ANALYTICAL SOLUTIONS OF SINGULAR ISOTHERMAL QUADRUPOLE LENS

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ABSTRACT

Using an analytical method, we study the singular isothermal quadrupole (SIQ) lens system, which is the simplest lens model that can produce four images. In this case, the radial mass distribution is in accord with the profile of the singular isothermal sphere lens, and the tangential distribution is given by adding a quadrupole on the monopole component. The basic properties of the SIQ lens have been studied in this Letter, including the deflection potential, deflection angle, magnification, critical curve, caustic, pseudo-caustic, and transition locus. Analytical solutions of the image positions and magnifications for the source on axes are derived. We find that naked cusps will appear when the relative intensity $k$ of quadrupole to monopole is larger than 0.6. According to the magnification invariant theory of the SIQ lens, the sum of the signed magnifications of the four images should be equal to unity, as found by Dalal. However, if a source lies in the naked cusp, the summed magnification of the left three images is smaller than the invariant 1. With this simple lens system, we study the situations where a point source infinitely approaches a cusp or a fold. The sum of the magnifications of the cusp image triplet is usually not equal to 0, and it is usually positive for major cusps while negative for minor cusps. Similarly, the sum of magnifications of the fold image pair is usually not equal to 0 either. Nevertheless, the cusp and fold relations are still equal to 0 in that the sum values are divided by infinite absolute magnifications by definition.

Key words: gravitational lensing: strong – methods: analytical

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1. INTRODUCTION

The singular isothermal sphere (SIS) lens is a very important lens model in strong gravitational lenses. The mass distribution of this lens yields a flat rotation curve. More practical is the singular isothermal ellipsoid lens model, derived by extending the mass distribution of the SIS lens into an elliptical distribution (Kassiola & Kovner 1993). There are also other generalized SIS lenses, with an SIS radial profile and an arbitrary tangential distribution (Evans & Witt 2001, 2003). The singular isothermal quadrupole (SIQ) lens is one of the generalized SIS lenses whose tangential distribution function is given by adding a quadrupole on the monopole component. Therefore, the SIQ lens is often regarded as an SIS lens perturbed by an external quadrupole shear. In general, critical curves and caustics can be divided into tangential and radial ones. Because the singular isothermal lens does not formally have a radial critical curve, the radial caustic associated with the singularity in the lens center is usually called pseudo-caustic (or cut). When a point source traverses such a pseudo-caustic, the image number changes by $\pm 1$, instead of $\pm 2$ for the general caustic (Evans & Wilkinson 1998).

The tangential caustic of a smooth quadrupole lens commonly comprises four cusps and four folds. The transition loci in the lens plane are also very important for understanding the spatial distribution of multiple images. The transition loci together with critical curves are the images of caustics. For detailed information about transition loci, please refer to Finch et al. (2002), An & Evans (2006), and Chu et al. (2013).

There are some important magnification relations for the multiple image lenses. For some specific lens models the magnification invariant means that the sum of signed magnifications for all lensed images of a given source is a constant (Witt & Mao 1995, 2000; Hunter & Evans 2001). It is very interesting and surprising that the invariants are independent of most of the model parameters. For the SIQ lens, the sum of the four magnifications equals unity (Dalal 1998; Dalal & Rabin 2001), i.e., $\sum_{i=1}^{4} \mu_i = 1$ (here and throughout $\mu$ is the signed magnification). In these previous works, the magnification invariant was calculated through polynomial coefficients to avoid deriving each magnification of the images. The cusp and fold relations are local magnification relations compared with the magnification invariant (Petters & Werner 2010). It has been found that if a point source approaches the cusp infinitely from the inner side of the tangential caustic, the three close images will have a relation with $R_{\text{cusp}} = \sum_{i=1}^{3} \mu_i / \sum_{i=1}^{3} |\mu_i| = 0$ (Blandford & Narayan 1986; Schneider et al. 1992; Mao & Schneider 1998). A similar magnification relation holds when the source lies near a fold caustic, however, the prerequisite is that the source position should not be very close to the cusp. If the source infinitely approaches the fold line, the fold image pair will also have a relation with $R_{\text{fold}} = \sum_{i=1}^{2} \mu_i / \sum_{i=1}^{2} |\mu_i| = 0$ (Blandford & Narayan 1986; Goldberg et al. 2010). In this Letter, we present analytical solutions and some interesting results for the SIQ lens.

