A REMARK ON JACOBI ENSEMBLE

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Abstract. We prove the large deviation principle for the supports of Jacobi ensembles following Guionnet’s method.

1. Introduction

Let $P(N), Q(N)$ be a sequence of two random projection matrices. Its statistical behavior can be understood by means of $P(N)Q(N)P(N)$, whose eigenvalue distribution is known to be a Jacobi ensemble in a natural setup. Hiai–Petz [4] proved the large deviation principle for empirical probability distributions of Jacobi ensembles. See [4] for further references on these facts.

In this short note, we will explain that the method of [3, Theorem 4.8] (related results are summarized therein) due to Guionnet that establishes the large deviation principle for the supports of $\beta$-ensembles in the large $N$ limit certainly works well for the supports of ($\beta$-)Jacobi ensembles too.

2. Jacobi ensemble

Let $P([0,1])$ be the set of all probability measures on $[0,1]$ with a metric $d$ inducing the weak topology.

Definition 2.1. For each $N \in \mathbb{N}$ and $n(N) \in \mathbb{N}$, $\kappa(N), \lambda(N) \in [0,\infty)$, let $P_N = P_{n(N),\kappa(N),\lambda(N)}$ be the probability measure on $[0,1]$ given by

$$\frac{1}{Z(N)} \exp \left( -2n(N) \sum_{i=1}^{n(N)} V_N(x_i) + 2 \sum_{1 \leq i < j \leq n(N)} \log |x_i - x_j| \right) \prod_{i=1}^{n(N)} 1_{[0,1]}(x_i) dx_i \quad (2.1)$$

with normalization constant $Z(N) = Z(N; n(N), \kappa(N), \lambda(N))$, where we define

$$V_N(x) = V_N^{n(N),\kappa(N),\lambda(N)} := -\frac{\kappa(N)}{2n(N)} \log x - \frac{\lambda(N)}{2n(N)} \log(1-x).$$

Remark 2.2. ($\beta$-Jacobi ensemble) A $\beta$-Jacobi ensemble is a probability distribution over $[0,1]^N$ whose density function (with respect to the Lebesgue measure) is proportional to

$$\prod_{i=1}^{N} x_i^{a(N)} (1-x_i)^{b(N)} \prod_{1 \leq i < j \leq N} |x_i - x_j|^\beta.$$

This measure is a special case of Definition 2.1 when $n(N) = N$, $\kappa(N) = 2a(N)/\beta$ and $\lambda(N) = 2b(N)/\beta$. 

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Throughout this note, we assume that
\[ n(N)/N \to \rho, \quad \kappa(N)/N \to \kappa, \quad \lambda(N)/N \to \lambda \quad (as \ N \to \infty) \quad (2.2) \]
for some \( \rho \in (0, \infty) \) and \( \kappa, \lambda \in [0, \infty) \). Let us recall Hiai–Petz’s result \[4\] Proposition 2.1] on LDP for the sequence \( P_N \).

**Proposition 2.3.** The following hold:

(i) The limit \( \lim_{N \to \infty} N^{-2} \log Z(N) \) exists and equals \( \rho^2 B(\kappa/\rho, \lambda/\rho) \) with the function \( B(s, t) \) in \[4\] proposition 2.2]

(ii) When \((x_1, \ldots, x_{n(N)})\) is distributed under \( P_N \), the empirical probability measure

\[ \hat{\mu}_{n(N)} := \frac{1}{n(N)} \sum_{i=1}^{n(N)} \delta_{x_i} \]

satisfies the large deviation principle in scale \( 1/N^2 \) with good rate function

\[ I(\mu) := -\rho^2 \int \int \log |x - y| d\mu(x) d\mu(y) \]
\[ -\rho \int_0^1 \kappa \log x + \lambda \log(1 - x) d\mu(x) + \rho^2 B(\kappa/\rho, \lambda/\rho) \]

for \( \mu \in \mathcal{P}([0, 1]) \). Moreover, there exists a unique minimizer \( \mu_0 \in \mathcal{P}([0, 1]) \) of \( I(\mu) \) with \( I(\mu_0) = 0 \).

