Algorithms for computing linear invariants of directed graphs

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Abstract

The present paper presents without proof, some algorithms which solve the problem next to be explained. (The proofs occur in a longer version, submitted 3-11-2005 to Linear Algebra and its Applications (Elsevier)—the author will be happy to E-mail this longer version in response to requests sent to jtowber@uic.edu.)

Call two pairs \((M, N)\) and \((M', N')\) of \(m \times n\) matrices over a field \(K\), simultaneously \(K\)-equivalent if there exist square invertible matrices \(S, T\) over \(K\), with \(M' = SMT\) and \(N' = SNT\). Kronecker [2] has given a complete set of invariants for simultaneous equivalence of pairs of matrices.

Associate in the natural way to a finite directed graph \(\Gamma\), with \(v\) vertices and \(e\) edges, an ordered pair \((M, N)\) of \(e \times v\) matrices of zeros and ones. It is natural to try to compute the Kronecker invariants of such a pair \((M, N)\), particularly since they clearly furnish isomorphism-invariants of \(\Gamma\). Let us call two graphs ‘linearly equivalent’ when their two corresponding pairs are simultaneously equivalent.

The purpose of the present paper, is to compute directly these Kronecker invariants of finite directed graphs, from elementary combinatorial properties of the graphs. A pleasant surprise is that these new invariants are purely rational —indeed, integral, in the sense that the computation needed to decide if two directed graphs are linearly equivalent only involves counting vertices in various finite graphs constructed from each of the given graphs—and does not involve finding the irreducible factorization of a polynomial over \(K\) (in apparent contrast both to the familiar invariant-computations of graphs furnished by the eigenvalues of the connection matrix, and to the isomorphism problem for general pairs of matrices.)
1 Introduction: First Statement of the Problem

Let \( \Gamma \) be a finite directed graph, with \( v \) vertices and \( e \) edges. Let us call \( \Gamma \) **reduced** if it contains no ‘parallel’ pairs of edges. (We allow ‘reduced’ graphs to contain ‘loops’, i.e. \( \Gamma \) may contain an edge which connects a vertex to itself.) (The invariants to be defined, will turn out basically to depend only on the ‘reduced’ version of \( \Gamma \). For the time being, however, we do not assume that \( \Gamma \) is reduced.)

If we choose (arbitrarily) orderings \( \langle E \rangle = (E_1, E_2, \cdots, E_e) \) of the edges of \( \Gamma \), and \( \langle V \rangle = (V_1, V_2, \cdots, V_v) \) of the vertices, we then obtain two \( e \times v \) matrices

\[
M = M(\Gamma, \langle E \rangle, \langle V \rangle), \quad N = N(\Gamma, \langle E \rangle, \langle V \rangle)
\]

of 0’s and 1’s, given by:

\[
M_{i,j} = \begin{cases} 
1 & \text{if } V_j \text{ is the initial vertex of } E_i \\
0 & \text{otherwise}
\end{cases}
\]

and similarly

\[
N_{i,j} = \begin{cases} 
1 & \text{if } V_j \text{ is the terminal vertex of } E_i \\
0 & \text{otherwise}
\end{cases}
\]

Now suppose we are given a second directed graph \( \Gamma' \), with the same numbers \( v \) of vertices, and \( e \) of edges as \( \Gamma \). Let \( \Gamma' \) be similarly associated with a pair \( (M', N') \) of \( e \times v \) matrices.

Clearly, the two following assertions are equivalent:

Ia) \( \Gamma \) and \( \Gamma' \) are isomorphic as directed graphs.

Ib) There exist \( e \times e \) (resp. \( v \times v \)) permutation matrices \( S \) (resp. \( T \)) such that

\[
M' = SMT, \quad N' = SNT.
\]

The purpose of the present paper is to consider what happens when we replace the question of when \( \Gamma \) and \( \Gamma' \) are related in the manner just described (a question which seems immensely difficult), by the much easier modification of this question, suggested by the following two definitions.

Let \( K \) be any field.

**Definition 1.1** Let \( M, N, M', N' \) be matrices over the ground-field \( K \), all of the same size \( m \times n \). We shall say that \( (M, N) \) and \( (M', N') \) are simultaneously **\( K \)-equivalent**, if there exist invertible square matrices \( S \) and \( T \) over \( K \), of sizes \( m \times m \) and \( n \times n \) respectively, such that

\[
M' = SMT \quad \text{and} \quad N' = SNT.
\]
Definition 1.2 Let $\Gamma$ and $\Gamma'$ be finite directed graphs, with the same number $v$ of vertices, and the same number $e$ of edges. Let the pair $(M, N)$ of $e \times f$ matrices of 0's and 1's be related to $\Gamma$ as described above (i.e. via 1), together with an arbitrary choice of the orderings $<_E$ and $<_V$. Let $(M', N')$ be similarly associated with $\Gamma'$. Then we say that $\Gamma$ and $\Gamma'$ are $K$-linearly equivalent, if $(M, N)$ is simultaneously $K$-equivalent to $(M', N')$.

We note that Def.1.2 is independent of the choice of vertex- and edge-orderings on $\Gamma$ and $\Gamma'$. We also note that the only difference between graph-isomorphism and $K$-linear equivalence, is that $S, T$ in Eqn.3) are required to be permutation-matrices for $\Gamma$ and $\Gamma'$ to be isomorphic, while they only need to be invertible matrices over $K$ for $K$-linear equivalence to hold.

Thus $K$-linear equivalence is a coarser equivalence relation than isomorphism; and the purpose of the present paper is to establish some easily computed combinatorial invariants of directed graphs, which furnish what the author hopes is an efficient decision-procedure for $K$-linear equivalence.

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2 A More Detailed Statement of the Problem

Let $K$ be a given field.

We next wish to review the details of Kronecker’s canonical form, for pairs of matrices over $K$ under simultaneous $K$-equivalence (cf.Def.1.2). The computational details involved in reduction to Kronecker’s canonical form, seem to the author to become clearer, if stated in terms of vector-spaces over $K$ and $K$-linear transformations, rather than in terms of matrices. Thus, let us consider the problem of classifying pairs of maps from one vector-space over $K$ into a second—i.e. of classifying diagrams

$$
\begin{array}{ccc}
E & \xrightarrow{\mu} & V \\
\downarrow \nu & & \downarrow \nu \\
V & & V
\end{array}
$$

(4)

where $E$ and $V$ are finite-dimensional vector-spaces over the ground-field $K$, and where $\mu$ and $\nu$ are $K$-linear transformations—such a diagram will be
referred to as a transformation-pair over $K$. We shall also use the notation $[\mu, \nu : E \to V]$ to refer to the transformation-pair (4).

Of course, to have a well-specified classification problem, we must specify precisely what meaning we wish to attach to ‘isomorphism’ between two such diagrams (4). We do this in the obvious way: by a $K$-isomorphism from $[\mu, \nu : E \to V]$ to $[\mu', \nu' : E' \to V']$ will be meant an ordered pair

$$(\alpha : E \to E', \beta : V \to V')$$

of $K$-linear isomorphisms, such that

$$\mu' \circ \alpha = \beta \circ \mu, \text{ and } \nu' \circ \alpha = \beta \circ \nu$$

(This is just another way of describing the earlier notion of simultaneous equivalence of pairs of matrices, the matrices being those that represent $\mu, \nu$ with respect to a choice of $K$-bases for $E$ and $V$.)

Thus one is led to study the classification problem for the category whose objects are diagrams (4), and whose morphisms are the $K$-isomorphisms just specified—let us denote this category by $\text{Tsf Pair}(K)$ . One slight further delay, before we finally get to Kronecker’s beautiful classification of diagrams (4): let us note how all this is connected (by a process of ‘linearization’) with directed graphs:

**Definition 2.1** Let $\Gamma$ be a finite directed graph, with vertex set $V$ and whose set of edges is $E$. We then construct a corresponding element

$$K(\Gamma) = [\mu_\Gamma, \nu_\Gamma : E_\Gamma \to V_\Gamma]$$

in $\text{Tsf Pair}(K)$ as follows:

We take $E_\Gamma$ to be the vector-space over $K$ freely generated by the basis $E$, $V_\Gamma$ to be similarly $K$-free on the set $V$ of vertices, while the $K$-linear transformations $\mu_\Gamma, \nu_\Gamma : E_\Gamma \to V_\Gamma$ are defined on the basis-vectors $e \in E$ for $E_\Gamma$, by:

$$\mu_\Gamma(e) = \text{initial vertex of } e, \nu_\Gamma(e) = \text{terminal vertex of } e.$$  

We shall refer to this transformation-pair $K(\Gamma)$ in $\text{Tsf Pair}(K)$ as the $K$-linearisation of $\Gamma$.  

