FEYNMAN-KAC REPRESENTATION FOR THE PARABOLIC ANDERSON MODEL DRIVEN BY FRACTIONAL NOISE

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Abstract. We consider the parabolic Anderson model driven by fractional noise:
\[ \frac{\partial}{\partial t} u(t, x) = \kappa \Delta u(t, x) + u(t, x) \frac{\partial}{\partial t} W(t, x) \quad x \in \mathbb{Z}^d, \quad t \geq 0, \]

where \( \kappa > 0 \) is a diffusion constant, \( \Delta \) is the discrete Laplacian defined by \( \Delta f(x) = \sum_{|y-x|=1} (f(y) - f(x)) \), and \( \{W(t, x) : t \geq 0\} \) is a family of independent fractional Brownian motions with Hurst parameter \( H \in (0, 1) \), indexed by \( \mathbb{Z}^d \). We make sense of this equation via a Stratonovich integration obtained by approximating the fractional Brownian motions with a family of Gaussian processes possessing absolutely continuous sample paths. We prove that the Feynman-Kac representation
\[ u(t, x) = \mathbb{E}_x \left[ u_0(X(t)) \exp \int_0^t W(ds, X(t-s)) \right], \]
is a mild solution to this problem. Here \( u_0(y) \) is the initial value at site \( y \in \mathbb{Z}^d \), \( \{X(t) : t \geq 0\} \) is a simple random walk with jump rate \( \kappa \), started at \( x \in \mathbb{Z}^d \) and independent of the family \( \{W(t, x) : t \geq 0\} \) and \( \mathbb{E} \) is expectation with respect to this random walk. We give a unified argument that works for any Hurst parameter \( H \in (0, 1) \).

1. Introduction

The parabolic Anderson model (PAM) named after the Nobel laureate physicist Philip W. Anderson, is the parabolic partial differential equation
\[ \frac{\partial}{\partial t} u(t, x) = \kappa \Delta u(t, x) + \xi(t, x) u(t, x), \quad x \in \mathbb{Z}^d, \quad t \geq 0, \]
where \( \kappa > 0 \) is a diffusion constant and \( \Delta \) is the discrete Laplacian defined by \( \Delta f(x) = \sum_{|y-x|=1} (f(y) - f(x)) \). The potential \( \{\xi(t, x)\}_{t,x} \) can be a random or deterministic field and even a Schwartz distribution.

The parabolic Anderson model which has been extensively studied, particularly in the last twenty years, has many applications and connections to problems in chemical kinetics, magnetic fields with random flow and the spectrum of random Schrödinger operators, to mention a few. The solution \( u(t, x) \) of (2) has also a population dynamics interpretation as the average number of particles at site \( x \) and time \( t \) conditioned on a realization of the medium \( \xi \), where the particles perform branching random walks in random media. In this case, the first right-hand-side term of (2) signifies the diffusion and the second term represents the birth/death of the particles. We refer to the classical work of Carmona and Molchanov [1] and the survey by Gärtner and König [2].

We consider the parabolic Anderson model with the potential \( \xi(t, x) := \frac{\partial}{\partial t} W(t, x) \) for \( x \in \mathbb{Z}^d \) and \( t \geq 0 \), where \( \{W(t, x) : t \geq 0\} \) is a family of independent fractional Brownian motions (fBM) of Hurst parameter \( H \), indexed by \( \mathbb{Z}^d \).

Key words and phrases. Feynman-Kac formula, parabolic Anderson model, stochastic heat equation, fractional Brownian motion, Malliavin calculus.

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As the paths of fBM are almost surely nowhere differentiable, this equation
doesn’t make sense a priori in the classical sense and we reformulate it in a mild
sense:

\[
\begin{aligned}
\left\{ \begin{array}{c}
\int_{0}^{t} \Delta u(s, x) \, ds + \int_{0}^{t} u(s, x) \, W(ds, x), \\
u(0, x) = u_0(x)
\end{array} \right. 
\end{aligned}
\]

where the stochastic integral is Stratonovich type in the sense that the fractional
Brownian motion is approximated by a family of smooth processes \( \{ W^\varepsilon \}_{\varepsilon > 0} \) and the integral \( \int u \, dW \) is defined by the limit of the family \( \{ \int u \, dW^\varepsilon \}_{\varepsilon > 0} \) as \( \varepsilon \) tends to zero. We assume that \( u_0(\cdot) \) is a bounded measurable function.

We will show that the following Feynman-Kac formula gives a solution to (3):

\[
\int_{0}^{t} W(ds, X(t - s)) = \mathbb{E}^x \left[ u_0(X(t)) \exp \int_{0}^{t} W(ds, X(t - s)) \right],
\]

where \( X(t) \) is a simple random walk with jump rate \( \kappa \), started at \( x \in \mathbb{Z}^d \) and independent of the family \( \{ W(t, x) : t \geq 0 \} \) and \( \mathbb{E}^x \) is expectation with respect to this random walk. Here the stochastic integral is nothing other than a summation. Indeed, suppose that \( \{ t_i \}_{i=1}^n \) are the jump times of the time-reversed random walk \( \{ X(t - s), s \in [0, t] \} \) with the additional convention \( t_0 := 0 \) and \( t_{n+1} := t \). Let also \( x_i \) for \( i = 0, \ldots, n \) be the value of \( X(t - \cdot) \) at time interval \( (t_i, t_{i+1}) \). Then we have

\[
\int_{0}^{t} W(ds, X(t - s)) = \sum_{i=0}^{n} (W(t_{i+1}, x_i) - W(t_i, x_i)).
\]

Carmona and Molchanov in their classical memoir [1] proved that for bounded \( u_0 \) and \( H = 1/2 \) i.e. standard Brownian motion, the Feynman-Kac formula [3] solves equation [3]. The asymptotic behavior of the Feynman-Kac expression [3] as the partition function of a directed polymer in a random environment has been studied in [13], but its connection with the PAM has not been investigated. The Feynman-
Kac representation for PAM on \( \mathbb{R}^d \) driven by fractional noise was established in [14] for Hurst parameters \( H \geq 1/2 \) and in [15] for \( H \geq 1/4 \). Our method is able to prove this property without any restriction on \( H \) due to the fact that in the discrete case one deals with locally constant random walk instead of Brownian motion which is only locally \( \alpha \)-Hölder continuous for \( \alpha < 1/2 \).

The paper is organized as follows:

In section 2 we collect some important background material that we will use in the
succeeding sections.

In section 3 we outline our methodology including the approximation scheme that
we apply to fractional Brownian motion. We show that the problem reduces to
demonstrating the convergence of three expressions \( u_\varepsilon, V_{1,\varepsilon} \) and \( V_{2,\varepsilon} \).

It section 4 we prove that piecewise-constant integrals with respect to the approximating processes introduced in section 3 converge to integrals with respect to fractional Brownian motion.

The remaining chapters are devoted to showing the convergence of \( u_\varepsilon, V_{1,\varepsilon} \) and \( V_{2,\varepsilon} \).

2. Preliminaries

A Gaussian random process \( W(\cdot) \) is called a fractional Brownian motion of Hurst
parameter \( H \in (0, 1) \), if it has continuous sample paths and its covariance function
is of the following form:

\[
\mathbb{E}[W(t)W(s)] = R_H(t, s) := \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H}).
\]
This process was first introduced by Kolmogorov in [3], but the term “Fractional Brownian motion” was coined by Mandelbrot and Van Ness in [3].

Let \( \{ W(t, x); t \in \mathbb{R}, x \in \mathbb{Z}^d \} \) be a family of independent fractional Brownian motions indexed by \( x \in \mathbb{Z}^d \) all with Hurst parameter \( H \).

Similar to [3], let \( \mathcal{H} \) be the Hilbert space defined by the completion of the linear span of indicator functions \( 1_{[0,t] \times \{x\}} \) for \( t \in \mathbb{R} \) and \( x \in \mathbb{Z}^d \) under the scalar product
\[
(1_{[0,t] \times \{x\}}, 1_{[0,s] \times \{y\}})_{\mathcal{H}} = R_H(t,s) \delta_{x,y},
\]
where \( \delta \) is the Kronecker delta. Here we assume the convention \( 1_{[t,0) \times \{x\}} := -1_{[0,t) \times \{x\}} \) for negative \( t \). The mapping \( W(1_{[0,t] \times \{x\}}) := W(t,x) \) can be extended to a linear isometry from \( \mathcal{H} \) onto the Gaussian Hilbert space spanned by \( \{ W(t,x): t \in \mathbb{R}, x \in \mathbb{Z}^d \} \).

Similar to [3], for any piecewise constant function \( X : \mathbb{R} \rightarrow \mathbb{Z}^d \), and every \( s \in \mathbb{R} \), \( x \in \mathbb{Z}^d \) and \( \varepsilon > 0 \) we define the following functions on \( \mathbb{R} \times \mathbb{Z}^d \):
\[
\begin{align*}
    g_{s,x}^\varepsilon (r,z) &:= \frac{1}{2\varepsilon} 1_{[s-\varepsilon,s+\varepsilon]}(r) \delta_{x,z}, \\
    g_{s,x}^X (r,z) &:= 1_{[0,\infty)}(r) \delta_{X(s-r),z} , \\
    g_{s,x}^{\varepsilon,X} (r,z) &:= \int_0^s \frac{1}{2\varepsilon} 1_{[s-\varepsilon,s+\varepsilon]}(r) \delta_{X(s-r),z} d\theta .
\end{align*}
\]

It can be easily shown that \( g_{s,x}^\varepsilon \), \( g_{s,x}^X \) and \( g_{s,x}^{\varepsilon,X} \) are all in \( \mathcal{H} \), and moreover
\[
\begin{align*}
    W(g_{s,x}^\varepsilon) &= \hat{W}_\varepsilon(s,x), \\
    W(g_{s,x}^X) &= \int_0^s W(d\theta, X(s-\theta)), \\
    W(g_{s,x}^{\varepsilon,X}) &= \int_0^s \hat{W}_\varepsilon(\theta, X(s-\theta)) d\theta,
\end{align*}
\]
where \( \hat{W}_\varepsilon(t,x) := \frac{1}{2\varepsilon} (W(t+\varepsilon,x) - W(t-\varepsilon,x)) \) for any \( t \in \mathbb{R} \) and \( x \in \mathbb{Z}^d \).

