Cylindrical Korteweg–de Vries solitons on a ferrofluid surface

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Abstract. Linear and nonlinear surface waves on a ferrofluid cylinder surrounding a current-carrying wire are investigated. Suppressing the Rayleigh-plateau instability of the fluid column by the magnetic field of a sufficiently large current in the wire axis-symmetric surface deformations are shown to propagate without dispersion in the long wavelength limit. Using multiple scale perturbation theory, the weakly nonlinear regime may be described by a Korteweg–de Vries equation with coefficients depending on the magnetic field strength. For different values, for the current in the wire hence different solutions such as hump- or hole-solitons may be generated. The possibility to observe these structures in experiments is also elucidated.

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1. Introduction

Solitons are among the most interesting structures in nature. Being configurations of continuous fields they retain their localized shape even after interactions and collisions. Observed originally long ago as stable moving humps in shallow water channels [1], they have been established since then in various physical systems including optical waveguides, crystal lattices, Josephson junctions, plasmas and spiral galaxies (for an introduction see [2]–[4]). Long lasting efforts to theoretically describe their intriguing properties have culminated in the development of the inverse scattering technique [5] which is among the most powerful methods to obtain exact solutions of nonlinear partial differential equations [6].

Particularly popular examples for solitons in hydrodynamic systems are the solutions of the Korteweg–de Vries (KdV) equation

$$
\partial_t u(x, t) + 6u(x, t)\partial_x u(x, t) + \partial^3_x u(x, t) = 0,
$$

(1)

with $x$ standing for a space coordinate and $t$ denoting time. With $u$ representing the surface elevation of a liquid in a shallow duct, this equation can be derived perturbatively from the Euler equation for the motion of an incompressible and inviscid fluid [7, 8]. The one-soliton solution of (1) is given by

$$
u(x, t) = \frac{c}{2} \text{sech}^2 \left( \frac{\sqrt{c}}{2} (x - ct) \right),
$$

(2)

which for all values of $c > 0$ describes a hump of invariable shape moving to the right with velocity $c$. The amplitude of the hump is given by $c/2$, whereas $L = 2/\sqrt{c}$ is a measure of its width.

A decisive prerequisite to derive (1) is that to linear order in the field $u$ the system under consideration admits travelling waves $u \sim e^{i(kx-\omega t)}$ with dispersion relation

$$
\omega = c_0 k + O(k^3) \quad \text{for } k \to 0,
$$

(3)
where $c_0$ denotes the phase velocity. Intuitively the invariant shape of the soliton solution may then be understood as the consequence of a delicate balance between nonlinearity and dispersion at higher orders of both $u$ and $k$ [2].

In the present paper, we investigate cylindrical solitons of KdV-type on the surface of a ferrofluid in the magnetic field of a current-conducting wire. In order to conserve the radial symmetry of the problem, we neglect gravity. A possible experimental realization of this situation is to surround the ferrofluid column with a non-magnetic fluid of the same density. In this case, the hydrodynamics of this fluid has to be treated as well.

Ferrofluids are stable suspensions of ferromagnetic nano-particles in Newtonian liquids and behave superparamagnetically in external magnetic fields [9]. In the standard setup of a horizontal layer of ferrofluid subject to a homogeneous magnetic field, an additional term proportional to $k^2$ shows up in the dispersion relation (3) [10], which inhibits the derivation of a KdV equation in this geometry. On the other hand, for a ferrofluid cylinder in the magnetic field of a current-carrying wire, the magnetic force may replace gravity and allows for dispersion free surface waves in the long wavelength limit. This in turn paves the way to derive a KdV equation for axis-symmetric surface deformations on the ferrofluid cylinder [11, 12].

Due to surface tension a long fluid cylinder is unstable to surface modulations resulting eventually in disconnected drops (Rayleigh-plateau instability). Before embarking on the study of travelling waves on the fluid surface therefore means have to be found to suppress this instability. Fortunately, this can also be accomplished with the help of the magnetic field [9].

An accurate experimental investigation of solitons in hydrodynamic systems is notoriously difficult due to the ubiquitous presence of dissipation. Most quantitative studies have been devoted to hump-solitons in shallow channels of water [13, 14], whereas recently also the detection of hole-solitons on the surface of mercury have been reported [15]. In our present setup either hole or hump solitary waves are possible depending on the value of the applied current. We therefore hope that the present theoretical work may also stimulate new experimental investigations.

The paper is organized as follows. In section 2, we collect the basic equations and boundary conditions. Section 3 is devoted to a linear stability analysis of a cylinder of ferrofluid in the magnetic field of a current-carrying wire. Here, we demonstrate the possibility to suppress the Rayleigh-plateau instability and establish the dispersion relation (3) for axis-symmetric surface waves. In section 4, we derive the KdV equation by multiple scale perturbation theory with details of the calculation relegated to two appendices. Section 5 provides the explicit form of the one- and two-soliton solution and gives some estimates for possible experimental realizations. Finally, section 6 contains some conclusions.

