Circulants and critical points of polynomials

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Abstract

We prove that for any circulant matrix $C$ of size $n \times n$ with the monic characteristic polynomial $p(z)$, the spectrum of its $(n-1) \times (n-1)$ submatrix $C_{n-1}$ constructed with first $n-1$ rows and columns of $C$ consists of all critical points of $p(z)$. Using this fact we provide a simple proof for the Schoenberg conjecture recently proved by R. Pereira and S. Malamud. We also prove full generalization of a higher order Schoenberg-type conjecture proposed by M. de Bruin and A. Sharma and recently proved by W.S. Cheung and T.W. Ng. in its original form, i.e. for polynomials whose mass centre of roots equals zero. In this particular case, our inequality is stronger than it was conjectured by de Bruin and Sharma. Some Schmeisser’s-like results on majorization of critical point of polynomials are also obtained.

1 Introduction

In [20] I. Schoenberg conjectured the following statement

Schoenberg’s conjecture. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the roots of a polynomial $p$ of degree $n \geq 2$ such that $\sum_{j=1}^{n} \lambda_j = 0$, and let $w_1, w_2, \ldots, w_{n-1}$ be the roots of the derivative $p'$. Then

$$
\sum_{k=1}^{n-1} |w_k|^2 \leq \frac{n-2}{n} \sum_{j=1}^{n} |\lambda_j|^2,
$$

where equality holds if, and only if, all $\lambda_j$ lie on a straight line.

Later M. de Bruin, K. Ivanov, and A. Sharma [5] and independently Katsoprinakis [15] showed that Schoenberg’s conjecture is equivalent to the following inequality

$$
\sum_{k=1}^{n-1} |w_k|^2 \leq \frac{1}{n^2} \left( \sum_{j=1}^{n} \lambda_j \right)^2 + \frac{n-2}{n} \sum_{j=1}^{n} |\lambda_j|^2,
$$

(1.1)

which was conjectured to be true for all complex polynomials with no restrictions on their roots, and equality holds if, and only if, all $\lambda_j$ lie on a straight line. The inequality (1.1) was proved by R. Pereira [18] and independently by S. Malamud [16, 17].

In [4] M. de Bruin and A. Sharma proposed the following higher order Schoenberg-type conjecture.

De Bruin and Sharma’s conjecture. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the roots of a polynomial $p$ of degree $n \geq 2$ such that $\sum_{j=1}^{n} \lambda_j = 0$, and let $w_1, w_2, \ldots, w_{n-1}$ be the roots of the derivative $p'$. Then

$$
\sum_{k=1}^{n-1} |w_k|^4 \leq \frac{n-4}{n} \sum_{j=1}^{n} |\lambda_j|^4 + \frac{2}{n^2} \left( \sum_{j=1}^{n} |\lambda_j|^2 \right)^2,
$$

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where equality holds if, and only if, all $\lambda_j$ lie on a straight line passing through the origin of the complex plane.

This conjecture was proved by W. Cheung and T. Ng in [6] with use their approach of companion matrices (they also give an alternative proof of Schoenberg’s conjecture). Later, in [7] they generalized this approach by one-rank perturbation technique. Note that in [7], W. Cheung and T. Ng partially rediscover the so-called “one-rank perturbation method” developed by Yu. Barkovsky in his PhD thesis [1] which, unfortunately, was only partially published. Some ideas of this method can be found among problems in the lecture notes [2]. From the point of view of this method it is easy to see that, indeed, the rank one perturbation approach of W. Cheung and T. Ng generalizes the approach of differentiators used by R. Pereiera in [18].

S. Malamud [16, 17] proved Schoenberg’s conjecture by using another technique (with the same underlying ideas, in fact, basing on the same one-rank perturbation method). To prove the conjecture he established the fact saying that for any polynomial $p$ of degree $n$, there exists a normal matrix $A$ with $p(z) = \det(zI - A)$ such that the spectrum of its $(n-1) \times (n-1)$-submatrix $A_{n-1}$ constructed with the first $n-1$ rows and the first $n-1$ columns consists of the roots of the derivative of $p$ (Proposition 4.2 in [17]).

In this work, we use Barkovsky’s one-rank perturbation method (in its simplified Cheung-Ng’s form) to prove that circulant matrices is an example to Malamud’s Proposition 4.2, thus giving explicit simple examples of differentiators for a certain class of matrices. Note that W. Cheung and T. Ng also constructed a differentiator explicitly. But they started from a diagonal matrix, so their differentiator has a more complicated form, thus as a consequence, to work with such a differentiator is not easy.

It is known [8] that if $p(z)$ is a complex polynomial, then there exists a circulant matrix

$$
C = \begin{pmatrix}
  c_0 & c_1 & c_2 & \ldots & c_{n-2} & c_{n-1} \\
  c_{n-1} & c_0 & c_1 & \ldots & c_{n-3} & c_{n-2} \\
  c_{n-2} & c_{n-1} & c_0 & \ldots & c_{n-4} & c_{n-3} \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  c_2 & c_3 & c_4 & \ldots & c_0 & c_1 \\
  c_1 & c_2 & c_3 & \ldots & c_{n-1} & c_0
\end{pmatrix}
$$

(1.2)

whose characteristic polynomial (up to a constant factor) is $p(z)$. It turned out (Theorem 2.1) that the derivative $p'(z)$ is the characteristic polynomial (up to a constant factor) of the submatrix $C_{n-1}$ constructed with the first $n-1$ rows and columns of $C$.

**Remark 1.1.** After this paper was submitted, we learned that the result of Theorem 2.1 was briefly mentioned in Section 6.3 of the work [12] without a detailed proof.

Let $\{w_1, w_2, \ldots, w_{n-1}\}$ be the eigenvalues of the matrix $C_{n-1}$. The entries of the matrix $C_{n-1}$ depend linearly [8] on the roots of the polynomial $p(z)$, so we can use Schur’s theorem [21] (Theorem 2.6) to estimate sums of squares of absolute values of $w_k$ to prove Schoenberg’s conjecture. Then applying the same theorem to the matrix $C_{n-1}^2$ we prove the following fact which generalizes Cheung and Ng’s theorem (de Bruijn and Sharma’s conjecture).

