Surprising Symmetries in Relativistic Charge Dynamics

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Abstract

The eigenspinor approach uses the classical amplitude of the algebraic Lorentz rotation connecting the lab and rest frames to study the relativistic motion of particles. It suggests a simple covariant extension of the common definition of the electric field: the electromagnetic field can be defined as the proper spacetime rotation rate it induces in the particle frame times its mass-to-charge ratio. When applied to the dynamics of a point charge in an external electromagnetic field, the eigenspinor approach reveals surprising symmetries, particularly the invariance of some field properties in the rest frame of the accelerating charge. The symmetries facilitate the discovery of analytic solutions of the charge motion and are simply explained in terms of the geometry of spacetime. Symmetries of the uniformly accelerated charge and electric dipole are also briefly discussed.

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I. INTRODUCTION

The electric field is commonly defined as the acceleration per unit charge-to-mass ratio of a small charge at rest. Lorentz transformations of the charge and the covariant electromagnetic field then imply the Lorentz-force equation. The eigenspinor (or rotor) approach to the motion of charged particles in electromagnetic fields suggests a definition of the electromagnetic field that is a simple extension of the textbook definition of the electric field. The extension, which is required to make the equation of motion for the eigenspinor linear, associates the electromagnetic field with the spacetime rotation rate of any charge in the field. It implies not only the Lorentz-force equation but also a spatial rotation in any magnetic field at the cyclotron frequency.

A couple of rather surprising symmetries of the dynamics of charges in the electromagnetic field result from the definition. The symmetries involve the invariance of field in a classical rest frame (commoving inertial frame) of charges moving in (a) a uniform constant field, and (b) a pulsed plane-wave field. This paper derives, examines, and illustrates these symmetries, showing that they result directly from the definition of the field and from the geometry of Minkowski spacetime. It also discusses symmetries in the field of a uniformly accelerating charge and the interaction of a uniformly accelerated physical dipole.

Symmetries simplify many physics problems and are often keys to finding analytical solutions. This has in fact been demonstrated for charges in a pulsed plane-wave field, where the symmetry, together with the powerful spinor and projector tools inherent in the eigenspinor approach, has allowed analytic solutions to be found for the relativistic motion of charged particles in propagating plane-wave pulses and in such pulses plus constant longitudinal fields, such as occur, for example, in autoresonant laser accelerators. It is important to understand the origin of the symmetries so that their applications to other problems can be anticipated. This is particularly true of relativistic symmetries that are often less obvious.

The eigenspinor is the amplitude of a Lorentz transformation relating the instantaneously commoving particle frame with the lab. It gives directly both the proper velocity and relative orientation of the particle frame and is related to the quantum wave function of the particle. It arises as a natural rotor or transformation element of the particle in the algebra of physical space (APS), Clifford’s geometric algebra of vectors in three-
dimensional Euclidean space.

In the following section, we review the spinorial form of Lorentz transformations that arises naturally in treatments of classical relativistic dynamics based on Clifford’s geometric algebra. A summary of essential features of APS is given in Appendix A. Section III describes the use of rotors to describe the motion of charges in external fields, and Sections IV and V present the dynamical symmetries for static and plane-wave fields, respectively. A couple of further symmetries are briefly mentioned in Section VI, followed by conclusions.

II. LORENTZ ROTORS

APS is the algebra of physical vectors and their products. It is isomorphic to both complex quaternions and to the even subalgebra of the spacetime algebra, and it is sometimes called the Pauli algebra because of its representation in which the unit vectors $e_k$ are replaced by the Pauli spin matrices $\sigma_k$. In fact, there are many possible matrix representations, but only the algebra is important. The algebra is quite simple, and we summarize its basic elements in the Appendix. More details can be found elsewhere.$[2, 7]$

In the algebra, simple, physical (“restricted”) Lorentz transformations are rotations in spacetime planes, and the Lorentz rotation of a spacetime vector $p$ such as the momentum of a charge is given by the algebraic product

$$p \rightarrow LpL^\dagger,$$

where $L = \exp (W/2)$ is an amplitude of the rotation called a Lorentz rotor, and $W$ gives both the spacetime plane and size of the rotation in that plane. The rotors are unimodular, that is unit elements of the algebra: $L\overline{L} = 1$.

