JACOB'S LADDERS, FACTORIZATION AND METAMORPHOSES
AS AN APPENDIX TO THE RIEMANN FUNCTIONAL
EQUATION FOR \( \zeta(s) \) ON THE CRITICAL LINE

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Abstract. In this paper we obtain a new set of metamorphoses of the oscillating Q-system by using the Euler’s integral. We split the final state of mentioned metamorphoses into three distinct parts: the signal, the noise and finally appropriate error term. We have also proved that the set of distinct metamorphoses of that class is infinite one.

1. Introduction and the first result

1.1. Let us remind the Riemann’s functional equation (1859)
\[
\zeta(1-s) = 2(2\pi)^s \cos \frac{\pi s}{2} \Gamma(s) \zeta(s), \quad s \in \mathbb{C} \setminus \{1\}.
\]

Remark 1. It is known that L. Euler discovered a formula equivalent to (1.1) for real values of the variable \( s \) in 1749, (comp. [2], pp. 23-26), however the proof in the Euler’s work is missing.

Next, if we put
\[
\chi(s) = \pi^{s-1/2} \frac{\Gamma\left(\frac{1}{2} - \frac{s}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \Rightarrow \chi(s)\chi(1-s) = 1,
\]
(1.2) (comp. [9], pp. 13, 16) then we obtain (see (1.1), (1.2)) that
\[
\frac{\zeta(1-s)}{\zeta(s)} = \chi(s).
\]

Remark 2. Now, in connection with (1.3) we use the following (E. Landau, [3], p. 30): the quotient
\[
\frac{\zeta(1-s)}{\zeta(s)}
\]
is expressed by the known function . . . (of course, known function is such one that is different from \( \zeta(s) \)).

1.2. However, in the case
\[
s = \frac{1}{2} + it, \quad 1-s = \frac{1}{2} - it = \left(\frac{1}{2} + it\right)^{\ast}
\]
we have (comp. (1.2), (1.4)) that
\[
\left|\chi\left(\frac{1}{2} + it\right)\right| = 1,
\]

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i.e. (see (1.3))

\[
\left| \frac{\zeta \left( \frac{1}{2} + it \right)}{\zeta \left( \frac{1}{2} + i\gamma \right)} \right| = 1, \ \forall t \neq \gamma : \ \zeta \left( \frac{1}{2} + i\gamma \right) = 0,
\]

(after continuation for \( t = \gamma \) this is valid for all \( t \in \mathbb{R} \)).

**Remark 3.** Consequently, on the critical line

\[
s = \frac{1}{2} + it
\]

the Riemann’s functional equation (1.3) gives only the trivial result (1.5) (of course, the main aim of the Riemann’s functional equation is the analytic continuation of \( \zeta(s) \) to \( \mathbb{C} \setminus \{1\} \)).

1.3. It is clear that in this situation our Remark 2 leads us to the following **Question.** about another method of sampling of the points

\[
t_2 > t_1 > 0
\]

(say) from a corresponding set that the quotient (comp. (1.3), (1.5))

\[
\left| \frac{\zeta \left( \frac{1}{2} + it_2 \right)}{\zeta \left( \frac{1}{2} + it_2 \right)} \right|, \ t_1, t_2 \neq \gamma
\]

is expressed by a known function.

There is method giving an answer to the Question – namely one answer from a set of many possibilities – mentioned method is our method of transformation by using the reversely iterated integrals (comp. [7], (4.1) – (4.19)).

1.4. In this paper we obtain, for example, the following **Formula1.** For every sufficiently big \( L \in \mathbb{N} \) and for every \( U \in (0, \pi) \) there are functions

\[
\alpha_0^4 = \alpha_0^4(L, U; a, b), \quad \alpha_1^4 = \alpha_1^4(L, U; a, b), \quad \beta_1^4 = \beta_1^4(L, U),
\]

(1.8)

such that

\[
\left| \frac{\zeta \left( \frac{1}{2} + i\alpha_1^4 \right)}{\zeta \left( \frac{1}{2} + i\beta_1^4 \right)} \right| \sim \arctan \left( \frac{\pi + \tan \left( \frac{L}{2} \right)}{\sqrt{\frac{a+b}{a+b} U}} \right) \frac{a + b \cos(\alpha_0^4)}{a + b}, \ L \to \infty,
\]

(1.9)

where

\[
\alpha_0^4 \in (2\pi L, 2\pi L + U), \quad \alpha_1^4, \beta_1^4 \in (2\pi L, 2\pi L + U),
\]

(1.10)

\[
(2\pi L, 2\pi L + U) \sim (2\pi L, 2\pi L + U),
\]
and the functions \((1.8)\) have the following property

\[\alpha_4^4 - \alpha_0^4 \sim (1 - c)\pi(2\pi L), \quad L \to \infty,\]

where \(c\) is the Euler’s constant and \(\pi(t)\) is the prime-counting function.

