Conformal regularization of Einstein’s field equations

Niklas Röhr and Claes Uggla

Department of Physics, University of Karlstad, S-651 88 Karlstad, Sweden
E-mail: Niklas.Rohr@kau.se and Claes.Uggla@kau.se

Received 14 April 2005, in final form 19 July 2005
Published 18 August 2005
Online at stacks.iop.org/CQG/22/3775

Abstract

To study asymptotic structures, we regularize Einstein’s field equations by means of conformal transformations. The conformal factor is chosen so that it carries a dimensional scale that captures crucial asymptotic features. By choosing a conformal orthonormal frame, we obtain a coupled system of differential equations for a set of dimensionless variables, associated with the conformal dimensionless metric, where the variables describe ratios with respect to the chosen asymptotic scale structure. As examples, we describe some explicit choices of conformal factors and coordinates appropriate for the situation of a timelike congruence approaching a singularity. One choice is shown to just slightly modify the so-called Hubble-normalized approach, and one leads to dimensionless first-order symmetric hyperbolic equations. We also discuss differences and similarities with other conformal approaches in the literature, as regards, e.g., isotropic singularities.

PACS numbers: 04.20.−q, 98.80.Jk, 04.20.Dw, 04.20.Ha

1. Introduction

The importance of conformal properties and causal structure in connection with Einstein’s field equations (EFEs) is well known, particularly as regards asymptotic structure, see e.g., [1–3] and references therein. It is also well known that scale-invariant, self-similar [4], solutions act as important building blocks for our understanding of the asymptotic properties of non-scale-invariant solutions, see, e.g., [5] and references therein. Indeed, the latter feature motivated the introduction of the so-called Hubble-normalized dynamical systems formulation of EFEs [6] (henceforth denoted as UEWE), which yielded regularized field equations in the neighbourhood of generic singularities and progress as regards our understanding, and ability to numerically handle [7, 8], asymptotic dynamics towards such singularities.

The purpose of this paper is to provide a framework that naturally captures all the above aspects. We thus combine the conformal and dynamical systems approaches to EFEs...
into a common geometric instrument; conformal transformations are used to obtain scale-invariant and thus dimensionless, regularized formulations of EFEs that naturally incorporate key asymptotic causal structures. The goal is to unravel features of the solution space and properties of solutions of EFEs. The relationship between conformal and asymptotic structures provides a systematic geometric foundation for finding suitable geometrically interpretable variables for dynamical systems analysis. This is to be contrasted with the more or less random ad hoc choices of variables that characterize the history of dynamical systems studies in general relativity.

The outline of the paper is as follows: in section 2, we discuss dimensionality under scale transformations and use this to restrict the choice of conformal factor and frame; subsequently, we give a set of conformal, dimensionless, field equations. As an example, we address asymptotic temporal structures associated with singularity formation in section 3. We then give some examples of useful conformal and temporal gauge choices: one that geometrizes and slightly modifies the Hubble-normalized approach used in, e.g., UEWE and [8], and one that can be extended and modified to a dimensionless autonomous first-order symmetric hyperbolic system. In section 4, we conclude with a discussion and a comparison with other work; we also outline our underlying philosophy.

2. Scales and conformal transformations

2.1. Scales and dimensions

Consider a spacetime \((M, \hat{g})\) where \(M\) is a suitably smooth four-dimensional manifold and \(\hat{g}\) is the physical Lorentzian metric field with signature \((-+,+,+).\) We use units \(c = 1 = 8\pi G\) so that all geometric properties can be dimensionally expressed in terms of a length scale. Let \(\ell\) be the unit of length, then each physical geometric field \(\Phi\) transforms under a scale transformation \(\ell' = S\ell,\) where \(S = \text{const},\) like \(\Phi' = S^q\Phi,\) where \(q\) defines the geometrical object’s dimension, see [9] which we refer to for further discussion.

General relativity is characterized by general coordinate covariance; coordinates \(x^\mu (\mu = 0, 1, 2, 3)\) are to be regarded as just labels for different spacetime events. Hence, in general, they do not carry any physical significance and thus it is natural to regard them as dimensionless, i.e., they have \(q = 0.\) This is in contrast to the spacetime interval \(ds^2\) which describes an invariant physical property and naturally has a weight \(q = 2.\) Since \(ds^2 = g_{\mu\nu} dx^\mu dx^\nu\) it follows that \(g_{\mu\nu}\) has \(q = 2.\) Hence, one has to take into account the specific positioning of indices for a geometric object when considering its dimensional weight; for example, in the case of the energy–momentum tensor: \(\hat{T}^\mu_\nu, \hat{T}^\mu_v, \hat{T}_\mu^\nu\) have dimensions \(q = -4, q = -2, q = 0,\) respectively.

Although coordinates are not dimensional in general, it is natural to assign them dimensional weight when they express invariant physical properties that occur in special cases, which tend to dominate the literature. For example, the radial area coordinate when one has spherical symmetry; proper time or length along a given timelike or spacelike congruence, respectively; the affine parameter along a geodesic congruence—these quantities all naturally carry weight \(q = 1.\) Another important special case is weak gravity where one has a Minkowski background. This background is preferably expressed in Minkowski coordinates with natural weight \(q = 1,\) since they constitute affine parameters of geodesics and thus express invariant features in the Minkowski spacetime.

For computational purposes, one needs to specify a frame and use components. This introduces an additional element since it is quite natural to assign different dimensions to
different choices of frames. There are three types of frames that make dimensional counting particularly easy for the components of geometric objects. (i) Coordinate frames, since the coordinates in general are to be regarded as dimensionless. (ii) Orthonormal frames (ONF): in this case, it is natural to regard the constant metric coefficients as dimensionless and instead let the orthonormal 1-forms carry dimension 1, while the dual vector fields have $q = -1$. This yields that connection components have $q = -1$ while ONF curvature components have $q = -2$. The ONF approach was taken as the starting point in UEWE, which we refer to for further discussions about dimensions in ONF contexts. (iii) The third choice is the one that is going to be explored in this paper—conformal ONF where the conformal factor carries the dimension.

We define a conformal ONF by

$$\hat{g} = \Psi^2 \eta_{ab} \omega^a \omega^b = \Psi^2 g,$$

where $\eta_{ab}$ are constants ($a, b = 0, 1, 2, 3$), and where $g = \eta_{ab} \omega^a \omega^b$ is an ‘unphysical’ metric expressed in an ONF. The conformal factor $\Psi > 0$, which is a function of the spacetime coordinates, is chosen so that it has dimension length, i.e. $q = 1$, so that the metric $g$ (and the 1-forms $\omega^a$ and their dual frame vectors $e_a$) becomes dimensionless.

