THE FROBENIUS-EULER FUNCTION AND ITS APPLICATIONS

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Abstract. In the present paper, we deal with Fourier-transformation of Frobenius-Euler polynomials. We shall give its applications by using infinite series. Our applications possess interesting properties which we state in this paper.

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1. Introduction

The ordinary Frobenius-Euler numbers are defined by means of the following generating function:

\[ \sum_{n=0}^{\infty} H_n(u) \frac{t^n}{n!} = e^{H(u)t} = \frac{1-u}{e^t-u}. \]

where, in the umbral calculus, \( H^n(u) \) is symbolically replaced by \( H_n(u) \) in the formal series expansion of

\[ e^{tH(u)} = \sum_{n=0}^{\infty} H_n(u) \frac{t^n}{n!}. \]

From expression of this definition, we state the following

\( (H(u) + 1)^n - uH_n(u) = \begin{cases} 1 - u & \text{if } n = 0, \\ 0 & \text{if } n \in \mathbb{N}, \end{cases} \)

where \( \mathbb{N} \) denotes the set of positive integers.

From (1.2), we note that

\[ H_0(u) = 1, H_1(u) = \frac{1}{1-u}, H_2(u) = \frac{1+u}{(1-u)^2}, \ldots. \]

The Frobenius Euler polynomials are also introduced as

\[ e^{xt} \frac{1-u}{e^t-u} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x, u). \]

By (1.1) and (1.3), we can find the following

\[ H_n(x, u) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} H_l(u). \]
By expression of (1.2), it is not difficult to show that the recurrence relation for the Frobenius-Euler numbers as follows:

\[
\sum_{l=0}^{n} \binom{n}{l} H_l(u) - u H_n(u) = (1 - u) \delta_{0,n}
\]

where \( \delta_{m,n} \) is the Kronecker delta, is defined by

\[
\delta_{m,n} = \begin{cases} 
1, & \text{if } m = n \\
0, & \text{if } m \neq n.
\end{cases}
\]

Thus, we easily procure the following:

\[
H_n(1, u) - u H_n(u) = 0 \quad (\text{for } n \in \mathbb{N}).
\]

Thus, we arrive the following lemma.

**Lemma 1.** For \( n \in \mathbb{N} \), we have

\[
H_n(1, u) = u H_n(u).
\]

Substituting \( u = -1 \) in the above lemma, it leads to

\[
H_n(-1, u) = E_n(1) = -E_n
\]

where \( E_n \) is called Euler numbers, as is well-known, Euler numbers are defined by the following generating function:

\[
\sum_{n=0}^{\infty} E_n \frac{t^n}{n!} = \frac{2}{e^t + 1}.
\]

(for more informations on this subjects, see[1-20]).

Recently, Fourier transformation of the special functions have been studied by many mathematicians (cf., [1], [2], [4], [13], [14], [16]). In [16], Luo gave Fourier expansions of Apostol-Bernoulli and Apostol-Euler polynomials and derived some integral representations of Apostol-Bernoulli and Apostol-Euler polynomials by using Fourier expansions. After, Bayad [1] introduced as theoretical identities of the Fourier transformation of the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials. Next, T. Kim also defined the Euler function which is Fourier transformation of Euler polynomials. We easily see that Kim’s method is different from Bayad and Luo. Actually, Kim’s paper [3, pp. 131-136] motivated us to write this paper. Thus, we also give Fourier transformation of Frobenius-Euler function by using Kim’s method. In this paper, we also show that this function is related to Lerch transcendent \( \Phi(z, s, a) \).

2. On the Frobenius-Euler function

In this section, we consider Frobenius-Euler function by using infinite series. For \( m \in \mathbb{N} \), the Fourier transformation of Frobenius-Euler function is introduced as

\[
H_m(x, u) = \sum_{n=-\infty}^{\infty} a_n^{(m)}(u) e^{(2n+1)\pi i x}, \quad \left( a_n^{(m)}(u) \in \mathbb{C} \right)
\]

where \( \mathbb{C} \) denotes the set of complex numbers and

\[
a_n^{(m)}(u) = \int_0^1 H_m(x, u) e^{-(2n+1)\pi i x} dx.
\]
By applying some technical method on (2.2), we procure the following

\[ a^{(m)}_n (u) = \left[ \frac{H_{m+1}(x, u)}{m+1} e^{-(2n+1)i\pi x} \right]_0^1 - \frac{(2n+1)\pi i}{m+1} \int_0^1 H_{m+1}(x, u) e^{-(2n+1)i\pi x} dx \]

\[ = - \frac{u + 1}{m+1} H_{m+1}(u) - \frac{(2n+1)\pi i}{m+1} a^{(m+1)}_n (u). \]

