BMS field theory and the open roads

Claudio Dappiaggi

Dipartimento di Fisica Nucleare e Teorica & INFN Sezione di Pavia
Via Bassi,6 I-27100 Pavia (Italy)
E-mail: claudio.dappiaggi@pv.infn.it

Abstract. We develop the basic ingredients of a field theory intrinsically defined on the null infinity boundary of an asymptotically flat spacetime. We also discuss some recent related results and open questions.

1. Introduction
The last decade witnessed several proposals to solve the long standing quest to formulate a satisfactory quantum version of Einstein theory. Compared to previous attempts, the most recent ones are characterized by the ubiquitous presence of a new paradigm: “the holographic principle”. Its origin can be traced back to the seminal paper by G. ’t Hooft [1] who remarked that one of the main obstruction in the quantization of general relativity lies in the existence of sources of extreme gravitational fields i.e. black holes. The peculiarity of such objects can be appreciated already studying their behaviour at a classical level and in particular the three laws of black hole thermodynamic. Within this context, the most striking property originates from the Bekenstein formula which relates the entropy $S$ of a black hole to a quarter$^1$ of the area of its event horizon. From a mere classical perspective this relation is rather counterintuitive since the entropy of a common physical system is proportional to the volume of the region of spacetime where it evolves. Furthermore, if we add to this reasoning, the remark that $S$ can be read as a measure of the degrees of freedom accessible to the system itself, we have the key ingredients leading ’t Hooft to claim the following: $S = \frac{A}{4}$ is an indicator that the information “lost” inside the black hole can be encoded on the event horizon by means of suitable theory with a density of data not exceeding the Planck one.

This statement is the original formulation of the holographic principle which has been later enhanced extending its validity to regions of spacetime different from black holes [2]. This idea which goes under the name of “covariant entropy conjecture” will not be described here in detail whereas we will focus on a related problem: is it possible to encode the information contained in the whole spacetime on a suitable codimension one submanifold which plays the same role as the event horizon in ’t Hooft conjecture? This question could be addressed either classically, from the perspective of general relativity [2] either at a quantum level constructing a lower dimensional quantum field theory which encodes the informations of the physical fields (gravitational included) living in the spacetime.

A concrete answer to this programme has been formulated in an AdS background and it goes under the name of AdS/CFT correspondence [3]; roughly speaking it states the existence of a

$^1$ In this paper we assume $c = G = \hbar = 1$. 
The duality between a type IIB superstring theory on $AdS_5 \times S^5$ and a $SU(N)$ super Yang Mills field theory leaving on the boundary of $AdS_5$. Though the Maldacena conjecture is widely accepted and, to a certain degree, it can be extended to asymptotically AdS spacetimes, it addresses only the specific scenario of negative cosmological constant $\Lambda$. It thus natural to wonder if the holographic principle could be implemented if we consider different kind of backgrounds and, in particular, we will deal with solutions of Einstein’s equations with $\Lambda = 0$ or more precisely with asymptotically flat spacetimes (see [5, 6] for different approaches to the same problem).

2. BMS field theory: formulation

In order to construct a theory encoding the data from any but fixed asymptotically flat spacetime, the first step consists on finding a suitable codimension one submanifold where to implement the holographic principle. In our scenario there is a natural candidate, i.e. $\mathcal{I}^\pm$, the future (or past) null infinity. Since it will play a key role in the forthcoming analysis, we give now some sketches of its main properties (see also [4] and section 2 in [11]); thus, a four dimensional Lorentzian manifold ($\tilde{M}^4, g_{\mu\nu}$), is called asymptotically flat at null infinity if it exists a second manifold ($M^4, g_{\mu\nu}$), an embedding $i: \tilde{M}^4 \rightarrow M^4$ and a real positive scalar function $\Omega$ - the compactification factor - such that

- $\mathcal{I}^\pm \cup i_0 = \partial \left[i(M^4)\right]$, being $i_0$ spatial infinity,
- $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ where $\Omega$ is smooth everywhere except at most $i_0$ where it is at least twice differentiable,
- $\Omega = 0$ on $\mathcal{I}^\pm$ but $d\Omega|_{\mathcal{I}} \neq 0$.

Bearing in mind this definition, it is straightforward to realize the the boundary structure of $\tilde{M}^4$ is geometrically characterized in $M^4$ by a triple of data namely, the loci $\mathcal{I}^+$ and $\mathcal{I}^-$ which are topologically equivalent to $S^2 \times \mathbb{R}$, the metric $h_{\mu\nu} = g_{\mu\nu}|_{\mathcal{I}}$ and the vector $n^\mu = \nabla^\mu \Omega|_{\mathcal{I}}$ which turns out to be complete on $\mathcal{I}$.

