NEW RESULTS ON DOUBLY ADJACENT PATTERN-REPLACEMENT EQUивALENCES

WILLIAM KUSZMAUL

Abstract. In this paper, we consider the family of pattern-replacement equivalence relations referred to as the "indices and values adjacent" case. Each such equivalence is determined by a partition $P$ of a subset of $S_c$ for some $c$. In 2010, Linton, Propp, Roby, and West posed a number of open problems in the area of pattern-replacement equivalences. Five, in particular, have remained unsolved until now, the enumeration of equivalence classes under the $\{123, 132\}$-equivalence, under the $\{123, 321\}$-equivalence, under the $\{123, 132, 213\}$ equivalence, and under the $\{123, 132, 213, 321\}$-equivalence. We find formulas for three of the five equivalences and systems of representatives for the equivalence classes of the other two. We generalize our results to hold for all replacement partitions of $S_3$, as well as for an infinite family of other replacement partitions. In addition, we characterize the equivalence classes in $S_n$ under the $S_c$-equivalence, finding a generalization of Stanley’s results on the $\{12, 21\}$-equivalence.

To do this, we introduce a notion of confluence that often allows one to find a representative element in each equivalence class under a given equivalence relation. Using an inclusion-exclusion argument, we are able to use this to count the equivalence classes under equivalence relations satisfying certain conditions.
1. INTRODUCTION

In this paper, we consider the family of pattern-replacement equivalence relations referred to as the "indices and values adjacent" case in [5]. Each such equivalence is determined by a partition $P$ of a subset of $S_c$ for some $c$. Before introducing our results, we provide background definitions.

**Definition 1.1.** A word is a series of letters, where each letter has an integer value.

*Example:* The word 13264 has letters 1, 3, 2, 6, and 4.

**Definition 1.2.** A Permutation of size $n$ is an $n$-letter word using each letter from 1 to $n$ exactly once.

*Example:* 14235 is a permutation of size 5.

**Definition 1.3.** $S_n$ denotes the set of permutations of size $n$.

*Example:* $S_3 = \{123, 132, 213, 231, 312, 321\}$.

**Definition 1.4.** The word $w = w_1w_2\cdots w_c$ forms the permutation $u = u_1u_2\cdots u_c \in S_c$ if for some integer $k$,

$$w_1w_2\cdots w_c = (u_1 + k)(u_2 + k)\cdots (u_c + k).$$

As shorthand, we may just say that $w$ forms $u$.

*Example:* 7968 forms the permutation 2413. An even simpler example is that 2413 forms the permutation 2413.

**Definition 1.5.** A replacement partition is a partition of a subset of $S_c$ for some $c$. Each replacement partition $P$ determines an equivalence relation which we refer to as the $P$-equivalence.

**Definition 1.6.** A hit is a contiguous subword of a permutation that forms $u$ for some $u \in S_c$. A $P$-hit is a contiguous subword of a permutation that forms a permutation in a partition $P$ of $S_c$.

*Example:* The subword 7968 of the permutation 157968324 forms the permutation 2413.

**Definition 1.7.** Given a pattern-replacement partition $P$, we define the $P$-equivalence on $S_n$ in the following manner. Given a $P$-hit $h$ in a permutation $w$ that forms the permutation $t \in P$, we are allowed to rearrange the letters within $h$ in any way such that the resulting word forms some $t'$ in the same part of $P$ as $t$. A class, or equivalence class, containing $w$ is the set of permutations that can be reached from $w$ by repeated rearrangements of the type just described. If two permutations $a$ and $b$ are in the same equivalence class, we say that $a$ is equivalent to $b$, or that $a \equiv b$.

**Definition 1.8.** Let $P$ be a replacement partition. Let $h$ be a $P$-hit in a permutation $w$ such that $h$ forms a permutation in the $i$th part of $P$. Rearranging the letters in $h$ in any way such that $h$ still forms a permutation in the $i$th part of $P$ is referred to as a $P$-rearrangement.

*Example:* Let $w = 1432657$. Then applying a $\{123, 213, 321\}\{312, 231\}$-rearrangement to the hit 432 allows us to rearrange the hit either as 234 or as 324. This results in the permutations 1234657 and 1324657 respectively.

**Definition 1.9.** The $P$-equivalence is the reflexive-transitive closure of $P$-rearrangement. In other words, the permutations that can be reached from a permutation $w$ by means of repeated $P$-rearrangements are considered equivalent to $w$ under the $P$-equivalence. The set of permutations equivalent to $w$ comprises an equivalence class.

*Example:* In Figure 1 we show a visual depiction of the equivalence class containing 123456 under the $\{123, 321\}$-equivalence. Each of the above permutations are considered equivalent under the relation. Two permutations are connected by a line segment if one can be reached from
the other by the rearrangement of a single \{123, 321\}-hit. For example, 125436 connects to 123456 because 543 forms the permutation 321 and thus can be rearranged to form the permutation 321, becoming 345.

We will refer to the type of equivalence relations studied in this project as \textbf{pattern-replacement} equivalence relations. In the literature, they are referred to either as doubly adjacent pattern-replacement equivalence relations or as the “indices and values adjacent” case.

In 2010, Linton, Propp, Roby, and West posed a number of open problems in the area of pattern-replacement equivalences \cite{Linton2010}. Five, in particular, have remained unsolved until now, the enumeration of equivalence classes under the \{123, 132\}-equivalence, under the \{123, 321\}-equivalence, under the \{123, 132, 213\} equivalence, and under the \{123, 132, 213, 321\}-equivalence.

