Global well–posedness of a class of singular hyperbolic Cauchy problems

Rahul Raju Pattar\textsuperscript{1} \cdot N. Uday Kiran\textsuperscript{1}

Received: 18 December 2021 / Accepted: 17 May 2022 / Published online: 27 May 2022
© The Author(s), under exclusive licence to Springer-Verlag GmbH Austria, part of Springer Nature 2022

Abstract
The goal of this paper is to establish a global well–posedness, cone condition and loss of regularity for singular hyperbolic equations with coefficients in $L^1((0, T]; C^\infty(\mathbb{R}^n)) \cap C^1((0, T]; C^\infty(\mathbb{R}^n))$ and Cauchy data in an appropriate Sobolev space tailored to a metric on the phase space. The coefficients are unbounded near the singular hyperplane $t = 0$ and polynomially growing as $|x| \to \infty$. The singular behavior is characterized by the blow–up rate of the coefficients and their first $t$-derivatives near $t = 0$. In order to study the interplay of the singularity in $t$ and unboundedness in $x$, we consider a class of metrics on the phase space. Our methodology relies on the use of the Planck function associated to the metric to subdivide the extended phase and to define an infinite order pseudodifferential operator for the conjugation. We also give some counterexamples.

Keywords Singular Hyperbolic Cauchy Problem \cdot Loss of Regularity \cdot Global Well-posedness \cdot Metric on the Phase Space \cdot Pseudodifferential Operators

Mathematics Subject Classification 35L81 \cdot 35L15 \cdot 35S05 \cdot 35B65 \cdot 35B30

1 Introduction

We consider the singular hyperbolic Cauchy problems of the form

$$u_{tt} + A(t, x, D_x)u_t + B(t, x, D_x)u = f, \quad (t, x) \in (0, T] \times \mathbb{R}^n, \quad T < \infty,$$


1 Department of Mathematics and Computer Science, Sri Sathya Sai Institute of Higher Learning, Puttaparthi, India

\textsuperscript{1} Dedicated to Bhagawan Sri Sathya Sai Baba on the Occasion of His 96th Birthday
where $u$ is a function of $t$, valued in a Sobolev space while $A$ and $B$ are linear partial differential operators some of whose coefficients tend to infinity in some sense as $t \to 0$. See [3] and the references therein for a discussion on singular hyperbolic Cauchy problems. A study of such problems is motivated by applications in physics [6], in the study of blow–up solutions of quasilinear problems [7, Section 4] and also in the study of certain degenerate Cauchy problems [3]. One of the prominent features of singular hyperbolic Cauchy problems is the loss of regularity index of the solutions in relation to the initial data defined in an appropriate Sobolev space.

In order to describe the loss of regularity and well–posedness results from the literature, consider the following model strictly hyperbolic equation

$$\left\{ \begin{array}{l}
\partial_t^2 u - a(t, x) \partial_x^2 u + \sum_{j+l=0}^1 b_{j,l}(t, x) \partial_x^j \partial_t^l u = f(t, x), \quad (t, x) \in (0, T] \times \mathbb{R}, \\
u(0, x) = f_1(x), \quad \partial_t(0, x) = f_2(x).
\end{array} \right. \tag{1.1}$$

The operator coefficients are in $L^1 ((0, T]; C^\infty(\mathbb{R})) \cap C^1 ((0, T]; C^\infty(\mathbb{R}))$ and the singular behavior of the above Cauchy problem is described by the following estimates

$$\begin{align}
|\partial_x^\beta \partial_t a(t, x)| &\leq C_\beta^{(1)} \omega(x)^2 \Phi(x)^{-|\beta|} \frac{1}{t^{qq}} |\ln t|^{(\gamma-1)I_q}, \\
|\partial_x^\beta a(t, x)| &\leq C_\beta^{(2)} \omega(x)^2 \Phi(x)^{-|\beta|} \frac{1}{t^{pp}} |\ln t|^{\gamma I_q}, \\
|\partial_x^\beta b_{j,l}(t, x)| &\leq C_\beta^{(3)} \omega(x)^j \Phi(x)^{-|\beta|} \frac{1}{t^{rr}}, \tag{1.2}
\end{align}$$

with $\beta \in \mathbb{N}_0$, $C_\beta^{(i)} > 0$, $i = 1, 2, 3$, $q \in [1, \infty)$, $p \in [0, 1)$, $p \leq q - 1$, $\gamma \in (0, \infty)$ and $r \in [0, 1)$. The function $I_q$ is such that $I_q \equiv 1$ if $q = 1$ else $I_q \equiv 0$. The functions $\omega(x)$ and $\Phi(x)$ are positive monotone increasing in $|x|$ such that $1 \leq \omega(x) \lesssim \Phi(x) \lesssim \langle x \rangle = (1 + |x|^2)^{1/2}$. They specify the structure of the differential equation in the space variable. These functions will be discussed in detail in Sect. 2.

In this work, our main interest is on the optimality of loss when the coefficients are singular in time and unbounded in space. In particular, our interest is either blow–up or infinitely many oscillations near $t = 0$ and polynomial growth in $x$. Such equations have a well-known behavior of a loss of derivatives when the coefficients are bounded in space together with all their derivatives (see for example [4, 5]). Along with the extension of the results to a global setting ($x \in \mathbb{R}^n$ and the coefficients are allowed to grow polynomially in $x$), we also investigate the behavior at infinity of the solution in relation to the coefficients. By a loss of regularity index of the solution in relation to the initial datum we mean a change of indices in an appropriate Sobolev space. See Table 1 for the well–posedness results from the literature in the context of singular hyperbolic Cauchy problems.

In [4], Cicognani discussed the well–posedness of (1.1) for the case $\omega(x) = \Phi(x) = 1$ and $p = 0$, $q = 1$, $\gamma = 1$, $r = 0$ in (1.2), where the author reports well–posedness in $C^\infty(\mathbb{R})$ for the Cauchy problem (1.1) with a finite loss of derivatives. Colombini
et al. [5] considered the Cauchy problem (1.1) with operator coefficients independent of \( x \) and singular behavior prescribed by the parameters \( p \in (0, 1), q \in (1, \infty) \) and \( r = 0 \). They report well-posedness in Gevrey space \( G^s \), \( 1 \leq s < \frac{q-p}{q-r} \), with infinite loss of derivatives. We study in [11] the case of \( p = 0, q \in (1, \frac{3}{2}) \) and \( r = 0 \) with generic structure functions \( \omega \) and \( \Phi \) in (1.2) and report infinite loss of both derivatives and decay.

In this paper our interest is in the operator coefficients that are \( L^1 \) integrable in \( t \) but singular near \( t = 0 \) and are of polynomial growth in \( x \). In particular, our interest is \( p \in \left[0, \frac{1}{2}\right), q \in \left(1, \frac{3}{2}\right), p \leq q - 1, r \in (0, 1) \) and polynomial growth in \( x \) prescribed by \( \omega(x), \Phi(x) \) in (1.2). An example of a coefficient \( a(t, x) \) satisfying (1.2) is given below.

**Example 1.1** Let \( T = 1, \kappa_1 \in [0, 1) \) and \( \kappa_2 \in (0, 1] \) such that \( \kappa_1 \leq \kappa_2 \). Then,

\[
a(t, x) = \langle x \rangle^{2\kappa_1} \left( 2 + \cos \langle x \rangle^{1-\kappa_2} \right) \left( \frac{1}{t^{1/4}} \left( 2 + \sin \left( \frac{1}{t^{1/8}} \right) \right) \right)
\]

satisfies the estimates (1.2) for \( \omega(x) = \langle x \rangle^{\kappa_1}, \Phi(x) = \langle x \rangle^{\kappa_2}, p = \frac{1}{4}, q = \frac{11}{8} \). The example shows that singular coefficients can also have infinitely many oscillations near \( t = 0 \).

In order to study the interplay between the singularity in time and unboundedness in space one needs to consider an appropriate metric on the phase space [9, 10]. In our case, we study the Cauchy problem (1.1) by considering the metrics

\[
g_{\Phi, k} = \Phi(x)^{-2} dx^2 + \langle \xi \rangle_k^{-2} d\xi^2,
\]

where \( \langle \xi \rangle_k = (k^2 + |\xi|^2)^{1/2} \), for sufficiently large \( k \) chosen appropriately. These metrics are discussed in Sect. 2.1. From the estimates (1.2), note that \( \omega \) and \( \Phi \) are associated with the weight and metric respectively, and they specify the structure of the differential equation in the space variable. These functions will be discussed in detail in Sect. 2.2. We report that the solution not only experiences a loss of derivatives but also a decay in relation to the initial datum defined in a Sobolev space modelled by the infinite order pseudodifferential operator

\[
e^{\Lambda(t) \Theta(x, D_x)}.
\]

Here \( \Lambda \in C([0, T]) \) and the symbol of the operator \( \Theta(x, D_x) \) is given by \( h(x, \xi)^{-1/\sigma} = (\Phi(x) \langle \xi \rangle_k)^{1/\sigma} \) where \( h(x, \xi) \) is the Planck function related to the metric \( g_{\Phi, k} \) in (1.3) and \( 3 \leq \sigma < (q - p)/(q - 1) \). The operator \( \Theta(x, D_x) \) explains the quantity of the loss by linking it to the metric on the phase space and the singular behavior while \( \Lambda(t) \) gives a scale for the loss. Hence, we call the conjugating operator as loss operator.

