LinkMirage: How to Anonymize Links in Dynamic Social Systems

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Abstract—Social network based trust relationships present a critical foundation for designing trustworthy systems, such as Sybil defenses, secure routing, and anonymous/censorship-resilient communications. A key issue in the design of such systems, is the revelation of users’ trusted social contacts to an adversary – information that is considered sensitive in today’s society.

In this work, we focus on the challenge of preserving the privacy of users’ social contacts, while still enabling the design of social trust based applications. First, we propose LinkMirage, a community detection based algorithm for anonymizing links in social network topologies; LinkMirage preserves community structures in the social topology while anonymizing links within the communities. LinkMirage considers the evolution of the social network topologies, and minimizes privacy leakage due to temporal dynamics of the system.

Second, we define metrics for quantifying the privacy and utility of a time series of social topologies with anonymized links. We analyze the privacy and utility provided by LinkMirage both theoretically, as well as using real world social network topologies: a Facebook dataset with 870K links and a large-scale Google+ dataset with 940M links. We find that our approach significantly outperforms the existing state-of-art.

Finally, we demonstrate the applicability of LinkMirage on real-world applications such as Sybil defenses, reputation systems, anonymity systems and vertex anonymity. We also prototype LinkMirage as a Facebook application such that real world systems can bootstrap privacy-preserving trust relationships without the cooperation of the OSN operators.

I. INTRODUCTION

Trust relationships between users in social networks can provide a foundation for the design of secure systems. Social networks have been leveraged in the design of anonymity systems [1]–[4], Sybil defenses [5]–[9], secure routing [10]–[12], spam mitigation [13] and secure reputation systems [14].

However, users’ social contacts are considered sensitive information in today’s society. For instance, popular OSNs such as Facebook, Google+ and LinkedIn all provide privacy controls to limit access to this information, and a majority of users are exercising these options [15]. Unfortunately, the social network based systems discussed above do not protect the privacy of users’ social contacts; this information is revealed to an adversary either explicitly, or implicitly via traffic analysis. The lack of privacy for users’ social contacts hinders the deployment of these social network based applications. For example, users facing Internet censorship, or users engaging in protest movements in Hong Kong/Ferguson/Ukraine would benefit from using their social trust relationships for anonymous and Sybil resilient communications, provided that the privacy of their social contacts is preserved.

How can we enable the design of social network based applications while preserving the privacy of users’ social contacts? Previous work mostly focuses on vertex anonymity [16] [17], which does not preserve privacy of social trust relationships and thus is orthogonal to our work. Prior work on link privacy is very limited and only considers the static social network topology [18]–[20]. However, social network topologies are dynamic, and evolve over time – this introduces a new privacy challenge, as adversaries can combine information available in multiple anonymized graphs to infer users’ social contacts.

In this work, we propose LinkMirage to address the challenge of preserving the privacy of users’ social contacts (edge/link privacy, and not vertex privacy) in dynamic social network based systems. LinkMirage preserves community structures while anonymizing links within communities. We also develop a Facebook application to make LinkMirage available for real world systems that leverage social trust relationships in a privacy-preserving manner, even without the cooperation of the OSN operators.

LinkMirage considers the evolution of social network topologies. Our key insight is the paradigm of selective graph perturbation, in which we first compare the clustering of consecutive graph snapshots to identify the changed and unchanged graph components, and then apply independent perturbations (noises) to only the changed components. For the unchanged components, we leverage the perturbations from the previous graph snapshot. Thus, by reusing information that is already available from previously perturbed graphs, our approach minimizes privacy leaks over time. Overall, our work makes the following contributions.

- First, we propose LinkMirage to protect link privacy in dynamic social networks. LinkMirage preserves community structures while anonymizing links within the communities. To account for network evolution, our method consists of two steps: (a) dynamic clustering and (b) selective perturbation. Dynamic clustering aims to cluster the current graph snapshot based on the clustering result for the previous graph snapshot, in order to identify the changed and the unchanged communities. Selective perturbation aims to apply independent noise to only the changed graph components, while reusing the randomness/noise for the unchanged graph components from the
previously perturbed graph. LinkMirage thus achieves a good balance between utility and privacy.

- Second, we use information theory and Bayesian inference to define several metrics for characterizing link privacy, when considering the prior information that an adversary may have for de-anonymizing links. We perform a worst case analysis of link privacy for LinkMirage both theoretically and using real world social network topologies. The experimental results for both a Facebook dataset (with 870K links) and a large-scale Google+ dataset (with 940M links) show the privacy benefits of LinkMirage over previous state of art. We also illustrate the relationships between these metrics to the conventional notion of differential privacy.

- Third, we extend static utility metrics in prior work to quantify the utility of a time series of perturbed graphs, based on which we also develop a new utility metric. This metric, called the temporal utility distance, considers the correlation between consecutive perturbed graphs. We analyze the temporal utility provided by LinkMirage both theoretically, as well as using real world social network (Facebook and Google+) topologies. We find that our algorithm provides benefits in both static and dynamic contexts. We also show that our utility metric is closely related to global properties of social networks, such as mixing time and the second largest eigenvalue modulus.

- Finally, we experimentally demonstrate the applicability of LinkMirage in real world applications, such as Sybil defenses, reputation systems, anonymous communication and vertex anonymity. LinkMirage enables the design of social network based systems and simultaneously preserves the privacy of users’ social contacts. It is interesting to note that the addition of noise as in our approach can improve the performance of certain applications. LinkMirage improves the detection performance in Sybil defenses and accelerates the performance of reputation systems like Yelp.

II. RELATED WORK AND BACKGROUND

At the very beginning, it is worth noting that we focus our research on protecting the link privacy between labeled vertices in a time series of social graphs. For real world applications that leverage social links, it is critical to know the identities of vertices. Thus approaches that protect vertex privacy in graphs (with unlabeled vertices) are not applicable to these systems, and are orthogonal to our work [16], [21].

A. Related work

1) Privacy with labeled vertices: Before discussing temporal graphs, we first introduce related work on link privacy for graph with labeled vertices.

One thread of research aims to preserve link privacy between labeled vertices by obfuscating the social graph, i.e., by adding/deleting edges and vertices [18]–[20]. These methods aim to randomize the structure of the social graph, while differing in the manner of adding noise. Hay et al. [18] perturb the graph by applying a sequence of $k$ edge deletions and $k$ edge insertions. Both the added and the deleted edges are selected with a uniform probability from the set of all edges, and are not based on the structural properties of the graph. Ying and Wu [20] further analyzed the spectral properties and link privacy of Hay et al.’s perturbation method. They also proposed a new perturbation method for preserving spectral properties, without analyzing its privacy performance.

Mittal et al. proposed a perturbation method in [19], which serves as the foundation for LinkMirage. Their method deletes all edges in the original graph, and replaces each edge with a fake edge that is sampled based on the structural properties of the graph. In particular, random walks are performed on the original graph to sample edges; Let $k$ denote the length of the random walk, which also serves as the privacy parameter. Consider each node $u$ in the original graph $G$ and its arbitrary neighbor $v$. The algorithm considers a random walk of length $k - 1$ starting from $v$, and samples the terminus point of the random walk $w$. Then, a perturbed graph $G'$ is constructed, in which the link $(u, w)$ is added (instead of the link $(u, v)$). Thus, the topology of the original graph is perturbed to provide link privacy, while still preserving the macro-level structural properties of the graph.

We will experimentally compare our method with the methods of Hay et al. [18] and Mittal et al. [19].

Another line of research aims to preserve link privacy [22] [23] by aggregating the vertices and edges into super vertices using community detection techniques. The privacy of links within each community is naturally protected by this perturbation method. However, such approaches do not permit fine grained utilization of graph properties, making it difficult to be applied to systems such as social network based anonymous communication and Sybil defenses.

2) Privacy with unlabeled vertices: Most previous work focuses on the setting of unlabeled vertices, which is orthogonal to our goals. We describe them for completeness.

Liu et al. [16] proposed $k$-anonymity for anonymizing unlabeled vertices which places at least $k$ vertices at an equivalent level. The following work in [24] [25] differ only in structural features that an adversary might use to partition vertices in the anonymized graphs. Further, Narayanan et al. [26] developed an efficient de-anonymization attack for the graph with unlabeled vertices, based on the structural relationships between the graph and another auxiliary graph.

Differential privacy provides a theoretical framework for adding noise to aggregate information such that users’ privacy is preserved. Sala et al. [27] applied differential privacy to privately publish social graphs. Their approach first projects the original graph into degree correlation statistics, and adds noise to these statistics such that differential privacy is preserved. Next, the perturbed degree correlation statistics are used to generate synthetic perturbed graphs using graph generation models. We note that this method results in graphs with unlabeled vertices, and is thus not applicable to our setting.

Finally, we consider anonymity in temporal graphs with unlabeled vertices. Bhagat et al. [28] consider dynamic social
networks where a time series of graphs are to be published, similar to the scenario in this work. They aim to stem privacy degradation over time by exploiting link prediction techniques to model the evolution of social networks. Their approach combines the predicted graph with the current graph to perform group-based anonymization, and reduces privacy loss over time. However, their method is based on the label-list perturbation algorithm [29], which considers unlabeled vertices. They do not provide any theoretical results for utility and privacy measurements.

In this work, we take a complimentary approach of leveraging past perturbed snapshots to minimize additional information revealed in the current perturbed snapshot. Furthermore, [17], [30] explored privacy degradation in vertex privacy schemes due to the release of multiple graph snapshots. These observations motivate our work, even though we focus on labeled vertices.

B. Basic Theory

In this section, we present background concepts and notations for temporal social graphs, which lead to a concise and elegant introduction to LinkMirage.

Let us denote a time series of social graphs as \(G_0, \cdots, G_T\). For each temporal graph \(G_t = (V_t, E_t)\), the set of vertices is \(V_t\) and the set of edges is \(E_t\). Here, the vertices are labeled as \(1, 2, \cdots, |V_t|\), where \(|V_t|\) denotes the number of vertices in time \(t\). This paper focuses on undirected graphs where all the \(|E_t|\) edges are symmetric, i.e. \((i, j) \in E_t\) iff \((j, i) \in E_t\).

