Observables and a Hilbert Space for Bianchi IX

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Abstract

We consider a quantization of the Bianchi IX cosmological model based on taking the constraint to be a self-adjoint operator in an auxiliary Hilbert space. Using a WKB-style self-consistent approximation, the constraint chosen is shown to have only continuous spectrum at zero. Nevertheless, the auxiliary space induces an inner product on the zero-eigenvalue generalized eigenstates such that the resulting physical Hilbert space has countably infinite dimension. In addition, a complete set of gauge-invariant operators on the physical space is constructed by integrating differential forms over the spacetime. The behavior of these operators indicates that this quantization preserves Wald’s classical result that the Bianchi IX spacetimes expand to a maximum volume and then recollapse.

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I. INTRODUCTION

Finite dimensional cosmological models are a mainstay of modern general relativity for the simple reason that they address many exciting conceptual issues without forcing the researcher to confront the complicated, infinite dimensional, and in the quantum context perturbatively non-renormalizable formalism of full 3+1 gravity. Our interest here will be the quantum theory and our motivation for studying this model is exactly that stated above. Again, we ask the historical questions (e.g., [1–4]): “What are the quantum observables?” “How are they constructed or even defined?” And, “On what Hilbert space, if any, do they act?”

A proposal for addressing these questions was introduced in [4] and applied to finite dimensional parameterized Newtonian systems, the free relativistic particle, and “separable, semi-bound” models (which include the Kantowski-Sachs and locally rotationally symmetric (LRS) Bianchi IX cosmological models, see [4]). For such systems, the approach was found to define a Hilbert space of physical states, to produce a complete set of gauge-invariant quantum operators, and to reproduce (in a certain sense) the classical feature that the semi-bound models describe spacetimes with homogeneous spatial hypersurfaces first expand and then recollapse. The strategy was based on that of DeWitt [5] and takes seriously the idea that observables in General Relativity are given by integrals of differential forms over the spacetime manifold. The analogues of such integrals were, in fact, performed in the quantum theory.

In order to implement this basic proposal, a mathematical framework must first be introduced. The main framework of [4] is based on that of Dirac [6] and considers an auxiliary Hilbert space in which gauge-dependent operators act. For the models in question, a physical inner product was induced on the generalized eigenvectors of the constraint by this auxiliary structure despite the fact that no solution of the constraint was normalizable in the auxiliary inner product.

However, [4] considered only relatively simple models. While some models addressed by
section V of [4] are not explicitly solvable even classically, the usual minisuperspace models (Bianchi I, Kantowsk-Sachs, and LRS Bianchi IX) that fall into this category are solvable and have even been deparameterized [7]. Also, the separable nature of these systems was explicitly used in [4] to construct the physical Hilbert space and to study convergence of the integrals that define the gauge invariant operators.

Our goal here is to apply these same ideas to the more complicated Bianchi IX, or Mixmaster, minisuperspace [8–10]. Like other Bianchi models, it describes a class of solutions to the vacuum Einstein equations that are foliated by a family of homogeneous spacelike hypersurfaces. This preferred family effectively reduces the gauge symmetry of full General Relativity to that of time reparameterizations.

Unlike the simpler models, the Bianchi IX cosmology has not been completely solved or deparameterized classically. In fact, it is often [11,12] considered to be an example of chaotic behavior in General Relativity. It therefore constitutes a much more stringent test of the proposal of [4]. We shall see that, by introducing new tools and a WKB-style self-consistent approximation, all of the corresponding results are obtained.

We begin our presentation by reviewing the method of [4] in section II. Section III then applies these techniques to the mixmaster model, beginning with a description of the quantum theory (section III A) and followed by a study of solutions to our constraint (III B) and a characterization of the physical Hilbert space (III C). One consequence of our results is that, within the validity to this approximation, we show that the space of solutions to the Hamiltonian constraint for Bianchi IX is infinite dimensional. Finally, a complete set of gauge-invariant operators is constructed and the recollapsing behavior of our quantum theory is verified in III D. We close with a summary of the results and a comparison with other work.
II. THE GENERAL APPROACH

Because our goal is to apply the techniques of [4] to the Bianchi IX minisuperspace, we now summarize the relevant points of [4] for the reader’s convenience. The general setting considered is that of a time reparameterization invariant system with Hamiltonian constraint:

\[ h = 0 \quad (2.1) \]

for some function \( h \) on the phase space \( \Gamma = \mathbb{R}^{2N} \) (see [4] for a more general treatment). Here, as in [4], we will take capital letters to stand for quantum operators and lower case latin letters to refer to their classical counterparts. For each coordinate \( q_i, p_j \) on \( \Gamma \), a family \( Q_i(t), P_j(t) \) of self-adjoint operators is introduced on an (auxiliary) Hilbert space \( \mathcal{H} = L^2(\mathbb{R}^N) \).

Now, choose some time independent symmetric factor ordering \( H(Q_i(t), P_j(t)) \) corresponding to \( h(q_i, p_j) \) and consider the associated self-adjoint operator \( H \). The families \( Q_i(t), P_j(t) \) are chosen to satisfy

\[ \frac{dA}{dt} = i[N(t)H, A] \quad (2.2) \]

for any \( A = A(Q_i(t), P_j(t)) \) and a family of lapse operators \( N(t) \) that are proportional to the identity on \( \mathcal{H} \); \( N(t) = n(t)\mathbb{1} \). The commutators of the families form a quantized version of a generalized Peierls bracket [13] and the formulation of [4] provides a notion of gauge transformations corresponding to time reparameterizations.

This structure is to be applied toward two goals: i) The construction of a physical Hilbert space \( \mathcal{H}_{phys} \) and ii) The construction of a complete set of gauge-invariant operators that commute with \( H \). Here, we use the same notion of a complete set as in [4]. That is, we say that a set of operators on \( \mathcal{H}_{phys} \) is complete if the corresponding set of phase space functions obtained in the classical limit separates the points of the phase space. For parameterized Newtonian systems and so-called “separable semi-bound” systems (which include
the Kantowski-Sachs and LRS Bianchi IX cosmological models), both of these objectives are attained in [4]. We will now see that they may be accomplished for the full-fledged Bianchi IX model as well.

The physical Hilbert space is to be defined as follows. We first find a convenient “energy basis” of states $|n,E\rangle$ for $E \in [\lambda, 0]$ (for some $\lambda < 0$) and $n \in \mathbb{Z}^+$ that satisfy $H|n,E\rangle = E|n,E\rangle$ and

$$\langle n', E'|n, E \rangle = \delta(E - E')\delta_{n,n'}$$

(2.3)

and span the subspace of $\mathcal{H}$ corresponding to the interval $[\lambda, 0]$ in the continuous spectrum of $H$. The physical space $\mathcal{H}_{phys}$ is to be the closure of the vector space generated by the symbols $\{|n,0\rangle\}$ for $n \in \mathbb{Z}^+$ with the physical inner product: $(|n',0\rangle, |n,0\rangle)_{phys} = \delta_{n,n'}$.

