CHANCE-CONSTRAINED MULTIPERIOD MEAN ABSOLUTE DEVIATION UNCERTAIN PORTFOLIO SELECTION

PENG ZHANG
School of Economics and Management
South China Normal University
Guangzhou 510006, China

(Communicated by Yuanguo Zhu)

Abstract. In this paper, we propose a new multiperiod mean absolute deviation uncertain chance-constrained portfolio selection model with transaction costs, borrowing constraints, threshold constraints and cardinality constraints. In proposed model, the return rate of asset is quantified by uncertain expected value and the risk is characterized by uncertain absolute deviation. The chance constraints are that the uncertain expected return of the portfolio selection is bigger than the preset return of investors under the given confidence level. Cardinality constraints limit the number of assets in the optimal portfolio and threshold constraints limit the amount of capital to be invested in each asset and prevent very small investments in any asset. Based on uncertain theories, the model is converted to a dynamic optimization problem. Because of the transaction costs and cardinality constraints, the multiperiod portfolio selection is a mix integer dynamic optimization problem with path dependence, which is NP hard problem. The proposed model is approximated to a mix integer dynamic programming model. A novel discrete iteration method is designed to obtain the optimal portfolio strategy, and is proved linearly convergent. Finally, an example is given to illustrate the behavior of the proposed model and the designed algorithm using real data from the Shanghai Stock Exchange.

1. Introduction. In 1952, Markowitz [29] provided the mean-variance portfolio selection problem. In proposed model, return is measured by the mean of the portfolio return, and risk is measured by the variance of the portfolio return. After that, numerous portfolio selection models have been developed to improve and extend the mean-variance method. A typical extension is Konno and Yamazaki (1991) [17], which employed absolute deviation to measure the risk of the portfolio return and formulated a mean-absolute deviation model. This model can cope with large-scale portfolio optimization because it can remove most of the difficulties associated with the classical Markowitz’s model while maintaining its advantages. When all the returns are normally distributed random variables, the authors showed that the mean-absolute deviation model gave essentially the same results as the mean-variance model.
In Markowitz’s theoretical framework, an implicit assumption is that future returns of securities can be correctly reflected by past performance. In other words, security returns should be represented by random variables whose characteristics such as expected value and risk may accurately be calculated based on the sample of available historical data. It keeps valid when large amounts of data are available such as in the developed financial market. However, investors often encounter the situation that there is a lack of data about security returns just like in an emerging market. In many cases, security returns are beset with ambiguity and vagueness. In particular, when little information is available, fuzzy approaches are, in general, more appropriate. Thus, it is worthwhile to use fuzzy set theory to investigate the uncertainty of financial markets. Following the widely used fuzzy set theory in Zadeh (1965) [44], researchers have realized that they could use the fuzzy set theory to investigate portfolio selection problems under uncertain environment. Numerous models have been proposed by using different approaches. Qin (2017) [35] proposed a random fuzzy mean absolute deviation portfolio selection model. Some researchers extend the single period fuzzy portfolio selection into multiperiod setting. By using experts’ judgments, Sadjadi et al. (2011) [37] formulated a fuzzy multiperiod portfolio selection model with different rates for borrowing and lending by using fuzzy set theory. Zhang et al (2012, 2014) [45,46], and Liu et al (2012, 2013) [26,27] respectively proposed several kinds of multiperiod fuzzy portfolio selection models. Zhang and Zhang (2014) [47] proposed a multiperiod mean absolute deviation fuzzy portfolio selection model with cardinality constraints.

Though possibility measure has been widely used in portfolio selection, it has limitation. One great limitation is that possibility measure is not self-dual. Using possibility measure, we can find that two fuzzy events with different occurring chances may have the same possibility value. In addition, whenever the possibility value of a portfolio return greater than a target value is lower than 1, the possibility value of the opposite event (i.e., the portfolio return less than or equal to the target value) is the maximum value of 1; or whenever the possibility value of a portfolio return less than or equal to a target value is lower than 1, the possibility value of the opposite event (i.e., the portfolio return greater than the target value) is the maximum value of 1. These results are quite awkward and will confuse the decision maker. Thus, Liu (2007) [24] proposed uncertainty theory to describe the subjective imprecise quantity. Based on this framework, much work is undertaken to develop the theory and related practical applications. Uncertainty theory is also applied to model the portfolio selection. Huang and Qiao (2012) [15] established a risk index model for uncertain portfolio selection. Qin and Kar (2013) [34] formulated an uncertain mean-variance model. As extensions, Zhang (2016) [48] proposed the multiperiod mean absolute deviation uncertain portfolio selection with transaction costs, borrowing constraints and threshold constraints.

Recently, scholars have been showing great enthusiasm in portfolio selection and tried to use chance-constrained programming to develop the theory of portfolio selection. Liu (2007) [24] developed a spectrum of general forms of fuzzy chance-constrained programming and a general uncertain chance-constrained programming. Huang (2006) [14] proposed a credibility-based chance-constrained portfolio selection model that considered only the basic objective of return, and did not consider traditional return-risk tradeoff of portfolio selection. To avoid concentrative investment, a risk constraint is added to the fuzzy chance-constrained portfolio selection model. Li et al. (2010) [22], Gupta et al. (2013) [12] have proposed a multiobjective
credibilistic model with fuzzy chance constraints of the portfolio selection problem. Omidi et al. (2017) [32] presents a neural network model for solving uncertain portfolio selection model with chance constraint.

Although the case of a long-term investment horizon is of greater importance in practice, much less has been done in that area. The first formulation of the multiperiod portfolio selection problem has already been given in the book of Markowitz (1959) [30]. Although it is heavily discussed in recent literature, to the best of our knowledge, a closed-form solution is not available in the general case up to now. Closed-form solutions are presented only under the assumption of independence, ie. Li and Ng (2000) [20] used dynamic programming approach to deal with the multiperiod mean variance portfolio selection problem by using the idea of embedding the problem in a tractable auxiliary problem. Then, they obtained breakthrough result, that is, the optimal mean-variance portfolio policy and the efficient frontier; Zhu et al. (2004) [50] incorporated a control of the probability of bankruptcy in the generalized mean variance formulation for multiperiod portfolio optimization; Yu et al. (2012) [43] discussed a dynamic portfolio optimization problem with risk control for the absolute deviation model; Zhu et al. (2004) [50] propose a multiperiod portfolio optimization problem with stochastic cash flows.

In many existing models are proposed on the framework of Markowitz’s mean variance portfolio selection problem within the game theoretic framework for a defined-contribution pension scheme member. For more general models, the solution is frequently determined by a numerical procedure i.e. van Binsbergen and Brandt (2007) [39] compared the numerical performance of value function iterations with portfolio weight iterations in the context of the simulation-based dynamic programming approach; Mansini et al. (2007) [28] presented multiperiod mean CVaR portfolio selection model; Gülpinar and Rustem (2007) [13] extend the multiperiod mean-variance optimization framework to worst-case design with multiple rival return and risk scenarios; Köksalan and Şakar (2016) [18] consider expected return, conditional value at risk, and liquidity criteria in a multiperiod portfolio optimization setting modeled by stochastic programming.

In many existing models are proposed on the framework of Markowitz’s mean variance with cardinality constraints, threshold constraints and so on. These real constraints come from real-world practice where the administration of a portfolio made up of many assets is clearly not desirable because of transaction costs, complexity of management, or policy of the asset management companies. Because of its practical relevance, the cardinality constrained Markowitz model, and some variations have been intensively studied in the last decade. Especially from the computational viewpoint, some researchers proposed exact solution methods, ie., Bertsimas and Shioda (2009) [2]; Li et al. (2006) [21]; Murray and Shek (2012) [31]; Cesarone et al. (2013) [3]; Cui et al. (2013) [9]; Le Thi et al. (2014) [38]. Since exact solution methods are able to solve only a fraction of practically useful LAM
The contribution of this work is as follows. We originally represent uncertain absolute deviation to measure portfolio risk, and propose a new multiperiod mean absolute deviation uncertain portfolio selection model with real constraints, i.e., borrowing constraints, transaction costs, threshold constraints and chance constraints. Based on uncertain theories, the model is converted to a dynamic optimization problem. Because of the transaction costs, the multiperiod portfolio selection is a dynamic optimization problem with path dependence, which is NP hard problem that is very difficult to solve. The proposed model is approximated to a dynamic programming model. A novel discrete iteration method is designed to obtain the optimal portfolio strategy, and is proved linearly convergent. We design a novel discrete iteration method for solution. Finally, we give an example to illustrate the idea of the model and demonstrate the effectiveness of the designed algorithm.

This paper is organized as follows. In Section 2, several concepts, properties of uncertain measure, the definitions of the uncertain mean and the uncertain absolute deviation are introduced, respectively. In Section 3, the borrowing constraints, transaction costs, threshold constraints and chance constraints are formulated into the multiperiod portfolio, and a new multiperiod uncertain portfolio selection model is proposed. A novel discrete iteration method is proposed to solve it in Section 4. In Section 5, a numerical example is also presented to illustrate the modeling idea and the effectiveness of the designed algorithm. Finally, some conclusions are given in Section 6.

2. Preliminaries. Let \( \Gamma \) be a nonempty set, and let \( A \) be a \( \sigma \)-algebra over \( \Gamma \). Each element of \( A \) is called an event. A set function is called an uncertain measure (Liu, 2007) [24] if and only if it satisfies

**Axiom 2.1.** (Normality) \( M(\Gamma) = 1 \);

**Axiom 2.2.** (Self-duality) \( M(A) + M(A^c) = 1 \) for any event \( A \);

**Axiom 2.3.** (Subadditivity) \( M(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} M(A_i) \) for any countable sequence of events \( \{A_i\} \).

It can be proven (Liu, 2002) [23] that any uncertain measure \( M \) is increasing.

**Definition 2.1.** (Liu, 2007) [24] Let \( \Gamma \) be a nonempty set, and let \( A \) be a \( \sigma \)-algebra over it. If \( M \) is an uncertain measure, then the triplet \( (\Gamma, A, M) \) is called an uncertainty space.

**Definition 2.2.** (Liu, 2007) [24] Uncertain variable \( \xi \) is defined as a measurable function from an uncertainty space \( (\Gamma, A, M) \) to the set of real numbers \( \mathbb{R} \). That is, for any Borel set \( B \), we have

\[
\{ \gamma \in \Gamma; \xi(\gamma) \in B \} \in A
\] (1)
Definition 2.3. (Liu, 2007) [24] Let \( \xi \) be an uncertain variable. Then the expected value of \( \xi \) is defined as

\[
E[\xi] = \int_0^{+\infty} M\{\xi \geq x\} dx - \int_{-\infty}^0 M\{\xi \leq x\} dx
\]

(provided that at least one of the two integrals is finite).

Based on Definition 2.3, Liu (2009) [25] deduced the following two theorems.

Theorem 2.4. (Liu (2009) [25]) Let \( \xi \) be an uncertain variable with finite expected value. Then, for any real numbers \( a \) and \( b \), it holds that

\[
E[a\xi + b] = aE[\xi] + b
\]

Theorem 2.5. (Linearity of Expected Value Operator, Liu (2009) [25]) Let \( \xi \) and \( \eta \) be independent uncertain variables with finite expected values. Then, for any real numbers \( a \) and \( b \), it holds that

\[
E[a\xi + b\eta] = aE[\xi] + bE[\eta]
\]

Definition 2.6. (Liu, 2007) [24] An uncertain variable \( \xi \) can be characterized by an uncertainty distribution which is a function \( \Phi : \mathbb{R} \rightarrow [0, 1] \) is defined as

\[
\Phi(t) = M\{\xi \leq t\}
\]

Definition 2.7. Let \( \xi \) be an uncertain variable with finite expected value \( e \). Then the absolute deviation of \( \xi \) is defined by

\[
AD(\xi) = E[|\xi - e|]
\]

If \( \xi \) is an uncertain variable with expected value \( e \), then its absolute deviation is used to measure the spread of its distribution about \( e \).

