ISOSPECTRAL COMMUTING VARIETY AND THE HARISH-CHANDRA $\mathcal{D}$-MODULE

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ABSTRACT. Let $\mathfrak{g}$ be a complex reductive Lie algebra with Cartan algebra $\mathfrak{t}$. Hotta and Kashiwara defined a holonomic $\mathcal{D}$-module $\mathcal{M}$, on $\mathfrak{g} \times \mathfrak{t}$, called Harish-Chandra module. We give an explicit description of $gr \mathcal{M}$, the associated graded module with respect to a canonical Hodge filtration on $\mathcal{M}$. The description involves the isospectral commuting variety, a subvariety $\mathfrak{X} \subset \mathfrak{g} \times \mathfrak{t} \times \mathfrak{t}$ which is a finite extension of the variety of pairs of commuting elements of $\mathfrak{g}$. Our main result establishes an isomorphism of $gr \mathcal{M}$ with the structure sheaf of the normalization of $\mathfrak{X}$. It follows, thanks to the theory of polarized Hodge modules, that the normalization of the isospectral commuting variety is Cohen-Macaulay and Gorenstein, confirming a conjecture of M. Haiman.

In the special case where $\mathfrak{g} = \mathfrak{gl}_n$, there is an open subset of the isospectral commuting variety that is closely related to the Hilbert scheme of $n$ points in $\mathbb{C}^2$. The sheaf $gr \mathcal{M}$ gives rise to a locally free sheaf on the Hilbert scheme. We show that the corresponding vector bundle is isomorphic to the Procesi bundle that plays an important role in the work of M. Haiman.

1. THE ISOSPECTRAL COMMUTING VARIETY

1.1. Reminder on commuting schemes. Let $G$ be a connected complex reductive group with Lie algebra $\mathfrak{g}$. We fix $T \subset G$, a maximal torus, and let $\mathfrak{t} = \text{Lie } T$ be the corresponding Cartan subalgebra of $\mathfrak{g}$. The group $G$ acts on $\mathfrak{g}$ via the adjoint action $G \ni g \mapsto \text{Ad } g(x)$.

We put $\mathfrak{G} := \mathfrak{g} \times \mathfrak{g}$ and let $G$ act diagonally on $\mathfrak{G}$. The commuting scheme $\mathfrak{C}$, of the Lie algebra $\mathfrak{g}$, is defined as the scheme-theoretic zero fiber of the commutator map $\kappa : \mathfrak{G} \rightarrow \mathfrak{g}$, $(x, y) \mapsto [x, y]$. Thus, $\mathfrak{C}$ is a closed $G$-stable subscheme of $\mathfrak{G}$ and, set-theoretically, one has $\mathfrak{C} = \{ (x, y) \in \mathfrak{G} \mid [x, y] = 0 \}$.

Given $x \in \mathfrak{g}$, we let $G_x = \{ g \in G \mid \text{Ad } g(x) = x \}$ and write $\mathfrak{g}_x = \text{Lie } G_x$ for the centralizer of $x$ in $\mathfrak{g}$. The isotropy group of a pair $(x, y) \in \mathfrak{G}$ under the $G$-diagonal action equals $G_{x,y} := G_x \cap G_y$. Let $\mathfrak{g}_{x,y} := \text{Lie } G_{x,y} = \mathfrak{g}_x \cap \mathfrak{g}_y$ denote the corresponding Lie algebra. We call an element $x \in \mathfrak{g}$, resp. $(x, y) \in \mathfrak{C}$, regular if we have $\dim \mathfrak{g}_x = \dim \mathfrak{t}$, resp. $\dim \mathfrak{g}_{x,y} = \dim \mathfrak{t}$. Write $\mathfrak{C}^r$, resp. $\mathfrak{C}^{rs}$, for the set of regular elements of $\mathfrak{g}$, resp. of $\mathfrak{C}$. Further, let $\mathfrak{C}^{rs} \subset \mathfrak{C}^r$ be the set of pairs $(x, y) \in \mathfrak{C}$ such that both $x$ and $y$ are regular semisimple elements.

Basic properties of the commuting scheme may be summarized as follows.

**Proposition 1.1.1.** (i) The set $\mathfrak{C}^{rs}$ is Zariski open and dense in $\mathfrak{C}$.

(ii) The smooth locus of the scheme $\mathfrak{C}$ equals $\mathfrak{C}^r$.

Here, part (i) is due to Richardson [Rii]. To prove (ii), one views the commutator map as a moment map $\kappa : T^* \mathfrak{g} \rightarrow \mathfrak{g}^*$. The result then follows easily from a general property of moment maps saying that $\ker (d\kappa)$, the kernel of the differential of a moment map $\kappa$, equals the annihilator of the image of $(d\kappa)^\top$, the dual map, cf. also [LO] Lemma 2.3.

Proposition 1.1.1 implies that $\mathfrak{C}$ is a generically reduced and irreducible scheme. It is a long standing open problem whether or not this scheme is reduced.
Given a scheme $X$, write $\mathcal{O}_X$ for the structure sheaf and put $\mathbb{C}[X] = \Gamma(X, \mathcal{O}_X)$. Let $X_{\text{red}}$ denote the reduced scheme corresponding to $X$. Given an algebraic group $K$ acting on $X$, let $\mathbb{C}[X]^K \subset \mathbb{C}[X]$ denote the subalgebra of $K$-invariants, and put $X//K := \text{Spec} \mathbb{C}[X]^K$.

Let $N(T)$ be the normalizer of $T$ in $G$ and $W = N(T)/T$ be the Weyl group. We put $\mathfrak{T} := t \times t$ and let $W$ act diagonally on $\mathfrak{T}$. Clearly, $\mathfrak{T} \subset \mathfrak{c}$. So, restriction of polynomial functions gives algebra maps

$$\text{res} : \mathbb{C}[\mathfrak{g}]^G \to \mathbb{C}[\mathfrak{c}]^G \to \mathbb{C}[\mathfrak{T}]^W.$$

(1.1.2)

The group $W$ acts freely on $\mathfrak{T}' := \mathfrak{c}' \cap \mathfrak{T}$, a Zariski open dense subset of $\mathfrak{T}$. Further, the assignment $(g, x, y) \mapsto (\text{Ad }g(x), \text{Ad }g(y))$ induces an isomorphism $G \times N(T) \mathfrak{T}' \to \mathfrak{c}'$. It follows, since the set $\mathfrak{c}'$ is dense in $\mathfrak{c}$ and the second map in (1.1.2) is surjective by a theorem of Joseph [10], that this map induces an isomorphism $\mathfrak{T}/W \cong [\mathfrak{c}'//G]_{\text{red}}$. That isomorphism is analogous to the isomorphism $t//G \cong g//G$ induced by the Chevalley isomorphism $\mathbb{C}[g]^G \cong \mathbb{C}[t]^W$. Thus, we have a diagram

$$\mathfrak{T} \to \mathfrak{T}/W \to [\mathfrak{c}'//G]_{\text{red}} \to \mathfrak{c}'//G \to \mathfrak{c}.$$  

(1.1.3)

In the special case where $g = g_{\mathfrak{s}l_2}$, the scheme $\mathfrak{c}'//G$ is known to be reduced, by [GG, Theorem 1.3]. It is expected to be reduced for any reductive Lie algebra $g$.

1.2. There is a natural $G \times W$-action on $g \times t$, resp. on $\mathfrak{g} \times \mathfrak{T}$, induced by the $G$-action on the first and $W$-action on the second factor. There is also a $\mathbb{C}^*$-action on $g \times t$ by dilations, and an induced $\mathbb{C}^* \times \mathbb{C}^*$-action on $\mathfrak{g} \times \mathfrak{T} = (g \times t) \times (g \times t)$.

The fiber product $\mathfrak{r} := g \times_{\mathbb{C}^*//G} t$, resp. $\mathfrak{r} \times_{\mathfrak{c}'//G} \mathfrak{T}$, is a closed $G \times W \times \mathbb{C}^*$-stable subscheme in $g \times t$, resp. $G \times W \times \mathbb{C}^* \times \mathbb{C}^*$-stable subscheme in $\mathfrak{g} \times \mathfrak{T}$. The first projection $\mathfrak{r} \to g$, resp. $\mathfrak{r} \times_{\mathfrak{c}'//G} \mathfrak{T} \to \mathfrak{c}'$ is a $G$-equivariant finite morphism. The group $W$ acts along the fibers of this morphism.

Lemma 1.2.1. (i) The set $\mathfrak{r}' := g' \times_{\mathfrak{c}'//G} t'$, resp. $\mathfrak{r}' := g' \times_{\mathfrak{c}'//G} \mathfrak{T}'$, is a smooth, irreducible, Zariski open and dense subset of $\mathfrak{r}$, resp. of $\mathfrak{r} \times_{\mathfrak{c}'//G} \mathfrak{T}$.

(ii) The first projection $\mathfrak{r}' \to g'$, resp. $\mathfrak{r}' \to \mathfrak{c}'$, is a Galois covering with Galois group $W$.

For the proof that $\mathfrak{r}'$ is irreducible and Zariski dense in $\mathfrak{c} \times_{\mathfrak{c}'//G} \mathfrak{T}$ see Corollary 6.2.3 of [6] below. All other statements of the above lemma are clear.

The scheme $\mathfrak{r}$ is known to be a reduced normal complete intersection in $g \times t$, cf. eg [BB]. On the contrary, the scheme $\mathfrak{r} \times_{\mathfrak{c}'//G} \mathfrak{T}$ is not reduced already in the case $g = g_{\mathfrak{s}l_2}$.

Definition 1.2.2. The isospectral commuting variety is defined as $\mathfrak{x} := [\mathfrak{c} \times_{\mathfrak{c}'//G} \mathfrak{T}]_{\text{red}}$, a reduced fiber product. Let $p_\mathfrak{x} : \mathfrak{x} \to \mathfrak{c}$, resp. $p_\mathfrak{T} : \mathfrak{x} \to \mathfrak{T}$, denote the first, resp. second, projection.

Remark 1.2.3. The isospectral commuting variety has been considered by M. Haiman, in [Ha2] §8, [Ha3] 7.2).

Lemma 1.2.1 shows that $\mathfrak{x}$ is an irreducible variety and that we may (and will) identify the set $\mathfrak{r}'$ with a Zariski open subset of $\mathfrak{x}$.

Let $q : T^*X \to X$ denote the cotangent bundle on a smooth variety $X$. The group $\mathbb{C}^*$ acts along the fibers of $q$ by dilations. An irreducible reduced subvariety $\Lambda \subset T^*X$ is said to be a Lagrangian cone if it is $\mathbb{C}^*$-stable and the tangent space to $\Lambda$ at any smooth point of $\Lambda$ is a Lagrangian vector subspace with respect to the canonical symplectic 2-form on $T^*X$.

We use an invariant bilinear form on $g$ to identify $g^*$ with $g$, resp. $t^*$ with $t$. This gives an identification $T^*(g \times t) = \mathfrak{g} \times \mathfrak{T}$. The $\mathbb{C}^*$-action along the fibers of $q$ corresponds to the action of the subgroup $\{1\} \times \mathbb{C}^* \subset \mathbb{C}^* \times \mathbb{C}^*$. 

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Let $\mathfrak{r}' := \{(x, t) \in \mathfrak{r} \mid x \in \mathfrak{g}^r\}$. This is a Zariski open and dense subset of $\mathfrak{r}$ which is contained in the smooth locus of $\mathfrak{r}$ (since the differential of the adjoint quotient map $\mathfrak{g} \rightarrow \mathfrak{g}/G$ has maximal rank at any point of $\mathfrak{r}'$). Let $N_{\mathfrak{r}'} \subset T^*(\mathfrak{g} \times \mathfrak{t})$ be the total space of the conormal bundle on $\mathfrak{r}'$. Thus, $\overline{N_{\mathfrak{r}'}}$, the closure of $N_{\mathfrak{r}'}$, is a Lagrangian cone in $T^*(\mathfrak{g} \times \mathfrak{t})$.

**Lemma 1.2.4.** In $T^*(\mathfrak{g} \times \mathfrak{t}) = \mathfrak{g} \times \mathfrak{T}$, we have $\overline{N_{\mathfrak{r}'}} = \mathfrak{x}$; in particular, $\mathfrak{x}$ is a Lagrangian cone.

**Proof.** We know that $\mathfrak{x}$ is a reduced and irreducible, $C^\infty$-stable subvariety. By Lemma 1.2.1, one has $\dim \mathfrak{y} = \dim \mathfrak{x}^N = \dim \mathfrak{g}^N = \dim (G \times \mathfrak{T}) = \dim \mathfrak{g} + \dim \mathfrak{t}$. Hence, $\dim \mathfrak{y} = 1/3 \dim T^*(\mathfrak{g} \times \mathfrak{t})$. We leave to the reader to check that tangent spaces to $\mathfrak{x}^N$ are isotropic subspaces with respect to the symplectic form on $T^*(\mathfrak{g} \times \mathfrak{t})$. It follows that $\mathfrak{x}$ is a Lagrangian cone. Further, set theoretically, one has $\bar{\mathfrak{g}}(\mathfrak{x}) = \mathfrak{g} \times G/\mathfrak{t} \mathfrak{r} = \mathfrak{r}$.

The above properties force $\mathfrak{x}$ to be equal to the closure of the total space of the conormal bundle on the smooth locus of $\mathfrak{r}$, cf. eg. [CG], Lemma 1.3.27.

### 1.3. An analogue of the Grothendieck-Springer resolution.

We recall a few definitions.

Let $\mathcal{B}$ be the flag variety, the variety of all Borel subalgebras $\mathfrak{b} \subset \mathfrak{g}$. For any pair $\mathfrak{b}, \mathfrak{b}'$ of Borel subalgebras, there is a canonical isomorphism $\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] \cong \mathfrak{b}'/[\mathfrak{b}', \mathfrak{b}']$, cf. eg. [CG], Lemma 3.1.26. This quotient is referred to as the ‘abstract Cartan algebra’.

Given a Borel subgroup $T \subset B \subset G$, we may identify $\mathcal{B} \cong G/B$. Write $\mathfrak{b} = \text{Lie } B$, resp. $\mathfrak{t} = \text{Lie } T$. The composite map $t \leftarrow \mathfrak{b} \rightarrow \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$ induces an isomorphism $t \rightarrow \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$. So, we will abuse the notation and write $\mathfrak{t}$ for the abstract Cartan algebra as well.

Motivated by Grothendieck and Springer, we introduce the following varieties

$$\bar{\mathfrak{g}} := \{(b, x) \in \mathcal{B} \times \mathfrak{g} \mid x \in \mathfrak{b}\}, \quad \text{resp. } \bar{\mathfrak{g}} := \{(b, x, y) \in \mathcal{B} \times \mathfrak{g} \times \mathfrak{g} \mid x, y \in \mathfrak{b}\}.$$

The first projection makes $\bar{\mathfrak{g}}$, resp. $\bar{\mathfrak{g}}$, a sub vector bundle of the trivial vector bundle $\mathcal{B} \times \mathfrak{g} \rightarrow \mathcal{B}$, resp. $\mathcal{B} \times \mathfrak{g} \times \mathfrak{g} \rightarrow \mathcal{B}$. One has a $G$-equivariant vector bundle isomorphism $\bar{\mathfrak{g}} \cong G \times _B \mathfrak{b}$, resp. $\bar{\mathfrak{g}} \cong G \times _B (\mathfrak{b} \times \mathfrak{b})$. Thus, $\bar{\mathfrak{g}}$ and $\bar{\mathfrak{g}}$ are smooth connected varieties. Further, the assignment $(b, x) \mapsto x \mod [\mathfrak{b}, \mathfrak{b}] \in \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$, resp. $(b, x, y) \mapsto (x \mod [\mathfrak{b}, \mathfrak{b}], y \mod [\mathfrak{b}, \mathfrak{b}])$, gives a well defined smooth morphism $\nu : \bar{\mathfrak{g}} \rightarrow \mathfrak{t}$, resp. $\nu : \bar{\mathfrak{g}} \rightarrow \mathfrak{t}$. There is also a projective morphism $\mu : \bar{\mathfrak{g}} \rightarrow \mathfrak{g}$, $(b, x) \mapsto (x, y)$. The map $\mu$ is known as the Grothendieck-Springer resolution.

**Notation 1.3.1.** Let $\mathcal{T}_X$ denote the tangent sheaf, resp. $K_X := \wedge^{\dim X} \mathcal{T}_X^*$ denote the canonical sheaf (or line bundle), on a smooth variety $X$.

**Proposition 1.3.2.**

(i) The image of the map $\mu \times \nu : \bar{\mathfrak{g}} \rightarrow \mathfrak{g} \times \mathfrak{t}$ is contained in $\mathfrak{r} = \mathfrak{g} \times _{\mathfrak{g}/G} \mathfrak{t}$. The resulting morphism $\pi : \bar{\mathfrak{g}} \rightarrow \mathfrak{r}$ is a resolution of singularities such that $\pi : \pi^{-1}(\mathfrak{r}) \rightarrow \mathfrak{r}'$ is an isomorphism.

(ii) The image of the map $\mu \times \nu : \bar{\mathfrak{g}} \rightarrow \mathfrak{g} \times \mathfrak{t}$ is contained in $\mathfrak{g} \times _{\mathfrak{g}/G} \mathfrak{T}$. The resulting map gives a resolution of singularities $\bar{\mathfrak{g}} \\rightarrow \mathfrak{g} \times _{\mathfrak{g}/G} \mathfrak{T}_{\mathfrak{G}}^{\text{red}}$.

(iii) The canonical bundle on $\bar{\mathfrak{g}}$ has a natural trivialization by a nowhere vanishing section $\omega \in K_{\bar{\mathfrak{g}}}$ such that one has $\mu^*(dx) = (\prod_{\alpha \geq 0} (\alpha, \nu)) \cdot \omega$, where the product in the RHS is taken over all positive roots $\alpha \in \mathfrak{t}^*$ and where $dx$ is a constant volume form on $\mathfrak{g}$.

