(v)$.

For example, the classical arithmetic is generated by the identity function.
Every ordered pair of arithmetics (\(\phi\)-arithmetic, \(\phi'\)-arithmetic) gives rise to a calculus, where \(\phi\)-arithmetic is used for arguments and \(\phi'\)-arithmetic is used for values of functions in the calculus. For a particular choice of \(\phi\) and \(\phi'\) the following calculi can be generated:

| Calculus   | \(\phi\) | \(\phi'\) |
|------------|-----------|-----------|
| classical  | I         | I         |
| geometric  | I         | exp       |
| anageometric | exp     | I         |
| bigeometric | exp     | exp       |

where \(I\) and \(\text{exp}\) denote the identity and exponential functions respectively.

**Definition 2.4.** (Geometric arithmetic)\[11, 10\]: The arithmetic generated by the exponential function is called geometric arithmetic.

**Definition 2.5.** (Geometric real number)\[24\]: The realm of the geometric arithmetic is the set of all positive real numbers which we denote by \(\mathbb{R}(G)\) and call it the set of geometric real numbers. Then,

\[
\mathbb{R}(G) = \{e^x : x \in \mathbb{R}\}.
\]

2.1. **Some properties of geometric arithmetic system:** Let us denote the geometric operations addition, subtraction, multiplication, and division by \(\oplus\), \(\ominus\), \(\otimes\), and \(\oslash\) respectively and the ordering relation by the usual symbol <. Some properties of the geometric arithmetic system are as follows\[11, 4\]: For all \(u, v \in \mathbb{R}(G)\), we have

(i) \((\mathbb{R}(G), \oplus, \otimes)\) is a field with geometric identity \(e\) and geometric zero \(1\).
(ii) Geometric addition: \(u \oplus v = \exp\{\ln u + \ln v\} = uv\).
(iii) Geometric subtraction: \(u \ominus v = \exp\{\ln u - \ln v\} = u/v\).
(iv) Geometric multiplication: \(u \otimes v = \exp\{\ln u \ln v\} = u^{\ln v}\).
(v) Geometric division: \(u \oslash v = u^{\frac{-1}{\ln v}}, v \neq 1\).
(vi) Since \(u < v\) if and only if \(\ln u < \ln v\), the usual < relation is taken as geometric ordering relation.
(vii) Geometric exponentiation: For a real number \(q\), we have \(u^{qG} = \exp(\ln u)^q = u^{\ln q}\).
(viii) \(e^n \ominus u = u \oplus u \oplus \cdots \oplus u = u^n\) \(n\) times.

(ix) Geometric modulus: \(|u|_G = \exp(\ln u)\) = \[
\begin{cases}
  u, & \text{when } u > 1 \\
  1, & \text{when } u = 1 \\
  \frac{1}{u}, & \text{when } 0 < u < 1
\end{cases}
\]

Thus, \(|u|_G\) is always greater than or equal to one.

(x) \(|u|_{2G} \geq 1\) for all \(u \in \mathbb{R}(G)\) and \(q \in \mathbb{R}\).
(xi) \(|u \otimes v|_G = |u|_G \otimes |v|_G|.
(xii) \(|u \oplus v|_G \leq |u|_G \oplus |v|_G|\) (Triangular inequality).
(xiii) The symbols \(\sum_g\) and \(\prod_g\) represent the geometric sum and geometric product of sequence of numbers respectively.

**Definition 2.6.** (Geometric normed space): Let \(X\) be a linear space over \(\mathbb{R}(G)\). Then \(X\) is said to be a normed space if there exists a function \(\|\cdot\|_G\) from \(X\) to \(\mathbb{R}(G)\) such that for all \(x, y \in X\) and \(a \in \mathbb{R}(G)\),

\[
\|x + y\|_G \leq \|x\|_G + \|y\|_G
\]

\[
\|ax\|_G = |a|_G \|x\|_G
\]

\[
\|x\|_G = 0 \iff x = 0
\]

\[
\|x\|_G \geq 0
\]
Lemma 2.11. Then the following inequality holds:

\[ \|x\|_G \geq 1. \]

Lemma 2.10. If and only if \( x = \theta \), where \( \theta \) is the zero element of \( X \).

Definition 2.7. The bigeometric derivative of a function \( f : \mathbb{R}(G) \to \mathbb{R}(G) \) at a point \( a \in \mathbb{R}(G) \) is given by

\[ [D_G f](a) = \lim_{x \to a} \frac{f(x) - f(a)}{\|x - a\|_G}. \]

2.2. Some useful results of bigeometric derivative:

(1) Let \( f'(a) \) be the classical derivative of a function \( f(x) \) at the point \( a \); then the bigeometric derivative of the function at the same point is given by

\[ [D_G f](a) = e^{f'(a)G}. \]

(2) Bigeometric derivative of some of the important functions are as follows:

- \( D_G e^x = e^x \cdot x > 0. \)
- \( D_G \ln(x) = e^{\frac{x}{\ln(x)}} \cdot x > 1. \)
- \( D_G \sin(x) = e^{x \cot(x)} \cdot x \in \cup_{n=0}^{\infty} (2n, (2n + 1)\pi). \)

(3) Let \( f \) and \( g \) be two functions which are defined in \( \mathbb{R}(G) \) and whose ranges are subsets of \( \mathbb{R}(G) \); then we have

- \( D_G (f(x) \odot g(x)) = D_G f(x) \odot D_G g(x). \)
- \( D_G (f(x) \odot g(x)) = D_G f(x) \odot D_G g(x). \)
- \( D_G (f(x) \odot g(x)) = (D_G f(x) \odot g(x)) \odot (D_G g(x) \odot f(x)). \)
- \( D_G (f(x) \odot g(x)) = \frac{(D_G f(x) \odot g(x)) \odot (D_G g(x) \odot f(x))}{g(x) \odot f(x)}. \)

Theorem 2.8. [5] The ordered pair \((\mathbb{R}(G), d_G)\) is a complete metric space with respect to the metric \( d_G \) defined by

\[ d_G(x, y) = |x \odot y|_G \]

for all \( x, y \in \mathbb{R}(G) \).

Lemma 2.9. [19] For \( a, b \in \mathbb{R} \) and \( p \in [1, \infty) \), the following inequality holds:

\[ |a + b|^p \leq 2^{p-1}(|a|^p + |b|^p). \]

Lemma 2.10. (Jessen’s inequality)[8]: Let \( (a_k) \) be a sequence in \( \mathbb{R} \) and \( 0 < p < q \), then the following inequality holds:

\[ \left( \sum_{k=1}^{n} |a_k|^q \right)^{1/q} \leq \left( \sum_{k=1}^{n} |a_k|^p \right)^{1/p}. \]

Lemma 2.11. (Geometric Minkowski’s inequality)[24]: Let \( p \geq 1 \) and \( a_k, b_k \in \mathbb{R}(G) \) for all \( k \in \mathbb{N} \). Then,

\[ \left( \sum_{k=1}^{\infty} |a_k \odot b_k|_G^{p_G} \right)^{(1/p_G)} \leq \left( \sum_{k=1}^{\infty} |a_k|_G^{p_G} \right)^{(1/p_G)} \odot \left( \sum_{k=1}^{\infty} |b_k|_G^{p_G} \right)^{(1/p_G)}. \]

Lemma 2.12. [4] \( \sup_k |x_k \odot x_{k+1}|_G < \infty \) if and only if (i) \( \sup_k e^{k^{-1}} \cdot |x_k|_G < \infty \) and (ii) \( \sup_k |x_k \odot e^{k(k+1)^{-1}} \cdot x_{k+1}|_G < \infty \) hold.
Putting \( \Delta^{-1}_G x_k \) instead of \( x_k \) in the above lemma, we get the following result.

