A Class of Backward Doubly Stochastic Differential Equations with Discontinuous Coefficients *

Qingfeng Zhu and Yufeng Shi

a School of Statistics and Mathematics, Shandong University of Finance, Jinan 250014, China
b School of Mathematics, Shandong University, Jinan 250100, China

May 17, 2010

Abstract

In this work the existence of solutions of one-dimensional backward doubly stochastic differential equations (BDSDEs in short) where the coefficient is left-Lipschitz in $y$ (may be discontinuous) and Lipschitz in $z$ is studied. Also, the associated comparison theorem is obtained.

keywords: backward doubly stochastic differential equations, backward stochastic integral, comparison theorem

1 Introduction

Nonlinear backward stochastic differential equations (BSDEs in short) have been independently introduced by Pardoux and Peng [14] and Duffie and Epstein [2]. Since then, BSDEs have been studied intensively. In particular, many efforts have been made to relax the assumption on the generator. For instance, Lepeltier and San Martin [10] have proved the existence of a solution for the case when the generator is only continuous with linear growth, and Jia [6, 7] studied the existence of BSDEs with left-Lipschitz coefficients. Another main reason is due to their enormous range of applications in such diverse fields as mathematical finance (see Duffie and Epstein [2] and Peng [17]), partial differential equations (see Peng [16]), stochastic optimal control and stochastic game (see Hamadene and Lepeltier [9]), nonlinear mathematical expectations (see Jiang and Chen [8] and Hu and Peng [4]), and so on.

A class of backward doubly stochastic differential equations (BDSDEs in short) was introduced by Pardoux and Peng [15] in 1994, in order to provide a probabilistic interpretation for the solutions of a class of semilinear stochastic partial differential equations (SPDEs in short). They have proved the existence

---

*This work is supported by National Natural Science Foundation of China Grant 10771122, Natural Science Foundation of Shandong Province of China Grant Y2006A08 and National Basic Research Program of China (973 Program, No.2007CB814900)

†Corresponding author, E-mail: yfshi@sdu.edu.cn
and uniqueness of solutions for BDSDEs under uniformly Lipschitz conditions. Since then, Shi et al. [19] have relaxed the Lipschitz assumptions to linear growth conditions. Bally and Matoussi [1] have given a probabilistic interpretation of the solutions in Sobolev spaces for semilinear parabolic SPDEs in terms of BDSDEs. Zhang and Zhao [21] have proved the existence and uniqueness of solution for BDSDEs on infinite horizons, and described the stationary solutions of SPDEs by virtue of the solutions of BDSDEs on infinite horizons. N’zi and Owo [13] have proved the existence of a solution for one dimensional BDSDEs when the coefficient is linear growth. Lin [11] has also proved the existence of a solution for one dimensional BDSDEs when the coefficient is bounded monotone. Recently, Ren et al. [18] and Hu and Ren [5] considered the BDSDEs driven by Levy process with Lipschitz coefficient and applications in SPDEs.

Unfortunately, most existence or uniqueness results of solution of BDSDEs need the generator be at least continuous, which is somehow too strong in some applications. Indeed, there are many SPDEs, in which the generator may be discontinuous, and these PDEs have associated existence results of solution (see Yoo [20] and Kim [9]). Thus, a natural and interesting problem is: can we establish the connections between SPDEs with discontinuous coefficient and BDSDEs? Of course, the first step should be to obtain the existence and uniqueness result of BDSDEs with discontinuous coefficient, next, to construct the connections such as stochastic Feynman-Kac formula. Under which conditions do the BDSDEs with discontinuous $g$ have adapted solution?

Because of their important significance to SPDEs, it is necessary to give intensive investigation to the theory of BDSDEs. In this paper we shall study one-dimensional BDSDEs

$$Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds + \int_t^T g(s,Y_s,Z_s)dB_s - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T,$$

where $\xi \in L^2(\Omega,\mathcal{F}_T,P)$, $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, and $g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^l$, may be discontinuous in $y$. Note that the integral with respect to $\{B_t\}$ is a “backward Itô integral” and the integral with respect to $\{W_t\}$ is a standard forward Itô integral. These two types of integrals are particular cases of the Itô-Sokorohod integral in Nualart and Pardoux [12]. In fact, we show that the one-dimensional BDSDE associated with $(f,g,T,\xi)$ has at least a solution if $f$ and $g$ satisfy the following conditions:

(H1) $f(t,\cdot,z)$ is left-continuous, and $f(t,y,\cdot)$ is Lipschitz continuous, i.e., there exists a constant $K > 0$, such that $|f(t,y,z_1) - f(t,y,z_2)| \leq K|z_1 - z_2|$, for all $t \in [0,T]$, $y \in \mathbb{R}$, $z_1, z_2 \in \mathbb{R}^d$.

(H2) There exist two BDSDEs with generators $f_1$, $f_2$ respectively, such that $f_1(t,y,z) \leq f(t,y,z) \leq f_2(t,y,z)$, for all $t \in [0,T]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, and for given $T$ and $\xi$, the equations $(f_1,g,T,\xi)$ and $(f_2,g,T,\xi)$ have at least one solution respectively, denoted by $(Y_{t}^1,Z_{t}^1)$, $i = 1,2$, where $Y_{t}^1 \leq Y_{t}^2$, for $t \in [0,T]$, a.s., a.e. Moreover, the processes $f_i(t,Y_{t}^i,Z_{t}^i)$ are square integrable.
(H3) \( f(t,\cdot,z) \) satisfies left Lipschitz condition in \( z \), i.e., \( f(t,y_1,z) - f(t,y_2,z) \geq -K(y_1 - y_2) \), for all \( t \in [0,T] \), \( y_1, y_2 \in \mathbb{R} \), and \( y_1, y_2, z \in \mathbb{R}^d \).

(H4) There exist constants \( c > 0 \) and \( 0 < \alpha < 1 \) such that \( |g(\omega,t,y_1,z_1) - g(\omega,t,y_2,z_2)|^2 \leq c|y_1 - y_2|^2 + \alpha|z_1 - z_2|^2 \), for all \( (\omega,t) \in \Omega \times [0,T] \), \( (y_1,z_1) \in \mathbb{R} \times \mathbb{R}^d \), \( (y_2,z_2) \in \mathbb{R} \times \mathbb{R}^d \).