3. MAIN RESULT

Define the effective potential

\[ V_{\text{eff}}(x) := V(x) - \int_0^1 \log |x - y| d\mu_0(y) - D_{\rho, \kappa, \lambda}, \]

where

\[ V(x) := -\frac{\kappa}{2\rho} \log x - \frac{\lambda}{2\rho} \log(1 - x), \]

and

\[ D_{\rho, \kappa, \lambda} := -B(\kappa/\rho, \lambda/\rho) - \int_0^1 V(x) d\mu_0(x). \]

Here are two lemmas.

**Lemma 3.1.** The effective potential \( V_{\text{eff}} \) satisfies the condition

\[ V_{\text{eff}}(x) \begin{cases} = 0 & \text{quasi-everywhere on supp}(\mu_0), \\ \geq 0 & \text{if } x \in [0, 1] \setminus \text{supp}(\mu_0), \end{cases} \]

where \( \text{supp}(\mu_0) \) denotes the support of \( \mu_0 \).

**Proof.** This immediately follows from \[5\] Theorem I.1.3]. (See the proof of \[4\] Proposition 2.1] too.) \( \square \)
Lemma 3.2. The probability measure \( \mathbb{Q}_N := \frac{1}{Z(N)} \mathbb{P}_{\mathcal{N}} \) on \([0,1]^{n(N)-1}\) is exactly

\[
\frac{1}{C(N)} \exp \left( -2n(N) \sum_{i=1}^{n(N)-1} V_{n(N),\kappa(N),\lambda(N)}(x_i) \right) + 2 \sum_{1 \leq i < j \leq n(N)-1} \log |x_i - x_j| \prod_{i=1}^{n(N)-1} 1_{[0,1]}(x_i)dx_i
\]

with \( C(N) = Z(N; n(N) - 1, \kappa(N), \lambda(N)) \).

Proof. This follows from \( V_{n(N),\kappa(N),\lambda(N)}(x) = \frac{n(N)-1}{n(N)} V_{n(N)-1,\kappa(N),\lambda(N)}(x) \). \( \square \)

Let us prove our main result.

Theorem 3.3. Define the probability measure \( \mathbb{P}_N \) on \([0,1]\) by

\[
\mathbb{P}_N(X) := \mathbb{P}_N(\{ (x_1, \ldots, x_{n(N)}) \in [0,1]^{n(N)}; \{x_1, \ldots, x_{n(N)}\} \cap X \neq \emptyset \})
\]

for any Borel subset \( X \) of \([0,1]\). Then, the sequence \( \mathbb{P}_N \) satisfies the large deviation principle

\[
\liminf_{N \to \infty} \frac{1}{N} \log \mathbb{P}_N(O) \geq - \inf_{x \in O} 2\rho_{\text{eff}}(x), \quad (3.1)
\]

\[
\limsup_{N \to \infty} \frac{1}{N} \log \mathbb{P}_N(F) \leq - \inf_{x \in F} 2\rho_{\text{eff}}(x). \quad (3.2)
\]

Lemma 3.2 shows that \( \inf_{x \in O} \rho_{\text{eff}}(x) = 0 \) if \( O \cap \text{supp}(\mu_0) \neq \emptyset \) for any open subset \( O \) of \([0,1]\). Thus, the “state space” of this large deviation principle should sit inside \([0,1]\)\( \setminus \text{supp}(\mu_0) \) rather than \([0,1]\).

The proof below essentially follows the idea in [3] Theorem 4.8, but we have to take care of some details (for example, the part of proving [3,6] follows [1] Lemma 2.6.7) instead.

Proof. The proof is divided into four steps.

