Falling into the Schwarzschild black hole. Important details.

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Abstract

The Schwarzschild space is one of the best studied spacetimes and its exhaustive considerations are easily accessible. Nevertheless, for some reasons it is still surrounded by a lot of misconceptions, myths, and “paradoxes”. In this pedagogical paper an attempt is made to give a simple (i.e., without cumbersome calculations) but rigorous consideration of the relevant questions. I argue that 1) an observer falling into a Schwarzschild black hole will not see “the entire history of the universe” 2) he will not cross the horizon at the speed of light 3) when inside the hole, he will not see the (future) singularity, and 4) the latter is not “central”.

1 Introduction

The Schwarzschild spacetime (alias maximally extended Schwarzschild spacetime, alias the Kruskal spacetime) is certainly one of the best studied solutions of the Einstein equations. A rare textbook in relativity does not dwell on that space, which is no surprise taking into account its importance and (relative) simplicity. So, one might think that no mysteries are harbouried there any longer, a careful reading of [1 §§31,32] being able to give the answer to almost any “silly” question. This, however, is not quite so. For a person who has not yet got used to the basic concepts of general relativity (equivalence of all coordinate systems, the impossibility of attaching a preferred extended reference system to an observer, etc.) the Schwarzschild space is

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fraught with pitfalls. Such a person encounters various “paradoxes” and “miracles” (exactly as in studying special relativity or quantum mechanics) and it takes some work to sort them out. Unfortunately, the areal of those paradoxes and miracles is not restricted to student internet forums and popular literature. They have infiltrated the semi-popular, research, and even pedagogical works. Thus one can find there the assertion that, just before crossing a black hole horizon, an astronaut in a single moment of his proper time will see the whole infinitely long evolution of the external universe [2, 3]. He will see how our Sun swells becoming a red giant, how the Earth skimming over the upper atmosphere of the dying Sun evaporates in its glare, and how the Sun later transforms into a white dwarf ... [3]. Elsewhere one reads that the astronaut will traverse the horizon at the speed of light [4] and after crossing the horizon he will see the “central singularity” [3]. The authors of these excerptions are all scholars of authority, so one can only pity a student reading all that.

Thus, it seems there is a need for a paper where the most puzzling properties of the Schwarzschild space would be illuminated in an as clear (but rigorous) manner as possible. In the following sections I treat — hopefully just in that manner — a few most “controversial” issues, which are: Will an observer falling into the black hole see the entire future of our universe? Will he cross the horizon at the speed of light? Will he see the singularity? Is that singularity “central”? (The answers to all four questions are negative). The reader is supposed to be familiar with only the basics of semi-Riemannian geometry. Units are used in which $G = c = 1$.

2 The locale

2.1 The geometry of the Schwarzschild spacetime

The simplest (i. e., non-rotating and uncharged) black hole is described, as everybody knows, by the spacetime $\mathcal{M}$:

$$ds^2 = 4m^2 \left\{ \frac{4}{xe^x} du dv + x^2 (d\theta^2 + \cos^2 \theta \, d\phi) \right\}, \quad (1)$$

$$u, v \in \mathbb{R}, \quad x > 0,$$
where \( m \) is a positive parameter (called the mass, see below) and \( x = x(u, v) \) is the function defined (implicitly) by the equation

\[
uv = (1 - x)e^x. \tag{2}
\]

The importance of \( \mathcal{M} \) — it is this spacetime that we shall call Schwarzschild’s — lies, of course, in the fact that it is a spherically symmetric solution to the vacuum Einstein equations and, moreover, by Birkhoff’s theorem it is the only such solution in the class of maximal globally hyperbolic spacetimes. It is often convenient to choose \( x \) (or \( r \), which is almost the same) as a new coordinate. That cannot be done in the entire \( \mathcal{M} \) (as follows, for example, from the fact that \( \nabla x(0, 0) = 0 \)) and we shall restrict ourselves to the region

\[
\mathcal{M}_*: \quad u < 0, \quad v > 0.
\]