Our methodology relies upon two important techniques: the subdivision of the extended phase space into two regions and conjugation of a first order system corresponding to the operator \( P \) in (3.1) by the loss operator. Both these techniques result in
Table 1  Well-posedness results for singular hyperbolic Cauchy problems. (**) refers to Theorem 3.1 of this paper

| Order of Singularity at $t = 0$ | Regularity in $t$ of coefficients | Growth in $x$ of coefficients | Loss of Regularity index for Solution | Ref. |
|-------------------------------|----------------------------------|-------------------------------|--------------------------------------|------|
| $p$                           | $q$                              | $r$                           | $\omega(x)$ $\Phi(x)$               |      |
| 0                             | 1                                | $(0, 1)$                       | $C^1((0, T])$                        |      |
| 0                             | 1                                | 1                              | $C^1((0, T])$                        |      |
| 0                             | 1                                | $[1, \infty)$                  | $C^2((0, T])$                        |      |
| $[0, 1)$                      | $(1, \infty)$                   | $-$                            | $C^1((0, T])$                        |      |
| $\left[0, \frac{1}{2}\right)$ | $(1, \frac{3}{2})$              | $-$                            | $C^1((0,T])$                         |      |

Arbitrarily small \[ 15\]

Finite \[ 4\]

Finite \[ 13\]

Ranging from Finite to Infinite \[ 14\]

Infinite \[ 5\]

Infinite **
the change of the metric governing the operator where the new metric is conformally equivalent to the one in (1.3). As seen in (5.3), the characteristic roots corresponding to the operator $P$ showcase a stronger singular behavior compared to the principal symbol. Due to the subdivision of the phase space, this results in the change of the metric as demonstrated in Lemma 5.1. This metric is of the form

$$\tilde{g}_{\phi,k}^{(1)} = (\Phi(x)\langle \xi \rangle_k)^{2\delta'} g_{\phi,k},$$

(1.5)

where $\delta' = -\frac{p}{q-p}$. On the other hand, our work in [11, Theorem 4.0.1] suggests that the conjugation by the loss operator changes the metric to

$$\tilde{g}_{\phi,k}^{(2)} = (\Phi(x)\langle \xi \rangle_k)^{2/\sigma} g_{\phi,k}.$$  

(1.6)

As our approach to establish well-posedness is based on the energy estimates, we consider the metric $\tilde{g}_{\phi,k}^{(2)}$ (as $\tilde{g}_{\phi,k}^{(1)} < \tilde{g}_{\phi,k}^{(2)}$) for the application of the sharp Gårding inequality [8, Theorem 18.6.14].

From the energy estimate used in proving the well-posedness we derive an optimal cone condition for the solution of the Cauchy problem (3.1) in Sect. 6. Though the characteristics of the operator $P$ in (3.1) are singular, the $L^1$ integrability of the singularity guarantees that the propagation speed is finite. The weight function governing the coefficients influences the geometry of the slope of the cone due to which the cone condition in our case is anisotropic.

In Sect. 7, we give a set of counterexamples showing how the lower order terms influence the well-posedness and the loss of regularity when $L^1$ integrability condition is violated. We cover various cases such as no loss, finite loss and nonuniqueness.

The paper is organized as follows. In Sect. 2, we describe the tools necessary for our analysis. In Sect. 3, we define a Cauchy problem of our interest and state the well-posedness result whose proof will be presented in Sect. 5. In Sect. 4, we define appropriate generalized parameter dependent symbol classes. In Sect. 6 we derive a cone condition, while in Sect. 7 we provide the set of examples.

2 Tools

In this section we introduce the main tools of this paper. The first part is devoted to a class of metrics on the phase space that govern the geometry of the symbols in our consideration while the second treats the properties of structure functions. In the third part, we define Sobolev spaces associated to the metric. Lastly, we devise a localization technique based on the Planck function associated to the metric.

2.1 Our choice of metric on the phase space

In this section, we review some notation and terminology used in the study of metrics on the phase space, see [9, Chapter 2] and [10] for details. Let us denote by $\sigma(X, Y)$
the standard symplectic form on $T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$: if $X = (x, \xi)$ and $Y = (y, \eta)$, then $\sigma$ is given by

$$\sigma(X, Y) = \xi \cdot y - \eta \cdot x.$$ 

We can identify $\sigma$ with the isomorphism of $\mathbb{R}^{2n}$ to itself such that $\sigma^* = -\sigma$, with the formula $\sigma(X, Y) = \langle \sigma X, Y \rangle$. Consider a Riemannian metric $g_X$ on $\mathbb{R}^{2n}$ (which is a measurable function of $X$) to which we associate the dual metric $g^\sigma_X$ by

$$g^\sigma_X(Y) = \sup_{0 \neq Y' \in \mathbb{R}^{2n}} \frac{\langle \sigma Y, Y' \rangle^2}{g_X(Y')} , \text{ for all } Y \in \mathbb{R}^{2n}.$$ 

Considering $g_X$ as a matrix associated to positive definite quadratic form on $\mathbb{R}^{2n}$, $g^\sigma_X = \sigma^* g^{-1}_X \sigma$. We define the Planck function [10] which plays a crucial role in the development of pseudodifferential calculus as

$$h_g(x, \xi) := \sup_{0 \neq Y \in \mathbb{R}^{2n}} \left( \frac{g_X(Y)}{g^\sigma_X(Y)} \right)^{1/2}.$$ 

The uncertainty principle is quantified as the upper bound $h_g(x, \xi) \leq 1$. In the following, we often make use of the strong uncertainty principle, that is, for some $\kappa > 0$, we have

$$h_g(x, \xi) \leq (1 + |x| + |\xi|)^{-\kappa}, \quad (x, \xi) \in \mathbb{R}^{2n}.$$ 

In general, we use the metrics of the form

$$g^\rho, r_{\Phi, k} = \left( \frac{\langle \xi \rangle_{k_2}^2}{\Phi(x) \tilde{\rho}_1} \right)^2 |dx|^2 + \left( \frac{\Phi(x) \tilde{\rho}_2}{\langle \xi \rangle_{k_1}^2} \right)^2 |d\xi|^2. \quad (2.1)$$

Here $\rho = (\rho_1, \rho_2)$, $\tilde{\rho} = (\tilde{\rho}_1, \tilde{\rho}_2)$ for $\rho_j, \tilde{\rho}_j \in [0, 1], j = 1, 2$ are such that $0 \leq \rho_2 < \rho_1 \leq 1$ and $0 \leq \tilde{\rho}_2 < \tilde{\rho}_1 \leq 1$. The Planck function associated to the metric in (2.1) is $\Phi(x) \tilde{\rho}_2 - \tilde{\rho}_1 \langle \xi \rangle_{k_1}^2 - \rho_1 \langle \xi \rangle_{k_1}^2 - \rho_1$.

2.2 Properties of the structure functions $\omega(x)$ and $\Phi(x)$

The functions $\omega(x)$ and $\Phi(x)$ are associated with weight and metric respectively. They specify the structure of the differential equation. As pseudodifferential calculus is the datum of the metric satisfying some local and global conditions. In our case, it amounts to the conditions on $\Phi$. The symplectic structure and the uncertainty principle also play a natural role in the constraints imposed on $\Phi$. So we consider $\Phi$ to be a monotone increasing function of $|x|$ satisfying the following conditions:
1 \leq \Phi(x) \lesssim 1 + |x| \quad \text{(sub-linear)}

|x - y| \leq r \Phi(y) \quad \Rightarrow \quad C^{-1} \Phi(y) \leq \Phi(x) \leq C \Phi(y) \quad \text{(slowly varying)}

\Phi(x + y) \lesssim \Phi(x)(1 + |y|)^s \quad \text{(temperate)}

for all \( x, y \in \mathbb{R}^n \) and for some \( r, s, C > 0 \).

For the sake of calculations arising in the development of symbol calculus related to the metrics \( g_{\Phi,k} \), we need to impose following additional conditions:

\[
|\Phi(x) - \Phi(y)| \leq \Phi(x + y) \leq \Phi(x) + \Phi(y), \quad \text{(Subadditive)}
\]

\[
|\partial^\beta \Phi(x)| \lesssim \Phi(x) \langle x \rangle^{-|\beta|},
\]

\[
\Phi(ax) \leq a \Phi(x), \quad \text{if } a > 1,
\]

\[
\Phi(ax) \leq \Phi(ax), \quad \text{if } a \in [0, 1],
\]

where \( \beta \in \mathbb{Z}^n_+ \). It can be observed that the above conditions are quite natural in the context of symbol classes. In our work, we need even the weight function \( \omega \) to satisfy the above stated properties of \( \Phi \). In order to arrive at an energy estimate using the sharp Gårding inequality (see Sect. 5.3 for details), we impose the following condition

\[
\omega(x) \lesssim \Phi(x), \quad x \in \mathbb{R}^n.
\]

2.3 Sobolev spaces

We now introduce the Sobolev space related to the metric \( g_{\Phi,k} \) that is suitable for our analysis.