For a gauge-invariant operator $A$ on $\mathcal{H}$ that commutes with $H$, we would then like to consider an operator $A_{phys}$ on $\mathcal{H}_{phys}$ given by:

$$A_{phys}|n,0\rangle = \sum_k a_{nk}(0)|k,0\rangle$$

(2.4)

where the coefficients $a_{nk}$ are defined by $A|n,E\rangle = \sum_k a_{nk}(E)|k,E\rangle$. This procedure is satisfactory when $a_{nk}(E)$ is continuous in $E$ on $[\lambda, 0]$ and in this case $A_{phys}$ is a symmetric bilinear form on $\mathcal{H}_{phys}$ when $A$ is symmetric on $\mathcal{H}$. In this way our physical inner product captures the classical reality conditions [14] to the same extent as the auxiliary inner product on $\mathcal{H}$.

Such gauge invariant operators were constructed in [4] by integrating over the parameter $t$ that labels the families of operators that appear in [22]. In particular, given two operators $A$ and $Z$ on $\mathcal{H}$, we follow [4] and consider

$$[A]_Z^L = \int_{-\infty}^{\infty} dt A(t)\left(\frac{\partial}{\partial t}\theta(Z(t) - \tau)\right)$$

(2.5)

and

$$[A]_Z^R = \int_{-\infty}^{\infty} dt \left(\frac{\partial}{\partial t}\theta(Z(t) - \tau)\right)A(t)$$

(2.6)
where $\theta(Z(t) - \tau)$ is the projection onto the non-negative spectrum of $Z(t) - \tau$. The object

$$[A]_{Z=\tau} = \frac{1}{2}([A]^L_{Z=\tau} + [A]^R_{Z=\tau}) \quad (2.7)$$

is formally symmetric and in the classical limit is simply a function on the space $\mathcal{S}$ of solutions that, when applied to a particular solution $s$, is a linear combination of the values of $A$ at the points along $s$ at which $Z = \tau$. Furthermore, when the integrals $2.5$ and $2.6$ converge, they define operators that commute with $H$. For appropriate systems and choices of $A$ and $Z$, this procedure was shown in [4] to produce a complete set of well-defined operators $([A]_{Z=\tau})_{phys}$ on $\mathcal{H}_{phys}$. Because the correspondence between the gauge invariant $A$ and the physical operator $A_{phys}$ is direct, we now drop the subscript $phys$ from these operators.

One final property derived in [4] is that a quantization of this type captures the classical “recollapsing behavior” of separable semi-bound cosmological models. That is, in contrast to quantization schemes based on the Klein-Gordon inner product [15], there is a quantum analogue of the classical result that these models expand and then recollapse. For such models, if $Z$ is taken to be the scale factor $\alpha = \ln(\det g)$ for the metric $g$ on a homogeneous spatial hypersurface, then the operators $[A]_{\alpha=\tau}$ vanish in the large $\tau$ limit. Because the corresponding classical quantities satisfy $[a]_{\alpha=\tau} \rightarrow 0$ as a direct consequence of the fact that a given solution $s$ will not reach arbitrarily large values of $\alpha$, we interpret

$$[A]_{\alpha=\tau} \rightarrow 0 \text{ as } \tau \rightarrow +\infty \quad (2.8)$$

as the quantum statement of recollapse. The Bianchi IX model also recollapses classically [16] and we shall see that, when quantized as above, it recollapses quantum mechanically as well.

III. THE QUANTIZATION OF BIANCHI IX

The discussion of [4] considered the parameterized Newtonian particle and separable semi-bound models, but our interest here is in the more complicated Bianchi IX minisuperspace. We will see that the same (qualitative) results can be obtained as we will now
construct \(i\) a non-trivial physical Hilbert space from generalized eigenvectors of the constraint operator as well as \(ii\) a complete set of gauge invariant operators and \(iii\) verify that our quantum theory leads to the recollapsing behavior derived classically in [16].

We follow much the same path as in section V of [4]. Section III A presents the system and sections III B and III C construct a convenient “energy” basis for the auxiliary space \(\mathcal{H}\) in terms of which the physical states can be simply characterized. This part of the analysis is carried out at the somewhat physical level of WKB-style self-consistent approximations. Section III D considers operators 2.7 of the form \(A_\alpha = \tau\) for \(\alpha = \ln(\det g)\) and \(g\) the three-metric on such a slice and finds that matrix elements of these operators converge for a complete set of appropriate \(A\) so such \([A]_{a=\tau}\) exist as bilinear forms on \(\mathcal{H}_{phys}\). We then modify the construction of \([A]_{a=\tau}\) by introducing a “level operator” \(L\) whose physical interpretation is less clear, but which improves the behavior of our operators. The modified \(\tilde{[A]}_{a=\tau}\) can then be proven to define (for appropriate \(A\)) a complete set of bounded operators on \(\mathcal{H}_{phys}\).

Both the forms \([A]_{a=\tau}\) and the operators \(\tilde{[A]}_{a=\tau}\) vanish as \(\tau \to \infty\), demonstrating that our model preserves the classical recollapsing behavior.

A. The Quantum System

The mixmaster model describes anisotropic homogeneous solutions to the 3 + 1 vacuum Einstein equations. For this case, the metric \(g_{ij}\) on a homogeneous slice [11] may be written in the form

\[
g_{ij} = e^{2\alpha}(\beta^+\beta^-)_{ij}
\]

(3.1)

for a matrix \(\beta_{ij}\) given by \(\text{diag}(\beta^+ + \sqrt{3}\beta^-, \beta^+ - \sqrt{3}\beta^-, -2\beta^+\)) which defines the anisotropy parameters \(\beta^\pm\) and the scale factor \(\alpha\). Taking the metric to be in this form fixes all of the gauge freedom except that of time reparameterizations.

Since the classical phase space is described by \((\alpha, \beta^+, \beta^-, p_\alpha, p_{\beta^+}, p_{\beta^-}) \in \mathbb{R}^6\) we take the auxiliary space \(\mathcal{H}\) to be \(L^2(\mathbb{R}^3, \delta^3x)\) following [4]. Our quantum theory will be built from
the configuration operators $\alpha, \beta\pm$ which act on functions $\psi \in L^2(\mathbb{R}^3)$ by multiplication as well as the derivative operators $P_\alpha = -i\hbar \frac{\partial}{\partial \alpha}$, $P_\beta = -i\hbar \frac{\partial}{\partial \beta}$ which are the corresponding momenta.

