Quantum Metrology in the Kerr Metric

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A surprising feature of the Kerr metric is the anisotropy of the speed of light. The angular momentum of a rotating massive object causes co- and counter-propagating light paths to move at faster and slower velocities, respectively. Based on this effect we derive ultimate quantum limits for the measurement of the Kerr rotation parameter $a$ using an interferometric set up. As a possible implementation, we propose a Mach-Zender interferometer to measure the “one-way height differential” time effect. We isolate the effect by calibrating to a dark port and rotating the interferometer such that only the direction dependent Kerr phase term remains. We also identify a flat metric where the observers see $c = 1$. We use this metric and the Lorentz transformations to calculate the same Kerr phase shift. We then consider non-stationary observers moving with Earth’s rotation, and find an additional phase from the classical relative motion.

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I. INTRODUCTION

The noise induced by a measurement device is fundamentally restricted by limits set by quantum mechanics. Quantum metrology is the study of these lower limits for the estimation of physical parameters \[1\]. Techniques in quantum metrology can assist in developing devices to measure the fundamental interplay between quantum mechanics and general relativity at state-of-the-art precision. A prime example is the detection of gravitational waves from black-hole mergers by LIGO \[2\].

Recently there have been investigations of how we can exploit quantum resources to measure space-time parameters such as the Schwarzschild radius $r_s$ and the Kerr parameter $a$ in the rotating Kerr metric \[3–6\]. Quantum communications were shown to be affected by the rotation of Earth \[7\]. However, more fundamental effects in general relativity induced by the Kerr metric were not analysed. One interesting feature of the Kerr metric is the anisotropy of the velocity of light. The rotating massive object causes co- and counter-propagating light to move at faster and slower velocities, respectively.

In this paper, we note that there is a phase shift of co-moving light beams at different radial positions in the Kerr metric. We use a Mach-Zender (MZ) interferometer to probe for this phase. We isolate the effect by calibrating to a dark port and rotating the interferometer and due to the anisotropy of $c$, only the Kerr phase term remains. From this, we can construct using Quantum Information techniques lower bounds for the variance of parameter estimation of $a$ \[3–6, 8\].

Locally, we can find a co-rotating frame in which the space-time is locally flat (“the zero angular momentum ring-riders”) \[9\]. We find that the locally measured velocity of light is $c = 1$ as expected in the flat metric. If an observer Alice compares the locally measured time with Bob who is a ring-rider at a different radius, there will be a disagreement of simultaneity of events. We also consider non-stationary observers that are moving in the rotational plane of Earth. As expected, we find an additional phase term from rotation and special relativistic time dilation. We find that this term is dominant compared to the Kerr phase. Finally, we compare the magnitude of the Kerr phase on Earth to light anisotropy from microwave resonator experiments \[10\].

This paper is organized as follows. We first introduce the full Kerr metric in Section \[1\]. In Section \[1A\] we solve for the null geodesic to determine the velocity of light in the equatorial plane. We find that there is an anisotropy in $c$. Next in Section \[1B\] we calculate the “height differential effect” which could be detected by a Mach-Zender interferometer.

In Section \[II\] we determine quantum limits of the Kerr space-time parameter $a$ for the height differential effect. In Section \[IIA\] we focus on the stationary Mach-Zender interferometer in the weak field limit and calculate the phase shift. We comment on how we can calibrate to a dark port and rotate the interferometer to isolate the Kerr phase. We compare the magnitude of the Kerr phase with the Schwarzschild phase for Earth parameters. In Section \[IV\] we identify the co-moving flat metric in which the so-called “ring-rider” measures $c = 1$. In Section \[V\] we demonstrate an alternative calculation using Lorentz transformations between stationary and ring-riders to find the phase detected at the output of the MZ interferometer. We also confirm that the “two-way” velocity of light is $c = 1$ as detected by a Michelson interferometer at rest in the Kerr metric. Furthermore, we consider the motion of non-stationary observers on the rotating Earth. In Section \[VI\] we consider an extreme black hole and we numerically find the full strong field solution of the Kerr phase. Finally, we conclude by commenting on the feasibility of detecting the light anisotropy.

