Sparse channel separation using random probes

Justin Romberg and Ramesh Neelamani

1 School of Electrical and Computer Engineering, Georgia Tech, Atlanta, GA, USA
2 ExxonMobil Exploration Company, Houston, TX, USA

E-mail: jrom@ece.gatech.edu and ramesh.neelamani@exxonmobil.com

Abstract

This paper considers the problem of estimating the channel response (or the Green’s function) between multiple source–receiver pairs. Typically, the channel responses are estimated one-at-a-time: a single source sends out a known probe signal, the receiver measures the probe signal convolved with the channel response and the responses are recovered using deconvolution. In this paper, we show that if the channel responses are sparse and the probe signals are random, then we can significantly reduce the total amount of time required to probe the channels by activating all of the sources simultaneously. With all sources activated simultaneously, the receiver measures a superposition of all the channel responses convolved with the respective probe signals. Separating this cumulative response into individual channel responses can be posed as a linear inverse problem. We show that channel response separation is possible (and stable) even when the probing signals are relatively short in spite of the corresponding linear system of equations becoming severely underdetermined. We derive a theoretical lower bound on the length of the source signals that guarantees that this separation is possible with high probability. The bound is derived by putting the problem in the context of finding a sparse solution to an underdetermined system of equations, and then using mathematical tools from the theory of compressive sensing. Finally, we discuss some practical applications of these results, which include forward modeling for seismic imaging, channel equalization in multiple-input multiple-output communication and increasing the field-of-view in an imaging system by using coded apertures.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

This paper gives a theoretical treatment to the problem of channel estimation in multiple-input multiple-output (MIMO) systems. The general scenario is illustrated in figure 1. A set of
Inverse Problems 26 (2010) 115015
J Romberg and R Neelamani

Figure 1. (a) Multiple sources on the left, multiple receivers on the right. The black arrow denotes
the direct arrival from source 2 to receiver 3. The colored arrows denote the indirect arrivals caused
by the different reflectors (denoted by small circles). (b) Channel response $h_{2,3}$ between source 2
and receiver 3.

$p$ sources emit different probe signals, which then travel through different channels and are
observed by $q$ receivers. We will assume that the channel between each source/receiver pair
is linear and time-invariant; if source $i$ sends the probe signal $\phi_i$, then receiver $j$ observes the
convolution $\phi_i \ast h_{i,j}$ of the probe signal with the corresponding channel response $h_{i,j}$. The
goal is to estimate all of these channel responses $h_{i,j}$ and to do so using the smallest total
amount of probing time.

We will focus on the discrete version of this problem. We assume that each channel
response $h_{i,j}$ has length $n$. If a single source $i$ emits a known probe sequence \{$\phi_i(1), \phi_i(2), \ldots, \phi_i(m)$\} of length $m$, receiver $j$ observes the linear convolution

$$y_{i,j}^{\text{lin}} = \phi_i \ast h_{i,j} = \begin{bmatrix}
\phi_i(1) & 0 & \cdots & 0 \\
\phi_i(2) & \phi_i(1) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\phi_i(n) & \phi_i(n-1) & \cdots & \phi_i(1) \\
\vdots & \vdots & \ddots & \vdots \\
\phi_i(m) & \phi_i(m-1) & \cdots & \phi_i(m-n+1) \\
0 & \phi_i(m) & \cdots & \phi_i(m-n+2) \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \phi_i(m)
\end{bmatrix} \cdot \begin{bmatrix}
h_{i,j}(1) \\
h_{i,j}(2) \\
\vdots \\
h_{i,j}(n)
\end{bmatrix} =: \Phi_i^{\text{lin}} h_{i,j}. \quad (1.1)$$

Throughout the paper, we will assume that the probing sequence is at least as long as the
channel, $m \geq n$ (this does not affect the discussion to much at this point, but will be important
below). With the probing sequence \{$\phi_i(t)$\} known, and given the observations $y_{i,j}^{\text{lin}}$, recovering
the channel response $h_{i,j}$ is a classical deconvolution problem. The inverse problem can be made well conditioned if $\phi_i$ is chosen carefully; if not, then the inversion can be regularized using some types of prior information about the channel.

We will measure the cost of the channel estimation by the amount of time we spend probing the channel, which we can see is proportional to $n + m - 1 = O(n + m)$, the number of rows in the linear system in (1.1). From a single source, we can estimate the response to all of the receivers by emitting a single probing sequence and solving (1.1) at each receiver $j$. If there are multiple sources, then typically the $p$ sources are activated one-at-a-time. In this case, the total amount of probing time required to estimate all of the $h_{i,j}$ is $O(p(n + m))$. In theory, $m$ could be made much smaller than $n$ in this situation, giving us a lower bound of $O(np)$ on the cost.

In this paper, we propose and rigorously analyze an alternative strategy for estimating the channels between each source–receiver pair. Our strategy will reduce the total amount of time spent on probing the channels by activating all of the sources simultaneously, and then jointly estimating the $h_{i,j}$. (This approach was first proposed in the context of seismic imaging in a related conference paper [28].) Now, of course, the sources will interfere with one another, and the receiver will observe the sum of each source convolved with its respective channel. With all $p$ sources active, the observations at receiver $j$ can be written as the following system of equations:

$$y_{lin,j} = \sum_{i=1}^{p} y_{lin,i,j} = \begin{bmatrix} \Phi_1^{lin} & \Phi_2^{lin} & \cdots & \Phi_p^{lin} \end{bmatrix} \begin{bmatrix} h_{1,j} \\ h_{2,j} \\ \vdots \\ h_{p,j} \end{bmatrix} =: \Phi^{lin}h_j. \quad (1.2)$$

The $\Phi^{lin}$ in (1.2) is the $(m + n - 1) \times np$ matrix formed by concatenating the source convolution matrices $\Phi_i^{lin}$ row-wise; the $h_j$ is the unknown $np$-vector consisting of the $p$ channel responses for the path between each source and receiver $j$. With all of the sources activated simultaneously, the total cost of the acquisition is $O(n + m)$, but now the channel responses interfere with one another. The question now is how long (quantified by $m$) the probing sequences must be to reliably ‘untangle’ the individual $h_{i,j}$ from the observations $y_{lin} = \Phi^{lin}h_j$. If the probing sequences are chosen carefully and in concert with one another, the system in (1.2) will be invertible for $m \approx np$, again making the total activation time $O(np)$. If we are interested in recovering all possible channels without making any assumptions about their structure, we of course cannot have $m < np$.

We will show that if the combined channel response $h_j$ is sparse, then the probing sequences can be significantly shorter than $np$ if they are random. This problem, along with the tools we will deploy to solve it, is closely related to the recent work in the field of compressive sensing (CS). The theory of CS states that vectors $x_0$ with $s$ non-zero components can be recovered from an underdetermined set of linear measurements $y = \Phi x_0$ if the matrix $\Phi$ is sufficiently diverse (the precise technical conditions are reviewed in detail in section 2). The essential contribution of this paper is to show that when the sequences $\{\phi_i(t)\}$, consist of independent and identically distributed Gaussian random variables, the matrix $\Phi^{lin}$ in (1.2) meets this diversity criterion for pulse lengths $m$ that are within a poly-logarithmic factor of the sparsity $s$. In particular, theorems 3.1 and 3.3 combined with proposition 2.1 shows that if the total number of significant components in $h_j$ is $s$, then it can be recovered from $y_{lin}$ for

$$m \gtrsim s \cdot \log^5(np),$$

reducing the total probing time to $O(n + s \log^5(np))$. When the channels are sparse, that is $s \ll np$, then the cost of estimating all of the channels is not much more than estimating a
single channel independently. Having the sources activated simultaneously introduces ‘cross-
talk’ between the different channels, but if each source uses a different random code, then
we can use the sparse structure of the channels to separate the cross-talk into its constituent
components.

While the general problem of recovering a sparse vector from an underdetermined system
of random equations has been studied in many different contexts in the CS literature, this
paper provides the first quantitative bounds for problems with the multichannel form of (1.2).
The system $\Phi^\text{lin}$ is random, since it is generated from the random probing sequences $\phi_i(t)$,
but it also has a great deal of structure; each of the concatenated $\Phi^\text{lin}_i$ are Toeplitz, with a total
of $(m + n - 1)n$ entries generated from only $m$ independent random variables. This type of
structure requires a far more intricate analysis than if the entries were all independent random
variables. Despite this additional structure, we will see that our theoretical bounds on the
number of required measurements exhibit the same linear-in-$s$ scaling as when the system is
chosen completely at random. It is also worth mentioning that matrices with the form of $\Phi^\text{lin}$
in (1.2) can be applied efficiently in an implicit manner, making them suitable for solving
large-scale problems. The cost of a matrix-vector multiplication is $O(np \log n)$, the same as
$p$ length-$n$ FFTs, and requires $O(np)$ storage.

In the remainder of this section, we will discuss some applications of the channel
separation problem and review recent related work. Section 2 provides an overview of
sparse recovery from underdetermined linear measurements. Section 3 carefully states our
main theorems, which provide a sufficient lower bound on the length of the probing signals
(in relation to the number of sources, the length of the channels and their sparsity) that allows
us to robustly recover $h_j$ from $y^\text{lin}$ using any number of sparse recovery algorithms. Proof of
these theorems is given in sections 4 and 5. The proofs rely heavily on estimates for random
sums of rank-1 matrices, which are overviewed in the appendix.

1.1. Applications

For further motivation, we discuss three specific scenarios in which this multichannel
separation problem arises.

Seismic exploration and forward modeling. Subsurface images of the earth are formed by
individually activating acoustic sources at different locations on the earth’s surface, and
then measuring the response to each individual source at a number of receiver locations.
From these recorded responses, a 3D subsurface model of the earth (consisting of the local
velocity of the propagating elastic waves) can be reconstructed using what is known as full
waveform inversion (FWI). Dense samplings for the positions of the acoustic sources led to
higher resolution reconstructions, but also longer field acquisitions and more computationally
intensive inversion.