Let \( G \) be a Gaussian Hilbert space, \( H \) a Hilbert space and \( W : H \rightarrow G \) a Hilbert space isometry between \( H \) and \( G \). By a Gaussian Hilbert space we mean a set of zero-mean Gaussian random variables which is a Hilbert space with respect to covariance as its inner product [3]. Define \( S \) as the space of random variables \( F \) of the form:
\[
F = f(W(\varphi_1), \ldots, W(\varphi_n)),
\]
where \( \varphi_i \in H \) and \( f \in C^\infty(\mathbb{R}^n) \) with \( f \) and all its partial derivatives having polynomial growth. The Malliavin derivative of \( F \) denoted by \( \nabla F \), is defined (see e.g. [3] [5] [9] [11]) as the \( H \)-valued random variable given by
\[
\nabla F := \sum_{i=1}^n \frac{\partial f}{\partial \varphi_i} (W(\varphi_1), \ldots, W(\varphi_n)) \varphi_i.
\]
The operator \( \nabla \) extends to the Sobolev space \( D^{1,2} \) which is defined as the closure of \( S \) with respect to the following norm [3 5]:
\[
\|F\|_{1,2} = \sqrt{E(F^2) + E(\|\nabla F\|_H^2)}.
\]
The divergence operator \( \delta \) is the adjoint of the derivative operator \( \nabla \), determined by the duality relationship [3 5]:
\[
E(\delta(u)F) = E(\langle \nabla F, u \rangle_H) \quad \text{for every } F \in D^{1,2}.
\]
The space of $H$-valued Malliavin derivable $L^2$ random variables with $L^2$ derivatives, denoted by $\mathbb{D}^{1,2}(H)$, is contained in the domain of $\delta$, and moreover for any $u \in \mathbb{D}^{1,2}(H)$, we have

$$\mathbb{E}(\delta(u)^2) \leq \mathbb{E}(\|u\|^2_H) + \mathbb{E}(\|\nabla u\|^2_{H \otimes H}).$$

For any random variable $F \in \mathbb{D}^{1,2}$ and $\varphi \in H$ the change of variable formula \cite{3, 5}:

$$F^\prime \mathbf{W}(\varphi) = \delta(F \varphi) + \langle \nabla F, \varphi \rangle_H.$$

For more on Malliavin calculus we refer to \cite{5, 9}.

We will use the following lemma in several occasions:

**Lemma 2.1.** Let $(M, \mathcal{M}, \mu)$ be a measure space and $B, B'$ be Banach spaces. Let also $\Lambda : B \to B'$ be a continuous linear operator and $f : M \to B$ a separably-valued measurable function, i.e. there exists a separable subspace $B_1$ of $B$ such that $f \in B_1$ almost surely. If $\int \|f\|_B d\mu < \infty$ then

$$\Lambda \int f d\mu = \int \Lambda f d\mu.$$

**Proof.** As $f$ is separably-valued, there exists \cite{5, 7} a sequence of simple functions $(u_n)_n$ of the form $\sum_i 1_{A_i} h_i$ with $A_i \in \mathcal{M}$ and $h_i \in B$ with the property that

$$\int \|u_n - f\|_B d\mu \to 0 \quad as \quad n \to \infty.$$

As $\Lambda$ is linear, it commutes with integration on $(u_n)_n$. As $\Lambda$ is continuous we have $\|\Lambda(x)\|_B \leq C\|x\|_{B'}$ for some positive constant $C$, so

$$\int \|\Lambda(u_n - f)\|_{B'} d\mu \leq C \int \|(u_n - f)\|_B d\mu$$

and also

$$\|\Lambda \int (u_n - f) d\mu\|_{B'} \leq C \int \|(u_n - f)\|_B d\mu$$

$$\leq C \int \|u_n - f\|_B d\mu.$$

Hence $\Lambda$ commutes with integration for $f$ too.\qed

3. Setting

As explained in the previous section we aim to approximate the fractional Brownian motions with a family of smooth Gaussian processes. There are two obvious ways to approximate a (fractional) Brownian motion. First the so-called Wong-Zakai approximation scheme \cite{13} which is the piecewise linear approximation of (fractional) Brownian motion paths. The second natural scheme is as follows: The time derivative of a fractional Brownian motion does not exist in the classical sense but only in the distributional sense. The idea is to approximate the ‘derivative’ of the fractional Brownian motion and then integrate it. Indeed we define the approximate derivative of $W(\cdot, x)$ as $\hat{W}(\cdot, x)$

$$\hat{W}(t, x) := \frac{1}{2\varepsilon} \left( W(t + \varepsilon, x) - W(t - \varepsilon, x) \right).$$

Proposition \cite{13} shows in particular that the integral of this family of Gaussian processes converges to fractional Brownian motion.

While the first scheme doesn’t seem to be easy to work with, the second one has been proved to be very suitable in our setting where we use the Wiener space technics and Malliavin calculus \cite{3}.
Now let first replace the fBM family \( \{ W(t,x) \} \) in equation (10) by a family of absolutely continuous functions \( \{ \xi(t,x) \} \), or equivalently replace the family of fractional noises \( \{ \frac{\partial}{\partial x} W(t,x) \} \) by a family of locally integrable functions \( \{ \xi(t,x) \} \) where \( \xi(t,x) = \int_0^t \xi(s,x)ds \) for every \( t \) and \( x \). Carmona and Molchanov in [1] showed that the Feynman-Kac formula

\[
\mathcal{F}(\Xi) := \mathbb{E}^x [u_o(X(t)) \exp \int_0^t \Xi(s,X(t-s))ds] = \mathbb{E}^x [u_o(X(t)) \exp \int_0^t \xi(s,X(t-s))ds]
\]

solves the PAM driven by the potential \( \{ \xi(t,x) \} \) if this expression is finite for every \( t \) and \( x \).

If we approximate every fractional Brownian motion \( W(t,x) \) by a family of stochastic processes \( \{ W^\varepsilon(t,x) \} \) which converge to \( W(t,x) \) and with the property that every \( W^\varepsilon(t,x) \) has absolutely continuous sample paths, we expect that \( \mathcal{F}(W^\varepsilon) \) should also converge \( \mathcal{F}(W) \). On the other hand, if we denote by \( u^\varepsilon \) the solution of equation (10) with \( W \) replaced by \( W^\varepsilon \), we also expect that \( u^\varepsilon \) should converge to the solution of (3) with the integral understood in the Stratonovich sense. The reason is that for the stochastic differential equations with Brownian motion or more generally semi-martingale terms, if the Brownian motions (semi-martingales) are approximated by a family of processes with absolutely continuous sample paths, the sequence of solutions converges to the Stratonovich solution of the original differential equation [3] [10]. Note that for each sample path of an such processes, a solution in the classical sense exists.

So we consider the approximation scheme of equation (10). In the rest of the paper, without any loss of generality we will assume that \( \kappa = 1 \). We also denote by \( \mathbb{E} \) the expectation with respect to the fractional Brownian field and by \( \mathbb{E}^x \) the expectation with respect to the random walk \( X(\cdot) \).

Let

\[
u^\varepsilon(t,x) := \mathbb{E}^x [u_o(X(t)) \exp \int_0^t W^\varepsilon(s,X(t-s))ds],
\]

where \( W^\varepsilon \) is defined in (10).

By lemma [5.4] we have \( \mathbb{E}|u^\varepsilon(t,x)| < \infty \) for every \( x \) and \( t \). So almost surely, \( u^\varepsilon(t,x) \) is finite for every \( x \) and \( t \). On the other hand, the sample paths of \( W^\varepsilon \) are locally integrable. So by the above mentioned theorem of Carmona and Molchanov [1] the field \( \{ u^\varepsilon(t,x) \} \) solves the following equation

\[
\left\{ \begin{array}{l}
\frac{\partial u^\varepsilon}{\partial t} = \Delta u^\varepsilon + u^\varepsilon W^\varepsilon \\
u^\varepsilon(0,x) = u_o(x).
\end{array} \right.
\]

We aim to show that (11) gives a solution to (3) with the Stratonovich integral

\[
\int_0^t u(s,x)W(ds,x)
\]

defined in the following natural manner which was also used in [3].

**Definition 3.1.** For a random field \( u = \{ u(t,x) : t \in \mathbb{R}, x \in \mathbb{Z}^d \} \), the Stratonovich integral

\[
\int_0^t u(s,x)W(ds,x)
\]

is defined [3] as the following \( L^2 \) limit (if it exists)

\[
\lim_{\varepsilon \to 0} \int_0^t u(s,x)W^\varepsilon(s,x)ds.
\]

Using the same methodology of [3] we will show that the Stratonovich integral of the Feynman-Kac formula (11) exists and moreover it satisfies (3).
Indeed equation (12) can be integrated to
\[(13) \quad u_\varepsilon(t, x) - u_0(x) = \int_0^t \Delta u_\varepsilon(s, x) ds + \int_0^t u_\varepsilon(s, x) \dot{W}_\varepsilon(s, x) ds \, .\]

Once we show that \(u_\varepsilon\) (given by (11)) converges to \(u\) (given by (4)) in \(L^2\) sense and uniformly in \(t \in [0, T]\) as \(\varepsilon\) goes down to zero, along with equation (13), it would imply the \(L^2\)-convergence of \(\int (u_\varepsilon W - u W)\) to some random variable. If moreover one shows that \(\int (u_\varepsilon W - u W)\) converges in \(L^2\) to zero, it would imply the convergence of \(\int (u W)\) and hence the existence of the Stratonovich integral \(\int u dW\). But this means that \(u\) satisfies equation (3).

Let \(g^\varepsilon_{s,x}\) be defined as in equation (5). So we have \(W(g^\varepsilon_{s,x}) = W_\varepsilon(s, x)\) and by the change of variable formula (9) we obtain
\[
\begin{align*}
u_\varepsilon(s, x) W_\varepsilon(s, x) - u(s, x) W_\varepsilon(s, x) &= \tilde{u}_\varepsilon(s, x) W(g^\varepsilon_{s,x}) \\
&= \delta(\tilde{u}_\varepsilon(s, x) g^\varepsilon_{s,x}) + \langle \nabla \tilde{u}_\varepsilon(s, x), g^\varepsilon_{s,x}\rangle_{\mathbb{H}} ,
\end{align*}
\]
where \(\tilde{u}_\varepsilon := u_\varepsilon - u\).

Hence it suffices to show that \(V_{1,\varepsilon} := \int_0^t \delta(\tilde{u}_\varepsilon(s, x) g^\varepsilon_{s,x})\, ds\) and \(V_{2,\varepsilon} := \int_0^t \langle \nabla \tilde{u}_\varepsilon(s, x), g^\varepsilon_{s,x}\rangle_{\mathbb{H}}\, ds\) both converge to zero as \(\varepsilon\) goes to zero. In sections 5, 6 and 7 we will deal with the convergence of \(u_\varepsilon, V_{1,\varepsilon}\) and \(V_{2,\varepsilon}\).