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II. KERR ROTATIONAL METRIC

A rotating massive body tends to drag the space-time with its rotation. The Kerr metric used to describe this space-time includes the Kerr rotation parameter \( "a" \) which quantifies the amount of space-time drag. The Kerr line element in Boyer-Lindquist coordinates \((t, r, \theta, \phi)\) is [9]:

\[
d s^2 = -(1 - \frac{r_s}{r}) dt^2 + \frac{\Delta}{\Sigma} dr^2 + \Sigma d\theta^2 + \Sigma(r^2 + a^2) d\phi^2 - \frac{2r_s a \sin^2 \theta}{\Sigma} d\phi dt
\]

(1)

Where \( \Delta := r^2 - r_s r + a^2 \), \( \Sigma := r^2 + a^2 \cos^2 \theta \) and \( a = \frac{J}{M} \) where \( J \) is the angular momentum of the object of mass \( M \). Note that the Schwarzschild radius \( r_s = \frac{2GM}{c^2} = 2M \) where we work in natural units for which \( c = 1 \) and \( G = 1 \). Compared with the Schwarzschild metric, the cross term \( d\tau d\phi \) introduces a coupling between the motion of the massive object and time, which leads to interesting effects.

When \( r_s = 0 \), the space-time is flat and reduces to \( ds^2 = -dt^2 + \frac{\Delta}{\Sigma} dr^2 + (r^2 + a^2) d\phi^2 \). At first glance, this metric doesn’t seem flat. However, we have used the oblong sphere coordinates \( x = \sqrt{r^2 + a^2} \sin \theta \cos \phi \), \( y = \sqrt{r^2 + a^2} \sin \theta \sin \phi \) and \( z = r \cos \theta \). We wish next to determine the tangential velocity of light close to the massive object as seen by a far-away observer.

A. Far-away velocity of light

In the equatorial plane (where \( \theta = \frac{\pi}{2} \)), for the null light geodesic, we set \( ds^2 = 0 \) and determine the solution for the tangential velocity of light as seen by the far-away observer:

\[
d s^2 = 0 = -(1 - \frac{r_s}{r}) dt^2 + \frac{\Delta}{\Sigma} dr^2 + (r^2 + a^2 + \frac{r_s a^2}{r}) d\phi^2 - \frac{2r_s a}{r} dt d\phi
\]

(2)

The definition of the measured radius differs from the Schwarzschild metric. The reduced circumference \( 2\pi R \) where \( R = \sqrt{r^2 + a^2 + \frac{r_s a^2}{r}} \) defines the measured radius by an observer from which \( r \) can be extracted. Thus, the tangential distance is \( dx = R d\phi = \sqrt{r^2 + a^2(1 + r_s/r)} d\phi \) and the light geodesic solution is:

\[
0 = -(1 - \frac{r_s}{r_s}) \dot{x}^2 - \frac{2r_s a}{r \sqrt{r^2 + a^2(1 + r_s/r)}} \dot{x}
\]

(3)

Where \( \dot{x} = \frac{dx}{dt} \):

\[
\frac{dx}{dt} = \frac{r_s a}{r \sqrt{r^2 + a^2(1 + r_s/r)}} \pm \sqrt{\frac{r_s^2 a^2}{r^4} \left(1 - \frac{a^2}{r^2} \right) + 1 - \frac{r_s}{r}}
\]

(4)

However, if \( \frac{a}{r} \ll 1 \) and \( \frac{r_s}{r} \ll 1 \) we have the weak field solution:

\[
\frac{dx}{dt} \approx \frac{r_s a}{2r} \pm \sqrt{\frac{a^2 r_s^2}{r^4} \left(1 - \frac{a^2}{r^2} \right) + \frac{a^2 r_s^2}{r^4}}
\]

(5)

Where we have used the Taylor expansion \( \sqrt{1 + x - y} \approx 1 + \frac{x}{2} - \frac{x^2}{8} \). We have two solutions representing counter- and co-rotating light. Notice that locally, \( \frac{dx}{dt} \approx (1 - \frac{\frac{a}{r}}{2} + \frac{\frac{r_s}{r}}{2}) \) can exceed 1 for the positive solution. However, we cannot naively use the Schwarzschild co-ordinate time in this curved metric. Later we will show that there is a locally flat metric where \( c = 1 \).

