The Geometry of Numbers in the Digital Age

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The interaction of geometry with other branches of mathematics has proven to be fruitful and mutually beneficial. Let us mention just four examples of this relationship: (1) The synthesis of geometry and algebra by Descartes and Fermat giving rise to analytic geometry; (2) The theory of dynamical systems by Henri Poincaré, which allowed the marriage of geometry and differential equations; (3) The use of cutting sequences by Hedlund and Morse to exemplify the powerful machinery of the symbolic dynamical systems theory that they created; and (4) The union between geometry and number theory accomplished by Hermann Minkowski. In this paper, we explore the amazing relationship between number theory and geometry.

In 1887, Hermann Minkowski submitted the paper "Räumliche Anschauung und Minima positiv definiter quadratischer 1 as a requisite to apply for a vacant position at the University of Bonn.

¹Spatial insight and minima of positive definite quadratic forms
The great J. Dieudonné considered this report as one that “...contains the first example of the method which Minkowski would develop some years later in his famous geometry of numbers” [2].

The Geometry of Numbers [4] is Minkowski’s posthumously published book pioneering the study of number theory problems in the realm of geometry. In it, Minkowski conceived geometry as a scenario to represent complicated problems in number theory and a platform to obtain clues about its solutions by means of the Räumliche Anschauung. A fundamental tool for this purpose is the set of points in the plane with integer coordinates, the so-called standard lattice [5].

One way that the standard lattice has been useful is in working out problems by simply giving an intuitive explanation of important known mathematical concepts. For example, Figure 1 shows the standard lattice and a line through the origin with an irrational slope. The fact that the line does not cross any other lattice points leads immediately to the definition of Dedekind’s cut. This same line can be used to define a Sturmian sequence, which is a biinfinite sequence using the symbols 0 and 1 obtained in the following way: adding a symbol 1 or a symbol 0 every time the line crosses a horizontal lattice line or a vertical lattice line, respectively. At the origin, the only point where the line crosses a horizontal and vertical lattice line simultaneously, we decide to add a word 01 rather than choosing between a symbol 0 or 1. This construction characterizes all the biinfinite sequences in the symbols 0 and 1 such that the differences in the number of 0’s between any two consecutive symbols 1’s is at most 1. Finally, as an illustration of the density of the rational numbers in the reals, let us consider the particular case when the line has a slope equal to the golden mean $\Phi = \frac{1 + \sqrt{5}}{2}$. Since it is known that the sequence $\frac{F_{n+1}}{F_n}$, where $F_n$ is the $n$-th term of the familiar Fibonacci sequence, converges to $\Phi$, then the family of lines (through the origin) having slopes equal to $\frac{F_{n+1}}{F_n}$ converges to the line having slope $\Phi$. Therefore, since the pairs of relatively prime numbers $(F_n, F_{n+1})$ determine the family of lines, we may say that there is a sequence of pairs of relatively prime numbers that represent the golden mean number.

In its origins in ancient Greece, geometry was an eminently visual discipline. With the increasing level of abstraction that it attained in the XIX century, it nearly left aside any pictorial
representation. Minkowski’s book does not have a single illustration. In this essay, we show that with the advent of digital computers, images can play again a central role in the understanding and development of geometry.

With this purpose, we show that when plotting a certain large collection of relatively prime numbers, a family of quadratic arcs emerges. The appearance of these arcs was unexpected and, to the best of our knowledge, has not been observed before. We divide this work into two sections. In the first section, we introduce the Bézout transformations and use them to generate a special family of relatively prime numbers. In the second section, we propose two arguments that justify the appearance of quadratic arcs in the graphs introduced in Section 1. Our first argument is algorithmic, and the second is algebraic.

![Figure 1: The standard lattice and a line with irrational slope](image)

1 Bézout transformation

Recall that an integer \( p \) greater than one is said to be prime if its only (positive) divisors are 1 and \( p \) itself. The fundamental theorem of arithmetic states that any positive integer greater than...
one is either prime or the product of prime numbers and that this decomposition is unique except for the order of the factors. Two positive integers \( p \) and \( q \) are said to be relatively prime if their only common (positive) divisor is 1. Notice that when \( p \) is a prime number, then every positive number \( q \) less that \( p \) is relatively prime to \( p \); however, if \( p \) is not prime, then the number of positive integers relatively prime to \( p \) is given by the well-known Euler totient function \( \phi(p) \). For example, if \( p \) is prime, then \( \phi(p) = p - 1 \).

If one draws dots in the Cartesian plane each representing a pair of relatively prime numbers, it is intriguing to discover a fractal-like picture; see, for example, [7]. In Figure 2 we draw all pairs of relatively prime numbers \((p, q)\) with \( 1 \leq q \leq p \leq 1000 \).