The adjacency matrix \(A_t\) and the transition probability matrix (TPM) \(P_t\) are closely related to \(G_t\). \(A_t\) is a binary matrix that directly represents the existence of edges of \(G_t\), where \(A_t(i, j) = 1\) if \((i, j) \in E_t\), otherwise \(A_t(i, j) = 0\). \(P_t\) is the transition probability matrix of the Markov chain on the vertices of \(G_t\). \(P_t\) measures the probability we follow an edge from one vertex to another, where \(P_t(i, j) = 1/\deg(i)\) (\(\deg(i)\) denotes the degree of vertex \(i\)) if \((i, j) \in E_t\), otherwise \(P_t(i, j) = 0\).

A random walk on the graph \(G_t\) is a sequence of vertices moving from a vertex \(v\) according to the transition probability matrix \(P_t\). Define the probability distribution of the random walk state at any given step \(k\) as \(\pi_t^{(k)}\) such that \(\pi_t^{(k)} = \pi_t^{(k-1)} \cdot P_t\). Therefore, we have \(\pi_t^{(0)} = P_t, \pi_t^{(1)} = P_t^2, \cdots\) where \(\pi_t^{(0)}\) is the initial state distribution and \(P_t^k\) is the \(k\)-step transition probability matrix of \(G_t\). Since the undirected and connected social graphs are irreducible and aperiodic, the corresponding Markov chain is ergodic. As \(k\) increases, the state distribution \(\pi_t^{(k)}\) will converge to a unique stationary distribution \(\pi_t\) which satisfies \(\pi_t = \pi_t \cdot P_t\) and \(\pi_t(i) = \deg(i)/|E_t|\).

To degrade leakage of link privacy, we make a time series of perturbed graphs \(\{G_t'\}_{t=0}^T\) instead of the original graph sequence \(\{G_t\}_{t=0}^T\) available, where \(G_t'\) denotes the perturbed graph of \(G_t\). Our goal is to construct a perturbed graph sequence that minimizes privacy leakage.

III. Structure-based dynamic perturbation: LinkMirage

A. Perturbation Goals

We envision that systems relying on trust relationships between users can bootstrap this information from online social network operators such as Facebook, Google+, Twitter or OSN applications, with access to the users’ social relationships. Since these systems may not protect the privacy of users’ social contacts, the OSN operators (or OSN applications) can perturb the social graph topology (by adding noise), before making it available.

Social graphs evolve from time to time, and applications would benefit from access to the most current version of the graph. Thus, the OSN operators (or OSN applications) can publish a sequence of perturbed graphs over time, mirroring the evolution of the social network topology.

However, publishing a time series of perturbed graphs raises a serious privacy challenge: an adversary can combine information available from multiple perturbed graphs over time to compromise the privacy of users’ social contacts. This leads to a trade-off between improving application performance via access to multiple graph snapshots, and the degradation of user privacy over time.

Consider a social graph series \(G_0 = (V_0, E_0), G_1 = (V_1, E_1), \cdots, G_T = (V_T, E_T)\). We want to transform the graph series as \(G_0' = (V_0, E_0'\), \(G_1' = (V_1, E_1'), \cdots, G_T' = (V_T, E_T')\), such that the vertices in \(G_t'\) remain the same as the original input graph \(G_t\), but the set of edges is perturbed to protect link privacy. Moreover, while perturbing the graph \(G_t\), the OSN operator has access to only the past snapshots in the time series (i.e. \(G_0, \cdots, G_{t-1}\)). Our perturbation goal is to balance the utility of social graph topologies and the privacy of users’ social contacts, across time.

Next, we discuss our dynamic perturbation algorithm LinkMirage. Later, we will define general metrics to quantify the utility and privacy of our algorithm, and analyze the metrics using real world social network topologies.

B. Dynamic Perturbation Algorithm: LinkMirage

Social networks evolve over time and a baseline approach is to perturb each graph snapshot independently. However, by combining information from multiple perturbed graphs, the adversary can deanonymize users’ social contacts.

Consider an extreme scenario where the graphs evolve very slowly, such that consecutive graph snapshots differ by just a few links/nodes. If perturbed graphs are generated independently for each snapshot, then the sequence of perturbed graphs provide significantly more observations to an adversary than just a single perturbed graph.

We argue that an effective perturbation method should take the evolution of the original graph sequence into consideration. For example, in the above scenario, consecutive snapshots can be perturbed similarly. Such an approach enhances privacy by correlating the noise addition over time, without degrading the utility of the perturbed topologies.
Community structures in evolving graphs by simultaneously evolving graphs; we refer the reader to algorithms. There are several methods in the literature to cluster.

There are two key steps in LinkMirage: dynamic clustering and perturbation. Noting that one node (the other two green nodes and one red node) and merge the remaining three red nodes to a big virtual node. Then, we cluster these new nodes, freed nodes and the remaining virtual node to detect communities in $G_t$.

Next, we compare the communities within $G_{t-1}$ and $G_t$, and identify the changed and unchanged subgraphs. For the unchanged subgraphs $C_1, C_2$, we set their perturbation at time $t$ to be identical to their perturbation at time $t-1$, denoted by $C'_1, C'_2$. For the changed subgraph $C_3$, we perturb it independently to obtain $C'_3$. Finally, we publish $G'_t = [C'_1, C'_2, C'_3]$. There are two key steps in LinkMirage: dynamic clustering and selective perturbation, which we describe in detail as follows.

1) Dynamic clustering: Dynamic clustering aims to find community structures in evolving graphs by simultaneously considering consecutive graph snapshots in its clustering algorithms. There are several methods in the literature to cluster evolving graphs; we refer the reader to [31] for a survey of the state-of-art approaches. For our perturbation mechanism, we adapt an approach [32] for clustering the graph $G_t$ using the clustering result for the previous graph $G_{t-1}$. This enables our perturbation mechanism to (a) exploit the link correlation/similarity in consecutive graph snapshots, and (b) reduce computation complexity by avoiding repeated clustering for unchanged links.

Clustering the graph $G_t$ from the clustering result of the previous graph $G_{t-1}$ requires a backtracking strategy. We use the maximum-modularity method [33] for clustering, which is hierarchical and thus easy to backtrack. Our backtrack strategy is to first maintain a history of the merge operations that led to the current clustering. When an evolution occurs, the algorithm backtracks over the history of merge operations, in order to incorporate the new additions and deletions in the graph.

More concretely, if the link between node $x$ and node $y$ is changed (added or deleted), we omit all the $m$-hop neighborhoods of $x$ and $y$ as well as $x$ and $y$ themselves from the clustering result of the previous timestamp, and then perform re-clustering. All the new nodes, the changed nodes and their $m$-hop neighbors, the remaining merged nodes in

Algorithm 1 LinkMirage, with dynamic clustering (steps 1-2) and selective perturbation (steps 3-6). The parameter $k$ denotes the random-walk steps as well as the perturbation level for each community. Here, $ch, un, in$ are short for changed, unchanged, inter-community, respectively.

Input: $\{G_t, G_{t-1}, G'_{t-1}\}$ if $t \geq 1$ or $\{G_t\}$ if $t = 0$;
Output: $G'_t$;
$G'_t, C_t =$null;
if $t=0$;
    cluster $G_0$ to get $C_0$;
    label $C_0$ as changed, i.e. $C_{0-ch} = C_0$;
else
/*dynamic clustering*/
    1. free the nodes within $m$ hops of the changed links;
    2. re-cluster the new nodes, the freed nodes and the remaining merged virtual nodes in $C_{(t-1)}$ to get $C_t$ ;
/*dynamic clustering*/
/*selective perturbation*/
    3. find the unchanged communities $C^\prime_{t-un}$ and the changed communities $C^\prime_{t-ch}$;
    4. let $G^\prime_{t-un} = G^\prime_{(t-1)-un}$;
    5. perturb $C^\prime_{t-ch}$ for $G^\prime_{t-ch}$ by the static perturbation method with perturbation parameter $k$;
    6. foreach community pair $a$ and $b$:
        if both of the communities belong to $C^\prime_{t-un}$
            $C'_{t-in}(a, b) = C'_{(t-1)-in}(a, b)$;
        else
            foreach marginal node $v_a(i)$ in $a$ and $v_b(j)$ in $b$ randomly add an edge $(v_a(i), v_b(j))$ with probability $\frac{\deg(v_a(i)) \deg(v_b(j)) v_a(i) v_b(j)}{\deg(v_a(i)) + \deg(v_b(j))}$ to $G^\prime_{t-in}(a, b)$;
/*selective perturbation*/
    return $G'_t = [C'_{t-ch}, G'_{t-un}, G'_{t-in}]$;

Fig. 1. Dynamic perturbation mechanism LinkMirage for $G_t$. Assume that $G_{t-1}$ has already been dynamically perturbed, based on dynamic clustering (step 1) and selective perturbation (step 2). Our mechanism analyzes the evolved graph $G_t$ (step 3) and aims to dynamically cluster $G_t$ based on the clustering result of the previous graph $G_{t-1}$. Dynamic clustering (step 4) first removes the $m$ hop neighborhood ($m = 2$) of new links (between green and blue nodes) in $G_t$ from the clusters in $G_{t-1}$, and merges nodes that remain in the clusters into a virtual node (large red node in step 4). Next, all new/free nodes and the merged virtual nodes are re-clustered to form communities. Finally, by comparing the communities in $G_{t-1}$ and $G_t$, we can implement selective perturbation (step 5), i.e. perturb the changed blue community independently and perturb the unchanged red & green communities in the same way as $G'_{t-1}$.
the previous clustering result would be considered as basic elements for clustering $G_t$.

For efficient implementation, we store the intermediate results of the hierarchical clustering process in a data structure, and upon link changes between $x, y$, we free the $m$-hop neighborhood of $x$ and $y$ from the stored data structure.