It should be noted that our conditions of this paper, without explicit growth constraint, is different from N’zi and Owo [13]; and without the monotone and bounded constraint, is different from Lin [11].

This paper is organized as follows. In Section 2 we formulate the problem accurately and give some preliminary results. Section 3 is devoted to the proof of the existence of solutions of BDSDEs. Finally, in Section 4 the comparison theorem is obtained.

2 Preliminaries

Let \((\Omega,\mathcal{F},P)\) be a probability space, and \(T > 0\) be an arbitrarily fixed constant throughout this paper. Let \(\{W_t;0 \leq t \leq T\}\) and \(\{B_t;0 \leq t \leq T\}\) be two mutually independent standard Brownian Motions with values in \(\mathbb{R}^d\) and \(\mathbb{R}^l\), respectively, defined on \((\Omega,\mathcal{F},P)\). Let \(\mathcal{N}\) denote the class of \(P\)-null sets of \(\mathcal{F}\). For each \(t \in [0,T]\), we define \(\mathcal{F}_t = \mathcal{F}_t^W \cup \mathcal{F}_t^B\), where for any process \(\{\eta_t\}\), \(\mathcal{F}_{s,t}^\eta = \sigma\{\eta_r - \eta_s; s \leq r \leq t\} \cup \mathcal{N}\). Note that \(\{\mathcal{F}_t; t \in [0,T]\}\) is neither increasing nor decreasing, so it does not constitute a common filtration.

We introduce the following notations:

\[
S^2([0,T];\mathbb{R}) = \{v_t, 0 \leq t \leq T, \text{ is a } \mathbb{R}\text{-valued, } \mathcal{F}_t\text{-measurable continuous process such that } E(\sup_{0 \leq t \leq T}|v_t|^2) < \infty\},
\]

\[
M^2(0,T;\mathbb{R}^n) = \{v_t, 0 \leq t \leq T, \text{ is a } \mathbb{R}^n\text{-valued, } \mathcal{F}_t\text{-measurable process such that } E\int_0^T |v_t|^2 dt < \infty\}.
\]

We use the usual inner product \(\langle \cdot, \cdot \rangle\) and Euclidean norm \(|\cdot|\) in \(\mathbb{R}, \mathbb{R}^l\) and \(\mathbb{R}^d\). All the equalities and inequalities mentioned in this paper are in the sense of \(dt \times dP\) almost surely on \([0,T] \times \Omega\).

**Definition 2.1** A pair of processes \((y,z) : \Omega \times [0,T] \rightarrow \mathbb{R} \times \mathbb{R}^d\) is called a solution of BDSDE \((\mathcal{I})\) if \((y,z) \in S^2([0,T];\mathbb{R}) \times M^2(0,T;\mathbb{R}^d)\) and satisfies BDSDE \((\mathcal{I})\).

Also we need one lemma, which is a special case of comparison theorem in Shi et al. [19].

**Lemma 2.2** Let \(f_1(s,y,z) = ly + mz\), \(f_2(s,y,z) = ly + mz\), where constants \(l,m \in \mathbb{R}\), and positive process \(\phi \in M^2(0,T;\mathbb{R})\), furthermore, \((y^i_t,z^i_t)_{t\in[0,T]} (i = 1, 2)\) are the solution to the following equations:

\[
y^i_t = \xi + \int_t^T (f_i(s,y^i_s,z^i_s) + \phi_s) ds + \int_t^T g(s,y^i_s,z^i_s) dB_s - \int_t^T z^i_s dW_s, \quad i = 1, 2.
\]
If \( \xi \in L^2(\Omega, \mathcal{F}_T, P) \) and \( \xi \geq 0 \), then \( y_i^t \geq 0 \), \( P \)-a.s., \( t \in [0, T] \), \( i = 1, 2 \).

**Remark 2.3** The assumptions \((H1)\) and \((H3)\) imply
\[
f(t, y_1, z_1) - f(t, y_2, z_2) \geq -K(y_1 - y_2) - K|z_1 - z_2|, \quad y_1, y_2 \in \mathbb{R}, z_1, z_2 \in \mathbb{R}^d.
\]

# 3 Existence of Solutions of BDSDEs

In this section, we will state and prove the existence of solutions of BDSDEs.

**Theorem 3.1** Under the assumptions \((H1)-(H4)\), then, if \( \xi \in L^2(\Omega, \mathcal{F}_T, P) \), BDSDE (1) has a solution \((Y_t, Z_t) \in S^2([0, T]; \mathbb{R}) \times M^2(0, T; \mathbb{R}^d)\).

At first, we denote that \((Y^j_t, Z^j_t)\) are the solutions of \((f_j, g, T, \xi)\), where \( j = 1, 2 \), that is
\[
Y^j_t = \xi + \int_t^T f_j(s, Y^j_s, Z^j_s) ds + \int_t^T g(s, Y^j_s, Z^j_s) dB_s - \int_t^T Z^j_s dW_s, \quad (2)
\]
where \( f_j \) satisfies \((H2)\) and \( f_j(t, Y^j_t, Z^j_t) \in M^2(0, T; \mathbb{R}) \). Now we construct a sequence of BDSDEs as follows:
\[
y^i_t = \xi + \int_t^T \left( f(s, y^{i-1}_s, z^{i-1}_s) - K(y^i_s - y^{i-1}_s) - K|z^i_s - z^{i-1}_s| \right) ds + \int_t^T g(s, y^i_s, z^i_s) dB_s - \int_t^T z^i_s dW_s, \quad (3)
\]
where \( i = 1, 2, \cdots \), and \((y^0_t, z^0_t) = (Y^1_t, Z^1_t)\). By Theorem 1.1 in Pardoux and Peng [15], BDSDEs (3) \((i = 1, 2, \cdots)\) have a unique adapted solution respectively if \( f(t, y^{i-1}_t, z^{i-1}_t) \in M^2(0, T; \mathbb{R}) \). For these equations, we have:

**Lemma 3.2** Under the assumptions \((H1)-(H4)\), for any positive integer \( i \), then, if \( \xi \in L^2(\Omega, \mathcal{F}_T, P) \), BDSDE (3) has a unique adapted solution \((y^i_t, z^i_t) \in S^2([0, T]; \mathbb{R}) \times M^2(0, T; \mathbb{R}^d)\) and \( Y^1_t \leq y^i_t \leq y^{i+1}_t \leq Y^2_t \), \( P \)-a.s., \( \forall t \in [0, T] \).

