On the Chiral Magnetic Effect in Soft-Wall AdS/QCD

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Abstract

The essence of the chiral magnetic effect is generation of an electric current along an external magnetic field. Recently it has been studied by Rebhan et al. within the Sakai–Sugimoto model, where it was shown to be zero. As an alternative, we calculate the chiral magnetic effect in soft-wall AdS/QCD and find a non-zero result with the natural boundary conditions. The mechanism of the dynamical neutralization of the chiral chemical potential via the string production is discussed in the dual two-form representation.
1 Introduction

Chiral magnetic effect \[1, 2, 3\] (CME) is best described as a generation of an electric current by a magnetic field in a topologically nontrivial background. The standard field-theoretical argumentation is the following. Let us consider QCD with massless quarks, so that left and right quarks can be dealt with independently, and suppose a chiral chemical potential \(\mu_5\) is present, accounting for a certain topologically nontrivial background. The topologically nontrivial field configuration changes chirality, and an external magnetic field \(\mathbf{B} = (0, 0, B)\) orders spins parallel to itself. Thus arises a non-zero vector current, which is given by Fukushima, Kharzeev and Warringa \[3\]

\[
J^V_3 = \frac{\mu_5 B}{2\pi^2} \equiv J_{FKW}. \tag{1}
\]

During recent years holography has become one of the main alternative tools for analyzing non-perturbative QCD. Different conductivities of quark matter, including chiral magnetic conductivity, have already been analyzed in a variety of holographic models. Electric conductivity in the D3/D7 model was examined by Karch and O’Bannon in \[4\]. Axial, ohmic and Hall conductivity in a magnetic field were calculated on the basis of the Kubo formula and correlator analysis for the Sakai-Sugimoto model in \[5, 6\]. One of the results for electric current in \[5\] is \(1/2\) of QCD weak coupling result \(1\).

An attempt to describe the chiral magnetic effect for the vector current in Sakai-Sugimoto model has been made recently in \[7\]. The result at zero frequency, where only the Yang–Mills part of the action was used, exactly amounts to the weak coupling QCD effect; non-zero frequencies have also been considered. In \[10\] a more sophisticated anomaly subtraction scheme was suggested. It was argued that if one uses the Bardeen term subtraction, then one gets zero effect for vector current, otherwise one gets \(2/3\) of the weak-coupling effect. The reason for adding the Bardeen term to the action was to cure the pathological behavior of the vector anomaly. Note that a similar effect for the axial current in a theory with the conventional chemical potential has been discussed in \[8, 9\].

Experimental status of the problem is discussed in \[15\]. Presently it is claimed that the effect is present, yet the exact proportionality coefficient \(c\) in \(J^V_3 = c \cdot \frac{\mu_5 B}{2\pi^2}\) cannot be inferred from it. Lattice estimates are also close to \(2/3\) \[11\] of weak-coupling effect. The discussion of the effect in the framework of NJL model can be found in \[12\]. Chiral magnetic effect at low temperature was considered in \[13\]. An analog of this effect is known in superfluid helium \[14\].

The present note aims at comparing the calculations of \[10\] to the chiral magnetic effect as derived in the framework of soft-wall AdS/QCD. The question whether the effect is present in a holographic model or not, turns out to be quite delicate. The paper is organized as follows. In Section 2 we consider the analysis
of the gauge sector of the soft-wall model and confirm the result of [10]. In Sec-

tion 3 we discuss the contribution of scalars and pseudoscalars and focus on their

boundary conditions in the 5d equations of motion. Section 4 is devoted to the dual

representation of the chiral chemical potential and the mechanism of its possible dy-

namical neutralization via the Schwinger-type process. The results of this note are

summarized in the Conclusion.

2 The soft-wall model

2.1 Gauge part of the action

Let us consider the gauge field sector of the soft-wall AdS/QCD model [18] taking

into account the Chern-Simons action \( \int A \wedge F \wedge F \). We begin our consideration with

an action of Abelian fields \( L \) and \( R \) with a coupling \( g_5 \) that has the following form:

\[
S = S_{YM}[L] + S_{YM}[R] + S_{CS}[L] - S_{CS}[R]
\]  

(2)

\[
S_{YM}[A] = -\frac{1}{8g_5^2} \int e^{-\phi} F \wedge *F = -\frac{1}{8g_5^2} \int dz \, d^4x \, e^{-\phi} \sqrt{g} F_{MN} F^{MN}
\]  

(3)

\[
S_{CS}[A] = -\frac{k \cdot N_c}{24\pi^2} \int A \wedge F \wedge F - \frac{1}{2} A \wedge A \wedge A \wedge F + \frac{1}{10} A \wedge A \wedge A \wedge A
\]  

(4)

Here \( k \) is an integer that scales the CS term and effectively the magnetic field. Canonical normalization of the CS term is \( k = 1 \), but it will be kept it for the sake of generality. The metric tensor is the following:

\[
d\sigma^2 = g_{MN} dX^M dX^N = \frac{R^2}{z^2} \eta_{MN} dX^M dX^N = \frac{R^2}{z^2} (-dz^2 + dx_\mu dx^\mu).
\]  

(5)