We also consider two other approaches for dynamically clustering evolving graphs but found them to be unsuitable for use in our perturbation mechanism. We refer interested readers to further information in the appendix.

2) Selective perturbation: After clustering $G_t$ based on $G_{t-1}$ using the dynamic clustering method, we perturb the $G_t$ based on $G_t, G_{t-1}$ and the perturbed $G’_{t-1}$. First, we compare the communities detected in $G_{t-1}$ and $G_t$, and classify them as changed or unchanged. Our unchanged classification does not require that the communities are exactly the same, but that the overlap among vertices/links exceeds a threshold. Our key idea is to keep the perturbation process for links in the unchanged communities to be identical to their perturbation in the previous snapshot. In this manner, we can preserve the privacy of these unchanged links to the largest extent; it is easy to see that alternate approaches would leak more information.

For the communities which are classified as changed, our approach is to perturb their links independently of the perturbation in the previous timestamp. This preserves the utility of perturbation algorithm. For independent perturbations, we leverage the static perturbation method of Mittal et al. in [19].

Finally, we need to interconnect the subgraphs/communities identified above. Suppose that $v_a$ nodes and $v_b$ nodes are connecting communities $a$ and $b$ respectively, and they construct an inter-community subgraph. For each marginal node $v_a(i) \in v_a$ and $v_b(j) \in v_b$ (here the marginal node in community $a(b)$ refers to the node that has neighbors in community $b(a)$), we randomly connect them with probability $\frac{\text{deg}(v_a(i)) \cdot \text{deg}(v_b(j))}{\text{deg}(v_a(i)) + \text{deg}(v_b(j)) + |E|}$. Here all the computations for $\text{deg}(\cdot), |E|$ only consider the marginal nodes. We can combine the perturbed links corresponding to the unchanged communities, changed communities, and inter-community subgraphs, to compute the output of our algorithm, i.e., graph $G’_t$. Detailed procedures are stated in Algorithm 1.

LinkMirage not only preserves the structural characteristics of the original graph series, but also protects the privacy of the users by randomizing the original links. As compared to prior work, our method provides stronger privacy and utility guarantees for evolving graphs.

Surprisingly, our approach of first isolating communities and then selectively perturbing them provides benefits even in a static context! This is because previous static approaches use a single parameter to control the privacy/utility trade-off. Thus, if we apply them to the whole graph using high privacy parameters, it would destroy graph utility (community structures). On the other hand, LinkMirage applies perturba-
tions selectively to communities; thus it is possible to use a very high privacy parameter in the perturbation process, while preserving community structures.

C. Visual depiction of algorithm

For our experiments, we consider a real world Facebook social network dataset [34]. This dataset contains friendship and interaction information (i.e. wall posts) among New Orleans regional network on Facebook, spanning from Sep 2006 to Jan 2009.

Here, we utilize the wall post interaction data which represents stronger trust relationships. This original dataset comprises of 46,952 nodes (users) connected by 876,993 edges (wall posts happening at different time are considered as different edges). We partitioned the dataset using three month intervals from Sep 2006 to Jan 2009 to construct a total of 9 social network instances, as shown in Table. I (repeated wall posts happening within three months are considered as the same edge). It is worth noting that the wall posts data experiences tremendous churn with less than 40% overlap for consecutive graphs. Since LinkMirage relies on the correlation between consecutive graphs, the evaluation of LinkMirage on the Facebook wall posts data is conservative. In actual applications, the graph sequence is likely to evolve in a much slower rate, which would show better performance by LinkMirage.

Fig. 2 depicts the outcome of our perturbation algorithm on the partitioned Facebook graph sequence with timestamp \( t = 3,4,5 \) (out of 9 snapshots), for varying perturbation parameter \( k \) (perturbation parameter for each community). For comparative analysis, we consider a baseline approach that applies static perturbation for each timestamp independently [19]. In the dynamic clustering step of our experiments, we free the two-hop neighborhoods of the changed nodes, i.e. \( m = 2 \).

The maximum-modularity clustering method yields two communities for \( G_3 \), three communities for \( G_4 \), and four communities for \( G_5 \). For the perturbed graphs, we use the same color for the vertices as in the original graph. This visualization allows us to see the change in community structures for perturbed graphs.

In both perturbation algorithms, we can see that for small perturbation parameter \( k \), the community structures (related to utility) are preserved, even though links are randomized. Even for high values of \( k \), LinkMirage preserves the macro-level structural characteristics of the graph, while randomizing the links within communities. On the other hand, for high values of \( k \), the static perturbation algorithm results in the loss of community structure, and appears to resemble a random graph. Thus, our approach of first isolating communities and applying perturbation at the level of communities has benefits even in a static context. Fig. 3 shows the privacy benefits of our perturbation algorithm for timestamps \( t = 4,5 \). We can see that LinkMirage reuses perturbed links (shown as black unchanged links) in unchanged communities (one unchanged community for \( t = 4 \) and two unchanged communities for \( t = 5 \)). Therefore, LinkMirage preserves the privacy of users’ social contacts by considering correlations among the social graph sequence. As we can see from Fig. 2, this benefit does not come at the cost of utility. In the following sections, we will formally quantify the privacy and utility properties of LinkMirage.

D. Facebook Application Deployment

To make the perturbed graphs available to other social trust based applications, a user needs to obtain the social relationships about his/her friends. To improve usability and adoption of our scheme, and to avoid dependance of OSN operators, we developed a Facebook application (available: https://apps.facebook.com/linkmirage/) that implements graph construction and perturbation. The work flow of the Facebook
application is as follows: (i) When a user visits the above URL, Facebook checks the credentials of the user, asks whether to grant the user’s friends permission, and then gets redirected to the application hosting server. (ii) The application server authenticates itself, and then queries Facebook for the information of the user’s friends who also use this application, and returns their information such as user’s id. The list of user’s friends can then be collected by the application server to construct a Facebook social graph. Leveraging LinkMirage, a perturbed graph would be available which preserves the link privacy of the users.

Real world systems such as Uproxy, Lantern and Kaleidoscope [35], as well as Sybil defenses [5]–[9], reputation system [14] and anonymity systems [1]–[4], can directly benefit from our Facebook application.

IV. PRIVACY

We now address the question of understanding link privacy of LinkMirage. We first consider adversaries with no prior information and analyze privacy performance. We further use Bayesian inference and information theory to characterize the privacy offered by LinkMirage for adversaries with prior information. Both theoretical analysis and experimental results with a Facebook dataset with 870K links and a large-scale Google+ dataset with 940M links show the benefits of LinkMirage over previous approaches. We also illustrate the relationship between our privacy metric and differential privacy.

A. Experimental Datasets

To illustrate how the temporal information degrades privacy, we consider two social network datasets. The first one is a large-scale Google+ dataset [36] whose temporal statistics are illustrated in Table. II. To the best of our known, this is the largest temporal dataset of social networks. The Google+ dataset is crawled from Jul. 2011 to Oct. 2011 which has 28,942,911 nodes and 947,776,172 edges. The dataset only considers link additions, i.e. all the edges in the previous graphs exist in the current graph. We partitioned the graph into 84 timestamps whose temporal characteristics are illustrated in Fig. 4. The second one is the 9-timestamp Facebook wall posts dataset [34] with temporal characteristics shown in Table. I.

B. Sampling Privacy

We first consider a simple situation where an adversary has no prior information about the original graphs. The adversary can infer the original graphs only from the published perturbed graphs. Recall that after community detection in our algorithm, we anonymize the links by leveraging the k-hop random walk.

Therefore, the perturbed graph G’ is actually a sampling of the k-hop graph Gk, where the k-hop graph Gk represents graph where all the k-hop neighbors in the original graph are connected. It is intuitive that larger difference between Gk and G’ represents better privacy. Here, we utilize the distance between the corresponding transition probability matrices ||P’ k – Pk||TV to measure this difference. We extend the definition of total variance from vector to matrix by averaging total variance distance of each row, i.e. ||P’ k – Pk||TV = \frac{1}{|V|} \sum_{v=1}^{|V|} ||P’ k(v) – Pk(v)||TV. where P’ k(v), Pk(v) denotes the v-th row of P’ k, Pk. We then formally define the sampling privacy as

**Definition 1.** The sampling privacy for a perturbed graph G’ t wrt the original graph G t and the perturbation parameter k is defined as Privacy\textsubscript{Sampling}(G t, G’ t, k) = ||P’ k – Pk||TV.

The adversary’s final objective is to obtain an estimated measurement of the original graph, e.g. the estimated transition probability matrix \hat{P} t which satisfies \hat{P} k = Pk. A straightforward manner to evaluate privacy is to compute the estimation error of the transition probability matrix i.e. ||P t – \hat{P} t||TV. We can derive the relationship between the sampling privacy and the estimation error as

**Theorem 1.** The sampling privacy is a lower bound of the estimation error, and

||P’ k – Pk||TV \leq k||P t – \hat{P} t||TV. (1)
struct a new perturbed graph
mation about the
is only
lective perturbation preserves correlation between consecutive
Fig. 5. (a)(b) show the temporal sampling privacy for the Google+ dataset and the Facebook dataset. The sampling privacy decreases as time evolves because more information is leaked with more perturbed graphs. Leveraging selective perturbation, LinkMirage achieves much better sampling privacy than the static baseline method.

\[ \| P_t^k - P'_t \|_{TV} = \| P_t^k - \hat{P}_t^k \|_{TV} \]
\[ = \frac{1}{2|V_t|} \sum_{v=1}^{|V_t|} \| P_t(v)P_t^{k-1} - \hat{P}_t(v)\hat{P}_t^{k-1} \|_1 \]
\[ \leq \frac{1}{2|V_t|} \sum_{v=1}^{|V_t|} \| P_t(v)P_t^{k-1} - \hat{P}_t(v)\hat{P}_t^{k-1} \|_1 \]
\[ + \frac{1}{2|V_t|} \sum_{v=1}^{|V_t|} \| P_t(v)P_t^{k-1} - \hat{P}_t(v)\hat{P}_t^{k-1} \|_1 \]
\[ \leq \frac{1}{2|V_t|} \sum_{v=1}^{|V_t|} \| P_t(v) - \hat{P}_t(v) \|_1 + \| P_t^{k-1} - \hat{P}_t^{k-1} \|_1 \]
\[ = k\| P_t - \hat{P}_t \|_{TV} \]
\[ \leq k\| P_t - \hat{P}_t \|_{TV} \]

We further consider the network evolution where the adversary can combine all the perviously perturbed graphs together to extract more \( k \)-hop information of the current graph. Under this situation, a strategic methodology for the adversary is to combine the perturbed graph series \( G_0',G_1',\ldots,G_t' \) to construct a new perturbed graph \( G_t' \), where \( G_t' = \bigcup_{i=0,1,\ldots,t} G_i \). The combined perturbed graph \( G_t' \) contains more information about the \( k \)-hop graph \( P_t^k \) than \( G_t' \). Correspondingly, \( \| P_t^k - P_t \|_{TV} \) would provide more information than \( \| P_t^k - \hat{P}_t \|_{TV} \).