The theoretical results in this paper suggest that these expenses can be mitigated by
activating the sources simultaneously using different random waveforms. In the field, this
approach reduces the amount of time required for the acquisition. Although the sources will
interfere with one another, the individual responses can be separated afterward by taking
advantage of the sparsity of each of the channel responses. The source waveforms will have
to be longer than if each of the $p$ sources were activated individually, but the net activation
time across all sources will be much smaller than $p$ individual channel probes.

Sparse models are common in seismic imaging [34, 35]. In practice, additional gains are realized by going beyond
the setting treated in this paper and modeling the channels jointly, viewing them as cross sections of a larger 3D image
(with an associated sparse transform) rather than as individual sparse channels; see [22], for example.
Sparse channel separation can also reduce the amount of computation required for the inversion. The most expensive step in wavefront inversion is testing a candidate model to see how well it fits the measurements that have been collected. This so-called forward modeling simulation consists of solving an extremely large PDE. The cost of this simulation is proportional to the length of the source signals (i.e. the number of time steps required), but does not depend at all on the number of sources that are active at one time—running a simulation with a single source active costs just as long per time step as with many sources active. If we simulate each of the \( p \) sources individually, we will need to run each simulation for \( O(n) \) time for a total cost of \( O(np) \) time steps (and the cost of each time step can be extremely high). If we simulate the sources simultaneously, then the number of time steps in the simulation can be \( O(n + s \log^5(np)) \). Given the results of the simulation with simultaneous active sources, we will of course have to recover the individual channel responses using some type of sparse recovery algorithm (solving the optimization program in (2.4) below, for example). But the computations required for this recovery are minor in comparison to the forward modeling simulation, especially given recent progress in optimization algorithms [2, 14, 18, 40] and the fact that the system \( \Phi_1 \) can be applied quickly using FFTs.

Source separation for seismic exploration is explored in further detail in the companion publications [28, 29] and in the independent work [21].

**Channel estimation in MIMO communications.** In wireless communications, transmitted signals often bounce off reflecting objects such as buildings and water bodies before reaching the receiver; this phenomenon is known as the multipath. Instead of the transmitted waveform, the receiver observes the convolution of the transmitted signal with the effective response of the wireless channel between the transmitter and the receiver. The channel response is characterized by the number of reflecting objects between the transmitter and receiver; if the number of reflecting objects is small, then the channel response is sparse. If the channel response is known, then the receiver can easily compensate for the multipath effect by deconvolving the channel response from the measured signal. Since the channel between a transmitter and a receiver can change with time, the channel response needs to be estimated periodically. The channel response is estimated by measuring the signal at the receiver when the transmitter emits a known probing sequence. If there are multiple transmitters and multiple receivers, we can save time by probing all of the channel pairs simultaneously, and separating the individual responses using sparse recovery. This approach is particularly useful when the channel is changing rapidly, a common problem in underwater acoustic communications [9].

**Coded aperture imaging.** In [24–26], an imaging architecture is introduced to increase the field-of-view (FOV) of a camera using coded apertures. A coded aperture is a series of small openings (apertures) whose net effect is to convolve a target image with a sequence (code) determined by the pattern in which these openings appear. Coded apertures offer a way around the classical trade-off between aperture size and image brightness; the multiple apertures overlay many copies of the image at slightly different shifts, making the image incident on the detector array ‘bright’ and easily recovered (via deconvolution) if the aperture code is chosen carefully (e.g. the MURA patterns in [16]).

The essential idea from [24] to increase the FOV without the sacrificing resolution is illustrated in figure 2: the image is broken into \( p \) subimages, each of which we would like to recover to a resolution of \( n \) pixels. Rather than measuring each subimage directly, which would require a detector array of size \( m = np \), we pass each image through its own coded aperture of size \( m \) and these coded subimages are combined onto a detector array of size \( m \). The task at hand, then, is to recover the full \( np \) pixel image from these \( m \) measurements.
Figure 2. Sketch of the architecture proposed in [24] for increasing the field of view of a camera. The image is broken into $p$ subimages, each subimage passes through a different coded aperture and the coded subimages are combined on the detector array. The effect of the coded aperture is to convolve the subimage with an associated code, and so (1.2) models the process mathematically.

This problem also conforms to our multichannel framework\(^4\)—in this case, we have $p$ sources and one receiver. Here, the known ‘probe signals’ are the coded aperture patterns and the unknown channels $h_{i,1}$ are the different subimages. The main results of this paper say that if the entire image is approximately $s$ sparse, then the size $m$ of the detector array needs to be only on the order of $s$ (within a log factor) rather than the full resolution $np$. If the images we are reconstructing are consecutive frames in a video sequence, the image sparsity can come from looking at the differences between consecutive frames.

1.2. Relationship to previous work

We have cast the multichannel separation problem as recovering a sparse vector from an underdetermined, random system of equations. This general problem has been studied extensively in recent literature under the name of compressive sensing (CS). The essential results from this field state that if we observe $y = \Phi x_0$, where $x_0$ is $s$-sparse and $\Phi$ is an $m \times N$ matrix that obeys a technical condition called the restricted isometry property (see (2.3) below), then $x_0$ can be recovered from $y$ even when $m \ll N$, and this recovery is stable in the presence of measurement noise, and robust against modeling error (i.e. it is effective even when $x_0$ is not perfectly sparse) [6, 7, 12]. Random matrices that obey this property for $m$ within a logarithmic factor of $s$ include matrices with independent entries [1, 7], matrices that have been subsampled from orthobases consisting of vectors whose energy is almost evenly distributed between their entries [33], as well as other matrices with more structured randomness [31, 38]. These measurement systems provide efficient encodings for $x_0$ because the number of measurements we need to make is roughly proportional to the number active elements. The results from section 3 states that matrices formed by concatenating a series of $p$ random convolutions are another such efficient encoding (with $N = np$).

Early work on signal processing algorithms using sparse models for channel estimation can be found in [10] and [15]. In [19], estimation of a single channel using a pulse consisting of a sequence of independent Gaussian random variables is explored; the mathematical results of [19] are framed in the language of CS, and the key recovery condition (the restricted isometry

\(^4\) Passing an image through a coded aperture has the effect of convolving it with a binary code. The theoretical results presented in this paper require the code to be Gaussian; this requirement was imposed so that each convolution could be diagonalized in the Fourier domain, which allows us to apply recent results from the theory of random matrices to prove theorems 3.1 and 3.3. In practice, we would expect there to be little difference between then Gaussian and binary cases.
property in (2.3) below) is established for pulse lengths of $O(n + s^2 \log n)$. The paper [30] shows that the recovery conditions can be improved to $O(n + s \log^3 n)$ when the observations are noiseless and the channel is exactly $s$-sparse. Using convolution with a random pulse to perform compressive sensing was also considered in the context of imaging in [31] and as a way to handle streaming data in [39]. Results for super-resolved radar imaging using ideas from CS can be found in [20]. In this paper, the underdetermined system arises not because we are subsampling a signal after it has been convolved with a pulse, but by combining the convolutions from multiple channels into one observed sequence.

Multichannel separation also bears some resemblance to the problem of finding the sparsest decomposition in a union of bases [5, 13, 17, 37]; this resemblance becomes even more pronounced when we recast the problem using circular convolution (see section 3.1) and take $m = n$. We can think of each convolution matrix as a different basis, and search for a way to write the measurements as a superposition of a small number of vectors chosen from these bases. In contrast to the previous work on this problem, the bases here are random and not quite orthogonal (the related paper [32] considers an alternative way to generate the random pulses so that each of the convolution matrices is exactly orthogonal).

2. Sparse recovery from underdetermined measurements

In the previous section, we set up multiple channel estimation as a linear inverse problem. Classically, these types of problems are solved using least-squares; the stability of the solution is almost completely characterized through the eigenvalues of $\Phi^*\Phi$. If for all $x \in \mathbb{R}^{np}$ we have

$$\|x\|_2^2 \leq \|\Phi x\|_2^2 \leq (1 + \delta)\|x\|_2^2$$

for some small $0 \leq \delta < 1$, then recovering $x$ from $\Phi x$ is well-posed and stable in the presence of noise. Of course, if $m < np$, then the system will be underdetermined, $\Phi$ will have a nontrivial null space, and the lower bound in (2.1) cannot hold. It appears that to simultaneously estimate all of the channel responses, the length $m$ of the probe sequence must exceed $np$.

Recent results from compressive sensing have stated that if the vector we are trying to recover is sparse, then a much weaker condition on $\Phi$ is sufficient for a well-posed, stable recovery. In particular, if (2.1) holds for all $2s$-sparse vectors $x$, rather than all $x \in \mathbb{R}^{np}$, then we will be able to recover $h_j$ from $r_j = \Phi h_j$ about as well as if we had observed the $s$ largest (most important) entries in $h_j$ directly.

We can make this precise in the following manner. Denote by $B^s_{\ell_2}$ the set of all vectors $x \in \mathbb{R}^{np}$ that are nonzero only on the set $\Gamma \subset \{1, \ldots, np\}$ and have unit $\ell_2$ norm. For a square matrix $A$, we define the $\| \cdot \|_s$ norm as

$$\|A\|_s = \sup_{|\Gamma| \leq s} \max_{x, y \in B^s_{\ell_2}} |y^*Ax|,$$

where the supremum is taken over all $s$-sparse vectors with unit energy. (We use * for the transpose of a real-valued vector or matrix, or conjugate-transpose for a complex-valued vector or matrix.) It is easy to see that if

$$\|I - \Phi^*\Phi\|_s \leq \delta_s$$

then

$$(1 - \delta_s)\|x\|_2^2 \leq \|\Phi x\|_2^2 \leq (1 + \delta_s)\|x\|_2^2$$

for all $s$-sparse $x$. (2.3)
Establishing (2.3), which has gone by the names *uniform uncertainty principle* and *restricted isometry property* in the CS literature, is the key for stable sparse recovery [3, 6–8, 27]. The following proposition gives us a concrete algorithm for recovering a sparse vector from measurements made by a matrix that satisfies (2.3).