**Proof.** Firstly, for \( i = 1 \), by \( Y^2_t \geq Y^1_t \) and \((H2)\), it follows that
\[
f_2(t, Y^2_t, Z^2_t) - f(t, Y^1_t, Z^1_t) \geq f(t, Y^2_t, Z^2_t) - f(t, Y^1_t, Z^1_t) \geq -K(Y^2_t - Y^1_t) - K|Z^2_t - Z^1_t|. \quad (4)
\]
Thus
\[
f_2(t, Y^2_t, Z^2_t) + K(Y^2_t - Y^1_t) + K|Z^2_t - Z^1_t| \geq f(t, Y^1_t, Z^1_t) \geq f_1(t, Y^1_t, Z^1_t),
\]
this implies that \( f(t, Y^1_t, Z^1_t) \in M^2(0, T; \mathbb{R}) \). By Theorem 1.1 in Pardoux and Peng [15], it follows that BDSDE (3) has a unique adapted solution \((y^1_t, z^1_t)\).
Now, by (3) and (2) when \( i = 1 \) and \( j = 1 \), we have

\[
y_t^1 - Y_t^1 = \int_t^T (-K(y_s^1 - Y_s^1) - K|z_s^1 - Z_s^1| + \Delta_s^1)ds \\
+ \int_t^T (g(s, y_s^1, z_s^1) - g(s, Y_s^1, Z_s^1))dB_s - \int_t^T (z_s^1 - Z_s^1)dW_s,
\]

where \( \Delta_s^1 := f(s, Y_s^1, Z_s^1) - f_1(s, Y_s^1, Z_s^1) \geq 0 \) and \( \Delta_s^1 \in M^2(0, T; \mathbb{R}) \). Applying Lemma 2.2, we know \( y_t^1 \geq Y_t^1 \), P-a.s., \( \forall t \in [0, T] \).

Again we consider (3) and (2) when \( i = 1 \) and \( j = 2 \), we have

\[
Y_t^2 - y_t^1 = \int_t^T (-K(Y_s^2 - y_s^1) - K|Z_s^2 - z_s^1| + \Delta_s^2)ds \\
+ \int_t^T (g(s, Y_s^2, Z_s^2) - g(s, y_s^1, y_s^1))dB_s - \int_t^T (Z_s^2 - z_s^1)dW_s,
\]

where \( \Delta_s^2 := f_2(s, Y_s^2, Z_s^2) - f(s, Y_s^1, Z_s^1) + K(Y_s^2 - Y_s^1) + K|Z_s^2 - Z_s^1| \geq 0 \) and \( \Delta_s^2 \in M^2(0, T; \mathbb{R}) \). Then from Lemma 2.2, we have

\[
Y_t^1 \leq y_t^1 \leq Y_t^2, \text{ P-a.s., } \forall t \in [0, T].
\]

Similarly, for \( i = 2 \), since \( Y_t^1 \leq y_t^1 \leq Y_t^2 \) and (H2), it follows that

\[
f_2(t, Y_t^2, Z_t^2) - f(t, y_t^1, z_t^1) \geq f(t, Y_t^2, Z_t^2) - f(t, y_t^1, z_t^1) \\
\geq -K(Y_t^2 - y_t^1) - K|Z_t^2 - z_t^1|.
\]

Thus

\[
f_2(t, Y_t^2, Z_t^2) + K(Y_t^2 - y_t^1) + K|Z_t^2 - z_t^1| \geq f(t, y_t^1, z_t^1).
\]

But

\[
f(t, y_t^1, z_t^1) \geq f_1(t, Y_t^1, Z_t^1) - K(y_t^1 - Y_t^1) - K|z_t^1 - Z_t^1|,
\]

this implies that \( f(t, y_t^1, z_t^1) \in M^2(0, T; \mathbb{R}) \), and BDSDE (3) has a unique adapted solution \((y_t^2, z_t^2)\). Using the similar method, we have

\[
y_t^1 \leq y_t^2 \leq Y_t^2, \text{ P-a.s., } \forall t \in [0, T].
\]

Finally, for \( i > 2 \), we assume that \( Y_t^1 \leq y_t^{i-1} \leq y_t^i \leq Y_t^2 \) and \( f(t, y_t^{i-1}, z_t^{i-1}) \in M^2(0, T; \mathbb{R}) \), we consider BDSDE (3) for \( i + 1 \), which can be written as

\[
y_{t+1}^{i+1} = \xi + \int_t^{T} \left( f(s, y_s^i, z_s^i) - K(y_s^{i+1} - y_s^i) - K|z_s^{i+1} - z_s^i| \right) ds \\
+ \int_t^{T} g(s, y_s^i, z_s^i)dB_s - \int_t^{T} z_s^i dB_s,
\]

Then by the similar argument as the case \( i = 2 \), we have

\[
f_2(t, Y_t^2, Z_t^2) + K(Y_t^2 - y_t^i) + K|Z_t^2 - z_t^i| \\
\geq f(t, y_t^i, z_t^i).
\]
Applying Itô’s formula to

This implies that \( f(t, y^i_t, z^i_t) \in M^2(0, T; \mathbb{R}) \), and BDSDE (6) has a unique adapted solution \((y^{i+1}_t, z^{i+1}_t)\). By Lemma 2.2 again, we have

\[ Y^1_t \leq y^i_t \leq y^{i+1}_t \leq Y^2_t, \quad \text{P.a.s., } \forall t \in [0, T]. \]

\[ \square \]

**Lemma 3.3** There exists a constant \( A > 0 \), such that

\[ \sup_i E \left[ \sup_{0 \leq t \leq T} |y^i_t|^2 + \int_0^T |z^i_t|^2 dt \right] < A. \]

**Proof.** From Lemma 3.2, we have

\[ \sup_i \left[ E \sup_{0 \leq t \leq T} |y^i_t|^2 \right] \leq E \left[ \sup_{0 \leq t \leq T} |Y^1_t|^2 \right] + E \left[ \sup_{0 \leq t \leq T} |Y^2_t|^2 \right] < \infty. \]