In Section 3, we find formulas for three of the five equivalences and systems of representatives for the equivalence classes of the other two. We generalize our results to hold for all replacement partitions of \(S_3\), as well as for an infinite family of other replacement partitions.

In order to do this, we first introduce the notion of \((P,C)\)-confluence in Section 2; when \((P,C)\)-confluence is satisfied by a partition \(P\), one can easily find a set of representative permutations for the equivalence classes in \(S_n\) under the \(P\)-equivalence. We construct a set of tools allowing us to often quickly prove that a pattern-replacement equivalence satisfies this confluence. Finally, we use the prior results in the section to present a formula for the number equivalence classes in \(S_n\) under pattern-replacement equivalences satisfying certain conditions.

In Section 4, we use our results to completely characterize the equivalence classes in \(S_n\) under the \(S_c\)-equivalence. Our results connect interestingly to Stanley’s results on the \{12, 21\}-equivalence \cite{Stanley1984}.

For this entire paper, fix \(n\) and \(c\) be positive integers such that \(c \leq n\). We will be considering equivalence classes of \(S_n\) under doubly adjacent pattern-replacement equivalences involving patterns of size \(c\).

Before continuing, we provide some useful background on the notion of a binary relation being \textbf{confluent}.

\textbf{Definition 1.10.} Let \(\rightarrow\) be a binary relation on a finite set \(A\) and \(\rightarrow'\) be the reflexive-transitive closure of \(\rightarrow\). We say that \(\rightarrow\) is \textbf{confluent} if

- there is no \(a, b \in A\) such that \(a \rightarrow' b\) and \(b \rightarrow' a\);
- for each connected component \(C\) of \(\rightarrow\) considered as a directed graph (with \(A\) as the set of nodes), there is a unique minimal node \(m \in C\). (i.e., there is a unique node \(m \in C\) such that for all \(c \in C\) with \(c \neq m\), we have that \(c \rightarrow' m\).)
The following lemma, known as the Diamond Lemma, is well known, and is easy to prove.

**Lemma 1.11.** Let \( \rightarrow \) be a binary relation on a finite set \( A \) such that \( a \rightarrow a \) is false for \( a \in A \). Assume the following is true.

1. There does not exist an infinite (possibly repeating) sequence \( a_1, a_2, \ldots \) such that for all \( i, \ a_i \in A, \) and \( a_i \rightarrow a_{i+1} \). In other words, \( \rightarrow \) terminates.
2. For all \( a, b, c \in A \) such that \( a \rightarrow b \) and \( a \rightarrow c \), there exists \( d \in A \) such that \( b \rightarrow d \) and \( c \rightarrow d \)

Then, \( \rightarrow \) is confluent.

## 2. General Results on a New Type of Confluence

In this section, we introduce the notion of a replacement partition \( P \) being \((P,C)\)-confluent. When a partition is \((P,C)\)-confluent, it is easy to characterize a set of permutations in \( S_n \) exactly one of which is in a given equivalence class in \( S_n \) under the \( P \)-equivalence. Such a permutation is referred to as a \((P,C)\)-root permutation. We provide three results which together allow one to often determine quickly that a pattern-replacement equivalence satisfies \((P,C)\)-confluence (Proposition 3.1, Theorem 2.16, Theorem 2.17). We then present a formula for the number of equivalence classes in \( S_n \) under the \( P \)-equivalence when \( P \) is \((P,C)\)-confluent and satisfies certain conditions (Theorem 2.18).

**Definition 2.1.** Let \( w \) be a word, each letter of which has a distinct value. The **tail size** of \( w \) is the smallest positive integer \( k \) such that the first \( k \) letters of \( w \) contain the \( k \) smallest letter values in \( w \).

**Example:** The tail size of 14238576 is 4 because 1423 contains 1, 2, 3, and 4.

**Definition 2.2.** Let \( w \) be a word, each letter of which has a distinct value.

- \( w \) is **right leaning** if the tail size of \( w \) is less than \( n \).
- \( w \) is **left leaning** if the tail size of a written backwards version of \( w \) is less than \( n \).
- \( w \) is **omni leaning** if it is neither left leaning nor right leaning.

**Example:** 14238576 is right leaning because its tail size is 4. Consequently, writing it backwards to get 67583241 gives us a left leaning permutation.

**Example:** One can check that 4213 is omni leaning.

The sets of right leaning, left leaning, and omni leaning permutations in \( S_n \) respectively are denoted by \( R_n, L_n, \) and \( O_n \). Note that the three sets do not intersect.

**Definition 2.3.** Let \( w \in S_n \), and \( h \) be a hit in \( w \). If \( h \) comprises all of \( w \) we say \( w \) is **left polarized**. Otherwise, let \( a \) be the average value of the letters in \( h \), and let \( w' \) be \( w \) except with the entire hit \( h \) replaced by a single letter whose value is \( a \). Let \( b \) be the letter in \( w' \) closest to \( a \) in value out of the (either one or two) letters immediately adjacent to \( b \) in \( w' \). If there is a tie, than pick the one on \( b \)'s left, although one can check it does not matter which we pick.