In the \( A_z = 0 \) gauge the YM action acquires the form

\[
S_{YM}[A] = -\frac{R}{4g_5^2} \int dz \, d^4x \left\{ \frac{e^{-\phi}}{z} A_\mu (\square \eta^{\mu\nu} - \partial^\mu \partial^\nu) A_\nu + A_\mu \partial_z \left( \frac{e^{-\phi}}{z} \partial_z \right) A_\mu \right\}
\]  

\[
+ \frac{R}{4g_5^2} \int d^4x \, \left. \frac{e^{-\phi}}{z} A_\mu \partial_z A_\mu \right|_{z=\infty} - \left. \frac{e^{-\phi}}{z} A_\mu \partial_z A_\mu \right|_{z=0}.
\]  

(6)

From the YM part of the action we get

\[
\frac{\delta S_{YM}[A]}{\delta A_\mu} = -\frac{R}{2g_5^2} \left\{ \frac{e^{-\phi}}{z} (\square \eta^{\mu\nu} - \partial^\mu \partial^\nu) A_\nu + \partial_z \left( \frac{e^{-\phi}}{z} \partial_z A_\mu \right) \right\}.
\]  

(7)
Varying the volume term of the action one gets
\[
\frac{\delta S_{CS}[A]}{\delta A_\mu} = \frac{k \cdot N_c}{2\pi^2} \epsilon^{\mu\nu\rho\sigma} \partial_\nu A_\rho F_{\sigma\nu}.
\] (8)

Taking into account \( R \frac{g_5^2}{\Delta} = \frac{N_c}{12\pi^2} \), one obtains the equations of motion for the fields L and R
\[
\partial_z \left( \frac{e^{-\phi(z)}}{z} \partial_\mu L^\mu \right) - 24k \epsilon^{\mu\nu\rho\sigma} \partial_\nu L_\rho \partial_\sigma L_\mu = 0
\] (9)
\[
\partial_z \left( \frac{e^{-\phi(z)}}{z} \partial_\mu R^\mu \right) + 24k \epsilon^{\mu\nu\rho\sigma} \partial_\nu R_\rho \partial_\sigma R_\mu = 0
\] (10)
with the following boundary conditions
\[
L_0(0) = \mu_L, \quad R_0(0) = \mu_R, \quad (11)
\]
\[
L_3(0) = j_L, \quad R_3(0) = j_R, \quad (12)
\]
\[
L_1(0, x_2) = -\frac{1}{2} x_2 B, \quad R_1(0, x_2) = -\frac{1}{2} x_2 B, \quad (13)
\]
\[
L_\mu(\infty) = R_\mu(\infty), \quad \partial_z L_\mu(\infty) = -\partial_z R_\mu(\infty), \quad (14)
\]
here \( \mu = \frac{1}{2}(\mu_L + \mu_R), \mu_5 = \frac{1}{2}(\mu_L - \mu_R) \), and \( j_{L,R} \) are the gauge field boundary values, a variation with respect to which gives the currents
\[
\frac{\delta S[L, R]}{\delta L_3(z = 0)} = \frac{1}{V_{4D}} \frac{\partial S[L, R]}{\partial j_L} = J_L, \quad (15)
\]
\[
\frac{\delta S[L, R]}{\delta R_3(z = 0)} = \frac{1}{V_{4D}} \frac{\partial S[L, R]}{\partial j_R} = J_R. \quad (16)
\]
Conditions (14) arise because both left- and right-handed gauge fields are associated with a single gauge field in the Sakai–Sugimoto model \[16, 17\]. In that model, regions of positive and negative values of the holographic coordinate \( \rho = 1/z \) correspond to left-handed D8 and right-handed \( \bar{D}8 \) branes respectively. Since the gauge field is smooth and continuous at \( \rho = 0 \), a boundary condition \( \frac{\delta S[L, R]}{\delta L_3(z = 0)} = \frac{1}{V_{4D}} \frac{\partial S[L, R]}{\partial j_L} = J_L \) is obtained at \( z = 1/\rho = \infty \). Furthermore, \( \frac{\delta S[L, R]}{\delta R_3(z = 0)} = \frac{1}{V_{4D}} \frac{\partial S[L, R]}{\partial j_R} = J_R \) may be understood as a zero boundary condition for the axial gauge field that is usually imposed at the black hole horizon, which in our zero-temperature case is located at \( z = \infty \).
Denoting $\beta = 12kB$ one can get the following set of e.o.m.’s
\[
\begin{align*}
\partial_z \left( \frac{e^{-\phi(z)}}{z} \partial_z L_0 \right) &= \beta \partial_z L_3, \\
\partial_z \left( \frac{e^{-\phi(z)}}{z} \partial_z L_1 \right) &= 0,
\end{align*}
\]
\[
\begin{align*}
\partial_z \left( \frac{e^{-\phi(z)}}{z} \partial_z L_0 \right) &= \beta \partial_z L_3, \\
\partial_z \left( \frac{e^{-\phi(z)}}{z} \partial_z R_0 \right) &= -\beta \partial_z R_3, \\
\partial_z \left( \frac{e^{-\phi(z)}}{z} \partial_z R_1 \right) &= 0.
\end{align*}
\]
\[
\begin{align*}
\partial_z \left( \frac{e^{-\phi(z)}}{z} \partial_z L_1 \right) &= 0, \\
\partial_z \left( \frac{e^{-\phi(z)}}{z} \partial_z R_1 \right) &= 0.
\end{align*}
\]
Solution is the following
\[
\begin{align*}
L_0(z) &= \mu_L + \left( \mu_5 - \frac{1}{2} j_5 \right) (e^{-|\beta|w(z)} - 1), \\
L_3(z) &= j_L - \left( \mu_5 - \frac{1}{2} j_5 \right) (e^{-|\beta|w(z)} - 1), \\
R_0(z) &= \mu_R - \left( \mu_5 + \frac{1}{2} j_5 \right) (e^{-|\beta|w(z)} - 1), \\
R_3(z) &= j_R - \left( \mu_5 + \frac{1}{2} j_5 \right) (e^{-|\beta|w(z)} - 1), \\
R_1(z, x_2) &= -\frac{1}{2} x_2 B, \\
L_1(z, x_2) &= -\frac{1}{2} x_2 B,
\end{align*}
\]
here $j = j_L + j_R, j_5 = j_L - j_R, \text{ and } w(z) = \int_0^z du \, e^{\phi(u)}, \quad \frac{e^{-\phi(z)}}{z} \, w'(z) = 1.$