Since $\Psi$ carries the dimensional scale, it follows that everything is compared with this scale in terms of dimensionless ratios. However, there exists no overall preferred global choice of $\Psi$ suitable for all possible situations; this is reminiscent of the coordinate issue—there exists no global coordinate choice for a general spacetime either. Precisely as in the coordinate case, one has to patch together a complete spacetime from regions where one has used different conformal factors. Instead of aiming for some global conformal choice, $\Psi$ is to be adapted to the particular local feature one is interested in. This can be some preferred structure associated with asymptotic features or a structure associated with special initial or boundary conditions or symmetries. Useful candidates are obtained from suitable functions of dimensional coordinates or scalars or quantities that preserve the defining structure of a boundary conditions or symmetries. Examples are, e.g., $\Psi = \hat{r}$, where $\hat{r}$ is proper time along a timelike congruence; $\Psi \propto \theta^{-1}$, where $\theta$ is the expansion of a null or timelike congruence ($q = -1$ for $\theta$); $\Psi = r$, where $r$ is the radial area coordinate in the case of spherical symmetry. We will use $\Psi = \hat{r}$, $\Psi \propto \theta^{-1}$, associated with a timelike congruence, as examples below in the context of asymptotic temporal properties towards spacetime singularities.

2.2. Conformal transformations and dimensionless field equations

Let us begin with some notation and definitions. Consider some arbitrary metric $g$ in some basis of vector fields $e_a$, with a dual basis of 1-forms $\{\omega^a\}$, i.e. $\langle \omega^a, e_b \rangle = \delta^a_b$. Let us further introduce a connection $\nabla_b e_a = \Gamma^c_{ab} e_c$, $\Gamma^a_{bc} = \langle \omega^a, \nabla_b e_c \rangle$, where $\nabla_b := \nabla_{e_b}$. Then we assume that the connection is (i) torsion free and (ii) metric:

(i) \[ \nabla_u v - \nabla_v u - [u, v] = 0, \]
(ii) \[ \nabla g = 0, \]

where $u$ and $v$ are two arbitrary vectors and $\nabla_u = u^a \nabla_a$. The components of the metric in the basis $\{e_a\}$ are given by $g_{ab} = g(e_a, e_b)$. Commutation functions $\epsilon^c_{ab}$ are defined by

$$[e_a, e_b] = \epsilon^c_{ab} e_c.$$  

The curvature operator, defined by $R(u, v) = [\nabla_u, \nabla_v] - \nabla_{[u,v]}$, yields the Riemann curvature tensor expressed in components as $R^c_{hbd} = \langle \omega^c, R(e_h, e_d)e_b \rangle$, while the Ricci curvature tensor and scalar, and the Einstein tensor, are defined by $R_{bd} = R_{db} = R^c_{hbd} R = R^a_{ab}, G^a_b = R^a_b - \frac{1}{2} \delta^a_b R$. The components of the connection and Riemann and Ricci
The curvature are
\[ \Gamma_{abc} = -\frac{1}{2} \left[ e_a(g_{bc}) - e_b(g_{ca}) - e_c(g_{ab}) + e_{abc} + e_{bca} - e_{cab} \right] \tag{4} \]
\[ R_{abcd} = 2e_a \Gamma^a_{[bd]} + 2\Gamma^a_{[f|b|d]} - \Gamma^a_{bf} e_{cd} \tag{5} \]
\[ R_{ab} = 2e_a \Gamma^a_{|b|c} + \Gamma^c_{dc} \Gamma^d_{ab} - \Gamma^c_{ad} \Gamma^d_{bc} \tag{6} \]
where \( \Gamma_{abc} = g_{ad} \Gamma^d_{bc} \). It follows that \( R_{abcd} = -R_{abdc}, R_{abcd} = -R_{bacd}, R_{abcd} = R_{cdab} \), and \( R_{a[bc]} = 0 \), \( \nabla_{[e} R_{f|b|d]} = 0 \), where the two last relations are the cyclic and Bianchi identities, respectively. See, e.g., [10] for further discussion.

Let us now consider a ‘physical’ metric \( \hat{g} \) conformally related to an ‘unphysical’ metric \( g \) according to \( \hat{g} = \hat{g}^{ab} \omega^a \omega^b = \Psi^2 g_{ab} \omega^a \omega^b = \Psi^2 g \), \( \hat{g} \) being a dual basis of 1-forms to the basis \( e^a \) :
\[ \langle \omega^a, e^b \rangle = \delta^{ab} \]. Then the connection and Ricci tensor of \( \hat{g} \) are related to the connection and Ricci tensor of \( g \) according to (easily derivable from equations (4) and (6); or see, e.g., [11])
\[ \hat{\Gamma}^a_{bc} = \Gamma^a_{bc} + 2\delta^a_{(b} r^{c)} - g_{bc} r^a \tag{8} \]
\[ \hat{R}_{ab} = R_{ab} - 2\Gamma_{(a|b)} + 2 r_a r_b - g_{ab}(\nabla r^e + 2r^2) \tag{9} \]
in the basis \( e_a \), where
\[ \frac{\Psi}{e_a} = e_a \Psi, \tag{10} \]
and \( r^a = g^{ab} r_b, r^2 = g^{ab} r_a r_b \) and \( \nabla_a r_b = e_a r_b - \Gamma^c_{ba} r_c \). We now express the unphysical metric \( R_{ab} \) in an ONF so that \( g_{ab} = \eta_{ab} = \text{diag}[-1, 1, 1, 1] \), and thus the physical metric is given in a conformal ONF: \( \hat{g}_{ab} = \Psi^2 \eta_{ab} \), see equation (1).

We take the frame variables \( e^a_\mu \), defined by
\[ e_a = e^a_\mu \frac{\partial}{\partial x^\mu}, \tag{11} \]
and the commutator functions \( e^a_{bc} \) as our basic variables, possibly supplemented by \( r_a \); we will give some examples in section 3.