(in Fig. 1, it is shown by dark gray). There the transition to the coordinates

\[
r \equiv 2mx, \quad t \equiv 2m \ln(-v/u) \tag{3}
\]

brings the metric (1) to a more customary form:

\[
ds^2 = -(1 - 2m/r)dt^2 + (1 - 2m/r)^{-1}dr^2 + r^2(d\theta^2 + \cos^2\theta\,d\phi) \tag{4}
\]

\[t \in \mathbb{R}, \quad r > 2m.
\]

To relate it to the everyday consider the region \( \mathcal{M}_{r_0} \subset \mathcal{M}_* \) defined by the inequality \( r > r_0 > 2m \). The region is spherically symmetric and asymptotically flat, and the metric there solves the source-free Einstein equations. So, \( \mathcal{M}_{r_0} \) (and — again by Birkhoff’s theorem — only \( \mathcal{M}_{r_0} \)) describes the universe outside a ball of radius \( r_0 \) and mass \( m \). (The equation for the \( r \)-coordinate of a radial geodesic parametrized by the proper time \( \tau \) is

\[
\ddot{r} = -\frac{m}{r^2}, \tag{5}
\]

where the dot stands for the derivative by \( \tau \). The comparison of this equation with the Newtonian one justifies our interpretation of \( m \) as the mass).

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1 \( \mathcal{M} \) cannot be extended, say, to the region \( x(u, v) < 0 \) because the scalar \( R_{abcd}R^{abcd} \) diverges at \( x \to 0 \).

2 It is easily found by varying the “geodesic Lagrangian” \( \mathcal{L} = -(1 - 2m/r)t^2 + (1 - 2m/r)^{-1}r^2 \) with respect to \( r \).
Figure 1: The sections $\phi = \text{const}, \theta = \text{const}$ of the Schwarzschild spacetime. (a). The dark gray region $\mathcal{M}_*$ is asymptotically flat, it is the region “outside of the Schwarzschild black hole”. By the light gray the regions $\mathcal{M}_-$ and $\mathcal{M}_+$ are shown, which are, respectively, expanding and contracting “universes”; (b). The gray region is the causal past of $\gamma$, i.e., the union of the causal pasts of all its points. This region includes all events that have ever been observed by $\gamma$.

The surfaces $u = 0$ and $v = 0$ (alias $x = 1$, alias $r = 2m$) bounding $\mathcal{M}_*$ are called horizons. It should be emphasized that the points of horizons have no “magic” properties; each of them has a small neighbourhood with exactly the same (in a qualitative sense) properties as a neighbourhood of any other point of any spacetime; the tidal forces here are finite, massive bodies move on timelike curves, the world lines of photons are null geodesics, etc. Now, what is there beyond the horizon? One might naively expect that since the horizon is a sphere (at each moment of time; we are discussing the section of the spacetime by some simultaneity surface $\mathcal{S}$), then what it bounds is a ball. Or rather a punctured ball, with a singularity at the center. That would perfectly fit the idea that the Schwarzschild solution “describes the field of a point mass (located at the center, the singular point of the metric)” \[5\]. The said idea goes back to classics of the pre-Kruskal epoch \[6\] and is amazingly widespread even today. It should be stressed therefore that the just drawn picture though not wrong ($\mathcal{S}$ can be chosen so as to justify it) is, nevertheless, grossly misleading. We shall see, in particular, that the term...
“central” is applicable to Schwarzschild’s singularity no more than, say, to Friedmann’s.

To perceive the real geometry of the region

\[ \mathcal{M}_+: \quad u > 0, \quad 0 < r < 2m, \]

(shadowed in light gray in Fig. 1a) it is instructive to introduce there the coordinates

\[ \eta \equiv 2mx, \quad l \equiv 2m \ln(v/u). \]  

The metric then takes the form

\[
\begin{align*}
ds^2 &= -(\frac{2m}{\eta} - 1)^{-1}d\eta^2 + \frac{2m}{\eta} - 1)dl^2 + \eta^2(d\theta^2 + \cos^2 \theta d\phi) \\
l &\in \mathbb{R}, \quad \eta \in (0, 2m).
\end{align*}
\]

Remark 1. The transformation (3) is singular at \( u = 0 \) and \( v = 0 \). Therefore it cannot be extended to \( \mathcal{M}_+ \). In other words, \((t, r)\) and \((\eta, l)\) are different coordinates. Unfortunately, this fact is overlooked sometimes, which leads to much confusion and the talk about “space and time swapping their roles” inside the black hole.