Theorem 2.8. Let \( \xi \) be an uncertain variable with finite expected value \( e \). Then its uncertain absolute deviation is defined as

\[
AD(\xi) = \int_e^{+\infty} (1 - \Phi(r)) dr - \int_{-\infty}^e \Phi(r) dr
\]

Proof. From the Definition 2.7 and Definition 2.3, it follows that

\[
AD(\xi) = E[|\xi - e|]
\]

\[
= \int_0^{+\infty} M(|\xi - e| \geq x) dx - \int_{-\infty}^0 M(|\xi - e| \leq x) dx
\]

\[
= \int_0^{+\infty} M(\xi \geq x) dx - \int_{-\infty}^0 M(\xi \leq x) dx
\]

\[
= \int_0^{+\infty} M(\xi \geq x) dx + \int_{-\infty}^0 M(\xi \leq x) dx
\]

\[
= \int_e^{+\infty} M(\xi \geq r) dr - \int_{-\infty}^e M(\xi \leq r) dr
\]

\[
= \int_e^{+\infty} (1 - M(\xi \leq r)) dr - \int_{-\infty}^e M(\xi \leq r) dr
\]

\[
= \int_e^{+\infty} (1 - \Phi(r)) dr - \int_{-\infty}^e \Phi(r) dr
\]

Thus, the proof of the theorem is ended.

Theorem 2.9. Let \( \xi \) be an uncertain variable with finite expected value \( e \). Then for any nonnegative real numbers \( \lambda \), it holds

\[
AD(\lambda \xi) = \lambda AD(\xi)
\]
Proof. From the Definition 2.7, it follows that
\[ AD(\lambda \xi) = E[| \lambda \xi - \lambda e |] = \lambda E[| \xi - e |] = \lambda AD(\xi) \]
Thus, the proof of the theorem is ended. \qed

**Theorem 2.10.** Let $\xi$ be an uncertain variable with finite expected value $e$. Then for any nonnegative real numbers $\lambda$, it holds
\[ AD(\lambda \xi + \eta) = \lambda AD(\xi) \] (9)

Proof. From the Definition 2.7, it follows that
\[ AD(\lambda \xi + \eta) = E[| \lambda \xi + \eta - (\lambda e + \eta) |] = E[| \lambda \xi - e |] = \lambda E[| \xi - e |] = \lambda AD(\xi) \]
Thus, the proof of the theorem is ended. \qed

If $r = (a, \alpha, \beta)$ be a triangular uncertain variable, then uncertainty distribution $\Phi(r)$ can be described as:
\[
\Phi(r) = \begin{cases}
0, & \text{if } r \leq a - \alpha, \\
\frac{r-(a-\alpha)}{2\alpha}, & \text{if } a - \alpha \leq r \leq a, \\
\frac{r+\beta-a}{2\beta}, & \text{if } a \leq r \leq a + \beta, \\
1, & \text{if } r \geq a + \beta.
\end{cases} \tag{10}
\]
The triangle uncertain variable is denoted by $r(a, \alpha, \beta)$, where $\alpha \geq 0$, $\beta \geq 0$.

**Definition 2.11.** (Liu 2007) [24]. Let $\xi$ be an uncertain variable with regular uncertainty distribution $\Phi(x)$, $\delta$ be the confidence level. Then the inverse function $\Phi^{-1}(\delta)$ is called the inverse uncertainty distribution of $\xi$.

**Definition 2.12.** (Liu 2007) [24]. Let $\xi(a, \alpha, \beta)$ be a triangle uncertain variable, $\delta$ be the confidence level, and uncertainty distribution $\Phi(\xi)$ be described as Equation 10. Then the inverse function
\[
\Phi^{-1}(\delta) = \begin{cases}
(1 - 2\delta)(a - \alpha) + 2\delta a, & \text{if } \delta \leq 0.5 \\
(2 - 2\delta)a + (2\delta - 1)(a + \beta), & \text{if } \delta \geq 0.5
\end{cases} \tag{11}
\]

**Theorem 2.13.** (Liu 2007) [24]. A function $\Phi^{-1}(\xi) : (0, 1) \to \mathbb{R}$ is an inverse uncertainty distribution if and only if it is a continuous and strictly increasing function with respect to $\xi$.

**Theorem 2.14.** (Liu 2007) [24]. Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If $f$ is a strictly increasing function, then $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ has an uncertainty distribution
\[
\Psi(t) = \sup_{f(t_1, t_2, \ldots, t_n) = t} \left( \min_{1 \leq i \leq n} \Phi_i(t_i) \right), \ t \in R \tag{12}
\]
and has an inverse uncertainty distribution
\[
\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)), \ 0 < \alpha < 1 \tag{13}
\]

**Theorem 2.15.** If $r = (a, \alpha, \beta)$ be a triangle uncertain variable, the expected value of $r$ can be given by:
\[
E(r) = a + \frac{\beta - \alpha}{4} \tag{14}
\]
Proof. From the Definition 2.3, it follows that

\[
E(r) = \int_0^{+\infty} M\{r \geq x\} \, dx - \int_{-\infty}^0 M\{r \leq x\} \, dx
\]

\[
= \int_0^{+\infty} (1 - M\{r \leq x\}) \, dx - \int_{-\infty}^0 M\{r \leq x\} \, dx
\]

\[
= \int_0^{+\infty} (1 - \Phi(r)) \, dr - \int_{-\infty}^0 \Phi(r) \, dr
\]

According to Equation 10, the right-hand side of Equation 15 is

\[
\int_0^{+\infty} (1 - \Phi(r)) \, dr - \int_{-\infty}^0 \Phi(r) \, dr
\]

\[
= \int_0^{a-\alpha} (1 - 0) \, dr + \int_{a-\alpha}^a \left(1 - \frac{r - (a - \alpha)}{2\alpha}\right) \, dr
\]

\[
+ \int_a^{a+\beta} \left(1 - \frac{r + a + \beta}{2\beta}\right) \, dr + \int_{a+\beta}^{+\infty} (1 - 1) \, dr
\]

\[
= a - \alpha + \frac{3\alpha}{4} + \frac{\beta}{4} = a + \frac{\beta - \alpha}{4}
\]

According to Equation 15 and Equation 16, we can get

\[
E(r) = a + \frac{\beta - \alpha}{4}
\]

Thus, the proof of the theorem is ended. 

Theorem 2.16. Let \( r = (a, \alpha, \beta) \) be a triangle uncertain variable, which \( E(r) = a + \frac{\beta - \alpha}{4} \). Then, the uncertain absolute deviation of \( \xi \) can be given by:

\[
AD(r) = \begin{cases} 
\frac{4a^2 - 12(a - \alpha)a + 4(\alpha + \beta) - 6(\alpha - \alpha)(\alpha + \beta) + 9(\alpha - \alpha)^2 + (\alpha + \beta)^2}{32\alpha}, & \text{if } \beta \leq \alpha \\
\frac{(3\beta + \alpha)^2}{32\beta}, & \text{if } \beta \geq \alpha
\end{cases}
\]

Proof. From the Theorem 2.8, it follows that

\[
AD(r) = \int_c^{+\infty} (1 - \Phi(r)) \, dr + \int_{-\infty}^c \Phi(r) \, dr
\]

\[
= \int_{a+\beta-a}^{+\infty} (1 - \Phi(r)) \, dr + \int_{-\infty}^{a+\beta-a} \Phi(r) \, dr
\]

If \( \beta \leq \alpha \), the right-hand side of Equation 18 is

\[
\int_{a+\beta-a}^{+\infty} (1 - \Phi(r)) \, dr + \int_{-\infty}^{a+\beta-a} \Phi(r) \, dr
\]

\[
= \int_{a+\beta-a}^b \left(1 - \frac{r - (a - \alpha)}{2\alpha}\right) \, dr + \int_a^{a+\beta} \left(1 - \frac{r + a + \beta}{2\beta}\right) \, dr
\]

\[
+ \int_{a+\beta}^{+\infty} (1 - 1) \, dr + \int_0^a 0 \, dr + \int_{a-\alpha}^{a+\beta-a} \frac{r - (a - \alpha)}{2\alpha} \, dr
\]

\[
= \frac{20a^2 - 28(a - \alpha)a - 12a(a + \beta) + 10(a - \alpha)(a + \beta) + 9(\alpha - \alpha)^2 + (\alpha + \beta)^2}{64\alpha}
\]

\[
+ \frac{\beta}{4} + \frac{(3\alpha + \beta)^2}{64\alpha}
\]
his/her wealth among the risky assets and one risk-free asset in financial market for trading. An investor wants to allocate his/her initial wealth \( W \) among \( n + 1 \) assets at the beginning of period 1, and obtains the final wealth at the end of period \( T \). He/She can reallocate his/her wealth among the \( n \) risky assets at the beginning of each of the following \( T \) consecutive investment periods. Suppose that the return rates of the \( n \) risky assets at each period are denoted as triangular uncertain variables, and the returns of portfolios among different periods are independent of each other. For the sake of description, let us first introduce the following notations. Let \( x_{i0} \) be the initial investment proportion of risky asset \( i \) at period 0, \( x_{it} \) be the investment proportion of risky asset \( i \) at period \( t \), \( x_{it} \) be the investment proportion of risk-free asset at period \( t \), where \( x_{it} = 1 - \sum_{i=1}^{n} x_{it} \), \( x_{it} \) be the portfolio at period \( t \), where \( x_{it} = (x_{it}, x_{1t}, x_{2t}, \ldots, x_{nt})' \) be the lower bound of the investment proportion of risk-free asset at period \( t \), where \( x_{it} \geq x_{jt} \), \( R_{it} \) be the return of risky asset \( i \) at period \( t \), \( r_{pt} \) be the return rate of the portfolio \( x_{it} \) at period \( t \), \( r_{it} \) be the borrowing rate of the risk-free asset at period \( t \), \( l_{it} \) be the lending rate of the risk-free asset at period \( t \), \( u_{it} \) be the upper bound constraints of \( x_{it} \), \( r_{Nt} \) be the net return rate of the portfolio \( x_{it} \) at period \( t \), \( W_{it} \) be the crisp form of the holding wealth at the beginning of period \( t \), \( c_{it} \) be the unit transaction cost of risky asset \( i \) at period \( t \), \( K \) be the desired number of risky assets in the portfolio at period \( t \).

### 3. The multiperiod portfolio selection model

Assume that there are \( n \) risky assets and one risk-free asset in financial market for trading. An investor wants to allocate his/her initial wealth \( W \) among \( n + 1 \) assets at the beginning of period 1, and obtains the final wealth at the end of period \( T \). He/She can reallocate his/her wealth among the \( n \) risky assets at the beginning of each of the following \( T \) consecutive investment periods. Suppose that the return rates of the \( n \) risky assets at each period are denoted as triangular uncertain variables, and the returns of portfolios among different periods are independent of each other. For the sake of description, let us first introduce the following notations. Let \( x_{i0} \) be the initial investment proportion of risky asset \( i \) at period 0, \( x_{it} \) be the investment proportion of risky asset \( i \) at period \( t \), \( x_{it} \) be the investment proportion of risk-free asset at period \( t \), where \( x_{it} = 1 - \sum_{i=1}^{n} x_{it} \), \( x_{it} \) be the portfolio at period \( t \), where \( x_{it} = (x_{it}, x_{1t}, x_{2t}, \ldots, x_{nt})' \) be the lower bound of the investment proportion of risk-free asset at period \( t \), where \( x_{it} \geq x_{jt} \), \( R_{it} \) be the return of risky asset \( i \) at period \( t \), \( r_{pt} \) be the return rate of the portfolio \( x_{it} \) at period \( t \), \( r_{it} \) be the borrowing rate of the risk-free asset at period \( t \), \( l_{it} \) be the lending rate of the risk-free asset at period \( t \), \( u_{it} \) be the upper bound constraints of \( x_{it} \), \( r_{Nt} \) be the net return rate of the portfolio \( x_{it} \) at period \( t \), \( W_{it} \) be the crisp form of the holding wealth at the beginning of period \( t \), \( c_{it} \) be the unit transaction cost of risky asset \( i \) at period \( t \), \( K \) be the desired number of risky assets in the portfolio at period \( t \).