We evaluate the sampling privacy of LinkMirage on both the Google+ dataset and the Facebook dataset. Here we perform our experiments based on a conservative assumption that a link always exists after it is introduced. The sampling privacy decreases with time since more information about the \( k \)-hop neighbors of the graph is leaked as shown in Fig. 5. Our selective perturbation preserves correlation between consecutive graphs, therefore leaks less information and achieves better privacy than the static baseline methods. For the Google+ dataset, the sampling privacy for the method of Mittal et al. is only \( \frac{1}{k} \) of LinkMirage after 84 timestamps.

C. Bayesian formulation for link privacy

Next, we consider adversaries with prior information about the original graph. We define the privacy of a link \( L_t \) (or a subgraph) in the \( t \)-th graph instance, as the probability of existence of the link (or a subgraph), computed by the adversary using its prior information \( W \), and the knowledge of the perturbed graph sequence \( G_0' \cdots G_t' \). Utilizing Bayesian inference, we have

**Definition 2.** For link \( L_t \) in a perturbed graph sequence \( G_0', G_1', \ldots, G_t' \), the adversary’s prior information \( W \), the Bayesian privacy is evaluated by the similarity between the posterior probability \( P(L_t | \{ G_i' \}_{i=0}^t \), \( W \) and the prior probability \( P(L_t | W) \), where the posterior probability is defined as

\[ P(L_t | \{ G_i' \}_{i=0}^t, W) = \frac{P(\{ G_i' \}_{i=0}^t | L_t, W) \times P(L_t | W)}{P(\{ G_i' \}_{i=0}^t | W)} \]

Smaller similarity implies better Bayesian privacy.

In the above expression, \( P(L_t | W) \) is the prior probability of the link, which can be computed based on the known structural properties of social networks, for example, by using link prediction algorithms. Note that \( P(\{ G_i' \}_{i=0}^t | W) \) is a normalization constant that can be analyzed by sampling techniques. The key challenge is to compute \( P(\{ G_i' \}_{i=0}^t | L_t, W) \).

For evaluation, we consider a special case where the adversary’s prior is the entire time series of original graphs except the link \( L_t \) (which is the link we want to quantify privacy for, and \( L_t = 1 \) denotes the existence of this link while \( L_t = 0 \) denotes the nonexistence of this link). Note that this is a very strong adversarial prior, which would lead to the worst-case analysis of link privacy. Denoting \( \{ \tilde{G}_t(L_t) \}_{i=0}^t \) as the prior which contains all the information except \( L_t \), we have the posterior probability of link \( L_t \) under the worst case is

\[ P(L_t | \{ G_i' \}_{i=0}^t, \{ \tilde{G}_t(L_t) \}_{i=0}^t) = \frac{P(\{ G_i' \}_{i=0}^t, L_t, \{ \tilde{G}_t(L_t) \}_{i=0}^t) \times P(L_t | \{ \tilde{G}_t(L_t) \}_{i=0}^t)}{P(\{ G_i' \}_{i=0}^t | \{ \tilde{G}_t(L_t) \}_{i=0}^t)} \]

where

\[ P(\{ G_i' \}_{i=0}^t | L_t, \{ \tilde{G}_t(L_t) \}_{i=0}^t) = P(\{ G_i' \}_{i=0}^t | G_0(L_t)) \times P(\{ G_i' \}_{i=0}^t | G_0(L_t), \{ \tilde{G}_t(L_t) \}_{i=0}^t) \]

The objective of perturbation algorithms is to make \( P(L_t | \{ G_i' \}_{i=0}^t, \{ \tilde{G}_t(L_t) \}_{i=0}^t) \) close to \( P(L_t | \{ \tilde{G}_t(L_t) \}_{i=0}^t) \).
i.e., the posterior probability should not differ much from the prior probability. The difference between the posterior probability and the prior probability represents the information leaked by the perturbation mechanism. Fig. 6 shows the posterior probability distribution for the whole Facebook graph sequence and the sampled Facebook graph sequence with 80% overlapping ratio, respectively. We can see that the posterior probability corresponding to LinkMirage is closer to the prior probability. In Fig. 6(b), taking the point where the link probability equals 0.1, the distance between the posterior CDF and the prior CDF for the static approach is a factor of 3 larger than LinkMirage ($k = 20$). We computed the prior probability using the link prediction method in [37]. Larger perturbation degree $k$ improves privacy and leads to smaller difference with the prior probability. Finally, by comparing Fig. 6(a) and Fig. 6(b), we can see that larger overlap in the graph sequence improves the privacy benefits of LinkMirage.

Next, we also compare with the work of Hay et al. in [18], which randomize the graph with $k$ real links deleted and another $k$ fake links introduced. The probability for a real link to be preserved in the anonymized graph is $\frac{k}{m}$, which should not be small otherwise the utility would not be preserved. Considering $\frac{k}{m} = 0.96$, the posterior probability for a link by the method of Hay et al. even without prior information would be 0.5, which is rather high compared with our dynamic method. LinkMirage provides significantly higher privacy than the work of Hay et al.

**D. Equivocation**

Considering the posterior probability of a link under the worst case $P(L_t|\{G^t_i\}_{i=0}^t, \{\tilde{G}_i(L_t)\}_{i=0}^t)$, we need to qualify the privacy metric. Since our goal is to reduce the information leakage of $L_t$ based on the perturbed graphs $\{G^t_i\}_{i=0}^t$ and the prior knowledge $\{G_i(L_t)\}_{i=0}^t$, we consider the metric of equivocation to quantify privacy. Shannon introduced the concept of equivocation as the conditional entropy of a private message given the observed variables [38]. Sankar et al. in [39] use equivocation as a measure of privacy of their data transformation. The aim is also to maximize the average equivocation of the unknown input $I$ given the observables $O$, i.e., $H(I|O)$ (where $H$ denotes entropy of a variable). Here, we define our metric for link privacy as the equivocation privacy.

**Definition 3.** The equivocation for a link $L_t$ in a perturbed graph $G^t_i$ wrt the original graph $G_i$ and the adversary’s prior information $\{\tilde{G}_i(L_t)\}_{i=0}^t$ is defined as

$$\text{Privacy}_{\text{equivocation}} = H(L_t|\{G^t_i\}_{i=0}^t, \{\tilde{G}_i(L_t)\}_{i=0}^t)$$

Then, we quantify the behavior of equivocation over time. For our analysis, we continue to consider the worst case prior of the adversary knowing the entire graph sequence except the link $L$. To make the analysis tractable, we add another condition that if the link $L$ exists, then it exists in all the graphs (link deletions are rare in real-world friendship networks). For a large-scale graph, only one link would not affect the clustering result.

**Theorem 2.** The equivocation privacy decreases with time,

$$H(L_t|\{G^t_i\}_{i=0}^t, \{\tilde{G}_i(L_t)\}_{i=0}^t) \geq H(L_t|\{G^{t+1}_i\}_{i=0}^t, \{\tilde{G}_i(L_t)\}_{i=0}^t)$$

(6)

The inequality follows from the theorem conditioning reduces entropy in [40]. Eq.6 shows that the equivocation would not increase as time evolves. The reason is that over time, multiple perturbed graphs can be used to infer additional information about the existence of link $L$.

Next, we theoretically show why LinkMirage has better privacy performance than the static method. For each graph $G_t$, denote the perturbed graphs using LinkMirage and the static method as $G^t_i, G^{t,s}_i$, respectively.

**Theorem 3.** The equivocation privacy for LinkMirage is greater than that for the static perturbation method, i.e.

$$H(L_t|\{G^t_i\}_{i=0}^t, \{\tilde{G}_i(L_t)\}_{i=0}^t) \geq H(L_t|\{G^{t,s}_i\}_{i=0}^t, \{\tilde{G}_i(L_t)\}_{i=0}^t)$$

(7)

**Proof:** In LinkMirage, the perturbation for the current graph $G_t$ is based on perturbation for $G_{t-1}$. Let us denote the
changed subgraph between \( G_t-1, G_t \) as \( G_t-\text{ch} \). Then we have

\[
H(L_t | \{G'_t\}^t_{i=0}, \{\tilde{G}_i(L_t)\}^t_{i=0}) = H(L_t | \{G'_t\}^t_{i=0}, G'_t-1, G'_t-\text{ch}, G'_t-s, \{\tilde{G}_i(L_t)\}^t_{i=0}) = H(L_t | \{G'_t\}^t_{i=0}, G'_t-s, \{\tilde{G}_i(L_t)\}^t_{i=0}) \geq H(L_t | \{G'_t\}^t_{i=0}, \{\tilde{G}_i(L_t)\}^t_{i=0}) \tag{8}
\]

where inequality (1) also comes from the theorem conditioning reduces entropy. The inequality (2) generalizes the inequality (1) from a snapshot \( t \) to the entire sequence. From Eq.8, we can see that LinkMirage offers superior equivocation compared to the static perturbation, and thus provides higher privacy. □

Next, we experimentally analyze our privacy metric of equivocation over time, i.e. \( H(L_t | \{G'_t\}^t_{i=0}, \{\tilde{G}_i(L_t)\}^t_{i=0}) \). Fig. 7 depicts the equivocation metric using the whole Facebook graph sequence and the sampled Facebook graph sequence with 80% overlap. We can see that the static perturbation leaks more information over time. In contrast, selective perturbation achieves significantly higher equivocation. In Fig. 7(a), after 9 snapshots, and using \( k = 5 \), the equivocation of the static perturbation method is roughly \( \frac{1}{10} \) of the equivocation of LinkMirage. This is because selective perturbation explicitly takes the temporal evolution into consideration, and stems privacy degradation via the selective perturbation step. Comparing Fig. 7(a) and Fig. 7(b), we can see that LinkMirage has more advantages for larger overlapped graph sequence.