If \( a \) and \( b \) are in increasing order in \( w' \), then \( h \) is **right polarized**. If \( a \) and \( b \) are in decreasing order in \( w' \), then \( h \) is **left polarized**.

**Example:** The hit 456 in the permutation \( w = 1456237 \) is left polarized. Here, \( a = \frac{4+5+6}{3} = 5 \) and thus \( w' = 15237 \). The two letters immediately adjacent to \( a \) in \( w' \) are 1 and 2. Since 2 is closer to 5 than is 1, \( b = 2 \). In \( w' \), \( a \) and \( b \) are in decreasing order, meaning \( h \) is left polarized.

**Definition 2.4.** A hit \( h \) is **backward** in a permutation \( w \) if \( h \) either

- is both right polarized and left leaning,
- or is both left polarized and right leaning,
Then straightening any two hits of \( P \) is as described in Proposition 2.10.

\( \text{Definition 2.5.} \) Let \( P \) be a partition of a subset \( S \) of \( S_\varepsilon \). Let \( C \subseteq S \). Then \( C \) is a \textbf{straightening set} of \( P \) if

- for each part of \( P \) containing at least one left leaning permutation, \( C \) contains exactly one left leaning permutation from that part;
- for each part \( C \) of \( P \) containing at least one right leaning permutation, \( C \) contains exactly one right leaning permutation from that part;
- and for each part of \( P \) containing only omni leaning permutation, \( C \) contains exactly one permutation from that part.

\textbf{Example:} \{12345, 21435, 54231\} is a straightening set of \{12345, 12354\} \{21435, 54231, 43251\}.

\( \text{Definition 2.6.} \) Let \( C \) be a straightening set of a partition \( P \). A \( P \)-hit \( h \) in \( w \in S_n \) is \( (P,C)\text{-straightened} \) if either

- \( h \) is forward and is in \( C \),
- or \( h \) is backward and is in \( C \), and there is no \( P \)-rearrangement that when applied to \( h \) results in \( h \) being forward.

In turn, \( (P,C)\text{-straightening} \) a \( P \)-hit \( h \) is defined as rearranging a \( P \)-hit to be \( (P,C)\)-straightened by means of a \( P \)-rearrangement.

\textbf{Example:} Let \( P = \{123, 213, 321\} \) and \( C = \{123, 321\} \). Consider the permutation 85671234. The hit 567 is right leaning but left polarized and is thus backwards. Consequently, \( (P,C)\)-straightening it rearranges it to be the forward hit 765, bringing us to the permutation 87651234. On the other hand, \{\{123, 213, 321\}, \{123, 321\}\}-straightening the hit 765 in this new permutation does nothing at all.

\( \text{Definition 2.7.} \) Let \( P \) be a partition of a subset of \( S_\varepsilon \) and \( C \) be a straightening set of \( P \). Let the binary relation \( \rightarrow \) on \( S_n \) be such that for \( w, w' \in S_n \), \( w \rightarrow w' \) exactly when \( w' \) can be reached from \( w \) by \( (P,C)\)-straightening a hit in \( w \) that was not already \( (P,C)\)-straightened. We refer to \( \rightarrow \) as the \( (P,C)\text{-straightening operator} \).

\( \text{Definition 2.8.} \) Let \( P \) be a partition of a subset of \( S_\varepsilon \) and \( C \) be a straightening set of \( P \). The partition \( P \) is said to be \( C\text{-confluent} \) if the \( (P,C)\)-straightening operator is confluent.

\( \text{Definition 2.9.} \) Let \( P \) be \( C \)-confluent. A permutation \( w \) is said to be a \( (P,C)\text{-root permutation} \) if it is the unique permutation in its equivalence class under the \( P \)-equivalence such that it can be reached from any other equivalent permutation by means of repeated \( (P,C)\)-straightenings.

\textbf{Example:} It turns out that \{123, 321\} is \{321, 123\}-confluent. Figure 3 demonstrates this for a particular equivalence class in \( S_4 \). An arrow between two permutations means one can be reached from the other under the \{123, 321\}-equivalence by a \{\{123, 321\}, \{321, 123\}\}-straightening. Observe how all paths lead to the permutation 123456 which is consequently the \{\{123, 321\}, \{321, 123\}\}-root permutation in the class.

\( \text{Proposition 2.10.} \) Let \( P \) be a partition of \( R_\varepsilon \) with parts \( P_1, P_2, \ldots \). Let \( C \) be the straightening set of \( P \) containing the lexicographically smallest permutation from each part of \( P \). Suppose that for any two hits \( h \) and \( h' \) in any permutation \( w \in S_n \) such that \( h' \) forms a permutation in \( P_r \), \( (P,C)\)-straightening \( h \) leaves the letters in \( h' \) still in a contiguous subword forming a permutation in \( P_r \). Then \( P \) is \( C\)-confluent.
Proof. Let $\rightarrow$ be the $(P, C)$-straightening operator. Since $a \rightarrow b$ implies that $b$ is lexicographically smaller than $a$, $\rightarrow$ terminates. Given $w \in S_n$, let $D_j(w)$ be the permutation $w'$ reached by $(P, C)$-straightening the $P$-hit beginning with $w_j$ in $w$ if such a $P$-hit exists and $w$ otherwise. Note that $D_j(D_j(w)) = D_j(D_j(D_j(w)))$ due to the restrictions we have imposed. Thus by the Diamond lemma, $P$ is $C$-confluent. \hfill \Box

We state the following lemmas without proof for the sake of brevity.