4. Approximation rate

In this section we prove the following theorem that establishes the approximation of \(W(ds)\) by \(W_\varepsilon(s)ds\). In the proof we will use some ideas of [3] as well as simple properties of random walk.

**Proposition 4.1.** Let \(t, T, t_1, t_2, \ldots, t_N\) be some positive real numbers with \(t_0 := 0 < t_1 < \cdots < t_N < t_{N+1} := t \leq T\) and \(X(\cdot)\) a jump function on \([0, t]\) with values in \(\mathbb{Z}^d\) and jump times \(\{t_1, \ldots, t_N\}\), i.e. \(X(s) = x_i \in \mathbb{Z}^d\) for \(s \in (t_i, t_{i+1}]\). Then
\[
\mathbb{E} \left[ \int_0^t \dot{W}_\varepsilon(s, X(s))\, ds - \int_0^t W(ds, X(s)) \right]^2 \leq CN^2 \varepsilon^\min\{2H, 1\} ,
\]
where \(C\) is a constant depending only on \(T\) and \(H\) and
\[
\int_0^t W(ds, X(s)) = \sum_{i=0}^N (W(t_{i+1}, x_i) - W(t_i, x_i)) .
\]

**Proof.** First we show that for every \(t_1\) and \(t_2\), \(t_1 < t_2 \leq T\), and any fractional Brownian motion \(W(\cdot)\) with Hurst parameter \(H \in (0, 1)\) we have
\[
\mathbb{E} \left[ W(t_2) - W(t_1) - \int_{t_1}^{t_2} \dot{W}_\varepsilon(\theta)d\theta \right]^2 \leq C \varepsilon^\min\{2H, 1\} ,
\]
where \(\dot{W}_\varepsilon\) is the symmetric \(\varepsilon\)-derivative of \(W\):
\[
\dot{W}_\varepsilon(t) := \frac{1}{2\varepsilon} \left( W(t + \varepsilon) - W(t - \varepsilon) \right)
\]
and \(C\) is some positive constant depending only on \(T\) and \(H\). We have to calculate and bound
\[
\mathbb{E} \left[ W(t_2) - W(t_1) - \int_{t_1}^{t_2} \dot{W}_\varepsilon(\theta)d\theta \right]^2 = \mathbb{E} \left[ W(t_2) - W(t_1) \right]^2
\]
\[
+ \int_{t_1}^{t_2} \int_{t_1}^{t_2} \mathbb{E} \left[ \dot{W}_\varepsilon(\theta) \dot{W}_\varepsilon(\eta) \right] d\theta d\eta - 2 \int_{t_1}^{t_2} \mathbb{E} \left[ \left( W(t_2) - W(t_1) \right) \dot{W}_\varepsilon(\theta) \right] d\theta .
\]
Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be the first and second terms on the right hand side of this equation and $\mathcal{S}_3$ be the third term without its $-2$ factor. Using the following equality

$$
E \left[ (W(a) - W(b)) (W(c) - W(d)) \right] = \frac{1}{2} \left| a - d \right|^{2H} + \left| b - c \right|^{2H} - \left| a - c \right|^{2H} - \left| b - d \right|^{2H}
$$

we have:

$$
\mathcal{S}_1 = \left| t_2 - t_1 \right|^{2H},
$$

$$
\mathcal{S}_2 = \int_{t_1}^{t_2} \int_{t_1}^{t_2} \frac{1}{8\varepsilon^2} \left[ |s - \eta + 2\varepsilon|^{2H} + |\eta - s + 2\varepsilon|^{2H} - 2|s - \eta|^{2H} \right] d\eta ds
$$

and

$$
\mathcal{S}_3 = \frac{1}{4\varepsilon} \int_{t_1}^{t_2} \left[ |t_2 - \theta + \varepsilon|^{2H} + |	heta - t_1 + \varepsilon|^{2H} - |t_2 - \theta - \varepsilon|^{2H} - |	heta - t_1 - \varepsilon|^{2H} \right] d\theta.
$$

We will show that both $\mathcal{S}_2$ and $\mathcal{S}_3$ converge to $|t_2 - t_1|^{2H}$.

**Step I: Limiting behavior of $\mathcal{S}_2$**

By a change of variable we can replace the integration interval with $[0, t_2 - t_1]$ with the integrand remaining intact. But as the integrand is symmetric in $s$ and $\eta$, we may calculate the integral over a triangular surface hence getting:

$$
\mathcal{S}_2 = \frac{2}{8\varepsilon^2} \int_{0}^{t_2-t_1} \int_{0}^{s} \left[ |s - \eta + 2\varepsilon|^{2H} + |\eta - s + 2\varepsilon|^{2H} - 2|s - \eta|^{2H} \right] d\eta ds.
$$

By a change of variable of $\gamma = s - \eta$ we get:

$$\label{eq:16} \mathcal{S}_2 = \frac{1}{4\varepsilon^2} \int_{0}^{t_2-t_1} \int_{0}^{\gamma} \left[ |\gamma + 2\varepsilon|^{2H} + |\gamma - 2\varepsilon|^{2H} - 2|\gamma|^{2H} \right] d\gamma d\varepsilon.
$$

We will show that $\mathcal{S}_2$ converges to $|t_2 - t_1|^{2H}$ with the following rate of convergence for $H < \frac{1}{2}$

$$\label{eq:17} \left| \mathcal{S}_2 - |t_2 - t_1|^{2H} \right| \leq 4(2\varepsilon)^{2H}
$$

and

$$\label{eq:18} \left| \mathcal{S}_2 - |t_2 - t_1|^{2H} \right| \leq C \varepsilon
$$

for $H > \frac{1}{2}$. Here $C$ is some constant depending only on $T$ and $H$. For the simplicity of notation let $t := t_2 - t_1$. Defining $g(s) := \int_{0}^{s} |r|^{2H} dr$, \eqref{eq:15} can be written as:

$$\label{eq:19} \mathcal{S}_2 = \frac{1}{4\varepsilon^2} \int_{0}^{t} \left[ g(s + 2\varepsilon) + g(s - 2\varepsilon) - 2g(s) \right] ds.
$$

As $g'$ is continuous everywhere and $g''(r) = 2H \operatorname{sgn}(r)|r|^{2H-1}$ is continuous everywhere except for the origin when $H < \frac{1}{2}$ and everywhere when $H \geq \frac{1}{2}$, this equation can be written as:

$$\label{eq:20} \mathcal{S}_2 = \frac{1}{4} \int_{-1}^{1} \int_{-1}^{1} \int_{0}^{t} g''(s + \xi \varepsilon + \eta \varepsilon) ds d\xi d\eta.
$$

Let $\Delta := \xi \varepsilon + \eta \varepsilon$ and first suppose that $H < \frac{1}{2}$.

**Case i) $\Delta \geq 0$:**

$$
\left| \int_{0}^{t} \left( g''(s + \Delta) - 2H s^{2H-1} \right) ds \right| = 2H \int_{0}^{t} \left( s^{2H-1} - (s + \Delta)^{2H-1} \right) ds = \left[ \frac{1}{2H} - (t + \Delta)^{2H} \right] + \Delta^{2H} \leq \Delta^{2H}.
$$
Case ii) \(-t < \Delta < 0\):
\[
\int_0^t (g''(s + \Delta) - 2H s^{2H-1})ds = -2H \int_0^{-\Delta} ((-s - \Delta)^{2H-1} + s^{2H-1})ds \\
+ 2H \int_{-\Delta}^t ((s + \Delta)^{2H-1} - s^{2H-1})ds.
\]
(21)

The first term equals \(-2|\Delta|^{2H}\) and the second term equals \((t + \Delta)^{2H} - t^{2H} + \Delta^{2H}\) which is bounded by \(2|\Delta|^{2H}\).

Case iii) \(\Delta \leq -t\):
\[
\int_0^t (g''(s + \Delta) - 2H s^{2H-1})ds \\
\leq 2H \int_{-\Delta}^0 ((-s - \Delta)^{2H-1} + s^{2H-1})ds = 2|\Delta|^{2H}.
\]
(22)

Noting that \(|\Delta| < 2\varepsilon\), inequality (17) is proved.

Now we consider the case of \(H \geq \frac{1}{2}\).

Case i) \(\Delta \geq 0\):
\[
\int_0^t (g''(s + \Delta) - 2H s^{2H-1})ds = 2H \int_0^t ((s + \Delta)^{2H-1} - s^{2H-1})ds \\
= 2H \int_0^t \int_0^\Delta (2H - 1)(s + \alpha)^{2H-2}d\alpha ds \\
= 2H \int_0^\Delta ((t + \alpha)^{2H-1} - \alpha^{2H-1})d\alpha.
\]
(23)

As \(2H - 1 < 1\) we have \((t + \alpha)^{2H-1} - \alpha^{2H-1} \leq t^{2H-1}\) which shows that the above integral is bounded by \(2H t^{2H-1}|\Delta|\) and hence by \(2HT^{2H-1}|\Delta|\).

Case ii) \(-t < \Delta < 0\): Equation (21) remains valid with its first term bounded by \(2|\Delta|^{2H}\) which is smaller than \(2|\Delta|\), assuming \(|\Delta| < 1\). As \(2H - 1 > 0\), the absolute value of the second term equals:
\[
2H \int_{-\Delta}^0 ((s + \Delta)^{2H-1} - s^{2H-1})ds \\
\leq 2H \int_{-\Delta}^0 ((t + \Delta)^{2H-1} - \Delta^{2H-1})d\Delta \\
\leq 2H \int_0^\Delta ((t + \alpha)^{2H-1} - \alpha^{2H-1})d\alpha.
\]

The last inequality is true because \(2H - 1 < 1\). So we get the bound \((2 + 2HT^{2H-1})|\Delta|\).

Case iii) \(\Delta \leq -t\): Equation (22) works without any change and we get the bound \(2|\Delta|^{2H} \leq 2|\Delta|\).

Noting \(|\Delta| \leq 2\varepsilon\) the proof of inequality (18) is complete with \(C = 2^H(2 + 2HT^{2H-1})\).

In the \(H \geq \frac{1}{2}\) regime we can establish the following alternative bound which will be used in section 5
\[
|G_2 - |t_2 - t_1|^{2H}| \leq 2|t_2 - t_1|^{2H + 1}e^{2H-1}.
\]
(24)

It is shown case by case

- For case i), using the first equality in equation (23) and noting \((s+\Delta)^{2H-1} - s^{2H-1} \leq \Delta^{2H-1}\) we have the bound \(2HT\Delta^{2H-1}\).
For case ii), the second term on the right hand side in (21) can be bounded by $2H(t - |\Delta|)|\Delta|^{2H-1} \leq 2Ht|\Delta|^{2H-1}$ and the first term by $2|\Delta|^{2H} \leq 2|\Delta|^{2H-1}$.