**Proposition 2.1 ([6]).** Let \( x_0 \) be an \( s \)-sparse vector, and \( \Phi \) be a matrix that obeys (2.3) with \( \delta_{C_1,s} \leq C_2 \), where \( C_1 \geq 1 \) and \( 0 < C_2 < 1 \) are constants. Given noisy observations \( y = \Phi x_0 + e \), where \( e \) is an error vector with norm at most \( \|e\|_2 \leq \epsilon \), the solution \( \tilde{x}_0 \) to the optimization program

\[
\min_{x} \|x\|_1 \quad \text{subject to} \quad \|\Phi x - y\|_2 \leq \epsilon
\]

will obey

\[
\|\tilde{x}_0 - x_0\|_2 \leq C_3 \epsilon,
\]

where \( C_3 \) is a known universal constant. In addition, if \( x_0 \) is a general non-sparse vector, then the solution to (2.4) obeys

\[
\|\tilde{x}_0 - x_0\|_2 \leq C_4 (\epsilon + s^{-1/2} \|x_0 - x_{0,s}\|_1),
\]

where \( x_{0,s} \) is the best \( s \)-sparse approximation to \( x_0 \); the nonzero components in \( x_{0,s} \) are the \( s \) largest components of \( x_0 \).

The constants in the theorem above are known to be small. For example, in [4] it is shown that we need \( C_1 = 2 \) and \( C_2 \leq \sqrt{2} - 1 \), and with \( C_2 = 1/4 \), we have \( C_4 \leq 6 \).

Proposition 2.1 states that although the recovery procedure (solving (2.4)) is nonlinear, it is exceptionally stable against both noise being added to the measurements and \( x_0 \) not being exactly sparse. First, note that if \( x_0 \) is exactly \( s \)-sparse (so that \( x_{0,s} = x_0 \)) and there is no measurements noise (\( \epsilon = 0 \)), then the right-hand side in (2.5) is zero and (2.4) recovers \( x_0 \) exactly. If measurement noise is added, then the error in the recovery scales with \( \epsilon \), the bound on the size of the noise error. If the signal is not sparse, then we can interpret \( s \) in (2.5) as the largest value for which the restricted isometry property (2.3) holds. In this case, the recovery error scales with (renormalized) \( \ell_1 \) error between the original signal \( x_0 \) and the best \( s \)-term approximation of \( x_0 \). That is, we can recover as good an approximation to \( x_0 \) by observing it through \( \Phi \) as we can by observing its \( s \) most significant components directly.

Similar stability results hold for recovery procedures other than \( \ell_1 \) minimization. In particular, in [27] and [3], it is shown that particular types of iterative thresholding algorithms can achieve essentially the same performance after a very reasonable number of iterations.

The main result of this paper, codified in theorems 3.1 and 3.3, is that the \( \Phi \) which arises in the multichannel separation problem will obey the restricted isometry property (2.3) for \( s \) almost proportional (within a log factor) to \( m \).

### 3. A multichannel separation theorem

#### 3.1. From linear to circular convolution

Rather than analyzing the spectral properties of \( \Phi^\text{im} \) in (1.2) directly, we will replace it with a slightly modified version whose components are submatrices of large circular matrices, and thus can be diagonalized in the Fourier domain, which simplifies the analysis considerably. To do this, we simply ‘pre-process’ the measurements by adding some of them together to create a slightly shorter observation vector.
To start, consider the single source measurements $y_{i,j}^{\text{lin}}$ in equation (1.1), with the pulse length $m$ exceeding the length of the channel $n$. Suppose that we add the first $n-1$ measurements to the last $n-1$ measurements to form

$$y_{i,j} = \begin{bmatrix} y_{i,j}^{\text{lin}}(n) \\ \vdots \\ y_{i,j}^{\text{lin}}(m) \\ y_{i,j}^{\text{lin}}(m+1) + y_{i,j}^{\text{lin}}(1) \\ \vdots \\ y_{i,j}^{\text{lin}}(m+n-1) + y_{i,j}^{\text{lin}}(n-1) \end{bmatrix} = \begin{bmatrix} \phi_i(n) & \phi_i(n-1) & \cdots & \phi_i(1) \\ \vdots & \vdots & & \vdots \\ \phi_i(m) & \phi_i(m-1) & \cdots & \phi_i(m-n+1) \\ \phi_i(1) & \phi_i(m) & \cdots & \phi_i(m-n+2) \\ \vdots & \vdots & & \vdots \\ \phi_i(n-1) & \phi_i(n-2) & \cdots & \phi_i(m) \end{bmatrix} \begin{bmatrix} h_{i,j}(1) \\ h_{i,j}(2) \\ \vdots \\ h_{i,j}(n) \end{bmatrix}.$$  

The matrix $\Phi_i$ consists of the first $n$ columns of an $m \times m$ circulant matrix with

$$r_i^* = [\phi_i(n) \cdots \phi_i(1) \phi_i(m) \cdots \phi_i(n+1)].$$

as the first row. As such, we can use the discrete Fourier transform to diagonalize $\Phi_i$. Let $F$ be the $m \times m$ normalized discrete Fourier matrix with entries

$$F(\omega, t) = \frac{1}{\sqrt{m}} e^{-2\pi j (\omega - 1)(t-1)/m},$$

and let $F_{(1:n)}$ denote the $m \times n$ matrix consisting of the first $n$ columns of $F$. Then

$$\Phi_i = F^* G_i F_{(1:n)}, \quad \text{with} \quad G_i = \text{diag}(\{g_i(\omega)\}_{\omega=1}^m).$$  

(3.1)

The vector $g_i(\omega)$ is the (re-normalized) Fourier transform of $r_i$:

$$g_i = m \cdot F r_i.$$

When all the sources are active simultaneously, we can perform the same manipulations on the composite linear system (1.2), combining the first $n-1$ entries in $y_{j}^{\text{lin}}$ with the last $n-1$ to yield

$$y_j = \begin{bmatrix} \Phi_1 & \Phi_2 & \cdots & \Phi_p \end{bmatrix} \begin{bmatrix} h_{1,j} \\ h_{2,j} \\ \vdots \\ h_{p,j} \end{bmatrix} = : \Phi h_j.$$  

(3.2)

As in (3.1), we can write $\Phi$ as

$$\Phi = F^* \begin{bmatrix} G_1 F_{(1:n)} & G_2 F_{(1:n)} & \cdots & G_p F_{(1:n)} \end{bmatrix}.$$  

(3.3)

We assume that each source emits an independent random waveform. That is, we take the probe samples $(\phi_i(t))_{i,t}$ to be iid Gaussian random variables with zero mean and variance $m^{-1}$ (so each probing waveform $\phi_i$ has unit energy in expectation). Since the $\phi_i(t)$ are iid
Gaussian, the corresponding Fourier transforms $g_i(\omega)$ are sequences of conjugate symmetric complex-valued Gaussian random variables:

$$g_i(\omega) \sim \begin{cases} 
\text{Normal}(0, 1) & \omega = 0, m/2 + 1 \\
\text{Normal}(0, 1/\sqrt{2}) + j \cdot \text{Normal}(0, 1/\sqrt{2}) & \omega = 2, \ldots, m/2
\end{cases}$$

and $g_i(\omega) = g_i(n - \omega + 2)^*$ for $\omega = n/2 + 2, \ldots, n$.

### 3.2. Recovery theorems

Our main theorem shows that the random matrix $\Phi_1$, generated from the random sequences $\{g_i(\omega)\}$ as in (3.2), is an approximate restricted isometry in expectation for $m \gtrsim s \log^5(np)$.

**Theorem 3.1.** Let $\Phi$ be as in (3.3). There exists constants $C_5$ and $C_6$ such that

$$E\|I - \Phi^* \Phi\|_s \leq \sqrt{\frac{C_5 \cdot s \cdot \log^2 s \cdot \log^5 (mp) \log(np)}{m}},$$

when

$$m \gtrsim C_6 \cdot s \cdot \log^2 s \cdot \log^7 (mp) \log(np).$$

It is straightforward to turn theorem 3.1 into a direct statement about the restricted isometry constants.

**Corollary 3.2.** There is a constant $C_7$ such that

$$E\|I - \Phi^* \Phi\|_s \leq \delta_s$$

when

$$m \gtrsim C_7 \delta_s^{-2} \cdot s \cdot \log^5 (np),$$

for any $0 < \delta_s \leq 1$, provided that $m \leq np$.

To see how (3.5) follows from (3.4), note that for $m \leq np$,

$$\log(mp) = \log m + \log p \leq 2 \log(np),$$

and for $s \leq np$,

$$\sqrt{\frac{C_5 \cdot s \cdot \log^2 s \cdot \log^2 (mp) \log(np)}{m}} \leq \sqrt{\frac{4C_5 \cdot s \cdot \log^5 (np)}{m}}.$$

As a result, we can choose $m$ as in (3.5).

Theorem 3.1 gives us a lower bound on the length of a pulse sufficient to endow, in expectation, $\Phi$ with certain restricted isometry constants. The following theorem gives us a lower bound for the length of the pulses that guarantees that $\Phi$ has certain isometry constants with high probability.