2. Basic equations

We consider a cylindrical column of ferrofluid surrounding a straight, thin, long, current-carrying wire under zero gravity. The ferrofluid is modelled as an incompressible, inviscid liquid of density $\rho$ and constant magnetic susceptibility $\chi$ surrounded by a vacuum. Although we will eventually be interested in the nonlinear evolution of the surface profile of the fluid, the assumption of a linear magnetization law $M = \chi H$ is quite reasonable for experimentally relevant parameters as will be discussed in section 5. We use cylindrical coordinates $(r, \theta, z)$, with the $z$-axis pointing

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Figure 1. Schematic plot of the system under consideration. A current-carrying wire is surrounded by a ferrofluid column with magnetic susceptibility $\chi$ and density $\rho$ (region ➀) under zero gravity. Region ➁ is a nonmagnetic medium of negligible density treated as vacuum. The dynamics of the deflection $\zeta$ of the surface from the perfect cylindrical shape with radius $R$ is the central quantity of interest. The vector $n$ denotes the normal on the free interface $R + \zeta(z, \theta, t)$.

along the wire (see figure 1). The magnetic field is given by

$$H = \frac{J}{2\pi r}e_\theta,$$  \hspace{1cm} (4)

where $J$ denotes the current through the wire. Due to the field a magnetization $M$ builds up in the ferrofluid. The corresponding magnetic force, $F_m = \mu_0 (M \nabla) H$ attracts the ferrofluid radially inward. The equilibrium free surface of the ferrofluid is hence cylindrical with the radius denoted by $R$. Deviations from this shape are parametrized by a function $\zeta(z, \theta, t)$ according to $r = R + \zeta(z, \theta, t)$.

The velocity field $v(r, \theta, z)$ inside the ferrofluid is determined by the continuity equation

$$\nabla \cdot v = 0,$$  \hspace{1cm} (5)

and by the Euler equation

$$\rho \partial_t v + \rho (v \nabla) v = -\nabla P + \mu_0 (M \nabla) H.$$  \hspace{1cm} (6)

Here $P(r, \theta, z)$ denotes the pressure. We will only consider situations in which the flow of the fluid is irrotational,

$$\nabla \times v = 0.$$  \hspace{1cm} (7)

It is convenient then to introduce a scalar potential for the velocity

$$v = \nabla \Phi,$$  \hspace{1cm} (8)

which due to (5) fulfils the Laplace equation

$$\Delta \Phi = 0.$$  \hspace{1cm} (9)
The Euler equation may now be integrated once to yield the Bernoulli equation
\[ \rho \partial_t \Phi + \frac{\rho}{2} (\nabla \Phi)^2 + P - \frac{\mu_0 \chi}{2} H^2 = \text{const}. \] (10)

The magnetic field has to obey the magnetostatic Maxwell equations [9]
\[ \nabla \cdot \mathbf{H} = 0, \quad \nabla \times \mathbf{H} = 0, \] (11)
both inside and outside the ferrofluid. Denoting the respective fields by \( \mathbf{H}_1 \) and \( \mathbf{H}_2 \) equations (11) allow the representations
\[ \mathbf{H}_1 = -\nabla \Psi_1 \quad \text{and} \quad \mathbf{H}_2 = -\nabla \Psi_2, \] (12)
with the scalar magnetic potentials \( \Psi_1 \) and \( \Psi_2 \) also fulfilling the Laplace equation
\[ \Delta \Psi_1 = 0 \quad \text{and} \quad \Delta \Psi_2 = 0. \] (13)

Equations (9), (10) and (13) are to be complemented by boundary conditions. On the hydrodynamic side we have, assuming no radial extension of the wire,
\[ \lim_{r \to 0} \partial_r \Phi = 0. \] (14)
Moreover at the free surface, we need to fulfill the kinematic condition
\[ \partial_t \zeta + \partial_z \Phi \partial_z \zeta + \frac{\partial_\theta \Phi \partial_\theta \zeta}{r^2} = \partial_r \Phi, \] (15)
as well as the pressure equilibrium [9]
\[ P = P_0 + \sigma K - \frac{\mu_0}{2} M_n^2. \] (16)
Here, \( \sigma \) is the surface tension, \( K := \nabla \cdot \mathbf{n} \) denotes the curvature of the free surface, and \( M_n \) is the magnetization perpendicular to the surface. The normal vector \( \mathbf{n} \) on the surface is given by
\[ \mathbf{n} = \frac{\nabla (r - \zeta(z, \theta, t))}{|\nabla (r - \zeta(z, \theta, t))|}. \] (17)
Note that \( \zeta \equiv 0 \) yields \( K = 1/R \) as it should be for the undisturbed cylinder.

