Improved discrimination of unitaries by entangled probes

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We consider the problem of discriminating among a set of unitaries by means of measurements performed on the state undergoing the transformation. We show that use of entangled probes improves the discrimination in the two following cases: i) for a set of unitaries that are the UIR of a group and, ii) for any pair of transformations provided that multiple uses of the channel are allowed.

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I. INTRODUCTION

Entanglement is perhaps the most distinctive ingredient of quantum mechanics. In the recent years it has been recognised that entanglement is a resource to improve processing of quantum information and to increase the speed of computation. In this paper, we address the use of entanglement as a resource to improve quantum measurements. In particular, we will deal with measurements that correspond to the estimation of the parameter $\theta$ labeling a unitary transformation $U_\theta$ which acts on a system described by the Hilbert space $\mathcal{H}$. Usually, the problem is faced by fixing an input state $|\psi\rangle \in \mathcal{H}$ that undergoes one of the $U_\theta$’s (Fig. 1), and then applying quantum estimation theory [2] to look for the POVM which is able to distinguish the possible output states $U_\theta|\psi\rangle$ with the minimum error probability $P_E$. In general, this error probability, or any other chosen figure of merit, will be a function of the input state $|\psi\rangle$, and one further optimizes on $|\psi\rangle$.

Here, we will consider the possibilities offered by the use of a bipartite input state $|E\rangle \in \mathcal{H} \otimes \mathcal{H}$ instead of the simpler local state $|\psi\rangle$. The transformation $U_\theta$ will act locally on $|E\rangle$ thus giving as output the state $|\Psi_\theta\rangle = U_\theta \otimes I|E\rangle$, as depicted in Fig. 1. We will show that such novel configuration can do better than local measurements in discriminating the unitaries.

In Section II we focus our attention on the discrimination of unitary transformations drawn from a unitary irreducible representation of a group (UIR), whereas in Section III we will treat the problem of distinguishing among two given unitaries. Section IV closes the paper with some concluding remarks.
FIG. 1. The parameter $\theta$ is estimated as the result of a unitary transformation $|\psi\rangle \rightarrow U_\theta |\psi\rangle$ (up figure). In this scenario the use of a possibly entangled input $|E\rangle$ in place of $|\psi\rangle$ is considered, with the unknown transformation $U_\theta$ acting locally on one Hilbert space only (down figure).

II. DISCRIMINATION AMONG A SET OF UNITARY TRANSFORMATIONS (UIR)

As a first example consider the problem of discriminating among the four unitary transformations given by the Pauli matrices $\{\sigma_i\}$ acting on a qubit. By applying these unitaries to a any local pure state $|\psi\rangle$, one gets the four non-orthogonal states $\sigma_j |\psi\rangle$, whereas for a maximally entangled input state one finds four maximally entangled states which are orthogonal, and thus exactly distinguishable, at least in principle. In fact, by adopting the notation $|E\rangle = \sum_{ij} E_{ij} |i\rangle |j\rangle$ that puts vectors $|E\rangle \in H \otimes H$ into correspondence with operators $E$ on $H$, a generic maximally entangled input state can be written as $1/\sqrt{d} |U\rangle$, with $U$ unitary. Thus, in the Pauli example the possible outputs are $1/\sqrt{2} |\sigma_i U\rangle$, and they are orthogonal since $\langle \sigma_i U | \sigma_j U \rangle = \text{Tr}[U^\dagger \sigma_i^\dagger \sigma_j U] = \delta_{ij}$. We notice that, basically, the same kind of configuration has been used for quantum dense coding. The generalization to a $d$-dimensional system corresponds to the problem of discriminating the $d^2$ unitary transformations $\displaystyle U(m, n) = \sum_{k=0}^{d-1} e^{2\pi i km/d} |k\rangle \langle k \oplus n|$, with $n$ and $m$ ranging in $0 \div d - 1$, and $\oplus$ denoting addition modulo $d$. Again, if the input is maximally entangled, we have orthogonal output states.

