The one-dimensional model of non-Newtonian hemodynamics

M A Verigina and G V Krivovichev
Faculty of Applied Mathematics and Control Processes Saint Petersburg State University, 7/9 Universitetskaya nab., Saint Petersburg, 199034, Russian Federation
E-mail: mari-veri701@yandex.ru, g.krivovichev@spbu.ru

Abstract. The one-dimensional model of the blood flow in large vessels is considered. The non-Newtonian nature of blood is modeled by the friction force and Boussinesq coefficient value. The power law rheological model of fluid is used. The comparison with the models of blood as ideal and Newtonian fluid is realized. The nonlinear problems of single-pulse propagation in a single vessel and in a vessel with bifurcation are considered. It is demonstrated, that non-Newtonian effects play an important role in the obtained solutions.

1. Introduction
The mathematical models of blood flow are extensively used for the simulation of different processes in cardiovascular system [1, 2]. For simulation of blood flow in a whole cardiovascular system and in vascular systems of the single organ the one-dimensional (1D) models, obtained by the averaging of three-dimensional Navier — Stokes system are considered [1, 3].

The blood can be considered as the heterogeneous suspension of formed cellular elements and the fluid component called plasma. Cellular components, mainly represented by erythrocytes, have a significant effect on the mechanical properties and caused the non-Newtonian properties of blood [10]. So, to taking into account the realistic behavior, the non-Newtonian effects can be included in the mathematical models of blood dynamics.

The rheological models, used for the description of blood mechanical properties, are usually restricted by the ideal and viscous Newtonian fluid. In the model of the ideal fluid, the effect of viscosity is neglected and the case of the flat velocity profile is considered. This model is widely used in the analytical investigations [4, 5]. The Newtonian model is used in most of the works. The velocity distribution for this model is parabolic and defined by the Poiseuille profile.

It must be noted, that due to the averaging procedure, presented in [1], in 1D models the rheology of blood is defined by the Boussinesq coefficient (computed from the velocity profile) and friction term, defined by the relation for the tangential stress tensor. The velocity profile is obtained after the solution of the stationary 1D problem for the flow in a tube with a circular cross-section. Unfortunately, in most of the papers, the value of Boussinesq coefficient is inconsistent with the used velocity profile or this profile is inconsistent with the used rheological model for blood. For example, it is ordinary to use the value, equal to unity (which corresponds to the case of ideal fluid) in the models, where blood is considered as a Newtonian fluid (for this model the coefficient is equal to 4/3) [6, 7, 8, 9]. The non-Newtonian properties of blood in
many models are taken into consideration only by the use of the profile, which is differed from the parabolic, while the frictional term, corresponding to the Newtonian model, is used [7, 8].

In the presented paper, we try to consider the model, where non-Newtonian nature of blood is taken into account by the proper considerations of the expressions for the frictional term, velocity profile and Boussinesq coefficient. Two nonlinear initial-boundary-value problems are considered for the comparison with the models, where blood is considered as an ideal and Newtonian fluid.

The paper has the following structure. In Section 2 the mathematical model of non-Newtonian hemodynamics is considered. In Section 3 two nonlinear test problems on the pulse propagation in a single vessel and in the vessel with bifurcation are considered. Some concluding remarks are made in Section 4.

2. Mathematical model of blood flow

The power law fluid is one of the simplest and widely-used non-Newtonian models of blood [10, 11]. The model is defined by the following relation:

\[ T = 2\mu D, \quad D = \frac{1}{2} (\nabla V + (\nabla V)^*) \Rightarrow \mu(I_2) = kI_2^{\frac{n-1}{n}} \]  \hspace{1cm} (1)

where \( T \) is a tangential stress tensor, \( D \) is a strain rate tensor, \( I_2 \) is the second invariant of tensor \( D \) and \( k,n \) are the parameters, obtained from experimental data.

Due to the incompressibility of blood [3], the blood dynamics is described by the following system:

\[ \nabla \cdot V = 0, \quad \rho \frac{dV}{dt} = \rho f + \nabla \cdot \Sigma, \]  \hspace{1cm} (2)

where \( V \) is a velocity, \( \rho \) is a constant density, \( \Sigma = -pI + T \) is a full stress tensor, \( I \) is a unity tensor, \( p \) is a pressure, \( f \) characterize the body force action (in the presented paper \( f = 0 \)).