Recall that the full Hamiltonian constraint for this model may be written \([10]\) as

$$H = -P_\alpha^2 + P_\beta^2 + e^{4\alpha}(V(\beta) - 1)$$

(3.2)

using a rescaled lapse $N(t) = (\frac{3\pi}{2})^{1/2} e^{-3\alpha} \tilde{N}(t)$ and Hamiltonian $H = (\frac{2}{3\pi})^{1/2} e^{3\alpha} \tilde{H}$. (where $\tilde{N}$ and $\tilde{H}$ are the usual ADM lapse and Hamiltonian of general relativity \([17]\) applied to the minisuperspace ansatz \([3.1]\)). Note that by writing \(3.2\) as a quantum operator, we have chosen a certain factor ordering of the classical expression. The explicit form of the potential is given by

$$V(\beta) = \frac{1}{3} e^{-8\beta^+} - \frac{4}{3} e^{-2\beta^+} \cosh 2\sqrt{3}\beta^- + 1 + \frac{2}{3} e^{4\beta^+} (\cosh 4\sqrt{3}\beta^- - 1)$$

(3.3)

which has a global minimum $V = 0$ at the origin $\beta^\pm = 0$ but has no other critical points.

The operator \(3.2\) is symmetric on $L^2(\mathbb{R}^3)$ and we shall assume that some suitable extension is in fact self-adjoint. This extension will also be written as $H$ and no distinction will be made between the two operators.

The operator $H$ of \(3.2\) is Hermitian and we use it to introduce a “time dependent” set of configuration bases $\{|\alpha, \beta^+, \beta^-; t\rangle\}$ which are related through

$$|\alpha, \beta^+, \beta^-; t\rangle = e^{i(t'-t)H} |\alpha, \beta^+, \beta^-; t\rangle.$$  

(3.4)

The (time dependent) direct product decomposition $\mathcal{H} = \mathcal{H}_{\alpha,t} \otimes \mathcal{H}_{\beta,t}$ induced by the factorization $|\alpha, \beta^+, \beta^-; t\rangle = |\alpha; t\rangle \otimes |\beta^+, \beta^-; t\rangle = |\alpha; t\rangle |\beta^+, \beta^-; t\rangle$ will be used heavily in what follows.

As our system is time reparametrization invariant and \(3.2\) is classically constrained to vanish, it should be emphasized that this $t$ merely serves as a label and is not the Newtonian time parameter of non-relativistic quantum mechanics. Nevertheless, it behaves similarly when used as a technical tool and we use it to construct the families $Q_i(t)$ and $P_j(t)$. These will be defined by $Q_i(0) = Q_i$, $P_j(0) = P_j$ and
\[ Q_i(t) = \exp \left( iH \int_0^t dt' N(t') \right) Q_i \left( -iH \int_0^t dt' N(t') \right) \]
\[ P_j(t) = \exp \left( iH \int_0^t dt' N(t') \right) P_j \left( -iH \int_0^t dt' N(t') \right) \]  
(3.5)

for some \( N(t) = n(t) \mathbb{1}, n(t) \in \mathbb{R} \). This “lapse operator” and its relation to gauge transformations are discussed in [4].

The generalized eigenvectors of the constraint (3.2) with eigenvalue zero will form the physical Hilbert space \( \mathcal{H}_{phys} \). In order to study solutions of \( H|\psi\rangle = 0 \), we will consider the operator

\[ H_1 = P_{\beta^+}^2 + P_{\beta^-}^2 + e^{4\alpha} (V(\beta) - 1) \]  
(3.6)

and for each \( t' \), introduce a family \( H^*(\alpha; t') \) of operators on \( \mathcal{H}_{\beta;t'} \) parametrized by \( \alpha \):

\[ H_1 = \int d\alpha |\alpha; t'\rangle \langle \alpha; t'| \otimes H^*(\alpha; t'). \]  
(3.7)

This \( H^*(\alpha; t') \) is a Hermitian operator on the Hilbert space \( \mathcal{H}_{\beta;t'} = L^2(\mathbb{R}^2) \) of square integrable functions of \( \beta^+ \) and \( \beta^- \). We similarly define \( P_{\beta^\pm}^* \) and \( \beta^{\pm*} \). Note that, for various \( \alpha \), the \( H^*(\alpha) \) differ only in the overall scale of the potential \( V^*(\beta^{\pm*}; \alpha) = e^{4\alpha} (V(\beta^{\pm*}) - 1) \) and that, because the region in which the potential is less than \( E \) for any \( E \in \mathbb{R} \) has finite area (see appendix A), the spectrum of these operators is entirely discrete [18] and the degeneracy of each spectral projection is finite.

Thus, we may label the eigenstates of \( H^*(\alpha; t') \) in \( \mathcal{H}_{\beta;t'} \) as \( |n; \alpha; t'\rangle_\beta \) for \( n \in \mathbb{Z}^+ \). The state \( |n; \alpha; t'\rangle_\beta \) is to have the eigenvalue \( E_n(\alpha) \) (which does not depend on \( t' \)) of \( H^*(\alpha; t') \) and these states are orthonormal: \( \beta\langle n'; \alpha; t'|n; \alpha; t'\rangle_\beta = \delta_{n,n'} \). Since the family \( H^*(\alpha; t') \) is analytic in \( \alpha \), the states \( |n; \alpha; t\rangle \) may also be chosen smooth in the label \( \alpha \) for each \( n \) [19] and we may express any state \( |\psi\rangle \) in \( \mathcal{H} \) as

\[ |\psi\rangle = \int d\alpha \sum_n f_n(\alpha)|\alpha\rangle |n; \alpha\rangle_\beta \]  
(3.8)

for some \( f_n(\alpha) \). Until [11], we will work at some fixed value of \( t' \) and suppress this label on the states, operators, and spaces.
B. Solutions of the Constraint

In the following, we operate at the somewhat physical level of self-consistent approximations and to use 3.8 to study the eigenvectors of $H$; that is, to solve $H|\psi\rangle = E|\psi\rangle$. We begin by considering the action of $H$ on 3.8:

$$H|\psi\rangle = \int \! d\alpha \sum_n \left( \hbar^2 \frac{\partial^2}{\partial \alpha^2} (f_n(\alpha)|n; \alpha\rangle + f_n(\alpha)E_n(\alpha)|n; \alpha\rangle) \right)|\alpha\rangle$$  \hfill (3.9)

Now, from the equation

$$(H^*(\alpha) - E_n(\alpha))|n; \alpha\rangle = 0$$  \hfill (3.10)

in $H_{\beta}$ it follows that the states $|n; \alpha\rangle_{\beta}$ may be chosen to satisfy

$$\frac{\partial}{\partial \alpha}|n; \alpha\rangle_{\beta} = \frac{1}{(H^*(\alpha) - E_n(\alpha))_{\perp}}(\sum_m M_{mn}|m; \alpha\rangle_{\beta}$$  \hfill (3.11)

where $\frac{1}{(H^*(\alpha) - E_n(\alpha))_{\perp}}$ is the inverse of $H^*(\alpha) - E_n(\alpha)$ except when acting on states $|m; \alpha\rangle_{\beta}$ with $E_m(\alpha) = E_n(\alpha)$ (i.e., states degenerate with $|n; \alpha\rangle_{\beta}$), which it annihilates.