III. EIGENSPINOR

The *eigenspinor* $\Lambda$ of a particle is the Lorentz rotor $L$ that transforms properties from the rest frame to the lab. By rest frame is meant the commoving inertial frame. The inertial frame instantaneously at rest with an accelerating particle is continuously changing. Given any known paravector $p_{\text{rest}}$ in the commoving particle frame, it is transformed to the lab by $p = \Lambda p_{\text{rest}}\Lambda^\dagger$. For example, the time axis $e_0$ in the rest frame becomes the dimensionless proper velocity $u = \Lambda e_0\Lambda^\dagger$ and $p = mcu$ is the momentum of the particle.
The time development of the eigenspinor takes the form

$$\dot{\Lambda} = \frac{1}{2} \Omega \Lambda, \quad (1)$$

where $\Omega = 2\dot{\Lambda}\Lambda$ is a spacetime plane giving the spacetime rotation rate of the particle and the dot indicates a derivative with respect to the proper time. We note that $\dot{\Lambda}$ and $\Lambda$ are orthogonal by virtue of the unimodularity of $\Lambda$. The proper-time derivative of the particle momentum is

$$\dot{p} = \frac{d}{d\tau} \left( \Lambda mc\Lambda^\dagger \right) = \frac{1}{2} \left( \Omega p + p\Omega^\dagger \right) \equiv \langle \Omega p \rangle_R,$$

where $\langle x \rangle_R = (x + x^\dagger)/2$ is the real (i.e., the hermitian) part of the element $x$. The Lorentz-force equation has exactly the same form:

$$\dot{p} = \langle eFu \rangle_R, \quad (2)$$

where $F = \frac{1}{2} F^{\mu\nu} \langle e_\mu e_\nu \rangle_V$ is the electromagnetic field. This follows from the eigenspinor equation of motion (1) for a spacetime rotation rate

$$\Omega = \frac{e}{mc} F,$$  

and this rotation rate suggests an explicit, relativistically covariant definition the electromagnetic field: $F$ is the spacetime rotation rate of the frame of a classical point charge per unit $e/mc$. For a charge at rest, this definition reduces to the usual definition of the electric field as the force per unit charge.

The identification (3) is not the only relation between $\Omega$ and $F$ that gives the Lorentz-force equation. The Lorentz force is independent of the magnetic field in the rest frame, and integration of the Lorentz-force equation gives the velocity and path of the particle, but not the orientation of its frame. The choice (3) satisfies the Lorentz force equation with a particular evolution of the orientation of the particle frame. We call the frame to which it refers the classical-particle frame. It is the simplest choice for $\Omega$ consistent with the Lorentz-force equation (2) and the only one for which $\Omega$ is independent of $\Lambda$ and for which the equation of motion (1) is therefore linear in $\Lambda$. A more general relation consistent with the Lorentz-force equation (2) is given in Appendix B, where it is shown that the choice (3) corresponds to the nonspinning frame of a particle with a $g$-factor of 2.

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IV. SURPRISING SYMMETRY 1: CHARGE IN UNIFORM FIELD

When $F$ is any uniform, constant electromagnetic field, the solution of (13) is

$$\Lambda (\tau) = \exp \left( \Omega \tau / 2 \right) \Lambda (0)$$

from which one generally gets a spacetime rotation (both a boost and a spatial rotation) of the particle frame in the plane of $F$. The field seen by the particle at proper time $\tau$ is

$$F_{\text{rest}} (\tau) = \bar{\Lambda} (\tau) F \Lambda (\tau)$$

$$= \bar{\Lambda} (0) \exp (-\Omega \tau / 2) F \exp (\Omega \tau / 2) \Lambda (0)$$

$$= F_{\text{rest}} (0)$$

since the plane $F$ is invariant under rotations in the plane itself. Thus the field seen by the accelerating charge is invariant!