Lemma 2.11. Let $w \in S_n$ and $h$ and $h'$ be overlapping hits of size $c$ in $w$. Then, $h$ and $h'$ are both forward and either are both left leaning or are both right leaning.

Lemma 2.12. Let $w$ and $w'$ be equivalent under the $P$-equivalence where $P$ partitions a subset of $S_c$. Suppose $w_i < w_j$ for $|i - j| \geq c$. Then, $w_i' < w_j'$.

Lemma 2.13. Let $w, w' \in S_n$ be such that $w'$ can be reached from $w$ by a $P$-rearrangement (where $P$ partitions a subset of $S_c$). Let $i$ and $j$ be such that $w_j$ is not in $h$. Then, $(w_i < w_j) = (w_i' < w_j')$.

Now we introduce a slightly less obvious Lemma.

Lemma 2.14. Let $U$ be a $D$-confluent partition. Then

1. and $(U, D)$-straightening any forward hit in a permutation results in a permutation with the same number of $U$-hits that are both backward and non-$(U, D)$-straightened.
2. $(U, D)$-straightening a non-$(U, D)$-straightened backward hit in a permutation results in a permutation with one fewer $U$-hits that are both backward and non-$(U, D)$-straightened;

Proof. Observe that (a) a backward $U$-hit cannot overlap any other $U$-hit (by Lemma [2.11]), and (b) $(U, D)$-straightening one $U$-hit cannot change whether another $U$-hit is right polarized or left polarized (by Lemma [2.13]).

By (a) and (b), $(U, D)$-straightening a forward $U$-hit in a permutation does not change the set of backward $U$-hits in the permutation. Also by (a) and (b), $(U, D)$-straightening a backwards non-$(U, D)$-straightened hit cannot rearrange another $(U, D)$-straightened backward hit to become non-$(U, D)$-straightened. \hfill \Box

Definition 2.15. Let $J, K, P$ be partitions of $A, B, C \subseteq S_c$ respectively such that $A \cap B = \{\}$ and $A \cup B = C$. We say that $P$ is a disjoint union of $J$ and $K$ if each part of $P$ is a disjoint union of some (possibly empty) part from $J$ and some (possibly empty) part from $K$.

Example: $\{1234, 2134, 3214\} \{4123\} \{2134, 2314\}$ is a disjoint union of $\{1234\} \{4123\} \{2134\}$ and $\{2134, 3214\} \{2314\}$. 

\[ \begin{array}{c}
143256 \\
\quad \\
123456 \\
\quad \\
125436 \\
\quad \\
321456 \\
\quad \\
321654 \\
\quad \\
\end{array} \]

Figure 2. Demonstration that $\{123, 321\}$ is $\{321, 123\}$-equivalent for equivalence class containing 123456.
**Theorem 2.16.** Let $P$ be a $C$-confluent partition of $R_c$ and $Q$ be a $C'$-confluent partition of $L_c$. Let $D = C \cup C'$ and $U$ be a disjoint union of $P$ and $Q$. Then $U$ is $D$-confluent.

**Proof.** Let $\rightarrow$ acting on $S_n$ be the $(U, D)$-straightening operator and $\rightarrow'$ be the reflexive-transitive closure of $\rightarrow$. Since $D$ is clearly a straightening set of $U$, it is sufficient to show that $\rightarrow$ is confluent. To do this, we will use the Diamond Lemma.

We begin by proving that $\rightarrow$ terminates. Suppose there exists an infinite chain $a_1 \rightarrow a_2 \rightarrow \ldots$. For each $i$, let $h_i$ be a $U$-hit of $a_i$. The $(U, D)$-straightening of which brings one from $a_i$ to $a_{i+1}$. Without loss of generality, an infinite number of $i$ such that $h_i$ is right leaning in $a_i$ (since otherwise, an infinite number of $i$ would satisfy that $h_i$ is left leaning).

Let $k$ be the number of backward $U$-hits in $a_1$. By Lemma 2.14 there are at most $k$ values of $i$ such that $h_i$ is backward in $a_i$. Let $j$ be the greatest such $i$.

Observe that for $i > j$, $h_i$ is forward in $a_i$. By Lemma 2.11, the action of $(U, D)$-straightening a right leaning $U$-hit commutes with the action of $(U, D)$-straightening a left leaning $U$-hit. Recall that there exists an infinite number of $i$ such that $h_i$ is right leaning in $a_i$, and thus also an infinite number of such $i$ where $i > j$. It follows that for a given value of $l$, there exists an infinite chain $b_1 \rightarrow b_2 \rightarrow \ldots$ such that $b_{i+1}$ is reached from $b_i$ by means of the $(U, D)$-straightening of a forward right leaning $U$-hit. However, since such hits are also $P$-hits and $P$ is $C$-confluent, the pigeonhole principle implies this cannot be true for $l > n!$, a contradiction.

We will now consider condition (2) of the Diamond Lemma. Suppose $a, b, c \in S_n$ are such that $a \rightarrow b$, $a \rightarrow c$, $a$ reaches $b$ by means of the $(U, D)$-straightening of the $U$-hit $h$, and $a$ reaches $c$ by means of the $(U, D)$-straightening of the $U$-hit $h'$. If $h$ and $h'$ do not overlap in $a$, then $(U, D)$-straightening $h$ and then $(U, D)$-straightening $h'$ in $a$ yields the same permutation as does $(U, D)$-straightening $h'$ and then $(U, D)$-straightening $h$. Suppose instead that $h$ and $h'$ overlap. By Lemma 2.11, we can assume without loss of generality that both are forward and right leaning. By the assumption that $P$ is $C$-confluent, it follows that there exists $d$ such that $b \rightarrow' d$ and $c \rightarrow' d$.

