Construction of nonsingular cosmological solutions in string theories

Shinji Tsujikawa

Research Center for the Early Universe, University of Tokyo, Hongo, Bunkyo-ku, Tokyo 113-0033, Japan
email: shinji@resceu.s.u-tokyo.ac.jp

Abstract

We study nonsingular cosmological scenarios in a general $D$-dimensional string effective action of the dilaton-modulus-axion system in the presence of the matter source. In the standard dilatonic Brans-Dicke parameter ($\omega = -1$) with radiation, we analytically obtain singularity-free bouncing solutions where the universe starts out in a state with a finite curvature and evolves toward the weakly coupled regime. We apply this analytic method to the string-gas cosmology including the massive state in addition to the leading massless state (radiation), with and without the axion. We numerically find bouncing solutions which asymptotically approach an almost radiation-dominant universe with a decreasing curvature irrespective of the presence of the axion, implying that inclusion of the matter source is crucial for the existence of such solutions for $\omega = -1$. In the theories with $\omega \neq -1$, it is possible to obtain complete regular bouncing solutions with a finite dilaton and curvature in both past and future asymptotics for the general dimension, $D$. We also discuss the case where dilatonic higher-order corrections are involved to the tree-level effective action and demonstrate that the presence of axion/modulus fields and the matter source does not significantly affect the dynamics of the dilaton-driven inflation and the subsequent graceful exit.
1 Introduction

String cosmology has continuously received much attention with the development of string theory as a possible candidate to unify all fundamental interactions in nature. For cosmologists it is very important to test the viability of string theory by extracting cosmological implications from it. In particular string cosmology has an exciting possibility to resolve the big-bang singularity which plagues in General Relativity.

Among the string-inspired cosmological scenarios proposed so far, the pre-big-bang (PBB) model based on the low energy, tree-level string effective action has been widely studied (see ref. for a recent review). In this scenario there exist two disconnected branches, one of which corresponds to the dilaton-driven inflationary stage and another of which is the Friedmann branch with decreasing curvature. Then string corrections to the effective action can be important around the high curvature regime where the branch change occurs. Recently proposed Ekpyrotic/Cyclic cosmologies have a similarity to the PBB scenario in the sense that the description in terms of the tree-level effective action breaks down around the collision of two branes in a five dimensional bulk.

When the universe evolves toward the strongly coupled, high curvature regime with growing dilaton, it is inevitable to implement higher-order corrections to the tree-level action. Indeed it was found that two branches can be smoothly joined by taking into account the dilatonic higher-order derivative and loop corrections in the context of PBB and Ekpyrotic scenarios. In the system where a modulus field is dynamically important rather than the dilaton, Antoniadis et al. showed that the big-bang singularity can be avoided by including the Gauss-Bonnet curvature invariant coupled to the modulus.

In contrast we can consider the case where the dilaton evolves toward the weakly coupled region. In the brane-gas cosmology and the recently proposed string-gas cosmology, the universe starts out from a dense and hot state filled with a gas of fundamental branes and strings in thermal equilibrium. These are originated from the work of Brandenberger and Vafa (see also ref. in an attempt to construct singularity-free cosmological scenarios using T-duality. In these models it is typically assumed that the cosmic evolution does not reach the strongly coupled regime in order to avoid the breakdown of the effective ten-dimensional background description (see ref. for the parameter regions where the effective ten-dimensional description of the physics is no longer valid). In particular, in the context of the string-gas cosmology, it was shown that only the large 3 dimensions expand and asymptotically approach the radiation dominant universe while the small 6 or 7 dimensions are kept undetectably small.

In order to obtain viable nonsingular cosmological scenarios in string theories, the string coupling and the scalar curvature are required to be finite during the cosmological evolution. In the standard tree-level action with the dilatonic Brans-Dicke parameter $\omega = -1$, it is generally difficult to construct nonsingular solutions in the absence of the matter source term. This comes from the fact that the evolution toward the strongly coupled regime reaches the curvature singularity in the tree-level action and that the evolution toward the weakly coupled regime does not have enough power to lead to bouncing solutions. Although bouncing solutions were found in ref. by including the axion field in the tree-level action, this corresponds to the case where the string coupling and the scalar curvature diverge in both past and future asymptotics. Therefore this can not be regarded as ideal singularity-free bouncing solutions unless some higher-order effects alter the dynamics of the system when the dilaton enters the
strong coupling regime.

We can take into account the radiation-like matter source which comes, for example, from the ideal string-gas [17] or the 5-form of the Ramond-Ramond sector of the type IIB superstring [24, 25, 26]. In the presence of the radiation matter and the axionic field in $d = 3$ spatial dimensions, Constantinidis et al. [24] showed that bouncing solutions can be obtained when the dilatonic Brans-Dicke parameter $\omega$ is negative (i.e., including the $\omega = -1$ case). When $-4/3 < \omega < 0$ these solutions exhibit divergent behavior of the scalar curvature as the dilaton evolves toward the large coupling regime ($\varphi \to \infty$), in which case Constantinidis et al. did not regard them as ideal nonsingular bouncing solutions. Meanwhile these solutions, in another asymptotic, tend to approach the radiation dominant universe with a decreasing curvature and a finite dilaton. Then it should be possible to have regular bouncing solutions if the initial state of the universe is in the weakly coupled region with a finite curvature as in the case of string/brane-gas cosmology.

In this paper we shall investigate singularity-free cosmological scenarios based on the effective action of the dilaton-modulus-axion system in the presence of the matter source term. We keep the dilatonic Brans-Dicke parameter $\omega$ arbitrary so that the effective action includes a wide variety of theories such as the $F$-theory [27, 28] or the multidimensional theory [29]. Indeed it was recently found that in the presence of the axion and the pure radiation complete regular bouncing solutions were found in both past and future asymptotics when $\omega < -3/2$ in the case of $d = 3$ spatial dimensions [25, 26]. With an application to string/brane-gas cosmology in mind, we shall extend the analysis to the general $d$ spatial dimensions with/without the axion and modulus fields. In fact there exist a number of bouncing solutions even for the tree-level action in the presence of the radiation source term. In particular we will make numerical analysis in the context of string-gas cosmology with and without the axion by including the massive state together with the leading massless state (radiation) and demonstrate the existence of nonsingular bouncing solutions even for $\omega = -1$.

When the system evolves toward the strongly coupled regime, it is likely that the evolution is altered by the higher-order corrections to the tree-level action. Therefore we shall also make numerical simulations for the case where the dilatonic high-order derivative and loop corrections are included in the dilaton-modulus-axion system with a matter source term. We will show that these higher-order corrections typically dominate the dynamics of the system, by which the big-bang singularity can be avoided as in the case where the only dilaton is present.

2 Model

Consider the following effective action with three scalar fields $\varphi$, $\chi$ and $\sigma$:

$$S = \int d^Dx \sqrt{-g_D} \left[ \frac{e^{-\varphi}}{2} \left( R - \omega (\nabla \varphi)^2 - (\nabla \chi)^2 - e^{-(n-1)\varphi} (\nabla \sigma)^2 \right) + L_m + L_c \right], \quad (1)$$

where $R$ is the scalar curvature in $D$-dimensions with metric $g_{\mu\nu}$ with $g_D = \det g_{\mu\nu}$. $L_m$ represents the matter source term and $L_c$ corresponds to the higher-order corrections to the tree-level action. The action (1) contains a variety of models derived from string theories. In what follows we shall briefly show such examples.

• Low energy effective string action
The lowest order $\bar{D}$-dimensional string effective action coming from the beta function of the string world-sheet is characterized by (1)

$$S_{\bar{D}} = \int d^\bar{D}x \sqrt{-\bar{g}_\bar{D}} \frac{e^{-\bar{\psi}}}{2} \left[ R_{\bar{D}} + (\nabla \bar{\psi})^2 - \frac{1}{12} H_{\mu \nu \lambda} H^{\mu \nu \lambda} \right],$$

(2)

where $\bar{\psi}$ is the dilaton, $H_{\mu \nu \lambda}$ is the antisymmetric tensor field. Let us compactify the $k = \bar{D} - 4$ dimensions by introducing a modulus field, $e^{k \chi}$. We define an effective dilaton in four dimensions as

$$\phi = \bar{\psi} - k \chi.$$

We also introduce a pseudo-scalar axion field $\sigma$ as

$$H_{\mu \nu \lambda} = e^{\phi} \epsilon_{\mu \nu \lambda \rho} \nabla^\rho \sigma$$

by taking into account the fact that $H_{\mu \nu \lambda}$ has only one degree of freedom in 4 dimensions (1). Then we get the action (1) with $D = 4$, $\omega = -1$, $\chi = \sqrt{k \chi}$, $n = -1$, $L_m = 0$ and $L_c = 0$.

**String/Brane-gas cosmology**

The string-gas cosmology (17) is an attempt to explain why and how the three spatial dimensions become large and other 6 or 7 are kept to be small. The bulk action in 10 or 11 dimensions is assumed to take the same form as (2), but we have additional source matters due to the presence of the ideal string-gas in thermal equilibrium. The simplest version of the string-gas cosmology (17) corresponds to $\omega = -1$, $\chi = 0$, $\sigma = 0$, $L_m \neq 0$ and $L_c = 0$ in (1). Instead of introducing a modulus field, one considers the evolution of one scale factor for $(D - 1)$ spatial dimensions or two scale factors for large and small dimensions (17). In this work we shall also analyze the case where the axion is present ($\sigma \neq 0$). Note that the brane-gas cosmology discussed in refs. (12, 13) belongs to this class with a different matter source when we use the low energy bulk action (2) with $\omega = -1$.