### 3.1. Return, risk and real constraints

In this section, we employ the uncertain mean value of the net return on the portfolio at each period to measure the return of portfolio. The risk on the return rate of portfolio at each period is quantified by the uncertain absolute deviation. The return rate of security \( i \) at period \( t \), \( R_{it} = (a_{it}, \alpha_{it}, \beta_{it}) \), is triangular uncertain variable for all \( i = 1, \ldots, n \) and \( t = 1, \ldots, T \).
The uncertain mean value of the portfolio \( x_t = (x_{ft}, x_{1t}, x_{2t}, \ldots, x_{nt})' \) at period \( t \) can be expressed as
\[
 r_{pt} = \sum_{i=1}^{n} E(R_{it})x_{it} + r_{ft}(1 - \sum_{i=1}^{n} x_{it}) \\
= \sum_{i=1}^{n} (\alpha_{it} + \frac{\beta_{it} - \alpha_{it}}{4})x_{it} + r_{ft}(1 - \sum_{i=1}^{n} x_{it}), t = 1, \ldots, T 
\] (21)

where \( r_{ft} = \begin{cases} 
 r_{lt}, & 1 - \sum_{i=1}^{n} x_{it} \geq 0 \\
 r_{bt}, & 1 - \sum_{i=1}^{n} x_{it} \leq 0 
\end{cases} \) when \( 1 - \sum_{i=1}^{n} x_{it} \geq 0 \), it denotes that lending is allowed on the risk-free asset; when \( 1 - \sum_{i=1}^{n} x_{it} \leq 0 \), it represents that borrowing is allowed on the risk-free asset.

The uncertain chance-constraint of the portfolio \( x_t = (x_{ft}, x_{1t}, x_{2t}, \ldots, x_{nt})' \) at period \( t \) can be expressed as
\[
 M \{ \sum_{i=1}^{n} R_{it}x_{it} + r_{ft}(1 - \sum_{i=1}^{n} x_{it}) \geq r_{0t} \} \geq \delta 
\] (22)

where \( r_{0t} \) is the preset value that investor excepts to obtain, \( \delta \) is confidence level.

According to Definition 2.12 and Theorem 2.14, the Equation 22 can be turned into as follows:
\[
 \sum_{i=1}^{n} \Phi^{-1}_{it}(\delta)x_{it} + r_{ft}(1 - \sum_{i=1}^{n} x_{it}) \geq r_{0t} 
\] (23)

Let the preset value be \( x_{ft}^b \), where \( x_{ft}^b \leq 0 \), the borrowing constraint of risk-free asset at period \( t \) is
\[
 x_{ft} = 1 - \sum_{i=1}^{n} x_{it} \geq x_{ft}^b 
\] (24)

We assume in the sequel that the transaction costs at period \( t \) is a V shape function of difference between the \( t \)th period portfolio \( x_t = (x_{ft}, x_{1t}, x_{2t}, \ldots, x_{nt})' \) and the \( t-1 \)th period portfolio \( x_{(t-1)} = (x_{f(t-1)}, x_{1(t-1)}, x_{2(t-1)}, \ldots, x_{n(t-1)})' \). The transaction cost for asset \( i \) at period \( t \) can be expressed by
\[
 C_{it} = c_{it} \mid x_{it} - x_{i(t-1)} \mid 
\] (25)

Hence, the total transaction costs of the portfolio \( x_t = (x_{ft}, x_{1t}, x_{2t}, \ldots, x_{nt})' \) at period \( t \) can be represented as
\[
 C_t = \sum_{i=1}^{n} c_{it} \mid x_{it} - x_{i(t-1)} \mid, t = 1, \ldots, T 
\] (26)

Thus, the net return rate of the portfolio \( x_t \) at period \( t \) can be denoted as
\[
 r_{Nt} = \sum_{i=1}^{n} (\alpha_{it} + \frac{\beta_{it} - \alpha_{it}}{4})x_{it} + r_{ft}(1 - \sum_{i=1}^{n} x_{it}) - \sum_{i=1}^{n} c_{it} \mid x_{it} - x_{i(t-1)} \mid, t = 1, \ldots, T 
\] (27)
Then, the crisp form of the holding wealth at the beginning of the period $t$ can be written as

$$W_{t+1} = W_t(1 + r_{Nt})$$

$$= W_t(1 + \sum_{i=1}^{n} (a_{it} + \frac{\beta_{it} - \alpha_{it}}{4}) x_{it} + r_{ft}(1 - \sum_{i=1}^{n} x_{it}) - \sum_{i=1}^{n} c_{it} \mid x_{it} - x_{i(t-1)} |),$$

for $t = 1, \ldots, T$ (28)

The absolute deviation of the portfolio $x_t$ can be expressed as

$$AD_t(x_t) = AD_t(r_{1t}x_{1t} + r_{2t}x_{2t} + \ldots + r_{nt}x_{nt})$$ (29)

To formulate cardinality constraints into the multiperiod portfolio model, zero-one decision variables are added as:

$$z_{it} = \begin{cases} 1, & \text{if any of asset } i \text{ of period } t(i = 1, \ldots, n; t = 1, \ldots, T) \text{ is held} \\ 0, & \text{otherwise} \end{cases}$$ (30)

where $\sum_{i=1}^{n} z_{it} \leq K$.

Threshold constraints limit the amount of capital to be invested in any risky asset and prevent very small investments in any risky asset. The threshold constraints of multiperiod portfolio selection can be expressed as

$$l_{it} \leq x_{it} \leq u_{it}$$ (31)

where $l_{it}$ and $u_{it}$ are the preset lower and upper bounds values of $x_{it}$, respectively.

3.2. The basic multiperiod portfolio optimization models. When the investors can give a tolerable level of risk at period $t$, and want to maximize the terminal wealth at the given level of risk, we get the multiperiod uncertain mean absolute deviation model with real constraints as follows:

$$\max \prod_{t=1}^{T} (\sum_{i=1}^{n} (a_{it} + \frac{\beta_{it} - \alpha_{it}}{4}) x_{it} + r_{ft}(1 - \sum_{i=1}^{n} x_{it}) - \sum_{i=1}^{n} c_{it} \mid x_{it} - x_{i(t-1)} |)$$

subject to

$$W_{t+1} = (1 + \sum_{i=1}^{n} (a_{it} + \frac{\beta_{it} - \alpha_{it}}{4}) x_{it} + r_{ft}(1 - \sum_{i=1}^{n} x_{it}) - \sum_{i=1}^{n} c_{it} \mid x_{it} - x_{i(t-1)} |))W_t \quad (a)$$

$$AD_t(r_{1t}x_{1t} + r_{2t}x_{2t} + \ldots + r_{nt}x_{nt}) \leq AD_{0t} \quad (b)$$

$$\sum_{i=1}^{n} \Phi_{it}^{-1}(\delta)x_{it} + r_{ft}(1 - \sum_{i=1}^{n} x_{it}) \geq r_{0t} \quad (c)$$

$$1 - \sum_{i=1}^{n} x_{it} \geq x_{it}^{h} \quad (d)$$

$$\sum_{i=1}^{n} z_{it} \leq K, z_{it} \in \{0, 1\}, i = 1, \ldots, n, t = 1, \ldots, T \quad (e)$$

$$l_{it}z_{it} \leq x_{it} \leq u_{it}z_{it}, i = 1, \ldots, n, t = 1, \ldots, T \quad (f)$$

where $AD_{0t}$ denotes the maximum risk level the investors can tolerate. The Model32 consists of an objective, namely, the maximization of the investors terminal wealth. Constraint (a) denotes the wealth accumulation constraint; constraint (b) indicates
the risk of portfolio selection at period \( t \) can not exceed preset value \( AD_{0t} \); constraint (c) states the chance constraint; constraint (d) indicates that the investment proportion of risk-free asset at period \( t \) must exceed the given lower bound; constraint (e) represents the desired number of assets in the portfolio must not exceed a given value \( K \); constraint (f) states the lower and upper of \( x_{it} \).

According to Qin et al. (2011) [33], if \( r_{1t}, r_{2t}, \ldots, r_{nt} \) are independent triangular uncertain variables, and \( x_{it} \geq 0, i = 1, \ldots, n, \)

\[
AD_t(r_{1t}x_{1t} + r_{2t}x_{2t} + \ldots + r_{nt}x_{nt}) = \sum_{i=1}^{n} x_{it} AD_t(r_{it})
\]

(33)

According to Equation 33, the Model 32 can be turned into as follows:

\[
\begin{align*}
\max & \prod_{t=1}^{T} \left( \sum_{i=1}^{n} (a_{it} + \frac{\beta_{it} - \alpha_{it}}{4}) x_{it} + r_{ft} (1 - \sum_{i=1}^{n} x_{it}) - \sum_{i=1}^{n} c_{it} | x_{it} - x_{i(t-1)} | \right) \\
& \begin{cases} \\
W_{t+1} = (1 + \left( \sum_{i=1}^{n} (a_{it} + \frac{\beta_{it} - \alpha_{it}}{4}) x_{it} \\
+ r_{ft} (1 - \sum_{i=1}^{n} x_{it}) - \sum_{i=1}^{n} c_{it} | x_{it} - x_{i(t-1)} | \right)) W_{t} \quad (a) \\
\sum_{i=1}^{n} x_{it} AD_t(r_{it}) \leq AD_{0t} \quad (b) \\
\sum_{i=1}^{n} \Phi_{it}^{-1}(\delta) x_{it} + r_{ft} (1 - \sum_{i=1}^{n} x_{it}) \geq r_{0t} \quad (c) \\
1 - \sum_{i=1}^{n} x_{it} \geq x_{ft}^{b} \quad (d) \\
\sum_{i=1}^{n} z_{it} \leq K, z_{it} \in \{0, 1\}, i = 1, \ldots, n, t = 1, \ldots, T \quad (e) \\
l_{it} z_{it} \leq x_{it} \leq w_{it} z_{it}, i = 1, \ldots, n, t = 1, \ldots, T \quad (f)
\end{cases}
\end{align*}
\]

(34)

where

\[
AD_t(r_{it}) = \begin{cases} \\
\frac{4a_{it}^{2} - 12(a_{it} - \alpha_{it})a_{it} + 4a_{it}(a_{it} + \beta_{it})}{32\alpha_{it}}, & \text{if } \beta_{it} \leq \alpha_{it} \\
\frac{6(a_{it} - \alpha_{it})(a_{it} + \beta_{it}) + 9(a_{it} - \alpha_{it})^{2} + (a_{it} + \beta_{it})^{2}}{32\alpha_{it}}, & \text{if } \beta_{it} \geq \alpha_{it}
\end{cases}
\]

(35)

4. Solution algorithm. In this section, the multiperiod mean absolute deviation uncertain portfolio selection problem with real constraints will be approximated into a mix integer dynamic programming problem with linear inequality constraints. A novel discrete iteration method will be proposed to solve the problem. The linear convergence of the method will be proved.