(iv) We have $K_{\bar{\mathfrak{g}}} = \wedge^{\dim \mathfrak{g}} q^* \mathcal{T}_{\mathfrak{g}}$, where $q : \bar{\mathfrak{g}} \rightarrow \mathfrak{g}$ stands for the first projection.

Part (i) of this proposition is well-known, cf. [BB], [CG]. Part (ii) is an immediate consequence of Lemma 6.1.1 of [CG] below. This lemma implies, in particular, that the image of the map $\mu$ equals the set of pairs $(x, y) \in \mathfrak{g}$ such that $x$ and $y$ generate a solvable Lie
subalgebra of \( g \). The statement in (ii) is a variation of results concerning the null-fiber of the adjoint quotient map \( \mathfrak{g} \to \mathfrak{g}//G \), see [RI2], [KW].

The descriptions of canonical bundles in Proposition 1.3.2(iii)-(iv) are straightforward. Finally, the equation of part (iii) that involves the section \( \omega \) appears eg. in [HK1], formula (4.1.4). This equation is, in essence, nothing but Weyl’s classical integration formula. \( \square \)

Let \( \mathcal{N} \) be the variety of nilpotent elements of \( g \) and put \( \widetilde{\mathcal{N}} := \{ (b, x) \in \mathcal{B} \times g \mid x \in [b, b] \} \).

One may restrict the commutator map \( \kappa \) to each Borel subalgebra \( b \subset g \) to obtain a morphism \( \tilde{\kappa} : \mathfrak{g} \to \widetilde{\mathcal{N}} \). \( (b, x, y) \mapsto (b, [x, y]) \). One has a commutative diagram, where the vertical map in the middle is known as the Springer resolution,

\[
\begin{aligned}
\mathcal{B} = \mu^{-1}(0) & \xrightarrow{\kappa = b \mapsto (b, 0)} \widetilde{\mathcal{N}} \xrightarrow{\tilde{\kappa}} \mathfrak{g} \\
\{0\} & \xleftarrow{\kappa = 0} \mathcal{N} \xleftarrow{\kappa} g \xrightarrow{\kappa} \mathfrak{g}
\end{aligned}
\]

(1.3.3)

We introduce the following closed subset in \( \tilde{\mathfrak{g}} \):

\[ \tilde{\mathfrak{X}} := \{(b, x, y) \in \mathcal{B} \times g \times g \mid x, y \in b, \ [x, y] = 0 \} = \tilde{\kappa}^{-1}(\varepsilon(\mathcal{B})). \]

(1.3.4)

Diagram (1.3.3) shows that the morphism \( \mu \) maps \( \tilde{\mathfrak{X}} \) to \( \mathfrak{c} \), resp. \( \mu \times \nu \) maps \( \tilde{\mathfrak{X}} \) to \( \mathfrak{c} \times \varepsilon//G \mathfrak{X} \).

Remark 1.3.5. Let \( \tilde{\mathfrak{X}}^{rs} := \tilde{\mathfrak{X}} \cap \nu^{-1}(\mathfrak{X}^T) \). This is an open subset of \( \tilde{\mathfrak{X}} \). One can show that \( \tilde{\mathfrak{X}}^{rs} \) is a smooth local complete intersection in \( \tilde{\mathfrak{g}} \) and the map \( \mu \times \nu \) induces an isomorphism \( \tilde{\mathfrak{X}}^{rs} \sim \mathfrak{X}^{rs} \). It is likely that the scheme \( \tilde{\mathfrak{X}} \) is not irreducible so that \( \tilde{\mathfrak{X}}^{rs} \) is not dense in \( \tilde{\mathfrak{X}} \).

1.4. A DG algebra. Let \( T := T_\mathcal{B} \) and write \( q : \widetilde{\mathcal{N}} \to \mathcal{B} \), resp. \( q : \tilde{\mathfrak{g}} \to \mathfrak{g} \), for the first projection. The cotangent space at a point \( b \in \mathcal{B} \) may be identified with the vector space \((g/b)^* \cong [b, b]\). This yields a natural isomorphism \( \widetilde{\mathcal{N}} \cong T^*\mathcal{B} \), of \( G \)-equivariant vector bundles on \( \mathcal{B} \), cf. [CC] ch. 3. For each \( n \geq 0 \), we put \( \mathfrak{A}^n = \wedge^n q^*T \), a coherent sheaf on \( \tilde{\mathfrak{g}} \). The sheaf \( q_*\mathfrak{A}^n \) is a quasi-coherent sheaf on \( \mathcal{B} \) that may be identified with the sheaf of sections of a \( G \)-equivariant vector bundle on \( \mathcal{B} \) with fiber \( \text{Sym}(b^* \otimes b^*) \otimes \wedge^n [b, b]^* \).

The sheaf \( q^*T^* \), on \( T^*\mathcal{B} \), has a canonical section \( s \). Using the isomorphisms \( T^*\mathcal{B} = \widetilde{\mathcal{N}} \) and \( \nu^*(q^*T^*) = q^*T^* \), we may view \( \nu^*s \) as a section of the sheaf \( q^*T^* \). Contraction with \( \nu^*s \) gives a morphism \( i_{\kappa, \nu} : \nu^*q^*T^* \to \nu^{-1}q^*T^* \). This makes \( \mathfrak{A}^s \) := \( (\nu^*q^*T^*, i_{\kappa, \nu}) \) a DG algebra, with differential \( i_{\kappa, \nu} \) of degree \((-1)\).

Now, view the commutator map as an element \( \kappa \in b^* \otimes b^* \otimes [b, b] = \text{Hom}(b \otimes b, [b, b]) \). The differential on the DG algebra \( \mathfrak{A}^s \), induced by \( i_{\kappa, \nu} \), acts by fiberwise contraction with the element \( \kappa \) as follows (for any \( k, m, n \geq 0 \)):

\[ i_{\kappa} : \text{Sym}^{k} b^* \otimes \text{Sym}^{m} b^* \otimes (\wedge^n [b, b]^*) \to \text{Sym}^{k+1} b^* \otimes \text{Sym}^{m+1} b^* \otimes (\wedge^{n-1} [b, b]^*). \]

(1.4.1)

The DG algebra \( \mathfrak{A}^s \) has an important self-duality property. Observe first that, in degree \( d := \dim \mathcal{B} \), the top nonzero degree of the algebra \( \mathfrak{A}^s \), we have \( \mathfrak{A}^d = K_{\tilde{\mathfrak{g}}} \), by Proposition 1.3.2(iv). Moreover, multiplication in \( \mathfrak{A}^s \) induces a canonical isomorphism of complexes

\[ \mathfrak{A}^{d-\text{c}} \sim \mathfrak{H}om_{\mathfrak{g}}(\mathfrak{A}^s, \mathfrak{A}^d) = \mathfrak{H}om_{\mathfrak{g}}(\mathfrak{A}^s, K_{\tilde{\mathfrak{g}}}). \]

(1.4.2)

Notation 1.4.3. Let \( D^b_{\text{coh}}(X) \) denote the bounded derived category of the category \( \text{Coh} X \), of coherent sheaves on a scheme \( X \). Given \( F \in D^b_{\text{coh}}(X) \), let \( \mathfrak{H}^j(F) \in \text{Coh} X \) denote the \( j \)th cohomology sheaf, resp. \( \mathbb{H}^j(X, F) \) denote the \( j \)th hyper-cohomology group, of \( F \).
Associated with diagram (1.3.3), there are natural derived functors

\[ D^b_{\text{coh}}(\mathcal{B}) \xrightarrow{\iota_*} D^b_{\text{coh}}(\mathcal{N}) \xrightarrow{L\Gamma^*} D^b_{\text{coh}}(\mathcal{G}) \xrightarrow{R(\mu \times \nu)_*} D^b_{\text{coh}}(\mathcal{G} \times \mathcal{G}). \]  

(1.4.4)

Contraction with the canonical section \(s\) yields a Koszul complex \(\cdots \rightarrow \wedge^3 q^*T \rightarrow \wedge^2 q^*T \rightarrow q^*T \rightarrow \iota_* \mathcal{O}_\mathcal{B} \rightarrow 0.\) This is the standard locally free resolution of the sheaf \(\iota_* \mathcal{O}_\mathcal{B}\), on \(\mathcal{N}\). Therefore, the DG algebra \(\mathcal{A}\) \(\cong \mathcal{R}^*(\wedge q^*T, \iota_* \mathcal{O}_\mathcal{B})\) provides a DG algebra model for the object \(L\Gamma^*(\iota_* \mathcal{O}_\mathcal{B}) \in D^b_{\text{coh}}(\mathcal{G}).\) The corresponding DG scheme \(\text{Spec} \, \mathcal{A}\) may be thought of as a ‘derived analogue’ of the scheme \(\tilde{X}\), in the sense of derived algebraic geometry. The coordinate ring of the DG scheme \(\text{Spec} \, \mathcal{A}\) is \(\text{RT}(\mathcal{G}, \mathcal{A})\), a DG algebra well defined up to quasi-isomorphism, cf. [H]. Hence, \(\mathcal{H}^*(\mathcal{G}, \mathcal{A})\), the hyper-cohomology of that DG algebra, acquires the canonical structure of a graded commutative algebra. The following result says that the DG scheme \(\text{Spec} \, \mathcal{A}\) is, in a sense, a ‘resolution’ of the isospectral commuting variety.

**Theorem 1.4.5.** We have \(\mathcal{H}^k(\mathcal{G}, \mathcal{A}) = 0\) for all \(k \neq 0\), and there is a canonical \(G \times \mathbb{C}^* \times \mathbb{C}^*\)-equivariant algebra isomorphism \(\mathcal{H}^0(\mathcal{G}, \mathcal{A}) \cong \mathbb{C}[X_{\text{norm}}]\), where \(X_{\text{norm}}\) is the normalization of the isospectral commuting variety.

The statement of Theorem 1.4.5 (in an equivalent form presented in Theorem 1.6.2 below) was suggested to me by Dmitry Arinkin.

### 1.5. Bigraded character formula.

Given a reductive group \(K\), a \(K\)-scheme \(X\) and a \(K\)-equivariant coherent sheaf \(\mathcal{F}\), on \(X\), we write \(\chi^K(\mathcal{F})\) for its equivariant Euler characteristic, a class in the Grothendieck group of rational \(K\)-modules. We identify the Grothendieck group of rational \(\mathbb{C}^* \times \mathbb{C}^*\)-modules with \(\mathbb{C}[q_1^{\pm 1}, q_2^{\pm 1}]\), a Laurent polynomial ring.

Recall that we have a natural \(G \times W \times \mathbb{C}^* \times \mathbb{C}^*\)-action on the variety \(X\). Since a reductive group action can always be lifted canonically to the normalization, cf. [Kr], §4.4, we obtain a \(G \times W \times \mathbb{C}^* \times \mathbb{C}^*\)-action on \(X_{\text{norm}}\) as well. This makes the coordinate ring \(\mathbb{C}[X_{\text{norm}}]\) a bigraded \(G \times W\)-module.

Let \(R_+ \subset t^*\) denote the set of weights of the adjoint \(t\)-action on the vector space \([b, b]^* \cong g/b\), and let \(\ell(-)\) denote the length function on the Weyl group \(W\).

**Corollary 1.5.1.** The bigraded \(T\)-character of the coordinate ring of the variety \(X_{\text{norm}}\) is given by the formula

\[
\chi_T^{\mathbb{C}^* \times \mathbb{C}^*}(O_{X_{\text{norm}}}) = \sum_{w \in W} (-1)^{\ell(w)} \prod_{\alpha, \beta_1, \beta_2, \gamma \in R_+} \frac{(1 - q_1 q_2 e^{w(\alpha)})}{(1 - q_1 e^{w(\beta_1)})(1 - q_2 e^{w(\beta_2)})(1 - e^{-w(\gamma)})}.
\]

Proof. For any locally finite representation \(E\) of a Borel subgroup \(B \subset G\), let \(E\) denote the corresponding induced \(G\)-equivariant vector bundle on \(\mathcal{B} = G/B\).

We let the group \(\mathbb{C}^*\) act on \(b\) by dilations. This gives a \(\mathbb{C}^* \times \mathbb{C}^*\)-action along the fibers, \(b \times b\), of the projection \(q : \mathcal{G} \rightarrow \mathcal{B}\). That makes \(\mathcal{G} a G \times \mathbb{C}^* \times \mathbb{C}^*\)-variety. For each \(m \geq 0\), the sheaf \(\mathcal{A}^m\) comes equipped with a natural \(G \times \mathbb{C}^* \times \mathbb{C}^*\)-equivariant structure. From the definition of the sheaf \(q, \mathcal{A}^*\), we get an equation, cf. (1.4.1):

\[
\chi^{G \times \mathbb{C}^* \times \mathbb{C}^*}(q, \mathcal{A}^*) = \sum_{k, m, n \geq 0} (-q_1 q_2)^n k_c m \cdot \chi^G(\text{Sym}^k b^* \otimes \text{Sym}^m b^* \otimes (\wedge^n [b, b]^*)).
\]  

(1.5.2)
Thanks to the $G \times \mathbb{C}^x \times \mathbb{C}^x$-equivariant isomorphism of Theorem [1.4.3], the LHS of equation (1.5.2) is equal to the bigraded character of the $G$-module $\mathbb{C}[X_{\text{norm}}]$. To obtain the formula of the corollary, one computes the RHS of equation (1.5.2) using the Atiyah-Bott fixed point formula for $G$-equivariant vector bundles on the flag variety, cf. e.g. [CG] §6.1.16. □

It would be very interesting to describe the structure of $\mathbb{C}[X_{\text{norm}}]$ as a $W$-module in terms of the DG algebra $\mathcal{A}$.

1.6. Main results. Given an irreducible, not necessarily reduced, scheme $X$ let $\psi : X_{\text{norm}} \to X_{\text{red}}$ denote the normalization of the corresponding reduced scheme $X_{\text{red}}$.

One may view the complex $(\mathcal{A}, i_{\text{red}})$ as an object of $D^b_{\text{coh}}(\mathcal{G})$. Thus, taking the (derived) direct image via the morphism $\mu \times \nu$, we get

$$R\Gamma(\mathcal{G}, \mathcal{A}) = R\Gamma(\mathcal{G} \times \mathcal{T}, R(\mu \times \nu)_* \mathcal{A}) = R\Gamma(\mathcal{G} \times \mathcal{T}, R(\mu \times \nu)_* L\mathcal{K}^*(i_{\text{red}}^* \mathcal{O}_\mathcal{G})).$$

(1.6.1)

Using that the scheme $\mathcal{G} \times \mathcal{T}$ is affine, we see from the above isomorphisms that Theorem [1.4.5] is equivalent to the following sheaf theoretic result

**Theorem 1.6.2.** The sheaves $\mathcal{H}^k(R(\mu \times \nu)_* L\mathcal{K}^*(i_{\text{red}}^* \mathcal{O}_\mathcal{G}))$ vanish for all $k \neq 0$ and there is an isomorphism $\mathcal{H}^0(R(\mu \times \nu)_* \mathcal{A}) \cong \psi_* \mathcal{O}_{X_{\text{norm}}}$ of sheaves of $\mathcal{O}_{\mathcal{G} \times \mathcal{T}}$-algebras.

This theorem will be deduced from Theorem [4.4.1] of [4.4] below.

The significance of Theorem 1.6.2 is due to the self-duality isomorphism (1.4.2). It follows, since $\mu \times \nu$ is a proper morphism, that the object $R(\mu \times \nu)_* \mathcal{A} \in D^b_{\text{coh}}(\mathcal{G} \times \mathcal{T})$ is isomorphic to its Grothendieck-Serre dual, up to a shift. Therefore, the vanishing statement in Theorem 1.6.2 implies, in particular, that the sheaf $\mathcal{H}^0(R(\mu \times \nu)_* \mathcal{A})$ is Cohen-Macaulay.

Thus, Theorem 1.6.2 yields the following result that confirms a conjecture of M. Haiman, [Ha3, Conjecture 7.2.3].

**Theorem 1.6.3.** The sheaf $\mathcal{O}_{X_{\text{norm}}}$ is Cohen-Macaulay and $X_{\text{norm}}$ is a Gorenstein variety with trivial canonical bundle. □

The composite map $p_{G} \circ \psi : X_{\text{norm}} \to X \to \mathcal{C}$ factors through a finite $G$-equivariant morphism $p : X_{\text{norm}} \to \mathcal{C}_{\text{norm}}$. We put $\mathcal{R} := p_* \mathcal{O}_{X_{\text{norm}}}$, a $G$-equivariant coherent sheaf of $\mathcal{O}_{\mathcal{C}_{\text{norm}}}$-algebras. The group $W$ acts along the fibers of $p$ and this gives a $W$-action on $\mathcal{R}$ by $\mathcal{O}_{\mathcal{C}_{\text{norm}}}$-algebra automorphisms. Write $\mathcal{R}^W$ for the subsheaf of $W$-invariant sections.

From Theorem 1.6.3 we deduce

**Proposition 1.6.4.** (i) The canonical morphism $\mathcal{O}_{\mathcal{C}_{\text{norm}}} \to \mathcal{R}^W$ is an isomorphism, equivalently, the map $p$ induces an isomorphism $X_{\text{norm}}/W \cong \mathcal{C}_{\text{norm}}$.

(ii) The sheaves $\mathcal{R}$ and $\mathcal{R}^W$ are Cohen-Macaulay; thus, $\mathcal{C}_{\text{norm}}$ is a Cohen-Macaulay variety.

(iii) The restriction of the sheaf $\mathcal{R}$ to $\mathcal{C}$, the smooth locus of $\mathcal{C}$, is a locally free sheaf. Each fiber of the corresponding algebraic vector bundle affords the regular representation of the group $W$.