By the similar argument as (11), we can deduce

\[
\begin{align*}
& f_2 (t, y^2_t, Z^2_t) + K (Y^2_t - y^i_t) + K |Z^2_t - z^i_t| \\
& \geq f(t, y^i_t, z^i_t) - f_1 (t, Y^1_t, Z^1_t) - K (y^i_t - Y^1_t) - K |z^i_t - Z^1_t|.
\end{align*}
\]

Then, we have

\[
\begin{align*}
& |f(t, y^i_t, z^i_t) - K (y^{i+1}_t - y^i_t) - K |z^{i+1}_t - z^i_t|| \\
& \leq |f_1(t, Y^1_t, Z^1_t) - K (y^i_t - Y^1_t) - K |z^i_t - Z^1_t|| + K |y^{i+1}_t - y^i_t| \\
& \quad + |f_2(t, Y^2_t, Z^2_t) - K (Y^2_t - y^i_t) - K |Z^2_t - z^i_t|| + K |z^{i+1}_t - z^i_t| \\
& \leq \sum_{j=1}^2 \left[ |f_j(t, Y^j_t, Z^j_t)| + K (|Y^j_t| + |Z^j_t|) \right] \\
& \quad + 3K (|y^i_t| + |z^i_t|) + K (|y^{i+1}_t| + |z^{i+1}_t|).
\end{align*}
\]

(7)

Applying Itô’s formula to \( |y^{i+1}_t|^2 \) for \( t \in [0, T] \), we deduce

\[
\begin{align*}
E \int_0^T |z^{i+1}_t|^2 dt \\
& \leq E |\xi|^2 + 2E \int_0^T y^{i+1}_t \cdot (f(t, y^i_t, z^i_t) - K (y^{i+1}_t - y^i_t) - K |z^{i+1}_t - z^i_t|) dt \\
& \quad + E \int_0^T |g(t, y^{i+1}_t, z^{i+1}_t)|^2 dt \\
& \leq E |\xi|^2 + 2E \int_0^T |y^{i+1}_t| \cdot |f(t, y^i_t, z^i_t) - K (y^{i+1}_t - y^i_t) - K |z^{i+1}_t - z^i_t|| dt \\
& \quad + E \int_0^T |g(t, y^{i+1}_t, z^{i+1}_t)|^2 dt \\
& \leq C_1 + \frac{1 - \alpha'}{4} E \int_0^T (|z^i_t|^2 + |z^{i+1}_t|^2) dt + E \int_0^T |g(t, y^{i+1}_t, z^{i+1}_t)|^2 dt,
\end{align*}
\]

6
with some constant $C_1 > 0$. Hereafter, $\forall n \geq 1$, $C_n$ will be some positive real constant. Then

$$E \int_0^T |z_i^{i+1}|^2 dt \leq C_2 + \frac{1-\alpha'}{4} E \int_0^T (|z_i|^2 + |z_i^{i+1}|^2) dt + \alpha' E \int_0^T |z_i^{i+1}|^2 dt.$$  

That is

$$E \int_0^T |z_i^{i+1}|^2 dt \leq C_3 + \frac{1}{3} E \int_0^T |z_i|^2 dt.$$  

This implies that $\sup E \int_0^T |z_i^{i+1}|^2 dt < \infty$, which yields that the quantities $\psi^{i+1}(t, y_t^{i+1}, z_t^{i+1}) = f(t, y_t^i, z_t^i) - K(y_t^{i+1} - y_t^i) - K|z_t^{i+1} - z_t^i|$ are uniformly bounded in $M^2(0, T; \mathbb{R})$. Set $C_0 = \sup E \int_0^T |\psi^{i+1}(t, y_t^{i+1}, z_t^{i+1})|^2 dt$. □

**Lemma 3.4** There exist processes $(y_t, z_t) \in S^2([0, T]; \mathbb{R}) \times M^2(0, T; \mathbb{R}^d)$ such that as $n \to \infty$

$$E \left[ \sup_{0 \leq t \leq T} |y_t^n - y_t|^2 + \int_0^T |z_t^n - z_t|^2 dt \right] \to 0.$$

**Proof.** By Lemma 3.2 it follows that there exists a process $y_t$ such that $y_t^n \stackrel{\text{P-a.s.}}{\to} y_t$, $\forall t \in [0, T]$, as $n \to \infty$, and $E \left[ \sup_{0 \leq t \leq T} |y_t|^2 \right] < \infty$. By the dominated convergence theorem, we get as $n \to \infty$,

$$E \int_0^T |y_t^n - y_t|^2 dt \to 0. \quad (8)$$

Coming back to (7) and by Lemma 3.2, we can deduce

$$\sup_n \left[ E \int_0^T |\psi^n(t, y_t^n, z_t^n)|^2 dt \right] < \infty. \quad (9)$$

Applying Itô's formula to $|y_t^n - y_t^m|^2$ for $t \in [0, T]$, taking expectation in both sides, we have

$$E(|y_t^n - y_t^m|^2) + E \int_0^T |z_t^n - z_t^m|^2 dt$$

$$= \quad 2E \int_0^T (y_t^n - y_t^m)(\psi^n(t, y_t^n, z_t^n) - \psi^m(t, y_t^m, z_t^m))dt$$

$$+ E \int_0^T |g(t, y_t^n, z_t^n) - g(t, y_t^m, z_t^m)|^2 dt$$

$$\leq \quad 4C_0 E \int_0^T |y_t^n - y_t^m|^2 dt^{1/2} + E \int_0^T (c|y_t^n - y_t^m|^2 + \alpha|z_t^n - z_t^m|^2) dt.$$  

Then

$$E \int_0^T |z_t^n - z_t^m|^2 dt$$

$$\leq \quad \frac{1}{1-\alpha} \{4C_0 E \int_0^T |y_t^n - y_t^m|^2 dt^{1/2} + cE \int_0^T |y_t^n - y_t^m|^2 dt \}.$$  

