Abstract  We introduce the calculus of Classical Transitions (CT), which extends the research line on the relationship between linear logic and processes to labelled transitions. The key twist from previous work is registering parallelism in typing judgements, by generalising linear logic judgements from one sequents to many (hypersequents). This allows us to bridge the gap between the structures of operators used as proof terms in previous work and those of the standard π-calculus (in particular parallel operator and restriction). The proof theory of CT allows for new proof transformations, which we show correspond to a labelled transition system (LTS) for processes. We prove that CT enjoys subject reduction and progress.

1 Introduction

Classical Processes (CP) [21] is a process calculus inspired by the correspondence between the session-typed π-calculus and linear logic [5], where processes correspond to proofs, session types (communication protocols) to propositions, and communication to cut elimination. Bridging process languages to linear logic paves the way to apply methods developed in one field to the other. This already worked for a few results, in both directions. For example, the proof theory of linear logic can be used to guarantee progress for processes [5, 21], and multiparty session types, originally developed for processes [12], inspired a generalisation of the standard cut rule to the composition of an arbitrary number of proofs, allowing for safe circular dependencies among proofs [8].

The hallmark of CP is that the semantics of processes is given by sound proof transformations in Classical Linear Logic (CLL). While this permits reusing the metatheory of linear logic “as is” to reason about process behaviour (e.g., cut elimination yields communication progress), it also exhibits some fundamental discrepancies with the key operators of the π-calculus [16].

Some discrepancies are syntactic. For example, the term for output of a linear name is $x[y].(P \mid Q)$, read “send $y$ over $x$ and proceed as $P$ in parallel to $Q$”. Notice that the term constructor for output here actually takes $x$, $y$, $P$ and $Q$ as parameters at the same time. This discrepancy is caused by adopting processes as proof terms for CLL: the typing rule for output (i.e., the $\otimes$ rule of CLL) checks that the processes respectively implementing the behaviours of $y$ ($P$) and of $x$ ($Q$) share no resources, by taking two premises ($P$ and $Q$). In general, there is no independent parallel term $P \parallel Q$ in the grammar of CP, and even if we added it as the mix rule suggested in the original presentation of CP [21], it would not allow
$P$ and $Q$ to communicate as in standard $\pi$-calculus. Synchronisation is governed instead by the restriction operator $(\nu xy)(P | Q)$ (we use the latest syntax for CP, from [6]), which links $x$ at $P$ with $y$ at $Q$ to enable communication. Again, parallel is mixed with another operator (restriction here), but in this case it means that $P$ and $Q$ will communicate.

The discrepancies carry over from syntax (and typing) to semantics. The rule for reducing an output with an input in CP is the following.

$$(\nu xy) (x[x'] (P \mid Q) \mid y(y') . R) \rightarrow (\nu x'y') (P \mid (\nu xy)(Q \mid R))$$

Notice how the rule needs to inspect the structure of the continuation of the output term ($P \mid Q$) to produce a typable structure for the resulting network, by nesting restrictions appropriately.

A consequence of the discrepancies is that CP still misses a labelled transition system (LTS) semantics. Keeping with our example, it is difficult to define a transition axiom for output, as in $x[y].(P \mid Q) \xrightarrow{x[y]} P \mid Q$, because it is not possible to type $P \mid Q$. Even if it were, we hit another problem when attempting to recreate the reduction above using transitions. Ideally, we should be able to define a rule that does not inspect the structure of processes, but only their observables, as follows.

$$P \xrightarrow{x[x']} P' \quad Q \xrightarrow{y(y')} Q'$$

$$\frac{}{(\nu xy)(P \mid Q) \xrightarrow{\tau} (\nu x'y')(\nu xy)(P' \mid Q')}$$

However, this is not possible because the restriction term in the result is not typable in (nor is in the syntax of) CP.

In this paper, we present the calculus of Classical Transitions (CT), an attempt at mending the discrepancies that we discussed. The key twist from CP to CT is to generalise the judgement form of CLL from one sequent to collections of sequents, called hypersequents [3]. Crucially, we use the separation of hypersequents to register the “parallelism” of propositions (as in, manipulated by separate proofs); in particular, we interpret the composition of hypersequents as parallel composition of processes. This allows us to redefine the typing rules of CP such that, whenever parallelism is required, we can guarantee it by looking at the structure of types (hypersequents) instead of the syntax of terms. Following this principle, the adaptation of all rules in CLL is straightforward. In CT, referring to our previous examples, the syntax for output is $x[y].P$, that for restriction is $(\nu xy) P$, and that for parallel composition is $P \mid Q$, and our typing rules follow this structure in the expected way.