This Letter is arranged as follows. In Section 2, some basic properties of the SIQ lens are presented, including the deflection potential, deflection angle, magnification, critical curve, caustic, pseudo-caustic, and transition locus. In Section 3, the analytical solutions of the image positions and magnifications for point source on the axes are calculated. Then, the general case with arbitrary source position is extended. In Section 4, the lost image and the magnification relations are investigated, including
the magnification invariant, cusp, and fold relations. Finally, conclusions and a discussion will be given in Section 5.

2. BASIC PROPERTIES OF THE SIQ LENS

Figure 1 shows the main curves of the SIQ lens in the source plane and lens plane. If a source lies in the astroid caustic, there are usually four images produced. They are distributed in the four image regions, which are divided by the critical curve and transition loci. Images 1 and 3 are positive, while images 2 and 4 are negative. The convergence (surface mass density in units of the critical surface mass density) of the SIQ lens is given by adding a quadrupole component on the SIS lens

\[ \kappa(\theta) = \theta_E^2 (1 + k \cos 2\phi), \]

where \((\theta, \phi)\) are the polar coordinates and \(\theta_E\) is the Einstein radius of the SIS lens. Here, \(k (0 \leq k \leq 1)\) is the intensity of the quadrupole relative to the monopole.

The deflection potential depends on the two-dimensional Poisson equation \(\nabla^2 \psi(\theta) = 2\kappa(\theta)\). By solving this differential equation in polar coordinates, one can derive the deflection potential

\[ \psi(\theta) = \theta_E \theta \left(1 - \frac{1}{3} k \cos 2\phi\right). \]

(2)

The scaled deflection angle \(\alpha = \nabla \psi(\theta)\) is the first derivation of deflection potential. Its radial and tangential components are

\[ \alpha_{\text{rad}} = \theta_E - \frac{1}{3} \theta_E k \cos 2\phi, \]

(3)

\[ \alpha_{\text{tan}} = \frac{2}{3} \theta_E k \sin 2\phi. \]

(4)

The shear can be calculated through the deflection potential \(\psi\). For this lens model, \(\gamma = \kappa\) (Chu et al. 2013). Consequently, the magnification in the lens plane is

\[ \mu = \frac{1}{(1 - \kappa)^2 - \gamma^2} = \frac{\theta}{\theta - \theta_E - \theta_E k \cos 2\phi}. \]

(5)

and therefore, the critical curve where \(\mu\) is infinite can be derived as

\[ \theta = \theta_E + \theta_E k \cos 2\phi. \]

(6)

The lens equation \(\beta = \theta - \alpha\), describing the transformation between the lens plane \((\theta, \phi)\) and the source plane \((\beta, \varphi)\), can be written in polar coordinates as (Chu et al. 2013)

\[ \beta^2 = (\theta - \alpha_{\text{rad}})^2 + \alpha_{\text{tan}}^2, \]

(7)

\[ \tan(\phi - \varphi) = \frac{\alpha_{\text{tan}}}{\theta - \alpha_{\text{rad}}}. \]

(8)

The major and minor cusps correspond to points with \(\cos 2\phi = \pm 1\) on the critical curve, so we can obtain \(\alpha_{\text{tan}} = 0\). Then, from Equations (3), (6), and (7), one can derive the distances from the major or minor cusps to the center of the source plane. Both of them are exactly

\[ \beta_{\text{cusp}} = \frac{4}{3} \theta_E k. \]

(9)

Putting Equations (3), (4), and (6) into Equation (8), and using the trigonometric functions, the phase relation between the corresponding points on critical curve and caustic can be obtained

\[ \tan^3 \phi = -\tan \varphi. \]

(10)

Inserting Equations (3), (4), and (6) into Equation (7), \(\beta\) of caustic can be derived as

\[ \beta = \frac{1}{2} \beta_{\text{cusp}} \sqrt{1 + 3 \cos^2 2\phi}. \]

Then, by substituting \(\phi\) with \(\varphi\) through Equation (10), the caustic is given as

\[ \beta = \beta_{\text{cusp}} \sqrt{\frac{1}{\tan^{2/3} \varphi} - 1} \frac{1 + \tan^{2/3} \varphi}{1 - \tan^{2/3} \varphi}. \]

(11)

It can be easily found that the caustic is symmetric respective to the axes \(\varphi = \pm \pi/4\).