The result follows from the Diamond Lemma. □

We now show that omni leaning permutations can be nicely inserted into arbitrary parts of partitions without screwing up the confluence properties of the partition.

**Theorem 2.17.** Let $P$ be a $C$-confluent partition of a subset $S$ of $S_c$. Let $o \in O_c$ such that $o$ is not in $S$.

1. Let $P'$ be $P$ except with $o$ added to one of the parts of $P$. Then $P'$ is $C$-confluent.
2. Let $P'$ be $P$ except with $o$ added as a part of size 1. Then $P'$ is $C \cup \{o\}$-confluent.

**Proof.** We only bother to prove (1), as (2) is straightforward. Suppose $P'$ is $P$ except with $o$ added to one of the parts of $P$. Let $\rightarrow$ acting on $S_n$ be the $(P', C)$-straightening operator. We will show that $\rightarrow$ satisfies the conditions of the Diamond Lemma.

Any $P'$-hit forming $o$ is backwards. Thus, by Lemma 2.14 if there exists an infinite chain $a_1 \rightarrow a_2 \rightarrow \cdots$, then there is some $k$ such that $a_k \rightarrow a_{k+1} \rightarrow \cdots$ and no $P'$-hits forming $o$ are used to go from $a_i$ to $a_{i+1}$ in the chain for $i \geq k$. But since $P$ is $C$-confluent, no such chain can exist. Thus $\rightarrow$ terminates.

Observe that a $P'$-hit forming $o$ cannot overlap any other $P'$-hit (Lemma 2.11). Since $P$ is $C$-confluent, it follows that condition (2) of the Diamond Lemma is satisfied by $\rightarrow$. Thus $\rightarrow$ is confluent. □
Theorem 2.18. Let \( P \) be a \( C \)-confluent partition of \( S_n \). Suppose that for \( P \)-hits \( h \) and \( h' \) in \( w \in S_n \), to overlap implies that at least one of \( h \) and \( h' \) is \((P,U)\)-straightened. Pick \( n \in \mathbb{N} \) and let \( k = n! - |P| \) where \( |P| \) is the number of parts in \( P \). Then the number of equivalence classes in \( S_n \) under the \( P \)-equivalence, which we will denote as \( f(n) \), is

\[
\sum_{j \geq 0} (-1)^j (n - c + j)!^2 k^j / j!(n - cj)!
\]

Proof. Let \( T_i \) be the number of \( w \in S_n \) such that \( w \) contains a \( P \)-hit \( h \) such that the first letter of \( h \) is in position \( i \) in \( w \) and \( h \) is not \((P,C)\)-straightened in \( w \). By the inclusion-exclusion principle,

\[
f(n) = n! + \sum_{j > 0} (-1)^j \sum_{S \subseteq [n], |S| = j} \left| \bigcap_{i \in S} T_i \right|.
\]

Note that for any \( P \)-hit \( h \) in \( w \in S_n \) such that \( h \) is not \((P,C)\)-straightened, there are exactly \( n! - |P| \) possibilities for \( h \) given the position of its first letter, the value of its smallest letter, and whether \( h \) is right polarized or left polarized. Let us calculate the number of ways to construct \( j \) non-overlapping \( P \)-hits and place them in a permutation so that none of them are \((P,C)\)-straightened.

Let \( \Gamma \) be the set containing exactly the \( j \) letters that will each be the smallest letter in one of our \( j \) \( P \)-hits. We claim there are \( (n - j(c - 1)) \) possibilities for \( \Gamma \). Since no two \((P,C)\)-straightened \( P \)-hits can overlap, the possibilities are exactly the sets of \( j \) integers from 1 to \( n - c + 1 \) such that no two of them are within \( c - 1 \) of each other in value. These sets, in turn, are in bijection with binary words containing \( n - j(c - 1) \) letters, \( j \) of which are ones. Indeed, given such a word \( w \), we may substitute each 1 with the letter 1 followed by \( c - 1 \) copies of the letter 2, bringing us to \( w' \). We then choose to stick the letter \( i \) in \( \Gamma \) exactly when the \( i \)th letter of \( w' \) is 1. The inverse bijection is easy to see.

Given \( \Gamma \), the relative positions of each of the first letters of each of the \( j \) \( P \)-hits along with the letters not in any of the \( j \) \( P \)-hits can be chosen in \( (n - j(c - 1))! \) ways. This, in turn, determines which hits are right polarized and left polarized. Given this choice, the arrangement of each individual hit can be chosen from \( n! - |P| \) possibilities. Hence

\[
\sum_{S \subseteq [n], |S| = j} \left| \bigcap_{i \in S} T_i \right| = \left( n - j(c - 1) \right)^j \left( n - j(c - 1) \right)! \left( n! - |P| \right)^j.
\]

Inserting this into the formula found by the inclusion-exclusion principle and slightly rearranging yields the desired result. \( \square \)

3. Some Immediate Applications of Our Results

In this section, we apply our results from Section 2 to all replacement partitions of \( S_3 \) (Theorem 3.1), on the way solving three open problems of Linton, Propp, Roby, and West [5]. We then apply our results to a particularly interesting infinite family of pattern-replacement equivalences (Theorem 3.3).