**Theorem 1.2.** Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the roots of a polynomial $p$ of degree $n \geq 2$, and let $w_1, w_2, \ldots, w_{n-1}$ be the roots of its derivative $p'$. Then

$$
\sum_{k=1}^{n-1} |w_k|^4 \leq \frac{n}{n} - \frac{6}{n} \sum_{j=1}^{n} |\lambda_j|^4 + \frac{1}{n^2} \left( \sum_{j=1}^{n} |\lambda_j|^2 \right)^2 + \frac{1}{n^2} \sum_{j=1}^{n} \lambda_j^2 - \frac{1}{n^2} \left( \sum_{j=1}^{n} \lambda_j \right)^2 +
$$

$$
+ \frac{2}{n} \sum_{j=1}^{n} |\lambda_j|^2 \lambda_j + \frac{1}{n} \sum_{j=1}^{n} \lambda_j^2 - \frac{4}{n^2} \sum_{j=1}^{n} |\lambda_j|^2 \sum_{j=1}^{n} \lambda_j \lambda_j^2,
$$

(1.3)

where equality holds if, and only if, all $\lambda_j$ lie on a straight line.

For the specific case considered in de Bruijn and Sharma’s conjecture, we obtain the following inequality which is stronger than the original conjecture.
Corollary 1.3. Let \( \lambda_1, \lambda_2, \ldots, \lambda_n \) be the roots of a polynomial \( p \) of degree \( n \geq 2 \), and let \( w_1, w_2, \ldots, w_{n-1} \) be the roots of its derivative \( p' \). If \( \sum_{j=1}^{n} \lambda_j = 0 \), then
\[
\sum_{k=1}^{n-1} |w_k|^4 \leq \frac{n-4}{n} \sum_{j=1}^{n} |\lambda_j|^4 + \frac{1}{n^2} \left( \sum_{j=1}^{n} |\lambda_j|^2 \right)^2 + \frac{1}{n^2} \left( \sum_{j=1}^{n} \lambda_j^2 \right)^2,
\]
where equality holds if, and only if, all \( \lambda_j \) lie on a straight line passing through the origin of the complex plane.

It is easy to see that de Bruin and Sharma’s conjecture follows from Corollary 1.3, since
\[
\sum_{j=1}^{n} \lambda_j^2 \leq \sum_{j=1}^{n} |\lambda_j|^2.
\]
As a by product, we obtain the fact that if \( p(z) \) is a monic polynomial and \( C \) is a corresponding circulant matrix such that \( p(z) = \det(zI - C) \), then for any \( \alpha \in \mathbb{C} \), the polynomial \( p(z) + \alpha p'(z) \) is the characteristic polynomial of the matrix
\[
\begin{pmatrix}
c_0 - \alpha & c_1 & c_2 & \ldots & c_{n-2} & c_{n-1} \\
c_n & c_0 & c_1 & \ldots & c_{n-3} & c_{n-2} \\
c_{n-2} & c_{n-1} & c_0 & \ldots & c_{n-4} & c_{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
c_2 & c_3 & c_4 & \ldots & c_0 & c_1 \\
c_1 & c_2 & c_3 & \ldots & c_{n-1} & c_0
\end{pmatrix}
\]
Thus, this matrix can be used to study and estimating the roots of the polynomial \( p(z) + \alpha p'(z) \).

Finally, using some facts on singular values of matrices we establish a new majorization inequality on the absolute values of roots of derivatives (critical points) of polynomials.

Theorem 1.4. Let \( p \) be a polynomial of degree \( n \geq 2 \) with zeroes \( \lambda_j, j = 1, \ldots, n \), and critical points \( w_k, k = 1, \ldots, n-1 \). If \( q \) be a polynomial with zeroes \( |\lambda_j|^2, j = 1, \ldots, n \), and critical points \( \eta_k, k = 1, \ldots, n-1 \), then
\[
(\Phi(|w_1|), \ldots, \Phi(|w_{n-1}|)) \prec_w (\Phi(\sqrt{\eta_1}), \ldots, \Phi(\sqrt{\eta_{n-1}})),
\]
for any increasing function \( \Phi : [0, +\infty) \to \mathbb{R} \) such that \( \Phi \circ \exp \) is convex on \( \mathbb{R} \).

Here the symbol \( \prec_w \) means that the set \( \{\sqrt{\eta_1}, \ldots, \sqrt{\eta_{n-1}}\} \) weakly majorizes the set \( \{|w_1|, \ldots, |w_{n-1}|\} \) (for definition of majorization see Section 4).

Theorem 1.4 applied to polynomials with positive roots provides the following curious corollary.

Corollary 1.5. Let \( p(z) \) be a real polynomials with positive roots \( \lambda_j, j = 1, \ldots, n \), and let \( \xi_k, k = 1, \ldots, n-1 \), be its critical points. If \( \eta_k, k = 1, \ldots, n-1 \), are the critical points of the polynomial \( q \) with roots \( \lambda_j^2 \), then
\[
(\Phi(\xi_1), \ldots, \Phi(\xi_{n-1})) \prec_w (\Phi(\sqrt{\eta_1}), \ldots, \Phi(\sqrt{\eta_{n-1}})),
\]
for any increasing function \( \Phi : [0, +\infty) \to \mathbb{R} \) such that \( \Phi \circ \exp \) is convex on \( \mathbb{R} \).

However, this corollary shows, at the same time, that Theorem 1.4 is weaker than the following theorem that was proved by Cheung and Ng [6] for \( p(0) = 0 \), and later by Pereira [19] for arbitrary polynomials.

Theorem 1.6 (Cheung-Ng-Pereira). Let \( p \) be a polynomial of degree \( n \geq 2 \) with zeroes \( \lambda_j, j = 1, \ldots, n \), and critical points \( w_k, k = 1, \ldots, n-1 \). If \( q \) be a polynomial with zeroes \( |\lambda_j|, j = 1, \ldots, n \), and critical points \( \xi_k, k = 1, \ldots, n-1 \), then
\[
(\Phi(|w_1|), \ldots, \Phi(|w_{n-1}|)) \prec_w (\Phi(\xi_1), \ldots, \Phi(\xi_{n-1})),
\]
for any increasing function \( \Phi : [0, +\infty) \to \mathbb{R} \) such that \( \Phi \circ \exp \) is convex on \( \mathbb{R} \).

