Cones and causal structures 
on topological and differentiable manifolds

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Abstract

General definitions for causal structures on manifolds of dimension $d + 1 > 2$ are presented for
the topological category and for any differentiable one.

Locally, these are given as cone structures via local (pointwise) homeomorphic or diffeomorphic
abstraction from the standard null cone variety in $\mathbb{R}^{d+1}$. Weak ($\mathcal{C}$) and strong ($\mathcal{C}^m$) local cone (LC)
structures refer to the cone itself or a manifold thickening of the cone respectively.

After introducing cone (C-)causality, a causal complement with reasonable duality properties
can be defined. The most common causal concepts of space-times are generalized to the present
topological setting. A new notion of precausality precludes inner boundaries within future/past cones.

LC-structures, C-causality, a topological causal complement, and precausality may be useful tools
in conformal and background independent formulations of (algebraic) quantum field theory and quantum gravity.

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I. Introduction

While classical general relativity usually employs a Lorentzian space-time metric, all genuine approaches to quantum gravity are free of such a metric background. This poses the question whether there still exists a notion of structure which captures some essential features of light cones and their mutual relations in manifolds in a purely topological manner without a priori recursion to a Lorentzian metric or a conformal class of such metrics. Below we will see that the answer is positive.

It is a well known folk theorem that the causal structure on a Lorentzian manifold determines its metric up to conformal transformations. In [1, 2] a path topology for strongly causal space-times was defined which then determined their differential, causal, and conformal structure. In [3] it was shown that the conformal class of a Lorentzian metric can be reconstructed from the characteristic surfaces of the manifold. Similarly [4] gives a nice proof that the null cones determine the Lorentzian metric (modulo global sign) up to a conformal factor. All these previous results already indicate that the notion of a causal structure could exist indeed in a different and possibly more general setting than that of Lorentzian space-times. However all the previously mentioned investigations in the literature assume a priori the existence of some undetermined Lorentzian metric and then show that it can be determined modulo conformal transformation uniquely by some other structure.

Motivated by the requirements on suitable structures for a theory of quantum gravity, in this paper new notions of causal structure are developed which do not assume a priori existence of any (Lorentzian) metric or conformal metric but rather work on arbitrary topological and differential manifolds.

In Section II weak (C) and strong (C\textsuperscript{m}) local cone (LC) structures are defined on any topological (or differentiable) manifold \(M\). These structures are given by continuous (or differentiable) families of pointwise homeomorphisms from the standard null cone variety in \(\mathbb{R}^{d+1}\) or a manifold thickening thereof respectively into \(M\). In the differentiable case it turns out that a strong LC structure implies the existence of a conformal Lorentzian metric, while a weak LC structure already implies its uniqueness should one exist. However the metric resulting from a strong LC structure contains only pointwise information about the asymptotic structure of the cone at the vertex. Within a given manifold thickening of the cone at a given point of \(M\), the cone in any neighborhood of the vertex need a priori not at all be related to the null structure spanned by the null geodesics of this metric. However, if such a relationship holds in some region, then all the cones in that region are consistent with each other and this way yield a notion of causality.

Section III provides precisely those definitions of causality which allow to formulate the consistency of different strength for cones at different points, in some or any open region in \(M\). Cone (C-) causality allows first of all the definition of a causal complement with reasonable properties. It enables us also to define in a topological (differentiable) manner spacelike, null, and timelike curves. We discuss C-causality also in the particular context of a fibration. Generalizations of the most common causality notions for space-times in purely topological terms are provided. In the case of Lorentzian manifolds these notions agree with the usual ones and they assume their usual hierarchy. Finally, precausality is defined as a notion which makes the future and the past of any cone homeomorphic to the future and the past of the standard cone \(\mathcal{C}\) in \(\mathbb{R}^{d+1}\) respectively.

The discussion points out some of the major open issues which require further investigation. It addresses also the issues of causal diffeomorphisms, foliations, and possible restrictions of the cone.
structure and causality from the manifold to an embedded graph therein, giving also a perspective for possible applications in conformal and background independent quantum field theories and quantum gravity.

Here and below a CAT manifold refers to a Hausdorff \((T_2)\) space with CAT structure, where CAT= \(C^0\) (the topological category) or CAT\(\subset C^1\) (any differentiable category). If CAT\(\subset C^1\), a CAT homeomorphism is a diffeomorphism and a CAT continuous map is a differentiable map. For differentiable categories we also define CAT\(_{-1}\) := \(C^r\) if CAT= \(C^{r+1}\), CAT\(_{-1}\) := \(C^\infty\) if CAT= \(C^{\infty}\), and CAT\(_{-1}\) := \(C^\omega\) if CAT= \(C^\omega\).

II. Local cone (LC) structures of manifolds

In this section we derive local notions of a cone structure on a topological \(d+1\)-dimensional manifold \(M\) (CAT\(\subset C^0\)). Let

\[
\mathcal{C} := \{x \in \mathbb{R}^{d+1} : x_0^2 = (x - x_0 e_0)^2\}, \mathcal{C}^+ := \{x \in \mathcal{C} : x_0 \geq 0\}, \mathcal{C}^- := \{x \in \mathcal{C} : x_0 \leq 0\}
\]

be the standard (unbounded double) light cone, and the forward and backward subcones in \(\mathbb{R}^{d+1}\), respectively.

