Non-perturbative states in the three-dimensional $\phi^4$ theory

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We study the spectrum of massive excitations of the three-dimensional $\phi^4$ and Ising models, in the broken-symmetry phase. Using a variational method, we show that the spectrum contains all the $0^+$ states that one expects from duality with the glueball spectrum of the $\mathbb{Z}_2$ gauge model. From the point of view of continuum $\phi^4$ theory, we show that at least one of the states we find has a non-perturbative origin.

1. DUALITY AND UNIVERSALITY

The three-dimensional Ising model is related to two other important three-dimensional theories: by duality, to the $\mathbb{Z}_2$ gauge model, and by universality, to $\phi^4$ theory.

Duality is an exact equality of partition functions: it means that the Ising model and the $\mathbb{Z}_2$ gauge model are different descriptions of the same physics. In particular, the broken symmetry phase of the Ising model is equivalent to the confined phase of the gauge model.

Universality tells us that the Ising model and $\phi^4$ theory behave in the same way when approaching the critical point. Universal quantities are the same in a critical region around the transition point. Indeed, the universal quantities of the Ising universality class have been predicted to great accuracy using perturbative $\phi^4$ theory (see e.g. Ref. [1]).

Let us apply the tools of universality and duality to the problem of determining the spectrum of massive excitations of the $3D$ Ising model in the broken symmetry phase. The glueball spectrum in the $\mathbb{Z}_2$ gauge model has been thoroughly studied numerically: it turns out to be a rich spectrum with many excitations in various angular momentum channels. On the other hand, we certainly do not expect to find an interesting spectrum in perturbative $\phi^4$ theory, which describes just one particle.

Therefore it seems that duality and universality lead to contradictory expectations about the spectrum of the Ising model and $\phi^4$ theory in the broken phase. This work clarifies these issues by a numerical evaluation of the spectrum of both models, performed with a new variational procedure.

2. MONTE CARLO DETERMINATION OF THE SPECTRUM

The spectrum of a model is extracted from Monte Carlo simulations by studying the long distance exponential decay of correlation functions. It is convenient to study time slice observables: for example if $\phi$ is the order parameter one defines

$$S(t) = \frac{1}{L^2} \sum_{x,y} \phi(t, x, y)$$

where $L$ is the lattice size in the $x$ and $y$ directions. One then studies connected correlators of the $S$ operator, the advantage being that they behave as a pure exponential in the long distance...
The crucial point is of course the choice of the operator basis, which must be carefully fine-tuned to obtain an efficient determination of the spectrum. Our choice is described in detail in Ref. [3]. Here we just mention that we included the standard magnetization Eq. (1) in the basis \{\hat{O}_t\} and that the other operators are defined on different length scales.

We simulated both the Ising model and the lattice regularized \(\phi^4\) theory in the broken symmetry phase, at various temperatures well inside the scaling region where universality is expected to hold. We considered the \(0^+\) channel only. It turns out that three states can be identified in this channel. \textit{Scaling} is perfectly satisfied since the ratios between the masses of the three states do not change with the temperature within the scaling region. \textit{Universality} is satisfied as well since we obtain compatible results for the ratios from the Ising and the \(\phi^4\) simulations. Therefore we can quote a single result for each mass ratio:

\[
\frac{m_2}{m_1} = 1.83(3) \quad (5)
\]  
\[
\frac{m_3}{m_1} = 2.45(10) \quad (6)
\]

Also \textit{duality} is satisfied since in the \(\mathbb{Z}_2\) gauge model one obtains mass ratios of 1.88(2) and 2.59(4) in the \(0^+\) channel of the glueball spectrum [4].

Note that the first excited state lies below the pair production threshold: this means that, in terms of continuum \(\phi^4\) theory, it cannot be of perturbative origin. On the other hand the state at 2.45 times the fundamental mass could well be a signature of the cut in the Fourier transform of the propagator induced by self interaction effects, and therefore a perturbative effect.

Let us analyze in more detail what we expect from perturbative \(\phi^4\) theory in this respect: a simple one-loop computation (see Ref. [5]) shows that the cut in the momentum space propagator implies for the time slice correlators the behavior

\[
\langle S(0)S(t) \rangle_c \sim c_1 e^{-m_1|t|} + c_2 e^{-m_2|t|} + \ldots
\]  

However the second term can be shown to be numerically indistinguishable from an exponential decay with mass \(\sim 2.4 m_1\). Therefore we conclude that while the state \(m_3\) could be explained as a perturbative self-interaction effect, the state \(m_2\) is certainly of non-perturbative origin.

3. BACK TO THE SPIN-SPIN CORRELATOR

In this section we apply the knowledge we have gained of the spectrum of the theory to the analysis of the (time slice) spin-spin correlator

\[
G(t) = \langle S(0)S(t) \rangle_c
\]  

It is useful to define the effective correlation length

\[
1/\xi_{eff}(t) = \log G(t) - \log G(t+1)
\]  

so that for \(t \to \infty \xi_{eff} \to 1/m\), \(m\) being the fundamental mass. Clearly if \(G(t)\) had a purely exponential behavior then \(\xi_{eff}(t)\) would be constant: the presasymptotic behavior of \(\xi_{eff}(t)\) depends on higher mass states and/or interaction effects. These effects are shown in Fig.1, where \(m \xi_{eff}\) is shown for various values of the temperature in the Ising model. The data from the various temperatures are perfectly compatible with each other, signaling that the presasymptotic behavior of \(\xi_{eff}(t)\) is a physical, scaling effect and not a lattice artifact.
In Fig. 2 we use our knowledge of the spectrum to describe the behavior of $\xi_{eff}(t)$: the Monte Carlo are compared to the perturbative prediction

$$\xi_{eff}(t) = c_1 \left[ e^{-m_1 t} + f_{cut}(t) \right]$$

(10) (dotted line), and to the curve

$$\xi_{eff}(t) = c_1 \left[ e^{-m_1 t} + f_{cut}(t) \right] + c_2 e^{-m_2 t}$$

(11) (solid line) where the constants $m_1$, $m_2$, $c_1$, $c_2$ are taken from our variational evaluation of the spectrum, and $f_{cut}$ is taken from one-loop perturbative calculations in the continuum theory (see Ref. [3]). The good agreement between this curve and the data suggests that in fact the third mass $m_3$ is not a new state but a perturbative interaction effect.

4. CONCLUSIONS

The main result of our analysis is that 3D $\phi^4$ theory has a rich spectrum of massive excitations that signals the existence of non-perturbative physics. This spectrum matches accurately the corresponding spectrum of the 3D Ising model, to which $\phi^4$ is related by universality, and the glueball spectrum of the 3D $\mathbb{Z}_2$ gauge model, related by duality to the Ising model.

We are currently investigating higher spin excited states, corresponding to higher spin glueballs, and the effect of non-perturbative physics on the field theoretic prediction of universal quantities. Another interesting development would be to investigate the same issues in other 3D universality classes, in particular in $N$-component $\phi^4$ theory.

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