We also compare with the work of Hay et al. in [18]. For the first timestamp, the probability for a real link to be preserved in the anonymized graph is \( \frac{1}{10} \). As time evolves, the probability would increase to \( 1 - \frac{1}{2^{10}} \). The corresponding equivocation is shown as the black dotted line in Fig. 7, which converges to 0 very quickly (for utility consideration, we consider \( \frac{1}{2^{10}} = 0.96 \)). Compared with the work of Hay et al, LinkMirage significantly improves privacy performance.

**E. Relationship with Differential Privacy**

Our Bayesian privacy analysis considers the worst case adversarial prior to infer the existence of a single link in the graph. Next, we uncover a novel relationship between this Bayesian privacy analysis, and differential privacy.

Differential privacy is a popular theory to evaluate the privacy of a perturbation mechanism [41], [42]. The framework of differential privacy defines local sensitivity of a function \( f \) on a specific dataset \( D \) as the maximal \( |f(D) - f(D')|_1 \) for all \( D' \) differing from \( D \) in at most one element:

\[
df = \max_{D'} |f(D) - f(D')|_1 \tag{9}
\]

Based on the theory of differential privacy, a mechanism that adds independent Laplacian noise \( df/\epsilon \) to \( f \), satisfies \( \epsilon \)-differential privacy. The degree of added noise, which determines the utility of the mechanism, depends on the local sensitivity. To achieve a good utility as well as privacy, the local sensitivity \( df \) should be as small as possible.

**Remark 1.** The requirement for good utility-privacy balance in differential privacy is equivalent to the objective of our Bayesian analysis under the worst case.

**Proof:** When considering differential privacy for dynamic graphs \( \{G_t\}^t_{i=0} \), we have

\[
f(D) = P(\{G'_t\}^t_{i=0} | \{\tilde{G}_i(L_t)\}^t_{i=0}, L_t = 1) \tag{10}
\]

\[
f(D') = P(\{G'_t\}^t_{i=0} | \{\tilde{G}_i(L_t)\}^t_{i=0}, L_t = 0)
\]

For a good privacy performance, we need

\[
P(\{G'_t\}^t_{i=0} | \{\tilde{G}_i(L_t)\}^t_{i=0}, L_t = 1) \approx P(\{G'_t\}^t_{i=0} | \{\tilde{G}_i(L_t)\}^t_{i=0}, L_t = 0) \tag{11}
\]

Recall the probability of \( \{G'_t\}^t_{i=0} \) given \( \{\tilde{G}_i(L_t)\}^t_{i=0} \) as

\[
P(\{G'_t\}^t_{i=0} | \{\tilde{G}_i(L_t)\}^t_{i=0}) = P(\{G'_t\}^t_{i=0} | \{\tilde{G}_i(L_t)\}^t_{i=0}, L_t = 1)P(L_t = 1 | \{\tilde{G}_i(L_t)\}^t_{i=0}) + P(\{G'_t\}^t_{i=0} | \{\tilde{G}_i(L_t)\}^t_{i=0}, L_t = 0)P(L_t = 0 | \{\tilde{G}_i(L_t)\}^t_{i=0}) \tag{12}
\]
It is easy to see that if Eq.11 holds, we have
\[
P(L_t | \{G'_t\}_{t=0}^T, \{G_t(L_t)\}_{t=0}^T) = \frac{P(\{G'_t\}_{t=0}^T | \{G_t(L_t)\}_{t=0}^T, L_t) \times P(L_t | \{\tilde{G}_t(L_t)\}_{t=0}^T)}{P(\{\tilde{G}_t(L_t)\}_{t=0}^T)} 
\]
which is the same as in Definition 2 and means that the posterior probability is similar to the prior probability, i.e. the perturbed graphs do not leak much information. 

Furthermore, [43] has formulated the objective of differential privacy as maximizing the equivocation. Based on the theoretical results in our (equivocation) analysis, LinkMirage can also increase the equivocation.

V. UTILITY

In this section, we develop formal metrics to characterize the utility of the perturbed graph sequence. Furthermore, we theoretically analyze the lower bound on utility for LinkMirage, uncover connections between our utility metric and structural properties of the graph sequence, and experimentally analyze our metric using the real world Google+ and Facebook datasets

A. Metrics

Previous work has only considered utility metrics for a static case, by comparing a single graph \(G_t\) with its perturbed version \(G'_t\). One intuitive global evaluation metric is the degree of vertices. It is interesting to find that the expected degree of each node in the perturbed graph is the same as the original degree (shown in Theorem 4). We defer the proof to the Appendix.

Theorem 4. The expected degree of each node after perturbation by LinkMirage is the same as in the original graph: 
\[
\forall v \in V_t, \mathbb{E}(\deg'(v)) = \deg(v), \text{ where } \deg'(v) \text{ denotes the degree of vertex } v \text{ in } G'_t. 
\]

Another global utility metric is modularity [44], which is closely related to community structure. A high modularity score indicates a sophisticated internal connection structure. Other global utility metrics are mixing time, second largest eigenvalue modulus, which will be analyzed in Section V-D.

All the above metrics evaluate the global utility performance. To understand the utility in a fine-grained level, previous work has considered utility from the perspective of each vertex. For example, the static utility distance (SUD) is defined as
\[
\text{SUD}(G_t, G'_t, l) = \sum_{v \in V_t} \frac{1}{|V_t|} d\left(\left\langle P_t^{(l)}(v), (P_t')^{(l)}(v)\right\rangle\right) 
\]
where \(d(\cdot)\) denotes the distance between two vectors.

Next, we extend this definition to capture the overall utility distance for a perturbed graph sequence.

Definition 4. The static utility distance of a perturbed graph sequence \(G'_0, G'_1, \cdots, G'_T\) wrt the original graph sequence \(G_0, G_1, \cdots, G_T\), and an application parameter \(l\) is defined as
\[
\text{SUD}(G_0, G_1 \cdots G_T, G'_0, G'_1 \cdots G'_T, l) = \frac{1}{T} \sum_{t=1}^{T} \text{SUD}(G_{t-1}, G_t, G'_{t-1}, G'_t, l) 
\]
\[
= \frac{1}{T} \sum_{t=1}^{T} \sum_{v \in V_t} \frac{1}{|V_t|} d\left(\left\langle P_t^{(l)}(v), (P_t')^{(l)}(v)\right\rangle\right) 
\]

Recall that LinkMirage considers the similarity/correlation between consecutive graphs. In order to account for the graph evolution, it is also natural to define the utility metric based on the consecutive graphs instead of just one instance. For time \(t - 1\) and \(t\), the difference between \(G_{t-1}\) and \(G_t\) reflects the structural evolution of the dynamic network. We consider a new metric for utility that captures the similarity of the structural evolution between consecutive graph instances in the original graph sequence \((G_{t-1}, G_t)\), and the perturbed graph sequence \((G'_{t-1}, G'_t)\).

Definition 5. The temporal utility distance (TUD) of a perturbed graph pair \(G'_{t-1}, G'_t\) wrt the original graph pair \(G_{t-1}, G_t\), and an application parameter \(l\) is defined as the statistical distance between \((G_t - G_{t-1})\) and \((G'_t - G'_{t-1})\). The statistical distance is computed from the perspective of all vertices \(v\) in the graph as the distance between the vectors \(P_t^{(l)}(v) - P_{t-1}^{(l)}(v)\) and \((P_t')^{(l)}(v) - (P'_{t-1})^{(l)}(v)\)

\[
\text{TUD}(G_{t-1}, G_t, G'_{t-1}, G'_t, l) = \text{Distance}(G_t - G_{t-1}, G'_t - G'_{t-1}, l) = \sum_{v \in V_{t-1} \cup V_t} \frac{1}{|V_t|} d\left(\left\langle P_t^{(l)}(v), (P_t')^{(l)}(v)\right\rangle\right) 
\]

\[
= \sum_{v \in V_{t-1} \cup V_t} d\left(\left\langle P_t^{(l)}(v), (P_t')^{(l)}(v)\right\rangle\right) \left[ \begin{array}{c} P_t^{(l)}(v) \\ -P_{t-1}^{(l)}(v) \end{array} \right] - \left[ \begin{array}{c} (P_t')^{(l)}(v) \\ -(P'_{t-1})^{(l)}(v) \end{array} \right] \right| |V_{t-1} \cup V_t| 
\]

Note that a lower value of \(\text{TUD}(G_{t-1}, G_t, G'_{t-1}, G'_t, l)\) corresponds to higher utility (we want the distance between probability distributions of original graph sequence and perturbed sequence to be low).

Our TUD definition intuitively captures the behavioral differences of \(l\) hop random walks between the original graph pair, and the perturbed graph pair. We note that random walks are closely linked to structural properties of social networks. In fact, a lot of social network based security applications such as Sybil defenses and anonymity systems directly perform random walks in their protocols. The parameter \(l\) is application specific; for applications that require access to fine grained local community structures, such as recommendation systems, the value of \(l\) should be small. For other applications that utilize coarse and macro community structure of the social graphs, such as Sybil defense mechanisms, \(l\) can be set to a larger value (typically around 10 according to [6]).

