UNIVERSALITY OF TRAP MODELS IN THE ERGODIC TIME SCALE

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Abstract. Consider a sequence of possibly random graphs $G_N = (V_N, E_N)$, $N \geq 1$, whose vertices’ weights have i.i.d. distributions $\{W^N_x : x \in V_N\}$ with a distribution belonging to the basin of attraction of an $\alpha$-stable law, $0 < \alpha < 1$. Let $X^N_t$, $t \geq 0$, be a continuous time random walk on $G_N$ which waits a mean $W^N_x$ exponential time at each vertex $x$. Under considerably general hypotheses, we prove that in the ergodic time scale this trap model converges in an appropriate topology to a $K$-process. We apply this result to a class of graphs which includes the hypercube, the $d$-dimensional torus, $d \geq 2$, random $d$-regular graphs and the largest component of super-critical Erdős-Rényi random graphs.

1. Introduction

Trap models were introduced to investigate aging, a nonequilibrium phenomenon of considerable physical interest [27, 12]. These trap models are defined as follows: consider an unoriented graph $G = (V, E)$ with finite degrees and a sequence of i.i.d. strictly positive random variables $\{W^N_x : x \in V\}$ indexed by the vertices. Let $\{X^N_t : t \geq 0\}$ be a continuous-time random walk on $V$ which waits a mean $W^N_x$ exponential time at site $x$, at the end of which it jumps to one of its neighbors uniformly.

The expected time spent by the random walk on a vertex $z$ is thus natural to regard the environment $W$ as a landscape of valleys or traps with depth given by the values of the random variables $\{W^N_z : z \in V\}$. As the random walk evolves, it explores the random landscape, finding deeper and deeper traps, and aging appears as a consequence of the longer and longer times the process remains at the same vertex.

The investigation of trap models in the time scale in which the deepest traps are visited was started in [15, 13], where the authors examined the asymptotic behavior of the random walk among traps in $\mathbb{Z}$ and in the complete graph. In the first case the random walk converges to a degenerated diffusion, while in the second it converges to the $K$-process, a continuous-time, Markov dynamics on $\mathbb{N}$ which hits any finite subset $A$ of $\mathbb{N}$ with uniform distribution. This latter result was extended in [17] to the hypercube and in [22] to the $d$-dimensional torus, $d \geq 2$.

In the present paper, we exhibit simple conditions that imply the convergence to the $K$-process in the scaling limit. Our conditions are general enough to include the hypercube and the torus, as well as random $d$-regular graphs and the largest component of the super-critical Erdős-Rényi random graphs. These are good examples to keep in mind throughout the text.

Let $\{G_N : N \geq 1\}$, $G_N = (V_N, E_N)$, be a sequence of possibly random, finite, connected graphs defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $V_N$ represents the set of vertices and $E_N$ the set of unoriented edges. Assume that the number of vertices, $|V_N|$, converges to $+\infty$ in $\mathbb{P}$-probability.

Assume that on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we are given an i.i.d collection of random variables $\{W^N_j : j \geq 1\}$, $N \geq 1$, independent of the random graph $G_N$ and whose common distribution belongs to the basin of attraction of an $\alpha$-stable law, $0 < \alpha < 1$. Hence, for all $N \geq 1$ and $j \geq 1$,

$$\mathbb{P}[W^N_j > t] = \frac{L(t)}{t^{\alpha}}, \quad t > 0,$$  \hspace{1cm} (1.1)

where $L$ is a slowly varying function at infinity.

For each $N \geq 1$, re-enumerate in decreasing order the weights $W^N_{1_N}, \ldots, W^N_{|V_N|}$: $\tilde{W}_j^N = W^N_{\sigma(j)}$, $1 \leq j \leq |V_N|$ for some permutation $\sigma$ of the set $\{1, \ldots, |V_N|\}$ and $\tilde{W}_j^N \geq \tilde{W}_{j+1}^N$ for $1 \leq j < |V_N|$. Let $(x^N_1, \ldots, x^N_{|V_N|})$ be a random enumeration of the vertices of $G_N$ and define $W^N_{x_j^N} = \tilde{W}_j^N$, $1 \leq j \leq |V_N|$, turning $G_N = (V_N, E_N, W^N)$ into a finite, connected, vertex-weighted graph.

Consider for each $N \geq 1$, a continuous-time random walk $\{X^N_t : t \geq 0\}$ on $V_N$, which waits a mean $W^N_x$ exponential time at site $x$, after which it jumps to one of its neighbors with uniform probability.

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The generator \( \mathcal{L}_N \) of this walk is given by:
\[
(\mathcal{L}_N f)(x) = \frac{1}{\deg(x)} \frac{1}{W^N_x} \sum_{y \sim x} [f(y) - f(x)]
\]
for every \( f : V_N \rightarrow \mathbb{R} \), where \( y \sim x \) means that \( \{x, y\} \) belongs to the set of edges \( E_N \) and where \( \deg(x) \) stands for the degree of \( x \): \( \deg(x) = \#\{y \in V_N : y \sim x\} \).

**Heuristics.** The main results of this article assert that, under fairly general conditions on the graph sequence \( G_N \), the random walk \( X^N_t \) converges in the ergodic time scale to a \( K \)-process. Let us now give an informal description of the above statement.

Given the graph sequence \( G_N \) and the associated weights \( W^N_x \), suppose that

1. A small number of sites supports most of the stationary measure of the process \( X^N_t \), see (18), and that we are able to find a sequence \( N \rightarrow \infty \) satisfying the following conditions:
   1. the ball \( B(x, \ell_N) \) around a typical point \( x \) has a volume much smaller than \( |V_N| \), see (31),
   2. starting outside of the above ball, the random walk ‘mixes’ before hitting its center \( x \), see (32) and
   3. the graphs \( G_N \) are transitive (or satisfy the much weaker hypothesis (33)).

Under the above conditions, we are able to show that
\[
X^N_t \text{ converges to a } K \text{-process,}
\]
introduced in (18) (29), after proper scaling, see Theorems (21) and (22).

Still on a heuristic level, let us give a brief explanation of why the above conditions should imply the stated convergence. Let \( M_N \) be a sequence of integers converging to \(+\infty\) slowly enough for the balls \( B(x^N_j, \ell_N) \), \( 1 \leq j \leq M_N \), to be disjoint. We call the vertices \( \{x^N_1, \ldots, x^N_{M_N}\} \) the deep traps and the remaining vertices \( \{x^N_{M_N+1}, \ldots, x^N_{|V_N|}\} \) the shallow traps. The idea is to decompose the trajectory of the random walk in excursions between the successive visits to the balls \( B(x^N_j, \ell_N) \).

Denote by \( v_{x_N}(x^N_j) \) the escape probability from \( x^N_j \). This is the probability that the random walk \( X^N_t \) starting from \( x^N_j \) attains the boundary of the ball \( B(x^N_j, \ell_N) \) before returning to \( x^N_j \). The random walk \( X^N_t \) starting from \( x^N_j \) visits \( x^N_j \) on average \( v_{x_N}(x^N_j)^{-1} \) times before it escapes. After escaping, it mixes and then it reaches a new deep trap with a distribution determined by the topology of the graph. This distribution does not depend on the last deep trap visited because the process has mixed before reaching the next trap. In an excursion between two deep traps, the random walk visits only shallow traps, which should not influence the asymptotic behaviour.

Hence, if the escape probabilities and the degrees of the random graph have a reasonable asymptotic behavior, see (33), we expect the random walk \( X^N_t \) to evolve as a Markov process on \( \{1, \ldots, M_N\} \) which waits at site \( j \) a mean \( W^N_{x^N_j} v_{x^N}(x^N_j)^{-1} \) exponential time, at the end of which it jumps to a point in \( \{1, \ldots, M_N\} \) whose distribution does not depend on \( j \). This latter process can be easily shown to converge to the \( K \)-process, proving the main result of this article.

There are several interesting examples of random graphs which are not considered in this article. Either because the assumptions (50) (53) fail or because they have not been proved yet. We leave as open problems the asymptotic behavior of a random walk among random traps on uniform trees on \( N \) vertices, on the critical component of an Erdős-Rényi graph, on Sierpinski carpets, on the giant component of the percolation cluster on a torus or on the invasion percolation cluster.

The article is organized as follows. In the next section we give a precise statement of our main results. In the following two sections we present some preliminary results on hitting probabilities and holding times of a random walk among random traps. In section 5 we present the topology in which the convergence to the \( K \)-process takes place and in Section 6 we construct a coupling between the random walk and a Markov process on the set \( \{1, \ldots, M\} \). This latter process can be seen as the trace of the \( K \)-process on the set \( \{1, \ldots, M\} \) and the coupling as the main step of the proof. In Section 7 we show that this latter process converges to the \( K \)-process. Putting together the assertions of Sections 6 6 6 7 7 7 we derive in Section 8 a result which provides sufficient conditions for the convergence to the \( K \)-process of a sequence of random walks among random traps on deterministic graphs. We adapt this result in Section 9 to random pseudo-transitive graphs and in Section 10 to graphs with asymptotically random conductances. We show in Section 11 that this latter class includes the largest component of a super-critical Erdős-Rényi graphs.
2. Notation and results

Recall the notation introduced in the previous section up to the subsection Heuristics. Denote by \( \nu_N \) the unique stationary distribution of the process \( \{X_t^N : t \geq 0\} \). An elementary computation shows that \( \nu_N \) is in fact reversible and given by

\[
\nu_N(x) = \frac{\text{deg}(x)W_N(x)}{Z_N}, \quad x \in V_N,
\]

where \( Z_N \) is the normalizing constant \( Z_N = \sum_{y \in V_N} \text{deg}(y)W_N(y) \).

For a fixed graph \( G_N \) and a fixed environment \( W = \{W_z : z \in V_N\} \), denote by \( P_X = P^{G_N,W}_x \), \( x \in V_N \), the probability on the path space \( D(\mathbb{R}_+, V_N) \) induced by the Markov process \( \{X_t^N : t \geq 0\} \) starting from \( x \). Expectation with respect to \( P_X \) is represented by \( E_x^N \). We denote sometimes \( X_t^N \) by \( X^N(x) \) to avoid small characters.

Let \( \{X_n^N : n \geq 0\} \) be the lazy embedded discrete-time chain in \( X_t^N \), i.e., the discrete-time Markov chain which jumps from \( x \) to \( y \) with probability \((1/2)\text{deg}(x)^{-1}\) if \( y \sim x \) and which jumps from \( x \) to \( x \) with probability \((1/2)\). Denote by \( \pi_N \) the unique stationary, in fact reversible, distribution of the skeleton chain, given by

\[
\pi_N(x) = \frac{\text{deg}(x)}{\sum_{y \in V_N} \text{deg}(y)}, \quad x \in V_N.
\]

For a subset \( B \) of \( V_N \), we denote by \( H_B \) the hitting time of \( B \) and by \( H_B^+ \) the return time to \( B \):

\[
H_B = \inf \{ t \geq 0 : X_t^N \in B \}, \quad H_B^+ = \inf \{ t \geq 0 : X_t^N \in B \mbox{ and } \exists s < t \mbox{ s.t. } X_s^N \notin B \}.
\]

When \( B \) is a singleton \( \{x\} \), we denote \( H_B \), \( H_B^+ \) by \( H_x \), \( H_x^+ \), respectively. We also write \( H_B \) (resp. \( H_B^+ \)) for the hitting time of a set \( B \) (resp. return time to \( B \)) for the discrete chain \( X_n^N \).

**K-processes.** To describe the asymptotic behavior of the random walk \( X_t^N \), consider two sequences of positive real numbers \( u = \{u_k : k \in \mathbb{N}\} \) and \( Z = \{Z_k : k \in \mathbb{N}\} \) such that

\[
\sum_{k \in \mathbb{N}} Z_k u_k < \infty, \quad \sum_{k \in \mathbb{N}} u_k = \infty.
\]

Consider the set \( \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\} \) of non-negative integers with an extra point denoted by \( \infty \). We endow this set with the metric induced by the isometry \( \phi : \overline{\mathbb{N}} \to \mathbb{R} \), which sends \( n \in \mathbb{N} \) to \( 1/n \) and \( \infty \) to 0. This makes the set \( \overline{\mathbb{N}} \) into a compact metric space.

In Section 4 based on [18], we construct a Markov process on \( \overline{\mathbb{N}} \), called the K-process with parameter \((Z_k, u_k)\) which can be informally described as follows. Being at \( k \in \mathbb{N} \), the process waits a mean \( Z_k \) exponential time, at the end of which it jumps to \( \infty \). Immediately after jumping to \( \infty \), the process returns to \( \mathbb{N} \). The hitting time of any finite subset \( A \) of \( \mathbb{N} \) is almost surely finite. Moreover, for each fixed \( n \geq 1 \), the probability that the process hits the set \( \{1, \ldots, n\} \) at the state \( k \) is equal to \( u_k/\sum_{1 \leq j \leq n} u_j \). In particular, the trace of the K-process on the set \( \{1, \ldots, n\} \) is the Markov process which waits at \( k \) a mean \( Z_k \) exponential time at the end of which it jumps to \( j \) with probability \( u_j/\sum_{1 \leq j \leq n} u_j \).

**Topology.** Between two successive sojourns in deep traps, the random walk \( X_t^N \) visits a short time interval several shallow traps. If we want to prove the convergence of the process \( X_t^N \) to a process which visits only the deep traps, we need to consider a topology which disregard short excursions. With this in mind, we introduce the following topology.

Fix \( T > 0 \). For any function \( f : [0, T] \to \mathbb{R} \) and any point \( t \in [0, T] \), we say that \( f \) is locally constant at \( t \) if \( f \) is constant in a neighborhood of \( t \). Let

\[
\mathcal{C}(f) = \{ t \in [0, T] ; f \text{ is locally constant in } t \},
\]

and \( \mathcal{D}(f) = \mathcal{C}(f)^c \). Notice that the set \( \mathcal{D}(f) \) is always closed. Let \( \Lambda \) denote the Lebesgue measure in \([0, T]\) and denote by \( M_0 \) the space of functions which are locally constant a.e., that is

\[
M_0 := \{ f : [0, T] \to \mathbb{R}; \Lambda(\mathcal{D}(f)) = 0 \}.
\]

We say that two locally constant functions \( f \) and \( g \in M_0 \) are equivalent, \( f \sim g \), if \( f(t) = g(t) \) for any \( t \notin \mathcal{D}(f) \cup \mathcal{D}(g) \). Note that if \( f \sim g \) then \( f = g \) almost everywhere. We show in Lemma 5.1 below that \( \sim \) is an equivalence relation in \( M_0 \).

Let \( M = M_0/\sim \) and make the space \( M \) into a metric space by introducing the distance

\[
d_T(f,g) = \inf_{A \in \mathcal{B}} \{ ||f - g||_{\infty,A^c} + \Lambda(A) \},
\]

where \( \mathcal{B} = \{ B \subset [0, T] ; B \text{ is Borel and } \Lambda(B^c) < \infty \} \).
where $\mathcal{B} = \mathcal{B}([0, T])$ is the set of Borel subsets of $[0, T]$, and $\|f - g\|_{\infty, A^c}$ stands for the supremum norm of $f - g$ restricted to $A^c$. Intuitively speaking, the distance between $f$ and $g$ is small if they are close to each other, except for a set of small measure.

We prove in Lemma 5.3 that $d_T$ is well defined and that it introduces a metric in $M$. With this metric, $M$ is separable but not complete.

**Main result.** Let $V = V_N = |V_N|$ and let $\Psi_N : V_N \to \{1, \ldots, V_N\}$ be the random function defined by $\Psi_N(x_N^j) = j$. The first main result of this article relies on three assumptions. We first require the sequence of invariant measures $\nu_N$ to be almost surely tight. Assume that for any increasing sequence $J_N$, with $\lim_N J_N = \infty$,

$$\lim_{N \to \infty} \mathbb{E}[\nu_N(\{x_1^N, \ldots, x_{\min(J_N, V_N)}^N\})] = 0. \quad (B0)$$

Denote by $B(x, \ell)$ the ball of radius $\ell$ centered at $x \in V_N$ with respect to the graph distance $d = d_N$ in $G_N$. Fix a sequence $\{\ell_N : N \geq 1\}$ of positive numbers, representing the radius of balls we place around each deep trap. Let $r$ be a vertex chosen uniformly among the vertices of $V_N$. We assume that

$$\lim_{N \to \infty} \mathbb{E}\left[\frac{|B(r, 2\ell_N)|}{V_N}\right] = 0. \quad (B1)$$

It follows from this condition that the number of vertices $V_N$ of the graph $G_N$ diverges in probability:

$$\lim_{N \to \infty} \mathbb{P}[V_N \geq K] = 0$$

for every $K \geq 1$.

Let $|\mu - \nu|_{TV}$ be the total variation distance between two probability measures $\mu, \nu$ defined on $V_N$, and let $t_{\text{mix}} = t_{\text{mix}}(N)$ be the mixing time of the discrete chain $\{X^N_n : n \geq 0\}$, see equation (4.33) in [25].

We assume that the typical point $\tau$ is not hit before the mixing time if one starts the random walk at distance at least $\ell_N$ from $\tau$. More precisely, we suppose that there exists an increasing sequence $L_N$, $\lim_{N \to \infty} L_N = \infty$, such that

$$\lim_{N \to \infty} \mathbb{E}\left[\sup_{y \in B((\tau, \ell_N))} \mathbb{P}_y[|t_{\text{mix}}| \leq L_N] \right] = 0. \quad (B2)$$

We finally introduce the notion of pseudo-transitive graphs, which includes the classical definition of transitive graphs but also encompasses other important examples such as random regular graphs, discussed in Proposition 6.3.

Consider a sequence of possibly random graphs $G_N = (V_N, E_N)$. We say that two subsets $A$, $B$ of $V_N$ with distinguished vertices $\tau \in A, \eta \in B$, are isomorphic, $(\tau, A) \equiv (\eta, B)$, if there exists a bijection $\varphi : A \to B$ with the property that $\varphi(\tau) = \eta$ and that for any $a, b \in A$, $\{a, b\}$ is an edge of $G_N$ if and only if $\{\varphi(a), \varphi(b)\}$ is an edge of $G_N$.

Let $\tau, \eta \in V_N$ be two vertices chosen independently and uniformly in $V_N$. We say that $G_N$ is pseudo-transitive for the sequence $\ell_N$, if

$$\lim_{N \to \infty} \mathbb{P}[(\tau, B(\tau, \ell_N)) \neq (\eta, B(\eta, \ell_N)) ] = 0. \quad (2.7)$$

Clearly, any sequence of transitive graphs is pseudo-transitive for any given sequence $\ell_N$.

For $x \in V_N$, let $v_\ell(x) = v_{\ell_N}^N(x)$ be the probability of escape from $x$:

$$v_\ell(x) = \mathbb{P}_x^N[|R(x, \ell)| < \ell^+],$$

and let $\{c_k : k \geq 1\}$ be the sequence defined by

$$c_k^{-1} = \inf\{t \geq 0 : \mathbb{P}[W^N_t > t] \leq k^{-1}\} \quad (2.8)$$

The constant $c_N^{-1}$ represents the typical size of $\max_{1 \leq k \leq N} W^N_k$, so that $c_N W^N_{\ell_j}$ for fixed $j$ is of order one.

**Theorem 2.1.** Fix a sequence of pseudo-transitive graphs $G_N$ with respect to a sequence $\ell_N$. Suppose that $B0$–$B2$ hold and that $\Psi_N(X_0^N)$ converges in probability to some $k \in \mathbb{N}$. Then, letting $\beta^{-1}_N = c_N v_{\ell_N}^N(x_1^N)$, we have that

$$(c_N W^N, \Psi_N(X_{\beta N}^N)) \text{ converges weakly to } (w, K),$$

where the sequence $w = (w_1, w_2, \ldots)$ is defined in [25] and where for each fixed $w$, $K$ is a $K$-process with parameter $(w, 1)$ starting from $k$. In the convergence, we adopted $L^1(\mathbb{N})$ topology in the first coordinate and $d_T$-topology in the second.
It is not difficult to show from the definition of the random sequence $w = (w_1, w_2, \ldots)$ that $w_1$ has a Fréchet distribution. In Section 9 we apply Theorem 2.1 to the hypercube, the $d$-dimensional torus, $d \geq 2$, and to a sequence of random $d$-regular graphs, $d \geq 3$.

The second main result of the article concerns graphs in which the assumption (2.7) of isometry of neighborhoods is replaced by an asymptotic independence and a second moment bound.

Assume that there exists a coupling $Q_N$ between the random graph $\{G_N : N \geq 1\}$ and a sequence of i.i.d random vectors $\{(D_k, E_k) : k \geq 1\}$ such that for every $K \geq 1$ and $\delta > 0$,

$$\lim_{N \to \infty} Q_N \left[ \max_{1 \leq j \leq K} \left| v_{\ell}(f_j)^{-1} - E_j^{-1} \right| > \delta \right] = 0,$$

$$\lim_{N \to \infty} Q_N \left[ \bigcup_{j=1}^K \{ \deg(f_j) \neq D_j \} \right] = 0,$$

$$Q_N[D_1 \geq 1, 0 < E_1 \leq 1] = 1, \quad E_{Q_N}[(D_1/E_1)^2] < \infty,$$

for one and therefore all $N \geq 1$, where $\ell = \ell_N$ is the radius of the balls placed around each trap and introduced right above (B1), and $f_1, \ldots, f_K$ is a collection of distinct vertices chosen uniformly in $V_N$.

We can now state our second main result, which can be seen as a generalization of Theorem 2.1.

**Theorem 2.2.** Fix a sequence of random graphs $G_N$. Suppose that (B0)–(B3) hold and that $\Psi_N(X_0^N)$ converges in probability to some $k \in \mathbb{N}$. Then, defining $\beta_N = c_N^{-1}$, we have that

$$(c_N \mathbf{W}_N, \Psi_N(X_0^N)) \text{ converges weakly to } (w, K_1),$$

where the sequence $w = (w_1, w_2, \ldots)$ is defined in (B4) and where for each fixed $w$, $K_1$ is a $K$-process starting from $k$ with parameter $(Z, u)$, where $Z_0 = w_k/E_k$ and $u_k = D_k E_k$. In the convergence, we adopted $L^1(\mathbb{N})$ topology in the first coordinate and $dt$-topology in the second.

In Section 11 we apply this result to the largest component of a super-critical Erdős-Rényi random graph. We expect this statement to be applicable in a wider context, such as random graphs with random degree sequences, or percolation clusters on certain graphs.

### 3. Hitting probabilities

We prove in this section general estimates on the hitting distribution of a random walk on a finite graph. These estimates will be useful in the description of the trace of our trap model on the deepest traps. Since $N$ will be kept fixed throughout the section, we omit $N$ from the notation almost everywhere.

Recall that we denote by $d = d_N$ the graph distance on $V_N$: $d(x, y) = m$ if there exists a sequence $x = z_0, z_1, \ldots, z_m = y$ such that $z_{i+1} \sim z_i$ for $0 \leq i \leq m - 1$, and if there do not exist shorter sequences with this property. For $x \in V_N$ and a subset $C$ of $V_N$, denote by $d(x, C)$ the distance from $x$ to $C$: $d(x, C) = \min_{y \in C} d(x, y)$. For $\ell \geq 1$, denote by $B(C, \ell)$ the vertices at distance at most $\ell$ from $C$: $B(C, \ell) = \{ x \in V_N : d(x, C) \leq \ell \}$ and let $R(C, \ell) = B(C, \ell)^c$. When the set $C$ is a singleton $\{x\}$, we write $B(x, \ell)$, $R(x, \ell)$ for $B(\{x\}, \ell)$, $R(\{x\}, \ell)$, respectively.

Fix $M \geq 1$, a subset $A = \{x_1, \ldots, x_M\}$ of $V_N$ and $\ell \geq 1$. Recall from Section 2 that we denote by $v_\ell(x)$, $x \in A$, the escape probability from $x$, and let $p(x, A)$ be the probability of reaching the set $A$ at $x$, when starting at equilibrium:

$$v_\ell(x) = P_x[H_{R(x, \ell)} < H_{x}^+] = P_{\pi_N}[X_N(H_A) = x], \quad (3.1)$$

where $\pi_N$ is the stationary state of the discrete-time chain $X_N^N$, introduced in (2.2).