The matrix $M_{mn}$ is antihermitian since $M_{nm}$ vanishes when $E_n = E_m$ and, for $E_n \neq E_m$, we have

$$M_{mn} = \langle m; \alpha| \frac{-4}{(H^*(\alpha) - E_n(\alpha))_{\perp}} e^{4\alpha}(V - 1)|n; \alpha\rangle = \langle m; \alpha| -4 \frac{e^{4\alpha}(V - 1)|n; \alpha\rangle}{E_m(\alpha) - E_n(\alpha)}$$  \hfill (3.12)

for which the numerator is Hermitian and the denominator is real and antisymmetric. We now use $M_{nm}$ to rewrite the derivative in 3.9 as

$$\frac{\partial}{\partial \alpha} \left( \sum_n f_n(\alpha)|n; \alpha\rangle_{\beta} \right) = \sum_n \left( \frac{\partial}{\partial \alpha} f_n(\alpha) + \sum_m f_m(\alpha)M_{nm} \right)|n; \alpha\rangle_{\beta}.$$  \hfill (3.13)

Our goal is to study generalized eigenvectors of $H$ for which the solutions $|\psi\rangle$ of 3.8 are at least delta-function normalizable. Thus, for almost every $\alpha$ the list of functions $f_n(\alpha)$ should give the components $(\mathcal{F}(\alpha))_n$ of an $\alpha$-dependent vector $\mathcal{F}(\alpha)$ in the Hilbert space.
of square summable sequences that satisfies \( \int d\alpha |\mathcal{F}(\alpha)|^2 = \langle \psi | \psi \rangle \). On this space, the coefficients \( M_{mn} \) define an antihermitian operator through \((M\mathcal{F})_n = \sum_m M_{nm}\mathcal{F}_m\). Let \( U(\alpha) \) be the corresponding unitary path ordered exponential \( \mathcal{P} \exp(-\int_{\alpha_0}^\alpha d\alpha' M(\alpha')) \) satisfying \( \frac{dU^{-1}}{d\alpha} = U^{-1} M \) and let \( \mathcal{F} = U\mathcal{g} \) for some \( \mathcal{g} \). Then we have

\[
\left( \frac{\partial}{\partial \alpha} f_n + \sum_m M_{nm} f_m \right) = (U \frac{\partial}{\partial \alpha} \mathcal{g})_n
\]

and we may write \( 3.3 \) as

\[
H|\psi\rangle = \int d\alpha \sum_n \left( h^2 \left( U \frac{\partial^2}{\partial \alpha^2} \mathcal{g} \right)_n + (U \mathcal{g})_n E_n(\alpha) \right) |n; \alpha\rangle.
\]

It follows that \( |\psi\rangle \) is an eigenvector of \( H \) with eigenvalue \( E \) exactly when the corresponding \( \mathcal{g} \) is a generalized \( L^2 \) function and satisfies

\[
h^2 \frac{\partial^2}{\partial \alpha^2} \mathcal{g} + (\tilde{E}(\alpha) \circ \mathcal{g}) = E \mathcal{g}.
\]

Here, \( \tilde{E}(\alpha) = U^{-1}(\alpha)\overline{E}(\alpha)U(\alpha) \) and \( \overline{E}(\alpha) \) is the operator on \( l^2 \) such that \( (\overline{E}(\alpha)\mathcal{f})_n = E_n(\alpha)\mathcal{f}_n \). We will show that for \( E \leq 0 \), there is one such solution for every \( \mathcal{g}_0 \in l^2 \).

The equation \( 3.16 \) to be solved for \( \mathcal{g}(\alpha) \) is essentially a one-dimensional time independent Schrödinger equation for a multiple component wavefunction with potential \( -\tilde{E}(\alpha) \) and energy \( -E \). Thus, existence, normalizability, and multiplicity of the solutions to \( 3.16 \) is related to whether this “potential” confines a (fictitious) particle to finite values of \( \alpha \) or lets such a particle escape to the asymptotic regions. Note that, as the potential \( V^*(\beta^*;\alpha) \) of \( H^*(\alpha) \) becomes more and more shallow for large negative \( \alpha \), \( E_n(\alpha) \to 0 \) as \( \alpha \to -\infty \) and our fictitious particle is unbound in this direction for \( E \leq 0 \). In particular, for large negative \( \alpha \), even the ground state energy of \( H^*(\alpha) \) is positive so that \( -E_n(\alpha) \) approaches 0 from below as \( \alpha \to -\infty \) and there can be no normalizable eigenvector with \( E \leq 0 \); all solutions to \( 3.16 \) with \( E \leq 0 \) are delta-function normalizable in the region \( \alpha \leq 0 \).

Solutions of \( 3.16 \) with \( E > 0 \) are difficult to analyze in the \( \alpha \to -\infty \) limit without a uniform bound on the rate at which \( E_n(\alpha) \) vanishes. Note, however, that the existence of such a uniform bound would show that the spectrum of \( H \) is discrete for \( E > 0 \) so that zero
is the largest continuum eigenvalue in dramatic contrast to the separable cases of [4]. A
WKB approximation of the type used below for \( \alpha \to +\infty \) might provide insight but is even
more difficult to control than at the positive \( \alpha \) end. As we only require knowledge of the
spectrum of \( H \) on some \([a,b]\) containing zero, we take \( a < 0 \) and \( b = 0 \) and do not concern
ourselves with the case \( E > 0 \).

The remaining task is to study the behavior of \( E_n(\alpha) \) for large, positive \( \alpha \). To do so, we
write \( \overline{\eta} \) in the form

\[
\overline{\eta} = \mathcal{P} \exp\left[ \int_0^\alpha d\alpha' N(\alpha') \right] \overline{\eta}_0 \tag{3.17}
\]

where \( V = \mathcal{P} \exp\left[ \int_0^\alpha d\alpha' N(\alpha') \right] \) is the path ordered exponential that satisfies \( \frac{\partial V}{\partial \alpha} = NV \). Our
equation \([3.16]\) can now be written as

\[
[N^2 + \dot{N}V + \dot{E}V] \overline{\eta}_0 = EV \overline{\eta}_0 \tag{3.18}
\]

where \( \dot{N} = dN/d\alpha \). In order to show that a solution exists for all \( \overline{\eta}_0 \), we will (approximately)
solve the matrix equation

\[
N^2 V + \dot{N}V + \dot{E}V - EV = 0 \tag{3.19}
\]

As in the finite dimensional WKB method, assume that \( N^2 >> \dot{N} \), so that \([3.19]\) becomes
\( N^2 = E \mathbb{1} - \dot{E} \). Since \( \dot{E} \) is a positive self-adjoint operator with known spectrum, a number
of square-roots \( \sqrt{E \mathbb{1} - \dot{E}} \) are easily defined. For the moment, let \( N \) be any one of these.