2.2 On-shell action and symmetry currents

Let us now compute the on-shell action with the gauge fields given by (20) for both left- and right-handed gauge fields. Its Yang–Mills part is given as
\[
S_{YM}[A] = -\int dz \, d^4x \, \frac{e^{-\lambda z^2}}{z} \frac{R}{8g_5^2} \eta^{AB} \eta^{MN} F_{AM} F_{BN}
\]
\[
= -\frac{R}{8g_5^2} \int dz \, d^4x \, \frac{e^{-\lambda z^2}}{z} \left\{ -2\eta^{\alpha\beta} \partial_z A_\alpha \partial_z A_\beta + \eta^{\alpha\beta} \eta^{\mu\nu} F_{\alpha\mu} F_{\beta\nu} \right\}
\]
\[
= -\frac{R}{4g_5^2} \frac{B^2}{4V_4} \int dz \, \frac{e^{-\lambda z^2}}{z}
\]
(21)

The Chern–Simons part of the action is
\[
S_{CS}[A] = \frac{k \cdot N_e}{6\pi^2} \int dz \, d^4x \, e^{\mu\nu\rho\sigma} A_\mu \partial_z A_\nu F_{\rho\sigma}
\]
\[
= \frac{k \cdot N_e}{3\pi^2} \int dz \, d^4x \, (A_0 \partial_z A_3 - A_3 \partial_z A_0) F_{12}.
\]
(22)

Recall that $w(z) = \int_0^z du \, e^{\phi(u)}, w(0) = 0, w(\infty) = \infty$. Then the solutions (20) have
the following form

\[ L_0(z) = \mu + \frac{1}{2} j_5 + \left( \mu_5 - \frac{1}{2} j_5 \right) e^{-|\beta| w(z)}, \quad L_3(z) = \mu_5 + \frac{1}{2} j - \left( \mu_5 - \frac{1}{2} j_5 \right) e^{-|\beta| w(z)}, \]

\[ R_0(z) = \mu + \frac{1}{2} j_5 - \left( \mu_5 + \frac{1}{2} j_5 \right) e^{-|\beta| w(z)}, \quad R_3(z) = \mu_5 + \frac{1}{2} j - \left( \mu_5 + \frac{1}{2} j_5 \right) e^{-|\beta| w(z)}, \]

\[ F_{12}^L = \frac{1}{2} B, \quad F_{12}^R = \frac{1}{2} B. \]  

(23)

Upon substituting (23) into (22) the on-shell CS action becomes

\[ S_{CS}[L] = \frac{k \cdot N_c}{2 \pi} \int dz \, d^4 x \, F_{12}^L (L_0 \partial z L_3 - L_3 \partial z L_0) \]

\[ = \frac{k \cdot N_c}{6 \pi^2} BV_{4D} \left( \mu_5 + \mu_5^2 - \frac{1}{2} \mu j_5 + \frac{1}{2} \mu_5 j - \frac{1}{4} j_5^2 - \frac{1}{4} j j_5 \right), \]  

(24)

and

\[ S_{CS}[R] = \frac{k \cdot N_c}{2 \pi} \int dz \, d^4 x \, F_{12}^R (R_0 \partial z R_3 - R_3 \partial z R_0) \]

\[ = \frac{k \cdot N_c}{6 \pi^2} BV_{4D} \left( \mu_5 - \mu_5^2 + \frac{1}{2} \mu j_5 - \frac{1}{2} \mu_5 j + \frac{1}{4} j_5^2 - \frac{1}{4} j j_5 \right). \]  

(25)

The symmetry currents \( J_L, J_R \) are equal to the partial derivatives of the action with respect to \( j_L, j_R \)

\[ J_L = \frac{1}{V_{4D}} \frac{\partial S}{\partial j_L} = \frac{1}{V_{4D}} \left( \frac{\partial j}{\partial j_L} \frac{\partial S}{\partial j} + \frac{\partial j_5}{\partial j_L} \frac{\partial S}{\partial j_5} \right) = \frac{1}{V_{4D}} \left( \frac{\partial S}{\partial j} + \frac{\partial S}{\partial j_5} \right), \]  

(26)

\[ J_R = \frac{1}{V_{4D}} \frac{\partial S}{\partial j_R} = \frac{1}{V_{4D}} \left( \frac{\partial j}{\partial j_R} \frac{\partial S}{\partial j} + \frac{\partial j_5}{\partial j_R} \frac{\partial S}{\partial j_5} \right) = \frac{1}{V_{4D}} \left( \frac{\partial S}{\partial j} - \frac{\partial S}{\partial j_5} \right), \]  