B. Height differential effect

Let’s consider a stationary observer in the Kerr metric sending co-moving beams of light that travel tangentially at velocities \( c_1 = 1 - v - \frac{\_s}{c^2 r} \) and \( c_2 = 1 - v - \frac{r_s}{c^2 r} \) at radii \( r_1 \) and \( r_2 = r_1 + h \) where \( h \) is the coordinate height. For simplicity we made the weak field approximation and only retained terms from Eq. [5] to first order in \( v = \frac{r_s}{r \sqrt{r^2 + a^2 + \frac{r_s a^2}{r}}} \). The light travels the distance \( L \) with time \( t_1 = \frac{L}{c_1} \). Similarly, the second observer measures the travel time \( t_2 = \frac{L}{c_2} \). The far-away observer agrees that the length \( L \) is the same for both. Thus the time delay to first order is

\[
\Delta t_r = \frac{L}{c_1} - \frac{L}{c_2} = L \left( \frac{1}{1 - v_1 - \frac{r_s}{c^2 r_1}} - \frac{1}{1 - v_2 - \frac{r_s}{c^2 r_2}} \right)
\]

\[
\approx L\left( r_s a \left( \frac{1}{R_1 r_1} - \frac{1}{R_2 r_2} \right) + \frac{L h r_s}{2 r_1 r_2} \right)
\]

(6)

\[
\approx L r_s ah (r_1 + r_2) + \frac{L h r_s}{2 r_1 r_2}
\]

Where we have ignored the cross term \( \frac{r_s a}{c^2 r_1} - \frac{r_s a}{c^2 r_2} \) since it is much smaller. This time delay can be incorporated into a Mach-Zender interferometric arrangement which can be rotated along its centre to measure the phase for \( +a \) and \( -a \) as will be discussed shortly.
III. QUANTUM LIMITED ESTIMATION OF THE KERR SPACE-TIME PARAMETER

Using these time delays, we want to determine the ultimate bound for estimating the Kerr metric parameter \(a\). The variance of an unbiased estimator is determined by the Quantum Cramer-Rao (QCR) bound \([8]\). In quantum information theory, for \(M\) number of independent measurements, the QCR bound for the linear phase estimator \(\phi\) is given by \((\Delta \phi)^2 \geq \frac{1}{MH(\phi)}\). Where \(H(\phi)\) is the Quantum Fisher Information which quantifies the amount of information obtainable about a parameter using the optimal measurement.

We have seen that we can measure the phase \(\Delta \phi = \omega \Delta t_c\) at different heights where \(\omega\) is the central frequency of the probe and \(\Delta t_c\) is given by Eq. [6]. The QCR bound for the Kerr rotation parameter is then:

\[
\frac{(\Delta a)^2}{a} \geq \frac{\gamma^2 r_c^2}{\omega L a r_s h (r_1 + r_2) \sqrt{MH(\Delta \phi)}}
\]  

(7)

In general \(r_1, r_2 >> h\) and therefore the Kerr parameter standard deviation scales as \((\Delta a)^2 \approx \frac{r_c^2}{\sqrt{N}}\).

A larger height difference \(h\) or length \(L\) reduces the noise limit. For coherent probe states undergoing linear phase evolution, \(H(\phi) = |\alpha|^2 = N\). Therefore, we have the standard quantum noise limit \(\frac{1}{\sqrt{N}}\), known as the conventional Heisenberg limit \([12, 13]\) or with \(\chi\) Kerr non-linearities the noise can scale as \(\frac{1}{\sqrt{N^\chi}}\) \([14, 15]\).

A. Mach-Zender interferometer

Let’s consider a physical system that can detect the discrepancy in the velocity of light from the differential height effect in the Kerr metric. We consider a Mach-Zender interferometer (see Fig. [1] that is stationary with respect to the centre of mass of a rotating planet. We will work in far-away time coordinates. Although the final implications will be the same, this is an approach where no assumption is made about how the speed of light is measured locally.

The measured phase of the bottom arm of the Mach-Zender interferometer is \(\Delta \phi_A = \omega \Delta t_A\) where \(\omega\) is the frequency of light measured locally at the source and \(\Delta t_A\) is the time as seen by a faraway observer, and \(\Phi\) is a local phase shifter. At \(r = r_A\) the faraway time \(\Delta t_A = \frac{c_A r_A}{r c_B}\) where \(c_A\) is the speed of light as measured by a faraway observer (see Eq. [5]) and \(L\) is the arm length also seen by a faraway observer. We have set both arm lengths to be the same. Thus, in the top arm the phase is \(\Delta \phi_B = \omega \Delta t_B = \omega \frac{L}{c_B}\).

We assume that \(dr = 0\) and the Mach-Zender interferometer arms are sufficiently small that the curvature is negligible. The tangential velocity of light depends on \(R\) and the sign of \(a\). The solution in the weak field limit is \(c^2 = \frac{dx}{dt} = R\frac{dx}{dt} \approx 1 \pm \frac{\gamma}{2 c_R} - \frac{c_s^2 r^2}{2 r} + \frac{\alpha^2 r^2}{2 r^2}\).