The proof theory of CT allows for new sound proof transformations w.r.t. CP, which we show correspond to labelled transitions for processes, yielding an LTS semantics. We show that CT enjoys subject reduction and progress (terms never get stuck, implying lack of deadlocks). Differently from CP, our progress result does not require any commuting conversions: actions are executed in place (just as in the LTS for the $\pi$-calculus), instead of being permuted inside or outside of parallel compositions as in [21]. Our semantics also evidences syntactically the
explicit resource management that CLL performs whenever server processes are replicated, which is hidden by “communicating” name substitutions in CP. We envision that bridging the gap that we discussed and giving an LTS semantics to CLL proofs (adapted to hypersequents) will push even further the successful research line that investigates the relationship between linear logic and processes.

2 Classical Transitions

We present Classical Transitions (CT), a strict generalisation of the latest version of the calculus of Classical Processes [6].

Processes In CT, programs are processes \((P, Q, R, \ldots)\) that communicate using channels names \((x, y, z, \ldots)\). Channels represent endpoints of sessions, as in [6, 20]. Processes are given by the grammar below; some terms include types \((A, B, C, \ldots)\) which will be discussed afterwards.

\[
P, Q ::= x[y].P \quad \text{output endpoint } y \text{ on } x \text{ and continue as } P
\]

\[
x(y).P \quad \text{input endpoint as } y \text{ from } x \text{ and continue as } P
\]

\[
x[\text{inl}].P \quad \text{select left on } x \text{ and continue as } P
\]

\[
x[\text{inr}].P \quad \text{select right on } x \text{ and continue as } P
\]

\[
x.\text{case}(P, Q) \quad \text{offer on } x \text{ to continue as } P \text{ (left) or } Q \text{ (right)}
\]

\[
x[A].P \quad \text{output type } A \text{ on } x \text{ and continue as } P
\]

\[
x[X].P \quad \text{input a type as } X \text{ in } P
\]

\[
x[] \quad \text{close endpoint } x \text{ and terminate}
\]

\[
x()\cdot P \quad \text{wait for } x \text{ to be closed and continue as } P
\]

\[
x \rightarrow y \quad \text{forward endpoint } x \text{ to } y
\]

\[
!x.(y)\cdot P \quad \text{server offering service } (y)\cdot P \text{ on } x
\]

\[
x[\text{use } y].P \quad \text{client service request}
\]

\[
x[\text{spawn } x^\prime].P \quad \text{client request spawn}
\]

\[
x[\text{disp}].P \quad \text{client dispose service}
\]

\[
(\nu x y)\cdot P \quad \text{link endpoints } x \text{ and } y \text{ in } P
\]

\[
P \mid Q \quad \text{parallel composition of processes } P \text{ and } Q
\]

\[
0 \quad \text{terminated process}
\]

We first discuss terms that are unchanged wrt CP. We use Wadler’s convention of denoting outputs with square brackets and inputs with round parentheses [21]. Term \(x[y].P\) denotes a process that sends a fresh name \(y\) over \(x\) and then proceeds as \(P\). Dually, term \(x(y).P\) receives a name \(y\) over \(x\) and then proceeds as \(P\). Thus both input and output actions bind their object in continuations, as in the internal \(\pi\)-calculus [19]—thanks to links, it is easy to recover free output as syntactic sugar, see [14]. Term \(x[]\) closes channel \(x\), and term \(x().P\) waits for \(x\) to be closed before continuing as \(P\). Terms \(x[\text{inl}].P\) and \(x[\text{inr}].P\) respectively select the left and right branch of a (binary) offer available over \(x\) before proceeding as \(P\). Dually, term \(x.\text{case}(P, Q)\) offers over \(x\) a choice between proceeding as \(P\) (left branch) or \(Q\) (right branch). Term \(x[A].P\) sends type \(A\) over \(x\) and term \(x(X).P\)
receives a type to replace $X$ with in the continuation $P$ (binding $X$ in $P$). Term $x \to y$ is a forwarding proxy: inputs on $x$ are forwarded as outputs on $y$ and vice versa.

