The Static String

Sergio Giardino

IMECC/Unicamp, Brazil

In this article the quantum fluctuation of a rigid and static string is reported to be identical to a free quantum particle. Solutions similar to this static string have already been found in the semi-classical quantization of pulsating strings, and our results show that the semi-classical quantization of pulsating strings is, in some cases, a perturbation of static strings. We also interpret the energy of the static string as a lower bound for the pulsating string and speculate about a description of quantum mechanics in terms of semi-classical string theory.

PACS numbers:

I. INTRODUCTION

The semi-classical quantization of a pulsating string in $AdS_5 \times S^5$ \cite{1,2} is executed in such a manner that the string is allowed to move along the radial dimension $\rho$ of $AdS_5$ and along the angular coordinates $\theta$ and $\phi$ of the $S^2$ sector of $S^5$. The ansatz $t = \tau$, $\rho = \rho(\tau)$, $\theta = \theta(\tau)$ and $\phi = m\sigma$ leads to the Nambu-Goto action

$$S = -m\sqrt{\lambda} \int dt \sin \theta \sqrt{\cosh^2 \rho - \dot{\rho}^2 - \dot{\theta}^2},$$

where $\tau$ is the proper time of the string and $\sigma$ is the length parameter of the string. From this action the canonical Hamiltonian is obtained, and it allows the wave-function and the energy spectrum to be calculated. However, the classical equations of motion mean that $\dot{\rho} = 0$, and thus the string does not move along this direction, instead moving in the spherical sector only. Although the classical string does not move in $AdS_5$, the simple fact that it might move along $\rho$ implies that there is a quantum effect associated with this direction. This is due

*Electronic address: giardino@ime.unicamp.br
to canonical momentum associated with \( \rho \) obtained from (5), and part of the Hamiltonian operator that describes quantum behavior and determines the quantum energy spectrum. The quantum effect of this radial degree of freedom is not important when considering high quantum numbers. On the other hand, it raises the question of a string with no motion along any coordinate that allows obtain a canonical Hamiltonian and consequently quantum fluctuations in its energy.

The possibility of a classical string being used to construct a quantum model enables us to establish a relation between string theory and quantum mechanics. The quantum harmonic oscillator has already been obtained from a classical pulsating string [1] in Minkowski space-time, and a generalization of this method has obtained a variety of quantum models in different space times [3]. In this article we go further and determine that the quantum free particle may be obtained from a classical static string.

On the other hand, quantum fluctuations do not depend only on the string; it is also determined by the topology and geometry of the space-time where the string is located. This fact has already been observed in the semi-classical quantization of pulsating strings in various backgrounds [4–9]. We develop this point further by calculating the differences in the quantum fluctuations of a static string in a \((2 + 1)\)-dimensional space both with and without the effect of the other coordinates of the space. This idea has already been explored in the case of pulsating strings and of free falling strings [3], and in this article we propose to extend it to static strings. The results indicate that the dimension of the space may indeed contribute to quantum fluctuations, but the \((2 + 1)\)-dimensional is interesting as a toy model in order to conceptualize an understanding of the problem.

This article is organized as follows: section (II) describes a classical static string in a \((2 + 1)\)-dimensional space-time and builds a general quantum model associated with it. In section (III), the embedding of a world-sheet of a pulsating string and of a free falling string in a 3—dimensional plane space are reviewed and the quantum model of static strings in these space-times is described using polar coordinates. In section (IV), quantum fluctuations in periodic coordinates are considered in spherical and toroidal spaces; however, the quantum discussion of the toroidal space is qualitative only. Section (V) contains our conclusions.
II. THE SEMI-CLASSICAL DESCRIPTION