**Definition 2.1** The Sobolev space \( H^{s,\varepsilon,\sigma}_{\Phi,k}(\mathbb{R}^n) \) for \( \sigma > 2, \varepsilon \geq 0 \) and \( s = (s_1, s_2) \in \mathbb{R}^2 \) is defined as

\[
H^{s,\varepsilon,\sigma}_{\Phi,k}(\mathbb{R}^n) = \left\{ v \in L^2(\mathbb{R}^n) : \Phi(x)^{s_2} \langle D \rangle_k^{s_1} \exp\{\varepsilon (\Phi(x) \langle D \rangle_k)^{1/\sigma}\} v \in L^2(\mathbb{R}^n) \right\}
\]

equipped with the norm \( \|v\|_{k,s,\varepsilon,\sigma} = \|\Phi(\cdot)^{s_2} \langle D \rangle_k^{s_1} \exp\{\varepsilon (\Phi(\cdot) \langle D \rangle_k)^{1/\sigma}\} v\|_{L^2} \). The operator \( \exp\{\varepsilon (\Phi(x) \langle D \rangle_k)^{1/\sigma}\} \) is an infinite order pseudodifferential operator with the symbol \( \exp\{\varepsilon (\Phi(x) \langle \xi \rangle_k)^{1/\sigma}\} \).

The subscript \( k \) in the notation \( H^{s,\varepsilon,\sigma}_{\Phi,k}(\mathbb{R}^n) \) is related to the parameter in the operator \( \langle D \rangle_k = (k^2 - \Delta)_{1/2} \).

2.4 Subdivision of the phase space

One of the main tools in our analysis is the division of the extended phase space \( J = [0, T] \times \mathbb{R}^{2n} \), where \( T > 0 \), into two regions using the Planck function \( h(x, \xi) = (\Phi(x) \langle \xi \rangle_k)^{-1} \) of the metric \( g_{\Phi,k} \) in (1.3). We use these regions in the proof of Theorem
3.1 (see Sect. 5.1) to handle the low regularity in \( t \). To this end we define the time splitting point \( t_{x, \xi} \), for a fixed \((x, \xi)\), as the solution to the equation

\[
t^q - p = N h(x, \xi),
\]

where \( N \) is a positive constant to be chosen appropriately later. Since \( 3 \leq \sigma < \frac{q-p}{q-1} \), we consider \( \delta \in (0, 1) \) such that

\[
\frac{1}{\sigma} = \frac{q-1 + \delta}{q-p}.
\]

Implying

\[
\gamma := 1 - \frac{1}{\sigma} = \frac{1 - \delta - p}{q-p}.
\]

Using \( t_{x, \xi} \) and (2.3) we define the interior region

\[
Z_{int}(N) = \{(t, x, \xi) \in J : 0 \leq t \leq t_{x, \xi}, \ |x| + |\xi| > N \}
\]

and the exterior region

\[
Z_{ext}(N) = \{(t, x, \xi) \in J : t_{x, \xi} < t \leq T, \ |x| + |\xi| > N \}
\]

We use these regions to define the parameter dependent global symbol classes in Sect. 4.

### 3 Statement of the main result

We consider the Cauchy problem

\[
\begin{aligned}
P(t, x, \partial_t, D_x) u(t, x) &= f(t, x), & D_x &= -i\nabla_x, & (t, x) &\in (0, T] \times \mathbb{R}^n, \\
 u(0, x) &= f_1(x), & \partial_t u(0, x) &= f_2(x),
\end{aligned}
\]

with the strictly hyperbolic operator

\[
P(t, x, \partial_t, D_x) = \partial_t^2 + b_0(t, x) \partial_t + a(t, x, D_x) + b(t, x, D_x)
\]

where

\[
a(t, x, \xi) = \sum_{i,j=1}^n a_{i,j}(t, x) \xi_i \xi_j 
\]

and

\[
b(t, x, \xi) = i \sum_{j=1}^n b_j(t, x) \xi_j + b_{n+1}(t, x).
\]

Here, the matrix \( (a_{i,j}(t, x)) \) is real symmetric for all \((t, x) \in (0, T] \times \mathbb{R}^n, a_{i,j} \in L^1((0, T]; C^\infty(\mathbb{R}^n)) \cap C^1((0, T]; C^\infty(\mathbb{R}^n)) \) and \( b_j \in L^1((0, T]; C^\infty(\mathbb{R}^n)) \). We have the following assumptions on \( a(t, x, \xi), b(t, x, \xi) \) and \( b_j(t, x), j = 0, n + 1 \):
\[ a(t, x, \xi) \geq C_0 \omega(x)^2 \langle \xi \rangle_k^2, \quad C_0 > 0, \]
\[ |\partial_\xi^\alpha \partial_x^\beta a(t, x, \xi)| \leq C |\alpha|! |\beta|! \sigma_1 \frac{1}{t^p} \omega(x)^2 \Phi(x)^{-|\beta|} \langle \xi \rangle_k^2 - |\alpha|, \]
\[ |\partial_\xi^\alpha \partial_x^\beta \partial_t a(t, x, \xi)| \leq C |\alpha|! |\beta|! \sigma_1 \frac{1}{t^q} \omega(x)^2 \Phi(x)^{-|\beta|} \langle \xi \rangle_k^2 - |\alpha|, \]
\[ |\partial_\xi^\alpha \partial_x^\beta b(t, x, \xi)| \leq C |\alpha|! |\beta|! \sigma_1 \frac{1}{t^r} \omega(x) \Phi(x)^{-|\beta|} \langle \xi \rangle_k^2 - |\alpha|, \]
\[ |\partial_x^\beta b_j(t, x)| \leq C |\beta|! \sigma_1 \frac{1}{t^r} \Phi(x)^{-|\beta|}, \quad j = 0, n + 1, \]

(3.2)

\( q \in \left(1, \frac{3}{2}\right), p \leq q - 1, r \in \left[0, \frac{1}{2}\right), s \in \left[0, 1\right), \sigma \leq \frac{(q - p)}{(q - 1)} \) and \( (t, x, \xi) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \). Note that \( C > 0 \) is a generic constant.

We now state the main result of this paper. Let \( e = (1, 1) \).

**Theorem 3.1** Consider the strictly hyperbolic Cauchy problem (3.1) satisfying the conditions in (3.2). Let the initial data \( f_j \) belong to \( H^{s+(2-j)e, \Lambda_j, \sigma}_{\Phi, k} \) and the right-hand side \( f \in C([0, T]; H^{s, \Lambda, \sigma}_{\Phi, k}) \), \( \Lambda_j > 0, j = 1, 2 \). Then, there exist a continuous function \( \Lambda(t) \) and positive constants \( \Lambda_0 \) and \( \delta^* \), such that there is a unique solution

\[ u \in C \left([0, T]; H^{s+e, \Lambda(t), \sigma}_{\Phi, k}\right) \cap C^1 \left([0, T]; H^{s, \Lambda(t), \sigma}_{\Phi, k}\right), \]

for \( \Lambda(t) < \Lambda^* = \min\{\Lambda_0, \Lambda_1, \Lambda_2\} \). More specifically, the solution satisfies an a priori estimate

\[ \sum_{j=0}^{1} \| \partial_j^j u(t, \cdot) \|_{\Phi, k; s+(1-j)e, \Lambda(t), \sigma} \]
\[ \leq C \left( \sum_{j=1}^{2} \| f_j \|_{\Phi, k; s+(2-j)e, \Lambda(0), \sigma} + \int_0^t \| f(\tau, \cdot) \|_{\Phi, k; s, \Lambda(\tau), \sigma} d\tau \right) \]

(3.3)

where \( 0 \leq t \leq T \leq (\delta^* \Lambda^*/\lambda)^{1/\delta^*} \), \( C = C_{\varepsilon} > 0 \) and \( \Lambda(t) = \frac{1}{\delta^*} \left(T^{\delta^*} - t^{\delta^*}\right) \) for a sufficiently large \( \lambda \).

**Remark 3.1** Observe that we have \( 3 \leq \sigma < (q - p)/(q - 1) \) where as in [5, Theorem 2], it is \( 1 \leq \sigma < (q - p)/(q - 1) \). The increase in the lower bound for \( \sigma \) is due to the application of sharp Gårding inequality in our context that dictates \( \sigma \geq 3 \). This is discussed in Sect. 5.3. Due to this increment in \( \sigma \), we have \( q \in \left(1, \frac{3}{2}\right) \).

### 4 Parameter dependent global symbol classes

We now define certain parameter dependent global symbols that are associated with the study of the Cauchy problem (3.1). Let \( m = (m_1, m_2) \in \mathbb{R}^2 \). Consider the metrics
Proposition 4.1 $G^{m_1,m_2}(\omega, \rho, \tilde{g}_{\Phi,k})$ is the space of all functions $a \in C^\infty(\mathbb{R}^{2n})$ satisfying
\[
|\partial_\xi^\alpha D_x^\beta a(x, \xi)| \leq C_{\alpha \beta} |(\xi)^{m_1-\rho_1|\alpha|+\rho_2|\beta|} \alpha\omega(x)^{m_2} \Phi(x)\tilde{\rho}_1|\beta|+\tilde{\rho}_2|\alpha| |
\] (4.1)

Since $g_{\Phi,k} \leq \tilde{g}_{\Phi,k}^{(1)} \leq \tilde{g}_{\Phi,k}^{(2)}$, we have
\[
G^{m_1,m_2}(\omega, g_{\Phi,k}) \subset G^{m_1,m_2}(\omega, \tilde{g}_{\Phi,k}^{(1)}) \subset G^{m_1,m_2}(\omega, \tilde{g}_{\Phi,k}^{(2)}).
\]

Let $\mu \geq 1$ and $\nu \geq 1$.