**Lemma 3.1.** Fix a subset $A = \{x_1, \ldots, x_M\}$ of $V$. For any $z \notin A$ and for any $L \geq 1$,

$$\sum_{j=1}^{M} | P_z[X_A = x_j] - p(x_j, A) | \leq 2 \left( 2^{-L} + P_z[H_A < Lt_{mix}] \right).$$

Moreover, if there exists $\ell \geq 1$ such that $d(x_a, x_b) > 2\ell + 1$ for $a \neq b$, then for all $L \geq 1$ and for all $1 \leq i \leq M$,

$$\sum_{j \neq i} | P_{x_i}[X_A = x_j] - v_\ell(x_i) p(x_j, A) | \leq 2 v_\ell(x_i) \max_{z \in R(A, \ell)} \left( 2^{-L} + P_z[H_A < Lt_{mix}] \right).$$

5
Proof. Fix a subset $A = \{x_1, \ldots, x_M\}$ of $V$ and $z \notin A$. By definition of the mixing time $t_{\text{mix}}$ and by the definition of the total variation distance,

$$
\sum_{j=1}^{M} \left| \mathbb{E}_z \left[ \mathbb{P}_{X(L_{\text{mix}})} \left[ X_{H_A} = x_j \right] \right] - \mathbb{P}_\pi \left[ X_{H_A} = x_j \right] \right|
$$

$$
= \sum_{j=1}^{M} \left| \sum_{w \in V} \left\{ \mathbb{P}_z \left[ X(L_{\text{mix}}) = w \right] - \pi(w) \right\} \mathbb{P}_w \left[ X_{H_A} = x_j \right] \right|
$$

$$
\leq 2 \left\| \mathbb{P}_z \left[ X(L_{\text{mix}}) = \cdot \right] - \pi(\cdot) \right\|_{TV} \leq 2 \cdot 2^{-L}.
$$

To prove the first claim of the lemma, apply the Markov property to get that

$$
\mathbb{P}_z \left[ X_{H_A} = x_j \right] \leq \mathbb{E}_z \left[ \mathbb{P}_{X(L_{\text{mix}})} \left[ X_{H_A} = x_j \right] \right] + \mathbb{P}_z \left[ X_{H_A} = x_j, \mathbb{H}_A \leq L_{\text{mix}} \right]
$$

and that

$$
\mathbb{P}_z \left[ X_{H_A} = x_j \right] \geq \mathbb{P}_z \left[ X_{H_A} = x_j, \mathbb{H}_A > L_{\text{mix}} \right]
$$

$$
= \mathbb{E}_z \left[ \mathbb{P}_{X(L_{\text{mix}})} \left[ X_{H_A} = x_j \right] \right] - \mathbb{E}_z \left[ \mathbb{P}_{X(L_{\text{mix}})} \left[ X_{H_A} = x_j \right], \mathbb{H}_A \leq L_{\text{mix}} \right].
$$

The triangular inequality together with the previous two bounds and the estimate presented in the beginning of the proof show that

$$
\sum_{j=1}^{M} \left| \mathbb{P}_z \left[ X_{H_A} = x_j \right] - \mathbb{P}_\pi \left[ X_{H_A} = x_j \right] \right| \leq 2 \left( 2^{-L} + \mathbb{P}_z \left[ \mathbb{H}_A < L_{\text{mix}} \right] \right).
$$

This proves the first claim of the lemma.

We turn now to the proof of the second claim of the lemma. Since $d(x_i, A \setminus \{x_j\}) > \ell$ and $i \neq j$, the expression inside the absolute value on the left hand side of the inequality can be written as

$$
\mathbb{P}_z \left[ X(H_A) = x_j \mid \mathbb{H}_{R(x_i, \ell)} < \mathbb{H}_A^+ \right] \nu(x_i) - \nu(x_j) p(x_j, A).
$$

The absolute value is thus bounded by

$$
\sum_{z \in V} \left\{ \mathbb{P}_z \left[ X(H_A) = x_j \right] - p(x_j, A) \left| \mathbb{P}_z \left[ \mathbb{H}_{R(x_i, \ell)} < \mathbb{H}_A^+, X(H_{R(x_i, \ell)}) = z \right] \right\}
$$

Since $d(x_a, x_b) > 2\ell + 1$, $a \neq b$, the set of vertices $z$ at distance $\ell + 1$ from $x_i$ is disjoint from $A$. Hence, by the first part of the proof, the sum over $j \neq i$ of this expression is bounded above by

$$
2 \nu(x_i) \max_{z \in R(A, \ell)} \left\{ 2^{-L} + \mathbb{P}_z \left[ \mathbb{H}_A < L_{\text{mix}} \right] \right\}
$$

for every $L \geq 1$. This proves the lemma. \(\square\)

Denote by $\mathcal{D}(f)$ the Dirichlet form of a function $f : V \to \mathbb{R}$:

$$
\mathcal{D}(f) = \frac{1}{2} \sum_{x \in V} \sum_{y \sim x} \frac{\nu(x)}{\deg(x) W_x} (f(x) - f(y))^2.
$$

For disjoint subsets $A$ and $B$ of $V$, denote by $\text{cap}(A, B)$ the capacity between $A$ and $B$:

$$
\text{cap}(A, B) = \inf_f \mathcal{D}(f),
$$

where the infimum is carried over all functions $f : V \to \mathbb{R}$ such that $f(x) = 1$ for $x \in A$, $f(y) = 0$, $y \in B$. Let $g : V \to [0, 1]$ be given by

$$
g_{A,B}(x) = \mathbb{P}_z[H_A \leq H_B].
$$

It is a known fact that

$$
\text{cap}(A, B) = \mathcal{D}(g_{A,B}) = \sum_{y \in A} \nu(y) W_y^{-1} \mathbb{P}_y[H_B < H_A^+].
$$

Note that we may replace in the above identity $H_B, H_A^+$ by $\mathbb{H}_B, \mathbb{H}_A^+$, respectively.

Take a set $A \subset V$ composed of $M$ points which are far apart and let $x$ be a point in $A$. In the next lemma, we are going to estimate the probability $p(x, A) = \mathbb{P}_z[X_{H_A} = x]$. This probability will be roughly proportional to $\deg(x) \nu(x)$. Let us first introduce a normalizing constant. For $\ell \geq 1$ and a finite subset $A$ of $V$, let

$$
\Gamma_\ell(A) = \sum_{x \in A} \deg(x) \nu(x).
$$
Lemma 3.2. Fix a subset $A = \{x_1, \ldots, x_M\}$ of $V$ such that $d(x_a, x_b) > 2\ell + 1$, $a \neq b$, for some $\ell \geq 1$. Then,

\[
\max_{1 \leq i \leq M} \left| p(x_i, A) - \frac{\deg(x_i) v(x_i)}{\Gamma_\ell(A)} \right| \leq 2 \max_{z \in R(A, \ell)} \left\{ 2^{-L} + P_z[H_A \leq L_{\text{mix}}] \right\}.
\]

Proof. Fix $1 \leq i \leq M$ and let $A_i = A \setminus \{x_i\}$. Since $\mathcal{D}(g(x_i), A_i) = \mathcal{D}(1 - g(x_i), A_i)$, by (3.2)

\[
\deg(x_i) P_{x_i, [H_A, H_A^+]} = \sum_{j \neq i} \deg(x_j) P_{x_j, [H_x, H_x^+]}.
\]

On the other hand, since $d(x_i, A_i) > \ell$, we obtain

\[
P_{x_i, [H_A^+, H_A^+]} = E_{x_i} \left[ 1 \{H_{R(x_i, \ell)} < H_x^+\} P_{X(R(x_i, \ell))} [H_A < H_x] \right]
= E_{x_i} \left[ 1 \{H_{R(x_i, \ell)} < H_x^+\} (1 - P_{X(R(x_i, \ell))} [X_{H_A} = x_i]) \right].
\]

Therefore,

\[
P_{x_i, [H_A^+, H_A^+]} - v(x_i) \left[ 1 - p(x_i, A) \right]
= E_{x_i} \left[ 1 \{H_{R(x_i, \ell)} < H_x^+\} \right] \left\{ p(x_i, A) - P_{X(R(x_i, \ell))} [X_{H_A} = x_i] \right\}.
\]

Since $d(x_a, x_b) > 2\ell + 1$, we may replace in the previous expression $X(R(x_i, \ell))$ by $X(H_A, \ell)$. By the first assertion of Lemma 3.3, the absolute value of the difference inside braces is less than or equal to $2 \max_{z \in R(A, \ell)} \left\{ 2^{-L} + P_z[H_A \leq L_{\text{mix}}] \right\}$ for every $L \geq 1$. Hence,

\[
\left| P_{x_i, [H_A^+, H_A^+]} - v(x_i) \left[ 1 - p(x_i, A) \right] \right| \leq 2 v(x_i) \max_{z \in R(A, \ell)} \left\{ 2^{-L} + P_z[H_A \leq L_{\text{mix}}] \right\}
\]

for every $L \geq 1$.

Similarly, from (3.3) one obtains that

\[
\deg(x_i) P_{x_i, [H_A, H_A^+]} = \sum_{j \neq i} \deg(x_j) E_{x_j} \left[ 1 \{H_{R(x_j, \ell)} < H_x^+\} P_{X(R(x_j, \ell))} [X_{H_A} = x_i] \right].
\]

It follows from this identity and the previous argument that

\[
\left| \deg(x_i) P_{x_i, [H_A, H_A^+]} - \sum_{j \neq i} \deg(x_j) v(x_j) p(x_i, A) \right|
\leq 2 \sum_{j \neq i} \deg(x_j) v(x_j) \max_{z \in R(A, \ell)} \left\{ 2^{-L} + P_z[H_A \leq L_{\text{mix}}] \right\}
\]

for all $L \geq 1$.

The two previous estimates yield the bound

\[
\left| \deg(x_i) v(x_i) [1 - p(x_i, A)] - \sum_{j \neq i} \deg(x_j) v(x_j) p(x_i, A) \right|
\leq 2 \sum_{j = 1}^M \deg(x_j) v(x_j) \max_{z \in R(A, \ell)} \left\{ 2^{-L} + P_z[H_A \leq L_{\text{mix}}] \right\}.
\]

To conclude the proof of the lemma, it remains to divide both sides of the inequality by $\Gamma_\ell(A)$. \qed

4. Holding Times of the Trace Process

We present in this section a general result on Markov chains computing the time spent by this chain on a subset of the state space. This will be useful later in proving that the time spent by the walk on the shallow traps can be disregarded.

Consider an irreducible continuous-time Markov process $\{X_t : t \geq 0\}$ on a finite state space $V$. Denote by $\{W_x : x \in V\}$ the mean of the exponential waiting times, by $\nu$ the unique stationary probability measure, and by $\{\tau_j : j \geq 0\}$ the sequence of jump times.

Denote by $P_x$, $x \in V$, the probability measure on the path space $D(\mathbb{R}_+, V)$ induced by the Markov process $X_t$ starting from $x$. Expectation with respect to $P_x$ is represented by $E_x$. For a probability measure $\mu$ on $V$, let $P_{\mu} = \sum_{x \in V} \mu(x) P_x$. 

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Fix a set $A \subset V$ and let $U$ be a stopping time such that for all $x \in A$,
\[ P_x[\tau \leq U] = 1, \quad P_x[H_{A \setminus \{x\}} \geq U] = 1, \quad E_x[U] < \infty. \]

$U = H_{R(A, \ell)}$ is the example to keep in mind, where $\ell$ is chosen so that $d(x, y) \geq 2\ell + 1$ for all $x \neq y \in A$. Let $S_A = U + H_A \circ \theta_U$ be the hitting time of the set $A$ after time $U$. Denote by $\nu(x)$ the probability that starting from $x$ the stopping time $U$ occurs before the process returns to $x$: $\nu(x) = P_x[U < H^+_x]$, which should be understood as an escape probability.

Let $D_k, k \geq 0$, be the time of the $k$-th return to $A$ after escaping: $D_0 = 0$, $D_1 = S_A$, $D_{k+1} = D_k + S_A \circ \theta_{D_k}, k \geq 1$. Clearly, if $X_0$ belongs to $A$, $\{X_{D_k} : k \geq 0\}$ is a discrete time Markov chain on $A$. On the other hand, by assumption $E_x[D_1] = E_x[U + H_A \circ \theta_U]$ is finite.

**Lemma 4.1.** The Markov chain $\{X_{D_k} : k \geq 0\}$ is irreducible. Moreover, for every $f : V \to \mathbb{R}$,
\[ \lim_{k \to \infty} \frac{1}{k} \int_0^{D_k} f(X_t) \, dt = \sum_{z \in A} \nu(z) E_z \left[ \int_0^{D_1} f(X_t) \, dt \right], \]
$P_\nu$-almost surely, where $\nu$ is the unique stationary state of the discrete time chain $\{X_{D_k} : k \geq 0\}$.

**Proof.** We first prove the irreducibility of the chain $\{X_{D_k} : k \geq 0\}$. Fix $x, y \in A$ and consider a self-avoiding path $x_0 = x, \ldots, x_n = y$ such that the discrete-time Markov chain associated to the Markov process $X_t$ jumps from $x_i$ to $x_{i+1}$, $0 \leq i < n$, with positive probability. Such path exists by the irreducibility of $X_t$. Let $x_j$ be the first state in the sequence $x_1, \ldots, x_n$ which belongs to $A$. Since $P_x[H_{A \setminus \{x\}} \geq U] = 1$,
\[ P_x[X_{D_1} = x_j] \geq P_x[X_{D_1} = x_j, Z_1 = x_1, \ldots, Z_j = x_j] = P_x[X_{U+H_A \circ \theta_U} = x_j, U \leq H_{A \setminus \{x\}}, Z_1 = x_1, \ldots, Z_j = x_j], \]
where $\{Z_n : n \geq 0\}$ is the discrete-time jump chain associated to the process $\{X_t : t \geq 0\}$. Since $U \geq \tau_1$, on the event $\{Z_1 = x_1, \ldots, Z_j = x_j\} \cap \{U \leq H_{A \setminus \{x\}}\}$, $U + H_A \circ \theta_U = \tau_j$. The previous probability is thus equal to
\[ P_x[Z_{\tau_j} = x_j, Z_1 = x_1, \ldots, Z_j = x_j] = P_x[Z_1 = x_1, \ldots, Z_j = x_j] > 0. \]

Repeating this argument for the subsequent states in the sequence $x_1, \ldots, x_n$ which belong to $A$, we prove that the chain $X_{D_k}$ is irreducible.

Fix a function $f : V \to \mathbb{R}$. Clearly,
\[ \frac{1}{k} \int_0^{D_k} f(X_t) \, dt = \frac{1}{k} \sum_{x \in A} \sum_{j=0}^{k-1} \int_{D_j}^{D_{j+1}} f(X_t) \, dt 1\{X_{D_j} = x\}. \]

For $x \in A$, let $K^+_1 = \min\{j \geq 0 : X_{D_j} = x\}$, $K^+_n = \min\{j > K^+_1 : X_{D_j} = x\}, n \geq 1$, and let $L^+_k = \#\{j < k : X_{D_j} = x\}$. With this notation, we can rewrite the previous sum as
\[ \frac{1}{k} \sum_{x \in A} \sum_{n=1}^{L^+_k} \int_{D_{K^+_n}}^{D_{K^+_n+1}} f(X_t) \, dt = \sum_{x \in A} \frac{L^+_k}{k} \frac{1}{L^+_k} \sum_{n=1}^{L^+_k} \int_{D_{K^+_n}}^{D_{K^+_n+1}} f(X_t) \, dt. \]

By the irreducibility of the chain $X_{D_k}$, for each $x \in A$, $L^+_k/k$ converges a.s. as $k \uparrow \infty$ to $\nu(x)$. Moreover, for each $x$, the variables $\int_{D_{K^+_n}}^{D_{K^+_n+1}} f(X_t) \, dt, n \geq 1$, are independent and identically distributed.

Hence, since $L^+_k \uparrow \infty$, by the law of large numbers, $P_\nu$-almost surely,
\[ \lim_{k \to \infty} \frac{1}{L^+_k} \sum_{n=1}^{L^+_k} \int_{D_{K^+_n}}^{D_{K^+_n+1}} f(X_t) \, dt = E_x \left[ \int_0^{D_1} f(X_t) \, dt \right]. \]

The lemma follows from the two previous convergences. \hfill $\square$

**Proposition 4.2.** The unique stationary state $\nu$ of the discrete-time Markov chain $\{X_{D_k} : k \geq 0\}$ satisfies
\[ \nu(x) = \nu(x) \nu(x) W^x_x^{-1} E_x[D_1] = \nu(x) \nu(x) W^x_x^{-1} \sum_y \nu(y) \nu(y) W^y_y^{-1}. \]
Moreover, for every \( g : V \rightarrow \mathbb{R} \),
\[
\sum_{x \in A} v(x) \nu(x) W_{x}^{-1} E_{x} \left[ \int_{0}^{D_{1}} g(X_{t}) \, dt \right] = \sum_{x \in V} g(x) \nu(x) .
\] (4.2)

**Proof.** Applying Lemma 4.1 to \( f = 1 \), we obtain that \( \mathbb{P}_{\nu} \)-almost surely
\[
\lim_{k \rightarrow \infty} \frac{D_{k}}{k} = \lim_{k \rightarrow \infty} \frac{1}{k} \int_{0}^{D_{k}} g(X_{t}) \, dt = \mathbb{E}_{\rho}[D_{1}] .
\] (4.3)

By Lemma 4.1 with \( f(y) = \mathbb{1}_{\{ y = x \}} \), we get that \( \mathbb{P}_{\nu} \)-almost surely
\[
\lim_{k \rightarrow \infty} \frac{1}{k} \int_{0}^{D_{k}} \mathbb{1}_{\{ X_{t} = x \}} \, dt = \rho(x) E_{x} \left[ \int_{0}^{D_{1}} \mathbb{1}_{\{ X_{t} = x \}} \, dt \right]
\]
because starting from \( y \neq x \), the process does not visit \( x \) before time \( D_{1} \). In particular, all terms on the right-hand side in the statement of Lemma 4.1 but the one \( z = x \), vanish. On the other hand, dividing and multiplying the expression on the left-hand side of the previous equation by \( D_{k} \), we obtain by the ergodic theorem and by (4.3) that
\[
\mathbb{E}_{\rho}[D_{1}] \nu(x) = \rho(x) E_{x} \left[ \int_{0}^{D_{1}} \mathbb{1}_{\{ X_{t} = x \}} \, dt \right].
\] (4.4)

The time spent at \( x \) before \( D_{1} \) is the time spent at \( x \) before \( U \) which is a geometric sum of independent exponential times. The success probability of the geometric is \( \nu(x) \) and the mean of the exponential distributions is \( W_{x} \). Hence, the right-hand side of the previous formula is equal to \( \rho(x) W_{x} / \nu(x) \). This proves the first identity in (4.1). To derive the second identity, note that \( \mathbb{E}_{\rho}[D_{1}] \) does not depend on \( x \), and it is therefore only a normalizing constant to make \( \rho \) into a probability distribution.

By the ergodic theorem, for every \( g : V \rightarrow \mathbb{R} \),
\[
\lim_{k \rightarrow \infty} \frac{1}{D_{k}} \int_{0}^{D_{k}} g(X_{t}) \, dt = \sum_{x \in V} g(x) \nu(x) .
\]

To conclude the proof of the proposition, it remains to show that the left hand side of this expression is equal to the left-hand side of (4.2). To this end, we will use the previous lemma.

For a function \( g : V \rightarrow \mathbb{R} \), by Lemma 4.1 for \( f = g \) and (4.3), we get
\[
\lim_{k \rightarrow \infty} \frac{1}{D_{k}} \int_{0}^{D_{k}} g(X_{t}) \, dt = \lim_{k \rightarrow \infty} \frac{k}{k} \int_{0}^{D_{k}} g(X_{t}) \, dt = \frac{1}{\mathbb{E}_{\rho}[D_{1}]} \mathbb{E}_{\rho} \left[ \int_{0}^{D_{1}} g(X_{t}) \, dt \right].
\]

To conclude the proof of the proposition, it suffices to use (4.1).

**Corollary 4.3.** We have that
\[
\mathbb{E}_{\rho}[D_{1}] = \frac{\mathbb{E}_{\rho}[W_{x} / \nu(x)]}{1 - \nu(V \setminus A)} .
\]

Furthermore, for any function \( g : V \rightarrow \mathbb{R} \),
\[
\mathbb{E}_{\rho} \left[ \int_{0}^{D_{1}} g(X_{t}) \, dt \right] = \mathbb{E}_{\nu}[g] \mathbb{E}_{\rho}[D_{1}] .
\]

**Proof.** We can write
\[
\mathbb{E}_{\rho}[D_{1}] = \mathbb{E}_{\rho} \left[ \int_{0}^{D_{1}} dt \right] = \mathbb{E}_{\rho} \left[ \int_{0}^{D_{1}} \mathbb{1}_{\{ X_{t} \in A \}} dt \right] + \mathbb{E}_{\rho} \left[ \int_{0}^{D_{1}} \mathbb{1}_{\{ X_{t} \notin A \}} dt \right].
\]

By the same reasoning as below (4.4), we conclude that the first expectation in the sum above equals \( \mathbb{E}_{\rho}[W_{x} / \nu(x)] \). To evaluate the second expectation, we use Proposition 4.2 with \( g = \mathbb{1}_{\{ V \setminus A \}} \) to conclude that
\[
\mathbb{E}_{\rho} \left[ \int_{0}^{D_{1}} \mathbb{1}_{\{ X_{t} \notin A \}} dt \right] = \mathbb{E}_{\rho}[D_{1}] \nu(V \setminus A) .
\]

Putting together the above equations, we conclude the proof of the first assertion of the corollary.

The second claim follows from the first identity in (4.1) and from (4.2). \( \square \)
5. Topology

We define in this section a topology on the space of locally constant functions, where our main convergence will take place.

Fix $T > 0$. For any non-empty set $F \subseteq [0, T]$ and any $\delta > 0$ define the ball $B(F, \delta)$ by

$$B(F, \delta) = \bigcup_{t \in F} \{ s \in [0, T] : |t - s| < \delta \} . \tag{5.1}$$

When $F$ is a singleton $\{t\}$, we simply write $B(t, \delta)$ instead of $B(\{t\}, \delta)$. For any function $f : [0, T] \to \mathbb{R}$ and any point $t \in [0, T]$, we say that $f$ is locally constant at $t$ if there exists $\delta > 0$ such that $f$ is constant in $B(t, \delta)$. Define the open set

$$\mathcal{C}(f) = \{ t \in [0, T] : f \text{ is locally constant in } t \}, \tag{5.2}$$

and let $\mathcal{D}(f)$ be the closed set $\mathcal{D}(f) = \mathcal{C}(f)^c$. Let $\Lambda$ denote the Lebesgue measure in $[0, T]$ and denote by $\mathcal{M}_0$ the space of locally constant functions:

$$\mathcal{M}_0 := \{ f : [0, T] \to \mathbb{R} : \Lambda(\mathcal{D}(f)) = 0 \} . \tag{5.3}$$

We say that two locally constant functions $f$ and $g \in \mathcal{M}_0$ are equivalent, $f \sim g$, if $f(t) = g(t)$ for any $t \in \mathcal{C}(f) \cap \mathcal{C}(g)$. Note that $f = g$ almost everywhere if $f \sim g$.

**Lemma 5.1.** The relation $\sim$ is an equivalence relation in $\mathcal{M}_0$.

**Proof.** The relation $\sim$ is clearly reflexive and symmetric. To prove that it is also transitive, fix three functions $f$, $g$ and $h$ in $\mathcal{M}_0$ such that $f \sim g$ and $g \sim h$. Fix $t \in \mathcal{C}(f) \cap \mathcal{C}(h)$. Since $t$ is a point where both $f$ and $h$ are locally constant, $f$ and $h$ are constant in a neighborhood of $t$. As $f = g$ almost surely and $h = g$ almost surely, we must have that $f(t) = h(t)$. \( \square \)

Let $\mathfrak{M} = \mathcal{M}_0 / \sim$ and make the space $\mathfrak{M}$ into a metric space by introducing the distance

$$d_T(f, g) = \inf_{A \in \mathcal{B}([0, T])} \left\{ \| f - g \|_{\infty,A} + \Lambda(A) \right\} , \tag{5.4}$$

where $\mathcal{B}([0, T])$ is the set of Borel subsets of $[0, T]$, and $\| f - g \|_{\infty,A}$ stands for the supremum norm of $f - g$ restricted to $A$.