The key consequence of the above construction is that $\mathcal{I}^\pm$ are null hypersurfaces and that, given a fixed asymptotically flat spacetime, its boundary structure is intrinsically characterized by the data $(\mathcal{I}, h_{\mu\nu}, n^\mu)$. Furthermore one can show that such a structure is also universal i.e., given any two asymptotically flat spacetimes, say $(M^4_1, g_{1\mu\nu})$ and $(M^4_2, g_{2\mu\nu})$, and their respective triples characterizing the boundary structure, say $(\mathcal{I}_1, h_{1\mu\nu}, n^1\mu)$ and $(\mathcal{I}_2, h_{2\mu\nu}, n^2\mu)$, it always exists a diffeomorphism

$$
\gamma : \mathcal{I}_1 \rightarrow \mathcal{I}_2 \quad \text{such that} \quad \gamma^* h_{2\mu\nu} = h_{1\mu\nu} \quad \wedge \quad \gamma^* n^1\mu = n^2\mu. \quad (1)
$$

According to the above discussion, both future and past null infinity represents a natural candidate where to implement the holographic principle since the intrinsic nature and the universality of their geometric structure allows us to discuss the boundary theory regardless of the specific bulk spacetime we wish to consider.

The next step in our discussion will be to construct a field theory on $\mathcal{I}$ which we only require to be invariant under the diffeomorphisms (1). These transformations can be made explicit if we introduce the so called Bondi frame $(u, \Omega, \theta, \varphi)$ where, besides $\Omega$, $\theta$ and $\varphi$ are the usual coordinates on $S^2$ and $u$ is the affine parameter spanning the null direction on $\mathcal{I}$. With this choice and up to a stereographic projection mapping $(\theta, \varphi)$ into $(z, \tilde{z})$ with $z = e^{i\varphi} \cot \theta$, (1) becomes [7]

$$
z \rightarrow z' = \Lambda z, \quad \text{and c.c.,} \quad \Lambda \in SL(2, \mathbb{C}) \quad (2)
$$

$$
u \rightarrow u' = K_\Lambda [z, \tilde{z}] [u + \alpha(z, \tilde{z})], \quad (3)
$$

\begin{itemize}
  \item $\mathcal{I}^\pm \cup i_0 = \partial \left[i(M^4)\right]$, being $i_0$ spatial infinity,
  \item $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ where $\Omega$ is smooth everywhere except at most $i_0$ where it is at least twice differentiable,
  \item $\Omega = 0$ on $\mathcal{I}^\pm$ but $d\Omega|_{\mathcal{I}} \neq 0$.
\end{itemize}

\textsuperscript{2} From now on we omit the apexes $\pm$ on $\mathcal{I}$ since they are not relevant for our discussion.
where $\alpha(z, \bar{z})$ is any smooth real scalar function over $S^2$ and

$$K_{\Lambda}[z, \bar{z}] = \frac{1 + |z|^2}{|az + b|^2 + |cz + d|^2},$$

where $\Lambda = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $ad - bc = 1$ and $a, b, c, d \in \mathbb{C}$.

A direct inspection of (2) and (3) shows that the diffeomorphism group - known as Bondi-Metzner-Sachs group (BMS) - of the boundary of any four dimensional asymptotically flat spacetime has the structure of a semidirect product (which we will indicate with $\ltimes$) i.e.

$$BMS_4 = SL(2, \mathbb{C}) \ltimes C^\infty(S^2),$$

where $N = C^\infty(S^2)$ is an abelian subgroup usually referred to as supertranslations.

The existence of such a big group of symmetry at null infinity is a clear signal that the role of gravity cannot be discarded even asymptotically when the spacetime metric tends to become Minkowskian and, thus, one could expect to retrieve the Poincaré group. Nonetheless, although (4) is an infinite dimensional group, we can still conceive to explicitly construct a BMS invariant field theory following a path similar to the one that led Wigner to construct all the kinematical and dynamical configurations of a Poincaré invariant field theory. In this latter context a free field is defined as a wave function transforming under a unitary and irreducible representation of $SL(2, \mathbb{C}) \ltimes T^4$ and, in our approach, we stick to this idea only substituting Poincaré with BMS invariance. In this paper we will not discuss all the techniques which allow us to concertise the above programme leaving instead an interested reader to [7, 11] where all the mathematical details are dealt with. Conversely, here, we only point out the key ingredients namely:

- the identification of a coordinate and a momenta space which, in a BMS language, are respectively $N = C^\infty(S^2)$ and its topological dual space $N^*$, i.e. the space of distributions over $S^2$. The reader should bear in mind that $N$ and $N^*$ play exactly the role of $T^4$ and $T^4^*$ in a Poincaré context.