**Remark 4.** The formula \((1.9)\) gives one answer to our Question (comp. \((1.7), (1.10)\)) in the direction outlined in Remark 2.

**Remark 5.** Let us notice explicitly that in the case \((1.7)\), i.e. on the critical line, we may suppose that the known function is every function, for example

\[f(|\zeta|), \quad \arg \zeta, \ldots\]

In this direction, we have obtained the following (see also corresponding formulae in the papers \([6] - [8], k = 1\) )

**Formula 2.**

\[\left|\frac{\zeta \left( \frac{1}{2} + i\beta_0^1 \right)}{\zeta \left( \frac{1}{2} + i\beta_1^1 \right)}\right| \sim \sqrt[4]{\zeta(2\sigma)} \left|\arg \zeta \left( \frac{1}{2} + it \right)\right|^{-l}, \quad \sigma \geq 1.\]

**Formula 3.**

\[\left|\frac{\zeta \left( \frac{1}{2} + i\beta_3^1 \right)}{\zeta \left( \frac{1}{2} + i\beta_1^1 \right)}\right| \sim \pi^l \sqrt[4]{\zeta(2\sigma)} \left|\int_0^{\alpha_0^1(T)} \arg \zeta \left( \frac{1}{2} + it \right) dt\right|^{-l}.\]

**Formula 4.**

\[\left|\frac{\zeta \left( \frac{1}{2} + i\alpha_1^4 \right)}{\zeta \left( \frac{1}{2} + i\alpha_0^4 \right)}\right| \sim \frac{\sqrt{\zeta(2\sigma)H}}{\sqrt{|\zeta \left( \frac{1}{2} + i\alpha_0^4 \right)|}}.\]

**Remark 6.** Of course, the results \((1.9), (1.11) - (1.13)\) are based on properties of the Jacob’s ladder \(\varphi_1(t)\) as follows:

\[\alpha_0^4 = \varphi_1^4(d) = \varphi_1(d) \in (2\pi L, 2\pi L + U),\]

\[\alpha_1^4 = \varphi_0^4(d) = d \in (2\pi L, 2\pi L + U),\]

\[\beta_1^4 = \varphi_0^4(e) = e \in (2\pi L, 2\pi L + U),\]

say (comp. \((1.14)\) and \((1.10)\)).

1.5. Let us remind - for completeness - that Jacob’s ladder

\[\varphi_1(t) = \frac{1}{2} \varphi(t)\]

has been introduced in our work \([4]\) (see also \([5]\)), where the function

\[\varphi(t)\]

is arbitrary solution of the non-linear integral equation

\[\int_0^{\mu(x(T))} Z^2(t) e^{-\frac{\pi}{T}t} dt = \int_0^{T} Z^2(t) dt,\]

where each admissible function \(\mu(y)\) generates the solution

\[y = \varphi(T; \mu) = \varphi(T), \quad \mu(y) \geq 7y \ln y.\]

The function \(\varphi_1(t)\) is called the Jacob’s ladder according to Jacob’s dream in Chu-mash, Bereishis, 28:12.
Remark 7. We have shown (see [4]), by making use of these Jacob’s ladders, that the classical Hardy-Littlewood integral (1918)

\[ \int_0^T |\zeta\left(\frac{1}{2} + it\right)|^2 \, dt \]

has - in addition to the Hardy-Littlewood expression (and other similar to that one) possessing an unbounded error at \(T \to \infty\) - the following infinite set of almost exact expressions

\[ \int_0^T |\zeta\left(\frac{1}{2} + it\right)|^2 \, dt = \varphi_1(T) \ln \varphi_1(T) + (c - \ln 2)\pi_1(T) + c_0 + O\left(\frac{\ln T}{T}\right), \quad T \to \infty, \]

where \(c\) is the Euler’s constant and \(c_0\) is the constant from the Titchmarsh-Kober-Atkinson formula.