**Proof.** The scheme $X_{\text{norm}}/W$ is reduced and integrally closed, as a quotient of an integrally closed reduced scheme by a finite group action, [K], §3.3. Further, by Lemma [1.2.1](ii), the map $p$ is generically a Galois covering with $W$ being the Galois group. It follows that the induced map $X/W \to \mathcal{C}_{\text{norm}}$ is finite and birational. Hence it is an isomorphism and (i) is proved.

Part (ii) follows from Theorem 1.6.3 since the property of a coherent sheaf be Cohen-Macaulay is stable under taking direct images by finite morphisms and also under taking direct summands. Finally, any coherent Cohen-Macaulay sheaf on a smooth variety is locally free, hence $\mathcal{R}|_{\mathcal{C}}$ is a locally free sheaf. Lemma [1.2.1](ii) implies that the fiber of the
corresponding vector bundle over any point of $C^r$ affords the regular representation of the group $W$. By continuity, the same holds for the fiber at any point of $C$, since $C^r$ is dense in $C$.

2. The Harish-Chandra $\mathcal{D}$-module

2.1. Algebraic definition of the Harish-Chandra module. Let $\mathcal{D}_X$ denote the sheaf of algebraic differential operators on a smooth algebraic variety $X$, and write $\mathcal{D}(X) := \Gamma(X, \mathcal{D}_X)$ for the algebra of global sections. Given an action on $X$ of an algebraic group $K$, we let $\mathcal{D}(X)^K \subset \mathcal{D}(X)$ denote the subalgebra of $K$-invariant differential operators.

We have $\mathcal{D}(g)$ and $\mathcal{D}(t)$, the algebras of polynomial differential operators on the vector spaces $g$ and $t$, respectively. The corresponding subalgebras of differential operators with constant coefficients may be identified with $\text{Sym}^a g$ (linear) vector field on $g$ an algebra homomorphism such that its restriction to $G$ is a linear map $a : g \to g, x \mapsto [a, x]$ as a (linear) vector field on $g$, that is, as a first order differential operator on $g$. The assignment $a \mapsto \text{ad}a$ gives a linear map $\text{ad} : g \to \mathcal{D}(g)$, with image $\text{ad} g$.

Harish-Chandra defined a ‘radial part’ map

$$\text{rad} : \mathcal{D}(g)^G \to \mathcal{D}(t)^W,$$

(2.1.1)

an algebra homomorphism such that its restriction to $G$-invariant polynomials, resp. to $G$-invariant constant coefficient differential operators, reduces to the Chevalley isomorphism $\mathbb{C}[g]^G \to \mathbb{C}[[t]]^W$, resp. $(\text{Sym} g)^G \to (\text{Sym} t)^W$.

Remark 2.1.2. According to the results of Wallach [Wa] and Levasseur-Stafford [LS1],[LS2], the radial part map (2.1.1) induces an algebra isomorphism

$$\text{rad} : \mathcal{D}(g)^G/[\mathcal{D}(g) \text{ad} g]^G = [\mathcal{D}(g)/\mathcal{D}(g) \text{ad} g]^G \to \mathcal{D}(t)^W.$$

We will use a special notation $\mathcal{D} := \mathcal{D}_{g \times t}$ for the sheaf of differential operators on the vector space $g \times t$. We have $\Gamma(g \times t, \mathcal{D}) = \mathcal{D}(g \times t) = \mathcal{D}(g) \otimes \mathcal{D}(t)$.

Definition 2.1.3. The Harish-Chandra module is a left $\mathcal{D}$-module defined as follows

$$\mathcal{M} := \mathcal{D} / \{\mathcal{D} \cdot (\text{ad} g \otimes 1) + \mathcal{D} \cdot \{u \otimes 1 - 1 \otimes \text{rad}(u), u \in \mathcal{D}(g)^G\}\}. \quad (2.1.4)$$

Remark 2.1.5. The above definition was motivated by, but is not identical to, the definition of Hotta and Kashiwara, see [HK1], formula (4.5.1). The equivalence of the two definitions follows from Lemma [4.1.2.3] of §4 below.

2.2. The order filtration on the Harish-Chandra module. Let $X$ be a smooth variety. The sheaf $\mathcal{D}_X$ comes equipped with an ascending filtration $F_{\text{ord}} \mathcal{D}_X$, by the order of differential operator. For the associated graded sheaf, one has a canonical isomorphism $\text{gr}^{\text{ord}} \mathcal{D}_X \cong q_* \mathcal{O}_{T^* X}$, where $q : T^* X \to X$ is the cotangent bundle projection.

Let $M$ be a $\mathcal{D}_X$-module. An ascending filtration $F_i M$ such that one has $F_i \mathcal{D}_X \cdot F_j M \subset F_{i+j} M, \forall i, j$, is said to be good if $\text{gr}^F M$, the associated graded module, is a coherent $q_* \mathcal{O}_{T^* X}$-module. In that case, there is a canonically defined coherent sheaf $\text{gr}^F M$, on $T^* X$, such that one has an isomorphism $\text{gr}^F M = q_* \text{gr}^F M$, of $q_* \mathcal{O}_{T^* X}$-modules. We write $[\text{Supp}(\text{gr}^F M)]$ for the support cycle of the sheaf $\text{gr}^F M$. This is an algebraic cycle in $T^* X$, a linear combination of the irreducible components of the support of $\text{gr}^F M$ counted with multiplicities. The cycle $[\text{Supp}(\text{gr}^F M)]$ is known to be independent of the choice of a good filtration on $M$, cf. [Bo].

Recall that $M$ is called holonomic if one has $\dim \text{Supp}(\text{gr}^F M) = \dim X$. In such a case, each irreducible component of $\text{Supp}(\text{gr}^F M)$, viewed as a reduced variety, is a Lagrangian
cone in $T^*X$. Holonomic $\mathcal{D}_X$-modules form an abelian category and the assignment $M \mapsto [\text{Supp}(\mathfrak{g}_F^I M)]$ is additive on short exact sequences of holonomic modules, cf. eg. [B9], [HTT]. From this, one obtains

**Lemma 2.2.1.** If $M$ is a holonomic $\mathcal{D}_X$-module such that the cycle $[\text{Supp}(\mathfrak{g}_F^I M)]$ equals the fundamental cycle of an irreducible variety taken with multiplicity 1, then $M$ is a simple $\mathcal{D}_X$-module.

According to formula (2.1.4) the Harish-Chandra module has the form $M = \mathcal{D}/\mathcal{I}$, where $\mathcal{I}$ is a left ideal of $\mathcal{D}$. The order filtration on $\mathcal{D}$ restricts to a filtration on $\mathcal{I}$ and it also induces a quotient filtration $F^\text{ord}_\mathcal{I} M$, on $\mathcal{D}/\mathcal{I}$. Using the identifications $T^*(\mathfrak{g} \times t) = \mathfrak{e} \times \mathfrak{t}$ and $\mathfrak{g}_F^I \mathcal{D} = \mathcal{O}_{\mathfrak{e} \times \mathfrak{t}}$, we have $\mathfrak{g}^\text{ord} M = \mathcal{O}_{\mathfrak{e} \times \mathfrak{t}}/\mathfrak{g}^\text{ord} \mathcal{I}$, where $\mathfrak{g}_F^I \mathcal{I}$ the associated graded ideal, is a subsheaf of ideals of $\mathcal{O}_{\mathfrak{e} \times \mathfrak{t}}$.

Let $J \subset \mathcal{O}_{\mathfrak{e} \times \mathfrak{t}}$ be the ideal sheaf of $\mathfrak{C} \times \mathbb{C}/G \mathfrak{t}$, a (non reduced) subscheme in $\mathfrak{e} \times \mathfrak{t}$. The following result provides a relation between the ideals of $\mathcal{O}_{\mathfrak{e} \times \mathfrak{t}}$ and $J$.

**Lemma 2.2.2.** One has inclusions $J \subset \mathfrak{g}_F^I \mathcal{I} \subset \sqrt{J}$, of ideals, and an equality $[\text{Supp}(\mathfrak{g}^\text{ord} M)] = [\mathcal{I}]$, of algebraic cycles in $\mathfrak{e} \times \mathfrak{t}$. In particular, $M$ is a nonzero simple holonomic $\mathcal{D}$-module.

**Proof.** To simplify notation, it will be convenient below to work with spaces of global sections rather than with sheaves. Thus, let $D := \mathcal{D}(\mathfrak{g} \times t) = \mathcal{D}(\mathfrak{g}) \otimes \mathcal{D}(t)$ be the algebra of polynomial differential operators on $\mathfrak{g} \times t$ so, one has $\mathfrak{g}^\text{ord} D = \mathcal{C}[\mathfrak{e} \times \mathfrak{t}]$. Let $I = \Gamma(\mathfrak{g} \times t, \mathcal{I})$, resp. $J = \Gamma(\mathfrak{e} \times \mathfrak{t}, J)$. The vector space $\mathfrak{g} \times t$ being affine, it is sufficient to prove an analogue of the lemma for the ideals $\mathfrak{g}^\text{ord} I$ and $J$.

Let $\kappa^* : \mathfrak{g} = \mathfrak{g}^* \to \mathcal{C}[\mathfrak{e}]$ be the pull-back morphism induced by the commutator map $\kappa$. By definition, one has $\mathcal{C}[\mathfrak{e}] = \mathcal{C}[\mathfrak{e}]/\mathcal{C}[\mathfrak{e}]\kappa^*(\mathfrak{g})$. Hence, we obtain

$$\mathcal{C}[\mathfrak{e} \times \mathfrak{e} // G \mathfrak{t}] = \mathcal{C}[\mathfrak{e}] \otimes \mathcal{C}[\mathfrak{e}] G \mathfrak{t} = \left( \mathcal{C}[\mathfrak{e}] / \mathcal{C}[\mathfrak{e}] \kappa^*(\mathfrak{g}) \right) \otimes_{\mathcal{C}[\mathfrak{e}] G} \mathcal{C}[\mathfrak{t}].$$

We conclude that $J$ is an ideal of the algebra $\mathcal{C}[\mathfrak{e} \times \mathfrak{t}] = \mathcal{C}[\mathfrak{e}] \otimes \mathcal{C}[\mathfrak{t}]$ generated by the following set

$$\mathfrak{S} := \kappa(\mathfrak{g}) \otimes 1 \cup \{ f \otimes 1 - 1 \otimes \text{res}(f) \mid f \in \mathcal{C}[\mathfrak{e}] G \}.$$

To compare the ideals $\mathfrak{g}^\text{ord} I$ and $J$ we introduce a subset of the algebra $D$ as follows:

$$S := \text{ad} \mathfrak{g} \otimes 1 \cup \{ u \otimes 1 - 1 \otimes \text{rad}(u) \mid u \in \mathcal{D}(\mathfrak{g}) \}.$$

**Claim 2.2.3.** We have $\mathfrak{S} = \{ \mathfrak{g}^\text{ord}(s) \mid s \in S \}$; in other words, the set $\mathfrak{S}$ is the set formed by the principal symbols of the elements of $S$.

To prove the claim, we observe that the map $\mathfrak{g}^\text{ord}(\text{ad}) : \mathfrak{g} \to \mathfrak{g}^\text{ord} D = \mathcal{C}[\mathfrak{e}]$ may be identified with the map $\kappa^*$ considered above. Further, the order filtration $F^\text{ord}_\mathcal{D}(\mathfrak{g})$, resp. $F^\text{ord}_{\mathcal{D}} \mathcal{D}(t)$, induces a filtration on $\mathcal{D}(t)^W$, resp. on $\mathcal{D}(\mathfrak{g}) G$. It follows from definitions that the radial part map (2.1.1) respects the filtrations and $\mathfrak{g}^\text{ord}(\text{rad})$, the associated graded map, is nothing but the algebra map $\text{res}$ in (1.1.2). This completes the proof of the claim.

Thus, we see that principal symbols of elements of the set $S$ generate the ideal $J$. On the other hand, formula (2.1.1) shows that the set $S$ is a set of generators of the left ideal $I$. This yields an inclusion $J \subset \mathfrak{g}^\text{ord} I$. Furthermore, it follows by a standard argument that, for any point $x \in \mathfrak{e} \times \mathfrak{t}$ where the ideal generated the principal symbols of elements of the set $S$ is reduced, one has an equality $\mathcal{O}_x \otimes \mathcal{C}[\mathfrak{e} \times \mathfrak{t}] J = \mathcal{O}_x \otimes \mathcal{C}[\mathfrak{e} \times \mathfrak{t}] \mathfrak{g}^\text{ord} I$, of ideals in the local ring $\mathcal{O}_x$ of the point $x$. In particular, this equality holds for any point $x \in \mathfrak{e}^\text{rs}$, by Lemma [1.2.1(i)]. Hence, any element $f \in \mathfrak{g}^\text{ord} I$, viewed as a function on $\mathfrak{e} \times \mathfrak{t}$, vanishes on the set $\mathfrak{e}^\text{rs}$. The set $\mathfrak{e}^\text{rs}$ being Zariski dense in $\mathfrak{e} \times \mathfrak{e} // G \mathfrak{t}$, by Lemma [1.2.1(i)], we deduce that the function $f$
vanishes on the zero set of the ideal \( J \). Hence, \( f \in \sqrt{J} \), by Hilbert’s Nullstellensatz. Thus, we have proved that \( \text{gr}^{\text{ord}} I \subset \sqrt{J} \).

Now, the inclusion \( J \subset \text{gr}^{\text{ord}} I \) implies that, set theoretically, one has \( \text{Supp}(\text{gr}^{\text{ord}} \mathcal{M}) \subset \mathcal{X} \). Further, the scheme \( \mathcal{C} \times_{\mathcal{C}/G} \mathcal{T} \) being generically reduced, by Lemma 1.2.1(i), the inclusion \( \text{gr}^{\text{ord}} I \subset \sqrt{J} \) implies that the fundamental cycle \( [\mathcal{X}] \) occurs in \( [\text{Supp}(\text{gr}^{\text{ord}} \mathcal{M})] \) with multiplicity one. Finally, the dimension of any irreducible component of the support of the sheaf \( \text{gr}^{\text{ord}} \mathcal{M} \) is \( \geq \dim \mathcal{X} \), since \( \mathcal{X} \) is a Lagrangian subvariety. We conclude that \( \mathcal{X} \) is the only irreducible component of \( \text{Supp}(\text{gr}^{\text{ord}} \mathcal{M}) \). Thus, we have \( [\text{Supp}(\text{gr}^{\text{ord}} \mathcal{M})] = [\mathcal{X}] \), and Lemma 2.2.1 completes the proof.

2.3. Hodge filtration. In his work [Sa], M. Saito defines, for any smooth algebraic variety \( X \) over \( \mathbb{Q} \), a semisimple abelian category \( \text{HM}(X) \), of polarized Hodge modules. The data of a Hodge module includes, in particular, a holonomic \( \mathcal{D}_X \)-module \( \mathcal{M} \), with regular singularities, and a good filtration \( F \), on \( \mathcal{M} \), called Hodge filtration. Abusing notation, we write \((M, F) \in \text{HM}(X)\), and let \( \text{gr}(M, F) \) denote the corresponding coherent sheaf on \( T^*X \).

There is a duality functor on the abelian category of holonomic \( \mathcal{D}_X \)-modules, called Verdier duality, cf. [HTT]. One has a similar duality functor \( \mathcal{D} : \text{HM}(X) \to \text{HM}(X) \). Saito shows that the assignment \((M, F) \mapsto \text{gr}(M, F)\) intertwines the functor \( \mathcal{D} \) with the standard Grothendieck duality on \( \mathcal{D}^{\text{coh}}_{\text{c}}(T^*X) \). Specifically, he proves that, for \((M, F) \in \text{HM}(X)\), the Grothendieck dual of the coherent sheaf \( \text{gr}(M, F) \) is isomorphic, up to shifts, to \( \text{gr}(\mathcal{D}(M, F)) \).

This implies, in particular, that the Grothendieck dual of \( \text{gr}(M, F) \), viewed as an object of \( \mathcal{D}^{\text{coh}}_{\text{c}}(T^*X) \), has a single nonvanishing cohomology sheaf, cf. (4.1). In other words, one has, see [Sa], Remark 5.1.4(7).

**Lemma 2.3.1.** For any \((M, F) \in \text{HM}(X)\), the \( O_{T^*X} \)-module \( \text{gr}(M, F) \) is Cohen-Macaulay.

Hotta and Kashiwara [HK1] gave a geometric construction of the Harish-Chandra module, to be explained in §4.2. That construction gives the Harish-Chandra module \( \mathcal{M} \) the natural structure of a polarized Hodge module on \( \mathfrak{g} \times t \) which is, in addition, isomorphic to its Verdier dual, up to a shift. Thus, there is a canonical Hodge filtration \( F_{\text{Hodge}}^* \mathcal{M} \), on \( \mathcal{M} \). Write \( \text{gr}^\text{Hodge} \mathcal{M} \) for the corresponding associated graded sheaf. Applying Lemma 2.3.1 and using the self-duality of \( \mathcal{M} \), we obtain, cf. also Lemma 2.2.2(ii),

**Corollary 2.3.2.** The sheaf \( \text{gr}^\text{Hodge} \mathcal{M} \) is a Cohen-Macaulay coherent \( O_{\mathfrak{g} \times \mathbb{T}} \)-module. This module is set-theoretically supported on \( \mathcal{X} \) and it is isomorphic to its Grothendieck dual, up to a shift.

In the previous subsection, we have considered another filtration \( F_{\text{ord}}^* \mathcal{M} \), the order filtration on the Harish-Chandra module. We do not know if the Hodge and the order filtrations on \( \mathcal{M} \) are equal or not. The result below, to be proved in §5.4, provides a partial answer.