4.1. The proposed model approximated to dynamic programming problem. The sub-problem of Model 34 at period \( t \) is as follows:

\[
\begin{align*}
\max & \prod_{t=1}^{T} \left( \sum_{i=1}^{n} (a_{it} + \frac{\beta_{it} - \alpha_{it}}{4}) x_{it} + r_{ft} (1 - \sum_{i=1}^{n} x_{it}) - \sum_{i=1}^{n} c_{it} | x_{it} - x_{i(t-1)} | \right) \\
& \begin{cases} \\
W_{t+1} = (1 + \left( \sum_{i=1}^{n} (a_{it} + \frac{\beta_{it} - \alpha_{it}}{4}) x_{it} \\
+ r_{ft} (1 - \sum_{i=1}^{n} x_{it}) - \sum_{i=1}^{n} c_{it} | x_{it} - x_{i(t-1)} | \right)) W_{t} \quad (a) \\
\sum_{i=1}^{n} x_{it} AD_t(r_{it}) \leq AD_{0t} \quad (b) \\
\sum_{i=1}^{n} \Phi_{it}^{-1}(\delta) x_{it} + r_{ft} (1 - \sum_{i=1}^{n} x_{it}) \geq r_{0t} \quad (c) \\
1 - \sum_{i=1}^{n} x_{it} \geq x_{ft}^{b} \quad (d) \\
\sum_{i=1}^{n} z_{it} \leq K, z_{it} \in \{0, 1\}, i = 1, \ldots, n, t = 1, \ldots, T \quad (e) \\
l_{it} z_{it} \leq x_{it} \leq w_{it} z_{it}, i = 1, \ldots, n, t = 1, \ldots, T \quad (f)
\end{cases}
\end{align*}
\]
Model 34 can be approximated into the following model:

\[
\begin{aligned}
&\sum_{i=1}^{n} x_{it} AD_t(r_{it}) \leq AD_{it} \\
&\sum_{i=1}^{n} \Phi_{it}^{-1}(\delta) x_{it} + r ft(1 - \sum_{i=1}^{n} x_{it}) \geq r_{0t} \\
&s.t. \quad 1 - \sum_{i=1}^{n} x_{it} \geq x_{ft}^b \\
&\sum_{i=1}^{n} z_{it} \leq K, z_{it} \in \{0,1\}, i = 1, \ldots, n, t = 1, \ldots, T \\
&l_{it} z_{it} \leq x_{it} \leq u_{it} z_{it}, i = 1, \ldots, n, t = 1, \ldots, T
\end{aligned}
\] (36)

Let \(x_{i(t-1)} = x_{i(t-1)}\) where \(x_{i(t-1)}\) is preset value, \(\sum_{i=1}^{n} x_{i(t-1)} = 1 - x_{ft}^b\), the Model 34 can be approximated into the following model:

\[
\max \prod_{t=1}^{T} \left( \sum_{i=1}^{n} \left(a_{it} + \frac{\beta_{it} - \alpha_{it}}{4}\right) x_{it} + r ft(1 - \sum_{i=1}^{n} x_{it}) - \sum_{i=1}^{n} c_{it} | x_{it} - x_{i(t-1)} | \right)
\]

\[
W_{t+1} = (1 + \left( \sum_{i=1}^{n} \left(a_{it} + \frac{\beta_{it} - \alpha_{it}}{4}\right) x_{it} \\
+ r ft(1 - \sum_{i=1}^{n} x_{it}) - \sum_{i=1}^{n} c_{it} | x_{it} - x_{i(t-1)} | \right)) W_t
\]

\[
\begin{aligned}
&\sum_{i=1}^{n} x_{it} AD_t(r_{it}) \leq AD_{it} \\
&\sum_{i=1}^{n} \Phi_{it}^{-1}(\delta) x_{it} + r ft(1 - \sum_{i=1}^{n} x_{it}) \geq r_{0t} \\
&s.t. \quad 1 - \sum_{i=1}^{n} x_{it} \geq x_{ft}^b \\
&\sum_{i=1}^{n} z_{it} \leq K, z_{it} \in \{0,1\}, i = 1, \ldots, n, t = 1, \ldots, T \\
&l_{it} z_{it} \leq x_{it} \leq u_{it} z_{it}, i = 1, \ldots, n, t = 1, \ldots, T
\end{aligned}
\] (37)

The sub-problem of Model 37 at period \(t\) is as follows:

\[
\max \sum_{i=1}^{n} \left(a_{it} + \frac{\beta_{it} - \alpha_{it}}{4}\right) x_{it} + r ft(1 - \sum_{i=1}^{n} x_{it}) - \sum_{i=1}^{n} c_{it} | x_{it} - x_{i(t-1)} |
\]

\[
\begin{aligned}
&\sum_{i=1}^{n} x_{it} AD_t(r_{it}) \leq AD_{it} \\
&\sum_{i=1}^{n} \Phi_{it}^{-1}(\delta) x_{it} + r ft(1 - \sum_{i=1}^{n} x_{it}) \geq r_{0t} \\
&s.t. \quad 1 - \sum_{i=1}^{n} x_{it} \geq x_{ft}^b \\
&\sum_{i=1}^{n} z_{it} \leq K, z_{it} \in \{0,1\}, i = 1, \ldots, n, t = 1, \ldots, T \\
&l_{it} z_{it} \leq x_{it} \leq u_{it} z_{it}, i = 1, \ldots, n, t = 1, \ldots, T
\end{aligned}
\] (38)

**Theorem 4.1.** Let the optimal solution and objective function value of Model 36 respectively be \(x^*\) and \(G(x^*)\). Let the optimal solution and objective function value of Model 38 respectively be \(x^{2*}\) and \(F(x^{2*})\). Then \(G(x^*) - G(x^{2*}) \leq 4 \max_{i=1}^{n} \{c_{it}\} (1 - x_{ft}^b)\).
Proof. Because the feasible solution set of the Model 36 is same as Model 38, \( x^{1*} \) and \( x^{2*} \) are, respectively the feasible solutions of Model 36 and Model 38. Then,
\[
G(x^{1*}) \geq G(x^{2*})
\]
and
\[
F(x^{2*}) \geq F(x^{1*})
\]
that is
\[
G(x^{1*}) + F(x^{2*}) \geq G(x^{2*}) + F(x^{1*})
\]
then
\[
G(x^{1*}) - G(x^{2*}) + F(x^{2*}) - F(x^{1*}) \geq 0
\]
(39)
The right-hand side of Equation (39) is
\[
G(x^{1*}) - G(x^{2*}) + F(x^{2*}) - F(x^{1*}) = \\
\left[ \sum_{i=1}^{n} c_{it} | x_{it}^{1*} - x_{i(t-1)} | - \sum_{i=1}^{n} c_{it} | x_{it}^{1*} - \overline{x}_{i(t-1)} | \right] \\
+ \left[ \sum_{i=1}^{n} c_{it} | x_{it}^{2*} - \overline{x}_{i(t-1)} | - \sum_{i=1}^{n} c_{it} | x_{it}^{2*} - x_{i(t-1)} | \right] \\
\leq 2 \sum_{i=1}^{n} c_{it} | x_{i(t-1)} - \overline{x}_{i(t-1)} |
\]
(40)
Because \( x_{i(t-1)} \geq 0, x_{it} \geq 0,
\[
2 \sum_{i=1}^{n} c_{it} | x_{i(t-1)} - \overline{x}_{i(t-1)} | \leq 2 \sum_{i=1}^{n} c_{it} x_{i(t-1)} + 2 \sum_{i=1}^{n} c_{it} \overline{x}_{i(t-1)} \\
= 2 \max_{i=1}^{n} \{ c_{it} \} \sum_{i=1}^{n} x_{it} + 2 \max_{i=1}^{n} \sum_{i=1}^{n} x_{i(t-1)} \\
\leq 2 \max_{i=1}^{n} \{ c_{it} \} (1 - x_{ft}^b) + 2 \max_{i=1}^{n} \{ c_{it} \} (1 - x_{ft}^b) \\
= 4 \max_{i=1}^{n} \{ c_{it} \} (1 - x_{ft}^b)
\]
so
\[
G(x^{1*}) - G(x^{2*}) \leq 4 \max_{i=1}^{n} \{ c_{it} \} (1 - x_{ft}^b)
\]
Which ends the proof. \( \square \)

Because \( \max_{i=1}^{n} \{ c_{it} \} \ll r_{ft} \), where asset \( i \in \) efficient asset set of portfolio,

4 \( \max_{i=1}^{n} \{ c_{it} \} (1 - x_{ft}^b) \) is small, that \( G(x^{1*}) - G(x^{2*}) \) is also small.

If \( c_{it} = 0.003, x_{ft}^b = 0.5 \), \( G(x^{1*}) - G(x^{2*}) \leq 4 \max_{i=1}^{n} \{ c_{it} \} (1 - x_{ft}^b) \leq 4 \times 0.003 \times 1.5 = 0.018.

4.2. The smallest and biggest value of state variable at every period. In Model 37, investors can choose \( W_t \) between \( W_t^{\min} \) and \( W_t^{\max} \). \( W_t^{\min} \) and \( W_t^{\max} \) can be respectively obtained as follows:

The investor considers to maximize the expected return of the portfolio at period \( t \),

\[
\max \sum_{i=1}^{n} \left( a_{it} + \frac{\beta_{it} - \alpha_{it}}{4} \right) x_{it} + r_{ft}(1 - \sum_{i=1}^{n} x_{it}) - \sum_{i=1}^{n} c_{it} | x_{it} - \overline{x}_{i(t-1)} |
\]
Let $y_{it} = |x_{it} - x_{i(t-1)}|$. Then the Model (41) can be turned into as follows.

$$\max \sum_{i=1}^{n} \left( a_{it} + \frac{\beta_{it} - \alpha_{it}}{4} \right) x_{it} + r_{ft} (1 - \sum_{i=1}^{n} x_{it}) - \sum_{i=1}^{n} c_{it} y_{it}$$

subject to

$$\sum_{i=1}^{n} x_{it} AD_{t}(r_{it}) \leq AD_{0t}$$
$$\sum_{i=1}^{n} \Phi^{-1}(\delta) x_{it} + r_{ft} (1 - \sum_{i=1}^{n} x_{it}) \geq r_{0t}$$
$$1 - \sum_{i=1}^{n} x_{it} \geq x_{ft}^{b}$$
$$\sum_{i=1}^{n} z_{it} \leq K, z_{it} \in \{0, 1\}, i = 1, \ldots, n, t = 1, \ldots, T$$
$$l_{it} z_{it} \leq x_{it} \leq u_{it} z_{it}, i = 1, \ldots, n, t = 1, \ldots, T$$
$$y_{it} \geq x_{it} - x_{i(t-1)}; y_{it} \geq - (x_{it} - x_{i(t-1)})$$

$x_{t}^{max}$ (the optimal solution $x_{t} = (x_{ft}, x_{1t}, x_{2t}, \ldots, x_{nt})'$) can be obtained solving Model (42) by the CPLEX. $r_{Nt}^{max}$ (the biggest of $\sum_{i=1}^{n} (a_{it} + \frac{\beta_{it} - \alpha_{it}}{4}) x_{it} + r_{ft} (1 - \sum_{i=1}^{n} x_{it}) - \sum_{i=1}^{n} c_{it} y_{it}$) can also be obtained. Then, $W_{t+1}^{max}$ can be obtained as follows:

$$W_{t+1}^{max} = W_{t}^{max} \left( 1 + \sum_{i=1}^{n} (a_{it} + \frac{\beta_{it} - \alpha_{it}}{4}) x_{it}^{max} + r_{ft} (1 - \sum_{i=1}^{n} x_{it}^{max}) \right)$$
$$- \sum_{i=1}^{n} c_{it} y_{it}^{max}, t = 1, \ldots, T$$

where $W_{1}$ is initial wealth, which is preset value.

The biggest value of the absolute deviation of the portfolio selection at period $t$ ($\sum_{i=1}^{n} x_{it}^{max} AD_{t}(r_{it})$) can be also obtained.

The investor only considers to minimize the absolute deviation of the portfolio at period $t$, that is, the smallest value of the $r_{Nt}$ can be obtained as follows:

$$\min \sum_{i=1}^{n} AD_{t}(r_{it}) x_{it}$$
modi-multiplication respectively, defined on \( \mathbb{Q} \) of non-empty set \( W \) where

The discrete iteration method.