Remark 8. The Jacob’s ladder \(\varphi_1(t)\) can be interpreted by our formula (see [4])

\[ T - \varphi_1(T) \sim (1 - c)\pi(T); \quad \pi(T) \sim \frac{T}{\ln T}. \]

where \(\pi(T)\) is the prime-counting function, as an asymptotically complementary function to

\[ (1 - c)\pi(T) \]

in the following sense

\[ \varphi_1(T) + (1 - c)\pi(T) \sim T, \quad T \to \infty. \]

2. Factorization, Oscillating Q-system and its Metamorphoses as a Generic Complement to the Riemann’s Functional Equation on the Critical Line

2.1. The oscillating Q-system was defined in our work [7], (2.1) as follows

\[ G(x_1, \ldots, x_k; y_1, \ldots, y_k) \overset{\text{def}}{=} \prod_{r=1}^k \left| \frac{\zeta\left(\frac{1}{2} + ix_r\right)}{\zeta\left(\frac{1}{2} + iy_r\right)} \right| = \prod_{r=1}^k \left| \frac{\sum_{n \leq \tau(x_r)} \frac{2}{\sqrt{n}} \cos \left(\psi(x_r) - x_r \ln n\right) + R(x_r)}{\sum_{n \leq \tau(y_r)} \frac{2}{\sqrt{n}} \cos \left(\psi(y_r) - y_r \ln n\right) + R(y_r)} \right|, \]

\[ \tau(t) = \sqrt{\frac{t}{2\pi}}, \quad R(t) = O(t^{-1/4}), \quad k \leq k_0 \in \mathbb{N}, \]

for corresponding sets (see [7], (2.2)) of the points

\( (x_1, \ldots, x_k), (y_1, \ldots, y_k). \)

Remark 9. It is clear that the definition relation (2.1) is based on simple generalization

\[ \left| \frac{\zeta\left(\frac{1}{2} + ix\right)}{\zeta\left(\frac{1}{2} + iy\right)} \right| \to \prod_{r=1}^k \left| \frac{\zeta\left(\frac{1}{2} + ix_r\right)}{\zeta\left(\frac{1}{2} + iy_r\right)} \right| \]

(comp. (1.7), (1.11) – (1.13)).

Let us remind some of the previous results playing the role of the Riemann’s functional equation on the critical line.
(A). There are the functions (see [7], (2.5))

\[
\begin{align*}
\alpha_r^2 &= \alpha^2_r(\sigma, T, \Theta, k, \varepsilon), \ r = 0, 1, \ldots, k, \\
\beta_r^2 &= \beta^2_r(T, \Theta, k), \ r = 1, \ldots, k,
\end{align*}
\]

for admissible

\[
\sigma, T, \Theta, k, \varepsilon
\]
such that the following factorization formula

\[
\prod_{r=1}^{k} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha_r^2 \right)}{\zeta \left( \frac{1}{2} + i\beta_r^2 \right)} \right| \sim \sqrt[\varepsilon]{\frac{\zeta(2\sigma)}{\zeta(\sigma + \imath \alpha_0^2(\sigma))}}, \ T \to \infty,
\]

(see [7], (2.6)) holds true, i.e. there is following set of metamorphoses of the oscillating Q-system (2.1):

\[
\begin{align*}
\prod_{r=1}^{k} \left| \frac{\sum_{n \leq r(\alpha^2_r)} \frac{2}{\sqrt{n}} \cos \left( \varphi(\alpha^2_r) - \alpha^2_r \ln n \right) + R(\alpha^2_r)}{\sum_{n \leq r(\beta^2_r)} \frac{2}{\sqrt{n}} \cos \left( \varphi(\beta^2_r) - \beta^2_r \ln n \right) + R(\beta^2_r)} \right| \\
\sim \sqrt[\varepsilon]{\frac{\zeta(2\sigma)}{\zeta(\sigma + \imath \alpha_0^2(\sigma))}}, \ \sigma > 1 + \varepsilon, \ T \to \infty
\end{align*}
\]

(see [7], (2.6)), where \(\mu(n)\) is the Möbius function.

(B). There are functions

\[
\begin{align*}
\alpha_r^3 &= \alpha^3_r(T, l, \varepsilon, k), \ r = 0, 1, \ldots, k, \ l \in \mathbb{N} \\
\beta_r^3 &= \beta^3_r(T, \varepsilon, k), \ r = 1, \ldots, k,
\end{align*}
\]

for admissible

\[
T, l, \varepsilon, k
\]
such that the following factorization formula

\[
\left| \int_0^{\alpha_0^3} \arg \zeta \left( \frac{1}{2} + it \right) \, dt \right| \sim \pi e^{\frac{1}{2}} \prod_{r=1}^{k} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha_r^3 \right)}{\zeta \left( \frac{1}{2} + i\beta_r^3 \right)} \right|^{-\frac{1}{4}}, \ T \to \infty
\]