This is hard to believe. Consider a familiar example: charge motion from rest in crossed $E$ and $B$ fields when $c^2 B^2 > E^2$.

At the top of the cycloid motion (see Fig. 1), the charge is moving at about twice the drift velocity (more precisely: $u = u^2_{\text{drift}}$) and the electric field in the unrotated charge frame has changed sign. However, the particle frame has also rotated about the lab $B$ direction by 180 degrees, so that the field it sees is unchanged. A similar result can be shown for any point on the trajectory.

![FIG. 1: Cycloid motion of charge in crossed fields.](image)
V. SURPRISING SYMMETRY 2: CHARGE IN PLANE WAVE

Directed plane waves are null flags of the form

\[ \mathbf{F} = \left(1 + \hat{k}\right) \mathbf{E}(s), \tag{4} \]

where \( \hat{k} \) is the propagation direction, which is perpendicular to the electric field \( \mathbf{E} \), assumed to be a known function of the Lorentz scalar \( s = \langle k\bar{x} \rangle_s = \omega t - \mathbf{k} \cdot \mathbf{x} \), where \( x \) is the spacetime position of the charge and \( k \) is a constant spacetime vector proportional to the spacetime wave vector. The null-flag form (4) is imposed by Maxwell’s equations for source-free space, \( \bar{\delta} \mathbf{F} = 0 \), since for any nontrivial field of the form \( \mathbf{F}(s) \) they imply \( \bar{k} \mathbf{F} = 0 \), and this implies that both \( k \) and \( \mathbf{F} \) are noninvertible and hence null: \( k\bar{k} = 0 = \mathbf{F}^2 \). The spacetime wave vector \( k = \omega \left(1 + \hat{k}\right)/c \) is called the flagpole of the null flag.

The equation of motion \( \mathbf{F}(s) \) can only be solved because of a remarkable symmetry. In the particle rest frame it is

\[ k_{\text{rest}} = \bar{\Lambda} k \bar{\Lambda}^\dagger \]

which is a rotation of \( k \) in the spacetime plane of \( \mathbf{F} \). But because \( k \) is null ( \( k\bar{k} = 0 \) ), it is not only in the null flag, it is also orthogonal to it, and therefore it is invariant under rotations in \( \mathbf{F} \) (see Fig. 2). Thus, while the charge is being accelerated by the plane wave, it continues to see a fixed wave paravector \( k_{\text{rest}} \).

FIG. 2: Null flag field. The spacetime wave vector \( k \) is in the flag plane but also orthogonal to it.
Consequently
\[ \dot{s} = \langle k \bar{u} \rangle_S = \omega_{\text{rest}} \]
is constant and the equation of motion reduces to
\[ \dot{\Lambda} = \omega_{\text{rest}} \frac{d\Lambda}{ds} = \frac{e}{2mc} \left( 1 + \hat{k} \right) E(s) \Lambda(0) \]
which is trivially integrated. Note that since \( \left( 1 - \hat{k} \right) \dot{\Lambda} = 0 \), we have
\[
\left( 1 + \hat{k} \right) E(s) \Lambda(\tau) = E(s) \left( 1 - \hat{k} \right) \Lambda(\tau) = E(s) \left( 1 - \hat{k} \right) \Lambda(0) = \left( 1 + \hat{k} \right) E(s) \Lambda(0).
\]

VI. OTHER SYMMETRIES OF THE ELECTROMAGNETIC FIELD

A. Boosts of Plane-Wave Fields

Any two null flags are related by a rotation and a dilation. Since propagating plane waves have the form of a null flag, and any inertial observer will see a propagating plane wave as a propagating plane wave, any boost of a propagating plane wave must be equivalent to a spatial rotation and dilation of that wave. This can be easily verified algebraically, where a less obvious symmetry is also demonstrated: the boost applied to the spacetime wave vector \( k \) is also equivalent to a spatial rotation and dilation, and the rotation angle and dilation factor are precisely the same as for the null flag.