Because the pseudo-caustic corresponds to the singular point in the lens center, its expression can be easily derived by setting \(\theta = 0\) in the lens equations (7) and (8). The transition locus can be calculated by putting Equation (11) into Equations (7) and (8) again. The pseudo-caustic and transition loci can be obtained in the form of parametric equations. However, their expressions are very complex.
3. POSITIONS AND MAGNIFICATIONS OF THE MULTIPLE IMAGES

Given a position \((\beta, \varphi)\) in the source plane, one can calculate the corresponding image positions \((\theta, \phi)\) in the lens plane through lens equations. It is relatively easier to calculate \(\phi\) first. Therefore we eliminate \(\theta\) through Equations (7) and (8) to obtain
\[
\left( \frac{\tan \phi - \tan \varphi}{1 + \tan \phi \tan \varphi} \right)^2 = \frac{\alpha^2_{\text{tan}}}{\beta^2 - \alpha^2_{\text{tan}}},
\]

where the tangential deflection angle can also be written as
\[
\alpha_{\text{tan}} = \frac{\beta_{\text{cusp}} \tan \beta}{1 + \tan^2 \phi}.
\]
Thus, the expression of the lens equation is changed with only one unknown quantity, \(\tan \phi\). Then we set \(\eta = \tan \phi\) and \(\lambda = \tan \varphi\) to get a simple expression
\[
\left( \frac{\eta - \lambda}{1 + \eta \lambda} \right)^2 = \frac{\beta^2_{\text{cusp}} \eta^2}{\beta^2 (1 + \eta^2)^2 - \beta^2_{\text{cusp}} \eta^2}.
\]

### 3.1. Source on the Major Axis

For the positive direction of the major axis in the source plane, it has \(\varphi = 0\) and \(\lambda = 0\). The left-hand side of the equal sign in Equation (12) turns into \(\eta^2\). Apparently, \(\eta = 0\) satisfies the equation, so we have two solutions \(\theta_2 = \pi\) and \(\theta_4 = 0\). In addition, after eliminating \(\lambda\), Equation (12) can be organized into
\[
\beta^2 \eta^4 + (2 \beta^2 - \beta^2_{\text{cusp}}) \eta^2 + \beta^2 - \beta^2_{\text{cusp}} = 0.
\]
Disregarding the unphysical solution \(\eta^2 = -1\), we have
\[
\eta^2 = \frac{\beta^2_{\text{cusp}} - \beta^2}{\beta^2}.
\]

Thus, \(\tan \phi = \pm \sqrt{\beta^2_{\text{cusp}} - \beta^2} / \beta\). The two solutions are invalid when \(\beta > \beta_{\text{cusp}}\). Substituting the four solutions of \(\phi\) into either Equation (7) or (8), the radii \(\theta\) will be
\[
\theta_2 = \theta_E - \frac{1}{4} \beta_{\text{cusp}} - \beta,
\]
\[
\theta_4 = \theta_E - \frac{1}{4} \beta_{\text{cusp}} + \beta,
\]
\[
\theta_{1,3} = \theta_E + \frac{1}{4} \beta_{\text{cusp}} + \frac{\beta^2}{2 \beta_{\text{cusp}}}.
\]

One can find that \(\theta_{1,3}\) are always larger than \(\theta_{2,4}\). Putting the four solutions of \(\phi\) into Equation (5), and substituting \(k\) using Equation (9), the magnifications are derived:
\[
\mu_2 = -\frac{4 \theta_E - \beta_{\text{cusp}} - 4 \beta}{4 \beta_{\text{cusp}} + 4 \beta},
\]
\[
\mu_4 = -\frac{4 \theta_E - \beta_{\text{cusp}} + 4 \beta}{4 \beta_{\text{cusp}} - 4 \beta},
\]
\[
\mu_{1,3} = \frac{4 \theta_E \beta_{\text{cusp}} + \beta^2_{\text{cusp}} + 2 \beta^2}{4 \beta^2_{\text{cusp}} - 4 \beta^2}.
\]

Finally, one can obtain the exact relation \(\mu_1 + \mu_2 + \mu_3 + \mu_4 = 1\).