4
**EXAMPLE 2.1:** Consider the directed graph $\Gamma_1$ furnished by the following diagram:

![Diagram of directed graph $\Gamma_1$]

Then the $K$-linearisation of $\Gamma_1$ is $[\mu, \nu : E \to V]$, where

$$E = K\{e_1, e_2, e_3, e_4\}, \quad V = K\{v_1, v_2, v_3\}$$

and where (all x's lying in $K$)

$$\mu(x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4) = (x_1 + x_2)v_1 + x_3v_2 + x_4v_3 \quad (5)$$

$$\nu(x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4) = (x_1 + x_4)v_2 + (x_2 + x_3)v_3$$

We now proceed to sketch Kronecker’s classification results, stated in terms of the category $\text{Tsf Pair}(K)$. These results will here be stated without proof (for which see [1], Chap.XII; [5], Chap.IX; or, [2]). Also, in [4] a direct proof is given for the classification of the category $\text{Lin Rel}(K)$ defined in §?? below; it is then easy (using the results obtained in §??) to deduce the classification of $\text{Tsf Pair}(K)$ from that of $\text{Lin Rel}(K)$.

In the first place, there is an obvious notion of direct sum defined as follows on the category $\text{Tsf Pair}(K)$: the direct sum

$$[\mu, \nu : E \to V] \oplus [\mu', \nu' : E' \to V']$$

of two objects, is the object

$$[\mu \oplus \mu', \nu \oplus \nu' : E \oplus E' \to V \oplus V']$$

There is a unique **zero object** in $\text{Tsf Pair}(K)$, namely

$$[0, 0 : 0 \to 0]$$

An object $[\mu, \nu : E \to V]$ in $\text{Tsf Pair}(K)$ will be called **indecomposable** if it is not the zero object, and is not isomorphic to the direct sum of two non-zero objects in $\text{Tsf Pair}(K)$.
Kronecker’s theory then furnishes us with the following five families of indecomposable objects in Tsf Pair(K), such that every indecomposable object in this category, is isomorphic to a unique object in this list:

In this listing, it will be convenient to denote by \( \{f_1^{(n)}, \ldots, f_n^{(n)}\} \) the usual standard basis for \( K^n \), so that

\[
f^{(n)} = (1, 0, \ldots, 0), f_1^{(n)} = (0, 1, \ldots, 0), \ldots, f_n^{(n)} = (0, 0, \ldots, 1)
\]

Also, we shall denote by \( I_n \) the identity map on \( K^n \).

**TYPE \( T^0_n \) (n = 1, 2, 3, \ldots)**

This is the object \([\mu_n, I_n : K^n \rightarrow K^n]\) where \( \mu_n \) is the nilpotent \( K \)-linear transformation on \( K^n \) which maps:

\[
\mu_n : f_1^{(n)} \mapsto f_2^{(n)} \mapsto \ldots \mapsto f_n^{(n)} \mapsto 0
\]

(6)

**TYPE \( T^n_0 \) (n = 1, 2, 3, \ldots)**

This is the object \([I_n, \mu_n : K^n \rightarrow K^n]\), where \( \mu_n \) is still given by (6).

**TYPE \( T_n \) (n = 0, 1, 2, \ldots)**

This is the object \([\kappa_n, \lambda_n : K^n \rightarrow K^{n+1}]\), where \( \kappa_n, \lambda_n \) are defined by:

\[
\kappa_n(f_1^{(n)}) = f_1^{(n+1)}, \kappa_n(f_2^{(n)}) = f_2^{(n+1)}, \ldots, \kappa_n(f_n^{(n)}) = f_n^{(n+1)}
\]

and

\[
\lambda_n(f_1^{(n)}) = f_1^{(n+1)}, \lambda_n(f_2^{(n)}) = f_2^{(n+1)}, \ldots, \lambda_n(f_n^{(n)}) = f_n^{(n+1)}
\]

(When \( n = 0 \), this is to be understood as the object \([0, 0 : 0 \rightarrow K]\))

**TYPE \( T^0_n \) (n = 0, 1, 2, \ldots)**

This is the object \([\kappa'_n, \lambda'_n : K^{n+1} \rightarrow K^n]\), where \( \kappa'_n, \lambda'_n \) are defined by:

\[
\kappa_n'(f_1^{(n+1)}) = f_1^{(n)}, \kappa_n'(f_2^{(n+1)}) = f_2^{(n)}, \ldots, \kappa_n'(f_n^{(n+1)}) = f_n^{(n)}, \text{ but } \kappa_n'(f_{n+1}^{(n+1)}) = 0;
\]

and

\[
\lambda_n'(f_1^{(n+1)}) = 0, \lambda_n'(f_2^{(n+1)}) = f_1^{(n)}, \lambda_n'(f_3^{(n+1)}) = f_2^{(n)}, \ldots, \lambda_n'(f_{n+1}^{(n+1)}) = f_n^{(n)}
\]

(When \( n = 0 \), this is to be understood as the special object \([0, 0 : K \rightarrow 0]\)—this special object plays a somewhat exceptional role, at several places in the investigations that follow.)

**TYPE \( S(p(X)) \) \)** Here \( n \) denotes a positive integer, and \( p(X) \)—subject to the condition \( p(X) \neq X \)—denotes a monic irreducible polynomial in the polynomial ring \( K[X] \) in one indeterminate \( X \) over \( K \). \( S(p(X)) \) is
then defined to be the object $[\xi, I(V) : V \to V]$ where $V = V(n, p) = K[X]/(p(X))^n$, where $I(V)$ is the identity map on $V$, and where $\xi$ is the $K$-endomorphism of $V$ given by multiplication by $X$.

Let us denote by $\mathcal{L}(K)$ the list (just described) of indecomposable objects in $\text{Tsf Pair}(K)$.

**Remark 2.2**

More generally, for any monic polynomial $F(X) \in K[X]$, it will be quite convenient for us to denote by $S(F(X))$ the transformation-pair

$$[\phi, I : K[X]//(F) \to K[X]//(F)],$$

where $\phi$ is induced via multiplication by $X$ (so that $F(\phi) = 0$). Note that, in particular, $S(X^n)$ is $0^nT_n$.

Of course, $S(F(X))$ is not in general indecomposable in $\text{Tsf Pair}(K)$; rather, if $F$ has the irreducible factorization in $K[X]$ given by

$$F = \prod_{i=1}^{n} p_i(X)^{n_i},$$

then $S(F(X)) = \bigoplus_{i=0}^{n} S(p_i(X)^{n_i})$ (7)

(In our decomposition in $\text{Tsf Pair}(K)$ of the $K$-linearizations of graphs, we shall begin by splitting off certain of these $S(F)$, and then make use of (7).)

In our present formulation, Kronecker’s theory asserts that every object $\pi$ in $\text{Tsf Pair}(K)$ is isomorphic to a direct sum of a finite number of indecomposable objects; and asserts that, given an indecomposable object $\kappa$ in the above list $\mathcal{L}(K)$, the multiplicity $[\pi : \kappa]$ with which $\kappa$ occurs in such a direct sum decomposition for $\pi$, is the same for all such decompositions. Thus, a complete list of invariants for objects in $\text{Tsf Pair}(K)$ is afforded by the collection of these multiplicities

$$\{|[\pi : \kappa]| \kappa \in \mathcal{L}(K)\}$$

(8)

(which are non-negative integers; for given $\pi$, all but finitely many of these integers vanish.)

We shall also use the following notation for the invariants associated to the first four infinite families in the above list $\mathcal{L}(K)$: for $\pi$ any transformation-pair over $K$, we shall set
\[
\begin{aligned}
\begin{cases}
t_n(\pi) := [\pi : T_n] &= \text{multiplicity of } T_n \text{ in } \pi \ (0 \leq n) \\
\text{and similarly} & \\
0t_n(\pi) := [\pi : 0^0 T_n] &= (1 \leq n) \\
t_n^0(\pi) := [\pi : 0^0 T_n^0] &= (0 \leq n) \\
0t_n^0(\pi) := [\pi : 0^0 T_n^0] &= (0 \leq n)
\end{cases}
\end{aligned}
\]

**A FURTHER HISTORICAL NOTE:**

The work of Kronecker just sketched, was based on the earlier classification results in Weierstrass’ 1867 paper [6]. Restating Weierstrass’ work in terms of transformation-pairs: in it were classified those transformation-pairs

\[ [\mu, \nu : E \to V] \]

which are **regular**, i.e. have both the following properties:

1) \( \dim(E) = \dim(V) \), and 2) \( \exists c \in K \text{ so } |\mu + cv| \neq 0 \)

Namely, Weierstrass showed (in our present terms) that every regular transformation-pair, is uniquely isomorphic to a direct sum of the indecomposable types

\[ 0^0 T_n \text{ and } T_n^0 \ (n > 1); \quad S(p^n) \quad p \text{ irreducible, } n > 0 \]

defined above.