**Theorem 3.3.** Let $\Phi$ and $\delta_s$ be as in theorem 3.1. There exist constants $C_8$ and $C_9$ such that

$$P\{\|I - \Phi^* \Phi\|_s > \delta_s\} \leq C_8 (np)^{-1}$$

when

$$m \gtrsim C_9 \delta_s^{-2} \cdot s \cdot \log^6 (np).$$

It is worth mentioning that we chose a probability of failure of $\sim (np)^{-1}$ mostly out of convenience. In fact, the probability can be made arbitrarily small by adjusting the constant
we could achieve a failure rate of \((np)^{-\alpha}\) for any \(\alpha > 1\) by making the constant in (3.6) \(C_{0\alpha}\).

The essential consequence of the next theorem is that for pulse lengths (3.6), we can simultaneously estimate the channel responses from all sources to receiver \(j\), which are concatenated in the vector \(h_j\), from either concatenated circular convolution observations \(\Phi h_j\) or concatenated linear convolution observations \(\Phi^{lin} h_j\). As linear convolution observations are more typical, we state our channel separation corollary in terms of \(\Phi^{lin}\).

**Corollary 3.4.** Suppose we observe \(y_j^{lin} = \Phi^{lin} h_j + e\), where \(\gamma^{lin}, \Phi^{lin}\) and \(h_j\) are as in (1.2) and \(e\) is an unknown vector of measurement errors with \(\|e\|_2 \leq \epsilon\). Take \(C_1, C_2\) and \(C_4\) as in proposition 2.1, and take \(m\) as in theorem 3.3 so that \(\delta_{C_4, s} \leq C_2\), where \(\delta_{C_4, s}\) is the isometry constant for the concatenated circulant matrix \(\Phi\) generated from \(\Phi^{lin}\) as in (3.2). Then the solution \(\tilde{h}_j\) to

\[
\min_{\tilde{h}} \|\tilde{h}\|_1 \quad \text{subject to} \quad \|\Phi^{lin} \tilde{h} - y_j^{lin}\|_2 \leq \epsilon
\]

is a close approximation to \(h_j\) in that

\[
\|\tilde{h}_j - h_j\|_2 \leq C_4(\sqrt{2} \epsilon + s^{-1/2} \|h_j - h_{j,s}\|_1), \tag{3.7}
\]

where \(h_{j,s}\) is the best \(s\)-term approximation to \(h_j\).

**Proof.** Theorem 3.3 coupled with proposition 2.1 give us robust reconstruction for observations made through the concatenated circulant system \(\Phi\). To establish the proposition, we will make a concrete connection between the solutions to the linear and circular convolution inverse problems.

First, we consider the case where there is no noise and \(h_j\) is perfectly \(s\)-sparse. Given the circular observations \(y = \Phi h_j\), we could solve (2.4) with \(\epsilon = 0\), making the constraints \(\Phi x = y\). With \(m\) as in (3.5), the solution \(\tilde{h}_j\) will be exactly \(h_j\) with high probability. Stated differently, there is no vector in the nullspace of \(\Phi\) that can be added to \(h_j\) that lowers the \(\ell_1\) norm. Since \(\text{Null}(\Phi^{lin}) \subset \text{Null}(\Phi)\), we could also solve (2.4) with \(\gamma^{lin}\) and \(\Phi^{lin}\) in place of \(y\) and \(\Phi\) and recover the signal exactly.

To make the connection when there is noise, we use the following proposition, which is contained in [4, 6], but is slightly stronger than proposition 2.1.

**Proposition 3.5.** Under the conditions of proposition 2.1, if \(d\) is any vector that satisfies \(\|x_0 + d\|_1 \leq \|x_0\|_1\) and \(\|\Phi d\|_2 \leq 2\epsilon\) (both of which must be true for \(d = x_0 - x_0\)), then

\[
\|d\|_2 \leq C \cdot (\epsilon + s^{-1/2} \|x_0 - x_{0,s}\|_1).
\]

Now suppose we solve (2.4) with observations \(y^{lin}\) and matrix \(\Phi^{lin}\), denoting the solution \(\tilde{h}_j^{lin}\) and set \(d = \tilde{h}_j^{lin} - h_j\). Since \(h_j\) is feasible, we will have both \(\|h_j + d\|_1 \leq \|h_j\|_1\) and \(\|\Phi^{lin} d\|_2 \leq 2\epsilon\). We can write \(\Phi = A \Phi^{lin}\), where \(A\) combines the first and last \(n - 1\) elements of a vector. Since the maximum singular value of \(A\) is \(\sqrt{2}\), we also have

\[
\|\Phi d\|_2 = \|A \Phi^{lin} d\|_2 \leq \sqrt{2} \|\Phi^{lin} d\|_2 \leq 2 \sqrt{2} \epsilon.
\]

Thus, the solution \(\tilde{h}_j^{lin}\) is as accurate as solving (2.4) with the circulant observations \(y_j\) and matrix \(\Phi\) with \(\epsilon\) increased by a factor of \(\sqrt{2}\). Thus, \(\tilde{h}_j\) will obey (3.7).
4. Proof of Theorem 3.1

The essential tool for establishing (3.4) is a variation (Lemma A.2) of a lemma due to Rudelson and Vershynin (Lemma A.1). Most of our efforts will go toward manipulating \( I - \Phi^*\Phi \) to put it in a form where we can apply Lemma A.2. The basic flow of the proof is to divide \( I - \Phi^*\Phi \) into several parts, each of which can be written as a sum of independent rank-1 matrices, and then apply the bounds in the appendix to each part. This process is not difficult, but it is somewhat laborious. To aid the exposition, we have divided the proof into steps, each one of which accomplishes a particular task.

We will not track the constants. We will use \( C \) to denote a constant that is independent of all the variables of interest (\( s, n, m, p \)); the particular value of \( C \) may change between instantiations. We will give a constant a label in the subscript if we want to refer to it later.

To start, we set \( Z = I - \Phi^*\Phi \).

\[(E1) \text{ Write } Z \text{ as a sum of rank-1 matrices. Recall from (3.3) that we can write the multichannel convolution matrix } \Phi \text{ as} \]

\[
\Phi = F^*[G_1 F_{(1:n)} \  \  G_2 F_{(1:n)} \  \  \ldots \  \  G_p F_{(1:n)}],
\]

where \( G_k \) are diagonal matrices consisting of the re-normalized Fourier transforms of the sources. We can write \( \Phi^*\Phi \) in matrix form as

\[
\Phi^*\Phi = \begin{bmatrix}
F^*_{(1:n)} G_1^* \\
F^*_{(1:n)} G_2^* \\
\vdots \\
F^*_{(1:n)} G_p^*
\end{bmatrix} F F^*[G_1 F_{(1:n)} \  \  G_2 F_{(1:n)} \  \  \ldots \  \  G_p F_{(1:n)}]
\]

\[
= \begin{bmatrix}
F^*_{(1:n)} \\
F^*_{(1:n)} \\
\vdots \\
F^*_{(1:n)}
\end{bmatrix} \begin{bmatrix}
G_1 G_1 & G_1 G_2 & \cdots & G_1 G_p \\
G_2 G_1 & G_2 G_2 & \cdots & G_2 G_p \\
\vdots & \vdots & \ddots & \vdots \\
G_p G_1 & G_p G_2 & \cdots & G_p G_p
\end{bmatrix}
\times
\begin{bmatrix}
F_{(1:n)} \\
F_{(1:n)} \\
\vdots \\
F_{(1:n)}
\end{bmatrix},
\]

where we have used the fact that \( FF^* = I \). We can compact this expression by introducing \( f_{k,\omega} \in \mathbb{C}^{np} \) as the vector which has column \( \omega \) of \( F^*_{(1:n)} \) in entries \((k-1)n+1, \ldots, kn\) and is zero elsewhere. Then we can rewrite \( \Phi^*\Phi \) as

\[
\Phi^*\Phi = \sum_{k=1}^{p} \sum_{j=1}^{p} \sum_{\omega=1}^{m} g_k(\omega)^* g_j(\omega) f_{k,\omega} f_{j,\omega}^*. \tag{4.1}
\]

Since

\[
\sum_{k=1}^{p} \sum_{\omega=1}^{m} f_{k,\omega} f_{k,\omega}^* = I, \tag{4.2}
\]

we can now write \( Z = I - \Phi^*\Phi \) as

\[
Z = \sum_{k=1}^{p} \sum_{\omega=1}^{m} (1 - |g_k(\omega)|^2) f_{k,\omega} f_{k,\omega}^* + \sum_{j \neq k} \sum_{\omega=1}^{m} g_k(\omega)^* g_j(\omega) f_{k,\omega} f_{j,\omega}^* \equiv H_1 + H_2. \tag{4.3}
\]
Noting that $E\|Z\|_s \leq E\|H_1\|_s + E\|H_2\|_s$, we will proceed by bounding each of $E\|H_1\|_s$ and $E\|H_2\|_s$ in turn.