**Lemma 5.2.** The distance $d_T$ is well defined and it introduces a metric in $\mathfrak{M}$.

**Proof.** Since $\mathcal{D}(h)$ has measure zero for any $h \in \mathcal{M}_0$, we can restrict the infimum to those sets $A$ whose complement $A^c$ is contained in $\mathcal{C}(f) \cap \mathcal{C}(g)$.

To see that $d_T$ is well defined, note that replacing $f$ by some $f' \sim f$ does not alter the value of $d_T(f, g)$, according to the previous remark. Symmetry of $d_T$ is also clear.

Now suppose that $d_T(f, g) = 0$. To prove that $f = g$ in $\mathfrak{M}$, i.e., that $f \sim g$, we need to show that $f(t) = g(t)$ for all $t \in \mathcal{C}(f) \cap \mathcal{C}(g)$. Fix such point $t$ and note that $f$ and $g$ are constant on a neighborhood $B(t)$ of $t$. Taking $\epsilon < \Lambda(B(t))$, since $d_T(f, g) = 0$, we can find a set $A_\epsilon$ in $\mathcal{B}([0, T])$ such that $\| f - g \|_{\infty,A_\epsilon} + \Lambda(A_\epsilon) \leq \epsilon$. Therefore, $B(t) \cap A_\epsilon^c \not= \emptyset$, so that $|f(t) - g(t)| \leq \epsilon$. Since this holds for arbitrary $\epsilon > 0$, $f(t) = g(t)$, as we wanted to prove.

Finally, to prove the triangular inequality, consider three functions $f$, $g$ and $h$ in $\mathfrak{M}$. For $\epsilon > 0$, let $A_{\epsilon}$, $B_{\epsilon}$ be sets in $\mathcal{B}([0, T])$ such that

$$d_T(f, g) \geq \| f - g \|_{\infty,A_{\epsilon}} + \Lambda(A_{\epsilon}) - \epsilon ,$$

$$d_T(g, h) \geq \| g - h \|_{\infty,B_{\epsilon}} + \Lambda(B_{\epsilon}) - \epsilon .$$

Since for any set $E$ in $\mathcal{B}([0, T])$, $\| f - h \|_{\infty,E} \leq \| f - g \|_{\infty,E} + \| g - h \|_{\infty,E}$, we have

$$d_T(f, h) \leq \| f - h \|_{\infty,A_{\epsilon}\cap B_{\epsilon}} + \Lambda(A_{\epsilon} \cup B_{\epsilon}) \leq d_T(f, g) + d_T(g, h) + 2\epsilon .$$

Since $\epsilon$ is arbitrary, $d_T$ is a metric. \( \square \)

The space $\mathfrak{M}$ is separable with respect to the metric $d_T$, but it is not complete. On the one hand, the set of piecewise constant functions for which the jump points and the range are in a dense countable set is dense in $\mathfrak{M}$. On the other hand, the function $f(t) = t$ which does not belong to $\mathfrak{M}$ can be arbitrarily approximated in the distance $d_T$ by functions in $\mathfrak{M}$.  

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Among all elements of an equivalent class, we choose one representative as follows. For each element \( f \) of \( \mathcal{M}_0 \), let \( \tilde{f} : [0, T] \to \mathbb{R} \) be given by

\[
\tilde{f}(t) = \frac{1}{2} \left\{ \liminf_{\delta \to 0} f(s) + \limsup_{\delta \to 0} f(s) \right\}.
\]  

(5.5)

When \( \liminf_{s \to t, s \in C(f)} f(s) = -\infty \) and \( \limsup_{s \to t, s \in C(f)} f(s) = +\infty \), we set \( \tilde{f}(t) = 0 \).

Clearly, \( \tilde{f} = f \) on \( \mathcal{C}(f) \) so that \( \mathcal{D}(\tilde{f}) \subset \mathcal{D}(f) \), where inclusion may be strict. In particular, \( \tilde{f} \) belongs to the equivalence class of \( f \): \( \tilde{f} \sim f \).

**Lemma 5.3.** We have that

\[
\limsup_{s \to t} f(s) = \limsup_{s \to t} g(s)
\]

whenever \( f \sim g \), with a similar identity if we replace \( \limsup \) by \( \liminf \). In particular, \( \tilde{f} = \tilde{g} \) if \( f \sim g \) and equation (5.5) distinguishes a unique representative for each equivalence class of \( \mathcal{M} \).

**Proof.** Consider two functions \( f, g \) in the same equivalence class of \( \mathcal{M} \). It is enough to show that

\[
\limsup_{s \to t} f(s) \leq \limsup_{s \to t} g(s) \quad \text{and} \quad \liminf_{s \to t} f(s) \geq \liminf_{s \to t} g(s).
\]

We prove the first inequality, the derivation of the second one being similar. There exists a sequence \( \{s_j : j \geq 1\} \) such that \( s_j \in \mathcal{C}(f) \), \( \lim_j s_j = t \),

\[
\limsup_{s \to t} f(s) = \lim_{j \to \infty} f(s_j).
\]

Since \( s_j \) belongs to \( \mathcal{C}(f) \), \( f \) is constant in an interval \( (s_j - \epsilon, s_j + \epsilon) \) and therefore in the interval \( I_j = (s_j - \epsilon, s_j + \epsilon) \cap (s_j - (1/j), s_j + (1/j)) \). Of course, \( I_j \subset \mathcal{C}(f) \). As \( \mathcal{D}(g) \) has Lebesgue measure 0, \( \mathcal{C}(g) \cap I_j = \emptyset \). Take an element \( s'_j \) of this latter set. Since \( I_j \) is contained in \( \mathcal{C}(f) \), \( s'_j \) belongs to \( \mathcal{C}(f) \cap \mathcal{C}(g) \) so that \( g(s'_j) = f(s'_j) \). Moreover, since \( f \) is constant in \( I_j \) and \( s_j, s'_j \) belong to \( I_j \), \( f(s_j) = f(s'_j) \). On the hand, \( \lim_j s'_j = t \) because \( s_j \) converges to \( t \) and \( |s'_j - s_j| < (1/j) \). Hence,

\[
\lim_{j \to \infty} f(s_j) = \lim_{j \to \infty} g(s'_j) \leq \limsup_{s \to t} g(s),
\]

which proves the lemma. \( \square \)

From now on when considering an equivalence class in \( \mathcal{M} \), we always refer to the representative defined by (5.3). For example, \( \|f\|_{\infty} \), \( f \in \mathcal{M} \), whose value may be different for two distinct functions in \( \mathcal{M}_0 \) belonging to the same equivalence class, means in reality \( \|\tilde{f}\|_{\infty} \).

In order to obtain a compactness criterion in \( \mathcal{M} \), we introduce the following modulus of continuity. For a measurable function \( f : [0, T] \to \mathbb{R} \) and \( \delta > 0 \), let

\[
\omega_{\delta}(f) = \Lambda(B(\mathcal{D}(f), \delta)).
\]

The modulus of continuity \( \omega_{\delta}(f) \) converges to 0 as \( \delta \to 0 \) if and only if \( f \) belongs to \( \mathcal{M}_0 \). We extend this definition to the space \( \mathcal{M} \). For an equivalence class \( f \in \mathcal{M} \), let

\[
\omega_{\delta}(f) = \Lambda(B(\mathcal{D}(\tilde{f}), \delta)).
\]

Lemma 5.3 ensures that the modulus of continuity is well defined, i.e., that \( \omega_{\delta}(f) = \omega_{\delta}(g) \) if \( f \) and \( g \) belong to the same equivalence class, because \( \tilde{f} = \tilde{g} \).

**Proposition 5.4.** A subset \( \mathcal{F} \subseteq \mathcal{M} \) is sequentially precompact with respect to \( d_{\mathcal{F}} \) if

\[
\sup_{f \in \mathcal{F}} \|f\|_{\infty} < \infty \quad \text{and} \quad \limsup_{\delta \to 0} \omega_{\delta}(f) = 0.
\]

(5.6)

**Proof.** For \( f \in \mathcal{F} \), define \( \ell_{\delta}^f(t) = \text{dist}(t, B(\mathcal{D}(\tilde{f}), \delta)^c) \). Since \( \ell_{\delta}^f \) is 1-Lipschitz for any \( f \in \mathcal{M} \) and any \( \delta > 0 \), the family \( \{\ell_{\delta}^f : f \in \mathcal{F}\} \) is equicontinuous. Fix a sequence \( f_n \) in \( \mathcal{F} \) and a sequence \( \{\delta_m : m \geq 1\} \) of positive numbers such that \( \lim_m \delta_m = 0 \). Since \( \sup_{f \in \mathcal{F}} \|f\|_{\infty} < \infty \), by a standard Cantor diagonal argument, we can extract a subsequence, still denoted by \( f_n \), for which, as \( n \to \infty \), \( \ell_{\delta_m}^{f_n} \) converges uniformly to some function \( \ell_{\delta_0}^f \) for every \( m \), and \( f_n(t) \) converges to some limit \( F(t) \) for any rational \( t \) in \([0, T]\).
Let $\epsilon_m = \limsup_{n \to \infty} \omega_{\delta_m}(f_n)$. By (5.6), $\lim_m \epsilon_m = 0$. Since $\ell^\delta_m$ converges uniformly to $\ell^\delta_m$ and since $\{\ell^\delta_m \neq 0\} = B(D(\hat{f}_n), \delta_m)$, 

$$
\Lambda(\ell^\delta_m \neq 0) \leq \limsup_{n \to \infty} \Lambda(\ell^\delta_m \neq 0) = \limsup_{n \to \infty} \omega_{\delta_m}(f_n) = \epsilon_m. \quad (5.7)
$$

We claim that for every $t \in [0, T]$ such that $\ell^\delta_m(t) = 0$ for some $m \geq 1$, there exist a neighborhood $N(t)$ of $t$ and an integer $n_0 \geq 1$ for which $\hat{F}$ is constant on $N(t) \cap \mathbb{Q}$ and $\hat{f}_n(t)$ is constant on $N(t)$ for $n \geq n_0$. We postpone the proof of this claim.

As $\lim_{m \to \infty} \epsilon_m = 0$, by (5.7) $\lim_m \Lambda(\ell^\delta_m \neq 0) = 0$. There exists therefore a subsequence $\{m(j) : j \geq 1\}$ such that $\sum_j \Lambda(\ell^\delta_{m(j)} \neq 0) < \infty$. Let $\Lambda = \cap_{k \geq 1} \bigcup_{j \geq k} \{\ell^\delta_{m(j)} \neq 0\}$ so that $\Lambda(A) = 0$. If $t$ belongs to the set $A^c$, which has full measure, $\ell^\delta_{m(j)}(t) = 0$ for some $j$. By the conclusions of the previous paragraph, there exist a neighborhood $N(t)$ of $t$ and an integer $n_0 \geq 1$ for which $\hat{F}$ is constant on $N(t) \cap \mathbb{Q}$ and $\hat{f}_n(t)$ is constant on $N(t)$ for $n \geq n_0$.

In view of the previous result we may define a function $\hat{F} : [0, T] \to \mathbb{R}$ which vanishes on the set $A$, and which on each element $t$ of the set $A^c$ is locally constant with value given by the value of $F$ on a rational point close to $t$. In particular, $A^c \subset \mathcal{C}(\hat{F})$ which ensures that $\hat{F}$ belongs to $\mathfrak{M}_0$. Moreover, it follows from the convergence of $\hat{f}_n$ to $\hat{F}$ on the rationals that $\hat{f}_n(t)$ converges to $\hat{F}(t)$. Since the set $A$ has Lebesgue measure $0$, $\hat{f}_n$ converges almost surely to $\hat{F}$. Therefore, by Egoroff theorem, $f_n$ converges to $\hat{F}$ with respect to the metric $d_T$.

To conclude the proof of the proposition, it remains to verify the assertion assumed in the beginning of the argument. Fix $t \in [0, T]$ and suppose that $\ell^\delta_m(t) = 0$ for some $m \geq 1$. In this case, since $\ell^\delta_m(t) \to 0$, $\lim_m \text{dist}(t, B(D(\hat{f}_n), \delta_m)^c) = 0$. Take a point $t_n$ in the compact set $B(D(\hat{f}_n), \delta_m)^c$ realizing this distance to conclude that there exists a sequence $\epsilon_n$ converging to $t$ for which $\ell^\delta_m(t_n) = 0$. As $\ell^\delta_m(t_n) = 0$, $\hat{f}_n$ is constant in the interval $(t_n - \delta_m, t_n + \delta_m)$. Therefore, the functions $\hat{f}_n$ are constant in a neighborhood $N(t)$ of $t$ for $n$ large enough. Since $\hat{f}_n$ converges on the rationals to $\hat{F}$, we conclude, as claimed, that $\hat{F}$ is constant in $N(t) \cap \mathbb{Q}$. □

Another topology which can be defined in the space $\mathfrak{M}$ corresponds to the projection of the Skorohod’s $M_2$ topology, which is generated by the Hausdorff distance between the graphs of the functions. For two equivalence classes $f$ and $g$ in $\mathfrak{M}$, define the distance $d_T^{(2)}(f, g)$ by 

$$d_T^{(2)}(f, g) := d_H(\Gamma_f, \Gamma_g), \quad (5.8)$$

where $\tilde{f}, \tilde{g}$ are the representative of the equivalence class of $f, g$,

$$\Gamma_f = \bigcup_{t \in [0, T]} \{t\} \times [\liminf_{s \to t} \tilde{f}(s), \limsup_{s \to t} \tilde{f}(s)],$$

and $d_H$ is the Hausdorff distance.

Recall the definition of the modulus of continuity $\omega_\delta(f)$ and note that $\omega_\delta(f) \geq 2\delta$ unless $f$ is constant. Denote by $B(f; r)$, $B^{(2)}(f; r)$ the ball of center $f$ and radius $r$ with respect to the metric $d_T$, $d_T^{(2)}$, respectively.

**Lemma 5.5.** For any equivalence class $f \in \mathfrak{M}$ and any $\delta > 0$, 

$$B^{(2)}(f; \delta) \subseteq B(f; \delta + \omega_{2\delta}(f)).$$

**Proof.** Fix $f \in \mathfrak{M}$, $\delta > 0$ and $g \in B^{(2)}(f; \delta)$. By definition of $d_T$,

$$d_T(f, g) = d_H(\tilde{f}, \tilde{g}) \leq \|\tilde{f} - \tilde{g}\|_\infty, B(D(\tilde{f}), 2\delta)^c + \Lambda(B(D(\tilde{f}), 2\delta)) = \|\tilde{f} - \tilde{g}\|_\infty, B(D(\tilde{f}), 2\delta)^c + \omega_{2\delta}(\tilde{f}).$$

In order to evaluate the first term above, fix $t \notin B(D(\tilde{f}), 2\delta)$ so that $\tilde{f}$ is constant in $B(t, 2\delta)$. In particular, $\Gamma_f \subset \Sigma = [0, t - 2\delta] \times \mathbb{R} \cup [0, T] \times \{f(t)\} \cup [t + 2\delta, T] \times \mathbb{R}$. Since $d_T^{(2)}(\tilde{g}, \tilde{f}) = d_T^{(2)}(g, f) \leq \delta$, by definition of the Hausdorff distance,

$$\delta \geq \text{dist}(t, \tilde{g}(t)), \Gamma_f \geq \text{dist}(t, \tilde{g}(t), \Sigma) = 2\delta \wedge |\tilde{f}(t) - \tilde{g}(t)|.$$

This implies that $|\tilde{f}(t) - \tilde{g}(t)| \leq \delta$ for every $t \notin B(D(\tilde{f}), 2\delta)$, which finishes the proof of the lemma. □
Consider a sequence \( \{Y_n : 1 \leq n \leq \infty\} \) of real-valued stochastic processes defined on some probability space \((\Omega, \mathcal{F}, P)\). Assume that the trajectories of each \( Y_n, 1 \leq n \leq \infty \), belong to \( \mathcal{M}_0 \) \( P \)-almost surely. This is the case, for instance, of continuous-time Markov chains taking values on a countable subset of \( \mathbb{R} \).

**Theorem 5.6.** Fix \( T > 0 \). If \( d_T^{(2)}(Y_n, Y_\infty) \) converges to 0 in probability as \( n \uparrow \infty \), then \( d_T(Y_n, Y_\infty) \) converges to 0 in probability as \( n \uparrow \infty \).

**Proof.** It is enough to show that for each \( \epsilon > 0 \), \( \lim_{n \rightarrow \infty} P[d_T(Y_n, Y_\infty) > 2\epsilon] = 0 \). Fix \( \delta < \epsilon \) so that the previous probability is bounded by \( P[d_T(Y_n, Y_\infty) > \epsilon + \delta] \). This latter probability is in turn less than or equal to

\[
P[d_T(Y_n, Y_\infty) > \epsilon + \delta, \omega_2(\tilde{Y}_\infty) \leq \epsilon] + P[\omega_2(\tilde{Y}_\infty) > \epsilon].
\]

Since \( Y_\infty \) has trajectories in \( \mathcal{M}_0 \) \( P \)-almost surely, the second term vanishes as \( \delta \downarrow 0 \). The first one is bounded by \( P[d_T(Y_n, Y_\infty) > \delta + \omega_2(\tilde{Y}_\infty)] \) which by the previous lemma is less than or equal to \( P[d_T^{(2)}(Y_n, Y_\infty) > \delta] \). By assumption, this term vanishes as \( n \uparrow \infty \). \( \square \)

Assume that in the probability space \((\Omega, \mathcal{F}, P)\) introduced before the statement of the previous theorem is also defined a sequence \( \{X_n : 1 \leq n < \infty\} \) of real-valued stochastic processes whose trajectories belong to \( \mathcal{M}_0 \) \( P \)-almost surely.

**Corollary 5.7.** Fix \( T > 0 \). If both \( d_T(X_n, Y_n) \) and \( d_T^{(2)}(Y_n, Y_\infty) \) converge to zero in probability as \( n \uparrow \infty \), then \( d_T(X_n, Y_\infty) \) also converges to zero in probability as \( n \uparrow \infty \).

### 6. Main result

We prove in this section that under certain assumptions the continuous time Markov process \( X_t^N \), introduced in Section 6, is close, in an appropriate time scale and with respect to the topology introduced in Section 2, to a simple random walk \( Y_t^N \) which only visits the set \( A_N \) of the deepest traps and which has identically distributed jump probabilities: \( p(x, y) = \rho_N(y), x, y \in A_N \). For such result we need, roughly speaking, the set of deepest traps \( A_N \)

- to support most of the stationary measure \( \nu \),
- to consist of well separated points,
- to be unlikely to be hit in a short time,
- to have comparable escape probabilities from different points.

The main result presented below holds in a more general context than the one described in Section 2. We suppose throughout this section that \( \{G_N : N \geq 1\} \) is a sequence of finite, connected, vertex-weighted graphs, where \( \{W_x^N : x \in V_N\} \) represents the positive weights. The vertices of \( V_N \) are enumerated in decreasing order of weights, \( V_N = \{x_1^N, \ldots, x_{|V_N|}^N\}, W_{x_j}^N \geq W_{x_{j+1}}^N, 1 \leq j \leq |V_N| - 1 \).

Denote by \( X_t^N \) the Markov process on \( V_N \) with generator given by (12). We do not assume that the depths \( W_{x_i}^N \) are chosen according to (11), but we impose some conditions presented below in (A0)–(A3).

We write in this section \( J_N \uparrow \infty \) to represent an increasing sequence of natural numbers \( \{J_N : N \geq 1\} \) such that \( \lim_{N \rightarrow \infty} J_N = \infty \). To keep notation simple, we sometimes omit the dependence on \( N \) of states, measures and sets.

Recall that \( \nu = \nu_N \), defined in (22), is the stationary measure of the random walk \( X_t^N \). Assume that \( \nu(B_N) \) vanishes asymptotically for any sequence of subsets \( B_N = \{x_1^N, \ldots, x_{J_N}^N\} \subset V_N \) such that \( J_N \uparrow \infty \):

\[
\lim_{N \rightarrow \infty} \nu_N(B_N^c) = 0. \tag{A0}
\]

Fix three sequences \( M_N \uparrow \infty, \ell_N \uparrow \infty \) and \( L_N \uparrow \infty \), \( M_N \leq |V_N| \). The sequence \( M_N \) represents the number of deep traps selected, and \( \ell_N \) a lower bound on the minimal distance among these deepest traps. We formulate three assumptions on these sequences. Let \( A_N = \{x_1^N, \ldots, x_{M_N}^N\} \) be the set of the deepest traps. We first require the deepest traps to be well separated:

\[
d(x_i^N, x_j^N) > 2\ell_N + 1, \quad 1 \leq i \neq j \leq M_N \tag{A1}
\]

for all \( N \) large enough. This condition, which is analogous to condition (B1), ensures that any path \( \{x_i^N = z_0, z_1, \ldots, z_m = x_j^N\} \) from \( x_i^N \) to \( x_j^N \) has a state \( z_k \) which belongs to \( R(A_N, \ell_N) \).

The second assumption is somehow related to (B3) and requires, as explained below, the different escape probabilities \( \nu_x, x \in A_N \), to have similar order of magnitude. For a subset \( B \) of \( V_N \), let \( \nu_B \) be
the measure $\nu$ conditioned on $B$:
\[
\nu_B(x) = \frac{W_x^N \deg(x)}{\sum_{y \in B} W_y^N \deg(y)}, \quad x \in B.
\]

Expectation with respect to $\nu_B$ is denoted by $E_{\nu_B}$.

We suppose that there exists a sequence $\{\beta_N : N \geq 1\}$ such that for any sequence of subsets $B_N = \{x_1^N, \ldots, x_{N_j}^N\} \subset A_N$ such that $|B_N| = J_N \uparrow \infty$
\[
\limsup_{N \to \infty} E_{\nu_B} \left[ \frac{W_x^N \beta_N \nu(x)}{\nu_B(x)} \right] < \infty, \quad \limsup_{N \to \infty} \frac{1}{|B_N|} E_{\nu_B} \left[ \frac{\beta_N \nu(x)}{W_x^N} \right] < \infty. \tag{A2}
\]

This hypothesis postulates essentially a law of large numbers for $\deg(x) \nu(x)$ and a bound for the sum of $(W_x^N)^2 \deg(x_j)/\nu(x_j)$.

In analogy with (B2), we will also assume that the hitting time of $A_N$ is much smaller than the mixing time of the discrete-time random walk on $G_N$. For $L \geq 1$ let
\[
\kappa_N = \kappa(L, M_N, \ell_N) = \max_x \max_{x \in A_N} \max_{z \in B(x, \ell_N)} P^N_x [H_z < \text{Lt}_{\text{mix}}^N]. \tag{6.1}
\]
Assume that for some sequence $L_N \uparrow \infty$,
\[
\lim_{N \to \infty} M_N^2 \kappa_N = 0, \quad \lim_{N \to \infty} M_N^2 2^{-L_N} = 0. \tag{A3}
\]

**Remark 6.1.** Consider three sequences $M_N \uparrow \infty, \ell_N \uparrow \infty$ and $L_N \uparrow \infty$ satisfying (A0)--(A2) and such that
\[
\lim_{N \to \infty} \kappa(L, M_N, \ell_N) = 0. \tag{6.2}
\]
Then, there exists a sequence $M_N' \uparrow \infty, M_N \leq M_N'$, for which the three sequence $M_N', \ell_N, L_N$ satisfy (A0)--(A3).

Indeed, it follows from (B2) and the fact that $L_N \uparrow \infty$ that there exists a sequence $K_N \uparrow \infty$ such that $\lim_{N \to \infty} K_N^2 2^{-L_N} = 0, \lim_{N \to \infty} K_N^2 \kappa(L, M_N, \ell_N) = 0$. Define a new sequence $M_N'$ by $M_N' = \min\{M_N, K_N\}$ and define $A_N'$ accordingly. Since $A_N' \subset A_N$ and $\kappa_N \leq \kappa_N$, (A0)--(A3) hold for the sequences $M_N', \ell_N, L_N$.