This work of Weierstrass is explained in Gantmacher([1]), pp.24-28, and also in Turnbull-Aitken([3]), pp.113–118 — both these books go from there to explain the extension of this work, in Kronecker’s 1890 paper [2], to the full category \( \text{Tsf Pair}(K) \) (as explained above) which involved adding the two infinite classes of non-regular indecomposable forms

\[ T_n \text{ and } 0^0 T_n^0 \ (n \geq 0) \].

**EXAMPLE 2.1 RE-VISITED:**

In Example 2.1 we computed the \( K \)-linearization of the directed graph \( \Gamma_1 \) pictured above. Let us, in this simple example, compute the associated Kronecker invariants:

We begin by looking inside \( K(\Gamma_1) \) for a copy of Type \( 0^0 T_1^0 \), i.e. of the indecomposable transformation-pair \([\mu, \nu : K^2 \to K]\) indicated by the following picture:

Using equations (5), we see first that $\mu e'_2 = 0$ implies that $e'_2$ is a scalar multiple of $e_1 - e_2$, say

$$e'_2 = e_1 - e_2, v' = \nu e'_2 = \nu(e_1 - e_2) = v_2 - v_3.$$ 

Then

$$\mu e'_1 = v' = v_2 - v_3, \nu e'_2 = 0$$

specify $e'_1$ uniquely as $e_1 - e_2 + e_3 - e_4$ This shows the Kronecker invariant $0^0_{11}(\Gamma_1) = [K(\Gamma_1) : 0^0_{11}]$ has the value 1.

We next look inside $K(\Gamma_1)$ for a copy of Type $T^0_1$. Thus, we try to find $e'' \in E = K\{e_1, e_2, e_3, e_4\}, v'' \in V = \{v_1, v_2, v_3\}$ such that $\mu e'' = v''$ and $\nu e'' = 0$. Using equations (5), it is readily seen that the most general solution to these equations is given by

$$e'' = \alpha(e_1 - e_4) + \beta(e_2 - e_3), v'' = \alpha(v_1 - v_3) + \beta(v_1 - v_2)$$

Almost any choice of $(\alpha, \beta)$ is satisfactory; choices which will fail to give a Kronecker decomposition are those for which $\beta = -\alpha$, so that $e''$ is a scalar multiple of $e'_2$ and our second object is contained inside the first.) Let us choose $\alpha = 1, \beta = 0$, so our copy of Type $T^0_1$ has

$$e'' = e_1 - e_4, v'' = v_1 - v_3$$

Finally, it is readily verified that a copy of $S(X-1)$ inside $K(\Gamma_1)$, i.e. a solution to $\mu e''' = \nu e''' = v'''$, is furnished by

$$e''' = e_1 - e_2 + e_3, v''' = v_2$$

It is easy to check that (with the choices indicated above) $\{e'_1, e'_2, e'', e'''\}$ are linearly independent over $K$, as are $\{v', v'', v'''\}$, so that $K(\Gamma_1)$ is the direct sum of the three sub-objects constructed (which could however have been selected for this purpose in infinitely many other ways), and we have:

$$K(\Gamma_1) \simeq 0^0_{11} \oplus T^0_1 \oplus S(X - 1)$$

(10)
**EXAMPLE 2.2:** Let us consider a second example (whose properties will be useful in the next section). Namely, let $S_2 = \{v_1, v_2\}$ be a set with two elements, and let $\Gamma_2$ be the directed graph, on these two vertices, which contains precisely one edge $e_{i,j}$ going from $v_i$ to $v_j$ for $i, j \in \{1, 2\}$.

The $K$-linearization is then

$$K(\Gamma_2) = [\mu_2, \nu_2 : E_2 \to V_2]$$

where $V_2$ is $K$-free on $v_1, v_2$, $E_2$ is $K$-free on the four edges $e_{1,1}, e_{1,2}, e_{2,1}, e_{2,2}$

and where

$$\mu_2(a_{1,1}e_{1,1} + a_{1,2}e_{1,2} + a_{2,1}e_{2,1} + a_{2,2}e_{2,2}) = (a_{1,1} + a_{1,2})v_1 + (a_{2,1} + a_{2,2})v_2$$
$$\nu_2(a_{1,1}e_{1,1} + a_{1,2}e_{1,2} + a_{2,1}e_{2,1} + a_{2,2}e_{2,2}) = (a_{1,1} + a_{2,1})v_1 + (a_{1,2} + a_{2,2})v_2$$

If $\alpha = a_{1,1}e_{1,1} + a_{1,2}e_{1,2} + a_{2,1}e_{2,1} + a_{2,2}e_{2,2}$ satisfies $\mu_2(\alpha) = \nu_2(\alpha) = 0$, then it follows that $a_{2,1} = -a_{1,1}, a_{1,2} = -a_{1,1}, a_{2,2} = a_{1,1}$, so $\text{Ker } \mu_2 \cap \text{Ker } \nu_2$ consists of multiples of

$$e_{1,1} - e_{1,2} - e_{2,1} + e_{2,2}.$$ 

This gives a copy inside $K(\Gamma_2)$ of

$$0^{T_0^a} = [0, 0 : K \to 0]$$

unique up to scalar multiple. The general results to be obtained in the sequel, will imply the Kronecker decomposition

$$K(\Gamma_2) \cong 0^{T_0^a} \oplus 0^{T_1^b} \oplus S(X - 1) \quad (11)$$

The reader may wish to prove this, directly from the definitions.

We now conclude this section by re-stating (in sharper focus) the question, sketched in the preceding section, (and discussed for the two preceding special cases), whose general solution will occupy the remainder of this paper:

**Given a directed graph $\Gamma$, how can we efficiently compute the Kronecker invariants**

$$\{ [K(\Gamma) : \kappa] | \kappa \in L(K) \}$$

**of the $K$-linearization of $\Gamma$?**
3 Reduction to the Case of Binary Relations

Our next step is an easy preliminary reduction of the problem just stated:

Namely, let $\Gamma$ be a finite directed graph; denote by $E_\Gamma$ resp. $V_\Gamma$ the set of edges resp. vertices of $\Gamma$. Let $\sim_\Gamma$ denote the equivalence-relation of ‘parallelism’ on the set of edges of $\Gamma$, so that if $e$ and $e' \in E_\Gamma$:

$e \sim_\Gamma e' \iff e$ and $e'$ have the same initial vertex, and also the same final vertex.

In §1 we defined $\Gamma$ to be \textit{reduced} if $\Gamma$ does not contain a pair $e, e'$ of distinct edges which are parallel. Even if $\Gamma$ does contain parallel edges, the following obvious construction produces a finite directed graph $\Gamma^{\text{red}}$ which is reduced:

We define the set $E_{\Gamma^{\text{red}}} := E_\Gamma / \sim_\Gamma$ of edges of $\Gamma^{\text{red}}$ to be the set of $\sim_\Gamma$-equivalence-classes; we also set $V_{\Gamma^{\text{red}}} := V_\Gamma$ i.e., $\Gamma^{\text{red}}$ is to have the same vertices as $\Gamma$. Finally, if $e$ is an edge of $\Gamma$, and if $\text{cls } e \in E_{\Gamma^{\text{red}}}$ is the $\sim_\Gamma$-equivalence-class containing $e$, then the initial (resp. terminal) vertex in $\Gamma^{\text{red}}$ of $\text{cls } e$ are defined to coincide with the initial (resp. terminal) vertex in $\Gamma$ of $e$.

Let us call $\Gamma^{\text{red}}$ the \textit{reduced form} of $\Gamma$. With the notation just explained, we have:

\textbf{Proposition 3.1} Let the equivalence-relation $\sim_\Gamma$ partition $E_\Gamma$ into $s$ equivalence classes, of cardinalities $n_1, n_2, \ldots, n_s$. Then the $K$-linearization $K(\Gamma)$ of $\Gamma$, is the direct sum of $K(\Gamma^{\text{red}})$ with

$$n_1 + n_2 + \cdots + n_s - s$$

(12)

copies of the irreducible type $^0T_0^0 = [0, 0 : K \to 0]$.