The governing dimensionless equations are the commutator equations, the Jacobi identities for \( e_a \), and EFES, which in the conformal ONF, \( e_a \), are given by
\[ 2 e_a e^b_\mu = e^c_{ab} e^\mu_c \tag{12} \]
\[ e^a_{[c} e^d_{b]} = e^d_{[a} e^e_{b]c} \tag{13} \]
\[ \hat{R}_{ab} = T_{ab} - \frac{1}{2} \eta_{ab} r^c r_c \tag{14} \]
where \( T_{ab} \) are the dimensionless components of the energy–momentum tensor in the conformal ONF (recall that the energy–momentum tensor \( T_{\mu \nu} \) is dimensionless, i.e. \( q = 0 \), and hence it is possible to make the identification \( \hat{T}_{ab} = T_{ab} \) in the conformal ONF), and now
\[ \hat{R}_{ab} = R_{ab} + U_{ab}, \quad U_{ab} := -2\nabla_{[a} r_{b]} + 2 r_a r_b - \eta_{ab}(\nabla r^e + 2r^2) \tag{15} \]
\[ \Gamma^a_{bc} = -\frac{1}{2} \eta^{ad} \left[ \eta_{ed} e^e_{bc} + 2\eta_{(a|b)} e^e_{c]d} \right]. \tag{16} \]
Note that \( 2\Gamma^a_{a(bc)} = -e_{abc} \) and \( \Gamma^a_{a(bc)} = -e_{(bc)a} \); one can thus use \( \Gamma^a_{bc} \) as variables instead of \( e^a_{bc} \).
When one has a non-trivial matter source the above equations have to be supplemented with appropriate matter equations, however, one always has local energy–momentum conservation: \( \nabla_a T^{ab} = 0 \). Recall that \( \hat{T}^{\mu\nu} \) has dimension \( q = -4 \). In the conformal ONF, we have \( \hat{T}^{ab} = \hat{g}^{ac} \hat{g}^{bd} \hat{T}_{cd} = \Psi^{-4} \hat{g}^{ac} \hat{g}^{bd} \hat{T}_{cd} = \Psi^{-4} T^{ab} \), where we have defined the dimensionless object \( T^{ab} := \hat{g}^{ac} \hat{g}^{bd} \hat{T}_{cd} \). This yields (see [12])

\[
e_b T^{ab} + \Gamma^a_{db} T^{db} + \Gamma^b_{db} T^{ad} + 2 \eta_a r^a - \rho^a \eta_a r^a = 0.
\]

(17)

It is of key importance to note that the equations for the dimensional variable \( \Psi, e_a/\Psi \), decouple from the above dimensionless equations (this is seen explicitly, but also follows directly from dimensional reasons). The components of \( r_a \) are either given functions of coordinates or functions of the dimensionless state space variables that depend on what type of choice of \( \Psi \) one has made, examples will be given below. This means that \( \Psi \) itself plays a ‘passive’ subsidiary role. It is \( \Psi \) that carries the scale that typically asymptotically leads to the blow up of dimensional quantities. Since this scale now has been factored out of the problem, leaving an asymptotically regularized dimensionless system, this drastically simplifies an asymptotic analysis. Once an asymptotic analysis has been accomplished for the essential dimensionless equations, the result can be used for a relatively simple asymptotic analysis of the decoupled equations for \( \Psi \), thus yielding a complete physical result. This geometric splitting of the problem into more easily handled problems is the conformal ONF approach’s main advantage.

It is of interest to relate the present variables to those that one uses in the ONF approach. In the latter approach, one uses a basis so that \( \hat{g} = \eta_{ab} \hat{\omega}^a \hat{\omega}^b \) and an associated dual basis \( \hat{e}_a, \langle \hat{\omega}^a, \hat{e}_b \rangle = \delta^a_b \). The variables in this approach are the frame variables \( \hat{e}_a^\mu \), defined by \( \hat{e}_a \equiv \hat{e}_a^\mu \partial/\partial x^\mu \), and the commutator variables \( \hat{c}_a^{bc} \), defined by \( [\hat{e}_b, \hat{e}_c] = \hat{c}_a^{bc} \hat{e}_a \). These variables are related to the present ones as follows:

\[
\hat{e}_a^\mu = \Psi^{-1} e_a^\mu, \quad e_a^\mu = \Psi \hat{e}_a^\mu
\]

(18)

\[
\hat{c}_a^{bc} = \Psi^{-1} (c_a^{bc} + \delta^a_{[b} r_{c]}); \quad c_a^{bc} = \Psi \hat{c}_a^{bc} - \delta^a_{[b} r_{c]}.
\]

(19)

where \( \hat{c}_a^{bc} = \langle \hat{\omega}^a, [\hat{e}_b, \hat{e}_c] \rangle \) and \( c_a^{bc} = \langle \omega^a, [e_b, e_c] \rangle \), i.e., the above equations are not tensor equations; instead they relate the components of the frame variables and the commutator functions of two conformally related sets of basis vector fields. The above relationships explicitly show that \( e_a^\mu \) and \( c_a^{bc} \) are dimensionless if \( \Psi \) has dimensional weight \( q = 1 \), and that everything is measured with respect to the scale carried by \( \Psi \).

To make this more concrete, we will consider a timelike congruence and give some examples of conformal and temporal gauge choices.

3. The 1+3 conformally orthonormal approach

3.1. 1+3 decomposition

We here adapt our formalism to a timelike reference congruence. We therefore choose a time coordinate along the congruence and align one of the basis vectors tangentially to it; this allows us to make a 1+3 split of the variables.

The main application in this paper is the use of conformal regularization towards a generic singularity. This means that there is a close connection with UEWE. Unfortunately, the notation in UEWE is not adapted to the conformal formalism at all, which suggests that it perhaps would be best to use new notation that is naturally associated with the conformal approach. Nevertheless, in this paper we adopt a notation that follows that of UEWE as closely.
as possible, since this emphasizes the close connection and simplifies a comparison between
UEWE and the present work, even though this leads to some awkwardness.

In UEWE, the starting point was the ONF formalism associated with the physical metric.
Since conformal transformations were not discussed in UEWE, the frame vectors did not have
hats. We therefore now drop the hats on the ONF vectors, i.e., \( \{ \hat{e}_a \} \rightarrow \{ e_a \} \). This causes a
problem for the conformal ONF vectors, however, these vectors correspond to the Hubble-
normalized vectors \( \partial_a \) in UEWE; we thus rename the conformal ONF vectors according to
\( \{ e_a \} \rightarrow \partial_a \), but in contrast to UEWE (associated with \( \Psi = H^{-1} \) where \( H \) is the Hubble
variable), \( \Psi \) is now any function with \( q = 1 \). A 1+3 split (in contrast to the 3+1 split done
in UEWE; see e.g. [13], and references therein) of the ONF and conformal ONF vectors and
variables yields

\[
e_a = M^{-1} \partial_i, \quad e_a = e_a^i (M_i \partial_i + \partial_i)
\]

(20)

\[
\partial_0 = M^{-1} \partial_i, \quad \partial_a = E_a^i (M_i \partial_i + \partial_i), \quad \alpha = 1, 2, 3; \quad i = 1, 2, 3,
\]

(21)

where \( e_0 (\partial_0) \) is the future-directed tangent to the physical (unphysical conformal) timelike
reference congruence; \( M (\mathcal{M}) \) is the physical (unphysical conformal) threading lapse function
and \( M_i \) the dimensionless (assuming dimensionless coordinates) threading shift 1-form.