The standard open interior and exterior of \(\mathcal{C}\) is defined as

\[
\mathcal{F} := \{x \in \mathbb{R}^{d+1} : x_0^2 > (x - x_0 e_0)^2\}, \mathcal{C} := \{x \in \mathbb{R}^{d+1} : x_0^2 < (x - x_0 e_0)^2\}.
\]

A manifold thickening with thickness \(m > 0\) is given as

\[
\mathcal{C}^m := \{x \in \mathbb{R}^{d+1} : |x_0^2 - (x - x_0 e_0)^2| < m^2\},
\]

The characteristic topological data of the standard cone is encoded in the topological relations of all its manifold subspaces (which includes in particular also the singular vertex \(O\)) and among each other.

Typical (CAT) manifold subspaces of \(\mathcal{C}\) are the standard future and past cones \(\mathcal{C}^\pm\), and the standard light rays

\[
l(n) := \{x \in \mathcal{C} : x_0 = (x, n)\},
\]

where \(n \in S^{d-1} \subset p\) is a normal direction in the \(d\)-dimensional hyperplane \(p := \{x \in \mathbb{R}^{d+1} : (x, y) = 0 \ \forall y \in a\}\) perpendicular to the cone axis \(a := \{x \in \mathbb{R}^{d+1} : x = \lambda e_0, \lambda \in \mathbb{R}\}\).

The topological relations between all the CAT manifold subspaces of the cone are the natural data which will be required to be conserved under a homeomorphism of the cone as a topological space into the manifold \(M\) at any point \(p\).

Let \(\tau\) denote the closed sets of the manifold topology of \(\mathcal{C} - O\). The set \(\mathcal{C}\) can either inherit the induced topology \(\tau_1\) from \(\mathbb{R}^{d+1}\) which is \(T_1\) but not \(T_2\) (Hausdorff) or it can be equipped with a more coarse subtopology defined in terms of closed sets as \(\tau_2 := \{\{0\} \cup V : V \in \tau\} \cup \{V \in \tau\}\) which is Hausdorff. However \(\tau_2\) places geometrically unnatural restrictions on possible submanifolds of \(\mathcal{C}\). Hence, unless specified otherwise, \(\mathcal{C}\) will be equipped with \(\tau_1\).

**Definition 1:** Let \(M\) be a CAT manifold. A (CAT) (null) cone at \(p \in \text{int}M\) is the homeomorphic image \(\mathcal{C}_p := \phi_p\mathcal{C}\) of a homeomorphism of topological spaces \(\phi_p : \mathcal{C} \rightarrow \mathcal{C}_p \subset M\) with \(\phi_p(0) = p\), such
that
(i) every (CAT) submanifold \( N \subset \mathcal{C} \) is mapped (CAT) homeomorphically on a submanifold \( \phi_p(N) \subset M \),

(ii) for any two submanifolds \( N_1, N_2 \subset \mathcal{C} \) there exist homeomorphisms \( \phi_p(N_1) \cap \phi_p(N_2) \cong N_1 \cap N_2 \) and \( \phi_p(N_1) \cup \phi_p(N_2) \cong N_1 \cup N_2 \) of (CAT) manifolds if these are (CAT) manifolds and of topological spaces otherwise, and

(iii) if \( \text{CAT} \subset C^1 \) then for any two CAT curves \( c_1, c_2 : ] - \epsilon, \epsilon [ \to \mathcal{C} \) with \( c_1(0) = c_2(0) = p \) it holds
\[ T_0 c_1 = T_0 c_2 \Rightarrow T_p(\phi_p \circ c_1)]_{-\epsilon,\epsilon} = T_p(\phi_p \circ c_2)]_{-\epsilon,\epsilon}. \]

Condition (iii) says that in the differentiable case the well defined notion of transversality of intersections at the vertex is preserved by \( \phi_p \).

On each homeomorphic cone \( \mathcal{C}_p \) at any \( p \in \text{int} M \), the topology \( \tau_1 \) or \( \tau_2 \) of \( \mathcal{C} \) yields under \( \phi_p \) likewise a non-Hausdorff \( T_1 \) topology \( \phi_p(\tau_1) \) or a \( T_2 \) one \( \phi_p(\tau_2) \). However, \( \phi_p \circ \tau_2 \) would unnaturally restrict the possible submanifolds of \( \mathcal{C} \), while \( \phi_p \circ \tau_1 \) is consistent with the topology induced from \( M \).

**Definition 2:** An (ultraceak) cone structure on \( M \) is an assignment \( \text{int} M \ni p \mapsto \mathcal{C}_p \) of a cone at every \( p \in \text{int} M \).

A cone structure on \( M \) can in general be rather wild with cones at different points totally unrelated unless we impose a topological connection between the cones at different points. Most naturally the connection is provided by continuity of the family \( \{ \mathcal{C}_p \} \). This allows to define a local cone (LC) structures.

**Definition 3:** Let \( M \) be a CAT manifold. A weak (\( \mathcal{C} \)) local cone (LC) structure on \( M \) is a cone structure which is (CAT) continuous i.e. \( \{ p \mapsto \mathcal{C}_p \} \) is a (CAT) continuous family.

Given a cone structure one wants to know first of all under which conditions, for given \( p \in \text{int} M \) an exterior and interior of the cone can be distinguished locally, i.e. for any open neighborhood \( U \ni p \) within \( (M - \mathcal{C}_p) \cap U \).

**Proposition 1:** Let \( \forall p \in \text{int} M \) exist open (CAT) submanifolds \( \mathcal{T}_p \) and \( \mathcal{E}_p \) such that the interior of \( M \) decomposes in the disjoint union \( \text{int} M = \mathcal{C}_p \cup \mathcal{T}_p \cup \mathcal{E}_p \).