We are looking for a static string in a $(2 + 1)$-dimensional section of space-time with the line element

$$ds^2 = -dt^2 + dx^2 + g^2 dy^2,$$

where $g = g(x)$ is a function. The classical motion can be studied by means of the Virasoro constraint,

$$-(t'^2 + \dot{x}^2 + \ddot{x}^2 + g^2(y'^2 + \dot{y}^2) = 0,$$

where the dot means a derivative relative to the proper time $\tau$ and the prime means a derivative in relation to the string length parameter $\sigma$. A static solution can be obtained if $\dot{x} = \dot{y} = 0$ and $r' = t'$, where the simplest choice is, of course, $r' = t' = 0$. It will also be ascertained that $y'$ is a constant, and then we must choose $y$ as an angular coordinate. Finally, (3) implies that $t'^2 = g^2 y'^2$, with $g$ calculated at the position of the string. Hence the final ansatz for the string is

$$t = \kappa \tau, \quad x = x(\tau) \quad \text{and} \quad y = m\sigma + y(\tau),$$

where $\kappa$ and $m$ are constants. However, in order to study the quantum corrections to the classical energy, the Nambu-Goto action is used. In the general case where every coordinate is a function of $\tau$ and $\sigma$, the Nambu-Goto action is

$$S = -\sqrt{\lambda} \int d\tau \int d\sigma \sqrt{(i x' - \dot{x} t')^2 + g^2 (i y' - \dot{y} t')^2 - g^2 (\dot{y} x' - \dot{x} y')^2},$$

and, according to the ansatz (4), (5) becomes

$$S = -\sqrt{\lambda} \int d\tau g y' \sqrt{t'^2 - \dot{x}^2}. $$

This classical action has no $\dot{y}$ dependence, which means there is no classical momentum associated with this coordinate. Consequently, no quantum fluctuation in this direction is observed. The canonical Hamiltonian in this case is expressed as

$$H = i \sqrt{\Pi^2 + \lambda y'^2 g^2}, \quad \text{where} \quad \Pi = \sqrt{\lambda} y' g \frac{\dot{x}}{t'^2 - \dot{x}^2},$$

is the canonical momentum for the radial direction. The potential term of the Hamiltonian is constant, thus we use the square of it in the Schrödinger equation, so that

$$(\Pi^2 + E^2) \Psi = \frac{E^2}{\kappa^2} \Psi,$$
where $\mathcal{E} = \sqrt{\lambda} \dot{t}$ and $\dot{t} = \kappa$ have been used as the classical energy of the string and $\dot{t}^2 = g^2 y'^2$ has been used from (3). Thus, the quantum solutions are free particles whose squared energy is given by the difference between the squared classical energy $\mathcal{E}^2$ and the squared quantum oscillation $\mathcal{E}^2/\kappa^2$,

$$E^2 = \left| \frac{\mathcal{E}^2}{\kappa^2} - \mathcal{E}^2 \right|. \quad (9)$$

The eigenvalue $\pm E^2$ can be either positive or negative, and hence the static string has the quantum effect of generating free oscillating particles and non-oscillating modes in every point in space. Quantum fluctuation depends on the function $g$, which determines the geometry of the space and the momentum operator $\hat{\Pi}$. Some possibilities for $g$ have already been discussed in the context of moving strings [3], and we bring them to the context of the static string in the next section.

III. POLAR COORDINATES

The $(2 + 1)$-dimensional geometries where the string lies have been classified according to the function $g$, with $x = r$ a radial coordinate [3]; the results are summarized in the table below.

| $g^2$       | radial rank | topology        |
|-------------|-------------|-----------------|
| $\ell^2 r^n$| $n = 1$     | $r \in [0, \infty)$ | non-physical |
|             | $n = 2$     | $r \in [0, \infty)$ | infinite plane |
|             | $n > 2$     | $r \in [0, \mathcal{R}]$ | finite cone |
| $\ell^2 r^{-n}$| $n = 1$     | $r \in [0, \infty)$ | non-physical |
|             | $n > 1$     | $r \in [\mathcal{R}, \infty)$ | punctured plane |