**Lemma 2.3.3.** (i) For each \( k \geq 0 \), one has inclusions \( j : F_{\text{ord}}^k \mathcal{M} \subset F_{\text{Hodge}}^k \mathcal{M} \), and the restriction of the induced map \( j : \text{gr}^{\text{ord}} j : \text{gr}^{\text{ord}} \mathcal{M} \to \text{gr}^\text{Hodge} \mathcal{M} \) to the open set \( \mathcal{X}^{rs} \) is an isomorphism.

(ii) The \( O_{\mathfrak{g} \times \mathbb{T}} \)-module \( \text{gr}^\text{Hodge} \mathcal{M} \) has a natural structure of commutative \( O_{\mathfrak{g} \times \mathbb{T}} \)-algebra such that the following composite map is a morphism of \( O_{\mathfrak{g} \times \mathbb{T}} \)-algebras

\[
O_{\mathfrak{g} \times \mathbb{T}}/\mathcal{I} = O_{\mathfrak{g} \times \mathbb{T}}/J \quad \xrightarrow{\text{Lemma 2.2.2(ii)}} \quad O_{\mathfrak{g} \times \mathbb{T}}/\mathcal{I} = \text{gr}^{\text{ord}} \mathcal{M} \quad \xrightarrow{\text{gr} j} \quad \text{gr}^\text{Hodge} \mathcal{M}.
\]

**Remark 2.3.5.** The Hodge filtration on \( \mathcal{M} \) that we use, in Lemma 2.3.3 in particular, differs by a degree shift from the one used by Saito. The precise normalization of our Hodge filtration will be specified in §5.4. Degree shifts clearly do not affect the validity of Lemma 2.3.1; hence, Corollary 2.3.2 holds with our normalization of the Hodge filtration as well.
2.4. The Hodge filtration on the Harish-Chandra module. Recall the normalization map \(\psi : \overline{X}_{\text{norm}} \to \overline{X}\).

One of the main results of the paper is the following description of the sheaf \(\text{gr}^{\text{Hodge}} M\).

**Theorem 2.4.1.** There is a natural \(O_{\mathfrak{g} \times \mathfrak{T}}\)-algebra isomorphism \(\psi_* O_{\overline{X}_{\text{norm}}} \cong \text{gr}^{\text{Hodge}} M\).

**Remark 2.4.2.** Note that the above theorem, combined with Corollary 2.3.3, implies Theorem 1.6.3 from §5.2 below. For the proof of Lemma 2.4.6 see Remark 2.4.5.

**Sketch of proof of Theorem 2.4.1.** Let \(X := \text{Spec}_{\mathfrak{g} \times \mathfrak{T}}(\text{gr}^{\text{Hodge}} M)\) be the relative spectrum of \(\text{gr}^{\text{Hodge}} M\), the latter being viewed, thanks to Lemma 2.3.3(ii), as a sheaf of commutative \(O_{\mathfrak{g} \times \mathfrak{T}}\)-algebras. Thus, \(X\) is a scheme equipped with a finite morphism \(X \to \mathfrak{g} \times \mathfrak{T}\) that factors through a morphism \(f : X \to \mathfrak{c} \times \mathfrak{c} / \mathfrak{T},\) by Lemma 2.3.3(ii). We know that \(\mathfrak{c} \times \mathfrak{c} / \mathfrak{T}\) is an irreducible and generically reduced scheme, by Lemma 1.2.1(i). Furthermore, the scheme \(X\) is Cohen-Macaulay, by Corollary 2.3.3(i). It follows that \(X\) is reduced. Therefore, the map \(f\) factors through a finite morphism \(f : X \to \bar{X}\).

Recall next that \(\bar{X}\) is irreducible (Lemma 1.2.1) and, we have \(f_* [X] = [\text{Supp}(\text{gr}^{\text{Hodge}} M)] = [\text{Supp}(\text{gr}^{\text{ord}} M)] = [\mathfrak{c}]\), by Lemma 2.2.2. The equality \(f_* [X] = [\mathfrak{c}]\), of algebraic cycles, forces \(X\) be irreducible and the map \(f : X \to \bar{X}\) be a finite birational isomorphism.

To proceed further we need the following technical

**Definition 2.4.3.** Let \(\mathfrak{c}^{rr}\) be the set of pairs \((x, y) \in \mathfrak{c}\) such that either \(x\) or \(y\) is a regular element of \(\mathfrak{g}\). We put \(\mathfrak{c}^{rr} := p_1^{-1}(\mathfrak{c}^{rr})\), a subset of \(\bar{X}\).

The following two lemmas explain the role of the set \(\mathfrak{c}^{rr}\) in the proof of Theorem 2.4.1.

**Lemma 2.4.4.**

(i) \(\mathfrak{c}^{rr}\) is a Zariski open subset and one has \(\dim(\mathfrak{c} \setminus \mathfrak{c}^{rr}) \leq \dim \mathfrak{c} - 2\);

(ii) The set \(\mathfrak{c}^{rr}\) is contained in the smooth locus of the variety \(\mathfrak{c}\) and it is Zariski dense in \(\mathfrak{c}\).

**Remark 2.4.5.**

(i) It is clear from Proposition 1.1.1 that the set \(\mathfrak{c}^{rr}\) is contained in the smooth locus of \(\mathfrak{c}\) and that it is dense in \(\mathfrak{c}\).

(ii) It is essential that in Lemma 2.4.4(ii) one takes the preimage of \(\mathfrak{c}^{rr}\) in \(\mathfrak{c}\); the preimage of \(\mathfrak{c}^{rr}\) in \(\mathfrak{c} \times \mathfrak{c} / \mathfrak{T}\), a nonreduced fiber product, is not smooth already for \(\mathfrak{g} = \mathfrak{s}_2\).

(iii) Definition 2.4.3 and Lemma 2.4.4 were motivated, in part, by [Ha1] Lemma 3.6.2.

**Lemma 2.4.6.** The morphism \(f\) induces an isomorphism \(f^{-1}(\mathfrak{c}^{rr}) \cong \mathfrak{c}^{rr}\).

Part (i) of Lemma 2.4.4 will be proved in §6.3 and part (ii) of Lemma 2.4.4 is Lemma 5.2.1(ii), of [5.2] below. For the proof of Lemma 2.4.6 see [5.5].

We can now complete the proof of Theorem 2.4.1. The map \(p_{\mathfrak{c}^{rr}}\), resp. \(f\), being finite, from Lemma 2.4.4(i) we deduce that the complement of the set \(\mathfrak{c}^{rr}\), resp. of the set \(f^{-1}(\mathfrak{c}^{rr})\), has codimension one in \(\mathfrak{c}\), resp. in \(X\). Hence, Lemma 2.4.4(ii) implies that the variety \(\mathfrak{c}\), resp. \(X\), is smooth in codimension 1. We conclude, since \(X\) is irreducible and Cohen-Macaulay, that \(X\) is normal and, moreover, \(f = \psi\) is the normalization map.

The above proof shows that the isomorphism of Theorem 2.4.1 fits into the following chain of \(O_{\mathfrak{g} \times \mathfrak{T}}\)-algebra maps, cf. Lemma 2.2.2(ii),

\[
\begin{align*}
&O_{\mathfrak{c} \times \mathfrak{c} / \mathfrak{T}} \longrightarrow \text{gr}^{\text{ord}} M \longrightarrow \mathfrak{c}^{-1} \longrightarrow O_{\mathfrak{c}^{rr}} \cong \psi_* O_{\overline{X}_{\text{norm}}}, \quad \overset{\text{Theorem 2.4.1}}{\longrightarrow} \text{gr}^{\text{Hodge}} M.
\end{align*}
\]
3. Relation to Work of M. Haiman

Throughout this section, we consider a special case where $G = GL_n$.

3.1. A vector bundle on the Hilbert scheme. Let $V = \mathbb{C}^n$ be the fundamental representation of the group $GL_n$, so $g = \text{End}_\mathbb{C} V$. For any $(x, y) \in \mathcal{C}$, one may view the vector space $g_{x,y}$ as an associative subalgebra of $\text{End}_\mathbb{C} V$. The group $G_{x,y}$ may be identified with the group of invertible elements of that subalgebra, hence, this group is connected.

Let $\mathcal{C}(x,y)$ denote the associative subalgebra of $\text{End}_\mathbb{C} V$ generated by the elements $x$ and $y$. Recall that a vector $v \in V$ is said to be a cyclic vector for a pair $(x, y) \in \mathcal{C}$ if one has $\mathbb{C}(x,y)v = V$. Let $\mathcal{C}^\circ$ be the set of pairs $(x, y) \in \mathcal{C}$ which have a cyclic vector. Part (i) of the following result is due to Neubauer and Saltman [NS] and part (ii) is well-known, cf. [NS].

**Lemma 3.1.1.**

(i) A pair $(x, y) \in \mathcal{C}$ is regular if and only if one has $\mathbb{C}(x,y) = g_{x,y}$.

(ii) The set $\mathcal{C}^\circ$ is a Zariski open subset of $\mathcal{C}^\circ$. For $(x, y) \in \mathcal{C}^\circ$, all cyclic vectors for $(x, y)$ form a single $G_{x,y}$-orbit, which is an open $G_{x,y}$-orbit in $V$. □

Next, introduce a variety of triples

$$\mathcal{S} = \{ (x, y, v) \in g \times g \times V \mid [x, y] = 0, \mathbb{C}(x, y)v = V \}.$$

It is easy to see that $\mathcal{S}$ is a smooth quasi-affine variety. We let $G$ act on $\mathcal{S}$ by $g : (x, y, v) \mapsto (gxg^{-1}, gyg^{-1}, gv)$. This action is free, by Lemma 3.1.1 and it is known that there exists a universal geometric quotient morphism $\rho : \mathcal{S} \to \mathcal{S}/G$. Furthermore, the variety $\mathcal{S}/G$ may be identified with $\text{Hilb} := \text{Hilb}^n(\mathbb{C}^2)$, the Hilbert scheme of $n$ points in $\mathbb{C}^2$, see [Na]. Writing $\delta$ for the projection $(x, y, v) \mapsto (x, y)$, one obtains a commutative diagram, cf. also [Ha2] §8,

$$\begin{array}{ccc}
\mathcal{S} & \xrightarrow{\rho} & \text{Hilb} \\
\downarrow \delta & & \downarrow \text{Hilb-Chow} \\
\mathcal{C}/W & \xleftarrow{\mathcal{S}} & \\
\end{array}$$

(3.1.2)

Let $\mathcal{G}$ be the scheme theoretic preimage of the diagonal in $\mathcal{S} \times \mathcal{S}$ under the morphism $G \times \mathcal{S} \to \mathcal{S} \times \mathcal{S}$, $(g, s) \mapsto (gs, s)$. Set theoretically, one has $\mathcal{G} = \{ (x, y, \gamma) \in g \times g \times G \mid (x, y) \in \mathcal{C}, \gamma \in G_{x,y} \}$. The group $G$ acts naturally on $\mathcal{G}$ and the projection $\mathcal{G} \to \mathcal{C}$, $(x, y, \gamma) \mapsto (x, y)$ is a $G$-equivariant map. The scheme $\mathcal{G}$ has the natural structure of a group scheme over $\mathcal{C}$. Lemma 3.1.1(i) implies that $\mathcal{G}|_{\mathcal{C}^\circ}$, the restriction of $\mathcal{G}$ to the regular locus of the commuting scheme, is a smooth commutative group scheme. We put $\mathcal{G}^\circ := \mathcal{G}|_{\mathcal{C}^\circ}$.

We define a $\mathcal{G}^\circ$-action along the fibers of the map $\delta$, see (3.1.2), by $(x, y, \gamma) : (x, y, v) \mapsto (x, y, \gamma v)$. Lemma 3.1.1(ii) implies that this $\mathcal{G}^\circ$-action on $\mathcal{S}$ is free. Moreover, the map $\delta$ makes $\mathcal{S}$ a $G$-equivariant $\mathcal{G}^\circ$-torsor over $\mathcal{C}^\circ$. In particular, $\delta$ is a smooth morphism.

We use the notation of diagram (3.1.2) and put $\mathcal{P} = (\rho, \delta^* \mathcal{R})^G$. By equivariant descent, one has a canonical isomorphism $\rho^* \mathcal{P} = \delta^* \mathcal{R}$. Proposition 1.6.4 yields the following result

**Corollary 3.1.3.** The sheaf $\mathcal{P}$ is a locally free coherent sheaf of commutative $O_{\text{Hilb}}$-algebras equipped with a natural $W$-action, by algebra automorphisms. The fibers of the corresponding algebraic vector bundle afford the regular representation of the group $W$. □

3.2. The isospectral Hilbert scheme. Let $\mathcal{W} := \text{Spec}_{\text{Hilb}}^\circ \mathcal{P}$ be the relative spectrum of $\mathcal{P}$, a sheaf of algebras on the Hilbert scheme. The scheme $\mathcal{W}$ comes equipped with a flat and finite morphism $\eta : \mathcal{W} \to \text{Hilb}$ and with a $W$-action along the fibers of $\eta$. We conclude that $\mathcal{W}$ is a reduced Cohen-Macaulay and Gorenstein variety.
One can interpret the construction of the scheme $\mathcal{M}$ in more geometric terms as follows. Let $X^0 := p^{-1}(C^0)$ and consider the following commutative diagram

$$
\begin{array}{ccccccccc}
X^0 & \xrightarrow{\bar{\sigma}} & X^0 \times_{C^0} S & \xrightarrow{h} & S \times_{\text{Hilb}} \mathcal{M} & \xrightarrow{\tilde{p}} & \mathcal{M} \\
p & & \downarrow{\bar{\rho}} & & \downarrow{\bar{\eta}} & & \downarrow{\eta} \\
C^0 & \xrightarrow{\delta} & S & \xrightarrow{\rho} & \text{Hilb} & & & & \\
\end{array}
$$

(3.2.1)

In this diagram, the morphisms $\delta$ and $\rho$ are smooth, resp. the morphisms $p$ and $\eta$ are finite and flat. The morphism $\bar{\delta}$, resp. $\bar{\rho}, \bar{\eta}$, is obtained from $\delta$, resp. from $\rho, p, \eta$, by base change. Hence, flat base change yields

$$
\delta^* \mathcal{R} = \delta^* p_* \mathcal{O}_{X^0} = \bar{\rho}_* \mathcal{O}_{X^0 \times_{C^0} S}, \quad \text{resp.} \quad \rho^* \mathcal{P} = \rho^* \eta_* \mathcal{O}_{\mathcal{M}} = \bar{\eta}_* \mathcal{O}_{S \times_{\text{Hilb}} \mathcal{M}}.
$$

(3.2.2)

We equip the scheme $S \times_{\text{Hilb}} \mathcal{M}$ with a $G$-action induced from the one on $S$, resp. equip $X^0 \times_{C^0} S$ with the $G$-diagonal action. Thus, $\bar{\rho}, \bar{\eta}$, is a $G$-equivariant finite and flat morphism. Since $\delta^* \mathcal{R} = \rho^* \mathcal{P}$, from (3.2.2) we deduce a canonical isomorphism $\bar{\rho}_* \mathcal{O}_{X^0 \times_{C^0} S} \cong \bar{\eta}_* \mathcal{O}_{S \times_{\text{Hilb}} \mathcal{M}}$, of $G$-equivariant sheaves of $\mathcal{O}_S$-algebras. This means that there is a canonical $G$-equivariant isomorphism $h : X^0 \times_{C^0} S \rightarrow S \times_{\text{Hilb}} \mathcal{M}$, the dotted arrow in diagram (3.2.1).

The scheme $X^0$ being Gorenstein, we deduce that the scheme $\mathcal{M}$ is Gorenstein as well.

The $W$-action on $X^0$ induces one on $X^0 \times_{C^0} S$. This $W$-action commutes with the $G$-diagonal action, hence, descends to $(X^0 \times_{C^0} S)/G$. The resulting $W$-action may be identified with the $W$-action on $\mathcal{M}$ that was defined earlier. The composite map $p \circ \bar{\sigma} : X^0 \times_{C^0} S \rightarrow X^0 \rightarrow \mathcal{X}$ descends to a $W$-equivariant morphism $\mathcal{M} \rightarrow \mathcal{X}$. We obtain the following diagram

$$
\begin{array}{cccccc}
X^0 & \xrightarrow{\bar{\sigma}} & X^0 \times_{C^0} S & \xrightarrow{\bar{\rho} \circ h} & (X^0 \times_{C^0} S)/G = \mathcal{M} & \xrightarrow{\sigma \times \eta} & \mathcal{X} \times_{\text{red}} \text{Hilb}.
\end{array}
$$

(3.2.3)

Following Haiman, one defines the isospectral Hilbert scheme as $[\text{Hilb} \times_{\mathcal{X}/W} \mathcal{X}]_{\text{red}}$, a reduced fiber product. We have

**Proposition 3.2.4.** The map $\sigma \times \eta$, on the right of (3.2.3), factors through an isomorphism

$$
\mathcal{M} \cong [\text{Hilb} \times_{\mathcal{X}/W} \mathcal{X}]_{\text{norm}}.
$$

In particular, the normalization of the isospectral Hilbert scheme is Cohen-Macaulay and Gorenstein.

**Proof.** It is clear that $\bar{\rho} \circ h(\bar{\sigma}^{-1}(X^{rs}))$ is a Zariski open subset of $\mathcal{M}$. Lemma [1.2.1(ii)] implies readily that the map $\sigma \times \eta$ restricts to an isomorphism $\bar{\rho} \circ h(\bar{\sigma}^{-1}(X^{rs})) \cong \text{Hilb} \times_{\mathcal{X}/W} \mathcal{X}$; in particular, $\sigma \times \eta$ is a birational isomorphism. The image of the map $\sigma \times \eta$ is contained in $[\text{Hilb} \times_{\mathcal{X}/W} \mathcal{X}]_{\text{red}}$ since the scheme $\mathcal{M}$ is reduced.