**Cosmology with p-brane or Dp-brane solitons**

The $p$-brane or Dp-brane are the solitonic degrees of freedom on which string endpoints live. These can be fundamentally important when the string coupling is large, since they are light in the high curvature regime (20, 30). In the string $\sigma$-model whose metric is minimally coupled to the $p$-brane, Duff et al. (31) showed that the effective action can be described by (1) with $\chi = 0$, $\sigma = 0$, $L_m \neq 0$ and

$$\omega = -\frac{(D - 1)(\bar{p} - 2) - \bar{p}^2}{(D - 2)(\bar{p} - 2) - \bar{p}^2},$$

(3)

where $\bar{p} = p + 1$. From this we have $\omega = -1$ for the string ($p = 1$). However $\omega$ depends upon the values of $d$ and $p$ for the $p$-branes with $p \neq 1$. Cosmology in this scenario was investigated in refs. (32, 33) in the absence of higher-order corrections to the tree-level action (see also ref. (34)).

**F-theory**

The superstring type IIB theory can be reformulated geometrically within the framework of 12 dimensional theory—called F-theory (27, 28). After the dimensional reduction from 12 to 10 dimensions, the following action may be obtained (26):

$$S = \frac{1}{2} \int d^{10}x \sqrt{-g_{10}} \left[ e^{-\varphi} \left( R_{10} + 3(\nabla \varphi)^2 - \frac{1}{12} F_{\mu \nu \lambda} F^{\mu \nu \lambda} - \frac{1}{8} F_{\mu \nu} F^{\mu \nu} \right) \right],$$

(4)
where the terms $F_{\mu\nu}$ and $F_{\mu\nu\lambda}$ come from the Ramond-Ramond (RR) sector of the type IIB superstring. This action is different from the standard string theory with $\omega = -1$. Making the similar dimensional reduction from 10 to 4 dimensions as [2] with the isotropization of the RR terms, we get the action (1) with $\omega = -3$, $\chi \neq 0$, $\sigma \neq 0$ ($n = -1$) and the radiation fluid term ($\mathcal{L}_m \neq 0$) coming from the 5-form of the RR sector. The axionic terms minimally coupled to the dilaton can also appear from the same RR sector [26]. This corresponds to the case of adding another field with $n = 0$ in the action (1).

- **Multidimensional cosmologies**

  The low energy limit of the string effective action may be reformulated in the context of multidimensional theories [36]. In its simplest form the multidimensional theories contain only the geometry, in which case one has $\omega = (1 - \tilde{d})/\tilde{d}$ with $\chi = \sigma = 0$ by the dimensional reduction ($\tilde{d}$ is the number of compactified dimensions). When a 2-form field or a conformal gauge field with $(\tilde{d} + 4)/2$-form is present, we have an axionic term $\sigma$ in eq. (1) with $n = -2/\tilde{d} + 1$ or $n = -2/\tilde{d}$, respectively. Therefore we keep the value of $n$ arbitrary so that multidimensional theories are also included in (1).

  Hereafter we shall analyze the cosmological solutions based on the action (1). For convenience we call the fields $\varphi$, $\chi$ and $\sigma$ as dilaton, modulus and axion, respectively, unless otherwise specified. We do not include the potential of scalar fields in the action. Extending the analysis of refs. [24, 25, 26] with $D = 4$, we will study the cosmological evolution in the general $D$-dimensional action with a time-dependent modulus field ($\dot{\chi} \neq 0$). In applying to string-gas or $p$-brane cosmologies, it is convenient to consider the case of the general dimension $D$. In addition the presence of the internal space (modulus) can alter the parameter range where the nonsingular solutions exist. We should also keep in mind that higher-order corrections to the tree-level action are inevitably important in the strongly coupled regime ($e^{\varphi} \gtrsim 1$). In later sections we shall numerically investigate the dynamics of the system in the action (1) when the dilatonic $\alpha'$ and loop corrections are taken into account ($\mathcal{L}_c \neq 0$).

  Let us consider the case where the $D = d + 1$ dimensional spacetime is described by a flat Friedmann-Robertson-Walker (FRW) metric with the line element

\[
 ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j, \tag{5}
\]

where $a(t)$ is the scale factor. In the context of the string (brane)-gas cosmology, it is possible to consider the case with two scale factors corresponding to the “large” and “small” dimensions. It was shown in ref. [17] that only the large dimensions can be dynamically important whereas the small dimensions are kept to be nearly constant. In such cases it is sufficient to analyze the case of the one scale factor described by eq. (5). However we will also discuss the case of two scale factors in sec. 5.

  Defining a new scalar field,

\[
 \phi = e^{-\varphi}, \tag{6}
\]

the variation of the action (1) yields the following equations of motion

\[
 H^2 = \frac{1}{d(d-1)\phi} \left( \omega \frac{\dot{\phi}^2}{\phi} + \phi \dot{\chi}^2 + \phi^n \dot{\sigma}^2 - 2dH\dot{\phi} + 2\rho + 2\rho_c \right), \tag{7}
\]

See ref. [35] for the construction of nonsingular cosmological solutions including the dilaton potential.
\[ \dot{H} = -\frac{1}{(d-1)\phi} \left( \omega \frac{\dot{\phi}^2}{\phi^2} + \phi \dot{\chi}^2 + \phi^n \dot{\sigma}^2 - H \ddot{\phi} + \ddot{\phi} + \rho + p + p_c + p_{c_c} \right), \quad (8) \]

\[ \ddot{\phi} + dH \dot{\phi} - \frac{\dot{\phi}}{2\omega} \left( \omega \frac{\dot{\phi}^2}{\phi^2} + \chi^2 + n \phi^n \dot{\sigma}^2 + R + \Delta_c \right) = 0, \quad (9) \]

\[ \ddot{\chi} + \left( dH + \frac{\dot{\phi}}{\phi} \right) \chi = 0, \quad (10) \]

\[ \ddot{\sigma} + \left( dH + n \frac{\dot{\phi}}{\phi} \right) \sigma = 0, \quad (11) \]

\[ \ddot{\rho} + dH (\rho + p) = 0, \quad (12) \]

where \( H \equiv \dot{a}/a \) is the Hubble rate. Here \( \rho \) and \( p \) stand for the energy and pressure density of the matter source, respectively. \( p_c \) and \( p_{c_c} \) correspond to the dilatonic higher-order corrections to the tree-level action with stress-energy tensor \( T_{\mu \nu} = (-\rho_c, p_c, p_{c_c}, p_{c_c}) \). \( \Delta_c \) comes from the variation of \( L_c \) with respect to \( \phi \).

The scalar curvature, \( R \), is expressed as

\[ R = d(d+1)H^2 + 2d\dot{H} \]

\[ = -\omega \frac{\dot{\phi}^2}{\phi^2} - \chi^2 - \phi^n \dot{\sigma}^2 + \frac{2}{(d-1)\phi} \left( -d^2 \dot{H} \dot{\phi} - \ddot{\phi} + \rho - dp + \rho_c - dp_{c_c} \right). \quad (13) \]

Substituting this relation for eq. (9), we get

\[ \ddot{\phi} + dH \dot{\phi} + \frac{(1-n)(d-1)\phi^n \dot{\sigma}^2 - 2(\rho - dp) - 2(\rho_c - dp_{c_c}) - (d-1)\phi \Delta_c}{2\{\omega(d-1) + d\}} = 0. \quad (14) \]

The equations (10) and (11) are easily integrated to give

\[ \dot{\chi} = \frac{A}{a^d \phi^n}, \quad \dot{\sigma} = \frac{B}{a^d \phi^n}, \quad (15) \]

where \( A \) and \( B \) are integration constants. Making use this relation and introducing a new time parameter,

\[ \tau \equiv \int a^{-d} dt, \quad (16) \]

the \( \phi \) equation (15) can be written in the form:

\[ \frac{d^2 \phi}{d\tau^2} + \frac{(1-n)(d-1)B^2}{2\{\omega(d-1) + d\}} \phi^{-n} - \frac{2(\rho - dp) + 2(\rho_c - dp_{c_c}) + (d-1)\phi \Delta_c}{2\{\omega(d-1) + d\}} a^2 d = 0. \quad (17) \]

In the absence of the correction \( L_c \), the last term of eq. (18) vanishes for \( \rho = p = 0 \) (no matter source) or \( \rho = dp \) (radiation). In these cases it is possible to obtain analytic solutions for the general spatial dimension \( d \) even in the presence of axion and modulus fields. Notice that the simplest pre-big-bang scenario [3] corresponds to the case of \( \rho = p = 0 \) without the axion field.

The radiation matter (\( \rho = dp \)) can appear naturally in the context of string-gas cosmology at finite temperature [17] or the 5-form existing in the RR sector of the type IIB superstring.
When $\rho = dp$, the integration of eq. (12) gives $\rho = \rho_0 a^{-(d+1)}$, with $\rho_0$ being a constant. Introducing a new parameter, $b = a \phi^{1/(d-1)}$, eq. (7) can be written for $L_c = 0$ as

$$d(d - 1) \phi^2 \left( \frac{b'}{b} \right)^2 = 2 \rho_0 b^{d-1} + A^2 + B^2 \phi^{1-n} + \frac{\omega(d - 1) + d}{d - 1} \phi^2,$$

where a prime denotes the derivative with respect to $\tau$. This equation includes the case without the matter source by setting $\rho_0 = 0$.

In the next two sections we shall derive the analytic solutions for the scale factor and the dilaton in the case of the radiation or no matter source without the correction $L_c$. For the general equations of state with $p = \gamma \rho$, it is not so easy to find analytic solutions due to the fact that the last term in eq. (18) does not vanish.$^2$ In sec. 5 we will analyze the dynamics of string-gas cosmology as an application by including the massive state (Kaluza-Klein and winding modes) in addition to the leading massless state (radiation), with and without the axion. In this case the string-gas state is not purely the radiation. Since it is difficult to proceed analytic approach when the higher-order correction $L_c$ is present, we shall make numerical analysis separately in sec. 6 for the general action (1). Hereafter we will consider the case of $d > 1$.