Where $\ell$ is a dimensional constant that gives $g$ the length dimension for each $n$,

$$\mathcal{R} = \left( \frac{2}{\ell n} \right)^{\frac{n}{n-2}} \quad \text{and} \quad \mathcal{R} = \left( \frac{\ell n}{2} \right)^{\frac{n}{n-2}}. \quad (10)$$

The above geometries were used for studying pulsating strings when $g^2 = \ell^2 r^n$ and free-falling strings when $g^2 = \ell^2 r^{-n}$. Both of these geometries support the static string, however the quantum fluctuations in each of them are different, therefore we analyze the cases separately, according to the function $g$. 
A. \( g^2 = \ell^2 r^n \)

In this case, the Schrödinger equation

\[
\Psi'' + \frac{n}{2r} \Psi' \pm E^2 \Psi = 0.
\]  

(11)

has different solutions for each sign above, and they are expressed in terms of Bessel functions

\[
\Psi_+ = \frac{1}{r^{n+1}} \left[ A J_{n-\frac{1}{2}}(E r) + B Y_{n-\frac{1}{2}}(E r) \right],
\]  

(12)

\[
\Psi_- = \frac{1}{r^{n+1}} \left[ C I_{n-\frac{1}{2}}(E r) + D K_{n-\frac{1}{2}}(E r) \right],
\]  

(13)

where the \( A, B, C \) and \( D \) are integration constants. The wave-function \( \Psi_+ \) describes oscillating modes so that \( E^2 / \kappa^2 > \mathcal{E}^2 \), and \( \Psi_- \) obviously corresponds to non-oscillating modes so that \( E^2 / \kappa^2 < \mathcal{E}^2 \). The quantum fluctuation energy spectrum can thus be expressed for each case as

\[
\mathcal{E}^2_+ = \kappa^2 (\mathcal{E}^2 + E^2), \quad \text{and} \quad \mathcal{E}^2_- = \kappa^2 (\mathcal{E}^2 - E^2),
\]  

(14)

where the discretization or the continuousness of the energy spectrum depends on the eigenvalue \( E^2 \). It can also be seen that \( \kappa^2 \mathcal{E}^2 \) is the zero point squared energy, so that in the case of \( \mathcal{E}^2_+ \) the classical energy is a lower bound and in the case of the \( \mathcal{E}^2_- \) the classical energy is an upper bound.

The free particle wave-functions are normalizable for the finite spaces where \( n > 2 \). However, as the Bessel functions \( Y \) and \( K \) are singular at \( r = 0 \), we make \( B = 0 \) and \( D = 0 \). The Bessel function \( I \) goes to infinity if \( r \to \infty \), hence for \( n = 2 \), where \( r \in [0, \infty) \), \( C = 0 \) as well, and thus non-oscillating solutions occur only for \( n > 2 \). However, in the \( n = 2 \) case the wave-function is not normalizable. This problem can be circumvented when the solutions are localizable \( [3] \), something that is obtained if the wave-function obey

\[
\int_0^\infty \Psi \Psi^* \sqrt{-G} \, dr = \delta(r),
\]  

(15)

where \( G \) is the determinant of the metric tensor. The Dirac delta function expressed in terms of Bessel functions in a \((d + 1)\)-dimensional space is given as

\[
\delta^{d+1}(\epsilon^2 - \eta^2) = \int_0^\infty dr \, r \, J_\mu(\epsilon \, r) \, J_\mu(\eta \, r),
\]  

(16)
and as the wave-function for \( n = 2 \) satisfies (16), it can indeed be understood as a free particle. The non-oscillating solutions are either non-localizable or non-normalizable for \( n = 2 \), and thus they have been disregarded. On the other hand, for \( n > 2 \), \( r \) is finite and \( \Psi_- \) may be admitted. Non-oscillating modes are known from the tunneling phenomenon, where they describe classically prohibited regions. They are not localizable, and certainly cannot be understood as particles.