\( i \) Then \( \mathcal{T}_p \) and \( \mathcal{E}_p \) can be topologically distinguished locally in any neighborhood of the vertex \( p \) if and only if for any neighborhood \( U \ni p \) it holds \( (\mathcal{T}_p|_U) \not\cong (\mathcal{E}_p|_U) \).

\( ii \) Given any neighborhood \( U \ni p \) assume \( \exists k \in \mathbb{N}_0 : \Pi_k(\mathcal{T}_p|_U) \not\cong \Pi_k(\mathcal{E}_p|_U) \). Then \( \mathcal{T}_p \) and \( \mathcal{E}_p \) can be topologically distinguished locally in any neighborhood of the vertex \( p \).

**Proof:** (i) follows from \( U - \mathcal{C}_p|_U = \mathcal{T}_p|_U \cup \mathcal{E}_p|_U \). (ii) holds because homotopy groups are topological invariants. \( \square \)

Note that, although \( \mathcal{T}_p = \phi_p(\mathcal{T}) \), \( \mathcal{T} \) and \( \mathcal{E} \) here need not be homeomorphic to \( \phi_p(\mathcal{T}) \) and \( \phi_p(\mathcal{E}) \) respectively. The notion of precausality (see below in Section III) is set up to ensure \( \mathcal{E}_p \cong \phi_p(\mathcal{E}) \).

A weak LC structure at each point \( p \in \text{int} M \) defines a characteristic topological space \( \mathcal{C}_p \) of codimension 1 which is Hausdorff everywhere but at \( p \). In particular \( \mathcal{C}_p \) does not contain any open \( U \ni p \) from the manifold topology of \( M \). However stronger structures can be defined as follows.

**Definition 4:** Let \( M \) be a CAT manifold. A (CAT) (manifold) thickened cone of thickness \( m > 0 \) at \( p \in \text{int} M \) is the (CAT) homeomorphic image \( \mathcal{C}_p^m := \phi_p \mathcal{C}^m \) of a (CAT) homeomorphism of manifolds \( \phi_p : \mathcal{C} \to \mathcal{C}_p \subset M \) with \( \phi_p(0) = p \).

Note that due to the manifold property the notion of a thickened cone is much more simple than that of a cone itself. It also clear that now the only consistent topology on \( \mathcal{C} \subset \mathcal{C}_p \) is \( \tau_1 \) and
correspondingly $\phi_p(\tau_1)$ on $\mathcal{C}_p \subset \mathcal{C}_p^m$.

**Definition 5:** A thickened cone structure on $M$ is an assignment $\text{int}M \ni p \mapsto \mathcal{C}_p^{m(p)}$ of a thickened cone at every $p \in \text{int}M$.

Note that in general the thickness $m$ can vary from point to point in $M$ Here $m : M \to \mathbb{R}_+$ is an a priori not necessarily continuous function. However an important case even more special than the continuous one is that of constant $m$.

**Definition 6:** A homogeneously thickened cone structure on $M$ is a thickened cone structure $\text{int}M \ni p \mapsto \mathcal{C}_p$ with constant thickness $m$.

Although homogeneity might be too restrictive, at least continuity of structures on $M$ is a natural assumption in the topological category.

**Definition 7:** Let $M$ be a CAT manifold. A strongly thickened ($\mathcal{C}^m$) LC structure on $M$ is a (CAT) continuous family of (CAT) homeomorphisms $\phi_p : \mathcal{C}^m \to \mathcal{C}^{m(p)} \subset M$ with $\phi_p(0) = p$ and such that the thickness $m$ is a CAT function on $M$.

In particular the conditions of (ii) in Proposition 1 apply for all manifolds of dimension $d + 1 > 2$ with a strong LC structure. While a weak LC structure at $p \in \text{int}M$ may not be able to distinguish $\mathcal{T}_p|U$ and $\mathcal{E}_p|U$ within any $U \ni p$.

**Theorem 1:** Let $M$ carry a strong LC structure. At any $p \in \text{int}M$ there exists an open $U \ni p$ such that:

For $d := \dim M - 1 > 0$ it is $|\Pi_0(\mathcal{T}_p|U)| = 2$ and $\Pi_{d-1}(\mathcal{E}_p|U) = \Pi_{d-1}(S^{d-1})$,

for $d > 1$ it is $\Pi_{d-1}(\mathcal{E}_p|U) = 0$ and $|\Pi_0(\mathcal{E}_p|U)| = 1$,

for $d = 1$ it is $\Pi_{d-1}(\mathcal{T}_p|U) = \Pi_{d-1}(\mathcal{E}_p|U) = \Pi_0(S^0)$, i.e. $|\Pi_0(\mathcal{T}_p|U)| = |\Pi_0(\mathcal{E}_p|U)| = 2$,

and in dimension $d = 0$ it is $\mathcal{T}_p = \mathcal{E}_p = \emptyset$.

**Proof:** For all $p \in \text{int}M$ the strong LC structure provides a thickened cone $\mathcal{C}_p^{m(p)}$. Since $m(p) > 0$, $\mathcal{C}_p^{m(p)}$ contains always a neighborhood $U \ni p$ homeomorphic to a neighborhood $\phi_p^{-1}(U) \ni 0$ of the standard cone which in any dimension has the desired properties.