We can see that the static utility distance is just a special case of Definition 5. In particular, it does not account for the temporal evolution of social networks. So far, we have discussed how to compute the temporal utility distance for a
perturbed graph pair. Next, we formally define the temporal utility distance (TUD) metric below.

### Definition 6

The temporal utility distance of a perturbed graph sequence \( G_0', G_1', \ldots, G_T' \) wrt the original graph sequence \( G_0, G_1, \ldots, G_T \), and an application parameter \( l \) is defined as the average temporal utility distance for each consecutive graph pair \( G_{t-1}' \cdot G_t \).

\[
\text{TUD}(G_0, G_1 \cdots G_T, G_0', G_1' \cdots G_T', l) = \frac{1}{T} \sum_{t=1}^{T} \text{TUD}(G_{t-1}, G_t, G_{t-1}', G_t', l)
\]

(17)

\[
= \frac{1}{T} \sum_{t=1}^{T} \left\| \begin{bmatrix} P_l^t \\ -P_{l-1}^t \end{bmatrix} - \begin{bmatrix} (P_l^t) \\ -(P_{l-1}^t) \end{bmatrix} \right\|_{TV}
\]

B. Upper Bound of SUD and TUD

LinkMirage aims to stem the degradation of link privacy over time. Usually, mechanisms that preserve privacy trade-off application utility. In the following, we will theoretically derive an upper bound on the SUD and TUD for our algorithm. This corresponds to a lower bound on utility that LinkMirage is guaranteed to provide.

### Theorem 5

The static utility distance of each graph instance by LinkMirage is upper bounded by \( 2l \) times the sum of the utility distance of each community \( \epsilon \) and the ratio cut \( \delta_t \) of \( G_t \), i.e.

\[
\text{SUD}(G_t, G_t', l) \leq 2l (\epsilon + \delta_t)
\]

(18)

To improve the readability, we defer the result to the Appendix. Based on the triangular inequality theorem, we have

\[
\text{TUD}(G_0, G_1 \cdots G_T, G_0', G_1' \cdots G_T', l)
\]

\[
= \frac{1}{T} \sum_{t=1}^{T} \left\| \begin{bmatrix} P_l^t \\ -P_{l-1}^t \end{bmatrix} - \begin{bmatrix} (P_l^t) \\ -(P_{l-1}^t) \end{bmatrix} \right\|_{TV}
\]

\[
\leq \frac{1}{2T} \sum_{t=1}^{T} \sum_{v=1}^{l} \left\| P_l^t(v) - (P_{l-1}^t) \right\|_{TV} + \left\| P_l^t - (P_{l-1}^t) \right\|_{TV}
\]

(19)

\[
= \frac{1}{2T} \sum_{t=1}^{T} \left( \text{SUD}(G_t, G_t', l) + \text{SUD}(G_{t-1}, G_{t-1}', l) \right)
\]

Combining Theorem 5 and Eq.19, we have

### Theorem 6

The temporal utility distance of a graph sequence by LinkMirage is upper bounded by \( 2l \) times the sum of the utility distance of each community \( \epsilon \) and the ratio cut \( \delta_t \) of \( G_t \), i.e.

\[
\text{TUD}(G_0, G_1 \cdots G_T, G_0', G_1' \cdots G_T', l) \leq 2l (\epsilon + \delta_t)
\]

(20)

where \( \delta_t \) denotes the number of inter-community links over the number of vertices \( 45 \), and each community \( C_k \) within \( G_t \) satisfies \( ||C_k - C_k'||_{TV} \leq \epsilon \).

While better privacy usually requires adding more noise to the original sequence to obtain the perturbed sequence, Theorem 5 and Theorem 6 bounds the temporal utility distance. Note that an upper bound on utility distance corresponds to a lower bound on utility of our algorithm; thus we can see that LinkMirage is guaranteed to provide a minimum level of performance.

In the derivation process, we do not take specific evolutionary pattern such as overlapped degree into consideration, therefore our theoretical upper bound is rather loose. Next, we will show that in practice, LinkMirage achieves smaller utility distance (higher utility) than the baseline approach of independent static perturbations.

### C. Utility Experiments Analysis

We analyze the utility performance of LinkMirage on the the large-scale Google+ (we limit our analysis to the first three timestamps) and the Facebook dataset.

First, we show the modularity of the perturbed graphs and the original graph (computed by the timestamp \( t = 3 \) in the Google+ dataset and the Facebook dataset, respectively). In Table III, we can see that LinkMirage preserves the modularity characteristics of the original graph, while the method of Mittal et al degrades this utility metric especially for larger perturbation parameter \( k \) (recall the visual intuition of LinkMirage in Fig. 2).

We show the static utility distance (by averaging the entire graph sequence using Eq.15) of the perturbed graphs, for varying perturbation degree \( k \) and application level parameter \( l \). Fig. 8 depicts the static utility distance for the Google+ and the Facebook graph sequences, averaged over individual graph instances. We can see that LinkMirage has lower static utility distance than that of Mittal et al. Thus our approach of first identifying community structures and then performing perturbation is of independent interest even in the static case.

We can also see that as \( k \) increases, the distance metric increases. This is natural since additional noise will increase the distance between probability distributions computed from the original and the perturbed graph series. As the application parameter \( l \) increases, the distance metric decreases. This illustrates that LinkMirage is more suited for security applications that rely on macro community structures, as opposed to applications that require exact information about one or two hop neighborhoods.

Fig. 9 illustrates the temporal utility distance of the perturbed graphs. We can see that LinkMirage also has better temporal utility performance. Results pertaining to varying \( k \) and \( l \) are similar to the previous case: increasing \( k \) results in larger distance and increasing \( l \) results in a smaller distance. Furthermore, by comparing Fig. 8(a) and Fig. 9(a), we can see that LinkMirage achieve better superiority wrt temporal utility than wrt static utility. The reason is that LinkMirage
and Theorem 16 in independent static perturbations. Comparing Fig. 8(b), the advantage of LinkMirage wrt temporal utility distance is greater for the Google+ dataset. The reason is that there are more overlapping links in the first three timestamps of Google+ dataset. In Fig. 9(a), LinkMirage decreases the temporal utility distance by a factor greater than 2 compared to the static perturbation method, even for larger values of $l$. Fig. 10 shows how the pairwise TUD between the consecutive snapshots evolves with time ($l = 5$) using the Facebook dataset. We can see the power of LinkMirage: pairwise utility distance over time is consistently smaller than the static approach. It is worth noting that the utility value for the first snapshot is computed using the static utility distance, since there is no previous snapshot to compare it with.

D. Metrics Analysis

The mixing time $\tau_{\epsilon}(G_t)$ measures the time required for the Markov chain to converge to its stationary distribution, and is defined as $\tau_{\epsilon}(G_t) = \min_{\epsilon} \max_{(r,v)} |P^\epsilon_t(v) - \pi_{\epsilon}| < \epsilon$. Based on the Perron-Frobenius theory, we denote the eigenvalues of $P_t$ as $1 = \mu_1(G_t) \geq \mu_2(G_t) \geq \cdots \geq \mu_{|V_t|}(G_t) \geq -1$.

The convergence rate of the Markov chain to $\pi_{\epsilon}$ is determined by the second largest eigenvalue modulus (SLEM) as $\mu(G_t) = \max \{ \mu_2(G_t), -\mu_{|V_t|}(G_t) \}$.

Since our utility distance is defined using the transition matrix $P_t$, the metric is closely related to structural properties of the graph sequence, as shown in Theorem 7 and Theorem 8.

**Theorem 7.** Let us denote the utility distance between the per-
turbed graph $G'_t$ and the original graph $G_t$ by $SUD(G_t, G'_t, l)$, then we have $\tau_{G'_t}(SUD(G_t, G'_t, \tau_{G'_t}(\epsilon)) - \epsilon) \geq \tau_{G_t}(\epsilon)$.

**Theorem 8.** Let us denote the second largest eigenvalue modulus (SLEM) of transition matrix $P_t$ of graph $G_t$ as $\mu_{G_t}$. We can bound the SLEM of a perturbed graph $G'_t$ using the mixing time of the original graph, and the utility distance between the graphs as $\mu_{G'_t} \geq 1 - \frac{\log n + \log SUD(G_t, G'_t, \tau_{G'_t}(\epsilon))}{\tau_{G_t}(\epsilon)}$.

**VI. APPLICATIONS**

In this section, we demonstrate the applicability and benefits of LinkMirage to social network based systems, from the perspective of Sybil defenses, reputation system, anonymous communication and vertex anonymity.

**A. Sybil Defenses**

We use SybilLimit [6] as representative protocol for Sybil defenses. We want to analyze how the use of a perturbed graph changes the Sybil detection performance of the system. Fig. 11(a) depicts the false positives (honest users misclassified as Sybils) wrt the random route length to improve communication on the Sybllimit protocol. Fig. 11(b) shows the final attack edges wrt the attack edges in the original topology. We can see that the false positive rate is much lower for the perturbed graphs than for the original graph, while the number of the attack edges stay the same for the original graph and the perturbed graphs. This shows the effectiveness of LinkMirage in real world Sybil defenses. The number of Sybil identities that an adversary can insert is given by $S = g' \cdot w'$ ($g'$ is the number of attack edges and $w'$ is the random route parameter). Since $g'$ stays almost invariant and the random route parameter $w'$ (for any desired false positive rate) is reduced, LinkMirage improves Sybil resilience and provides the privacy of the social trust relationships such that Sybil defense protocols continue to be applicable (similar to static approaches whose Sybil-resilience performance have been demonstrated in previous work).