\textbf{PROOF:} Let $E_1, \cdots, E_s$ be the $\sim_\Gamma$-equivalence classes, and let

$E_i = \{e^i_1, \cdots, e^i_{n_i}\}$ for $1 \leq i \leq s$.

Then set

$E := \{e^1_1, e^2_1, \cdots, e^s_1\}$
and
\[ f^i_j := e^i_j - e^i_1 \in E_\Gamma \text{ for } 1 \leq i \leq s, 2 \leq j \leq n_i. \]

It is readily verified that \( \Gamma := (E, V) \) forms a sub-graph of \( \Gamma \) isomorphic to \( \Gamma^{red} \), and that \( K(\Gamma) \) is the direct sum of \( K(\Gamma) \) with the \( \sum(n_i - 1) \) sub-objects

\[ [0, 0 : Kf^i_j \to 0] \cong 0 \quad (1 \leq i \leq s, 2 \leq j \leq n_i). \]

This proves the asserted proposition.

**CAUTION:** It does not follow from the Prop. 3.1 just proved, that the multiplicity \( 0l_0^i (K(\Gamma)) \) with which \( 0T^0_0 \) occurs as a direct summand in \( K(\Gamma) \) is given by \( 12 \)—because also \( K(\Gamma^{red}) \) may contain one or more summands of type \( 0T^0_0 \). (See Example 2.2 above for an example of a reduced graph whose \( K \)-linearization contains a direct summand of type \( 0T^0_0 \).)

Because of the proposition Prop. 3.1 just proved, we may as well restrict to reduced directed graphs. If \( \Gamma \) is a reduced directed graph, with vertex-set \( V \), we may as well consider the set \( E \) of edges to be a sub-set of \( V \times V \), and this is the viewpoint which we shall adopt for the remainder of this paper. Then \( E \subseteq V \times V \) endows our set-up with the familiar structure of a binary relation \( E \) on a finite set \( V \).

This modified point of view, is a bit better adapted to the combinatorial constructions in the next section. It makes available the various basic operations on the set of binary relations on a given set \( S \)—e.g., given binary operations \( R, R' \) on \( S \), we may form the following further binary relations on \( S \):

- \( R \cup R' \), \( R \cap R' \), the converse relation \( R^{-1} \), and the composite \( RR' \).

—operations which will make possible the constructions of the next section.

Just to clarify our terminology: for the rest of this paper, a **binary relation** is an ordered pair

\[ \Gamma = (R, V) \text{ where } R \subseteq V \times V. \]

—in the present work, “binary relation” is never taken in the more general sense of a subset \( R \subseteq S \times S' \) with \( S \) and \( S' \) distinct sets.

\( \Gamma \) is **finite** if \( V \) is a finite set. We shall sometimes vary our language by writing instead

\[ \Gamma = (R \subseteq V \times V) \]

For \( v \) and \( v' \) in \( V \), the notations \((v, v') \in R \) and \( vRv' \) are synonymous.
Let us spell out how the problem being studied looks with this minor change in language:

Let $\Gamma = (R, V)$ denote a binary relation on a finite set $V$. This corresponds to the reduced directed graph, whose edge-set coincides with $R$, whose vertex-set coincides with $V$, and such that the ordered pair $(v, v')$ in $R$ is regarded as an edge, with initial vertex $v$, and terminal vertex $v'$. The $K$-linearization $K(\Gamma)$ of this directed graph consists of the transformation-pair

$$[\mu_\Gamma, \nu_\Gamma : KR \to KV] \in \text{Tsf Pair}(K)$$

where $KV$ (resp. $KR$) is the vector-space over $K$ free on the basis $V$ (resp. $R$), and where the $K$-linear maps $\mu_\Gamma, \nu_\Gamma$ are defined on basis-vectors $(v, v') \in R$ by

$$\mu_\Gamma(v, v') = v, \ \nu_\Gamma(v, v') = v'$$

For each $\kappa \in \mathcal{L}(K)$ we shall denote by $[\Gamma : \kappa]$ the multiplicity with which the indecomposable object $\kappa$ appears as a direct summand in $K(\Gamma)$.

The fundamental problem studied in this paper, may now be re-stated as the computation, for finite binary relations, of these integers

$$\{[\Gamma : \kappa]| \kappa \in \mathcal{L}(K)\}$$

In the next section, we define some combinatorial constructions on binary relations $R$, which will enable us to accomplish this.

4 Left and Right Contractions of Binary Relations

Let $\Gamma = (R, S)$ denote a binary relation on a finite set $S$. (We consider $S$ to be part of the given structure $\Gamma$, i.e. if $S$ is a proper subset of $S'$, $(R, S)$ and $(R, S')$ count as distinct binary relations.)

**Definition 4.1** Let $\sim$ be an equivalence relation on $S$. We then denote by $S/\sim$ the set of $\sim$-equivalence classes, and by

$$\Gamma/\sim = (R/\sim, S/\sim)$$

the binary relation on $S/\sim$ specified by

$$\alpha(R/\sim) \beta \iff \exists a \in \alpha, b \in \beta \text{ with } aRb \text{ (for } \alpha, \beta \in S/\sim)$$

Also, we denote by $\pi(\Gamma, \sim)$ the canonical epimorphism

$$\pi(\Gamma, \sim) : S \to S/\sim$$

which maps each $s \in S$ into the $\sim$-equivalence class containing $s$. 

13
Definition 4.2  We denote by
\[ l \equiv \sim (\Gamma) \]
the equivalence relation on \( S \) generated by \( R^{-1}R \), i.e. the smallest equivalence relation on \( S \) such that
\[ xRy \text{ and } xRy' \Rightarrow y \overset{l}{\sim} y' ; \]
and we define the left contraction of \( R \) to be the binary relation
\[ C_L(\Gamma) := \Gamma/ \overset{l}{\sim} \text{ on the set } V/ \overset{l}{\sim} . \]

Similarly, we denote by
\[ \overset{r}{\sim} (\Gamma) \]
the equivalence relation on \( S \) generated by \( RR^{-1} \), i.e. the smallest equivalence relation on \( S \) such that
\[ xRy \text{ and } x'Ry \Rightarrow x \overset{r}{\sim} x' ; \]
and we define the right contraction of \( \Gamma \) to be the binary relation
\[ C_R(\Gamma) := \Gamma/ \overset{r}{\sim} \text{ on the set } V/ \overset{r}{\sim} . \]

Remark 4.3
If we denote the converse relation \((R^{-1}, S)\) to \( \Gamma \) by \( \Gamma^{-1} \), where (for all \( s \) and \( s' \) in \( S \)),
\[ sR^{-1}s' \iff s'Rs , \]
then clearly
\[ C_l(\Gamma^{-1}) = (C_r\Gamma)^{-1} \text{, and } C_r(\Gamma^{-1}) = (C_l\Gamma)^{-1} . \]

Example 4.4
Consider the binary relation \( \Gamma_3 \) given by this diagram:
\[ \Gamma_3 = (R_3, S_3) = \]

Then \(4R_3\,2\), \(4R_3\,3\) imply \(2 \sim 3\), and the \(\sim\)-equivalence classes are \(\{1\}\), \(\{2, 3\}\), \(\{4\}\).

Thus the left contraction of \((R_3, S_3)\) has three vertices, and may be drawn as

\[ C_l(R_3, S_3) = \]

Similarly, \(1R_3\,2\) and \(4R_3\,2\) imply \(1 \sim 4\), and the right contraction of \((R_3, S_3)\) is given by

\[ C_r(R_3, S_3) = \]

5 Statement of the Main Theorem

The present section contains a number of statements, which together furnish an algorithm which constitutes a complete solution to the computational problem stated in Sections 1 and 2. Note: No proofs are presented in this section, or the next, in order not to interrupt the exposition of these results—results which will be stated in this section, and will be illustrated by some specific computations in the next section. Following these two sections, the remainder of the present paper then contains the proofs of these results.

In this section, \(\Gamma = (R, S)\) will denote a finite binary relation (i.e., a reduced directed graph.)
Proposition 5.1 Let $\Gamma = (R, S)$ be a binary relation; then there is a natural graph-isomorphism\footnote{Actually, this may be strengthened, to assert a closer identity between $C_l C_r \Gamma$ and $C_r C_l \Gamma$; cf. Th. 7.12. This stronger form is useful in checking computations such as those in \S 6.} between $C_l C_r \Gamma$ and $C_r C_l \Gamma$.