Theorem 3.1. (A) Let \( P \) be a partition of \( S_3 \). Then there exists a straightening set \( C \) of \( P \) such that \( P \) is \( C \)-confluent. (B) Furthermore, if 123, 132, and 213 are not all in the same part of \( P \) and 321, 312, and 231 are not all in the same part of \( P \), then the number of equivalence classes in \( S_n \) under the \( P \)-equivalence can be established with Theorem 2.18.
Proof. By Proposition 2.18, \( \{123, 132\}\{213\} \) is \( \{123, 213\}\)-confluent, \( \{123, 213\}\{132\} \) is \( \{123, 132\}\)-confluent, \( \{213, 132\}\{123\} \) is \( \{213, 123\}\)-confluent, and \( \{123, 213, 132\} \) is \( \{123\}\)-confluent. (These are all the partitions of \( R_3 \); note that \( O_3 \) is empty.) Note that only in the final of these cases does the partition not satisfy the requirements for Theorem 2.18 to be applied. By symmetry, similar statements can be said for each partition of \( L_3 \). Since every partition of \( S_3 \) is the disjoint union of a partition of \( L_3 \) and a partition of \( R_3 \), (A) follows from Theorem 2.18. Since two hits of size three of size 3 in a permutation \( w \) can only overlap if either both are in \( L_3 \) or both are in \( R_3 \) (Lemma 2.11), (B) follows as well. \( \Box \)

Remark 3.2. Note that Theorem 3.1 resolves three open problems, the enumeration of equivalence classes under each of the \( \{123, 132\}\)-equivalence, the \( \{123, 321\}\)-equivalence, and the \( \{123, 132, 321\}\)-equivalence. It also makes progress on two additional open problems, the enumerations of equivalence classes under the \( \{123, 132, 213\}\)-equivalence and \( \{123, 132, 213, 321\}\)-equivalence; in each case, Theorem 3.1 allows one to characterize a set of permutations \( (P,C)\)-root permutations), exactly one of which appears in each equivalence class in \( S_n \).

Theorem 3.3. Let \( a_1, a_2, a_3, \ldots, a_k \in L_c \cup O_c \) and \( b_1, b_2, b_3, \ldots, b_k \in R_c \cup O_c \) such that \( a_i \neq b_j \) for all \( i, j \). Let \( P \) be the partition \( \{a_1, b_1\}\{a_2, b_2\}\cdots\{a_k, b_k\} \). The number of equivalence classes in \( S_n \) under the \( P\)-equivalence is

\[
\sum_{j \geq 0} (-1)^j (n-c+j)!^2 k_j^\frac{1}{j! (n-c)!}.
\]

Proof. It is trivial that \( \{a_1\}\{a_2\}\{a_3\} \cdots \{a_k\} \) is \( \{a_1, a_2, a_3, \ldots, a_k\}\)-confluent and that \( \{b_1\}\{b_2\}\{b_3\} \cdots \{b_k\} \) is \( \{b_1, b_2, b_3, \ldots, b_k\}\)-confluent. By Theorem 2.16, \( \{a_1, b_1\}\{a_2, b_2\}\cdots\{a_k, b_k\} \) (which we will refer to as \( P \)) is \( \{a_1, b_1, a_2, b_2, a_3, b_3, \ldots, a_k, b_k\}\)-confluent. Observe that if two \( P\)-hits overlap, they either form \( a_i \) and \( a_j \) for some \( i, j \), or they form \( b_i \) and \( b_j \) for some \( i, j \) (by Lemma 2.11). In either case, both \( P\)-hits are forward and thus \( (P, \{a_1, b_1, a_2, b_2, a_3, b_3, \ldots, a_k, b_k\})\)-straightened. Thus, pretending that all the permutations in \( S_c \) not in \( P \) are each in a part of size one of \( P \), we can apply Theorem 2.18 to obtain the desired formula. \( \Box \)

Remark 3.4. Observe that Theorem 3.3 counts equivalence classes in \( S_n \) under the \( P \) equivalence for at least \( \sum_j \binom{c+j}{j}\frac{1}{j!} \) choices of \( P \subseteq S_c \) for a given \( c \). This is easy to see by allowing only half of the omni leaning permutations in \( S_c \) to be assigned to the \( a_i \) and allowing only the other half to be assigned to the \( b_i \) (so the choices for \( a_i \) come from exactly half of the permutations in \( S_c \), and similarly for \( b_i \)).

Remark 3.5. It should be surprising that equivalence relations with different class structures often have the same number of classes in \( S_n \).

For example, by Theorem 3.1, for all partitions \( P \) of \( S_3 \) of the form \( \{a, b\}\{c, d\}\{e\}\{f\} \) the \( P \)-equivalence yields the same enumeration. Yet it is easy to see that the structure of the equivalence classes created under such equivalence differs greatly between equivalence relations.

In fact, Theorem 3.3 provides an infinite number of examples of this phenomenon.