4.3. period weighted digraph can be obtained. Finally, \( k \) be introduced first. Then, the method of finding the longest path of the multi-
the starting point to the ending point, will be proposed, and some definitions will
solving the longest path of the above multiperiod weighted digraph, which is from

\[ \begin{align*}
\sum_{t=1}^{n} \Phi_{it}(\delta)x_{it} + r_{ft}(1 - \sum_{i=1}^{n} x_{it}) & \geq r_{it} \\
1 - \sum_{i=1}^{n} x_{it} & \geq y_{ft} \\
\sum_{i=1}^{n} z_{it} & \leq K, z_{it} \in \{0, 1\}, i = 1, \ldots, n, t = 1, \ldots, T \\
l_{it}z_{it} & \leq u_{it}z_{it}, i = 1, \ldots, n, t = 1, \ldots, T
\end{align*}\]  

\( \sum_{i=1}^{n} \) \( x_{it} \) (the optimal solution \( x_{t} = (x_{ft}, x_{1t}, x_{2t}, \ldots, x_{nt})' \)) can be obtained solving Model 44 by the CPLEX. Simultaneously, \( r_{Nt}^{\min} \) (the smallest of \( \sum_{i=1}^{n} (a_{it} + \beta_{it} - \alpha_{it})x_{it} + r_{ft}(1 - \sum_{i=1}^{n} x_{it}) - \sum_{i=1}^{n} c_{it} \mid x_{it} - x_{it}' \) ) is also obtained. Then, \( W_{t+1}^{\min} \) can be obtained as follows:

\[ W_{t+1}^{\min} = W_{t}^{\min} + \left( a_{it} + \frac{\beta_{it} - \alpha_{it}}{4} \right) x_{it}^{\min} + r_{ft}(1 - \sum_{i=1}^{n} x_{it}^{\min}) - \sum_{i=1}^{n} c_{it} \mid x_{it}^{\min} - x_{it}' \), \( t = 1, \ldots, T \)

where \( W_{1} \) is initial wealth, which is preset value.

The smallest value of the absolute deviation of the portfolio selection at period

\( t \) \( (\sum_{i=1}^{n} x_{it}^{\min} AD_{t}(r_{it})) \) can be also obtained.

4.3. The discrete iteration method. In this section, the max-plus algebra,
which is proposed by Heidergott et al. (2006) [16], will be used. The method solving
the longest path of the above multiperiod weighted digraph, which is from
the starting point to the ending point, will be proposed, and some definitions will
be introduced first. Then, the method of finding the longest path of the multi-
period weighted digraph can be obtained. Finally, \( k + 1 \) th iteration method will
be presented.

**Definition 4.2.** A semi-field (or semi-ring) is a triplet \( A = \{Q, \oplus, \otimes\} \) consisting of non-empty set \( Q \) and two binary operations \( \oplus \) and \( \otimes \), called modi-addition and modi-multiplication respectively, defined on \( Q \), such that for all \( a_{1}, a_{2}, a_{3} \) in \( Q \)

(i) each of the operations \( \oplus \) and \( \otimes \) is commutative

\[ a_{1} \oplus a_{2} = a_{2} \oplus a_{1}, a_{1} \otimes a_{2} = a_{2} \otimes a_{1} \]

(ii) each of the operations \( \oplus \) and \( \otimes \) is associative

\[ (a_{1} \oplus a_{2}) \oplus a_{3} = a_{1} \oplus (a_{2} \oplus a_{3}), (a_{1} \otimes a_{2}) \otimes a_{3} = a_{1} \otimes (a_{2} \otimes a_{3}) \]

(iii) the operation \( \otimes \) is distributive with respect to the operation \( \oplus \)

\[ a_{1} \otimes (a_{2} \oplus a_{3}) = a_{1} \otimes a_{2} \oplus a_{1} \otimes a_{3} \]

(iv) there exists an element in \( Q \) which is the zero element, denoted by \( \varepsilon \), such that for all \( a \) in \( Q \), we have

\[ \varepsilon \oplus a = a, \varepsilon \otimes a = a \]

If there exists an element denoted by \( e \) in a semi-field \( A = \{Q, \oplus, \otimes\} \), such that
for all \( a \) in \( Q \), we have

\[ e \otimes a = a \]

then \( e \) is called an identity element of the semi-field.
Definition 4.3. \( a_1, a_2 \in R \), in semi-field \( A = \{ R, max, + \} \), then \( \varepsilon = -\infty \) and \( e = 0 \).

Definition 4.4. Let us consider matrices \( A_{n \times n} = (a_{ij})_{n \times n}, B_{n \times n} = (b_{ij})_{n \times n}, a_{ij}, b_{ij} \in R \), in semi-field \( A = \{ R, max, + \} \). Then \( C = A \oplus B, C_{n \times n} = (c_{ij})_{n \times n} \), where \( c_{ij} = \max\{a_{ij}, b_{ij}\} \).

Definition 4.5. Let us consider matrices \( A_{n \times m} = (a_{ij})_{n \times m}, B_{m \times k} = (b_{ij})_{m \times k}, a_{ij}, b_{ij} \in R \), in semi-field \( A = \{ R, max, + \} \). Then \( C = A \oplus B, C_{n \times k} = (c_{ij})_{n \times k} \), where \( c_{ij} = \max\{a_{ij}, b_{ij}\} \).

The Model 37 is a mix integer dynamic programming problem with linear inequality constraints, the optimal solution can be obtained by the dynamic programming recursive relationship. In this section, a novel discrete iteration method is proposed. The method is as follows: Firstly, according to the network approach, discretizes the state variables and transforms the model into multiperiod weighted digraph. Secondly, uses the max-plus algebra to solve the largest path that is the admissible solution. Thirdly, based on the admissible solution, continues iterating until the two admissible solutions are real near. Finally, the method is proved linearly convergent.

The state variable \( W_t \) of the period \( t \) is discretized into four intervals of same widths from the smallest value to the biggest one. It means that there are five discrete values for the state variable in every period. In this way, Model 37 is transformed into a multiperiod weighted digraph as shown in Figure 1. The investment period, the value of the objective function of the period \( t \) and a discrete value of the state variable are respectively represented by the stage, the weight of the period \( t \) and the point of the multiperiod weighted digraph.

![Figure 1. The multiperiod weighted digraph](image)

In this section, a discrete iteration method will be proposed to solve the Model 37.

Step 1. The discrete state variables at period \( t(t = 2, \ldots, T + 1) \) can be obtained by discretizing the interval value of \( W_t^{\max} - W_t^{\min} \) into four equalities. That is

\[
W_{it} = W_t^{\min} + \frac{W_t^{\max} - W_t^{\min}}{4} \cdot i, i = 0, \ldots, 4
\]
Step 2. The weight of the arcs in Figure 1 can be obtained following three steps:

Step 2.1. The net expected return of the portfolio $r_{N1}(1, j)$, $j = 1, \ldots, 5$, can be obtained as follows:

$$r_{N1}(1, j) = \frac{W_{j2}}{W_1} - 1$$

Step 2.2. The net expected return of portfolio $r_{Nt}(j, k)$ at period $t$, which $j$ is the number of the point at period $t, j = 1, \ldots, 5$ and $k$ is the number of the point at period $t + 1, k = 1, \ldots, 5$, can be obtained as follows:

$$r_{Nt}(j, k) = \frac{W_{kt+1}}{W_{jt}} - 1$$

Step 2.3. The weights of the side at period $t$, which the values of the objective function $F_1(1, j)$ and $F_t(j, k)$ in Figure 1, can be obtained as follows:

When $r_{Nt}(k, l)$ is known, the sub-problem at period $t$ of the Model 37 can be turned into

$$\max \left( \sum_{i=1}^{n} \left( a_{it} + \frac{\beta_{it} - \alpha_{it}}{4} \right) x_{it} + r_{ft}(1 - \sum_{i=1}^{n} x_{it}) - \sum_{i=1}^{n} c_{it} \mid x_{it} - x_{it(t-1)} \right)$$

$$\begin{align*}
&\sum_{i=1}^{n} (a_{it} + \frac{\beta_{it} - \alpha_{it}}{4}) x_{it} + r_{ft}(1 - \sum_{i=1}^{n} x_{it}) - \sum_{i=1}^{n} c_{it} \mid x_{it} - x_{it(t-1)} \geq r_{Nt}(k, l) \\
&\sum_{i=1}^{n} x_{it} AD_t(r_{it}) \leq AD_{0t} \\
&\sum_{i=1}^{n} \Phi_{it}^{-1}(\delta) x_{it} + r_{ft}(1 - \sum_{i=1}^{n} x_{it}) \geq r_{0t} \\
&1 - \sum_{i=1}^{n} x_{it} \geq x^b_{ft} \\
&\sum_{i=1}^{n} z_{it} \leq K, z_{it} \in \{0, 1\}, i = 1, \ldots, n, t = 1, \ldots, T \\
&l_{it} x_{it} \leq x_{it}, i = 1, \ldots, n, t = 1, \ldots, T \\
&s.t.
\end{align*}$$

Let $y_{it} = |x_{it} - x_{it(t-1)}|$. Then the Model 46 can be turned into as follows.

$$\max \left( \sum_{i=1}^{n} \left( a_{it} + \frac{\beta_{it} - \alpha_{it}}{4} \right) x_{it} + r_{ft}(1 - \sum_{i=1}^{n} x_{it}) - \sum_{i=1}^{n} c_{it} y_{it} \right)$$

$$\begin{align*}
&\sum_{i=1}^{n} (a_{it} + \frac{\beta_{it} - \alpha_{it}}{4}) x_{it} + r_{ft}(1 - \sum_{i=1}^{n} x_{it}) - \sum_{i=1}^{n} c_{it} y_{it} \geq r_{Nt}(k, l) \\
&\sum_{i=1}^{n} x_{it} AD_t(r_{it}) \leq AD_{0t} \\
&\sum_{i=1}^{n} \Phi_{it}^{-1}(\delta) x_{it} + r_{ft}(1 - \sum_{i=1}^{n} x_{it}) \geq r_{0t} \\
&1 - \sum_{i=1}^{n} x_{it} \geq x^b_{ft} \\
&\sum_{i=1}^{n} z_{it} \leq K, z_{it} \in \{0, 1\}, i = 1, \ldots, n, t = 1, \ldots, T \\
&l_{it} x_{it} \leq x_{it}, i = 1, \ldots, n, t = 1, \ldots, T \\
&y_{it} \geq x_{it} - x_{it(t-1)} \\
&y_{it} \geq -(x_{it} - x_{it(t-1)}) \\
&s.t.
\end{align*}$$
$x_t^*$ (the optimal solution $x_t = (x_{t1}, x_{t2}, x_{t3}, \ldots, x_{tn})'$) can be obtained solving the Model 47 by CPLEX. Simultaneously, the objective function value $F_t(j, k)$ can be obtained as follows:

$$F_t(j, k) = \sum_{i=1}^{n}(a_{it} + \frac{\beta_{it} - \alpha_{it}}{4})x_{it}^* + r_{ft}(1 - \sum_{i=1}^{n}x_{it}^*) - \sum_{i=1}^{n}c_{it}y_{it}^*.$$  

**Step 3.** Calculation of the longest path of the multiperiod weighted digraph  

According to Definition 4.3, the longest path $F^{(1)}$ of the multiperiod weighted digraph of the first iteration can be obtained as follows:

$$F^{(1)} = F_1^{(1)} \otimes F_2^{(1)} \otimes \ldots \otimes F_T^{(1)}$$  

where $F_1^{(1)} = (F_1^{(1)}(1,j))_{1 \times 5}, F_2^{(1)} = (F_2^{(1)}(i,j))_{5 \times 5} \ldots, F_T^{(1)} = (F_T^{(1)}(i,j))_{5 \times 5}.$

**Step 4.** The discrete iteration method $k + 1$ th iteration can be described as follows:  

Let the longest path of the $k$ th iteration be $W_1 \rightarrow W_{i_2}^{(k)} \rightarrow W_{i_3}^{(k)} \rightarrow \ldots \rightarrow W_{i_{T+1}}^{(k)}$, then the optimal solution of the longest path $F^{(k+1)}$ also is the approximately optimal solution of the Model 37. Otherwise turn Step 2.

**Step 4.1.** $W_2^{\min}$ and $W_{i_2}^{(k)}$, $W_{i_3}^{(k)}$ and $W_2^{\max}$ are discretized into two same internals, respectively. The five discrete points of $S_2$, i.e., $W_2^{\min}, W_2^{k+1}, W_2^{k+1}, W_2^{k+1}, W_2^{\max}$ can be obtained.

**Step 4.2.** Based on $(W(k)_{i_3}, \ldots, W(k)_{i_{T+1}})$, using the same method, the state variables from period 3 to period $T + 1$ are respectively discretized into the five points. The utility of period $t$ can also be obtained.