**Corollary 2.13.** [17] The following assertions (i) and (ii) are equivalent.

(i) \( \sup_k |\Delta^{-1}_G x_k \odot \Delta^{-1}_G x_{k+1}|_G < \infty. \)

(ii) (a) \( \sup_k e^{-i} \odot |\Delta^{-1}_G x_k|_G < \infty. \)

(b) \( \sup_k |\Delta^{-1}_G x_k \odot e^{i+1} \odot \Delta^{-1}_G x_{k+1}| < \infty. \)

**Lemma 2.14.** [17] If \( \sup_k e^{-i} \odot |\Delta^{-1}_G x_k|_G < \infty, \) then \( \sup_k e^{-i+1} \odot |\Delta^{-1}_G x_k|_G < \infty \) for all \( i, m \in \mathbb{N} \) and \( 1 \leq i < m. \)

**Corollary 2.15.** [17] If \( \sup_k e^{-i} \odot |\Delta^{-1}_G x_k|_G < \infty, \) then \( \sup_k e^{-m} \odot |x_k|_G < \infty. \)

### 3. Main results

In this section, we define some bigeometric Cesàro difference sequence spaces. Let \( (x_k) \) be a sequence in the set \( \mathbb{R}(G) \) of geometric real numbers; then the first order geometric difference operator \( \Delta_G \) is defined by \( \Delta_G x_k = x_k \odot x_{k+1} \) and the \( m \)-th order geometric difference operator \( \Delta^m_G \) is defined by \( \Delta^m_G x_k = \Delta^{m-1}_G (x_k \odot x_{k+1}). \)

Thus \( \Delta^m_G x_k = G \sum_{v=0}^{m} (\odot e)^v \odot e^{(v)} \odot x_{k+v}. \) It is easy to verify that the geometric difference operator is a linear operator. Let \( w(G) \) denotes the set of all sequences in \( \mathbb{R}(G) \). Then the set \( w(G) \) is a linear space over \( R(G) \) with respect to the operations

(a) \( \oplus : w(G) \times w(G) \to w(G) \)

defined by \( ((x_k), (y_k)) \to (x_k) \oplus (y_k) = (x_k \oplus y_k) \) and

(b) \( \odot : \mathbb{R}(G) \times w(G) \to w(G) \)

defined by \( ((x_k), (y_k)) \to (x_k) \odot (y_k) = (x_k \odot y_k). \)

We introduce the following sequence spaces in bigeometric calculus as follows:

\[
C^G_p(\Delta^m_G) = \left\{ x = (x_k) \in w(G) : \mathbb{G} \sum_{n=1}^{\infty} (e \odot e^n) \odot \mathbb{G} \sum_{k=1}^{n} \Delta^m_{G} x_k \right\}_{\mathbb{G}}^p < \infty \right\}
\]

for \( 1 \leq p < \infty \) and

\[
C^G_\infty(\Delta^m_G) = \left\{ x = (x_k) \in w(G) : \sup_n \left| (e \odot e^n) \odot \mathbb{G} \sum_{k=1}^{n} \Delta^m_{G} x_k \right|_G < \infty \right\}.
\]

**Lemma 3.1.** For \( a, b \in \mathbb{R}(G) \) and \( p \in [1, \infty) \), the following inequality holds:

\[
|a \oplus b|_G^p \leq e^{(p-1)} \odot (|a|_G^p \oplus |b|_G^p).
\]

**Proof.** Let \( u = |a \oplus b|_G^p \); then \( u = |ab|_G^p \).

Using the definition of geometric modulus, we have

\[
u = (\exp|\ln(ab)|_{G})^p.
\]
Again using the definition of geometric exponentiation, we get
\[
\begin{align*}
u &= \exp [\exp [\ln(ab)]]^p \\
&= \exp[\ln(ab)]^p
\end{align*}
\]
(3.3)
\[\ln u = |\ln a + \ln b|^p.\]

By applying Lemma 2.9 in (3.3), we get
\[
\begin{align*}
\ln u &\leq 2^{p-1}(|\ln a|^p + |\ln b|^p) \\
&= 2^{p-1}[\ln \exp \{|\ln(\ln ab)|\}]^p + \ln \exp \{|\ln(\ln b)|\}^p
\end{align*}
\]

Using the definition of geometric modulus, we have
\[
\ln u = 2^{p-1}\ln \exp \{|\ln(\ln ab)|\}^p + \ln \exp \{|\ln(\ln b)|\}^p
\]

Again using the definition of geometric exponentiation, we get
\[
\begin{align*}
\ln u &= 2^{p-1}|\ln|a_G|^p + |\ln|b_G|^p| \\
&= 2^{p-1}\ln (|a_G|^p|b_G|^p) \\
&= 2^{p-1}\ln (|a_G|^p|b_G|^p).
\end{align*}
\]

That is,
\[
u \leq e^{2^{p-1}\ln(|a_G|^p|b_G|^p)}.\]

Thus, we have
\[
|a \oplus b|_{G}^{p_G} \leq e^{2^{p-1} \circ (|a|_{G}^{p_G} \oplus |b|_{G}^{p_G})}.
\]

\[\square\]

**Theorem 3.2.** The sets \( C^G_p(\Delta^n_G) \) and \( C^G_\infty(\Delta^n_G) \) are linear subspaces of \( w(G) \).

**Proof.** As the geometric difference operator \( \Delta^n_G \) is linear, then using Lemma 3.1 it is easy to prove that the sets \( C^G_p(\Delta^n_G) \) and \( C^G_\infty(\Delta^n_G) \) are linear subspaces of \( w(G) \). \[\square\]

**Theorem 3.3.** The linear spaces \( C^G_p(\Delta^n_G) \) and \( C^G_\infty(\Delta^n_G) \) are normed spaces with respect to the norms
\[
\|x\|_p^G = \sum_{i=1}^{m}|x_i|_G \oplus \left( \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} \left( e \odot e^n \right) \odot \sum_{k=1}^{n} \Delta^n_G x_k \right) \right)_{G}^{(\frac{1}{p})_G}
\]
(3.4)
and
\[
\|x\|_\infty^G = \sum_{i=1}^{m}|x_i|_G \oplus \sup_{n} \left( e \odot e^n \right) \odot \sum_{k=1}^{n} \Delta^n_G x_k \right)_{G}
\]
(3.5)
respectively.