Definition 5.2 For $m, n$ any non-negative integers, we may define (using the preceding Prop. 5.1) a binary relation

$$C_l^m C_r^n \Gamma = (mR^n, mS^n)$$

(which we take to be $\Gamma$ if $m = n = 0$.) Then by $\gamma_{m,n}(\Gamma)$ will be meant the cardinality of the set $mS^n$.

Example 5.3

Let us re-examine Example 4.4 in Section 4. For the binary relation $\Gamma_3$ in this example, we have

$$\gamma_{0,0}(\Gamma_3) = 4, \gamma_{1,0}(\Gamma_3) = 3, \gamma_{0,1}(\Gamma_3) = 3.$$ 

It is readily verified that

$$C_l^n \Gamma_3 \cong C_l \Gamma_3, \text{ and } C_r^n \Gamma_3 \cong C_l \Gamma_3 \text{ for all } n \geq 1.$$ 

whence $\gamma_{n,0}(\Gamma_3) = \gamma_{0,n}(\Gamma_3) = 3$ for $n \geq 1$.

$C_l C_r \Gamma_3 \cong C_r C_l$ is given by

$$\{1, 4\} \longrightarrow \{2, 3\}$$

and since this graph is stable under the actions of $C_l$ and $C_r$, we see that

$$\gamma_{m,n}(\Gamma_3) = 2 \text{ for } m \geq 1, n \geq 1.$$ 

Theorem 5.4 (Main Theorem, Part One) Let $\Gamma = (R, S)$ be a finite binary relation, with $K$-linearization $K(\Gamma)$. Then:

for $n \geq 1$,

$$0t_n(K(\Gamma)) = \gamma_{n+1,n-1} - \gamma_{n,n} - \gamma_{n+1,n} + \gamma_{n,n+1}$$

(13)

$$t_n^0(K(\Gamma)) = \gamma_{n-1,n+1} - \gamma_{n,n} - \gamma_{n-1,n} + \gamma_{n,n-1}$$

(14)

$$= \gamma_{n+1,n-1} - \gamma_{n,n} - \gamma_{n,n+1} + \gamma_{n+1,n}$$
For $0_{t}^{0}$—still with the restriction $n > 0$—we must distinguish two cases, according as $n$ is even or odd:

\begin{align*}
0_{t}^{0}(K(\Gamma)) &= \gamma_{p,p} - \gamma_{p,p+1} - \gamma_{p+1,p} + \gamma_{p+1,p+1} \\
0_{t}^{0}(K(\Gamma)) &= \gamma_{p,p-1} - \gamma_{p,p} - \gamma_{p+1,p-1} + \gamma_{p+1,p}
\end{align*}

(15)

(16)

This still leaves open the multiplicities in $K(\Gamma)$ of the indecomposable types $T_n$ and $S(n, p(X))$—also, $0_{t}^{0}(\Gamma)$ remains open. To determine these, we first need the following definition (and also, of course, proof of the facts assumed by this definition—as already promised, all such proofs will be supplied in subsequent sections of this paper.)

**Definition 5.5** Let $\Gamma = (R, S)$ be a finite binary relation. Then the graphs $C_{\Gamma}^{m}C_{\Gamma}^{n}$ all coincide (up to natural isomorphisms) for $m, n$ sufficiently large, and their common value will be called the $C$-stabilization of $\Gamma$, and denoted by $C^{\infty,\infty}\Gamma$.

The proof of the following proposition is postponed until sections 7–12:

**Proposition 5.6** Let $\Gamma = (R, S)$ be a finite binary relation. Then $C^{\infty,\infty}\Gamma$ is in a unique way the disjoint union of a finite number of sub-graphs of the following types:

**TYPE** $C_N$ ($N \geq 1$) This is an $N$-cycle consisting of $N$ vertices $v_1, v_2, \ldots, v_N$, with relation consisting of the $N$ edges

$$(v_1, v_2), (v_2, v_3), \ldots, (v_{N-1}, v_N), (v_N, v_1)$$

(If $N = 1$ this consists of a single vertex $v$, together with a loop $(v, v)$.)

**TYPE** $L_N$ ($N \geq 1$) This consists of $N$ vertices $v_1, v_2, \ldots, v_N$, with relation consisting of the set of $N - 1$ edges

$$(v_1, v_2), (v_2, v_3), \ldots, (v_{N-1}, v_N)$$

(If $N = 1$ this consists of a single vertex $v$, with the set of edges empty.)

We are now ready to state a second part of the main theorem Th.5.4

**Theorem 5.4 (Main Theorem, Part Two)** Let $\Gamma = (R \subseteq S \times S)$ be a finite binary relation, with $K$-linearization $K(\Gamma)$. With the notation of Prop.5.6, let $C^{\infty,\infty}\Gamma$ be the disjoint union of $p$ graphs $L(m_1), L(m_2), \ldots, L(m_p)$, together with $q$ cyclic graphs $C(n_1), C(n_2), \ldots, C(n_q)$. Then:
a) The indecomposable summands of Type $T_n$ which occur in the Kronecker decomposition of $L(\Gamma)$ are given (multiplicity included) by the list

$$T_{m_1}, T_{m_2}, \ldots, T_{m_p}$$

—in other words, for every positive integer $n$, $t_n(K(\Gamma))$ equals the number of times $n$ is repeated in the sequence $m_1, m_2, \ldots, m_p$.

b) The direct sum of those indecomposable summands of type $S(p(X)^n)$ in the Kronecker decomposition of $K(\Gamma)$, is isomorphic to

$$\bigoplus_{i=1}^{q} S(X^{n_i} - 1)$$

(17)

(a fact which furnishes the summands of type $S(p(X)^n)$ via Eqn. 7.)

There is, however, one Kronecker multiplicity still undetermined by the first two parts of Th. 5.4, namely the multiplicity $0^0_0(\Gamma)$ in $K(\Gamma)$ of the rather exceptional type

$$^0_0 \tau_0^0 = [0, 0 : K \rightarrow 0].$$

(What makes this type unusual, is perhaps, that it is the only indecomposable type in Kronecker’s list $L(K)$, which is not reduced, i.e. not a linear relation, in the sense explained in Section ?? below). Note that this type is not covered by equations (15) and (16) (which are explicitly stated to be restricted to $0^n_0$ with $n > 0$).

There will next be presented two different methods for the computation of this remaining Kronecker multiplicity $0^0_0(\Gamma)$, one of which uses all the other multiplicities of $\Gamma$. A nice check on the computation is then furnished by the requirement that these two methods yield the same result.

The first of these methods is given by:

Proposition 5.7 Let

$$\tau = [\mu, \nu : E \rightarrow V]$$

be a transformation-pair over $K$; then

$$0^0_0(\tau) = \dim(Ker \mu \cap Ker \nu).$$

In order to explain the second method for computing $0^0_0(\Gamma)$, we first need the concept of edge-number and vertex-number:

Definition 5.8 Let $\tau$ be an indecomposable transformation-pair over $K$ in Kronecker’s list $L(K)$; then the edge-number $E(\tau)$ and the vertex-number $V(\tau)$ of $\tau$, are defined as follows:
• For \( n > 0 \),
\[
\mathcal{E}(0T_n) = \mathcal{V}(0T_n) = n, \quad \mathcal{E}(T_n^0) = \mathcal{V}(T_n^0) = n
\]

• For \( n \geq 0 \),
\[
\mathcal{E}(T_n) = n, \quad \mathcal{V}(T_n) = n + 1.
\]

• For \( n \geq 0 \),
\[
\mathcal{E}(0T_n^0) = n + 1, \quad \mathcal{V}(0T_n^0) = n
\]

• For \( n > 0 \), and \( p \) a monic irreducible in \( K[X] \) with \( p \neq X \),
\[
\mathcal{E}(S(p^n)) = n \cdot \deg p = \mathcal{V}(S(p^n))
\]

Also, if \( \Gamma = (R \subseteq S \times S) \) is a finite binary relation, we define its
**edge-number** \( \mathcal{E}(\Gamma) \) to be the cardinality of \( R \), and its **vertex-number** \( \mathcal{V}(\Gamma) \)
to be the cardinality of \( S \):
\[
\mathcal{E}(\Gamma) = \#(R) \quad \text{and} \quad \mathcal{V}(\Gamma) = \#(S).
\]