The lapse, \( M = \Psi \mathcal{M} \), \( \mathcal{M} = \Psi^{-1} \mathcal{M} \), and the dimensionless shift vector are associated
with gauge freedom while \( e_a^i \) and \( E_a^i \) are regarded as dynamical variables, related by

\[
e_a^i = \Psi^{-1} E_a^i, \quad E_a^i = \Psi e_a^i.
\]

(22)

The commutators are decomposed according to

\[
[e_0, e_a] = \dot{u}_a e_0 - \left[ H \delta_a^\beta + \sigma_a^\beta + \epsilon_a^\beta \gamma (\omega^\gamma + \Omega^\gamma) \right] e_\beta
\]

(23)

\[
[e_a, e_\beta] = 2 \epsilon_{\alpha \beta \gamma} \omega^\alpha e_0 + (2 \delta_{[a} \delta_{\beta]}^\gamma + \epsilon_{\alpha \beta \gamma} n^{\gamma}) e_{\gamma}
\]

(24)

\[
[\partial_0, \partial_a] = U_a \partial_0 - \left[ \mathcal{H} \delta_a^\beta + \Sigma_a^\beta + \epsilon_a^\beta \gamma (W^\gamma + R^\gamma) \right] \partial_\beta
\]

(25)

\[
[\partial_a, \partial_\beta] = 2 \epsilon_{\alpha \beta \gamma} W^\gamma \partial_0 + (2 \Lambda_{[a} \delta_{\beta]}^\gamma + \epsilon_{\alpha \beta \gamma} N^{\gamma}) \partial_{\gamma},
\]

(26)

where the above decompositions imply the following definitions:

\[
H = -\frac{1}{2} \epsilon_0^a a_a, \quad \sigma_{ab} = -\epsilon_0^\gamma (a_0 \delta_{\beta \gamma}), \quad \dot{u}_a = \epsilon_0^0 a_a,
\]

(27)

\[
\omega_a = \frac{1}{2} \epsilon_{a \beta \gamma} c_0^\beta b_0^\gamma, \quad \omega_a + \Omega_a = \frac{1}{2} \epsilon_{a \beta \gamma} c_0^\beta b_0^\gamma,
\]

(28)

\[
\mathcal{H} = -\frac{1}{2} \epsilon_0^a a_a, \quad \Sigma_{ab} = -\epsilon_0^\gamma (a_0 \delta_{\beta \gamma}), \quad \dot{U}_a = \epsilon_0^0 a_a,
\]

(29)

\[
W_a = \frac{1}{2} \epsilon_{a \beta \gamma} c_0^\beta b_0^\gamma, \quad \dot{W}_a + R_a = \frac{1}{2} \epsilon_{a \beta \gamma} c_0^\beta b_0^\gamma,
\]

(30)

\[
N^{ab} = \frac{1}{2} \epsilon_{a \mu \nu} (c_{0 \beta} b_{0 \gamma}) \mu \nu, \quad A_a = \frac{1}{2} \epsilon_{a \beta \gamma} b_0^\gamma.
\]

where (\( \cdot \)) represents trace-free symmetrization. Here, \( H = \frac{1}{2} \theta \) is the Hubble variable with \( \theta \) being the expansion; \( \sigma_{ab} \) is the shear; \( \dot{u}_a \) is the acceleration; \( \omega_a \) is the rotation; \( \Omega_a \) is the Fermi rotation—all quantities are associated with the congruence of which \( e_0 \) is the tangent
vector field; \( \rho_{ab}, \alpha_{a} \) are spatial commutator functions, which describe the 3-curvature when \( \omega_a = 0 \); for a more detailed description see, e.g., [5, 13]. Analogous interpretations hold for the conformal quantities. In the above formulæ, we have adhered to the conventions used in [14].
Equations (19), (27)–(30) yield the following relationship between the ONF and conformal ONF commutator function variables:

\[ \mathcal{H} = \Psi H - r_0, \quad \Sigma_{\alpha\beta} = \Psi \sigma_{\alpha\beta}, \] (31)

\[ \dot{U}_a = \Psi \dot{u}_a - r_a, \quad W_a = \Psi \omega_a, \] (32)

\[ N^{\alpha\beta} = \Psi n^{\alpha\beta}, \quad A_a = \Psi a_a + r_a, \] (33)

\[ W^a + R^a = \Psi (\omega^a + \Omega^a). \] (34)

Let us now focus on the conformal ONF approach. From the above definitions and equation (16), it follows that the 1+3 split connection components of the conformal metric are given by

\[ \Gamma_a^{00} = \dot{U}_a, \quad \Gamma_a^{0\beta} = \mathcal{H} \delta_a^\beta + \Sigma_{a\beta} - \epsilon_a^{\beta\gamma} W^\gamma, \] (35)

\[ \Gamma_a^{\alpha\beta} = \epsilon_a^{\gamma\delta} (\partial_\gamma - A_\gamma), \quad \Gamma_a^{\alpha\beta\gamma} = 2 A_{(a}^{\beta} R_{\beta)\gamma} + \epsilon_a^{\beta\delta} N^{\delta\gamma}. \] (36)

Instead of referring to \( H, \Sigma_{\alpha\beta}, \dot{U}_a, W_a, R_a, A_a, N^{\alpha\beta} \) as commutator function variables, one may refer to them as connection variables, since they describe \( \Gamma^{a}_{bc} \) as well as \( c^{a}_{bc} \).

The commutator equations can be written succinctly as follows:

\[ 0 = (\partial_a + \dot{U}_a)\partial_0 - (\delta_a^\beta \partial_0 - F_a^\beta) \partial_\beta, \] (37)

\[ 0 = 2 W_a \partial_0 - C_a^\beta \partial_\beta, \] (38)

where

\[ F_a^\beta := c_a^{\beta} \dot{u}_a = -\mathcal{H} \delta_a^\beta - \Sigma_{a\beta} - \epsilon_a^{\beta\gamma} (W^\gamma + R^\gamma), \] (39)

\[ C_a^\beta := \epsilon_a^{\gamma\delta} (\partial_\gamma - A_\gamma) - N_a^\beta, \] (40)

and this suggests that the equations can be written concisely using the above notation.

It is natural to divide the equations into gauge equations and dynamical equations, and to further subdivide the latter into evolution equations and constraints (if the temporal frame derivative, \( \partial_0 \), does not appear in a dynamical equation we refer to it as a constraint equation, even though the spatial frame derivatives \( \partial_\alpha \) contain the partial time derivative \( \partial_t \)).