We have chosen the co-moving direction such that \(c_A \approx 1 - \frac{c_A}{r_A R_A} - \frac{c_A}{2 r_A} + \frac{\alpha^2 r^2}{2 r_A^2}\) and \(c_B \approx 1 - \frac{c_B}{r_B R_B} - \frac{c_B}{2 r_B} + \frac{\alpha^2 r^2}{2 r_B^2}\).

The phase is thus:

![Fig. 1: A Mach-Zender interferometer of length L and height h stationary above the rotating planet. \(\Phi\) is a phase shifter in the bottom arm to calibrate the interferometer to a dark port of zero intensity.](image1)

![Fig. 2: Measured phase differences of \(L = 1\) m and \(h = 1\) m Mach-Zender interferometer in co- and counter-mov- ing directions (blue and red respectively). Black line is in the Schwarzschild metric with \(a = 0\). We use the values for the Earth’s Schwarzschild radius \(r_s = 9 \text{ mm}\), rotation parameter \(a = 3.9 \text{ m}\) and the operating frequency of light \(\omega = k = 2 \times 10^6\) m\(^{-1}\) corresponding to 500 nm measured locally at the source.](image2)
\[ \Delta \phi_{MZ} - \Phi = \omega \left( \frac{L}{c_B} - \frac{L}{c_A} \right) \]
\[ \approx \omega L \left( (1 + \frac{r_s a}{r_B R_B} + \frac{r_s^2 a}{2 r_B^2 R_B} + \frac{r_s^2 a^2 r_s^2}{2 r_B^4 B}) - (1 + \frac{r_s a}{r_A R_A} + \frac{r_s^2 a}{r_A^2 R_A} + \frac{a^2 r_s^2}{2 r_A^4 A}) \right) \]
\[ \approx \omega L \left( (\Omega_B R_B - \Omega_A R_A) - \frac{r_s h}{2 r_A R_B} + r_s(\Omega_B - \Omega_A) \right) \]
\[ + \frac{a^2 r_s^2}{2 r_B^4} - \frac{a^2 r_s^2}{2 r_A^4 A} \]  

Where we have used the Taylor expansion \( \frac{1}{1-x-y} \approx 1 + x + y + 2xy \) and the approximations \( \frac{r_s}{r_B} \ll 1 \) and \( \frac{a}{r} \ll 1 \). Note that for the vertical arms, the accumulated phases are equal for the vertical arms, the accumulated phases are equal to 0, and the operating frequency of light \( c = 10^4 \text{ nm} \) then the order of magnitude of the dominant term for the Kerr rotating effect is \( |\Delta \phi_{Kerr}^l| \approx kL(\frac{r_s a}{r_B R_B} - \frac{r_s a}{r_A R_A}) \approx \frac{k a L n h}{r_B} \approx 3 \times 10^{-16} \).

Conversely, the Schwarzschild time dilation effect is of the order \( \Delta \phi_{Schwarzschild} = \frac{\omega L r_s}{2 r_B} \approx 2.2 \times 10^{-10} \). Note that the term \( \omega L r_s(\Omega_B - \Omega_A) \approx \omega L \delta r x 10^{-22} \) is too small and can be neglected. Similarly, the term \( \omega L \left( \frac{a^2 r_s^2}{2 r_B^4} - \frac{a^2 r_s^2}{2 r_A^4 A} \right) \approx 1 \times 10^{-30} \) can be neglected in further calculations.

**MZ interferometer calibration.** We set the total phase shift \( \Delta \phi_{MZ} = 0 \) and thus the phase shifter \( \Phi \) balances the interferometer to the dark port. We can see in Fig. 2 the phase of the interferometer if we were positioned in the co- and counter-moving directions. Thus, we can rotate the Mach-Zender interferometer with angle \( \pi \) around its vertical axis and measure the axis sign dependence directly. Since only the sign of \( a \) changes and \( \Phi \) stays the same then we have,

\[ \Delta \phi_{MZ} - \Phi \approx 2 \omega L (\Omega_A R_A - \Omega_B R_B - r_s(\Omega_B - \Omega_A)) \]
\[ \approx 2 |\Delta \phi_{Kerr}^l| \]  

Therefore, we have a signal which only depends on \( a \). Without the anisotropy of light speed, there would be no signal and the phase would remain a dark port.