**Proof.** Here we prove the theorem for the space \( C^G_p(\Delta^n_G), |x|_p^G \) leaving the proof of other space as the proof runs on the parallel lines. Let \( x = (x_k), y = (y_k) \in C^G_p(\Delta^n_G) \) and \( a \in R(G) \).

We first show that \( |x|_G^G \geq 1 \) for all \( x \in C^G_p(\Delta^n_G) \). We know that geometric modulus is always greater than or equal to one, so
\[
|x_i|_G \geq 1 \text{ for all } i = 1, 2, \ldots, m
\]
and
\[
\left( e \odot e^n \right) \odot \sum_{k=1}^{n} \Delta^n_G x_k \right)_{G} \geq 1 \text{ for all } n.
\]
Using the property 2.1(x) of geometric arithmetic and taking geometric summations, we get

\[ (3.6) \quad \sum_{i=1}^{m} |x_i|_G \geq 1 \quad \text{and} \quad \left( \sum_{n=1}^{\infty} (e \otimes e^n) \otimes \sum_{k=1}^{n} \Delta_G^m x_k \right)_{G}^{pG} \geq 1. \]

That is,

\[ (3.7) \quad \|x\|_p^G = \sum_{i=1}^{m} |x_i|_G \oplus \left( \sum_{n=1}^{\infty} (e \otimes e^n) \otimes \sum_{k=1}^{n} \Delta_G^m x_k \right)_{G}^{pG} \geq 1 \]

for all \( x \in C_p^G(\Delta_G^m) \).

Next we show that \( \|x\|_p^G = 1 \iff x = (1, 1, \ldots) \). Let \( x = (1, 1, \ldots) \). Then clearly \( \|x\|_p^G = 1 \). Conversely let \( x = (x_k) \in C_p^G(\Delta_G^m) \) be such that \( \|x\|_p^G = 1 \). Then

\[ (3.8) \quad \sum_{i=1}^{m} |x_i|_G \oplus \left( \sum_{n=1}^{\infty} (e \otimes e^n) \otimes \sum_{k=1}^{n} \Delta_G^m x_k \right)_{G}^{pG} = 1. \]

Using 2.1(ii), (3.6) and (3.8), we have

\[ \sum_{i=1}^{m} |x_i|_G = 1 \quad \text{and} \quad \left( \sum_{n=1}^{\infty} (e \otimes e^n) \otimes \sum_{k=1}^{n} \Delta_G^m x_k \right)_{G}^{pG} = 1. \]

That is,

\[ |x_1|_G \cdots |x_m|_G = 1 \]

and

\[ (e \otimes e^1) \otimes \sum_{k=1}^{1} \Delta_G^m x_k \right)_{G}^{pG} \cdot \left( e \otimes e^2 \right) \otimes \sum_{k=1}^{2} \Delta_G^m x_k \right)_{G}^{pG} \cdots = 1. \]

Since the left hand side of the above equalities are products of the terms having magnitude either greater than or equal to one, we have

\[ |x_i|_G = 1 \quad \text{for all} \quad i = 1, 2, \ldots, m \quad \text{and} \quad \left( e \otimes e^n \right) \otimes \sum_{k=1}^{n} \Delta_G^m x_k \right)_{G}^{pG} = 1 \quad \text{for all} \quad n. \]

That is,

\[ (3.9) \quad x_i = 1 \quad \text{for all} \quad i = 1, 2, \ldots, m \]

\[ (3.10) \quad \text{and} \quad \sum_{k=1}^{n} \Delta_G^m x_k = 1 \quad \text{for all} \quad n. \]

Taking \( n = 1 \) in (3.10), we get

\[ (3.11) \quad \Delta_G^m x_1 = \sum_{v=0}^{m} (e)_{vG} \otimes (e(v))_G \otimes x_{1+v} = 1. \]

Putting the values of \( x_1, \ldots, x_m \) from (3.9) in (3.11), we have \( x_{m+1} = 1 \). Similarly if we take \( n = 2, 3, \ldots \) successively in (3.10), then we get \( x_{m+2} = x_{m+3} = \cdots = 1 \). Thus \( x = (x_k) = (1, 1, \ldots) \).
Next we show that \( \|a \odot x\|_p^G = |a|_G \odot \|x\|_p^G \) for all \( a \in \mathbb{R}(G) \) and \( x \in C_p^G(\Delta_m^G) \). We consider
\[
\|a \odot x\|_p^G = \left( \sum_{i=1}^{m} |a|_G \odot |x_i|_G \right) \odot \left( \sum_{n=1}^{\infty} |e \odot e^n|_G \odot \sum_{k=1}^{n} \Delta_{G}^m(a \odot x_k) \right)_{\mathbb{R}(G)}^{(1/p)_G}
\]
Since \((\mathbb{R}(G),\oplus, \odot)\) is a field and the operator \(\Delta_m^G\) is linear, we get
\[
\|a \odot x\|_p^G = \sum_{i=1}^{m} |a|_G \odot |x_i|_G \oplus \left( \sum_{n=1}^{\infty} |e \odot e^n|_G \odot \sum_{k=1}^{n} \Delta_{G}^m x_k \right)_{\mathbb{R}(G)}^{(1/p)_G}
\]
From the property of geometric modulus, we get
\[
\|a \odot x\|_p^G = \sum_{i=1}^{m} |a|_G \odot |x_i|_G \oplus \left( \sum_{n=1}^{\infty} |e \odot e^n|_G \odot \sum_{k=1}^{n} \Delta_{G}^m x_k \right)_{\mathbb{R}(G)}^{(1/p)_G}
\]
\[
= |a|_G \odot \|x\|_p^G.
\]
Finally we show geometric subadditivity in \( C_p^G(\Delta_m^G) \). Consider
\[
\|x + y\|_p^G = \sum_{i=1}^{m} |x_i + y_i|_G \oplus \left( \sum_{n=1}^{\infty} |e \odot e^n|_G \odot \sum_{k=1}^{n} \Delta_{G}^m (x_k + y_k) \right)_{\mathbb{R}(G)}^{(1/p)_G}
\]
From geometric triangular inequality and geometric Minkowski’s inequality, we have
\[
\|x + y\|_p^G \leq \left( \sum_{i=1}^{m} |x_i + y_i|_G \oplus \sum_{i=1}^{m} |y_i|_G \right) \oplus \left( \sum_{n=1}^{\infty} |e \odot e^n|_G \odot \sum_{k=1}^{n} \Delta_{G}^m x_k \right)_{\mathbb{R}(G)}^{(1/p)_G}
\]
\[
\oplus \left( \sum_{n=1}^{\infty} |e \odot e^n|_G \odot \sum_{k=1}^{n} \Delta_{G}^m y_k \right)_{\mathbb{R}(G)}^{(1/p)_G}
\]
\[
\leq \|x\|_p^G \oplus \|y\|_p^G.
\]
Thus \( C_p^G(\Delta_m^G) \) is a normed linear space. Similarly following the similar lines, one can prove that \( C_p^G(\Delta_m^G) \) is also a normed

\[\square\]

**Theorem 3.4.** The normed spaces \( C_p^G(\Delta_m^G) \) and \( C_p^G(\Delta_m^G) \) are Banach spaces with respect to the norms (3.4) and (3.5) respectively.