**Gauge equations:**

\[ \partial_0 M_a = F_a^\beta M_\beta + (\partial_a + \dot{U}_a) M^{-1} \] (41)

\[ 0 = C_a^\beta M_\beta - 2 M^{-1} W_a. \] (42)

**Evolution equations:**

\[ \partial_0 E_a^i = F_a^\beta E_\beta^i \] (43)

\[ \partial_0 \mathcal{H} = -\mathcal{H}^2 - \frac{1}{3} \Sigma_{a\beta} \Sigma^{a\beta} + \frac{2}{3} \mathcal{W}^2 + \frac{1}{3} (\partial_a + \dot{U}_a - 2 A_a) \dot{U}_a - \frac{1}{6} (T_{00} + T_a^a) + \frac{1}{3} \mathcal{U}_0 \] (44)

\[ \partial_0 \Sigma_{a\beta} = -3 \mathcal{H} \Sigma_{a\beta} - \epsilon^{\gamma\delta} (2 \Sigma_{\beta\gamma} R^\delta - N_{\beta\gamma}) \dot{U}_\delta - 2 W_a R_{\beta a} \]

\[ + (\partial_a + A_a + \dot{U}_a) \dot{U}_\beta - \frac{3}{2} S_{a\beta} + T_{(a\beta)} - U_{(a\beta)} \] (45)

\[ \partial_0 W_a = -(3 \mathcal{H} \delta_a^\beta + F_a^\beta) W_\beta + \frac{1}{3} C_a^\beta U_\beta \] (46)

\[ \partial_0 A_a = F_a^\beta A_\beta - \frac{1}{3} (\partial_\beta + \dot{U}_\beta) (3 \mathcal{H} \delta_a^\beta + F_a^\beta) \] (47)

\[ \partial_0 N^{a\beta} = -(3 \mathcal{H} \delta_{\gamma}^{(a} + 2 F_{\gamma}^{(a}) N^{\beta)\gamma} + (\partial_\gamma + \dot{U}_\gamma) \epsilon^{\gamma\delta(a} F_a^{\beta)}). \] (48)
Constraint equations:

\[ 0 = C_a^{\beta} E_\beta \]

\[ 0 = 3H^2 - \frac{1}{2} \Sigma_{\alpha \beta} + W^2 - 2W_a R^a + \frac{1}{2} 3 \mathcal{R} - T_{00} + \frac{1}{2} (U_{00} + U^\alpha \alpha) \]

\[ 0 = -2 \partial_a \mathcal{H} + (\partial_{\beta} - 3 A_{\beta}) \Sigma_{\alpha}^{\beta} + \epsilon_a^{\beta \gamma} (\Sigma_\gamma^{\delta} N_{\delta \gamma} + 2U_\beta W_\gamma) + C_a^{\beta} W_\beta - T_{0a} + U_{0a} \]

\[ 0 = (\partial_{\beta} - 3 A_{\beta}) U_a^{\beta} + \epsilon_a^{\beta \gamma} \partial_\beta A_\gamma - 2F_a^{\beta} W_\beta \]

\[ 0 = (\partial_{\alpha} - U_a) W_\alpha, \]

where

\[ 3 \mathcal{S}_{\alpha \beta} = \partial_{(\alpha} A_{\beta)} - (\partial_\gamma - 3 A_\gamma) N_{\delta (\alpha} \epsilon_{\beta)}^{\gamma} + B_{(\alpha \beta)} \]

\[ 3 \mathcal{R} = 4 \partial_a A^a - 6 \mathcal{A}^3 - \frac{1}{2} B_a^a \]

\[ B_{ab} = 2 N_{ab} N_{\gamma \beta} - N_{\gamma \gamma} N_{\alpha \beta} \]

\[ U_{00} = -3(\partial_0 + \mathcal{H}) r_0 + [\delta_\beta^{(\beta} (\partial_{\beta} + 2 r_\beta)] + 3 \mathcal{U}^\gamma - 2 A^\gamma r_\gamma \]

\[ U_{0a} = -2(\partial_0 - r_\alpha) r_0 + 2(\partial_0 + \mathcal{H} a^{\beta} + \Sigma_\alpha^{\beta} + \epsilon_a^{\beta \gamma} W_\gamma) r_\beta \]

\[ U_{(ab)} = 2 \left[ \Sigma_{a \beta} r_0 - (\partial_{(\alpha} + A_{(a} r_\beta) - \epsilon_\gamma^{(a} N_\gamma^{\beta)} r_\gamma + r_{(a} r_\beta) \right] \]

\[ U_{00} + U^a_a = 6(2 \mathcal{H} + r_0) r_0 - 2[\delta_\gamma^{(\gamma} (2 \mathcal{H} + r_\beta) - 4 A^\gamma r_\gamma], \]

where we have used the notation \( v_a v^a = v^2 \). If \( M_a = 0 = W_a \), then \( 3 \mathcal{R} \) and \( 3 \mathcal{S}_{ab} \) are the curvature scalar and trace-free part of the Ricci tensor, respectively, of the conformal 3-metric.

A conformal 1+3 split of the equations for \( T_{ab} \) yields (to avoid clashes with the notation in UEWE, we will refrain from explicitly splitting \( T_{ab} \) in terms of its irreducible parts, but see the discussion below)

\[(\partial_0 + 3 \mathcal{H}) T_{00} + 3 \mathcal{H} T_{0a} - (\partial_\beta + 2U_\beta - 2 A_\beta) T_{0}^{\beta} + \Sigma_\beta^{\alpha} T_{\alpha}^{\beta} + C_0 = 0 \]

\[(\partial_0 + 4 \mathcal{H}) T_{0a} - U_a T_{00} - A_a T_{0}^{\beta} + \Sigma_\beta^{\alpha} T_{0}^{\beta} - (\partial_\beta + U_\beta - 3 A_\beta) T_{a}^{\beta} + \epsilon_a^{\beta \gamma} [N_\beta^{\delta} T_{\delta \gamma} - (W_\gamma - R_\gamma) T_{00}] + C_a = 0, \]

where

\[ C_0 := (T_{00} + T_{0}^{\beta}) r_0 - 2 T_{0}^{\beta} r_\beta \]

\[ C_a := 2T_{0a} r_0 - (T_{00} - T_{0}^{\beta}) r_\alpha - 2 T_{a}^{\beta} r_\beta. \]