**Theorem 5.4 (Main Theorem, Part Three)** Let \( \Gamma = (R \subseteq S \times S) \) be a
finite binary relation, with \( K \)-linearization \( K(\Gamma) \). Let
\[
K(\Gamma) = \bigoplus_{\tau \in \mathcal{L}(K)} \Gamma : \tau \tau
\]
be the Kronecker decomposition of \( K(\Gamma) \) with respect to Kronecker’s list
\( \mathcal{L}(K) \) of indecomposable \( K \)-transformation-pairs; then
\[
\mathcal{E}(\Gamma) = \sum_{\tau \in \mathcal{L}(K)} [\Gamma : \tau] \mathcal{E}(\tau), \quad (18)
\]
and
\[
\mathcal{V}(\Gamma) = \sum_{\tau \in \mathcal{L}(K)} [\Gamma : \tau] \mathcal{V}(\tau). \quad (19)
\]

**NOTE:** By definition, \( 0T_0^0 \) has edge-number 1, so we may solve eqn. (18)
for \( 0t_0^0(\Gamma) \) in terms of the other Kronecker multiplicities of \( K \Gamma \) ( which we
may regard as already determined by Parts One and Two of Th.5.4), thus obtaining:
\[
0_t_0^0 = \mathcal{E}(\Gamma) - \sum_{n>0} n(0t_n + t_n^0) - \sum_{n>0} n \cdot t_n -
- \sum_{n>0} (n + 1 \cdot t_n^0 - \sum_{p,n>0} n \cdot \deg p \cdot [K(\Gamma) : S(p^n)]
\]

19
with all multiplicities being evaluated for $K(\Gamma)$. Also, since $^0T_0^0$ has vertex-number 0, we may also rewrite eqn. (19) in the form

$$V(\Gamma) = \sum_{n>0} n(0^0t_n + t_n^0) + \sum_{n \geq 0} (n + 1) \cdot t_n + \sum_{n>0} n \cdot v_n^0 + \sum_{p,n>0} n \cdot \deg p \cdot [K(\Gamma) : S(p^n)]$$

(again, with all multiplicities being evaluated for $K(\Gamma)$), and in this form it provides a nice check for the multiplicities computed using Parts One and Two of Th 5.4.

Clearly, once proved, the three parts of Th 5.4 completely solve the problem formulated in Sections 1 and 2. The remainder of this paper (with the exception of the next section) will be devoted to the proof of Th 5.4 together with the proofs of: Prop 5.1, the assumptions underlying Definition 5.5, Prop 5.6, and Prop 5.7.

Let us note the following immediate consequence of Th 5.4:

**Theorem 5.8** Let $\Gamma, \Gamma'$ be two finite binary relations; then necessary and sufficient that $\Gamma$ and $\Gamma'$ be linearly equivalent (i.e. that the transformation-pairs $K(\Gamma)$ and $K(\Gamma')$ be isomorphic) is that, first,

$$\gamma_{i,j}(\Gamma) = \gamma_{i,j}(\Gamma') ,$$

for all natural numbers $i, j$ with $|i - j| \leq 2$; secondly, that $C^\infty,\infty(\Gamma)$ and $C^\infty,\infty(\Gamma')$ have the same decompositions (in accordance with Prop 5.4 above) into pieces of types $L_N$ and $C_N$; and thirdly that $\Gamma$ and $\Gamma'$ have the same number of edges. (We may also express this by saying, that $\{\gamma_{i,j}(\Gamma)\}$, the type of $C^\infty,\infty(\Gamma)$—i.e., the number of graphs $C(i), L(j)$ in its decomposition according to Prop 5.4—and the natural number $E(\Gamma)$, form a complete set of linear equivalence invariants for a binary relation $\Gamma$.)

**Note:** For given $\Gamma$, these linear invariants of $\Gamma$ are completely independent of $K$.

\[2\] Of course, if $L(N)$ is one of the graphs of type $L$ occurring in the decomposition of $C^\infty,\infty(\Gamma)$, we must factor $X^N - 1$ over $K$ to obtain the precise Kronecker decomposition of $K(\Gamma)$—but this factorization is not needed to obtain a complete list of $K$-linear invariants of $\Gamma$.  

20
6 Some Illustrative Examples

For the purpose of better understanding the algorithm explained in the preceding section, we shall in this section apply it (without proof) to a few specific examples. The remainder of the paper after this section, then establishes the correctness of this algorithm.

Note that the $\gamma_{m,n}$ which occur in equations (13), (14), (15), (16) of Th. 5.4 all satisfy

$$|m - n| \leq 2.$$  

We shall call a lattice-point $(m,n)$ **suitable** if it satisfies (22). It will be convenient to fit the various suitable $(m,n)$ into a single picture as follows:

For a given finite binary relation $\Gamma$, we obtain the **contraction-diagram** $D(\Gamma)$ for $\Gamma$ by inserting the integers $\gamma_{m,n}$ for $(m,n)$ in the above diagram. For example, for the graph $\Gamma_3$ of Example 4.4, the pictures in that Example show that

$$\gamma_{0,0}(\Gamma_3) = 4, \gamma_{1,0}(\Gamma_3) = 3 = \gamma_{0,1}(\Gamma_3)$$

Also, it will readily be verified that

$$(C_l)^2\Gamma_3 = C_l\Gamma_3, (C_r)^2\Gamma_3 = C_r\Gamma_3$$

and that

$$C_lC_r\Gamma_3 \cong \Gamma_2 \cong C_rC_l\Gamma_3$$

where $\Gamma_2$ is the graph in Example 2.2; so we have

$$\gamma_{2,0}(\Gamma_3) = \gamma_{0,2}(\Gamma_3) = 3, \gamma_{1,1}(\Gamma_3) = 2.$$  

Thus the contraction-diagram for $\Gamma_3$ is:

$$D(\Gamma_3) = \begin{array}{c}
\begin{array}{cccc}
4 & 3 & 2 & 2 \\
3 & 2 & 2 & 3 \\
2 & 2 & 3 & 3 \\
2 & 2 & 3 & 3 \\
3 & 2 & 2 & 3 \\
4 & 3 & 2 & 2 \\
\end{array}
\end{array}$$
where all the intersections with no integer marked, are assigned to 2. The reader may wish to verify, as an exercise, the following somewhat abbreviated contraction-diagrams for the graphs $\Gamma_1$ and $\Gamma_2$ in Examples 2.1 and 2.2:

$$D(\Gamma_1) = \begin{array}{c}
\begin{array}{ccc}
& 2 & \\
2 & & 1 \\
& 2 & \\
\end{array}
\end{array}, \quad D(\Gamma_2) = \begin{array}{c}
\begin{array}{c}
1 \\
\end{array}
\end{array}$$

Unfortunately, the three graphs $\Gamma_1, \Gamma_2, \Gamma_3$ studied up to this point, seem too small to show all the features to be illustrated. It’s time for a slightly heftier example: Consider the binary relation

$$\Gamma_4 = (R_4, S_4)$$

on the set $S_4$ of integers from 0 to 17, represented by the following diagram:

Let us next compute the left contraction

$$C_l \Gamma_4 = ^1(\Gamma_4)^0$$

of this binary relation:
We first list all cases where two arrows emerge from the same vertex: they are furnished by

$$1 \leftarrow 0 \rightarrow 11, \ 6 \leftarrow 5 \rightarrow 14, \ 8 \leftarrow 7 \rightarrow 15, \ 5 \leftarrow 14 \rightarrow 15$$

together with the 3 arrows emerging from 2 to 3, 8, 12.
Thus the equivalence relation $\sim^{l=1}_{r=0}$ is here generated by

$$1 \sim^l 11, \ 6 \sim^l 14, \ 8 \sim^l 15, \ 5 \sim^l 15 \text{ and } 3 \sim^l 8 \sim^l 12 \sim^l 3$$

yielding the 12 $\sim^l$-equivalence-classes

$$\{0\}, \ \{1, 11\}, \ \{2\}, \ \{3, 5, 8, 12, 15\}, \ \{4\}, \ \{6, 14\}, \ \{7\}, \ \{9\}, \ \{10\}, \ \{13\}, \ \{16\}, \ \{17\}, \ .$$
which make up the vertices of the left-contracted binary relation $C^L_4 \Gamma^M_4$. Hence $\gamma_{1,0}(\Gamma_4) = 12$. In order to iterate this contraction process, and so compute the higher $\gamma$’s, we must also compute the contracted relation $C^L_4 R^M_4$ on these 12 equivalence classes; using Def.4.1 we obtain the following diagram:

$$ C^L_4 \Gamma^M_4 =: \{4\} $$

$$ \{2\} \rightarrow \{3, 5, 8, 12, 15\} \rightarrow \{16\} \rightarrow \{17\} $$

$$ \{0\} \rightarrow \{1, 11\} \rightarrow \{13\} \rightarrow \{6, 14\} \rightarrow \{7\} \rightarrow \{9\} \rightarrow \{10\} $$

A similar straightforward process can be used to compute the right contraction $C^R_4 \Gamma_4$. Here, we must begin by listing all cases where two arrows converge to the same vertex —for $\Gamma_4$, one such example is

$$ 11 \to 12 \leftrightarrow 2, \text{ whence } 11 \sim 2. $$

The integers from 0 to 17 then divide into 12 equivalence-classes with respect to $\sim^L (\Gamma_4)$, namely

$$ \{0\}, \{1\}, \{2, 4, 7, 11, 14\}, \{3\}, \{5, 13\}, \{6\}, \{8\}, \{9, 17\}, \{10\}, \{12\}, \{15\}, \{16\} $$

so that $\gamma_{0,1}(\Gamma) = 12$.