- the complete classification of unitary and irreducible representations (irreps.) for (4) which has been studied in the seventies in [9] by means of Mackey’s theory of induced representations.

In particular the paradigm of Mackey’s approach is that any irrepp. of a group $G$ can be constructed starting from an irrepp. of a suitable subgroup $H$ of $G$ which, in a BMS scenario, is $H = N \ltimes L$ where $L \subset SL(2, \mathbb{C})$. In particular the plethora of all possible subgroups $L$ has been classified in [9] and the most notables are the connected ones i.e. $SU(2)$, $SO(2)$ and $\Delta$, the double cover of the two dimensional Euclidean group.

Furthermore, in a (super)momenta frame, we can define a BMS free field or induced wave function as a map:

$$\tilde{\psi} : O = \frac{SL(2, \mathbb{C})}{L} \ni N^* \rightarrow \mathcal{H},$$

where $\mathcal{H}$ is a suitable target Hilbert space and where $\tilde{\psi}$ transforms under the action of any element $g = (\Lambda, \alpha) \in SL(2, \mathbb{C}) \ltimes N$ as

$$\left(\Lambda \tilde{\psi}\right)(p) = \sqrt{\frac{d\mu(\Lambda p)}{d\mu(p)}}D(\omega^{-1}(p)\Lambda \omega(\Lambda^{-1} p))\tilde{\psi}(\Lambda^{-1} \beta),$$

$$\left(\alpha \tilde{\psi}\right)(p) = e^{i(p, \alpha)}\tilde{\psi}(p),$$

where $D(\Lambda)$ is a unitary irrep. of $L$, $d\mu(p)$ is a suitable measure on the orbit $O$, $\omega$ is a global section of the bundle $\rho : SL(2, \mathbb{C}) \rightarrow O$ whereas we refer to [7] for the definition of the $SL(2, \mathbb{C})$-action on the point $p$. 

The expression \((p, \alpha)\) is the canonical evaluation of a distribution over \(S^2\) with a test function \(\alpha\); to clarify its origin, the reader should bear in mind that, within Wigner approach, the orbit (and thus the point \(p\)) of a free field is supposed to live on a finite dimensional space \(SL(2, \mathbb{C})_L\) which is embedded in the space of supermomenta \(N^*\) with the following construction: pick \(\beta\), any distribution over \(S^2\) such that \(L\beta = \beta\), and then consider as orbit the set of points/distributions obtained as \(\beta = SL(2, \mathbb{C})\beta\).

To conclude we want to point out that it is also possible to associate a notion of mass to the induced wave function. We will not discuss here the mathematical details referring for an interested reader to [7]; suffice to say that it is possible to univocally associate to each element \(\beta \in N^*\) a unique four vector, \(\pi[\beta]\), which transforms under the action of an \(SL(2, \mathbb{C})\) element \(\Lambda\) as \(\pi'[\beta] = \Lambda^\mu_\nu \pi[\beta]\). Furthermore the quantity \(m^2 = \eta_{\mu\nu} \pi[\beta]_\mu \pi[\beta]_\nu\) represents a Casimir invariant for the unitary irrep. of the full BMS group and thus it is justified to interpret \(m^2\) as the squared mass of a BMS free field which is strictly positive if \(L = SU(2)\) in (5) whereas if \(L = SO(2)\) or \(\Delta\) it can also take a vanishing value.

An alternative and more common point of view in physical theories consists instead on introducing the so-called covariant wave function i.e. a wave function transforming in a (super)momenta frame under a unitary but non necessary irreducible representation of \(SL(2, \mathbb{C})\) i.e. a map
\[
\psi : N^* \longrightarrow \mathcal{H}', \tag{6}
\]
where \(\mathcal{H}'\) is a suitable target Hilbert space and where \(\psi\) transforms under the action of any \(g = (\Lambda, \alpha) \in SL(2, \mathbb{C}) \ltimes N\) as
\[
[U(g)\psi](\beta) = e^{i(\beta, \alpha)} \tilde{D}(\Lambda)\psi(\Lambda^{-1}\beta),
\]
where \(\tilde{D}(\Lambda)\) is a unitary but non necessary irreducible representation of \(SL(2, \mathbb{C})\).