The boundary conditions for the magnetic field assume the form [9]
\[ \lim_{r \to 0} \partial_r \Psi_1 = 0, \quad \lim_{r \to \infty} \partial_r \Psi_2 = 0. \] (18)
At the free surface we have
\[ \mathbf{n} \cdot \nabla (\Psi_2 - (1 + \chi) \Psi_1) = 0, \quad \Psi_2 - \Psi_1 = 0. \] (19)
Equations (19) describe the feedback of the flow of the ferrofluid on to the magnetic field.
It is convenient to introduce dimensionless units. We measure all lengths in units of the cylinder radius $R$, and use the replacements

\[
t \to \sqrt{\frac{R^3 \rho}{\sigma}} t, \quad \Phi \to \sqrt{\frac{R \sigma}{\rho}} \Phi, \quad P \to \frac{\sigma}{R} P, \quad \Psi \to \frac{J}{2\pi} \Psi.
\] (20)

The overall magnetic field strength which can be externally controlled by changing the current $J$ is then characterized by the dimensionless magnetic Bond number

\[
Bo := \frac{\mu_0 \chi J^2}{4\pi^2 \sigma R}.
\] (21)

Using the Bernoulli equation (10), the pressure equilibrium (16) at the free surface $r = 1 + \zeta(z, \theta, t)$ is given by

\[
\partial_t \Phi + \frac{1}{2} (\nabla \Phi)^2 + \nabla \cdot \mathbf{n} - \frac{Bo}{2} (\chi (n \cdot \nabla \psi_1)^2 + (\nabla \psi_1)^2) = 1 - \frac{Bo}{2}.
\] (22)

Here, the reference pressure $P_0$ in (16) has been chosen such that $\Phi \equiv 0$, $\zeta \equiv 0$ is a solution of (22).

3. Linear stability analysis

In this section, we study the linear stability of the cylindrical interface given by $\zeta \equiv 0$, $\Phi \equiv 0$, $\psi_1 = \psi_2 = 0$. To this end, we introduce small perturbations $\zeta(\theta, z, t)$, $\phi(r, \theta, z, t)$, $\psi_1(r, \theta, z, t)$ and $\psi_2(r, \theta, z, t)$ of the surface profile, velocity potential and magnetic potentials respectively and linearize the basic equations and their boundary conditions in these perturbations. From the translational invariance along the $z$-axis and equations (9) and (13) together with the boundary conditions (14) and (18), it follows that these perturbations are of the form

\[
\zeta(\theta, z, t) = C_n \exp(in\theta + ikz + pt), \tag{23}
\]

\[
\phi(r, \theta, z, t) = D_n I_n(kr) \exp(in\theta + ikz + pt), \tag{24}
\]

\[
\psi_1(r, \theta, z, t) = A_n I_n(kr) \exp(in\theta + ikz + pt), \tag{25}
\]

\[
\psi_2(r, \theta, z, t) = B_n K_n(kr) \exp(in\theta + ikz + pt). \tag{26}
\]

Here, $k$ denotes the wavenumber in $z$-direction, $n \in \mathbb{Z}$ characterizes the azimuthal modulations, and $p$ is the growth rate. The $A_n$, $B_n$, $C_n$ and $D_n$ are constants (with their dependence on $k$ and $p$ suppressed) and $I_n(k)$ and $K_n(k)$ denote modified Bessel functions of order $n$ [16].
Using the linearization of \((19)\), we may express \(A_n\) and \(B_n\) in terms of \(C_n\) according to

\[
A_n = \imath n \chi \frac{K_n(k)}{I_n(k) K_n'(k) - \mu_r I_n'(k) K_n(k)} C_n, \\
B_n = \imath n \chi \frac{I_n(k) K_n'(k) - \mu_r I_n'(k) K_n(k)}{I_n(k) K_n'(k)} C_n,
\]

where the prime denotes differentiation with respect to the argument. In addition the linearized version of \((15)\) gives

\[
D_n = \frac{p}{k I_n'(k)} C_n. \tag{28}
\]

Finally, linearizing \((22)\) we find

\[
\partial_t \phi - \partial^2_\theta \zeta - \partial^2_z \zeta + (Bo - 1) \zeta + Bo \partial_\theta \psi_1 = 0, \tag{29}
\]

which when combined with \((27)\) and \((28)\) yields the dispersion relation

\[
p_n^2(k) = k^2 \frac{I_n'(k)}{I_n(k)} (1 - n^2 - Bo - k^2) + \frac{n^2 \chi Bo}{[I_n(k) K_n'(k) / I_n'(k) K_n(k)] - (1 + \chi)}. \tag{30}
\]

The reference state of a cylindrical column becomes unstable if combinations of \(k, n\) and \(Bo\) exist for which \(p_n^2\) is positive.

