Abstract

A small quantum scattering system (the microsystem) is studied in interaction with a large quantum system (the macrosystem) described by unknown stochastic variables. The interaction between the two systems is diagonal for the microsystem in a certain basis, and it leads to an imprint on the macrosystem. Moreover, the interaction is assumed to involve only small transfers of energy and momentum between the two systems (as compared to typical energies/momenta within the microsystem). This makes it suitable to carry out the analysis in scattering theory, where the transition amplitude for the whole system factorizes. The interaction taking place within the macrosystem is assumed to depend on the stochastic variables in such a way that, on the average, no particular channel is favoured. The result is then, in the thermodynamic limit of the macrosystem, that the whole system bifurcates and the microsystem ends up in a state described by one of the basis vectors (in the mentioned basis). The macrosystem ends up in an entangled state tied to this basis vector. For the ensemble of macrosystems, the interaction with the microsystem leads, on the average, to the usual decoherence and diagonal density matrix for the microsystem. The macrosystem can be interpreted as representing a measurement device for performing a measurement on the microsystem. The whole discussion is carried out within quantum mechanics itself without any modification or generalization.
1 Introduction: the problem

The aim of this paper is to analyse a microscopic quantum event in a microscopic quantum system $\mu$ together with a related interaction with a macroscopic system $M$, not known in any detail and therefore described by stochastic variables. The intention is to model a measurement process, where $M$ is a measurement device for performing a measurement on $\mu$.

Let us assume that an observable $A$ with non-degenerate eigenstates $|j\rangle_\mu$ is to be measured. The interaction between $\mu$ and $M$ must then be such that the state $|j\rangle_\mu$ of $\mu$ makes an imprint on $M$. We assume this to take place without the state of $\mu$ being changed.

In the discussion of measurement it has often been assumed that the process within $\mu$ can be analysed independently of the interaction between $\mu$ and $M$ (assumption $X$), although the final state of $\mu$ is registered only through measurement, i.e., only through the interaction between $\mu$ and $M$.

If the interaction between $\mu$ and $M$ is analysed under the assumption $X$ of a given final state from the process within $\mu$, and if the interaction between $\mu$ and $M$ is of the kind just described, then in general, a reduction of the state of $\mu$ into an eigenstate of $A$ cannot take place without violating the superposition principle [1]. This situation has led to a common notion that there may be two kinds of interaction, (i) a microscopic interaction within the system $\mu$ itself, obeying a linear Schrödinger equation, and (ii) another type of interaction between $\mu$ and $M$. A possibility to combine both of these effects is to have a non-linear equation of motion [2] to [5], where the non-linearities become non-negligible only in interactions involving mesoscopic or macroscopic objects.

In this paper we show that a non-linearity arises within quantum mechanics itself through extension of the system considered, to include also $M$.

All since the famous Einstein-Podolski-Rosen paper [6], quantum entanglement phenomena have shown consequences that have been regarded as counter-intuitive. In this analysis, we use S-matrix theory to investigate the rôle of quantum entanglement between $\mu$ and $M$ through a final-state interaction, that gives a factor in the overall transition matrix.

The point of view taken here, not accepting the assumption $X$, is that it is necessary to analyse the interaction within $\mu$ and the interaction between $\mu$ and $M$ as a whole. As pointed out already, the interaction between $\mu$ and $M$ is assumed to give $M$ an imprint from $\mu$ without changing the eigenstate of the observable $A$ for $\mu$. It then also leads to an entanglement of $\mu$ with the (metastable) system $M$. The stochastic variables characterizing $M$ may have an enhancing or inhibiting influence on the transitions within $M$ to a final state. Therefore, the different initial states of the metastable system $M$, described by stochastic variables, compete on an unequal basis to reach the final state, and the ensemble of final states can have a very different composition from that of the initial states.

The system $M$ should not only be metastable; it should also be unbiased. We take this to mean that the corresponding enhancement factors and inhibition factors of $M$ occur with the same frequency in the initial state.

Since we are studying the process of internal interaction and measurement as a whole, it is natural to conduct the analysis within the framework of scattering theory. Moreover, in the limit of low energy and momentum transfer, the $\mu$-$M$ interaction factorizes in the scattering amplitude
(see Appendix 1) and hence also in the transition probability per unit time. The factor from $\mu$-$M$ interaction depends on $\mu$ only through its outgoing state.

The stochastic variables of $M$ are introduced through a stepwise mapping procedure, thus going from the situation of the microsystem $\mu$ by itself to a situation where $\mu$ interacts with the system $M$ in the thermodynamic limit, i.e., in the limit of an infinite number of stochastic variables.

This mapping is non-linear, and the non-linearity can be understood in the formalism of perturbation theory, where the internal $\mu$ interaction has to appear in Feynman diagrams mixed with the $\mu$-$M$ interaction (Appendix 2).

This mapping turns out to be a bifurcation process, describable in the probability simplex of $\mu$ as a random walk (brownian motion), ending up in one of the corners. The ensemble of such walks is then a diffusion process with the corners of the simplex as attractors.

A random-walk process is not new in this context. It results from a non-linear dynamics like the one suggested in [5]. It must be emphasized that the model in this paper is an $S$-matrix model not describing the detailed dynamics. As mentioned already, the mapping here interpolates between a situation without the system $M$ and a situation with the system $M$ where $M$ has many degrees of freedom (the thermodynamic limit).

In the next section, the scattering process is described for the situation of $\mu$ without $M$. The modifications due to a final-state interaction with $M$ is described in Section 3. The stochastic variables are introduced in Section 4, and their influence on the whole process is taken into account. Section 5 describes the mapping procedure, the resulting random-walk or diffusion process and the thermodynamic limit of $M$. In Section 6, a simplified model is shown in detail. In Section 7, we show, how correlations between stochastic variables of $M$ can build up through entanglement with $\mu$. In Section 8, we indicate how the measurement process can be interpreted as an evolution process in Darwinian terms. Finally, we state briefly a few conclusions in the last section.