To conclude this section, we wish to emphasize that the covariant and induced wave functions do represent the same physical entities provided that we impose on the former suitable constraints in order to be fully equivalent to a free field [7, 8]. In a generic scenario this request translates in three equations i.e.

- an orbit equation which restricts the support of (6) to that of (5) i.e.
  \[ [\beta - SL(2, \mathbb{C})\beta]\psi(\beta) = 0, \]
  where \(\beta\) is a distribution over \(S^2\) such that \(L\beta = \beta\),
- a mass equation selecting a specific value of the mass for (6)
  \[ [\eta_{\mu\nu} \pi[\beta]_\mu \pi[\beta]_\nu - m^2] \psi(\beta) = 0, \]
- an orthoparallel equation
  \[ \rho(\beta)\psi(\beta) = \psi(\beta), \]
  which selects by means of the orthoprojector \(\rho\) the unitary representation \(D(\Lambda)\) of \(L\) in (5) contained in \(\tilde{D}(\Lambda)\) in (6).

3. Reconstruction of bulk data: a conjecture
In the previous section we have discussed the key ingredients in order to define the kinematical and dynamical configurations of a BMS invariant free field. The universal nature of the intrinsic symmetry group of an asymptotically flat spacetime has the advantage that we can construct a unique field theory regardless of the specific bulk background. Nonetheless this represents at the same time the main disadvantage from an holographic perspective because we lack a prescription
on how to reconstruct the data from a fixed bulk spacetime starting only from the boundary counterpart. Furthermore, in the Wigner construction we followed, a major setback lies in the absence a priori of any connection between the support of the wave function (either covariant or induced) and spacetime points or momenta. In a Poincaré invariant context, this is a minor nuisance since the role of $N$ and $N^*$ in the previous section is played respectively by $T^4$ (four dimensional translation group) and by $T^4^*$; it is thus immediate to identify $T^4$ with $\mathbb{R}^4$ and $T^4^*$ with $\mathbb{R}^4$ by means of the canonical isomorphism between vectors and covectors induced by the Minkowski metric $\eta_{\mu\nu}$.

Conversely, in a BMS scenario, it is unconceivable to identify in a suitable way elements in $N$ or $N^*$ with points on $\mathbb{I}$ and our proposal to holographically reconstruct the bulk originates from this obstruction. As a matter of fact, if we wish to give a physical interpretation to points lying in $N^*$, i.e distributions over $S^2$, or, more properly, in $N = C^\infty(S^2)$, we need to adopt a slightly uncommon perspective. The starting point for our novel point of view consists on an alternative formulation of general relativity (nonetheless fully equivalent to it) which calls for substituting the metric as the fundamental variable in Einstein’s theory. In this approach, which goes under the name of null surface formulation of general relativity, we consider two generic points $x^\mu, x'^\mu$ of the spacetime here required to be asymptotically flat. If we impose $x'^\mu \in \mathbb{I}^+$ (or equivalently $\mathbb{I}^-$), and if we introduce the Bondi reference frame $(u, \Omega, \theta, \varphi)$ in such a way that $x'^\mu = (u, 0, \theta, \varphi)$, the light cone equation $L(x^\mu, x'^\mu) = 0$ connecting $x^\mu$ and $x'^\mu$ can be inverted as

$$L(x^\mu, u, \theta, \varphi) = 0 \implies u = Z(x^\mu, \theta, \varphi).$$

The set of all possible inverse $Z$-functions - called “cut functions” - is neither a priori unique nor differentiable but $Z(x^\mu, \theta, \varphi)$ can be chosen as an alternative fundamental field in Einstein theory. Leaving the details of this formulation to [10] and references therein, suffice to say for our purposes that all the geometrical quantities characterizing the spacetime can be written in terms of the cut functions and, more importantly, each $Z(x^\mu, \theta, \varphi)$ becomes a suitable neighbourhood of $\mathbb{I}$ a smooth real scalar function i.e., holding fixed the bulk point $x^\mu$, $Z$ lies in $N = C^\infty(S^2)$.

Starting from these premises, in [11], we have proposed to combine the null surface formulation of general relativity with BMS field theory in order to concretely implement the holographic paradigm. In particular we conjectured that, not only the geometrical information of a fixed bulk spacetime is encoded in the supertranslations determined by means of the null surface construction, but also the data from matter fields are encoded in the BMS wave functions whose support (which lies in the set of supertranslation or of supermomenta) is compatible with the prescriptions given by the light cone equation.

Unfortunately, in this paper, we cannot discuss a detailed example supporting the above idea and we refer, for an interested reader, to [11] where the Feynman propagator for a massive scalar field in Minkowski spacetime has been reconstructed only by means of the data of a BMS scalar Klein-Gordon field.