The paper is organized as follows. In Section 2 we remind to the reader some necessary facts on spectra of circulants and prove our main Theorem 2.1. Section 3 is devoted to the proof of Theorem 1.2 on circulants. In Section 4 we discuss the usefulness of circulants for proving theorems on majorization of the absolute values of critical points of polynomials. In particular, we prove Theorem 1.4. The last Section 5 we discuss other possibilities to use Theorem 2.1 for localization of roots of polynomials and pose some problems that can be of interest in this sense.
2 Circulants

A matrix of the form

\[
C = \begin{pmatrix}
c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\
c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\
c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
c_2 & c_3 & c_4 & \cdots & c_0 & c_1 \\
c_1 & c_2 & c_3 & \cdots & c_{n-1} & c_0
\end{pmatrix}
\] (2.1)

is called circulant. The polynomial

\[
f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1} x^{n-1}
\]

is called the associated polynomial of the matrix \(C\). It is very well-known [8] that the eigenvalues \(\lambda_j, j = 1, \ldots, n\), of the matrix \(C\) can be expressed as follows

\[
\lambda_j = c_0 + c_1 \omega_j^{-1} + c_2 \omega_j^{-2} + \cdots + c_{n-1} \omega_j^{-n+1} = f(\omega_j^{-1}), \quad j = 1, \ldots, n,
\] (2.2)

where

\[
\omega_k = e^{\frac{2\pi i}{n}}k, \quad k = 0, 1, \ldots, n-1,
\]

are the \(n^{th}\) roots of unity. The formula (2.2) shows that for any set \(\{\lambda_1, \ldots, \lambda_n\}\) of \(n\) complex numbers, there exist at most \(n!\) (number of permutations) different circulant matrices with the spectrum at these numbers (counting multiplicities). At the same time, if we fix the arrangement of these numbers, for example, if we arrange the numbers \(\lambda_j\) as follows

\[
|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|,
\] (2.3)

then there exists exactly one circulant matrix whose spectrum \(\{\lambda_1, \ldots, \lambda_n\}\) satisfies the equations (2.2) and have the prescribed arrangement (e.g. satisfy the condition (2.3)). This follows from the simple fact that the system (2.2) is not singular, since its matrix

\[
\begin{pmatrix}
\omega_0 & \omega_1 & \cdots & \omega_{n-2} & \omega_{n-1} \\
\omega_0^2 & \omega_1^2 & \cdots & \omega_{n-2}^2 & \omega_{n-1}^2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\omega_0^{n-2} & \omega_1^{n-2} & \cdots & \omega_{n-2}^{n-2} & \omega_{n-1}^{n-2} \\
\omega_0^{n-1} & \omega_1^{n-1} & \cdots & \omega_{n-2}^{n-1} & \omega_{n-1}^{n-1}
\end{pmatrix}
\] (2.4)

has determinant equal to \(n\). With a fixed arrangement of eigenvalues, to the eigenvalue \(\lambda_j\) there corresponds the following eigenvector

\[
u_j = \begin{pmatrix} 1 \\ \omega_j^{-1} \\ \omega_j^{-2} \\ \vdots \\ \omega_j^{n-1} \end{pmatrix},
\] (2.5)

so that

\[Cu_j = \lambda_j u_j, \quad j = 1, \ldots, n.\]

Moreover,

\[
\langle u_k, u_j \rangle = n \delta_{kj}, \quad k, j = 1, \ldots, n,
\] (2.6)

where \(\langle \cdot, \cdot \rangle\) is the inner product in \(\mathbb{C}^n\), and \(\delta_{kj}\) is the Kronecker symbol.
Let us denote by $C_{n-1}$ the following $(n-1) \times (n-1)$ matrix constructed with first $n-1$ rows and columns of the circulant $C$:

\[
C_{n-1} = \begin{pmatrix}
c_0 & c_1 & c_2 & \ldots & c_{n-3} & c_{n-2} \\
c_{n-1} & c_0 & c_1 & \ldots & c_{n-4} & c_{n-3} \\
c_{n-2} & c_{n-1} & c_0 & \ldots & c_{n-5} & c_{n-4} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
c_3 & c_4 & c_5 & \ldots & c_0 & c_1 \\
c_2 & c_3 & c_4 & \ldots & c_{n-1} & c_0
\end{pmatrix}
\]  

(2.7)

Now we are in a position to establish one of the main results of the present work revealing an interesting and deep property of circulant matrices.

**Theorem 2.1.** For a given complex circulant matrix $C$ with the monic characteristic polynomial $p(z)$, the spectrum of its submatrix $C_{n-1}$ coincides with the set of all critical points of $p(z)$ (counting multiplicities).

**Proof.** Let $\lambda_j, j = 1, \ldots, n$, be the roots of the polynomial $p(z)$. Suppose first that $p'(\lambda_j) \neq 0$ and $\lambda_j \neq 0, \quad j = 1, \ldots, n,$

and construct the following rational function

\[
R(z) = \frac{1}{n} \frac{zp'(z)}{p(z)} = 1 + \frac{1}{n} \sum_{j=1}^{n} \frac{\lambda_j}{z - \lambda_j}.
\]

Let $u$ be the vector as follows

\[
u \overset{\text{def}}{=} \frac{1}{n} \sum_{j=1}^{n} u_j = \frac{1}{n} \sum_{j=1}^{n} \begin{pmatrix}
1 \\
\omega_j \\
\omega_j^2 \\
\vdots \\
\omega_j^{n-1}
\end{pmatrix} = \begin{pmatrix}
1 \\
0 \\
0 \\
\vdots \\
0
\end{pmatrix},
\]

(2.8)

where $u_j, j = 1, \ldots, n$, are the eigenvectors of the matrix $C$ defined in (2.5), and let $H$ be the following one-rank matrix

\[
H \overset{\text{def}}{=} u \otimes u = \begin{pmatrix}
1 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0
\end{pmatrix},
\]

thus $\forall x \in \mathbb{C}^n, \quad Hx = \langle x, u \rangle u$,

so that

\[
Hu_j = u, \quad j = 1, \ldots, n,
\]

as it follows from (2.8) and (2.6) (or from the definition of the matrix $H$). Also it is clear that $Hu = u$, so $H^2 = H$, and the operator

\[
P = I - H
\]

is a projector to the subspace

\[
\{ x \in \mathbb{C}^n : \langle x, u \rangle = 0 \}.
\]

as a difference of the identity operator and the projector $H$ to the subspace $\text{span}\{u\}$.