2 The underlying scattering process

We assume that the microscopic quantum system $\mu$ has an internal dynamics taking it from an initial state $|0\rangle_\mu$, which is an eigenstate of the free Hamiltonian $H_0$,

$$H_0|0\rangle_\mu = E_0|0\rangle_\mu, \quad \mu\langle 0|0\rangle_\mu = 1,$$

(1)

to a certain final state $|\Psi\rangle_\mu$, which is also an eigenstate of $H_0$, but different from the initial state (for instance localized in a different spatial region),

$$H_0|\Psi\rangle_\mu = E|\Psi\rangle_\mu, \quad \mu\langle \Psi|\Psi\rangle_\mu = 1, \quad \mu\langle 0|\Psi\rangle_\mu = 0.$$

(2)

Consider an orthonormal basis for the final states,

$$|j\rangle_\mu, \quad j = 1, 2, ..., n; \quad \mu\langle j|0\rangle_\mu = 0; \quad \mu\langle j|k\rangle_\mu = \delta_{jk},$$

(3)
formed by states that are degenerate eigenstates of $H_0$ and non-degenerate eigenstates of an observable $A = A^\dagger$ acting on $\mu$,

\begin{align*}
H_0 |j\rangle_\mu &= E |j\rangle_\mu; \\
A |j\rangle_\mu &= a_j |j\rangle_\mu, \quad a_j^* = a_j, \quad a_j \neq a_k \text{ for } j \neq k; \\
A |0\rangle_\mu &= a_0 |0\rangle_\mu, \quad a_0^* = a_0, \quad a_0 \neq a_k.
\end{align*}

Let the final state in this basis be

$$|\Psi\rangle_\mu = \sum_{j=1}^{n} \Psi_j |j\rangle_\mu; \quad \sum_{j=1}^{n} |\Psi_j|^2 = 1.$$  \hspace{1cm} (5)

Because of (4), the observable $A$ commutes with $H_0$, and it is suitable for describing an outgoing state. Clearly the internal interaction Hamiltonian for $\mu$, $H_I$, does not commute with $A$,

$$[H_I, A] \neq 0.$$  \hspace{1cm} (6)

Then the S-matrix elements are proportional to the components of $|\Psi\rangle_\mu$ in (5),

$$\mu \langle j | S | 0 \rangle_\mu = \mu \langle j | M | 0 \rangle_\mu \delta (E - E_0),$$  \hspace{1cm} (7)

with

$$\mu \langle j | M | 0 \rangle_\mu = \sqrt{\Gamma} \Psi_j.$$  \hspace{1cm} (8)

This is the transition probability amplitude for a transition from the initial state $|0\rangle_\mu$ to the final state $|j\rangle_\mu$ of $\mu$.

The transition probability per unit time for this transition is then

$$(2\pi)^{-1} \mu \langle j | M | 0 \rangle_\mu^2 \delta (E - E_0) = (2\pi)^{-1} \Gamma |\Psi_j|^2 \delta (E - E_0),$$  \hspace{1cm} (9)

and the total transition probability per unit time for a transition to any of the final states is

$$(2\pi)^{-1} \Gamma \delta (E - E_0).$$  \hspace{1cm} (10)

The density operator of the initial state for given energy is

$$\rho_0 = |0\rangle_\mu \langle 0|; \quad \text{Tr} \rho_0 = 1.$$  \hspace{1cm} (11)

The corresponding density operator of the final state is

$$\rho_f = \Gamma^{-1} M |0\rangle_\mu \langle 0 | M^\dagger = |\Psi\rangle_\mu \langle \Psi | = \sum_{j,k=1}^{n} \Psi_j \Psi_k^* |j\rangle_\mu \langle k|; \quad \text{Tr} \rho_f = 1.$$  \hspace{1cm} (12)
3 Changes due to soft final-state interaction

We now introduce the interaction between the particles in the outgoing states (the eigenstates of $A$) of $\mu$ and the large system $M$. We assume this interaction to involve only very small momentum and energy transfers. As pointed out already, the interaction between $\mu$ and $M$ and within $M$ is assumed to be such that an eigenstate $|j\rangle_\mu$ of $A$ for $\mu$ does not change but makes an imprint on $M$. We assume the initial states of $M$ to be $|0;\varepsilon\rangle_M$, where '0' denotes 'no imprint' and where $\varepsilon$ is a set of stochastic variables to describe the (unknown) structure of $M$. We assume the variables $\varepsilon$ to be so defined that they are constants of motion. The corresponding final states can then be denoted by $|j;\varepsilon\rangle_M$, where 'j' denotes an imprint on $M$.

Here the assumption is made that the measurement interaction copies the state of $\mu$ without changing it. For the class of measurements where the state of $\mu$ is changed, a generalized formalism is needed. Such a generalization seems to be rather straightforward however.

Thus the initial state of the combined system $\mu + M$ is

$$|0\rangle_\mu \otimes |0;\varepsilon\rangle_M,$$

and a corresponding set of final states are

$$|j\rangle_\mu \otimes |j;\varepsilon\rangle_M.$$  \hspace{1cm} (14)

The soft (low momentum-transfer) character of the $\mu$-$M$ interaction implies a factorization of this interaction. This is well-known and has been known for a long time \cite{7}, but we show the factorization for soft electromagnetic interaction in Appendix 1. Thus, the element of the new $M$-matrix denoted by $\overline{M}$, is then a modified version of (8),

$$\left( |\mu\langle j| \otimes M \langle j;\varepsilon| \right) \overline{M} \left( |0\rangle_\mu \otimes |0;\varepsilon\rangle_M \right) = b_j(\varepsilon) \sqrt{\Gamma} \Psi_j.$$  \hspace{1cm} (15)

The factor $b_j(\varepsilon)$ gives the modification due to final-state interaction, to be discussed in more detail below.