4. Elementary particles and holography

Another scenario where BMS field theory could play a prominent role lies in the quest to solve the long standing problem of finding a fully satisfactory definition in a curved background of elementary particles. According to the common rationale, these are defined as wave functions transforming under a unitary irreducible representation of the Poincaré group. The latter represents a good symmetry of Minkowski background but, whenever gravity is switched on, such invariance disappears and consequently also the notion of elementary particle.

In quantum field theory over curved background, such obstruction is considered to be inevitable since each solution of Einstein equation has its own different degree of symmetry always much smaller than the Poincaré one. Nonetheless, if we consider the set of asymptotically
flat spacetime, we have shown in section 2 that, at null infinity, the notion of BMS group is universal and independent from the bulk content. Thus we could think to $SL(2,\mathbb{C}) \ltimes C^\infty(S^2)$ as a good candidate to substitute the Poincaré group defining an elementary particle in terms of BMS invariance.

Such proposal, first appeared in [12], is nonetheless unsatisfactory due to two key problems: from one side the alternative definition for an elementary particle is intrinsically asymptotic i.e. it holds only on $\mathbb{I}^\pm$ where the BMS group is defined. On the other side, the analysis in [9] teaches us that the possible unitary and irreducible representations for the BMS group are a plethora compared to the Poincaré counterpart and there is no physical explanation neither experimental evidence for their existence. Nonetheless, in [13], we realized that the free BMS $SU(2)$ and $\Delta$ free fields are related to the Poincaré counterpart not only at a kinematical but also at a dynamical level i.e. we demonstrated the existence of a one-to-one correspondence between the covariant phase spaces (i.e. the set of wave functions satisfying the equations of motion) in both scenarios. This result, in combination with the conjecture stated in the previous section, suggested us to define an elementary particle as a wave function transforming under a unitary and irreducible representation of the BMS group and holographically encoding the bulk data [14]. It is straightforward to show that the above statement reduces to the common one in a Minkowski background where the problem of the little groups in excess vanishes since they do not encode any bulk data. Nonetheless it is imperative to test this definition in a more rigorous mathematical context such as the algebraic formulation of quantum field theory. A preliminary analysis in [11] suggests that the proposed point of view could be correct though we do not have yet an overall complete proof and this problem is currently under investigation.

Acknowledgments
This research has been supported by a grant from the Department of Theoretical and Nuclear Physics - Pavia University. The author is grateful to V. Moretti and N. Pinamonti for the fruitful and long explanations on the algebraic approach to quantum field theory.

References
[1] G. ’t Hooft, “Dimensional reduction in quantum gravity,” arXiv:gr-qc/9310026,
[2] R. Bousso, “The holographic principle”, Rev. Mod. Phys. 74 (2002) 825,
[3] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, “Large N field theories, string theory and gravity,” Phys. Rept. 323 (2000) 183,
[4] R.M. Wald, “General relativity” (1984) Chicago Univ. Press,
[5] J. de Boer and S. N. Solodukhin, “A holographic reduction of Minkowski space-time,” Nucl. Phys. B 665 (2003) 545,
[6] E. Alvarez, J. Conde and L. Hernandez, “Goursat’s problem and the holographic principle,” Nucl. Phys. B 689 (2004) 257,
[7] C. Dappiaggi, V. Moretti and N. Pinamonti, “Rigorous steps towards holography in asymptotically flat spacetimes,” arXiv:gr-qc/0506069,
[8] G. Arcioni and C. Dappiaggi, “Exploring the holographic principle in asymptotically flat spacetimes via the BMS group,” Nucl. Phys. B 674 (2003) 553,
[9] P.J. McCarthy, “The Bondi-Metzner-Sachs in the nuclear topology” Proc. R. Soc. London A343 (1975) 489,
[10] S. Frittelli, C. Kozameh, E. T. Newman, “GR via characteristic surfaces” J. Math. Phys. 36 (1995) 4984,
[11] C. Dappiaggi, “BMS field theory and holography in asymptotically flat space-times,” JHEP 0411 (2004) 011,
[12] P. J. M. McCarthy, “Asymptotically Flat Space-Times And Elementary Particles,” Phys. Rev. Lett. 29 (1972) 817,
[13] G. Arcioni and C. Dappiaggi, “Holography in asymptotically flat space-times and the BMS group,” Class. Quant. Grav. 21 (2004) 5655,
[14] C. Dappiaggi, “Elementary particles, holography and the BMS group,” Phys. Lett. B 615 (2005) 291.