At any interior point $p \in \text{int}M$ the open exterior $\mathcal{E}_p$ and the open interior $\mathcal{T}_p$ of the cone $\mathcal{C}_p$ are locally topologically distinguishable for $d > 1$, indistinguishable for $d = 1$, and empty for $d = 0$. With a strong LC structure $\mathcal{T}_p|U \neq \mathcal{E}_p|U \forall U \ni p \iff d + 1 > 2$, whence locally in any neighborhood $U \ni p$ the interior and exterior of $\mathcal{C}_p \cap U$ at $p$ in $U$ has an intrinsic invariant meaning. $\mathcal{C}_p|U$ divides $U - \mathcal{C}_p|U$ in three open submanifolds, a non-contractable exterior $\mathcal{E}_p|U$, plus two contractable connected components of $\mathcal{T}_p =: \mathcal{T}_p|U \cup \mathcal{E}_p|U$, the local future $\mathcal{F}_p|U$ and the local past $\mathcal{P}_p|U$ with $\partial \mathcal{T}_p|U = \mathcal{C}_p^+|U$ where $\mathcal{C}_p^+ := (\phi_p \mathcal{C}^+)$ and $\partial \mathcal{E}_p|U = \mathcal{C}_p^-|U$ where $\mathcal{C}_p^- := \phi_p \mathcal{C}^-$ respectively. This rises also the question if and how $\mathcal{T}_p$ and $\mathcal{E}_p$ or their local restriction to $U \ni p$ can be distinguished. This problem is solved by a topological $\mathbb{Z}_2$ connection (see also Section III below).

Given a strong LC structure, a local (conformal) metric can always be proven to exist on any differentiable manifold $M$ with $\text{CAT} \subset \mathcal{C}^1$. Within such CAT, any metric $\eta$ on $\mathbb{R}^{d+1}$ can be restricted to $\mathcal{C}^m$ and pulled back pointwise along $(\phi_p)^{-1}$ to a metric $g$ on $\mathcal{C}_p^{m(p)}$. The CAT continuity of the family $\{p \mapsto \mathcal{C}_{p}^{m(p)}\}$ implies CAT$_{-1}$ continuity of the family $\{p \mapsto g|\mathcal{C}_p^{m(p)}\}$. So we can extract a CAT$_{-1}$ continuous metric $\{p \mapsto g_p\}$.

Here we are interested in particular only in Lorentzian metrics which are locally compatible with a (weak or strong) LC structure. The Minkowski metric $\eta$ is locally compatible with the cone $\mathcal{C}$ in the sense that $\eta_0(v, v) = 0 \iff v \in T_0N$, with arbitrary submanifold $N \subset \mathcal{C} \subset \mathbb{R}^{d+1}$ such that
(0, v) ∈ TN. Correspondingly, a Lorentzian metric g is said to be \textit{locally compatible} with an LC structure \( p \mapsto \mathcal{C}_p \), iff, with \( \mathcal{C}_p \supset \phi_p(N) \cong N \), it holds

\[
g_p(V(p), V(p)) = 0 \iff V(p) \in T_p\phi_p(N), \forall p \in \text{int}M,
\]
i.e. locally at any vertex the cone determines the characteristic null directions in the tangent space.

On the other hand, the cone structure poses an equivalence relation on Lorentzian metrics which are compatible with the LC structure. Given any such metric \( g \), the corresponding equivalence class \([g]\) is the conformal class of \( g \). We summarize the existence and uniqueness result as follows:

**Proposition 2:** Given a strong LC structure on a (CAT) manifold,

(i) there always exist a (CAT\(-1\)) Lorentzian metric \( g \) on \( M \) compatible with the LC structure.

(ii) the conformal class \([g]\) of LC compatible metrics is uniquely determined by the LC structure.

The existence of a conformal Lorentzian metric is guaranteed by a \textit{strong} LC structure, but not by a weak one. However, since conditions \([3.2]\) needs only the existence of the tangent bundle of \( \mathcal{C}_p \), uniqueness is assured already by a differentiable \textit{weak} LC structure.

Although at each \( p \in \text{int}M \) a CAT strong LC structure on \( M \) admits a conformal class \([g]\) of CAT\(-1\) Lorentzian metrics \( g \) with characteristic directions in \( T_pM \) tangential to \( \mathcal{C}_p \), away from the vertex \( p \) the cones of the LC structure need not at all be compatible with the null structure of any conformal metric \([g]\). This reflect the fact that, apart from its local vertex structure, a strong LC structure is still much more flexible than a conformal structure. For any \( q \neq p \) the tangent directions given by \( T_q\mathcal{C}_p \) need a priori not be related to tangent directions of null curves of \( g \), since the cone (or its thickening) at \( p \) is in general unrelated to that at \( q \). The need for compatibility conditions between cones at different points motivates the introduction of some of the causality structures in open regions of \( M \) introduced later in the following section.

### III. Causality structures on manifolds

Given a (weak or strong) LC structure one wants to know first of all under which conditions, for given \( p \in \text{int}M \) an exterior and interior of the cone can be distinguished within the complement \( M - \mathcal{C}_p \). This problem is the global analogue of the local one which was answered by Proposition 1 and Theorem 1 above.

**Proposition 3:** Assume that at \( p \in \text{int}M \) there are open (CAT) submanifolds \( \mathcal{I}_p \) and \( \mathcal{E}_p \) such that the interior of \( M \) decomposes into the disjoint union \( \text{int}M = \mathcal{I}_p \cup \mathcal{E}_p \cup \mathcal{C}_p \). Assume \( \exists k \in \mathbb{N}_0 : \Pi_k(\mathcal{I}_p) \neq \Pi_k(\mathcal{E}_p) \). Then \( \mathcal{I}_p \) and \( \mathcal{E}_p \) can be topologically distinguished.