This means that for a given set of stochastic variables $\varepsilon$ the final state of the combined system, corresponding to (5) is proportional to

$$\overline{M} \left( |0\rangle_\mu \otimes |0;\varepsilon\rangle_M \right) = \sqrt{\Gamma} \sum_{j=1}^{n} b_j(\varepsilon) \Psi_j |j\rangle_\mu \otimes |j;\varepsilon\rangle_M,$$

which, in general, is not normalized.

In any scattering process, there is emission of soft radiation of photons and gravitons, which factorizes like in the description given here. Since this radiation is not detected, a summation over final states is necessary. In (9) this means an integration over how the energy $E_0$ is shared by the soft radiation and the other particles of $\mu$. We can leave out the $\delta$-function, and use (11) and (12) for the density operator, which also makes normalization simpler. The energy $\delta$-function for the integrated process involving $\mu$ and $M$ is $\delta(E + \Delta E - E_0)$, where $E$ is the final energy of $\mu$, and where $\Delta E$ is the (very small) energy transfer from $\mu$ to $M$. Thus $E$ is approximately equal to $E_0$. 

5
The equations for the density operators corresponding to (11) and (12), are now
\[ \rho_0(\varepsilon) = (|0\rangle_\mu \otimes |0; \varepsilon\rangle_M) (|0\rangle \otimes M \langle 0; \varepsilon|) \] (17)
for the initial state, and
\[ \rho_f(\varepsilon) = \Gamma^{-1} M \rho_0(\varepsilon) M^\dagger = \] 
\[ \left( \sum_{l=1}^{n} |b_l(\varepsilon)|^2 |\Psi_l|^2 \right)^{-1} \sum_{j,k=1}^{n} b_j(\varepsilon)b_k(\varepsilon)^* \Psi_j(\varepsilon) \Psi_k^*(\varepsilon) (|j\rangle_\mu \otimes |j; \varepsilon\rangle_M) (|k\rangle \otimes M \langle k; \varepsilon|) \] (18)
for the final state, where
\[ \Gamma(\varepsilon) = \Gamma \sum_{l=1}^{n} |b_l(\varepsilon)|^2 |\Psi_l|^2 \] (19)
is the new probability per unit time for a transition.

We notice that the density operator (18) is non-linear in \(|\Psi_j|^2\). This is quite natural, since
in the total scattering operator, the internal interaction in \(\mu\), described by the S-matrix elements
given by (7) and (8), and the \(\mu\)-\(M\) interaction appear mixed to all orders, and a perturbation
expansion would show these non-linearities as diagrammatically described in Appendix 2. How-
ever, due to the factorization (16), the internal \(\mu\)-interaction only appears through its \(M\)-matrix elements (8).

4 The stochastic variables and their influence

For simplicity we assume that the stochastic variables describing \(M\) are sign factors,
\[ \varepsilon = (\varepsilon_{jx}); \quad \varepsilon_{jx} = \pm 1; \quad j = 1, 2, \ldots, n; \quad x = 1, 2, \ldots, X \text{ where } X \gg 1. \] (20)
and that they contribute to \(b_j(\varepsilon)\) with random enhancement/inhibition factors
\[ b_j(\varepsilon) = G \prod_{x=1}^{X} \prod_{l=1}^{n} \left( 1 + \eta_{lx} \left( \delta_{lj} - \frac{1}{2} \right) \varepsilon_{lx} \right); \quad \eta_{lx} = \eta_{xl}, \quad 0 < \eta_{lx} << 1, \] (21)
which can be ascribed to \(\varepsilon\)-dependence in the interactions within \(M\). Then, to second order in the \(\eta\)'s,
\[ |b_j(\varepsilon)|^2 = \chi B_j(\varepsilon); \]
\[ B_j(\varepsilon) = \prod_{x=1}^{X} \prod_{l=1}^{n} \left( 1 + \eta_{lx} \left( 2\delta_{lj} - 1 \right) \varepsilon_{lx} \right); \quad \chi = |G|^2 \prod_{x=1}^{X} \prod_{l=1}^{n} \left( 1 + \frac{1}{4} \eta_{lx}^2 \right). \] (22)
Moreover, we assume the ensemble of incoming states of \(M\) to be unbiased. Thus, all values
of \(\varepsilon\) are equally probable initially, each with a probability \(2^{-nX}\). Due to the \(\varepsilon\)-dependent en-
hancement/inhibition factors, in the ensemble of final states, however, the transition probabilities
differ. The probability for a transition taking place from an initial state, where $M$ is described by $\varepsilon$ is

$$P(\varepsilon) = \frac{\mathcal{T}(\varepsilon)}{\sum_{\varepsilon'} \mathcal{T}(\varepsilon')} = 2^{-nX} \sum_{j=1}^{n} |\Psi_j|^2 B_j(\varepsilon); \quad \sum_{\varepsilon} P(\varepsilon) = 1. \quad (23)$$

The final-state density operator (18) can now be written

$$\rho_f(\varepsilon) = \left( \sum_{l=1}^{n} |\Psi_l|^2 B_l(\varepsilon) \right)^{-1} \sum_{j,k=1}^{n} \sqrt{B_j(\varepsilon) B_k(\varepsilon)} \Psi_j \Psi_k^\dagger (|j\rangle_\mu \otimes |j;\varepsilon\rangle_M) (\langle j | \otimes M_{k;\varepsilon}). \quad (24)$$