This symmetry allows one to derive results for waves obliquely incident on plane conductors in terms of the normally incident case, and to express wave-guide modes as boosted standing waves.

B. Field of Uniformly Accelerated Charge

Consider a point charge in hyperbolic motion tracing out the world line \( r(\tau) \):
\[
\begin{align*}
    r &= \exp(c\tau \bar{e}_3) = \gamma (\bar{e}_3 + \beta) \\
    u &= c^{-1} \dot{r} = \bar{e}_3 r = \gamma \left( 1 + \beta \bar{e}_3 \right).
\end{align*}
\]

Here, \( r^0 \equiv ct = \gamma \beta = \sinh c\tau \) is the local coordinate time, \( \beta \) is the speed of the charge in units of \( c \), and \( \gamma = (1 - \beta^2)^{-1/2} \). The unit of length is \( l_0 = c^2 / |a_r| = 1 \), where \( a_r = c \bar{u} \bar{u} = c^2 \bar{e}_3 \) is
the acceleration in the rest frame. We note \( u \bar{u} = 1 = -r \bar{r} \) and \( r \bar{u} = e_3 \). The field position is \( x = x^\mu e_\mu = x_0 + x \), and the relative position \( R = x - r \) is lightlike:

\[
R \bar{R} = R_0^2 - R^2 = x \bar{x} - 1 - 2 \langle x \bar{r} \rangle_S = 0.
\]

This is the retarded condition that gives the retarded proper time \( \tau \) in terms of \( x \). The Liénard-Wiechert field [2]

\[
F = \frac{K c e}{\langle x \bar{u} \rangle_S^3} \left( \langle R \bar{u} \rangle_V + \frac{1}{2c} \bar{R} u \bar{R} \right)
\]

is the sum of the boosted Coulomb field and the acceleration field. Neither field by itself satisfies Maxwell’s equations; only the sum does. For the hyperbolic motion [3], the total field reduces to

\[
F = -\frac{K c e}{2 \langle x \bar{u} \rangle_S^3} (e_3 + x e_3 \bar{x}).
\]

At the instant \( t = 0 \), \( x = x \) and \( F \) is purely real: the magnetic field vanishes throughout space.

The boosted Coulomb field and the acceleration field separately have magnetic parts, but their sum cancels everywhere at \( t = 0 \), as it must by the equivalence principle. It is surprising that part of the essentially \( R^{-1} \) radiation field can be canceled by the Coulomb term. The usual \( R^{-2} \) dependence of the Coulomb term has a \( R^{-1} \) behavior because the proper velocity \( u \) at the retarded time grows linearly in \( R \), and this makes the cancellation possible. The electric field lines are curved away from the direction of acceleration. In the equivalent case of the uniform gravitational field, one would ascribe the curvature of the field lines to being a result of the gravitational field. The interesting implications for the interpretation of of the radiative field will be explored elsewhere.

The dynamical symmetry in this case arises when we consider the interaction of two opposite charges that form a dipole. Both charges have the same hyperbolic motion [5] but are held displaced a small distance from each other along a direction perpendicular to \( e_3 \). At \( t = 0 \), both charges are instantaneously at rest and each interacts with the purely electric field of the other. However, because of the curvature of the electric field lines, there is a net force in the direction of the acceleration. This force is readily evaluated and corresponds to a reduction in the gravitational force on the dipole that it would experience in a gravitational field. One also expects a reduction in the mass of the dipole arising from the attractive electromagnetic interaction between the charges. It is easily confirmed that
at small separations the resultant reduction in gravitational force equals to the upward lift arising from the curved electric-field lines.