As done in, e.g., UEWE, the energy–momentum tensor can be 1+3 split according to

\[ \tilde{T}_{ab} = \tilde{\rho} \tilde{u}_a \tilde{u}_b + 2 \tilde{\rho} \tilde{q}_b + \tilde{p} \tilde{h}_{ab} + \tilde{\kappa}_{ab}, \]

where \( \tilde{u}_a \tilde{u}_a = 0, \tilde{u}_a \tilde{q}^a = 0, \tilde{\kappa}_{ab} = 0 \). In [5] and UEWE, the following normalization was introduced: \( (\Omega, P, Q^a, \Pi_{ab}) = (\tilde{\rho}, \tilde{p}, \tilde{q}^a, \tilde{\kappa}_{ab})/(3H^2) \); the reason for this is that this yields the standard definition of the important cosmological dimensionless density parameter \( \Omega \). However, from a conformal geometric perspective it follows that if one wants to use \( H^{-1} \) as a conformal factor, then the natural normalization factor is \( H^{-2} \). This then suggests the following new definitions in the Hubble-normalization case: \( (D, P, Q^a, \Pi_{ab}) = (\tilde{\rho}, \tilde{p}, \tilde{q}^a, \tilde{\kappa}_{ab})/H^2 \), associated with the irreducible 1+3 decomposition of \( T_{ab} \), and hence \( \Omega = D/3 \); alternatively, one may use \( (\sqrt{3} H)^{-1} \) as a conformal factor, but that changes the conventions with respect to [5] and UEWE as regards the connection variables.
3.2. Examples of conformal and gauge choices

Associated with a choice of a conformal normalization factor $\Psi$, there exists a natural temporal gauge choice—the conformal ‘proper time gauge’, $M = 1$, which in the 3+1 case $M = 0 = W_a$ reduces to the conformal Gauss gauge (see [3] for a discussion about the use of conformal Gauss coordinates to cover large spacetime domains), however, it is of course not necessary to choose this gauge. Instead of a general discussion about conformal, frame and coordinate freedom, we will focus on a few examples. Since the emphasis in this paper is on the conformal approach, we will divide our discussion in terms of conformal choices; we consider two such choices—conformal Hubble-normalization and conformal proper time normalization.

3.2.1. Conformal Hubble-normalization. The first example is given by

$$\Psi = H^{-1},$$

(66)

where $H = \frac{1}{3} \theta$ is the physical Hubble variable and $\theta$ is the physical expansion, defined by $\theta = \tilde{\nabla} \hat{a}^a$, where $\hat{a} = \hat{e}_0$ (in the notation of section 2). In this case, we define the physical deceleration parameter $q$ (not to be confused with the scale weight $q$) and logarithmic spatial frame derivative $r_a$ by

$$\partial_0 H = -(1 + q) H, \quad \partial_a H = -r_a H,$$

(67)

i.e., in terms of $r_a$, we have $r_0 = 1 + q = -\partial_0 H / H$ and $r_a = -\partial_a H / H$, which combined with equations (57)–(60) determine $U_{ab}$.

Equations (31)–(34) and the above definitions yield

$$\mathcal{H} = -q, \quad \dot{U}_a = \dot{U}_a^H - r_a, \quad A_a = A_a^H + r_a,$$

(68)

where $\dot{U}_a^H := \dot{u}_a / H$, $A_a^H := a_a / H$, while the other variables are just the usual Hubble-normalized variables used in, e.g., UEWE.

One can choose to let $q$ and $r_a$ be determined by the Raychaudhuri equation (44) (the time derivative of $q$ drops out when $\mathcal{H} = -q, r_0 = 1 + q$ are inserted in (44)) and the Codacci constraint (51) (the spatial derivatives of $q$ drop out), respectively. However, since $-q$ is just one of the connection variables in the conformal formulation, it is quite natural to extend the normalized state space to include $q$ and $r_a$ (which is also connected to the present formalism through its link to the gauge quantity $\dot{U}_a$) as independent variables, something which has been found to be quite useful, see, e.g., [8, 15].

Setting $M = 1$ yields the separable volume gauge, see UEWE, and if one in addition sets $M_a = 0 = W_a$ one obtains the inverse mean curvature gauge, which in the present context can be interpreted as the conformal Gauss gauge associated with $\Psi = H^{-1}$, something which is also reflected in that the congruence is conformally geodesic: $\dot{U}_a = 0$. This further emphasizes the geometric nature of the present approach and the preference of using $\dot{U}_a$ instead of $\dot{U}_a^H$.

Note that with the current choice of conformal factor, and a conformal Gauss coordinate choice, the present formulation reduces to the ‘standard’ Hubble-normalized formulation in the spatially homogeneous (SH) case, since $\dot{U}_a = 0 = r_a$, i.e., the present formulation yields a natural geometric generalization of the Hubble-normalized SH case, discussed extensively in, e.g., [5]. For the same reason, the present approach reduces to that used in, e.g., UEWE for the SH part of the so-called silent boundary, where the attractor for generic singularities resides (this is also the case for the subset associated with isotropic singularities [16]). Hence, the description of the attractor for a generic singularity in the present geometric formulation is identical to that in UEWE—the asymptotic regularization properties are generically identical.
3.2.2. Conformal proper time normalization. The second example uses the physical proper time $\hat{t}$ along a timelike reference congruence as the conformal factor:

$$\Psi = \hat{t}. \quad (69)$$

The time variable is subsequently reparametrized so that one obtains a dimensionless time variable, $\hat{t}$, according to

$$t = \ln(\hat{t}/\hat{t}_0), \quad \hat{t} = \hat{t}_0 e^t, \quad (70)$$

where $\hat{t}_0$ is some reference time; it follows that the new time variable is just the conformal proper time, since $\mathcal{M} = 1$, and hence $\partial_0 = \partial/\partial t$ and $r_a = (1, 0, 0, 0)$. This leads to

$$U_{00} = -3H, \quad U_{0a} = 2\dot{U}_a, \quad U_{(a\beta)} = 2\Sigma_{a\beta}, \quad U_{00} + U^a_a = 6(2H + 1), \quad (71)$$

and thus one obtains a first-order autonomous system of equations. This can be seen explicitly, but again this also follows from dimensional reasons: $\hat{t}$ is the only varying quantity that carries dimension and hence $\hat{t}$ cannot appear in the dimensionless equations, and therefore the same holds for $t$; neither does the normalization affect the essential first-order structure of the usual dimensional ONF approach.

In [17–19], it was shown that by extending the ONF approach to also include the curvature tensor and the Bianchi identities one can obtain a first-order symmetric hyperbolic system, if one uses proper time along a timelike congruence (this was shown for a perfect fluid with a barotropic equation of state by using proper time along the fluid congruence). It is of course also possible to extend the current conformal ONF approach similarly, as will be discussed in the following section. Since $\Psi = \hat{t} = \hat{t}_0 e^t$ does not modify the principal parts of the equations of a curvature extended formulation, we conclude that it is possible to extend the present ‘minimal’ formalism and obtain a dimensionless autonomous first order symmetric hyperbolic system for the $\hat{t}$-normalized equations; incidentally, this system is of course well-posed.