Now, suppose we have a set of unitary transformations $\{U_g\}$, $g \in G$ that form a (projective) representation of the group $G$, i. e. $U_g U_h = \omega(g, h) U_{gh}$, where $\omega(g, h)$ is a phase factor satisfying the Jacobi associativity constraints, namely that $\omega(gh, l) = \omega(g, hl) \omega(h, l)$ and $\omega(g, g^{-1}) = \omega(g, e) = 1$, for $g, h, l \in G$, $e$ being the identity element. We will consider the case in which such a representation is irreducible (UIR), i.e. there are no subspaces of $H$ invariant for the action of all the $U_g$. This was also the case of the preceding example, with $\{U(m, n)\}$ a UIR of the group $\mathbb{Z}_d \times \mathbb{Z}_d$. Given a UIR, from Schur’s lemma it follows that for each operator $O$ on $H$ one has

$$\left[ U_g O U^\dagger_g \right]_G = \text{Tr}[O] I ,$$

(1)
where \([f(g)]_G\) denotes the group averaging \([f(g)]_G \doteq \sum_{g \in G} \mu(g) f(g)\), with \(\mu(g) = \frac{d}{|G|}\), \(d = \dim(\mathcal{H})\), and \(|G|\) the cardinality of \(G\). Eq. (1) can be generalized to the continuous case by defining group averaging as \([f(g)]_G \doteq \int_G \mu(dg) f(g)\), \(\mu(dg)\) being a properly normalized invariant measure on the group \(G\).

In order to show that entanglement is of help in improving the discrimination, and to quantify this improvement, we now consider several state-related parameters. First of all, as in the first two examples, one can see that the dimension of the Hilbert space \(\mathcal{H}_{\text{out}}\) spanned by the output states is larger for an entangled input than for factorized states. In fact, \(\dim(\mathcal{H}_{\text{out}})\) can be calculated as the rank of the operator

\[
O = \left[ |\Psi_g\rangle \langle \Psi_g| \right]_G = \left[ U_g \otimes I |E\rangle \langle E| U_g^\dagger \otimes I \right]_G,
\]

where \(\Psi_g = U_g E\). By means of Eq. (1) one has \(O = I \otimes \text{Tr}_1 [E] \langle E| = I \otimes (E^\dagger E)^T\), so that

\[
\dim(\mathcal{H}_{\text{out}}) = d \times \text{rank}(E^\dagger E),
\]

i.e. the output space is enlarged by a factor equal to the Schmidt number \([\mathbb{I}]\) of the input state. Indeed, since probing the operation with a bipartite entangled system gives access to a larger Hilbert space we have, literally, more room for improvement. In the following, we refine these concepts, and give conditions under which an entangled scheme is convenient.

The Schmidt number is only a coarse measure of the amount of entanglement stored in \(|E\rangle\rangle\), and the dimension of the output space is only indirectly connected to the distinguishability of the outputs. A more refined goodness criterion is given by the Holevo’s information \(\chi\) of the set of output states, all taken with the same probability \(p(g) = 1/|G|\) (or \(p(dg) = \mu(dg)/\mu(G)\) in the continuous case), this quantity is an upper bound for the accessible information \([\mathbb{I}]\). Denoting by \(S(\rho) = -\text{Tr} \rho \log \rho\), the von Neumann entropy of \(\rho\), the Holevo’s information \(\chi\) reads

\[
\chi = S \left( \frac{1}{\mu(G)} \left[ |\Psi_g\rangle \langle \Psi_g| \right]_G \right) - \frac{1}{\mu(G)} \left[ S(|\Psi_g\rangle \langle \Psi_g|) \right]_G =
\]

\[
= S \left( \frac{1}{\mu(G)} I \otimes E^T E^* \right) =
\]

\[
= \frac{d}{\mu(G)} \log \mu(G) + \frac{d}{\mu(G)} S(E^T E^*),
\]

and thus the bound is increased by an amount proportional to the degree of entanglement \(S(E^T E^*)[\mathbb{I}]\) of the input state \(|E\rangle\rangle\) (recall that for discrete groups \(\mu(G) = d\)).