4. THE \( S_c \)-EQUIVALENCE

One infinite family of equivalence relations in particular, is worth studying in greater depth. The \( S_c \) equivalence has strictly fewer equivalence classes than any equivalence defined using another replacement partition of \( S_c \). Consequently, enumerating the equivalence classes under the
$S_c$-equivalence is an important task. In this section, we make progress on this by completely characterizing the equivalence classes. In the case of the $\{12, 21\}$-equivalence ($c = 2$), Stanley [3] enumerated and characterized the equivalence classes in $S_n$ and found representative elements for each one. Surprisingly, in our work, we will find a set of representative elements (the $(S_c, \{123 \cdots c, c \cdots 321\})$-root permutations) that in the case of the $\{12, 21\}$-equivalence differ from the already found one.

**Definition 4.1.** We will refer to the permutation $123 \cdots c$ as $\alpha$ and to the permutation $c(c-1)(c-2) \cdots 1$ as $\hat{\alpha}$.

**Example:** When $c = 3$, the $S_c$-equivalence is the $\{123, 132, 213, 231, 312, 321\}$-equivalence, $\alpha = 123$, and $\hat{\alpha} = 321$.

**Lemma 4.2.** For any two $S_c$-hits $h$ and $h'$ in a permutation $w \in S_n$ such that $h'$ forms a permutation in $R_c$, $(R_c, \{\alpha\})$-straightening $h$ leaves the letters in $h'$ still forming a permutation in $R_c$.

**Proof.** This is because rearranging the letters of a contiguous subword of a right leaning $S_c$-hit to be in increasing order will still yield a right leaning $S_c$-hit.

**Lemma 4.3.** For any two $S_c$-hits $h$ and $h'$ in a permutation $w \in S_n$, $(S_c, \{\alpha, \hat{\alpha}\})$-straightening $h$ leaves the letters in $h'$ still forming a permutation in $S_c$.

**Proof.** By lemma [2.11], we only need to consider the case where $h$ and $h'$ are either both left leaning or both right leaning. Without loss of generality, we can assume they are both right leaning (and since they overlap, are both in $R_c$). The Lemma thus follows from Lemma [4.2].

**Theorem 4.4.** The $S_c$-equivalence is $(\alpha, \hat{\alpha})$-confluent.

We provide two proofs of this.

**Proof.** By Lemma 4.2 and Proposition 3, the $R_c$-equivalence is $\{\alpha\}$-confluent. By symmetry, the $L_c$-equivalence is $\{\hat{\alpha}\}$-confluent. It follows from Theorem 2.16 that the $(L_c \cup R_c)$-equivalence is $\{\alpha, \hat{\alpha}\}$-confluent. It follows from Theorem 2.17 that the $S_c$-equivalence is $\{\alpha, \hat{\alpha}\}$-confluent.

The following alternative proof is an interesting application of the Diamond Lemma.

**Proof.** Let $\rightarrow$ be the $(S_c, \{\alpha, \hat{\alpha}\})$-straightening operator.

Observe that $(S_c, \{\alpha, \hat{\alpha}\})$-straightening a $S_c$-hit in a permutation $w$ does not take any other $S_c$-hits that were $\{\alpha, \hat{\alpha}\}$-straightened and rearrange them to not be so. (Observation (1))

As a consequence of Observation (1), $\{\alpha, \hat{\alpha}\}$-straightening a $S_c$-hit in a permutation $w$ results in a permutation $w'$ with strictly more $\{\alpha, \hat{\alpha}\}$-straightened $S_c$-hits. Since the number of such hits a permutation can have is bounded, $\rightarrow$ terminates.

As an additional consequence of Observation (1) and Lemma 4.3, the actions of $(S_c, \{\alpha, \hat{\alpha}\})$-straightening the $S_c$-hit beginning in position $i$ commutes with the action of $(S_c, \{\alpha, \hat{\alpha}\})$-straightening the $S_c$-hit in position $i'$ for any given $i$ and $i'$. Thus $\rightarrow$ satisfies condition (2) of the Diamond Lemma. By the Diamond Lemma, $\rightarrow$ is confluent.

**Theorem 4.5.** For a given equivalence class of $S_n$ under the $S_c$-equivalence, let $w$ be the unique $(\{S_c\}, \{\alpha, \hat{\alpha}\})$-root permutation in the class. (Such a $w$ exists by Theorem 4.4) Then $w$ is of the form $v_1v_2 \cdots v_k$ such that

1. each $v_i$ is an increasing or decreasing word of consecutive integers.
2. for all $u \equiv w$, $u$ is of the form $v'_1v'_2 \cdots v'_k$ such that $v'_i \equiv v_i$. 


Definition 4.6. Given \( w \in S_n \), we define \( v_i(w) \) to be the unique \( v_i \) defined in the proof of Theorem 4.5.

Observe that Stanley found a result similar to Theorem 4.5 in the case of the \( \{12\}\{21\} \)-equivalence in the proof of Theorem 4.5.

By the previous theorem, all that remains in characterizing the equivalence classes in \( S_n \) under the \( S_c \)-equivalence is to characterize the equivalence class containing the \( t \)-hit containing both letters from \( \alpha \) and \( \beta \). By symmetry, it is sufficient to characterize the equivalence class containing the identity permutation in \( S_n \), 123 \( \cdots \) \( n \).

Definition 4.7. Let \( w \) be a word of size \( n \), each letter of which has a distinct value. Recall that the tail size of \( w \) is the smallest positive integer \( k \) such that the first \( k \) letters of \( w \) contain the \( k \) smallest letter values in \( w \).