**Proof.** We prove the theorem for the space \( C_p^G(\Delta_m^G) \) only because the proof for the space \( C_p^G(\Delta_m^G) \) runs along the same line. Let \((x^r)\) be a Cauchy sequence in \( C_p^G(\Delta_m^G) \), where \( x^r = (x_1^r, x_2^r, \ldots) \in C_p^G(\Delta_m^G) \) for each \( r = 1, 2, \ldots \). Then,
\[
\|x^r \odot x^t\|_{\infty}^G \to 1
\]
as \( r \) and \( t \) tends to \( \infty \). That is,
\[
\sum_{i=1}^{m} |x_i^r \odot x_i^t|_G \oplus \sup_{n} |e \odot e^n|_G \odot \sum_{k=1}^{n} \Delta_{G}^m (x_k^r \odot x_k^t) \to 1
\]
as $r$ and $t$ tends to $∞$. This implies that
\[
G \sum_{i=1}^{m} |x_i^r \otimes x_i^t|_G \xrightarrow{G} 1
\]
as $r$ and $t$ tends to $∞$ because of the property 2.1(ix). Consequently,

\[
|x_i^r \otimes x_i^t|_G \xrightarrow{G} 1 \quad (3.12)
\]
for $i = 1, 2, \ldots, m$ and

\[
|e \otimes e^n|_G \xrightarrow{G} 1 \quad (3.13)
\]
for all $n \in \mathbb{N}$ when $r$ and $t$ tends to $∞$. Putting $n = 1, 2, \ldots$ in (3.13) and applying (3.12), we get

\[
|x_k^r \otimes x_k^t|_G \xrightarrow{G} 1
\]
for each $k$ when $r$ and $t$ tends to infinity. Thus, for each $k$ the sequence $(x_k^r)_{r=1}^{∞}$ is a Cauchy sequence in $R(G)$. Since $R(G)$ is complete, the sequence $(x_k^r)_{r=1}^{∞}$ converges, that is $x_k^r \xrightarrow{G} x_k$ (say) for each $k$ as $r$ tends to infinity. As $(x^r)$ is a Cauchy sequence, there exists a natural number $N$ for each $\epsilon > 1$ such that

\[
\|x^r \otimes x^t\|_G < \epsilon
\]
for all $r, t \geq N$. Hence,

\[
G \sum_{i=1}^{m} |x_i^r \otimes x_i^t|_G < \epsilon \quad (3.14)
\]
for all $r, t \geq N$. Fix $r$ and let $t$ tends to infinity in (3.14), we get

\[
|e \otimes e^n|_G \xrightarrow{G} 1 \quad (3.15)
\]
for all $r \geq N$. This shows that

\[
\|x^r \otimes x\|_G < \epsilon^2
\]
for all $r \geq N$. Thus, the sequence $(x^r)$ converges to the sequence $x = (x_k)$. Now, we need to show that the sequence $x = (x_k) \in C^G_∞(\Delta^m_G)$. For this, we consider

\[
\|e \otimes e^n|_G \xrightarrow{G} 1 \quad (3.16)
\]
\[
\|x^r \otimes x^t\|_G < \epsilon^2
\]
and let $\epsilon > 1$.

From the Inequalities (3.15) and keeping in view that the sequence $(x^N_k) \in C^G_∞(\Delta^m_G)$, we conclude that $(x_k) \in C^G_∞(\Delta^m_G)$. Therefore, the space $C^G_∞(\Delta^m_G)$ is a Banach space.
Lemma 3.5. Let \((a_k)\) be a sequence in \(R(G)\) and \(0 < p < q\), then the following inequality holds:

\[
\left( \sum_{k=1}^{n} |a_k|^{q} \right)^{\frac{1}{q}} \leq \left( \sum_{k=1}^{n} |a_k|^{p} \right)^{\frac{1}{p}}.
\]

Proof. Let

\[
u = \left( \sum_{k=1}^{n} |a_k|^{q} \right)^{\frac{1}{q}}.
\]

Converting the above equation in classical arithmetic, we get

\[
\ln \nu = \left( \sum_{k=1}^{n} \ln |a_k|^q \right)^{\frac{1}{q}}.
\]

By Lemma 2.10, we have

\[
\ln \nu \leq \left( \sum_{k=1}^{n} \ln |a_k|^p \right)^{\frac{1}{p}}.
\]

Now converting this inequality in geometric arithmetic, we get

\[
u \leq \left( \sum_{k=1}^{n} |a_k|^{p} \right)^{\frac{1}{p}}.
\]

This proves the Lemma.

We now prove some inclusion relations.

Theorem 3.6. If \(1 \leq p < q < \infty\), then the inclusion \(C_p^G(\Delta m - 1) \subset C_q^G(\Delta m)\) holds.

Proof. The proof of this theorem easily follows from Lemma 3.5. So, we omit it.

Theorem 3.7. If \(1 \leq p < \infty\), then the inclusion \(C_p^G(\Delta m - 1) \subset C_p^G(\Delta m)\) holds strictly.

Proof. Let a sequence \(x = (x_k) \in C_p^G(\Delta m - 1)\). Now we consider

\[
\left| (e \odot e^n) \odot \sum_{k=1}^{n} \Delta_{G}^{m-1} x_k \right|_G.
\]

The triangular inequality suggests that

\[
\left| (e \odot e^n) \odot \sum_{k=1}^{n} \Delta_{G}^{m-1} x_k \right|_G
\]

\[
\leq \left| (e \odot e^n) \odot \sum_{k=1}^{n} \Delta_{G}^{m-1} x_k \right|_G + \left| (e \odot e^n) \odot \sum_{k=1}^{n} \Delta_{G}^{m-1} x_{k+1} \right|_G.
\]
From Lemma 3.1, we have

\[
\left| (e \otimes e^n) \otimes \sum_{k=1}^{n} \Delta^m_G x_k \right|_G^P \leq M \left\{ \left| (e \otimes e^n) \otimes \sum_{k=1}^{n} \Delta^{m-1}_G x_k \right|_G^P \oplus \left| (e \otimes e^n) \otimes \sum_{k=1}^{n} \Delta^{m-1}_G x_{k+1} \right|_G^P \right\},
\]

where \( M = e^{2p-1} \). Taking geometric summation from \( n = 1 \) to \( s \), we get

\[
\sum_{n=1}^{s} \left| (e \otimes e^n) \otimes \sum_{k=1}^{n} \Delta^m_G x_k \right|_G^P \leq M \left\{ \sum_{n=1}^{s} \left| (e \otimes e^n) \otimes \sum_{k=1}^{n} \Delta^{m-1}_G x_k \right|_G^P \oplus \sum_{n=1}^{s} \left| (e \otimes e^n) \otimes \sum_{k=1}^{n} \Delta^{m-1}_G x_{k+1} \right|_G^P \right\}.
\]