- In case iii), using the first equality in (22) it can be bounded by $4Ht|\Delta|^{2H-1}$.

So we have the bound $2((2H + 1)|\Delta|^{2H-1} \leq 2t(2H + 1)\varepsilon^{2H-1}$.

**Step II: Limiting behavior of $\mathcal{S}_3$**

By setting $t := t_2 - t_1$ and two changes of variables, $\mathcal{S}_3$ can be written as

$$
\frac{2}{4\varepsilon} \int_0^t \left(|\theta + \varepsilon|^{2H} - |\theta - \varepsilon|^{2H}\right) d\theta = \frac{1}{2\varepsilon} \int_0^t \int_{-\varepsilon}^{\varepsilon} 2H|\theta + \alpha|^{2H-1} d\alpha d\theta.
$$

So

$$
(\mathcal{S}_3 - t^{2H}) = \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \int_0^t 2H(|\theta + \alpha|^{2H-1} - \alpha^{2H-1}) d\theta d\alpha.
$$

Let’s first assume $\varepsilon \leq t$. Let’s break this integral into three sub-integrals:

$$
\int_0^{+\varepsilon} \int_0^t \ldots + \int_{-\varepsilon}^{0} \int_0^{-\alpha} \ldots + \int_{-\varepsilon}^{0} \int_{-\alpha}^t \ldots
$$

and call them $A$, $B$ and $C$, respectively.

We bound these terms separately for $H \leq \frac{1}{2}$ and $H > \frac{1}{2}$.

First suppose $H \leq \frac{1}{2}$.

$$
|A| = \frac{1}{2\varepsilon} \int_0^{+\varepsilon} \int_0^t 2H[\theta^{2H-1} - (\theta + \alpha)^{2H-1}] d\theta d\alpha
$$

$$
= \frac{1}{2\varepsilon} \int_0^{+\varepsilon} \left[\alpha^{2H} - (\alpha + t)^{2H} + t^{2H}\right] d\alpha
$$

$$
\leq \frac{1}{2\varepsilon} \int_0^{+\varepsilon} \alpha^{2H} d\alpha = \frac{1}{2(2H + 1)} \varepsilon^{2H}.
$$

For the second term we have

$$
|B| \leq \frac{1}{2\varepsilon} \int_{-\varepsilon}^0 \int_0^{-\alpha} 2H[\theta^{2H-1} + (\theta + \alpha)^{2H-1}] d\theta d\alpha
$$

$$
= \frac{1}{\varepsilon} \int_{-\varepsilon}^0 (\alpha)^{2H} d\alpha = \frac{1}{2H + 1} \varepsilon^{2H}.
$$

Finally:

$$
|C| = \frac{1}{2\varepsilon} \int_{-\varepsilon}^0 \int_{-\alpha}^t 2H[(\theta + \alpha)^{2H-1} - \theta^{2H-1}] d\theta d\alpha
$$

$$
= \frac{1}{2\varepsilon} \int_{-\varepsilon}^0 [(t + \alpha)^{2H} - t^{2H} + (\alpha)^{2H}] d\alpha
$$

$$
\leq \frac{1}{2\varepsilon} \int_{-\varepsilon}^0 (\alpha)^{2H} d\alpha = \frac{1}{2(2H + 1)} \varepsilon^{2H}.
$$

So for $H \leq \frac{1}{2}$:

$$
|\mathcal{S}_3 - t^{2H}| \leq \frac{2}{2H + 1} \varepsilon^{2H}.
$$
Now for $H > \frac{1}{2}$, we again examine each of the terms:

$$|A| = \frac{1}{2\varepsilon} \int_0^{+\varepsilon} \int_0^t 2H \left[ (\theta + \alpha)^{2H - 1} - (\theta - \alpha)^{2H - 1} \right] d\theta \, d\alpha$$

$$= \frac{H}{\varepsilon} \int_0^{+\varepsilon} \int_0^t \int_0^\alpha (2H - 1)(\theta + \xi)^{2H - 2} d\xi \, d\theta \, d\alpha \tag{27}$$

$$= \frac{H}{\varepsilon} \int_0^{+\varepsilon} \int_0^t \int_0^\alpha [(t + \xi)^{2H - 1} - (\xi - \alpha)^{2H - 1}] d\xi \, d\theta \, d\alpha$$

$$\leq \frac{H}{\varepsilon} \int_0^{+\varepsilon} \int_0^t \int_0^\alpha (\xi - \alpha)^{2H - 1} d\xi \, d\theta \, d\alpha$$

As equation (4) remains valid for $H > \frac{1}{2}$, we have:

$$|B| \leq \frac{1}{2H + 1} \leq \frac{1}{2H + 1}.$$

For $|C|$ we use the same trick as in (27):

$$|C| = \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \int_0^t 2H \left[ (\theta + \alpha)^{2H - 1} - (\theta + \alpha)^{2H - 1} \right] d\theta \, d\alpha$$

$$= \frac{H}{\varepsilon} \int_{-\varepsilon}^{\varepsilon} \int_0^{-\alpha} \int_0^t (2H - 1)(\theta + \xi)^{2H - 2} d\xi \, d\theta \, d\alpha \tag{28}$$

$$= \frac{H}{\varepsilon} \int_{-\varepsilon}^{\varepsilon} \int_0^{-\alpha} [(t + \xi)^{2H - 1} - (\xi - \alpha)^{2H - 1}] d\xi \, d\theta \, d\alpha$$

$$\leq \frac{H}{\varepsilon} \int_{-\varepsilon}^{\varepsilon} \int_0^{-\alpha} (\xi - \alpha)^{2H - 1} d\xi \, d\theta \, d\alpha$$

Now we address the case where $\varepsilon > t$. Here we need to break the integral in (25) into three sub-integrals:

$$\int_{-\varepsilon}^{\varepsilon} \int_0^t \int_0^\alpha + \int_0^t \int_0^{-\alpha} + \int_{-\varepsilon}^{\varepsilon} \int_0^t \int_{-\alpha}^{\alpha} \cdots$$

Let’s call the terms as $A', B', C', D'$, respectively.

One can check easily that the same procedures used for bounding $A$ and $C$ work for $A'$ and $C'$. For $B'$ and $D'$ we have

$$|B'| \leq \frac{1}{2\varepsilon} \int_{-t}^t \int_0^{-\alpha} 2H \left[ (\theta + \alpha)^{2H - 1} - (\theta - \alpha)^{2H - 1} \right] d\theta \, d\alpha,$$

and

$$|D'| \leq \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \int_0^{-t} 2H \left[ (\theta + \alpha)^{2H - 1} - (\theta - \alpha)^{2H - 1} \right] d\theta \, d\alpha,$$

Hence

$$|B'| + |D'| \leq |B|.$$

So in brief the same bounds found above for $|S_3 - \varepsilon^2H|$ for the case $\varepsilon \leq t$ remain valid for the case $\varepsilon > t$ too. So inequality (13) is proved.
Now we turn back to the proof of proposition 4.1. we have:

\[
\mathbb{E}\left| \int_0^t \tilde{W}_\varepsilon(s, X(s)) ds - \int_0^t W(ds, X(s)) \right|^2 \\
\leq \mathbb{E}\{ \left( \sum_{i=0}^{N} W(t_{i+1}) - W(t_{i}) - \int_{t_i}^{t_{i+1}} \dot{W}_\varepsilon(\theta) d\theta \right)^2 \} \\
\leq C_1 (N+1)^2 \varepsilon^{\min\{2H,1\}} \leq C_2 N^2 \varepsilon^{\min\{2H,1\}} .
\]

\[\square\]

5. Convergence of \( u_\varepsilon \)

In this section, using simple random walk properties we prove that \( \tilde{u}_\varepsilon \) and its Malliavin derivative both converge to zero in \( L^2 \).

**Proposition 5.1.** \( \tilde{u}_\varepsilon := u_\varepsilon - u \) converges to 0 in \( \mathbb{D}^{1,2} \) uniformly in \([0, T] \), i.e.

\[
\sup_{s \in [0, T]} \mathbb{E}\left[ |\tilde{u}_\varepsilon(s, x)|^2 + \| \nabla \tilde{u}_\varepsilon(s, x) \|^2 \right] \rightarrow 0 \quad \text{as} \quad \varepsilon \downarrow 0 .
\]

Let \( X : [0, T] \rightarrow \mathbb{Z}^d \) be a piecewise constant function on the lattice \( \mathbb{Z}^d \) with jump times \( t_1 < t_2 < \cdots < t_N \). Let also \( t_0 := 0 \) and \( t_{N+1} := T \). For any given \( \delta > 0 \) we may chop up \([0, T]\) into calm periods and rough ones. A calm period is defined as an interval in which all the consecutive jumps are at least \( \delta \) apart, and a rough period as one in which all the consecutive jumps are at most \( \delta \) apart. We additionally require that these intervals begin with a jump and end with another.

We also define \( R \) as the number of jumps in \([0, T]\) that are within \( \delta \) distance of their previous one. In other words, \( R \) is defined to be the cardinality of \( \{ i \mid t_i - t_{i-1} < \delta, t_i \leq T \} \)

**Lemma 5.2.** Consider a Poisson process with intensity \( \lambda \) and let \( R(=R_T) \) be defined for any sample path of the Poisson process as above. Then for any given \( \delta > 0 \), we have

\[ \mathbb{P}(R \geq n) \leq (C\delta)^n , \]

where \( C \) is a constant that depends only on \( T \) and \( \lambda \).