After the making of some assumptions [1], the system (1)–(2) can be averaged and the resulted 1D system is written as:

\[ \frac{\partial A}{\partial t} + \frac{\partial Q}{\partial z} = 0, \quad \frac{\partial Q}{\partial t} + \frac{\partial}{\partial z} \left( \frac{Q^2}{A} \right) + \frac{A \partial P}{\rho \partial z} + K\frac{Q|Q|^{n-1}}{A^{\frac{2n-1}{n}}} = 0, \]  \hspace{1cm} (3)

where \( A = A(t, z) \) is the cross-section area, \( Q = Q(t, z) \) is a flow rate, \( P = P(t, z) \) is a pressure, corresponds to the vessel cross-section at point \( z \), \( \alpha \) is a Bouissinesq coefficient, \( K \) is calculated as:

\[ K = -\frac{2ks'(1)s'(1)^{n-1}y^{\frac{n+1}{n}}}{\rho}, \]

where \( s = s(y) \) is the dimensionless velocity profile, \( y \) is the dimensionless radius. For the rheological model (1) \( s(y) \) is presented as:

\[ s(y) = \frac{3n+1}{n+1} \left( 1 - y^{1+\frac{1}{n}} \right). \]

The Boussinesq coefficient is computed as [1]:

\[ \alpha = \frac{\int \frac{s^2 d\sigma}{A}}, \]

where \( S \) is the vessel cross-section. For the considered model, the expression for \( \alpha \) is written as:

\[ \alpha = \frac{3n+1}{2n+1}. \]
The model (3) must be closed by the equation-of-state \( P = P(A) \). In the presented paper, the widely used equation is considered [7]:

\[
P - P_{\text{ext}} = P_{\min} + \frac{\beta}{A_{\min}}(\sqrt{A} - \sqrt{A_{\min}}), \quad \beta = \frac{4}{3}\sqrt{\pi El},
\]

where \( P_{\text{ext}} \) is the external pressure, \( P_{\min} \) and \( A_{\min} \) are the diastolic pressure and cross-section area respectively, \( E \) is the elastic modulus, \( h \) is the vessel wall thickness.

System (3) can be rewritten in the quasilinear form:

\[
\frac{\partial U}{\partial t} + H(U)\frac{\partial U}{\partial z} = g(U),
\]

where \( U = (A, Q)^T \).

For the practical computations, the initial and boundary conditions are stated for the system (3) and obtained problems are solved numerically. In the presented study the parametric Lax - Wendroff scheme is used for the computations.

3. Test problems

In this section, two nonlinear test problems are considered for the comparison of the proposed model with the models, where the blood is considered as an ideal and viscous Newtonian fluid.

3.1. Single pulse propagation in a straight reflection-free vessel

The presented problem is described in details in [7], where all parameter values are presented. This problem simulates the propagation of a narrow Gaussian–shaped wave in a single vessel with constant parameters and non-reflecting outflow boundary condition.

Let \( L \) is a vessel length. The inflow and outflow points correspond to \( z = 0 \) and \( z = L \) respectively. At the initial moment, the following conditions are stated:

\[ A(0, z) = A_{\min}, \quad Q(0, z) = Q_0, \]

where \( Q_0 = Q_L(0), \) where \( Q_L(t) \) is the function, which defines the flow rate at \( z = 0 \).

At \( z = 0 \), the value of \( Q \) is defined as \( Q(t, 0) = Q_L(t) \), where the expression for \( Q_L(t) \) is presented at [7], and compatibility condition along the characteristic of system (3), corresponds to the negative eigenvalue \( \lambda_1 \) of matrix \( H \), is stated:

\[
l_1 \left( \frac{\partial U}{\partial t} + H(U)\frac{\partial U}{\partial z} \right) = l_1 f,
\]

where \( l_1 \) is the left eigenvector, corresponding to \( \lambda_1 \). For practical computations, (4) is discretized and with the condition for \( Q \) at \( z = 0 \), is used for the computation of \( A(t, 0) \).

At \( z = L \), the compatibility condition along the characteristic corresponds to positive eigenvalue \( \lambda_2 \), is stated:

\[
l_2 \left( \frac{\partial U}{\partial t} + H(U)\frac{\partial U}{\partial z} \right) = l_2 f,
\]

where \( l_2 \) is the left eigenvector, corresponding to \( \lambda_2 \), and the non-reflecting condition is stated:

\[
l_1 \left( \frac{\partial U}{\partial t} - f \right) = 0.
\]

After the discretization, (5) and (6) formed an algebraic system for the finding the approximate values of \( Q(t, L) \) and \( A(t, L) \).
Figure 1. The plots of dimensionless pressure for different rheological models of blood for the problem of single pulse wave propagation: blue line — a case of the ideal fluid; red line — a case of the Newtonian fluid; green line — a case of the non-Newtonian fluid.

Figure 2. The computational domain for the problem of single pulse propagation in a vessel with bifurcation.