Let us continue this process, computing the binary relation

$$ C^L_4 C^M_4(\Gamma_4) = (R^L_4, R^M_4) $$

for the first few $(L, M)$ which are ‘suitable’, i.e., for which $|L - M| \leq 2$. We obtain a steadily coarsening collection of partitionings $S^{L,M}_4$ of

$$ S_4 = \{0, 1, \cdots, 17\} $$

of which the first three are given above, while also

$$ S^{2,0}_2 = \{\{0\}, \{1, 11\}, \{2, 3, 5, 7, 8, 12, 15\}, \{4, 6, 9, 13, 14, 16\}, \{10\}, \{17\}\} $$

$$ S^{1,1}_2 = \{\{0\}, \{1, 2, 4, 6, 7, 11, 14\}, \{3, 5, 8, 12, 13, 15\}, \{9, 17\}, \{10\}, \{16\}\} $$

$$ S^{0,2}_2 = \{\{0, 1, 3, 5, 6, 13\}, \{2, 4, 7, 11, 12, 14\}, \{8, 16\}, \{9, 17\}, \{10\}, \{15\}\} $$

$$ S^{2,1}_2 = \{S_4 \setminus \{0, 10\}, \{0\}, \{10\}\} $$

$$ S^{1,2}_2 = S^{1,3}_2 = \{S_4 \setminus \{9, 10, 17\}, \{9, 17\}, \{10\}\} $$

$$ S^{2,2}_2 = S^{2,3}_2 = S^{2,4}_2 = \{S_4 \setminus \{10\}, \{10\}\} $$

$$ S^{3,1}_2 = \{S_4 \setminus \{0\}, \{0\}\} $$

—and where all further $S^{L,M}_4$ with $(L, M)$ suitable consist of the single equivalence class

$$ S_4 = \{0, 1, \cdots, 17\} $$
Thus, we obtain the values of $\gamma_{m,n}(\Gamma_4)$ in the following contraction-diagram for $\Gamma_4$:

(D(\Gamma_4) = ⫷

(abbreviated by the same convention as before, whereby all ‘suitable’ vertices not shown or not labelled, are understood to have assigned as labels the minimal $\gamma$-value, which here is 1).

Another useful further result of this computation, is that — as a special case of Prop 5.6 — the graphs $C^L_mC^M_r(\Gamma_4)$ repeatedly contracted from $\Gamma_4$ eventually stabilize at the graph $C^{\infty,\infty}(\Gamma_4) = C_1$, consisting of one vertex, and one loop at that vertex.

Thus, we have computed $D(\Gamma_4)$ and $C^{\infty,\infty}(\Gamma_4)$; it only remains to note the number

$E(\Gamma_4) = 23$

of edges in $\Gamma_4$, to have obtained a complete set of linear equivalence invariants, according to Theorem 5.8.

NOTE: A slight strengthening of Prop 5.1 furnishes a useful repeated check during such computations, which works as follows: Consider, for example, the computation of

$\gamma_{2,2}(\Gamma_4) = \text{number of vertices in } C^L_2C^2_r(\Gamma_4)$.

The check in question consists in noting that we have two ways to construct the ‘multi-contracted’ graph in question, namely as $C^L_l(C^L_lC^2_r)$ and as $C^r_r(C^L_2C^2_r)$, and also in noting that, if as above we compute these graphs in terms of partitionings of $S_4$, then we have not only graph-isomorphism between them (as asserted by Prop 5.1), but even more: actual identity of the partitionings involved (whence the binary relations involved, as determined by Def 4.1, also coincide). This stronger version of Prop 5.1 is actually what is proved below, in §??.
Having thus obtained a complete set of linear equivalence invariants for \( \Gamma_4 \), the assertions of the preceding § tell us how to compute from them, the Kronecker invariants for the \( K \)-transformation pair \( K(\Gamma_4) \). Let us now follow the instructions for doing so:

We first direct our attention to Part One of Th.5.4 which expresses some of the Kronecker invariants —namely

\[
0^0 t_n(K(\Gamma)), 0^0 t_n^0(K(\Gamma)), 0^0 t_n^0(K(\Gamma)) \quad (\text{all with } n > 0)
\]

—in terms of the \( \gamma_{m,n}(\Gamma) \), i.e. of the contraction-diagram for \( \Gamma \). Equations \( \{13\} - \{16\} \) of this theorem, may perhaps be more easily visualized in term of the contraction-diagram, as follows.

At this point, the reader is asked to examine carefully the picture (\( \ast \)) at the beginning of this section. We shall see that this picture may usefully be sub-divided into smaller parts according to three different methods. We begin with the simplest method of sub-division:

**SUB-DIVISION ONE:**

This sub-divides the picture (\( \ast \)) into the coordinate squares of the \((m, n)\) lattice—as follows:

If \( S \) is one of these squares, with vertices

\[
a = (a_1, a_2), b = (b_1, b_2), c = (c_1, c_2), d = (d_1, d_2)
\]

as indicated in the above figure, let us define the \( \Gamma \)-content of \( S \) to be

\[
|S|_\Gamma := \gamma_a(\Gamma) - \gamma_b(\Gamma) - \gamma_d(\Gamma) + \gamma_c(\Gamma)
\]
For instance, the Γ-content of the square $A_1$ is

$$|A_1|_\Gamma = \gamma_{0,0}(\Gamma) - \gamma_{1,0}(\Gamma) - \gamma_{0,1}(\Gamma) + \gamma_{1,1}(\Gamma)$$

which by equation (15) is equal to the Kronecker invariant $0t_1^0(\Gamma \Gamma)$. We may similarly visualize the remaining information in equation (15): which tells us that the Γ-contents of the central squares $A_1, A_2, A_3, \ldots$ in the preceding Figure* $A$, are equal respectively to the Kronecker invariants

$$0t_1^0(\Gamma \Gamma), 0t_3^0(\Gamma \Gamma), 0t_5^0(\Gamma \Gamma), \ldots$$

with odd subscripts.

Applying this method to the particular case $\Gamma_4$, we see that values for $0t_{2i+1}^0$ (odd subscripts) are furnished by the central squares $A_1, A_2, A_3, \ldots$ in $D(\Gamma_4)$, as follows:

$$0t_1^0(\Gamma \Gamma_4) = |A_1|_{\Gamma \Gamma_4} = 18 + 6 - 12 - 12 = 0, 0t_3^0(\Gamma \Gamma_4) = 6 - 3 - 3 + 2 = 2$$

while

$$0t_5^0(\Gamma \Gamma_4) = 2 - 2 - 1 + 1 = 0, 0t_7^0(\Gamma \Gamma_4) = 0t_9^0(\Gamma \Gamma_4) = \ldots = 1 - 1 - 1 + 1 = 0$$

Similarly, applying these facts to the abbreviated contraction-diagrams computed above for the binary relations $\Gamma_1, \Gamma_2, \Gamma_3$, we obtain

$$0t_1^0(\Gamma \Gamma_1) = |A_1|_{\Gamma \Gamma_1} = 3 - 2 - 1 + 1 = 1, 0t_3^0(\Gamma \Gamma_2) = 2 - 1 - 1 + 1 = 1,$$

$$0t_5^0(\Gamma \Gamma_3) = 4 - 3 - 3 + 2 = 0.$$  

This is in agreement with the fact (obtained by direct $ad$ $hoc$ methods in §2) that each of $K(\Gamma_1), K(\Gamma_2)$ has in its Kronecker decomposition, exactly one direct summand of type $0T_1^0$.