### 3 Low energy tree-level action without axion ($\sigma = 0$ and $L_c = 0$)

Let us first analyze the low energy tree-level action without the axion ($L_c = 0$ and $\sigma = 0$). When $\rho = p = 0$ (no matter source) or $\rho = dp$ (radiation), eq. (18) is integrated to give

$$\phi = c \tau,$$

where $c$ is a constant. We can set another integration constant to be zero without loss of generality. Since $\phi = e^{-\varphi}$ is required to be larger than zero, we have $\tau > 0$ for $c > 0$ and $\tau < 0$ for $c < 0$. The string coupling, $g_s^2 = e^{\phi} = \phi^{-1}$, diverges as $\tau \to 0$.

Since we are now considering the case with $B = 0$, eq. (19) yields

$$\int \frac{db}{b \sqrt{Mb^{d-1} + N}} = \pm \int \frac{d\tau}{\phi},$$

where

$$M = \frac{2 \rho_0}{d(d - 1)}, \quad N = \frac{1}{d(d - 1)} \left[ A^2 + \frac{\omega(d - 1) + d}{d - 1} c^2 \right].$$

The integral in the l.h.s. of eq. (21) is different depending on the sign of $N$. For example, $N$ is positive in the standard string theory ($\omega = -1$) with $d > 1$. However, $N$ can be negative for the theories such as the solitonic $p$-brane/D$p$-brane or multidimensional theories. In what follows we shall discuss the positive and negative $N$ cases separately.$^2$ Note, however, that it is still possible to obtain analytic solutions for some special equations of state in the absence of the axion and the modulus$^3$.  

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3.1 Case of $N > 0$

When $N > 0$ the integration of eq. (21) leads to

$$\left(\sqrt{M} b^{d-1} + N + \sqrt{N}\right)^2 = \frac{1}{\xi^2}.$$  \hspace{1cm} (23)

where

$$\xi = \xi_0 (c \tau)^q \quad \text{with} \quad q = \pm \frac{\sqrt{N}}{2c} (d - 1).$$ \hspace{1cm} (24)

Here $\xi_0$ is a positive integration constant. Then the analytic solution for the scale factor can be written as

$$a = \left(\frac{2\sqrt{N}}{\sqrt{\phi (\xi - M/\xi)}}\right)^{2/(d-1)}$$

$$= f(\phi)^{2/(d-1)} \quad \text{with} \quad f(\phi) = \frac{2\sqrt{N}}{\xi_0 \phi^{1/q} - (M/\xi_0) \phi^{-q}}.$$ \hspace{1cm} (25)

This indicates that the evolution of the scale factor is determined in terms of the field $\phi$ for the case (20). The simplest pre-big-bang scenario with a constant modulus corresponds to the case of $A = 0$, $B = 0$, $M = 0$, $d = 3$ and $\omega = -1$, thereby yielding $q = \pm \sqrt{3}/6$ and

$$a = \frac{2\sqrt{N}}{\xi_0} \phi^{(-3+\sqrt{3})/6}. \hspace{1cm} (27)$$

This represents an inflationary solution with decreasing $\phi$ from the weakly coupled regime ($\phi \to \infty$) to the strongly coupled regime ($\phi \to 0$). Note that the solution eventually reaches the curvature singularity as $\phi \to 0$. If we consider the case of increasing $\phi$, the solution presented in eq. (27) describes a contracting universe. In this case the solution does not connect to our expanding branch.

The situation changes when the radiation is taken into account ($M \neq 0$). Differentiating eq. (26) with respect to $\phi$, we have

$$\frac{df(\phi)}{d\phi} = -\sqrt{N} (1 + 2q) \frac{\xi_0 \phi^{\frac{1}{2} + q} - (1 - 2q)(M/\xi_0) \phi^{\frac{1}{2} - q}}{(\xi_0 \phi^{q} - (M/\xi_0) \phi^{-q})^2}.$$ \hspace{1cm} (28)

This means that it is possible to have bouncing solutions with the increase of $\phi$ as long as $-1/2 < q < 0$. In this case, if the condition $(1 + 2q)\xi_0 \phi^{\frac{1}{2} + q} > (1 - 2q)(M/\xi_0) \phi^{\frac{1}{2} - q}$ is satisfied initially with a finite value of the dilaton ($\phi_i \neq 0$), the universe contracts at the initial stage. This is followed by a bounce due to the growth of the $(1 - 2q)(M/\xi_0) \phi^{\frac{1}{2} - q}$ term. The bounce occurs when the field $\phi$ passes at

$$\phi_* = \left(\frac{1 - 2q}{1 + 2q} \frac{M}{\xi_0^2}\right)^{1/(2q)}.$$ \hspace{1cm} (29)

The universe begins to expand for $\phi > \phi_*$ with the increase of $\phi$. As found from eq. (26) the scale factor goes to infinity as $\phi$ approaches the critical value,

$$\phi_c = \left(\frac{M}{\xi_0}\right)^{1/(2q)} \hspace{1cm} (30)$$
Let us find the asymptotic behavior of $a$ around $\phi = \phi_c$. Setting $\phi = \phi_c - c\Delta \tau$ with $\Delta \tau$ being a small time parameter, we find the scale factor \[26\] is proportional to $(\Delta \tau)^{-2/(d-1)}$. Since the cosmic time $t$ is related with $\tau$ by the relation $dt/d\tau = a^d$, the evolution of the scale factor is described by
\[
a \propto t^{2/(d+1)} \quad \text{(for} \quad \phi \to \phi_c) ,
\]
which shows the radiation dominant behavior.

In the limit of $\phi \to \phi_c$, we have $t \to \infty$ as $\Delta \tau \to 0$. Then the scalar curvature asymptotically approaches zero as $R \propto t^{-2}$ for $\phi \to \phi_c$. In addition the string coupling, $g_s^2 = 1/\phi$, is also finite in this case. This means that nonsingular bouncing solutions with finite curvature and dilaton can be obtained by taking into account the radiation in the low energy tree-level string action. Notice that we are not considering the regime around the strong coupling limit ($\phi \to 0$). The universe is assumed to start out from the region with finite dilaton and curvature, as in the string/brane-gas cosmologies. Then the dilaton evolves toward the weakly coupled region (see fig. \[1\]), which asymptotically approaches the radiation dominant universe given by \[31\]. In sec. 5 we will numerically investigate the existence of such bouncing solutions in string-gas cosmology taking into account the massive state in addition to the leading massless state.

These bouncing solutions exist for $-1/2 < q < 0$, whose condition is written as
\[
\omega < -\frac{A^2}{c^2} . \quad \text{(32)}
\]
In the absence of the modulus, this condition yields $\omega < 0$. Note, however, that the large initial value of the modulus restricts the range for the existence of the bouncing solutions.

If the condition, $(1+2q)\xi_0^2 \phi^2 + q < (1-2q)(M/\xi_0)\phi^{1-q}$, is satisfied initially with $-1/2 < q < 0$, the scale factor \[26\] grows from the beginning with the increase of $\phi$. Then it asymptotically approaches the radiation dominant solution given by eq. \[31\]. This solution was numerically found as well in the context of string-gas cosmology \[17\]. Note that when $|q| \geq 1/2$ we have either ever expanding or contracting solutions.

It is possible to obtain bouncing solutions provided the condition \[32\] is satisfied and the dilaton $\varphi$ decreases. In contrast, it is worth mentioning the case where the dilaton evolves toward the strongly coupled regime. In this case, the scalar curvature does not necessarily converge in the limit of $\phi \to 0$. Around the region of $\phi = 0$, the scale factor is estimated as $a \propto \phi^{-1+2q} d^{-\frac{1}{d+1}}$ for $-1/2 < q < 0$. Combining this relation with $dt/d\tau = a^d$, we have
\[
t = t_0 + \tilde{c} \tau^{-\frac{1+2q}{d+1}} , \quad \text{(33)}
\]
where $t_0$ and $\tilde{c}$ are constants. Then the scale factor around $\phi = 0$ is expressed as
\[
a \propto (t - t_0)^{\frac{1+2q}{d+1}} . \quad \text{(34)}
\]
When $q > -1/(2d)$, one has $t \to \infty$ as $\tau \to 0$ from eq. \[33\]. Since the power in eq. \[34\] is positive in this case, the universe expands for $\tau \to 0$. Meanwhile we have $t \to t_0$ as $\tau \to 0$ for $q < -1/(2d)$. Since we are considering the case of $q > -1/2$, the power in eq. \[34\] is negative for $-1/2 < q < -1/(2d)$, thereby leading to the growth of $a$ as $t \to t_0$.

Since the scalar curvature evolves as $R \propto (t - t_0)^{-2}$ for the solution \[34\], it asymptotically decreases toward zero only for $q > -1/(2d)$. When $-1/2 < q < -1/(2d)$, $R$ diverges in the
Figure 1: A nonsingular bouncing solution in the low energy tree-level action without the axion in the presence of the radiation ($\rho = dp$). With the growth of $\phi = e^{-\varphi}$, the universe contracts and exhibits the bounce at $\phi = \phi_*$. After that the scale factor begins to grow and the system asymptotically approaches the radiation dominant universe as $\phi \to \phi_c$.

limit of $\phi \to 0$. Therefore, when the system approaches the strongly coupled regime, we have bouncing solutions with finite curvature for $q > -1/(2d)$, i.e.,

$$\omega < -1 - \frac{1}{d} - \frac{A^2}{c^2}. \quad (35)$$

This does not include the string case ($\omega = -1$). Combining (35) with the condition $N > 0$, we have $-d^{-1} - \frac{A^2}{c^2} < \omega < -1 - \frac{1}{d} - \frac{A^2}{c^2}$. When $d = 3$ and $A = 0$ (no modulus), this condition yields $-3/2 < \omega < -4/3$.