**E2** Bound $E\|H_1\|_s$. We start by making the random variables in the expression for $H_1$ symmetric. Let $H_1'$ be an independent copy of $H_1$ created from an independent Gaussian sequence $\{g'_k(\omega)\}_{k,\omega}$, and set

$$Y = H_1 - H_1' = \sum_{k=1}^p \sum_{\omega=1}^m (|g'_k(\omega)|^2 - |g_k(\omega)|^2) f_{k,\omega} f_{k,\omega}' .$$

(4.4)

Our strategy is to control $\|Y\|_s$ and use that fact $E\|H_1\|_s \leq E\|Y\|_s$, since

$$E\|H_1\|_s = E\|H_1 - EH_1'\|_s \quad \text{(} H_1' \text{ is zero mean)}$$

$$= E\|E[H_1 - H_1'\|H_1]\|_s \quad \text{(independence, } E[H_1'H_1] = E[H_1]\text{)}$$

$$\leq E\|E[H_1 - H_1'\|H_1]\|_s \quad \text{(Jensen’s inequality)}$$

$$= E\|H_1 - H_1'\|_s \quad \text{(iterated expectation)} .$$

Next, we randomize the sum in (4.4). $Y$ has the same distribution as

$$Y' = \sum_{k=1}^p \sum_{\omega=1}^m \varepsilon_k(\omega) (|g'_k(\omega)|^2 - |g_k(\omega)|^2) f_{k,\omega} f_{k,\omega}' ,$$

(5.5)

where $\{\varepsilon_k(\omega)\}_{k,\omega}$ is an independent Rademacher sequence—the $\varepsilon_k(\omega)$ are iid and take values of $\pm 1$ with equal probability. Note that

$$E\|Y\|_s = E\|Y'/_s = E\|E[Y'/_s | g_k(\omega), g'_k(\omega)]\| .$$

Third, apply lemma A.1 with $v_{k,\omega} = \parallel g_k'(\omega)^2 - |g_k(\omega)|^2 \parallel^{1/2} f_{k,\omega}$. We define the random variable $B$ as

$$B := \max_{k,\omega} \max \{ |g_k(\omega)|, |g'_k(\omega)| \} \geq \max \{ |g_k(\omega)|, |g'_k(\omega)| \}^{1/2} ,$$

(4.6)

and note that

$$\|v_{k,\omega}\|_\infty \leq \max_{k,\omega} \max \{ |g_k(\omega)|, |g'_k(\omega)| \} \parallel f_{k,\omega}\|_\infty \leq B / \sqrt{m} .$$

With the $\{g_k(\omega)\}$, $\{g'_k(\omega)\}$ fixed, lemma A.1 (with $M = B / \sqrt{m}$) states that

$$E\|Y'/_s | g_k(\omega), g'_k(\omega)\| \leq \sqrt{ \frac{C \cdot s \cdot L(s, n, m, p)}{m} } \cdot B$$

$$\cdot \left( \sum_{k=1}^p \sum_{\omega=1}^m (|g'_k(\omega)|^2 - |g_k(\omega)|^2) f_{k,\omega} f_{k,\omega}' \right)^{1/2} ,$$

where $L(s, n, m, p) = \log^2(s) \log(np) \log(mp)$—to make things more compact, we will abbreviate this with $L$, and remember that the quantity depends on the sparsity, length of the channel, length of the pulse and number of channels. Then by the Cauchy–Schwarz inequality

$$E\|Y\|_s \leq \sqrt{ \frac{C \cdot s \cdot L}{m} } \cdot (E[B^2])^{1/2} \cdot \left( E \left\{ \sum_{k=1}^p \sum_{\omega=1}^m (|g'_k(\omega)|^2 - |g_k(\omega)|^2) f_{k,\omega} f_{k,\omega}' \right\}^{1/2} \right) .$$

(4.7)

We can estimate $E[B^2]$ as follows. $B^2$ is the maximum of the $|g_k(\omega)|^2$, $|g'_k(\omega)|^2$, which are chi-squared random variables of degree 2 (when $2 \leq \omega \leq m/2$) or 1 (when $\omega = 1, m/2 + 1$). In either case, $E|g_k(\omega)|^2 = 1$ and

$$P[|g_k(\omega)|^2 > u] \leq e^{-u} .$$
and since there are \((m/2 + 1) \cdot p \cdot 2 = (m + 2)p\) unique magnitudes among the \(|g_k(\omega)|^2, |g'_k(\omega)|^2\),
\[
P(B^2 > u) \leq \min(1, (m + 2)p \cdot e^{-u}). \tag{4.8}
\]
Since \(B^2\) is a positive random variable
\[
E[B^2] = \int_0^\infty P(B^2 > u) \, du
\leq \log((m + 2)p) + (m + 2)p \int_0^\infty e^{-u} \, du
= \log((m + 2)p) + 1. \tag{4.9}
\]
Combining this with the fact that
\[
E\left\| \sum_{k=1}^m \sum_{\omega} \left( |g_k(\omega)|^2 - |g_k(\omega)|^2 \right) f_{k,\omega} f_{k,\omega}^* \right\|_s \leq 2E\left\| \sum_{k=1}^m \sum_{\omega} |g_k(\omega)|^2 f_{k,\omega} f_{k,\omega}^* \right\|_s,
\]
the bound in (4.7) becomes
\[
E\|Y\|_s \leq \sqrt{\frac{C \cdot s \cdot L \log(mp)}{m}} \cdot \left( E\left\| \sum_{k=1}^m \sum_{\omega} (1 - |g_k(\omega)|^2) f_{k,\omega} f_{k,\omega}^* \right\|_s + 1 \right)^{1/2}
\]
Using (4.2) and the fact that \(E\|I\|_s = 1\) yields
\[
E\|Y\|_s \leq \sqrt{\frac{C \cdot s \cdot L \log(mp)}{m}} \cdot (E\|H_1\|_s + 1)^{1/2}
\leq \sqrt{\frac{C \cdot s \cdot L \log(mp)}{m}} \cdot (E\|Y\|_s + 1)^{1/2}. \tag{4.10}
\]
Relabeling the constant in the expression above as \(C_{10}\), taking
\[
m \geq C_{10} \cdot s \cdot L \log(mp).
\]
allows us to invoke lemma B.1 with \(\beta = E\|Y\|_s, \alpha = \sqrt{C_{10} s L \log(mp)/m} \leq 1\), and \(c = 0\) to arrive at
\[
E\|H_1\|_s \leq E\|Y\|_s \leq C_{11} \cdot \sqrt{\frac{s \cdot L \log(mp)}{m}}. \tag{4.11}
\]

**(E3) Decouple \(H_2\).** Set
\[
H_2 = \sum_{j \neq k} \sum_{\omega} g_k(\omega)^* g_j(\omega) f_{k,\omega} f_{j,\omega}^*, \tag{4.12}
\]
where \(\{g'_j(\omega)\}\) is an independent sequence of random variables with the same distribution as \(\{g_k(\omega)\}\). We can now control \(\|H_2\|_s\) by controlling \(\|H_2^*\|_s\), because
\[
E\|H_2\|_s \leq C_{12} E\|H_2^*\|_s. \tag{4.13}
\]
For the proof of (4.13), see [11, section 3.1], which also provides a precise value for the constant \(C_{12}\).
(E4) Add back the diagonal. Write
\[
H'_2 = \sum_{j=1}^{p} \sum_{k=1}^{p} \sum_{a=1}^{m} g_k(\omega)^* g'_j(\omega) f_{k,\omega} f^*_{j,\omega} - \sum_{k=1}^{p} \sum_{a=1}^{m} g_k(\omega)^* g'_k(\omega) f_{k,\omega} f^*_{k,\omega}
\]
\[
:= H_3 + H_4. \tag{4.14}
\]

(E5) Bound $E\|H_4\|_s$. Denoting the angle of the complex number $g_k(\omega)^* g'_k(\omega)$ as $\theta_k(\omega)$, $H_4$ has the same distribution as
\[
H'_4 = \sum_k \sum_{\omega} \epsilon_k(\omega) u_{k,\omega} v^*_{k,\omega}, \quad u_{k,\omega} = e^{i\theta_k(\omega)/2}|g_k(\omega)| f_{k,\omega}, \quad v_{k,\omega} = e^{-j\theta_k(\omega)/2}|g'_k(\omega)| f^*_{k,\omega},
\]
where $\{\epsilon_k(\omega)\}$ is an independent Rademacher sequence. With $B$ as in (4.6), it follows that
\[
\|u_{k,\omega}\|_\infty \leq B/\sqrt{m} \quad \text{and} \quad \|v_{k,\omega}\|_\infty \leq B/\sqrt{m}.
\]

With the $\{g_k(\omega), g'_k(\omega)\}$ fixed, we apply lemma A.2 to get
\[
E[\|H'_4\|_s | \{g_k(\omega), \{g'_k(\omega)\}\}] \leq \sqrt{\frac{C \cdot s \cdot L}{m}} \cdot B
\]
\[
\cdot \left( \left\| \sum_{k=1}^{p} \sum_{\omega=1}^{m} |g_k(\omega)|^2 f_{k,\omega} f^*_{k,\omega} \right\|_s^{1/2} + \left\| \sum_{k=1}^{p} \sum_{\omega=1}^{m} |g'_k(\omega)|^2 f_{k,\omega} f^*_{k,\omega} \right\|_s^{1/2} \right).
\]

As in (4.7), we use the law of iterated expectation and the Cauchy–Schwarz inequality to remove the conditioning
\[
E[\|H'_4\|_s] \leq \sqrt{\frac{C \cdot s \cdot L}{m}} \cdot (E[B^2])^{1/2}
\]
\[
\cdot \left( \left( \left\| \sum_{k=1}^{p} \sum_{\omega=1}^{m} |g_k(\omega)|^2 f_{k,\omega} f^*_{k,\omega} \right\|_s^{1/2} + \left\| \sum_{k=1}^{p} \sum_{\omega=1}^{m} |g'_k(\omega)|^2 f_{k,\omega} f^*_{k,\omega} \right\|_s^{1/2} \right)^2 \right)^{1/2}.
\]

The $\{g_k(\omega)\}$ and $\{g'_k(\omega)\}$ are identically distributed, and so using Jensen’s inequality
\[
E[\|H'_4\|_s] \leq \sqrt{\frac{C \cdot s \cdot L}{m}} \cdot (E[B^2])^{1/2} \cdot \left( E \left( \left\| \sum_{k=1}^{p} \sum_{\omega=1}^{m} |g_k(\omega)|^2 f_{k,\omega} f^*_{k,\omega} \right\|_s^{1/2} \right)^2 \right)^{1/2}.
\]

Using the bound in (4.10) in Step E2, we have
\[
E \left( \left\| \sum_{k=1}^{p} \sum_{\omega=1}^{m} |g_k(\omega)|^2 f_{k,\omega} f^*_{k,\omega} \right\|_s^{1/2} \right) \leq \left( E \left( \left\| \sum_{k=1}^{p} \sum_{\omega=1}^{m} (1 - |g_k(\omega)|^2) f_{k,\omega} f^*_{k,\omega} \right\|_s + 1 \right)^{1/2} \right)
\]
\[
= (E[\|H_1\|_s] + 1)^{1/2}.
\]