**Step 4.3.** The longest path of the $k + 1$ th iteration $F^{(k+1)}$ and another feasible solution can be obtained as follows:

$$F^{(k+1)} = F_1^{(k+1)} \otimes F_2^{(k+1)} \otimes \ldots \otimes F_T^{(k+1)}$$  

where $F_1^{(k+1)} = (F_1^{(k+1)}(1,j))_{1 \times 5}, F_2^{(k+1)} = (F_2^{(k+1)}(i,j))_{5 \times 5} \ldots, F_T^{(k+1)} = (F_T^{(k+1)}(i,j))_{5 \times 5}.$

If $| F^{(k+1)} - F^{(k)} | \leq 10^{-6}$, then the optimal solution of the longest path $F^{(k+1)}$ also is the approximately optimal solution of the Model 37. Otherwise turn Step 2.

4.4. Convergence property of the discrete iteration method.

**Theorem 4.6.** The discrete approximate iteration method is linearly convergent.

**Proof.** Let the longest path in period 1 be $F_1^{\max}(1,j_2)$, and the longest path in period $t$ be $F_t^{\max}(i_t, j_{t+1}), t = 2, ..., T$. Then the upper bound of the solution of Model 37 is

$$F_1^{\max}(1,j_2) \times F_2^{\max}(i_2, j_3) \times \ldots \times F_T^{\max}(i_T, j_{T+1})$$

The longest path of the multiperiod weighted digraph of the $k$ th iteration $F^{(k)}$ is obtained as follows:

$$F^{(k)} = F_1^{(k)} \otimes F_2^{(k)} \otimes \ldots \otimes F_T^{(k)}$$  

where $F_1^{(k)} = (F_1^{(k)}(1,j))_{1 \times 5}, F_2^{(k)} = (F_2^{(k)}(i,j))_{5 \times 5} \ldots, F_T^{(k)} = (F_T^{(k)}(i,j))_{5 \times 5}.$

Let the longest path of the $k$ th iteration be $W_1 \rightarrow W_{i_2}^{(k)} \rightarrow W_{i_3}^{(k)} \ldots \rightarrow W_{i_{T+1}}^{(k)}$, Using the Step 4, the multiperiod weighted digraph of the $k + 1$ th iteration can be
obtained, which is the value of the longest path of the multiperiod weighted digraph of the $k+1$th iteration, can be obtained by the Equation 50. So $F^{(k+1)} \geq F^{(k)}$. The solution is becoming bigger and bigger. Because the solutions of Model 37 are increasing sets and there is an upper bound of the solution of Model 37. Then, the discrete iteration method is convergent.

Let the optimal value of period $t$ of Model 37 be $F^*_t$, $F_t^{(k)}$ be the optimal solution of the $k$th iteration at period $t$.

Because the objective function of Model 37 is linear, then $\left| F_t^{(k+1)} - F_t^* \right| \leq \left| F_t^{(k)} - F_t^* \right|$. Because $F_t^* \geq F_t^{(k+1)}, F_t^* \geq F_t^{(k)}$, then $F_t^* - F_t^{(k+1)} \leq F_t^* - F_t^{(k)}$, ie.,

$$0 \leq \frac{\sum_{i=1}^{T} |F_i^{(k+1)} - F_i^*|}{\sum_{i=1}^{T} |F_i^{(k)} - F_i^*|} \leq 1.$$ So the discrete iteration method is linearly convergent.

Thus, the proof of the Theorem 4.6 is ended. $\square$

5. **Numerical example.** In this section, a numerical example is given to express the idea of the proposed model. Assume that an investor chooses thirty stocks from Shanghai Stock Exchange for his investment. The stocks codes are respectively $S_1, \ldots, S_{30}$. He/She intends to make five periods of investment with initial wealth $W_1 = 1$ and his wealth can be adjusted at the beginning of each period. He/she assumes that the returns, risk and turnover rates of the thirty stocks at each period are represented as triangular uncertain numbers. We collect historical data of them from April 2006 to March 2017 and set every three months as a period to handle the historical data. By using the estimation method in Vercher et al. (2007) [40] to handle their historical data, the triangular uncertain distributions of the return rates of assets at each period can be obtained as shown in Appendix A. According to Equation 34 and Appendix A, $AD_t(r_{it})(i = 1, \ldots, 30; t = 1, \ldots, 5)$ can be obtained as shown in Appendix B.

Suppose that the transaction costs of assets of the two periods investment take the same value $c_{tt} = 0.003(t = 1, \ldots, 30; t = 1, \ldots, 5)$, the lower bound of the investment proportion of risk-free asset $x_{jt}^b = -0.4$, the borrowing rate of the risk-free asset $r_{bt} = 0.017$, the lending rate of the risk-free asset $r_{lt} = 0.009, t = 1, \ldots, 5$, the lower $l_{it} = 0$ and upper bound constraints $u_{it} = 0.2(i = 1, \ldots, 30; t = 1, \ldots, 5)$, the desired number of risky assets in the portfolio $K = 2, \ldots, 9$ at period $t$, $t = 1, \ldots, 5$, confidence level $\delta = 95\%$ or 99%.

In case when the $K = 3, AD_t = 0.02, r_{0t} = 0.1, \delta = 95\%$, the multiperiod uncertain mean absolute deviation portfolio selection model maximizing the terminal utility is set as follows:

If $K = 3, AD_t = 0.02, r_{0t} = 0.1, \delta = 95\%$, the optimal solution of Model 37 will be obtained as the Table 1 using the discrete iteration method.

When $K = 3, AD_t = 0.02, r_{0t} = 0.1, \delta = 95\%$, the optimal investment strategy at period 1 is $x_{31} = 0.2, x_{131} = 0.1715808, x_{171} = 0.2, x_{f1} = 0.4284192$ and being the rest of variables equal to zero, which means investor should allocate his initial wealth on asset 3, asset 13, asset 17, risk-free asset and otherwise asset by the proportions of 20%, 17.15808%, 20%, 42.84192% and being the rest of variables equal to zero among the thirty stocks, respectively. From Table 1, the optimal investment strategy at period 2, period 3, period 4 and period 5 can also be obtained. In this case, the available terminal wealth is 1.549717.
Table 1. The optimal solution when $K = 3, AD_t = 0.02, r_{0t} = 0.1, \delta = 95\%$

| Asset $i$ | $t$ | The optimal investment proportions |
|-----------|-----|----------------------------------|
| Asset 3   | 1   | 0.2 | Asset 13 | 0.171581 |
| Asset 17  | 0.2 |     | Asset 17 | 0.2 | $x_{f1}$ | 0.428419 |
| Asset 22  | 0.2 |     | Asset 29 | 0.2 | $x_{f2}$ | 0.4 |
| Asset 25  | 0.2 | 0.2 | Asset 22 | 0.2 | $x_{f3}$ | 0.450594 |
| Asset 30  | 0.2 | 0.2 | Asset 22 | 0.2 | $x_{f4}$ | 0.430612 |

In case when the $K = 6, AD_t = 0.02, r_{0t} = 0.1, \delta = 95\%$, the multiperiod uncertain mean absolute deviation portfolio selection model maximizing the terminal utility is set as follows:

If $K = 6, AD_t = 0.02, r_{0t} = 0.1, \delta = 95\%$, the optimal solution of Model 37 will be obtained as the Table 2 using the discrete iteration method.

Table 2. The optimal solution when $K = 6, AD_t = 0.02, r_{0t} = 0.1, \delta = 95\%$

| Asset $i$ | $t$ | The optimal investment proportions |
|-----------|-----|----------------------------------|
| Asset 3   | 1   | 0.2 | Asset 13 | 0.004973 |
| Asset 17  | 0.2 |     | Asset 17 | 0.2 | $x_{f1}$ | 0.195027 |
| Asset 22  | 0.2 | 0.2 | Asset 29 | 0.2 | $x_{f2}$ | 0.329614 |
| Asset 25  | 0.2 | 0.2 | Asset 22 | 0.2 | $x_{f3}$ | 0.339219 |
| Asset 30  | 0.2 | 0.2 | Asset 22 | 0.2 | $x_{f4}$ | 0.344693 |
| Asset 30  | 0.2 | 0.2 | Asset 22 | 0.2 | $x_{f5}$ | 0.15403 |

When $K = 6, AD_t = 0.02, r_{0t} = 0.1, \delta = 95\%$, the optimal investment strategy at period 1 is $x_{31} = 0.2, x_{131} = 0.00497357, x_{171} = 0.2, x_{221} = 0.2, x_{251} = 0.2, x_{f1} = 0.195026643$ and being the rest of variables equal to zero, which means investor should allocate his initial wealth on asset 3, asset 13, asset 17, asset 22, asset 25, risk-free asset and otherwise asset by the proportions of 20%, 0.4973357%, 20%, 20%, 20%, 19.5026643% and being the rest of variables equal to zero among the thirty stocks, respectively. From Table 2, the optimal investment strategy at period 2, period 3, period 4 and period 5 can also be obtained. In this case, the available terminal wealth is 1.568211.

From Table 1 and Table 2, when $K$ increases, the terminal wealth also increases, and the proportion of risk-free asset decreases.

In case when the $K = 6, AD_t = 0.03, r_{0t} = 0.1, \delta = 95\%$, the multiperiod uncertain mean absolute deviation portfolio selection model maximizing the terminal utility is set as follows:

If $K = 6, AD_t = 0.03, r_{0t} = 0.1, \delta = 95\%$, the optimal solution of Model 37 will be obtained as the Table 3 using the discrete iteration method.

When $K = 6, AD_t = 0.03, r_{0t} = 0.1, \delta = 95\%$, the optimal investment strategy at period 1 is $x_{31} = 0.2, x_{131} = 0.1825933, x_{171} = 0.2, x_{221} = 0.2, x_{251} = 0.2, x_{f1} = 0.0174067$ and being the rest of variables equal to zero, which means investor should allocate his initial wealth on asset 3, asset 13, asset 17, asset 22, asset 25, risk-free asset and otherwise asset by the proportions of 20%, 18.25933%,
Table 3. The optimal solution when $K = 6, AD_t = 0.03, r_{0t} = 0.1, \delta = 95\%$

| $t$ | Asset | Asset 13 | Asset 17 | Asset 22 | Asset 25 | $f_1$ | $f_2$ |
|-----|-------|----------|----------|----------|----------|-------|-------|
| 1   | Asset3 | 0.2      | 0.182593 | 0.2      | 0.2      | 0.017407 | 0.017407 |
| 2   | Asset15 | 0.2      | 0.2      | 0.2      | 0.2      | 0.121612 | 0.078388 |
| 3   | Asset3 | 0.2      | 0.2      | 0.2      | 0.2      | 0.04580153 | 0.154198 |
| 4   | Asset6 | 0.196736 | 0.2      | 0.2      | 0.2      | 0.003265 | 0.003265 |
| 5   | Asset15 | 0.2      | 0.2      | 0.2      | 0.2      | -0.16864 | -0.16864 |

Where $W_6$ is denoted the terminal wealth of the portfolio.

In the used data sets, the experiments in this paper correspond to the values of $K$ in the interval $[2, 9]$. It can be seen that, as will be seen in Table 4, the terminal wealth becomes bigger when $2 \leq K \leq 8$, become larger; the terminal wealth is same, when $K \geq 8$; there is no feasible solution, when $0 \leq K \leq 2$; which reflects the influence of $K$ on portfolio selection.

If $AD_t = 0.07, r_{0t} = 0.15, \delta = 95\%, K = 2, \ldots, 9$, the optimal solution of Model 37 will be obtained as the Table 4 using the discrete iteration method.