Hence, in the applications, if one is able to prove (B2), one can redefine the sequence $M_N$ to obtain (A3) which is the condition assumed in the main result of this section. Moreover, if a sequence $M_N$ satisfies conditions (A1), (A2), (B2), then any sequence $M_N' \uparrow \infty$ which increases to infinity with $N$ at a slower pace than $M_N, M_N' \leq M_N$, also satisfies these three conditions. The same observation holds for the sequence $L_N$. Hence, in the applications, both sequences shall increase very slowly to infinity, in a way that (A3) is fulfilled, and all the problem rests on the identification of a convenient space scale $\ell_N$, large for the process to mix before returning to a state, as required in condition (B2), but not too large, to permit a good description of a ball of radius $\ell_N$ and a good estimate of the escape probability $\nu(x)$.

Let $\rho_N$ be the probability measure on the set $A_N$ given by
\[
\rho_N(x_j) = \frac{\deg(x_j) \nu_L(x_j)}{\sum_{1 \leq i \leq M_N} \deg(x_i) \nu_L(x_i)}, \tag{6.3}
\]
where $\nu_L(x_j) = \nu_L^{N_j}(x_j)$ is the escape probability introduced in (A1). By (2.4), $\rho_N$ can also be written as
\[
\rho_N(x_j) = \frac{\nu(x_j) \nu(x_j) W_x^{\ell_N - 1}}{\sum_{1 \leq i \leq M_N} \nu(x_i) \nu(x_i) W_x^{\ell_N}}, \tag{6.4}
\]
which corresponds to (A4) with $U = H_{R(A_N, \ell_N^N)}$.

For each $N \geq 1$, consider the continuous-time Markov process $\{Y_t^N : t \geq 0\}$ on $A_N$ defined as follows. While at $x \in A_N$ the process waits a mean $W_x^N/\nu_L(x)$ exponential time at the end of which it jumps to $y \in A_N$ with probability $\rho_N(y)$. Note that the jump distribution is independent of the current state and that the process may jump to its current state since we did not impose $y$ to be different from $x$. Moreover, the probability measure $\nu^N(x)/\nu^N(A_N)$ is the (reversible) stationary state of the Markov chain $\{Y_t^N : t \geq 0\}$.

We are now in a position to state the main result of this paper, from which we will deduce Theorems 6.1 and 6.2.
Theorem 6.2. Suppose that conditions (A0)–(A3) are in force. Then, for every \( N \geq 1 \), there exists a coupling \( Q_N \) between the stationary, continuous-time Markov chain \( \{Y^N_{\beta_N t} : t \geq 0\} \) described above, and the Markov chain \( \{X^N_{\beta_N t} : t \geq 0\} \) such that \( Q_N[X^N_0 = Y^N_0 = y] = \rho(y), y \in A_N, \) and
\[
\lim_{N \to \infty} Q_N[d_T(X^N_{\beta_N t}, Y^N_{\beta_N t}) > \delta] = 0
\]
for every \( T \geq 0 \) and \( \delta > 0 \), where \( d_T \) stands for the distance introduced in (5.3).

Corollary 6.3. Consider a sequence \( \{z^N : N \geq 1\} \), \( z^N \in A_N \), such that
\[
\liminf_{N \to \infty} \rho_N(z^N) > 0.
\]
The statement of Theorem 6.2 remains in force if in the assumptions we replace the property \( Q_N[X^N_0 = Y^N_0 = y] = \rho(y), y \in A_N, \) by the property \( Q_N[X^N_0 = Y^N = z^N] = 1. \)

Proof. The assertion of this corollary follows from Theorem 6.2 by conditioning on the event \( X^N_0 = z^N. \)

Theorem 6.2 follows from Lemmas 6.4, 6.5 and Proposition 6.6 below. Theorem 6.2 asserts that the process \( X^N_{\beta_N t} \) is close to the process \( Y^N_{\beta_N t} \) which jumps at rate \( \beta_{\nu}v(x)/W^N_x \). If this latter expression is not of order one, the asymptotic behavior of \( Y^N_{\beta_N t} \) will not be meaningful and our approximation of \( X^N_{\beta_N t} \) by \( Y^N_{\beta_N t} \) devoid of interest. Hence, in the applications we expect
\[
\beta_N \approx \frac{W^N_x}{v(x)}.
\]

Lemma 6.4. Assume that hypotheses (A0)–(A3) are in force. Then, there exists a subset \( B_N = \{x^N_1, \ldots, x^N_{J_N}\} \subset A_N \) such that,
\[
\lim_{N \to \infty} \kappa_N M_N E_{\rho} \frac{\beta_N}{\nu(B^c_N)} = 0, \quad (6.5)
\]
\[
\lim_{N \to \infty} \nu(B^c_N) = 0, \quad (6.6)
\]
\[
\limsup_{N \to \infty} \frac{\beta_N}{E_{\rho} \nu(B^c_N)} = 0. \quad (6.7)
\]

Proof. We start proving (6.5). By definition of the probability measure \( \rho_N \) this expression is equal to
\[
\kappa_N M_N \frac{1}{E_{\nu}(\beta_N v(x))}.
\]
This term vanishes as \( N \uparrow \infty \) in view of (A3) and (A2) with \( B_N = A_N. \)

By (A0), \( \nu(A^c_N) \) vanishes as \( N \uparrow \infty. \) There exists, therefore, a sequence \( K_N \uparrow \infty \) such that
\[
\lim_{N \to \infty} K_N \nu(A^c_N) = 0.
\]
Let \( B_N = \{x^N_1, \ldots, x^N_{J_N}\}, \) where \( J_N = \min\{M_N, K_N\} \) so that \( |B_N| \nu(A^c_N) \to 0. \)

The second assertion of the lemma follows from assumption (A0) because \( J_N \uparrow \infty. \) Moreover, as
\[
\beta_N E_{\rho} \left[ \frac{W^N_x}{v(x)} \right]^{-1} \rho(B_N) \leq E_{\nu} \left[ \frac{\beta_N v(x)}{W^N_x} \right],
\]
by (A2) and by definition of the set \( B_N, \) we have that
\[
\limsup_{N \to \infty} \frac{\beta_N}{E_{\rho} \nu(B^c_N)} = 0
\]
for some finite constant. This concludes the proof of the lemma. \( \square \)
Lemma 6.5. Assume that conditions (A2), (A3), (6.5)–(6.7) are in force. Then, there exists a sequence \( \{K_N : N \geq 1\} \) such that
\[
\lim_{N \to \infty} K_N M_N 2^{-LN} = 0, \quad \lim_{N \to \infty} K_N M_N^2 \kappa_N = 0, \quad (6.8)
\]
\[
\lim_{N \to \infty} K_N \nu(V_N \setminus A_N) \beta_N \nu(A_N) \rho \left( \frac{W_x^N}{v(x)} \right) = 0, \quad (6.9)
\]
\[
\lim_{N \to \infty} K_N \beta_N \nu(A_N) \rho \left( \frac{W_x^N}{v(x)} \right) = 0, \quad (6.10)
\]
\[
\lim_{N \to \infty} K_N \frac{K_N^2 \nu(V_N \setminus A_N)}{\nu(A_N)} \rho \left( \frac{W_x^N}{v(x)} \right) = 0, \quad (6.11)
\]
\[
\lim_{N \to \infty} K_N \beta_N \nu(A_N) \rho \left( \frac{W_x^N}{v(x)} \right) = 0. \quad (6.12)
\]

**Proof.** In view of (A3), there exists a sequence \( \psi_N \uparrow \infty \) such that \( \psi_N M_N^3 2^{-LN} \), \( \psi_N M_N^3 \kappa_N \) vanish as \( N \uparrow \infty \). We may choose this sequence \( \psi_N \) so that the limits in (6.8)–(6.10) hold when multiplied by \( \psi_N \), as well as the one in (6.7) when multiplied by \( \psi_N^2 \). Given this sequence \( \psi_N \), let
\[
K_N = \frac{\psi_N \beta_N \nu(v(x))}{W_x^N} = \psi_N \beta_N \nu(A) \left[ \frac{v(x)}{v(x)} \right].
\]

Conditions (6.5) follow the definition of \( \psi_N \) and from (A2), while condition (6.9) follows from (6.5) since \( B_N \subseteq A_N \). To verify (6.10), it is enough to remember that \( \rho(x) W_x^N v(x)^{-1} = v(x) \) and to recall (6.6). Condition (6.11) follows from assumptions (6.7), (6.8) and the definition of \( K_N \). Finally, condition (6.12) requires \( \psi_N \) to diverge. \( \square \)

**Proposition 6.6.** Suppose that conditions (A1), (A2), (6.8)–(6.12) are in force. Then, for every \( N \geq 1 \), there exists a coupling \( Q_N \) between the stationary, continuous-time Markov chain \( \{Y_{N,t}^N : t \geq 0\} \) on \( A_N \) with mean \( W_x^N/\beta_N v(x) \) exponential times and uniform jump probabilities \( p_N(x,y) = \rho(y), \quad x, y \in A_N \), and the Markov chain \( \{X_{N,t}^N : t \geq 0\} \) such that \( Q_N[X_N^N = Y_0^N] = \rho(y), \quad y \in A_N \), and
\[
\lim_{N \to \infty} Q_N\left[\delta_T(X_{N,t}^N,Y_{N,t}^N) > \delta\right] = 0
\]
for every \( T \geq 0 \) and \( \delta > 0 \), where \( \delta_T \) stands for the distance introduced in (5.4).

**Proof.** Recall the definition of the sequence of stopping times \( \{D_k : k \geq 0\} \) introduced in Section 4 with \( U = H_{R(A_N, \ell)} \). Since by (A1) \( R(A_N, \ell) \neq \emptyset \) and since the state space is finite and irreducible, \( E_{x}[U] < \infty \) for all \( x \in A \). It also follows from assumption (A1) that \( P_x[H_{A \setminus \{x\}} \geq U] = 1 \) for all \( x \in A \).

Therefore, by Lemma 4.1 and Proposition 4.2 the discrete-time Markov chain \( X_{N,t}^0 \) is irreducible and its unique stationary state is the measure \( \rho \) defined in (6.3).

We start the construction of the measure \( Q_N \) by coupling the discrete skeleton of the chain \( Y_{N,t}^N \), and by coupling the waiting times of the chain \( Y_{N,t}^N \) with the times spent by \( X_N^0 \) at each site of \( A_N \). This follows from Lemma 5.2 which presents an estimate of the distance between the measure \( \rho \) and the measure \( p(\cdot, A) \), from Lemma 5.1 and from the strong Markov property at time \( H_{R(A, \ell)} \)
\[
\sup_{y \in A} \left\| P_y[X_{N,t}^0 = y] - \rho(\cdot) \right\|_{TV} \leq (M_N + 1)(2^{-LN} + N \kappa_N) =: a_N. \quad (6.13)
\]

Let \( \sigma_0 = 0 \) and denote by \( \{\sigma_i : i \geq 1\} \) the jump times of the chain \( Y_{N,t}^N \), including among these jumps the ones to the same site. We couple the initial state \( X_0^N \) and \( Y_0^N \) so that \( Q_N[X_0^N = Y_0^N] = 1 \).

As \( Y_{N,t}^N \) is distributed according to \( \rho \), by (6.13) we can couple \( X_{N,t}^0 \) and \( Y_{N,t}^N \) in a way that they coincide with probability at least \( 1 - a_N \). Moreover, conditioned on \( X_{N,0}^0 = x \), the number of visits of \( X_{N,t}^0 \) to the point \( x \) between times \( D_i \) and \( D_{i+1} \) is a geometric random variable with success probability \( v(x) \), so that
\[
\int_{[D_i, D_{i+1})} 1(X_{N,t} = x) \, dt
\]
is an exponential random variable with expectation \( W_x^N v(x) \). This is also the distribution of the time that \( Y_{N,t}^N \) spends in \( x \). Proceeding by induction and using the strong Markov property at times \( D_i \) (for
$X_t^N$ and $\sigma_t$ (for $Y_t^N$), we obtain a coupling $Q_N$ between $X_t^N$ and $Y_t^N$ such that

$$Q_N \left[ \begin{array}{l}
X_t^N = Y_t^N, \\
\int_{D_i}^{D_{i+1}} 1\{X_t^N = X_j^N\} dt = \sigma_i - \sigma_{i+1}
\end{array} \right] \geq 1 - K_N a_N,$$

where $K_N$ is the sequence introduced in Lemma 5.5. Denote the event appearing in the previous formula by $\mathcal{G}$. By 6.12,

$$\lim_{N \to \infty} Q_N[\mathcal{G}^c] = 0. \quad (6.14)$$

We claim that the coupling $Q_N$ defined above satisfies the statement of the theorem. To estimate the distance between the processes $X_t^N$ and $Y_t^N$, we introduce a third process $\bar{X}_t^N$ close to $X_t^N$ in the distance $d_T$. Following 2, consider the process $X_t^N$ defined by

$$\bar{X}_t^N = X_t^N \left( \text{sup} \{s \leq t : X_s^N \in A_N\} \right). \quad (6.15)$$

The (non-Markovian) process $\bar{X}_t^N$ indicates the last site in $A_N$ visited by $X_s^N$ before time $t$. We adopt for $X_t^N$ the same convention agreed for the process $Y_t^N$ and consider that the process $\bar{X}_t^N$ jumped from $y \in A_N$ to $y$ at time $t'$ if the process $X_t^N$ being at $y$ at time $t < t'$, reached $R(A_N, \ell)$ and then returned to $y$ at time $t'$ before hitting another site $z \in A_N \setminus \{y\}$. With this convention, the jump times of the process $\bar{X}_t^N$ are exactly the stopping times $\{D_i : i \geq 1\}$.

We assert that for every $T > 0$ and $\delta > 0$,

$$\lim_{N \to \infty} P_\rho[d_T(\bar{X}_t^N, X_t^N) > \delta] = 0. \quad (6.16)$$

Fix $T > 0$ and $\delta > 0$. By definition of the process $X_t^N$,

$$d_T(\bar{X}_t^N, X_t^N) \leq \frac{1}{\beta_N} \int_0^{\beta_N T} 1\{X_t^N \notin A_N\} dt. \quad (6.17)$$

Therefore,

$$P_\rho[d_T(\bar{X}_t^N, X_t^N) > \delta] \leq \frac{1}{\beta_N \delta} E_\rho \left[ \int_0^{D_K_N} 1\{X_t^N \notin A_N\} dt \right] + P_\rho[D_{K_N} \leq \beta_N T].$$

Let us define

$$\Delta_N := \int_0^{D_K_N} 1\{X_t^N \notin A_N\} dt = D_{K_N} - \sigma_{K_N}. \quad (6.18)$$

This quantity will appear a couple of times in the computations below. By 6.18, $P_\rho[D_{K_N} \leq \beta_N T]$ vanishes as $N \uparrow \infty$ because $\sigma_{K_N} \leq D_{K_N}$. On the other hand, by definition of the process $\bar{X}_t^N$ and by stationarity,

$$\frac{1}{\beta_N \delta} E_\rho[\Delta_N] = \frac{K_N}{\beta_N \delta} E_\rho \left[ \int_0^{D_{\ell_1}} 1\{X_t^N \notin A_N\} dt \right].$$

By Corollary 6.2, the previous expression equals

$$\frac{K_N}{\beta_N \delta} E_\rho[\nu_N(V_N \setminus A_N)] = \frac{K_N}{\beta_N \delta} \frac{E_\rho[W_N]}{E_\rho[\nu_N(A_N)]}. \quad (6.19)$$

By 6.5, this expression vanishes as $N \uparrow \infty$. This proves 6.19.

Now we turn into the estimation of the distance between $X_t^N$ and $Y_t^N$. On the event $\mathcal{G}$, the first $K_N$ jumps of the processes $X_t^N$ and $Y_t^N$ are the same, and the process $Y_t^N$ is always “ahead” of $X_t^N$ in the sense that $\bar{X}_t^N$ spends more time at each site than $Y_t^N$. We need to show that the delay between $\bar{X}_t^N$ and $Y_t^N$ is small. Let $B_N = \{x_1^N, \ldots, x_M^N\} \subseteq A_N$ be the set appearing in conditions 6.10 and 6.11, and let $\mathcal{N}_N$ be the number of times the process $Y_t^N$ visits $B_N$ before $\sigma_{K_N}$:

$$\mathcal{N}_N := \# \{j < K_N : Y_{\sigma_j}^N \in B_N\}. \quad (6.20)$$

Denote by $\mathcal{G}_1$ the event $\mathcal{G} \cap \{\sigma_{K_N} \geq \beta_N T\}$. Since we have that $d_T(\bar{X}_t^N, Y_t^N) = \beta_N^{-1} d_{\sigma_{K_N}}(\bar{X}_t^N, Y_t^N)$, on the set $\mathcal{G}_1$, $d_T(\bar{X}_t^N, Y_t^N) \leq \beta_N^{-1} d_{\sigma_{K_N}}(\bar{X}_t^N, Y_t^N)$. Therefore, on the set $\mathcal{G}_1$,

$$d_T(\bar{X}_t^N, Y_t^N) \leq \frac{1}{\beta_N} \sum_{j=1}^{K_N} \int_{\sigma_{j-1}}^{\sigma_j} 1\{Y_t^N \neq \bar{X}_t^N\} dt$$

$$\leq \frac{1}{\beta_N} \int_0^{\sigma_{K_N}} 1\{Y_t^N \notin B_N\} dt + \frac{1}{\beta_N} \sum_{j=1}^{K_N} \int_{\sigma_{j-1}}^{\sigma_j} 1\{Y_t^N \in B_N, Y_t^N \neq \bar{X}_t^N\} dt.$$
We claim that each integral in the second term of the previous sum is bounded by $\Delta_N$. Indeed, the total delay of the process $X^N_N$ with respect to the process $Y^N_N$ in the interval $[0, \sigma_{K_N}]$ is $D_{K_N} - \sigma_{K_N} = \Delta_N$. On the other hand, either the length of time interval $[\sigma_{j-1}, \sigma_j]$ is bounded by $\Delta_N$, in which case the claim is trivial, or the length is greater than $\Delta_N$. In this latter situation, since the total delay between $Y$ and $X$ in the interval $[0, \sigma_{K_N}]$ is $\Delta_N$, $D_{j-1} - \sigma_{j-1} \leq \Delta_N$ for $1 \leq j \leq K_N$. Hence, in the interval $[\sigma_{j-1} + \Delta_N, \sigma_j)$ we have that $Y_t = Y_i$. This proves our assertion. In conclusion, if one recalls the definition of $\mathcal{G}_N$, on the set $\mathcal{G}_1$, 

$$d_T(X^N_N, Y^N_N, Y^N_N \leq B_N) \leq \frac{1}{\beta_N} \int_0^{\sigma_{K_N}} 1\{Y^N_t \notin B_N\} dt + \frac{1}{\beta_N} \Delta_N \mathcal{G}_N.$$ 

In conclusion, 

$$Q_N[d_T(X^N_N, Y^N_N, Y^N_N \leq B_N) > \delta] \leq Q_N[\mathcal{G}_1] + \frac{2}{\beta_N \delta} Q_N\left[\int_0^{\sigma_{K_N}} 1\{Y^N_t \notin B_N\} dt\right] + Q_N[\Delta_N \mathcal{G}_N > (1/2)\delta \beta_N].$$

The first term vanishes as $N \uparrow \infty$ by (6.14) and (6.15). By Tchebychev and Cauchy-Schwarz inequalities, 

$$P[ZW > \delta] = P[\sqrt{ZW} > \sqrt{\delta}] \leq (\delta^{-1} E[Z|E[W]])^{1/2}$$

for any pair of nonnegative random variables $Z, W$. Therefore, the sum of the second and third terms is bounded by 

$$\frac{2K_N}{\beta_N \delta} E_{\rho} \left[\frac{W^N_{v(x)}}{\nu_v(x)} \mathbb{1}\{x \notin B_N\}\right] + \sqrt{\frac{2Q_N[\Delta_N]Q_N[\mathcal{G}_N]}{\beta_N \delta}}.$$

Since $Q_N[\mathcal{G}_N] = K_N \rho(B_N)$, by (6.14) this expression is less than or equal to 

$$\frac{2K_N}{\beta_N \delta} E_{\rho} \left[\frac{W^N_{v(x)}}{\nu_v(x)} \mathbb{1}\{x \notin B_N\}\right] + \sqrt{\frac{2Q_N[\Delta_N]Q_N[\mathcal{G}_N]}{\beta_N \delta}}.$$

By assumptions (6.10) and (6.11), this expression vanishes as $N \uparrow \infty$.

To conclude the proof of the theorem it remains to show that 

$$\lim_{N \to \infty} Q_N[\sigma_{K_N} \leq \beta_N T] = 0. \quad (6.18)$$

For any random variable $Z$ and any $T \geq 0$ such that $E[Z] \geq 2T$, by Tchebycheff inequality we have that 

$$P[Z < T] \leq \frac{4 \text{Var}(Z)}{E[Z]^2}.$$

Note that 

$$Q_N[\sigma_{K_N}] = K_N E_{\rho} \left[\frac{W^N_{v(x)}}{\nu_v(x)}\right], \quad \text{Var}_{Q_N}(\sigma_{K_N}) \leq 2 K_N E_{\rho} \left[\frac{W^N_{v(x)}}{\nu_v(x)}\right]^2,$$

and that, by assumption (6.12), $K_N E_{\rho}[W^N_{v(x)}] \geq 2 \beta_N T$ for $N$ sufficiently large. By the previous elementary inequality, 

$$Q_N[\sigma_{K_N} \leq \beta_N T] \leq \frac{8 E_{\rho} \left[\frac{W^N_{v(x)}}{\nu_v(x)}\right]^2}{K_N E_{\rho} \left[\frac{W^N_{v(x)}}{\nu_v(x)}\right]^2} \leq \frac{8 \beta_N}{K_N E_{\rho} \left[\frac{W^N_{v(x)}}{\nu_v(x)}\right]^2}.$$ 

By assumption (6.12), the first term of this expression vanishes as $N \uparrow \infty$. The second one is equal to 

$$E_{\nu_A} \left[\frac{W^N_{v(x)}}{\beta_N v(x)}\right].$$

By (A2) this expression is bounded uniformly in $N$. This concludes the proof of (6.18) and the one of Proposition 6.6. 

In the previous proposition, instead of starting from the stationary measure $\rho_N$, we may also start from any state $z^N$, provided its asymptotic $\rho_N$-measure does not vanish with $N$, as in the hypothesis of Corollary 6.3.

The following remark will be important when proving Theorem 2.2.

**Remark 6.7.** Assumption (A0) has only been used in Lemma 6.4 to prove the existence of a sequence of subsets $B_N$ satisfying (6.13), (6.15). In particular, Theorem 6.2 remains in force if hypothesis (A0) is replaced by the existence of a sequence $I_N \leq M_N$, $I_N \uparrow \infty$, for which $B_N = \{x_1^N, \ldots, x_{N^2}^N\}$ satisfies (6.13) and such that 

$$\lim_{N \to \infty} |B_N| \nu(A^N) = 0. \quad (6.19)$$
7. \textbf{K-processes}

We introduce in this section K-processes, a class of strong Markov processes on $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ with one fictitious state. We refer to [13] for historical remarks and to [29] for a detailed presentation and the proofs omitted here. The main result of this section presents sufficient conditions for the convergence of a sequence of finite-state Markov processes to a K-process.

Throughout this section we fix two sequences of positive real numbers $\{u_k : k \in \mathbb{N}\}$ and $\{Z_k : k \in \mathbb{N}\}$. The first sequence represents the ‘entrance measure’ and the second one the ‘hopping times’ of the K-process. The only assumption we make over these sequences is that

$$\sum_{k \in \mathbb{N}} Z_k u_k < \infty.$$  \hfill (7.1)

However, the process will be more interesting in the case

$$\sum_{k \in \mathbb{N}} u_k = \infty.$$  \hfill (7.2)

If this sum is finite, the K-process associated to the sequences $u_k$ and $Z_k$ corresponds to a Markov process on $\mathbb{N}$ with no fictitious state.

Consider the set $\overline{\mathbb{N}}$ of non-negative integers with an extra point denoted by $\infty$. We endow this set with the metric induced by the isometry $\phi: \mathbb{N} \to \mathbb{R}$ which sends $n \in \mathbb{N}$ to $1/n$ and $\infty$ to 0. This makes the set $\overline{\mathbb{N}}$ into a compact metric space. We use the notation $\text{dist}(x, y) = |\phi(y) - \phi(x)|$ for this metric.

For each $k \in \mathbb{N}$, define independent Poisson process $\{N^k_t : t \geq 0\}$ with jump rate given by $u_k$.