As \( s \to \infty \), we obtain

\[
\sum_{n=1}^{\infty} \left| (e \otimes e^n) \otimes \sum_{k=1}^{n} \Delta^m_G x_k \right|_G^P \leq M \left\{ \sum_{n=1}^{\infty} \left| (e \otimes e^n) \otimes \sum_{k=1}^{n} \Delta^{m-1}_G x_k \right|_G^P \oplus \sum_{n=1}^{\infty} \left| (e \otimes e^n) \otimes \sum_{k=1}^{n} \Delta^{m-1}_G x_{k+1} \right|_G^P \right\} < \infty.
\]

Hence, the inclusion \( C^G_p(\Delta^m_G) \subset C^G_p(\Delta^{m-1}_G) \) holds. To show strictness of the inclusion, we consider the sequence \( x = (e^{k^{m-1}}) \). Then

\[
\Delta^{m-1}_G x_k = G \sum_{v=0}^{m} (-1)^v e^{(k-v)^{m-1}} \otimes e^{(k-v)^{m-1}}.
\]

Converting the above equation into classical arithmetic, we get

\[
\Delta^m_G x_k = \sum_{v=0}^{m} (-1)^v (k-v)^{m-1} = e^{m(k-1)} = e^0 = 1.
\]

Then

\[
(3.16) \quad \sum_{n=1}^{\infty} \left| (e \otimes e^n) \otimes \sum_{k=1}^{n} \Delta^m_G x_k \right|_G^P = 1 < \infty.
\]

This shows that \( x = (e^{k^{m-1}}) \in C^G_p(\Delta^m_G) \). Now we show that \( x = (e^{k^{m-1}}) \notin C^G_p(\Delta^{m-1}_G) \). Converting \( \Delta^{m-1}_G x_k \) into classical arithmetic, we get

\[
\Delta^{m-1}_G x_k = e^{(k-1)^{m-1}} = e^{(-1)^{m-1}(m-1)!}.
\]

Then

\[
\sum_{n=1}^{\infty} \left| (e \otimes e^n) \otimes \sum_{k=1}^{n} \Delta^{m-1}_G x_k \right|_G^P = \sum_{n=1}^{\infty} \left| e^{1/n} \otimes \sum_{k=1}^{n} e^{(-1)^{m-1}(m-1)!} \right|_G^P = \sum_{n=1}^{\infty} \left| e^{1/n} \otimes e^{(-1)^{m-1}(m-1)!} \right|_G^P.
\]
the sets $\upsilon C$ and $\upsilon C$. If

$$\sum_{n=1}^{\infty} \left| e_n \right| = \sup_{x} \| x \|_G = \infty.$$  

(3.17)

This implies that $x = (e_k \cdot e^{k-1}) \not\in C^G_p(\Delta^{m-1}_G)$. Thus $x = (e_k \cdot e^{k-1})$ belongs to $C^G_p(\Delta^{m-1}_G)$ but does not belong to $C^G_p(\Delta^{m-1}_G)$. Hence the inclusion is strict.

Similarly, the inclusion $C^G_{\infty}(\Delta^{m-1}_G) \subset C^G(\Delta^{m-1}_G)$ also holds strictly and strictness can be seen by considering the sequence $x = (e_k \cdot e^{k-1})$ that belongs to $C^G_{\infty}(\Delta^{m-1}_G)$ but does not belong to $C^G_{\infty}(\Delta^{m-1}_G)$.

4. Dual spaces and matrix transformations

In this section, we determine $\alpha$-dual of the space $C^G_{\infty}(\Delta^{m-1}_G)$ and study some matrix transformations. The $\alpha$- and $\beta$- duals of a sequence space $X$ in bigeometric calculus are denoted by $X^\alpha$ and $X^\beta$ and defined as

$$X^\alpha = \left\{ a = (a_k) \in w(G) : \sum_{k=1}^{\infty} |a_k \cdot x_k|_G < \infty \text{ for all } (x_k) \in X \right\},$$

and

$$X^\beta = \left\{ a = (a_k) \in w(G) : \sum_{k=1}^{\infty} a_k \cdot x_k \text{ converges for all } (x_k) \in X \right\}.$$

respectively. We note that if two spaces $X$ and $Y$ are such that $X \subseteq Y$, then $Y^\alpha \subseteq X^\alpha$ and $Y^\beta \subseteq X^\beta$. For $X = C^G_p(\Delta^{m}_G)$ or $C^G_{\infty}(\Delta^{m}_G)$, we define an operator $v : X \to X$ by $v(x) = (1, \ldots, 1, x_{m+1}, x_{m+2}, \ldots)$ for all $x = (x_k) \in X$. Consider the sets $vC^G_p(\Delta^{m}_G)$ and $vC^G_{\infty}(\Delta^{m}_G)$ as follows:

$$vC^G_p(\Delta^{m}_G) = \left\{ x = (x_k) : x \in C^G_p(\Delta^{m}_G) \text{ and } x_1 = x_2 = \cdots = x_m = 1 \right\},$$

and

$$vC^G_{\infty}(\Delta^{m}_G) = \left\{ x = (x_k) : x \in C^G_{\infty}(\Delta^{m}_G) \text{ and } x_1 = x_2 = \cdots = x_m = 1 \right\}.$$

Lemma 4.1. If $x = (x_k) \in vC^G_{\infty}(\Delta^{m}_G)$, then $\sup_{k} e^{k-1} \circ |\Delta^{m-1}_G x_k|_G < \infty$.

Proof. Let $x = (x_k) \in vC^G_{\infty}(\Delta^{m}_G)$, then

$$\sup_{n} \left| (e \circ e^{n}) \circ G \sum_{k=1}^{n} \Delta^{m}_G x_k \right|_G < \infty.$$  

(4.1)

Now, we consider

$$\sum_{k=1}^{n} \Delta^{m}_G x_k = \sum_{k=1}^{n} \Delta^{m-1}_G (x_k \oplus x_{k+1})$$

$$= \sum_{k=1}^{n} (\Delta^{m-1}_G x_k \oplus \Delta^{m-1}_G x_{k+1})$$

$$= \Delta^{m-1}_G x_1 \oplus \Delta^{m-1}_G x_{n+1}.$$
Using the property 2.1(xi), we have

Putting this value in the Inequality (4.1), we get

Using the property 2.1(xi), we have

Since \( e \circ e^n = e^{n-1} > 1 \), so we get

Now,

From (4.2) and (4.3), we get

Replacing \( n \) by \( k \), we get the result.

**Lemma 4.2.** If a sequence \( x = (x_k) \in \nu C_G^G(\Delta^m_G) \), then \( \sup_k e^{k-m} \circ |x_k|_G < \infty \).

**Proof.** Let \( x = (x_k) \in \nu C_G^G(\Delta^m_G) \), then from Lemma 4.1 \( \sup_k e^{k-1} \circ |\Delta^m_G x_k|_G < \infty \), which then by Corollary 2.15 turns out to be \( \sup_k e^{k-m} \circ |x_k|_G < \infty \).