Similar to (4.11) (but with different constants) we see that $m \geq C \cdot s \cdot L \log(mp)$ implies $E[\|H_1\|_s] \leq 1$. As we reasoned in (4.9) above, we will also have $E[B^2] \leq \log((m+2)p) + 1$, and so there are constants $C_{13}, C_{14}$ such that
\[
E[\|H_4\|_s] \leq C_{14} \cdot \sqrt{\frac{s \cdot L \log(mp)}{m}} \tag{4.16}
\]
when
\[
m \geq C_{13} \cdot s \cdot L \log(mp).
\]
\( H_3 \) has the same distribution as
\[
H_3 = \sum_{k=1}^{m} \sum_{j=1}^{m} \sum_{a=1}^{p} \epsilon_{ao} g_k(\omega) g_j^*(\omega) f_{k,ao} f_{j,ao}^*
\]
\[
= \sum_{a=1}^{p} \epsilon_{ao} \left( \sum_{k=1}^{m} g_k(\omega)^* f_{k,ao} \right) \left( \sum_{j=1}^{m} g_j^*(\omega) f_{j,ao}^* \right)
\]
\[
= \sum_{a=1}^{p} \epsilon_{ao} u_{ao} v_{ao}^*
\]
where
\[
u_{ao} = \sum_{j=1}^{m} g_j^*(\omega) f_{j,ao}^*, \quad u_{ao} = \sum_{k=1}^{m} g_k(\omega)^* f_{k,ao}.
\]

The \( f_{k,ao} \) have disjoint support for different values of \( k \), so \( \|u_{ao}\|_\infty, \|v_{ao}\|_\infty \leq B/\sqrt{m} \) where \( B \) is defined as in (4.6). Also note that
\[
\sum_{a=1}^{p} \epsilon_{ao} u_{ao}^* = \sum_{a=1}^{p} \sum_{k=1}^{m} \sum_{j=1}^{m} g_k(\omega)^* g_j(\omega) f_{k,ao} f_{j,ao}^*,
\]
and so recalling (4.1), we find that \( \sum_{a=1}^{p} u_{ao}^* u_{ao} \) and \( \sum_{a=1}^{p} v_{ao} v_{ao}^* \) are independent realizations of \( \Phi^* \Phi \). Lemma A.2 and Cauchy–Schwarz tell us that
\[
E\|H_3\|_s = E\|H_3\|_s \leq \sqrt{C \cdot s \cdot L'} \cdot (E[B^2])^{1/2} \cdot (E[\Phi^* \Phi])^{1/2}
\]
\[
\leq C_{15} \cdot \sqrt{s \cdot L' \log(mp)} \cdot (E[Z] + 1)^{1/2},
\]
where \( L' := L'(s, n, m, p) := \log^2 s \log m \log np \). Since \( L'(s, n, m, p) < L(s, n, m, p) \), we can replace \( L' \) with \( L \) above.

**Collect the results.** To summarize, we have shown that
\[
E\|Z\|_s \leq E\|H_1\|_s + E\|H_2\|_s \quad \text{(from (4.3) in step E1)}
\]
\[
\leq E\|H_1\|_s + C_{12} E\|H_2\|_s \quad \text{(from (4.13) in step E3)}
\]
\[
\leq E\|H_1\|_s + C_{12} C_{15} E\|H_3\|_s + C_{12} C_{15} E\|H_3\|_s \quad \text{(from (4.14) in step E4)}.
\]
For \( m \geq \max(C_{10}, C_{13}) \cdot s \cdot L \log(mp) \), we also have the bounds
\[
E\|H_1\|_s \leq C_{11} \sqrt{s \log(mp) \log(mp)} \quad \text{(from (4.11) in step E2)}
\]
\[
E\|H_2\|_s \leq C_{14} \sqrt{s \log(mp)} \quad \text{(from (4.16) in step E5)}
\]
\[
E\|H_3\|_s \leq C_{15} \sqrt{s \log(mp)} \cdot (E[Z] + 1)^{1/2} \quad \text{(from (4.18) in step E6)}.
\]
Thus,
\[
E\|Z\|_s \leq C \sqrt{s \log(mp)} \cdot (2 + \sqrt{E[Z] + 1}).
\]
Using lemma B.1, we see that there is indeed a constant \( C_6 \) such that when \( m \geq C_6 \cdot s \cdot L \log(mp) \), we will have
\[
E\|I - \Phi^* \Phi\|_s = E\|Z\|_s \leq \sqrt{C \cdot s \cdot L \log(mp)}.
\]
5. Proof of theorem 3.3

We begin with a brief overview of the steps we will use to establish theorem 3.3. We will use the same decomposition of $Z$ as in section 4: dividing $Z$ into $Z = H_1 + H_2$, decoupling $H_2$ to get $H'_2$, and then dividing $H'_2$ into $H'_2 = H_3 + H_4$. The essential idea is that we have estimated the means of $\|H_1\|_s$, $\|H_3\|_s$, and $\|H_4\|_s$ in the previous section; we will use these estimates and the concentration inequality in lemma A.4 to derive a tail bound for each of these components in turn.

The main nuisance is that while we can write $H_1$, $H_3$ and $H_4$ as sums of independent random rank-1 matrices, the norms of these matrices are not bounded (as Gaussian random variables are not bounded). To handle this, we define the random variable $B$ as in section 4

$$B = \max_{k,\omega} \max\{|g_k(\omega)|, |g'_k(\omega)|\},$$

and then derive an estimate for $\|Z\|_s$ conditioned on the event $M = \{B^2 \leq M\}$,

where we will choose $M$ so that $\mathbb{P}[M] = 1$ and $\mathbb{E}[M]$ to denote expectation and probability conditioned on the event $M$ occurring.

We start by decomposing the tail bound as

$$\mathbb{P}[\|Z\|_s > \delta] \leq \mathbb{P}[\|H_1\|_s > \delta/2] + \mathbb{P}[\|H_2\|_s > \delta/2].$$

Step P1 below bounds $\mathbb{P}[\|H_1\|_s > \delta/2]$. Step P2 decouples the sum for $H_2$ to get

$$\mathbb{P}[\|H_2\|_s > \delta/2] \leq C\mathbb{P}[\|H'_2\|_s > \delta/2C].$$

Steps P3 and P4 then condition on $M$,

$$\mathbb{P}[\|H'_2\|_s > \delta/2C] \leq C(M\{\|H'_2\|_s > \delta/2C\} + \mathbb{P}[M^c]),$$

divide $H'_2$ into $H'_2 = H_3 + H_4$,

$$\mathbb{P}_M[\|H'_2\|_s > \delta/2C] \leq \mathbb{P}_M[\|H_3\|_s > \delta/4C] + \mathbb{P}_M[\|H_4\|_s > \delta/4C],$$

and then bound $\mathbb{P}_M[\|H_3\|_s > \delta/4C]$ and $\mathbb{P}_M[\|H_4\|_s > \delta/4C]$ in turn. These individual results are unified to finally establish the theorem in step P5.

We will control each probability with a parameter $\gamma$, which can be selected as $0 < \gamma < 1/2$, and derive a bound for $m$ so that the total probability of failure is $O(\gamma)$.

(P1) Tail bound for $\|H_1\|_s$. Recall the definitions of $Y = H_1 - H'_1$ and $Y'$, which has the same distribution as $Y$, from (4.4) and (4.5). We can develop a tail bound for $\|H_1\|_s$ from a tail bound for $\|Y\|_s$ (or $\|Y'\|_s$) by following [23, section 6.1]. For any $a, \lambda > 0$

$$\mathbb{P}[\|H_1\|_s < a] \leq \mathbb{P}[\|H_1\|_s > a + \lambda] \leq \mathbb{P}[\|Y\|_s > \lambda].$$

In particular, if we take $a = 2E\|H_1\|_s = 2E\|H'_1\|_s$, we will have $\mathbb{P}[\|H_1\|_s < a] \geq 1/2$, since the median of a positive random variable is no more than twice its mean, and so

$$\mathbb{P}[\|H_1\|_s > 2E\|H_1\|_s + \lambda] \leq 2\mathbb{P}[\|Y\|_s > \lambda] = 2\mathbb{P}[\|Y'\|_s > \lambda].$$

To bound the right-hand side, we first condition on $M$:

$$\mathbb{P}[\|Y'\|_s > \lambda] \leq \mathbb{P}[M]\{\|Y'\|_s > \lambda\} + \mathbb{P}[M^c].$$

Conditioned on $M$, each term in the sum that comprises $Y'$ has the bounded norm, and so we can apply lemma A.4 with $V_{k,\omega} = |g_k(\omega)|^2 = |g'_k(\omega)|^2 = |f_{k,\omega}|^2 |f_{k,\omega}^*|^2$, noting that

$$\|V_{k,\omega}\|_s \leq M \cdot \|f_{k,\omega}f_{k,\omega}^*\|_s = M \cdot s/m.$$
This yields
\[ P_M \left\{ \|Y\|_s > C \left( u \mathbb{E}_M \|Y\|_s + t \frac{Ms}{m} \right) \right\} \leq e^{-u^2} + e^{-t}. \]

From (4.11), we know that
\[ \mathbb{E}_M \|Y\|_s \leq E \|Y\|_s \leq C \sqrt{\frac{sLM}{m}} \]
when \( m \geq C \cdot sL \log(mp) \). Plugging this into the expression for the tail bound gives us
\[ P_M \left\{ \|Y\|_s > C \left( u \sqrt{\frac{sLM}{m}} + t \frac{Ms}{m} \right) \right\} \leq e^{-u^2} + e^{-t}. \]