![Figure 2](image.png)

Figure 2: Pairs \((a, b)\) of relatively prime numbers with \( p = 1000 \) and \( 0 < a \leq b \leq p \). A detail of the graph is shown in the box on the top left-hand side.

We began to be interested in particular pairs of relatively prime numbers; namely, for a fixed a
positive integer \( p \) greater than 1, we associated to each pair \((p, q)\) of relatively prime numbers its corresponding Bézout coefficients \((a, b)\), that is, the unique pair of integers such that \(0 < a \leq p\), \(0 \leq b \leq q\) and that satisfy the Bézout identity

\[
aq - bp = 1.
\] (1)

We refer to [6, Theorem 5.1] for a proof of their existence. For convenience, let us denote by \(B_1(p, q)\) the Bézout coefficients of \((p, q)\). Therefore, for example, \(B_1(6, 5) = (5, 4)\) and \(B_1(5, 2) = (3, 1)\).

Let us now mention the Farey sequences as a motivation for restricting our attention to pairs of relatively prime numbers \((p, q)\) and their corresponding Bézout coefficients \(B_1(p, q) = (a, b)\). A Farey sequence \(F_p\) of order \(p\) is the set of all reduced nonnegative fractions less than or equal to 1 and with denominators less than or equal to \(p\), arranged in increasing order. For example,

\[
F_5 = \{0/1, 1/5, 1/4, 1/3, 2/5, 1/2, 3/5, 2/3, 3/4, 4/5, 1/1\}
\]

and

\[
F_6 = \{0/1, 1/6, 1/5, 1/4, 1/3, 2/5, 1/2, 3/5, 2/3, 3/4, 4/5, 5/6, 1/1\}
\]

For a fixed positive integer \( p \), notice that there is a one-to-one correspondence between the elements \(b/a\) in \(F_p\) and the points \((a, b)\), which belong to the set of all pairs of relatively prime numbers with \(0 < b \leq a \leq p\). For example, if we graph all points \((a, b)\) such that \(b/a\) belongs to \(F_p\) with \(p = 1000\), then we recover Figure 2.

Farey sequences have remarkable properties. One such property is the following. Given \(q/p \in F_p\), the predecessor \(b/a\) of \(q/p\) in \(F_p\) satisfies Bézout’s relation \(aq - bp = 1\) [1, Theorem 5.3, pp 98]. Hence, if \(b/a\) is the predecessor of \(q/p\) in the Farey sequence \(F_p\), then \(B_1(p, q) = (a, b)\). Hence, in other words, we are proposing to highlight neighbors of the elements of the form \(q/p\) in the Farey sequence \(F_p\). Now, we already pointed out that the Bézout coefficients \(B_1(p, q) = (a, b)\) of \((p, q)\)
correspond to the predecessor $b/a$ of $q/p$ in $F_p$. Analogously, we observed that the successor $d/c$ of $q/p$ in $F_p$ satisfies the equality
\[
eq -1 \tag{2}
\]
and $0 < c \leq p$ and $0 < d \leq q$. We remark that the $q/p$ in $F_p$ is the mediant sum of its neighbors $b/a$ and $d/c$, that is, $q/p = (b + d)/(a + c)$, see [1] Theorem 5.2, pp 98).

Hence, for a pair of relatively prime numbers $(p, q)$, we are motivated to introduce a second Bézout transformation $B_{-1}$, analogous to the Bézout transformation $B_1$, defined as $B_{-1}(p, q) = (c, d)$, where $(c, d)$ is the unique pair of integers that satisfy Equation (2) and $0 < c \leq p$ and $0 < d \leq q$. Hence, if $B_1(p, q) = (a, b)$, then $B_{-1}(p, q) = (p - a, q - b)$.

With this terminology, it will be convenient to introduce the transformation $B_0$ on a pair of positive integers $(p, q)$ to be defined as a unique pair of relatively prime numbers $(r, s)$ such that $rq - sp = 0$. Therefore, $B_0$ is just the identity when evaluated in a pair of relatively prime numbers. Finally, let $F$ denote the flip transformation, that is, $F(p, q) = (q, p)$ for every pair $(p, q)$.

**Figure 3:** Plotting points $B_1(p, q)$ with $0 < q < p = 10^6$ relatively prime.

**Figure 4:** Plotting points $B_1(p, q)$ and $B_1(q, p)$ with $0 < q < p = 10^6$ relatively prime.

**Figure 5:** Plotting points $B_{\pm 1}(p, q)$ and $B_{\pm 1}(q, p)$ with $0 < q < p = 10^6$ relatively prime.

It became apparent that these transformations hold a number of symmetries. In Figures 3
we show how plotting $B_1(p, q)$ together with $B_{-1}(p, q)$ and $B_{\pm 1}(q, p)$ generates a graph containing some intriguing arcs. There are properties that explain some of these symmetries:

(a) $F$ is invertible and $F^{-1} = F$.