7
Thus \( \{z^n_t\} \) is a Cauchy sequence in \( M^2(0,T;\mathbb{R}^d) \), therefore \( \{z^n_t\}_{n=1}^{\infty} \) converges in \( M^2(0,T;\mathbb{R}^d) \), to a limit \( z_t \), we have

\[
E \int_0^T |z^n_t - z_t|^2 dt \to 0. \tag{10}
\]

Applying Itô’s formula to \( |y^n_s - y^m_s|^2 \) for \( s \in [t,T] \), we have

\[
|y^n_t - y^m_t|^2 + \int_t^T |z^n_s - z^m_s|^2 ds
\]

\[
= 2 \int_t^T (y^n_s - y^m_s)(\psi^n(s, y^n_s, z^n_s) - \psi^m(s, y^m_s, z^m_s)) ds
\]

\[
+ \int_t^T |g(s, y^n_s, z^n_s) - g(s, y^m_s, z^m_s)|^2 ds - 2 \int_t^T (y^n_s - y^m_s) \cdot (z^n_s - z^m_s) dW_s
\]

\[
+ 2 \int_t^T (y^n_s - y^m_s) \cdot (g(s, y^n_s, z^n_s) - g(s, y^m_s, z^m_s)) dB_s.
\]

Taking supremum and expectation, by Young’s inequality, we get

\[
E \left[ \sup_{0 \leq t \leq T} |y^n_t - y^m_t|^2 \right]
\]

\[
\leq 2 \left[ E \int_0^T (y^n_s - y^m_s)^2 ds \right]^{1/2} \left[ E \int_0^T (\psi^n(s, y^n_s, z^n_s) - \psi^m(s, y^m_s, z^m_s))^2 ds \right]^{1/2}
\]

\[
+ E \int_0^T |g(s, y^n_s, z^n_s) - g(s, y^m_s, z^m_s)|^2 ds
\]

\[
+ 2E \sup_{0 \leq t \leq T} \left| \int_t^T (y^n_s - y^m_s) \cdot (z^n_s - z^m_s) dW_s \right|
\]

\[
+ 2E \sup_{0 \leq t \leq T} \left| \int_t^T (y^n_s - y^m_s) \cdot (g(s, y^n_s, z^n_s) - g(s, y^m_s, z^m_s)) dB_s \right|. \tag{11}
\]

By Burkholder-Davis-Gundy’s inequality, we deduce

\[
E( \sup_{0 \leq t \leq T} \left| \int_t^T (y^n_s - y^m_s) \cdot (g(s, y^n_s, z^n_s) - g(s, y^m_s, z^m_s)) dB_s \right|)
\]

\[
\leq kE(\int_0^T |y^n_s - y^m_s|^2 ds \cdot |g(s, y^n_s, z^n_s) - g(s, y^m_s, z^m_s)|^2 ds)^{1/2}
\]

\[
\leq kE(( \sup_{0 \leq t \leq T} |y^n_t - y^m_t|^2 )^{1/2} (\int_0^T |g(s, y^n_s, z^n_s) - g(s, y^m_s, z^m_s)|^2 ds)^{1/2})
\]

\[
\leq 2k^2 E \int_0^T |y^n_s - y^m_s|^2 ds + \frac{1}{8} E(\sup_{0 \leq t \leq T} |y^n_t - y^m_t|^2)
\]

\[
+ 2k^2 \alpha E \int_0^T |z^n_s - z^m_s|^2 ds. \tag{12}
\]
In the same way, we have
\[
E(\sup_{0 \leq t \leq T} \left| \int_t^T (y^n_s - y^m_s) \cdot (z^n_s - z^m_s) dW_s \right|) \\
\leq \frac{1}{8} E(\sup_{0 \leq t \leq T} |y^n_t - y^m_t|^2) + 2k^2 \int_0^T |z^n_s - z^m_s|^2 ds. \tag{13}
\]

From (12), (13) and (11), it follows that
\[
E\left[ \sup_{0 \leq t \leq T} |y^n_t - y^m_t|^2 \right] \\
\leq 4C_0 \left[ E \int_0^T (y^n_s - y^m_s)^2 ds \right]^{1/2} + 2c(4k^2 + 1) E \int_0^T |y^n_s - y^m_s|^2 ds \\
+ 2(\alpha + 4k^2(\alpha + 1)) \int_0^T |z^n_s - z^m_s|^2 ds.
\]

Then from (8) and (10), we can deduce
\[
E\left[ \sup_{0 \leq t \leq T} |y^n_t - y^m_t|^2 \right] \to 0
\]
as \(n, m \to \infty\). Obviously, the process \(y_t\) belongs to \(S^2([0,T]; \mathbb{R})\).

By (H1), (8) and (11), it follows that there exists a subsequence (we still denote by \(n\)) such that as \(n \to \infty\),
\[
\psi^n(t, y^n_t, z^n_t) - f(t, y_t, z_t) \to 0, \quad dt \times dP\text{-a.s.} \tag{14}
\]
□

Now we are in the position to give the proof of Theorem 3.1.

**Proof of Theorem 3.1** By (10) and the continuity of the stochastic integral, we get
\[
\sup_{0 \leq t \leq T} \left| \int_t^T z^n_s dW_s - \int_t^T z_s dW_s \right| \to 0 \quad \text{in probability},
\]
\[
\sup_{0 \leq t \leq T} \left| \int_t^T g(s, y^n_s, z^n_s) dB_s - \int_t^T g(s, y_s, z_s) dB_s \right| \to 0 \quad \text{in probability}.
\]

So there exist a subsequence (we still denote by \(\{n\}\)) such that as the convergence is \(P\)-almost surely.