**IV. ZERO ANGULAR MOMENTUM OBSERVER METRIC**

The velocity of light in the Kerr metric has two solutions depending on the direction of measurement. However, locally we expect observers to isotropically measure \( c = 1 \). It would be useful to identify a reference frame in which the cross terms \( d\phi dt \) vanish and where locally we obtain a flat space-time metric with \( c = 1 \). To determine this transformation, we consider the Killing vectors \( \partial_t \) and \( \partial_\phi \) that are responsible for two conserved quantities along the geodesic. These are the energy:

\[ E = -k_\mu u^\mu = -g_{\mu \nu} u^\nu = -p_t = (1 - \frac{r_s}{r}) \frac{dt}{d\tau} + \frac{r_s a}{r} \frac{d\phi}{d\tau} \]

And the angular momentum:

\[ L = g_{\phi \mu} u^\mu = -\frac{r_s a}{r} \frac{dt}{d\tau} + (r^2 + a^2 + \frac{r_s^2 a^2}{r^2}) \frac{d\phi}{d\tau} \]

When we set \( L = 0 \) then we have that \( \frac{d\phi}{d\tau} = \frac{r_s a}{r^2} \). Thus there remains an angular motion even with zero angular momentum. The interpretation here is that the rotating space-time drags an object close to the rotating mass, as seen by a far-away observer. If we are co-rotating in the zero angular momentum reference frame \( d\phi / d\tau \) then the metric cross terms \( d\phi dt \) cancel out and becomes:

\[ ds^2 = -\left(1 - \frac{r_s}{r} + \frac{r_s^2 a^2}{r^2 R_B^2}\right) dt^2 + R^2 d\phi^2 \]

This is known as the zero angular momentum observer (ZAMO) metric [16]. Taking \( d\tau_{ring} = \sqrt{1 - \frac{r_s}{r} + \Omega R^2} dt \) we have that \( ds^2 = dt_{ring}^2 - R^2 d\phi_{ring}^2 \) giving a locally flat metric for the ring-rider in which \( c = 1 \). We seek the metric in stationary shell coordinates: \( ds^2 = dt_s^2 - R_s^2 d\phi_{ring}^2 \) where obviously again \( c = 1 \). Locally, however, there is a lack of simultaneity between events in the shell metric and events in the ring-rider metric (and hence far-away events). This is the source of the anisotropy of the speed of light. We have from the Lorentz transformation that a space-like event implies \( dt_{ring} = \gamma (dt_s - v dx_s) = 0 \) where \( v = \Omega R \) and \( dx_s = R_s d\phi_{ring} \), thus \( dt_s = v R_s d\phi_{ring} \). From the equivalence of the line elements we have \(-R^2 d\phi_{ring}^2 = v^2 R_s^2 d\phi_{ring}^2 - R_s^2 d\phi_{ring}^2 \) and therefore the ring-rider radius is \( R = R_s \sqrt{1 - \Omega^2 R_s^2} \) implying it is Lorentz contracted.

We have redefined the coordinate times of the respective ring-rider as \( dt = \sqrt{1 - \frac{r_s}{r} + \frac{r_s^2 a^2}{r^2 R_B^2}} dt \). However, the term between ring-riders separated by \( h \) height is \( \frac{r_s^2}{2} - \frac{r_s^2}{2} = \Omega_B^2 \frac{r_B^2}{2} - \Omega_A^2 \frac{r_A^2}{2} = \frac{1}{2} \left( \frac{r_s a}{r_B R_B} \right)^2 - \frac{1}{2} \left( \frac{r_s a}{r_A R_A} \right)^2 \approx 10^{-38} \) compared to the Schwarzschild term \( \frac{r_s h}{r_B R_B} \approx 10^{-16} \). Clearly this is far too small to be detectable between ring-riders. The advantage of the ring-rider frame is that we can use Lorentz transformations to the stationary
The speed of light is

observer frame to determine the much more significant height differential effect.

V. RING-RIDER PERSPECTIVE

We have previously shown that in the ZAMO flat metric the speed of light is \( c = 1 \). It is helpful in understanding the physics of our estimation protocols to consider them from the perspective of ring-rider observers. This is also a convenient method to generalize to non-stationary interferometers.