We now claim that $zp'(z) = n \det(zI - PC)$. Indeed, since

\[
(zI - C)^{-1}u_j = \frac{u_j}{z - \lambda_j}, \quad j = 1, \ldots, n,
\]

and

\[
HCu_j = \lambda_j u,
\]

then

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one has for $z \neq \lambda_j, j = 1, \ldots, n$,

$$HC(zI - C)^{-1} u = \frac{1}{n} \sum_{j=1}^{n} HC u_j = \frac{1}{n} \sum_{j=1}^{n} \frac{\lambda_j}{z - \lambda_j} u = (R(z) - 1) u.$$ 

From this identity it is clear that the matrix $HC(zI - C)^{-1}$ has the form

$$HC(zI - C)^{-1} = \left( \begin{array}{cccc} R(z) - 1 & \times & \ldots & \times \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \end{array} \right),$$

so the only (possibly) nonzero entries of this matrix lie on the first row. Thus, we have

$$\det(zI - PC) = \det([zI - C] + HC) = \det(zI - C) \det(I + HC(zI - C)^{-1}) = p(z)(R(z) - 1) = \frac{1}{n} zp'(z),$$

so

$$zp'(z) = n \det(zI - PC).$$

On the other hand,

$$n \det(zI - PC) = nz \det(zI - C_{n-1}),$$

where $C_{n-1}$ is defined in (2.7). Therefore,

$$p'(z) = n \det(zI - C_{n-1}).$$

If $p(z)$ has simple roots one of which is equal to zero, the we consider a polynomial $\tilde{p}(z) = p(z - \varepsilon), \varepsilon \neq 0$, so that $\tilde{p}'(z) = p'(z - \varepsilon)$. Then we find for $\tilde{p}(z)$ a circulant $\tilde{C}$ such that $\tilde{p} = \det(zI - \tilde{C})$ and $\tilde{p}'(z) = n \det(zI - \tilde{C}_{n-1})$. Changing variables $z \rightarrow z + \varepsilon$ gives us $p(z) = \det(zI - C)$, where $C = \tilde{C} - \varepsilon I$, and $p'(z) = n \det(zI - C_{n-1})$.

Finally, if $p(z)$ has multiple roots, then one can approximate $p(z)$ by a sequence of polynomials $p_m(z)$ with simple roots, so $p_m(z) \rightarrow p(z)$ and $p_m'(z) \rightarrow p'(z)$ as $m \rightarrow +\infty$ uniformly on compact sets. Clearly, in this case the corresponding circulants $C^{(m)}$ such that $p_m(z) = \det(zI - C^{(m)})$ tend to a circulant matrix $C$ such that $p(z) = \det(zI - C)$ and $p'(z) = n \det(zI - C_{n-1})$, as required.

\[ \square \]

**Remark 2.2.** In the proof of Theorem 2.1, we used the technique from the work [7] by Cheung and Ng, but, indeed, the fact proved in Theorem 2.1 is a simple consequence of a more general fact established in [1] and [3, Section 2].

The circulant matrix $C$ is normal [8], that is,

$$CC^* = C^* C.$$ 

Here the matrix $C^*$ adjoint to $C$ is also a circulant. Its eigenvalues are $\bar{\lambda}_j$ with the same corresponding eigenvectors $u_j$ defined in (2.5). Let us denote $A \overset{def}{=} CC^*$. The matrix $A$ is a circulant as well

$$A = \left( \begin{array}{cccccc} a_0 & a_1 & a_2 & \ldots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \ldots & a_{n-3} & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \ldots & a_{n-4} & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_2 & a_3 & a_4 & \ldots & a_0 & a_1 \\ a_1 & a_2 & a_3 & \ldots & a_{n-1} & a_0 \end{array} \right),$$

(2.9)

where

$$a_0 = \sum_{j=0}^{n-1} |c_j|^2$$

(2.10)

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and

\[ a_k = \sum_{j=0}^{k-1} c_j \bar{c}_{n-k+j} + \sum_{j=0}^{n-k-1} c_{k+j} \bar{c}_j. \]  \hspace{1cm} (2.11)

Here bar means the complex conjugation. Clearly,

\[ a_k = \bar{a}_{n-k}, \quad k = 1, \ldots, n-1, \quad \text{and} \quad a_0 \in \mathbb{R}. \]  \hspace{1cm} (2.12)

The eigenvalues of the matrix $A$ are $|\lambda_j|^2$, $j = 1, \ldots, n$.

Generally speaking, the matrix $C_{n-1}$ is not a circulant and is not normal. However, it is possible to describe the case when it is normal. The following theorem is a consequence of a theorem due to Ky Fan and G. Pall [14, Theorem 2] which deals with arbitrary normal matrices and its principal submatrices.

**Theorem 2.3.** The submatrix $C_{n-1}$ of a circulant $C$ is normal if, and only if, all the eigenvalues of $C$ lie on a straight line.

This theorem also follows from [18, Proposition 3.4], since $C_{n-1}$ is a differentiator. It can also be deduced from the result of Farenick et al [9, Theorem 1.1] (see also [11]) that fully describes the structure of all finite normal Toeplitz matrices. However, we prove Theorem 2.3 here in order to reveal some specific properties of circulants and their submatrices. To prove it, we need the following simple fact whose proof we provide here for completeness.

**Proposition 2.4.** The circulant $C$ has all eigenvalues real if, and only if, it is self-adjoint or, equivalently, if its coefficients satisfy the reflection property

\[ c_0 \in \mathbb{R}, \quad c_{n-k} = \bar{c}_k, \quad k = 1, \ldots, n. \]  \hspace{1cm} (2.13)

**Proof.** It is clear that $C$ is self-adjoint if, and only if, the reflection property (2.13) holds. So, this property implies the reality of the spectrum of $C$.

Conversely, if all the eigenvalues $\lambda_j$ of $C$ are real, then from (2.2) one has

\[ \bar{\lambda}_j = \bar{c}_0 + \bar{c}_{n-1} \omega_{j-1} + \bar{c}_{n-2} \omega_{j-1}^2 + \cdots + \bar{c}_1 \omega_{j-1}^{n-1} = \lambda_j, \quad j = 1, \ldots, n. \]

Since the matrix (2.4) of this system is non-singular, the solution is unique (since the arrangement of $\lambda_j$ is supposed to be fixed), so it must satisfy the condition (2.13). \hfill \square

If $C$ has only real eigenvalues, so it is self-adjoint, then its (principal) submatrix $C_{n-1}$ is also self-adjoint, and has only real eigenvalues. The reality of the eigenvalues of $C_{n-1}$ also follows from Theorem 2.1 but it does not guarantee, generally speaking, that $C_{n-1}$ is self-adjoint.