So from above, it leads to the following

\[ a^{(m+1)}_n (u) = \left[ a^{(m)}_n + \frac{H_{m+1}(u)}{m+1} \right] \frac{m+1}{(2n+1)\pi i}. \]

By continuing this process, becomes as follows:

\[(2.3a) \quad a^{(m)}_n (u) = \left[ a^{(1)}_n (u) + (u + 1) \frac{H_2(u)}{2} \right] \frac{m!}{((2n+1)\pi i)^{m-1}} + (u + 1) \left[ \frac{1}{(2n+1)\pi i} H_m(u) + \frac{m}{((2n+1)\pi i)^2} H_{m-1}(u) + \ldots + \frac{m!}{4!((2n+1)\pi i)^{m-3}} \right]. \]

We want to note that

\[ \lim_{u \to -1} a^{(m)}_n (u) = a^{(m)}_n (-1) := a^{(m)}_n \]

where \(a^{(m)}_n \in \mathbb{C}\) is defined by Kim in [3] as follows:

\[ a^{(m)}_n = \frac{m!}{((2n+1)\pi i)^{m-1}} a^{(1)}_n (u). \]

By using (2.1) and (2.3a), we readily derive the following

\[ H_m(x, u) = \sum_{n=-\infty}^{\infty} \left\{ \left[ a^{(1)}_n (u) + (u + 1) \frac{H_2(u)}{2} \right] \frac{m!}{((2n+1)\pi i)^{m-1}} + (u + 1) \left[ \frac{1}{(2n+1)\pi i} H_m(u) + \frac{m}{((2n+1)\pi i)^2} H_{m-1}(u) + \ldots + \frac{m!}{4!((2n+1)\pi i)^{m-3}} \right] \right\} e^{(2n+1)\pi ix}. \]

From this, we can state the following

\[ H_m(x, u) = \sum_{n=-\infty}^{\infty} \left[ a^{(1)}_n (u) + (u + 1) \frac{H_2(u)}{2} \right] \frac{m!e^{(2n+1)\pi ix}}{((2n+1)\pi i)^{m-1}} + (u + 1) \sum_{n=-\infty}^{\infty} \left[ \frac{1}{(2n+1)\pi i} H_m(u) + \frac{m}{((2n+1)\pi i)^2} H_{m-1}(u) + \ldots + \frac{m!}{4!((2n+1)\pi i)^{m-3}} H_1(u) \right] e^{(2n+1)\pi ix} \]

After some calculations on the above equation, we have the following

\[ H_m(x, u) = \sum_{n=-\infty}^{\infty} \left[ \frac{u + 1}{u - 1} \frac{1}{(2n+1)\pi i} + \frac{2}{((2n+1)\pi i)^2} + \frac{1}{2} \left( \frac{u + 1}{1 - u} \right)^2 \right] \frac{m!e^{(2n+1)\pi ix}}{((2n+1)\pi i)^{m-1}} + (u + 1) \sum_{n=-\infty}^{\infty} \left[ \frac{1}{(2n+1)\pi i} \frac{d^k}{dx^k} H_m(x, u) \right]_{x=0} e^{(2n+1)\pi ix} \]

As a result, we conclude the following theorem.
Theorem 1. For \( m \in \mathbb{N} \) and \( 0 \leq x < 1 \), we have

\[
H_m(x, u) = m! \sum_{n=-\infty}^{\infty} \left[ \frac{u+1}{u-1(2n+1)\pi i} + \frac{2}{((2n+1)\pi i)^2} + \frac{1}{2} \left( \frac{u+1}{1-u} \right)^2 \right] \frac{\text{e}^{(2n+1)\pi ix}}{((2n+1)\pi i)^m-1} 
+ (u+1) \sum_{k=0}^{m-1} \frac{1}{(2\pi i)^{k+1}} \frac{d^k}{dx^k} H_m(x, u) |_{x=0} \sum_{n=-\infty}^{\infty} \frac{\text{e}^{(2n+1)\pi ix}}{(n+\frac{1}{2})^{k+1}}.
\]

Considering generating functions of Euler and Frobenius-Euler polynomials, we reach the following corollary.