### 3.2. Source on the Minor Axis

For the positive direction of the minor axis in the source plane, it has \(\varphi = \pi/2\) and \(\lambda = \infty\). The left-hand side of the equal sign in Equation (12) turns into \(1 / \eta^2\). Apparently, \(\eta = \infty\) is the solution; in other words \(\phi_1 = \pi/2, \phi_3 = -\pi/2\). In addition, after eliminating \(\lambda\), Equation (12) can be organized into
\[
(\beta^2_{\text{cusp}} - \beta^2) \eta^4 + (\beta^2_{\text{cusp}} - 2 \beta^2) \eta^2 - \beta^2 = 0.
\]
Disregarding the unphysical solution \(\eta^2 = -1\), we have
\[
\eta^2 = \frac{\beta^2}{\beta^2_{\text{cusp}} - \beta^2}.
\]
Thus, \(\tan \phi_{2,4} = \pm \beta / \sqrt{\beta^2_{\text{cusp}} - \beta^2}\). The two solutions are also invalid when \(\beta > \beta_{\text{cusp}}\). Putting the four solutions of \(\phi\) into either Equation (7) or (8), the radii \(\theta\) will be
\[
\theta_1 = \theta_E + \frac{1}{4} \beta_{\text{cusp}} + \beta,
\]
\[
\theta_3 = \theta_E - \frac{1}{4} \beta_{\text{cusp}} - \beta,
\]
\[
\theta_{2,4} = \theta_E - \frac{1}{4} \beta_{\text{cusp}} - \frac{\beta^2}{2 \beta_{\text{cusp}}}.
\]
As before, \(\theta_{1,3}\) are always larger than \(\theta_{2,4}\). Putting the four solutions of \(\phi\) into Equation (5), and substituting \(k\) using Equation (9), the magnifications are derived:
\[
\mu_1 = \frac{4 \theta_E + \beta_{\text{cusp}} + 4 \beta}{4 \beta_{\text{cusp}} + 4 \beta},
\]
\[
\mu_3 = \frac{4 \theta_E + \beta_{\text{cusp}} - 4 \beta}{4 \beta_{\text{cusp}} - 4 \beta},
\]
\[
\mu_{2,4} = -\frac{4 \theta_E \beta_{\text{cusp}} - \beta^2_{\text{cusp}} - 2 \beta^2}{4 \beta^2_{\text{cusp}} - 4 \beta^2}.
\]
Again, one can obtain the exact relation \(\mu_1 + \mu_2 + \mu_3 + \mu_4 = 1\).

### 3.3. Source Position for the General Case

For the general case that a point source lies at an arbitrary position, Equation (12) can be reorganized into
\[
(\eta^2 + 1)[\eta^4 - 2 \lambda \eta^3 - \gamma'(1 + \lambda^2) \eta^2 - 2 \lambda \eta + \lambda^2] = 0,
\]
where \(\gamma = \sqrt{\beta^2_{\text{cusp}} - \beta^2} / \beta\). The redundant solution \(\eta^2 = -1\) also appears as before. This quartic equation looks complex but in principle can be solved analytically. It has four analytical solutions, and each of them is a function of \(\lambda(\phi)\) and \(\gamma(\beta)\). As has been derived above, when \(\lambda = 0\) or \(\lambda = \infty\), we have solutions on the axes where \(\eta = \pm \gamma\) or \(\eta = \pm 1 / \gamma\). Woldesenbet & Williams (2012) have also studied angular distributions of the four images of the SIQ lens intensively. If \(\phi = \pi/4\), and \(\lambda = 1\), then the quartic equation in Equation (29) turns into
\[
\eta^4 - 2 \eta^3 - 2 \gamma^2 \eta^2 - 2 \eta + 1 = 0.
\]
Although the form is very simple, the analytical solutions are still complex.
Alternatively, the solutions of Equation (29) can be calculated numerically. Then, using Equation (8) again, one can obtain the result of radii

\[ \theta = \theta_E + \frac{\beta_{\text{cusp}} \eta^3 + 3\lambda \eta^2 + 3\eta + \lambda}{4} = \frac{\theta_E \eta^3 - \lambda \eta^2 - \eta - \lambda}{4\beta_{\text{cusp}}(\eta^3 + \lambda)} . \] (31)