**Theorem 4.3.** \([\nu C_G^G(\Delta^m_G)]^a = \left\{ a = (a_k) : G \sum_{k=1}^{\infty} e^{k-m} \circ |a_k|_G < \infty \right\} \).

**Proof.** We consider \( U = \left\{ a = (a_k) : G \sum_{k=1}^{\infty} e^{k-m} \circ |a_k|_G < \infty \right\} \). Let \( a = (a_k) \in U \); then for any \( x = (x_k) \in \nu C_G^G(\Delta^m_G) \), we have

which is finite because of Lemma 4.2. Since \( x = (x_k) \) is arbitrary, we conclude that \( a = (a_k) \in [\nu C_G^G(\Delta^m_G)]^a \). Hence,

(4.4) \( U \subseteq [\nu C_G^G(\Delta^m_G)]^a \).
Conversely, let \( a \in [vC_G^\infty(\Delta_G^m)]^\alpha \). Then, \( G \sum_{k=1}^\infty |a_k \otimes x_k|_G < \infty \) for each \( x = (x_k) \in vC_G^\infty(\Delta_G^m) \). Now, consider the sequence \( x = (x_k) \) that is defined by

\[
(4.5) \quad x_k = \begin{cases} 1, & k \leq m \\ e_k^m, & k > m \end{cases}
\]

The sequence given by (4.5) belongs to \( vC_G^\infty(\Delta_G^m) \). Hence, \( G \sum_{k=m+1}^\infty |a_k \otimes e_k^m|_G < \infty \). Consequently,

\[
G \sum_{k=1}^\infty e_k^m \otimes a_k |_G = G \sum_{k=1}^m e_k^m \otimes a_k |_G + G \sum_{k=m+1}^\infty e_k^m \otimes a_k |_G < \infty.
\]

This implies that \( a \in U \). Thus, (4.6) \( [vC_G^\infty(\Delta_G^m)]^\alpha \subseteq U \).

Inclusions (4.4) and (4.6) prove the theorem. \( \Box \)

**Theorem 4.4.** \( [C_G^\infty(\Delta_G^m)]^\alpha = [vC_G^\infty(\Delta_G^m)]^\alpha \).

**Proof.** Since \( vC_G^\infty(\Delta_G^m) \subseteq C_G^\infty(\Delta_G^m) \), we have \([C_G^\infty(\Delta_G^m)]^\alpha \subseteq [vC_G^\infty(\Delta_G^m)]^\alpha \). Conversely, let \( a = (a_k) \in [vC_G^\infty(\Delta_G^m)]^\alpha \), then \( G \sum_{k=1}^\infty |a_k \otimes x_k|_G < \infty \) for all \( x = (x_k) \in vC_G^\infty(\Delta_G^m) \). Now, consider any sequence \( x' = (x'_k) \in C_G^\infty(\Delta_G^m) \), then the corresponding sequence \( (1, 1, \ldots, 1, x'_{m+1}, x'_{m+2}, \ldots) \in vC_G^\infty(\Delta_G^m) \) and \( G \sum_{k=m+1}^\infty |a_k \otimes x'_k|_G < \infty \). Thus,

\[
G \sum_{k=1}^\infty |a_k \otimes x'_k|_G = G \sum_{k=1}^m |a_k \otimes x'_k|_G + G \sum_{k=m+1}^\infty |a_k \otimes x'_k|_G < \infty.
\]

for all \( x = (x'_k) \in C_G^\infty(\Delta_G^m) \). Therefore, the sequence \( a = (a_k) \in [C_G^\infty(\Delta_G^m)]^\alpha \) and \([vC_G^\infty(\Delta_G^m)]^\alpha \subseteq [C_G^\infty(\Delta_G^m)]^\alpha \). Hence, the result. \( \Box \)

Let us denote the spaces of bounded, convergent and absolutely \( p \)-summable sequences in bigeometric calculus by \( l_G^\infty \), \( e_G^p \) and \( l_p^G \) respectively. Then the next result tells us that under certain conditions on the matrix \( A \), which transforms \( l_G^\infty \) or \( e_G^p \) to \( C_p^G(\Delta_G^m) \). We state the theorem as follows:

**Theorem 4.5.** Let \( E = l_G^\infty \) or \( e_G^p \) and \( A = (a_{nk}) \) be an infinite matrix whose entries are geometric real numbers, then \( A \in (E, C_p^G(\Delta_G^m)) \) for \( 1 \leq p < \infty \) if and only if

(i) \( G \sum_{k=1}^\infty |a_{nk}|_G < \infty \) and

(ii) \( B \in (E, l_p^G) \)

hold, where \( B = (b_{nk}) = (e^{1/i} \otimes (\Delta_G^{m-1} a_{1k} \otimes \Delta_G^{m-1} a_{i+1,k})) \).

**Proof.** The sufficiency part of the theorem is trivial. To prove the necessity part, let us suppose that the matrix \( A \in (E, C_p^G(\Delta_G^m)) \) for \( 1 \leq p < \infty \). Then the series \( A_n(x) = G \sum_{k=1}^\infty a_{nk} \otimes x_k \) converges for all \( n \) and for all \( x = (x_k) \in E \) and the sequence
where the polynomials Lagrange’s form of classical Hermite polynomial is given by

\[ (A_n(x)) \in C_p^G(\Delta^m_G) \]. As the series \( G \sum_{k=1}^{\infty} a_{nk} \otimes x_k \) converges for all \( x = (x_k) \in E \), the sequence \( (a_{nk})_k \in E^G = I^G_1 \). Thus the condition (i) follows. Since the sequence \( (A_n(x)) \in C_p^G(\Delta^m_G) \), we get

\[
G \sum_{i=1}^{\infty} e^{\frac{1}{i}} \otimes G \sum_{n=1}^{i} \Delta^m_G A_n(x) \biggm|_G^{pG} = G \sum_{i=1}^{\infty} e^{\frac{1}{i}} \otimes G \sum_{n=1}^{i} (\Delta^{m-1}_G A_n(x) \otimes \Delta^{m-1}_G A_{n+1}(x)) \biggm|_G^{pG} = G \sum_{i=1}^{\infty} e^{\frac{1}{i}} \otimes G \sum_{k=1}^{\infty} (\Delta^{m-1}_G a_{1k} \otimes \Delta^{m-1}_G a_{i+1,k}) \otimes x_k \biggm|_G^{pG} = G \sum_{i=1}^{\infty} e^{\frac{1}{i}} \otimes G \sum_{k=1}^{\infty} (\Delta^{m-1}_G a_{1k} \otimes \Delta^{m-1}_G a_{i+1,k}) \otimes x_k \biggm|_G^{pG} < \infty.
\]

Now if the matrix \( B = (b_{ik}) \) is such that \( b_{ik} = \left( e^{\frac{1}{i}} \otimes (\Delta^{m-1}_G a_{1k} \otimes \Delta^{m-1}_G a_{i+1,k}) \right) \), then we have \( G \sum_{i=1}^{\infty} e^{\frac{1}{i}} \otimes G \sum_{k=1}^{\infty} b_{ik} \otimes x_k \biggm|_G^{pG} < \infty \). Hence \( B = (b_{ik}) \in (E, I^G_p) \), so the condition (ii) follows.