Let us now for simplicity specialize the temporal reference congruence to be non-rotating, $M_a = 0 = W_a$, so that we obtain proper time normalized equations and a conformal Gauss coordinate system, for which $U_a = 0$ and hence $U_{0a} = 0$. Moreover, let us consider a generic initial spacelike singularity and let us specialize the time coordinate so that it becomes a simultaneous bang function, i.e., $\hat{t} = 0$ at big bang, and hence $t \to -\infty$ towards the singularity. We thus take the synchronous coordinates used by Belinski, Khalatnikov and Lifshitz (BKL) [20] as the starting point (see also [21] for a discussion about the existence of such coordinates) and obtain a dimensionless formulation that brings us particularly close to the work of BKL, which therefore can be interpreted directly in terms of the dimensionless state space picture the present formulation gives rise to.

It may seem that the latest approach is superior to the Hubble-normalized one, however, both have advantages and disadvantages. The advantage of the Hubble-normalized approach is that one essentially uses the expansion which appears prominently in the singularity theorems and that one decouples the dimensional variable $H$. This implies that $H$ carries the dimensional constant of integration, which we denote as the scale parameter even though it is a function in general, when one has a scale-invariant source. The advantages of the conformal proper time normalization approach is that it yields a first order system which may be extended to a first-order symmetric hyperbolic system and that one obtains a formulation closely related to that of BKL, which facilitates comparisons. A disadvantage is that one does not decouple a variable that carries the scale parameter. The difference of the two approaches as regards the last aspect can be illustrated by the Kasner subset.

Let us for simplicity only consider the vacuum case (as discussed in UEWE, if one has a source one may have additional test fields such as the 3-velocity of a fluid). In the Hubble-normalized approach, the Kasner subset is defined by setting all variables to zero except...
the shear which satisfies \(1 - \frac{1}{6} \Sigma_{a\beta} \Sigma^{a\beta} = 0\) (and \(q = 2\) if we consider the \(r_a\) extension), which yields the so-called Kasner sphere, see UEWE. On the other hand, when we use the conformal proper time normalization, with a simultaneous bang function, then \(\mathcal{H}\) and \(\Sigma_{a\beta} \neq 0\). Setting \(r_a = (1, 0, 0, 0)\) and all other variables to zero, apart from \(\mathcal{H}\) and \(\Sigma_{a\beta}\), leads to that equation (50) yields \((1 + \mathcal{H})^2 - \frac{1}{6} \Sigma_{a\beta} \Sigma^{a\beta} = 0\), i.e., we obtain a cone with \(\mathcal{H} = -1\) as apex. If we consider expanding models, then \(\mathcal{H} > 0\); for Kasner \(\mathcal{H} = \mathcal{H}_0 = \text{const} > 0\), and hence we obtain a Kasner sphere for each value of \(\mathcal{H}_0\). This illustrates that in contrast to the Hubble-normalized formulation, we obtain a scale parameter as a constant of integration in the proper time normalized formulation, something that somewhat complicates the description of the structure of the attractor for generic singularities.

Implicit in the above discussion is also the need to choose a ‘dominant’ quantity as the conformal factor in order to obtain asymptotically regular and well-behaved equations, e.g., if we had used the inverse of a component of \(n^{a\beta}\) as the conformal factor many state space variables would have blown up towards an approach to Kasner (and towards a generic singularity). Thus, e.g., for a generic singularity one needs to use a conformal factor that goes to zero at least as fast as \(H^{-1}\) in order for the state space variables to remain finite; however, it is preferable if \(\Psi H\) remains finite towards the singularity, i.e., it is preferable to have a ‘marginally dominant’ conformal factor that leads to finite state space variables, without all of them going to zero, and well-behaved field equations.

4. Discussion

In this paper, we have used conformal transformations in order to obtain dimensionless regularized field equations that allow one to extract asymptotic features and properties about the solution space of general relativity. The conformal factor is to be chosen so that it captures a characteristic scale associated with asymptotic structure so that all the state space variables form dimensionless ratios with respect to this scale. In this paper, we have used a ‘minimal’ approach; however, we could have extended our formalism to also include the curvature, in particular the Weyl curvature, and the Bianchi identities, as done in, e.g., [13, 18]. Since the Weyl curvature is conformally invariant, this implies that one should use the Weyl curvature in a conformal ONF, i.e., in contrast to Friedrich’s conformal approach, see [17] and references therein, the Weyl tensor is not to be scaled with the conformal factor in our approach—it suffices to express it in a conformal ONF. We here note that even though the conformal factor enters our equations implicitly in the combination \(e_a \Psi / \Psi = r_a\) (in the notation of section 2), the components of \(r_a\) stay finite when \(\Psi \to 0\), if \(\Psi\) is chosen as an appropriate marginally dominant scalar that carries dimension \(q = 1\). This leads to a coupled system of regular dimensionless field equations, since the equations (differential or algebraically trivial, depending on the choice of \(\Psi\)) for the dimensional \(\Psi\) decouple; furthermore, it is the reduced dimensionless system that carries the essential dynamics, since \(\Psi\) can be obtained afterwards once the equations of the reduced system have been solved.

Note that the components of the Weyl tensor in an ONF (which have dimension \(q = -2\)) and the components of the Weyl tensor in a conformal ONF (dimension \(q = 0\) when the conformal factor has dimension \(q = 1\)) only differ by the square of the conformal factor. In [13, 5], the Weyl tensor was normalized with \((\sqrt[3]{H})^{-2}\) as the scale factor, since the same factor was used to normalize \(\tilde{T}^{ab}\), as discussed previously. However, we now see that from a conformal point of view the natural normalization factor is just the square of the conformal factor, which in the Hubble-normalization case is \(H^{-2}\).

Our choice of conformal factor is also quite different from that used in conformal approaches to isotropic singularities, see e.g. [22, 23]. There the motivation for the conformal
factor is a purely mathematical one; choose a conformal factor so that regular expressions for the covariant coordinate components of the 3-metric at the singularity are obtained. This typically leads to a conformal factor that does not have any particular dimension, indeed, the dimensional weight is different for different matter sources (not surprisingly, increasingly complicated dimensional conformal properties lead to increasingly messy subsequent mathematical analysis—this is why, e.g., the dimensionally simple case of a perfect fluid with radiation as equation of state is relatively easy to treat). The present approach uses a strategy that is almost the opposite. The conformal factor is always chosen to carry the dimensional weight and for isotropic singularities, see [16], as well as for typical timelines for generic singularities, all the components of the covariant 3-metric blow up, and this is a very good thing! Instead the focus is on the components of the spatial frame vectors which determine the contravariant components of the 3-metric; these components all go to zero, and this directly reflects the asymptotic causal properties towards the singularity—asymptotic silence, see UEWE and [7]. The present conformal approach emphasizes the geometrical content of the discussion about asymptotic silence in UEWE and [7] even further due to the connection between causal and conformal properties. In the present approach, the focus is on the conformal state space which is extended to include the so-called silent boundary where all components of the contravariant 3-metric are zero. This extension then allows one to use the state space picture to perturb the structure on the silent boundary into the physical state space, and thus derive physical results about asymptotic spacetime properties.