**Proof:** \( \text{int}M - \mathcal{C}_p = \mathcal{I}_p \cup \mathcal{E}_p \), and homotopy groups are topological invariants. \( \square \)

In particular the conditions of Proposition 3 apply for \( d + 1 > 2 \) in particular to all manifolds with the following topological structure:

**Example 1:** Let in any dimension \( d := \dim M - 1 > 0 \) at any \( p \in \text{int}M \) be \( |\Pi_0(\mathcal{I}_p)| = 2 \) and \( |\Pi_{d-1}(\mathcal{E}_p)| = |\Pi_{d-1}(S^{d-1})|, \) for \( d > 1 \) be \( |\Pi_{d-1}(\mathcal{I}_p)| = 0 \) and \( |\Pi_{d-1}(\mathcal{E}_p)| = 1 \), For \( d = 1 \) be \( |\Pi_{d-1}(\mathcal{I}_p)| = \Pi_{d-1}(\mathcal{E}_p) = \Pi_0(S^0) \), i.e. \( |\Pi_0(\mathcal{I}_p)| = |\Pi_0(\mathcal{E}_p)| = 2 \), and in dimension \( d = 0 \) be \( \mathcal{I}_p = \mathcal{E}_p = \emptyset \) at any \( p \in \text{int}M \). Then in particular \( \mathcal{I}_p \neq \mathcal{E}_p \iff d + 1 > 2 \). The open exterior \( \mathcal{E}_p \) and the open interior \( \mathcal{I}_p \) of the cone \( \mathcal{C}_p \) at any interior point \( p \in \text{int}M \) are topologically distinct for \( d > 1 \), indistinguishable for \( d = 1 \), and empty for \( d = 0 \).
In the case of Example 1, \( \mathcal{C}_p \) divides \( M - \mathcal{C}_p \) in three open submanifolds, a non-contractable exterior \( \mathcal{E}_p \), plus two contractable connected components of \( \mathcal{F}_p := \mathcal{F}_p \cup \mathcal{P}_p \), the future \( \mathcal{F}_p \) and the past \( \mathcal{P}_p \) with \( \partial \mathcal{F}_p = \mathcal{C}_p^+ := \phi_p \mathcal{C}^+ \) and \( \partial \mathcal{P}_p = \mathcal{C}_p^- := \phi_p \mathcal{C}^- \) respectively. This rises also the question if and how \( \mathcal{F}_p \) and \( \mathcal{P}_p \) can be distinguished.

Let \( M \) be differentiable and \( \tau \) be any vector field \( M \rightarrow TM \) such that at any \( p \in \text{int}M \) its orientation agrees with that of \( \phi_p(a) \). Such a orientation vector field comes naturally along with a (\( \text{CAT}_{-1} \)) \( \mathbb{Z}_2 \)-connection on \( M \) which allows to compare the orientation \( \tau(p) \) at different \( p \in \text{int}M \). Given a strong LC structure on \( M \), the \( \mathbb{Z}_2 \)-connection is in fact provided via continuity of \( p \mapsto T_p \phi_p(a) \). Then \( \tau \) is tangent to an integral curve segment through \( p \) from \( \mathcal{P}_p \) to \( \mathcal{F}_p \). In particular, \( \mathcal{F}_p \) and \( \mathcal{P}_p \) are distinguished from each other by a consistent \( \tau \)-orientation on \( M \).

If \( M \) is not differentiable, in order to distinguish continuously \( \mathcal{P}_p \) from \( \mathcal{F}_p \) on \( \text{int}M \) it remains just to impose a topological \( \mathbb{Z}_2 \)-connection on \( \text{int}M \) a fortiori.

In order to obtain a more specific causal structure it remains to identify natural consistency conditions for the intersections of cones of different points. In order to define topologically timelike, lightlike, and spacelike relations, and a reasonable causal complement, we introduce the following causal consistency conditions on cones.

**Definition 8:** \( M \) is (locally) cone causal or C-causal in an open region \( U \), if it carries a (weak or strong) LC structure and in \( U \) the following relations between different cones in \( \text{int}M \) hold:

1. For \( p \neq q \) one and only one of the following is true:
   1. \( q \) and \( p \) are causally timelike related, \( q \ll p : \iff q \in \mathcal{F}_p \land p \in \mathcal{P}_q \) (or \( p \ll q \))
   2. \( q \) and \( p \) are causally lightlike related, \( q \bowtie p : \iff q \in \mathcal{C}_p^+ - \{p\} \land p \in \mathcal{C}_q^- - \{q\} \) (or \( p \bowtie q \)),
   3. \( q \) and \( p \) are causally spacelike related, i.e. relatively spacelike to each other, \( q \bowtie p : \iff q \in \mathcal{E}_p \land p \in \mathcal{E}_q \).

2. Other cases (in particular non-symmetric ones) do not occur.

\( M \) is locally C-causal, if it is C-causal in any region \( U \subset M \). \( M \) is C-causal if conditions (1) and (2) hold \( \forall p \in \mathcal{E} \).

Let \( M \) be C-causal in \( U \). Then, \( q \ll p \iff \exists r : q \in \mathcal{P}_r \land p \in \mathcal{F}_r \), and \( q \bowtie p \iff \exists r : q \in \mathcal{C}_r^+ \land p \in \mathcal{C}_r^- \).

If an open curve \( \mathbb{R} \ni s \mapsto c(s) \) or a closed curve \( S^1 \ni s \mapsto c(s) \) is embedded in \( M \), then in particular its image is \( \text{im}(c) \equiv c(\mathbb{R}) \cong \mathbb{R} \) or \( \text{im}(c) \equiv c(S^1) \cong S^1 \) respectively, whence it is free of self-intersections. Such a curve is called spacelike : \( \iff \forall p \equiv c(s) \in \text{im}(c) \exists \epsilon : c||s-\epsilon,s+\epsilon|\{s\} \in \mathcal{E}_c(s) \), and timelike : \( \iff \forall p \equiv c(s) \in \text{im}(c) \exists \epsilon : c||s-\epsilon,s+\epsilon|\{s\} \in \mathcal{F}_c(s) \).