With the probability distribution (23), the average over $M$ and $\varepsilon$ of (24) is

$$\langle \rho_{\mu f}(\varepsilon) \rangle_{\varepsilon} = \sum_{\varepsilon} P(\varepsilon) \text{Tr}_M[\rho_f(\varepsilon)] =$$

$$= 2^{-nX} \sum_{\varepsilon} \sum_{j,k=1}^{n} \sqrt{B_j(\varepsilon) B_k(\varepsilon)} \Psi_j \Psi_k^\dagger (|j\rangle_\mu \otimes |j;\varepsilon\rangle_M) (\langle j | \otimes M_{k;\varepsilon}). \quad (25)$$

Here $\theta_\mu \to 0$ in the limit of large $X$, since, to second order in the $\eta$'s,

$$2^{-nX} \sum_{\varepsilon} \sqrt{B_j(\varepsilon) B_k(\varepsilon)} =$$

$$= 2^{-nX} \sum_{\varepsilon} \prod_{x=1}^{X} \prod_{l=1}^{n} \left( 1 + \eta_{lx} (\delta_{lj} + \delta_{lk} - 1) \varepsilon_{lx} - \frac{1}{2} \eta_{lx}^2 (\delta_{lj} + \delta_{lk} - 2\delta_{lj}\delta_{lk}) \right) =$$

$$= \delta_{jk} + (1 - \delta_{jk}) \prod_{x=1}^{X} \left( 1 - \frac{1}{2} \eta_{lx}^2 - \frac{1}{2} \eta_{kx}^2 \right), \quad (26)$$

which goes to $\delta_{jk}$ in the limit of large $X$. Thus, in the thermodynamical limit of $M$,

$$\langle \rho_{\mu f}(\varepsilon) \rangle_{\varepsilon} = \sum_{j=1}^{n} |\Psi_j|^2 |j\rangle_\mu \langle j |, \quad (27)$$

i.e., in the ensemble of final states, the mean of the density matrix for $\mu$ becomes diagonal with the elements $|\Psi_j|^2$. This is the standard decoherence result for averaging over the ensemble of measurements.

In the next section, we shall study the development of the density matrix itself,

$$\rho_{\mu f}(\varepsilon) = \text{Tr}_M[\rho_f(\varepsilon)] \quad (28)$$

for varying $\varepsilon$, i.e., we shall study the whole ensemble of measurement processes, not just ensemble averages.
5 A mapping procedure in the stochastic variables

To go from the situation with only $\mu$ present to the situation with a system $M$ in its thermodynamical limit, we shall use a mapping procedure where the $X$th step goes from $n(X - 1)$ to $nX$ stochastic variables. We let $X$ increase from 0 to a large value, $X >> 1$.

We then get a recursive equation for the elements of the density matrix (24),

$$
(\rho_{fX})_{jk} = \left( \sum_{t=1}^{n} |\Psi_t|^2 B_t(\xi_X) \right)^{-1} \sqrt{B_j(\xi_X)B_k(\xi_X)} \Psi_j \Psi_k^* =
$$

$$
= (\rho_{f(X-1)})_{jk} \frac{(1 + \eta_{jX}\xi_{jX} + \eta_{kX}\xi_{kX} - \sum_{t=1}^{n} \eta_{tX}\xi_{tX}) \left(1 - \frac{1}{2}(1 - \delta_{jk})(\eta_{jX}^2 + \eta_{kX}^2)\right)}{1 + 2 \sum_{t'=1}^{n}(\rho_{f(X-1)})_{tt'}\eta_{t'X}\xi_{t'X} - \sum_{t'=1}^{n} \eta_{t'X}\xi_{t'X}}.
$$

Thus, the change in the density matrix in this step of the mapping is

$$
\Delta(\rho_{fX})_{jk} = (\rho_{fX})_{jk} - (\rho_{f(X-1)})_{jk}
$$

$$
= (\rho_{f(X-1)})_{jk} \frac{\eta_{jX}\xi_{jX} + \eta_{kX}\xi_{kX} - 2 \sum_{t=1}^{n} (\rho_{f(X-1)})_{tt'}\eta_{t'X}\xi_{t'X} - \frac{1}{2}(1 - \delta_{jk})(\eta_{jX}^2 + \eta_{kX}^2)}{1 + 2 \sum_{t'=1}^{n}(\rho_{f(X-1)})_{tt'}\eta_{t'X}\xi_{t'X} - \sum_{t'=1}^{n} \eta_{t'X}\xi_{t'X}}.
$$

The probability for a certain

$$\xi_X = \langle \xi_{jX} \rangle; \quad j = 1, 2, ..., n$$

in the $X$th step, given all previous steps, is

$$
2^{-n} \left( 1 + 2 \sum_{t'=1}^{n} (\rho_{f(X-1)})_{tt'}\eta_{t'X}\xi_{t'X} - \sum_{t'=1}^{n} \eta_{t'X}\xi_{t'X} \right).
$$

For the average values and the variation of (28) we get, suppressing subscripts $f$ and $X$ [4],

$$
\langle \Delta \rho_{jk} \rangle = -\frac{1}{2}(1 - \delta_{jk})(\eta_j^2 + \eta_k^2)\rho_{jk};
$$

$$
\langle \Delta \rho_{jk} \Delta \rho_{lm} \rangle = \rho_{jk}\rho_{lm} \left( \delta_{jl}\eta_j\eta_l + \delta_{jm}\eta_j\eta_m + \delta_{kl}\eta_k\eta_l + \delta_{km}\eta_k\eta_m - 2(\rho_{jj}\eta_j^2 + \rho_{kk}\eta_k^2 + \rho_{ll}\eta_l^2 + \rho_{mm}\eta_m^2) + 4 \sum_{s=1}^{n} \rho_{ss}\eta_s^2 \right). \quad (33)
$$