Now, recall that \( \dot{E}(\alpha) = U^{-1}(\alpha) \dot{E}(\alpha) U(\alpha) \). It follows that \( N = U^{-1}(\alpha)(\sqrt{E \mathbb{1} - \dot{E}}) U(\alpha) \)
and we have

\[
\frac{\partial}{\partial \alpha}(UV) - [\frac{\partial U}{\partial \alpha} U^{-1}] UV = U \frac{\partial V}{\partial \alpha} = \sqrt{E \mathbb{1} - \dot{E}} UV \tag{3.20}
\]

which has the solution \( V = U^{-1} \mathcal{P} \exp\left[ \int_0^\alpha (\sqrt{E \mathbb{1} - \dot{E}}) - M \right] \). This form of \( V \) will be convenient
for the approximations below.

We now reason as follows. Consider again \( H^*(\alpha) \) and the associated eigenvalue problem

\[
E_n \psi_n = -\hbar^2 \Delta \psi_n + e^{4\alpha}(V - 1) \psi_n. \tag{3.21}
\]
Here, $\Delta$ is the two-dimensional Laplacian and $\psi_n$ is the wavefunction $\psi_n = \langle \beta_+, \beta_- | n \rangle$. Since (3.21) may also be written as

$$\frac{E_n}{\hbar^2} \psi_n = -\Delta \psi_n + \frac{e^{4\alpha}}{\hbar^2} (V - 1) \psi_n$$

the ratio $E_n/\hbar^2$ must depend only on $\hbar$ and $\alpha$ through $e^{4\alpha}/\hbar^2$. Similarly, we must have $e^{-4\alpha} E_n = f(e^{4\alpha}/\hbar^2)$ for some function $f$ and it follows that $\alpha \to \infty$ is essentially just the model’s classical limit. However, since the potential $(V - 1)e^{4\alpha}$ has only a single critical point (which is, in fact, a global minimum) we expect that as $\hbar \to 0$ every energy level falls toward this minimum. Thus, $e^{-4\alpha} E_n \to -1$ as $\alpha \to \infty$.

This will in turn imply that, for large $\alpha$, the $n$th state is concentrated near $\beta^\pm = 0$. Since $E_n = \langle n | H^* | n \rangle$, $-\Delta \geq 0$, and $(V - 1) \geq -1$ as $\alpha \to \infty$, the behavior of $E_n(\alpha)$ implies that $\langle n | (V - 1) | n \rangle \to -1$. But $(V - 1) = -1$ only at the origin, so we must have $\int_{\mathbb{R}^2 - B_{\epsilon}} |\psi_n|^2 \to 0$ for every $\epsilon$-ball $B_{\epsilon}$ centered at $\beta^\pm = 0$.

For small enough $\epsilon$, the potential $V - 1$ within $B_{\epsilon}$ is essentially that of a (rotationally symmetric) Harmonic oscillator ($V \approx 8e^{4\alpha}[\beta^2 + \beta^{-2}]$). Thus, since $E_n \to -e^{4\alpha}$ for every $n$, for each $N > 0$ there is some $\alpha$ for which $H^*(\alpha)$ may be replaced by the rotationally invariant Harmonic oscillator Hamiltonian $-\Delta + 8e^{4\alpha}(\beta_+^2 + \beta_-^2) - e^{4\alpha}$ when considering states with $n \leq N$ to some accuracy $\epsilon_0$. At this point, it is convenient to replace the label $n$ with the two occupation numbers $n_1, n_2$ of the oscillator so that $E_{n_1, n_2}(\alpha) \approx -e^{4\alpha} + 2\sqrt{2}e^{2\alpha}\hbar(n_1 + n_2 + 1)$. Similarly, we find $\langle m_1, m_2 | (V - 1) | n_1, n_2 \rangle \approx c_{(m_1, m_2)(n_1, n_2)} e^{-2\alpha}$, for $c_{(m_1, m_2)(n_1, n_2)}$ independent of $\alpha$ and vanishing for $|m_1 - n_1| > 2$ or $|m_2 - n_2| > 2$, so that

$$M_{(m_1, m_2)(n_1, n_2)} \approx \frac{e^{2\alpha} c_{(m_1, m_2)(n_1, n_2)}}{e^{2\alpha}\hbar(m_1 + m_2 - n_1 - n_2)}$$

which is also independent of $\alpha$ in this regime. On the other hand, $\sqrt{E \mathbb{1} - E} \to u e^{2\alpha}$ for some $u^2 = \mathbb{1}$ so that we may neglect $M$ in comparison to this term and write

$$V \approx U^{-1} \exp(u e^{2\alpha}).$$

---

1Thanks to Charles W. Misner for pointing this out.
Note in particular that \( N^2 \sim e^{4\alpha} \) while \( \dot{N} \sim e^{2\alpha} \) so that for large \( \alpha \) we do indeed have \( N^2 >> \dot{N} \).

C. Eigenvectors and the Physical Hilbert Space

While 3.24 (approximately) solves 3.19, the question arises of which solutions \( V(\alpha)\overline{g}_0 \) are (delta-function) normalizable. Suppose first that \( u \) has some eigenvector \( \overline{g}_0 \) with eigenvalue +1. Then, the corresponding solution \( \overline{g} = W\overline{g}_0 \) has diverging \( l^2 \) norm \( (|\overline{g}|^2) \) as \( \alpha \to \infty \). However, for any eigenvector \( \overline{g}_0 \) of \( u \) with eigenvalue \( -1 \), the corresponding \( \overline{g} = W\overline{g}_0 \) is square integrable on the region \( \alpha \geq 0 \) \((\int_{\alpha \geq 0} |\overline{g}|^2 < \infty)\) for all square summable \( \overline{g}_0 \). We thus take \( u = -1 \) for the general normalizable solution.

For large negative \( \alpha \), we already know that the behavior of our solutions is that of a free particle. It therefore follows that there is one delta-function normalizable solution to 3.16 for each \( \overline{g}_0 \in l^2 \) and every \( E \leq 0 \) so that the continuous spectrum of \( H \) includes the interval \((-\infty, 0]\). This verifies that \( H \) is of the type described in section II and provides one of the labels \((E)\) of our useful basis of states.