Step 1. \( \rho_{\text{eff}} \) is a good rate function: Note that the effective potential \( \rho_{\text{eff}} \) is lower semicontinuous on \([0,1]\). Indeed, \( \rho_{\text{eff}}(x) = \sup_{M > 0} \rho_{\text{eff}}^M(x) \) where

\[
\rho_{\text{eff}}^M(x) := (V(x) \wedge M) - \int_{0}^{1} \log((x - y) \vee M^{-1})d\mu_0(y) - D_{\rho,\kappa,\lambda}
\]

for any \( M > 0 \), which is clearly continuous on \([0,1]\). Thus, the effective potential \( \rho_{\text{eff}}(x) \) is lower semicontinuous. It immediately follows that \( \{x \in [0,1]; \rho_{\text{eff}}(x) \leq K\} \) is closed and hence compact for any \( K \in \mathbb{R} \).

Define

\[
\gamma_N(X) := Q_N \left[ \int_X \exp \left( -2n(N)V_N(\xi) + 2(n(N) - 1) \int_{0}^{1} \log |\xi - \eta|d\mu_n(N-1)(\eta) \right) d\xi \right],
\]

where \( Q_N[\cdot] \) denotes the expectation with respect to \( Q_N \) (see Lemma 3.2).

Step 2. A large deviation lower bound for \( \gamma_N \): Let \( O \) be an open subset of \([0,1]\). Since \( O \cap (0,1) \) is open in \( \mathbb{R} \), we choose and fix an arbitrary \( x \in O \cap (0,1) \) so that \([x - \delta, x + \delta] \subset O \cap (0,1) \) for all sufficiently small \( \delta > 0 \). We fix such a \( \delta > 0 \) for a while. In what follows, we write \( V_N = V_{n(N),\kappa(N),\lambda(N)} \) for short as in §2.
We have
\[
\gamma_N \left( O \cap (0, 1) \right) \geq \\
\mathbb{Q}_N \left[ \int_{x - \delta}^{x + \delta} \exp \left( -2n(N)V_N(\xi) + 2(n(N) - 1) \int_0^1 \log |\xi - \eta| d\hat{\mu}_{n(N) - 1}(\eta) \right) d\xi \right]
\]
with \( \hat{\mu}_{n(N) - 1} = (n(N) - 1)^{-1} \sum_{i=1}^{n(N)} \delta_{x_i} \). Since \( V_N(\xi) \) converges to \( V(\xi) \) uniformly on \( [x - \delta, x + \delta] \), for any \( \varepsilon > 0 \), there exists an \( N_0 \in \mathbb{N} \), such that \( N \geq N_0 \) implies \( V_N(\xi) \leq V(\xi) + \varepsilon \) for all \( \xi \in [x - \delta, x + \delta] \). Hence
\[
\gamma_N \left( O \cap (0, 1) \right) \geq \\
\exp \left( -2n(N)V(\xi) + \varepsilon + 2(n(N) - 1) \int_0^1 \log |\xi - \eta| d\hat{\mu}_{n(N) - 1}(\eta) \right) \mathbb{Q}_N \left[ \int_{x - \delta}^{x + \delta} \exp \left( 2(n(N) - 1) \int_0^1 \log |\xi - \eta| d\hat{\mu}_{n(N) - 1}(\eta) \right) d\xi \right].
\]