The first equation here describes a drift of the non-diagonal elements to zero. For the diagonal elements in the probability simplex,

$$
p = (p_1, p_2, ..., p_n);
$$

$$
p_j = \rho_{jj} > 0, \quad \sum_{k=1}^{n} p_k = 1, \quad (34)
$$
we get the following random walk \([4,8]\),

\[
\langle \Delta p_j \rangle = 0;
\]

\[
\langle \Delta p_j \Delta p_k \rangle = 4 p_j p_k \left( \delta_{jk} \eta_j^2 - p_j \eta_j^2 - p_k \eta_k^2 + \sum_{l=1}^{n} \eta_l^2 \right).
\]

(35)

Going to the case of a continuous (suitably normalized) step variable \(X\) for the ensemble of random walks, this leads to the following diffusion equation \([4,8]\) in the simplex (34),

\[
\frac{\partial}{\partial X} F(p, X) = \frac{1}{2} \frac{\partial^2}{\partial p_j \partial p_k} \left( p_j p_k \left( \delta_{jk} \eta_j^2 - p_j \eta_j^2 - p_k \eta_k^2 + \sum_{l=1}^{n} \eta_l^2 \right) \right) F(p, X),
\]

(36)

where we have introduced the density function for the ensemble in the simplex,

\[
F(p, X) \geq 0 \quad \int dp F(p, X) = 1;
\]

\[
F(p, 0) = \delta(p; p_0); \quad p_0 = (|\Psi_1|^2, |\Psi_2|^2, \ldots |\Psi_n|^2).
\]

(37)

Here \(dp\) is the \((n-1)\)-dimensional normalized volume element of the simplex, and \(\delta(p; p_0)\) is the corresponding \(\delta\)-function.

The entropy function in the probability simplex,

\[
S(p) = - \sum_{j=1}^{n} p_j \ln p_j
\]

has an \(X\)-dependent mean value

\[
\overline{S}(X) = \int dp F(p, X) S(p).
\]

(39)

The diffusion equation in \(X\) (36) implies

\[
\frac{\partial}{\partial X} \overline{S}(X) = \]

\[
= \frac{1}{2} \int dp F(p, X) \sum_{j,k=1}^{n} \frac{\partial^2 S(p)}{\partial p_j \partial p_k} p_j p_k \left( \delta_{jk} \eta_j^2 - p_j \eta_j^2 - p_k \eta_k^2 + \sum_{l=1}^{n} \eta_l^2 \right) = \]

\[
= - \int dp F(p, X) \sum_{j=1}^{n} \eta_j^2 p_j (1 - p_j) \leq 0,
\]

(40)

which means that the diffusion continues with decreasing entropy, asymptotically approaching a distribution over states with

\[
p_j (1 - p_j) = 0 \quad \text{all } j.
\]

(41)
i.e., a distribution with support only in the corners of the probability simplex \([4,8]\). Thus a random walk of repeated mappings (35) ends up in one of the corners.

As can be shown from (36), similar to (40), the mean of \(p_j\)

\[
\bar{p}_j(X) = \int dp F(p, X)p_j
\]  

(42)

stays constant,

\[
\frac{\partial}{\partial X} \bar{p}_j(X) = 0
\]  

(43)

in agreement with (35). Thus, the probability to reach the \(m\)th corner, \(p_m = 1\), is given by the original probability before the mappings, for \(X = 0\), i.e., \(|\Psi_m|^2\). Let \(\delta_m(p)\) be the \(\delta\)-function for the \(m\)th corner. Then

\[
F(p, \infty) = \sum_{m=1}^{n} |\Psi_m|^2\delta_m(p).
\]  

(44)

For the ensemble of mapping procedures, the chain of mappings is thus a diffusion process starting from the unimodal distribution in (37) and bifurcating into the \(n\)-modal distribution (44).

For the single case, i.e., a single measurement, the procedure always ends up in one of the corners of the probability simplex for \(\mu\), that means in an eigenstate of \(A\).

### 6 A simple model

In this section we give a simple model with a two-state microsystem as an example where the mathematics can be carried out in detail.

Let us go back to (23) and (24) with \(n = 2\) and \(\varepsilon_{1x} = -\varepsilon_{2x}(x = 1, 2, ..., X)\). Furthermore, we let all \(\eta_{lx}\) have the same value \(\frac{1}{2}\eta\). Let \(X_j\) be the number of \(\varepsilon_{jx}\) (varying \(x\)) that are equal to +1 \((X_1 + X_2 = X)\). There are \(\frac{X!}{X_1!X_2!}\) \(\varepsilon_X\)’s of this kind and with this constellation, the probability for a transition is, according to (23) and (22),

\[
P(X_1, X_2) = \frac{X!}{X_1!X_2!} \left[ |\Psi_1|^2 \left( \frac{1 + \eta}{2} \right)^{X_1} \left( \frac{1 - \eta}{2} \right)^{X_2} + |\Psi_2|^2 \left( \frac{1 - \eta}{2} \right)^{X_1} \left( \frac{1 + \eta}{2} \right)^{X_2} \right],
\]  

(45)

with normalization

\[
\sum_{X_1+X_2=X} P(X_1, X_2) = 1.
\]  

(46)

The diagonal elements of the density matrix are (see (24))

\[
p_1(X_1, X_2) = \frac{|\Psi_1|^2(1 + \eta)^{X_1}(1 - \eta)^{X_2}}{|\Psi_1|^2(1 + \eta)^{X_1}(1 - \eta)^{X_2} + |\Psi_2|^2(1 - \eta)^{X_1}(1 + \eta)^{X_2}}
\]

\[
p_2(X_1, X_2) = \frac{|\Psi_2|^2(1 - \eta)^{X_1}(1 + \eta)^{X_2}}{|\Psi_1|^2(1 + \eta)^{X_1}(1 - \eta)^{X_2} + |\Psi_2|^2(1 - \eta)^{X_1}(1 + \eta)^{X_2}}
\]

\[
p_1(X_1, X_2) + p_2(X_1, X_2) = 1.
\]  