We now move to terms that are new or changed wrt CP.

Term $!x.(y)P$ is a server that offers on $x$ a replicable process $P$, where $y$ is bound in $P$. Server channels ($x$ in the server term) are typed with the exponential connective $!$ of CLL, which guarantees that channel $x$ can be used at will (zero, one, or many times). The number of times that a server channel is used is determined by the process connected to the server (the client). We thus interpret the server channel as offering three possibilities, which can be selected from by clients (if you like, this can be seen as a variant or tagged union type). Specifically, term $x[\text{use } y].P$ requests the server connected to $x$ to use its replicable process exactly once, and to continue communicating with the latter on $y$. Term $x[\text{spawn } x'].P$ requests the server on $x$ to duplicate itself and to make the new server accessible over $x'$. Finally, term $x[\text{disp}].P$ (for “dispose”) terminates the server—i.e., it informs it that it will be used zero times.

In CP, only the action for using a server once has an explicit term (our $x[\text{use } y].P$). Duplication and disposal are visible only from the proof used to type a process. This yields a slightly unexpected reduction semantics, where a process may communicate with a server (e.g., for its disposal) without consuming any syntactic term (disposal is an explicit communication in CP, between the proof of the client and that of the server). We chose to make all client terms explicit in CT, for two reasons. First, we will see that this allows processes to represent faithfully the structure of the proof with which they are typed, since now all rule applications have a corresponding term constructor. Second, when we will formulate the LTS semantics of CT in the next section, we shall see that all three client invocations (usage, duplication, and disposal) correspond to transitions with observable actions. Having explicit client terms thus allows us to give transition rules in an SOS style: client actions will arise from syntactically corresponding terms, as usual (which would not be possible with the “silent” treatment of duplication and disposal in CP).

A restriction term $(\nu xy)P$ connects endpoints $x$ and $y$ to form a session, allowing the two endpoints to communicate—and binding the names $x$ and $y$ to $P$. This term was originally introduced in [20] for the session-typed $\pi$-calculus. Later, it was adopted in CP [6], but with the arity problem discussed in the Introduction. Our term, instead, is exactly the same as that in [20], which is logically reconstructed in a precise way for the first time here. CT also has the standard parallel composition term $P \parallel Q$, and the terminated process term $0$. We extend the terminology to terms that are parallel compositions of $0$, i.e., we say that a process is terminated if it is a parallel composition of $0$ terms.

**Types**

There are two kinds of types in CT: channel types (also called session types) and process types.

Channel types $(A, B, C, \ldots)$ are standard CLL propositions. They are defined by the following grammar, where $X$ ranges over atomic propositions.
A, B ::= A ⊗ B send A, proceed as B | A ⊨ B receive A, proceed as B
| A ⊕ B select A or B | A & B offer A or B
| 0 unit for ⊗ | ⊤ unit for ⊘
| ?A client request | ↓ unit for ⊞
| ∃X.A existential | ∀X.A universal
| X atomic proposition | X⊥ dual of atomic prop.

Types on the left-hand column are for outputs and types in the right-hand column for inputs. Connectives on the same row are respective duals, e.g., ⊗ and ⊞ are dual of each other. We assume the standard notion of duality of CLL, writing A⊥ for the dual of A. Duality proceeds homomorphically and replaced connectives with their duals, for example (A ⊗ B)⊥ = A⊥ ⊞ B⊥. In ∃X.A and ∀X.A, the type variable X is bound in A. We write ftv(A) for the set of free type variables in A, and B{A/X} to denote substitution of A for X in B.

Process types are CLL hypersequents (Γ, Δ, ...), i.e., collections (multisets) of CLL sequents (γ, δ, ...). Their grammar is given in the following.

γ, δ ::= x1 : A1, ..., xn : An
Γ, Δ ::= γ1 | ··· | γn

The separator | used in hypersequents indicates that the sequents in it are independent. The side-conditions on the right are standard: we require channel names to be disjoint in both a single sequent and among all sequents in the same hypersequents. We write n(γ) for the channel names in γ. For convenience of exposition, we assume that free type variables are never shadowed by bound ones, e.g., we assume that X ∉ ftv(Γ | γ) whenever we write Γ | γ, ∀X.B. Both sequents and hypersequents allow for exchange, which we apply silently in the remainder. Likewise, we assume unit laws for empty sequents. As usual for linear logic, they do not allow for implicit weakening or contraction, which are managed explicitly by typing rules using exponentials.