Note that C-causality of \( M \) forbids a multiple refolding intersection topology for any two cones. In particular along any timelike curve the future/past cones do not intersect, because otherwise there would exist points which are simultaneously timelike and lightlike related. Continuity then implies that future/past cones in fact foliate the part of \( M \) which they cover. Hence, if there exists a fibration \( \mathbb{R} \leftrightarrow \text{int}M \rightarrow \Sigma \), then C-causality implies that the future/past cones foliate in particular on any fiber. In fact, given a fibration, one could define also a weaker form of causality just by the foliating property of all future/past cones on each fiber. (Physically this situation corresponds to ultralocal classical clocks. Quantum uncertainty of the fiber would require to take appropriate ensemble averages over some bundle of neighboring fibers which then contains in particular spacelike related vertices on the fibers of the bundle. Then the corresponding future or past cones intersect for sure, and even timelike related ones of different fibers may intersect !) C-causality however requires more, namely the future/past cones of all timelike related vertices should be non-intersecting, not only those in a particular fiber.
Therefore C-causality allows also a reasonable definition of a causal complement.

**Definition 9:** For any open set $S$ in a C-causal manifold $M$ the *causal complement* is defined as

$$S^\perp := \bigcap_{p \in \text{cl}S} \mathcal{E}_p,$$

(3.1)

where $\text{cl}S$ denotes the closure in the topology induced from the manifold. Although the causal complement is always open, it will in general not be a contractable region even if $S$ itself is so.

Assume $p$ and $q$ are timelike related, $p \in \mathcal{P}_q$ and $q \in \mathcal{P}_p$. $\mathcal{K}_p^q := \mathcal{P}_p \cap \mathcal{P}_q$ is the bounded open double cone between $p$ and $q$. Given any bounded open $K$ such that $\exists p, q \in M : K = \mathcal{P}_p \cap \mathcal{P}_q$, we set $i^+(K) := \{q\}$, $i^-(K) := \{p\}$, and $r^0(K) := C_p^+ \cap C_q^-$. For any $\mathcal{K}_p^q \subset M$ let $\text{cl}_c(\mathcal{K}_p^q)$ be its causal closure.

Since C-causality prohibits relative refolding of cones, it also ensures that $(\mathcal{K}_p^q)^\perp = \mathcal{K}_p^q$, i.e. the causal complement is a duality operation on double cones.

The open double cones of a C-causal manifold $M$ generate a topology, called the *double cone topology* which is a genuine generalization of the usual Alexandrov topology for strongly causal space-times. For strongly causal space-times the Alexandrov topology is equivalent to the manifold topology \[4, 5\]. When $M$ fails to be locally causal the double cone topology may be coarser than the manifold topology.

Let us discuss now possible natural relations that can appear between two double cones $\mathcal{K}_1$ and $\mathcal{K}_2$ of a C-causal manifold. First there is the case $\mathcal{K}_1 \cap \mathcal{K}_2 = \emptyset$ which corresponds to causally unrelated sets. For $\mathcal{K}_1 \cap \mathcal{K}_2 \neq \emptyset$, the intersection is such that $\mathcal{K}_1 \cup \mathcal{K}_2 - \mathcal{K}_1 \cap \mathcal{K}_2$ is either given by two disconnected pieces or it is connected. In the latter case we distinguish whether $\partial \mathcal{K}_1 \cap \partial \mathcal{K}_2$ is empty or not. It is in the former case that one of $\mathcal{K}_1$ and $\mathcal{K}_2$ will be contained in the other.

Local C-causality does a priori not preclude other more pathological possibilities. However it is possible to define in a purely topological manner more refined causality notions.

**Definition 10:** Let $M$ be a C-causal manifold.

(i) $M$ is *globally hyperbolic* : $\Leftrightarrow \text{cl}_c \mathcal{K}_p^q$ compact $\forall p, q \in M$

(ii) $M$ is *causally simple* : $\Leftrightarrow \text{cl}_c \mathcal{K}_p^q$ closed $\forall p, q \in M$

(iii) $M$ is *causally continuous* : $\Leftrightarrow$ $M$ is distinguishing and both $\mathcal{P} : p \mapsto \mathcal{P}_p$ and $\mathcal{P} : q \mapsto \mathcal{P}_q$ are continuous

(iv) $M$ is *stably causal* : $\Leftrightarrow$ $M$ admits a $C^0$ function $f : M \to \mathbb{R}$ strictly monotonously increasing along each future directed nonspace-like curve (global time function)

(v) $M$ is *strongly causal* : $\Leftrightarrow$ the topology generated by $\{\mathcal{K}_p^q\}_{p,q \in M}$ is equivalent to the manifold topology of $M$

(vi) $M$ is *distinguishing* : $\Leftrightarrow \mathcal{P}_p = \mathcal{P}_q \Rightarrow p = q$ and $\mathcal{P}_p = \mathcal{P}_q \Rightarrow p = q$

(vii) $M$ is *causal* : $\Leftrightarrow$ every closed curve in $M$ is not nonspace-like

(viii) $M$ is *chronological* : $\Leftrightarrow$ every closed curve in $M$ is not timelike