Facing the problem with a maximum likelihood strategy, the optimal covariant POVM that discriminates among the \{\(|\Psi_g\rangle\rangle\}\) takes the form \([\mathbb{I}]\)

\[
\Pi_g = \mu(g)(U_g \otimes I) P(U_g^\dagger \otimes I),
\]

\(^1\)\(S(E^T E^*)\) represents the entropy of the partial traces of \(|E\rangle\rangle\), which indeed is the measure of entanglement for pure states.
with \( P \geq 0 \) a positive operator on \( \mathcal{H} \otimes \mathcal{H} \) normalized as \( \text{Tr}_1[P] = I \). By covariance, the likelihood – i.e. the probability of getting an outcome \( g \) when the state is \( |\Psi_g\rangle \) – is proportional to \( \langle E|P|E\rangle \leq d \), where the bound comes from the normalization condition on \( P \), which limits the largest possible eigenvalue of \( P \) below \( d \). Again, the optimality (saturation of the bound) is reached for a maximally entangled input state, i.e. for \( E = d^{-\frac{1}{2}}U \), with \( U \) unitary, and \( P = |U\rangle \langle U| \). The optimality of a maximally entangled input state for the estimation of unitaries in \( SU(d) \) has also been noticed in Ref. [3].

Since the overlap of two states is the only parameter that determines their distinguishability, we will consider the average overlap \( \Omega(E) \) of all the couples of states in \( \{ |\Psi_g\rangle \} \): the lower is \( \Omega(E) \) the better will be the overall distinguishability. One has

\[
\Omega(E) = \frac{1}{2\mu(G)} \left[ \frac{1}{\langle \langle \Psi_g | \Psi_g \rangle \rangle} \right]_{G \times G} = \frac{1}{2\mu(G)} \left[ \langle \langle E | \Psi_g \rangle \rangle \langle \langle E | E \rangle \rangle \right]_{G} = \frac{1}{2\mu(G)} \langle \langle E | I \otimes (E^T E^*) | E \rangle \rangle = \frac{1}{2\mu(G)} \langle \langle E | E E^\dagger E \rangle \rangle = \frac{1}{2\mu(G)} \text{Tr}[(E^\dagger E)^2]. \tag{6}
\]

In order to analyze the properties of \( \Omega(E) \), we have to briefly recall the definition of the “majorization” relation between entangled pure states and its physical meaning. Given two states \( |A\rangle \) and \( |B\rangle \) in \( \mathcal{H} \otimes \mathcal{H} \), let \( \lambda_A^j \) and \( \lambda_B^j \) be the vectors of eigenvalues of \( A^\dagger A \) and \( B^\dagger B \) respectively, sorted in descending order. We say that \( |A\rangle \prec |B\rangle \) iff

\[
\sum_{j=1}^{k} (\lambda_A^j)_j \leq \sum_{j=1}^{k} (\lambda_B^j)_j , \quad \text{for each } k \leq d . \tag{7}
\]

The physical meaning of this partial ordering relation has been clarified in Ref. [3]: \( |A\rangle \) can be transformed into \( |B\rangle \) by local operations and classical communication if and only if \( |A\rangle \prec |B\rangle \). Our average overlap \( \Omega(E) \) is a so called “Schur convex function” of the eigenvalues of \( E^\dagger E \), namely if \( |A\rangle \prec |B\rangle \) then \( \Omega(A) \leq \Omega(B) \). Since any maximally entangled state is majorized by any other state, it is clear that the minimum overlap is found in correspondence with \( |E\rangle \) maximally entangled, and any manipulation of such a state can only increase \( \Omega(E) \), thus reducing the distinguishability, and, as a consequence, the sensitivity of the measurement.

### III. Discrimination between two unitary transformations

Let us suppose that we have to distinguish among two unitaries \( U_1 \) and \( U_2 \). Given an input state \( |\psi\rangle \), one optimizes over the possible measurements, and the minimum error probability in discriminating \( U_1|\psi\rangle \) and \( U_2|\psi\rangle \) [3] is given by

\[
P_E = \frac{1}{2} \left[ 1 - \sqrt{1 - |\langle \psi | U_2^\dagger U_1 |\psi\rangle|^2} \right] , \tag{8}
\]

so that one has to minimize the overlap \( |\langle \psi | U_2^\dagger U_1 |\psi\rangle| \) with a suitable choice of \( |\psi\rangle \). Chosing as a basis the eigenvectors \( \{|j\rangle\} \) of \( U_2^\dagger U_1 \), and writing \( |\psi\rangle = \sum_j \psi_j |j\rangle \), we define
\[ z_\psi = \langle \psi | U_2^\dagger U_1 | \psi \rangle = \sum_j |\psi_j|^2 e^{i\gamma_j}, \]  