Thus, an observer after crossing the horizon finds himself in a “universe” with not quite usual properties. The “space” of that universe (i.e., the surface \( \mathcal{S} \) given in this case by the equation \( \eta = \text{const} \)) is a homogeneous cylinder \( \mathbb{R}^1 \times S^2 \). It is spherically symmetric, but not isotropic, the distinguished direction being that along the \( l \)-axis. At the same time, even though the surfaces \( \eta = \text{const}, \ l = \text{const} \) are spheres, one should not call the \( l \)-coordinate “radial”, because the space is invariant w. r. t. translations in that direction. Note that the space has neither a singularity, nor a centre.

With time the geometry of \( \mathcal{M}_+ \) changes. This fact is not surprising — the Schwarzschild space as a whole is non-static, even though it has a static [as is seen from (4)] region \( \mathcal{M}_* \). The radius of the cylinders \( \mathcal{S} \) falls and it is its vanishing at \( \eta = 0 \) that is referred to as the Schwarzschild singularity.

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3 This homogeneous, anisotropic universe is a special case of the Kantowski-Sachs spacetime [2].

4 The isometries \( \sigma_A^* : t \mapsto t + A \), that act on \( \mathcal{M}_* \) can be extended to the isometries \( \sigma_A : u \mapsto e^{-\frac{A}{2m}}u, \ v \mapsto e^{\frac{A}{2m}}v \) acting on the entire \( \mathcal{M} \), but in \( \mathcal{M}_+ \) the orbits of the group \( \sigma_A \) are spacelike.

5 The case in point is the “upper singularity” in Fig. 1; the other one is, of course, in the past.
Evidently, for any observer in $\mathcal{M}_+$ the singularity is in the future and, in particular, nobody (on whichever side of the horizon) can ever observe it.

Remark 2. The surfaces of simultaneity could be chosen differently, of course. For example, they could be defined by the equation $u + v = \text{const}$ instead of $\eta = \text{const}$. In such a case $\mathcal{M}$ would appear as an evolving wormhole [1]. The throat of the wormhole lies in $\mathcal{M}_{\pm}$ and connects two asymptotically flat isometric “universes” — ours and $\mathcal{M}'$. As can be seen from Fig. [1] the two universes are causally disconnected, but a traveller from one of them is allowed to see some events in the other (though not before traversing the horizon).

Often it is $\mathcal{M}_+$ that is called Schwarzschild’s space and $t, r$ — Schwarzschild’s coordinates, while $\mathcal{M}$ and $u, v$ are called Kruskal–Szekeres’. The coordinates $u, v$ cover the entire manifold. And, in studying the radial motion, when only the sections $\phi = \text{const}, \theta = \text{const}$ matter, their additional advantage is that the metric of those sections takes the form

$$ds^2 = -F(r)du dv, \quad F = 16m^2x^{-1}e^{-x},$$

which simplifies significantly the analysis of their causal structure. A curve in the $(u, v)$-plane is causal (i.e., can be the world line of a particle) if and only if in all its points the angle between its tangent and the vertical (i.e., the line $u - v = \text{const}$) is $\leq 45^\circ$. Thus the set of all points from which signals can come to a point $p$ — this set is called the causal past of $p$ — is the down-directed angle with the vertex in $p$ and the sides parallel to the $u$- and $v$-axes. And the causal future of $p$, i.e., the set of all points at which $p$ can be seen, is the angle vertical to that.

We shall consider only the $(u, v)$-plane taking into account that the non-radial motion complicates the analysis without adding anything qualitatively new. So, by a “signal” or “motion”, etc., from now on we understand a radially propagating signal or “radial motion”, etc.