For \(Bo = 0\), we find back the well-known Rayleigh-plateau instability accomplished by radially symmetric modes with \(n = 0\). Modes with higher values of \(n\) are not able to destabilize the fluid cylinder.

Since one has for all \(k\)

\[
\frac{I_n'(k)}{I_n(k)} > 0 \quad \text{and} \quad \frac{I_n(k) K_n'(k)}{I_n'(k) K_n(k)} < 0, \tag{31}
\]

we infer from \((30)\) that \(p_n^2(k)\) is a monotonically decreasing function of the magnetic Bond number \(Bo\). The magnetic field hence always stabilizes the cylindrical surface. Consequently it may change the qualitative behaviour of the system only due to its influence on the \(n = 0\) modes.

For \(n = 0\), the dispersion relation reads

\[
p_0^2(k) = k^2 \frac{I_1(k)}{I_0(k)} (1 - Bo - k^2). \tag{32}
\]

It is displayed for several values of \(Bo\) in figure 2. From \((32)\), we see that the Rayleigh-plateau instability for a ferrofluid column will be suppressed by a sufficiently strong magnetic field fulfilling \(Bo > 1\). Using typical parameter values as \(\chi = 1.2\), \(\sigma = 0.03 \text{ J m}^{-2}\) and a fluid radius of \(R = 1\text{ cm}\), the system remains stable if the current exceeds the threshold \(J_c \simeq 89\text{ A}\).

It is instructive to investigate the dispersion relation \((30)\) in the long wavelength limit \((k \ll 1)\). Using the expansion of the modified Bessel functions for small arguments [16], we get

\[
\frac{I_0'(k)}{I_0(k)} \sim \frac{k^3}{2} - \frac{k^5}{16}, \tag{33}
\]
Figure 2. Plot of the square of the growth rate of axis symmetric distortions as a function of the wavenumber $k$ as given by (32) for magnetic Bond numbers $Bo = 0, 0.3, 0.7, 1.0$ and $Bo = 1.3$ (from top to bottom). The inset shows a magnification of the region around $k = 0$. For $Bo > 1$, one has $p^2 < 0$ for all $k$ and no instability occurs.

\[
\frac{I_n'(k)}{I_n(k)} \sim \frac{n}{k} \quad \text{if } n > 0, \quad \text{(34)}
\]

\[
\frac{I_0'(k)K_0'(k)}{I_0(k)K_0(k)} \sim \frac{2}{k^2 \log k}, \quad \text{(35)}
\]

\[
\frac{I_n'(k)K_n'(k)}{I_n(k)K_n(k)} \sim -1 \quad \text{if } n > 0, \quad \text{(36)}
\]

and hence find

\[
p_0^2(k) = \frac{1 - Bo}{2} k^2 - \frac{9 - Bo}{16} k^4 + O(k^6), \quad \text{(37)}
\]

\[
p_n^2(k) = n(1 - n^2 - Bo) - n^2 \frac{\chi Bo}{\chi + 2} + O(k^2) \quad \text{if } n > 0. \quad \text{(38)}
\]

Therefore for $n = 0$ and $Bo > 1$ the system exhibits surface waves $\zeta(z, t) \sim \exp(i(kz - \omega t)$ with dispersion relation

\[
\omega(k) = \sqrt{\frac{Bo - 1}{2} k \left(1 - \frac{1}{16} \frac{Bo - 9}{Bo - 1} k^2\right)} + O(k^2). \quad \text{(39)}
\]

The important point for what follows is that these surface waves become *dispersion free*, $\omega = c_0 k$, in the long wavelength limit $k \to 0$. The phase velocity is given by

\[
c_0 = \sqrt{\frac{Bo - 1}{2}}. \quad \text{(40)}
\]

The situation is hence analogous to the shallow water equations which form the starting point for the derivation of the KdV equation in a rectangular duct [8]. Note that no such waves are possible for $n > 1$, cf (38).

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4. KdV equation

In the previous section, we have seen that the system under consideration admits to linear order in the surface deflection $\zeta$ cylindrical, axis symmetric surface waves with no dispersion in the long wavelength limit $k \to 0$. From the experience with plane surface waves on shallow water [8], it is hence tempting to investigate whether at higher orders in $k$ and $\zeta$ nonlinear waves may be obtained for which the effects of nonlinearity and dispersion exactly balance each other. This could then give rise to axis symmetric soliton solutions in the present cylindrical geometry.