Typing Typing judgements in CT have the form P ⊩ Γ and read “process P uses channels according to Γ”. We omit empty (hyper)sequents. We say that a process P is well-typed whenever P ⊩ Γ for some hypersequent Γ. The rules for deriving typing judgements are displayed in Figure 1.

Typing rules associate types to channels by looking at how channels are used in process terms. Rule selection is structural on the syntax of processes, in the sense that it depends only on the outermost constructor of a process term. The typing rules of CT are those of CLL, adapted from sequents to hypersequents as expected [3]. The key twists that we introduce are the structural rules mix and cut. Rule mix types the parallel composition of two processes, by combining their types as a hypersequent. Previous presentations of rule mix (e.g., [21]) do not use hypersequents, thus losing the information that the resources in the two premises of the rule are independent. This information is crucial to our reformulation of rule cut, which types a restriction connecting endpoints x and y by requiring that the types of x and y are respective duals (as usual in CLL).
Figure 1. Classical transitions, typing rules.

and are used by parallel components of the process (new in CT). The latter condition, which we can check thanks to hypersequents, makes the rule sound without having to inspect the structure of the restricted process. By comparison, the standard cut rule of linear logic requires two separate proofs as premises, yielding the restriction term constructor \((\nu xy)(P|Q)\) that we discussed in the Introduction. Our rule \(\otimes\) is reformulated from CLL using the same intuition for rule cut (the original rule requires two separate proofs for \(A\) and \(B\) respectively). This yields a logical reconstruction of the expected output term from the internal \(\pi\)-calculus [19].

Rule \(\text{ax}\) types a forwarding proxy between endpoints \(x\) and \(y\) by requiring that the types of \(x\) and \(y\) are respective duals. This ensures that any message on \(x\) can be safely forwarded to \(y\), and \textit{vice versa}. All rules for typing channels enforce linear usage, aside from client requests (typed with the exponential connective \(?\)), for which contraction and weakening are allowed. Contraction (rule \(c\)) allows for multiple client requests for the same server channel, and weakening (rule \(w\)) for clients that do not use a server. Thus, CT exposes syntactically that CLL yields a calculus where servers are resources managed explicitly by clients.

All other rules are standard. Rule \(\text{mix}_0\) was introduced to CP in [2].

Proposition 2.1 below formalises that all sequents in a provable hypersequent are independent, in the sense that they are independently provable. endpoints.

**Proposition 2.1.** If \(\vdash \Gamma | \gamma\) then, \(\vdash \gamma\).

Intuitively, this confirms that the parallel composition of sequents in hypersequents denotes non-interference. Different sequents can indeed interact only
when connected by rule \textsc{cut}, which then merges the interacting sequents together (since they now depend on each other).

Proposition 2.2 below states that syntax and typing of CT form a strict generalisation of CP. The proof theory itself is a strict extension of CLL since, e.g., \( P \vdash 1 \otimes \bot \) is provable in CT but not in CLL.

**Proposition 2.2.** If \( P \vdash \gamma \) in CP then \( P \vdash \gamma \) in CT but not vice versa.

## 3 Semantics

We now move to defining a semantics for CT in terms of a labelled transition system (LTS). The key novelty of our approach is viewing proofs as states of the LTS, and proof transformations as transitions. More specifically, we will show that the proof theory of CT can be given a labelled semantics in the SOS style \cite{18}, by viewing:

- inference rules as operations of a (sorted) signature;
- proofs as terms generated by this signature;
- (labelled) proof transformations as (labelled) transitions;
- and a specification of rules for deriving proof transformations as an SOS specification.

Then, a semantics for CT processes in terms of an SOS specification is obtained simply by reading off how the SOS specification of proof transformations manipulate the processes that they type.

We illustrate the intuition behind the LTS for proof transformations. Consider the proof for a judgement \( x().P \vdash \gamma, x: \bot \). By the strict correspondence between term constructors and typing rules, the proof necessarily has the following shape.

\[
P \vdash \gamma \\
x().P \vdash \gamma, x: \bot \\
\]

We can view rule \( \bot \) as the outermost operation used in the proof. Then, the proof of \( P \vdash \gamma \) is an argument of the operation, which is also parametric in channel \( x \).