(27)

\[ J = \frac{2}{V_{4D}} \frac{\partial S}{\partial j}, \quad J_5 = \frac{2}{V_{4D}} \frac{\partial S}{\partial j_5}. \]  

(28)

As can be seen from (21), the YM part of the action does not depend on the current sources. The CS part, on the other hand, equals

\[ S_{CS} = S_{CS}[L] - S_{CS}[R] = \frac{k \cdot N_c}{6 \pi^2} BV_{4D} \left( 2 \mu_5^2 - \frac{1}{2} j_5^2 + \mu_5 j - \mu j_5 \right). \]  

(29)

From eqs. (28,29) one obtains

\[ J = \frac{k \cdot N_c}{3 \pi^2} B \mu_5, \]  

(30)

\[ J_5 = \frac{k \cdot N_c}{3 \pi^2} B (\mu + j_5). \]  

(31)
If one sets \( k = 1 \), a standard normalization of the CS action (4) is recovered and the result (30) is in agreement with [10] without the Bardeen counterterm. The axial supercurrent introduced in [10] is an equivalent of our \( j_5 \). If it is interpreted as a source for the axial current it has to be set to zero. Minimizing the action with respect to it is analogous to setting the axial current (31) to zero. It is interesting that the answer does not depend on \( j \), which probably justifies its absence in [10].

2.3 The divergence of the vector current

In this section the general formula for the left and right symmetry currents \( J_{L,R} \) will be derived.

We are going to use a different approach to calculating currents, yet the results will be identical to the previous section. The current definition is the following (for the current \( J_{L,R} \) it is the same as in (28)):

\[
J_{L}^{\mu}(x) = \frac{\delta S}{\delta L_{\mu}(z = 0, x)}, \quad J_{R}^{\mu}(x) = \frac{\delta S}{\delta R_{\mu}(z = 0, x)}. \tag{32}
\]

The variation of the action \( \delta S = \delta S_{YM} + \delta S_{CS} \) can be split into two parts – one proportional to the equations of motion and one reducible to a surface term

\[
\begin{align*}
\delta S_{YM}[A] &= \delta S_{YM}^\text{sol}[A] - \frac{R}{2 g_5^2} \int d^4 x \left( e^{\phi(z)} \partial_z A^\mu \delta A_\mu \right)_{z=0}, \\
\delta S_{CS}[A] &= \delta S_{CS}^\text{sol}[A] + \frac{k \cdot N_c}{6 \pi^2} \int d^4 x \, \epsilon^{\mu\nu\rho\sigma} A_\nu F_{\rho\sigma} \delta A_\mu \bigg|_{z=0}. \tag{33, 34}
\end{align*}
\]

The 5D parts are equal to zero on-shell, so that one gets

\[
\begin{align*}
J_{L}^{\mu}(x) &= - \frac{R}{2 g_5^2} \frac{e^{-\phi(z)}}{z} \partial_z L^\mu + \frac{k \cdot N_c}{6 \pi^2} \epsilon^{\mu\nu\rho\sigma} L_{\nu} F_{\rho\sigma}^L, \\
J_{R}^{\mu}(x) &= - \frac{R}{2 g_5^2} \frac{e^{-\phi(z)}}{z} \partial_z R^\mu - \frac{k \cdot N_c}{6 \pi^2} \epsilon^{\mu\nu\rho\sigma} R_{\nu} F_{\rho\sigma}^R. \tag{35, 36}
\end{align*}
\]

Recalling that \( \frac{R}{g_5^2} = \frac{N_c}{12 \pi^2} \) and \( V_\mu = L_\mu + R_\mu \) the following expression for the divergence of the vector current is obtained

\[
\begin{align*}
\partial_\mu J^\mu &= \partial_\mu (J_{L}^{\mu} + J_{R}^{\mu}) = - \frac{N_c}{24 \pi^2} \frac{e^{-\phi(z)}}{z} \partial_\mu \partial_\nu V^\nu + \frac{k \cdot N_c}{6 \pi^2} \epsilon^{\mu\nu\rho\sigma} (\partial_\mu L_\nu F_{\rho\sigma}^L - \partial_\mu R_\nu F_{\rho\sigma}^R) \\
&= - \frac{N_c}{24 \pi^2} \frac{e^{-\phi(z)}}{z} \partial_\mu \partial_\nu V^\nu + \frac{k \cdot N_c}{3 \pi^2} \epsilon^{\mu\nu\rho\sigma} \partial_\mu V_\nu \partial_\rho A_\sigma. \tag{37}
\end{align*}
\]
To express the divergence of the vector field $V_\mu$ another equation of motion generated by $\frac{\delta S}{\delta A_z}$ will be needed

$$\frac{\delta S_{YM}[A]}{\delta A_z} = \frac{R}{2g^2} \frac{e^{-\phi(z)}}{z} \partial_\mu \partial_\mu A^\mu = \frac{N_c}{24\pi^2} \frac{e^{-\phi(z)}}{z} \partial_\mu \partial_\mu A^\mu,$$

$$\frac{\delta S_{CS}[A]}{\delta A_z} = -\frac{kN_c}{24\pi^2} \frac{\delta}{\delta A_z} \int d^4x \ d\bar{z} \ \epsilon^{\mu \nu \rho \sigma} (A_\nu F_{\mu \rho} F_{\sigma \nu} - 4 A_\mu F_{\nu} F_{\rho \sigma}) =$$

$$= -\frac{kN_c}{2\pi^2} \epsilon^{\mu \nu \rho \sigma} \partial_\mu A_\nu \partial_\rho A_\sigma. \quad (38)$$