Definition 4.2 $AG^{m_1,m_2}(\omega, \rho, \tilde{g}_{\Phi,k})$ is the space of all functions $a \in C^\infty(\mathbb{R}^{2n})$ satisfying (4.1) with $C_{\alpha \beta} = C|\alpha|+|\beta|((\alpha)!)^\mu((\beta)!)^\nu$ for some $C > 0$.

Definition 4.3 $AG_{\sigma}^{m_1,m_2}(\omega, \rho, \tilde{g}_{\Phi,k})$ is the space of all functions $a \in C^\infty(\mathbb{R}^{2n})$ satisfying (4.1) when $h(x, \xi) \leq C_1|\alpha|^{-\sigma}$ with $C_{\alpha \beta} = C_2|\alpha|+|\beta|((\alpha)!)^\mu((\beta)!)^\nu$ for some positive constants $C_1, C_2 > 0$.

We denote the set of operators with symbols in $G^{m_1,m_2}(\omega, \rho, \tilde{g}_{\Phi,k})$ and $AG^{m_1,m_2}(\omega, \rho, \tilde{g}_{\Phi,k})$ by $OPG^{m_1,m_2}(\omega, \rho, \tilde{g}_{\Phi,k})$ and $OPAG^{m_1,m_2}(\omega, \rho, \tilde{g}_{\Phi,k})$, respectively. As far as the calculus of these pseudodifferential operators are concerned we refer to [12, Appendix II & III], [10, Section 6.3] and [1, Appendix].

In our analysis, we require the following conjugation result.

Proposition 4.1 Let $e^{\Lambda(t)\Theta(x,D_x)}$ be as in (1.4) and $a(x, \xi) \in AG_{\sigma}^{m_1,m_2}(\omega, \tilde{g}_{\Phi,k}^{(1)})$. Then, there exists $\Lambda_0 > 0$ such that for $\Lambda(t) > 0$ with $\Lambda(t) < \Lambda_0$,
\[
e^{\Lambda(t)\Theta(x,D_x)}a(x, D_x)e^{-\Lambda(t)\Theta(x,D_x)} = a(x, D_x) + \sum_{j=1}^3 r_{\Lambda}^{(j)}(t, x, D_x)
\]
where the symbols of $r_{\Lambda}^{(j)}(t, x, D_x)$ for $j = 1, 2, 3$ are in $C([0, T]; AG_{\sigma}^{m_1,m_2}(\omega, \tilde{g}_{\Phi,k}^{(1)}), C([0, T]; AG_{\sigma}^{m_1,m_2}(\omega, \tilde{g}_{\Phi,k}^{(1)}), \text{ and } C([0, T]; AG_{\sigma}^{m_1,m_2}(\omega, \tilde{g}_{\Phi,k}^{(1)}), respectively.

Proof Noting the fact that $\tilde{g}_{\Phi,k}^{(1)} \leq \tilde{g}_{\Phi,k}^{(2)}$, the proof follows in similar lines to [11, Theorem 4.0.1].

Observe that the derivatives of $\sqrt{a(t, x, \xi)}$, characteristic roots of operator $P$ in (3.1) show stronger singularity compared to $a(t, x, \xi)$ due to the singularity. Thus, to handle the singular behavior of the characteristics, we have the following symbol classes.
In this section, we give a proof of the main result, Theorem 3.1. There are three key

Definition 4.4 \( AG^m_{\sigma} \{ l; \delta \}^{(1)}_{N} (\omega, g_{\Phi,k}^{\rho,\tilde{\rho}}) \) for \( l \in \mathbb{R} \) and \( \delta \in [0, 1) \) is the space of all functions \( a \in C^1((0, T]; G^m_{\rho,\tilde{\rho}}(\omega, g_{\Phi,k}^{\rho,\tilde{\rho}})) \) satisfying

\[
|\frac{\partial}{\partial t} D^\beta \xi a(t, x, \xi)| \leq C[|\alpha|+|\beta|]a!(\beta!^\sigma \langle \xi \rangle_k^{m-\rho_1|\alpha|+\rho_2|\beta|} \omega(x)^m_2 \Phi(x)^{-\tilde{\rho}_1|\beta|+\tilde{\rho}_2|\alpha|} \left( \frac{1}{t} \right)^{\delta_1 l},
\]

for all \( (t, x, \xi) \in Z_{int}(N) \) and for some \( C > 0 \) where \( \alpha, \beta \in \mathbb{N}_0^n \).

Definition 4.5 \( AG^m_{\sigma} \{ l_1, l_2, l_3; \delta_1, \delta_2 \}_N^{(2)} (\omega, g_{\Phi,k}^{\rho,\tilde{\rho}}) \) for \( l_1, l_2, l_3 \in \mathbb{R}, l_2 \in \{0, 1\} \) and \( \delta_1 \in [0, 1), \delta_2 \in (1, 3/2) \) is the space of all functions \( a \in C^1((0, T]; G^m_{\rho,\tilde{\rho}}(\omega, g_{\Phi,k}^{\rho,\tilde{\rho}})) \) satisfying

\[
|\frac{\partial}{\partial t} D^\beta \xi a(t, x, \xi)| \leq C[|\alpha|+|\beta|]a!(\beta!^\sigma \langle \xi \rangle_k^{m-\rho_1|\alpha|+\rho_2|\beta|} \omega(x)^m_2 \Phi(x)^{-\tilde{\rho}_1|\beta|+\tilde{\rho}_2|\alpha|} \left( \frac{1}{t} \right)^{\delta_1 l_1+l_2(|\alpha|+|\beta|)+\delta_2 l_2}
\]

for all \( (t, x, \xi) \in Z_{ext}(N) \) and for some \( C_{\alpha\beta} > 0 \) where \( \alpha, \beta \in \mathbb{N}_0^n \).

Given a \( t \)-dependent global symbol \( a(t, x, \xi) \), we can associate a pseudodifferential operator \( Op(a) = a(t, x, D_x) \) to \( a(t, x, \xi) \) by the following oscillatory integral

\[
a(t, x, D_x)u(t, x) = \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot \xi} a(t, x, \xi)u(t, y)d\xi
\]

\[
= (2\pi)^{-n} \int_{\mathbb{R}^{n}} e^{ix\cdot \xi} a(t, x, \xi) \hat{u}(t, \xi)d\xi,
\]

where \( d\xi = (2\pi)^{-n}d\xi \) and \( \hat{u} \) is the Fourier transform of \( u \) in the space variable. The calculus for the operators with symbols of form \( a(t, x, \xi) = a_1(t, x, \xi) + a_2(t, x, \xi) \) such that

\[
a_1 \in AG^{\tilde{m}_1,\tilde{m}_2}_\sigma \{ \tilde{I}; \delta \}_N^{(1)} (\omega, g_{\Phi,k}^{\rho,\tilde{\rho}}), \quad a_2 \in AG^m_{\sigma} \{ l_1, l_2, l_3; \delta_1, \delta_2 \}_N^{(1)} (\omega, g_{\Phi,k}^{\rho,\tilde{\rho}}),
\]

for \( N_1 \geq N_2 \), can be readily built by following the similar standard arguments given in [13, Appendix], [12, Appendix II & III] and [1, Appendix].

5 Proof of theorem 3.1

In this section, we give a proof of the main result, Theorem 3.1. There are three key steps in the proof. First, we factorize the operator \( P(t, x, \partial_t, D_x) \). To this end, we begin with modifying the coefficients of the principal part by performing an excision so that the resulting coefficients are regular at \( t = 0 \). Second, we reduce the original Cauchy
problem to a Cauchy problem for a first order system (with respect to $\partial_t$). Lastly, using sharp Gårding’s inequality we arrive at the $L^2$ well−posedness of a related auxiliary Cauchy problem, which gives well−posedness of the original problem in the weighted Sobolev spaces $H^{s,e,\sigma}_{\Phi,k}$.

5.1 Factorization

Consider the operator $a(t, x, D_x)$ defined in (3.1). We modify its symbol $a(t, x, \xi)$ in $Z_{\text{int}}(2)$, by defining

$$\tilde{a}(t, x, \xi) = \varphi(t\Phi(x)|\xi|)\omega(x)^2|\xi|^2 + (1 - \varphi(t\Phi(x)|\xi|))a(t, x, \xi)$$

(5.1)

for $\varphi \in C^\infty(\mathbb{R})$, $0 \leq \varphi \leq 1$, $\varphi = 1$ in $[0, 1]$, $\varphi = 0$ in $[2, +\infty)$. Note that $(a - \tilde{a}) \in AG^{2,2}_\sigma(\{1; p\}_{\text{int}}, 2(\omega, g_{\phi,k})$ and $(a - \tilde{a}) \sim 0$ in $Z_{\text{ext}}(2)$. This implies that $t^p(a - \tilde{a})$ for $t \in [0, T]$ is a bounded and continuous family in $AG^{2,2}_\sigma(\omega, g_{\phi,k})$. Observe that $a - \tilde{a}$ is $L^1$ integrable in $t$, i.e.,

$$\int_0^T |(a - \tilde{a})(t, x, \xi)|dt \leq \kappa_0^2\omega(x)^2|\xi|^2 \int_0^T (2\Phi(x)|\xi|)^{(q-p)/2} \frac{1}{t^p} dt \leq (\Phi(x)|\xi|)^{2-(1-p)/(q-p)}.$$  

(5.2)

as $\omega(x) \lesssim \Phi(x)$.