A. Stationary Mach-Zender above rotating massive object

Let's consider the Mach-Zender interferometer from the reference frames of the ring-riders. The ring-riders are in the flat metric (see Fig. 3). Therefore, for each ring-rider, we can use the Lorentz Transformations. We maintain for now the weak field approximations that \( \frac{v}{c} << 1 \) and \( \frac{\Omega}{c} << 1 \) such that the Mach-Zender interferometer is far enough away from the centre of the massive body. Taking into account special relativity, a stationary observer would measure the travel time of light \( t'_1 = \gamma(t_1 + v_A x_A) = \gamma(L + v_A L) = \sqrt{1 + \frac{v_A^2}{c^2}} L \approx (1 + v_A + \frac{v_A^2}{2}) L \) where \( v_A = \Omega_A R_A \) is the relative velocity

between the ring-rider and stationary observer at \( R_A \) and \( t_1 = L \) is the travel time in the ZAMO flat metric. Note that the stationary observer as seen by the ring-rider is travelling in the negative \( x \) direction. Similarly, for the ring-rider at \( R_B \), \( t'_2 = \sqrt{1 + \frac{v_B}{1 - v_B^2}} L \approx (1 + v_B + \frac{v_B^2}{2}) L \) where \( v_B = \Omega_B R_B \). For an observer at \( r = \infty \), we use the coordinate times of the ZAMO metric. Since the coordinate times are

\[
\begin{align*}
t'_1 &= \frac{t'_1}{\sqrt{1 - \frac{v_A}{\gamma}} + \Omega_A^2 R_A^2} \approx (1 + \frac{v_A}{2r_A} - \frac{v_A^2}{2}) L (1 + v_A + \frac{v_A^2}{2}) \\
t'_2 &= \frac{t'_2}{\sqrt{1 - \frac{v_B}{\gamma}} + \Omega_B^2 R_B^2} \approx (1 + \frac{v_B}{2r_B} - \frac{v_B^2}{2}) L (1 + v_B + \frac{v_B^2}{2})
\end{align*}
\]

and

Thus the time delay is

\[
\Delta t = t'_2 - t'_1 = L ((1 + \frac{r_B}{2r_A} - \frac{v_B^2}{2}) L (1 + \Omega_B R_B)) - (1 + \frac{r_A}{2r_A} - \frac{v_A^2}{2}) L (1 + \Omega_A R_A)) \approx L (\Omega_B R_B - \Omega_A R_A - \frac{r_A h}{2r_A r_B} + \frac{r_A}{2} (\Omega_B - \Omega_A))
\]

These calculations are equivalent with using the velocity of light obtained from using the Kerr Metric in far-away coordinates in Eq. \ref{15}.

B. Michelson interferometer

Given that the far-away observer sees an anisotropic speed of light it is instructive to ask why a local Michelson interferometer fails to see an effect. A stationary observer sends a light beam tangential to the equator that bounces off a mirror \( L \) distance away and returns to the observer. The time delay in this signal arm would be \( \Delta t_{\text{signal}} = \frac{L}{\sqrt{1 - \frac{v}{c}^2}} (1 + \frac{v}{c}^2) \approx 2L (1 + \frac{v}{c}^2) \). The reference arm perpendicular to the equator is approximately the Schwarzschild local time as found in Eq. \ref{13} (see Appendix A). This is the same phase as the signal arm \( \Delta t_{\text{refl}} \approx 2L (1 + \frac{v}{c}^2) \). Thus the total phase difference is \( \Delta \phi_{\text{Michelson}} = 0 \), implying that the speed of light is \( c = 1 \) locally and isotropic, as expected from the special theory of relativity. From the point of view of the far-away observer, although the speed of light is anisotropic, they find the “two-way” speed, to the mirror and back, is the same in each direction, leading to no phase shift. It may seem a contradiction with the results of the height differential effect, which requires \( c \) to be anisotropic to see a signal in the MZ interferometer. However, this is due to a difference in the amount of space-time dragging at different radial positions in the Kerr metric that the MZ interferometer measures non-locally.
C. Non-stationary co-moving observers on Earth

In an experiment conducted say on Earth, the rotation of the non-stationary Earth observers must be taken into account. Our previous calculations have considered only a stationary Mach-Zender interferometer with the Earth rotating beneath. However, let’s consider the bottom arm of the MZ interferometer on Earth’s surface with the tangential velocity \( v_A’ = \Omega_E R_A - \Omega_A R_A \) and the top arm co-moving at \( v_B’ = \Omega_E R_B - \Omega_B R_B \) with the same angular velocity \( \Omega_E \) of Earth. This relative velocity between observers introduces an additional time dilation.