Since (H1), (H3) and \(Y^1_t \leq y_t \leq Y^2_t\), we can deduce from the similar argument as (14) that \(f(t, y_t, z_t) \in M^2([0,T];\mathbb{R})\). In view of (10) and (14), then by the dominated convergence theorem, passing to a subsequence (we still denote by \(\{n\}\)), we have
\[
\int_0^T |\psi^n(t, y^n_t, z^n_t) - f(t, y_t, z_t)| dt \to 0, \quad P\text{-a.s.}
\]
Hence, passing to the limit, as $i \to \infty$ on both sides of (3), we can get

$$y_t = \xi + \int_t^T f(s, y_s, z_s)ds + \int_t^T g(s, y_s, z_s)dB_s - \int_t^T z_s dW_s.$$ 

It is obvious that $(y_t, z_t)$ is a solution of BDSDE (1) under (H1)-(H4). □

**Remark 3.5** Although the solution $(y_t, z_t)$ we get in Theorem 3.1 is constructed by approximating from below, we cannot get that the constructed solution is the minimal solution of BDSDE (1), in the sense that for any other solution $(Y_t, Z_t)$ of BDSDE (1), we have $y_t \leq Y_t$ under (H1)-(H4). This is because we cannot compare the generators of BDSDE (1) and BDSDE (3) such that Lemma 2.2 cannot be used to compare the solutions of BDSDE (1) and BDSDE (3).

In order to get the minimal solution of BDSDE (1), in the following of this paper, we replace (H2) by

(H5) $|f(t, y, 0)| \leq |f(t, 0, 0)| + K|y|,$  

$\forall (t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d,$ and $E \int_0^T |f(t, 0, 0)|^2 dt < \infty.$

**Remark 3.6** (i) The assumptions (H1) and (H5) imply

$$|f(t, y, z)| \leq K|y| + K|z| + |f(t, 0, 0)|, y \in \mathbb{R}, z \in \mathbb{R}^d, t \in [0, T].$$

(ii) It is obvious that (H5) is a special case of (H2).

We construct a sequence of BDSDEs as follows:

$$Y_0^i = \xi + \int_t^T (-K|Y_0^i| - K|Z_0^i| - |f(s, 0, 0)|)ds + \int_t^T g(s, Y_0^i, Z_0^i)dB_s - \int_t^T Z_0^i dW_s, \quad (15)$$

$$Y_0^{i+1} = \xi + \int_t^T (f(s, Y_i^0, Z_i^0) - K(Y_i^{i+1} - Y_i^i) - K|Z_i^{i+1} - Z_i^i|)ds + \int_t^T g(s, Y_i^{i+1}, Z_i^{i+1}) dB_s - \int_t^T Z_i^{i+1} dW_s, \quad (16)$$

where $i = 0, 1, 2, \ldots$. Besides the above equations, we also need the other BDSDE

$$Y_t^0 = \xi + \int_t^T (K|Y_s^0| + K|Z_s^0| + |f(s, 0, 0)|)ds + \int_t^T g(s, Y_s^0, Z_s^0)dB_s - \int_t^T Z_s^0 dW_s. \quad (17)$$

By Theorem 1.1 in Pardoux and Peng [15], BDSDEs (15), (16) $(i = 0, 1, \ldots)$ and (17) have a unique adapted solution respectively. For these solutions mentioned above, with the technique similarly to Lemma 3.2, we can obtain the following properties:
Lemma 3.7 Under the assumptions (H1) and (H3)-(H5), the following properties hold true:

(i) For any positive integer $i$, $Y_{i+1}^i \geq Y_i^i \geq Y_0^0$, P-a.s., $\forall t \in [0,T]$.

(ii) For any positive integer $i$, $Y_i^i \leq Y_t^t$, P-a.s., $\forall t \in [0,T]$.

Lemma 3.7 implies that the sequence of solutions of BDSDEs (15) and (16) is increasing and have upper bound by the solution of (17), that is

$$Y_0^0 \leq Y_i^i \leq Y_{i+1}^i \leq Y_t^t,$$  P-a.s., $\forall t \in [0,T], i = 0, 1, 2, \cdots$  \hspace{1cm} (18)

Furthermore, we get the existence of the minimal solution of BDSDEs.

Theorem 3.8 Under the assumptions (H1) and (H3)-(H5), then, if $\xi \in L^2(\Omega, \mathcal{F}_T, P)$, BDSDE (17) has a solution $(Y_t, Z_t) \in S^2([0,T]; \mathbb{R}) \times M^2(0,T; \mathbb{R}^d)$. Moreover, there is a minimal solution $(\bar{Y}_t, \bar{Z}_t)$ of BDSDE (17) in the sense that, for any other solution $(Y_t, Z_t)_{t \in [0,T]}$ of BDSDE (17), we have $Y_t \geq \bar{Y}_t$, P-a.s., $\forall t \in [0,T]$.

Proof. Similarly to the arguments in Theorem 3.4, the existence of a solution of (17) can be obtained easily. We want to prove the existence of a minimal solution of (17). Let $(Y_t, Z_t)_{t \in [0,T]}$ be a solution of BDSDEs (17). For $i = 0$, we have

$$Y_t - Y_0^0 = \int_t^T (f(s, Y_s, Z_s) + K|Y_s^0| + K|Z_s^0| + |f(s, 0, 0)|)ds$$
$$+ \int_t^T (g(s, Y_s, Z_s) - g(s, Y_0^0, Z_0^0))dB_s - \int_t^T (Z_s - Z_0^0)dW_s$$
$$= \int_t^T (-K|Y_s - Y_0^0| - K|Z_s - Z_0^0| + \Psi_s^0)ds$$
$$+ \int_t^T (g(s, Y_s, Z_s) - g(s, Y_0^0, Z_0^0))dB_s - \int_t^T (Z_s - Z_0^0)dW_s$$

where

$$\Psi_s^0 := K|Y_s - Y_0^0| + K|Z_s - Z_0^0| + K|Y_0^0| + K|Z_0^0| + |f(s, 0, 0)| - f(s, Y_s, Z_s)$$
$$\geq K|Y_s| + K|Z_s| + |f(s, 0, 0)| - f(s, Y_s, Z_s) \geq 0,$$

From Lemma 2.2 we have $Y_t \geq Y_0^0$, P-a.s., $\forall t \in [0,T]$.