Let $\tau(t, x, \xi) = \sqrt{a(t, x, \xi)}$. Denote the indicator functions for the regions $Z_{\text{int}}(N_1)$ and $Z_{\text{ext}}(N_2)$ by $\chi_1(N_1)$ and $\chi_2(N_2)$, respectively. It is easy to note that

(i) $\tau(t, x, \xi)$ is $G_{\omega}$-elliptic symbol of order $(1, 1)$ i.e. there is $C > 0$ such that for all $(t, x, \xi) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$ we have

$$|\tau(t, x, \xi)| \geq C\omega(x)|\xi|.$$  

(ii) $\tau \in AG^{1,1}_\sigma(\{0; 1\}_{\text{int}}(\omega, g_{\phi,k}) + AG^{1,1}_\sigma(\{1/2, 1, 0; p, 0\}_{\text{int}}(\omega, g_{\phi,k})$. More precisely, for $|\alpha| + |\beta| > 0$,

$$|\tau(t, x, \xi)| \leq C_0\omega(x)|\xi|\left(\chi_1(1) + \chi_2(1)t^{-p/2}\right),$$

$$|\partial^\alpha D^\beta_x \tau(t, x, \xi)| \leq C_{\alpha\beta}\omega(x)\Phi(x)^{-|\beta|}(|\xi|^{1-|\alpha|}\left(\chi_1(1) + \chi_2(1)t^{-p(|\alpha|+|\beta|)}\right)).$$  

(5.3)

(iii) $\partial_t \tau$ is such that for $|\alpha| + |\beta| > 0$ we have

$$|\partial_t \tau(t, x, \xi)| \sim 0 \text{ in } Z_{\text{int}}(1),$$

$$|\partial_t \tau(t, x, \xi)| \leq C_0\omega(x)|\xi|\left(\chi_1(2)\omega(x)|\xi|t^{-p} + \chi_2(1)t^{-q}\right),$$

$$|\partial^\alpha D^\beta_x \partial_t \tau(t, x, \xi)| \leq C_{\alpha\beta}\omega(x)\Phi(x)^{-|\beta|}(|\xi|^{1-|\alpha|}$$

$$\times \left(\chi_1(2)\omega(x)|\xi|t^{-q} + \chi_2(1)t^{-q}\right)t^{-p(|\alpha|+|\beta|)}.$$
By the definition of the time splitting point and the subdivision of the phase space, we see that
\[
\begin{align*}
|\partial_t \tau(t, x, \xi)| &\sim 0 \text{ in } \mathbb{Z}_{\text{int}}(1), \\
|\partial^\alpha_\xi D^\beta_x \partial_t \tau(t, x, \xi)| &\leq C_{\alpha\beta} \chi_1(2) \omega(x) \Phi(x)^{-|\beta|} \langle \xi \rangle^{1-|\alpha|} t^{-q} t^{-p(|\alpha|+|\beta|)}
\end{align*}
\]
(5.4)
for $|\alpha| + |\beta| \geq 0$. Hence, $\partial_t \tau \sim 0$ in $\mathbb{Z}_{\text{int}}(1)$ and $\partial_t \tau \in AG^{1,1}_\sigma\{0, 1, 1; p, q\}^{(2)}(\omega, g_{\Phi, k})$.

From the above properties of $\tau$ and by the definition of $\tilde{a}$ in (5.1), we have the following two lemmas.

**Lemma 5.1** Let $\tilde{g}_{\Phi, k}^{(1)}$ be as in (1.5) and $\delta$ as in (2.3). Then,

(i) $\tau \in L^\infty([0, T]; AG^{1+\delta/2,1}_\sigma(\omega \Phi^{\delta/2}, \tilde{g}_{\Phi, k}^{(1)}))$,

(ii) $t^{1-\delta} \tau \in C([0, T]; AG^{1,1}_\sigma(\omega, \tilde{g}_{\Phi, k}^{(1)}))$,

(iii) $t^{-1} \tau \in C([0, T]; AG^{-1,-1}_\sigma(\omega, \tilde{g}_{\Phi, k}^{(1)}))$,

(iv) $t^{1-\delta} \partial_t \tau \in C([0, T]; AG^{1+1/\sigma,1}_\sigma(\omega \Phi^{1/\sigma}, \tilde{g}_{\Phi, k}^{(1)}))$.

**Proof** The first claim follows from (5.3) and the observation that in $\mathbb{Z}_{\text{ext}}(1)$

\[
\left(\frac{1}{t}\right)^p \leq \left(\frac{\Phi(x) \langle \xi \rangle^k}{N}\right)^{p/(q-p)},
\]
(5.5)
while the second and third claims are straightforward consequences of (5.3) and (5.5). The fourth claim follows from (5.5) and the following estimate in $\mathbb{Z}_{\text{ext}}(1)$

\[
\frac{1}{t^q} = \frac{1}{t^{1-\delta}} \frac{1}{t^{(q-p)/\sigma}} \leq \frac{1}{t^{1-\delta}} \left(\frac{\Phi(x) \langle \xi \rangle^k}{N}\right)^{1/\sigma}.
\]

□

**Lemma 5.2** Let $\delta$ as in (2.3). Then,

(i) $t^{1-\delta}(a(t, x, D_x) - \tilde{a}(t, x, D_x)) \in C([0, T]; OPAG^{1+1/\sigma,1+1/\sigma}_\sigma(\omega, g_{\Phi, k}))$,

(ii) $\tilde{a}(t, x, D_x) - \tau(t, x, D_x)^2 \in L^\infty([0, T]; OPAG^{1,1}_\sigma(\omega, \tilde{g}_{\Phi, k}^{(1)}))$,

(iii) $t^r b(t, x, D_x) \in C([0, T]; OPAG^{1,1}_\sigma(\omega, g_{\Phi, k})$)
Proof The proof is a consequence of the fact that in $Z_{\text{int}}(2)$

$$|\partial_\alpha^\beta \partial_\xi^\gamma (a - \tilde{a})(t, x, \xi)| \leq C_{\alpha \beta} \chi_1(2) \omega(x)^{2 \Phi(x) - |\beta| |\xi|_k^{2 - |\alpha|}} \frac{1}{t^p}$$

$$\leq C_{\alpha \beta} \chi_1(2) \omega(x)^{1+1/\sigma} \Phi(x) - |\beta| |\xi|_k^{1+1/\sigma - |\alpha|} \frac{1}{t^{1-\delta}} \frac{1}{\xi^{1-\delta}}$$

$$\leq C_{\alpha \beta} \chi_1(2) \omega(x)^{1+1/\sigma} \Phi(x) - |\beta| |\xi|_k^{1+1/\sigma - |\alpha|} \frac{1}{t^{1-\delta}} \frac{1}{\xi^{1-\delta}}.$$

The second and third claims follow directly from the definitions of $\tilde{a}(t, x, D_x)$ and $b(t, x, D_x)$. □

Let us define $\delta^* > 0$ as

$$\delta^* = \min\{\delta, 1 - r, 1 - p\}. \quad (5.6)$$

We are interested in the factorization of the operator $P(t, x, \partial_t, D_x).$ This leads to

$$P = (\partial_t - i \tau(t, x, D_x)) (\partial_t + i \tau(t, x, D_x)) + b_0(t, x) \partial_t + (a - \tilde{a} + a_1)(t, x, D_x)$$

where the operator $a_1(t, x, D_x)$ is such that, for $t \in [0, T]$,

$$a_1 = -i[\partial_t, \tau] + \tilde{a} - \tau^2 + b$$

and $t^{1-\delta^*} a_1(t, x, D_x) \in OPA\mathcal{G}_{\sigma, 1}^{1+1/\sigma, 1}(\omega \Phi^{1/\sigma} \delta_{\Phi,k}).$

### 5.2 First order pseudodifferential system

We will now reduce the operator $P$ to an equivalent first order $2 \times 2$ pseudodifferential system. The procedure is similar to the one used in [4, 13, 15]. To achieve this, we introduce the change of variables $U = U(t, x) = (u_1(t, x), u_2(t, x))^T$, where

$$\begin{cases} 
  u_1(t, x) = (\partial_t + i \tau(t, x, D_x) + b_0(t, x) \partial_t + (a - \tilde{a} + a_1)(t, x, D_x)u(t, x), \\
  u_2(t, x) = \omega(x) \langle D_x \rangle_k u(t, x) - H(t, x, D_x)u_1,
\end{cases} \quad (5.7)$$

and the operator $H$ with the symbol $\sigma(H)(t, x, \xi)$ is such that

$$\sigma(H)(t, x, \xi) = \frac{i}{2} \omega(x) \langle \xi \rangle_k \left(1 - \varphi\left(\frac{t \Phi(x) \langle \xi \rangle_k}{3}\right)\right).$$