We have to keep in mind that the description of the tree-level action ceases to be valid as $\phi$ evolves toward the strong coupling region ($\phi \to 0$). The higher-order corrections to the tree-level action are expected to be important in this regime. On the contrary, nonsingular bouncing solutions toward the weakly coupled regime shown in fig. 1 can be regarded as ideal ones. In addition the dilaton and the scalar curvature do not diverge, provided that the universe starts out from the region with finite $\varphi$ and $R$.

3.2 Case of $N < 0$

Let us next consider the case of $N < 0$. For example, when $d = 3$ and $A = 0$, this condition corresponds to $\omega < -3/2$. The only difference relative to the $N > 0$ case is the integral in the l.h.s. of eq. (21). For negative $N$ we have

$$\arctan\frac{Mb^{d-1} + N}{|N|} = \log \xi_0 |c\tau + \phi_0|^q \quad \text{with} \quad q = \pm \frac{\sqrt{|N|}}{2c} (d - 1), \quad (36)$$
where $\xi_0$ is a positive constant. Then the scale factor can be written as

$$a = \left( \sqrt{\frac{|N|}{M}} \frac{1}{\sqrt{\phi \cos (\log \xi_0 \phi^q)}} \right)^{2/(d-1)}. \quad (37)$$

This indicates that nonsingular bouncing solutions exist in the interval

$$\frac{1}{\xi_0} \exp \left[ \left( 2j - \frac{1}{2} \right) \pi \right] < \phi^q < \frac{1}{\xi_0} \exp \left[ \left( 2j + \frac{1}{2} \right) \pi \right], \quad (38)$$

where $j$ is an integer. From eq. (37) we find that the bounce occurs at $\cos(\log \xi_0 \phi^q) = q + \sqrt{q^2 + 1}$, i.e.,

$$\phi_* = \frac{1}{\xi_0^{1/q}} \exp \left[ \frac{1}{q} \arccos \left( q + \sqrt{q^2 + 1} \right) \right]. \quad (39)$$

Then we have $q < 0$ from the condition, $|\cos(\log \xi_0 \phi^q)| \leq 1$. The asymptotic behavior of the scale factor is given by $a \propto t^{2/(d+1)}$ as the dilaton approaches $\phi \to (1/\xi_0^{1/q}) \exp[(2j \pm 1/2)\pi/q]$. In addition it is easy to show that the scalar curvature asymptotically decreases in the limit of $\phi \to (1/\xi_0^{1/q}) \exp[(2j \pm 1/2)\pi/q]$. Therefore the bouncing solution (37) is characterized by the radiation dominant universe in both asymptotics with finite values of the dilaton and the scalar curvature. We have a sequence of solutions (38) corresponding to the each integer value of $j$. This is different from the case of $N > 0$ where singularity-free solutions with finite dilaton exist only for $\phi_i < \phi < \phi_c$ ($\phi_i$ is the initial value). The key point is that the dilaton can be controlled without entering the strongly coupled region with $\phi \sim 0$.

In the absence of the axion with $\rho = dp$ (or $\rho = p = 0$) considered above, the dilaton evolves as the free massless field ($\phi'' = 0$). However the situation changes when the axion is not dynamically negligible.

4 Low energy tree-level action with axion ($\sigma \neq 0$ and $L_c = 0$)

In this section we implement the effect of the axion and discuss the difference of the dynamics compared to the case of $\sigma = 0$. First, the evolution of the dilaton is altered by the existence of the second term in eq. (18). When $L_c = 0$ and $\rho = dp$ (or $\rho = p = 0$), we find a conserved quantity, $E$, from eq. (18):

$$\frac{1}{2} \phi'^2 + \frac{(d-1)B^2}{2(\omega(d-1)+d)} \phi^{1-n} = E. \quad (40)$$

Making use of this relation, eq. (17) is reduced to the integral form (21) with

$$M = \frac{2\rho_0}{d(d-1)}, \quad N = \frac{1}{d(d-1)} \left[ A^2 + \frac{\omega(d-1)+d}{d-1} E \right]. \quad (41)$$

The only difference relative to the $\sigma = 0$ case is that $c^2$ in eq. (22) is replaced for $E$.

The integral in the l.h.s. of eq. (21) can be performed as in the previous section depending on the sign of $N$. In order to integrate the r.h.s. of eq. (21), it is required to make further classifications depending on the sign of the second term in eq. (40).
4.1 $\omega > -d/(d-1)$

When $\omega > -d/(d-1)$, it is convenient to introduce a new time parameter, $\eta$, satisfying

$$\phi = \phi_0 (\sin \eta)^{2/(1-n)} \quad \text{with} \quad \phi_0 = \left( \frac{E \omega (d-1) + d}{(d-1)B^2} \right)^{1/(1-n)}.$$  \hfill (42)

From eq. (10) we find the relation $d\phi/d\tau = \pm \sqrt{E} \cos \eta$. Together with eq. (11) we obtain

$$\frac{d\tau}{d\eta} = \pm \frac{2}{(1-n)\sqrt{E}} \phi_0 (\sin \eta)^{\frac{1+n}{1-n}}.$$  \hfill (43)

Making use of this relation it is easy to perform the integral (21). Notice that we are now considering the case with $N > 0$ since this is automatically satisfied for $\omega > -d/(d-1)$. The scale factor is expressed by the form (23) with

$$\xi = \xi_0 \left| \tan \frac{\eta}{2} \right|^q, \quad q = \pm \frac{d-1}{1-n} \sqrt{\frac{N}{E}}.$$  \hfill (44)

Namely the explicit form of $a$ can be written in terms of $\eta$ :

$$a = \left( \frac{2\sqrt{N}}{\sqrt{\phi_0 (\sin \eta)^{1/(1-n)}} \left\{ \xi_0 |\tan \eta/2|^q - (M/\xi_0) |\tan \eta/2|^{-q} \right\}} \right)^{2/(d-1)}.$$  \hfill (45)

The evolution of the scale factor is well understood by considering the behavior of $\xi$ and $\phi$ for $0 < \eta < \pi$ (or the region $2j\pi < \eta < (2j+1)\pi$ with $j$ being an integer).

Let us first investigate the case of $n < 1$ and $q < 0$. It is sufficient to analyze negative values of $q$ only, since the function $\xi$ is symmetric with respect to $\eta = \pi/2$ for the change of $q \rightarrow -q$. As found from eq. (25), $\xi$ is larger than $\sqrt{M}$. It is clear from eq. (26) that the universe expands as $\xi$ approaches $\xi_c \equiv \sqrt{M}$ (note that $\phi$ is finite in this limit). Consider the asymptotic behavior of the scale factor when $\xi$ evolves toward $\xi_c = \xi_0 |\tan(\eta_c/2)|^q$ with $M \neq 0$. Setting $\eta = \eta_c - \Delta \eta$ in eq. (25), we find that the scale factor evolves as $a \propto (\Delta \eta)^{-\frac{1}{1-q}}$ around $\eta = \eta_c$. Making use of the relation (15) and $d\tau/dt = a^{-d}$, we have $a \propto t^{2/(d+1)}$ with $t \rightarrow \infty$ as $\Delta \eta \rightarrow 0$. This represents the radiation dominant universe with decreasing curvature ($R \rightarrow 0$) as is similar to the case (31).

In another limit $\eta \rightarrow 0$, one has $\phi \rightarrow 0$ and $\xi \rightarrow \infty$ for $n < 1$ and $q < 0$. Since the evolution of the scale factor is approximately described by $a \propto \eta^{-\frac{2}{d-1}(1+n)+q}$ around $\eta = 0$, we get an expanding solution for $q > -\frac{1}{1-n}$. From eq. (11) and (43) this condition yields

$$\omega < \frac{A^2}{E},$$  \hfill (46)

which is similar to eq. (32). Following the similar process by which eqs. (33) and (34) are derived, we get the evolution of the scale factor around $\eta = 0$ as

$$a \propto (t - t_0)^{\frac{1+q(1-n)}{1+dq(1-n)}}.$$  \hfill (47)

Here $t = t_0 + \tilde{c} \eta^{-\frac{2}{d-1}(1+n)+dq}$ with $t_0$ and $\tilde{c}$ being integration constants. It is easy to show that the scalar curvature decreases toward zero for $q > -\frac{1}{d(1-n)}$, whereas $R$ diverges for $q < -\frac{1}{d(1-n)}$. 


If we want to construct bouncing solutions where the dilaton evolves toward $\phi = 0$ with finite curvature, this requires the condition $q > -\frac{1}{d(1-n)}$, i.e.,

$$\omega < -1 - \frac{1}{d} \frac{A^2}{E},$$

which is analogous to eq. (35). Combining this with the condition $\omega > -\frac{d}{(d - 1)}$, we have $-\frac{3}{2} < \omega < -\frac{4}{3}$ for $d = 3$ and $A = 0$ (no modulus), as shown in ref. [24]. Notice that the presence of the modulus makes the allowed range of $\omega$ narrower.

Again we should caution that the string coupling, $g_s^2 = \phi^{-1}$, diverges in the asymptotic limit $\phi \to 0$. In this sense these solutions can not be regarded as ideals ones, unless some higher-order effects are taken into account in the strong coupling regime. Meanwhile when the universe begins from a state with finite dilaton and curvature satisfying $\phi_i > 1$, it is possible to have nonsingular bouncing solutions provided that the solutions approach the region around $\xi = \xi_c$. In this case the dilaton is finite for the range $\phi_i < \phi < \phi_c$ with $\phi_c = \phi_0 \left( \sin \eta_c \right)^{2/(1-n)}$. The system asymptotically approaches the radiation dominant universe with decreasing curvature. One different point relative to the $\sigma = 0$ case is that the axion tends to work for the dilaton to turn back toward the strong coupling regime. Nevertheless the string coupling does not diverge as long as the dilaton approaches the value $\phi = \phi_c$ instead of $\phi = 0$.