In this section, we show that it is indeed possible to derive a KdV equation for the surface deflection $\zeta(z, t)$ by using a multiple scale perturbation theory similar to the case of rectangular geometry. To this end, we first observe that for an axis symmetric free surface the magnetic field problem decouples from the hydrodynamics and we have the exact result

$$\Psi_1 = \theta. \tag{41}$$

This in turn implies $\nabla \Psi_1 = (0, 1/r, 0)$ and therefore $\mathbf{n} \cdot \nabla \Psi_1 = 0$. Using moreover the explicit expression for $\mathbf{n}$ in terms of $\zeta(z, t)$ resulting from (17), we get from (22)

$$\partial_t \Phi + \frac{1}{2} [\partial_r (\partial_r \Phi)^2 + (\partial_z \Phi)^2] + \frac{[1 + (\partial_z \zeta)^2 / 1 + \zeta] - \partial_r^2 \zeta}{[1 + (\partial_z \zeta)^2]^{3/2}} - \frac{1}{2} \frac{Bo}{(1 + \zeta)^2} = 1 - \frac{Bo}{2}. \tag{42}$$

The kinematic condition (15) simplifies to

$$\partial_t \zeta + \partial_z \Phi \partial_z \zeta = \partial_r \Phi. \tag{43}$$

The KdV equation appears in the limit of small nonlinearity, $\zeta \ll 1$, and small dispersion, $k \ll 1$ with the proper balance between these two ingredients occurring for $\zeta = O(k^2)$. To make this combined limit explicit, we introduce a small parameter, $\epsilon$, and use the rescalings

$$z \to \frac{z}{\sqrt{\epsilon}}, \quad r \to r, \quad \zeta \to \epsilon \zeta, \quad t \to \frac{t}{c_0 \sqrt{\epsilon}}, \quad \Phi \to \sqrt{\epsilon} c_0 \Phi, \tag{44}$$

where $c_0$ is defined by (40). To derive the KdV equation, we will need the two basic equations (42) and (43) up to order $\epsilon^2$. Plugging (44) into these equations, we find to the required order

$$\partial_t \Phi + \frac{1}{2} [\epsilon (\partial_z \Phi)^2 + (\partial_r \Phi)^2] + 2 \zeta - \frac{\epsilon}{2 c_0^2} (3 Bo - 2) \zeta^2 - \frac{\epsilon}{c_0^2} \partial_r^2 \zeta = 0, \tag{45}$$

and

$$\epsilon \partial_t \zeta + \epsilon^2 \partial_r \Phi \partial_r \zeta = \partial_r \Phi. \tag{46}$$

To get a suitable expansion for the velocity potential $\Phi$ we note that from the Laplace equation (9) and the boundary condition (14), one may derive the following representation for $\Phi(r, z, t)$ (see appendix A)

$$\Phi(r, z, t) = \sum_{m=0}^{\infty} \frac{r^{2m} e^{-m}}{(2^m m!)^2} \partial_r^{2m} \Phi_0(z, t), \tag{47}$$

with the so far undetermined function $\Phi_0(z, t)$.
Using this expansion for $\Phi$ in (45) and (46) and observing that both equations hold at the interface, i.e. for $r = 1 + \epsilon \zeta$, we get to the desired order in $\epsilon$

$$\partial_t \Phi_0 + 2 \zeta = \frac{\epsilon}{4} \partial_t \partial_z^2 \Phi_0 - \frac{\epsilon}{2} (\partial_z \Phi_0)^2 + \frac{\epsilon (3B_0 - 2)}{2c_0^2} \zeta^2 + \frac{\epsilon}{c_0^2} \partial_z^2 \zeta,$$  

(48)

and

$$\partial_t \zeta + \frac{1}{2} \partial_z^2 \Phi_0 = -\epsilon \partial_z \Phi_0 \partial_z \zeta - \frac{\epsilon}{2} \zeta \partial_z \Phi_0 + \frac{\epsilon}{16} \partial_z^4 \Phi_0.$$  

(49)

It is convenient to differentiate (48) with respect to $z$ and to introduce the $z$-component of the velocity of the ferrofluid $u = \partial_z \Phi$. We then find the final set of equations to determine $\zeta$ and $u$

$$\partial_t u + 2 \partial_z \zeta = \epsilon \left( \frac{1}{4} \partial_t \partial_z^3 u - u \partial_z u + \frac{3B_0 - 2}{c_0^2} \zeta \partial_z \zeta + \frac{1}{c_0^2} \partial_z^3 \zeta \right),$$  

(50)

and

$$\partial_t \zeta + \frac{1}{2} \partial_z u = \epsilon \left( -u \partial_z \zeta - \frac{1}{2} \zeta \partial_z u + \frac{1}{16} \partial_z^3 u \right).$$  

(51)