Furthermore, using Equations (5) and (31), one can calculate the magnifications

\[ \mu = \frac{(4\theta_E + \beta_{\text{cusp}} \eta^3 - (4\theta_E - 3\beta_{\text{cusp}})\lambda \eta^2 + (4\theta_E + 3\beta_{\text{cusp}}) \eta - (4\theta_E - \beta_{\text{cusp}})\lambda)}{4\beta_{\text{cusp}}(\eta^3 + \lambda)} . \] (32)

4. THE LOST IMAGE AND MAGNIFICATION RELATIONS

The naked cusps in the quadrupole lens often make problems more complex. However, they are also very intriguing. By setting \( \phi = 0 \) (\( \phi = \pi/2 \)) and \( \theta = 0 \) in Equation (7), one can find that, on the major (minor) axis, the distance from the pseudo-caustic to the center of the source plane is

\[ \beta_{\text{pseu}} = \theta_E \mp \frac{1}{3} \theta_E \kappa . \] (33)

Therefore, the distance on the major axis is smaller than that on the minor axis. Comparing with \( \beta_{\text{cusp}} \) in Equation (9), one can find that, on the minor axis, \( \beta_{\text{pseu}} \) is always larger than or equal to \( \beta_{\text{cusp}} \), which means the naked cusp never appears near the minor cusp. However, on the major axis, \( \beta_{\text{pseu}} \) can be smaller than \( \beta_{\text{cusp}} \) as long as \( k > 0.6 \), which means that the naked cusp can appear near the major cusp.

It is very interesting to study the lost image when a source is located in the naked cusp. In this case, \( k > 0.6 \) and \( \beta_{\text{pseu}} < \beta < \beta_{\text{cusp}} \), the value \( \theta_E \) of Equation (15) will be smaller than 0, while the value \( \mu_2 \) of Equation (18) will be larger than 0. Figure 2 shows the magnifications of different images when \( k = 0.6 \). In the left panel, the intersection point of the blue curve and dashed line \( \mu = 0 \) sits just on the cusp. If \( k < 0.6 \), the intersection point will move to the right. If \( k > 0.6 \), the intersection point will move to the left instead. Therefore, when naked cusps appear, the magnification \( \mu_2 \) of the lost image has a positive sign.

Naked cusps will never appear on the minor axis. However, as shown in Figure 1, if a source continues to move out along the axis, it will also traverse the pseudo-caustic and result in an image disappearing in the lens center. After the source traverses the pseudo-caustic on the minor axis, the value \( \theta_E \) of Equation (24) will be smaller than 0, while the value \( \mu_3 \) of Equation (27) will be larger than 0. From Equations (9) and (33), one can find on the minor axis, when \( k = 0.6 \), \( \beta_{\text{pseu}} = 1.5 \beta_{\text{cusp}} \), which is also shown by the intersection point of the red curve and dashed line \( \mu = 0 \) in the right panel of Figure 2.

In Section 3, we have proven that the relation \( \sum_{i=1}^{4} \mu_i = 1 \) is always valid on the axes, even for \( \beta > \beta_{\text{cusp}} \). However, this equation includes magnifications of the lost images. As long as a source lies outside of the astroid caustic or the pseudo-caustic, the number of the images will be less than four, so the sum of the magnifications of the total images is no longer equal to 1. When a source is located in the naked cusp, there are three images left, including two with positive parity and one with negative parity, and the sum of magnifications will be smaller than the invariant 1.

According to the magnification invariant, when a source approaches the cusp infinitely, the summed magnification of the triple infinite images is usually not equal to 0, except that the finite magnification of the fourth image equals 1. The sum of magnifications of the triple images for major and minor cusps can be written as

\[ \sum_{i=1}^{3} \mu_i = \frac{\pm4\theta_E + 3\beta_{\text{cusp}}}{4\beta_{\text{cusp}} + 4\beta} . \] (34)