This corresponds to the term constructor \( x().(\cdot) \) in the syntax of CT processes—which in this case takes \( P \) as parameter, i.e., the term corresponding to the proof of the premise. Thus, this operation is the proof equivalent of the term constructor \( x().(\cdot) \) in the syntax of CT processes, which denotes an observable action. Term constructors like this, also called action prefixes, are typically assigned a transition rule in process calculi. Therefore, this correspondence points at the transition axiom below (we box proofs for readability).

\[
\frac{P \vdash \gamma}{x().P \vdash \gamma, x: \bot} \\
\]

The label identifies the prefix constructor (i.e., rule name and parameter) and its syntax is inspired to common syntax for labels of action prefixes in process calculi.
Labels:

- $x[]$: close $x$
- $x[x']$: send $x'$ on $x$
- $x[\text{inl}]$: send select left
- $x[\text{inr}]$: send select right
- $x[A]$: send type $A$ on $x$
- $x[\text{use } x']$: open session on $x$ as $x'$
- $x[\text{spawn } x']$: request spawn as $x'$
- $x[\text{disp}]$: request dispose

Transitions:

\[
\begin{align*}
\text{Figure 2. Classical Transitions, process labelled transition system.}
\end{align*}
\]
calculi. By reading proof terms off the rule above we obtain the axiom below.

\[ x().P \xrightarrow{\{\}} P \]

This axiom defines the semantics of the constructor \( y()(-) \) as one would expect.

Following this methodology, we derive an LTS for proofs in CT, and reading its process part we obtain the LTS of CT processes given in Figure 2. We discuss the transition rules in the remainder of this section, by discussing the proof transformations that they originate from.

### 3.1 Multiplicatives and \( \text{mix} \)

The multiplicative fragment of CT is formed by the rules \( \otimes, \smallotimes, 1 \) and \( \bot \) together with the structural rules \( \text{mix}, \text{mix}_0 \) and \( \text{cut} \). Observe that rules from the first group have the “action prefix” form described above (we stretch the definition by regarding \( y()[]: 1 \) as \( y()[]: 0 \)). The corresponding axioms are given below.

\[
\frac{P \vdash \gamma, x': A \mid \delta, x: B}{x[x']: A \otimes B} \quad \frac{P \vdash \gamma, x': A \otimes B}{P \vdash \gamma, x': A \mid \delta, x: B}\]

\[
\frac{P \vdash \gamma, y': A, y: B}{y(y'): A \smallotimes B} \quad \frac{y(y'): A \smallotimes B}{P \vdash \gamma, y': A, y: B}\]

\[
\frac{x[] \vdash x: 1}{x[]: 1 \xrightarrow{0} \emptyset}\]

We extend the notion of duality from types to labels; for a label \( \alpha \) we write \( \alpha^\perp \) for any label that describes an action that is dual to that of \( \alpha \). Observe that label duality is not an involution but a binary relation: \( x[x']: A \otimes B \) and \( y(y')': A^\perp \smallotimes B^\perp \) are termed dual regardless of the name parameters, likewise for \( x[]: 1 \) and \( y(): \bot \).

The derivation rules associated to rule \( \text{mix} \) are listed below.

\[
\frac{P \vdash \Gamma}{P' \vdash \Gamma'} \quad \frac{P \vdash \Gamma Q \vdash \Delta}{P | Q \vdash \Gamma | \Delta} \quad \frac{P \vdash \Gamma \quad Q \vdash \Delta \quad \text{bn}(\alpha) \cap \text{fn}(Q) = \emptyset}{P' | Q \vdash \Gamma' | \Delta} \quad \text{PAR}_1
\]

\[
\frac{P' \vdash \Gamma' \quad Q \vdash \Delta \quad \text{bn}(\alpha) \cap \text{fn}(Q) = \emptyset}{P' | Q \vdash \Gamma' | \Delta} \quad \text{PAR}_2
\]
Rules \(\text{PAR}_1\) and \(\text{PAR}_2\) transform one of the two parallel components composed by rule \(\text{MIX}\) given that the transformation preserves non-interference in the result (disjointness of names). This condition follows from well-formedness of hypersequents \(\Gamma \mid \Delta, \Gamma' \mid \Delta\) but must be explicitly listed as a premise if we read off only the process part of the rule, yielding exactly the process transition that one would expect for internal \(\pi\)-calculus. Rule \(\text{SYN}\) pairs dual transformations of parallel components.