### 2.2 Schwarzschild and free-falling observers

To analyze the fall into the black hole let us consider two observers separating into a point $s$, see Fig. [1b]. One of them, let us label him $\alpha$, after the parting moves with constant $r, \phi$ and $\theta$. Such observers — we shall call them Schwarzschild — are at rest in the Schwarzschild coordinates, in which the metric does not depend on time. It would be quite untrue, however, to
regard Schwarzschild observers as “immobile” or, at least, inertial. Their world lines are not geodesics; so the observers experience an acceleration $a$, the fact well known (empirically) to the reader as all of us are, to high accuracy, Schwarzschild observers with $r = R$, in the metric with $m = M$, and each of us moves — in the instantaneously comoving system — with the acceleration $a \approx 9.8 \text{ m/s}^2$.

The second observer, $\gamma$, falls freely, i.e., his world line is a radial geodesic $\gamma(\tau)$ with $\dot{r}(0) \leq 0$. The most important fact about $\gamma$ is that at some moment $\tau_h < \infty$ of its proper time it unavoidably meets the horizon.

**Proof.** As follows from (5) the function $x(\tau)$ is convex. At the same time

$$x(0) > 1, \quad \dot{x}(0) \leq 0.$$ 

Hence, there is $\tau_h > 0$ such that $x(\tau_h) = 1$. So, we only have to prove that $\tau$ takes all values in $[0, \tau_h]$. In other words, the observer $\gamma$ must reach the horizon, if he lives long enough, and our task is to prove that he does not cease to exist before his clock shows $\tau_h$. Note that this follows neither from (5), nor from any general considerations: one could imagine, for example, that $\gamma$ approaches the horizon like $\mu$ in Fig.1 and leaves $M_*$ as $\tau \to a \in (0, \tau_h]$. To exclude such a possibility notice that as long as $\gamma$ stays in $M_*$ the coordinate $v$ on it obeys the following assessment

$$\ddot{v} = -(\ln F)_v \dot{v}^2 = (1 - x^{-2})v^{-1} \dot{v}^2 < v^{-1} \dot{v}^2, \quad (7)$$

where the first equality is the $v$-component of the geodesic equation and the second follows from the simple chain

$$(\ln F)_v = (\ln F)'(vu)_v/(vu)' = v^{-1} \ln' F/\ln' |vu| = v^{-1}(1 - x^2)/x^2,$$

in which we have made use of (2). Both $v$ and $\dot{v}$ are positive in $M_*$, and from (7) it follows immediately that

$$v(\tau) \leq c_1 e^{c_2 e^{c_1 \tau}},$$

where $c_{1,2}$ are some constants. Consequently, until $\gamma$ leaves $M_*$, $v(\tau)$ is bounded on any interval. \qed

Once $\gamma$ enters $M_*$, it cannot cross the horizon back and inevitably terminates at the singularity ($r \to 0$ as $\tau \to \tau_0 < \infty$). Note that the same is true
for any causal curve — geodesic or not — just because it has to stay within the right angle with the vertex in its (arbitrary) point and the sides parallel to the \( u \)- and \( v \)-axes.

All the abovesaid looks quite elementary. However, for the reasons discussed in the Introduction we should discuss in more detail two aspects of \( \gamma \)'s history.

### 3 The velocity at the horizon

It is common knowledge that an object similar to the black hole exists in Newtonian physics too. If a ball has a sufficiently large mass and small radius, the escape velocity \( V_e \) may equal the speed of light. But a body falling on such a ball — with the zero initial speed — from infinity would land just with \( V_e \). Perhaps, it is such reasoning that gave rise to a popular belief that a body crosses the horizon at the speed of light. Is it true?

At first glance — yes. Indeed, consider a family of observers \( \mathcal{N} \): each member \( n_\tau \) meets \( \gamma \) in the corresponding point — in \( \gamma(\tau) \) — and measures \( \gamma \)'s velocity in his, member’s, proper reference system. By the proper reference system we here understand a perfectly local and well-defined entity — an orthonormal tetrad in \( \gamma(\tau) \)

\[
\{ \mathbf{e}_{(i)}(\tau) \}, \quad i = 0, \ldots, 3,
\]

with the vector \( \mathbf{e}_{(0)} \) tangent to the world line of \( n_\tau \) (thus, instead of a family of observers we could speak about a tetrad field along \( \gamma \)). Denote now by \( \mathbf{v}(\tau) \) the 3-velocity of \( \gamma \) as measured by \( n_\tau \), i.e., found in the basis \( \{ \mathbf{e}_{(i)}(\tau) \} \).