The energy of oscillating modes can either be discrete or continuous, and the energy spectra of non-oscillating solutions are always continuous. For \( n = 2 \) the energies are always continuous, and for \( n > 2 \) continuous and discrete energies coexist. The Bessel function \( J \) for \( n > 2 \) determines quantum solutions if a root occurs at the border \( r = \mathcal{R} \) of the space. Then there are quantized and non-quantized solutions, so that the eigenvalues for the quantized solutions obey

\[
E_N = \frac{R^{(N)}}{\mathcal{R}},
\]

so that \( N \in \mathbb{N} \) and \( R^{(N)} \) is a root of \( J \). As \( \mathcal{R} \) and \( R^{(N)} \) are fixed for each \( N \), there is just one value of \( E \) which obeys (17), and then this class of solutions is quantized.

\[B. \quad g^2 = \ell^2 r^{-n}\]

This case is very similar to the preceding one, and the Schrödinger equation

\[
\Psi'' - \frac{n}{2r} \Psi' \mp E^2 \Psi = 0
\]

also has its solutions given in terms of Bessel functions

\[
\Psi_+ = r^{n+2} \left[ A J_{\frac{n+2}{4}}(Er) + B Y_{\frac{n+2}{4}}(Er) \right],
\]

and

\[
\Psi_+ = r^{n+2} \left[ C I_{\frac{n+2}{4}}(Er) + D K_{\frac{n+2}{4}}(Er) \right],
\]

where \( A, B, C \) and \( D \) are integration constants. \( \Psi_+ \) is non-normalizable but is localizable in the sense of (15) for either \( B = 0 \) or \( A = 0 \). \( \Psi_- \) is evanescent and normalizable for \( C = 0 \).

There are discrete quantum solutions for \( \Psi_+ \) when a root of the Bessel function \( J \) or \( Y \) occurs at the border of the space, where \( r = \mathcal{R} \), and the energies for the discrete quantized solutions obey

\[
E_N = \frac{R^{(N)}}{\mathcal{R}} , \quad \text{so that} \quad N \in \mathbb{N} \quad \text{and} \quad \mathcal{Z} = \{ J, Y \}.
\]
where \( N \in \mathbb{N} \) and \( R_2^{(N)} \) is a root of either \( J \) or \( Y \). \( \Psi_+ \) allows the Bessel function \( Y \) because the singular point \( r = 0 \) is out of the space. Then, there are more allowed quantum wavefunctions than in the preceding case.

These solutions, for both of \( g^2 \), enable us to state that the oscillating quantum fluctuations of the static string are free particles. Thus, the quantum problems of the harmonic oscillator and the free particle can be described in terms of the semi-classical approximation of string theory. In this sense, we can speculate that there is a correspondence between semi-classical string theory and quantum mechanics. We note in these cases the usefulness of string theory as an instrument to describe different physical theories where there is no string at all, something that has already been developed in the context of \( AdS/CFT \) correspondence. If this hypothesis is correct, the role of string theory in physics remains an open question: it can either be a mathematical framework or a fundamental theory in which strings are a physical reality.

IV. ANGULAR COORDINATES

With space coordinates angular and periodic in (6), different topologies can be studied, and we consider here the sphere and the torus. The classical string on a sphere has been studied in the context of a pulsating string [1], and we extend it to the static case. The string on a torus is a new possibility, and we are interested in pointing out the difference of the quantum fluctuations due to the topology change in relation to the spherical case.

A. STRING ON A SPHERE

The sphere is obtained for \( x = \theta \), \( y = \varphi \) and \( g = \sin \theta \). The static string ansatz for this geometry is

\[
t = \kappa \tau, \quad \theta = \theta(\tau) \quad \text{and} \quad \varphi = m \sigma, \quad (22)
\]

and the Nambu-Goto action, the Hamiltonian and the Schrödinger equation comes straight from (8). However, contrary to the former cases, the \( \hat{\Pi}^2 \) operator changes with the dimension of space-time because the Laplacian operator changes with the determinant of the metric. If we use a \((2 + 1)\)-dimensional space, we obtain

\[
\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d}{d\theta} \Psi \right) \pm E^2 \Psi = 0, \quad (23)
\]
and if a ten dimensional space is used, the result is

\[
\frac{1}{\sin^3 \theta \cos \theta} \frac{d}{d\theta} \left( \sin^3 \theta \cos \theta \frac{d}{d\theta} \Psi \right) \pm E^2 \Psi = 0. \tag{24}
\]