**Proof.** Let \( A \) be the event of having at least one jump in \([0, t]\) which is within \( \delta \) of a previous one and \( B \) be the event of having at least one jump in \([0, \delta]\). Let also \( N(t) \) be the number of jumps in \([0, t]\) and \( t_0 := 0 \). We have

\[
\mathbb{P}(A \cup B) \leq \sum_{k=1}^{\infty} \mathbb{P}(t_k - t_{k-1} < \delta \text{ and } t_{k-1} < t ) \\
= \sum_{k=1}^{\infty} \mathbb{P}(t_k - t_{k-1} < \delta \mid t_{k-1} < t ) \mathbb{P}(t_{k-1} < t ) \\
= (1 - e^{-\lambda \delta}) \sum_{k=1}^{\infty} \mathbb{P}(t_{k-1} < t ) \\
= (1 - e^{-\lambda \delta}) \sum_{k=1}^{\infty} \mathbb{P}(N(t) \geq k ) \\
= (1 - e^{-\lambda \delta}) \left( \mathbb{E}(N(t)) + 1 \right) .
\]

Using the fact that the expectation of \( N(t) \) is \( \lambda t \) and noting the inequality \( 1 - e^{-\lambda \delta} \leq \lambda \delta \), we get \( \mathbb{P}(A \cup B) \leq C_t \delta \), where \( C_t = \lambda \delta (1 + t\lambda) \). In particular \( C_t \) is increasing in \( t \).
Now we define $\sigma_1$ as the first jump time that is within $\delta$ of the previous one, i.e. $\sigma_1 := \inf\{t_k > 0 : t_k - t_{k-1} < \delta\}$. Having defined $\sigma_n$ we define $\sigma_{n+1}$ as the first jump time after $\sigma_n$ that is within $\delta$ of the previous one, i.e. $\sigma_{n+1} := \inf\{t_k > \sigma_n : t_k - t_{k-1} < \delta\}$. We have

$$P(\sigma_{i+1} < T | \sigma_i) \leq \begin{cases} 0 & \text{if } \sigma_i \geq T \\ C_{T-\sigma_i} & \text{if } \sigma_i < T. \end{cases}$$

As $C_i$ is an increasing function in $t$ we have the following uniform bound:

$$P(\sigma_{i+1} < T | \sigma_i) \leq (C_T \delta)1_{\{\sigma_i < T\}}. $$

So

$$P(\sigma_{i+1} < T) = E[P(\sigma_{i+1} < T | \sigma_i)] \leq (C_T \delta)P(\sigma_i < T). $$

So by induction

$$P(\sigma_k < T) \leq (C_T \delta)^k.$$ 

Now noticing that $R \geq n$ implies $\sigma_n < T$, we get

$$P(R \geq n) \leq P(\sigma_n < T) \leq (C_T \delta)^n. $$

Lemma 5.3. For a Poisson process of intensity $\lambda$ and for any given $\delta > 0$, let $L$ be the total length of its rough periods in $[0,T]$ and $K$ be the number of rough periods in $[0,T]$. Then there exists a constant $C$ depending only on $T$ and $\lambda$ such that

$$P(K \geq n) \leq (C\delta)^n $$

and

$$P(L \geq n\delta) \leq (C\delta)^n.$$

Proof. As $L < R\delta$ and $K \leq R$, any of $L \geq n\delta$ or $K \geq n$ implies $R \geq n$. The result follows from the previous lemma. \qed

Now we are ready to prove the following lemma.

Lemma 5.4. For any $p \geq 1$, there exists $M > 0$ such that $E|u(t,x)|^p$ is bounded uniformly in $(t,x) \in (0,M] \times [0,T] \times \mathbb{Z}^d$. $E|u(t,x)|^p$ is also bounded uniformly in $(t,x) \in (0,T] \times \mathbb{Z}^d$.

Proof. First consider $E|u(t,x)|^p$.

$$E|u(t,x)|^p \leq \|u_o\|_{\mathcal{L}^p}^p \mathbb{E} \exp\left[p \int_0^t W(ds, X(t-s))\right]$$

$$= \|u_o\|_{\mathcal{L}^p}^p \mathbb{E} \exp\left(\frac{p^2}{2} \text{var}\left[\int_0^t W(ds, X(t-s))\right]\right).$$

So it is enough to find a uniform bound on $\text{var}\left[\int_0^t W(ds, X(t-s))\right]$. For any sample path $X(\cdot)$ of simple random walk on $\mathbb{Z}^d$ let $t_1 < t_2 < \cdots < t_N$ be the jump times of the reversed path $X(t-\cdot)$ and $x_1, x_2, ..., x_{N+1}$ be its values. Let also $t_0 := 0$ and $t_{N+1} := t$. We have

$$\text{var}\left[\int_0^t W(ds, X(t-s))\right] = \text{var}\left[\sum_{i=1}^{N+1} \int_{t_{i-1}}^{t_i} W(ds, x_i)\right]$$

$$= \text{var}\left[\sum_{i=1}^{N+1} W(t_i, x_i) - W(t_{i-1}, x_i)\right].$$
For $H \geq \frac{1}{2}$ we have

$$\text{var}\left[\sum_{i=1}^{N+1} W(t_i, x_i) - W(t_{i-1}, x_i)\right]$$

$$\leq (N + 1) \sum_{i=1}^{N+1} \text{var}[W(t_i, x_i) - W(t_{i-1}, x_i)]$$

$$= (N + 1) \sum_{i=1}^{N+1} (t_i - t_{i-1})^2H \leq (N + 1)^{1-2H}t^{2H}.$$  

As $N$ is a Poisson random variable, $\mathbb{E}\exp(CN)$ is finite for any constant $C$.

For $H \leq \frac{1}{2}$ we use the well-known property that disjoint increments of a fractional Brownian motion with Hurst parameter less than half are negatively correlated. So we have

$$\text{var}\left[\sum_{i=1}^{N+1} W(t_i, x_i) - W(t_{i-1}, x_i)\right] \leq \sum_{i=1}^{N+1} \text{var}[W(t_i, x_i) - W(t_{i-1}, x_i)]$$

$$= \sum_{i=1}^{N+1} (t_i - t_{i-1})^{2H} \leq (N + 1)^{1-2H}t^{2H}.$$  

In the last inequality we have used the fact that for $H \leq \frac{1}{2}$, the expression $x_2^{2H} + x_2^{2H} + \cdots + x_n^{2H}$ achieves its maximum when all $x_i$’s are equal and the maximum is hence $m^{1-2H}(\sum_{i=1}^{n} x_i^{2H})$.

Again as $N$ is Poisson, $\mathbb{E}\exp(C^{\alpha}N)$ is finite for any constants $C$ and $\alpha \leq 1$.

Now let us consider $E|u_\varepsilon(t, x)|^p$:

$$E|u_\varepsilon(t, x)|^p \leq \|u_0\|_\infty^p \mathbb{E} \mathbb{P} \exp\left[p \int_0^t \dot{W}_\varepsilon(s, X(t - s))ds\right]$$

$$(29)$$

$$= \|u_0\|_\infty^p \mathbb{E} \mathbb{P} \exp\left(\frac{p^2}{2} \text{var}[\int_0^t \dot{W}_\varepsilon(s, X(t - s))ds]\right).$$

Again we need to distinguish between $H$ larger and less than half.

When $H$ is larger than a half, $\text{var}\left(\int_{t_{i-1}}^{t_i} \dot{W}_\varepsilon(s)ds\right)$ being equal to $\mathcal{G}_2$ introduced in section 4 is bounded by $(t_2 - t_1)^{2H} + 2(t_2 - t_1)(2H + 1)\varepsilon^{2H-1}$ by inequality (24).

With the above notation

$$\text{var}\left[\int_0^t \dot{W}_\varepsilon(s, X(t - s))ds\right] = \text{var}\left[\sum_{i=1}^{N+1} \int_{t_{i-1}}^{t_i} \dot{W}_\varepsilon(s, x_i)ds\right]$$

$$\leq (N + 1) \sum_{i=1}^{N+1} \text{var}\left(\int_{t_{i-1}}^{t_i} \dot{W}_\varepsilon(s, x_i)ds\right)$$

$$\leq (N + 1) \sum_{i=1}^{N+1} \left((t_{i+1} - t_i)^{2H} + 2(t_{i+1} - t_i)(2H + 1)\varepsilon^{2H-1}\right)$$

$$\leq (N + 1) \left(t^{2H} + 2(2H + 1)\varepsilon^{2H-1}\right).$$

Again we get a multiple of $N$ and hence a finite bound.

When $H \leq \frac{1}{2}$, the situation is more complicated. Let $\{t_i\}_{i=1}^N$ be the increasingly ordered jump times of $\{X(t - s); s \in [0, t]\}$ with additional convention of $t_0 := 0$ and $t_{N+1} := t$. We decompose $[0, t]$ into calm and rough periods of $X(t - \cdot)$ with respect to $\delta = 2\varepsilon$. Let increasingly enumerate the set of indices $\{i; t_i - t_{i-1} \geq \delta\}$ as $\{t_{i_k}\}_k$. In other words, we single out and enumerate those time intervals $[t_i - 1, t_i]$
whose length is larger than or equal to \( \delta = 2\varepsilon \). It is evident that such intervals constitute the calm periods. Let also \( \{Y_k\}_k \) be the integral of \( \dot{W}_\varepsilon(\cdot, x_{i_k}) \) over the time interval \([t_{i_k-1}, t_{i_k}]\), \( i_k := \int_{t_{i_k-1}}^{t_{i_k}} W_\varepsilon(s, x_{i_k}) \, ds \). Let also \( Z \) be the sum of the integrals over all rough periods. Using equation (29), Cauchy-Schwarz and the simple inequality \( \mathbb{E}(X + Y)^2 \leq 2\mathbb{E}X^2 + 2\mathbb{E}Y^2 \), we have

\[
\mathbb{E}|u_\varepsilon(t, x)|^p \leq \|u_\varepsilon\|^p_{\infty} \mathbb{E}^\varepsilon \exp\left(\frac{\varepsilon^2}{2} \mathbb{E}(Z + \sum_k Y_k)^2\right)
\leq \|u_\varepsilon\|^p_{\infty} \left[\mathbb{E}^\varepsilon \exp\left(2p^2 \mathbb{E}(Z^2)\right)\right]^{1/2} \mathbb{E}^\varepsilon \left[2p^2 \mathbb{E}\left(\sum_k Y_k^2\right)\right]^{1/2}.
\]

Once again we will use the negativeness of the covariance of disjoint increments of a fractional Brownian motion with Hurst parameter less than half.

First we consider the integral over the rough periods, i.e. the first term above. Let \( I \) be the union of all the rough intervals in \([0, t]\).

We notice that for \( \alpha, \beta \in [0, t] \), a fractional Brownian motion \( W(\cdot) \) of Hurst parameter \( H \leq 1/2 \) we have

\[
\mathbb{E}\dot{W}_\varepsilon(\alpha)\dot{W}_\varepsilon(\beta) \leq 0 \text{ for } |\alpha - \beta| \geq 2\varepsilon,
\]

which is nothing but the negative correlation of non-overlapping increments of a fBM, and

\[
\mathbb{E}\dot{W}_\varepsilon(\alpha)\dot{W}_\varepsilon(\beta) \leq \frac{4(4\varepsilon)^{2H}}{(2\varepsilon)^2} \text{ for } |\alpha - \beta| < 2\varepsilon,
\]

which is easily followed by a simple calculation.