With \( E_\delta(V) := \sup \{|V(x) - V(y)|; |x - y| < \delta\} \) \( (n.b., x \text{ has been fixed}) \), it follows that
\[
\gamma_N \left( O \cap (0, 1) \right) \geq \\
\exp \left( -2n(N)V(x) + E_\delta(V) + \varepsilon + 2(n(N) - 1) \int_0^1 H_{x,\delta}(\eta) d\hat{\mu}_{n(N) - 1}(\eta) \right),
\]
where
\[
H_{x,\delta}(\eta) := \int_{x - \delta}^{x + \delta} \log |\xi - \eta| \frac{d\xi}{2\delta}.
\]
Note that \( H_{x,\delta}(\eta) \) can explicitly be calculated and turns out to be a continuous function \( (in \eta) \) over \( \mathbb{R} \). Thanks to Lemma 5.2 and Theorem 2.3 \( (n.b., (n(N) - 1)/N \rightarrow \rho \text{ as } N \rightarrow \infty) \), we see that \( \hat{\mu}_{n(N) - 1} \) weakly converges to \( \mu_0 \) almost surely with \( \mathbb{Q}_N \), and hence
\[
R(\delta, N) := \int_0^1 H_{x,\delta}(\eta) d\hat{\mu}_{n(N) - 1}(\eta) - \int_0^1 H_{x,\delta}(\eta) d\mu_0(\eta) \rightarrow 0
\]
as \( N \rightarrow \infty \) almost surely with \( \mathbb{Q}_N \).

Observe that
\[
\gamma_N \left( O \cap (0, 1) \right) \geq 2\delta \exp \left( -2n(N)V(x) + E_\delta(V) + \varepsilon + 2(n(N) - 1) \int_0^1 H_{x,\delta}(\eta) d\mu_0(\eta) + 2(n(N) - 1)R(\delta, N) \right),
\]
and hence
\[
\liminf_{N \rightarrow \infty} \frac{1}{N} \log \gamma_N \left( O \cap (0, 1) \right) \geq -2\rho(V(x) + E_\delta(V) + \varepsilon) + 2\rho \int_0^1 H_{x,\delta}(\eta) d\mu_0(\eta)
\]
(3.3)
holds by (2.2).
Write $F_\eta(x) := \int_0^x \log |\xi - \eta| \, d\xi$. We observe that
\[
\lim_{\delta \to 0} \int_0^1 H_{x,\delta}(\eta) \, d\mu_0(\eta) = \lim_{\delta \to 0} \int_0^1 \frac{F_\eta(x + \delta) - F_\eta(x - \delta)}{2\delta} \, d\mu_0(\eta)
= \lim_{\delta \to 0} \int_0^1 \frac{F_\eta(x + \delta) - F_\eta(x)}{2\delta} + \frac{F_\eta(x - \delta) - F_\eta(x)}{2(-\delta)} \, d\mu_0(\eta)
= \int_0^1 \lim_{\delta \to 0} \frac{F_\eta(x + \delta) - F_\eta(x)}{2\delta} + \frac{F_\eta(x - \delta) - F_\eta(x)}{2(-\delta)} \, d\mu_0(\eta)
= \int_0^1 \log |x - \eta| \, d\mu_0(\eta),
\]
where we used the dominated convergence theorem in the third line and the fundamental theorem of calculus in the fourth line. Therefore, taking the limit of $\lim sup_{\delta \to 0}$ as $\delta \to 0$, we have
\[
\lim_{N \to \infty} \frac{1}{N} \log \gamma_N \left( O \cap (0,1) \right) \geq -2\rho \left( V(x) - \int_0^1 \log |x - y| \, d\mu_0(y) \right) - 2\rho \varepsilon.
\]
Since $\varepsilon > 0$ can arbitrary be small, we have, for any $x \in X \cap (0,1)$,
\[
\liminf_{N \to \infty} \frac{1}{N} \log \gamma_N \left( O \cap (0,1) \right) \geq -2\rho \left( V(x) - \int_0^1 \log |x - y| \, d\mu_0(y) \right).
\]
Since $V(x) - \int_0^1 \log |x - y| \, d\mu_0(y) = \infty$ if $x \in \{0,1\}$, we conclude that
\[
\liminf_{N \to \infty} \frac{1}{N} \log \gamma_N (O) \geq -2\rho \inf_{x \in X} \left( V(x) - \int_0^1 \log |x - y| \, d\mu_0(y) \right). \tag{3.4}
\]