In fact, we have shown that all states in \( \mathcal{H} \) are of the form

\[
|\psi(\overline{g}_0(E))\rangle = \int dE|\overline{g}_0(E), E\rangle = \int dE \sum_n (\overline{g}_0(E))_n |\chi_n, E\rangle
\]  

(3.25)

where \( |\overline{g}_0, E\rangle \) is the solution of \( H|\psi\rangle = E|\psi\rangle \) given by \( \overline{g}(\alpha) = V_E(\alpha)\overline{g}_0 \) for \( \overline{g}_0 \in l^2 \) and \((\overline{\chi}_n)_m = \delta_{n,m}\). If, for convenience, we now refer to the states \( |\chi_n, E\rangle \) as \( |n, E\rangle \) then we have found a basis for \( \mathcal{H} \) such that \( H|n, E\rangle = E|n, E\rangle \) and \( \langle n, E|n', E'\rangle = \delta(E - E')\delta_{n,n'} \). The physical Hilbert space \( \mathcal{H}_{phys} \) may now be constructed as in [4] as the closure of the span of \( \{|n, 0\}\) with respect to the inner product \((|n, 0\rangle, |m, 0\rangle)_{phys} = \delta_{n,m}\).

D. Physical Operators

Having constructed \( \mathcal{H}_{phys} \), we now address both the formation of physical operators on this space as in [4] through integration over \( t \) and the verification of recollapsing behavior.
We first construct a complete set of operators that i) has a straightforward physical interpretation in the classical limit, ii) are defined as bilinear forms on the physical space, and iii) satisfy the recollapse criterion 2.8 as bilinear forms. We will then regularize these operators using a new operator $L$ whose physical interpretation is less than clear. The regularized operators, however, form a complete set of bounded operators on the physical Hilbert space which satisfy 2.8 in the sense of strong convergence; that is, convergence on a dense set of states [20].

Consider then the objects $[A]_{\alpha=\tau}$ of 2.7. From [4], the action of these operators on $|n,0\rangle$ may be written as

$$
(|n^*,0), [A]_L^{\alpha=\tau}|n,0\rangle)_{\text{phys}} = 2\pi i \langle n^*,0|A[H,\theta(\alpha - \tau)]|n,0\rangle
$$

and

$$
(|n^*,0), [A]^R_{\alpha=\tau}|n,0\rangle)_{\text{phys}} = 2\pi i \langle n^*,0|[H,\theta(\alpha - \tau)]A|n,0\rangle
$$

(3.26)

(3.27)

Now, for any bounded operator $B$ on $\mathcal{H}$, let

$$
B_{\text{reg}} = \frac{1}{H + i}\theta(\alpha - \tau')B\theta(\alpha - \tau')\frac{1}{H - i}.
$$

(3.28)

This $B_{\text{reg}}$ is bounded since $H$ is self-adjoint and is Hermitian when $B$ is. As in [4], the operator $[B_{\text{reg}}]_{\alpha=\tau}$ has the same classical limit as $[B]_{\alpha=\tau}$ for $\tau' < \tau$ since, when the constraint is satisfied, the factors of $\frac{1}{\hbar \pm i}$ combine to have no effect and the extra step functions $\theta(\alpha - \tau')$ are classically irrelevant for $\tau > \tau'$.

To see that the $[B_{\text{reg}}]_{\alpha=\tau}$ give a complete set of operators, note that $B$ may be chosen to be any bounded function of $\beta^\pm$ or of $P_{\beta^\pm}$. By also inserting into $B$ appropriate functions of $1 \pm \text{sign}(P_\alpha)$ as in [4], we may construct objects $[B_{\text{reg}}]_{\alpha=\tau}$ whose classical limits give arbitrary (bounded) functions of $\beta^\pm$ or $P_{\beta^\pm}$ evaluated at point where $\alpha = \tau$ and either $P_\alpha > 0$ or $P_\alpha < 0$. However, from the result of [10] each classical solution has $\alpha = \tau$ only twice (if at all), once with $P_\alpha > 0$ and once with $P_\alpha < 0$. Thus, since we consider the $[B_{\text{reg}}]_{\alpha=\tau}$ for all real $\tau$, this is a complete set of operators for the Bianchi IX model.
Because all of the solutions to \(3.10\) that were used to build \(\mathcal{H}_{phys}\) are square-integrable over the region \(\alpha \geq 0\) (see section III C), the states \(\theta(\alpha - \tau')|n^*, 0\rangle\) and \(\theta(\alpha - \tau)|n, 0\rangle\) are normalizable in \(\mathcal{H}\). It follows that the physical matrix elements \((|n^*, 0\rangle, [B_{reg}]_{\alpha=\tau}|n, 0\rangle)_{phys}\) are well-defined and finite since, for example,

\[
(|n^*, 0\rangle, [B_{reg}]_{\alpha=\tau}^L|n, 0\rangle)_{phys} = 2\pi \langle n^*, 0 | \theta(\alpha - \tau') \left[B\theta(\alpha - \tau') \frac{H}{\mathcal{H} - i}\right]\theta(\alpha - \tau)|n, 0\rangle \quad (3.29)
\]

and \(B\theta(\alpha - \tau) \frac{H}{\mathcal{H} - i}\) is bounded on \(\mathcal{H}\). This means that integrals defining \([B_{reg}]_{\alpha=\tau}\) converge as bilinear forms on \(\mathcal{H}_{phys}\). Also, since \(|\theta(\alpha - \tau)|n, 0\rangle|^2 \to 0\) as \(\tau \to \infty\), these objects converge to zero as forms and verify the classical recollapsing behavior.

Now, a modification of \(2.5\) and \(2.6\) leads to objects that can be proven to be bounded operators on \(\mathcal{H}_{phys}\). To determine if some \([A]_{\alpha=\tau}\) is a well-defined operator, it must be checked whether the norms of the states obtained by acting with \([A]_{\alpha=\tau}\) on physical states are finite. Thus, from \(3.26\) and \(3.27\), we are interested in sums of the form

\[
\sum_{n^*} |\langle n^*, 0; t'|\mathcal{O}|n, 0; t'\rangle|
\]

for \(\mathcal{O} = [A, H]\theta(\alpha - \tau)\) and \(\mathcal{O} = \theta(\alpha - \tau)[H, A]\). Equivalently, we can replace \([A, H]\) in \(3.30\) with \(AH\) or \(HA\) as appropriate since \(H\) vanishes on the physical subspace.