**Proof of Theorem 2.3.** If all $\lambda_j$ lie on a straight line in the complex plane, then they have the form

\[ \lambda_j = \alpha \eta_j + \beta, \quad j = 1, \ldots, n, \]

where $\alpha, \beta \in \mathbb{C}$, and $\eta_j \in \mathbb{R}$. Therefore, the matrix $C$ can be represented as follows

\[ C = \alpha \bar{C} + \beta I, \]

where $\bar{C}$ is a circulant matrix whose eigenvalues are the numbers $\eta_j$. By Theorem 2.4, $\bar{C}$ is self-adjoint. Consequently, $C_{n-1}$ has the form

\[ C_{n-1} = \alpha \bar{C}_{n-1} + \beta I_{n-1}. \]  \hspace{1cm} (2.14)

where $I_{n-1}$ is the $(n-1) \times (n-1)$ identity matrix, and $\bar{C}_{n-1}$ is self-adjoint as principal submatrix of $\bar{C}$. It is easy to check now that $C_{n-1}$ is normal.

Conversely, suppose that $C_{n-1}$ is normal. Let us denote

\[ B \overset{\text{def}}{=} C_{n-1} \bar{c}_{n-1}. \]  \hspace{1cm} (2.15)

The matrix $B$ is self-adjoint, and its entries have the form

\[ b_{lk} = a_{k-l} - c_{n-l} \bar{c}_{n-k}, \quad 1 \leq l \leq k \leq n-1. \]  \hspace{1cm} (2.16)
At the same time, the entries \( \tilde{b}_{kl} \) of the self-adjoint matrix

\[
\tilde{B} \overset{\text{def}}{=} C_{n-1}^* C_{n-1}.
\]  

have the form

\[
\tilde{b}_{lk} = a_{k-l} - c_k \overline{c_l}, \quad 1 \leq l \leq k \leq n-1.
\]  

Thus, from normality of the matrix \( C_{n-1} \) it follows that \( B = \tilde{B} \). This equality together with (2.16)–(2.18) implies

\[
|c_k| = |c_{n-k}|, \quad k = 1, \ldots, n-1,
\]

and

\[
c_{n-k} \overline{c}_k = c_k \overline{c}_{n-k}, \quad 1 \leq l < k \leq n-1.
\]

If we put \( c_k = r_k e^{i \varphi_k}, \ \varphi_k \in [0, 2\pi), \ k = 1, \ldots, n-1 \), then the identities (2.19)–(2.20) imply

\[
\varphi_k + \varphi_{n-k} = \varphi_l + \varphi_{n-l}, \quad 1 \leq l < k \leq n-1.
\]

Denoting \( 2\theta \overset{\text{def}}{=} \varphi_k + \varphi_{n-k} \) for all \( k = 1, \ldots, n-1 \), and

\[
\tilde{c}_k \overset{\text{def}}{=} -\theta + \varphi_k, \quad k = 1, \ldots, n-1,
\]

we get

\[
\tilde{c}_k + \tilde{c}_{n-k} = -2\theta + \varphi_k + \varphi_{n-k} = 0, \quad k = 1, \ldots, n-1,
\]

so the numbers \( \tilde{c}_k \overset{\text{def}}{=} c_k e^{-i\theta} \) satisfy the following conditions

\[
\tilde{c}_k = \overline{c}_{n-k}, \quad k = 1, \ldots, n-1.
\]

This means that we can represent the matrix \( C_{n-1} \) as follows

\[
C_{n-1} = e^{i\theta} \tilde{C}_{n-1} + c_0 I_{n-1},
\]

where the matrix

\[
\tilde{C}_{n-1} = \begin{pmatrix}
0 & \hat{c}_1 & \hat{c}_2 & \cdots & \hat{c}_{n-3} & \hat{c}_{n-2} \\
\hat{c}_{n-1} & 0 & \hat{c}_1 & \cdots & \hat{c}_{n-4} & \hat{c}_{n-3} \\
\hat{c}_{n-2} & \hat{c}_{n-1} & 0 & \cdots & \hat{c}_{n-5} & \hat{c}_{n-4} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\hat{c}_3 & \hat{c}_4 & \hat{c}_5 & \cdots & 0 & \hat{c}_1 \\
\hat{c}_2 & \hat{c}_3 & \hat{c}_4 & \hat{c}_5 & \cdots & \hat{c}_{n-1} & 0
\end{pmatrix}
\]

is self-adjoint by (2.21). Thus the initial matrix \( C \) has the form

\[
C = e^{i\theta} \tilde{C} + c_0 I,
\]

where \( \tilde{C} \) is self-adjoint. It is clear now that all eigenvalues of \( C \) lie on a straight line. \( \square \)

For completeness we prove here the following simple and curious fact. The usefulness of this statement is discussed in Section 5.

**Proposition 2.5.** The matrix \( \sqrt{CC^*} \) is a (unique) circulant matrix whose eigenvalues are \( |\lambda_j|, \ j = 1, \ldots, n \) for a fixed arrangement of \( |\lambda_j| \).

**Proof.** As we mentioned above, there exists a unique circulant \( \tilde{C} \) whose eigenvalues are \( |\lambda_j| \) with a fixed arrangement, say, satisfying the inequalities (2.3). By Proposition 2.4, \( \tilde{C} \) is self-adjoint, so it is positive semidefinite self-adjoint matrix. It is clear that \( \tilde{C}^2 = CC^* \), since the eigenvalues of the circulants \( CC^* \) coincide up to their arrangement. The positive semidefinite square root of the positive semidefinite self-adjoint matrix \( CC^* \) is unique [10, Theorem 7.2.6], therefore \( \tilde{C} = \sqrt{CC^*} \), as required. \( \square \)

Finally, let us recall the following theorem due to Schur [21] which is to be of use in the next Section.

**Theorem 2.6.** If \( T = (t_{kj})_{k,j=1}^n \) is an \( n \times n \) matrix with eigenvalues \( \mu_1, \mu_2, \ldots, \mu_n \), then

\[
\sum_{j=1}^n |\mu_j|^2 \leq Tr(TT^*) = \sum_{k,j=1}^n |t_{kj}|^2,
\]

where equality holds if, and only if, \( A \) is normal.
3 Schoenberg’s and de Bruin and Sharma’s conjectures

In this Section we show that both Schoenberg’s conjecture and de Bruin and Sharma’s conjecture, are a simple consequence of Theorem 2.6 and 2.1. Moreover, we generalize de Bruin and Sharma’s conjecture here to arbitrary polynomials.

