Regularization of Point Vortices Pairs for the Euler Equation in Dimension Two

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Abstract

In this paper, we construct stationary classical solutions of the incompressible Euler equation approximating singular stationary solutions of this equation. This procedure is carried out by constructing solutions to the following elliptic problem

\[
\begin{cases}
-\varepsilon^2 \Delta u = \sum_{i=1}^{m} \chi_{\Omega_i^+}(u - q - \frac{\kappa_i^+}{2\pi} \ln \frac{1}{\varepsilon})^p \\
- \sum_{j=1}^{n} \chi_{\Omega_j^-}(q - \frac{\kappa_j^-}{2\pi} \ln \frac{1}{\varepsilon} - u)^p, & x \in \Omega, \\
u = 0, & x \in \partial\Omega,
\end{cases}
\]

where \( p > 1 \), \( \Omega \subset \mathbb{R}^2 \) is a bounded domain, \( \Omega_i^+ \) and \( \Omega_j^- \) are mutually disjoint subdomains of \( \Omega \) and \( \chi_{\Omega_i^+} \) (resp. \( \chi_{\Omega_j^-} \)) are characteristic functions of \( \Omega_i^+ \) (resp. \( \Omega_j^- \)), \( q \) is a harmonic function. We show that if \( \Omega \) is a simply-connected smooth domain, then for any given \( C^1 \)-stable critical point of Kirchhoff–Routh function \( W(x_1^+, \ldots, x_m^+, x_1^-, \ldots, x_n^-) \) with \( \kappa_i^+ > 0 (i = 1, \ldots, m) \) and \( \kappa_j^- > 0 (j = 1, \ldots, n) \), there is a stationary classical solution approximating stationary \( m + n \) points vortex solution of incompressible Euler equations with total vorticity \( \sum_{i=1}^{m} \kappa_i^+ - \sum_{j=1}^{n} \kappa_j^- \). The case that \( n = 0 \) can be dealt with in the same way as well by taking each \( \Omega_j^- \) as an empty set and set \( \chi_{\Omega_j^-} \equiv 0, \kappa_j^- = 0 \).

1. Introduction and Main Results

The incompressible Euler equations

\[
\begin{cases}
v_t + (v \cdot \nabla)v = -\nabla P, \\
\nabla \cdot v = 0,
\end{cases}
\]
describe the evolution of the velocity $\mathbf{v}$ and the pressure $P$ in an incompressible flow. In $\mathbb{R}^2$, the vorticity of the flow is defined by $\omega := \nabla \times \mathbf{v} = \partial_1 v_2 - \partial_2 v_1$, which satisfies the equation

$$\omega_t + \mathbf{v} \cdot \nabla \omega = 0.$$  

The velocity $\mathbf{v}$ of an incompressible fluid in two dimensions admits a stream function $\psi$ such that $\mathbf{v} = J \nabla \psi = (\partial_2 \psi, -\partial_1 \psi)$, where $J$ denotes the symplectic matrix $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

By the definitions, $\psi$ is a solution of the Poisson equation $-\Delta \psi = \omega$.

Suppose that $\omega$ is known, then the velocity $\mathbf{v}$ can be recovered by the following Biot–Savart law

$$\mathbf{v} = \omega * \frac{1}{2\pi} \frac{-Jx}{|x|^2}.$$  

One special singular solution of Euler equations is given by $\omega = \sum_{i=1}^{m} \kappa_i \delta_{x_i(t)}$ ($\kappa_i \neq 0$ is called the strength of the $i^{th}$ vortex $x_i$), which is related to

$$\mathbf{v} = -\sum_{i=1}^{m} \kappa_i \frac{J(x - x_i(t))}{2\pi |x - x_i(t)|^2}.$$  

The positions of the vortices $x_i : \mathbb{R} \to \mathbb{R}^2$ satisfy the following Kirchhoff law

$$\kappa_i \frac{dx_i}{dt} = J \nabla x_i \mathcal{W},$$  

where $\mathcal{W}$ is the so called Kirchhoff–Routh function defined by

$$\mathcal{W}(x_1, \ldots, x_m) = \frac{1}{2} \sum_{i \neq j}^{m} \frac{\kappa_i \kappa_j}{2\pi} \log \frac{1}{|x_i - x_j|}.$$  

For a simply connected bounded domain $\Omega \subset \mathbb{R}^2$, let $v_n$ be the normal component of the velocity $\mathbf{v}$ on $\partial \Omega$, that is $v_n(x) = \mathbf{v}(x) \cdot \nu(x)$, where $\nu(x)$ is the unit outward normal on $\partial \Omega$ at $x \in \partial \Omega$. Then by $\nabla \cdot \mathbf{v} = 0$, $\int_{\partial \Omega} v_n = 0$. It turns out that the Kirchhoff–Routh function for the bounded domain $\Omega$ is associated with the Green function and $v_n$. Suppose that $\mathbf{v}_0$ is the unique harmonic field whose normal component on the boundary $\partial \Omega$ is $v_n$. If $\Omega$ is simply-connected, then $\mathbf{v}_0$ can be represented by $\mathbf{v}_0 = J \nabla \psi_0$, and $\psi_0$ is determined up to a constant by

$$\begin{cases} -\Delta \psi_0 = 0, & \text{in } \Omega, \\ -\frac{\partial \psi_0}{\partial \tau} = v_n, & \text{on } \partial \Omega, \end{cases}$$  

where $\frac{\partial \psi_0}{\partial \tau}$ denotes the tangential derivative on $\partial \Omega$. The Kirchhoff–Routh function associated to the vortex dynamics then is given by (see Lin [23])