(b) $B_{-1} \circ F = F \circ B_1$. Here, the symbol $\circ$ denotes composition of functions.

(c) $B_{-1} + B_1 = B_0$. Here, the addition is the usual component-wise addition of pairs of numbers, that is, $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$. This operation defines a group structure on $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$. This property corresponds to the mediant sum of Farey sequences.

Furthermore, if $B_1(p, q) = (a, b)$ and $q < p$, then it follows that

(d) $B_{-1}(p, p - q) = (a, a - b)$

(e) $B_1(p, a) = (q, b)$

(f) $B_1(q, p) = (q - b, p - a)$

To plot the Bézout coefficients in Figures 2, 3, and 4, the extended Euclidean algorithm is used. The extended Euclidean algorithm uses the Euclidean algorithm repeatedly to produce the coefficients for the Bézout identity. Specifically, for $(p, q)$ relatively prime and positive, it produces a pair of integer coefficients $(x, y)$ satisfying the following:

$$xq + yp = 1.$$ 

Note that then, one of the coefficients $x$ or $y$ must be positive and the other one negative or zero. If $y > 0$, then $x \leq 0$, and we obtain $B_{-1}(p, q) = (|x|, y)$, while if $y \leq 0$ and $x > 0$, then $B_1(p, q) = (x, |y|)$. If we obtained a pair $(a, b) = B_{-1}(p, q)$, we calculate a new pair applying the transformation $a' = p - a$ and $b' = q - b$ so that $(a', b') = B_1(p, q)$. Similarly, if $(a, b) = B_1(p, q)$ then $B_{-1}(p, q) = (p - a, q - b)$.
Finally, we notice that, defining $B_1(p, q) = (aq/|q|, bp/|p|)$, where $B_1(|p|, |q|) = (a, b)$, one extends the Bezout transformation $B_1$ to all pairs of relatively prime integer numbers. Additionally, $B_{-1}$ may be extended to all pairs of relatively prime integer numbers $p$ and $q$, by the formula $B_{-1}(p, q) = (a'q/|q|, b'p/|p|)$ where $B_{-1}(|p|, |q|) = (a', b')$. Analogously, $B_0$ may be defined to all pairs $(a, b) \neq (0, 0)$ of integers by defining $B_0(a, b) = (rp/|p|, sq/|q|)$, where $B_0(|a|, |b|) = (r, s)$.

Thus, for a fixed positive number $p$ and for each $0 < q < p$ relatively prime to $p$, we plot the points of the form $B_{\pm 1}(p, q)$ together with those of the form $B_{\pm 1}(q, p)$ and their corresponding points in the other three quadrants, as described above. We call this graph the Bézout graph for $p$. For $p = 1000000$, see Figure 6 and for $p = 250000$, see Figure 7. To discard the possibility of an optical artifact, we include Figure 8, where the scale is 1:1 and $p = 512$. It is interesting to notice that the arcs that seem to be forming in these figures appear when $p$ has at least one repeated prime in its prime factorization. For example, for $p = 317811 = (3)(13)(29)(281)$, which incidentally is also a Fibonacci number, it has a Bézout graph with no discernible arcs, while for $p = 46368 = (2^5)(3^2)(7)(23)$, also a Fibonacci number, the arcs show up. See Figure 9.
Figure 6: Bézout graph for $p = 1000000$. 
2 The Geometry of relative prime numbers

We are going to give two arguments to justify the appearance of the quadratic arcs that appear to be forming in Figures 6, 7 and 8. Our first argument is algorithmic, while the second is algebraic.
Figure 8: A 1:1 scale graph of Bézout coefficients of pairs $(p, q)$ of relatively prime numbers with $p = 512$.

2.1 Algorithmic argument

Let us begin by fixing $p$ as a positive integer and suppose that $q < p$ is a positive integer relatively prime to $p$. Let $B_1(p, q) = (a, b)$. We next show how to derive an arithmetic sequence of relatively prime pairs $\{(p, q + nt)\}_{n \in \mathbb{Z}}$ so that their Bézout coefficients satisfy a certain quadratic relation. In fact, when fixing $p$ and repeating this exercise for all $q$ relatively prime to $p$, the graph obtained by plotting the corresponding Bézout coefficients of the pairs in the sequence turns out to recover some of the arcs in Figure [6]. We point out that while this argument exhibits quite clearly why the points in Figure [6] are points of a quadratic function, the common difference $t$ in the arithmetic sequence $q + nt$ is sometimes too large to be useful since we are restricted to integers $q$ less than $p$.  