Further, Corollary [3.1.3] and Lemma [2.4.4] imply that the scheme $\mathcal{M}$ is Cohen-Macaulay and smooth in codimension 1. We conclude that $\mathcal{M}$ is a normal scheme which is birational and finite over $[\text{Hilb} \times_{\mathcal{X}/W} \mathcal{X}]_{\text{red}}$. This yields the isomorphism of the proposition. The Gorenstein property of $\mathcal{M}$ follows, since $X$ is Gorenstein, from the isomorphism $\mathcal{M} = (X^0 \times_{C^0} S)/G$, as has been already mentioned earlier. 

We remark that Haiman has proved in [Ha1] that the isospectral Hilbert scheme is, in fact, normal. Assuming this result, Proposition [3.2.4] implies that the isospectral Hilbert scheme is Cohen-Macaulay and Gorenstein, the main result of [Ha1]. Unfortunately, our method does not seem to yield an independent proof of normality of the isospectral Hilbert scheme, while
the proof of normality given in [Ha1] is based on the ‘polygraph theorem’, a key technical result of [Ha1].

As a consequence of Haiman’s work, we deduce

**Corollary 3.2.5.** The vector bundle $\mathcal{P}$, on $\text{Hilb}$, is isomorphic to the Procesi bundle, cf. [Ha1]. □

### 4. Geometric construction of the Harish-Chandra module

#### 4.1. Filtered $\mathcal{D}$-modules

Following G. Laumon and M. Saito, we consider the category $F\mathcal{D}_X\text{-mod}$, of left $\mathcal{D}_X$-modules $M$ equipped with a good filtration $F$. This is an exact (not abelian) category and there is an associated derived category. Let $D^b_{coh}(F\mathcal{D}_X\text{-mod})$ be the full triangulated subcategory of the derived category whose objects are isomorphic to bounded complexes $(M, F)$, of filtered $\mathcal{D}_X$-modules, such that each cohomology group $\mathcal{H}^i(M, F) \in F\mathcal{D}_X\text{-mod}$ is a coherent $\mathcal{D}_X$-module. The assignment $(M, F) \mapsto \text{gr}^F M$ gives a functor $F\mathcal{D}_X\text{-mod} \to \mathcal{O}_X\text{-mod}$ that can be extended to a triangulated functor $\mathcal{D}_X\text{-mod} \to D^b_{coh}(F\mathcal{D}_X\text{-mod})$

Associated with a proper morphism $f : X \to Y$, of smooth varieties, there is a natural direct image functor $f_!$, resp. $f_*^R$, between bounded derived categories of coherent left, resp. right, $\mathcal{D}$-modules on $X$ and $Y$, respectively, cf. [Bo]. For a left $\mathcal{D}_X$-module $M$, one has $f_! M = K_Y^{-1} \otimes_{\mathcal{O}_Y} f_*^R(K_X \otimes_{\mathcal{O}_X} M)$.

The direct image functor can be upgraded to a functor $D^b_{coh}(F\mathcal{D}_X\text{-mod}) \to D^b_{coh}(F\mathcal{D}_Y\text{-mod})$, $(M, F) \mapsto f_!(M, F)$, between filtered derived categories, cf. [Sa], [La]. The latter functor commutes with the associated graded functor $\text{dgr}(-)$. We will only need a special case of this result for maps of the form $f : X \times Y \to Y$, the projection along a proper variety $Y$. In this case, one has a diagram

$$T^*Y \xrightarrow{\text{pr}} X \times T^*Y \xrightarrow{\iota_{X \times T^*Y \to T^*(X \times Y)}} T^*X \times T^*Y = T^*(X \times Y). \quad (4.1.1)$$

Here, $\iota : X \hookrightarrow T^*X$ denotes the zero section and we put $\iota_{X \times T^*Y \to T^*(X \times Y)} := \iota \otimes \text{Id}_{T^*Y}$.

The relation between the functors $\text{dgr}(-)$ and $f_!$ is provided by the following result

**Theorem 4.1.2 ([La], [Sa]).** Let $f : X \times Y \to Y$ be the second projection where $X$ is proper. Then, for any $(M, F) \in D^b_{coh}(F\mathcal{D}_{X \times Y}\text{-mod})$, in $D^b_{coh}(T^*Y)$, there is a functorial isomorphism

$$\text{dgr}(f_!(M, F)) = \text{R pr}_!\left(\left(K_X \otimes \mathcal{O}_{T^*Y}\right) \otimes_{\mathcal{O}_{X \times T^*Y}} L\iota_{X \times T^*Y \to T^*(X \times Y)}^{L} \text{dgr}(M, F)\right). \quad \square$$

Let $f : X \to Y$ be a proper morphism and $(M, F)$ a filtered $\mathcal{D}_X$-module. Each cohomology of the filtered complex $f_!(M, F)$ is a $\mathcal{D}_Y$-module $\mathcal{H}^i\left(f_!(M, F)\right)$ that comes equipped with an induced filtration. However, Theorem 4.1.2 is not sufficient, in general, for describing $\text{gr} \mathcal{H}^i\left(f_!(M, F)\right)$, the associated graded sheaves. To explain this let, more generally,

$$E : \ldots \to E^{k-1} \xrightarrow{d_{k-1}} E^k \xrightarrow{d_k} E^{k+1} \xrightarrow{d_k} \ldots,$$

be an arbitrary filtered complex in an abelian category. Then, one has an induced filtration on each cohomology group $H^k(E)$, $k \in \mathbb{Z}$, and there is a spectral sequence $H^i(\text{gr} E) \Rightarrow \text{gr} H^j(E)$. The filtered complex $E$ is said to be strict if the morphism $d_k : F_j E^k \to \text{Im} d_k \cap F_j E^{k+1}$ is surjective, for any $k, j \in \mathbb{Z}$. For a strict filtered complex, the spectral sequence degenerates and one has a canonical isomorphism $\text{gr} H^i(E) \cong H^i(\text{gr} E)$.

Now, let $f : X \to Y$ be a projective morphism of smooth varieties. A polarized Hodge module on $X$ may be viewed as an object $(M, F) \in D^b_{coh}(F\mathcal{D}_X\text{-mod})$. The corresponding
direct image, \( f^!(M, F) \in D^b_{coh}(F \mathcal{D}_Y \text{-mod}) \), may be thought of as a filtered bounded complex of \( \mathcal{D}_Y \)-modules. One of the main results of Saito’s theory reads, see [Sa] Theorem 5.3.2:

**Theorem 4.1.3.** For any \((M, F) \in \mathrm{HM}(X)\) and any projective morphism \( f : X \to Y \), the filtered complex \( f^!(M, F) \) is strict and each cohomology group \( \mathcal{H}^j(f^!(M, F)) \) has a natural structure of a polarized Hodge module on \( Y \).

In the situation of the theorem, we refer to the induced filtration on \( \mathcal{H}^j(f^!(M, F)) \), \( j = 0, 1, \ldots \), as the Hodge filtration and let \( \mathfrak{g}^{Hodge} \mathcal{H}^j(f^!(M, F)) \) denote the associated graded coherent sheaf on \( T^*Y \). Similar notation will be used for right \( \mathcal{D} \)-modules.

4.2. The Hotta-Kashiwara construction. The sheaf \( O_{\mathfrak{g}} \) has an obvious structure of a holonomic left \( \mathcal{D}_\mathfrak{g} \)-module. So, one has \( \int_{x \times u} O_{\mathfrak{g}} \), the direct image of this \( \mathcal{D}_\mathfrak{g} \)-module via the proper morphism \( x \times u \). Each cohomology group \( \mathcal{H}^k(\int_{x \times u} O_{\mathfrak{g}}) \) is a holonomic \( \mathcal{D}_{\mathfrak{g} \times \mathfrak{t}} \)-module set theoretically supported on the variety \( \mathfrak{r} \subset \mathfrak{g} \times \mathfrak{t} \).

Hotta and Kashiwara proved the following important result, [HK1], Theorem 4.2.

**Theorem 4.2.1.** For any \( k \neq 0 \), we have \( \mathcal{H}^k(\int_{x \times u} O_{\mathfrak{g}}) = 0 \), and there is a natural isomorphism of \( \mathcal{D} \)-modules \( \mathcal{H}^0(\int_{x \times u} O_{\mathfrak{g}}) \cong \mathcal{D}/I' \), where \( I' \subset I \) is the following left ideal

\[
I' := \mathcal{D} \cdot (\text{ad} \otimes 1) + \mathcal{D} \cdot \{ P - \text{rad}(P) \mid P \in \mathbb{C}[\mathfrak{g}]^G \} + \mathcal{D} \cdot \{ Q - \text{rad}(Q) \mid Q \in (\text{Sym} \mathfrak{g})^G \}.
\]

(4.2.2)

An alternative, more direct, proof of a variant of Theorem 4.2.1 may be found in [HK2].

It is clear, by comparing formulas (2.1.4) and (4.2.2), that, using the notation of Lemma 2.2.2, one has an inclusion \( I' \subset I \), of left ideals.

**Lemma 4.2.3.** We have \( I' = I \).

**Proof.** According to [HK1], Lemma 4.6.1, one has that \( \mathcal{D}/I' \) is either a simple \( \mathcal{D} \)-module or else \( \mathcal{D}/I' = 0 \). We conclude, since \( \mathcal{D}/I = \mathcal{M} \neq 0 \) by Lemma 2.2.2(ii), that the natural surjection \( \mathcal{D}/I' \to \mathcal{D}/I = \mathcal{M} \) must be an isomorphism.

The canonical section \( \omega \), cf. Proposition 1.3.2(iii), gives an element in \( \mathcal{H}^0(\int_{x \times u} K_{\mathfrak{g}}) \). Therefore, writing \( dx \), resp. \( dt \), for constant volume forms on \( \mathfrak{g} \), resp. \( \mathfrak{t} \), one can view \( (dx \otimes dt)^{-1} \otimes \omega \) as a section of \( \mathcal{H}^0(\int_{x \times u} O_{\mathfrak{g}}) \). The proof of the isomorphism \( \mathcal{H}^0(\int_{x \times u} O_{\mathfrak{g}}) \cong \mathcal{D}/I' \), see [HK1], §4.7, combined with Lemma 4.2.3 implies the following

**Corollary 4.2.4.** The assignment \( u \mapsto u[(dx \otimes dt)^{-1} \otimes \omega] \) yields an isomorphism \( \mathcal{M} \to \mathcal{H}^0(\int_{x \times u} O_{\mathfrak{g}}) \).

**Remark 4.2.5.** It follows from the corollary that the section \( (dx \otimes dt)^{-1} \otimes \omega \) is annihilated by all differential operators of the form \( u \otimes 1 - 1 \otimes \text{rad} u, \ u \in \mathcal{D}(\mathfrak{g})^G \). This is a strengthening of [HK1], formulas (4.7.2)-(4.7.3).

4.3. It will be convenient to factor the map \( \mu \otimes \nu : \tilde{\mathfrak{g}} \to \mathfrak{g} \times \mathfrak{t} \) as a composition of a closed imbedding \( \epsilon : \tilde{\mathfrak{g}} \subset \mathfrak{B} \times \mathfrak{g} \times \mathfrak{t} \), \( (b, x) \mapsto (b, x, \nu(x)) \) and a projection \( f : \mathfrak{B} \times \mathfrak{g} \times \mathfrak{t} \to \mathfrak{g} \times \mathfrak{t} \), along the first factor. The image of the imbedding \( \epsilon \) is a smooth subvariety in \( \epsilon(\tilde{\mathfrak{g}}) \subset \mathfrak{B} \times \mathfrak{g} \times \mathfrak{t} \).

Therefore, \( E := \int_f^R K_{\tilde{\mathfrak{g}}} \) is a simple holonomic right \( \mathcal{D} \)-module on \( \mathfrak{B} \times \mathfrak{g} \times \mathfrak{t} \). This \( \mathcal{D} \)-module is generated by the section \( \epsilon_* \omega \). So, we have a surjective morphism \( \gamma : \mathfrak{D}_{\mathfrak{B} \times \mathfrak{g} \times \mathfrak{t}} \to E, \ 1 \mapsto \epsilon_* \omega \), of right \( \mathfrak{D}_{\mathfrak{B} \times \mathfrak{g} \times \mathfrak{t}} \)-modules. Applying the functor \( \int_f^R \) one obtains a morphism \( \int_f^R \gamma : \int_f^R \mathfrak{D}_{\mathfrak{B} \times \mathfrak{g} \times \mathfrak{t}} \to \int_f^R E, \) of complexes of right \( \mathfrak{D}_{\mathfrak{g} \times \mathfrak{t}} \)-modules.
Lemma 4.3.1. All nonzero cohomology sheaves of the complex $\int_f^R \mathcal{D}_g \times \mathfrak{g} \times t$ vanish and one has an isomorphism $\mathcal{H}^0(\int_f^R \mathcal{D}_g \times \mathfrak{g} \times t) \cong \mathcal{D}_g$ of right $\mathcal{D}_g \times \mathfrak{g}$-modules.

Proof. Let $\mathcal{T}$ denote the tangent sheaf on $\mathcal{B}$. There is a standard Koszul complex $\mathcal{K}^*$, with terms $\mathcal{K}^j = \mathcal{D}_g \otimes \mathcal{O}_g \wedge^j \mathcal{T}$, that gives a resolution of the structure sheaf $\mathcal{O}_g$, cf. [1, 4]. Thus, using that $H^0(\mathcal{B}, \mathcal{O}_\mathcal{B}) = \mathbb{C}$ and $H^k(\mathcal{B}, \mathcal{O}_\mathcal{B}) = 0$ for any $k \neq 0$, by the definition of direct image, we get

$$\int_f^R \mathcal{D}_g \times \mathfrak{g} \times t = R\Gamma(\mathcal{B}, \mathcal{K}^*) \otimes_{\mathcal{D}_g} \mathcal{D}_g \times \mathfrak{g} \times t = R\Gamma(\mathcal{B}, \mathcal{O}_\mathcal{B}) \otimes_{\mathcal{D}_g} \mathcal{D}_g \times \mathfrak{g} \times t = \mathcal{D}_g.$$ 

We have $\int_{\mu \times \nu} \mathcal{O}_g = \int_f^R \mathcal{I}_e \mathcal{O}_g = K^{-1}_{\mathfrak{g} \times t} \otimes_{\mathcal{O}_g^{\mathfrak{g} \times t}} \int_f^R \mathcal{E}$. Hence, from Theorem 4.2.1 and Corollary 4.2.4, we deduce that $\mathcal{H}^j(\int_f^R \mathcal{E}) = 0$ for all $j \neq 0$ and, moreover, one has an isomorphism $K^{-1}_{\mathfrak{g} \times t} \otimes \mathcal{H}^0(\int_f^R \mathcal{E}) \cong \mathcal{M}$.

We identify the sheaf $\mathcal{D}_g$ with $\mathcal{D}_g^{op}$ via the trivialization of the canonical bundle on $\mathfrak{g} \times t$ provided by the section $dx \otimes dt$. Let $\tilde{\gamma} := \mathcal{H}^0(\int_f^R \mathcal{E})$ denote the map of 0-th cohomology groups induced by the morphism $\int_f^R \mathcal{E}$. Thus, using Lemma 4.3.1, we obtain a chain of morphisms of left $\mathcal{D}_g \times \mathfrak{g}$-modules

$$\mathcal{D}_g \times \mathfrak{g} = K^{-1}_{\mathfrak{g} \times t} \otimes \mathcal{H}^0(\int_f^R \mathcal{D}_g \times \mathfrak{g} \times t) \xrightarrow{\tilde{\gamma}} K^{-1}_{\mathfrak{g} \times t} \otimes \mathcal{H}^0(\int_f^R \mathcal{E}) = \mathcal{H}^0(\int_{\mu \times \nu} \mathcal{O}_g) = \mathcal{M}.$$ 

(4.3.2)

It is straightforward to see, using the explicit isomorphism of Corollary 4.2.4, that the composite morphism in (4.3.2) is nothing but the natural projection $\mathcal{D}_g \times \mathfrak{g} \to \mathcal{D}_g$. 

4.4. Let $\Lambda$ be the total space of the conormal bundle on $e(\mathfrak{g}) \subset \mathcal{B} \times \mathfrak{g} \times t$. Thus, $\Lambda$ is a smooth Lagrangian cone in $T^*(\mathcal{B} \times \mathfrak{g} \times t) = T^*\mathfrak{g} \times \mathfrak{g} \times t\times \mathfrak{g}$. We will view the structure sheaf $\mathcal{O}_\Lambda$ as a coherent sheaf on $T^*\mathfrak{g} \times \mathfrak{g} \times t\times \mathfrak{g}$ supported on $\Lambda$.

We consider diagram (4.1.1) in the case where $X = \mathcal{B}$ and $Y = \mathfrak{g} \times t$.