(47)
The distribution (45) is a sum of two distributions with weights $|\Psi_1|^2$ and $|\Psi_2|^2$ and with peaks at $X_1 = X^{1+\eta}$ and $X_1 = X^{1-\eta}$, respectively. The width of each peak is $\frac{1}{2}\sqrt{X}$, and the distance between the peaks is $\eta X$. For given $\eta << 1$ and with $X \lesssim \frac{1}{4}\eta^{-2}$, the peaks are fused into one. With increasing $X$, they begin to separate at $X \approx \frac{1}{4}\eta^{-2}$. Beyond this, the distance exceeds the width, and for $X >> \eta^{-2}$, the two peaks become distinct.

We thus take the limit

$$X >> \eta^{-2} >> 1,$$

and introduce continuous variables,

$$z = \frac{X_1 - \frac{1}{2}X}{X\eta}, \quad Z = 2X\eta^2;$$

$$p_j(z) = p_j(X_1, X_2); \quad q(z) \, dz = P(X_1, X_2), \quad \int dz \, q(z) = 1. \tag{49}$$

Here $z$ is a pointer variable, and $Z$ is a variable for the approach to the thermodynamic limit of $M$. We then get

$$q(z) = q_1(z) + q_2(z),$$

$$q_1(z) = \sqrt{\frac{Z}{\pi}} |\Psi_1|^2 e^{-Z(z-\frac{1}{2})^2},$$

$$q_2(z) = \sqrt{\frac{Z}{\pi}} |\Psi_2|^2 e^{-Z(z+\frac{1}{2})^2}, \tag{50}$$

and

$$p_1(z) = \frac{q_1(z)}{q(z)}, \quad p_2(z) = \frac{q_2(z)}{q(z)}. \tag{51}$$

In the limit of $Z \rightarrow \infty$,

$$q_1(z) = |\Psi_1|^2 \delta \left(z - \frac{1}{2}\right), \quad p_1 \left(\frac{1}{2}\right) = 1, \quad p_1 \left(-\frac{1}{2}\right) = 0,$$

$$q_2(z) = |\Psi_2|^2 \delta \left(z + \frac{1}{2}\right), \quad p_2 \left(\frac{1}{2}\right) = 0, \quad p_2 \left(-\frac{1}{2}\right) = 1, \tag{52}$$

in agreement with the previous result (44).

To get an indication how fast the distribution $q(z)$ in (50) becomes bimodal with increasing $Z$, we can follow the approach to zero of the average of the geometric mean of the two probabilities $p_1$ and $p_2$,

$$\langle \sqrt{p_1 p_2} \rangle = \int_{-\infty}^{\infty} dz \, q(z) \sqrt{p_1(z)p_2(z)} = |\Psi_1|\,|\Psi_2| \sqrt{\frac{Z}{\pi}} \int_{-\infty}^{\infty} dz \, e^{-Z(z^2 + \frac{1}{4})} =$$

$$= |\Psi_1|\,|\Psi_2| \, e^{-\frac{Z}{4}} = |\Psi_1|\,|\Psi_2| \, e^{-\frac{1}{2}X\eta^2}. \tag{53}$$

Now $X$ is the number of stochastic parameters and they can be chosen as the number of components of the state of $M$, i.e., as the number of dimensions of the Hilbert space of $M$. This
grows exponentially with the number of degrees of freedom, i.e., with the number of particles $N$ involved. Thus, the separation indicator (53) goes to zero with growing $N$ like

$$\alpha \exp[-\beta \eta^2 e^{\gamma N}], \quad (54)$$

where $\alpha$, $\beta$ and $\gamma$ are positive parameters.

This means that the stochasticity necessary for the selection process of a final state — and a measurement result — is provided by relatively few particles in the beginning of the chain of interactions within $M$. How correlations can develop when there are several such chains will be briefly discussed in the next section.

7 Correlations

We go back to the same kind of model as in Section 6 but with only two stochastic signs, $\varepsilon_{11}$ and $\varepsilon_{21}$ and with $\eta_{11} = \eta_{21} = \eta$, i.e., only one step. Then the distribution over $(\varepsilon_{11}, \varepsilon_{21})$ corresponding to (45) is

$$P(\varepsilon_{11}, \varepsilon_{21}) = \frac{1}{4} \left(1 + \eta \left(|\Psi_1|^2 - |\Psi_2|^2\right) \left(\varepsilon_{11} + \varepsilon_{21}\right) + \eta^2 \varepsilon_{11} \varepsilon_{21}\right). \quad (55)$$

This clearly leads to a positive correlation between $\varepsilon_{11}$ and $\varepsilon_{21}$.

$$\langle \varepsilon_{11} \varepsilon_{21} \rangle = \eta^2. \quad (56)$$

Thus, entanglement of different, initially independent, parts of $M$, with the quantum system $\mu$, leads to this kind of correlation through their common influence on transition probability per unit time.