**Theorem 3.1** (Pereira [18], Malamud [17]). Let \( \lambda_1, \lambda_2, \ldots, \lambda_n \) be the roots of a polynomial \( p \) of degree \( n \geq 2 \), and let \( w_1, w_2, \ldots, w_{n-1} \) be the roots of the derivative \( p' \). Then

\[
\sum_{k=1}^{n-1} |w_k|^2 \leq \frac{1}{n^2} \left( \sum_{j=1}^{n} \lambda_j \right)^2 + \frac{n-2}{n} \sum_{j=1}^{n} |\lambda_j|^2,
\]

where equality holds if, and only if, all \( \lambda_j \) lie on a straight line.

**Proof.** Let \( C \) be a circulant matrix whose eigenvalues are \( \lambda_j, \ j = 1, \ldots, n \). From Theorems 2.1 and 2.6, we have

\[
\sum_{k=1}^{n-1} |w_k|^2 \leq \text{Tr}(C_{n-1}C_{n-1}^*) = (n-1)|c_0|^2 + (n-2) \sum_{k=1}^{n-1} |c_k|^2 = |c_0|^2 + (n-2) \sum_{k=0}^{n-1} |c_k|^2.
\] (3.1)

On the other hand, by normality of the circulant \( C \), one has

\[
\sum_{j=1}^{n} |\lambda_j|^2 = n \sum_{k=0}^{n} |c_k|^2.
\] (3.2)

Moreover,

\[
c_0 = \frac{1}{n} \sum_{j=1}^{n} \lambda_j.
\] (3.3)

Now (3.1)–(3.3) together with Theorem 2.6 imply the theorem. \( \square \)

We are now in a position to prove the full generalization of de Bruin and Sharma’s conjecture.

**Theorem 1.1.** Let \( \lambda_1, \lambda_2, \ldots, \lambda_n \) be the roots of a polynomial \( p \) of degree \( n \geq 2 \), and let \( w_1, w_2, \ldots, w_{n-1} \) be the roots of its derivative \( p' \). Then

\[
\sum_{k=1}^{n-1} |w_k|^4 \leq \frac{n-6}{n} \sum_{j=1}^{n} |\lambda_j|^4 + \frac{1}{n^2} \left( \sum_{j=1}^{n} |\lambda_j|^2 \right)^2 + \frac{1}{n^2} \sum_{j=1}^{n} \lambda_j^2 - \frac{1}{n^2} \left( \sum_{j=1}^{n} \lambda_j \right)^2 + \frac{2}{n} \sum_{j=1}^{n} |\lambda_j|^2 \left| \lambda_j + \frac{1}{n} \sum_{j=1}^{n} \lambda_j \right| - \frac{4}{n^3} \sum_{j=1}^{n} |\lambda_j|^2 \sum_{j=1}^{n} \lambda_j
\] (3.4)

where equality holds if, and only if, all \( \lambda_j \) lie on a straight line.

**Proof.** Since the eigenvalues of the matrix \( C_{n-1}^2 \) are \( w_k^2, k = 1, \ldots, n-1 \), from Theorem 2.6 we have

\[
\sum_{k=1}^{n-1} |w_k|^4 \leq \text{Tr} (C_{n-1}C_{n-1}^*C_{n-1}^*C_{n-1}^*) = \text{Tr}(C_{n-1}C_{n-1}^*C_{n-1}^*C_{n-1}^*) = \text{Tr}(B\tilde{B}),
\]

where \( B \) and \( \tilde{B} \) are defined in (2.15)–(2.16) and in (2.17)–(2.18), respectively. To calculate \( \text{Tr}(B\tilde{B}) \), let us note first that

\[
B = A_{n-1} - D \quad \text{and} \quad \tilde{B} = A_{n-1} - \tilde{D},
\]

where \( A_{n-1} \) is the \( (n-1)^{th} \) leading principal submatrix of the circulant \( A \) defined in (2.9), while

\[
D = \begin{pmatrix}
|c_{n-1}|^2 & c_{n-1}c_{n-2} & \ldots & c_{n-1}c_1 \\
-c_{n-1}c_{n-1} & |c_{n-2}|^2 & \ldots & c_{n-2}c_1 \\
-c_{n-1}c_{n-1} & c_{n-3}c_{n-3} & \ldots & c_{n-3}c_1 \\
\vdots & \vdots & \ddots & \vdots \\
c_{1}c_{n-1} & c_{1}c_{n-3} & \ldots & |c_1|^2 \\
c_{1}c_{n-1} & c_{1}c_{n-3} & \ldots & c_{1}c_1
\end{pmatrix}
\]

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and

\[
\tilde{D} = \begin{pmatrix}
|c_1|^2 & c_2 & c_3 & \cdots & c_{n-2} & c_{n-1} & c_1 \\
|c_2|^2 & |c_2|^2 & c_3 & \cdots & c_{n-3} & c_{n-2} & c_1 \\
|c_3|^2 & c_2 & |c_3|^2 & \cdots & c_{n-4} & c_{n-3} & c_1 \\
& \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
|c_{n-2}|^2 & c_{n-3} & c_{n-4} & \cdots & |c_{n-2}|^2 & c_{n-3} & c_1 \\
|c_{n-1}|^2 & c_{n-2} & c_{n-3} & \cdots & c_{n-4} & \vdots & \vdots \\
\end{pmatrix}
\]

Since \( A \) is self-adjoint, \( A_{n-1} \) is self-adjoint, as well. Thus we have

\[
\text{Tr}(B\tilde{B}) = \text{Tr}(A^2_{n-1}) + \text{Tr}(D\tilde{D}) - \text{Tr}(A_{n-1}\tilde{D}) - \text{Tr}(A_{n-1}D).
\]