Throughout this section we fix two sequences of positive real numbers $\{u_k : k \in \mathbb{N}\}$ and write

$$t \in [0, \infty) = \mathbb{R}_+ \cup \{\infty\},$$

where the case in which

$$\sum_{i=1}^\infty u_i = 1$$

holds. We summarize in the next result the main properties of the process $X^y$. Its proof can be found in [29] or adapted from [18] where the case in which $u_k = 1$ for all $k \geq 1$ is examined. Recall that we denote by $H_A$ the hitting time of a set $A$ and that $Z_\infty = 0$.

**Theorem 7.1.** For any $y \in \overline{\mathbb{N}}$, the process $\{X^y(t) : t \geq 0\}$ is a strong Markov process on $\overline{\mathbb{N}}$ with right-continuous paths with left limits. Being at $k \in \mathbb{N}$, the process waits a mean $Z_k$ exponential time at the end of which it jumps to $\infty$. For any finite subset $A$ of $\mathbb{N}$, $H_A$ is a.s. finite and

$$\mathbb{P}[X^y(H_A) = j] = \frac{u_j}{\sum_{i \in A} u_i}, \quad j \in A.$$  \hfill (7.3)

We investigate in this section the convergence of a sequence of Markov processes in finite state spaces towards the process $X^y(t)$. Let $\{M_N : N \geq 1\}$ be a sequence of integers such that $M_N \uparrow \infty$, and consider the sequences of positive real numbers

$$u_k^N, \quad Z_k^N, \quad 1 \leq k \leq M_N, \quad N \geq 1.$$  \hfill (7.4)

In analogy with (7.3), we define processes $X^y_N(t)$ with ‘entrance measure’ given by $u_k^N$ and ‘hopping times’ given by $Z_k^N$. For $N \geq 1$, let $T_0^N$, $T_i^N$, $N^N_i$, and $\sigma_i^N$, $1 \leq k \leq M_N$, $i \geq 1$, be defined as above and write

$$\Gamma^y_N(t) = Z_0^N T_0^N + \sum_{k=1}^{M_N} Z_k^N \sum_{i=1}^{N^N_i} T_i^N,$$

for $1 \leq y \leq M_N$

and

$$X^y_N(t) = \begin{cases} y & \text{if } 0 \leq t < Z_0^N T_0^N, \\ k & \text{if } \Gamma^y_N(\sigma_i^N) \leq t < \Gamma^y_N(\sigma_i^N) \text{ for some } i \geq 1. \end{cases}$$  \hfill (7.5)
One can easily see that the process $X^y_k$ is a continuous-time càdlàg, Markov chain over $\{1, \ldots, M\}$. The order in which the points $\{1, \ldots, M\}$ are visited by $X^y_k$, after the starting position, is given by the order of the times $\gamma^{-1}$. From this fact we can conclude that the law of $X^y_k$ is characterized by the following properties:

- The state space is $\{1, \ldots, M\}$ and the process starts from $y$ almost surely.
- The process $X^y_k$ remains at any site $k$ an exponential time with mean $Z^N_k$, after which it jumps to a site $j$ with probability $u^N_j / \sum_{1 \leq i \leq M} u^N_i$.

**Remark 7.2.** Note that the dynamics of the process $X^y_k$ does not change if one replaces the vector $\{u_k^N : 1 \leq k \leq M\}$ by the vector $\{\gamma N u_k^N : 1 \leq k \leq M\}$ for some $\gamma > 0$. In particular, when applying the theorem below we may multiply the sequence $u_k^N$ by a constant $\gamma$ to ensure the convergence of $\gamma N u_k^N$ to $u_k$.

The main result of this section is stated below. Recall from [15, (5.2)] the definition of the Skorohod’s $J_1$ topology.

**Theorem 7.3.** Assume that for every $k \in \mathbb{N}$

$$
\lim_{N \to \infty} \left( Z^N_k, u^N_k \right) = (Z_k, u_k) \quad (7.6)
$$

and that

$$
\lim_{m \to \infty} \limsup_{N \to \infty} \sum_{k=0}^{M} Z^N_k u^N_k = 0.
$$

Then, for any given $y \in \mathbb{N}$, $X^y_k$ converges weakly, as $N \uparrow \infty$, towards $X^y$ in the Skorohod’s $J_1$ topology.

**Proof.** The proof is a modification of the one of Lemma 3.11 in [18]. We first couple the Poisson point processes used to define $\Gamma^y_k$ and $\Gamma^y$. In some probability space $(\Omega, \mathcal{A}, \mathbb{Q})$ we construct a collection $\{X^k : k \in \mathbb{N}\}$ of Poisson point processes in $\mathbb{R}_+ \times \mathbb{R}_+$ with respect to the Lebesgue measure. Let $N^k(u, t)$ be the number of points falling in the rectangle $[0, t] \times [0, u]$. For fixed $k \in \mathbb{N}$ and $u \geq 0$, $N^k(u, \cdot)$ is distributed as a Poisson counting process with rate $u$. Define $\Gamma^y$ and $\Gamma^y_k$ as before, but using these coupled arrival processes, with corresponding intensities $u_k$ and $u^N_k$. Moreover, we also use the same jump clocks $\{T^k : k \in \mathbb{N}, i \geq 1\}$ in their constructions.

Fix an integer $m \in \mathbb{N}$ and denote by $\{S^m_i : i \geq 1\}$ the arrival times of the process $N^1(u_1, \cdot) + N^2(u_2, \cdot) + \cdots + N^m(u_m, \cdot)$, with $S^m_0 = 0$. Fix $T \geq 0$ and let $L^m_T = \inf\{i \geq 1; \Gamma^y_k(S^m_i) \geq T \text{ for every } N \geq 1\}$.

Since $(Z^N_1, u^N_1)$ converges to $(Z_1, u_1)$ and since $\Gamma^y_k(s) \geq \sum_{1 \leq i \leq N^1(u^N_1, s)} Z^N_1 T^1_i$, by the law of large numbers the above infimum is finite.

Since the sequence $\{u_k : k \in \mathbb{N}\}$ is not summable, there exists a random integer $m'$ large enough so that almost surely

$$
\sum_{k=m+1}^{m'} Z^N_k \sum_{j=0}^{N^k(u_k, S^m_{k+1})} T^k_j > 0, \quad i = 0, \ldots, L^m_T, \quad (7.7)
$$

where $f(s-)$ stands for the left limit at $s$ of a càdlàg function $f$.

Since $u^N_k$ converges to $u_k$, almost surely there exists $N(m)$ such that

$$
N^k(u^N_k, t) = N^k(u_k, t) \quad (7.8)
$$

for all $1 \leq k \leq m$, $0 \leq t \leq S^m_{L^m_T}$ and all $N \geq N(m)$. By possibly increasing $N(m)$ we can also assume that,

$$
\inf_{N \geq N(m)} \sum_{k=m+1}^{m'} Z^N_k \sum_{j=0}^{N^k(u^N_k, S^m_{k+1})} T^k_j > 0, \quad i = 0, \ldots, L^m_T. \quad (7.9)
$$

It follows from (7.8) that the arrival times $S^m_i$ are the same for the process $X^y$ and $X^y_k$. Furthermore, by (7.7), (7.9), on each interval $(S^m_i, S^m_{i+1})$ there is at least one arrival of a Poisson process $N^k(u_k, \cdot)$ for some $k > m$ and one arrival for a Poisson process $N^k(u^N_k, \cdot)$ for some $k > m$. In particular, in the time interval $[\Gamma^y(S^m_i), \Gamma^y(S^m_{i+1})]$ (resp. $[\Gamma^y(N^1(S^m_i)), \Gamma^y(N^1(S^m_{i+1}))])$, $0 \leq i < L^m_T$, the process $X^y$ (resp. $X^y_k$) performs an excursion in the set $\{1, \ldots, m\}$, while on each time interval $[\Gamma^y(S^m_i), \Gamma^y(S^m_{i+1})]$ (resp. $[\Gamma^y_k(S^m_i), \Gamma^y_k(S^m_{i+1})]$), $1 \leq i < L^m_T$, the processes $X^y$ and $X^y_k$ sit on the same site of $\{1, \ldots, m\}$.
For \( N \geq N(m) \), define the time changes \( \lambda^m_N : [0, \Gamma^y_N(S^m_L Y_T)] \to \mathbb{R}_+ \) by

\[
\lambda^m_N(t) = \frac{Z_y}{Z_y} t \quad \text{for} \quad 0 \leq t < Z_y N T_0.
\]

For \( 0 \leq i \leq L^m_T - 1 \), let

\[
\lambda^m_N(t) = \Gamma^y(S^m_i) + \frac{\Gamma^y(S^m_{i+1}) - \Gamma^y(S^m_i)}{\Gamma^y_N(S^m_{i+1}) - \Gamma^y_N(S^m_i)} [t - \Gamma^y_N(S^m_i)]
\]

if \( \Gamma^y_N(S^m_i) \leq t \leq \Gamma^y_N(S^m_{i+1}) \) and let

\[
\lambda^m_N(t) = \Gamma^y(S^m_{i+1}) + \frac{\Gamma^y(S^m_{i+1}) - \Gamma^y(S^m_i)}{\Gamma^y_N(S^m_{i+1}) - \Gamma^y_N(S^m_i)} [t - \Gamma^y_N(S^m_i)]
\]

if \( \Gamma^y_N(S^m_{i+1}) \leq t \leq \Gamma^y_N(S^m_i) \).

In view of our previous discussion,

\[
dist(X^y(\lambda^m_N(t)), X^y_N(t)) \leq (1 + m)^{-1}, \quad \text{for every } t \leq T.
\]

(7.10)

Indeed, whenever \( X^y(\lambda^m_N(t)) \) differs from \( X^y_N(t) \), they are both above \( m \), and the diameter of the set \( \{m + 1, m + 2, \ldots\} \) is given by \( (m + 1)^{-1} \).

We claim that \( \lambda^m_N \) is close to the identity: for any \( \delta > 0 \),

\[
\lim \limsup_N Q \left[ \sup_{0 \leq t \leq T} |\lambda^m_N(t) - t| > \delta \right] = 0.
\]

(7.11)

To prove this claim, fix \( m \geq 1 \) and note that

\[
\sup_{0 \leq t \leq T} |\lambda^m_N(t) - t| \leq \max_{0 \leq s \leq L^m_T} \left\{ |\Gamma^y(S^m_i) - \Gamma^y_N(S^m_i)| \lor |\Gamma^y(S^m_i) - \Gamma^y_N(S^m_i)| \right\}.
\]

By construction, the right hand side is bounded above by

\[
|Z^N_y - Z_y| T_0 + \sum_{k=1}^m |Z^N_k - Z_k| \sum_{j=1}^{N^k(u_k, S^m_L Z^m_T)} T^k_j + \sum_{k=m+1}^{\infty} \sum_{j=1}^{N^k(u_k, S^m_L Z^m_T)} T^k_j.
\]

(7.12)

For each fixed \( m \), the first two terms vanish almost surely as \( N \) goes to infinity. To estimate the other two terms note that \( L^m_T \geq L^{m+1}_T \), that \( S^m_{i+1} \leq L^m_T \), and that \( N^k \), \( \{T^k_j : j \geq 1\} \) are independent of \( S^m_L T^m_T \) for \( k > m \). In particular, for \( k > m \) and \( u > 0 \),

\[
E_Q \left[ \sum_{j=1}^{N^k(u_k, S^m_L Z^m_T)} T^k_j \right] = u \; E_Q[S^m_{L^1_T}] \leq u \; E_Q[S^1_{L^1_T}].
\]

Last expectation is bounded because \( S^1_{L^1_T} \) is defined through a Poisson process. Therefore, as \( Z_k u_k \) is summable in \( k \), the third term in \( \lambda^m_N \), which does not depend on \( N \), has finite expectation and converges to zero almost surely and in \( L^1(Q) \) as \( N \) tends to infinity. Similarly,

\[
E_Q \left[ \sum_{k=m+1}^{M_N} \sum_{j=m}^{N^k(u_k, S^m_L Z^m_T)} T^k_j \right] \leq \sum_{k=m+1}^{M_N} Z^N_k u_k \; E_Q[S^1_{L^1_T}].
\]

By assumption, this expression vanishes as \( N \uparrow \infty \) and then \( m \uparrow \infty \). This proves that \( \lambda^m_N \) holds in fact in \( L^1(Q) \).

As a consequence of (7.11), one can extract a sequence \( m_N \) growing slowly enough such that

\[
\sup_{0 \leq t \leq T} |\lambda^m_N(t) - t| \text{ converges to zero in probability as } N \uparrow \infty.
\]

This, together with (7.10), provides the two conditions of Proposition 5.3 (c) in [15]. Hence, \( X^y_N \) converges in probability to \( X^y \) in the Skorohod’s \( J_1 \) topology as \( N \) tends to infinity.
8. Scaling limit of trap models

In this section we join the results of the last three sections to establish the asymptotic behaviour of random walks on vertex-weighted graphs.

Throughout this section, we restrict our attention to weights given by an i.i.d. sequence of random variables in the basin of attraction of an α-stable distribution, as in [1]. Let us first collect some consequences of this choice of random variables. In particular we obtain the convergence of the environment to a limiting distribution.

Recall that α ∈ (0, 1) is the parameter of the stable distribution. Let λ be the measure on ℝ × (0, ∞) given by \( \lambda = \alpha w^{-(1+\alpha)} \mathrm{d}x \mathrm{d}w \). Denote by \( \{(z_i, \hat{w}_i) \in \mathbb{R} \times (0, \infty) : i \geq 1\} \) the marks of a Poisson point process of intensity λ independent of the sequence of graphs \( G_N : N \geq 1 \) and defined on a probability space \((\mathcal{Y}, \mathcal{F}, \mathbb{P})\). Define the random measure ζ on ℝ by

\[
\zeta = \sum_{i \geq 1} \hat{w}_i \delta_{z_i} ,
\]

and let \( \zeta_t = \zeta([0, t]), t \geq 0 \), be the \( \zeta \)-measure of the interval \([0, t] \). Let \( F : [0, \infty) \rightarrow [0, \infty) \) be defined by

\[
P[\zeta_t > F(t)] = \mathbb{P}[W_N^t > t], \quad t \geq 0 .
\]

The function \( F \) is non-decreasing and right-continuous. Denote its right-continuous generalized inverse by \( F^{-1} \) and let

\[
\hat{\tau}_N^i = F^{-1}(\mathcal{Y}^{1/\alpha} [\zeta_i/\zeta(i-1)/\zeta]) , \quad 1 \leq i \leq \mathcal{V} .
\]

Denote by \( \tau_N^i, 1 \leq i \leq \mathcal{V} \), the sequence \( \hat{\tau}_N^i \) in decreasing order: \( \hat{\tau}_N^i = \tau_N^{\sigma(i)} \) for some permutation \( \sigma \) of \( \{1, \ldots, \mathcal{V}\} \) and \( \tau_N^i \geq \tau_N^{i+1} \).

By [10 Proposition 3.1], \( \{\tau_N^i : 1 \leq i \leq \mathcal{V}\} \) has the same distribution as \( \{W_N^i : x \in V_N\} \). Therefore, \( (\tau_N^1, \ldots, \tau_N^\mathcal{V}) \) has the same distribution as \( (W_N^1, \ldots, W_N^\mathcal{V}) \). Moreover, since \( \mathcal{V}_N = |V_N| \rightarrow \infty \) \( \mathbb{P} \)-almost surely, the same result implies that \((\mathbb{P} \times \mathbb{P})\)-almost surely,

\[
\lim_{N \rightarrow \infty} \sum_{j \geq 1} |c_j \tau_N^j - w_j| = 0 ,
\]

where \( W = \{w_i : i \geq 1\} \) represents the weights in decreasing order of the measure \( \zeta \) restricted to \([0, 1]\):

\[
w_1 = \max\{\hat{w}_i : z_i \in [0, 1]\} ,
\]

\[
w_{j+1} = \max\{\hat{w}_i : z_i \in [0, 1], \hat{w}_j \notin \{w_1, \ldots, w_j\}\} , \quad j \geq 1 ,
\]

and \( \{c_k : k \geq 1\} \) is the sequence defined by [23].

Recall the definition of the function \( \Psi_N \) introduced just before the statement of Theorem 7.1.

**Theorem 8.1.** Let \( G_N = (V_N, E_N) \) be a sequence of finite vertex-weighted graphs fulfilling assumptions (A0)–(A2) for some sequences \( M_N, \ell_N \). Assume, furthermore, that there exist sequences \( L_N \uparrow \infty \), \( \{\beta_N : N \geq 1\} \) and \( \{\gamma_N : N \geq 1\} \) such that

\[
\lim_{N \rightarrow \infty} \kappa(L_N, M_N, \ell_N) = 0 ,
\]

and such that

\[
\lim_{N \rightarrow \infty} \left( \frac{W_N^x}{\beta_N v_N(x_j)} , \gamma_N v_N(x_j) \deg(x_j) \right) = (Z_j, u_j) , \quad \text{for all } j \geq 1 ,
\]

\[
\lim_{m \rightarrow \infty} \limsup_{N \rightarrow \infty} \sum_{j=m}^{M_N} \frac{W_N^x}{\beta_N v_N(x_j)} \gamma_N v_N(x_j) \deg(x_j) = 0 .
\]

Suppose, finally, that \( \Psi_N(X_0^N) \) converges weakly to \( k \in \mathbb{N} \). Then, for every \( T > 0 \), the Markov chain \( \{\Psi_N(X_t^N) : 0 \leq t \leq T\} \) converges to the \( K \)-process with parameters \( (Z_j, u_j) \) starting from \( k \), in the topology introduced in Section 5.

**Proof.** Repeating the arguments presented below (5.2), we obtain a new sequence \( M'_N \) for which (A3) holds, as well as (8.0) with \( M'_N \) instead of \( M_N \). Denote this new sequence by \( M_N \). Under assumptions (A0)–(A3), Theorem 6.2 furnishes a coupling between the random walk \( X_{\beta_N}^N \) and a Markov process \( Y_{\beta_N}^N \) on \( \{1, \ldots, M_N\} \) whose \( d_T \)-distance converges to 0 in probability. In view of Remark 7.2 and by Theorem 7.3 under conditions (8.0), the Markov process \( Y_{\beta_N}^N \) converges to the \( K \)-process with parameters \( (Z_j, u_j) \) in the Skorohod’s \( J_1 \) topology. By Skorohod’s representation theorem, there exists a probability space in which this convergence take place almost surely. It remains to apply Corollary 5.7. \( \square \)
In view of Remark 6.7 we may replace condition (A0) by assumptions 6.6 and (6.11).

**Theorem 8.2.** Let \( G_N = (V_N, E_N) \) be a sequence of finite vertex-weighted graphs fulfilling assumptions (A1)–(A3) for some sequences \( M_N, \ell_N, L_N \). Assume that there exists a sequence of subsets \( B_N = \{x_1^N, \ldots, x_{I_N}^N\} \), \( I_N \leq M_N, I_N \uparrow \infty \), satisfying (6.6), (6.11). Suppose, furthermore, that condition (6.10) is in force and that \( I_N(x_0^N) \) converges weakly to \( k \in \mathbb{N} \). Then, for every \( T > 0 \), the Markov chain \( \{\Psi_N(X_{\tau_{2t}}^N) : 0 \leq t \leq T\} \) converges to the K-process with parameters \( (Z_j, u_j) \) starting from \( k \), in the topology introduced in Section 3.

**Proof.** By Remark 6.7 there exists a coupling between the random walk \( X_{\tau_{2t}}^N \), and a Markov process \( Y_{\tau_{2t}}^N \) on \( \{1, \ldots, M_N\} \) whose \( d_\tau \)-distance converges to 0 in probability. By Theorem 7.3, in the topology introduced in Section 3, the Markov process \( Y_{\tau_{2t}}^N \) converges to the \( K \)-process with parameters \((Z_j, u_j) \) in the Skorohod’s \( J_1 \) topology. By Skorohod representation theorem, there exists a probability space in which this convergence take place almost surely. It remains to apply Corollary 6.9.

\[ \square \]

9. PSEUDO-TRANSITIVE GRAPHS

We prove in this section Theorem 2.1 inspired by Theorem 8.2 and we apply this result to some pseudo-transitive graphs. The assumptions (A1)–(A3), (6.6), (6.19), (6.6) simplify in this context because the degree and the escape probability from the deep traps do not depend on the specific vertex.

**Proof of Theorem 2.1.** Fix an increasing sequence \( \ell_N \) and a sequence of pseudo-transitive graphs \( G_N \) with respect to the sequence \( \ell_N \). We first derive some consequences of assumptions (B0)–(B2) and (6.6).

It follows from these hypotheses that there exists an increasing sequence \( M_N \uparrow \infty \) such that
\[
\lim_{N \to \infty} M_N^2 \mathbb{E} \left[ \frac{B(x, 2\ell_N)}{V_N} \right] = 0, \quad \lim_{N \to \infty} M_N \mathbb{P}[x, B(x, \ell_N) \neq (y, B(y, \ell_N))] = 0,
\]
\[
\lim_{N \to \infty} M_N \mathbb{E} \left[ \sup_{y \notin B(x, \ell_N)} P_y [H_x \leq L_N t_{mix}] \right] = 0, \quad \lim_{N \to \infty} M_N 2^{-L_N} = 0.
\]
where we replaced \((x, B(x, \ell_N)) \neq (y, B(y, \ell_N)) \) by \( B(x, \ell_N) \neq B(y, \ell_N) \).

Let \( \Sigma_N^1, 1 \leq j \leq 3 \) be the events
\[
\Sigma_N^1 = \bigcap_{1 \leq i \neq j \leq M_N} \{B(x_i^N, \ell_N) \cap B(x_j^N, \ell_N) = \emptyset\},
\]
\[
\Sigma_N^2 = \bigcup_{j=1}^{M_N} \{(x_1^N, B(x_1^N, \ell_N)) \equiv (x_j^N, B(x_j^N, \ell_N))\},
\]
\[
\Sigma_N^3 = \left\{\max_{1 \leq j \leq M_N} \sup_{y \notin B(x_j^N, \ell_N)} P_y [H_{x_j^N} \leq L_N t_{mix}] \leq M_N^{-1}\right\}.
\]

In the places where the vertices of the graph appear, as in the definition of the set \( \Sigma_N^2 \), the sequence \( M_N \) obtained above has to be replaced by \( \min \{M_N, V_N\} \), where \( V_N \) stands for the number of vertices of the random graph \( G_N \). It is easy to see that all three events have probability asymptotically equal to one. We prove this assertion for \( \Sigma_N^1 \) and leave to the reader the proof for the other two. By definition, \( \mathbb{P}[(\Sigma_N^1)^c] \) is bounded above by
\[
\sum_{1 \leq \rho \neq \sigma \leq M_N} \mathbb{P}[B(x_\rho^N, \ell_N) \cap B(x_\sigma^N, \ell_N) \neq \emptyset] \leq M_N^2 \mathbb{P}[B(x_1^N, \ell_N) \cap B(x_2^N, \ell_N) \neq \emptyset]
\]
because \( x_1^N, \ldots, x_N^N \) is uniformly distributed. By this same reason, conditioning on \( x_1^N \), we obtain that the right-hand side is equal to
\[
M_N^2 \mathbb{E} \left[ \frac{|B(x_1^N, 2\ell_N)| - 1}{V_N - 1} \right],
\]
which vanishes as \( N \uparrow \infty \) in view of the definition of the sequence \( M_N \).

Let \( A_N = \{x_1^N, \ldots, x_M^N\} \). By hypothesis (B0), \( \nu_N(A_N) \) converges to 0 in \( \mathbb{P} \)-probability. In particular, there exists a deterministic sequence \( I_N \uparrow \infty, I_N \leq M_N \), such that \( I_N \nu_N(A_N) \) converges to 0 in \( \mathbb{P} \)-probability. Let \( B_N = \{x_1^N, \ldots, x_{I_N}^N\} \). Since \( I_N \uparrow \infty \), by hypothesis (B0), \( \nu_N(B_N) \) converges to 0 in \( \mathbb{P} \)-probability. Therefore, there exists a sequence \( \epsilon_N \downarrow 0 \) for which
\[
\lim_{N \to \infty} \mathbb{P} \left[ \nu_N(B_N) + I_N \nu_N(A_N^c) \geq \epsilon_N \right] = 0.
\]
Let $\Sigma_N = \{\nu_N(B_{n0}) + I_N\nu_N(A_{n0}) < \epsilon_N\}$.