Table 4. the optimal terminal wealth and risk of the portfolio when $AD_t = 0.07, r_{0t} = 0.15, \delta = 95\%, K = 2, \ldots, 9$

| $K$ | $W_6$ |
|-----|-------|
| 2   | 1.505725 |
| 3   | 1.70608 |
| 4   | 1.923746 |
| 5   | 2.158884 |
| 6   | 2.408406 |
| 7   | 2.659334 |
| 8   | 2.659334 |

When $AD_t = 0.07, K = 3, r_{0t} = 0.18, \delta = 95\%$, the optimal investment strategy at period 1 is $x_{121} = 0.2, x_{131} = 0.2, x_{281} = 0.2, x_{f1} = 0.4$ and being the rest of variables equal to zero, which means investor should allocate his initial wealth on asset 12, asset 13, asset 28, risk-free asset and otherwise asset by the proportions of 20%, 20%, 20%, 40% and being the rest of variables equal to zero among the thirty stocks, respectively. From Table 5, the optimal investment strategy at period 2, period 3, period 4 and period 5 can also be obtained. In this case, the available terminal wealth is 1.704487.

If $AD_t = 0.07, K = 3, r_{0t} = 0.18, \delta = 99\%$, the optimal solution of Model 37 will be obtained as the Table 6 using the discrete iteration method.
Table 5. The optimal solution when $AD_t = 0.07, K = 3, r_{0t} = 0.18, \delta = 95\%$

| t  | Asset1 | Asset12 | Asset13 | Asset16 | Asset17 | Asset28 | Asset29 | $x_1$  | $x_2$  | $x_3$  | $x_4$  |
|----|--------|---------|---------|---------|---------|---------|---------|--------|--------|--------|--------|
| 1  | 0.2    | 0.2     | 0.2     | 0.2     | 0.2     | 0.2     | 0.2     |        |        |        |        |
| 2  | 0.2    | 0.2     | 0.2     | 0.2     | 0.2     | 0.2     | 0.2     |        |        |        |        |
| 3  | 0.2    | 0.2     | 0.2     | 0.2     | 0.2     | 0.2     | 0.2     |        |        |        |        |
| 4  | 0.2    | 0.2     | 0.2     | 0.2     | 0.2     | 0.2     | 0.2     |        |        |        |        |
| 5  | 0.2    | 0.2     | 0.2     | 0.2     | 0.2     | 0.2     | 0.2     |        |        |        |        |

Table 6. The optimal solution when $AD_t = 0.07, K = 3, r_{0t} = 0.18, \delta = 99\%$

| t  | Asset1 | Asset12 | Asset13 | Asset16 | Asset17 | Asset28 | Asset29 | $x_1$  | $x_2$  | $x_3$  | $x_4$  |
|----|--------|---------|---------|---------|---------|---------|---------|--------|--------|--------|--------|
| 1  | 0.2    | 0.2     | 0.2     | 0.2     | 0.2     | 0.2     | 0.2     |        |        |        |        |
| 2  | 0.2    | 0.2     | 0.2     | 0.2     | 0.2     | 0.2     | 0.2     |        |        |        |        |
| 3  | 0.2    | 0.2     | 0.2     | 0.2     | 0.2     | 0.2     | 0.2     |        |        |        |        |
| 4  | 0.2    | 0.2     | 0.2     | 0.2     | 0.2     | 0.2     | 0.2     |        |        |        |        |
| 5  | 0.2    | 0.2     | 0.2     | 0.2     | 0.2     | 0.2     | 0.2     |        |        |        |        |

When $K = 3, AD_t = 0.03, r_{0t} = 0.18, \delta = 99\%$, the optimal investment strategy at period 1 is $x_{131} = 0.2, x_{161} = 0.2, x_{281} = 0.2, x_{f1} = 0.4$ and being the rest of variables equal to zero, which means investor should allocate his initial wealth on asset 13, asset 16, asset 28, risk-free asset and otherwise asset by the proportions of 20%, 20%, 20%, 40% and being the rest of variables equal to zero among the thirty stocks, respectively. From Table 3, the optimal investment strategy at period 2, period 3, period 4 and period 5 can also be obtained. In this case, the available terminal wealth is 1.679096.

When the different confidence level $\delta$ is preset, the corresponding optimal strategies and terminal wealth can also be derived. The detailed results are shown in Table 5 and Table 6. For some rational investors, they might not only consider the expectations the terminal wealth of the portfolio, but also concern about the risk, and we can see when the confidence level $\delta$ increases, the terminal wealth become smaller.

6. Conclusions. In this paper, we use the uncertain mean value and the absolute deviation to measure the return and the risk of the multiperiod portfolio, respectively. A new multiperiod portfolio optimization model with transaction cost, borrowing constraints, threshold constraints, cardinality constraints and chance constraints is proposed. Based on the uncertain theories, the proposed model is transformed into a mix integer dynamic optimization problem. Because of the transaction costs and cardinality constraints, the multiperiod portfolio selection is a mix integer dynamic optimization problem with path dependence, which is NP hard problem that is very difficult to solve. The proposed model is approximated to a mix integer dynamic programming model. A novel discrete iteration method is designed to obtain the optimal portfolio strategy, and is proved linearly convergent.
Table 7. The uncertain return rates on assets of five periods investment.

|   | Asset 1       | Asset 2       | Asset 3       |
|---|--------------|--------------|--------------|
| 1 | 0.143        | 0.105        | 0.115        |
| 2 | 0.1449       | 0.0881       | 0.1136       |
| 3 | 0.1458       | 0.08         | 0.1127       |
| 4 | 0.1516       | 0.062        | 0.107        |
| 5 | 0.1532       | 0.0609       | 0.1054       |

Table 8. The uncertain return rates on assets of five periods investment.

|   | Asset 4       | Asset 5       | Asset 6       |
|---|--------------|--------------|--------------|
| 1 | 0.1172       | 0.0731       | 0.0613       |
| 2 | 0.1203       | 0.0743       | 0.0782       |
| 3 | 0.1255       | 0.0749       | 0.0734       |
| 4 | 0.1274       | 0.0733       | 0.0701       |
| 5 | 0.1289       | 0.0633       | 0.067        |

Table 9. The uncertain return rates on assets of five periods investment.

|   | Asset 7       | Asset 8       | Asset 9       |
|---|--------------|--------------|--------------|
| 1 | 0.0798       | 0.0572       | 0.1694       |
| 2 | 0.0907       | 0.0643       | 0.175        |
| 3 | 0.0992       | 0.0555       | 0.15         |
| 4 | 0.1029       | 0.0551       | 0.1462       |
| 5 | 0.1069       | 0.0534       | 0.1423       |

Finally, we give an example to illustrate the idea of the model and demonstrate the effectiveness of the designed algorithm. The results obtained are highly valuable in both theory and practice for solving the portfolio selection with uncertain return problems in engineering.

Due to changes of situation in financial markets and investors preferences towards risk, most of the applications of multiperiod portfolio optimization involve maximizing the terminal wealth for a given level of risk at period $t$. In addition, the investors can use VaR, CVaR, et al. to measure the risk of portfolio selection. How do the investors make a correct multiperiod decision? It will be very important for a real multiperiod portfolio selection problem. So the multiperiod portfolio models based on the uncertain means and other risk will be some future directions on the proposed approach in solving real life problems.

Appendix A. The codes of thirty stocks are respectively $S_1(600000)$, $S_2(600005)$, $S_3(600015)$, $S_4(600016)$, $S_5(600019)$, $S_6(600028)$, $S_7(600030)$, $S_8(600036)$, $S_9(600048)$, $S_{10}(600050)$, $S_{11}(600104)$, $S_{12}(600362)$, $S_{13}(600519)$, $S_{14}(600900)$, $S_{15}(601088)$, $S_{16}(601111)$, $S_{17}(601166)$, $S_{18}(601168)$, $S_{19}(601318)$, $S_{20}(601328)$, $S_{21}(601390)$, $S_{22}(601398)$, $S_{23}(601600)$, $S_{24}(601601)$, $S_{25}(601628)$, $S_{26}(601857)$, $S_{27}(601919)$, $S_{28}(601939)$, $S_{29}(601988)$, $S_{30}(601998)$. The triangle uncertain distributions $\xi_{it} = (a_{it}, \alpha_{it}, \beta_{it})$ of the return rates of assets at each period can be obtained as shown in Table 7 to Table 16.

Appendix B. According Table 7 to Table 16, and Equation 34, $AD_t(R_{it})(i = 1, \ldots, 30; t = 1, \ldots, 5)$ can be obtained as shown in Table 17 to Table 20.
Table 10. The uncertain return rates on assets of five periods investment.

|    | Asset 10 | Asset 11 | Asset 12 |
|----|---------|---------|---------|
| 1  | 0.0377  | 0.0325  | 0.0414  |
| 2  | 0.0460  | 0.0318  | 0.0321  |
| 3  | 0.0469  | 0.0314  | 0.0309  |
| 4  | 0.0492  | 0.0318  | 0.0298  |
| 5  | 0.0575  | 0.0403  | 0.1282  |

Table 11. The uncertain return rates on assets of five periods investment.

|    | Asset 13 | Asset 14 | Asset 15 |
|----|---------|---------|---------|
| 1  | 0.2049  | 0.1244  | 0.1331  |
| 2  | 0.2102  | 0.1162  | 0.1277  |
| 3  | 0.2194  | 0.1236  | 0.1186  |
| 4  | 0.2225  | 0.1248  | 0.1154  |
| 5  | 0.2238  | 0.1029  | 0.1142  |

Table 12. The uncertain return rates on assets of five periods investment.

|    | Asset 16 | Asset 17 | Asset 18 |
|----|---------|---------|---------|
| 1  | 0.0645  | 0.0622  | 0.4419  |
| 2  | 0.0625  | 0.2795  | 0.2306  |
| 3  | 0.0656  | 0.0514  | 0.2276  |
| 4  | 0.0747  | 0.0464  | 0.4785  |
| 5  | 0.0835  | 0.0506  | 0.2096  |

Table 13. The uncertain return rates on assets of five periods investment.

|    | Asset 19 | Asset 20 | Asset 21 |
|----|---------|---------|---------|
| 1  | 0.25    | 0.0746  | 0.0896  |
| 2  | 0.0916  | 0.0716  | 0.0634  |
| 3  | 0.0928  | 0.0706  | 0.0622  |
| 4  | 0.094   | 0.057   | 0.069   |
| 5  | 0.0852  | 0.0472  | 0.06   |

Table 14. The uncertain return rates on assets of five periods investment.

|    | Asset 22 | Asset 23 | Asset 24 |
|----|---------|---------|---------|
| 1  | 0.1043  | 0.1197  | 0.0664  |
| 2  | 0.117   | 0.0618  | 0.0739  |
| 3  | 0.12    | 0.0623  | 0.0708  |
| 4  | 0.1373  | 0.0656  | 0.0674  |
| 5  | 0.1254  | 0.0488  | 0.0654  |

Table 15. The uncertain return rates on assets of five periods investment.

|    | Asset 25 | Asset 26 | Asset 27 |
|----|---------|---------|---------|
| 1  | 0.0569  | 0.1432  | 0.1024  |
| 2  | 0.1021  | 0.0652  | 0.0591  |
| 3  | 0.1037  | 0.0567  | 0.0574  |
| 4  | 0.1044  | 0.0314  | 0.0567  |
| 5  | 0.109   | 0.0243  | 0.0521  |
Table 16. The uncertain return rates on assets of five periods investment.

|  | Asset 28 |  |  | Asset 29 |  |  | Asset 30 |  |  |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.1551 | 0.1128 | 0.0498 | 0.0994 | 0.1248 | 0.0674 | 0.0674 | 0.1355 | 0.0854 |
| 2 | 0.1392 | 0.0789 | 0.0969 | 0.1123 | 0.0641 | 0.0406 | 0.1037 | 0.0636 | 0.0438 |
| 3 | 0.1395 | 0.0679 | 0.0956 | 0.1134 | 0.0648 | 0.0488 | 0.1048 | 0.0645 | 0.0426 |
| 4 | 0.1426 | 0.0379 | 0.0924 | 0.1157 | 0.0416 | 0.0464 | 0.106 | 0.0574 | 0.0414 |
| 5 | 0.147 | 0.0364 | 0.088 | 0.1175 | 0.0312 | 0.0446 | 0.1061 | 0.035 | 0.0413 |