The solutions for (23) and (24) with the plus sign have already been used to semi-classically quantize the pulsating string in Lunin-Maldacena space-time [4], respectively in terms of Legendre and Jacobi polynomials, so that

\[
\Psi_{\text{Legendre}} = \sqrt{2N + 1} P_N(\cos \theta) \quad \text{and} \quad \Psi_{\text{Jacobi}} = 2 \sqrt{N + 1} P_N^{(0,1)}(1 - \cos^2 \theta), \tag{25}
\]

where \( N \in \mathbb{N} \). They are oscillating functions whose eigenvalues are respectively

\[
E^2_{\text{Legendre}} = N(N + 1) \quad \text{and} \quad E^2_{\text{Jacobi}} = 4N(N + 2). \tag{26}
\]

On the other hand, the \(-E^2\) case in (23) is written in terms of Legendre functions with complex index, the conical function \( P_{-\frac{1}{2} + i\tau} \), where \( \tau = \sqrt{4E^2 - 1} \) and the energy is not discrete in this case. These functions are normalizable, and then they can be used as wavefunctions for the non-oscillating modes. On the other hand, the solution for (24) with \(-E^2\) is hypergeometric function \( F\left(1 - q, 1 + q; 1, 1; \frac{1 - \cos \theta}{2}\right) \), with \( q = \sqrt{E^2 - 1} \). This function generates a divergent hypergeometric series and consequently cannot be a normalizable wave-function.

The results show that the former pulsating string semi-classical quantization results have indeed been obtained from perturbation of the static string wave-function, and then this is a more fundamental situation. The results also demonstrate that the effect of other dimensions of space are in fact sensible, which leads to very different energy spectra (26) and also determines whether the non-oscillating modes exist or not.

**B. STRING ON A TORUS**

The embedding coordinates of the torus in a three-dimensional space are

\[
x^1 = (R + \rho \cos \theta) \cos \phi, \quad x^2 = (R + \rho \cos \theta) \sin \phi, \quad \text{and} \quad x^3 = \rho \sin \theta, \tag{27}
\]

where \( R \) and \( \rho \) are the radii of the torus so that \( R > \rho \). The range of both of the angular coordinates is \([0, 2\pi]\), \( \theta \) is the angular coordinate using \( \rho \) as radius and \( \phi \) is the angular
coordinate using $R$ as its radius. From (27), the metric of the toroidal $(2 + 1)$-dimensional space-time is
\[ ds^2 = -dt^2 + \rho^2 d\theta^2 + \left( R + \rho \cos \theta \right)^2 d\phi^2. \] (28)

Using the usual ansatz
\[ t = \kappa \tau \quad \theta = \theta(\tau) \quad \text{and} \quad \phi = m\sigma, \] (29)
the Virasoro constraint
\[ -\kappa^2 + \rho^2 \dot{\theta}^2 + \left( R + \rho \cos \theta \right)^2 \phi'^2 = 0 \] (30)
is satisfied for $\dot{\theta} = 0$, and then the string is static. Using $x = \rho \theta$, $y = \phi$ and $g = R + \rho \cos \theta$ in the Nambu-Goto action (6), the Schrödinger equation for the free particle follows with the determinant of the metric $\sqrt{-G} = R + \rho \cos \theta$. Changing the variable so that $\chi = \frac{1 + \cos \theta}{2}$, we obtain a Heun-type equation
\[ \Psi_{xx} + \left[ \frac{1}{2} \left( \frac{1}{\chi} - \frac{1}{1 - \chi} \right) + \frac{1}{\chi + \frac{R - \rho}{2\rho}} \right] \Psi_x \pm \frac{E^2}{\chi(1 - \chi)} \Psi = 0, \] (31)
where the index $x$ means a derivative with respect to this coordinate and (31) is a Heun-type equation. The solution for $+E^2$ is periodic, oscillating and orthogonal function with appropriate choice of the parameters. When the eigenvalue is $-E^2$, the periodic and oscillating character disappear, and then the static string in the torus also has the qualitative features that were found in the static string on the sphere. The toroidal case differs from the spherical case in topology only. This geometrical difference is hence responsible for the change in the wave-function and in the energy spectrum, and is then another example of the influence of space-time on the quantum behavior of a string. The discussion of the quantum fluctuations of the static string in the toroidal space is qualitative because a detailed treatment of the quantum solutions of (31) is complicated enough to deserve an independent study.