Using the Lorentz transformations, a stationary observer observer would measure the travel time of light
\[
t^\prime_1 = \gamma(t_1 + v_A x_A) = \gamma(L + v_A’ L) = \frac{1 + v_A’}{1 - v_A’} L \approx (1 + v_A’ + \frac{v_A^2}{2}) L.
\]
Similarly, for the moving observer at \( R_B \),
\[
t^\prime_2 = \gamma(t_2 + v_B x_B) = \gamma(L + v_B’ L) = \frac{1 + v_B’}{1 - v_B’} L \approx (1 + v_B’ + \frac{v_B^2}{2}) L.
\]
For an observer at \( r = \infty \), we use the coordinate times of the ZAMO metric,
\[
t^\prime_A = \frac{t_1}{\sqrt{1 - \frac{v_A^2}{c^2}}} \quad \text{and} \quad t^\prime_B = \frac{t_2}{\sqrt{1 - \frac{v_B^2}{c^2}}}.
\]
Thus:
\[
\Delta t = t^\prime_B - t^\prime_A
\]
\[
= L((1 + \frac{r_A}{2 R_B} - \frac{(\Omega_B R_B)^2}{2})(1 + v_B’ + \frac{v_B^2}{2})
- (1 + \frac{r_A}{2 R_A} - \frac{(\Omega_A R_A)^2}{2})(1 + v_A’ + \frac{v_A^2}{2}))
\]
\[
\approx L(R_A h + v_B’ - v_A’ + \frac{r_A v_B’}{2 R_B} - \frac{r_A v_A’}{2 R_A} + \frac{v_B^2}{2} - \frac{v_A^2}{2})
\]
\[
\approx \Delta t_{MZ} + \frac{\Omega_E h L}{2} + \frac{\Omega_B^2 h L(R_A + R_B)}{2}
\]

(16)

Where we have neglected the terms \((\Omega_A R_A)^2\) and \((\Omega_B R_B)^2\). The term \(\Omega_E h L\) is a classical effect due to the relative motion of the observers but the term \(\Omega_B^2 h L(R_A + R_B)/2\) is the higher order correction due to special relativity. We calibrate the MZ interferometer such that the total phase \(\Delta \phi_{MZ} = 0\) and then we rotate it. The only remaining terms in Eq. 16 are linear with the rotation. Thus the new phase is
\[
\Delta \phi^\prime_{MZ} = 2 \Delta \phi_{Kerr} + 2 \omega_0 \Omega_E h L
\]
(17)

The Kerr phase varies inversely with \( r^3 \), and thus in principle can be distinguished from the classical effect.

D. Probing the Kerr phase on Earth using MZ interferometer

An interesting calculation is to estimate how compact an object with Earth mass and spin would need to be such that the Kerr term was dominant over the effect of the spin. The relative velocity term is \( |\Delta \phi_{Rotation}| = \omega_0 L \Omega_E h \approx 5 \times 10^{-7} \) for a fixed interferometer with the angular frequency of the Earth \( \Omega_E = 7.2 \times 10^{-5} \) Hz. To determine \( a \), we need to isolate it from the dominant effect of Earth’s rotation.

We can vary the position of the interferometer while keeping its size constant. The contribution from the rotation term \( \Delta \phi_{Rotation} \approx \omega_0 L \Omega_E h L \) is approximately constant. We want to determine at what radial position the Kerr effect becomes dominant. This occurs when \(\Delta \phi_{Kerr} \geq \Delta \phi_{Rotation}. \) Therefore, \(\omega_0 L \frac{r_A}{r_B} = \omega_0 L \Omega_E h \) which implies that \(r_B = (\frac{r_A}{r_B})^{1/3} \approx 5 \) km. Note that the condition \( \frac{a}{R_E} \ll 1 \) is still satisfied. In Fig. 4, we have the same interferometer over a range of positions extend-
ing 2 km around the point at which the Kerr phase becomes significant. Clearly an Earth bound measurement is very far from this condition. However, for a compact object such as a neutron star of the same Schwarzschild radius it is possible in principle.

VI. EXTREME BLACK HOLES

To explore the strong field situation, let’s now lower our stationary Mach-Zender interferometer close to a black hole. We can no longer use the approximations $\frac{r}{c^2} << 1$ and $\frac{r}{c^2} << 1$. We must use the full solution of $c_A$ and $c_B$ as in Eq. [3]. We can see in Fig. [3] for a black hole of Schwarzschild radius $r_s = 10$ km and angular momentum $a = \frac{5}{2}$, the phase difference for a co- (red) and counter- (blue) direction Mach Zender interferometer. The two directions of the Mach-Zender interferometer become increasingly distinguishable as it gets closer to the event horizon at $r = r_s$.

VII. CONCLUSION

We have determined the quantum limits of estimating the Kerr parameter which arises from the anisotropy of the speed of light. We propose a stationary Mach-Zender interferometer that can directly measure the Kerr parameter at the event horizon at $r = r_s$.