Corollary 1. Taking \( u = -1 \), we have Fourier transformation of Euler function, which is defined by Kim in [3] as follows:

\[
H_m(x, -1) = E_m(x) = 2m! \sum_{n=-\infty}^{\infty} \frac{\text{e}^{(2n+1)\pi ix}}{((2n+1)\pi i)^m+1} \text{ (for } 0 \leq x < 1). \]

The Lerch transcendent \( \Phi(z, s, a) \) is the analytic continuation of the series

\[
(2.5) \quad \Phi(z, s, a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}
\]

which converges for \( a \in \mathbb{C}\setminus\mathbb{Z}_0^- \), \( s \in \mathbb{C} \) when \( |z| < 1 \); \( \Re(s) > 1 \) when \( |z| = 1 \) where \( \mathbb{Z}_0^- = \mathbb{Z}^- \cup \{0\} \), \( \mathbb{Z}^- = \{-1, -2, -3, ...\} \). Lerch transcendent \( \Phi(z, s, a) \) is the proportional not only Riemann zeta function, Hurwitz zeta function, the Dirichlet’s eta function but also Dirichlet beta function, the Legendre chi function, the polylogarithm, the Lerch zeta function (for details, see [15], [17]).

We now want to indicate that \( \sum_{n=-\infty}^{\infty} \frac{\text{e}^{(2n+1)\pi ix}}{((2n+1)\pi i)^m} \) is closely related to Lerch transcendent \( \Phi(z, s, a) \). So, we compute as follows:

\[
\sum_{n=-\infty}^{\infty} \frac{\text{e}^{(2n+1)\pi ix}}{((2n+1)\pi i)^m} = \frac{1}{(2\pi i)^m} \sum_{n=-\infty}^{\infty} \frac{\text{e}^{(2n+1)\pi ix}}{(n+\frac{1}{2})^m} = \frac{1}{(2\pi i)^m} \sum_{n=-\infty}^{\infty} \frac{\text{e}^{(2n+1)\pi ix}}{(n+\frac{1}{2})^m} + \frac{1}{(2\pi i)^m} \sum_{n=0}^{\infty} \frac{\text{e}^{(2n+1)\pi ix}}{(n+\frac{1}{2})^m}
\]

After some applications on the above equation, we procure the following

\[
(2.6) \quad \sum_{n=-\infty}^{\infty} \frac{\text{e}^{(2n+1)\pi ix}}{((2n+1)\pi i)^m} = -\frac{e^{\pi ix}}{(\pi i)^m} + \frac{(-1)^m e^{\pi ix}}{(2\pi i)^m} \Phi\left( e^{-2\pi ix}, m, -\frac{1}{2} \right).
\]

By using Theorem 1 and (2.6), we give the following theorem.
Theorem 2. The following equality holds true:
\[
H_m(x, u) = m \frac{u + 1}{u - 1} \left( -\frac{e^{\pi ix}}{(\pi i)^m} + \frac{(-1)^m e^{\pi ix}}{(2\pi i)^m} + \Phi \left( e^{-2\pi ix}, m, -\frac{1}{2} \right) \right) + 2m! \left( -\frac{e^{\pi ix}}{(\pi i)^{m+1}} + \frac{(-1)^{m+1} e^{\pi ix}}{(2\pi i)^{m+1}} + \Phi \left( e^{-2\pi ix}, m + 1, -\frac{1}{2} \right) \right) + \frac{1}{2} \left( \frac{u + 1}{u - 1} \right)^2 \left( -\frac{e^{\pi ix}}{(\pi i)^{m-1}} + \frac{(-1)^{m-1} e^{\pi ix}}{(2\pi i)^{m-1}} + \Phi \left( e^{-2\pi ix}, m - 1, -\frac{1}{2} \right) \right) + (u + 1) \sum_{k=0}^{m-4} \frac{d^k}{dx^k} H_m(x, u) \bigg|_{x=0} \left( -\frac{e^{\pi ix}}{(\pi i)^{k+1}} + \frac{(-1)^{k+1} e^{\pi ix}}{(2\pi i)^{k+1}} + \Phi \left( e^{-2\pi ix}, k + 1, -\frac{1}{2} \right) \right).
\]

For \( u = -1 \) on the above theorem, we have the following corollary.