Step 3. A large deviation upper bound for $\gamma_N$: Let $F$ be closed subset of $[0,1]$ and define
\[
\Phi_N^L(\xi, \mu) := V_N(\xi) \land L - \frac{n(N)}{n(N) - 1} \int_0^1 \log(|\xi - \eta| \lor L^{-1}) \, d\mu(\eta)
\]
on $[0,1] \times \mathcal{P}([0,1])$ for any $L > 1$. Then we have
\[
-\Phi_N^L \leq -(V_N \land L) \leq C \tag{3.5}
\]
for some $C \in \mathbb{R}$, since $\log(|\xi - \eta| \lor L^{-1}) \leq 0$ on $[\xi, \eta] \in [0,1] \times [0,1]$.

Observe that
\[
\gamma_N(F) \leq Q_N \left[ \int_F \exp \left( -2n(N)\Phi_N^L(\xi, \mu_{n(N)-1}) \right) \, d\xi \right].
\]
Choose and fix an arbitrarily small $\delta > 0$. Dividing the integration range of $Q_N$ into two parts $\{\mu_{n(N)-1} \in \{d(\cdot, \mu_0) \leq \delta\}\}$ and $\{\mu_{n(N)-1} \in \{d(\cdot, \mu_0) > \delta\}\}$ and using (3.5), we obtain that
\[
\gamma_N(F) \leq \exp \left( -2n(N) \inf_{(\xi, \mu) \in F \times \{d(\cdot, \mu_0) \leq \delta\}} \Phi_N^L(\xi, \mu) \right)
+ e^{2n(N)C} Q_N \left( \mu_{n(N)-1} \in \{d(\cdot, \mu_0) > \delta\} \right),
\]
and hence
\[
\limsup_{N \to \infty} \frac{1}{N} \log \gamma_N(F) \leq \max \left\{ \limsup_{N \to \infty} -2n(N) \inf_{(\xi, \mu) \in F \times \{d(\cdot, \mu_0) \leq \delta\}} \Phi_N^L(\xi, \mu), \right.
\]
\[
\left. \limsup_{N \to \infty} 2n(N)C + \limsup_{N \to \infty} \frac{1}{N} \log Q_N \left( \mu_{n(N)-1} \in \{d(\cdot, \mu_0) > \delta\} \right) \right\}
\]
by \([2,3]\). By Theorem \([3,3]\) together with Lemma \([3,2]\) we have
\[
\limsup_{N \to \infty} \frac{1}{N} \log \mathcal{Q}_N \left( \hat{\mu}_{n(N)^{-1}} \in \{ d(\cdot, \mu_0) > \delta \} \right) = -\infty,
\]
and hence
\[
\limsup_{N \to \infty} \frac{1}{N} \log \gamma_N (F) \leq -2\rho \liminf_{N \to \infty} \inf_{(\xi, \mu) \in F \times \{ d(\cdot, \mu_0) \leq \delta \}} \Phi^\xi_N (\xi, \mu).
\]
By \([2,2]\), \(\Phi^\xi_N (\xi, \mu)\) converges to \(V(\xi) \wedge L - \int_0^1 \log(\xi - \eta \mid V \wedge L^{-1})d\mu(\eta)\) uniformly on \([0,1]\) as \(N \to \infty\), and thus
\[
\limsup_{N \to \infty} \frac{1}{N} \log \gamma_N (F) = -2\rho \inf_{(\xi, \mu) \in F \times \{ d(\cdot, \mu_0) \leq \delta \}} \left( V(\xi) \wedge L - \int_0^1 \log(\xi - \eta \mid V \wedge L^{-1})d\mu(\eta) \right),
\]
Since \(V(\xi) \wedge L - \int_0^1 \log(\xi - \eta \mid V \wedge L^{-1})d\mu(\eta)\) is continuous on \([0,1] \times \mathcal{P}([0,1])\), we observe that
\[
\inf_{(\xi, \mu) \in F \times \{ d(\cdot, \mu_0) \leq \delta \}} \left( V(\xi) \wedge L - \int_0^1 \log(\xi - \eta \mid V \wedge L^{-1})d\mu(\eta) \right) \to \inf_{\xi \in F} \left( V(\xi) \wedge L - \int_0^1 \log(\xi - \eta \mid V \wedge L^{-1})d\mu_0(\eta) \right)
\]
as \(\delta \to 0\) (see the proof of \([2]\) Lemma 4.1.6(a)). Thus,
\[
\limsup_{N \to \infty} \frac{1}{N} \log \gamma_N (F) \leq -2\rho \inf_{\xi \in F} \left( V(\xi) - \int_0^1 \log(\xi - \eta \mid V \wedge L^{-1})d\mu_0(\eta) \right),
\]
and letting \(L \to \infty\) we conclude that
\[
\limsup_{N \to \infty} \frac{1}{N} \log \gamma_N (F) \leq -2\rho \inf_{\xi \in F} \left( V(\xi) - \int_0^1 \log(\xi - \eta \mid V \wedge L^{-1})d\mu_0(\eta) \right).
\]
(3.6)