If the source is located exactly on the cusp, Equation (34) turns into \((3/8)(\pm (1/k))\). For the major cusp, it is apparently larger than 0, while for the minor cusp, it is smaller than 0 and can be equal to 0 only when \( k = 1 \). If \( k \) is very small, the summed magnification can be a very large value. Nevertheless, after they are divided by the sum of absolute magnifications, they will become as small as 0 and the cusp relation is still valid with \( R_{\text{cusp}} = 0 \).
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The sum of the magnifications of fold image pair can be derived from the summed value of the two images which have finite magnifications. It is similar in that the sum of the magnifications of fold image pair is usually not equal to 0, except that the summed magnification of two finite images is equal to 1. Obviously, it is hardly satisfied in ordinary situations. The summed magnification of the infinite fold image pair is finite, as long as the point source on the fold is not infinitely close to the cusp. For the SIQ lens, the summed magnification of the fold image pair changes continuously along the fold line. In general, closer to the major cusp, the summed magnification will be smaller, while closer to the minor cusp, the magnification will be larger. After the sum of the magnifications is divided by the sum of the absolute magnifications, it will turn into 0 and will be larger. After the sum of the magnifications is divided by the sum of the absolute magnifications, it will turn into 0 and will be larger. After the sum of the magnifications is divided by the sum of the absolute magnifications, it will turn into 0 and will be larger. After the sum of the magnifications is divided by the sum of the absolute magnifications, it will turn into 0 and will be larger. After the sum of the magnifications is divided by the sum of the absolute magnifications, it will turn into 0 and will be larger. After the sum of the magnifications is divided by the sum of the absolute magnifications, it will turn into 0 and will be larger.

In general, closer to the major cusp, the summed magnification of the fold image pair changes continuously along the fold line. For the SIQ lens, the summed magnification of the fold image pair changes continuously along the fold line. The summed magnification of the infinite fold image pair is 0. Rather, in the case where a source on the fold is infinitely approaching the cusp, the value of the fold relations will be \( R_{\text{fold}} = \mp 1/3 \) for the major and minor cusps, respectively (Keeton et al. 2005; Aazami & Natarajan 2006).

5. CONCLUSIONS AND DISCUSSION

In this work, we mainly focus on the SIQ lens model, which can produce the simplest critical curve and caustic among all of the four-image lenses. The four-image lens systems are very important and are very common in the observations of lensed quasars (Kovner 1987; Rusin & Tegmark 2001). Using analytical methods, we study some basic properties of the SIQ lens in the polar coordinate, including the deflection potential, deflection angle, magnification, critical curve, caustic, pseudo-caustic, and transition locus. Analytical solutions of image positions and magnifications for sources located on axes are derived. Consequently, we verify the magnification invariant theory for the source on the axes of the SIQ lens, which is that the sum of magnifications of the four images is equal to 1. If the relative intensity \( k \) of the quadrupole to the monopole is larger than 0.6, naked cusps will appear. It is found that if the source is located in the naked cusp, there are only two positive images and one negative image left, and the sum of the magnifications of the triplet will be smaller than the invariant 1.

In previous works, when the positions or the magnifications of the cusp or fold images are calculated, the higher order terms of the Fermat potential are usually ignored (Schneider et al. 1992; Zakharov 1995; Congdon et al. 2008). In popular views, for a point source infinitely approaching a cusp or a fold, the summed magnifications of a cusp triplet or fold pair are considered to be 0, which are independent of lens models (Zakharov 1995; Aazami & Petters 2009; Werner 2009). The summed magnifications are higher order infinitesimals compared to the magnifications of the cusp or fold images. Nevertheless, when the magnifications of these images are infinite, the higher order terms of the Fermat potential can bring a significant effect. Through this simple lens model, we found that when the point source infinitely approaches the cusp from the inner side, the summed magnification of the triple images is usually not equal to 0. For the major cusp of the SIQ lens, the summed magnification is larger than 0, while for the minor cusp, it is smaller than 0 and can be equal to 0 only when \( k = 1 \). Similarly, if a point source approaches the fold infinitely, the summed magnification of the fold double images is usually not equal to 0 either. In general, for the fold image pair of a smooth lens, the closer to the major cusp, the smaller the summed magnification will be, while the closer to the minor cusp, the larger the magnification will be. However, after the summed values are divided by the sum of absolute magnifications, they are equal to 0, so that the cusp and fold relations are still valid. These results derived from the SIQ lens may be important for explaining the anomalous flux ratio problem of the lensed quasars.

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