Alternatively to rule \(\text{SYN}\), one may combine sets of all transformations instead of just duals, delegating pairing to transition rules for \(\text{CUT}\). This would yield a “true concurrency” interpretation of CT instead of the standard semantics, which we leave to future work.

Communication under \(\text{CUT}\) is modelled by transitions derived with the rules below, one for each type of dual labels.

These transformations do not interact with the context nor have any effect on the type and are hence labelled with \(\tau\) as common for process calculi. Moreover, they correspond to cut elimination in CP [21, Fig. 3]. Finally, unrestricted actions are
modelled by simply propagating transitions as formalised by the rule below.

\[
\begin{array}{c}
P \vdash \Gamma \mid \gamma, x : A \mid \delta, y : A^\perp \quad \implies \quad P' \vdash \Gamma' \mid \gamma', x : A \mid \delta', y : A^\perp \quad \text{x, y } \notin n(\alpha) \\
\frac{P \vdash \Gamma \mid \gamma, x : A \mid \delta, y : A^\perp \quad \text{(}\nu xy)P \vdash \Gamma \mid \gamma, \delta \quad \text{CUT}}{P' \vdash \Gamma' \mid \gamma', x : A \mid \delta', y : A^\perp \quad \text{(}\nu xy)P' \vdash \Gamma' \mid \gamma', \delta' \quad \text{CUT}}
\end{array}
\]

3.2 Additives

The transition rules modelling (left) selection and choice are given below and are obtained following the reasoning discussed above.

\[
\begin{array}{c}
P \vdash \gamma, x : A \\
\text{x[inl].} P \vdash \gamma, x : A \oplus B \quad \oplus_1 \quad P \vdash \gamma, x : A
\end{array}
\]

\[
\begin{array}{c}
P \vdash \gamma, y : A \quad Q \vdash \gamma, y : B \\
\text{y.case}(P, Q) \vdash \gamma, y : A \& B \quad \& \quad y[inl] : A \& 1 \\
P \vdash \gamma, x : A \oplus B \mid \delta, y : A^\perp \& B^\perp \quad \bot \{x[inl] : A \oplus 1, y[inl] : A^\perp \& 1 \}
\end{array}
\]

\[
\begin{array}{c}
P \vdash \Gamma \mid \gamma, x : A \oplus B \mid \delta, y : A^\perp \& B^\perp \\
\frac{P' \vdash \Gamma \mid \gamma, x : A \oplus B \mid \delta, y : A^\perp \& B^\perp \quad \text{(}\nu xy)P' \vdash \Gamma' \mid \gamma, \delta \quad \text{CUT}}{P' \vdash \Gamma \mid \gamma, x : A \oplus B \mid \delta, y : A^\perp \& B^\perp \quad \text{(}\nu xy)P' \vdash \Gamma' \mid \gamma, \delta \quad \text{CUT}}
\end{array}
\]

Rules for right selection are symmetric.

\[
\begin{array}{c}
P \vdash \gamma, x : B \\
\text{x[inr].} P \vdash \gamma, x : A \oplus B \quad \oplus_2 \quad P \vdash \gamma, x : B
\end{array}
\]

\[
\begin{array}{c}
P \vdash \gamma, y : A \quad Q \vdash \gamma, y : B \\
y.case(P, Q) \vdash \gamma, y : A \& B \quad \& \quad y[inr] : A \& 2 \\
Q \vdash \gamma, y : B
\end{array}
\]
3.3 Links

There is one transition for \( \text{ax} \), defined by the axiom below (akin to the axiom for 1).

\[
\frac{x \rightarrow y \vdash x: A^\perp, y: A}{\text{AX}} \quad \frac{x \rightarrow y: \text{ax} \ A}{x \rightarrow y: \text{ax} \ A \rightarrow 0 \cdot .}
\]

Rules \( \text{AX}_1 \) and \( \text{AX}_2 \) below correspond to cuts of dual uses of rule \( \text{AX} \).