We turn now into the proof of the theorem which relies on Theorem 8.2. Recall the definition of the random weights $\tau_j^N$, $1 \leq j \leq \nu_N$, introduced at the beginning of Section 8. Since $\{\tau_j^N : 1 \leq j \leq \nu_N\}$ has the same distribution as $\{W_j^N : 1 \leq j \leq N\}$, we may replace the latter random weights by the former and assume that the random walk $X_j^N$ evolves among random traps with depth $\tau_j^N$ instead of $W_j^N$.

To show that the pair $(c_n\tau^N, \Psi_n(X_{\tau^N}))$ converges weakly to $(w, K_1)$, it is enough to show that any subsequence $\{N_j : j \geq 1\}$ possesses a sub-subsequence $n$ such that $(c_n\tau^N, \Psi_n(X_{\tau^N}))$ converges to $(w, K_1)$. Fix, therefore, a subsequence $N_j$.

By (8.3), the ordered sequence $(c_{N_j} \tau_{N_j}^1, \ldots, c_{N_j} \tau_{N_j}^{N_j})$ converges almost surely in $L^1(\Omega)$ to $w = (w_1, w_2, \ldots)$. This proves the weak convergence of the first coordinate. Let $\Sigma_{N_j} = \cap_{1 \leq k \leq N_j} \Sigma_n$. There exists a sub-subsequence, denoted by $n$, for which

$$\mathbb{P}\left[ \bigcup_{n \geq 1} \bigcap_{n \geq n_0} \Sigma_n \right] = 1.$$ 

We affirm that all assumptions of Theorem 8.2 hold on the set $\cup_{n_0 \geq 1} \cap_{n \geq n_0} \Sigma_n$. Indeed, recall that $\beta_n^{-1} = c_n \nu_n^p(x_1^p)$. Condition (A1) follows from the definition of the set $\Sigma_n$. On the set $\Sigma_n$, the escape probabilities $v_j(x_n^j)$ and the degrees $\deg(x_n^j)$ are all the same for $1 \leq j \leq M_n$. In particular, by definition of the sequence $\beta_n$, condition (A2) becomes

$$\limsup_{n \to \infty} \sum_{j=1}^{J_n} \frac{c_n (\tau_j^N)^2}{\sum_{j=1}^{J_n} \tau_j^N} < \infty, \quad \limsup_{n \to \infty} \frac{1}{\sum_{j=1}^{J_n} c_n \tau_j^N} < \infty,$$

for all sequences $J_n$ such that $J_n \leq M_n$, $J_n \uparrow \infty$. Since the sequence $\tau_j^N$ is decreasing, the first ratio is bounded by $c_n \tau_1^N$, and these bounds are a consequence of (8.3). Condition (A3) follows from the definition of the sequence $M_n$ and from the definition of the set $\Sigma_n$. Conditions (A6), (B19) follow from the definition of the set $\Sigma_n$. Finally, on the set $\Sigma_n$, $v_j(x_n^j) \deg(x_n^j)$, $1 \leq j \leq M_n$, is constant and the hypotheses (8.3) with $\gamma_n = [v_j(x_n^j) \deg(x_n^j)]^{-1}$ and $(Z_j, u_j) = (w_j, 1)$ follow from (8.3). This proves the affirmation.

We may now apply Theorem 8.2 to conclude that the Markov chain $\Psi_n(X_{\beta_n}^N)$ converges to the $K$-process with parameters $(w_j, 1)$ starting from $k$, in the topology introduced in Section 5. This concludes the proof of Theorem 8.1.

We conclude this section with some examples of graphs satisfying the assumptions of Theorem 8.1.

9.1. **Hypercube.**

We prove in this subsection the convergence of the trap model on the $n$-dimensional hypercube towards the $K$-process associated to constant entrance measure. This result has been established in [17] under the stronger Skorohod’s $J_1$ topology with a different approach. Here we give a proof as an application of Theorem 8.1.

Let $N = 2^n$, $n \geq 1$, and let $G_N$ be the $n$-dimensional hypercube $\{0, 1\}^n$ with edges connecting any two points that differ by only one coordinate. By estimate (6.15) in [25], $\tau_{\text{mix}}^N \ll n^2$.

**Proposition 9.1.** The assumptions of Theorem 8.1 are in force for the hypercube $G_N$ with $\tau_N = \log_2(N)/10 = n/10$.

**Proof.** Since the graph is transitive, condition (2.24) is satisfied and (B0) follows from (8.3). To estimate the ratio in (131) note that $|B(0, 2\ell_N)|/\nu_N$ is equal to the probability that the sum of $n$ Bernoulli(1/2) independent random variables is less than or equal to $2\ell_N = n/5$. By the law of large numbers, this probability vanishes as $n \uparrow \infty$.

To show that (132) is in force, we could compare the distance $d(0, X_t)$ with an Ehrenfest’s urn, see [25] Section 2.3, and proceed with a calculation based on a birth and death chain. For simplicity, we give instead a reference implying the result. By Lemmas 3.6 (i) and 3.2 (i) of [10], with $m(N) = N^2$ and $a = 1$, there exists a finite constant $C_0$ independent of $n$ such that

$$\sup_{y \in B(0, \ell_N)} \mathbb{P}_y[H_0 \leq n^2] \leq C_0 \left( \frac{n^2}{N} + \left( \frac{n}{N/10} \right)^{-1} n^{1/2} \log(n) \right),$$

which vanishes as $n \uparrow \infty$, proving (132). \qed
To complete the description of the asymptotic behavior of the trap model on the hypercube, it remains to determine the time scale $\beta_N$. By a computation based on a birth-and-death chain, the escape probability converges to 1 as $N \to \infty$, and therefore $\lim N \beta_N e_N = 1$.

9.2. Discrete torus for $d \geq 2$. In this subsection the graph $G_N$ stands for the $d$-dimensional discrete torus $\mathbb{T}_N^d = (\mathbb{Z}/N\mathbb{Z})^d$, $d \geq 2$, endowed with nearest neighbors edges. By [24] Theorem 5.5,

$$t_{\text{mix}}^N \leq C_0 N^2$$

for some $C_0 = C_0(d)$. This constant may change from line to line, but will only depend on $d$.

We proved in [22] that in this context the trap model converges to the $K$-process. The next proposition shows that this result follows from Theorem 2.1.

Proposition 9.2. The assumptions of Theorem 2.1 are in force for the $d$-dimensional torus $G_N$ with

$$\ell_N = \begin{cases} N^{1/2} / \log^{1/4} N & d \geq 3, \\ \log^2 N & d = 2, \\ \log^{1/4} N & d = 2. \end{cases}$$

Proof. Since the graph is transitive, condition (2.7) is satisfied and (B0) follows from (23). On the other hand, assumption (B1) is clearly in force by definition of $\ell_N$. It remains to check hypothesis (B2). Recall the definition of the sequence $L_N$. The case $d \geq 3$ follows directly from Lemma 3.1 of [34], and we focus on the case $d = 2$. Fix $x \in \mathbb{T}_N^d$ and $z \notin B(x, \ell_N)$. If $\Pi$ stands for the canonical projection from $\mathbb{Z}^2$ to $\mathbb{T}_N^2$ and $P_z$ for the probability corresponding to the symmetric nearest neighbor discrete time random walk on $\mathbb{Z}^2$,

$$P_z[\Pi \circ \mathbb{H}_x < L_N t_{\text{mix}}^N] = P_z[\Pi^{-1}(z) < L_N t_{\text{mix}}^N].$$

We may bound the previous probability by

$$P_z[\Pi \circ \mathbb{H}_x < L_N t_{\text{mix}}^N] \leq L_N t_{\text{mix}}^N + \sum_i P_z[\Pi \circ \mathbb{H}_x < L_N t_{\text{mix}}^N],$$

where the sum is performed over all sites $x_i$ in the pre-image of $x$ which belong to the ball $B(z, N \log^{1/4} N)$. The first term can be bounded using the estimate (9.1) for the mixing time and an exponential Doob inequality since each component of the random walk is a martingale. This argument shows that the first term is bounded by $4 \exp\{-a \log^{1/4} N\}$ for some $a > 0$. Since there are no more than $C_0 \sqrt{\log N}$ terms in the sum, the second expression in the previous decomposition is bounded above

$$C_0 \sqrt{\log N} P_0[\Pi \circ \mathbb{H}_x < L_N t_{\text{mix}}^N],$$

where $x$ is a site at distance $\ell_N$ from the origin. Decomposing this probability according to whether the random walk reached the ball with radius $N \log^{1/4} N$ before time $C_0 N^2 \log^{1/4} N$ or not, and recalling the argument employed to bound the first term in (9.2), we conclude that the previous expression is bounded by

$$C_0 \sqrt{\log N} e^{-a \log^{1/4} N} + C_0 \sqrt{\log N} P_0[\Pi \circ \mathbb{H}_x < \mathbb{H}_{B(0,N \log^{1/4} N)c}].$$

for some finite constant $C_0$ and some positive $a$. By [23] Proposition 1.6.7] and the reversibility of the random walk, the second term is less than or equal to

$$C_0 \sqrt{\log N} \left(1 - \frac{\log \ell_N}{\log (N \log^{1/4} N)} + \frac{C_0}{\log^2 N} \right) \leq C_0 \log^{-1/4} N,$$

which proves condition (B2). □

To complete the description of the asymptotic behavior of the trap model on the discrete torus $\mathbb{T}_N^d$, it remains to determine the time scale $\beta_N$. Let $v_d$, $d \geq 3$, be the escape probability of a simple random walk on $\mathbb{Z}^d$, and let

$$\beta_N' = \begin{cases} \frac{1}{v_d^N} \frac{2}{(2\pi)} \log(N) & d = 2, \\ \frac{1}{v_d^N} v_d^{-1} & d \geq 3. \end{cases}$$

In view of the definition of $\beta_N$ and of [24] Theorem 1.6.6], $\lim N \to \infty \beta_N / \beta_N' = 1.$
9.3. Random d-regular graphs. In this subsection we consider a sequence of graphs \( G_N \) with \( N \) vertices satisfying the following three assumptions.

(G1) \( G_N \) is d-regular for some \( d \geq 3 \);

(G2) There is a constant \( \alpha > 0 \) such that for any vertex \( x \) of \( V_N \), the ball \( B(x, \alpha \log N) \) contains at most one cycle;

(G3) The spectral gap \( \lambda_N \) of the continuous time random walk on \( G_N \) is bounded below by some positive constant: \( \lambda_N \geq \gamma > 0 \) for all \( N \geq 1 \).

It follows from [11, Remark 1.4] that these three hypotheses hold, with probability approaching 1 as \( N \uparrow \infty \), for a sequence of random d-regular graphs on \( N \) vertices. They are also satisfied by the so-called Lubotzky-Phillips-Sarnak graphs [26].

By [30, p. 328, under conditions (G1) and (G3)], the mixing time \( t^N_{\text{mix}} \) is bounded above by \( C_0 \log N \) for some finite constant \( C_0 \).

Proposition 9.3. Let \( \{G_N : N \geq 1\} \) be a sequence of random graphs defined on some probability space \((\Omega, \mathcal{F}, \mathbb{P})\) satisfying the assumptions (G1)–(G3) with a \( \mathbb{P} \)-probability converging to 1 as \( N \uparrow \infty \).

Then, the conditions of Theorem 2.1 are fulfilled with \( L_N = \log N \) and \( \ell_N = \alpha' \log N \) for some \( \alpha' < \min\{\alpha, [2 \log(d-1)]^{-1}\} \), where \( \alpha \) is the constant appearing in condition (G2).

Proof. Condition (B0) follows from assumption (G1) and (G3). The rest of the proof is based on estimates obtained in [11].

By [11, Lemma 6.1] with \( \Delta = \ell_N \), the probability that a ball \( B(x, \ell_N) \) is not a tree is bounded by \((d-1)^{-(\alpha-\alpha') \log N} \).

Let \( \Sigma_N \) be the event
\[
\Sigma_N = \{B(x^N_1, \ell_N) \text{ and } B(x^N_2, \ell_N) \text{ are disjoint trees}\}.
\]

We claim that \( \mathbb{P}[\Sigma_N] \) converges to 1 as \( N \uparrow \infty \). Indeed, if \( \Sigma_N \) stands for the event that \( B(x^N_1, \ell_N) \), \( B(x^N_2, \ell_N) \) are trees, in view of the estimate of the previous paragraph, \( \mathbb{P}[\Sigma_N] \) is bounded by \( 2(d-1)^{-(\alpha-\alpha') \log N} \) which vanishes as \( N \uparrow \infty \). On the other hand, since \( |B(x_1, r)| \leq 4(d-1)^r \) for any ball in a d-regular graph and since \( x^N_1, x^N_2 \) are uniformly distributed,
\[
\mathbb{P}[B(x_1, \ell_N) \cap B(x_2, \ell_N) \neq \emptyset] \leq 4 \frac{(d-1)^{2\ell_N}}{N}.
\]

As \( \alpha' < [2 \log(d-1)]^{-1} \), this expression vanishes as \( N \uparrow \infty \). This proves the claim and assumption (2.7).

It remains to examine the escape probability appearing in condition (B2). It follows from the bound for the mixing time presented just before the statement of the proposition and from our choice of the sequence \( L_N \) that
\[
\mathbb{P} \left[ \|x\| < L_N t^N_{\text{mix}} \right] \leq \mathbb{P} \left[ \|x\| < C_0 (\log N)^2 \right].
\]

By [11, Lemma 3.4] with \( r = 0 \) and \( s = \alpha' \log N \), the previous expression for \( z \notin B(x, \ell_N) \), is bounded by \( C_0 N^{-a} \) for some finite constant \( C_0 \) and some positive \( a > 0 \). This concludes the proof of the proposition. \( \square \)

We conclude this section computing the scaling factor \( \beta_N \) in the context of graphs satisfying assumptions (G1)–(G3). On the event (9.3), which has asymptotic probability equal to one, \( B(x_1, \ell_N) \) is a d-regular tree so that
\[
v_{t^N_{\text{mix}}} (x_1) = \frac{d-2}{d-1} \left( \frac{1}{1 - (d-1)^{-\ell_N}} \right).
\]

In particular, \( \lim_{N \to \infty} \beta_N c_N = (d-1)/(d-2) \).

10. Graphs with asymptotically random conductances

We prove in this section Theorem 2.2. The proof follows the one of Theorem 2.1. However, the absence of regularity of the graph requires some extra effort in establishing (A2).

Recall the coupling \( Q_N \) defined in (33) between the random graph \( G_N \) and the sequence of i.i.d. random vectors \( \{(D_j, E_j) : j \geq 1\} \). We extend this coupling \( Q_N \) to a coupling \( Q \) between all random graphs \( G_N \) and the sequence of i.i.d. random vectors \( \{(D_j, E_j) : j \geq 1\} \) using \( Q_N \) as the conditional probability:
\[
Q[ G_N = G | \{(D_j, E_j) : j \geq 1\} ] = Q_N [ G_N = G | \{(D_j, E_j) : j \geq 1\} ],
\]
with the further condition that the graphs \( G_N, N \geq 1 \), are conditionally independent, given \( \{(D_j, E_j) : j \geq 1\} \). Include in the probability space just defined the random measure \( \zeta \) introduced in (31) which
is associated to the marks of a Poisson point process independent from the variables \((D_j, E_j)\) and from the random graphs \(G_N\). The probability measure on this new space is still denoted by \(\Omega\).

Recall the definition of the random weights \(\hat{\tau}^N_j; 1 \leq j \leq V_N\), introduced in Section 2. Since \(\{\hat{\tau}^N_j; 1 \leq j \leq V_N\}\) has the same distribution as \(\{W^N_j; 1 \leq j \leq |V_N|\}\), we may replace the latter random weights by the former and assume that the random walk \(X^N_t\) evolves among random traps with depth \(\tau^N_j\) instead of \(W^N_j\).

Since \(w_j\) is a.s. summable, since by (B3) \(D_1/E_1\) has finite \(\Omega\)-expectation and since the sequences \(\{w_j\}\) and \(\{(D_j, E_j)\}\) are independent,
\[
\sum_{j \geq 1} w_j \frac{D_j}{E_j} \text{ is } \Omega\text{-almost surely finite.} \tag{10.1}
\]
By the strong law of large numbers, almost surely
\[
\frac{1}{n} \sum_{j=1}^n D_j/E_j \leq C_1 \tag{10.2}
\]
for all large enough \(n\), where \(C_1 = 2E_0[D_1/E_1]\).

By hypotheses (B1)-(B3), there exists an increasing sequence \(M_N \uparrow \infty\) such that
\[
\lim_{N \to \infty} M^2_N E_0 \left[ \frac{|B(\{\hat{\tau}^N_j\})|}{V_N} \right] = 0, \quad \lim_{N \to \infty} M_N^2 2^{-L_N} = 0, \\
\lim_{N \to \infty} \Omega \left[ \max_{1 \leq j \leq M_N} \left| \nu_{\{\hat{\tau}^N_j\}}^{-1} - E_j^{-1} \right| > M_N^{-2} \right] = 0, \\
\lim_{N \to \infty} \Omega \left[ \sum_{j=1}^{M_N} \{\deg(\hat{\tau}^N_j) \neq D_j\} \right] = 0, \\
\lim_{N \to \infty} M^2_N E_0 \left[ \sup_{y \in B(\{\hat{\tau}^N_j\}, L_N)} \mathbb{P}_y \left[ H^N_1 \leq L_N t_{\text{mix}} \right] \right] = 0.
\]
As before, in the places where the vertices of the graph appear, as in the definition of the set \(\Sigma_1\), the sequence \(M_N\) obtained above has to be replaced by \(\min\{M_N, V_N\}\), where \(V_N\) stands for the number of vertices of the random graph \(G_N\).

Let \(\Sigma_N^1, 1 \leq j \leq 4\) be the events
\[
\Sigma_1^1 = \bigcap_{1 \leq i \neq j \leq M_N} \{B(x^N_i, \ell_N) \cap B(x^N_j, \ell_N) = \emptyset\}, \\
\Sigma_2^N = \left\{ \max_{1 \leq j \leq M_N} \left| \nu_{\{\hat{\tau}^N_j\}}^{-1} - E_j^{-1} \right| \leq M_N^{-2} \right\}, \\
\Sigma_3^N = \bigcap_{j=1}^{M_N} \{\deg(x^N_j) = D_j\}, \\
\Sigma_4^N = \left\{ M^2_N \max_{1 \leq j \leq M_N} \sup_{y \in B(x^N_j, \ell_N)} \mathbb{P}_y \left[ H^N_{\ell_N} \leq L_N t_{\text{mix}} \right] \leq M_N^{-1} \right\}.
\]
Similarly to what was done in the proof of Theorem 2.1 we can show that these events have probability asymptotically equal to one. In the places where the vertices of the graph appear, as in the definition of the set \(\Sigma_N\), the sequence \(M_N\) obtained above has to be replaced by \(\min\{M_N, V_N\}\), where \(V_N\) stands for the number of vertices of the random graph \(G_N\).

By (B3), we may replace the sequence \(M_N\) by a possibly random increasing sequence \(M^*_N \leq \min\{M_N, V_N\}, M^*_N \uparrow \infty\) \(\Omega\)-a.s., still denoted by \(M_N\), for which all the previous estimates hold and such that for all \(N \geq 1\),
\[
\sum_{j \geq 1} |c_N \tau^N_j - w_j| \leq M^*_N^2. \tag{10.3}
\]
By hypothesis (B0), even though the sequence \(M_N\) is random, the expectation \(\mathbb{E} [\nu_N(x^N_1, x^N_{M_N})^\ell]\) vanishes as \(N \uparrow \infty\). Let \(A_N = \{x^N_1, \ldots, x^N_{M_N}\}\). As in the proof of Theorem 2.1, presented in the previous sections, using again hypothesis (B0) we construct a set \(B_N = \{x^N_1, \ldots, x^N_{M_N}\}, |B_N| = I_N\), and a sequence \(\epsilon_N > 0\) for which
\[
\lim_{N \to \infty} \Omega \left[ \nu_N(B^N_N) + I_N \nu_N(A^N_N) \geq \epsilon_N \right] = 0.
\]
Let $\Sigma_N^n = \{\nu_N(B_n^N) + I_N \nu_N(A_n^N) \leq \epsilon_N\}$.

To show that the pair $(c_n \tau^n, \Psi_N(X^n_N))$ converges weakly to $(\nu, K_t)$, it is enough to show that any subsequence $\{N_j : j \geq 1\}$ possesses a sub-subsequence $n$ such that $(c_n \tau^n, \Psi_N(X^n_{1bn}))$ converges to $(\nu, K_t)$.

Fix, therefore, a subsequence $N_j$. By (8.3), the ordered sequence $(c_n \tau^n_j, \ldots, c_n \tau^n_{N_j})$ converges almost surely in $L^1(\mathbb{N})$ to $\nu = (w_1, w_2, \ldots)$. This proves the weak convergence of the first coordinate. Let $\Sigma_{N_j} = \cap_{1 \leq k \leq k \Sigma_{N_j}}$. There exists a sub-subsequence, denoted by $n$, for which

$$\Omega \left[ \bigcup_{n \geq 1} \bigcap_{n \geq n_0} \Sigma_n \right] = 1.$$

We affirm that all assumptions of Theorem 8.2 hold on the event $\cup_{n \geq 1} \cap_{n \geq n_0} \Sigma_n$ intersected with the ones in (10.11), (10.2) and (10.3). Indeed, condition (A1) follows from the definition of the set $\Sigma_1^n$. Similarly to the proof of Theorem 8.1, condition (A3) follows from the definition of the sequence $M_n$ and the set $\Sigma_3^n$. Conditions (6.5), (6.10) follow from the definition of the set $\Sigma_5^n$.

We turn to condition (A2). Recall that $\beta_n = c_n^{-1}$. Fix a sequence $J_n \uparrow \infty$ such that $J_n \leq M_n$, and let $B_n = \{x_1^n, \ldots, x_{J_n}^n\}$. Since we replaced the weights $W_{j}^n$ by $\tau_j^n$, the first expectation appearing in this hypothesis can be rewritten as

$$\frac{\sum_{1 \leq j \leq J_n} c_n \tau_j^n \deg(x_j^n)}{\sum_{1 \leq j \leq J_n} c_n \tau_j^n \deg(x_j^n)}.$$

By definition of the set $\Sigma_3^n$, we may replace $\deg(x_j^n)$ by $D_j$. Since $\tau^n_j$ is decreasing, by definition of the set $\Sigma_3^n$, the numerator is bounded by

$$c_n \tau^n_j \sum_{j=1}^{J_n} c_n \tau_j^n D_j + c_n \tau^n_j \sum_{j=1}^{J_n} c_n \tau_j^n D_j.$$

The second term divided by the denominator in (10.4) is less than or equal to $c_n \tau^n_j M_n^{-2}$, which goes to 0 as $n \to \infty$ in view of (10.4). Also, by (10.3), the first term is bounded by

$$c_n \tau^n_j \sum_{j=1}^{J_n} w_j D_j E_j + c_n \tau^n_j \frac{1}{M_n} \max_{1 \leq j \leq J_n} D_j E_j.$$

Since the denominator in (10.4) is bounded below by $c_n \tau^n_j D_j \geq c_n \tau^n_j$, the first condition in (A2) follows from (10.1), (10.2).

The second condition of assumption (A2) can be written as

$$\frac{1}{J_n} \sum_{1 \leq j \leq J_n} \deg(x_j^n) \cdot \deg(x_j^n) \cdot \deg(x_j^n).$$

By definition of the set $\Sigma_3^n$, we may replace $\deg(x_j^n)$ by $D_j$. The sum in the denominator is bounded below by $c_n \tau^n_j D_j \geq c_n \tau^n_j$, which is uniformly bounded. Since the escape probability is bounded by one and since $\sum_{1 \leq j \leq J_n} E_j$ is bounded by one, the numerator is less than or equal to $\sum_{1 \leq j \leq J_n} D_j / E_j$, whose average by (10.2) is bounded.