If a manifold carries a Lorentzian metric, we saw in Section II above that this is locally compatible with a strong LC structure. Beyond that, it is an interesting question under which conditions a Lorentzian metric is *compatible* with some LC structure. The Minkowski metric $\eta$ is compatible with the cone $C$ in the sense that $\eta_v(v, v) = 0 \Leftrightarrow (x, v) \in T^*C := \bigcup_{y \in \mathcal{E}} T_yC$ where $T_yC := \bigcup_{y \in N \subset \mathcal{E}} T_yN \subset \mathbb{R}^{d+1}$ and the latter union is over all (differentiable) 1-dimensional submanifolds $N \subset C$ passing through $y$, with all their tangent spaces embedded as linear submanifolds with common origin within
the common embedding space $\mathbb{R}^{d+1}$. Hence, for $y \neq 0$, the fibers $T_y C \cong \mathbb{R}^d$ are all usual isomorphic tangent spaces, while the only non-standard fiber $T_0 C \cong C \subset \mathbb{R}^{d+1}$ reproduces the $d$-dimensional cone itself, which is the local model of its own singularity. Correspondingly, a Lorentzian metric $g$ is said to be compatible with some LC structure $p \mapsto C_p$, iff

$$g_q(V(q), V(q)) = 0 \iff \forall p \in M : q \in C_p \Rightarrow V(q) \in T_q C_p := (\phi_p)_* T_{\phi_p^{-1}(q)} C = \bigcup_{\phi_p^{-1}(q) \in N \subset C} T_q\phi_p(N),$$

(3.2)

where the latter union is over all (differentiable) 1-dimensional submanifolds $N \subset C$ passing through $\phi_p^{-1}(q)$, and the latter identity holds with tangent push forward $(\phi_p)_* T_y N := T_{\phi_p(y)} \phi_p(N)$. Therefore, with (3.2) the cones are the characteristic surfaces of the Lorentzian metric. As pointed out above, (3.2) does not hold in general. However one might search for sufficient and necessary causality conditions such that this compatibility holds. A systematic investigation of this point is beyond our present investigations. Let us rather assure the correspondence of the causality notions of Def. 10 to the usual ones in the case of a Lorentzian space-time.

**Theorem 2:** Let $M$ carry a smooth Lorentzian metric $g$. Then the Lorentzian metric determines a C-causal structure. If a C-causal structure of $M$ is related to a Lorentz metric, then the definitions (i)-(viii) agree with the standard definitions and the following chain of implications of properties of $M$ holds: globally hyperbolic $\Rightarrow$ causally simple $\Rightarrow$ causally continuous $\Rightarrow$ stably causal $\Rightarrow$ strongly causal $\Rightarrow$ distinguishing $\Rightarrow$ causal $\Rightarrow$ chronological.

**Proof:** Given a smooth Lorentzian metric $g$ the cones determined by the null structure $[g]$ respect the relations of Def. 8, because otherwise there would exist some singularities. For (v) in the case of Lorentzian manifolds see [3], for the other notions and for the chain of implications see [7].

Finally let us define a condition which excludes the existence of singularities or internal boundaries within the future and past cones.

**Definition 11:** Let $M$ carry a (weak or strong) LC structure.

(i) $M$ is precausal in an open region $U \subset M$, if the $d+1$-parameter CAT family $\{\phi_p\}_{p \in U}$ of CAT homeomorphisms $\phi_p : \mathbb{R}^{d+1} \supset V \to U$ is such that at any $p \in U$ it is $C_p|_V = \phi_p C|_V$, and any CAT submanifold of $C_p$ or $(M - C_p) \cap U$ is a CAT homeomorphic image of $C$ or $(\mathbb{R}^{d+1} - C) \cap V$ respectively. $M$ is locally precausal iff it is precausal in any open region $U \subset M$.

(ii) $M$ is precausal if it is locally precausal such that in the CAT $d + 1$-parameter family $\{\phi_p\}_{p \in U}$ any CAT homeomorphism extends also to a homeomorphism of the interior $\phi_p : C \to C_p$.

**IV. Discussion and perspective**

Above we presented topological definitions of local (i.e. pointwise) cone (LC) structures for a general topological or differentiable manifold $M$ of dimension $d + 1 > 2$ and notions of causality on $M$ in a purely topological manner. It is remarkable that such definitions are possible, whence the usual recursion to a Lorentzian metric becomes redundant.

Proposition 1 gives criteria which locally distinguish the exterior and the interior of the cone at any point from each other. Proposition 3 and Example 1 provide concrete global topological conditions for $M$ in order to allow the relative distinction of interior and exterior of all its cones.
Minkowski space is obviously a manifold which satisfies the conditions for topologically distinguished interior and exterior according to Example 1. It is however a priori not clear what for each given category CAT of manifolds is the largest class of manifolds with the topological structure described in Example 1.

We saw that a global consistent distinction between future and past cones requires just a topological $\mathbb{Z}_2$-connection. Note that, as an important possible application, the canonical approach to quantum gravity comes always along with such a connection. In fact the canonical configuration variables for oriented manifolds may be there be chosen as $\text{SO}(d + 1)$-connections.

The presented LC structures, C-causality, and other our purely topological causality notions provide some alternative to the poset approach [9, 10, 11] for defining causality in quantum theories of quantum gravity. While that approach is based on a much weaker local notion of causality on sets, which essentially involves only a partial ordering, the present definition gives the possibility to work with local definition of causality on differentiable manifolds which still captures the essential notions for curves in a C-causal manifold to be lightlike, timelike or spacelike without the need of an underlying Lorentzian structure. For any set $S$ in a C-causal manifold a topological notion of a causal complement $S^\perp$ is given by (3.1). Any double cone $\mathcal{K}$ in a C-causal manifold then has the duality property $\mathcal{K}^{\perp\perp} = \mathcal{K}$.