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We obtain the arithmetic sequences as follows. Let $w$ be the smallest positive integer such that $pw$ is a perfect square. Let $B_0(\sqrt{pw}, q-a) = (r, s)$. Put $t = r\sqrt{pw}$. Hence,

$$B_1(p, q + nt) = (a - nt, b - swn - r^2wn^2).$$

For example, for $p = 1024$, let $q = 817$. We compute $B_1(p, q) = (465, 371)$. In this case, $w = 1$, since $p$ is a perfect square. Then, $B_0(\sqrt{pw}, q-a) = B_0(32, 352) = (1, 11) = (r, s)$. Hence, the common difference is $t = 32$. We list some of the obtained points in Table 1 and illustrate them in Figure 10.

### 2.2 Algebraic argument

We now give a second argument to explain the formation of the remarkable symmetries presented in Figures 6, 7 and 8. This argument, unlike the given in the previous section, is algebraic in nature. Let $p$ be a positive integer. Recall that the set $\mathbb{Z}_{p^*}$, which consists of all positive integers
Table 1: The points of the form $B_1(1024, 817 + 32n)$ in the right column belong to a quadratic arc.

| $n$ | $(p, q + nt)$ | $B_1(p, q + nt)$ |
|-----|---------------|------------------|
| −2  | (1024, 753)   | (529, 389)       |
| −1  | (1024, 785)   | (497, 381)       |
| 0   | (1024, 817)   | (465, 371)       |
| 1   | (1024, 849)   | (433, 359)       |
| 2   | (1024, 881)   | (401, 345)       |
| 3   | (1024, 913)   | (369, 329)       |

less than $p$ and relatively prime to $p$, forms an Abelian group of order $\phi(p)$, with the operation multiplication module $p$, which we shall denote by $\cdot$; the reader is referred to [3, Theorem 2.4.7, pp. 62] for more details about this group. Notice that the group unit in $\mathbb{Z}_p^*$ is the number 1. As an example, suppose $p = 9$; then, we have $\mathbb{Z}_9^* = \{1, 2, 4, 5, 6, 7, 8\}$, and the product $2 \cdot 5$ in $\mathbb{Z}_9^*$ is equal to 1 since $(2)(5) = 10 \equiv 1 \mod 9$. Similarly, $4 \cdot 6 = 6$ in $\mathbb{Z}_9^*$.

We now introduce the following function. Let $p$ be a positive integer. We define the function

$$\theta_p : \mathbb{Z}_p^* \rightarrow \mathbb{Z}_p^*$$

by $\theta_p(q) = a$ if $B_1(p, q) = (a, b)$; in other words, $\theta_p(q)$, with $q \in \mathbb{Z}_p^*$, is the projection on the first coordinate of $B_1(p, q)$. Recall the notation $a \equiv b \mod p$ means that $p$ divides the difference $a - b$. An equivalent way to define $\theta_p(q)$ is as $\theta_p(q) = q^{-1}$, the inverse of $q$ in $\mathbb{Z}_p^*$, that is, as the unique integer $0 < \theta_p(q) < p$ such that $q \theta_p(q) \equiv 1 \mod p$. In other words, the function $\theta_p$ maps an invertible element of the ring $\mathbb{Z}_p$ to its inverse in $\mathbb{Z}_p$. Finally, since Euler’s theorem [3, Theorem 2.4.8, pp. 63] states that $q^{\phi(p)} \equiv 1 \mod p$, one may also think of $\theta_p(q)$ as the remainder of dividing $q^{\phi(p)-1}$ by $p$. For example, when $(p, q) = (6, 5)$, we have $\phi(6) = 2$, so the remainder of dividing $5^{\phi(6)-1}$ by 6 is equal to 5. Thus, $\theta_6(5) = 5$, and therefore, the first component of $B_1(6, 5)$ is 5, as was obtained in the example after Bézout’s identity, Equation (1). Incidentally, the integer $\frac{q^{\phi(p)} - 1}{p}$, obtained from Euler’s theorem, is known as Euler’s quotient.
Since it has been observed that $\theta_p(q) = q^{-1}$, it follows that $\theta_p$ is an automorphism of $\mathbb{Z}_p^*$. An automorphism of a finite group can be regarded as a rule that permutes the elements of the group in such a way that geometry, in terms of algebra, is preserved. Hence, it can be argued that the symmetries exhibited in Figures 6, 7 and 8 are somehow the symmetries preserved by the group automorphism $\theta_p$. Indeed, essentially, we are plotting points of the form $B_1(p, q) = \left(\theta_p(q), \frac{q \theta_p(q) - 1}{p}\right)$, with $q \in \mathbb{Z}_p^*$. Thus, the first component of $B_1(p, q)$ is nothing but the image of an automorphism, while the second component may be regarded as an Euler quotient.


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