It remains to establish (8.6) with $\gamma_n = 1$, $Z_j = w_j / E_j$ and $u_j = E_j D_j$. The convergence of the first term follows from (10.3), the definition of $\Sigma_3^n$ and $\Sigma_4^n$ and the fact that the variables $E_j$ are bounded by one. The second part of (8.6) amounts to estimate

$$\sum_{j=m}^{M_n} \tau^n_j \deg(x_j^n) = \sum_{j=m}^{M_n} \tau^n_j D_j \leq \sum_{j=m}^{M_n} w_j (D_j / E_j) + \frac{1}{M_n} \max_{1 \leq j \leq M_n} D_j (D_j / E_j),$$

where the identity follows from the definition of $\Sigma_3^n$ and the inequality from (10.3) and the boundedness of $E_j$. The first term on the right hand side vanishes in view of (10.1) and the second one by (10.2). This concludes the proof of the Theorem.

11. Supercritical Erdős-Rényi random graphs

We show in this section that super-critical Erdős-Rényi random graphs satisfy the assumptions of Theorem 2.2. Let $\mathcal{F}_N$ be the set of vertices $\mathcal{F}_N = \{1, \ldots, N\}$. For $\lambda > 1$ fixed, let $\{\xi_{x,y} : x, y \in \mathcal{F}_N\}$ be i.i.d. Bernoulli($\lambda / N$) random variables constructed in a probability space $(\Omega, A, \mathbb{P})$. The Erdős-Rényi random graph is defined as $G_N = (\mathcal{F}_N, \mathcal{E}_N)$, where $\mathcal{E}_N$ is the random set of edges given by $\{(x, y) : \xi_{x,y} = 1\}$. Throughout this section, $c_j, C_j, j \geq 0$, represent positive constants depending on $\lambda$.
and sometimes on further parameters, the first ones being typically small and the last ones large. Next result can be found in [11, Theorem 2.3.2].

**Theorem 11.1.** There is a constant \( c_0 \) such that with \( \mathbb{P} \)-probability converging to one as \( N \) tends to infinity, there is a unique component \( C_{\text{max}} \) in \((\mathcal{F}_N, \mathcal{E}_N)\) with \( |C_{\text{max}}| > c_0 \log N \). Moreover, there exists \( 0 < v_\lambda < 1 \) such that

\[
\lim_{N \to \infty} \mathbb{P} \left[ \left| \frac{|C_{\text{max}}|}{N} - v_\lambda \right| > \epsilon \right] = 0,
\]

for all \( \epsilon > 0 \).

We will be interested in analyzing the trap model in \( C_{\text{max}} \), providing another interesting example for which our machinery can be applied. For the sake of simplicity we shall assume that the common distribution of the traps \( \{W_j^N : j \geq 1\} \) is \( \alpha \)-stable. More precisely, recall the definition of the variables \( \hat{\tau}_i^N, 1 \leq i \leq N \), introduced in (8.2) with \( V = N \) and \( F(t) = t \). We assume in this section that \( W_i^N = \hat{\tau}_i^N, 1 \leq i \leq N \).

Let \( V_N = C_{\text{max}} \) be the random set of vertices and let \( E_N = \{ \{x, y\} \subset V_N : \{x, y\} \in \mathcal{E}_N \} \) be the random set of edges of the random graph \( G_N \). In contrast with the previous examples presented in Section 6 the number of vertices of the random graph \( G_N \) is also random. The weights are distributed as follows. Given \( V_N \), re-enumerate the weights \( W_j^N, 1 \leq j \leq |V_N| \), in decreasing order and denote by \( \hat{W}_j^N \) the new sequence, so that \( \hat{W}_j^N \geq \hat{W}_{j+1}^N, 1 \leq j < |V_N| \), \( \hat{W}_{\sigma(j)}^N = W_j^N \) for some permutation \( \sigma \) of \( V_N \). Randomly enumerate the vertices of \( V_N \), obtaining a vector \( (x_1^N, \ldots, x_{|V_N|}^N) \), and set \( W_{x_i}^N = \hat{W}_i^N \). Given this random vertex-weighted graph, we examine the continuous-time random walk \( X_t^N \) on \( G_N \) with generator given by (12).

Note that to define the random weights \( W_j^N = \hat{\tau}_j^N \) we divided the interval \([0, 1]\) in \( N \) sub-intervals instead of dividing it in \( |V_N| \) intervals. In particular, in contrast with the examples of Section 6 \( N^{-1/\alpha} W_{x_i}^N \) does not converge to a Fréchet distribution, but so does \( v_\lambda^{-1/\alpha} N^{-1/\alpha} W_{x_i}^N \), where \( v_\lambda \) is given by Theorem 11.1.

In the rest of this section, we prove that the assumptions of Theorem 2.2 are fulfilled. By Theorem 11.1, the number of vertices converges in probability to \( +\infty \). To establish (3.10), fix a sequence \( J_N \uparrow \infty \) and denote by \( \mathcal{W}_N^1, \ldots, \mathcal{W}_N^N \) the sequence \( W_1^N, \ldots, W_N^N \) enumerated in decreasing order. Note that \( \mathcal{W}_j^N \geq \hat{W}_j^N, 1 \leq j \leq |V_N| \). By (8.3) and (2.8), for every \( \epsilon > 0 \),

\[
\lim_{N \to \infty} \mathbb{P} \left[ \sum_{j \geq 1} |N^{-1/\alpha} \mathcal{W}_j^N - w_j| \geq \epsilon \right] = 0.
\]

Since \( \sum_{j \geq J_N} w_j \) vanishes almost surely as \( N \uparrow \infty \), if \( \Sigma_N^0 \) stands for the event \( \sum_{j \geq J_N} N^{-1/\alpha} \mathcal{W}_j^N \leq 1 \),

\[
\lim_{N \to \infty} \mathbb{P} \left[ \Sigma_N^0 \right] = 1.
\]

Denote by \( \Sigma_N^1 \) the event \( \{|V_N - v_\lambda N| \leq \epsilon N\} \) for some \( 0 < \epsilon < \min\{v_\lambda, 1 - v_\lambda\} \). By Theorem 11.1, \( \mathbb{P}[\Sigma_N^1] \to 1 \). In conclusion, to prove (3.10) we need to show that

\[
\lim_{N \to \infty} \mathbb{E} \left[ v_N \left( \{x_1, \ldots, x_{\min\{J_N, |V_N|\}}\} \right) \mathbf{1}[\Sigma_N^0 \cap \Sigma_N^1] \right] = 0.
\]

By definition of \( v_N \), and since all vertices in \( V_N \) have degree at least equal to one,

\[
v_N \left( \{x_1, \ldots, x_{\min\{J_N, |V_N|\}}\} \right) \leq \sum_{j=J_N+1}^{|V_N|} W_{x_j}^N \deg(x_j).
\]

Since \( \mathcal{W}_j^N \geq \hat{W}_j^N, 1 \leq j \leq |V_N| \),

\[
\sum_{j=J_N+1}^{|V_N|} W_{x_j}^N \deg(x_j) \leq \sum_{j=J_N+1}^{|V_N|} \mathcal{W}_j^N \deg(x_j) \leq \sum_{j=J_N+1}^N \mathcal{W}_j^N \deg(x_j),
\]

if \( x_{|V_N|+1}, \ldots, x_N \) represents a random enumeration of the vertices of \( \mathcal{F}_N \) which do not belong to the largest component. On the set \( \Sigma_N^1 \), \( W_{x_i}^N \geq \max_{1 \leq k \leq c_2 N} W_k^N \), where \( c_2 = v_\lambda - \epsilon \). This latter variable as well as the variables \( \mathcal{W}_j^N \) depend only on the Poisson point process defined at the beginning of Section...
This expectation is equal to

\[ \frac{\sum_{j=W_{j+1}}^N \omega_j \deg(x_j)}{W_{j+1}} 1_{\{\Sigma_N^0 \}} \leq \frac{\sum_{j=W_{j+1}}^N \omega_j \deg(x_j)}{\max_{1 \leq k \leq c_N} \deg(x_j)} \sum_{j=W_{j+1}}^N 1_{\{\Sigma_N^0 \}} \mathbb{E} \left[ \frac{1}{|V_N|} \right] . \]

We first estimate the conditional expectation and then the remaining expression. Since the law of the graph \( \mathcal{G}_N \) is independent of the \( \sigma \)-algebra \( \mathcal{M} \), the previous conditional expectation is equal to

\[ \mathbb{E}[\deg(x_j) 1_{\{\Sigma_N^0 \}}] . \]

By construction if \( j \leq |V_N|, \deg(x_j) \) has the same distribution as \( \deg(x_k) \) for \( 1 \leq k \leq |V_N| \). Therefore, for a fixed \( j \), the previous expectation is bounded by

\[ \sum_{\ell \leq j} \mathbb{E}[\{ |V_N| = \ell \}] \frac{1}{|V_N|} \sum_{\ell \leq j} \deg(y) + \sum_{\ell \geq j} \mathbb{E}[\{ |V_N| = \ell \}] \frac{1}{|V_N|} \sum_{y \in V_N} \deg(y) , \]

where the sum is carried over all \( \ell \) such that \( |\ell - \nu_N| \leq \epsilon N \). Estimating the denominators by the worst case, we get that the sum is less than or equal to

\[ \frac{1}{\min\{\nu_N - \epsilon, 1 - \epsilon - \nu_N\}} \mathbb{E}\left[ \frac{1}{|V_N|} \sum_{\ell \geq 1} \deg(y) \right] . \]

This expectation is equal to \( \lambda \).

It remains to estimate the expectation involving the weights. On the set \( \Sigma_N^0 \), \( \sum_{j=W_{j+1}}^N \omega_j \leq N^{1/\alpha} \). On the other hand, using the notation introduced in \( \mathcal{S} \), \( \max_{1 \leq k \leq c_N} \sum_{N^{1/\alpha} \leq \deg(x_k) \geq w(\lambda)} \), where \( w(\lambda) = \max_i \hat{w}_i \), and where the maximum is carried over all indices \( i \) such that \( z_i \leq c\lambda \). Hence,

\[ \mathbb{E}\left[ \frac{1}{\max_{1 \leq k \leq c_N} \deg(x_k)} \sum_{j=W_{j+1}}^N \omega_j \right] \leq \mathbb{E}\left[ \frac{1}{w(\lambda)} \right] . \]

Since \( w' = w(\lambda)/c^{1/\alpha}_N \) has a Fréchet distribution, \( P(w' \leq t) = \exp\{-1/t^\alpha\} \), this expectation is finite, which proves condition \( \mathcal{S} \).

The results of this section should still hold if we require the variables \( W_j \) to belong to the domain of attraction of an \( \alpha \)-stable law and to satisfy the bound

\[ \limsup_{N \to \infty} \mathbb{E}\left[ c_N \sup_{1 \leq i \leq N} W_i^{-1} \right] < +\infty , \]

where \( c_N \) has been introduced in \( \mathcal{S} \).

To understand the asymptotic law of the escape probabilities, we need to introduce a related branching process. Let \( \mathcal{T} \) be the random tree obtained by the Galton-Watson process with offspring distribution Poisson(\( \lambda \)) and denote its law by \( \mathcal{P} \). Since \( \lambda \) is assumed to be greater than one, the event that \( \mathcal{T} \) is infinite has positive \( \mathcal{P} \)-probability, [14 Theorem 2.1.4]. We denote by \( \mathcal{Q} \) the root of \( \mathcal{T} \).

We first show that the neighborhood of a random point in the Erdős-Rényi graph looks like the neighborhood of \( 0 \) in \( \mathcal{Q} \). This is made precise as follows. We write \( (x, G) \) for a graph with a marked vertex \( x \). We say that \( (x, G) \) is isometric to \( (x', G') \) if there exists an isometry between \( G \) and \( G' \), sending \( x \) to \( x' \). As an abuse of notation, we consider \( \mathcal{T} \subseteq \mathcal{Q}_N \) both as a set of vertices and as the corresponding induced subgraph of \( \mathcal{Q}_N \).

**Proposition 11.2.** Let \( 0 < \gamma < (3 \log \lambda)^{-1} \). There exist constants \( C_1 \) and \( N_0 = N_0(\lambda, \gamma) \) such that given a random point \( z \in \mathcal{Q}_N \), we can find a coupling \( Q_N \) between the random graph \( \mathcal{G}_N \) under \( \mathcal{P} \) and the Galton-Watson tree \( \mathcal{T} \) under \( \mathcal{P} \) such that for all \( N \geq N_0 \),

\[ Q_N \left[ (z, B(z, \gamma \log N)) \right] \text{ is isometric to } (0, B(0, \gamma \log N)) \geq 1 - C_1 N^{3 \gamma \log \lambda - 1} . \]

**Proof.** We follow an argument similar to the one in [14 Section 2.2]. Assume, without loss of generality, that \( z = 1 \) and define an exploration of the cluster \( C_1 \) containing 1 in the following way. Let \( S_0 = \{2, 3, \ldots, N\} \), \( I_0 = \{1\} \) and \( R_0 = 0 \). These sets represent respectively the ‘susceptible’, the ‘infected’
and the ‘removed’ sites. Define a discrete time evolution by
\begin{align*}
R_{t+1} &= R_t \cup I_t, \\
I_{t+1} &= \{ y \in S_t; \xi_{x,y} = 1 \text{ for some } x \in I_t \}, \\
S_{t+1} &= S_t \setminus I_{t+1}.
\end{align*}

Note that the cluster $c_t$ is given by $\cup_{s=t}^\infty I_s$ and that $B(1,r) = \cup_{r=1}^\infty I_r$.

In order to couple the above exploration process with a Galton-Watson branching process, we introduce a new set of independent Bernoulli($\lambda/N$) random variables $\zeta^t_{x,y}$, $t \geq 1$, $x \geq 1$, $1 \leq y \leq N$. Let $Z_0 = 1$ and
\begin{equation}
Z_{t+1} = \sum_{x \in I_t} \xi_{x,y} + \sum_{x \in I_t} \sum_{y \in S_t} \zeta^t_{x,y} + \sum_{x=N+1}^{N+Z_t-[I_t]} \sum_{y=1}^{\theta^t_{x,y}}. \tag{11.1}
\end{equation}

The first term in the above sum can be written as $|I_{t+1}| + C_{t+1}$, where $C_{t+1}$ represents the number of ‘collisions’ occurring in the exploration process, that is, individuals in $I_{t+1}$ connected to more than one individual in $I_t$. The second term stands for the ‘immigrants’ introduced to compensate the fact that $|S_t| < N$, and the third term for children of individuals that are not in $I_t$.

It is easy to check that the process $\{Z_t : t \geq 0\}$ is a branching process with offspring distribution $\text{Binomial}(N, \lambda/N)$. Let $T'$ be the random tree associated with $Z_t$. More precisely, if $x$ is the $i$-th individual in the $t$-th generation of $T'$, the number of offsprings of $x$ will be given by
\begin{equation}
\begin{cases}
\sum_{y \in S_t} \xi_{x,y} + \sum_{y \in S_t \setminus S_t} \zeta^t_{x,y} & \text{if } i \leq |I_t|, \\
\sum_{y=1}^{N} \zeta^t_{x,y} & \text{otherwise}.
\end{cases}
\end{equation}

It is immediate to check that $Z_t$ is the size of the $t$-th generation of $T'$ and that $Z_t \geq |I_t|$.

On the event $Z_s = |I_s|$, $1 \leq s \leq t$, there were no collisions and no immigrants. Therefore, in this event the subgraph $(1, B(1, t))$ of $\mathcal{G}_N$ is isometric to the subgraph $(\emptyset, B(\emptyset, t))$ of $T'$. Hence, by [14, Theorem 2.2.2] with $t = c \log N$, there exist a constant $C_1 < \infty$ and a coupling $Q'$ between $\mathcal{G}_N$ and $T'$ such that with probability at least $1 - C_1 N^{3\gamma \log \lambda - 1}$, $(1, B(1, t))$ is isometric to $(\emptyset, B(\emptyset, t))$.

**Claim A**: Let $0 < \gamma < (3 \log \lambda)^{-1}$. There exist $n_0$ and a coupling $Q''$ between the tree $T'$ with $\text{Binomial}(N, \lambda/N)$ offsprings and the tree $T$ with Poisson($\lambda$) offsprings, such that, with probability at least $1 - C_1 N^{3\gamma \log \lambda - 1}$, $(\emptyset, B(\emptyset, \gamma \log N))$ (in $T'$) is isometric to $(\emptyset, B(\emptyset, \gamma \log N))$ (in $T$) for $N \geq n_0$.

It is well known that a Poisson($\lambda$) random variable $Y$ can be coupled with a $\text{Binomial}(N, \lambda/N)$ random variable $Y'$ in a way that
\begin{equation}
P[Y = Y'] \geq 1 - 2\lambda^2 N^{-1}, \tag{11.2}
\end{equation}
see for instance [13, Chapter 2.6] or [23, Theorem 1] for a bound on the total variation distance and [25, Chapter 4] for a connection between total variation distance and coupling. On the other hand, by [1, Theorem 4], there exist $\theta = \theta(\lambda) > 0$ and $C_3$ such that for any $t$, $A \geq 0$,
\begin{equation}
P\{Z_t \geq A\} \leq e^{\theta (Z_t/A)} \leq e^{-\theta A} e^{\theta (Z_t/A)} \leq C_3 e^{-\theta A}.
\end{equation}

This bound permits to estimate the volume of the subgraph $B(\emptyset, \gamma \log N)$ of $T$. Fix $\gamma \in (0, 1)$. Since $|B(\emptyset, \gamma \log N)| = \sum_{0 \leq t \leq \gamma \log N} Z_t$, we have that
\begin{equation}
P\{|B(\emptyset, \gamma \log N)| \geq N^{3\gamma \log \lambda}\}
\leq \sum_{t=0}^{\gamma \log N} P|Z_t| \geq N^{2\gamma \log \lambda} \leq \sum_{t=0}^{\gamma \log N} P|Z_t| \geq N^{\gamma \log \lambda} \tag{11.3}
\end{equation}

for all $N \geq N_0(\lambda, \gamma)$. Therefore, applying the previous estimate, we conclude that for every $0 < \gamma < 1$, there exist $C_3 < \infty$ and $N_0(\lambda, \gamma) < \infty$ such that
\begin{equation}
P\{|B(\emptyset, \gamma \log N)| \geq N^{3\gamma \log \lambda}\} \leq C_3 \exp\{-\theta N^{\gamma \log \lambda}\}.
\end{equation}

In the proof of the previous lemma we also obtained a bound on the size of a ball $B(z, \gamma \log N)$ around a typical point $z$. \hfill \Box

In the proof of the previous lemma we also obtained a bound on the size of a ball $B(z, \gamma \log N)$ around a typical point $z$. \hfill \Box
Corollary 11.3. For any $0 < \gamma < (3\log \lambda)^{-1}$, there exist a finite constant $C_2$ and an integer $N_0$, depending only on $\lambda$ and $\gamma$, such that for any random point $z \in \{1, \ldots, N\}$, 
\[
P\left[|B(z, \gamma \log N)| \geq N^{3\gamma \log \lambda}\right] \leq C_2 N^{3\gamma \log \lambda - 1}
\] for all $N \geq N_0$.

As required in [133], we extend the local isometry obtained in Proposition 11.2 to various balls in the random graph $\mathcal{G}_N$.

Corollary 11.4. Fix positive numbers $b$ and $\gamma$ such that $0 < 2b + 6\gamma \log \lambda < 1$. There exist constants $C_0, N_0$, depending only on $\lambda$ and $\gamma$, and a coupling $Q'$ between the random graph $\mathcal{G}_N$ and $N^b$ independent Galton-Watson trees $\mathcal{T}_i$, $1 \leq i \leq N^b$, such that for all $N \geq N_0$,
\[
Q'[\mathcal{G}_N] \leq C_0 N^{2b + 6\gamma \log \lambda - 1},
\]
where $\mathcal{B}$ is the event “The balls $(z_i, B(z_i, \gamma \log N))$, $1 \leq i \leq N^b$, are disjoint and isometric to $(\varnothing, B(\varnothing, \gamma \log N))$”, and $z_1, \ldots, z_{N^b}$ are sites randomly chosen in $\{1, \ldots, N\}$.

Proof. Choose randomly $N^b$ sites on $\{1, \ldots, N\}$, denoted by $z_1, \ldots, z_{N^b}$. By Proposition 11.2 for $N$ large, there is a coupling $Q'$ between independent Erdős-Rényi random graphs $\mathcal{G}_N$, $1 \leq i \leq N^b$, and independent Galton-Watson trees $\mathcal{T}_i$ in a way that with probability at least $1 - C_1 N^b N^{3\gamma \log \lambda - 1}$ each ball $(z_i, B(z_i, \gamma \log N))$ in $\mathcal{G}_N$ is isomorphic to $(\varnothing, B(\varnothing, \gamma \log N))$ in $\mathcal{T}_i$.

We construct an Erdős-Rényi-distributed graph $\mathcal{G}_N$ which is partially determined by the above $\mathcal{G}_N$'s. We first explore the ball $B(z_i, \gamma \log N)$ in $\mathcal{G}_N$. Every edge $\{x, y\}$ revealed during this exploration is open in $\mathcal{G}_N$ if and only if it is open in $\mathcal{G}_N$. Then we proceed by exploring $B(z_i, \gamma \log N)$ in $\mathcal{G}_N$ observing only that we do not reassign values to edges in $\mathcal{G}_N$ that were already established in the previous step. After proceeding with this exploration for $i = 1, \ldots, N^b$, we assign the remaining edges of $\mathcal{G}_N$ independently.

It is clear from the above exploration procedure that the graph $\mathcal{G}_N$ is distributed as an Erdős-Rényi random graph. Moreover, on the event $A$ defined as “the balls $B(z_i, \gamma \log N)$, $i = 1, \ldots, N^b$, are pairwise disjoint in $\{1, \ldots, N\}$”, we have that $(z_i, B(z_i, \gamma \log N))$ in $\mathcal{G}_N$ is isomorphic to the corresponding pair in $\mathcal{G}_N$. Consequently they will be isomorphic to $(\varnothing, B(\varnothing, \gamma \log N))$ in $\mathcal{T}_i$. Therefore, to prove the corollary, it remains to estimate $Q'[\mathcal{G}_N]$.

Since all the vertices are indistinguishable, $Q'[\mathcal{G}_N]$ is bounded by 
\[
N^{2b} Q'[B(z_1, \gamma \log N) \cap B(z_2, \gamma \log N) \neq \varnothing] = N^{2b} Q'[z_1 \in B(z_2, 2\gamma \log N)] .
\]
Since $z_2$ is independent of $z_1$, this latter probability is bounded by 
\[
Q'[B(z_2, 2\gamma \log N)] \geq N^{6\gamma \log \lambda} + \frac{1}{N} N^{6\gamma \log \lambda} .
\]
By Corollary 11.3 for $N$ large, the first term is bounded above by $C_2 N^{6\gamma \log \lambda - 1}$ for some finite constant $C_2$. Hence,
\[
Q'[\mathcal{G}_N] \leq C_2 N^{2b + 6\gamma \log \lambda - 1} ,
\]
which proves the corollary.

It is a well known fact that conditioned on being infinite, $\mathcal{T}$ is $\mathcal{P}$-a.s. transient, see Theorem 3.5 and Corollary 5.10 in [28]. We denote by $v_{\varnothing}$ the probability that a simple random walk starting at $\varnothing$ never returns to this site, the so called escape probability. As we will show, the distribution of $v_{\varnothing}$ under $\mathcal{P}$ is close to that of the probability that a random walk on the giant component $\mathcal{C}_{\max}$ of the random graph $\mathcal{G}_N$ escapes from a certain neighborhood of a random vertex.

Since the isometry obtained in Corollary 11.3 is local, we need a tool to show that looking at a neighborhood of $\varnothing \in \mathcal{T}$ we can obtain precise estimates on the escape probability $v_{\varnothing}$. The next result plays a central role in this respect. Denote by $\Delta_l$, $l \geq 0$, the points of the $l$-th generation of a tree: $\Delta_l = B(\varnothing, l) \setminus B(\varnothing, l - 1)$.

For a fixed tree $\mathcal{T}$, we denote by $P_y$, $y \in \mathcal{T}$ the probability induced by the discrete-time simple random walk on $\mathcal{T}$ starting from $y$.

