Thermodynamic chaos and infinitely many critical exponents in the Baxter-Wu model

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Abstract

The mechanisms leading to thermodynamic chaos in the Baxter-Wu model is considered. We compare the Baxter-Wu model with triangular antiferromagnets and discuss the difficulties related to the modeling of thermodynamic chaos by disordered models. We also discuss how to overcome the problem of infinitely many order parameters. Then we consider the Baxter-Wu model in a complex magnetic field and show the existence of infinitely many critical exponents in this model.

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1 Introduction

In contrast to microscopic/classical chaos the physics behind thermodynamic chaos is far from being well understood. Defined to be the macroscopic state of a system with chaotically broken translational symmetry, this phenomena is one of the main characteristics of the glassy systems\cite{1}. Traditionally these systems investigated by introducing randomness in the microscopic parameters. Although in many cases this can be physically justified, it leaves the origin of randomness questionable, especially when microscopic parameters in turn are measured from macroscopic response of the system. Recent progress in low temperature physics shows that frustration and competing interactions are more important for thermodynamic chaos than the randomness itself\cite{2}. Thus the interest to the deterministic models of Statistical Mechanics (SM) where frustration and competing interactions are not the result of randomness.

The antiferromagnetic Ising model on triangular lattices is a classical example of the model incorporating both frustration and competing interactions\cite{3,4}. In 2d, exact solution in the absence of magnetic field reveals that this model is not capable to exhibit thermodynamic chaos\cite{5}. Recently, we investigated the antiferromagnetic Ising model in a magnetic field on Husimi tree, which being approximation to regular lattices, allows to preserve frustrative nature of antiferromagnetic interaction\cite{6}. Although this model has an interesting phase structure, no thermodynamic chaos has been found. The other approximations preserving frustration also show no sign of thermodynamic chaos in triangular antiferromagnets\cite{7}. The reason is the following: frustration on a triangle results macroscopic degeneracy of the ground state, including chaotic configurations, but due to symmetry of the Hamiltonian these configurations are not observable at the macroscopic level. On the other hand, the magnetic field, which competes with antiferromagnetic interaction completely confines frustrations. Thus a special symmetry breaking field, other than uniform, is necessary for observing thermodynamic chaos in triangular antiferromagnets.

The Baxter-Wu model in a magnetic field is one of the models where frustration and competing interactions present simultaneously\cite{8}. The thermodynamic chaos found in this model by Monroe\cite{9} has been investigated in some details by the present authors\cite{10,11}. Universal transition to chaos is one of the interesting phenomena which has been found. The main dif-
difficulties which one faces is that it is necessary to introduce infinitely many order parameters to describe the system in a chaotic phase. It is interesting to note that similar problem exist also in disordered models of spin glasses, namely one needs to introduce Parisi’s order parameter functional or local Edwards-Anderson order parameters in order to describe the system in a glassy phase [12].

Fortunately, for the deterministic models one can use the thermodynamic formalism of dynamical systems to infer the universal characteristics in chaotic phases. In particular, distribution of the local Lyapunov exponents shows universal scaling behavior, similar to one found for a logistic map [10, 11]. Although similar universalities have been also found in the distribution of the local Edwards-Anderson order parameters of disordered models [13], as we shall see, in many cases disordered models are not capable to model thermodynamic chaos.

In this paper we investigate the Baxter-Wu model in a complex magnetic field. At this point, it is interesting to draw some parallels between Quantum Mechanics/Quantum Field Theory (QFT) and SM. Investigation of the singularities of the scattering matrix in a complex energy/momentum plane reveals an information about bounded states and resonants, which otherwise is difficult to obtain [14]. But in contrast to scattering matrix, singularities of thermodynamic quantities are not necessarily isolated. In fact, as it has been proven by Lee and Yang in 1952 [15], zeros of the partition function of the 2d Ising model in a complex magnetic field lie on a curve, which tend to the real axis at the phase transition point and become dense in the thermodynamic limit. Soon after, Huang in his seminal textbook of SM [16] rises a question about possibility of having SM model, zeros of the partition function of which pinch the real axis at some range instead of a single point.

Recently, we investigated the Baxter-Wu model in a complex temperature plane [17]. It has been shown that the Fisher zeros of this model densely fills a domain near the real axis. As we shall see, the Lee-Yang zeros of the Baxter-Wu model satisfy the criterion mentioned by Huang. This allows us to prove the existence of infinitely many critical exponents in this model.

The paper is organized as follows. In Section 2 after introducing the Baxter-Wu model we discuss principal differences in modeling thermodynamic chaos by deterministic and disordered models. In Section 3 we present our numerical results for the phase structure of the Baxter-Wu model in a complex magnetic field and discuss the role of thermodynamic chaos on it.
In Section 4 we present our conclusions.