Note that by the definition of $H$, $\text{supp } \sigma(H) \cap \text{supp } \sigma(a - \tilde{a}) = \emptyset$ and we have

$$\sigma(2iH(t, x, D_x) \circ \tau(t, x, D_x)) \sim 0, \quad \text{in } Z_{\text{int}}(3),$$

$$\sigma(2iH(t, x, D_x) \circ \tau(t, x, D_x)) = \omega(x) \langle \xi \rangle_k (1 + \sigma(K_1)), \quad \text{in } Z_{\text{ext}}(3),$$

 Springer
where \( \sigma(K_1) \in AG_{\sigma}^{1,-1}[0; \; 1 \times 1; \; p, q]_N^{(2)}(\omega, g_{\Phi,k}) + AG_{\sigma}^{1,1}[0, 1; \; p, 0]_N^{(2)}(\omega, g_{\Phi,k}) \). Then, the equation \( Pu = f \) is equivalent to the first order \( 2 \times 2 \) system:

\[
LU = (\partial_t - D + A_0 + A_1)U = F,
\]

\[
U(0, x) = (f_2 + i\tau(0, x, D_x) f_1, \Phi(x)(D_x) f_1)^T,
\]

(5.8) where

\[
F = (f(t, x), -H(t, x, D_x) f(t, x))^T,
\]

\[
D = \text{diag}(i\tau(t, x, D_x), -i\tau(t, x, D_x)),
\]

\[
A_0 = \begin{pmatrix}
B_0 H & B_0 \\
-H B_0 H & H B_0
\end{pmatrix} = \begin{pmatrix}
\mathcal{R}_1 & B_0 \\
-B_3 & \mathcal{R}_2
\end{pmatrix},
\]

\[
A_1 = \begin{pmatrix}
B_1 H + B_3 & B_1 + B_4 \\
B_2 - H B_3 & i[M, \tau] M^{-1} - H(B_1 + B_3)
\end{pmatrix}.
\]

The operators \( M, M^{-1}, B_0, B_1 \) and \( B_2 \) are as follows

\[
M = \omega(x)\langle D_x \rangle_k, \quad M^{-1} = \langle D_x \rangle_k^{-1} \omega(x)^{-1},
\]

\[
B_0 = (a(t, x, D_x) - \tilde{a}(t, x, D_x))\langle D_x \rangle_k^{-1} \omega(x)^{-1},
\]

\[
B_1 = \begin{pmatrix}
-i \partial_t \tau(t, x, D_x) + \tilde{a}(t, x, D_x) - \tau(t, x, D_x)^2 + b(t, x, D_x)\langle D_x \rangle_k^{-1} \omega(x)^{-1},
\end{pmatrix}
\]

\[
B_2 = 2iH \tau - M + i[M, \tau] M^{-1} H + i[\tau, H] - H B_1 H + \partial_t H,
\]

\[
B_3 = b_0(1 - i\lambda M^{-1} H), \quad B_4 = i b_0 \lambda M^{-1}.
\]

Here \( b_0 = b_0(t, x) \) is as in (3.1). By the definition of operator \( H \), we have \( B_0 H = \mathcal{R}_1, H B_0 = \mathcal{R}_2, H B_0 H = \mathcal{R}_3 \) for \( \mathcal{R}_j \in G^{\infty, \infty}(\omega, g_{\Phi,k}), \; j = 1, 2, 3 \), and the operator \( 2iH \tau - M \) is such that

\[
\sigma(2iH \tau - M) = \begin{cases}
-\omega(x)\langle \xi \rangle_k, & \text{in } Z_{int}(3), \\
\omega(x)\langle \xi \rangle_k \sigma(K_1), & \text{in } Z_{ext}(3).
\end{cases}
\]

Since \( 2p \leq q \), we have

\[
AG_{\sigma}^{1,0}[0, 1; \; p, q]_N^{(2)}(\omega, g_{\Phi,k}) \subset AG_{\sigma}^{0,0}[1, 1; \; p, q]_N^{(2)}(\omega, g_{\Phi,k}).
\]

The symbols of operators \( D, A_0 \) and \( A_1 \) are in the following symbol classes

\[
\sigma(D) \in \left[ AG_{\sigma}^{1,1}[0; \; 0]_N^{(1)}(\omega, g_{\Phi,k}) + AG_{\sigma}^{1,1}[1, 1; \; p, 0]_N^{(2)}(\omega, g_{\Phi,k}) \right],
\]

\[
\sigma(A_0) \in \left[ AG_{\sigma}^{1,1}[1; \; p]_N^{(1)}(\omega, g_{\Phi,k}) + AG_{\sigma}^{\infty, \infty}[0, 0; \; 0, 0]_N^{(2)}(\omega, g_{\Phi,k}) \right],
\]

\[
\sigma(A_1) \in \left[ AG_{\sigma}^{1,1}[0; \; 0]_N^{(1)}(\omega, g_{\Phi,k}) + AG_{\sigma}^{0,0}[1; \; r]_N^{(1)}(\omega, g_{\Phi,k}) \right] + AG_{\sigma}^{0,0}[0, 1; \; p, q]_N^{(2)}(\omega, g_{\Phi,k})
\]

(5.9)
and thus, by Lemmas 5.1 - 5.2 and the choice of \( \delta^* \) as in (5.6),
\[
\begin{align*}
&\quad t^{1-\delta^*} \sigma(A_0(t)) \in C \left( [0, T]; AG_{\sigma}^{1/\sigma, 1/\sigma} (\omega, g_{\Phi,k}) \right), \\
&\quad t^{1-\delta^*} \sigma(A_1(t)) \in C \left( [0, T]; AG_{\sigma}^{1/\sigma, 1/\sigma} (\omega, \bar{g}_{\Phi,k}) \right).
\end{align*}
\] (5.10)

As \( g_{\Phi,k} \leq \bar{g}_{\Phi,k}^{(1)} \leq \bar{g}_{\Phi,k}^{(2)} \), from (5.10) we have
\[
\begin{align*}
&\quad t^{1-\delta^*} \sigma(A_0(t)), \ t^{1-\delta^*} \sigma(A_1(t)) \in C \left( [0, T]; AG_{\sigma}^{1/\sigma, 1/\sigma} (\omega, \bar{g}_{\Phi,k}^{(2)}) \right).
\end{align*}
\] (5.11)

Let us choose \( \lambda > 0 \) as large as possible so that
\[
|\sigma(A_0(t))| + |\sigma(A_1(t))| \leq \frac{\lambda}{t^{1-\delta^*}} (\Phi(x) \langle \xi \rangle_k)^{1/\sigma}.
\] (5.12)

\[ 5.3 \text{ Energy estimate} \]

In this section, we prove the estimate (3.3). Note that it is sufficient to consider the case \( s = (0, 0) \) as the operator \( \Phi(x)s_2(D)^s_1 L(D)^{-s_1} \Phi(x)^{-s_2} \), where \( s = (s_1, s_2) \) is the index of the weighted Sobolev space, satisfies the same hypotheses as \( L \).

In the following, we establish some lower bounds for the operator \( D - A_0 - A_1 \). The symbol \( d(t, x, \xi) \) of the operator \( D(t) + D^*(t) \) is such that
\[
d \in AG_{\sigma}^{0,0} \{0; 0\}^{(1)} (\omega, g_{\Phi,k}) + AG_{\sigma}^{0,0} \{1/2, 1, 0; p, q\}^{(2)} (\omega, g_{\Phi,k}).
\]

It follows from the definition of \( \delta^* \) and Lemma 5.1 that
\[
\quad t^{1-\delta^*} d \in C([0, T]; AG_{\sigma}^{0,0} (\omega, \bar{g}_{\Phi,k}^{(1)})).
\]

Thus
\[
2 \text{Re} \langle DU, U \rangle_{L^2} \geq -\frac{C_1}{t^{1-\delta^*}} \langle U, U \rangle_{L^2}, \quad C_1 > 0.
\] (5.13)

To control lower order terms, we make the following change of variable
\[
V(t, x) = e^{\Lambda(t) \Theta(x, D_x)} U(t, x),
\] (5.14)

where \( \Lambda(t) = \frac{\lambda}{2\pi} (T^{\delta^*} - t^{\delta^*}) \) with \( \lambda \) as in (5.12), and the operator \( \Theta(x, D_x) \) is as in (1.4). From [11, Corollary 4.0.4],
\[
e^{\pm \Lambda(t) \Theta(x, D_x)} e^{\mp \Lambda(t) \Theta(x, D_x)} = I + R^{(\pm)}(t, x, D_x),
\]
where for sufficiently large \( k \) the operators \( I + R^{(\pm)}(t, x, D_x) \) are invertible. Let us denote the operators \( I + R^{(+)}(t, x, D_x) \), \( I + R^{(-)}(t, x, D_x) \) and \( e^{\pm \Lambda(t) \Theta(x, D_x)} \) by
Global well–posedness of a class of singular hyperbolic Cauchy problems

Observe that

\[ t = e^{-\Lambda(t)} \Theta(x, D_x) V(t, x) \]

Then, we have

\[ \tilde{\sigma} \]

Theorem 18.6.14] for the metric

\[ \tilde{\omega}, \tilde{g}_{\Phi, k} \]

Choosing \( \lambda \) sufficiently large, we obtain

\[ \text{Re} \left( \left( \frac{\lambda}{t^{1-\delta}} \Theta(x, D_x) + B \right) V, V \right) \geq -C_2 \| V \|_{L^2}, \quad C_2 > 0. \] (5.15)

The above estimate is the result of application of sharp Gårding inequality, see [8, Theorem 18.6.14] for the metric \( \tilde{g}_{\Phi, k}^{(2)} \) with the Planck function \( \Phi(x) \langle \xi \rangle_k \frac{1}{\pi} \). It is important to note that the application of sharp Gårding inequality requires \( \sigma \geq 3 \).