This shows that for \( \alpha, \beta \in [0, t] \), there are only two possibilities: either \( \dot{W}_\varepsilon(\alpha, X(t-\alpha)) \) and \( \dot{W}_\varepsilon(\beta, X(t-\beta)) \) have negative correlation or they are uncorrelated, depending on whether \( X(t-\alpha) \) is the same as \( X(t-\beta) \) or not. So we have

\[
\mathbb{E}(Z^2) = \mathbb{E}\left[\int I \dot{W}_\varepsilon(\alpha, X(t-\alpha)) \, d\alpha \int I \dot{W}_\varepsilon(\beta, X(t-\beta)) \, d\beta\right]
\leq \int \int \mathbb{E}[\dot{W}_\varepsilon(\alpha, X(t-\alpha))\dot{W}_\varepsilon(\beta, X(t-\beta))] 1_{|\alpha - \beta| < 2\varepsilon} \, d\beta \, d\alpha
\leq \frac{2\varepsilon^{2H}}{\varepsilon^2} \int \int \mathbb{E}\left[\dot{W}_\varepsilon(\alpha)\dot{W}_\varepsilon(\beta)\right] 1_{|\alpha - \beta| < 2\varepsilon} \, d\beta \, d\alpha
\leq \frac{2\varepsilon^{2H}}{\varepsilon^2} \left(4\varepsilon\right) \int \frac{1}{\alpha} \, d\alpha
\leq \frac{8\varepsilon^{2H-1}}{L},
\]

where \( L \) is the total length of rough periods, i.e. the length of \( I \).

So

\[
\mathbb{E}^\varepsilon \exp\left(2p^2 \mathbb{E}(Z^2)\right) \leq \mathbb{E}^\varepsilon \exp\left(16p^2\varepsilon^{2H} L/\varepsilon\right).
\]

As \( L/\varepsilon \) has exponential tail by lemma [5.3], the above expectation is finite for \( \varepsilon \) small enough.

For the second term, \( \mathbb{E}\left(\sum_k Y_k^2\right) \), observe that the length of each time interval \([t_{i_k-1}, t_{i_k}]\) is larger than \( 2\varepsilon \) which means the distance of every two non-neighboring such intervals is at least \( 2\varepsilon \). But this means that only consecutive \( Y_k \)'s can be positively correlated because for any two intervals \( I_1 \) and \( I_2 \) that are at least \( 2\varepsilon \) apart, the integrals \( \int_{I_1} \dot{W}_\varepsilon(s) \, ds \) and \( \int_{I_2} \dot{W}_\varepsilon(s) \, ds \) are negatively correlated which in
turn is a consequence of concavity of $H$. So
\[ E\left(\sum_k Y_k^4\right) \leq E(Y_1^2) + 2E(Y_1Y_2) + E(Y_2^2) + 2E(Y_2Y_3) + E(Y_3^2) + \ldots \]
\[ + 2E(Y_{n-1}Y_n) + E(Y_n^2) \]
\[ \leq 2E(Y_1^2) + 3E(Y_2^2) + 3E(Y_2^2) + \ldots + 3E(Y_{n-1}^2) + 2E(Y_n^2) \]
\[ \leq 3 \sum_k E(Y_k^2). \]

In the first inequality we have used the fact that for non-consecutive $Y_i$ and $Y_j$, their covariance $E(Y_iY_j)$ is negative and in the last inequality we have used $2E(XY) \leq E(X^2) + E(Y^2)$. Using equation (17) we have
\[ \text{var} \left[ \int_{t_i}^{t_{i+1}} \tilde{W}_c(s) \, ds \right] \leq (t_{i+1} - t_i)^{2H} + 4(2\varepsilon)^{2H}. \]

So noting $m \leq N$, where $N$ denotes the number of jumps in $[0,t]$ and using the fact that $x_1^{2H} + x_2^{2H} + \cdots + x_m^{2H}$ is bounded by $m^{1-2H}(\sum_i x_i)^{2H}$ for $H \leq 1/2$ which is a consequence of concavity of $(\cdot)^{2H}$, we get
\[ E\left(\sum_k Y_k^2\right) \leq 3 \sum_{k=1}^m (t_{ik} - t_{ik-1})^{2H} + 4(2\varepsilon)^{2H} \]
\[ \leq 3m^{1-2H}(\sum_{k=1}^m (t_{ik} - t_{ik-1}))^{2H} + 12m(2\varepsilon)^{2H} \]
\[ \leq 3(N+1)^{1-2H}\varepsilon^{2H} + 12(N+1)(2\varepsilon)^{2H}. \]

\[ \square \]

**Proof of proposition 5.1** We give the same argument used in [3]. Since $u_0$ is bounded, for simplicity and without any loss of generality we drop it from now on.

For $p \geq 1$ arbitrary, using the inequalities $|e^a - e^b| \leq (e^a + e^b)|a - b|$ and $(a + b)^n \leq 2^{n-1}(a^n + b^n)$ and also Hölder’s and Jensen’s inequalities we get
\[ \mathbb{E}[|u^\varepsilon(t,x) - u(t,x)|]^p \]
\[ = \mathbb{E}[\mathbb{E}[e^{W(g_{t,x}^\varepsilon})} - e^{W(g_{t,x})}]^p] \]
\[ \leq \mathbb{E}[\mathbb{E}[e^{W(g_{t,x}^\varepsilon})} - e^{W(g_{t,x})}]^p] \]
\[ \leq \mathbb{E}\left(\mathbb{E}[e^{2pW(g_{t,x}^\varepsilon})} + e^{2pW(g_{t,x})}]^{2p}\right)^{1/2} \mathbb{E}\mathbb{E}[W(g_{t,x}^\varepsilon) - W(g_{t,x})]^2]^{1/2} \]
\[ \leq C \left(\mathbb{E}[e^{2pW(g_{t,x}^\varepsilon})} + e^{2pW(g_{t,x})}]^{2p}\right)^{1/2} \mathbb{E}[W(g_{t,x}^\varepsilon) - W(g_{t,x})]^2, \]

where in the second inequality we have used the fact that for Gaussian random variables all the $n$-norms are equivalent to 2-norm.

So by applying lemma 5.4 and proposition 4.1 we obtain
\[ \sup_{t \in [0,T]} \mathbb{E}|\tilde{u}^\varepsilon(t,x)|^2 \to 0 \quad \text{as} \quad \varepsilon \downarrow 0. \]

For the convergence of $\nabla \tilde{u}^\varepsilon$, we use the fact that for a separably-valued $\mathbb{D}^{1,2}$-valued random variable $f \in \mathbb{L}^1(\mathbb{D}^{1,2})$ with $\mathbb{X}$ a probability space independent of the underlying Gaussian space of $\mathbb{D}^{1,2}$, we have $\mathbb{E}\nabla f = \nabla \mathbb{E}f$ provided that $\mathbb{E}(\|f\|_{\mathbb{D}^{1,2}}) < \infty$, where the expectations are taken with respect to $\mathbb{X}$. This follows
from lemma \[2.1\]

So we have

\[ \nabla u(t, x) = E \left[ g_{t,x}^\varepsilon \mathcal{W}(g_{t,x}^\varepsilon) \right] \]

\[ \nabla u(t, x) = E \left[ g_{t,x}^\varepsilon \mathcal{W}(g_{t,x}^\varepsilon) \right] . \]

So

\[ E \| \nabla u(t, x) - \nabla u(t, x) \|_{\mathcal{H}}^2 \]

\[ = E \left| g_{t,x}^\varepsilon \mathcal{W}(g_{t,x}^\varepsilon) - \nabla u(t, x) \|_{\mathcal{H}}^2 \right| \]

\[ \leq 2E \mathbb{E} \left( \left| g_{t,x}^\varepsilon \mathcal{W}(g_{t,x}^\varepsilon) - g_{t,x}^\varepsilon \mathcal{W}(g_{t,x}^\varepsilon) \right| \right) \]

\[ + 2E \mathbb{E} \left( \left| \nabla u(t, x) - \mathcal{W}(g_{t,x}^\varepsilon) \right| \right) . \]

If we apply the Schwartz inequality and note that \( \| g_{t,x}^\varepsilon - g_{t,x}^\varepsilon \|_{\mathcal{H}}^2 = E \left( \mathcal{W}(g_{t,x}^\varepsilon) - \mathcal{W}(g_{t,x}) \right)^2 \), along with fact that for Gaussian random variables all norms are equivalent to the 2-norm, using equation \([3]\), lemma \([5-4]\) and proposition \([1.1]\) we get

\[ \sup_{t \in [0,T]} E \| \nabla \tilde{u}_\varepsilon(t, x) \|_{\mathcal{H}}^2 \to 0 \quad \text{as} \quad \varepsilon \downarrow 0. \]

\[ \square \]

6. Convergence of \( V_{1,\varepsilon} \)

For \( V_{1,\varepsilon} \) we use basically the same proof as in \([3]\). As one can easily show that

\[ \int_0^T \| \tilde{u}_\varepsilon(s, x) g_{s,x}^\varepsilon \|_{\mathcal{H}} ds < \infty , \]

where \( \mathbb{D}^{1,2}(\mathcal{H}) \) denotes the Sobolev space of \( \mathcal{H} \)-valued \( \mathcal{L}^2 \) random variables with \( \mathcal{L}^2 \) Malliavin derivatives, we can apply lemma \([2.1]\) to get:

\[ V_{1,\varepsilon} = \delta(\psi_\varepsilon) , \]

where

\[ \psi_\varepsilon := \int_0^t \tilde{u}_\varepsilon(s, x) g_{s,x}^\varepsilon ds. \]

So using inequality \([3]\), we have

\[ E(\| V_{1,\varepsilon} \|^2) = E(\delta(\psi_\varepsilon)^2) \leq E(\| \psi_\varepsilon \|^2_\mathcal{H}) + E(\| \nabla \psi_\varepsilon \|^2_\mathcal{H} \otimes \mathcal{H}). \]