(see [8], (2.4)) holds true, i.e. there is following set of metamorphoses of the oscillating Q-system (2.1):

\[
\begin{align*}
\prod_{r=1}^{k} \left| \frac{\sum_{n \leq r(\alpha^3_r)} \frac{2}{\sqrt{n}} \cos \left( \varphi(\alpha^3_r) - \alpha^3_r \ln n \right) + R(\alpha^3_r)}{\sum_{n \leq r(\beta^3_r)} \frac{2}{\sqrt{n}} \cos \left( \varphi(\beta^3_r) - \beta^3_r \ln n \right) + R(\beta^3_r)} \right| \\
\sim \pi' \sqrt{\frac{\alpha_0^3}{\sigma}} \left| \int_0^{\alpha_0^3} \arg \zeta \left( \frac{1}{2} + it \right) \, dt \right|, \ T \to \infty,
\end{align*}
\]

(see [8], (4.6)).
we begin with (see [8], (4.11)), then we obtain the set of metamorphoses (2.5) in reverse direction:

We have obtained the first set of metamorphoses of the primeval multiform (C).

already metamorphosed into almonds ripened (comp. Chumash, Bamidbar, 17:23). that is the bud of the Aaron staff (corresponding to J.acob’s ladder . . .

in our paper [6]. The corresponding results expressed in terms of oscillating Q-system (2.1) are (see [6], (1.7), (2.5)): the factorization formula

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(B1). If we rewrite the formula (2.4) as follows

where

\( m = m(\alpha_0^3), \mu_m < \alpha_0^3 < \mu_{m+1}; S_1(\mu_m) = 0 \)

(see [8], (4.11)), then we obtain the set of metamorphoses (2.6) in reverse direction: we begin with

that is the Aaron staff (say),

that is the bud of the Aaron staff (corresponding to \( w = \alpha_0^3 \))

\[ \sim \left| \frac{\sum_{n \leq \tau(\alpha_0^2)} \frac{2}{\sqrt{n}} \cos\{\theta(\alpha_0^2) - \alpha_0^3 \ln n\} + R(\alpha_0^3)}{\sum_{n \leq \tau(\beta_0^2)} \frac{2}{\sqrt{n}} \cos\{\theta(\beta_0^2) - \beta_0^3 \ln n\} + R(\beta_0^3)} \right| \]

already metamorphosed into almonds ripened (comp. Chumash, Bamidbar, 17:23).

(C). We have obtained the first set of metamorphoses of the primeval multiform

\[ G(x_1, \ldots, x_k) = \prod_{r=1}^{k} \left| \zeta\left(\frac{1}{2} + i x_r\right) \right| \]

in our paper [6]. The corresponding results expressed in terms of oscillating Q-system (2.1) are (see [6], (1.7), (2.5)): the factorization formula

(2.6)

\[ \prod_{r=1}^{k} \left| \frac{\zeta\left(\frac{1}{2} + i x_r\right)}{\zeta\left(\frac{1}{2} + i y\right)} \right| \sim \sqrt{\frac{2\pi}{H}} \frac{1}{\zeta\left(\frac{1}{2} + i \alpha_0^3\right)} \]

and corresponding set of metamorphoses of the oscillating Q-system

(2.7)

\[ \prod_{r=1}^{k} \left| \frac{\sum_{n \leq \tau(\alpha_0^2)} \frac{2}{\sqrt{n}} \cos\{\theta(\alpha_0^2) - \alpha_0^3 \ln n\} + R(\alpha_0^3)}{\sum_{n \leq \tau(\beta_0^2)} \frac{2}{\sqrt{n}} \cos\{\theta(\beta_0^2) - \beta_0^3 \ln n\} + R(\beta_0^3)} \right| \sim \frac{1}{\sqrt{T}} \frac{1}{\sqrt{\sum_{n \leq \tau(\alpha_0^3)} \frac{2}{\sqrt{n}} \cos\{\theta(\alpha_0^3) - \alpha_0^3 \ln n\} + R(\alpha_0^3)}} \], \( T \to \infty \).
(D). Moreover, the sequences
\[
\{\alpha^n_r\}_{r=0}^k, \ \{\beta^n_r\}_{r=1}^k, \ n = 1, 2, 3
\]
have the following universal property:
\[
\alpha^n_r - \alpha^n_r \sim (1 - c)\pi(T), \ r = 0, 1, \ldots, k - 1,
\]
\[
\beta^n_r - \beta^n_r \sim (1 - c)\pi(T), \ r = 1, \ldots, k - 1, \ k \geq 2,
\]
(comp. [10], [17], Remark 8 and 9).