The estimate (3.3) on the solution \( u \) can be established by proving that the function \( V(t, x) \) satisfies the a priori estimate

\[ \| V(t) \|_{L^2}^2 \leq C \left( \| V(0) \|_{L^2}^2 + \int_0^t \| F_1(\tau, \cdot) \|_{L^2}^2 d\tau \right), \quad t \in [0, T], C > 0. \] (5.16)

The Cauchy problem for the operator \( L_1 \) is given by

\[
\begin{aligned}
\partial_t V(t, x) &= \left( D - \frac{\lambda}{t^{1-\delta}} \Theta(x, D_x) - B \right) V(t, x) + F_1(t, x), \\
V(0, x) &= e^{\Lambda(0) \Theta(x, D_x)} U(0, x)
\end{aligned}
\] (5.17)

Observe that

\[
\begin{aligned}
\partial_t \| V(t) \|_{L^2}^2 &= 2 \text{Re} \langle \partial_t V, V \rangle_{L^2} \\
&= 2 \text{Re} \langle D V, V \rangle_{L^2} - 2 \text{Re} \left( \left( \frac{\lambda}{t^{1-\delta}} (\Phi(x) \langle D_x \rangle_k)^{1/\sigma} + B \right) V, V \right)_{L^2} \\
&\quad + 2 \text{Re} \langle F_1, V \rangle_{L^2}.
\end{aligned}
\]
From (5.13) and (5.15) we have
\[
\frac{d}{dt} \| V(t) \|_{L^2}^2 \leq \frac{C}{t^{1-\delta^*}} \| V(t) \|_{L^2} + C \| F_1(t, \cdot) \|_{L^2}.
\]
We apply Gronwall’s lemma and obtain that
\[
\| V(t) \|_{L^2}^2 \leq C' e^{\frac{T \delta^*}{\delta^*}} \left( \| V(0) \|_{L^2}^2 + \int_0^t \| F_1(\tau, \cdot) \|_{L^2}^2 d\tau \right).
\]
This proves the well–posedness of the Cauchy problem (5.17). Note that the solution \( U \) to (5.8) belongs to \( C \left( [0, T] ; H^{s, \Lambda(t), \sigma} \right) \). Returning to our original solution \( u = u(t, x) \) we obtain the estimate (3.3) with
\[
u \in C \left( [0, T] ; H^{s+e, \Lambda(t), \sigma} \right) \cap C^1 \left( [0, T] ; H^{s, \Lambda(t) \sigma} \right).
\]
This concludes the proof.

6 Cone condition

Existence and uniqueness follow from the a priori estimate established in the previous section. It now remains to prove the existence of cone of dependence.

We note here that the \( L^1 \) integrability of the characteristics plays a crucial role in arriving at the finite propagation speed. The implications of the discussion in [16, Sections 2.3 & 2.5] to the global setting suggest that if the Cauchy data in (3.1) are such that \( f \equiv 0 \) and \( f_1, f_2 \) are supported in the ball \( |x| \leq R \), then the solution to Cauchy problem (3.1) is supported in the ball \( |x| \leq R + c^* \omega(x)^{1-\frac{p}{2}} \). The constant \( c^* \) is such that the quantity \( c^* \omega(x)^{1-\frac{p}{2}} \) dominates the characteristic roots, i.e.,
\[
c^* = \sup \left\{ \sqrt{a(t, x, \xi)} \omega(x)^{-1} t^\frac{p}{2} : (t, x, \xi) \in [0, T] \times \mathbb{R}_x^n \times \mathbb{R}_\xi^n, |\xi| = 1 \right\}.
\]
Note that the support of the solution increases as \( |x| \) increases since \( \omega(x) \) is a monotone increasing function of \( |x| \).

In the following we prove the cone condition for the Cauchy problem (3.1). Let \( K(x^0, t^0) \) denote the cone with the vertex \( (x^0, t^0) \):
\[
K(x^0, t^0) = \{ (t, x) \in [0, T] \times \mathbb{R}^n : |x - x^0| \leq c^* \omega(x)(t^0 - t)^{1-\frac{p}{2}} \}.
\]
Observe that the slope of the cone is anisotropic, that is, it varies with both \( x \) and \( t \).

**Proposition 6.1** The Cauchy problem (3.1) has a cone dependence, that is, if
\[
f \big|_{K(x^0, t^0)} = 0, \quad f_i \big|_{K(x^0, t^0) \cap \{ t = 0 \}} = 0, \quad i = 1, 2,
\]
\( \square \) Springer
then
\[
\left. u \right|_{K(x^0, t^0)} = 0. 
\] (6.3)

**Proof** Consider \( t^0 > 0, \ C^* > 0 \) and assume that (6.2) holds. We define a set of operators \( P_\varepsilon(t, x, \partial_t, D_x), \ 0 \leq \varepsilon \leq \varepsilon_0 \) by means of the operator \( P(t, x, \partial_t, D_x) \) in (3.1) as follows
\[
P_\varepsilon(t, x, \partial_t, D_x) = P(t + \varepsilon, x, \partial_t, D_x), \ t \in [0, T - \varepsilon_0], \ x \in \mathbb{R}^n,
\]
and \( \varepsilon_0 < T - t^0 \), for a fixed and sufficiently small \( \varepsilon_0 \). For these operators we consider Cauchy problems
\[
P_\varepsilon v_\varepsilon = f, \quad t \in [0, T - \varepsilon_0], \ x \in \mathbb{R}^n,
\]
\[
\partial_t^k v_\varepsilon(0, x) = f_k(x), \quad k = 1, 2.
\]

Note that \( v_\varepsilon(t, x) = 0 \) in \( K(x^0, t^0) \) and \( v_\varepsilon \) satisfies an a priori estimate (3.3) for all \( t \in [0, T - \varepsilon_0] \). Further, we have
\[
P_{\varepsilon_1}(v_{\varepsilon_1} - v_{\varepsilon_2}) = (P_{\varepsilon_2} - P_{\varepsilon_1})v_{\varepsilon_2}, \quad t \in [0, T - \varepsilon_0], \ x \in \mathbb{R}^n,
\]
\[
\partial_t^k(v_{\varepsilon_1} - v_{\varepsilon_2})(0, x) = 0, \quad k = 1, 2.
\]

Since our operator is of second order, for the sake of simplicity we denote \( b_j(t, x) \), the coefficients of lower order terms, as \( a_{0,j}(t, x), 1 \leq j \leq n \), while \( b_0(t, x) \) and \( b_{n+1}(t, x) \) are denoted as \( a_{1,0}(t, x) \) and \( a_{0,0}(t, x) \), respectively. Let \( a_{i,0}(t, x) = 0, \ 2 \leq i \leq n \). Substituting \( s - \varepsilon \) for \( s \) in the a priori estimate, we obtain
\[
\sum_{j=0}^1 \left\| \partial_t^j(v_{\varepsilon_1} - v_{\varepsilon_2})(t, \cdot) \right\|_{\Phi, k; s - \varepsilon, \Lambda(t), \sigma} \leq C \int_0^t \left\| (P_{\varepsilon_2} - P_{\varepsilon_1})v_{\varepsilon_2}(\tau, \cdot) \right\|_{\Phi, k; s - \varepsilon, \Lambda(\tau), \sigma} d\tau
\]
\[
\leq C \int_0^t \sum_{i,j=0}^n \left\| (a_{i,j}(\tau + \varepsilon_1, x) - a_{i,j}(\tau + \varepsilon_2, x))D_{ij}v_{\varepsilon_2}(\tau, \cdot) \right\|_{\Phi, k; s - \varepsilon, \Lambda(\tau), \sigma} d\tau,
\] (6.4)

where \( D_{00} = I, D_{10} = \partial_t, D_{i0} = 0, i \geq 2, D_{0j} = \partial_{x_j}, j \neq 0 \) and \( D_{ij} = \partial_{x_i}\partial_{x_j}, i, j \neq 0 \). Using the Taylor series approximation in \( \tau \) variable, we have
\[
|a_{i,j}(\tau + \varepsilon_1, x) - a_{i,j}(\tau + \varepsilon_2, x)| = \left| \int_{\tau+\varepsilon_2}^{\tau+\varepsilon_1} (\partial_t a_{i,j})(r, x)dr \right|
\]
\[
\leq \omega(x)^2 \left| \int_{\tau+\varepsilon_2}^{\tau+\varepsilon_1} \frac{dr}{r^q} \right|
\]
\[
\leq \omega(x)^2 |E(\tau, \varepsilon_1, \varepsilon_2)|,
\]
where
\[ E(\tau, \varepsilon_1, \varepsilon_2) = \frac{1}{q-1} \left( (\tau + \varepsilon_1)^{-q+1} - (\tau + \varepsilon_2)^{-q+1} \right). \]

Note that \( \omega(x) \lesssim \Phi(x) \) and \( E(\tau, \varepsilon, \varepsilon) = 0 \). Then right-hand side of the inequality in (6.4) is dominated by
\[ C \int_0^t |E(\tau, \varepsilon_1, \varepsilon_2)\|v_{\varepsilon_2}^{(\tau, \cdot)}\|_{\Phi, k; \varepsilon+\varepsilon, \Lambda(\tau), \sigma} d\tau, \]
where \( C \) is independent of \( \varepsilon \). By definition, \( E \) is \( L_1 \)-integrable in \( \tau \).