It is easy to see that

\[
\text{Tr}(A^2_{n-1}) = a_0^2 + (n-2) \sum_{k=0}^{n-1} |a_k|^2 = a_0^2 + (n-2)e_0,
\]

where

\[
e_0 = \sum_{k=0}^{n-1} |a_k|^2
\]

is the diagonal entry of the circulant matrix

\[
E \overset{\text{def}}{=} AA^* = \begin{pmatrix}
e_0 & e_1 & e_2 & \cdots & e_{n-2} & e_{n-1} \\
e_{n-1} & e_0 & e_1 & \cdots & e_{n-3} & e_{n-2} \\
e_{n-2} & e_{n-1} & e_0 & \cdots & e_{n-4} & e_{n-3} \\
& \vdots & \vdots & \ddots & \vdots & \vdots \\
e_2 & e_3 & e_4 & \cdots & e_0 & e_1 \\
e_1 & e_2 & e_3 & \cdots & e_1 & e_0
\end{pmatrix}
\]

whose eigenvalues are \( |\lambda_j|^4 \), \( j = 1, \ldots, n \), so

\[
e_0 = \frac{1}{n} \sum_{j=1}^{n} |\lambda_j|^4.
\]  \hspace{1cm} \text{(3.5)}

It is clear that

\[
\text{Tr}(D\tilde{D}) = \left| \sum_{k=1}^{n-1} c_k c_{n-k} \right|^2.
\]
Furthermore, for \( Tr(A_{n-1}D) \) we have by (2.10)-(2.11)

\[
Tr(A_{n-1}D) = a_0 \sum_{k=1}^{n-1} |c_{n-k}|^2 + \sum_{l=1}^{n-1} c_{n-l} \sum_{k=1}^{n-1} c_{n-k} a_{n-l+k} + \sum_{l=1}^{n-2} c_{n-l} \sum_{k=1}^{n-1} c_{n-k} a_{k-l} = \\
= a_0(a_0 - |c_0|^2) + \sum_{l=1}^{n-1} l-1 \sum_{k=1}^{n-1} c_{n-l} c_{n-l+k} a_{n-k} + \sum_{l=1}^{n-2} \sum_{k=1}^{n-1} c_{n-l} c_{n-k-l} a_k = \\
= a_0(a_0 - |c_0|^2) + \sum_{k=1}^{n-1} a_k \left[ \sum_{j=0}^{k-1} c_j c_{n-k-j} + \sum_{j=1}^{n-k-1} c_{j+k} c_j \right] - c_0 \sum_{k=1}^{n-1} a_k c_k - c_0 \sum_{k=1}^{n-1} a_k c_{n-k} = \\
= a_0^2 + \sum_{k=1}^{n-1} |a_k|^2 - a_0 |c_0|^2 - \sum_{k=1}^{n-1} a_k [ c_0 c_k + c_0 c_{n-k} ] = \\
= \epsilon_0 - a_0 |c_0|^2 - \sum_{k=1}^{n-1} a_k [ c_0 c_k + c_0 c_{n-k} ].
\]

Analogously, we have

\[
Tr(A_{n-1}\tilde{D}) = \epsilon_0 - a_0 |c_0|^2 - \sum_{k=1}^{n-1} a_k [ c_0 c_k + c_0 c_{n-k} ],
\]

so

\[
Tr(B\tilde{B}) = a_0^2 + (n-4)\epsilon_0 + \left| \sum_{k=1}^{n-1} c_k c_{n-k} \right|^2 + 2a_0 |c_0|^2 + 2 \sum_{k=1}^{n-1} a_k [ c_0 c_k + c_0 c_{n-k} ],
\]

(3.6)

On another hand, from (2.2) we obtain

\[
c_k = \frac{1}{n} \sum_{j=1}^{n} \omega_{j-1}^{-k} \lambda_j, \quad k = 1, \ldots, n-1,
\]

(3.7)

so

\[
\sum_{k=1}^{n-1} c_k c_{n-k} = \frac{1}{n^2} \sum_{j=1}^{n} \sum_{l=1}^{n} \lambda_j \lambda_l \sum_{k=1}^{n} \omega_{j-1}^{-k} \omega_{l-1}^{-k} = \frac{n-1}{n^2} \sum_{j=1}^{n} \lambda_j^2 - \frac{2}{n} \sum_{1 \leq l < j \leq n} \lambda_j \lambda_l = \frac{1}{n} \sum_{j=1}^{n} \lambda_j^2 - \frac{1}{n^2} \left( \sum_{j=1}^{n} \lambda_j \right)^2.
\]

Therefore,

\[
\left| \sum_{k=1}^{n-1} c_k c_{n-k} \right|^2 = \left| \frac{1}{n} \sum_{j=1}^{n} \lambda_j^2 - \frac{1}{n^2} \left( \sum_{j=1}^{n} \lambda_j \right)^2 \right|^2.
\]

(3.8)

Recall that the eigenvalues of the matrix \( A \) defined in (2.9) are \( |\lambda_j|^2 \), \( j = 1, \ldots, n \), so analogously to (2.2), one has

\[
|\lambda_j|^2 = a_0 + a_1 \omega_{j-1} + a_2 \omega_{j-1}^2 + \cdots + a_{n-1} \omega_{j-1}^{n-1}, \quad j = 1, \ldots, n.
\]

(3.9)
Finally, from (3.7) and (3.9) we obtain the following
\[ a_0|c_0|^2 + \sum_{k=1}^{n-1} a_k \left[ c_0 \bar{c}_k + \bar{c}_0 c_{n-k} \right] = -a_0|c_0|^2 + \frac{1}{n} \sum_{k=0}^{n-1} a_k \sum_{j=1}^{n} j \lambda_j c_0 + \lambda_j \bar{c}_0 = \]
\[ = -a_0|c_0|^2 + \frac{1}{n} \sum_{j=1}^{n} |\lambda_j|^2 \left[ \lambda_j c_0 + \lambda_j \bar{c}_0 \right] = -a_0|c_0|^2 + \frac{1}{n} \sum_{j=1}^{n} |\lambda_j|^2 \left[ |\lambda_j + c_0|^2 - |c_0|^2 - |\lambda_j|^2 \right] = \]
\[ = \frac{1}{n} \sum_{j=1}^{n} |\lambda_j|^2 |\lambda_j + c_0|^2 - 2a_0|c_0|^2 - c_0. \]

Now the inequality (3.4) follows from (3.6), (3.8), (3.10), and from (2.10), (3.3), (3.5).

4 Majorization of critical points of polynomials

Theorem 2.1 allows also to obtain some facts on majorization of critical points of polynomials or simplify their proofs. Before we show this, we need to present the definition of majorization.

For any vector \( x = (x_1, \ldots, x_n) \), denote by \( (x_1^[1], \ldots, x_n^[1]) \) the following rearrangement of the components of \( x \)
\[ x_1^[1] \geq x_2^[1] \geq \cdots \geq x_n^[1]. \]

**Definition 4.1.** For any two vectors \( a = (a_1, \ldots, a_n) \) and \( b = (b_1, \ldots, b_n) \) from \( \mathbb{R}^n \), we say that \( b \) weakly majorizes \( a \), and denote this as \( a \prec_w b \) if
\[ \sum_{j=1}^{m} a_{[j]} \leq \sum_{j=1}^{m} b_{[j]}, \quad m = 1, \ldots, n. \]

Moreover, \( a \) is said to be strongly majorized by \( b \) if, in addition, for \( m = n \), one has
\[ \sum_{j=1}^{n} a_{[j]} = \sum_{j=1}^{n} b_{[j]} \]

The following fact was established in [13].