3. Theorem: factorization as an analogue of the Riemann’s functional equation on the critical line

3.1. We use the following Euler’s integral (see [1], pp. 134, 135)

\[
\int \frac{d\varphi}{a + b \cos \varphi} = \frac{1}{\sqrt{a^2 - b^2}} \arctan \left( \frac{a - b}{\sqrt{a^2 - b^2}} \frac{\varphi}{2} \right), \ \varphi \in (0, \pi),
\]
\[
a + b > 0, a^2 - b^2 > 0 \implies a > |b|.
\]

We obtain immediately from (3.1) the following formula

\[
\int_{2\pi L + U}^{2\pi L} \frac{d\varphi}{a + b \cos \varphi} =
\]

\[
= \frac{1}{a + b} U \arctan \left( \frac{\sqrt{a^2 - b^2} \tan \frac{U}{2}}{\sqrt{a + b}} \right), \ L \in \mathbb{Z}, \ U \in (0, \pi).
\]

3.2. Now, if we use our method of transformation (see [7], (4.1) – (4.19)) in the case of the formula (3.2) then we obtain the following statement.

**Theorem.** Let

\[
[2\pi L, 2\pi L + U] \rightarrow \left[ \frac{1}{\pi L}, \frac{1}{\pi L + U} \right], \ldots, \left[ \frac{k}{\pi L}, \frac{k}{\pi L + U} \right]
\]

where

\[
\left[ \frac{r}{\pi L}, \frac{r}{\pi L + U} \right], \ r = 1, \ldots, k, \ k \leq k_0 \in \mathbb{N}
\]

are reversely iterated segments corresponding to the first segment in (3.3) and \( k_0 \) be arbitrary and fixed number. Then there is a sufficiently big

\[
T_0 = T_0(a, b) > 0
\]
such that for every

\[
L > \frac{1}{2\pi T_0}
\]

and for every admissible \( L, U, k \) there are functions

\[
\alpha^4_r(L, U; a, b), \ r = 0, 1, \ldots, k,
\]
\[
\beta^4_r(L, U), \ r = 1, \ldots, k,
\]

(3.4)
\[
\alpha^4_r, \beta^4_r \neq \gamma : \zeta \left( \frac{1}{2} + i\gamma \right) = 0
\]
such that
\[
\prod_{r=1}^{k} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha_r^4 \right)}{\zeta \left( \frac{1}{2} + i\beta_r^4 \right)} \right|^2 \sim \\
\arctan \left( \frac{a + b \cos \alpha_0^4}{\sqrt{a^2 - b^2}} \right) \frac{a + b \alpha_0^4}{\sqrt{a^2 - b^2}} \quad , L \to \infty ,
\]
(3.5)
of course
\[
G(\alpha^4, \beta^4) \sim \sqrt{\ldots} \quad \iff \quad \{G(\alpha^4, \beta^4)\}^2 \sim \ldots .
\]
Moreover, the sequences
\[
\{\alpha_r^4\}_{r=0}^{k}, \quad \{\beta_r^4\}_{r=1}^{k}, \quad n = 1, 2, 3
\]
have the following properties
\[
2\pi L < \alpha_0^4 < \alpha_1^4 < \ldots < \alpha_k^4 ,
\]
\[
2\pi L < \beta_1^4 < \beta_2^4 < \ldots < \beta_k^4 ,
\]
(3.6)
\[
\alpha_0^4 \in (2\pi L, 2\pi L + U) ,
\]
\[
\alpha_r^4, \beta_r^4 \in (2\pi L, 2\pi L + U) , \quad r = 1, 2, \ldots, k ,
\]
(3.7)
\[
\alpha_{r+1}^4 - \alpha_r^4 \sim (1 - c)\pi(2\pi L) , \quad r = 0, 1, \ldots, k - 1 ,
\]
\[
\beta_{r+1}^4 - \beta_r^4 \sim (1 - c)\pi(2\pi L) , \quad r = 1, \ldots, k - 1 , \quad k \geq 2 ,
\]
where
\[
\pi(T) \sim \frac{T}{\ln T} , \quad T \to \infty
\]
is the prime-counting function and \(c\) is the Euler’s constant.

3.3. Now, let us notice the following.

Remark 10. The asymptotic behavior of the following sets
\[
\{\alpha_r^4\}_{r=0}^{k}
\]
is as follows: if \(L \to \infty\) then the points of every set \(\{\alpha_r^4\}_{r=0}^{k}\) recede unboundedly each from other and all together recede to infinity. Hence, at \(L \to \infty\) each of the sets \(\{\alpha_r^4\}_{r=0}^{k}\) behaves as one-dimensional Friedmann-Hubble universe.