We now solve these equations perturbatively using the ansatz

$$\zeta(z, t, \tau) = \zeta_0(z, t, \tau) + \epsilon \zeta_1(z, t, \tau) + O(\epsilon^2),$$

$$u(z, t, \tau) = u_0(z, t, \tau) + \epsilon u_1(z, t, \tau) + O(\epsilon^2).$$

(52)

where we have introduced a second, slow time variable $\tau := \epsilon t$. Plugging these expansions into (50) and (51), we find to zeroth order in $\epsilon$

$$L \begin{pmatrix} u_0 \\ \zeta_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

(53)

where the linear operator $L$ is given by

$$L = \begin{pmatrix} \partial_t & \frac{1}{2} \partial_z \\ \frac{1}{4} \partial_t & 2 \partial_z \end{pmatrix}.$$  

(54)

The solution are dispersion free travelling waves of d’Alembert form

$$u_0(z, t, \tau) = 2f(z - t, \tau), \quad \zeta_0(z, t, \tau) = f(z - t, \tau),$$

(55)

with a so far unspecified function $f(x, \tau)$ where we have restricted ourselves to waves travelling to the right.

To order $\epsilon$ we find

$$L \begin{pmatrix} u_1 \\ \zeta_1 \end{pmatrix} = \begin{pmatrix} -\partial_t u_0 + \frac{1}{4} \partial_t \partial_z^2 u_0 - u_0 \partial_z u_0 + \frac{3B_0 - 2}{c_0^2} \zeta_0 \partial_z \zeta_0 + \frac{1}{c_0^2} \partial_z^3 \zeta_0 \\ -\partial_t \zeta_0 - u_0 \partial_z \zeta_0 - \frac{1}{2} \zeta_0 \partial_z u_0 + \frac{1}{16} \partial_z^3 u_0 \end{pmatrix}.$$  

(56)

This inhomogeneous equation involves again the linear operator $L$ which is singular, cf (53). Hence the inhomogeneity of this equation must be orthogonal to the zero eigenspace of the
adjoint operator \( L^+ \). The determination of \( L^+ \) and the projection of the right-hand side of (56) on to the eigenfunction of \( L^+ \) with eigenvalue zero is done in appendix B. The solvability condition for (56) finally acquires the form

\[
\partial_t f + \frac{2Bo - 3}{4c_0^2} f \partial_z f + \frac{Bo - 9}{32c_0^2} \partial^3_z f = 0.
\] (57)

Using (55), denoting \( \zeta_0 \) simply by \( \zeta \) and reversing the scalings (44) then yields the following KdV equation for the surface deflection \( \zeta(z, t) \)

\[
\partial_t \zeta + c_0 \partial_z \zeta + \frac{2Bo - 3}{4c_0} \zeta \partial_z \zeta + \frac{Bo - 9}{32c_0} \partial^3_z \zeta = 0.
\] (58)

When discussing the implications of this equation, one has to keep in mind that it is valid for small \( \zeta \) only.

5. Results

Equation (58) is of the form

\[
\partial_t \zeta + c_0 \partial_z \zeta + c_1 \zeta \partial_z \zeta + c_2 \partial^3_z \zeta = 0,
\] (59)

with the coefficients

\[
c_0 = \sqrt{\frac{Bo - 1}{2}}, \quad c_1 = \frac{2Bo - 3}{4c_0} \quad \text{and} \quad c_2 = \frac{Bo - 9}{32c_0},
\] (60)

all depending on the magnetic field strength \( Bo \). From section 3, we know that we must have \( Bo > 1 \) since otherwise the fluid cylinder is susceptible to the Rayleigh-plateau instability. Hence both \( c_1 \) and \( c_2 \) may change sign for allowed values of \( Bo \).

The one-soliton solution of (59) is of the form (cf (2))

\[
\zeta^{(1)}(z, t) = \frac{3c}{c_1} \text{sech}^2 \left( \sqrt{\frac{c}{4c_2}} (z - (c + c_0)t) \right),
\] (61)

where \( c \ll 1 \) is a free constant having the same sign as \( c_2 \). For \( Bo < 9 \), we have hence \( c < 0 \) and the soliton has a slightly smaller velocity than the linear waves. If \( Bo < 3/2 \) also \( c_1 < 0 \) and therefore the amplitude of the soliton is positive, i.e. we have a hump-soliton as shown in figure 3(a). For \( 3/2 < Bo < 9 \) on the other hand \( c_1 > 0 \) and consequently (61) describes a depression or hole-soliton as depicted in figure 3(b). Finally, for \( Bo > 9 \) we have \( c_2 > 0 \), hence \( c > 0 \), and also \( c_1 > 0 \). The soliton amplitude is therefore positive again and its velocity is now slightly larger than that of the corresponding linear waves.