Proposition 11.5. There exist constants $c_1, c_2$, depending only on $\lambda$, such that, for every $l \geq 1$,
\[
\mathbb{P}\left[\sup_{y \in \Delta_l} P_y[H_{\varnothing} < \infty] \geq \exp\{-c_1 l\}\right] \leq \exp\{-c_2 l\}.
\]

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Proof. Throughout the proof of this lemma, given a rooted tree $T$ and a vertex $y \in T$, we denote by $T_y$ the subtree formed by the root $y$ together with the descendants of $y$ in $T$.

The idea is to show that in the path between $y$ and $\emptyset$ there are many tunnels from which the random walk can escape to infinity. In order to properly define these tunnels, we need to introduce some extra notation. For an arbitrary tree $T$ rooted at $\emptyset$, we define the tree $T^{\text{tail}}$, obtained by adding a vertex $\emptyset'$ which is connected to $\emptyset$ by an edge. This extra element should be regarded as the ancestor of $\emptyset$. In the proof, we use the notation $P_{\emptyset}^{T}$ to specify on which tree the random walk is defined.

For a given $\delta > 0$ and a tree $T$ with root $\emptyset$, we say that $T$ satisfies the property $Q^\delta$ if

$$P_{\emptyset}^{T^{\text{tail}}} [H_{\emptyset'} = \infty] \geq \delta.$$  

In other words, the property $Q^\delta$ is saying that a random walk on $T^{\text{tail}}$ has probability at least $\delta$ of never hitting the ancestor $\emptyset'$ of the root $\emptyset$.

It is clear from (11.3) that for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon, \lambda) > 0$ such that

$$P[T \text{ does not satisfy } Q^\delta] \leq q + \epsilon,$$  

(11.5)

where $q$ is the extinction probability: $q = P[T \text{ is finite}]$.

If $y$ is in the $l$th generation of $T$, we write $\emptyset = y_0, y_1, \ldots, y_l = y$ to denote the unique simple path connecting $\emptyset$ to $y$. Moreover, we denote by $\Gamma(y)$ the number of elements $y_k$, $0 \leq k < l$, having at least one descendant $y'_k \neq y_{k+1}$ such that $T_{y'_k}$ satisfies $Q^\delta$.

We can now use (11.5) together with in [21, Lemma 1] to conclude that there exist constants $c_3$ and $c_4$ such that

$$P[\exists y \in \Delta_l \text{ such that } \Gamma(y) < c_3 l] \leq \exp(-c_4 l).$$

To conclude the proof of the lemma it remains to show that there exists $c_1 > 0$ for which the event 

"$\exists y \in \Delta_l$ such that $P_y[H_{\emptyset} < \infty] \geq \exp(-c_1 l)$" is contained in the event 

"$\exists y \in \Delta_l$ such that $\Gamma(y) < c_3 l$".

Assume that all points $z$ in generation $l$ of $T$ are such that $\Gamma(z) \geq c_3 l$ and fix a point $y \in \Delta_l$. Recall the definition of $y_0, \ldots, y_l$ given above and consider a subsequence $k_j$, $1 \leq j \leq c_3 l$, for which $y_{k_j}$ has a descendant $y'_{k_j} \neq y_{k_j+1}$ such that $T_{y'_{k_j}}$ satisfies $Q^\delta$. These points are the entrance to the tunnels $T_{y'_{k_j}}$ that we have referred to in the beginning of the proof.

Let $T_-$ be the subtree of $T$ with all the descendants of $y_{k_j}$ removed, $1 \leq j \leq c_3 l$, with the exception of $y_{k_j+1}$ and $y'_{k_j}$. An argument based on flows or capacities shows that $P_{\emptyset}^{T_[-]} [H_{\emptyset} < \infty] \leq P_{\emptyset}^{T_[-]}[H_{\emptyset} < \infty] \leq P_{y_{k_{m-1}}} [H_{\emptyset} < \infty]$ where $m = c_3 l$. By the strong Markov property,

$$P_{\emptyset}^{T_[-]} [H_{\emptyset} < \infty] \leq P_{y_{k_{m-1}}} [H_{y_{k_{m-1}} - 1} < \infty] P_{y_{k_{m-1}}} [H_{\emptyset} < \infty].$$

Since $T_{y_{k_{m-1}}}$ satisfies $Q^\delta$ and since we removed all descendants of $y_{k_{m-1}}$, $P_{y_{k_{m-1}}} [H_{y_{k_{m-1}}} = \infty] \geq (1/3)P_{y_{k_{m-1}}} [H_{y_{k_{m-1}}} = \infty] \geq \delta/3$. Hence, the previous expression is bounded by

$$[1 - (\delta/3)] P_{y_{k_{m-1}}} [H_{\emptyset} < \infty].$$

Iterating this argument $m - 1$ times we finally get that $P_{\emptyset}^{T_[-]} [H_{\emptyset} < \infty]$ is bounded by $[1 - (\delta/3)]^{c_3 l - 1}$, which concludes the proof of the lemma.

Proposition 11.5 permits to approximate the inverse of the escape probability $v_{\emptyset}$ by a local quantity. Fix a infinite tree $T$ and $m \geq 1$. Let $v^{(m)}_{\emptyset}$ be the probability to escape from $B(\emptyset, m)$, $v^{(m)}_{\emptyset} = P_{\emptyset}^{T^{\text{tail}}} [H_{B(\emptyset, m)} > H_{B(\emptyset, m)}]$. Recall from [25, Chapter 9] the notion of flow and energy of a flow. Since $|T| = \infty$, we can define a trivial unit flow from $\emptyset$ to $B(\emptyset, m)^c$ which has energy equal to $m$. Hence, by Proposition 9.5 and Theorem 9.10 of [25],

$$v^{(m)}_{\emptyset} \geq (d_{\emptyset} m)^{-1},$$  

(11.6)

where $d_{\emptyset}$ is the degree of the root.

**Corollary 11.6.** There exist positive constants $c_1$ and $c_2$, depending only on $\lambda$, such that

$$P[|\Delta_l| < \exp(c_1 l) \mid \Delta_l \neq \emptyset] \leq \exp(-c_2 l)$$

for every $l \geq 1$.  

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Proof. For a tree with at least \( l \) generations, let \( \mathcal{G}_l \) be the graph obtained by identifying all points in \( \Delta_l \), naming this vertex \( z_l \). All other sites are left untouched, and the number of vertices of this new graph is \(|B(\emptyset, l)| - |\Delta_l| + 1\). We know that
\[
|\Delta_l|/d_\emptyset = \pi(z_l)/\pi(\emptyset),
\]
where \( \pi \) stands for the stationary measure of a simple random walk on \( \mathcal{G}_l \). The ratio in the right hand side of the above equation can be estimated using the escape probabilities from these two points. If \( P^\emptyset_{z_l} x \in \mathcal{G}_l \), stands for the probability on the path space induced by a discrete-time random walk on \( \mathcal{G}_l \) starting from \( x \),
\[
\frac{\pi(z_l)}{\pi(\emptyset)} = P^\emptyset_{z_l} [H_{z_l} < H^+_\emptyset] / P^\emptyset_{z_l} [H^+_\emptyset < H^+_{z_l}].
\]
We may couple the random walk on \( \mathcal{G}_l \) with a random walk on the tree in such a way that \( P^\emptyset_{z_l} [H_{z_l} < H^+_\emptyset] = P_\emptyset [H_{z_l} < H^+_\emptyset] \) and that \( P^\emptyset_{z_l} [H^+_\emptyset < H^+_{z_l}] \leq \max_{y \in \Delta_l} P_y [H^+_\emptyset < H^+_y] \). By [11.6], \( P_\emptyset [H_{\Delta_l} < H^+_\emptyset] \geq (d_\emptyset l)^{-1} \). Putting together all previous estimates, we get that on the set \( \Delta_l \neq \emptyset \),
\[
|\Delta_l|^{-1} \leq l \max_{y \in \Delta_l} P_y [H^+_\emptyset < H^+_{\Delta_l}] \leq l \max_{y \in \Delta_l} P_y [H^+_\emptyset < \infty]. \tag{11.7}
\]
Since there is a positive probability that a super-critical tree survives, the probability appearing in the statement of the lemma is bounded by \( C_0 P[|\Delta| < \exp(c_1 l), \Delta_l \neq \emptyset] \). By [11.7], this probability is bounded by \( C_0 P[l \max_{y \in \Delta_l} P_y [H^+_\emptyset < \infty] \geq \exp(-c_2 l)] \), which is bounded by \( \exp(-c_2 l) \) by Proposition [11.5].

Corollary 11.7. For any \( 0 < \gamma < 1 \), there exist positive constants \( c_0 \) and \( N_0 \geq 1 \), depending only on \( \gamma \) and \( \lambda \), such that for all \( N \geq N_0 \),
\[
P\left[ \frac{1}{v_\emptyset} - \frac{1}{v_\emptyset} \geq d_\emptyset N^{-c_0} |T| = \infty \right] \leq N^{-c_0},
\]
where \( d_\emptyset \) represents the degree of \( \emptyset \) and \( v_\emptyset = P_\emptyset [H^+_\emptyset > H_B(\emptyset, \gamma \log N)] \).

Proof. Fix \( 0 < \gamma < 1 \) and an infinite tree \( \mathcal{T} \). To keep notation simple, let \( B = B(\emptyset, \gamma \log N) \) and let \( \partial B \) be the set of points in \( B^c \) which have a neighbor in \( B \). By the strong Markov property,
\[
P_\emptyset [H^+_B > H^+_B] \geq P_\emptyset [H^+_B = \infty] \geq P_\emptyset [H^+_B > H_B] \inf_{x \in \partial B} P_x [H^+_x = \infty].
\]
Inverting these terms, we obtain
\[
0 \leq \frac{1}{v_\emptyset} - \frac{1}{v_\emptyset} \leq \frac{1}{v_\emptyset} \left( \frac{1}{\inf_{x \in \partial B} P_x [H^+_x = \infty]} - 1 \right).
\]
By Proposition [11.5] with \( l = \gamma \log N \), there exists constants \( c_1, c_2 > 0 \), depending on \( \lambda \), such that on a set with probability at least \( 1 - N^{-\gamma c_2} \) the previous infimum is bounded below \( 1 - N^{-\gamma c_1} \). Since \( (1 - x)^{-1} \leq 1 + 2x \) for \( x \in (0, 1/2) \), there exists \( N_0 = N_0(\gamma, \lambda) \) such that for \( N \geq N_0 \),

\[
\frac{1}{v_\emptyset} - \frac{1}{v_\emptyset} \leq \frac{2}{N^{c_0}} \frac{1}{v_\emptyset}.
\]
Estimate [11.6] permits to conclude the proof of the corollary, changing the values of the exponents if necessary.

Corollary 11.8. Let \( \mathcal{T} \) be a Galton-Watson tree with Poisson(\( \lambda \)) offsprings, \( \lambda > 1 \). Then, there exist finite constants \( c_0, C_0 \) and \( s_0 < \infty \), depending only on \( \lambda \), such that
\[
P[(v_\emptyset)^{-1} \geq s | |T| = \infty] \leq C_0 \exp(-c_0 \sqrt{s})
\]
for all \( s \geq s_0 \).

Proof. Since \( \mathcal{T} \) is super-critical, the probability appearing in the statement of the lemma is bounded by \( C_3 P[(v_\emptyset)^{-1} \geq s, |T| = \infty] \) for some finite constant \( C_3 \) depending only on \( \lambda \). Fix an integer \( n \geq 1 \). By the strong Markov property, \( v_\emptyset \) is bounded below by \( P_\emptyset [H_B \leq H^+_B] \inf_{y \in B} P_y [H^+_\emptyset = \infty] \), where \( B = B(\emptyset, n) \). Therefore, \( P[(v_\emptyset)^{-1} \geq s, |T| = \infty] \) is less than or equal to
\[
P[P_\emptyset [H_B \leq H^+_B]^{-1} \geq s/2, |T| = \infty] + P[\inf_{y \in B} P_y [H^+_\emptyset = \infty] \leq 1/2].
\]
By [11.6], \( P_\emptyset [H_B \leq H^+_B] \geq (d_\emptyset n)^{-1} \). The previous expression is thus bounded by
\[
P[d_\emptyset n \geq s/2] + P[\sup_{y \in B} P_y [H^+_\emptyset < \infty] \geq 1/2].
\]
Set $n = \sqrt{s}$, recall that $d_{i}$ has a Poisson($\lambda$) distribution. Apply an exponential Tchebychev inequality to estimate the first term. By Proposition 11.5 with $l = \sqrt{s}$, the second term is bounded by $\exp\{-c_{2}\sqrt{s}\}$ provided $s$ is large enough.

The following corollary allows us to bound the quantity $\epsilon_{N}$ appearing in (6.6) and (6.7).

**Corollary 11.9.** Fix an arbitrary vertex $y \in \{1, \ldots, N\}$ and $0 < \gamma < (3 \log \lambda)^{-1}$. Then, there exists positive constants $c_{0}$ and $N_{0} \geq 1$, depending only on $\gamma$ and $\lambda$, such that for all $N \geq N_{0}$,

$$\mathbb{P}\left[ \sup_{z \in B(y, \gamma \log N)} \mathbb{P}_{z}[H_{y} \leq \log^4 N] > N^{-c_{0}} \right] \leq N^{-c_{0}}.$$  

**Proof.** Denote by $\partial_{A}$ the internal boundary of a set $A$; $\partial_{A} = \{ x \in A : d(x, A^{c}) = 1 \}$. Fix $0 < \gamma < (3 \log \lambda)^{-1}$. By Propositions 11.2 and 11.3 there exist positive constants $c_{1}, c_{2}$ and $C_{1}$, depending only on $\lambda$, such that

$$\mathbb{P}\left[ \sup_{z \in \partial_{B}} \mathbb{P}_{z}[H_{y} \leq H_{B_{r}}] > N^{-c_{1}} \right] \leq C_{1} N^{c_{1}} + N^{-c_{2}},$$

where $c = 3 \gamma \log \lambda - 1$ and $B = B(y, \gamma \log N)$.

Assume that $\sup_{z \in \partial_{B}} \mathbb{P}_{z}[H_{y} \leq H_{B_{r}}] \leq N^{-c_{1}}$. We claim that in this case

$$\sup_{z \in B} \mathbb{P}_{w}[H_{y} \leq \log^4 N] \leq N^{-c_{1}} + \sup_{z \in B} \mathbb{P}_{z}[H_{y} \leq \log^4 N - 1].$$

Iterating this estimate $\log^4 N$ times, we conclude the proof of the corollary. It is enough, therefore, to prove (11.8). By the strong Markov property, $\mathbb{P}_{z}[H_{y} \leq \log^4 N]$ is bounded by $\sup_{w \in \partial_{B}} \mathbb{P}_{w}[H_{y} \leq \log^4 N]$. If $\{ H_{y} < H_{B_{r}} \}$, by the initial assumption we may bound the probability by $N^{-c_{1}}$. This gives the first term on the right hand side of (11.8). On the other hand, on the set $\{ H_{y} > H_{B_{r}} \}$, $H_{y} = H_{B_{r}} + y_{0} \theta_{H_{B_{r}}}$ and $H_{y} \circ \theta_{H_{B_{r}}} \leq \log^4 N - 1$. Hence, by the strong Markov property, for every $w \in \partial_{B}$,

$$\mathbb{P}_{w}[H_{y} \leq \log^4 N, H_{B_{r}} < H_{y}] \leq \mathbb{P}_{w}[H_{B_{r}} < H_{y}] \sup_{z \in B} \mathbb{P}_{z}[H_{y} \leq \log^4 N - 1],$$

which proves (11.8) and the corollary. \hfill \Box

We conclude this section deriving the scaling limit of the random walk $X_{t}^{N}$ on the giant component of the super-critical Erdős-Rényi random graph.

**Theorem 11.10.** Consider the trap model $X_{t}^{N}$ on the largest component $C_{\text{max}}$ of the Erdős-Rényi random graph with traps $W_{t}^{N}$, $x \in C_{\text{max}}$, as described in the beginning of this section. Assume that $\Psi_{N}(X_{t}^{N})$ converges in probability to some $K \in \mathbb{N}$. Let $\beta_{N} = (\psi_{\lambda}N)^{1/\alpha}$. Then,

$$\left( \beta_{N}^{-1} W_{N}, \Psi_{N}(X_{t}^{N}) \right) \text{ converges weakly to } \left( w, K_{t} \right),$$

where $w$ is the sequence defined in (8.3) and where, for each fixed $w$, $K_{t}$ is a $K$-process starting from $k$ with parameter $(Z, u)$, where $Z_{k} = u_{k}/E_{k}$ and $u_{k} = D_{k}E_{k}$. Here, $(D_{k}, E_{k})$, $k \geq 1$ is an i.i.d. sequence, distributed as $(d_{z}, v_{z})$ under $P_{y}$. $|T| = \infty$. The above convergence refers to the $L^{1}$-topology in the first coordinate and $d_{T}$-topology in the second.

**Proof.** We need to establish conditions (B0)–(B3) for the above sequence of graphs and to apply Theorem 2.2. Condition (B0) has been proven in the beginning of this section. The main difficulty in checking the remaining hypotheses comes from the fact that we are dealing with the giant component $C_{\text{max}}$, which has a random size, instead of the whole set $\{1, \ldots, N\}$ as in the above lemmas and propositions.

In order to prove (B1), let $t_{N} = (\gamma/2) \log N$ with $\gamma$ satisfying the conditions of Corollary 11.4. Since the term inside the expectation in (B1) is bounded by one, the expectation in (B1) is less than or equal to

$$\mathbb{E}\left[ \frac{1}{C_{\text{max}}} \sum_{x \in C_{\text{max}}} |B(x, 2t_{N})| \right] \leq \mathbb{P}[C_{\text{max}} < (\psi_{\lambda}/2)N] + \frac{4}{(\psi_{\lambda})^{2}} \mathbb{E}\left[ \sum_{x=1}^{N} |B(x, 2t_{N})| \right].$$

By Theorem 11.1 the first term vanishes as $N \uparrow \infty$, while by Proposition 11.2 and Corollary 11.3 the second term vanishes. This proves that condition (B1) is fulfilled.
By [7] with high probability the mixing time of a random walk on \( \mathcal{C}_{\max} \) is less than or equal to \( C_0 \log^2 N \) for some finite constant \( C_0 \). Choosing \( L_N = C_0^{-1} \log^2 N \), the hypothesis \([B2]\) becomes a direct consequence of Corollary \([11.9]\). It is indeed enough to condition the event appearing in the statement of Corollary \([11.9]\) on the set that \( y \) belongs to \( \mathcal{C}_{\max} \) and to recall from Theorem \([11.1]\) that the giant component has a positive density with probability converging to 1.

It remains to check \([B3]\). Let \( Q_N \) be the coupling between the random graph \( \mathcal{G}_N \) and \( N^b \) independent Galton-Watson trees \( \mathcal{T}_i \) constructed in Corollary \([11.4]\). We assume that these trees are the first \( N^b \) trees of an infinite i.i.d. sequence of Galton-Watson trees.

Fix \( K \geq 1 \) and let \( f_1, f_2, \ldots, f_K \) be the first \( K \) points \( z_i \) which belong to \( \mathcal{C}_{\max} \): \( f_1 = z_1 \) if \( z_1 \in \mathcal{C}_{\max} \) and \( z_i \notin \mathcal{C}_{\max} \) for \( 1 \leq i < j \), and so on. It is clear that \( f_1, \ldots, f_K \) is uniformly distributed among all possible choices and that the probability of not finding \( K \) points in \( \mathcal{C}_{\max} \) among \( N^b \) points uniformly distributed in \( \mathcal{G}_N \) converges to 0.

Let \( \eta_i, 1 \leq i \leq K \), be the first \( K \) indices of trees \( \mathcal{T}_i \) which are infinite and let \( (D_j, E_j) \) be the degree and the escape probabilities \((d_{x_{\mathcal{G}_N}}, v_{x_{\mathcal{G}_N}})\) in \( \mathcal{T}_{\eta_i} \). Note that the vectors \((D_j, E_j)\) are independent and identically distributed and that \( Q_N[(D_1, E_1) \in A] = P\{(d_{x_{\mathcal{G}_N}}, v_{x_{\mathcal{G}_N}}) \in A | |\mathcal{T}| = \infty\} \). In particular, by Corollary \([11.8]\) and Schwarz inequality the last two conditions in \([B3]\) are fulfilled.

Let \( A_N \) be the event “the graphs \((\eta_i, B(\eta_i, \gamma \log N)) \), \( 1 \leq i \leq K \), are isometric to the graphs \((\eta_i, B(\eta_i, \gamma \log N)) \), \( 1 \leq i \leq K \).” In view of Corollary \([11.8]\) on the set \( A_N \), the first two conditions in \([B3]\) are fulfilled. To conclude the proof of condition \([B3]\) it remains to show that
\[
\lim_{N \to \infty} P[\mathcal{A}^c_N] = 0. \tag{11.9}
\]

We define six sets \( \mathcal{S}_{N,j} \), \( 0 \leq j \leq 5 \), such that \( \cap_{0 \leq j \leq 5} \mathcal{S}_{N,j} \subset A_N \) and then prove that each of this set has asymptotic full measure. Recall that \( b \) and \( \gamma \) satisfy the assumptions of Corollary \([11.4]\) and let \( \mathcal{S}_{N,0} = \emptyset \). Since \( \{(\eta_i)_{i=1}^K = (\eta_j)_{j=1}^K\} \cap \mathcal{S}_{N,0} \subset A_N \), we need to impose further restriction to guarantee that \( \eta_1 = \eta_2, 1 \leq i \leq K \).

Let \( \mathcal{S}_{N,1} = \{\delta(\mathcal{C}_{\max}) \geq \gamma \log N\} \), let \( \mathcal{S}_{N,2} = \{|\mathcal{C}_{\max}| \geq K\} \) and let \( \mathcal{S}_{N,3} \) be the event “every three \( \mathcal{T}_i \), \( 1 \leq i \leq \log N \), with diameter greater or equal to \( \gamma \log N \) survives”. On \( \mathcal{S}_{N,0} \cap \mathcal{S}_{N,1} \cap \mathcal{S}_{N,2} \cap \mathcal{S}_{N,3} \), the graphs \((\eta_i, B(\eta_i, \gamma \log N)) \), \( 1 \leq i \leq K \), are coupled to infinite trees.

It remains to guarantee that there is no infinite tree coupled with a graph \((z_i, B(z_i, \gamma \log N))\) whose root \( z_i \) does not belong to \( \mathcal{C}_{\max} \). Let \( \mathcal{S}_{N,4} \) be the event “Every tree \( \mathcal{T}_i \), \( 1 \leq i \leq \log N \), with diameter greater of equal than \( \gamma \log N \) has at least \( N^\delta \) elements among the first \( \gamma \log N \) generations”, and let \( \mathcal{S}_{N,5} \) be the event “Every connected subset of \( \mathcal{G}_N \) with more than \( N^\delta \) elements is contained in \( \mathcal{C}_{\max} \).” On \( \mathcal{S}_{N,0} \cap \mathcal{S}_{N,1} \cap \mathcal{S}_{N,2} \cap \mathcal{S}_{N,3} \), all infinite trees \( \mathcal{T}_i \), \( 1 \leq i \leq \log N \), are coupled with graphs whose root belongs to \( \mathcal{C}_{\max} \).

Putting together the previous assertions, we get that \( \cap_{0 \leq j \leq 5} \mathcal{S}_{N,j} \subset A_N \), as claimed. We next show that each event introduced above has asymptotic full probability. By Corollary \([11.3]\) \( P[\mathcal{S}_{N,0}] \) vanishes, by Theorem \([11.1]\) and by Corollary \([11.3]\), \( P[\mathcal{S}_{N,1}] \), and by Theorem \([11.4]\) \( P[\mathcal{S}_{N,2}] \), \( P[\mathcal{S}_{N,3}] \) vanishes. By Corollary \([11.6]\) \( P[\mathcal{S}_{N,4}] \) vanishes for some \( \delta > 0 \), and by Theorem \([11.1]\) \( P[\mathcal{S}_{N,5}] \) vanishes. Finally, by Corollary \([11.6]\) there exists \( \delta = \delta(\gamma, \lambda) > 0 \) with the following property. A tree which has diameter \( \gamma \log N \) has at least \( N^\delta \) elements at generation \( \gamma \log N \) with probability converging to \( K \). Since from each element of the generation \( \gamma \log N \) descends an independent super-critical tree which has positive probability to survive, \( P[\mathcal{S}_{N,5}] \) vanishes.

\[\square\]

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