In order to improve the convergence of \(3.30\), we consider the new “level operator” \(L = L_0 \oplus 1^\perp\) where the direct sum corresponds to the decomposition \(\mathcal{H} = \mathcal{H} \oplus \mathcal{H}^\perp\) of \(\mathcal{H}\) into \(\mathcal{H}\) and its orthogonal complement. The operator \(1^\perp\) is the identity operator on \(\mathcal{H}^\perp\) and \(L_0\) is defined by

\[
L_0|n, E\rangle = n|n, E\rangle
\]

so that \(L\) is self-adjoint. Now, instead of \(2.5\) and \(2.6\), consider the integrals

\[
\frac{[A]_{\alpha=\tau}^L}{L} = \int_{-\infty}^{\infty} f(L(t))A(t) \frac{\partial}{\partial t}\theta(\alpha(t) - \tau)f(L(t))
\]

\[
\frac{[A]_{\alpha=\tau}^R}{L} = \int_{-\infty}^{\infty} f(L(t)) \frac{\partial}{\partial t}\theta(\alpha(t) - \tau)A(t)f(L(t))
\]

\(3.32\)

where \(f\) is an as yet unspecified function that vanishes for large \(L\). Here, \(L(t) = L\) since \([H, L] = 0\) from \(3.31\), so that \(3.32\) amounts to multiplying expressions \(2.5\) and \(2.6\) by \(f(L)\).
on the left and on the right. As before, \( \tilde{A}^L_{\alpha=\tau} = \tilde{A}^R_{\alpha=\tau} \) for self-adjoint \( A \), \( \tilde{A}_{\alpha=\tau} \equiv \frac{1}{2}(\tilde{A}^L_{\alpha=\tau} + \tilde{A}^R_{\alpha=\tau}) \), and

\[
(|n^*, 0\rangle, \tilde{A}^L_{\alpha=\tau}|n, 0\rangle)_{phys} = -i2\pi\langle n^*, 0|f(L)A\theta(\alpha - \tau)f(L)|n, 0\rangle \\
(|n^*, 0\rangle, \tilde{A}^R_{\alpha=\tau}|n, 0\rangle)_{phys} = i2\pi\langle n^*, 0|f(L)\theta(\alpha - \tau)H\hat{A}(L)|n, 0\rangle
\]

(3.33)

Again, let \( B_{reg} = \frac{1}{\pi^2}\theta(\alpha - \tau')B\theta(\alpha - \tau')\frac{1}{\pi^2} \). For this case, the matrix elements in (3.33) may be written

\[
(|n^*, 0\rangle, [\tilde{B}_{reg}]_{\alpha=\tau}|n, 0\rangle)_{phys} = -2\pi f(n^*)f(n)|\psi_{n^*}(\tau')|B\theta(\alpha - \tau')H\frac{H}{H - i}|\psi_n(\tau)|
\]

(3.34)

where \( |\psi_n(\tau)\rangle = \theta(\alpha(\tau') - \tau)|n, 0\rangle \) is a state in \( \mathcal{H} \) with finite norm \( z_\tau(n) \). Let us choose \( f \) such that \( |f(n)| \leq \frac{1}{n\pi z_\tau(n)} \) and \( \tau' \) such that \( \tau' < \tau \). Thus, \( f \) may depend on \( \tau' \) but not on \( \tau \). Since \( z_\tau(n) \) is a nonincreasing function of \( \tau \), we have \( z_\tau(n) \leq z_{\tau'}(n) \), and the matrix elements (3.34) are bounded in absolute value by

\[
2\pi |f(n)f(n^*)| z_\tau(n)z_{\tau'}(n^*)||B|| \leq \frac{||B||}{nn^*} \frac{z_\tau(n)z_{\tau'}(n^*)}{2\pi}
\]

(3.35)

where \( ||B|| \) is the operator norm of \( B \) on \( \mathcal{H} \).

The physical norms of \( [A]^{L,R}_{\alpha=\tau}|n, 0\rangle \) are then bounded by

\[
||[\tilde{B}_{reg}]^L_{\alpha=\tau}|n, 0\rangle||^2_{phys} \leq \frac{z_\tau(n)}{z_{\tau'}(n)} \sum_{n^*} \frac{1}{(n^*)^2} (2\pi)^2 \\
||[\tilde{B}_{reg}]^R_{\alpha=\tau}|n, 0\rangle||^2_{phys} \leq \sum_{n^*} \frac{1}{(n^*)^2} \frac{z_\tau(n^*)}{z_{\tau'}(n^*)} (2\pi)^2
\]

(3.36)

and \( \tilde{B}_{reg} \) is a bounded operator on \( \mathcal{H}_{phys} \). Furthermore, the action of \( \tilde{B}_{reg}^L_{\alpha=\tau} \) on any of the states \( |n, 0\rangle \) vanishes as \( \tau \to \infty \) since both bounds in (3.36) become zero in this limit. This is clear for the first bound and follows for the second from the facts that the sum over \( n^* \) converges and that \( z_\tau(n^*) \) is a positive nonincreasing function of \( \tau \) that vanishes as \( \tau \to \infty \). It follows that \( \tilde{B}_{reg}^L_{\alpha=\tau} \) converges strongly to zero and, in fact, \( \tilde{B}_{reg}^R_{\alpha=\tau} \) converges uniformly to zero.

Again, in the classical limit and on the constraint surface, the factors of \( \frac{1}{n\pi^2} \) cancel and the factors of \( \theta(\alpha - \tau') \) have no effect for \( \tau' \leq \tau \). In addition, since \( [L, H] = 0 \), \( L \) is conserved
along a classical solution and, for a given solution \( s \), 
\[
[f(l)\frac{1}{\pi+1}\theta(\alpha - \tau')b\theta(\alpha - \tau')\frac{1}{\pi+1}f(l)]_{\alpha=\tau}(s)
\]
tends to zero for large \( \tau \) if and only if \( [b]_{\alpha=\tau} \rightarrow 0 \) as \( \tau \rightarrow \infty \). We thus conclude that our model describes a quantum theory of recollapsing cosmologies.

Since \( l \) is some function on the phase space, it can be expressed in terms of \( \beta^\pm \) and \( p_{\beta^\pm} \). But \( b \) may be an arbitrary bounded function of \( \beta^\pm \) and \( p_{\beta^\pm} \) so that \( \{[f(l)]^2b\} \) is still sufficient to separate points on the phase space. In this way, we have constructed a complete set of quantum operators for the Bianchi IX minisuperspace.

IV. DISCUSSION

As claimed in the introduction, we have applied the techniques of [1] to the Bianchi IX cosmological model and thereby construct a quantum theory. Specifically, a Hilbert space structure on the generalized eigenvalues of \( H \) was induced by the auxiliary space \( H_{aux} \), a complete set of observables was constructed, and recollapsing behavior was derived, all despite the complicated and perhaps chaotic nature of the model.