(9)

where \( e^{i\gamma_j} \) are the eigenvalues of \( U_2^\dagger U_1 \). The normalization condition for \( |\psi\rangle \) is \( \sum_j |\psi_j|^2 = 1 \), so that the subset \( K(U_2^\dagger U_1) \subset \mathbb{C} \) described by \( z_\psi \) for varying \( |\psi\rangle \) is the convex polygon having the points \( e^{i\gamma_j} \) as vertices. The minimum overlap

\[ r(U_2^\dagger U_1) = \min_{||\psi||=1} |\langle \psi | U_2^\dagger U_1 | \psi \rangle| \]  

(10)

is the distance of \( K(U_2^\dagger U_1) \) from \( z = 0 \). This geometrical picture indicates in a simple way what is the best one can do in discriminating \( U_1 \) and \( U_2 \): if \( K \) contains the origin then the two unitaries can be exactly discriminated, otherwise one has to find the point of \( K \) nearest to the origin, and the minimum probability of error is related to its distance from the origin. Once the optimal point in \( K \) is found, the optimal states \( \psi \) are those corresponding that point through Eq. (9).

\[ \gamma^- \]
\[ \gamma^+ \]

FIG. 2. \( r \) is the minimum distance between the origin and the polygon \( K \)

If \( \Delta(U_2^\dagger U_1) \) is the angular spread of the eigenvalues of \( U_2^\dagger U_1 \) (referring to Fig. 2, it is \( \Delta = \gamma^+ - \gamma^- \)), from Eq. (8) for \( \Delta < \pi \) one has

\[ P_E = \frac{1}{2} - \frac{1}{2} \sqrt{1 - \cos^4 \Delta}, \]  

(11)

whereas for \( \Delta \geq \pi \) one has \( P_E = 0 \) and the discrimination is exact.

Given \( U_1 \) and \( U_2 \) non exactly discriminable, one is interested in understanding wheter or not an entangled input state could be of some use. The answer is negative, in fact using entanglement translates the problem into the one of distinguishing between \( U_1 \otimes I \) and \( U_2 \otimes I \), thus one has to analyze of the polygon \( K(U_2^\dagger U_1 \otimes I) \). Since \( U_2^\dagger U_1 \otimes I \) has the same eigenvalues as \( U_2^\dagger U_1 \), the polygons \( K(U_2^\dagger U_1 \otimes I) \) and \( K(U_2^\dagger U_1) \) are exactly the same, so that they lead to the same minimum probability of error.

The situation changes dramatically if \( N \) copies of the unitary transformation are used, as depicted in Fig. 3: here one has to compare the “performance” of \( K(U_2^\dagger U_1) \) to the one of \( K((U_2^\dagger U_1)^\otimes N) \).
FIG. 3. When distinguishing between two unitaries $U = U_{1,2}$ it is possible to achieve perfect discrimination even for nonorthogonal $U_1$ and $U_2$ for sufficiently large number $N$ of copies of the unitary transformation, using a $N$-partite entangled state as in figure (see text).

Since $\Delta((U_1^U U_2^U)^{\otimes N}) = \min\{N \times \Delta(U_1^U U_2^U), 2\pi\}$, it is clear that there will be an $\bar{N}$ such that $U_1^{\otimes \bar{N}}$ and $U_2^{\otimes \bar{N}}$ will be exactly discriminable. This same result has been demonstrated in Ref. [12] starting from a different approach.

IV. CONCLUSIONS

We have shown that the use of entangled states as a probe provides an effective scheme to discriminate among a set of unitary transformations. We have analyzed the discrimination of a set of unitaries being the UIR of a group, showing that entanglement is always useful. We have also considered the discrimination between two generic transformations, where it is possible to achieve perfect discrimination even for nonorthogonal $U_1$ and $U_2$ for sufficiently large number $N$ of copies of the unitary transformation, if a $N$-partite entangled state is available. The present results for the discrimination of a discrete set of unitaries can be generalized to the continuous case [13], i.e., to the estimation of parameters. In this case entanglement improves the performances of the measurement scheme also in presence of losses.

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