Step 4. Transition from \(\gamma_N\) to \(\widehat{\gamma}_N\): We first claim that
\[
\frac{\gamma_N(X)}{\gamma_N([0,1])} \leq \widehat{\gamma}_N(X) \leq n(N) \gamma_N(X) / \gamma_N([0,1]) \quad (3.7)
\]
This follows from the following two observations:
\[
\mathbb{P}_N \left( \{ (x_1, \ldots, x_{n(N)} \in [0,1]^{n(N)}; x_{n(N)} \in X) \} \right)
\]
\[
\leq \widehat{\mathbb{P}_N} (X) \leq n(N) \mathbb{P}_N \left( \{ (x_1, \ldots, x_{n(N)} \in [0,1]; x_{n(N)} \in X) \} \right)
\]
and
\[
\mathbb{P}_N \left( \{ (x_1, \ldots, x_{n(N)} \in [0,1]^{n(N)}; x_{n(N)} \in X) \} \right) = \frac{1}{Z(N)} \int_{[0,1]^{n(N)-1}} \int_X \exp \left( -2n(N) \sum_{i=1}^{n(N)} V_N(x_i) + 2 \sum_{1 \leq i < j \leq n(N)} \log |x_i - x_j| \right) \prod_{i=1}^{n(N)} dx_i = \frac{C(N)}{Z(N) C(N)} \int_{[0,1]^{n(N)-1}} \left( \int_X \exp \left( -2n(N) V_N(\xi) + 2 \sum_{i=1}^{n(N)-1} \log |x_i - \xi| \right) d\xi \right)
\]
\[
\times \exp \left( -2n(N) \sum_{i=1}^{n(N)-1} V_N(x_i) + 2 \sum_{1 \leq i < j \leq n(N)-1} \log |x_i - x_j| \right) \prod_{i=1}^{n(N)-1} dx_i = \frac{\gamma_N(X)}{\gamma_N([0,1])}
\]
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by $\gamma_N([0,1]) = Z(N)/C(N)$.

By (3.4), (3.6) and Lemma 3.1 we have

$$\lim_{N \to \infty} \frac{1}{N} \log \gamma_N([0,1]) = -2\rho \inf_{\xi \in [0,1]} \left( V(\xi) - \int_0^1 \log |\xi - \eta| d\mu_0(\eta) \right) = -2\rho D_{\rho,\kappa,\lambda}.$$ 

This and $\lim_{N \to \infty} \frac{1}{N} \log n(N) = 0$ (thanks to (2.2)) enable us to derive (3.1) and (3.2) from (3.4) and (3.6), respectively. Hence we have completed the proof. □

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