We say that if \( w \) is empty, it has no irreducible blocks.

Otherwise, let \( k \) be the tail size of \( w \), let \( a \) be a word made of the first \( k \) letters of \( w \), and let \( b \) be a word made of the final \( n-k \) letters of \( w \). We say that the irreducible blocks of \( w \) are \( B_1(w) = a \) and \( B_i(w) = B_{i-1}(b) \) for \( i > 1 \) satisfying that \( B_{i-1}(b) \) exists.

Definition 4.8. A permutation is \( c \)-toothed if the following is true.

1. For each \( B_i(w) \), \( |B_i(w)| < c \) (where \( |B_i(w)| \) is the number of letters in \( B_i(w) \));
2. there is some sequence of consecutive integers \( i = i_1, \ldots, i_t \) such that \( \sum_{i \in I} |B_i(w)| = c \).

Example: Figure 4 is a visual depiction of the irreducible block decomposition of the permutation \( w = 3124657 \). Note that \( |B_1(w)| = 3 \), \( |B_2(w)| = 1 \), \( |B_3(w)| = 2 \), and \( |B_4(w)| = 1 \). Consequently, \( w \) is 3-toothed, 4-toothed, and 7-toothed. However, \( w \) is not 2-toothed since \( |B_1(w)| > 2 \).

Theorem 4.9. Let \( w \in S_n \). Then \( w \equiv \text{id}_n \) exactly if \( w \) is \( c \)-toothed.

Proof. Observe that every \( c \)-toothed permutation in \( S_n \) except for the identity contains at least one \( S_c \)-hit not forming \( \alpha \). Also note that \( (S_c, \{\alpha, \hat{\alpha}\}) \)-straightening a \( S_c \)-hit in a \( c \)-toothed permutation in \( S_n \) is exactly the same as rearranging the \( S_c \)-hit to form \( \alpha \). Finally, observe that rearranging a \( S_c \)-hit in a \( c \)-toothed permutation to form \( \alpha \) yields another \( c \)-toothed permutation. Thus it follows from Theorem 4.4 that every \( c \)-toothed permutation in \( S_n \) is equivalent to \( \text{id}_n \) under the \( S_c \)-equivalence.

It is straightforward to check that every permutation in \( S_n \) that is equivalent to \( \text{id}_n \) under the \( S_c \)-equivalence must also be \( c \)-toothed.

Definition 4.10. Let \( T_{c,n} \) be the set of \( c \)-toothed permutations in \( S_n \). Let \( T_c \) be the set \( \{ |T_{c,j}| : j > 0 \} \).

---

1 We follow the convention that \( \text{id}_n = 123 \cdots n \).
It follows from Theorem 4.9 and Theorem 4.5 that the size of any equivalence class under the $S_c$-equivalence is a product of elements of $T_c$. Consequently, it is interesting to study $|T_{c,n}|$. We do so for $c = 3$.

**Theorem 4.11.** For $n \geq 3$, $|T_{3,n}|$ is the value of the coefficient of $x^n$ in the generating function

\[
F = \frac{x}{1 - x - x^2 - 3x^3} - \frac{1}{1 - x^2}.
\]

**Proof.** Let $F_n$ be the number of permutations in $S_n$ comprising only irreducible blocks of size $\leq 3$. If the final irreducible block is of size 1, the final letter of such a permutation is $n$. If the final irreducible block is of size 2, the final two letters are $n(n-1)$. If the final irreducible block is of size 3, the final three letters are $n(n-1)(n-2)$, or $n(n-2)(n-1)$, or $(n-1)n(n-2)$. Thus $F_n = F_{n-1} + F_{n-2} + 3F_{n-3}$ for $n > 1$, $F_{n<1} = 0$, and $F_1 = 1$. Thus if $F$ is the generating function satisfying $[x^n]F = F_n$, then $F = xF + x^2F + 3x^3F + x$. Rearranging yields

\[
F = \frac{x}{1 - x - x^2 - 3x^3}.
\]

Now consider the permutations in $S_n$ comprising only of irreducible blocks of size $\leq 3$ but containing no consecutive irreducible blocks whose sizes add to 3. For $n \geq 3$, it is easy to see that such a permutation can only exist when $n$ is even and that it is of the form 21436587...$n$ (comprising only of irreducible blocks of size two). Thus for $n \geq 3$, $|T_{3,n}|$ is the value of the coefficient of $x^n$ in the generating function

\[
F - \frac{1}{1 - x^2} = \frac{x}{1 - x - x^2 - 3x^3} - \frac{1}{1 - x^2}.
\]

\[\square\]

5. **Conclusion and Directions of Future Work**

We present three directions of future work.

1. Is it interesting to extend the notion of confluence introduced in this paper to the case where instead of $(P,C)$-straightening $P$-hits, one $(P,C)$-straightens contiguous subwords of some fixed size. When this size of $c$, this reduces to the notion of $(P,C)$-straightening we have used in this paper.
(2) Is there a formula counting the number of classes in $S_n$ under the $S_c$-equivalence? Such a formula would provide a lower bound for the number of equivalence classes under an arbitrary doubly adjacent pattern-replacement equivalence.

(3) Is there a formula for $|T_{c,n}|$? Such a formula is of interest since the size of any equivalence class under the $S_c$-equivalence is a product of elements of $T_c = \{T_{c,j} : j > 0\}$. 
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