The corresponding e.o.m.’s assume the form:

$$\frac{e^{-\phi(z)}}{z} \partial_\mu \partial_\mu L^\mu = 12k \epsilon^{\mu \nu \rho \sigma} \partial_\mu L_\nu \partial_\rho L_\sigma,$$

$$\frac{e^{-\phi(z)}}{z} \partial_\mu \partial_\mu R^\mu = 12k \epsilon^{\mu \nu \rho \sigma} \partial_\mu R_\nu \partial_\rho R_\sigma,$$

$$\frac{e^{-\phi(z)}}{z} \partial_\mu \partial_\mu V^\mu = 12k \epsilon^{\mu \nu \rho \sigma} \partial_\mu V_\nu \partial_\rho A_\sigma. \quad (39)$$

Thus the divergence in (37) equals:

$$\partial_\mu J^\mu = \frac{kN_c}{6\pi^2} \epsilon^{\mu \nu \rho \sigma} \partial_\mu V_\nu \partial_\rho A_\sigma. \quad (40)$$

### 2.4 The Bardeen counterterm

The Bardeen counterterm has a dimensionless prefactor $c$ that is determined by the following condition

$$\partial_\mu J^\mu_{subtracted} = \partial_\mu J^\mu + \partial_\mu J^\mu_{Bardeen} = 0, \quad (41)$$

where the counterterm has the form

$$S_{Bardeen} = c \int d^4x \epsilon^{\mu \nu \rho \sigma} L_\mu R_\nu (F_{\rho \sigma}^L + F_{\rho \sigma}^R). \quad (42)$$

It may be interpreted as a surface counterterm in the spirit of the holographic renormalization. In our case

$$S_{Bardeen} = -2c \int d^4x (L_0 R_3 - L_3 R_0) \partial_\nu V_\nu \bigg|_{z=0} = 2cB V_{4D} (L_0 R_3 - L_3 R_0) \bigg|_{z=0} =$$

$$= 2cB V_{4D} (\mu_L j_R - \mu_R j_L) = 2cB V_{4D} (\mu_5 j - \mu j_5). \quad (43)$$

hence

$$J_{Bardeen} = 4cB \mu_5, \ J_{Bardeen} 5 = -4cB \mu. \quad (44)$$
The general expression for the currents is the following:

\[ \delta S_{\text{Bardeen}} = \int d^4x \left( J_{\text{Bardeen} \ L}^\mu \delta L_\mu + J_{\text{Bardeen} \ R}^\mu \delta R_\mu \right) \]

\[ J_{\text{Bardeen} \ L}^\mu = 2c\epsilon^{\mu\nu\rho\sigma} \left( R_\nu \partial_\rho R_\sigma + 2R_\nu \partial_\rho L_\sigma + L_\nu \partial_\rho R_\sigma \right), \]

\[ J_{\text{Bardeen} \ R}^\mu = -2c\epsilon^{\mu\nu\rho\sigma} \left( L_\nu \partial_\rho L_\sigma + 2L_\nu \partial_\rho R_\sigma + R_\nu \partial_\rho L_\sigma \right), \]

\[ J_{\text{Bardeen}}^\mu = 2c\epsilon^{\mu\nu\rho\sigma} \left( R_\nu \partial_\rho R_\sigma - L_\nu \partial_\rho L_\sigma - 3L_\nu \partial_\rho R_\sigma + 3R_\nu \partial_\rho L_\sigma \right). \quad (45) \]

The divergence of the Bardeen current equals:

\[ \partial_\mu J_{\text{Bardeen}}^\mu = -2c\epsilon^{\mu\nu\rho\sigma} \partial_\mu V_\nu \partial_\rho A_\sigma \]

(46)

Based on (40, 41, 46) one gets

\[ c = -\frac{kN_c}{12\pi^2}. \quad (47) \]

As a result, the subtracted current turns out to be zero (30, 44, 47):

\[ J_{\text{subtracted}} = J + J_{\text{Bardeen}} = \frac{kN_c}{3\pi^2} B_\mu 5 + 4c B_\mu 5 = \frac{kN_c}{3\pi^2} B_\mu 5 + 4 \left( -\frac{kN_c}{12\pi^2} \right) B_\mu 5 = 0. \quad (48) \]

3 Scalars and Pseudoscalars

3.1 Kinetic term and potential

Let us consider now the scalar–pseudoscalar sector, which was first omitted from our considerations. The bilinear part is

\[ S_X = \int d^4xdz \ e^{-\phi} \ R^3 \left[ \frac{1}{x^3} (D^\mu X)^\dagger D^\mu X + \frac{3}{x^5} |X|^2 \right] \quad (49) \]

where \( D_\mu = \partial_\mu X - iL_\mu X + iXR_\mu \); field \( X \) is related to pion field via

\[ X = \exp \left( i\frac{\pi^a t^a}{f_\pi} \right) \frac{1}{2} v(z). \quad (50) \]

What is crucial for our case is that there is a scalar interaction with gauge fields. It can be thought of in two different ways. If one works in \( A_z = 0 \) gauge (both \( L_z = 0 \) and \( R_z = 0 \)), than pion is identified with phase of field \( X \) as in (50), and interaction term is

\[ S_{XAA} = \frac{N_c}{24\pi^2} \text{tr} \int d^4xdz \ \partial_\mu V_\nu \partial_\lambda V_\rho \frac{\partial_\alpha \pi}{f_\pi} \epsilon^{\mu\nu\lambda\rho\alpha}. \quad (51) \]
If, however, one does not impose this gauge, then the holonomy of the axial field $\int A_z \, dz$ is itself the pion field, and the term (51) arises directly from Chern–Simons. Note there is no double-counting here: when dealing with Chern-Simons solely (as was the case in the Sakai-Sugimoto model of [10]), $A_z$ can always be set to zero. This is impossible without inducing a phase of $X$ in the true AdS/QCD model by Karch–Katz–Son–Stephanov [18] which we work in.