2 The Baxter-Wu model

The Hamiltonian of the Baxter-Wu model in a magnetic field has the following form:

\[ H = -J_3 \sum_{\triangle} \sigma_i \sigma_j \sigma_k - h \sum \sigma_i \]  

(1)

where \( \sigma_i \in \{-1; +1\} \) are Ising variables, \( J_3 \) and \( h \) are three-site interaction strength and magnetic field respectively. The first sum goes over all triangles and the second one over all sites. Like two-site interacting Ising model this model has multiple applications and is one of the few models of SM exactly solvable 2d. For the first time the solution has been found by Baxter and Wu in 1973 by using Bethe Ansatz method[8]. Later, it has been shown that this solution is a particular case of more general solution of the eight-vertex model[18, 19]. Other interesting relations between Baxter-Wu model and other exactly solvable SM models can be obtained using generalized star-triangle relations[19, 20]. Recently, the solution by the Bethe Ansatz method has been generalized to include more general boundary conditions and an interesting conjecture in the framework of Conformal Field Theory has been proposed, viz. that the Baxter-Wu model and 4-state Potts model share the same operator contents[22].

On a single triangle the ground state of the three-site interaction consist of configuration where all spins aligned at the same direction (up or down depending on the sigh of \( J_3 \)) and of configurations obtained from this one by reserving the spins at arbitrary two sites of the triangle. This makes the ground state of the Baxter-Wu model highly degenerate. In particular, an arbitrary alignment of the spins along arbitrary direction can be achieved starting from the uniform configuration without altering the total energy of the system. By encoding these configurations in binary sequences it becomes clear that the ground state of the Baxter-Wu model involve uniform, modulated, as well as chaotic configurations. Note that third spin in the Hamiltonian of Baxter-Wu model acts like a (pseudo) random two-site interaction strength.
As we mentioned in the introduction, the macroscopic degeneracy itself is not sufficient for thermodynamic chaos. In addition to macroscopic degeneracy a symmetry breaking field is necessary, which will pick up a particular chaotic configuration of the ground state when averaging over all configurations. As we shall see, in the Baxter-Wu model the magnetic field with opposite sign to $J_3$ satisfies this criteria.

To that end, let us consider the Baxter-Wu model on the Husimi tree, so that frustrative nature of the three-site interaction can be preserved and at the same time analytical expression for thermodynamic quantities can be obtained, even in the presence of magnetic field.

The magnetization at the central site of the Husimi tree as a function of temperature $T$, magnetic field $h$ and three-site coupling $J_3$ is given by

$$m_n(T, h, J_3) = \frac{\mu x_n^\gamma - 1}{\mu x_n^\gamma + 1}$$

(2)

where $n$ numbers the generation in the hierarchies of Husimi trees, $\gamma$ is equal to the twice the coordination number and $x_n$ is given by the following recurrent relation

$$x_n = f(x_{n-1}), \quad f(x) = \frac{z\mu x^{2(\gamma-1)} + 2\mu x^{\gamma-1} + z}{\mu^2 x^{2(\gamma-1)} + 2z\mu x^{\gamma-1} + 1}$$

(3)

where $z = e^{2J_3/kT}$ and $\mu = e^{2h/kT}$. Initial condition for the recurrent relation (3) depends on the boundary conditions (e.g. $x_0 = 1$ corresponds to the free boundary condition).

It is interesting to point out that the recurrent relation (3) enters also in the formulas for the expectation values of quantum mechanical operators in some field theoretical models.

Depending on $T$, $h$ and $J_3$ the attractor of the map consist of a stable point, periodic cycles or strange attractors, so that our system is in uniform (paramagnetic or ferromagnetic), modulated or glassy phases respectively (Figure 1).

Note that as far as the dynamic of the map (3) is concerned there is no difference whether it describes the dynamic of microscopic quantities or the distribution of macroscopic one. The only difference is in the interpretation of the results, so that in the former case it describes time evolution of microscopic quantities, whereas in the later case site to site variation of
macroscopic quantities. For instance, the map which is involved in the formulas for magnetization of the ANNNI model on the Bethe lattice natural arises in the microscopic dynamic of a neuron with non-monotonic transfer function\[24\].

Taking into account the discussion in the introduction, one can see that the large variety of phases, which are typical in low temperature physics and biophysics are the result of collective effects of frustrations and competing interactions.

The main difficulties lie in the parameter space where the system exhibits thermodynamic chaos. In the uniform or modulated phases one can use conventional order parameters, such as magnetization, but at every bifurcation point the system undergoes a continuous phase transition and their number doubles. Thus in a chaotic phase we end up with infinitely many order parameters.

To find a possible solution to this problem one can use an invariant measure $P(m)$ of the map (3) to define e.g. $q = \int m P(m)$ as an order parameter. But, as it is well known, like in axiomatic QFT or in stochastic dynamical systems, in general, even the first few momentums are not sufficient for describing the system. Thus we have to use the thermodynamic formalism of dynamical systems for the complete description of the system\[25, 26\].