Theorem 4.4.1. All nonzero cohomology sheaves of the object $R\text{pr}_* L^i_\mathcal{B} \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \times t \to T^*\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \times t \mathcal{O}_\Lambda \in D^b_{\text{coh}}(\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \times t)$ vanish and, we have

$$\mathcal{H}^0(R\text{pr}_* L^i_\mathcal{B} \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \times t \to T^*\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \times t \mathcal{O}_\Lambda) = \mathcal{H}^i_{\text{Hodge}} \mathcal{M}.$$ 

Proof. In section 4.3 we have introduced the right $\mathcal{D}_g$-module $E = \int_f^R K^{\alpha \mathfrak{g}}$. This module comes equipped with the natural structure of a polarized Hodge module. One has $\mathcal{H}^i_{\text{Hodge}} E = q^*_\Lambda(e, K^{\alpha \mathfrak{g}})$, where $q_\Lambda : \Lambda \to T^*(\mathcal{B} \times \mathfrak{g} \times t) \to \mathcal{B} \times \mathfrak{g} \times t$ denotes the composite map. Let $M := K^{-1}_{\mathfrak{g} \times t} \otimes E$, be the corresponding left $\mathcal{D}_g \times \mathfrak{g} \times t$-module. Since the canonical bundle on $\mathfrak{g}$, resp. on $\mathfrak{g} \times t$, is trivial we find

$$K^{\mathfrak{g}} \otimes L^i(\mathcal{H}^j_{\text{Hodge}} M) = K^{\mathfrak{g}} \otimes L^i q^*_\Lambda(e, K^{\alpha \mathfrak{g}} \otimes K^{-1}_{\mathfrak{g} \times \mathfrak{g} \times t}) = K^{\mathfrak{g}} \otimes L^i q^*_\Lambda(K^{\mathfrak{g}} \otimes \mathcal{O}_{\mathfrak{g} \times t}) = L^i \mathcal{O}_\Lambda,$$

where we have used simplified notation $K^{\mathfrak{g}} \otimes (\mathfrak{g}) = (K^{\mathfrak{g}} \otimes \mathcal{O}_{\mathfrak{g} \times t}) \otimes \mathcal{O}_{\mathfrak{g} \times t} \otimes \mathfrak{g}$ and $\nu := t_{\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \times t} = (K^{\mathfrak{g}} \otimes \mathcal{O}_{\mathfrak{g} \times t}) \otimes \mathcal{O}_{\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \times t}$. From the above isomorphisms, applying Theorem 4.1.3 to the left $\mathcal{D}_g \times \mathfrak{g} \times t$-module $M$, we get $\mathcal{D}_{\mathfrak{g} \times \mathfrak{g} \times t}(\int_f^R M) = R\text{pr}_* L^i \mathcal{O}_\Lambda$. Thus, since $\int_{\mu \times \nu} \mathcal{O}_g = \int_f(\int_f \mathcal{O}_g) = \int_f M$, we deduce $\mathcal{D}_{\mathfrak{g} \times \mathfrak{g} \times t}(\int_{\mu \times \nu} \mathcal{O}_g) = R\text{pr}_* L^i \mathcal{O}_\Lambda$.

Now, the sheaf $\mathcal{O}_g$ has the natural structure of a polarized Hodge left $\mathcal{D}_g$-module. Thanks to Theorem 4.1.3, we obtain an isomorphism $\mathcal{H}^i_{\text{Hodge}} \mathcal{H}^j(\int_{\mu \times \nu} \mathcal{O}_g) = \mathcal{H}^j(R\text{pr}_* L^i \mathcal{O}_\Lambda)$, for any $j \in \mathbb{Z}$. Theorem 4.2.1 completes the proof. 

□
5. Completing the proof of Theorem 5.1.1

5.1. Symplectic geometry interpretation. Let \( \Phi : \tilde{N} \simeq T^*\mathcal{B} \) be an isomorphism obtained by composing the natural \( G \)-equivariant vector bundle isomorphism \( \tilde{N} \cong T^*\mathcal{B} \), used in 5.4 with the sign involution along the fibers of the vector bundle \( T^*\mathcal{B} \).

Recall the setup and notation of 1.3. The following result gives an interpretation of the variety \( \tilde{\mathcal{G}} \) in terms of symplectic geometry.

**Proposition 5.1.1.** The map \( \Psi = (\Phi \circ \kappa) \times \mu \times \nu : \tilde{\mathcal{G}} \to T^*\mathcal{B} \times \mathcal{G} \times \Sigma \) gives an isomorphism of \( \mathcal{G} \) with the variety \( \Lambda \subset T^*\mathcal{B} \times \mathcal{G} \times \Sigma \), the total space of the conormal bundle to the subvariety \( e(\mathcal{G}) \).

**Proof.** Fix \( b \in \mathcal{B} \) and \( x \in \mathcal{B} \). So, \( (b, x) \in \tilde{\mathcal{G}} \) and the corresponding point in \( \mathcal{B} \times \mathcal{G} \times \mathfrak{t} \) is given by the triple \( u = (b, x, x \mod [b, b]) \). The fiber of the vector bundle \( T(\mathcal{B} \times \mathcal{G} \times \mathfrak{t}) \) at the point \( u \) may be identified with the vector space \( (\mathcal{G}/\mathcal{B}) \times \mathfrak{t} \). It is easy to verify that the fiber of the sub vector bundle \( T(\mathcal{G}) \) at \( u \) is equal to the following subspace
\[
\{ (\alpha \mod b, [\alpha, x] + \beta, \beta \mod [b, b]) \in (\mathcal{G}/\mathcal{B}) \times \mathfrak{t} \mid \alpha \in \mathcal{G}, \beta \in b \}.
\]

Now, write \( (-,-) \) for an invariant bilinear form on \( \mathcal{G} \) and use it to identify the fiber of the vector bundle \( T^*(\mathcal{B} \times \mathcal{G} \times \mathfrak{t}) \) at \( u \) with the vector space \([b, b] \times \mathfrak{t} \). Let \((n, y, h) \in [b, b] \times \mathfrak{t} \) be a point of that vector space. Such a point belongs to the fiber of \( \Lambda \) over \( u \) if and only if the following equation holds
\[
\langle \alpha, n \rangle + \langle [\alpha, x] + \beta, y \rangle + \langle \beta \mod [b, b], h \rangle = 0 \quad \forall \alpha \in \mathcal{G}, \beta \in b. \tag{5.1.2}
\]

Taking \( \alpha = 0 \) and applying equation \( (5.1.2) \) we get \( \langle [b, b], y \rangle = 0 \). Hence, \( y \in b \) and \( h = \beta \mod [b, b] \). Next, for any \( \alpha \in \mathcal{G} \), we have \( \langle [\alpha, x], y \rangle = \langle \alpha, [x, y] \rangle \). Hence, for \( \beta = 0 \) and any \( n \in [b, b], y \in b \), equation \( (5.1.2) \) gives
\[
0 = \langle \alpha, n \rangle + \langle [\alpha, x], y \rangle = \langle \alpha, n + [x, y] \rangle \quad \forall \alpha \in \mathcal{G}.
\]

It follows that \( n + [x, y] = 0 \). We conclude that the fiber of \( \Lambda \) over \( u \) is equal to
\[
\{ (-[x, y], y, y \mod [b, b]) \in [b, b] \times \mathfrak{t} \mid y \in b \}. \tag{5.1.3}
\]

We have a projection \( pr_{12} : \tilde{\mathcal{G}} \to \tilde{\mathcal{G}}, (b, x, y) \to (b, x) \), along the last factor. We see that the vector space \( (5.1.3) \) is nothing but the image of set \( pr_{12}^{-1}(b, x) \) under the map \( \Psi \). This proves that the latter map gives an isomorphism \( \mathcal{G} \to \Lambda \) as well as the commutativity of the diagram of the proposition. \( \square \)

Let \( pr_{\Lambda \to T^*\mathcal{B}} : \Lambda \to T^*\mathcal{B} \), resp. \( pr_{\Lambda \to \mathcal{G} \times \Sigma} : \Lambda \to \mathcal{G} \times \Sigma \), denote the restriction to \( \Lambda \) of the projection of \( T^*\mathcal{B} \times \mathcal{G} \times \Sigma \) to the first, resp. along the first, factor. By the definition \( \Psi = (\Phi \circ \kappa) \times \mu \times \nu \), of the map \( \Psi \), one has a commutative diagram
\[
\begin{array}{ccc}
\mathcal{G} \times \Sigma & \xleftarrow{\mu \times \nu} & \tilde{\mathcal{G}} \xrightarrow{\tilde{\kappa}} \tilde{N} \\
\downarrow {\text{Id}} & & \downarrow \Phi \\
\mathcal{G} \times \Sigma & \xleftarrow{pr_{\Lambda \to T^*\mathcal{B}}} & \Lambda \xrightarrow{pr_{\Lambda \to T^*\mathcal{B}}} T^*\mathcal{B}.
\end{array}
\tag{5.1.4}
\]

From this diagram, we deduce, by Proposition 5.1.1, that the map \( \Psi \) induces an isomorphism between the scheme \( \tilde{\mathcal{X}} = \tilde{\kappa}^{-1}(\nu(\mathcal{B})) \), see \( 1.3.4 \) and \( \Lambda \cap (\mathcal{B} \times \mathcal{G} \times \Sigma) = pr_{\Lambda \to T^*\mathcal{B}}^{-1}(\nu(\mathcal{B})) \).
5.2. Recall the open set \( \mathcal{C}^{tr} \subset \mathcal{C} \), see Definition 2.4.3. Thus, \( \tilde{\mathcal{X}}^{tr} := \mu^{-1}(\mathcal{C}^{tr}) \) is a Zariski open subset of \( \tilde{\mathcal{X}} \), resp. \( \mathcal{X}^{tr} \) is a Zariski open subset of \( \mathcal{C} \times_{\mathcal{C} \times G} \mathcal{X} \), and we have

**Lemma 5.2.1.**

(i) The differential of the morphism \( \tilde{\kappa} : \tilde{\mathcal{O}} \to \tilde{N} \) is surjective at any point \((b, x, y) \in \tilde{\mathcal{X}}^{tr} \); in particular, the set \( \tilde{\mathcal{X}}^{tr} \) is contained in the smooth locus of the scheme \( \tilde{\mathcal{X}} \).

(ii) The set \( \mathcal{X}^{tr} \) is contained in the smooth locus of the variety \( \mathcal{X} \).

**Proof.** Let \( x \in b \) and write \( \text{ad}_b x \) for the map \( b \mapsto [b, b], u \mapsto [u, x] \). We have \( \dim b - \dim g_x \leq \dim b - \dim \ker(\text{ad}_b x) = \dim \text{Im}(\text{ad}_b x) \leq \dim[\mathcal{X}^t, \mathcal{X}] \). If \( x \in \mathcal{X}^t \), then we have \( \dim g_x = \dim t \) and from the above inequalities we deduce \( \dim[\mathcal{X}^t, \mathcal{X}] \). It follows that one has \( \text{Im}(\text{ad}_b x) = [b, b] \). Thus, for \( x \in \mathcal{X}^t \), the map \( \text{ad}_b x \) is surjective.

Next, let \( y \in b \) be another element. The differential of the commutator map \( \kappa : b \times b \to [b, b] \) at the point \((x, y) \in b \times b \) is a linear map \( d_{b, x, y} \kappa : b \oplus b \to [b, b] \) given by the formula \( d_{b, x, y} \kappa : (u, v) \mapsto \text{ad}_b x(u) + \text{ad}_b y(v) \). In particular, we see that \( \text{Im}(\text{ad}_b x) \subseteq \text{Im}(d_{b, x, y} \kappa) \).

Now, let \((b, x, y) \in \tilde{\mathcal{X}}^{tr} \). Without loss of generality, we may assume that \( x \) is a regular element of \( \mathcal{X} \). Thus, the map \( \text{ad}_b x \) is surjective. By the preceding paragraph, we deduce that the map \( d_{b, x, y} \kappa \) is surjective as well. Part (i) follows from this by \( G \)-equivariance.

To prove (ii), we use Lemma 1.2.4 to identify \( \mathcal{X} \) with \( N_{\mathcal{X}} \). Let \( g : N_{\mathcal{X}} \to \mathcal{X} \) denote the projection. It is clear that the set \( g^{-1}(r^t) = N_r \) is contained in the smooth locus of \( \mathcal{X} \).

To complete the proof of the lemma, write \( \mathcal{C}^{tr} = \mathcal{C}_1^{tr} \cup \mathcal{C}_2^{tr} \) where \( \mathcal{C}_1^{tr} \) is the set of pairs \((x_1, x_2) \in \mathcal{C} \) such that \( x_i \in \mathcal{X}^t \). It is immediate from Proposition 1.3.2(i) that one has \( p_{r^t}^{-1}(\mathcal{C}_1^{tr}) \subset g^{-1}(r^t) \). We conclude that the set \( p_{r^t}^{-1}(\mathcal{C}_1^{tr}) \) is contained in the smooth locus of \( \mathcal{X} \).

We may use Proposition 5.1.1 to identify the set \( \tilde{\mathcal{X}}^{tr} \subset \tilde{\mathcal{X}} \subset \tilde{\mathcal{O}} \) with a subset of the set \( \Lambda \cap (\mathcal{B} \times \mathcal{G} \times \mathcal{X}) \subset T^*\mathcal{B} \times \mathcal{G} \times \mathcal{X} \). Let \( \pi \) denote the restriction of the proper morphism \( \mu \times \nu \) to the subset \( \tilde{\mathcal{X}}^{tr} \).

**Corollary 5.2.2.**

(i) The intersection \( \Lambda \cap (\mathcal{B} \times \mathcal{G} \times \mathcal{X}) \) is transverse at any point of the set \( \tilde{\mathcal{X}}^{tr} \).

(ii) The map \( \pi \) induces an isomorphism \( \tilde{\mathcal{X}}^{tr} \cong \mathcal{X}^{tr} \).

**Proof.** Part (i) is equivalent to the statement of Lemma 5.2.1(i). To prove (ii), let \( \mathcal{U} \) denote the preimage of \( \mathcal{C}^{tr} \) under the first projection \( \mathcal{C} \times_{\mathcal{C} \times G} \mathcal{X} \to \mathcal{C} \). It is clear that the morphism \( \pi \) maps \( \tilde{\mathcal{X}}^{tr} \) to \( \mathcal{U} \). We claim that the resulting map \( \pi : \tilde{\mathcal{X}}^{tr} \to \mathcal{U} \) is a set theoretic bijection. Indeed, it is surjective, since the image of this map contains the set \( \mathcal{C}^{tr} \subset \mathcal{C} \times_{\mathcal{C} \times G} \mathcal{X} \) and \( \pi \) is a proper morphism. To prove injectivity, we interpret the assignment \((b, x, y) \mapsto (x, y) \) as the map \( \pi \times \text{Id} : g \times \mathcal{G} \to \mathcal{X} \). This last map gives a bijection between the set \( g \times \mathcal{G} \) and its preimage in \( \tilde{\mathcal{X}}^{tr} \), thanks to Proposition 1.3.2(i). Our claim follows.

Next, we observe that since \( \tilde{\mathcal{X}}^{tr} \) is a smooth scheme by Lemma 5.2.1(i) the scheme theoretic image of \( \tilde{\mathcal{X}}^{tr} \) under the morphism \( \pi \) is actually contained in \( \mathcal{X}^{tr} = U_{red} \). The reduced scheme \( \mathcal{X}^{tr} \) is smooth by Lemma 2.4.3. Thus, we have a proper morphism \( \pi : \tilde{\mathcal{X}}^{tr} \to \mathcal{X}^{tr} \), between smooth varieties, which is a set theoretic bijection. Such a morphism is necessarily an isomorphism, and part (ii) follows.

There is an alternative proof that the morphism \( \pi : \tilde{\mathcal{X}}^{tr} \to \mathcal{X}^{tr} \) is étale based on symplectic geometry. In more detail, put \( X = \mathcal{B} \) and \( Y = \mathcal{G} \times \mathcal{X} \) and let \( \epsilon : \mathcal{G} \to X \times Y \) be the imbedding used in 4.3. We have smooth locally closed subvarieties \( r^t \subset \mathcal{G} \times \mathcal{X} \) and \( Z := \epsilon(r^{-1}(r^t)) \subset X \times Y \), respectively. Using the notation of the proof of Lemma 5.2.1(ii), we can write \( p_{r}^{-1}(\mathcal{C}_1^{tr}) \subset q^{-1}(r^t) = N_{r^t} \). Further, we may use Proposition 5.1.1 to identify \( \mathcal{U} \) with an open subset of \( N_{\mathcal{X}} \), the total space of the conormal bundle on \( Z \).
We know that the projection $X \times Y \to Y$ induces an isomorphism $Z \to r^\ast$, by Proposition 1.3.2(i). Now, one can prove a general result saying that, in this case, the map $N_Z \to N_Y$, induced by the projection $T^\ast \times T^\ast Y \to T^\ast Y$, is étale at any point where $N_Z$ meets the subvariety $X \times T^\ast Y \subset T^\ast (X \times Y)$ transversely.

To complete the proof we observe that this transversality condition holds for any point of the set $\mu^{-1}(c'_i) \subset N_Z$, thanks to part (i) of the Corollary.

\[ \square \]

5.3. Let $A, C, C'$ be a triple of commutative algebras. Given a pair $A \to C$ and $A \to C'$, of algebra homomorphisms, one may view $C$ and $C'$ as $A$-algebras. Therefore, the tensor product $C \otimes_A C'$ has the natural structure of a commutative $A$-algebra. One can also form $C \otimes_A C'$, a derived tensor product. The latter may be viewed as a commutative DG algebra which is concentrated in nonpositive degrees and has differential of degree +1. This DG algebra is well defined up to homotopy, cf. eg. [Hi] for details. By definition, one has $H^{-\ast}(C \otimes_A C') = \text{Tor}_A^C(C, C')$, where we use the cohomological, rather than homological, grading on the left hand side. The Tor-group above is a graded commutative algebra with degree zero component being equal to $C \otimes_A C'$.

The above construction localizes. Thus, given a smooth variety $X$ and a pair $\mathcal{E}_X$ and $\mathcal{E}'_X$, of coherent $O_X$-algebras, one has a sheaf $\mathcal{E}_X \otimes O_X \mathcal{E}'_X$, of DG $O_X$-algebras, well defined up to quasi-isomorphism.