The values of power law model parameters $n$ and $k$ can be defined at the experiments. In the presented paper, the values from [10] are used: $k = 0.17 \text{ s}^n \cdot \text{dyn}/\text{sm}^2$, $n = 0.708$. These values corresponds to $\alpha \approx 1.29$. The results of computations are compared with the models, correspond to the case of blood as an ideal fluid ($k = 0$, $\alpha = 1$) and viscous Newtonian fluid ($k = 0.045 \text{ s} \cdot \text{dyn}/\text{sm}^2$, $n = 1$, $\alpha = 4/3$). At fig. 1, the results for several time moments are presented. As it can be seen, the maximum damping takes place for the case of non-Newtonian model. It must be noted, that the results at the case of the ideal fluid are the same, as presented at paper [7].
Figure 3. Plots of the flow rate for the problem of the single pulse wave propagation at a vessel with a bifurcation at the inlet of the parent vessel (a) and at the bifurcation point (b): 1 — ideal fluid; 2 — Newtonian fluid; 3 — non-Newtonian fluid
3.2. Single pulse propagation in a vessel with a single bifurcation

In the presented problem, the computational domain is constructed from three segments of vessel connected to make a single symmetric bifurcation (fig. 2). All parameter values and geometric characteristics are taken from the paper of Xiu and Sherwin [12]. The initial condition for the parent vessel (vessel 1 at fig. 2) and daughter branches (vessels 2 and 3 at fig. 2) are taken from [12]. At the inlet of the parent vessel, the value of the flow rate $Q$ and non-reflecting condition are stated. At the outlets of the daughter branches conditions (5)–(6) are stated.

At the bifurcation point, the condition for the conservation of mass across the bifurcation:

$$Q_1 = Q_2 + Q_3,$$

and the conditions for the total pressure continuity:

$$P_2 = P_1, \quad P_3 = P_1,$$

are stated, as the following three compatibility conditions:

$$I_2(U_1) \left( \frac{\partial U_1}{\partial t} + \lambda_2(U_1) \frac{\partial U_1}{\partial z} \right) = I_2(U_1)f(U_1),$$

$$I_1(U_2) \left( \frac{\partial U_2}{\partial t} + \lambda_1(U_2) \frac{\partial U_2}{\partial z} \right) = I_1(U_2)f(U_2),$$

$$I_1(U_3) \left( \frac{\partial U_3}{\partial t} + \lambda_1(U_3) \frac{\partial U_3}{\partial z} \right) = I_1(U_3)f(U_3).$$

The system (7)–(11), after the discretization of (9)–(11), consists of six algebraic equations on six variables. At every time moment, the system is solved by the Newton iterative method.

At fig. 3 the results of the computations are presented. As it can be seen, the reflection of the wave from the bifurcation point takes place and the damping is realized only for the models of blood as a viscous fluid. The Newtonian and non-Newtonian cases differ from each other. The maximum damping is realized for the non-Newtonian model.

4. Conclusion

In the presented paper the 1D mathematical model of blood flow dynamics with the non-Newtonian properties taken into account is realized. The rheological model of the power law fluid is used. After the solutions of the test problems from [7] and [12] it is demonstrated, that non-Newtonian effects play an important role and obtained results differ from the Newtonian case.

References

[1] Formaggia L, Lamponi D and Quarteroni A 2003 *J. Eng. Math.* 47 251
[2] Quarteroni A, Manzoni A and Vergara C 2017 *Acta Numer.* 26 365
[3] Bunicheva A, Mukhin S, Sosnin N and Khruelenko A 2015 *Comp. Math. Math. Phys.* 55 1381
[4] Il’yin O 2019 *Wave Motion* 84 56
[5] Spiller C, Toro E, Vasquez-Cendon M and Contarino C 2017 *Appl. Math. Comput.* 303 178
[6] Wang X, Fullana J and Lagree P 2015 *Comp. Meth, Biomech. Biomed. Eng.* 18 1704
[7] Boileau E, Nithiarasu P, Blanco P, Muller L, Fossan F, Hellevik L, Donders W, Huberts W, Willemet M and Alastruey J 2015 *Int. J. Num. Meth. Biomed. Eng.* 31 e02732
[8] Carson J and van Loon R 2017 *Int. J. Num. Meth. Biomed. Eng.* 33 e02837
[9] Li G, Delestre O and Li Y 2018 *Int. J. Num. Meth. Fluids* 86 491
[10] Karimi S, Dabagh M, Vasava P, Dadvar M, Dabir B and Jalali P 2014 *J. Non-New. Fluid Mech.* 207 42
[11] Razavi A, Shirani E and Sadeghi M 2011 *J. Biomech.* 44 2021
[12] Xiu D and Sherwin S 2007 *J. Comp. Phys.* 226 1385