Taking action (51) and differentiating it over $F_{z3}$ one gets the following contribution to current

$$J_{XAA} = \frac{N_c}{2\pi^2} \frac{1}{3} B \frac{\partial_0 \pi(x)}{f_\pi}.$$  (52)

The special care concerns the boundary conditions. We would like to argue that the linear in time “rotating” boundary conditions are appropriate. Let us remind the PCAC relation connecting the axial current and the pion field

$$\bar{\Psi} \gamma_\nu \gamma_5 \Psi \leftrightarrow f_\pi \partial_\nu \pi$$  (53)

which implies that we add the following term in the pion lagrangian

$$\mu_5 f_\pi \partial_0 \pi.$$  (54)

This term changes the pion canonical momentum and condition of the vanishing of the canonical momentum yields the rotating boundary condition

$$P = \partial_0 \pi + \mu_5 f_\pi = 0$$  (55)

Collecting all the terms we get

$$J_{\text{full, subtracted}} = J + J_{\text{Bardeen}} + J_{XAA} = \frac{1}{3} J_{\text{FKW}}$$
$$J_{\text{full, nonsubtracted}} = J + J_{XAA} = J_{\text{FKW}}$$  (56)

### 3.2 Chern-Simons action with scalars

The result of the previous section can be justified from a somewhat different point of view. Let us once more consider the Chern–Simons action

$$S_{CS} = \frac{-k N_c}{24 \pi^2} \left( \int L \wedge dL \wedge dL - \int R \wedge dR \wedge dR \right).$$  (57)

Its gauge transformation ($L \rightarrow L + d\alpha_L$, $R \rightarrow R + d\alpha_R$) is proportional to a surface term

$$S_{CS} \rightarrow S_{CS} + \frac{-k N_c}{24 \pi^2} \left( \int d\alpha_L \wedge dL \wedge dL - \int d\alpha_R \wedge dR \wedge dR \right).$$  (58)
While in the standard field theory this is satisfactory, in our particular consideration this boundary term is nonzero and the gauge invariance is violated. When the component $A_z$ is gauged out, one has to introduce in some other way the pion back into the Chern–Simons action.

One may proceed in the following way. An explicitly gauge invariant Chern–Simons term is defined as

$$\bar{S}_{CS} = \frac{-k N_c}{24 \pi^2} \left( \int (L + d\phi_L) \wedge dL \wedge dL - \int (R + d\phi_R) \wedge dR \wedge dR \right), \quad (59)$$

where $\phi_{L,R}$ are scalar fields that transform so as to keep the combinations within the brackets invariant, $\phi_{L,R} \to \phi_{L,R} - \alpha_{L,R}$. This means that the combination $f_\pi (\phi_R - \phi_L)$ may be associated with the five-dimensional pion in the gauge in which $A_z$ is set to zero.

As in the previous section arguments can be made in favor of setting the scalar fields proportional to the chemical potentials on the ultraviolet holographic boundary at $z = 0$

$$\phi_{L,R} \big|_{z=0} = -\frac{1}{2} \mu_{L,R} \cdot t. \quad (60)$$

To clarify the infrared behavior ($z \to \infty$) of the scalars let us turn once more to the Sakai–Sugimoto model [16, 17], in which the infrared region is the area where the $D8$ and $\bar{D}8$ branes connect. There the Chern–Simons action is a single integral over both $D8$ and $\bar{D}8$ branes

$$S_{CS} \sim \int A \wedge dA \wedge dA, \quad (61)$$

where the holographic coordinate $\rho = 1/z$ runs from $-\infty$ (right $\bar{D}8$ brane) to $\infty$ (left $D8$ brane) and the gauge field $A$ is associated with the left-handed field $L$ of the Karch–Katz–Son–Stephanov model at $\rho > 0$ and with the right-handed field $R$ at $\rho < 0$.

We might undertake a similar procedure of making this action explicitly gauge invariant by adding a single scalar $\phi$

$$\bar{S}_{CS} \sim \int (A + d\phi) \wedge dA \wedge dA, \quad (62)$$

and this scalar field will be analogous to $\phi_L$ ($\phi_R$) for positive (negative) values of $\rho$.

Now, since the field $\phi$ is smooth and continuous at $\rho = 0$ a boundary condition is obtained for the $\phi_{L,R}$ fields in our setup

$$\text{for all } x_\mu \phi_L \big|_{z=\infty} = \phi_R \big|_{z=\infty}. \quad (63)$$
It is quite analogous to the condition (14) from the third section of this paper.

In what follows it will be assumed that the gauge fields are adiabatically tuned out in the temporal positive and negative infinities.