Take \( u = \sqrt{\log 1/\gamma}, t = \log 1/\gamma \) to get
\[ P_M \left\{ \|Y\|_s > \lambda \right\} \leq 2\gamma, \quad \lambda = C \left( \sqrt{\frac{sLM \log(1/\gamma)}{m}} + \log(1/\gamma) \frac{sM}{m} \right). \tag{5.1} \]

With this value of \( \lambda \), we can use the bound (4.11) for \( E \|H\|_s \) to get
\[ 2E\|H\|_s + \lambda \leq C \left( \sqrt{\frac{sLM \log(1/\gamma)}{m}} + \sqrt{\frac{sLM \log(1/\gamma)}{m}} + \log(1/\gamma) \frac{sM}{m} \right). \]

Since we are choosing \( m \) to make all three terms above less than 1, the middle term will dominate. We see that there is a constant \( C_{16} \) such that
\[ m \geq C_{16} \cdot \delta_s^{-2} \cdot s \cdot LM \log(1/\gamma) \]
implies
\[ 2E\|H\|_s + \lambda \leq \delta_s/2, \]
and hence
\[ P \{ \|H\|_s > \delta_s/2 \} \leq 4\gamma + 2P[M^c]. \tag{5.2} \]

(P2) **Decouple \( H_2 \).** In Step E3, we saw that we could decouple \( H_2 \) and add back the diagonal, giving us the decomposition (4.14). We can also derive a tail bound for \( \|H_2\|_s \) using the fact that
\[ P[\|H_2\|_s \geq \lambda] \leq C_{17} \cdot P[\|H_2^s\|_s \geq \lambda/C_{17}] \]
for a universal constant \( C_{17} \), where \( H_2^s \) is the ‘decoupled’ version of \( H_2 \) given by (4.12) (for proof of this and an explicit value for \( C_{17} \), see [11, section 3.4]). We will decompose \( H_2 = H_3 + H_4 \) as in (4.14) and proceed by finding tail bounds for \( \|H_3\|_s \) and \( \|H_4\|_s \) conditioned on \( M \).

(P3) **Conditional tail bound for \( \|H_4\|_s \).** We start with the tail bound for \( \|H_4\|_s \). Recall that \( H_4 \) has the same distribution as \( H_4^s \) in (4.15). Using (4.16) from step E5, we can bound the conditional mean
\[ E_M \|H_4^s\|_s \leq E \|H_4^s\|_s / P[M] \leq C \sqrt{\frac{sL \log(mp)}{m}}. \]

Recall that we can write \( H_4^s \) as a random sum of rank-1 matrices as shown in (4.15). Conditioned on \( M \),
\[ \|u_{k_0}v_{k_0}^*\|_s \leq M \|f_{k_0}f_{k_0}^*\|_s = M \cdot s/m. \]
We now apply the concentration inequality (A.7) as before with 
\( u = \sqrt{\log(C_{17}/\gamma)} \) and 
\( t = \log(C_{17}/\gamma) \):

\[
P_M \left\{ \| H'_3 \|_s > C \left( \frac{s \cdot LM \log(C_{17}/\gamma)}{m} + \frac{s \cdot M \log(C_{17}/\gamma)}{m} \right) \right\} \leq 2\gamma/C_{17}.
\]

Since we are making both terms on the right-hand side inside the probability brackets less than 1, the first one will dominate. Thus, there exists a constant \( C_{18} \) so that

\[
m \geq C_{18} \cdot \delta_s^2 \cdot s \cdot LM \log(1/\gamma)
\]

implies

\[
P_M \{ \| H'_3 \|_s > \delta_s/4C_{17} \} \leq 2\gamma/C_{17},
\]

and finally

\[
P_M \{ \| H_3 \|_s > \delta_s/4C_{17} \} \leq 2\gamma/C_{17}
\]

since \( H_3 \) and \( H'_3 \) have the same distribution.

(P4) **Conditional tail bound for \( \| H_3 \|_s \).** As in step E6, \( H_3 \) has the same distribution as

\[
H'_3 = \sum_{a=1}^{m} \epsilon_a u_a v^*_a,
\]

with \( u_a, v_a \) as in (4.17). In (4.18) in step E6, we showed that

\[
E \| H'_3 \|_s \leq C \sqrt{ \frac{sL \log(mp)}{m} } (E \| Z \|_s + 1)^{1/2},
\]

and in (4.19), we showed that \( E \| Z \|_s < 1 \) for \( m \geq CsL \log(mp) \). So for this range of \( m \), we have \( E \| H'_3 \|_s \leq C m^{-1/2} (sL \log(mp))^{1/2} \) and so

\[
E_M \| H'_3 \|_s \leq \frac{E \| H'_3 \|_s}{P(M)} \leq C \sqrt{ \frac{s \cdot L \log(mp)}{m} }.
\]

Conditioned on \( M \),

\[
\| u_a v^*_a \|_s \leq \sup_{|\Gamma| \leq s} \| y^* u_a \| \cdot \sup_{|\Gamma| \leq s} \| x^* v_a \| \leq M \cdot s/m.
\]

We apply the concentration inequality (A.7) with 
\( u = \sqrt{\log(C_{17}/\gamma)} \) and 
\( t = \log(C_{17}/\gamma) \), yielding

\[
P_M \left\{ \| H_3 \|_s > C \left( \frac{s \cdot L \log(mp) \log(C_{17}/\gamma)}{m} + \frac{s \cdot M \log(C_{17}/\gamma)}{m} \right) \right\} \leq 2\gamma/C_{17}.
\]

Below we will see that we can take \( M \sim \log(mp/\gamma) \); this means that there exists a constant \( C_{19} \) such that

\[
m \geq C_{19} \cdot \delta_s^2 \cdot s \cdot LM \log(1/\gamma)
\]

implies

\[
P_M \{ \| H_3 \|_s > \delta_s/4C_{17} \} \leq 2\gamma/C_{17}.
\]
(P5) Collect the tail bounds.

We have shown that

\[
P \{ \| Z \|_s > \delta_s \} \leq P \{ \| H_1 \|_s > \delta_s / 2 \} + P \{ \| H_2 \|_s > \delta_s / 2 \}
\leq P \{ \| H_1 \|_s > \delta_s / 2 \} + C_{17} P \{ \| H_2 \|_s > \delta_s / 2 C_{17} \}
\leq P \{ \| H_1 \|_s > \delta_s / 2 \} + C_{17} P \{ \| H_2 \|_s > \delta_s / 2 C_{17} \} + C_{17} P \{ \| M' \| \}
\leq P \{ \| H_1 \|_s > \delta_s / 2 \} + C_{17} P \{ \| H_2 \|_s > \delta_s / 2 C_{17} \} + C_{17} P \{ \| M' \| \}
\]

Combining the results from steps P1, P3 and P4, we find that for any \( 0 < \gamma < 1 / 2 \), there is a constant \( C_{20} \) such that when \( m \geq C_{20} \cdot \delta_s^{-2} \cdot s \cdot L \cdot \log(1 / \gamma) \)
we will have all of the following:

\[
P \{ \| H_1 \|_s > \delta_s / 2 \} \leq 4 \gamma + 2 P \{ \| M' \| \} \quad \text{(from (5.2) in step P1)}
C_{17} P \{ \| H_2 \|_s > \delta_s / C_{17} \} \leq 2 \gamma \quad \text{(from (5.3) in step P3)}
C_{17} P \{ \| H_3 \|_s > \delta_s / 4 C_{17} \} \leq 2 \gamma \quad \text{(from (5.4) in step P4)}
\]

It remains to fix \( M \). As \( P \{ B^2 > M \} \leq \min(1, (m + 2) \cdot e^{-M}) \)—recall (4.8)—choosing
\( M = \log(C_{17}(m + 2) / \gamma) \) \( \Rightarrow \) \( C_{17} P \{ \| M' \| \} \leq \gamma \).

With this choice of \( M \) (and assuming that \( C_{17} \geq 2 \)),
\[
P \{ \| Z \|_s > \delta_s \} \leq 10 \gamma \quad \text{(5.5)}
\]
when
\[
m \geq C \cdot \delta_s^{-2} \cdot s \cdot L \cdot \log(mp / \gamma) \cdot \log(1 / \gamma). \quad \text{(5.6)}
\]

We establish the theorem by taking \( \gamma = C(n p)^{-1} \) and noting that
\( L \cdot \log(mp / \gamma) \cdot \log(1 / \gamma) \leq \log^6 mp \)
and so taking \( m \) as in (3.6) will guarantee (5.6) and hence (5.5).

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Appendix A. Random matrices

A.1. Random sums of rank-1 matrices

The theoretical results in this paper depend crucially on our ability to estimate the size of the \( \| \cdot \|_s \) norm of random matrices that can be written as the sum of independent rank-1 matrices:

\[
\left\| \sum_{i=1}^{m} \epsilon_i u_i v_i^* \right\|_s
\]

where the \( u_i \) and \( v_i \) are vectors in \( \mathbb{C}^n \) and the \( \{ \epsilon_i \} \) are iid Bernoulli random variables taking the values \( \pm 1 \) with equal probability. Taking \( U, V \) as the \( n \times m \) matrices with the \( v_i, u_i \) as columns, and letting \( \Sigma \) be the diagonal matrix with \( \Sigma_{ii} = \epsilon_i \), (A.1) can be written more compactly as \( \| U \Sigma V^* \|_s \).
In [33], Rudelson and Vershynin provided a bound for the expectation of (A.1) when \( U = V \). The following is lemma 3.8 in [33].

**Lemma A.1.** Let the vectors \( v_i \) and the matrices \( V \) and \( \Sigma \) be defined as above, and suppose that \( \|v_i\|_\infty \leq M \). Then for some constant \( C \),

\[
E\|V \Sigma^*\|_s \leq C \cdot M \cdot s^{1/2} \cdot \log s \sqrt{\log m \log n \cdot \|VV^*\|_1^{1/2}}.
\]  

(A.2)

The following is the analogous result for the more general case when \( U \neq V \).