**Theorem 4.6.** Let \( E = I^G_\infty \) or \( e^G \) and \( A = (a_{nk}) \) be an infinite matrix whose entries are geometric real numbers, then \( A \in (E, C^G_p(\Delta^m_G)) \) if and only if

(i) \( G \sum_{k=1}^{\infty} |a_{nk}|^G < \infty \) and

(ii) \( B \in (E, I^G_\infty) \)

hold, where \( B = (b_{ik}) = \left( e^{\frac{1}{i}} \otimes (\Delta^{m-1}_G a_{1k} \otimes \Delta^{m-1}_G a_{i+1,k}) \right) \).

**Proof.** The proof of this theorem runs along the similar lines as that of the Theorem 4.5. So, we omit details of the proof. \( \square \)

5. **Bigeometric Hermite interpolation**

In this section, we study how bigeometric calculus is useful to interpolate any function that is defined in \( \mathbb{R}(G) \) and whose range lies in \( \mathbb{R}(G) \). We give an interpolating formula in bigeometric calculus analogous to the Hermite interpolating formula in classical calculus. Let the values of a function \( f \) and its derivative \( f' \) are defined at \( n+1 \) distinct points \( x_0, x_1, \ldots, x_n \) on the interval \([a, b]\); then the Lagrange’s form of classical Hermite polynomial is given by

\[
p(x) = \sum_{i=0}^{n} Q_i(x)f(x_i) + \sum_{i=0}^{n} \bar{Q}_i(x)f'(x_i),
\]

where the polynomials

\[
Q_i(x) = [1 - 2L^2_{n,i}(x)]L^2_{n,i}(x),
\]

\[
\bar{Q}(x) = (x - x_i) L^2_{n,i}(x)
\]
and the Lagrange’s polynomials $L_{n,i}$ are defined by

$$L_{n,i}(x) = \frac{(x-x_0)(x-x_1)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_0)(x_i-x_1)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)}.$$ 

We derive an equivalent interpolating polynomial in bigeometric calculus.

**Theorem 5.1.** Let $f$ be a function such that $f(x_i)$ and $D_G f(x_i)$ for $i = 0, 1, \ldots, n$ are defined at each of the points $x_0, x_1, \ldots, x_n$ in the geometric interval $[a, b]$. Then there is a unique bigeometric polynomial, $p_G$, of geometric degree at most $2n + 1$ such that $p_G(x_i) = f(x_i)$ and $D_G\{p_G(x_i)\} = D_G f(x_i)$ for each $i = 0, 1, \ldots, n$.

**Proof.** Define bigeometric polynomials $H_i(x)$ and $\hat{H}_i(x)$ of geometric degree $2n + 1$ as

$$H_i(x) = [e \circ e^2 \circ D_G\{T_{n,i}(x_i)\} \circ (x \circ x_i)] \circ T_{n,i}^{2G}(x)$$

and

$$\hat{H}_i(x) = (x \circ x_i) \circ T_{n,i}^{2G}(x),$$

where $T_{n,i}(x)$ is defined by

$$T_{n,i}(x) = \frac{(x \circ x_0) \circ (x \circ x_1) \ldots \circ (x \circ x_{i-1}) \circ (x \circ x_{i+1}) \circ \ldots \circ (x \circ x_n)}{(x_i \circ x_0) \circ (x_i \circ x_1) \circ \ldots \circ (x_i \circ x_{i-1}) \circ (x_i \circ x_{i+1}) \circ \ldots \circ (x_i \circ x_n)^G}.$$

Clearly,

$$H_i(x_j) = \begin{cases} e; & i = j \\ 1; & \text{otherwise} \end{cases}, \quad \hat{H}_i(x_j) = 1 \quad \text{for all } i \text{ and } j,$$

$$D_G\{H_i(x_j)\} = 1 \quad \text{for all } i \text{ and } j, \quad D_G\{\hat{H}_i(x_j)\} = \begin{cases} e; & i = j \\ 1; & \text{otherwise} \end{cases}.$$ 

Now we consider the polynomial

$$p_G(x) = \sum_{i=0}^{n} H_i(x) \circ f(x_i) \circ T_{n,i}^{2G}(x) \circ \sum_{i=0}^{n} \hat{H}_i(x) \circ D_G f(x_i).$$

Then, $p_G(x_i) = f(x_i)$ and $D_G\{p_G(x_i)\} = D_G f(x_i)$. This proves the existence part of the theorem. To show uniqueness of the polynomial $p_G$, if possible suppose there is another polynomial $q_G$ of geometric multiplicity at most $2n + 1$ such that $q_G \neq p_G$, $q_G(x_i) = f(x_i)$ and $D_G\{q_G(x_i)\} = D_G f(x_i)$ for $i = 0, 1, \ldots, n$. Then the polynomial $r(x) = p_G(x) \oplus q_G(x)$ will be of geometric degree at most $2n + 1$ with $r(x_i) = 1$ and $D_G r(x_i) = 1$. Thus, each $x_i$ is a geometric root of $r(x)$ with geometric multiplicity $2$. Therefore, $r(x)$ has $2n + 2$ geometric roots, whereas its geometric degree is at most $2n + 1$. This shows that $r(x) \equiv 1$. That is, both the polynomials $p_G$ and $q_G$ are equal. This proves the theorem. 

**Construction of Newton’s form of bigeometric Hermite interpolation formula:** We construct Newton’s form of bigeometric Hermite interpolation formula. Let $z_i = x_{[i/2]}$, where $[.]$ denotes the greatest integer function as follows. That is, $z_0 = x_0, z_1 = x_0, z_2 = x_1, z_3 = x_1, z_4 = x_2, z_5 = x_2$ and so on. We
define the Newton’s form of bigeometric Hermite interpolation formula as follows:

\[ p_G(x) = f(x_0) \oplus_G \sum_{k=1}^{2n+1} f_G[z_0, z_1, \ldots, z_k] \otimes_G \prod_{i=0}^{k-1} (x \ominus z_i) \]

\[ = f(x_0) \oplus f_G[x_0, x_0] \circ (x \ominus x_0) \oplus f_G[x_0, x_0, x_1] \circ (x \ominus x_0)^2 \oplus \cdots \]

\[ \oplus f_G[x_0, x_0, x_1, x_1] \circ (x \ominus x_1) \oplus \cdots \]

\[ \oplus f_G[x_0, x_0, x_1, x_1, \ldots, x_n, x_n] \circ (x \ominus x_0)^2 \circ (x \ominus x_1)^2 \circ \cdots \circ (x \ominus x_{n-1})^2 \circ (x \ominus x_n), \]

where

\[ f_G[z_i, z_{i+1}] = D_G f(z_i), \text{ when } z_i = z_{i+1} \]

\[ f_G[z_i, z_{i+1}] = \frac{f(z_{i+1}) \ominus f(z_i)}{z_{i+1} \ominus z_i} G = \left[ \frac{f(z_{i+1})}{f(z_i)} \right]^{\ln \left( \frac{z_{i+1}}{z_i} \right) \ominus z_{i+1} \ominus z_i}, \text{ when } z_i \neq z_{i+1} \]

and

\[ f_G[z_i, z_{i+1}, \ldots, z_{i+k}] = f_G[z_{i+1}, \ldots, z_{i+k}] \ominus f_G[z_i, \ldots, z_{i+k-1}] \]

for \( k \geq 2 \).