\[
\frac{P \vdash \Gamma \ | \ x: A^\perp, y: A | \ \gamma, z: A^\perp}{x \rightarrow y: \text{ax} \ A \rightarrow} \quad \frac{Q \vdash \Gamma \ | \ \gamma, z: A^\perp}{Q \vdash \Gamma \ | \ \gamma, x: A^\perp}
\]

\[
\frac{P \vdash \Gamma \ | \ x: A^\perp, y: A | \ \gamma, z: A^\perp}{(\nu y) P \vdash \Gamma \ | \ \gamma, x: A^\perp \ \tau \quad \frac{Q \vdash \Gamma \ | \ \gamma, x: A^\perp}{Q \vdash \Gamma \ | \ \gamma, x: A^\perp}}
\]

\[
\frac{P \vdash \Gamma \ | \ \gamma, w: A | \ x: A^\perp, y: A}{x \rightarrow y: \text{ax} \ A \rightarrow} \quad \frac{Q \vdash \Gamma \ | \ \gamma, w: A}{Q \vdash \Gamma \ | \ \gamma, y: A^\perp}
\]

\[
\frac{P \vdash \Gamma \ | \ \gamma, w: A | \ x: A^\perp, y: A}{(\nu w) P \vdash \Gamma \ | \ \gamma, y: A \ \tau \quad \frac{Q \vdash \Gamma \ | \ \gamma, y: A^\perp}{Q \vdash \Gamma \ | \ \gamma, y: A^\perp}}
\]

3.4 Exponentials

In CT, clients can interact with servers in three ways: they request a service, dispose of a server, or request server duplication. Requesting a service is modelled by the following two (dual) axioms and rule (for cut elimination).

\[
\frac{P \vdash \gamma, y: A}{x[\text{use } y], P \vdash \gamma, x: ?A \ \rightarrow \ x[\text{use } y]: ?A \rightarrow} \quad \frac{P \vdash \gamma, y: A}{P \vdash \gamma, y: A}
\]

\[
\frac{(\nu x)(y) P \vdash \gamma, x: !A \ \rightarrow \ x(\text{use } y): !A \rightarrow} \quad \frac{P \vdash \gamma, y: A}{P \vdash \gamma, y: A}
\]
Server disposal is captured by the following dual axioms and rule.

$$P \vdash \gamma, x : !A \mid \delta, y : ?A$$

$$Q \vdash \gamma, x' : A \mid \delta', y' : A$$

$$P \vdash \gamma, x : !A \mid \delta, y : ?A$$

$$Q \vdash \gamma, x' : A \mid \delta', y' : A$$

$$\frac{P \vdash \gamma, x : !A \mid \delta, y : ?A}{(\nu x y) P \vdash \gamma, \delta} \text{ CUT}$$

Finally, server duplication is defined by the following dual axioms and rule. We use the convention of adding a prime to indicate a homomorphic copy where every free name $z$ is replaced with $z'$, likewise for (hyper)sequents.

$$P \vdash \gamma, x : ?A, x' : ?A$$

$$x[\text{spawn } x']. P \vdash \gamma, x : ?A$$

$$P \vdash \gamma, x : ?A, x' : ?A$$
One may wonder why server dependencies are handled by rules !w and !c and not by axioms for !, there are two main reasons: The first is that in this way resource use is observable from labels as well as the type in the underlying proof-transformation. The second is adherence with CP where the contraction rule is applied outside cuts (cf. [21, Fig. 3]) meaning that dependencies are (silently) duplicated outside the restriction where they are needed—as in rule !c. We observe however, that it is possible to move resource management to axioms for server disposal and spawning once rule for external (i.e. at the hypersequent level) contraction is added to the type system (external weakening is derivable in this case).

3.5 Polymorphism

Polymorphism is achieved by communicating types, according to the following transition axioms and rule.
4 Metatheory: Subject Reduction, Progress, Termination

The transition rules of CT are derived from proof transformations that preserve provability. Observe that all rules for \( \tau \)-transitions enjoy type preservation: preserve the types in the judgement in the conclusion remain unchanged. All the other transitions, for observable actions, preserve provability: types change depending on the action(s), but the concluding judgement remains valid. Thus we immediately obtain the following theorem.

**Theorem 4.1 (Subject Reduction).** Let \( P \vdash \Gamma \) and \( P \overset{\alpha}{\rightarrow} Q \). Then:

- \( \alpha = \tau \) if and only if \( Q \vdash \Gamma \);
- \( \alpha \neq \tau \) if and only if \( Q \vdash \Delta \) for some \( \Delta \neq \Gamma \).