**Theorem 4.2 (Ky Fan).** Let \( M \) be a complex \( n \times n \) matrix, and \( \hat{M} = \frac{1}{2}(M + M^*) \). Then
\[ (\text{Re } \lambda_1(M), \ldots, \text{Re } \lambda_n(M)) \prec_w (\lambda_1(\hat{M}), \ldots, \lambda_n(\hat{M})), \]
where \( \lambda_j(M) \) and \( \lambda_j(\text{Re } M) \), \( j = 1, \ldots, n \), are eigenvalues of the matrices \( M \) and \( \hat{M} \), respectively.

Now if \( p(z) = \det(zI - C) = \prod_{j=1}^{n} (z - \lambda_j) \) for some circulant matrix, then numbers \( \text{Re } \lambda_j, \ j = 1, \ldots, n \), are the eigenvalues of the matrix \( H = \frac{1}{2}(C + C^*) \) which is a circulant as well. By Theorem 2.1, the eigenvalues \( \xi_k, \ k = 1, \ldots, n-1 \), of the matrix \( H_{n-1} = \frac{1}{2}(C_{n-1} + C_{n-1}^*) \) are the critical points of the polynomial \( \prod_{j=1}^{n} (z - \text{Re } \lambda_j) \), and by Ky Fan’s theorem one has
\[ (\text{Re } w_1, \ldots, \text{Re } w_{n-1}) \prec_w (\xi_1, \ldots, \xi_{n-1}). \]

The inequalities (4.1) were conjectured by Katsoprinakis [15] and proved by Pereira in [18]. In fact, this proof is similar to the original proof of Pereira. The only difference is that we have an explicit differentiator \( C_{n-1} \) which allows us to clarify some ideas.

The circulant approach also allows us to establish the following theorem mentioned in Introduction.

**Theorem 1.4.** Let \( p \) be a polynomial of degree \( n \geq 2 \) with zeroes \( \lambda_j, \ j = 1, \ldots, n \), and critical points \( w_k, \ k = 1, \ldots, n-1 \). If \( q \) be a polynomial with zeroes \( |\lambda_j|^2, \ j = 1, \ldots, n \), and critical points \( \eta_k, \ k = 1, \ldots, n-1 \), then
\[ (\Phi([w_1]), \ldots, \Phi([w_{n-1}])) \prec_w (\Phi(\sqrt{n_1}), \ldots, \Phi(\sqrt{n_{n-1}})), \]
for any increasing function \( \Phi : [0, +\infty) \to \mathbb{R} \) such that \( \Phi \circ \exp \) is convex on \( \mathbb{R} \).
Proof. For certainty, let us fix the arrangement of the numbers $\lambda_j$, $j = 1, \ldots, n$, say, as in (2.3), and construct the unique circulant $C$ with the spectrum $\{\lambda_1, \ldots, \lambda_n\}$. By Proposition 2.5, the unique circulant matrix with eigenvalues $|\lambda_j|^2$, $j = 1, \ldots, n$, is $A = CC^*$. By Theorem 2.1, the spectrum of the matrix $A_{n-1}$ is $\{\xi_1, \ldots, \xi_{n-1}\}$, while the spectrum of $C_{n-1}$ is $\{w_1, \ldots, w_{n-1}\}$. By construction, we have $A_{n-1} = C_{n-1}C_n^*$. By Theorem 2.1, the spectrum of the matrix $A_{n-1}$ is $\{\eta_1, \ldots, \eta_{n-1}\}$, while the spectrum of $C_{n-1}$ is $\{w_1, \ldots, w_{n-1}\}$. By construction, we have $A_{n-1} = C_{n-1}C_n^*$, so the numbers $\sqrt{\eta_k}$, $k = 1, \ldots, n-1$, are the singular values (see e.g. [10, p. 135]) of matrix $C_{n-1}$. If now we arrange the numbers $w_k$ and $\eta_k$, $k = 1, \ldots, n-1$, as follows

$$|w_1| \geq |w_2| \geq \cdots \geq |w_{n-1}|$$

and

$$|\eta_1| \geq |\eta_2| \geq \cdots \geq |\eta_{n-1}|,$$

that is always possible, then [10, Theorem 3.3.13] implies (4.2).

But as we noticed in Introduction, this theorem is weaker than Theorem 1.6 that was established by Cheung and Ng [6] for $p(0) = 0$, and later by Pereira [19] for arbitrary polynomials. It remains unknown whether it is possible to prove Theorem 1.6 by the circulant approach.

5 Questions and open problems

In this Section, we enumerate some problems related to circulants and root location of polynomials.

1) If $p(z)$ is a characteristic polynomial of a circulant matrix $C$, then by Schur’s theorem, for the critical points $w_k$ of $p(z)$, one has

$$\sum_{k=1}^{n-1} |w_k|^{2m} \leq Tr(C_{n-1}^m(C_n^*)^m).$$

However, even for $m = 3$ it is rather a technically difficult problem to express $Tr(C_{n-1}^m(C_n^*)^m)$ via roots of the polynomial $p(z)$. We are sure that, anyway, the case $m = 3$ can be covered at least for the situation when the sum of roots of $p(z)$ equals zero ($c_0 = 0$). But we postpone solution of this problem, since so far, we do not see that anything new can be found here from methodological point of view. Nevertheless, estimate for $\sum_{k=1}^{n-1} |w_k|^6$ (stronger than the one provided by Schmeisser’s theorem) would be quite interesting.

In view of the theory presented in [1, 2, 18, 17, 6, 7] and in the present paper, it is interesting to know whether it is possible to estimate

$$\sum_{k=1}^{n-1} |w_k|^{2m+1},$$

for $m = 0, 1, 2, \ldots$ In this case, Proposition 2.5 may be of use.

2) The relations between the eigenvalues of a circulant $C$ and its entries $c_k$ are well known. It would be interesting to find relations between $c_k$ and the critical points of the polynomial $\det(zI - C)$. The eigenvectors of the matrix $C_{n-1}$ are of particular interest.

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