We assume that $Y_t \geq Y_{i+1}^i$, we will prove that $Y_t \geq Y_{i+1}^i$. We have

$$Y_t - Y_{i+1}^i = \int_t^T (-K|Y_s - Y_{i+1}^i| - K|Z_s - Z_{i+1}^i| + \Psi_s^{i+1})ds$$
$$+ \int_t^T (g(s, Y_s, Z_s) - g(s, Y_{i+1}^i, Z_{i+1}^i))dB_s - \int_t^T (Z_s - Z_{i+1}^i)dW_s$$

where

$$\Psi_s^{i+1} := f(s, Y_s, Z_s) - f(s, Y_{i+1}^i, Z_{i+1}^i) + K(Y_{i+1}^i - Y_i^i) + K|Z_{i+1}^i - Z_i^i|$$
$$+ K(Y_{i+1}^i - Y_s) + K|Z_{i+1}^i - Z_s|$$
$$\geq f(s, Y_s, Z_s) - f(s, Y_{i+1}^i, Z_{i+1}^i) + K(Y_s - Y_i^i) + K|Z_s - Z_i^i| \geq 0,$$
then, \( Y_t \geq Y_{t+}^{i+1}, \) P-a.s., \( \forall t \in [0,T] \). This implies \( Y_t \geq Y_{t+}, \) P-a.s., \( \forall t \in [0,T] \). □

In order to get the upper bound of solution of BDSDE (11), besides (15), we also need the following BDSDE:

\[
\begin{align*}
\bar{Y}_t^{i+1} &= \xi + \int_t^T \left( f(s, \bar{Y}_s^i, \bar{Z}_s^i) - K(\bar{Y}_s^i + 1 - \bar{Y}_s^i) + K(\bar{Z}_s^i + 1 - \bar{Z}_s^i) \right) ds \\
&\quad + \int_t^T g(s, \bar{Y}_s^i, \bar{Z}_s^i) dB_s - \int_t^T \bar{Z}_s^i dW_s.
\end{align*}
\]  (19)

For any positive integer \( i \), it is obvious that BDSDE (19) has a unique adapted solution. By similar procedures, we get the following result:

**Theorem 3.9** Under the assumptions \((H1)\) and \((H3)-(H5)\), and assuming \( \{\bar{Y}_t^i, \bar{Z}_t^i\}_{t \in [0,T]} \) are the solutions of BDSDEs (19), then

(i) \( \bar{Y}_t^0 \leq \bar{Y}_t^{i+1} \leq \bar{Y}_t^0, \) P-a.s., \( \forall t \in [0,T], i = 0, 1, \ldots, \)

(ii) \( \{\bar{Y}_t^i, \bar{Z}_t^i\}_{t \in [0,T]} \) converge in \( \mathbb{S}^2([0,T]; \mathbb{R}) \times M^2(0,T; \mathbb{R}^d) \) to a limit \( \{\bar{Y}_t, \bar{Z}_t\}_{t \in [0,T]} \), which is the upper bound of solution of (11), in the sense that, for any other solution \( \{Y_t, Z_t\}_{t \in [0,T]} \) of BDSDE (11), we have \( \bar{Y}_t \geq Y_t, \) P-a.s., \( \forall t \in [0,T] \).

**Remark 3.10** It is uncertain whether the upper bound \( \{\bar{Y}_t, \bar{Z}_t\}_{t \in [0,T]} \) of solution of (11) is the solution of (11).

### 4 Comparison Theorem

The comparison theorem is an important and effective technique in the theory of BDSDEs. Shi et al. [19] have proved the comparison theorem for the solutions of BDSDEs with Lipschitz coefficients. As an application, they showed the existence of a solution for one dimensional BDSDEs when the coefficient is continuous with linear growth. In this section, we generalize the comparison theorem to the case where the coefficient is left-Lipschitz in \( y \) (may be discontinuous) and Lipschitz in \( z \).

**Theorem 4.1** (Comparison theorem) Let \( \{Y_t^i, Z_t^i\}_{t \in [0,T]} \) \( (i = 1, 2) \) be the minimal solutions of the following BDSDEs

\[
\begin{align*}
Y_t^1 &= \xi^1 + \int_t^T f_1(s, Y_s^1, Z_s^1) ds + \int_t^T g(s, Y_s^1, Z_s^1) dB_s - \int_t^T Z_s^1 dW_s, \quad \quad \quad (21) \\
Y_t^2 &= \xi^2 + \int_t^T f_2(s, Y_s^2, Z_s^2) ds + \int_t^T g(s, Y_s^2, Z_s^2) dB_s - \int_t^T Z_s^2 dW_s, \quad \quad \quad (22)
\end{align*}
\]

respectively, where \( f_1, f_2, g \) satisfy \((H1)\) and \((H3)-(H5)\), \( \xi^1, \xi^2 \in L^2(\Omega, \mathcal{F}_T, P) \), and \( \xi^1 \geq \xi^2, \) a.s., \( f_1(t, y, z) \geq f_2(t, y, z), \) a.s., \( \forall (t, y, z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d \). Then \( Y_t^1 \geq Y_t^2, \) P-a.s., \( \forall t \in [0,T] \).
then, for any positive integer \(i\), we have

\[
y_i^0 = \xi^2 + \int_t^T (-K|y_s^0| - K|z_s^0|)ds + \int_t^T g(s, y_s^0, z_s^0)dB_s - \int_t^T z_s^0dW_s. \tag{25}
\]

First, we prove \(Y_t^1 \geq y_t^0\), P.a.s., \(\forall t \in [0, T]\). From (21) and (25), we have

\[
Y_t^1 - y_t^0 = \xi^1 - \xi^2 + \int_t^T (f_1(s, Y_s^1, Z_s^1) + K|y_s^1| + K|z_s^1| + K)ds
\]

\[
+ \int_t^T (g(s, Y_s^1, Z_s^1) - g(s, y_s^0, z_s^0))dB_s - \int_t^T (Z_s^1 - z_s^0)dB_s
\]

\[
= \int_t^T (-K|Y_s^1 - y_s^0| - K|Z_s^1 - z_s^0| + \theta_s^0)ds
\]

\[
+ \int_t^T (g(s, Y_s^1, Z_s^1) - g(s, y_s^0, z_s^0))dB_s - \int_t^T ((Z_s^1 - z_s^0)dW_s
\]

where

\[
\theta_s^0 := K|Y_s^1 - y_s^0| + K|Z_s^1 - z_s^0| + K|y_s^0| + K|z_s^0| + K - f_2(s, Y_s^2, Z_s^2)
\]

\[
\geq K|Y_s^1| + K|Z_s^1| + K + f_1(s, Y_s^1, Z_s^1) \geq 0,
\]

we know \(Y_t^1 \geq y_t^0\), P.a.s., \(\forall t \in [0, T]\).