When $n > 1$, both $\xi$ and $\eta$ diverge either in the limit of $\eta \to \pi$ or $\eta \to 0$, depending on the sign of $q$. Since the asymptotic evolution of the scale factor exhibits the contracting behavior, it is not possible to have bouncing solutions for $n > 1$.

At the final of this subsection, it is worth mentioning the case without the matter source ($M = 0$). When $n < 1$ and $q < 0$, the scale factor approaches the asymptotic solution given by eq. (47) in both limits of $\eta \to 0$ and $\eta \to \pi$. Therefore if the condition (48) is not satisfied, the scalar curvature diverges in both past and future asymptotics together with the infinite string coupling. The $\omega = -1$ case is contained in this class. The low energy tree-level string action ($\omega = -1$) with an axion leads to bouncing solutions as found in refs. [21, 22, 23], but these correspond to the solutions with divergent behavior of the dilaton and the curvature in past/future asymptotics. When the condition (48) is satisfied, the curvature asymptotically decreases but the string coupling diverges as $\eta \to 0$ and $\eta \to \pi$. From these arguments, we find that inclusion of the matter source ($M \neq 0$) is crucial for the construction of nonsingular bouncing solutions with finite dilaton and curvature even in the presence of the axion. In sec. 5 we will analyze the dynamics of the system with and without axion in the context of string-gas cosmology.

4.2 $\omega < -d/(d - 1)$

Let us next proceed to the case of $\omega < -d/(d - 1)$. Introducing a new time parameter, $\eta$, defined by

$$\phi = \phi_0 (\sin \eta)^{2/(1-n)} \quad \text{with} \quad \phi_0 = \left( -E \{\omega(d-1) + d \} \right)^{1/(1-n)} \left( (d-1)B^2 \right),$$

the relation between $\tau$ and $\eta$ yields

$$\frac{d\tau}{d\eta} = \pm \frac{2}{(1-n)\sqrt{E}} \phi_0 (\sin \eta)^{1+n}.$$

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In this case the value of $N$ in eq. \textsuperscript{11} can be either positive or negative\textsuperscript{3}.

### 4.2.1 Case of $N > 0$

The positive value of $N$ can be realized in the presence of the modulus. Integrating eq. \textsuperscript{21}, the scale factor is given by the form \textsuperscript{25} with

$$\xi = \xi_0 \left| \tan \frac{\eta}{2} \right|^q, \quad q = \pm \frac{d - 1}{1 - n} \sqrt{\frac{|N|}{E}}. \quad (51)$$

Consider the evolution of $a$ for $q > 0$ with $\eta > 0$. When $n < 1$ both $\xi$ and $\phi$ increase as $\eta \to \infty$, in which case the universe contracts asymptotically. In the case of $n > 1$, $\phi$ decreases toward zero, whereas $\xi$ approaches $\xi \to \xi_0$ as $\eta \to \infty$. In this asymptotic region, the scale factor evolves as

$$a \propto e^{\left(\frac{2}{d-1}\eta\right)^n} \propto (t - t_0)^\frac{2}{d-1-n} \quad \text{with} \quad t = t_0 + \tilde{c} e^{-\left(\frac{(1+n)(d-1)-2d}{(1-n)(d-1)}\right)^{2/d-1}}, \quad (52)$$

where $t_0$ and $\tilde{c}$ are constants. Since the power of the scale factor is larger than unity for $n > 1$, this represents an inflationary solution. Notice that this is achieved due to the presence of the modulus field. The curvature asymptotically decreases for $1 < n < \frac{d+1}{d-1}$, whereas $R$ diverges for $n > \frac{d+1}{d-1}$. In both cases the string coupling diverges in the limit of $\eta \to \infty$. The scale factor grows as the radiation dominant universe in another asymptotic, $\xi \to \sqrt{M}$. In this limit the scalar curvature decreases toward zero with a finite string coupling. Therefore it is possible to obtain nonsingular bouncing solutions for $n > 1$ if the universe starts out from the region with finite dilaton and curvature and evolves toward the weakly coupled regime. When the universe begins around the region $\xi \sim \sqrt{M}$ and evolves toward $\phi = 0$, we also have the bouncing solutions that asymptotically approach the inflationary solution \textsuperscript{52}. However, since the solutions enter the strong coupling region in the latter case, this indicates the limitation of using the tree-level action.

When $q < 0$ the scale factor approaches zero as $\eta \to \infty$ or $\eta \to 0$, depending on the sign of $q$. This suggests a difficulty to obtain bouncing solutions for $q < 0$. In summary we can have regular bouncing solutions for $q > 0$ and $n > 1$.

### 4.2.2 Case of $N < 0$

When $N < 0$ the integration of eq. \textsuperscript{21} leads to

$$a = \left( \sqrt{\frac{|N|}{M}} \frac{1}{\sqrt{\phi \cos \xi}} \right)^{2/(d-1)}, \quad (53)$$

where

$$\xi = \log \left( \xi_0 \left| \tan \frac{\eta}{2} \right|^q \right), \quad q = \pm \frac{d - 1}{1 - n} \sqrt{\frac{|N|}{E}}. \quad (54)$$

\textsuperscript{3}In the absence of the modulus, $N$ is negative.
From this we find that bouncing solutions emerge in the interval \( \eta_i < \eta < \eta_f \) with

\[
\eta_i = \log \left[ \frac{\varepsilon_0^{1/p} + \exp \{(2j - 1/2)\pi/p\}}{\varepsilon_0^{1/p} - \exp \{(2j - 1/2)\pi/p\}} \right], \quad \eta_f = \log \left[ \frac{\varepsilon_0^{1/p} + \exp \{(2j + 1/2)\pi/p\}}{\varepsilon_0^{1/p} - \exp \{(2j + 1/2)\pi/p\}} \right]. \tag{55}
\]

Setting \( \eta = \eta_i + \Delta \eta \) and \( \eta = \eta_f - \Delta \eta \) around \( \eta = \eta_i, \eta_f \), the scale factor evolves as

\[
a \propto \left( \Delta \eta \right)^{-2/(d-1)} \propto (t - t_0)^{-\frac{2}{d-1}}, \tag{56}
\]

where \( t_0 \) is a constant. When \( n > 1 \) or \( n < 1/d \), the solution \( 55 \) represents the decelerating expansion of the universe with decreasing curvature \( (R \rightarrow 0) \). The standard axion coupling in \( d = 3 \) dimensions \( (n = -1) \) is included in this case, and the solutions exhibit the radiation dominant behavior with \( a \propto (t - t_0)^{1/2} \). For the general axion coupling with \( n \neq -1 \), however, the evolution of the scale factor is not asymptotically radiation dominant. This means that the axion affects the dynamics of the system even in the asymptotic limit. In the case of \( 1/d < n < 1 \), we have bouncing solutions with growing curvature in both asymptotics, which means that the solutions are not ideal ones.

We have many sequences of intervals with \( \eta_i < \eta < \eta_f \), during each of which the string coupling and the curvature are kept finite. As in the case of \( N < 0 \) with no axion, nonsingular bouncing solutions can be constructed without entering the strong coupling region around \( \phi = 0 \).

## 5 Application to string-gas cosmology

In this section we shall consider the matter source term derived from the free energy of the ideal string-gas associated with type IIA/B string theory compactified on a square \( T^9 \)-torus, simple product of nine circles. It was recently shown that it is possible to construct the string-gas cosmological model where some dimensions \( (d) \) start to expand while the remaining dimensions are kept small \( 17 \). In this work we include the effect of the axion. We also wish to investigate the validity of the analytic solutions in the presence of the large and small dimensions. In the region where the temperature, \( T = 1/\beta \), is below the Hagedorn one, \( T_H = 2\sqrt{2}\pi \), the free energy of a string-gas in the canonical ensemble takes the following form \( 37, 17 \):

\[
F^{(d)}(\beta) = -\frac{V_d}{2\pi \sqrt{\alpha'}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{d\tau}{\tau_2^{(3+d)/2}} \left[ \Lambda(r; \tau) \right]^{[0] - \infty} \sum_{\tilde{p}=1} e^{-\frac{q_2 p^2}{4\pi^2}} |M_2|^2(\tilde{\tau}), \tag{57}
\]

where \( V_d = a^d \) is the volume of the large \( d \) dimensions with the scale factor \( a \) and

\[
\Lambda(r; \tilde{\tau}) = \sum_{\tilde{p}=1}^{+\infty} \sum_{m_i, n_i = -\infty}^{+\infty} q^{\frac{\tilde{p}^2 (m_i + n_i \tau)^2}{2}} q^{\frac{\tilde{p}^2 (m_i - n_i \tau)^2}{2}}, \tag{58}
\]

represents the contributions of the whole Kaluza-Klein (KK) and winding modes along the small dimensions. Here \( \tilde{\tau} \) is the modular parameter of the world-sheet torus and \( q = \exp(2i\pi \tilde{\tau}) \). The sum in terms of \( \tilde{p} \) in eq. (57) runs only over positive odd numbers and corresponds to taking the correct quantum statistic for bosons and fermions.