Let us recall that using SM one anticipates to get rid of the large amount microscopic degrees of freedom and to avoid the solution of the complicated microscopic dynamics involving thermostat. The thermodynamic formalism of dynamical systems, on the other hand, allows one to obtain the quantities which describes the dynamical systems, e.g. the spectrum of Lyapunov exponents or generalized dimensions\[27\]. By applying thermodynamic formalism of dynamical systems we do not anticipate to get rid of infinitely many order parameters. Instead, we obtain an information about our system in terms of Lyapunov exponents or generalized dimensions. That is the universalities in the distribution of these quantities which allows to avoid the measurement of infinitely many order parameters\[11, 28\].

To compare deterministic and disordered models of thermodynamic chaos let us consider the problem of modeling thermodynamic chaos in a computer. In order to distinguish the modulated phase with large period from a chaotic one we have to simulate the deterministic model on a very large lattice. On the other hand, we can simulate this system on a small lattice by a disordered model with appropriately chosen measure for random parameters. But
since strange attractors of dynamical systems with a few exceptions (e.g. hyperbolic dynamical systems) have infinitely many invariant measures, disordered models are not capable to model thermodynamic chaos at all temperatures and boundary conditions.

It is interesting to note that thermodynamic or special chaos can be observed in deterministic systems by changing boundary conditions only.

3 Lee-Yang singularities

An attractive feature of disordered models is that it is believed that only a few set of critical exponents are needed to describe different phase transitions taking place in spin glasses. Complex temperature/field analysis is one of the tools used among many others to extract the critical exponents.

Zeros of the partition function in the complex temperature/field plane coincide with the Julia set of renormalization group map and provide information about phase transition points and critical exponents. In particular, the density on the curve on which partition function zeros lie and the angle which this curve make with the real axis are directly related to the critical exponents.

The fact that the system is in complex temperature and/or magnetic field does not mean that this system is unphysical or nonunitary. In fact, there are well know examples in the literature where duality or star-triangle relation map the system with real temperature and magnetic field into a system with complex temperature and/or magnetic field.

Zeros of the partition function of the Baxter-Wu model on the Husimi tree satisfy the following relation

\[ \mu x^\gamma + 1 = 0 \]  

on the attractor of the map.

In Figure 2 we plot the zeros of the partition function in a complex field for different temperatures. One can see at Figure 2a that zeros of the partition function approach to only a few set of critical points located on the real axis, whereas at low temperatures (Figures 2b) the Lee-Yang singularities densely fill domains near real axis which include patches of the real axis. This indicates a condensation of the phase transition points. In these domains there are infinitely many different ways leading to a given point in the real
axis, which shows the existence of infinitely many critical exponents in the Baxter-Wu model. The existence of infinitely many critical exponents can be seen also from Figure 1. Since at every bifurcation point one can define critical exponents in terms of staggered magnetizations, in a chaotic phase there are infinitely many critical exponents.

We reiterate that in a real experiment there is no need to measure neither infinitely many order parameters nor infinitely many critical exponents.

4 Conclusion

In this work we studied thermodynamic chaos in the Baxter-Wu model. Although Baxter-Wu model shares many features of triangular antiferromagnets, the external magnetic field competes with the three-site interaction leaving the ground state highly degenerate, a phenomena which makes chaos observable at the macroscopic level.

This phenomena is well known in disordered models of spin glasses, where similar result can be obtained by introducing randomness in the microscopic parameters (e.g. magnetic field or interaction strength). Nevertheless, disordered models, in general, fail to model thermodynamic chaos, since probabilistic measure (which in disordered models we have to chose phenomenologically) is not unique and varies depending on temperature and boundary conditions.

Investigation of the Lee-Yang singularities revealed the existence of infinitely many critical exponents in the Baxter-Wu model. The problem with infinitely many order parameters and critical exponents is that in a real experiment one have to perform infinitely many measurements. Notice that a similar problem exist also in nonrenormalizable QFT, where nonrenormalizable interaction leads to infinitely many counter terms and corresponding coupling constants. As we mentioned, for the Baxter-Wu model we can use the thermodynamic formalism of dynamical systems, which allows us to completely describe the systems in chaotic phases by a few set of universal quantities (e.g. the slope in the distribution of the Lyapunov exponents). We hope that, duality relation between Baxter-Wu model and $Z(2)$ gauge symmetric model involving nonrenormalizable three-plaquette interaction, which we recently found, will help us to understand the connection between nonrenormalizable QFT and thermodynamic chaos.
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Figure 1: Plots of $m$ versus $h$ for different temperatures $T$ ($\gamma = 4, J_3 = -1$).

a - $T = 3$, b - $T = 1.3$, c - $T = 1.15$, d - $T = 0.7$. 

m

h

1

n

0

h

1
Figure 2: Lee-Yang singularities of the Baxter-Wu model ($\gamma = 4$). a - $z = 6.05$, b - $z = 7.0$. 