We can prove this formula by considering the polynomial

\[ p_G(x) = f(x_0) \oplus \sum_{k=1}^{2n+1} A_k \otimes_G \prod_{i=0}^{k-1} (x \ominus z_i) \]

and its bigeometric derivative

\[ D_G[p_G(x)] = \sum_{k=1}^{2n+1} A_k \otimes D_G \left\{ \prod_{i=0}^{k-1} (x \ominus z_i) \right\}, \]

where \( A_k \) for \( k = 1, 2, \ldots, 2n + 1 \) are constants to be determined. Putting the values of \( x_i \) for \( i = 0, 1, 2, \ldots, n \) in the equations (5.2) and (5.3), we get \( A_k = f_G[z_0, z_1, \ldots, z_k] \).

Next we illustrate below the construction of bigeometric Hermite interpolating polynomial by giving some examples.

**Example 5.2.** Let the values of a function \( f(x) \) and its derive \( f'(x) \) are given as shown in the table.

| \( x \) | \( e \) | \( e^2 \) |
|-------|-------|-------|
| \( f(x) \) | \( e^2 \) | \( e^4 \) |
| \( f'(x) \) | \( 2e \) | \( 2e^2 \) |

We will construct the bigeometric Hermite interpolating polynomial \( p_G(x) \). We first compute the value of \( D_G f(x) \) at the given points by using the formula \( D_G f(x) = e^x \frac{f''(x)}{f'(x)} \).

| \( x \) | \( e \) | \( e^2 \) |
|-------|-------|-------|
| \( f(x) \) | \( e^2 \) | \( e^4 \) |
| \( D_G f(x) \) | \( e^2 \) | \( e^2 \) |

The divided difference table for bigeometric Hermite interpolation is as follows:
From Newton’s form of bigeometric Hermite interpolation formula, we get
\[ p_G(x) = f(x_0) \oplus f_G[x_0, x_0] \odot (x \ominus x_0) \]
\[ = e^2 \oplus e^2 \odot (x \ominus e) \]
\[ = e^2 e^{2 \ln(x/e)} = x^2. \]

**Example 5.3.** Let the values of the function \( f(x) = \ln x \) are given as shown in the table. The data in the following table have been taken from [4].

| \( x \) | 3   | 6   | 12  | 24  |
|---------|-----|-----|-----|-----|
| \( f(x) \) | 1.0986 | 1.7918 | 2.4849 | 3.1781 |

| \( x \) | 3   | 6   | 12  | 24  |
|---------|-----|-----|-----|-----|
| \( f(x) \) | 1.0986 | 1.7918 | 2.4849 | 3.1781 |

| \( x \) | 3   | 6   | 12  | 24  |
|---------|-----|-----|-----|-----|
| \( D_G f(x) \) | 2.4849 | 1.7474 | 1.4954 | 1.3698 |

The divided difference table for bigeometric Hermite interpolation is as follows:

| \( x \) | \( f(x) \) | 1st | 2nd | 3rd | 4th | 5th | 6th | 7th |
|---------|-------------|-----|-----|-----|-----|-----|-----|-----|
| \( x_0 = 3 \) | 1.0986 |   | 2.4849 |   |   |   |   |   |
| \( x_0 = 3 \) | 1.0986 | 0.7445 |   | 2.0254 | 1.1257 |   |   |   |
| \( x_1 = 6 \) | 1.7918 | 0.8082 | 0.9613 |   | 1.7474 | 1.0658 | 1.0139 |   |
| \( x_1 = 6 \) | 1.7918 | 0.8828 | 0.9944 | 1.0150 | 1.4954 | 1.0230 | 1.0026 |   |
| \( x_2 = 12 \) | 2.4849 | 0.9048 | 0.9908 | 0.9987 | 1.6028 | 1.0362 | 1.0053 | 1.0014 |
| \( x_2 = 12 \) | 2.4849 | 0.9338 | 0.9944 |   | 1.4954 | 1.0230 | 1.0026 |   |
| \( x_3 = 24 \) | 3.1781 | 0.9435 |   | 1.4261 | 1.0150 |   |   |   |
| \( x_3 = 24 \) | 3.1781 |   |   |   |   |   |   |   |
From Newton's form of bigeometric Hermite interpolation formula, we get
\[ p_C(x) = f(x_0) \oplus f_C[x_0, x_0] \ominus (x \ominus x_0) \oplus f_C[x_0, x_0, x_1] \ominus (x \ominus x_0)^2 \oplus f_C[x_0, x_0, x_1, x_1] \]
\[ \ominus (x \ominus x_0)^3 \ominus f_C[x_0, x_0, x_1, x_1, x_2] \ominus (x \ominus x_0)^3 \ominus (x \ominus x_1)^3 \ominus f_C[x_0, x_0, x_1, x_1, x_2, x_2] \ominus (x \ominus x_0)^3 \ominus (x \ominus x_1)^3 \ominus (x \ominus x_2)^3 \ominus f_C[x_0, x_0, x_1, x_1, x_2, x_2, x_3] \ominus (x \ominus x_0)^3 \ominus (x \ominus x_1)^3 \ominus (x \ominus x_2)^3 \ominus (x \ominus x_3) \]
\[ = 1.0986 \oplus 2.4849 \ominus (x \ominus 3) \bigoplus 0.7445 \ominus (x \ominus 3)^2 \bigoplus 1.1257 \ominus (x \ominus 3)^3 \bigoplus \]
\[ (x \ominus 6) \bigoplus 0.9613 \ominus (x \ominus 3)^2 \bigoplus (x \ominus 6)^2 \bigoplus 1.0139 \ominus (x \ominus 3)^3 \bigoplus \]
\[ (x \ominus 6)^2 \bigoplus (x \ominus 12) \bigoplus 0.9959 \ominus (x \ominus 3)^3 \bigoplus (x \ominus 6)^2 \bigoplus (x \ominus 12)^2 \bigoplus \]
\[ 1.0014 \ominus (x \ominus 3)^2 \bigoplus (x \ominus 6)^2 \bigoplus (x \ominus 12)^2 \bigoplus (x \ominus 24). \]

Figure 1 shows that the graph of interpolating polynomial \( p_C(x) \) almost overlapping the graph of \( \ln(x) \) in the interval \([3, 24]\).

6. Conclusion

In this paper, we have introduced some sequence spaces in bigeometric calculus, determined their \( \alpha \)-duals and studied matrix transformations of these spaces. We have also derived an interpolating formula in bigeometric calculus and shown some related examples.

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