For the first right hand side term we have

\[ E(\| \psi_\varepsilon \|^2_\mathcal{H}) \]

\[ = \int_0^t \int_0^t E(\tilde{u}_\varepsilon(s_1, x) \tilde{u}_\varepsilon(s_2, x)) (g_{s_1,x}^\varepsilon, g_{s_2,x}^\varepsilon) ds_1 ds_2 \]

\[ \leq M_1 \int_0^t \int_0^t \left| E(\tilde{W}_\varepsilon(s_1, x) \tilde{W}_\varepsilon(s_2, x)) \right| ds_1 ds_2 , \]

where \( M_1 = \sup_{s \in [0,t]} E|\tilde{u}_\varepsilon(s, x)|^2 \). Here taking the integration out of the inner product is justified by once more using lemma \([2.1]\)

\[ \int_0^t \int_0^t E(\tilde{W}_\varepsilon(s_1, x) \tilde{W}_\varepsilon(s_2, x)) ds_1 ds_2 \]

being the same as the term \( \mathcal{Q}_2 \) in equation \([15]\), is uniformly upper-bounded using equations \([17]\) and \([18]\). On the other hand, \( M_1 \) goes to zero as \( \varepsilon \downarrow 0 \). So it follows that \( E(\| \psi_\varepsilon \|^2_\mathcal{H}) \) converges to zero.
For the second term, applying lemma 2.4 to the derivative operator and inner product we get
\[
\mathbb{E}(\|\nabla \psi_\varepsilon\|^{2}_{\mathcal{H} \otimes \mathcal{H}})
\]
\[
= \mathbb{E}(\langle \nabla \int_{0}^{t} \bar{u}_\varepsilon(s,1,x) g^\varepsilon_{s_1,x} ds_1, \nabla \int_{0}^{t} \bar{u}_\varepsilon(s_2,2,x) g^\varepsilon_{s_2,x} ds_2 \rangle)
\]
\[
= \mathbb{E}(\int_{0}^{t} \langle \nabla (\bar{u}_\varepsilon(s_1,1,x)) \otimes g^\varepsilon_{s_1,x}, \nabla (\bar{u}_\varepsilon(s_2,2,x)) \otimes g^\varepsilon_{s_2,x} \rangle ds_1 ds_2)
\]
\[
\leq M_2 \int_{0}^{t} \int_{0}^{t} \mathbb{E}(|\langle g^\varepsilon_{s_1,x}, g^\varepsilon_{s_2,x} \rangle|) ds_1 ds_2
\]
where \(M_2 = \sup_{s \in [0,t]} \mathbb{E}(\|\nabla \bar{u}_\varepsilon(s,x)\|^2_{\mathcal{H}}).
\)
The same argument given for the first term above shows that \(\mathbb{E}(\|\nabla \psi_\varepsilon\|^{2}_{\mathcal{H} \otimes \mathcal{H}})\) also converges to zero as \(\varepsilon\) goes down to zero.

7. Convergence of \(V_{2,\varepsilon}\)

Establishing the convergence of \(V_{2,\varepsilon}\) is more involved. First applying lemma 2.4 to \(u\) and \(u_\varepsilon\) for the derivative operator we get
\[
\nabla u_\varepsilon(s,x) = \mathbb{E}^x[u_0(X(s)) e^{W(g^\varepsilon_{s_1,x})} g^\varepsilon_{s_2,x}]
\]
and
\[
\nabla u(s,x) = \mathbb{E}^x[u_0(X(s)) e^{W(g_{s_1,x})} g_{s_2,x}].
\]
Let
\[
A^X(s,x) := u_0(X(t)) e^{W(g_{s_2,x})}
\]
and
\[
A^\varepsilon(X)(s,x) := u_0(X(s)) e^{W(g^\varepsilon_{s_2,x})}.
\]
Hence we have
\[
V_{2,\varepsilon} = \int_{0}^{t} \langle \nabla u_\varepsilon(s,x) - \nabla u(s,x), g^\varepsilon_{s_2,x} \rangle ds
\]
\[
= \int_{0}^{t} \mathbb{E}^x[(A^X(s,x) g_{s_2,x} - A^\varepsilon(X)(s,x) g^\varepsilon_{s_2,x}, g^\varepsilon_{s_2,x})] ds
\]
\[
= \int_{0}^{t} \mathbb{E}^x[(A^X - A^\varepsilon(X)) g^\varepsilon_{s_2,x}, g^\varepsilon_{s_2,x}] + \mathbb{E}^x[(A^X g^\varepsilon_{s_2,x} - A^\varepsilon(X) g_{s_2,x})] ds
\]
\[
= \int_{0}^{t} \mathbb{E}^x[(A^X - A^\varepsilon(X))(g^\varepsilon_{s_2,x}, g^\varepsilon_{s_2,x})] ds.
\]
Let
\[
P_{1,\varepsilon} := \int_{0}^{t} \mathbb{E}^x[(A^X - A^\varepsilon(X))(g^\varepsilon_{s_2,x}, g^\varepsilon_{s_2,x})] ds
\]
and
\[
P_{2,\varepsilon} := \int_{0}^{t} \mathbb{E}^x[A^X(g^\varepsilon_{s_2,x} - g_{s_2,x})] ds.
\]
So we will show in two steps that each of these terms converge to zero in $\mathcal{L}^2$.

**Step 1: Convergence of $P_{1,\varepsilon}$.** For the first term, using Hölder inequality for $1/p + 1/q = 1$, we have

$$
\mathbb{E}^x |(A^X - A^{\varepsilon,X})(g^{\varepsilon,X}, g^x)| \leq (\mathbb{E}^x |AX - A^{\varepsilon,X}|^q)^{1/q} (\mathbb{E}^x |(g^{\varepsilon,X}, g^x)|^p)^{1/p}.
$$

In fact equation (30) also proves that for any $p \geq 1$

$$
\sup_{s \in [0,t]} \mathbb{E}^x |A^X(s, x) - A^{\varepsilon,X}(s, x)|^p \longrightarrow 0 \quad \text{as} \quad \varepsilon \downarrow 0.
$$

So if we can show that $\mathbb{E}^x |(g^{\varepsilon,X}, g^x)|^p$ is bounded by some constant which depends only on $H$ and $t$ we are done because then

$$
\mathbb{E} \left( \int_0^t \mathbb{E}^x \left( (A^X - A^{\varepsilon,X})(g^{\varepsilon,X}, g^x) \right) \, ds \right)^2
\leq \mathbb{E} \left( \int_0^t (\mathbb{E}^x |A^X - A^{\varepsilon,X}|^q)^{1/q} (\mathbb{E}^x |(g^{\varepsilon,X}, g^x)|^p)^{1/p} \, ds \right)^2
\leq \int_0^t \mathbb{E} (\mathbb{E}^x |A^X - A^{\varepsilon,X}|^q)^{2/q} \, ds,
$$

where $\leq$ means less than up to a constant. So either $q > 2$, where we get $\int_0^t (\mathbb{E}^x |A^X - A^{\varepsilon,X}|^q)^{2/q} \, ds$ as an upper bound or $q \leq 2$, where we get the upper bound $\int_0^t \mathbb{E}^x |A^X - A^{\varepsilon,X}|^2 \, ds$.

Let $\{t_i\}_{i=1}^n$ be the jump times of the path $X(\cdot)$ up to time $s$, $t_0 := 0$ and $t_n := s$. Let then $J$ be the set of indices $j$ for which $X(\cdot)$ stays at site $x$ in the time interval $[t_j, t_{j+1}]$. Now applying the definitions (30)-(7) we get

$$
\langle g^{\varepsilon,X}, g^x \rangle = \sum_{i \in J} \int_{s-t_i}^{s-t_{i+1}} \frac{1}{2}\epsilon \int_{\theta - \varepsilon, \theta + \varepsilon} 1_{[\theta - \varepsilon, \theta + \varepsilon]}(\theta) \, d\theta
\leq \frac{1}{2\epsilon} \int_{s-t_i}^{s-t_{i+1}} 1_{[s-\varepsilon, s+\varepsilon]}(\theta) \, d\theta
= \frac{1}{2\epsilon} \sum_{i \in J} \int_{s-t_i}^{s-t_{i+1}} \mathbb{E} [(W_{\theta+\varepsilon} - W_{\theta-\varepsilon})(W_{s+\varepsilon} - W_{s-\varepsilon})] \, d\theta
= \frac{1}{2\epsilon} \sum_{i \in J} \int_{s-t_i}^{s-t_{i+1}} [(\gamma + 2\epsilon)^{2H} + |\gamma - 2\epsilon|^{2H} - 2\gamma^{2H}] \, d\gamma,
$$

where $\{W_t\}_t$ is a fractional Brownian motion of the same Hurst parameter $H$. We split this expression into two terms

$$
\Gamma_1 := \frac{1}{2\epsilon^2} \int_0^{t_1} [(\gamma + 2\epsilon)^{2H} + |\gamma - 2\epsilon|^{2H} - 2\gamma^{2H}] \, d\gamma,
$$

and

$$
\Gamma_2 := \frac{1}{2\epsilon^2} \sum_{i \in J, j \geq 2} \int_{t_i}^{t_{i+1}} [(\gamma + 2\epsilon)^{2H} + |\gamma - 2\epsilon|^{2H} - 2\gamma^{2H}] \, d\gamma.
$$

For the first term, using the same reasoning as in (19) and (20), we have

$$
\Gamma_1 = \frac{1}{8} \int_{-1}^1 \int_{-1}^1 f''(t_1 + \xi \varepsilon + \eta \varepsilon) \, d\xi \, d\eta,
$$

where $f(s) := \int_0^s |r|^{2H} \, dr$ and hence $f''(r) = 2H \text{sgn}(r) |r|^{2H-1}$. Letting $\Delta := \xi \varepsilon + \eta \varepsilon$ and noting that $t_1$ is exponentially distributed, we have

$$
\mathbb{E}^x |f''(t_1 + \Delta)|^p \leq 2H \int_0^{t_1} |t_1 + \Delta|^{(2H-1)p} \, dt_1.
$$
As we can restrict ourselves to \( \varepsilon \leq 1 \) and hence \( |\Delta| \leq 1 \) and as \( 0 < s < t \), we have
\[
\int_0^s |t_1 + \Delta|^{(2H-1)p} dt_1 \leq \int_{-1}^{t+1} |t_1|^{(2H-1)p} dt_1.
\]
So if we choose \( p > 1 \) such that \( (2H-1)p > -1 \), we get a finite bound on \( \mathbb{E}^x |f''(t_1 + \Delta)|^p \) and hence a bound on \( \mathbb{E}^x \Gamma_1^p \) that only depends on \( t \) and \( H \).