To get some impression of the accessibility of the solution in experiments, the results for the following parameter sets may be helpful. For a ferrofluid with \( \chi = 1.2, \ \rho = 1.12 \text{ g cm}^{-3} \) and \( \sigma = 0.03 \text{ J m}^{-2} \) forming a cylinder of radius \( R = 1 \text{ cm} \) a current \( I = 100 \text{ A} \) corresponds to \( Bo \simeq 1.27 \). A soliton with amplitude \( A = 2 \text{ mm} \) has then a velocity of \( U = 1.8 \text{ cm s}^{-1} \) and...
the width of the hump is about $L = 20$ cm. This soliton will hence be difficult to observe in an experiment. For a current of $I = 294$ A corresponding to $Bo \simeq 11$ the extension reduces for the same amplitude to $L = 1.6$ cm with the velocity increasing to $U = 12.3$ cm s$^{-1}$. A hole-soliton with amplitude $A = -2$ mm, velocity $U = 8.4$ cm s$^{-1}$, and width $L = 2.1$ cm can be realized with a current of $I = 235$ A corresponding to $Bo \simeq 7$. The latter two solitons are shown schematically in figure 3. Both should be easily observable experimentally. Note that for such values of the current the magnitude of error assuming $\chi = \text{const}$. of the magnetization is about 5%.

A two-soliton solution may be derived using, e.g., Hirota’s method [18]. Depending on the value of the magnetic bond number, one may combine either two hump- or two hole-solitons. The case of two hump-solitons is described by the solution

$$
\zeta^{(2)}(z, t) = \frac{8 \gamma_2^2 \xi_1^2 + \gamma_1^2 \xi_2^2 + (\gamma_1 - \gamma_2)^2 \xi_1 \xi_2}{(1 + \xi_1 + \xi_2 + (\gamma_1 - \gamma_2/\gamma_1 + \gamma_2)^2 \xi_1 \xi_2)^2},
$$

(62)

where

$$
\gamma_i^2 = \frac{3c_i}{c_1} \quad \text{and} \quad \xi_i = \exp \left( \sqrt{\frac{c_i}{c_2}} (z - z_{0i} - (c_i + c_0)t) \right), \quad (63)
$$

for $i = 1, 2$.

A snapshot of the solution is displayed in figure 4, its time evolution is characterized by figure 5. The main feature is the passing of the slower soliton by the faster one. After the interaction process the two solitons re-emerge undisturbed which is the defining property of a soliton solution. An animated version of the two-soliton solution is shown in a movie for parameter values as in figure 4.

6. Conclusion

In the present paper, we have investigated nonlinear waves on the cylindrical surface of a ferrofluid surrounding a current-carrying wire under zero gravity. We have shown that for a sufficiently large current a KdV equation for axis-symmetric surface distortion can be derived. Accordingly
In order to observe these solitons, first of all the ubiquitous Rayleigh-plateau instability has to be suppressed. This can be accomplished by the magnetic field, if the current exceeds a critical value which for experimentally relevant parameters is of about 100 A.

We have shown that non axis-symmetric perturbations of the surface always disperse whereas axis-symmetric ones propagate almost dispersion-free if the wavelength is very large. Using the fact that for axis-symmetric surface deflections, the magnetic field problem decouples completely from the hydrodynamic part and a KdV equation can be derived. The parameters in this equation depend on the magnetic field strength which gives rise to qualitatively different soliton solutions like hump- and hole-solutions for different values of the current in the wire.

The one- and two-soliton solutions were discussed in detail and conditions for their experimental realization were given.

It should be noted that several approximations were used in our theoretical analysis. First of all the derivation of the KdV equation is perturbative and therefore approximate as is typical for the theoretical discussion of solitons in hydrodynamic systems. Furthermore, we have neglected the hydrodynamic influence of a non-magnetic fluid surrounding the magnetic column which is necessary to ensure zero gravity in experiments. Finally, viscosity was neglected throughout the system under consideration is well suited to experimentally investigate cylindrical solitons of KdV type.
the analysis since the KdV equation results from the Euler equation describing inviscid fluids. In experiments, one will hence always see a damping of the soliton solutions with time due to dissipation by viscous shear flow [13]–[15]. It is an attractive idea to counter-balance these viscous losses by appropriately chosen time-dependent magnetic fields; however, we were not able to find a suitable geometry for this idea to become operative. In any case, a theoretical analysis aiming at this goal has to go beyond the quasi-static version of ferro-hydrodynamics employed in the present analysis and has to include magneto-dissipative couplings, see e.g. [19].