Remark 11. Next, we express the result (3.5) in connection with (1.1), (1.3), Remark 2 and Remark 9 in the form
\[
\prod_{r=0}^{k} \left| \zeta \left( \frac{1}{2} + i\alpha_r^4 \right) \right| \sim \sqrt{\chi_4(U, \alpha_0^4)} \prod_{r=1}^{k} \left| \zeta \left( \frac{1}{2} + i\beta_r^4 \right) \right| ,
\]
(3.9)
where \(\chi_4\) stands for the right-hand side of (3.5). It is quite clear that this formula if an analogue of the Riemann’s function equation on the critical line.

Remark 12. Of course, the first result (1.9) is a particular case of (3.5).
4. **On a set of metamorphoses that correspond to the formula (3.5)**

4.1. Let us remind the spectral form of the Riemann-Siegel formula

\[
Z(t) = \sum_{n \leq \tau(x_r)} \frac{2}{\sqrt{n}} \cos \{t \omega_n(x_r) + \psi(x_r)\} + O(x_r^{-1/4}),
\]

\[\tau(x_r) = \sqrt{\frac{x_r}{2\pi}},\]

\[t \in [x_r, x_r + V], \quad V \in (0, x_r^{1/4}),\]

and similarly for \(x_r \rightarrow y_r\), where

\[T_0 < 2\pi L < x_r, y_r\]

(see [4], (6.1), comp. [8], (4.4) and Remark 6 ibid).

**Remark 13.** We call the expressions

\[\sum_{n \leq \tau(x_r)} \frac{2}{\sqrt{n}} \cos \{t \omega_n(x_r) + \psi(x_r)\}, \ldots\]

as the Riemann’s oscillators with:

(a) the amplitude

\[\frac{2}{\sqrt{n}}\]

(b) the incoherent local phase-constant

\[\psi(x_r) = -\frac{x_r}{2} - \frac{\pi}{8},\]

(c) the non-synchronized local time

\[t = t(x_r) \in [x_r, x_r + V],\]

(d) the local spectrum of the cyclic frequencies

\[\{\omega_n(x_r)\}_{n \leq \tau(x_r)}, \quad \omega_n(x_r) = \ln \frac{\tau(x_r)}{n}, \ldots\]

**Remark 14.** Now, we see that the Q-system, where of course

\[|\zeta \left(\frac{1}{2} + it\right)| = |Z(t)|,
\]

is really complicated oscillating system (comp. (2.1), (4.1), (4.3)).

4.2. Consequently, we have (see (2.1), (3.5), (4.1), Remark 14 and (4.3)) the following.

**Corollary 1.** The following set of metamorphoses

\[\prod_{r=1}^{b} \left| \sum_{n \leq \tau(\alpha_r^4)} \frac{2}{\sqrt{n}} \cos \{\alpha_r^4 \omega_n(\alpha_r^4) + \psi(\alpha_r^4)\} + R(\alpha_r^4) \right| \sim \frac{\arctan \left(\sqrt{\frac{\pi - \tan \frac{\pi}{2}}{\pi + \tan \frac{\pi}{2}}} a + b \cos(\alpha_0^4)\right)}{\sqrt{\pi + \tan \frac{\pi}{2}}} \quad a + b, \quad L \to \infty.
\]

corresponds to the factorization formula (3.5).
4.3. Now, let us notice the following.

Remark 15. By Theorem, there are control functions \( \text{Golem's shem} \) of the set of metamorphoses \( (4.4) \) of the oscillating Q-system \( (2.1) \), (see also \( (4.1), (4.3) \)).

Remark 16. The mechanism of metamorphosis is as follows. Let (comp. \( (3.4) \) and \( [7], (2.2) \))

\[
M^3_k = \{\alpha^4_1, \ldots, \alpha^4_k\}, \\
M^4_k = \{\beta^4_1, \ldots, \beta^4_k\},
\]

(4.5)

where, of course, (comp. \( [7], (2.12) \))

\[
M^3_k \subset M^1_k \subset (T_0, +\infty)^k, \\
M^4_k \subset M^2_k \subset (T_0, +\infty)^k.
\]

(4.6)

Now, if we obtain, after random sampling such points (comp. conditions \( [7], (2.2) \)) that

\[
(x_1, \ldots, x_k) = (\alpha^4_1, \ldots, \alpha^4_k) \subset M^3_k, \\
(y_1, \ldots, y_k) = (\beta^4_1, \ldots, \beta^4_k) \subset M^4_k,
\]

(4.7)

(see \( (4.5), (4.6) \)), then – at the points \( (4.7) \) – the Q-system \( (2.1) \) changes its old form (=chrysalis) into its new form (=butterfly) and the last is controlled by the function \( \alpha_0^4 \).