If \( \mathcal{N} \) is chosen (at \( \tau < \tau_h \), of course) to be the set of Schwarzschild observers, then for a radial \( \gamma \) it can be shown, see [5, (102.7)], that

\[
|\mathbf{v}| = \sqrt{1 - \xi(x-1)/x},
\]

where \( \xi \) is a positive constant which depends on the choice of \( \gamma \). Thus

\[
|\mathbf{v}| \to 1 \quad \text{as} \quad \tau \to \tau_h - 0. \quad (8)
\]

It is this fact that is interpreted sometimes as attainment of the speed of light by a falling body and thereby as self-inconsistency of general relativity, see, e.g., [4].
The falseness in that interpretation is that $|v|$ is assumed to be continuous in $\tau$. In fact, however, the properties of $|v|$ depend heavily on the choice of $N$ (in this sense $v(\tau)$ characterizes $N$ rather than $\gamma$). And in the case under consideration, when $N$ cannot be complemented in a continuous way by an observer meeting $\gamma$ in $h$ (such an observer would have to move with the speed of light), the vector
\[ e_{(0)} = (v\partial_v - u\partial_u)/|v\partial_v - u\partial_u| = (v\partial_v - u\partial_u)/\sqrt{32m^2(1 - 1/x)} \]
has obviously no limit as $\tau \to \tau_h - 0$. So, $v(\tau)$ could have been continuous in $\tau_h$ only by a miracle. In other words, \(\Box\) does not imply $|v(\tau_h)| = 1$.

Actually the vector in $h$ tangent to $\gamma$ is timelike. This has nothing to do with relativity, or even with the metric under discussion, but follows from a fundamental geometric fact: a geodesic timelike in a point ($s$ in this case) is timelike in all points. Thus in any orthonormal basis (i.e., in a proper reference system of any observer located in $h$) $\gamma$ crosses the horizon moving slower than light.

4 What will the falling observer see?

Another widely met statement is “From the point of view [or ‘in the reference system’, or ‘as measured by the clock’] of a remote observer it takes infinite time for a body to reach the horizon”. In this section I argue that contrary to the first impression it is possible to give a meaning to that statement and even in three different ways (and, indeed, in the literature all three meanings can be met). The statement deserves a detailed analysis, because one of the three interpretations is simply wrong.

The problem, in essence, is that an observer’s clock measures the observer’s proper time $\tau$, i.e., for an observer with the world line $\alpha = x^i(\xi)$ it measures the quantity
\[ \tau(\xi) = \int_0^\xi \sqrt{g_{nk}\dot{x}^n\dot{x}^k} \, d\xi' \]
(the dot here is a derivative with respect to $\xi'$) and no reasonable way is seen to make the clock measure a time interval between events lying off $\alpha$ (in our case those are the events $s$ and $h$). Normally this causes no problems, because we can pick any coordinate system and measure the time by using
it. In doing so we need not bother to interpret the thus defined time as “true”, or “measured by the clock of this or that observer”. Hence, the first way to interpret the above-mentioned statement is to reformulate it: “It takes infinite Schwarzschild time for a body to reach the horizon”, or, more strictly (since the Schwarzschild coordinates do not cover $h$): “Between $s$ and $h$ there are events on $\gamma$ with arbitrarily large $t'$. The latter statement is trivially true, see Fig. 1.

Remark 3. Replacement of the words “in the Schwarzschild coordinates” with “from the point of view of a remote observer” is quite common. The point is that light signals sent by a Schwarzschild observer at regular intervals $\Delta \tau$ of his proper time are received by another Schwarzschild observer also at regular intervals $\Delta \tau'$ (whatever are the corresponding intervals $\Delta t$ are $\Delta t'$ of the coordinate time). Which enables one to “synchronize the clocks” throughout the entire $M_*$, i.e., to introduce the time coordinate by requiring that $\Delta t = \Delta t'$ (and that is how the Schwarzschild time can, indeed, be defined). The similarity of this procedure to that used for building a reference frame in special relativity (a purely illusive similarity, of course, since $\Delta \tau \neq \Delta \tau'$ even though the observers are “at rest” w. r. t. each other) can mislead one into the idea that the Schwarzschild coordinates are “more physical” than the others and, in particular, an interval of $t$ is exactly “the time by the clock of a remote observer”. It is this deeply non-relativistic idea that makes — actually simple — properties of $M$ look paradoxical. One such paradox has been already considered, another is considered below, and two more are presented in Fig. 2.