If we consider two sets of stochastic variables connected to independent parts of the measurement apparatus $M$, such as different detectors in an Einstein-Podolsky-Rosen experiment, and let them be described (in the limit of many variables) by the continuous variables $z$ and $u$ as in the previous section, then the equations corresponding to (50) and (51) are

$$q(z, u) = q_1(z, u) + q_2(z, u),$$

$$q_1(z, u) = \frac{\sqrt{ZU}}{\pi} |\Psi_1|^2 e^{-Z(z-\frac{1}{2})^2} e^{-U(u-\frac{1}{2})^2},$$

$$q_2(z, u) = \frac{\sqrt{ZU}}{\pi} |\Psi_2|^2 e^{-Z(z+\frac{1}{2})^2} e^{-U(u+\frac{1}{2})^2} \quad (57)$$

and

$$p_1(z, u) = \frac{q_1(z, u)}{q(z, u)}; \quad p_2(z, u) = \frac{q_2(z, u)}{q(z, u)}.$$

In the limit $Z, U \rightarrow \infty$, this implies total correlation between the pointer variables $z$ and $u$. 12
8 A Darwinian perspective

One can look upon the measurement process, i.e., the mapping of the out state of the scattering process within $\mu$, on the final state, including registration of the measurement result, as an evolution process, taking place with the density matrix for final states as the relevant phenotype. There is no external influence (from outside $\mu$ and $M$) needed to describe this process: scattering + measurement is to be viewed as an integrated process.

It should be emphasized that the out state of the scattering process itself is not realized in the actual case due to the interaction between the microsystem $\mu$ and the measurement apparatus $M$. The mapping is thus a mapping from a situation without $M$ to a situation with $M$ present. The whole evolution process represents one single measurement.

The replicators in the evolution process are the individuals (the final states), each represented by its genetic code, for the $X$th generation, $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_X)$.

The genetic information is inherited from one generation to the next without any mutation. In the replication from the $(X-1)$th generation to the $X$th generation, new genetic material $\varepsilon_X$ is added, bringing in new variability. The phenotype of interest is described by the corresponding generation of the density matrix, reduced to $\mu$, $\rho_{f,X} = \text{Tr}_M[\rho_{f,X}]$. The fitness is described by $\sum_{l=1}^n |\Psi_l|^2 B_l(\varepsilon_X)$.

In the evolution, the change in phenotype in the $X$th generation is described by (30). The fitness, normalized to a probability distribution over the additional genetic information (31), is given by (32). This leads to a drift and a random walk as described by (33). The non-diagonal elements drift towards zero, step by step, and the diagonal elements take part in a random walk without drift as described by (35), with the corners (the eigenstates of the observable) as attractors.

9 Conclusions

The result just obtained means that the interaction between $\mu$ and $M$ has the characteristics of a measurement process with $\mu$ being measured upon by means of $M$.

In the final transition matrix, there is a non-linear dependence on the partial transition amplitude for $\mu$ in isolation as described in Section 2 (the modulus squared of the "wave function" $|\Psi_j|^2$). The non-linearity in $|\Psi_j|^2$ (in (24), (29) or (35)) comes from the mixing of the two kinds of interaction: the internal interaction within $\mu$, and the final-state interaction between $\mu$ and $M$. The consequence of abandoning the assumption X, referred to in the introduction, is to treat these two interactions as one whole within scattering theory. The largely unknown state of $M$ was brought in through the stochastic variables $\varepsilon$. This was done stepwise through the mapping procedure of Section 5. At no point was there a break away from quantum mechanics, here applied in the form of quantum scattering theory. The property of $M$ to be unbiased was brought in through the factor (21) together with the assumption of an even à priori initial-state probability distribution over $\varepsilon$.

Correlations between the stochastic variables of $M$, related through entanglement with $\mu$, build up, as described in Section 7. This is clearly essential for an analysis of causality concepts.
The mechanism presented here for the selection process in the $\mu$-$M$ interaction also opens new questions. One is what the process looks like in space and time. Here this question is hidden behind scattering theory involving infinite time. Another is the question indicated above, how a useful concept of causality could be defined and applied. A third question is how to understand in more detail the unbiased metastable state of $M$ and the origin of the factor (21).

The present work is in the tradition of references [2] to [5] and [8]. The ambition is to let the density matrix for the final state describe the result of a single measurement and not only the ensemble of measurements. The random walk connected to this is characterized by the changes of the density matrix for $\mu$, given by (33) and, for the diagonal elements, by (35). The first equation (33) leads to a decoherence for the ensemble of density matrices through a drift of the non-diagonal elements to zero; the second leads to a random walk towards the corners of the probability simplex, i.e., to a reduction of the wave function.

Mathematically, the mechanism described here is very close to the diffusion process of Gisin and Percival, but instead of time, we use system extension, and we stay within linear quantum mechanics. However a space-time analysis of the process suggested here would probably lead to a diffusion process of the kind that Gisin and Percival [5] have suggested.

Finally, a small comment on the ambitions of Einstein and Bohr revealed in their discussion on measurement in quantum mechanics.

Einstein did not like the idea that God plays dice behind the scene. He might have accepted a dice-throwing that can be explained in a statistical way. Here the source of randomness is not lack of microcausality but stochasticity of an object with many unknown degrees of freedom. Through entanglement and enhancement/inhibition, this stochasticity leads to correlation build-up and finally to a total bifurcation into different final states of $\mu$, the eigenstates of the measured observable $A$.

Niels Bohr emphasized the necessity of having a measuring instrument of a classical nature. Here the selection process (reduction of the wave packet) takes place only in the thermodynamical (classical) limit of the system $M$.

In the language of Dennett [9], the cranes of quantum mechanics seem adequate for analysing the measurement process, and no skyhook, such as a randomness without an identifiable source or a many-world interpretation or a non-linear extension of quantum mechanics, seems to be needed.

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Appendix 1: Soft photon exchange

Soft-photon exchange and emission is an old and well-known example of factorizable processes in quantum electrodynamics. When this was properly understood, the picture of scattering was drastically changed in the sense that no non-forward scattering of charged particles takes place without soft-photon emission. Later, this was identified as coherent radiation from a classical current describing the charged particles.