Remark 17. Now, it should be clear that the set of metamorphoses of oscillating Q-system also belongs to the family of analogues of the Riemann’s functional equation on the critical line.

5. ON DECOMPOSITION OF THE RESULT OF METAMORPHOSES \( (4.4) \) INTO THREE PARTS: SIGNAL, NOISE AND ERROR TERM

In this section we use the terminology from the theory of signal processing.

5.1. Let us remind (see \( [7], (4.11) \)) that

\[
\tilde{Z}^2(t) = \left| \frac{\zeta \left( \frac{1}{2} + it \right)}{\omega(t)} \right|^2, \\
\omega(t) = \left\{ 1 + O \left( \frac{\ln \ln t}{\ln t} \right) \right\} \ln t.
\]

Since in our case

\[
t \longrightarrow 2\pi L,
\]

then (comp. \( [7], (4.11), (4.12) \))

\[
\tilde{Z}^2(\alpha_0^4) = \frac{\zeta \left( \frac{1}{2} + i\alpha_0^4 \right)}{\left\{ 1 + O \left( \frac{\ln \ln L}{\ln L} \right) \right\} \ln t}, \ldots
\]
Remark 18. Consequently, the primary form of the asymptotic formula for metamorphoses (4.4) is as follows

\[
\prod_{r=1}^{k} \left| \frac{\sum_{n \leq \tau(\alpha_r^4)} \frac{2}{\sqrt{n}} \cos\{\alpha_r^4 \omega_n(\alpha_r^4) + \psi(\alpha_r^4)\} + R(\alpha_r^4)}{\sum_{n \leq \tau(\beta_r^4)} \frac{2}{\sqrt{n}} \cos\{\beta_r^4 \omega_n(\beta_r^4) + \psi(\beta_r^4)\} + R(\beta_r^4)} \right| \sim \left\{ 1 + \mathcal{O}\left( \frac{\ln \ln L}{\ln L} \right) \right\} \frac{\arctan\left( \frac{a}{a+b} \tan \frac{U}{2} \right)}{\sqrt{\frac{a-b}{a+b} U^2}} \frac{a+b}{a+b} \cos(\alpha_0^4) + \mathcal{O}\left( \frac{\ln \ln L}{\ln L} \right), \quad L \to \infty.
\]

(5.1)

Since the last two factors on the right-hand side of (5.1) are bounded functions for all

\[
U \in (0, \pi), \quad L > \frac{1}{2\pi T_0}
\]

(for all fixed admissible \(k, a, b\), see (3.4)), then we obtain from (5.1) the following.

Corollary 2.

\[
\prod_{r=1}^{k} \left| \frac{\sum_{n \leq \tau(\alpha_r^4)} \frac{2}{\sqrt{n}} \cos\{\alpha_r^4 \omega_n(\alpha_r^4) + \psi(\alpha_r^4)\} + R(\alpha_r^4)}{\sum_{n \leq \tau(\beta_r^4)} \frac{2}{\sqrt{n}} \cos\{\beta_r^4 \omega_n(\beta_r^4) + \psi(\beta_r^4)\} + R(\beta_r^4)} \right| = \frac{a}{a+b} \arctan\left( \frac{\frac{a}{a+b} \tan \frac{U}{2}}{U} \right) + \mathcal{O}\left( \frac{\ln \ln L}{\ln L} \right), \quad L \to \infty.
\]

(5.2)

5.2. Let us remind (see (3.3)), that

\[
(5.3)
\]

\[
\alpha_0^4 = \alpha_0^4(L, U; k; a, b) = \alpha_0^4(L, U)
\]

for admissible and fixed \(k, a, b\).

(a) We see that the first function on the right-hand side of (5.2) is the L-th member

\[
f(2\pi L + U) = g_L(U) = \frac{a}{a+b} \arctan\left( \frac{\frac{a}{a+b} \tan \frac{U}{2}}{U} \right), \quad U \in (0, \pi)
\]

(5.4)

\[
g_L(U) = g_{L'}(U), \quad \forall \ L, L' > \frac{1}{2\pi},
\]

\[
2\pi L + U \in [2\pi L, 2\pi L + \pi)
\]

of the stationary sequence

\[
\{g_L(U)\}_{L>T_0/2\pi}, \quad U \in (0, \pi).
\]

(5.5)

By (5.4), (5.5) the corresponding signal is defined.