Some advantages of conformal invariance in the quantization in minisuperspace models of higher-dimensional Einstein gravity have been pointed out in [12, 13]. In particular, factor ordering problems can be resolved uniquely this way. For a more general background independent quantum theory the restriction of local diffeomorphisms to those consistent with a causal structure, say e.g. a LC structure, on the whole manifold might appear too restrictive. After all a strong LC structure implies already the existence of a conformal metric, whence diffeomorphisms may be restricted locally to conformal ones. Nevertheless note that even a strong LC structure is much more flexible than a conformal metric structure. The local cones of different vertices might refold away from their vertices with rather complicated intersection topologies while a CAT continuous conformal metric within its (regular!) domain does not admit refolding singularities of the characteristic surfaces, each of the which is spanned out by all the null geodesics passing through a given vertex. Of course refolding and the associated singularities should be a topic of further more systematic investigations elsewhere.

The canonical approach to field quantization usually employs a foliation $\Sigma \leftrightarrow \text{int}M \rightarrow \mathbb{R}$. This rises the question when this is consistent with a (C-)causal structure. This may roughly be answered as follows: A CAT foliation of $M$ may be said to be consistent with a C-causal structure, if for any open set $O$ in a CAT slice $\Sigma \subset M$ there exists a double cone $\mathcal{K} \subset \text{int}M$ such that $i^0(\mathcal{K}) \subset \partial(\mathcal{K} \cap \Sigma)$ (compare also Section III, below Def. 9).

Consider now such a double cone $\mathcal{K}$ in $M$ with $O := \mathcal{K} \cap \Sigma$ and $\partial O = i^0(\mathcal{K})$ and a diffeomorphism $\phi$ in $M$ with pullbacks $\phi^\Sigma \in \text{Diff}(\Sigma)$ to $\Sigma$ and $\phi^\mathcal{K} \in \text{Diff}(\Sigma)$ to $\mathcal{K}$. If $\phi(\mathcal{K}) = \mathcal{K}$, it can be naturally identified with an element of $\text{Diff}(\mathcal{K})$. ($\phi|_{\mathcal{K} - \mathcal{K}} = id_{\mathcal{K} - \mathcal{K}}$ is a sufficient but not necessary condition for that to be true.) If in addition $\phi(\Sigma) = \Sigma$ then also $\phi(O) = O$, and $\phi|_O$ is a diffeomorphism of $O$.

When $M$ is homeomorphic to $\Sigma \times \mathbb{R}$ it is straightforward to extend the above from a single hypersurface $\Sigma \subset M$ to a foliation of $M$ via a 1-parameter set of embeddings $\Sigma \leftrightarrow M$.

For a canonical approach to quantum gravity, one might want to work with a restriction of the causal structure to cones with their vertices on a given graph $\Gamma$ within a slice $\Sigma$ of a foliation. A
given topological (differentiable) causal structure, selects particular causal homeomorphisms (diffeomorphisms) which preserve it. A strong LC-structure on all of $M$ already implies the existence of a conformal metric structure and a requirement of compatibility with that metric would reduce the local covariance group to local conformal diffeomorphisms. One might however also weaken the LC and causal structure of the manifold by considering in any leaf $\Sigma$ of a given foliation only cones with vertices on $\Gamma \subset \Sigma$ instead on all of $\Sigma$. A natural choice for $\Gamma$ is the dual graph of a triangulation. Then the cones have to CAT vary along the edges, but at least for $\text{CAT} \supset C^\infty$ the cones at the vertices of the graph can be freely ascribed. Consequently, a geometry constructed on that basis will be invariant under diffeomorphisms much more general than conformal ones.

Let us however also emphasize that, although the existence of a local conformal metric is guaranteed by a strong LC structure, it is a priori not obvious that this metric should play any significant rôle. Then however also the need to restrict diffeomorphisms to those compatible with the conformal metric may be questioned. One might eventually expect that within some approach to quantum geometry a cone at a vertex $p \in O \subset \Sigma$ should be replaced by an appropriate average over cones with vertices within some region $O$ of minimal Heisenberg uncertainty. Then the flexibility of the weak and strong LC structures makes them interesting concepts and potential ingredients for a possible definition of quantum causality too. Presently however this is still matter of many speculations.

Classically, the existence of a local metric requires only the differentiable structure in an arbitrary small neighborhood of the vertex, and the defined LC structures fix the preferred null directions only locally at each vertex. With sufficiently strong notions of causality (e.g. $C$-causality above) the null structures of this metric may become consistent with the global structure of cones of the LC structure. Note that in the case of a given Lorentz metric null geodesics lie on cones, and with sufficiently strong causality, e.g. global hyperbolicity, these cones have to be consistent with respect to each other and under variation of the vertex without refolding into each other (i.e. in particular without conjugate points).

For Lorentzian manifolds there is a hierarchy of common notions of causality which have been generalized above. Provided our definitions of causality are sufficiently natural it should be possible to prove (at least parts) of this hierarchy in the more general topological setting. However a complete investigation of the mutual relations between different topological causality concepts is beyond the scope and goal of the present paper.

It should be emphasized that the above was just brief demonstration of the possibility to introduce notions of cones and causality on CAT topological manifolds without a metric. In particular, weak and strong LC structures, $C$-causality, precausality, and some generalizations of the most common notions of causality have been obtained. However the investigation is far from complete. It remains for future work to develop the topological approach to causal structures on manifolds further, to investigate better some of its implications, and last not least to demonstrate its applicability in background independent formulations of algebraic quantum field theory and quantum gravity.

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