We find that the “two-way” velocity of light is isotropic and $c = 1$ as measured by a Michelson interferometer. However, our Mach-Zender interferometer is no longer a dark port after it is rotated by $\pi$ because of the combined effect of the anisotropy of light and the difference in the amount of space-time dragging in the radial position. On Earth, we have to consider non-stationary observers which adds an additional classical phase that dominates the Kerr phase. A precise enough experiment would distinguish between these two phases.

Recent experiments using microwave resonators have been able to detect the anisotropy of light with a precision of $\Delta c/c \approx 10^{-17}$ [10]. Our Mach-Zender interferometer predicts a change in the speed of light due to the Kerr metric of $\Delta c/c = \frac{4a}{r_s^2} \approx 10^{-20}$. In principle, future devices need only increase precision by 3 orders of magnitude to measure the Kerr phase on a small scale Mach-Zender interferometer. Using coherent probe states, the noise of the phase is the standard noise limit (SNL) $\Delta \phi \geq \frac{1}{\sqrt{M N}}$. For $M = 10$ GHz measurements [17], this suggests that $N = 10^{22} - 10^{26}$ per light pulse. This would imply extremely high power, which is one of the current limiting factor to increasing phase sensitivity.

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Appendix A: Proper length perpendicular to the equator

Let’s consider the proper length perpendicular to the equator. The Kerr metric away from the equator is [21]:

\[ ds^2 = -(1 - \frac{r_s}{r})dt^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 \quad (A1) \]

Where \( \theta \) is the azimuth in spherical coordinates, and \( \Sigma = r^2 + a^2 \cos \theta^2 \).

Therefore, we set \( dt = 0 \), \( dr = 0 \) and \( d\phi = 0 \) and get the proper distance \( d\sigma = \sqrt{r^2 + a^2 \cos^2 \theta} d\theta \). However, we expand around the equator where \( \Theta = \theta - \frac{\pi}{2} = 0 \) and thus we make the small angle approximation \( \cos \theta = \sin \Theta \approx \Theta \) since the arm length \( L << R \). We have \( ds \approx \sqrt{r^2 + a^2 \Theta^2} d\Theta \approx rd\Theta \). The velocity of light is given by solving the null geodesic:

\[ ds^2 = 0 = -(1 - \frac{r_s}{r} + \frac{a^2}{r})dt^2 + (r^2 + a^2 \cos^2 \theta)d\theta^2 \]

\[ = -(1 - \frac{r_s}{r})dt^2 + r^2 d\Theta^2 \quad (A2) \]

And thus the proper length is:

\[ \Delta \sigma_{Reff} = \frac{2L}{\sqrt{1 - \frac{r_s}{r}}} \approx 2L(1 + \frac{r_s}{2r}) \quad (A3) \]

Which is the same as in the Schwarzschild metric.

Appendix B: Extremal black holes

Let’s consider the full solution to the speed of light without any weak field approximations. The phase is therefore:

\[ \Delta \phi = \omega(t_B - t_A) = kL \left( \frac{1}{c_B} - \frac{1}{c_A} \right) \quad (B1) \]

Where

\[ c_B = \frac{r_B \sqrt{r_B^2 + a^2(1 + r_s/r_B)}}{\sqrt{r_B^2 + a^2(1 + r_s/r_B^2)}} \pm \frac{r_s a}{r_B \sqrt{r_B^2 + a^2(1 + r_s/r_B^2)}} + (1 - \frac{r_s}{r_B}) \]. Using units of \( r_s \),

\[ a \to a' r_s, \; r_A \to r_A' r_s \; \text{and} \; r_B \to r_B' r_s \; \text{and} \; r_B \to r_B' r_s \; \text{and} \; r_B \to r_B' r_s \]. This simplifies to

\[ c_B = \frac{1}{r_B' \sqrt{r_B^2 + a^2(1 + 1/r_B') + (1 - \frac{1}{r_B})}} \]

Let’s consider the values of an almost extremal black hole with \( r_s = 10 \text{ km}, \; a' = \frac{4}{5} \; \text{with} \; r_B' = r_A' + h' \) where \( h' = \frac{1}{10000} \) since \( h = 1 \text{ m} \).

FIG. 6: Difference in exact phase as determined numerically for full solution of \( c \) (red) and weak field approximation (blue). (Note the extremal black hole parameters \( r_s = 10 \text{ km}, \; h' = 10^{-4} \) and \( a' = \frac{4}{5} \))

FIG. 7: Difference in exact phase as determined numerically for full solution of \( c \) (red) and weak field approximation (blue) for Earth parameters. (Note that \( r_s = 9 \text{ mm}, \; h' = 111 \) and \( a' = 433 \))