Finally, since the literature on the mixmaster cosmology is extensive, a short comparison of section III with previous work is presented below. Note first that the observables studied here are, in spirit, much like those of [21] (and suggested by Moncrief), which describe anisotropies and momenta at the maximal value of \( \alpha \). Such objects are easily written down in the formalism used here by choosing \( Z \) in \( \ref{Z} \) and \( \ref{P} \) to be \( P_\alpha \) and \( \tau \) to be zero, since \( P_\alpha \) vanishes classically on the maximal volume surface. Note that [21] also defines observables away from maximal volume which, for volumes suitably close to maximal are (exactly) unitarily related. This “unitary evolution” then breaks down at a finite separation from maximal volume. In our case, because the physical operators \( [A]_{\alpha=\tau} \) vanish for large \( \tau \), we also find that the \( [A]_{\alpha=\tau} \) and \( [A]_{\alpha=\tau'} \) are not unitarily related when the separation between \( \tau \) and \( \tau' \) is large. On the other hand, there is no reason to think that \textit{any} of the \( [A]_{\alpha=\tau} \) of [11] are unitarily related or that there should be a preferred finite separation at which such a unitary evolution breaks down. Thus, we expect these quantizations to be inequivalent.
We now single out the recent construction of exact solutions to the quantum constraint by Moncrief and Ryan \cite{22} for more detailed discussion. Such solutions were found earlier by Kodama \cite{23} for the constraint written in terms of Ashtekar variables, but the work of \cite{22} uses the variables of section \textsection{III} so that the comparison with our approach is direct. A similar comment applies to the super-symmetric solutions of Graham \cite{24}.

However, the quantum states of \cite{22} do not solve the constraint \ref{3.2} as they are related to a different factor ordering of the classical expression. Instead, they solve

\begin{equation}
0 = H'\psi = \frac{\partial^2}{\partial\alpha^2}\psi - B\frac{\partial}{\partial\alpha}\psi - \frac{\partial^2}{(\partial\beta^+)^2}\psi - \frac{\partial^2}{(\partial\beta^-)^2}\psi + e^{4\alpha}(V - 1)\psi
\end{equation}

for some real $B$ where, following \cite{22}, we have dropped the $\hbar$’s.

Thus, our setting does not coincide with that of \cite{22} and some effort will be needed to reconcile them. In particular, \cite{22} does not consider an auxiliary Hilbert space and we must introduce one here for comparison with section \textsection{III}. Note that \cite{22} provides explicit solutions only for $B = -6 \neq 0$ and that for $B \in \mathbb{R}$, $B \neq 0$, the differential operator above is not Hermitian in the Hilbert space $L^2(\mathbb{R}^3, d\alpha d\beta d\beta^-)$. This operator is, however, self-adjoint in the Hilbert space $L^2(\mathbb{R}^3, e^{-B\alpha}d\alpha d\beta^+ d\beta^-)$, which we will therefore take to be our auxiliary space.

A rescaling of the wavefunction $\psi = e^{B\alpha/2}\phi$ leads to the equivalent constraint

\begin{equation}
0 = H''\phi = \frac{\partial^2}{\partial\alpha^2}\phi - B^2\frac{\partial^2}{4}\phi - \frac{\partial^2}{(\partial\beta^+)^2}\phi - \frac{\partial^2}{(\partial\beta^-)^2}\phi + e^{4\alpha}(V - 1)\phi
\end{equation}

on $\phi \in L^2(\mathbb{R}^3, d^3x)$. Note that the arguments of \textsection{III} apply as well to \ref{4.2} as to \ref{3.2}, the only difference being a relative shift of $E_n(\alpha)$ by $B^2/4$. Thus, the behavior at large $\alpha$ is unchanged and all energies are “bound” at this end. The difference appears at large negative $\alpha$, where the behavior of our wavefunction reduces to that of a free particle. If $H''\phi = E\phi$, then this free particle has energy $-(E + B^2/4)$. Thus, the (probable) upper bound on the continuous spectrum moves from zero to $-B^2/4$. In particular, the solutions described by \cite{22} vanish rapidly as $\alpha \to +\infty$ but become asymptotically constant as $\alpha \to -\infty$. The rescaled wavefunction $\phi$ then behaves as $e^{-B\alpha/2}$ for large negative $\alpha$ so that such solutions are normalizable whenever $B < 0$ such as when $B = -6$.  

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Thus, zero is now in the *discrete* spectrum of the constraint and the construction of III C is unnecessary since the relevant eigenvectors form a subspace of the auxiliary space and so inherit the inner product directly. Also, because the spectrum is discrete, from [4] we expect that the integrals 2.5 and 2.6 which attempt to define our gauge-invariant operators will now fail to converge.

Finally, if we allow $B$ to be imaginary then the original operator $H'$ is self-adjoint on $L^2(\mathbb{R}^2, d^3x)$. The same analysis follows as for $B$ real, but now the boundary between the discrete and continuous spectrum moves to $E = |B|^2/4 > 0$ so that zero again lies in the continuous spectrum. Solutions of the form described by [23] would then correspond to elements of $\mathcal{H}_{phys}$ of [II] and the construction of gauge-invariant operators proceeds as in [II].

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**APPENDIX A: FINITENESS OF THE REGION $V < E$**

We now quickly derive the fact that the area $A(E)$ of the region of the $\beta^+, \beta^-$ plane on which $V$ of 3.3 is less than $E$ is finite for any $E$ in $\mathbb{R}$. To do so, we use the $2\pi/3$ discrete rotational symmetry of $V$ and its well-known “triangular” shape [I]. Recall that this “triangle” points along the $\beta^+$ axis.

For large positive $\beta^+$, let $h^E(\beta_+)$ be the value of $\beta_-$ for which $V(\beta_+, \beta_-) = E$. In this part of the plane, there are always two solutions $\pm h^E(\beta_+)$ and we take $h^E(\beta_+)$ to be the positive one. $A(E)$ is then finite if we have...
\[
\int_{\lambda}^{\infty} h^E(\beta_+) d\beta_+ < \infty \quad (A1)
\]

for any finite \(\lambda\).

From \[3.3\], it follows that
\[
V(\beta_+, \beta_-) < -\frac{4}{3} e^{-2\beta_+} + \frac{2}{3} e^{4\beta_+} (\cosh 4\sqrt{3}\beta_- - 1) \quad (A2)
\]
so that \(h^E(\beta_+)\) is less than the \(h^E_0(\beta_+)\) defined by
\[
0 = -2e^{-2\beta_+} + e^{4\beta_+} (\cosh 4\sqrt{3}h^E_0(\beta_+) - 1) \quad (A3)
\]
Clearly, \(h^E_0 \to 0\) for large \(\beta_+\) so that, if \(\beta_+\) is large enough, \(\cosh 4\sqrt{3}h^E_0 - 1 \geq \frac{1}{2} [3(2^3)(h^E_0)^2] \).

It follows that \(h^E \leq h^E_0 \leq \frac{1}{6} e^{-6\beta_+}\) and that \(A1\) holds.
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