**Lemma A.2.** Let \( V, \Sigma \) be as in lemma A.1, and let \( U \) be another \( n \times m \) matrix whose maximum entry is less than \( M \). Then for some constant \( C \),

\[
E\|U \Sigma^*\|_s \leq C \cdot M \cdot s^{1/2} \cdot \log s \sqrt{\log m \log n \cdot \left(\|VV^*\|_1^{1/2} + \|UU^*\|_1^{1/2}\right)}.
\]

(A.3)

**Proof.** As in [33], we can bound \( E\|U \Sigma^*\|_s \) by the supremum of a Gaussian random process. Letting \( \{g_i\} \) be a sequence of iid Gaussian random variables with zero mean and unit variance, we have

\[
E\|U \Sigma^*\|_s = \mathbb{E} \sup_{|\Gamma| \leq s} \left| \sum_{i=1}^m g_i \langle x_a, u_i \rangle \langle v_i, x_b \rangle \right|.
\]

We now apply the Dudley inequality (see, for example, [36, chapter 2]), which states that for a Gaussian process \( G(x) \) indexed by a set \( x \in T \), the expected maximum value of \( G \) over \( T \) obeys

\[
E \sup_{x \in T} |G(x)| \leq C \int_0^\infty \log^{1/2} N(T, d, t) \, dt.
\]

(A.4)

Above, the \( N(T, d, t) \) are the so-called \( t \)-covering numbers for \( T \) under the pseudo-metric \( d(x, y) = \sqrt{\mathbb{E}[|G(x) - G(y)|^2]} \). To define these more carefully, recall that a set \( C \) is called a \( t \)-cover of \( T \) if all \( x \in T \) are within a distance \( t \) of some point in \( C \):

\[
\sup_{x \in T} \inf_{q \in C} d(x, q) \leq t.
\]

The covering number \( N(T, d, t) \) is the size of the smallest such cover:

\[
N(T, d, t) = \inf_{\text{\(t\)-cover of } T} |\mathcal{C}|.
\]

The process in (A.3) is indexed by two vectors \( x_a, x_b \), so here

\[
G(x_a, x_b) = \sum_i g_i \langle x_a, u_i \rangle \langle v_i, x_b \rangle \quad \text{and} \quad T = \bigcup_{|\Gamma| \leq s} B^x_\Gamma \odot B^v_\Gamma,
\]

with the metric \( d \) given by

\[
d((x_a, x_b), (y_a, y_b)) = \left( \sum_{i=1}^m (\langle x_a, u_i \rangle \langle v_i, x_b \rangle - \langle y_a, u_i \rangle \langle v_i, y_b \rangle)^2 \right)^{1/2}.
\]
We can bound this distance using
\[
\begin{align*}
\text{d}&\left((x_a, x_b), (y_a, y_b)\right) = \frac{1}{2} \left( \sum_{i=1}^{m} (\langle x_a + y_a, u_i \rangle \langle v_i, x_b - y_b \rangle + \langle x_a - y_a, u_i \rangle \langle v_i, x_b + y_b \rangle)^2 \right)^{1/2} \\
&\leq \frac{1}{2} \cdot \max_i \left( |\langle u_i, x_a - y_a \rangle|, |\langle v_i, x_b - y_b \rangle| \right) \\
&\times \left( \sum_{i=1}^{m} (|\langle x_a + y_a, u_i \rangle| + |\langle x_b + y_b, v_i \rangle|)^2 \right)^{1/2} \\
&\leq R \cdot \max_i \left( |\langle u_i, x_a - y_a \rangle|, |\langle v_i, x_b - y_b \rangle| \right),
\end{align*}
\]
where
\[
R^2 = \frac{1}{4} \sup_{(x_a, x_b) \in T} \sum_{i=1}^{m} (|\langle x_a + y_a, u_i \rangle| + |\langle x_b + y_b, v_i \rangle|)^2
\]
\[
\leq \frac{1}{4} \sup_{(x_a, x_b) \in T} \left( \sum_{i=1}^{m} |\langle x_a + y_a, u_i \rangle|^2 + \sum_{i=1}^{m} |\langle x_a + y_a, v_i \rangle|^2 + 2 \sum_{i=1}^{m} |\langle x_a + y_a, u_i \rangle| \cdot |\langle x_b + y_b, v_i \rangle| \right)
\]
\[
\leq \|UU^*\|_s + \|VV^*\|_s + \frac{1}{2} \sup_{(x_a, x_b) \in T} \left( \sum_{i=1}^{m} |\langle x_a + y_a, u_i \rangle|^2 \right)^{1/2} \cdot \left( \sum_{i=1}^{m} |\langle x_b + y_b, v_i \rangle|^2 \right)^{1/2}
\]
\[
\leq \|UU^*\|_s + \|VV^*\|_s + 2 \|UU^*\|_s^{1/2} \|VV^*\|_s^{1/2}
\]
\[
= \left( \|UU^*\|_s^{1/2} + \|VV^*\|_s^{1/2} \right)^2 =: R^2.
\]

Defining the norms
\[
\|x\|_U = \max_i |\langle x, u_i \rangle| \quad \text{and} \quad \|x\|_V = \max_i |\langle x, v_i \rangle|,
\]
our bound on the metric becomes
\[
d\left((x_a, x_b), (y_a, y_b)\right) \leq R' \max_i \left( \|x_a - y_a\|_U, \|x_b - y_b\|_V \right).
\]

Now let
\[
T' = \bigcup_{|t| \leq t} B_2^t (A.5)
\]
and note that \( T \subset T' \otimes T' \), and so \( N(T, d, t) \leq N(T' \otimes T', d, t) \). If \( C_1 \) is a \( t \)-cover for \( T' \) under the metric \( \| \cdot \|_U \) and \( C_2 \) is a \( t \)-cover for \( T' \) under the metric \( \| \cdot \|_V \), then \( C_1 \otimes C_2 \) is a \( t \)-cover for \( T' \otimes T' \) under the metric \( \max(\| \cdot \|_U, \| \cdot \|_V) \). Hence,
\[
N(T, d, t) \leq N(T', R' \| \cdot \|_U, t) \cdot N(T', R' \| \cdot \|_V, t)
\]
and
\[
\int_0^\infty \log^{1/2} N(T, d, t) \, dt \leq R' \int_0^\infty \log^{1/2} N(T', \| \cdot \|_U, t) \, dt + R' \int_0^\infty \log^{1/2} N(T', \| \cdot \|_V, t) \, dt. \quad (A.6)
\]

We can now apply estimates for the covering numbers in (A.6) that were developed in [33], where the following is shown.
Proposition A.3 ([33]). Let $x_1, \ldots, x_m$ be vectors in $\mathbb{C}^n$ with $\|x_i\|_\infty \leq M$, and define the norm $\|x\|_X = \max_i |\langle x, x_i \rangle|$. For $T'$ as in (A.5),
\[
\int_0^\infty \log^{1/2} N(T', \|\cdot\|_X, t) \, dt \leq C \cdot M \sqrt{s} \cdot \log s \log^{1/2} m \log^{1/2} n
\]
for some constant $C$.

Combining this proposition with (A.6) yields
\[
\mathbb{E}\|U^*/\Sigma V\|_s \leq C \cdot R' \cdot M \cdot \sqrt{s} \cdot \log s \log^{1/2} m \log^{1/2} n,
\]
establishing the lemma.

A.2. A concentration inequality

The following is a specialized version of [23, theorem 6.17] and appears in the following form in [38, proposition 19].

Lemma A.4. Let $V_1, \ldots, V_m$ be a sequence of square matrices with $\|V_i\|_s \leq M$, and let $\{\epsilon_i\}$ be a Rademacher sequence. Set $V = \sum_i \epsilon_i V_i$. Then for all $u, t \geq 1$
\[
\mathbb{P}\{|\|V\|_s \geq C(u\mathbb{E}\|V\|_s + tM)\| \leq e^{-u^2} + e^{-t}.
\] (A.7)

Appendix B. A simple inequality

Lemma B.1. Fix $\alpha \leq 1$ and $c \geq 0$. If
\[
\beta \leq \alpha(c + \sqrt{\beta + 1}) \quad \text{for} \quad \beta \geq 0,
\] (B.1)
then
\[
\beta \leq \alpha(c + 1/2 + \sqrt{c + 5/4}) \quad \text{for} \quad \beta \geq 0.
\]

Proof. Let $x = (\beta + 1)^{1/2}$; note that $x$ is a monotonic function of $\beta$. Then (B.1) becomes
\[
x^2 - 1 \leq \alpha(c + x) \quad \Rightarrow \quad x^2 - \alpha x - (\alpha c + 1) \leq 0.
\]
The polynomial on the left is strictly increasing when $x \geq \alpha/2$. Since $\alpha \leq 1$ and $x \geq 1$ for $\beta \geq 0$, it is strictly increasing over the entire domain of interest. Thus,
\[
x^2 - \alpha x - (\alpha c + 1) \leq 0 \quad \Rightarrow \quad x \leq \frac{\alpha + \sqrt{\alpha^2 + 4(\alpha c + 1)}}{2}.
\]
Substituting $(\beta + 1)^{1/2}$ back in for $x$, we get
\[
\beta + 1 \leq \frac{\alpha^2}{4} + \frac{\alpha\sqrt{\alpha^2 + 4(\alpha c + 1)}}{2} + \frac{\alpha^2 + 4(\alpha c + 1)}{4},
\]
and so
\[
\beta \leq \alpha \left(\frac{c + \frac{\alpha}{2} + \sqrt{\alpha^2 + 4(\alpha c + 1)}}{2}\right) \leq \alpha(c + 1/2 + \sqrt{c + 5/4})
\]
when $\alpha < 1$. □
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