Now, let $i_Y : Y \hookrightarrow X$, resp. $i_Z : Z \hookrightarrow X$, be closed embeddings of smooth subvarieties. It is known that, in $D^b_{\text{coh}}(X)$, there are natural isomorphisms

\[ (i_Y)_! L_i^\ast_Y[(i_Z)_! O_Z] \cong (i_Y)_! O_Y \otimes_{O_X} (i_Z)_! O_Z \cong (i_Z)_! L_i^\ast_Y[(i_Y)_! O_Y]. \]  

(5.3.1)

We apply the discussion above in the case where $\mathcal{E}_X = (i_Y)_! O_Y$ and $\mathcal{E}'_X = (i_Z)_! O_Z$. Thus, the object at the middle of formula (5.3.1) may be viewed as a DG $O_X$-algebra. There are, in fact, DG $O_X$-algebra structures on two other objects in (5.3.1) as well (the DG algebra $\mathcal{O}'$ from (1.4) is an example of this). Furthermore, the isomorphisms in (5.3.1) are DG $O_X$-algebra quasi isomorphisms.

We are interested in the special case where $X = T^\ast \mathcal{B} \times \mathcal{B} \times \Sigma$. Write $pr_{T^\ast \mathcal{B}}$, resp. $pr_{\mathcal{B} \times \Sigma}$ for the projection of the variety $T^\ast \mathcal{B} \times \mathcal{B} \times \Sigma$ to the first, resp. along the first, factor. We will identify $\mathcal{B}$ with a subvariety of $T^\ast \mathcal{B}$ via the zero section $\iota : \mathcal{B} \hookrightarrow T^\ast \mathcal{B}$.

Now, in the setting of diagram (5.3.1), we put $Y := \Lambda$ and $Z := \mathcal{B} \times \mathcal{B} \times \Sigma$. To simplify notation, we will identify the structure sheaf $O_Y$, resp. $O_Z$, with the corresponding $O_X$-module $(i_Y)_! O_Y$, resp. $(i_Z)_! O_Z$. Using the notation of diagram (5.1.4), we can write $pr_{\Lambda \to T^\ast \mathcal{B}} = pr_{T^\ast \mathcal{B} \times \mathcal{B} \times \Sigma} \circ \iota$. We deduce a chain of DG algebra quasi-isomorphisms

\[ L_i^\ast (O_{\mathcal{B} \times \mathcal{B} \times \Sigma}) = L_i^\ast (pr_{T^\ast \mathcal{B} \times \mathcal{B} \times \Sigma} \circ \iota_\ast (O_{\mathcal{B} \times \mathcal{B} \times \Sigma}) = L (pr_{T^\ast \mathcal{B} \times \mathcal{B} \times \Sigma} \circ \iota_\ast (O_{\mathcal{B} \times \mathcal{B} \times \Sigma})) = L (pr_{\Lambda \to T^\ast \mathcal{B} \times \mathcal{B} \times \Sigma} \circ \iota_\ast (O_{\mathcal{B} \times \mathcal{B} \times \Sigma})). \]

Next, we use the isomorphism $\widetilde{\Phi} : \widetilde{N} \to T^\ast \mathcal{B}$ to identify the zero section $\iota : \mathcal{B} \hookrightarrow \widetilde{N}$ with the zero section $\iota : \mathcal{B} \hookrightarrow T^\ast \mathcal{B}$. Thus, from the above chain of isomorphisms, by commutativity of diagram (5.1.4), we get

\[ R (pr_{\Lambda \to \mathcal{B} \times \Sigma})_! L_i^\ast (O_{\mathcal{B} \times \mathcal{B} \times \Sigma}) = R (pr_{\Lambda \to \mathcal{B} \times \Sigma})_! L (pr_{\Lambda \to T^\ast \mathcal{B} \times \mathcal{B} \times \Sigma} \circ \iota_\ast (O_{\mathcal{B} \times \mathcal{B} \times \Sigma})). \]

(5.3.2)

Using the notation of Theorem 4.4.1, we can write $i_Z = i_{\mathcal{B} \times \mathcal{B} \times \Sigma \to T^\ast \mathcal{B} \times \mathcal{B} \times \Sigma}$, resp. $pr_{\mathcal{B} \times \Sigma} = pr$. Now, we apply the functor $(R pr_{\mathcal{B} \times \Sigma})_! (\ast)$ to each of the 3 objects of our diagram (5.3.1). Combining the isomorphisms in (5.3.1) with (5.3.2), we obtain the following DG algebra
quasi-isomorphisms

\[ R\mu_* L\tilde{\kappa}^*(i_*\mathcal{O}_\mathcal{G}) \cong R(pr_{\mathcal{G} \times \mathcal{T}})_*(\mathcal{O}_{\mathcal{G} \times \mathcal{T}} \otimes_{\mathcal{O}_{T^* \mathcal{G} \times \mathcal{T}}}^{L} \mathcal{O}_\Lambda) \cong R\text{pr}_* L\tilde{\iota}_* \mathcal{O}_{\mathcal{G} \times \mathcal{T}} \to_{T^* \mathcal{G} \times \mathcal{T}} \mathcal{O}_\Lambda. \]

Note that \( L\tilde{\kappa}^*(i_*\mathcal{O}_\mathcal{G}) = \mathcal{A}' \) is the DG algebra considered in [1.4.1]. Thus, Theorem 1.6.2 follows from the above isomorphisms and Theorem 4.4.1 cf. also §5.5.

5.4. Proof of Lemma 2.3.3 First of all, we specify our normalization of the Hodge filtration on the Harish-Chandra module \( \mathcal{M} \). To this end, we recall the setting of §4.3. Thus, we have a simple holonomic right \( \mathcal{D}_{\mathcal{G} \times \mathcal{T}} \)-module \( E \), with support \( \epsilon(\mathfrak{g}) \), and a surjective homomorphism \( \gamma : \mathcal{D}_{\mathcal{G} \times \mathcal{T}} \to E \), of right \( \mathcal{D}_{\mathcal{G} \times \mathcal{T}} \)-modules. The order filtration on \( \mathcal{D}_{\mathcal{G} \times \mathcal{T}} \), induces, via the projection \( \gamma \), a quotient filtration on \( E \). The resulting filtration is known to be equal to the Hodge filtration on \( E \) up to a shift, since \( \epsilon(\mathfrak{g}) \) is a smooth submanifold in \( \mathcal{G} \times \mathfrak{g} \times \mathfrak{t} \). We choose the normalization of the Hodge filtration on \( E \) so that the two filtrations coincide.

The filtration on \( E \), resp. on \( \mathcal{D}_{\mathcal{G} \times \mathfrak{g} \times \mathfrak{t}} \), makes \( \int_R E \), resp. \( \int_R \mathcal{D}_{\mathcal{G} \times \mathfrak{g} \times \mathfrak{t}} \), a filtered complex. We define the Hodge filtration on \( \mathcal{M} \) to be the filtration that comes from the induced filtration on \( \mathcal{H}^0(\int_R E) \) via the isomorphism \( \mathcal{M} = K^{-1}_{\mathfrak{g} \times \mathfrak{t}} \otimes \mathcal{H}^0(\int_R E) \), cf. (4.3.2).

Proof of part (i). With the above choice of normalization, the map \( \gamma \) becomes a morphism of filtered \( \mathcal{D} \)-modules. It follows that the induced morphism \( \int_R \gamma : \int_R \mathcal{D}_{\mathcal{G} \times \mathfrak{g} \times \mathfrak{t}} \to \int_R E \) is a morphism of filtered complexes.

The proof of Lemma 4.3.1 shows that the filtration on the \( \mathcal{D} \)-module \( \mathcal{H}^0(\int_R \mathcal{D}_{\mathcal{G} \times \mathfrak{g} \times \mathfrak{t}}) \), induced by the filtered structure on \( \int_R \mathcal{D}_{\mathcal{G} \times \mathfrak{g} \times \mathfrak{t}} \), goes, under the isomorphism of the Lemma, to the standard order filtration on the sheaf \( \mathcal{D}_{\mathcal{G} \times \mathfrak{t}} \). It follows that all the maps in (4.3.2) are filtration preserving. Thus, writing \( \tilde{\gamma} : \mathcal{G} \times \mathfrak{t} \to \mathcal{M} \) for the composite map in (4.3.2), we get \( \tilde{\gamma}(\mathcal{D}^\ord_{\mathcal{G} \times \mathfrak{t}}) \subset \mathcal{E}^\Hodge_{\mathcal{M}} \), for any \( k \in \mathbb{Z} \). Now, according to the remark after formula (4.3.2), the map \( \tilde{\gamma} \) is the natural projection \( \mathcal{G} \times \mathfrak{t} \to \mathcal{G} \times \mathfrak{t}/\mathcal{I} = \mathcal{M} \). This yields the inclusions of part (i) of Lemma 2.3.3 since the order filtration \( \mathcal{F}^\ord \mathcal{M} \), on the Harish-Chandra module, was defined as the quotient filtration on \( \mathcal{G} \times \mathfrak{t}/\mathcal{I} \).

Proof of part (ii). Recall the setup of the proof of Theorem 4.4.1 and introduce simplified notation \( \iota := \iota_{\mathcal{G} \times \mathfrak{g} \times \mathcal{T}} \to_{T^* \mathcal{G} \times \mathcal{T}} \mathcal{O}_{\mathcal{G} \times \mathcal{T}} \). The DG algebra structure on the object on the right of (5.3.3) induces an \( \mathcal{O}_{\mathcal{G} \times \mathcal{T}} \)-algebra structure on \( \mathcal{H}^0(R\text{pr}_* L\iota^* \mathcal{O}_\Lambda) \), the 0-th cohomology sheaf. The \( \mathcal{O}_{\mathcal{G} \times \mathcal{T}} \)-algebra structure on \( \mathcal{L}^\Hodge_{\mathcal{M}} \) referred to in part (ii) of Lemma 2.3.3 is induced by that on \( \mathcal{H}^0(R\text{pr}_* L\iota^* \mathcal{O}_\Lambda) \) via the isomorphism of Theorem 4.4.1.

Observe next that we have \( R\text{pr}_* L\iota^* \mathcal{O}_{T^* \mathcal{G} \times \mathfrak{g} \times \mathcal{T}} = R\text{pr}_* (\mathcal{O}_{\mathcal{G}} \boxtimes \mathcal{O}_{\mathcal{G} \times \mathcal{T}}) = \mathcal{O}_{\mathcal{G} \times \mathcal{T}} \), since \( H^k(\mathcal{D}, \mathcal{O}_\mathcal{G}) = 0 \) for all \( k > 0 \), cf. proof of Lemma 4.3.1. One may upgrade the above to a quasi-isomorphism \( \mathcal{O}_{\mathcal{G} \times \mathcal{T}} \xrightarrow{\sim} R\text{pr}_* L\iota^* \mathcal{O}_{T^* \mathcal{G} \times \mathfrak{g} \times \mathcal{T}} \), of DG algebras. Thus, applying the functor \( R\text{pr}_* L\iota^*(-) \) to an obvious restriction morphism \( \mathcal{O}_{T^* \mathcal{G} \times \mathfrak{g} \times \mathcal{T}} \to \mathcal{O}_{\mathcal{G} \times \mathcal{T}} \) yields DG algebra morphisms

\[ \mathcal{O}_{\mathcal{G} \times \mathcal{T}} \xrightarrow{\sim} R\text{pr}_* L\iota^* \mathcal{O}_{T^* \mathcal{G} \times \mathfrak{g} \times \mathcal{T}} \to R\text{pr}_* L\iota^* \mathcal{O}_{\mathcal{G} \times \mathcal{T}}. \]  

(5.4.1)

Applying further the functor \( \mathcal{H}^0(\cdot) \) we obtain a chain of \( \mathcal{O}_{\mathcal{G} \times \mathcal{T}} \)-algebra morphisms

\[ u : \mathcal{O}_{\mathcal{G} \times \mathcal{T}} \xrightarrow{\sim} \mathcal{H}^0(R\text{pr}_* L\iota^* \mathcal{O}_{T^* \mathcal{G} \times \mathfrak{g} \times \mathcal{T}}) \to \mathcal{H}^0(R\text{pr}_* L\iota^* \mathcal{O}_{\mathcal{G} \times \mathcal{T}}) \to \mathcal{H}^0(R\text{pr}_* L\iota^* \mathcal{O}_{\mathcal{G} \times \mathcal{T}}) \to \mathcal{L}^\Hodge_{\mathcal{M}}, \]  

(5.4.2)

where the last map is an algebra morphism thanks to our definition of the algebra structure on \( \mathcal{L}^\Hodge_{\mathcal{M}} \).
Let $u$ be the composite morphism in (5.4.2). Further, let $u_{\text{lem}}$ denote the composite morphism involved in the statement of Lemma 2.3.3(ii) and let $\text{res}: O_{\mathfrak{g} \times \mathfrak{T}} \to O_{\mathfrak{g} \times \mathfrak{T}/O_{\mathfrak{T}}}$ be the natural restriction map. It is not hard to see from the proof of part (i) of the lemma (repeat arguments in the proof of Theorem 1.4.4) in a simpler case where the $\mathcal{D}$-module $E$ is replaced by $\mathcal{D}_{\mathfrak{g} \times \mathfrak{T}}$ one has $u = u_{\text{lem}} \circ \text{res}$. We know that both $u$ and $\text{res}$ are algebra maps. It follows that $u_{\text{lem}}$ is an algebra map as well.

5.5. Proof of Lemma 2.4.6. We choose a Zariski open subset $U \subset \mathfrak{g} \times \mathfrak{T}$ such that one has $\mathfrak{X} \cap U = \mathfrak{X}^{rr}$. We claim that $\mathcal{M}(R\text{pr}_* L^i \mathcal{O}_\Lambda)|_U = 0$ for all $k \neq 0$ and, moreover, the restriction of the composite morphism in (5.4.1) to the set $U$ yields a quasi-isomorphism

\[ O_{\mathfrak{X}^{rr}} = O_{\mathfrak{g} \times \mathfrak{T}}|_{\mathfrak{X}^{rr}} \xrightarrow{\sim} \mathcal{M}(R\text{pr}_* L^i \mathcal{O}_\Lambda)|_U. \]  

(5.5.1)

To prove the claim, recall the setting of formula (5.3.3) and the DG algebra $\mathcal{A}^r$, see [1.4]. Let $\tilde{U} := \mu^{-1}(U) \subset \mathfrak{g}$, and put $\tilde{\mathfrak{X}}^{rr} := \mu^{-1}(\mathfrak{X}^{rr}) = \mathfrak{X} \cap \tilde{U}$. Further, view $\mathcal{M}^0 = O_{\mathfrak{g}}$ as a DG algebra with zero differential. The transversality result from part (i) of Corollary 5.2.2 implies that we have $\mathcal{M}^k(L\text{R}^*(\iota_* O_{\mathfrak{g}}))|_{\tilde{U}} = 0$, $\forall k > 0$ and, moreover, that the morphism $O_{\tilde{U}} \to L\text{R}^*(\iota_* O_{\mathfrak{g}})|_{\tilde{U}}$ induced by the DG algebra imbedding $\mathcal{M}^0|_{\tilde{U}} \hookrightarrow \mathcal{A}^r|_{\tilde{U}}$ descends to a quasi-isomorphism $O_{\mathfrak{X}^{rr}} \xrightarrow{\sim} L\text{R}^*(\iota_* O_{\mathfrak{g}})|_{\tilde{U}}$.

Let $\pi := (\mu \times \nu)|_{\tilde{U}}$. We apply the functor $R\pi_*$ to the above quasi-isomorphism and use Corollary 5.2.2(ii). We deduce that the composite of canonical morphisms $O_{\tilde{U}} \to R\pi_* O_{\tilde{U}} \to R\pi_*(L\text{R}^*(\iota_* O_{\mathfrak{g}}))|_{\tilde{U}}$ descends to a quasi-isomorphism $O_{\mathfrak{X}^{rr}} \xrightarrow{\sim} (R\pi_* L^i \mathcal{O}_\Lambda)|_U$. Next, we use the isomorphism between the objects on the left and on the right of formula (5.3.3). Thus, we have $(R\text{pr}_* L^i \mathcal{O}_\Lambda)|_U \cong (R\pi_* L^i \mathcal{O}_\Lambda)|_U$. By the conclusion of the previous paragraph, we obtain that the canonical map $w$ : $O_{\tilde{U}} \to (R\text{pr}_* L^i \mathcal{O}_\Lambda)|_U$ descends to a quasi-isomorphism $O_{\mathfrak{X}^{rr}} \xrightarrow{\sim} (R\pi_* L^i \mathcal{O}_\Lambda)|_U$. The morphism $w$ here corresponds, via the isomorphisms from (5.3.5), to the restriction to $U$ of the composite morphism in (5.4.1). It follows that the map $u$, in (5.4.2), descends to an isomorphism $u|_U : O_{\mathfrak{X}^{rr}} \xrightarrow{\sim} \mathcal{M}(R\text{pr}_* L^i \mathcal{O}_\Lambda)|_U$. This proves our claim that (5.5.1) is an isomorphism.

Now, the proof of Lemma 2.3.3(ii) shows that the map $X = \text{Spec}(\mathcal{O}_{\mathfrak{g}}^{\text{Hodge}} \mathcal{M}) \to \mathfrak{g} \times \mathfrak{T}$ is induced by the $O_{\mathfrak{g} \times \mathfrak{T}}$-algebra morphism $u$, in (5.4.2). The above map factors through the map $f : X \to \mathfrak{X}$. Thus, the map $f : f^{-1}(\mathfrak{X}^{rr}) \to \mathfrak{X}^{rr}$ may be identified with the map induced by the composite algebra morphism $u|_U : O_{\mathfrak{X}^{rr}} \to \mathcal{M}(R\text{pr}_* L^i \mathcal{O}_\Lambda)|_U \to \mathcal{O}_{\mathfrak{g}}^{\text{Hodge}} \mathcal{M}|_U$. Formula (5.5.3) says that the latter morphism is an isomorphism, completing the proof of Lemma 2.4.6.

6. Some technical results

6.1. We write $x = h_x + n_x$ for the Jordan decomposition of an element $x \in \mathfrak{g}$. We say that $(x, y) \in \mathfrak{g}$ is a semisimple pair if both $x$ and $y$ are semisimple elements of $\mathfrak{g}$.