The \( M_2 \) factor in eq. (57) is expanded in powers of \( q \), as

\[
M_2(\tilde{\tau}) = \frac{\theta_2(\tilde{\tau})^4}{\eta(\til\tau)^{12}} = \sum_{N_m=0}^{\infty} D(N_m) q^{N_m}, \tag{59}
\]

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where $\theta_2$ and $\bar{\eta}$ are modular functions on the torus (see ref. [38]). Here the value of $N_m$ corresponds to each string mass level with the degeneracy factor, $D(N_m)$. When we integrate the $\tau_1$ and $\tau_2$ integrals in eq. (57), it is convenient to extract the term with $N_m = \bar{N}_m = m_i = n_i = 0$ that corresponds to the purely massless state. Denoting this massless state (radiation) as $\psi$ one has [17]

$$F_{rad}^{(d)} = -\frac{d^2}{2\pi} D(0)^2 \Gamma \left( \frac{d+1}{2} \right) (4\pi)^{(d+1)/2} \zeta(d+1) (1 - 2^{-(d+1)}) \beta^{-d-1},$$

(60)

where $\Gamma$ and $\xi$ are the Gamma function and the Riemann zeta-function, respectively. From this the energy $E_{rad}^{(d)}$ and pressure $P_{rad}^{(d)}$ along the $d$ spatial dimensions are evaluated as

$$E_{rad}^{(d)} = F_{rad}^{(d)} + \beta \frac{\partial F_{rad}^{(d)}}{\partial \beta} = \frac{d a^d}{2\pi} D(0)^2 \Gamma \left( \frac{d+1}{2} \right) (4\pi)^{(d+1)/2} \zeta(d+1) (1 - 2^{-(d+1)}) \beta^{-d-1},$$

(61)

$$P_{rad}^{(d)} = -\frac{1}{d} \frac{\partial F_{rad}^{(d)}}{\partial (\ln a)} = \frac{F_{rad}^{(d)}}{d},$$

(62)

which represents the equation of state for radiation in $d$ spatial dimensions.

The pressure along the $(9 - d)$-small dimensions is vanishing ($F_{rad}^{(9-d)} = 0$) for the pure massless state. In this case the evolution of the small dimensions is trivial and can be kept small around the duality symmetric radius, $r = 1$ [17]. Therefore it is sufficient to consider the evolution of only the large $d$-dimensions with a constant radius of the small dimension for the pure massless state. Introducing the shifted dilaton, $\psi \equiv \varphi - d\mu$, with $a = e^\mu$, $\omega = -1$ and $\dot{\chi} = 0$, eqs. (7)-(9) yield

$$-d\mu^2 + \dot{\varphi}^2 - e^{-(n-1)\varphi} \sigma^2 = e^\psi E,$$

(63)

$$\ddot{\mu} - \dot{\psi} \dot{\mu} - \frac{1-n}{2} e^{-(n-1)\varphi} \sigma^2 = \frac{1}{2} e^\psi P,$$

(64)

$$\ddot{\psi} - d\mu^2 - \frac{1+n}{2} e^{-(n-1)\varphi} \sigma^2 = \frac{1}{2} e^\psi E.$$

(65)

Note that the energy and pressure are related with $\rho$ and $p$ in eqs. (7)-(9), as $E = 2a^d \rho$ and $P = 2a^d p$. Since the pure massless state satisfies the equation of state with $\rho = dp$ from eq. (62), the analytic method given in the previous section can be applied for this case.

We are now considering the case of $\omega = -1$ and $A = 0$, thereby yielding a positive value of $\lambda$ from eq. (22). In the absence of the axion we have $q = \pm \sqrt{3}/6$ from eq. (24). Since the negative value of $q$ satisfies $-1/2 < q < 0$, bouncing solutions can be obtained in this case as shown in sec. 3.1 [see fig. 1]. This is achieved by choosing negative finite initial values of $\dot{\mu}$, in which case the scalar curvature is also finite. The scale factor exhibits a bounce at $\phi = \phi_s$ given by (29), after which the system approaches the radiation dominant universe described by eq. (31) as $\phi \rightarrow \phi_c$. In order to confirm these behaviors, we have numerically solved the background equations (63)-(65) together with the equation for $\beta$:

$$\frac{d}{dt} S = \frac{d}{dt} \left( \frac{\beta^2 \partial F^{(d)}_{rad}}{\partial \beta} \right) = 0.$$

(66)

Here we assumed an adiabatic evolution corresponding to the conservation of the entropy $S$. For the pure massless state [30] the inverse temperature, $\beta = 1/T$, is proportional to the scale...
factor \(a\) as in the standard radiation-dominated universe. We show in figs. 2 and 3 one example of the bouncing solution as dotted curves for the pure massless state without the axion. The dilaton continues to evolve toward the weakly coupled regime as expected, and the asymptotic behavior of the scale factor is found to be radiation dominant.

Notice that the scale factor grows from the beginning for positive initial values of \(\dot{\mu}\). In this case the time derivative of the dilaton, \(\dot{\varphi} = \dot{\psi} + d\dot{\mu}\), can be positive even when \(\dot{\psi}\) is negative initially. Since we are considering the weak coupling regime where \(e^\varphi \ll 1\) is satisfied, it is typical that the r.h.s. of eq. (63) is negligible relative to the first and second terms. In the absence of the axion, \(\dot{\psi}\) is roughly estimated as \(\dot{\psi} \simeq -\sqrt{d}\dot{\mu}\). Then \(\dot{\varphi}\) is positive for \(d > 1\), thereby leading to the growth of the dilaton toward the strongly coupled regime. We argue that positive initial values of \(\dot{\mu}\) are not so welcome in order to avoid the breakdown of the string-gas description at finite temperature. In contrast, the bouncing solutions (i.e., \(\dot{\mu} < 0\) initially) correspond to the decreasing dilaton toward the weakly coupled regime.

When the massive effect is taken into account, one has an extra source term in the small dimensions as well as the extra energy and pressure in eqs. (63)-(65). Let us study the leading terms that have an explicit dependence on the radius of the small dimensions \((r = e^\nu)\) in the infinite sums appearing in eq. (67). Taking the first KK and winding modes along a small direction with \(p = 1\), i.e., the terms with \(\{N_m = N_m = 0, m_i = (1, 0, \ldots, 0), n_i = 0\}\) (as well as \(m_i\) and \(n_i\) exchanged), plus the remaining \(8 - d\) inequivalent permutations, the pressure along the small dimensions is described as

\[
P^{(9-d)} = V_d C(\beta) \beta \left[\frac{1}{r^{(d+3)/2}} K_{(d-1)/2} \left(\frac{\beta}{r}\right) - r^{(d+3)/2} K_{(d-1)/2}(\beta r)\right].
\]

Here \(K_{(d-1)/2}\) is the modified Bessel function, \(C(\beta)\) is defined as \(C(\beta) = (2\pi/\beta)^{(d+1)/2}(18 - 2d)D(0)^2/\pi\) with \(D(0) = 16\) being a string degeneracy factor. We have an extra energy and pressure along the large dimensions due to the massive state in addition to (61) and (62). See ref. [17] for the explicit forms of these terms.

Numerically we solved the dilaton gravity equations (63)-(65) and making use of the approximate relation

\[
\dot{\nu} - \psi \ddot{\nu} = \frac{1}{2} e^\psi P^{(9-d)}.
\]

Since we are considering the case where the axion exists only for the large \(d\) dimensions, we do not include its effect along the small dimensions. The pressure \(P^{(9-d)}\) completely vanishes at the self-dual radius \(r = 1\) due to the compensation of KK and winding modes. Therefore the small dimensions are kept small around \(r = 1\) provided that the initial values of \(\nu\) and \(\dot{\nu}\) are close to zero. We have done numerical simulations for a variety of initial conditions and found that the massive effect for the large dimensions is weak as long as the small dimensions stay nearly constant around \(r = 1\) (see figs. 2 and 3 as an example). Although the massive state gives rise to the extra energy and pressure, the dominant contribution comes from the pure massless state. For the ideal case where the small dimensions are kept small around \(r = 1\), the system is well approximated by the state of pure radiation.

Eliminating the energy \(E\) from eqs. (63) and (65) and making use of the approximate relation \(E \simeq dP\) in eq. (64), we have

\[
\dot{f} = \frac{1}{2} dl^2 + \frac{1}{2} f^2 + \frac{1}{2} nB^2 e^{(n+1)\varphi} a^{2d}.
\]
Figure 2: The evolution of the scale factor $a$ in the case of $d = 3$ with or without the axion for the initial conditions $\dot{\mu}_0 = -0.03$, $\mu_0 = 3.0$, $\dot{\nu}_0 = -0.005$, $\nu_0 = 0.01$, $\psi_0 = -14$, $\beta_0 = 17$. The massive effect is taken into account together with the radiation for the solid curves, whereas the dotted curve corresponds to the pure radiation case without the axion.

Figure 3: The evolution of the dilaton $\varphi$ in the case of $d = 3$ for the same initial conditions shown in fig. 2. The massive effect is taken into account together with the radiation for the solid curves, whereas the dotted curve corresponds to the pure radiation case without the axion.
Figure 4: Phase space trajectories of the solutions of string-gas cosmology for $d = 3$. We implement the effect of the massive state together with the leading massless state. In the absence of the axion (solid curves), the solutions asymptotically approach the line given by $l/f = -1/d$, which represents the radiation dominant universe. The expansion rate $l$ gets larger when the axion is included (dotted curves), but the solutions eventually approach the radiation dominant universe.

$$l \simeq -\frac{1}{2} f^2 + f l + \frac{d(1 - n) - 1}{2d} B^2 e^{(n+1)\varphi} \frac{a^{2d}}{a^{2d}},$$

(70)

where $f$ and $l$ are defined as $f \equiv \dot{\psi}$ and $l \equiv \dot{\lambda}$. In the absence of the axion ($B = 0$), it was shown in refs. [19, 17] that the line $l = -f/d$ is an attractor for the massless state. From fig. 4 we find that the solutions without the axion (solid curves) asymptotically approach the line $l = -f/d$ even in the presence of the massive state.