VII. CONCLUSION

Much beautiful symmetry in electrodynamics can best be appreciated in a relativistically covariant formulation of electrodynamics such as APS that emphasizes the geometry of spacetime while maintaining clear relationships to the space and time components seen by any observer. The extension of the definition of the electric field to the covariant electromagnetic field $F$ in terms of the spacetime rotation rate of the classical charged-particle frame leads to new symmetries that are powerful tools for solving some problems in relativistic dynamics.

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Appendix A: Summary of APS

The structure of APS is entirely determined by the axiom that the square of any vector is its length squared: $v^2 \equiv vv = v \cdot v$, together with the usual associative and distributive laws for the sums and products of square matrices. For example, it follows directly from the axiom that aligned vectors, which are proportional to each other, commute, and that every nonzero vector has a inverse: $v^{-1} = v/v^2$. In particular, unit vectors such as $e_k$, $k = 1, 2, 3$, are their own inverses, and an explicit operator that transforms the vector $v$ into $w$ can be written $wv^{-1}$. By replacing $v$ in the axiom by the sum of perpendicular vectors, one sees that perpendicular vectors anticommute. Indeed, $e_2e_1 = -e_1e_2$ is called a bivector, and when operating from the left on any vector $v = v_xe_1 + v_ye_2$ in the $e_1e_2$ plane, it gives another vector, related to the original by a $\pi/2$ rotation in the plane: $e_2e_1v = v_xe_2 - v_ye_1$. To rotate $v$ by the angle $\phi$ in the plane, we can multiply by $\exp(e_2e_1\phi) = \cos \phi + e_2e_1 \sin \phi$, where the Euler relation follows by power-series expansions when one notes that $(e_2e_1)^2 = -1$. 

The bivector $e_2 e_1$ thus generates rotations in the $e_2 e_1$ plane. A rotation of a general vector $u = u_x e_1 + u_y e_2 + u_z e_3$ by $\phi$ in the $e_2 e_1$ plane can be expressed by what is called a spin transformation,

$$u \rightarrow R u R^\dagger,$$

where $R = \exp (e_2 e_1 \phi / 2)$ is a rotor and $R^\dagger = \exp \left[ (e_2 e_1)^\dagger \phi / 2 \right] = \exp (e_1 e_2 \phi / 2)$ is its reversion, obtained by reversing the order of vector factors. The notation reflects the fact that in any matrix rotation in which the basis vectors are hermitian, reversion corresponds to hermitian conjugation. Note that in this sense, $R$ is also unitary: $R^\dagger = R^{-1}$, and it follows that the bivector $e_2 e_1$ is itself invariant under rotations in the $e_2 e_1$ plane.

A general element of APS can contain scalar, vector, bivector, and trivector parts. The unit trivector $e_1 e_2 e_3$ is invariant under any rotations, commutes with vectors and hence all elements of APS, and squares to $-1$. It is called the pseudoscalar of the algebra and can be identified with the unit imaginary $i$. The linear space of APS is thus eight-dimensional space over the reals and contains several subspaces, including the original vector space, spanned by $\{e_1, e_2, e_3\}$, the bivector space, spanned by $\{e_2 e_3, e_3 e_1, e_1 e_2\}$, the complex field, spanned by $\{1, e_1 e_2 e_3 = i\}$, and direct sums of subspaces such as paravector space, spanned by $\{1, e_1, e_2, e_3\}$. Paravectors are sums of scalars plus vectors. The linear space of APS can be viewed as paravector space over the complex field.

The Euclidean metric of physical space induces a Minkowski spacetime metric on paravector space. This is seen by noting that the square of a paravector $p = p^0 + \mathbf{p}$ is generally not a scalar, but that $p \bar{p} = (p^0)^2 - \mathbf{p}^2$ always is, where $\bar{p} \equiv p^0 - \mathbf{p}$ is the Clifford conjugate of $p$. It is convenient to denote $e_0 = 1$ so that paravectors can be written with the Einstein summation convention as $p = p^\mu e_\mu$. The scalar product of paravectors $p$ and $q$ is then the scalar-like part of the product $p \bar{q}$:

$$\langle p \bar{q} \rangle_S = \frac{1}{2} (p \bar{q} + \bar{p} q) = p^\mu q^\nu \eta_{\mu\nu}$$

where the metric tensor $\eta_{\mu\nu} = \langle e_\mu \bar{e}_\nu \rangle_S$ is exactly that of Minkowski spacetime. It is therefore natural to use paravectors as spacetime vectors, where the scalar part of the paravector represents the time component of the spacetime vector. If $\langle p \bar{q} \rangle_S = 0$, the paravectors $p$ and $q$ are orthogonal to each other. The inverse of a paravector $p$ is $p^{-1} = \bar{p} / p \bar{p}$, but this exists only if $p$ is not null: $p \bar{p} \neq 0$. An explicit algebraic operator that transforms $p$ into $q$ is $qp^{-1} = q\bar{p} / p \bar{p}$.
Any two non-collinear paravectors $p, q$ determine a plane in spacetime represented by the biparavector $\langle pq \rangle_V \equiv \frac{1}{2} (pq - qp)$. Biparavector space is six dimensional and is spanned by $\{ \langle e_\mu \bar{e}_\nu \rangle_V \}_{0 \leq \mu < \nu \leq 3}$. It is a direct sum of the vector and bivector spaces of APS. Biparavectors generate rotations in spacetime, and these are the physical (restricted) Lorentz transformations, which we may also call Lorentz rotations. Such rotations preserve the scalar product of paravectors and can be generally written in the same form as a spatial rotation:

$$p \rightarrow LpL^\dagger,$$

where $L = \pm \exp(W/2)$ is a Lorentz rotor and $W = \frac{1}{2} W^{\mu\nu} \langle e_\mu \bar{e}_\nu \rangle_V$ is a biparavector. If $W$ contains only bivector parts, $L = \bar{L}^\dagger$ is unitary and gives a spatial rotation; if $W$ contains only vector parts, $L = L^\dagger$ and gives a boost. In all cases, $L$ is unimodular: $LL^\dagger = 1$.

Because of the unimodularity, the Lorentz rotation of a biparavector takes the form

$$pq \rightarrow LpqL^\dagger = Lpq\bar{L},$$

and in particular, the biparavector for the spacetime plane of a Lorentz rotation is invariant under that rotation.

**Appendix B: General Rotation Rate for a Spinning Particle**

As discussed above, the Lorentz-force equation determines the path of a charge, starting with a given position and velocity, in an electromagnetic field $F$, but it does not give the orientation of its frame. The eigenspinor $\Lambda$ gives both the path and the orientation. Mathematically, the acceleration $\dot{u}$ depends only on the real part of $\Omega$ in the particle rest frame, whereas $\Lambda$ depends on both the real and imaginary parts. However, if the charge also possesses a spin with an associated magnetic moment, that spin will precess in a magnetic field, and this precession constrains the evolution of the orientation. The most general relation consistent with the Lorentz-force equation \(^2\) can be expressed

$$\Omega = \frac{e}{mc} \left[ F + \frac{g - 2}{4} (F - uF^\dagger \bar{u}) \right] + \omega_0 S,$$

where $g$ is the $g$-factor for the spin and $\omega_0$ is the spin rate in the spin plane given by the unit bivector $S$. This can be shown \(^1\) to give the well-known BMT equation \(^12\) for the motion of a classical point charge with spin. The last term on the RHS represents the rotation
rate associated with the spin, while the second term on the RHS contains a \( u \) dependence that makes the eigenspinor equation of motion \( (1) \) nonlinear. The simpler result \( (3) \) is the spacetime rotation rate of what we may call the classical particle frame: a non-spinning frame tied to the point charge. Because \( \Omega (3) \) does not depend on the velocity of the charge, it gives a linear equation of motion \( (1) \). By comparison to the general expression \( (6) \), the spacetime rotation rate classical particle frame is that of a particle with \( \omega_0 = 0 \) and \( g = 2 \).

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