Let us simplify the modification of the Chern–Simons action

$$S_{\phi AA} = \bar{S}_{CS} - S_{CS}$$

which happens to be a surface term

$$S_{\phi AA} = -\frac{kN_c}{24\pi^2} \left( \int d\phi_L \wedge dL \wedge dL - \int d\phi_R \wedge dR \wedge dR \right)$$

$$= \frac{kN_c}{6\pi^2} B \left( \int dz \ d^4x \ \partial_t \phi_L \partial_z L_3 - \int dz \ d^4x \ \partial_t \phi_R \partial_z R_3 \right).$$

(64)

If the surface terms that arise at temporal infinities are neglected, the following is obtained

$$S_{\phi AA} = \frac{kN_c}{6\pi^2} B \left[ (j_L \partial_t \phi_L - j_R \partial_t \phi_R) \bigg|_{z=0} \right.$$  

$$+ \left( \mu_5 + \frac{1}{2} j_L + \frac{1}{2} j_R \right) \left( \partial_t \phi_L - \partial_t \phi_R \right) \bigg|_{z=\infty} \right].$$

(65)

Due to the boundary condition (63) the second term vanishes and another contribution to the current is found

$$J_{\phi AA} = \frac{kN_c}{6\pi^2} B \mu_5.$$  

(66)

The total current now equals

$$J_{full, \ subtracted} = J + J_{Bardeen} + J_{\phi AA} = \frac{1}{3} \frac{kN_c}{2\pi^2} B \mu_5.$$  

(67)

$$J_{full, \ nonsubtracted} = J + J_{\phi AA} = \frac{kN_c}{2\pi^2} B \mu_5.$$  

(68)

Here is a summary of all the contributions to the chiral magnetic effect.

| Term in the action | Yang–Mills | Chern–Simons | Bardeen counterterm | Scalars in CS |
|--------------------|------------|--------------|---------------------|-------------|
|                     | bulk       | boundary     | bulk                | boundary    |
| Contribution to the current | $-\frac{1}{3} \frac{N_c}{2\pi^2} B \mu_5$ | $\frac{1}{3} \frac{N_c}{2\pi^2} B \mu_5$ | $\frac{1}{3} \frac{N_c}{2\pi^2} B \mu_5$ | $\frac{2}{3} \frac{N_c}{2\pi^2} B \mu_5$ |

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| Action taken into account | Total | Total without scalars |
|---------------------------|-------|-----------------------|
| Resulting current, in terms of $\frac{N_c}{2\pi^2}B\mu_5$ | $\frac{1}{3}$ | $1$ | $0$ | $\frac{2}{3}$ |

That is, we see that with the Bardeen counterterm excluded we have reproduced the initial result for the CME current. The natural question concerns the relation of our answer with the one obtained in the Sakai-Sugimoto model. Naively both models describe the same system, however in the soft-wall model we have extracted the contribution from the scalars explicitly. The point which yields the different answers follows from the nontrivial boundary conditions imposed on the meson fields in the CME situation. If we take them carefully into account, the additional term has to be considered in the Sakai-Sugimoto model as well. Note that the question concerning the necessity of Bardeen counterterm is very subtle and deserves the additional study. It is the account of this term that results in an answer for the axial current in a theory with a vector chemical potential which is different from [8,9].

4 On the dynamical neutralization of $\mu_5$

In this Section we make a comment concerning the dual interpretation of the chiral chemical potential and mention the nonperturbative effect of its dynamical neutralization most clearly seen in the dual formulation. Let us first remind the expression for the Goldstone-Wilczek current [19] for the fermions in the external pseudoscalar and electromagnetic fields

$$J_{GW,\mu} = \epsilon_{\mu\nu\alpha\beta} \partial_\nu \phi F_{\alpha\beta}$$

That is, the chiral magnetic effect in a generic situation can emerge from the pseudoscalar field linear in time, $\phi = \mu_5 t$.

Consider now the dual representation of the pseudoscalar field in $D = 4$ which reads as

$$\partial_\mu \phi \propto \epsilon_{\mu\nu\alpha\beta} \partial_\nu B_{\alpha\beta}$$

where $B_{\mu\nu}$ is the antisymmetric two-form field which in $D = 4$ has only one nontrivial degree of freedom. The non-vanishing constant $\mu_5$ means that we have constant curvature of the two-form field $H = dB \propto \mu_5$. Actually $\mu_5$ corresponds to the "magnetic" part of the curvature $H$.

The string is charged with respect to the two-form field, that is, at $\mu_5 \neq 0$ the issue of magnetic string production via Schwinger mechanism can be posed. This process of closed string production screens the initial value of $\mu_5$. To calculate the probability rate one can consider the Euclidean classical solution for the string in the
external field. The classical solution corresponds to the spherical worldsheet and the surface tension term is compensated by the volume term from the external field

$$S_{eff}(R) = \pi R^2 T - 4/3 \pi R^3 H$$  \hspace{1cm} (71)

where $T$ is the effective tension of the string. The extremization of the effective action yields the critical radius $R$ of the bounce configuration

$$R_0 = \frac{T}{2H}. \hspace{1cm} (72)$$

That is, the non-perturbative probability rate to create the closed string reads as

$$w \propto \exp\left(-\frac{\pi T^3}{12H^2}\right) \hspace{1cm} (73)$$

To justify the validity of the semi-classical analysis the action calculated at the bounce configuration is assumed to be large. To this aim let us note that the magnetic string can be considered as the D2 brane wrapped over one internal direction $[20]$ and its tension tends to vanish at the $T < T_c$. That is at small temperatures the situation is similar to the absolute instability of the electric field in the massless QED. In the case under consideration the $\mu_5$ background at $T < T_c$ is similarly unstable with respect to the magnetic string production and the very introduction of the chiral chemical potential seems to be impossible.