V. CONCLUSION

In this paper we introduced a classical circular string that remains at a definite position in the space. This object can be semi-classically quantized in various space-times, and a quantum free particle is then obtained in each case. The difference in relation to the
usual quantum free particle is the split between oscillating and non-oscillating modes. The oscillating modes are the usual quantum free particles and their energies are higher than the classical energies of static strings, while the non-oscillating modes have lower energy than the classical energy of the static string. The results also demonstrate that static string wavefunctions have already been applied in the semi-classical quantization of pulsating strings. In fact, the known quantum pulsating string in a sphere is a static string perturbation, and hence the static string it is possibly a more fundamental object than a pulsating string.

Furthermore, the results give rise to some fundamental questions about the nature of string theory. The quantum harmonic oscillator and the quantum free particle can be obtained semi-classically from string theory, and it is important to comprehend to what extent quantum mechanics may be reconstructed from string theory. If every result from quantum mechanics can be recovered, then string theory is either a more fundamental theory or a very powerful mathematical framework. However, string theory may describe only some quantum results, and so the theories are in fact distinct frameworks with some overlap in specific limits. The extension of this overlap between string theory and quantum mechanics seems to be, from the author’s standpoint, a very important question that needs to be answered in order to understand the physical content of these theories.

Acknowledgements

Sergio Giardino is grateful for the financial support of Capes and for the facilities provided by the IFUSP.

[1] J. A. Minahan. “Circular semiclassical string solutions on $AdS_5 \times S^5$". Nucl. Phys., B648:203–214, (2003) hep-th/0209047.
[2] J. Engquist; J. A.Minahan; K. Zarembo. “Yang-Mills duals for semiclassical strings on $AdS_5 \times S^5$. JHEP, 0311:063, (2003) hep-th/0310188.
[3] S. Giardino. Semi-classical strings in $(2 + 1)$–dimensional backgrounds. (2013) arXiv:1303.7167[hep-th].
[4] Sergio Giardino and Victor Rivelles. Pulsating Strings in Lunin-Maldacena Backgrounds. JHEP, 1107:057, (2011) arXiv:1105.1353[hep-th].
[5] D. Arnaudov; H. Dimov; R. C. Rashkov. On the pulsating strings in $AdS_5 \times T^{1,1}$. *J.Phys.*, **A44**:495401, (2011) arXiv:1006.1539[hep-th].

[6] D. Arnaudov; H. Dimov; R. C. Rashkov. “On the pulsating strings in Sasaki-Einstein spaces”. *AIP Conf.Proc.*, **1301**:51–58, (2010) arXiv:1007.3364[hep-th].

[7] M. Beccaria; G. V. Dunne; G. Macorini; A. Tirziu; A. A. Tseytlin. “Exact computation of one-loop correction to energy of pulsating strings in $AdS_5 \times S^5$”. *J.Phys.*, **A44**:015404, (2011) arXiv:1009.2318[hep-th].

[8] H. Dimov; R. C. Rashkov. “Generalized pulsating strings”. *JHEP*, **0405**:068, (2004) hep-th/0404012.

[9] M. Smedback. “Pulsating strings on $AdS_5 \times S^5$”. *JHEP*, **0407**:004, (2004) hep-th/0405102.