Table 17. The uncertain absolute deviation of assets of five periods investment.

|  | Asset 9 |  |  | Asset 10 |  |  | Asset 11 |  |  |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.0574 | 0.0185 | 0.044 | 0.076 | 0.0563 | 0.0228 | 0.0573 | 0.0994 |
| 2 | 0.0368 | 0.0186 | 0.0435 | 0.0708 | 0.0615 | 0.0143 | 0.036 | 0.0765 |
| 3 | 0.0363 | 0.016 | 0.0408 | 0.0647 | 0.0589 | 0.015 | 0.0357 | 0.074 |
| 4 | 0.0362 | 0.0156 | 0.0398 | 0.062 | 0.0588 | 0.0162 | 0.0399 | 0.0704 |
| 5 | 0.0356 | 0.0123 | 0.0405 | 0.0587 | 0.0543 | 0.0195 | 0.0335 | 0.0688 |

Table 18. The uncertain absolute deviation of assets of five periods investment.

|  | Asset 17 |  |  | Asset 18 |  |  | Asset 19 |  |  |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.0285 | 0.0666 | 0.0469 | 0.0366 | 0.0119 | 0.0229 | 0.024 | 0.025 |
| 2 | 0.0361 | 0.0392 | 0.0338 | 0.0342 | 0.0453 | 0.0339 | 0.0319 | 0.0197 |
| 3 | 0.0393 | 0.0608 | 0.0217 | 0.0349 | 0.0443 | 0.0333 | 0.0311 | 0.0203 |
| 4 | 0.0417 | 0.0801 | 0.0278 | 0.0278 | 0.0435 | 0.0333 | 0.0346 | 0.0229 |
| 5 | 0.0322 | 0.0735 | 0.0269 | 0.0214 | 0.04 | 0.0287 | 0.0373 | 0.0255 |

Table 19. The uncertain absolute deviation of assets of five periods investment.

|  | Asset 25 |  |  | Asset 26 |  |  | Asset 27 |  |  |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.024 | 0.0456 | 0.0568 | 0.0434 | 0.0279 | 0.0379 |
| 2 | 0.0254 | 0.0545 | 0.0638 | 0.0441 | 0.0273 | 0.0233 |
| 3 | 0.0285 | 0.0542 | 0.0599 | 0.0411 | 0.0272 | 0.0217 |
| 4 | 0.0224 | 0.0482 | 0.0581 | 0.0426 | 0.0288 | 0.0257 |
| 5 | 0.0196 | 0.047 | 0.0564 | 0.0411 | 0.0292 | 0.0191 |

Table 20. The uncertain absolute deviation of assets of five periods investment.

REFERENCES

[1] K. P. Anagnostopoulos and G. Mamanis, The mean-variance cardinality constrained portfolio optimization problem: an experimental evaluation of five multiobjective evolutionary algorithms, Expert Systems with Applications, 38 (2011), 14208-14217.
[2] D. Bertsimas and R. Shiода, Algorithm for cardinality-constrained quadratic optimization, Computational Optimization and Applications, 43 (2009), 1–12.

[3] F. Cesaroni, A. Scozzari and F. Tardella, A new method for mean-variance portfolio optimization with cardinality constraints, Annals of Operations Research, 205 (2013), 213–234.

[4] Z. Chen, G. Li and Y. Zhao, Time-consistent investment policies in Markovian markets: A case of mean-variance analysis, Journal of Economic Dynamics and Control, 40 (2014), 293–316.

[5] Z. Chen, J. Liu, G. Li and Z. Yan, Composite time-consistent multi-period risk measure and its application in optimal portfolio selection, Journal of Economic Dynamics and Control, 24 (2016), 515–540.

[6] X. Y. Cui, D. Li, S. Y. Wang and S. S. Zhu, Better than dynamic mean-variance: Time inconsistency and free cash flow stream, Mathematical Finance, 22 (2012), 346–378.

[7] X. Y. Cui, X. Li and D. Li, Unified framework of mean-field formulations for optimal multi-period mean-variance portfolio selection, IEEE Transactions on Automatic Control, 59 (2014), 1833–1844.

[8] X. Y. Cui, D. Li and X. Li, Mean variance policy for discrete time cone-constrained markets: time consistency in efficiency and the minimum-variance signed supermartingale measure, Mathematical Finance, 27 (2017), 471–504.

[9] X. T. Cui, X. J. Zheng, S. S. Zhu and X. L. Sun, Convex relaxations and MIQCQP reformulations for a class of cardinality-constrained portfolio selection problems, Journal of Global Optimization, 56 (2013), 1409–1423.

[10] G. F. Deng, W. T. Lin and C. C. Lo, Markowitz-based portfolio selection with cardinality constraints using improved particle swarm optimization, Expert Systems with Applications, 39 (2012), 4558–4566.

[11] J. J. Gao, D. Li, X. Y. Cui and S. Y. Wang, Time cardinality constrained mean-variance dynamic portfolio selection and market timing: A stochastic control approach, Automatica, 54 (2015), 91–99.

[12] N. Gülpinar and B. Rustem, Worst-case robust decisions for multi-period mean-variance portfolio optimization, European Journal of Operational Research, 183 (2007), 981–1000.

[13] P. Gupta, M. Inuiguchi, M. Kumar Mehlawat and G. Mittal, Multiobjective credibilistic portfolio selection model with fuzzy chance-constraints, Information Sciences, 229 (2013), 1–17.

[14] B. Heidergott, G. J. Olsder and J. V. Woude, Max Plus at Work: Modeling and Analysis of Synchronized Systems: A Course on Max-Plus Algebra and Its Applications, Princeton University Press, 2006.

[15] X. Huang, Fuzzy chance-constrained portfolio selection, Applied Mathematics and Computation, 177 (2006), 500–507.

[16] X. Huang and L. Qiao, A risk index model for multi-period uncertain portfolio selection, Information Sciences, 217 (2012), 108–116.

[17] M. Köksalan and C. T. Şakar, An interactive approach to stochastic programming-based portfolio optimization, Annals of Operations Research, 245 (2016), 47–66.

[18] H. Konno and H. Yamazaki, Mean absolute portfolio optimization model and its application to Tokyo stock market, Management Science, 37 (1991), 519–531.

[19] C. J. Li and Z. F. Li, Multi-period portfolio optimization for asset-liability management with bankrupt control, Applied Mathematics and Computation, 218 (2012), 11196–11208.

[20] D. Li and W. L. Ng, Optimal dynamic portfolio selection: Multiperiod mean-variance formulation, Mathematical Finance, 10 (2000), 387–406.

[21] D. Li, X. Sun and J. Wang, Optimal lot solution to cardinality constrained mean-variance formulation for portfolio selection, Mathematical Finance, 16 (2006), 83–101.

[22] X. Li, Z. Qin and L. Yang, A chance-constrained portfolio selection model with risk constraints, Applied Mathematics and Computation, 217 (2010), 949–951.

[23] B. Liu, Theory and Practice of Uncertain Programming, Physics-verlag, Heidelberg, 2002.

[24] B. Liu, Uncertainty Theory, 2nd edition, Springer-Verlag, Berlin, 2007.
[25] B. Liu, Some research problems in uncertainty theory, *Journal of Uncertain Systems*, 3 (2009), 3–10.

[26] Y. J. Liu, W. G. Zhang and W. J. Xu, Fuzzy multi-period portfolio selection optimization models using multiple criteria, *Automatica*, 48 (2012), 3042–3053.

[27] Y. J. Liu, W. G. Zhang and P. Zhang, A multi-period portfolio selection optimization model by using interval analysis, *Economic Modelling*, 33 (2013), 113–119.

[28] R. Mansini, W. Ogryczak and M. G. Speranza, Conditional value at risk and related linear programming models for portfolio optimization, *Annals of Operations Research*, 152 (2007), 227–256.

[29] H. M. Markowitz, Portfolio selection, *Journal of Finance*, 7 (1952), 77–91.

[30] H. M. Markowitz, *Portfolio Selection: Efficient Diversification of Investments*, New York, Wiley, 1959.

[31] W. Murray and H. Shek, A local relaxation method for the cardinality constrained portfolio optimization problem, *Computational Optimization and Applications*, 53 (2012), 681–709.

[32] F. Omidi, B. Abbasi and A. Nazemi, An efficient dynamic model for solving a portfolio selection with uncertain chance constraint models, *Journal of Computational and Applied Mathematics*, 319 (2017), 43–55.

[33] Z. Qin, M. Wen and C. Gu, Mean-absolute deviation portfolio selection model with fuzzy returns, *Iranian Journal of Fuzzy Systems*, 8 (2011), 61–75.

[34] Z. Qin and S. Kar, Single-period inventory problem under uncertain environment, *Journal of Applied Mathematics and Computing*, 219 (2013), 9630–9638.

[35] Z. Qin, Random fuzzy mean-absolute deviation models for portfolio optimization problem with hybrid uncertainty, *Applied Soft Computing*, 56 (2017), 597–603.

[36] R. Ruiz-Torrubiano and A. Suarez, Hybrid approaches and dimensionality reduction for portfolio selection with cardinality constraints, *IEEE Computational Intelligence Magazine*, 5 (2010), 92–107.

[37] S. J. Sadjadi, S. M. Seyedhosseini and Kh. Hassanlou, Fuzzy multi period portfolio selection with different rates for borrowing and Lending, *Applied Soft Computing*, 11 (2011), 3821–3826.

[38] H. A. Le Thi and M. Moeini, Long-short portfolio optimization under cardinality constraints by difference of convex functions algorithm, *Journal of Optimization Theory and Applications*, 161 (2014), 199–224.

[39] J. H. van Binsbergen and M. Brandt, Solving dynamic portfolio choice problems by recursing on optimized portfolio weights or on the value function?, *Computational Economics*, 29 (2007), 355–367.

[40] E. Vercher, J. Bermudez and J. Segura, Fuzzy portfolio optimization under downside risk measures, *Fuzzy Sets and Systems*, 158 (2007), 769–782.

[41] M. Woodside-Oriakhi, C. Lucas and J. E. Beasley, Heuristic algorithms for the cardinality constrained efficient frontier, *European Journal of Operational Research*, 213 (2011), 538–550.

[42] H. Wu and Y. Zeng, Equilibrium investment strategy for defined-contribution pension schemes with generalized mean-variance criterion and mortality risk, *Insurance: Mathematics and Economics*, 64 (2015), 396–408.

[43] M. Yu and S. Y. Wang, Dynamic optimal portfolio with maximum absolute deviation model, *Journal of Global Optimization*, 53 (2012), 363–380.

[44] L. A. Zadeh, Fuzzy sets, *Information and Control*, 8 (1965), 338–353.

[45] W. G. Zhang, Y. J. Liu and W. J. Xu, A possibilistic mean-semivariance-entropy model for multi-period portfolio selection with transaction costs, *European Journal of Operational Research*, 222 (2012), 341–349.

[46] W. G. Zhang, Y. J. Liu and W. J. Xu, A new fuzzy programming approach for multi-period portfolio optimization with return demand and risk control, *Fuzzy Sets and Systems*, 246 (2014), 107–126.
[47] P. Zhang and W. G. Zhang, Multiperiod mean absolute deviation fuzzy portfolio selection model with risk control and cardinality constraints, Fuzzy Sets and Systems, 255 (2014), 74–91.

[48] P. Zhang, Multiperiod mean absolute deviation uncertain portfolio selection, Industrial Engineering & Management Systems, 15 (2016), 63–76.

[49] Z. Zhou, H. Xiao, J. Yin, X. Zeng and L. Lin, Pre-commitment vs. time-consistent strategies for the generalized multi-period portfolio optimization with stochastic cash flows, Insurance: Mathematics and Economics, 68 (2016), 187–202.

[50] S. S. Zhu, D. Li and S. Y. Wang, Risk control over bankruptcy in dynamic portfolio selection: A generalized mean-variance formulation, IEEE Transactions on Automatic Control, 49 (2004), 447–457.

Received August 2017; 1st revision December 2017; final revision December 2018.

E-mail address: zhangpeng300478@aliyun.com