Now for the second term, \( \Gamma_2 \), let
\[
f^\varepsilon (\gamma) := \frac{1}{4\varepsilon^2} [(\gamma + 2\varepsilon)^{2H} + |\gamma - 2\varepsilon|^{2H} - 2\gamma^{2H}].
\]
We have \( |f^\varepsilon (\gamma)| \leq 18\gamma^{2H-2} \) because either \( \gamma \leq 4\varepsilon \) which implies that \( |\gamma - 2\varepsilon|^{2H} \leq (2\varepsilon)^{2H} \) and \( (\gamma + 2\varepsilon)^{2H} \leq (6\varepsilon)^{2H} \) and hence \( |f^\varepsilon (\gamma)| \leq 18\gamma^{2H-2} \) or \( \gamma > 4\varepsilon \) in which case we may write \( f^\varepsilon (\gamma) \) as the following
\[
f^\varepsilon (\gamma) = \frac{1}{4} \int_{-1}^{1} \int_{-1}^{1} 2H(2H - 1)(\gamma + \xi \varepsilon + \eta \varepsilon)^{2H-2} d\xi d\eta.
\]
Letting again \( \Delta := \xi \varepsilon + \eta \varepsilon \), we have \( |\Delta| \leq 2\varepsilon \) and so
\[
(\gamma + \Delta)^{2H-2} \leq \gamma^{2H-2}(1 + \Delta/\gamma)^{2H-2} \leq 2^{2H-2} \gamma^{2H-2},
\]
which gives \( |f^\varepsilon (\gamma)| \leq 8\gamma^{2H-2} \).
So we have
\[
\Gamma_2 \leq \int_{-1}^{s} |f^\varepsilon (\gamma)|d\gamma \leq \int_{-1}^{s} \gamma^{2H-2} d\gamma.
\]
So \( \Gamma_2 \) is bounded (up to a constant) by either \( t_1^{2H-1} \) for \( H < \frac{1}{2} \), or \( s^{2H-1} \) for \( H > \frac{1}{2} \). The case \( H = \frac{1}{2} \) can also be treated easily using the inequality \( \ln(x) \leq x^\alpha \) for any \( \alpha \) positive. So as \( (2H-1)p > -1 \), \( \mathbb{E}^x \Gamma_2^p \) can be bounded by a constant only dependant on \( t \) and \( H \). So this completes the proof showing that \( \mathbb{E}^x |(g^{\varepsilon,X}, g^\varepsilon)|^p \leq C \), for some \( p > 1 \) and \( C \) a constant only dependant on \( t \) and \( H \).

**Step II: Convergence of \( P_{2,\varepsilon} \).** For establishing the convergence of \( P_{2,\varepsilon} \) we will use the dominated convergence theorem.

In ‘step I’ we showed that
\[
\langle g^{\varepsilon,X}, g^\varepsilon \rangle = \frac{1}{2} \sum_{i \in J} \int_{t_i}^{t_{i+1}} f^\varepsilon (r) dr,
\]
where \( f^\varepsilon \) is defined in (33).

Now let \( \{t_i\}_{i=0}^{n+1} \) and \( J \) be as in ‘step I’, i.e. \( \{t_i\}_{i=1}^{n} \) be the jump times of the path \( X(\cdot) \) up to time \( s \), \( t_0 := 0 \) and \( t_n := s \) and \( J \) the set of indices \( j \) for which \( X(\cdot) \) stays at site \( x \) in the time interval \([t_j, t_{j+1}]\). So we have
\[
\langle g^{X}, g^{\varepsilon} \rangle = \sum_{i \in J} \langle 1_{[s-t_i, s-t_i]}(r) \delta_x (s-r) \rangle + \frac{1}{2\varepsilon} 1_{[s-\varepsilon, s+\varepsilon]}(r) \delta_x (z) \rangle
\]
\[
= \sum_{i \in J} \langle 1_{[s-t_{i+1}, s-t_i]}(r) \delta_x (s-r) \rangle + \frac{1}{2\varepsilon} 1_{[s-\varepsilon, s+\varepsilon]}(r) \delta_x (z) \rangle
\]
\[
= \frac{1}{4\varepsilon} \left[ |t_{i+1} + \varepsilon|^{2H} - |t_i + \varepsilon|^{2H} + |t_i - \varepsilon|^{2H} - |t_{i+1} - \varepsilon|^{2H} \right]
\]
\[
= \frac{1}{4\varepsilon} \left[ |t_1 + \varepsilon|^{2H} - |t_1 - \varepsilon|^{2H} \right] + \frac{1}{2} \sum_{i \in J, i > 1} \int_{t_i}^{t_{i+1}} h^\varepsilon (r) dr,
\]
where
\[
h^\varepsilon (r) := \frac{2H}{2\varepsilon} \left[ |r + \varepsilon|^{2H-1} - \text{sgn}(r-\varepsilon)|r-\varepsilon|^{2H-1} \right].
\]
We will show that \((g^X, g^\varepsilon) - (g^{\varepsilon,X}, g^\varepsilon)\) converges to zero. For doing so we shall show that 

\[
\frac{1}{t_i} \left( |t_1 + \varepsilon|^{2H} - |t_1 - \varepsilon|^{2H} \right) - \frac{1}{t_i} \int_0^{t_i} f^\varepsilon(r) \, dr
\]

converges to zero and that every \(\int_{t_{i+1}}^{t_i} (h^\varepsilon - f^\varepsilon)(r) \, dr\) also converges to zero.

By equations (31) and (32), we have

\[
\int_0^{t_i} f^\varepsilon(r) \, dr = \frac{1}{4} \int_{-1}^{1} \int_{-1}^{1} 2H \text{sgn}(\rho + \varepsilon) |\rho + \varepsilon + \varepsilon|^{2H-1} \, d\rho \, d\varepsilon.
\]

So for a fixed positive \(t_1\) this converges to \(2Ht_1^{2H-1}\). On the other hand \(\frac{1}{t_i} \left( |t_1 + \varepsilon|^{2H} - |t_1 - \varepsilon|^{2H} \right)\) also converges to \(\frac{1}{2} 2Ht_1^{2H-1}\).

For \(\int_{t_{i+1}}^{t_i} (h^\varepsilon - f^\varepsilon)(r) \, dr\), we will show that \(h^\varepsilon - f^\varepsilon\) converges to zero and then apply the dominated convergence theorem. Using (36) it can be easily shown that

\[
\lim_{\varepsilon \to 0} f^\varepsilon(r) = 2H(2H - 1)r^{2H-2}.
\]

By simply recognizing the definition of derivative we have

\[
\lim_{\varepsilon \to 0} h^\varepsilon(r) = 2H(2H - 1)r^{2H-2}.
\]

So it remains to find an integrable \(\varepsilon\)-independent upper bound. As shown in the paragraph following (33), \(f^\varepsilon(r)\) is bounded by \(18s^{2H-2}\) and for \(f^\varepsilon(r)\), restricting \(\varepsilon\) to be less than \(t_1/2\), where \(t_1\) is the first index in \(J\) after 1, we have for all \(r \geq t_1\)

\[
h^\varepsilon(r) = \frac{1}{2} 2H(2H - 1) \int_{-1}^{1} |\rho + \varepsilon|^{2H-2} \, d\rho.
\]

But then as \(|\rho + \varepsilon|^{2H-2} \leq \left(\frac{\varepsilon}{2}\right)^{2H-2}\) it gives \(8\varepsilon^{2H-2}\) as an upper bound on \(h^\varepsilon\). This completes the proof for convergence to zero of \((g^X, g^\varepsilon) - (g^{\varepsilon,X}, g^\varepsilon)\).

Now, for applying the dominated convergence theorem to \(P_{2,2}\) we only need to find an \(\varepsilon\)-independent upper bound \(G\) on \((g^X, g^\varepsilon) - (g^{\varepsilon,X}, g^\varepsilon)\) having the property that 
\(E(\int_0^1 E^\varepsilon(G))^2 < \infty\). For \((g^{\varepsilon,X}) - (g^{\varepsilon,X}, g^\varepsilon)\) such an upper bound has been established in step 1 above. It remains to find an upper bound on \((g^X, g^\varepsilon)\).

For \(2H - 1 \geq 0\) the situation is quite trivial because using equation (36) we easily get

\[
(g^X, g^\varepsilon) = \frac{1}{2} \sum_{\varepsilon \in J} \int_{t_1}^{t_{i+1}} h^\varepsilon(r) \, dr.
\]

When \(2H - 1 \geq 0\), equation (36) remains valid for any value of \(\varepsilon\) and \(r\). As for any \(\varepsilon \leq 1\) we have

\[
\int_{-1}^{1} |\rho + \varepsilon|^{2H-2} \, d\rho \leq \int_{-1}^{1} |\rho|^{2H-2} \, d\rho,
\]

hence we get an upper bound dependant only on \(t\) and \(H\).

So we consider now the case of \(2H - 1 < 0\). For \(2H < 1\) and any \(r > 0\) we have

\[
\rho(r) := \frac{1}{4\varepsilon} (|r + \varepsilon|^{2H} - |r - \varepsilon|^{2H}) \leq 2r^{2H-1}.
\]

This is true because either \(r \leq 2\varepsilon\) in which case

\[
\rho(r) \leq \frac{1}{4\varepsilon} ((3\varepsilon)^{2H} - \varepsilon^{2H}) \leq \varepsilon^{2H-1} \leq 2r^{2H-1},
\]
or $r > 2\varepsilon$, where we have

$$
\rho(r) \leq \frac{1}{4} \int_{-1}^{1} 2H (r + \varepsilon u)^{2H-1} \, dr
$$

$$
\leq \frac{1}{4} \int_{-1}^{1} \left( \frac{1}{2} \right)^{2H-1} \, dr \leq r^{2H-1}.
$$

So by (35) we have

$$
|\langle g^X, g^\varepsilon \rangle| \leq 2 \sum_{i \in J}(t_i^{2H-1} + t_{i+1}^{2H-1}) \leq 2Nt_i^{2H-1},
$$

where $N$ is the number of jumps in $[0, t]$.

Applying the Hölder inequality with

$$
\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1
$$

we have

$$
E^x |A^X(g^X, g^\varepsilon)| \leq (E^x |A^X|^q)^{1/q} (E^x N^r)^{1/r} (E^x t_i^{(2H-1)p})^{1/p}.
$$

So we just need to pick a $p > 1$ with $(2H-1)p+1 > 0$, in which case the exponential distribution of $t_i$ implies

$$
E^x t_i^{(2H-1)p} \leq \int_0^s t_i^{(2H-1)p} \, dt_i = s^{(2H-1)p+1} \leq t^{(2H-1)p+1}.
$$

In fact the proof of lemma [5.4] also shows that for any $q \geq 1$, $E^x |A^X|^q$ is uniformly bounded in $0 \leq s \leq t$. As $N$ has a Poisson distribution $E^x N^r$ is also finite.

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