**B. Social networks in Reputation Systems**

Next, we demonstrate the application of LinkMirage to reputation systems such as Yelp. Yelp formulates a social network, which is available in http://www.yelp.com/dataset challenge. We consider the largest connected component in the Yelp social graph, which consists of 28,977 users and 301,676 edges. The Yelp social network aims to improve propagation of reviews and news. Therefore, we consider the averaging Hitting time of the graph as the utility of reputation systems. The Hitting time here represents the time taken for a business that was originally discovered by user $i$ to propagate to user $j$ [46]. Fig. 12 shows the relationships between the utility (Hitting time) and the perturbation parameter $k$. We can see that larger perturbation parameter would result in smaller Hitting time, because larger perturbation parameter results in a more random graph. By randomizing the links within communities, our dynamically perturbed graphs have smaller hitting time than that of the original graphs (corresponding to $k = 0$). Therefore, LinkMirage improves performance of Yelp through accelerating information propagation in the Yelp social graph, while preserving privacy for social contacts.

**C. Anonymous Communication**

As a concrete application, we consider the problem of anonymous communication [1]. Systems for anonymous communication, such as the Tor network, aim to improve user’s privacy by hiding the communication link between the user and the remote communicating entity. Nagaraja et al. and others [2]–[4] have suggested that the security of anonymity systems can be improved by leveraging users’ trusted social contacts. However, it is difficult to realize the design of such systems, since relay operators and users of the Tor network are hesitant to publicly reveal their trusted social contacts. We envision that our work can be a key enabler for the design of such social network based systems, while preserving the privacy of users’ social contacts.

Since the popular Tor network is a low-latency anonymity system (does not introduce timing delays), we restrict our analysis to low-latency anonymity systems that leverage social links, such as the Pisces protocol [4]. Similar to the Tor protocol, users in Pisces rely on proxy servers and onion routing for anonymous communication. However, the relays involved in the onion routing path are chosen by performing a random walk on a trusted social network topology. Recall that LinkMirage provides lower temporal utility distance (higher utility) as compared to static perturbations. We now show that this translates into improved anonymity over time, by performing an analysis of the degradation of user anonymity over multiple graph snapshots. For each graph snapshot, we consider a worst case anonymity analysis as follows: if a users’ neighbor in the social topology is malicious, then over multiple communication rounds (within that graph instance) its anonymity will be compromised using state-of-art traffic analysis attacks [47]. Now, suppose that all of a user’s neighbors in the first graph instance are honest. As the perturbed graph sequence evolves, there is further potential for degradation of user anonymity since in the subsequent instances, there is a chance of the user connecting to a malicious neighbor. Suppose the probability for a node to be malicious is $f$. Denote $n_1(v)$ as the distinct neighbors of node $v$ at time $t$. For a temporal graph sequence, the number of the union neighbors
Fig. 11. (a) show the false positive rate for sybil defenses. We can see that the perturbed graphs have lower false positive rate than the original graph. Random route length is directly proportional to the number of Sybil identities that can be inserted in the system. (b) show the final attack edges in the perturbed topologies as a function of attack edges in the original topology. The final attack edges are roughly the same for the perturbed graphs and the original graphs.

Fig. 13. The worst case probability of de-anonymizing users’ communications ($f = 0.1$). Over time, LinkMirage provides better anonymity compared to static approaches.

\[
\cup_{k=0}^t n_k(v) \text{ of } v \text{ increases with time, and the probability for } v \text{ to be attacked under the worst case is:}
\]

\[
P_{t}^{\text{attack}}(v) = 1 - (1 - f)^{|\cup_{k=0}^t n_k(v)|}
\]  

(21)

Fig. 13 depicts the degradation of worst case anonymity as a function of number of perturbed topologies. We can see that the attack probability for our method is lower than the static approach. This is because over consecutive graph instances, the users’ social neighborhood has higher similarity as compared to the static approach, reducing potential for anonymity degradation. Therefore, LinkMirage can provide better security for anonymous communication, and other trust-based applications.

D. Vertex Anonymity

To further explore the benefits of LinkMirage, we also try to apply LinkMirage to anonymize vertices, i.e. to publish a perturbed topology without labeling any vertex. In [48], the authors model the anonymization as a sampling process where the sampling probability $p$ denotes the probability of an edge in the original graph $G_o$ to exist in the anonymized graph $G_a$. LinkMirage can also be applied for vertex anonymity, where the perturbed graph $G'$ (corresponding to the anonymized graph $G_a$) is sampled from the $k$-hop graph $G^k$ (corresponding to $G_o$). Many de-anonymization methods [26] have been proposed to recover the identities of anonymized vertices based on the structural similarity between the anonymized graph $G_a$ and a given auxiliary graph $G_a$.

Ji et al. in [48] derived a theoretical bound for the number of nodes needed for perfect de-anonymization, and found that a weaker bound is needed with larger sampling probability $p$. Larger $p$ implies that the anonymized graph $G'$ is topologically more similar to $G$, making it easier to enable a perfect de-anonymization. When considering social network evolution, the sampling probability $p$ can be estimated as $\frac{|E(G^k_0, \ldots, G^k_t)|}{|E(G^k_0, \ldots, G^k_t)|}$, where $E(G^k_0, \ldots, G^k_t)$ are the edges of the perturbed graph sequence, and $E(G^k_0, \ldots, G^k_t)$ are the edges of the $k$-hop graph sequence. Compared with the static baseline approach of Mittal et al., LinkMirage selectively reuses information from previously perturbed graphs, thus leading to smaller overall sampling probability $p$, which makes it harder to perfectly de-anonymize the graph sequence. For example, the average sampling probability $p$ for the Google+ temporal dataset (with $k = 2$) is 0.431, 0.973 for LinkMirage and the static method respectively. For the Facebook temporal dataset (with $k = 3$), the average sampling probability $p$ is 0.00012, 0.00181 for LinkMirage and the static method respectively. Therefore, LinkMirage is more resilient against de-anonymization attacks even when applied to Vertex anonymity.

VII. CONCLUSION

We propose a dynamic perturbation method LinkMirage for preserving link privacy in evolving graphs. For successive graph instances, LinkMirage first performs joint clustering of the graph instances (dynamic clustering) to identify the changed and unchanged community structures. Next, LinkMirage minimizes information leakage by selectively perturbing only the changed communities independently, and reusing previous perturbation for the unchanged communities. We also developed a Facebook application to realize our algorithm without relying on the OSN operators.

We formally defined the sampling privacy to consider an adversary with no prior information, and further analyze privacy by leveraging Bayesian inference and information theory to consider an adversary with certain prior information. We also show connections between our privacy definitions with
differential privacy. For evaluating utility, we not only extend the static utility metrics for multiple timestamps, but also propose the notion of temporal utility, and show the connection between these metrics and global graph properties.

We analyzed the link privacy and utility of LinkMirage both theoretically and experimentally using real-world Facebook dataset (with 870K links) and the large-scale Google+ dataset (with 940M links). We found that LinkMirage significantly outperforms baseline static techniques in terms of both link privacy and utility. In fact, our approach of first identifying community structures and then applying perturbations is of interest even in the context of static topologies. Our work enables the deployment of real world applications such as social trust based Sybil defenses, reputation systems, and anonymity systems while preserving the privacy of users’ social contacts.

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VIII. APPENDIX

A. Rejected Ideas for Dynamic Community Detection

We considered two other approaches for dynamically clustering evolving graphs, but found them to be unsuitable for use in our perturbation mechanism. One approach to dynamic clustering involves performing community detection at each time stamp independently, and to then establish relationships between communities to track their evolution. We found that this approach suffers from performance issues induced by
inherent randomness in clustering algorithms, in addition to increased computational complexity.

Another interesting approach is to combine multiple graph snapshots into a single coupled graph. The coupled graph is constructed by adding edges between the same nodes across different graphs. Clustering can be performed on the single coupled graph. We found that the clustering performance is almost the same as the degree distribution in the original graph.

Fig. 14. Degree distribution of nodes in Facebook interaction dataset and the expected degree distribution of nodes in the perturbed graphs by LinkMirage. We can see that the expected degree distribution in the perturbed graphs are almost the same as the degree distribution in the original graph.

B. Proof of Theorem 4: Expectation of Perturbed Degree

According to Theorem 3 in [19], we have

$$E(\deg'_{com}(v)) = \deg(v)$$

where \(\deg'_{com}(v)\) denotes the degree of \(v\) after perturbation within community.

Then we consider the random perturbation for inter-community subgraphs. Since the probability for an edge to be chosen is \(\frac{\deg(v_a(i))\deg(v_b(j))(|v_a| + |v_b|)}{|E_{ab}|(|v_a| + |v_b|)}\), the expected degree after inter-community perturbation satisfies

$$E(\deg'_{inter}(v_a(i))) = \sum_j \deg(v_a(i))\deg(v_b(j))(|v_a| + |v_b|)$$

$$= \deg(v_a(i))$$

Combine the two expectations in Eq.22 and Eq.23, we have

$$E(\deg' (v)) = \deg(v)$$

Next, we show experimental results validating the above theorem. Figure 5 depicts the node degrees of the Facebook dataset with timestamp \(t = 3\), and the expected node degrees of the corresponding perturbed graphs, with respect to different perturbation parameter \(k\). We can see that the degree distributions are nearly identical for all graphs, validating our theoretical results.

C. Proof of Theorem 5: Upper Bound for the Static Utility Distance

To prove Theorem 5, we first introduce some notations and concepts. We consider two perturbation methods in the derivation process below. The first method is our dynamic perturbation method, which takes the graph evolution into consideration. The second method is the intermediate method, where we only implement dynamic clustering without selective perturbation. That is to say, we cluster \(G_t\), then perturb each community by the static method and each inter-community subgraphs by randomly connecting the marginal nodes, independently. We denote the perturbed graphs corresponding to the dynamic, the intermediate method as \(G'_t, G''_t\) respectively. Similarly, we denote the perturbed TPM for the two approaches as \(P'_t, P''_t\). To simplify the derivation process, we partition the proof into two stages. In the first stage, we derive the SUD upper bound for the intermediate perturbation method. In the second stage, we derive the relationship between \(G'_t\) and \(G''_t\). Results from the two stages can be combined to find the TUD upper bound of the dynamically perturbed graph sequence.