The sequence \( v_{\varepsilon_k}, k = 1, 2, \ldots \) corresponding to the sequence \( \varepsilon_k \to 0 \) is in the space
\[ C \left( [0, T^*]; H_{\Phi, k}^{\varepsilon, \Lambda(t), \sigma} \right) \cap C^1 \left( [0, T^*]; H_{\Phi, k}^{\varepsilon-\varepsilon, \Lambda(t), \sigma} \right), \quad T^* > 0, \]
and \( u = \lim_{k \to \infty} v_{\varepsilon_k} \) in the above space and hence, in \( D'(K(x^0, t^0)) \). In particular,
\[ \langle u, \varphi \rangle = \lim_{k \to \infty} \langle v_{\varepsilon_k}, \varphi \rangle = 0, \quad \forall \varphi \in D(K(x^0, t^0)) \]
gives (6.3) and completes the theorem. \( \square \)

7 Counterexamples

In this section, we show by a set of counterexamples how the lower order terms of operator \( P \) in (3.1) influence the loss of regularity and well–posedness when \( L_1 \) integrability condition is violated. We cover various cases such as no loss, finite loss and nonuniqueness. All the examples in this section correspond to the case \( \omega(x) = \Phi(x) = 1, \ x \in \mathbb{R} \) in (1.2).

Following example shows that one can encounter finite loss when the coefficients of lower order terms are not \( L_1 \) integrable.

**Example 7.1**
\[
\begin{aligned}
\left\{ \begin{array}{l}
\left( \partial_t^2 - \partial_x^2 + \frac{1}{2t} (\partial_t - (4m + 1)\partial_x) \right) u(t, x) = 0, \\
u(0, x) = u_0(x), \quad \partial_t u(0, x) = (4m + 1)\partial_x u_0(x),
\end{array} \right.
\end{aligned}
\]
(7.1)

for some \( m \in \mathbb{N}_0 \). Note that the above problem corresponds to the case \( p = q = 0 \) and \( r = 1 \) in (3.2). The solution to the above Cauchy problem is given by
\[ u(t, x) = \sum_{j=0}^m C_j^{(m)} t^j \partial_x^j u_0(x + t), \]
for $C^{(m)}_j$ of the form

$$C_0 = 1, \quad C^{(m)}_j = \frac{(-2)^j}{j!} \frac{(m)_j}{(-\frac{1}{2})_j}, \quad j \geq 1,$$

where $(y)_j, y \in \mathbb{R}$, is the $j^{th}$ falling factorial of $y$ [18] given by

$$(y)_j = y(y-1) \cdots (y-j+1).$$

For example, when $m = 2$, $u(t, x)$ is of the form

$$u(t, x) = u_0(x + t) + 8t \partial_x u_0(x + t) + \frac{16}{3} t^2 \partial_x^2 u_0(x + t).$$

We observe loss of derivatives: if $u_0 \in H^s$, then $u(t, \cdot) \in H^{s-m}$ where $H^s$ is the usual Sobolev space with $s \in \mathbb{R}$.

From Theorem 3.1 we see that $L^1$ integrability condition on lower order terms is sufficient to ensure that they do not influence the well–posedness and loss of regularity of the solution. The following example shows that the condition is not necessary.

**Example 7.2**

$$\left\{ \begin{array}{l}
\left( \partial_t^2 - \partial_x^2 - \frac{2}{t} \partial_x \right) u(t, x) = 0 \\
u(0, x) = 0, \quad \partial_t u(0, x) = u_0(x).
\end{array} \right. \quad (7.2)$$

This corresponds to the case $p = q = 0$ and $r = 1$ in (1.2). The solution to the above Cauchy problem is given by

$$u(t, x) = tu_0(x + t).$$

The following example demonstrates that when the coefficient of the top order term is oscillatory but in $C^1((0, T]) \cap W^{1,1}((0, T])$ and that of the lower order term is in $C^1((0, T]) \cap L^1((0, T])$, one may have no loss.

**Example 7.3**

$$\left\{ \begin{array}{l}
\left( \partial_t^2 - (2 + \sin \sqrt{t})^2 \partial_x^2 - \frac{\cos \sqrt{t}}{2 \sqrt{t}} \partial_x \right) u(t, x) = 0 \\
u(0, x) = u_0(x), \quad \partial_t u(0, x) = 2\partial_x u_0(x).
\end{array} \right. \quad (7.3)$$

This corresponds to the case $p = 0, q = \frac{1}{2}$ and $r = \frac{1}{2}$ in (1.2). The solution to the above Cauchy problem is given by

$$u(t, x) = u_0 \left( x + \int_0^t (2 + \sin \sqrt{s}) ds \right).$$
The following example demonstrates that one may encounter nonuniqueness when the lower order terms are not $L^1$ integrable.

**Example 7.4**

\[
\begin{cases}
\left( \partial_t^2 - \partial_x^2 - \frac{1}{t} (\partial_t + 3\partial_x) \right) u(t, x) = 0 \\
u(0, x) = 0, \quad \partial_t u(0, x) = 0.
\end{cases}
\]  
(7.4)

This corresponds to the case $p = q = 0$ and $r = 1$ in (1.2). The solution to the above Cauchy problem is given by

\[u(t, x) = t^2 u_0(x + t),\]

for any function $u_0(x)$.

**Remark 7.1** Observe that the form of solutions to the Cauchy problems in all the above examples suggests that the Cauchy data only propagates along one characteristic i.e., they are all one-way waves [2].

**Acknowledgements** The first author is funded by the University Grants Commission, Government of India, under its JRF and SRF schemes.

**References**

1. Ascanelli, A., Cappiello, M.: Hölder continuity in time for SG hyperbolic systems. J. Differ. Equ. **244**, 2091–2121 (2008)
2. Bschorr, O., Raida, H.-J.: Factorized One-Way Wave Equations. Acoustics **3**, 717–722 (2021)
3. Carroll, R.W., Showalter, R.E.: Singular and Degenerate Cauchy Problems. Mathematics in Science and Engineering, vol. 127. [Harcourt Brace Jovanovich, Publishers]. Academic Press, New York (1976)
4. Cicognani, M.: The Cauchy problem for strictly hyperbolic operators with non-absolutely continuous coefficients. Tsukuba J. Math. **27**, 1–12 (2003)
5. Colombini, F., Del Santo, D., Kinoshita, T.: Well-posedness of the Cauchy problem for a hyperbolic equation with non-Lipschitz coefficients. Ann. Scuola. Norm.-Sci. Ser. **5** 1, 327–358 (2002)
6. Galstian, A., Yagdjian, K.: Microlocal Analysis for Waves Propagating in Einstein & de Sitter Space-time. Math. Phys. Anal. Geom. **17**, 223–246 (2014)
7. Hirosawa, F.: Loss of regularity for the solutions to hyperbolic equations with non-regular coefficients-an application to Kirchhoff equation. Mathematical Methods in the Applied Sciences **26**, 783–799 (2003)
8. Hörmander, L.: The Analysis of Linear Partial Differential Operators III. Springer-Verlag, Heidelberg, Berlin (1985)
9. Lerner, N.: Metrics on the Phase Space and Non-Selfadjoint Pseudo-Differential Operators. Birkhäuser Basel, Switzerland (2010)
10. Nicola, F., Rodino, L.: Global Pseudo-differential Calculus on Euclidean Spaces. Birkhäuser Basel, Switzerland (2011)
11. Pattar, R.R., Uday Kiran, N.: Global well-posedness of a class of strictly hyperbolic Cauchy problems with coefficients non-absolutely continuous in time. Bulletin des Sciences Mathématiques **171**, 103037 (2021)
12. Pattar, R.R., Uday Kiran, N.: Global well-posedness of a class of strictly hyperbolic Cauchy problems with coefficients non-absolutely continuous in time. **arXiv:2007.07153v3**. (Extended version of [11] with Appendix II & III) (2021)
13. Pattar, R.R., Kiran, N.U.: Strictly hyperbolic Cauchy problems on $\mathbb{R}^n$ with unbounded and singular coefficients. Annali dell’Università di Ferrara **68**, 11–45 (2022)
14. Pattar, R.R., Kiran, N.U.: Energy estimates and global well-posedness for a broad class of strictly
hyperbolic Cauchy problems with coefficients singular in time 13, 9 (2022)
15. Pattar, R.R., Kiran, N.U.: Strictly hyperbolic equations with coefficients sublogarithmic in time,
arXiv:2111.11701 (2021)
16. Rauch, J.: Hyperbolic Partial Differential Equations and Geometric Optics. Graduate studies in math-
ematics. American Mathematical Society, Rhode Island, USA (2012)
17. Yagdjian, K.: The Cauchy Problem for Hyperbolic Operators: Multiple Characteristics. Wiley, Micro-
Local Approach (1997)
18. https://mathworld.wolfram.com/FallingFactorial.html

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps
and institutional affiliations.