Theorem 4.1 formalises that \( \tau \)-transitions of a process \( P \) have no dependencies on the context, since they do not influence the type of \( P \). This matches the intuition of LTS semantics for the \( \pi \)-calculus, where \( \tau \)-transitions capture internal “unobservable” moves. In CT, performing unobservable moves coincides with type-preserving proof transformations.

CT also enjoys progress, in the sense that well-typed programs are either terminated or admit a transition.

**Theorem 4.2 (Progress).** If \( P \vdash \Gamma \), then either \( P \) is terminated or there exist \( \alpha \) and \( Q \) such that \( P \overset{\alpha}{\rightarrow} Q \).

CT enjoys weak termination. The only source of divergence in CT is the term for replicated processes, more precisely when a server is not paired (cut) with its client. Then, we can observe an infinite number of replications. However, we can also decide at any moment to choose the transition that visibly disposes of the server. In the following, \( \overset{\alpha}{\rightarrow}^* \) is the transitive closure of \( \overset{\alpha}{\rightarrow} \), i.e., \( P \overset{\alpha}{\rightarrow}^* Q \) means \( P_1 \overset{\alpha_1}{\rightarrow} \cdots \overset{\alpha_n}{\rightarrow} Q \) for some \( n \) (possibly zero, in which case \( P = Q \) and \( \overset{\alpha}{\rightarrow} \) is empty).

**Theorem 4.3 (Weak Termination).** If \( P \vdash \Gamma \), then there exist \( \overset{\alpha}{\rightarrow} \) and \( Q \) such that \( P \overset{\alpha}{\rightarrow} Q \) and \( Q \) is terminated.
5 Related Work

Since its inception, linear logic was described as the logic of concurrency [10]. Correspondences between the proof theory of linear logic and variants of the $\pi$-calculus emerged soon afterwards [1, 4], by interpreting linear logic propositions as types for channels. Later, linearity inspired also the seminal theories of linear types for the $\pi$-calculus [13] and session types [11]. Even though the two theories do not use exactly linear logic, the work by Dardha et al. [9] shows that the link is still strong enough that session types can be encoded into linear types.

It took more than ten years for a formal correspondence between linear logic and (a variant of) session types to emerge, with the seminal paper by Caires and Pfenning [5]. This then inspired the development of Classical Processes (CP) by Wadler [21], which we have already discussed in the Introduction. We have extended this line of work to labelled transition systems.

The idea of extending linear logic to hypersequents for typing processes is not new. Specifically, in [7], the multiplicative-additive fragment of intuitionistic linear logic is extended with hypersequents to type choreographies (global descriptions of process communications). Differently, CT is based on CLL, uses hypersequents to type processes, and deals also with exponentials and polymorphism. The major difference is that the rules for manipulating hypersequents are different w.r.t. [7]. In particular, hypersequents can be formed in [7] only when sequents share resources (compare to our $\text{mix}$, which requires the opposite), and these resource sharings are then explicitly tracked using an additional connection modality (which is not present in CT). As a consequence, the work in [7] does not support an LTS for the same reasons that we discussed for CP in the Introduction. Adding a connection modality to CT might be interesting for providing an LTS semantics to the choreographies studied in [7].

6 Conclusions

We believe that CT might spark a new line of research on the observable behaviour of processes typed with the theory of linear logic. An immediate direction would be studying behavioural theories for CT. For example, standard bisimulation techniques have not been applied to these calculi yet, and instead new definitions of behavioural equivalences based on logical relations or denotational semantics have been devised for linear logic proof terms [2, 17]. The main reason is that linear logic exhibits more concurrent behaviour than usual, which makes contextual equivalence more coarse than in the $\pi$-calculus. For example, the processes $x().y().P$ and $y().x().P$ would be distinguished by standard (strong) bisimulation, but they are contextually equivalent because typing enforces $x$ and $y$ to be connected to separate parallel processes, as observed in [2]. We believe that adding delayed actions (or variations thereof) bridge this gap, e.g., these two processes in particular would now be equated by bisimulation. A thorough investigation of the relation between bisimulation and contextual equivalence in CT is left to future work.
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