When the axion is included in the large $d$ dimensions, the evolution of the scale factor and the dilaton should be changed as discussed in sec. 3.1. As one example we show in figs. 2 and 3 the evolution of $a$ and $\varphi$ for $d = 3$ and $n = -1$. The initial conditions are chosen to be $\dot{\varphi}_i < 0$ and $e^{\dot{\varphi}_i} \ll 1$. The dilaton does not continue to evolve toward the weakly coupled region [see eq. (42)] except for the initial stage, unlike the case without the axion. In fact, we find from fig. 3 that the dilaton changes to evolve toward the strong coupling regime after the initial decrease. The presence of the axion provides additional source terms in eqs. (69) and (70), thereby leading to the different evolution of the scale factor as shown in fig. 2. When $n = -1$, the growth of the scale factor is faster than in the case without the axion due to the positive last term in the r.h.s. of eq. (70). Nevertheless we find from fig. 4 that the solutions eventually approach the line $l = -f/d$ as $f$ and $l$ decrease toward zero. This means that the dilaton approaches a finite value with the radiation dominant universe ($a \propto t^{2/(d+1)}$ as $\xi \to \xi_c = \sqrt{M}$), as discussed in sec. 4.1. In addition the scalar curvature decreases to zero as $\xi \to \xi_c$.

Since we are now dealing with the case of $\omega = -1$ and $A = 0$, the condition (46) is satisfied whereas (48) is not. The condition (48) corresponds to the bouncing solutions where the scalar curvature $R$ converges in the limit of $\phi \to 0$, whereas $R$ diverges in the present case in this limit
(\omega = -1, A = 0). Nevertheless, when the universe starts out from a weak coupling regime with a finite curvature, it is possible to construct bouncing solutions that asymptotically approach the radiation dominant universe with finite \( R \) and \( \varphi \) irrespective of the presence of the axion as found in sec. 4.1 for the pure massless case. The examples plotted in figs. 2 and 3 show the existence of nonsingular bouncing solutions even when the condition (48) is not fulfilled. Notice that in the presence of the axion but without the radiation-like source matter \( R \) and \( \varphi \) diverge in both asymptotic limits. This indicates that inclusion of the matter source such as the ideal string-gas is crucial to construct the bouncing solutions without singularities in asymptotic future. It will be interesting to extend our analysis to other matter sources which appear e.g., in the \( p \)-brane (D\( p \)-brane) cosmology.

We also made numerical analysis for larger initial values of the dilaton which are close to \( e^\varphi \sim 1 \). We found that the system tends to evolve toward the strongly coupled regime with the growth of \( \varphi \) toward infinity. The scalar curvature shows divergent behavior together with the unbounded growth of the dilaton. Nevertheless we have to keep in mind that the string-gas description based on the free energy (57) ceases to be valid when the system enters the regime of strong coupling and large curvatures. In addition we need to consider the higher-order corrections to the tree-level effective action in this regime. In the next section we will discuss the dynamics of the system when the dilatonic higher-order corrections are taken into account.

6 Inclusion of higher-order corrections to the tree-level action

In order to understand the effect of the higher order corrections to the tree-level action, let us consider the dilaton-dependent corrections, given by (8)

\[
L_c = -\frac{1}{2} \alpha' \lambda \xi(\varphi) \left[ c_1 R_{GB}^2 + c_2 (\nabla \varphi)^4 \right],
\]

(71)

where \( \xi(\varphi) \) is a general function of \( \varphi \), \( \lambda \) is an additional parameter depending on the species of strings. For example, we have \( \lambda = -1/4, -1/8 \) for the Bosonic and Heterotic string, respectively. The Gauss-Bonnet term, \( R_{GB}^2 = R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} \), has the property of keeping the order of the gravitational equations of motion unchanged. The dilatonic corrections \( L_c \) are the sum of the tree-level \( \alpha' \) corrections (\( i = 0 \)) and the quantum \( i \)-loop corrections (\( i = 1, 2, 3, \cdots \)), with the function \( \xi(\varphi) \) given by (9, 10, 39)

\[
\xi(\varphi) = - \sum_{n=0}^{\infty} C_i e^{(i-1)\varphi},
\]

(72)

where \( C_i \) (\( i \geq 1 \)) are the coefficients of \( i \)-loop corrections with \( C_0 = 1 \).

If the universe starts out from a sufficiently weak coupling regime (\( e^\varphi \ll 1 \)), it can reach the high curvature regime while the string coupling is still small. In this case the \( \alpha' \) correction (\( n = 0 \)) becomes important before the quantum loop corrections begin to work. Gasperini et al. (8) showed that inclusion of the \( \alpha' \) correction gives rise to the fixed points in the (\( \dot{\varphi}, H \)) plane when only the dilaton is present. This leads to an inflationary solution driven by the constant \( H \) and the growing \( \varphi \). Although the successful transition from this dilaton-driven inflationary phase to a decelerated FRW era is difficult to be achieved only in the presence of the \( \alpha' \) correction, this problem can be solved by implementing the quantum loop corrections. In fact Brustein and Madden (9) showed that inclusion of the quantum loop corrections can lead to the successful graceful exit due to the violation of the null energy condition.
Here we wish to analyze the dynamics of the system in the presence of the modulus, axion and the stringy matter source. Let us consider the case of \( d = 3 \) and \( \omega = -1 \). The source terms due to \( \mathcal{L}_c \) in eqs. (7)-\( \mathbf{(9)} \) are given by \( \mathbf{(39)-(40)} \)

\[
\rho_c = \sum_{i=0}^{n} C_i \{ \rho_c \}_i, \quad p_c = \sum_{i=0}^{n} C_i \{ p_c \}_i, \quad \Delta_c = \sum_{i=0}^{n} C_i \{ \Delta_c \}_i, \quad (73)
\]

where

\[
\{ \rho_c \}_i = \alpha' \lambda \varphi e^{(i-1)\varphi} \left\{ -24c_1(i-1)H^3 + 3c_2 \varphi^3 \right\}, \quad (74)
\]

\[
\{ p_c \}_i = \alpha' \lambda \varphi e^{(i-1)\varphi} \left\{ 8c_1(i-1)H \left[ (i-1)\varphi^2 H + \varphi H + 2\varphi (H + H^2) \right] + c_2 \varphi^4 \right\}, \quad (75)
\]

\[
\{ \Delta_c \}_i = \alpha' \lambda \varphi e^{(i-1)\varphi} \left\{ 24c_1(i-1)H^2 (H + H^2) - 3c_2 \varphi^2 \left[ 4\varphi + 4\varphi H + (i-1)\varphi^2 \right] \right\}. \quad (76)
\]

Substituting these expressions for eqs. (7)-(\( \mathbf{(9)} \)), we get the equations for \( \dot{H} \) and \( \ddot{\varphi} \):

\[
\left[ 6 - 24a' \lambda \varphi H^2 \sum_{i=0}^{n} C_i(i-1) e^{i\varphi} \right] \dot{H} = -12H^2 + \omega \varphi^2 - 6H \omega \varphi - 2\omega \varphi - \chi^2
\]

\[
- n e^{-(n-1)\varphi} \sigma^2 + \alpha' \lambda \sum_{i=0}^{n} C_i e^{i\varphi} \left\{ 24c_1(i-1)H^4 - 3c_2 \varphi^2 [4\varphi + 4H \varphi + (i-1)\varphi^2] \right\}, \quad (77)
\]

\[
\left[ 2 - 8a' \lambda \varphi H^2 \sum_{i=0}^{n} C_i(i-1) e^{i\varphi} \right] \ddot{\varphi} = -4H \varphi + 4H + 6H^2 + (\omega + 2) \varphi^2 + \chi^2
\]

\[
+ e^{-(n-1)\varphi} \sigma^2 - \alpha' \lambda \sum_{i=0}^{n} C_i e^{i\varphi} \left\{ 8c_1(i-1)H [(i-1)\varphi^2 H + 2\varphi (H + H^2)] + c_2 \varphi^4 \right\} + 2e^{\varphi} p. \quad (78)
\]

In the case of \( \chi = \sigma = 0 \) and \( \rho = p = 0 \), the solutions approach the string phase characterized by constant \( H \) and \( \varphi \), provided that the system begins from the weakly coupled regime \( (e^{\varphi} \ll 1) \). These values are obtained by setting \( \dot{H} = 0 \) and \( \ddot{\varphi} = 0 \) in eqs. (77) and (78) with \( i = 0 \). The effect of the quantum loop corrections \( (i \neq 0) \) begin to work around \( e^{\varphi} \sim 1 \), which can lead to the graceful exit toward the decelerating FRW branch. The number of e-folds, \( N_e \equiv \ln a \), before the graceful exit is approximately estimated as \( N_e \simeq |\varphi_i|/2 \), where \( |\varphi_i| \) is the initial value of the dilaton.

Let us study the case of \( \omega = -1 \), \( \chi \neq 0 \) and \( \sigma \neq 0 \), and \( \rho = p = 0 \). In the dilaton-driven phase corresponding to the lowest order PBB cosmology, the evolution of \( a \) and \( \phi \) is described by \( a \propto (-t)^{1/\sqrt{3}} \) and \( \phi = e^{-\varphi} \propto (-t)^{3/4+1} \). From eq. (16) \( \dot{\chi} \) and \( \dot{\sigma} \) evolve as \( \dot{\chi} \propto (-t)^{-1} \) and \( \dot{\sigma} \propto (-t)^{3/4-n(3+1)} \). Therefore \( \dot{\chi} \) grows during the dilaton-driven phase, whereas \( \dot{\sigma} \) decreases for \( n < (3 - \sqrt{3})/2 \). The growth of \( \dot{\chi} \) can change the dynamics of the system through eqs. (77) and (78). For a fixed value of the initial Hubble rate, the initial \( \dot{\varphi} \) gets larger from the constraint equation (17) in the presence of the modulus. We also find from eq. (78) that the increase of \( \dot{\chi} \) leads to the faster growth of \( \dot{\varphi} \). The axion has a similar effect for \( n > (3 - \sqrt{3})/2 \).