We assume that \(Y_t^1 \geq y_t^i\), P.a.s., \(\forall t \in [0, T]\), from (21) and (23), we have

\[
Y_t^{i+1} - y_t^i = \xi^1 - \xi^2 + \int_t^T (f_1(s, Y_s^1, Z_s^1) - f_2(s, y_s^i, z_s^i) + K(y_s^{i+1} - y_s^i)
\]

\[
+ K|z_s^{i+1} - z_s^{i+1}|)ds
\]

\[
+ \int_t^T (g(s, Y_s^1, Z_s^1) - g(s, y_s^{i+1}, z_s^{i+1}))dB_s - \int_t^T (Z_s^{i+1} - z_s^{i+1})dB_s
\]

\[
= \xi^1 - \xi^2 + \int_t^T (-K|Y_s^1 - y_s^{i+1}| - K|Z_s^1 - z_s^{i+1}| + \theta_s^{i+1})ds
\]

\[
+ \int_t^T (g(s, Y_s^1, Z_s^1) - g(s, y_s^{i+1}, z_s^{i+1}))dB_s - \int_t^T (Z_s^{i+1} - z_s^{i+1})dB_s
\]

where

\[
\theta_s^{i+1} := K(Y_s^1 - y_s^{i+1}) + K|Z_s^1 - z_s^{i+1}| + f_1(s, Y_s^1, Z_s^1) - f_2(s, y_s^i, z_s^i)
\]

\[
+ K(y_s^{i+1} - y_s^i) + K|z_s^{i+1} - z_s^{i+1}|
\]

\[
\geq f_1(s, Y_s^1, Z_s^1) - f_2(s, y_s^i, z_s^i) + K(Y_s^1 - y_s^i) + K|Z_s^1 - z_s^i| \geq 0,
\]

then, for any positive integer \(i\), we have \(Y_t^i \geq y_t^i\), P.a.s., \(\forall t \in [0, T]\). From Theorem 3.5 we have \(\{y_t^i, z_t^i\}_{t \in [0, T]}\) converges to \(\{Y_t^1, Z_t^1\}_{t \in [0, T]}\) P.a.s. \(\square\)
Remark 4.2 we replace (H1) by

\((H6)\) \(f(t,\cdot,z)\) is right-continuous, and \(f(t,y,\cdot)\) is Lipschitz continuous.

We can deduce

(i) Under the assumptions (H2)-(H4) and (H6), the same result in Theorem 3.1 holds true.

(ii) Under the assumptions (H3)-(H6), we can prove the existence and comparison results of the maximal solutions of BDSDE (1).

References

[1] V. Bally, A. Matoussi, Weak solutions for SPDEs and backward doubly stochastic differential equations, J. Theoret. Probab. 14 (2001) 125–164.

[2] D. Duffie, L. Epstein, Stochastic differential utilities, Econometrica 60 (1992) 354–439.

[3] S. Hamadene, J-P. Lepeltier, Backward equations, stochastic control and zero-sum stochastic differential games, Stoch. Stoch. Rep. 54 (1995) 221–231.

[4] M. Hu, S. Peng, On representation theorem of G-expectations and paths of G-Brownian motion, Acta Mathematicae Applicatae Sinica 25 (2009) 539–546.

[5] L. Hu, Y. Ren, Stochastic PDEs with nonlinear Neumann boundary conditions and generalized backward doubly stochastic differential equations driven by Lévy processes, J. Comput. Appl. Math. 229 (2009) 230–239.

[6] G. Jia, A generalized existence theorem of BSDEs, C. R. Acad. Sci. Paris, Ser. I 342 (2006) 685–688.

[7] G. Jia, On existence of backward stochastic differential equations with left-Lipschitz coefficient, Chin. J. of Contemp. Math. 28 (2007) 345–354. (in Chinese)

[8] L. Jiang, Z. Chen, A result on the probability measures dominated by g-expectation, Acta Mathematicae Applicatae Sinica 20 (2004) 1–6.

[9] K. Kim, \(L_p\) estimates for SPDE with discontinuous coefficients in domains, Electron. J. Probab. 10 (2005) 1–20.

[10] J.P. Lepeltier, J. San Martin, Backward stochastic differential equations with continuous coefficients, Statist. Probab. Lett. 34 (1997) 425–430.

[11] Q. Lin, A class of backward doubly stochastic differential equations with non-Lipschitz coefficients, Statist. Probab. Lett. 79 (2009) 2223–2229.

[12] D. Nualart, E. Pardoux, Stochastic calculus with anticipating integrands, Probab. Theory Related Fields 78 (1988) 535–581.

[13] M. N’zi, J.M. Owo, Backward doubly stochastic differential equations with discontinuous coefficients, Statist. Probab. Lett. 79 (2009) 920–926.

[14] E. Pardoux, S. Peng, Adapted solution of a backward stochastic differential equation, Systems Control Lett. 14 (1990) 55–61.

[15] E. Pardoux, S. Peng, Backward doubly stochastic differential equations and systems of quasilinear parabolic SPDEs, Probab. Theory Related Fields 98 (1994) 209–227.
[16] S. Peng, *Probabilistic interpretation for systems of quasilinear parabolic partial differential equations*, Stoch. Stoch. Rep. **37** (1991) 61–74.

[17] S. Peng, *Filtration consistent nonlinear expectations and evaluations of contingent claims*, Acta Mathematicae Applicatae Sinica **20** (2004) 1–24.

[18] Y. Ren, A. Lin, L. Hu, *Stochastic PDIEs and backward doubly stochastic differential equations driven by Lévy processes*, J. Comput. Appl. Math. **223** (2009) 901–907.

[19] Y. Shi, Y. Gu, K. Liu, *Comparison theorems of backward doubly stochastic differential equations and applications*, Stoch. Anal. Appl. **23** (2005), 97–110.

[20] H. Yoo, *$L_p$-estimate for stochastic PDEs with discontinuous coefficients*, Stoch. Anal. Appl. **17** (1999) 678–711.

[21] Q. Zhang, H. Zhao, *Stationary solutions of SPDEs and infinite horizon BDSDEs*, J. Funct. Anal. **252** (2007) 171–219.