We may also use Figure $*A$ (together with the concept of Γ-content) to visualize equation (16). In these terms, this equation asserts two things:

A) The non-central off-diagonal squares in Figure $*A$ occur in pairs

$$B_1 \text{ and } B_1'; \ B_2 \text{ and } B_2'; \ B_3 \text{ and } B_3'; \ldots$$

for which both elements in a pair have the same Γ-content:

$$|B_n|_\Gamma = |B'_n|_\Gamma \text{ for all } n \geq 1.$$  

B) These common Γ-contents, are equal respectively to the Kronecker invariants

$$0t_2^0(\Gamma \Gamma), 0t_4^0(\Gamma \Gamma), 0t_6^0(\Gamma \Gamma), \ldots$$
with *even* subscripts.

For example, here is what we get if we apply these assertions to the contraction-diagram for $\Gamma_4$:

$$|B_1|_{\Gamma_4} = 12 - 6 - 6 + 3 = |B'_1|_{\Gamma_4} = 3 = 0_{t_2}(K\Gamma_4)$$

and all $0_{t_2_p}(K\Gamma_4)$ with $p > 1$ are 0—e.g.,

$$|B_2|_{\Gamma_4} = 3 - 3 - 2 + 2 = |B'_2|_{K\Gamma_4} = 3 - 2 - 2 + 1 = 0 = 0_{t_4}(K\Gamma_4)$$

So much for equations (15) and (16). In order to find a similar interpretation of equation (13), which furnishes the values of the multiplicities $t_n$, it is necessary to sub-divide a portion of the picture ($\ast$) at the beginning of this section, into quadrilaterals by a second method. Here, unlike Sub-Division One, where the entire diagram ($\ast$) was covered by the quadrilaterals $A_i, B_i, B'_i$, we are led by equation (13) to introduce quadrilaterals $C_1, C'_1, C_2, C'_2, C_3, C'_3, \ldots$

which cover roughly $2/3$ of ($\ast$), as follows. The quadrilaterals $C_1, C'_1; C_2, C'_2; C_3, C'_3; \ldots$ in this figure are all parallelograms, with sides either vertical or parallel to the indicated $n$-axis. (In addition to these quadrilaterals, the picture also contains a number of isosceles right triangles, which seem irrelevant to our present purposes.)

**SUB-DIVISION TWO:**

If $Q$ is one of these quadrilaterals, with vertices

$$a = (a_1, a_2), \ a' = (a'_1, a'_2), \ b = (b_1, b_2), \ b' = (b'_1, b'_2)$$

as indicated in the above figure, let us define the $\Gamma$-content of $Q$ to be

$$|Q|_{\Gamma} := \gamma_a(\Gamma) + \gamma_{a'}(\Gamma) - \gamma_b(\Gamma) - \gamma_{b'}(\Gamma),$$

27
For instance, the $\Gamma$-content of $C_1$ is

$$|C_1|_\Gamma = \gamma_{0,2}(\Gamma) + \gamma_{1,0}(\Gamma) - \gamma_{0,1}(\Gamma) - \gamma_{11}(\Gamma)$$

while that of $C'_1$ is

$$|C'_1|_\Gamma = \gamma_{1,2}(\Gamma) + \gamma_{2,0}(\Gamma) - \gamma_{1,1}(\Gamma) - \gamma_{2,1}(\Gamma)$$

and equation (13) (for $n = 1$) asserts that these two expressions are equal, and that their common value is the Kronecker multiplicity $^0t_1(\Gamma T)$. We may similarly visualize the remaining information in equation (13), which asserts that, for all positive $n$, $C_n$ and $C'_n$ have the same $\Gamma$-content, and that this equals the Kronecker multiplicity $^0t_n$ of $^0T_n$ in $\Gamma T$: 

$$|C_n|_\Gamma = |C'_n|_\Gamma = ^0t_n(\Gamma T)$$

For the particular case $\Gamma_4$, these facts become:

$$|C_1|_\Gamma = 5 + 12 - 12 - 6 = 0 = 3 + 6 - 6 - 3 = |C'_1|$$

giving the value 0 for $^0t_1(\Gamma T_4)$; and similarly

$$^0t_2(\Gamma T_4) = 1, \ ^0t_n(\Gamma T_4) = 0 \text{ for } n > 2$$

The final information remaining to be discussed in Th. 5.4 is that furnished by equation (14), which we shall visualize in terms of the following SUB-DIVISION THREE:

Here, we have the diagram (*) sub-divided into isosceles right triangles, together with a collection of paired parallelograms

$$D_1 \text{ and } D'_1; \ D_2 \text{ and } D'_2; \ D_3 \text{ and } D'_3; \text{ etc.}$$
each of whose sides is either vertical, or parallel to the indicated m-axis. If R is one of these parallelograms, with vertices as indicated, we define the \( \Gamma \)-content of \( R \) to be

\[
|R|_\Gamma := \gamma_a(\Gamma) + \gamma_a'(\Gamma) - \gamma_b(\Gamma) - \gamma_b'(\Gamma)
\]

It is readily verified that equation (14) is equivalent to asserting that the two following statements hold for all positive integers \( n \) and all binary relations \( \Gamma \):

Firstly, the two paired parallelograms \( D_n \) and \( D'_n \) of this third subdivision, have the same \( \Gamma \)-content.

Secondly, this common \( \Gamma \)-content is equal to the Kronecker invariant \( t_0^n(\Gamma) \) of \( K\Gamma \):

\[
|D_n|_\Gamma = |D'_n|_\Gamma = t_0^n(\Gamma)
\]

Applying this to the particular case \( \Gamma_4 \), we obtain

\[
|D_3|_{\Gamma_4} = 2 + 1 - 1 - 1 = 1 = |D'_3|_{\Gamma_4}, \text{ so } t_0^3(\Gamma) = 1
\]

while all other \( t_0^n(\Gamma_{\Gamma_4}) \) vanish.

This exhausts the information provided by combining Part One of Th.5.4 with our knowledge of \( D(\Gamma_4) \). We next turn to Part Two of Th.5.4 applying this to the result obtained above, that the contraction-stabilization of \( \Gamma_4 \) is the graph consisting of a single loop on one vertex, we see that all \( t_n(\Gamma_4) \) are 0, and that the only indecomposable summand of \( K\Gamma_4 \) of type \( S(p^n) \), is \( S(X - 1) \) with multiplicity 1.

This leaves only one further indecomposable type to examine in relation to \( K\Gamma_4 \), namely the exceptional type \( 0T_0^0 \). To sum up our work on \( \Gamma_4 \) up to this point, we have obtained for \( K\Gamma_4 \) the following non-zero multiplicities:

\[
0t_2^0 = 3, \ 0t_3^0 = 2, \ 0t_2 = 1, \ 0t_3 = 1, ; [K\Gamma_4 : S(X - 1)] = 1
\]

with all other multiplicities (except possibly that of \( 0T_0^0 \)) being 0. Note that these computations indeed check with eqn. (21):

\[
\mathcal{V}(\Gamma_4) = 18 = 2\mathcal{V}(0t_2^0) + 3\mathcal{V}(0t_3^0) + \mathcal{V}(0t_2) + \mathcal{V}(t_3^0) + 1 \cdot (\deg(X - 1)) = 2(3) + 3(2) + 2 + 3 + 1
\]

Applying Part Three of Th.5.4 in the form of eqn. (20), and utilizing the previously obtained and checked results for the other Kronecker multiplicities for \( K\Gamma_4 \), together with the observation that \( \Gamma_4 \) has 23 edges, we obtain

\[
0t_0^0(\Gamma_4) = 23 - 1 \cdot (2 + 3) - 0 - (3)(3) - (4)(2) - (1)(1) = 0
\]
Thus, finally, we obtain the Kronecker decomposition
\[ K\Gamma_4 \cong 3^0T_2^0 \oplus 2^0T_3^0 \oplus T_3^0 \oplus S(X-1). \]

The reader is invited to check similarly, using Th.5.4, that the Kronecker decompositions for the binary relations explained in Examples 2.1, 2.2 and 4.4 are:
\[ K\Gamma_1 \cong 0T_1^0 \oplus T_1^0 \oplus S(X-1), \quad K\Gamma_2 \cong 0T_0^0 \oplus T_1^0 \oplus S(X-1), \]
and
\[ K\Gamma_3 \cong 0T_1^0 \oplus T_1^0 \oplus S(X^2-1) \]
(where the first of these decompositions agrees with the result \( \text{[10]} \) obtained earlier by other methods.)

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