The growth of \( \dot{\chi} \) (and \( \dot{\sigma} \) for \( n > (3 - \sqrt{3})/2 \)) terminates once the system enters the string phase characterized by the constant \( H \) and \( \varphi \) \( (H_0 = 0.616 \) and \( \varphi_0 = 1.40 \) for \( d = 3 \)). During this phase, we have \( a \propto e^{H_0 t} \) and \( \varphi \propto \varphi_0 e^t \), thereby yielding \( \dot{\chi} \propto e^{-(3H_0 - \dot{\varphi}_0) t} \) and \( \dot{\sigma} \propto e^{-(3H_0 - n\dot{\varphi}_0) t} \). The fixed point \( (H_0, \varphi_0) \) exists in the range of \( 3H_0 > \dot{\varphi}_0 \) in order to allow for a subsequent graceful exit \( \mathbf{(40)} \). Therefore \( \dot{\chi} \) begins to decrease once the system enters the string phase, and the situation is the same for \( \dot{\sigma} \) as long as \( n < 3H_0/\dot{\varphi}_0 \). In standard low-energy effective
Figure 5: The evolution of $H$ and $\dot{\varphi}$ as a function of the number of e-folds for $d = 3$, $\rho = p = 0$ and $n = -1$ with initial conditions $\varphi_i = -130$ and $H_i = 1.5 \times 10^{-3}$. We include the quantum corrections up to $i = 2$ with coefficients $C_1 = 1$ and $C_2 = -10^{-3}$ together with the $\alpha'$ correction ($C_0 = 1$). The dotted curves correspond to the evolution without the modulus and axions fields, whereas these fields are involved in the solid curves with initial conditions $\dot{\chi}_i = 10^{-2}$ and $\dot{\sigma}_i = 10^{-2}$. We do not find a significant difference even when $\chi$ and $\sigma$ are present.

Figure 6: The evolution of the dilaton $\varphi$ in the presence of the modulus and axions fields for the same initial conditions as in fig.5. The initial dilaton-driven phase ($0 < t < 100$) is followed by the string phase with a linearly growing dilaton ($100 < t < 200$), after which the solutions connect to the decelerating FRW branch. **Inset:** The evolution of $\dot{\chi}$ and $\dot{\sigma}$. Both $\dot{\chi}$ and $\dot{\sigma}$ are exponentially suppressed during the string phase.
string theory \[21\], we have \( n = -1 \) for the axion coupling, in which case \( \dot{\sigma} \) is exponentially suppressed during the string phase and its energy density becomes very small relative to the dilaton. Typically \( n \) is smaller than unity not only in low energy effective string theory but also in multidimensional models with gauge fields or in the higher dimensional conformal gauge field theory \[24\]. In such cases the effect of the axion is not dynamically important when the correction \( (71) \) is present.

We numerically solved the equations \( (77) \) and \( (78) \) for \( \omega = -1 \) and \( n = -1 \) together with the initial conditions satisfying \( e^{\phi} \ll 1 \), \( \dot{\chi_i} < 1 \) and \( \dot{\sigma_i} < 1 \). In the presence of the modulus and the axion, the string phase with constant \( H \) and \( \dot{\varphi} \) is typically reached earlier than in the case only with the dilaton. Nevertheless the difference is not significant as seen from fig. \[5\] In both cases the quantum loop corrections begin to work around \( e^{\phi} \sim 1 \), after which the solutions connect to the decelerating FRW branch. When only the \( \alpha' \) and one-loop corrections are present \( (C_0 = 1 \) and \( C_1 = 1) \), the dilaton and the curvature exhibit unbounded growth after the branch change as shown in ref. \[9\] in the single field case. If the two-loop correction with negative coefficient \( C_2 \) is included in eq. \( (72) \), this can overwhelm the one-loop correction when the dilaton becomes sufficiently large. In fact, as shown in fig. \[6\] the evolution of the dilaton becomes mild by including the two-loop correction.

Even when the matter source is present \( (\rho \neq 0, p \neq 0) \), the dynamics of the system is dominated by the correction \( (71) \). The effect of the matter is involved in the last term of eq. \( (78) \). Since we are considering the initial conditions corresponding to the weakly coupled region \( (e^{\phi_i} \ll 1) \), the \( \alpha' \) correction tends to dominate over the last term in eq. \( (78) \). Then the system soon approaches the string phase with constant \( H \) and \( \dot{\varphi} \). During the string phase, \( \rho \) and \( p \) are rapidly suppressed due to the exponential growth of the scale factor. This means that the matter source term is practically negligible around the graceful exit. For example, the pressure of the string-gas discussed in the previous section evolves as \( p \propto a^{-4} \propto e^{-2\varphi} \) in the string phase, thereby leading to the exponential decrease of the last term of eq. \( (78) \). In fact we have done numerical simulations for the free energy \( (60) \) in the context of string-gas cosmology and found that the evolution of the system is almost the same as shown in figs. \[5\] and \[6\]. The energy density and pressure of the string-gas are exponentially suppressed, together with the rapid decrease of the temperature \( (T \propto a^{-1}) \) before the graceful exit.

We also investigated the case where the dilatonic Brans-Dicke parameter \( \omega \) does not equal to \(-1\). When \( \omega \) lies in the range \(-3/2 < \omega < 0\) with \( d = 3 \) and \( A = 0 \), the qualitative behavior is mostly the same as in the \( \omega = -1 \) case. The difference is that the values of \( H \) and \( \dot{\varphi} \) in the attractor region get bigger for smaller \( |\omega| \), thereby leading to larger amount of e-foldings. However, when \( \omega < -3/2 \), we find that the system is typically unstable and tends to fall into a singularity. Although nonsingular solutions can be obtained when \( \omega \) is close to \(-3/2\) and \( e^{\varphi} \) is not too much smaller than unity, the parameter range of these solutions is restricted to be very narrow. This suggests that inclusion of the correction \( L_c \) is not necessarily welcome for the construction of regular solutions for \( \omega < -3/2 \).

7 Summary and discussions

In this paper we have studied the construction of nonsingular cosmological solutions in a general \( D \)-dimensional effective action derived from string theories. In particular we tried to find singularity-free solutions analytically for the dilaton-modulus-axion system in the presence of
the “stringy” matter source term. We also keep the dilatonic Brans-Dicke parameter $\omega$ arbitrary so that the action contains a variety of theories such as the $F$-theory and the multidimensional theory. In the tree-level action without higher-order $\alpha'$ and loop corrections, we found the following results.

- Even for the general $(d+1)$-dimensional action with modulus and axion fields, the system is integrable for the flat FRW background if the matter source is radiation ($\rho = dp$) or there is no matter ($\rho = p = 0$).

- In the case of $\omega = -1$, we find nonsingular bouncing solutions where the universe starts out from the state of the finite scalar curvature and asymptotically approaches the radiation dominant stage with decreasing curvature and finite string coupling. This situation naturally appears in the context of string-gas cosmology.

- In the theories where $\omega$ is largely negative, we have a sequence of regular bouncing solutions where the string coupling and the scalar curvature remain finite. This was already pointed out in ref. [25, 26] for $d = 3$, but we extended the analysis to the case of the general spatial dimension $d$ with modulus fields.

- As a “stringy” matter source, we considered the ideal string-gas in thermal equilibrium taking into account the massive state coming from Kaluza-Klein and winding modes in addition to the leading massless state. Numerically we obtained nonsingular bouncing solutions where the “large” 3 dimensions asymptotically approach the almost radiation dominant universe irrespective of the presence of the axion. This indicates that inclusion of the radiation-like source is crucial for the existence of such regular solutions.

While bouncing solutions can be obtained in the tree-level action with the radiation-like matter source, it can happen that the system enters the strongly coupled region. In this case it is inevitable to implement the higher-order corrections to the tree-level action. Therefore we also analyzed the case where the dilatonic higher-derivative and loop corrections are taken into account for the dilaton-modulus-axion system with some matter source. We find that the dynamics of the system for $\omega = -1$ is not significantly changed by including scalar fields or matters other than the dilaton. The system approaches the string phase characterized by constant $H$ and $\dot{\varphi}$, after which the graceful exit is realized through the dominance of higher-order loop corrections. Typically the energy densities of the modulus, axion, and the matter source are exponentially suppressed once the dilaton-driven inflation begins in the string phase. We also find that the system tends to be unstable for the case of large negative $\omega$.

There are several open issues that we do not study but that deserve further investigation. While bouncing solutions can be obtained in the tree-level action with the radiation-like matter source, it should be fairly easy to extend our analysis to the closed/open FRW background with general $D = d + 1$ dimensions, as was already done for the case of $d = 3$ without the modulus [26]. In addition it is worth analyzing the anisotropic background in order to check the generality of the singularity avoidance. In fact a new singularity—called the determinant singularity—appears in the presence of anisotropies when the action contains a Gauss-Bonnet term [41]. It is of interest to investigate whether the singularity solutions we found are stable by including small anisotropies.

The validity of our nonsingular cosmological scenarios may be investigated further by analyzing the power spectra of produced density perturbations. Recent observations of WMAP
suggest that the spectrum of scalar perturbations is close to scale-invariant. In the simplest tree-level pre-big-bang scenario, the curvature perturbation $R$ is blue-tilted with a spectral index $n \simeq 4$, which is incompatible with observations. This situation does not change so much even involving the higher-order corrections to the tree-level action. Nevertheless it is possible to have nearly scale-invariant spectra of the axion perturbations depending on the expansion rate of the modulus field. Making use of the “curvaton” mechanism, there remains a possibility to explain the observationally supported flat spectra if the axion plays the role of the curvaton (see ref.). In addition, originating from the proposal of Ekpyrotic cosmology, there are a number of controversial arguments about the choice of the gauge conditions for scalar perturbations generated in bouncing cosmologies. It is certainly of interest to analyze the evolution and the spectra of scalar perturbations in our nonsingular bouncing models by taking into account the curvaton perturbations.

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