Here we choose a simple case to show the factorization of soft photon emission or exchange. We consider an outgoing electron (charge \(-e\), mass \(m\)) with final momentum \(p\), described by a spinor \(u(p)\),

\[
p^2 = m^2; \quad u(p)(ip \cdot \gamma + m) = 0 \tag{A1.1}
\]

after emitting two soft photons with momenta \(k_1, k_2\) and polarizations \(e_1, e_2\),

\[
k_1^2 = k_2^2 = 0; \quad k_1 \cdot e_1 = k_2 \cdot e_2 = 0;
|k_1|, |k_2| \ll m \tag{A1.2}
\]

In the evaluation of the Feynman diagram (Fig. 1), the spinor \(u(p)\) is then changed into a term proportional to

\[
e^2 e^{-\frac{1}{2} \frac{p \cdot k_1 + p \cdot k_2}{p \cdot k_1 + p \cdot k_2}} u(p) \left[ e_1 \cdot \gamma \frac{ip \cdot \gamma + m}{(p + k_1)^2 + m^2} e_2 \cdot \gamma + (1 \leftrightarrow 2) \right] \frac{ip \cdot (k_1 + k_2) \cdot \gamma + m}{(p + k_1 + k_2)^2 + m^2} =
\]

\[
e^2 \frac{1}{2(p \cdot k_1 + p \cdot k_2)} u(p) \left[ e_1 \cdot \gamma (ip \cdot \gamma + m) e_2 \cdot \gamma (ip \cdot \gamma + m) \right] \left( \frac{1}{p \cdot k_1} + \frac{1}{p \cdot k_2} \right) u(p) = (s(k_1) \cdot e_1)(s(k_2) \cdot e_2) u(p), \tag{A1.3}
\]

where

\[
s_{\mu}(k) = -e \frac{ip_{\mu}}{p \cdot k} = -e \int_0^\infty dt \int d^3 x e^{i(k \cdot x - |k| t)} \delta^3 \left( x - \frac{P}{p_0} t \right) \frac{p_\mu}{p_0}. \tag{A1.4}
\]

is the Fourier transform of the current of a classical point charge \(-e\) moving from \(x=0\) at time zero with velocity \(p/p_0\). The rest of the diagram is unchanged in the limit of small \(k_1, k_2\).

Equation (A1.3) states that the emission of the two photons is described by one independent scalar emission factor for each photon. The corresponding holds for two photons being absorbed by the electron.

For \(m\) photons, use is made of the identity,

\[
\sum_{(i_1 i_2 \ldots i_m)} a_{i_1} (a_{i_1} + a_{i_2}) \ldots (a_{i_1} + a_{i_2} + \ldots + a_{i_m}) = \frac{1}{a_1 a_2 \ldots a_m}. \tag{A1.5}
\]

There is also a factor \((m!)^{-1}\). Summation over photon states and over \(m\), then gives rise to a coherent state generated by the classical current (A1.4). Different external particles contribute independent factors.
This is our basis for changing (8) into (15).

**Remark A1.1:** The non-linear terms of the kind deriving from the denominator in (24) were not treated in the old literature of soft radiation. The reason was that these terms did not contribute to the infrared divergences, i.e., the divergences connected to the limit of zero photon mass, and were therefore not needed for regularizing the theory.

The dependence of the transition probabilities on the detailed geometry of soft photon detection is not very strong. Therefore soft photon radiation did not immediately suggest non-linearities that could otherwise have shown a way to handle the measurement process.

**Remark A1.2:** It is possible to show that the inclusion of soft photon emission leads to decoherence in the sense that the non-diagonal elements of the density matrix $\rho_f$ disappear. If the time development is analysed, one can see that this is a very slow process due to the relative weakness of the electromagnetic interaction. In the case of a soft electromagnetic field coupled to measurement device $M$, the large number of degrees of freedom available in $M$ compensates for this restriction. (Moreover, the stochastic interaction within $M$ leads to the selection process, described by a bifurcating random walk, resulting in a definite measuring result.)

\[
\begin{align*}
&\begin{array}{c}
\text{Figure 1: Feynman diagrams corresponding to (A1.3) for emission of two photons characterized by polarizations and momenta, } e_1, k_1 \text{ and } e_2, k_2. \\
\end{array}
\end{align*}
\]
Appendix 2: Non-linearity in "wave-function" $\Psi$ of overall S-matrix

The amplitude for a transition of $\mu$ from the initial state $|0\rangle_\mu$ to an outgoing state $|j\rangle_\mu$ can be illustrated by the diagram $a$ of Fig. 2. Here $\mu$ is represented by two particles in the ingoing and the outgoing states, and the scattering amplitude for the transition within $\mu$ is represented by a shaded circle. The interaction between $\mu$ and $M$ taking place in the outgoing state is included in diagram $b$ of Fig. 2 for the modified transition amplitude and represented by double lines.

The normalized total S-matrix element is obtained through diagrams to all orders of mixed interactions ($S$ and $S^\dagger$ within $\mu$ together with $\mu$-$M$ interaction) as in diagram $c$. This gives rise to a non-linear dependence on the internal transitions amplitudes $\Psi_j$ for $\mu$, here playing the role of wave function components.

\[ \sum_{C,D} \left[ A \quad C' \quad \begin{array}{c} \mu \end{array} \quad A \quad C \right] + \left[ A \quad C' \quad B \quad D \quad C \quad D \right] \]

\[ \sum_{C,D'} \left[ A \quad C' \quad \begin{array}{c} \mu \end{array} \quad A \quad C' \quad \begin{array}{c} \mu \end{array} \quad A \quad C \quad A \quad C \right] + \left[ A \quad C' \quad B \quad D' \quad C \quad D \right] \]

+ ...

**Figure 2:** S-matrix diagrams for scattering in $\mu$, $A + B \rightarrow C + D$, without $M$ (diagram $a$) and with $C$ and $D$ interacting with $M$ (diagram $b$). The sum over diagrams to all orders (as indicated in $c$) should be taken into account.