Corollary 2. The following identity
\[
E_m(x) = 2m! \left( -\frac{e^{\pi ix}}{(\pi i)^{m+1}} + \frac{(-1)^{m+1} e^{\pi ix}}{(2\pi i)^{m+1}} + \Phi \left( e^{-2\pi ix}, m + 1, -\frac{1}{2} \right) \right)
\]
is true.

Setting \( x = 1 \) in Theorem 1, we obtain

(2.7)
\[
H_m(1, u) = -m! \sum_{n=-\infty}^{\infty} \left[ \frac{u + 1}{u - 1} \frac{1}{(2n + 1) \pi i} + \frac{2}{((2n + 1) \pi i)^2} + \frac{1}{2} \left( \frac{u + 1}{1 - u} \right)^2 \right] \frac{1}{((2n + 1) \pi i)^{m-1}} - (u + 1) \sum_{k=0}^{m-4} \frac{d^k}{dx^k} H_m(x, u) \bigg|_{x=0} \sum_{n=-\infty}^{\infty} \frac{1}{(n + \frac{1}{2})^{k+1}}.
\]

By expressions of (2.7) and Lemma 1, we easily see the following corollary.

Corollary 3. The following identity holds true:

\[
u H_m(u) = -m! \sum_{n=-\infty}^{\infty} \left[ \frac{u + 1}{u - 1} \frac{1}{(2n + 1) \pi i} + \frac{2}{((2n + 1) \pi i)^2} + \frac{1}{2} \left( \frac{u + 1}{1 - u} \right)^2 \right] \frac{1}{((2n + 1) \pi i)^{m-1}} - (u + 1) \sum_{k=0}^{m-4} \frac{d^k}{dx^k} H_m(x, u) \bigg|_{x=0} \sum_{n=-\infty}^{\infty} \frac{1}{(n + \frac{1}{2})^{k+1}}.
\]

Now, by using Kim’s method in [3], we discover the following

(2.8a)
\[
\frac{1}{1 - ue^{-t}} = \sum_{n=0}^{\infty} u^n e^{-nt} = \sum_{n=0}^{\infty} u^n (e^{-t})^n = \sum_{n=0}^{\infty} u^n \left( \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!} \right)^n = \sum_{n=0}^{\infty} \left( \sum_{a_1 + a_2 + \ldots + a_n = n} \frac{n!}{(a_1)! (a_2)! \ldots (a_l)!} (-1)^{a_1+2a_2+\ldots} \right) u^{a_1+2a_2+\ldots}
\]
Let \( p(i, j) : a_1 + 2a_2 + \ldots = i, a_1 + a_2 + \ldots = j \), from expression of (2.8a), we compute as follows:

\[
\frac{1}{1 - ue^{-t}} = \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} u^n \sum_{p(m,n)} \frac{n!}{(a_1)! (a_2)! \ldots (a_m)! (1)^{a_1} (2)^{a_2} \ldots (m)^{a_m}} \right) \frac{t^{a_1+2a_2+\ldots+ma_m}}{
\sum_{m=0}^{\infty} (-1)^m \left( \sum_{n=0}^{m} n!u^n \sum_{p(m,n)} \frac{m!}{(a_1)! (a_2)! \ldots (a_m)! (1)^{a_1} (2)^{a_2} \ldots (m)^{a_m}} \right) \frac{t^m}{m!}.
\]

where \( s_2(m, n) \) is the second kind stirling number.

Via the definition of Frobenius-Euler numbers, we readily derive the following

\[
\frac{1}{1 - ue^{-t}} = \frac{u^{-1} - u^{-1}}{u^{-1} - 1} = \frac{1}{1 - u} \sum_{m=0}^{\infty} (-1)^m H_m \left( u^{-1} \right) \frac{t^m}{m!}.
\]

By comparing the coefficients of \( \frac{t^n}{n!} \) on the both sides of (2.9a) and (2.10), we reach the following theorem.

**Theorem 3.** The following equality holds true:

\[
\frac{1}{1 - u} \sum_{m=0}^{\infty} \frac{t^m}{m!} = \sum_{n=0}^{m} n!u^n s_2(m, n).
\]

**Corollary 4.** Substituting \( u = -1 \) in the above theorem, we get the following, which is defined by Kim [3]

\[
E_m = 2 \sum_{n=0}^{m} n! (-1)^n s_2(m, n).
\]

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