Consequently, by the first function on the right-hand side of (5.2) deterministic signal is expressed (see (5.5).
(b) The main factor in the second member is the following function
\[ \cos(\alpha_0^4), \quad \alpha_0^4 \in \alpha_0^4(L, U), \]
where (comp. [1.14], [5.3])
\[ \alpha_0^4 = \varphi_1(d) \in (2\pi L, 2\pi L + U), \quad d = d(L, U), \]
and \( \varphi_1(d) \) is the value of the Jacob’s ladder. That is, the distribution of the values
\[ \alpha_0^4 \in (2\pi L, 2\pi L + U), \quad U \in (0, \pi) \]
we may suppose as very complicated.
Consequently, the second function we shall characterize as noise – a non-useful part of the signal. The noise may be controlled by variation of the parameter \( b \),
\[ a > |b| \]
(see (6.1)), i.e. by abatement of \( |b| \).
(c) The third function we shall call (fine) error term, since
\[ O\left(\frac{\ln \ln L}{\ln L}\right) \xrightarrow{L \to \infty} 0. \]

Remark 19. Hence, the final state of metamorphoses in (5.2) is split into three parts: signal, noise and (fine) error term.

6. The set of distinct metamorphoses in (4.4)

Of course, there is a point \( U_0 \in (0, \pi) \) such that (see (5.3))
\[ g_L(U)|_{U=U_0} \neq 0, \quad \forall L > \frac{T_0}{2\pi}, \]
i.e.
\[ g_L(U) \neq 0, \quad U \in O_{\delta}(U_0) = (U_0 - \delta, U_0 + \delta), \]
\forall U', U'' \in O_{\delta}(U_0), U' \neq U'' \Rightarrow g_L(U') \neq g_L(U'') \]
for suitable \( \delta > 0 \).
Next, we shall suppose that there are such
\[ U_1, U_2 \in O_{\delta}(U_0), \quad U_1 \neq U_2, \]
that
\[ \alpha_1^r(U_1, L) = \alpha_1^r(U_2, L), \quad r = 0, 1, \ldots, k, \]
\[ \beta_1^r(U_1, L) = \beta_1^r(U_2, L), \quad r = 1, \ldots, k \]
for all
\[ L > \tilde{L} > \frac{T_0}{2\pi}. \]
In this case we obtain, by comparison of the formulas (5.1), for \( U_1, U_2 \) that
\[
\frac{\arctan \left( \sqrt{\frac{a-b}{a+b}} \tan \frac{U_1}{2} \right)}{\sqrt{\frac{a-b}{a+b} U_1}} = \frac{\arctan \left( \sqrt{\frac{a-b}{a+b}} \tan \frac{U_2}{2} \right)}{\sqrt{\frac{a-b}{a+b} U_2}}.
\]
\[ = \left\{ 1 + O\left(\frac{\ln \ln L}{\ln L}\right) \right\} \frac{\arctan \left( \sqrt{\frac{a-b}{a+b}} \tan \frac{U_1}{2} \right)}{\sqrt{\frac{a-b}{a+b} U_1}}. \]
i.e. in the limit case we obtain the equality
\[
\arctan \left( \sqrt{\frac{a+b}{a+b}} \tan \frac{U_1}{2} \right) = \arctan \left( \sqrt{\frac{a+b}{a+b}} \tan \frac{U_2}{2} \right)
\]
that contradicts with (6.1).
Hence, we have that for every
\[ U', U'' \in O_\delta(U_0), \quad U' \neq U'' \]
there is an infinite subsequence
\[ \{ \bar{L} \} \subset \{ L \}, \quad \bar{L} > \tilde{L} \]
such that (comp. (6.2))
\[
(\alpha_1^4(U', \bar{L}), \alpha_1^4(U', \bar{L}), \ldots, \alpha_k^4(U', \bar{L}),
\beta_1^4(U', \bar{L}), \ldots, \beta_k^4(U', \bar{L})) \\
\neq (\alpha_1^4(U'', \bar{L}), \alpha_1^4(U'', \bar{L}), \ldots, \alpha_k^4(U'', \bar{L}),
\beta_1^4(U'', \bar{L}), \ldots, \beta_k^4(U'', \bar{L})) , \quad \forall \bar{L} \in \{ \bar{L} \}.
\]
Consequently, by (6.3) we have the following.

**Corollary 3.** There is an infinite set of distinct metamorphoses (6.4).

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