In this paper, we have used temporal asymptotic aspects associated with singularities as an example and made contact with other work to illustrate the usefulness of our approach. However, it is our belief that the current formalism may be useful for all types of asymptotics in general relativity: temporal, null and spacelike; for non-isolated and isolated systems. Indeed, we already know that it is useful for future temporal asymptotes in SH contexts, since it contains the Hubble-normalized formalism which has already proven to be useful in this regard. However, it should be pointed out that the Hubble-normalization did not lead directly to regularized equations towards the future for the general SH models, some additional manipulations were needed in order to obtain asymptotic results, but the Hubble-normalization provided the first key step [24]. A similar situation is expected for null infinity. The present formalism is expected to yield direct results for special cases, but not the most general ones where additional manipulations will be necessary. The current formalism could have been used as the starting point in the work [25], using the radial area coordinate \( r \) as a conformal factor, which yielded asymptotic results as regards spacelike asymptotes, for non-isolated and isolated systems, in the context of static spherically symmetric spacetimes (in [25] there existed two relevant scales; however, several scales can be handled by first using the conformal transformation to take care of an overall scale and then making additional variable transformations that form further ratios, which one by one quotient out the other scales).

However, the main reason for believing that our proposed approach may be a useful ingredient in future studies of asymptotics perhaps comes from the simplicity and naturalness of the main ideas—summarized as follows:

(i) Consider a marginally dominant dimensional scale that captures some key asymptotic features.

(ii) Use conformal transformations to geometrically quotient out and decouple this scale so that all remaining quantities represent dimensionless ratios with respect to that scale.

(iii) Use the obtained reduced dimensionless regularized field equations on an extended state space (i.e., include asymptotic limits if they occur on the boundary of the original dimensionless state space) to derive and describe asymptotic properties.
Thus, dynamical systems approaches in general relativity, based on regularized dimensionless field equations, have found their place in a familiar conformal geometric setting.

Acknowledgments

It is a pleasure to thank Lars Andersson, Henk van Elst, Woei Chet Lim and John Wainwright for many helpful and stimulating asymptotically silent discussions. CU is supported by the Swedish Research Council.

References

[1] Penrose R and Rindler W 1986 Spinors and Space-Time Vol 2: Spinor and Twistor Methods in Space-Time Geometry (Cambridge: Cambridge University Press)
[2] Hawking S W and Ellis G F R 1973 The Large Scale Structure of Space-Time (Cambridge: Cambridge University Press)
[3] Friedrich H 2002 Conformal Einstein evolution The Conformal Structure of Space-Time (Lecture Notes in Physics) ed J Frauendiener and H Friedrich (Berlin: Springer) (Preprint gr-qc/0209018)
[4] Carr B J and Coley A A 1999 Self-similarity in general relativity Class. Quantum Grav. 16 R31
[5] Wainwright J and Ellis G F R 1997 Dynamical Systems in Cosmology (Cambridge: Cambridge University Press)
[6] Uggla C, van Elst H, Wainwright J and Ellis G F R 2003 The past attractor in inhomogeneous cosmology Phys. Rev. D 68 103502
[7] Andersson L, van Elst H, Lim W C and Uggla C 2005 Asymptotic silence of generic cosmological singularities Phys. Rev. Lett. 94 051101
[8] Garfinkle D 2004 Numerical simulations of generic singularities Phys. Rev. Lett. 93 161101
[9] Eardley D M 1974 Self-similar spacetimes: geometry and dynamics Commun. Math. Phys. 37 287
[10] Misner C W, Thorne K A and Wheeler J A 1973 Gravitation (San Francisco, CA: Freeman)
[11] Stephani H, Kramer D, MacCallum M A H, Hoenselaers C and Herlt E 2003 Exact Solutions to Einstein’s Field Equations 2nd edn (Cambridge: Cambridge University Press)
[12] Wu Z C 1981 Self-similar cosmological models Gen. Rel. Grav. 13 625
[13] van Elst H and Uggla C 1997 General relativistic 1+3 orthonormal frame approach Class. Quantum Grav. 14 2673
[14] Ellis G F R and van Elst H 1999 Cosmological models Theoretical and Observational Cosmology (Cargèse Lectures 1998) ed M Lachièze-Rey (Dordrecht: Kluwer) p 1 (Preprint gr-qc/9812046)
[15] Garfinkle D and Gundlach C 2005 Well-posedness of the scale-invariant tetrad formulation of the vacuum Einstein equations Preprint gr-qc/0501031
[16] Lim W C, van Elst H, Uggla C and Wainwright J 2004 Asymptotic isotropization in inhomogeneous cosmology Phys. Rev. D 69 103507
[17] Friedrich H 1998 Evolution equations for gravitating ideal fluid bodies in general relativity Phys. Rev. D 57 2317
[18] van Elst H and Ellis G F R 1999 Causal propagation of geometrical fields in relativistic cosmology Phys. Rev. D 59 024013
[19] van Elst H, Ellis G F R and Schmidt B G 2000 On the propagation of jump discontinuities in relativistic cosmology Phys. Rev. D 62 104023
[20] Belinski V A, Khalatnikov I M and Lifshitz E M 1982 A general solution of the Einstein equations with a time singularity Adv. Phys. 31 639
[21] Wald R M and Yip P 1981 On the existence of simultaneous synchronous coordinates in spacetimes with spacelike singularities J. Math. Phys. 22 2659
[22] Goode S W and Wainwright J 1985 Isotropic singularities in cosmological models Class. Quantum Grav. 2 99
[23] Anguige K and Tod K P 1999 Isotropic cosmological singularities I: polytropic perfect fluid spacetimes Ann. Phys., NY 276 257
[24] Wainwright J, Hancock M and Uggla C 1999 Asymptotic self-similarity breaking at late times in cosmology Class. Quantum Grav. 16 2577
[25] Heinzel J M, Röhr N and Uggla C 2003 Dynamical systems approach to relativistic spherically symmetric static perfect fluid models Class. Quantum Grav. 20 4567