The fact by itself that two events are separated by an infinite coordinate interval is vapid: it is true for any pair of causally related events if the time coordinate is chosen appropriately. A more meaningful — since geometric — statement can be made if we turn from a relation between two events ($h$ and $s$) to a relation between an event and an observer ($h$ and $\alpha$, respectively), or two observers ($\alpha$ and $\gamma$). Indeed, notice that the whole $\alpha$ lies in the causal future of the segment of $\gamma$ bounded by $s$ and $h$. Physically this means that all his — infinite — life $\alpha$ will be able to receive signals sent by his falling comrade before the latter reached the horizon. $\alpha$ may interpret this fact in two ways depending on which spacelike surfaces he chooses as the surfaces of

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6 Because $M_*$ is static.
7 The signals though must be sent more and more frequently and with more and more blue photons.
Figure 2: (a). The waist of a free falling victim must cross the horizon at some moment. But his head at that moment still makes a regular observer (it moves slower than light w. r. t. a Schwarzschild observer). So, does it mean that in “the head’s reference system” the feet have not yet reached the horizon? (b). “From the point of view” of a Schwarzschild observer with the coordinate $r_0$ his distance to the horizon is $\Delta = \int_{2m}^{r_0} (1 - 2m/r)^{-1/2} dr$. Evidently $\Delta \to 0$ as $r_0 \to 2m$ and hence at some $r_0$ he will be only, say, 10 cm far from it. What can prevent the observer from simply stretching a hand and touching the horizon?

1. If the surfaces are more or less horizontal in Fig. 1 (for instance, events are regarded simultaneous if they have the same value of $u + v$), then $\alpha$ will find nothing unusual in receiving the messages from $\gamma$. Exactly as we speak of the light of a distant star coming to us for years after the star died, $\alpha$ could say that due to a huge — and growing — delay he keeps receiving signals from $\gamma$ centuries after the latter actually traversed the horizon.

2. One can choose, however, the simultaneity surfaces to be more and more tilted (cf. the surfaces $t = \text{const}$ in Fig. 1). This — rather exotic — choice would mean that $\alpha$ considers the information received with every signal as more and more fresh. And he would be quite consistent claiming that the fall is infinitely long.

There is, however, another — opposite, in a sense — approach to what should be called infinitely long by a remote observer’s clock. Imagine a point
$p$ which contains the entire $\alpha$ in its causal past. An observer $\omega$, if his world line passes through $p$, would see the entire (infinite) history of $\alpha$: he will see $\alpha$ aging, his sun swelling and reddening, its protons decaying, etc. In his turn, $\alpha$ would be able at any moment to send a message to $\omega$ (though maybe not to receive a response). All in all it would be quite legitimate to say that $\omega$ reaches $p$ in infinite time by $\alpha$’s clock.

In this just formulated sense the statement that $\gamma$’s falling time is infinite is wrong. Indeed, as is seen from Fig. [1]b, $\alpha$ — like any other Schwarzschild observer — leaves the causal past of $h$ and of the entire $\gamma$, too. So, the falling observer will not see the entire future of the Universe. Moreover, he will see nothing at all beyond the shadowed region in Fig. [1]b. In particular, the last event in $\alpha$’s life observed by $\gamma$ before the latter submerges into $\mathcal{M}_*$, is $d$.

Remark 4. In Reissner-Nordström and Kerr black holes under their event horizons (which are quite similar to Schwarzschild’s) there is another remarkable surface — the Cauchy horizon. And that horizon does have the property in discussion: an astronaut falling into the black hole reaches the Cauchy horizon in a finite proper time and crosses it in a point $p$ that contains in its causal past the whole “external universe”. Such an astronaut, indeed, will be able to see the death of stars and galaxies, see, e. g., [3].

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