We finally note that our system is an experimentally accessible realization of the introductory example for a soliton given in chapter 1.4 of the book by Lamb [20]. There, an incompressible fluid inside a cylinder made of independent elastic rings is considered. The rings are supposed to deform axis-symmetrically in reaction to the fluid pressure. However, although confining the liquid tightly they must be uncoupled in order not to sustain elastic waves by themselves. Gravity is neglected altogether. Using the conservation of mass and momentum of the fluid and linear elasticity for the rings, it is then possible to derive a KdV equation for axis-symmetric deformations of the rings. As we have shown in the present paper, the somewhat unrealistic properties of the elastic rings can be mimicked by a cylindrical magnetic field if the fluid to be confined is a ferrofluid.

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Appendix A. Expansion of $\Phi(r, z, t)$

After the rescalings (44), the Laplace equation for the velocity potential $\Phi$ takes the form

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Phi}{\partial r} \right) + \epsilon \frac{\partial^2 \Phi}{\partial z^2} = 0.$$  

(A.1)

Representing $\Phi$ as a power series in $r$

$$\Phi(r, z, t) = \sum_m r^m \Phi_m(z, t),$$  

(A.2)

we find

$$\sum_m r^m [(m + 2)^2 \Phi_{m+2} + \epsilon \partial_z^2 \Phi_m] = 0,$$

(A.3)

leading to the recursion relation

$$\Phi_{m+2} = -\epsilon \frac{\partial_z^2 \Phi_m}{(m + 2)^2}.$$  

(A.4)
Because of the boundary condition (14), we have
\[ \sum_{m} m r^{m-1} \Phi_m = 0, \] (A.5)
implying \( \Phi_1 = 0 \). From (A.4) we hence find \( \Phi_{2m+1} = 0 \) for all \( m \). The velocity potential may therefore be expressed entirely in terms of \( \Phi_0 \) and its derivatives
\[ \Phi(r, z, t) = \sum_{m=0}^{\infty} r^{2m} e^m \frac{(-1)^m}{(2m)!^2} \partial_z^{2m} \Phi_0(z, t) \] (A.6)
\[ = \Phi_0(z, t) - \frac{e r^2}{4} \partial_z^2 \Phi_0(z, t) + \frac{e^2 r^4}{64} \partial_z^4 \Phi_0(z, t) + O(e^3), \] (A.7)
which coincides with (47).

**Appendix B. The solvability condition**

Under the usual scalar product
\[ \langle \bar{\Psi}|\Psi \rangle = \lim_{Z,T \to \infty} \frac{1}{4ZT} \int_{-Z}^{Z} \int_{-T}^{T} dz dt \bar{\Psi}^* \cdot \Psi, \] (B.1)
with \( \Psi = (u, \zeta) \) we find for \( L^+ \)
\[ L^+ = \begin{pmatrix} -\partial_t & -\frac{1}{2} \partial_z \\ -2\partial_z & -\partial_t \end{pmatrix}. \] (B.2)
The complete eigenmode to zero eigenvalue of \( L^+ \) is hence given by
\[ \bar{u}_0(z, t) = \bar{f}(z - t) - \bar{g}(z + t), \quad \bar{z}_0(z, t) = 2(\bar{f}(z - t) + \bar{g}(z + t)), \] (B.3)
where \( \bar{f} \) and \( \bar{g} \) are arbitrary functions of a single argument. Setting the projection of the right-hand side of (56) on this mode equal to zero we find
\[ 0 = \lim_{Z,T \to \infty} \frac{1}{4ZT} \int_{-Z}^{Z} \int_{-T}^{T} dz dt \left[ (-2\partial_t f + \frac{Bo}{c_0} f \partial_z f - \frac{Bo - 5}{4c_0^3} \partial_z^3 f)(\bar{f} - \bar{g}) \\
+ 2(-\partial_t f - 3 f \partial_z f + \frac{1}{8} \partial_z^3 f)(\bar{f} + \bar{g}) \right] \\
= \lim_{Z,T \to \infty} \frac{1}{4ZT} \int_{-Z}^{Z} \int_{-T}^{T} dz dt \left[ \left( -4\partial_t f - \frac{2Bo - 3}{c_0^3} f \partial_z f - \frac{Bo - 9}{8c_0^3} \partial_z^3 f \right) \bar{f} \\
- \left( \frac{4Bo - 3}{c_0^2} f \partial_z^2 f + \frac{3Bo - 11}{8c_0^2} \partial_z^3 f \right) \bar{g} \right]. \] (B.4)
The part of the integrand involving \( \bar{g} \) may be written in the form \( \bar{g}(z + t) \partial_z F(z - t, \tau) \). Substituting \( \xi = z - t, \eta = z + t \) one realizes that these terms do not contribute for \( Z, T \to \infty \). Since moreover \( \bar{f} \) is an arbitrary function of its argument (B.4) implies (57).
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