However, it is reasonable to discuss this process above the phase transition. Two features of this region are relevant. First, the tension of the magnetic string above the transition point is finite $[20]$ and the semi-classical calculation above is reasonable and the notion of the chiral chemical potential can be introduced. Secondly, the problem becomes effectively three-dimensional and the two-form field in $D = 3$ becomes non-dynamical. This means that the value of the pseudoscalar can only jump crossing the string worldsheets. It provides a kind of capacitor for the strings.

Usually the tunneling is assumed to proceed at zero energy, however having in mind that in the heavy-ion collisions the induced tunneling at nonzero energy has to be considered at equal footing. There is some peculiarity concerning the induced string production in the external two-form field. The point is that the process involves the intermediate Minkowski region before the Euclidean part of the path similar to the induced false vacuum decay in the $D > 2$. The Minkowski region solution corresponds to the oscillating string which at the transition point has to be glued with the Euclidean solution. Such a two-step solution has been considered in $[24]$ in a slightly different context. Let us also note that at some quantized values of the energy the resonant tunneling takes place and the probability rate for the string production increases dramatically. The example of the resonant tunneling has been considered in $[24]$. 

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It was suggested in [2] that non-vanishing value of the chiral chemical potential is due to the instantons. The discussion of this section implies that the instantons are not directly related to $\mu_5$ since they do not provide the constant curvature of the two-form field and the different nonperturbative solutions like magnetic string could be relevant. The role of the instantons in the two-form field theory is different and they yield the effective mass for the fluctuations of the two-form field similar to the generation of the axion mass in the dual representation [25].

5 Conclusion

In this note the holographic derivation of the chiral magnetic effect has been revisited in the soft-wall AdS/QCD model. Unlike the estimate via the Sakai-Sugimoto model [10], in soft-wall model the effect is present under reasonable boundary conditions; the difference between our model and one in [10] is the presence of an additional contribution from the scalar part of the action. Putting it loosely, scalars act as “catalysts” for the effect, the value of which is determined however not by those, but by the Chern–Simons term. Thus the effect is still topological in its nature, as it is within the standard paradigm; though triggered by scalars, it is a robust prediction in the sense of its independence on the Lagrangian of the scalars. Notice that the effect is trivially absent in the D3/D7 model due to the different form of the Chern-Simons term.

The dual representation of the chiral chemical potential has been suggested and an effect which depends on $\mu_5$ non-analytically has been mentioned. This non-perturbative Schwinger-type effect of string creation is responsible for the dynamical neutralization of the chiral chemical potential and we have emphasized its strong dependence on the temperature.

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Note Added

Since the completion of our paper there have been two new papers on the similar subject. The paper [26] deals with the proper introduction of the chiral chemical potential in the theory under consideration. It was argued that $\mu_5$ has to be coupled
to the conserved chiral current, that is the initial fermionic current has to be modified by the anomalous contribution. To compare this argument with our approach note that our additional contribution involving the scalar does the same job. Indeed, we have argued that vanishing of the canonical momentum of the scalar implies that the scalar field has a constant time gradient proportional $\mu_5$. Substituting this expression for the gradient of the scalar into our additional CS term we get the expression for the conserved current discussed in [26].

In the other paper [27] it has been argued that a nontrivial contribution to the holographic CME in the Lagrangian without scalars comes from the singular gauge configurations at the horizon. This statement is still to be clarified. Comparing this point to our approach with the scalar field we can assume that the nontrivial effect due to this field may somehow be related with the singular solutions in [27] and could influence the choice of the gauge. This point certainly needs further clarification.

It is well-known from the study of the triangle diagram that there is an ambiguity in the regularization which allows to obtain either conserved vector or axial currents. In the standard situation demanding the conserved vector current we get the anomaly in the axial current. In the study of the chiral magnetic effect we can assume that the axial current is conserved instead of introducing the chiral chemical potential. It is this unusual viewpoint that amounts to the discussion on the role of the Bardeen counterterm.

In our model we focus on the scalar Goldstone-Wilczek contribution to the vector current which is familiar in the chiral theory. This GW contribution has been overlooked in the previous papers on this issue. For generality we have presented the different answers which correspond to different ways to account for the Bardeen counterterm.

Rubakov calculates the value of current for a differently defined chemical potential. Namely, his “axial chemical potential” is related to a conserved chiral charge, whereas ours is not. The difference manifests itself in whether we include the Bardeen counterterm into the calculation – it should be taken into account if we treat the axial chemical potential as a temporal component of a gauge field.

On the other hand, if the Bardeen counterterm is left out, our result may be compared with Rubakov’s. In that case our result agrees with both the weak-coupling and with Rubakov’s results. Furthermore, the scalar field contribution in our calculation corresponds to the anomalous term in the conserved chiral charge according to Rubakov.

The said ambiguity is that between choice of different models, not within our model itself; the coefficients in the action of both our and Rubakov’s model are topologically fixed.

Our calculations allow to extract an expression for the axial current $J_5 = -\frac{1}{3} \frac{N_c\mu}{2\pi^2 B}$ (with the Bardeen counterterm left out) which is different from one in the paper by
Metlitski and Zhitnitsky. This is not surprising since in their paper they consider the regularization corresponding to the conserved vector current which is necessary to introduce the standard, not the chiral chemical potential.

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