Lemma 6.1.1. (i) For $(x, y) \in \mathfrak{g}$, the following conditions are equivalent

- There exists a semisimple pair $(h_1, h_2) \in \mathfrak{T}$ such that, for any polynomial $f \in \mathbb{C}[\mathfrak{g}]^{G}$, we have $f(x, y) = f(h_1, h_2)$.
- There exists a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ such that $x, y \in \mathfrak{b}$.

(ii) If $(x, y) \in \mathfrak{c}$ then the $G$-diagonal orbit of the pair $(h_x, h_y)$ is the unique closed $G$-orbit contained in the closure of the $G$-diagonal orbit of $(x, y)$.

Proof. Let $t \subset \mathfrak{b}$ be Cartan and Borel subalgebras of $\mathfrak{g}$ and let $T$ be the maximal torus corresponding to $t$. Clearly, there exists a suitable one parameter subgroup $\gamma : \mathbb{C}^\times \to T$ such that, for any $t \in T$ and $n \in [\mathfrak{b}, \mathfrak{b}]$, one has $\lim_{n \to 0} \text{Ad}(\gamma(t))(t + n) = t$.
To prove (i), let \((x, y) \in \mathfrak{G}\). We can write \(x = h_1 + n_1, \ y = h_2 + n_2\), where \(h_i \in \mathfrak{t}\) and \(n_i \in [\mathfrak{b}, \mathfrak{b}]\). We see from the above that the pair \((h_1, h_2)\) is contained in the closure of the \(G\)-orbit of the pair \((x, y)\) hence, for any \(f \in \mathbb{C}[\mathfrak{G}]^G\), we have \(f(x, y) = f(h_1, h_2)\).

Conversely, let \((x, y) \in \mathfrak{G}\) be such that \(f(x, y) = f(h_1, h_2)\) holds for any \(f \in \mathbb{C}[\mathfrak{G}]^G\). The group \(G_{h_1, h_2}\) is a reductive subgroup of \(G\). Hence, the \(G\)-diagonal orbit of \((h_1, h_2)\) is contained in \(\mathfrak{G}\), cf. e.g. [COV]. It follows that this \(G\)-orbit is the unique closed orbit that is contained in the closure of the \(G\)-diagonal of the pair \((x, y)\), since \(G\)-invariant polynomials on \(\mathfrak{G}\) separate closed \(G\)-orbits. By the Hilbert-Mumford criterion, we deduce that there exists a suitable one parameter subgroup \(\gamma : \mathbb{C}^* \to G\) such that, conjugating the pair \((h_1, h_2)\) if necessary, on gets \(\lim_{t \to 0} \text{Ad} \gamma(z)(x, y) = (h_1, h_2)\).

Now, let \(a\) be the Lie subalgebra of \(\mathfrak{g}\) generated by the elements \(x\) and \(y\). We deduce from the above that, for any \(u \in [a, a]\), one has \(\lim_{t \to 0} \text{Ad} \gamma(z)(u) = 0\). This implies that \(u\) is a nilpotent element of \(\mathfrak{g}\). Thus, \([a, a]\) is a nilpotent Lie algebra, by Engel’s theorem. We conclude that \(a\) is a solvable Lie algebra. Hence, there exists a Borel subalgebra \(\mathfrak{b}\) such that \(a \subset \mathfrak{b}\), and (i) is proved.

To prove (ii), we observe that the elements \(h_x, h_y, n_x, n_y\) generate an abelian Lie subalgebra of \(\mathfrak{g}\). Hence, there exists a Borel subalgebra \(\mathfrak{b}\) such that we have \(h_x, h_y, n_x, n_y \in \mathfrak{b}\). We may choose a Cartan subalgebra \(\mathfrak{t} \subset \mathfrak{b}\) such that \(h_x, h_y \in \mathfrak{t}\). Then, the argument at the beginning of the proof shows that the pair \((h_x, h_y)\) is contained in the closure of the \(G\)-diagonal orbit of \((x, y)\).

\[\square\]

6.2. Below, we will have to consider several reductive Lie algebras at the same time. To avoid confusion, we write \(\mathbb{C}(\mathfrak{l})\) for the commuting scheme of a reductive Lie algebra \(\mathfrak{l}\), and use similar notation for other objects associated with \(\mathfrak{l}\).

Let \(\mathfrak{M}(\mathfrak{l})\) be the nilpotent commuting variety of \(\mathfrak{l}\), the variety of pairs of commuting nilpotent elements of \(\mathfrak{l}\), equipped with reduced scheme structure. It is clear that we have \(\mathfrak{M}(\mathfrak{l}) = \mathfrak{M}(\mathfrak{l}') \subset \mathfrak{l}' \times \mathfrak{l}'\), where \(\mathfrak{l}' :=[\mathfrak{l}, \mathfrak{l}]\) is the derived Lie algebra of the reductive Lie algebra \(\mathfrak{l}\). According to [77], the irreducible components of nilpotent commuting variety are parametrized by the conjugacy classes of distinguished nilpotent elements of \(\mathfrak{l}'\). The irreducible component corresponding to such a conjugacy class is equal to the closure in \(\mathfrak{l}' \times \mathfrak{l}' = T^\circ(\mathfrak{l}')\) of the total space of the conormal bundle on that conjugacy class. It follows, in particular, that the dimension of each irreducible component of the variety \(\mathfrak{M}(\mathfrak{l})\) equals \(\dim \mathfrak{l}'\).

Now, we fix a reductive connected group \(G\) with Lie algebra \(\mathfrak{g}\), and a Cartan subalgebra \(\mathfrak{t} \subset \mathfrak{g}\). Recall that the centralizer of an element of \(\mathfrak{t}\) is called a standard Levi subalgebra of \(\mathfrak{g}\). Let \(S\) be the set of standard Levi subalgebras. Given a standard Levi subalgebra \(\mathfrak{l}\), let \(t_1\) denote the center of \(\mathfrak{l}\), and let \(\mathfrak{S}_1 \subset \mathfrak{S}\), denote the set of elements \((t_1, t_2) \in \mathfrak{S}\) such that \(\mathfrak{g}_{t_1, t_2} = \mathfrak{l}\). This is clearly a Zariski open subset of \(\mathfrak{S} \times t_1\).

Recall the isospectral commuting variety \(\mathfrak{X}(\mathfrak{g})\) and the projection \(p_x : \mathfrak{X}(\mathfrak{g}) \to \mathfrak{S}\). For each standard Levi subalgebra \(\mathfrak{l}\), of \(\mathfrak{g}\), we put \(\mathfrak{X}_1(\mathfrak{g}) := (p_x)^{-1}(\mathfrak{S}_1)\), and view this set as a reduced scheme. Let \(L\) denote the Levi subgroup of \(G\) such that \(\mathfrak{l} = \text{Lie } L\).

**Lemma 6.2.1.** For any standard Levi subalgebra \(\mathfrak{l}\), the following assignment

\[
G \times \mathfrak{S} \times \mathfrak{S}_1 \ni g \times (y_1, y_2) \times (t_1, t_2) \mapsto \left( \text{Ad } g(y_1 + t_1), \text{Ad } g(y_2 + t_2) \right) \times (t_1, t_2) \in \mathfrak{S} \times \mathfrak{S}_1
\]

induces a \(G\)-equivariant isomorphism

\[
(G \times_L \mathfrak{M}(\mathfrak{l})) \times \mathfrak{S}_1 \cong \mathfrak{X}_1(\mathfrak{g}).
\]  

(6.2.2)
The map \( p_\alpha : \mathcal{X}_l(g) \to \tilde{\mathcal{X}}_l \) goes, under the isomorphism, to the second projection \((G \times L \text{Nil}(l)) \times \tilde{\mathcal{X}}_l \to \tilde{\mathcal{X}}_l \). Thus, all irreducible components of the set \( \mathcal{X}_l(g) \) have the same dimension and are in one-to-one correspondence with the distinguished nilpotent conjugacy classes in the Lie algebra \( l \).

**Proof.** It is clear that any commuting semisimple pair is \( G \)-conjugate to a pair of elements of \( g \). Lemma 6.1.1(ii) implies that, for any \((x_1, x_2, t_1, t_2) \in \mathcal{C}(g) \times \mathcal{C}(g)/G \mathcal{T} \) and any polynomial \( f \in \mathbb{C}[\mathcal{C}(g)]^G \), we have \( f(x_1, x_2) = f(h_{x_1}, h_{x_2}) \). We know that \( W \)-invariant polynomials separate \( W \)-orbits in \( \mathcal{T} \) and that the restriction map (1.1.2) is surjective. We deduce that the pair \((h_{x_1}, h_{x_2})\) is \( W \)-conjugate to the pair \((t_1, t_2)\). It follows that the pair \((x_1, x_2)\) is \( G \)-conjugate to a pair of the form \((t_1 + y_1, t_2 + y_2)\), for some \((y_1, y_2) \in \text{Nil}_l \). The isomorphism of the lemma easily follows from this.

We are now ready to complete the proof of Lemma 1.2.1.

**Corollary 6.2.3.** The set \( \mathcal{X}^{rs} \) is Zariski dense in \( \mathcal{X} \).

**Proof.** The set \( \mathcal{X}^{rs} \) is obviously dense in \( \mathcal{X} \), hence, the closure of the set \( \mathcal{C}^{rs}(g) \) contains \( \mathcal{C}^s(g) \), the set of all semisimple pairs in \( \mathcal{C}(g) \). It follows that the closure of the set \( \mathcal{X}^{rs}(g) \) in \( \mathcal{X} \) contains the set \( p_\alpha^{-1}(\mathcal{C}^s(g)) \). Thus, to prove the Corollary, it suffices to show that, for any \( w \in W \) and any \((t_1, t_2) \in \tilde{\mathcal{X}}_l \), the quadruples \((x_1, x_2, t_1, t_2) \in (p_\alpha)^{-1}(t_1, t_2) \) such \((x_1, x_2) \in \mathcal{C}^s(g) \) form a dense subset of the fiber \((p_\alpha)^{-1}(t_1, t_2)\).

To prove this last statement, we use the isomorphism of Lemma 6.2.1. We see from the isomorphism that it suffices to show that \( \text{Nil}(l) \), the nilpotent commuting variety of any standard Levi algebra \( l \), is contained in the closure of the set \( \mathcal{C}^s(l) \). But we have \( \mathcal{C}^s(l) \supseteq \mathcal{C}^{rs}(l) \) and the set \( \mathcal{C}^{rs}(l) \) is dense in \( \mathcal{C}(l) \), by Proposition 4.1.1(i); explicitly, this is Corollary 4.7 from [R1]. The result follows.

6.3. **Proof of Lemma 2.4.4**. Let \( \mathcal{C}_1^{rr} \), resp. \( \mathcal{C}_2^{rr} \), be the set of pairs \((x_1, x_2) \in \mathcal{C}(g) \) such that \( x_1 \), resp. \( x_2 \), is a regular element of \( g \). By Definition 2.4.3, we have \( \mathcal{C}^{rr} = \mathcal{C}_1^{rr} \cup \mathcal{C}_2^{rr} \).

**Lemma 6.3.1.** The set \( \mathcal{C}^{rr} \) is a Zariski open subset contained in the smooth locus of the scheme \( \mathcal{C} \).

**Proof.** For any \((x, y) \in \mathcal{C}_1^{rr} \), we have \( \dim(g_x \cap g_y) \leq \dim g_x = \dim t \), since \( x \) is a regular element. Hence \((x, y)\) is a smooth point of \( \mathcal{C} \), by Proposition 1.1.1(ii). Further, the set \( \mathcal{C}^{rr}_1 \) is the preimage of the set of regular elements under the first projection \( \mathcal{C} \to g \times g \to g \). It follows that \( \mathcal{C}^{rr}_1 \) is open in \( \mathcal{C} \). Similar arguments apply to the set \( \mathcal{C}^{rr}_2 \), hence, also to \( \mathcal{C}^{rr} \).

Next, let \( l \subseteq g \) be a proper standard Levi subalgebra of \( g \). Then, \( t_l \), the center of \( l \), has codimension 1 in \( t \) iff \( l \) is a minimal Levi subalgebra of \( g \). In that case, there is a root \( \alpha \in \mathfrak{t}^* \), in the root system \((g, \mathfrak{t})\), such that one has \( t_l = \ker \alpha \). This is a hyperplane in \( t \) and we have \( l = t_l \oplus \mathfrak{s}t_2 \), where \( \mathfrak{s}t_2 = \mathfrak{l} \) is the standard \( \mathfrak{sl}_2 \)-subalgebra of \( g \) associated with the root \( \alpha \). Recall further that any nonzero element \( x \in \mathfrak{s}t_2 \) is regular and that elements \( x, y \in \mathfrak{s}t_2 \) commute iff they are proportional to each other (where the zero element is declared to be proportional to any element).

**Lemma 6.3.2.** We have \( \dim(\mathcal{C} \setminus \mathcal{C}^{rr}) \leq \dim \mathcal{C} - 2 \).

**Proof.** We have the projection \( p_\alpha : \mathcal{X}(g) \to \mathcal{C}(g) \) and, for any standard Levi subalgebra \( l \subseteq g \), put \( \mathcal{C}_l(g) = p_\alpha(\mathcal{X}_l(g)) \), where we use the notation of Lemma 6.2.1. Thus, one has \( \mathcal{C}(g) = \bigcup_{l \in S} \mathcal{C}_l(g) \). Note that, for a pair of standard Levi subalgebras \( l_1, l_2 \subseteq g \), the corresponding pieces \( \mathcal{C}_{l_1} \) and \( \mathcal{C}_{l_2} \) are equal whenever the Levi subalgebras \( l_1 \) and \( l_2 \) are conjugate in \( g \); otherwise these two pieces are disjoint.
Let \( l \) be a standard Levi subalgebra in \( \mathfrak{g} \). We are interested in the dimension of the set \( \mathcal{C}_l(\mathfrak{g}) \setminus \mathcal{C}^{rr} \). From the isomorphism of Lemma 2.4.4 we deduce that

\[
\dim(\mathcal{C}_l(\mathfrak{g}) \setminus \mathcal{C}^{rr}) = \dim(G/L) + \dim \left( \left( \mathfrak{m}(l) \times \mathfrak{t}_l \right) \setminus \mathcal{C}^{rr} \right).
\]

We put \( d_l := \dim(\mathfrak{m}(l) \times \mathfrak{t}_l) - \dim \left( \left( \mathfrak{m}(l) \times \mathfrak{t}_l \right) \setminus \mathcal{C}^{rr} \right) \), a nonnegative integer. Since \( \dim(\mathcal{C}_l(\mathfrak{g}) = \dim(G/L) + \dim \mathfrak{t} + \dim \mathfrak{t} \) and \( \dim(\mathfrak{m}(l) \setminus \mathfrak{t}_l) = \dim \mathfrak{t}' + 2 \dim \mathfrak{t}_l \), we find

\[
\dim \mathcal{C} - \dim(\mathcal{C}_l(\mathfrak{g}) \setminus \mathcal{C}^{rr}) = \dim \mathfrak{t} + \dim \mathfrak{t} - \dim \left( \left( \mathfrak{m}(l) \times \mathfrak{t}_l \right) \setminus \mathcal{C}^{rr} \right)
= \dim \mathfrak{t} + \dim \mathfrak{t} + d_l - \dim \mathfrak{t}' - 2 \dim \mathfrak{t}_l = \dim(t/t_l) + d_l.
\]

To prove the lemma, we will show that, for any standard Levi subalgebra \( l \subset \mathfrak{g} \), the set \( \mathcal{C}_l(\mathfrak{g}) \setminus \mathcal{C}^{rr} \) has codimension \( \geq 2 \) in \( \mathcal{C}(\mathfrak{g}) \). We see from (6.3.3) that this holds automatically whenever \( \dim(t/t_l) \geq 2 \). Thus, the only nontrivial cases are: \( \dim(t/t_l) = 0 \) or \( \dim(t/t_l) = 1 \).

In the first case we have \( t = t' \), so \( l = \mathfrak{t} \). In this case, \( \mathfrak{t}_l = \mathfrak{t} \) and \( \mathfrak{m}(l) = \{0\} \). Thus, the set \( \mathfrak{t} \setminus \mathcal{C}^{rr} \) consists of the pairs \( (h_1, h_2) \in \mathfrak{t} \) such that neither \( h_1 \) nor \( h_2 \) is regular. Therefore, each of these two elements belongs to some root hyperplane in \( t \), that is, belongs to a finite union of codimension 1 subspaces in \( t \). We conclude that \( d_l = \dim(\mathfrak{t} \setminus \mathcal{C}^{rr}) \leq \dim \mathfrak{t} - 2 \), hence, \( d_l \geq 2 \).

Next, put \( Z := \mathfrak{m}(l) + \mathfrak{t}_l \). To complete the proof of the lemma, we must show that, in the case where \( \dim(t/t_l) = 1 \), the set \( Z \setminus \mathcal{C}^{rr} \) has codimension \( \geq 1 \) in \( Z \). To this end, pick \( h \in \mathfrak{t}_l \). Then, the centralizer of \( h \) in \( \mathfrak{g} \) equals \( l \). Therefore, for any nilpotent element \( n \in t' \), the element \( h + n \) is a regular element of \( \mathfrak{g} \). Clearly, \( (n, n) \in \mathfrak{m}(l) \) and we have \( (h + n, h + n) \in Z \cap \mathcal{C}^{rr} \). Hence, \( Z \cap \mathcal{C}^{rr} \) is a nonempty Zariski open subset of \( Z \). Thus, the set \( Z \setminus \mathcal{C}^{rr} \) is a closed proper subset of \( Z \). Since \( Z \) is irreducible, we conclude that the set \( Z \setminus \mathcal{C}^{rr} \) has codimension \( \geq 1 \) in \( Z \), and we are done.

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