1) The first stage:

**Theorem 9.** When \(l = 1\), the static utility distance for each graph instance after intermediate perturbation is bounded by the sum of the utility distance of each community \(\epsilon\) and the ratio cut \(\delta_t\) of \(G_t\).

$$\text{SUD}(G_t, G''^t, 1) \leq \epsilon + \delta_t$$

**Proof:** Denoting the communities as \(C_1, C_2, \ldots, C_{K_t}\) and the inter-community subgraphs as \(C_{12}, C_{13}, \ldots\), we have

$$P_t = \left( \begin{array}{cccc} P_{t(1,1)} & P_{t(1,2)} & \cdots & P_{t(1,K_t)} \\ P_{t(2,1)} & P_{t(2,2)} & \cdots & P_{t(2,K_t)} \\ \vdots & \vdots & \ddots & \vdots \\ P_{t(K_t,1)} & P_{t(K_t,2)} & \cdots & P_{t(K_t,K_t)} \end{array} \right)$$

For \(l = 1\), \(\text{SUD}(G_t, G''^t, 1)\) can be represented as

$$\|P_t - P''^t\|_{TV} = \left\| \begin{array}{cccc} P_{t(1,1)} - P'_{t(1,1)} & \cdots & P_{t(1,K_t)} - P'_{t(1,K_t)} \\ P_{t(2,1)} - P'_{t(2,1)} & \cdots & P_{t(2,K_t)} - P'_{t(2,K_t)} \\ \vdots & \ddots & \vdots \\ P_{t(K_t,1)} - P'_{t(K_t,1)} & \cdots & P_{t(K_t,K_t)} - P'_{t(K_t,K_t)} \end{array} \right\|_{TV}$$

$$= \frac{1}{|V_t|} \sum_{k=1}^{K_t} |V_t(k)||P_{t(k,k)} - P'_{t(k,k)}|_{TV}$$

$$+ \frac{1}{|V_t|} \sum_{k,j=1,k \neq j}^{K_t} |E_t(k,j)||P_{t(k,j)} - P'_{t(k,j)}|_{TV}$$

$$\leq \epsilon + \delta_t$$

Here, \(\delta_t\) represents the ratio cut of the graph [45], and

$$\delta_t = |E_{t-in}|/|V_t| = \sum_{k,j=1,k \neq j}^{K_t} |E_t(k,j)|/|V_t|$$
Theorem 9 describes the utility bound of the intermediate perturbation method with $l = 1$. To generalize the result to arbitrary $l$, we need the following Lemma.

**Lemma 1.** For arbitrary matrix $P$ and $Q$, we have

$$\|P^l - Q^l\|_{TV} \leq l \|P - Q\|_{TV}$$  \hspace{1cm} (29)

*Proof:*

$$\|P^2 - Q^2\|_{TV} = \frac{1}{2|V|} \sum_{v=1}^{|V|} \|p_vP - q_vQ\|_1$$

$$\leq \frac{1}{2|V|} \sum_{v=1}^{|V|} \|p_vP - q_vP\|_1 + \|q_vP - q_vQ\|_1$$

$$\leq \frac{1}{2|V|} \sum_{v=1}^{|V|} \|p_v - q_v\|_1 + \|P - Q\|_{TV}$$

$$= 2\|P - Q\|_{TV}$$  \hspace{1cm} (30)

Generalizing Lemma 1, we have

$$\|P^l - Q^l\|_{TV} \leq l \|P - Q\|_{TV}$$  \hspace{1cm} (31)

Combining Theorem 9 and Lemma 1, we obtain the utility upper bound of the intermediate perturbation method.

**Theorem 10.** For arbitrary $l$, the static utility distance of each graph instance with intermediate perturbation is bounded by $l$ times the sum of utility distance of each community $\epsilon$ and the ratio cut $\delta_l$ of $G_t$, i.e.

$$\text{SUD}(G_t, G_t^l, 1) \leq l(\epsilon + \delta_l)$$  \hspace{1cm} (32)

*Proof:*

the static utility bound for arbitrary $l$ becomes

$$\text{SUD}(G_t, G_t^l, 1) \leq l\|P_l - P_l^l\|_{TV} \leq l(\epsilon + \delta_l)$$  \hspace{1cm} (33)

2) The second stage: The first stage only considers the utility performance of the intermediate perturbation. The following Lemma 2 generalizes the utility analysis of intermediate perturbation to the proposed dynamic perturbation.

**Lemma 2.** When $l = 1$, the static utility distance of the dynamic perturbation method is upper bounded by the sum of the static utility distance of the intermediate perturbation, the utility distance of each community $\epsilon$ and the ratio cut $\delta_l$ of $G_t$, i.e.

$$\text{SUD}(G_t, G_t^l, 1) \leq \text{SUD}(G_t, G_t^l, 1) + \epsilon + \delta_l$$  \hspace{1cm} (34)

*Proof:*

First denote that there are $K^*_t$ out of $K_t$ clusters that are considered as changed, which would be perturbed independently. And $K^*_t$ out of $K_t$ clusters are considered as unchanged, i.e. their perturbation would follow the perturbation manner in $G_t^{l-1}$. Also, to simplify derivation, we use $P_{l(k)}$ instead of $P_{l(k),k}$ to represent the TPM of the $k$-th community.

Then, we have

$$\text{SUD}(G_t, G_t^l, 1) = \|P_l - P_l^l\|_{TV} \leq \frac{\sum_{k=1}^{K^*_t} \|P_{l(k)} - P_{l(k)}^l\|_{TV} + \sum_{j=1}^{K^*_u} \|P_{l(j)} - P_{l(j)}^l\|_{TV}}{|K_t|} + \delta_l$$

$$\leq \frac{1}{|K_t|} \sum_{k=1}^{K^*_t} \|P_{l(k)} - P_{l(k)}^l\|_{TV} + \sum_{j=1}^{K^*_u} (\|P_{l(j)} - P_{l(j-1)}\|_{TV}$$

$$+ |P_{l(j-1)} - P_{l(j)}| + |P_{l(j)} - P_{l(j)}^l| + \|P_{l(j)}^l - P_{l(j)}\|_{TV})) + \delta_l$$

$$= \sum_{k=1}^{K^*_t} \|P_{l(k)} - P_{l(k)}^l\|_{TV} + \frac{|K_t|\epsilon_0}{|K_t|} + \delta_l$$

$$\leq \text{SUD}(G_t, G_t^l, 1) + \epsilon + \delta_l$$  \hspace{1cm} (35)

where $\epsilon_0$ denotes the threshold to classify a community as changed or unchanged. The last inequality comes from the assumption that usually $\epsilon_0 \leq \epsilon$.

Then we can prove the upper bound of static utility distance for the dynamic perturbation method in Theorem 5.

**Proof of Theorem 5:** based on Lemma 1, Lemma 2 and Eq.33, we have

$$\text{SUD}(G_t, G_t^l, l) = \|P_l - (P_l^l)^l\|_{TV}$$

$$\leq l\|P_l - P_l^l\|_{TV}$$

$$\leq l\|P_l - P_l^l\|_{TV} + l(\epsilon + \delta_l)$$

$$= 2l(\epsilon + \delta_l)$$  \hspace{1cm} (36)

Therefore, we have Theorem 5 proved.

D. Proof of Theorem 7: Relating SUD with Mixing Time

From the definition of total variation distance, we have

$$\|P^\mu(G_t^l) - \pi\|_{TV} + \|P^\mu(G_t) - \pi\|_{TV} \geq \|P^\mu(G_t^l) - P^\mu(G_t)\|_{TV}$$\hspace{1cm}(37)

Taking the maximum over all vertices, we have

$$\max \|P^\mu(G_t^l) - \pi\|_{TV} + \max \|P^\mu(G_t^l) - \pi\|_{TV} \geq \max \|P^\mu(G_t^l) - P^\mu(G_t)\|_{TV}$$\hspace{1cm}(38)

So, for $t \geq \tau_G(\epsilon)$

$$\max \|P^\mu(G_t^l) - \pi\|_{TV} \geq \max \|P^\mu(G_t^l) - P^\mu(G_t)\|_{TV} + \max \|P^\mu(G_t^l) - \pi\|_{TV}$$

$$\geq \frac{\sum_{v=1}^{|V|} \|P^\mu(G_t^l) - P^\mu(G_t)\|_{TV} - \pi\|_{TV} - \epsilon}{|V_t|}$$

$$\leq \text{SUD}(G_t, G_t^l, \tau_G(\epsilon)) - \epsilon$$  \hspace{1cm} (39)

Therefore, we have

$$\tau_G(\epsilon) \leq \text{SUD}(G_t, G_t^l, \tau_G(\epsilon) - \epsilon) \geq \tau_G(\epsilon)$$  \hspace{1cm} (40)

E. Proof of Theorem 8: Relating SUD with SLEM

We now sketch the proof of the above theorem. It is known that for undirected graphs, the second largest eigenvalue modulus is related to the mixing time of the graph as follows.

$$\tau_G(\epsilon) \leq \frac{\log n + \log \frac{1}{1 - \mu_G}}{1 - \mu_G}$$  \hspace{1cm} (41)
From the above equation, we can bound the SLEM in terms of the mixing time as follows:

\[
1 - \frac{\log n + \log (\frac{1}{\tau_G})}{\tau_{G'}} \leq \mu_{G'}(42)
\]

Replacing \( \epsilon \) with \( \text{SUD}(G_t, G'_t, \tau_G(\epsilon)) - \epsilon \), we have that:

\[
1 - \frac{\log n + \log \text{SUD}(G_t, G'_t, \tau_G(\epsilon)) - \epsilon}{\tau_{G'}(\text{SUD}(G_t, G'_t, \tau_G(\epsilon)) - \epsilon)} \leq \mu_{G'}(43)
\]

Finally, we leverage \( \tau_{G'}(\text{SUD}(G_t, G'_t, \tau_G(\epsilon)) \geq \tau_G(\epsilon) \) in the above equation, to get:

\[
\mu_{G'} \geq 1 - \frac{\log n + \log \text{SUD}(G_t, G'_t, \tau_G(\epsilon)) - \epsilon}{\tau_{G'}(\epsilon)} (44)
\]